MAXIMAL $k$-EDGE-COLORABLE SUBGRAPHS, VIZING’S THEOREM, AND TUZA’S CONJECTURE

GREGORY J. PULEO

Abstract. We prove that if $M$ is a maximal $k$-edge-colorable subgraph of a multigraph $G$ and if $F = \{ v \in V(G) : d_M(v) \leq k - \mu(v) \}$, then $d_F(v) \leq d_M(v)$ for all $v \in V(G)$ with $d_M(v) < k$. (When $G$ is a simple graph, the set $F$ is just the set of vertices having degree less than $k$ in $M$.) This implies Vizing’s Theorem as well as a special case of Tuza’s Conjecture on packing and covering of triangles. A more detailed version of our result also implies Vizing’s Adjacency Lemma for simple graphs.

1. Introduction

A proper $k$-edge-coloring of a multigraph $G$ without loops is a function $\psi : E(G) \rightarrow [k]$ such that $\psi(e) \neq \psi(f)$ whenever $e$ and $f$ are distinct edges sharing an endpoint (or both endpoints), where $[k] = \{1, \ldots, k\}$. A graph is $k$-edge-colorable if it admits a proper $k$-edge-coloring. We will tacitly assume in the rest of this paper that all multigraphs under consideration are loopless.

A fundamental theorem concerning edge-coloring is Vizing’s Theorem [30]. Given a multigraph $G$, we write $\mu_G(v, w)$ for the number of edges joining two vertices $v$ and $w$, and we write $\mu_G(v)$ for $\max_{w \in V(G)} \mu_G(v, w)$. When the graph $G$ is understood, we omit the subscripts. We also write $\Delta(G)$ for the maximum degree of $G$ and $\mu(G)$ for $\max_{v \in V(G)} \mu(v)$. Vizing’s Theorem can then be stated as follows:

Theorem 1.1 (Vizing [30]). If $G$ is a multigraph and $k \geq \Delta(G) + \mu(G)$, then $G$ is $k$-edge-colorable.

Following the notation of [27], let $\Delta^\mu(G) = \max_{v \in V(G)}[d(v) + \mu(v)]$. Since $\Delta^\mu(G) \leq \Delta(G) + \mu(G)$ for any multigraph $G$, and since this inequality is sometimes strict, the following theorem of Ore [22] strengthens Theorem 1.1.

Theorem 1.2 (Ore [22]). If $G$ is a multigraph and $k \geq \Delta^\mu(G)$, then $G$ is $k$-edge-colorable.

In this paper, we prove the following generalization of Theorem 1.2. Here, when $F \subset V(G)$, we write $d_F(v)$ for $\sum_{w \in F} \mu(v, w)$, and when $M \subset E(G)$, we write $d_M(v)$ for the total number of $M$-edges incident to $v$.

Theorem 1.3. Let $G$ be a multigraph, let $k \geq 1$, and let $M$ be a maximal $k$-edge-colorable subgraph of $G$. If $F = \{ v \in V(G) : d_M(v) \leq k - \mu(v) \}$, then for every $v \in V(G)$ with $d_M(v) < k$, we have $d_F(v) \leq d_M(v)$.

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Theorem 1.13 is easiest to understand in the case of simple graphs, where \( \mu(v) = 1 \) for all \( v \). In this case, \( F \) is just the set of all vertices with fewer than \( k \) colors present on the incident edges, that is, the set of all vertices missing at least one color.

It is also instructive to consider Theorem 1.13 in the cases \( k = 1 \) and \( k = 2 \). Since a maximal matching in a graph \( G \) is just a maximal 1-edge-colorable subgraph of \( G \), the \( k = 1 \) case of Theorem 1.13 just states the observation that the set of vertices left uncovered by a maximal matching is independent.

In the case \( k = 2 \), we can observe that in a maximal 2-edge-colorable subgraph \( M \subset G \), every component of \( M \) is an even cycle or a path (possibly a 1-vertex path), and the vertices of \( F \) are the endpoints of the path components. Theorem 1.13 then states that \( G[F] \) induces a graph consisting of a matching together with possibly some isolates, where all vertices isolated in \( M \) are also isolated in \( G[F] \). This conclusion is not difficult to prove directly, as the maximality of \( M \) implies that the only \( G \)-edges among the vertices of \( F \) are edges that join the endpoints of the same path, if this would yield an odd cycle.

For \( k > 2 \), no simple characterization of \( k \)-edge colorable graphs is known, so a direct appeal to the structure of \( M \) is not possible. However, Theorem 1.13 still yields the following corollary.

**Corollary 1.4.** If \( G \) is a simple graph, \( M \) is a maximal \( k \)-edge-colorable subgraph of \( G \), and \( F \) is the set of vertices with fewer than \( k \) incident \( M \)-edges, then \( \Delta(G[F]) \leq k - 1 \).

To see that Theorem 1.13 implies Theorem 1.2, observe that if \( k \geq \Delta^\mu(G) \) and \( M \) is a maximal \( k \)-edge-colorable subgraph of \( G \), then \( F = V(G) \), so Theorem 1.13 states that \( d_M(v) \geq d_G(v) \) for every vertex \( v \). As \( M \) is a subgraph of \( G \), this implies \( M = G \), so that \( G \) is \( k \)-edge-colorable. In Section 6 we show that Theorem 1.13 also implies a multigraph version of a strengthening of Vizing’s Theorem due to Lovasz and Plummer [20] and to Berge and Fournier [5].

In order to prove Theorem 1.13 we actually prove a more technical version of the theorem, with a somewhat stronger conclusion. This version of Theorem 1.13 is similar to Vizing’s Adjacency Lemma, and we explore the connection in more detail in Section 5.

**Definition 1.5.** Given a multigraph \( G \), a subgraph \( M \subset G \), and an integer \( k \geq 1 \), for each \( v \in V(G) \) we define vertex sets \( F(v) \) and \( U(v) \) by

\[
F(v) = \{ w \in N(v) : d_M(w) \leq k - \mu_G(v, w) \}, \\
U(v) = \{ w \in F(v) : \mu_M(v, w) < \mu_G(v, w) \}.
\]

We also write \( d^F(v) \) for \( d_{F(v)}(v) \), that is, \( d^F(v) \) is the total number of edges from \( v \) to the vertices in \( F(v) \). The superscript here is meant to emphasize that the \( F \) in this notation is a set depending on \( v \), rather than being a fixed set as in Theorem 1.13 Figure 1 illustrates the definition of \( F(v) \) and \( U(v) \).

**Theorem 1.6.** Let \( G \) be a multigraph, let \( k \geq 1 \), and let \( M \) be a maximal \( k \)-edge-colorable subgraph of \( G \). For every \( v \in V(G) \) with \( d_M(v) < k \), we have

\[
d^F(v) \leq d_M(v) - \sum_{w \in U(v)} (k - d_M(w) - \mu_G(v, w)).
\]

\(^{1}\)The letter \( F \) is meant to evoke the word “deficient”, the letter \( D \) being unavailable since it is used in a different context in this paper.
Figure 1. Illustration of $F(v)$ and $U(v)$ for a vertex $v$, in the case $k = 4$. Thick edges denote edges in $M$; vertices have no incident edges aside from those pictured.

Note that since $U(v) \subset F(v)$ by definition, we have $d_M(w) \leq k - \mu_G(v, w)$ for all $w \in U(v)$, so that each term $k - d_M(w) - \mu_G(v, w)$ in the above sum is nonnegative. Furthermore, when $F_0$ is the set defined in Theorem 1.3, we see that $(N(v) \cap F_0) \subset F(v)$ for all $v \in V(G)$. Thus, Theorem 1.6 indeed strengthens Theorem 1.3.

We now consider a conjecture of Tuza regarding packing and covering of triangles.

**Definition 1.7.** Given a graph $G$, let $\tau(G)$ denote the minimum size of an edge set $X$ such that $G - X$ is triangle-free, and let $\nu(G)$ denote the maximum size of a set of pairwise edge-disjoint triangles in $G$.

It is easy to show that $\nu(G) \leq \tau(G) \leq 3\nu(G)$: if $S$ is a largest set of pairwise edge-disjoint triangles, then to make $G$ triangle-free we must delete at least one edge from each triangle of $S$, and on the other hand deleting all edges contained in triangles of $S$ will always make $G$ triangle-free. Tuza conjectured a stronger upper bound.

**Conjecture 1.8** (Tuza’s Conjecture [28, 29]). $\tau(G) \leq 2\nu(G)$ for all graphs $G$.

Tuza’s Conjecture is sharp, if true; as observed by Tuza [29], equality in the upper bound is achieved by any graph whose blocks are all isomorphic to $K_4$, among other examples. The best general upper bound on $\tau(G)$ in terms of $\nu(G)$ is due to Haxell [12], who showed that $\tau(G) \leq 2.87\nu(G)$ for all graphs $G$. Tuza’s Conjecture has been studied by many authors, who proved the conjecture for special classes of graphs [14, 18, 23, 25, 26] or studied various fractional relaxations of the conjecture [6, 13, 15, 18].

A major theme of the author’s previous work on Tuza’s Conjecture [23] is to reduce questions about triangle packings to questions about matchings, since matchings are very well understood. To further pursue this idea, we study the conjecture on graphs of the form $I_k \lor H$, where $I_k$ is an independent set of size $k$, $H$ is a triangle-free graph, and the join $G_1 \lor G_2$ of two graphs $G_1$ and $G_2$ is obtained from the disjoint union of $G_1$ and $G_2$ by adding all possible edges between $V(G_1)$ and $V(G_2)$. Each triangle of $I_k \lor H$ consists of an edge in $H$ together with a vertex of $I_k$; thus, triangle packings in $I_k \lor H$ correspond to partial $k$-edge-colorings of $H$. In Section 4, we prove a similar correspondence for edge sets whose deletion results in a triangle-free graph, and we use Theorem 1.3 to prove the following special case of Tuza’s Conjecture.

**Theorem 1.9.** If $H$ is triangle-free and $k \geq 1$, then $\tau(I_k \lor H) \leq 2\nu(I_k \lor H)$. 

A similar idea, restricted to $k = 1$, appears in [15] and [6], where $H$ is taken to be a triangle-free Ramsey graph with small independence number, and graphs of the form $I_1 \lor H$ are used as sharpness examples for upper bounds on $\tau(G)$.

The rest of the paper is structured as follows. In Section 2 we prove Theorem 1.6, which implies Theorem 1.6. In Section 3 we use Theorem 1.3 to prove a stronger version of Theorem 1.2. In Section 4 we use Theorem 1.3 to prove Theorem 1.9; we also state a conjecture arising naturally from the tools used in this proof. Finally, in Section 5 we show that Theorem 1.6 implies the simple-graph case of Vizing’s Adjacency Lemma.

2. Proof of Theorem 1.6

Let $y \in V(G)$ with $d_M(y) < k$ be given. We will show that $d^F(y) \leq d_M(y) - \sum_{z \in U(y)} (k - d_M(z) - \mu_G(y,z))$. Fix a proper $k$-edge-coloring $\psi$ of $M$. We use a family of auxiliary multidigraphs defined by Kostochka [17].

**Definition 2.1.** For distinct $w, z \in V(G)$, let $\psi(w, z)$ be the set of colors used by $\psi$ on edges joining $w$ and $z$. (If there are no edges joining $w$ and $z$, then $\psi(w, z) = \emptyset$.) For each $w \in V(G)$, let $\psi(w)$ be the set of all colors used on edges incident to $w$, and let $O(w) = [k] \setminus \psi(w)$.

Observe that $O(y)$ is nonempty, since $d_M(y) < k$.

**Definition 2.2.** For each $u \in U(y)$, let $H_u$ be the multidigraph with vertex set $N_M(y) \cup \{u\}$, where the number of arcs $\mu_{H_u}(w, z)$ from $w$ to $z$ is given by

$$\mu_{H_u}(w, z) = |O(w) \cap \psi(y, z)|.$$  

Kostochka proved the following useful properties of the digraphs $H_u$, under the hypothesis that $M + yu$ has no $k$-edge-coloring. By the maximality of $M$, this hypothesis holds in our context as well. Recall that $v$ is reachable from $u$ in the digraph $H_u$ if $H_u$ contains a directed path from $u$ to $v$.

**Lemma 2.3** (Kostochka [17]). If $v$ is reachable from $u$ in $H_u$, then $O(v) \cap O(y) = \emptyset$.

**Definition 2.4.** When $\alpha$ and $\beta$ are colors, an $[\alpha, \beta]$-path is a path in $M$ whose edges (under the coloring $\psi$) are alternately colored $\alpha$ and $\beta$. For $v, w \in V(M)$, an $[\alpha, \beta](v, w)$-path is an $[\alpha, \beta]$-path whose endpoints are $v$ and $w$.

**Lemma 2.5** (Kostochka [17]). If $v$ is reachable from $u$ in $H_u$, then for each $\alpha \in O(y)$ and each $\beta \in O(v)$, there is an $[\alpha, \beta](y, v)$-path.

Kostochka [17] focused on studying graphs for which the maximal $k$-edge-colorable subgraph $M$ consists of all edges of $G$ except a single edge $yu$, and therefore focused on a single digraph $H_u$. In contrast, we work with graphs for which $M$ may be much smaller, and therefore wish to work with many of these digraphs simultaneously, which is facilitated by the following definitions.

**Definition 2.6.** Say that $z \in N_M(y)$ is remote if for all $u \in U(y)$, the vertex $z$ is not reachable from $u$ in $H_u$. For each $w \in U(y) \cup N_M(y) \cup \{y\}$, define $C(w)$ as follows: if $w$ is remote, then $C(w) = \psi(y, w)$, and otherwise $C(w) = O(w)$. (In particular, $C(y) = O(y)$.)

**Observation 2.7.** Since, by definition, $F(y) \subset N_G(y)$, we have $F(y) \subset U(y) \cup N_M(y)$, so that $C(z)$ is defined for every $z \in F(y)$. If $z$ is remote and $z \in F(y)$, then in particular, $z \notin U(y)$, so $\mu_M(y, z) = \mu_G(y, z)$. 
Our next lemma strengthens Claim 3 of Kostochka [17]. It can also be viewed as generalizing Lemma 1 of Andersen [1] to the context of more than one uncolored edge.

**Lemma 2.8.** For all distinct \( w, z \in N_M(y) \cup \{ y \} \), we have \( C(w) \cap C(z) = \emptyset \).

**Proof.** If \( w = y \) and \( z \) is remote, then \( C(y) \cap C(z) = O(y) \cap \psi(y, z) = \emptyset \). If \( w = y \) and \( z \) is not remote, then Lemma 2.3 implies that \( C(y) \cap C(z) = O(y) \cap O(z) = \emptyset \). Hence we may assume that \( y \notin \{ w, z \} \).

If \( w, z \) are both remote, then since \( \psi \) is a proper coloring, we see that \( C(w) \cap C(z) = \psi(y, w) \cap \psi(y, z) = \emptyset \).

If \( z \) is remote and \( w \) is not remote, then there is some \( u \in U(y) \) such that \( w \) is reachable from \( u \) in \( H_u \), while \( z \) is not reachable from \( u \), so that \( H_u \) has no arc \( wz \). By the definition of \( H_u \), this implies that \( C(w) \cap C(z) = O(w) \cap \psi(y, z) = \emptyset \).

Thus, we may assume that neither \( w \) nor \( z \) is remote. Let \( \alpha \in O(y) \) and suppose that there is some \( \beta \in O(w) \cap O(z) \). Let \( P \) be the unique maximal \([\alpha, \beta]\)-path starting at \( y \). Lemma 2.3 implies that both \( w \) and \( z \) are the other endpoint of \( P \), which is impossible. Hence \( C(w) \cap C(z) = O(w) \cap O(z) = \emptyset \). \( \square \)

Now we complete the proof of Theorem 1.6. First we argue that \( |C(z)| \geq \mu_G(z, y) \) for all \( z \in F(y) \). If \( z \) is remote, then by Observation 2.7 all edges from \( z \) to \( y \) are colored, hence \( |C(z)| = \mu_G(z, y) \). If \( z \) is not remote, then since \( z \in F(y) \), we have \( |C(z)| = |O(z)| = k - d_M(z) \geq \mu_G(z, y) \).

Lemma 2.8 implies that \( \sum_{z \in F(y)} |C(z)| \leq k - |C(y)| \), so we have

\[
\begin{align*}
d^F(y) &= \sum_{z \in F(y)} \mu_G(z, y) \\
&\leq \sum_{z \in F(y)} |C(z)| - \sum_{z \in U(y)} (|C(z)| - \mu_G(z, y)) \\
&\leq k - |C(y)| - \sum_{z \in U(y)} (|C(z)| - \mu_G(z, y)) \\
&= k - |O(y)| - \sum_{z \in U(y)} (|C(z)| - \mu_G(z, y)) \\
&= d_M(y) - \sum_{z \in U(y)} (|C(z)| - \mu_G(z, y)).
\end{align*}
\]

Now for \( z \in U(y) \) we have

\[
|C(z)| = |O(z)| = k - d_M(z),
\]

so we conclude that

\[
d^F(y) \leq d_M(y) - \sum_{z \in U(y)} [k - d_M(z) - \mu_G(z, y)].
\]

3. **Forests of Maximum Degree**

Following the notation of Anstee and Griggs [2], given a multigraph \( G \), we define \( G_\Delta \) to be the subgraph of \( G \) induced by the vertices of maximum degree. We also define \( G_{\Delta, \mu} \) to be the subgraph of \( G \) induced by the vertices which have both maximum degree and maximum multiplicity. (Possibly no such vertices exist, as
occurs when \( \Delta^\mu(G) < \Delta(G) + \mu(G) \); in this case, we consider \( G_{\Delta,\mu} \) to be a graph with no vertices and no edges.)

The following theorems give conditions on \( G_\Delta \) or \( G_{\Delta,\mu} \) which imply the stronger claim that \( G \) can be properly edge-colored with fewer colors than Theorem 1.1 would require.

**Theorem 3.1** (Berge–Fournier [5]). If \( k \geq \Delta(G) + \mu(G) - 1 \) and \( G_{\Delta,\mu} \) has no edges, then \( G \) is \( k \)-edge-colorable.

**Theorem 3.2** (Lovasz–Plummer [20] and Berge–Fournier [5]). If \( \mu(G) = 1 \), \( k \geq \Delta(G) \), and \( G_\Delta \) is a forest, then \( G \) is \( k \)-edge-colorable.

In this section, we use Theorem 1.1 to prove the following common generalization of Theorem 3.1 and Theorem 3.2. Let \( G^* \) be the subgraph of \( G \) induced by all vertices \( v \) such that \( d(v) + \mu(v) = \Delta^\mu(G) \). Note that \( G^* \) only differs from \( G_{\Delta,\mu} \) when \( \Delta^\mu(G) < \Delta(G) + \mu(G) \). For such graphs, Theorem 1.2 implies that \( G \) is \((\Delta(G) + \mu(G) - 1)\)-edge-colorable without any further restriction on the graph structure, while the following theorem implies that \( G \) can be edge-colored with fewer colors if it satisfies certain restrictions on \( G^* \).

**Theorem 3.3.** If \( k \geq \Delta^\mu(G) - 1 \) and \( G^* \) has no cycle of length greater than 2, then \( G \) is \( k \)-edge-colorable.

Equivalently, the hypothesis of Theorem 3.3 is that merging parallel edges in \( G^* \) should yield a forest. As in the proof of Theorem 1.2 from Theorem 1.3, our proof will not make explicit reference to any particular edge-coloring, only to maximal \( k \)-edge-colorable subgraphs of \( G \).

*Proof.* Fixing a value of \( k \), we use induction on \(|E(G^*)|\), with base case when \( G^* \) has no edges or when \( k = \Delta^\mu(G) \). If \( k \geq \Delta^\mu(G) \) then Theorem 1.2 immediately implies that \( G \) is \( k \)-edge-colorable. Thus, we may assume that \( k = \Delta^\mu(G) - 1 \).

Suppose that \( G^* \) has no edges. By Theorem 1.2, \( G - V(G^*) \) is \( k \)-edge-colorable. Among all \( k \)-edge-colorable subgraphs of \( G \) containing \( E(G - V(G^*)) \), choose \( M \) to be maximal. The only possible edges in \( E(G) - E(M) \) are edges incident to vertices of \( G^* \).

Let \( F = \{ v \in V(G) : d_M(v) \leq k - \mu_G(v) \} \), as in Theorem 1.3. For all \( v \in V(G) - V(G^*) \), we have \( d_G(v) + \mu_G(v) < \Delta^\mu(G) \), hence

\[
d_M(v) \leq d_G(v) \leq k - \mu_G(v),
\]

and so \( V(G) - V(G^*) \subset F \).

Now consider any \( v \in V(G^*) \). Since \( G^* \) has no edges, we have \( d_G(v) = d_F(v) \), so if \( d_M(v) < d_G(v) \), then Theorem 1.3 yields the contradiction \( d_F(v) > d_M(v) \geq d_F(v) \). Thus, \( E(G) - E(M) \) has no edge incident to any vertex of \( G^* \). By the choice of \( M \), this implies that \( M = G \). This proves the claim when \( G^* \) has no edges.

Now suppose that \( G^* \) contains some edges. Let \( v \) be a “leaf vertex” in \( G_{\Delta,\mu} \), that is, choose a vertex \( v \) that has exactly one neighbor \( w \) in \( G_{\Delta,\mu} \), possibly with \( \mu(v,w) > 1 \). Let \( M \) be the graph obtained from \( G \) by removing one copy of the edge \( vw \).

We claim that \( M \) is \( k \)-edge-colorable. If \( \Delta^\mu(M) < \Delta^\mu(G) \), then Theorem 1.2 implies that \( M \) is \( k \)-edge-colorable. On the other hand, if \( \Delta^\mu(M) = \Delta^\mu(G) \), then \( V(M^*) = V(G^*) \cup \{ v, w \} \), so the induction hypothesis implies that \( M \) is \( k \)-edge-colorable.
Now if $G$ is not $k$-edge-colorable, then $M$ is a maximal $k$-edge-colorable subgraph of $G$. Let $F = \{v \in V(G): d_M(v) \leq k - \mu(v)\}$, as in Theorem 1.3 As before, $V(G) - V(G^*) \subset F$. Furthermore, as $v$ and $w$ are both incident to the uncolored edge $vw$, we have $v, w \in F$. Thus all neighbors of $v$ lie in $F$, since $v$ has no other neighbor in $G^*$. Theorem 1.3 now yields the contradiction $d_F(v) = d_G(v) \geq d_M(v) \geq d_F(v)$. It follows that $G$ is $k$-edge-colorable.

4. Tuza’s Conjecture

In this section, we consider only simple graphs.

**Definition 4.1** (Fink–Jacobson [10, 11]). For positive integers $k$, a vertex set $D \subset V(G)$ is $k$-dependent if the induced subgraph $G[D]$ has maximum degree at most $k - 1$. A vertex set $D$ is $k$-dominating if $|N(v) \cap D| \geq k$ for all $v \in V(G) - D$.

**Definition 4.2.** For any set $D \subset V(G)$ and any $k \geq 1$, define $\phi_k(D) = k|D| - |E(G[D])|$, and define $\phi_k(G) = \max_{D \subset V(G)} \phi_k(D)$. A $k$-optimal set is a $k$-dependent set achieving this maximum value of $\phi_k$.

The notation $\phi_k(D)$ is borrowed from the survey paper [7], but the function $\phi_k$ appears to have first been studied by Favaron [9], who proved that every $k$-optimal set is $k$-dominating, thereby answering a question posed by Fink and Jacobson [10, 11]. While [9, 7] considered sets which maximize $\phi_k$ only over $k$-dependent vertex sets, rather than considering a maximum over all vertex sets as we do here, the following lemma shows that this does not change the maximum value achieved.

**Lemma 4.3.** If $G$ is a graph and $T \subset V(G)$, then for any $k \geq 1$, there is a $k$-dependent subset $D \subset T$ such that $\phi_k(D) \geq \phi_k(T)$. In particular, every graph has a $k$-optimal set.

**Proof.** If $T$ is not $k$-dependent, then there is some $v \in T$ with $d_T(v) \geq k$; now $\phi_k(T - v) \geq \phi_k(T)$. Repeatedly removing such vertices yields the desired $k$-dependent subset. □

**Definition 4.4.** For a graph $G$ and $k \geq 1$, let $\alpha'_k(G)$ denote the largest number of edges in a $k$-edge-colorable subgraph of $G$.

**Theorem 4.5.** If $H$ is triangle-free, then

$$\nu(I_k \vee H) = \alpha'_k(H), \text{ and}$$

$$\tau(I_k \vee H) = k|V(H)| - \phi_k(H).$$

**Proof.** Let $G = I_k \vee H$. We first show that $\nu(G) = \alpha'_k(H)$. Let $S$ be a maximum set of edge disjoint triangles in $G$. For each $v \in I_k$, let $S_v = \{T \in S: v \in T\}$. Since each triangle in $G$ consists of exactly one vertex of $I_k$ together with an edge in $H$, we can write $S$ as the disjoint union $S = \bigcup_{v \in I_k} S_v$. Since the triangles in $S_v$ are edge-disjoint, no two triangles in $S_v$ can share a common vertex $w \in V(H)$: if this were the case, they would intersect in the edge $vw$. Hence the edges of $S_v$ that lie in $H$ form a matching $M_v$ in $H$. Since the triangles in $S$ are edge-disjoint, it follows that the matchings $M_v$ are pairwise disjoint, so $\bigcup_{v \in I_k} M_v$ is a $k$-edge-colorable subgraph of $H$ having size $\nu(G)$. Therefore, $\nu(G) \leq \alpha'_k(H)$.

On the other hand, if $H_0$ is a maximum $k$-edge-colorable subgraph of $H$, then we can write $E(H_0) = M_1 \cup \cdots \cup M_k$, where each $M_i$ is a matching. Let $v_1, \ldots, v_k$
be the vertices of $I_k$, and for $i \in [k]$, let $S_i = \{v_i w z: w z \in M_i\}$. Now $\bigcup_{i \in [k]} S_i$ is a family of $\alpha'_k(G)$ pairwise edge-disjoint triangles in $G$, so $\nu(G) \geq \alpha'_k(G)$.

Next we show that $\tau(G) = k |V(H)| - \phi_k(H)$. Let $D$ be a $k$-optimal subset of $V(H)$, and define an edge set $X$ by

$$X = E(G[D]) \cup \{vw: v \in I_k \text{ and } w \in V(H) - D\}.$$ 

Clearly, $|X| = |E(G[D])| + k(|V(H)| - |D|)$, which rearranges to $|X| = k |V(H)| - \phi_k(H)$, since $D$ is $k$-optimal. We claim that $G - X$ is triangle-free. Let $T$ be any triangle in $G$; we may write $T = uwv$, where $uw \in E(H)$ and $v \in I_k$. If $u \notin D$, then $vu \in X$, and likewise for $w$. On the other hand, if $u, w \in D$, then $uw \in X$. Hence $G - X$ is triangle-free, so $\tau(G) \leq k |V(H)| - \phi_k(H)$.

Conversely, let $X$ be a minimum edge set such that $G - X$ is triangle-free. For each $v \in I_k$, let $C_v = \{w \in V(H): vw \in X\}$. We transform $X$ so that all the sets $C_v$ are equal: pick $v^* \in I_k$ to minimize $|C_{v^*}|$, and define $X_1$ by

$$X_1 = (X \cap E(H)) \cup \{vw: w \in C_{v^*}\}.$$ 

Now $G - X_1$ is triangle-free: if $vwz$ is a triangle in $G - X_1$, then $v^*wz$ is a triangle in $G - X$, contradicting the assumption that $G - X$ is triangle-free. Furthermore, by the minimality of $|C_{v^*}|$, we have $|X_1| \leq |X|$.

Therefore, $|X_1| = |X| = \tau(G)$. Let $D = V(H) - C_{v^*}$. Since $G - X_1$ is triangle-free, we have $E(G[D]) \subset X_1$, and so

$$|X_1| \geq |E(G[D])| + k |V(H) - D| = k |V(H)| - \phi_k(D).$$ 

As $\phi_k(D) \leq \phi_k(H)$, we conclude that $\tau(G) = \tau(X_1) \geq k |V(H)| - \phi_k(H)$. \hfill \Box

**Theorem 4.6.** For any graph $G$ and any $k \geq 1$, $2\alpha'_k(G) \geq k |V(G)| - \phi_k(G)$.

*Proof.* Let $M$ be a maximal $k$-edge-colorable subgraph of $G$, and let $F = \{v \in V(G): d_M(v) < k\}$. By the degree-sum formula and Theorem 1.3, we have

$$2 |E(M)| = \sum_{v \in V(G)} d_M(v)$$

$$= k |V(G)| - k |F| + \sum_{v \in F} d_M(v)$$

$$\geq k |V(G)| - k |F| + \sum_{v \in F} d_F(v)$$

$$\geq k |V(G)| - k |F| + |E(G[F])|$$

$$= k |V(G)| - \phi_k(F) \geq k |V(G)| - \phi_k(G).$$ \hfill \Box

**Corollary 4.7.** If $H$ is triangle-free and $k \geq 1$, then $\tau(I_k \lor H) \leq 2\nu(I_k \lor H)$.

The problem of finding lower bounds on $\alpha'_k(G)$ has been studied by several authors [3, 8, 16, 21, 24], usually with the goal of finding approximation algorithms. While Theorem 4.6 gives a lower bound on $\alpha'_k(G)$, the same bound applies even for “small” maximal $k$-edge-colorable subgraphs of $G$, and therefore typically will not be sharp.

We close this section with a conjecture concerning $k$-optimal sets which would furnish an alternative proof of Theorem 4.6. First consider the case $k = 1$. A
1-optimal set in a graph $G$ is just a maximum independent set of $G$, and a 1-edge-colorable subgraph is just a matching. The following theorem of Berge therefore relates 1-optimal sets and 1-edge-colorable subgraphs of $G$.

**Theorem 4.8 (Berge [4]).** An independent set $D$ is a maximum independent set if and only if, for every independent set $T$ disjoint from $D$, there is a matching of $T$ into $D$.

**Corollary 4.9.** If $D$ is a maximum independent set in a graph $G$, then $G$ has a matching that covers every vertex of $V(G) - D$.

**Proof.** Let $M_1$ be a maximal matching in $V(G) - D$, and let $S$ be the set of vertices in $V(G) - D$ not saturated by $M_1$. Since $M_1$ is a maximal matching, $S$ is an independent set. By Theorem 4.8, there is a matching $M_2$ of $S$ into $D$. Thus, $M_1 \cup M_2$ is a matching that covers $V(G) - D$. \qed

Corollary 4.9 suggests the following generalization to higher values of $k$.

**Conjecture 4.10.** If $D$ is a $k$-optimal set in a graph $G$, then $G$ has a $k$-edge-colorable subgraph $M$ such that $d_M(v) = k$ for all $v \in V(G) - D$.

Conjecture 4.10 would be, in some sense, a converse to Corollary 1.4, which states that for every maximal $k$-edge-colorable subgraph $M$, the set of vertices having $M$-degree less than $k$ is a $k$-dependent set. We also remark that Lovasz’s ($g, f$)-factor theorem [19] implies the following weaker version of Conjecture 4.10.

**Proposition 4.11.** If $D$ is a $k$-optimal set in a graph $G$, then $G$ has a subgraph $M$ of maximum degree at most $k$ such that $d_M(v) = k$ for all $v \in V(G) - D$.

5. **Theorem 1.6 and Vizing’s Adjacency Lemma**

Say that an edge $e$ in a multigraph $G$ is critical if $\chi'(G - e) < \chi'(G)$. Vizing’s Adjacency Lemma [30] was originally formulated for simple graphs $G$ with $\chi'(G) = \Delta(G) + 1$ such that every edge is critical. The following multigraph formulation of Vizing’s Adjacency Lemma [30] was given by Andersen [1].

**Lemma 5.1 (Andersen [1]).** Let $G$ be a graph with $\chi'(G) = \max_{e \in V(G)}[d(v) + \mu(v)]$, and let $xy$ be a critical edge of $G$. If $t = d(x) + \mu(x, y)$, then $y$ has at least $\chi'(G) - t + 1$ neighbors $z$ other than $x$ such that $d(z) + \mu(y, z) = \chi'(G)$.

We show that the simple graph case of Lemma 5.1 follows from Theorem 1.6. However, the fully general multigraph case requires a more detailed analysis, and in that case it seems we can do no better than rewording the proof given by Andersen [1]; thus, we consider only simple graphs.

**Proof of Lemma 5.1 for simple graphs.** In the simple graph case, we have $\chi'(G) = \Delta(G) + 1$. Let $k = \chi'(G) - 1 = \Delta(G)$ and let $M = G - xy$. By hypothesis, $M$ is a maximal $k$-edge-colorable subgraph of $G$. For $z \in N_G(y) - \{x\}$, we have $d_M(z) = d_G(z)$, so if $d(z) < \Delta(G)$, then $z \in F(y)$. Furthermore, $x \in F(y)$, since $d_M(x) + 1 = d_G(x) \leq k$.

It follows that if $z \in N_G(y) - F(y)$, then $z$ is a neighbor of $y$ other than $x$ such that $d(z) + \mu(y, z) = \chi'(G)$. Thus, the desired claim follows if we can show that $|N_G(y) - F(y)| \geq \chi'(G) - t + 1$. 


Since \( x \in F(y) \) and since \( xy \) is the only uncolored edge in the graph, we have \( U(y) = \{x\} \). Therefore, the conclusion of Theorem 1.6 yields

\[
d^F(y) \leq d_M(y) - (k - d_M(x) - 1) = d_M(y) - (k - d_G(x)).
\]

Since \( d_G(y) = d_M(y) + 1 \), this rearranges to

\[
d_G(y) - d^F(y) \geq k - d_G(x) + 1 = \chi'(G) - t + 1.
\]

Since \( G \) is a simple graph, we have |\( N_G(y) - F(y) \)| = \( d_G(y) - d^F(y) \), so we are done. \( \square \)

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Coordinated Science Lab, University of Illinois at Urbana-Champaign. Now at Department of Mathematics and Statistics, Auburn University.