HOMOTOPY COLIMITS OF NILPOTENT SPACES

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Abstract. We show that cellular approximations of nilpotent Postnikov stages are always nilpotent Postnikov stages, in particular classifying spaces of nilpotent groups are turned into classifying spaces of nilpotent groups. We use a modified Bousfield-Kan homology completion tower $z_kX$ whose terms we prove are all $X$-cellular for any $X$. As straightforward consequences, we show that if $X$ is $K$-acyclic and nilpotent for a given homology theory $K$, then so are all its Postnikov sections $P_n X$, and that any nilpotent space for which the space of pointed self-maps $\map_*(X, X)$ is “canonically” discrete must be aspherical.

1. Introduction

The Postnikov tower of a nilpotent space $X$ is considered classically as a mean to reconstruct its Postnikov sections $P_n X$ out of Eilenberg-Mac Lane spaces (basic homotopical building blocks) by principal fibration sequences (see Example 2.3 for what we mean by Postnikov sections). Eventually the space $X$ itself is recovered as homotopy limit of the Postnikov tower.

In this work we change the perspective and show that, if $X$ is nilpotent, then its Postnikov sections can be constructed out of $X$ by means of wedges, homotopy push-outs, and telescopes (see Theorem 6.4). This is certainly not true for arbitrary spaces and, when $X$ is nilpotent, it allows us to deduce strong properties of its Postnikov sections. Let us first mention a few of these consequences and then present our methods and techniques. We provide a topological strengthening of the classical Serre class statements on the relation between homotopy and homology groups, see [25].

7.10 Theorem. Let $K$ be a reduced homology theory.

(1) Assume $X$ is $K$-acyclic. If $P_n X$ is nilpotent, then it is also $K$-acyclic.

(2) Assume $X$ is nilpotent. Then $\prod_{k \geq 1} K(\pi_k X, k)$ is $K$-acyclic if and only if $\prod_{k \geq 1} K(H_k(X, \mathbb{Z}), k)$ is $K$-acyclic.

(3) If $K(G, 1)$ is $K$-acyclic, then so is $K(G/\Gamma_n G, 1)$ for any $n$.

In [2] and [3, Lemma 5.3] Bousfield proved his celebrated result known as the “key lemma”. All his results related to understanding the failure of preservation of fibrations by localizations depended on it. The key lemma implies for example that, if $X$ is simply connected, then the map $\pi_n: \map_*(X, X) \rightarrow \text{Hom}(\pi_n X, \pi_n X)$
is a weak equivalence if and only if $X$ is weakly equivalent to $K(\pi_n X, n)$. If $X$ is not simply connected, then this is far from being true. In this article we offer an extension of this last result to spaces with a nilpotent fundamental group.

**Theorem.** Let $X$ be a connected space whose fundamental group $\pi_1 X$ is nilpotent. Assume that the map $\pi_1 : map_*(X, X) \to \text{Hom}(\pi_1 X, \pi_1 X)$ is a weak equivalence. Then $X$ is weakly equivalent to $K(\pi_1 X, 1)$.

For arbitrary fundamental groups, this fails as illustrated in [20, Example 2.6] by a space $X$ with $\pi_1 X \cong \Sigma_3$ whose universal cover is the homotopy fiber of the degree 3 map on the sphere $S^3$. We come back to this space in Example 7.6.

To prove the above results we use cellularization techniques. Looking at spaces through the eyes of a given space $A$ via the pointed mapping space $map_*(A, -)$ is the central idea ([15], [5]). Recently cellularization has found applications in other contexts: Dwyer, Greenlees, and Iyengar used it to investigate duality in stable homotopy, [16], see also [20]; periodicity phenomena in unstable homotopy theory are the subject of [9]; cellularity has also been intensively studied in group theory, see for example [18] for an early general reference, and [1] for a recent point of view on finite (simple) groups; in [22] Kiessling computed explicitly the cellular lattice of certain perfect chain complexes, refining the Bousfield lattice.

Classically $A$ is the zero-sphere $S^0$ and we are doing standard homotopy theory. In this context the cellularization of a space is nothing else than the cellular approximation of the space, the best approximation built out of $S^0$ and its suspensions $S^n$ via pointed homotopy colimits. Changing the sphere for another space $A$ allows us to modify our point of view and we do now $A$-homotopy theory. The cellularization functor $\text{cell}_A : \text{Spaces}_* \to \text{Spaces}_*$ on pointed spaces retains from a space its essence from the point of view of $A$. A space of the form $\text{cell}_A Y$ is $A$-cellular in the sense that it can be constructed from $A$ and its suspensions by pointed homotopy colimits. When a space $X$ is $A$-cellular we write $X \gg A$.

It turns out that to prove the above theorems we need to show that $\text{cell}_A$ behaves nicely on nilpotent Postnikov sections of any space. A result of the first author [5, Section 7] says that the structure map $\text{cell}_A X \to X$ is a principal fibration. Hence, the cellularization of a nilpotent space is always nilpotent, which hints at the tame behavior of $\text{cell}_A$ on nilpotent spaces. We show in particular in Corollary 7.2 that if $X$ is a nilpotent $n$-Postnikov stage (see the end of Section 2), then so is $\text{cell}_A X$. When $n = 2$, this result is in perfect accord with a purely group-theoretical result, [18, Theorem 1.4 (1)], showing that in the category of groups any cellularization of a nilpotent group is nilpotent. To understand how cellularization functors affect Postnikov sections, we rely on the following statement which we regard as the main result of this paper.

**Theorem.** Let $X$ be a space. If $P_n X$ is nilpotent, then $P_n X \gg X$, namely any Postnikov section is $X$-cellular.

This theorem is a consequence of preservation of polyGEM by cellularization functors. A space is called a 1-polyGEM if it is a GEM, i.e. a product of Eilenberg-MacLane spaces $K(A_i, i)$ for abelian groups $A_i$. It is called an $n$-polyGEM, for $n > 1$, if it is weakly equivalent to a retract of the homotopy fiber of a map $f : X \to Y$ where $X$ is $(n-1)$-polyGEM and $Y$ is a GEM. We prove in Theorem 7.3 that cellularization functors turn $n$-polyGEMs into $n$-polyGEMs.
To prove the preservation of polyGEMs we need detection tools for polyGEMs which behave well with respect to cellularity. It is well known that a space \( X \) is a GEM if and only if it is a retract of \( ZX \), where \( ZX \) is the Dold-Thom free abelian group construction on \( X \), see for example [12, Definition 3.5], the simplicial version of the infinite symmetric product \( SP^\infty X \) (see also [13]). We need similar functors that detect the property of being a polyGEM. As a first attempt one might try to consider the functors \( Z_kX \) in the Bousfield-Kan completion tower with respect to the integral homology [4]. Although the spaces in this tower are polyGEMs, they are very special polyGEMs, so called flat [23]. Thus instead of general polyGEMs, the functors \( Z_kX \) detect only flat polyGEMs. Furthermore we do not know the needed cellularity properties of \( Z_kX \).

It turns out that for our purposes, instead of the classical Bousfield-Kan completion tower, one should consider the modified Bousfield-Kan tower \( \{z_kX\} \) constructed by the second author in [17]. We do that in Section 5. This tower has two key properties that are essential in our work. First, it detects polyGEMs.  

5.4 Proposition. A space \( W \) is a polyGEM if and only if it is a retract of \( z_nW \) for some \( n \).

Second, the functors in the modified tower are cellular, a property that we are unable to prove for the functors in the classical Bousfield-Kan tower.

5.5 Proposition. For all \( k \geq 0 \) the co-augmented functor \( z_k \) is cellular. In particular \( z_kX \) is \( X \)-cellular for any \( X \).

We define and study cellular functors in Section 4. The functor \( z_k \) being cellular tells us much more than the fact that \( z_kX \) is \( X \)-cellular. It says that the homotopy cofiber of the augmentation \( X \to z_kX \) is \( \Sigma X \)-acyclic, so that \( z_kX \) can be built from \( X \) starting from \( X \) and adding higher cells \( \Sigma^iX \) for \( i \geq 1 \). This is how Bousfield’s original ideas about his key lemma come into play.

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2. Notation and set-up

In this section we will recall basic definitions related to cellularity and acyclicity of spaces and state their fundamental properties. For more information about these notions we refer the reader to [15].

The category of pointed simplicial sets with the standard simplicial model structure is denoted by \( \text{Spaces}_* \). Its objects are called pointed spaces or simply spaces, and morphisms are called maps. The space of maps between two pointed spaces \( X \) and \( Y \) is denoted by \( \text{map}_*(X,Y) \).

We say that a class \( \mathcal{C} \) of pointed spaces is closed under weak equivalences if when \( X \) belongs to \( \mathcal{C} \), then so does any pointed space weakly equivalent to \( X \). We say that \( \mathcal{C} \) is closed under homotopy colimits if, for any functor \( F : I \to \text{Spaces}_* \) whose values belong to \( \mathcal{C} \), the homotopy colimit \( \text{hocolim}_IF \) in \( \text{Spaces}_* \) also belongs to \( \mathcal{C} \). A class of pointed spaces \( \mathcal{C} \) which is closed both under weak equivalences and homotopy colimits is called cellular.

Cellular classes were called closed classes in [15], but we believe that the name “cellular” is more descriptive in our context. For a class \( \mathcal{C} \) to be cellular it is sufficient if it is closed under weak equivalences, arbitrary wedges and homotopy
push-outs. This is so since all pointed homotopy colimits can be built by repeatedly using these two special cases. Furthermore a retract of a member of a cellular class also belongs to the cellular class, [15, 2.D.1.5], where recall that $X$ is a retract of $Y$ if there are maps $f : X \to Y$ and $r : Y \to X'$ whose composition $rf$ is a weak equivalence.

The symbol $\mathcal{C}(A)$ denotes the smallest cellular class in $\text{Spaces}_*$ containing a given space $A$. If $X$ belongs to $\mathcal{C}(A)$, then we write $X \gg A$ and say that $X$ is $A$-cellular or that $A$ builds $X$. For example, $\mathcal{C}(S^0)$ consists of all pointed spaces and $\mathcal{C}(S^n)$ of all $(n - 1)$-connected pointed spaces.

A weaker notion than cellularity is given by acyclicity (see [5] for the origin of the terminology). For a map $f : X \to Y$ of pointed spaces, $\text{Fib}(f)$ denotes the homotopy fiber of $f$ over the base point. A cellular class $\mathcal{C}$ is called acyclic if, for any map $f : X \to Y$ such that $Y$ and $\text{Fib}(f)$ belong to $\mathcal{C}$, the space $X$ belongs to $\mathcal{C}$. We also say that $\mathcal{C}$ is closed under extensions by fibrations.

Given a pointed space $A$, the symbol $\overline{\mathcal{C}}(A)$ denotes the smallest acyclic class in $\text{Spaces}_*$ containing $A$. If a space $X$ belongs to $\overline{\mathcal{C}}(A)$, then we write $X > A$ and say that $X$ is $A$-acyclic. There is an obvious inclusion $\mathcal{C}(A) \subset \overline{\mathcal{C}}(A)$ which in general is proper. For example, if $p$ is a prime number and $G$ a finite $p$-group, $K(G, 1)$ is always $K(\mathbb{Z}/p, 1)$-acyclic; however, $K(G, 1)$ is $K(\mathbb{Z}/p, 1)$-cellular if and only if $G$ is generated by elements of order $p$, see [19, Section 4] for details.

In [5], the first author proved that cellularity can be detected by means of a universal property:

2.1. **Theorem.** A pointed space $X$ is $A$-cellular if and only if, for any map $f$ between fibrant spaces such that $\text{map}_*(A, f)$ is a weak equivalence, then $\text{map}_*(X, f)$ is also a weak equivalence.

The present paper deals with possible values of the $A$-cellular approximation or $A$-cellular cover of spaces. The existence and basic properties of these cellular cover functors $\text{cell}_A$ are guaranteed by the following result proved in [15, Section 2], see also [5].

2.2. **Theorem.** Let $A$ be a pointed space.

(A) There is a natural fibration $c_{A,X} : \text{cell}_AX \to X$ in $\text{Spaces}_*$ such that:
   - $\text{cell}_AX$ preserves weak equivalences;
   - $\text{cell}_AX$ is $A$-cellular;
   - the map $\text{map}_*(A, c_{A,X})$ is a weak equivalence.

(B) A pointed space $X$ is $A$-cellular if and only if the map $c_{A,X} : \text{cell}_AX \to X$ is a weak equivalence.

The map $c_{A,X} : \text{cell}_AX \to X$, given by Theorem 2.2, is called the $A$-cellular cover of $X$ and the functor $\text{cell}_A : \text{Spaces}_* \to \text{Spaces}_*$ the $A$-cellularization.

2.3. **Example.** Let $S^{n+1}$ be an $(n+1)$-dimensional sphere. The $S^{n+1}$-cellular cover $c_{S^{n+1},X} : \text{cell}_{S^{n+1}}X \to X$ coincides with the $n$-connected cover and fits into a fibration sequence:

$$\text{cell}_{S^{n+1}}X \xrightarrow{c_{S^{n+1},X}} X \xrightarrow{p_{n,X}} P_nX$$

where $p_{n,X} : X \to P_nX$ is the $n$-th Postnikov section. We call a space $X$ an $n$-Postnikov stage if the map $p_{n,X} : X \to P_nX$ is a weak equivalence, that is, if $\pi_iX = 0$ for $i \geq n + 1$. Thus a 0-Postnikov stage is homotopically discrete, and a connected 1-Postnikov stage is an Eilenberg–Mac Lane space $K(\pi, 1)$. 


Similarly to the $n$-connected cover, for any $A$, the map $c_{A,X}: \text{cell}_A X \to X$ is always a principal fibration, \cite[Corollary 20.7]{5}.

3. Basic cellular inequalities

This section contains fundamental cellular and acyclic inequalities. The idea is to use these inequalities as basic moves to obtain more involved ones. Although our basic result is an inequality $P_n X \gg X$ for a nilpotent $P_n X$, we will use on the way many general inequalities that hold without any nilpotency condition. The following results describe our basic dictionary of cellular inequalities, which in most cases admit direct proofs.

Our first statements, formulated for cellular inequalities $\gg$, hold also as stated for the weak inequality $>$.  

3.1. Proposition. Let $A$ and $X$ be pointed spaces.

1. If $X \gg A$, then, for any space $E$, $E \wedge X \gg E \wedge A$.
2. If $X$ is connected, then $\Omega X \gg A$ if and only if $X \gg S^1 \wedge A$.
3. If $X \gg A$ and $A$ is connected, then $\Omega X \gg \Omega A$.
4. For any $X$ and $n \geq 1$, $\Omega^n (S^n \wedge X) \gg X$.

Proof. The first cellular inequality is the content of \cite[Theorem 4.3]{7}, while the acyclic inequality follows from the cellular one and the fact that, for any space $A$, there is a space $B$ such that $C(A) = C(B)$, see \cite[Corollary 6.2]{10}. The cellular inequality (2) is proved in \cite[Theorem 10.8]{5} and the acyclic one is \cite[Theorem 18.5]{5}. The inequalities (3) and (4) are easy consequences of (1) and (2).

Our second set of inequalities concerns homotopy fibers and cofibers of a map $f: X \to Y$. The last one tells us that the “fiber of the cofiber” is close to the space $X$ from a cellular point of view.

3.2. Proposition. Let $f: X \to Y$ be a map to a connected pointed space $Y$.

1. $\text{Cof}(f) \gg S^1 \wedge \text{Fib}(f)$.
2. For any $E$, $\text{Fib}(E \wedge f) \gg E \wedge \text{Fib}(f)$.
3. If $\alpha: Y \to \text{Cof}(f)$ is a homotopy cofiber of $f$, then $\text{Fib}(\alpha) \gg X$.

Proof. Inequality (5) is \cite[Proposition 10.5]{5} while (6) follows from the case of a circle \cite, and induction on a cell decomposition. Finally (7) is \cite[Corollary 9.A.10]{15} (see also \cite[Proposition 4.5.(4)]{5}).

In fibration and cofibration sequences we can sometimes relate the cellularity of the spaces involved in the sequence. The main difficulty for a fibration sequence $F \to E \to B$ is that in general we cannot extract information about the total space of the type $E \gg A$ knowing that $B, F \gg A$.

3.3. Proposition. Let $Z \to X \to Y$ be either a cofibration or a fibration sequence.

1. If $Z \gg A$ and $Y \gg A$, then $X \gg A$.
2. If $Z \gg A$ and $Y > S^1 \wedge A$, then $X \gg A$.
3. If $Z > A$ and $X \gg A$, then $Y \gg A$.

Proof. For a fibration sequence inequality (8) holds by definition of the relation $>$. The cofibration sequence case follows from (7) and the fibration sequence case. Inequality (9) is \cite[Corollary 20.2]{5} for a fibration sequence, and the cofibration case follows again from (7) and the fibration sequence case. Finally, inequality (10) may be deduced from statements (5) and (9).
The following inequalities, that put in relation fiber and cofiber from the cellular point of view, will be especially relevant in the rest of the paper.

3.4. Proposition.

(11) Let $X$ be a connected pointed space and $n \geq 1$. Let $e_{n,X} : X \to \Omega^n(S^n \wedge X)$ be the adjoint to id: $S^n \wedge X \to S^n \wedge X$. Then $\text{Fib}(e_{n,X}) \gg S^1 \wedge \Omega X \wedge \Omega X$ and $\text{Cof}(e_{n,X}) \gg X \wedge X$.

Proof. If $n = 1$, then the statement for the homotopy fiber is \cite{6} Theorem 7.2. We proceed by induction on $n$. Let $n > 1$. The map $e_{n,X}$ factors as the composition:

$$
X \xrightarrow{e_{n-1,X}} \Omega^{n-1}(S^{n-1} \wedge X) \xrightarrow{\Omega^{n-1}(e_{1,S^{n-1} \wedge X})} \Omega^n(S^n \wedge X)
$$

which leads to a fibration sequence:

$$
\text{Fib}(e_{n-1,X}) \to \text{Fib}(e_{n,X}) \to \text{Fib}(\Omega^{n-1}(e_{1,S^{n-1} \wedge X}))
$$

and a cofibration sequence:

$$
\text{Cof}(e_{n-1,X}) \to \text{Cof}(e_{n,X}) \to \text{Cof}(\Omega^{n-1}(e_{1,S^{n-1} \wedge X}))
$$

Let us analyze $\text{Fib}(\Omega^{n-1}(e_{1,S^{n-1} \wedge X})) = \Omega^n(\text{Fib}(e_{1,S^{n-1} \wedge X}))$. According to statements (2) and (3) the following cellular relation holds:

$$
\Omega^{n-1}(S^1 \wedge \Omega(S^{n-1} \wedge X) \wedge \Omega(S^{n-1} \wedge X)) \gg \Omega^n(S^1 \wedge S^{n-2} \wedge X \wedge S^{n-2} \wedge X)
$$

For this last space $\Omega^{n-1}(S^{2n-3} \wedge X \wedge X)$, by the same argument, we have the following relation:

$$
\Omega^{n-1}(S^{2n-3} \wedge X \wedge X) \gg S^{n-2} \wedge X \wedge X \gg S^n \wedge \Omega X \wedge \Omega X \gg S^2 \wedge \Omega X \wedge \Omega X
$$

Thus $\text{Fib}(e_{n,X}) \gg S^1 \wedge \Omega X \wedge \Omega X$ is a consequence of the inductive step and statement (9). The case of the homotopy cofiber is an analogous induction based on the proof of \cite{6} Theorem 7.2, which analyzes the James construction. \hfill \Box

We arrive now at more delicate inequalities involving diagrams. Consider a homotopy push-out square:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{h} \\
C & \xleftarrow{k} & D
\end{array}
$$

Let $\beta : \text{Fib}(g) \to \text{Fib}(h)$ and $\gamma : A \to T := \text{holim}(C \xrightarrow{k} D \xleftarrow{h} B)$ be the maps induced by the commutativity of this square.

3.5. Theorem.

(12) If $C$ and $D$ are connected, then $\text{Fib}(h) \gg \text{Fib}(g)$.

(13) If $\text{Fib}(k)$ is connected, then $\text{Cof}(\beta) \gg S^1 \wedge \text{Fib}(g) \wedge \Omega \text{Fib}(k)$.

(14) If $B$, $C$, $T$ and $\text{Fib}(\gamma)$ are connected, then $\text{Fib}(\gamma) > S^1 \wedge \Omega \text{Fib}(f) \wedge \Omega \text{Fib}(g)$. 

6
Proof. Inequality (12) is [7, Theorem 3.4]. To check (13), observe that according to Puppe’s theorem we can form the following homotopy-push out square of homotopy fibers:

\[
\begin{array}{c}
\text{Fib}(g) \\
\downarrow h' \\
\text{Fib}(hf) \\
\downarrow g' \\
\text{Fib}(k) \\
\downarrow \Delta[0]
\end{array}
\]

where the map \( h' : \text{Fib}(g) \rightarrow \text{Fib}(hf) \) is a homotopy fiber of \( g' : \text{Fib}(hf) \rightarrow \text{Fib}(k) \). Let \( \text{Fib}(k) \hookrightarrow R\text{Fib}(k) \) be a weak equivalence into a fibrant space. Since the composition \( g'h' \) factors through a contractible space, by taking the cofibers of \( h' \) and \( \beta \), we can form a new homotopy push-out square:

\[
\begin{array}{c}
\text{Cof}(h') \\
\downarrow g'' \\
\text{Cof}(\beta) \\
\downarrow R\text{Fib}(k) \\
\Delta[0]
\end{array}
\]

By Ganea’s theorem [21], \( \text{Fib}(g'') \simeq S^1 \wedge \text{Fib}(g) \wedge \Omega\text{Fib}(k) \). Under the assumption that \( \text{Fib}(k) \) is connected, we can then apply statement (13) to this last homotopy push-out square to get \( \text{Cof}(\beta) \gg S^1 \wedge \text{Fib}(g) \wedge \Omega\text{Fib}(k) \). Finally, inequality (14) is [11, Theorem 5.1]. □

4. Cellular functors and Bousfield key lemma

An essential tool to understand the failure of preservation of fibrations by localizations and cellularizations is the so-called “key lemma” of Bousfield [3, Theorem 5.3] and Dror Farjoun [15]. In its original form it states that if, for connected pointed spaces \( X \) and \( Y \), \( \text{map}_* (X,Y) \) is weakly equivalent to a discrete space and \( \pi_1 Y \) acts trivially on the set of components \( \pi_0 \text{map}_* (X,Y) \), then the Hurewicz map \( h_X : X \rightarrow ZX \) induces a weak equivalence between \( \text{map}_* (X,Y) \) and \( \text{map}_* (ZX,Y) \).

The space \( ZX \) is the simplicial abelian group freely generated by the simplices of \( X \). It is the simplicial model for the Dold-Thom infinite symmetric product \( SP^\infty X \simeq \prod K(H_0(X; \mathbb{Z}), n) \), see for example [12, 3.11].

The assumptions of the key lemma are equivalent to the statement that the inclusion \( S^1 \vee X \hookrightarrow S^1 \times X \) induces a weak equivalence between \( \text{map}^*_n (S^1 \vee X, Y) \) and \( \text{map}^*_n (S^1 \times X, Y) \) which means that \( Y \) is local with respect to \( S^1 \vee X \hookrightarrow S^1 \times X \).

The key lemma states therefore that, for a connected space \( X \), the Hurewicz map \( h_X : X \rightarrow ZX \) is an \( L_{S^1 \vee X} \simeq L_{S^1 \times X} \)-equivalence (see [15]). One way to prove this lemma is to show a stronger statement: the map \( h_X : X \rightarrow ZX \) is constructed inductively starting with \( X \) and taking push-outs along the inclusion \( S^1 \vee X \hookrightarrow S^1 \times X \) or its suspensions. Since the cofiber of the map \( S^1 \vee X \hookrightarrow S^1 \times X \) is \( S^1 \wedge X \), it follows that \( \text{Cof}(h_X) \gg S^1 \wedge X \) (in fact a stronger relation \( \text{Cof}(h_X) \gg S^1 \wedge X \) holds in this case) [15, Section 4.A]. Furthermore if \( X \) is simply connected, the same argument can be used to show that the Hurewicz map \( h_X : X \rightarrow ZX \) is constructed inductively starting with \( X \) and taking push-outs along the inclusion
$S^2 \vee X \leftrightarrow S^2 \times X$ or its suspensions. By the Ganea theorem [21], $\text{Fib}(S^2 \vee X \leftrightarrow S^2 \times X) \simeq S^1 \wedge \Omega S^2 \wedge \Omega X$. Thus, if $X > S^1 \wedge A$, for a connected $A$, then:

$$\text{Fib}(S^2 \vee X \leftrightarrow S^2 \times X) > S^1 \wedge S^1 \wedge \Omega X > S^1 \wedge S^1 \wedge A = S^2 \wedge A$$

We can then use Theorem 3.5.(12) to infer that $\text{Fib}(h_X) > S^2 \wedge A$. These two properties of the Hurewicz map play an important role in this paper and therefore we are going to give them a name. Recall that a co-augmented functor $K : \text{Spaces}_* \to \text{Spaces}_*$ is equipped with a natural transformation $\mu : X \to K(X)$ between $\text{id} : \text{Spaces}_* \to \text{Spaces}_*$.

4.1. **Definition.** A co-augmented functor $\mu : X \to K(X)$ is called cellular if:

(a) $\text{Cof}(\eta_X : X \to K(X)) > S^1 \wedge X$ for any connected $X$;

(b) $\text{Fib}(\eta_X : X \to K(X)) > S^2 \wedge A$ for any $X > S^1 \wedge A$ and connected $A$.

The following is a consequence of Proposition 3.3.(9).

4.2. **Corollary.** If $K$ is a co-augmented cellular functor, then $K(X)$ is $X$-cellular for any $X$. □

4.3. **Example.** The natural unit map $X \to \Omega^n(S^n \wedge X)$ given by the “loop-suspension” adjunction defines a cellular functor, and so does the map $X \to QX = \Omega^\infty \Sigma^n X$. The co-augmentation $X \to SP^n X = X^n/\Sigma_n$ to the $n$-symmetric space satisfies the requirement (a) of [15, Section 4.A], and one might ask if this map is also cellular. We believe it is, although we do not have a clear argument at the time of writing this paper. Our key example of a co-augmented cellular functor is given by the Hurewicz map $h_X : X \to \mathbb{Z}X$.

How can we construct new co-augmented cellular functors out of old ones? For that we are going to use the following proposition. Let $f : X \to Y$ be a map. Take its homotopy cofiber $\alpha : Y \leftrightarrow \text{Cof}(f)$ and the homotopy fiber $\text{Fib}(\alpha) \to Y$ of $\alpha$. These maps fit into the following commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\text{Fib}(\alpha)} & Y \\
\downarrow \simeq & & \downarrow \alpha \\
\Delta[0] \sim CX & \xrightarrow{\sim} & \text{Cof}(f)
\end{array}
$$

where the front square is a push-out, the right back square is a pull-back, and the indicated maps are weak equivalence, fibrations and cofibrations. The map $\overline{f} : X \to \text{Fib}(\alpha)$ is called the comparison map.

4.4. **Proposition.** Let $f : X \to Y$ be a map, $\overline{f}$ as above, and $A$ be a connected space.

(15) If $X$ is connected and $\text{Cof}(f)$ is simply connected, then $\text{Cof}(\overline{f}) \gg S^1 \wedge X$.

(16) If $\text{Fib}(f)$ is simply connected and $X > S^1 \wedge A$, then $\text{Fib}(\overline{f}) > S^2 \wedge A$.

Proof. We prove (15) first. By the assumption $\text{Cof}(f)$ is 1-connected. Thus its loop space, which is the homotopy fiber of $CX \to \text{Cof}(f)$, is connected. We can
therefore apply Theorem 3.5.(13) to the following homotopy push-out square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
CX & \xrightarrow{\alpha} & \text{Cof}(f)
\end{array}
\]

to get \(\text{Cof}(f) \gg S^1 \wedge X \wedge \Omega^2 \text{Cof}(f) \gg S^1 \wedge X\).

To prove (16), assume \(X > S^1 \wedge A\). This implies \(X\) is 1-connected. According to Proposition 3.2.(7) \(\text{Fib}(\alpha) \gg X\) and thus \(\text{Fib}(\alpha)\) is also 1-connected. The hypothesis of Theorem 3.5.(14) is thus satisfied and we get \(\text{Fib}(f) > S^1 \wedge \Omega X \wedge \Omega \text{Fib}(f)\). Since \(\text{Fib}(f)\) is 1-connected, \(\Omega \text{Fib}(f)\) is connected and we can conclude \(\text{Fib}(f) > S^1 \wedge \Omega X \wedge \Omega \text{Fib}(f) > S^2 \wedge \Omega X \wedge A\). □

Given a co-augmented functor \(K\), the above construction allows us to construct a new functor \(\overline{K}: \text{Spaces} \rightarrow \text{Spaces}\). For any space \(X\), the co-augmentation \(\mu_X: X \rightarrow K(X)\) induces namely a map \(\overline{\mu_X}: X \rightarrow \overline{K}(X)\).

Here is the key statement of this section:

4.5. **Proposition.** Let \(\mu_X: X \rightarrow K(X)\) be a co-augmented functor. Assume:

- If \(X\) is connected, then \(\text{Cof}(\mu_X)\) is simply connected.
- If \(X\) is simply connected, then so is \(K(X)\) and \(\pi_2(\mu_X)\) is an epimorphism.

Then \(\overline{K}\) is a cellular co-augmented functor and \(\overline{K}(X)\) is \(X\)-cellular for any \(X\).

**Proof.** Requirement (a) of Definition 4.1 follows from Proposition 4.4.(15) and requirement (b) from Proposition 4.4.(16). The fact that \(\overline{K}(X)\) is \(X\)-cellular is a direct consequence of 4.2. □

For example the assumptions of Proposition 4.5 are satisfied if \(K\) is cellular:

4.6. **Corollary.** If \(K\) is a co-augmented cellular functor, then so is \(\overline{K}\). □

5. **THE MODIFIED BOUSFIELD-KAN TOWER**

The aim of this section is to show that the co-augmented functors \(\mu_{k,X}: X \rightarrow z_k X\) in a modified version of the integral Bousfield-Kan completion tower, as defined by the second author in [17], are cellular (see Definition 4.1). The modified tower was built originally as an elementary construction that models the pro-homology type of any space by a tower of much simpler spaces called polyGEMs.

5.1. **Definition.** A 1-polyGEM is defined to be a GEM, i.e., a space weakly equivalent to a product of abelian Eilenberg-MacLane spaces. For \(n \geq 2\), an \(n\)-polyGEM is a space which is weakly equivalent to a retract of the homotopy fiber of a map from an \((n-1)\)-polyGEM to a GEM. A space is a **polyGEM** if it is an \(n\)-polyGEM for some integer \(n\).

PolyGEMs are examples of nilpotent spaces. They are in a sense universal such examples:

5.2. **Proposition.** A connected space \(X\) is nilpotent if and only if, for any \(n \geq 1\), the \(n\)-th Postnikov section \(P_n X\) is a polyGEM.
Proof. If $X$ is nilpotent its Postnikov tower admits a refinement by principal fibrations whose fibers are Eilenberg-MacLane spaces. Conversely, if $P_n X$ is a polyGEM, it is nilpotent. Hence its fundamental group is nilpotent and acts nilpotently on all homotopy groups. This is true for all integers $n$, so $X$ itself is nilpotent. □

How can we detect that a space is a polyGEM? This can be done using the modified version of the integral Bousfield-Kan homology completion tower:

Recall from [17] the inductive construction of the tower. For $k = 0$, $z_0 X = Z X$ and $\mu_0, X : X \to z_0 X$ is the Hurewicz map $h X : X \to Z X$. For $k \geq 0$, the space $z_k X$ is the homotopy fiber of the composition:

$$z_k X \xrightarrow{\alpha} \text{Cof}(\mu k, X) \xrightarrow{\text{hCof}(\mu k, X)} Z \text{Cof}(\mu k, X)$$

and the map $\mu_{k+1}, X : X \to z_{k+1} X$ fits into the following commutative diagram:

$$X \xrightarrow{\mu_{k, X}} z_k X \xrightarrow{\mu_{k+1, X}} z_{k+1} X \xrightarrow{\alpha} \text{Cof}(\mu k, X) \xrightarrow{h \text{Cof}(\mu k, X)} Z \text{Cof}(\mu k, X)$$

Observe that $z_0 X$ is a GEM by definition, $z_1 X$ is a 2-polyGEM as it is the homotopy fiber of a map between GEMs, and, more generally $z_k X$ is a $(k+1)$-polyGEM for any $k \geq 0$, as it is by induction the homotopy fibre of a map from a $k$-polyGEM to a GEM. Moreover, this new tower mimics the behavior of the classical Bousfield-Kan tower in the following sense:

5.3. Proposition. For any space $X$ the maps $\mu_{k, X} : X \to z_k X$ induce a pro-homology and cohomology isomorphism between the constant tower $X$ and the modified integral Bousfield-Kan tower $(\cdots \to z_1 X \to z_0 X)$. Moreover, if $X$ is a polyGEM, then this map of towers induces a pro-isomorphism on pro-homotopy groups.

Proof. The pro-homology isomorphism (and therefore also pro-cohomology isomorphism) holds by [17 Theorem 2.2]. As a corollary, [17 Proposition 2.13], the map of towers induces a pro-isomorphism on pro-homotopy groups for any polyGEM, i.e. the kernel and cokernel are pro-isomorphic to zero. □

We can now state our detection principle.

5.4. Proposition. A space $W$ is a polyGEM if and only if it is a retract of $z_n W$ for some $n$.

Proof. Since $z_n W$ is a polyGEM, by definition so is any of its retracts. That proves one implication.
Assume now that \( W \) is a polyGEM. The maps \( \mu_{k,W} : W \to z_kW \) induce a pro-cohomology equivalence as we just have seen in Proposition 5.3. Thus, for a fibrant GEM \( P \), the maps of mapping spaces \( \mu_{k,W} : (z_kW, P) \to \text{map}(W, P) \) induces an ind-homotopy equivalence. By induction, the same holds for any fibrant polyGEM \( P \). To obtain the desired retraction, we can now use this ind-homotopy equivalence when \( P \) is a fibrant replacement of \( W \).

We conclude this section with its main result.

5.5. Proposition. For all \( k \geq 0 \), the co-augmented functor \( z_k \) is cellular. In particular \( z_kX \) is \( X \)-cellular for any \( X \).

Proof. The proof is by induction on \( k \). The cellularity of \( z_0 = Z \) was already discussed in Section 4. Assume that \( k > 0 \). Let us denote by \( \beta : z_kX \to z_{k+1}X \) the left vertical map in the diagram (1). This map fits into a commutative triangle:

\[
\begin{array}{ccc}
X & \xrightarrow{\mu_{k,X}} & z_kX \\
\downarrow & & \downarrow \\
\mu_{k+1,X} & \xrightarrow{\beta} & z_{k+1}X
\end{array}
\]

which exhibits \( \mu_{k+1,X} \) as a composition of two maps, yielding both a cofibration and a fibration sequence:

\[
\text{Cof}(\mu_{k,X}) \to \text{Cof}(\mu_{k+1,X}) \to \text{Cof}(\beta) \quad \text{Fib}(\mu_{k,X}) \to \text{Fib}(\mu_{k+1,X}) \to \text{Fib}(\beta)
\]

Assume now that \( X \) is connected. Since the functor \( z_k \) is cellular (by induction), \( \text{Cof}(\mu_{k,X}) > S^1 \wedge X \) and we get:

\[
\Gamma\text{Cof}(\mu_{k,X}) := \text{Fib} \left( \text{Cof}(\mu_{k,X}) \xrightarrow{h_{\text{Cof}(\mu_{k,X})}} \text{ZCof}(\mu_{k,X}) \right) > S^2 \wedge X
\]

From the cellularity of \( Z \) (part (b) of Definition 4.1), it follows that:

\[
\text{Fib}(\beta : z_kX \to z_{k+1}X) \simeq \Omega\text{Fib}(\mu_{k,X}) > \Omega(S^2 \wedge X) > S^1 \wedge X
\]

Consequently \( \text{Cof}(\beta) > S^1 \wedge \text{Fib}(\beta) > S^2 \wedge X \) (see Proposition 3.3 (8)). As \( z_k \) is cellular (see Corollary 4.6), we also have \( \text{Cof}(\mu_{k,X}) > S^1 \wedge X \). These last two inequalities imply \( \text{Cof}(\mu_{k+1,X}) > S^1 \wedge X \) which is requirement (a) of Definition 4.1.

Assume now that \( X > S^1 \wedge A \) for a connected space \( A \). Since \( z_k \) is cellular, \( \text{Fib}(\mu_{k,X}) > S^2 \wedge A \). We have already seen that \( \text{Fib}(\beta) > S^1 \wedge X \), and hence \( \text{Fib}(\beta) > S^2 \wedge A \). These inequalities imply \( \text{Fib}(\mu_{k+1,X}) > S^2 \wedge A \), which is requirement (b) of Definition 4.1. This concludes the induction step and the proof of the proposition.

6. Cellularity of Postnikov sections

Recall that the Postnikov sections and highly connected covers are the most basic occurrences of nullifications and cellular covers: \( P_{S^{n+1}}X \) is the \( n \)-th Postnikov section \( P_nX \) and \( \text{cell}_{S^{n+1}}X \) is the \( n \)-connected cover. We are ready now to prove the main theorem of this article: a space builds any of its nilpotent Postnikov sections, even if the space itself is not nilpotent. This is in contrast with highly connected covers as the following examples illustrates.
6.1. Example. In general, an $n$-connected cover $\mathrm{cell}_{S^{n+1}}X$ – even of a nilpotent space $X$ – is not $X$-cellular. Consider for example $K(\mathbb{Z}, 2) \vee K(\mathbb{Z}, 2)$ whose cellular class is that of $K(\mathbb{Z}, 2)$. Its 2-connected cover is $S^3$ and the 3-sphere is not $K(\mathbb{Z}, 2)$-cellular, in fact not $K(\mathbb{Z}, 2)$-acyclic. To see this let $M(\mathbb{Z}/2, 3)$ be the double suspension of $\mathbb{R}P^2$. This is a finite complex so, by the Sullivan conjecture, $[24]:$

$$\mathrm{map}_*(K(\mathbb{Q}/\mathbb{Z}, 1), M(\mathbb{Z}/2, 3)) \simeq *$$

The space $K(\mathbb{Q}/\mathbb{Z}, 1)$ is the homotopy fiber of the map $K(\mathbb{Z}, 2) \to K(\mathbb{Q}, 2)$ induced by the inclusion $\mathbb{Z} \subset \mathbb{Q}$. By the above consequence of the Sullivan conjecture we then have:

$$\mathrm{map}_*(K(\mathbb{Z}, 2), M(\mathbb{Z}/2, 3)) \simeq \mathrm{map}_*(K(\mathbb{Q}, 2), M(\mathbb{Z}/2, 3))$$

However, since $K(\mathbb{Q}, 2)$ is a rational space, $\mathrm{map}_*(K(\mathbb{Q}, 2), M(\mathbb{Z}/2, 3))$ is contractible and consequently so is $\mathrm{map}_*(K(\mathbb{Z}, 2), M(\mathbb{Z}/2, 3))$. This shows that $M(\mathbb{Z}/2, 3)$ can not be $K(\mathbb{Z}, 2)$-acyclic. As $M(\mathbb{Z}/2, 3) > S^3$, $S^3$ can not be $K(\mathbb{Z}, 2)$-acyclic either.

In the proof of our main result the following cellular and acyclic properties of polyGEMs play key roles. In Example 6.1 we have seen that in general the $n$-connected cover $\mathrm{cell}_{S^{n+1}}X$ of a nilpotent space $X$ can fail to be $X$-acyclic. This however can not happen when $X$ is a polyGEM:

6.2. Proposition. Let $W$ be a polyGEM. Then, for any $n \geq 0$:

1. $\mathrm{cell}_{S^n} W > W$;
2. $K(\pi_n W, n) > W$;
3. $P_n W \gg W$ and in particular $K(\pi_1 W, 1) \gg W$;
4. $C(W) = C(ZW) = C(\prod_{k \geq 0} K(\pi_k W, k))$

6.3. Lemma. Let $X$ be a space. For any $k \geq 0$ and $n \geq 0$, $\mathrm{cell}_{S^n} z_k X > X$.

Proof. If $X$ is not connected, then the lemma is clear as all spaces are $X$-acyclic. Assume $X$ is connected. The proof is by induction on $k$. For $k = 0$, the space $z_0 X = ZX$ is a GEM. Thus for any $n \geq 0$, $\mathrm{cell}_{S^n} ZX$ is a retract of $ZX$ and since $ZX$ is $X$-cellular, then so is $\mathrm{cell}_{S^n} ZX$.

Assume $k > 0$. Since $X$ is connected, then so is $z_k X$ and hence $\mathrm{cell}_{S^n} z_k X$ and $\mathrm{cell}_{S^n} z_k X$ are weakly equivalent to $z_k X$, which is $X$-acyclic (even cellular) by Proposition 6.2. Assume $n \geq 2$ and form the following commutative diagram where the horizontal sequences are fibration sequences and the left and right vertical maps are cellular covers (see [15], Proposition E2)):

$$
\begin{array}{ccc}
\mathrm{cell}_{S^n} z_k X & \longrightarrow & \mathrm{cell}_{S^n+1} Z \mathrm{CoF}(\mu_{k-1}, X) \\
\downarrow & & \downarrow \\
z_k X & \longrightarrow & z_{k-1} X
\end{array}
$$

Explicitly, the space $E$ is the $(n - 1)$-connected cover of the homotopy pull-back of the right hand side pull-back diagram. Since $z_{k-1}$ is a cellular functor (see Proposition 6.2), $\mathrm{CoF}(\mu_{k-1}, X) > S^1 \wedge X$. This, together with the case $k = 0$, gives $\mathrm{cell}_{S^n+1} Z \mathrm{CoF}(\mu_{k-1}, X) > \mathrm{CoF}(\mu_{k-1}, X) > S^1 \wedge X$ or equivalently by Proposition 6.2, $2 \Omega \mathrm{cell}_{S^n+1} Z \mathrm{CoF}(\mu_{k-1}, X) > X$. To show $\mathrm{cell}_{S^n} z_k X > X$ it is therefore enough to prove $E > X$ (see Proposition 6.3(8)).
The space $E$ is $(n - 1)$-connected and the map $E \to z_{k-1}X$ induces an isomorphism on homotopy group $\pi_i$ for $i \geq n + 1$. We have thus a fibration sequence:

$$\text{cell}_{S^{n+1}} z_{k-1}X \to E \to K(\pi_n E, n)$$

By the inductive assumption $\text{cell}_{S^{n+1}} z_{k-1}X > X$. The inequality $E > X$ will then follow once we show $K(\pi_n E, n) > X$.

Let $G = \pi_n z_{k-1}X$, $H = \pi_n \text{Cof}(\mu_{k-1, X})$ and $f : G \to H$ be the group homomorphism induced on $\pi_n$ by the map $z_{k-1}(X) \to \text{Cof}(\mu_{k-1, X})$. By the inductive assumption $\text{cell}_{S^n z_{k-1}X} > X$ and $\text{cell}_{S^{n+1} z_{k-1}X} > X$. These spaces fit into the following fibration sequence:

$$\text{cell}_{S^{n+1} z_{k-1}X} \to \text{cell}_{S^n z_{k-1}X} \to K(G, n),$$

It follows that $K(G, n) > X$. As $K(H, n)$ is a retract of $\text{Cof}(\mu_{k-1, X})$, we also have $K(H, n) > \text{Cof}(\mu_{k-1, X}) > S^1 \wedge X$. These inequalities imply:

$$K(\text{Ker}(f), n) \times K(\text{Coker}(f), n - 1) \simeq \text{Fib}(f, n) : K(G, n) \to K(H, n)) > X.$$

Hence, as a retract of $\text{Fib}(f, n)$, the space $K(\text{Ker}(f), n)$ is also $X$-acyclic. The long exact sequences in homotopy for the fibrations in the above diagram allow us to identify $\pi_n E$ with $K(f : G \to H)$. We conclude that $K(\pi_n E, n) > X$. □

Proof of Proposition 6.2. If $W$ is not connected, then all the four statements are clear. Assume then that $W$ is connected.

1: Since $W$ is a polyGEM, Proposition 5.2 implies that it is a retract of $z_k W$ for some $k$. By functoriality, $\text{cell}_S W$ is then a retract of $\text{cell}_{S^n} z_k W$ and we conclude by Lemma 6.3 that $\text{cell}_S W > W$.

2: This is a consequence of (1) and the fact that we have a fibration sequence:

$$\text{cell}_{S^{n+1}} W \to \text{cell}_{S^n} W \to K(\pi_n W, n).$$

3: For $n = 0$ the result is immediate as $P_0 W$ is a retract of $W$. Let $n \geq 1$. In this case the statement follows from (1) and Proposition 5.2(10) applied to the fibration sequence $\text{cell}_{S^{n+1}} W \to W \to P_n W$.

4: We start by showing by induction that $z_k W > Z W$. For $k = 0$ there is nothing to prove. Assume that $k \geq 1$. Recall that $z_k W$ fits into a fibration sequence:

$$\Omega \text{Cof}(\mu_{k-1, W}) \to z_k W \to z_{k-1} W$$

As $z_{k-1}$ is cellular (see Proposition 5.2), $\text{Cof}(\mu_{k-1, W}) > S^1 \wedge W$ and hence $\text{ZCof}(\mu_{k-1, W}) > Z(S^1 \wedge W)$ which implies $\Omega \text{ZCof}(\mu_{k-1, W}) > Z W$. By induction we also have $z_{k-1} W > Z W$. We can then conclude $z_k W > Z W$. Note that in the above argument we did not use the assumption that $W$ is a polyGEM and hence this acyclic inequality is true for an arbitrary space. However in the case $W$ is a polyGEM, there is an integer $k$ for which $W$ is a retract of $z_k W$. For such a $k$ we have then the following relations $W \gg z_k W > Z W \gg W$ which proves the equality $\overline{C(W)} = \overline{C(ZW)}$.

The inequality $\prod K(\pi_k W, k)) > W$ follows from statement (2). To conclude that the acyclic classes $\overline{C(W)}$ and $\overline{C(\prod K(\pi_k W, k))}$ coincide what remains is the proof of the relation $ZW > \prod K(\pi_k W, k)$. For any $n \geq 0$, the inequality $K(H_n(W), n) \gg$
\[ \mathbb{Z}(P_n W) \text{ holds since } W \text{ and } P_n W \text{ have isomorphic } n\text{-th integral homology groups by the Whitehead Theorem. Therefore:} \]
\[ K(H_n(W), n) \gg \mathbb{Z}(P_n W) \gg P_n W > \prod_{k \geq 0} K(\pi_k W, k) \]

This implies that \( \mathbb{Z}W > \prod_{k \geq 0} K(\pi_k W, k) \).

**6.4. Theorem.** Let \( X \) be a space. If \( P_n X \) is nilpotent, then \( P_n X \gg X \).

**Proof.** If \( X \) is not connected, then the conclusion is clear. Assume thus that \( X \) is connected and \( P_n X \) is nilpotent, which by Proposition 6.2 means that \( P_n X \) is a polyGEM. Proposition 6.1 implies the existence of an integer \( k \) and of a map \( r: z_k P_n X \to W \) into a fibrant space \( W \) such that the composition with the augmentation \( \mu_{k, P_n X}: P_n X \to z_k P_n X \) is a weak equivalence. We can then form the following diagram which is commutative if we remove the dotted arrows and in which the symbol \( p \) denotes the relevant functorial \( n\)th Postnikov section maps:

\[
\begin{array}{ccc}
X & \xrightarrow{p} & P_n X \\
\downarrow{\eta_X} & & \downarrow{\eta_{P_n X}} \\
\underset{z_k P_n X}{z_k X} & \xrightarrow{z_k P} & \underset{z_k P_n X}{z_k P_n X} \\
\downarrow{p_{z_k X}} & & \downarrow{p} \\
P_n z_k X & \xrightarrow{P_n z_k} & P_n z_k P_n X \\
\end{array}
\]

The maps represented by the dotted arrows, which make the entire diagram homotopy commutative, exist by the universal property of the Postnikov sections. We can therefore conclude that \( P_n X \) is a retract of \( P_n z_k X \). We thus have the following cellular inequalities:
\[ P_n X \gg P_n z_k X \gg z_k X \gg X \]
where the second cellular inequality follows from Proposition 6.2 (3) together with the fact that \( z_k X \) is a polyGEM and the last one is given in Proposition 5.5. \( \square \)

The following is a particular case of Theorem 6.4 for \( n = 1 \). Hints that this result could hold motivated the present work.

**6.5. Corollary.** If \( \pi_1(X) \) is nilpotent, then \( K(\pi_1 X, 1) \gg X \). \( \square \)

We can also use Theorem 6.4 to get a Serre class-type statement that describes a global relation between the integral homology groups and the homotopy groups of a nilpotent space. No spectral sequence is needed in our proof, even though it seems that one could also obtain the mutual acyclicity of the homotopy and homology groups by a spectral sequence argument.

**6.6. Corollary.** If \( X \) is nilpotent, then \( C(\mathbb{Z}(X)) = C(\prod_{k \geq 0} K(\pi_k X, k)) \).

**Proof.** If \( X \) is not connected, the statement is clear. Assume thus \( X \) is connected. Even without the nilpotency assumption on \( X \), for any \( n \geq 0 \), we have:
\[ K(H_n(X), n) = K(H_n(P_n X), n) \gg \mathbb{Z}(P_n X) \gg P_n X > \prod_{k \geq 0} K(\pi_k X, k) \]

Consequently \( \mathbb{Z}(X) > \prod_{k \geq 0} K(\pi_k X, k) \). For the opposite inequality, we need the assumption \( X \) is nilpotent, which according to Proposition 6.2 is equivalent.
to $P_nX$ being a polyGEM for any $n \geq 0$. We can then use Theorem 6.4 and Corollary 6.2 to obtain $K(\pi_nX, n) > P_{n+1}X \gg X$ which, for $n > 1$, implies $K(\pi_nX, n) \gg \mathbb{Z}(K(\pi_nX, n)) \gg \mathbb{Z}(X)$. For $n = 1$, since $\pi_1X$ is nilpotent, we also have $K(\pi_1X, 1) > \mathbb{Z}(X)$. This shows $\prod_{k \geq 0} K(\pi_kX, k) > \mathbb{Z}(X)$. □

7. Applications

In this section we state various consequences of Proposition 5.5 and Theorem 6.4. We start with the preservation of polyGEMS by general cellularizations:

7.1. Theorem. If $X$ is a polyGEM, then so is $\text{cell}_A X$ for any space $A$.

Proof. Assume $X$ is a polyGEM. It is thus a retract of $z_nX$ (see 5.4), for some $n$, and hence we can form the following diagram with the indicated maps being weak equivalences and which, if the dotted arrow is removed, is commutative by functoriality of the constructions:

Since $z_n$ is a cellular functor (see Proposition 5.5), $z_n X \gg \text{cell}_A X \gg A$. We can then use the universal property of the $A$-cellular cover $c_A z_nX$ to get the existence of the dotted arrow that makes the entire diagram homotopy commutative. Commutativity of this diagram shows that $\text{cell}_A X$ is a retract of $z_n X \gg \text{cell}_A X$ and hence by Proposition 6.4, $\text{cell}_A X$ is a polyGEM. □

An analogous result to Theorem 7.1 holds also for finite nilpotent Postnikov stages.

7.2. Corollary. If $X$ is a nilpotent $n$-Postnikov stage, then so is $\text{cell}_A X$ for any $A$.

Proof. Assume $p_nX : X \to P_nX$ is a weak equivalence and $P_nX$ is nilpotent. Consider the following diagram where the indicated maps are weak equivalences and, if the dotted arrow is removed, it is commutative by functoriality of the constructions:

Since $X$ is a polyGEM (see 5.2), then so is $\text{cell}_A X$ by Theorem 7.1. We can then use Proposition 6.2(3) to conclude that $P_n \text{cell}_A X \gg \text{cell}_A X \gg A$. The universal property of the cellular cover $c_A P_nX$ gives the existence of the dotted arrow making the above diagram homotopy commutative. This implies that the map $p_{n, \text{cell}_A X} : \text{cell}_A X \to P_n \text{cell}_A X$ induces a monomorphism on all homotopy
groups. As it also induces an epimorphism, it is a weak equivalence and cell\(_{A}X\) is an \(n\)-Postnikov stage.

Theorem 7.1 can be used to give a description of all cellular covers of the classifying space of a nilpotent group in terms of the group theoretical covers [18, 8, 1]:

**7.3. Corollary.** Assume that \(G\) is a nilpotent group and \(A\) a connected space. Then cell\(_{A}K(G, 1) \simeq K(\text{cell}_{\pi_{1}A}G, 1)\) where cell\(_{\pi_{1}A}G\) is the group theoretical \(\pi_{1}A\)-cellularization of \(G\) (see [18, 8, 1]).

**Proof.** Corollary 7.2 implies that cell\(_{A}K(G, 1) \simeq K(H, 1)\) where \(H\) is a nilpotent group. Furthermore we claim that since \(K(H, 1)\) is \(A\)-cellular, the group \(H\) is \(\pi_{1}A\)-cellular. To see this note that, by the Seifert-van Kampen Theorem, the collection of all connected spaces with \(\pi_{1}A\)-cellular fundamental group is a cellular class. Since it contains \(A\), it has to include the smallest cellular collection \(C(A)\), and in particular it contains \(K(H, 1)\).

By the universal property of the cellularization and Theorem 2.2 (A), the following map is a weak equivalence:

\[
\text{map}_{\pi_{1}}(A, c_{A,K(G,1)}): \text{map}_{\pi_{1}}(A, K(H, 1)) \to \text{map}_{\pi_{1}}(A, K(G, 1))
\]

Thus, on \(\pi_{1}\), we get that the homomorphism \(\pi_{1}c_{A,K(G,1)}: H \to G\) induces a bijection:

\[
\text{Hom}(\pi_{1}A, \pi_{1}c_{A,K(G,1)}): \text{Hom}(\pi_{1}A, H) \cong \text{Hom}(\pi_{1}A, G)
\]

This homomorphism \(\pi_{1}c_{A,K(G,1)}: H \to G\) is therefore the \(\pi_{1}A\)-cellularization and \(H\) is isomorphic to cell\(_{\pi_{1}A}G\).

By [18, Theorem 1.4 (2)], the group theoretical cellularizations of a finite nilpotent group \(N\) are always subgroups of \(N\). It is sometimes possible to compute all possible such cellularizations.

**7.4. Example.** Let \(D_{2^{n}}\) denote the dihedral group of order \(2^{n}\) for \(n \geq 2\). This is the group of symmetries of a regular polygon with \(2^{n-1}\) sides and it is nilpotent of class \(n - 1\). The third author showed in [19, Proposition 5.1] that \(K(D_{2^{n}}, 1)\) is \(\mathbb{Z}/2\)-cellular. We have hence only two possible cellularizations, cell\(_{A}K(D_{2^{n}}, 1)\) can be contractible or cell\(_{A}K(D_{2^{n}}, 1) \simeq K(D_{2^{n}}, 1)\). The latter is obtained for example for \(A = K(\mathbb{Z}/2, 1)\). For \(n \leq 3\), we were able to perform these computations by hand, but already for \(n = 4\) we do not know of a direct calculation of all cellularizations of \(K(D_{16}, 1)\) without showing first that they must be \(K(G, 1)\)’s.

We can also strengthen Corollary 6.5 Instead of assuming that the fundamental group of \(X\) is nilpotent we make it nilpotent by taking the quotient by some stage of the lower central series.

**7.5. Corollary.** For any \(n \geq 1\), \(K(\pi_{1}X/\Gamma_{n}\pi_{1}X, 1) \gg X\).

**Proof.** Let \(G = \pi_{1}X\). If \(X\) is not connected, the corollary is clear. Assume \(X\) is connected. In this case according to Corollary 7.3 cell\(_{X}K(G/\Gamma_{n}G, 1) \simeq K(\text{cell}_{G}(G/\Gamma_{n}G, 1)). \) However \(G/\Gamma_{n}G\) is a \(G\)-cellular group (see [8, Proposition 7.1.(3)]). We can then conclude cell\(_{X}K(G/\Gamma_{n}G, 1) \simeq K(G/\Gamma_{n}G, 1)\), which proves that \(K(G/\Gamma_{n}G, 1)\) is \(X\)-cellular.

The statement of the next result does not involve cellularity, however we do not know of a proof which does not use our techniques. This is the extension to
nilpotent fundamental groups of the Bousfield Key Lemma we presented in the introduction. We recalled the precise statement of the key lemma at the beginning of Section 4. It implies for example that, if $X$ is simply connected, then the map $\pi_n : \text{map}_*(X, X) \to \text{Hom}(\pi_n X, \pi_n X)$ is a weak equivalence if and only if $X$ is weakly equivalent to $K(\pi_n X, n)$. If $X$ is not simply connected, then the situation is much more complicated.

7.6. Example. The $K(\mathbb{Z}/2, 1)$-cellularization of $K(\Sigma_3, 1)$ has been computed in [20, Example 2.6]. It is a space $X$ whose fundamental group is the symmetric group $\Sigma_3$ and its universal cover is the homotopy fiber of the degree 3 map on the sphere $S^3$. In particular its homotopy groups are non trivial in infinitely many degrees. By the universal property of the cellularization we have weak equivalences of mapping spaces

$$\text{map}_*(X, X) \simeq \text{map}_*(X, K(\Sigma_3, 1)) \simeq \text{Hom}(\Sigma_3, \Sigma_3)$$

The mapping space of pointed self-maps of $X$ is homotopically discrete, but $X$ is far from being a $K(G, 1)$. This example also shows that the cellularization of non-nilpotent spaces can become quite complicated.

7.7. Theorem. Let $X$ be a connected space whose fundamental group $\pi_1 X$ is nilpotent. Assume that the map $\pi_1 : \text{map}_*(X, X) \to \text{Hom}(\pi_1 X, \pi_1 X)$ is a weak equivalence. Then $X$ is weakly equivalent to $K(\pi_1 X, 1)$.

Proof. The assumptions imply that $\text{map}_*(X, P_1 X) : \text{map}_*(X, X) \to \text{map}_*(X, P_1 X)$ is a weak equivalence. Thus $X \simeq \text{cell}_X K(\pi_1 X, 1)$, which by Corollary 7.3 means that $X \simeq K(\pi_1 X, 1)$. □

Here is another way to restate this result. If the first Postnikov section $X \to K(G, 1)$ induces a weak equivalence on pointed mapping spaces $\text{map}_*(X, X) \simeq \text{map}_*(X, K(G, 1))$, then $X$ is a $K(G, 1)$. We also offer a version for higher Postnikov sections. The very same argument as in the proof of Theorem 7.7 can be used to show:

7.8. Corollary. Let $X$ be a connected space whose $n$-th Postnikov section $P_n X$ is nilpotent. Assume that the map $P_n : \text{map}_*(X, X) \to \text{map}_*(P_n X, P_n X)$ is a weak equivalence. Then $P_n X : X \to P_n X$ is a weak equivalence, i.e. $X$ is an $n$-Postnikov stage. □

Here is another application of Theorem 6.4 and the characterization of cellularity in Theorem 2.1. Notice however that the statement is not true for a general space $X$. Neither is the analogous statement for higher connected covers of a nilpotent $X$ as we have seen in Example 6.1.

7.9. Corollary. Assume $X$ is nilpotent. If $\text{map}_*(X, Y)$ is contractible, then so is $\text{map}_*(P_n X, Y)$ for any $n \geq 0$. □

In the last part of this section we offer a few results which state simple - but not obvious - homological properties of nilpotent spaces. All are immediate consequences of our main theorem.

7.10. Theorem. Let $K$ be a reduced homology theory.
(1) Assume $X$ is $K$-acyclic. If $P_nX$ is nilpotent, then it is also $K$-acyclic.

(2) Assume $X$ is nilpotent. Then $\prod_{k\geq 1} K(\pi_kX, k)$ is $K$-acyclic if and only if $\prod_{k\geq 1} K(H_kX, k)$ is $K$-acyclic.

(3) If $K(G, 1)$ is $K$-acyclic, then so is $K(G/\Gamma_nG, 1)$ for any $n$. □

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