Weak and variational entropy solutions to the system describing steady flow of a compressible reactive mixture

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Abstract

We consider a system of partial differential equations which describes steady flow of a compressible heat conducting chemically reacting gaseous mixture. We extend the result from Giovangigli, Pokorný, Zatorska (2015) in the sense that we introduce the variational entropy solution for this model and prove existence of a weak solution for $\gamma > \frac{4}{3}$ and existence of a variational entropy solution for any $\gamma > 1$. The proof is based on improved density estimates.

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1 Introduction

Chemically reacting mixtures appear in many real-life situations, especially in chemical engineering (\cite{18}), combustion (\cite{20}), description of some atmospheric phenomena (\cite{19}) and many others. There are many models of mixtures which can be derived from different general physical models depending on the phenomena which we want to study. We may start from molecular theories like the kinetic theory, statistical mechanics and thermodynamics or from the macroscopic theories like continuum physics and continuum thermodynamics.
Here, we rely on the latter. We continue the program started in [12] which was applied to a special situation for the steady problem in [6].

More precisely, we investigate a system of partial differential equations describing steady flow of chemically reactive, heat conducting, gaseous mixture. The system, which composes of the steady compressible Navier–Stokes–Fourier system coupled with the balance of mass fractions, reads

\[
\begin{align*}
\text{div} (\rho \mathbf{u}) &= 0, \\
\text{div} (\rho \mathbf{u} \otimes \mathbf{u}) - \text{div} \mathbb{S} + \nabla \pi &= \rho \mathbf{f}, \\
\text{div} (\rho E \mathbf{u}) + \text{div} (\pi \mathbf{u}) + \text{div} \mathbf{Q} - \text{div} (\mathbb{S} \mathbf{u}) &= \rho \mathbf{f} \cdot \mathbf{u}, \\
\text{div} (\rho Y_k \mathbf{u}) + \text{div} \mathbf{F}_k &= m_k \omega_k, \quad k \in \{1, \ldots, n\}.
\end{align*}
\]

(1)

In the above equations \( \mathbb{S} \) denotes the viscous part of the stress tensor, \( \pi \) the internal pressure of the fluid, \( \mathbf{f} \) the external force, \( E \) the specific total energy, \( \mathbf{Q} \) the heat flux, \( \omega_k \) the molar production rate of the \( k \)-th species, \( \mathbf{F}_k \) the diffusion flux of the \( k \)-th species and \( m_k \) the molar mass of the \( k \)-th species which we assume to be equal, hence, without loss of generality

\[
m_1 = \ldots = m_n = 1.
\]

(2)

System (1) is supplemented by the no-slip boundary conditions for the velocity

\[
\mathbf{u}|_{\partial \Omega} = \mathbf{0},
\]

(3)

together with

\[
\mathbf{F}_k \cdot \mathbf{n}|_{\partial \Omega} = 0,
\]

(4)

and the Robin boundary condition for the heat flux

\[
- \mathbf{Q} \cdot \mathbf{n} + L(\vartheta - \vartheta_0) = 0
\]

(5)

which means that the heat flux through the boundary is proportional to the difference of the temperature inside \( \Omega \) and the known external temperature \( \vartheta_0 \). The coefficient \( L \) describes thermal insulation of the boundary and for simplicity we assume it to be constant.

We further prescribe the total mass of the mixture

\[
\int_{\Omega} \rho \, dx = M > 0.
\]

(6)

The mass fractions \( Y_k, k \in \{1, \ldots, n\} \), are defined by \( Y_k = \frac{\omega_k}{\vartheta} \). Thus, by definition, they satisfy

\[
\sum_{k=1}^{n} Y_k = 1.
\]

(7)

Concerning the chemical production rates, we assume them to be sufficiently regular, bounded functions of \( \rho, \vartheta \) and \( Y_k \) such that

\[
\omega_k \geq 0 \quad \text{for} \quad Y_k = 0.
\]
We also assume
\[ \omega_k \geq -CY_k^r \] for some \( C, r > 0 \), \( (8) \)
which means that a species cannot decrease faster than proportionally to some positive power of its fraction (a possible natural choice is \( r = 1 \)). The stress tensor \( \mathbb{S} \) is given by the Newton rheological law as
\[ \mathbb{S} = \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^t - \frac{2}{3} \text{div} \mathbf{u} \mathbb{I} \right] + \nu (\text{div} \mathbf{u}) \mathbb{I}, \] \( (9) \)
where \( \mu = \mu(\vartheta) > 0, \nu = \nu(\vartheta) \geq 0 \), Lipschitz continuous functions in \( \mathbb{R}^+ \), are the shear and bulk viscosity coefficients, respectively, on which we assume
\[ \mu(1 + \vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta), \quad 0 \leq \nu(\vartheta) \leq \overline{\nu}(1 + \vartheta) \] \( (10) \)
for some positive constants \( \mu, \bar{\mu}, \bar{\nu} \), and \( \mathbb{I} \) is the identity matrix.

1.1 Thermodynamic relations

Pressure and internal energy. We consider the pressure \( \pi = \pi(\varrho, \vartheta) \) with following form
\[ \pi = \pi(\varrho, \vartheta) = \pi_c(\varrho) + \pi_m(\varrho, \vartheta), \] \( (11) \)
where the molecular pressure \( \pi_m \) obeys the Boyle law
\[ \pi_m = \sum_{k=1}^{n} \varrho Y_k \vartheta = \varrho \vartheta. \] \( (12) \)
It represents the pressure for an ideal mixture of \( n \) species, with molar masses equal to 1. Moreover, without loss of generality, the gaseous constant equals one. The first component of \( \pi \), \( \pi_c \), is the so called cold pressure. We assume it in the form
\[ \pi_c = \varrho^\gamma, \quad \gamma > 1. \] \( (13) \)
Indeed, a more general form of the cold pressure may be treated. The only important assumptions are that \( \pi_c(\varrho) \sim \varrho^\gamma \) for \( \varrho \) large, \( \pi \in C([0, \infty)) \cap C^1((0, \infty)) \), strictly increasing in \( \mathbb{R}^+ \).

The specific total energy \( E \) is a sum of the specific kinetic and specific internal energies
\[ E = E(\varrho, \mathbf{u}, \vartheta, Y_1, \ldots, Y_n) = \frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta, Y_1, \ldots, Y_n). \]
The internal energy consists of two components corresponding to the components of the pressure
\[ e = e_c(\varrho) + e_m(\vartheta, Y_1, \ldots, Y_n), \]
where the cold energy $e_c$ and the molecular internal energy $e_m$ are given by

$$e_c = \frac{1}{\gamma - 1} \varrho^{\gamma-1}, \quad e_m = \sum_{k=1}^{n} Y_k c_k = \vartheta \sum_{k=1}^{n} c_{vk} Y_k.$$  

Here, $c_{vk}$ are the mass constant-volume specific heats and can be different for different species. Under our assumption, the constant-pressure specific heat, denoted by $c_{pk}$, equals

$$c_{pk} = c_{vk} + 1, \quad (14)$$

and both $c_{vk}$ and $c_{pk}$ are assumed to be constant.

**Entropy.** According to the second law of thermodynamics, there exists a differentiable function called the *specific entropy of the mixture* $s(\varrho, \vartheta, Y_1, \ldots, Y_n)$. It can be expressed in terms of the partial specific entropies $s_k = s_k(\varrho, \vartheta, Y_k)$ of the $k$-th species

$$s = \sum_{k=1}^{n} Y_k s_k. \quad (15)$$

The Gibbs formula relates the differential of entropy to the differential of energy, total density and mass fractions as follows

$$\vartheta Ds = Dc + \pi D \left( \frac{1}{\varrho} \right) - \sum_{k=1}^{n} g_k D Y_k, \quad (16)$$

with the Gibbs functions

$$g_k = h_k - \vartheta s_k. \quad (17)$$

Here $h_k = h_k(\vartheta)$, $s_k = s_k(\varrho, \vartheta, Y_k)$ denote the specific enthalpy and the specific entropy of the $k$-th species, respectively, with the following exact forms

$$h_k = c_{pk} \vartheta, \quad s_k = c_{vk} \log \vartheta - \log \varrho - \log Y_k,$$

and we assume

$$-\sum_{k=1}^{n} g_k \omega_k \geq 0. \quad (18)$$

The cold pressure and the cold energy correspond to isentropic processes. Using (16) it is possible to derive an equation for the specific entropy $s$

$$\text{div} \left( \varrho s u \right) + \text{div} \left( \frac{Q}{\vartheta} - \sum_{k=1}^{n} \frac{g_k}{\vartheta} F_k \right) = \sigma, \quad (19)$$

where $\sigma$ is the entropy production rate

$$\sigma = \varepsilon : \nabla u - \frac{Q \cdot \nabla \vartheta}{\vartheta^2} - \sum_{k=1}^{n} F_k \cdot \nabla \left( \frac{g_k}{\vartheta} \right) - \sum_{k=1}^{n} g_k \omega_k \frac{\vartheta}{\vartheta}. \quad (20)$$
1.2 The form of transport fluxes

**Heat flux.** The heat flux $Q$ consists of two terms. The first one represents the transfer of energy due to the species molecular diffusion and the second one the Fourier law,

$$Q = \sum_{k=1}^{n} h_k F_k + q, \quad q = -\kappa \nabla \vartheta,$$

where $\kappa = \kappa(\vartheta)$ is the thermal conductivity coefficient on which we assume

$$\kappa(1 + \vartheta^m) \leq \kappa(\vartheta) \leq \kappa(1 + \vartheta^m)$$

for some constants $m, \kappa, \kappa > 0$.

**Diffusion flux.** The diffusion flux of the $k$-th species $F_k$ is given by

$$F_k = -\rho Y_k \sum_{l=1}^{n} D_{kl}^\rho \nabla Y_l,$$

where $D_{kl}^\rho = D_{kl}^\rho(\rho, \vartheta, Y_1, \ldots, Y_n)$, $k, l = 1, \ldots, n$ are the multicomponent diffusion coefficients. The coefficients $\rho D_{kl}^\rho$ depend only on $\vartheta$ and $Y_1, \ldots, Y_n$ (see [5]), therefore we introduce another matrix

$$\mathbb{D} = (D_{kl})_{k,l=1}^{n} = (\rho D_{kl}^\rho)_{k,l=1}^{n} = (D_{kl}(\vartheta, Y_1, \ldots, Y_n))_{k,l=1}^{n}.$$

We denote by $N(\mathbb{D})$ the nullspace of the matrix $\mathbb{D}$, $R(\mathbb{D})$ its range, and $\bar{Y}^\perp$ is the orthogonal complement of $\bar{Y}$. The diffusion matrix $\mathbb{D}$ has the following properties which are discussed in [5] Chapter 7:

$$\begin{align*}
\mathbb{D} &= \mathbb{D}^t, \quad N(\mathbb{D}) = \mathbb{R}\bar{Y}, \quad R(\mathbb{D}) = \bar{Y}^\perp, \\
\mathbb{D} &\text{ is positive semidefinite over } \mathbb{R}^n.
\end{align*}$$

Note that we assumed $\bar{Y} = (Y_1, \ldots, Y_n)^t > 0$.

Furthermore, the matrix $\mathbb{D}$ is homogeneous of a non-negative order with respect to $Y_1, \ldots, Y_n$ and $D_{ij}$ are differentiable functions of $\vartheta, Y_1, \ldots, Y_n$ for any $i, j \in \{1, \ldots, n\}$ such that

$$|D_{ij}(\vartheta, \bar{Y})| \leq C(\bar{Y})(1 + \vartheta^a)$$

for some $a \geq 0$. Denoting $\bar{U} = (1, \ldots, 1)^t$, the form of $F_k$ implies in particular $\{F_k\}_{k=1}^{n} \in \bar{U}^\perp$ which yields

$$\sum_{k=1}^{n} F_k = 0.$$

Therefore, since the species equations must sum to the continuity equation, we obtain

$$\sum_{k=1}^{n} \omega_k = 0.$$
1.3 Entropy production rate

Due to (24) the matrix $\mathbb{D}$ is positive definite over $\overrightarrow{U}^\perp$. As we shall see now, this property is connected with the positivity of entropy production rate $\sigma$ defined in (20). Indeed, we have

$$\nabla \left( \frac{g_k}{\vartheta} + c_{pk} \log \vartheta \right) = \nabla \log p_k,$$

where $p_k = \varrho Y_k \vartheta$. Therefore (20) may be rewritten in the following form

$$\sigma = S : \nabla \mathbf{u} + \frac{\kappa |\nabla \vartheta|^2}{\vartheta} - \sum_{k=1}^n \mathbf{F}_k \cdot \nabla (\log p_k) - \sum_{k=1}^n \frac{g_k \omega_k}{\vartheta}. \tag{28}$$

Let us have a look on the structure of the third term. We have

$$-\sum_{k=1}^n \mathbf{F}_k \cdot \nabla (\log p_k) = -\sum_{k=1}^n \frac{\mathbf{F}_k}{p_k} \cdot \nabla p_k$$

$$= -\sum_{k=1}^n \mathbf{F}_k \cdot \left( \frac{\nabla Y_k}{Y_k} + \frac{\nabla (\varrho \vartheta)}{\varrho \vartheta} \right) \quad \text{[due to (25)]} \tag{29}$$

$$= \sum_{k,l=1}^n D_{kl} \nabla Y_l \cdot \nabla Y_k \geq c \sum_{k=1}^n \frac{|\nabla Y_k|^2}{Y_k} \geq c \sum_{k=1}^n |\nabla Y_k|^2,$$

where we have used the fact that $\partial_x \tilde{Y} \in \overrightarrow{U}^\perp$ for all $i \in \{1, 2, \ldots, n\}$ due to (7) (cf. [5, Lemma 7.6.1]) Note that the last inequality is due to the fact that $Y_k \leq 1$, therefore the second last term contains additional information about the mass fractions, but we do not exploit it. Now, (28), (29), (9) together with (18) yields $\sigma \geq 0$.

2 Weak and variational entropy solutions. Main Results.

We are now in a position to formulate the definition of weak solutions to our system.

**Definition 1.** We say the set of functions $(\varrho, \mathbf{u}, \vartheta, \tilde{Y})$ is a weak solution to problem (1–6) with assumptions stated above, provided

- $\varrho \geq 0$ a.e. in $\Omega$, $\varrho \in L^{6\gamma/5}(\Omega)$, $\int_{\Omega} \varrho \, dx = M$
- $\mathbf{u} \in W^{1,2}_0(\Omega)$, $\varrho|\mathbf{u}|$ and $\varrho|\mathbf{u}|^2 \in L^{\frac{6}{5}}(\Omega)$
- $\vartheta \in W^{1,2}(\Omega) \cap L^{3m}(\Omega)$, $\varrho \vartheta$, $\varrho \vartheta|\mathbf{u}|$, $\mathbf{S} \mathbf{u}$, $\kappa|\nabla \vartheta| \in L^1(\Omega)$
- $\tilde{Y} \in W^{1,2}(\Omega)$, $Y_k \geq 0$ a.e. in $\Omega$, $\sum_{k=1}^n Y_k = 1$ a.e. in $\Omega$, $\mathbf{F}_k \cdot \mathbf{n}|_{\partial \Omega} = 0$
and the following integral equalities hold

- the weak formulation of the continuity equation
  \[ \int_{\Omega} \rho \mathbf{u} \cdot \nabla \psi \, dx = 0 \]  \hspace{1cm} (30)
  holds for any test function \( \psi \in C^\infty(\Omega) \);
- the weak formulation of the momentum equation
  \[ - \int_{\Omega} (\rho (\mathbf{u} \otimes \mathbf{u}) : \nabla \varphi - S : \nabla \varphi) \, dx - \int_{\Omega} \pi \text{div} \varphi \, dx = \int_{\Omega} \rho \mathbf{f} \cdot \varphi \, dx \]  \hspace{1cm} (31)
  holds for any test function \( \varphi \in C^\infty_0(\Omega) \);
- the weak formulation of the species equations
  \[ - \int_{\Omega} Y_k \rho \mathbf{u} \cdot \nabla \psi \, dx - \int_{\Omega} F_k \cdot \nabla \psi \, dx = \int_{\Omega} \omega_k \psi \, dx \]  \hspace{1cm} (32)
  holds for any test function \( \psi \in C^\infty(\Omega) \) and for all \( k = 1, \ldots, n \);
- the weak formulation of the total energy balance
  \[ - \int_{\Omega} \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \rho e \right) \mathbf{u} \cdot \nabla \psi \, dx + \int_{\Omega} \kappa \nabla \vartheta \cdot \nabla \psi \, dx - \int_{\Omega} \left( \sum_{k=1}^n h_k \mathbf{F}_k \right) \cdot \nabla \psi \, dx \]
  \[ = \int_{\Omega} \rho \mathbf{f} \cdot \mathbf{u} \psi \, dx - \int_{\Omega} (\mathbf{S} \mathbf{u}) \cdot \nabla \psi \, dx + \int_{\Omega} \pi \mathbf{u} \cdot \nabla \psi \, dx - \int_{\partial \Omega} L(\vartheta - \vartheta_0) \psi \, dS \]  \hspace{1cm} (33)
  holds for any test function \( \psi \in C^\infty(\Omega) \).

The admissible range of \( \gamma \) in the pressure law \([13]\) for which we are able to show existence of weak solutions in the above sense is limited mostly by the terms \( \rho |\mathbf{u}|^2 \mathbf{u} \) and \( \mathbf{S} \mathbf{u} \) in the weak formulation of total energy balance. Therefore, following \([13], [15]\) we introduce a slightly more general notion of variational entropy solutions to system \([\Pi]\) which consist in replacing the weak formulation of the total energy balance by the weak formulation of the entropy inequality.

**Definition 2.** We say the set of functions \((\rho, \mathbf{u}, \vartheta, \mathbf{Y})\) is a variational entropy solution to problem \((\Pi)\) with assumptions stated above, provided

- \( \rho \geq 0 \) a.e. in \( \Omega \), \( \rho \in L^{s\gamma}(\Omega) \) for some \( s > 1 \), \( \int_{\Omega} \rho \, dx = M \)
- \( \mathbf{u} \in W^{1,2}_0(\Omega) \), \( \rho \mathbf{u} \in L^\frac{6}{5}(\Omega) \)
- \( \vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega) \), \( r > 1 \), \( \rho \vartheta, \mathbf{S} : \frac{\nabla \vartheta}{\vartheta}, \kappa \frac{\nabla \vartheta}{\vartheta^2}, \kappa \frac{\nabla \vartheta}{\vartheta} \in L^1(\Omega) \), \( \frac{1}{\vartheta} \in L^1(\partial \Omega) \)
- \( \mathbf{Y} \in W^{1,2}(\Omega) \), \( Y_k \geq 0 \) a.e. in \( \Omega \), \( \sum_{k=1}^n Y_k = 1 \) a.e. in \( \Omega \), \( \mathbf{F}_k \cdot \mathbf{n}|_{\partial \Omega} = 0 \)

\( L^{3m}(\Omega) \) abbreviates \( L^{3m}(\Omega) \).
satisfy equations (30) – (32), the following entropy inequality

\[
\int_{\Omega} \frac{S}{\vartheta} \cdot \nabla \mu \psi \, dx + \int_{\Omega} \frac{|\nabla \vartheta|^2}{\vartheta^2} \psi \, dx - \int_{\Omega} \sum_{k=1}^{n} \omega_k (c_{pk} - c_{vk} \log \vartheta + \log Y_k) \psi \, dx \\
+ \int_{\Omega} \psi \sum_{k=1}^{n} D_{kl} \nabla Y_k \cdot \nabla Y_l \, dx + \int_{\Omega} \frac{L}{\vartheta} \vartheta_0 \psi \, dS \leq \int_{\Omega} \frac{\kappa \vartheta}{\vartheta^2} \cdot \nabla \psi \, dx - \int_{\Omega} \varrho \mu \cdot \nabla \psi \, dx \\
- \int_{\Omega} \log \vartheta \left( \sum_{k=1}^{n} F_k \varrho_{vk} \right) \cdot \nabla \psi \, dx + \int_{\Omega} \left( \sum_{k=1}^{n} F_k \log Y_k \right) \cdot \nabla \psi \, dx + \int_{\partial \Omega} L \psi \, dS \tag{34}
\]

for all non-negative \( \psi \in C^\infty(\Omega) \) and the global total energy balance (i.e. (33) with \( \psi \equiv 1 \))

\[
\int_{\partial \Omega} L (\vartheta - \vartheta_0) \, dS = \int_{\Omega} \varrho f \cdot \mu \, dx. \tag{35}
\]

Formally, the entropy inequality (34) is nothing but a weak formulation of (19). We will return to this in the part devoted to the formulation of the approximate solution, where we deduce the approximate entropy (in)equality from the approximate internal energy balance and the approximate momentum balance. Note, however, that we have here inequality instead of equality. This is a consequence of the fact that for sequences of functions which do not converge strongly but only weakly in some spaces we are not able to ensure the corresponding limit passages and we are obliged to use only the weak lower semicontinuity in some terms. Note further that (34) does not contain all terms from (19), some of them are missing. These terms are formally equal to zero due to assumptions that \( \omega_k \) and \( F_k \) sum up to zero. We removed them from the formulation of the entropy inequality due to the fact that we cannot exclude the situation that \( \varrho = 0 \) in some large portions of \( \Omega \) (with positive Lebesgue measure), thus \( \log \varrho \) is not well defined there. However, the variational entropy solution still has the property that any sufficiently smooth variational entropy solution in the sense above is a classical solution to our problem, provided the density is strictly positive in \( \Omega \).

We are now in position to formulate our main result.

**Theorem 1.** Let \( \gamma > 1 \), \( M > 0 \), \( m > \max\{ \frac{2}{3}, \frac{2}{3(\gamma - 1)} \} \), \( a < \frac{3m}{2} \). Let \( \Omega \in C^2 \). Then there exists at least one variational entropy solution to our problem above. Moreover, \( (\varrho, \mu) \) is the renormalized solution to the continuity equation.

In addition, if \( m > \max\{ 1, \frac{2\gamma}{3(3\gamma - 3)} \}, \gamma > \frac{4}{3}, a < \frac{3m-2}{2} \), then the solution is a weak solution in the sense above.

The second part of the main result, dealing with weak solutions, is an improvement of the result from [6]. This is connected with the fact that we will use finer estimates of the density before the last limit passage.

Note further that the assumptions on \( \gamma \) and \( m \) in both variational entropy and weak solution correspond to those which ensure the existence of the corresponding type of a
solution for the steady compressible Navier–Stokes–Fourier system, cf. [13]. Finally, recall that the pair \((\rho, u)\) is a renormalized solution to the continuity equation provided \(u \in W^{1,2}(\Omega), \rho \in L^{\frac{5}{2}}(\Omega)\) and for any \(b \in C^1(0, \infty) \cap C([0, \infty)), b'(z) = 0\) for \(z \geq M\) for some \(M > 0\)

\[
\int_{\Omega} \left( b(\rho) \text{div} \psi + (b(\rho) - b'(\rho)\rho) \text{div} u \psi \right) \, dx = 0
\]

for all \(\psi \in C^\infty(\Omega)\).

The weak solutions for the compressible Navier–Stokes equations were for the first time considered in the seminal monography by P.L. Lions [11]. Their existence was shown for \(\gamma > \frac{9}{5}\). Using more precise estimates of the density, the result was subsequently improved in the papers [4], [8] and [7] to reach the existence of weak solutions for \(\gamma > 1\). The theory was applied to the compressible Navier–Stokes–Fourier system in the series of papers [14], [15] (here, the notion of variational entropy solutions in the steady case was introduced) and [9]. See also [13] for further details.

The system of equations describing the flow of chemically reacting, heat conducting gaseous mixture was considered firstly in the evolutionary case in the context of variational entropy solutions in [3], however, with Fick’s law. A more general multicomponent diffusion flux was in the context of weak solutions considered in [12] and in the steady regime in [6].

3 Approximation

Following [6] we will prove our main results introducing five steps of approximation. The first four are connected with small parameters \(\delta > \varepsilon > \lambda > \eta > 0\) and the last one, connected with a positive integer \(N\), is the Galerkin approximation for the velocity.

Precisely, we introduce the approximation of diffusion flux \(F_k:\)

\[
J_k = -\sum_{l=1}^{n} Y_k Y_l \widehat{D}_{kl}(\vartheta, \vec{Y}) \nabla Y_l / Y_l - (\varepsilon (\varrho + 1) Y_k + \lambda) \nabla Y_k / Y_k,
\]

with

\[
\widehat{D}_{kl}(\vartheta, \vec{Y}) = \frac{1}{(\sigma Y + \varepsilon) r} D_{kl}(\vartheta, \vec{Y}),
\]

where \(\sigma Y = \sum_{k=1}^{n} Y_k\). The reason for this notation is that, unless we let \(\lambda \to 0^+\), it is not clear whether \(\sigma Y = 1\). We only know that \(Y_k \geq 0\). Furthermore, we introduce a regularization of the stress tensor

\[
S_{\eta} = \frac{\mu_{\eta}(\vartheta)}{1 + \eta \vartheta} \left[ \nabla u + (\nabla u)^t - \frac{2}{3} \text{div} u \mathbb{I} \right] + \frac{\nu_{\eta}(\vartheta)}{1 + \eta \vartheta} (\text{div} u) \mathbb{I},
\]

where \(\mu_{\eta}, \nu_{\eta}\) are standard mollifications of the viscosity functions. Next,

\[
\kappa_{\delta, \eta} = \kappa_{\eta} + \delta \vartheta^B + \delta \vartheta^{-1}
\]
is a regularization of heat conductivity coefficient with $B > 0$ sufficiently large which will be determined later and $\kappa^0$ is the mollification of the heat conductivity. Compared to [6] we introduce a minor modification in the approximation, namely we approximate the fractional entropies with

$$s_k^\lambda = c_{\nu k} \log \vartheta - \log Y_k - \log (\varrho + \sqrt{\lambda}).$$

(40)

Analogously, we denote

$$g_k^\lambda = c_{\nu k} \vartheta - \vartheta s_k^\lambda, \quad s^\lambda = \sum_{k=1}^n Y_k s_k^\lambda.$$

This modification will enable us to pass to the limit with $\lambda$ in the weak formulation of the entropy inequality, on the other hand it is harmless for crucial a priori estimates for the full approximation.

We are now ready to formulate the approximate problem involving five above mentioned parameters. Let $\{w_n\}_{n=1}^\infty$ be an orthogonal basis of $W_0^{1,2}(\Omega)$ such that $w_i \in W^{2,q}(\Omega)$ for $q < \infty$ (we can take for example eigenfunctions of the Laplace operator with Dirichlet boundary conditions). At the level of full approximation we want to show existence of a set of functions $(\varrho, \eta, \lambda, \varepsilon, \delta, u, Y, \vartheta, N, \eta, \lambda, \varepsilon, \delta)$ (from now on we skip the indices) such that

- the approximate continuity equation

$$\varepsilon \varrho + \operatorname{div} (\varrho u) = \varepsilon \Delta \varrho + \varepsilon \vartheta, \quad \nabla \varrho \cdot n|_{\partial \Omega} = 0,$$

(41)

where $\bar{\varrho} = \frac{M}{|\Omega|}$, is satisfied pointwisely

- the Galerkin approximation for the momentum equation (note that the convective term reduces to the standard form provided $\operatorname{div} (\varrho u) = 0$, even in the weak sense)

$$\int_{\Omega} \left( \frac{1}{2} \varrho u \cdot \nabla u \cdot w - \frac{1}{2} \varrho (u \otimes u) : \nabla w + \eta \cdot \nabla w \right) \, dx$$

$$- \int_{\Omega} (\pi + \delta \vartheta^2 + \delta \vartheta^2) \operatorname{div} w \, dx = \int_{\Omega} \varrho f \cdot w \, dx$$

(42)

is satisfied for each test function $w \in X_N$, where $u \in X_N$, $X_N = \operatorname{span} \{w_i\}_{i=1}^N$, and $\beta > 0$ is large enough

- the approximate species mass balance equations

$$\operatorname{div} J_k = \omega_k + \varepsilon \bar{\omega}_k - \varepsilon Y_k \vartheta - \operatorname{div} (Y_k \varrho u) + \varepsilon \operatorname{div} (Y_k \nabla \varrho) - \sqrt{\lambda} \log Y_k, \quad J_k \cdot n|_{\partial \Omega} = 0,$$

(43)

are satisfied pointwisely, where $\sum_{k=1}^n \bar{\varrho}_k = \bar{\varrho}$, for example we take $\bar{\varrho}_k = \frac{\bar{\varrho}}{n}$.
• the approximate internal energy balance

\[-\text{div} \left( \kappa_{\delta,\eta} \frac{\varepsilon + \vartheta}{\vartheta} \nabla \vartheta \right) = -\text{div} (\varrho e\mathbf{u}) - \pi \text{div} \mathbf{u} + \frac{\delta}{\vartheta} + \mathbb{S}_\eta : \nabla \mathbf{u} \]

\[+ \delta \varepsilon (\beta \varrho^{\beta - 2} + 2) |\nabla \varrho|^2 - \text{div} \left( \vartheta \sum_{k=1}^{n} c_{vk} \mathbf{J}_k \right) \]

(44)

with the boundary condition

\[\kappa_{\delta,\eta} \frac{\varepsilon + \vartheta}{\vartheta} \nabla \vartheta \cdot \mathbf{n}|_{\partial \Omega} + (L + \delta \vartheta^{B - 1})(\vartheta - \vartheta_0^n) + \varepsilon \log \vartheta + \lambda \vartheta^B \log \vartheta = 0 \]

(45)

is satisfied pointwisely, where \(\vartheta_0^n\) is a smooth, strictly positive approximation of \(\vartheta_0\) and \(\kappa_{\delta,\eta}\) is as above.

It remains to formulate the approximate entropy inequality for the purpose of showing existence of variational entropy solutions. Note that the entropy inequality (or rather equality on this level of approximation) is not an additional assumption, but a consequence of the approximate relations above.

Remark 1. Note that there is one more change with respect to paper [6], namely we have in (44) in the last term on the right-hand side \(\sqrt{\lambda}\) instead of \(\lambda\). This is connected with the limit passage \(\lambda \to 0^+\) in the weak formulation of the entropy inequality. It is an easy matter to check that the proof in [6] would work also for this approximation.

3.1 Approximate entropy inequality

We now deduce the form of the approximate entropy inequality. Even though the computations below are rather formal (and require certain regularity of all functions), it can be verified that the regularity enjoyed by the approximate solutions is enough for the entropy equality to hold.

Recalling the form of internal energy and pressure we observe that

\[\text{div} (\varrho e\mathbf{u}) + \pi \text{div} \mathbf{u} = \varrho \mathbf{u} \cdot \nabla \left( \frac{\vartheta^{\gamma - 1}}{\gamma - 1} + \vartheta \sum_{k=1}^{n} c_{vk} Y_k \right) + e \text{div} (\varrho \mathbf{u}) + (\varrho^{\gamma - 1} + \vartheta) \varrho \text{div} \mathbf{u} \]

\[= \varrho \mathbf{u} \cdot \nabla \left( \vartheta \sum_{k=1}^{n} c_{vk} Y_k \right) - \vartheta \mathbf{u} \cdot \nabla \varrho + (\varrho^{\gamma - 1} + \vartheta + e) \text{div} (\varrho \mathbf{u}).\]

Therefore, multiplying the approximate internal energy balance (44) by \(\frac{\vartheta}{\vartheta}\) and integrating
over $\Omega$ we get
\[
\int_{\Omega} \kappa_{\delta,\eta}(\varepsilon + \vartheta) \nabla \vartheta \cdot \nabla \psi \, dx - \int_{\Omega} \kappa_{\delta,\eta}(\varepsilon + \vartheta) |\nabla \vartheta|^2 \psi \, dx \\
+ \int_{\partial \Omega} \frac{\psi}{\vartheta} \left[ (L + \delta \vartheta B^{-1})(\vartheta - \psi_0^2) + \varepsilon \log \vartheta + \lambda \vartheta B/2 \log \vartheta \right] dS \\
- \int_{\Omega} \sum_{k=1}^{n} h_k J_k \cdot \nabla \left( \frac{\psi}{\vartheta} \right) \, dx + \int_{\Omega} \frac{\psi}{\vartheta} \left[ \varrho u \cdot (\nabla \vartheta \sum_{k=1}^{n} c_{\vartheta k} Y_k + \vartheta \sum_{k=1}^{n} c_{\vartheta k} \nabla Y_k) - \vartheta u \cdot \nabla \varrho \right] \, dx \\
+ \int_{\Omega} \varepsilon(\Delta \vartheta + \bar{\vartheta} - \bar{\vartheta}) (\vartheta \gamma^{-1} + \varepsilon + \theta) \frac{\psi}{\vartheta} \, dx - \int_{\Omega} \frac{\delta \psi}{\vartheta^2} \, dx - \int_{\Omega} \frac{\psi \bar{\vartheta} : \nabla u}{\vartheta} \, dx \\
- \int_{\vartheta} \vartheta \left[ \sum_{k=1}^{n} (\varepsilon + 1) Y_k + \lambda \nabla Y_k \right] \cdot \nabla \left( \frac{\psi}{\vartheta} \right) \, dx - \int_{\Omega} \frac{\delta \varepsilon (\beta \vartheta^{-2} + 2)|\nabla \vartheta|^2 \psi}{\vartheta} \, dx = 0. \quad (46)
\]

Taking the sum over $k$ of the approximate species equations (43) multiplied by $-\frac{g_k \psi}{\vartheta}$ we get
\[
\int_{\Omega} \psi \sum_{k=1}^{n} \left( Y_k \varrho u \cdot \nabla \left( \frac{g_k^k \psi}{\vartheta} \right) \right) \, dx + \int_{\Omega} \sum_{k=1}^{n} \left( \frac{g_k^k}{\vartheta} Y_k \right) \varrho u \cdot \nabla \psi \, dx + \int_{\Omega} \sum_{k=1}^{n} J_k \cdot \nabla \left( \frac{g_k^k \psi}{\vartheta} \right) \, dx \\
- \varepsilon \int_{\Omega} \sum_{k=1}^{n} Y_k \nabla \vartheta \cdot \nabla \left( \frac{g_k \psi}{\vartheta} \right) \, dx - \sqrt{\lambda} \int_{\Omega} \frac{\psi}{\vartheta} \sum_{k=1}^{n} \log Y_k g_k^k \, dx + \varepsilon \int_{\Omega} \sum_{k=1}^{n} (\bar{\vartheta}_k - Y_k \vartheta) \frac{g_k^k \psi}{\vartheta} \, dx \\
= - \int_{\Omega} \sum_{k=1}^{n} \frac{g_k^k \omega_k \psi}{\vartheta} \, dx. \quad (47)
\]

The definition of $g_k^k$ yields
\[
B1 + B2 = - \int_{\Omega} \left( \sum_{k=1}^{n} c_{pk} J_k \right) \cdot \nabla \psi \, dx + \int_{\Omega} \psi \left( \sum_{k=1}^{n} c_{pk} J_k \right) \cdot \nabla \log \vartheta \, dx \\
+ \int_{\Omega} \nabla \psi \cdot \left( \sum_{k=1}^{n} J_k g_k^k \right) \, dx + \int_{\Omega} \psi \sum_{k=1}^{n} J_k \cdot \nabla \left( \frac{g_k^k}{\vartheta} \right) \, dx \\
= - \int_{\Omega} \left( \sum_{k=1}^{n} J_k s_k^k \right) \nabla \psi \, dx + \int_{\Omega} \left( \sum_{k=1}^{n} c_{pk} J_k \right) \cdot \nabla (\log \vartheta) \psi \, dx \\
+ \int_{\Omega} \psi \sum_{k=1}^{n} J_k \cdot \nabla \left( \frac{g_k^k}{\vartheta} \right) \, dx,
\]
and
\[
C = \int_{\Omega} \psi \varrho u \cdot \left( \sum_{k=1}^{n} Y_k \nabla \left( \frac{g_k^k}{\vartheta} \right) \right) \, dx - \int_{\Omega} \psi \text{div} \left( \varrho u \sum_{k=1}^{n} Y_k (c_{pk} - s_k^k) \right) \, dx.
\]
Rewriting the second term by virtue of the approximate continuity equation

\[ C = \int_{\Omega} \psi \mathbf{u} \cdot \left( \sum_{k=1}^{n} Y_k \nabla \left( \frac{g_k}{\vartheta} \right) \right) \, dx - \int_{\Omega} \psi \mathbf{u} \cdot \sum_{k=1}^{n} c_{pk} \nabla Y_k \, dx \\
+ \int_{\Omega} \psi \mathbf{u} \cdot \nabla \left( \sum_{k=1}^{n} Y_k s_k^\lambda \right) \, dx - \int_{\Omega} \psi \text{div} (\mathbf{u}) \sum_{k=1}^{n} Y_k (c_{pk} - s_k^\lambda) \, dx. \]

Finally we have

\[ D = \int_{\Omega} \psi \mathbf{u} \cdot \left( \sum_{k=1}^{n} c_{vk} \nabla Y_k \right) \, dx + \int_{\Omega} \psi \mathbf{u} \cdot \left( \sum_{k=1}^{n} \nabla (c_{vk} \log \vartheta) Y_k \right) \, dx - \int_{\Omega} \psi \mathbf{u} \cdot \nabla (\log \varrho) \, dx. \]

Substituting \( c_{vk} \log \vartheta = s_k^\lambda + \log(\varrho + \sqrt{\lambda}) + \log Y_k \) to the second term yields

\[ C + D = \int_{\Omega} \psi \mathbf{u} \cdot \left( \sum_{k=1}^{n} \nabla s_k^\lambda Y_k \right) \, dx - \int_{\Omega} \psi \mathbf{u} \cdot \left( \sum_{k=1}^{n} \nabla Y_k c_{vk} \right) \, dx + \int_{\Omega} \psi \mathbf{u} \cdot \nabla s^\lambda \, dx \\
+ \int_{\Omega} \psi \mathbf{u} \cdot \left( \sum_{k=1}^{n} \nabla Y_k c_{vk} \right) \, dx + \int_{\Omega} \psi \mathbf{u} \cdot \left( \sum_{k=1}^{n} \nabla s_k^\lambda Y_k \right) \, dx \\
+ \int_{\Omega} \psi \mathbf{u} \cdot \left( \left( \sum_{k=1}^{n} Y_k \right) \nabla \log(\varrho + \sqrt{\lambda}) - \nabla \log \varrho \right) \, dx \\
+ \int_{\Omega} \psi \mathbf{u} \cdot \left( \sum_{k=1}^{n} \nabla Y_k \right) \, dx - \int_{\Omega} \sum_{k=1}^{n} \psi Y_k (c_{pk} - s_k^\lambda) \text{div} (\mathbf{u}) \, dx \\
\]  
\[ = \int_{\Omega} \psi \mathbf{u} \cdot \nabla s^\lambda \, dx + \int_{\Omega} \psi \mathbf{u} \cdot \left( \left( \sum_{k=1}^{n} Y_k \right) \nabla \log(\varrho + \sqrt{\lambda}) - \nabla \log \varrho \right) \, dx \\
- \int_{\Omega} \sum_{k=1}^{n} \psi Y_k c_{pk} \text{div} (\mathbf{u}) \, dx + \int_{\Omega} \psi \sum_{k=1}^{n} (Y_k s_k^\lambda) \text{div} (\mathbf{u}) \, dx. \tag{48} \]

Integrating in the first term by parts we get

\[ \int_{\Omega} \psi \mathbf{u} \cdot \nabla s^\lambda \, dx = - \int_{\Omega} \psi s^\lambda \text{div} (\mathbf{u}) \, dx - \int_{\Omega} s^\lambda \mathbf{u} \cdot \nabla \psi \, dx. \]

The first term cancels with the last term from (48) and applying the approximate continuity equation yields

\[ C + D = - \int_{\Omega} s^\lambda \mathbf{u} \cdot \nabla \psi \, dx + \int_{\Omega} \psi \mathbf{u} \cdot \left[ \left( \sum_{k=1}^{n} Y_k \right) \nabla \log(\varrho + \sqrt{\lambda}) \right. \\
\left. - \nabla \log \varrho \right] \, dx - \varepsilon \int_{\Omega} \psi \sum_{k=1}^{n} Y_k c_{pk} (\Delta \varrho + \bar{\varrho} - \varrho) \, dx. \tag{49} \]
With the above considerations we are ready to formulate the approximate entropy inequality which at this stage can be still written as equality. Namely, adding (46) and (47) we arrive at

\[
\int \frac{\kappa \eta (\varepsilon + \dot{\theta}) \nabla \dot{\theta} \cdot \nabla \psi}{\dot{\theta}^2} \, dx - \int \frac{\kappa \eta (\varepsilon + \dot{\theta}) |\nabla \dot{\theta}|^2}{\dot{\theta}^2} \psi \, dx \\
+ \int_{\partial \Omega} \frac{\psi}{\dot{\theta}} \left[ (L + \delta \dot{\theta}B^{-1})(\dot{\theta} - \dot{\theta}_0^\eta) + \varepsilon \log \dot{\theta} + \lambda \dot{\theta}B/2 \log \dot{\theta} \right] \, dS \\
- \int \frac{\delta \psi}{\dot{\theta}^2} \, dx - \int \frac{\psi \delta \eta \cdot \nabla \mathbf{u}}{\dot{\theta}} \, dx + \int \frac{\varepsilon (\Delta \dot{\theta} + \ddot{\theta} - \ddot{\theta})(\dot{\theta}^{-1} + e + \theta) \psi}{\dot{\theta}} \, dx \\
+ \int \sum_{k=1}^n \frac{g_k \lambda \omega_k \psi}{\dot{\theta}} \, dx - \int \frac{g \lambda \psi \cdot \nabla \psi}{\dot{\theta}} \, dx \\
- \int \left( \sum_{k=1}^n \mathbf{J}_k s_k^\lambda \right) \cdot \nabla \psi \, dx + \int \frac{\psi \nabla (\log \dot{\theta}) \cdot \left( \sum_{k=1}^n c_{pk} \mathbf{J}_k \right)}{\dot{\theta}} \, dx + \int \frac{\psi \sum_{k=1}^n \mathbf{J}_k \cdot \nabla \left( g_k^\lambda \right)}{\dot{\theta}} \, dx
\]

\[
\int J = - \int \left( \sum_{k=1}^n \hat{\mathbf{F}}_k s_k^\lambda \right) \cdot \nabla \psi \, dx + \int \frac{\psi \nabla (\log \dot{\theta}) \cdot \left( \sum_{k=1}^n c_{pk} \hat{\mathbf{F}}_k \right)}{\dot{\theta}} \, dx + \int \frac{\psi \sum_{k=1}^n \hat{\mathbf{F}}_k \cdot \nabla \left( g_k^\lambda \right)}{\dot{\theta}} \, dx \\
- \int \psi \sum_{k=1}^n c_{pk} (\varepsilon (\dot{\theta} + 1) Y_k + \lambda) \frac{\nabla Y_k}{Y_k} \cdot \nabla \log \dot{\theta} \, dx + \int \psi \sum_{k=1}^n (\varepsilon (\dot{\theta} + 1) Y_k + \lambda) \frac{\nabla Y_k}{Y_k} c_{pk} \nabla \log \dot{\theta} \, dx
\]

\[
+ \int \sum_{k=1}^n (\varepsilon (\dot{\theta} + 1) Y_k + \lambda) s_k^\lambda \frac{\nabla Y_k}{Y_k} \cdot \nabla \psi \, dx - \int \psi \sum_{k=1}^n (\varepsilon (\dot{\theta} + 1) Y_k + \lambda) \frac{\nabla Y_k}{Y_k} \cdot \nabla (\log (\dot{\theta} + \sqrt{\lambda})) \, dx \\
- \int \psi \sum_{k=1}^n (\varepsilon (\dot{\theta} + 1) Y_k + \lambda) \left| \frac{\nabla Y_k}{Y_k} \right|^2 \, dx. \tag{51}
\]

Recalling (14) we have

\[
J_1 = - \int \psi \sum_{k=1}^n (\varepsilon (\dot{\theta} + 1) Y_k + \lambda) \frac{\nabla Y_k}{Y_k} \cdot \nabla \log \dot{\theta} \, dx. \tag{52}
\]
The second last term in (50) reads
\[
- \int \sum_{k=1}^{n} (\varepsilon (\varphi + 1) Y_k + \lambda) \frac{\nabla Y_k}{Y_k} \cdot \nabla \psi \, dx + \int \psi \sum_{k=1}^{n} (\varepsilon (\varphi + 1) Y_k + \lambda) \frac{\nabla Y_k}{Y_k} \cdot \nabla \log \vartheta \, dx.
\]

Now, the second term above cancels with \( J_1 \).

For the purpose of the passage to the limit it is better to rewrite the above formulation in the following way, using the fact that \( \sum_{k=1}^{n} \mathcal{F}_k = 0 \) and \( \sum_{k=1}^{n} \omega_k = 0 \)

\[
\int \frac{\psi S_{\eta}}{\vartheta} \nabla \cdot \nabla \psi \, dx + \int \frac{\kappa_{\delta, \eta}}{\vartheta^2} |\nabla \vartheta|^2 \psi \, dx - \int \omega_k (c_{pk} - c_{vk} \log \vartheta + \log Y_k) \psi \, dx
\]
\[
+ \int \frac{\delta \psi}{\vartheta^2} \, dx - \int \psi \sum_{k=1}^{n} \mathcal{F}_k \cdot \nabla \log Y_k \, dx + \int \frac{\psi}{\vartheta} (L + \delta \vartheta^{B-1}) \vartheta_0 \, dS
\]
\[
+ \int \frac{\delta \varepsilon (\beta \vartheta^2 + 2)|\nabla \vartheta|^2 \psi}{\vartheta} \, dx + \int \psi \sum_{k=1}^{n} (\varepsilon (\varphi + 1) Y_k + \lambda) \frac{\nabla Y_k}{Y_k} \cdot \nabla \psi \, dx
\]
\[
= \int \frac{\kappa_{\delta, \eta}}{\vartheta^2} (\varepsilon + \vartheta) |\nabla \vartheta|^2 \psi \, dx - \int \vartheta \sum_{k=1}^{n} (c_{vk} \log \vartheta - \log Y_k) \mathcal{F}_k \cdot \nabla \psi \, dx
\]
\[
+ \int \psi \vartheta \left( \left( \sum_{k=1}^{n} Y_k \nabla \log (\varphi + \sqrt{\lambda}) - \nabla \log \varphi \right) \right) \, dx - \int \psi \sum_{k=1}^{n} Y_k c_{pk} (\Delta \vartheta + \bar{\vartheta} - \varphi) \, dx
\]
\[
+ \int \frac{\psi}{\vartheta} \left( (L + \delta \vartheta^{B-1}) \vartheta + \varepsilon \log \vartheta + \lambda \vartheta^{B/2} \log \vartheta \right) \, dS - \int \sum_{k=1}^{n} Y_k \nabla \vartheta \cdot \nabla \left( \frac{g_k^\lambda \psi}{\vartheta} \right) \, dx
\]
\[
- \sqrt{\lambda} \int \left( \left( \sum_{k=1}^{n} g_k^\lambda \log Y_k \right) \psi \right) \, dx + \int \varepsilon (\Delta \vartheta + \bar{\vartheta} - \varphi)(\vartheta^{-1} + e + \vartheta) \psi \, dx
\]
\[
+ \varepsilon \int \sum_{k=1}^{n} (\tilde{g}_k - Y_k \varphi) \frac{g_k^\lambda \psi}{\vartheta} \, dx - \int \sum_{k=1}^{n} (\varepsilon (\varphi + 1) Y_k + \lambda) \frac{\nabla Y_k}{Y_k} \cdot \nabla \psi \, dx
\]
\[
+ \int \sum_{k=1}^{n} (\varepsilon (\varphi + 1) Y_k + \lambda) s_k^\lambda \frac{\nabla Y_k}{Y_k} \cdot \nabla \psi \, dx - \int \psi \sum_{k=1}^{n} (\varepsilon (\varphi + 1) Y_k + \lambda) \frac{\nabla Y_k}{Y_k} \cdot \nabla \log (\varphi + \sqrt{\lambda}) \, dx.
\]

(53)

Letting formally \( \eta \to 0^+ \), \( \lambda \to 0^+ \), \( \varepsilon \to 0^+ \) and \( \delta \to 0^+ \), we obtain (34) with equality. However, in rigorous limit passages we will have to apply the weak lower semicontinuity of norms leading to inequality instead of the equality.

### 3.2 Existence of solutions for the Galerkin approximation.

The existence of a solution can be proved exactly as in [6, Theorem 5.2]. The proof is based on the following ideas:
the existence is proved by means of a version of the Schauder fixed point theorem for a suitably defined operator

instead of the temperature $\vartheta$ and the mass fractions $Y_k$ we look for their logarithms to ensure their positiveness

the a priori estimates are deduced from the entropy inequality \((53)\) with $\psi \equiv 1$, the “total” energy balance integrated over $\Omega$ (i.e. \((42)\) with $w = u$ and the internal energy balance \((44)\) integrated over $\Omega$), the approximate continuity equation \((41)\) and the Galerkin approximation of the momentum balance \((12)\) with $w = u$.

We can verify the following result

**Theorem 2.** Let $\delta$, $\varepsilon$, $\lambda$ and $\eta$ be positive numbers and $N$ a positive integer. Let $\Omega \in C^2$. Then there exists a solution to system \((41–44)\) such that 

\begin{align*}
\rho &\in W^{2,q}(\Omega) \quad \forall \, q < \infty, \quad \rho \geq 0 \quad \text{in} \quad \Omega, \\
\varrho &\in X_N, \quad \tilde{Y} \in W^{1,2}(\Omega) \quad \forall q < \infty, \quad \tilde{Y} > 0 \quad \text{a.e. in} \quad \Omega \quad \text{and} \quad \vartheta \in W^{2,q}(\Omega) \quad \forall q < \infty, \quad \vartheta \geq C(N) > 0. 
\end{align*}

Moreover, this solution satisfies the entropy equation \((53)\) and the following estimate

\begin{align}
\sqrt{\lambda} \sum_{k=1}^{n} \left( \|Y_k\|_{1,2} + \left\| \frac{\nabla Y_k}{Y_k} \right\|_2 + \lambda^{-1/4} \| \log Y_k \|_2 \right) + \sum_{k=1}^{n} \left( \left\| \frac{\nabla Y_k}{Y_k} \right\|_1 + \left\| \nabla \vartheta^{B/2} \right\|_2 + \left\| \frac{\nabla \vartheta}{\vartheta} \right\|_2 \right) \\
+ \left\| \frac{\nabla \varrho}{\sqrt{\varrho + \sqrt{\lambda}}} \right\|_2 + \| \vartheta^{-2} \|_1 + \| \vartheta \|_{B,\partial \Omega} + \left\| \frac{\log \vartheta}{\vartheta} \right\|_{1,\partial \Omega} + \left\| \nabla^2 \varrho \right\|_2 + \| u \|_{1,2} + \| \nabla \varrho \|_6 \leq C,
\end{align}

where $C$ is independent of $N$.

Note that the bound on $\log Y_k$ in $L^2$ appears in \((54)\) due to the presence of the term $-\sqrt{\lambda}(\int_{\Omega} \sum_{k=1}^{n} g_k^\lambda \log Y_k) \frac{1}{\varrho} \, dx$ on the right-hand side of \((53)\). The term $\frac{\left\| \nabla \vartheta \right\|^2}{\varrho + \sqrt{\lambda}}$ appears due to the 7th term on the right-hand side of \((53)\).

**Remark 2.** In the entropy inequality particular attention should be paid to terms containing logarithms, since at the level of approximation we should avoid infinities in the entropy formulation. We overcome this difficulty constructing the approximate temperature and $Y_k$ as exponential functions and possible singularities in $\log \varrho$ are avoided due to definition of $s_k^\lambda$ \((41)\). Thus we know that all the quantities in the approximate entropy equation \((53)\) are finite. However, we must control that these terms remain finite throughout all passages below.

### 4 Limit passages I

In this section we will study the limit passages $N \to \infty$, $\eta \to 0^+$, $\lambda \to 0^+$ and $\varepsilon \to 0^+$. Most of the arguments will be similar to \([6]\) and the references therein, therefore we will mostly skip them and we will concentrate mostly on the new aspect, i.e. the entropy (in)equality which must hold (possibly modified) after each limit passage.
4.1 Limit passages $N \to \infty$ and $\eta \to 0$

We start with $N \to \infty$. At this stage the estimates copy exactly [6], hence we may follow the arguments there. Note that, except the quadratic term in $\nabla u_N$ on the right-hand side (rhs) of the internal energy balance (44), the limit passages are easy to perform. To get also the convergence of this term we use the fact that due to the $\eta$-approximation of the stress tensor we may use as test function $u$ in the limit version of the momentum equation (42) and get

$$\lim_{N \to \infty} \int_{\Omega} S_{\eta}(\vartheta_N, \nabla u_N) : \nabla u_N \, dx = \int_{\Omega} S_{\eta}(\vartheta, \nabla u) : \nabla u \, dx$$

due to the energy equality. This equality even implies that $\nabla u_N \to \nabla u$ strongly in $L^2(\Omega)$, however, we do not use this information here.

Next we deal with the entropy inequality. In the first two terms in (53) we use the weak lower semicontinuity of $L^2$ norm with respect to weak convergence in $L^2$ (see [14] for details). We have to restrict ourselves to non-negative test functions $\psi$ and get

$$\lim_{N \to \infty} \int_{\Omega} S^N_{\eta} \frac{\nabla u_N}{\vartheta_N} \psi \, dx \geq \int_{\Omega} S_{\eta} \frac{\nabla u}{\vartheta} \psi \, dx \quad (55)$$

and

$$\lim_{N \to \infty} \int_{\Omega} \kappa \delta_{\eta} (\epsilon + \vartheta_N) |\nabla \vartheta_N|^2 \psi \, dx \geq \int_{\Omega} \kappa \delta_{\eta} (\epsilon + \vartheta) |\nabla \vartheta|^2 \psi \, dx. \quad (56)$$

In the other terms we can pass to the limit due to estimates (54), however, we comment some of the limits in more details. Notice that in the 8th term on the rhs the part with log $\rho$ does not cause any troubles due to the control of $\log Y_k$ in $L^6$. However, in the subsequent limit passages, we will have to use another argument here. Similarly we may treat all other terms containing $\log Y_k$. The terms containing log $\rho$ are either multiplied by $\rho$, or they are in fact in the form $\log(\rho + \sqrt{\lambda})$ and cause no troubles at this moment. Therefore the entropy inequality (we loose equality here) of the form (53) holds true. Note only that the test functions $\psi$ must be non-negative and we have inequality ($\leq$) instead of the equality sign in (53).

The next step is the passage $\eta \to 0^+$. Since we have no information to ensure the strong convergence of the quadratic term on the rhs of the internal energy balance (44), we have to replace it by the total energy inequality. To this aim, we sum (44) with the kinetic energy balance, i.e. (42) with the test function $w = u \psi$ (this was not possible on
the level of Galerkin approximation), and we obtain

\[- \int_{\Omega} \left[ \rho e + \frac{1}{2} \rho |u|^2 + (\pi + \delta \varrho^\beta + \delta \varrho^2) \right] u \cdot \nabla \psi \, dx \]

\[= \int_{\Omega} \left( S_\eta u \cdot \nabla \psi + \delta \varrho^{-1} \psi \right) \, dx + \int_{\Omega} \kappa_{\delta, \eta} \frac{\varepsilon + \varrho}{\eta} \nabla \vartheta \cdot \nabla \psi \, dx \]

\[+ \int_{\partial \Omega} \left[ (L + \delta \vartheta B^{-1})(\vartheta - \varrho_{0}^\eta) + \varepsilon \log \vartheta + \lambda \varrho \frac{\varrho}{\vartheta} \log \vartheta \right] \psi \, dS \]

\[\sum_{k=1}^{n} c_{\varrho k} \int_{\Omega} \left[ \delta \sum_{l=1}^{n} Y_k \Delta D_{kl} \nabla Y_l \cdot \nabla \psi + \vartheta(\varrho + 1) Y_k + \lambda \frac{\nabla Y_k}{Y_k} \cdot \nabla \psi \right] \, dx \]

\[= \int_{\Omega} \rho f \cdot u \psi \, dx + \frac{\delta}{\beta - 1} \int_{\Omega} (\varepsilon \varrho \varrho^{-1} \psi + \varrho^\beta u \cdot \nabla \psi - \varepsilon \varrho^2 \psi) \, dx \]

\[+ \delta \int_{\Omega} (2\varepsilon \varrho \varrho \psi + \varrho^2 u \cdot \nabla \psi - 2\varepsilon \varrho^2 \psi) \, dx \quad (57)\]

for all \(\psi \in C^\infty(\Omega)\). Now it is easy to pass to the limit in (57), similarly as in [6]. The limit passage in the other equalities (continuity equation, momentum equation and the species balance) is easy to perform.

On the level of entropy inequality this limit passage does not entail any additional difficulties with respect to the previous limit passage, since we have all the previous estimates. Therefore we pass to the limit directly and get inequality of the type (53), where we have inequality instead of equality and we remove all indices \(\eta\).

### 4.2 Limit passage \(\lambda \to 0\)

Here we still dispose of estimates (54). Note, however, that the estimate of \(u\) in \(W^{1,2}\) uniformly in \(\lambda\) does not follow from the kinetic energy balance (which is not anymore available) but from the entropy inequality. Furthermore, we loose the uniform control of \(\log Y_k\) and \(Y_k\) in \(W^{1,2}\). Nonetheless, see [6, Formula (6.12)], we can verify that

\[\| \sum_{k=1}^{n} \nabla Y_k \|_{2} + \| \sum_{k=1}^{n} Y_k - 1 \|_{6} \leq C(\lambda) \sim \sqrt{\lambda} \to 0 \quad \text{for} \quad \lambda \to 0. \quad (58)\]

This bound, together with (54), implies

\[\sum_{k=1}^{n} \| \nabla Y_k \|_{\frac{12}{7}} \leq C \quad (59)\]

with \(C\) independent of \(\lambda\). The above estimates combined with (54) allow to pass to the limit in the continuity, momentum, species and total energy balances. We have

- the approximate continuity equation

\[\varepsilon \varrho + \text{div} (\varrho u) = \varepsilon \Delta \varrho + \varepsilon \varrho, \quad \nabla \varrho \cdot n|_{\partial \Omega} = 0 \quad (60)\]
• the weak formulation of the approximate momentum equation

\[
\int_{\Omega} \left( \frac{1}{2} \rho \mathbf{u} \cdot \nabla \mathbf{u} \cdot \varphi - \frac{1}{2} \frac{\partial}{\partial \varphi} (\mathbf{u} \otimes \mathbf{u}) : \nabla \varphi - \nabla \varphi - S : \nabla \varphi \right) \, dx
- \int_{\Omega} (\pi + \delta \varphi + \delta \varphi^2) \text{div} \varphi \, dx = \int_{\Omega} \rho \mathbf{f} \cdot \varphi \, dx
\]

for all \( \varphi \in C^\infty_0(\Omega) \)

• the weak formulation of the approximate species balance equations

\[
\int_{\Omega} \left( \varepsilon Y_k \varphi - Y_k \rho \mathbf{u} \cdot \nabla \varphi + \sum_{l=1}^n Y_k \hat{D}_{kl} \nabla Y_l \cdot \nabla \varphi \right) \, dx
= \int_{\Omega} \left[ \omega_k \varphi - \varepsilon \varphi \nabla Y_k \cdot \nabla \varphi + \varepsilon \varphi (Y_k \nabla \rho) \varphi - \varepsilon \nabla Y_k \cdot \nabla \varphi + \varepsilon \Omega_k \varphi \right] \, dx,
\]

for all \( \varphi \in C^\infty(\Omega) \) \( (k = 1, 2, \ldots, n) \)

• the weak formulation of the approximate total energy equation

\[
- \int_{\Omega} \left[ \rho \mathbf{e} + \frac{1}{2} \rho |\mathbf{u}|^2 + (\pi + \delta \varphi + \delta \varphi^2) \right] \mathbf{u} \cdot \nabla \varphi \, dx
- \int_{\Omega} \left( \mathbf{S} \mathbf{u} \cdot \nabla \varphi + \delta \varphi^{-1} \varphi \right) \, dx + \int_{\Omega} \kappa \delta \frac{\varphi}{\varphi} \nabla \varphi \cdot \nabla \varphi \, dx
+ \int_{\partial \Omega} \left[ (L + \delta \varphi^2 B^{-1}) (\varphi - \varphi_0) + \varepsilon \log \varphi \right] \psi \, dS
+ \int_{\Omega} \left[ \varrho \sum_{k,l=1}^n c_{vk} Y_k \hat{D}_{kl} \nabla Y_l \cdot \nabla \varphi + \varrho \sum_{k=1}^n \varepsilon (\varrho + 1) c_{vk} \nabla Y_k \cdot \nabla \varphi \right] \, dx
= \int_{\Omega} \rho \mathbf{f} \cdot \mathbf{u} \varphi \, dx + \frac{\delta}{\beta - 1} \int_{\Omega} \left( \varepsilon \delta \varphi \varphi^{-1} \varphi \mathbf{u} \cdot \nabla \varphi - \varepsilon \beta \varphi^2 \varphi \right) \, dx
+ \delta \int_{\Omega} \left( 2 \varepsilon \delta \varphi \varphi + \varepsilon \varphi^2 \varphi \right) \, dx
\]

for all \( \varphi \in C^\infty(\Omega) \)

Next we consider the limit passage in the entropy inequality. The terms on the left-hand side (lhs) can be treated as in the previous limit passage. We only have to pay attention to the terms containing \( \log \rho \) and \( \log Y_k \). In the former, we use the approximation \( s_k^\lambda \). Namely, we have

\[
\int_{\Omega} \varphi \mathbf{u} \left[ \left( \sum_{k=1}^n Y_k \right) \nabla \log (\varrho + \sqrt{\lambda}) - \nabla \log \varrho \right] \, dx = \int_{\Omega} \mathbf{u} \cdot \nabla \varrho \left[ \left( \sum_{k=1}^n Y_k \right) \frac{\varrho}{\varrho + \sqrt{\lambda}} - 1 \right] \, dx \to 0.
\]

The next term we should look at is the last term on the rhs. After passage with \( \lambda \) the part with \( \varepsilon \) will vanish due to (58) and (44). Thus it is enough to treat the second term which reads

\[
\int_{\Omega} \varphi \frac{\lambda}{\varrho + \sqrt{\lambda}} \nabla \varrho \cdot \sum_{k=1}^n \frac{\nabla Y_k}{Y_k}
\]
and tends to 0 as $\frac{\lambda}{e^{\sqrt{\lambda}}}$ → 0 and the rest is bounded in $L^1$. To show convergence of the second last term on the rhs, we use the bound of $\frac{\nabla Y_k}{Y_k}$ and $\log Y_k$ in $L^2$ from (54) (and, indeed, also other bounds coming from there). Note that it is exactly here, where we need the $\sqrt{\lambda}$ instead of $\lambda$ in (43) to ensure that the $\lambda$ part of this term converges to zero as $\lambda \to 0^+$. The part with $\varepsilon$ and $\log Y_k$ is also complicated, as we miss any estimate of $\log Y_k$ which does not blow up when $\lambda \to 0^+$. To this reason, we write

$$
\varepsilon \int \sum_{k=1}^{n} (\varrho + 1) \log Y_k \nabla Y_k \cdot \nabla \psi \, dx = -\varepsilon \int \sum_{k=1}^{n} (\varrho + 1) \nabla Y_k \cdot \nabla \psi \, dx 
$$

$$
- \varepsilon \int \sum_{k=1}^{n} Y_k \log Y_k \nabla \varrho \cdot \nabla \psi \, dx - \varepsilon \int \sum_{k=1}^{n} (\varrho + 1) Y_k \log Y_k \Delta \psi \, dx 
$$

$$
+ \varepsilon \int \sum_{k=1}^{n} (\varrho + 1) Y_k \log Y_k \nabla \psi \cdot \mathbf{n} \, dS. \tag{64}
$$

Now it is easy to let $\lambda \to 0^+$ in all terms in (64). The remaining terms coming from $s^\lambda_k$ cause no troubles. The term with $\log(\varrho + \sqrt{\lambda})$ tends to zero as $\sum_{k=1}^{n} \nabla Y_k$ goes to zero faster than $\log \lambda$ blows up; the other term with $\log \vartheta$ is well defined.

Finally, the form of internal energy and (58) imply that after passing with $\lambda$ we have

$$
\int \varepsilon (\Delta \varrho + \bar{\varrho} - \varrho)(\varrho^{\gamma-1} + e + \vartheta) \psi \, dx 
$$

$$
= \varepsilon \frac{\gamma}{\gamma - 1} \int \psi \varrho^{\gamma-1} (\Delta \varrho + \bar{\varrho} - \varrho) \, dx + \varepsilon \int \psi (\Delta \varrho + \bar{\varrho} - \varrho) \sum_{k=1}^{n} c_{pk} Y_k \, dx. 
$$

The second term cancels with the 5th term on the rhs of (53). We can therefore pass to
the limit with \( \lambda \) obtaining

\[
\int \frac{\psi \mathbf{S} \cdot \nabla \mathbf{u}}{\psi} \, dx + \int \kappa \frac{(\varepsilon + \vartheta) \left| \nabla \vartheta \right|^2}{\mathbf{u}} \, dx - \int \psi \sum_{k=1}^{n} \omega_k (c_{pk} - c_{vk} \log \vartheta + \log Y_k) \psi \, dx
\]

\[
+ \int \frac{\delta \psi}{\vartheta^2} \, dx + \int \psi \sum_{k=1}^{n} \sum_{l=1}^{n} \hat{D}_{kl} \nabla Y_l \nabla Y_k \, dx + \int \frac{\psi}{\vartheta} (L + \delta \vartheta^{B-1}) \partial_0 \, dS
\]

\[
+ \int \frac{\delta \varepsilon (\beta \vartheta^{3-2} + 2) \left| \nabla \vartheta \right|^2 \psi}{\vartheta} \, dx + \int \psi \sum_{k=1}^{n} \varepsilon (\varrho + 1) \left| \nabla Y_k \right|^2 \, dx
\]

\[
+ \varepsilon \gamma \int \frac{\psi}{\vartheta} \left| \nabla \vartheta \right|^2 \, dx + \varepsilon - 1 \int \frac{\psi}{\vartheta} \vartheta \, dx
\]

\[
\leq \int \kappa \frac{(\varepsilon + \vartheta) \left| \nabla \vartheta \right|^2 \partial \psi}{\mathbf{u}} \, dx - \int \psi \mathbf{u} \cdot \nabla \psi \, dx - \int \sum_{k=1}^{n} (c_{vk} \log \vartheta - \log Y_k) \hat{\mathbf{F}}_k \cdot \nabla \psi \, dx
\]

\[
+ \int \frac{\psi}{\vartheta} ((L + \delta \vartheta^{B-1}) \vartheta + \varepsilon \log \vartheta) \, dS - \varepsilon \int \sum_{k=1}^{n} Y_k \nabla \vartheta \cdot \nabla \left( \frac{g_k \psi}{\vartheta} \right) \, dx
\]

\[
- \varepsilon \gamma \int \vartheta^{-1} \nabla \vartheta \cdot \partial \vartheta \, dx + \varepsilon \gamma \int \frac{\psi}{\vartheta} \vartheta^{-1} \nabla \vartheta \cdot \partial \vartheta \, dx + \varepsilon \gamma - 1 \int \frac{\psi}{\vartheta} \vartheta \vartheta^{-1} \, dx
\]

\[
+ \varepsilon \frac{M}{|\Omega|} \int \psi \sum_{k=1}^{n} (c_{pk} - c_{vk} \log \vartheta + 1_{\vartheta > 1} \log \vartheta) \, dx
\]

\[
- \varepsilon \int \sum_{k=1}^{n} Y_k \vartheta \frac{g_k \psi}{\vartheta} \, dx + \varepsilon \int \sum_{k=1}^{n} (\vartheta + 1) \nabla Y_k \cdot \nabla \psi \, dx
\]

\[
+ \varepsilon \int \sum_{k=1}^{n} Y_k \log Y_k \nabla \vartheta \cdot \nabla \psi \, dx + \varepsilon \int \sum_{k=1}^{n} (\vartheta + 1) Y_k \log Y_k \Delta \psi \, dx
\]

\[
- \varepsilon \int \sum_{k=1}^{n} (\vartheta + 1) Y_k \log Y_k \nabla \psi \cdot \mathbf{n} \, dS + \varepsilon \int \sum_{k=1}^{n} (\vartheta + 1) c_{vk} \log \vartheta \nabla Y_k \cdot \nabla \psi \, dx, \quad (65)
\]

where we have integrated by parts the term \( \int \frac{\psi}{\vartheta} \vartheta^{-1} \Delta \vartheta \, dx \), used the fact that \( \log Y_k \leq 0 \) for \( \lambda = 0 \), \( \log \vartheta < 0 \) for \( \vartheta < 1 \) and \( \hat{D}_{kl} \) is defined in (37).

### 4.3 Limit passage \( \varepsilon \to 0 \)

First of all, we have the following estimates independent of \( \varepsilon \):

\[
\sqrt{\varepsilon} \left( \left\| \nabla Y_k \right\|_2 + \left\| \vartheta \right\|_{\vartheta^{1/2}} + \left\| \vartheta \right\|_2 \right) + \sum_{k=1}^{n} \left( \left\| Y_k \right\|_{1,2} + \left\| Y_k \right\|_{\infty} + \left\| \vartheta \right\|_{\vartheta^{1/2}} + \left\| \vartheta \right\|_{B,\partial \Omega}
\]

\[
+ \left\| \vartheta \right\|_{3,\Omega} + \left\| \vartheta^{-1} \right\|_1 + \left\| \nabla \vartheta \right\|_{\vartheta^{1/2}} + \left\| \vartheta^{-1} \right\|_{1,\partial \Omega} + \left\| \mathbf{u} \right\|_{1,2} \right) \leq C \left( 1 + \int \vartheta \mathbf{u} \, dx \right). \quad (66)
\]
These estimates follow from the entropy inequality and the total energy balance, both with the test function $\psi \equiv 1$, and the continuity equation. At this stage we cannot dispose of the estimates on the density (except the $L^1$ bound due to given mass) since they depend on $\varepsilon$. We have to show some estimates of the density which will imply that the rhs of (66) can be controlled.

Note that the momentum equation is in fact the same as in the case of the compressible Navier–Stokes–Fourier system studied in [14], so we may apply the same technique to obtain the so called Bogovskii-type of estimates. Following [14], we use as test function in (31) the function $\phi$, solution to
\[
\text{div } \phi = \rho^{\frac{2}{3}} - \frac{1}{|\Omega|} \int_{\Omega} \rho^{\frac{2}{3}} \, d x, \quad \phi|_{\partial \Omega} = 0.
\]
For more information on the Bogovskii operator, we refer the reader to e.g. [16, Lemma 3.17]. In consequence of this testing we may obtain the additional bound on $\rho$, namely
\[
\|\rho\|^{\frac{2}{3}} \leq C,
\]
which allows to estimate the rhs of (66). Now we can proceed with the limit passage. Note that the estimates of the density do not imply the compactness of it, however, using the DiPerna–Lions renormalization technique applied on the continuity equation and the consequences of the effective viscous flux identity, as it is well-known in the case of compressible Navier–Stokes(–Fourier) system, we may show the strong convergence of the densities in $L^p$ for any $p < \beta$. As we have to repeat this procedure also in the final limit passage we present the crucial steps there, referring for more details to [16] or to [14] in the case of heat–conducting fluid. Therefore we have after the limit passage $\varepsilon \to 0^+$

- the continuity equation
  \[
  \int_{\Omega} \rho u \cdot \nabla \psi = 0
  \]
  for all $\psi \in C^\infty(\overline{\Omega})$

- the weak formulation of the approximate momentum equation
  \[
  \int_{\Omega} \left( - \rho (u \otimes u) : \nabla \varphi - \mathbb{S} : \nabla \varphi \right) \, dx \\
  - \int_{\Omega} (\pi + \delta \rho^3 + \delta \rho^2) \text{div } \varphi \, dx = \int_{\Omega} \rho f \cdot \varphi \, dx
  \]
  for all $\varphi \in C^\infty_0(\Omega)$

- the weak formulation of the approximate species balance equations
  \[
  \int_{\Omega} \left( - Y_k \rho \mathbf{u} \cdot \nabla \psi + \sum_{l=1}^n Y_k D_{kl} \nabla Y_l \cdot \nabla \psi \right) \, dx = \int_{\Omega} \omega_k \psi \, dx
  \]
for all $\psi \in C^\infty(\Omega) \ (k = 1, 2, \ldots, n)$

- the weak formulation of the approximate total energy equation

\[
\begin{align*}
& - \int_{\Omega} \left[ \rho \varepsilon + \frac{1}{2} \rho |u|^2 + (\pi + \delta \vartheta^2 + \delta q^2) \right] u \cdot \nabla \psi \, dx \\
& - \int_{\Omega} \left( S u \cdot \nabla \psi + \delta \vartheta^{-1} \psi \right) \, dx + \int_{\Omega} \kappa \delta \vartheta \cdot \nabla \psi \, dx \\
& + \int_{\partial \Omega} \left[ (L + \delta \vartheta^{B-1}) (\vartheta - \vartheta_0) \right] \psi \, dS + \int_{\Omega} \vartheta \sum_{k,l=1}^{n} c_{vk} Y_k D_{kl} \nabla Y_l \cdot \nabla \psi \, dx \\
& = \int_{\Omega} \varepsilon \psi \cdot u \, dx + \frac{\delta}{\beta - 1} \int_{\Omega} \vartheta^2 u \cdot \nabla \psi \, dx + \delta \int_{\Omega} \vartheta^2 u \cdot \nabla \psi \, dx
\end{align*}
\]

for all $\psi \in C^\infty(\Omega)$

Next we deal with the limit passage in the entropy inequality. The lhs does not cause any troubles: we use the weak lower semicontinuity of certain terms or simply cancel some non-negative terms. Most of the terms are easy to treat, the only difficult one is in fact the term

\[
\varepsilon \gamma \int_{\Omega} \frac{\vartheta^{\gamma-1}}{\vartheta^2} \nabla \vartheta \cdot \nabla \vartheta \psi \, dx
\]

which must be controlled by the lhs (in fact, already at the moment when we want to deduce the $\varepsilon$-independent estimates). However, using the fact that $\varepsilon \ll \delta$ and $\beta$ is sufficiently high we may estimate it by

\[
\frac{1}{4} \int_{\Omega} \frac{\delta \varepsilon (\beta \vartheta^{\beta-2} + 2) |\nabla \vartheta|^2 \psi}{\vartheta} \, dx + \frac{1}{4} \int_{\Omega} \frac{\kappa \delta (\varepsilon + \vartheta) |\nabla \vartheta|^2 \psi}{\vartheta^2} \, dx,
\]

in particular by the part $\delta \vartheta^{-1}$ in $\kappa \delta$. The other terms are easy to treat and we end up with

\[
\begin{align*}
\int_{\Omega} \frac{\psi \nabla u}{\vartheta} \, dx + \int_{\Omega} \kappa \delta \frac{|\nabla \vartheta|^2 \psi}{\vartheta^2} \, dx & - \int_{\Omega} \sum_{k=1}^{n} \omega_k (c_{pk} - c_{vk} \log \vartheta + \log Y_k) \psi \, dx \\
& + \int_{\Omega} \frac{\delta \psi}{\vartheta^2} \, dx + \int_{\Omega} \psi \sum_{k=1}^{n} \sum_{l=1}^{n} D_{kl} \nabla Y_l \nabla Y_k \, dx + \int_{\partial \Omega} \frac{\psi}{\vartheta} (L + \delta \vartheta^{B-1}) \vartheta_0 \, dS \\
& \leq \int_{\Omega} \frac{\kappa \delta \nabla \vartheta \cdot \nabla \psi}{\vartheta} \, dx - \int_{\Omega} gs u \cdot \nabla \psi \, dx - \int_{\Omega} \sum_{k=1}^{n} (c_{vk} \log \vartheta - \log Y_k) F_k \cdot \nabla \psi \, dx \\
& \quad + \int_{\partial \Omega} (L + \delta \vartheta^{B-1}) \psi \, dS. \tag{71}
\end{align*}
\]

5 Limit passage $\delta \to 0$

In the final limit passage we can distinguish three steps. The first is in fact a direct application of the method from [13], where we refer for details. In the second step we
derive new pressure estimates using the approach from [15]. In fact, we clarify here one estimate in more details, cf. [13]. We can therefore pass to the limit in the equations and the entropy inequality, however, we are not able to identify the weak limits in the terms which are non-linear in the density. To this aim, we finally show the strong convergence of the density using the techniques developed for compressible Navier–Stokes system (which is possible as the momentum and continuity equations are indeed the same).

5.1 Estimates independent of \( \delta \)

Unlike the previous sections, we will denote throughout this section by \((\rho_\delta, \mathbf{u}_\delta, \vartheta_\delta, \mathbf{Y}_\delta)\) the solution corresponding to \(\delta > 0\), while \((\rho, \mathbf{u}, \vartheta, \mathbf{Y})\) will denote the (weak or strong) limits of the corresponding functions when \(\delta \to 0^+\). Furthermore, \(\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)^T\), similarly for \(\mathbf{Y}_\delta\).

5.1.1 Estimates from the entropy inequality

From the total energy balance (70) tested by a constant function we derive

\[
\|\vartheta_\delta\|_{1,\partial \Omega} + \delta\|\vartheta^B_\delta\|_{1,\partial \Omega} \leq C\left(1 + \left|\int_{\Omega} \rho_\delta \mathbf{u}_\delta \cdot \mathbf{f} \, dx\right| + \delta\|\vartheta^{-1}_\delta\|_1\right).
\]

Next, the entropy inequality (71) with \(\psi \equiv C\) yields

\[
\|\nabla \mathbf{Y}_\delta\|_2^2 + \|\nabla \vartheta^m_\delta\|_2^2 + \|\mathbf{u}_\delta\|_2^2 + \|\vartheta^{-1}_\delta\|_{1,\partial \Omega}
+ \delta \left(\|\nabla \vartheta^B_\delta\|_2^2 + \|\nabla \vartheta^{-1}_\delta\|_2^2 + \|\vartheta^{-2}_\delta\|_1 + \|\vartheta^{B-2}_\delta\|_{1,\partial \Omega}\right) \leq C(1 + \delta\|\vartheta^{B-1}_\delta\|_{1,\partial \Omega}).
\]

Recall also that we know \(0 \leq (Y_k)_\delta \leq 1, \ k = 1, 2, \ldots, n\). In order to get rid of the \(\delta\)-dependent terms in the above estimates we apply once again Bogovskii-type estimates, this time testing the momentum equation by a solution to

\[
\text{div} \mathbf{\phi} = \rho_\delta - \frac{M}{|\Omega|}, \quad \mathbf{\phi}|_{\partial \Omega} = 0.
\]

It is an easy matter to verify the bound (see also [14])

\[
\delta\|\rho_\delta\|_{\beta+1}^{\beta-4} \leq C.
\]

Applying this estimate to (73) and (72) we can get rid of most of \(\delta\)-terms obtaining

\[
\|\nabla \mathbf{Y}_\delta\|_2^2 + \|\mathbf{Y}_\delta\|_\infty + \|\nabla \vartheta^m_\delta\|_2 + \|\mathbf{u}_\delta\|_{1,\partial \Omega} + \|\vartheta^{-1}_\delta\|_{1,\partial \Omega}
+ \delta\left(\|\nabla \vartheta^B_\delta\|_2^2 + \|\nabla \vartheta^{-1}_\delta\|_2^2 + \|\vartheta^{-2}_\delta\|_1 + \|\vartheta^{B-2}_\delta\|_{1,\partial \Omega}\right) \leq C
\]

and

\[
\|\vartheta_\delta\|_{3m} \leq C\left(1 + \left|\int_{\Omega} \rho_\delta \mathbf{u}_\delta \cdot \mathbf{f} \, dx\right|\right).
\]

See also [10] for similar computations in the case of a more complex dependence of the viscosity on the temperature.
5.1.2 Local pressure estimates

The second step consist in derivation of δ-independent estimates for the density. This is the core estimate which finally will allow us to get a bound $\gamma > \frac{4}{3}$ for weak solutions and $\gamma > 1$ for variational entropy solutions. Here we follow the idea of local pressure estimates introduced in several papers by Plotnikov and Sokolowski (see [17]), Novotný and Březina ([1]) and Frehse, Steinhauer and Weigant ([4]) and applied to the compressible Navier–Stokes–Fourier system in [15]; see also [13] for further information.

For $b > 1$ let us denote

$$A = \int_{\Omega} \rho_0^b |\mathbf{u}_0|^2 \, dx.$$  

Applying Hölder’s inequality to the rhs of (75) we get

$$\|\vartheta_\delta\|_{3m} \leq C (1 + A^{\frac{1}{3m}}).$$  

(76)

Next we apply once again Bogovskii-type estimate to show

Lemma 3. We have for $1 < s \leq \frac{2b}{b+2}$, $s \leq \frac{6m}{2+3m}$, $m > \frac{2}{3}$ and $b \geq 1$

$$\int_{\Omega} \rho_0^s \, dx + \int_{\Omega} \rho_0^{(s-1)\gamma} \pi(\rho_\delta, \delta) \, dx + \int_{\Omega} (\rho_\delta |\mathbf{u}_\delta|^2)^s \, dx + \delta \int_{\Omega} \rho_\delta^{\beta+(s-1)\gamma} \, dx \leq C (1 + A^{\frac{4s-3}{3m}}).$$  

(77)

Proof. We sketch the main steps referring to [15] for more details. Testing the momentum equation with $\phi$ solving

$$\text{div} \, \phi = \rho_\delta^{(s-1)\gamma} - \frac{1}{|\Omega|} \int_{\Omega} \rho_\delta^{(s-1)\gamma} \, dx, \quad \phi|_{\partial\Omega} = 0$$

we obtain

$$\int_{\Omega} \rho_\delta^{(s-1)\gamma} \pi(\rho_\delta, \delta) \, dx + \delta \int_{\Omega} \rho_\delta^{(s-1)\gamma} (\rho_\delta^\beta + \rho_\delta^2) \, dx = \frac{1}{|\Omega|} \int_{\Omega} \pi(\rho_\delta, \delta) \, dx \int_{\Omega} \rho_\delta^{(s-1)\gamma} \, dx$$

$$+ \frac{\delta}{|\Omega|} \int_{\Omega} (\rho_\delta^\beta + \rho_\delta^2) \, dx \int_{\Omega} \rho_\delta^{(s-1)\gamma} \, dx - \int_{\Omega} \rho_\delta (\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla \phi \, dx + \int_{\Omega} \mathbf{S}(\delta_\delta, \nabla \mathbf{u}_\delta) : \nabla \phi \, dx$$

$$- \int_{\Omega} \rho \mathbf{f} : \phi \, dx = I_1 + I_2 + I_3 + I_4 + I_5.$$  

(78)

We have to estimate the rhs. The most restrictive terms are $I_3$, giving the restriction on $s$, and $I_4$ which leads to the other restrictions, especially to $m > \frac{2}{3}$. For more details see [15].

Now we come to the core of our estimates. The idea is to test the momentum equation by a cleverly chosen function involving the distance from the boundary to find a bound

$$\sup_{x_0 \in \Omega} \int_{\Omega} \frac{\pi(\rho_\delta, \delta)}{|x - x_0|^\alpha} \, dx \leq C$$
for some $\alpha > 0$ with $C$ independent from $\delta$. We have to use different test functions distinguishing 3 cases: $x_0$ far from the boundary, $x_0$ at the boundary and finally $x_0$ close to the boundary. The first two cases are treated in details in [15], therefore we only recall the results here. The third case is most delicate and has not been presented so far and some ideas can be only found in [13].

The case of $x_0$ far from the boundary is the easiest. We test the momentum equation (68) with

$$\varphi^0(x) = \frac{x - x_0}{|x - x_0|^\alpha} \tau^2$$

(79)

with $\tau \equiv 1$ in $B_{R_0}(x_0)$, $\tau \equiv 0$ outside $B_{2R_0}(x_0)$ with $R_0$ as below, $|\nabla \tau| \leq \frac{C}{R_0}$. Calculating directly the derivatives of $\varphi$ we obtain (see [15, Lemma 3.5] or [13]):

**Lemma 4.** Let $x_0 \in \Omega$, $R_0 < \frac{1}{3} \text{dist}(x_0, \partial \Omega)$. Then

$$\int_{B_{R_0}(x_0)} \frac{\pi(\varphi^0, \varphi^0) + \delta(\varphi^0 + \varphi^0)}{|x - x_0|^\alpha} dx$$

(80)

$$\leq C \left( 1 + \|\pi(\varphi^0, \varphi^0)\|_1 + \|\mathbf{u}_0\|_{1,2}(1 + \|\partial_x\|_{3m}) + \|\partial_\delta\mathbf{u}_0\|_2^2 \right),$$

provided

$$\alpha < \min \left\{ \frac{3m - 2}{2m}, 1 \right\}.$$ (81)

Next we treat the case $x_0 \in \partial \Omega$. This time we use in (68) a test function

$$\varphi^1(x) = d(x)\nabla d(x)(d(x) + |x - x_0|^\alpha)^{-\alpha}$$

(82)

where $a = \frac{2}{1 - \alpha}$ and $d(x)$ is a function which behaves like $\text{dist}(x, \partial \Omega)$ near the boundary and it is a $C^2(\Omega)$ function. It can be shown (see [15, Lemma 3.5] or [13]) that $\varphi^1 \in W_0^{1,q}(\Omega)$ for $1 \leq q < \frac{3 - \alpha}{\alpha}$ and

$$\partial_j \varphi^1_i(x) = \frac{d(x)\partial_j^2 d(x)}{(d(x) + |x - x_0|^\alpha)^\alpha} + \frac{(1 - \alpha)d(x) + |x - x_0|^\alpha}{2(d(x) + |x - x_0|^\alpha)^{1+\alpha}} \partial_j d(x) \partial_j d(x)$$

$$+ \frac{(1 - \alpha)d(x) + |x - x_0|^\alpha}{2(d(x) + |x - x_0|^\alpha)^{1+\alpha}} \partial_j d(x) - \mu^i(x)) \partial_j d(x) - \mu^j(x))$$

$$+ \frac{2(d(x) + |x - x_0|^\alpha)^{1+\alpha}}{\alpha d(x) \partial_j d(x) \partial_j (|x - x_0|^\alpha) - \partial_i d(x) \partial_j (|x - x_0|^\alpha)}$$

(83)

$$- \frac{2(d(x) + |x - x_0|^\alpha)^{1+\alpha}}{\alpha d(x) \partial_j d(x) \partial_j (|x - x_0|^\alpha)}$$

$$- \frac{2(d(x) + |x - x_0|^\alpha)^{1+\alpha}}{\alpha d(x) \partial_j d(x) \partial_j (|x - x_0|^\alpha)},$$

where

$$\mu^i(x) = \alpha d(x)((1 - \alpha)d(x) + |x - x_0|^\alpha)^{-1} \partial_i (|x - x_0|^\alpha),$$

$i = 1, 2, 3$. These properties of $\varphi^i$ enable to show the estimate ([15], Lemma 3.6):
Lemma 5. Under the assumptions above, we have for $\alpha < \frac{9m-6}{9m-2}$, $x_0 \in \partial \Omega$ and $R_0$ sufficiently small (uniformly with respect to $x_0$)

$$\int_{B_{R_0}(x_0) \cap \Omega} \frac{\pi(g_\delta, \vartheta_\delta) \delta(g_\delta^2 + g_\delta^2)}{|x - x_0|^\alpha} \, dx \leq C(1 + \|\pi(g_\delta, \vartheta_\delta)\|_1 + (1 + \|\vartheta_\delta\|_{3m}) \|u_\delta\|_{1,2} + \|g_\delta|u_\delta|^2\|_1).$$  

(84)

Now we come to the most delicate part of the estimate. Notice that in Lemma 4 the ball is separated from the boundary, therefore we have to treat separately the case of $x_0 \in \Omega$ which is close to the boundary. This gap was not commented in the original papers, here we fill it using a carefully chosen test function vanishing at the boundary, which enables us to reach with the ball up to the boundary. Precisely, we show the following

Lemma 6. Assume that $x_0 \in \Omega$ is such that $\text{dist}\{x_0, \partial \Omega\} = 5\varepsilon$ for some $0 < \varepsilon \ll 1$ and $\alpha < \frac{9m-6}{9m-2}$. Then

$$\int_{\Omega} \frac{\pi(g_\delta, \vartheta_\delta) \delta(g_\delta^2 + g_\delta^2)}{|x - x_0|^\alpha} \, dx \leq C(1 + \|\pi(g_\delta, \vartheta_\delta)\|_1 + \|u_\delta\|_{1,2} + \|\vartheta_\delta\|_{3m} + \|g_\delta|u_\delta|^2\|_1).$$  

(85)

Proof. We use again the function $\varphi^1$ defined in (82). From (83) we see that

$$\int_{\Omega} g_\delta(u_\delta \otimes u_\delta) : \nabla \varphi^1 \, dx \geq C_1 \int_{\Omega} \frac{g_\delta u_\delta \cdot \nabla d^2}{(d(x) + |x - x_0|^a)^\alpha} \, dx - C_2 \int_{\Omega} g_\delta|u_\delta|^2 \, dx$$

(86)

and

$$\int_{\Omega} [\pi(g_\delta, \vartheta_\delta) + \delta(g_\delta^2 + g_\delta^2)] \text{div} \varphi^1 \, dx \geq C_1 \int_{\Omega} \frac{\pi(g_\delta, \vartheta_\delta) + \delta(g_\delta^2 + g_\delta^2)}{(d(x) + |x - x_0|^a)^\alpha} \, dx$$

$$- C_2 \int_{\Omega} (\pi(g_\delta, \vartheta_\delta) + \delta(g_\delta^2 + g_\delta^2)) \, dx.$$  

(87)

The form of $\nabla \varphi^1$ in (83) imply for $q < \frac{3-a}{a}$ that $\|\varphi^1\|_{1,q} \leq C$ independently of the distance from the boundary. However, we have

$$\frac{1}{(d(x) + |x - x_0|^a)^\alpha} \geq \frac{C}{|x - x_0|^\alpha}$$

only for $x \in \Omega \setminus B_\varepsilon(x_0)$. Therefore (86) and (87) does not provide estimate for $\frac{\pi}{|x - x_0|^\alpha}$ in $B_\varepsilon(x_0)$ where we need an additional estimate. To this end we introduce additional function which behaves like $\varphi^0$ defined in (79), but additionally vanishes on the boundary. To combine these requirements we define it in a following way:

$$\varphi^2(x) = \begin{cases} 
\frac{|x - x_0|}{|x - x_0|^\alpha} \left(1 - \frac{1}{2^\alpha} \right), & |x - x_0| < \varepsilon, \\
(x - x_0) \frac{1}{|x - x_0|^\alpha} - \frac{1}{(|x - x_0| + d(x))^\alpha}, & |x - x_0| > \varepsilon, d(x) > \varepsilon, \\
(x - x_0) \frac{1}{|x - x_0|^\alpha} - \frac{1}{(|x - x_0| + d(x))^\alpha}, & |x - x_0| > \varepsilon, d(x) \leq \varepsilon.
\end{cases}$$

(88)
First of all, we easily verify that
\[ \varphi^2 \in W^{1,q}_0(\Omega) \quad \text{for all} \quad 1 \leq q < \frac{3}{\alpha} \]
with the norm bounded independently of \( \varepsilon \). Indeed, the singularity in \( \varphi^2 \) and its derivatives appears only in \( B_\varepsilon(x_0) \), where we have
\[ \nabla \varphi^2 \sim \nabla \frac{x - x_0}{|x - x_0|^\alpha} \sim \frac{1}{|x - x_0|^\alpha}, \]
which yields the above limitation on \( q \). Now we can verify that
\[ \int_\Omega \rho_\delta (u_\delta \otimes u_\delta) : \nabla \varphi^2 \, dx \geq K_1 \int_{B_\varepsilon(x_0)} \frac{\varphi_\delta |u_\delta|^2}{|x - x_0|^\alpha} \, dx \\
- K_2 \int_{\{d(x) \leq \varepsilon\}} \frac{\varphi_\delta (u_\delta \cdot \nabla d)^2}{(d(x) + |x - x_0|^\alpha)} \, dx \\
- K_3 \int_\Omega \varphi_\delta |u_\delta|^2 \, dx \quad \text{(90)} \]
and
\[ \int_\Omega \left( \pi(\varphi_\delta, \varphi_\delta) + \delta(\varphi_\delta^2 + \varphi_\delta^2) \right) \, dx \geq K_1 \int_{B_\varepsilon(x_0)} \frac{\pi(\varphi_\delta, \varphi_\delta) + \delta(\varphi_\delta^2 + \varphi_\delta^2)}{|x - x_0|^\alpha} \, dx \\
- K_3 \int_\Omega \left( \pi(\varphi_\delta, \varphi_\delta) + \delta(\varphi_\delta^2 + \varphi_\delta^2) \right) \, dx \quad \text{(91)} \]
Notice that the first term on the rhs of (91) is exactly the one which we were missing (the estimate outside \( B_\varepsilon(x_0) \) is given by (87)). After all these considerations we can test (68) with
\[ \varphi = K \varphi^1 + \varphi^2 \]
where \( K \) is a sufficiently large constant. Then the first term on the rhs of (86) which has a good sign compensates the second term on the rhs of (90) and we conclude (85) provided \( \alpha < \frac{9m-6}{9m-2} \) which completes the proof. \( \square \)

Combining Lemmas 4–6 we conclude

**Proposition 7.** Assume \( \alpha < \frac{9m-6}{9m-2} \). Then
\[ \sup_{x_0 \in \Omega} \int_\Omega \frac{\pi(\varphi_\delta, \varphi_\delta) + \delta(\varphi_\delta^2 + \varphi_\delta^2)}{|x - x_0|^\alpha} \, dx \leq C \left( 1 + \|\pi(\varphi_\delta, \varphi_\delta)\|_1 + \|u_\delta\|_{1,2}(1 + \|\varphi_\delta\|_{3m}) + \|\varphi_\delta|u_\delta|^2\|_1 \right) \quad \text{(92)} \]

Using the above pressure estimate we show

**Lemma 8.** Let \( 1 \leq b < \gamma, \alpha < \frac{9m-6}{9m-2} \) and \( \alpha > \frac{3b-2\gamma}{b} \). Then
\[ A = \int_\Omega \varphi_\delta^b |u_\delta|^2 \, dx \leq C \|u_\delta\|^2_{1,2} \left( 1 + \|\pi(\varphi_\delta, \varphi_\delta)\|_1 + \|u_\delta\|_{1,2}(1 + \|\varphi_\delta\|_{3m}) + \|\varphi_\delta|u_\delta|^2\|_1 \right)^\frac{b}{2} \quad \text{(93)} \]
This is exactly Lemma 3.7 from [15], however we sketch the proof here to show the application of (92) which is not evident. First using interpolation inequality we show
\[
\int_\Omega \frac{\varrho^b_\delta}{|x - x_0|} \, dx \leq C \left( 1 + \|\pi(\varrho_\delta, \vartheta_\delta)\|_1 + \|u_\delta\|_1,2(1 + \|\vartheta_\delta\|_{3m}) + \|\varrho_\delta u_\delta\|^2 \right)^{\frac{b}{\gamma}}.
\]
Next we introduce \( h \) as a solution to \( \Delta h = \varrho^b_\delta \), \( h|_{\partial \Omega} = 0 \), and represent it with the Green function to obtain
\[
\|h\|_\infty \leq C \sup_{x_0 \in \Omega} \int_\Omega \frac{\varrho^b_\delta(x)}{|x - x_0|} \, dx.
\]
The definition of \( h \) yields
\[
A \leq C\|\nabla u_\delta\|_2 \left( \int_\Omega |u_\delta|^2 |\nabla h|^2 \, dx \right)^{\frac{1}{2}},
\]
and integrating by parts the last integral we get (93). We are now ready to show the following

Lemma 9. Let \( \gamma > 1 \), \( m > \frac{2}{3} \) and \( m > \frac{2}{9} \gamma - 1 \). Then there exists \( s > 1 \) such that \( \varrho_\delta \) is bounded in \( L^{s\gamma}(\Omega) \) and \( \pi(\varrho_\delta, \vartheta_\delta) \), \( \varrho_\delta |u_\delta| \) and \( \varrho_\delta |u_\delta|^2 \) are bounded in \( L^s(\Omega) \). Moreover, if \( \gamma > \frac{4}{3} \) and \( m > 1 \) for \( \gamma \geq \frac{12}{7} \) and \( m > \frac{2\gamma}{3(3\gamma - 4)} \) for \( \gamma \in \left( \frac{4}{3}, \frac{12}{7} \right) \), we can take \( s > \frac{6}{5} \).

Proof. Interpolation inequality yields
\[
\|\pi(\varrho_\delta, \vartheta_\delta)\|_1 \leq C \left( \left( \int_\Omega \varrho_\delta^{s\gamma} \, dx \right)^{\frac{1}{s}} + \left( \int_\Omega \varrho_\delta^{(s-1)\gamma} \pi(\varrho_\delta, \vartheta_\delta) \, dx \right)^{\frac{1}{(s-1)\gamma+1}} \times \left( \int_\Omega \vartheta_\delta \, dx \right)^{\frac{(s-1)\gamma}{(s-1)\gamma+1}} \right).
\]
Therefore, combining Lemmas 3 and 8 and applying (76) we show
\[
A \leq C \left( 1 + A^{\frac{4s-3}{6s-2s}} + A^{\frac{1}{6s-4}(1 + \frac{8s-7}{(s-1)\gamma+1})} \right)^{\frac{b}{\gamma}}.
\]
In order to get the statement of the Lemma we need
\[
\frac{4s - 3}{s} \frac{1}{3b - 2\gamma} < 1 \quad \text{and} \quad \frac{1}{6b - 4} \left( 1 + \frac{8s - 7}{(s-1)\gamma+1} \right) < 1
\]
for a certain \( s > 1 \) and \( 1 \leq b < \gamma \). Collecting these and other assumptions from this section we get the statement for \( \pi \) and \( \varrho_\delta u_\delta \) and the result for \( \varrho_\delta |u_\delta|^2 \) follows from
\[
\|\varrho_\delta u_\delta\|_s \leq \|\varrho_\delta\|^{\frac{1}{s}} \|\varrho_\delta|u_\delta|^2\|^{\frac{1}{s}}.
\]
If we require \( s > \frac{6}{5} \), we get more restrictions, see [15] or [13].

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In order to pass to the limit in the total energy balance, we have to show that

$$\lim_{\delta \to 0^+} \delta \| \varrho_\delta \|_{\frac{6}{5}}^\frac{2}{5} = 0.$$  \hspace{1cm} (94)

To this end we use the Bogovskii-type estimates of the momentum equation (68) with $\varrho_\delta^\frac{1}{\beta+\eta}$, $\eta > 0$. Assuming additionally that

$$\varrho_\delta |u_\delta|^2$$

is bounded in some $L^q(\Omega)$, $q > \frac{6}{5}$,

we deduce (see [15] or [13] for details)

$$\delta \| \varrho_\delta \|_{\frac{6}{5}}^\frac{2}{5} \leq C$$

for some $\eta > 0$ which yields (94) due to interpolation of $L^\frac{6}{5}(\Omega)$ between $L^1(\Omega)$ and $L^\frac{6}{5}(\Omega)$.

5.2 Limit passage

5.2.1 Limit passage based on a priori estimates

Collecting the estimates obtained so far we have the following convergences

$$u_\delta \to u \quad \text{in} \quad W^{1,2}_0(\Omega),$$

$$\varrho_\delta \to \varrho \quad \text{in} \quad L^{\gamma}(\Omega), \ \gamma > 1, \ m > \frac{2}{3}, \ m > \frac{2}{9} \gamma - 1,$$

$$\vartheta_\delta \to \vartheta \quad \text{in} \quad W^{1,r}(\Omega), \ r = \min \left\{ 2, \frac{3m}{m+1} \right\},$$

$$\vartheta_\delta \to \vartheta \quad \text{in} \quad L^q(\Omega), \ q < 3m,$$

$$\vartheta_\delta \to \vartheta \quad \text{in} \quad L^q(\partial \Omega), \ q < 2m,$$

$$Y_\delta \to Y \quad \text{in} \quad W^{1,2}(\Omega),$$

$$Y_\delta \to Y \quad \text{in} \quad L^q(\Omega), \ q < \infty.$$  \hspace{1cm} (95)

These allow to pass to the limit in the continuity equation, momentum equation, species balance equations and entropy inequality to obtain

$$\int_{\Omega} \varrho u \cdot \nabla \psi \, dx = 0$$ \hspace{1cm} (96)

for all $\psi \in C^\infty(\overline{\Omega})$,

$$- \int_{\Omega} (\varrho (u \otimes u) : \nabla \phi + S : \nabla \phi) \, dx - \int_{\Omega} (\varrho \vartheta + \varrho \vartheta') \text{div} \phi \, dx = \int_{\Omega} \varrho f \cdot \phi \, dx$$ \hspace{1cm} (97)

for all $\phi \in C_0^\infty(\Omega)$,

$$- \int_{\Omega} Y_k \varrho u \cdot \nabla \psi \, dx + \int_{\Omega} Y_k \sum_{l=1}^n D_{kl} \nabla Y_l \cdot \nabla \psi \, dx = \int_{\Omega} \omega_k \psi \, dx$$ \hspace{1cm} (98)
for all $\psi \in C^\infty(\Omega)$, $k = 1, 2, \ldots, n$, and

$$
\int_\Omega \frac{\psi \mathbf{S} : \nabla \mathbf{u}}{\vartheta} \, dx + \int_\Omega \kappa \frac{\vert \nabla \vartheta \vert^2}{\vartheta^2} \psi \, dx - \int_\Omega \sum_{k=1}^n \psi (c_{pk} - c_{vk} \log \vartheta + \log Y_k) \omega_k \, dx \\
+ \int_\Omega \psi \sum_{k,l=1}^n D_{kl} \nabla Y_k \nabla Y_l \, dx + \int_{\partial \Omega} \frac{\psi}{\vartheta} L \partial_0 \, dS \leq \int_\Omega \frac{\kappa \nabla \vartheta \cdot \nabla \psi}{\vartheta} \, dx - \int_\Omega \frac{\vartheta \mathbf{u} \cdot \nabla \psi}{\vartheta} \, dx \\
- \int_\Omega \sum_{k=1}^n F_k \cdot (c_{vk} \log \vartheta - \log Y_k) \nabla \psi \, dx + \int_{\partial \Omega} \psi L \, dS \tag{99}
$$

for all non-negative $\psi \in C^\infty(\Omega)$. In order to pass in the species equations we need to assume $D_{kl}(\vartheta, \cdot) \leq C(1 + \vartheta^a)$ for $a < \frac{3m}{2}$, no further restrictions on $\gamma$ are needed.

However, in order to pass in the total energy balance we need $s > \frac{6}{5}$ in (95). This requirement combined with other assumptions from this section yields (see Lemma 9) $\gamma > \frac{4}{3}$ and

$$
m > 1 \quad \text{for} \quad \gamma \geq \frac{12}{7}, \\
m > \frac{2\gamma}{3(3\gamma - 4)} \quad \text{for} \quad \gamma \in \left(\frac{4}{3}, \frac{12}{7}\right). \tag{100}
$$

Furthermore, we also need $a < \frac{3m}{2} - 2$. Under these restrictions we can pass to the limit also in the total energy balance to get

$$
- \int_\Omega \left[ \vartheta \partial_0 \sum_{k=1}^n c_{vk} Y_k + \frac{1}{2} \vartheta \mathbf{|u|^2} + \vartheta \partial_0 + \frac{\gamma}{\gamma - 1} \vartheta^\gamma \right] \mathbf{u} \cdot \nabla \psi \, dx - \int_\Omega \mathbf{S} \mathbf{u} \cdot \nabla \psi \, dx \\
+ \int_\Omega \kappa \nabla \vartheta \cdot \nabla \psi \, dx + \int_{\partial \Omega} L (\vartheta - \vartheta_0) \psi \, dS \\
+ \int_\Omega \partial_0 \sum_{k,l=1}^n c_{vk} Y_k D_{kl} \nabla Y_l \cdot \nabla \psi \, dx = \int_{\partial \Omega} \vartheta \mathbf{f} \cdot \mathbf{u} \psi \, dS. \tag{101}
$$

### 5.2.2 Strong convergence of the density

In order to finish the proof of Theorem 1 we have to get rid of the weak limits in (96)–(101) denoted by bars. For this purpose we need to show that

$$
\varrho_\delta \to \varrho \quad \text{strongly in} \quad L^1(\Omega).
$$

Here we apply the techniques developed for compressible Navier–Stokes system which consist in testing the momentum equation with appropriately chosen test function leading to so called effective viscous flux identity. As the momentum equation is in our case essentially the same we can repeat this approach. We skip the details as this is already standard in the theory of compressible flows, however for the sake of completeness we recall the main steps.
Step 1. Effective viscous flux identity. Consider

\[ T_k(z) = kT \left( \frac{z}{k} \right), \quad T(z) = \begin{cases} 
  z & \text{for } 0 \leq z \leq 1, \\
  \text{concave on } (0, \infty), \\
  2 & \text{for } z \geq 3.
\end{cases} \]

Using as a test function \( \zeta(x) \nabla \Delta^{-1}(1_\Omega T_k(\vartheta_\delta)) \) in the approximate momentum equation (68) and \( \zeta(x) \nabla \Delta^{-1}(1_\Omega T_k(\vartheta)) \) in its limit version (97) with \( \zeta(x) \in C_0^\infty(\Omega) \) we get the identity (for the proof see [14], Lemma 12 with \( T_k(\vartheta) \) instead of \( \vartheta \)):

\[
\begin{align*}
  &\lim_{\delta \to 0^+} \int_\Omega \zeta(x) \left( \pi(\vartheta_\delta, \vartheta) T_k(\vartheta_\delta) - \mathcal{S}(\vartheta_\delta, \nabla u_\delta) : \mathcal{R}[1_\Omega T_k(\vartheta_\delta)] \right) \, dx \\
  &= \int_\Omega \zeta(x) \left( \pi(\vartheta, \vartheta) T_k(\vartheta) - \mathcal{S}(\vartheta, \nabla u) : \mathcal{R}[1_\Omega T_k(\vartheta)] \right) \, dx \\
  &\quad + \lim_{\delta \to 0^+} \int_\Omega \zeta(x) \left( T_k(\vartheta_\delta) u_\delta \cdot \mathcal{R}[1_\Omega \vartheta_\delta u_\delta] - \vartheta_\delta (u_\delta \otimes u_\delta) : \mathcal{R}[1_\Omega T_k(\vartheta_\delta)] \right) \, dx \\
  &\quad - \int_\Omega \zeta(x) \left( T_k(\vartheta) u \cdot \mathcal{R}[1_\Omega u] - \vartheta (u \otimes u) : \mathcal{R}[1_\Omega T_k(\vartheta)] \right) \, dx,
\end{align*}
\]

where \( \mathcal{R} \) denotes the double Riesz operator, \( (\mathcal{R}[v])_{ij} = (\nabla \otimes \nabla^{-1})_{ij} v = \mathcal{F}^{-1} \left[ \frac{\xi i \eta j}{|\xi|^s} \mathcal{F}(v)(\xi) \right] \) with \( \mathcal{F} \) the Fourier transform, and we used that div \( (\vartheta_\delta u_\delta) = \text{div} (\vartheta u) = 0 \). We recall some auxiliary results we will apply. The first one is (see [21], Theorem 10.27)

Lemma 10 (Commutators I). Let \( U_\delta \rightharpoonup U \) in \( L^p(\mathbb{R}^3) \), \( v_\delta \rightharpoonup v \) in \( L^q(\mathbb{R}^3) \), where

\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1.
\]

Then

\[
v_\delta \mathcal{R}[U_\delta] - \mathcal{R}[v_\delta] U_\delta \rightharpoonup v \mathcal{R}[U] - \mathcal{R}[v] U
\]

in \( L^s(\mathbb{R}^3) \).

The second is (see Theorem 10.28 in [21])

Lemma 11 (Commutators II). Let \( w \in W^{1,r}(\mathbb{R}^3) \), \( z \in L^p(\mathbb{R}^3) \), \( 1 < r < 3 \), \( 1 < p < \infty \), \( \frac{1}{r} + \frac{1}{p} - \frac{1}{3} < \frac{1}{s} < 1 \). Then for all such \( s \) we have

\[
\| \mathcal{R}[wz] - w \mathcal{R}[z] \|_{a,s,R^3} \leq C \| w \|_{1,r,R^3} \| z \|_{p,R^3},
\]

where \( \frac{2}{3} = \frac{1}{s} + \frac{1}{p} - \frac{1}{r} \). Here, \( \| \cdot \|_{a,s,R^3} \) denotes the norm in the Sobolev–Slobodetskii space \( W^{a,s}(\mathbb{R}^3) \).

Finally we have ([9, Lemma 6]):

Lemma 12. Let \( \Omega \) bounded, \( f_\delta \to f \) in \( L^1(\Omega) \), \( g_\delta \to g \) in \( L^1(\Omega) \) and \( f_\delta g_\delta \rightharpoonup h \) in \( L^1(\Omega) \). Then \( h = fg \).
We have the following identity
\[ g(u \otimes u) : \mathcal{R}[T_k(\varrho)] = \sum_{i,j=1}^{3} u_i (\mathcal{R}[T_k(\varrho)])_{ij} g_{ij}, \] (103)
which is uniformly bounded in \( L^p \) for \( 1 \leq p \leq s \). Moreover, we have
\[ g_\delta u_\delta \to g_{uv}, \quad g_\delta u_\delta \otimes u_\delta \to g u \otimes u \quad \text{in } L^1. \]
Therefore, applying Lemma 10 with
\[ v_\delta = T_k(g_\delta) \to \overline{T_k(\varrho)} \quad \text{in } L^q(\mathbb{R}^3), \quad q < \infty \text{ arbitrary} \]
\[ U_\delta = g_\delta u_\delta \to g_{uv} \quad \text{in } L^p(\mathbb{R}^3), \quad \text{for certain } p > 1, \]
and Lemma 12 with
\[ f_\delta = T_k(g_\delta) \mathcal{R}[1_\Omega g_\delta u_\delta] - g_\delta u_\delta \mathcal{R}[1_\Omega T_k(g_\delta)], \quad g_\delta = \zeta u_\delta \]
we obtain
\[ \int_\Omega \zeta(x) u_\delta \cdot (T_k(g_\delta) \mathcal{R}[1_\Omega g_\delta u_\delta] - g_\delta \mathcal{R}[1_\Omega T_k(g_\delta)] u_\delta) \, dx \]
\[ \to \int_\Omega \zeta(x) u \cdot (\overline{T_k(\varrho)} \mathcal{R}[1_\Omega g u] - g \mathcal{R}[1_\Omega \overline{T_k(\varrho)}] u) \, dx. \]
This convergence in view of (102) and (103) gives
\[ \lim_{\delta \to 0^+} \int_\Omega \zeta(x) \left( \pi(g_\delta, \varrho_\delta) T_k(g_\delta) - S(\varrho_\delta, \nabla u_\delta) : \mathcal{R}[1_\Omega T_k(g_\delta)] \right) \, dx \]
\[ = \int_\Omega \zeta(x) \left( (p(\varrho, \vartheta) \overline{T_k(\varrho)} - S(\vartheta, \nabla u) : \mathcal{R}[1_\Omega \overline{T_k(\varrho)}]) \right) \, dx. \] (104)
Next we can write
\[ \int_\Omega \zeta(x) \tilde{S}(\vartheta, \nabla u) : \mathcal{R}[1_\Omega T_k(\varrho)] \, dx = \lim_{\delta \to 0^+} \int_\Omega \zeta(x) \left( \frac{4}{3} \mu(\vartheta_\delta) + \zeta(\vartheta_\delta) \right) \text{div } u_\delta T_k(g_\delta) \, dx \]
\[ + \lim_{\delta \to 0^+} \int_\Omega T_k(\vartheta_\delta) \left( \mathcal{R} : \left[ \zeta(x) \mu(\vartheta_\delta) (\nabla u_\delta + (\nabla u_\delta)^T) \right] \right) \]
\[ - \zeta(x) \mu(\vartheta_\delta) \mathcal{R} : \left[ \nabla u_\delta + (\nabla u_\delta)^T \right] \, dx, \] (105)
where \( \mathcal{R} : \mathcal{A} := \sum_{i,j=1}^{3} (\nabla \otimes \nabla \Delta^{-1})_{ij} \mathcal{A}_{ij} \) for a tensor valued function \( \mathcal{A} \). Applying Lemma 11 to the second term in (105) we finally obtain the effective viscous flux identity:
\[ \frac{p(\varrho, \vartheta) T_k(\varrho) - \left( \frac{4}{3} \mu(\vartheta) + \zeta(\vartheta) \right) \overline{T_k(\varrho)} \text{div } u}{\overline{T_k(\varrho)}} = \frac{p(\varrho, \vartheta) T_k(\varrho) - \left( \frac{4}{3} \mu(\vartheta) + \zeta(\vartheta) \right) \overline{T_k(\varrho)} \text{div } u}{\overline{T_k(\varrho)}}. \] (106)
Step 2. Renormalized continuity equation. In the next step we verify that \((\varrho, u)\) satisfies the renormalized continuity equation. For this purpose we introduce the oscillations defect measure:

\[
\text{osc}_q[\varrho_\delta \to \varrho](Q) = \sup_{k > 1} \left( \limsup_{\delta \to 0^+} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^q \, dx \right). \tag{107}
\]

Applying (106) we show ([15, Lemma 4.5]):

Lemma 13. Let \((\varrho_\delta, u_\delta, \vartheta_\delta)\) satisfy

\[
\begin{align*}
\varrho_\delta &\rightharpoonup \varrho \quad \text{in } L^1(\Omega), \\
u_\delta &\rightharpoonup u \quad \text{in } L^r(\Omega), \\
abla u_\delta &\rightharpoonup \nabla u \quad \text{in } L^r(\Omega), \quad r > 1.
\end{align*} \tag{108}
\]

Assume further that \(m > \max\{\frac{2}{2(\gamma - 1) + \frac{2}{3}}\}\). Then there exists \(q > 2\) such that

\[
\text{osc}_q[\varrho_\delta \to \varrho](\Omega) < \infty. \tag{109}
\]

Moreover,

\[
\limsup_{\delta \to 0^+} \int_\Omega \frac{1}{1 + \vartheta} |T_k(\varrho_\delta) - T_k(\varrho)|^{q+1} \, dx \leq \int_\Omega \frac{1}{1 + \vartheta} \left( \overline{p(\varrho, \vartheta)} T_k(\varrho) - \overline{p(\varrho, \vartheta)} T_k(\varrho) \right) \, dx. \tag{110}
\]

It is known (see [2, Lemma 3.8]) that (108) together with (109) implies that \((\varrho, u)\) satisfies the renormalized continuity equation.

Step 3. Strong convergence of the density. As \((\varrho, u)\) and \((\varrho_\delta, u_\delta)\) satisfy the renormalized continuity equation, in particular we have

\[
\int_\Omega T_k(\varrho) \text{div } u \, dx = 0, \quad \int_\Omega T_k(\varrho_\delta) \text{div } u_\delta \, dx = 0
\]

and the second identity implies

\[
\int_\Omega \overline{T_k(\varrho)} \text{div } u \, dx = 0.
\]

Therefore using (106) we get

\[
\int_\Omega \frac{1}{3 \mu(\vartheta) + \xi(\vartheta)} \left( \overline{p(\varrho, \vartheta)} T_k(\varrho) - \overline{p(\varrho, \vartheta)} T_k(\varrho) \right) \, dx \]

\[
= \int_\Omega (T_k(\varrho) - \overline{T_k(\varrho)}) \text{div } u \, dx \to_{k \to \infty} 0,
\]

which together with (110) implies

\[
\lim_{k \to \infty} \limsup_{\delta \to 0^+} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^q \, dx = 0
\]

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with $q$ as in Lemma 13. It remains to use the fact that
\[ \| \rho_5 - \rho \|_1 \leq \| \rho_5 - T_k(\rho_5) \|_1 + \| T_k(\rho_5) - T_k(\rho) \|_1 + \| T_k(\rho) - \rho \|_1, \]
which yields strong convergence of the density in $L^1$, therefore also in $L^p$ for $1 \leq p < s\gamma$.

The above strong convergence of the density allows to remove all the bars in (97), (99) and (101). Collecting all the assumptions on $m$ we see that the most restrictive constraint is $m > \frac{2}{2(\gamma-1)}$ and for weak solutions we must take into account $m > \max\{1, \frac{2\gamma}{3(\gamma-4)}\}$. This completes the proof of Theorem 1.

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