Pseudo-rotations and holomorphic curves

Erman Çineli¹ · Viktor L. Ginzburg¹ · Başak Z. Gürel²

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Abstract
We prove a variant of the Chance–McDuff conjecture for pseudo-rotations: under certain additional conditions, a closed symplectic manifold which admits a Hamiltonian pseudo-rotation must have deformed quantum product and, in particular, some non-zero Gromov–Witten invariants. The only assumptions on the manifold are that it is weakly monotone and that its minimal Chern number is at least two. The conditions on the pseudo-rotation are expressed in terms of the linearized flow at one of the fixed points and are hypothetically satisfied for most (but not all) pseudo-rotations.

Keywords  Pseudo-rotations · Periodic orbits · Hamiltonian diffeomorphisms · Floer homology · Quantum product · Gromov–Witten invariants

Mathematics Subject Classification  53D40 · 37J10 · 37J45

Contents

1 Introduction ............................................. 2
2 Main results ............................................. 4
  2.1 Definitions ............................................ 5
    2.1.1 Base group ........................................ 5
    2.1.2 Loop contribution ................................... 7
  2.2 Detecting the quantum product ................................. 7

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Erman Çineli
scineli@ucsc.edu

Başak Z. Gürel
basak.gurel@ucf.edu

¹ Department of Mathematics, UC Santa Cruz, Sant Cruz, CA 95064, USA
² Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA
1 Introduction

We show that a closed symplectic manifold, which has minimal Chern number at least two and admits a Hamiltonian pseudo-rotation satisfying certain mild additional conditions, must have non-vanishing Gromov–Witten invariants and, moreover, its quantum product is deformed, i.e., different from the intersection product.

To put this result in perspective, recall that by the Conley conjecture, for many symplectic manifolds every Hamiltonian diffeomorphism has infinitely many periodic points. Obviously, the conjecture requires some additional assumptions on the manifold: an irrational rotation of $S^2$ about the $z$-axis has only two periodic points: these are the fixed points – the Poles. In a similar vein, the conjecture fails for some other manifolds such as complex projective spaces, Grassmannians and flag manifolds, symplectic toric manifolds, and most of the coadjoint orbits of compact Lie groups. In fact, the conjecture fails for all manifolds admitting a Hamiltonian circle (or torus) action with isolated fixed points – a generic element of the circle or the torus gives rise to a Hamiltonian diffeomorphism with finitely many periodic points.

On the conjecture side, these counterexamples are comparatively rare and in a series of works easily spending three decades and contributed by many, the Conley conjecture has been proved in many cases. The state of the art is that it holds for $M$ unless there exists $A \in \pi_2(M)$ such that $\langle \omega, A \rangle > 0$ and $\langle c_1(TM), A \rangle > 0$; see [9,16] and also [15] for further references and a thorough discussion. In particular, the conjecture holds whenever $M$ is symplectically aspherical or negative monotone.

Yet, these purely topological conditions leave aside a more subtle question about symplectic topological criteria for the Conley conjecture to hold. In that realm, the outstanding problem, referred to as the Chance–McDuff conjecture, is that whenever the Conley conjecture fails some Gromov–Witten invariants of $M$ are non-zero. It is well-known that there is a strong connection between the symplectic topology of $M$ (e.g., Gromov–Witten invariants or the quantum product) and the dynamics (periodic
orbits) of Hamiltonian diffeomorphisms \( \varphi \) of \( M \). However, this connection is explored and usually utilized only in one direction: from symplectic topology to dynamics. The difficulty in proving the Chance–McDuff conjecture lies in that it requires going in the opposite direction and this is a much less understood problem. Till now the only work along these lines was [25] where it is shown that a symplectic manifold admitting a Hamiltonian circle action is uniruled, i.e., has a non-zero Gromov–Witten invariant with one of the homology classes being the point class.

In this context, pseudo-rotations are, roughly speaking, Hamiltonian diffeomorphisms with a finite and minimal possible number of periodic points. (Actual definitions vary, but all of them reflect the same idea; see [17].) In particular, pseudo-rotations are counterexamples to the Conley conjecture and, in fact, they are the only counterexamples known to date. (See [35] and also [11] for some relevant recent results.) Every known Hamiltonian diffeomorphism \( \varphi \) with finitely many periodic points is a pseudo-rotation in a very strong sense: all periodic points of \( \varphi \) are its fixed points, they are elliptic, and all iterates \( \varphi^k \) are non-degenerate. This is the definition we adopt here.

Pseudo-rotations occupy a distinguished place in dynamical systems theory far and mainly beyond the Hamiltonian setting. They can have extremely interesting dynamics. For instance, there are examples of ergodic Hamiltonian pseudo-rotations and even of pseudo-rotations with a finite number of ergodic measures. Such pseudo-rotations are obtained by the so-called conjugation method which requires the manifold to have a circle or torus action; see [3,13,23]. In fact, in all known examples, a manifold which admits a pseudo-rotation also admits a circle or torus action. This, combined with the results from [25], was the main motivation for the Chance–McDuff conjecture. Recently it has been understood that symplectic topological methods are well suited for studying Hamiltonian pseudo-rotations; [6–8,17,18].

Here we prove a variant of the Chance–McDuff conjecture for pseudo-rotations. Namely, we show that, under certain additional conditions, a manifold \( M \) that admits a pseudo-rotation \( \varphi \) must have deformed quantum product and, in particular, some non-vanishing Gromov–Witten invariants. The only assumptions on \( M \) are that it is weakly monotone and that \( N > 1 \), where \( N \) is the minimal Chern number. The conditions on \( \varphi \) are more involved and phrased in terms of the linearized flow at one of its one-periodic orbits. One may expect these conditions to be met for the majority (although certainly not all) of pseudo-rotations.

Our method uses only a minimal input from symplectic topology. However, on the unexpected side, it relates combinatorics of integer partitions to the pair-of-pants product and the regularity of zero-energy pair-of-pants curves in Floer theory; see Sects. 3.3 and 4.2. We use these curves to capture non-vanishing Gromov–Witten invariants. Identifying the quantum and Floer homology, we show by purely combinatorial means that in many instances there are abundant zero-energy pair-of-pants curves corresponding to long products in quantum homology. These long products would vanish if the quantum product were undeformed. The underlying idea can be best illustrated by the example of an irrational rotation of \( S^2 \).

**Example 1.1 (Irrational Rotations of \( S^2 \)).** Let \( \varphi \) be an irrational rotation of \( S^2 \) by an angle \( \theta \), where \( \pi < \theta < 2\pi \). The fixed points of \( \varphi \) are the North Pole \( y \) and the
South Pole $x$. The iterates $y^k$ and $x^k$ are also the only periodic points of $\varphi$. We equip these points with trivial cappings. Then $\mu(y) = 1$ and $\mu(x) = -1$. Thus, when we identify the Floer complex $\text{CF}_*(\varphi)$ with the quantum homology $\text{HQ}_*(S^2)[-1]$, the North Pole $y$ represents the fundamental class $[S^2]$ and the South Pole $x$ represents $[pt]$. On the other hand, $\mu(x^2) = -3$. Thus $[x^2]$ represents the class $q[S^2]$, where $q$ is the generator of the Novikov ring. (With our conventions $|q| = -4$.) There exists exactly one pair-of-pants curve from $(x, x)$ to $x^2$ – the constant curve. Assuming that this pair-of-pants curve is regular, which indeed is the case (see Corollary 3.2 and [34]), we have

$$x \ast x = x^2 + \ldots,$$

where $\ast$ is the quantum product and the dots stand for capped periodic orbits with action strictly smaller than the action of $x^2$. (In fact, it is easy to see that no such orbits enter this identity.) In any event, no cancellations can happen on the right-hand side and we conclude that

$$[pt] \ast [pt] = q[S^2] + \ldots \neq 0.$$

On the other hand, if the quantum product were not deformed (i.e., agreed with the intersection product) we would obviously have $[pt] \ast [pt] = 0$. (Moreover, we see that $\text{GW}_A([pt], [pt], [pt]) \neq 0$, where $A$ is the “positive” generator of $H_2(S^2; \mathbb{Z})$.)

This method, which shares some common elements with [34], readily lends itself to several generalizations to be explored elsewhere. First of all, by using other, more sophisticated algebraic structures one can certainly alter the requirements on the pseudo-rotation or, perhaps, even eliminate these requirements entirely. (See [10,36,37] for some related results based on [34,38–40] and asserting that the quantum Steenrod square is deformed when $M$ is monotone and admits a pseudo-rotation.) Secondly, under favorable circumstances, the method allows one to obtain more specific information about the quantum homology algebra of $M$ although the combinatorics of the problem quickly gets rather involved. For instance, this method is used in [4] to give a dynamics characterization of $\mathbb{C}P^2$ and, on the quantum homology level, of $\mathbb{C}P^n$.

The paper is organized in a somewhat counter-logical fashion. In Sect. 2 we give necessary definitions and state main results. Preliminary material from symplectic topology is discussed in Sect. 3. In Sect. 4 we introduce extremal partitions – the key combinatorial ingredient of the proofs – and reduce the main results of the paper to combinatorial problems. Extremal partitions are studied in detail in Sect. 5, where we prove the combinatorial counterparts of the main theorems and thus complete their proofs.

## 2 Main results

To detect the quantum product, our method requires imposing some additional conditions on a pseudo-rotation $\varphi$. These requirements are often, but not always, satisfied
and are expressed in terms of the linearized flow $\Phi = D\phi^t|_{\bar{x}}$ along a capped one-periodic orbit $\bar{x}$ of $\phi^t$. In this section, we first formulate these conditions and then state the main results of the paper, deliberately opting to work with requirements which are easier to state rather than more general.

2.1 Definitions

We start by introducing several symplectic linear algebra invariants associated with the linearized time one-map (or the flow) at a one-periodic orbit $x$ of a Hamiltonian diffeomorphism $\phi$. In the discussion below, the reader should think that $P = D\phi|_x$ and $\Phi^t$ is the linearized flow $D\phi^t|_{\bar{x}}$ along a capped one-periodic orbit $\bar{x}$ and that $N$ is the minimal Chern number of $M$.

A word is also due on the nomenclature used in this section: on Conditions A, B1 and B2. The reason for this labeling is that the role of Condition A is distinctly different from that of Conditions B1 or B2. Condition A is used to detect long non-vanishing products in the quantum homology, while Conditions B1 and B2 ensure that these products are not essentially the products of the fundamental class with itself.

2.1.1 Base group

Consider an elliptic and non-degenerate symplectic transformation $P \in \text{Sp}(2n)$ and let $\tilde{P} \in \text{Sp}(2n)$ be isospectral to $P$ and semi-simple. In other words, we require that all eigenvalues of $\tilde{P}$ are unit, 1 is not an eigenvalue, and there exists a family of non-degenerate tranformations $P_t \in \text{Sp}(2n)$ connecting $P_0 = P$ and $P_1 = \tilde{P}$ such that all $P_t$ have the same spectrum, and $\tilde{P}$ is diagonalizable, i.e., $\mathbb{R}^{2n}$ splits into a sum of $n$ invariant symplectic subspaces. (Then each of these subspaces is a plane and on it $\tilde{P}$ is conjugate to a rotation.) It is easy to see that such a transformation $\tilde{P}$ exists and, in fact, can be taken arbitrarily close to $P$; see, e.g., [14, Lemma 5.1]. Of course, $\tilde{P}$ is not unique.

Since $\tilde{P}$ is elliptic and semi-simple, it is symplectically conjugate to a unitary transformation and, as a consequence, the closure of the sequence $\{\tilde{P}^k | k \in \mathbb{N}\}$ is a compact abelian subgroup of $\text{Sp}(2n)$, which we denote by $\Gamma$ (or $\Gamma(P)$ or $\Gamma(x)$ when $P = D\phi|_x$), and call the base group. By construction, $\Gamma$ is monothetic and thus $\Gamma_0 / \Gamma_0$ is finite cyclic, where $\Gamma_0$ is the connected component of the identity in $\Gamma$. Clearly, up to symplectic conjugation, $\Gamma$ is independent of the choice of $\tilde{P}$.

Alternatively, $\Gamma$ can be described as follows. Since $P$ is elliptic, all eigenvalues of $P$ lie on the unit circle. Let

$\bar{\theta} := (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n = S^1_1 \times \ldots \times S^1_n$

be the collection of Krein–positive eigenvalues of $P$, ordered in an arbitrary way; see, e.g., [1, Sect. 1.3] or [24,32]. Then $\Gamma$ is naturally isomorphic to the subgroup of the torus $\mathbb{T}^n$ generated by $\bar{\theta}$, i.e., to the closure of the sequence $\{k\bar{\theta} | k \in \mathbb{N}\}$ in $\mathbb{T}^n$. For a suitable choice of a complex structure on $\mathbb{R}^{2n}$ we can think of $\mathbb{T}^n$ as the maximal torus in $\text{U}(n)$ containing $\tilde{P} = \bar{\theta}$. Note that $\dim \Gamma \geq 1$ when $P$ is strongly non-degenerate,
i.e., all iterates \( P^k, k \in \mathbb{N} \), are non-degenerate. (The converse is not true.) The key point here is that the “index theory” for \( \tilde{P} \) is the same as for \( P \), but the group generated by \( P \) in \( \text{Sp}(2n) \) is not compact, unless \( P \) is semi-simple, and is much harder to work with.

**Example 2.1** Assume that \( \varphi \) is a true rotation (i.e., \( \varphi \) generates a compact subgroup \( G \) of \( \text{Ham}(M) \), see Example 2.9) or that it is obtained from such a rotation by the conjugation method. Let \( P = D\varphi|_x \), where \( x \) is a fixed point of \( \varphi \). Then \( P \) is automatically semi-simple and, for a true rotation, \( \Gamma' \) is the image of \( G \) in \( \text{Sp}(T_x M) \) under the natural representation of \( G \) on \( T_x M \).

**Definition 2.2** (Condition A) For a fixed \( r \in \mathbb{N} \), the transformation \( P \) (or the subgroup \( \Gamma = \Gamma(P) \) or the orbit \( x \)) satisfies Condition A if there exist \( r \) points \( \vec{\theta}_1, \ldots, \vec{\theta}_r \) in \( \Gamma \) such that
\[
\sum_{i=1}^{r} \lambda_{ij} < 1 \quad \text{for all } j = 1, \ldots, n, \tag{2.1}
\]
where we set
\[
\vec{\theta}_i = \left( e^{2\pi \sqrt{-1}\lambda_{i1}}, \ldots, e^{2\pi \sqrt{-1}\lambda_{in}} \right) \quad \text{with } 0 < \lambda_{ij} < 1.
\]

To see what this means geometrically, let us identify \( \mathbb{T}^n \) with the product \( I_1 \times \ldots \times I_n \) of \( n \) intervals each of which is \([0, 1]\). Then (2.1) determines the standard open simplex \( \Delta \) in the cube \( I'_n \) and Condition A is equivalent to that \( \Gamma'^r \) intersects the region in \( (\mathbb{T}^n)^r \) obtained from the product of \( r \) copies of \( \Delta \) by rearranging the coordinates. It is clear that Condition A is independent of the choice of \( \tilde{P} \) or the ordering of the eigenvalues of \( P \).

**Example 2.3** (Toric \( \Phi \)). Assume that \( \Phi \) is toric, i.e., by definition \( \dim \Gamma = n \) or equivalently \( \Gamma = \mathbb{T}^n \); see Sect. 2.3.1. Then Condition A is automatically satisfied and \( P \) is elliptic, semi-simple and strongly non-degenerate. More generally, Condition A is met when \( \Gamma \) contains a one-parameter subgroup of the form \( t \mapsto (a_1 t, \ldots, a_n t), t \in \mathbb{R} \), with \( a_i > 0 \) for all \( i \); see Example 4.2.

We will see later that Condition A is in some sense satisfied for “most” of the elliptic transformations \( P \).

Next, denote by \( \mu_\Gamma \in \text{H}^1(\Gamma; \mathbb{Z}) \) the restriction of the Maslov class to \( \Gamma \). In other words, consider the codimension-one cocycle in \( \Gamma \) which is the sum of \( n \) cocycles obtained by setting the \( i \)th coordinate \( \theta_i \in S^1_i \) in \( \mathbb{T}^n \) equal to 1 and co-oriented by the counterclockwise orientation of \( S^1_i \). (We are assuming here that \( \Gamma \) is not contained in any of the subtori \( \theta_i = 1 \).) Then \( \mu_\Gamma \) is the cohomology class of this cocycle. Note that the mean index \( \hat{\mu}(\gamma) \) of a loop \( \gamma \) in \( \Gamma \) is \( 2\mu_\Gamma(\gamma) \).

**Definition 2.4** (Condition B1) For a fixed \( N \in \mathbb{N} \), the transformation \( P \) (or the subgroup \( \Gamma = \Gamma(P) \) or the orbit \( x \)) satisfies Condition B1 if \( \mu_\Gamma \) is not divisible by \( N \), i.e., there exists a loop \( \gamma \) in \( \Gamma \) such that \( N \nmid \mu_\Gamma(\gamma) \).
Example 2.5 Assume that $P$ is toric, i.e., $\Gamma = \mathbb{T}^n$. Then Condition B1 is automatically satisfied when $N > 1$. On the other hand, let $\Gamma$ be the circle $t \mapsto (a_1 t, \ldots, a_n t)$, $t \in S^1$, where $a_i \in \mathbb{Z}$ are non-zero and relatively prime. Then $\mu_\Gamma = a_1 + \ldots + a_n$ in $H^1(\Gamma; \mathbb{Z}) \cong \mathbb{Z}$. Thus Condition B1 is satisfied if and only if $N \not| (a_1 + \ldots + a_n)$. Finally, note that this condition is never met when $N = 1$.

We say that a path $\Phi : [0, 1] \to \text{Sp}(2n)$ satisfies Conditions A and B1 if the end-point $\Phi(1)$ satisfies these conditions. We will elaborate on Conditions A and B1 in Sects. 2.3 and 5.1.

2.1.2 Loop contribution

Conditions A and B1 are expressed entirely in terms of the linear map $P$ or the group $\Gamma$. However, in our case more information is available – this is the linearized flow along $x$ – and in this section we utilize it.

Consider a strongly non-degenerate path $\Phi : [0, 1] \to \text{Sp}(2n)$, which we view as an element of $\tilde{\text{Sp}}(2n)$, with end-point $P = \Phi(1)$. When $P$ is semi-simple we can decompose $\Phi$ as the concatenation (or product) of a loop $\phi$ and a direct sum of $n$ “short rotations” $t \mapsto \exp(\pi \sqrt{-1} \lambda t)$, where $t \in [0, 1)$ and $|\lambda| < 1$; cf. [18, Sect. 4]. When $P$ is not semi-simple we need to add an isospectral path $P_t$ to this decomposition as in the previous section. In either case, as is easy to see, the free homotopy class of the loop $\phi$ is uniquely determined by $\Phi$. Equivalently, the mean index $\hat{\mu}(\phi)$ is well defined. Set $\text{loop}(\Phi) := \hat{\mu}(\phi)$ and call $\text{loop}(\Phi)$ the loop part of $\Phi$. (Note that $\text{loop}(\Phi)$ is necessarily even and equal to twice the Maslov class of $\phi$.)

Definition 2.6 (Condition B2) For a fixed $N \in \mathbb{N}$, the path $\Phi$ (or a capped orbit $\bar{x}$) satisfies Condition B2 if $\Gamma$ is connected and there exists a convex neighborhood $V$ of $0 \in \mathbb{T}^n$ whose intersection with $\Gamma$ is connected and an iterate $\Phi^k(1) \in V$ such that $2N \not| \text{loop}(\Phi^k)$.

Roughly speaking, one should expect $N - 1$ out of $N$ randomly taken paths $\Phi$ to satisfy this condition. On the other hand, Condition B2 (just as Condition B1) is never satisfied when $N = 1$.

2.2 Detecting the quantum product

Let $(M^{2n}, \omega)$ be a closed weakly monotone symplectic manifold with minimal Chern number $N$. Fix a ground ring $\mathbb{F}$, suppressed in the notation; e.g., $\mathbb{F} = \mathbb{Z}$ or $\mathbb{Z}_2$ or $\mathbb{Q}$. For our purposes it is convenient to adopt the following definition; cf. [17].

Definition 2.7 (Pseudo-rotations) A Hamiltonian diffeomorphism $\varphi : M \to M$ is called a pseudo-rotation (over $\mathbb{F}$) if $\varphi$ is strongly non-degenerate, and the differential in the Floer complex of $\varphi^k$ over $\mathbb{F}$ vanishes for all $k \in \mathbb{N}$.

The differential in the Floer complex depends on the almost complex structure, but it is easy to see that its vanishing is a well-defined condition. Note also that for a pseudo-rotation all periodic orbits are automatically one-periodic and that an iterate of
a pseudo-rotation is again a pseudo-rotation. Definition 2.7 is slightly different from the one in [17] although it captures the same phenomenon. We refer the reader to that paper for a detailed discussion of various definitions of a pseudo-rotation. Finally, note that in some of our results the non-degeneracy requirement can be somewhat relaxed, but not entirely omitted.

**Example 2.8** Assume that \( \varphi \) is strongly non-degenerate and all its periodic orbits are elliptic. Then \( \varphi \) is a pseudo-rotation. All known to date pseudo-rotations are of this type.

**Example 2.9** (True rotations) Assume that \( \varphi \) is a true rotation, i.e., by definition \( \varphi \) generates a compact (but not finite) subgroup \( G \) of \( \text{Ham}(M) \). Then \( G \) is necessarily a compact Lie group by [29], and hence its connected component \( G_0 \) of the identity is a torus. It is then a standard fact that \( \varphi \) is strongly non-degenerate if and only if its periodic points are isolated and if and only if it has finitely many periodic orbits; see [19]. Furthermore, the resulting \( G_0 \)-action on \( M \) is Hamiltonian. (This is ultimately a consequence of some deep results, starting with [5] characterizing Hamiltonian diffeomorphisms as symplectomorphisms with zero flux and then the flux conjecture proved in [27]; see also [21,22].) In particular, strongly non-degenerate true rotations are among pseudo-rotations. All other known examples of pseudo-rotations are obtained from such true rotations by the conjugation method, [3,13,23].

Recall that the established cases of the Conley conjecture discussed in the introduction limit the class of manifolds that can possibly admit pseudo-rotations or more generally Hamiltonian diffeomorphisms with finitely many periodic orbits, [9,16]. (Namely, when \( M \) admits a pseudo-rotation there exists \( A \in \pi_2(M) \) such that \( \langle \omega, A \rangle > 0 \) and \( \langle c_1(TM), A \rangle > 0 \). In particular, \( \omega|_{\pi_2(M)} \neq 0 \) and \( c_1(TM)|_{\pi_2(M)} \neq 0 \).) The following simple result, specific to pseudo-rotations, further narrows down the class of such manifolds.

**Proposition 2.10** Assume that \( M^{2n} \) admits a pseudo-rotation. Then \( N \leq 2n \), where \( N \) is the minimal Chern number of \( M \).

A similar result has been recently proved in [36] under slightly less restrictive conditions and by a different method. The upper bound from Proposition 2.10 is extremely unlikely to be sharp: \( N \leq n + 1 \) for all known manifolds admitting pseudo-rotations. We defer the proof of the proposition to Sect. 3.2.1.

Denote by \( \text{HQ}_*(M) \) the (small) quantum homology of \( M \), by \( * \) the quantum product, and by \( |\alpha| \) the degree of an element \( \alpha \in \text{HQ}_*(M) \). Recall that the quantum product is said to be deformed if it is not equal to the intersection product.

The main result of the paper is the following.

**Theorem 2.11** Assume that \( M^{2n} \) admits a pseudo-rotation \( \varphi \) with an elliptic fixed point \( x \) which, for some \( r \in \mathbb{N} \), satisfies Condition A and also Condition B1 or, for some capping, Condition B2. Then there exist \( r \) elements \( \alpha_1, \ldots, \alpha_r \) in \( \text{HQ}_*(M) \) of even degree such that

\[
\alpha_1 \ast \ldots \ast \alpha_r \neq 0
\]  

(2.2)
and

\[ |\alpha_i| \not\equiv 2n \mod 2N \text{ for all } i = 1, \ldots, r. \quad (2.3) \]

This theorem is proved in Sect. 4.2. The key ingredient of the argument is a combinatorial result of independent interest (Theorem 4.7) concerning certain long products in \( \tilde{\text{Sp}}(2n) \) maximizing the defect of the Conley–Zehnder- or Maslov-type quasimorphism. Here, we have tacitly assumed that \( N \), for which Condition B1 or B2 is satisfied, is the minimal Chern number of \( M \). However, the theorem holds for any \( N \) meeting this requirement, which is actually a stronger result, and (2.3) gets easier to satisfy as \( N \) grows.

**Corollary 2.12** Assume that the conditions of Theorem 2.11 are satisfied with \( N \) being the minimal Chern number of \( M \) and \( r \geq n + 1 \). Then the quantum product is deformed and, in particular, some Gromov–Witten invariants of \( M \) are non-zero.

Note that here we could have as well required that \( N \geq 2 \); for Conditions B1 and B2 are never satisfied when \( N = 1 \). We will see from the results in Sect. 2.3 that, while involved, the conditions of Theorem 2.11 and Corollary 2.12 are satisfied in many cases. In fact, one can expect them to be met for a majority of pseudo-rotations unless \( N = 1 \). There are, however, exceptions: when a pseudo-rotation is “too symmetric” the conditions might not be met for any fixed point; see Remark 2.13 below.

**Proof of Corollary 2.12** Write \( \alpha_i = \sum_j f_j \alpha_{ij} \), where \( f_j \) is an element of the Novikov ring of degree \( j \) and \( \alpha_{ij} \in \text{H}_{\text{even}}(M) \). With our conventions \( |f_j| \) is divisible by \( 2N \); see Sect. 3. By (2.3), \( |\alpha_{ij}| < 2n \). Thus, if \( \ast \) is equal to the cup product, we necessarily have

\[ \alpha_1 \ast \ldots \ast \alpha_r = 0 \]

when \( r \geq n + 1 \); for \( \alpha_{ij} \) is not proportional to the fundamental class. \( \square \)

**Remark 2.13** While the conditions of Corollary 2.12 are probably met in most cases unless \( N = 1 \), there are some exceptions. For instance, let \( R_0 : S^2 \to S^2 \) be the rotation by \( \theta \not\equiv 2\pi \mathbb{Q} \). Then the diagonal map \( \varphi = (R_0, R_0) : S^2 \times S^2 \to S^2 \times S^2 \) does not have a periodic orbit \( x \) meeting the requirements of the corollary or of Theorem 2.19 below, which is sharper. Thus, for this pseudo-rotation, our method in its present form does not detect the deformed quantum product. (However, one can show that even in this case there are non-trivial Gromov–Witten invariants coming from zero energy pair-of-pants curves).

**Remark 2.14** (True rotations) Theorem 2.11 and Corollary 2.12 are not obvious even for true rotations. However, then \( M \) admits a non-degenerate Hamiltonian circle action and the results from [25] guarantee non-vanishing of certain Gromov–Witten invariants. (In general, the argument in [25] does not require non-degeneracy and seems to take no advantage of it.) A direct comparison of the results from [25] and this paper is not straightforward. One could expect that for non-degenerate true rotations the results from [25] would be stronger than, say, Theorem 2.11, but surprisingly this does not seem to be the case. It appears that the two methods in general detect different Gromov–Witten invariants.
2.3 Particular cases and refinements

In this section we discuss some particular cases and refinements of Theorem 2.11 and Corollary 2.12.

2.3.1 Toric pseudo-rotations

One case when the conditions of Theorem 2.11 are automatically satisfied is when a pseudo-rotation behaves like a generic element of the Hamiltonian $\mathbb{T}^n$-action on a toric symplectic $2n$-dimensional manifold.

**Definition 2.15** (Toric Pseudo-rotations) A pseudo-rotation $\phi$ of a closed symplectic manifold $M^{2n}$ is said to be *toric* if it has a fixed point $x$ with $\dim \Gamma(x) = n$.

Note that in this case $x$ is necessarily strongly non-degenerate.

**Corollary 2.16** Assume that $M$ admits a toric pseudo-rotation and $N > 1$. Then the quantum product is deformed and, in particular, some Gromov–Witten invariants of $M$ are non-zero.

This corollary is an immediate consequence of Examples 2.3 and 2.5 showing that Conditions A and B1 are automatically satisfied, and, of course, of Theorem 2.11. Moreover, then the theorem can be refined as follows:

**Theorem 2.17** Assume that $M^{2n}$ admits a toric pseudo-rotation. Then, for every $r \geq 1$, there exists $\alpha \in HQ_{2n-2}(M)$ such that $\alpha^r \neq 0$.

The proof of this result is quite similar to the proof of Theorem 2.11; see Sect. 4.2.

**Remark 2.18** Although the condition that a pseudo-rotation is toric appears generic – and it is indeed generic in $P = D\varphi_x$ – it is probably quite restrictive. Assume, for instance, that a toric pseudo-rotation is a true rotation; see Example 2.9. Then, as is easy to see from Example 2.1, $M^{2n}$ is necessarily a toric symplectic manifold and $\phi$ (or some iterate of it) is a topological generator of a Hamiltonian $\mathbb{T}^n$-action. One can expect only very few manifolds to have toric pseudo-rotations even among manifolds admitting pseudo-rotations, although this expectation is based more on the lack of knowledge and examples than on serious evidence. Note however that the proof of Theorem 2.17 provides, at least in principle, a way to obtain detailed information about the quantum product structure for $M$ and it would be interesting to compare it with the quantum product for toric manifolds.

2.3.2 Pseudo-rotations in dimension four

When $\dim M = 4$, i.e., $n = 2$, which we assume throughout this section, Theorem 2.11 and Corollary 2.12 can be further refined. Here we focus on detecting that the quantum product is deformed, although with some more work the method can be used to get specific information about the structure of the quantum product under minor assumptions on the base group.
When \( n = 2 \), the dimension of the base group \( \Gamma \) is either 0 or 1 or 2. If \( \dim \Gamma = 0 \), some iteration of \( P \) is necessarily degenerate. When \( \dim \Gamma = 2 \), Theorem 2.17 applies. Hence, here we concentrate on the case where \( \dim \Gamma = 1 \). Then the connected component \( \Gamma_0 \) of the identity in \( \Gamma \) is given by the equation

\[
s_1\theta_1 + s_2\theta_2 = 0
\]

on \( \mathbb{T}^2 \) with angular coordinates \((\theta_1, \theta_2)\), where \( s_1 \) and \( s_2 \) are relatively prime integers. When \( P \) is non-degenerate, \( s_1 \neq 0 \) and \( s_2 \neq 0 \). We refer to the ratio \( s = -s_1/s_2 \) as the slope of \( \Gamma \).

By Proposition 2.10, to admit a pseudo-rotation the manifold \( M \) must have minimal Chern number \( N \leq 4 \). When \( N = 1 \) our method does not detect the quantum product. The case of \( N = 4 \) is very improbable and not considered here. The following theorem gives a rather precise criterion for \( N = 2 \) and 3.

**Theorem 2.19** Assume that \( \dim M = 4 \) and \( M \) admits a pseudo-rotation with an elliptic fixed point \( x \) such that \( \dim \Gamma(x) = 1 \). Assume furthermore that \( N = 2 \) and \( s \neq \pm 1, 3, 1/3, -2, -1/2 \) or \( N = 3 \) and \( s \neq -1, \pm 2, \pm 1/2 \). Then the quantum product is deformed.

Thus, in dimension four (with \( N \geq 2 \)), the quantum product is deformed whenever \( \Gamma_0 \) is not one of these eight undesirable subgroups. Here, of course, the case of \( N = 2 \) is by far most interesting; the only example of a 4-manifold with \( N = 3 \) which admits a pseudo-rotation known to us is \( \mathbb{CP}^2 \). The proof of the theorem is based on detecting the product of length \( r = 3 \) with (2.3) satisfied; see Sect. 4.2. In a similar vein, to detect a product of any length \( r \) it would be sufficient to rule out only a finite number of subgroups. Finally, note that the requirements of Theorem 2.19 are strictly weaker than Conditions A and B1. For instance, Condition B1 holds if and only if \( s_1 + s_2 \) is odd. (However, Condition A in dimension four is met for every \( r \) by all but a finite number of subgroups \( \Gamma \) of positive dimension; see Remark 5.7.) One can think of Theorem 2.19 as an additional proof of concept result: ultimately the method should enable one to treat many more cases than covered by Theorem 2.11, although the combinatorics of the proof might get rather involved.

# 3 Preliminaries

In this section we set the conventions and notation used in the paper and briefly recall several definitions and facts from symplectic topology relevant for the proofs. We also prove Proposition 2.10 and the regularity results for zero-energy pair-of-pants curves.

## 3.1 Conventions and notation

Throughout the paper \((M^{2n}, \omega)\) is a closed symplectic manifold, which, to avoid foundational issues, we will always assume to be weakly monotone in the sense of [20]. The minimal Chern number, i.e., the positive generator of the group \( \langle c_1(TM), \pi_2(M) \rangle \subset \mathbb{Z} \), is denoted by \( N \). (When this group is zero, \( N = \infty \).)
A Hamiltonian diffeomorphism is the time-one map $\varphi = \varphi_H$ of the time-dependent flow $\varphi^t_H$ of a 1-periodic in time Hamiltonian $H : S^1 \times M \to \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. The Hamiltonian vector field $X_H$ of $H$ is defined by $i_{X_H}\omega = -dH$. Such time-one maps form the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of $M$. In what follows, it will be convenient to view Hamiltonian diffeomorphisms as elements of the universal covering $\tilde{\text{Ham}}(M, \omega)$.

Let $x : S^1 \to M$ be a contractible loop. A capping of $x$ is an equivalence class of maps $A : D^2 \to M$ such that $A|_{S^1} = x$. Two cappings $A$ and $A'$ of $\bar{x}$ are equivalent if the integrals of $\omega$ and $c_1(TM)$ over the sphere obtained by attaching $A$ to $A'$ are equal to zero. A capped closed curve $\bar{x}$ is, by definition, a closed curve $x$ equipped with an equivalence class of cappings, and the presence of capping is always indicated by a bar.

The action of a Hamiltonian $H$ on a capped closed curve $\bar{x} = (x, A)$ is

$$A_H(\bar{x}) = -\int_A \omega + \int_{S^1} H_t(x(t))\,dt.$$ 

The space of capped closed curves is a covering space of the space of contractible loops, and the critical points of $A_H$ on this space are exactly the capped one-periodic orbits of $X_H$.

The $k$-periodic points of $\varphi_H$ are in one-to-one correspondence with the $k$-periodic orbits of $H$, i.e., of the time-dependent flow $\varphi^t_H$. Recall also that a $k$-periodic orbit of $H$ is called simple or prime if it is not iterated. Clearly, the action functional is homogeneous with respect to iteration: $A_{H^k}(\bar{x^k}) = kA_H(\bar{x})$, where $\bar{x^k}$ is the $k$th iteration of the capped orbit $\bar{x}$. (The capping of $\bar{x^k}$ is obtained from the capping of $\bar{x}$ by taking its $k$-fold cover branched at the origin.)

A $k$-periodic orbit $x$ of $H$ is said to be non-degenerate if the linearized return map $d\varphi^t_H : T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. We call $x$ strongly non-degenerate if all iterates $x^k$ are non-degenerate. A Hamiltonian $H$ is non-degenerate if all its one-periodic orbits are non-degenerate and $H$ is strongly non-degenerate if all periodic orbits of $H$ (of all periods) are non-degenerate.

Let $\bar{x}$ be a non-degenerate capped periodic orbit. The Conley–Zehnder index $\mu(\bar{x}) \in \mathbb{Z}$ is defined, up to a sign, as in [31,32]. In this paper, we normalize $\mu$ so that $\mu(\bar{x}) = n$ when $x$ is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian. The mean index $\bar{\mu}(\bar{x}) \in \mathbb{R}$ measures, roughly speaking, the total angle swept by certain (Krein–positive) unit eigenvalues of the linearized flow $d\varphi^t_H|_{\bar{x}}$ with respect to the trivialization associated with the capping; see [24,32]. The mean index is defined even when $x$ is degenerate and depends continuously on $H$ and $\bar{x}$ in the obvious sense. Furthermore,

$$|\bar{\mu}(\bar{x}) - \mu(\bar{x})| \leq n.$$ 

The mean index is homogeneous with respect to iteration: $\bar{\mu}(\bar{x^k}) = k\bar{\mu}(\bar{x})$. For an uncapped orbit $x$, the mean index $\bar{\mu}(x)$ is well defined as an element of $S^1_{2N} := \mathbb{R}/2N\mathbb{Z}$. Likewise, when $x$ is non-degenerate, the Conley–Zehnder index $\mu(x)$ is well defined as an element of $\mathbb{Z}/2N\mathbb{Z}$. 


3.2 Floer homology and the pair-of-pants product

3.2.1 Floer homology

Let $\varphi = \varphi_H$ be a non-degenerate Hamiltonian diffeomorphism, which we will view as an element of $\tilde{\text{Ham}}(M)$. Fixing a ground ring $\mathbb{F}$ (e.g., $\mathbb{Z}_2$ or $\mathbb{Q}$) and an almost complex structure, both of which will be suppressed in the notation, we denote by $\text{CF}_*(\varphi)$ and $\text{HF}_*(\varphi)$ the Floer complex and homology of $\varphi$; see, e.g., [20,26,31]. The exact definition of the differential on $\text{CF}_*(\varphi)$ is immaterial for our purposes, but it is essential that $\text{CF}_*(\varphi)$ is generated by the capped one-periodic orbits $\bar{x}$ of $H$ and graded by the Conley–Zehnder index. Furthermore, $\text{CF}_*(\varphi)$ and $\text{HF}_*(\varphi)$ are filtered by the action of $H$. We have the canonical isomorphism

$$\text{HF}_*(\varphi) \cong \text{HQ}_*(M)[−n],$$

(3.1)

where $\text{HQ}_*(M)$ is the quantum homology of $M$; see, e.g., [26,31] and references therein.

The total homology $\text{HQ}_*(M)$ and $\text{HF}_*(\varphi)$ and the complex $\text{CF}_*(\varphi)$ are modules over a Novikov ring $\Lambda$; and $\text{HQ}_*(M) \cong H_*(M) \otimes \Lambda$ (as a module). There are several choices of $\Lambda$; see, e.g., [26]. A specific choice is inessential for our purposes, but we prefer to think of $\Lambda$ as a certain quotient of the group algebra of $\pi_2(M)$, accounting for the equivalence of cappings; see, e.g., [20]. Then $\Lambda$ naturally acts on $\text{CF}_*(\varphi)$ by recapping. We denote by $|\alpha|$ the degree of $\alpha$ in $\text{HQ}_*(M)$ or $\text{HF}_*(\varphi)$. Thus the fundamental class $[M]$ has degree $2n$ in $\text{HQ}_*(M)$ and $n$ in $\text{HF}_*(\varphi)$ and the point class $[pt]$ has degree zero in $\text{HQ}_*(M)$ and $-n$ in $\text{HF}_*(\varphi)$.

When $\varphi$ is a pseudo-rotation we have natural isomorphisms

$$\text{CF}_*(\varphi) \cong \text{HF}_*(\varphi) \cong \text{HQ}_*(M)[−n].$$

(3.2)

Any iterate $\varphi^k$ is then also a pseudo-rotation, and hence

$$\text{CF}_*(\varphi^k) \cong \text{HF}_*(\varphi^k) \cong \text{HQ}_*(M)[−n].$$

With the notation set, we are in a position to prove Proposition 2.10.

**Proof of Proposition 2.10** Let $\varphi$ be a pseudo-rotation of $M^{2n}$. By passing to an iterate of $\varphi$ if necessary, we can ensure that for every fixed point $x$ of $\varphi$ all elliptic eigenvalues of $D\varphi_x : T_x M \to T_x M$ are close to 1 and that for any capping of $x$ the loop part of $D\varphi_x$, viewed as an element of $\tilde{\text{Sp}}(T_x M)$, is divisible by $2N$; see Sect. 2.1.2 for the definition.

By (3.2), $\text{HF}_*(\varphi^k)$ and hence $\text{CF}_*(\varphi^k)$ are supported within $[−n, n] + 2N\mathbb{Z}$, i.e., for any fixed point $\tilde{x}^k$ of $\varphi^k$ with any capping its Conley–Zehnder index is within this union of the intervals $[−n, n] + 2Nj$. Without loss of generality we may assume that $N \geq n + 1$; for $n + 1 \leq 2n$. As a consequence, these intervals are disjoint.

Let $\tilde{x}$ be a fixed point of $\varphi$ capped so that $\mu(\tilde{x}) = n$. Such a point necessarily exists because $\text{HQ}_{2n}(M) \neq 0$. Since $x$ is non-degenerate, $\hat{\mu}(x) > 0$ and thus $\mu(\tilde{x}^k) \to \infty$;
cf. [32]. Furthermore, \( \text{loop}(D\varphi_{\bar{x}}) = 0 \) as is not hard to see from the condition that \( N \geq n + 1. \) (Otherwise we would have \( \mu(\bar{x}) \geq 2N - n > n. \)) Then
\[
0 \leq \mu(\bar{x}^{k+1}) - \mu(\bar{x}^k) \leq 2n.
\]
(These inequalities follow from Proposition 5.1 and also are easy to prove directly.) Therefore, we have
\[
\mu(\bar{x}^k) \in [n + 1, 3n]
\]
for some \( k \in \mathbb{N}. \) For this value to be in the support of the Floer homology, we must have \( 2N - n \leq 3n, \) i.e., \( N \leq 2n. \)

\[\Box\]

### 3.2.2 Pair-of-pants product

Recall that for a pair of Hamiltonian diffeomorphisms \( \varphi \) and \( \psi \) we have the pair-of-pants product
\[
HF_*(\varphi) \otimes HF_*(\psi) \to HF_*(\varphi \psi).
\]
This product, which we denote by \( \ast \), has degree \( -n \), i.e., \( |\alpha \ast \beta| = |\alpha| + |\beta| - n. \) Under the identification (3.1), the pair-of-pants product turns into the quantum product on \( HQ_*(M) \) also denoted by \( \ast \), which is a certain deformation of the intersection product on \( H_*(M) \) and has degree \( -2n. \) As a consequence, setting \( \varphi = \varphi_1 \ldots \varphi_r \), we also have the pair-of-pants product
\[
HF_*(\varphi_1) \otimes \ldots \otimes HF_*(\varphi_r) \to HF_*(\varphi),
\]
which agrees with the quantum product, and, in particular,
\[
HF_*(\varphi^{k_1}) \otimes \ldots \otimes HF_*(\varphi^{k_r}) \to HF_*(\varphi^{k})
\]
where \( k_1 + \ldots + k_r = k \) and
\[
|\alpha_1| + \ldots + |\alpha_r| - |\alpha_1 \ast \ldots \ast \alpha_r| = (r - 1)n.
\]

Referring the reader to, e.g., [2, 26, 28] for a detailed treatment of the pair-of-pants product (see also [33] for a different and more modern approach) and skipping over some nuances, we only mention here few relevant points. There are several ways to describe the pair-of-pants product at the level of complexes and any of them is suitable for our purposes as long as it respects the action filtration. (Thus, for instance, the construction from [28] does not meet this requirement, but the one in [2] does.)

The product
\[
CF_*(\varphi_1) \otimes \ldots \otimes CF_*(\varphi_r) \to CF_*(\varphi)
\]
“counts” the number of solutions \( u: \Sigma \to M \) of a suitably defined Floer equation, where the domain \( \Sigma \) is the \((r+1)\)-punctured sphere; see, e.g., [2,26]. In other words, consider capped one-periodic orbits \( \bar{x}_j \) of \( \varphi_i \) and a capped one-periodic orbit \( \bar{y} \) of \( H \). Let \( \mathcal{M} \) be the moduli space of such solutions \( u \) “connecting” \( \bar{x}_1, \ldots, \bar{x}_r \) to \( \bar{y} \). The virtual dimension of \( \mathcal{M} \) is

\[
\dim \mathcal{M} = \mu(\bar{x}_1) + \ldots + \mu(\bar{x}_r) - \mu(\bar{y}) - (r - 1)n. \tag{3.3}
\]

Assume that this dimension is zero. Then \( \bar{y} \) enters the product \( \bar{x}_1 \ast \ldots \ast \bar{x}_r \) with the coefficient equal to the number of points (counted with signs if \( \mathbb{F} \neq \mathbb{Z}_2 \)) in the moduli space of such \( u \) “connecting” \( \bar{x}_1, \ldots, \bar{x}_r \) to \( \bar{y} \), provided that a certain regularity condition is met. This condition, which we will touch upon in the next section, is satisfied for generic maps \( \varphi_i \).

With or without regularity, we necessarily have

\[
\mathcal{A}_{H_1}(\bar{x}_1) + \ldots + \mathcal{A}_{H_r}(\bar{x}_r) - \mathcal{A}_H(\bar{y}) = E(u) \geq 0,
\]

where \( E(u) \) is the energy of \( u \), the Hamiltonian \( H_i \) generates \( \varphi_i \) and \( H \) generates \( \varphi \); see [2, Eq. (3-18)]. (The choice of \( H \) depends on the Hamiltonians \( H_i \).) In particular, \( E(u) = 0 \) if and only if

\[
\mathcal{A}_{H_1}(\bar{x}_1) + \ldots + \mathcal{A}_{H_r}(\bar{x}_r) = \mathcal{A}_H(\bar{y}). \tag{3.4}
\]

In turn, this is the case if and only if \( x_1(0) = \ldots = x_r(0) \), the loop \( y \) is the concatenation of the loops \( x_i \) and \( u \) maps \( \Sigma \) onto \( y \). Without loss of generality we may assume that the orbits \( x_i \) are constant; see, e.g., [14, Sect. 2.3]. Then (3.4) holds if and only \( E(u) = 0 \) and if and only if \( u \) is a constant map.

If the regularity condition is not satisfied, as is often the case for \( \varphi_i = \varphi_i^{k_i} \), one replaces the maps \( \varphi_i \) by small perturbations \( \varphi'_i \). Since \( \varphi_i \) is non-degenerate there is a one-to-one correspondence between one-periodic orbits of \( \varphi_i \) and \( \varphi'_i \) and also a canonical isomorphism \( CF_*(\varphi_i) \cong CF_*(\varphi'_i) \). However, this isomorphism effects the action filtration.

### 3.3 Regularity for zero-energy solutions

Our goal in this section is to show that zero index, zero energy pair-of-pants solutions of the Floer equation are automatically regular. Thus let \( x \) be a strongly non-degenerate one-periodic orbit of \( H \) and let \( u: \Sigma \to M \) be the zero energy solution asymptotic to \( \tilde{x}^{k_1} \ldots \tilde{x}^{k_r} \) and \( \tilde{x}^k \) where \( k_1 + \ldots + k_r = k \). As has been mentioned above, we may assume that \( x \) is a constant one-periodic orbit, and hence \( u \) is a constant solution of the Floer equation mapping \( \Sigma \) to \( x \). Denote by \( D: \mathcal{E}^1 \to \mathcal{E}^0 \) the linearized Floer operator along \( u \). Here \( \mathcal{E}^1 \) is the space of, say, \( W^{1,p} \)-sections of \( u^*TM \) with \( p > 1 \) and \( \mathcal{E}^0 \) is the space of \( L^p \)-sections. The operator \( D \) has the form \( \tilde{\partial} + S \), where \( S \) is an automorphism of \( u^*TM \), and is Fredholm due to the non-degeneracy assumption.

**Proposition 3.1** We have \( \ker D = 0 \).
Proposition 3.1 is quite standard and has several predecessors. A variant of the proposition for Floer cylinders is established in [31, Sect. 2.3] and for closed holomorphic curves in [26, Lemma 6.7.6]. Perhaps the easiest way to prove the proposition is by adapting the argument from [34, p. 971]. Namely, let us pass to Lagrangian Floer theory by using the graph construction. Then $D$ turns into the Cauchy–Riemann operator (with the complex structure in the target space parametrized by the domain) and a solution $\xi$ of the equation $D\xi = 0$ becomes a zero-energy holomorphic map into $T_x M \oplus T_x M$. Such a curve is necessarily constant and then $\xi = 0$ since $\xi$ is globally in $W^{1,p}$.

Recall that $u$ is regular when $D$ is onto, i.e., coker $D = 0$, and that the Fredholm index of $D$ is given by (3.3):

$$\dim \ker D - \dim \coker D = \mu(\bar{x}^{k_1}) + \ldots + \mu(\bar{x}^{k_r}) - \mu(\bar{x}^k) - (r - 1)n.$$ 

Thus coker $D = 0$ whenever the index of $D$ is zero and we have proved

Corollary 3.2  Assume that

$$\mu(\bar{x}^{k_1}) + \ldots + \mu(\bar{x}^{k_r}) - \mu(\bar{x}^k) - (r - 1)n = 0,$$

i.e., $k_1 + \ldots + k_r = k$ is an extremal partition (see Definition 4.1). Then the zero energy solution is automatically regular.

4 From extremal partitions to the quantum product

4.1 Extremal partitions: the first look

The notion central to the combinatorial part of the proof of the main results is that of an extremal partition.

4.1.1 Definitions and basic facts

Fix a path $\Phi \in \tilde{Sp}(2n)$. For the sake of simplicity, we will assume that $\Phi$ is elliptic and strongly non-degenerate, i.e., the iterate end-point $\Phi^k(1)$ is non-degenerate for all $k \in \mathbb{N}$.

Definition 4.1 (Extremal partitions) A partition $k_1 + \ldots + k_r = k$, $k_i \in \mathbb{N}$, of length $r \geq 2$ is said to be extremal (with respect to $\Phi$) if

$$\mu(\Phi^{k_1}) + \ldots + \mu(\Phi^{k_r}) - \mu(\Phi^k) = (r - 1)n.$$  

(4.1)

We will show that the existence of an extremal partition is equivalent to Condition A; see Proposition 5.3. Deferring a detailed discussion of extremal partitions to Sect. 5.1, we only mention here two simple facts. Consider the defect

$$D = D(\Phi_1, \ldots, \Phi_r) := \sum \mu(\Phi_i) - \mu(\Phi_1 \cdot \ldots \cdot \Phi_r)$$  

(4.2)
of the “Conley–Zehnder quasimorphism”, where we have assumed that all \( \Phi_r \) and all partial products \( \Phi_1 \cdot \ldots \cdot \Phi_\ell, \ell \leq r \), are non-degenerate. Then, as is shown in [12],

\[ |D| \leq (r - 1)n. \]

(The non-degeneracy requirement is essential.) We will further discuss this fact and give a short proof in Sect. 5.3; see Proposition 5.1. In particular, for any partition \( k_1 + \ldots + k_r = k, k_i \in \mathbb{N} \), we have

\[ \mu(\Phi^{k_1}) + \ldots + \mu(\Phi^{k_r}) - \mu(\Phi^k) \leq (r - 1)n \quad (4.3) \]

as long as the products are non-degenerate; cf. [34, Lemma 5.10]. Thus extremal partitions maximize the defect; hence, the term.

Furthermore, \( D \) depends only on the end-points \( \Phi_1(1), \ldots, \Phi_r(1) \). Indeed, composing one of the maps \( \Phi_i \) with a loop changes both terms in (4.2) by the mean index of the loop. In particular, the left-hand side of (4.2) is completely determined by \( \Phi(1) \) and, of course, the partition. In other words, whether or not \( k_1 + \ldots + k_r = k \) is an extremal partition is a feature of \( \Phi(1) \), but the indices \( \mu(\Phi^{k_i}) \) depend on the path \( \Phi \). For the sake of brevity we set \( \Gamma(\Phi) := \Gamma(\Phi(1)) \).

**Example 4.2** Assume that \( \Phi \) is the direct sum of \( n \) counterclockwise rotations \( \exp(2\pi \sqrt{-1} \lambda_i t) \), where \( \lambda_i > 0 \) are small and \( t \in [0, 1] \). Then \( \mu(\Phi^r) = n \) as long as \( r \max \lambda_i < 1 \), and \( 1 + \ldots + 1 = r \) is an extremal partition with (4.1) taking the form \( rn - (r - 1)n = n \).

**Example 4.3** Assume that \( \Phi \) is toric, i.e., \( \dim \Gamma(\Phi) = n \). Then \( \Phi \) admits extremal partitions of arbitrarily large length. We will prove this fact in Sect. 5.1, but it is also not hard to see this directly as a consequence of Example 4.2.

**Example 4.4** Assume that \( \Phi \) is the sum of a clockwise rotation \( \exp(-2\pi \sqrt{-1} \lambda t) \), \( t \in [0, 1] \), where \( \lambda > 0 \), and the counterclockwise rotation by the same angle. Then \( \mu(\Phi^k) = 0 \) for all \( k \in \mathbb{N} \) and \( \Phi \) does not admit extremal partitions.

**Example 4.5** Assume that \( \Phi \) is the clockwise rotation \( \exp(-2\pi \sqrt{-1} \lambda t) \), \( t \in [0, 1] \), where \( \lambda > 0 \), or the direct sum of such rotations by the same angle. Then \( \Phi \) also admits extremal partitions. This follows, for instance, from Example 4.2 and Proposition 5.2 (i) or can be verified directly.

**Remark 4.6** The condition that \( \Phi \) is elliptic imposed above is in some sense redundant: non-elliptic symplectic maps simply do not admit extremal partitions. This fact readily follows from Proposition 5.2 and is also easy to prove directly.

### 4.1.2 Combinatorial results: the existence of extremal partitions

As one can guess already from (3.3) giving the dimension of the relevant moduli spaces, extremal partitions are intimately related to certain products in quantum homology; see Theorem 4.12. However, to conclude from this that the quantum product is deformed
one needs to have additional information about the classes involved in the product, which in our context is a combinatorial problem. For instance, the proof of Theorem 2.11 hinges on the following result.

**Theorem 4.7** (Extremal partition theorem) Let $\Phi \in \tilde{Sp}(2n)$ be elliptic and strongly non-degenerate. Assume that for some $r \in \mathbb{N}$ the linear symplectic map $\Phi(1)$ satisfies Condition A and also, for some $N \in \mathbb{N}$, Condition B1 or Condition B2. Then there exists an extremal partition $k_1 + \ldots + k_r = k$ with respect to $\Phi$ such that

$$\mu(\Phi^{k_i}) \not\equiv n \mod 2N \text{ for all } i = 1, \ldots, r.$$  \hfill (4.4)

Note that here, as in Theorem 2.11, we could have required that $N \geq 2$, since Conditions B1 and B2 are never satisfied when $N = 1$. It is also worth pointing out again that in this theorem Conditions A and B1 or B2 play very different roles. Condition A is necessary and sufficient to guarantee the existence of an extremal partition (cf. Proposition 5.3), while Condition B1 or B2 is used to establish (4.4).

In a similar vein, Theorem 2.17 relies on the combinatorics of extremal partitions in the toric case.

**Theorem 4.8** Assume that $\Phi$ is toric, i.e., $\Gamma(\Phi) = \mathbb{T}^n$. Then, for every $r \geq 1$, there exists an extremal partition $m + \ldots + m = k$ of length $r$ (i.e., $r \cdot m = k$) such that

$$\mu(\Phi^m) \equiv n - 2 \mod 2N.$$  \hfill (4.5)

Note that $\Phi$ is then strongly non-degenerate and all eigenvalues of $\Phi(1)$ are necessarily distinct; cf. Example 2.3. In particular, $\Phi(1)$ is automatically semi-simple if $\Gamma = \mathbb{T}^n$.

**Remark 4.9** An immediate consequence of the proof of Theorem 4.8 is that the assertion of the theorem also holds whenever Conditions A and B2 are satisfied for the connected component of the identity $\Gamma_0(\Phi)$. (The same is true for Theorem 2.11.) We will use this fact in Sect. 5.2 in the proof of Theorem 2.19.

Finally, Theorem 2.19 is also a consequence of the following combinatorial result.

**Theorem 4.10** Let $\Phi \in \tilde{Sp}(4)$ be elliptic and strongly non-degenerate, and such that $\dim \Gamma(\Phi) = 1$. Assume furthermore that the slope $s \neq \pm 1, 3, 1/3, -2, -1/2$ when $N = 2$ or $s \neq -1, \pm 2, \pm 1/2$ when $N = 3$. Then there exists an extremal partition of length 3 such that

$$\mu(\Phi^{k_i}) \not\equiv 2 \mod 2N \text{ for } i = 1, 2, 3.$$  \hfill (4.6)

**Remark 4.11** This theorem is more precise than Theorem 4.7 and it gives essentially a necessary and sufficient condition in dimension four. Namely, assume that $\Gamma$ is connected and its slope is “black-listed” in Theorem 4.10. Then there exists $\Phi \in \tilde{Sp}(4)$ such that $\Gamma(\Phi) = \Gamma$ and there are no extremal partitions satisfying (4.6). However, in the setting of the theorem, $\Phi$ still satisfies Condition A, and if Condition B2 holds the desired partitions exist.
The conditions of these theorems are satisfied for most (but not all) strongly non-degenerate, elliptic $\Phi \in \widehat{Sp}(2n)$.

4.2 Combinatorics of extremal partitions and the quantum product

In this section we establish the main result of the paper, Theorem 2.11, as an easy consequence of Theorem 4.7 and Corollary 3.2, and also Theorems 2.17 and 2.19. The proofs of all three theorems follow the same path and can be rephrased as a general argument reducing the problem to a combinatorial question.

Recall that since $\varphi$ is a pseudo-rotation, for every $k \in \mathbb{N}$ we have canonical identifications (3.2):

$$CF_* (\varphi^k) \cong HF_* (\varphi^k) \cong HQ_* (M)[-n],$$

where we view $\varphi$ as an element of $\widehat{Ham}(M)$ rather than $Ham(M)$.

**Theorem 4.12** Let $\bar{x}$ be a capped one-periodic orbit of a pseudo-rotation $\varphi$, and let $k_1 + \ldots + k_r = k$ be an extremal partition of length $r$ with respect to $\Phi := D\varphi|\bar{x}$. Using (3.2), set $\alpha_i = [\bar{x}^{k_i}] \in HQ_* (M)$. Then $|\alpha_i| = n + \mu (\Phi^{k_i})$ and (2.2) holds:

$$\alpha_1 * \ldots * \alpha_r \neq 0.$$  

Note that in the setting of this theorem $x$ is automatically elliptic; see Remark 4.6. Theorems 2.11 and 2.19 immediately follow from this general result and Theorems 4.7 and 4.10, combined with the observation that all iterated indices $\mu (\Phi^{k_i})$ have the same parity when $\Phi$ is elliptic.

**Proof of Theorem 4.12** The argument is based on Example 1.1. Clearly $|\alpha_i| = n + \mu (\Phi^{k_i})$. Thus we only need to verify (2.2), i.e., that

$$[\bar{x}^{k_1}] * \ldots * [\bar{x}^{k_r}] \neq 0,$$

where we have now identified the quantum product with the pair-of-pants product

$$HF_* (\varphi^{k_1}) \otimes \ldots \otimes HF_* (\varphi^{k_r}) \rightarrow HF_* (\varphi^k).$$

Consider small non-degenerate perturbations $\varphi_{k_i}$ of $\varphi^{k_i}$ such that on the level of Floer complexes the regularity condition is satisfied for the pair-of-pants product

$$CF_* (\varphi_{k_1}) \otimes \ldots \otimes CF_* (\varphi_{k_r}) \rightarrow CF_* (\varphi_k),$$

where $\varphi_k := \varphi_{k_r} \circ \ldots \circ \varphi_{k_1}$. Note that $\varphi_k$ is also a small perturbation of $\varphi^k$, and we have canonical isomorphisms of the Floer complexes

$$CF_* (\varphi_{k_i}) = CF_* (\varphi^{k_i})$$

and

$$CF_* (\varphi_k) = CF_* (\varphi^k).$$
Furthermore, by Corollary 3.2, we can make these perturbations such that $\varphi_{ki} = \varphi^k_i$ near $x$ and, as a consequence, $\varphi_k = \varphi^k$ on a small neighborhood of $x$. (Here it is convenient to assume that $x$ is a constant one-periodic orbit – this can always be achieved by composing $\varphi$ with a contractible loop; see, e.g., [14, Sect. 2.3].) Thus $\bar{x}^k_i$ is still a capped one-periodic orbit of $\varphi_{ki}$ and $\bar{x}^k$ is a capped periodic orbit of $\varphi_k$. Let us redenote these orbits as $\bar{x}^k_i$ and $\bar{x}^k$, respectively.

The only zero-energy pair-or-pants curves are constant; see Sect. 3.2.2. Thus the constant curve is the only curve from $(\bar{x}^k_1, \ldots, \bar{x}^k_r)$ to $\bar{x}^k$. Furthermore, consider the moduli space of such curves. This moduli space has virtual dimension zero and the constant curve from $(\bar{x}^k_1, \ldots, \bar{x}^k_r)$ to $\bar{x}^k$ is regular by Corollary 3.2. Therefore,

$$\bar{x}^k_1 \ast \ldots \ast \bar{x}^k_r = \bar{x}^k + \ldots,$$

(4.8)

where the dots stand for capped periodic orbits of $\varphi_k$ with action strictly smaller than the action of $\bar{x}^k$. As a consequence,

$$[\bar{x}^{k_1}] \ast \ldots \ast [\bar{x}^{k_r}] = [\bar{x}^k] + \ldots,$$

where the dots represent again some cohomology classes generated by the orbits with action strictly smaller than the action of $\bar{x}^k$. Hence, the right-hand side is non-zero. This proves (4.7) and concludes the proof of the theorem. □

Remark 4.13 Choosing the perturbations $\varphi_{ki}$ equal to $\varphi^k_i$ near $x$ is convenient but not really necessary. Since the constant pair-of-pants curve from $(\bar{x}^{k_1}, \ldots, \bar{x}^{k_r})$ to $\bar{x}^k$ is regular, it will persist under a small perturbation turning into exactly one non-constant small energy curve. This is enough to separate the action of $\bar{x}^k$ from the actions of other periodic orbits on the right-hand side of the product (4.8); cf. [16, Prop. 2.2].

Proof of Theorem 2.17 The result easily follows from Theorem 4.12. We only need to make sure that in this case we can take the product of $r$ equal elements of degree $2n - 2$. Let $x$ be a one-periodic orbit of $\varphi$ such that $\dim \Gamma(x) = n$. Let $\Phi = D\varphi^t|_{\bar{x}}$, where we have used an arbitrary capping of $x$, and let $m$ be as in Theorem 4.8. Then $m + \ldots + m = rm$ is an extremal partition for $\Phi$. Although in general the degree of $[\bar{x}^m]$ need not be equal to $2n - 2$, we have $\mu(\bar{x}^m) = n - 2$ in $\mathbb{Z}_{2N}$ by (4.5). (Recall that the Conley–Zehnder index of an un-capped orbit is well-defined as an element of $\mathbb{Z}_{2N}$.) Denote by $\bar{y}$ the orbit $x^m$ capped so that $\mu(\bar{y}) = n - 2$. Then $[\bar{y}]^r = f \cdot [\bar{x}^m]^r$ for some $f \neq 0$ in the Novikov ring, and $[\bar{x}^m]^r \neq 0$ by Theorem 4.12. It follows that $\alpha^r \neq 0$ and $|\alpha| = 2n - 2$, where $\alpha = [\bar{y}]$. □

5 Study of extremal partitions

5.1 General properties

The notion of an extremal partition is certainly interesting by itself. In this section we establish some of their general properties, recalling some of the facts already mentioned in Sect. 4.1 and going slightly farther than is strictly speaking necessary for
applications to our main results. We start with the following general result concerning the defect of “the Conley–Zehnder quasimorphism”:

**Proposition 5.1** (Cor. 3.5 in [12]). For any two non-degenerate elements \( \Phi \) and \( \Psi \) of \( \widetilde{Sp}(2n) \), we have

\[
|\mu(\Psi \Phi) - \mu(\Psi) - \mu(\Phi)| \leq n.
\]

This upper bound is sharp and the non-degeneracy requirement is essential. The proposition in particular implies a sharp upper bound for the defect of several Conley–Zehnder (or Maslov-) type quasimorphisms on \( \widetilde{Sp}(2n) \); see Remark 5.9. For the sake of completeness, we give a short and elementary proof of the proposition in Sect. 5.3. (Note also that the regularity arguments from Sect. 3.3 can be turned into an analytical proof of Proposition 5.1.)

As a consequence, for any partition \( k_1 + \ldots + k_r = k \), \( k_i \in \mathbb{N} \), we have (4.3), i.e.,

\[
\mu(\Phi^{k_1}) + \ldots + \mu(\Phi^{k_r}) - \mu(\Phi^k) \leq (r - 1)n,
\]

as long as the products are non-degenerate. (Another way to state this fact is that the function \( k \mapsto \mu(\Phi^k) - n \) is sub-additive.) Thus extremal partitions maximize the left-hand side of this inequality. This upper bound is again sharp and non-degeneracy is essential.

**Proposition 5.2** (Properties of extremal partitions). Let \( \Phi \in \widetilde{Sp}(2n) \) and \( \Psi \in \widetilde{Sp}(2n') \).

(i) A partition \( k_1 + \ldots + k_r = k \) is extremal for \( \Phi \) if and only if it is extremal for \( \phi \Phi \) for any loop \( \phi \) in \( Sp(2n) \). Thus the property to be extremal depends only on \( \Phi(1) \in Sp(2n) \).

(ii) A partition \( k_1 + \ldots + k_r = k \) is extremal for \( \Phi^m \) if and only if \( mk_1 + \ldots + mk_r = mk \) is extremal for \( \Phi \).

(iii) A partition \( k_1 + \ldots + k_r = k \) is extremal for \( \Phi \oplus \Psi \) if and only if it is simultaneously extremal for \( \Phi \) and \( \Psi \).

(iv) Assume that \( k_1 + \ldots + k_r = k \) and \( \ell_1 + \ldots + \ell_s = k_1 \) are extremal partitions for \( \Phi \). Then \( \ell_1 + \ldots + \ell_s + k_2 + \ldots + k_r = k \) is also an extremal partition for \( \Phi \). Conversely, assume that \( k_1 + \ldots + k_r = k \) is an extremal partition. Then for any \( 1 \leq s \leq r \), the sum \( k_1 + \ldots + k_s =: m \) of the first \( s \) terms and the sum \( m + k_{s+1} + \ldots + k_r = k \) are also extremal partitions.

**Proof** Recall that \( k_1 + \ldots + k_r = k \) is an extremal partition for \( \Phi \) if (4.1) holds:

\[
\mu(\Phi^{k_1}) + \ldots + \mu(\Phi^{k_r}) - \mu(\Phi^k) = (r - 1)n.
\]

Replacing \( \Phi \) by \( \phi \Phi \) adds \( k_i \hat{\mu}(\phi) \) and \( k \hat{\mu}(\phi) \) to the terms on the left and hence does not effect the sum. This proves (i). Assertion (ii) is obvious from the definition.

By additivity of the Conley–Zehnder index, an extremal partition of \( \Phi \) and \( \Psi \) is also an extremal partition for \( \Phi \oplus \Psi \). Conversely, if a partition is not extremal for \( \Phi \) or/and \( \Psi \), the condition (4.1) becomes a strict inequality by Proposition 5.1. Adding up these
inequalities for $\Phi$ and $\Psi$ we obtain a strict inequality for $\Phi \oplus \Psi$. This concludes the proof of (iii).

In one direction, Assertion (iv) is also clear from the definition. To prove the converse, consider an extremal partition $k_1 + \ldots + k_r = k$. For $s < r$, set $m = k_1 + \ldots + k_s$. We need to show that the partitions $k_1 + \ldots + k_s = m$ and $m + k_{s+1} + \ldots + k_r = k$ are also extremal. Assume not. Then, by Proposition 5.1, we have

$$\mu(\Phi^k) + \ldots + \mu(\Phi^{k_s}) - \mu(\Phi^m) \leq (s - 1)n$$

and

$$\mu(\Phi^m) + \mu(\Phi^{k_{s+1}}) + \ldots + \mu(\Phi^{k_r}) - \mu(\Phi^k) \leq (r - s)n,$$

where at least one of the inequalities is strict by the assumption. Combining these inequalities, we conclude that (4.1) is also strict and thus the original partition is not extremal.

The role of Condition A in our method is clarified by the next result.

**Proposition 5.3** An elliptic element $\Phi \in \tilde{Sp}(2n)$ admits an extremal partition of length $r$ if and only if $\Gamma(\Phi)$ satisfies Condition A.

We emphasize that both Condition A and the existence of extremal partitions are in fact properties of $\Phi(1) \in Sp(2n)$; see Proposition 5.2 (i).

**Example 5.4** (Toric $\Phi$ revisited). Assume that $\dim \Gamma = n$, i.e., $\Gamma = \mathbb{T}^n$. Then $\Phi$ admits extremal partitions of arbitrarily large length; cf. Example 2.3.

**Proof of Proposition 5.3** The sequence of iterated indices $\mu(\Phi^k), k \in \mathbb{N}$, does not change under an isospectral deformation of $\Phi$. Hence we can assume that $\Phi(1)$ is semi-simple and view it as a topological generator of $\Gamma$.

Assume first that Condition A is satisfied: there exist $r$ points $\tilde{\theta}_1, \ldots, \tilde{\theta}_r$ in $\Gamma$ such that (2.1) holds:

$$\sum_{i=1}^r \lambda_{ij} < 1 \text{ for all } j = 1, \ldots, n,$$

where $\tilde{\theta}_i = (e^{2\pi \sqrt{-1} \lambda_{i1}}, \ldots, e^{2\pi \sqrt{-1} \lambda_{in}})$ with $0 < \lambda_{ij} < 1$. For $i = 1, \ldots, r$, set

$$\Psi_i(t) = (e^{2\pi \sqrt{-1} \lambda_{11} t}, \ldots, e^{2\pi \sqrt{-1} \lambda_{in} t}), \quad t \in [0, 1]. \quad (5.1)$$

Then $\mu(\Psi_i) = n$ and also $\mu(\Psi_1 \ldots \Psi_r) = n$ by Condition A. Therefore,

$$\sum \mu(\Psi_i) - \mu(\Psi_1 \ldots \Psi_r) = (r - 1)n.$$

The end-points $\tilde{\theta}_i = \Psi_i(1)$ can be approximated arbitrarily well by the iterates $\Phi^{k_i}(1)$ for some $k_i \in \mathbb{N}$. In other words, for a loop $\phi_i$, the element $\phi_i \Phi^{k_i}$ can be made
arbitrarily close to $\Psi_i$ and the product of the paths $\phi_i \Phi^{k_i}$ can be made arbitrarily close to $\Psi_1 \ldots \Psi_r$. Note that this product has the form $\phi \Phi^k$, where $k = k_1 + \ldots + k_r$ and $\phi$ is the product of the loops $\phi_i$. (The group $\pi_1(\text{Sp}(2n))$ is in the center of $\tilde{\text{Sp}}(2n)$.) Recalling that the defect depends only on the end-points, we see that

$$
\sum \mu(\Phi^{k_i}) - \mu(\Phi^k) = \sum \mu(\Psi_i) - \mu(\Psi_1 \ldots \Psi_r) = (r - 1)n.
$$

Conversely, assume that $k_1 + \ldots + k_r = k$ is an extremal partition for $\Phi$. Set $\tilde{\theta}_i = \Phi^{k_i}(1)$. It suffices to show that (2.1) holds, where $\lambda_{ij}$ are as above. Define the paths $\Psi_i$ by (5.1) and set $\Psi = \Psi_1 \ldots \Psi_r$. (By slightly perturbing $\Phi$ we can ensure that $\Psi$ is non-degenerate.) Then $\mu(\Psi_i) = n$ and

$$
\sum \mu(\Psi_i) - \mu(\Psi) = \sum \mu(\Phi^{k_i}) - \mu(\Phi^k) = (r - 1)n
$$

by (4.1). Thus $\mu(\Psi) = n$.

Clearly,

$$
\Psi(t) = (e^{2\pi \sqrt{-1} \lambda_{1,t}}, \ldots, e^{2\pi \sqrt{-1} \lambda_{n,t}}), \quad t \in [0, 1],
$$

where

$$
\lambda_j = \sum_{i=1}^r \lambda_{ij} > 0.
$$

All intersection points of this path with the Maslov cycle are positive. The condition that $\lambda_j < 1$ for all $j$ is equivalent to that the only intersection of $\Psi$ with the Maslov cycle is at $t = 0$ which, in turn, is equivalent to that $\mu(\Psi) = n$.

Condition A is hard to visualize and verify directly and this is where the following criterion comes handy. Denote by $\Pi_r$ the open cube $(0, 1/r)^n$ in the torus $\mathbb{T}^n$ identified with the quotient of the cube $[0, 1]^n$.

**Proposition 5.5** Assume that codim $\Gamma \leq 1$. Then Condition A is satisfied for $\Gamma$ if and only if $\Gamma \cap \Pi_r \neq \emptyset$.

In dimension four, this gives a general necessary and sufficient condition for Condition A to be satisfied:

**Corollary 5.6** Assume that $n = 2$ and dim $\Gamma \geq 1$. Then Condition A is satisfied if and only if $\Gamma \cap \Pi_r \neq \emptyset$.

Here of course the case of dim $\Gamma = 1$ is most interesting: when dim $\Gamma = 2$, Condition A obviously holds; see Example 5.4.

**Remark 5.7** If $n = 2$, for every $r$ we have $\Gamma \cap \Pi_r \neq \emptyset$ for all but a finite number of subgroups $\Gamma \subset \mathbb{T}^2$ of positive dimension. For instance, assume that $\Gamma$ is connected and $r = 3$ – this is the minimal value of $r$ needed in dimension four to detect the
quantum product. Then $\Gamma \cap \Pi_3 = \emptyset$ if and only if the slope $s = -1, -2, -1/2$. This is no longer true when $n \geq 3$, but even then this requirement is met for a majority of subgroups.

**Proof of Proposition 5.5** In one direction the assertion is obvious and requires no additional conditions on $\Gamma$. Thus we need to show that $\Gamma \cap \Pi_r \neq \emptyset$ whenever Condition A holds.

Throughout the proof we will view the cube $C = [0, 1]^n \subset \mathbb{R}^n$ as the fundamental domain for $T_n$. Let $H$ be the inverse image in $\mathbb{R}^n$ of the connected component of the identity in $\Gamma$, and $L_q$ stand for connected components of the inverse image $L$ of $\Gamma$ in $C$ under the natural maps $C \to \mathbb{P}^n$. Clearly, each $L_q$ is the intersection of a hyperplane parallel to $H$ with $C$. Among these denote by $L_0$ be the component closest to 0.

If $0 \in L_0$, we obviously have $L_0 \cap \Pi_r \neq \emptyset$ for all $r$ and the proof is finished. Thus we can assume that $0 \notin L_0$, i.e., $L_0$ is a positive distance from 0.

Denote by $\tilde{\lambda}_i \in C$ the inverse image of the point $\tilde{\theta}_i$ from Condition A. Thus, in the notation for that condition,

$$\tilde{\lambda}_i = (\lambda_{i1}, \ldots, \lambda_{in})$$

and (2.1) holds for all $j = 1, \ldots, n$, i.e.,

$$\sum_{i=1}^{r} \lambda_{ij} < 1.$$ 

Note that the points $\tilde{\lambda}_i$ may lie on different components $L_q$.

Consider the segment $Y_i = \{t\tilde{\lambda}_i \mid t \in [0, 1]\}$ connecting 0 and $\tilde{\lambda}_i$. The intersection $\tilde{\lambda}'_i \in Y_i \cap L$ that is closest to zero lies on $L_0$. Denote the components of $\tilde{\lambda}'_i$ by $\lambda_j'$. Then $\lambda'_{ij} = t\lambda_{ij}$ with $t \in (0, 1]$, and hence

$$\sum_{i=1}^{r} \lambda_{ij}' \leq \sum_{i=1}^{r} \lambda_{ij} < 1.$$ 

Let

$$\tilde{\lambda}' = \frac{1}{r} \sum_{i=1}^{r} \tilde{\lambda}_i'$$

be the mean or the “center of mass” of the points $\tilde{\lambda}_i'$. Then $\tilde{\lambda}' \in L_0$, since all $\tilde{\lambda}_i' \in L_0$ and $L_0$ is convex. As a consequence, the projection $\tilde{\theta}'$ of $\tilde{\lambda}'$ to $\mathbb{P}^n$ is in $\Gamma$. Furthermore, for every component $\lambda_j'$ of $\tilde{\lambda}'$ we have

$$\lambda_j' = \frac{1}{r} \sum_{i=1}^{r} \lambda_{ij}' \leq \frac{1}{r} \sum_{i=1}^{r} \lambda_{ij} < \frac{1}{r}.$$
Therefore, $\vec{\theta}' \in \Gamma \cap \Pi_r$. \hfill $\Box$

**Remark 5.8** It is very unlikely that Proposition 5.5 holds in other settings without significant constraints on $\Gamma$ and $r$. However, some partial results are certainly feasible. For instance, assuming that $\Gamma$ is connected, one could expect the proposition to hold, perhaps under some additional (un-)divisibility conditions on $\mu_\Gamma$ and $r$.

5.2 Proofs of Theorems 4.7, 4.8 and 4.10

In this section we establish the combinatorial results underlying the main theorems of the paper.

**Proof of Theorem 4.7** As in the proof of Proposition 5.3, we can require $\Phi$ to be semi-simple; thus $\Phi(1) \in \Gamma$. Throughout the proof we will assume that Condition A is satisfied. Thus, by Proposition 5.3, there exists an extremal partition $\ell_1 + \ldots + \ell_r = \ell$ of length $r$. Our goal is to modify it if necessary, creating a new extremal partition $k_1 + \ldots + k_r = k$ such that (4.4) holds:

$$\mu(\Phi^{k_i}) \not\equiv n \mod 2N \text{ for all } i = 1, \ldots, r.$$

Assume first that Condition B1 is satisfied, i.e., there exists a loop $\gamma$ in $\Gamma$ such that $N \not| \mu(\gamma) = \hat{\mu}(\gamma)/2$. Here we view $\gamma$ as a loop in $\text{Sp}(2n)$ and, in particular, as an element of $\tilde{\text{Sp}}(2n)$.

We claim that $\gamma$ can be approximated arbitrarily well by the elements of the form $\phi\Phi^m$, where $\phi$ is a loop in $\text{Sp}(2n)$ with $2N \not| \hat{\mu}(\phi)$.

To prove this, observe that we can take a one-dimensional subgroup in $\Gamma$ as $\gamma$. Then there exists an element $\tilde{\psi}$ in the inverse image of $\Gamma$ in $\tilde{\text{Sp}}(2n)$ such that $\tilde{\psi}^{2N} = \gamma$. Let $\tilde{\psi}$ be its image in $\Gamma$. Since $\Phi(1)$ generates $\Gamma$, we can approximate $\psi$ by the powers $\Phi^s(1)$ arbitrarily well. As a consequence, we can approximate $\tilde{\psi}$ arbitrarily well in $\tilde{\text{Sp}}(2n)$ by the elements of the form $\phi_0^s\Phi^m$ where $\phi_0$ is a loop. Hence, the elements $\phi_0^{2N}\phi_0^{ns}$ approximate $\gamma$. Then $\hat{\mu}(\phi_0^{2N}) = 2N\hat{\mu}(\phi_0)$ and it remains to set $\phi = \phi_0^{2N}$ and $m = 2Ns$.

With the claim established, we are ready to modify the partition $\ell_1 + \ldots + \ell_r = \ell$. If $\mu(\Phi^{\ell_i}) \not\equiv n \mod 2N$ we simply set $k_i = \ell_i$. If $\mu(\Phi^{\ell_i}) \equiv n \mod 2N$ we replace $\ell_i$ by $k_i = \ell_i + m$. Then

$$\mu(\Phi^{k_i}) = \mu(\Phi^{\ell_i}\Phi^m).$$

Making the approximation accurate enough, we have

$$\mu(\Phi^{\ell_i}\Phi^m) = \mu(\Phi^{\ell_i}\phi^{-1}\gamma) = \mu(\Phi^{\ell_i}) + \hat{\mu}(\phi^{-1}) + \hat{\mu}(\gamma).$$

Here the first term is congruent to $n$ modulo $2N$, the second term is divisible by $2N$ and the last term is not divisible by $2N$. Thus $\mu(\Phi^{k_i}) \not\equiv n \mod 2N$.

It remains to show that the new partition is still extremal. The modification results in replacing $\Phi^{\ell_i}$ by $\Phi^{k_i}$ which is approximately a product of $\Phi^{\ell_i}$ with a loop and
likewise $\Phi^k$ is approximately the product of $\Phi^\ell$ with a loop. It is clear that if the approximations are accurate enough, depending only on $\Phi^{\ell_1}(1)$ and their products, the partition will remain extremal.

Next assume that Condition B2 is satisfied: $\Gamma$ is connected and there exists a convex neighborhood $V$ of $0 \in \mathbb{T}^n$ whose intersection with $\Gamma$ is connected and an iterate $\Phi^k(1) \in V$ such that $2N \nmid \text{loop}(\Phi^k)$. In addition, we can also require that $N \mid \mu_\Gamma$, i.e., Condition B1 fails.

Since $\Gamma$ is connected and Condition A is a feature of $\Gamma$, replacing the original $\Phi$ by $\Phi^k$, we can assume that $\ell_1 + \ldots + \ell_r = \ell$ is an extremal partition for $\Phi$ where $\Phi(1) \in V \cap \Gamma$ and $2N \nmid \text{loop}(\Phi)$ and $2N \mid \ell_i$ for all $i$. This is the partition we will change to a new extremal partition $k_1 + \ldots + k_r = k$ such that (4.4) holds.

As in the first part of the proof, we set $k_i = \ell_i$, i.e., no modification is needed, if $\mu(\Phi^{\ell_1}) \equiv n \mod 2N$. In the rest of the argument we describe how to change $\ell_i$ when $\mu(\Phi^{\ell_1}) \not\equiv n \mod 2N$.

Since $\Gamma$ is connected, for every $i = 1, \ldots, r$, any arithmetic progression contains an infinite subsequence $k_{ij} \to \infty$ such that $\Phi^{k_{ij}}(1) \to \Phi^{\ell_i}(1)$ as $j \to \infty$. Thus, setting $k_i = k_{ij}$, we will assume in what follows that $\Phi^{k_i}(1)$ is sufficiently close to $\Phi^{\ell_i}(1)$.

We claim that

$$\mu(\Phi^{k_i}) = k_i \text{loop}(\Phi) + d_i, \quad \text{where } d_i \equiv \mu(\Phi^{\ell_i}) \mod 2N.$$  

In particular, the residue of $d_i$ in $\mathbb{Z}_{2N}$ is independent of $k_i$.

Indeed, let us write $\Phi \in \hat{\text{Sp}}(2n)$ as the product $\phi \xi$, where $\phi$ is a loop and $\xi$ is a short path; see Sect. 2.1.2. Then $\text{loop}(\Phi) = \hat{\mu}(\phi)$ and

$$\mu(\Phi^{k_i}) = k_i \hat{\mu}(\phi) + \mu(\xi^{k_i}).$$

Let $\zeta$ be a geodesic in $\Gamma$ connecting the origin to $\Phi^{k_i}(1)$. We have

$$\mu(\Phi^{k_i}) = k_i \hat{\mu}(\phi) + d_i, \quad \text{where } d_i = \mu(\zeta) + \hat{\mu}(\xi^{k_i} \zeta^{-1}).$$

Here $\xi^{k_i} \zeta^{-1}$ is a loop in $\Gamma$, and hence $2N \mid \hat{\mu}(\xi^{k_i} \zeta^{-1})$ since Condition B1 is assumed to fail. Let $\tilde{\zeta}$ be the geodesic close to $\zeta$, connecting the origin to $\Phi^{\ell_i}$. Such a geodesic exists once $\Phi^{k_i}(1)$ is close to $\Phi^{\ell_i}(1)$, and $\mu(\tilde{\zeta}) = \mu(\zeta)$. Then, it is not hard to see that $2N \mid \hat{\mu}(\Phi^{\ell_i} \tilde{\zeta}^{-1})$ from the condition that $2N \mid \ell_i$, and therefore $d_i \equiv \mu(\Phi^{\ell_i}) \mod 2N$.

Now we are in a position to modify the partition $\ell_1 + \ldots + \ell_r = \ell$. Namely, when $\mu(\Phi^{\ell_i}) \equiv n \mod 2N$, we take $k_i \in 2N\mathbb{N} + 1$ such that $\Phi^{k_i}(1)$ is sufficiently close to $\Phi^{\ell_i}(1)$. Then $d_i \equiv n \mod 2N$ and

$$\mu(\Phi^{k_i}) = k_i \text{loop}(\Phi) + d_i \equiv \text{loop}(\Phi) + n \not\equiv n \mod 2N.$$  

To show that $k_1 + \ldots + k_r =: k$ is again an extremal partition one argues exactly as in the first part of the proof. □
**Proof of Theorem 4.8** Since $\Gamma = \mathbb{T}^n$, all eigenvalues of $\Phi(1)$ are necessarily distinct and in particular $\Phi(1)$ is semi-simple. Furthermore, the orbit $\Phi(1)^k, k \in \mathbb{N}$ is dense on $\mathbb{T}^n$. Hence for a suitable iterate $\Phi^k$, the end point $\Phi^k(1)$ is the sum of arbitrarily small rotations $\exp(\pi \sqrt{-1} \lambda_i)$, where $0 < \lambda_i \ll |\lambda_n|$ for $i = 1, \ldots, n - 1$ and $\lambda_n < 0$. Iterating again to bring $\exp(\pi \sqrt{-1} \lambda_n)$ close to $1 \in S^1$ while the other components stay small due to the inequality between the eigenvalues, we can ensure that $\Phi^m(1)$ is a sum of arbitrarily small rotations, which we still denote by $\exp(\pi \sqrt{-1} \lambda_i)$ with all $\lambda_i > 0$, and such that $\text{loop}(\Phi^m) = -2 + d$ with $2N | d$. We have

$$\mu(\Phi^m) = n - 2 + d \equiv n - 2 \mod 2N$$

and, as long as $r \max |\lambda_i| < 2$,

$$\mu(\Phi^m) = n + r(d - 2).$$

Furthermore, $m + \ldots + m = rm =: k$ is an extremal partition, for

$$r \mu(\Phi^m) - (r - 1)n = n + r(d - 2) = \mu(\Phi^m).$$

**Proof of Theorem 4.10** As in the proof of Proposition 5.5, we view $C = [0, 1]^2 \subset \mathbb{R}^2$ as the fundamental domain for $\mathbb{T}^2$ and let $L_q$ stand for the connected components of the inverse image $L$ of $\Gamma$ in $C$ under the map $C \rightarrow \mathbb{T}^2$. We first investigate when the Condition A fails. Since $L$ always intersects $\Pi_3 = (0, 1/3)^2 \subset C$ when the slope $s$ is positive, we assume that $s < 0$ and write $s = -s_1/s_2$ where $s_1, s_2 \in \mathbb{N}$ are relatively prime. If $\Gamma$ is connected, $L$ divides parallel boundary components of $C$ into $s_i$ equal segments (and more if $\Gamma$ is not connected). As a consequence, if $s_1 \geq 3$ or $s_2 \geq 3$ there exists $L_q \subset L$ such that $L_q \cap \Pi_3 \neq \emptyset$. Note that if $\Gamma$ is connected and $s = -1, -1/2, -2$, then $L \cap \Pi_3 = \emptyset$ and Condition A fails by Corollary 5.6 (cf. Remark 5.7). In other words, Condition A is satisfied for a connected group $\Gamma$ if and only if $s$ is not one of these values.

In the remaining part of the proof we assume that Condition A is satisfied, i.e., $s \neq -1, -1/2, -2$, but Condition B1 fails and Condition B2 fails for the connected component of identity in $\Gamma$; cf. Remark 4.9. In particular, for every loop $\gamma$ in $\Gamma$ we have $2N \mid \mu(\gamma)$. Let $V$ be a convex neighborhood of $0 \in \mathbb{T}^2$ such that $V \cap \Gamma$ is connected and let $k \in \mathbb{N}$ be such that $\Phi^k(1) \in V \cap \Gamma$. By assumption, $2N \mid \text{loop}(\Phi^k)$. We replace $\Phi$ by $\Phi^k$ and $\Gamma(\Phi)$ by $\Gamma(\Phi^k)$, while keeping the notation $\Phi$ for the iterated map.

Below we will show that for any $k \in \mathbb{N}$ the index $\mu(\Phi^k)$ only depends (modulo $2N$) on the connected component $L_q \subset L$ where the end point $\Phi^k(1)$ is, and, furthermore, for any consecutive connected components $L_{q_1}, L_{q_2}$ of $L$, the index is different modulo $2N$. Here $L_{q_1}$ is consecutive to $L_{q_2}$ if $L_{q_1} \neq L_{q_2}$ and there is no other connected component of $L$ which is strictly closer to $L_{q_1}$ in $C$ than $L_{q_2}$.

Indeed, let $L_q \subset L$ and $k_1, k_2 \in \mathbb{N}$ be such that $\Phi^{k_1}(1)$ and $\Phi^{k_2}(1)$ are in $L_q$. Since $2N \mid \text{loop}(\Phi)$, the difference

$$d := \mu(\Phi^{k_1}) - \mu(\Phi^{k_2}) \mod 2N$$


is equal (modulo $2N$) to the mean index $\hat{\mu}(\gamma)$ of a loop $\gamma \subset \Gamma$. Since Condition B1 fails, $2N$ divides $\hat{\mu}(\gamma)$ and hence $d \equiv 0 \mod 2N$. To prove the second assertion, suppose that $\Phi^{k_1}(1)$ and $\Phi^{k_2}(1)$ are on two consecutive components $L_{q_1}$ and $L_{q_2}$. Let $\gamma$ be a path in $\Gamma$ connecting $\Phi^{k_2}(1)$ and $\Phi^{k_1}(1)$, and let $\eta$ be the shortest path in $C$ from $\Phi^{k_1}(1)$ to $\Phi^{k_2}(1)$. We have

$$d \equiv \hat{\mu}(\gamma \sharp \eta) \mod 2N,$$

where $\gamma \sharp \eta \subset \mathbb{T}^2$ is the loop obtained by concatenating $\gamma$ and $\eta$. If $d \equiv 0 \mod 2N$, by shifting the loop $\gamma \sharp \eta$, we see that the index is constant (modulo $2N$) as a function of $L_q$, which means that $L$ is connected, i.e., $s = \pm 1$.

With these observations in mind, we are ready to prove the theorem. Assume first that $s > 0$ and either $s_1 > 3$ or $s_2 > 3$. Then $L \cap \Pi_3$ has at least two connected components, and for at least one of them the index is different from $2 \mod 2N$. Combining this with the assumption that Condition B1 fails finishes the proof for positive slopes: For $s > 0$ the assertion of the theorem holds if $N = 2$ and $s \neq 1, 3, 1/3$, or $N = 3$ and $s \neq 2, 1/2$. (When $s = 1$ and $N = 3$, Condition B2 is satisfied.)

When $s < 0$ we will give different arguments for $N = 2$ and $N = 3$. In both cases there will be no other slope to rule out other than $s = -1, -2, -1/2$ (for which Condition A fails).

The case of $N = 2$ Since we are assuming that the Condition B1 fails and thus $s_1 - s_2$ is even, both $s_1, s_2$ are odd. The index on the connected components of $L$ that contain $0 \in \mathbb{T}^2$ is equal to $0 \mod 4$. Then since $s_1$ and $s_2$ are odd, the index is $0 \mod 4$ on the connected component that is closest to $0 \in C$. We conclude that for $s < 0$ and $N = 2$ the assertion holds if $s \neq -1, -2, -1/2$.

The case of $N = 3$ If either $s_1 \geq 6$ or $s_2 \geq 6$, then $L \cap \Pi_3$ has two connected components and the proof is finished as in the $s > 0$ case. It remains to check the following slopes: $s = -1/4, -4, -2/5, -5/2$. (Here we are again using the assumption that the Condition B1 fails.) A direct computation shows that when $s$ is from the list above, one of the connected components of $L$ that contains $0 \in \mathbb{T}^2$ intersects $\Pi_3$. Furthermore, on such a component, as in the $N = 2$ case, the index is equal to $0 \mod 6$. We again conclude that for $s < 0$ and $N = 3$ the assertion holds if $s \neq -1, -2, -1/2$. □

5.3 Proof of Proposition 5.1

Consider two elements $\Phi$ and $\Psi$ of $\widetilde{Sp}(2n)$. Our goal is to prove the upper bound

$$|D| = |\mu(\Psi \Phi) - \mu(\Psi) - \mu(\Phi)| \leq n$$

on the absolute value of the defect $D$, where $\Phi$ and $\Psi$ and the product $\Psi \Phi$ (or rather the end-points $\Phi(1)$ and $\Psi(1)$ and their product) are non-degenerate. In fact, we only need to show that $D \leq n$, for the opposite inequality $D \geq -n$ follows by replacing $\Phi$ by $\Phi^{-1}$ and $\Psi$ by $\Psi^{-1}$.
It is convenient to recast the question in terms of the Robbin–Salamon index $\mu_{RS}$; see [30]. For any path $\Phi: [0, 1] \to \mathrm{Sp}(2n)$ denote by $\mathrm{gr}(\Phi)$ the path traced by the graph of $\Phi(t)$ in the Lagrangian Grassmannian of the twisted product
\[
\bar{\mathbb{R}}^{2n} \times \mathbb{R}^{2n} = (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega_0 \times \omega_0),
\]
where $\omega_0$ is the standard symplectic structure on $\mathbb{R}^{2n}$. For a non-degenerate element $\Phi \in \tilde{\mathrm{Sp}}(2n)$, we have
\[
\mu(\Phi) = \mu_{RS}(\mathrm{gr}(\Phi), \triangle),
\]
where $\triangle$ is the diagonal in $\bar{\mathbb{R}}^{2n} \times \mathbb{R}^{2n}$. Furthermore,
\[
D = \mu_{RS}(\mathrm{gr}(\Psi(1)\Phi), \triangle) - \mu_{RS}(\mathrm{gr}(\Phi), \triangle),
\]
since the Robbin–Salamon index is additive under concatenation of paths. The index is invariant under linear symplectic maps. Hence,
\[
\mu_{RS}(\mathrm{gr}(\Phi), \triangle) = \mu_{RS}(\mathrm{gr}(\Psi(1)\Phi), \mathrm{gr}(\Psi(1)))
\]
and
\[
D = \mu_{RS} \left( \begin{array}{c} \mathrm{gr}(\Psi(1)\Phi), \triangle \end{array} \right)_{L_1} - \mu_{RS} \left( \begin{array}{c} \mathrm{gr}(\Psi(1)\Phi), \mathrm{gr}(\Psi(1)) \end{array} \right)_{L_2}.
\]
In other words, in the notation introduced by the underbraces, we see that $D$ can be expressed as the difference
\[
s(L_1, L_2; \Lambda(0), \Lambda(1)) := \mu_{RS}(\Lambda, L_1) - \mu_{RS}(\Lambda, L_2),
\]
which is independent of the path $\Lambda$ connecting $\Lambda(0)$ to $\Lambda(1)$ and called the Hörmander index; see [30, Thm. 3.5]). Below we utilize this path independence to bound the defect $D$ from above.

Choose a Lagrangian complement $N$ to $\triangle$ which is transverse to $\mathrm{gr}(\Psi(1))$ and $\mathrm{gr}(\Psi(1)\Phi(1))$, and identify $\triangle \cong N$ using a symplectic basis. Every Lagrangian subspace transverse to $N$ can be written as the graph of a symmetric matrix with respect to the splitting $\triangle \times N$. Indeed, observe that the graph of a linear map $S: \triangle \to N$ is Lagrangian if and only if
\[
\omega_0((u, Su), (v, Sv)) = \langle u, Sv \rangle - \langle Su, v \rangle = 0
\]
for every $u, v \in \triangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{2n}$. This is equivalent to the condition that $S$ is symmetric.
Let $A: \triangle \to N$ and $B: \triangle \to N$ be such that $\text{gr}(A) = \text{gr}(\Psi(1))$ and $\text{gr}(B) = \text{gr}(\Psi(1)\Phi(1))$. Then

$$D = s(\triangle, \text{gr}(\Psi(1)); \text{gr}(\Psi(1)), \text{gr}(\Psi(1)\Phi(1)))$$

$$= s(\text{gr}(0), \text{gr}(A); \text{gr}(A), \text{gr}(B)).$$

Next, applying [30, Thm. 3.5] (the first equality) and [30, Lemma 5.2] (the second equality) to the right hand side, we see that

$$D = \frac{1}{2} \left[ \text{sgn}(B) - \text{sgn}(A) - \text{sgn}(B - A) \right]$$

$$= \frac{1}{2} \text{sgn}(B^{-1} - A^{-1})$$

$$\leq n,$$

since $\text{sgn}(S)/2 \leq n$ for any symmetric matrix $S$. This completes the proof of the proposition. \hfill \Box

**Remark 5.9** (Defect of the Conley–Zehnder type quasimorphisms). As an immediate consequence of Proposition 5.1, one obtains upper bounds on the defect $D$ of several types of Maslov or Conley–Zehnder quasimorphisms. Namely, it readily follows from the proposition that $|D| \leq 4n$ for the mean index and the upper or lower semi-continuous extensions of the Conley–Zehnder index. The proof of the proposition yields the upper bound $|D| \leq 3n$ for the Robbin–Salamon index.

**References**

1. Abbondandolo, A.: Morse Theory for Hamiltonian Systems, Chapman & Hall/CRC Research Notes in Mathematics, vol. 425. Chapman & Hall/CRC, Boca Raton, FL (2001)
2. Abbondandolo, A., Schwarz, M.: Floer homology of cotangent bundles and the loop product. Geom. Topol. 14, 1569–1722 (2010)
3. Anosov, D.V., Katok, A.B.: New examples in smooth ergodic theory. Ergodic diffeomorphisms, (in Russian). Trudy Moskov. Mat. Obšč. 23, 3–36 (1970)
4. Banik, M.: On the dynamics characterization of complex projective spaces, J. Fixed Point Theory Appl., 22(1), Paper No. 24, 8 pp. (2020)
5. Banyaga, A.: Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique. Comment. Math. Helv. 53, 174–227 (1978)
6. Bramham, B.: Periodic approximations of irrational pseudo-rotations using pseudoholomorphic curves. Ann. Math. 181, 1033–1086 (2015)
7. Bramham, B.: Pseudo-rotations with sufficiently Liouvillean rotation number are $C^0$-rigid. Invent. Math. 199, 561–580 (2015)
8. Bramham, B., Hofer, H.: First steps towards a symplectic dynamics. Surv. Differ. Geom. 17, 127–178 (2012)
9. Çineli, E.: Conley conjecture and local Floer homology. Arch. Math. (Basel) 111, 647–656 (2018)
10. Çineli, E., Ginzburg, V.L., Gürel, B.Z.: From pseudo-rotations to holomorphic curves via quantum Steenrod squares. Int. Math. Res. Not. IMRN (2020). https://doi.org/10.1093/imrn/rmaa173
11. Çineli, E., Ginzburg, V.L., Gürel, B.Z.: Another look at the Hofer–Zehnder conjecture, Preprint arXiv:2009.13052
12. De Gosson, M., De Gosson, S., Piccione, P.: On a product formula for the Conley–Zehnder index of symplectic paths and its applications. Ann. Global Anal. Geom. 34, 167–183 (2008)
13. Fayad, B., Katok, A.: Constructions in elliptic dynamics. Ergodic Theory Dyn. Syst. 24, 1477–1520 (2004)
14. Ginzburg, V.L.: The Conley conjecture. Ann. Math. 172, 1127–1180 (2010)
15. Ginzburg, V.L., Gürel, B.Z.: The Conley conjecture and beyond. Arnold Math. J. 1, 299–337 (2015)
16. Ginzburg, V.L., Gürel, B.Z.: Conley conjecture revisited. Int. Math. Res. Not. IMRN (2017). https://doi.org/10.1093/imrn/rnx137
17. Ginzburg, V.L., Gürel, B.Z.: Hamiltonian pseudo-rotations of projective spaces. Invent. Math. 214, 1081–1130 (2018)
18. Ginzburg, V.L., Gürel, B.Z.: Pseudo-rotations vs. rotations, Preprint arXiv:1812.05782
19. Guillemin, V., Ginzburg, V., Karshon, Y.: Cobordisms and Hamiltonian Group Actions, Mathematical Surveys and Monographs, 98. American Mathematical Society, Providence, RI (2002)
20. Hofer, H., Salamon, D.: Floer homology and Novikov rings. In: The Floer Memorial Volume, Progr. Math., vol. 133, Birkhäuser, Basel, 483–524, (1995)
21. Lalonde, F., McDuff, D., Polterovich, L.: On the flux conjectures. In: Geometry, Topology, and Dynamics (Montreal, PQ, 1995), 69–85, CRM Proc. Lecture Notes, 15, Amer. Math. Soc., Providence, RI (1998)
22. Lalonde, F., McDuff, D., Polterovich, L.: Topological rigidity of Hamiltonian loops and quantum homology. Invent. Math. 135, 369–385 (1999)
23. Le Roux, F., Seyfaddini, S.: The Anosov–Katok method and pseudo-rotations in symplectic dynamics, Preprint arxiv:2010.06237
24. Long, Y.: Index Theory for Symplectic Paths with Applications. Birkhäuser Verlag, Basel (2002)
25. McDuff, D.: Hamiltonian $S^1$-manifolds are uniruled. Duke Math. J. 146, 449–507 (2009)
26. McDuff, D., Salamon, D.: J-Holomorphic Curves and Symplectic Topology. Colloquium publications, vol. 52. AMS, Providence, RI (2004)
27. Ono, K.: Floer–Novikov cohomology and the flux conjecture. Geom. Funct. Anal. 16, 981–1020 (2006)
28. Piunikhin, S., Salamon, D., Schwarz, M.: Symplectic Floer–Donaldson theory and quantum cohomology, in Contact and Symplectic Geometry (Cambridge, 1994), Publ. Newton Inst., vol. 8, Cambridge University Press, Cambridge, 171–200 (1996)
29. Repovš, D., Ščepin, E.V.: A proof of the Hilbert-Smith conjecture for actions by Lipschitz maps. Math. Ann. 308, 361–364 (1997)
30. Robbin, J., Salamon, D.: The Maslov index for paths. Topology 32, 827–844 (1993)
31. Salamon, D.A.: Lectures on Floer homology, in Symplectic Geometry and Topology, IAS/Park City Math. Ser., vol. 7, Am. Math. Soc., Providence, RI, 143–229 (1999)
32. Salamon, D., Zehnder, E.: Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. Commun. Pure Appl. Math. 45, 1303–1360 (1992)
33. Seidel, P.: A biased view of symplectic cohomology, in Current Developments in Mathematics, vol. 2006, pp. 211–253, Int. Press, Somerville, MA (2008)
34. Seidel, P.: The equivariant pair-of-pants product in fixed point Floer cohomology. Geom. Funct. Anal. 25, 942–1007 (2015)
35. Shelukhin, E.: On the Hofer–Zehnder conjecture, Preprint arXiv:1905.04769
36. Shelukhin, E.: Pseudorotations and Steenrod squares, Preprint arXiv:1905.05108; to appear in J. Mod. Dyn
37. Shelukhin, E.: Pseudo-rotations and Steenrod squares revisited, Preprint arXiv:1909.12315
38. Shelukhin, E., Zhao, J.: The $\mathbb{Z}/(p)$-equivariant product-isomorphism in fixed point Floer cohomology, Preprint arXiv:1905.03666
39. Wilkins, N.: Quantum Steenrod squares and the equivariant pair-of-pants in symplectic cohomology, Preprint arXiv:1810.02738
40. Wilkins, N.: A construction of the quantum Steenrod squares and their algebraic relations. Geom. Topol. 24, 885–970 (2020)

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