Weight distribution of cosets of small codes with good dual properties

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Abstract

The bilateral minimum distance of a binary linear code is the maximum $d$ such that all nonzero codewords have weights between $d$ and $n - d$. Let $Q \subset \{0, 1\}^n$ be a binary linear code whose dual has bilateral minimum distance at least $d$, where $d$ is odd. Roughly speaking, we show that the average $L_\infty$-distance – and consequently the $L_1$-distance – between the weight distribution of a random coset of $Q$ and the binomial distribution decays quickly as the bilateral minimum distance $d$ of the dual of $Q$ increases. For $d = \Theta(1)$, it decays like $n^{-\Theta(d)}$. On the other $d = \Theta(n)$ extreme, it decays like $e^{-\Theta(d)}$. It follows that, almost all cosets of $Q$ have weight distributions very close to the binomial distribution. In particular, we establish the following bounds. If the dual of $Q$ has bilateral minimum distance at least $d = 2t + 1$, where $t \geq 1$ is an integer, then the average $L_\infty$-distance is at most $\min\{\left(\frac{e \ln \frac{n}{2t}}{2t}\right)^t \frac{2t}{n} e^{-\frac{t}{2}}, \sqrt{2}e^{-\frac{t}{10}}\}$. For the average $L_1$-distance, we conclude the bound $\min\{(2t + 1) \left(\frac{e \ln \frac{n}{2t}}{2t}\right)^t \frac{2t}{n} e^{-\frac{t}{2}} - 1, \sqrt{2}(n+1)e^{-\frac{t}{10}}\}$, which gives nontrivial results for $t \geq 3$. We given applications to the weight distribution of cosets of extended Hadamard codes and extended dual BCH codes. Our argument is based on Fourier analysis, linear programming, and polynomial approximation techniques.

1 Introduction

The weight distribution of a random linear codes is well approximated by the binomial distribution $inom{n}{w}$ (see [MS77], page 287, and Lemma 7.1 in this paper). For nonrandom codes, the binomiality of the weight distribution has been extensively studied in the high rate regime, and applied to rate-1 BCH codes. Strong approximation results were established in the literature assuming that the code dual has good distance properties. Let $Q \subset \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code of dimension $k$ and let $Q^\perp$ be the dual of $Q$. Let $d'$ be the minimum distance of $Q^\perp$, and let $\sigma$ be the width of $Q^\perp$, i.e., the minimum integer $\sigma$ such that $\|y\| - n/2 \leq \sigma/2$ for each nonzero $y \in Q^\perp$. Let $m_Q(0), m_Q(1), \ldots, m_Q(n)$ be the weight distribution of $Q$. That is, $m_Q(w)$ is the fraction of codewords of $Q$ of weight $w$ for $w = 0, \ldots, n$. The results in the literature can be divided in two categories: those assuming that the dual width $\sigma$ is small, and those assuming the weaker condition that the dual minimum distance $d'$ is large.

Assuming that the dual width $\sigma = o(n)$ and the rate $r$ is high (e.g., $r$ close to 1), bounds of the form $m_Q(w) = \binom{n}{w}(1 + E_n)$ were established in [S71, KFL85, S90, KL99], where $E_n$ is the

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approximation error term. This approach was initiated by Sidelnikov [S71] who verified that the error term $|E_w| \leq n^{-0.1}$ if $r$ appropriately tends to 1. The bound on $|E_w|$ was later improved in [KFL85, S90, KL99], yielding exponential decay in $n$ for a certain range of $w$.

In [KL87, KL93, KL98, ABL01], upper bounds on $m_Q(w)$ were established assuming that the dual distance $d'$ is large (linear in $n$) and the rate $r$ is high (the bounds at least require that $r$ does not go to zero as the block length $n$ increases). Assuming that $d' = an$, for some constant $0 < \alpha < 1/2$, the bounds are of the form $m_Q(w) = O(\binom{n}{w}(\sqrt{n})$ if $|w - n/2| \leq \beta$, for some constant $0 < \beta < 1/2$ which increases with $\alpha$. Unlike upper bounds on the dual width, lower bounds on the dual distance do not lead to lower bounds on the weight distribution (e.g., the code consisting of even weight strings has dual distance $n - 1$ but $m_Q(w) = 0$ for all odd $w$). All the above bounds use MacWilliams’s identity [Mac63] and bounds on Krawtchouk polynomials.

In [Tie90, Tie91, S94, SS93], lower bounds on the dual distance have been also used to derive upper bounds on the covering radius of the code, which is related to the width of the weight distribution. Another related work is [GR08], where worst case bounds on the moments of the weight distribution of cosets of dual BCH codes were derived based on the minimum distance of the dual code.

1.1 Contribution

In contrast with the above works on the binomiality of the weight distribution of codes, we focus in this paper on the low rate regime, and we study the weight distribution of a random coset of the code rather than of the code itself. Our bounds are for codes with small dual width $\sigma$, but rather than formulating the statements in terms of dual width, we use the equivalent notion of dual bilateral minimum distance. Define the bilateral minimum distance of an $\mathbb{F}_2$-linear code $C \subset \mathbb{F}_2^n$ to be the maximum $d$ such that $d \leq |y| \leq n - d$, for each nonzero $y \in C$. Thus, the bilateral minimum distance $d$ of the dual $Q^\perp$ of $Q$ is related to the width $\sigma$ of $Q^\perp$ via $d = n/2 - \sigma/2$. For technical convenience, we choose to express our results in terms of $d$ rather than $\sigma$. We derive bounds which hold for values of $d$ as small as $d = 3$ and as large as $d = \Theta(n)$. Note that the $d = \Theta(1)$ regime typically corresponds to linear codes of size $n^{\Theta(1)}$ (for random codes). On the other extreme, the $d = \Theta(n)$ regime typically corresponds to linear codes of size $2^{\Theta(n)}$.

Roughly speaking, we show that the average $L_\infty$-distance $- \perp$ and consequently the $L_1$-distance between the weight distribution of a random cosets of $Q$ and the binomial distribution decays quickly as the dual bilateral minimum $d$ increases. For $d = \Theta(1)$, it decays like $n^{-\Theta(d)}$. On the other $d = \Theta(n)$ extreme, it decays like and $e^{-\Theta(d)}$. In particular, we establish the following bounds.

**Theorem 1.1** Let $Q \subseteq \{0,1\}^n$ be an $\mathbb{F}_2$-linear code whose dual has bilateral minimum distance at least $d = 2t + 1$, where $t \geq 1$ is an integer. For each coset $Q + u$ of $Q$, where $u \in \mathbb{F}_2^n$, consider its weight distribution $m_{Q+u}(0), \ldots, m_{Q+u}(n)$. That is, $m_{Q+u}(w)$ is the fraction of vectors in $Q + u$ of weight $w$ for $w = 0, \ldots, n$. Denote the uniform distribution on $\{0,1\}^n$ by $U_n$. If $t \geq 1$,

a) (Small dual distance bound) $E_{w \sim U_n} \|m_{Q+u} - bin_n\|_\infty \leq (e \ln \frac{n}{t^2})^t (\frac{2n}{t^2})^{\frac{t}{2}}$.

Thus, for $t = \Theta(1)$, $E_{w \sim U_n} \|m_{Q+u} - bin_n\|_\infty = O\left(\frac{2n}{t^2}\right)$.

b) (Large dual distance bound) $E_{w \sim U_n} \|m_{Q+u} - bin_n\|_\infty \leq \sqrt{2} e^{-\frac{t}{16}}.$
If $t \geq 3$,

c) (Small dual distance bound) $E_{u \sim U_n \setminus \mathcal{H}} \|m_{Q+u} - \bin n\|_1 \leq (2t + 1) \left( e \ln \frac{w}{n} \right)^t \left( \frac{2t}{n} \right)^{t-1}$.

Thus, for $t = \Theta(1)$, $E_{u \sim U_n \setminus \mathcal{H}} \|m_{Q+u} - \bin n\|_1 = O(\frac{(\ln n)^t}{n^{t-1}})$.

d) (Large dual distance bound) $E_{u \sim U_n \setminus \mathcal{H}} \|m_{Q+u} - \bin n\|_1 \leq \sqrt{2(n+1)e^{-\frac{t}{n}}}$.

For $n$ sufficiently large, the bounds in (a) and (c) are better than those in (b) and (d) as long as $\frac{d}{n} < \delta^*$, where $\delta^* \approx 0.003446$.

The above results are best understood in the $d = \Theta(1)$ regime, which typically corresponds to codes of size $n^{\Theta(1)}$. The weight distribution of a random linear code $Q$ of size $n^{\Theta(1)}$ is $O(n^{-\Theta(1)})$ close to the binomial distribution (see Lemma 7.1). In the nonrandom case, the weight distribution of $Q$ may not be arbitrarily close to the binomial distribution even if the bilateral minimum distance of $Q$ is large. The simplest example is probably the extended Hadamard code $Q = H \cup (H + \vec{1}) \subset \mathbb{F}_2^n$, where $H$ is the $(2^n - 1, r, 2^{r-1})$-Hadamard code, $\vec{1} \in \mathbb{F}_2^n$ is the all ones vector, $n = 2^r - 1$, and $r \geq 2$ is an integer. It is not hard to see that the size of $Q$ is $2(n+1)$ and its bilateral minimum distance is greater than $d = 3$ (see Section 3 for details). It has only 4 possible weights $0, \frac{n+1}{2}, \frac{n}{2},$ and $n$, thus $\|m_Q - \bin n\|_\infty = \Theta(1)$ and $\|m_Q - \bin n\|_1 = \Theta(1)$. However, by Part (a) of Theorem 1.1 with $t = 1$, we have $E_{u \sim U_n \setminus \mathcal{H}} \|m_{Q+u} - \bin n\|_1 = O(\frac{\ln n}{\sqrt{n}})$. It follows that for almost all cosets $Q + u$ of $Q$, we have $\|m_{Q+u} - \bin n\|_\infty = O(\frac{\ln n}{\sqrt{n}})$. Another example is the extended dual BCH code $Q$ of size $(n+1)^t$, where $t \geq 3$ is a constant. Its bilateral minimum distance is at least $2t + 1$ but $\|m_Q - \bin n\|_1 = \Theta(1)$ for all constant values of $t$ (see Section 3 for details). However, by Part (a) of Theorem 1.1 we have $E_{u \sim U_n \setminus \mathcal{H}} \|m_{Q+u} - \bin n\|_1 = O(\frac{(\ln n)^t}{n^{t-1}})$, hence for almost all cosets, we have $\|m_{Q+u} - \bin n\|_1 = O(\frac{(\ln n)^t}{n^{t-1}})$.

1.2 Proof technique

At a high level, the proof of Theorem 1.1 is based on Fourier analysis, linear programming, and polynomial approximation techniques. Our argument is not based on Krawtchouk polynomials, which naturally arise when studying a property of the code given dual constraints. In our problem, we are studying an average over cosets given dual constraints. This lead us to a different type of approximations based on Taylor approximation of the exponential function. Unlike the above $L_\infty$ and $L_1$ statement in Theorem 1.1 our key technical result (Theorem 2.4) is a mean-square-error statement. Using a squared norm enabled us to go the Fourier domain via Parseval’s equality. As in Delsarte’s linear programming approach [Del73], Linear programming naturally arise as a relaxation of the problem of optimizing over codes subject to dual constraints to optimizing over probability distributions satisfying those constraints. Compared to the classical LP relaxations of coding problems, our relaxation does not require the non-negativity of the Fourier transom of the probability distribution. We use the code linearity before the relaxation via Parseval’s equality and a subtle application of MacWilliams’s identity.

1.3 Motivation

The original motivation behind the work reported in this paper was the problem of explicitly constructing for each constant $c > 0$ (a family of) polynomial-size subset $S \subset \mathbb{F}_2^n$ which are
pseudobinomial in the sense that for all $u \in \mathbb{F}_2^n$, the $L_1$-distance between the weight distribution of the translation of $S$ by $u$ is $n^{-c}$-close to the binomial distribution in the $L_1$-sense. One consequence of the result in this paper is that polynomial-size linear codes with good dual properties achieve this goal for almost all $u \in \mathbb{F}_2^n$ (e.g., extended dual BCH codes). We believe that the original goal which requires the stronger condition “for all $u \in \mathbb{F}_2^n$” is not achievable using linear codes (see Section 3.3). In a recent paper [Baz14], we studied the notion of pseudobinomiality in the context of small-bias probability distributions. A probability distribution $\mu$ on $\{0,1\}^n$ is called is $\delta$-biased if $|E_\mu \chi_z| \leq \delta$ for each nonzero $z \in \{0,1\}^n$, where $\chi_z(x) \overset{\text{def}}{=} (-1)^{\sum_i x_i z_i}$ [NN93]. Note that linear codes give rise to highly biased probability distributions because of their defining linear constraints. Namely, if $\mu_Q$ is the probability distribution resulting from choosing a uniformly random element of an $\mathbb{F}_2$-linear code $Q$, then $E_{\mu_Q} \chi_z = 1$ for each $z \in Q \setminus \{0\}$. If instead of a linear code we have a $\delta$-biased probability distribution on $\{0,1\}^n$, using a much simpler argument, bounds similar to those in Theorem 1 hold: $E_{u \sim U_n} ||\mu_u - \text{bin}_n||_1 \leq \delta \sqrt{n} + 1$, where $\mu_u$ is the weight distribution of the $\mathbb{F}_2$-translation of $\mu$ by $u$, i.e., $\mu_u(x) = \mu(x : x + u = w)$ (see Corollary 6.2 in [Baz14]). The result in this paper can be seen as an extension of this bound to biased distributions. We elaborate on the comparison with small-bias probability distributions in Section 6.

1.4 Paper outline

In Section 2 we formally state our results and we reduce them to a mean-square error statement. We give in Section 3 applications to the weight distribution of cosets of extended Hadamard codes and extended dual BCH codes. In Section 3.3 we conjecture that there are no small codes such that the weight distribution of all cosets is arbitrarily close to the binomial distribution in the $L_1$-norm. In Section 4 we give some Fourier transform preliminaries used in the proof. The proof of our main technical result is in Section 5. In Section 6 we compare with small-bias probability distributions. In Section 7 we compare our average $L_1$-approximation error with random codes.

2 Statement of the main result

If $x \in \{0,1\}^n$, the weight of $x$, which we denote by $|x|$, is the number of nonzero coordinates of $x$. If $C \neq \emptyset \subset \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code ($\mathbb{F}_2$ is the finite field structure on $\{0,1\}$), the minimum distance of $C$ is the minimum weight of a nonzero codeword. Define the bilateral minimum distance of $C$ as the maximum $d$ such that $d \leq |y| \leq n - d$, for each nonzero $y \in C$. Note that, by definition, we must have $d \leq n/2$. In most of our proposition, we will assume that $d \geq 3$, thus $n \geq 6$. A related notion is the width of a code $C$ (e.g., [S90]), which is defined as the minimum integer $\sigma$ such that $||x| - n/2| \leq \sigma/2$ for each nonzero $x \in C$. Thus, $d = n/2 - \sigma/2$. For technical convenience, we choose to express our results in terms of bilateral minimum distance rather than width.

If $A \subset \{0,1\}^n$, let $\mu_A$ denote the probability distribution on $\{0,1\}^n$ uniformly distributed on $A$, i.e.,

$$
\mu_A(x) \overset{\text{def}}{=} \begin{cases} 
\frac{1}{|A|} & \text{if } x \in A \\
0 & \text{otherwise.}
\end{cases}
$$

1In some works, e.g., [KFLS], the all ones vector is allowed in $C$, i.e., the width of a $C$ is the minimum integer $\sigma$ such that $||x| - n/2| \leq \sigma/2$ for each $x \in C$ other than zero and the all ones vector. We will not adopt this exception as our result rely on the all ones vector not being present in the dual code.
We will denote the uniform distribution on \( \{0, 1\}^n \) by \( U_n \), i.e., \( U_n = \mu_{\{0,1\}^n} \). We will use the notation \( [0:n] \stackrel{\text{def}}{=} \{0, \ldots, n\} \). If \( \mu \) is a probability distribution on \( \{0, 1\}^n \), let \( \overline{\mu} \) be the corresponding weight distribution on \( [0:n] \), i.e., for all \( w \in [0:n] \),

\[
\overline{\mu}(w) \stackrel{\text{def}}{=} \mu(x \in \{0, 1\}^n : \left| x \right| = w).
\]

Thus, if \( A \subset \{0, 1\}^n \), then \( \overline{\mu}_A(w) \) is the fraction of elements of \( A \) of weight \( w \). Note that \( \overline{\mu}_A(w) = m_A(w) \) in terms of the introductory notations used in Section 1. Let \( \text{bin}_n \) denote the binomial distribution on \( [0:n] \), i.e., \( \text{bin}_n(w) = \binom{n}{w} 2^n \), and note that \( \text{bin}_n = \overline{U}_n \). Finally, we will denote expectation with respect to a probability distribution \( \mu \) by \( E_\mu \).

The following theorem is a restatement of Parts (a) and (b) of Theorem 1.1.

**Theorem 2.1 (L\(_\infty\)-bound)** Let \( Q \subseteq F_2^n \) be an \( F_2 \)-linear code whose dual \( Q^\perp \) has bilateral minimum distance at least \( d = 2t + 1 \), where \( t \geq 1 \) is an integer. Then, we have the bounds:

a) (Small dual distance bound)

\[
E_{u \sim U_n} \left\| \overline{\mu}_{Q+u} - \text{bin}_n \right\|_{\infty} \leq \left( e \ln \frac{n}{2t} \right)^{t} \left( \frac{2t}{n} \right)^{\frac{t}{2}}.
\]

b) (Large dual distance bound)

\[
E_{u \sim U_n} \left\| \overline{\mu}_{Q+u} - \text{bin}_n \right\|_{\infty} \leq \sqrt{2} e^{-\frac{t}{10}}.
\]

An immediate consequence of the above is the following corollary, which is restatement of Parts (c) and (d) of Theorem 1.1.

**Corollary 2.2 (L\(_1\)-bound)** Let \( Q \subseteq F_2^n \) be an \( F_2 \)-linear code whose dual \( Q^\perp \) has bilateral minimum distance at least \( d = 2t + 1 \), where \( t \geq 3 \) is an integer. Then, we have the bounds:

a) (Small dual distance bound)

\[
E_{u \sim U_n} \left\| \overline{\mu}_{Q+u} - \text{bin}_n \right\|_{1} \leq (2t + 1) \left( e \ln \frac{n}{2t} \right)^{t} \left( \frac{2t}{n} \right)^{\frac{t}{2} - 1}.
\]

b) (Large dual distance bound)

\[
E_{u \sim U_n} \left\| \overline{\mu}_{Q+u} - \text{bin}_n \right\|_{1} \leq \sqrt{2(n + 1)} e^{-\frac{t}{10}}.
\]

**Proof:** The bounds follow from Theorem 2.1 since \( \left\| \overline{\mu}_{Q+u} - \text{bin}_n \right\|_{1} = \sum_{w=0}^{n} \left| \overline{\mu}_{Q+u}(w) - \text{bin}_n(w) \right| \leq (n + 1) \left\| \overline{\mu}_{Q+u} - \text{bin}_n \right\|_{\infty} \). Note that in (a) we used the bound \( n + 1 \leq d = \frac{n}{2t} \) (which holds for all \( d \geq 2 \) and \( n \geq 1 \) such that \( d \leq n + 1 \)), hence \( (n + 1) \left( \frac{2t}{n} \right)^{\frac{t}{2}} \leq (2t + 1) \left( \frac{2t}{n} \right)^{\frac{t}{2} - 1} \). Finally, note that the claim hold for all \( t \geq 1 \), but (a) gives nontrivial bounds for \( t \geq 3 \). ■
Corollary 2.3 Let $Q \subseteq \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code whose dual $Q^\perp$ has bilateral minimum distance at least $d$, where $d \geq 7$ is odd. Assume that $d = \Theta(1)$. Then for each $\epsilon > 0$, $E_{w \sim U_n} \|\mu_{Q+u} - \text{bin}_n\|_1 \leq n^{-\frac{d-6}{4}}$, for $n$ large enough. Hence, for each $\xi > 0$, for all but at most an $n^{-\frac{d-6}{4}}$-fraction of the cosets $\{Q + u\}_u$, we have $\|\mu_{Q+u} - \text{bin}_n\|_1 \leq n^{-\frac{d-6}{4}}$, for $n$ large enough.

Proof: This follows from Part (a) of Corollary 2.2 and Markov Inequality. □

Our main technical result is Theorem 2.4 below which unlike the previous statements is a mean-square-error statement.

Theorem 2.4 (Mean-square-error bound) Let $Q \subseteq \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code whose dual $Q^\perp$ has bilateral minimum distance at least $d = 2t + 1$, where $t \geq 1$ is an integer. If $0 \leq \theta < 2\pi$, define $e_\theta : \{0,1\}^n \to \mathbb{C}$ by $e_\theta(x) = e^{i\theta|x|}$. Then, for each $0 \leq \theta < 2\pi$, we have the bounds:

a) (Small dual distance bound)

$$E_{w \sim U_n} |E_{\mu_{Q+u}e_\theta} - E_{U_n e_\theta}|^2 \leq \left( e \ln \frac{n}{2t} \right)^2 \left( \frac{2t}{n} \right)^t$$

b) (Large dual distance bound)

$$E_{w \sim U_n} |E_{\mu_{Q+u}e_\theta} - E_{U_n e_\theta}|^2 \leq 2e^{-\frac{t}{\pi}}.$$  

The proof of Theorem 2.4 is in Section 5. We establish below Theorem 2.1 using Theorem 2.4.

Proof of Theorem 2.1 using Theorem 2.4. If $w \in [0 : n]$, define the indicator function $I_w : \{0,1\}^n \to \{0,1\}$ by $I_w(x) = 1$ iff $|x| = w$. Thus, $\text{bin}_n(w) = E_{U_n I_w}$ and $\mu_{Q+u}(w) = E_{\mu_{Q+u}I_w}$. For each $b \in [0 : n]$, we have the character sum identity

$$\sum_{a=0}^{n-1} e^{2\pi i a b/n} = \begin{cases} n & \text{if } b = 0 \\ 0 & \text{otherwise} \end{cases}$$

It follows that for each $x \in \{0,1\}^n$,

$$I_w(x) = \frac{1}{n} \sum_{a=0}^{n-1} e^{2\pi i (|x| - w) a/n} = \sum_{a=0}^{n-1} \alpha_a e_{\theta_a}(x)$$

where $\alpha_a = \frac{1}{n} e^{-2\pi i a w/n}$ and $\theta_a = \frac{2\pi a}{n}$. Note that $\sum_a |\alpha_a| = 1$. Thus, for all $w \in [0 : n]$ and $u \in \{0,1\}^n$, we have

$$|\mu_{Q+u}(w) - \text{bin}_n(w)| = |E_{\mu_{Q+u}I_w} - E_{U_n I_w}| = \left| \sum_a \alpha_a (E_{\mu_{Q+u}e_{\theta_a}} - E_{U_n e_{\theta_a}}) \right|$$

$$\leq \sum_a |\alpha_a| |E_{\mu_{Q+u}e_{\theta_a}} - E_{U_n e_{\theta_a}}|.$$
By Jensen’s inequality, \( (E_{u \sim U_n}|E_{\mu_{Q+u}}e_\theta - E_{U_n}e_\theta|^2)^2 \leq E_{u \sim U_n}|E_{\mu_{Q+u}}e_\theta - E_{U_n}e_\theta|^2 \), or any \( 0 \leq \theta < 2\pi \). It follows that:

\[
E_{u \sim U_n}\|\mu_{Q+u} - \text{bin}_{n}\|_\infty = E_{u \sim U_n}\max_w |\mu_{Q+u}(w) - \text{bin}_n(w)|
\]

\[
\leq E_{u \sim U_n}\sum_a |\alpha_a| |E_{\mu_{Q+u}}e_{\theta a} - E_{U_n}e_{\theta a}|
\]

\[
= \sum_a |\alpha_a| E_{u \sim U_n}|E_{\mu_{Q+u}}e_{\theta a} - E_{U_n}e_{\theta a}|
\]

\[
\leq \max_{\theta} E_{u \sim U_n}|E_{\mu_{Q+u}}e_{\theta} - E_{U_n}e_{\theta}|^2 \quad (\text{since } \sum_a |\alpha_a| = 1)
\]

\[
\leq \max_{\theta} \sqrt{E_{u \sim U_n}|E_{\mu_{Q+u}}e_{\theta} - E_{U_n}e_{\theta}|^2}.
\]

Theorem 2.1 then follows from Theorem 2.4.

3 Applications

In this section, we apply Theorem 2.1 to the weight distribution of cosets of extended Hadamard codes, and Corollary 2.2 to the weight distribution of cosets of extended dual BCH code. We conclude with a conjecture that there are no small codes such that the weight distribution of all cosets is arbitrarily close to the binomial in the \( L_1 \)-norm.

A natural construction of codes with large dual bilateral minimum distance from codes with large minimum distance is the following.

Lemma 3.1 If \( n \) is odd and \( D \subset \mathbb{F}_2^n \) is an \( \mathbb{F}_2 \)-linear code of minimum distance at least \( d \) such that the all ones vector \( \vec{1} \in D \). Then \( Q \overset{\text{def}}{=} D^\perp \cup (D^\perp + \vec{1}) \) is a code whose dual has bilateral minimum distance at least \( d \).

Proof: Since \( \vec{1} \in D \), we have \( y + \vec{1} \in D \) for each \( y \in D \), hence \( n - |y| = |y + 1| \geq d \) if \( y \neq \vec{1} \). Thus, if \( C \subset D \) is an \( \mathbb{F}_2 \)-linear code such that \( \vec{1} \not\in C \), then the bilateral minimum distance of \( C \) is at least \( d \). Let \( C \) be the set of even weight codewords of \( D \). Thus, \( \vec{1} \not\in C \) since \( n \) is odd, hence the bilateral minimum distance of \( C \) is at least \( d \). The dual of \( C \) is the \( \mathbb{F}_2 \)-linear code generated by \( D^\perp \) and \( \vec{1} \), i.e., \( C^\perp = D^\perp \cup (D^\perp + \vec{1}) \).

3.1 Extended Hadamard code

Let \( n = 2^r - 1 \), where \( r \geq 2 \) is an integer, and let \( D \) be the \( (2^r - 1, 2^r - 1 - r, 3) \)-Hamming code. Thus, \( D^\perp \) is the \( (2^r - 1, r, 2^{r-1}) \)-Hadamard code, and \( Q = D^\perp \cup (D^\perp + \vec{1}) \) is the extended Hadamard code of size \( 2^{r+1} = 2(n + 1) \). The all-ones vector \( \vec{1} \in D \) since all the codewords of the Hadamard code \( D^\perp \) have even weight (the weight is either 0 or \( \frac{n+1}{2} = 2^{r-1} \)). Thus, by Lemma 3.1, the dual of \( Q \) has bilateral minimum distance at least \( d = 3 \). The weight distribution of \( Q \) is given by

\[
\mu_Q(w) = \begin{cases} 
\frac{1}{2} - O\left(\frac{1}{n}\right) & \text{if } w = \frac{n-1}{2} \text{ or } \frac{n+1}{2} \\
O\left(\frac{1}{n}\right) & \text{if } w = 0 \text{ or } n \\
0 & \text{otherwise.}
\end{cases}
\]

Note also that if \( d \) is odd, which is the case in the examples below, then the bilateral minimum distance of \( C \) is at least \( d + 1 \). Nevertheless, we will use the lower bound \( d \) on the bilateral minimum distance since Theorem 2.1 and Corollary 2.2 assume that \( d \) is odd.
Thus, $\|m_Q - \text{bin}_n\|_{\infty} = \Theta(1)$ and $\|m_Q - \text{bin}_n\|_1 = \Theta(1)$. However, by Part (a) of Theorem 2.1 with $t = 1$, we have
\[
E_{u \sim U_n} \|m_{Q+u} - \text{bin}_n\|_{\infty} \leq (\sqrt{2}e)^{\frac{\ln \left(\frac{n}{2}\right)}{\sqrt{n}}} = O\left(\frac{\ln n}{\sqrt{n}}\right).
\]
Note that Corollary 2.2 is not useful here since it gives no nontrivial bounds for $t \geq 3$.

### 3.2 Extended dual BCH code

Let $\mathbb{F}_{2^r}$ be the finite field with $2^r$ elements and $\mathbb{F}_{2^r}^*$ be the set of nonzero elements of $\mathbb{F}_{2^r}$. If $r \geq 2$ and $t \geq 1$ are integers such that $2t - 2 < 2^{r/2}$, let $n = 2^r - 1$, and consider the BCH code $BCH(t, r) \subset \mathbb{F}_2^n$:

\[
BCH(t, r) = \{(f(a))_{a \in \mathbb{F}_2^*} : f \in \mathbb{F}_{2^r}[x] \text{ s.t. } \deg(f) \leq 2^r - 2t - 1\} \cap \mathbb{F}_{2}^{F_{2^r}}.
\]

We have (see [MS77]):

- a) $\dim(BCH(t, r)) = 2^r - 1 - rt$
- b) The minimum distance of $BCH(t, r)$ is at least $2t + 1$
- c) (Weil-Carlitz-Uchiyama Bound) For each non-zero codeword $x \in BCH(t, r)^{\perp}$, we have $|x| - 2^{r-1} \leq (t - 1)2^{r/2}$, hence $|x| - (n + 1)/2 \leq (t - 1)\sqrt{n} + 1$. Note that the condition $2t - 2 < 2^{r/2}$ is equivalent to $t < \frac{1}{4}\sqrt{n + 1} + 1$. Let $D = BCH(r, t)$ and note that $\mathbb{F}_{2^r}^* \subset D$ (for $f = 1$). Consider the dual BCH code $D^{\perp} \subset \mathbb{F}_2^n$ and note that $|D^{\perp}| = 2^{rt} = (n + 1)^t$. Then $Q = D^{\perp} \cup (D^{\perp} + \mathbb{I})$ is the extended dual BCH code of size $2(n + 1)^t$. By Lemma 3.1, the dual of $Q$ has bilateral minimum distance at least $d = 2t + 1$.

**Lemma 3.2** If $t = \Theta(1)$, then $\|\mu_Q - \text{bin}_n\|_1 = \Theta(1)$.

However, it follows from Part (a) of Corollary 2.2 that
\[
E_{U \sim U_n} \|\mu_{Q+u} - \text{bin}_n\|_1 \leq (2t + 1) \left(\frac{e \ln \frac{n}{2t}}{t}\right)^\frac{2t}{n} \geq t \left(\frac{2t}{n}\right)^{\frac{2t}{n} - 1} = O\left(\frac{\ln t}{n^{\frac{2t}{n} - 1}}\right)
\]
for $t = O(1)$. For $t \geq 3$ (i.e., $d \geq 7$), the decay bypasses the $\Theta(1)$ error floor in Lemma 3.2.

**Proof of Lemma 3.2** By the Weil-Carlitz-Uchiyama Bound, we have $\mu_Q(w) = 0$ if $w \neq 0, n$ and $|w - (n + 1)/2| > (t + 1)\sqrt{n} + 1$. Thus,
\[
\|\mu_Q - \text{bin}_n\|_1 \geq \binom{n}{w} = n : |w - \frac{n+1}{2}| > (t + 1)\sqrt{n} + 1
\]
\[
\geq \binom{n}{\frac{n+1}{2} + (t + 1)\sqrt{n} + 1} \leq \binom{n}{\frac{n+1}{2} + (t + 2)\sqrt{n} + 1} (\text{since } \binom{n}{\frac{n+1}{2} + (t + 2)\sqrt{n} + 1} < n, \text{ for } n \text{ large enough})
\]
\[
\geq \binom{n}{\frac{n+1}{2} + (t + 2)\sqrt{n} + 1} \geq \sqrt{\frac{2(n + 1)}{\pi n}} e^{-2(t+2)^2} (1 - o(1)),
\]
using de Moivre-Laplace normal approximation of the binomial $\binom{n}{w} = \sqrt{\frac{2}{\pi}} e^{-2(w^2/2)} (1 - o(1))$, which holds if $|w - n/2| = o(n^{2/3})$ (e.g., [Fel68] page 184).
3.3 Conjectures

It follows from the Extended dual BCH code example that polynomial-size codes may have a coset whose weight distribution is bounded away from the binomial distribution in the $L_1$-norm by a constant error floor, even if the code has a large dual bilateral distance. We believe that this error floor is not due to the weakness of the “large dual bilateral distance” requirement, but it is simply due to the small-size and the linearity of the code.

**Conjecture 3.3** For each constant $t > 0$, there is a constant $\epsilon > 0$ such that, for $n$ large enough, for each $\mathbb{F}_2$-linear code $Q \subset \mathbb{F}_2^n$ of size at most $n^t$, there exists a coset $u + Q$ of $Q$ for some $u \in \mathbb{F}_2^n$ such that $\|\mu_{Q+u} - \text{bin}_n\|_1 \geq \epsilon$.

That is, polynomial-size linear codes do not behave like arbitrary random subsets of $\{0,1\}^n$ of the same size. We leave the question of proving or disproving the conjecture open. A stronger conjecture is the following.

**Conjecture 3.4** Let $Q \subset \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code of size $n^t$, where $t = \Theta(1)$. Then the covering radius of $Q$ is at least $n/2 - O(\sqrt{n})$. That is, for each constant $t > 0$, there exists a constant $c > 0$ such that, for $n$ large enough, for each $\mathbb{F}_2$-linear code $Q \subset \mathbb{F}_2^n$ of size at most $n^t$, there exists $u \in \mathbb{F}_2^n$ such that the distance between $u$ and each codeword of $Q$ is at least $n/2 - \sqrt{cn}$.

The fact that this a stronger conjecture follows from computations similar to those in the proof of Lemma 3.2. Note that the covering radius of random polynomial-size subset of $\{0,1\}^n$ is $\Theta(\sqrt{n \log n})$.

4 Fourier transform preliminaries

The study of error correcting codes using using Harmonic analysis methods dates back to MacWilliams [Mac63] (see also [LMN93] for and the references therein). We give below some preliminary notions used in the proof.

Identify the hypercube $\{0,1\}^n$ with the group $\mathbb{Z}_2^n = (\mathbb{Z}/2\mathbb{Z})^n$. The characters of the abelian group $\mathbb{Z}_2^n$ are $\{\chi_z\}_{z \in \mathbb{Z}_2^n}$, where $\chi_z : \{0,1\}^n \to \{-1,1\}$ is given by $\chi_y(x) = (-1)^{\langle x, y \rangle}$, and $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$.

Consider the $\mathbb{C}$-vector space $L(\mathbb{Z}_2^n)$ of complex valued functions $\mathbb{Z}_2^n \to \mathbb{C}$ endowed with the inner product $\langle , \rangle$ associated with the uniform distribution on $\{0,1\}^n$:

$$\langle f, g \rangle = E_{\mathbb{U}_n} f \overline{g} = \frac{1}{2^n} \sum_x f(x) \overline{g(x)},$$

where $\overline{\cdot}$ is the complex conjugation operator.

The characters $\{\chi_z\}_{z}$ form an orthonormal basis of $L(\mathbb{Z}_2^n)$, i.e., for each $z, z' \in \{0,1\}^n$,

$$\langle \chi_z, \chi_{z'} \rangle = \begin{cases} 1 & \text{if } z = z' \\ 0 & \text{if } z \neq z'. \end{cases}$$

If $f \in L(\mathbb{Z}_2^n)$, its Fourier transform $\hat{f} \in L(\mathbb{Z}_2^n)$ is given by the coefficients of the unique expansion of $f$ in terms of $\{\chi_z\}_{z}$:

$$f(x) = \sum_z \hat{f}(z) \chi_z(x) \quad \text{and} \quad \hat{f}(z) = \langle f, \chi_z \rangle = E_{\mathbb{U}_n} f \chi_z.$$
Define ∆:

Proof: since χf = Lemma 4.2 function is another exponential function.

If tion (Lemma 4.1 let We need the following basic lemma which follows from Parseval’s equality.

Fourier transform of the exponential function. The Fourier transform of the exponential function is another exponential function.

Lemma 4.2 Let r be complex number and \( g_r : \{0,1\}^n \to \mathbb{C} \) be given by \( g_r(x) = r^{|x|} \). Then \( \hat{g}_r(z) = \left( \frac{1+r}{2} \right)^n \left( \frac{1-r}{1+r} \right)^{|z|} \). Moreover, if \( r = e^{i\theta} \), then \( \hat{g}_r(z) = e^{in\theta/2} \left( \cos \frac{\theta}{2} \right)^n \left( -i \tan \frac{\theta}{2} \right)^{|z|} \).

The degree of \( f \) is the smallest degree of a polynomial \( p \in \mathbb{C}[x_1, \ldots, x_n] \) such that \( p(x) = f(x) \) for all \( x \in \{0,1\}^n \). Equivalently, in terms of the Fourier transform \( \hat{f} \) over \( \mathbb{C} \), the degree of \( f \) is equal to the maximal weight of \( z \in \mathbb{Z}_2^n \) such that \( \hat{f}(z) \neq 0 \).

If \( f, g \in \mathbb{L}(\mathbb{Z}_2^n) \), we have

\[ \langle f, g \rangle = 2^n \langle \hat{f}, \hat{g} \rangle = \sum_z \hat{f}(z)\hat{g}(z). \] (1)

Parseval’s equality. A special case of (1) is Parseval’s equality:

\[ E_{U_n}|f|^2 = \sum_z |\hat{f}(z)|^2 = ||\hat{f}||^2. \] (2)

We need the following basic lemma which follows from Parseval’s equality.

**Lemma 4.1** Let \( \mu \) be a probability distribution on \( \{0,1\}^n \). For each \( u \in \{0,1\}^n \), define the translation (mod 2) \( \sigma_u \mu \) of \( \mu \) by \( u \) to be the probability distribution on \( \{0,1\}^n \) given by \( (\sigma_u \mu)(x) = \mu(x + u) \). If \( f : \{0,1\}^n \to \mathbb{C} \), then

\[ E_{u \sim U_n}|E_{\sigma_u \mu}f - E_{U_n} f|^2 = \sum_{z \neq 0} |\hat{f}(z)|^2 (E_{\mu} \chi_z)^2. \]

**Proof:** Define \( \Delta : \{0,1\}^n \to \mathbb{R} \) by \( \Delta(u) = E_{\sigma_u \mu} f - E_{U_n} f \). Consider the Fourier expansion of \( f = \sum_z \hat{f}(z) \chi_z \). Thus,

\[ \Delta(u) = E_{y \sim \mu} \sum_z \hat{f}(z) \chi_z(y + u) - E_{U_n} f = \sum_z \chi_z(u) \hat{f}(z) E_{\mu} \chi_z - E_{U_n} f = \sum_{z \neq 0} \chi_z(u) \hat{f}(z) E_{\mu} \chi_z \]

since \( \chi_0 = 1 \) and \( \hat{f}(0) = E_{U_n} f \). Hence \( \hat{\Delta}(0) = 0 \) and \( \hat{\Delta}(z) = \hat{f}(z) E_{\mu} \chi_z \) for each \( z \neq 0 \). The lemma then follows from Parseval’s equality.

**Fourier transform and linear codes.** If \( Q \subset \mathbb{F}_2^n \) is an \( \mathbb{F}_2 \)-linear code and \( Q^\perp \) is its dual, then for each \( z \in \mathbb{F}_2^n \), we have

\[ \sum_{y \in Q} \chi_z(y) = \begin{cases} |Q| & \text{if } z \in Q^\perp \\ 0 & \text{otherwise}. \end{cases} \] (3)
It follows that

\[ E_{\mu_Q}^{z} = \begin{cases} 
1 & \text{if } z \in Q^\perp \\
0 & \text{otherwise.}
\end{cases} \tag{4} \]

Finally, we need MacWilliams’s identity:

**Lemma 4.3 (MacWilliams’s identity \cite{Mac63})** Let \( Q \subset \mathbb{F}_2^n \) be an \( \mathbb{F}_2 \)-linear code and \( s \) be a complex number, then

\[ \sum_{z \in Q^\perp} s^{|z|} = (1 + s)^n E_{x \in Q} \left( \frac{1+s}{1+s} \right)^{|x|}. \]

Since the proof is direct given the above machinery, we add it for completeness.

**Proof:** Let \( r = \frac{1-s}{1+s} \). By Lemma 4.2, \( g_r : g_r(x) = \sum_{z} \left( \frac{1+r}{1+r} \right)^n \left( \frac{1+r}{1+r} \right)^{|z|} \chi_z(x) \). It follows from (4) that \( E_{x \in Q} g_r(x) = \left( \frac{1+r}{1+r} \right)^n \sum_{z \in Q^\perp} \left( \frac{1+r}{1+r} \right)^{|z|} \chi_z(x) \). MacWilliams’s identity thus follows from the relations

\[ 1+r = s \quad \text{and} \quad \frac{1+r}{1+r} = \frac{1}{1+r}. \]

\[ \square \]

## 5 Proof of Theorem 2.4

**Definition 5.1** If \( 0 \leq \theta < 2\pi \), define \( e_\theta : \{0,1\}^n \to \mathbb{C} \) by \( e_\theta(x) \) def = \( e^{i\theta |x|} \). If \( Q \subset \mathbb{F}_2^n \) is an \( \mathbb{F}_2 \)-linear code, define \( \Delta_{Q,\theta} : \{0,1\}^n \to \mathbb{C} \) by \( \Delta_{Q,\theta}(u) \) def = \( E_{u \in Q} e_\theta - E_{u \in Q^\perp} e_\theta \).

We restate below Theorem 2.4 in terms of \( \Delta_{Q,\theta} \).

**Theorem 2.4 (Mean-square-error bound)** Let \( Q \subseteq \mathbb{F}_2^n \) be an \( \mathbb{F}_2 \)-linear code whose dual \( Q^\perp \) has bilateral minimum distance at least \( d = 2t+1 \), where \( t \geq 1 \) is an integer. Then, for each \( 0 \leq \theta < 2\pi \), we have the bounds:

a) (Small dual distance bound)

\[ E_{U_n} |\Delta_{Q,\theta}|^2 \leq \left( e \ln \frac{n}{2t} \right)^{2t} \left( \frac{2t}{n} \right)^t \]

b) (Large dual distance bound)

\[ E_{U_n} |\Delta_{Q,\theta}|^2 \leq 2 e^{-\frac{t}{n}}. \]

At a high level, our proof technique is as follows. First we show in Lemma 5.2 that

\[ E_{U_n} |\Delta_{Q,\theta}|^2 = E_{w \sim \mu_{Q^\perp}} c^w - \left( \frac{c+1}{2} \right)^n, \]

where \( c = \cos \theta \). Note that \( \left( \frac{c+1}{2} \right)^n = E_{w \sim \text{bin}_n} c^w \). Lemma 5.2 exploits the linearity of the code in subtle manner. The starting point is to express \( E_{U_n} |\Delta_{Q,\theta}|^2 \) in terms of the squared norm of the Fourier coefficients of \( e_\theta \) and the dual of \( Q \) using Lemma 4.1 applied to \( f = e_\theta \). We establish Lemma 5.2 using the expression of the Fourier transform of \( e_\theta \) in Lemma 4.2 and using MacWilliams’s identity to go back to the original domain. The argument seems convoluted since after going the Fourier domain, we use MacWilliams’s identity to go back to the original domain. The catch is that MacWilliams’s identity is used after Parseval’s equality (Lemma 4.1) is based
on Parseval’s equality) which involves squaring the norm of the Fourier coefficients of $e_\theta$. The claim can be established without going to the Fourier domain by algebraically exploiting the code linearity but the proof is more complicated.

Then we bound $E_{w \sim \mu Q} c^w - (\cos \frac{\theta}{2})^n$ by ignoring the code linearity, and maximizing over the choice of probability distributions $\mu$ on $\{0, 1\}^n$ such that $E_{\mu} \chi_z = 0$ for all nonzero $z \in \{0, 1\}^n$ such that $|z| \leq d - 1$ or $|z| \geq n - d + 1$. This property is satisfied by $\mu_Q$ as it has bilateral minimum distance at least $d$ (see (4)). The constraints on $\mu$ define a linear program. Due to the symmetry of the problem, we note in Lemma 5.5 that the dual linear program is $\min_h E_{\text{bin}, n} h$, where we are optimizing on the all functions $h : [0 : n] \to \mathbb{R}$ such that:

i) $h$ can be expressed as $h(w) = f(w) + (-1)^w g(w)$, for some polynomials $f(x), g(x) \in \mathbb{R}[x]$ each degree at most $d - 1$

ii) $h(w) \geq c^w - \left(\frac{c+1}{2}\right)^n$, for all $w \in [0 : n]$.

That is $E_{U_n} |\Delta_{Q, \theta}|^2 \leq \min_h E_{\text{bin}, n} h$. We construct $h$ in Lemma 5.6. Let $k = d - 1 = 2t$. We use three different constructions of $h$ depending on whether $c^* \leq c \leq 1$, $-1 \leq c \leq -c^*$, or $|c| < c^*$, where $c^*$ is a parameter which will optimized. If $c^* \leq c \leq 1$, we construct $h$ by truncating the Taylor series expansion of $c^w$ around $n/2$ to obtain a polynomial of degree $k$. If $-1 < c \leq c^*$, we write $c^w = (-1)^w |c|^w$ and we suitably construct $h$ using $(-1)^w$ and the first $k$ terms of the Taylor series expansion of $|c|^w$ around $n/2$. If $|c| < c^*$, we set $h$ to a degree $k$ polynomial of the form $h(w) = a(n/2 - w)^k + b$, where $a, b > 0$ are suitably chosen parameters. The constructions rely on the fact that $k$ is even.

**Lemma 5.2** Let $Q \subsetneq \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code and $0 \leq \theta < 2\pi$. Then

$$E_{U_n} |\Delta_{Q, \theta}|^2 = E_{w \sim \mu Q} (\cos \theta)^w - \left(\frac{\cos \theta + 1}{2}\right)^n.$$  

Note that if $\cos \theta = 0$ and $w = 0$, then $0^0 = 1$ is interpreted as the limit of $(\cos \theta)^0$ as $\cos \theta$ tends to 0.

**Proof:** Applying Lemma 4.1 to $f = e_\theta$, we get

$$E_{U_n} |\Delta_{Q, \theta}|^2 = \sum_{z \neq 0} |\hat{e}(z)|^2 (E_{\mu_Q} \chi_z)^2.$$  

We know from (4) that

$$E_{\mu_Q} \chi_z = \begin{cases} 1 & \text{if } z \in Q^\perp \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$E_{U_n} |\Delta_{Q, \theta}|^2 = \sum_{z \neq 0 \in Q^\perp} |\hat{e}(z)|^2 = \sum_{z \in Q^\perp} |\hat{e}(z)|^2 - |\hat{e}(0)|^2. \tag{5}$$

By Lemma 4.2 applied with $r = e^{i\theta}$, we have $\hat{e}(z) = e^{in\theta/2} (\cos \frac{\theta}{2})^n (-i \tan \frac{\theta}{2})^{|z|}$. Thus,

$$\sum_{z \in Q^\perp} |\hat{e}(z)|^2 = (\cos \frac{\theta}{2})^{2n} \sum_{z \in Q^\perp} \left(\frac{\sin \frac{\theta}{2}}{\tan \frac{\theta}{2}}\right)^{|z|}.$$
Proof: Let $\theta = \tan \theta / 2$, we obtain

$$
\sum_{z \in Q^\perp} |e_\theta(z)|^2 = (\cos \theta / 2)^{2n} \left(1 + \left(\tan \theta / 2\right)^2\right)^n \sum_{x \in Q} \left(1 - \left(\tan \theta / 2\right)^2 \right)^{|x|} = E_{x \in Q} \left(\cos \theta \right)^{|x|},
$$

(6)

since $(\cos \theta / 2)^2 (1 + \left(\tan \theta / 2\right)^2) = (\cos \theta / 2)^2 + (\sin \theta / 2)^2 = 1$ and $\frac{1 - \left(\tan \theta / 2\right)^2}{1 + \left(\tan \theta / 2\right)^2} = \sin \theta / 2 - (\cos \theta / 2)^2 = \cos \theta$.

Finally, applying Lemma 4.2 again with $r = \theta$ and $z = 0$, we get $|e_\theta(0)|^2 = (\cos \theta / 2)^{2n} = (\cos \theta / 2 + 1)^n$. Replacing with $[\theta]$ in (5), we obtain

$$
E_{U_n}|\Delta_{Q,\theta}|^2 = \sum_{w=0}^{n} \frac{|\{y \in Q : |y| = w\}|}{|Q|} (\cos \theta)^w - \left(\frac{\cos \theta + 1}{2}\right)^n.
$$

Lemma 5.3 Let $Q \subseteq F_2^n$ be an $F_2$-linear code whose dual $Q^\perp$ has bilateral minimum distance at least $d \geq 1$.

a) If $r : \{0,1\}^n \to \mathbb{R}$ can be expressed as $r(x) = p(x) + (-1)^{|x|} q(x)$ for some $p, q : \{0,1\}^n \to \mathbb{R}$ each of degree (see section 4) at most $d - 1$, then $E_{\mu Q} r = E_{U_n} r$.

b) If $h : [0 : n] \to \mathbb{R}$ can be expressed as $h(w) = f(w) + (-1)^w g(w)$ for some polynomials $f(x), g(x) \in \mathbb{R}[x]$ each of degree at most $d - 1$, then $E_{\mu Q} h = E_{bin, u}$.  

Proof:

(a) By (4), we have

$$
E_{\mu Q} \chi_z = \begin{cases} 
1 & \text{if } z \in Q^\perp \\
0 & \text{otherwise}
\end{cases}
$$

Since $Q^\perp$ has bilateral minimum distance at least $d$, then $d \leq |z| \leq n - d$, for each nonzero $z \in Q^\perp$. Thus, $E_{\mu Q} \chi_z = 0$ for each nonzero $z \in \{0,1\}^n$ such that $|z| \leq d - 1$ or $|z| \geq n - d + 1$.

Consider the Fourier expansions $p(x) = \sum_{z : |z| \leq d-1} \hat{p}(z) \chi_z(x)$ and $q(x) = \sum_{z : |z| \leq d-1} \hat{q}(z) \chi_z(x)$.

Since $(-1)^{|x|} \chi_z(x) = \chi_{z + \bar{1}}(x)$ and $|z + \bar{1}| = n - |z|$, where $\bar{1} \in \{0,1\}^n$ is the all ones vector, we obtain

$$
r(x) = \sum_{z : |z| \leq d-1} \hat{q}(z) \chi_z(x) + \sum_{z : |z| \geq n - d + 1} \hat{r}(z + \bar{1}) \chi_z(x)
$$

It follows that $E_{\mu Q} r = \hat{q}(0) = E_{U_n} r$.

(b) Let $p, q, r : \{0,1\}^n \to \mathbb{R}$ be given by $p(x) = f(|x|), q(x) = g(|x|)$, and $r(x) = p(x) + (-1)^{|x|} q(x)$. Thus, $p$ and $q$ are each of degree at most $d - 1$ and $r(x) = h(|x|)$. It follows from (a) that $E_{\mu Q} r = E_{U_n} r$. Since $r$ is a symmetric function (i.e., $r(x)$ depends only on the weight $|x|$ of $x$), we have $E_{\mu Q} r = E_{\mu Q} h$ and $E_{U_n} r = E_{bin, u} h$. It follows that $E_{\mu Q} h = E_{bin, u} h$. 

\[\Box\]
Definition 5.4 If $-1 \leq c \leq 1$, define $H_c^{(n)} : [0 : n] \rightarrow \mathbb{R}$ by

$$H_c^{(n)}(w) \equiv e^w - \left(\frac{c + 1}{2}\right)^n.$$ 

Here again, if $c = 0$ and $w = 0$, then $0^0 = 1$ is interpreted as the limit of $c^0$ as $c$ tends to 0.

Note that $E_{w \sim \text{bin}} c^w = \left(\frac{c + 1}{2}\right)^n$.

Lemma 5.5 (LP duality) Let $Q \subseteq \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code whose dual $Q^\perp$ has bilateral minimum distance at least $d \geq 1$, and let $0 \leq \theta < 2\pi$. Then

$$E_{U_n}[\Delta_{Q,\theta}]^2 \leq \min_h E_{\text{bin}} h,$$

where we are optimizing on the all functions $h : [0 : n] \rightarrow \mathbb{R}$ such that:

i) $h$ can be expressed as $h(w) = f(w) + (-1)^w g(w)$, for some polynomials $f(x), g(x) \in \mathbb{R}[x]$ each degree at most $d - 1$

ii) $h(w) \geq H_c^{(n)}(w)$ for all $w \in [0 : n]$.

Proof: Using Lemma 5.2, $E_{U_n}[\Delta_{Q,\theta}]^2 = E_{\mu_Q} H_c^{(n)}$. Since $h \geq H_{\cos \theta}$, we have $E_{\mu_Q} H_c^{(n)} \leq E_{\mu_Q} h$. By Lemma 5.3, $E_{\mu_Q} h = E_{\text{bin}} h$. It follows that $E[\Delta_{Q,\theta}]^2 \leq E_{\text{bin}} h$.

Note that the above argument is weak LP duality. The converse also holds, in the sense that it not hard to verify that the following linear programs are duals of each others:

$$\min_h E_{\text{bin}} h = \max_{\gamma} E_{\gamma} H_{\cos \theta}^{(n)}$$

where the min is over all functions $h : [0 : n] \rightarrow \mathbb{R}$ satisfying (i) and (ii), and the max is over all probability distributions $\gamma$ on $[0 : n]$ such that $E_{\gamma} h = E_{\text{bin}} h$ for each $h$ satisfying (i).

Theorem 2.4 follows from Lemma 5.5 and Lemma 5.6 below. Note that Lemma 5.6 assumes that $n \geq 2d$, which must be the case by the definition of bilateral minimum distance.

Lemma 5.6 (Construction) Let $n$ and $t$ be integers such that $t \geq 1$ and $n \geq 2d$, where $d = 2t + 1$.

a) For all $-1 \leq c \leq 1$, there exist polynomials $f(x), g(x) \in \mathbb{R}[x]$ each degree at most $d - 1$, such that with $h(w) = f(w) + (-1)^w g(w)$, we have $h(w) \geq H_c^{(n)}(w)$ for all $w \in [0 : n]$, and

$$E_{\text{bin}} h \leq \left(c \ln \frac{n}{2t}\right)^{2t} \left(\frac{2t}{n}\right)^t.$$  

b) For all $-1 \leq c \leq 1$, there exist polynomials $f(x), g(x) \in \mathbb{R}[x]$ each degree at most $d - 1$, such that with $h(w) = f(w) + (-1)^w g(w)$, we have $h(w) \geq H_c^{(n)}(w)$ for all $w \in [0 : n]$, and

$$E_{\text{bin}} h \leq 2e^{-\frac{t}{2}}.$$
Proof: Let \( k = d - 1 = 2t \), thus \( k \) is even and \( k \geq 2 \). Let \( c^* = e^{-2\frac{k}{n}\beta} \), where \( \beta > 0 \) is a parameter which we will optimize later. We use three different constructions depending on the value of \( c \): \( c^* \leq c \leq 1 \), \(-1 \leq c \leq -c^* \), and \( |c| < c^* \). At the end of the proof, we will set \( \beta = \ln \frac{n}{k} \) to establish (a), and we will set \( \beta \) to a constant to establish (b).

**Case 1:** Assume that \( c^* \leq c \leq 1 \). Consider the Taylor approximation of the exponential around 0:

\[
e^x = \sum_{i=0}^{k-1} \frac{x^i}{i!} + e^\alpha \frac{x^k}{k!}
\]

for some number \( \alpha \) between 0 and \( x \). We will use it to approximate \( c^w \) around \( n/2 \):

\[
c^w = c^{n/2}e^{w-n/2} = c^{n/2}e^{(n/2-w)\ln \frac{1}{e}}
\]

\[
= a_c(w) + c^{n/2}e^{\alpha_c(w)}b_c(w),
\]

(7)

where:

i) \( a_c(w) = c^{n/2} \sum_{i=0}^{k-1} \frac{1}{i!} \left( \frac{n}{2} - w \right) \ln \frac{1}{e} \)

ii) \( b_c(w) = \frac{1}{k!} \left( \frac{n}{2} - w \right) \ln \frac{1}{e} \)

iii) \( \alpha_c(w) \) is a number between 0 and \( \left( \frac{n}{2} - w \right) \ln \frac{1}{e} \).

Thus, for all \( w \in [0 : n] \), we have \( \alpha_c(w) \leq \frac{n}{2} \ln \frac{1}{e} \), and hence \( c^{n/2}e^{\alpha_c(w)} \leq 1 \). Moreover, \( c^{n/2}e^{\alpha_c(w)} \geq 0 \). Thus, \( 0 \leq c^{n/2}e^{\alpha_c(w)} \leq 1 \).

Since \( k \) is even, \( b_c(w) \geq 0 \) for all \( w \). Therefore, \( c^{n/2}e^{\alpha_c(w)}b_c(w) \leq b_c(w) \) for all \( w \in [0 : n] \) (since by (iii), \( c^{n/2}e^{\alpha_c(w)} \leq 1 \)). Accordingly, \( c^w \leq a_c(w) + b_c(w) \), for all \( w \in [0 : n] \). Let

\[
h(w) = a_c(w) + b_c(w) - \left( \frac{c + 1}{2} \right)^n.
\]

Thus, \( \deg(h) = k = d - 1 \) and \( h(w) \geq H_c(w) \).

Since \( E_{w \sim \text{bin}_n} c^w = \left( \frac{c+1}{2} \right)^n \), we have

\[
E_{\text{bin}_n} h = E_{w \sim \text{bin}_n} a_c(w) + b_c(w) - c^w
\]

\[
= E_{w \sim \text{bin}_n} \left( 1 - c^{n/2}e^{\alpha_c(w)} \right)b_c(w)
\]

\[
\leq E_{w \sim \text{bin}_n} b_c(w) \quad (\text{since } c^{n/2}e^{\alpha_c(w)} \geq 0)
\]

\[
= \frac{1}{k!} \left( \frac{n}{2} \ln \frac{1}{e} \right)^k E_{w \sim \text{bin}_n} \left( \frac{n/2 - w}{n/2} \right)^k
\]

\[
\leq \frac{1}{k!} (k\beta)^k E_{w \sim \text{bin}_n} \left( \frac{n/2 - w}{n/2} \right)^k \quad (\text{since } c \geq c^* = e^{-2\frac{k}{n}\beta})
\]

Using Lemma [5.7](#) below, we obtain

\[
E_{\text{bin}_n} h \leq \frac{1}{k!} (k\beta)^k \left( \frac{k}{n} \right)^{k/2}
\]

\[
\leq \frac{1}{2} (e\beta)^k \left( \frac{k}{n} \right)^{k/2}
\]

(8)

by Stirling Approximation: \( k! \geq \sqrt{2\pi k^{k+1/2}}e^{-k} \geq 2k^ke^{-k} \), which holds for all for \( k \geq 1 \).
Lemma 5.7 For each even \( k \geq 2 \), we have

\[
E_{w \sim \text{bin}_n} \left( \frac{n/2 - w}{n/2} \right)^k \leq \left( \frac{k}{n} \right)^{k/2}.
\]

Proof: We have

\[
E_{w \sim \text{bin}_n} \left( \frac{n/2 - w}{n/2} \right)^k = E_{x \sim U_n} \left( \frac{\sum_{i=1}^n (-1)^{x_i}}{n} \right)^k \quad \text{(since } \frac{n}{2} - |x| = \frac{1}{2} \sum_i (-1)^{x_i})
\]

\[
= \frac{1}{n^k} \sum_{i_1, \ldots, i_k \in \{n\}} E_{x \sim U_n} \left( -1 \right)^{\sum_{i=1}^k x_{i_t}} = \frac{A_{n,k}}{n^k},
\]

where \( A_{n,k} \) is the number of \( k \)-tuples \((i_1, \ldots, i_k) \in [n]^k\) such that \((i_1, \ldots, i_k)\) can be partitioned into \( k/2 \) disjoint pairs of equal integers. Thus, \( A_{n,k} \leq (n(k - 1)n(k - 3)n(k - 5) \ldots n \leq (nk)^{k/2}.\) ▼

Case 2: Assume that \(-1 \leq c \leq -c^*\). Thus, \( c^w = (-1)^w |c|^w \) and \( c^* \leq |c| \leq 1 \). We use the Taylor approximation of \(|c|^w\) in Case 1 (Equation (7)):

\[
|c|^w = a_{|c|}(w) + |c|^{n/2} e^{\alpha_{|c|}(w)} b_{|c|}(w),
\]

where \( a_{|c|} \) and \( b_{|c|} \) and \( \alpha_{|c|} \) are as given in (i),(ii), and (iii) above with \( |c| \) instead of \( c \). Hence

\[
c^w = (-1)^w a_{|c|}(w) + (-1)^w |c|^{n/2} e^{\alpha_{|c|}(w)} b_{|c|}(w).
\]

Let

\[
h(w) = (-1)^w a_{|c|}(w) + b_{|c|}(w) - \left( \frac{c + 1}{2} \right)^n.
\]

We have \( \deg(a_{|c|}) = k - 1 < d - 1 \) and \( \deg(b_{|c|}) = k = d - 1.\)

Since \( k \) is even, \( b_{|c|}(w) \geq 0 \) for all \( w \). We know from (iii) that \( 0 \leq |c|^{n/2} e^{\alpha_{|c|}(w)} \leq 1 \), for all \( w \in [0 : n] \). Therefore, \((-1)^w |c|^{n/2} e^{\alpha_{|c|}(w)} b_{|c|}(w) \leq b_{|c|}(w) \) for all \( w \in [0 : n] \). Accordingly \( c^w \leq (-1)^w a_{|c|}(w) + b_{|c|}(w) \), for all \( w \in [0 : n] \). Therefore, \( H_c(w) \leq h(w).\)

To bound \( E_{\text{bin}_n} h \), we proceed as in Case 1. The only difference is that we get an extra factor of \( 2 \). We have \( E_{w \sim \text{bin}_n} c^w = \left( \frac{c+1}{2} \right)^n \), thus

\[
E_{\text{bin}_n} h = E_{w \sim \text{bin}_n} (-1)^w a_{|c|}(w) + b_{|c|}(w) - c^w
\]

\[
= E_{w \sim \text{bin}_n} (1 - (-1)^w |c|^{n/2} e^{\alpha_{|c|}(w)}) b_{|c|}(w)
\]

\[
\leq 2 E_{w \sim \text{bin}_n} b_{|c|}(w) \quad \text{(since } 0 \leq |c|^{n/2} e^{\alpha_{|c|}(w)} \leq 1)\)
\]

\[
= 2 \frac{1}{k!} \left( \frac{n}{2} \ln \frac{1}{|c|} \right)^k E_{w \sim \text{bin}_n} \left( \frac{n/2 - w}{n/2} \right)^k
\]

\[
\leq 2 \frac{1}{k!} (k\beta)^k E_{w \sim \text{bin}_n} \left( \frac{n/2 - w}{n/2} \right)^k \quad \text{(since } |c| \geq c^* = e^{-2\frac{k}{n}\beta})
\]

\[
\leq (e\beta)^k \left( \frac{k}{n} \right)^{k/2} \quad \text{(by arguing as in Case 1).} \quad (9)
\]
Case 3: Assume that $|c| < c^*$. Then

$$H_c^{(n)}(w) = e^w - \left(\frac{c + 1}{2}\right)^n < e^w \leq c^*w = e^{-2kw^2/n}.$$

Note that $c + 1 > 0$ since $|c| < c^* < 1$.

Let $0 < \gamma < 1/2$ be a parameter which will be optimized. If $w \geq \gamma n$, we use the bound $H_c^{(n)}(w) \leq e^{-2\beta\gamma k}$. If $w < \gamma n$, we use the bound $H_c^{(n)}(w) \leq 1$. Let

$$h(w) = \frac{1}{(1 - 2\gamma)^k} \left(\frac{n/2 - w}{n/2}\right)^k + e^{-2\beta\gamma k}.$$

Thus, if $w \geq \gamma n$, we have $H_c^{(n)}(w) \leq e^{-2\beta\gamma k} \leq h(w)$ (recall that $k$ is even, hence $(n/2 - w)^k \geq 0$).

If $w < \gamma n$, i.e., $1 - 2\gamma < \frac{n/2 - w}{n/2}$, then $1 < \frac{1}{(1 - 2\gamma)^k} \left(\frac{n/2 - w}{n/2}\right)^k$, hence $H_c^{(n)}(w) \leq 1 < h(w)$. It follows that $h$ is a degree $k = d - 1$ polynomial such that $h \leq H_c^{(n)}$. Moreover,

$$E_{\text{bin},h} = \frac{1}{(1 - 2\gamma)^k} E_{w\sim\text{bin}} \left(\frac{n/2 - w}{n/2}\right)^k + e^{-2\beta\gamma k}$$

(by Lemma 5.7). (10)

The bound in (9) in Case 2 is twice that in (8) in Case 1, hence we can focus on Cases 2 and 3, i.e., (9) and (11).

First, we establish the bound in Part (a). Set $\beta = \ln \frac{n}{k}$ and $\gamma = \frac{1}{4}$, hence (9) reduces to $(e \ln \frac{n}{k})^k \left(\frac{k}{n}\right)^{k/2}$, and (10) reduces to $2^k \left(\frac{k}{n}\right)^{k/2} + \left(\frac{k}{n}\right)^{k/2} \leq (e \ln \frac{n}{k})^k \left(\frac{k}{n}\right)^{k/2}$ if $\frac{n}{k} \geq 4$. It follows that, if $\frac{n}{k} \geq 4$, then – in each of the 3 cases – there exists $h$ such that

$$E_{\text{bin},h} \leq (e \ln \frac{n}{k})^k \left(\frac{k}{n}\right)^{k/2} = (e \ln \frac{n}{k})^k \left(\frac{2t}{n}\right)^t.$$

We can ignore the $\frac{n}{k} \geq 4$ assumption since the bound $(e \ln \frac{n}{k})^k \left(\frac{k}{n}\right)^{k/2} = ((e \ln \frac{n}{k})^2)^{k/2} \geq 1$, for $\frac{n}{k} \leq 212$. Moreover, since $H_c^{(n)}(w) \leq 1$ for all $w$ and $c$, by setting $h = 1$, we can trivially achieve $E_{\text{bin},h} = 1$.

To establish Part (b), set $\beta = \frac{1}{e(1-2\gamma)^2}$ and $\gamma = 0.107$, hence (10) reduces to $\frac{1}{(1 - 2\gamma)^k} \left(\frac{k}{n}\right)^{k/2}$, thus (10) is the larger bound. Since $k = d - 1$ and $d \leq n/2$, we have $\frac{k}{n} \leq \frac{1}{2}$. It follows that (10) is at most

$$\frac{1}{(1 - 2\gamma)^k} \left(\frac{1}{2}\right)^{k/2} + e^{-2\beta\gamma k} = e^{-(0.1057..)k} + e^{-(0.1001..)k} \leq 2e^{-k/10} = 2e^{-t/5}.$$
6 Comparison with small-bias probability distributions

In this section, we compare with bound in Corollary 6.2 in [Baz14], which is the analogue of Corollary 2.2 for small-bias probability distributions. Roughly speaking, a probability distribution on \( \{0,1\}^n \) has small bias if it looks like the uniform distribution for all all parity functions on subsets of the \( n \) input variables. More formally, let \( \delta \geq 0 \). A probability distribution \( \mu \) on \( \{0,1\}^n \) is \( \delta \)-biased if \( |E_{\mu} \chi_z| \leq \delta \) for each nonzero \( z \in \{0,1\}^n \) [NN93]. Recall that if \( \mu \) is a probability distribution on \( \{0,1\}^n \) and \( u \in \{0,1\}^n \), then the \( \mathbb{F}_2 \)-translation \( \sigma_u \mu \) of \( \mu \) by \( u \) is the probability distribution on \( \{0,1\}^n \) given by \( (\sigma_u \mu)(x) = \mu(x + u) \). The bound in Corollary 6.2 in [Baz14] is \( E_{u \sim U_n} \|\sigma_u \mu - \text{bin}_n\|_1 \leq \delta \sqrt{n+1} \), i.e., the average \( L_1 \)-distance between the binomial distribution and the weight distribution of the translation of \( \mu \) by a random vector in \( \{0,1\}^n \) is at most \( \delta \sqrt{n+1} \). The key behind this bound is following observation which is inspired by the paper of Viola [Vio08] (the argument used to establish Lemma 3 in [Vio08]).

Lemma 6.1 ([Baz14, Lemma 6.1]) If \( f : \{0,1\}^n \to \mathbb{C} \) and \( \mu \) be a \( \delta \)-biased probability distribution on \( \{0,1\}^n \), then

\[
E_{u \sim U_n}|E_{\sigma_u \mu} f - E_{U_n} f|^2 \leq \delta^2(E_{U_n}|f|^2 - |E_{U_n} f|^2).
\]

The proof follows from Parseval’s equality and the definition of small bias. For completeness, we derive it below using Lemma 4.1 (which is also based on Parseval’s equality).

Proof: By Lemma 4.1

\[
E_{u \sim U_n}|E_{\sigma_u \mu} f - E_{U_n} f|^2 = \sum_{z \neq 0} |\hat{f}(z)|^2(E_{\mu} \chi_z)^2 \leq \delta^2 \sum_{z \neq 0} |\hat{f}(z)|^2.
\]

We have \( \hat{f}(0) = E_{U_n} f \) and, by Parseval’s equality (2), \( E_{U_n}|f|^2 = \sum_z |\hat{f}(z)|^2 \). It follows that \( \sum_{z \neq 0} |\hat{f}(z)|^2 = E_{U_n}|f|^2 - |E_{U_n} f|^2 \).

Lemma 6.1 is the analog of Theorem 2.4 for small-bias spaces. The proof of Lemma 6.1 is significantly simpler. It is based on the fact \( |E_{\mu} \chi_z| \leq \delta \) for each nonzero \( z \in \{0,1\}^n \). In this context, the key weakness of linear codes is that \( E_{\mu Q} \chi_z = 1 \) for all \( z \) in the dual \( Q^\perp \), which is huge for small codes. On the other hand, the fact that \( E_{\mu Q} \chi_z = 0 \) for all \( z \not\in Q^\perp \) is essential for the correctness Theorem 2.4 in the sense that its proof breaks down if we ignore the linearity of the code and focus on the constraint that \( E_{\mu Q} \chi_z = 0 \) for each nonzero \( z \in \{0,1\}^n \) of weight less than \( d \) or larger than \( n - d \). Finally, we note that Lemma 6.1 is more general than Theorem 2.4 as it gives good bounds for any \( f : \{0,1\}^n \to \mathbb{C} \) whose variance is not very large. On the other hand, Theorem 2.4 is specific to \( f(x) = e^{i\theta|x|} \) (it can also be used to obtain good bounds for symmetric functions (i.e., \( f(x) \) depends on the weight \( |x| \) of \( x \)) with small \( L_\infty \)-norms).

7 Comparison with random codes

In this section, we compare the bound in Corollary 2.2 on the average \( L_1 \)-distance \( E_{u \sim U_n}\|\mu_{Q+u} - \text{bin}_n\|_1 \), over the random choice of coset \( Q + u \), to the average \( L_1 \)-distance \( E_Q\|\mu_{Q+u} - \text{bin}_n\|_1 \), when \( u \in \mathbb{F}_2^n \) is fixed and the code \( Q \) is chosen at random. We focus on small codes, and namely codes of polynomial-size.

Let \( Q \subseteq \mathbb{F}_2^n \) be a random \( \mathbb{F}_2 \)-linear code of size \( N \) (where \( N \) is a power of 2). Then \( |E_Q \mu_{Q}(w) - \text{bin}_n(w)| \leq 1/N \) for \( w = 0,\ldots,n \) (see [MS77], page 287). More generally, it is not hard to establish the following estimates.
**Lemma 7.2** Let $N$ be a function of $n$ such that $N$ is a power of 2, $N = w(1)$, and $N = o(2^n)$. Fix any $u \in \mathbb{F}_2^n$. Let $\Gamma = E_Q ||\mu_{Q+u} - bin_n||^2_2$, where $Q \subset \mathbb{F}_2^n$ is an $\mathbb{F}_2$-linear code of size $N$ chosen uniformly at random. Then, $\Gamma = \frac{1+o(1)}{N}$ and $\Gamma \leq E_Q ||\mu_{Q+u} - bin_n||_1 \leq \sqrt{(n+1)\Gamma}$.

The proof is in Appendix A. Assume that code is of polynomial-size, i.e., $N = n^c$, where $c = \Theta(1)$. To compare with random codes, we need following simple variation of the Gilbert-Varshamov bound.

**Lemma 7.2** Let $c > 1$ be a constant such that $n^c$ is a power of 2. Then, for $n$ large enough, almost all $\mathbb{F}_2$-linear codes $Q \subset \mathbb{F}_2^n$ of size $n^c$ have dual bilateral minimum distance at least $d = \lceil c \rceil - 1$.

The proof of Lemma 7.2 is below. The following bound follows from Lemma 7.2 and Corollary 2.2.

**Corollary 7.3** Let $c \geq 8$ be a constant such that $[c]$ is even and $N = n^c$ is a power of 2. Then, for each $\epsilon > 0$, for $n$ large enough and for almost all $\mathbb{F}_2$-linear codes $Q \subset \mathbb{F}_2^n$ of size $N$, we have $E_{u \sim U_n} ||m_{Q+u} - bin_n||_1 = O(\frac{n^{1.5+\epsilon}}{N^{1/2}})$.

**Proof:** By Lemma 7.2 for $n$ large enough, almost all codes $Q \subset \mathbb{F}_2^n$ of size $n^c$ have dual bilateral minimum distance at least $d = \lceil c \rceil - 1$. Since $\lceil c \rceil \geq 8$ is even, $d = \lceil c \rceil - 1 = 2t + 1$, for some integer $t \geq 3$. It follows from Corollary 2.2 that $E_{u \sim U_n} ||m_{Q+u} - bin_n||_1 = O(\frac{(lnn)^{t}}{n^{1.5+\epsilon}}) = O(\frac{n^{1.5+\epsilon}}{N^{1/2}})$ because $t/2 - 1 = \lceil c \rceil / 4 - 1.5 \geq c/4 - 1.5$.

We can compare the upper bound $E_{u \sim U_n} ||m_{Q+u} - bin_n||_1 = O(\frac{n^{1.5+\epsilon}}{N^{1/2}})$ to the lower bound $E_Q ||\mu_{Q+u} - bin_n||_1 \geq \Gamma = \Theta(\frac{1}{N})$ and the upper bound $E_Q ||\mu_{Q+u} - bin_n||_1 \leq \sqrt{(n+1)\Gamma} = \Theta(\frac{n^{1/2}}{N^{1/2}})$ of random codes. The bounds differ by the $O(n^{1.5+\epsilon})$ factor and the exponent of $N$. The exponent of $N$ in Corollary 7.3 comes from the exponent $\frac{1}{2}$ in Corollary 2.2. It is not clear what is the optimal exponent. We leave the question open.

**Proof of Lemma 7.2** Note that $d \geq 1$ since $c > 1$. Choose the generator matrix $G_{k \times n}$ of the dual code $Q^\perp$ uniformly at random, where $k = n - c \log_2 n$. The probability $p$ that there exists a nonzero $x \in \mathbb{F}_2^n$ such that the weight of $xG$ is less than $d$ or larger than $n - d$ is at most $2 \times (2^k - 1) \times V(d - 1)/2^n$, where $V(d - 1)$ is the volume of the Hamming ball of radius $d - 1$. We have $V(d - 1) \leq n^{d-1} + 1 \leq 2n^{d-1}$, thus $p \leq 4n^{-c}n^{d-1} = 4n^{-c+\lceil c \rceil - 2} < n^{-1}$, for $n$ large enough.

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Appendix

A Proof of Lemma 7.1

For convenience we repeat the statement of Lemma 7.1 here.

**Lemma 7.1** Let $N$ be a function of $n$ such that $N$ is a power of 2, $N = w(1)$, and $N = o(2^n)$. Fix any $u \in \mathbb{F}_2^n$. Let $\Gamma = E_Q[|\bar{\mu}_{u} - bin_n|_2]$, where $Q \subset \mathbb{F}_2^n$ is an $\mathbb{F}_2$-linear code of size $N$ chosen uniformly at random. Then, $\Gamma \leq E_Q[|\bar{\mu}_{u} - bin_n|_1] \leq \sqrt{(n + 1)\Gamma}$.

**Proof:** The bound $E_Q[|\bar{\mu}_{u} - bin_n|_1] \leq \sqrt{(n + 1)\Gamma}$ follows from Jensen’s inequality applied to $g(Q,w) = \bar{\mu}_{u}(w) - bin_n(w)$, and $E_Q[|g(Q,w)|^2] \leq E_Q[|g(Q,w)|]^2$, and the bound $E_Q[|\bar{\mu}_{u} - bin_n|_1] \geq \Gamma$ follows from the fact that for each $Q$, $||\bar{\mu}_{u}(w) - bin_n(w)||_1 \geq ||\bar{\mu}_{u}(w) - bin_n(w)||_2^2$ because $|\bar{\mu}_{u}(w) - bin_n(w)|_1 \leq 1$ for each $w \in [0:n]$.

To establish the estimate $\Gamma = \frac{1+o(1)}{N}$, for each $w \in [0:n]$, define $f_w : \{0,1\}^n \rightarrow \mathbb{R}$ by $f_w(y) = I_w(y + u) - bin_n(w)$, where $I_w : \{0,1\}^n \rightarrow \{0,1\}$ is the indicator function given by $I_w(x) = 1$ iff $|x| = w$. Thus, $\bar{\mu}_{u}(w) - bin_n(w) = E_{y \sim \mathcal{Q}} f_w(y)$ and $\Gamma = \sum_w \Gamma_w$, where $\Gamma_w = E_Q(E_{y \sim \mathcal{Q}} f_w(y))^2$.

Fix any $w \in [0:n]$ and note that $E_{U_n} f_w = 0$. We have

$$(E_{y \sim \mathcal{Q}} f_w(y))^2 = E_{y,y' \sim \mathcal{Q}} f_w(y) f_w(y') = \frac{1}{N^2} \sum_{y,y' \in Q} f_w(y) f_w(y').$$

Each nonzero $y \in \{0,1\}^n$ belongs to $Q$ with probability $p_N = \frac{N-1}{2^n}$. The $y = 0$ case is special as 0 must be in the code. Moreover, for $N \geq 4$, for any distinct $y,y' \neq 0$, the events $\{y \in Q\}$ and $\{y' \in Q\}$ are independent. Therefore,

$$\Gamma_w = E_Q(E_{y \sim \mathcal{Q}} f_w(y))^2 = \frac{1}{N^2} \left( \sum_{y,y' \neq 0 : y \neq y'} p_N^2 f_w(y) f_w(y') + \sum_{y \neq 0} p_N f_w(y)^2 + 2 \sum_{y \neq 0} p_N f_w(y) f_w(0) + f_w(0)^2 \right). \quad (11)$$

Since $E_{U_n} f_w = 0$, we have $\sum_{y \neq 0} f_w(y) = -f_w(0)$. Moreover, $\frac{1}{2^n} \sum_{y,y'} f_w(y) f_w(y') = (E_{U_n} f_w)^2 = 0$, thus

$$\sum_{y,y' \neq 0 : y \neq y'} f_w(y) f_w(y') = -\sum_{y \neq 0} f_w(y)^2 - 2 \sum_{y \neq 0} f_w(y) f_w(0) - f_w(0)^2.$$

Replacing in (11), we obtain

$$\Gamma_w = \left( \frac{p_N}{N^2} - \frac{p_N^2}{N} \right)^2 \sum_{y \neq 0} f_w(y)^2 + 2 \left( \frac{p_N}{N^2} - \frac{p_N^2}{N} \right)^2 \sum_{y \neq 0} f_w(y) f_w(0) + \left( \frac{1}{N^2} - \frac{p_N}{N} \right)^2 f_w(0)^2$$

$$= \left( \frac{p_N}{N^2} - \frac{p_N^2}{N} \right)^2 \sum_{y} f_w(y)^2 - 2 \left( \frac{p_N}{N^2} - \frac{p_N^2}{N} \right)^2 f_w(0)^2 + \left( \frac{1}{N^2} - \frac{p_N}{N} \right)^2 f_w(0)^2$$

$$= \frac{a}{N} E_{U_n} f_w^2 + \frac{b}{N^2} f_w(0)^2,$$
where \( a = \frac{2^w}{N}(p_N - p_N^2) = (1 - \frac{1}{N})^{1-p_N^2} \), and \( b = 1 + 2p_N^2 - 3p_N \). Note that since \( N = w(1) \) and \( N = o(2^n) \), and hence \( p_N = o(1) \), we have \( a = 1 - o(1) \) and \( b = 1 - o(1) \). Now,

\[
E_{\mathcal{U}_n} f_w^2 = E_{y \sim \mathcal{U}_n} I_w(y + u)^2 - \text{bin}_n(w)^2 = \text{bin}_n(w) - \text{bin}_n(w)^2
\]

and \( f_w(0)^2 = (I_w(u) - \text{bin}_n(w))^2 \). It follows that \( \sum_w E_{\mathcal{U}_n} f_w^2 = 1 - \sum_w \text{bin}_n(w)^2 \) and

\[
\sum_w f_w(0)^2 = (1 - \text{bin}_n(|u|))^2 + \sum_{w \neq |u|} \text{bin}_n(w)^2 = 1 - 2\text{bin}_n(|u|) + \sum_w \text{bin}_n(w)^2.
\]

Using the identity, \( \sum_w \binom{n}{w}^2 = \binom{2n}{n} \), we get \( \sum_w \text{bin}_n(w)^2 = \text{bin}_{2n}(n) \). Therefore,

\[
\Gamma = \frac{a}{N} (1 - \text{bin}_{2n}(n)) + \frac{b}{N^2} (1 - 2\text{bin}_n(|u|) + \text{bin}_{2n}(n))
\]

\[
= \frac{a}{N} \left( 1 - O(n^{-1/2}) \right) + \frac{b}{N^2} \left( 1 \pm O(n^{-1/2}) \right)
\]

\[
= \frac{1 \pm o(1)}{N}.
\]

\[\square\]