ABSTRACT. We study the $L^1 - L^\infty$ dispersive estimate of the inhomogeneous fourth-order Schrödinger operator $H = \Delta^2 - \Delta + V(x)$ with zero energy obstructions in $\mathbb{R}^3$. For the related propagator $e^{-itH}$, we prove that for $0 < t \leq 1$, then $e^{-itH} P_{ac}(H)$ satisfies the $|t|^{-3/4}$-estimate. For $t > 1$, we prove that: 1) if zero is a regular point of $H$, then $e^{-itH} P_{ac}(H)$ satisfies the $|t|^{-3/2}$-dispersive estimate. 2) if zero is a resonance of $H$, there exists a time dependent operator $F_t$ such that $e^{-itH} P_{ac}(H) - F_t$ satisfies the $|t|^{-3/2}$-dispersive estimate. 3) if zero is a resonance and/or an eigenvalue of $H$, then there exists a time dependent operator $G_t$ such that $e^{-itH} P_{ac}(H) - G_t$ satisfies the $|t|^{-3/2}$-dispersive estimate. Here $F_t$ and $G_t$ satisfy $|t|^{-1/2}$-dispersive estimates.

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1. Introduction

In this paper we consider the $L^1 - L^\infty$ dispersive estimates of the inhomogeneous fourth-order Schrödinger operator

$$H = \Delta^2 - \Delta + V(x), \quad H_0 = \Delta^2 - \Delta$$

in $L^2(\mathbb{R}^3)$, where $V(x)$ is a real-valued function satisfies $|V(x)| \leq (1 + |x|)^{-\beta}$ with some $\beta > 0$.

It was known that the fourth-order Schrödinger equation was introduced by Karpman [Kar94, Kar96] and Karpman and Shagalov [KS00] to take into account the role of small fourth-order
dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. These equations are defined as follows

\[ i \partial_t u + \Delta u + \varepsilon \Delta^2 u + |u|^{2p} u = 0, \quad u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}, \quad \varepsilon \in \mathbb{R}. \quad (1.1) \]

When \( \varepsilon = 0, d = 2 \) and \( p = 1 \), this corresponds to the canonical model. When \( 2 < dp < 4 \), Karpman and Shagalov [KS00] showed, among other things, that the waveguides induced by the nonlinear Schrödinger equation become stable when \( |\varepsilon| \) is taken sufficiently large. Then equation (1.2) is predominantly governed by the corresponding homogeneous fourth-order equation

\[ i \partial_t u + \varepsilon \Delta^2 u + |u|^{2p} u = 0. \quad (1.2) \]

The homogeneous fourth-order Schrödinger equation (1.2) has been widely investigated in [MZ16, MXZ11, PS10, MXZ09, FIP02]. The inhomogeneous fourth-order equation (1.1) has been recently investigated by [RWZ16, Seg15, PX13, Seg11, JPS10] and references therein.

In this paper, we are interested in the inhomogeneous fourth-order operator \( H = \Delta^2 - \Delta + V(x) \) which is originally from nonlinear problem (1.1). For the homogeneous fourth-order operator \( \Delta^2 + V \), linear dispersive estimates have recently been studied in [EGT19, GT19, FSY18, FWY18]. In [FSY18, FWY18], they obtained the Kato-Jensen decay estimates of \( e^{it(\Delta^2 + V)} \) with presence of zero resonance or zero eigenvalue in \( \mathbb{R}^d \) with \( d \geq 5 \). In [EGT19, GT19], they established the \( L^1 - L^\infty \) dispersive estimates of \( e^{it(\Delta^2 + V)} \) with presence of zero resonance or zero eigenvalue in three and four dimensions respectively. However, for \( d = 1, 2 \) and \( d \geq 5 \), the \( L^1 - L^\infty \) dispersive estimates of \( e^{it(\Delta^2 + V)} \) is still open up to my knowledge. The \( L^1 - L^\infty \) dispersive estimates of the inhomogeneous fourth-order operators \( \Delta^2 \pm \Delta + V \) with nonvanish potential has very few results so far. In this sequel, we will restrict our attention to the \( L^1 - L^\infty \) dispersive estimates of \( H = \Delta^2 - \Delta + V \) in 3-dimension with zero energy obstructions. In the coming paper, we will study the \( L^1 - L^\infty \) dispersive estimates for the inhomogeneous operator \( \Delta^2 + \Delta + V \). The operator \( \Delta^2 + \Delta + V \) is more complicated than \( H = \Delta^2 - \Delta + V \). It has two finite thresholds \(-1/4\) and \(0\) while \( H = \Delta^2 - \Delta + V \) has only zero threshold.

For free propagators \( e^{-it(\Delta^2 + \varepsilon \Delta)} \) with \( \varepsilon \in \{-1, 0, 1\} \), Ben-Artzi, Koch and Saut in [BAKS00] established the pointwise estimates for kernels of \( e^{-it(\Delta^2 + \varepsilon \Delta)} \). Let \( I_1(t, x) \) be the kernel of \( e^{-it(\Delta^2 - \Delta)} \). If \( 0 < t \leq 1 \) or \(|x| \geq t\), then

\[ |D^\alpha I_1(t, x)| \leq c t^{-(d+|\alpha|)/4} \left( 1 + t^{-1/4} |x| \right)^{(|\alpha| - d)/3}, \quad x \in \mathbb{R}^d. \]

If \(|x| \leq t \) and \( t \geq 1 \), then

\[ |D^\alpha I_1(t, x)| \leq c t^{-(d+|\alpha|)/2} \left( 1 + t^{-1/2} |x| \right)^{|\alpha|}, \quad x \in \mathbb{R}^d. \]

The above pointwise kernel estimates imply the \( L^1 - L^\infty \) dispersive estimates of \( e^{-itH_0} \):

- For \( 0 < t \leq 1 \),
  \[ \|e^{-itH_0}\|_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \leq c |t|^{-d/4}. \quad (1.3) \]

- For \( t > 1 \),
  \[ \|e^{-itH_0}\|_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \leq c |t|^{-d/2}. \quad (1.4) \]

Note that the \( L^1 - L^\infty \) estimate of free propagator \( e^{-it(\Delta^2 - \Delta)} \) has different time bound for \( t > 1 \) and \( 0 < t \leq 1 \). In contrast to the homogeneous operators, the \( L^1 - L^\infty \) estimate of Laplacian \(-\Delta\) and bi-harmonic operator \( \Delta^2 \), the time bounds of small time \( 0 < t \leq 1 \) and large time \( t > 1 \) are the same while the inhomogeneous operator \( H_0 = \Delta^2 - \Delta \) is not.
For Schrödinger type operator $H = \Delta^2 - \Delta + V$ with real-valued, polynomial decaying potential $V(x)$, we will establish the $L^1 - L^{\infty}$ dispersive estimate in $\mathbb{R}^3$ for the perturbed propagator $e^{-ith}$ with the presence of zero energy obstructions, i.e. the distributional solutions to $H\psi = 0$ with $\psi \in L^2_{-1/2}(\mathbb{R}^3)$. We provide a full classification of the zero energy obstructions: zero is a regular point, zero is a resonance and zero is an eigenvalue of $H$ with the presence of zero energy obstructions, i.e. the distributional solutions to $e^{iH} = 0$ with $\psi \in L^2(\mathbb{R}^3)$. There exists distributional solutions $\psi \in L^2(\mathbb{R}^3)$ such that $H\psi = 0$ and $\psi$ is a regular point of $H$, see Lemma [3.3, 3.7] below. Following the terminology of Kato in [JN79], we say zero is a regular point of $H$ means zero is not an eigenvalue nor a resonance of $H$. For $H = \Delta^2 - \Delta + V$ in $\mathbb{R}^3$, we say zero is a resonance of $H$ if there exists distributional solutions $\psi \in L^2_{-1/2}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)$ such that $H\psi = 0$. Here $L^2_{-1/2}(\mathbb{R}^3) = \cap_{\nu < -\frac{1}{2}} L^2_\nu(\mathbb{R}^3)$ which contains $L^2(\mathbb{R}^3)$ as a true subset. Actually, the property of $H = \Delta^2 - \Delta + V$ at zero energy behaves similar to Schrödinger operator $-\Delta + V$. Recall the classification of zero energy obstructions for $-\Delta + V$ (under suitable assumptions on $V$) in $\mathbb{R}^3$, Kato in [JN79] also discussed zero energy as regular point, resonance and eigenvalue of $-\Delta + V$. Furthermore, the definition of zero resonance for $H$ matches Kato’s zero resonance definition of $-\Delta + V$ in [JN79].

Before comparing $H = \Delta^2 - \Delta + V$ with $-\Delta + V$ and $\Delta^2 + V$, it is necessary to state our main theorem: the $L^1 - L^{\infty}$ dispersive estimates of $e^{-ith}$ with zero energy obstructions. Let $P_{ac}(H)$ denotes the projection onto the absolutely continuous spectrum space of $H$.

**Theorem 1.1.** Considering $H = \Delta^2 - \Delta + V$ with real-valued potential satisfying $|V(x)| \leq (1 + |x|)^{-\beta}$ for some $\beta > 7$. Assume that $H$ has no positive embedded eigenvalues.

For $0 < t \leq 1$, we have

$$\|e^{-ith}P_{ac}(H)u\|_{L^\infty(\mathbb{R}^3)} \lesssim |t|^{-3/4} \|u\|_{L^1(\mathbb{R}^3)}.$$  

For $1 < t \in \mathbb{R}$, then we have:

(1) If $0$ is a regular point of $H$, then

$$\|e^{-ith}P_{ac}(H)u\|_{L^\infty(\mathbb{R}^3)} \lesssim |t|^{-3/2} \|u\|_{L^1(\mathbb{R}^3)}.$$  

(2) If $0$ is a resonance of $H$, then

$$\|e^{-ith}P_{ac}(H)u - F_t u\|_{L^\infty(\mathbb{R}^3)} \lesssim |t|^{-3/2} \|u\|_{L^1(\mathbb{R}^3)},$$

where $F_t$ is a time dependent operator which satisfies $\|F_t\|_{L^1 \to L^\infty} \leq |t|^{-1/2}$.

(3) If $0$ is a resonance and/or an eigenvalue of $H$, then

$$\|e^{-ith}P_{ac}(H)u - G_t u\|_{L^\infty(\mathbb{R}^3)} \lesssim |t|^{-3/2} \|u\|_{L^1(\mathbb{R}^3)},$$

where $G_t$ is a time dependent operator which satisfies $\|G_t\|_{L^1 \to L^\infty} \leq |t|^{-1/2}$.

**Remark 1.2.** The presence of zero energy obstruction of $H = \Delta^2 - \Delta + V$ has no effect on the $L^1 - L^{\infty}$ estimate for $0 < t \leq 1$. Indeed, for $0 < t \leq 1$, we obtain the natural $|t|^{-3/4}$-time bound for $H = \Delta^2 - \Delta + V$ whether zero is regular or not. This can be confirmed by the dispersive estimate (1.3) for free propagator $e^{-ith_0}$.

**Remark 1.3.** It is known that positive embedded eigenvalues may exist for $H = \Delta^2 - \Delta + V$ even with compactly supported smooth potential. Note that $\Delta(r^{-1}e^{-br}) = b^2r^{-1}e^{-br}$ for $r = |x|$ and any $b > 0$. The absence of embedded eigenvalue is a common standing assumption in dispersive equation papers. Furthermore, we also notice that for a general selfadjoint operator $\mathcal{H}$ on $L^2(\mathbb{R}^d)$, if $\mathcal{H}$ has a simple embedded eigenvalue $\lambda_0$. Costin and Soffer in [CS01] have proved that $\mathcal{H} + \epsilon W$ can kick off the eigenvalue in a small interval around $\lambda_0$ under certain small perturbation of potential.
The result of $L^1-L^{\infty}$ dispersive estimates for Schrödinger operator $-\Delta + V$ is quite rich and thoroughly. The fundamental work is [ISS91]. Journé, Soffer and Sogge first proved the Schrödinger propagator $e^{-it(-\Delta + V)}$ satisfies
\[ \|e^{-it(-\Delta + V)}P_{ac}(-\Delta + V)\|_{L^1(\mathbb{R}^d) \to L^{\infty}(\mathbb{R}^d)} \leq |t|^{-d/2}, \quad d \geq 3, \]
with zero is a regular point by using commutator methods. Later, Rodnianski and Schlag in [RS04] proved the bound (1.5) for $-\Delta + V$ with rough and time-dependent potentials in 3-dimension by making full use of the kernel of the free propagator $e^{it\Delta}$ and the uniformly Sobolev estimates. Goldberg and Schlag in [GS04] proved (1.5) for $d = 1, 3$ in the regular case by using resolvent methods. In [ES04], Erdoğan and Schlag established the $L^1-L^{\infty}$ dispersive estimates of $-\Delta + V$ with zero energy obstructions in 3-dimension. For convenience of comparison, here we list their result briefly. Under suitable decaying assumption on $|V(x)|$ (see [ES04, Theorem 1.1, 1.2]):

- If there is a resonance at energy zero but zero is not an eigenvalue of $-\Delta + V$, then
  \[ \|e^{-it(-\Delta + V)}P_{ac}(-\Delta + V) - F_i^1\|_{L^1(\mathbb{R}^d) \to L^{\infty}(\mathbb{R}^d)} \leq |t|^{-3/2}. \]
- If there is a resonance at energy zero and/or zero is an eigenvalue of $-\Delta + V$, then
  \[ \|e^{-it(-\Delta + V)}P_{ac}(-\Delta + V) - F_i^2\|_{L^1(\mathbb{R}^d) \to L^{\infty}(\mathbb{R}^d)} \leq |t|^{-3/2}. \]

Here $F_i^1, F_i^2$ are time-dependent, finite rank operators satisfying $\|F_i\|_{L^1 \to L^{\infty}} \leq |t|^{-1/2}, \quad j = 1, 2$. It is worth pointing out that Goldberg in [Gol06] proved estimate (1.5) when $d = 3$ for $-\Delta + V$ with almost critical potential. Comparing with the dispersive estimate of $H = \Delta^2 - \Delta + V$ in Theorem [1.1] one can conclude that $H = \Delta^2 - \Delta + V$ behaves similar to Schrödinger operator $-\Delta + V$ in the sense of $L^1-L^{\infty}$ dispersive estimates for large time $t > 1$. More results about the dispersive estimates for the Schrödinger operator $-\Delta + V$, see [GG17, GG15, EG13, CCV11, Sch07, ES06, GV06, Sch05, Yaj05, Yaj95] and references therein.

Recently, Erdoğan, Green and Toprak in [EGT19] established the $L^1-L^{\infty}$ dispersive estimates for $\Delta^2 + V$ with zero energy obstruction in 3-dimension. In order to compare $H = \Delta^2 - \Delta + V$ with $\Delta^2 + V$, we state the results in [EGT19] briefly. In [EGT19], under the assumptions that $\Delta^2 + V$ has no positive embedded eigenvalue and suitable decaying assumption on $V(x)$, they proved that (see [EGT19, Theorem 1.1])

- For $0 < t \leq 1$, then
  \[ \|e^{-it(\Delta^2 + V)}P_{ac}(\Delta^2 + V)\|_{L^1(\mathbb{R}^d) \to L^{\infty}(\mathbb{R}^d)} \leq |t|^{-3/4}. \]
- For $t > 1$, if zero is regular or there is a first kind resonance at zero of $\Delta^2 + V$, then
  \[ \|e^{-it(\Delta^2 + V)}P_{ac}(\Delta^2 + V)\|_{L^1(\mathbb{R}^d) \to L^{\infty}(\mathbb{R}^d)} \leq |t|^{-3/4}. \]
- For $t > 1$, if there is a second kind or third kind resonance at zero of $\Delta^2 + V$, then there exist time dependent operators $F_i^3$ and $F_i^4$, respectively, such that
  \[ \|e^{-it(\Delta^2 + V)}P_{ac}(\Delta^2 + V) - F_i^j\|_{L^1(\mathbb{R}^d) \to L^{\infty}(\mathbb{R}^d)} \leq |t|^{-3/4}, \quad j = 3, 4, \]
  \[ \text{where } F_i^3, F_i^4 \text{ satisfy } \|F_i^j\|_{L^1 \to L^{\infty}} \leq |t|^{-1/4}, \quad j = 3, 4. \]

Notice that the third kind resonance at zero of $\Delta^2 + V$ is actually zero eigenvalue according to the definition and classification of threshold subspaces in [EGT19].

Comparing to our main Theorem [1.1] for $0 < t \leq 1$, $H = \Delta^2 - \Delta + V$ and $\Delta^2 + V$ satisfy the same $|t|^{-3/4}$-time bound of the $L^1-L^{\infty}$ dispersive estimate. However when $t > 1$ and both in the
regular case, we obtain \(|t|^{-3/2}\)-time decay for \(H = \Delta^2 - \Delta + V\) while \(\Delta^2 + V\) satisfies \(|t|^{-3/4}\)-time decay. Further, when \(t > 1\) and both in the zero eigenvalue case, we obtain \(|t|^{-1/2}\)-time decay for \(H = \Delta^2 - \Delta + V\) while \(\Delta^2 + V\) satisfies \(|t|^{-1/4}\)-time decay. In addition, the \(L^1 - L^\infty\) dispersive estimates of \(\Delta^2 + V\) in 4-dimension, please see [GT19]. For the remaining dimensional cases, up to my knowledge, it is still open so far.

In this paper, we study the \(L^1 - L^\infty\) dispersive estimates for the inhomogeneous operator \(H = \Delta^2 - \Delta + V\) which can be used to study asymptotic stability of solitons and other related problems for nonlinear dispersive equations. As usual, we use spectrum representation theorem to write

\[
e^{-itH}P_{ac}(H) = \frac{1}{2\pi i} \int_0^\infty e^{-it\lambda} [R_\lambda^+(\lambda) - R_\lambda^-(\lambda)] d\lambda,
\]

Here, the difference of the perturbed resolvents provides the spectral measure by Stone’s formula. Let \(\lambda \in \mathbb{R}^+\), we define the limiting resolvent operators by

\[
R_0^0(\lambda) := R_0(\lambda \pm i0) = \lim_{\epsilon \downarrow 0} \left( H_0 - (\lambda \pm i\epsilon) \right)^{-1};
\]

\[
R_\lambda^0(\lambda) := R_V(\lambda \pm i0) = \lim_{\epsilon \downarrow 0} \left( H - (\lambda \pm i\epsilon) \right)^{-1}.
\]

For free resolvent \(R_0(z) := (H_0 - z)^{-1}\), we have the following splitting identity:

\[
R_0(z) = \frac{1}{2\sqrt{1/4 + z}} \left[ (-\Delta + \frac{1}{2} - \sqrt{1/4 + z})^{-1} - (-\Delta + \frac{1}{2} + \sqrt{1/4 + z})^{-1} \right]. \tag{1.6}
\]

Here we denote \(R_\Lambda\) be the Schrödinger resolvent \(R_\Lambda(\sqrt{1/4 + z} - \frac{1}{2}) = (-\Delta + \frac{1}{2} - \sqrt{1/4 + z})^{-1}\). Since \(H_0 = \Delta^2 - \Delta\) is essentially self-adjoint and \(\sigma_{ac}(H_0) = [0, \infty)\), by Weyl’s criterion \(\sigma_{ess}(H) = [0, \infty)\) for a sufficiently decaying potential. Note that, by basic calculation, for \(z \in \mathbb{C} \setminus [0, \infty)\) with \(0 < \arg(z) < 2\pi\) we have \(\text{Im}(\sqrt{1/4 + z} - 1/2)^{1/2} > 0\). Using identity (1.6), for \(\lambda > 0\) we have

\[
R_0^\pm(\lambda) = \frac{1}{2\sqrt{1/4 + \lambda}} \left[ R_\Lambda^\pm(\sqrt{1/4 + \lambda} - \frac{1}{2}) - R_\Lambda(\frac{1}{2} - \sqrt{1/4 + \lambda}) \right]. \tag{1.7}
\]

Note that \(R_\Lambda(-\frac{1}{2} - \sqrt{1/4 + \lambda}) \in B(L^2, L^2)\) since \(-\Delta\) has nonnegative spectrum. Further, by Agmon’s limiting absorption principle, see [Agm75], \(R_\Lambda^\pm(\sqrt{1/4 + \lambda} - \frac{1}{2})\) is well-defined between weighted \(L^2\) spaces. Therefore, \(R_\lambda^0(\lambda)\) is also well-defined in weighted \(L^2\) spaces. For the perturbed resolvent \(R_\lambda^0(\lambda)\), we will extend this property to \(R_\lambda^0(\lambda)\) in Section 5.

In the literature, in order to obtain the asymptotic properties of the spectral measure of \(H = \Delta^2 - \Delta + V\) when \(\lambda\) close to zero and \(\infty\), we need to derive the low energy asymptotic expansion of \(R_\lambda^0(\lambda)\) when \(\lambda \to 0\) and the high energy decay estimate for \(R_\lambda^0(\lambda)\) when \(\lambda \to \infty\). For free resolvent \(R_0^0(\lambda)\), we apply identity (1.7) to get the low energy asymptotic expansion and high energy decay estimate by using the known results of Laplacian (see e.g. [JK79]). For Schrödinger operator \(-\Delta + V\), Kato, Jensen and Nenciu’s series work [JK79, Jen80, Jen84, JN01, JN04] has established quite a standard approach to obtain the low energy asymptotic expansion and high energy decay estimates. For general operator \(P(D) + V\), Murata in [Mur82] already gave the asymptotic expansion at threshold point for a class of \(P(D) + V\) which includes \(H = \Delta^2 - \Delta + V\). Here we do not follow Murata’s result in [Mur82] since Jensen and Nenciu’s processes in [JN01] is more directly and succinctly.
For perturbed resolvent $R^\pm_0(\lambda)$, we apply the symmetric resolvent identity:

$$R^\pm_0(\lambda) = R^\pm_0(\lambda) - R^\pm_0(\lambda)v\left(M^\pm(\lambda)\right)^{-1}vR^\pm_0(\lambda)$$

where $M^\pm(\lambda) = U + vR^\pm_0(\lambda)v$, $v(x) = |V(x)|^{1/2}$ and

$$U(x) = \begin{cases} 1, & V(x) \geq 0; \\ -1, & V(x) < 0. \end{cases}$$

Since in the low frequency portion of $H = \Delta^2 - \Delta + V$, the major operator is $-\Delta + V$. Thus Jensen and Nenciu’s approach should work. Hence making use of the approach established in [JN01], we obtain the low energy asymptotic expansion of $R^\pm_0(\lambda)$ with the presence of zero energy resonance or zero eigenvalue of $H = \Delta^2 - \Delta + V$. Further, we identified the zero energy resonance space.

The paper is organized as follows. In Section 2, we show asymptotic expansion for the free resolvent and establish the natural dispersive bound for the free propagator. In Section 3, we derive expansions for the perturbed resolvent around the threshold with the presence of zero energy obstruction and identified the zero energy resonance space. In Section 4, we utilize these expansions to prove the low energy part dispersive estimates in Theorem 1.1. In the last section, we develop the high energy decay estimates and the limiting absorption principle for the perturbed resolvent. Then apply the high energy decay estimates to prove the high energy part dispersive estimates in Theorem 1.1.

2. The free evolution

In this section we obtain expansions for the free resolvent operators $R^\pm_0(\lambda)$ using identity (1.6) and the representation of the free Schrödinger resolvent. Using resolvent expansions, we establish dispersive estimates for the free evolution $e^{-itH_0}$.

Before deducing the expansions, we first introduce some notations for the convenience reading. We define the weighted $L^2$ spaces:

$$L^2_s(\mathbb{R}^3) := \left\{ f : (1 + |\cdot|^2)^{s}f \in L^2(\mathbb{R}^3) \right\}, \ s \in \mathbb{R}.$$  

For any $s, s' \in \mathbb{R}$, $B(s, s')$ denotes the family of bounded linear operators from $L^2_s(\mathbb{R}^3)$ to $L^2_{s'}(\mathbb{R}^3)$. For an operator $\mathcal{E}(\lambda)$, we write $\mathcal{E}(\lambda) = O_1(\lambda^{-a})$ if its kernel $\mathcal{E}(\lambda; x, y)$ satisfies:

$$\sup_{x, y \in \mathbb{R}^3, t \geq 0} \left| t^a |E(\lambda; x, y)| + \lambda^{a+1} |\partial_x \mathcal{E}(\lambda; x, y)| \right| < \infty.$$  

Similarly, we use the notation $\mathcal{E}(\lambda) = O_1(\lambda^{-a}g(x, y))$ if $\mathcal{E}(\lambda; x, y)$ satisfies:

$$|\mathcal{E}(\lambda; x, y)| + \lambda |\partial_x \mathcal{E}(\lambda; x, y)| \lesssim \lambda^a g(x, y).$$

Next, we show the process of deducing the asymptotic expansions. Recall the expression of the free Schrödinger resolvents in 3-dimension (see e.g. [JK79]),

$$R^\pm_0(\eta^2; x, y) = \frac{e^{\pm i|\eta||x-y|}}{4\pi|x-y|}, \quad (2.1)$$

where $\eta = \left( \sqrt{1/4 + \lambda} - \frac{1}{2} \right)^{1/2}$. Therefore, by (1.7),

$$R^\pm_0(\lambda; x, y) = \frac{1}{1 + 2\eta^2} \left( \frac{e^{\pm i|\eta||x-y|}}{4\pi|x-y|} - \frac{e^{-\sqrt{1+\eta^2}|x-y|}}{4\pi|x-y|} \right). \quad (2.2)$$
**Proposition 2.1.** For the free evolution $e^{-itH_0}$, denotes $\mathcal{R}_0(\eta; x, y) = R_0^+(\lambda; x, y) - R_0^-(\lambda; x, y)$. Then we have the following uniformly bounds:

For $1 < t \in \mathbb{R}$, we have

$$
\sup_{x, y \in \mathbb{R}^3} \left| \int_0^\infty e^{-it(\eta^4 + \eta^2)} \mathcal{R}_0(\eta; x, y) (4\eta^3 + 2\eta) \, d\eta \right| \lesssim |t|^{-3/2}. \tag{2.3}
$$

For $0 < t \leq 1$, we have

$$
\sup_{x, y \in \mathbb{R}^3} \left| \int_0^\infty e^{-it(\eta^4 + \eta^2)} \mathcal{R}_0(\eta; x, y) (4\eta^3 + 2\eta) \, d\eta \right| \lesssim |t|^{-3/4}. \tag{2.4}
$$

**Proof.** Note that, for the difference of the free resolvents we have

$$
|\mathcal{R}_0(\eta; x, y)| = \frac{\eta}{1 + 2\eta^2} \left| \frac{e^{\eta|y-x|} - 1 + 1 - e^{-\eta|y-x|}}{4\pi \eta |x-y|} \right| \lesssim \frac{\eta}{1 + 2\eta^2}, \tag{2.5}
$$

uniformly in $x, y$ by the mean value theorem. Further,

$$
\left| \frac{d}{d\eta} \mathcal{R}_0(\eta; x, y) \right| \leq \frac{1}{1 + 2\eta^2} \left| \frac{e^{\eta|y-x|} + e^{-\eta|y-x|}}{4\pi} \right| + \frac{4\eta}{1 + 2\eta^2} \left| \frac{e^{\eta|y-x|} - e^{-\eta|y-x|}}{4\pi |x-y|} \right| \lesssim \frac{1}{1 + 2\eta^2}. \tag{2.6}
$$

For large time $t > 1$ case, we have

$$
\left| \int_0^\infty e^{-it(\eta^4 + \eta^2)} \mathcal{R}_0(\eta; x, y) (4\eta^3 + 2\eta) \, d\eta \right| \\
\leq \frac{1}{|t|} \left| \int_0^\infty e^{-it(\eta^4 + \eta^2)} \mathcal{R}_0(\eta; x, y) \, d\eta \right| + \frac{1}{|t|} \left| \int_0^\infty e^{-it(\eta^4 + \eta^2)} \frac{d}{d\eta} \mathcal{R}_0(\eta; x, y) \, d\eta \right| \\
\leq \frac{1}{|t|} \left| \int_0^\infty e^{-it(\eta^4 + \eta^2)} \frac{e^{\eta|y-x|} + e^{-\eta|y-x|}}{1 + 2\eta^2} \, d\eta \right| + \frac{1}{|t|} \int_0^\infty \left| \frac{e^{\eta|y-x|} - e^{-\eta|y-x|}}{1 + 2\eta^2} \right| \frac{4\eta^2}{4\pi |x-y|} \, d\eta |t|^{-3/2}, \ t > 1.
$$

In the last inequality we use the van der Corput’s lemma, see e.g. [Ste93].

For small time $0 < t \leq 1$, we have

$$
\int_0^\infty e^{-it(\eta^4 + \eta^2)} \mathcal{R}_0(\eta; x, y) (4\eta^3 + 2\eta) \, d\eta = I_1(t; x, y) + I_2(t; x, y)
$$

where

$$
I_1(t; x, y) = \int_0^{1/4} e^{-it(\eta^4 + \eta^2)} \mathcal{R}_0(\eta; x, y) (4\eta^3 + 2\eta) \, d\eta;
$$

$$
I_2(t; x, y) = \int_{1/4}^\infty e^{-it(\eta^4 + \eta^2)} \mathcal{R}_0(\eta; x, y) (4\eta^3 + 2\eta) \, d\eta.
$$

For $I_1(t; x, y)$, we have

$$
\left| I_1(t; x, y) \right| \lesssim \int_0^{1/4} \left| \mathcal{R}_0(\eta; x, y)(4\eta^3 + 2\eta) \right| \, d\eta \lesssim |t|^{-3/4}, \ 0 < t \leq 1.
$$
For $I_2(t; x, y)$, integral by parts, we have
\[
|I_2(t; x, y)| \leq \frac{1}{|t|} \left| e^{-i(t\theta + \eta \gamma)} R_0(\eta; x, y) \right|_{t=1/4}^\infty + \frac{1}{|t|} \int_{1/4}^\infty \left| \frac{d}{d\eta} R_0(\eta; x, y) \right| d\eta \\
\leq \frac{|t|^{-3/4}}{|t|^{1/2} + 2} + \frac{1}{|t|(1 + |t|^{-1/4})} \leq |t|^{-3/4}, \quad 0 < t \leq 1.
\]

3. The asymptotic expansions of resolvent $R_0^\pm(\lambda)$

In this section, we aim to obtain the low energy asymptotic expansion of $R_0^\pm(\lambda)$. For $\lambda$ near the only threshold point zero, we use the symmetric resolvent identity:
\[
R_0^\pm(\lambda) = R_0^\pm(\lambda) - R_0^\pm(\lambda) v(M^\pm(\lambda))^{-1} vR_0^\pm(\lambda)
\]
where $M^\pm(\lambda) = U + vR_0^\pm(\lambda) v$, $v(x) = |V(x)|^{1/2}$ and
\[
U(x) = \begin{cases} 
1, & V(x) \geq 0; \\
-1, & V(x) < 0.
\end{cases}
\]

3.1. Asymptotic expansion of $R_0^\pm(\lambda)$ near $\lambda = 0$. Recall the kernel of $R_0^\pm(\lambda)$:
\[
R_0^\pm(\lambda; x, y) = \frac{1}{1 + 2\eta^2} \left[ \frac{e^{i\eta|x-y|}}{4\pi|x-y|} - \frac{e^{\sqrt{1+\eta^2}|x-y|}}{4\pi|x-y|} \right],
\]
where $\eta = (\sqrt{1/4 + \lambda} - 1/2)^{1/2}$. Expand each term into Taylor series at $\eta = 0$, we have:

**Lemma 3.1.** For $\lambda > 0$ and $\eta = (\sqrt{1/4 + \lambda} - 1/2)^{1/2}$, we have the following formally expansions of $R_0^\pm(\lambda)$ as $\lambda \downarrow 0$:
\[
R_0^\pm(\lambda) = G_0 \pm i\eta G_1 + \eta^2 G_2 \pm i\eta^3 G_3 + \eta^4 G_4 + O(\eta^5|x-y|^4),
\]
where $G_j, j = 0, 1, 2, 3, 4$ are operators given by the following kernels
\[
\begin{align*}
G_0(x, y) &= \frac{1}{4\pi|x-y|} - \frac{e^{-i|x-y|}}{4\pi|x-y|}; \\
G_1(x, y) &= \frac{1}{4\pi}; \\
G_2(x, y) &= \left( \frac{|x-y|}{8\pi} + \frac{e^{-i|x-y|}}{8\pi} \right) - \left( \frac{1}{2\pi|x-y|} - \frac{e^{-i|x-y|}}{2\pi|x-y|} \right); \\
G_3(x, y) &= \frac{1 - e^{-i|x-y|}}{\pi|x-y|} + \frac{|x-y| - e^{-i|x-y|}}{4\pi} + \left( \frac{|x-y|^2}{96\pi} - \frac{1 + |x-y|}{32\pi} e^{-i|x-y|} \right).
\end{align*}
\]
Note that $G_0(x, y)$ is actually the kernel of free resolvent $R_0(0) := (-\Delta)^{-1} - (-\Delta + 1)^{-1}$. Furthermore, $G_0 \in B(s, -s')$ with $s, s' > 1/2$ and $s + s' > 2$. For $j = 1, 2, 3, 4$, $G_j \in B(s, -s')$ with $s, s' > j + 1/2$. The operator with kernel $|x-y|^4$ belongs to $B(s, -s')$ with $s, s' > 11/2$.

**Proof.** Note that $|x-y|^j$ with $j = 0, 1, 2, 3, 4$, one obtain
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |x|)^{-2s'}|x-y|^{2j}(1 + |y|)^{-2s}dxdy < \infty
\]
for $s, s' > j + 3/2$. Since $(4\pi|x-y|)^{-1}$ is the kernel of $(-\Delta)^{-1}$, we know $(-\Delta)^{-1} \in B(s, -s')$ and $(-\Delta)^{-1}$ is compact in $B(s, -s')$ with $s + s' > 2$ by [Jen80, Lemma 2.3].

3.2. **Asymptotic expansion of** \( R^+_\nu(\lambda) \) **near** \( \lambda = 0 \). In order to obtain the asymptotic expansions of \( R^+_\nu(\lambda) \) near zero threshold, we need to derive the expansions for \( (M^+\nu(\lambda))^{-1} \) by identity (3.1). The behavior of \( (M^+\nu(\lambda))^{-1} \) as \( \lambda \to 0 \) depends on the type of resonances at zero energy, see Definition 3.3 below. We get these expansions case by case and establish their contribution to the spectral measure in Stone’s formula. From the expansions of free resolvent \( R^+_0(\lambda) \) in the weighted spaces \( B(s,-s') \), see Lemma 3.1, we have the following expansions for \( M^\pm(\lambda) \).

**Lemma 3.2.** Let \( P = \nu(v,\psi)||V||_{L^2(R^3)}^{-1} \) denote the orthogonal projection onto the span of \( v \). Assume that \( v(x) \leq (1 + |x|)^{-\beta/2} \) with some \( \beta > 9 \). Then for \( 0 < \lambda < 1 \) in \( B(0,0) \), we have

\[
M^\pm(\lambda) = T_0 + i\frac{||V||_{L^2}}{4\pi} \eta P + \eta^2 vG_2v + i\eta^3 vG_3v + O_1(\eta^4 v(x)|x-y|^3v(y))
\]  

(3.4)

where \( T_0 = U + vG_0v \).

**Proof.** From Lemma 3.1 we need only to show \( |G_4(x,y)| \leq |x-y|^3 \). Since

\[
\frac{1 - e^{-|x-y|}}{\pi|x-y|} \leq \begin{cases} 1, & |x-y| < 1; \\ |x-y|^{-1}, & |x-y| \geq 1,
\end{cases}
\]

thus the lemma holds by the representation of \( G_4(x,y) \). \( \square \)

In order to get the asymptotic expansions of \( (M^+\nu(\lambda))^{-1} \) at \( \lambda = 0 \), we deal with zero in three cases: regular (not resonance nor eigenvalue), resonance and eigenvalue.

**Definition 3.3.** i) If \( T_0 = U + vG_0v \) is invertible on \( L^2(R^3) \), we say zero is a regular point of \( H = \Delta^2 - \Delta + V \). ii) If \( T_0 \) is not invertible and \( T_1 = S_1PS_1 \) is invertible on \( S_1L^2(R^3) \), we say zero is a resonance of \( H \). Here \( S_1 \) is the Riesz projection onto \( \ker(T_0) \). iii) If \( T_0 \) is not invertible and \( T_2 = S_2vG_2vS_2 \) is invertible on \( S_2L^2(R^3) \), we say zero is an eigenvalue of \( H \). Here \( S_2 \) is the Riesz projection onto \( \ker(T_1) \).

**Remark 3.4.** i) Note that \( S_2 \leq S_1 \) and \( S_1 \) is of finite rank. Since \( vG_0v \) is a compact operator in \( L^2(R^3) \) by the proof of Lemma 3.1, thus \( T_0 \) is a compact perturbation of \( U \). Hence, the Fredholm alternative theorem guarantees that \( S_1 \) is a finite-rank projection. ii) \( PS_2 = S_2P = 0 \). iii) If \( 0 \neq S_2 = S_1 \), then zero is both eigenvalue and resonance of \( H = \Delta^2 - \Delta + V \). Otherwise, zero is purely an eigenvalue of \( H \) provided \( 0 \neq S_2 < S_1 \). iv) Denotes \( D_0 = (T_0 + S_1)^{-1} \) and \( D_1 = (T_1 + S_2)^{-1} \), then \( S_1D_0 = D_0S_1 = S_1 \) and \( S_2D_1 = D_1S_2 = S_2 \).

For better understanding zero threshold, we proceed to establish the relationship between the spectral subspaces \( S_1L^2(R^3) \), \( S_2L^2(R^3) \) and distributional solutions to \( H\psi = 0 \). For any \( s_0 \in \mathbb{R} \), denotes \( L^{s_0}_{-1/2}(R^3) = \cap_{s < s_0} L^2(R^3) \). Especially, \( L^2(R^3) \subseteq L^{s_0}_{-1/2}(R^3) \).

**Lemma 3.5.** Assume \( v(x) \leq (1 + |x|)^{-s} \) with some \( s > 3/2 \). If \( \phi \in S_1L^2(R^3) \setminus \{0\} \), then \( \phi = Uv\psi \) where \( \psi \in L^2_{-1/2}(R^3) \) satisfies \( H\psi = 0 \) in the distributional sense, and

\[
\psi(x) = -G_0v\phi = -\int_{\mathbb{R}^3} \left( \frac{1}{4\pi|x-y|} - \frac{e^{-|x-y|}}{4\pi|x-y|} \right)v(y)\phi(y)dy.
\]

Conversely, if \( \psi \in L^2_{-1/2}(R^3) \) satisfies \( H\psi = 0 \), then \( \phi = Uv\psi \in S_1L^2(R^3) \).
Lemma 3.6. Assume $v \in \mathcal{L}_2^1$ and $ψ \in \mathcal{L}_2^{1/2}(\mathbb{R}^3)$. Then $ψ \in \mathcal{L}_2^{1/2}(\mathbb{R}^3)$.

Recall that $G_0 = (\Delta - 1)^{-1}(-1 + 1)\psi$. Hence $U \psi + vG_0 \psi = U \psi + vU \psi - v \psi = 0$. 

Lemma 3.7. Assume $v(x) \leq (1 + |x|)^{-s}$ with some $s > 3/2$. If $ψ \in \mathcal{L}_2^1(\mathbb{R}^3)$ satisfies $H \psi = 0$ in the distributional sense. Conversely, if $ψ \in \mathcal{L}_2^1(\mathbb{R}^3)$ satisfies $H \psi = 0$, then $ψ = U \psi + v \psi$. 

Proof. For $φ \in \mathcal{L}_2^1(\mathbb{R}^3)$, then $U \psi + vG_0 \psi = 0$ which implies $ψ = U \psi(G_0 \psi) = U \psi$. Since $ψ = -G_0 \psi$, then

$$|ψ| \leq \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} |φ(y)dy| = (\Delta)^{-1}(ψ).$$

Since $(-\Delta)^{-1} \in \mathcal{L}_2^1 \setminus \{0\}$ for $s, s' > 1/2$, we have $ψ \in \mathcal{L}_2^{1/2}(\mathbb{R}^3)$. 

If $ψ \in \mathcal{L}_2^{1/2}(\mathbb{R}^3)$ satisfies $H \psi = 0$, we show $(U \psi + vG_0 \psi) = 0$. Since $0 = H \psi = (\Delta - 1)\psi + vU \psi$ for any tiny $ε > 0$, we show $(1 + |x|)\psi \in \mathcal{L}_2^{1/2}(\mathbb{R}^3)$. 

Proof. Since $ψ \in \mathcal{L}_2^1(\mathbb{R}^3)$ and $0 = \langle S_1P, φ, ψ \rangle = \langle P, φ, ψ \rangle$. Thus $P \psi = 0$. Since $S_2 \leq S_1$, then $H \psi = 0$ and $ψ(x) = -\int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} v(y)φ(y)dy + \int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{4\pi|x-y|} v(y)φ(y)dy$. 

Note that $ψ_2(x) = (-\Delta + 1)(1 + |x|)\psi \in \mathcal{L}_2^1(\mathbb{R}^3)$, since $(-\Delta + 1)(1 + |x|)\psi \in \mathcal{L}_2^1(\mathbb{R}^3)$. 

Since $P \psi = 0$, we show $(1 + |x|)\psi \in \mathcal{L}_2^1(\mathbb{R}^3)$ by the boundedness of $(-\Delta)^{-1}$ in $B(s, s')$, see [Jen80, Lemma 2.3]. Thus $ψ_1 \in \mathcal{L}_2^{1/2}(\mathbb{R}^3) = \mathcal{L}_2^1(\mathbb{R}^3)$ for any tiny $ε > 0$.

If $ψ \in \mathcal{L}_2^1(\mathbb{R}^3)$ satisfies $H \psi = 0$, we show $P \psi = 0$. Since $ψ \in \mathcal{L}_2^1(\mathbb{R}^3) \subset \mathcal{L}_2^{1/2}(\mathbb{R}^3)$, we have $L^2(\mathbb{R}^3) \ni ψ = ψ_1 + ψ_2$. Since $ψ_2 = (-\Delta + 1)(1 + |x|)\psi \in \mathcal{L}_2^1(\mathbb{R}^3)$, thus $ψ_1 \in \mathcal{L}_2^1(\mathbb{R}^3)$. However, $ψ_1(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1 + |x| - |x-y|}{(1 + |x|)|x-y|} v(y)φ(y)dy$. 

and $|ψ_1(x)| \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1 + |x| - |x-y|}{(1 + |x|)|x-y|} v(y)φ(y)dy \leq (1 + |x|)^{-1}(1 + |x|)\psi \in \mathcal{L}_2^1(\mathbb{R}^3)$. 

which implies $\frac{1}{4\pi} \int_{\mathbb{R}^3} v(y)φ(y)dy \in \mathcal{L}_2^1(\mathbb{R}^3)$. Hence $\int_{\mathbb{R}^3} v(y)φ(y)dy = 0$. 

Lemma 3.7. Assume $v(x) \leq (1 + |x|)^{-s}$ with some $s > 3/2$, then $\ker(S_2G_2vS_2) = \{0\}$. 


Proof. For $\phi \in \ker(S_2vG_2vS_2)$, then $\phi \in S_2L^2(\mathbb{R}^3)$, thus $P\phi = 0$ which implies $vG_1v\phi = 0$ by Lemma 3.6 and the definition of $P$. Let $\zeta = (\sqrt{1/4 + z - 1/2})^{1/2}$. Therefore, by (1.6) we have
\[
0 = \langle S_2vG_2vS_2\phi, \ \phi \rangle = \langle G_2v\phi, \ v\phi \rangle
\]
\[
= \lim_{\zeta \to 0} \left( \frac{R_0(\zeta) - G_0 - iG_1\zeta}{\zeta^2} v\phi, \ v\phi \right)
\]
\[
= \lim_{\zeta \to 0} \left( \frac{R_0(\zeta^4 + \zeta^2) - G_0}{\zeta^2} v\phi, \ v\phi \right)
\]
\[
= \lim_{\zeta \to 0} \frac{1}{\zeta^2} \left( \frac{1}{\zeta^4 + \zeta^2 - (\zeta^4 + \zeta^2)} - \frac{1}{\zeta^4 + \zeta^2} \right) \nu_0(\xi), \ \nu_0(\xi) \right)
\]
\[
= \int_{\mathbb{R}^3} \frac{\nu_0(\xi)^2}{(\xi^4 + \xi^2)^2} d\xi.
\]
Here we used the dominated convergence theorem as $\zeta \to 0$ with $\Re(\zeta^4 + \zeta^2) < 0$ (by choose $0 < |z| < 1$ with $\Re(z) < 0$) in the last identity. Hence we have $v\phi = 0$ since $v\phi \in L^1$. Note that $\phi \in S_2L^2 \subset S_1L^2$, thus $\phi = Uv(-G_0v\phi) = 0$ by Lemma 3.6.

In the rest of this section, we aim to obtain suitable expansions for $(M^\pm(\lambda))^{-1}$ as $\lambda \to 0$ in the three cases: zero is a regular point, zero is a resonance and zero is an eigenvalue. Recall that $\eta = (\sqrt{1/4 + \lambda - 1/2})^{1/2}$. Thus $\lambda \to 0$ equals $\eta \to 0$.

Theorem 3.8. i) If zero is a regular point of $H = \Delta^2 - \Delta + V$ with $|V(x)| \leq (1 + |x|)^{-\beta}$ for some $\beta > 3$, then
\[
(M^\pm(\lambda))^{-1} = T_0^{-1} + i \frac{||V||_{L^1}}{4\pi} T_0^{-1} PT_0^{-1} \eta + O_1(\eta^2)
\]
in $B(0,0)$ as $\eta \to 0$.

ii) If zero is a resonance of $H = \Delta^2 - \Delta + V$ with $|V(x)| \leq (1 + |x|)^{-\beta}$ for some $\beta > 5$, then
\[
(M^\pm(\lambda))^{-1} = \mp i \frac{4\pi}{||V||_{L^1}} S_1(S_1PS_1)^{-1} S_1 \eta^{-1} + \left( D_0 + \frac{16\pi^2}{||V||_{L^2}^2} S_1(S_1PS_1)^{-1} S_1 S_2vG_2vS_1(S_1PS_1)^{-1} S_1 \right)
\]
\[
- \left( D_0PS_1(S_1PS_1)^{-1} S_1 + S_1(S_1PS_1)^{-1} S_1 PD_0 \right) + O_1(\eta)
\]
in $B(0,0)$ as $\eta \to 0$.

iii) If zero is a resonance and / or an eigenvalue of $H = \Delta^2 - \Delta + V$ with $|V(x)| \leq (1 + |x|)^{-\beta}$ for some $\beta > 7$, then
\[
(M^\pm(\lambda))^{-1} = \eta^{-2} S_2(S_2vG_2vS_2)^{-1} S_2 + \eta^{-1} A_{\pm}^1 + A_{\pm}^3 + O_1(\eta)
\]
in $B(0,0)$ as $\eta \to 0$. Here $A_{\pm}^1$ and $A_{\pm}^3$ are Hilbert-Schmidt operators.

The following lemma is the main tool to deduce the asymptotic expansions of $(M^\pm(\lambda))^{-1}$.

Lemma 3.9. ([1N01, Lemma 2.1]) Let $M$ be a closed operator on a Hilbert space $\mathcal{H}$ and $S$ be a projection. Suppose $M + S$ has a bounded inverse. Then $M$ has a bounded inverse if and only if $M_1 := S - S(M + S)^{-1} S$

has a bounded inverse in $S \mathcal{H}$, and in the case
\[
M^{-1} = (M + S)^{-1} + (M + S)^{-1} S M_1^{-1} S(M + S)^{-1}.
\]
Proof of Theorem 5.8. Since \( M^-(\lambda) = \overline{M^-(\lambda)} \), thus we only deal with \( M^+ (\lambda) \) below.

In the regular case, \( T_0 = U + vG_0V \) is invertible on \( L^2(\mathbb{R}) \) and

\[
M^+(\lambda) = T_0 + i\frac{\|V\|_L^2}{4\pi} P\eta + O(\eta^3).
\]

Writing \( (M^+(\lambda))^{-1} \) into Neumann series, then

\[
(M^+(\lambda))^{-1} = \left(1 + i\frac{\|V\|_L^2}{4\pi} T_0^{-1} P\eta + O(\eta^2)\right)^{-1} T_0^{-1} = T_0^{-1} - i\frac{\|V\|_L^2}{4\pi} T_0^{-1} PT_0^{-1} \eta + O(\eta^2).
\]

If zero is a resonance of \( H \), then \( T_0 + S_1 \) is invertible since \( S_1 \) is the Riesz projection onto \( \ker(T_0) \) and \( T_0 \) is self-adjoint. Since

\[
M^+(\lambda) = T_0 + i\frac{\|V\|_L^2}{4\pi} P\eta + \eta^2 vG_2v + O(\eta^3),
\]

applying Lemma 5.9 to \( M^+(\lambda) \) with projection \( S_1 \), then

\[
(M^+(\lambda) + S_1)^{-1} = (M^+(\lambda) + S_1)^{-1} S_1 \left(M^+(\lambda) + S_1\right)^{-1} S_1 (M^+(\lambda) + S_1)^{-1} \quad \text{(3.5)}
\]

where \( M^+_1(\lambda) = S_1 - S_1 (M^+(\lambda) + S_1)^{-1} S_1 \). Writing \( (M^+(\lambda) + S_1)^{-1} \) into Neumann series, we have

\[
(M^+(\lambda) + S_1)^{-1} = \left(1 + i\frac{\|V\|_L^2}{4\pi} \eta \eta D_0 + \eta^2 D_0 vG_2 v + O(\eta^3)\right)^{-1} D_0
\]

\[
= D_0 - i\frac{\|V\|_L^2}{4\pi} \eta D_0 P D_0 - \eta^2 \left(D_0 vG_2 v D_0 + \frac{\|V\|_L^2}{16}\right) D_0 P D_0 + O(\eta^3)
\]

where \( D_0 = (T_0 + S_1)^{-1} \). Note that \( \text{rank}(P) = 1 \) since \( P \) is a projection onto the span of \( v \). Thus \( PD_0 = \rho P \) with \( \rho = \text{trace}(PD_0) \). Using \( S_1 D_0 = D_0 S_1 = S_1 \), then

\[
M^+_1(\lambda) = i\frac{\|V\|_L^2}{4\pi} \eta S_1 P S_1 + \eta^2 \left(S_1 vG_2 v S_1 + \frac{\|V\|_L^2}{16}\right) \rho S_1 P S_1 + O(\eta^3).
\]

In the resonance case, \( T_1 = S_1 P S_1 \) is invertible, thus

\[
(M^+_1(\lambda))^{-1} = \frac{4\pi}{i\|V\|_L^2} \eta^{-1} \left(S_1 P S_1 + \frac{4\pi}{i\|V\|_L^2} \eta S_1 vG_2 v S_1 + \frac{\|V\|_L^2}{16}\rho S_1 P S_1 + O(\eta^2)\right)^{-1}
\]

\[
= \frac{4\pi}{i\|V\|_L^2} \eta^{-1} T_1^{-1} + \frac{16\pi^2}{\|V\|_L^2} T_1^{-1} \left(S_1 vG_2 v S_1 + \frac{\|V\|_L^2}{16}\rho S_1 P S_1\right) T_1^{-1} + O(\eta).
\]

Substituting the Neumann series of \( (M^+(\lambda) + S_1)^{-1} \) and \( (M^+(\lambda))^{-1} \) into identity (3.5), we obtain

\[
(M^+(\lambda))^{-1} = -i\frac{4\pi}{\|V\|_L^2} S_1 T_1^{-1} S_1 \eta^{-1} + \left(D_0 + \frac{16\pi^2}{\|V\|_L^2} S_1 T_1^{-1} S_1 vG_2 v S_1 T_1^{-1} S_1\right)
\]

\[
- \left(D_0 P S_1 T_1^{-1} S_1 + S_1 T_1^{-1} S_1 P D_0\right) + O(\eta).
\]

If zero is an eigenvalue of \( H \), then \( S_1 P S_1 \) is not invertible but \( S_1 P S_1 + S_2 \) is invertible. Denotes

\[
\tilde{M}^+_1(\lambda) = S_1 P S_1 + \frac{4\pi}{i\|V\|_L^2} \eta S_1 vG_2 v S_1 + \frac{\|V\|_L^2}{16\rho S_1 P S_1} + O(\eta^2)
\]
then \( \left( M_1^+(\lambda) \right)^{-1} = \frac{4\pi}{i||V||_L^2} \eta^{-1} \left( \tilde{M}_1^+(\lambda) \right)^{-1} \). Applying Lemma 3.9 to \( \tilde{M}_1^+(\lambda) \) with projection \( S_2 \), then

\[
\left( \tilde{M}_1^+(\lambda) \right)^{-1} = \left( M_1^+(\lambda) + S_2 \right)^{-1} + \left( M_1^+(\lambda) + S_2 \right)^{-1} S_2 \left( M_1^+(\lambda) + S_2 \right)^{-1}
\]

with \( M_2^+(\lambda) = S_2 - S_2 \left( M_1^+(\lambda) + S_2 \right)^{-1} S_2 \). Writing \( \left( \tilde{M}_1^+(\lambda) + S_2 \right)^{-1} \) into Neumann series, then

\[
\left( \tilde{M}_1^+(\lambda) + S_2 \right)^{-1} = \left( 1 + \frac{4\pi}{i||V||_L^2} \eta D_1 \left( S_1 vG_2 vS_1 + \frac{||V||_L^2}{16\pi^2} \rho S_1 PS_1 \right) + O(\eta^2) \right)^{-1} D_1
\]

\[
= D_1 - \frac{4\pi}{i||V||_L^2} \eta D_1 \left( S_1 vG_2 vS_1 + \frac{||V||_L^2}{16\pi^2} \rho S_1 PS_1 \right) D_1 + O(\eta^2)
\]

where \( D_1 = (S_1 PS_1 + S_2)^{-1} \). Using \( S_2 D_1 = D_1 S_2 = S_2 \), we get

\[
M_2^+(\lambda) = \frac{4\pi}{i||V||_L^2} \eta S_2 \left( S_1 vG_2 vS_1 + \frac{||V||_L^2}{16\pi^2} \rho S_1 PS_1 \right) S_2 + O(\eta^2) = \frac{4\pi}{i||V||_L^2} \eta S_2 vG_2 vS_2 + O(\eta^2).
\]

Since \( S_2 vG_2 vS_2 \) is invertible by Lemma 3.7 then

\[
\left( M_2^+(\lambda) \right)^{-1} = \frac{i||V||_L^2}{4\pi} \eta^{-1} \left( T_2 vG_2 vS_2 \right)^{-1} + O(1).
\]

Substituting the expansions back step by step, we obtain

\[
\left( M^+(\lambda) \right)^{-1} = \eta^{-2} S_2 \left( S_2 vG_2 vS_2 \right)^{-1} S_2 + \eta^{-1} A_{1+} + A_0^+ + O(\eta)
\]

with \( A_{1+} \), \( A_0^+ \) are Hilbert-Schmidt operators independent of \( \eta \).

\[\square\]

4. The perturbed evolution for low energy

In this section, our aim is to study the \( L^1 - L^\infty \) dispersive estimates of perturbed evolution \( e^{-itH} \) for small energy. Here small energy means the spectral variable \( \lambda \) is near the threshold energy \( \lambda = 0 \). The presence of zero resonance or zero eigenvalue affect the asymptotic behavior of the perturbed resolvent \( R_0^+(\lambda) \) as \( \lambda \to 0 \). The effect of the presence of zero energy resonance or zero eigenvalue is only felt in the small energy regime.

**Lemma 4.1.** For \( \lambda > 0 \) and \( \eta = (\sqrt{1/4 + \lambda} - 1/2)^{1/2} \), then

\[
\sup_{x,y \in \mathbb{R}^3} \left| R_0^+(\lambda; x, y) \right| \leq \frac{\eta + \sqrt{1 + \eta^2}}{1 + 2\eta^2} \tag{4.1}
\]

and

\[
\sup_{x,y \in \mathbb{R}^3} \left| \frac{d}{d\eta} R_0^+(\lambda; x, y) \right| \leq \frac{\eta(\eta + \sqrt{1 + \eta^2})}{(1 + 2\eta^2)^2} + \frac{1 + \eta(1 + \eta^2)^{-1/2}}{1 + 2\eta^2}. \tag{4.2}
\]

**Proof.** Recall that

\[
R_0^+(\lambda; x, y) = \frac{1}{1 + 2\eta^2} \left[ e^{\frac{\pm i\eta|x-y|}{4\pi|x-y|}} - \frac{e^{-\sqrt{1+\eta^2}|x-y|}}{4\pi|x-y|} \right] \tag{4.3}
\]

where \( \eta = (\sqrt{1/4 + \lambda} - 1/2)^{1/2} \). By the mean value theorem, then

\[
\left| R_0^+(\lambda; x, y) \right| \leq \frac{1}{1 + 2\eta^2} \left| \frac{e^{\frac{\pm i\eta|x-y|}{4\pi|x-y|}} - 1}{4\pi|x-y|} \right| + \frac{1 - e^{-\sqrt{1+\eta^2}|x-y|}}{4\pi|x-y|} \leq \frac{\eta + \sqrt{1 + \eta^2}}{1 + 2\eta^2}.
\]
Furthermore, we have
\[
\frac{d}{d\eta} R^\pm_0(\lambda; x, y) = \frac{4\eta}{(1 + 2\eta^2)^2} \left[ \frac{e^{i\eta|x-y|}}{4\pi|x-y|} - \frac{e^{-\sqrt{1+\eta^2}|x-y|}}{4\pi|x-y|} \right]
+ \frac{1}{1 + 2\eta^2} \left[ \frac{\pm ie^{i\eta|x-y|}}{4\pi} + \frac{\eta(1 + \eta^2)^{-1/2}e^{-\sqrt{1+\eta^2}|x-y|}}{4\pi} \right]
\leq \frac{\eta(\eta + \sqrt{1 + \eta^2})}{(1 + 2\eta^2)^2} + \frac{1}{1 + 2\eta^2} \lesq \frac{1}{1 + 2\eta^2}.
\]

\[\square\]

In what follows, we use the smooth, even low energy cut-off \(\chi_t\) defined by \(\chi_t(\lambda) = 1\) if \(|\lambda| < \lambda_0\) and \(\chi_t(\lambda) = 0\) if \(|\lambda| > 2\lambda_0\) for some constant \(0 < \lambda_0 \leq 1\). For the high energy part, we use the complementary cut-off \(\chi_h(\lambda) = 1 - \chi_t(\lambda)\).

**Proposition 4.2.** For \(H = \Delta^2 - \Delta + V\) with \(|V(x)| \lesq (1 + |x|)^{-\beta}\) for some \(\beta > 3\). Assume that \(H\) has no positive embedded eigenvalue. If 0 is a regular point of \(H\). For \(t > 1\), then
\[
\sup_{x, y \in \mathbb{R}^3} \left| \int_0^{r^{-1/2}} e^{-i\eta(t^4 + \eta^2)} \chi_t(\eta) \left[ R^+_V - R^-_V \right] (\eta^4 + \eta^2; x, y) (4\eta^3 + 2\eta) \, d\eta \right| \lesq |t|^{-3/2}.
\]

For \(0 < t \leq 1\), then
\[
\sup_{x, y \in \mathbb{R}^3} \left| \int_0^{r^{-1/4}} e^{-i\eta(t^4 + \eta^2)} \chi_t(\eta) \left[ R^+_V - R^-_V \right] (\eta^4 + \eta^2; x, y) (4\eta^3 + 2\eta) \, d\eta \right| \lesq |t|^{-3/4}.
\]

**Proof.** Recall that
\[
R^+_V(\lambda) = R^+_0(\lambda) - R^+_0(\lambda) v \left( M^\pm(\lambda) \right)^{-1} v R^+_0(\lambda).
\]

It remains to prove the second term satisfies the above bounds by Proposition 2.1. In the regular case, then \((M^\pm(\lambda))^{-1} = T^+_0 + O^\pm(\eta)\) in \(B(0,0)\) by Theorem 3.8. Furthermore, since \(L^\infty(\mathbb{R}^3) \subset L^2_{-\delta}(\mathbb{R}^3)\) for \(s > 3/2\), then
\[
\left\| R^+_0(\lambda) v \left( M^\pm(\lambda) \right)^{-1} v R^+_0(\lambda) \right\|_{L^1 \rightarrow L^\infty} \lesq \left\| R^+_0(\lambda) \right\|_{L^2 \rightarrow L^\infty} \left\| v \left( M^\pm(\lambda) \right)^{-1} v \right\|_{L^2_{-\delta} \rightarrow L^2_{-\delta}} \left\| R^+_0(\lambda) \right\|_{L^1 \rightarrow L^2_{-\delta}} (4.4)
\]

By estimates (2.5), (2.6) and Lemma 4.1 for \(t > 1\) then
\[
\left| \int_0^{r^{-1/2}} e^{-i\eta(t^4 + \eta^2)} \left[ R^+_0 v T^+_0 v R^+_0 - R^-_0 v T^+_0 v R^-_0 \right] (\eta^4 + \eta^2) (4\eta^3 + 2\eta) \, d\eta \right|
\lesq \left| \int_0^{r^{-1/2}} e^{-i\eta(t^4 + \eta^2)} \left[ R^+_0 v T^+_0 v R^+_0 - R^-_0 v T^+_0 v R^-_0 \right] (\eta^4 + \eta^2) \, d(\eta^4 + \eta^2) \right|
+ \left| \int_0^{r^{-1/2}} e^{-i\eta(t^4 + \eta^2)} \left[ R^+_0 v T^+_0 v R^+_0 - R^-_0 v T^+_0 v R^-_0 \right] (\eta^4 + \eta^2) \, d(\eta^4 + \eta^2) \right|
\lesq \frac{1}{|t|} \left| \left( R^+_0 - R^-_0 \right) v T^+_0 v R^+_0 \right| (\eta^4 + \eta^2) \right|^{-1/2} \left| \int_0^{r^{-1/2}} \frac{d}{d\eta} \left( (R^+_0 - R^-_0) v T^+_0 v R^+_0 \right) (\eta^4 + \eta^2) \big| \, d\eta \right|
\lesq |t|^{-1} \left| \eta \right|^{-1/2} \left| \int_0^{r^{-1/2}} \left| 1 + \eta \right| \, d\eta \right| \lesq |t|^{-3/2}.
\]
For the remaining term, we have

\[
\left| \int_0^{\tau^{-1/2}} e^{-it(\eta^4 + \eta^2)} \left[ R_0^\pm(\eta^4 + \eta^2) vO^\pm(\eta) vR_0^\pm(\eta^4 + \eta^2) \right] (4\eta^3 + 2\eta) \ d\eta \right| \\
\leq \frac{1}{|t|} \left| \int_0^{\tau^{-1/2}} e^{-it(\eta^4 + \eta^2)} \left[ R_0^\pm(\eta^4 + \eta^2) vO^\pm(\eta) vR_0^\pm(\eta^4 + \eta^2) \right] \ d\eta \right| \\
+ \frac{1}{|t|} \int_0^{\tau^{-1/2}} \left| \frac{d}{d\eta} \left[ R_0^\pm(\eta^4 + \eta^2) vO^\pm(\eta) vR_0^\pm(\eta^4 + \eta^2) \right] \right| \ d\eta \\
\leq |t|^{-3/2} + \frac{1}{|t|} \int_0^{\tau^{-1/2}} |1 + \eta| d\eta \lesssim |t|^{-3/2}.
\]
For $0 < t \leq 1$, by Lemma 4.1 and inequality (4.4), then

$$\left| \int_0^{t^{1/4}} e^{-it(\eta^4 + \eta^2)} R^+_0(\eta^4 + \eta^2) v(M^+(\eta^4 + \eta^2))^{-1} vR^+_0(\eta^4 + \eta^2)(4\eta^3 + 2\eta) \, d\eta \right|$$

$$\leq \int_0^{t^{1/4}} (4\eta^3 + 2\eta) R^+_0(\eta^4 + \eta^2) v(M^+(\eta^4 + \eta^2))^{-1} vR^+_0(\eta^4 + \eta^2) \, d\eta$$

$$\leq \int_0^{t^{1/4}} (4\eta^3 + 2\eta) \frac{\eta + \sqrt{1 + \eta^2}}{1 + 2\eta^2} (\eta^{-1} + 1 + \eta) \frac{\eta + \sqrt{1 + \eta^2}}{1 + 2\eta^2} \, d\eta$$

$$\leq \int_0^{t^{1/4}} (1 + \eta + \eta^2) \, d\eta \lesssim |t|^{-3/4}.$$  

For $t > 1$, we only need to deal with the term $R^+_0(\lambda)v(M^+(\lambda))^{-1} vR^+_0(\lambda)$ by Proposition 2.1 and the symmetric resolvent identity (3.1). Since

$$R^+_0(\eta^4 + \eta^2) v(M^+(\eta^4 + \eta^2))^{-1} vR^+_0(\eta^4 + \eta^2) - R^+_0(\eta^4 + \eta^2) v(M^-(-\eta^4 + \eta^2))^{-1} vR^+_0(\eta^4 + \eta^2)$$

$$= R^+_0(\eta^4 + \eta^2) v(M^+_{-1\eta^{-1}}) vR^+_0(\eta^4 + \eta^2) - R^+_0(\eta^4 + \eta^2) v(M^-_{-1\eta^{-1}}) vR^+_0(\eta^4 + \eta^2)$$

$$+ R^+_0(\eta^4 + \eta^2) vM_0 vR^+_0(\eta^4 + \eta^2) - R^+_0(\eta^4 + \eta^2) vM_0 vR^+_0(\eta^4 + \eta^2)$$

$$+ R^+_0(\eta^4 + \eta^2) vO^-(\eta) vR^+_0(\eta^4 + \eta^2) - R^+_0(\eta^4 + \eta^2) vO^-(-\eta) vR^+_0(\eta^4 + \eta^2)$$

$$:= I(\eta; x, y) + II(\eta; x, y) + III(\eta; x, y).$$

For $II(\eta; x, y)$ and $s > 3/2$, we have

$$\left\| \left[ (R^+_0 - R^+_0) vM_0 vR^+_0 \right] (\eta^4 + \eta^2) \right\|_{L^1 \rightarrow L^\infty} \leq \left\| R^+_0 - R^+_0 \right\|_{L^2 \rightarrow L^\infty} \left\| vM_0 \right\|_{L^2 \rightarrow L^\infty} \left\| R^+_0 \right\|_{L^1 \rightarrow L^2}$$

$$\leq \left\| R^+_0 - R^+_0 \right\|_{L^1 \rightarrow L^\infty} \left\| M_0 \right\|_{L^2 \rightarrow L^2} \left\| R^+_0 \right\|_{L^1 \rightarrow L^\infty}$$

$$\leq \frac{\eta}{1 + 2\eta^2} \left( \eta + \sqrt{1 + \eta^2} \right) \lesssim \eta(1 + 2\eta^2)^{-3/2}.$$

Similarly, by Lemma 4.1 we have

$$\left\| \frac{d}{d\eta} \left[ (R^+_0 - R^+_0) vM_0 vR^+_0 \right] (\eta^4 + \eta^2) \right\|_{L^1 \rightarrow L^\infty} \leq \frac{\eta + \sqrt{1 + \eta^2}}{(1 + 2\eta^2)^2}.$$  

Thus we get

$$\left| \int_0^{t^{1/2}} e^{-it(\eta^4 + \eta^2)} II(\eta; x, y) (4\eta^3 + 2\eta) \, d\eta \right|$$

$$\lesssim \left| \int_0^{t^{1/2}} e^{-it(\eta^4 + \eta^2)} \left[ R^+_0 - R^+_0 \right] (\eta^4 + \eta^2) vM_0 vR^+_0(\eta^4 + \eta^2) \, d(\eta^4 + \eta^2) \right|$$

$$+ \left| \int_0^{t^{1/2}} e^{-it(\eta^4 + \eta^2)} R^+_0(\eta^4 + \eta^2) vM_0 v\left[ R^+_0 - R^+_0 \right] (\eta^4 + \eta^2) \, d(\eta^4 + \eta^2) \right|.$$
that the projection $S$ eigenvalue of $H$. Proof. Proposition 4.4. For $H = \Delta^2 - \Delta + V$ with $|V(x)| \leq (1 + |x|)^{-7}$ for some $\beta > 7$. Assume that $H$ has no positive embedded eigenvalue. If $0$ is a resonance and $f$ an eigenvalue of $H$.

For $t > 1$, then

$$\sup_{x,y \in \mathbb{R}^3} \left| \int_0^{1/2} e^{-it(\eta^4 + \eta^2)} \chi_t(\eta) \left[ R^+_t - R^{-}_t \right] (\eta^4 + \eta^2; x, y) (4\eta^3 + 2\eta) \, d\eta - G(x, y) \right| \lesssim |t|^{-3/2} \quad (4.5)$$

where $G$ is a time dependent finite rank operator satisfying $\|G\|_{L^1 \to L^\infty} \lesssim |t|^{-1/2}$.

For $0 < t \leq 1$, then

$$\sup_{x,y \in \mathbb{R}^3} \left| \int_0^{1/4} e^{-it(\eta^4 + \eta^2)} \chi_t(\eta) \left[ R^+_t - R^{-}_t \right] (\eta^4 + \eta^2; x, y) (4\eta^3 + 2\eta) \, d\eta \right| \lesssim |t|^{-3/4} \quad (4.6)$$

Proof. Recall that 0 is both resonance and eigenvalue of $H$ if $S_2 = S_1 \neq 0$. 0 is purely an eigenvalue of $H$ if $S_2 < S_1$ strictly, see Remark 3.4. Here we deal with the case $S_2 < S_1$. The
following argument also holds for $S_2 = S_1$. If zero is an eigenvalue of $H$, by Theorem \ref{thm:h}, then
\[ (M^+(\eta^4 + \eta^2))^{-1} = \eta^{-2}A_{-2} + \eta^{-1}A_{-1}^+ + A_0^+ + O_1(\eta). \]
where $A_{-2} = S_2(S_2^*G_2S_2)^{-1}S_2$. Applying the similar argument as in Theorem \ref{thm:4.3} we know for $0 < t \leq 1$, the term $\eta^{-1}A_{-1}^+$, $A_0^+$ and the remaining term $O_1(\eta)$ satisfy the expected estimates \eqref{eq:4.6}. For $t > 1$, $A_0^+$ and $O_1(\eta)$ contributes $|t|^{-3/2}$ in estimate \eqref{eq:4.5}. The term $\eta^{-1}A_{-1}^+$ contributes $|t|^{-3/2}$ in estimate \eqref{eq:4.5}.

Next, we show the first term $\eta^{-2}A_{-2}$ satisfies the estimates we needed. By \eqref{eq:4.4} and Lemma \ref{lem:4.1} then
\[
\left| \int_0^{-1/2} e^{-it(\eta^4 + \eta^2)} \left[ R_0^+(\eta^4 + \eta^2)v(\eta^{-2}A_{-2})vR_0^+(\eta^4 + \eta^2) - R_0^+(\eta^4 + \eta^2)v(\eta^{-2}A_{-2})vR_0^-(\eta^4 + \eta^2) \right] d(\eta^4 + \eta^2) \right| \leq \int_0^{-1/2} e^{-it(\eta^4 + \eta^2)} \left[ R_0^+(\eta^4 + \eta^2)v(\eta^{-2}A_{-2})vR_0^+(\eta^4 + \eta^2) - R_0^+(\eta^4 + \eta^2)v(\eta^{-2}A_{-2})vR_0^-(\eta^4 + \eta^2) \right] \left| (4\eta^3 + 2\eta) \right| d\eta \leq \int_0^{-1/2} \left| (4\eta^3 + 2\eta) \right| d\eta \leq \int_0^{-1/2} 1 d\eta \leq |t|^{-1/2}. \]

Similarly, we get the expected estimates for $0 < t \leq 1$ with $|t|^{-1/4}$. Let
\[
G(t; x, y) = F(t; x, y) + \int_0^{|t|^{-1/2}} e^{-it(\eta^4 + \eta^2)} \left[ R_0^+(\eta^4 + \eta^2)v(\eta^{-2}A_{-2})vR_0^+(\eta^4 + \eta^2) - R_0^+(\eta^4 + \eta^2)v(\eta^{-2}A_{-2})vR_0^-(\eta^4 + \eta^2) \right] (4\eta^3 + 2\eta) d\eta.
\]
Thus $G(t; x, y)$ satisfies $\|G\|_{L^1 \to L^\infty} \leq |t|^{-1/2}$ for $t > 1$. Furthermore, $S_2$ is finite rank by Remark \ref{rem:3.4}. Hence $G$ is finite rank. 

\section{The perturbed evolution for high energy}

In this part, we aim to show the proof of dispersive bound for high energy portion of the perturbed evolution. This section include two subsections. In the first subsection, under the assumption that $H$ has no positive embedded eigenvalue, we prove decay estimates for $R_v^\pm(\lambda)$ with $\lambda \to \infty$ in $B(s, -s)$ using the limiting absorption principle. In the second subsection, we show the high energy bounds of Theorem \ref{thm:1.1}.

\subsection{High energy decay estimates of $R_v(z)$}

Denote by $C^+ = \{ \text{Im} z > 0 \}$ the open upper half complex plane, and by $C^- = \{ \text{Im} z < 0 \}$ the open lower half complex plane. Define $\Xi$ be the disjoint union of $C^+$ and $C^-$ with the identified points $z \leq 0$. Denote $V_0 := \min\{V(x), x \in \mathbb{R}^d\}$. Since for any $\lambda \in \mathbb{R} \setminus \{ V_0, +\infty \}, H - \lambda = \Delta^2 - \Delta + V - \lambda > 0$, then for the resolvent set $\rho(H)$ of $H$, we have $C \setminus \{ V_0, +\infty \} \subseteq \rho(H)$. 


Theorem 5.1. Let \( k = 0, 1, 2, 3, \ldots \), and \( |V(x)| \lesssim (1 + |x|)^{-\beta} \) with \( \beta > k + 1 \). Assuming that \( H \) has no positive embedded eigenvalue. For any \( s > k + 1/2 \), then \( \mathcal{R}_V^{(k)}(z) \in B(s, -s) \) is continuous for \( z \in \Xi \setminus [V_0, 0] \). Further, the bound

\[
\|\mathcal{R}_V^{(k)}(z)\|_{B(s, -s)} = O(|z|^{-3+3k/4}) \tag{5.1}
\]

holds as \( z \to \infty \) in \( \Xi \setminus [V_0, 0] \).

Proof. The theorem follows by Proposition 5.4 and Lemma 5.8 below.

Using the high energy decay estimates for free resolvent \( R_0(z) \), we first prove estimate (5.1) for \( z \in \Xi \setminus [V_0, \infty) \) by perturbation arguments. Then, we extend \( z \) to the positive real axis applying the limiting absorption principle. For the free resolvent we have:

Proposition 5.2. For \( z \in C \setminus [0, +\infty), k = 0, 1, 2, \ldots \) and any \( s, s' > k + \frac{1}{2} \), then

\[
\|\mathcal{R}_V^{(0)}(z)\|_{B(s, s')} \lesssim |z|^{-3+3k/4}, \quad |z| \to \infty. \tag{5.2}
\]

Proof. For free resolvent of Laplacian \( \mathcal{R}(-\Delta; z) : = (-\Delta - z)^{-1} \), it is known that

\[
\|\mathcal{R}^{(k)}(-\Delta; z)\|_{B(s, s')} \lesssim |z|^{-1+k/2}, \quad |z| \to \infty \tag{5.3}
\]

holds, see e.g. [KK12 Theorem 16.1]. For \( k = 0 \), by (1.6) and (5.3), then

\[
\|R_0(z)\|_{B(s, s')} = \left| \frac{1}{2 \sqrt{1/4 + z}} \right| \left\| - \Delta + \frac{1}{2} - \sqrt{1/4 + z}^{-1} \right\|_{B(s, s')} - \left| \frac{1}{2 \sqrt{1/4 + z}} \right| \left\| - \Delta + \frac{1}{2} + \sqrt{1/4 + z}^{-1} \right\|_{B(s, s')} \lesssim |z|^{-3/4}, \quad |z| \to \infty.
\]

Now we check decay estimate (5.2) for \( k \neq 0 \). For any \( s, s' > 1/2 + k \), then

\[
\|R_0^{(k)}(z)\|_{B(s, s')} \lesssim \left| \frac{d^k_1}{d z^k_1} \right| \left| \frac{d^k_2}{d z^k_2} \right| \left\| - \Delta + \frac{1}{2} - \sqrt{1/4 + z}^{-1} \right\|_{B(s, s')} \left\| - \Delta + \frac{1}{2} + \sqrt{1/4 + z}^{-1} \right\|_{B(s, s')} \lesssim |z|^{-1/2-k}
\]

where \( \zeta = \sqrt{1/4 + z} \) and \( k = k_1 + k_2 (k_1, k_2 \geq 0) \).

Now, we prove the high energy decay estimate of perturbed resolvent \( R_0^{(k)}(z) \).

Lemma 5.3. Let \( V(x) \) satisfies that \( |V(x)| \lesssim (1 + |x|)^{-\beta} \) with some \( \beta > 2 \). Then operators \( vR_0(z)v \in B(0, 0) \) are compact for \( z \in C \setminus [0, +\infty) \). Moreover, \( M(z) = U + vR_0(z)v \) are invertible in \( L^2(\mathbb{R}^3) \) for \( z \in C \setminus [V_0, +\infty) \).
Proof. By the free resolvent identity \( (1.6) \), we only need to show \( \psi(\Delta - \frac{1}{2} \pm \sqrt{1/4 + z})^{-1} \) are compact in \( L^2(\mathbb{R}^3) \). Recall that the kernel \( k(\xi; x, y) \) of \( (\Delta - \xi)^{-1} \):

\[
k(\xi; x, y) = e^{-i\sqrt{\xi} |x-y|/(4\pi |x-y|)}, \quad \text{Im} \sqrt{\xi} > 0.
\]

Thus, under the assumption on \( V \), one can check that the Hilbert-Schmidt norm

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v(x)k(\xi; x, y)v(y)|^2 \, dx \, dy < \infty.
\]

Next, we show that \( M(z) = U + vR_0(z)v \) is invertible by Fredholm’s alternative theorem. We claim that \( (H - z)\psi = 0 \) only has trivial solution in \( L^2(\mathbb{R}^3) \). In fact, for \( z \in \mathbb{C} \setminus \mathbb{R} \), if \( \psi \neq 0 \), then

\[
\text{Im}(H - z)\psi, \psi = -\text{Im} z(\psi, \psi) \neq 0.
\]

Since \( ((H - z)\psi, \psi) = (H\psi, \psi) - z(\psi, \psi) \) and \( (H\psi, \psi) \in \mathbb{R} \), hence \( (H - z)\psi \neq 0 \) if \( \psi \neq 0 \). For \( z \in \mathbb{C} \setminus \mathbb{R} \) and \( \text{Re}(z) < V_0 \), then

\[
\text{Re}(H - z)\psi, \psi \geq \text{Re}(V_0 - z)(\psi, \psi) \neq 0
\]

provided \( \psi \neq 0 \). Thus \( (H - z)\psi \neq 0 \) if \( \psi \neq 0 \).

Let \( \psi = Uv\phi \). Then \( M(z)\phi = 0 \) if and only if \( (H - z)\psi = 0 \). Thus \( M(z)\phi = 0 \) only has zero solution. Hence, Fredholm’s alternative theorem tells that \( M(z) = U + vR_0(z)v \) is invertible. \( \square \)

**Proposition 5.4.** Let \( k = 0, 1, 2, \ldots \), and \( |V(x)| \leq (1 + |x|)^{-\beta} \) with \( \beta > 2k + 2 \). For large \( z \in \mathbb{C} \setminus \{V_0, +\infty\} \) and any \( s, s' > k + 1/2 \), then

\[
\|R^{(k)}_{V}(z)\|_{B(s, -s')} \leq C(\sigma, k)(|z|^{-3(3\beta)/4}), \quad |z| \to \infty. \tag{5.4}
\]

**Proof.** For \( k = 0 \), by identity \( (3.1) \) and Proposition \( 5.2 \) the above bound \( (5.4) \) holds by the uniformly boundedness of \( M(z)^{-1} \) for large \( z \in \mathbb{C} \setminus \{V_0, +\infty\} \) in \( L^2 \). It is equivalent to prove that for large \( z \in \mathbb{C} \setminus \{0, +\infty\} \),

\[
\|f\|_{L^2(\mathbb{R}^3)} \leq C\left\|(U + VR_0(z)v)f\right\|_{L^2(\mathbb{R}^3)}.
\]

In fact, by the triangle inequality we have

\[
\|f\|_{L^2} - \|VR_0(z)v f\|_{L^2} \leq \left\|(U + VR_0(z)v)f\right\|_{L^2} \leq \|f\|_{L^2} + \left\|VR_0(z)v f\right\|_{L^2}.
\]

By the decay estimate \( (5.2) \), for \( |z| \) large enough, then

\[
\|VR_0(z)v f\|_{L^2} \leq C(s, a)|z|^{-\beta} \|f\|_{L^2} \leq \frac{1}{4} \|f\|_{L^2}.
\]

For \( k \geq 1 \), differentiating \( (3.1) \) \( k \)-times in \( z \), then

\[
R^{(k)}_{V}(z) = R^{(k)}_{0}(z) - \sum_{k_1 + k_2 + k_3 = k} R^{(k)}_{0}(z)v \frac{d^{k_2}}{dz^{k_2}}[M(z)^{-1}]vR^{(k_3)}_{0}(z). \tag{5.5}
\]

Note that the derivative term \( \frac{d^{j} x}{dz^{j}}[M(z)^{-1}] \) is a linear combination of terms such as \( [M(z)^{-1}]^j M^{(\ell)}(z) \) with \( 0 \leq j, \ell < k_2 \). By the representation of \( M(z) \), we know \( M^{(\ell)}(z) = VR^{(\ell)}_{0}(z)v \) for \( \ell \geq 1 \). Since \( v(x)(1 + |x|)^{\beta + 1/2} \in L^\infty \) under the assumption of \( V(x) \), thus \( (5.4) \) holds by mathematical induction. \( \square \)

Now, we show the limiting absorption principle for \( R_{V}(z) \). For the resolvent of free Schrödinger operator \( R(\Delta; \xi) \), we have:
Lemma 5.5. ([JK79] Theorem 8.1) Let $k = 0, 1, 2, \ldots$. If $s > k + 1/2$, then $R^k(-\Delta; \zeta) \in B(s, -s)$ is continuous in $\zeta \in \Xi \setminus \{0\}$. Further, the boundary value

$$R^k(-\Delta; \lambda \pm i0) = \lim_{\varepsilon \downarrow 0} R^k(-\Delta; \lambda \pm i\varepsilon) \in B(s, -s)$$

exists for any $\lambda \in (0, +\infty)$. The decay estimate (5.3) can extend $\zeta \in \mathbb{C} \setminus [0, +\infty)$ to $\zeta \in \Xi \setminus \{0\}$.

By the resolvent identity (1.6), for the free resolvent $R_0(z)$, we have:

Corollary 5.6. Let $k = 0, 1, 2, \ldots$. For $s > k + 1/2$, then $R^k_0(z) \in B(s, -s)$ is continuous in $z \in \Xi \setminus \{0\}$. Further, the boundary value

$$R^k_0(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} R^k_0(\lambda \pm i\varepsilon) \in B(s, -s)$$

exists for any $\lambda \in (0, +\infty)$, and the bound

$$\|R^k_0(\zeta)\|_{B(s, -s)} = O(|z|^{-(3+3k)/4})$$

holds as $z \to \infty$ in $\Xi \setminus \{0\}$.

Next, under the spectral assumption that $H$ has no positive embedded eigenvalues, we prove that the boundary value $R^+_V(\lambda)$ exists on $\lambda \in (0, +\infty)$. Recall that $V_0$ is the minimal value of potential function $V$. If $V_0 \geq 0$, then the segment $[V_0, 0] = \{0\}$.

Lemma 5.7. Let $|V(x)| \leq (1 + |x|)^{-\beta}$ with some $\beta > 2$. For $\lambda \geq 0$, then $vR^+_V(\lambda)v \in B(0, 0)$ are compact.

Proof. Recall the kernel of $R^+_0(\lambda)$:

$$R^+_0(\lambda; x, y) = \frac{1}{1 + 2\eta^2} \left( e^{\eta|x-y|} \frac{e^{-\sqrt{1+\eta^2}|x-y|}}{4\pi|x-y|} - e^{-\sqrt{1+\eta^2}|x-y|} \frac{e^{\eta|x-y|}}{4\pi|x-y|} \right),$$

where $\eta = \left( \sqrt{1/4 + \lambda} - 1/2 \right)^{1/2}$. Thus for $\lambda \geq 0$, then $|R^+_0(\lambda; x, y)| \leq \frac{1}{2\pi|x-y|}$, Since the Hilbert-Schmidt norm of $v(x)|x-y|^{-1}v(y)$ is finite under the assumption of $V$, thus $vR^+_V(\lambda)v$ are compact. \(\Box\)

Lemma 5.8. For $H = \Delta^2 - \Delta + V$, assume that $H$ has no positive embedded eigenvalue.

(1) Let $|V(x)| \leq (1 + |x|)^{-\beta}$ with some $\beta > 2$. For $s, s' > 1/2$, then $R_V(z) \in B(s, -s')$ is continuous for $z \in \Xi \setminus (\Sigma \cup \{0\})$. Furthermore, the boundary value

$$R^+_V(\lambda) = \lim_{\varepsilon \downarrow 0} R_V(\lambda \pm i\varepsilon) \in B(s, -s')$$

exists for $\lambda \in \sigma_r(H) \setminus (\Sigma \cup \{0\})$.

(2) Let $|V(x)| \leq (1 + |x|)^{-\beta}$ with some $\beta > 2$. Assume that $0$ is a regular point of $H$. For $s, s' > 1$, then the function $R_V(z) \in B(s, -s')$ defined on $z \in \Xi \setminus \Sigma$ is continuous at $z = 0$.

Proof. The conclusions follow from Lemma 5.7 and symmetric resolvent identity (3.1) provided

$$[U + vR_0(\lambda \pm i\varepsilon)v]^{-1} \to [U + vR_0(\lambda \pm i0)v]^{-1}, \quad \varepsilon \downarrow 0.$$

The convergence holds if and only if both limit operators $U + vR_0(\lambda \pm i0)v : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ are invertible. According to Lemma 5.7 and Fredholm’s alternative theorem, it is enough to show that $[U + vR_0(\lambda \pm i0)v]\phi = 0$ only admits zero solution in $L^2(\mathbb{R}^3)$. Note that $[U + vR_0(\lambda \pm i0)v]\phi = 0$ equals $(H - \lambda)\psi = 0$ with $\psi = U\phi$. Thus $\phi = 0$ under the assumptions that zero is a regular point of $H$ and $H$ has no positive eigenvalue. \(\Box\)
5.2. **Proof of dispersive estimate for large energy.** To deal with the large energy part, we use the resolvent identity:

\[ R^+_V(\lambda) = R^+_0(\lambda) - R^+_0(\lambda)V R^+_0(\lambda) + R^+_0(\lambda)VR^+_0(\lambda)VR^+_0(\lambda). \]  

(5.7)

From Proposition 2.1, we know that the first summand in (5.7) satisfies the expected estimates. Therefore, it suffices to establish the expected bounds for the last two summands in (5.7). Denote

\[ R^+_0(\eta; x, y) := \left[ R^+_0(\lambda)VR^+_0(\lambda) \right](x, y); \]

\[ R^-_V(\eta; x, y) := \left[ R^+_0(\lambda)VR^+_0(\lambda)VR^+_0(\lambda) \right](x, y). \]

Recall that \( \lambda = \eta^4 + \eta^2 \). For the term \( R^+_V(\eta; x, y) \), we will use the estimates of \( R^+_V(\lambda) \) in the weighted spaces \( B(s, -s) \) for some suitable \( s > 0 \). However, when \( t > 1 \) then \( R^+_V(\lambda) \) satisfies different estimates for \( 1 < \eta < t^{-1/2} \) and for \( \eta > 1 \) by the low energy asymptotic expansions Theorem 3.8 and the high energy decay estimates Theorem 5.1. Thus for the term \( R^+_V(\eta; x, y) \), when \( t > 1 \), we split \( t^{-1/2} < \eta < \infty \) into middle part \( t^{-1/2} < \eta < t \) and large part \( \eta > t \).

For small time \( 0 < t \leq 1 \) and large time \( t > 1 \) with high energy \( \eta > t \), we have:

**Proposition 5.9.** For \( H = \Delta^2 - \Delta + V \) with \( |V(x)| \lesssim (1 + |x|)^{-\beta} \) for some \( \beta > 3 \). Assume that \( H \) has no positive embedded eigenvalue. For \( t > 1 \), then

\[ \sup_{x, y \in \mathbb{R}^3} \left| \int^\infty_{t^{1/4}} e^{-i(\eta^4 + \eta^2)} \left[ R^+_V(\lambda; x, y) - R^-_V(\lambda; x, y) \right] (4\eta^3 + 2\eta) \ d\eta \right| \lesssim |t|^{-3/4}. \]

For \( 0 < t \leq 1 \), then

\[ \sup_{x, y \in \mathbb{R}^3} \left| \int^\infty_{t^{1/4}} e^{-i(\eta^4 + \eta^2)} \left[ R^+_V(\lambda; x, y) - R^-_V(\lambda; x, y) \right] (4\eta^3 + 2\eta) \ d\eta \right| \lesssim |t|^{-3/4}. \]

For large time \( t > 1 \) with middle energy \( t^{-1/2} < \eta < t \), we have:

**Proposition 5.10.** For \( H = \Delta^2 - \Delta + V \) with \( |V(x)| \lesssim (1 + |x|)^{-\beta} \) for some \( \beta > 0 \). Assume that \( H \) has no positive embedded eigenvalue.

1. If 0 is a regular point of \( H \), let \( \beta > 3 \), then

\[ \sup_{x, y \in \mathbb{R}^3} \left| \int^t_{t^{1/2}} e^{-i(\eta^4 + \eta^2)} \left[ R^+_V(\lambda; x, y) - R^-_V(\lambda; x, y) \right] (4\eta^3 + 2\eta) \ d\eta \right| \lesssim |t|^{-3/2}, \ t > 1. \]

2. If 0 is a resonance of \( H \), let \( \beta > 5 \), then

\[ \sup_{x, y \in \mathbb{R}^3} \left| \int^t_{t^{1/2}} e^{-i(\eta^4 + \eta^2)} \left[ R^+_V(\lambda; x, y) - R^-_V(\lambda; x, y) \right] (4\eta^3 + 2\eta) \ d\eta \right| \lesssim |t|^{-1}, \ t > 1. \]

3. If 0 is a resonance and / or eigenvalue of \( H \), let \( \beta > 7 \), then

\[ \sup_{x, y \in \mathbb{R}^3} \left| \int^t_{t^{1/2}} e^{-i(\eta^4 + \eta^2)} \left[ R^+_V(\lambda; x, y) - R^-_V(\lambda; x, y) \right] (4\eta^3 + 2\eta) \ d\eta \right| \lesssim |t|^{-1/2}, \ t > 1. \]

Since \( \|R^+_V(\lambda)\|_{B(s, -s)} \) is continuous for \( \lambda > 0 \) with suitable \( s > 0 \), thus \( \|R^+_V(\eta^4 + \eta^2)\|_{B(s, -s)} \) for \( t^{-1/2} < \eta < t \) is controlled by \( \|R^+_V(\eta^4 + \eta^2)\|_{B(s, -s)} \) for \( \eta \) near zero or \( \eta \) approaches \( \infty \). However, the presence of zero resonance or eigenvalue affect the asymptotic property of \( R^+_V(\lambda) \), see Theorem 3.8. Thus for the middle energy \( t^{-1/2} < \eta < t \) (\( t > 1 \)), \( R^+_V(\eta^4 + \eta^2) \) satisfies the following estimates:
Lemma 5.11. For \( H = \Delta^2 - \Delta + V \) with \(|V(x)| \leq (1 + |x|)^{-\beta} \) for some \( \beta > 0 \). Assume that \( H \) has positive embedded eigenvalue. Then
\[
\left\| R^\pm_\eta(\eta^4 + \eta^2) \right\|_{B(s,-s)} \lesssim \begin{cases} 
1, & 0 \text{ is regular;} \\
\eta^{-1}, & 0 \text{ is a resonance;} \\
\eta^{-2}, & 0 \text{ is a resonance and/or eigenvalue;}
\end{cases}
\]
and
\[
\left\| \frac{d}{d\eta} R^\pm_\eta(\eta^4 + \eta^2) \right\|_{B(s,-s)} \lesssim \begin{cases} 
1, & 0 \text{ is regular;} \\
\eta^{-2}, & 0 \text{ is a resonance;} \\
\eta^{-3}, & 0 \text{ is a resonance and/or eigenvalue.}
\end{cases}
\]

Here we choose \( \beta > 3 \), \( s > 3/2 \) if \( 0 \) is a regular point, \( \beta > 5 \), \( s > 5/2 \) if \( 0 \) is a resonance and \( \beta > 7 \), \( s > 7/2 \) if \( 0 \) is a resonance and/or eigenvalue.

Proof. Recall that
\[
R^\pm_\eta(\eta^4 + \eta^2) = R^\pm_0(\eta^4 + \eta^2) - R^\pm_0(\eta^4 + \eta^2)v(\beta^\pm(\eta^4 + \eta^2))^{-1} vR^\pm_0(\eta^4 + \eta^2)
\]
where \( v(x) = |V(x)|^{1/2} \). Thus
\[
\frac{d}{d\eta} R^\pm_\eta(\eta^4 + \eta^2) = \left[ \frac{d}{d\eta} R^\pm_0(\eta^4 + \eta^2) \right] - \left[ \frac{d}{d\eta} R^\pm_0(\eta^4 + \eta^2)v(\beta^\pm(\eta^4 + \eta^2))^{-1} vR^\pm_0(\eta^4 + \eta^2) \right]
\]
Further, since
\[
\frac{d}{d\eta}(\beta^\pm(\eta^4 + \eta^2))^{-1} = \frac{d}{d\eta}(U + vR^\pm_0(\eta^4 + \eta^2)v)^{-1} = (\beta^\pm(\eta^4 + \eta^2))^{-1} \left[ \frac{d}{d\eta} R^\pm_0(\eta^4 + \eta^2) \right] v(\beta^\pm(\eta^4 + \eta^2))^{-1}.
\]
Substituting asymptotic expansions of \((\beta^\pm(\eta^4 + \eta^2))^{-1}\) and \(\frac{d}{d\eta} R^\pm_0(\eta^4 + \eta^2)\) (see Lemma 3.1 and Theorem 8.8) into the preceding identity, then we obtain the asymptotic expansion of \((\beta^\pm(\eta^4 + \eta^2))^{-1}\) as \( \eta \to 0 \). Especially, when \( 0 \) is a resonance and/or eigenvalue, then
\[
\frac{d}{d\eta}(\beta^\pm(\eta^4 + \eta^2))^{-1} = (\eta^{-2} S_2 (S_2 v G_2 v S_2)^{-1} S_2 + \eta^{-1} A^\pm_{-1} + A^\pm_0 + O(\eta))v \left( \pm i G_1 + 2\eta G_2 + O(\eta^2) \right) v
\]
Recall that \( PS_2 = S_2 P = 0 \). Thus the term of \( \eta^{-4} \) is
\[
S_2 (S_2 v G_2 v S_2)^{-1} S_2 v G_1 v S_2 (S_2 v G_2 v S_2)^{-1} S_2 = \frac{||V||_4}{4\pi} S_2 (S_2 v G_2 v S_2)^{-1} S_2 P S_2 (S_2 v G_2 v S_2)^{-1} S_2 = 0.
\]
By Lemma 4.1, then
\[
\sup_{\eta \geq 0} \left\| R^\pm_0(\eta^4 + \eta^2) \right\|_{B(s,-s)} \leq 1, \quad \sup_{\eta \geq 0} \left\| \frac{d}{d\eta} R^\pm_0(\eta^4 + \eta^2) \right\|_{B(s,-s)} \leq 1.
\]
Note that \((\beta^\pm(\eta^4 + \eta^2))^{-1}\) exists in \( L^2 \). Hence the lemma holds by Theorem 8.8 and 5.1. \( \square \)
Next, we give the proof of Proposition 5.9 and 5.10. For the second term $R_0^\pm(\eta; x, y)$ in (5.7), we have:

**Lemma 5.12.** Let $|V(x)| \leq (1 + |x|)^\beta$ for some $\beta > 3$. For $t > 1$, then

$$\sup_{x, y, \eta \in \mathbb{R}^3} \left| \int_{t^{-1/2}}^\infty e^{-i(\eta^4 + \eta^2)( \eta; x, y) (4\eta^3 + 2\eta)} \, d\eta \right| \lesssim |t|^{-3/2}. $$

For $0 < t \leq 1$, then

$$\sup_{x, y, \eta \in \mathbb{R}^3} \left| \int_{t^{-1/4}}^\infty e^{-i(\eta^4 + \eta^2)( \eta; x, y) (4\eta^3 + 2\eta)} \, d\eta \right| \lesssim |t|^{-3/4}. $$

**Proof.** By Lemma 4.1 then

$$|R_0^\pm(\eta; x, y)| \lesssim \|V\|_{L^1(\mathbb{R}^3)} \left( \sup_{x, y, \lambda \in \mathbb{R}^3} |R_0^\pm(\lambda; x, y)| \right)^2 \lesssim \frac{1}{1 + 2\eta^2},$$

and

$$\left| \frac{d}{d\eta} R_0^\pm(\eta; x, y) \right| \lesssim \|V\|_{L^1(\mathbb{R}^3)} \left( \sup_{x, y, \lambda \in \mathbb{R}^3} \left| \frac{d}{d\eta} R_0^\pm(\lambda; x, y) \right| \right) \left( \sup_{x, y, \lambda \in \mathbb{R}^3} \left| R_0^\pm(\lambda; x, y) \right| \right) \lesssim \frac{1 + \eta}{(1 + 2\eta^2)^2}. $$

For $0 < t \leq 1$, integrating by parts, then

$$\left| \int_{t^{-1/4}}^{\infty} e^{-i(\eta^4 + \eta^2)( \eta; x, y) (4\eta^3 + 2\eta)} \, d\eta \right| \lesssim \frac{1}{|t|} e^{-i(\eta^4 + \eta^2)( \eta; x, y)} \int_{t^{-1/4}}^{\infty} \left| \frac{d}{d\eta} R_0^\pm(\eta; x, y) \right| \, d\eta \lesssim \frac{1}{|t| (1 + |t|^{-1/2})} \lesssim |t|^{-3/4}, \ 0 < t \leq 1.$$ 

For $t > 1$, we have:

$$\left| \int_{t^{-1/2}}^{\infty} e^{-i(\eta^4 + \eta^2)( \eta; x, y) (4\eta^3 + 2\eta)} \, d\eta \right| \lesssim \left| \int_{t^{-1/2}}^{\infty} e^{-i(\eta^4 + \eta^2)} \left[ R_0^+ - R_0^- \right](\eta; x, y) (4\eta^3 + 2\eta) \, d\eta \right|$$

$$\lesssim \left| \int_{t^{-1/2}}^{\infty} e^{-i(\eta^4 + \eta^2)} \left( (R_0^+ - R_0^-) V R_0^\pm \right)(\eta; x, y) \, d(\eta^4 + \eta^2) \right|$$

$$+ \left| \int_{t^{-1/2}}^{\infty} e^{-i(\eta^4 + \eta^2)} \left[ R_0^- V (R_0^+ - R_0^-) \right](\eta; x, y) \, d(\eta^4 + \eta^2) \right|$$

$$:= I(t) + II(t).$$
For the first term $I(t)$, integrating by parts, then

$$I_1(t) \lesssim \frac{1}{|t|} \left| e^{-it(q^4 + q^5)} \left[ (R_0^+ - R_0^-) VR_0^+ \right](\eta; x, y) \right|_{t=-1/2}^{|t|} + \frac{1}{|t|} \int_{-1/2}^{|t|} e^{-it(q^4 + q^5)} \frac{d}{d\eta} \left[ (R_0^+ - R_0^-) VR_0^+ \right](\eta; x, y) d\eta$$

$$\lesssim \frac{1}{|t|} \left| \frac{1}{(1 + 2\eta^2)^2} \int_{\mathbb{R}^3} \left( \frac{e^{i\eta|y-x|} - e^{-i\eta|y-x|}}{4\pi|y-x|} \right) V(x_1) \left( \frac{e^{i\eta|y-x|} - e^{-\sqrt{1 + \eta^2}|y-x|}}{4\pi|y-x|} \right) dx_1 \right|_{\eta=1/2}^{|t|}$$

$$+ \frac{1}{|t|} \left| \int_{-1/2}^{|t|} e^{-it(q^4 + q^5)} \frac{d}{d\eta} \left[ (R_0^+ - R_0^-) VR_0^+ \right](\eta; x, y) d\eta \right|$$

$$\leq |t|^{-1/2} \|V\|_{L^1(\mathbb{R}^3)} + \frac{1}{|t|} \left| \int_{-1/2}^{|t|} e^{-it(q^4 + q^5)} \frac{d}{d\eta} \left[ (R_0^+ - R_0^-) VR_0^+ \right](\eta; x, y) d\eta \right|.$$

Now, we aim to show that

$$\sup_{x,y \in \mathbb{R}^3} \left| \int_{-1/2}^{|t|} e^{-it(q^4 + q^5)} \frac{d}{d\eta} \left[ (R_0^+ - R_0^-) VR_0^+ \right](\eta; x, y) d\eta \right| \lesssim |t|^{-1/2}, \quad t > 1.$$
For the remaining terms, similarly, we have

\[
\left| \int_{t-1/2}^{\infty} e^{-it(q^4+\eta^2)} \frac{i}{(1+2\eta^2)^2} \int_{\mathbb{R}^3} \left( e^{i|yx|} + e^{-i|yx|} \right) V(x_1) \left( e^{i|yx|} - e^{-\sqrt{1+\eta^2}|yx|} \right) dx_1 d\eta \right|
\]

\[
= \left| \int_{\mathbb{R}^3} \int_{t-1/2}^{\infty} \left( e^{-it(q^4+\eta^2)} + e^{-it(q^4+\eta^2)} \right) \left( e^{i|yx|} - e^{-\sqrt{1+\eta^2}|yx|} \right) d\eta V(x_1) dx_1 \right| \leq |t|^{-1/2}.
\]

Applying van der Corput’s lemma again, we also obtain

\[
\left| \int_{t-1/2}^{\infty} e^{-it(q^4+\eta^2)} \frac{1}{(1+2\eta^2)^2} \int_{\mathbb{R}^3} \left( e^{i|yx|} - e^{-i|yx|} \right) V(x_1) \left( i e^{i|yx|} + \eta(1+\eta^2)^{-1/2} e^{-\sqrt{1+\eta^2}|yx|} \right) dx_1 d\eta \right|
\]

\[
\leq \left| \int_{\mathbb{R}^3} \int_{t-1/2}^{\infty} \left( e^{-it(q^4+\eta^2)} - e^{-it(q^4+\eta^2)} \right) \frac{\eta(1+\eta^2)^{-1/2}}{16\pi^2(1+2\eta^2)^2} dx_1 d\eta \right|
\]

\[
+ \left| \int_{t-1/2}^{\infty} e^{-it(q^4+\eta^2)} \int_{\mathbb{R}^3} \frac{\eta(1+\eta^2)^{-1/2}}{16\pi^2(1+2\eta^2)^2} \left( e^{i|yx|} - e^{-i|yx|} \right) V(x_1) e^{-\sqrt{1+\eta^2}|yx|} dx_1 d\eta \right|
\]

\[
\leq |t|^{-1/2}.
\]

Hence, we get \( I(t) \leq |t|^{-3/2} \) for \( t > 1 \). Applying the similar arguments to \( II(t) \) as for \( I(t) \), we can obtain \( II(t) \leq |t|^{-3/2} \) for \( t > 1 \).

For the third term \( R^+_y(\eta; x, y) \) in (5.7), we have:

**Lemma 5.13.** Let \(|V(x)| \leq (1 + |x|)^\beta \) for some \( \beta > 3 \). For \( t > 1 \), then

\[
\sup_{x,y \in \mathbb{R}^3} \left| \int_t^{\infty} e^{-it(q^4+\eta^2)} \left[ R^+_y - R^-_y \right] \eta \right| (4\eta^3 + 2\eta) \right) dx \right| \leq |t|^{-3/2}.
\]

For \( 0 < t \leq 1 \), then

\[
\sup_{x,y \in \mathbb{R}^3} \left| \int_{t^{-1/4}}^{\infty} e^{-it(q^4+\eta^2)} R^+_y(\eta; x, y) (4\eta^3 + 2\eta) \right| dx \right| \leq |t|^{-3/4}.
\]

**Proof.** Note that \( L^\infty(\mathbb{R}^3) \subset L^{2-s}(\mathbb{R}^3) \) for any \( s > 3/2 \). From Lemma 4.1 and Theorem 5.1, then

\[
\left\| R^+_y(\eta) \right\|_{L^1 \rightarrow L^\infty} \leq \left\| R^+_y(\lambda) \right\|_{L^1 \rightarrow L^\infty} \left\| VR^+_y(\lambda)V \right\|_{L^{2-s} \rightarrow L^2} \left\| R^+_0(\lambda) \right\|_{L^1 \rightarrow L^{2-s}}
\]

\[
\leq \left\| R^+_y(\lambda) \right\|_{L^1 \rightarrow L^\infty} \left\| VR^+_y(\lambda)V \right\|_{L^{2-s} \rightarrow L^2} \left\| R^+_0(\lambda) \right\|_{L^1 \rightarrow L^{2-s}} \left\| V \right\|_{L^{2-s} \rightarrow L^2} \left\| R^+_0(\lambda) \right\|_{L^1 \rightarrow L^{2-s}}
\]

\[
\leq \left( \frac{\eta + \sqrt{1+\eta^2}}{1+2\eta^2} \right)^2 \left( \eta^4 + \eta^2 \right)^{-3/4}
\]

\[
\leq \frac{1}{1+2\eta^2} \frac{\eta^{-3/2}}{(1+\eta^2)^{3/4}} \leq \frac{1}{\eta^{3/2}(1+\eta^2)^{3/4}} \leq \eta^{-1}.
\]
Similarly, for $1 < \eta$, we have

$$
\left\| \frac{d}{d\eta} R^e_V(\eta) \right\|_{L^1 \to L^\infty} \lesssim \left\| \frac{d}{d\eta} R^e_0(\lambda) \right\|_{L^1_t L^\infty_x} \left\| V \right\|_{L^2_t L^4_x} \left\| R^e_V(\lambda) \right\|_{L^1_t L^2_x} \left\| V \right\|_{L^2_t L^4_x} \left\| R^e_0(\lambda) \right\|_{L^1_t L^2_x} + \left\| R^e_V(\lambda) \right\|_{L^1_t L^\infty_x} \left\| V \right\|_{L^2_t L^4_x} \left\| R^e_0(\lambda) \right\|_{L^1_t L^2_x} \left\| V \right\|_{L^2_t L^4_x} \left\| R^e_0(\lambda) \right\|_{L^1_t L^2_x}
$$

\[
\lesssim \left( \frac{\eta(\eta + \sqrt{1 + \eta^2})}{(1 + 2\eta^2)^2} + \frac{1}{\eta^2} \frac{\eta^3}{1 + 2\eta^2} \right) \left( \eta^4 + \eta^2 \right)^{-3/2} \left( \eta^4 + 2\eta \right) + \frac{1}{\eta^2(1 + \eta^2)^{3/2}} \leq \eta^{-2}.
\]

Integrating by parts, for $0 < t \leq 1$ then

$$
\left| \int_{t^{1/4}}^{\infty} e^{-i(t^4 + \eta^2)} R^e_V(\eta; x, y) d(\eta^4 + \eta^2) \right| \leq \frac{1}{|t|^4} \left| e^{-i(t^4 + \eta^2)} R^e_V(\eta; x, y) \right|_{t^{1/4}}^{\infty} + \frac{1}{|t|} \int_{t^{1/4}}^{\infty} \left| \frac{d}{d\eta} R^e_V(\eta; x, y) \right| d\eta 
$$

$$
\leq \frac{1}{|t|} \left| \left( \frac{\eta^4 + 2\eta}{\eta^2(1 + \eta^2)^{3/4}} \left| e^{-i(t^4 + \eta^2)} R^e_V(\eta; x, y) \right| d\eta \right| \leq |t|^{-3/4}.
$$

For $t > 1$, we have

$$
\left| \int_{t}^{\infty} e^{-i(t^4 + \eta^2)} R^e_V(\eta; x, y) d(\eta^4 + \eta^2) \right| \leq \int_{t}^{\infty} \left( 4\eta^3 + 2\eta \right) R^e_V(\eta; x, y) d\eta 
$$

$$
\leq \int_{t}^{\infty} \left( \frac{4\eta^3 + 2\eta}{\eta^2(1 + \eta^2)^{3/4}} \right) d\eta \leq |t|^{-3/2}.
$$

\[\Box\]

**Proof of Proposition 5.10** It is enough to show that $\left[ (R^+_0 - R^-_0) VR^+_V VR^-_V \right](\eta; x, y)$ and $\left[ R^+_0 VR^+_V V(R^+_0 - R^-_0) \right](\eta; x, y)$ satisfy the corresponding estimates in Proposition 5.10. Here, we show the proof for $\left[ (R^+_0 - R^-_0) VR^+_V VR^-_V \right](\eta; x, y)$ since $\left[ R^+_0 VR^+_V V(R^+_0 - R^-_0) \right](\eta; x, y)$ holds by the same arguments.

Integrating by parts, then

$$
\left| \int_{t^{1/2}}^{\infty} e^{-i(t^4 + \eta^2)} \left( (R^+_0 - R^-_0) VR^+_V VR^-_V \right)(\eta; x, y) \left( 4\eta^3 + 2\eta \right) d\eta \right| 
$$

$$
\lesssim \frac{1}{|t|^4} \left| e^{-i(t^4 + \eta^2)} \left( (R^+_0 - R^-_0) VR^+_V VR^-_V \right)(\eta; x, y) \right|_{t^{1/2}}^{\infty} + \frac{1}{|t|} \int_{t^{1/2}}^{\infty} e^{-i(t^4 + \eta^2)} \left| \frac{d}{d\eta} \left( (R^+_0 - R^-_0) VR^+_V VR^-_V \right)(\eta; x, y) \right| d\eta 
$$

$$
\lesssim \frac{1}{|t|} \left| \left( (R^+_0 - R^-_0) VR^+_V VR^-_V \right)(\eta; x, y) \right|_{t^{1/2}}^{\infty} + \frac{1}{|t|} \int_{t^{1/2}}^{\infty} \left| \frac{d}{d\eta} \left( (R^+_0 - R^-_0) VR^+_V VR^-_V \right)(\eta; x, y) \right| d\eta 
$$

\[
\leq \frac{1}{|t|} \left| \right| \left( (R^+_0 - R^-_0) VR^+_V VR^-_V \right)(\eta; x, y) \right|_{t^{1/2}}^{\infty} + \frac{1}{|t|} \int_{t^{1/2}}^{\infty} \left| \frac{d}{d\eta} \left( (R^+_0 - R^-_0) VR^+_V VR^-_V \right)(\eta; x, y) \right| d\eta.
\]

$$
\leq |t|^{-2} + \frac{1}{|t|} \left| \right| \left( (R^+_0 - R^-_0) VR^+_V VR^-_V \right)(\eta; x, y) \right|_{t^{1/2}}^{\infty} + \frac{1}{|t|} \int_{t^{1/2}}^{\infty} \left| \frac{d}{d\eta} \left( (R^+_0 - R^-_0) VR^+_V VR^-_V \right)(\eta; x, y) \right| d\eta.
$$
By Lemma 4.1 then
\[
\left\| (R_0^+ - R_0^-) VR^\pm V R_0^\pm \right\|_{L^1 \to L^\infty} \leq \left\| R_0^+ - R_0^- \right\|_{L^2 \to L^\infty} \left\| VR^\pm V \right\|_{L^2 \to L^2} \left\| R_0^\pm \right\|_{L^1 \to L^2}.
\]
Similarly, we have
\[
\left\| \frac{d}{d\eta} (R_0^+ - R_0^-) VR^\pm V R_0^\pm \right\|_{L^1 \to L^\infty} \leq \left\| \frac{d}{d\eta} (R_0^+ - R_0^-) \right\|_{L^2 \to L^\infty} \left\| VR^\pm V \right\|_{L^2 \to L^2} \left\| R_0^\pm \right\|_{L^1 \to L^2} + \frac{\eta + \sqrt{1 + \eta^2}}{1 + 2\eta^2} \left\| R_0^\pm \right\|_{L^1 \to L^2}.
\]
Thus by Lemma 5.11 for \( t > 1 \) we have
\[
\frac{1}{|t|} \left| (R_0^+ - R_0^-) VR^\pm V R_0^\pm (t^{-1/2}; x, y) \right| \leq \begin{cases} |t|^{-3/2}, & \text{0 is regular;} \\ |t|^{-1}, & \text{0 is a resonance;} \\ |t|^{-1/2}, & \text{0 is a resonance and/or eigenvalue.} \end{cases}
\]
When 0 is a resonance, then
\[
\frac{1}{|t|} \int_{r^{-1/2}}^{t} \left| \frac{d}{d\eta} (R_0^+ - R_0^-) VR^\pm V R_0^\pm (\eta; x, y) \right| d\eta \leq \frac{1}{|t|} \int_{r^{-1/2}}^{t} \frac{\eta + \sqrt{1 + \eta^2}}{1 + 2\eta^2} \eta^{-1} d\eta \leq |t|^{-1}, \quad t > 1.
\]
When 0 is a resonance and/or an eigenvalue, then
\[
\frac{1}{|t|} \int_{r^{-1/2}}^{t} \left| \frac{d}{d\eta} (R_0^+ - R_0^-) VR^\pm V R_0^\pm (\eta; x, y) \right| d\eta \leq \frac{1}{|t|} \int_{r^{-1/2}}^{t} \eta^{-2} d\eta \leq |t|^{-1/2}, \quad t > 1.
\]
However, when 0 is a regular point, we will show that
\[
\frac{1}{|t|} \int_{r^{-1/2}}^{t} e^{-it(\eta^4 + \eta^2)} \left| \frac{d}{d\eta} (R_0^+ - R_0^-) VR^\pm V R_0^\pm (\eta; x, y) \right| d\eta \leq |t|^{-3/2}, \quad t > 1.
\]

\[\square\]

**Lemma 5.14.** If 0 is a regular point of \( H \), for \( t > 1 \), then
\[
\sup_{x, y \in \mathbb{R}^3} \left| \int_{r^{-1/2}}^{t} e^{-it(\eta^4 + \eta^2)} \left| \frac{d}{d\eta} (R_0^+ - R_0^-) VR^\pm V R_0^\pm (\eta; x, y) \right| d\eta \right| \leq |t|^{-1/2}.
\]

**Proof.** Since
\[
R^\pm_V = R^\pm_0 - R^\pm_0 v M^\pm (\eta^4 + \eta^2)^{-1} v R^\pm_0,
\]
thus it is enough to show that
\[
\sup_{x,y\in \mathbb{R}^3} \left| \int_{t^{-1/2}}^t e^{-it(\xi^2 + \eta^2)} \frac{d}{d\eta} \left[ (R_0^+ - R_0^-) VR_0^+ VR_0^+ \right] (\eta; x, y) \ d\eta \right| \lesssim |t|^{-1/2} \tag{5.8}
\]
and
\[
\sup_{x,y\in \mathbb{R}^3} \left| \int_{t^{-1/2}}^t e^{-it(\xi^2 + \eta^2)} \frac{d}{d\eta} \left[ (R_0^+ - R_0^-) VR_0^+ (M^\pm)^{-1} vR_0^+ VR_0^+ \right] (\eta; x, y) \ d\eta \right| \lesssim |t|^{-1/2}. \tag{5.9}
\]

Applying the similar arguments to \(\frac{d}{d\eta} \left[ (R_0^+ - R_0^-) VR_0^+ VR_0^+ \right] (\eta; x, y)\) as what we treat the term \((R_0^+ - R_0^-) VR_0^+\) in the proof of Lemma 5.12 we can get estimate (5.8) holds. Next, we apply the van der Corput’s lemma to show (5.9) holds. Denotes
\[
\mathcal{R}_M(\eta; x, y) = \left[ (R_0^+ - R_0^-) VR_0^+ \left( \frac{d}{d\eta} (M^\pm)^{-1} \right) vR_0^+ VR_0^+ \right] (\eta; x, y);
\]
\[
\mathcal{R}_1(\eta; x, y) = \left[ \left( \frac{d}{d\eta} (R_0^+ - R_0^-) \right) VR_0^+ (M^\pm)^{-1} vR_0^+ VR_0^+ \right] (\eta; x, y);
\]
\[
\mathcal{R}_2(\eta; x, y) = \left[ (R_0^+ - R_0^-) V \left( \frac{d}{d\eta} R_0^+ \right) v(M^\pm)^{-1} vR_0^+ VR_0^+ \right] (\eta; x, y);
\]
\[
\mathcal{R}_3(\eta; x, y) = \left[ (R_0^+ - R_0^-) VR_0^+ v(M^\pm)^{-1} v \left( \frac{d}{d\eta} R_0^+ \right) VR_0^+ \right] (\eta; x, y);
\]
\[
\mathcal{R}_4(\eta; x, y) = \left[ (R_0^+ - R_0^-) VR_0^+ v(M^\pm)^{-1} vR_0^+ V \left( \frac{d}{d\eta} R_0^+ \right) \right] (\eta; x, y).
\]

By Lemma 4.1 and 5.11 for \(s > 3/2\), then
\[
\sup_{x,y\in \mathbb{R}^3} \left| \mathcal{R}_M(\eta; x, y) \right| \lesssim \left\| R_0^+ - R_0^- \right\|_{L^2 \to L^\infty} \left\| VR_0^+ \right\|_{L^2 \to L^2} \left\| \frac{d}{d\eta} (M^\pm)^{-1} \right\|_{L^2 \to L^2} \left\| vR_0^+ V \right\|_{L^4 \to L^4} \left\| R_0^+ \right\|_{L^1 \to L^\infty} 
\lesssim \frac{\eta (1 + \eta)}{(1 + 2\eta^2)^2}.
\]

Similarly, we have
\[
\sup_{x,y\in \mathbb{R}^3} \left| \mathcal{R}_j(\eta; x, y) \right| \lesssim \frac{\eta (1 + \eta)}{(1 + 2\eta^2)^2}, \quad j = 2, 3;
\]
\[
\sup_{x,y\in \mathbb{R}^3} \left| \mathcal{R}_j(\eta; x, y) \right| \lesssim \frac{1 + \eta}{(1 + 2\eta^2)^2}, \quad j = 1, 4.
\]

Hence \(\sup_{\eta \geq 0} \sup_{x,y\in \mathbb{R}^3} \left| \mathcal{R}_j(\eta; x, y) \right| \lesssim 1\) for \(j = M, 1, 2, 3, 4\). Furthermore, we have
\[
\sup_{x,y\in \mathbb{R}^3} \left| \frac{d}{d\eta} \mathcal{R}_M(\eta; x, y) \right| \lesssim \sum_{j=1}^4 \sup_{x,y\in \mathbb{R}^3} \left| \mathcal{R}_j(\eta; x, y) \right| + \left\| R_0^+ - R_0^- \right\|_{L^1 \to L^\infty} \left\| (1 + |\cdot|) V R_0^+ V \right\|_{L^2 \to L^2} 
\times \left\| \left( \frac{d^2}{d\eta^2} (M^\pm)^{-1} \right) \right\|_{L^2 \to L^2} \left\| vR_0^+ V \right\|_{L^2 \to L^\infty} \left\| R_0^+ \right\|_{L^1 \to L^\infty} 
\lesssim \frac{\eta (1 + \eta)}{(1 + 2\eta^2)^2} + \frac{1 + \eta}{(1 + 2\eta^2)^2} \in L^1(\mathbb{R}).
\]
Here, since 0 is a regular point of $H$, by Theorem 3.8, then $\|(M^\pm)^{-1}\|_{L^2 \to L^2} \leq 1$. Thus
\[
\|\frac{d^2}{d\eta^2}(M^\pm)^{-1}\|_{L^2 \to L^2} \leq \|(M^\pm)^{-1}\|_{L^2 \to L^2} \left(\frac{d^2}{d\eta^2} R_0^\pm\right) v(M^\pm)^{-1} \|_{L^2 \to L^2} + \left(\frac{d}{d\eta} R_0^\pm\right) v(M^\pm)^{-1} \|_{L^2 \to L^2} 
\leq 1 + \|v(x)|x-y| v(y)\|_{L^1(R^3) \to L^2(R^3)} \|(M^\pm)^{-1}\|_{L^2 \to L^2} \leq 1.
\]

For $\mathcal{R}_j(\eta; x, y)$, $j = 2, 3$, we also have
\[
\sup_{x,y \in \mathbb{R}^3} \left| \frac{d}{d\eta} \mathcal{R}_j(\eta; x, y) \right| \leq \frac{\eta(1 + \eta)}{(1 + 2\eta^2)^2} + \frac{1 + \eta}{(1 + 2\eta^2)^2} \in L^1(\mathbb{R}).
\]

By the van der Corput’s lemma, we obtain for $j = M, 2, 3$:
\[
\sup_{x,y \in \mathbb{R}^3} \left| \int_{t^{-1/2}}^t e^{-i(\eta^2 + \eta^2)} \mathcal{R}_j(\eta; x, y) \, d\eta \right| \leq |t|^{-1/2}.
\]

For $\mathcal{R}_1(\eta; x, y)$, since
\[
\mathcal{R}_1(\eta; x, y) = \int_{\mathbb{R}^1} \frac{d}{d\eta} \left( (R_0^+ - R_0^-)(\eta; x, x_1) V(x_1) R_0^+(\eta; x_1, x_2) v(x_2) (M^\pm)^{-1}(\eta; x_2, x_3) v(x_3) \right) 
\times R_0^+(\eta; x_3, x_4) V(x_4) \, dx_1 dx_2 dx_3 dx_4 
\]
\[
= \int_{\mathbb{R}^1} \left( e^{i\eta(x-x_1)} + e^{-i\eta|x-x_1|}\right) V(x_1) R_0^+(\eta; x_1, x_2) v(x_2) (M^\pm)^{-1}(\eta; x_2, x_3) v(x_3) 
\times R_0^+(\eta; x_3, x_4) V(x_4) \, dx_1 dx_2 dx_3 dx_4 
\]
\[
= \int_{\mathbb{R}^1} \left( e^{i\eta(x-x_1)} + e^{-i\eta|x-x_1|}\right) M(\eta; x_1, x_2, x_3, x_4, y) \, dx_1 dx_2 dx_3 dx_4 
\]
\[= \int_{\mathbb{R}^1} \left( e^{i\eta(x-x_1)} + e^{-i\eta|x-x_1|}\right) M(\eta; x_1, x_2, x_3, x_4, y) \, dx_1 dx_2 dx_3 dx_4 
\]
\[
|\int_{t^{-1/2}}^t e^{-i(\eta^2 + \eta^2)} \mathcal{R}_1(\eta; x, y) \, d\eta| = \int_{\mathbb{R}^1} \int_{t^{-1/2}}^t \left( e^{-i\eta^2 + \eta^2 - \eta|x-x_1|/t} + e^{-i\eta^2 + \eta^2 + \eta|x-x_1|/t} \right) 
\times M(\eta; x_1, x_2, x_3, x_4, y) \, d\eta dx_1 dx_2 dx_3 dx_4 
\]
Then apply the van der Corput’s lemma to the two integrals on the right side, we obtain
\[
\sup_{x,y \in \mathbb{R}^3} \left| \int_{t^{-1/2}}^t e^{-i(\eta^2 + \eta^2)} \mathcal{R}_1(\eta; x, y) \, d\eta \right| \leq |t|^{-1/2}.
\]

For $\mathcal{R}_4(\eta; x, y)$, the above estimate also holds by the similar processes as for $\mathcal{R}_1(\eta; x, y)$. \qed

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