CONFORMALLY EQUIVARIANT QUANTIZATION FOR SPINNING PARTICLES

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Abstract. This work takes place over a conformally flat spin manifold \((M, \mathfrak{g})\). We prove existence and uniqueness of the conformally equivariant quantization valued in spinor differential operators, and provide an explicit formula for it when restricted to first order operators. The Poisson algebra of symbols is realized as a space of functions on the supercotangent bundle \(M = T^*M \oplus \Pi TM\), endowed with a symplectic form depending on \(\mathfrak{g}\). It admits two different actions of the conformal Lie algebra: one tensorial and one Hamiltonian. They are intertwined by the uniquely defined conformally equivariant superization, for which an explicit formula is given. This map allows us to classify all the conformal supercharges of the spinning particle in terms of conformal Killing tensors with mixed symmetry. They are generated by the totally symmetric and skew-symmetric ones. Higher symmetries of the Dirac operator are obtained by quantization of the conformal supercharges.

1. Introduction

Whereas there exists a pseudo-classical model for the spinning particle, due to Berezin and Marinov [4], quantization is scarcely developed in that setting. We propose in this paper a natural extension of the conformally equivariant quantization for spinning particles, so that it is valued in spinor differential operators.

1.1. The prototypical example of quantization is the one of cotangent bundles \(T^*M\), endowed with their canonical symplectic structure. The quantum phase space \(\mathcal{H}\) is then given by the Hilbert space of square integrable functions on \(M\). The functions on \(T^*M\) are expected to have a linear operator on \(\mathcal{H}\) as quantum counterpart. In particular, we call quantization of \(T^*M\) a linear map \(Q : \text{Pol}(T^*M) \to \mathcal{D}(M)\), valued in differential operators, which is the inverse of a full symbol map. The algebra of polynomial fiberwise functions \(\text{Pol}(T^*M)\), or equivalently the algebra of symmetric tensors \(\Gamma(STM)\), identifies to the graded algebra of symbols associated to \(\mathcal{D}(M)\). Via the principal symbol map, it inherits of a Poisson bracket and an Hamiltonian action of vector fields, which coincide with the natural bracket and \(\text{Vect}(M)\)-action on \(\text{Pol}(T^*M) \cong \Gamma(STM)\).

The celebrated Weyl quantization of \(T^*M\) is characterized, if \(M = \mathbb{R}^n\), by its equivariance property under the action of the symplectic affine Lie algebra \(\mathfrak{sp}(2n, \mathbb{R}) \times \mathbb{R}^n\) on \(T^*M\). Focusing rather on equivariance under a Lie algebra \(\mathfrak{g}\) acting by vector fields on \(M\), Duval, Lecomte

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and Ovsienko have introduced the concept of $\mathfrak{g}$-equivariant quantization. In particular, they prove its existence and uniqueness for $\mathfrak{g}$ the projective [27] or the conformal Lie algebra [12], acting locally on manifolds $M$ which are projectively or conformally flat respectively. It has been intensely developed since then, see e.g. [5].

A deep motivation to build quantizations is to obtain a correspondence between classical and quantum symmetries whenever there exists one, and an efficient way to measure the quantum anomalies else. For a free massive scalar particle on $(M, g)$, this amounts to determine whether the first integrals of the geodesic flow of $g$, given by symmetric Killing tensors, can be quantized into differential operators commuting with the Laplacian. This has been investigated up to second order by Carter, via his ad hoc minimal quantization procedure [10]. There is no quantum anomalies in a number of examples, see e.g. [14], in particular if $(M, g)$ is Ricci-flat. Over a conformally flat manifold, the situation is now well-understood thanks to the conformally equivariant quantization introduced above. Indeed, the latter establishes a bijection between the classical and quantum symmetries of a free massless particle [33], i.e. between the constants of motion along the null geodesic flow, given by conformal Killing symmetric tensors, and the higher symmetries of Laplacian, studied by Eastwood [16]. Moreover, this correspondence extends to the massive case and leads ultimately to new quantum integrable systems [1, 15].

1.2. To develop the spin counterpart of the preceding picture, we suppose that $(M, g)$ is a spin manifold, with spinor bundle $S$. A quantum particle is now described by a square integrable section of $S$ and we focus on the algebra of differential operators $D(M, S)$ acting on such sections. Weyl quantization and conformally equivariant quantization [8] have been generalized to that setting, but with the usual algebra of symbols $\text{Pol}(T^*M) \otimes C^\infty(M) \Gamma(\text{Cl}(M, g))$ as source space. We rather follow Getzler [18], which uses in addition the filtration of the Clifford bundle, to derive a supercommutative bigraded algebra of symbols from the usual one. This superalgebra identifies to the one of tensors $\Gamma(STM \otimes \Lambda T^*M)$ or equivalently to the algebra $O(M)$ of fiberwise polynomial functions on the supercotangent bundle $\mathcal{M} = T^*M \oplus \Pi T M$, where $\Pi$ denotes the reverse parity functor. This supermanifold is precisely the phase space introduced by Berezin and Marinov [4] to deal with pseudo-classical spinning particles on $M$. Besides, the author proved in [31] the existence of a Hamiltonian filtration on $D(M, S)$, assigning order 2 to spinor covariant derivatives $\nabla_X$, with $X \in \text{Vect}(M)$, and order 1 to the Clifford elements $\gamma(\xi)$, with $\xi \in \Omega^1(M)$. This filtration is compatible with the commutator of $D(M, S)$ and induces a Poisson bracket and a new gradation on $O(M)$. This grading stems from the identification of $\mathcal{M}$ with the graded manifold $T^*[2]M \oplus T[1]M$, whereas the Poisson bracket comes from the symplectic structure on $\mathcal{M}$ induced by the metric $g$. The latter follows from a general construction, due to Rothstein in the super setting [37] and to Roytenberg in the graded setting [38]. In addition, the obtained symplectic structure on $\mathcal{M}$ corresponds to the one stemming from the Lagrangian of a free pseudo-classical spinning particle on $(M, g)$ [36, 26], so that the graded Poisson algebra $O(M)$ is a classical counterpart to $D(M, S)$. 
This is confirmed by the geometric quantization scheme, which associates the Hilbert space of square integrable spinors to the supercotangent bundle $\mathcal{M}$.

We name quantizations of $\mathcal{M}$ the linear maps $Q : \mathcal{O}(\mathcal{M}) \rightarrow \mathcal{D}(M, S)$ which are the inverse of a full symbol map for the Hamiltonian filtration of $\mathcal{D}(M, S)$. The Weyl quantization developed by Getzler and Voronov is precisely of this type. In particular, it enables to quantize supercharges linear in momenta, built from Killing forms, into symmetries of the Dirac operator $\mathcal{D}$.

However, Weyl quantization fails to quantize conformal supercharges into Dirac symmetry operators, even the simplest ones built from conformal Killing forms.

Our aim is to build a family $(Q^{\lambda, \mu})_{\lambda, \mu \in \mathbb{R}}$ of quantizations of the supercotangent bundle which are conformally equivariant, assuming that the base manifold $(M, g)$ is conformally flat. Such map $Q^{\lambda, \mu}$ allows, for right parameters $(\lambda, \mu)$, to quantize all the conformal supercharges into higher symmetries of the Dirac operator, i.e. into operators $D_1 \in \mathcal{D}(M, S)$ such that $\mathcal{D}D_1 = D_2 \mathcal{D}$ for some $D_2 \in \mathcal{D}(M, S)$. By the way, we also build a family $(\mathcal{S}^\delta)_{\delta \in \mathbb{R}}$ of so-called conformally equivariant superizations. For the right $\delta$, the map $\mathcal{S}^\delta$ establishes a correspondence between conformal Killing tensors with mixed symmetry, lying in $\Gamma(STM \otimes \Lambda T^*M)$, and conformal supercharges. Applications to separability of the Dirac equation and integrability of spinning particle motion are expected. This work is a first step towards the study of the algebra structure of the higher symmetries of the Dirac operator, which should be of interest for higher spin field theories.

1.3. We detail now the content of the present paper.

In Section 2, we introduce the needed definitions for conformally equivariant quantization of $\mathcal{M}$ to make sense. In particular, we have to introduce the actions on its source and target space of the Lie algebra $g$ of conformal Killing vector fields. We assume that $(M, g)$ is conformally flat and of signature $(p, q)$ so that $g \cong \mathfrak{o}(p + 1, q + 1)$. Its action on the spinor bundle $S$ is given by the Kosmann’s Lie derivative of spinors, which admits a one parameter deformation obtained by tensoring $S$ with a real power of the determinant bundle, i.e. with a density bundle. We denote by $(D^{\lambda, \mu})_{\lambda, \mu \in \mathbb{R}}$ the family of $g$-modules defined by the induced deformed adjoint actions of $g$ on $\mathcal{D}(M, S)$. For given weights $\lambda, \mu$, the action of $g$ on $D^{\lambda, \mu}$ induces two different actions on $\mathcal{O}(\mathcal{M})$. The first one is Hamiltonian and induced by the principal symbol map associated to the Hamiltonian filtration of $\mathcal{D}(M, S)$, it depends only on the shift $\delta = \mu - \lambda$ and leads to the $g$-module denoted $\mathcal{S}^\delta$. It should be understood as the module of classical observables. The latter $g$-action does not preserve the bigradation of $\mathcal{O}(\mathcal{M}) \cong \Gamma(STM \otimes \Lambda T^*M)$, but via the principal tensorial symbol map, it induces a second $g$-action on $\mathcal{O}(\mathcal{M})$ which does. We denote by $\mathcal{T}^\delta$ this extra $g$-module, which identifies to the tensor module $\Gamma(STM \otimes \Lambda T^*M)$ up to twist by appropriate density bundles. Then, we define the conformally equivariant superization $\mathcal{S}^\delta : \mathcal{T}^\delta \rightarrow \mathcal{S}^\delta$, inverse to a full tensorial symbol map, and the conformally equivariant quantization $Q^{\lambda, \mu} : S^\delta \rightarrow D^{\lambda, \mu}$, inverse to a full Hamiltonian symbol map. The name superization comes from the inclusion $\text{Pol}(T^*M) \subset \mathcal{T}^0$. 


as \( \text{Vect}(M) \)-modules, hence to each classical observable in \( \text{Pol}(T^*M) \) corresponds via \( \mathcal{S}^0 \) a spinning analog in \( S^0 \).

In Section 3, we introduce the building blocks of the maps \( \mathcal{S}^\delta \) and \( Q^{\lambda,\mu} \), which are the isometric invariant operators on \( T^\delta \). They are classified thanks to the Weyl’s theory of invariants [12]. In addition, we compute the Casimir operators of the three \( \mathfrak{g} \)-modules \( T^\delta \), \( S^\delta \) and \( D^{\lambda,\mu} \).

In Section 4, we prove our main theorem: existence and uniqueness of the conformally equivariant superization \( \mathcal{S}^\delta : T^\delta \to S^\delta \) and quantization \( Q^{\lambda,\mu} : S^\delta \to D^{\lambda,\mu} \), except for critical weights \( \delta = \mu - \lambda \). The proof follows the one of existence and uniqueness of conformally equivariant quantization in the scalar setting [12]. The only tricky point is the diagonalization of the Casimir operator of \( T^\delta \), which proves to be equivalent to the harmonic decomposition of the polynomial space \( \mathcal{S}\mathbb{R}^n \otimes \Lambda \mathbb{R}^n \). After deriving it, we discover that it was the purpose of [23]. They coincide with the weights \( \delta \) for which there exists conformally invariant operators on \( T^\delta \) (resp. \( S^\delta \)), which strictly lowers the degree. The classification of such operators allows us to show that critical weights are positive, so that existence and uniqueness of \( \mathcal{S}^\delta \) and \( Q^{\lambda,\mu} \) hold in the most usual case \( \delta = \mu - \lambda = 0 \).

In Section 5, we compute explicit formulæ for \( \mathcal{S}^\delta \) and \( Q^{\lambda,\mu} \). Following the method in the reference [13], we get local formulae for the conformally equivariant superization. Then, we obtain such formulæ for the conformally equivariant quantization also, when it is valued in first order operators. After that, we derive global expressions in terms of the Levi-Civita connection of \( \mathfrak{g} \) for both maps. Contrary to the scalar case, none of them is conformally invariant, i.e. depend only on the conformal class of \( \mathfrak{g} \). This is due to the fact that the symplectic form on \( M \) depends on \( \mathfrak{g} \) and therefore the Hamiltonian \( \mathfrak{g} \)-action on \( S^\delta \) also. Nevertheless, their composition \( Q^{\lambda,\mu} \circ \mathcal{S}^\delta \) leads to a conformally invariant map, which identifies to the conformally equivariant quantization valued in \( D^{\lambda,\mu} \) that one can deduce from the general work [8].

In Section 6, we classify the symmetries of the free massless spinning particle over conformally flat manifolds. In the pseudo-classical case, they are the conformal supercharges, which generalize the symmetries of the massive particle studied in [19, 40, 26]. They arise as the conformally equivariant superization of conformal Killing hook tensors, the latter including the purely symmetric and skew-symmetric ones. After quantization, the conformal supercharges correspond to the higher symmetries of the Dirac operator, and we recover in particular the ones given by first order operators classified in [3], and used for integrability purpose in [9].

In all that paper, we work over a pseudo-Riemannian manifold \((M, \mathfrak{g})\), with \( \mathfrak{g} \) a metric of signature \((p, q)\). That manifold is supposed to be spin and of even dimension \( n = p + q \). We freely use the Einstein’s summation convention. The tensor products of \( \mathcal{C}^\infty(M) \)-modules are understood over the algebra \( \mathcal{C}^\infty(M) \).
2. Spinor differential operators and their symbols

In order to keep a self-contained expository, this section presents a recollection of our previous work [31]. Namely, we introduce the \( \mathfrak{osp}(p+1, q+1) \)-modules structures of the space of differential operators acting on weighted spinors as well as the ones of its two spaces of symbols, and we compare these three modules.

2.1. The supercotangent bundle. Let \( E \) be a vector bundle over the smooth manifold \( M \) and \( d \) an integer. We denote by \( E[d] \) the \( \mathbb{N} \)-graded manifold with base \( M \) and structural sheaf \( \mathcal{O}(E[d]) \), which identifies to the graded sheaf of complexified sections of \( SE^* \) if \( d \) is even and of \( \Lambda E^* \) if \( d \) is odd. The sections of \( S^k E^* \), or \( \Lambda^k E^* \), receive a degree \( kd \). Such a formalism allows to encode symmetric contravariant tensors, i.e. sections of the bundle \( S^TM \), as functions on \( T^*\mathcal{C}M \), the order of tensors being twice the degree of corresponding functions. Analogously, the algebra of complexified differential forms \( \Omega^C(M) \) is viewed as the algebra of functions on \( T^*\mathcal{C}M \), the degree of forms equating the one of functions.

Both types of tensors are encompassed into the algebra of functions of the supercotangent bundle of \( M \), defined as the fiber product \( M = T^*\mathcal{C}M \times_M T\mathcal{C}M \). The gradation of its structural sheaf is given by the following subspaces, for \( \ell \in \mathbb{N} \),

\[
\mathcal{O}_{\ell}(M) = \bigoplus_{2k+\kappa = \ell} \mathcal{O}_{k,\kappa}(M) \quad \text{and} \quad \mathcal{O}_{k,\kappa}(M) = \operatorname{Pol}_k(T^*M) \otimes_{\mathcal{C}^\infty(M)} \Omega^\kappa_C(M),
\]

where \( \operatorname{Pol}_k(T^*M) \cong \Gamma(S^k TM) \) is the space of functions on \( T^*M \) of degree \( k \) in the fiber variables and \( \Omega^\kappa_C(M) \) is the space of complex differential forms of degree \( \kappa \). Starting from a local coordinate system \( (x^i) \) on \( M \), we can build a natural one \( (x^i, p_i, \xi^i) \) on \( \mathcal{M} \), with \( p_i \) identifying to the partial derivative \( \partial_i := \partial/\partial x^i \) and \( \xi^i \) to the differential 1-form \( dx^i \).

Rothstein has classified the even symplectic structures on supermanifolds in [37]. Accordingly, a symplectic structure on \( \mathcal{M} \) is equivalent to the three following piece of data: a symplectic form on \( T^*M \), a metric on the vector bundle \( TM \), i.e. a metric on \( M \), and a compatible connection. In particular, the Rothstein results lead to the following proposition.

**Proposition 2.1.** [31] Let \( (M, g) \) be a pseudo-Riemannian manifold and \( h \in \mathbb{R} \). Its supercotangent bundle \( \mathcal{M} = T^*[2]M \times_MT^*[1]M \) admits an exact symplectic structure \( \omega = d\alpha \), which reads in local natural coordinates as

\[
\alpha = p_i dx^i + \frac{h}{2i} g_{ij} \xi^i d^\nabla \xi^j,
\]

with \( d^\nabla \) the covariant differential associated to the Levi-Civita connection.

The introduction of the factor \( h/i \) into the symplectic form \( \omega \) deserves some explanations. First, from a physical point of view, a symplectic form should have the dimension of an action. Assigning no dimension to the odd variables \( \xi \) forces us to insert a constant with the dimension of an action in front of the \( \xi \)-term. We choose \( h \) since our aim is quantization. To deal with pseudo-classical spinning particles [4, 35, 30, 26], \( h \) should be replaced by a characteristic
action of the studied system. Second, the symplectic form should be real. For quantization purpose, we define the real structure on $\mathcal{M}$ by the involution anti-automorphism

$$
\tilde{\cdot} : \mathcal{O}(\mathcal{M}) \to \mathcal{O}(\mathcal{M})
$$

equal to the identity on coordinates. In particular, we get $\tilde{\xi}^i \tilde{\xi}^j = -\xi^i \xi^j$ and $\alpha, \omega$ are then real.

The symplectic form $\omega$ is of degree 2, hence $(\mathcal{M}, \omega)$ fits also into the classification of graded symplectic manifold of degree 2, performed in [38] by Roytenberg. The Poisson bracket associated to $\omega$ lowers the degree by 2 and is completely determined, thanks to Leibniz property, by the following equalities

$$
\{X,Y\} = \nabla_{[X,Y]} + \frac{\hbar}{2i} R^k_{ij} \xi_k \tilde{\xi}^j X^i Y^j, \quad \{X,\xi\} = \nabla_X \xi, \quad \{\xi,\xi'\} = \frac{i}{\hbar} g^{-1}(\xi,\xi'),
$$

where $X, Y \in \text{Pol}_1(T^*M)$ identify to vector fields, $\xi, \xi'$ are 1-forms on $M$, $g^{-1}$ is the metric induced by $g$ on $T^*M$ and $(R^k_{ij})_{ij} = [\nabla_i, \nabla_j]$ are the components of its Riemann tensor.

2.2. Spinor differential operators. Let $(M, g)$ be a pseudo-Riemannian spin manifold of even dimension $n$. Its spinor bundle $S$ satisfies $\text{End} S \cong \text{Cl}(M, g)$, where $\text{Cl}(M, g)$ is the complex Clifford bundle of $(M, g)$. We introduce the usual isomorphism

$$
\gamma : \Omega^C(M) \to \Gamma(\text{Cl}(M, g)),
$$

which satisfies $\gamma(\xi) \gamma(\xi') + \gamma(\xi') \gamma(\xi) = -2g^{-1}(\xi,\xi')$ for any $\xi, \xi' \in \Omega^C_1(M)$. The algebra $\mathcal{D}(M, S)$ of differential operators acting on sections of $S$ is an algebra filtered by the order of derivations, over the subalgebra $\Gamma(\text{End} S) \cong \Gamma(\text{Cl}(M, g))$ of zeroth order operators. The latter admits also a filtration, leading to one filtration on each subspace $\mathcal{D}_k(M, S)$ of $k$th order operators. Resorting to a spinor connection on $S$, we can then define a naive bifiltration of $\mathcal{D}(M, S)$ by the following subspaces, indexed by $k \in \mathbb{N}$ and $\kappa \leq n$,

$$
\mathcal{D}_{k,\kappa}^\nabla(M, S) = \text{span}\{\gamma(\xi_1) \cdots \gamma(\xi_m) \nabla_{X_1} \cdots \nabla_{X_m} | m \leq k \text{ and } m \leq \kappa\},
$$

where $\xi_i$ pertains to $\Omega^C_1(M)$ and $\nabla_{X_i}$ is the spinor covariant derivative, induced by the connection on $S$, along the vector field $X_i$. Such a bifiltration behaves badly with respect to the commutator, since $[\nabla_{X_i}, \nabla_{X_j}]$ is of degree 2 in $\gamma(\xi)$. Moreover, it depends on the choice of connection, as $\nabla_X = X^i \partial_i + X^i \Gamma^k_{ij} \gamma(\xi_j) \gamma(\xi_k)$. These difficulties are overcome by the following definition, introduced in [31],

$$
\mathcal{D}_{k,\kappa}(M, S) = \text{span}\{\gamma(\xi_1) \cdots \gamma(\xi_m) \nabla_{X_1} \cdots \nabla_{X_m} | m \leq k \text{ and } 2m + \mu \leq 2k + \kappa\}.
$$

For such a bifiltration, if $A$ and $B$ are two operators of orders $(k, \kappa)$ and $(k', \kappa')$, then their product $AB$ is of order $(k + k', \kappa + \kappa')$ and their commutator $[A, B]$ splits into two terms of orders $(k + k' - 1, \kappa + \kappa')$ and $(k + k', \kappa + \kappa' - 2)$. Thus, the associated bigraded algebra, defined as the direct sum over $(k, \kappa) \in \mathbb{N} \times \llbracket 0, n \rrbracket$ of the subspaces

$$
\text{bigr}_{k,\kappa} \mathcal{D}(M, S) = \mathcal{D}_{k,\kappa}(M, S)/\mathcal{D}_{k-1,\kappa}(M, S) + \mathcal{D}_{k,\kappa-1}(M, S),
$$

where
is a graded-commutative algebra. In addition to the usual filtration by the order of derivations, the above bifiltration allows us to introduce another one, given by the following increasing sequence

\[(2.5)\]

\[D_{\ell}(M, S) = \bigcup_{2k + \kappa = \ell} D_{k, \kappa}(M, S).\]

Thus, \(\nabla_X\) and \(\gamma(\xi)\gamma(\xi')\) are considered of same order. This filtration is compatible with the commutator so that the associated graded algebra \(\text{gr}D(M, S)\) is endowed with a Poisson bracket of degree \(-2\). In consequence, it is referred thereafter as the Hamiltonian filtration.

**Proposition 2.2.** We get the following isomorphisms

\[\text{bigr}D(M, S) \cong O(M) \cong \Gamma(STM \otimes \Lambda T^*M),\]
\[\text{gr}D(M, S) \cong O(M),\]

of respectively bigraded algebras and graded Poisson algebras.

Accordingly, we introduce the Hamiltonian principal symbol maps defined by the following projections,

\[(2.6)\]

\[\sigma_{\ell} : D_{\ell}(M, S) \to O_{\ell}(M),\]

which are normalized by \(\sigma_2(\nabla_{\partial_i}) = p_i\) and \(\sigma_1(\gamma(\xi_i)) = \left(\frac{\hbar}{2}\right)^{1/2} \xi_i\) to satisfy the usual relations:

\[(2.7)\]

\[\sigma_{\ell}(A)\sigma_{\ell'}(B) = \sigma_{\ell + \ell'}(AB)\quad \text{and}\quad \{\sigma_{\ell}(A), \sigma_{\ell'}(B)\} = \sigma_{\ell + \ell' - 2}(\{A, B\}),\]

for all differential operators \(A, B\) of Hamiltonian orders \(\ell\) and \(\ell'\).

By definition \(|\Lambda|^{\lambda} := |\Lambda|^\top T^*M|^{\otimes \lambda}\) is the trivial line bundle of \(\lambda\)-densities, the 1-densities being the natural objects for integration. Resorting to a pseudo-Hermitian inner product \(\langle \cdot, \cdot \rangle_S\) on \(S\), we obtain then a pairing between sections of \(S \otimes |\Lambda|^{\lambda}\) and \(S \otimes |\Lambda|^{\mu}\) if \(\lambda + \mu = 1\). Consequently, the adjoint of an operator \(D\) between the latter vector bundles is defined by

\[(2.8)\]

\[\int_M \langle \phi, D\psi \rangle_S = \int_M \langle D^\dagger \phi, \psi \rangle_S,\]

for all compactly supported sections \(\phi, \psi\) of \(S \otimes |\Lambda|^{\lambda}\). We choose the map \(\gamma\), introduced in (2.4), such that it intertwines the conjugation on \(M\) and the adjoint operation on \(\Gamma(\text{Cl}(M, g))\).

### 2.3. The conformal Lie algebra \(g\).

Let \((M, g)\) be a pseudo-Riemannian manifold of signature \((p, q)\). A conformal Killing vector field on \(M\) is a vector field \(X\) whose Lie derivative action preserves the direction of the metric, i.e. \(L_X g\) is proportional to \(g\). By definition, \((M, g)\) is conformally flat if \(g = F\eta\) locally, where \(F\) is a positive function and \(\eta\) is the flat metric of same signature \((p, q)\). Then, the Lie algebra \(g\) of local conformal Killing vector fields of \((M, g)\) is of maximal dimension and identifies to \(\mathfrak{o}(p + 1, q + 1)\). On \((\mathbb{R}^{p,q}, F\eta)\), the Lie algebra \(g \cong \mathfrak{o}(p + 1, q + 1)\) of conformal Killing vector fields is given explicitly, in terms of its...
generators, by
\begin{align}
X_i &= \partial_i, \\
X_{ij} &= x_i \partial_j - x_j \partial_i, \\
X_0 &= x^i \partial_i, \\
\tilde{X}_i &= x_j x^i \partial_i - 2x_i x^j \partial_j,
\end{align}
(2.9)
for \(i, j = 1, \ldots, n\), the indices being lowered using \(\eta\). The latter vector fields provide the local realization of \(\mathfrak{g}\) on \((M, g)\), endowed with a \([1]\)-gradation via the polynomial degree of the coefficients. With respect to the metric \(\eta\), \(\mathfrak{g}_{-1}\) is the Lie subalgebra of translations, \(\mathfrak{g}_0\) splits into the Lie subalgebra \(o(p, q)\) of rotations and its center, generated by the Euler vector field \(X_0\), and \(\mathfrak{g}_1\) is the subspace of conformal inversions. We denote the Lie algebra of isometries by \(\mathfrak{e}(p, q) := o(p, q) \ltimes \mathfrak{g}_{-1}\), and the one of similitudes by \(\mathfrak{ce}(p, q) := \mathfrak{R} \ltimes \mathfrak{e}(p, q)\).

2.4. The \(\mathfrak{g}\)-module of tensorial symbols \(T^\delta\). Let us remind that any vector field \(X\) on \(M\) admits a canonical lift to the linear frame bundle via its first order jet \((\partial_i X^j)\). This leads to an action of \(\text{Vect}(M)\) by Lie derivatives on any associated bundle, e.g. \(\ell^\lambda_X = X^i \partial_i + \lambda \partial_i X^i\) on \(\lambda\)-densities. The tensor bundle \(STM \otimes \Lambda^* M\) is also associated to the principal frame bundle, therefore it admits an action of \(\text{Vect}(M)\) by Lie derivatives. Under the identification of the latter tensors with functions on the supercotangent bundle \(\mathcal{M}\), the Lie derivative is given by the following lift of \(\text{Vect}(M)\) to \(\mathcal{M}\)
\begin{equation}
\text{Vect}(M) \ni X \mapsto \hat{X} := X^i \partial_i + (\partial_i X^j)(\xi^i \partial_{\xi^j} - p_j \partial_{p_i}),
\end{equation}
(2.10)
where \((x^i, p_i, \xi^i)\) denotes a local natural coordinate system on \(\mathcal{M}\) and \((\partial_i, \partial_{p_i}, \partial_{\xi^i})\) are the corresponding partial derivatives. Clearly, this \(\text{Vect}(M)\)-action preserves the bigradation of the tensor algebra \(\Gamma(STM \otimes \Lambda^* M)\).

**Definition 2.3.** We denote by \(T^\delta\) the vector bundle \(\bigoplus_{\kappa \leq n} STM \otimes \Lambda^\kappa T^*C\mathcal{M} \otimes |\Lambda|^\delta-n\). The bigraded module of sections \(T^\delta = \Gamma(T^\delta)\) is endowed with the natural action of \(X \in \text{Vect}(M)\), given on the \(\kappa\)-component by
\[ L_X^\delta = \hat{X} \otimes \text{Id} + \text{Id} \otimes \ell_X^{\delta-\kappa/n}. \]

The elements of \(T^\delta\) will be referred to as tensorial symbols.

The usual space of symbol of \(\mathcal{D}(M, S)\) is its graded algebra for the filtration by the order of derivations. It identifies to \(\text{Pol}(T^*M) \otimes \Gamma(\text{Cl}(M, g))\). The ideal of Clifford relations in the tensor bundle is stable under conformal transformations, hence we get an action of the Lie algebra \(\mathfrak{g}\) of conformal Killing vector fields on \(\Gamma(\text{Cl}(M, g))\). This action preserves the filtration of \(\Gamma(\text{Cl}(M, g))\) and the corresponding action on its associated graded algebra \(\Omega^\mathbb{C}(M)\) is easily identified. This leads to the following proposition, justifying the introduction of \(T^\delta\).
Proposition 2.4. Let \((M, g)\) be a pseudo-Riemannian manifold and \(\Sigma := \xi^i \partial_i\) be the Euler vector field of \(T[1]M\). We get an isomorphism of \(g\)-modules

\[
\text{Id} \otimes |\text{vol}| \mathfrak{H}_\gamma : \mathfrak{T}^\delta \rightarrow \Gamma(STM \otimes |\Lambda|^\delta) \otimes \Gamma(\text{Cl}(M, g)),
\]

which depends only on the conformal class of \(g\).

2.5. The \(g\)-module of Hamiltonian symbols \(S^\delta\). In contradistinction with the cotangent bundle case, the natural lift \((2.10)\) of \(\text{Vect}(M)\) by Lie derivatives do not lead to Hamiltonian vector fields on \(\mathcal{M}\). Besides, the preservation of the potential 1-form \(\alpha\), see \((2.2)\), is not a strong enough condition to determine a unique lift of \(X \in \text{Vect}(M)\) to \(\mathcal{M}\). In \([31]\), we ask in addition for preservation of the direction of the 1-form \(\beta = g_{ij} \xi^i dx^j\). Both conditions can be satisfied only for vector fields \(X \in g\), in that case they fix a unique lift

\[
g \ni X \mapsto \hat{X} := \hat{X} + \sum_{\lambda} \frac{\hbar}{2} \left( R^k_{\lambda ij} \xi_k X^j + \nabla_i (\partial_{[j} \xi_{k]} \right) \partial_{\lambda},
\]

the brackets denoting skew-symmetrization. The action of \(\hat{X}\) clearly preserves the gradation \(\mathcal{O}(\mathcal{M}) = \bigoplus_{\ell \in \mathbb{N}} \mathcal{O}[\ell](\mathcal{M})\) defined in \((2.1)\).

Definition 2.5. We denote by \(S^\delta\) the graded \(g\)-module \(\mathcal{O}(\mathcal{M}) \otimes \Gamma(|\Lambda|^\delta)\) of Hamiltonian symbols, where the action of \(X \in g\) is given by

\[
\mathcal{L}_X^\delta = \hat{X} \otimes \text{Id} + \text{Id} \otimes \ell_X^\delta,
\]

and the gradation is inherited from the one on \(\mathcal{O}(\mathcal{M})\).

For \(2k + \kappa = \ell\), the spaces \(S^\delta_{k,\kappa} := \mathcal{O}_{k,\kappa}(\mathcal{M}) \otimes \Gamma(|\Lambda|^\delta)\) provide a gradation of the homogeneous component \(S^\delta_{[\ell]} := \mathcal{O}_{[\ell]}(\mathcal{M}) \otimes \Gamma(|\Lambda|^\delta)\) along the decomposition \((2.1)\) of the latter space. But only the associated increasing filtration by the \(p\)-degree,

\[
S^\delta_{k-\kappa, n} \subset S^\delta_{k-\kappa, n+1} \subset \cdots \subset S^\delta_{[\ell]},
\]

is preserved by the \(g\)-action \(\mathcal{L}^\delta\) (by convention, if an index is not a natural integer then we get the null space). Accordingly, we can define the associated graded \(g\)-module \(\text{gr}S^\delta\), which turns to be isomorphic to \(T^\delta\). The principal tensorial symbol maps are defined by the usual projections, and resorting to \(\text{gr}S^\delta \cong T^\delta\), they read as

\[
\varepsilon_{k,\kappa} : \bigoplus_{j \leq \frac{k(n-k)}{2}} S^\delta_{k-j,\kappa+2j} \rightarrow T^\delta_{k,\kappa},
\]

2.6. The \(g\)-module of spinor differential operators \(D^{\lambda,\mu}\). Concerning Lie derivatives, the case of the spinor bundle is more involved than for natural bundles like \(TM\). Following Kosmann, we choose \(L_X = \nabla_X + \gamma(dX^\delta)/4\) as Lie derivative \([25]\), with \(X^\delta\) the 1-form deduced from \(X\) by the metric \(g\). This formula gives a representation of Lie algebras only if restricted to conformal Killing vector fields \(X \in g\). We introduce \(L_X^\lambda := L_X \otimes \text{Id} + \text{Id} \otimes \ell_X^\delta\), the induced Lie derivative on \(\Gamma(S \otimes |\Lambda|^\lambda)\).
Definition 2.6. We denote by $D^\lambda\mu$ the $g$-module $D(M; S \otimes |\Lambda|^\lambda, S \otimes |\Lambda|^\mu)$ of differential operators. The action of $g$ on one of its element $A$ is given by the following Lie derivative

$$L^\lambda\mu_X A = L^\mu_X A - A L^\lambda_X.$$

The bifiltration on the underlying space of $D^\lambda\mu$ turns to be preserved by the above $g$-action $L^\lambda\mu$. Hence it induces $g$-modules structure on $\text{bigr}D^\lambda\mu$, $\text{gr}D^\lambda\mu$ and on the usual symbol space $\Gamma(STM \otimes |\Lambda|^\mu-) \otimes \Gamma(\text{Cl}(M, g))$. We compare them below.

Proposition 2.7. For $\delta = \mu - \lambda$, we have the following isomorphisms of (bi)graded $g$-modules: $\text{gr}D^\lambda\mu \cong S^\delta$ and $\text{bigr}D^\lambda\mu \cong \text{gr}S^\delta \cong \Gamma^k STM \otimes |\Lambda|^\mu- \otimes \Gamma(\text{Cl}(M, g))$.

2.7. Explicit formulae for actions of $g$. We restrict here to the local model $(\mathbb{R}^{P,q}, F\eta)$ of a conformally flat manifold to provide explicit formulae for $L^\lambda\mu_X$, $L^\mu_X$ and $L^\lambda_X$. The supercotangent bundle admits then natural cartesian coordinates $(x^i, p_i, \xi^i)$ which transform tensorially, $p_i$ identifying to $\partial_i$ and $\xi^i$ to $dx^i$. The potential 1-form $\alpha$ of $M$ is written in terms of such coordinates in equation (2.2). By definition, it reads simply as $\alpha = \beta_i dx^i + \frac{1}{2}\eta_{ij} \xi^i d\xi^j$ in Darboux coordinates $(x^i, \beta_i, \xi^i)$. We deduce that $(\xi^i)$ form an orthonormal frame of $T^*M$ and that $\beta_i = p_i + \frac{h}{2i}\Gamma^i_{\xi^k} \xi^k \xi^i$. In terms of spinor differential operators, these coordinates satisfy $p_i = \sigma_2(\nabla_i)$ and $\gamma(\xi^i) \gamma(\xi^j) + \gamma(\xi^j) \gamma(\xi^i) = -2F^{-1} \eta^{ij}$, whereas $\beta_i = \sigma_2(\partial_i)$ and $\gamma(\xi^i) \gamma(\xi^j) + \gamma(\xi^j) \gamma(\xi^i) = -2\eta^{ij}$. Using both coordinate systems, we introduce a linear isomorphism,

$$\mathcal{F}: T^\delta \rightarrow S^\delta$$

$$P^i_{j_1\ldots j_k}(x) \xi^j_1 \ldots \xi^j_k p_{i_1} \ldots p_{i_k} \rightarrow |\text{vol}_\eta|^\frac{1}{2} P^i_{j_1\ldots j_k}(x) \tilde{\xi}^j_1 \ldots \tilde{\xi}^j_k \tilde{p}_{i_1} \ldots \tilde{p}_{i_k},$$

where $\text{vol}_\eta$ is the canonical volume form $dx^1 \wedge \ldots \wedge dx^n$ attached to the metric $\eta$. Remark that $|\text{vol}_\eta|^{1/n} \xi^i = |\text{vol}_g|^{1/n} \xi^i$, where $\text{vol}_g$ is the volume form associated to the metric $g = F\eta$. In addition, we introduce another linear isomorphism, the normal ordering,

$$\mathcal{N}: S^\delta \rightarrow D^\lambda\mu$$

$$P^i_{j_1\ldots j_k}(x) \tilde{\xi}^j_1 \ldots \tilde{\xi}^j_k \tilde{p}_{i_1} \ldots \tilde{p}_{i_k} \rightarrow P^i_{j_1\ldots j_k}(x) \frac{\gamma^j}{\sqrt{2}} \ldots \frac{\gamma^j}{\sqrt{2}} \partial_i \partial_1 \ldots \partial_i \partial_k,$$

where $\gamma^j$ denotes $\gamma(\xi^j)$ and $\delta = \mu - \lambda$.

Proposition 2.8. The maps $\mathcal{F}$ and $\mathcal{N}$ are isomorphisms of $c(c(p,q))$-modules, and are right inverses of respectively the symbol maps $\varepsilon_{k,\kappa}$ on $T^\delta_{k,\kappa}$, see (2.12), and $(\frac{h}{i})^{\ell/2} \sigma_\ell$ on $S^\delta_{[\ell]}$, see (2.6). Moreover, for any $\xi \in g$, we have

$$\mathcal{F}^{-1} L^\delta_X \mathcal{F} = L^\delta_X - \frac{h}{2i} \xi_k \varepsilon_k^i (\partial_i \partial_j X^k) \partial_{p_i},$$

$$\mathcal{N} \mathcal{F}^{-1} L^\delta_X \mathcal{F} = \mathcal{F}^{-1} L^\delta_X \mathcal{F} + \frac{h}{4i} (\partial_i \partial_k X^j) \left(-2p_i \partial_{p_j} + \chi^j_i \right) \partial_{p_k} - \frac{h}{4} \lambda \partial_j (\partial_i X^i) \partial_{p_j},$$

where $\chi^j_i = \xi^j \partial_{\xi^i} - \xi^i \partial_{\xi^j} + \frac{1}{2} \partial_{\xi^j} \partial_{\xi^i}$. 

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The explicit expressions of the actions $L_{X}^{\delta}$ and $\mathcal{L}_{X}^{\lambda,\mu}$ for all $X \in \mathfrak{g}$, are deduced from the Lie derivatives $L_{X}^{\delta}$, given below for $X$ a generator of $\mathfrak{g}$, cf. (2.9),

\[
\begin{align*}
\mathbb{L}_{X_{i}}^{\delta} &= \partial_{i}, \\
\mathbb{L}_{X_{ij}}^{\delta} &= x_{i}\partial_{j} - x_{j}\partial_{i} - (p_{i}\partial_{p_{j}} - p_{j}\partial_{p_{i}}) + \xi_{i}\partial_{\xi_{j}} - \xi_{j}\partial_{\xi_{i}}, \\
\mathbb{L}_{X_{0}}^{\delta} &= x^{i}\partial_{i} - p_{i}\partial_{p_{i}} + \delta n, \\
\mathbb{L}_{X_{i}}^{\lambda} &= (x_{j}x^{j}\partial_{i} - 2x_{i}x^{j}\partial_{j}) + (-2p_{i}x_{j}\partial_{p_{j}} + 2x_{i}p_{j}\partial_{p_{j}} + 2p_{k}x^{k}\partial_{p_{i}}) \\
&\quad + 2x_{j}\xi^{j}\partial_{\xi_{i}} - 2\xi_{i}x^{j}\partial_{\xi_{j}} - 2n\delta x_{i}.
\end{align*}
\]

2.8. Definitions of the $\mathfrak{g}$-equivariant superization and quantization. The supercotangent bundle $(M, \omega)$ of $(M, g)$ proves to be the phase space for a classical spinning particle on $(M, g)$. The Hamiltonian action of $\mathfrak{g}$ turns its space of functions $\mathcal{O}(M)$ into the $\mathfrak{g}$-module $S^{\delta}$, with $\delta = 0$. It can be interpreted as the space of classical observables for a spinning particle, whereas the space of quantum observables of such a particle is known to be the space of differential operators acting on spinors, or more precisely the $\mathfrak{g}$-module $D^{\lambda,\mu}$ for $\lambda = \mu = \frac{1}{2}$. This justifies the name quantization in the following definition.

**Definition 2.9.** Let $\lambda, \mu \in \mathbb{R}$ and $\delta = \mu - \lambda$. A conformally equivariant quantization is an isomorphism of $\mathfrak{g}$-modules, $Q^{\lambda,\mu} : S^{\delta} \to D^{\lambda,\mu}$, which is a right inverse of the principal Hamiltonian symbol maps defined in (2.6): $\sigma_{\ell} \circ Q^{\lambda,\mu} = (\hbar^{2})^{\ell/2} \text{Id}$ on $S_{[\ell]}^{\delta}$ for all integers $\ell$. Moreover, $Q^{\lambda,\mu}$ preserves the usual filtration by the order of derivations.

Such a quantization extends the geometric quantization of the supercotangent bundle obtained in [31], which is also conformally equivariant but defined only on symbols of Hamiltonian degree at most 2. Moreover, $Q^{\lambda,\lambda}$ gives rise to a deformation quantization of $S^{0}$, the star-product being defined as usual by pull-back of the product on $D^{\lambda,\lambda}$ via $Q^{\lambda,\lambda}$. Notice that Fedosov’s deformation quantization of symplectic supermanifolds has been investigated in [7].

Regarding $T^{\lambda}$, this is a $\mathfrak{g}$-module of geometric significance, defined as a twist by densities of the tensorial $\text{Vect}(M)$-module $\Gamma(STM \otimes \Lambda^{n}T^{*}\mathbb{R}^{n})$. If $\delta = 0$, it contains $\text{Poi}(T^{*}M)$ as a submodule, which can be interpreted as the algebra of classical observables for a (non-spinning) particle on $(M, g)$, so that the following map deserves the name superization.

**Definition 2.10.** A conformally equivariant superization is an isomorphism of $\mathfrak{g}$-modules, $\mathcal{S}^{\delta} : T^{\delta} \to S^{\delta}$, which preserves the Hamiltonian gradation (2.1) and the filtration by the $p$-degree (2.1). Moreover, it is a right inverse of the principal tensorial symbol maps defined in (2.12): $\varepsilon_{k,\kappa} \circ \mathcal{S}^{\delta} = \text{Id}$ on $T_{k,\kappa}^{\delta}$ for all integers $k$ and $\kappa \leq n$.

Let $\mathcal{E} = p_{i}\partial_{p_{i}}$ and $\Sigma = \xi^{i}\partial_{\xi_{i}}$ be the Euler operators of the vector bundles $T^{*}M$ and $T[1]M$. By proposition 2.3 the map $Q^{\lambda,\mu} \circ \mathcal{S}^{\delta} \circ ((\hbar/\iota)^{-\mathcal{E}} \otimes (\sqrt{2})^{\Sigma})^{-1}$ is a $\mathfrak{g}$-module morphism

$\Gamma(STM \otimes |\Lambda^{\delta}|) \otimes \Gamma(\text{End}S) \to D(M; S \otimes |\Lambda^{\lambda}, S \otimes |\Lambda^{\mu}|)$,

which is a right inverse of the usual principal symbol maps. Consequently, it gives a $\mathfrak{g}$-equivariant quantization in the sense of [8].
3. INVARIANT DIFFERENTIAL OPERATORS ON THE SUPERCOTANGENT BUNDLE

In all this section we work over the local model \((\mathbb{R}^{p,q}, F\eta)\) of a conformally flat manifold, so that we get global actions of the Lie algebras \(\mathfrak{c}(p,q) \leq \mathfrak{c}^\ast(p,q) \leq \mathfrak{g}\) on \(T^\delta\). They integrate into actions of the corresponding Lie groups: \(\mathsf{E}(p,q)\) of isometries of the metric \(\eta\), \(\mathsf{CE}(p,q)\) of conformal affine transformations and \(G := \mathsf{O}(p+1, q+1)\) which acts only locally. Up to the isomorphisms \(\mathcal{F}\) and \(\mathcal{N}\), defined in (2.13) and (2.14), conformally equivariant superizations and quantizations are given by \(\mathfrak{c}(p,q)\)-equivariant endomorphisms of the tensorial symbol space \(T^\delta = \Gamma(T^\delta)\). We study here their building blocks.

**Definition 3.1.** We denote by \(\mathcal{D}^{\delta, \delta}(\mathcal{M})\) the space of differential operators on \(\mathcal{M}\), endowed with the natural local \(G\)-action on operators from \(T^\delta\) to \(T^\delta\). For \(H\) a Lie subgroup of \(G\), the subspace of (locally) \(H\)-invariant differential operators is denoted by \(\mathcal{D}^{\delta, \delta}(\mathcal{M})^H\).

As \(G\)-modules, we have \(\mathcal{D}^{\delta, \delta}(\mathcal{M}) \cong \mathcal{D}(\mathbb{R}^n; T^\delta, T^\delta)\). Regarding this identification, we name fiberwise operators those linear operators which act only along the fibers of the tensor bundle \(T\).

3.1. **Euclidean invariants.** The superspace \(\mathbb{R}^{2n|p,q}\), \(m = p + q\), is endowed with a canonical Poisson bivector \(\Pi = \sum_{a=1}^n \partial_a \wedge \partial_{a+n} + \sum_{a=2n+1}^{2n+p} \partial_a \wedge \partial_a - \sum_{a=2n+p+1}^{2n+m} \partial_a \wedge \partial_a\) in a cartesian coordinate system \((x^a)_{a=1,\ldots,2n+m}\). The (super) Heisenberg Lie algebra \(\mathfrak{h}(2n|p,q)\) and the orthosymplectic Lie algebra \(\mathfrak{spo}(2n|p,q)\) are defined as Lie subalgebras of the Poisson algebra of polynomial functions over \(\mathbb{R}^{2n|p,q}\). Namely, the first one is given by the constant and linear functions, and the second one by the quadratic one. For \(m = 0\), we recover the usual definitions of the Heisenberg and symplectic Lie algebras, and \(\mathfrak{spo}(0|p,q)\) identifies to the orthogonal Lie algebra \(\mathfrak{o}(p,q)\).

Resorting to Weyl theory of invariants [42], we get below a slight generalization of the Howe duality between the Lie algebras \(\mathfrak{o}(p,q)\) and \(\mathfrak{spo}(2|1, 1)\) in \(\mathfrak{spo}(2|n, n)\), \(n = p + q\). See e.g. [24, 28] for more informations on the latter dual pair. As before, we denote by \((x^i, p_i, \xi^i)\) the natural cartesian coordinates on \(\mathcal{M}\) and by \((\partial_i, \partial_{p_i}, \partial_{\xi_i})\) the associated derivatives.

**Proposition 3.2.** Let \(\mathcal{E} = p_i \partial_{p_i}, \Sigma = \xi^i \partial_{\xi_i}\), and \(\delta \in \mathbb{R}\). The subalgebra of isometry invariant differential operators on \(\mathcal{M}\) satisfies \(\mathcal{D}^{\delta, \delta}(\mathcal{M})^{\mathcal{E}(p,q)} \cong \mathfrak{U}(\mathfrak{spo}(2|1, 1) \ltimes \mathfrak{h}(2|1, 1))\) for \(n \geq 3\). For all \(n\), it is generated by the fiberwise operators

\[
R = \eta^{ij} p_i p_j, \quad E = \mathcal{E} + \frac{\mathcal{F}}{2}, \quad T = \eta_{ij} \partial_{p_i} \partial_{p_j}, \quad \Sigma = \Sigma - \frac{\mathcal{F}}{2},
\]

\[
Q = \xi^i p_i, \quad \delta = \eta_{ij} \xi^i \partial_{p_j}, \quad \delta^* = \eta^{ij} p_i \partial_{\xi_j}, \quad Q^* = \partial_{\xi_i} \partial_{p_i},
\]

generating the Lie algebra \(\mathfrak{spo}(2|1, 1)\), and by the differential operators

\[
G = \eta^{ij} p_i \partial_j, \quad D = \partial_i \partial_{p_i}, \quad L = \eta^{ij} \partial_i \partial_j,
\]

\[
d = \xi^i \partial_i, \quad d^* = \eta^{ij} \partial_{\xi_i} \partial_{\xi_j},
\]

generating the Lie algebra \(\mathfrak{h}(2|1, 1)\).
Proof. We resort to Weyl quantization of $T^*\mathcal{M}$ which is $\text{spo}(4n|n,n)\ltimes\mathfrak{h}(4n|2p,2q)$-equivariant. It establishes a correspondence between the $\mathfrak{e}(p,q)$-invariants of $\mathcal{D}^{\delta,\delta}((\mathcal{M})$ and of $\text{Pol}(T^*\mathcal{M})$, which admits $(x^i,\hat{x}_i,p_i,\hat{p}_i,\xi^i,\hat{\xi}_i)$ as generators. Regarding the action of $\mathfrak{e}(p,q)$ on $\mathcal{M}$, given by (2.17), the $\mathfrak{e}(p,q)$-invariants of $\text{Pol}(T^*\mathcal{M})$ reduce to the $\mathfrak{o}(p,q)$-invariant polynomials on the space $(\mathbb{R}^n)^* \times (\mathbb{R}^n)^* \times \mathbb{R}^n \times (\mathbb{H}^n)^* \times (\mathbb{H}^n)^*$ with coordinates $(\hat{x}_i,p_i,\hat{p}_i,\xi^i,\hat{\xi}_i)$. By the Weyl theorem [42, theorem 2.9.A, p.53] these invariants split into even and odd invariants: the even ones are generated by the 15 scalar products between these five types of coordinates, and the odd ones are constructed from the determinant and are not invariant under the group $\mathfrak{o}(p,q)$. So we are left with the even invariants. The two squares of odd variables $\xi$ and $\hat{\xi}$ vanish and only 13 non-vanishing scalar products remain. The 8 ones without the variable $\hat{x}$ can be identified to a basis of the quadratic polynomials on $T^*\mathbb{R}^{1|1}$, while the five other ones can be identified to a basis of those of degrees 0 and 1. They correspond by Weyl quantization to the 8 operators (3.1) and the 5 operators (3.2) respectively. By equivariance property of Weyl quantization we get that the operators (3.1) generate the Lie algebra $\mathfrak{spo}(2|1,1)$, the operators (3.2) generate the Lie algebra $\mathfrak{h}(2|1,1)$, and together they generate the algebra of even $\mathfrak{e}(p,q)$-invariants, or equivalently of $\mathfrak{e}(p,q)$-invariants, in $\mathcal{D}^{\delta,\delta}((\mathcal{M})$.

Assuming $n \geq 3$, we get $d(\hat{x}^i\hat{x}_i)\wedge d(\hat{x}^p\hat{p}_i)\wedge\ldots\wedge d(\xi^i\hat{\xi}_i) \neq 0$. Thus the 13 obtained operators are algebraically independent and $\mathcal{D}^{\delta,\delta}((\mathcal{M})|\mathfrak{E}(p,q))$ is then isomorphic to the enveloping algebra $\mathfrak{U}(\mathfrak{spo}(2|1,1) \ltimes \mathfrak{h}(2|1,1))$. $\square$

All of the 13 operators in (3.1) and (3.2) identify to well-known operators on tensors. The three operators $R, \mathcal{E}, T$, restricted to $\Gamma(ST\mathbb{R}^n)$, correspond to the metric, the Euler operator and the trace. Together with $G, D, L$, which generalize the gradient, the divergence and the Laplacian, they have been introduced in [12] [13] as building blocks of the conformally equivariant quantization of cotangent bundles. The operator $\Sigma$ is the Euler operator on $\Gamma(\Lambda T^*\mathbb{R}^n)$ and the remaining operators square to zero and identify to well-known (co-)differentials. Indeed, $d, d^*$ are the usual de Rham (co-)differentials on $\Gamma(\Lambda T^*\mathbb{R}^n)$ and $\delta, \delta^*$ are Koszul type (co-)differentials on the whole space $\Gamma(ST\mathbb{R}^n \otimes \Lambda T^*\mathbb{R}^n)$. As for $Q, Q^*$, they are (co-)differentials introduced by Manin to give a cohomological description of the Berezinian [29]. Notice that $Q$ is also a function on $\mathcal{M}$: this is the principal symbol of the Dirac operator. The notation $Q$ is borrowed from physics where it plays the role of a supercharge [19]. Let us mention that similar algebras of invariant operators on tensor (and spinors) have been investigated in [20] [21] [22], over a background $(\mathcal{M}, g)$ of constant curvature.

The bold operators with a $*$-exponent lower the degree in $\xi$ by one, while the other ones rise it by one. The next proposition shows that, indeed, $\delta^*, Q^*$ are the codifferentials of $\delta, Q$.

**Proposition 3.3.** On $T^\delta$, the operators $\frac{1}{\delta+\Sigma}\delta\delta^*, \frac{1}{\delta+\Sigma}\delta^*\delta$ and $\frac{1}{n+\Sigma-QQ^*}Q^*Q, \frac{1}{n+\Sigma-QQ^*}QQ^*$ are two pairs of complementary projections, which lead to the decompositions

$$T^\delta = \ker\delta \oplus \ker\delta^* \quad \text{and} \quad T^\delta = \ker Q \oplus \ker Q^*.$$
Moreover, we have \( \ker \delta = \im \delta \), \( \ker \delta^* = \im \delta^* = \delta^* \ker \delta \) and \( \ker Q = \im Q \), \( \ker Q^* = \im Q^* = Q^* \ker Q \).

Proof. A straightforward computation leads to \( [\delta^*, \delta] = E + \Sigma \), hence \( \frac{1}{\varepsilon + \Sigma} \delta^* \delta \) is a projection whose complementary projection is \( \frac{1}{\varepsilon + \Sigma} \delta^* \delta \). Hence, we have \( \ker \frac{1}{\varepsilon + \Sigma} \delta^* \delta = \im \frac{1}{\varepsilon + \Sigma} \delta^* \delta \) and a complementary space in \( T^\delta \) is provided by \( \im \frac{1}{\varepsilon + \Sigma} \delta^* \delta = \ker \frac{1}{\varepsilon + \Sigma} \delta^* \delta \).

We compute now the latter four spaces. Using the above commutation relation, we first get that \( \delta^* \delta^* = (E + \Sigma) \delta^* \). The equalities of their kernels and of their images imply that \( \ker \frac{1}{\varepsilon + \Sigma} \delta^* \delta = \ker \delta^* \) and \( \im \frac{1}{\varepsilon + \Sigma} \delta^* \delta = \im \delta^* \). Similarly, we have \( \delta \delta^* \delta = (E + \Sigma) \delta \), which leads to \( \im \frac{1}{\varepsilon + \Sigma} \delta \delta^* \delta = \im \delta \) and \( \ker \frac{1}{\varepsilon + \Sigma} \delta \delta^* \delta = \ker \delta \). Combining all these equalities, we deduce the results concerning \( \delta, \delta^* \) as well as the announced decomposition of \( T^\delta \). The case of \( Q, Q^* \) is analogous. □

3.2. Conformal invariants. The \( \epsilon(p, q) \) action on \( T^\delta \) is independent of \( \delta \), hence the subspace of \( \epsilon(p, q) \)-invariants in \( D^\delta,\delta (\mathcal{M}) \) identifies to the previously determined one in \( D^\delta,\delta (\mathcal{M}) \). In contradistinction, the action of the dilation vector field depends on the shift of weight \( \delta' - \delta \). It turns out that all the \( \epsilon(p, q) \)-invariant generators are also \( \alpha(p, q) \)-invariants for the shift indicated in (3.3). Moreover, the fiberwise operators are automatically invariant under conformal inversions. The situation for the 13 generators of \( D^\delta,\delta (\mathcal{M})^{\epsilon(p,q)} \) is sum up in the following table,

| \( \alpha(p, q) \)-invariant operators | \( \beta \) | \( \delta \) | \( Q \) | \( E, D, \Sigma \) | \( Q, \delta^*, d, d' \) | \( R, G, L \) |
| \( g \)-invariant operators | \( T \) | \( \delta \) | \( Q \) | \( E, \Sigma \) | \( Q, \delta^* \) | \( R \) |

Restricting now to \( \delta = \delta' \), we deduce that any \( CE(p, q) \)-invariant operator is obtained from arbitrary powers of the operators \( E, D, \Sigma \) and monomials of the form

\[
R^r Q^q (\delta^*)^\beta d^\gamma (d^*)^\gamma' G^\alpha L^1 \delta \delta^* (Q^*)^\alpha' T^t,
\]

where \( 2r + 2q + 2l + \alpha + \beta + \gamma + \gamma' - \delta' - \alpha' - 2t = 0 \) and the exponents of odd operators are equal to 0 or 1, since they have null square. From the table (3.3) we deduce that the fiberwise \( G \)-invariant operators are of the same form but do not involve any of the 5 operators in (3.2). In fact, they exhaust all the \( G \)-invariant differential operators on \( \mathcal{M} \). The proof is similar to the one concerning differential operators acting on contravariant symmetric tensors \( \Gamma(ST \mathbb{R}^n) \), provided in [11] Lemma 3.3. Therefore, we get the following result.

**Proposition 3.4.** The algebra of conformal invariants \( D^\delta,\delta (\mathcal{M})^G \) is generated by

\[
\mathcal{E}, \Sigma, RT, QQ^*, \delta^* \delta, Q \delta, \delta^* Q^*.
\]

Moreover, \( \mathcal{E} \) is in the center and \( \mathcal{E}, \Sigma, RT, QQ^* + \delta^* \delta \) generate an abelian subalgebra.

Proof. As conformal invariants operators are all fiberwise, they coincide with the \( CE(p, q) \)-invariant fiberwise operators. We deduce from equation (3.4) that they are generated by \( \mathcal{E}, \Sigma \) and \( R^r Q^q (\delta^*)^\beta \delta \delta^* (Q^*)^\alpha' T^t \) for \( 2r + \alpha + \beta - \beta' - \alpha' - 2t = 0 \). Since the exponents of
odd operators are equal to 0 or 1, we obtain the announced generators in (3.5), plus $R\delta Q^*$ and $Q\delta^* T$.

We trivially check that $\mathcal{E}$ is in the center and that $\Sigma$ commute to $RT$, $QQ^*$ and $\delta^* \delta$. A direct computation shows that $\frac{1}{2}[RT, QQ^*] = -\frac{1}{2}[RT, \delta^* \delta] = [QQ^*, \delta^* \delta] = R\delta Q^* - Q\delta^* T$. Therefore, $\mathcal{E}, \Sigma, RT, QQ^*, \delta^* \delta$ generate a commutative algebra. Moreover, the sum of the three commutators of $[RT, QQ^*]$ with $RT$, $QQ^*$, $\delta^* \delta$, leads to $R\delta Q^* + Q\delta^* T$, hence the seven operators given in (3.5) generate indeed $D^{b, \delta}(\mathcal{M})^G$.

3.3. **Three Casimir operators.** Given a representation $\rho : \mathfrak{g} \to \text{End}(V)$, we remind that the Killing form $B$ of $\mathfrak{g}$ induces a particular $\mathfrak{g}$-invariant operator on $V$, called the Casimir operator. It is defined by $C_\rho = B^\alpha \beta \rho(X_\alpha)\rho(X_\beta)$, where $B^\alpha \beta$ is the inverse of the Gram matrix of the Killing form in the basis $(X_\alpha)$ of $\mathfrak{g}$. Hence, for each of the three representations of $\mathfrak{g}$ on $T^\delta$, $S^\delta$ and $D^{\lambda, \mu}$, we get a Casimir operator. We can pull back the ones of $S^\delta$ and $D^{\lambda, \mu}$ to $T^\delta$, via the isomorphisms $\mathcal{F}$ and $\mathcal{N}$ defined in (2.13) and (2.14) respectively. We get then $\mathcal{E}(\rho, q)$-invariant operators on $T^\delta$, which can be written in terms of the generators listed in (3.1) and (3.2). The Casimir operator of $T^\delta$ being $\mathfrak{g}$-invariant it can more specifically be written in terms of the operators in (3.5).

**Proposition 3.5.** Let $\delta = \mu - \lambda$. The three $\mathfrak{g}$-modules $T^\delta$, $S^\delta$ and $D^{\lambda, \mu}$ admit Casimir operators. After pull-back by the local isomorphisms $\mathcal{F}$ and $\mathcal{N}$, they read on $T^\delta$ respectively as,

\begin{align}
C_T &= \hat{C} + \Sigma (\Sigma - n) + 2(QQ^* + \delta^* \delta) - 2\mathcal{E}, \\
C_S &= C_T + 2\hbar \frac{d}{d\delta}, \\
C_D &= C_T + \hbar \left( GT - 2 \left( \mathcal{E} + n\lambda + \frac{1}{2} \right) D + 2d\delta + d^* \delta + dQ^* + \frac{1}{2} d^* Q^* \right),
\end{align}

where $\hat{C} = RT + [n(\delta - 1) - \mathcal{E}] \mathcal{E} - n^2 \delta(\delta - 1)$ is the Casimir operator of the $\mathfrak{g}$-module $\Gamma(\text{STR}^n \otimes |\Lambda|^\delta)$, obtained in [12].

**Proof.** To compute this three Casimir operators, we use the basis of $\mathfrak{g}$ introduce in (2.9) and the computation of the Gram matrix of the Killing form performed in [12], which leads to

$$C_\rho = \frac{1}{2} \eta^{ik} \eta^{jl} \rho(X_{ij}) \rho(X_{kl}) - \rho(X_0)^2 - \frac{1}{2} \eta^{ij} \rho(X_i) \rho(X_j) - \frac{1}{2} \eta^{ij} \rho(X_i) \rho(X_j),$$

for any representation $\rho$ of $\mathfrak{g}$. Replacing $\rho$ by successively the three representations $L^\delta$, $L^\delta$ and $L^{\lambda, \mu}$ and resorting to the given explicit expressions of the actions of the generators (2.9) of $\mathfrak{g}$ leads to the result, after straightforward but rather lengthy computations.

4. **Main results**

We work on a spin manifold $(M, g)$ of dimension $n$ and signature $(p, q)$, that is assumed to be conformally flat except in section 4.1.
4.1. Irreducible decomposition of the tensorial symbol bundle. Since the \(O(p,q)\)-invariant operators introduced in (3.1) are fiberwise, they generalize to arbitrary pseudo-Riemannian manifold \((M, g)\), up to replacing the metric \(\eta\) by \(g\). Their commutation relations, given in (A.4), remain the same.

In the case of usual differential operators, the symbols are sections of the vector bundle \(STM\). The decomposition of the latter into \(O(p,q)\)-homogeneous irreducible bundles stems from the harmonic decomposition of its fibers, isomorphic to \(S\mathbb{R}^n\). It coincides with the decomposition of \(S\mathbb{R}^n\) into joint eigenspaces of the \(O(p,q)\)-invariant operators \(E\) and \(RT\). For weighted spinor differential operators, the space of tensorial symbols is \(T^\delta = \Gamma(T^\delta)\), cf. definition 2.3. The decomposition of \(T^\delta\) into \(O(p,q)\)-homogeneous irreducible bundles relies on the one of its fibers, \(S\mathbb{R}^n \otimes \Lambda(R^n)^*\), into irreducible representations of \(O(p,q)\). This generalization of harmonic decomposition has been carried out in [23]. We recover it here independently as the decomposition into joint eigenspaces of the commuting \(O(p,q)\)-invariant operators \(E, \Sigma, RT, QQ^* + \delta^* \delta\), obtained in proposition 3.4. To take into account the weight \(\delta\), we rather regard these bundles as \(CO(p,q)\)-homogeneous ones. Resorting to previous notation, we introduce the projection \(\Pi_0 : \ker T^2 \to \ker T\), explicitly given by \(\text{Id} - \frac{1}{4n+2\epsilon}RT\), and the operator \(Q_0 = \Pi_0 \circ Q\).

**Theorem 4.1.** The vector bundle \(T^\delta\) admits an irreducible decomposition into \(CO(p,q)\)-homogeneous bundles, which correspond in terms of modules of sections to the decomposition

\[
T^\delta = \bigoplus_{k \in \mathbb{N}, \kappa \leq n, s \leq [k/2]} \bigoplus_{\alpha, \beta \in \{0,1\}} T^\delta_{k,\kappa,s;\alpha\beta},
\]

where

\[
T^\delta_{k,\kappa,s;\alpha\beta} = R^\epsilon(Q_0)^\alpha(\delta^*)^\beta \left( T^\delta_{k-2s,\kappa-\alpha+\beta} \cap (\ker T \cap \ker \delta \cap \ker Q^*) \right).
\]

Those modules are \(g\)-stable and form eigenspaces of the Casimir operator \(C_T\) with eigenvalues

\[
\gamma_{k,\kappa,s;ab} = \gamma_{k,s} + \kappa(n - 2) + 2(\alpha + \beta - 1)(k - 2s) + 2(\beta - \alpha)\kappa + 2\alpha(n - 2\beta),
\]

where \(\gamma_k = 2s[n + 2(k - s - 1)] + 2k[1 + n(\delta - 1) - k] - n^2\delta(\delta - 1)\) is the eigenvalue of the Casimir operator \(\hat{C}\), see (3.6). The multiplicity in the decomposition (4.1) is at most two, the only isomorphic modules occurring via

\[
T^\delta_{k,\kappa,s;10} \xrightarrow{\delta^* Q} T^\delta_{k-2s,\kappa-101}, \quad \text{and} \quad T^\delta_{k,\kappa,s;00} \xrightarrow{Q_0 \delta^* T} T^\delta_{k,\kappa,s-1;11},
\]

for all \(k,\kappa, s\) such that source and target spaces are well-defined.

**Proof.** We first obtain the decomposition (4.1) as the decomposition of \(T^\delta\) into joint eigenspaces for the four commuting operators \(E, \Sigma, RT, QQ^* + \delta^* \delta\). The first three ones lead to a straightforward generalization of the usual harmonic decomposition,

\[
T^\delta = \bigoplus_{k,\kappa,s} R^\epsilon \left( T^\delta_{k,\kappa} \cap \ker T \right).
\]
It remains then to decompose the space $\ker T$ into eigenspaces of $QQ^* + \delta^* \delta$. This is tricky since $RT, \delta^* \delta$ and $QQ^*$ do not commute. The proposition 3.3 states that $\frac{1}{\Sigma - \Sigma} \delta \delta^*$ is a projection on $\ker \delta$ along $\delta^* \ker \delta$. Since $[T, \delta \delta^*] = 2\delta Q^*$, the latter projection preserves the space $\ker Q^* \cap \ker T$, but not $\ker T$. The decomposition (4.1) relies then on the equality $\ker T = \bigoplus_{\alpha = 0,1} (Q_0)_{\alpha} \ker T \cap \ker Q^*$. To get it, we use the identity $\ker Q^* \cap \ker T = Q^* \ker T$, deduced from proposition 3.3 and we prove that there exists an operator $A$ such that following exact sequence is split

$$0 \longrightarrow \ker Q^* \cap \ker T \longrightarrow \ker T \longrightarrow \frac{AQ^*}{Q_0} \longrightarrow Q^* \ker T \longrightarrow 0.$$  

The table (A.5) leads to the equality $Q^* Q_0 = (n + \Sigma - \Sigma) - \frac{1}{(n + 2)(\Sigma - 1)} \delta \delta^*$ on $Q^* \ker T$, so that it defines an automorphism of this space. In consequence, this operator admits an inverse $A$, satisfying $AQ^* Q_0 = \text{Id}$ on $Q^* \ker T$. This proves (4.1).

Straightforward computations lead to the eigenvalues of the Casimir operator $C_T$ on the spaces $T_{k,\kappa,s;\alpha \beta}^\delta$. Restricted to such a space, the eight generators of the fiberwise $O(p,q)$-invariant operators, defined in (B.1), are either null or have zero kernel. Since $O(p,q)$ is semi-simple and $T_{k,\kappa,s;\alpha \beta}^\delta$ is a finite dimensional $C^\infty(M)$-module, the latter is the module of sections of an $O(p,q)$-homogeneous irreducible bundle on $M$. Clearly, this is also a $CO(p,q)$-homogeneous bundle. As the modules $T_{k,\kappa,s;\alpha \beta}^\delta$ are joint eigenspaces for the four operators $\Sigma, RT, QQ^* + \delta^* \delta$, the only isomorphisms between them come from the remaining fiberwise $CO(p,q)$-invariant operators listed in Proposition 3.4. The isomorphisms (4.4) follow. □

Notice that the modules of zero degree in the odd variables are of the form $T_{k,0,s;01}^\delta$. The eigenvalue of the Casimir operator $C_T$ on them reduces as expected to the one of $\hat{C}$.

### 4.2. Classification of conformally invariant operators on $T^\delta$

In paragraph 3.2, we have determined all the conformal invariant differential operators acting on $T^\delta$, over the local model $(\mathbb{R}^{p,q}, F_\eta)$. We turn now to those which are not necessarily differential. It is proved in [27] that an operator $\Gamma(S^k T^p) \rightarrow \Gamma(S^l T^p)$, with $k, l \in \mathbb{N}$, is local if it is invariant under translations and dilation. This result extends straightforwardly to operators acting from $T^\delta$ to $T^{\delta'}$, meaning that $\text{ce}(p,q)$-invariant operators are differential if restricted to functions of bounded degree in $p$. However, they are not necessarily differential on the whole space $T^\delta$, like the operator $(\Sigma + 1)^{-1}$. This does not impact the proof of proposition 3.2 and such operators are generated by the $\text{ce}(p,q)$-invariant operators given in (3.1) and (3.2). To determine those which are $g$-invariant is more tricky, there exist indeed non fiberwise $g$-invariant operators.
To classify them, we restrict ourselves to irreducible homogeneous bundles. According to the previous paragraph, we have the following commutative diagram of $g$-modules

\[
\begin{array}{ccc}
T^\delta_{k,\kappa,\alpha,\beta} & \longrightarrow & T^\delta'_{k',\kappa',\alpha',\beta'} \\
\delta^\alpha(Q^\gamma)^\beta_{T^\delta} & | & R^\alpha(Q_0)^\gamma(\delta')^\beta' \\
T^\delta_{k_0,\alpha_0,0,00} & \longrightarrow & T^\delta'_{k_0',\alpha_0',0,00}
\end{array}
\]

for well-chosen indices on the bottom part. The vertical arrows being isomorphisms, the general classification of conformally invariant operators acting on $T^\delta$ boils down to the one on its irreducible pieces of the form $T^\delta_{k,\kappa,0;00}$. They are given by certain linear combinations of $\epsilon(p,q)$-invariant operators of the type

\[
d_0^\ell(d_0^\gamma)G_0^\ell L_0^\ell D_0^\ell,
\]

where the indexes $0$ and $0$ denote respectively the restriction and corestriction to $\ker T$ and to $\ker T \cap \ker Q' \cap \ker \delta$. We determine all such $g$-invariant operators below. Using their equivalent description as morphisms of generalized Verma modules, their classification can also be derived from the general statements in [5, 6].

**Theorem 4.2.** Let $k, k'$ be non-negative integers, $0 \leq \kappa, \kappa' \leq n$, $\delta, \delta' \in \mathbb{R}$, and let us define $j = n(\delta' - \delta)$. The space of conformally invariant operators $D(T^\delta_{k,\kappa,0;00}, T^\delta'_{k',\kappa',0;00})^G$ over $(\mathbb{R}^{p,q}, \eta)$ is either trivial or of dimension $1$, generated by

- $D_0^\delta$ if $k' - k = -d$, $\kappa' = \kappa$, $j = 0$ and $\delta = 1 + \frac{2k-d}{n}$,
- $G_0^\ell$ if $k' - k = g$, $\kappa' = \kappa$, $j = 2g$ and $\delta = -\frac{g}{n}$,
- $L_\ell$ if $k' = k$, $\kappa' = \kappa$, $j = 2\ell$ and $\delta = \frac{1}{2} + \frac{k}{n}$,
- $d_0$ if $k' = k$, $\kappa' = \kappa$, $j = 1$ and $\delta = 1 + \frac{k}{n}$,
- $d_0^\gamma$ if $k' = k$, $\kappa' = \kappa$, $j = 0$ and $\delta = 1 + \frac{k}{n}$,

where the operator $L_\ell$ is of the form $\sum_{\gamma=0,1} a_{\varepsilon,j}d_0^\gamma(d_0^\varepsilon)\sum_{j=0}^{\ell} G_0^\ell L_0^\ell D_0^\ell$ for certain coefficients $a_{\varepsilon,j} \in \mathbb{R}$. If $n$ is even, we get also

- $d_0L_\ell$ if $k = n/2 + \ell$, $k' = k$, $\kappa' = \kappa$, $j = 2\ell + 1$ and $\delta = \frac{1}{2} + \frac{k}{n}$,
- $d_0^\gamma L_\ell$ if $k = n/2 - \ell$, $k' = k$, $\kappa' = \kappa$, $j = 2\ell + 1$ and $\delta = \frac{1}{2} + \frac{k}{n}$.

**Proof.** We follow the proof given by the author in [3], in the case of the cotangent bundle. Let $A \in D(T^\delta_{k,\kappa,0;00}, T^\delta'_{k',\kappa',0;00})^G$. According to (4.7), it is of the form $A = d_0^\gamma(d_0^\varepsilon)G_0^\ell L_0^\ell D_0^\ell$ for a certain operator $L_\ell$ satisfying $L_\ell = a_{\varepsilon,j}d_0^\gamma(d_0^\varepsilon)G_0^\ell L_0^\ell D_0^\ell$, where $g, d, \ell$ are integers, $\gamma, \gamma' = 0, 1$, and $a_{\varepsilon,j}$ are reals indexed by $\varepsilon = 0, 1$ and $j = 0, \ldots, \ell$. Moreover, we can ask for $a_{0,0} \neq 0$ so that $\ell$ is optimal.

We reduce now further the expression of $A$ by its invariance under conformal inversions. The action of $X_i$ on the powers of the five operators entering into $A$ is computed in the
appendix, it follows that

\[ [A, L^*_X] \in E_D \partial_{\rho_i} \oplus \Pi_0 \partial_{\xi_i} E_G \oplus E_{d^*} \Pi_0 \partial_{\xi_i} \oplus \Pi_0 \xi_i E_d \oplus \partial_i E_L, \]

where \( \Pi_0 \) is the projection onto the space \( \ker T \cap \ker Q^* \cap \ker \delta \), and \( E_G, E_D, E_L, E_d, E_{d^*} \) are vector spaces generated by the five operators \( d_0, d^*_0, G_0, D_0, L_0 \). The independence of the monomials in \( A \) with different powers of \( L_0 \) together with the vanishing of the five components of \( [A, L^*_X] \) lead then to the result. E.g., the vanishing of the component in \( E_D \partial_{\rho_i} \) of the higher degree term in \( L_0 \) of \( [A, L^*_X] \) reads as

\[ [D^0_\ell, L^*_X] = 0, \]

and from the relations \((A.3)\) we deduce that necessarily \( \delta = 1 + \frac{2k - d - 1}{n} \) if \( d \neq 0 \). Along the same reasoning we get that: \( G^0_0, d^*_0, \) and \( (d^*_0)^\gamma \) are \( g \)-invariant. Therefore, we have \( \delta + \frac{2}{n} = \frac{1-g}{n} \) if \( g \neq 0 \), \( \delta + \frac{2(k+g)}{n} = 1 + \frac{k-n}{n} \) if \( \gamma = 1 \) and \( \delta + \frac{2(k+g)+\gamma}{n} = \frac{k+n}{n} \) if \( \gamma' = 1 \). As the operator \( A \) is \( g \)-invariant, the same holds for \( L_\ell \). Since the component in \( \partial_i E_L \) of the higher degree term in \( L_0 \) of \( [L_\ell, L^*_X] \) vanishes, equations \((A.3)\) lead us to \( \delta = \frac{1}{2} + \frac{k-n}{n} \) if \( \ell \neq 0 \). Then, straightforward but lengthy computations show that there exist unique reals \( a_{\varepsilon,j} \) such that \( L_\ell \) is conformally invariant.

If \( n \) is odd, the five values found for \( \delta \) are incompatible two by two, so only one of the five exponents can be non zero. If \( n \) is even, the values of \( \delta \) for two non-vanishing exponents among \( \gamma, \gamma', \ell \) can be compatible for constrained value of \( \kappa \). The result follows. \( \square \)

The theorem being of local nature and depending only of the conformal class of the metric, it extends straightforwardly to conformally flat manifolds \((M, g)\). The local expressions of the \( g \)-invariant operators remain the same if we choose a conformal coordinate system, but to obtain their covariant formulæ is a non trivial task.

4.3. Existence and uniqueness of the \( g \)-equivariant quantization and superization. In the seminal paper \([12]\), existence and uniqueness of the conformally equivariant quantization of cotangent bundles was proved, resorting to diagonalization of the Casimir operators of the modules of differential operators and of their symbols. Thanks to theorem \([1.1]\) we can apply the same method to prove existence and uniqueness of equivariant superization and quantization of supercotangent bundles, introduced in definitions \([2.9] \) and \([2.10] \). This is our main result.

**Theorem 4.3.** Let \((M, g)\) be a conformally flat manifold. There exist two sets \( I^{\delta}, I^{\mu} \subset \mathbb{Q} \) such that:

1. conformally equivariant superization \( G^\delta : T^\delta \to S^\delta \), exists and is unique if \( \delta \notin I^{\delta} \),
2. conformally equivariant quantization \( Q^{\lambda,\mu} : S^\delta \to D^{\lambda,\mu} \), exists and is unique if \( \delta \notin I^{\mu} \).

Moreover, if \( \lambda + \mu = 1 \), the quantization intertwines conjugation \( (2.3) \) and adjoint operation \( (2.8) \), namely \( Q^{\lambda,\mu}(\cdot) = Q^{\lambda,\mu}(\cdot)^t \).
Proof. The two results can be proved in the same way, following \[12\]. We focus on the second one, as we will provide a stronger statement for superization in theorem 5.1.

Existence and uniqueness of $Q^{\lambda,\mu}$ is a result of local nature so that we can work over the local conformal model $(\mathbb{R}^{p,q}, F\eta)$, with $F$ a positive function. We denote by $\widetilde{C}_S = F \circ C_S \circ F^{-1}$ the Casimir operator of $S^\delta$ and by $\widetilde{C}_D = N \circ F \circ C_D \circ F^{-1} \circ N^{-1}$ the one of $D^{\lambda,\mu}$. They are computed in proposition 3.5. If a conformally equivariant quantization $Q^{\lambda,\mu}$ exists then it intertwines the two former Casimir operators, i.e. $Q^{\lambda,\mu} \circ \widetilde{C}_S = \widetilde{C}_D \circ Q^{\lambda,\mu}$. As a consequence, each eigenvector of $\widetilde{C}_S$ is mapped by $Q^{\lambda,\mu}$ to an eigenvector of $\widetilde{C}_D$ of same eigenvalue and same principal symbol.

**Lemma 4.4.** Let $k$ be a positive integer and $N$ be an operator on $T^\delta$ which lowers by one the degree in $p$. For $\delta \in \mathbb{R}$ out of a finite subset of $\mathbb{Q}$, to each eigenvector $P_k \in T^\delta_{k,k+s,0,0}$ of $C_T$ corresponds a unique eigenvector of $C_T + N$ of the form $P_k + P$ with $P$ of degree less than $k$.

**Proof.** We use the notation of the lemma’s statement. By hypothesis, we have the equality $(C_T + N)(P_k + P) = \gamma(P_k + P)$ for $\gamma$ a real number. This implies that $\gamma = \gamma_{k,k,s,0,0}$, the eigenvalue of $C_T$ on $T^\delta_{k,k,s,0,0}$, and $P = P_{k-1} + \ldots + P_0$ with $P_l$ of degree $l$ in $p$, satisfying $NP_l = (\gamma_{k,k,s,0,0} - C_T)P_{l-1}$. Then, $P$ is uniquely determined by $P_k$, as soon as $\gamma_{k,k,s,0,0}$ is distinct of the eigenvalues of $C_T$ on $\bigoplus_{0 \leq j \leq k-1} T^\delta_j$. In view of their expressions (4.3), this is true except for a finite number of rational values of $\delta$.

The lemma can be applied to the Casimir operators $C_S$ and $C_D$. Starting with an eigenvector $P_k$ of $C_T$ we get unique eigenvectors $P^S$ and $P^D$ of $C_S$ and $C_D$. In consequence, a conformally equivariant quantization should satisfy $Q^{\lambda,\mu} : F(P^S) \mapsto N \circ F(P^D)$. Since the Casimir operator $C_T$ is diagonalizable on the natural symbol space $T^\delta$, this specifies a unique quantization map, which turns out to be a $g$-equivariant quantization.

Let $\lambda + \mu = 1$. As the linear map $N$ intertwines conjugation and the adjoint operation, the map $P \mapsto Q^{\lambda,\mu}(P)\ f$ is a right inverse to the principal symbol map $\sigma$ on homogeneous symbols. Moreover, we easily check that the actions of $g$ on $S^\delta$ and $D^{\lambda,\mu}$ commute with the conjugation when they are pulled-back to $T^\delta$. Consequently, the map defined above is a $g$-equivariant quantization and by uniqueness it is equal to $Q^{\lambda,\mu}$. We deduce that $Q^{\lambda,\mu}$ intertwines conjugation and adjoint operation.

This theorem extends to the supercotangent bundle the main result of [12], giving existence and uniqueness of the conformally equivariant quantization on the cotangent bundle. If $\delta \notin F^\delta \cup F^\Omega$, we can compose the two previous maps and get existence and uniqueness of a $g$-equivariant map $T^\delta \mapsto D^{\lambda,\mu}$ which is a right inverse of the usual symbol map. As explained in Paragraph 2.8 this gives a particular instance of the general result in [8] about AHS-equivariant quantization.

The significance of the subsets of exceptional values of the weight $\delta$ in the context of equivariant quantization has been revealed in [34], following previous results in [39].
correspond to the existence of $g$-invariant operators on the initial module, which are not fiberwise. The proof in [34] generalizes straightforwardly to this context.

**Theorem 4.5.** [34] The $g$-equivariant superization exists and is unique on $T^\delta_{k,k,s;\alpha\beta}$ if and only if there is no non-trivial $g$-invariant operator from this module to $\bigoplus_{1 \leq \ell < k} T^\delta_{k-\ell,k+2\ell}$.

The $g$-equivariant quantization exists and is unique on $S^\delta_{k,k,s;\alpha\beta}$ if and only if there is no non-trivial $g$-invariant operator from this module to $\bigoplus_{0 \leq \ell < k, 0 \leq \ell' \leq \ell, \kappa} S^\delta_{k-\ell,k-2\ell'}$.

From the Theorem 4.2, providing the classification of conformally invariant operators on $T^\delta$, we easily deduce the following corollary.

**Corollary 4.6.** The subsets of critical values $I^S$ and $I^Q$ for the conformally equivariant superization and quantization are included into $\frac{1}{n}N^\ast$.

5. Explicit formulae for the $g$-equivariant quantization and superization

5.1. Local formulae. To write down explicit local formulae for the maps $S^\delta$ and $Q^{\lambda,\mu}$, we restrict to the local model $(\mathbb{R}^{p,q}, F_\eta)$. Thus, we can use the Euclidean invariants (for the metric $\eta$) classified in section 3. The obtained formulae being the only conformally equivariant ones, they are globally defined as a whole on arbitrary conformally flat manifolds $(M, g)$, but their building blocks are only locally defined, as they require the flat metric $\eta$.

The superization is easier to deal with, and we obtain an explicit formula for it on any component $T^\delta_{k,k,s;\alpha\beta}$ of $T^\delta$. We write the formula in a shape which illustrates Theorem 4.5. As previously, the 0 index denotes restriction and corestriction of operators to the space $\ker T$ and $\Pi_0$ is the projection onto that space.

**Theorem 5.1.** The conformally equivariant superization exists on $T^\delta_{k,k,s;\alpha\beta}$ if and only if $\delta \notin I^S_{k,k,s;\alpha\beta}$, given below. Then it is unique, and reads as

\[
S^\delta = \mathcal{F} \circ \left( \text{Id} + \frac{\hbar}{i} c_d \right)
\times \left[ d\delta + c_D R^sQ\delta(D + \frac{1}{k + \kappa - n\delta} dQ^*)T^s + c_G R^{s-1}Q\delta(G_0 + \frac{1}{k + \kappa - n\delta} d\delta^s_0)T^s \right].
\]

Denoting by $\rho_{k,s} = \prod_{s'=1}^s 2s'(n + 2(k - 2s + s' - 1))$ the eigenvalue of $R^sT^s$ on $T^\delta_{k,k,s}$, and by $a_{k,s} = \frac{n + 2(k - s)}{n + 2(k - s + 1)}$ an auxiliary factor, we have

\[
\begin{align*}
c_d &= -\frac{1}{k + \kappa + 1 - n\delta}, \\
c_D &= \frac{1}{(2(k - s - 1) + n(1 - \delta))\rho_{k,s}a_{k,s}}, \\
c_G &= \frac{2s}{(2s - n\delta)\rho_{k,s}}.
\end{align*}
\]
The non-existence cases $\delta \in T^S_{k,k,s;\alpha\beta}$ correspond to the existence of $\mathfrak{g}$-invariant operators with source space $T^\delta_{k,k,s;\alpha\beta}$. They are classified below.

| $\delta$ | $\mathfrak{g}$-invariant operator | source space $T^\delta_{k,k,s;\alpha\beta}$ |
|----------|-----------------------------------|-------------------------------------------|
| $k+\kappa \pm \frac{1}{n}$ | $[d\delta + cD \ldots + cG \ldots]$ | $\alpha\beta \neq 10, \kappa \leq (n-2), 1 \leq k$ |
| $2s \frac{n}{n}$ | $R^{s-1}Q\delta (G_0 + \frac{1}{k+\kappa-n\delta}d\delta_0)T^s$ | $\alpha = 0, 1 \leq s, \kappa \leq (n-2)$ |
| $1 + \frac{2(k-s-1)}{n}$ | $R^sQ\delta_0(D + \frac{1}{k+\kappa-n\delta}dQ^\ast)T^s$ | $\beta = 1, \kappa \leq (n-2), 2 \leq k$ |
| $k+\kappa \frac{n}{n}$ | $R^sQ\delta dQ^\ast T^s$ | $\alpha\beta = 11, 1 \leq \kappa \leq (n-1), 2 \leq k$ |
| | $R^{s-1}Q\delta d\delta^\ast T^s$ | $\alpha\beta = 00, 1 \leq s, 1 \leq \kappa \leq (n-1)$ |

**Proof.** The $\mathfrak{g}$-equivariance of $S^\delta$ reads as $S^\delta \circ L_X^\delta = L_X^\delta \circ S^\delta$ for all $X \in \mathfrak{g}$. As $F$ is a section of the principal tensorial symbol map $\varepsilon_{k,k}$ which is $\varepsilon(p,q)$-invariant, we have $S^\delta = F \circ (1d + N)$ with $N a \varepsilon(p,q)$-invariant operator from $T^\delta_{k,k,s;\alpha\beta}$ to $\bigoplus_{1 \leq \ell < k} T^\delta_{k-\ell, k+2\ell}$. In view of the form of those invariant operators, $N$ is a linear combination of the following linearly independent operators

$$d\delta, \quad R^sQ\delta DT^s, \quad R^{s-1}Q\delta G_0 T^s, \quad R^sQ\delta dQ^\ast T^s, \quad R^{s-1}Q\delta d\delta_0 T^s.$$  

Indeed, there is no $\varepsilon(p,q)$-invariant operator lowering the degree in $\xi$ by 4, and above are all those which lower the degree in $\xi$ by 2 and rise the degree in $p$ by one.

Using the comparison (2.15) between the action of $\mathfrak{g}$ on $S^\delta$ and $T^\delta$, the $\mathfrak{g}$-equivariance of $S^\delta$ reads then as

$$[N, L_X^\delta] = -2\frac{h}{i} \zeta_i \delta,$$

for all $i = 1, \ldots, n$. The actions of the conformal inversion $\bar{X}_i$ on the operators $G, D, d, d^\ast$ are given in (A.2). From them and the conformal invariance of $R^sT^s, Q\delta$, we deduce that, on $T^\delta_{k,k,s}^\ast$,

$$[d\delta, L_X^\delta] = 2(k + \kappa + 1 - n\delta)\zeta_i \delta + \frac{2}{p_{k,s}} R^sQ\delta \partial_{p_i} T^s + \frac{4s}{p_{k,s}} R^sQ\delta p_{i} T^s,$$

$$[R^sQ\delta DT^s, L_X^\delta] = 2(2(k - s - 1) + n(1 - \delta))R^sQ\delta \partial_{p_i} T^s + 2R^sQ\delta \zeta_i Q^\ast T^s,$$

$$[R^{s-1}Q\delta G_0 T^s, L_X^\delta] = 2(2s - n\delta)R^{s-1}Q\delta \Pi_{0p_i} T^s - 2R^{s-1}Q\delta \zeta_i \delta^\ast T^s.$$

The result follows then from the relation $R\partial_{p_i} T^s + 2s\Pi_{0p_i} T^s = 2s\Pi_{0p_i} T^s + (a_{k,s})^{-1} R\partial_{p_i} T^s$. \hfill $\square$

The conformally equivariant superization admits also a global formula on $T^\delta$ given below, as a drawback it does not allow to link easily the critical weights with conformally invariant operators.
Proposition 5.2. On $T^\delta$, the conformally equivariant superization is given by

$$S^\delta = F \circ \left( \text{Id} + \frac{\hbar}{i} C_d \left[ d\delta + Q\delta(n\delta \mathbb{D} - G T) C_G \right] \right),$$

where

$$\begin{cases}
(C_d)^{-1} = -(E + \Sigma - n\delta), \\
(C_G)^{-1} = R T + n\delta(2(E - 1) + n(1 - \delta)),
\end{cases}$$

$$\begin{cases}
\mathbb{D} = D + \frac{1}{\epsilon + \Sigma + 1 - n\delta} dQ^*, \\
G = G + \frac{1}{\epsilon + \Sigma + 1 - n\delta} d\delta^*.
\end{cases}$$

Proof. Taking advantage of the proof of the previous theorem, we just have to show that

$$N = \frac{\hbar}{i} C_d \left[ d\delta + Q\delta(n\delta \mathbb{D} - G T) C_G \right],$$

satisfies the relation (5.3). This follows from the next computations on $T^\delta$, relying again on (A.2) and the conformal invariance of $Q\delta$,

$$\begin{align*}
[d\delta, L_{\xi_i}] &= 2(E + \Sigma - n\delta)\xi_i \delta + Q\delta \partial_{p_i}, \\
[Q\delta \mathbb{D}, L_{\xi_i}] &= 2Q\delta (\partial_{p_i} (2(E - 1) + n(1 - \delta)) - p_i T), \\
[Q\delta G T, L_{\xi_i}] &= 2Q\delta (- n\delta p_i T + \partial_{p_i} R T).
\end{align*}$$

□

We turn now to the conformally equivariant quantization. Explicit computations are much harder and we restrict ourselves to $S^\delta_{\leq 1}$, the space of symbols of first order in $p$.

Theorem 5.3. Let $\lambda, \mu \in \mathbb{R}$ such that $\mu - \lambda = \delta$. For $n\delta \neq 1, \ldots, n + 1$, there exists a unique conformally equivariant quantization $Q^{\lambda,\mu} : S^\delta_{\leq 1} \to D_1^{\lambda,\mu}$, which reads as

$$Q^{\lambda,\mu} = N \circ F \circ \left( \text{Id} + \frac{\hbar}{i} \left[ c_D(\Sigma) D + c_\delta(\Sigma) d^* \delta + c_d(\Sigma) dQ^* + c_*(\Sigma) d^* Q^* \right] \right) \circ F^{-1},$$

the coefficients depending on the odd Euler operator $\Sigma = \xi_i \partial_{\xi_i}$ as follows

$$\begin{cases}
\begin{aligned}
c_D(\Sigma) &= \frac{2n\lambda + 1}{2n(1-\delta) + 2}, \\
c_\delta(\Sigma) &= \frac{n(1-\delta-2\lambda)}{\epsilon^2 - n(1-\delta)(2n(1-\delta) + 2)}, \\
c_d(\Sigma) &= -\frac{n(1-\delta-2\lambda)}{\epsilon^2 - n(1-\delta)(2n(1-\delta) + 2)}, \\
c_*(\Sigma) &= \frac{1}{4(\epsilon^2 - n(1-\delta))}.
\end{aligned}
\end{cases}$$
The remaining critical cases, \( n\delta = 1, \ldots n + 1 \), correspond to the existence of a \( \mathfrak{g} \)-invariant operator on certain subspaces \( T^\delta_{1,\kappa;0;\alpha\beta} \) as below (5.8)

| \( \delta \) | \( \mathfrak{g} \)-invariant operator | source space \( T^\delta_{1,\kappa;0;\alpha\beta} \) s.t. \( \mathcal{Q}^{\lambda,\lambda+\delta} \) exists |
|---|---|---|
| \( \frac{1}{n} \) | \( dQ^* \) | \( \kappa = 1, \alpha = 1 \) \( \lambda = \frac{n-1}{2n} \) |
| \( \frac{\ell}{n} \) | \( dQ^* \) | \( \kappa = \ell, \alpha = 1 \) none |
| \( 2 \leq \ell \leq n \) | \( d^*\delta \) | \( \kappa = n - \ell, \alpha\beta = 01 \) none |
| \( \frac{n+1}{n} \) | \( D + \frac{1}{2n-1}dQ^* - \frac{1}{2n+1}d^*\delta \) | \( \kappa = 0, \ldots, n \) \( \lambda = \frac{-1}{2n} \) |

For the two critical values \( \delta = \frac{1}{n}, \frac{n+1}{n} \), the quantization \( \mathcal{Q}^{\lambda,\lambda+\delta} \) exists for a unique value of \( \lambda \) and is given by, respectively,

\[
\mathcal{Q}^{\frac{n-1}{2n}, \frac{n+1}{n}} = \mathcal{N} \circ \mathcal{F} \circ \left( \text{Id} + \frac{\hbar}{i} \left[ c_D(\Sigma) D + c_\epsilon(\Sigma) d^*Q^* \right] \right) \circ \mathcal{F}^{-1},
\]

\[
\mathcal{Q}^{-\frac{1}{2n}, \frac{2n+1}{2n}} = \mathcal{N} \circ \mathcal{F} \circ \left( \text{Id} + \frac{\hbar}{i} \left[ c_\delta(\Sigma) d^*\delta + c_d(\Sigma) dQ^* + c_\epsilon(\Sigma) d^*Q^* \right] \right) \circ \mathcal{F}^{-1}.
\]

It is unique on \( \mathfrak{S}^\delta(T^\delta_{1,\kappa;0;\alpha\beta}) \), up to prospective \( \mathfrak{g} \)-invariant operators on that space, obtained from the list above via the superization \( \mathfrak{S}^\delta \).

**Proof.** The \( \mathfrak{g} \)-equivariance of \( \mathcal{Q}^{\lambda,\mu} \) reads as \( \mathcal{Q}^{\lambda,\mu} \circ L^\delta_X = L^\lambda_X \circ \mathcal{Q}^{\lambda,\mu} \) for all \( X \in \mathfrak{g} \). As \( \mathcal{N} \) and \( \mathcal{F} \) are sections of the principal Hamiltonian and tensorial symbol maps which are \( \mathfrak{c}(p,q) \)-invariant, we have \( \mathcal{Q}^{\lambda,\mu} = \mathcal{N} \circ \mathcal{F} \circ (\text{Id} + \mathcal{N}) \) with \( \mathcal{N} \) a \( \mathfrak{c}(p,q) \)-invariant operator from \( T^\delta_{1,\kappa;0;\alpha\beta} \) to \( \bigoplus_{\kappa' \leq \kappa+2} T^\delta_{0,\kappa'} \). In view of the form (3.4) of those invariant operators, \( \mathcal{N} \) is a linear combination of the following linearly independent operators,

\[
D, \quad d^*\delta, \quad dQ^*, \quad d^*Q^*,
\]

as all the other ones carry at least two derivations in the \( p \) variables, or do not lower the Hamiltonian order, like \( d\delta \). In consequence, \( \mathcal{Q}^{\lambda,\mu} \) is of the form specified by (5.6).

Using the comparisons (2.15) and (2.16) between the action of \( \mathfrak{g} \) on \( T^\delta, S^\delta \) and \( D^{\lambda,\mu} \), the \( \mathfrak{g} \)-equivariance of \( \mathcal{Q}^{\lambda,\mu} \) reads then as

\[
[N, L^\delta_X] = \frac{\hbar}{i} (2n\lambda \partial_{p_i} + \xi^i Q^* - \delta \partial_{\xi_i} + \frac{1}{2} \partial_{\xi_i} Q^*),
\]
for all \(i = 1, \ldots, n\). From the computations \((A.2)\) in appendix we deduce the following equalities in \(T^\delta_{k,\kappa}\),

\[
\begin{align*}
[d^*\delta, L^\delta_{X_i}] &= 2(\Sigma - n(1 - \delta))(\delta \partial_{\xi_i} - \partial_{\mu'}), \\
[d\Omega^*, L^\delta_{X_i}] &= 2(\Sigma - n\delta)\xi_i \Omega^*, \\
[d^*\Omega^*, L^\delta_{X_i}] &= 2(-\Sigma + n(1 - \delta))\partial_{\xi_i} \Omega^*, \\
[D, L^\delta_{X_i}] &= 2\delta \partial_{\xi_i} - 2\xi_i \Omega^* + 2n(1 - \delta)\partial_{\mu'}.
\end{align*}
\]

By substitution in the \(g\)-equivariance condition \((5.9)\), this determines a unique operator \(N\), as specified by \((5.7)\). The remaining statements of the theorem easily follow. \(\square\)

**Remark 5.4.** From the table \((5.2)\), we deduce that the conformally equivariant superization \(\mathcal{G}^\delta\) exists on each source space of the \(g\)-invariant operators listed in \((5.8)\). Consequently, they can be transported as \(g\)-invariant operators on \(S^\delta\), in accordance with theorem \((4.5)\).

**Remark 5.5.** Each of the \(g\)-invariant operators in tables \((5.2)\) and \((5.8)\) can be restricted and corestricted as an operator: \(T^\delta_{k,\kappa,\kappa',\alpha,\beta} \to T^\delta_{k',\kappa',\alpha',\beta'}\). Using the fiberwise invariant operators like in expression \((4.7)\), it corresponds to one of the \(g\)-invariant operator entering the classification of Theorem \((4.2)\). For table \((5.2)\), they are respectively \(d_0, G_0, D_0, d_0\) and for table \((5.8)\) \(d_0, d_0^*, d_0, d_0^*, d_0, d_0^*, d_0, D_0\).

### 5.2. Covariant formulæ and the curved case.

We would like now to devise formulæ in covariant terms for the equivariant superization and quantization and extend them over arbitrary conformal manifolds. In \([13]\), this was done in a conformally invariant fashion for the conformally equivariant quantization of cotangent bundles. The situation here is more involved, and only the composition \(Q^\lambda_{\mu} \circ \mathcal{G}^\delta\) admits a conformally invariant prolongation.

Starting with a pseudo-Riemannian spin manifold \((M, g)\), endowed with the Levi-Civita connection, we get covariant derivations on all associated bundles to the principal bundle of spin frames. For \(X \in \text{Vect}(M)\), we denote by \(\nabla^A_X\) the one acting on \(\Gamma(S \otimes |\Lambda|)\), and by \(\partial^\nabla_X\) the one acting on \(T^\delta = \Gamma(T^\delta)\). The latter derivative should be interpreted as the horizontal covariant derivative on the supercotangent bundle \(\mathcal{M}\). It allows to generalize the 13 invariant operators introduced in Proposition \((3.2)\) as global operators over any pseudo-Riemannian manifold \((M, g)\), replacing the flat metric \(\eta\) by \(g\) and the derivatives \(\partial_i\) by the covariant ones \(\partial_i^\nabla\). Their commutation relations remain the same than in the flat case, see \((A.4)\), except between the five ones containing covariant derivatives. Accordingly, we denote them with a superscript \(\nabla\),

\[
(5.10) \quad G^\nabla = g^{ij}p_i \partial_j^\nabla, \quad D^\nabla = \partial_p \partial_i^\nabla, \quad L^\nabla = \partial_i^\nabla g^{ij} \partial_j^\nabla, \quad d^\nabla = \xi^i \partial_i^\nabla, \quad d^\nabla = g^{ij} \partial_i^\nabla \partial_j^\nabla.
\]

With such a modification, the formulæ \((5.4)\) and \((5.6)\) provide covariant expressions which are proved to coincide, in the conformally flat case, with the conformally equivariant superization and quantization, respectively.
Proposition 5.6. Let \((M, g)\) be a pseudo-Riemannian spin manifold, \(P \in \mathcal{T}^\delta_1\), \(P_1 \in \mathcal{S}^\delta_1\) and \(\lambda, \mu \in \mathbb{R}\) such that \(\mu - \lambda = \delta\). Using notation of \((5.5)\) and \((5.7)\), we define the applications
\[
\mathcal{S}^\lambda_\delta(P) = |\text{vol}_g|^\frac{\delta}{2} \left( P + \frac{\hbar}{i} C_d \left[ d^\nabla \delta + Q \delta (n \delta \nabla - \nabla \nabla) C_G \right] P \right)
\]
and
\[
Q^\lambda_\delta(P_1) = \frac{\hbar}{i(\sqrt{2})^\Sigma} \gamma(P_1) \nabla^\lambda_i
\]
\[
+ \frac{\hbar}{i(\sqrt{2})^\Sigma} |\text{vol}_g|^\frac{\delta}{2} \gamma \left( c_d(\Sigma) D^\nabla P_1 + c_d(\Sigma) d^\nabla \delta P_1 + c_d(\Sigma) d^\nabla Q^* P_1 + c_d(\Sigma) d^\nabla Q^* P_1 \right).
\]
These maps coincide with, respectively, the conformally equivariant superization and quantization if \((M, g)\) is conformally flat.

Proof. Let \((M, g)\) be a conformally flat manifold and \(x\) a point in \(M\). We prove that \(\mathcal{S}^\lambda_\delta(P)\) and \(\mathcal{S}^\delta(P)\) are equal at \(x\). By conformal flatness of \(g\), there exists a neighborhood of \(x\) and conformal coordinates such that \(g_{ij} = F \eta_{ij}\) for \(F\) a positive function and \(\eta\) the flat metric. Moreover, up to a conformal change of coordinates we can assume that all the first derivatives of \(F\) at \(x\) vanish. This is a classical result, which can be derived from the infinitesimal action of inversions \(L_x \eta = -4x_i \eta\). Using such a coordinate system, we can write down both \(\mathcal{S}^\lambda_\delta(P)\) and \(\mathcal{S}^\delta(P)\) at \(x\). They coincide since the covariant formula \((5.11)\) involves only first order derivations. The same argument applies for the quantization maps. \(\square\)

Theorem 5.7. Let \((M, g)\) be a pseudo-Riemannian spin manifold, \(P \in \mathcal{T}^\delta_1\) and \(\lambda, \mu \in \mathbb{R}\) such that \(\mu - \lambda = \delta\). The following map
\[
Q^\lambda_\mu(P) = -\frac{\hbar}{i(\sqrt{2})^\Sigma} |\text{vol}_g|^\frac{\delta}{2} \gamma(P^i) \nabla^\lambda_i
\]
\[
+ \frac{\hbar}{i(\sqrt{2})^\Sigma} |\text{vol}_g|^\frac{\delta}{2} \gamma \left( c_0 d^\nabla \delta P + c_1 D^\nabla P + c_2 d^\nabla Q^* P + c_3 d^\nabla Q^* P \right),
\]
is conformally invariant, i.e. depends only on the conformal class of the metric \(g\), if and only if the real coefficients \(c_0, c_1, c_2, c_3, c_4\) are such that \(Q^\lambda_\mu\) is equal to the composition \(Q^\lambda_\mu \circ \mathcal{S}^\delta_\lambda\). But neither \(\mathcal{S}^\lambda_\delta\) nor \(Q^\lambda_\mu\) are conformally invariant.

Proof. We suppose that \(g\) and \(\hat{g}\) are two metrics conformally related through \(\hat{g} = F g\), with \(F\) a positive function on \(M\). The Christoffel symbols of their Levi-Civita connections satisfy
\[
\hat{\Gamma}^k_{ij} - \Gamma^k_{ij} = \frac{1}{2F} \left( F_i \delta^k_j + F_j \delta^k_i - F^k_{ij} \right),
\]
where \(F_i = \partial_i F\) and \(F^k = g^{ik} \partial_k F\). From the usual expressions of the spinor and the horizontal covariant derivatives in terms of Christoffel symbols, we deduce
\[
\hat{\nabla}^\lambda_j - \nabla^\lambda_j = -\frac{1}{8F} \gamma_{\lambda j} F_i - \frac{n \lambda}{2F} F_j,
\]
\[
\hat{\delta}^\nabla_i - \delta^\nabla_i = -\frac{1}{2F} \left( F_i E + F_k \partial_p \partial_{p_k} - F^j p_j \partial_p \right) - \frac{1}{2F} \left( F_k \Sigma^k_{ij} \partial_{\xi_i} - F^j \xi_i \partial_{\xi_j} \right) - \frac{n \delta}{2F} F_i,
\]
the latter acting on $T^\delta$. From the conformal invariance of $|\text{vol}_g|^\frac{2}{n} g$, and of $\delta, Q^*$ as operators from $T^\delta$ to $T^{\delta-1/n}$, we deduce the expression of each of the six terms involved in the map $Q^\lambda_{\nu} - Q^\lambda_{\nu}$:

$$|\text{vol}_g|^\frac{2}{n}g(P^j)(\hat{\nabla}_j^\lambda - \nabla_j^\lambda) = -|\text{vol}_g|^\frac{2}{n}g\left(\left[\delta \xi^i - \frac{1}{2} \xi^i Q^* + \frac{1}{2} \delta \partial_{\xi_i} + n \lambda \partial_{p_i}\right]P\right) \frac{F_i}{2F},$$

$$c_0 \left(\hat{d} - d^\nabla\right) \delta = c_0 \left((\Sigma - n \delta) \xi^i \delta\right) \frac{F_i}{2F},$$

$$c_1 \left(D_{\hat{\nabla}} - D_{\nabla}\right) = c_1 \left(\delta \partial_{\xi_i} - \xi^i Q^* + n(1 - \delta) \partial_{p_i}\right) \frac{F_i}{2F},$$

$$c_2 \left(d^*\hat{\nabla} - d^*\nabla\right) \delta = c_2 \left((\Sigma - n(1 - \delta)) \delta \partial_{\xi_i}\right) \frac{F_i}{2F},$$

$$c_3 \left(d\hat{\nabla} - d\nabla\right) Q^* = c_3 \left((\Sigma - n \delta) \xi^i Q^*\right) \frac{F_i}{2F},$$

$$c_4 \left(d^*\hat{\nabla} - d^*\nabla\right) Q^* = c_4 \left((\Sigma - n(1 - \delta)) Q^* \partial_{\xi_i}\right) \frac{F_i}{2F}.$$  

The first equality holds for all $P \in T^\delta_1$ whereas the other ones should be understood as equality of operators acting on $T^\delta_1$. The equality $Q^\lambda_{\nu}(P) = Q^\lambda_{\nu}(P) = 0$, for all $P \in T^\delta_1$, is then equivalent to a linear system in the five coefficients entering the definition of $Q^\lambda_{\nu}$. Solving this system leads exactly to $Q^\lambda_{\nu} = Q^\lambda_{\nu} \circ \mathcal{G}_\Sigma$, as wanted.

Since $c_0 = c_d(\Sigma) = \frac{1}{\Sigma - m}$, the preceding computations lead to

$$\mathcal{G}_\Sigma^\delta(P) = F^{\Sigma/2} \left(\mathcal{G}_\Sigma^\delta(P) - \frac{h}{2F} \xi^i \delta P\right),$$

and then neither $\mathcal{G}_\Sigma^\delta$ nor $Q^\lambda_{\nu}$ are conformally invariant.

Let $A$ be a first order differential operator acting on the tensor bundle $STM \otimes \Lambda^* M$, for $(M, g)$ a conformally flat manifold. Then, $g$-equivariance of $A$ implies its conformal invariance, whenever the $g$-actions are the natural ones on tensors [17]. But the Hamiltonian action is not of this type, as seen in (2.15). This explains the lack of conformal invariance of the conformally equivariant superization.

6. Symmetries of spinning particles

We classify all the symmetries of free massless spinning particles over conformally flat manifolds, both in the classical and quantum cases. They arise via the conformally equivariant superization and quantization of conformal Killing hook tensors, that we introduce over an arbitrary pseudo-Riemannian manifold.

6.1. Reminder on the non-spinning case. A classical particle on a pseudo-Riemannian manifold $(M, g)$ admits the cotangent flow of $T^* M$ as phase space. In the free case, its motion is described as the Hamiltonian flow of $R = g^{ij} p_i p_j$, which projects onto the geodesic flow of $(M, g)$. The symmetries, or conserved quantities, of such a system are the functions
which Poisson commute with $R$. If the particle is in addition massless, it moves along the null cone, characterized by $R = 0$. Then, the symmetries are functions $K$ which are required to Poisson commute with $R$ on the null cone, i.e. such that $\{R, K\} \in (R)$ where $(R)$ denotes the ideal generated by $R$ in $\text{Pol}(T^*M)$. Moreover, the functions in $(R)$ are considered as trivial symmetries since they vanish on the null cone. Through the identification of symmetric tensors $\Gamma(STM)$ and functions $\text{Pol}(T^*M)$, we get the following classical result, where $T$ denotes the operator $\partial_p \partial_{p^i}$ and $\Pi_0$ the projection on traceless tensors.

**Proposition 6.1.** The symmetries of a free massless particle on $(M, g)$ are given by conformal Killing symmetric tensors. The latter are symmetric tensors $K$ of order $k \in \mathbb{N}$, characterized by one of the three equivalent conditions

\[
\begin{align*}
\{R, K\} &\in (R) \text{ and } TK = 0, \\
\Pi_0 G \nabla K &= 0, \\
\Pi_0 \nabla_{(i_0} K_{i_1 \cdots i_k)} &= 0,
\end{align*}
\]

where the round brackets denote symmetrization.

Roughly speaking, a quantum massless free particle is given by an harmonic function of the conformal Laplacian $\Delta = \nabla_i g^{ij} \nabla_j + \frac{n-2}{4(n-1)} \text{Scal}$, where $\text{Scal}$ denotes the scalar curvature. The latter is a conformally invariant operator if considered as an element of the space $\mathcal{D}^{\lambda, \mu} := \mathcal{D}(M; |A|^{\lambda}, |A|^\mu)$ for $\lambda = \frac{n-2}{2n}, \mu = \frac{n+2}{2n}$. Following Eastwood, we introduce higher symmetries of $\Delta$.

**Definition 6.2.** Let $\lambda = \frac{n-2}{2n}, \mu = \frac{n+2}{2n}$ and let $(\Delta) = \{A \Delta | A \in \mathcal{D}^{\mu, \lambda}\}$ be the left ideal generated by the conformal Laplacian in $\mathcal{D}^{\lambda, \mu}$. A higher symmetry of $\Delta$ is a class of differential operators $[D_1] \in \mathcal{D}^{\lambda, \mu}/(\Delta)$, such that $\Delta D_1 = D_2 \Delta$, for some $D_2 \in \mathcal{D}^{\mu, \mu}$.

Such symmetries preserve the kernel of $\Delta$ and are quantum analogs of the conformal Killing symmetric tensors. Indeed, denoting by $\mathcal{K}_0$ the space of conformal Killing symmetric tensors and by $\mathcal{A}_0$ the space of higher symmetries of $\Delta$, we have the following theorem.

**Theorem 6.3.** [16][33] Let $(M, g)$ be a conformally flat manifold. The conformally equivariant quantization descends to an isomorphism of g-modules $\mathcal{Q}^{\lambda, \lambda} : \mathcal{K}_0 \to \mathcal{A}_0$, mapping conformal Killing symmetric tensors to higher symmetries of $\Delta$.

6.2. **Conformal Killing hook tensors.** We call hook tensors the tensors whose symmetry is described by a hook Young diagram. They correspond to elements in the tensor algebra $T^0$ and are referred as $(k, \kappa)$-tensors if they lie in $T^0_{k,\kappa}$. Thus, $(k,0)$-tensors are symmetric and $(0,\kappa)$-tensors are skew-symmetric. We define in this section the notion of conformal Killing hook tensors, so that it generalizes both conformal Killing symmetric tensors and conformal Killing forms, defined in the usual way below.

**Definition 6.4.** A skew-symmetric tensor $\eta$ of order $\kappa$ is conformal Killing, if it satisfies

\[
\nabla_X \eta = \frac{1}{\kappa + 1} \langle X, d\eta \rangle + \frac{1}{n - \kappa + 1} X^b \wedge \delta^* \eta,
\]
for all \( X \in \text{Vect}(M) \). Moreover, \( d^* \) and \( d \) denote the de Rham (co-)differentials, and \( X^\flat \) is the dual 1-form of \( X \) through the metric \( g \).

We would like to characterize the space of conformal Killing forms as the kernel of a conformally invariant operator. In addition to the invariant operators introduced in table (A.1) and their curved analogs (B.10), we resort to the conformally invariant projections \( \Pi_0 : T^k \to \bigoplus_{k,\kappa} T_{k,\kappa,0;0}^k \) and \( \Pi_{01} : T^k \to \bigoplus_{k,\kappa} T_{k,\kappa,0;01}^k \). The images of both projections are respectively equal to the space \( \ker \delta \cap \ker Q^* \cap \ker T \) and its image by \( \delta^* \).

**Proposition 6.5.** The operator \( \Pi_0 \circ G^\nabla : T^{-\frac{n}{2}} \to T^\frac{n}{2} \) is conformally invariant and satisfies \( \delta^* \Pi_0 G^\nabla = \Pi_0 G^\nabla \delta^* \). On the space \( \ker T \cap \ker Q^* \cap \ker \delta \), it is equal to

\[
\Pi_0 \circ G^\nabla = G^\nabla - \frac{1}{\mathcal{E} + \Sigma} \delta^* d^\nabla - \frac{1}{n + \mathcal{E} - \Sigma} Qd^\nabla - \frac{1}{n + 2\mathcal{E} - 4} R^\nabla,
\]

and it vanishes else, i.e. on all the spaces \( T_{\frac{n}{\kappa},s;0}^k \) with \( s \neq 0 \) or \( \alpha\beta \neq 0 \). The kernel of \( \Pi_{01} G^\nabla \delta^* \) restricted to \( T_{\frac{n}{\kappa},0;00}^k \) is the space of conformal Killing \( \kappa \)-forms.

**Proof.** The operator \( \Pi_0 \circ G^\nabla \) is \( g \)-invariant in the conformally flat case, as proved in theorem (1.22). Moreover, it is the composition of the Levi-Civita covariant derivative with the projection on some irreducible homogeneous subbundle, so that, by Fegan’s work [17], it is invariant by a conformal change of metric over a general manifold \((M, g)\).

Since \( G^\nabla \) commutes with \( \delta^* \), the equality \( \delta^* \Pi_0 = \Pi_0 \delta^* \) leads to \( \delta^* \Pi_0 G^\nabla = \Pi_0 G^\nabla \delta^* \). Moreover, \( G^\nabla \) commutes with \( R \) and \( Q \), so that the vanishing of \( \Pi_0 \circ G^\nabla \) on \( T_{\frac{n}{\kappa},s;0}^k \) if \( s \neq 0 \) or \( \alpha\beta \neq 0 \) is obvious. In the remaining case, the expression of \( \Pi_0 \circ G^\nabla \) follows from the vanishing of its composition with \( T, Q^* \) and \( \delta \). The coefficients are computed thanks to the table (A.1) of commutation relations.

From proposition (3.3) we deduce the equality \( \ker \delta^* \cap \ker \delta = \{0\} \). Therefore the kernel of \( \Pi_{01} G^\nabla \delta^* \) on \( T_{\frac{n}{\kappa},0;00}^k \) is equal to the one of \( \Pi_0 \circ G^\nabla \). Let \( \eta \in T_{\frac{n}{\kappa},0;00}^k \), and \( X \in \text{Vect}(M) \). We get

\[
g_{ij} X^i_\delta \rho_j (\Pi_0 \circ G^\nabla \eta) = \nabla X \eta - \frac{1}{\mathcal{E} + \Sigma} X^i \delta_\xi d^\nabla \eta - \frac{1}{n + \mathcal{E} - \Sigma} g_{ij} X^i \xi j d^\nabla \eta.
\]

Since \( d^\nabla \) and \( d^\nabla \) identify to de Rham (co-)differentials on differential forms, \( \eta \) has to be a conformal Killing form to lie in the kernel of \( \Pi_0 \circ G^\nabla \). As \( X \) is arbitrary, the converse statement clearly holds.

**Definition 6.6.** Let \( k \in \mathbb{N}^* \) and \( \kappa \in [0, n] \). A conformal Killing hook tensor of degree \((k, \kappa)\) is an element in \( \ker T \cap \ker Q^* \cap \delta^* = \bigoplus_{k,\kappa} T_{k,\kappa,0;01}^k \), which lies in the kernel of \( \Pi_0 \circ G^\nabla \).

From the proposition (6.1) we deduce that conformal Killing \((k, 0)\)-tensors are usual conformal Killing symmetric tensors of order \( k \). The proposition (6.5) shows that conformal Killing \((1, \kappa - 1)\)-tensors identify to conformal Killing \( \kappa \)-forms through the operator \( \delta^* \). Therefore, the latter definition extends both notions. As it is the kernel of a first order operator, the space \( \mathcal{K} := \ker(\Pi_0 \circ G^\nabla) \) of conformal Killing hook tensors is a subalgebra of \( T^9 \).
6.3. **Pseudo-classical spinning particles.** The phase space of a spinning particle on a pseudo-Riemannian manifold \((M, g)\) is its supercotangent bundle \((\mathcal{M}, \omega)\), the spin being represented by a quadratic function in the Grassmann variables, \(S = S_{ij}\xi^i\xi^j\). It automatically transforms in the right way under the action of the orthogonal group. This Hamiltonian model is equivalent to the well-known Lagrangian one developed in \[4\], it is the classical counterpart of the quantum description of spinning particles as spinor fields \[31\]. In the free case, the equations of motion of the particle are again given by the Hamiltonian flow of \(R\). Denoting by \(x\) the parameterized trajectory, we get in particular

\[
\nabla_\dot{x}\dot{x}^i = -\frac{1}{2}g^{jl}(R_{a_bkl}^a\xi^b)\dot{x}^k,
\]

\[
\nabla_\dot{x}\xi^a = 0,
\]

where \((R_{a_bkl}^a)\) denote the components of the Riemann tensor. The second equation shows that the spin is parallelly transported, whereas the first one is an analog of Papapetrou’s equation \[35\], which generalizes the geodesic equation for an extended object with spin. The deviation from geodesic motion is due to a coupling between the curvature and the spin. Generally, the particle’s spin is spacelike, that is expressed by \(p_i\xi^i = 0\). See \[36, 26\] for further informations.

The classical counterpart of the Dirac operator is its principal Hamiltonian symbol, given by \(Q = p_i\xi^i\), which Poisson squares to \(R = \{Q, Q\}\). Thus, all symmetries of the Hamiltonian flow of \(Q\) are also symmetries for the one of \(R\). Following \[19\], we call them supercharges.

**Definition 6.7.** A supercharge is an element \(K\) of \(\mathcal{O}(\mathcal{M})\) such that \(\{Q, K\} = 0\). A conformal supercharge is an element \(K\) of \(\mathcal{O}(\mathcal{M})\) such that \(\{Q, K\} \in (Q, R)\), where \((Q, R)\) is the ideal in \(\mathcal{O}(\mathcal{M})\) generated by \(Q\) and \(R\).

Among the conformal supercharges stand all the elements in the ideal \((Q, R)\), which are considered as trivial supercharges since they vanish if \(Q = 0 = R\). Consequently, the relevant space of conformal supercharges is the Poisson algebra

\[
\mathcal{SC} := \{K \in \mathcal{O}(\mathcal{M})|\{Q, K\} \in (Q, R)\}/(Q, R),
\]

which identifies to a symplectic reduction of the Poisson algebra \(\mathcal{O}(\mathcal{M})\).

We classify below all the conformal supercharges in a close spirit to proposition \[6.1\] This generalize \[19, 40\], where supercharges are built from Killing forms.

**Theorem 6.8.** If \((M, g)\) is a conformally flat manifold, the conformally equivariant superiznation induces an isomorphism of \(g\)-modules

\[
\mathcal{S}^0 : \mathcal{K} \longrightarrow \mathcal{SC},
\]

mapping conformal Killing hook tensors to conformal supercharges. On general manifolds, for \(K \in \mathcal{K}\), we have the following equivalence

\[
\mathcal{S}^0_\mathcal{K}(K) \in \mathcal{SC} \iff \Pi_{\mathcal{O}_1}(d\nabla)^2K = 0.
\]

This condition reads explicitly as: \(\Pi_{\mathcal{O}_1}(\xi^k\xi^lW_{bkl}^a\partial_{\rho_a}K) = 0\), with \(W\) the Weyl tensor.
Proof. Let $K \in T^0$. If $K$ factorizes through $R$ or $Q$, then so does $\mathcal{S}_V^0(K)$ according to (5.11) and we get a trivial symmetry. Hence, according to the decomposition (4.2), we get that $K \in \ker T \cap \ker Q^*$. We compute

\[
\{Q, \mathcal{S}_V^0(K)\} = (d^V - i \frac{\delta^*}{\hbar}) \circ (\text{Id} - \frac{\hbar}{i} \frac{1}{\mathcal{E} + \Sigma} d^V \delta)(K) + H,
\]

where $\mathcal{E}$ is the electric field. In view of the commutation relations given in table (A.4), the first equation reads as

\[
\{Q, \mathcal{S}_V^0(K)\} = -\frac{i}{\hbar} \delta^* K + \left(d^V + \frac{1}{\mathcal{E} + \Sigma} \delta^* d^V \delta - \frac{\hbar}{i} \frac{1}{\mathcal{E} + \Sigma} (d^V)^2 \delta\right)(K) + H,
\]

(6.4)

with $H \in (R, Q)$. By the proposition 3.3 we have $\text{im} \delta \cap \text{im} \delta^* = \{0\}$. Therefore, $\mathcal{S}_V^0(K)$ is a conformal supercharge if and only if $\delta^* K \in (R, Q)$ and $\delta \left(G^V - d^V \delta^* - \frac{\hbar}{i} (d^V)^2 \right)(K) \in (R, Q)$. In view of the commutation relations given in table (A.4), the first equation reads as $\delta^* K = 0$, which is automatic if $K \in K$. Assuming that $\delta^* K = 0$, the second equation is equivalent to $\Pi_{01} G^V K = \frac{1}{i} \Pi_{01} (d^V)^2 K$. In the conformally flat case the right hand side vanishes, and we get the announced isomorphism (6.2). For a general metric $g$, we deduce the equivalence (6.3) for $K \in K$. More precisely, the equation $\Pi_{01} (d^V)^2 K = 0$ reduces to $\Pi_{01} \circ (\xi^k \xi^l R^a_{bkl} \partial_a \partial_b) (K) = 0$ and thanks to the projection $\Pi_{01}$, all the traces of the Riemann tensor can be removed, leading to the Weyl tensor. The latter vanishes in the conformally flat case. \hfill \Box

6.4. **Symmetries of the Dirac operator.** The Dirac operator $\mathcal{D}$ is conformally invariant if viewed as an operator in $\mathcal{D}^\lambda,\mu$ for $\lambda = \frac{n-1}{2m}$ and $\mu = \frac{n+1}{2m}$. We introduce its higher symmetries following the Laplacian case.

**Definition 6.9.** Let $\lambda = \frac{n-1}{2m}$, $\mu = \frac{n+1}{2m}$ and let $(\mathcal{D}) = \{A\mathcal{D} | A \in \mathcal{D}^{\mu,\lambda}\}$ be the left ideal generated by the Dirac operator in $\mathcal{D}^{\lambda,\mu}$. A higher symmetry of $\mathcal{D}$ is a class of differential operators $[D_1] \in \mathcal{D}^{\lambda,\lambda}/(\mathcal{D})$, such that $\mathcal{D}_1 D_1 = D_2 \mathcal{D}$, for some $D_2 \in \mathcal{D}^{\mu,\mu}$.

We denote by $\mathcal{A}$ the algebra of higher symmetries of $\mathcal{D}$, it is recovered as the kernel of the following conformally invariant operator

\[
\text{QHS} : \mathcal{D}^{\lambda,\lambda}/(\mathcal{D}) \rightarrow \mathcal{D}^{\lambda,\mu}/(\mathcal{D})
\]

\[
[D] \mapsto [\mathcal{D} D]
\]

(6.5)

where QHS stands for Quantum Higher Symmetries. In the space $\mathcal{D}^{\lambda,\lambda}/(\mathcal{D})$, the class $[\mathcal{D} D]$ is clearly equal to the one of the commutator $[\mathcal{D}, D]$. Hence, using the grading (2.1) on $S^\delta$ and the corresponding order (2.5) on $\mathcal{D}^{\lambda,\mu}$, we deduce from the property (2.7) of the principal Hamiltonian symbol maps that

\[
\mathcal{D}^{\lambda,\lambda}_{[k]}/(\mathcal{D}) \xrightarrow{\text{QHS}} \mathcal{D}^{\lambda,\mu}_{[k+1]}/(\mathcal{D})
\]

\[
\sigma_k \quad \sigma_{k+1}
\]

(6.6)

\[
\frac{\mathcal{S}^0_{[k]}}{(Q, R)} \xrightarrow{\{Q, R\}} \mathcal{S}^{\frac{3}{2}}_{[k+1]}/(Q, R)
\]
is a commutative diagram of $\mathfrak{g}$-modules. Thus, we have that $\sigma_k(D) \in \mathcal{K}$ if $D$ is a higher
symmetry of $\mathcal{D}$. In particular, those of first order have a conformal Killing form as principal
symbol, through the operator $\delta^*$. They have all been classified by Benn and Kreiss [3], and
turn to be given by the conformally equivariant quantization. More generally, the latter leads
us to the classification of all higher symmetries, like for the Laplacian.

**Theorem 6.10.** Let $\lambda = \frac{n-1}{2n}$, $\mu = \frac{n+1}{2n}$ and $(M, g)$ be a conformally flat manifold. We have
the following isomorphisms of $\mathfrak{g}$-modules

$$Q^{\lambda, \lambda} : \mathcal{SC} \to \mathcal{A} \quad \text{and} \quad Q^{\lambda, \lambda} \circ \mathcal{S}^0 : \mathcal{K} \to \mathcal{A}$$

which establish a correspondence between conformal Killing hook tensors, conformal super-
charges and higher symmetries of $\mathcal{D}$. Those of first order are given by

$$Q^{\lambda, \lambda} \circ \mathcal{S}^0(\delta^* K) = \frac{\hbar}{i(\sqrt{2})^\kappa} \left( g^{ij} \gamma(K_i) \nabla^\lambda - \frac{1}{2\kappa} \gamma(d^\nabla K) \right) + \frac{n(\kappa - n) + 2}{2(n+1)(\kappa - n)} \gamma(d^\nabla K),$$

where $\kappa$ runs over $0, \ldots, n - 1$ and $K$ runs over the space of conformal Killing $\kappa + 1$-forms.

**Proof.** The idea of the proof is to use the conformally equivariant quantization to identify the
kernel $\mathcal{A}$ of the operator $Q\mathcal{H}$, see [6.8], to the one of $\{Q, \cdot\}$ on $\mathcal{S}^0/(Q, R)$, which is $\mathcal{SC}$ (cf.
diagram (6.8)). We will need the three following lemmas.

**Lemma 6.11.** The ideal $(Q, R)$ is stable by the conformally equivariant superizations $\mathcal{S}^0$
and $\mathcal{S}^\mp$. Moreover, we have $Q^{\lambda, \lambda}((Q, R)) = (\mathcal{D})$ and there exists a conformally equivariant
quantization $Q^{\lambda, \mu}$ such that $Q^{\lambda, \mu}((Q, R)) = (\mathcal{D})$.

**Proof.** The explicit formula (5.4) leads directly to the result for the superizations. Let
$\lambda' = \lambda, \mu$. According to corollary 4.6 the conformally equivariant quantizations $Q^{\lambda, \lambda}$ and
$Q^{\mu, \lambda'}$ exist and are unique. The conformal invariance of $\mathcal{D} \in D^{\mu, \lambda}$ ensures that $PQ \mapsto Q^{\lambda, \lambda}(P)\mathcal{D} \in D^{\lambda, \lambda'}$ defines a conformally equivariant map on $(Q)$. Easy computations prove
that it conformally extends to $(Q, R)$ via $\mathcal{S}^0(PR) \mapsto Q^{\mu, \lambda'}(PQ)\mathcal{D}$ for $P \notin (Q)$. If $\lambda' = \lambda$
this map coincides with $Q^{\lambda, \lambda}$ by uniqueness. If $\lambda' = \mu$, this defines a conformally equivariant
quantization on $(Q, R)$, which can be extended to $\mathcal{S}^\mp = (Q, R) \oplus \ker Q^* \cap \ker T$. Indeed, the
only obstructions to existence of $Q^{\lambda, \mu}$ are given by the conformally invariant operators $GT$
and $G\delta Q^*$ which vanishes on $\ker Q^* \cap \ker T$. The obtained factorization formulae for $Q^{\lambda, \lambda}$
and $Q^{\lambda, \mu}$ lead to the result.

According to the preceding lemma, the quotient spaces $T^\delta/(Q, R)$ inherit a $\mathfrak{g}$-module
structure from the one of $T^\delta$.

**Lemma 6.12.** Let $A : T^0/(Q, R) \to T^\mp/(Q, R)$ be a conformally invariant operator. Then,
extcept maybe on $T^0_{0,0} \simeq C^\infty(M)$, the operator $A$ is proportional to a linear combination of $\delta^*$
and $\Pi_{01}G\delta$, where $\Pi_{01}$ is the projection onto $\ker \delta : \ker Q^* \cap \ker T$.

**Proof.** Notice that, as $\mathfrak{g}$-modules, $T^0/(Q, R) \simeq \ker Q^* \cap \ker T$ splits into $\ker \delta : \ker Q^* \cap \ker T$ by
proposition 3.3. We easily deduce from theorem 4.2 that there is a unique $\mathfrak{g}$-invariant operator
with source space $\ker \delta \cap \ker Q^* \cap \ker T = \bigoplus_{k,\kappa} T^0_{k,\kappa,0,00}$, given by $\delta^*$, and a unique one with source space $\ker \delta^* \cap \ker Q^* \cap \ker T = \bigoplus_{k,\kappa} T^0_{k,\kappa,0,01}$, given by $\Pi_0 G\delta$.

According to lemma 6.11 the superizations $S^0, \mathfrak{S}^{1\frac{1}{2}}$ descend to the quotient spaces $S^\delta : T^\delta/(Q, R) \to S^\delta/(Q, R)$ for $\delta = 0, 1/n$. We get a similar result for the quantizations $Q^{\lambda,\lambda'} : S^\delta/(Q, R) \to D^{\lambda,\lambda'}/(D)$ for $\lambda' = \lambda + \delta$ and $\delta = 0, 1/n$.

**Lemma 6.13.** The operator $A$ defined by the following commutative diagram

\[
\begin{array}{ccc}
S^0/(Q, R) & \xrightarrow{\{Q, \cdot\}} & S^{\frac{1}{2}}/(Q, R) \\
\mathfrak{S}^0 & \downarrow & \mathfrak{S}^{1\frac{1}{2}} \\
T^0/(Q, R) & \xrightarrow{A} & T^{1\frac{1}{2}}/(Q, R)
\end{array}
\]

satisfies $A = -\frac{i}{\hbar} \delta^* + \Pi_0 G\delta$.

**Proof.** We have to prove that $\{Q, S^0(K)\} - \mathfrak{S}^{1\frac{1}{2}}(AK) \in (Q, R)$ for $K \in \ker Q^* \cap \ker T$. By proposition 3.3 the latter space splits into $\ker \delta \oplus \ker \delta^*$ and lead to two cases. As $(M, g)$ is conformally flat, we can deduce from the computation (6.1) that $\{Q, S^0(K)\} - (\frac{1}{\hbar}\delta^* + d)(K) \in (Q, R)$ in the first case $K \in \ker \delta$, and that $\{Q, S^0(K)\} - (\frac{1}{\hbar}\delta^* + \delta G) \in (Q, R)$ in the second case $K \in \ker \delta^*$. Moreover, by proposition 3.3 the projection on $\ker \delta^*$ reads as $\frac{1}{\epsilon + \omega^2} \delta^*$, hence we get $\frac{1}{\epsilon + \omega^2} \delta G - \Pi_0 G\delta K \in (Q, R)$ if $K \in \ker \delta^*$. The formula (5.1) giving the superization allows then to conclude in both cases. □

We are now ready to prove the theorem. According to lemma 6.11 we get the following commutative diagram of $\mathfrak{g}$-modules

\[
\begin{array}{ccc}
D^{\lambda,\lambda}/(D) & \xrightarrow{\text{QHS}} & D^{\lambda,\mu}/(D) \\
\mathfrak{Q}^{\lambda,\lambda} & \downarrow & \mathfrak{Q}^{\lambda,\mu} \\
S^0/(Q, R) & \xrightarrow{\text{CHS}} & S^{\frac{1}{2}}/(Q, R) \\
\mathfrak{S}^0 & \downarrow & \mathfrak{S}^{1\frac{1}{2}} \\
T^0/(Q, R) & \xrightarrow{A} & T^{1\frac{1}{2}}/(Q, R)
\end{array}
\]

where CHS and $A$ are conformally invariant operators. The diagram (6.6) leads to $\text{QHS} = \{Q, \cdot\} + B$, where $B$ does not rise the Hamiltonian degree, contrary to $\{Q, \cdot\}$ which rises it by one. By lemma 6.12 we know the form of $A$ and together with lemma 6.13 we deduce that the only possibility is $\text{QHS} = \{Q, \cdot\}$ and $A = -\frac{i}{\hbar} \delta^* + \Pi_0 G\delta$.

The kernel of $A$ is exactly the space $\mathcal{K}$, the one of $\{Q, \cdot\}$ is $\mathcal{S}C$ and the one of QHS is $A$. Moreover, the latter diagram proves that they are related by $\mathfrak{S}^0$ and $\mathfrak{Q}^{\lambda,\lambda}$. The formula (5.12) of the quantization $Q^{\lambda,\lambda}$ ends the proof. □
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Appendix A.

We collect here informations on the 13 generators of the \(E(p,q)\)-invariant operators on \(D(\mathcal{M})\) introduced in Proposition 3.2.

A.1. In the following table we recall their definitions together with their interpretation as operators on \(\Gamma(S\mathcal{T}M \otimes \Lambda^* T^* M)\), in the flat case \((\mathcal{M}, g) = (\mathbb{R}^p,q, \eta)\).

\[
\begin{align*}
R &= \eta^{ij} p_i p_j && \text{metric} & D &= \partial_{p_i} \partial_{p_i} & \text{divergence} \\
\mathcal{E} &= p_i \partial_{p_i} && \text{even Euler operator} & G &= \eta^{ij} p_i \partial_{j} & \text{gradient} \\
T &= \eta_{ij} \partial_{p_i} \partial_{p_j} && \text{trace} & L &= \eta^{ij} \partial_{j} \partial_{j} & \text{Laplacian} \\
\Sigma &= \xi^i \partial_{\xi^i} && \text{odd Euler operator} & d &= \xi^i \partial_{i} & \text{de Rham differential} \\
\delta &= \eta_{ij} \xi^i \partial_{p_j} && \text{Koszul differential} & d^* &= \partial_{\xi^i} \partial_{i} & \text{de Rham codifferential} \\
\delta^* &= \eta^{ij} p_i \partial_{\xi^j} && \text{Koszul codifferential} & d &= \xi^i \partial_{i} & \text{Berezin differential} \\
Q &= p_i \xi^i && \text{Berezin codifferential} & Q^* &= \partial_{\xi^i} \partial_{p_i} & \text{or symbol of } \mathcal{D}
\end{align*}
\]

A.2. We compute the action of the inversion \(\bar{X}_i\), see (2.9), on the five generators of \(\mathfrak{h}(2|1,1)\) viewed as operators \(A : T^g \to T^{g'}\) with \(\delta'\) chosen according to the table (3.3). Explicitly, this action reads as \([A, L^*_{\bar{X}_i}] := A L_{\bar{X}_i}^\delta - L_{\bar{X}_i}^\delta A\) and we get

\[
\begin{align*}
[D, L^\delta_{\bar{X}_i}] &= 2(2\mathcal{E} + n(1 - \delta)) \partial_{p_i} - 2p_i T + 2\delta \partial_{\xi^i} - 2\xi^i Q^*, \\
[G, L^*_{\bar{X}_i}] &= -2n\delta p_i + 2R \partial_{p_i} + 2Q \partial_{\xi^i} - 2\xi^i \delta^*, \\
[L, L^*_{\bar{X}_i}] &= 2(2\mathcal{E} + n(1 - 2\delta)) \partial_i + 4G \partial_{p_i} - 4p_i D + 4d \partial_{\xi^i} - 4\xi^i d^*, \\
[d, L^*_{\bar{X}_i}] &= 2(\mathcal{E} + \Sigma - 1 - n\delta) \xi^i + 2Q \partial_{p_i} - 2p_i \delta, \\
[d^*, L^*_{\bar{X}_i}] &= 2(\mathcal{E} - \Sigma - 1 + n(1 - \delta)) \partial_{\xi^i} - 2p_i Q^* + 2\delta^* \partial_{p_i}.
\end{align*}
\]

We introduce \(\Pi_0\), the conformally invariant projection on \(\ker T \cap \ker Q^* \cap \ker \delta\), and denote by an index \(\mathbf{0}\) the five generators of \(\mathfrak{h}(2|1,1)\) restricted and costricted to that space. Then,
the action of the inversion $\tilde{X}_i$ on their powers, acting on $T^\delta_{k,\kappa,0,0,0}$, reads as

$$
\left[ D^d_{\delta, \tilde{X}_i} \right] = 2d(2k - d + n(1 - \delta)) \partial_{p_i} D^{d - 1}_0,
$$

$$
\left[ G^d_{\delta, \tilde{X}_i} \right] = -2g(g + n\delta) \Pi_0 p_i G^{d - 1}_0,
$$

(A.3)

$$
\left[ L^\ell_{\delta, \tilde{X}_i} \right] = 2\ell \left( (2(k - \ell) + n(1 - 2\delta)) \partial_{\xi_i} + 4(G\partial_{p_i} + d\partial_{\xi_i} - p_i D - \xi_i d^*) \right) L^{\ell - 1}_0,
$$

$$
\left[ d_0, \tilde{L}^\ast_{\xi_i} \right] = 2(k + \kappa - n\delta) \Pi_0 \xi_i,
$$

$$
\left[ d^*_0, \tilde{L}^\ast_{\xi_i} \right] = 2(k - \kappa + n(1 - \delta)) \Pi_0 \partial_{\xi_i}.
$$

A.3. Recall that $E = \mathcal{E} + \frac{\nu}{2}$, and $\Sigma = \Sigma - \frac{\nu}{2}$. We sum up all the commutation relations between the previous 13 operators, they generate the super Lie algebra $\mathfrak{sp}(2|1,1) \ltimes \mathfrak{h}(2|1,1)$.

(A.4)

| $R$ | $E$ | $T$ | $\Sigma$ | $\delta$ | $\delta^*$ | $Q$ | $Q^*$ | $D$ | $G$ | $L$ | $\bar{d}$ | $\bar{d}^*$ |
|-----|-----|-----|---------|---------|----------|-----|-------|-----|-----|-----|-------|--------|
| $R$ | $0$ | $-2R$ | $-4E$ | $0$ | $-2Q$ | $0$ | $0$ | $-2\delta^*$ | $-2G$ | $0$ | $0$ | $0$ |
| $E$ | $2R$ | $0$ | $2T$ | $0$ | $-\delta$ | $\delta^*$ | $Q$ | $-Q^*$ | $-D$ | $G$ | $0$ | $0$ |
| $T$ | $4E$ | $-2T$ | $0$ | $0$ | $0$ | $2Q^*$ | $2\delta$ | $0$ | $0$ | $2D$ | $0$ | $0$ |
| $\Sigma$ | $0$ | $0$ | $0$ | $\delta$ | $-\delta^*$ | $Q$ | $-Q^*$ | $0$ | $0$ | $0$ | $d$ | $-d^*$ |
| $\delta$ | $2Q$ | $\delta$ | $0$ | $-\delta$ | $E + \Sigma$ | $0$ | $T$ | $0$ | $d$ | $0$ | $0$ | $D$ |
| $\delta^*$ | $0$ | $-\delta^*$ | $-2Q^*$ | $\delta^*$ | $E + \Sigma$ | $0$ | $R$ | $0$ | $d^*$ | $0$ | $0$ | $G$ |
| $Q$ | $0$ | $-Q$ | $-2\delta$ | $-Q$ | $0$ | $R$ | $0$ | $E - \Sigma$ | $-d$ | $0$ | $0$ | $G$ |
| $Q^*$ | $2\delta^*$ | $Q^*$ | $0$ | $Q^*$ | $T$ | $0$ | $E - \Sigma$ | $0$ | $0$ | $d^*$ | $0$ | $D$ |
| $D$ | $2G$ | $D$ | $0$ | $0$ | $0$ | $d^*$ | $d$ | $0$ | $0$ | $L$ | $0$ | $0$ |
| $G$ | $0$ | $-G$ | $-2D$ | $0$ | $-d$ | $0$ | $0$ | $-d^*$ | $-L$ | $0$ | $0$ | $0$ |
| $L$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\bar{d}$ | $0$ | $0$ | $0$ | $-d$ | $0$ | $G$ | $0$ | $D$ | $0$ | $0$ | $0$ | $L$ |
| $\bar{d}^*$ | $0$ | $0$ | $0$ | $d^*$ | $D$ | $0$ | $G$ | $0$ | $0$ | $0$ | $L$ | $0$ |

We denote by a zero index the operators $\delta^*, \delta, Q, Q^*$ restricted and corestricted to the kernel of the operator $T$. They satisfy the following commutation relations:

(A.5)

$$
\begin{align*}
\{Q_0, \delta^* \} &= 0 & \{Q_0, \delta \} &= 0 & \{\delta, \delta^* \} &= -4cQ_0^\delta_0 & \{\delta, Q^*_0 \} &= (n + \mathcal{E} - \Sigma) - 4c\delta^*_0\delta_0 \\
\{\delta^*_0, \delta \} &= 0 & \{\delta^*_0, Q^*_0 \} &= \Sigma + \mathcal{E} - 4cQ_0Q^*_0 & \{\delta_0, Q^*_0 \} &= -4c\delta^*_0Q^*_0 \\
\{\delta_0, \delta^*_0 \} &= -4cQ_0\delta_0 & \{\delta_0, Q^*_0 \} &= \Sigma + \mathcal{E} - 4cQ_0Q^*_0 & \{Q^*_0, Q^*_0 \} &= 0 \\
\{Q^*_0, \delta \} &= (n + \mathcal{E} - \Sigma) - 4c\delta^*_0\delta_0 & \{Q^*_0, Q^*_0 \} &= -4c\delta^*_0Q^*_0 & \{Q^*_0, \delta^* \} &= 0
\end{align*}
$$

where $c = \frac{1}{2(n + 2(\mathcal{E} - 1))}$ comes from the coefficient of $RT$ in $\Pi_0$, but with $\mathcal{E} - 1$ instead of $\mathcal{E}$ as the commutation with $\delta$ or $Q^*$ lowers by 1 the degree in $p$. 

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