RELATION ALGEBRAS OF SUGIHARA, BELNAP, MEYER, CHURCH

R. L. KRAMER, R. D. MADDOX

Abstract. Sugihara’s relation algebra is a complete atomic proper relation algebra that contains chains of relations isomorphic to Sugihara’s original matrix. Belnap’s relation algebra (better known as the Point Algebra) is a proper relation algebra containing chains of relations isomorphic to the 2-element and 4-element Sugihara matrices. In addition, the crystal matrices, Meyer’s RM84 matrices, and Church’s matrices are definitional subreducts of proper relation algebras. The axioms of R-mingle are analyzed as statements about binary relations.

1. Introduction

Definition 1. $S = (S, ∨, ∧, →, ∼)$ is a Sugihara matrix or Sugihara chain if the elements of $S$ form a chain under a linear ordering $≤$, $A ∨ B$ and $A ∧ B$ are the maximum and minimum of $A, B ∈ S$ in this ordering, $∼$ is an involution that inverts the ordering,

$$A ≤ B \text{ iff } ∼B ≤ ∼A,$$

and $A → B$ is $∼A ∨ B$ if $A ≤ B$, otherwise $∼A ∧ B$. An element $A ∈ S$ is said to be designated if $∼A ≤ A$.

In what Anderson and Belnap [1, p. 337] “have accordingly come to think of ... as the Sugihara matrix”, the elements are the set $\mathbb{Z}^*$ of non-zero integers and $∼(i) = −i$ for every non-zero $i ∈ \mathbb{Z}^*$. This is the Sugihara matrix $S_{\mathbb{Z}^*}$ (named by Meyer [1, p. 414]). Having described $S_{\mathbb{Z}^*}$, Anderson and Belnap [1, p. 337] make this suggestion, “(Or one might insert 0 between -1 and +1, counting it designated.)” The resulting matrix is Meyer’s $S_{\mathbb{Z}}$ [1, p. 414], which has a fixed point under $∼$, namely 0 = $∼0$. No such fixed point occurs in the original matrix of Sugihara [17] (described in [1, p. 335–6]). In this matrix, the elements are ordered like two copies of the integers, one after the other. Perhaps its name should be $S_{\mathbb{Z}+\mathbb{Z}}$ (where + denotes ordinal addition).

In $S_{\mathbb{Z}}$, we review some concepts from the theory of binary relations, including proper relation algebras. In $S_{\mathbb{Z}}$, a proper relation algebra $S_I$ is defined for every $I ⊆ \mathbb{Z}$. When $I = \mathbb{Z}$, we obtain Sugihara’s proper relation algebra $S_\mathbb{Z}$. The elements $S_I$ are binary relations on an underlying base set $U_I$. They form a complete atomic Boolean algebra under union $∪$, intersection $∩$, and complementation $¬$. The identity relation $\text{Id}_I$ on the base set $U_I$ is an atom of $S_I$. All the other atoms
(the diversity atoms) are dense linear orderings of $U_I$ without endpoints. Thus every relation in $\mathcal{S}_I$ is either a union of a set of pairwise-disjoint dense endpoint-free linear orderings of $U_I$, or else the union of such a relation with $\text{Id}_I$. (See Lemma 1 in SS9.)

In $\mathcal{S}_I$ there are two chains of binary relations, linearly ordered by inclusion and closed under union, intersection, residuation, and converse-complementation. Such chains are Sugihara matrices, with union as Sugihara’s maximum, intersection as minimum, converse-complementation as Sugihara’s order-reversing involution, and residuation of binary relations as Sugihara’s implication. In Sugihara’s relation algebra $\mathcal{SZ}$ these chains are both isomorphic to Sugihara’s original matrix $\mathcal{S}_{Z+Z}$ (see SS11). The designated elements of Sugihara’s matrix match up with the relations in Sugihara’s relation algebra that contain the identity relation. Similarly, Belnap’s proper relation algebra (which has been known for the last few decades as “the Point Algebra”) is a finite relation algebra on an 8-element set of binary relations. It is a definitional reduct of Belnap’s $M_0$ matrices, and it contains chains isomorphic to the 2-element and the 4-element Sugihara matrices (see SS3). Likewise, the crystal matrices (SS4), Church’s matrices (SS6), Meyer’s RM84 matrices (SS7), and odd Sugihara matrices (SS8) are definitional subreducts of small finite proper relation algebras.

The relevance logic RM, or $R$-mingle, is complete for Sugihara matrices, so by Theorem 1 the axioms of RM are axioms for a class of algebras of binary relations. Accordingly, in SS12 the axioms and theorems of RM are viewed as properties of binary relations. For example, the axiom $A \to A$ asserts that the binary relation $A$ is a subset of itself, and the crucial mingle axiom $A \to (A \to A)$ of RM asserts that $A$ is a transitive relation. The contraposition axiom $(A \to \sim B) \to (B \to \sim A)$ asserts that the relations $A$ and $B$ commute under relative multiplication, and $(A \to \sim A) \to \sim A$ asserts that $A$ is a dense relation.

2. Binary relations

Definition 2. For any set $U$, called the base set, the Cartesian square of $U$ is

$$U^2 = U \times U = \{ \langle u, v \rangle : u, v \in U \}.$$

Relations $R, S, \cdots$ on $U$ are subsets of $U^2$. The identity relation and diversity relation on $U$ are

$$\text{Id} = \{ \langle u, u \rangle : u \in U \},$$

$$\text{Di} = \{ \langle u, v \rangle : u, v \in U, u \neq v \}.$$

The union of $R$ and $S$ is

$$R \cup S = \{ \langle u, v \rangle : \langle u, v \rangle \in R \text{ or } \langle u, v \rangle \in S \}.$$

The intersection of $R$ and $S$ is

$$R \cap S = \{ \langle u, v \rangle : \langle u, v \rangle \in R \text{ and } \langle u, v \rangle \in S \}.$$

The complement of $R$ with respect to $U^2$ is

$$R' = R \setminus (U^2).$$

The converse of $R$ is

$$R^{-1} = \{ \langle v, u \rangle : \langle u, v \rangle \in R \}.$$
The converse-complement of $R$ with respect to $U^2$ is
\[ \sim R = R^{-1} \setminus (U^2) = R^{-1}. \]

The relative product of $R$ and $S$ is
\[ R|S = \{ \langle u, v \rangle : \text{for some } w, \langle u, w \rangle \in R \text{ and } \langle w, v \rangle \in S \}. \]

The residual of $R$ and $S$ is
\[ R \rightarrow S = R^{-1}|S = \{ \langle u, v \rangle : \text{for all } w, \text{ if } \langle w, u \rangle \in R \text{ then } \langle w, v \rangle \in S \}. \]

The relation algebra of binary relations on $U$ is
\[ \Re(U) = \langle \{ R : R \subseteq U^2 \}, \cup, \cap, \sim, |, |^{-1}, \text{Id} \rangle. \]

The class of proper relation algebras is the closure of
\[ \{ \Re(U) : U \text{ is a set} \} \]
under the formation of direct products and subalgebras.

A definitional reduct of an algebra $A$ is an algebra obtained by possibly omitting some of the fundamental operations of $A$ and possibly adding some operations that are term-definable in $A$. A definitional subreduct is a subalgebra of a definitional reduct.

3. Belnap’s relation algebra

$\Re(Q)$ and $\Re(Z)$ are the proper relation algebras of binary relations on the rationals $Q$ and the integers $Z$, respectively. $\Re(Q)$ has an 8-element subalgebra that we call (just in this paper) “Belnap’s relation algebra” because its definitional reduct, the Belnap $M_0$ matrices [2], was used to prove the variable-sharing property for relevance logic. The operations $\cup$ and $\cap$ are retained; $\sim$, $\mid$, and $\sim^{-1}$ are deleted; $\sim$ and $\rightarrow$ are added. These last two are term-definable according to Definition 2.

From the 1980s onward there has developed an extensive literature in which Belnap’s relation algebra is known as the Point Algebra, because among its eight relations are the three ways that two points on the real line can be related to each other: either equal ($=$), or the first to the left of the second ($<$), or to the right ($>$); for recent work see Bodirsky [4].

The Belnap matrices were presented as a lattice with tables for additional operations $\rightarrow$ and $\sim$; see [2], [11 pp. 197–8, 252–3], and [6, p. 101–2]. The observation that the Belnap matrices are a definitional reduct of a proper relation algebra occurs in [11], [3, p. 117], and [12, Theorem 4.1] (the source of its description here).

Definition 3. $\mathfrak{B}$ is Belnap’s relation algebra, where
\[ \mathfrak{B} = \langle M_0, \cup, \cap, \sim, |, |^{-1}, \text{Id} \rangle, \]
\[ M_0 = \{ \emptyset, \text{Id}, R, L, R \cup \text{Id}, L \cup \text{Id}, R \cup L, Q^2 \}, \]
\[ \text{Id} = \{ (q, q) : q \in Q \}, \]
\[ L = \{ (q, r) : q < r \in Q \}, \]
\[ R = \{ (q, r) : q > r \in Q \}. \]

The set of designated elements is $\{ \text{Id}, R \cup \text{Id}, L \cup \text{Id}, Q^2 \}$. 
$M_0$ is closed under $\cup, \cap, \neg, \vert$, and $^{-1}$, as is required for the definition of $\mathbb{B}$ to make sense. The operations in the Belnap matrices are $\cup, \cap, \rightarrow$, and $\sim$. There are four chains inside Belnap’s relation algebra that are closed under $\cup, \cap, \rightarrow$ and $\sim$, thus forming Sugihara matrices, namely,

\[
\emptyset \subset L \subset L \cup \text{id} \subset \mathbb{Q}^2, \quad L \subset L \cup \text{id}, \quad \emptyset \subset R \subset R \cup \text{id} \subset \mathbb{Q}^2, \quad R \subset R \cup \text{id}.
\]

The chains of length two appear in Belnap’s proof [2, p. 145] that relevance logic has the variable-sharing property, i.e., $A \rightarrow B$ cannot be a theorem unless $A$ and $B$ are formulas sharing at least one propositional variable.

The proof proceeds by first noting that if the variables of $A$ are mapped to relations in $\{L, L \cup \text{id}\}$, then, by the closure of this set under the operations, $A$ must also be mapped to one of these two relations. If the variables of $B$ do not occur in $A$, they may be mapped to $\{R, R \cup \text{id}\}$, and so will $B$. But $A \rightarrow B = \emptyset$ and $\emptyset$ not designated, so $A \rightarrow B$ is not valid because it fails to contain the identity relation.

4. Sugihara’s relation algebra

**Definition 4.** If $q \in \mathbb{Q}_\mathbb{Z}$, we say that $q$ is **eventually zero** if there exists some $n$ in the domain $\mathbb{Z}$ of $q$ such that $q_i = 0$ for all $i > n$. For every $I \subseteq \mathbb{Z}$, $U_I$ is the set of functions from $\mathbb{Z}$ to $\mathbb{Q}$ that are eventually zero and non-zero only on $I$:

\[
U_I = \{q: q \in \mathbb{Q}_\mathbb{Z}, (\exists n \in \mathbb{Z})(\forall i > n)(q_i = 0), (\forall i \in \mathbb{Z})(q_i \neq 0 \implies i \in I)\}.
\]

A function in $U_I$, called a **sequence**, looks like

\[
\cdots q_{-n} \cdots q_{-3} q_{-2} q_{-1} q_0 q_1 q_2 q_3 \cdots q_n \cdots 0 0 0 \cdots
\]

$\mathcal{G}_I$ is the subalgebra of $\mathcal{R}(U_I)$ completely generated (using arbitrary unions and intersections, together with the operations $\cap, \neg$, and $^{-1}$) by

\[
\mathcal{A}t_I = \{|\text{id}_I\} \cup \bigcup_{n \in I} \{L_n^I, R_n^I\},
\]

where $\text{id}_I = \{\langle q, q \rangle : q \in U_I\}$, and for every $n \in I$,

\[
L_n^I = \{\langle q, r \rangle : q, r \in U_I, q_n < r_n, \text{ and } q_i = r_i \text{ whenever } n < i\},
\]

\[
R_n^I = \{\langle q, r \rangle : q, r \in U_I, q_n > r_n, \text{ and } q_i = r_i \text{ whenever } n < i\}.
\]

**Sugihara’s relation algebra** is $\mathcal{G}_\mathbb{Z}$.

The condition on $I$ in (1) asserts that if $q \in U_I$ has a non-zero entry, it occurs at an index in $I$ (and $I$ is not empty). This cannot happen if $I = \emptyset$. Therefore $U_\emptyset = \{q\}$ where $q_i = 0$ for every $i \in \mathbb{Z}$, i.e., $q = \langle 0 : i \in \mathbb{Z} \rangle$. On the other hand, if $I = U_\mathbb{Z}$, then the condition on $I$ in (1) always holds, so $U_\mathbb{Z}$ is the set of all $\mathbb{Z}$-indexed sequences of rationals that are eventually zero. $U_\mathbb{Z}$ is the base set for Sugihara’s relation algebra; its elements are relations on $U_\mathbb{Z}$.

$\mathcal{R}(U_\emptyset)$ is the 2-element proper relation algebra whose relations are $\emptyset$ and $\text{id}_\emptyset = \{\langle q, q \rangle \} = (U_\emptyset)^2$. There are no relations $L_n^I$ and $R_n^I$ in this case, hence $\mathcal{A}t_\emptyset = \{|\text{id}_\emptyset\}$ and $\mathcal{G}_\emptyset = \mathcal{R}(U_\emptyset)$. This algebra has a 2-element Sugihara chain, namely $\langle \emptyset, (U_I)^2 \rangle$.

For the case in which $I$ is a singleton we consider $I = \{0\}$. There is a natural isomorphism from $U_{\{0\}}$ onto $\mathbb{Q}$ that sends each sequence $q \in U_{\{0\}}$ to its value at 0.
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Consequently \( \text{Re}(U_{\{0\}}) \) is isomorphic to \( \text{Re}(\mathbb{Q}) \) and \( \mathcal{S}_{\{0\}} \) is isomorphic to Belnap’s relation algebra \( \mathfrak{B} \),

\[
\text{Re}(U_{\{0\}}) \cong \text{Re}(\mathbb{Q}), \quad \mathcal{S}_{\{0\}} \cong \mathfrak{B}.
\]

For the case in which \( I \) is finite, let \( I = \{0, \ldots, n-1\} \). There are many isomorphisms between \( U_{\{0,\ldots,n-1\}} \) and \( \mathbb{Q}^n \), such as the one that sends a sequence in \( U_{\{0,\ldots,n-1\}} \) to its restriction to \( \{0, \ldots, n-1\} \). These isomorphisms extend to isomorphisms between \( \text{Re}(U_{\{0,\ldots,n-1\}}) \) and \( \text{Re}(\mathbb{Q}^n) \). In this way, the construction here subsumes the one used in the proof of [12, Theorem 6.2], where it is shown that every finite Sugihara matrix of even cardinality \( 2n + 2 \) is a definitional sub-reduct of the proper relation algebra \( \text{Re}(\mathbb{Q}^n) \). The infinite Sugihara matrix \( \mathcal{S}_{\mathbb{Z}} \) is also covered by this construction, for \( \mathcal{S}_{\mathbb{Z}} \) is a definitional sub-reduct of \( \mathcal{S}_{\{i : 0 < i \in \mathbb{Z}\}} \). (At the bottom of [3, p. 122], a computation suggesting that this is not possible is apparently in error.)

5. THE CRYSTAL MATRICES

To illustrate the concepts and give an application of this construction to other matrices of relevance logic, we pay special attention to the case in which \( I \) is a 2-element set. Let \( I = \{0, 1\} \), so \( U_I \cong \mathbb{Q}^2 \). There are exactly three relation algebras whose five atoms are the identity element and four diversity atoms, namely, two transitive dense atoms and their converses. In the numbering system of [10], these algebras are \( 2_{83}, 29_{83}, \) and \( 43_{83} \). \( \mathcal{S}_{\{0,1\}} \) is isomorphic to the first one:

\[
\mathcal{S}_{\{0,1\}} \cong 2_{83}.
\]

so \( 2_{83} \) is a proper relation algebra. Under this isomorphism, the multiplication table for the relative products of the atoms of \( \mathcal{S}_{\{0,1\}} \) is shown in Table \( \text{I} \). To get an explicit description of this representation, we correlate \( 0 \in I \) with the horizontal \( x \)-axis, and \( 1 \in I \) with the vertical \( y \)-axis, in the \( xy \)-plane of calculus and analytic geometry. Two points are related by \( L_0^I \) if we can get from the first to the second by motion to the right (by increasing the \( x \)-coordinate and keeping the same \( y \)-coordinate). The image of the origin under \( L_0^I \) is the positive part of the \( x \)-axis. The image of the origin under \( R_0^I \) is the negative part of the \( x \)-axis. \( L_1^I \) is upward motion (increase the \( y \)-coordinate and allow any new \( x \)-coordinate). The image of the origin under \( L_1^I \) is the upper half plane (excluding the \( x \)-axis). The image of the origin under \( R_1^I \) is the lower half plane (excluding the \( x \)-axis). With these images it is easy to visualize the relative products in Table \( \text{I} \).

| \( X \mid Y \) | \( \text{Id}_I \) | \( L_0^I \) | \( R_0^I \) | \( L_1^I \) | \( R_1^I \) |
| --- | --- | --- | --- | --- | --- |
| \( \text{Id}_I \) | \( \text{Id}_I \) | \( L_0^I \) | \( R_0^I \) | \( L_1^I \) | \( R_1^I \) |
| \( L_0^I \) | \( L_0^I \) | \( L_0^I \) | \( L_0^I \cup R_0^I \) | \( L_1^I \) | \( R_1^I \) |
| \( R_0^I \) | \( R_0^I \) | \( L_0^I \cup R_0^I \) | \( R_0^I \) | \( L_1^I \) | \( R_1^I \) |
| \( L_1^I \) | \( L_1^I \) | \( L_1^I \) | \( L_1^I \) | \( Q^2 \) | \( Q^2 \) |
| \( R_1^I \) | \( R_1^I \) | \( R_1^I \) | \( R_1^I \) | \( Q^2 \) | \( R_1^I \) |

Table 1. Table of relative products of atoms of \( \mathcal{S}_{\{0,1\}} \)
are attributed to R. K. Meyer, and are also presented in [15, p. 250], [5, p. 65–6] and [6, p. 95]. All three algebras 2_{83}, 2_{93}, and 4_{38} contain copies of Meyer’s crystal matrices. The elements of the crystal matrices in \( S\{0,1\} \) are these binary relations:

\[
\text{Cr} = \{\emptyset, L_1^1, L_1^1 \cup L_0^1, L_1^1 \cup R_0^1, L_1^1 \cup L_0^1 \cup R_0^1, D_i\},
\]

where \( D_i = L_1^1 \cup R_1^1 \cup L_0^1 \cup R_0^1 \). The relations in \( \text{Cr} \) form an 8-element chain, partially ordered by inclusion, hence \( \text{Cr} \) is closed under the relativizations of \( \cup \) and \( \cap \) to \( D_i \). For closure under \( \rightarrow \) and \( \sim \) we explicitly relativize these operations to \( D_i \):

\[
\sim' X = D_i \cap \sim X, \quad X \rightarrow' Y = D_i \cap (X \rightarrow Y).
\]

The four crystal matrices are the tables for the actions of \( \cup \), \( \cap \), \( \rightarrow' \), and \( \sim' \) on \( \text{Cr} \). Because of this description of Meyer’s crystal matrices as a definitional subreduct of \( S\{0,1\} \), we could refer to \( S\{0,1\} \) as Meyer’s crystal relation algebra. In the sequence of finite proper relation algebras \( S\{0\}, S\{0,1\}, \ldots, \) Belnap’s is first, Meyer’s is second.

### 6. Church’s Matrices

A similar treatment can be applied to Church’s lattice (or matrices) [15, p. 379] or diamond [16, p. 277]. We’ll use the proper relation algebra 2_7 from [10] and call it Church’s relation algebra. The smallest representation of Church’s algebra requires a 6-element set. Assuming \( V \) and \( W \) are disjoint 3-element sets, let

\[
U = V \cup W,
\]

\[
\text{Id} = \{(u, u) : u \in U\},
\]

\[
A = (V^2 \cup W^2) \cap D_i,
\]

\[
B = (V \times W) \cup (W \times V).
\]

Then \( \{\text{Id}, A, B\} \) is a partition of \( U^2 \) and is the set of atoms of an 8-element proper relation algebra (called 2_7 in [10]). The table of relative products of atoms in Church’s relation algebra is

|   | Id | A   | B   |
|---|----|-----|-----|
| Id | Id | A   | B   |
| A  | A  | Id  | A   |
| B  | B  | B   | Id  |

The set of binary relations for Church’s matrices/lattice/diamond is

\[
\text{Ch} = \{A \cup B, A, B, \emptyset\}.
\]

The matrices for \( \lor, \land, \rightarrow, \) and \( \sim \) are given by the action of \( \cup, \cap, \rightarrow' \), and \( \sim' \) on \( \text{Ch} \), respectively. Notice that since all the relations involved are symmetric (they are their own converses) the interpretation of \( \sim \) in this case boils down to complementation. The Church matrix for the logical connective \( \rightarrow \), computed according to the operation \( \rightarrow' \) defined in (4), is

| \( \rightarrow' \) | A \( \lor B \) | A   | B   | \emptyset |
|-----------------|-------------|-----|-----|----------|
| A \( \lor B \)  | A \( \land A \) | \emptyset | \emptyset | \emptyset |
| A               | A \( \lor B \) | A   | B   | \emptyset |
| B               | A \( \lor B \) | \emptyset | A   | \emptyset |
| \emptyset       | A \( \lor B \) | A \( \lor B \) | A \( \lor B \) | A \( \lor B \) |
7. Meyer’s RM84 matrices

Meyer’s RM84 matrices (see [1] p. 334 and [15] p. 253) record the action of \( \cup \), \( \cap \), \( \rightarrow \), and \( \sim \) on a set of 8 binary relations on a 7-element set [12, Theorem 4.2]. Those 8 relations form the proper relation algebra, called 3\( ^3 \), whose unique representation on a 7-element set sends one of the atoms to addition of 1, 2, or 4 modulo 7. Remarkably, this relation algebra has no representation on any 8-element set, but does have (increasingly many) representations on sets of cardinality 9 or more [10, SS56.8]. Since Meyer’s RM84 matrices are a reduct of the proper relation algebra 3\( ^3 \), we could refer to 3\( ^3 \) as Meyer’s RM84 relation algebra. Its table of relative products of atoms is

|   | \( \text{Id} \) | \( A \) | \( A^{-1} \) |
|---|---|---|---|
| \( \text{Id} \) | \( \text{Id} \) | \( A \) | \( A^{-1} \) |
| \( A \) | \( A \cup A^{-1} \) | \( U^2 \) |
| \( A^{-1} \) | \( U^2 \) | \( A \cup A^{-1} \) |

\( U = \{0, \ldots, 6\} \)
\( \text{Id} = \{\langle n, n \rangle : n \in U\} \)
\( A = \{\langle n, n + 7 \rangle : n \in U, m \in \{1, 2, 4\}\} \)
\( A^{-1} = \{\langle n, n + 7 \rangle : n \in U, m \in \{3, 5, 6\}\} \)

8. Odd Sugihara matrices

Finite Sugihara matrices of odd cardinality must have a fixed point under \( \sim \), \( \neg \), a formula equivalent to its own negation. Sugihara matrices without such fixed points were called “normal” by Meyer [1] p. 400, so normal Sugihara matrices are not odd. Recall the Sugihara chains in Belnap’s algebra—

\[
\emptyset \subset L \subset L \cup \text{Id} \subset D_i \cup \text{Id} \\
L \subset L \cup \text{Id} \\
R \subset R \cup \text{Id}
\]

If we simply erase “\( \cup \text{Id} \)” (more formally, relativize to \( D_i \)), then two of the relations become the same in each chain, leaving chains of length 1 and 3—

\[
\emptyset \subset L \subset D_i \\
L, \ R
\]

The ones of length 3 are the RM3 matrices [1] p. 470, [15] p. 92. They are Sugihara chains because they are closed under \( \cup, \cap, \rightarrow' \), and \( \sim' \). Indeed, \( L \) and \( R \) are fixed points under \( \sim' \). Similarly, Meyer’s crystal relation algebra contains Sugihara chains of length 5, closed under \( \cup, \cap, \rightarrow' \) and \( \sim' \). This observation applies to all finite odd Sugihara chains. Hence, by [12] Theorem 6.2, every finite Sugihara chain is a definitional subreduct of a proper relation algebra; the chain is closed under \( \cup, \cap, \rightarrow \) and \( \sim \) if it has even length, and under \( \cup, \cap, \rightarrow' \) and \( \sim' \) if odd. The observation also applies to \( S_Z \). Sugihara’s original matrix \( S_{Z+Z} \) will have a fixed point under \( \sim' \) if we relativize to \( D_i \) and add \( L_{(-\infty, \infty)} \) to \( C_{Z} \) in Definition 9.

9. The structure of \( S_I \)

From Definition 4 we immediately conclude that \( S_I \) is a complete atomic proper relation algebra, and it is a subalgebra of \( \text{Re}(U_I) \). It turns out that \( \cap, \rightarrow, \neg, \sim' \) are not needed to generate \( S_I \), because
Lemma 1. $\mathcal{A}_I$ is the set of atoms of $\mathcal{S}_I$, and the universe of $\mathcal{S}_I$ is the set of relations on $U_I$ that are unions of subsets of $\mathcal{A}_I$:

$$\mathcal{S}_I = \left\{ \bigcup X : X \subseteq \mathcal{A}_I \right\} \cup \cap, \cup, |, \sim \right\}.$$ 

Proof. It is easy to see that the relations in $\mathcal{A}_I$ are pairwise disjoint and their union is $U_I \times U_I$. Indeed, any two sequences $q, r \in U_I$ are either equal everywhere (are in the identity relation $\text{Id}_I$), or differ somewhere, in which case there is (since they are both eventually zero) a largest integer $n$ where they differ. Since they differ, one of them is not zero, so $n \in I$. The ordering of the rationals is linear, either $q_n < r_n$ or $q_n > r_n$ but not both, so the pair $(q, r)$ must be in the relation $L_n^I$ or $R_n^I$ but not both.

Since the relations in $\mathcal{A}_I$ are a partition of $U_I \times U_I$, the joins of subsets of $\mathcal{A}_I$ form a complete atomic Boolean algebra. What remains is to verify that this set of relations is closed under the formation of converses and relative products. The rules governing converse are

$$\left( L_n^I \right)^{-1} = R_n^I, \quad \text{Id}_I^{-1} = \text{Id}_I, \quad \left( \bigcup_{j \in J} X_j \right)^{-1} = \bigcup_{j \in J} (X_j)^{-1},$$

i.e., conversion reverses left and right, fixes identity, and sends the union of relations to the union of their converses.

For closure under relative multiplication, we reason as follows. Assume $q, r, s$ are three distinct sequences in $U_I$. Each sequence must differ from the other two, so the cardinality of $\{q_n, r_n, s_n\}$ cannot be 1 for every $n \in \mathbb{Z}$. One the other hand, since $q, r, s$ are all eventually zero, the number of elements in $\{q_n, r_n, s_n\}$ will eventually be constantly 1, since $\{q_n, r_n, s_n\} = \{0\}$ whenever $n$ is large enough. Hence there is an integer $n$ at which $\{q_n, r_n, s_n\}$ contains either exactly three or exactly two elements (hence $n \in I$ because they can’t all be zero) and $|\{q_i, r_i, s_i\}| = 1$ for all $i > n$ (they all agree beyond $n$). Although any two of these three sequences are equal beyond $n$, any pair of them could also agree beyond an integer strictly smaller than $n$. In the first case, when $\{q_n, r_n, s_n\}$ has exactly three elements, they must form a chain under the dense linear ordering $<$ on the rationals, and, since they all agree beyond $n$, we may choose $x, y, z$ such that $\{x, y, z\} = \{q, r, s\}$, we have $x_n \neq y_n = z_n$ and $1 = \{|x_i, y_i, z_i\| \}$ for every $i > n$. If $x_n < y_n = z_n$, then $x L_n^I y$ and $x L_n^I z$, while if $x_n > y_n = z_n$, then $x R_n^I y$ and $x R_n^I z$. Now $y$ and $z$ are distinct, but they agree beyond $n$ and also agree at $n$. Hence they disagree at some $j < n$, and agree beyond $j$, in which case $y L_j^I z$ or $y R_j^I z$. However, we may assume $x, y, z$ were chosen so that $y L_j^I z$. This yields two more cases: $y L_j^I z$ and either $x R_n^I y, z$ or $x L_n^I y, z$, where $j < n$. The three cases are

$$\begin{align*}
(6) \quad & x L_n^I y L_n^I z, \quad x L_n^I z \\
(7) \quad & x L_n^I y L_j^I z, \quad x L_j^I z, \quad (j < n) \\
(8) \quad & x R_n^I y L_j^I z, \quad x R_j^I z, \quad (j < n)
\end{align*}$$

From the fact that these are the only possible cases, we proceed to deduce all the rules for computing products of pairs of atoms in the Sugihara relation algebra.
First we consider the products of atoms with the identity relation.

(9) \( \text{Id}_I|\text{Id}_I = \text{Id}_I \) \( L^I_n|\text{Id}_I = \text{Id}_I|L^I_n = L^I_n \) \( R^I_n|\text{Id}_I = \text{Id}_I|R^I_n = R^I_n \)

We start with proof of \( \text{Id}_I|L^I_n = L^I_n \). Assume \( \langle q, r \rangle \in \text{Id}_I|L^I_n \). Then there is some \( s \) such that \( \langle q, s \rangle \in \text{Id}_I \) and \( \langle s, r \rangle \in L^I_n \). The latter two statements tell us that \( q = s \) and \( s \leq r \), from which we conclude \( q \leq r \) by congruence property of equality (equal objects have the same properties), hence \( \langle q, r \rangle \in L^I_n \), showing that \( \text{Id}_I|L^I_n \subseteq L^I_n \). For the opposite inclusion, we assume \( \langle q, r \rangle \in L^I_n \) and note that by choosing \( s = q \) we get \( \langle q, s \rangle \in \text{Id}_I \) and \( \langle s, r \rangle \in L^I_n \), i.e., \( \langle q, r \rangle \in \text{Id}_I|L^I_n \). Thus \( \text{Id}_I \subseteq \text{Id}_I|L^I_n \). Combining this with \( \text{Id}_I|L^I_n \subseteq \text{Id}_I \), we obtain the desired equality. The other equations have similar proofs.

Next we show the product of a diversity atom with itself is itself.

(10) \( L^I_n|L^I_n = L^I_n \) \( R^I_n|R^I_n = R^I_n \)

Assume \( \langle q, r \rangle \in L^I_n|L^I_n \), so there is some \( s \in U_I \) such that \( \langle q, s \rangle \in L^I_n \) and \( \langle s, r \rangle \in L^I_n \). It follows that \( q < r, s, r \) all agree beyond \( n \). We also have \( q_n < r_n \) by the transitivity of the ordering \( < \) on \( Q \), so \( \langle q, r \rangle \in L^I_n \). This shows \( L^I_n|L^I_n \subseteq L^I_n \). For the opposite inclusion, assume \( \langle q, r \rangle \in L^I_n \). Then \( q < r \) and \( q, r \) agree beyond \( n \). By the density of \( < \), we may choose \( s \in U_I \) so that \( s, q \) agree beyond \( n \) and has some intermediate value \( s_n \) (such as the average of \( q_n \) and \( r_n \)) so that \( q_n < s_n < r_n \). The values of \( s \) on arguments in \( I \) and smaller than \( n \) are arbitrary.)

Now we introduce notation for relations in \( \mathfrak{S}_I \). For any \( n, m \in I \) let

\[
L^I_{[n,m]} = \bigcup \{ L^I_\kappa : n \leq \kappa \leq m, \kappa \in I \} \\
L^I_{(n,\infty]} = \bigcup \{ L^I_\kappa : \kappa \in I \} \\
L^I_{(-\infty,n]} = \bigcup \{ L^I_\kappa : n \geq \kappa \in I \} \\
L^I_{(-\infty,\infty)} = \bigcup \{ L^I_\kappa : \kappa \in I \}
\]

Note that \( L^I_{[n,m]} = \emptyset \) if \( n > m \), and \( L^I_{[n,n]} = L^I_n \). The same notation is used with converses. The rules for converse (9) imply

\[
(L^I_{[n,m]})^{-1} = R^I_{[m,n]} \\
(L^I_{(n,\infty]})^{-1} = R^I_{(n,\infty)} \\
(L^I_{(-\infty,n]})^{-1} = R^I_{(-\infty,n]} \\
(L^I_{(-\infty,\infty)})^{-1} = R^I_{(-\infty,\infty)}
\]

The product of a diversity atom with its converse is the union of all diversity atoms with equal or smaller index.

(11) \( R^I_n|L^I_n = L^I_n|R^I_n = \text{Id}_I \cup L^I_{(-\infty,n]} \cup R^I_{(-\infty,n]} \)

The product of two diversity atoms with distinct indices \( n, m \) is the one with the larger index.

(12) \( L^I_m|L^I_n = L^I_n|L^I_m = R^I_m|L^I_n = L^I_n|R^I_m = L^I_n \) if \( m < n \)

(13) \( R^I_m|R^I_n = R^I_n|R^I_m = R^I_m|L^I_n = L^I_m|R^I_n = R^I_n \) if \( m < n \)

A pair in a product on the left side of one of the equations in (11), (12), and (13) gives rise to a triple that falls into exactly one of the three cases (6), (7), and (8), showing the pair is in the particular relation on the right side. For the converse, a pair in the relation on the right has to be shown to be part of a triple exemplifying the left hand side.

To do this for (11), assume \( \langle q, r \rangle \in R^I_n|L^I_n \). Then for some \( s \in U_I \), \( \langle q, s \rangle \in R^I_n \) and \( \langle s, r \rangle \in L^I_n \), hence \( \langle q, r \rangle \in L^I_n \). From \( \langle q, s \rangle \in L^I_n \) and \( \langle s, r \rangle \in L^I_n \) we conclude that either \( q = r \) or we are in case (6) or (7) with \( s = x \) and \{\( q, r \}\} = \{y, z\}. If
s = q then ⟨q, s⟩ is in \( \text{Id}_I \), one of the relations in the union on the right side of (11). If \( q \neq r \) then ⟨q, r⟩ is in some diversity atom whose index, according to (6) and (7), must be either \( n \) or some smaller integer \( j < n \). The pair ⟨q, r⟩ thus belongs to one of the relations on the right. For the converse, assume ⟨q, r⟩ ∈ \( L^t_{m} \cup R^t_{m} \) and \( m \leq n \). We will find \( s \in U_I \) such that ⟨q, s⟩ ∈ \( R^t_{n} \) and ⟨s, r⟩ ∈ \( L^t_{n} \). Since \( q \) and \( r \) agree beyond \( m \), they agree beyond \( n \) as well. Choose values for \( s \in U_I \) so that \( s \) agrees with \( q \) and \( r \) beyond \( n \). (Values at arguments in \( I \) and smaller than \( n \) may be anything.) At \( n \) we choose a rational \( s_n \) that is strictly smaller than both \( q_n \) and \( r_n \) (such as \( s_n = \min(q_n, r_n) - 1 \)). Here we are using the fact that the ordering of the rationals does not have any endpoints. From \( s_n < q_n, s_n < r_n \), and their agreement beyond \( n \) we get ⟨s, q⟩ ∈ \( L^t_{n} \) and ⟨s, r⟩ ∈ \( L^t_{n} \), hence ⟨q, s⟩ ∈ \( R^t_{n} \), hence ⟨q, r⟩ ∈ \( R^t_{n} \). This shows \( R^t_{n} \cup L^t_{n} = \text{Id}_I \cup L^t_{(-\infty,n]} \cup R^t_{(-\infty,n]} \), one of the two equations in (11). The other one has a similar proof, and the proofs for (12) and (13) are somewhat simpler.

**Lemma 2.** \( \mathcal{G}_I \) is commutative.

**Proof.** Relative multiplication distributes over arbitrary unions, so if \( X, Y \subseteq \mathcal{A}t_I \) then

\[
\bigcup X \bigcup Y = \bigcup \{ x : x \in X, y \in Y \} = \bigcup \{ y : x \in X, y \in Y \} = \bigcup Y \bigcup X.
\]

**Lemma 3.** In \( \mathcal{G}_I \) we have, for all \( n, m \in \mathbb{Z} \),

\[
\begin{align*}
(14) & \quad L^t_{(-\infty,n]} L^t_{(-\infty,m]} = L^t_{(-\infty,\max(n,m)]} = L^t_{(-\infty,n]} \cup L^t_{(-\infty,m]} \\
(15) & \quad R^t_{(-\infty,n]} R^t_{(-\infty,m]} = R^t_{(-\infty,\max(n,m)]} = R^t_{(-\infty,n]} \cup R^t_{(-\infty,m]} \\
(16) & \quad L^t_{[n,\infty]} L^t_{[m,\infty)} = L^t_{[\max(n,m),\infty)} = L^t_{[n,\infty]} \cap L^t_{[m,\infty)} \\
(17) & \quad R^t_{[n,\infty]} R^t_{[m,\infty)} = R^t_{[\max(n,m),\infty)} = R^t_{[n,\infty]} \cap R^t_{[m,\infty)} \\
(18) & \quad R^t_{(-\infty,n]} R^t_{[m,\infty)} = R^t_{[m,\infty)} \\
(19) & \quad R^t_{(-\infty,n]} R^t_{(-\infty,m]} = R^t_{(-\infty,\infty)} \cup \text{Id}_I \cup L^t_{(-\infty,-m]} \\
(20) & \quad \text{if } n < m \text{ then } L^t_{(-\infty,n]} R^t_{[m,\infty)} = R^t_{[m,\infty)} \\
(21) & \quad \text{if } n \geq m \text{ then } L^t_{(-\infty,n]} R^t_{[m,\infty)} = \text{Id}_I \cup L^t_{(-\infty,n]} \cup R^t_{(-\infty,m]} \\
(22) & \quad L^t_{(-\infty,n]} R^t_{[m,\infty)} = \begin{cases} R^t_{[m,\infty)} & \text{if } n < m \\ R^t_{(-\infty,\infty)} \cup \text{Id}_I \cup L^t_{(-\infty,-m]} & \text{if } n \geq m \end{cases}
\]

**Proof.** In the computations proving (14)–(22) we use rules (5), (9)–(13), and the fact that relative multiplication distributes over arbitrary unions of relations, noted by “dist”. (14) holds because

\[
L^t_{(-\infty,n]} L^t_{(-\infty,m]} = \bigcup \{ L^t_{\kappa} \cap L^t_{\lambda} : n \geq \kappa \in I, m \geq \lambda \in I \} \quad \text{defs, dist}
\]

\[
= \bigcup \{ L^t_{\max(\kappa,\lambda)} : n \geq \kappa \in I, m \geq \lambda \in I \} \quad \text{(12)}
\]

\[
= L^t_{(-\infty,\max(n,m)]} \quad \text{defs}
\]
Taking converses of both sides in (14) gives (15). For (16), applying converse to (16) gives (17). For (18), if $n < m$, then

$$R_{(-\infty, \infty)}^I | R_{(m, \infty)}^I = \bigcup \{ R_{(m, \infty)}^I : \kappa \in I, m \leq \lambda \in I \}$$

Applying converse to (16) gives (17). For (18),

$$R_{(-\infty, \infty)}^I | R_{(m, \infty)}^I = \bigcup \{ R_{(m, \infty)}^I : \kappa \in I, m \leq \lambda \in I \}$$

For (19),

$$R_{(-\infty, \infty)}^I | L_{(-\infty, m]}^I = \bigcup \{ R_{(-\infty, \infty)}^I, \kappa \in I, m \geq \lambda \in I \}$$

For (20), if $n < m$ then

$$L_{(-\infty, n]}^I | R_{(m, \infty)}^I = \bigcup \{ L_{(-\infty, n]}^I, \kappa \in I, n \leq \lambda \in I \}$$

For (21), if $n \geq m$ then

$$L_{(-\infty, n]}^I | R_{m}^I = \bigcup \{ L_{(-\infty, n]}^I, m > \lambda \in I \}$$

Next we prove (22). The first case is by (20). If $n \geq m$ then

$$L_{(-\infty, n]}^I | R_{(m, \infty)}^I = L_{(-\infty, n]}^I | R_{(m, \infty)}^I \cup L_{(-\infty, n]}^I | R_{(n+1, \infty)}^I$$
= \bigcup_{m \leq \lambda \leq n, \lambda \in I} L^I_{(-\infty, n]} R^I_{\{m, n]\} \cup R^I_{[n+1, \infty)} \tag{20}

= \bigcup_{m \leq \lambda \leq n, \lambda \in I} \left( |\lambda| \cup L^I_{(-\infty, n]} \cup R^I_{(-\infty, \lambda)} \right) \cup R^I_{[n+1, \infty)} \tag{21}

= L^I_{(-\infty, n]} \cup \lambda \cup R^I_{(-\infty, n]} \cup R^I_{[n+1, \infty)} \text{ defs
}

= R^I_{(-\infty, \infty)} \cup \lambda \cup L^I_{(-\infty, n]} \text{ defs
}

\square

10. Sugihara chains

Definition 5. For every $I \subseteq \mathbb{Z}$,

$$C_I = \{S^I_n : n \in I\} \cup \{T^I_n : n \in I\},$$

where, for every $n \in \mathbb{Z}$,

$$S^I_n = R^I_{[-n, \infty)}, \quad T^I_n = R^I_{(-\infty, \infty)} \cup \lambda \cup L^I_{[-n+\infty, \infty]}.$$

Together with $\emptyset$ and $(U_I)^2$, the relations in $C_I$ form a chain under inclusion.

$\emptyset \subseteq \cdots \subseteq S^I_{n+1} \subseteq \cdots \subseteq S^I_1 \subseteq S^I_0 \subseteq S^I_1 \subseteq \cdots \subseteq S^I_{n+1} \subseteq \cdots \subseteq\n
\subseteq \cdots \subseteq T^I_{n+1} \subseteq \cdots \subseteq T^I_1 \subseteq T^I_0 \subseteq T^I_1 \subseteq \cdots \subseteq T^I_{n+1} \subseteq \cdots \subseteq (U_I)^2$.

At one extreme, we have $U_{\emptyset} = \{\{0: n \in \mathbb{Z}\}\}$, $C_{\emptyset} = \emptyset$, and the chain reduces to $\emptyset \subseteq (U_I)^2$. At the other extreme, the order type of $C_{\mathbb{Z}}$ alone (without the endpoints) is $\omega^* + \omega + \omega^* + \omega$, the same as $S_{\mathbb{Z}+\mathbb{Z}}$. The designated elements in $S_{\mathbb{Z}+\mathbb{Z}}$ are the ones corresponding to the second larger copy of $\omega^* + \omega$. Here the set of designated elements is $\{T^I_n : n \in \mathbb{Z}\}$, i.e., the ones in $C_{\mathbb{Z}}$ that contain the identity relation.

Theorem 1. For every $I \subseteq \mathbb{Z}$, $\langle C_I, \cup, \cap, \to, \sim \rangle$ is a Sugihara chain. In particular, $

\langle C_{\mathbb{Z}}, \cup, \cap, \to, \sim \rangle$ is isomorphic to the original Sugihara matrix $S_{\mathbb{Z}+\mathbb{Z}}$.

Proof. First, note that

$$\overline{(S^I_m)}^{-1} = \overline{(R^I_{(-n, \infty)})^{-1}} = \overline{L^I_{(-n, \infty)}} = R^I_{(-\infty, \infty)} \cup \lambda \cup L^I_{[-n+1, \infty]} = T^I_{-n}.$$}

It follows that converse-complementation is an order-reversing involution when restricted to $C_I$ since

$$S^I_n \subseteq S^I_m \text{ iff } n < m \text{ iff } -m < -n \text{ iff } T^I_{-m} \subseteq T^I_{-n} \text{ iff } \overline{(S^I_m)^{-1}} \subseteq \overline{(S^I_n)^{-1}}.$$

Since $\emptyset^{-1} = (U_I)^2$, converse-complementation is also an order-reversing involution when restricted to $C_I \cup \{\emptyset, (U_I)^2\}$. Next we show, for all $n, m \in I$,

$$S^I_n \cap S^I_m = S^I_n \cap S^I_m, \quad S^I_n \cap T^I_m = T^I_m \setminus S^I_n \quad \text{ if } n \leq -m, \quad T^I_m \setminus T^I_n = T^I_m \cap T^I_n.$$

For (23) and (24) we have

$$S^I_n \cap S^I_m = R^I_{[-n, \infty)} \cap R^I_{[-m, \infty)} \text{ defs
For (25) we start with the observation that

Multiplying this out yields these nine products.

Taking the union of the relations on the right gives us

so (25) holds. In the Sugihara matrix, → is defined by

\[ A \rightarrow B = \begin{cases} 
\sim A \lor B & \text{if } A \leq B, \\
\sim A \land B & \text{if } B > A. 
\end{cases} \]

Since \( \sim \) is an order-reversing involution of the linear ordering \( \leq \), \( \land \) is minimum, and \( \lor \) is maximum, we can substitute \( \sim B \) for \( B \) and apply \( \sim \) to both sides to obtain the equivalent formula

\[ \sim(A \rightarrow \sim B) = \begin{cases} 
A \land B & \text{if } A \leq \sim B, \\
A \lor B & \text{if } \sim B > A. 
\end{cases} \]

To show residuation acts like Sugihara’s \( \rightarrow \), we will use the latter equation. From the definitions of converse-complementation and residuation for binary relations,
we get
\[ \sim(A \to \sim B) = \sim \left( \frac{A^{-1} \sim B}{A^{-1} | B^{-1}} \right)^{-1} = \left( \frac{A^{-1} | B^{-1}}{A^{-1} \sim B} \right) = B | A, \]
hence all we need to show, for any \( A, B \in \mathcal{C}_I \), is
\[ B | A = \begin{cases} A \cap B & \text{if } A \subseteq \sim B, \\ A \cup B & \text{if } \sim B \subseteq A. \end{cases} \]

Because of commutativity, there are just three cases that arise by substituting into (26) when \( n, m \in I, A \in \{S_n^I, T_n^I\} \), and \( B \in \{S_m^I, T_m^I\} \).

Case 1. \( A = S_n^I, B = S_m^I, \sim B = T_{-m}^I \). In this case, (26) holds because, by (23),
\[ B | A = S_m^I | S_n^I = \begin{cases} S_n^I \cap S_m^I & \text{if } S_n^I \subseteq T_{-m}^I \text{ (true)} \\ S_n^I \cup S_m^I & \text{if } T_{-m}^I \subseteq S_n^I \text{ (false)} \end{cases} = S_n^I \cap S_m^I = A \cap B. \]

Case 2. \( A = T_n^I, B = T_m^I, \sim B = S_{-m}^I \). By (25) we have
\[ B | A = T_m^I | T_n^I = \begin{cases} T_n^I \cap T_m^I & \text{if } T_n^I \subseteq S_{-m}^I \text{ (false)} \\ T_n^I \cup T_m^I & \text{if } S_{-m}^I \subseteq T_n^I \text{ (true)} \end{cases} = T_n^I \cup T_m^I = A \cup B. \]

Case 3. \( A = S_n^I, B = T_m^I, \sim B = S_{-m}^I \).
\[ B | A = T_m^I | S_n^I = \begin{cases} S_n^I & \text{if } n \leq -m \\ T_m^I & \text{if } n > -m \end{cases} = \begin{cases} S_n^I \cap T_m^I & \text{if } S_n^I \subseteq S_{-m}^I \\ S_n^I \cup T_m^I & \text{if } S_{-m}^I \subseteq S_n^I \end{cases} = \begin{cases} A \cap B & \text{if } A \subseteq \sim B, \\ A \cup B & \text{if } \sim B \subseteq A. \end{cases} \]

This completes the proof of Theorem 1. \( \square \)

The set of converses of a Sugihara chain form another Sugihara chain. Applying this observation to \( \mathcal{C}_I \), we let
\[ S_n^{I'} = L_{[-n, \infty)}, \]
\[ T_n^{I'} = L_{(-\infty, \infty)} \cup \{ \text{id}_I \cup R_{(-\infty, n-1]} \}, \]
\[ \mathcal{C}_I' = \{ S_n^{I'} : n \in I \} \cup \{ T_n^{I'} : n \in I \}. \]
Then \( \mathcal{C}_I' \) is the other copy of the Sugihara chain in \( \mathcal{S}_I \). Observe that
\[ \mathcal{X}_I = \mathcal{C}_I \cup \mathcal{C}_I' \cup \{ \text{id}_I \cup \text{id}_I, \sim \text{id}_I, \emptyset, U^2 \} \]
is closed under union, intersection, converse-complementation, and residuation. \( \mathcal{X}_I \) is also closed and commutative under relative multiplication. All the relations in \( \mathcal{X}_I \) are dense. The only non-transitive relation in \( \mathcal{X}_I \) is \( \sim \text{id}_I \).
(R1) \[ A \rightarrow A \]
(R2) \[(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \]
(R3) \[A \rightarrow ((A \rightarrow B) \rightarrow B)\]
(R4) \[(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)\]
(R5) \[A \land B \rightarrow A\]
(R6) \[A \land B \rightarrow B\]
(R7) \[(A \rightarrow B) \land (A \rightarrow C) \rightarrow (A \rightarrow B \land C)\]
(R8) \[A \rightarrow A \lor B\]
(R9) \[B \rightarrow A \lor B\]
(R10) \[(A \rightarrow C) \land (B \rightarrow C) \rightarrow (A \lor B \rightarrow C)\]
(R11) \[A \land (B \lor C) \rightarrow (A \lor B) \lor C\]
(R12) \[(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)\]
(R13) \[\sim \sim A \rightarrow A\]
(R14) \[A \rightarrow (A \rightarrow A)\]

Table 2. Axioms of RM

11. R-MINGLE

11.1. Completeness. RM is the logic called R-mingle, created by Dunn and McCall from Anderson and Belnap’s relevance logic R by adding the mingle axiom \[ A \rightarrow (A \rightarrow A) \]; see [1] SS8.15, SS27.1.1. The rules of deduction for both R and RM are Adjunction (to infer \(A \land B\) from \(A\) and \(B\)) and modus ponens (to infer \(B\) from \(A \rightarrow B\) and \(A\)). An axiom set (in current notation) for RM from [1, p. 341] is shown in Table 2. If \(S\) is a Sugihara matrix and the connectives of RM are interpreted as the corresponding operations (with the same name) in \(S\), then any function from the propositional variables of RM to elements of \(S\) extends uniquely to a map from formulas to elements of \(S\). A formula is valid in \(S\) if it is sent to a designated value by every such mapping.

R. K. Meyer [1, Corollary 3.1, p. 413] proved that the theorems of RM are exactly those formulas that are valid in all finite Sugihara matrices. For the proof he used Sugihara matrices with even cardinality. His result was employed for the completeness theorem [12 Theorem 6.2]. R. K. Meyer [1, Corollary 3.5, p. 414] also proved that the theorems of RM are exactly those formulas valid in the Sugihara matrix \(S_{2^*}\). Either of these results, together with Theorem [1] implies the completeness of RM with respect to the following class of algebras.

Definition 6. Let \(K_{RM}\) be the class of algebras of the form

\[ \mathcal{R} = (K, \cup, \cap, \rightarrow, \sim), \]

where

- \(K\) is a set of binary relations on a set \(U\),
- \(K\) is closed under \(\cup, \cap, \rightarrow,\) and \(\sim\),
- \(A|A = A\) for every \(A \in K\) (\(A\) is dense and transitive),
- \(A|B = B|A\) for all \(A, B \in K\).
Any function from the propositional variables to relations in \( K \) extends uniquely to a valuation that maps formulas to relations in \( K \) and is a homomorphism from the algebra of formulas into \( K \). A formula is valid in \( K \) if it is sent by every valuation to a relation containing the identity relation on \( U \). A formula is valid in \( K_{RM} \) if it is valid in every algebra in \( K_{RM} \).

From Theorem 1 and Meyer’s results we get the completeness theorem.

**Theorem 2** ([12, Theorem 6.2(iii)]). \( RM \) is the set of formulas valid in \( K_{RM} \).

**Proof.** The validity of the axioms of \( RM \) in \( K_{RM} \) is explored in detail in the next section. Validity is preserved by Adjunction, for if \( \text{Id} \subseteq A \) and \( \text{Id} \subseteq B \) then \( \text{Id} \subseteq A \cap B \), and validity is preserved by *modus ponens*, for if \( \text{Id} \subseteq A \rightarrow B \) and \( \text{Id} \subseteq A \), then \( \text{Id} = \text{Id} \cap \text{Id} \subseteq A \cap (A \rightarrow B) \subseteq B \) by Lemma 5 in the next section. Therefore all theorems of \( RM \) are valid in \( K_{RM} \).

For the converse, suppose \( A \) not a theorem of \( RM \). By [1, Corollaries 3.1, 3.5], the formula \( A \) fails in every Sugihara matrix with at least as many designated elements as propositional variables in \( A \), such as the Sugihara matrices \( S_{Sz} \) and \( S_{Hz} \). By Theorem 1 these are algebras in \( K_{RM} \) in which \( A \) is not valid. \( \square \)

11.2. **Related results.** [3, Theorem 5.8] asserts that four systems of logic (one of which is \( RM \)) are incomplete with respect to two classes of algebras: \( K_{RM} \) and the class of dense, commutative, transitive, proper relation algebras.

The first claim, that \( RM \) is incomplete for \( K_{RM} \), is corrected by [12, Theorem 6.2] (and Theorem 2 above).

The second claim, that \( RM \) is incomplete for the class of dense, commutative, transitive, proper relation algebras, holds because this class is complete for classical propositional calculus. It is incomplete for \( RM \) because \( RM \) does not contain all tautologies: \( A \rightarrow (B \rightarrow A) \) is missing, for example.

There are (up to isomorphism) exactly two transitive proper relation algebras on a given set, namely \( \mathfrak{Re}(\emptyset) \), the 1-element proper relation algebra, and \( \mathfrak{Re}(\{a\}) \), the 2-element proper relation algebra. The reason for this is that if the diversity relation on a set is transitive then it is empty. Indeed, if \( a, b \in U \) and \( a \neq b \), then \( (a, b) \) and \( (b, a) \) are in the diversity relation, and \( (a, a) \) belongs to the relative product of the diversity relation with itself, but \( (a, a) \) does not belong to the diversity relation.

Hence the diversity relation on a set with two or more elements is not transitive. The diversity relation on a 1-element set \( U = \{a\} \) is empty (and transitive), so the proper relation algebra of all binary relations on \( U \) has exactly two relations, \( \emptyset \) and \( \text{Id} = \{(a, a)\} \), and its equational theory is essentially the same as that of Boolean algebras.

More generally, a relation algebra is Boolean iff it is transitive [3, Lemma 2.3]. Boolean relation algebras are Boolean algebras with \( \cap \) repeated as \( | \). Thus transitive proper relation algebras are dense and commutative. The completeness of transitive proper relation algebras with respect to classical propositional calculus accounts for some of the other incompleteness results, such as [3, Theorem 5.7(2)] (where one can replace “dense, commuting and transitive PRAs” with “transitive proper relation algebras” or “Boolean relation algebras”).

12. **Formulas as relations; semantic analysis of RM**

12.1. Interpreting formulas as relations might be called “relational semantics”, but this term has long been used with other meanings. Routley-Meyer semantics are
Table 3. Axioms of $\text{RM}$ as inclusions between binary relations

| (R1)  | $A \subseteq A$       |
|-------|----------------------|
| (R2)  | $A \rightarrow B \subseteq (B \rightarrow C) \rightarrow (A \rightarrow C)$ |
| (R3)  | $A \subseteq (A \rightarrow B) \rightarrow B$ |
| (R4)  | $A \rightarrow (A \rightarrow B) \subseteq A \rightarrow B$ |
| (R5)  | $A \cap B \subseteq A$ |
| (R6)  | $A \cap B \subseteq B$ |
| (R7)  | $(A \rightarrow B) \cap (A \rightarrow C) \subseteq A \rightarrow B \cap C$ |
| (R8)  | $A \subseteq A \cup B$ |
| (R9)  | $B \subseteq A \cup B$ |
| (R10) | $(A \rightarrow C) \cap (B \rightarrow C) \subseteq A \cup B \rightarrow C$ |
| (R11) | $A \cap (B \cup C) \subseteq (A \cap B) \cup C$ |
| (R12) | $A \rightarrow \neg B \subseteq B \rightarrow \neg A$ |
| (R13) | $\neg \neg A \subseteq A$ |
| (R14) | $A \subseteq A \rightarrow A$ |

called “relational” because the central part of a relevant model structure is a ternary relation. Similarly, “binary relational semantics” applies to $[7]$, because Dunn’s $\text{RM}$ model structures have a binary accessibility relation instead of a ternary one. The correspondence with Sugihara matrices is detailed at the end of $[7, SS7]$. The binary relation in the $\text{RM}$ model structure obtained from a Sugihara matrix is inclusion between sets of proper filters that contain the designated elements of the Sugihara matrix. This binary accessibility relation corresponds precisely to the inclusion between relations in the Sugihara chain in $\mathfrak{S}_I$.

In ternary and binary relational semantics, there are model structures with ternary and binary relations between objects with no further structure, and formulas are interpreted as sets of these objects. In our analysis these objects do have structure—they are binary relations—and formulas are interpreted as (unions of) sets of binary relations, $i.e.$, as binary relations. Our name for this is “formulas as relations”.

12.2. Any formula can be analyzed as a binary relation or as a statement about binary relations, because

**Lemma 4 (D2 Theorem 5.1(17)).** If $A, B \subseteq U^2$ then

$$\text{Id} \subseteq A \rightarrow B \iff A \subseteq B.$$  

According to Lemma 4, the validity of each axiom of $\text{RM}$ can be equivalently expressed as an inclusion between binary relations. This is done in Table 3. The logical connectives $\wedge$ and $\vee$ have been converted to their set-theoretical counterparts $\cap$ and $\cup$. The symbols for the logical connectives $\rightarrow$ and $\sim$ have not been changed, and should be read in this context as operations on binary relations, rather than logical connectives. Evidently (R1), (R5), (R6), (R8), (R9), and (R11) are true under the set-theoretical meanings we have assigned to the symbols in these
formulas; see [12] Theorem 5.1(32), (33), (34), (36), (37), (39)], respectively. To analyze the remaining eight formulas we repeat some lemmas from [12], this time with proofs. It will also be convenient to first calculate a few residuals. As is customary in the theory of relation algebras, we use “1” to denote the largest relation, in this case \(1 = U^2\). In the calculation of residuals we use the definitions of \(\rightarrow\) and \(\sim\), plus these facts:

\[
\emptyset = X|\emptyset = \emptyset = 1^{-1} \\
Id = Id^{-1} \\
X = \Id|X = X|Id = X = (X^{-1})^{-1} \\
X^{-1} = (X^{-1})^{-1} \\
(X|Y)^{-1} = Y^{-1}|X^{-1}
\]

For all \(A \subseteq 1\) we have

\[
A \rightarrow 1 = A^{-1}|1 = A^{-1}|\emptyset = \emptyset = 1 \\
\emptyset \rightarrow A = \emptyset^{-1}|A = \emptyset|A = \emptyset = 1 \\
A \rightarrow \Di = A^{-1}|\Di = A^{-1}|\Id = A^{-1} = \sim A \\
\Id \rightarrow A = \Id^{-1}|\Id = \Id|A = \Id = A
\]

**Lemma 5 ([12] Theorem 5.1(17)].** For all \(A, B \subseteq 1\),

\[
A|(A \rightarrow B) \subseteq B.
\]

**Proof.**

\[
A^{-1}|(B \cap A^{-1}) = \emptyset \\
\overline{B} \cap A|A^{-1} = \emptyset \\
\overline{B} \cap A|(A \rightarrow B) = \emptyset \\
A|(A \rightarrow B) \subseteq B
\]

\[
\emptyset \cap X = \emptyset \\
X^{-1}|Y \cap Z = \emptyset \iff Y \cap X|Z = \emptyset \\
X \cap Y = \emptyset \iff Y \subseteq X
\]

\[\square\]

**Lemma 6 ([12] Theorem 5.1(18)].** For all \(A, B, C \subseteq 1\),

\[
A \rightarrow (B \rightarrow C) = (B|A) \rightarrow C.
\]

**Proof.**

\[
A \rightarrow (B \rightarrow C) = A^{-1}|B^{-1}|C \\
= A^{-1}|(B^{-1}|C) \\
= (A^{-1}|B^{-1})|C \\
= (B|A)^{-1}|C \\
= (B|A) \rightarrow C
\]

\[
\text{def of } \rightarrow
\]

\[
X = X \\
X|(Y|Z) = (X|Y)|Z \\
X^{-1}|Y^{-1} = (Y|X)^{-1}
\]

\[\square\]
**Lemma 7** ([12] Theorem 5.1(21), (22))). For all $A, B \subseteq 1$,
\[ A \subseteq B \implies B \rightarrow C \subseteq A \rightarrow C, \]
\[ A \subseteq B \implies C \rightarrow A \subseteq C \rightarrow B. \]

**Proof.**

\[
\begin{align*}
A \subseteq B & \quad \text{Hyp} \\
A^{-1} \subseteq B^{-1} & \quad X \subseteq Y \iff X^{-1} \subseteq Y^{-1} \\
A^{-1}|C \subseteq B^{-1}|C & \quad X \subseteq Y \implies X|Z \subseteq Y|Z \\
B^{-1}|C \subseteq A^{-1}|C & \quad X \subseteq Y \iff \overline{Y} \subseteq \overline{X} \\
B \rightarrow C \subseteq A \rightarrow C & \quad \text{def of } \rightarrow \\
A \subseteq B & \quad \text{Hyp} \\
\overline{B} \subseteq \overline{A} & \quad X \subseteq Y \iff \overline{Y} \subseteq \overline{X} \\
C^{-1}|\overline{B} \subseteq C^{-1}|\overline{A} & \quad X \subseteq Y \implies Z|X \subseteq Z|Y \\
C^{-1}|\overline{A} \subseteq C^{-1}|\overline{B} & \quad X \subseteq Y \iff \overline{Y} \subseteq \overline{X} \\
C \rightarrow A \subseteq B \rightarrow A & \quad \text{def of } \rightarrow
\end{align*}
\]

\[\square\]

12.3. (R7) and (R10) are true for all binary relations—

**Lemma 8** ([12] Theorem 5.1(35), (38))). For all $A, B, C \subseteq 1$,
\[
(A \rightarrow B) \cap (A \rightarrow C) \subseteq A \rightarrow B \cap C,
\]
\[
(A \rightarrow C) \cap (B \rightarrow C) \subseteq A \cup B \rightarrow C.
\]

**Proof.** Here is a proof of (R7).

1. \( \langle x, y \rangle \in (A \rightarrow B) \cap (A \rightarrow C) \) \quad Hyp.
2. \( \langle x, y \rangle \in A \rightarrow B \) \quad 1, def of \( \cap \)
3. \( \langle x, y \rangle \in A \rightarrow C \) \quad 1, def of \( \cap \)
4. \( \langle w, x \rangle \in A \) \quad Hyp.
5. \( \langle w, y \rangle \in B \) \quad 2, 4, def of \( \rightarrow \)
6. \( \langle w, y \rangle \in C \) \quad 3, 4, def of \( \rightarrow \)
7. \( \langle w, y \rangle \in B \cap C \) \quad 5, 6, def of \( \cap \)
8. \( \langle x, y \rangle \in B \cap C \) \quad 4–7, def of \( \rightarrow \)

12.4. (R13) is always true—

**Lemma 9** ([12] Theorem 5.1(40))). For all $A \subseteq 1$,
\[ \sim \sim A = A. \]

**Proof.**

\[ \sim \sim A = (A^{-1})^{-1} \quad \text{def of } \sim \]
\[\begin{align*}
&= (A^{-1})^{-1} \\
&= (X^{-1})^{-1} = X
\end{align*}\]

12.5. Up to this point we have only encountered \textbf{RM} formulas that are valid for all binary relations, namely, (R1), (R5)–(R11), and (R13). The remaining five formulas do not hold for all relations, but will hold under conditions on the relations that occur in them, and in some cases are equivalent to those conditions. We will now analyze (R2), (R3), (R4), (R12), and (R14), beginning with (R2).

By [12, Theorem 5.1(55)], (R2) holds whenever \(B \rightarrow C\) and \(A \rightarrow B\) commute, but it also holds under a weaker hypothesis; see Lemma \(\textbf{10}\) below. (R2) holds in every commutative proper relation algebra, but fails in non-commutative proper relation algebras. The smallest examples have 16 relations. On the other hand, (R2) is valid when recast as a rule of inference, i.e., if \(A \rightarrow B\) is valid (contains \(\text{Id}\)) then \((B \rightarrow C) \rightarrow (A \rightarrow C)\) is also valid [12, Theorem 5.1(29)]. This follows immediately from Lemma \(\textbf{4}\) and Lemma \(\textbf{7}\).

\textbf{Lemma 10.} For all \(A, B, C \subseteq 1\), if \((B \rightarrow C)\{A \rightarrow B\} \subseteq (A \rightarrow B)\{B \rightarrow C\}\) then

\[A \rightarrow B \subseteq (B \rightarrow C) \rightarrow (A \rightarrow C)\]

\textbf{Proof.} From the assumption we get

\[A\{B \rightarrow C\}\{A \rightarrow B\} \subseteq A\{A \rightarrow B\}\{B \rightarrow C\}\]

by the monotonicity and associativity of \(| |\), so

\[\subseteq B\{B \rightarrow C\} \quad \text{Lemma } \textbf{5} \quad |\text{is monotonic}\]

\[\subseteq C \quad \text{Lemma } \textbf{5}\]

and

\[C \rightarrow C \subseteq A\{B \rightarrow C\}\{A \rightarrow B\} \rightarrow C \quad \text{Lemma } \textbf{7}\]

\[= (A \rightarrow B) \rightarrow (A\{B \rightarrow C\} \rightarrow C) \quad \text{Lemma } \textbf{5}\]

\[= (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \quad \text{Lemma } \textbf{5}\]

so, since \(\text{Id} \subseteq C \rightarrow C\), it follows that

\[\text{Id} \subseteq (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\]

hence, by Lemma \(\textbf{4}\)

\[A \rightarrow B \subseteq (B \rightarrow C) \rightarrow (A \rightarrow C)\]

\[\square\]

The assumption and conclusion of Lemma \(\textbf{10}\) are not equivalent. To see this, we first write (R2) as equivalent inclusions:

\[A \rightarrow B \subseteq (B \rightarrow C) \rightarrow (A \rightarrow C)\]

\[A^{-1}B \subseteq (B^{-1}C)^{-1}A^{-1}C \quad \text{def of } \rightarrow\]

\[A^{-1}B \subseteq (B^{-1}C)^{-1}A^{-1}C \quad \overline{X} = X\]
Suppose \( C = 1 \). Then \( B \to C = 1 \) and this last inclusion holds for all \( A, B \). The hypothesis becomes

\[
1 | (A \to B) \subseteq (A \to B) | 1.
\]

This will fail whenever \( A \to B \) is not empty and has a domain that is not all of \( U \).

12.6. By [12, Theorem 5.1(54)], (R3) holds whenever \( A \) and \( A \to B \) commute. In fact, it holds under a weaker hypothesis to which it is not equivalent.

**Lemma 11.** For all \( A, B \subseteq 1 \), if \( (A \to B) | A \subseteq A | (A \to B) \), then

\[ A \subseteq (A \to B) \to B. \]

**Proof.**

\[
\begin{align*}
A | (A \to B) & \subseteq B & \text{Lemma 5} \\
(A \to B) | A & \subseteq B & \text{Hyp, } \subseteq \text{ is transitive} \\
\overline{B} \cap (A \to B) | A & = \emptyset & X \subseteq Y \iff \overline{Y} \cap X = \emptyset \\
(A \to B)^{-1} | (\overline{B}) \cap A & = \emptyset & Y \cap X | Z = \emptyset \iff X^{-1} | Y \cap Z = \emptyset \\
A & \subseteq (A \to B)^{-1} | (\overline{B}) & X \cap Y = \emptyset \iff X \subseteq \overline{Y} \\
A & \subseteq (A \to B) \to B & \text{def of } \to
\end{align*}
\]

Next we show the conclusion of Lemma 11 does not imply the hypothesis. Note that the last five lines in the proof are equivalent, so we want to choose \( A, B \) such that

\[
(A \to B) | A \not\subseteq A | (A \to B),
\]

\[
(A \to B) | A \subseteq B.
\]

We can do this by first setting \( B = 1 \), obtaining new goals

\[
\begin{align*}
1 | A = (A \to 1) | A \not\subseteq A | (A \to 1) = A | 1, \\
1 | A = (A \to 1) | A \subseteq 1.
\end{align*}
\]

Since the second inclusion always holds, we can achieve our goal by letting \( A \subseteq 1 \) be any relation whose domain is not all of \( U \).

12.7. By [12, Theorem 5.1(56)], (R4) holds whenever \( A \) is a dense relation, i.e., whenever \( A \subseteq A | A \). The density of \( A \) is equivalent to assuming (R4) holds for all \( B \). Here are four forms of (R4).

\[
\begin{align*}
A \to (A \to B) & \subseteq A \to B \\
(A | A) \to B & \subseteq A \to B & \text{Lemma 5} \\
(A | A)^{-1} | B & \subseteq A^{-1} | B & \text{def of } \to \\
A^{-1} | B & \subseteq (A | A)^{-1} | B & X \subseteq Y \iff \overline{Y} \subseteq X
\end{align*}
\]
Thus (R4) holds whenever \( A \) is dense, since \( \sim^{-1} \) and \( | \) are both monotonic. But if we choose \( A, B \) so that \( A|1 \subseteq B \), then
\[
A^{-1}|B| \subseteq A^{-1}|A| \subseteq I = \emptyset,
\]
so (R4) holds, whether \( A \) is dense or not. On the other hand, if (R4) holds when \( B = \sim \text{id} \), then \( A \) is dense and (R4) takes the form \( (A \to \sim A) \to \sim A \), which is valid iff \( A \to \sim A \subseteq \sim A \). This last inclusion can be transformed via the definitions of \( \sim \) and \( \to \) into \( A \subseteq A \text{id} \), i.e., \( A \) is dense. Thus, assuming (R4) is valid for all relations implies that \( A \) is dense.

12.8. Axiom (R12) is Contraposition. By [12, Theorem 5.1(53)], (R12) holds whenever \( A|B = B|A \), but it is actually equivalent to \( A|B \subseteq B|A \).

Lemma 12. For all \( A, B \subseteq 1 \),
\[
A \to \sim B \subseteq B \to \sim A \iff A|B \subseteq B|A.
\]

Proof. Each of the following statements is equivalent to the one above it for the reason given to its right.

\[
\begin{align*}
A \to \sim B \subseteq B \to \sim A & \quad \text{def of } \to \\
A^{-1}|B^{-1} \subseteq B^{-1}|A^{-1} & \quad \text{def of } \subseteq \\
A^{-1}|B^{-1} \subseteq B^{-1}|A^{-1} & \quad X = X \\
B^{-1}|A^{-1} \subseteq A^{-1}|B^{-1} & \quad X \subseteq Y \iff \overline{Y} \subseteq \overline{X} \\
(B^{-1}|A^{-1})^{-1} \subseteq (A^{-1}|B^{-1})^{-1} & \quad X \subseteq Y \iff X^{-1} \subseteq Y^{-1} \\
A|B \subseteq B|A & \quad (X^{-1}|Y^{-1})^{-1} = Y|X
\end{align*}
\]

12.9. By [12, Theorem 5.1(63)], (R14) holds if \( A \) is a transitive relation, but (R14) is actually equivalent to the transitivity of \( A \).

Lemma 13. For all \( A \subseteq 1 \), \( A \) is transitive if and only if \( A \subseteq A \to A \).

Proof. Because of the central importance of (R14) to \( \text{RM} \), we write out two proofs, the first involving individuals, the second in the calculus of relations.

1. \( A \) is transitive Hyp.
2. \( \langle x, y \rangle \in A \) Hyp.
3. \( \langle z, x \rangle \in A \) Hyp.
4. \( \langle z, y \rangle \in A \) 1, 2, 3, def of transitivity
5. \( \langle x, y \rangle \in A \to A \) 3–4, def of \( \to \)
6. \( A \subseteq A \to A \) 2–5, def of \( \subseteq \)

1. \( A \subseteq A \to A \) Hyp.
2. \( \langle x, y \rangle \in A \) Hyp.
3. \( \langle y, z \rangle \in A \) Hyp.
4. \( \langle y, z \rangle \in A \to A \) 1, 3, def of \( \subseteq \)
| 5. \( (x, z) \in A \) & 2, 4, def of \( \rightarrow \) \\
6. \( A \) is transitive & 2–5, def of transitivity \\

Each of the following statements is equivalent to the one above it for the reason given to its right.

\[
\begin{align*}
A \subseteq A &\rightarrow A \\
A \subseteq A^{-1} | A &\text{ def of } \rightarrow \\
A^{-1} | A \subseteq A &\text{ def of } \rightarrow \\
A^{-1} | A \cap A = \emptyset &\text{ X } \subseteq Y \iff Y \subseteq X \\
A \cap A = \emptyset &\text{ X } \subseteq Y \iff X \cap Y = \emptyset \\
A | A \subseteq A &\text{ X } \cap Y = \emptyset \iff Y \subseteq X \\
A &\text{ is transitive } \text{ def of transitivity} \\
\end{align*}
\]

12.10. The axioms of \( \text{RM} \) are classified with respect to their meanings as axioms about a set \( K \) of binary relations on a set \( U \).

- (R1), (R5), (R6), (R7), (R8), (R9), (R10), (R11), and (R13) hold for all binary relations.
- (R2) and (R3) hold if the relations in \( K \) commute under \( | \), but neither of them is equivalent to the commutativity of \( K \).
- (R4) is equivalent to the density of all relations in \( K \).
- (R12) is equivalent to the commutativity of \( K \).
- (R14) is equivalent to the transitivity of all relations in \( K \).

13. Concluding remarks

We have examined some proper relation algebras and have found in them copies of the Sugihara matrices, Belnap’s \( M_0 \) matrices, Meyer’s crystal matrices, Meyer’s \( \text{RM84} \) matrices, and Church’s matrices. This shows that some finite algebras of importance in the relevant enterprise [15, SS9.7] are sets of binary relations closed under union, intersection, residuation, and converse-complementation, or their relativizations to diversity relations. There are completeness results for these algebras, such as Theorem [2, 12 Theorem 6.2] (for \( \text{RM} \)), [3 Theorems 9.8.1, 9.8.3] (the crystal matrices are characteristic for CL [3 p. 114]), and [6 Theorems 9.8.4, 9.8.6] (the Belnap \( M_0 \) matrices are characteristic for \( \text{BM} \) [6 p. 128]), that are illuminated by a semantic analysis of formulas as binary relations, carried out partly in [12 Theorem 5.1] and in some detail for the axioms of \( \text{RM} \) in this paper.

Relevance logics are a rich source of problems for the theory of relation algebras, as the deep work of Kowalski [8, 9] and Mikulás [13] demonstrate. Are there more matrices in the relevance logic literature that are definitional subreducts of proper relation algebras? What properties do these proper relation algebras have? Consider the three 5-atom relation algebras that contain the crystal matrices. We presented a representation of \( 2^{83} \), and we know one for \( 43_{83} \), but is \( 29_{83} \) representable? It seems likely. Unlike the other two, \( 29_{83} \) is not commutative. This is but one instance of the following
Problem 1. If a relation algebra is finite, integral, possibly commutative, and every one of its diversity atoms is dense and transitive and distinct from its converse, must that algebra be representable?

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Department of Mathematics, 396 Carver Hall, Iowa State University, Ames, IA 50011, U.S.A.
E-mail address: maddux@iastate.edu