A General Inequality for Packings of Boxes

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Abstract

Keller packings and tilings of boxes are investigated. Certain general inequality measuring a complexity of such systems is proved. A straightforward application to the unit cube tilings is given:

Let \( \mathbf{m} = (m_1, \ldots, m_d) \) be a vector with integer coordinates such that \( m_i \geq 2 \) for every \( i \). Let \( \mathbf{mZ}^d = m_1 \mathbb{Z} \times \cdots \times m_d \mathbb{Z} \). We identify the \( d \)-dimensional torus \( \mathbb{R}^d / \mathbf{mZ}^d \) with \( \mathbb{T}^d_m = [0, m_1) \times [0, m_2) \times \cdots \times [0, m_d) \). The addition \( \oplus \) in \( \mathbb{T}^d_m \) is transferred from \( \mathbb{R}^d / \mathbf{mZ}^d \); that is, \( x \oplus y \in \mathbb{T}^d_m \) is the only element satisfying the congruence \( x \oplus y \equiv x + y \pmod{\mathbf{m}} \). A family \([0,1]^d \oplus T = \{[0,1]^d \oplus t : t \in T\}\) of translates of the unit cube \([0,1]^d\) is said to be a cube tiling of \( \mathbb{T}^d_m \) if it is a partition of \( \mathbb{T}^d_m \). Let us set

\[
p_i(T) = \{t_i \mod 1 : t \in T\}
\]

for \( i \in \{1, \ldots, d\} \). A straightforward consequence of our inequality reads that if \( \mathbf{m} = (n, n, \ldots, n) \), then

\[
\sum_i |p_i(T)| \leq \frac{n^d - 1}{n - 1}
\]

The inequality is tight. The extremal cube tilings for which it becomes an equality are classified.

Keywords: packing of boxes, cube tiling, polybox.

1 Introduction

Let \( \mathbf{m} = (m_1, \ldots, m_d) \) be a vector with integer coordinates such that \( m_i \geq 2 \) for every \( i \). Let \( \mathbf{mZ}^d = m_1 \mathbb{Z} \times \cdots \times m_d \mathbb{Z} \). We identify the \( d \)-dimensional torus \( \mathbb{R}^d / \mathbf{mZ}^d \) with \( \mathbb{T}^d_m = [0, m_1) \times [0, m_2) \times \cdots \times [0, m_d) \). The addition \( \oplus \) in \( \mathbb{T}^d_m \) is transferred from \( \mathbb{R}^d / \mathbf{mZ}^d \); that is, \( x \oplus y \in \mathbb{T}^d_m \) is the only element satisfying the congruence \( x \oplus y \equiv x + y \pmod{\mathbf{m}} \). A family \([0,1]^d \oplus T = \{[0,1]^d \oplus t : t \in T\}\) of translates of the unit cube \([0,1]^d\) is said to be a cube tiling of \( \mathbb{T}^d_m \) if it is a partition of \( \mathbb{T}^d_m \). Let us define the parameters of this tiling:

\[
p_i(T) = \{t_i \mod 1 : t \in T\}
\]

for \( i \in \{1, \ldots, d\} \). The following problem can be derived from Dutour and Itoh [4, Conjecture 5.4]:
Let \( m = (2, 2, \ldots, 2) \). Show that \( \sum_i |p_i(T)| \leq 2^d - 1 \). Moreover, prove that the equality takes place only in the case of tilings that are obtained by a special lamination construction.

One of our goals is to show that for any integer \( n \geq 2 \), if \( m = (n, n, \ldots, n) \), then

\[
\sum_i |p_i(T)| \leq \frac{n^d - 1}{n - 1}
\]

and the inequality is tight. The extremal cube tilings for which this inequality becomes an equality are classified (Theorem 6). Theorem 6 appears to be a consequence of a more general result concerning the so-called Keller packings and partitions (Theorem 3).

2 Keller families

Cube tilings have attracted a great deal of attention lasting for over one hundred years mainly in connection with Minkowski’s and Keller’s conjectures, e. g., [2, 3, 6, 10, 11, 13, 15, 16, 22, 23, 18, 19, 20, 21, 24, 25, 26, 27]. They are also related to Fuglede’s conjecture [7]. In particular, it appeared quite recently that they are in a 1–1 correspondence with the so-called exponential bases in \( L^2([0, 1]^d) \), e.g., [8, 17].

Kearnes and Kiss [9] stated the following combinatorial problem, which, as they mentioned, was completely unrelated to algebraic questions they were considering:

“Problem 5.5. Let \( A_1, \ldots, A_k \) be nonempty sets. If the rectangle \( A = A_1 \times \cdots \times A_k \) is partitioned into less than \( 2^k \) rectangular subsets, does it follow that there exists a rectangular subset in this partition which has full extent in some direction \( i \)?”

This problem is settled in [1]. It is proved that a minimal partition of the rectangle \( A \) into rectangular subsets without full extent in any direction has to have \( 2^k \) elements. Such partitions are thoroughly investigated in [5, 14]. It appears that they are combinatorially equivalent to unit cube tilings of the torus \( T_m^k \), where \( m = (2, \ldots, 2) \). This observation suggest that even if one investigates unit cube tilings it can be profitable to work with more abstract structures suggested by Problem 5.5. We follow this suggestion in the present paper.

Let \( [0,1]^d + T = \{ [0,1]^d + t : t \in T \} \) be a cube tiling of \( \mathbb{R}^d \). As observed by Keller [10], for every pair \( s, t \in T \) of different vectors there is \( i \) such that \( s_i - t_i \) is a non-zero integer (see also [8]). Obviously, there is a counterpart of this result for cube tilings of tori, since they can be interpreted as periodic tilings of \( \mathbb{R}^d \). A thorough inspection shows that Keller’s result is a consequence of the fact that if we take two unit segment partitions \([0,1]+S\) and \([0,1]+T\) of \( \mathbb{R} \), then \( T = S \) or for every pair of proper subsets \( P \subset S, Q \subset T \) the union of \([0,1]+P\) does not coincide with the union of \([0,1]+Q\). This observation motivates our further discussion.
2.1 Partitions

A partition \( \pi \) of a set \( U \) is non-trivial, if \( \pi \neq \{U\} \). Two partitions \( \pi \) and \( \pi' \) are independent if \( \{U\} = \pi \lor \pi' \); that is, \( \{U\} \) is a unique partition coarser than these two. Consequently,

**Lemma 1** If partitions \( \pi \) and \( \pi' \) of \( U \) are independent, \( \rho \) is a proper subfamily of \( \pi \) and \( \rho' \) is a proper subfamily of \( \pi' \), then the union of \( \rho \) and that of \( \rho' \) do not coincide.

In particular, it follows from Lemma 1 that if \( \Pi \) is a family of pairwise independent partitions of \( U \), then for any set \( A \) belonging to the union of \( \Pi \) there is a unique partition \( \pi \) such that \( A \in \pi \).

A family \( \Pi \) of partitions of \( U \) is complete if every partition \( \rho \) of \( U \) whose parts belong to the union of \( \Pi \) is a member of \( \Pi \). It is unital if \( \{U\} \) is its member.

**Examples 1** (\( \alpha_1 \)) Let \( U \) be a non-empty set. Then \( \Pi = \{\{A, U \setminus A\}: A \subseteq U, A \neq \{U, \emptyset\}\} \cup \{\{U\}\} \) is pairwise independent and unital. In general, it is not complete.

(\( \alpha_2 \)) This example has been already mentioned. Let \( U = \mathbb{T}_m = [0, m) \), where \( m \) is a positive integer. For every \( t \in [0, 1) \), let us define \( \pi_t = \{[0, 1) \lor t \lor k \mid k \in \{0, \ldots, m - 1\}\} \). Then \( \Pi = \{\pi_t : t \in [0, 1) \cup \{\{U\}\}\} \) is a complete and unital family of pairwise independent partitions of \( U \). The family of all non-trivial partitions belonging to \( \Pi \) is simply the family of all unit segment tilings of \( U \).

(\( \alpha_3 \)) Let us fix a hexagon \( H \) in \( \mathbb{R}^2 \) with three consecutive edges removed. Clearly \( \mathbb{R}^2 \) can be tiled by translates of \( H \). In fact, each such a tiling is a partition of \( \mathbb{R}^2 \). Let us assume that \( \pi \) is a member of \( \Pi \) if and only if \( \pi \) is a tiling of \( \mathbb{R}^2 \) by translates of \( H \) or \( \pi \) is equal to a trivial partition of \( \mathbb{R}^2 \). Again \( \Pi \) is complete, unital and consists of pairwise independent partitions. Tilings discussed here are infinite. Since they are periodical, one can define corresponding finite tilings of two-dimensional tori by translates of \( H \).

2.2 Boxes. Keller’s condition

Let \( X_i, i \in \{1, \ldots, d\} \), be a sequence of sets such that each of them has at least two elements. Let \( \Pi_i \) be a finite family of pairwise independent finite partitions of \( X_i \). Let \( X = X_1 \times \cdots \times X_d \) and \( \mathcal{U}_i \) be the union of \( \Pi_i \). Let us define the family of boxes \( \mathcal{B} = \mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_d \):

\[
K = K_1 \times \cdots \times K_d \in \mathcal{B} \quad \text{if and only if} \quad K_1 \in \mathcal{U}_1, \ldots, K_d \in \mathcal{U}_d.
\]

We say that a box \( K \in \mathcal{B} \) is proper, if \( K_i \neq X_i \), for every \( i \). Otherwise, \( K \) is improper. The triple \((X, \Pi, \mathcal{B})\) is called a system of \( d \)-boxes. Occasionally, we abuse slightly the terminology referring to \( \mathcal{B} \) alone as the system of boxes.

Now, we demonstrate an abstract version of Keller’s result concerning unit-cube tilings of tori.

**Proposition 2** Let \((X, \Pi, \mathcal{B})\) be a system of \( d \)-boxes. Let \( \mathcal{G} \subseteq \mathcal{B} \) be a partition of \( X \). If \( \Pi_i \) is complete for every \( i \in \{1, \ldots, d\} \), then \( \mathcal{G} \) satisfies Keller’s condition:
If $K, L \in \mathcal{G}$ and $K \neq L$, then there is $i$ such that $K_i, L_i$ are different parts of a partition $\pi \in \Pi_i$.

Proof. Clearly, we may assume that each $\Pi_i$ is unital. Suppose now that our proposition is false. Then we can choose such a partition $\mathcal{G}$, among those violating Keller’s condition, that there is a pair $K, L \in \mathcal{G}$ so that $\{K, L\}$ violates Keller’s condition and the set $I = \{i: K_i \cap L_i = \emptyset\}$ is minimal in the sense of inclusion. Let us pick any $s \in I$. Let $\pi \in \Pi_s$ be chosen so that $L_s \in \pi$. Let us define a ‘pile’ $\mathcal{P} = \{G \in \mathcal{G}: G_s \in \pi\}$. Let us fix any $x = (x_1, \ldots, x_d)$ belonging to a member $H$ of $\mathcal{P}$. Let

$$\ell = \{x_1\} \times \cdots \times \{x_{s-1}\} \times X_s \times \{x_{s+1}\} \times \cdots \times \{x_d\}.$$ 

Let $\mathcal{G}_\ell = \{G \in \mathcal{G}: G \cap \ell \neq \emptyset\}$. Since the sets $G \cap \ell$, for $G \in \mathcal{G}_\ell$, form a partition of $\ell$, the sets $(G \cap \ell)_s = G_s$, for $G \in \mathcal{G}_\ell$, form a partition $\rho$ of $X_s$. As $H \in \mathcal{G}_\ell$, it follows that $\pi$ and $\rho$ share the same part $H_s$. Since $\Pi_s$ is complete, $\rho$ has to belong belong to $\Pi_s$. As $\pi$ and $\rho$ are not independent, they have to coincide. Let $P$ be the union of $\mathcal{P}$. It is clear that $P = \bigcup_{x \in P} \ell_x$. Therefore, we can imagine $P$ as a kind of cylinder. For every box $B \in \mathcal{B}$ we can define a new box $B^\alpha$, by replacing factor $B_s$ by $X_s$. Let $\mathcal{H} = \{G^\alpha: G \in \mathcal{P} \text{ and } G_s = L_s\}$. The union of $\mathcal{H}$ coincides with $P$. Therefore, $(\mathcal{G} \setminus \mathcal{P}) \cup \mathcal{H}$ is a partition of $X$. Moreover $L^\alpha$ and $K$ are different and $J = \{i: K_i \cap L^\alpha_i = \emptyset\} = I \setminus \{s\}$, which contradicts the minimality of $I$. □

Let $\mathcal{G}$ be a non-empty subfamily of a system of boxes $\mathcal{B}$. We call it Keller family if it satisfies Keller’s condition. Clearly, Keller families consist of disjoint boxes, therefore, one can find justified to refer to them as Keller packings.

A set $G \subseteq X$ which is the union of a Keller family $\mathcal{G} \subseteq \mathcal{B}$ is called a polybox while $\mathcal{G}$ itself is called a suit for $G$. A polybox can have more than one suit. The suit $\mathcal{G}$ is proper if each box $K \in \mathcal{G}$ is proper; otherwise, it is improper.

Remark 1 Systems of $d$-boxes $(X, \Pi, \mathcal{B})$ such that $(\Pi_i: i = 1, \ldots, d)$ are as described in Example 1 $\alpha_1$ are investigated in [5,14].

3 Main results

Let $(X, \Pi, \mathcal{B})$ be a system of $d$-boxes. Let $i \in \{1, 2, \ldots, d\}$ and let $\mathcal{V}$ be a non-empty subfamily of $\mathcal{U}_i = \bigcup \Pi_i$. For every $\mathcal{G} \subseteq \mathcal{B}$, we define the restriction $\mathcal{G}\mid \mathcal{V}$ of $\mathcal{G}$ with respect to $\mathcal{V}$ by the equality

$$\mathcal{G}\mid \mathcal{V} = \{K \in \mathcal{G}: K_i \in \mathcal{V}\}.$$ 

If $\mathcal{V}$ is a singleton of $A$, then we write $\mathcal{G}\mid A$ rather than $\mathcal{G}\mid [A]$.

Let $\pi \in \Pi_i$ be a non-trivial partition. A Keller family $\mathcal{G}$ is laminated with respect to $\pi$ if $\mathcal{G}\mid \pi$ coincides with $\mathcal{G}$. It is $\{i\}$-laminated if it is laminated with respect to a non-trivial partition from $\Pi_i$. Finally, the family $\mathcal{G}$ is laminated if it is $j$-laminated for some $j \in \{1, \ldots, d\}$. 


A polybox $W \subseteq X$ is said to be an $i$-cylinder or shortly a cylinder if for every $x \in W$ the line
$\ell = \{x_1\} \times \cdots \times \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \cdots \times \{x_d\}$ is contained in $W$.

A Keller family $\mathcal{G}$ is said to be an $i$-pile or shortly pile if it is a suit for an $i$-cylinder $W$ and is
laminated with respect to a non-trivial partition $\pi \in \Pi_i$. For every $A \in \pi$, let us define

$\mathcal{G}^A = \{K_1 \times \cdots \times K_{i-1} \times X_i \times K_{i+1} \times \cdots \times K_d : K \in \mathcal{G}\} | A$.

We call $\mathcal{G}^A$ an elementary aggregate of the pile $\mathcal{G}$ (with respect to $A$). It is clear that $\mathcal{G}^A$ is a suit for $W$.

Let $i \in \{1, \ldots, d\}$. Let $\pi \in \Pi_i$ be a non-trivial partition, and $\mathcal{G} \subseteq \mathcal{B}$ be a Keller family. We write that $\pi$ is present in $\mathcal{G}$ if there is $K \in \mathcal{G}$ such that $K_i \in \pi$. Partition $\pi$ is hidden in $\mathcal{G}$ if $\mathcal{G}|\pi$ is a suit for an $i$-cylinder. Moreover, $\pi$ is said to be exposed in $\mathcal{G}$ if it is present and not hidden in $\mathcal{G}$.

We denote by $C_i(\mathcal{G})$ the family of all partitions $\pi \in \Pi_i$ which are hidden in $\mathcal{G}$. Let

$c_i(\mathcal{G}) = \sum_{\pi \in C_i(\mathcal{G})} (|\pi| - 1)$ and $c(\mathcal{G}) = \sum_{i=1}^{d} c_i(\mathcal{G})$.

We shall need a recursive definition of a multipile. Let $\mathcal{G} \subseteq \mathcal{B}$ be a Keller family. Suppose that one of the following two conditions is satisfied:

1. there is a box $K \in \mathcal{B}$ such that $\mathcal{G} = \{K\}$;
2. there is $i$ and non-trivial $\pi \in \Pi_i$ such that:
   - $\mathcal{G}$ is an $i$-pile laminated with respect to $\pi$,
   - for every $A \in \pi$, $\mathcal{G}|A$ is a multipile,
   - for every $k \neq i$ and every $A, B \in \pi$, if $A \neq B$, then $C_k(\mathcal{G}|A) \cap C_k(\mathcal{G}|B) = \emptyset$.

Then $\mathcal{G}$ is a multipile.

REMARK 2 It follows easily by induction that each multipile is a suit for a box.

THEOREM 3 Let $(X, \Pi, \mathcal{B})$ be a system of boxes. Suppose $\mathcal{G} \subseteq \mathcal{B}$ is a Keller family. Then

$c(\mathcal{G}) \leq |\mathcal{G}| - 1$.

The inequality becomes an equality if and only if $\mathcal{G}$ is a multipile.

We precede the proof of the theorem by two lemmas.

LEMMA 4 Let $(X, \Pi, \mathcal{B})$ be a system of boxes. Let $\mathcal{G}, \mathcal{G}' \subseteq \mathcal{B}$ be two suits for the same polybox $G$. If $\pi \in \Pi_i$ is exposed in $\mathcal{G}$, then it is exposed in $\mathcal{G}'$. 
Proof. Since \( \pi \) is exposed, \( \mathcal{G}|\pi \) is non-empty, and is not an \( i \)-pile. Therefore, there is \( u \in X \) such that the line \( \ell = \{u_1\} \times \cdots \{u_{i-1}\} \times X_i \times \{u_{i+1}\} \times \cdots \times \{u_d\} \) intersects some of the members of the family \( \mathcal{G}|\pi \) but is not covered by this family. Let \( \rho = \{ A_i : A \in \mathcal{G} \land A \cap \ell \neq \emptyset \} \). Since \( \mathcal{G} \) is a Keller family, all elements of \( \rho \) have to be parts of one partition belonging \( \Pi_i \). Now, since \( \pi \) and \( \rho \) have elements in common and \( \Pi \) is pairwise independent, \( \rho \subset \pi \). Let \( R \) be the union of \( \rho \). Then \( G \cap \ell = \{u_1\} \times \cdots \{u_{i-1}\} \times R \times \{u_{i+1}\} \times \cdots \times \{u_d\} \). Let \( \rho' = \{ A_i : A \in \mathcal{G}' \land A \cap \ell \neq \emptyset \} \). Since \( \mathcal{G}' \) is a suit for \( G \), we deduce that the union of \( \rho' \) is \( R \). By Lemma 1 it follows that \( \rho' \subseteq \pi \). Therefore, \( \pi \) is exposed in \( \mathcal{G}' \).

As an immediate corollary we have:

**Lemma 5** Let \( (X, \Pi, \mathcal{B}) \) be a system of boxes. Let \( \mathcal{G}, \mathcal{G}' \subseteq \mathcal{B} \) be two suits for the same polybox \( G \). If \( \pi \in \Pi_i \) is hidden in \( \mathcal{G} \) and present in \( \mathcal{G}' \), then it is hidden in \( \mathcal{G}' \).

**Proof of the theorem.** We shall proceed by induction with respect to the cardinality of \( \mathcal{G} \).

Suppose that \( \pi \in \Pi_i \) is hidden in \( \mathcal{G} \). Then \( \mathcal{C} = \mathcal{G}|\pi \) is an \( i \)-pile. Fix an \( A \in \pi \). Then \( \mathcal{C} \) and \( \mathcal{C}^A \) are suits for the same \( i \)-cylinder. Observe that the family \( \mathcal{G}' = (\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{C}^A \) is kellerian. To this end it suffices to pick \( K \in \mathcal{G} \setminus \mathcal{C} \) and \( L \in \mathcal{C}^A \), and show that \( \{K, L\} \) is kellerian. By the definition of \( \mathcal{C}^A \), the box \( L' = L_1 \times \cdots \times L_{i-1} \times A_i \times L_{i+1} \times \cdots \times L_d \) is a member of \( \mathcal{C} \). Therefore, \( \{K, L'\} \) is kellerian as a subfamily of \( \mathcal{G} \). It follows from the definition of \( \mathcal{G} \) that \( K_i \not\subset \pi \) while \( L'_i = A \in \pi \). Consequently, there is \( j \neq i \) such that \( K_j \) and \( L'_j = L_j \) are different parts of the same partition \( \tau \in \Pi_j \), which proves that \( \{K, L\} \) is kellerian as expected. It is clear now that \( \mathcal{G} \) and \( \mathcal{G}' \) are suits for the same polybox.

As an immediate consequence of the definition of \( \mathcal{G}' \) one has

\[
C_i(\mathcal{G}) = C_i(\mathcal{G}') \cup \{ \pi \},
\]  

(1) where the \( \cup \) symbol denotes the disjoint union.

Suppose now that \( k \neq i \) and \( \rho \in \Pi_k \) is hidden in \( \mathcal{G} \). If \( \rho \) is present in \( \mathcal{G}' \), then, by Lemma 5 it is hidden in \( \mathcal{G}' \). If it is not present in \( \mathcal{G}' \), then there is such a \( B \in \pi \setminus \{ A \} \) that \( \rho \) is present in \( \mathcal{G}|B \), which is equivalent to saying that \( \rho \) is present in \( \mathcal{C}^B \). Let \( \mathcal{G}'' = (\mathcal{G} \setminus \mathcal{C}) \cup \mathcal{C}^B \). It is clear now that \( \rho \) is hidden in \( \mathcal{G}'' \). Since \( \rho \) is not present in \( \mathcal{G} \), it cannot be present in \( \mathcal{G} \setminus \mathcal{C} \). Consequently, \( \rho \) is hidden \( \mathcal{C}^B \), which readily implies that it is hidden in \( \mathcal{G}|B \). Therefore, we have just shown that for every \( k \neq i \)

\[
C_k(\mathcal{G}) = C_k(\mathcal{G}') \cup \bigcup_{B \in \pi \setminus \{ A \}} C_k(\mathcal{G}|B).
\]  

(2) 

By (1) we have,

\[
c_i(\mathcal{G}) = c_i(\mathcal{G}') + |\pi| - 1.
\]  

(3) 

By (2),

\[
c_k(\mathcal{G}) \leq c_k(\mathcal{G}') + \sum_{B \in \pi \setminus \{ A \}} c_k(\mathcal{G}|B).
\]  

(4) 

Consequently,

\[
c(\mathcal{G}) \leq c_i(\mathcal{G}') + |\pi| - 1 + \sum_{k \neq i} \left( c_k(\mathcal{G}') + \sum_{B \in \pi \setminus \{ A \}} c_k(\mathcal{G}|B) \right).
\]  

(5)
Since \( c_i(\mathcal{G}|B) = 0 \), we get
\[
c(\mathcal{G}) \leq c(\mathcal{G}') + |\pi| - 1 + \sum_{B \in \pi \setminus \{A\}} c(\mathcal{G}|B)).
\] (6)
As \( |\mathcal{G}'| < |\mathcal{G}| \) and \( |\mathcal{G}|B| < |\mathcal{G}| \), we conclude by induction
\[
c(\mathcal{G}) \leq |\mathcal{G}'| - 1 + |\pi| - 1 + \sum_{B \in \pi \setminus \{A\}} (|\mathcal{G}|B| - 1) = |\mathcal{G}'| - 1 + \sum_{B \in \pi \setminus \{A\}} |\mathcal{G}|B|.
\] (7)
By the definition of \( \mathcal{G}' \), we have \( |\mathcal{G}'| = |\mathcal{G}\setminus \mathcal{C}| + |\mathcal{C}A| \). Since \( |\mathcal{C}A| = |\mathcal{G}|A | \) we obtain
\[
c(\mathcal{G}) \leq |\mathcal{G}\setminus \mathcal{C}| - 1 + \sum_{B \in \pi} |\mathcal{G}|B| = |\mathcal{G}\setminus \mathcal{C}| - 1 + |\mathcal{C}| = |\mathcal{G}| - 1.
\] (8)
It remains to discuss the case of equality. The fact that the equality holds for each multipile is an easy part of the proof, and as such is left to the reader.

If \( c(\mathcal{G}) = |\mathcal{G}| - 1 \), then all the above inequalities become equalities. In particular, (4) becomes an equality. This equality combined with (2) readily implies that the sets \( C_k(\mathcal{G}|B) \), for \( B \in \pi \setminus \{A\} \) are pairwise disjoint. Since \( A \) is arbitrary, we conclude that

(\(\beta\)) For every \( k \neq i \), the indexed family \( (C_k(\mathcal{G}|B) : B \in \pi) \) consists of pairwise disjoint members.

As (3) and (7) become equalities and \( A \) can be replaced by any element of \( \pi \), it follows that \( c(\mathcal{G}|B) = |\mathcal{G}|B| - 1 \) for \( B \in \pi \). By induction hypothesis, the sets \( \mathcal{G}|B \) are multipiles. This fact combined with (\(\beta\)) and the definition of \( \mathcal{C} \) leads to the conclusion that \( \mathcal{C} = \mathcal{G}|\pi \) is a multipile. Thus, in the case \( \mathcal{G} = \mathcal{C} \) there is nothing more to prove. Suppose now that \( \mathcal{G} \) is different from \( \mathcal{C} \). By Remark 2, \( \mathcal{G} \) is a suit for a box \( K \in \mathcal{B} \). Let us set \( \mathcal{G}''' = (\mathcal{G}\setminus \mathcal{C}) \cup \{K\} \). Clearly, \( \mathcal{G}''' \) and \( \mathcal{G} \) are suits for the same polybox. Suppose that \( \rho \in C_k(\mathcal{G}) \) for a certain \( k \neq i \). If \( \rho \) is present in \( \mathcal{G}\setminus \mathcal{C} \), then by Lemma 5 it is hidden in \( \mathcal{G}''' \). Clearly, \( C_k(\mathcal{G}'''\subseteq C_k(\mathcal{G}) \). If \( \rho \) is not present in \( C_k(\mathcal{G}\setminus \mathcal{C}) \), then it has to be hidden in some of the sets \( \mathcal{G}|B \), where \( B \in \pi \). Consequently,
\[
C_k(\mathcal{G}) = C_k(\mathcal{G}'''\cup \bigcup_{B \in \pi} \mathcal{G}|B).
\] (9)
Suppose now that \( \rho \in C_k(\mathcal{G}''' \). Since \( \pi \) is nontrivial, \( |\pi| > 1 \). By (\(\beta\)), there is \( A \in \pi \) such that \( \rho \notin C_k(\mathcal{G}|A) \). Since (3) becomes an equality for this particular \( A \) as for any other, the right-hand side expression of the equality (2) consists of disjoint sets. Therefore, \( \rho \) cannot belong to any of the sets \( C_k(\mathcal{G}|B), B \in \pi \). Thus, the right-hand side expression of the equality (5) consists of disjoint sets. As an immediate consequence,
\[
\sum_{k \neq i} c_k(\mathcal{G}) = \left( \sum_{k \neq i} c_k(\mathcal{G}''' \right) + \sum_{B \in \pi} c(\mathcal{G}|B).
\]
As we already know, \( c(\mathcal{G}) = |\mathcal{G}| - 1 \) implies \( c(\mathcal{G}|B) = |\mathcal{G}| - 1 \). Moreover, it follows from the definition of \( \mathcal{G}''' \) that \( c_i(\mathcal{G}) = c_i(\mathcal{G}'') + (|\pi| - 1) \). Thus,

\[
|\mathcal{G}| - 1 = c(\mathcal{G}'') + |\pi| - 1 + \sum_{B \in \pi} (|\mathcal{G}|B - 1) = c(\mathcal{G}''') + |\mathcal{G}| - 1.
\]

And

\[
c(\mathcal{G}''') = |\mathcal{G} \setminus \mathcal{G}| = |\mathcal{G}'''| - 1.
\]

Since \( |\mathcal{G}'''| < |\mathcal{G}| \), we deduce by induction that \( \mathcal{G}''' \) is a multipile. In particular, it is a \( j \)-pile for certain \( j \neq i \). Then \( \mathcal{G} \) is a \( j \)-pile. As we have already discussed the case where \( \mathcal{G} \) is a pile, the proof is complete. \( \square \)

**Remark 3** If \( \mathcal{G} \subseteq B \) is a partition of \( X \), then, by Proposition 2, it is automatically Kellerian. In particular, each unit cube tiling \([0, 1)^d + T\) of \( \mathcal{T}_m \) is Kellerian.

Now we are ready to prove our main result concerning cube tilings.

**Theorem 6** Let \([0, 1)^d \oplus T = \{[0, 1)^d \oplus t : t \in T\}\) be a cube tiling of \( \mathcal{T}_m \). Let

\[
p_i(T) = \{t_i \mod 1 : t \in T\}
\]

for \( i \in \{1, \ldots, d\} \). If \( m = (n, n, \ldots, n) \) for an integer \( n \geq 2 \), then

\[
p(T) := \sum_i |p_i(T)| \leq \frac{n^d - 1}{n - 1}.
\]

The equality takes place if and only if \([0, 1)^d \oplus T\) is a multipile.

**Proof.** Let \( X_i = \mathcal{T}_m = [0, m) \) for \( i \in \{1, \ldots, d\} \). Let \( \Pi_i \) be as described in Example 2. Let \((X, \Pi, \mathcal{B})\) be the corresponding system of \( d \)-boxes. Let \( \mathcal{G} = [0, 1)^d \oplus T \). Clearly, \( \mathcal{G} \subseteq \mathcal{B} \). By Proposition 2, \( \mathcal{G} \) satisfies Keller’s condition. Therefore, Theorem 3 applies to \( \mathcal{G} \).

For every \( i \in \{1, \ldots, d\} \), we have \( c_i(\mathcal{G}) = (n - 1)|p_i(T)| \). Moreover, \( |\mathcal{G}| = n^d \). The inequality (11) follows as an immediate consequence of Theorem 3, so does the case of equality. \( \square \)

An abstract version of Theorem 6 holds true as well.

**Theorem 7** Let \((X, \Pi, \mathcal{B})\) be a system of \( d \)-boxes. Suppose there is an integer \( n \) such that for every \( i \in \{1, \ldots, d\} \) all nontrivial partitions belonging to \( \Pi_i \) are of cardinality \( n \). Suppose \( \mathcal{G} \subseteq \mathcal{B} \) is a Keller partition of \( X \) which consists of proper boxes. Then

\[
\sum_i |C_i(\mathcal{G})| \leq \frac{n^d - 1}{n - 1}.
\]

The equality takes place if and only if \( \mathcal{G} \) is a multipile.
Clearly, the proof reduces to showing that

\[ |\mathcal{G}| = n^d. \]  

(12)

We begin by discussing a certain general construction, which is instrumental in the theory of Keller packings.

We may assume that each \( \Pi_i, i \in \{1, \ldots, d\} \), is unital. Let \( Y_i = \prod_{\pi \in \Pi_i} \pi \), for \( i \in \{1, \ldots, d\} \). Let \( A \in \mathcal{U}_i \), where as before \( \mathcal{U}_i = \bigcup \Pi_i \). Since \( \Pi_i \) is a family of pairwise independent partitions, there is exactly one \( \rho \in \Pi_i \) such that \( A \in \rho \). Let us define \( \hat{A} = \prod_{\pi \in \Pi_i} \hat{\pi} \subseteq Y_i \) by the formula

\[ \hat{A}_\pi = \left\{ \begin{array}{ll} \pi, & \text{if } \pi \neq \rho, \\ \{A\}, & \text{if } \pi = \rho. \end{array} \right. \]

Observe that for every partition \( \pi \in \Pi_i \), the family \( \hat{\pi} = \{\hat{A} : A \in \pi\} \) is a partition of \( Y_i \). Moreover, the family of partitions \( \hat{\Pi}_i = \{\hat{\pi} : \pi \in \Pi_i\} \) is pairwise independent. It is also complete even if \( \Pi \) is not. It is easily seen that we have.

For every pair \( A, B \in \mathcal{U}_i \), if \( A \neq B \), then \( A \) and \( B \) belong to the same partition \( \pi \in \Pi_i \) if and only if \( \hat{A}, \hat{B} \) are disjoint.

Let \( X_k = Y_1 \times \cdots \times Y_k \times X_{k+1} \times \cdots \times X_n \). For \( C \in \mathcal{B} \), let us define \( C^k = \hat{C}_1 \times \cdots \times \hat{C}_k \times C_{k+1} \times \cdots \times C_n \). If \( \mathcal{B} \) is a subfamily of \( \mathcal{B} \), then \( \mathcal{B}^k = \{C^k : C \in H\} \).

Let \( G \) be a polybox. One can wish to describe the corresponding mapping \( G \mapsto G^k \) for polyboxes. Suppose \( \mathcal{G} \subseteq \mathcal{B} \) is a suit for \( G \). One can expect that \( G^k \) can be defined as the union of \( \mathcal{G}^k \). Then one has to check that this union is independent of the choice of the suit \( \mathcal{G} \). This is done by induction. We prove only the first step, as the subsequent steps are mere repetitions.

For every \( x = (x_2, \ldots, x_d) \in X_2 \times \cdots \times X_d \) we define two lines \( \ell_x = X_1 \times \{x_2\} \times \cdots \times \{x_d\} \) and \( \ell_1^x = Y_1 \times \{x_2\} \times \cdots \times \{x_d\} \). Suppose first that \( G \cap \ell_x \neq \emptyset \). Let \( \mathcal{G}_x = \{A \in \mathcal{G} : A \cap \ell_x \neq \emptyset\} \). It is a Keller family as a subfamily of \( \mathcal{G} \). Therefore, there is \( \pi \in \Pi_1 \) such that \( \pi = \{A_1 : A \in \mathcal{G}_x\} \). Then \( \hat{\pi} \) is a partition of \( Y_1 \) and \( \{\ell_1^x \cap \hat{A} : A \in \mathcal{G}_x\} \) is a partition of \( \ell_1^x \). Thus \( G^1 \cap \ell_1^x = \ell_1^x \) which is evidently independent of the choice of \( \mathcal{G} \). Suppose now that \( G \cap \ell_x = \emptyset \). Then, by Lemma 1 there is a unique \( \pi \in \Pi_1 \) such that \( \rho = \{A_1 : A \in \mathcal{G}_x\} \) is a proper subfamily of \( \pi \). In fact, \( \pi \) is determined by the union of \( \rho \) which coincides with the image of \( G \cap \ell_x \) under the projection of \( X \) onto \( X_1 \). Therefore, the set \( \hat{\rho} = \{A_1 : A \in \mathcal{G}_x\} \) is independent of the choice of \( \mathcal{G} \). And since the image of \( G^1 \cap \ell_1^x \) under the projection of \( X^1 \) onto \( Y_1 \) coincides with the union of \( \hat{\rho} \), the set \( G^1 \cap \ell_1^x \) is also independent of the choice of \( \mathcal{G} \).

For \( C \in \mathcal{B} \), \( \mathcal{G} \subseteq \mathcal{B} \), and a polybox \( G \) let us set: \( \hat{C} = C^d \), \( \hat{\mathcal{G}} = \mathcal{G}^d \), \( \hat{G} = G^d \). We can summarize all we have shown as follows:

\((\gamma_1)\) For every pair \( C, D \in \mathcal{B} \), if \( C \neq D \), then \( \{C, D\} \) satisfies Keller’s condition if and only if \( \hat{C}, \hat{D} \) are disjoint.

\((\gamma_2)\) If \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are suits for a polybox \( G \), then \( \hat{\mathcal{G}}_1 \) and \( \hat{\mathcal{G}}_2 \) are suits for \( \hat{G} \).
Now, we can complete the proof of Theorem \[ \text{7} \] easily. Since every \( G \in \mathcal{G} \) is a proper box and for every \( i \in \{1, \ldots, d\} \) and every nontrivial \( \pi \in \Pi_i \) the cardinality of \( \pi \) is \( n \), it follows by the definition of \( \hat{G} \) that \( |\hat{G}| = \frac{1}{n^d} |Y| \). As \( \mathcal{G} \) and \( \{X\} \) are Keller partitions of \( X \), it follows that \( \mathcal{G} \) is a (Keller) partition of \( \hat{X} = Y \). Clearly, \( |\mathcal{G}| = |\hat{G}| \). Therefore,

\[
|Y| = \sum_{G \in \mathcal{G}} |\hat{G}| = \frac{|\mathcal{G}| |Y|}{n^d},
\]

which implies \([12]\). \( \square \)

### 4 Concluding remarks

1. Theorem \([5]\) addresses a particular kind of tori: \( m = (n, \ldots, n) \). It is of some interest to extend this result to arbitrary \( m = (m_1, \ldots, m_d) \). The following question arises:

Suppose \( p = p(T) \) attains its maximum for a tiling \( \mathcal{G} = [0, 1]^d \oplus T \) of \( \mathbb{T}_m^d \). Is it true that \( \mathcal{G} \) must be a multipile?

One can easily check that if the tiling \( \mathcal{G} \) is a multipile, then there is an ordering \( i_1, i_2, \ldots, i_d \) of the set \( \{1, \ldots, d\} \) such that

\[
p(T) = 1 + m_{i_1} m_{i_2} \cdots + m_{i_1} m_{i_2} \cdots m_{i_d-1}.
\]

Therefore, if the answer to our question is affirmative and the ordering \( i_1, i_2, \ldots, i_d \) is chosen so that \( m_{i_1} \geq m_{i_2} \geq \cdots \geq m_{i_d} \), then the latter equality expresses the maximum value of \( p \).

2. To convince the reader of the usefulness of the construction presented in the preceding section, we offer the following immediate consequence of \((\gamma_1)\) and \((\gamma_2)\):

Let \( \mathcal{G}_1, \ldots, \mathcal{G}_s \) and \( \mathcal{H}_1, \ldots, \mathcal{H}_s \) be two pairwise disjoint families that are contained in a system of boxes \( \mathcal{B} \). Suppose that for each \( i \) the families \( \mathcal{G}_i \) and \( \mathcal{H}_i \) are suits for the same polybox. Moreover, suppose that \( \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_s \) is a suit for a polybox \( G \). Then \( \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_s \) is a suit for \( G \) as well.

Similar constructions have been widely exploited in \([14]\) Proposition 2.6, Section 10].

3. Kiesielewicz \([12]\) is concerned with the problem of finding unit cube partitions \([0, 1]^d \oplus T \) of \( \mathbb{T}_m^d \), where \( m = (2, \ldots, 2) \), for which the parameters \( p(T) \) are equal and as large as possible. The simplest version of Theorem \([5]\) is already mentioned there \([12]\) Theorem 4.2, pp. 95, 104], and attributed to the present author.
References

[1] N. Alon, T. Bohman, R. Holzman, D. J. Kleitman, On partitions of discrete boxes, Discrete Math. 257 (2002) 255–258.

[2] K. Corrádi, S. Szabó, A combinatorial approach for Keller’s conjecture, Period. Math. Hungar. 21 (1990) 95–100.

[3] J. Debroni, J. D. Eblen, M. A. Langston, W. Myrvold, P. Shor, D. Weerapurage, A complete resolution of the Keller maximum clique problem, in: Proceedings of the Twenty-Second Annual ACM–SIAM Symposium on Discrete Algorithms, 2010.

[4] M. Dutour Sikirić, Y. Itoh, Combinatorial cube packings in the cube and the torus, European J. Combin. 31 (2010) 517–534.

[5] J. Grytczuk, A. P. Kisielewicz, K. Przesławski, Minimal partitions of a box into boxes, Combinatorica 24 (2004) 605–614.

[6] G. Hajós, Über einfache und mehrfache Bedeckung des n-dimensionalen Raumes mit einem Würfelgitter, Math. Z. 47 (1941) 427–467.

[7] B. Fuglede, Commuting Self-Adjoint Partial Differential Operators and a Group Theoretic Problem, J. Func. Anal. 16 (1974) 101-121.

[8] A. Iosevich, S. Pedersen, Spectral and tiling properties of the unit cube, Inter. Math. Res. Notices 16 (1998) 819–828.

[9] K. A. Kearnes, E. W. Kiss, Finite algebras of finite complexity, Discrete Math. 207 (1999) 89–135.

[10] O.-H. Keller, Über die lückenlose Erfüllung des Raumes Würfeln, J. Reine Angew. Math. 163 (1930) 231–248.

[11] O.-H. Keller, Ein Satz über die lückenlose Erfüllung des 5- und 6-dimensionalen Raumes mit Würfeln, J. Reine Angew. Math. 177 (1937) 61–64.

[12] A. P. Kisielewicz, Partitions and balanced matchings on an n-dimensional cube, European J. Combin. 4 (2014) 93–107.

[13] A. P. Kisielewicz, M. Łysakowska, On Keller’s Conjecture in Dimension Seven, Electron. J. Combin. 22 (2015) P1.16.

[14] A. P. Kisielewicz, K. Przesławski, Polyboxes, cube tilings and rigidity, Discrete Comput. Geom. 40 (2008) 1–30.

[15] A. P. Kisielewicz, K. Przesławski, The coin exchange problem and the structure of cube tilings, Electron. J. Combin. 19 (2012) P26.

[16] M. N. Kolountzakis, The study of translational tiling with Fourier Analysis, Fourier Analysis and Convexity 131–187, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston MA 2004.
[17] J. C. Lagarias, J. A. Reeds, Y. Wang, Orthonormal bases of exponentials for the \( n \)-cube, Duke. Math. J. 103 (2000) 25–37.

[18] M. Łysakowska, K. Przesławski, On the structure of cube tilings and unextendible systems of cubes in low dimensions, European J. Combin. 32 (2011) 1417–1427.

[19] M. Łysakowska, K. Przesławski, Keller’s conjecture on the existence of columns in cube tilings of \( \mathbb{R}^n \), Adv. Geom. 12 (2012) 329–352.

[20] J. C. Lagarias, P. W. Shor, Keller’s cube-tiling conjecture is false in high dimensions, Bull. Amer. Math. Soc. 27 (1992) 279–287.

[21] J. C. Lagarias, P. W. Shor, Cube tilings and nonlinear codes, Discrete Comput. Geom. 11 (1994) 359–391.

[22] J. Mackey, A cube tiling of dimension eight with no facesharing, Discrete Comput. Geom. 28 (2002) 275–279.

[23] H. Minkowski, Diophantische Approximationen, Teubner, Leipzig 1907.

[24] O. Perron, Über lückenlose Ausfüllung des \( n \)-dimensionalen Raumes durch kongruente Würfel I, II, Math. Z. 46 (1940) 1–26, 161–180.

[25] S. K. Stein, S. Szabó: Algebra and Tiling: Homomorphisms in the Service of Geometry. American Mathematical Association, Washington 1994.

[26] S. Szabó, Cube tilings as contributions of algebra to geometry. Beiträge Algebra Geom. 34 (1993) 63–75.

[27] Ch. Zong, The Cube: A Window to Convex and Discrete Geometry. Cambridge Tracts in Mathematics 168 (2006).

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