Lie symmetries and similarity solutions for the generalized Zakharov equations

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ABSTRACT

The theory of Lie point symmetries is applied to study the generalized Zakharov system with two unknown parameters. The system reduces into a three-dimensional real value functions system, where we find that admits five Lie point symmetries. From the resulting point, we focus on these which provide travel-wave similarity transformation. The reduced system can be integrated while we remain with a system of two second-order nonlinear ordinary differential equations. The parameters of the latter system are classified in order the equations to admit Lie point symmetries.

Exact travel-wave solutions are found, while the generalized Zakharov system can be described by the one-dimensional Ermakov-Pinney equation.

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Key Words and Phrases: Symmetries; Travel-wave; Plasma; Zakharov system.

1 Introduction

One side, it is usual practice for many years to develop a mathematical model to study the nature of the real life phenomena. In which differential equations are playing a crucial role to describe the phenomena very clearly. Another side, many researchers are trying to solve the differential equations by using various methods. Especially, Sophus Lie, during the period 1872 – 1899 [1, 2, 3, 4, 5, 6, 7], developed and applied Symmetry Analysis method successfully to solve the differential equations. Without any ansatz, the Symmetry Analysis directly can apply in a systematic way to
derive solutions of differential equations. Later, this theory was exploited by the Russian school with L. V. Ovsiannikov [8, 9, 10] that since 1960 over the explicit construction of solutions of any sort of problems, even complicated, of mathematical physics. During the last few decades, Lie’s theory has been given much attention on both theoretical and applied point of view [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

Algorithm of Lie’s theory to differential equations is completely algorithmic involves tedious calculations even for linear differential equations with constant coefficients. To overcome difficulties we have access to powerful Computer Algebra Systems (CAS) like Maple and Mathematica (commercial), etc. which are enabling us to do the calculations rapidly. In the past decades, many types of symmetries have been proposed in the literature such as approximate symmetries [31, 32], generalized symmetries [22, 10], and nonlocal symmetries [22, 15, 24, 33] to quote a few.

The novelty of the Lie’s theory is that provides a systematic way to treat nonlinear differential equations. The main application of Lie’s theory are summarized to: the determination of similarity transformations which are used to reduce the differential equation; to determine invariant solutions, also known as similarity solutions, construct conservation laws and write linearize differential equations [22, 15, 24, 33].

Applications of Lie symmetries cover a wide range of physical and natural systems, from classical mechanics [38, 39], fluid dynamics [40, 41], plasma physics [42], optics [43], gravitational physics [44, 45], financial mathematics [46, 47] and other areas of applied mathematics.

In this work are interesting on the symmetry analysis for the generalized Zakharov equations [48] of plasma physics [49]

\begin{align}
  iU_t + U_{xx} - UV - (|U|^{2m})U &= 0 \\
  V_{tt} - V_{xx} - (|U|^{2m})_{xx} &= 0
\end{align}

where the complex function $U(t, x)$ is the envelope of the high-frequency electric field and the real function $V(t, x)$ plasma density measured from its equilibrium value. Parameters $n, m$ are arbitrary in our consideration, while they describe the nonlinear self-interaction in the high-frequency subsystem which corresponds to a self-focusing effect in plasma physics [50]. For the system (1.1), (1.2) travel-wave solutions were studied in [50].

In the following we focus on the Lie point symmetries for the system (1.1), (1.2) while the free parameters $n, m$ will be determined by Lie’s theory, such that the resulting system to admit additional symmetries, that approach is

\footnote{In this work, for the calculation of the symmetries we use the Mathematica add-on Sym [53, 54].}
inspired by the Ovsiannikov’s classification scheme. The plan of the paper is as follows.

In Section 2 we briefly discuss the theory of Lie point symmetries. The Lie point symmetries of the generalized Zakharov system are derived in Section 3. We find that the system admits a five dimensional Lie algebra, we use the Lie symmetries to study the existence of travel wave solutions. The reduced system is classified according to the admitted Lie point symmetries. Surprisingly, we find the generalized Zakharov equations can be reduced into the Ermakov-Pinney equation. Finally, in Section 4 we summarize our results and we draw our conclusions.

2 Lie’s Theory

Suppose consider an equation

\[ \phi(t, x, y; u, v, u_t, v_t, u_x, v_x, u_y, v_y, u_{tt}, v_{tt}, u_{tx}, v_{tx}, u_{ty}, v_{ty}, u_{xx}, u_{xy}, \ldots) = 0, \]

(2.1)

where \( t, x, y \) are the set of independent variables and \( u, v \) are dependent variables. Infinitesimal point transformation for each variables is defined as in the following manner,

\[
\begin{align*}
\tilde{t}(t, x, y, \epsilon) &= t + \epsilon \xi^1(t, x, y) + o(\epsilon^2) = t + \epsilon X t + o(\epsilon^2), \\
\tilde{x}(t, x, y, \epsilon) &= x + \epsilon \xi^2(t, x, y) + o(\epsilon^2) = x + \epsilon X x + o(\epsilon^2), \\
\tilde{y}(t, x, y, \epsilon) &= y + \epsilon \xi^3(t, x, y) + o(\epsilon^2) = y + \epsilon X y + o(\epsilon^2), \\
\tilde{u}(t, x, y, \epsilon) &= u + \epsilon \eta^1(t, x, y) + o(\epsilon^2) = u + \epsilon X u + o(\epsilon^2), \\
\tilde{v}(t, x, y, \epsilon) &= v + \epsilon \eta^2(t, x, y) + o(\epsilon^2) = v + \epsilon X v + o(\epsilon^2),
\end{align*}
\]

where \( X \) is called infinitesimal generator which is denoted by

\[ X = \xi^1(t, x, y) \partial_t + \xi^2(t, x, y) \partial_x + \xi^3(t, x, y) \partial_y + \eta^1(t, x, y) \partial_u + \eta^2(t, x, y) \partial_v. \]

Based of the theory the invariant condition for (2.1) is given by

\[ \phi(t, x, y; u, v) = \phi(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}) \]

It is well known that one can reduce the order of the differential equations as well as number of independent variables by using the infinitesimal generator which is known as symmetries of (2.1).

Therefore, if \( X \) is a Lie point symmetry for equation \( \phi \equiv 0 \), then the following condition is true

\[ X^{[k]} \phi = \lambda \phi, \quad \mod \phi = 0 \]
where $\lambda$ is an arbitrary function, and $X^{[k]}$ is the $k$-th extension of $X$ in the jet-space.

3 Symmetries of Generalized Zakharov Equation

We write the dynamical system (1.1), (1.2) as follows

\[
\begin{aligned}
-q_t + p_{xx} - pv - p(p^2 + q^2) = 0 \\
p_t + q_{xx} - qv - q(p^2 + q^2) = 0 \\
v_{tt} - v_{xx} - 2m(p^2 + q^2)^{m-1}(pp_{xx} + qq_{xx} + p_x^2 + q_x^2) - 4m(m-1)(p^2 + q^2)^{m-2}(pp_x + qq_x)^2 = 0
\end{aligned}
\] (3.1)

after substituting the transformations $U = p + iq$ and $V = v$.

The latter real valued system admits five Lie point symmetries

\[
\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}, \\
\Gamma_2 &= \frac{\partial}{\partial x}, \\
\Gamma_3 &= p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}, \\
\Gamma_4 &= qt\frac{\partial}{\partial p} - pt\frac{\partial}{\partial q} + \frac{\partial}{\partial v}, \\
\Gamma_5 &= -qt^2\frac{\partial}{\partial p} + pt^2\frac{\partial}{\partial q} - 2t\frac{\partial}{\partial v}.
\end{aligned}
\] (3.2)

On the other hand, if we use the variables $U = Re^{i\theta}$, the Lie symmetries are simplified as

\[
\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}, \\
\Gamma_2 &= \frac{\partial}{\partial x}, \\
\Gamma_3 &= \frac{\partial}{\partial \theta}, \\
\Gamma_4 &= t\frac{\partial}{\partial \theta} + \frac{\partial}{\partial v}, \\
\Gamma_5 &= t^2\frac{\partial}{\partial \theta} - 2t\frac{\partial}{\partial v}.
\end{aligned}
\] (3.3)

In the following we continue our analysis by applying the Lie point symmetries which provide travel-wave solutions.

3.1 Travel Wave solution

The traveling wave solution of (3.1) can be constructed by taking linear combination $\Gamma_1$ and $\Gamma_2$. The new canonical variable is $r = x - ct$. Therefore, the system of PDE (3.1) reduces to system of ODE as follows
\[ p'' + cq' - p[v + (p^2 + q^2)^n] = 0 \quad (3.4) \]
\[ q'' - cp' - q[v + (p^2 + q^2)^n] = 0 \quad (3.5) \]
\[ (c^2 - 1)v'' - 2m(p^2 + q^2)^{m-1}(pp'' + qq'' + p'^2 + q'^2) - 
4m(m - 1)(p^2 + q^2)^{m-2}(pp' + qq')^2 = 0 \quad (3.6) \]

where prime represents the derivative with respect to \( r \).

The solution of (3.6) is given by

\[ v = \frac{(p^2 + q^2)^m}{c^2 - 1} + (c_1 r + c_2) \quad (3.7) \]

By substituting the expression of \( v \) in equations (3.4) and (3.5) we have

\[ p'' + cq' = p \left( \frac{(p^2 + q^2)^m}{c^2 - 1} + (c_1 r + c_2) + (p^2 + q^2)^n \right) \quad (3.8) \]
\[ q'' - cp' = q \left( \frac{(p^2 + q^2)^m}{c^2 - 1} + (c_1 r + c_2) + (p^2 + q^2)^n \right) \quad (3.9) \]

The above equations in general are having a rotational symmetry which is symmetry \( \Gamma_3 \). Now we have to solve only the equations (3.8) and (3.9).

Divide equations (3.8) and (3.9) then we have

\[ \frac{p'' + cq'}{q'' - cp'} = \frac{p}{q} \quad (3.10) \]

This implies that

\[ qp'' - pq'' + c(pp' + qq') = 0 \quad (3.11) \]
\[ d(qp' - pq') + \frac{c}{2} d(p^2 + q^2) = 0 \quad (3.12) \]
\[ \frac{qp' - pq'}{p^2 + q^2} + \frac{c}{2} = 0 \quad (3.13) \]
\[ d(\arctan \left( \frac{p}{q} \right)) + \frac{c}{2} = 0 \quad (3.14) \]

From where we determine the solution

\[ \frac{p}{q} = \tan \left( c_3 - \frac{c}{2} r \right) \]

However, for specific values of the free parameters, \( c_1, c_2, n \) and \( m \), the dynamical system (3.8), (3.9) admits additional Lie point symmetries.
3.1.1 Case I: $c_1 = 0$

If $c_1 = 0$ then (3.8) and (3.9) is simplified as

\[
p'' + cq' = p \left( \frac{(p^2 + q^2)^m}{c^2 - 1} + c_2 + (p^2 + q^2)^n \right) \tag{3.15}
\]

\[
q'' - cp' = q \left( \frac{(p^2 + q^2)^m}{c^2 - 1} + c_2 + (p^2 + q^2)^n \right) \tag{3.16}
\]

The solutions of (3.15) and (3.16) are expressed by the functions

\[
p = R(r) \sin[\theta(r)] \tag{3.17}
\]

\[
q = R(r) \cos[\theta(r)] \tag{3.18}
\]

where

\[
\theta(r) = \int \frac{c_3}{R(t)^2} dt + \frac{ct}{2} + c_4
\]

and $R(t)$ can be found from the following equation

\[
R'' = -\frac{1}{4} c^2 R + c_2 R + \frac{R^{2m+1}}{c^2 - 1} + R^{2n+1} + \frac{c_3^2}{R^3} \tag{3.19}
\]

where '$' represents the derivative with respect to $r$.

The trivial symmetry of (3.19) is $\partial_r$. Therefore the above equation reduced as follows. Let $R(r) = R$ and $R' = \phi(R)$ then (3.19)

\[
\phi \frac{d\phi}{dR} = -\frac{1}{4} c^2 R + c_2 R + \frac{R^{2m+1}}{c^2 - 1} + R^{2n+1} + \frac{c_3^2}{R^3} \tag{3.20}
\]

where $\phi$ is a function of $R$. The solution of (3.20) is given by

\[
\phi = \pm \sqrt{\left( c_2 - \frac{c^2}{4} \right) R^2 + \frac{R^{2m+2}}{(c^2 - 1)(m + 1)} + \frac{R^{2n+2}}{n + 1} - \frac{c_3^2}{R^2} + c_5} \tag{3.21}
\]

that is

\[
\int \sqrt{\left( c_2 - \frac{c^2}{4} \right) R^2 + \frac{R^{2m+2}}{(c^2 - 1)(m + 1)} + \frac{R^{2n+2}}{n + 1} - \frac{c_3^2}{R^2} + c_5} \ dR = r - r_0. \tag{3.22}
\]
3.1.2 Case II: $c_1 = c_2 = 0$ and $m = n = -2$

If $c_1 = c_2 = 0$ and $m = n = -2$ then equations (3.8) and (3.9) reduces to

\[
p'' + cq' = \left( \frac{c^2}{c^2 - 1} \right) \frac{p}{(p^2 + q^2)^2} \tag{3.23}
\]
\[
q'' - cp' = \left( \frac{c^2}{c^2 - 1} \right) \frac{q}{(p^2 + q^2)^2} \tag{3.24}
\]

The symmetries of the above equations are given by

\[
\Gamma_7 = \partial_r
\]
\[
\Gamma_8 = p\partial_q - q\partial_p
\]
\[
\Gamma_9 = (cp\sin[cr] + cq\cos[cr])\partial_p + (cq\sin[cr] - cp\cos[cr])\partial_q - 2\cos[cr]\partial_r
\]
\[
\Gamma_{10} = (cq\sin[cr] - cp\cos[cr])\partial_p + (cp\sin[cr] + cq\cos[cr])\partial_q + 2\sin[cr]\partial_r
\]

(3.25)

Surprisingly, the Lie point symmetries $\Gamma_7$, $\Gamma_9$ and $\Gamma_{10}$ for the $SL(2, R)$ Lie algebra, which means the resulting system can be reduced to the Ermakov-Pinney system \[51, 52, 53\].

Indeed under, the change of variables

\[
p = R(r) \sin\left[\int \frac{c_3}{R(t)^2} dt + \frac{ct}{2} + c_4\right] \tag{3.26}
\]
\[
q = R(r) \cos\left[\int \frac{c_3}{R(t)^2} dt + \frac{ct}{2} + c_4\right] \tag{3.27}
\]

we find the analytic solution which is expressed in closed form functions

\[
R(t) = \pm \sqrt{c_4 \left( (c^2 - 1)c^2 + c_3^2c^2 - c_4^2 \exp[-2ict] + 2cc_3c_4^2 \exp[-ict] - c^2c_4^2c_5^2 \right)} \frac{c_4 \exp[-\frac{ic}{2}]}{cc_4 \exp[-\frac{ic}{2}]}. \tag{3.28}
\]

3.1.3 Case III: $m = n = -2$

If $m = n = -2$ then (3.8) and (3.9) reduces to

\[
p'' + cq' = \left( \frac{c^2}{c^2 - 1} \right) \frac{p}{(p^2 + q^2)^2} + (c_1 r + c_2)p \tag{3.28}
\]
\[
q'' - cp' = \left( \frac{c^2}{c^2 - 1} \right) \frac{q}{(p^2 + q^2)^2} + (c_1 r + c_2)q \tag{3.29}
\]

The solution of (3.28) and (3.29) are given by
\[ p = R(r) \sin[\theta(r)] \] (3.30)
\[ q = R(r) \cos[\theta(r)] \] (3.31)

where
\[ \theta(r) = \int \frac{c_3}{R(t)^2} dt + \frac{ct}{2} + c_4 \]

and \( R(t) \) can be found from the following equation
\[ R'' = \left( c_1 r + c_2 - \frac{c^2}{4} \right) R + \left( \frac{c^2}{(c^2 - 1)} + c_3^2 \right) \left( \frac{1}{R^3} \right) \]

the latter equation is nothing else than the Ermakov-Pinney equation, while its symmetries are
\[ \Gamma_{11} = c_1^{\frac{1}{3}} AiryAi \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] AiryAi' \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] R \partial_R \]
\[ + AiroyAi \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right]^3 \partial_r \]
\[ \Gamma_{12} = c_1^{\frac{1}{3}} AiryBi \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] AiryBi' \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] R \partial_R \]
\[ + AiryBi \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right]^3 \partial_r \]
\[ \Gamma_{13} = c_1^{\frac{1}{3}} \left( AiryAi \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] AiryBi' \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] \right) \]
\[ + AiryBi \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] AiryAi' \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] R \partial_R \]
\[ + 2AiryAi \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] AiryBi \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] \partial_r \]

Finally, the solution of (3.29) is expressed in terms of the Airy function as follows
\[ R(r) = c_6 AiryAi \left[ \frac{4(c_1 r + c_2) - c^2}{4c_1^{\frac{3}{2}}} \right] \] (3.32)

with a condition \( c_3 = \pm \frac{c}{\sqrt{1 - c^2}} \)

4 Conclusion

The algebraic properties of the generalized Zakharov equations was the subject of this study. We wrote the Zakharov equations as a system of three
differential equations of real values functions. For the latter system we determined the infinitesimal transformations which leave invariants the Zakharov equations. The admitted Lie point symmetries are five. From the Lie symmetries we found a similarity transformation which provides reductions which lead to travel-wave solutions. The reduced system has been classified according to the admitted Lie point symmetries, where we found surprisingly that the Ermakov-Pinney equation can describe the generalized Zakharov equations. This work demonstrate the application of the Lie point symmetries in applied mathematics and in example in plasma physics. In a future work we plan to investigate the nature of the conservation laws provided by the symmetry vectors.

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