THE MAJORIZATION FIXED POINT PRINCIPLE
AND APPLICATION TO NONLINEAR INTEGRAL EQUATIONS

1. Introduction

The successive approximations method allows us to solve problems concerning existence and uniqueness of fixed points of wide classes of operators. The classical result in this field, such as Banach – Caccioppoli principle together with some its modification and generalisations, is applicable to operators satisfying Lipschitz condition with a small coefficient or, in other words, to operators with the compression property. However, the successive approximations method works well for other classes of operators that are not compressions. In particular, the well known Kantorovich fixed point principle [3] for differentiable operators deals with operators that, in general, are not compression; moreover, this principle covers some cases when Banach – Caccioppoli principle is nonapplicable.

Recall that Banach – Caccioppoli fixed point principle deals with operators in complete metric spaces. Kantorovich fixed point principle deals only with operators in Banach spaces; moreover, it is applicable only to differentiable operators. In this article we consider some modification of Kantorovich fixed point principle that covers nondifferentiable operators; some variants of this modification were used by P.P. Zabreiko in nineties; in [6] an almost final variant of this principle was offered. The variant given in the present article is related to its essential complement; in this variant we describe the exact (unimprovable) estimates of the internal and external radius of the domain of existence of a unique fixed point of the operator under consideration. In addition we present new apriori and aposteriori error estimates for successive approximations to the corresponding fixed point.

Some applications of the new fixed point principle to nonlinear integral operators of different types are given as well.

2. Majorized mappings principle

Let us consider the equation

\[ x = Ax, \]  

where \( A \) is an operator defined in a ball \( B[x_0, R] = \{ x : \| x - x_0 \| \leq R \} \) of a Banach space \( X \) (\( x_0 \in X \)).

**Definition 1.** Operator \( A \) satisfies the variable Lipschitz condition in the ball \( B[x_0, R] \) with a nonnegative in \([0, R]\) function \( k(\cdot) \), if the following conditions are fulfilled:

\[ \| Ax_1 - Ax_2 \| \leq k(r) \| x_1 - x_2 \|, \]  

where

\[ \| x_1 - x_0 \| \leq r, \quad \| x_2 - x_0 \| \leq r, \quad 0 < r \leq R. \]

The basic part of the theorem presented below for smooth operators \( A \) is given in [3]. Here we present the theorem for both smooth and nonsmooth operators. To start with let us introduce some notation.
First of all let \( a_+(\cdot) \) and \( a_-(\cdot) \) be the functions
\[
a_\pm(r) = a \pm \int_0^r k(t) \, dt, \quad \text{where} \quad a = \|Ax_0 - x_0\|. \tag{3}
\]

In what follows we call functions \( a_\pm(r) \) the majorant functions of the operator \( A \).

If function \( a_+(\cdot) \) has fixed points on the interval \([0, R]\) let us denote the smallest among them by \( r^* \). Let us also denote as \( r_* \) the smallest fixed point of the function \( a_-(\cdot) \). Finally, let
\[
r^{**} = \sup_{r^* < r \leq R} \{ r : a_+(r) < r \} \tag{4}
\]
(provided that the set under the sup sign is nonempty). Let also
\[
L(x_0, r^*, r^{**}) = \begin{cases} 
\{ x : r^* < \|x_0 - x\| < r^{**} \} & \text{if } a_+(R) \geq R, \\
\{ x : r^* < \|x_0 - x\| \leq r^{**} \} & \text{if } a_+(R) < R,
\end{cases} \tag{5}
\]
and
\[
L[x_0, r_*, r^*] = \{ x : r_* \leq \|x_0 - x\| \leq r^* \}. \tag{6}
\]

The main theorem is:

**Theorem 1.** Let operator \( A \) be defined in the ball \( B[x_0, R] \) of a Banach space \( X \) \((x_0 \in X)\) and satisfies the variable Lipschitz condition \((2)\) in the ball \( B[x_0, R] \) with a nonnegative in the interval \([0, R]\) function \( k(\cdot) \). Let the functions \( a_\pm(\cdot) \) have fixed points in the interval \([0, R]\). Then operator \( A \) has a unique fixed point \( x^* \in L[x_0, r_*, r^*] \) and it is also unique in each ball \( B[x_0, r] \) where \( r^* \leq r < r^{**} \) (i.e. there are no fixed points in \( B[x_0, r_*, r^*] \cup L[x_0, r^*, r^{**}] \)).

This majorized mappings theorem is a modification of the successive approximations method. It is easy to see that conditions of Banach – Caccioppoli theorem fulfill the conditions of Theorem 1.

Let us discuss advantages of Theorem 1.

First of all, Theorem 1 uses the variable Lipschitz condition instead of existence condition of continuous derivative, that essentially extends the class of mappings that can be anaylzed by means of this theorem. Second, the method reflected in Theorem 1 gives a method of finding out a real differentiable function having fixed point. And finally third, the method offered here is convenient for the comparison of the majorized mappings theorems with Banach – Caccioppoli principle.

Figures 1 – 3 show the relationship between the majorized mappings principle and Banach – Caccioppoli principle in the general situation. Let us denote: \( BC\)-zone, \( U\)-zone, \( E\)-zone — a set of radii \( r \) of the balls, where Banach – Caccioppoli principle of the fixed point can be applied (\( BC \)), where the uniqueness \((U)\) and existence \((E)\) is valid. Banach – Caccioppoli principle of fixed point can be applied in the ball \( B[x_0, r] \), where radius \( r \) should satisfy the inequality:
\[
r^* \leq r < r_{cr}, \quad \text{where} \quad r_{cr} = \inf_{k(r) = 1} r.
\]

So according to the Banach – Caccioppoli theorem fixed point \( x^* \) of an operator \( A \) lies in the ball \( B[x_0, r^*] \) and is unique in each ball \( B[x_0, r] \), where \( r^* \leq r < r_{cr} \). But due to Theorem 1 we can conclude more: fixed point \( x^* \) of an operator \( A \) lies in the domain \( L[x_0, r_*, r^*] \) and is unique in each ball \( B[x_0, r] \), where \( r^* \leq r < r^{**} \) for figures 1–2 and \( r^* \leq r \leq r^{**} \) for figure 3 (thus \( 0 \leq r < r^{**} \) is the uniqueness \((U)\) zone for figures 1–2 and \( 0 \leq r \leq r^{**} \) for figure 3).
Figure 1

Figure 2

Figure 3

Figure 4

Figure 4 shows a case when Banach – Caccioppoli theorem can not be applied. But the application of the principle of majorized mappings does not cause any difficulties. So in this case operator $A$ is neither compressive, nor expanding in the ball $B[x_0, r]$, where $r = r^* = r_{cr} = R$.

One may find a substantial part of the proof of Theorem 1 in [7]. Here we give the full proof. To prove the theorem we need the following

**Lemma 1.** Let operator $A$ that is defined in the ball $B[x_0, R] = \{x : \|x - x_0\| \leq R\}$ of a Banach space $X$ ($x_0 \in X$) satisfy the variable Lipschitz condition (2) in the ball $B[x_0, R]$ with a
nonnegative in the interval $[0, R]$ function $k(\cdot)$. Then the following inequality is valid:

$$\|A(x + h) - Ax\| \leq \int_{r}^{r+\delta} k(t) dt, \quad (\|x - x_0\| \leq r, \|h\| \leq \delta, r + \delta \leq R). \quad (7)$$

Lemma follows from the following obvious chain of inequalities

$$\|A(x + h) - Ax\| \leq \sum_{j=1}^{s} \left\|A \left(x + \frac{j}{s}h\right) - A \left(x + \frac{j-1}{s}h\right)\right\| \leq \sum_{j=1}^{s} k \left(r + \frac{j}{s}\delta\right) \frac{\delta}{s}$$

and passage to the limit while $s \to \infty$.

**Proof**

1) First of all let us prove that successive approximations

$$r_{n+1} = a_+(r_n) \quad (r_0 = 0, n = 0, 1, \ldots), \quad (8)$$

form a convergent sequence. Let us note that (3) implies

$$a'_+(r) = k(r) \geq 0, \quad r \in [0, R],$$

by virtue of (2). So function $a_+$ does not decrease in the interval $[0, R]$ and $r_n$ makes sense for any $n$, and moreover

$$r_n \leq r^* \quad (n = 0, 1, \ldots), \quad (9)$$

where $r^*$ is the smallest root (existence is presupposed suggested in Theorem 1) of the equation

$$r = a_+(r). \quad (10)$$

Really, for $n = 0$ inequality (2) is evident, and if it is proved for $n = k$, then from $r_k \leq r^*$ we get $a_+(r_k) \leq a_+(r^*)$ due to the monotonicity of $a_+(\cdot)$, i.e. $r_{k+1} \leq r^*$ and by induction the inequality (9) is proved for any $n$.

Using monotonicity of $a_+(\cdot)$ once again, one can prove by induction the monotonicity of the sequence $\{r_n\}$. Really $r_n \leq r_{n+1}$ implies $r_{n+1} = a_+(r_n) \leq a_+(r_{n+1}) = r_{n+2}$, where the inequality $0 = r_0 \leq r_1$ is obvious.

So far we established existence of the limit

$$r^* = \lim_{n \to \infty} r_n.$$ 

Due to (8) and continuity of $a_+(\cdot)$ $r^*$ that is a root to equation (10). And moreover $r^*$ is the smallest in $[0, R]$ root due to (2).

Let

$$x_{n+1} = Ax_n, \quad (n = 0, 1, \ldots), \quad (11)$$

where $x_0$ is the center of the ball $B[x_0, R]$. Let us prove that all the elements (11) make sense and form convergent sequence.

For $n = 0$ equality is obvious due to (3)

$$\|x_1 - x_0\| = \|Ax_0 - x_0\| = a = a_+(0) = a_+(r_0) = r_1,$$
then \(x_1 \in B[x_0, R]\). Suppose that we have already proved that \(x_1, x_2, \ldots, x_n \in B[x_0, R]\), and that
\[
\|x_{k+1} - x_k\| \leq r_{k+1} - r_k \quad (k = 0, 1, \ldots, n - 1). \tag{12}
\]
Then using lemma (1) we have
\[
\|x_{n+1} - x_n\| = \|Ax_n - Ax_{n-1}\| \leq \int_{r_{n-1}}^{r_n} k(t)dt = a_+(r_n) - a_+(r_{n-1}) = r_{n+1} - r_n.
\]
So (11) is proved for \(k = n\) and \(x_{n+1} \in B[x_0, R]\) is also proved due to
\[
\|x_{n+1} - x_0\| \leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_1 - x_0\| \leq (r_{n+1} - r_n) + (r_n - r_{n-1}) + \cdots + (r_1 - r_0) = r_{n+1} \leq R.
\]
Thus, inclusion \(x_k \in B[x_0, R]\) and estimation (12) are established for all \(k = 0, 1, \ldots\) by induction.

Then due to (12)
\[
\|x_{n+p} - x_n\| \leq \|x_{n+p} - x_{n+p-1}\| + \cdots + \|x_{n+1} - x_n\| \leq (r_{n+p} - r_{n+p-1}) + \cdots + (r_{n+1} - r_n) = r_{n+p} - r_n,
\tag{13}
\]
which implies convergence of the sequence \(\{x_n\}\). Let us denote
\[x^* = \lim_{n \to \infty} x_n.
\]
Passing to the limit in (11) and taking into account continuity of the operator \(A\), we get
\[x^* = Ax^*,\]
i. e. \(x^*\) is a root of equation (1).

Moreover inequality (12) implies the inequality
\[
\|x^* - x_n\| \leq r^* - r_n \quad (n = 0, 1, \ldots),
\]
which gives an estimation of the convergence speed.

Let us prove nonexistence of a fixed point of operator \(A\) in the ball \(B[x_0, r_*]\). Let \(\|x^* - x_0\| = r_0\).
Let us estimate \(\|x^* - x_0\|\) from below:
\[
\|x^* - x_0\| = \|Ax^* - x_0\| \geq \|Ax_0 - x_0\| - \|Ax^* - Ax_0\|, \tag{14}
\]
Due to Lemma 1
\[
\|Ax^* - Ax_0\| \leq \int_0^{r_0} k(t)dt.
\]
Then using equality (3) we get from (14):
\[
\|x^* - x_0\| \geq a - \int_0^{r_0} k(t)dt = a_-(r_0).
\]
It implies
\[ a_-(r_0) \leq r_0. \] (15)

It is easy to see that inequality (15) is valid for all \( r_0 \geq r_* \), where \( r_* \) is the point of intersection of the graph of the function \( \tilde{r} = a_-(r) \) and bisectrix \( \tilde{r} = r \). It immediately implies, that operator \( A_+ \) does not have any fixed point in the ball \( B[x_0, r_*] \).

Thus, \( r_* \leq r_0 \leq r^* \), in other words fixed point \( x* \) of the operator \( A_+ \) lies in the ring \( L[x_0, r_*, r^*] \).

The theorem is proved. ■

**Theorem 2.** Let all conditions of the theorem 1 be fulfilled. Then:

1) successive approximations \( \{\xi_n\} \):

\[ \xi_{n+1} = A\xi_n \quad (n = 0, 1, \ldots) \] (16)

with the initial approximation \( \xi_0 \in B[x_0, r^*] \cup L(x_0, r^*, r^{**}) \) are defined for any \( n \) and converge to the fixed point \( x^* \);

2) the following estimates are valid

\[ \|x^* - \xi_n\| \leq r^* + \rho_n - 2r_n \quad (n = 0, 1, \ldots), \] (17)

\[ \|\xi_{n+1} - \xi_n\| \leq \rho_{n+1} + \rho_n - 2r_n \quad (n = 0, 1, \ldots), \] (18)

where \( \{r_n\} \) are successive approximations from Theorem 1 and

\[ \rho_{n+1} = a_+(\rho_n) \quad (n = 0, 1, \ldots), \] (19)

where the initial approximation is \( \rho_0 = \|\xi_0 - x_0\| \), and \( \rho_0 \geq r_0 = 0 \).

**Proof**

1) Consider successive approximations (16) (the initial approximation \( \xi_0 \) is an arbitrary element from \( B[x_0, r^*] \cup L(x_0, r^*, r^{**}) \)). It is easy to see at Figure 5 and Figure 6 that if \( \rho_0 \geq r_0 \), then \( \rho_n \geq r_n \) for any \( n = 1, 2, \ldots \). Note also, that the sequence \( \{\rho_n\} \) is increasing to \( r^* \) if \( \rho_0 < r^* \) and is decreasing to \( r^* \) if \( \rho_0 > r^* \); in the case \( \rho_0 = r^* \) all terms in the sequence \( \{\rho_n\} \) coincide with \( r^* \).
Literally in the same way as in the proof of Theorem 1 we prove that the sequence \( \{\rho_n\} \) has a limit equal to \( \rho^* \), moreover \( \rho^* \) (that is the root of the equation \((10)\)) coincides with \( r^* \) (\( \rho^* = r^* \)).

Now let us prove that successive approximations sequence \( \{\xi_n\} \) converges and consequently gives a root of equation \((1)\). We have

\[
\|\xi_1 - x_1\| = \|A\xi_0 - Ax_0\|
\]

and due to Lemma 1 we get

\[
\|\xi_1 - x_1\| \leq \int_{r_0}^{\rho_0} k(t)dt = a_+(\rho_0) - a_+(r_0) = \rho_1 - r_1,
\]

and

\[
\|\xi_1 - x_0\| \leq \|\xi_1 - x_1\| + \|x_1 - x_0\| \leq (\rho_1 - r_1) + (r_1 - r_0) \leq \rho_1 \leq R,
\]

Clearly \( \xi_1 \in B[x_0, R] \).

The further reasoning goes by induction. Suppose that

\[
\xi_k \in B[x_0, R], \quad \|\xi_k - x_k\| \leq \rho_k - r_k \quad (k = 0, 1, \ldots, n).
\]

Then \( \xi_{n+1} - x_{n+1} = A\xi_n - Ax_n \). Let us use Lemma 1 again:

\[
\|\xi_{n+1} - x_{n+1}\| = \|A\xi_n - Ax_n\| \leq \int_{r_n}^{\rho_n} k(t)dt = a_+(\rho_n) - a_+(r_n) = \rho_{n+1} - r_{n+1},
\]

and then

\[
\|\xi_{n+1} - x_0\| \leq \|\xi_{n+1} - x_{n+1}\| + \|x_{n+1} - x_0\| \leq (\rho_{n+1} - r_{n+1}) + (r_{n+1} - r_0) \leq \rho_{n+1} \leq R
\]

consequently \( \xi_{n+1} \in B[x_0, R] \).

We conclude by induction that \((20)\) is valid for \( k = 1, 2, \ldots \).

Since sequences \( \{r_n\} \) and \( \{\rho_n\} \) have a common limit equal to \( r^* \) it follows that convergence of the sequence \( \{x_n\} \) implies convergence of the sequence \( \{\xi_n\} \) due to \((20)\) and the equality

\[
\lim_{n \to \infty} \xi_n = \lim_{n \to \infty} x_n = x^*.
\]

So it is proved, that the sequence of successive approximations converges to \( x^* \) with any initial approximation \( \xi_0 \in B[x_0, R] \). It implies the uniqueness of a root of the equation \((1)\). Thus, part 1) of the theorem is proved.

2) Let us prove estimate \((17)\). Let us note

\[
\|x^* - \xi_n\| \leq \|x^* - x_n\| + \|x_n - \xi_n\| \quad (n = 0, 1, \ldots).
\]

Therefore due to \((13)\) and \((20)\) we get

\[
\|x^* - \xi_n\| \leq \|x^* - x_n\| + \|x_n - \xi_n\| \leq (r^* - r_n) + (\rho_n - r_n),
\]

and, since \( \rho_n > r_n \), it follows that

\[
\|x^* - \xi_n\| \leq r^* + \rho_n - 2r_n \quad (n = 0, 1, \ldots).
\]
Estimate (17) is proved

Let us prove the estimate (18). Using Lemma 1 we have

$$\|\xi_{n+1} - \xi_n\| \leq \|\xi_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - \xi_n\|$$

$$\leq \rho_{n+1} - r_{n+1} + r_{n+1} - r_n + \rho_n - r_n = \rho_{n+1} + \rho_n - 2r_n.$$ 

So, estimate (18) is proved, and part 2) of the theorem is proved as well. Theorem is proved completely. ■

3. Examples

1) Let $X$ be a Banach space. Let us consider the Lemarié-Resset ([5], see also [7]) equation

$$x = \eta + T(x, \ldots, x),$$

(21)

where $\eta \in X$ and operator $T$ is an $m$-linear ($m \geq 2$) continuous operator, defined on $X$. As is well known, the operator $T$ satisfies the Lipshitz condition:

$$\|Tx_1 - Tx_2\| \leq Cmr^{m-1}\|x_1 - x_2\| \quad (x_1, x_2 \in B[x_0, r], \quad 0 < r \leq \infty),$$

i.e. the operator $T$ satisfies (2) with $k(r) = Cmr^{m-1}$; here $C$ is the norm of the $m$-linear operator $T$.

Now we can calculate the majorant functions:

$$a_{\pm}(r) = a \pm \int_0^r k(t)dt = a \pm C \int_0^r mt^{m-1}dt = a \pm Cr^m, \quad a = \|\eta\|.$$

Thus, the following equation

$$a + Cr^m = r.$$  (22)

allows us to present the solvability and uniqueness conditions for equation (21).

It is easy to solve this equation for $m = 2$. It is also possible to find a solution for each $m = 3, 4, \ldots$ But the solution in the general form for any $m$ can not be determined. It is clearly seen from Figure 7 that the graph of the function $a_{\pm}(\cdot)$ depends on the value $a = \|\eta\|$, i.e. the quantity of roots of equation (22) also depends on the value $a$, and the condition of the root existence is

$$a \leq a_{cr}, \quad \text{where} \quad a_{cr} = \left(\frac{1}{Cm}\right)^{\frac{1}{m-1}} m - 1.$$

(23)

Thus in the case when condition (23) is satisfied equation (21) has a unique root $x^* \in L[x_0, r^*, r^*]$, moreover, the operator $A$ has no fixed points in the set $B[0, r_s] \cup L(x_0, r^*, r^{**})$.

Let us note that results and reasoning presented above allows us to see more than Lemarié-Resset theorems do: the domain of solution existence can be found more precisely than it is done by Lemarié-Resset.

2) Let us consider a nonlinear integral equation of the Hammershtein mixed type (see [8])

$$x(t) = f(t) + \lambda \sum_{j=1}^m \int_a^b k_j(t, s)h_j(x(s))ds,$$  (24)

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where the kernel $k_j(t, s)$, for each $j$, is a measurable function with respect to variables $t, s \in [a, b]$, $h_j$ is a continuous function, $\lambda$ is a parameter, $f$ is a given function and $x$ is an unknown function. This equation was investigated in [8].

First of all let us consider equation (24) in the space $C[a, b]$ of continuous functions on $[a, b]$. Let us assume that the functions $h_j (j = 1, \ldots, m)$ satisfy the conditions

$$|h_j(y_1) - h_j(y_2)| \leq w_j(r)|y_1 - y_2| \quad (|y_1|, |y_2| \leq r, \quad 0 < r \leq R, \quad w_j(r) = 0),$$

where functions $w_j(r)$ are nondecreasing. Further, let the kernels $k_j(t, s)$ ($j = 1, \ldots, m$) define linear integral operators $K_j$ in the space $C$; this means that each kernel $k_j(t, s)$ is Lebesgue integrable with respect to $s$ in $[a, b]$ with $t \in [a, b]$,

$$\sup_{a \leq t \leq b} \int_a^b |k_j(t, s)| \, ds < \infty;$$

and, each function

$$\tilde{k}_j(t, s) = \int_a^s k_j(t, \varsigma) \, d\varsigma,$$

continuously depends on $t$ in average, i.e.

$$\lim_{t \to \tau} \int_a^b |\tilde{k}_j(t, s) - \tilde{k}_j(\tau, s)| \, ds = 0.$$

In addition, we have

$$\|K_j\| = \sup_{a \leq t \leq b} \int_a^b |k_j(t, s)| \, ds < \infty.$$

Figure 7
Under these conditions the operator

$$Ax(t) = f(t) + \lambda \sum_{j=1}^{m} \int_{a}^{b} k_j(t, s) h_j(x(s)) \, ds$$  (25)

acts in the space $C$ and satisfies the variable Lipschitz condition in the ball $B[0, R]$ with the function:

$$k(r) = |\lambda| \sum_{j=1}^{m} \|K_j\| w_j(r).$$  (26)

The functions (3) for the situation under consideration are defined by

$$a_\pm(r) = |\lambda| \left( a \pm \int_{0}^{r} \sum_{j=1}^{m} w_j(t) \|K_j\| \, dt \right).$$

Theorems 1, 2 allow us to formulate conditions of solvability of equation (24), define the ring where this solution lays and estimate the rate of convergence of successive approximations.

Now let us consider equation (24) in the space $L_p[a, b]$. It seems that results from the space $C[a, b]$ for the equation under consideration can be easily transferred to the space $L_p[a, b]$. But this is not true. The appropriate estimates can be obtained if Lipschitz conditions for the nonlinearities $h_j(u)$ are of the special form. Moreover these conditions are true only in the case when the nonlinearities $h_j(u)$ are defined for all $u \in \mathbb{R}$ and has power growth with respect to the variables $u$.

Let us assume that there exist nonnegative constants $(\xi, \eta)$ such that the following inequality is valid:

$$|h_j(u_1) - h_j(u_2)| \leq \left( \xi + \eta r^{\frac{-q_j}{p_j}} \right)|u_1 - u_2| \quad (|u_1|, |u_2| \leq r, \ 0 < r < \infty).$$  (27)

Then each operator $H_jx(t) = h_j(x(t)), j = 1, ..., m,$ acts from $L_p[a, b]$ to $L_{q_j}[a, b]$ and satisfies in each ball $B_r(L_p[a, b])$ the variable Lipschitz condition:

$$\|H_j(x_1) - H_j(x_2)\|_{L_{q_j}} \leq \tilde{h}_j(r) \|x_1 - x_2\|_{L_p} \quad (\|x_1\|_{L_p}, \|x_2\|_{L_p} \leq r, \ 0 < r < \infty),$$  (28)

where

$$\tilde{h}_j(r) = \inf_{(\xi, \eta) \in T(H_j)} \left\{ \xi (b - a)^{\frac{-q_j}{p_j}} + \eta r^{\frac{-q_j}{q_j}} \right\},$$  (29)

here $T(H_j)$ is the set of pairs $(\xi, \eta)$ satisfying (27).

In order to prove (29) it is sufficient to verify that

$$\tilde{h}_j(r) \leq \left\{ \xi (b - a)^{\frac{-q_j}{p_j}} + \eta r^{\frac{-q_j}{q_j}} \right\}$$  (30)

for arbitrary $(\xi, \eta) \in T(H_j)$. Remark that (27) implies

$$|h_j(\psi_1(s)) - h_j(\psi_2(s))| \leq \left( \xi + \eta \max \{|\psi_1(s)|, |\psi_2(s)|\}^{\frac{-q_j}{q_j}} \right)|\psi_1(s) - \psi_2(s)|, \quad \psi_1(s), \psi_2(s) \in L_p,$$
and, further,
\[
\|H_j \psi_1 - H_j \psi_2\|_{L_{qj}} \leq \left( \xi(b - a)^{p - q_j} + \eta \| \max \{|\psi_1|, |\psi_2|\} \|_{L_p} \right) \|\psi_1 - \psi_2\|_{L_p}, \quad \psi_1(s), \psi_2(s) \in L_p.
\] (31)

If \(\|\psi_1\|_{L_p}, \|\psi_2\|_{L_p} \leq r\) then \(\max \{|\psi_1|, |\psi_2|\} \|_{L_p} \leq 2^{1/2} r\) and the latter inequality implies only the estimate
\[
\|H_j \psi_1 - H_j \psi_2\|_{L_{qj}} \leq \left( \xi(b - a)^{p - q_j} + 2^{p - q_j} \eta r^{p - q_j} \right) \|\psi_1 - \psi_2\|_{L_p},
\]
and this estimate is worse than (30). Nevertheless, (31) implies (30).

Indeed, let \(\|x_1\|_{L_p}, \|x_2\|_{L_p} < r\) and \(\delta > 0\) such that \(\|x_1\|_{L_p}, \|x_2\|_{L_p} \leq r - \delta\). Let \(N\) be an integer such that \(2r < N \delta\). Set
\[
\psi_j = \left( 1 - \frac{j}{N} \right) x_1 + \frac{j}{N} x_2, \quad j = 0, 1, \ldots, N.
\]
Then
\[
\|H_j x_1 - H_j x_2\|_{L_{qj}} \leq \sum_{j=1}^{M} \|H_j \psi_j - H_j \psi_{j-1}\|_{L_{qj}}
\]
and, due to (31),
\[
\|H_j x_1 - H_j x_2\|_{L_{qj}} \leq \left( \frac{1}{N} \sum_{j=1}^{M} \left( \xi(b - a)^{p - q_j} + \eta \| \max \{|\psi_{j-1}|, |\psi_{j}|\} \|_{L_p} \right) \right) \|x_1 - x_2\|_{L_p}.
\]
Moreover, \(\|\psi_{j-1} - \psi_j\|_{L_{p+}} \leq \frac{1}{N} \|x_1 - x_2\|_{L_p} \leq 2r \frac{1}{N} < \delta\). Therefore,
\[
\|\max \{|\psi_{j-1}|, |\psi_j|\} \|_{L_p} = \|\psi_{j-1}\| + \max \{0, |\psi_j| - |\psi_{j-1}|\} \|_{L_p} \leq r - \delta + \|\psi_{j-1}\| \leq r
\]
and, hence,
\[
\|H_j x_1 - H_j x_2\|_{L_{qj}} \leq \left( \xi(b - a)^{p - q_j} + \eta r^{p - q_j} \right) \|x_1 - x_2\|_{L_p}.
\]
Thus, (30) holds true in the case when \(\|x_1\|_{L_p}, \|x_2\|_{L_p} < r\). The standard passage to the limit proves the validity of (30) for all \(\|x_1\|_{L_p}, \|x_2\|_{L_p} \leq r\). In [1] it is presented a different proof of (30) under the condition that (27) holds.

Further let us assume that for each \(j = 1, \ldots, m\) the kernel \(k_j(t, s)\) is measurable with respect to \(t, s\) and lies in the Zaanen space: \(k_j(t, s) \in Z(q_j, p') (p' = p/(p - 1))\). Recall [4] that the Zaanen space \(Z(\alpha, \beta)\) is the space of measurable functions \(z(t, s)\) with two variables \(t, s \in [a, b]\) for which the integrals
\[
\int_{a}^{b} \int_{a}^{b} z(t, s)x(s)y(t) \mathrm{d}s \mathrm{d}t, \quad x(t) \in L_\alpha, y(t) \in L_\beta,
\]
do exist; the norm in this space is defined by the formula
\[
\|z\|_{Z(\alpha, \beta)} = \sup_{\|x\|_{L_\alpha}, \|y\|_{L_\beta} \leq 1} \int_{a}^{b} \int_{a}^{b} |z(t, s)x(s)y(t)| \mathrm{d}s \mathrm{d}t.
\] (32)
Of course, this norm of a function $z(t,s)$ is equal to the norm of the linear integral operator $Z$ with the kernel $|z(t,s)|$ as an operator between the spaces $L_\alpha$ and $L_{\beta'}$, $\beta' = \beta/(\beta - 1)$. Some methods of calculation and estimation of this norm for various $\alpha$ and $\beta$ are gathered in [4].

Under these assumptions operator (25) satisfies the variable Lipschitz condition in the ball $B_r(L_p[a,b])$ with the function $k(\cdot)$:

$$k(r) = \sum_{j=1}^{m} \tilde{h}_j(r) \|k_j\|_{Z(q_j,p')} \quad \text{where } \tilde{h}_j(r) \text{ is defined in (29).}$$

Thus, the majorant functions of the operator $A$ are defined by the equations

$$a_\pm(r) = \lambda \left( a \pm \int_0^r \sum_{j=1}^{m} \tilde{h}_j(\varrho) \|k_j\|_{Z(q_j,p')} d\varrho \right).$$

And in this case, Theorems 1, 2 allow us to formulate conditions of solvability of equation (24), define the ring where this solution lays and estimate the rate of convergence of successive approximations.

3) Let us consider the nonlinear integral equation

$$x(t) = \int_a^b K(t,s,x(s),x(t)) \, ds, \quad (33)$$

where the function $K(t,s,u,v)$ is a measurable function with respect to the variables $t,s$ and is continuous with respect to the variables $u,v$ and $x$ is the unknown function.

First of all let us consider equation (33) in the space $C[a,b]$. Let us assume that the function $K$ satisfies the following condition:

$$|K(t,s,u_1,v_1) - K(t,s,u_2,v_2)| \leq l(t,s,r)|u_1 - u_2| + m(t,s,r)|v_1 - v_2|$$

$$\left( |u_1|, |u_2|, |v_1|, |v_2| \leq r, \quad 0 < r \leq \infty \right),$$

where $l(t,s,r)$ and $m(t,s,r)$ are nonnegative and nondecreasing functions in $[a,b] \times [a,b] \times [0,R]$.

Then the operator

$$Ax(t) = \int_a^b K(t,s,x(s),x(t)) \, ds, \quad (34)$$

satisfies the variable Lipschitz condition in the ball $B[x_0,R]$ with the nonnegative nondecreasing function:

$$k(r) = \max_{a \leq t \leq b} \int_a^b \left( l(t,s,r) + m(t,s,r) \right) ds \quad \text{where } l(t,s,r) \text{ is defined in (25).}$$

In this case,

$$a_\pm(r) = a \pm \int_0^r \max_{a \leq t \leq b} \int_a^b \left( l(t,s,\varrho) + m(t,s,\varrho) \right) d\varrho \, d\varrho.$$
Now let us consider equation (33) in the space $L_p[a, b]$. As in the previous example, it seems that the results from the space $C[a, b]$ for the equation under consideration can be easily transferred to the space $L_p[a, b]$. However, for this example, just as in the previous one we ought to consider only the nonlinearity satisfying the Lipschitz condition of special type. Moreover, we can deal only with the case when the nonlinearity $K(t, s, u, v)$ is defined for all $u, v \in \mathbb{R}$ and has power growth with respect to the variables $u$ and $v$.

Let us assume that the following conditions are satisfied:

$$\|K(t, s, u_1, v_1) - K(t, s, u_2, v_2)\| \leq \left( \sum_{j=0}^{\mu} a_j(t, s)r^{\theta_j} \right) |u_1 - u_2| + \left( \sum_{k=0}^{\nu} b_k(t, s)r^{\vartheta_k} \right) |v_1 - v_2|$$

$$(|u_1|, |u_2| \leq r, \; 0 = \theta_0 < \theta_1 < \ldots < \theta_{\mu} \leq p - 1, \; 0 \leq \vartheta_0 < \vartheta_1 < \ldots < \vartheta_{\nu} \leq p),$$

where $a_j(t, s)$ lies in the Zaanen space: $a_j(t, s) \in Z(\frac{p}{1+\theta_j}, p')$ ($p' = p/(p-1)$), and $b_k(t, s)$ lies in the Zaanen space: $b_j(t, s) \in Z(\frac{p}{1+\vartheta_k}, p')$ ($p' = p/(p-1)$).

This inequality implies

$$|Ax_1(t) - Ax_2(t)| \leq \sum_{j=0}^{\mu} \int_a^b a_j(t, s)r(s)^{\theta_j}|x_1(s) - x_2(s)|\, ds + \sum_{k=1}^{\nu} \int_a^b b_j(t, s)r(s)^{\vartheta_k} ds |x_1(t) - x_2(t)|,$$

where $r(s) = \sup \{|x_1(s)|, |x_2(s)|\}$. Repeating the argument used in the previous example, we have

$$\|Ax_1(t) - Ax_2(t)\|_{L_p} \leq \left( \sum_{j=1}^{\mu} \|a_j\|_{Z(\frac{p}{1+\theta_j}, p')}r^{\theta_j} + \sum_{k=0}^{\nu} \|b_k\|_{Z(\frac{p}{1+\vartheta_k}, 1)}r^{\vartheta_k} \right) \|x_1(t) - x_2(t)\|_{L_p}.$$ 

Thus operator (34) satisfies the variable Lipschitz condition in the ball $B[0, R]$ with the non-negative nondecreasing function

$$k(r) = \sum_{j=1}^{\mu} \|a_j\|_{Z(\frac{p}{1+\theta_j}, p')}r^{\theta_j} + \sum_{k=0}^{\nu} \|b_k\|_{Z(\frac{p}{1+\vartheta_k}, 1)}r^{\vartheta_k}.$$ 

Thus by means of the function $k(\cdot)$ we can define the functions

$$a_{\pm}(r) = a \pm \left( \sum_{j=1}^{\mu} \|a_j\|_{Z(\frac{p}{1+\theta_j}, p')} \frac{r^{1+\theta_j}}{1+\theta_j} + \sum_{k=0}^{\nu} \|b_k\|_{Z(\frac{p}{1+\vartheta_k}, 1)} \frac{r^{1+\vartheta_k}}{1+\vartheta_k} \right).$$

4) Let us consider the nonlinear integral equation

$$x(t) = F\left(t, x(t), \int_a^b K(t, s, x(s))\, ds \right),$$

where $F(t, u, v)$ is a continuous with respect to the variables $u, v$ function for fixed $t$ and is continuous with respect to the variable $t$ and $x$ is the unknown function.
The operator

\[ Ax(t) = F\left(t, x(t), \int_a^b K(t, s, x(s)) \, ds\right), \quad (36) \]

has the form

\[ Ax = F(x, Bx), \quad (37) \]

where \( F \) is the superposition operator \( F(x, y)(t) = F(t, x(t), y(t)) \), and

\[ Bx(t) = \int_a^b K(t, s, x(s)) \, ds. \quad (38) \]

First of all let us consider equation (35) in the space \( C[a, b] \). Let us assume that the operator \( K \) satisfies the following condition:

\[
|K(t, s, u)| \leq n_0(t, s, r) \quad (|u| \leq r), \\
|K(t, s, u_1) - K(t, s, u_2)| \leq n(t, s, r)|u_1 - u_2| \quad (|u_1|, |u_2| \leq r), \\
(\|u_1\|, \|u_2\| \leq r, |v_1|, |v_2| \leq \rho, 0 < r, \rho \leq \infty)
\]

where \( n(t, s, r) \) and \( n_0(t, s, r) \) are the functions that are nonnegative in \([a, b] \times [a, b] \times [0, R]\), nondecreasing with respect to \( r, \rho \) and measurable with respect to \( t, s \).

Then the following inequality is valid for the operator \( B \):

\[
|Bx(t)| \leq \int_a^b n_0(t, s, \|x\|) \, ds \quad (\|x\| \leq r)
\]

and

\[
|Bx_1(t) - Bx_2(t)| \leq \int_a^b n(t, s, \|x_1 - x_2\|) \, ds \quad (\|x_1\|, \|x_2\| \leq r).
\]

Further assume that

\[
|F(t, u_1, v_1) - F(t, u_2, v_2)| \leq l(t, r, \rho)|u_1 - u_2| + m(t, r, \rho)|v_1 - v_2|, \quad |u_1|, |u_2| \leq r, |v_1|, |v_2| \leq \rho, \quad (39)
\]

where \( l(t, r, \rho) \) and \( m(t, r, \rho) \) are the functions that are nonnegative in \([0, R] \times [0, R]\), nondecreasing with respect \( r, \rho \) and measurable with respect to \( t \). Then the superposition operator \( F(x, y)(t) = F(t, x(t), y(t)) \) satisfies the inequality

\[
|F(t, x_1, y_1) - F(t, x_2, y_2)| \leq m(t, r, \rho)\|x_1 - x_2\| + n(t, r, \rho)\|y_1 - y_2\| \quad \|x_1\|, \|x_2\| \leq r, \|y_1 - y_2\| \leq \rho.
\]

As a result, the operator \( A \) satisfies the Lipschitz condition

\[
\|Ax_1 - Ax_2\| \leq \sup_{a \leq t \leq b} \left(l\left(t, r, \int_a^b n_0(t, s, r) \, ds\right) + m\left(t, r, \int_a^b n_0(t, s, r) \, ds\right) \int_a^b n(t, s, r) \, ds\right)\|x_1 - x_2\|,
\]
i.e. it satisfies the variable Lipschitz condition with nonnegative and nondecreasing in \([0, R]\) function

\[
k(r) = \sup_{a \leq t \leq b} \left( l(t, r, \int_a^b n_0(t, s, r)ds) + m(t, r, \int_a^b n_0(t, s, r)ds) \int_a^b n(t, s, r)ds \right).
\]

Thus, the majorant functions \(a_\pm(r)\) for the operator \(A\) are defined by the equation

\[
a_\pm(r) = a \pm \int_0^r \sup_{a \leq t \leq b} l(t, r, \int_a^b n_0(t, s, r)ds) + m(t, r, \int_a^b n_0(t, s, r)ds) \int_a^b n(t, s, r)ds \, d\theta.
\]

Now let us consider equation (43) in the space \(L_p[a, b]\). First, we assume that

\[
|K(t, s, u)| \leq \sum_{j=0}^\mu a_j(t, s)|u|^{\theta_j} \quad (|u| \leq r, \ 0 \leq \theta_0 < \theta_1 < \ldots < \theta_\mu \leq p),
\]

and

\[
|K(t, s, u_1) - K(t, s, u_2)| \leq \sum_{k=0}^\nu b_k(t, s)r^{\vartheta_k} |u_1 - u_2|
\]

\((|u_1|, |u_2| \leq r, \ 0 \leq \vartheta_0 < \vartheta_1 < \ldots < \vartheta_\nu \leq p - 1)\).

Here \(a_j(t, s) \in Z(\frac{p}{\theta_j}, q'), \ b_k(t, s) \in Z(\frac{p}{\vartheta_k}, q')\). Then

\[
\|Kx\|_{L_q} \leq \sum_{j=0}^\mu \left\| \int_\Omega a_j(t, s)|x(s)|^{\theta_j}ds \right\|_{L_q} \leq \sum_{j=0}^\mu \|a_j\|_{Z(\frac{p}{\theta_j}, q')}r^{\theta_j}.
\] (40)

and

\[
\|Kx_1 - Kx_2\|_{L_q} \leq \sum_{k=0}^\nu \left\| \int_a^b b_k(t, s)r^{\vartheta_k} |x_1(s) - x_2(s)|ds \right\|_{L_q}
\]

\[
\leq \sum_{k=0}^\nu \|b_k(t, s)\|_{Z(\frac{p}{\vartheta_k}, q')}r^{\vartheta_k} \|x_1 - x_2\|_{L_p}.
\] (41)

(in the proof of (41) the argument used in Example 2 is applied).

Further, assume that

\[
|F(t, u_1, v_1) - F(t, u_2, v_2)| \leq c|u_1 - u_2| + (\mu(t) + \nu^{\frac{q-p}{p}})|v_1 - v_2|,
\]

\[
|v_1|, |v_2| \leq \rho, \ 0 < \rho < \infty, \ \mu(t) \in L_{\frac{qp}{q-p}}.
\] (42)

Then

\[
\|F(x_1, y_1) - F(x_2, y_2)\|_{L_p} \leq c\|x_1 - x_2\|_{L_p} + \left( \|\mu\|_{L_{\frac{qp}{q-p}}} + \nu^{\frac{q-p}{p}} \right) \|y_1 - y_2\|_{L_q}
\]

(again in the proof of this inequality the argument used in Example 2 is applied) and furthermore

\[
\|F(x_1, y_1) - F(x_2, y_2)\|_{L_p} \leq c\|x_1 - x_2\|_{L_p} + \inf_{(\mu, \nu) \in T(F)} \left( \|\mu\|_{L_{\frac{qp}{q-p}}} + \nu^{\frac{q-p}{p}} \right) \|y_1 - y_2\|_{L_q},
\]

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where $T(F)$ is the set of pairs $(\mu, \nu)$ for which inequality (42) holds.

Summing up all these inequalities we get

$$\|Ax_1 - Ax_2\|_{L_p} \leq \left( c + \inf_{(\mu, \nu) \in T(F)} \left( \|\mu\|_{L_{\frac{qp}{q-p}}} + \nu \left( \sum_{j=0}^{\mu} \|a_j\|_{Z(\frac{p}{q-j}, q')}^{\frac{q-p}{p}} \right) \right) \right) \times \nu \sum_{k=0}^{\nu} \|b_k(t, s)\|_{Z(\frac{p}{1+\varepsilon k}, q')}^{r_{\theta_k}} \|x_1 - x_2\|_{L_p}.$$

As a result, we have the following formulas for the Lipschitz constant $k(r)$ and majorant functions $a_{\pm}(r)$ of the operator $A$:

$$k(r) = c + \inf_{(\mu, \nu) \in T(F)} \left( \|\mu\|_{L_{\frac{qp}{q-p}}} + \nu \left( \sum_{j=0}^{\mu} \|a_j\|_{Z(\frac{p}{q-j}, q')}^{\frac{q-p}{p}} \right) \right) \times \nu \sum_{k=0}^{\nu} \|b_k(t, s)\|_{Z(\frac{p}{1+\varepsilon k}, q')}^{r_{\theta_k}}$$

$$a_{\pm}(r) = a$$

$$\pm \left( cr + \int_{0}^{r} \inf_{(\mu, \nu) \in T(F)} \left( \|\mu\|_{L_{\frac{qp}{q-p}}} + \nu \left( \sum_{j=0}^{\mu} \|a_j\|_{Z(\frac{p}{q-j}, q')}^{\frac{q-p}{p}} \right) \right) \sum_{k=0}^{\nu} \|b_k(t, s)\|_{Z(\frac{p}{1+\varepsilon k}, q')}^{r_{\theta_k}} \right) \, \rho \, d\rho.$$

5. Conclusion

The examples considered above can easily be generalized up to nonlinear operator equations with unknown functions defined on a measurable space $\Omega$ with $\sigma$-finite measure and taking values in a finite dimensional spaces. The reasoning presented in the article reflects the fact that different solvability and uniqueness results can be essentially strengthened on the base of a deeper analysis of the Lipschitz condition. Note also that in the case when $k(r)$ does not depend on $r$ the majoration fixed point principle is reduced to the Banach – Caccioppoli principle. At last, recall that the analogue of majoration fixed point principle is not valid for operators in arbitrary complete metric spaces.

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