DOUBLY SLICE KNOTS AND OBSTRUCTION TO LAGRANGIAN CONCORDANCE.

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Abstract. In this short note we observe that a result of Eliashberg and Polterovitch allows to use the doubly slice genus as an obstruction for a Legendrian knot to be a slice of a concordance from the trivial Legendrian knot with maximal Thurston-Bennequin invariant to itself. This allows to obstruct concordances from the Pretzel knot $P(3, -3, -m)$ when $m \geq 4$ to the unknot. Those examples are of interest because the Legendrian contact homology algebra cannot be used to obstruct such a concordance.

1. Introduction and results

Definition 1. Let $\Lambda^-, \Lambda^+$ be two Legendrian knots in $S^3$. A Lagrangian concordance from $\Lambda^-$ to $\Lambda^+$ is a Lagrangian submanifold $\Sigma \in \mathbb{R} \times S^3$ diffeomorphic to a cylinder and such that for some $T > 0$,

1. $((-\infty, -T) \times S^3) \cap \Sigma = (-\infty, -T) \times \Lambda^-$ and
2. $((T, +\infty) \times S^3) \cap \Sigma = (T, +\infty) \times \Lambda^+$

We denote $\Lambda^- \prec \Lambda^+$ if there is a concordance from $\Lambda^-$ to $\Lambda^+$.

Up to now it is not known if this relation induces a partial order on the set of Legendrian isotopy classes of Legendrian knots. It is of course well defined, transitive and reflexive but it is not known if $\Lambda^- \prec \Lambda^+$ and $\Lambda^+ \prec \Lambda^-$ implies that $\Lambda^-$ and $\Lambda^+$ are Legendrian isotopic. There is a related notion of decomposable Lagrangian concordance (denoted $\prec_{\text{dec}}$) that are concordances built from elementary combinatorial moves. The projection to $\mathbb{R}$ of such a concordance only has critical points of index 0 and 1 hence are ribbon. It follows from recent work of Agol [Ago22] that if $\Lambda^- \prec_{\text{dec}} \Lambda^+$ and $\Lambda^+ \prec_{\text{dec}} \Lambda^-$ then $\Lambda^-$ and $\Lambda^+$ are smoothly isotopic. Since Lagrangian concordances preserve both the Thurston-Bennequin number (TB) and the rotation number it implies that the relation $\prec_{\text{dec}}$ is a partial order on the class of Legendrian knots whose isotopy classes are simple.

In this note we are concerned with Legendrian knots $\Lambda$ such that $\Lambda_0 \prec \Lambda \prec \Lambda_0$ where $\Lambda_0$ is the maximal TB Legendrian unknot. The Legendrian knot $\Lambda_0$ is fillable by an exact Lagrangian disc. Thus it follows from the fact that a Lagrangian disc can be perturbed to a symplectic disc (because it is an open manifold) and a combination of results of Rudolph [Rud83] and Boileau-Orévkov [BO01] that the concordance realising $\Lambda_0 \prec \Lambda$ is ribbon. On the other hand in [CNS16, Theorem 3.2] it is shown that the concordance realising $\Lambda \prec \Lambda_0$ cannot be decomposable, actually following the proof one observes that they prove that this concordance cannot be ribbon (which is also implied by a recent result of Zemke [Zem19] or the above mentioned result from [Ago22]). In [CNS16] they give a list of Lagrangianly slice Legendrian knots for which the existence of a concordance to $\Lambda_0$ cannot be obstructed using Legendrian contact homology. The reason is that their Chekanov-Eliashberg algebras are stable tame isomorphic to the one of $\Lambda_0$ (see [EN, Appendix A] for explicit computation). The family in question consists of some $TB = -1$ Legendrian realisations of pretzel knots $P(3, -3, -m)$, for $m \geq 4$. Denote $\Lambda_m$ such a Legendrian
knot, see Figure 1 for its front projection. There are concordances $\Lambda_0 \prec \Lambda_m$ for all $m \geq 4$, which can be easily constructed via some elementary moves on the front.

In this note we make the following observation.

**Theorem 1.** There is no concordance $\Lambda_m \prec \Lambda_0$.

Theorem 1 follows from a result of Eliashberg-Polterovitch [EP96] that we recall here (rephrased in the language that fits our purpose):

**Theorem 2.** Let $D$ be a filling of $\Lambda_0$, then $D$ is Hamiltonian isotopic to the standard filling of $\Lambda_0$.

Indeed this result leads to obstructions for Legendrian knots to exist as middle slices of a concordance from $\Lambda_0$ to $\Lambda_0$. The symplectic flavour of the result leads to obstructions from Legendrian contact homology and variations of it, this has been used in [Cha15], [NRSS17] and [Wu] for instance. But the unknottedness of the disc in [EP96] has topological implications that to our knowledge have not been used in order to study Lagrangian cobordisms. Recall that a smooth knot $K$ is doubly slice if there is an unknotted 2-sphere $S$ in $\mathbb{R}^4$ such that $S \cap S^3 = K$.

As an immediate corollary of Theorem 2 we obtain:

**Theorem 3.** If there are concordances $\Lambda_0 \prec \Lambda$ and $\Lambda \prec \Lambda_0$, then $\Lambda$ is doubly slice.

Now the main computation of this note shows that:

**Theorem 4.** The pretzel knots $P(3, -3, -m)$ for $m \geq 4$ are not doubly slice.

This implies Theorem 1.

**Remark 1.** For the case $m = 3$, the knot $\Lambda_3$ has topological type $m(9_{46})$ and it has been shown previously by the first author in [Cha15] that there is no concordance from $\Lambda_3$ to $\Lambda_0$. In this case computations using Legendrian contact homology or other modern tools for the case are still necessary because it turns out that the knot $m(9_{46})$ is doubly slice. The fact that the only one to be doubly slice in this family happens to be the only one to have a rich Legendrian contact homology algebra is both fortunate and puzzling.
Remark 2. In [LM15] the authors give an (almost complete) classification of doubly slice knots up to 12 crossings. From their work, the pretzel knots $P(3, -3, -4) = m(10_{140})$, $P(3, -3, -5) = 11n_{139}$ and $P(3, -3, -6) = 12n_{582}$ are not doubly slice.

2. Proof of Theorem 4

2.1. Case $P(3, -3, -2k - 1)$, $k \geq 2$. In this case the pretzel knots are odd because they have an odd number of parameters which are all odd. McDonald in [McD20], using work of Issa and McCoy in [IM20], proves the following result:

Theorem 5. [McD20] Theorem 3] For $K$ an odd pretzel knot, the following are equivalent:

- $\Sigma_2(S^3, K)$ embeds in $S^4$;
- $K$ is a doubly slice pretzel knot;
- $K$ is a mutant of $P(a, -a, a, -a, \ldots, a)$ for some odd $a$.

It is noted in [McD20] that it follows from the proof of [IM20] Theorem 1.11] that an odd pretzel knot which is a mutant of $P(a, -a, a, -a, \ldots, a)$ for some odd $a$ has for parameters a permutation of $(a, -a, a, -a, \ldots, a)$. One concludes that the pretzel knots $P(3, -3, -2k - 1)$ for $k \geq 2$ are not doubly slice.

2.2. Case $P(3, -3, -2k)$, $k \geq 2$. For these knots we obstruct doubly sliceness using the signature function as in [LM15] for $P(3, -3, -6)$. Given a knot $K$ and a Seifert matrix $A$ for $K$, the Levine-Tristram signature of $K$ is a function $\sigma_K : S^1 \to \mathbb{Z}$ whose value at $\omega \in S^1$ is given by the signature of the matrix

$$(1 - \omega)A + (1 - \omega)A^T$$

where the signature is the number of positive eigenvalues minus the number of negative eigenvalues. If $K$ is slice, then $\sigma_K(\omega) = 0$ away of the roots of the Alexander polynomial of $K$. Moreover, if $K$ is doubly slice then $\sigma_K(\omega)$ vanishes for all $\omega \in S^1$. This is discussed in [LM15] Proposition 2.2 and follows from the fact that when $K$ is doubly slice then there exists a Seifert matrix for $K$ that is hyperbolic (see [Sum71]). Seifert matrices for the knots $P(3, -3, -2k)$ have been computed in [IM20] Section 3. We denote $A_k$ a Seifert matrix for $P(3, -3, -2k)$. We have:

$$A_k = \begin{pmatrix}
-1 & -1 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -k
\end{pmatrix}.$$

It turns out that the Alexander polynomial of these knots is the same, given by $(t^2 - t + 1)^2$ and has for roots the sixth root of unity and its conjugate. One can then compute that $\sigma_K(\omega) \neq 0$ for $K = P(3, -3, -2k)$ and $\omega = e^{i\pi/5}$. Indeed a direct computation shows that the matrix $(1 - \omega)A_k + (1 - \omega)A_k^T$ has rank 5 and hence cannot have even signature (one can compute that its signature is actually $-1$). Thus $K$ is not doubly slice which concludes both the proofs of Theorems 4 and 1.

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