Note on explicit form of entanglement entropy in the RST model

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Abstract

For an evaporating black hole which is a radiation-black hole combined system, we express the entanglement entropy and the Page time in terms of the conformal time in the RST model. The entropy change of the black hole is nicely written in terms of Hawking flux. Integrating the first law of thermodynamics, we can obtain the decreasing black-hole entropy and the increasing radiation entropy, and the entanglement entropy for this system based on the Page argument. We also obtain analytically the critical temperature to release black-hole information, which corresponds to the Page time, and discuss the relation between the conserved total entropy and information recovering of the black hole in this model.

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Bekenstein have suggested that the entropy of a black hole is proportional to the area of the horizon \([1–3]\), and subsequently Hawking’s discovery has led to the result that a black hole has thermal radiation with the temperature \(T_H = \kappa_H/2\pi\) \([4]\), where \(\kappa_H\) is the surface gravity at the event horizon. It has also been claimed that a black hole would eventually disappear completely through thermal radiation, which gives rise to information loss problem \([5]\). However, if Hawking radiation plays a role of carrier of information, information will come out so slowly until the Page time \([6]\) when the entanglement entropy becomes maximum such that the dimension of radiation equals to that of the black hole in the Hilbert space. When the dimension of radiation is larger than that of the black hole, information is naturally contained in radiation. Moreover, it has been shown that in Ref. \([6]\) the above statistical analysis can be realized in the Callan-Giddings-Harvey-Strominger (CGHS) model \([7]\) by taking into account the classical metric along with the corresponding constant temperature which is independent of black hole mass so that radiation does not reflect the back reaction of the geometry. Now, one may ask how to incorporate the back reaction of the geometry based on the Page model.

In this work, we are going to study the RST model \([8, 9]\) to take into account back reaction of the geometry, which yields naturally the time-dependent geometry. The essential difficulty is to identify the time-dependent temperature which is quite awkward in standard thermodynamics. So we would like to assume that a radiation-black hole combined system is in equilibrium at each time such that the radiation temperature measured by the fixed observer at the future null infinity is identified with the black hole temperature. Then, the thermodynamic first law is also read off from the differential form of the energy conservation law \([10]\), so that the entropy change of the black hole is nicely written in terms of Hawking flux. Integrating the first law of thermodynamics, we can obtain the decreasing black hole entropy and the increasing radiation entropy, and the entanglement entropy for this system based on the Page argument \([6]\). So, the total entropy is always constant while the total information is not conserved locally because of the time-dependent entanglement entropy; however, it is expected that the total information is recovered after complete evaporation of the black hole.

Now, let us start with the RST model given by action \([8, 9]\),

\[
S = \frac{1}{\pi} \int d^2 x \left[ -\frac{1}{\kappa} \partial_+ \chi \partial_- \chi + \frac{1}{\kappa} \partial_+ \Omega \partial_- \Omega + \lambda^2 e^{(2/\kappa)(\chi-\Omega)} + \frac{1}{2} \sum_{i=1}^N \partial_+ f_i \partial_- f_i \right],
\]
and the two constraints, \( \kappa t_\pm = -(1/\kappa)\partial_\pm \chi \partial_\pm \chi + \partial_\pm^2 \chi + (1/\kappa)\partial_\pm \Omega \partial_\pm \Omega + \frac{1}{2} \sum_{i=1}^{N} \partial_\pm f_i \partial_\pm f_i \), where \( \chi = \kappa \rho - (\kappa/2) \phi + e^{-2\phi} \) and \( \Omega = (\kappa/2) \phi + e^{-2\phi} \). The equations of motion derived from the action \( (1) \) can be exactly solved. In the Kruskal coordinates where \( \chi = \Omega \), the evaporating black hole formed by an incoming shock wave of \( T_{++}^q = [M/(\lambda x_0^+)] \delta (x^+ - x_0^+) \) is described by the solution of \( \Omega (x^+, x^-) = -\lambda^2 x^+ x^- - \frac{\kappa}{4} \ln (-\lambda^2 x^+ x^-) - \frac{M}{\lambda x_0^+} (x^+ - x_0^+) \Theta (x^+ - x_0^+) \), where the linear dilaton vacuum is chosen for \( x^+ < x_0^+ \). The energy-momentum tensors of quantum matter are \( T_{\pm \pm}^q = \kappa [\partial_\pm^2 \rho - (\partial_\pm \rho)^2 - t_\pm (x^\pm)] \) and \( T_{\pm -}^q = -\kappa \partial_+ \partial_- \rho \) [7]. The unknown functions \( t_\pm (x^\pm) \) reflect the nonlocal property of the effective action. An asymptotically static coordinate can be obtained from the coordinate transformations defined by \( x^+ = (1/\lambda) e^{\lambda \sigma^+} \) and \( x^- = -(1/\lambda) e^{-\lambda \sigma^-} - (M/\lambda^2) e^{-\lambda \sigma_0^+} \Theta (\sigma^+ - \sigma_0^+) \), where \( \sigma_0^+ = \lambda^{-1} \ln (\lambda x_0^+) \).

On the other hand, from the covariant conservation law, one can get the ordinary conserved quantity by expanding the metric and dilaton fields around the linear dilaton vacuum. Then, the linearized equation of motion becomes \( G_{\mu \nu}^{(1)} = T_{\mu \nu} - G_{\mu \nu}^{(2)} \) [11], where \( T_{\mu \nu} \) is a classical energy-momentum tensor, \( G_{\mu \nu}^{(1)} \) is the linear perturbed part of \( G_{\mu \nu} \), and \( G_{\mu \nu}^{(2)} \) is the rest. Then, one can choose the time and space coordinate so that it is easy to show that the linearized equation of motion identically satisfies the ordinary conservation law, \( \partial_\mu G^{(1)\mu \nu} = 0 \), by the use of the linearized Bianchi identity [12]. It implies that the current defined as \( J^\mu = T^{\mu 0} - G^{(2) \mu 0} \) satisfies the ordinary conservation law \( \partial_\mu J^\mu = 0 \). Thus we can define the Bondi mass \( B(\sigma^-) \) which is the energy evaluated along the null line [13], \( B(\sigma^-) = \frac{1}{2} \int_{-\infty}^{\infty} d\sigma^+ G^{(1) \sigma^+ \sigma^-} \left| \right. \) while the ADM mass is calculated at the spatial infinity as \( E_{\text{ADM}} (t) = \int_{-\infty}^{\infty} dq G^{(1)\sigma^+ \sigma^-} (t, q) \) [14]. Using the integrated form of the linearized equation of motion, the difference between the ADM mass and the Bondi mass can be obtained as [10]

\[
E_{\text{ADM}} (t) - B(\sigma^-) = \int_{-\infty}^{\sigma^-} d\sigma^- \left( T_{--}^{qt} \right) \bigg|_{\sigma^+ \to \infty}, \tag{2}
\]

where we took the vanishing classical energy-momentum tensor. Next, the integrated Hawking flux is given by \( H(\sigma^-) = \int_{-\infty}^{\sigma^-} d\sigma^- h(\sigma^-) \), where the Hawking flux is \( h(\sigma^-) = T_{--}^{qt} \bigg|_{\sigma^+ \to \infty}. \) The Hawking flux is simply reduced to the boundary function as \( h(\sigma^-) = -t_-(\sigma^-) \) since \( \sigma^\pm \) is a quasi-static coordinate system at infinity, and so the fields approach the linear dilaton vacuum at \( \sigma^+ \to \infty \). In this black hole, the Hawking radiation is written as \( h(\sigma^-) = (\kappa \lambda^2 / 4)[1 - (1 + (M/\lambda)e^{\lambda(\sigma^- - \sigma_0^+)})^{-2}] \) [7]. Note that the Bondi mass is the remaining energy after quantum-mechanical Hawking radiation have been emitted from the
system. So it is plausible to regard the Bondi mass as a black hole mass in the quantum back-reacted model. From Eq. (2), we can get the conservation law as

\[ B(\sigma^-) + H(\sigma^-) = M, \]

where \( M \) is the ADM mass. The energy can be conserved in this evaporating black-hole system so that the Bondi energy plus the Hawking radiation should be equal to the initial infalling energy by the scalar fields.

Now, we will assume the radiation-black hole combined system as a thermal equilibrium system for each conformal time \( \sigma^- \) in order to apply the thermodynamic first law. Let us first relate the Hawking flux to the Hawking temperature in analogy with the static case \([6, 15]\), then one can read off the black hole temperature \( T(\sigma^-) \) from the Hawking radiation by identifying

\[ h(\sigma^-) = \kappa \pi^2 T^2(\sigma^-), \]

which yields

\[ T(\sigma) = \frac{\lambda}{2\pi} \left[ 1 - \frac{1}{\left(1 + \frac{M}{\lambda} e^{\lambda(\sigma^- - \sigma^+_0)}\right)^2} \right]^{1/2}. \]

Note that it vanishes at \( \sigma^- \to -\infty \) since the black hole did not radiate yet and the well-known Hawking temperature is recovered as \( T_H = \lambda/2\pi \) at \( \sigma^- \to \infty \) which is compatible with the previous static results \([6, 8, 15]\). Using the differential form of the energy conservation law (3), the change of the black hole entropy can be written as

\[ \Delta S_h = S_h(\sigma^-) - S_h^0 = \int \frac{dB}{T}, \]

\[ = -\pi \sqrt{\kappa} \int_{-\infty}^{\sigma^-} \sigma^- \sqrt{h(\sigma^-)}, \]

where \( S_h^0 \) denotes the entropy of the black hole at \( \sigma^- \to -\infty \). The entropy change is essentially due to Hawking radiation such that the entropy of the black hole is decreasing. From Eq. (6) the entropy is calculated as

\[ S_h(\sigma^-) = \frac{2\pi M}{\lambda} - \frac{\pi \kappa}{2} \left[ \sec^{-1} \gamma(\sigma^-) + \ln \left( \gamma(\sigma^-) + \sqrt{\gamma^2(\sigma^-) - 1} \right) \right], \]

where \( \gamma(\sigma^-) = 1 + (M/\lambda) e^{\lambda(\sigma^- - \sigma^+_0)} \) and we employed the fact that the entropy of the black hole is given by \( S_h^0 = 2\pi M/\lambda \) at the initial time of \( \sigma^- \to -\infty \) since the entropy of the
black hole starts with the maximum thermal entropy of the area law, and at the same time the Hawking temperature (5) is zero. As time goes on, the black hole entropy is decreasing according to the increasing Hawking temperature which amounts to \( T_H = \frac{\lambda}{2\pi} \) at \( \sigma^- \rightarrow \infty \).

Note that in the conventional thermodynamic analysis, the black hole entropy and the temperature are given as 
\[
S = \frac{2\pi M}{\lambda}, \quad T_H = \frac{\lambda}{2\pi}.
\]

On the other hand, for a system consisting of the black-hole subsystem and the radiation subsystem, the entanglement entropy is given by the Page argument as [6]

\[
S_{\text{ent}}(\sigma^-) \simeq \begin{cases} 
S_r - \frac{1}{2}e^{S_r - S_h}, & \text{for } S_r \leq S_h \\
S_h - \frac{1}{2}e^{S_h - S_r}, & \text{for } S_r \geq S_h 
\end{cases}
\]

for \( S_h, S_r \gg 1 \), where \( S_h \) and \( S_r \) are the black-hole entropy and the radiation entropy, respectively. Note that the total entropy of the system is preserved such that it is given as \( S_r + S_h = 2\pi M/\lambda \). The entanglement entropy (8) has a maximum value at the Page time when the black hole emits a half of its initial Bekenstein-Hawking entropy, i.e., \( S_r = \frac{\pi M}{\lambda} \).

Using Eq. (7), we can write the entanglement entropy explicitly in terms of \( \sigma^- \), and it becomes

\[
S_{\text{ent}}(\sigma^-) \simeq \frac{\pi \kappa}{2} \left[ \sec^{-1} \gamma(\sigma^-) + \ln \left( \gamma(\sigma^-) + \sqrt{\gamma^2(\sigma^-) - 1} \right) \right] - \frac{1}{2} \left[ \left( \gamma(\sigma^-) + \sqrt{\gamma^2(\sigma^-) - 1} \right) \exp \left( \sec^{-1} \gamma(\sigma^-) - \frac{2M}{\kappa\lambda} \right) \right]^{-\pi \kappa} \quad (9)
\]

for \( \sigma^- \leq \sigma^-_c \) and

\[
S_{\text{ent}}(\sigma^-) \simeq \frac{2\pi M}{\lambda} - \frac{\pi \kappa}{2} \left[ \sec^{-1} \gamma(\sigma^-) + \ln \left( \gamma(\sigma^-) + \sqrt{\gamma^2(\sigma^-) - 1} \right) \right] - \frac{1}{2} \left[ \left( \gamma(\sigma^-) + \sqrt{\gamma^2(\sigma^-) - 1} \right) \exp \left( \sec^{-1} \gamma(\sigma^-) - \frac{2M}{\kappa\lambda} \right) \right]^{-\pi \kappa} \quad (10)
\]

for \( \sigma^- \geq \sigma^-_c \). Note that the entanglement entropy becomes maximum at the conformal time of \( \sigma^-_c \) which comes from the maximization of the entanglement entropy formally given as the closed form of \( \gamma(\sigma^-_c) \cos[\ln(\gamma(\sigma^-_c) + \sqrt{\gamma^2(\sigma^-_c) - 1}) - 2M/(\kappa\lambda)] = 1 \).

As a result, we have obtained the decreasing black-hole entropy, the increasing radiation entropy, the entanglement entropy, and the Page time in terms of the conformal time in the exactly soluble RTS model. Moreover, we can find a Page temperature at the Page time since \( \sigma^-_c \) was identified so that it becomes \( T(\sigma^-_c) \) from Eq. (5). In other words, information is significantly leased above the critical temperature of \( T(\sigma^-_c) \). On the other hand, the total
entropy is conserved but the the entanglement entropy is time-dependent so that the total information is not conserved locally but it can be conserved when the entanglement entropy vanishes after complete evaporation of the black hole.

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