Criterion of Bari basis property for $2 \times 2$ Dirac-type operators with strictly regular boundary conditions

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Abstract
The paper is concerned with the Bari basis property of a boundary value problem associated in $L^2([0,1];\mathbb{C}^2)$ with the following $2 \times 2$ Dirac-type equation for $y = \text{col}(y_1, y_2)$:

$$L_u(Q)y = -iB^{-1}y' + Q(x)y = \lambda y, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad b_1 < 0 < b_2,$$

with a potential matrix $Q \in L^2([0,1];\mathbb{C}^{2\times2})$ and subject to the strictly regular boundary conditions $U y :={U_1, U_2}y = 0$. If $b_2 = -b_1 = 1$, this equation is equivalent to one-dimensional Dirac equation. We show that the normalized system of root vectors $\{f_n\}_{n \in \mathbb{Z}}$ of the operator $L_u(Q)$ is a Bari basis in $L^2([0,1];\mathbb{C}^2)$ if and only if the unperturbed operator $L_u(0)$ is self-adjoint. We also give explicit conditions for this in terms of coefficients in the boundary conditions. The Bari basis criterion is a consequence of our more general result: Let $Q \in L^p([0,1];\mathbb{C}^{2\times2})$, $p \in [1,2]$, boundary conditions be strictly regular, and let $\{g_n\}_{n \in \mathbb{Z}}$ be the sequence biorthogonal to the normalized system of root vectors $\{f_n\}_{n \in \mathbb{Z}}$ of the operator $L_u(Q)$. Then,

$$\{\|f_n - g_n\|_2\}_{n \in \mathbb{Z}} \in (\ell^p(\mathbb{Z}))^* \iff L_u(0) = L_u(0)^*.$$

These abstract results are applied to noncanonical initial-boundary value problem for a damped string equation.

KEYWORDS
Bari basis property, damped string equation, Dirac-type systems, equidistribution theorem, regular and strictly regular boundary conditions

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1 | INTRODUCTION

Consider the following first-order system of differential equations:

$$L y = -iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, y_2), \quad x \in [0,1],$$

(1.1)
where

\[ B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad b_1 < 0 < b_2 \quad \text{and} \quad Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix} \in L^2([0,1]; \mathbb{C}^{2 \times 2}). \tag{1.2} \]

If \( B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \), system (1.1) is equivalent to the Dirac system (see the classical monographs [24, 37]).

Let us associate linearly independent boundary conditions

\[ U_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + a_{j3}y_1(1) + a_{j4}y_2(1) = 0, \quad j \in \{1, 2\}, \tag{1.3} \]

with system (1.1), and denote as \( L_U(Q) := L_U(Q) \) an operator, associated in \( \mathcal{S} := L^2([0,1]; \mathbb{C}^2) \) with the boundary value problem (BVP) (1.1)–(1.3). It is defined by differential expression \( L \) on the domain

\[ \text{dom}(L_U(Q)) = \{ f \in AC([0,1]; \mathbb{C}^2) : Lf \in \mathcal{S}, U_1(f) = U_2(f) = 0 \}. \tag{1.4} \]

To the best of our knowledge, the spectral properties of the general \( n \times n \) system of the form (1.1) with a nonsingular diagonal \( n \times n \) matrix \( B \) with complex entries and a potential matrix \( Q(\cdot) \) of the form

\[ B = \text{diag}(b_1, \ldots, b_n) \in \mathbb{C}^{n \times n} \quad \text{and} \quad Q(\cdot) = (q_{j,k}(\cdot))_{j,k=1}^n \in C^1([0,1]; \mathbb{C}^{n \times n}) \tag{1.5} \]

have first been investigated by G.D. Birkhoff and R.E. Langer [7]. Namely, they introduced the concepts of regular and strictly regular boundary conditions (1.3) and investigated the asymptotic behavior of eigenvalues and eigenfunctions of the corresponding operator \( L_U(Q) \). Moreover, they proved a pointwise convergence result on spectral decompositions of the operator \( L_U(Q) \) corresponding to the BVP (1.1)–(1.3) with regular boundary conditions.

The completeness property in \( L^2([0,1]; \mathbb{C}^n) \) of the system of root vectors of BVP for the general \( n \times n \) system of the form (1.1) with matrices \( B = \text{diag}(b_1, \ldots, b_n) \) and \( Q \in L^1([0,1]; \mathbb{C}^{n \times n}) \) was established for the first time by M.M. Malamud and L.L. Oridoroga in [35, 36] for a wide class of BVPs, although for the \( 2 \times 2 \) Dirac system, it was proved earlier by V.A. Marchenko (see [37, Chapter 1.3]). As a development of [35, 36], in [1, 2, 26, 27], completeness conditions for nonregular and even degenerate boundary conditions were found with applications to dissipative and normal operators. In the joint paper [27], the authors and M.M. Malamud also established the Riesz basis property with parentheses of the system of root vectors for different classes of BVPs for the \( n \times n \) system with arbitrary \( B \) of the form (1.5) and \( Q \in L^\infty([0,1]; \mathbb{C}^{n \times n}) \). Note also that BVP for the \( 2m \times 2m \) Dirac equation (\( B = \text{diag}(\pm I_m, I_m) \)) was investigated in [39] (Bari–Markus property for Dirichlet BVP with \( Q \in L^2([0,1]; \mathbb{C}^{2m \times 2m}) \)) and in [22, 23] (Bessel and Riesz basis properties on abstract level).

The Riesz basis property in \( L^2([0,1]; \mathbb{C}^2) \) of BVP (1.1)–(1.3), that is, of the operator \( L_U(Q) \), for the \( 2 \times 2 \) Dirac system \((b_2 = -b_1 = 1)\) with various assumptions on the potential matrix \( Q \) was investigated in numerous papers (see [5, 11–13, 20, 25, 28, 43, 50, 51] and references therein). The case of separated boundary conditions and \( Q \in C^1([0,1]; \mathbb{C}^{n \times n}) \) was treated first by P. Djakov and B. Mityagin [11] and later by A. Baskakov, A. Derbushev, and A. Shcherbakov [5]. Shortly after, P. Djakov and B. Mityagin [13] extended these results to general regular boundary conditions.

The most complete result on the Riesz basis property for \( 2 \times 2 \) Dirac and Dirac-type systems, respectively, with \( Q \in L^1([0,1]; \mathbb{C}^{2 \times 2}) \) and strictly regular boundary conditions was obtained independently by different methods and at the same time by A.M. Savchuk and A.A. Shkalikov [43] and by the author and M.M. Malamud [25, 28] (in [25] the sketches of the proof are given). The case of regular boundary conditions is treated in [43] for the first time. Other proofs were obtained later on in [28, 44, 45] (see also recent survey [46] and references therein).

In [28], results for the Dirac operator were also applied to the Timoshenko beam model. In general, one can show that dynamic generators of many physical models governed by systems of linear partial differential equation (PDE) are similar to certain first-order differential operators. Hence spectral properties of such operators are of significant importance in the study of stability of solutions and corresponding \( C_0 \)-semigroups of many physical models. In particular, in Section 7, we establish an explicit connection between \( 2 \times 2 \) Dirac-type operators (with \( b_1 \neq -b_2 \)) on the one hand and certain noncanonical initial-BVP for a damped string on the other hand. This allows us to apply results on the Bari and Riesz basis property (see Definition 1.2) for Dirac-type operators obtained here and in [28] to the damped string equation.
Recall (see [28, section 5]), that boundary conditions (1.3) are regular, if and only if they are equivalent to the boundary conditions

\[ \hat{U}_1(y) = y_1(0) + by_2(0) + a y_1(1) = 0, \quad \hat{U}_2(y) = dy_2(0) + cy_1(1) + y_2(1) = 0, \]  

(1.6)

with certain \( a, b, c, d \in \mathbb{C} \) satisfying \( ad - bc \neq 0 \). Recall also that regular boundary conditions (1.3) are called strictly regular, if and only if \( \{(a - d)^2 \neq -4bc\} \) is true. In particular, antiperiodic \( (a = d = 0, b = c = 1) \) boundary conditions are regular but not strictly regular for Dirac system, while they become strictly regular for the Dirac-type system if \( -b_1, b_2 \in \mathbb{N} \) and \( b_2 - b_1 \) is odd.

Note in this connection that periodic and antiperiodic (necessarily nonstrictly regular) BVP for \( 2 \times 2 \) Dirac and Sturm–Liouville equations have also attracted certain attention during the last decade. For instance, a criterion for the system of root vectors of the periodic BVP for \( 2 \times 2 \) Dirac equation to contain a Riesz basis (without parentheses!) was obtained by P. Djakov and B. Mityagin in [12] (see also recent papers [31, 32] by A.S. Makin and the references therein). It is also worth mentioning that F. Gesztesy and V. Tkachenko [15, 16] for \( q \in L^2[0, \pi] \) and P. Djakov and B. Mityagin [12] for \( q \in W^{-1,2}[0, \pi] \) established by different methods a criterion for the system of root vectors to contain a Riesz basis for Sturm–Liouville operator \( -d^2/dx^2 + q(x) \) on \( [0, \pi] \) (see also survey [30]).

Let us emphasize that the proof of the Riesz basis property in [5, 11, 13, 39] substantially relies on the Bari–Markus property: the quadratic closeness in \( L^2([0,1]; \mathbb{C}^2) \) of the spectral projectors of the operators \( L_{U}(Q) \) and \( L_{U}(0) \). Assuming boundary conditions to be strictly regular, let \( \{f_n\}_{n \in \mathbb{Z}} \) and \( \{f_n^0\}_{n \in \mathbb{Z}} \) be the systems of root vectors of the operators \( L_{U}(Q) \) and \( L_{U}(0) \), respectively. Then, Bari–Markus property states the implication: \( Q \in L^2 \Rightarrow \sum_{n \in \mathbb{Z}} \|f_n - f_n^0\|_2^2 < \infty \). Later, this property was generalized to the case \( Q \in L^p([0,1]; \mathbb{C}^{2 \times 2}) \), \( p \in [1,2] \), in [19, 29, 41–43]. The most complete results in this direction were established in the joint paper [29] by the author and M.M. Malamud. One of these results reads as follows.

**Theorem 1.1** (Theorem 7.15 in [29]). Let \( \mathcal{K} \in L^p([0,1]; C^{2 \times 2}) \) be a compact set for some \( p \in [1,2] \), let \( Q, \tilde{Q} \in \mathcal{K} \) and boundary conditions (1.3) be strictly regular. Then, for some normalized systems of root vectors \( \{f_n\}_{n \in \mathbb{Z}} \) and \( \{f_n^0\}_{n \in \mathbb{Z}} \) of the operators \( L_{U}(Q) \) and \( L_{U}(\tilde{Q}) \), the following uniform relations hold for \( Q, \tilde{Q} \in \mathcal{K} \):

\[
\sum_{|n|>N} \|f_n - \tilde{f}_n\|_p \leq C\|Q - \tilde{Q}\|_p, \quad p \in (1,2], \quad 1/p' + 1/p = 1, \quad (1.7)
\]

\[
\sum_{|n|>N} (1 + |n|)^{p-2} \|f_n - \tilde{f}_n\|_\infty \leq C\|Q - \tilde{Q}\|_p, \quad p \in (1,2], \quad (1.8)
\]

\[
\lim_{n \to \infty} \sup_{Q, \tilde{Q} \in \mathcal{K}} \|f_n - \tilde{f}_n\|_\infty = 0, \quad p = 1. \quad (1.9)
\]

Here and throughout the paper, we denote by \( \|f\|_s \), the \( L^s \)-norm of the element \( f \) of a scalar, vector, or matrix \( L^s \)-space.

Emphasize, that the proof of the estimates (1.7)–(1.8) is based on the deep Carleson–Hunt theorem. Note, however, that these estimates with \( \| \cdot \|_p \)-norm instead of \( \| \cdot \|_\infty \)-norm can be proved in a more direct way, which is elementary in character. Note also that these results substantially rely on transformation operators method that goes back to [28, 33, 34].

Recall that the concepts of Riesz bases and bases quadratically close to the orthonormal bases were introduced by N.K. Bari in [4]. Results of this fundamental paper can also be found in the classical monograph [17] where a basis quadratically close to the orthonormal basis is called a Bari basis. Let us recall the definition of Riesz and Bari bases following [17, section IV].

**Definition 1.2.** (i) A sequence of vectors \( \{f_n\}_{n \in \mathbb{Z}} \) in a separable Hilbert space \( \mathcal{H} \) is called a **Riesz basis** if it admits a representation \( f_n = T e_n, n \in \mathbb{N} \), where \( \{e_n\}_{n \in \mathbb{Z}} \) is an orthonormal basis in \( \mathcal{H} \) and \( T : \mathcal{H} \to \mathcal{H} \) is a bounded operator with bounded inverse.
(ii) A sequence of vectors \( \{ f_n \}_{n \in \mathbb{Z}} \) in a separable Hilbert space \( \mathcal{H} \) is called a **Bari basis** if it is quadratically close to some orthonormal basis \( \{ e_n \}_{n \in \mathbb{Z}} \) in \( \mathcal{H} \), that is,

\[
\sum_{n \in \mathbb{Z}} \| f_n - e_n \|_\mathcal{H}^2 < \infty.
\] (1.10)

Bases of subspaces with the property similar to (1.10) were studied in detail by A.S. Markus [38]. Bari basis property for different classes of differential operators was studied in [3, 8, 52]. Note, however, that to the best of our knowledge, the question of whether a system of root vectors of the operator \( L_U(Q) \) forms a Bari basis has not been studied before. Namely, results of papers [5, 11, 13, 25, 29, 39, 43] in the case of \( Q \in \mathcal{L}^2 \) and strictly regular boundary conditions establish quadratic closeness of systems of root vectors \( \{ f_n \}_{n \in \mathbb{Z}} \) and \( \{ f_0^n \}_{n \in \mathbb{Z}} \), but whether \( \{ f_n \}_{n \in \mathbb{Z}} \) is quadratically close to some orthonormal basis \( \{ e_n \}_{n \in \mathbb{Z}} \) remained an open question. The goal of this paper is to close this gap. One of our main results establishes the criterion for the system of root vectors of the operator \( L_U(Q) \) to form a Bari basis and reads as follows.

**Theorem 1.3.** Let boundary conditions (1.6) be strictly regular and let \( Q \in \mathcal{L}^2([0,1];\mathbb{C}^{2 \times 2}) \). Then, some normalized system of root vectors of the operator \( L_U(Q) \) is a Bari basis in \( \mathcal{L}^2([0,1];\mathbb{C}^2) \) if and only if the operator \( L_U(0) \) is self-adjoint. The latter holds if and only if the coefficients \( a, b, c, d \) in boundary conditions (1.6) satisfy the following relations:

\[
|a|^2 + \beta |b|^2 = 1, \quad |c|^2 + \beta |d|^2 = \beta, \quad a\overline{c} + \beta b\overline{d} = 0, \quad \beta := -b_2/b_1 > 0. \tag{1.11}
\]

In this case, every normalized system of root vectors of the operator \( L_U(Q) \) is a Bari basis in \( \mathcal{L}^2([0,1];\mathbb{C}^2) \).

Combining Theorem 1.3 with the results of the previous papers [5, 11, 13, 25, 29, 39, 43] concerning the Riesz basis property, we get the following surprising result.

**Corollary 1.4.** Let \( Q \in \mathcal{L}^2([0,1];\mathbb{C}^{2 \times 2}) \) and let boundary conditions (1.6) be strictly regular but not self-adjoint, that is, condition (1.11) does not hold. Then, each normalized system of root vectors of the operator \( L_U(Q) \) is a **Riesz basis but not a Bari basis** in \( \mathcal{L}^2([0,1];\mathbb{C}^2) \).

To demonstrate Corollary 1.4, let us list some explicit examples of strictly regular but not self-adjoint boundary conditions (1.6) (see Remark 3.7 for more examples of strictly regular boundary conditions):

1. Separated non–self-adjoint conditions: \( a = d = 0, bc \neq 0 \), and either \( \beta |b|^2 \neq 1 \) or \( |c|^2 \neq \beta \). Such conditions are always strictly regular.
2. Quasi-periodic conditions \( (b = c = 0 \text{ and } ad \neq 0) \) satisfying \( |d| \neq |a|^\beta \). The last condition implies both strictly regularity (see conditions (3.17) and (3.19) below) and non–self-adjointness, which is possible only if \( |a| = |d| = 1 \).
3. Boundary conditions with \( b = 0 \) and \( acd \neq 0 \) are always non–self-adjoint since \( a\overline{c} + \beta b\overline{d} = a\overline{c} \neq 0 \). They are strictly regular whenever \( |d| \neq |a|^\beta \) (see conditions (3.17) and (3.19) below for criterion of strict regularity). Similar effect happens when \( c = 0 \) and \( abd \neq 0 \).
4. Similarly, boundary conditions with \( a = 0 \) and \( bcd \neq 0 \) are always non–self-adjoint. In the case of Dirac operator \( (-b_1 = b_2 = 1) \), they are strictly regular if and only if \( d^2 \neq -4bc \). For the general Dirac-type operator \( (-b_1 \neq b_2) \), their strict regularity is much harder problem (see conditions (3.18) and (3.20) below for criterion of strict regularity in some cases).

Corollary 1.4 implies that in each of the listed cases normalized system of root vectors of the operator \( L_U(Q) \) is a Riesz basis but not a Bari basis in \( \mathcal{L}^2([0,1];\mathbb{C}^2) \).

## 2 DEFINITIONS AND FORMULATIONS OF THE MAIN RESULTS

Let us recall the following abstract criterion for Bari basis property.

...
Proposition 2.1 [17, Theorem VI.3.2]. A complete system \( \mathfrak{S} = \{f_n\}_{n \in \mathbb{Z}} \) of unit vectors in a separable Hilbert space \( \mathfrak{H} \) forms a Bari basis if and only if there exists a sequence \( \{g_n\}_{n \in \mathbb{Z}} \) biorthogonal to \( \mathfrak{S} \) that is quadratically close to \( \mathfrak{S} \):

\[
\sum_{n \in \mathbb{Z}} \|f_n - g_n\|_\mathfrak{H}^2 \leq \infty, \quad (f_n, g_m)_\mathfrak{H} = \delta_{nm}, \quad n, m \in \mathbb{Z}. \tag{2.1}
\]

Based on this abstract criterion, we will introduce a generalization of Bari basis concept. Let \( p \in [1, 2] \) and \( p' = p/(p - 1) \in [2, \infty] \). It is well known that for the dual space of \( \ell^p := \ell^p(\mathbb{Z}) \), we have

\[
(\ell^p(\mathbb{Z}))^* \cong \ell^{p'}(\mathbb{Z}), \quad p \in (1, 2], \quad \text{and} \quad (\ell^2(\mathbb{Z}))^* \cong c_0(\mathbb{Z}). \tag{2.2}
\]

For simplicity, we identify \((\ell^p(\mathbb{Z}))^* \) with \( \ell^p(\mathbb{Z}) \) for \( p \in (1, 2] \) and with \( c_0(\mathbb{Z}) \) for \( p = 1 \), respectively. For example, \( \{a_n\}_{n \in \mathbb{Z}} \in (\ell^p(\mathbb{Z}))^* \) for \( p > 1 \) means that \( \sum_{n \in \mathbb{Z}} |a_n|^{p'} < \infty \). With this in mind, we can extend Definition 1.2(ii) using equivalence from Proposition 2.1 to a more general concept of closeness of sequences \( \{f_n\}_{n \in \mathbb{Z}} \) and \( \{g_n\}_{n \in \mathbb{Z}} \).

Definition 2.2. Let \( p \in [1, 2] \), let \( \mathfrak{S} := \{f_n\}_{n \in \mathbb{Z}} \) be a complete minimal sequence of unit vectors in a separable Hilbert space \( \mathfrak{H} \) and let \( \mathfrak{S} := \{g_n\}_{n \in \mathbb{Z}} \) be its (unique) biorthogonal sequence: \( (f_n, g_m)_\mathfrak{H} = \delta_{nm}, n, m \in \mathbb{Z} \). A sequence \( \mathfrak{S} \) is called a Bari \((\ell^p)^*\)-sequence if it is \((\ell^p)^*\)-close” to its biorthogonal sequence \( \mathfrak{S} \), that is, \( \left\{ \|f_n - g_n\|_\mathfrak{S} \right\}_{n \in \mathbb{Z}} \in (\ell^p)^* \). In view of (2.2) it means that

\[
\sum_{n \in \mathbb{Z}} \|f_n - g_n\|_\mathfrak{S}^p < \infty \quad \text{if} \quad p \in (1, 2], \quad \text{and} \quad \lim_{n \to \infty} \|f_n - g_n\|_\mathfrak{S} = 0 \quad \text{if} \quad p = 1. \tag{2.3}
\]

For brevity, we will call Bari \((\ell^1)^*\)-sequence as Bari \(c_0\)-sequence and Bari \((\ell^p)^*\)-sequence as Bari \(\ell^{p'}\)-sequence for \( p \in (1, 2] \).

Proposition 2.1 implies that the notion of Bari \(\ell^2\)-sequence coincides with the notion of Bari basis. Note also that every Bari \((\ell^p)^*\)-sequence is Bari \(c_0\)-sequence. We specifically chose the word “sequence” because it is not clear if Bari \(c_0\)-property from Definition 2.2 is not equivalent to more conventional formulation of \(c_0\)-closeness of \(\{f_n\}_{n \in \mathbb{Z}}\) to a certain orthonormal basis \(\{e_n\}_{n \in \mathbb{Z}}\) even if \(\{f_n\}_{n \in \mathbb{Z}}\) is already a Riesz basis.

Remark 2.3. Note that Bari \(c_0\)-property from Definition 2.2 is not equivalent to more conventional formulation of \(c_0\)-closeness of \(\{f_n\}_{n \in \mathbb{Z}}\) to a certain orthonormal basis \(\{e_n\}_{n \in \mathbb{Z}}\) even if \(\{f_n\}_{n \in \mathbb{Z}}\) is already a Riesz basis. Indeed, in this case, \(f_n = e_n + K e_n\), where \(K\) and \((I + K)^{-1}\) are bounded operators in \(\mathfrak{H}\). Hence, \(\|f_n - e_n\|_\mathfrak{S} = \|K e_n\|_\mathfrak{S}\). It is easily seen that \(g_n = ((I + K)^{-1})^* e_n = e_n - (I + K)^{-1} K^* e_n\), and hence \(\|g_n - e_n\| \to 0\) as \(n \to \infty\) is equivalent to \(\|K^* e_n\| \to 0\) as \(n \to \infty\). If \(K\) is not compact, then \(\lim_{n \to \infty} \|K e_n\| = 0\) is generally not equivalent to \(\lim_{n \to \infty} \|K^* e_n\| = 0\) for a given orthonormal basis \(\{e_n\}_{n \in \mathbb{Z}}\).

Let us also recall the notion of the system of root vectors of an operator with compact resolvent. First, we recall a few basic facts regarding the eigenvalues of a compact, linear operator \(T \in B(\mathfrak{H})\) in a separable complex Hilbert space \(\mathfrak{H}\). The geometric multiplicity, \(m_g(\lambda_0, T)\), of an eigenvalue \(\lambda_0 \in \sigma_p(T)\) of \(T\) is given by \(m_g(\lambda_0, T) := \dim(\ker(T - \lambda_0))\).

The root subspace of \(T\) corresponding to \(\lambda_0 \in \sigma_p(T)\) is given by

\[
R_{\lambda_0}(T) = \{f \in \mathfrak{H} : (T - \lambda_0)^k f = 0 \text{ for some } k \in \mathbb{N}\}. \tag{2.4}
\]

Elements of \(R_{\lambda_0}(T)\) are called root vectors. For \(\lambda_0 \in \sigma_p(T)\), \(\{0\}\), the set \(R_{\lambda_0}(T)\) is a closed linear subspace of \(\mathfrak{H}\) whose dimension equals the algebraic multiplicity, \(m_a(\lambda_0, T)\), of \(\lambda_0, m_a(\lambda_0, T) := \dim(R_{\lambda_0}(T)) < \infty\).

Assuming for simplicity that \(0 \notin \sigma_p(T)\), denote by \(\{\lambda_j\}_{j=1}^{n_j}\) the sequence of (nonzero) eigenvalues of \(T\) and let \(n_j\) be the algebraic multiplicity of \(\lambda_j\). By the system of root vectors of the operator \(T\), we mean any sequence of the form \(\bigcup_{j=1}^{\infty} \{e_{j,k}\}_{k=1}^{n_j}\), where \(\{e_{j,k}\}_{k=1}^{n_j}\) is a basis in \(R_{\lambda_j}(T), n_j = m_a(\lambda_j, T) < \infty\). The system or root vectors of the operator \(T\) is called normalized if \(\|e_{j,k}\|_\mathfrak{H} = 1, j \in \mathbb{N}, k \in \{1, \ldots, n_j\}\).

We are particularly interested in the case where \(A\) is a densely defined, closed, linear operator in \(\mathfrak{H}\) whose resolvent is compact, that is, \(R_{\lambda}(\lambda) := (A - \lambda)^{-1} \in \mathcal{B}_c(\mathfrak{H}), \lambda \in \rho(A)\). Clearly, \(0 \notin \sigma_p(R_{\lambda}(\lambda))\). Via the spectral mapping theorem, all eigenvalues of \(A\) correspond to eigenvalues of its resolvent \(R_{\lambda}(\lambda), \lambda \in \rho(A)\), and vice versa. Hence, we use the same
notions of root vectors, root subspaces, geometric and algebraic multiplicities associated with the eigenvalues of \( A \), and the system of root vectors of \( A \).

Now we are ready to formulate the main result of this paper, which involves notions of Bari \((\ell^p)^*\)-sequences and \( c_0 \)-sequences from Definition 2.2.

**Theorem 2.4.** Let boundary conditions (1.6) be strictly regular and let \( Q \in L^p([0,1];\mathbb{C}^{2\times2}) \) for some \( p \in [1, 2] \). Then, some normalized system of root vectors of the operator \( L_U(Q) \) is a Bari \((\ell^p)^*\)-sequence in \( L^2([0,1];\mathbb{C}^2) \) if and only if the operator \( L_U(0) \) is self-adjoint, that is, when relations (1.11) hold for the coefficients \( a, b, c, d \) in boundary conditions (1.6). In this case, every normalized system of root vectors of the operator \( L_U(Q) \) is a Bari \((\ell^p)^*\)-sequence in \( L^2([0,1];\mathbb{C}^2) \).

As an immediate consequence of Theorem 2.4, we get Theorem 1.3: the criterion of Bari basis property for Dirac-type operator \( L_U(Q) \) with \( L^2 \)-potential and strictly regular boundary conditions.

Let us briefly comment on the proof of our main result, Theorem 2.4. First, we apply Theorem 1.1 to reduce the Bari \((\ell^p)^*\)-property of the system of root vectors of operator \( L_U(Q) \) with strictly regular boundary conditions to a certain explicit condition in terms of the eigenvalues \( \{\lambda_n^0\}_{n \in \mathbb{Z}} \) of the operator \( L_U(0) \), which reads as follows for the case \( p = 1 \).

**Proposition 2.5.** Let \( Q \in L^1([0,1];\mathbb{C}^{2\times2}) \) and boundary conditions (1.3) be strictly regular. Then, some normalized systems of root vectors \( \{f_n\}_{n \in \mathbb{Z}} \) of the operator \( L_U(Q) \) is a Bari \( c_0 \)-sequence in \( L^2([0,1];\mathbb{C}^2) \) if and only if:

\[
|b_1|c| + b_2|b| = 0, \quad \lim_{n \to \infty} \text{Im}\lambda_n^0 = 0 \quad \text{and} \quad \lim_{n \to \infty} z_n = |bc|,
\]

where

\[ z_n := \left(1 + d \exp(-ib_2\lambda_n^0)\right)\left(1 + a \exp(ib_1\lambda_n^0)\right), \]

and \( \{\lambda_n^0\}_{n \in \mathbb{Z}} \) is the sequence of the eigenvalues of the operator \( L_U(0) \), counting multiplicity.

With condition (2.5) established, the main difficulty arises in reducing this condition to the desired explicit condition (1.11). In this connection, recall that the sequence \( \{\lambda_n^0\}_{n \in \mathbb{Z}} \) of the eigenvalues of the operator \( L_U(0) \) coincides with the sequence of zeros of characteristic determinant

\[ \Delta_0(\lambda) = d + ae^{ib_1\lambda} + (ad - bc)e^{ib_2\lambda} + e^{ib_2\lambda}. \]

If \( b_2/b_1 \in \mathbb{Q} \) (where \( \mathbb{Q} \) is the set of rational numbers), then the sequence \( \{\lambda_n^0\}_{n \in \mathbb{Z}} \) has a simple explicit form: It is the union of arithmetic progression that lies on the lines parallel to the real axis, which simplifies the problem a lot.

The case \( b_2/b_1 \notin \mathbb{Q} \) is much more complicated. Namely, if \( |a| + |d| > 0 \) and \( bc \neq 0 \), there is no explicit description of the spectrum of the operator \( L_U(0) \). Nevertheless, we were able to establish equivalence of (2.5) and (1.11) using Weyl’s equidistribution theorem (see [49, Theorem 4.2.2.1]). It implies the following crucial property of zeros of \( \Delta_0(\cdot) \).

**Proposition 2.6.** Let \( b_2/b_1 \notin \mathbb{Q} \) and boundary conditions (1.6) be regular, that is, \( ad - bc \neq 0 \). Let \( \{\lambda_n^0\}_{n \in \mathbb{Z}} \) be the sequence of zeros of the characteristic determinant \( \Delta_0(\cdot) \) counting multiplicity. Then, each of the sequences \( \{\exp(ib_1\lambda_n^0\})_{n \in \mathbb{Z}} \) and \( \{\exp(ib_2\lambda_n^0\})_{n \in \mathbb{Z}} \) has an infinite set of limit points.

This result was key for proving equivalence of (2.5) and (1.11), which in turn implies our main result, Theorem 2.4, and its main corollary, Theorem 1.3.

### 3 Regular and Strictly Regular Boundary Conditions

In this section, we recall known properties of BVP (1.1)–(1.3) subject to regular or strictly regular boundary conditions from [28]. Let us set

\[
A := \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}, \quad A_{jk} := \begin{pmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{pmatrix}, \quad J_{jk} := \det(A_{jk}), \quad j, k \in \{1, \ldots, 4\}.
\]
Let
\[
\Phi(\cdot, \lambda) = \begin{pmatrix}
\varphi_{11}(\cdot, \lambda) & \varphi_{12}(\cdot, \lambda) \\
\varphi_{21}(\cdot, \lambda) & \varphi_{22}(\cdot, \lambda)
\end{pmatrix} =: \begin{pmatrix}
\Phi_1(\cdot, \lambda) & \Phi_2(\cdot, \lambda)
\end{pmatrix}, \quad \Phi(0, \lambda) = I_2,
\] (3.2)
be a fundamental matrix solution of the system (1.1), where \(I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Here, \(\varphi_k(\cdot, \lambda)\) is the \(k\)th column of \(\Phi(\cdot, \lambda)\).

The eigenvalues of the problem (1.1)–(1.3) counting multiplicity are the zeros (counting multiplicity) of the characteristic determinant
\[
\Delta_0(\lambda) := \det \begin{pmatrix}
U_1(\Phi_1(\cdot, \lambda)) & U_1(\Phi_2(\cdot, \lambda)) \\
U_2(\Phi_1(\cdot, \lambda)) & U_2(\Phi_2(\cdot, \lambda))
\end{pmatrix}. \tag{3.3}
\]

Inserting (3.2) and (1.3) into (3.3), setting \(\varphi_{jk}(\lambda) := \varphi_{jk}(1, \lambda)\), and taking notations (3.1) into account, we arrive at the following expression for the characteristic determinant
\[
\Delta_0(\lambda) = J_{12} + J_{34} e^{i(b_1 + b_2)\lambda} + J_{32} \varphi_{11}(\lambda) + J_{14} \varphi_{12}(\lambda) + J_{42} \varphi_{21}(\lambda) + J_{13} \varphi_{22}(\lambda). \tag{3.4}
\]

If \(Q = 0\), we denote a fundamental matrix solution as \(\Phi^0(\cdot, \lambda)\). Clearly

\[
\Phi^0(x, \lambda) = \begin{pmatrix}
e^{ib_1x\lambda} & 0 \\
0 & e^{ib_2x\lambda}
\end{pmatrix} =: \begin{pmatrix}
\varphi^0_{11}(x, \lambda) & \varphi^0_{12}(x, \lambda) \\
\varphi^0_{21}(x, \lambda) & \varphi^0_{22}(x, \lambda)
\end{pmatrix} =: \begin{pmatrix}
\Phi^0_1(x, \lambda) & \Phi^0_2(x, \lambda)
\end{pmatrix}, \quad x \in [0, 1], \ \lambda \in \mathbb{C}. \tag{3.5}
\]

Here, \(\Phi^0_k(\cdot, \lambda)\) is the \(k\)th column of \(\Phi^0(\cdot, \lambda)\). In particular, the characteristic determinant \(\Delta_0(\cdot)\) becomes
\[
\Delta_0(\lambda) = J_{12} + J_{34} e^{i(b_1 + b_2)\lambda} + J_{32} e^{ib_1\lambda} + J_{14} e^{ib_2\lambda}. \tag{3.6}
\]

In the case of Dirac system \((B = \text{diag}(-1, 1))\), this formula is simplified to
\[
\Delta_0(\lambda) = J_{12} + J_{34} + J_{32} e^{-i\lambda} + J_{14} e^{i\lambda}. \tag{3.7}
\]

Let us recall the definition of regular boundary conditions.

**Definition 3.1.** Boundary conditions (1.3) are called **regular** if
\[
J_{14} J_{32} \neq 0. \tag{3.8}
\]

Let us recall one more definition (cf. [21]).

**Definition 3.2.** Let \(\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}}\) be a sequence of complex numbers. It is called **incompressible** if for some \(d \in \mathbb{N}\), every rectangle \([t - 1, t + 1] \times \mathbb{R} \subset \mathbb{C}\) contains at most \(d\) entries of the sequence, that is,
\[
\text{card} \{n \in \mathbb{Z} : |\text{Re} \lambda_n - t| \leq 1\} \leq d, \quad t \in \mathbb{R}. \tag{3.9}
\]

Recall that \(D_r(z) \subset \mathbb{C}\) denotes the disc of radius \(r\) with a center \(z\).

Let us recall certain important properties from [28] of the characteristic determinant \(\Delta(\cdot)\) in the case of regular boundary conditions.

**Proposition 3.3** [28, Proposition 4.6]. Let the boundary conditions (1.3) be regular. Then, the characteristic determinant \(\Delta_0(\cdot)\) of the problem (1.1)–(1.3) given by (3.4) has infinitely many zeros \(\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}}\) counting multiplicities and
\[
|\text{Im} \lambda_n| \leq h, \quad n \in \mathbb{Z}, \quad \text{for some } h \geq 0. \tag{3.10}
\]

Moreover, the sequence \(\Lambda\) is incompressible and can be ordered in such a way that the following asymptotical formula holds:
\[
\text{Re} \lambda_n = \frac{2\pi n}{b_2 - b_1}(1 + o(1)) \quad \text{as } n \to \infty. \tag{3.11}
\]
Clearly, the conclusions of Proposition 3.3 are valid for the characteristic determinant \( \Delta_0(\cdot) \) given by (3.6). Let \( \Lambda_0 = \{ \lambda_n^0 \}_{n \in \mathbb{Z}} \) be the sequence of its zeros counting multiplicity. Let us order the sequence \( \Lambda_0 \) in a (possibly nonunique) way such that \( \Re \lambda_n^0 \leq \Re \lambda_{n+1}^0, \quad n \in \mathbb{Z} \). Let us recall an important result from [25, 28] and [43] concerning asymptotic behavior of the eigenvalues.

**Proposition 3.4** (Proposition 4.7 in [28]). Let \( Q \in L^1([0,1];\mathbb{C}^{2 \times 2}) \) and let boundary conditions (1.3) be regular. Then, the sequence \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) of zeros of \( \Delta_Q(\cdot) \) can be ordered in such a way that the following asymptotic formula holds:

\[
\lambda_n = \lambda_n^0 + o(1), \quad \text{as } n \to \infty, \quad n \in \mathbb{Z}. \tag{3.12}
\]

To define strictly regular boundary conditions, we need the following definition.

**Definition 3.5.** (i) A sequence \( \Lambda := \{ \lambda_n \}_{n \in \mathbb{Z}} \) of complex numbers is said to be **separated** if for some positive \( \tau > 0 \),

\[
|\lambda_j - \lambda_k| > 2\tau \quad \text{whenever } j \neq k. \tag{3.13}
\]

In particular, all entries of a separated sequence are distinct.

(ii) The sequence \( \Lambda \) is said to be **asymptotically separated** if for some \( N \in \mathbb{N} \), the subsequence \( \{ \lambda_n \}_{|n|>N} \) is separated.

Let us recall a notion of strictly regular boundary conditions.

**Definition 3.6.** Boundary conditions (1.3) are called **strictly regular**, if they are regular, that is, \( J_{14}J_{32} \neq 0 \), and the sequence of zeros \( \lambda_0 = \{ \lambda_n^0 \}_{n \in \mathbb{Z}} \) of the characteristic determinant \( \Delta_0(\cdot) \) is asymptotically separated. In particular, there exists \( n_0 \) such that zeros \( \{ \lambda_n^0 \}_{|n|>n_0} \) are geometrically and algebraically simple.

It follows from Proposition 3.4 that the sequence \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) of zeros of \( \Delta_Q(\cdot) \) is asymptotically separated if the boundary conditions are strictly regular.

Let boundary conditions (1.3) be regular. We can rewrite them in the following canonical form following [28, Section 5]:

\[
\begin{cases}
\hat{U}_1(y) = y_1(0) + b y_2(0) + a y_1(1) = 0, \\
\hat{U}_2(y) = d y_2(0) + c y_1(1) + y_2(1) = 0,
\end{cases} \tag{3.14}
\]

with some \( a, b, c, d \in \mathbb{C} \) such that \( ad - bc \neq 0 \). The characteristic determinants \( \Delta_0(\cdot) \) and \( \Delta(\cdot) \) take the form

\[
\Delta_0(\lambda) = d + ae^{(b_1+b_2)\lambda} + (ad - bc)e^{ib_1\lambda} + e^{ib_2\lambda}, \tag{3.15}
\]

\[
\Delta(\lambda) = d + ae^{(b_1+b_2)\lambda} + (ad - bc)\varphi_{11}(\lambda) + \varphi_{22}(\lambda) + c\varphi_{12}(\lambda) + b\varphi_{21}(\lambda). \tag{3.16}
\]

**Remark 3.7.** Let us list some types of **strictly regular** boundary conditions (3.14). In all of these cases except 4b, the set of zeros of \( \Delta_0 \) is a union of finite number of arithmetic progressions.

1. Regular boundary conditions (3.14) for Dirac operator \((-b_1 = b_2 = 1)\) are strictly regular if and only if \((a - d)^2 \neq 4bc\).
2. Separated boundary conditions \((a = d = 0, bc \neq 0)\) are always strictly regular.
3. Let \( b_2/b_1 \in \mathbb{Q} \), that is, \( b_1 = -n_1b_0, b_2 = n_2b_0, n_1, n_2 \in \mathbb{N}, b_0 > 0, \) and \( \gcd(n_1, n_2) = 1 \). Since \( ad \neq bc \), \( \Delta_0(\cdot)e^{-ib_1\lambda} \) is a polynomial in \( e^{ib_2\lambda} \) of degree \( n_1 + n_2 \) with nonzero roots. Hence, boundary conditions (3.14) are strictly regular if and only if this polynomial does not have multiple roots. Let us list some cases with explicit conditions.

(a) [28, Lemma 5.3] Let \( ad \neq 0 \) and \( bc = 0 \). Then, boundary conditions (3.14) are strictly regular if and only if

\[
b_1 \ln |d| + b_2 \ln |a| \neq 0 \quad \text{or} \quad n_1 \arg(-d) - n_2 \arg(-a) \notin 2\pi\mathbb{Z}. \tag{3.17}
\]
(b) In particular, antiperiodic boundary conditions \((a = d = 1, b = c = 0)\) are strictly regular if and only if \(n_1 - n_2\) is odd. Note that these boundary conditions are not strictly regular in the case of a Dirac system.

(c) \([28, \text{Proposition 5.6}]\) Let \(a = 0, bc \neq 0\). Then, boundary conditions (3.14) are strictly regular if and only if
\[
\begin{align*}
n_1^{n_1}n_2^{n_2}(-d)^{n_1+n_2} &\neq (n_1 + n_2)^{n_1+n_2}(-bc)^{n_2}.
\end{align*}
\] (3.18)

4. Let \(\alpha := -b_2/b_1 \not\in \mathbb{Q}\). Then, the problem of strict regularity of boundary conditions is generally much more complicated. Let us list some known cases:

(a) \([28, \text{Lemma 5.3}]\) Let \(ad \neq 0\) and \(bc = 0\). Then, boundary conditions (3.14) are strictly regular if and only if
\[
b_1 \ln |d| + b_2 \ln |a| \neq 0.
\] (3.19)

(b) \([28, \text{Proposition 5.6}]\) Let \(a = 0\) and \(bc, d \in \mathbb{R} \setminus \{0\}\). Then, boundary conditions (3.14) are strictly regular if and only if
\[
d \neq -(\alpha + 1)(|bc|^\alpha - \alpha)^{-1}.\] (3.20)

It is well known that the biorthogonal system to the system of root vectors of the operator \(L_U(Q)\) coincides with the system of root vectors of the adjoint operator \(L^*_U(Q) := (L_U(Q))^*\) after proper normalization. In this connection, we give the explicit form of the operator \(L^*_U(Q)\) in the case of boundary conditions (3.14).

**Lemma 3.8.** Let \(L_U(Q)\) be an operator corresponding to the problem (1.1), (3.14). Then, the adjoint operator \(L^*_U(Q)\) is given by the differential expression (1.1) with \(Q^*(x) = \begin{pmatrix} 0 & Q_2(x) \\ Q_1(x) & 0 \end{pmatrix}\) instead of \(Q\) and the boundary conditions
\[
\begin{align*}
U_{*1}(y) &= ay_1(0) + y_1(1) + \beta^{-1}cy_2(1) = 0, \\
U_{*2}(y) &= \beta by_1(0) + y_2(0) + dy_2(1) = 0,
\end{align*}
\] (3.21)
where as before \(\beta = -b_2/b_1 > 0\). That is, \(L^*_U(Q) = L_{U^*}(Q^*)\). Moreover, boundary conditions (3.21) are regular (strictly regular) simultaneously with boundary conditions (3.14).

**Corollary 3.9.** The operator \(L_U(0)\) corresponding to the problem (1.1), (3.14) with \(Q = 0\) is self-adjoint if and only if
\[
a = d \bar{u}, \quad d = a \bar{u}, \quad b = -\beta^{-1}c \bar{u}, \quad c = -\beta b \bar{u}, \quad u := ad - bc \neq 0,\] (3.22)
which in turn is equivalent to (1.11).

**Proof.** Boundary conditions (3.14) and (3.21) can be rewritten in a matrix form as
\[
\begin{pmatrix} y_1(0) \\ y_2(1) \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(0) \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} \bar{a} & \beta^{-1}\bar{c} \\ \beta \bar{b} & \bar{d} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(1) \end{pmatrix} + \begin{pmatrix} y_1(1) \\ y_2(0) \end{pmatrix} = 0,
\] (3.23)
respectively. Hence boundary conditions (3.14) and (3.21) are equivalent if and only if
\[
\begin{pmatrix} \bar{a} & \beta^{-1}\bar{c} \\ \beta \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{u} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},
\] (3.24)
which is equivalent to (3.22).

On the other hand, we can rewrite conditions (3.14) as
\[
Cy(0) + Dy(1) = 0, \quad C = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}, \quad D = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}.
\] (3.25)
According to [26, Lemma 5.1] (see also [27, Lemma 5.1]), operator $L_U(0)$ with boundary conditions rewritten as (3.25) is self-adjoint if and only if $CBC^* = DBD^*$. Straightforward calculations show that

$$b_1^{-1}CBC^* = b_1^{-1}\left(\begin{array}{cc} b_1 + b_2 & b_1 b_d \\ b_2 & b_2 |d| \end{array}\right) = \left(\begin{array}{cc} 1 - \beta |b|^2 & -\beta b_d \\ -\beta b_d & -\beta |d|^2 \end{array}\right),$$

$$b_1^{-1}DBD^* = b_1^{-1}\left(\begin{array}{cc} b_1 |a|^2 & b_1 a c \\ b_1 c^2 + b_2 & b_2 d \end{array}\right) = \left(\begin{array}{cc} |a|^2 & a c \\ a c & |c|^2 - \beta \end{array}\right).$$

Hence, $CBC^* = DBD^*$ is equivalent to the condition (1.11). It is interesting to note that establishing equivalence of (1.11) and (3.22) directly is somewhat tedious.

4 | PROPERTIES OF THE SPECTRUM OF THE UNPERTURBED OPERATOR

In this section, we obtain some properties of the sequence $\{\lambda_0^0\}_{n \in \mathbb{Z}}$ of the characteristic determinant $\Delta_0(\cdot)$ in the case of regular boundary conditions (3.14) that will be needed in Section 5 to study Bari $c_0$-property of the system of root vectors of the operator $L_U(0)$ (see Definition 2.2). Recall that $x_n \asymp y_n$, $n \in \mathbb{Z}$, means that there exists $C_2 > C_1 > 0$ such that $C_1 |y_n| \leq |x_n| \leq C_2 |y_n|$, $n \in \mathbb{Z}$. We start the following simple property of zeros of $\Delta_0(\cdot)$.

**Lemma 4.1.** Let boundary conditions (3.14) be regular and $\Lambda_0 := \{\lambda_0^0\}_{n \in \mathbb{Z}}$ be the sequence of zeros of $\Delta_0(\cdot)$ counting multiplicity. Set

$$e_{1n} := e_{1,n} := e^{ib_1\lambda_0^0}, \quad e_{2n} := e_{2,n} := e^{-ib_2\lambda_0^0}, \quad n \in \mathbb{Z}. \tag{4.1}$$

(i) Let $bc \neq 0$. Then,

$$\begin{align*}
1 + ae_{1n} &\asymp 1, \\
1 + de_{2n} &\asymp 1, \\
\lambda_{n} &\in \mathbb{Z}.
\end{align*} \tag{4.2}$$

(ii) Let boundary conditions (3.14) be strictly regular. Then,

$$\begin{align*}
|1 + ae_{1n}|^2 + |1 + de_{2n}|^2 &\asymp 1, \\
\lambda_{n} &\in \mathbb{Z}.
\end{align*} \tag{4.3}$$

**Proof.** Note that

$$\Delta_0(\lambda) = (1 + ae^{ib_1\lambda}) (d + e^{ib_2\lambda}) - bc \cdot e^{ib_1\lambda} = e^{ib_1\lambda} (1 + ae^{ib_1\lambda}) (1 + de^{-ib_2\lambda}) - bc \cdot e^{ib_1\lambda}, \quad \lambda \in \mathbb{C}. \tag{4.4}$$

Since $\Delta(\lambda_0^0) = 0$, $n \in \mathbb{Z}$, then with notation (4.1), we have

$$(1 + ae_{1n})(1 + de_{2n}) = bce_{1n}e_{2n}, \quad n \in \mathbb{Z}. \tag{4.5}$$

According to Proposition 3.3, relation (3.10) holds. Hence,

$$e_{jn} \asymp 1, \quad n \in \mathbb{Z}, \quad j \in \{1, 2\}. \tag{4.6}$$

(i) Since $bc \neq 0$, then combining (4.5) with (4.6) yields the following estimate with some $C_3 > C_2 > C_1 > 0$,

$$C_3 > C_2 |1 + ae_{1n}| \geq |(1 + ae_{1n})(1 + de_{2n})| = 2|bc| \cdot |e_{1n}e_{2n}| > C_1, \quad |n| \in \mathbb{Z}, \tag{4.7}$$

which proves the first relation in (4.2). The second relation is proved similarly.

(ii) If $bc \neq 0$, then (4.3) is implied by (4.2). Let $bc = 0$. In this case, $ad \neq 0$ and $\Delta_0(\lambda) = e^{ib_1\lambda} (1 + ae^{ib_1\lambda}) (1 + de^{-ib_2\lambda})$. It is clear that $\Lambda_0 = \Lambda_0^1 \cup \Lambda_0^2$, where $\Lambda_0^1 = \{\lambda_0^0\}_{n \in \mathbb{Z}}$ and $\Lambda_0^2 = \{\lambda_2^0\}_{n \in \mathbb{Z}}$ are the sequences of zeros of the first and second factors, respectively. Clearly, these sequences constitute arithmetic progressions lying on the lines, parallel to the real axis. More precisely,

$$\lambda_{1,n}^0 = \frac{\text{arg}(-a^{-1}) + 2\pi n}{b_1} + i\ln|a|b_1, \quad \lambda_{2,n}^0 = \frac{\text{arg}(-d) + 2\pi n}{b_2} - i\ln|d|b_2. \tag{4.8}$$
for \( n \in \mathbb{Z} \). Since boundary conditions \((3.14)\) are strictly regular, the union of these arithmetic progressions \( \Lambda_0 = \Lambda_0^1 \cup \Lambda_0^2 \) is asymptotically separated. It is easily seen that, in fact, \( \Lambda_0 \) is separated: If \( b_2/b_1 \notin \mathbb{Q} \), then \( \Lambda_0 \) is periodic and if \( b_2/b_1 \notin \mathbb{Q} \), then arithmetic progressions \( \Lambda_0^1 \) and \( \Lambda_0^2 \) necessarily lie on different parallel lines. This implies the following asymptotic relations:

\[
1 + de^{-ib_2 \lambda_0^1} \asymp 1, \quad 1 + ae^{ib_2 \lambda_0^1} \asymp 1, \quad n \in \mathbb{Z}.
\] (4.9)

Since \( \Lambda_0 = \Lambda_0^1 \cup \Lambda_0^2 \), relations (4.9) trivially imply (4.3).

Throughout the rest of the section, we will denote by \( \langle x \rangle := x - \lfloor x \rfloor \) the fractional part of \( x \in \mathbb{R} \). To treat the tricky case of \( \beta = -b_2/b_1 \notin \mathbb{Q} \), we need Weyl's equidistribution theorem (see [49, Theorem 4.2.2.1]). More precisely, we need the following consequence.

Lemma 4.2. Let \( \beta \in \mathbb{R} \setminus \mathbb{Q} \) and \( 0 \leq a < b \leq 1 \). Then, for any \( \varepsilon > 0 \), there exists \( M_{a,b, \varepsilon} > 0 \) such that for \( M \in \mathbb{N} \), we have:

\[
\text{card}\{m \in \{-M, \ldots, M\} : \langle \beta m \rangle \in [a, b]\} \leq 2(b-a+\varepsilon)M, \quad M \geq M_{a,b, \varepsilon}.
\] (4.10)

First, let us recall some simple properties of the sequences that have a finite set of limit points. For brevity, we denote the cardinality of the limit points set of a bounded sequence \( \{z_n\}_{n \in \mathbb{Z}} \subset \mathbb{C} \) as \( \#\lim\{z_n\}_{n \in \mathbb{Z}} \).

Lemma 4.3. The following statements hold:

(i) Let \( \{a_n\}_{n \in \mathbb{Z}} \subset \mathbb{C} \) be bounded, \( f \) be continuous on \( \cup_{|n|>N} \mathbb{D}(a_n) \) for some \( \varepsilon > 0 \) and \( N > 0 \), and \( \#\lim\{a_n\}_{n \in \mathbb{Z}} = m \in \mathbb{N} \). Then, \( \#\lim\{f(a_n)\}_{n \in \mathbb{Z}} \leq m \).

(ii) Let \( \{a_n\}_{n \in \mathbb{Z}} \subset \mathbb{C} \) and \( \{b_n\}_{n \in \mathbb{Z}} \subset \mathbb{C} \) be bounded sequences and let \( \#\lim\{a_n\}_{n \in \mathbb{Z}} = m_a \) and \( \#\lim\{b_n\}_{n \in \mathbb{Z}} = m_b \in \mathbb{N} \). Then, \( \#\lim\{a_n + b_n\}_{n \in \mathbb{Z}} \leq m_a m_b \) and \( \#\lim\{a_n b_n\}_{n \in \mathbb{Z}} \leq m_a m_b \).

(iii) Let \( y_n \in [0,1) \), \( n \in \mathbb{Z} \), and let \( \#\lim\{\sin(2 \pi y_n)\}_{n \in \mathbb{Z}} = m \in \mathbb{N} \). Then, \( \#\lim\{y_n\}_{n \in \mathbb{Z}} \leq 2m + 1 \).

(iv) Let \( a, b \in \mathbb{R} \), \( \{x_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) be bounded and \( \#\lim\{x_n\}_{n \in \mathbb{Z}} = m \in \mathbb{N} \). Then, \( \#\lim\{(ax_n + b)\}_{n \in \mathbb{Z}} \leq m + 1 \).

The following result of Diophantine approximation nature plays a crucial role in treating the tricky case of \( b_2/b_1 \notin \mathbb{Q} \).

Lemma 4.4. Let \( b_1, b_2 \in \mathbb{R} \setminus \{0\} \) and \( b_2/b_1 \notin \mathbb{Q} \). Further, let \( \{\alpha_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) be an incompressible sequence such that

\[
\text{card}\{n \in \mathbb{Z} : |\alpha_n| \leq M\} \geq \gamma M, \quad M \geq M_0,
\] (4.12)

for some \( \gamma, M_0 > 0 \). Then, one of the sequences \( \{\sin(b_1 \alpha_n)\}_{n \in \mathbb{Z}} \) and \( \{\sin(b_2 \alpha_n)\}_{n \in \mathbb{Z}} \) has an infinite set of limit points.

Proof. Assume the contrary. Namely, let

\[
\#\lim\{\sin(b_1 \alpha_n)\}_{n \in \mathbb{Z}} = m_1 \in \mathbb{N} \quad \text{and} \quad \#\lim\{\sin(b_2 \alpha_n)\}_{n \in \mathbb{Z}} = m_2 \in \mathbb{N}.
\]

Let us set

\[
b_1 \alpha_n = 2\pi(k_n + \delta_n), \quad k_n := \frac{b_1 \alpha_n}{2\pi} \in \mathbb{Z}, \quad \delta_n = \frac{b_1 \alpha_n}{2\pi} \in [0,1).
\] (4.13)

It is clear that \( \sin(2\pi \delta_n) = \sin(b_1 \alpha_n) \). Hence, by Lemma 4.3(iii),

\[
\#\lim\{\delta_n\}_{n \in \mathbb{Z}} \leq 2m_1 + 1.
\] (4.14)
It is clear from (4.13) that
\[ b_2 \alpha_n = 2\pi (\beta k_n + \beta \delta_n), \quad n \in \mathbb{Z}, \quad \beta := b_2 / b_1 \notin \mathbb{Q}. \]

The same reasoning as above shows that
\[ \# \lim \{ u_n \}_{n \in \mathbb{Z}} \leq 2m_2 + 1, \quad u_n := \langle \beta k_n + \beta \delta_n \rangle. \]

Further, combining (4.14) with Lemma 4.3(iv) implies that
\[ \# \lim \{ v_n \}_{n \in \mathbb{Z}} \leq 2m_1 + 2, \quad v_n := \langle \beta \delta_n \rangle, \quad n \in \mathbb{Z}. \] (4.15)

Finally, note that \( \langle \beta k_n \rangle = \langle u_n - v_n \rangle, n \in \mathbb{Z} \). Hence, by parts (ii) and (iv) of Lemma 4.3, the sequence \( \{ \langle \beta k_n \rangle \}_{n \in \mathbb{Z}} \) has exactly \( p \leq (2m_2 + 1)(2m_1 + 2) + 1 \) limit points \( 0 \leq x_1 < \ldots < x_p \leq 1 \).

Let \( \varepsilon > 0 \) be fixed. Then, there exists \( N_\varepsilon \in \mathbb{N} \) such that
\[ \langle \beta k_n \rangle \in I_\varepsilon := [0, 1) \cap \bigcup_{j=1}^p (x_j - \varepsilon, x_j + \varepsilon), \quad |n| \geq N_\varepsilon. \] (4.16)

Since \( \beta \notin \mathbb{Q} \), Lemma 4.2 implies that
\[ \text{card}(J_{\varepsilon, M}) \leq 6p\varepsilon M, \quad M \geq M_\varepsilon, \quad M \in \mathbb{N}, \quad \text{where} \]
\[ J_{\varepsilon, M} := \{ m \in \{-M, \ldots, M\} : \langle \beta m \rangle \in I_\varepsilon \}, \quad M \in \mathbb{N}, \] (4.17)

For \( M_\varepsilon := \max \left\{ M_{x_j - \varepsilon, x_j + \varepsilon} : j \in \{1, \ldots, p\} \right\} \),
let \( M \in \mathbb{N} \) and consider the set
\[ K_{\varepsilon, M} := \{ |n| \geq N_\varepsilon : |k_n| \leq M \} \subset \mathbb{Z}. \]

It is clear from (4.13) and inequality \( |\lfloor x \rfloor| < |x| + 1 \) that
\[ K_{\varepsilon, M} \supset \{ |n| \geq N_\varepsilon : |\alpha_n| \leq \tilde{M} \}, \quad \tilde{M} := \frac{2\pi(M - 1)}{|b_1|}. \]

Hence, if \( \tilde{M} \geq M_0 \), condition (4.12) implies that
\[ \text{card}(K_{\varepsilon, M}) \geq \gamma \tilde{M} - 2N_\varepsilon + 1 \geq \gamma_1 M, \quad M \geq \tilde{M}_\varepsilon, \] (4.19)

with \( \gamma_1 := \pi \gamma |b_1^{-1}| > 0 \) and some \( \tilde{M}_\varepsilon \geq M_\varepsilon \). Condition (4.16) and definition (4.18) of \( J_{\varepsilon, M} \) imply that for \( n \in K_{\varepsilon, M} \), we have \( k_n \in J_{\varepsilon, M} \). Since \( \{ \alpha_n \}_{n \in \mathbb{Z}} \) is incompressible, then so is \( \{ k_n \}_{n \in \mathbb{Z}} \). Hence multiplicities \( d_m := \text{card}\{ n \in \mathbb{Z} : k_n = m \} \) are bounded, \( d_m \leq d, m \in \mathbb{Z}, \) for some \( d \in \mathbb{N} \). Hence, for every \( m \in J_{\varepsilon, M} \), there are at most \( d \) values of \( n \in K_{\varepsilon, M} \) for which \( k_n = m \). Combining this observation with the estimate (4.17), we arrive at
\[ \text{card} (K_{\varepsilon, M}) \leq d \text{ card} (J_{\varepsilon, M}) \leq 6d \varepsilon M. \] (4.20)

Now picking \( \varepsilon > 0 \) such that \( 6d \varepsilon M < \gamma_1 \) and \( M > \tilde{M}_\varepsilon \), we see that cardinality estimates (4.19) and (4.20) contradict each other, which finishes the proof. \[ \square \]

**Remark 4.5.** It is clear from the proof of Lemma 4.4 that the statement remains valid if we relax condition (4.12) to only hold for \( M \in M \subset \mathbb{N} \), where \( M \) is some fixed unbounded subset of \( \mathbb{N} \).

To apply Lemma 4.4, we first need to establish property 4.4 for the sequence \( \{ \Re \lambda_0^n \}_{n \in \mathbb{Z}} \). It easily follows from the asymptotic formula (3.11).
Lemma 4.6. Let the boundary conditions (1.3) be regular. Then, for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that
\[ \text{card } \{ n \in \mathbb{Z} : |Re \lambda_0^n| \leq N \} \geq \frac{N}{\sigma + \varepsilon}, \quad N \geq N_\varepsilon, \quad \sigma := \frac{\pi}{b_2 - b_1} > 0. \] (4.21)

Proof. Asymptotic formula (3.11) for $\{\lambda_0^n\}_{n \in \mathbb{Z}}$ implies that $|Re \lambda_0^n| \leq (2\sigma + \varepsilon)|n|$, $|n| \geq n_\varepsilon$, for some $n_\varepsilon \in \mathbb{N}$. Hence,
\[ \mathbb{Z} \cap \left[ \left(-\frac{N}{2\sigma + \varepsilon}, -n_\varepsilon \right] \cup \left[n_\varepsilon, \frac{N}{2\sigma + \varepsilon} \right) \right] \subseteq \{ n \in \mathbb{Z} : |Re \lambda_0^n| \leq N \}, \]
for $N \geq (2\sigma + \varepsilon)n_\varepsilon$. Taking cardinalities in this inclusion implies
\[ \text{card } \{ n \in \mathbb{Z} : |Re \lambda_0^n| \leq N \} \geq \frac{N}{\sigma + \varepsilon} + n_\varepsilon, \quad N \geq N_\varepsilon, \] (4.22)
with $N_\varepsilon := 2(\sigma/\varepsilon + 1)(2\sigma + \varepsilon)n_\varepsilon$.

Combining two previous results leads to the following important property of zeros of characteristic determinant $\Delta_0(\cdot)$, which coincides with Proposition 2.6 and is formulated again for reader’s convenient.

Proposition 4.7. Let $b_2/b_1 \notin \mathbb{Q}$ and boundary conditions (1.6) be regular, that is, $u := ad - bc \neq 0$. Let $\{\lambda_0^n\}_{n \in \mathbb{Z}}$ be the sequence of zeros of the characteristic determinant $\Delta_0(\cdot)$ counting multiplicity. Then, each of these sequences $\{\exp(i b_1 \lambda_0^n)\}_{n \in \mathbb{Z}}$ and $\{\exp(i b_2 \lambda_0^n)\}_{n \in \mathbb{Z}}$ has infinite set of limit points.

Proof. (i) First, let $bc = 0$. Then according to the proof of Lemma 4.1, zeros of the characteristic determinant $\Delta_0(\cdot)$ are simple and split into two separated arithmetic progressions $\Lambda_0^1 = \{\lambda_0^n\}_{n \in \mathbb{Z}}$ and $\Lambda_0^2 = \{\lambda_0^n\}_{n \in \mathbb{Z}}$ given by (4.8). Let $k \in \{1, 2\}$ and $j = 2/k$. Since $E(z) = e^{2\pi i z}$ is periodic with period 1, we have for $n \in \mathbb{Z}$,
\[ \exp(ib_k \lambda_0^{jk,n}) = \exp(2\pi i nb_k/b_j + \omega_{k,j,a,d}) = \exp(2\pi i \langle nb_k/b_j \rangle + \omega_{k,j,a,d}), \] (4.23)
where $\omega_{k,j,a,d}$ is an explicit constant that can be derived from (4.8). Since $b_k/b_j \notin \mathbb{Q}$, then by the classical Kronecker theorem, the sequence $\{\langle nb_k/b_j \rangle\}_{n \in \mathbb{Z}}$ is everywhere dense on $[0,1)$. This implies that the sequence $\{\exp(ib_k \lambda_0^{jk,n})\}_{n \in \mathbb{Z}}$ has infinite set of limit points, which finishes the proof in this case.

(ii) Now, let $bc \neq 0$ and assume the contrary: One of the sequences $\{\exp(ib_1 \lambda_0^n)\}_{n \in \mathbb{Z}}$ and $\{\exp(ib_2 \lambda_0^n)\}_{n \in \mathbb{Z}}$ has a finite set of limit points. For definiteness, assume that
\[ \# \lim\{\exp(ib_1 \lambda_0^n)\}_{n \in \mathbb{Z}} = m_1 \in \mathbb{N}. \] (4.24)

Recall that $e_{1n} := \exp(ib_1 \lambda_0^n)$ and $e_{2n} := \exp(-ib_2 \lambda_0^n)$, $n \in \mathbb{Z}$, and also set
\[ \lambda_0^n = \alpha_n + i\beta_n, \quad \alpha_n := Re \lambda_0^n, \quad \beta_n := Im \lambda_0^n, \quad n \in \mathbb{Z}. \] (4.25)

It is clear that
\[ |e_{1n}| = |\exp(ib_1 \lambda_0^n)| = \exp(-b_1 \operatorname{Im} \lambda_0^n) = \exp(-b_1 \beta_n), \quad n \in \mathbb{Z}, \] (4.26)
It follows from (4.26), (4.24), (4.6), and Lemma 4.1(i), applied with $f_1(z) = -b_1^{-1} \log |z|$, that
\[ \# \lim\{\beta_n\}_{n \in \mathbb{Z}} = \# \lim\{-b_1^{-1} \log |e_{1n}|\}_{n \in \mathbb{Z}} \leq m_1. \] (4.27)
In turn, since $e_{1n} = |e_{1n}|e^{ib_1 \alpha_n} \asymp 1$, $n \in \mathbb{Z}$, then by Lemma 4.3(i) applied with $f_2(z) = \operatorname{Im} z/|z|$ we have
\[ \# \lim\{\sin (b_1 \alpha_n)\}_{n \in \mathbb{Z}} = \# \lim\{\operatorname{Im} e_{1n}/|e_{1n}|\}_{n \in \mathbb{Z}} \leq m_1. \] (4.28)
Since $bc \neq 0$ and boundary conditions (3.14) are regular, then Lemma 4.1(i) implies (4.2). Recall that $u := ad - bc \neq 0$. Since $\Delta(\lambda_n^0) = 0$, $1 + ae_{1n} \neq 0$ and boundary conditions (3.14) are regular, then Lemma 4.1(i) implies (4.2). Recall that $u := ad - bc \neq 0$.

\begin{align*}
1 + de_{2n} + ae_{1n} + e_{1n}e_{2n} = 0, \quad e_{2n} = -d + eu_{1n}, \quad e_{1n} = -1 + de_{2n}. 
\end{align*}

(4.29)

Since $e_{1n} \geq 1$, $e_{2n} \geq 1$, $n \in \mathbb{Z}$, relations (4.2) and (4.29) imply that

\begin{align*}
d + eu_{1n} \geq 1, \quad a + eu_{2n} \geq 1, \quad n \in \mathbb{Z}. 
\end{align*}

(4.30)

Hence, $f_3(z) := -\frac{1 + az}{d + uz}$ is continuous in the neighborhood of $\{e_{1n}\}_{n \in \mathbb{Z}}$. Combining this with Lemma 4.3(i), the second identity in (4.29), and relation (4.24) we arrive at

\begin{align*}
\# \lim\{e_{2n}\}_{n \in \mathbb{Z}} = \# \lim\{f_3(e_{1n})\}_{n \in \mathbb{Z}} \leq m_1. 
\end{align*}

(4.31)

Similarly to (4.28) we get

\begin{align*}
\# \lim\{\sin(b_2\alpha_n)\}_{n \in \mathbb{Z}} = \# \lim\{|\text{Im}e_{2n}/|e_{2n}|\}_{n \in \mathbb{Z}} \leq m_1. 
\end{align*}

(4.32)

Since boundary conditions (3.14) are regular, Proposition 3.3 implies that the sequence $\{\alpha_n\}_{n \in \mathbb{Z}}$ is incompressible and Lemma 4.6 implies the estimate (4.21), which in turn yields the estimate (4.12) for $\{\alpha_n\}_{n \in \mathbb{Z}}$ with $\gamma = \frac{1}{2\pi} = \frac{b_2-b_1}{2\pi}$. Since $b_2/b_1 \notin \mathbb{Q}$, byLemma 4.4, one of sequences $\{\sin(b_1\alpha_n)\}_{n \in \mathbb{Z}}$ and $\{\sin(b_2\alpha_n)\}_{n \in \mathbb{Z}}$ has infinite set of limit points. This contradicts relations (4.28) and (4.32) and finishes the proof.

\section{Bari $C_0$-Property of the System of Root Vectors of the Unperturbed Operator}

In this section, assuming boundary conditions (3.14) to be strictly regular, we show that the system of root vectors of the operator $L_U(0)$ is a Bari $C_0$-sequence in $L^2([0,1];\mathbb{C}^2)$ if and only if the operator $L_U(0)$ is self-adjoint. Since eigenfunctions of $L_U(0)$ in their “natural form” are not normalized, we need the following simple practical criterion of Bari $C_0$-property.

\begin{lemma}
Let $\tilde{\mathfrak{G}} = \{f_n\}_{n \in \mathbb{Z}}$ be a complete minimal system of vectors in a Hilbert space $\mathfrak{H}$. Let also $\{g_n\}_{n \in \mathbb{Z}}$ be “almost biorthogonal” to $\tilde{\mathfrak{G}}$. Namely, $(f_n, g_m) = 0$, $n \neq m$, $(f_n, g_n) \neq 0$, $n, m \in \mathbb{Z}$. Then, the normalized system

$\tilde{\mathfrak{G}} : = \{\tilde{f}_n\}_{n \in \mathbb{Z}}, \quad \tilde{f}_n := \frac{1}{\|f_n\|}f_n, \quad n \in \mathbb{Z},$

is a Bari $C_0$-sequence in $\mathfrak{H}$ (see Definition 2.2) if and only if

\begin{align*}
\frac{\|f_n\|_{\mathfrak{H}} \cdot \|g_n\|_{\mathfrak{H}}}{(f_n, g_n)_{\mathfrak{H}}} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty. 
\end{align*}

(5.1)

\end{lemma}

\begin{proof}
For brevity, we set $\|\cdot\| := \|\cdot\|_{\mathfrak{H}}$ and $(\cdot, \cdot) := (\cdot, \cdot)_{\mathfrak{H}}$. It is clear that for the system $\mathfrak{G} : = \{g_n\}_{n \in \mathbb{Z}}$ that is biorthogonal to $\tilde{\mathfrak{G}}$, we have

\begin{align*}
(f_n, g_m) = \delta_{nm}, \quad n, m \in \mathbb{Z}; \quad g_n = \frac{\|f_n\|}{(f_n, g_n)_{\mathfrak{H}}} \cdot g_n, \quad n \in \mathbb{Z}. 
\end{align*}

(5.2)

Relations (5.2) imply that

\begin{align*}
\|\tilde{f}_n - \tilde{g}_n\|^2 = \|\tilde{f}_n\|^2 - (\tilde{f}_n, \tilde{g}_n) - (\tilde{f}_n, \tilde{g}_n) + \|\tilde{g}_n\|^2 = \|\tilde{g}_n\|^2 - 1 = \frac{\|f_n\|^2 \cdot \|g_n\|^2}{(f_n, g_n)_{\mathfrak{H}}} - 1. 
\end{align*}

(5.3)

Hence, systems $\tilde{\mathfrak{G}}$ and $\mathfrak{G}$ are $C_0$-close if and only if condition (5.1) holds.
\end{proof}
The following simple property of compact operators with asymptotically simple spectrum will be also useful in the next section.

**Lemma 5.2.** Let $T$ be an operator with compact resolvent in a separable Hilbert space $\mathcal{H}$ and let $\{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of its eigenvalues counting multiplicities. Let also $p \in [1, 2]$. Assume that for some $N \in \mathbb{N}$ eigenvalues $\lambda_n$, $|n| \geq N$, are algebraically simple. Then, if some normalized system of root vectors of the operator $T$ is a Bari $(\ell^p)^*$-sequence in $\mathcal{H}$, then every normalized system of root vectors of the operator $T$ is a Bari $(\ell^p)^*$-sequence in $\mathcal{H}$.

**Proof.** Let $\mathbf{F} = \{f_n\}_{n \in \mathbb{Z}}$ be a normalized system of root vectors of the operator $T$, which is a Bari $(\ell^p)^*$-sequence in $\mathcal{H}$. By definition, the system $\mathbf{F}$ is complete and minimal in $\mathcal{H}$. Let $\mathbf{G} = \{g_n\}_{n \in \mathbb{Z}}$ be its (unique) biorthogonal system. Further, let $\tilde{\mathbf{F}} = \{\tilde{f}_n\}_{n \in \mathbb{Z}}$ be any other normalized system of root vectors of the operator $T$. Since eigenvalue $\lambda_n$, $|n| \geq N$, is algebraically simple, then $\dim(R_{\lambda_n}(T)) = 1, |n| \geq N$. Hence $\tilde{f}_n = \alpha_n f_n, |n| \geq N$, for some $\alpha_n \in \mathbb{T} : = \{z \in \mathbb{C} : |z| = 1\}$. It is clear that $\tilde{\mathbf{F}}$ is also complete and minimal and for its biorthogonal system $\tilde{\mathbf{G}} = \{\tilde{g}_n\}_{n \in \mathbb{Z}}$, we have that $\tilde{g}_n = \alpha_n^{-1} g_n = \alpha_n g_n, |n| \geq N$, since $|\alpha_n| = 1$. Hence, $\|f_n - g_n\|_{\mathcal{H}} = \|\alpha_n \cdot (f_n - g_n)\|_{\mathcal{H}} = \|f_n - g_n\|_{\mathcal{H}}, |n| \geq N$. This implies that $\{\|f_n - g_n\|_{\mathcal{H}}\}_{n \in \mathbb{Z}} = \{\|f_n - g_n\|_{\mathcal{H}}\}_{n \in \mathbb{Z}} \in (\ell^p)^*$ and finishes the proof. □

**Remark 5.3.** Let $A$ be a self-adjoint operator with compact resolvent. Then, every normalized system of root vectors is an orthonormal basis in $L^2([0,1];\mathbb{C}^2)$ and coincides with its biorthogonal sequence. This implies that every normalized system of root vectors of the operator $A$ is a Bari $c_0$-sequence.

To study norms $\|f_n^0\|_2$ and $\|g_n^0\|_2$ of the eigenvectors of the operators $L_U(0)$ and $L_U^*(0)$, we first need to obtain some properties of simple integrals $\int_0^1 |e^{\pm 2ib\lambda x}| dx, j \in \{1, 2\}, \lambda \in \mathbb{C}$.

**Lemma 5.4.** Denote for $j \in \{1, 2\}$ and $\lambda \in \mathbb{C}$:

$$E_j^\pm(\lambda) : = \int_0^1 |e^{\pm 2ib\lambda x}| dx = \int_0^1 e^{\pm 2b\lambda x} \pm \mu \lambda x - 1 $$

(5.4)

Then, the following estimate holds:

$$E_j^+(\lambda)E_j^-(\lambda) - 1 \geq \frac{(b_j \lambda \mu)^2}{3}, \quad j \in \{1, 2\}, \lambda \in \mathbb{C}. $$

(5.5)

In particular, $E_j^+(\lambda)E_j^-(\lambda) - 1 > 0$ if $\mu \lambda \neq 0$.

**Proof.** Let $h \geq 0$. It is clear that

$$E_j^\pm(\lambda) = f(\mp 2b \lambda \mu), \quad \text{where} \quad f(x) := \frac{e^x - 1}{x} = 1 + \frac{x}{2} + O(x^2), \quad |x| < h. $$

(5.6)

It follows from Taylor expansion of $e^x$ that for $x \in \mathbb{R}$,

$$f(x)f(-x) = \frac{e^x - 1}{x} \cdot \frac{e^{-x} - 1}{-x} = \frac{e^x + e^{-x} - 2}{x^2} = 2 \sum_{k=1}^\infty \frac{x^{2k-2}}{(2k)!} \geq 1 + \frac{x^2}{12}. $$

(5.7)

Estimate (5.5) now immediately follows from (5.6) and (5.7). □

First we establish the Bari $c_0$-property criterion in a special case $b = c = 0$.

**Proposition 5.5.** Let boundary conditions (3.14) be strictly regular with $b = c = 0$, that is, they are of the form

$$y_1(0) + ay_1(1) = dy_2(0) + y_2(1) = 0, \quad ad \neq 0. $$

(5.8)

Then, some normalized system of root vectors of the operator $L_U(0)$ is a Bari $c_0$-sequence in $L^2([0,1];\mathbb{C}^2)$ (see Definition 2.2) if and only if $|a| = |d| = 1$. 

Proof. (i) If $|a| = |d| = 1$ (and $b = c = 0$), then by Corollary 3.9, the operator $L_U(0)$ with boundary conditions (5.8) is self-adjoint. Remark 5.3 now finishes the proof.

(ii) Now assume that some normalized system of root vectors of the operator $L_U(0)$ is a Bari $c_0$-sequence in $L^2([0, 1]; \mathbb{C}^2)$. Since boundary conditions (5.8) are strictly regular, then by definition, eigenvalues of the operator $L_U(0)$ are asymptotically simple. Hence by Lemma 5.2, every normalized system of root vectors of the operator $L_U(0)$ is a Bari $c_0$-sequence in $L^2([0, 1]; \mathbb{C}^2)$.

According to the proof of Lemma 4.1, the eigenvalues of the operator $L_U(0)$ are simple and split into two separated arithmetic progressions $\Lambda_1 = \{\lambda_1^n, n \in \mathbb{Z}\}$ and $\Lambda_2 = \{\lambda_2^n, n \in \mathbb{Z}\}$ given by (4.8). It is easy to verify that the vectors

$$f_{1,n}^0(x) = \left( e^{ib_1\lambda_1^n x}, 0 \right), \quad g_{1,n}^0(x) = \left( e^{ib_1\lambda_1^n x}, 0 \right), \quad n \in \mathbb{Z},$$

(5.9)

are the eigenvectors of the operators $L_U(0)$ and $L_U^*(0)$ corresponding to the eigenvalues $\lambda_1^n$ and $\overline{\lambda}_1^n$, and the vectors

$$f_{2,n}^0(x) = \left( 0, e^{ib_2\lambda_2^n x} \right), \quad g_{2,n}^0(x) = \left( 0, e^{ib_2\lambda_2^n x} \right), \quad n \in \mathbb{Z},$$

(5.10)

are the eigenvectors of the operators $L_U(0)$ and $L_U^*(0)$ corresponding to the eigenvalues $\lambda_2^n$ and $\overline{\lambda}_2^n$, respectively. It is clear that

$$\left( f_{j,n}^0, g_{k,m}^0 \right)_2 = \delta_{j,k} \delta_{n,m}, \quad j, k \in \{1, 2\}, \quad n, m \in \mathbb{Z}. \quad (5.11)$$

Thus, the union system $\mathcal{F} : = \{f_{1,n}^0\}_{n \in \mathbb{Z}} \cup \{f_{2,n}^0\}_{n \in \mathbb{Z}}$ is the system of root vectors of the operator $L_U(0)$ and $\mathcal{G} : = \{g_{1,n}^0\}_{n \in \mathbb{Z}} \cup \{g_{2,n}^0\}_{n \in \mathbb{Z}}$ is biorthogonal to it. Hence, normalization of the system $\mathcal{F}$ is a Bari $c_0$-sequence in $L^2([0, 1]; \mathbb{C}^2)$. According to Lemma 5.1, we have

$$\alpha_{j,n} := \frac{||f_{j,n}^0||_2 \cdot ||g_{j,n}^0||_2}{||f_{j,n}^0, g_{j,n}^0||_2}_2 \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty, \quad j \in \{1, 2\}. \quad (5.12)$$

Let $j = 1$. Then, taking into account Lemma 5.4 and formula (5.11), we have

$$\alpha_{1,n}^2 = \frac{||f_{1,n}^0||_2^2 \cdot ||g_{1,n}^0||_2^2}{||f_{1,n}^0, g_{1,n}^0||_2^2}_2 = E_1^+(\lambda_1^n)E_1^-(\lambda_1^n) \geq 1 + \frac{1}{3} |b_2\text{Im}\lambda_1^n|^2, \quad n \in \mathbb{Z}. \quad (5.13)$$

It follows from (4.8) that $b_1\text{Im}\lambda_1^n = \ln |a|$. Since $\alpha_{1,n} \rightarrow 1$ as $n \rightarrow \infty$, formula (5.13) implies that $\ln |a| = 0$, or $|a| = 1$. Similarly considering the case $j = 2$, we conclude that $|d| = 1$, which finishes the proof.

In the following intermediate result, we reduce condition (5.1) of Bari $c_0$-property of the system of root vectors of the operator $L_U(0)$ to an explicit condition in terms of eigenvalues $\{\lambda_0^n\}_{n \in \mathbb{Z}}$ of $L_U(0)$.

**Proposition 5.6.** Let boundary conditions (3.14) be strictly regular and let one of the parameters $b$ or $c$ in them be nonzero, $|b| + |c| > 0$. Let $\{\lambda_0^n\}_{n \in \mathbb{Z}}$ be the sequence of the eigenvalues of the operator $L_U(0)$ counting multiplicities. Then, some normalized system of root vectors of the operator $L_U(0)$ is a Bari $c_0$-sequence in $L^2([0, 1]; \mathbb{C}^2)$ (see Definition 2.2) if and only if the following conditions hold:

$$|c| = \beta |b|, \quad \lim_{n \rightarrow \infty} \text{Im} \lambda_0^n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} z_n = |bc|, \quad (5.14)$$

where

$$z_n := \frac{1 + de^{-ib_2\lambda_0^n}}{1 + ae^{ib_2\lambda_0^n}}. \quad (5.15)$$

**Proof.** Without loss of generality, we can assume that $b \neq 0$. By definition of strictly regular boundary conditions, there exists $n_0 \in \mathbb{N}$ such that eigenvalues $\lambda_0^n$ of $L_U(0)$ for $|n| > n_0$ are algebraically simple and separated from each other.
According to the proof of Theorem 1.1 in [28] vector-functions $f_{n}^{0}(\cdot)$ and $g_{n}^{0}(\cdot)$, $|n| > n_{0}$, of the following form:

$$
\begin{align*}
  f_{n}^{0}(x) & := \left( \frac{b e^{i b_{1} \lambda_{0}^{n} x}}{(1 + a e^{i b_{2} \lambda_{0}^{n}}) e^{i b_{2} \lambda_{0}^{n} x}} \right), \\
  g_{n}^{0}(x) & := \left( \frac{1 + d e^{-i b_{1} \lambda_{0}^{n}}}{-\beta b e^{-i b_{2} \lambda_{0}^{n}}} \right).
\end{align*}
$$

(5.16)

are nonzero eigenvectors of the operators $L_{U}(0)$ and $L_{U}^{*}(0)$ corresponding to the eigenvalues $\lambda_{0}^{n}$ and $\overline{\lambda}_{0}^{n}$ for $|n| > n_{0}$, respectively. Let $f_{n}^{0}(\cdot)$ and $g_{n}^{0}(\cdot)$ be some root vectors of operators $L_{U}(0)$ and $L_{U}^{*}(0)$ corresponding to the eigenvalues $\lambda_{0}^{n}$ and $\overline{\lambda}_{0}^{n}$ for $|n| \leq n_{0}$. Clearly, $\mathfrak{F} := \{f_{n}^{0}\}_{n \in \mathbb{Z}}$ is a system of root vectors of the operator $L_{U}(0)$ and $\Theta := \{g_{n}^{0}\}_{n \in \mathbb{Z}}$ is the corresponding system for the adjoint operator $L_{U}^{*}(0)$. Let us show that normalization of $\mathfrak{F}$ is a Bari $c_{0}$-sequence in $L^{2}([0,1];\mathbb{C}^{2})$ if and only condition (5.14) holds. Since eigenvalues of the operator $L_{U}(0)$ are asymptotically simple, Lemma 5.2 will imply the statement of the proposition. Clearly, $\Theta$ is almost biorthogonal to $\mathfrak{F}$. Hence, Lemma 5.1 implies that normalization of $\mathfrak{F}$ is a Bari $c_{0}$-sequence in $L^{2}([0,1];\mathbb{C}^{2})$ if and only if condition (5.1) holds.

Set for brevity $E_{\pm}^{\pm} j_{n} := E_{\pm}^{\pm} j(\lambda_{0}^{n})$, $j \in \{1,2\}$, $n \in \mathbb{Z}$, where $E_{\pm}^{\pm} j(\lambda)$ is defined in (5.4). On account of this notation and notation (4.1), we get after performing straightforward calculations:

$$
\begin{align*}
  \|f_{n}^{0}\|_{2}^{2} &= |b|^{2} E_{1n}^{+} + |1 + a e_{1n}|^{2} E_{2n}^{+}, \\
  \|g_{n}^{0}\|_{2}^{2} &= |1 + d e_{2n}|^{2} E_{1n}^{-} + \beta^{2} |b|^{2} E_{2n}^{-}, \\
  (f_{n}^{0}, g_{n}^{0})_{2} &= b((1 + d e_{2n}) + \beta(1 + a e_{1n})).
\end{align*}
$$

(5.17)–(5.19)

Since boundary conditions (3.14) are strictly regular, it follows from the proof of Theorem 1.1 in [28] that the following estimate holds:

$$
(f_{n}^{0}, g_{n}^{0})_{2} \approx \Delta'(\lambda_{0}^{n}) \approx 1, \quad |n| > n_{0}.
$$

(5.20)

Hence, condition (5.1) is equivalent to

$$
\|f_{n}^{0}\|_{2}^{2} \cdot \|g_{n}^{0}\|_{2}^{2} - |(f_{n}^{0}, g_{n}^{0})_{2}|^{2} \to 0 \quad \text{as} \quad n \to \infty.
$$

(5.21)

On account of (5.17)–(5.19), we get

$$
\begin{align*}
  \|f_{n}^{0}\|_{2}^{2} \cdot \|g_{n}^{0}\|_{2}^{2} - |(f_{n}^{0}, g_{n}^{0})_{2}|^{2} &= (|b|^{2} \cdot E_{1n}^{+} + |1 + a e_{1n}|^{2} \cdot E_{2n}^{+}) \cdot \left( |1 + d e_{2n}|^{2} E_{1n}^{-} + \beta^{2} |b|^{2} E_{2n}^{-} \right) \\
  &- |b|^{2} |1 + d e_{2n}) + \beta(1 + a e_{1n})|^{2} = \tau_{1,n} + \tau_{2,n} + \tau_{3,n}, \quad |n| > n_{0},
\end{align*}
$$

(5.22)

where

$$
\begin{align*}
  \tau_{1,n} & := |b|^{2} \cdot |1 + d e_{2n}|^{2} \cdot (E_{1n}^{+} E_{1n}^{-} - 1), \\
  \tau_{2,n} & := \beta^{2} |b|^{2} \cdot |1 + a e_{1n}|^{2} \cdot (E_{2n}^{+} E_{2n}^{-} - 1), \\
  \tau_{3,n} & := \beta^{2} |b|^{4} E_{1n}^{+} E_{2n}^{-} + |z_{n}|^{2} \cdot E_{2n}^{+} E_{1n}^{-} - 2 \beta |b|^{2} \cdot \text{Re} z_{n},
\end{align*}
$$

(5.23)–(5.25)

where $z_{n} = (1 + d e_{2n}) (1 + a e_{1n})$ is defined in (5.15). According to Proposition 3.3, $|\text{Im} \lambda_{0}^{n}| \leq h$, $n \in \mathbb{Z}$, for some $h > 0$. Hence, terms $|1 + d e_{2n}|$, $|1 + a e_{1n}|$, $|z_{n}|$, $E_{1n}^{+}$, and $E_{2n}^{-}$ are all bounded for $n \in \mathbb{Z}$.

First assume that $\text{Im} \lambda_{0}^{n} \to 0$ as $n \to \infty$. Then, it is clear from (5.6) that

$$
|e_{jn}| \to 1 \quad \text{and} \quad E_{jn}^{\pm} \to 1 \quad \text{as} \quad n \to \infty, \quad j \in \{1,2\}.
$$

(5.26)

Hence, $\tau_{1,n} + \tau_{2,n} \to 0$ as $n \to \infty$, while $\tau_{3,n} \to 0$ as $n \to \infty$ if and only if

$$
\tau_{4,n} := \beta^{2} |b|^{4} + |z_{n}|^{2} - 2 \beta |b|^{2} \cdot \text{Re} z_{n} = |z_{n} - \beta |b|^{2}|^{2} \to 0 \quad \text{as} \quad n \to \infty.
$$

(5.27)
It follows from (4.5) and (5.26) that \(|z_n| = |bc| \cdot |e_{1n}e_{2n}| \to |bc|\) as \(n \to \infty\). Hence, since \(b \neq 0\),

\[
(\tau_{4n} \to 0 \text{ as } n \to \infty) \iff (|c| = \beta |b| \text{ and } z_n \to |bc| \text{ as } n \to \infty).
\] (5.28)

Now if condition (5.14) holds, then (5.28) and previous observations on \(\tau_{1n}, \tau_{2n}, \tau_{3n}, \tau_{4n}\) imply the desired condition (5.21).

Now assume that condition (5.21) holds. It follows from (5.5) and (5.6) that

\[
0 \leq E_{jn}^+ E_{jn}^- - 1 \asymp |\lambda_0^n|^2, \quad n \in \mathbb{Z}, \quad j \in \{1, 2\}.
\] (5.29)

Since \(b \neq 0\) and \(\beta > 0\), relations (5.23)–(5.24) and (5.29) combined with Lemma 4.1 imply that \(\tau_{1n} \geq 0, \tau_{2n} \geq 0, \tau_{1n} + \tau_{2n} \asymp |\lambda_0^n|^2, |n| > n_0\).

Hence, \(\tau_{3n} \geq 0\), \(n \in \mathbb{Z}\). Since \(\tau_{1n} + \tau_{2n} + \tau_{3n} = \|f_0^n\|_2^2 - \|f_0^n, g_0^n\|_2^2 \to 0\) as \(n \to \infty\), then \(\tau_{jn} \to 0\) as \(n \to \infty, j \in \{1, 2, 3\}\). Then condition (5.30) implies that \(\lambda_0^n \to 0\) as \(n \to \infty\). Combining this with the fact that \(\tau_{3n} \to 0\) as \(n \to \infty\), implies that \(\tau_{4n} \to 0\) as \(n \to \infty\), where \(\tau_{4n}\) is defined in (5.27). Now, equivalence (5.28) finishes the proof.

In the next result, we reduce part of the condition (5.14) to an explicit condition on the coefficients \(a, b, c, d\) in the boundary conditions (3.14) in the difficult case \(b_2/b_1 \notin \mathbb{Q}\).

**Lemma 5.7.** Let boundary conditions (3.14) be regular, that is, \(u := ad - bc \neq 0\). Let also \(\beta = -b_2/b_1 \notin \mathbb{Q}\). Let \(\{\lambda_0^n\}_{n \in \mathbb{Z}}\) be a sequence of zeros of the characteristic determinant \(\Delta_0(\cdot)\) counting multiplicity. Let

\[
bc \neq 0, \quad |a| + |d| > 0, \quad \text{Im} \lambda_0^n \to 0 \quad \text{and} \quad z_n \to |bc| \quad \text{as} \quad n \to \infty,
\] (5.32)

where \(z_n\) is defined in (5.15). Then,

\[
|a| = |d| > 0, \quad u = ad - bc = d/\bar{a} \quad \text{and} \quad adbc < 0.
\] (5.33)

**Proof.** Since \(\text{Im} \lambda_0^n \to 0\) as \(n \to \infty\), \(|e_{1n}| \to 1\) as \(n \to \infty\). Further, since boundary conditions are regular, \(b_2/b_1 \notin \mathbb{Q}\), and \(bc \neq 0\), all considerations in the proof of Proposition 4.7 are valid. Since \(u - ad = -bc\), the second relation in (4.29) implies:

\[
z_n = (1 + de_{2n}) \cdot (1 + ae_{1n}) = \left(1 - d\frac{1 + ae_{1n}}{d + ue_{1n}}\right) \cdot (1 + ae_{1n}) = \frac{-bce_{1n}(1 + \bar{a}e_{1n})}{d + ae_{1n}} = \frac{-bc(e_{1n} + \bar{a}|e_{1n}|^2)}{d + ae_{1n}}, \quad n \in \mathbb{Z}.
\] (5.34)

Recall that \(d + ae_{1n} \asymp 1, n \in \mathbb{Z}\), as established in (4.30). Since \(z_n \to |bc|\) and \(|e_{1n}| \to 1\) as \(n \to \infty\), (5.34) implies that

\[
|bc|(d + ue_{1n}) + bc(e_{1n} + \bar{a}) \to 0 \quad \text{as} \quad n \to \infty,
\] (5.35)

or

\[
(|bc|u + bc)e_{1n} + |bc|d + b\bar{a} \to 0 \quad \text{as} \quad n \to \infty.
\] (5.36)

But by Proposition 4.7, the sequence \(\{e_{1n}\}_{n \in \mathbb{Z}}\) has an infinite set of limit points. Hence, relation (5.36) is possible only if

\[
|bc|u = -bc \quad \text{and} \quad |bc|d = -bc\bar{a}.
\] (5.37)

Since \(|a| + |d| > 0\) and \(bc \neq 0\), the second relation in (5.37) implies that \(|a| = |d| > 0\) and that \(bc\bar{a}d = -|bc||d|^2 < 0\). This implies the first and the third relations in (5.33). Further, combining both relations in (5.37) implies the second relation in (5.33): \(u = d/\bar{a} = -bc/|bc|\), which finishes the proof. \(\square\)
Now we are ready to state the main result of this section.

**Theorem 5.8.** Let boundary conditions (3.14) be strictly regular. Then, some normalized system of root vectors of the operator \( L_U(0) \) is a Bari \( c_0 \)-sequence in \( L^2([0, 1]; \mathbb{C}^2) \) (see Definition 2.2) if and only the operator \( L_U(0) \) is self-adjoint. The latter holds if and only if coefficients \( a, b, c, d \) from boundary conditions (3.14) satisfy the following relations:

\[
|a|^2 + \beta |b|^2 = 1, \quad |c|^2 + \beta |d|^2 = \beta, \quad a\bar{c} + \beta b\bar{d} = 0, \quad \beta := -b_2/b_1 > 0.
\]

(5.38)

In this case, every normalized system of root vectors of the operator \( L_U(0) \) is a Bari \( c_0 \)-sequence in \( L^2([0, 1]; \mathbb{C}^2) \).

**Proof.** (i) If conditions (5.38) hold, then by Corollary 3.9 the operator \( L_U(0) \) with boundary conditions (5.8) is self-adjoint. Remark 5.3 now finishes the proof.

(ii) Now assume that some normalized system of root vectors of the operator \( L_U(0) \) forms a Bari basis in \( L^2([0, 1]; \mathbb{C}^2) \). If \( b = c = 0 \), then Proposition 5.5 yields that \(|a| = |d| = 1\), in which case, operator \( L_U(0) \) is self-adjoint. This finishes the proof in this case.

Now let \(|b| + |c| \neq 0\). Proposition 5.6 implies that relations (5.14) take place. In particular, \(|c| = \beta |b|\). Consider three cases.

**Case A.** Let \( b_1/b_2 \in \mathbb{Q} \). In this case, \( b_1 = -m_1 b_0, b_2 = m_2 b_0 \), where \( b_0 > 0, m_1, m_2 \in \mathbb{N} \). Set \( m = m_1 + m_2 \). Since \( ad \neq bc \), \( \Delta_0(\lambda) e^{-ib_1\lambda} \) is a polynomial in \( e^{ib_0\lambda} \) of degree \( m \) with nonzero roots \( e^{ib_0\mu_k}, \mu_k \in \mathbb{C}, k \in \{1, \ldots, m\}, \) counting multiplicities. Hence, the sequence of zeros \( \{\lambda_0(n)\}_{n \in \mathbb{Z}} \) of \( \Delta_0(\cdot) \) is a union of arithmetic progressions \( \{\mu_k + 2\pi n\}_{n \in \mathbb{Z}} \), \( k \in \{1, \ldots, m\} \). Clearly, \( \rho \lambda_0(n) = \rho \mu_k/b_0 \) for some \( k \in \{1, \ldots, m\} \). It is clear, that if \( \rho \mu_k \neq 0 \), for some \( k \in \{1, \ldots, m\} \), then \( \rho \lambda_0(n) \) does not tend to 0 as \( n \to \infty \). Hence, \( \rho \lambda_0(0) = 0, n \in \mathbb{Z} \). This implies that

\[
E^{\pm}_{j,n} = \int_0^1 \left| e^{\pm 2ib_1\lambda_0^2(n)x} \right| dx = 1, \quad n \in \mathbb{Z}, \quad j \in \{1, 2\}.
\]

(5.39)

It is clear that \( e^{-ib_2\lambda_0^2(n)} = (e^{-ib_0\mu_k})^{m_2} \) for some \( k \in \{1, \ldots, m\} \), \( n \in \mathbb{Z} \), and \( \{\lambda_0(n)\}_{n \in \mathbb{Z}} \) is a periodic sequence. Hence, the sequence \( \{e^{-ib_2\lambda_0^2(n)}\}_{n \in \mathbb{Z}} \) is periodic. Similarly the sequence \( \{e^{ib_2\lambda_0^2(n)}\}_{n \in \mathbb{Z}} \) is periodic. Hence, the sequence

\[
\{z_n\}_{n \in \mathbb{Z}}, \quad z_n = \left(1 + de^{-ib_2\lambda_0^2(n)}(1 + ae^{ib_1\lambda_0^2(n)})\right) = (1 + de^{2ib_2})(1 + ae_n),
\]

(5.40)

is periodic. Since \( z_n \to |bc| \) as \( n \to \infty \) and \( |c| = \beta |b| \), it implies that \( z_n = |bc| = \beta |b|^2, n \in \mathbb{Z} \). It now follows from (5.22)–(5.27) and (5.39) that

\[
\|f_n^0\|_2^2 \cdot \|g_n^0\|_2^2 - \|(f_n^0, g_n^0)\|_2^2 = \tau_{4,n} = |z_n - \beta |b|^2|_2 = 0, \quad n \in \mathbb{Z}.
\]

(5.41)

Taking into account formula (5.3), we see that the normalized eigenvectors \( \bar{f}_n \) and \( \bar{g}_n \) of the operators \( L_U(0) \) and \( L_U^*(0) \) corresponding to the common eigenvalue \( \lambda_n^0 = \lambda_n^0 \) are equal for all \( n \in \mathbb{Z} \), which implies that \( L_U(0) = L_U^*(0) \).

**Case B.** Let \( a = d = 0 \). Then, \( z_n = 1, n \in \mathbb{Z} \). Since \( z_n \to |bc| \) as \( n \to \infty \), then \( |bc| = 1 \). Combined with \( |c| = \beta |b| \), this implies the desired condition (3.58), and finishes the proof in this case.

**Case C.** Finally, let \( b_1/b_2 \notin \mathbb{Q} \), \( |a| + |d| > 0 \), and \( bc \neq 0 \). Since \( \rho \lambda_0^2(n) \to 0 \) and \( z_n \to |bc| \) as \( n \to \infty \), Lemma 5.7 implies condition (5.33). In particular, \( |a| = |d| > 0 \) and \( adbc < 0 \). Since, in addition, \( |c| = \beta |b| > 0 \),

\[
-\rho \bar{a} \bar{c} \bar{b} \bar{d} = |ad \cdot bc| = |d|^2 \cdot |\beta |b|^2 = \beta |b|^2 \bar{a} \cdot \bar{d}.
\]

(5.42)

Since \( \rho \bar{d} \neq 0 \), this implies that \( \rho \bar{a} \rho \bar{c} \rho \bar{d} = 0 \) and coincides with the third condition in (5.38). Further, the second relation in (5.33), combined with relations (5.42), \(|a| = |d| \), and \(|c| = \beta |b| \), implies that

\[
0 = (-ad + bc + \bar{d}/\bar{a}) \bar{b} \bar{d} = -ad\bar{b} + |bc|^2 + \frac{ad\bar{b}}{|a|^2} = |ad| \cdot |bc| + |bc|^2 - \frac{|ad| \cdot |bc|}{|a|^2}
\]

\[
= |bc|(|ad| + |bc| - 1) = |bc|(|a|^2 + \beta |b|^2 - 1) = |bc|(|d|^2 + \beta^{-1} |c|^2 - 1).
\]

(5.43)

Since \( bc \neq 0 \), relation (5.43) implies the first and second relations in (5.38), which finishes the proof.
6 | THE PROOF OF THE MAIN RESULT

This section is devoted to the proof of the main result of the paper, Theorem 2.4. Throughout the section, we use the following notations:

$$\mathcal{H} := L^2([0, 1]; \mathbb{C}^2), \quad \| \cdot \| := \| \cdot \|_2 = \| \cdot \|_{\mathcal{H}}, \quad \text{and} \quad (\cdot, \cdot) := (\cdot, \cdot)_2 = (\cdot, \cdot)_{\mathcal{H}}. \quad (6.1)$$

First we need the following trivial corollary from Theorem 1.1.

**Corollary 6.1.** Let $Q \in L^p([0, 1]; \mathbb{C}^{2 \times 2})$ for some $p \in [1, 2]$ and boundary conditions (1.3) be strictly regular. Let $\mathcal{F} := \{f_n\}_{n \in \mathbb{Z}}$ be a system of root vectors of the operator $L_Q$ such that $\|f_n\| \asymp 1$, $n \in \mathbb{Z}$. Then, there exists a system of root vectors $\tilde{\mathcal{F}} := \{\tilde{f}_n\}_{n \in \mathbb{Z}}$ of the operators $L_{U(0)}$ and $L_{U(Q)}$, respectively, such that $\\{\|f_n - \tilde{f}_n\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$. Hence, $\\{\|f_n - \tilde{f}_n\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$. By Proposition 3.4, eigenvalues of $L_{U(Q)}$ are asymptotically simple. Hence, vectors $f_n$ and $\tilde{f}_n$, $|n| \geq N$, are proportional for some $N \in \mathbb{N}$, that is, $f_n = \alpha_n \tilde{f}_n$, $|n| \geq N$, for some $\alpha_n \in \mathbb{C}$. Let us set

$$\tilde{\mathcal{F}}_0 := \{\tilde{f}_n^0\}_{n \in \mathbb{Z}}, \quad f_n^0 := \left\{ \begin{array}{ll} \alpha_n \tilde{f}_n^0, & |n| \geq N, \\ \|f_n - f_n^0\|_{\ell^p}, & |n| < N. \end{array} \right. \quad (6.2)$$

It is clear that $\tilde{\mathcal{F}}_0$ is a system of root vectors of the operator $L_{U(0)}$ and $\|f_n^0\| = \|f_n\|$, $n \in \mathbb{Z}$. Moreover, $\|f_n - f_n^0\| = |\alpha_n| \cdot \|f_n - f_n^0\|_{\ell^p}, |n| \geq N$. Since $|\alpha_n| = \|f_n\| \asymp 1$, $|n| \geq N$, and $\{\|f_n - f_n^0\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$, then $\{\|f_n - f_n^0\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$, which finishes the proof. □

Now we are ready to prove our main result on Bari $(\ell^p)^*$-property.

**Proof of Theorem 2.4.** Recall that $Q \in L^p([0, 1]; \mathbb{C}^{2 \times 2})$ for some $p \in [1, 2]$. Also note that if $L_U(0)$ is self-adjoint, then Theorem 5.8 implies conditions (1.11) on the coefficients from boundary conditions (3.14).

(i) First assume that the operator $L_U(0)$ is self-adjoint and let $\mathcal{F} := \{f_n\}_{n \in \mathbb{Z}}$ be some normalized system of root vector of the operators $L_{U(Q)}$. By Corollary 6.1, there exists normalized system of root vectors $\mathcal{F}_0 := \{f_n^0\}_{n \in \mathbb{Z}}$ of the operator $L_{U(0)}$ such that $\{\|f_n - f_n^0\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$. Since $L_U(0)$ is self-adjoint, then $\{f_n^0\}_{n \in \mathbb{Z}}$ is an orthonormal basis in $\mathcal{F}$. If $p = 2$, then the proof would be already finished since $\mathcal{F}$ is $\ell^2$-close to the orthonormal basis $\mathcal{F}_0$. But as Remark 2.3 shows for $p \in [1, 2)$, the $(\ell^p)^*$-closeness to the orthonormal basis is not equivalent to the Bari $(\ell^p)^*$-property.

To this end, let $\mathcal{O} := \{g_n\}_{n \in \mathbb{Z}}$ be the system of vectors in $\mathcal{H}$ that is biorthogonal to the system $\mathcal{F}$ as we need to prove that $\{\|f_n - g_n\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$. Clearly, $\mathcal{O}$ is not normalized system of root vectors of the adjoint operator $L_U^*(Q)$. Since $L_U(0)$ is self-adjoint then by Lemma 3.8, we have $L_{U(Q)}^*(Q) = L_{U(Q^*)}$. Using Corollary 6.1 in the “opposite” direction, we can find a normalized system of root vectors $\tilde{\mathcal{O}} := \{\tilde{g}_n\}_{n \in \mathbb{Z}}$ of the operator $L_{U(Q^*)}$ such that $\{\|\tilde{g}_n - f_n^0\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$. Therefore, $\{\|\tilde{g}_n - f_n^0\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$. Since both systems $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are root vector systems of the operator $L_{U(Q^*)}^*(Q) = L_{U(Q^*)}^*$ and eigenvalues of $L_{U(Q^*)}^*$ are asymptotically simple due to Proposition 3.4, vectors $g_n$ and $\tilde{g}_n$, $|n| \geq N$, are proportional for some $N \in \mathbb{N}$. Since $(f_n, g_n) = 1$, $n \in \mathbb{Z}$, it follows that $\tilde{g}_n = (f_n, \tilde{g}_n)g_n$, $|n| \geq N$. Note that if $f, g \in \mathcal{H}$ and $\|f\| = 1$, then

$$|(f, g) - 1|^2 = |(f, g)|^2 + 1 - 2 \text{Re}(f, g) \leq \|f\|^2 \|g\|^2 + 1 - 2 \text{Re}(f, g) = \|f\|^2 + \|g\|^2 - 2 \text{Re}(f, g) = \|f - g\|^2. \quad (6.3)$$

Since $\|f_n\| = 1$, $n \in \mathbb{Z}$, (6.3) implies that $|(f_n, \tilde{g}_n) - 1| \leq \|f_n - \tilde{g}_n\|$, $n \in \mathbb{Z}$. Hence for $|n| \geq N$, we have,

$$\|\tilde{g}_n - g_n\| = \|(f_n, \tilde{g}_n)g_n - g_n\| = |(f_n, \tilde{g}_n) - 1| \cdot \|g_n\| \leq \|f_n - \tilde{g}_n\| \cdot \|g_n\|. \quad (6.4)$$

By the main result of [25, 28, 43], the system $\mathcal{F} = \{f_n\}_{n \in \mathbb{Z}}$ is a Riesz basis in $\mathcal{H}$. Hence so is its biorthogonal system $\mathcal{O} = \{g_n\}_{n \in \mathbb{Z}}$. This in particular implies that $\|g_n\| \asymp 1$, $n \in \mathbb{Z}$. Since $\{\|f_n - \tilde{g}_n\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$ and $\|g_n\| \asymp 1$, $n \in \mathbb{Z}$, then inequality (6.4) implies that $\{\|\tilde{g}_n - g_n\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$, which in turn implies the desired inclusion $\{\|f_n - g_n\|\}_{n \in \mathbb{Z}} \in (\ell^p)^*$.
(ii) Now assume that some normalized system of root vectors \( \mathfrak{F} := \{ f_n \}_{n \in \mathbb{Z}} \) of the operator \( L_U(Q) \) is a Bari \((\ell^p)^*\)-sequence in \( \mathfrak{H} \). By definition \( \{ \| f_n - g_n \| \}_{n \in \mathbb{Z}} \in (\ell^p)^* \), where \( \mathfrak{G} := \{ g_n \}_{n \in \mathbb{Z}} \) is a system biorthogonal to \( \mathfrak{F} \) in \( \mathfrak{H} \). Clearly, \( \mathfrak{G} \) is a system of root vectors of the adjoint operator \( L_U^*(Q) \). By Corollary 6.1, there exists normalized system of root vectors \( \mathfrak{F}_0 := \{ f_0^n \}_{n \in \mathbb{Z}} \) of the operator \( L_U(0) \) such that \( \{ \| f_n - f_0^n \| \}_{n \in \mathbb{Z}} \in (\ell^p)^* \). Similarly, there exists (possibly not normalized) system of root vectors \( \mathfrak{G}_0 := \{ g_0^n \}_{n \in \mathbb{Z}} \) of the operator \( L_U^*(0) = L_{U^*}(0) \) such that \( \{ \| g_n - g_0^n \| \}_{n \in \mathbb{Z}} \in (\ell^p)^* \). It is clear, now that \( \{ \| f_0^n - g_0^n \| \}_{n \in \mathbb{Z}} \in (\ell^p)^* \).

Let \( \mathfrak{G}_0 := \{ g_0^n \}_{n \in \mathbb{Z}} \) be a system biorthogonal to \( \mathfrak{F}_0 \). As in part (i), \( \mathfrak{G}_0 \) is a Riesz basis in \( \mathfrak{H} \) and \( \{ f_0^n - g_0^n \| \}_{n \in \mathbb{Z}} \in (\ell^p)^* \) implies that \( \{ \| f_0^n - g_0^n \| \}_{n \in \mathbb{Z}} \in (\ell^p)^* \). Hence,

\[
\| g_0^n - g_0^n \| = \| f_0^n - g_0^n \| - 1 \| f_0^n - g_0^n \| \leq \| f_0^n - g_0^n \| \cdot \| g_0^n \|. \quad (6.5)
\]

Since \( \mathfrak{G}_0 \) is a Riesz basis, \( \| g_0^n \| \approx 1 \), \( n \in \mathbb{Z} \). Thus, inequality (6.5) and inclusion \( \{ \| f_0^n - g_0^n \| \}_{n \in \mathbb{Z}} \in (\ell^p)^* \) imply that \( \{ \| f_0^n - g_0^n \| \}_{n \in \mathbb{Z}} \in (\ell^p)^* \), which means that the normalized root vectors system \( \mathfrak{F}_0 = \{ f_0^n \}_{n \in \mathbb{Z}} \) of the operator \( L_U(0) \) is a Bari \((\ell^p)^*\)-sequence and, in particular, is a Bari \( c_0 \)-sequence. Theorem 5.8 now implies that the operator \( L_U(0) \) is self-adjoint and finishes the proof.

\[\Box\]

7 APPLICATION TO AN NONCANONICAL STRING EQUATION

In this section, we show the connection of \( 2 \times 2 \) Dirac type operators with a noncanonical string equation with \( u_{xx} \) term, and apply our results on Riesz and Bari basis property.

Consider the following noncanonical hyperbolic equation on a complex-valued function \( u(x, t) \) defined for \( x \in [0, 1] \) and \( t \in [0, \infty) \):

\[
u_{tt} - (\beta_1 + \beta_2)u_{xt} + \beta_1 \beta_2 u_{xx} + a_1(x)u_x + a_2(x)u_t = 0,
\]

(7.1)

with the boundary conditions

\[
u(0, t) = 0, \quad h_0 u_0(0, t) + h_1 u_x(0, t) + h_2 u_t(1, t) = 0, \quad t \in [0, \infty),
\]

(7.2)

and initial conditions

\[
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in [0, 1].
\]

(7.3)

Here, \( \beta_1, \beta_2 \) are constants and

\[
\beta_1 < 0 < \beta_2, \quad a_1, a_2 \in L^1[0, 1], \quad h_0, h_1, h_2 \in \mathbb{C}, \quad |h_1| + |h_2| > 0.
\]

(7.4)

If \( -\beta_1 = \beta_2 = \rho^{-1} > 0 \) and \( h_0 = 0 \), the initial-BVP (7.1)–(7.3) governs the small vibrations of a string of length \( 1 \) and density \( \rho \) with the presence of a damping coefficient \( a_2(x) \); the string is fixed at the left end \( (x = 0) \), while the right end \( (x = 1) \) is damped with the coefficient \( h_2/h_1 \in \mathbb{C} \cup \{\infty\} \). Functions \( u_0 \) and \( u_1 \) represent the initial position and velocity of the string, respectively.

If \( -\beta_1 \neq \beta_2 \), one can use linear transform of the variables \( x \) and \( t \) to reduce it to a classical string equation, but with damping that depends on \( t \) and nonclassical initial and boundary conditions: Initial condition will be on a segment nonparallel to the \( x \)-axis \( (t = 0) \), while boundary conditions will be on the rays nonparallel to the \( t \)-axis \( (x = 0) \).

Recall that \( W^{1,p}[0, 1] \), \( p \geq 1 \), denotes the Sobolev space of absolutely continuous functions with the finite norm

\[
\| f \|_{W^{1,p}[0, 1]}^p := \int_0^1 (|f(x)|^p + |f'(x)|^p) \, dx < \infty.
\]

(7.5)

For convenience, we introduce the following notations:

\[
\tilde{W}^{1,p}[0, 1] := \{ f \in W^{1,p}[0, 1] : f(0) = 0 \}, \quad \tilde{H}^1_{0,1} := \tilde{W}^{1,2}[0, 1],
\]

(7.6)

where \( p \in [1, \infty) \).
The noncanonical initial-BVP (7.1)–(7.3) of a damped string can be transformed into an abstract Cauchy problem in a Hilbert space $\mathfrak{H}$ of the form

$$\mathfrak{H} := \tilde{H}^1_0[0,1] \times L^2[0,1],$$

(7.7)

with the inner product

$$\langle f, g \rangle_{\mathfrak{H}} := \int_0^1 \left( f_1'(x) \cdot \overline{g_1(x)} + f_2(x) \cdot \overline{g_2(x)} \right) dx,$$

(7.8)

where $f = \text{col}(f_1, f_2)$, $g = \text{col}(g_1, g_2) \in \mathfrak{H}$.

Now the new representation of the problem (7.1)–(7.2) reads as follows:

$$\dot{Y}(t) = i\mathcal{L}Y(t), \quad Y(t) := \begin{pmatrix} u(\cdot, t) \\ u_t(\cdot, t) \end{pmatrix}, \quad t \geq 0, \quad Y(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

(7.9)

where the linear operator $\mathcal{L} : \text{dom}(\mathcal{L}) \to \mathfrak{H}$ is defined by

$$\mathcal{L}y = \mathcal{L} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -i \begin{pmatrix} \beta_1 \beta_2 y_1'' + (\beta_1 + \beta_2)y_2' - a_1 y_1' - a_2 y_2 \\ y_2 \end{pmatrix},$$

(7.10)

where $y = \text{col}(y_1, y_2) \in \text{dom}(\mathcal{L})$, with

$$\text{dom}(\mathcal{L}) = \{ y = \text{col}(y_1, y_2) \in \mathfrak{H} : y_1' \in W^{1,1}[0,1], \mathcal{L}y \in \mathfrak{H}, \quad h_0 y_1'(0) + h_1 y_1'(1) + h_2 y_2(1) = 0 \}. \quad (7.11)$$

It is clear from the definition of $\mathcal{L}$ and $\text{dom}(\mathcal{L})$ that for $y = \text{col}(y_1, y_2) \in \text{dom}(\mathcal{L})$ we have: $y_1 \in \tilde{W}^{1,1}_0[0,1]$ and $y_2 \in \tilde{H}^1_0[0,1]$. In particular, $y_1(0) = y_2(0) = 0$.

Spectral properties of the operator $\mathcal{L}$ play important role in the study of stability of solutions of the corresponding string equation. For example, Riesz basis property of the root vectors system of $\mathcal{L}$ guarantees the exponential stability of the corresponding $C_0$-semigroup. The Riesz basis property and behavior of the spectrum of the operator $\mathcal{L}$ have been studied in numerous papers (see [6, 9, 10, 14, 18, 40, 47, 48] and references therein).

Let us show that the operator $\mathcal{L}$ is similar to a certain $2 \times 2$ Dirac-type operator $L_U(\mathcal{Q})$. Since many spectral properties are preserved under similarity transform, known spectral properties for $2 \times 2$ Dirac-type operators will translate to corresponding properties of the dynamic generator $\mathcal{L}$.

To this end, we need to introduce some notations. Set

$$B := \text{diag}(b_1, b_2), \quad b_1 := \beta_1^{-1}, \quad b_2 := \beta_2^{-1},$$

(7.12)

$$Q(x) := \frac{i}{b_2 - b_1} \begin{pmatrix} 0 & w(x) \cdot \left( b_2^2 a_1(x) + b_2 a_2(x) \right) \\ -w(x) \cdot \left( b_1^2 a_1(x) + b_1 a_2(x) \right) & 0 \end{pmatrix},$$

(7.13)

where

$$w(x) := w_1(x)w_2(x), \quad w_j(x) := \exp \left( \frac{b_1 b_2}{b_2 - b_1} \int_0^x (b_j a_1(t) + a_2(t)) dt \right), \quad x \in [0,1], \quad j \in \{1, 2\}. \quad (7.14)$$

Note, that $w_1(\cdot)$, $w_2(\cdot)$ are well defined and $Q \in L^1([0,1], \mathbb{C}^{2 \times 2})$ in view of condition (7.4). Finally, let

$$U_1(y) := y_1(0) + y_2(0) = 0,$$

(7.15)

$$U_2(y) := b_1 h_0 y_1(0) + b_2 h_0 y_2(0) + (b_1 h_1 + h_2)w_1^{-1}(1)y_1(1) + (b_2 h_1 + h_2)w_2(1)y_2(1) = 0,$$

(7.16)

be boundary conditions for a Dirac operator $L_U(\mathcal{Q})$. Here, $w_1(\cdot)$, $w_2(\cdot)$ are given by (7.14).

**Proposition 7.1.** Operator $\mathcal{L}$ is similar to the $2 \times 2$ Dirac-type operator $L_U(\mathcal{Q})$ with the matrix $B$ given by (7.12), the potential matrix $Q(\cdot)$ given by (7.13), and boundary conditions $U \cdot y = \{ U_1, U_2 \} y = 0$ given by (7.15)–(7.16).
Proof. We will transform the operator $\mathcal{L}$ into the desired operator $L_{\mathcal{L}}(Q)$ via series of similarity transformations.

**Step 1.** Define

$$V_0 : \mathcal{H} \to L^2([0,1]; \mathbb{C}^2) \quad \text{as} \quad V_0 y := \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{H}. \quad (7.17)$$

Since $\frac{d}{dx}$ isometrically maps $\tilde{H}^1_0[0,1] = \{ f \in W^{1,2}[0,1] : f(0) = 0 \}$ onto $L^2[0,1]$, then the operator $V_0$ is bounded with bounded inverse. It is easy to verify that

$$L_1 y := V_0 L V_0^{-1} y = -i \begin{pmatrix} -\bar{\beta}_2 y_1' + (\beta_1 + \bar{\beta}_2) y_2' - a_1 y_1 - a_2 y_2 \\ y_2' \end{pmatrix}, \quad (7.18)$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \text{dom}(L_1) := V_0 \text{dom}(\mathcal{L}) = \{ y \in W^{1,1}([0,1]; \mathbb{C}^2) : \quad (7.19) \}$$

$$L_1 y \in L^2([0,1]; \mathbb{C}^2), \quad y_2(0) = 0, \quad h_0 y_1(0) + h_1 y_1(1) + h_2 y_2(1) = 0, \quad (7.20)$$

in view of (7.11) and definition of $\tilde{H}^1_0[0,1]$. Thus, the operator $\mathcal{L}$ is similar to the operator $L_1$,

$$L_1 y = -i B_1 y' + Q_1(x) y,$$

with the domain $\text{dom}(L_1)$ given by (7.19), and the matrices $B_1, Q_1(\cdot)$, given by

$$B_1 := \begin{pmatrix} 0 & 1 \\ -\beta_1 \beta_2 & \beta_1 + \beta_2 \end{pmatrix}, \quad Q_1(x) := \begin{pmatrix} 0 & 0 \\ i a_1(x) & i a_2(x) \end{pmatrix}. \quad (7.21)$$

Note, that $Q_1 \in L^1([0,1], \mathbb{C}^{2\times2})$ in view of condition (7.4).

**Step 2.** Next, we diagonalize the matrix $B_1$. To this end, let

$$V_1 := \begin{pmatrix} 1/\beta_1 & 1/\beta_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ 1 & 1 \end{pmatrix}, \quad \text{and so} \quad V_1^{-1} = \frac{1}{b_2 - b_1} \begin{pmatrix} -1 & b_2 \\ 1 & -b_1 \end{pmatrix}, \quad (7.22)$$

where $b_1$ and $b_2$ are defined in (7.12). We easily get after straightforward calculations that

$$V_1^{-1} B_1 V_1 = \text{diag}(\beta_1, \beta_2) = \text{diag}(b_1^{-1}, b_2^{-1}) = B^{-1}, \quad (7.23)$$

$$V_1^{-1} Q_1(x) V_1 = \frac{i}{b_2 - b_1} \begin{pmatrix} b_1 b_2 a_1(x) + b_2 a_2(x) & b_2 a_1(x) + b_2 a_2(x) \\ -b_2^2 a_1(x) - b_1 a_2(x) & b_2 b_1 a_1(x) - b_2 a_2(x) \end{pmatrix} =: Q_2(x), \quad x \in [0,1]. \quad (7.24)$$

Note, that $Q_2 \in L^1([0,1], \mathbb{C}^{2\times2})$ in view of condition (7.4). Introducing bounded operator $V_1 : y \to V_1 y$ in $L^2([0,1]; \mathbb{C}^2)$, noting that it has a bounded inverse, and taking into account (7.23) and (7.24), we obtain

$$L_2 y := V_1^{-1} L_1 V_1 y = -i V_1^{-1} B_1 V_1 y' + V_1^{-1} Q_1(x) V_1 y = -i B^{-1} y' + Q_2(x) y, \quad y \in V_1^{-1} \text{dom}(L_1) =: \text{dom}(L_2), \quad (7.25)$$

where

$$\text{dom}(L_2) = \{ y \in W^{1,1}([0,1]; \mathbb{C}^2) : \quad L_2 y \in L^2([0,1]; \mathbb{C}^2), \quad y_1(0) + y_2(0) = 0, \quad b_1 h_0 y_1(0) + b_2 h_0 y_2(0) + (b_1 h_1 + h_2) y_1(1) + (b_2 h_1 + h_2) y_2(1) = 0, \quad (7.26)$$

with account of formula (7.19) for the domain $\text{dom}(L_1)$ and the formula (7.22) for the matrix $V_1$. 

Step 3. In this step, we make potential matrix $Q_2$ to be off-diagonal. To this end, let $\tilde{Q}_2$ be a diagonal of $Q_2$, that is,

$$\tilde{Q}_2(x) = \begin{pmatrix} c_1 b_2 a_1(x) + b_2 a_2(x) & 0 \\ 0 & -c_1 b_2 a_1(x) - b_1 a_2(x) \end{pmatrix}.$$ 

Let $V_2(\cdot)$ be a solution of the initial value problem

$$-iB^{-1}V_2'(x) + \tilde{Q}_2(x)V_2(x) = 0, \quad V_2(0) = I_2. \quad (7.27)$$

It is easily seen that

$$V_2(x) := \begin{pmatrix} w_1(x) & 0 \\ 0 & w_2(x) \end{pmatrix}, \quad x \in [0,1], \quad (7.28)$$

where $w_1(\cdot)$, $w_2(\cdot)$ are defined in (7.14). Let us introduce operator $\mathcal{L}_2(\cdot):y \mapsto V_2(x)y$ in $L^2([0,1];\mathbb{C}^2)$. Since $a_1, a_2 \in L^1[0,1]$, the operator $\mathcal{L}_2$ is bounded and has a bounded inverse. Combining relation (7.27), definition (7.13) of $Q$, and definition (7.14) of $w$, we get

$$\begin{align*}
\mathcal{L}_3 y &= \mathcal{L}_2^{-1} \mathcal{L}_2 \mathcal{L}_2 y = -i[V_2(x)]^{-1}B^{-1}V_2(x)y' + [V_2(x)]^{-1}(-iB^{-1}V_2'(x) + Q_2(x)V_2(x))y \\
&= -iB^{-1}y' + [V_2(x)]^{-1}(Q_2(x) - \tilde{Q}_2(x))V_2(x)y = -iB^{-1}y' + Q(x)y, \quad y \in V_2^{-1}(\text{dom}(L_2)) =: \text{dom}(L_3). \quad (7.29)
\end{align*}$$

It is clear from the definition of $V_2$ that $\text{dom}(L_3)$ coincides with $\text{dom}(L_U(Q))$ defined via (7.15)–(7.16). Hence, $L_3 = L_U(Q)$. Combining all the steps of the proof, one concludes that $L$ is similar to $L_U(Q)$.

Combining Proposition 7.1 with our previous results for $2 \times 2$ Dirac-type operators, we obtain the Riesz basis property and analogous of Bari basis property for the dynamic generator $L$ of the noncanonical initial-BVP (7.1)–(7.3) for a damped string equation. Part (i) of the following result improves known results in the literature on the Riesz basis property for the operator $L$ in the case $-\gamma_1 = -\gamma_2$, $\aleph_1 \equiv 0, h_0 = 0$ (see [6, 9, 10, 14, 18, 40, 47, 48] and references therein). Part (ii) shows the application of one of our main results, Theorem 1.3.

**Theorem 7.2.** (i) Let parameters of the damped string equation satisfy relaxed conditions (7.4),

$$\beta_2 h_2 + h_1 \neq 0, \quad \beta_2 h_2 + h_1 \neq 0, \quad (7.30)$$

and let in addition boundary conditions (7.15)–(7.16) be strictly regular. Then, the system of root vectors of the operator $L$ forms a Riesz basis in $\mathfrak{D} = \mathbb{H}^1_0[0,1] \times L^2[0,1]$. (ii) Let in addition $a_1, a_2 \in L^2[0,1]$. Let also $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2$ be the operators defined in the steps of the proof of Proposition 7.1. Then, the system of root vectors of the operator $L$ is quadratically close in $\mathfrak{D}$ to a system of the form $\{V_0^{-1}V_1 \mathcal{V}_2 \mathcal{V}_2 e_n\}_{n \in \mathbb{Z}}$, where $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis in $L^2([0,1];\mathbb{C}^{2 \times 2})$, if and only if boundary conditions (7.15)–(7.16) are self-adjoint, which is equivalent to the condition

$$h_0 = 0, \quad \beta_1 = -\beta_2, \quad \int_0^1 \text{Im} a_2(t) dt = \beta_2 \log \left| \frac{\beta_2 h_2 + h_1}{\beta_2 h_2 - h_1} \right|. \quad (7.31)$$

**Proof.** First, let us transform boundary conditions (7.15)–(7.16) to a canonical form (1.6) assuming condition (7.30). For this, we multiply the first condition $U_1$ by $b_1 h_0$ and subtract from $U_2$ and then multiple the second condition $U_2$ by $(b_2 h_1 + h_2)^{-1} w_2^{-1}(1)$. Boundary conditions (7.15)–(7.16) will take the form

$$\begin{cases}
\check{U}_1(y) = y_1(0) + y_2(0) = 0, \\
\check{U}_2(y) = dy_2(0) + cy_1(1) + y_2(1) = 0,
\end{cases} \quad (7.32)$$

where

$$d = \frac{(b_2 - b_1) h_0}{(b_2 h_1 + h_2) w_2(1)} \quad \text{and} \quad c = \frac{b_1 h_1 + h_2}{(b_2 h_1 + h_2) w(1)}. \quad (7.33)$$
Here, \( w, w_1, w_2 \) are given by (7.14). In particular,
\[
w(1) := \exp \left( \frac{b_1 b_2}{b_2 - b_1} \int_0^1 ((b_1 + b_2)a_1(t) + 2a_2(t)) dt \right).
\] (7.34)

(i) Proposition 7.1 implies that the operator \( \mathcal{L} \) is similar to the operator \( L_U(Q) \) with the matrix \( B \) given by (7.12), the potential matrix \( Q(\cdot) \) given by (7.13), and boundary conditions \( U_y = \{U_1, U_2\} y = 0 \) given by (7.15)–(7.16). Note that condition (7.30) implies regularity of boundary conditions (7.15)–(7.16). In addition, they are strictly regular by the assumption. Hence, operator \( L_U(Q) \) has compact resolvent and by Proposition 3.4, its eigenvalues are asymptotically simple and separated. Moreover, Theorem 1.1 from [28] implies that the system of root vectors of the operator \( L_U(Q) \) forms a Riesz basis in \( L^2([0,1];\mathbb{C}^2) \). Similarity of \( \mathcal{L} \) and \( L_U(Q) \) implies the same properties for \( \mathcal{L} \) in the space \( \mathfrak{H} \), which finishes the proof of part (i).

(ii) Since \( a_1, a_2 \in L^2[0,1] \), it follows that \( Q \in L^2([0,1];\mathbb{C}^{2 \times 2}) \). Since boundary conditions (7.32) are strictly regular by Theorem 1.3, any and every system of root vectors of the operator \( L_U(Q) \) forms a Bari basis in \( L^2([0,1];\mathbb{C}^2) \) if and only if boundary conditions (7.32) are self-adjoint, which in turn is equivalent to conditions (1.11). Since \( a = 0 \) and \( b = 1 \), (1.11) is equivalent to
\[
d = 0, \quad b_1 = -b_2, \quad |c| = 1.
\] (7.35)

Since \( \tilde{\beta}_1 = b_1^{-1} \) and \( \tilde{\beta}_2 = b_2^{-1} \), this in turn is equivalent to (7.31).

Let us set \( V := Y_0^{-1}Y_1Y_2 \) and let \( \{V f_n\}_{n \in \mathbb{Z}} \) be some system of root vectors of the operator \( L_U(Q) \). It follows from the proof of Proposition 7.1 that \( \{V f_n\}_{n \in \mathbb{Z}} \) is a system of root vectors of the operator \( \mathcal{L} \). Hence, \( \{f_n\}_{n \in \mathbb{Z}} \) is quadratically close to an orthonormal basis \( \{e_n\}_{n \in \mathbb{Z}} \) in \( L^2([0,1];\mathbb{C}^2) \) if and only if \( \{V f_n\}_{n \in \mathbb{Z}} \) is quadratically close to \( \{Ve_n\}_{n \in \mathbb{Z}} \) in \( \mathfrak{H} \). This completes the proof. \( \square \)

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REFERENCES

[1] A. V. Agibalova, A. A. Lunyov, M. M. Malamud, and L. L. Oridoroga, Completeness property of one-dimensional perturbations of normal and spectral operators generated by first order systems, Integral Equations Operator Theory 91 (2019), Article number 37, 35 p.
[2] A. V. Agibalova, M. M. Malamud, and L. L. Oridoroga, On the completeness of general boundary value problems for \( 2 \times 2 \) first-order systems of ordinary differential equations, Methods Funct. Anal. Topology 18 (2012), no. 1, 4–18.
[3] B. P. Allahverdiev, Eigenvalue problems for a non-self-adjoint Bessel-type operators in limit-point case, Math. Methods Appl. Sci. 37 (2014), no. 18, 2946–2951.
[4] N. K. Bari, Biorthogonal systems and bases in Hilbert space, (Russian) Moskov. Gos. Univ. Učenye Zapiski Matematika 148 (1951), no. 4, 69–107.
[5] A. G. Baskakov, A. V. Derbushev, and A. O. Shcherbakov, The method of similar operators in the spectral analysis of non-self-adjoint Dirac operators with non-smooth potentials, Izv. Math. 75 (2011), no. 3, 445–469.
[6] A. Benaddi and B. Rao, Energy decay rate of wave equations with indefinite damping, J. Differential Equations 161 (2000), no. 2, 337–357.
[7] G. D. Birkhoff and R. E. Langer, The boundary problems and developments associated with a system of ordinary differential equations of the first order, Proc. Amer. Acad. Arts Sci. 58 (1923), 49–128.
[8] B. Bodenstorfer, A. Dijksma, and H. Langer, Dissipative eigenvalue problems for a Sturm-Liouville operator with a singular potential, Proc. Roy. Soc. Edinburgh Sect. A 130A (2000), no. 6, 1237–1257.
[9] S. Cox and E. Zuazua, The rate at which energy decays in a damped string, Comm. Partial Differential Equations 19 (1994), no. 1-2, 213–243.
[10] S. Cox and E. Zuazua, The rate at which energy decays in a string damped at one end, Indiana Univ. Math. J. 44 (1995), no. 2, 545–573.
[11] P. Djakov and B. Mityagin, Bari-Markus property for Riesz projections of ID periodic Dirac operators, Math. Nachr. 283 (2010), no. 3, 443–462.
[12] P. Djakov and B. Mityagin, Criteria for existence of Riesz bases consisting of root functions of Hill and ID Dirac operators, J. Funct. Anal. 263 (2012), no. 8, 2300–2332.
[13] P. Djakov and B. Mityagin, Unconditional convergence of spectral decompositions of ID Dirac operators with regular boundary conditions, Indiana Univ. Math. J. 61 (2012), no. 1, 359–398.
[14] F. Gesztesy and H. Holden, The damped string problem revisited, J. Differential Equations 251 (2011), no. 4-5, 1086–1127.
Ya.V.Mykytyuk and D.V.Puyda, Bari-Markus property for Dirac operators

A.S.Markus, Baribases of subspaces

S.Hassi and L.Oridoroga, Theorem of completeness for a Dirac-type operator with generalized

M.Shubov, Basis property of eigenfunctionsofnonself-adjoint operator pencilsgenerated by the equation of nonhomogeneous damped string.

Spectral analysis of a one-dimensional Dirac system with summable potential and a Sturm-Liouville operator with distribution coefficients, (Russian) Sovrem. Mat. Fundam. Napravl. 66 (2020), no. 3, 373–530.

M. Shubov, Basis property of eigenfunctions of nonself-adjoint operator pencils generated by the equation of nonhomogeneous damped string. Integral Equations Operator Theory 25 (1996), no. 3, 289–328.

B. M. Levitan and I. S. Sargsyan, Sturm-Liouville and Dirac operators

V. É. Katsnel’son, Exponential bases in $L^2$, Funct. Anal. Appl. 5 (1971), no. 1, 31–38.

V. M. Kurbanov and A. M. Abdullayeva, Bessel property and basicity of the system of root vector-functions of Dirac operator with summable coefficient, Oper. Matrices 12 (2018), no. 4, 943–954.

V. M. Kurbanov and G. R. Gadzhieva, Bessel inequality and the basis property for $2m \times 2m$ Dirac-type system with an integrable potential, Differ. Eq. 56 (2020), no. 5, 573–584.

B. M. Levitan and I. S. Sargsyan, Sturm-Liouville and Dirac operators, Translated from the Russian. Mathematics and Its Applications (Soviet Series), 59. Kluwer Academic Publishers Group, Dordrecht, 1991.

A. A. Lunyov and M. M. Malamud, On the Riesz basis property of the root vector system for Dirac-type $2 \times 2$ systems, Dokl. Math. 90 (2014), no. 2, 556–561.

A. A. Lunyov and M. M. Malamud, On spectral synthesis for dissipative Dirac-type operators, Integral Equations Operator Theory 90 (2014), 79–106.

A. A. Lunyov and M. M. Malamud, On the completeness and Riesz basis property of root subspaces of boundary value problems for first order systems and applications, J. Spectr. Theory 5 (2015), no. 1, 17–70 (arXiv:1401.2574).

A. A. Lunyov and M. M. Malamud, On the Riesz basis property of root subspaces of boundary value problems for first order systems and applications, J. Math. Anal. Appl. 441 (2016), 57–103 (arXiv:1504.04954).

A. A. Lunyov and M. M. Malamud, Stability of spectral characteristics of boundary value problems for $2 \times 2$ Dirac type systems. Applications to the damped string, J. Differential Equations 313 (2022), 633–742 (arXiv:2012.11170).

A. S. Makin, On summability of spectral expansions corresponding to the Sturm-Liouville operator, Inter. J. Math. and Math. Sci. 2012 (2012) 843562.

A. S. Makin, On convergence of spectral expansions of Dirac operators with regular boundary conditions, Math. Nachr. 295 (2022), no. 1, 189–210.

A. S. Makin, Regular boundary value problems for the Dirac operator, Dokl. Math. 101 (2020), no. 3, 214–217.

M. M. Malamud, Similarity of Volterra operators and related questions of the theory of differential equations of fractional order, Trans. Moscow Math. Soc. 55 (1994), 57–122.

M. M. Malamud, Questions of uniqueness in inverse problems for systems of differential equations on a finite interval, Trans. Moscow Math. Soc. 60 (1999), 173–224.

M. M. Malamud and L. L. Oridoroga, Completeness theorems for systems of differential equations, Funct. Anal. Appl. 34 (2000), no. 4, 308–310.

M. M. Malamud and L. L. Oridoroga, On the completeness of root subspaces of boundary value problems for first order systems of ordinary differential equations, J. Funct. Anal. 263 (2012), 1939–1980.

V. A. Marchenko, Sturm-Liouville operators and applications, Operator Theory: Advances and Applications, vol. 22, Birkhäuser Verlag, Basel, 1986.

A. S. Markus, Bari bases of subspaces, Math. Notes 5 (1969), no. 4, 277–281.

Ya. V. Mykytyuk and D. V. Puyda, Bari-Markus property for Dirac operators, Mat. Stud. 40 (2013), no. 2, 165–171.

L. Rzepnicki, The basis property of eigenfunctions in the problem of a nonhomogeneous damped string, Opuscula Math. 37 (2017), no. 1, 141–165.

L. Rzepnicki, Asymptotic behavior of solutions of the Dirac system with an integrable potential, Integral Equations Operator Theory 93 (2021), no. 55, 24 p.

I. V. Sadovnichaya, Uniform asymptotics of the eigenvalues and eigenfunctions of the Dirac system with an integrable potential, Differ. Equ. 52 (2016), no. 8, 1000–1010.

A. M. Savchuk and A. A. Shkalikov, The Dirac operator with complex-valued summable potential, Math. Notes 96 (2014), no. 5-6, 777–810.

A. M. Savchuk and I. V. Sadovnichaya, The Riesz basis property of subspaces for a Dirac system with summable potential, Dokl. Math. 91 (2015), no. 3, 309–312.

A. M. Savchuk and I. V. Sadovnichaya, The Riesz basis property with brackets for the Dirac system with a summable potential, J. Math. Sci. (N. Y.) 233 (2018), no. 4, 514–540; translated from Sovrem. Mat. Fundam. Napravl. 58 (2015), 128–152 (in Russian).

A. M. Savchuk and I. V. Sadovnichaya, Spectral analysis of a one-dimensional Dirac system with summable potential and a Sturm-Liouville operator with distribution coefficients, (Russian) Sovrem. Mat. Fundam. Napravl. 66 (2020), no. 3, 373–530.

L. Rzepnicki, Asymptotic behavior of solutions of the Dirac system with an integrable potential, Integral Equations Operator Theory 93 (2021), no. 55, 24 p.
[48] M. Shubov, Nonself-adjoint operators generated by the equation of a nonhomogeneous damped string, Trans. Amer. Math. Soc. 349 (1997), no. 11, 4481–4499.

[49] E. Stein and R. Shakarchi, Fourier analysis. An introduction, Princeton Lectures in Analysis, I. Princeton University Press, Princeton, NJ, 2003.

[50] I. Trooshin and M. Yamamoto, Riesz basis of root vectors of a nonsymmetric system of first-order ordinary differential operators and application to inverse eigenvalue problems, Appl. Anal. 80 (2001), 19–51.

[51] I. Trooshin and M. Yamamoto, Spectral properties and an inverse eigenvalue problem for nonsymmetric systems of ordinary differential operators, J. Inverse Ill-Posed Probl. 10 (2002), no. 6, 643–658.

[52] P. E. Zhidkov, On the Bari basis property of the eigenfunction system of a nonlinear integro-differential equation, Differ. Equ. 38 (2002), no. 9, 1260–1267.

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