On the Fourier transform for a symmetric group homogeneous space

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Abstract

By using properties of the Young orthogonal representation, this paper derives a simple form for the Fourier transform of permutations acting on the homogeneous space of $n$-dimensional vectors, and shows that the transform requires $2n - 2$ multiplications and the same number of additions.

Key words: Symmetric group, Fourier transform, complexity.

1. Introduction

Let $S_n$ denote the symmetric group on $n$ elements, and $S_n^n$ the subgroup fixing the $n$-th element. This paper derives a simplification for the Fourier transform of $S_n$ acting on $I_n = \{1, 2, \ldots, n\}$, or equivalently, the coset space $S_n/S_n^n$. Fourier analysis of permutations on $I_n$ is important for the statistical analysis of ranked data [1], pattern matching, and other applications.

To put the aim of the paper in context, it is useful to consider the ordinary Fourier transform. Let $\mathcal{H}$ be the $n \times n$ unitary matrix with entries $\mathcal{H}_{k,\ell} = (\sqrt{n})^{-1}e^{-j2\pi(k\ell)/n}$. Then $X = \mathcal{H}x$ is the discrete Fourier transform of the vector $x$. If $\Delta_d$ is the translation operator that sends $n \mapsto (n + d) \mod n$, and $\Phi_d$ is the phase shift matrix $\text{diag}[1, e^{2\pi d/n}, \ldots, e^{2\pi d(n-1)/n}]$, then

$$\mathcal{H}\Delta_d x = \Phi_d X. \quad (1)$$

Similarly, the permutation Fourier transform presented below converts permutations on $I_n$ to group representation “phase” shifts.
Fast Fourier transforms on the groups $S_n$ and their homogeneous spaces have been studied previously. In particular, by applying the method of Clausen [2], Maslen and Rockmore [3, Thm 6.5] give an upper bound for the number of operations (either multiplications or additions) on $S_n/S_n^m$ as $n^3 - n^2$. Maslen [4, Thm 3.5] improves the bound on the same space to show that, at most, $3n(n - 1)/2$ operations are necessary. This paper shows that $2n - 2$ operations are sufficient.

2. Background for this paper

We use standard results for permutations [5]. An adjacent transposition is the permutation $\tau_k = (k, k + 1)$ that exchanges the $k$-th and $(k + 1)$-th elements but leaves all others unchanged. Every permutation may be written as a product of adjacent transpositions.

The Fourier transform on $S_n$ relies on the group’s irreducible unitary representations, with “frequencies” given by arithmetic partitions. Let $\nu = (n_1, \ldots, n_q)$ be a partition of $n$ with $n_i \geq n_{i+1}$ and $n_1 + \ldots + n_q = n$; we write $\nu \vdash n$. For every $\nu \vdash n$ there exists an irreducible representation, denoted $D_\nu$. For example, when $\nu = (n)$, we have $D_{(n)}(\sigma) = 1$ for all $\sigma \in S_n$. For other $\nu$, we use the Young orthogonal representation (YOR) to construct the matrices. The Fourier transform of $f : S_n \to \mathbb{C}$ is

$$F(\nu) = \sum_{\sigma \in S_n} f(\sigma)D_\nu(\sigma), \quad \nu \vdash n.$$  

(2)

For each $\nu$, the coefficient $F(\nu)$ is a $n_\nu \times n_\nu$ matrix. If $f(\sigma) = g(\delta \sigma)$, i.e., $f$ and $g$ are left translates of each other, then, in a manner similar to (1), we obtain that $G(\nu) = D_\nu(\delta)^T F(\nu)$. Of particular interest in this paper is the “fundamental frequency” of the transform given by the partition $\phi = (n - 1, 1)$. The $(n - 1)^2$ entries of $D_\phi$ are obtained from the YOR as described in detail below.

It suffices to describe $D_\phi$ on the adjacent transpositions $\{\tau_k\}$, for those generate $S_n$. Let $D_\phi(\tau_1)$ be the $(n-1)$-dimensional matrix $\text{diag}[1, 1, \ldots, 1, -1]$. For any $m$, let $I_m$ denote the $m$-dimensional identity matrix, and for $k = 2, \ldots, n - 1$, let $R_k$ be the $2 \times 2$ symmetric matrix

$$R_k = \begin{bmatrix} -\frac{1}{k} & \sqrt{1 - \frac{1}{k^2}} \\ \sqrt{1 - \frac{1}{k^2}} & \frac{1}{k} \end{bmatrix}.$$  

(3)
Now, for $k = 2, \ldots, n - 1$, define $D_\phi(\tau_k)$ to be the symmetric, block-diagonal, matrix

\[
D_\phi(\tau_k) = \begin{bmatrix}
I_{n-k-1} & 0 & 0 \\
0 & R_k & 0 \\
0 & 0 & I_{k-2}
\end{bmatrix}.
\]

(4)

It may be verified that the matrices $\{D_\phi(\tau_k)\}$ satisfy the Coexeter relations [5, pg 88], and generate the irreducible YOR for partition $\phi = (n - 1, 1)$. Furthermore, note that the decomposition of each $\sigma \in S_n^n$ into $\{\tau_k\}$ excludes $\tau_{n-1}$. Therefore, from (4), it follows that, with $\oplus$ denoting matrix direct sum and $O_{n-2}(\sigma)$ a $(n - 2)$-dimensional orthogonal matrix,

\[
D_\phi(\sigma) = 1 \oplus O_{n-2}(\sigma), \quad \text{for } \sigma \in S_n^n.
\]

(5)

3. Fourier analysis on the homogeneous space

Our goal is to simplify (2) for functions defined on $I_n$. We may extend each $f$ defined on $I_n$ to a corresponding function $\tilde{f}$ on $S_n$ by $\tilde{f}(\sigma) = f(\sigma(n))$. Note that $\tilde{f}$ is constant on left cosets of $S_n^n$ and, therefore, “band-limited”.

**Proposition 3.1.** Given any complex-valued function $f$ defined on $I_n$, the Fourier coefficients $\tilde{F}(\nu)$ of the function $\tilde{f}$ on $S_n$ defined by $\tilde{f}(\sigma) = f(\sigma(n))$ are such that $\tilde{F}(\nu) = 0$ unless $\nu = (n)$ or $\nu = \phi = (n - 1, 1)$.

**Proof.** Since $\tilde{f}(\sigma\delta) = \tilde{f}(\sigma)$ for $\delta \in S_n^n$, we have by (2) that $\tilde{F}(\nu) = \tilde{F}(\nu)D_\nu(\delta)^t$. By averaging both sides over $S_n^n$, we get $\tilde{F}(\nu) = \tilde{F}(\nu)Z(\nu)$ where

\[
Z(\nu) = \frac{1}{(n-1)!} \sum_{\delta \in S_n^n} D_\nu(\delta)^t.
\]

(6)

Now, the Branching Rule [5, Thm 2.8.3] shows that for $\nu = \phi = (n - 1, 1)$ and $\nu = (n)$, the representation $D_\nu$ reduces on the subgroup $S_n^n$ to contain the constant representation, and that no other irreducible representation does so. By orthogonality, those matrix entries that are not constant on $S_n^n$ must sum to zero over the subgroup. Therefore $Z(\nu) = 0$ if $\nu$ is not $(n)$ or $\phi$. \hfill $\Box$

If $Z(\phi)_{i,j}$ is the $(i, j)$-th element, then from (5) we have $Z(\phi)_{1,1} = 1$, and, by orthogonality, $Z(\phi)_{i,j} = 0$ for all other $(i, j)$. Since $\tilde{F}(\phi) = \tilde{F}(\phi)Z(\phi)$ we obtain that $\tilde{F}(\phi)$ is zero except possibly in the leftmost column. Hence, the Fourier transform need only be calculated for the partition $(n)$, and for
the \( n - 1 \) entries in the left most column of \( D_\phi \). Let \( \mathcal{F} \) denote the linear transformation taking any \( n \)-dimensional vector \( x \) on \( I_n \) to its \( n \) Fourier transform coefficients \( \tilde{X}(\{(n)\}) \), and the leftmost column entries \( \tilde{X}(\phi)_{i,1} \) for \( i = 1, 2, \ldots, n - 1 \). We write
\[
\tilde{X} = \mathcal{F} x
\] (7)
to express the transform, now viewed a matrix operation. The transform (7) requires at most \( n^2 \) multiplications and \( n(n - 1) \) additions. We show below that, in fact, \( 2n - 2 \) operations of each kind are sufficient.

Our result relies on the following \( n \times n \) matrix \( U \), whose shape is similar to a “reverse” upper Hessenberg matrix:
\[
U = \begin{pmatrix}
+1 & +1 & +1 & \cdots & +1 & +1 \\
-1 & -1 & -1 & \cdots & -1 & (n - 1) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & 2 & 0 & \cdots & 0 \\
-1 & +1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\] (8)

Define \( A = (UU^t)^{-1/2} \), and let
\[
T = AU.
\] (9)

It is easily seen that \( T \) is an orthogonal matrix, and that \( A \) is diagonal with entries \( \{\alpha_k\}_{k=1}^n \), with \( \alpha_1 = A_{1,1} = 1/\sqrt{n} \), and for \( k > 1 \), we have
\[
\alpha_k = A_{k,k} = \frac{1}{\sqrt{(n-k+1)(n-k+2)}}.
\] (10)

Let \( x \) be any complex-valued \( n \times 1 \) vector, and let \( X = T x \). To each \( \sigma \in S_n \), let the \( n \times n \) matrix \( P(\sigma) \) be the permutation matrix obtained from the identity \( I_n \) with rows permuted by \( \sigma \), i.e., \( P(\sigma)_{i,j} = I_{\sigma(i),j} \). Note that \( \sigma \mapsto P(\sigma) \) is an antihomomorphism: \( P(\sigma\delta) = P(\delta)P(\sigma) \). To see that, note that for any \( \alpha \) we have \( P(\alpha)e_k = e_{\alpha^{-1}(k)} \) where \( e_k \) is \([0, \ldots, 0, 1, 0, \ldots, 0]^t\) with 1 in the \( k \)-th position. If \( x, y \) are \( n \times 1 \) vectors, and \( y = P(\sigma)x \), then \( y(i) = x(\sigma(i)) \) since \( P(\sigma) \) is the permutation operator on column vectors.

We now establish the following result, comparable to eq. (1).

**Theorem 3.2.** For every \( \sigma \in S_n \) and all \( n \times 1 \) vectors \( x \), we have that
\[
TP(\sigma)x = \left[1 \oplus D_\phi(\sigma)^t\right] X
\]
Proof. We start by proving for any adjacent transposition $\tau_k$ that

$$TP(\tau_k)T^t = 1 \oplus D_\phi(\tau_k) = 1 \oplus D_\phi(\tau_k)^t$$  \hspace{1cm} (11)

Note from (9), (10), the $m$-th row of $T$ for $m > 1$ sums to zero, with the form

$$[-\alpha_m, -\alpha_m, \ldots, -\alpha_m, (n - (m - 1))\alpha_m, 0, \ldots, 0].$$ \hspace{1cm} (12)

The product $TP(\tau_k)$ is the same as $T$ but with columns $k, k + 1$ swapped. By (12), we see that the only rows of $TP(\tau_k)$ that are affected by the column swap are as follows: for $k = 1$, row $n$ is modified; and for $k > 1$, rows $n - (k - 1), n - (k - 2)$ are modified. Therefore the product $TP(\tau_k)T^t$ is the same as the identity $I$ in all entries with the following exceptions: when $k = 1$, we have that $[TP(\tau_1)T^t]_{nn} = -2\alpha_n^2 = -1$; and when $k > 1$, we have that the $2 \times 2$ submatrix, whose upper-left corner indices are $(n - (k - 1), n - (k - 1))$, has the symmetric form

$$\begin{bmatrix}
-(k + 1)\alpha_{n-(k-1)} & (k^2 - 1)\alpha_{n-(k-2)} \\
(k^2 - 1)\alpha_{n-(k-2)} & (k - 1)\alpha_{n-(k-2)}^2
\end{bmatrix}$$ \hspace{1cm} (13)

Substituting from (11), we find that the above simplifies to $R_k$ as defined earlier in (3), thus verifying (11) for $k = 1, 2, \ldots, n - 1$.

For the general case, note that every $\sigma \in S_n$ may be written as a product of adjacent transpositions $\sigma = \tau_{k_1} \cdots \tau_{k_m}$. Since $\sigma \mapsto P(\sigma)$ is an anti-homomorphism, we have that

$$P(\sigma) = P(\tau_{k_1} \cdots \tau_{k_m}) = P(\tau_{k_m}) \cdots P(\tau_{k_1}).$$ \hspace{1cm} (14)

Applying a similarity transformation with $T$ yields

$$TP(\sigma)T^t = TP(\tau_{k_m})T^t \cdots TP(\tau_{k_1})T^t.$$ \hspace{1cm} (15)

On applying (11) we establish the theorem:

$$TP(\sigma)T^t = \left[1 \oplus D_\phi(\tau_{k_m})^t\right] \cdots \left[1 \oplus D_\phi(\tau_{k_1})^t\right] = 1 \oplus D_\phi(\sigma)^t.$$ \hspace{1cm} (16)

Note that for the Fourier transform in (7), we also have

$$\mathcal{F}P(\sigma)x = [1 \oplus D_\phi(\sigma)]^t\tilde{X}$$

from the translation property. Since this is true for all vectors $x$, we must have $\mathcal{F} = [\lambda_1 I_1 \oplus \lambda_2 I_{n-2}]T$. To see that, note that $\mathcal{F} = CT$ for some matrix $C$, and, by applying the Theorem above, we see that $C$ commutes with all matrices $1 \oplus D_\phi$; the result now follows from Schur’s lemma [5, pg 23].
3.1. Computation of the transform

The equality $T = AU$, combined with the matrix structure in (3), simplifies computation. Let $a_{x}(n) = x(1)$, $a_{x}(n-1) = x(1) + x(2)$, ..., $a_{x}(1) = x(1) + x(2) + \cdots + x(n)$. Computing all $\{a_{x}(k)\}$ values requires $n-1$ additions due to recursion. If $\hat{X} = UX$ then $\hat{X}(1) = a_{x}(1)$, $\hat{X}(2) = (n-1)*x(n) - a_{x}(2)$, ..., $\hat{X}(n) = x(2) - a_{x}(n)$. Hence, if $a_{x}$ has been computed, computing $\hat{X}$ requires $(n-2)$ multiplies and $(n-1)$ additions. Now, since $X = T x = A\hat{X}$, and $A$ is diagonal, we see that computing $X$ from $\hat{X}$ requires an additional $n$ multiplications. In total, computing the transform $X = Tx$ requires $2n-2$ multiplications and $2n-2$ additions. Note that computing $\bar{X} = Fx = CAUx$ does not require any extra computation as we may premultiply the diagonal matrix $C$ with $A$.

4. Conclusions

This paper describes a simplification of the Fourier transform on $S_{n}/S_{n}^{n}$, and shows that the transform requires $2n-2$ multiplications and the same number of additions.

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