LOCAL WELL-POSEDNESS FOR THE MAXWELL-CHERN-SIMONS-HIGGS SYSTEM IN FOURIER-LEBESGUE SPACES

HARTMUT PECHER
FACHBEREICH MATHEMATIK UND NATURWISSENSCHAFTEN
BERGISCHE UNIVERSITÄT WUPPERTAL
GAUSSSTR. 20
42119 WUPPERTAL
GERMANY
E-MAIL PECHER@UNI-WUPPERTAL.DE

Abstract. We consider local well-posedness for the Maxwell-Chern-Simons-Higgs system in Lorenz gauge for data with minimal regularity assumptions in Fourier-Lebesgue spaces \( \hat{H}^{s,r} \), where \( \|u\|_{\hat{H}^{s,r}} := \|\langle \xi \rangle^s \hat{u}(\xi)\|_{L^{r',r}} \), and \( r \) and \( r' \) are dual exponents. We show that the gap between this regularity and the regularity with respect to scaling shrinks in the case \( r > 1, r \to 1 \) compared to the classical case \( r = 2 \).

1. Introduction and main results

The Lagrangian of the (2+1)-dimensional Maxwell-Chern-Simons-Higgs model which was proposed in [LLM] is given by

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho + D_\mu \phi D^\mu \phi + \frac{1}{2} \partial_\mu N \partial^\mu N - \frac{1}{2} (e |\phi|^2 + \kappa N - ev^2)^2 - e^2 N^2 |\phi|^2
\]

in Minkowski space \( \mathbb{R}^{1+2} = \mathbb{R}_t \times \mathbb{R}^2 \) with metric \( g_{\mu\nu} = \text{diag}(1, -1, -1) \). We use the convention that repeated upper and lower indices are summed, Greek indices run over 0,1,2 and Latin indices over 1,2. Here

\[
D_\mu := \partial_\mu - ie A_\mu \quad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu
\]

Here \( F_{\mu\nu} : \mathbb{R}^{1+2} \to \mathbb{R} \) denotes the curvature, \( \phi : \mathbb{R}^{1+2} \to \mathbb{C} \) and \( N : \mathbb{R}^{1+2} \to \mathbb{R} \) are scalar fields, and \( A_\mu : \mathbb{R}^{1+2} \to \mathbb{R} \) are the gauge potentials. \( e \) is the charge of the electron and \( \kappa > 0 \) the Chern-Simons constant, \( v \) is a real constant. We use the notation \( \partial_\mu = \frac{\partial}{\partial x_\mu} \), where we write \( (x^0, x^1, \ldots, x^n) = (t, x^1, \ldots, x^n) \) and also \( \partial_0 = \partial_t \). \( \epsilon^{\mu\nu\rho} \) is the totally skew-symmetric tensor with \( \epsilon^{012} = 1 \).

The corresponding Euler-Lagrange equations are given by

\[
\partial_\lambda F^{\lambda\rho} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} + 2e Im(\phi D^\rho \phi) = 0 \quad (1)
\]

\[
D_\mu D^\mu \phi + U_N(|\phi|^2, N) = 0 \quad (2)
\]

\[
\partial_\mu \partial^\mu N + U_N(|\phi|^2, N) = 0 \quad (3)
\]

2020 Mathematics Subject Classification: 35Q40, 35L70

Key words and phrases: Maxwell-Chern-Simons-Higgs, local well-posedness, Lorenz gauge
where
\[ U_N(|\phi|^2, N) = (\epsilon|\phi|^2 + \kappa N - ev^2)\phi + e^2N^2\phi \]
\[ U_N(|\phi|^2, N) = \kappa(\epsilon|\phi|^2 + \kappa N - ev^2) + 2e^2N|\phi|^2. \]

(1) can be written as follows
\[ -\Delta A_0 + \partial_t(\partial_1 A_1 + \partial_2 A_2) + \kappa F_{12} + 2eIm(\phi D^\nu \phi) = 0 \quad (4) \]
\[ (\partial_1^2 - \partial_2^2)A_1 - \partial_1(\partial_1 A_0 - \partial_2 A_2) - \kappa F_{02} + 2eIm(\phi D^\nu \phi) = 0 \quad (5) \]
\[ (\partial_1^2 - \partial_2^2)A_2 - \partial_2(\partial_1 A_0 - \partial_2 A_1) + \kappa F_{01} + 2eIm(\phi D^\nu \phi) = 0. \quad (6) \]

The initial conditions are
\[ A_\nu(0) = a_{\nu 0}, \quad (\partial_1 A_\nu)(0) = a_{\nu 1}, \quad \phi(0) = \phi_0, \quad (\partial_1 \phi)(0) = \phi_1 \quad (7) \]
\[ N(0) = N_0, \quad (\partial_1 N)(0) = N_1. \]

The Gauss law constraint (1) requires the initial data to fulfill the following condition:
\[ \Delta a_{00} - \partial_t a_{11} - \partial_2 a_{21} - \kappa(\partial_1 a_{20} - \partial_2 a_{10}) - 2eIm(\phi D^\nu _1 \phi_1) + 2e^2a_{00}|\phi_0|^2 = 0. \quad (8) \]

The energy \(E(t)\) of (1), (2), (3) is (formally) conserved, where
\[ E(t) = \int \frac{1}{2} \sum F_{0i}(x, t) + \frac{1}{2} F_{12}(x, t) \]
\[ + \sum |D_\mu \phi(x, t)|^2 + \sum |\partial_\mu N(x, t)|^2 + U(|\phi|^2, N)(x, t)dx \]
with
\[ U(|\phi|^2, N) = \frac{1}{2}(\epsilon|\phi|^2 + \kappa N - ev^2)^2 + e^2N^2|\phi|^2. \]

There are two possible natural asymptotic conditions to make the energy finite: either the "nontopological" boundary condition \((\phi, N, A) \to (0, \frac{v^2}{2}, 0)\) as \(|x| \to \infty\) or the "topological" boundary condition \((|\phi|^2, N, A) \to (v^2, 0, 0)\) as \(|x| \to \infty\). We decide to study the "nontopological" boundary condition. Replacing \(N\) by \(N - \frac{v^2}{2}\), and denoting it again by \(N\) we obtain \((\phi, N, A) \to (0, 0, 0)\) as \(|x| \to \infty\), thus leading to solutions in standard Sobolev spaces, and in (2), (3) we now have
\[ U_n(|\phi|^2, N) = (\epsilon|\phi|^2 + \kappa N + \frac{ev^2}{\kappa})\phi \quad (9) \]
\[ U_N(|\phi|^2, N) = \kappa(\epsilon|\phi|^2 + \kappa N + 2\kappa^2 N + \frac{ev^2}{\kappa})|\phi|^2 \quad (10) \]

For the "topological" boundary condition the problem can also be reduced to a system for \((\phi, N, A)\) which fulfills \((\phi, N, A) \to (0, 0, 0)\) as \(|x| \to \infty\) for a modified function \(\phi\), if one makes the assumption that \(\phi \to \lambda \in \mathbb{C}\) as \(|x| \to \infty\) with \(|\lambda| = v\). In this case one simply replaces \(\phi\) by \(\phi - \lambda\). For details we refer to Yuan’s paper [8].

The equations (1), (2), (3) are invariant under the gauge transformations
\[ A_\mu \to A'_\mu = A_\mu + \partial_\lambda A_\mu, \phi \to \phi' = \exp(i\epsilon\chi)\phi, D_\mu \to D'_\mu = D_\mu - i\epsilon A'_\mu. \]

We consider exclusively the Lorenz gauge \(\partial_\mu A_\mu = 0\), so that we have to assume that the data fulfill \(\partial_\mu a_\mu = 0\).

We want to prove local well-posedness of the Cauchy problem for (1), (2), (3) for data in Fourier-Lebesgue spaces \(\tilde{H}^{s, r}\) for \(1 < r \leq 2\) with minimal regularity assumptions. These spaces are defined by its norm \(\|u\|_{\tilde{H}^{s, r}} = \|\langle \xi \rangle^s \hat{u}(\xi)\|_{L^{r'}}\) for dual exponents \(r\) and \(r'\).
Chae-Chae [CC] assumed $(\phi_0, \phi_1) \in H^2 \times H^1$ and proved local and even global well-posedness using energy conservation. This was improved by J. Yuan [Y] to $(\phi_0, \phi_1), (a_{\mu_0}, a_{\mu_1}), (N_0, N_1) \in H^s \times H^{s-1}$ with $s > \frac{3}{2}$, which obtained a local solution $\phi, A_\mu, N \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1})$, which is unique in a suitable subset of $X^{s,b}$-type. Using energy conservation this solution exists globally, if $s \geq 1$.

In [2] we further lowered down the regularity of the data to $(\phi_0, \phi_1) \in H^s \times H^{s-1}$, $(a_{\mu_0}, a_{\mu_1}) \in H^{2s-\frac{2}{3}} \times H^{2s-\frac{4}{3}}$, $(N_0, N_1) \in H^s \times H^{-\frac{2}{3}}$ on condition that $s > \frac{1}{2}$. We obtain a local solution $\phi \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1}), A_\mu \in C^0([0, T], H^{2s-\frac{2}{3}}) \cap C^1([0, T], H^{2s-\frac{4}{3}}), N \in C^0([0, T], H^s) \cap C^1([0, T], H^{-\frac{2}{3}})$, which is unique in a suitable subspace of $X^{s,b}$-type.

Whereas Chae-Chae only used standard energy type estimates Yuan applied bilinear Strichartz type estimates which were given in the paper of d’Ancona, Foschi and Selberg [AFS]. We also use this type of estimates but additionally take advantage of a crucial null condition of the term $A_\mu \partial^\mu \phi$ in the wave equation for $\phi$. This was detected by Klainerman-Machedon [KM] and Selberg-Tesfahun [ST] for the Maxwell-Klein-Gordon equations and also by Selberg-Tesfahun [ST1] for the corresponding problem for the Chern-Simons-Higgs equations. In chapter 3 we prove bilinear estimates for the null forms and for general bilinear terms in generalized Bourgain-Klainerman-Machedon spaces $H^s$ and $(X^s_{b,2})$ based on estimates by Foschi and Klainerman [FK], Grünrock [G], Grigoryan-Nahmod [GN] and Grigoryan-Tanguay [GT].

The critical space for the data with respect to scaling is $\hat{H}^{\frac{2}{3}-1,r}$ for $\phi_0$, $a_{\mu_0}$ and $N_0$ (cf. Remark 1 below). Thus in the case $r = 2$ there is still a gap, which amounts to $\frac{1}{2}$ for $\phi_0$ and $N_0$ and $\frac{1}{3}$ for $a_{\mu_0}$. In order to shrink this gap we leave the $L^2$-based Sobolev spaces for the data and consider instead data in Fourier-Bochner spaces $\hat{H}^{s,r}$ which coincide with the classical Sobolev spaces $H^s$ for $r = 2$. We are especially interested in the case $r > 1$, but close to 1. In this case the minimal regularity assumptions for the data $(\phi(0), (\partial_t \phi)(0)) \in \hat{H}^{s,r} \times \hat{H}^{s-1,r}$, $(A_\mu(0), (\partial_t A_\mu)(0)) \in \hat{H}^{1,r} \times \hat{H}^{1-1,r}$ and $(N(0), (\partial_t N)(0)) \in \hat{H}^{m,r} \times \hat{H}^{m-1,r}$ are $(s, l, m) = (\frac{2}{3}, \frac{1}{2} + \frac{m}{2})$ (cf. Theorem [13]). Thus the gap for $(s, l, m)$ shrinks to $(\frac{2}{3}, \frac{1}{2}, \frac{1}{2})$.

We now formulate our main results. One easily checks that a solution of (1), (2), (3) (with (9), (10)) under the Lorenz condition

$$\partial^\mu A_\mu = 0$$

also fulfills the following system

$$(\Box + 1) A_0 = -\kappa F_{12} - 2e Im(\phi \overline{D_0 \phi}) + A_0$$

$$(\Box + 1) A_i = -\kappa e^{ij} F_{0j} - 2e Im(\phi \overline{D_j \phi}) + A_i$$

$$(\Box + 1) \phi = 2ie A_0 \phi \overline{\phi} - 2e A^j \partial_j \phi - e^2 A^j A_j \phi + e^2 A_0^2 \phi - U_\phi(\phi^2, N) + \phi$$

$$(\Box + 1) N = -U_N(\phi^2, N) + N.$$
Defining $A_{\mu,\pm} = \frac{1}{2} (A_{\mu} \pm i^{-1} (\nabla)^{-1} \partial_t A_{\mu}) \Leftrightarrow A_{\mu} = A_{\mu, +} + A_{\mu, -}, \partial_t A_{\mu} = i(\nabla)(A_{\mu, +} - A_{\mu, -})$

$\phi_{\pm} = \frac{1}{2} (\phi \pm i^{-1} (\nabla)^{-1} \partial_t \phi) \Leftrightarrow \phi = \phi_{+} + \phi_{-}, \partial_t \phi = i(\nabla)(\phi_{+} - \phi_{-})$

$N_{\pm} = \frac{1}{2} (N \pm i^{-1} (\nabla)^{-1} \partial_t N) \Leftrightarrow N = N_{+} + N_{-}, \partial_t N = i(\nabla)(N_{+} - N_{-})$

we obtain the equivalent system

$$(i\partial_t \pm (\nabla))A_{0,\pm} = \pm 2^{-1}(\nabla)^{-1}(H.S. \ of \ (12))$$ (16)

$$(i\partial_t \pm (\nabla))A_{1,\pm} = \pm 2^{-1}(\nabla)^{-1}(H.S. \ of \ (13))$$ (17)

$$(i\partial_t \pm (\nabla))\phi_{\pm} = \pm 2^{-1}(\nabla)^{-1}(H.S. \ of \ (14))$$ (18)

$$(i\partial_t \pm (\nabla))N_{\pm} = \pm 2^{-1}(\nabla)^{-1}(H.S. \ of \ (15))$$ (19)

Denoting the Fourier transform $\mathcal{F}$ with respect to space as well as to space and time by $\hat{\cdot}$ the operator $(\nabla)^{\alpha}$ is defined by $\mathcal{F}((\nabla)^{\alpha}f)(\xi) = \langle \xi \rangle^{\alpha}f(\xi)$, where $\langle \cdot \rangle := (1 + | \cdot |^2)^{\frac{1}{2}}$. Define $a \pm := a \pm \epsilon$ for $\epsilon > 0$ sufficiently small.

We obtain the following result:

**Theorem 1.1.** Let $1 < r \leq 2$ and $s, l, m \geq 1$. Assume $s \geq \frac{25}{160} \frac{1}{4}, l \geq \frac{13}{8} \frac{1}{2}, m \geq \frac{13}{8} \frac{1}{2}, s - 1 \leq l \leq s + 1, s - 1 \leq m \leq s + 1$ and $2l - s > \frac{7}{8} - 1, 2s - l > \frac{7}{8} - 1$. Assume

$\phi_{0} \in \hat{H}^{s,r}, \phi_{1} \in \hat{H}^{s-1,r}, a_{\mu 0} \in \hat{H}^{l,r}, a_{\mu 1} \in \hat{H}^{l-1,r}$ \hspace{1em} ($\mu = 0, 1, 2$), \hspace{1em} $n_{0} \in \hat{H}^{m,r}, n_{1} \in \hat{H}^{m-1,r}$.\hspace{1em} There exists $T > 0$ such that the system $(12), (13), (14), (15)$ with $(9), (10)$ and Cauchy conditions

$\phi(0) = \phi_{0}, \partial_{t} \phi(0) = \phi_{1}, A_{\mu}(0) = a_{\mu 0}, \partial_{t} A_{\mu}(0) = a_{\mu 1}, N(0) = N_{0}, \partial_{t} N(0) = N_{1}$

has a unique local solution

$\phi \in X^{s, \frac{1}{2} +}_{+}[0, T] + X^{s, \frac{1}{2} +}_{-}[0, T]\hspace{1em} A_{\mu} \in X^{l, \frac{1}{2} +}_{+}[0, T] + X^{l, \frac{1}{2} +}_{-}[0, T]\hspace{1em} N \in X^{m, \frac{1}{2} +}_{+}[0, T] + X^{m, \frac{1}{2} +}_{-}[0, T].$

These spaces are defined in Definition 1.3 below, It has the properties

$\phi \in C^{0}([0, T], \hat{H}^{s, r}) \cap C^{1}([0, T], \hat{H}^{s, r})$ \hspace{1em} $A_{\mu} \in C^{0}([0, T], \hat{H}^{l, r}) \cap C^{1}([0, T], \hat{H}^{l-1, r})$ \hspace{1em} $N \in C^{0}([0, T], \hat{H}^{m, r}) \cap C^{1}([0, T], \hat{H}^{m-1, r}).$

This result is proven in section 4.

The following theorem was proven in [P].

**Theorem 1.2.** Let $r = 2$. Assume that $1 > s > \frac{1}{2}, l = 2s - \frac{1}{2}, m = \frac{1}{2}$, Then the statements of Theorem 1.1 remain true.

By "interpolation" between the results of Theorem 1.1 for $r = 1$ and Theorem 1.2 ($r = 2$) we obtain the following improvement.

**Corollary 1.1.** Let $1 < r \leq 2$. The statements of Theorem 1.1 remain true in the case $s = \frac{13}{8} - \frac{5}{8} \epsilon, l = \frac{5}{8} - \frac{5}{8} + \epsilon$ and $m = \frac{5}{8} - \frac{5}{8} + \epsilon$, where $\epsilon > 0$ is sufficiently small.
This is the minimal regularity for the data which is admissible by our method. Other combinations of $s$, $t$ and $m$ are certainly possible, but we do not pursue this.

In \[\mathbb{P}\] we obtained the following theorem as a consequence of Theorem 1.2. The same proof applies for Theorem 1.1 and Corollary 1.1.

**Theorem 1.3.** Let the assumptions of Theorem 1.1 or Theorem 1.2 be satisfied. Moreover assume that the data fulfill

$$
\Delta a_{60} - \partial_1 a_{11} - \partial_2 a_{21} - \kappa (\partial_1 a_{20} - \partial_2 a_{10}) = -2 e \text{Im}(\phi_0 \overline{\phi}_1) + 2 e^2 a_{00} |\phi_0|^2 = 0
$$

and

$$
\partial^\mu a_\mu = 0.
$$

The solution of Theorem 1.1 or Theorem 1.2 is the unique solution of the Cauchy problem for the system

$$
\partial_\lambda F^{\lambda \rho} + \frac{\kappa}{2} \mu^{\mu \rho} F_{\mu \nu} + 2 e \text{Im}(\phi \overline{\partial^\rho \phi}) = 0
$$

$$
D_\mu D^\mu \phi + U^\gamma(|\phi|^2, N) = 0
$$

$$
\partial_\mu \partial^\mu N + U_N(|\phi|^2, N) = 0,
$$

where

$$
U^\gamma(|\phi|^2, N) = (e|\phi|^2 + \kappa N)\phi + e^2 (N + \frac{ar{e}^2}{\kappa})^2 \phi
$$

$$
U_N(|\phi|^2, N) = \kappa (e|\phi|^2 + \kappa N) + 2 e^2 (N + \frac{ar{e}^2}{\kappa}) |\phi|^2
$$

with initial conditions

$$
A_\mu(0) = a_{\mu 0}, \partial_\lambda A_\mu(0) = a_{\mu 1}, \phi_0 = 0, \partial_\tau \phi(0) = \phi_1, N(0) = N_0, \partial_\tau N(0) = N_1,
$$

which fulfills the Lorenz condition \(\partial^\mu A_\mu = 0\).

**Definition 1.1.** Let \(1 \leq r \leq 2\), \(s, b \in \mathbb{R}\). We recall, that the Fourier-Lebesgue space \(H^{s, r}\) is defined by its norm \(\|u\|_{H^{s, r}} = \|\langle \xi \rangle^s \hat{u}(\xi)\|_{L^r}\) for dual exponents \(r\) and \(r'\). The wave-Sobolev spaces \(H^{s, b}_{r, b}\) are the completion of the Schwarz space \(S(\mathbb{R}^{1+3})\) with norm

$$
\|u\|_{H^{s, b}_{r, b}} = \|\langle \xi \rangle^s \langle \tau - \xi \rangle^b \hat{u}(\tau, \xi)\|_{L^r_{\tau, \xi}}
$$

where \(r'\) is the dual exponent to \(r\). We also define \(H^{s, b}_{r, b}[0, T]\) as the space of the restrictions of functions in \(H^{s, b}_{r, b}\) to \([0, T] \times \mathbb{R}^3\). Similarly we define \(X_{s, b, \pm}^{r'}\) with norm

$$
\|\phi\|_{X_{s, b, \pm}^{r'}} := \|\langle \xi \rangle^s (\tau \pm |\xi|)^b \hat{\phi}(\tau, \xi)\|_{L^r_{\tau, \xi}}
$$

and \(X_{s, b, \pm}^{r, b}[0, T]\).

In the case \(r = 2\) we denote \(H^{s, b} = H^{s, b, b}\) and similarly \(X_{s, b, \pm}^{2} = X_{s, b, \pm}^{2, b}\). Remark that \(\|u\|_{X_{s, b, \pm}^{2, b}} \leq \|u\|_{H^{s, b}_{2, b}}\) for \(b \leq 0\) and the reverse estimate for \(b \geq 0\).

**Remark 1:** The system (12)-(15) is not scaling invariant, but ignoring lower order terms we can write it schematically as

$$
\Box A = \phi \nabla \phi + A \phi^2
$$

$$
\Box \phi = A \nabla \phi + A^2 \phi + N \phi^2 + \phi^3
$$

$$
\Box N = N \phi^2.
$$

This system is invariant under the scaling

$$
A_{\lambda}(x, t) = \lambda A(\lambda x, \lambda t), \phi_{\lambda}(x, t) = \lambda \phi(\lambda x, \lambda t), N_{\lambda}(x, t) = \lambda N(\lambda x, \lambda t).
$$
This implies
\[
\|A_\lambda(0, \cdot)\|_{\dot{H}^{1+r}} = \lambda^{1+l-\frac{4}{p}} \|a_0\|_{\dot{H}^{l+r}},
\]
\[
\|\phi_\lambda(0, \cdot)\|_{\dot{H}^{1+r}} = \lambda^{1+l-\frac{4}{q}} \|\phi_0\|_{\dot{H}^{l+r}}
\]
\[
\|N_\lambda(0, \cdot)\|_{\dot{H}^{m+r}} = \lambda^{1+m-\frac{4}{q}} \|N_0\|_{\dot{H}^{m+r}}.
\]
Here \(\|u\|_{\dot{H}^{r'}} := \|\xi^r u(\xi)\|_{L^{r'}}\), where \(r\) and \(r'\) are dual exponents.

Thus the critical data space with respect to scaling for \(A(0)\), \(\phi(0)\), \(N(0)\) (in dimension 2) is \(\dot{H}^{\frac{4}{p}-1-r}\). If we consider Theorem 1.1 in the case \(r = 1\), there remains a gap between this space and our minimal assumptions in Theorem 1.1, namely \(A(0) \in H^{\frac{4}{p}}, \phi(0) \in \dot{H}^{\frac{4}{q}}, N(0) \in H^{\frac{4}{q}}\). Thus the gap amounts to \(\frac{2}{3}\) for \(s\) and \(\frac{1}{3}\) for \(l\) and \(m\).

However, we remark that this gap is considerably smaller compared to the gap for the case \(r = 2\) in Theorem 1.2, which amounts to \(\frac{1}{2}\) for \(s\) and \(m\) and to \(\frac{1}{4}\) for \(l\).

2. Preliminaries

We start by collecting some fundamental properties of the solution spaces. We rely on [G]. The spaces \(X_{s,b,\pm}\) are Banach spaces with \(S\) as a dense subspace. The dual space is \(X_{s,b,\pm}^r\), where \(\frac{1}{r} + \frac{1}{r'} = 1\). The complex interpolation space is given by
\[(X_{s_0,b_0,\pm}, X_{s_1,b_1,\pm}^{r_1})_{\theta} = X_{s,b,\pm}^r,\]
where \(s = (1-\theta)s_0 + \theta s_1\), \(\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}\), \(b = (1-\theta)b_0 + \theta b_1\). Similar properties have the space \(H^r_{s,b}\).

If \(u = u_+ + u_-\), where \(u_{\pm} \in X_{s,b,\pm}^r\), then \(u \in C(\{0\} \times [0,T], \dot{H}_{s}^{\frac{4}{p}\pm})\), if \(b > \frac{1}{2}\).

The "transfer principle" in the following proposition, which is well-known in the case \(r = 2\), also holds for general \(1 < r < \infty\) (cf. [GN], Prop. A.2 or [G], Lemma 1). We denote \(\|u\|_{L^p_t(\dot{L}^q_x)} := \|\hat{u}\|_{L^p_t(\dot{L}^q_x)}\).

**Proposition 2.1.** Let \(1 \leq p, q \leq \infty\). Assume that \(T\) is a bilinear operator which fulfills
\[
\|T(e^{\pm i t D} f_1, e^{\pm i t D} f_2)\|_{L^p_t(\dot{L}^q_x)} \lesssim \|f_1\|_{\dot{H}^{1+r}} \|f_2\|_{\dot{H}^{2-r}}
\]
for all combinations of signs \(\pm_1, \pm_2\), then for \(b > \frac{1}{2}\) the following estimate holds:
\[
\|T(u_1, u_2)\|_{L^p_t(\dot{L}^q_x)} \lesssim \|u_1\|_{H^s_{1,b}} \|u_2\|_{H^s_{2,b}}.
\]

The following general local well-posedness theorem was given by [G], Thm. 1.

**Theorem 2.1.** Let \(N_\pm(u) := N_\pm(u_+, u_-)\) be a multilinear functions. Assume that for given \(s \in \mathbb{R}\), \(1 < r < \infty\) there exist \(b > \frac{1}{4}\) such that the estimate
\[
\|N_\pm(u)\|_{X_{s,b-1,\pm}^r} \leq \omega(\|u\|_{X_{s,b}^r})
\]
is valid with a nondecreasing function \(\omega\), where \(\|u\|_{X_{s,b}^r} := \|u_--\|_{X_{s,b}^r} + \|u_+\|_{X_{s,b}^r}\).

Then there exist \(T = T(\|u_0\|_{\dot{H}^s}), r > 0\) and a unique solution \((u_+, u_-) \in X_{s,b,\pm}^r[0,T] \times X_{s,b-\pm}^r[0,T]\) of the Cauchy problem
\[
\partial_t u_{\pm} \pm i(\nabla) u = N_\pm(u) , \quad u_{\pm}(0) = u_{0,\pm} \in \dot{H}^{s,r}.
\]
This solution is persistent and the mapping data upon solution \((u_{0,\pm}) \mapsto (u_+, u_-) \in \dot{H}^{s,r} \times \dot{H}^{s,r} \rightarrow X_{s,b,\pm}^r[0,T_0] \times X_{s,b,-\pm}^r[0,T_0]\) is locally Lipschitz continuous for any \(T_0 < T\).
3. Bilinear estimates

The standard null forms are given by
\[ Q_{\alpha\beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v. \]

Let \( q_{\alpha\beta}(u, v) := Q_{\alpha\beta}(|\nabla|^{-1}u, |\nabla|^{-1}v) \), were \( |\nabla|^{-1} \) has Fourier symbol \( |\xi|^{-1} \). The proof of the following bilinear estimates relies on estimates given by Foschi and Klainerman [FK].

**Lemma 3.1.** Assume \( 0 \leq \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \geq \frac{1}{b} \) and \( b > \frac{1}{4} \). The following estimate applies
\[ \| q_{12}(u, v) \|_{L_{t\xi}^b} \lesssim \| u \|_{X_{\alpha_1, b, \pm 1}} \| v \|_{X_{\alpha_2, b, \pm 2}}. \]

**Proof.** Because we use inhomogeneous norms it is obviously possible to assume \( \alpha_1 + \alpha_2 = \frac{1}{b} \). Moreover, by interpolation we may reduce to the case \( \alpha_1 = \frac{1}{2} \), \( \alpha_2 = 0 \).

The left hand side of the claimed estimate equals
\[ \| \mathcal{F}(q_{12}(u, v)) \|_{L_{t\xi}^b} = \| \int q_{12}(\eta, \xi - \eta) \tilde{u}(\lambda, \eta) \tilde{v}(\tau - \lambda, \xi - \eta) d\lambda d\eta \|_{L_{t\xi}^b}. \]

Let now \( u(t, x) = e^{\pm iD} \hat{u}[\alpha_1](x) \), \( v(t, x) = e^{\pm iD} \hat{v}[\alpha_2](x) \), so that
\[ \tilde{u}(\tau, \xi) = c\theta(\tau \mp 1 |\xi|) \hat{u}[\alpha_1](\xi), \quad \tilde{v}(\tau, \xi) = c\theta(\tau \mp 2 |\xi|) \hat{v}[\alpha_2](\xi). \]

This implies
\[ \| \mathcal{F}(q_{12}(u, v)) \|_{L_{t\xi}^b} = c^2 \| \int q_{12}(\eta, \xi - \eta) \hat{u}[\alpha_1](\eta) \hat{v}[\alpha_2](\xi - \eta) \delta(\lambda \mp 1 |\eta|) \delta(\tau - \lambda \mp 2 |\xi - \eta|) d\lambda d\eta \|_{L_{t\xi}^b}. \]

By symmetry we only have to consider the elliptic case \( \pm_1 = \pm_2 = + \) and the hyperbolic case \( \pm_1 = +, \pm_2 = - \).

**Elliptic case.** We obtain by [FK], Lemma 13.2:
\[ |q_{12}(\eta, \xi - \eta)| \leq \frac{|\eta|(|\xi - \eta| + |\xi - \eta|)}{|\eta|(|\xi - \eta| - |\xi|)} \lesssim \frac{|\xi|^{\frac{1}{2}}|\eta| + |\xi - \eta| - |\xi|) \frac{1}{2}}{|\eta| \frac{1}{2} |\xi - \eta| \frac{1}{2}}. \]

By Hölder’s inequality we obtain
\[ \| \mathcal{F}(q_{12}(u, v)) \|_{L_{t\xi}^b} \lesssim \int \left[ \frac{|\xi|^{\frac{1}{2}}|\tau| - |\xi|^{\frac{1}{2}}}{\frac{1}{2} |\tau| - \frac{1}{2} |\xi - \eta|} \delta(\tau - |\eta| - |\xi - \eta|) |\hat{u}[\alpha_1](\eta)| |\hat{v}[\alpha_2](\xi - \eta)| d\eta \right]_{L_{t\xi}^b}. \]

where
\[ I = |\xi|^{\frac{1}{2}}|\tau| - |\xi|^{\frac{1}{2}} \left( \int \delta(\tau - |\eta| - |\xi - \eta|) |\eta|^{-1 - \frac{1}{2}} |\xi - \eta|^{-\frac{1}{2}} d\eta \right) \frac{1}{2}. \]

We want to prove \( \sup_{\tau, \xi} I \lesssim 1 \). By [FK], Lemma 4.3 we obtain
\[ \int \delta(\tau - |\eta| - |\xi - \eta|) |\eta|^{-1 - \frac{1}{2}} |\xi - \eta|^{-\frac{1}{2}} d\eta \sim \tau^A ||\tau| - |\xi||^B, \]

where \( A = \max(1 + \frac{1}{2}, \frac{3}{2}) - 1 - r = -\frac{1}{2} \) and \( B = 1 - \max(1 + \frac{1}{2}, \frac{3}{2}) = -\frac{1}{2} \). Using \( |\xi| \leq |\tau| \) this implies
\[ I \lesssim |\xi|^{\frac{1}{2}}|\tau| - |\xi|^{\frac{1}{2}} \tau^{-\frac{1}{2}} ||\tau| - |\xi||^{\frac{1}{2}} \leq 1. \]
Hyperbolic case. We start with the following bound (cf. [FK], Lemma 13.2):

\[ |q_{12}(\eta, \xi - \eta)| \leq \frac{|(\xi - \eta)_{2} - (\eta - (\xi - \eta))_{1}|}{|\eta| |\xi - \eta|} \lesssim \frac{|\xi|^\frac{1}{2}(|\xi| - |\eta| - |\eta - \xi|)|^{\frac{1}{2}}}{|\eta|^{\frac{1}{2}} |\xi - \eta|^{\frac{1}{2}}}, \]

so that similarly as in the elliptic case we have to estimate

\[ I = |\xi|^\frac{1}{2} |\tau| - |\xi| |\xi - \eta| \left( \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-\frac{1}{2}} |\xi - \eta|^{-\frac{1}{2}} d\eta \right)^{\frac{1}{2}}. \]

In the subcase \(|\eta| + |\xi - \eta| \leq 2|\xi|\) we apply [FK], Prop. 4.5 and obtain

\[ \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-\frac{1}{2}} |\xi - \eta|^{-\frac{1}{2}} d\eta \sim |\xi|^{A} ||\xi| - |\tau||^{B}. \]

where in the subcase \(0 \leq \tau \leq |\xi|\) we obtain \(A = \max(\frac{3}{2}, \frac{3}{2}) - 1 - r = \frac{1}{2} - r\) and \(B = 1 - \max(\frac{3}{2}, \frac{3}{2}) = -\frac{1}{2}\), which implies

\[ I \lesssim |\xi|^\frac{1}{2} |\tau| - |\xi||\xi - \eta|^{-\frac{1}{2}} d\eta. \]

Similarly in the subcase \(-|\xi| \leq \tau \leq 0\) we obtain \(A = \max(1 + \frac{3}{2}, \frac{3}{2}) - 1 - r = -\frac{1}{2}\), \(B = 1 - \max(1 + \frac{3}{2}, \frac{3}{2}) = -\frac{1}{2}\), which implies

\[ I \sim |\xi|^\frac{1}{2} |\tau| - |\xi||\xi - \eta|^{-\frac{1}{2}} |\eta|^{-\frac{1}{2}} d\eta \lesssim 1. \]

In the subcase \(|\eta| + |\xi - \eta| \geq 2|\xi|\) we obtain by [FK], Lemma 4.4:

\[ \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-\frac{1}{2}} |\xi - \eta|^{-\frac{1}{2}} d\eta \sim |\xi|^{-1} |\xi - \eta|^{-\frac{1}{2}} d\eta. \]

We remark that in fact the lower limit of the integral can be chosen as \(2\) by inspection of the proof in [FK]. The integral converges, because \(|\tau| \leq |\xi|\) and \(r > 1\). This implies the bound

\[ I \lesssim |\xi|^\frac{1}{2} |\tau| - |\xi||\xi - \eta|^{-\frac{1}{2}} |\tau| + |\xi|^{-\frac{1}{2}} |\xi|^\frac{1}{2} |\xi - \eta|^{-1} \lesssim 1. \]

Summarizing we obtain

\[ \|q_{12}(u, v)\|^{H_{0,0}^{\frac{1}{2}}} \lesssim \|D^\frac{1}{2} u_{0}^{+1}\|_{L^{r'}} \|v_{0}^{+2}\|_{L^{r'}}. \]

By the transfer principle Prop. 2.1, we obtain the claimed result. \(\square\)

In a similar manner we can also estimate the nullform \(q_{0j}(u, v)\).

**Lemma 3.2.** Assume \(0 \leq \alpha_{1}, \alpha_{2} \), \(\alpha_{1} + \alpha_{2} \geq \frac{1}{r} \) and \(b > \frac{1}{r} \). The following estimate applies

\[ \|q_{0j}(u, v)\|^{H_{b,0}^{\frac{1}{2}}} \lesssim \|u\|_{X_{b,0}^{\alpha, \pm1}} \|v\|_{X_{b,0}^{\alpha, \pm2}}. \]

**Proof.** Again we may reduce to the case \(\alpha_{1} = \frac{1}{r}\) and \(\alpha_{2} = 0\). Arguing as in the proof of Lemma 3.1, we use in the elliptic case the estimate (cf. [FK], Lemma 13.2):

\[ |q_{0j}(\eta, \xi - \eta)| \lesssim \frac{|(\eta)| + |(\xi - \eta)| - |(\xi)|}{\min(|(\eta)|^{\frac{1}{2}}, |(\xi - \eta)|^{\frac{1}{2}})}. \]

In the case \(|\eta| \leq |\xi - \eta|\) we obtain

\[ I = ||\tau| - |\xi||^{\frac{1}{2}} \left( \int \delta(\tau - |\eta| - |\xi - \eta|) |\eta|^{-1} d\eta \right)^{\frac{1}{2}} \sim ||\tau| - |\xi||^{\frac{1}{2}} |\tau|^{\frac{1}{2}} - |\xi|^{\frac{1}{2}} = 1, \]
because $A = \max(1 + \frac{r}{2}, \frac{3}{2}) - 1 - \frac{r}{2} = 0$ and $B = 1 - \max(1 + \frac{r}{2}, \frac{3}{2}) - \frac{r}{2} = -\frac{r}{2}$.

In the case $|\eta| \geq |\xi - \eta|$ we obtain

$$I = \||\tau| - |\xi||^{\frac{1}{2}} \left( \int \delta(|\tau - |\eta| - |\xi - \eta|) |\eta|^{-1} |\xi - \eta|^{-\frac{r}{2}} d\tau \right) \right|^\frac{1}{2}$$

$$\sim \||\tau| - |\xi||^{\frac{1}{2}} ||\tau||^{\frac{r}{2}} (1 + \log \|\tau\|^{-|\xi| - |\xi - \eta|} \right)^\frac{1}{2},$$

where $A = \max(1, \frac{r}{2}, \frac{3}{2}) - 1 - \frac{r}{2} = \frac{1}{2} - \frac{r}{2}$ and $B = -\frac{1}{2}$, so that

$$I \lesssim ||\tau| - |\xi||^{\frac{1}{2}} ||\tau||^{\frac{r}{2}} |\tau - |\xi||^{-\frac{r}{2}} \lesssim 1.$$ 

In the hyperbolic case we obtain by [FK], Lemma 13.2:

$$|g_{03}(\eta, \xi - \eta)| \lesssim |\xi| \frac{|\eta - |\eta| - |\eta - \xi||}{|\eta - |\eta|}$$

and argue exactly as in the proof of Lemma 3.1. The proof is completed as before.

\[ \square \]

**Lemma 3.3.** Let $1 < r \leq 2$. Assume $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 > \frac{1}{2} + \frac{r}{2}$, $b_1, b_2 > \frac{1}{r}$, $b_1 + b_2 > \frac{3}{2r}$. Then the following estimate applies:

$$\|uv\|_{H^r_{\alpha, \alpha}} \lesssim \|u\|_{H^r_{\alpha_1, b_1}} \|v\|_{H^r_{\alpha_2, b_2}}.$$

**Proof.** This follows from [GT], Prop. 3.1 by summation over the dyadic parts.

\[ \square \]

**Lemma 3.4.** If $\alpha_1, \alpha_2, b_1, b_2 \geq 0$, $\alpha_1 + \alpha_2 > \frac{2}{r}$ and $b_1 + b_2 > \frac{1}{r}$ the following estimate applies:

$$\|uv\|_{H^r_{\alpha, \alpha}} \lesssim \|u\|_{H^r_{\alpha_1, b_1}} \|v\|_{H^r_{\alpha_2, b_2}}.$$

**Proof.** We may assume $\alpha_1 = \frac{2}{r} + \frac{1}{r}$, $\alpha_2 = 0$, $b_1 = \frac{1}{r} + \frac{1}{r}$, $b_2 = 0$ (or similarly $b_1 = 0$, $b_2 = \frac{1}{r}$). By Young’s and Hölder’s inequalities we obtain

$$\|uv\|_{H^r_{\alpha, \alpha}} = \|\hat{u}\|_{L^r_{\xi, \tau}} \|\hat{v}\|_{L^r_{\xi, \tau}} \lesssim \|\hat{u}\|_{L^r_{\xi, \tau}} \|\hat{v}\|_{L^r_{\xi, \tau}}$$

$$\lesssim \|\xi\|^{-\frac{r}{2}} \|\tau - |\xi||^{-\frac{r}{2}} \|\xi||^{\frac{r}{2}} \|\tau - |\xi||^{\frac{r}{2}} \|\hat{u}\|_{L^r_{\xi, \tau}} \|\hat{v}\|_{L^r_{\xi, \tau}} \lesssim \|u\|_{H^r_{\alpha_1, b_1}} \|v\|_{H^r_{\alpha_2, b_2}}.$$ 

\[ \square \]

**Lemma 3.5.** Let $1 < r \leq 2$, $0 \leq \alpha_1, \alpha_2$ and $\alpha_1 + \alpha_2 \geq \frac{1}{r} + b$, $b > \frac{1}{r}$. Then the following estimate applies:

$$\|uv\|_{H^r_{\alpha, \alpha}} \lesssim \|u\|_{H^r_{\alpha_1, b_1}} \|v\|_{H^r_{\alpha_2, b_2}}.$$

**Proof.** We may assume $\alpha_1 = \frac{1}{r} + b$, $\alpha_2 = 0$. We apply the ”hyperbolic Leibniz rule” (cf. [AFS], p. 128):

$$\||\tau| - |\xi|| \lesssim ||\rho| - |\eta|| + ||\tau - \rho|| - |\xi - \eta|| + b_+(\xi, \eta),$$

where

$$b_+(\xi, \eta) = |\eta| + |\xi - \eta| - |\xi|, \quad b_- (\xi, \eta) = |\xi| - ||\eta|| - |\xi - \eta||.$$

Let us first consider the term $b_+(\xi, \eta)$ in (28). Decomposing as before $uv = u_+ v_+ + u_+ v_- + u_- v_+ + u_- v_-$, where $u_+(t) = e^{\pm i t D} f, v_+(t) = e^{\pm i t D} g$, we use

$$\hat{u}_\pm (\tau, \xi) = c \delta (\tau \mp |\xi|) \hat{f}(\xi), \quad \hat{v}_\pm (\tau, \xi) = c \delta (\tau \mp |\xi|) \hat{g}(\xi)$$

and

$$\hat{u}_0 (\tau, \xi) = c \delta (\tau) \hat{f}(\xi), \quad \hat{v}_0 (\tau, \xi) = c \delta (\tau) \hat{g}(\xi)$$

for some constant $c$.

\[ \square \]
and have to estimate
\[ \| \int b^h_\pm (\xi, \eta) \delta(\tau - |\eta| + |\xi - \eta|) \hat{f}(\xi) \hat{g}(\xi - \eta) d\eta \|_{L^r_x}^r \]
\[ = \| \int |\tau - |\xi||^b \delta(\tau - |\eta| + |\xi - \eta|) \hat{f}(\xi) \hat{g}(\xi - \eta) d\eta \|_{L^r_x}^r \]
\[ \lesssim \sup_{\tau, \xi} I \| \mathcal{D}^{\pm + b} f \|_{L^r_x} \| \hat{g} \|_{L^\infty} . \]
Here we used Hölder’s inequality, where
\[ I = |\tau| - |\xi||^b (\int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-1-br} d\eta)^{\frac{1}{r}} . \]

In order to obtain \( I \lesssim 1 \) we first consider the elliptic case \( \pm_1 = \pm_2 = + \) and use \([FK]\), Prop. 4.3. Thus
\[ I \sim |\tau| - |\xi||^b \| |\tau| - |\xi||^\frac{1}{r} = |\tau| - |\xi||^b |\tau| - |\xi||^{-b} = 1 \]
with \( A = \max(1 + br, \frac{3}{2}) - (1 + br) = 0 \) and \( B = 1 - \max(1 + br, \frac{3}{2}) = -br \). Next we consider the hyperbolic case \( \pm_1 = +, \pm_2 = - \).
First we assume \( |\eta| + |\xi - \eta| \leq 2|\xi| \) and use \([FK]\), Prop. 4.5 which gives
\[ \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-1-br} d\eta \sim |\xi|^A |\tau| - |\tau|^{\frac{1}{r}} \lesssim 1 . \]
If \( -|\xi| \leq \tau \leq 0 \) we obtain \( A = \max(1+br, \frac{3}{2}) - (1+br) = 0 \), \( B = 1 - \max(1+br, 2) = -br \), which implies \( I \lesssim 1 \).
Next we assume \( |\eta| + |\xi - \eta| \geq 2|\xi| \), use \([FK]\), Lemma 4.4 and obtain
\[ I \sim |\tau| - |\xi||^b (\int \delta(\tau - |\eta| - |\xi - \eta|) |\eta|^{-1-br} d\eta)^{\frac{1}{r}} \]
\[ \sim |\tau| - |\xi||^b (|\tau| - |\xi||^{-\frac{1}{r}}) |\tau| + |\xi||^{-\frac{1}{r}} \int_{-\infty}^{\infty} (|\xi| x + \tau)^{-br} (|\xi| x - \tau)(x^2 - 1)^{-\frac{1}{2}} dx)^{\frac{1}{r}} \]
\[ \sim |\tau| - |\xi||^b . \]
\[ \cdot (|\tau| - |\xi||^{-\frac{1}{r}}) |\tau| + |\xi||^{-\frac{1}{r}} \int_{-\infty}^{\infty} (x + \frac{\tau}{|\xi|})^{-br} (x - \frac{\tau}{|\xi|})(x^2 - 1)^{-\frac{1}{2}} dx \cdot |\xi|^{-br})^{\frac{1}{r}} . \]
This integral converges, because \( \tau \leq |\xi| \) and \( b > \frac{1}{r} \). This implies
\[ I \lesssim \| |\tau| - |\xi||^b \| \| \tau | + |\xi||^{-\frac{1}{r}} |\xi|^{-b} \lesssim 1 , \]
using \( |\tau| \lesssim |\xi| \).

By the transfer principle we obtain
\[ \| B^h_\pm (u, v) \|_{X^r_{s,0}} \lesssim \| u \|_{X^r_{s+b,0, \pm_1}} \| v \|_{X^r_{s+b,0, \pm_2}} . \]
Here \( B^h_\pm \) denotes the operator with Fourier symbol \( b_\pm \).
Consider now the term \( \| \rho - |\eta| \| \) (or similarly \( \| \tau - \rho - |\xi - \eta| \| \)) in (23). We have to prove
\[ \| u D^h v \|_{H^s_0} \lesssim \| u \|_{X^r_{s+b,0, \pm_1}} \| v \|_{X^r_{s+b,0, \pm_2}} , \]
which is implied by
\[ \| uv \|_{H^s_0} \lesssim \| u \|_{X^r_{s,b,0, \pm_1}} \| v \|_{X^r_{s,b,0, \pm_2}} . \]
This results from Lemma 3.3 because \( \alpha_1 + \alpha_2 \geq \frac{1}{r} + b > \frac{3}{r} \), which completes the proof. \( \square \)
As a consequence we obtain the following estimate for cubic nonlinearities.

**Lemma 3.6.** Let $1 < r < 2$, $b > \frac{1}{2}$, $s_0 \geq 1$, $s_0 \leq s_1 + 1$, $2s_1 - s_0 > \frac{2}{r} - 1$ and $s_1 > \frac{11}{8r} - \frac{1}{2}$. Then the following estimate applies:

\[
\|u v w\|_{H^{s_1-1,0}_r} \lesssim \|u\|_{H^{s_1,b}_r} \|v\|_{H^{s_1,b}_r} \|w\|_{H^{s_0,b}_r}.
\]

**Proof.** By Lemma 3.5 we obtain

\[
\|u v w\|_{H^{s_1-1,0}_r} \lesssim \|u\|_{H^{s_1,b}_r} \|v\|_{H^{s_1,b}_r} \|w\|_{H^{s_0,b}_r},
\]

if $\alpha_1 + \alpha_2 > 2r$ and $r > \frac{1}{r}$, and by Lemma 3.3

\[
\|u v w\|_{H^{0,0}_r} \lesssim \|u\|_{H^{s_1-1,0}_r} \|v\|_{H^{s_1,b}_r} \|w\|_{H^{s_0,b}_r},
\]

if $\alpha_1 + \alpha_2 > \frac{2}{r}$, $b_1, b_2 > \frac{1}{2r}$ and $b_1 + b_2 > \frac{2}{r}$. By interpolation this implies

\[
\|u v w\|_{H^{s_1-1,0}_r} \lesssim \|u\|_{H^{s_1,b}_r} \|v\|_{H^{s_1,b}_r} \|w\|_{H^{s_0,b}_r},
\]

if $\alpha_1 + \alpha_2 > \frac{2}{r}$ and $b > \frac{1}{r}$. Using first Lemma 3.3 again and then the last inequality we obtain

\[
\|u v w\|_{H^{s_1-1,0}_r} \lesssim \|u\|_{H^{s_1,b}_r} \|v\|_{H^{s_1,b}_r} \|w\|_{H^{s_0,b}_r},
\]

where $k + 1 > \frac{2}{2r}$ and $2s_1 - k > \frac{2}{2r}$, which requires $s_1 > \frac{11}{8r} - \frac{1}{2}$ and $k \geq s_0 - 1$, thus $2s_1 - s_0 > \frac{2}{r} - 1$.

\[\square\]

## 4. Proof of Theorem 1.1

**Proof of Theorem 1.1.** An application of the contraction mapping is by an obvious generalization of Theorem 2.1 to systems reduced to suitable multilinear estimates of the right hand sides of (1), (2) and (3).

The linear terms are easily treated and therefore omitted here.

We now consider the right hand side of (16):

\[-2e^{i\text{Im}(\phi D_0 \phi)} = -2e^{i\text{Im}(\phi_+ + \phi_-)\langle -i \nabla \rangle (\overline{\phi_+} - \overline{\phi_-})} - 2e^{2A_0} |\phi|^2.
\]

By Lemma 3.3 we obtain

\[
\|\phi \pm 1\langle \nabla \rangle \phi \pm 2\|_{H^{s_1-1,0}_r} \lesssim \|\phi \pm 1\|_{X^{s_1-1,0}_r} \|\langle \nabla \rangle \phi \pm 2\|_{X^{s_1-0,0}_r},
\]

if $2s - l > \frac{2}{r}$ and $s \geq l - 1$, where here and in the following $\pm 1$ and $\pm 2$ denote independent signs.

The cubic term is handled as follows. By Lemma 3.3 we obtain

\[
\|A_0 |\phi|^2\|_{H^{s_1-1,0}_r} \lesssim \|A_0\|_{H^{s_0,b}_r} \|\phi\|_{H^{s_1,b}_r},
\]

because $s \geq l - 1$, $s > \frac{11}{8r} - \frac{1}{2}$ and $2s - l > \frac{2}{r} - 1$.

The right hand side of (17) can be handled in the same way.

It remains to consider the right hand side of (18). We start with the most interesting quadratic term, where a null condition comes into play, namely $2ic A_0 \partial^\mu \phi$.

Defining the modified Riesz transforms $R_j := \langle \nabla \rangle^{-1} \partial_j$ and splitting $A_j$ into divergence-free and curl-free parts and a smooth remainder we obtain

\[A_j = A^{df}_j + A^{cf}_j + \langle \nabla \rangle^{-2} A_j,
\]

where

\[A^{df}_j = R_1 (R_1 A_2 - R_2 A_1) \quad A^{df}_2 = R_1 (R_2 A_1 - R_1 A_2)
\]

\[A^{cf}_j = -R_1 (R_1 A_1 + R_2 A_2) \quad A^{cf}_2 = -R_2 (R_1 A_1 + R_2 A_2).
\]
we obtain the sufficient estimate
\[ l > m > 1 \]
if follows from Lemma 3.1, if
\[ \partial_l A_0 = \nabla \cdot A, \]
to write
\[ A^\text{cf}, \nabla \phi = -\langle \nabla \rangle^{-2} \partial_l (\partial_l A_0) \partial_l \phi = -\partial_l (\langle \nabla \rangle^{-1} R^l A_0) \partial_l \phi. \]

We can also write
\[ A_0 \partial_l \phi = -\langle \nabla \rangle^{-2} \partial_l \partial_l^l A_0 \partial_l \phi + \langle \nabla \rangle^{-2} A_0 \partial_l \phi \]
\[ = -\partial_l (\langle \nabla \rangle^{-1} R^l A_0) \partial_l \phi + \langle \nabla \rangle^{-2} A_0 \partial_l \phi. \]
Combining the above identities, we get
\[ -A_0 \partial_l \phi + A^\text{cf} \cdot \nabla \phi = Q_{10}(\langle \nabla \rangle^{-1} R^l A_0, \phi) - \langle \nabla \rangle^{-2} A_0 \partial_l \phi. \]
Next, we consider the second term. We have
\[ A^\text{cf} \cdot \nabla \phi = (\langle \nabla \rangle^{-2} \partial_l (\partial_l A_2 - \partial_2 A_1)) \partial_l \phi - \langle \nabla \rangle^{-2} \partial_l (\partial_l A_2 - \partial_2 A_1) \partial_2 \phi \]
\[ = -Q_1(\langle \nabla \rangle^{-2} (\partial_l A_2 - \partial_2 A_1), \phi) \]
\[ = -Q_12 (\langle \nabla \rangle^{-1} (R^1 A_2 - R^2 A_1), \phi) . \]
Thus, we have shown
\[ A^\mu \partial_\mu \phi = -Q_12 (\langle \nabla \rangle^{-1} (R^1 A_2 - R^2 A_1), \phi) \]
\[ + Q_{10}(\langle \nabla \rangle^{-1} R^l A_0, \phi) + \langle \nabla \rangle^{-2} A^\mu \partial_\mu \phi. \]

Our aim is to prove the following estimate:
\[ \| A_{\mu, \pm 1} \partial_\mu \phi \|_{H_{\pm 1, 0}^r} \lesssim \sum_{\mu} A_{\mu, \pm 1} \| \langle \nabla \rangle \phi \| \| \langle \nabla \rangle x_{\pm 1, \pm 2} \|. \]
We first estimate the last term on the right hand side of (29). By Lemma 3.3 we obtain the sufficient estimate
\[ \| (\langle \nabla \rangle^{-2} A_{j, \pm 1} \partial_\mu \phi \|_{H_{\pm 1, 0}^r} \lesssim \| A_{j, \pm 1} \| \| \langle \nabla \rangle x_{\pm 1, \pm 2} \| \| \partial_\mu \phi \|_{H_{\pm 1, 0}^r} \],
if \( l > \frac{3}{2} - 2 \).
The estimate for the first two terms on the right hand side of (29) reduces to
\[ \| Q_12(A_{\mu, \langle \nabla \rangle \phi}) \|_{H_{\pm 1, 0}^r} + \| Q_{10}(A_{\mu, \langle \nabla \rangle \phi}) \|_{H_{\pm 1, 0}^r} \lesssim \| A_{\mu} \| \| \langle \nabla \rangle x_{\pm 1, \pm 2} \| \| \langle \nabla \rangle \phi \|_{H_{\pm 1, 0}^r} \|
which follows from Lemma 3.1 if \( l \geq s - 1 \) and \( l \geq \frac{1}{r} \).
Concerning \( A^\mu_{\alpha, \phi} \) we obtain by Lemma 3.6:
\[ \| A_{\mu}^\alpha \phi \|_{H_{\pm 1, 0}^r} \lesssim \| A_{\mu} \| H_{\alpha}^r \| \phi \| H_{\alpha}^r \] provided \( l > \frac{13}{3} - \frac{1}{2} \), \( s \leq l + 1 \) and \( 2l - s > \frac{7}{4} - 1 \).
For the terms \( |\phi|^2 \phi \) and \( N^2 \phi \) we obtain similarly
\[ \| \phi |^2 \phi \|_{H_{\pm 1, 0}^r} \lesssim \| \phi \|^3_{H_{\pm 1, 0}^r} ,
if \( s > \frac{13}{3} - \frac{1}{2} \), and
\[ \| N^2 \phi \|_{H_{\pm 1, 0}^r} \lesssim \| N \| H_{\pm 1, 0}^r \| \phi \| H_{\pm 1, 0}^r ,
if \( m > \frac{13}{3} - \frac{1}{2} \), \( s \leq m + 1 \) and \( 2m - s > \frac{7}{4} - 1 \).
The term \( N \phi \) can be handled even more easily.
Finally we have to consider the terms on the right hand side of (19). The term \( N |\phi|^2 \) (and similarly \( |\phi|^2 \)) is treated by Lemma 3.4 as follows:
\[ \| N |\phi|^2 \|_{H_{m, 0}^r} \lesssim \| N \| H_{m, 0}^r \| \phi \|^2_{H_{m, 0}^r} ,

if \( s > \frac{13}{8} r - \frac{1}{2}, \) \( m \leq s + 1 \) and \( 2s - m > \frac{7}{4} r - 1. \) The proof of Theorem 1.1 is now complete. \( \square \)

**Proof of Corollary 1.1.** Corollary 1.1 follows by an easy calculation using the interpolation properties of the spaces \( X_{r,s,b}^+, \) by interpolation between the bi- and trilinear estimates for the nonlinearities, which are given above for \( r = 1+ \) and in \( \[P\] \) for \( r = 2. \) \( \square \)

**References**

[AFS] P. d’Ancona, D. Foschi and S. Selberg: *Product estimates for wave-Sobolev spaces in 2 + 1 and 1 + 1 dimensions.* Contemporary Mathematics 526 (2010), 125-150

[CC] D. Chae and M. Chae: *The global existence in the Cauchy problem of the Maxwell-Chern-Simons-Higgs system.* J. Math. Phys. 43(2002), 5470-5482

[FK] D. Foschi and S. Klainerman: *Bilinear space-time estimates for homogeneous wave equations.* Ann. Sc. ENS. 4. serie, 33 (2000), 211-274

[G] A. Grünrock: *An improved local well-posedness result for the modified KdV equation.* Int. Math. Res. Not. (2004), no.61, 3287-3308

[GN] V. Grigoryan and A. Nahmod: *Almost critical well-posedness for nonlinear wave equation with \( Q^{\mu\nu} \) null forms in 2D.* Math. Res. Letters 21 (2014), 313-332

[GT] V. Grigoryan and A. Tanguy: *Improved well-posedness for the quadratic derivative nonlinear wave equation in 2D.* J. Math. Anal. Appl. 475 (2019), 1578-1595

[LLM] C. Lee, K. Lee and H. Min: *Self-dual Maxwell-Chern-Simons solitons.* Phys. Letters B 252 (1990), 79-83

[KM] S. Klainerman and M. Machedon: *On the Maxwell-Klein-Gordon equation with finite energy.* Duke Math. J. 74 (1994), 19-44

[P] H. Pecher: *Local solutions with infinite energy of the Maxwell-Chern-Simons-Higgs system in Lorenz gauge.* Discrete Contin. Dyn. Syst. 36 (2016), 2193–2204.

[ST] S. Selberg and A. Tesfahun: *Finite energy global well-posedness of the Maxwell-Klein-Gordon system in Lorenz gauge.* Comm. PDE 35 (2010), 1029-1057

[ST1] S. Selberg and A. Tesfahun: *Global well-posedness of the Chern-Simons-Higgs equations with finite energy.* Discrete Cont. Dyn. Syst. 33 (2013), 2531-2546

[Y] J. Yuan: *On the well-posedness of Maxwell-Chern-Simons-Higgs system in the Lorenz gauge.* Discrete Cont. Dyn. Syst. 34 (2014), 2389-2403