Data-Driven Control of Stochastic Systems: An Innovation Estimation Approach

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Abstract—Recent years have witnessed a booming interest in the data-driven paradigm for predictive control. However, under noisy data ill-conditioned solutions could occur, causing inaccurate predictions and unexpected control behaviours. In this article, we explore a new route toward data-driven control of stochastic systems through active offline learning of innovation data, which gives an answer to the critical question of how to derive an optimal data-driven model from a noise-corrupted dataset. A generalization of the Willems’ fundamental lemma is developed for non-parametric representation of input-output innovation trajectories, provided realizations of innovation are precisely known. This yields a model-agnostic unbiased output predictor and paves the way for data-driven receding horizon control, whose behaviour is identical to the “oracle” solution of certainty-equivalent model-based control with measurable states. For efficient innovation estimation, a new low-rank subspace identification algorithm is developed. Numerical simulations show that by actively learning innovation from input-output data, remarkable improvement can be made over present formulations, thereby offering a promising framework for data-driven control of stochastic systems.

Index Terms—Data-driven control, Linear systems, Learning-based control, Subspace identification

I. INTRODUCTION

THE ever-growing availability of data has been an enabling factor of contemporary developments in versatile engineering realms. In control engineering, the idea of learning from data is not only of current interest but also rich in history, where system identification has played a central role [1]. Notably, a common practice is to fit a parametric model first based on empirical data and then perform control design with the identified nominal model [2], [3].

In modern control applications, the explosive system complexities pose a great challenge to the traditional identification-for-control scheme, since accurate modeling is costly and heavily relies on engineering expertise [3], [4]. It is thus attractive to design controllers directly from raw data with the intermediate identification step bypassed. From this new perspective, a paradigm shift has been driven towards data-driven end-to-end solutions to simulation and control design [5]. A mainstream of research builds upon the pioneering result rooted in the behavioural theory [6], [7], which became known later as the Willems’ fundamental lemma [8]. It roughly states that in the deterministic case, all admissible trajectories from a linear system are precisely expressible as linear combinations of known trajectories, provided past input data are sufficiently informative. With this result, it suffices to use raw data-based constraints in lieu of parametric state-space models for description of system dynamics.

This trait becomes more fascinating when it comes to optimal trajectory tracking tasks. In this regard, the Data-enabled Predictive Control (DeePC) [9], an important control algorithm inspired from the fundamental lemma, establishes itself as a data-driven counterpart of conventional model-based predictive control. The appeal of DeePC arises from several aspects. First, output prediction can be carried out based on past trajectories only, even if state measurements are not available. Thus, both the cumbersome identification of parametric models and the construction of an online state observer are no longer needed. Besides its easy implementation, there is an additional subtlety that the bias induced by inadequate model fitting can be effectively circumvented [10], leading to competitive performance with model predictive control (MPC). Thanks to these merits, DeePC has found miscellaneous applications across domains, such as power systems [11]–[14], motion control [15]–[17], and smart buildings [18].

On the theoretical side, rigorous analysis of closed-loop stability of DeePC was given by [19], [20]. We refer to the recent survey paper [21] for a comprehensive review.

Strictly speaking, the philosophy of substituting state-space models with data-based trajectory representations, in the spirit of the fundamental lemma, is only valid in an ideal deterministic environment. In real-life applications, the fidelity of direct data-driven control is greatly challenged by the pervasive uncertain disturbance and noise corruption. Specifically, the data-driven model becomes highly ill-conditioned and over-fitted, thereby causing inflated variances in trajectory predictions of stochastic systems. To robustify DeePC against uncertainty, a series of remedies have been put forward. The earliest use of norm-based regularization was suggested in [9], which also admits a robust optimization interpretation [13], [22]. To tackle the inconsistency of empirical regularizers in noise-free cases, a projection-based regularizer was elaborated in [23]. We also note a recent distributionally robust formulation to hedge against ambiguity [24]. In [25], a statistical signal matrix model is built based on the maximum-likelihood principle. Yet, the effect of noise in data-driven control is still not completely understood.

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understood. Particularly, it remains unclear how to derive an optimal data-driven model from noisy data recorded from a stochastic system, as pointed out by various previous works [21], [24], [26], [27].

In this article, we seek to give a clear answer to this critical question. Through offline learning of innovation, a systematic approach to data-driven control of linear time-invariant (LTI) stochastic systems is developed. Starting from the innovation form of stochastic systems, a derivative of the Willems’ fundamental lemma is developed for data-based representation of system trajectories, provided that realizations of innovation are perfectly known. Despite generalizing the classic fundamental lemma immediately, this result bears profound implications. Importantly, it leads to a model-free output predictor of stochastic system, which is unbiased and optimal. It clarifies when and how a data-based description is tantamount to the “oracle” formulation in a stochastic setting. Precisely, in order to perform optimally, extra innovation data equations shall be added in the classical input-output data equations. This implies that solutions to known remedies of DeePC are substantially underdetermined in handling noisy data, which offers deep insight into their performance gap with respect to the oracle solution. The application of optimal output predictor in control design is also straightforward. Due to the unbiasedness and optimality, the incurred control behaviour is identical to the oracle certainty-equivalent control with measurable states.

It is noteworthy that in [28], [29], a similar stochastic variant of the fundamental lemma was proposed, which utilizes polynomial chaos expansions to describe system noise. By contrast, our proposal tackles innovation models with both system and output noise, which is thus much more general. Moreover, the nice properties of innovations enable a seamless integration with receding horizon control, where preceding innovations can be handily updated using the one-step prediction error, as well as the design of efficient offline estimation algorithms.

Since the assumption of precise innovation realizations is unrealistic, estimates of innovation from noise-corrupted datasets have to be used in data-driven control. The optimality condition highlights the peculiar value of attaining high-precision estimates in narrowing the gap between DeePC and the oracle solution. Hence, we delve into this issue in the second part of this article. Closest to explicit innovation estimation is the classic two-step closed-loop subspace identification [30], [31], where Markov parameters of a Kalman predictor are first estimated using least squares and innovations can be approximated as the prediction error. This simple strategy indeed adapts to data-driven control; however, extra preceding data are a prerequisite and over-fitting could occur. The recent nuclear-norm subspace identification (N2SID) algorithm [32] enables to alleviate over-fitting and yield a low-complexity model by means of nuclear-norm regularization. Yet, the nuclear-norm may undesirably suppress dominating eigenvalues and hyper-parameter tuning is also a non-trivial task. To overcome these drawbacks, we put forward a low-rank subspace identification (LRSID) algorithm for innovation estimation with improved accuracy. It directly enforces the low-rank constraint in place of the nuclear-norm penalty in N2SID. As a result, it only requires to know an upper-bound of system order. To efficiently solve the optimization problem, a tailored alternating direction of multipliers method (ADMM) algorithm [33] is devised.

In a nutshell, we present a systematic framework for data-driven control in the stochastic setting, which consists of a new data-driven control formulation including innovation and a tailored subspace approach to structured innovation estimation. Admittedly, one can modularly embed innovation estimates from other identification methods in the proposed DeePC scheme. We comprehensively investigate the efficacy of our DeePC scheme combined with various innovation estimates. Simulation examples show that by offline learning of innovation, improvement over popular empirical regularization can be generally achieved under noisy data. This highlights the value of proactive estimation of innovation in data-driven control. While using the proposed LRSID method, the highest accuracy of innovation estimation is attained, which secures a superior control performance.

The layout of this article is organized as follows. In Section II, the Willems’ fundamental lemma and its applications are reviewed. In Section III, the generalized fundamental lemma is formalized and a generalized DeePC formulation is presented. Section IV describes the LRSID algorithm for efficient innovation estimation. Case study results are reported in Section V, followed by concluding remarks in Section VI.

**Notation:** We denote by $\mathbb{Z}$ ($\mathbb{Z}^+$) the set of (positive) integers. The identity matrix of size $s \in \mathbb{Z}^+$ is $I_s \in \mathbb{R}^{s \times s}$. $I_s \in \mathbb{R}^s$ is the $s$-vector of all ones. $1_s \in \mathbb{R}^s$ denotes the unit vector whose $i$th element is one and others are zero. $X^\top$ and $\|X\|_F$ denote the Moore-Penrose inverse and the Frobenius norm of matrix $X$, respectively. The inner product between $X$ and $Y$ is defined as $\langle X, Y \rangle = \text{Tr}(X^\top Y)$. For a linear mapping $\Pi$, $\Pi^*$ denotes its adjoint operator.

Given a sequence $\{x(i)\}_{i=1}^N \in \mathbb{R}^n$, let $i, j \in \mathbb{N}^+$, $i \leq j \leq N$ and then $x_{[i:j]}$ denotes the restriction of $x$ to the interval $[i, j]$ as:

$$x_{[i:j]} = \begin{bmatrix} x(i) & x(i+1) & \cdots & x(j) \end{bmatrix}^\top,$$

which can also be expressed as $x_{[i:j]} = \text{col}(x(i), \ldots, x(j))$.

A block Hankel matrix of depth $s \in \mathbb{Z}^+$ can be constructed from $x_{[i:j]}$ via the following defined block Hankel matrix operator as:

$$H_s(x_{[i:j]}) = \begin{bmatrix} x(i) & x(i+1) & \cdots & x(j-s) \\ x(i+1) & x(i+2) & \cdots & x(j-s+1) \\ \vdots & \vdots & \ddots & \vdots \\ x(i+s-1) & x(i+s) & \cdots & x(j) \end{bmatrix}.$$

The lag of a state-space model $(A, B, C, D)$ is denoted as $\ell(A, B, C, D)$. Given $s \in \mathbb{Z}^+$, the extended observability matrix of $(A, C)$ is denoted as

$$O_s(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{s-1} \end{bmatrix}.$$

We define the class of lower block-triangular Toeplitz matrices of depth $s$ with block entries $p \times m$ as $T_s^{p \times m}$. The lower
block-triangular Toeplitz matrix constructed from a state-space model \((A, B, C, D)\) is defined as:

\[
T_s(A, B, C, D) = \begin{bmatrix}
D & & \\
CB & D & \\
& \ddots & \ddots \\
& & CA^{n-3}B & \dotsc & D
\end{bmatrix},
\]

where \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}\) and \(D \in \mathbb{R}^{p \times m}\) have compatible dimensions. Given a vector \(\theta \triangleq \mathrm{vec}(D^T (CB)^T \dotsc (CA^{n-2}B)^T)^T \in \mathbb{R}^{p \times m}\) defined by the quadruple \((A, B, C, D)\), the mapping from \(\theta\) to a lower block-triangular Toeplitz matrix is denoted as \(T_s(\theta) = T_s(A, B, C, D)\). Given a vector \(\theta' \triangleq \mathrm{vec}((CB)^T \dotsc (CA^{n-2}B)^T)^T \in \mathbb{R}^{p \times (s-1)m}\) defined by the tuple \((A, B, C)\), the mapping from \(\theta'\) to a lower block-triangular Toeplitz matrix is denoted as \(T_s(\theta') = T_s(A, B, C, 0)\).

II. PRELIMINARIES

In this section, the Willems’ fundamental lemma will be reviewed briefly at first, followed by its applications in deterministic systems and stochastic systems.

A. A Brief Overview on MPC

A discrete-time linear time-invariant (LTI) system subject to process and measurement noise is given by:

\[
\begin{align*}
x(k+1) &= Ax(k) + Bu(k) + w(k) \\
y(k) &= Cx(k) + Du(k) + v(k)
\end{align*}
\]  

where \(x(k) \in \mathbb{R}^n\), \(u(k) \in \mathbb{R}^m\) and \(y(k) \in \mathbb{R}^p\) stand for state, input and output, respectively. \(w(k) \in \mathbb{R}^m\) and \(v(k) \in \mathbb{R}^p\) denote process and measurement noises, respectively. The following assumptions are made.

**Assumption 1:** \(w(k), v(k)\) are zero-mean white Gaussian noises with covariances \(\Sigma_w\) and \(\Sigma_v\), respectively. In addition, they are mutually independent and uncorrelated with \(u(k)\).

**Assumption 2:** The system \((A, B, C)\) is observable and \((A, B, \Sigma_{w^{1/2}})\) is controllable.

The parametric state-space equations in \((1)\) are used to predict future trajectories, thereby being a key ingredient in model-based control design. Given system matrices \((A, B, C, D)\), a standard certainty-equivalent optimal control problem is formulated as:

\[
\begin{align*}
\min_{u, \hat{y}} \mathcal{J}(u, \hat{y}) \\
\text{s.t.} \quad &\hat{x}(k+1) = A\hat{x}(k) + Bu(k), \quad k = 1, \dotsc, L_f \\
&\hat{y}(k) = C\hat{x}(k) + Du(k), \quad k = 1, \dotsc, L_f \\
&\hat{x}(1) = x_{\text{ini}} \\
&u = \text{col}(u(1), \dotsc, u(L_f)), \quad u(k) \in U_k, \quad k = 1, \dotsc, L_f \\
&\hat{y} = \text{col}(\hat{y}(1), \dotsc, \hat{y}(L_f)), \quad y(k) \in \mathcal{Y}_k, \quad k = 1, \dotsc, L_f
\end{align*}
\]

where \(U_k\) and \(\mathcal{Y}_k\) denote stage-wise input and output constraints. The objective is the standard quadratic cost function

\[
\mathcal{J}(u, \hat{y}) = \sum_{k=1}^{L_f} ||\hat{y}(k) - r(k)||_R^2 + ||u(k)||_Q^2,
\]

where \(R, Q \succ 0\) are input and output cost matrices, \(r(k)\) is a reference signal and \(L_f\) is the control horizon.

B. DeePC for Deterministic LTI Systems

In the noise-free case \(w(k) = 0, v(k) = 0\), the celebrated Willems’ fundamental lemma [8] essentially yields a data-driven non-parametric description of true system behaviour in \((1)\). More precisely, any realistic input-output trajectory recorded from the deterministic system is expressible as a linear combination of columns of the Hankel matrix, which is formed with a sufficiently rich input-output trajectory. We recall the conventional notion of persistent excitation, which characterizes data informativity underpinning the fundamental lemma.

**Definition 1 (Persistent Excitation, [8], [34]):** A sequence \(x_{[i,j]}\) is said to be persistently exciting of order \(s\) if the Hankel matrix \(H_s(x_{[i,j]})\) has full row rank.

**Theorem 1 (fundamental lemma, [8]):** Assume the LTI system \((1)\) without noise is controllable, from which an input-state-output trajectory \(\{u_d(i), x_d(i), y_d(i)\}_{i=1}^N\) is recorded offline. If \(N \geq (m+1)(L+n) - 1, \quad L \geq \ell(A, B, C, D)\) and \(u_{d,[1,N]}\) is persistently exciting of order \((L+n)\), then we have:

(a) The offline data matrix

\[
\begin{bmatrix}
X_d \\
U_d
\end{bmatrix} = \begin{bmatrix}
\mathcal{H}_1(x_{d,[1,N-L+1]}) \\
\mathcal{H}_L(u_{d,[1,N]})
\end{bmatrix}
\]

has full row rank.

(b) An input-output data trajectory \(\{u(i), y(i)\}_{i=1}^L\) is generated from the system if and only if there exists a vector \(g \in \mathbb{R}^{N-L+1}\) satisfying the following relation:

\[
\begin{bmatrix}
\mathcal{H}_L(u_{d,[1,N]}) \\
\mathcal{H}_L(y_{d,[1,N]})
\end{bmatrix} g = \begin{bmatrix}
U_d \\
Y_d
\end{bmatrix} g = \begin{bmatrix}
u_{[1:L]} \\
y_{[1:L]}
\end{bmatrix}.
\]

A practical consequence of Theorem 1 is that simulation and control can be performed solely based on raw data \(U_d\) and \(Y_d\), even if the true system \((A, B, C, D)\) is unknown [35]. To do so, available samples are partitioned into past and future sections, formally defined as:

\[
\begin{align*}
U_p &= U_d(1 : mL_p, :) \in \mathbb{R}^{mL_p \times (N-L+1)}, \\
U_f &= U_d(mL_p + 1 : mL, :) \in \mathbb{R}^{mL_f \times (N-L+1)}, \\
u_p &= u_{[1:L_p]} \in \mathbb{R}^{mL_p}, \\
u_f &= u_{[L_p+1:L]} \in \mathbb{R}^{mL_f},
\end{align*}
\]

where \(L = L_p + L_f\) and likewise for \(Y_p, Y_f, y_p, y_f\). Thus, \((5)\) is rewritten as

\[
\begin{bmatrix}
U_p \\
Y_p \\
U_f \\
Y_f
\end{bmatrix} g = \begin{bmatrix}
u_p \\
y_p \\
u_f \\
y_f
\end{bmatrix}.
\]

Then, if prerequisites in Theorem 1 are stipulated and \(L \geq \ell(A, B, C, D)\), then the future output \(y_f\) is unique given \(u_p, y_p, u_f, y_f\), which can be calculated via Algorithm 1 in [35]. Thus, it suffices to utilize raw historical data instead of an explicit
model to predict future output trajectory. This leads to direct data-driven control design as a counterpart of \((\ref{eq:2})\) \cite{9}:

\[
\begin{align*}
\min_{u_f, y_f} & \ J(u_f, y_f) \\
\text{s.t.} & \ \text{Eq. (7), } u_f \in \mathcal{U}, \ y_f \in \mathcal{Y}
\end{align*}
\tag{8}
\]

where \(\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_p, \mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_L\) are Cartesian products of sets. As compared to \((\ref{eq:2})\), a collection of data-based constraints appear in lieu of model equations, where the vector \(g\) is a decision variable making a bridge between historical data and online prediction.

\subsection*{C. Remedies for Stochastic LTI Systems}

Note that in the noise-free case, the equation \((\ref{eq:5})\) is under-determined. When data are contaminated by noise, i.e. \(w(k) \neq 0, v(k) \neq 0\), Theorem III(b) no longer holds exactly. In addition, the linear equation \((\ref{eq:7})\) is highly under-determined for \(N\) sufficiently large. In this case, solving for \(g\) becomes even ill-conditioned and a default choice is to use the Moore-Penrose inverse as a remedy:

\[
\hat{g}_{\text{pinv}}^* = \begin{bmatrix} U_p & Y_p & U_f \end{bmatrix} \begin{bmatrix} U_p \ Y_p \ U_f \end{bmatrix}^\dagger \begin{bmatrix} u_p \ y_p \ u_f \end{bmatrix},
\tag{9}
\]

which is the least-norm solution to the linear equations

\[
\begin{bmatrix} U_p & Y_p & U_f \end{bmatrix} \begin{bmatrix} g \ y_p \ u_f \end{bmatrix} = \begin{bmatrix} u_p \ y_p \ u_f \end{bmatrix}. \tag{10}
\]

This essentially gives rise to the classical Subspace Predictive Control (SPC) scheme \cite{36,37}, which relies on a data-driven predictor linear in \(\{u_p, y_p, u_f\}\):

\[
\hat{y}_f = Y_f \hat{g}_{\text{pinv}} = Y_f \begin{bmatrix} U_p & Y_p & U_f \end{bmatrix} \begin{bmatrix} u_p \ y_p \ u_f \end{bmatrix}. \tag{11}
\]

Here, we adopt \(\hat{y}_f\) in lieu of \(y_f\) to stress the extrapolating nature of the predictor. The data-based optimal control problem is then recast as:

\[
\begin{align*}
\min_{u_f, \hat{y}_f} & \ J(u_f, \hat{y}_f) \\
\text{s.t.} & \ \text{Eq. (11), } u_f \in \mathcal{U}, \ \hat{y}_f \in \mathcal{Y}.
\end{align*}
\tag{12}
\]

An interesting observation is \(R_{\text{proj}}(\hat{g}_{\text{pinv}}^*) = 0\) and thus the SPC solution \((\ref{eq:2})\) will not be penalized by \((\ref{eq:14})\). This assures consistency, that is, under perfect data there is no bias in the regularized solution to \((\ref{eq:13})\). In contrast, heterogeneous solutions to \((\ref{eq:10})\) will be penalized by norm-based regularizers, which is an unwanted effect from an identification perspective. In addition, it was pointed out by \cite{23} that problem \((\ref{eq:13})\) with \(R_{\text{proj}}(\cdot)\) can be regarded as a convex relaxation of the SPC problem \((\ref{eq:12})\) for \(\lambda\) sufficiently small.

\section{Generalized Fundamental Lemma and DeepPC Based on Noisy Data}

In this section, a generalization of Willems’ fundamental lemma for stochastic systems is developed. Based on that, a generalized DeepPC with innovation constraints is derived.

Under Assumptions 1 and 2 the state-space model \((\ref{eq:1})\) has an innovation form:

\[
\dot{x}(k+1) = Ax(k) + Bu(k) + Ke(k) \tag{16a}
\]

\[
y(k) = C\hat{x}(k) + Du(k) + e(k) \tag{16b}
\]

where \(K\) is the Kalman gain, \(\hat{x}(k)\) is the one-step forward predictor of the state \(x(k)\) in \((\ref{eq:1})\) and \(e(k) \in \mathbb{R}^p\) denotes the innovation process which is a zero-mean white noise. The following assumption is a direct consequence of Assumptions 1 and 2.

\begin{Assumption} The pair \((A, [B K])\) is controllable and the pair \((A, C)\) is observable. Meanwhile, the eigenvalues of \(A - KC\) are strictly inside the unit circle.
\end{Assumption}

\subsection*{A. Generalized Fundamental Lemma and Data-Driven Optimal Predictor}

Next, we present a generalized fundamental lemma in the context of innovation models, useful for exact predictions of future trajectories by fully utilizing innovation information. To proceed, we need to clarify the informativity condition for both inputs and innovations. Recall that if \(u_{[1:N]}\) is persistently exciting of order \(s\) and the innovation \(e(k)\) is a white noise uncorrelated with all inputs, then the stacked matrix

\[
\begin{bmatrix} H_s(u_{[1:N]}) \\ H_s(e_{[1:N]}) \end{bmatrix}
\]

has full row rank almost surely. Hence, the following assumption is stipulated by the white noise property of \(e(k)\), based on which the generalized fundamental lemma is framed.

\begin{Assumption} (Persistent Excitation): The compound input signal \(\bar{u}(k) = [u(k)^T, e(k)^T]^T\) is persistently exciting of order \((L + n)\).
\end{Assumption}

\begin{Theorem} (Generalized Fundamental Lemma): Consider the finite-dimension LTI system of innovation form \((\ref{eq:16})\). Let \(\{u_1(i), \bar{x}_d(i), y_1(i), e_d(i)\}_{i=1}^N\) is an input-state-output-innovation sequence generated from the system. Under Assumptions 3 and 4 and assuming \(N \geq (m + p + 1)(L + n) - 1\) and \(L \geq \ell(A, B, C, D)\), the following statements hold.
\begin{enumerate}[a.)]
\item The matrix
\[
\begin{bmatrix} \hat{X}_d \\ \hat{U}_d \\ \hat{E}_d \end{bmatrix} = \begin{bmatrix} H_1(u_{[1:N-L+1]}^N) \\ H_L(u_{[1:N]}^L) \\ H_L(e_{[1:N]}^L) \end{bmatrix}
\tag{17}
\]
\end{enumerate}
\end{Theorem}
has full row rank.

(b) Any input-output-innovation trajectory \( \{u(i), y(i), e(i)\}_{i=1}^{L} \) is from the system (16) if and only if a vector \( g \in \mathbb{R}^{N-L+1} \) exists, such that:
\[
\begin{bmatrix}
U_d \\
Y_d \\
E_d
\end{bmatrix} \quad g = \begin{bmatrix} u \\ y \\ e \end{bmatrix}.
\]

(c) \( \text{rank}(\begin{bmatrix} Y_d^T & U_d^T & E_d^T \end{bmatrix}^T) = n + mL + pL \).

Proof: The proofs of statements (a) and (b) are straightforward by taking \( \bar{u}(k) \) as a compound input and then invoking [38, Theorem 1] for state-space models, which are omitted here for brevity. To prove (c), it is noted that
\[
\begin{bmatrix}
0 & I & 0 \\
\Gamma_T & H_d^u & H_d^e \\
0 & I & 0
\end{bmatrix}
\begin{bmatrix}
X_d \\
U_d \\
Y_d \\
E_d
\end{bmatrix} = \begin{bmatrix}
U_d \\
Y_d \\
E_d
\end{bmatrix}
\]

where the matrix of left multiplication has full column rank due to \( L \geq \ell(A, B, C, D) \). It then follows that
\[
\text{rank}(\begin{bmatrix} Y_d^T & U_d^T & E_d^T \end{bmatrix}^T) = n + mL + pL
\]
which completes the proof.

The significance of Theorem 2 lies in that, any input-output-innovation trajectory from the stochastic system (16) can be precisely represented by raw data \( \{Y_d, U_d, E_d\} \), provided past innovation data \( E_d \) are exactly known. By contrast, the data equation (5) built solely upon input-output data essentially suffers from being more under-determined. As a meaningful derivative of the fundamental lemma, Theorem 2 sheds light on the pivotal role of innovation realizations in data-driven control of stochastic systems. In what follows, we will exploit its usage in trajectory prediction in a purely data-driven setting, by temporarily assuming exact value of \( E_d \). We first establish the optimal predictor of \( y_f \) on the basis of exactly known state-space matrices \((A, B, C, D, K)\). Similar to (6), Hankel matrices \( E_p \) and \( E_f \) can be defined by partitioning \( E_d \).

Theorem 3: Assuming \( L_p \geq \ell(A, B, C, D) \) and under Assumption 3 the following data equation holds:
\[
\begin{align*}
y_f &= \Delta_{L_p} Y_p \\
&+ \Gamma_{L_p}(s_p A_K^L \Gamma_{L_p}^u + \Delta_{L_p}^u) Y_p \\
&+ \Gamma_{L_p}(s_p A_K^L \Gamma_{L_p}^u H_{L_p}^u) U_p + H_{L_p}^u U_f \\
&- \Gamma_{L_p} A_K^L \Gamma_{L_p}^u H_{L_p}^e E_p + H_{L_p}^e E_f,
\end{align*}
\]

where
\[
\begin{align*}
H_{L_p}^u &= T_s(A, B, C, D), \\
H_{L_p}^e &= T_s(A, K, C, I), \\
\Gamma_s &= O_s(A, C), \\
A_K &= -K C, \\
B_K &= B - K D, \\
\Delta_s(A, B) &= \begin{bmatrix} A^{-1} B & \cdots & AB B \end{bmatrix}, \\
\Delta_{L_p}^u &= \Delta_s(A_K, K), \\
\Delta_{L_p}^v &= \Delta_s(A_K, B_K).
\end{align*}
\]

Given \( u_p, u_f, y_p, e_p \), the optimal unbiased prediction of \( y_f \) is given by:
\[
\hat{y}_f = \Gamma_{L_f}(s_p A_K^L \Gamma_{L_f}^u + \Delta_{L_f}^u) y_p \\
+ \Gamma_{L_f}(s_p A_K^L \Gamma_{L_f}^u H_{L_f}^u) u_p + H_{L_f}^u u_f \\
- \Gamma_{L_f} A_K^L \Gamma_{L_f}^u H_{L_f}^e e_p + H_{L_f}^e e_f.
\]

Proof: Using the extended observability matrix and Markov parameters, the following extended state-space models can be built [39]:
\[
\begin{align*}
Y_f &= \Gamma_{L_f} \hat{X}_f + H_{L_f}^e U_f, \\
Y_p &= \Gamma_{L_p} \hat{X}_p + H_{L_p}^u U_p + H_{L_p}^e E_p,
\end{align*}
\]

where
\[
\hat{X}_p = H_1(\hat{x}_{d,1,N-L_p-L+1}), \\
\hat{X}_f = H_1(\hat{x}_{d,1,N-L_p-L+1}).
\]

Because \( L_p \geq \ell(A, B, C, D) \) and \((A, C)\) is observable, \( \Gamma_{L_p} \) has full column rank, which yields the identity
\[
\hat{X}_p = \Gamma_{L_p}^u (Y_p - H_{L_p}^e U_p - H_{L_p}^e E_p).
\]

Meanwhile, by eliminating \( e(k) \) in (16a) through iterations, we obtain
\[
\hat{X}_f = \hat{A}_L^p \hat{X}_p + \Delta_{L_p}^u U_p + \Delta_{L_p}^v Y_p.
\]

Substituting (25) into (26), we arrive at
\[
\hat{X}_f = \hat{A}_L^p \hat{X}_p + \Delta_{L_p}^u U_p + \Delta_{L_p}^v Y_p
\]

Then inserting (27) into (23) yields (20), which can be extended to every admissible input-output-innovation trajectory \( \{u_p, u_f, y_p, y_f, e_p, e_f\} \) from the stochastic system (16):
\[
\begin{align*}
y_f &= \Gamma_{L_f}(s_p A_K^L \Gamma_{L_f}^u + \Delta_{L_f}^u) y_p \\
&+ \Gamma_{L_f}(s_p A_K^L \Gamma_{L_f}^u H_{L_f}^u) u_p + H_{L_f}^u u_f \\
&- \Gamma_{L_f} A_K^L \Gamma_{L_f}^u H_{L_f}^e e_f + H_{L_f}^e e_f.
\end{align*}
\]

It is evident that (22) yields the optimal unbiased predictor \( \hat{y}_f \) conditioned on \( \{u_p, u_f, y_p, e_p\} \), which completes the proof.

Now we are ready to show that, building upon Theorems 2 and 5 a data-driven optimal predictor of future trajectories emanating from the current state can be constructed by solving linear equations. This gives a clear answer to the critical question of how to derive an optimal data-driven model from a noise-corrupted dataset.

Theorem 4 (Data-driven optimal trajectory prediction):
Suppose \( L_p \geq \ell(A, B, C, D) \) and the premise underlying Theorem 2 is stipulated. Consider any past input-output-innovation trajectory and the future input sequence. If \( g \) solves the system of linear equations
\[
\begin{bmatrix}
U_p \\
Y_p \\
E_p \\
U_f \\
E_f
\end{bmatrix} = \begin{bmatrix} u_p \\ y_p \\ e_p \\ u_f \\ 0 \end{bmatrix},
\]
then the vector \( \hat{y}_f = Y_f g \) is an optimal predictor of future outputs satisfying (22). Conversely, if \( \hat{y}_f \) is the optimal predictor
conditioned on \( \{u_p, u_f, y_p, e_p\} \), there always exists a vector \( g \) such that

\[
\begin{bmatrix}
U_p \\
Y_p \\
E_p \\
U_f \\
Y_f \\
E_f
\end{bmatrix} \begin{bmatrix}
u_p \\
y_p \\
e_p \\
u_f \\
y_f \\
0
\end{bmatrix} = 0.
\] (30)

**Proof:** Multiplying (20) by \( g \) and invoking (29) yield the desired sufficiency verdict. The necessity is straightforward due to Theorem 2(b) and the fact that the trajectory \( \{u_p, u_f, y_f, e_p, 0\} \) can be generated from the true system (16).

In the stochastic setup, precise predictions for future outputs become impossible. Indeed, the multi-step forecast in (22) is the best one can make, while necessary and sufficient conditions of assuring unbiasedness in a data-dependent way are highlighted by Theorem 4.

As compared to (10) in SPC, the system of linear equations (29) contains two extra blocks of constraints, which encode innovation information and strengthen our understanding of data-driven control of stochastic systems. On the one hand, it is conceptually beneficial to add more hard/soft constraints to mitigate the noise effect. This justifies, via a different lens, the effect of regularization heuristics prevailing recent literature on DeePC. On the other hand, whenever innovation realizations are perfectly known together with input-output data, the initial state does not have to be available and the best performance becomes accessible. This also gives better insight into the performance gap between regularized DeePC and “oracle” MPC, as can be frequently evidenced from recent literature.

Additionally, the theoretical basis formed in Theorem 4 allows to analyze the rationality and pitfalls of some remedies for DeePC, including SPC [36, 37] as well as the projection-based regularizer [24]. The findings are formalized below.

**Theorem 5:** For raw data recorded from (16), both the SPC solution \( g_{pinv} \) and the optimal solution to (13) with the projection-based regularizer \( R(g) = R_{proj}(g) \) and \( \lambda \to +\infty \) satisfy

\[
\begin{bmatrix}
U_p \\
Y_p \\
U_f \\
E_f
\end{bmatrix} \begin{bmatrix}
u_p \\
y_p \\
u_f \\
0
\end{bmatrix} = 0,
\] (31)

where \( \hat{E}_f = Y_f - \hat{Y}_f = Y_f(I - \Pi) \) denotes the least-square prediction error of \( Y_f \) using \( U_p, Y_p, U_f \).

**Proof:** It can be easily seen that both solutions satisfy linear equations (10). In addition, for the former SPC solution, it holds that

\[
\hat{E}_f g_{pinv}^* = Y_f(I - \Pi) \begin{bmatrix}
U_p \\
Y_p \\
U_f
\end{bmatrix}^\dagger \begin{bmatrix}
u_p \\
y_p \\
u_f
\end{bmatrix} = 0.
\] (32)

For the latter regularized solution, \( \hat{E}_f g^* = 0 \) is due to the enforced constraint \( (I - \Pi)g^* = 0 \). Summarizing these arguments yields the result.

**Remark 1:** In view of (20), \( \hat{E}_f \) essentially yields an approximation of residuals \( H_{L_f}^T E_f - \Gamma_{L_f} A_K^T \Gamma_{L_p}^T H_{L_p}^T E_p \) in the least-square sense. As a result, the constraint \( \hat{E}_f g = 0 \) ensured by both solutions can be conceived as imposing a necessary condition of \( E_p g = 0 \) and \( E_f g = 0 \). This can exert, more or less, certain shrinkage onto \( E_f g \), which justifies the effect of these remedial formulations. On the other side, the lack of sufficiency to ensure both \( E_p g = e_p \) and \( E_f g = 0 \) in (29) makes the solution \( g \) less constrained and inflates the variance in trajectory predictions. In particular, the equality \( E_p g = e_p \) involving innovation realizations \( e_p \) is ignored, which could be a main cause for performance gaps with respect to the oracle solution.

**Remark 2:** Another subtle issue within (20) is that a simple least-square fit of \( Y_f \) using \( U_p, Y_p, U_f \) substantially encloses extra and unnecessary parameters in the lower block-triangular matrix \( H_{L_f}^T \), thereby failing to preserve the causality. Previous analysis has shown that even if there exists guarantee on asymptotic efficiency of estimation, such non-causality can cause high variance and over-fitting under moderate sample size [40, 41]. As a consequence, both SPC and the regularized solution are prone to a similar issue.

### B. DeePC with Innovation Constraints

Based on the optimal predictor given in Theorem 4, a generalized DeePC formulation for stochastic LTI systems is motivated as a data-driven counterpart of (2). Specifically, the optimization problem is expressed as:

\[
\begin{align*}
\min_{u_f, \hat{y}_f, g} & \quad J(u_f, \hat{y}_f) \\
\text{s.t.} & \quad \text{Eq. (30)}, \ u_f \in U, \ \hat{y}_f \in Y.
\end{align*}
\] (33)

It can be easily concluded that solving (33) and (2) with known system matrices yields equivalent control performance under certain assumptions.

**Corollary 1 (Equivalence in Optimal Control Performance):** Consider the stochastic system (16). Under the assumptions behind Theorem 4, the model-based problem (2) with exactly known initial state \( x_{init} \) and the data-based problem (33) with exact innovation data have the same optimal control decision \( u_f^* \) and optimal output prediction \( \hat{y}_f^* \).

Next, we shed light on some desired properties of the receding horizon control (RHC) scheme by iteratively solving (33). The implementation procedure is summarized below.

- **Step 1.** By solving (33), the optimal control sequence and predicted output \( \{u(k + j), \hat{y}(k + j)\}_{j=0}^{L_f - 1} \) are obtained at time \( k \).
- **Step 2.** Implement the control action \( u(k) \) to steer the system.
- **Step 3.** Sample the output \( y(k) \) and update \( e_p \). Move to the next instance \( k \to k + 1 \) and repeat the procedure from Step 1 using updated \( u_p, y_p \) and \( e_p \). Notice there is an identity matrix in the first block of \( H_{L_f}^T \) in (28). So, if \( \hat{y}_f = Y_f g \) is an optimal unbiased predictor (assuming \( \hat{E}_d \) and \( e_p \) exactly known), the innovation

\[
\begin{align*}
e_f(1) & = y_f(1) - \hat{y}_f(1) \\
& = y(k) - \hat{y}_f(1)
\end{align*}
\] (34)
can be precisely computed upon observing $y(k)$ at the current snapshot. This yields a moving-window update of past innovation:

$$e_p \triangleq \left[ e_p^{[2:L_p]} \right]$$

which is still exact at the next instance. Loosely speaking, if initialized optimally, the entire moving horizon scheme will always yield optimal control decisions at future sampling times, thereby manifesting recursive optimality even without knowing the true system state. Such a property is formalized below.

**Corollary 2 (Recursive Optimality):** Under the assumptions behind Theorem 4 if the RHC scheme based on (33) is initialized with exact $E_d$ and $e_p$ at $k = 0$, the updated innovation vector $e_p$ in (35) remains exact for all $k \in \mathbb{Z}^+$ and the resulting control decision $u_f^*$ remains optimal.

We now turn our attention to more practical circumstances with inexact innovation information. Unlike inputs and outputs that can be directly measured, innovations have to be estimated from input-output data. Thus, by replacing $\{E_d, e_p\}$ with their imperfect estimates $\{\hat{E}_d, \hat{e}_p\}$, the optimal control problem becomes:

$$\min_{u_f, \hat{y}_f} \mathcal{J}(u_f, \hat{y}_f) \quad \text{s.t.} \quad g = \begin{bmatrix} U_p \\ Y_p \\ \hat{E}_d \\ U_f \\ Y_f \\ \hat{E}_f \end{bmatrix} \begin{bmatrix} u_p \\ y_p \\ \hat{e}_p \\ u_f \\ \hat{y}_f \\ 0 \end{bmatrix}, \quad u_f \in \mathcal{U}, \ \hat{y}_f \in \mathcal{Y}. \quad (36)$$

Note that the rule (35) still holds promise in online updating $\hat{e}_p$ recursively, while both $\hat{E}_p$ and $\hat{E}_f$ shall be estimated in a preliminary stage. Hence, we arrive at a holistic data-driven framework for implementing the proposed DeePC in a moving horizon setting, as profiled in Fig. 1. The entire schematic consists of an offline phase and the online RHC phase, where the former is devoted to actively disentangling innovations from raw historical data $\{U_d, Y_d\}$ without knowing system matrices. This can be achieved by leveraging a variety of tools from subspace identification, which will be detailed in the next section.

Before closing this section, we discuss an important robust variant arising from (36). Due to the imperfection in $\hat{E}_p$, $\hat{E}_f$ and $\hat{e}_p$, it is risky to enforce related linear constraints, which may incur infeasibility problems. In this regard, robustification can be made to put up with infeasible solutions, which has been a popular choice in data-driven control, see e.g. [13], [19], [22], [42]. To adopt a similar idea in the concerned problem, it is reasonable to introduce auxiliary slack variables for innovation-based constraints and regularize them in the cost function. This stimulates a pragmatic relaxation of problem (36):

$$\min_{u_f, \hat{y}_f, \hat{e}_p, \delta_f} \mathcal{J}(u_f, \hat{y}_f) + \lambda_p \|\delta_p\|^2 + \lambda_f \|\delta_f\|^2$$

s.t. $g = \begin{bmatrix} U_p \\ Y_p \\ \hat{E}_d \\ U_f \\ Y_f \\ \hat{E}_f \end{bmatrix} \begin{bmatrix} u_p \\ y_p \\ \hat{e}_p + \delta_p \\ u_f \\ \hat{y}_f \\ \delta_f \end{bmatrix}, \quad u_f \in \mathcal{U}, \ \hat{y}_f \in \mathcal{Y}. \quad (37)$$

where $\lambda_p > 0$ and $\lambda_f > 0$ are tuning parameters. Such a regularization heuristic is shown to admit a min-max reformulation that minimizes the worst-case performance under bounded uncertainty [13], [22], [42]. It is also interesting to see that even if $y(k)$ is contaminated by noise, the above constraints regarding input and output data are not relaxed. This is essentially due to the exploration of innovation information, differing vastly from usual regularized forms of DeePC where the output constraint $y_p g = y_p$ is robustified.

**IV. LOW-RANK SUBSPACE IDENTIFICATION FOR INNOVATION ESTIMATION**

Given exact innovation information, the best control performance is achievable. This hints at the immense value of attaining high-precision innovation estimates prior to online control, particularly in narrowing the gap between DeePC and the oracle solution. Oriented by this, we revisit relevant methods in subspace identification and propose an efficient algorithm towards innovation estimation.

**A. Innovation-Based Subspace Identification**

In subspace identification methods (SIMs), the focus is primarily concentrated on estimating model parameters...
(A, B, C, D) [43], whereas estimates of innovations are more like an intermediate or a by-product of the identification routine. A streamline of SIMs much related to innovation estimation is the two-step procedure for closed-loop identification [30], [31], whereby Markov parameters of a Kalman predictor are first estimated using a vector auto-regressive with exogenous input (VARX) model and then parameters are retrieved via a model reduction procedure [44]. Specifically, the fitting error of a VARX model gives estimates of innovations. To see this, we convert (16) into an equivalent observer form:

\[
\begin{cases}
\hat{x}(k + 1) = A_K \hat{x}(k) + B_K u(k) + K y(k) \\
y(k) = C \hat{x}(k) + Du(k) + e(k).
\end{cases}
\] (38)

Using standard recursions, one obtains:

\[
\hat{x}(k) = A_K^0 \hat{x}(k - \rho) + D u(k) + e(k),
\]

where \(A_K^0 \hat{x}(k - \rho) \approx 0\) for a sufficiently large \(\rho \in \mathbb{Z}^+\) due to Assumption 3. It immediately follows that

\[
y(k) = \Phi(y)(y(k-[\rho-1, k-1])) + \Phi_u u(k-[\rho-1, k-1]) + D u(k) + e(k),
\]

where \(\Phi(y) \triangleq C \Delta y_u \Phi_u \triangleq C \Delta y_u\) are block matrices encompassing Markov parameters of the observer [38]. It can be further arranged in a matrix form:

\[
\mathcal{H}_1(y_{1:N}) \approx \Phi(y) \mathcal{H}_p(y_{1-[\rho-1:N-1]}) + \Phi_u \mathcal{H}_p(u_{1-[\rho-1:N-1]}) + D \mathcal{H}_u(y_{1:N}) + \mathcal{H}_e(e_{1:N}).
\] (40)

To estimate Markov parameters \(\{\Phi, \Phi_u\}\) and \(D\), one can solve the following least-squares regression (LSR) [31], [44]:

\[
\min_{\Phi, \Phi_u, D} \left\| \mathcal{H}_1(y_{1:N}) - \left[ \begin{array}{c} \Phi(y) \\ \Phi_u \\ D \end{array} \right] \left[ \begin{array}{c} \mathcal{H}_p(y_{1-[\rho-1:N-1]}) \\ \mathcal{H}_p(u_{1-[\rho-1:N-1]}) \\ \mathcal{H}_u(y_{1:N}) \end{array} \right] \right\|_F^2.
\] (41)

In this article, the residuals of (41) are of particular interest since they yield an explicit approximation of \(\mathcal{H}_1(e_{1:N})\), based on which \(\tilde{E}_d\) can be readily constructed [45]. This is a simple and convenient strategy for innovation estimation that can be integrated with the proposed DeePC scheme. In [31], it was found that residuals of fitting a VARX model is less sensitive to ill-conditioning than parameter estimates themselves, which reveals, from a different perspective, the rationality of the proposed DeePC scheme with innovation estimation. In order to reduce approximation errors, a large \(\rho > n\) is generally needed in LSR, which stipulates extra input-output data \(\{u(1-\rho), y(1-\rho), \ldots, u(-1), y(-1), u(0), y(0)\}\) preceding \(\{U_d, Y_d\}\).

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B. Low-Rank Subspace Identification Approach

Note that in LSR, there is no restriction on the “complexity” of the observer realized with Markov parameters in the VARX model. It is known that a low-order stochastic system is associated with a low-complexity Kalman filter. This has inspired N2SID [32], an improved formulation of subspace identification that features a trade-off between minimizing the one-step prediction error and finding an observer of lowest order. The nuclear norm regularization is used as a convex relaxation of rank constraints to enforce a low-order model structure, which alleviates over-fitting and improves identification accuracy. Besides, N2SID enjoys the freedom from using instrumental variables and input-output data in addition to \(\{U_d, Y_d\}\), which certainly provides a viable option for innovation estimation. Yet, in order to strike a desirable balance, one has to judiciously select the regularization parameter. Meanwhile, the nuclear norm may shrink non-zero dominating eigenvalues, exerting a negative impact on the prediction performance.

Driven by the pursuit of high-fidelity estimates, we seek to address above issues and present the LRSID algorithm, which avoids unnecessary shrinkage effect on non-zero eigenvalues. Another merit is that the usage of regularization parameter can be avoided and it only needs knowledge of an upper-bound of system order.

Based on the observer form (38), a well-known structured data equation alternative to (40) is given by:

\[
Y_d = \Gamma_L \hat{X}_d + \Theta_u U_d + \Theta_y Y_d + E_d,
\] (42)

where \(\Gamma_L = \mathcal{O}_L(A_K, C)\) is the extended observability matrix of (38) and \(\Theta_u = \mathcal{T}_L(A_K, B_K, C, D), \Theta_y = \mathcal{T}_L(A_K, K, C, 0)\) are lower block-triangular Toeplitz matrices. We remember that \(\hat{X}_d = \mathcal{H}_1(\hat{x}_{d,[1:N-L+1]}), U_d\) and \(Y_d\) are block Hankel matrices defined in (45) and \(E_d\) is defined analogously.

The data equation (42) is highly structured. In particular, when \(N \geq (m+p+1)(L+n) - 1\) and \(L \geq \ell(A, B, C, D)\), as assumed in Theorem 4, the term \(\Gamma_L \hat{X}_d\) is rank-deficient, i.e.

\[
\text{rank}(\Gamma_L \hat{X}_d) = n.
\] (43)

Then the question is how to derive a simple regression model based on (42) while respecting the low-rank constraint (43). By defining the output prediction

\[
\hat{y}_d = \text{col}(\hat{y}_d(1), \hat{y}_d(2), \ldots, \hat{y}_d(N))
\]

\[
\approx y_d - e_d,
\]

the data equation (42) can be translated to

\[
\hat{Y}_d \triangleq \mathcal{H}_L(\hat{y}_d) = Y_d - E_d
\]

\[
= \Gamma_L \hat{X}_d + \Theta_u U_d + \Theta_y Y_d.
\] (45)

Based on this, a prediction error minimization problem subject to low-rank constraint is developed for innovation estimation:

\[
\min_{\hat{y}_d, Z, \Theta_u, \Theta_y} \frac{1}{2} \|y_d - \hat{y}_d\|^2
\]

s.t. \(\mathcal{H}_L(\hat{y}_d) = Z + \Theta_u U_d + \Theta_y Y_d\) where \(Z \in \mathcal{M}_n\), \(\Theta_u \in \mathcal{T}^{p,m}_L\), \(\Theta_y \in \mathcal{T}^{p,p}_L\)

\[
\mathcal{M}_n = \{ Z \in \mathbb{R}^{p \times (N-L+1)} : \text{rank}(Z) = n \}.
\]
Then the augmented Lagrangian function of (47) can be
entails the explicit knowledge about the system order
but unnecessarily shrinks the rest, which compromise the
Moreover, the nuclear norm sets certain eigenvalues to zero
rank constraint in (47) is dropped by adding the nuclear

c L_n. 

Remark 3: In the problem formulation of N2SID [32], the
rank constraint in (47) is dropped by adding the nuclear
norm penalty in the cost function. Despite a convex relaxation
that is computationally favorable, it requires tedious tuning of regularization parameter across a widespread range.
Moreover, the nuclear norm sets certain eigenvalues to zero
but unnecessarily shrinks the rest, which compromise the
prediction performance. By contrast, our formulation
entails the explicit knowledge about the system order n. When n
itself is unknown but an upper-bound n_u > n is available,
one can still solve (47) by replacing n with n_u. As to be
empirically shown later, the solution still exhibits a desirable
accuracy if the upper-bound is not too conservative. Indeed,
knowing an upper-bound is not restrictive because it is also
inherently dictated by Assumption 4 to ensure the persistency of excitation in DeePC.

C. A Tailored ADMM Algorithm

Rank minimization is essentially \textit{NP}-hard [46], so off-the-
shelves numerical solvers are no longer applicable to problem
in LRSID. Next, we present a tailored ADMM solution
algorithm, which is a modification of the routine for
N2SID [47]. For ease of reproduction, we have made available
the source code of ADMM at \url{https://github.com/wangy1bo-png/LRSID}.

For brevity, let us define the linear operator:

\[
Q(\hat{y}_d, \theta_u, \theta_y) = H_L(\hat{y}_d) - T_L(\theta_u) U_d - T'_L(\theta_y) Y_d. \tag{48}
\]

Then the augmented Lagrangian function of (47) can be
written as:

\[
\mathcal{L}(\hat{y}_d, Z, \theta_u, \theta_y, \Lambda) = \frac{1}{2}||\hat{y}_d - \hat{y}_d||^2 - (\Lambda, Q(\hat{y}_d, \theta_u, \theta_y) - Z) + \frac{\beta}{2}||Q(\hat{y}_d, \theta_u, \theta_y) - Z||_F^2 , \tag{49}
\]

where \( \Lambda \) is a dual variable and \( \beta \) is a positive penalty param-
eter. The ADMM routine iterates between minimizing \( \mathcal{L} \) over \( Z \)
subject to the low-rank constraint, minimizing of \( \mathcal{L} \) over
\( \{\hat{y}_d, \theta_u, \theta_y\} \) and a simple update for \( \Lambda \). The implementation
procedure is detailed in Algorithm 1 where each step admits
an analytic solution. It is noteworthy that to warm-start the
algorithm, a number of well-established SIMs are available,
such as PBSID [30], N4SID [48], MOESP [49], to name only a few.

Optimization over \( Z \): Note that the augmented Lagrangian

\[
\textbf{Algorithm 1 ADMM for Low-Rank Subspace Identification}
\]

\textbf{Input}: Input \( u_d \) and output \( y_d \), penalty parameter \( \beta \) and its
maximum \( \beta_{\text{max}} \). Upper-bound \( n_u \) of system order. Hyper-
parameters \( \tau > 0, \eta > 1 \).

\textbf{Output}: Estimated innovation sequence \( \hat{e}_d = y_d - \hat{y}_d \).

1: Construct \( U_d \) and \( Y_d \). Use off-the-shelf SIMs to initialize \( \theta_u^0, \theta_y^0 \) and \( \hat{y}_d^0 \).
2: \textbf{repeat}
3: Update \( Z \) via (50).
4: Update \( \theta_u, \theta_y \) and \( \hat{y}_d \) via (53).
5: Update \( \Lambda \) via (54).
6: Update \( \beta := \min(\eta/\beta, \beta_{\text{max}}) \);
7: \textbf{until} convergence

\( \mathcal{L} \) is quadratic in \( Z \). Then we have:

\[
Z^* = \arg \min_{Z \in \mathcal{M}_n} \mathcal{L}(\hat{y}_d, Z, \theta_u, \theta_y, \Lambda)
= \arg \min_{Z \in \mathcal{M}_n} \frac{1}{2}||Z||_F^2 - \left(Z, Q(\hat{y}_d, \theta_u, \theta_y) - \Lambda \right) + \frac{\beta}{2}||Z||_F^2 . \tag{50}
\]

According to [50], the optimal solution is obtained by truncat-
ing the singular values, which gives the update of \( Z \) in the
ith iteration:

\[
Z^{i+1} = U^i \text{diag}\{\sigma_1, \ldots, \sigma_n, 0, \ldots, 0\}(V^i)^T \tag{51}
\]

based on the singular value decomposition:

\[
Q(\theta_u^i, \theta_y^i, \hat{y}_d^i) - Z^i = U^i \Sigma^i (V^i)^T , \tag{52}
\]

where \( \{\sigma_1, \ldots, \sigma_n\} \) are the largest \( n \) singular values in the
diagonal matrix \( \Sigma^i \).

Optimization over \( \{\hat{y}_d, \theta_u, \theta_y\} \): We define \( \theta = \text{col}(\hat{y}_d, \theta_u, \theta_y) \) as the collection of all non-constrained primal variables.
Due to the existence of Hankel and Toeplitz matrices, the operator \( Q \) affine in \( \theta \) is intricate and it is difficult
to explicitly write out related coefficient matrices and vectors.
The fast Fourier transform (FFT) technique, a trick useful for
accelerating matrix computations in SIM literature [47], [51],
[52], is leveraged to address this issue. Eventually, minimizing
over \( \theta \) yields the following least-squares update:

\[
\theta^{i+1} = \begin{bmatrix} \hat{y}_d^{i+1} \\ \theta_u^{i+1} \\ \theta_y^{i+1} \end{bmatrix} = (M_u + \beta M)^{-1}[Q^*(\Lambda^i + \beta Z^i) + b] , \tag{53}
\]

where a self-contained derivation of (53) and detailed expres-
sions of \( M, M_u, Q^*(\cdot) \) and \( b \) can be found in the Appendix.

Optimization over \( \Lambda \): In each step, the dual variable is
updated using gradient ascent:

\[
\Lambda^{i+1} = \Lambda^i - \tau \beta [Q(\hat{y}_d^{i+1}, \theta_u^{i+1}, \theta_y^{i+1}) - Z^{i+1}] , \tag{54}
\]

where \( \tau > 0 \) is a step size parameter.
V. NUMERICAL STUDIES

In this section, we carry out simulated case studies to demonstrate the efficacy of the proposed innovation estimation approach to DeePC of stochastic systems, as well as the LRSID method.

A. Simulation Setup

Consider the stochastic LTI system (1) with:

\[
A = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix},
C = \begin{bmatrix} 0 & 1.4114 \end{bmatrix}, \quad D = 0,
\]

where \(n = 2\), \(m = p = 1\), \(w(k) \sim \mathcal{N}(0, \sigma_w^2 I_2)\) and \(v(k) \sim \mathcal{N}(0, \sigma_v^2 I_2)\). A square wave with a period of 100 and amplitude of 2, contaminated by a zero-mean white noise with variance 0.01, is used as the persistently exciting input \(u(k)\) in an open-loop condition. To recover the ground-truth of \(e(k)\), we first calculate \(K\) by solving the Riccati equation and then perform state estimation with the Kalman filter (16). Having \(e(k)\) at hand, the signal-to-noise ratio (SNR) is computed to quantify the noise level of the dataset:

\[
\text{SNR} = 10 \log_{10} \frac{\text{var}(y(k))}{\text{var}(e(k))}.
\]

By varying \(\sigma_w\) and \(\sigma_v\) proportionally, three cases are created that correspond to low, medium and high noise levels, i.e. SNR = 20, 30, 40dB. Under each level of SNR, an offline dataset \(\{U_d, Y_d\}\) is generated, where size parameters of Hankel matrices are \(L_p = 20, L_f = 25, N = 140\).

B. Offline Innovation Estimation Performance

We investigate the performance of the following innovation estimation approaches.

1) **LRSID**: The proposed low-rank approach assuming an exact system order of \(n = 2\). Hyper-parameters in ADMM are chosen as \(\beta = 1, \eta = 1.1, \tau = 1\) and \(\beta_{\text{max}} = 1 \times 10^6\). The whole procedure is initialized with PBSID [30].

2) **LRSIDu**: LRSID with known upper-bound \(n_u = 4\). The rest settings are identical to LRSID.

3) **N2SID** [47]: The best regularization parameter is selected from a grid of values. The rest setup follows the default in [47].

4) **LSR** [45]: The simple method by solving the least squares model [41] with \(\rho = 15\) and extra data samples prior to \(\{U_d, Y_d\}\).

The following index is defined to evaluate the estimation accuracy:

\[
\text{error} = \frac{||\hat{E}_d - E_d||_F}{||E_d||_F} \times 100\%,
\]

where the Hankel matrix \(\hat{E}_d\) is constructed from the estimates of four approaches and the ground-truth \(E_d\) stems from the model-based Kalman filter.

For a comprehensive evaluation, 100 Monte Carlo simulations are carried out under each level of SNR, based on which innovations are estimated using above methods and the performance is evaluated with (57). The results are visualized in Fig. 2.

![Fig. 2. Comparison of innovation estimation with different approaches at different noise level with 100 simulations](image)

in Fig. 2 It can be seen that N2SID even cannot catch up with the trivial LSR, while the proposed LRSID yields the most accurate estimation. This reveals the negative effect of nuclear norm penalty as well as the benefit of low-rank constraints in innovation estimation. It is noticeable that LRSIDu merely shows a slight decrease in accuracy based on an upper-bound, while still outperforming other two methods. This implies that the proposed LRSID can still be useful when an upper-bound of system order is know but not too conservative.

C. Online Predictive Control Performance

Next, we close the loop in an RHC manner based on the following control strategies.

1) **Oracle**: The MPC scheme by solving (2) with perfect information, which assumes known system matrices \((A, B, C, D)\) and full measurement of \(x(k)\). It yields the best performance that is achievable in an RHC scheme and thus acts as a benchmark.
2) **LRSID-PC**: The proposed DeePC scheme by solving (37), where \( \hat{E}_d \) is estimated with LRSID.

3) **LRSIDu-PC**: The proposed DeePC scheme by solving (37), where \( \hat{E}_d \) is estimated with LRSIDu.

4) **N2SID-PC**: The proposed DeePC scheme by solving (37), where \( \hat{E}_d \) is estimated with N2SID.

5) **LSR-PC**: The proposed DeePC scheme by solving (37), where \( \hat{E}_d \) is estimated with LSR.

6) **SPC** [36], [37]: The classical SPC scheme as described in [12].

7) **PBR** [24]: Regularized DeePC in (13) with PBR (14), where \( \lambda \) is optimally selected from a grid of values within \([10^{-3}, 10^1]\).

In the RHS schemes of DeePC with innovation constraints, \( \hat{Y}_p \) is initialized to zero and updated iteratively according to (35). The input and output cost matrices are chosen as \( Q = 1 \) and \( R = 0.01 \). Meanwhile, we set \( \mathcal{U} = \mathbb{R}^{m_L} \) and \( \mathcal{Y} = \mathbb{R}^{p_L} \). For a comprehensive assessment of control performance, 100 Monte Carlo runs are carried out. In each run, a sinusoidal signal \( \sin(2\pi t/N_c) \), \( t = 1, 2, ..., N_c \) is specified as the reference to be tracked, where \( N_c = 60 \) is the simulation duration. The control performance is quantified by the following cost function:

\[
J_{ctrl} = \sum_{k=1}^{N_c} \left| \sum_{k=1}^{N_c} \frac{u(k)}{2} + \sum_{k=1}^{N_c} \frac{y(k)}{2} \right|^2, \tag{58}
\]

where \( u(k) \) is the control decision made by each control strategy and \( y(k) \) denotes the stimulated true output. All optimal control problems are successfully solved using the OSQP package [53] via the YALMIP interface [54].

The closed-loop control results of different approaches are summarized in Fig. 3 and Table I. It can be seen that when combined with either innovation estimate, the proposed DeePC scheme performs remarkably better than the generic PBR and SPC methods and much more closely to the oracle solution under all noise levels. Under a higher noise level, the superiority of the proposed DeePC scheme becomes more pronounced, as confirmed by Table I. This underlines the prominent value of innovation information in enhancing data-driven control performance for stochastic systems. Furthermore, LRSID-PC achieves the best performance amongst four innovation-dependent DeePC methods, which substantially owes to a high offline estimation accuracy. Hence, it is worthwhile to attain high-precision estimates offline as long as the computational cost is affordable. Another empirical observation is that the performance of LRSIDu-PC is competitive with that of LRSID-PC, which shows the applicability of the proposed LRSID method when only knowing an upper-bound of \( n \).

### D. Effect of Regularization

The above results regarding the proposed DeePC scheme are based on [36] without regularization. Next, we investigate the effect of the robustified version (37) of LRSID-PC by varying \( \{\lambda_p, \lambda_f\} \). Besides, the conventional regularized DeePC [9] is considered:

\[
\min_{g, u_f, \hat{y}_f, \delta_y} \mathcal{J}(u_f, \hat{y}_f) + \lambda_g ||g||^2 + \lambda_p ||\delta_y||^2
\]

\[
s.t. \begin{bmatrix} U_p \ Y_p \ U_f \ Y_f \end{bmatrix} = \begin{bmatrix} u_p \ y_p + \delta_y \ y_f \\ u_f \ \hat{y}_f \end{bmatrix}, \ u_f \in \mathcal{U}, \ \hat{y}_f \in \mathcal{Y}, \tag{59}
\]

where two regularizers are added based on \( g \) and the residual of \( Y_p g = y_p \). We repeat the control experiment in the previous subsection under the high noise level (SNR = 20dB) by trying every regularization parameter from a grid of values in \([10^{-4}, 10^4]\). The induced landscapes of control performance index are portrayed in Fig. 3. It can be seen that for our regularized problem [37], there is a flat-floored basin in the cost function where a satisfactory control performance is acquirable. This showcases the ease of calibrating \( \lambda_p \) and...


\[ \frac{1}{N} H^H \text{diag}(F_h \hat{y}_d)G_h, \]

\[ \frac{1}{N} F_h^H \text{diag}(H_hZG_h^H), \]

---

**TABLE I**

| SNR = 20dB | SNR = 30dB | SNR = 40dB |
|------------|------------|------------|
| \( J_u \)  | \( J_y \)  | \( J_u \)  | \( J_y \)  | \( J_u \)  | \( J_y \)  |
| Oracle     | 0.96 ± 0.04 | 0.75 ± 0.15 | 0.96 ± 0.04 | 0.27 ± 0.06 | 0.95 ± 0.02 | 0.18 ± 0.02 |
| LRSID-PC   | 0.98 ± 0.08 | 0.94 ± 0.30 | 0.96 ± 0.05 | 0.42 ± 0.55 | 0.95 ± 0.02 | 0.20 ± 0.04 |
| N2SID-PC   | 1.33 ± 0.11 | 1.30 ± 0.46 | 1.08 ± 0.18 | 0.59 ± 0.17 | 1.01 ± 0.03 | 0.34 ± 0.05 |
| LSR-PC     | 1.29 ± 0.43 | 1.49 ± 0.36 | 1.08 ± 0.18 | 0.55 ± 0.36 | 0.99 ± 0.09 | 0.26 ± 0.14 |
| PBR        | 0.62 ± 0.33 | 6.79 ± 5.23 | 0.57 ± 0.27 | 5.57 ± 4.55 | 0.85 ± 0.24 | 1.73 ± 1.86 |
| SPC        | 1.28 ± 2.35 | 16.32 ± 50.48 | 0.73 ± 0.69 | 9.61 ± 26.18 | 1.03 ± 0.44 | 2.91 ± 5.98 |

---

**VI. CONCLUSIONS**

In this paper, we presented a novel data-driven control framework for stochastic systems through active learning of innovation realizations. A generalized fundamental lemma was put forward for data-driven representation of system behaviour, which then motivated a model-free optimal output predictor in the stochastic setting. On this basis, a new DeePC formulation with additional innovation constraints was derived. Given exact realizations innovation, the resulting control behaviour proves to coincide with the oracle model-based control with perfect information. This highlights the particular value of attaining high-precision innovation estimates offline. Driven by such implication, an efficient LRSID algorithm was developed that shows better accuracy than known subspace identification methods. Numerical simulations demonstrated significant advantages of the proposed innovation-based DeePC scheme over state-of-the-art data-driven methods in the presence of noisy data. When combined with the proposed LRSID for innovation estimation, the new DeePC scheme led to a superior control performance that is closest to the oracle performance. Therefore, attaining high-fidelity innovation estimates offline eventually pays off in online data-driven control of stochastic systems.

We mention a few directions that deserve future investigation. In the case study, we have made an incorrect yet practical initialization of proposed DeePC scheme by setting \( \hat{\epsilon}_p = 0 \). Nevertheless, the resulting performance is desirable and not too far away from the oracle. Thus, closed-loop stability and robustness of the proposed scheme is an interesting and open problem. Meanwhile, based on the regularized version (37), other stochastic and robust variants are worth attempting and their connections to known stochastic MPC and robust MPC algorithms are to be explored.

---

**APPENDIX**

We shall first clarify some notations needed in the Appendix. The set of complex numbers is denoted by \( \mathbb{C} \). Given a matrix \( X, X(i : j, k : l) \) denotes the submatrix that contains the entries in \( X \) from the \( i \)th row to the \( j \)th row and from the \( k \)th column to the \( l \)th column, and \( X(j : -1 : i, k : l) \) denotes the submatrix obtained by taking rows \( j \) through \( i \) in reverse order. The conjugate matrix of a matrix \( Z \in \mathbb{C}^{m \times n} \) is indicated by either \( \bar{Z} \) or \( Z^H \). \( X \odot Y \) and \( X \otimes Y \) denote, respectively, the Hadamard product and the Kronecker product of two matrices.

With the aid of FFT, the block Hankel matrix operator over the vector \( \hat{y}_d \) and its adjoint over matrix \( Z \in \mathbb{R}^{p_L \times (N-L+1)} \) can be expressed as [55]:

\[ \mathcal{H}_T(\hat{y}_d) = \frac{1}{N} H^H \text{diag}(F_h \hat{y}_d)G_h, \]

\[ \mathcal{H}_T^*(Z) = \frac{1}{N} F_h^H \text{diag}(H_hZG_h^H), \]
where
\[ H_h = F^N (:, (N - L + 1) : N) \otimes I_n, \quad F_h = F^N \otimes I_n, \]
\[ G_h = F^N (:, N - L + 1 : -1 : 1) \otimes I_n, \]
and \( F^N \in \mathbb{C}^{N \times N} \) denotes the FFT matrix of size \( N \).

The mappings \( T_L(\theta_u) \) and \( T_L^*(\theta_y) \) and the corresponding adjoint operators can be expressed as [56]:
\[
\begin{align*}
T_L(\theta_u) &= \frac{1}{2L-1} H_{t,u}^H \text{diag}(F_{t,u} \theta_u) G_{t,u}, \\
T_L^*(\theta_y) &= \frac{1}{2L-1} H_{t,y}^H \text{diag}(F_{t,y} \theta_y) G_{t,y}, \\
T_L^*(Z) &= \frac{1}{2L-1} H_{t,y}^H \text{diag}(H_{t,y} Z G_{t,y}^H), \\
T_L^{*,*}(Z) &= \frac{1}{2L-1} H_{t,y}^H \text{diag}(H_{t,y} Z G_{t,y}^H),
\end{align*}
\]

where
\[
\begin{align*}
H_{t,u} &= 1_m \otimes (F^L_{-1}(:, L : 2L - 1) \otimes I_n), \\
H_{t,y} &= 1_p \otimes (F^L_{-1}(:, L : 2L - 1) \otimes I_n), \\
F_{t,u} &= I_m \otimes (F^L_{-1}(:, L : 2L - 1) \otimes I_n), \\
F_{t,y} &= I_p \otimes (F^L_{-1}(:, L : 2L - 1) \otimes I_n), \\
G_{t,u} &= \sum_{i=1}^m i_1^m \otimes ((F^L_{-1}(:, 1 : L) \otimes I_n) \otimes (1_{m_1}^m)^\top), \\
G_{t,y} &= \sum_{i=1}^p i_1^p \otimes ((F^L_{-1}(:, 1 : L) \otimes I_n) \otimes (1_{i_1}^p)^\top).
\end{align*}
\]

Thus, the adjoint of \( Q \) is expressed as:
\[
Q^*(Z) = \begin{bmatrix}
-\frac{1}{2L-1} F_{t,u}^H \text{diag}(H_{t,y} Z G_{t,y}^H) \\
\frac{1}{2L-1} F_{t,u}^H \text{diag}(H_{t,y} Z U_d Y_d G_{t,y}^H) \\
\end{bmatrix}.
\]

Another important ingredient is the positive semi-definite matrix \( M \), which is defined by:
\[
Q^*(Q(\theta)) = M \theta. \]

It turns out that \( M \) has a block structure:
\[
M = \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{12} & M_{22} & M_{23} \\
M_{13} & M_{23} & M_{33}
\end{bmatrix},
\]
\[
M_{11} = \frac{1}{N^2} F_h^H ((H_h H_h^H) \otimes \text{diag}(G_h G_h^H)) F_h,
\]
\[
M_{12} = -\frac{1}{N(2L-1)} F_h^H ((H_h H_h^H) \otimes \text{diag}(G_h U_d Y_d G_h^H)) F_{t,u},
\]
\[
M_{13} = -\frac{1}{N(2L-1)} F_h^H ((H_h H_h^H) \otimes \text{diag}(G_h Y_d G_h^H)) F_{t,y},
\]
\[
M_{22} = \frac{1}{(2L-1)^2} F_{t,u}^H ((H_{t,u} H_{t,u}^H) \otimes \text{diag}(G_{t,u} U_d Y_d G_{t,u}^H)) F_{t,u},
\]
\[
M_{23} = \frac{1}{(2L-1)^2} F_{t,u}^H ((H_{t,u} H_{t,u}^H) \otimes \text{diag}(G_{t,u} U_d Y_d G_{t,u}^H)) F_{t,y},
\]
\[
M_{33} = \frac{1}{(2L-1)^2} F_{t,y}^H ((H_{t,y} H_{t,y}^H) \otimes \text{diag}(G_{t,y} Y_d G_{t,y}^H)) F_{t,y}.
\]

Then we have
\[
\theta^* = \arg \min_\theta \frac{1}{2} \begin{bmatrix}
\hat{y}_d^\top & \hat{y}_d & \hat{y}_d - \langle Q(\hat{y}_d, \theta_u, \theta_y), \beta Z + \Lambda \rangle & + \frac{\beta}{2} \langle Q(\hat{y}_d, \theta_u, \theta_y), Q(\hat{y}_d, \theta_u, \theta_y) \rangle
\end{bmatrix} \theta
\]
\[
= \arg \min_\theta \frac{1}{2} \theta^\top (M_\theta + \beta M) \theta - [Q^*(\Lambda + \beta Z) + b]^\top \theta
\]

where
\[
M_\theta = \begin{bmatrix} I_N & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} y_d \end{bmatrix}.
\]

This yields the least-squares solution [53].

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