Research Article

Weighted Multilinear Hardy Operators on Herz Type Spaces

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This paper focuses on the bounds of weighted multilinear Hardy operators on the product Herz spaces and the product Morrey-Herz spaces, respectively. We present a sufficient condition on the weight function that guarantees weighted multilinear Hardy operators to be bounded on the product Herz spaces. And the condition is necessary under certain assumptions. Finally, we extend the obtained results to the product Morrey-Herz spaces.

1. Introduction

Let \( f \) be a nonnegative integral function on \( \mathbb{R}^+ \). The Hardy operator \( H \) is defined by
\[
Hf(x) = \frac{1}{x} \int_0^x f(t) \, dt, \quad x > 0
\]
for the operator \( H \), Hardy et al. [1] proved that the inequality
\[
\|Hf\|_{L^p} \leq \frac{p}{p-1} \|f\|_{L^p}
\]
(1)
holds, where \( 1 < p < \infty \) and the constant \( p/(p-1) \) is the best possible. Usually, we call (1) classical Hardy’s inequality. With the development of analysis theory, many types of Hardy’s inequalities have been discussed. For example, a quite number of papers dealt with the various generalizations, numerous variants, and applications of Hardy’s inequalities in the past few years. On the detailed discussions of Hardy’s inequalities, we choose to refer to [2, 3].

In 1984, Carton-Lebrun and Fosset [4] gave the definition of the weighted Hardy operator \( H_\omega \) to be
\[
H_\omega f(x) := \int_0^1 f(tx) \omega(t) \, dt, \quad x \in \mathbb{R}^n,
\]
(2)
where \( \omega : [0, 1] \to (0, \infty) \) is a measurable function and \( f \) is a complex-valued measurable function on \( \mathbb{R}^n \). It is obvious that \( H_\omega \) degenerates into the classical Hardy operator \( H \) when \( \omega \equiv 1, n = 1 \), and \( f \) is defined on \( \mathbb{R}^1 \). In addition, we call the adjoint operator of \( H_\omega \) the weighted Cesàro average \( G_\omega f \).

And the definition of \( G_\omega f \) is
\[
(G_\omega f)(x) = \int_0^1 f\left(\frac{x}{t}\right) t^{-n} \omega(t) \, dt.
\]
(3)
When \( \omega \equiv 1 \) and \( n = 1 \), \( G_\omega \) becomes the classical Cesàro operator \( G \)
\[
Gf(x) = \begin{cases} \int_x^\infty f(y) \, dy, & x > 0, \\ \int_{-\infty}^x f(y) \, dy, & x < 0 \end{cases}
\]
(4)
It is easy to get that \( H_\omega \) and \( G_\omega \) satisfy
\[
\int_{\mathbb{R}^n} (H_\omega f)(x) g(x) \, dx = \int_{\mathbb{R}^n} f(x) (G_\omega g)(x) \, dx,
\]
(5)
when \( f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n), 1 < p < \infty, 1/p + 1/q = 1 \). This means that \( H_\omega \) and \( G_\omega \) satisfy the commutative rule
\[
H_\omega G_\omega = G_\omega H_\omega.
\]
Under certain conditions on \( \omega \), Carton-Lebrun and Fosset [4] proved that \( H_\omega \) maps \( L^p(\mathbb{R}^n) \) into itself. They also pointed out that the operator \( H_\omega \) commutes with the Hilbert transform when \( n = 1 \) and with certain Calderón-Zygmund singular integral operators including the Riesz
transform when \( n \geq 2 \). Refer to Xiao [5] for the further extension of the results above; see also [6, 7].

Since Herz space is a natural generalization of weighted Lebesgue spaces with power weights, researchers are also interested in studying the boundedness of \( H_\omega \) on Herz spaces. To make the description more clear below, we review the definition of the Herz spaces now. In the following definitions, \( \chi_k = \chi_{C_k}, C_k = B_k \setminus B_{k-1} \), and \( B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \} \), for \( k \in \mathbb{Z} \), \( C_k = \delta C_k \), \( C_0 = B_0 \), and \( \chi_k = \chi_{C_k} \), for \( k \in \mathbb{N} \) and \( \chi_E \) is the characteristic function of a set \( E \).

Definition 1 (see [8]). Let \( \alpha \in \mathbb{R}, 0 < p \leq \infty, 0 < q < \infty \).

1. The homogeneous Herz spaces \( K^\alpha_p(\mathbb{R}^n) \) is defined by

\[
K^\alpha_p(\mathbb{R}^n) = \left\{ f \in L^p_\text{loc}(\mathbb{R}^n \setminus \{0\}) : \| f \|_{K^\alpha_p(\mathbb{R}^n)} < \infty \right\},
\]

where

\[
\| f \|_{K^\alpha_p(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \| f \chi_k \|_{L^p(\mathbb{R}^n)} \right\}^{1/p}.
\]

(2) The inhomogeneous Herz spaces \( K^\alpha_p(\mathbb{R}^n) \) is defined by

\[
K^\alpha_p(\mathbb{R}^n) = \left\{ f \in L^p_\text{loc}(\mathbb{R}^n) : \| f \|_{K^\alpha_p(\mathbb{R}^n)} < \infty \right\},
\]

where

\[
\| f \|_{K^\alpha_p(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \| f \chi_k \|_{L^p(\mathbb{R}^n)} \right\}^{1/p}.
\]

with usual modifications made when \( p = \infty \) or \( q = \infty \).

In [9], Wu gave the definition of Morrey-Herz spaces.

Definition 2. Let \( \alpha \in \mathbb{R}, 0 < p \leq \infty, 0 < q < \infty \), and \( \lambda \geq 0 \).

1. The homogeneous Morrey-Herz space \( MK^\alpha_{p,q}(\mathbb{R}^n) \) is defined by

\[
MK^\alpha_{p,q}(\mathbb{R}^n) := \left\{ f \in L^q_\text{loc}(\mathbb{R}^n \setminus \{0\}) : \| f \|_{MK^\alpha_{p,q}(\mathbb{R}^n)} < \infty \right\},
\]

where

\[
\| f \|_{MK^\alpha_{p,q}(\mathbb{R}^n)} := \sup_{k \in \mathbb{Z}} 2^{-k\lambda} \left\{ \sum_{k=-\infty}^{k} 2^{k\alpha p} \| f \chi_k \|_{L^q(\mathbb{R}^n)} \right\}^{1/p}.
\]

(2) The inhomogeneous Morrey-Herz space \( MK^\alpha_{p,q}(\mathbb{R}^n) \) is defined by

\[
MK^\alpha_{p,q}(\mathbb{R}^n) := \left\{ f \in L^q_\text{loc}(\mathbb{R}^n) : \| f \|_{MK^\alpha_{p,q}(\mathbb{R}^n)} < \infty \right\},
\]

where

\[
\| f \|_{MK^\alpha_{p,q}(\mathbb{R}^n)} := \sup_{k \in \mathbb{N}} 2^{-k\lambda} \left\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \| f \chi_k \|_{L^q(\mathbb{R}^n)} \right\}^{1/p}.
\]

with usual modifications made when \( p = \infty \) or \( q = \infty \).

From the above definitions, it is not difficult to note that \( K^\alpha_p(\mathbb{R}^n) = K^0_{p,q}(\mathbb{R}^n) = L^p(\mathbb{R}^n), K^\alpha_{p,p}(\mathbb{R}^n) = K^1_{p,p}(\mathbb{R}^n) = L^p(x^p dx) \) for all \( 0 < p \leq \infty \) and \( \alpha \in \mathbb{R} \). Moreover, the homogenous Herz spaces \( K^\alpha_{q,q}(\mathbb{R}^n) \), the homogenous Morrey-Herz spaces \( MK^\alpha_{p,q}(\mathbb{R}^n) \), and the Morrey spaces \( M^{q,\lambda}(\mathbb{R}^n) \) (see [10, 11]) satisfy \( MK^\alpha_{p,q}(\mathbb{R}^n) = K^\alpha_{q,q}(\mathbb{R}^n), MK^{q,\lambda}(\mathbb{R}^n) \subset MK^\alpha_{p,q}(\mathbb{R}^n) \). In [12], Liu and Fu discussed the boundedness of the weighted Hardy operator \( H_\omega \) on the Herz space \( K^\alpha_{q,q}(\mathbb{R}^n) \). Conditions on the weighted function \( \omega \) were presented to guarantee that \( H_\omega \) is bounded on \( K^\alpha_{p,q}(\mathbb{R}^n) \). They also estimated the corresponding operator norm. And in [13], Fu and Liu further extended the results of [12] to the Morrey-Herz space \( MK^\alpha_{p,q}(\mathbb{R}^n) \).

In the past few years, the properties of multilinear operators have also been extensively studied by researchers. There are two reasons for this. First, the multilinear operators are the generalization of the linear ones, and its study makes the research contents of analysis theory more rich. Second, the multilinear operators naturally appear in analysis. The study of multilinear operators is traced to the multilinear singular integral operator theory (see [14]). For more detailed studies on multilinear operators, the readers refer to [15-18] and the references therein. Recently, we have studied the boundedness of weighted multilinear Hardy operators \( \mathcal{H}^m_\omega \) on the product of Lebesgue spaces and central Morrey spaces in [19]. Based on these results and inspired by the results of [12, 13], this paper further concerns the boundedness of \( \mathcal{H}^m_\omega \) on the product Herz spaces and the product Morrey-Herz spaces. We first recall the definition of \( \mathcal{H}^m_\omega \).

Definition 3. Let \( m \in \mathbb{N}, m \geq 2 \), and

\[
\omega : [0,1] \times [0,1] \times \cdots \times [0,1] \to [0,\infty)
\]

be an integrable function. The weighted multilinear Hardy operator \( \mathcal{H}^m_\omega \) is defined by

\[
\mathcal{H}^m_\omega(f_\lambda)(x) := \int_{\omega(t_1,\ldots,t_m)} \left( \prod_{i=1}^{m} f_i(t_i x) \right) \omega(t) dt,
\]

where \( \mathbf{t} := (f_1, \ldots, f_m), \omega(\mathbf{t}) := \omega(t_1, t_2, \ldots, t_m), dt := dt_1 \cdots dt_m \), and \( f_i(t_1, \ldots, t_m) \) are complex-valued measurable functions on \( \mathbb{R}^n \). When \( m = 2 \), \( \mathcal{H}^m_\omega \) is referred to as bilinear.

In accordance with the case of \( H_\omega \), we also recall the definition of the weighted multilinear Cesaro operator \( \mathcal{G}_\omega \) that is the adjoint operator of \( \mathcal{H}^m_\omega \).
Definition 4. Let \( m \in \mathbb{N} \), \( m \geq 2 \), and \( \omega : [0, 1] \times [0, 1]^{m-1} \rightarrow [0, \infty) \) be an integrable function. The weighted multilinear Cesàro operator \( \mathcal{G}_\omega^m \) is defined by

\[
\mathcal{G}_\omega^m (f) (x) = \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} f_i \left( \frac{x}{t_i} \right) (t_i)^m \right) \omega \left( \frac{t_i}{\tilde{t}} \right) d\tilde{t},
\]

where \( f_i \) are measurable complex-valued functions on \( \mathbb{R}^n \), \( 1 \leq i \leq m \).

Note that \( \mathcal{H}_\omega^m \) and \( \mathcal{G}_\omega^m \) do not satisfy the following commutative rule:

\[
\int_{\mathbb{R}^n} (\mathcal{H}_\omega^m f) (x) g (x) \, dx = \int_{\mathbb{R}^n} f (x) (\mathcal{G}_\omega^m g) (x) \, dx.
\]

This is different from the case of \( \mathcal{H}_\omega \) and \( \mathcal{G}_\omega \).

The paper is organized as follows. In Section 2, we present the estimate of the boundedness of \( \mathcal{H}_\omega^m \) on the product Herz spaces. In Section 3, we give the estimates of the boundedness of \( \mathcal{G}_\omega^m \) on the product Morrey-Herz spaces.

2. Boundedness of \( \mathcal{H}_\omega^m \) on the Product of Herz Spaces

We give the first main result of this paper.

Theorem 5. Let \( \alpha, \alpha_1, \ldots, \alpha_m \in \mathbb{R}, 1 < p, p_1, q, q_1 < \infty \) and \( \alpha_1 + \cdots + \alpha_m = \alpha, 1/q_1 + \cdots + 1/q_m = 1/q, 1/p_1 + \cdots + 1/p_m = 1/p, i = 1, 2, \ldots, m \). Then \( \mathcal{H}_\omega^m \) is bounded from \( K_{q_1}^{\alpha_1, p_1} \times \cdots \times K_{q_m}^{\alpha_m, p_m} \) to \( K_{q}^{\alpha, p} \) if

\[
\int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} t_i^{-(\alpha_i n/q_i)} \right) \omega \left( \frac{t_i}{\tilde{t}} \right) d\tilde{t} < \infty.
\]

Conversely, if \( \alpha_1 = \cdots = \alpha_m = (1/m) \alpha, q_1 = mq, p_1 = mp, i = 1, 2, \ldots, m \), and \( \mathcal{H}_\omega^m \) is bounded from \( K_{q_1}^{\alpha_1, p_1} \times \cdots \times K_{q_m}^{\alpha_m, p_m} \) to \( K_{q}^{\alpha, p} \), then (18) holds. Moreover, in this case, one has

\[
\| \mathcal{H}_\omega^m \|_{K_{q_1}^{\alpha_1, p_1} \times \cdots \times K_{q_m}^{\alpha_m, p_m} \rightarrow K_{q}^{\alpha, p}} = \int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} t_i^{-(\alpha_i n/q_i)} \right) \omega \left( \frac{t_i}{\tilde{t}} \right) d\tilde{t}.
\]

Similarly, we have the following result for the operator \( \mathcal{G}_\omega^m \).

Theorem 6. Let \( \alpha, \alpha_1, \ldots, \alpha_m \in \mathbb{R}, 1 < p, p_1, q, q_1 < \infty \), and \( \alpha_1 + \cdots + \alpha_m = \alpha, 1/q_1 + \cdots + 1/q_m = 1/q, 1/p_1 + \cdots + 1/p_m = 1/p, i = 1, 2, \ldots, m \). If \( \omega \) satisfies

\[
\int_{0 < t_1, t_2, \ldots, t_m < 1} \left( \prod_{i=1}^{m} t_i^{-(\alpha_i n(1-1/q_i))} \right) \omega \left( \frac{t_i}{\tilde{t}} \right) d\tilde{t} < \infty,
\]

then \( \mathcal{G}_\omega^m \) is bounded from \( K_{q_1}^{\alpha_1, p_1} \times \cdots \times K_{q_m}^{\alpha_m, p_m} \) to \( K_{q}^{\alpha, p} \).
\[
\begin{align*}
&\leq \int_0^1 \left( \|f_1 x_k m^{-1}\|_{L^n} + \|f_1 x_k m + \|f_2 x_{k+1}\|_{L^n} \right) \\
&\quad \times \left( \|f_2 x_{k+1} - 1\|_{L^n} + \|f_2 x_{k+1}\|_{L^n} \right) t_1^{-n/q} \\
&\quad \times t_2^{-n/q} \omega(t_1, t_2) dt_1 dt_2.
\end{align*}
\]

(23)

For \(1/p_1 + 1/p_2 = 1/p\) and \(\alpha_1 + \alpha_2 = \alpha\), Hölder inequality and Minkowski inequality give

\[
\|\mathcal{H}_\omega^2 (f_1, f_2)\|_{K_q}^{1/p} = \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha} \|\mathcal{H}_\omega^2 (f_1, f_2) x_k\|_{L_q(R^n)}^{p} \right)^{1/p}
\]

\[
\leq \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha} \left( \int_0^1 \left( \|f_1 x_k m^{-1}\|_{L^n} + \|f_1 x_k m + \|f_2 x_{k+1}\|_{L^n} \right)^{p} \\
\quad \times \left( \|f_2 x_{k+1} - 1\|_{L^n} + \|f_2 x_{k+1}\|_{L^n} \right)^{p} \\
\quad \times t_1^{-n/q} t_2^{-n/q} \omega(t_1, t_2) dt_1 dt_2 \right) \right)^{1/p}
\]

\[
\leq \int_0^1 \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha} \|f_1 x_k m^{-1}\|_{L^n} + \|f_1 x_k m + \|f_2 x_{k+1}\|_{L^n} \right)^{p} \\\n\quad \times \left( \|f_2 x_{k+1} - 1\|_{L^n} + \|f_2 x_{k+1}\|_{L^n} \right)^{p} \\\n\quad \times t_1^{-n/q} t_2^{-n/q} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p}
\]

(24)

From the above inequality, we have that the first conclusion in Theorem 5 holds.

On the other hand, we suppose that \(\mathcal{H}_\omega^2\) is bounded from \(K_{q_1}^{\alpha_1} \times K_{q_2}^{\alpha_2}\) to \(K_q^{\alpha}\) and \(\mathcal{H}_\omega^2\) has the operator norm \(\|\mathcal{H}_\omega^2\|_{K_{q_1}^{\alpha_1} \times K_{q_2}^{\alpha_2} \to K_q^{\alpha}}\). For any \(0 < \epsilon < 1\), let

\[
f_1(x) = \begin{cases} 0, & |x| \leq 1, \\
|x|^{-\alpha_1 - n/q_1 - \epsilon}, & |x| > 1. \end{cases}
\]

\[
f_2(x) = \begin{cases} 0, & |x| \leq 1, \\
|x|^{-\alpha_2 - n/q_2 - \epsilon}, & |x| > 1. \end{cases}
\]

(25)

Obviously, \(\|f_1 x_k\|_{L_q(R^n)} = \|f_2 x_k\|_{L_q(R^n)} = 0\), when \(k = 0, -1, -2, \ldots\). So, for any positive integer \(k\), it is easy to get that

\[
\|f_1 x_k\|_{L_q(R^n)} = \left( \int_{|x|^2}^{\infty} |x|^{-\alpha_1 - n/q_1 - \epsilon} dx \right)^{1/q_1} = 2^{-(\alpha_1 + \epsilon)k} \frac{S_n (2^{(\alpha_1 + \epsilon)q_1} - 1)}{(\alpha_1 + \epsilon) q_1}^{1/q_1}.
\]

(26)

where \(S_n = m n^{n/2} / \Gamma(1 + n/2)\). A simple computation gives

\[
\|f_1\|_{K_{q_1}^{\alpha_1} (R^n)} = 2^{-\epsilon} \left( \frac{1}{1 - 2^{-\epsilon q_1}} \right)^{1/p_1} S_n \left( \frac{2^{(\alpha_1 + \epsilon)q_1} - 1}{(\alpha_1 + \epsilon) q_1} \right)^{1/q_1}.
\]

(27)

Similarly, we obtain

\[
\|f_2\|_{K_{q_2}^{\alpha_2} (R^n)} = 2^{-\epsilon} \left( \frac{1}{1 - 2^{-\epsilon q_2}} \right)^{1/p_2} S_n \left( \frac{2^{(\alpha_2 + \epsilon)q_2} - 1}{(\alpha_2 + \epsilon) q_2} \right)^{1/q_2}.
\]

(28)
When $|x| \leq 1$ and $t \in [0, 1]$, we have $|tx| \leq 1$. So, $\mathcal{H}_\omega^2(f_1, f_2)(x) = 0$ in this case based on (25). When $|x| > 1$, we get

$$
\mathcal{H}_\omega^2(f_1, f_2)(x) = \int_{|x|>1} f_1(t_1x) f_2(t_2x) \omega(t_1, t_2) dt_1 dt_2
$$

(29)

Let $\delta = \varepsilon^{-1} > 1$. It is easy to find a positive integer $l$ such that $2^{l-1} \leq \delta < 2^l$. Thus, we have

$$
\|\mathcal{H}_\omega^2(f_1, f_2)\|_{K_q^{p,r}} \leq \sum_{k=1}^\infty 2^{kq} \left\|\mathcal{H}_\omega^2(f_1, f_2) \chi_k(x)\right\|_{L_t^p(\mathbb{R}^p)}
$$

$$
= \sum_{k=1}^\infty 2^{kq} \int_{|x|>1} \left|\int_{|x|>1} f_1(t_1x) f_2(t_2x) \omega(t_1, t_2) dt_1 dt_2 \right|^q dx \bigg|^{1/p}
$$

$$
\leq C_0 \int_{|x|>1} \left(\int_{|x|>1} \left|\int_{|x|>1} f_1(t_1x) f_2(t_2x) \omega(t_1, t_2) dt_1 dt_2 \right|^q dx \right) dx
$$

$$
\leq C_0 \int_{|x|>1} \left(\int_{|x|>1} \left|\int_{|x|>1} f_1(t_1x) f_2(t_2x) \omega(t_1, t_2) dt_1 dt_2 \right|^p dx \right) dx
$$

Since $q_1 = q_2 = 2q$, $\alpha_1 = \alpha_2 = (1/2)\alpha$ and $1/p = 1/p_1 + 1/p_2$, $p_1 = p_2 = 2p$, we have

$$
\|\mathcal{H}_\omega^2(f_1, f_2)\|_{K_q^{p,r}} \leq \sum_{k=1}^\infty 2^{kq} \left\|\mathcal{H}_\omega^2(f_1, f_2) \chi_k(x)\right\|_{L_t^p(\mathbb{R}^p)}
$$

$$
\leq C_0 \int_{|x|>1} \left(\int_{|x|>1} \left|\int_{|x|>1} f_1(t_1x) f_2(t_2x) \omega(t_1, t_2) dt_1 dt_2 \right|^p dx \right) dx
$$

$$
\leq \left\|\mathcal{H}_\omega^2(f_1, f_2) \chi_k(x)\right\|_{L_t^p(\mathbb{R}^p)} \omega(t_1, t_2) dt_1 dt_2
$$

This gives

$$
\|\mathcal{H}_\omega^2(f_1, f_2)\|_{K_q^{p,r}} \leq \|\mathcal{H}_\omega^2(f_1, f_2)\|_{K_q^{p,r}} \omega(t_1, t_2) dt_1 dt_2
$$

(32)

Using the boundedness of $\mathcal{H}_\omega^2$ and its operator norm yields

$$
\|\mathcal{H}_\omega^2(f_1, f_2)\|_{K_q^{p,r}} \leq \|\mathcal{H}_\omega^2(f_1, f_2)\|_{K_q^{p,r}} \omega(t_1, t_2) dt_1 dt_2
$$

(33)

Letting $\varepsilon \to 0^+$ in (33) gives

$$
\int_0^1 t_{1-\varepsilon}^{\alpha_1-n/q_1} t_{2-\alpha_2-n/q_2} \omega(t_1, t_2) dt_1 dt_2
$$

(34)

Thus, (18) holds.

Since $\mathcal{K}_P^0(R^n) = \mathcal{K}_P^0(R^n) = L^p(R^n)$ and $\mathcal{K}_P^{\alpha_1,\alpha_2}(R^n) = L^p(|x|^\alpha dx)$ for all $0 < p < \infty$, we deduce the following corollaries from Theorems 5 and 6.

**Corollary 7.** Let $\alpha, \alpha_1, \ldots, \alpha_2 \in \mathbb{R}$, $1 < p, p_i < \infty$ and $\alpha_1 + \ldots + \alpha_n = \alpha$, $1/p_1 + \ldots + 1/p_n = 1/p$, $i = 1, 2, \ldots, m$. Then $\mathcal{H}_\omega^m$ is bounded from $L^p(|x|^\alpha dx)$ to $L^p(|x|^\alpha dx)$ if

$$
\int_{a_1 < \ldots < a_m < 1} \left(\prod_{i=1}^m t_i^{(a_1+n)/p_i}\right) \omega(i) d\vec{f} < \infty.
$$

(35)

Conversely, if $\alpha_1 = \ldots = \alpha_n = (1/m)\alpha$, $p = mp, i = 1, 2, \ldots, m$, and $\mathcal{H}_\omega^m$ is bounded from $L^p(|x|^\alpha dx)$ to $L^p(|x|^\alpha dx)$ if

$$
\int_{a_1 < \ldots < a_m < 1} \left(\prod_{i=1}^m t_i^{(a_1+n)/p_i}\right) \omega(i) d\vec{f} < \infty.
$$

(30)
Let \( p > 0 \), then (35) holds. Moreover, in this case, one has

\[
\|f\|_{L^p(\mathbb{R})} \leq C \|f\|_{\mathcal{M}(\mathbb{R})}
\]

where \( C \) is a constant depending on \( p \) and \( \mathcal{M} \). This result is known as the boundedness of \( \mathcal{M} \) on \( L^p \).

3. Boundedness of \( \mathcal{H}_m \) on the Product of Morrey-Herz Spaces

Let us consider the case \( m = 2 \). Then \( \mathcal{H}_m \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) if \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). This result is known as the boundedness of \( \mathcal{H}_m \) on \( L^p \).

Conversely, when \( \alpha_1 = \cdots = \alpha_m = (1/m)\lambda \), \( \lambda_1 = \cdots = \lambda_m = \lambda \), \( q_1 = mq \), and \( p_i = mp \), \( i = 1, 2, \ldots, m \), if \( \mathcal{G}_m \) is bounded from \( MK^{\alpha_1,\lambda}_1 \times \cdots \times MK^{\alpha_m,\lambda}_m \) to \( MK^{\lambda}_p \), then (41) holds and

\[
\|f\|_{MK^{\alpha_1,\lambda}_1 \times \cdots \times MK^{\alpha_m,\lambda}_m} \leq C \|f\|_{MK^{\lambda}_p}
\]

where \( C \) is a constant depending on \( p \) and \( \mathcal{M} \). This result is known as the boundedness of \( \mathcal{H}_m \) on the product of Morrey-Herz Spaces.

For the operator \( \mathcal{G}_m \), we have a corresponding result.
\[ \leq \sup_{k \in \mathbb{Z}} 2^{-k_3 \lambda_3} \int_0^1 \left\{ \sum_{k=-\infty}^{\infty} 2^{k_3 \lambda_3} \left( \left\| f_1 \mathcal{X}_{k+m-1} \right\|_{L^{p_1}} + \left\| f_1 \mathcal{X}_{k+m} \right\|_{L^{p_1}} \right) \right\}^{1/p_1} \]

\[ \times \left( \sum_{k=-\infty}^{\infty} 2^{k_3 \lambda_3} \left( \left\| f_2 \mathcal{X}_{k+m-1} \right\|_{L^{p_1}} + \left\| f_2 \mathcal{X}_{k+m} \right\|_{L^{p_1}} \right) \right)^{1/p_2} \]

\[ \times t_1^{-\alpha/2} t_2^{-\alpha/2} \omega(t_1, t_2) dt_1 dt_2 \]

\[ \leq \| f_1 \|_{M_1^{\alpha, \lambda_1}} \| f_2 \|_{M_1^{\alpha, \lambda_2}} \]

\[ \times \left( \int_0^1 (2^{-(\alpha+\lambda_1)} \lambda_1 + 2^{-(\alpha+\lambda_2)} \lambda_2 + \lambda_3) \right)^{1/p_1} \]

\[ \times \left( \int_0^1 \left| x \right|^{-\alpha+n/2} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p_2} \]

\[ \leq \| f_1 \|_{M_1^{\alpha, \lambda_1}} \| f_2 \|_{M_1^{\alpha, \lambda_2}} \]

\[ \times \left( \int_0^1 \left| x \right|^{-\alpha+n/2} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p_2} \]

\[ \leq \left( \int_{\mathbb{R}^n} \left| x \right|^{-\alpha+n/2} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p_2} \]

\[ \times \left( \int_0^1 \left| x \right|^{-\alpha+n/2} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p_2} \]

\[ \times \omega(t_1, t_2) dt_1 dt_2 \]

\[ \leq \left( \int_{\mathbb{R}^n} \left| x \right|^{-\alpha+n/2} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p_2} \]

\[ \times \left( \int_0^1 \left| x \right|^{-\alpha+n/2} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p_2} \]

\[ \times \omega(t_1, t_2) dt_1 dt_2 \]

\[ \leq \left( \int_{\mathbb{R}^n} \left| x \right|^{-\alpha+n/2} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p_2} \]

\[ \times \left( \int_0^1 \left| x \right|^{-\alpha+n/2} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p_2} \]

\[ \times \omega(t_1, t_2) dt_1 dt_2 \]

On the other hand, we define

\[ f_1(x) = |x|^{-(\alpha+n/2)-\lambda_1}, \quad x \in \mathbb{R}^n, \]

\[ f_2(x) = |x|^{-(\alpha+n/2)-\lambda_2}, \quad x \in \mathbb{R}^n. \]

When \( \alpha_1 \neq \lambda_1 \) and \( \alpha_2 \neq \lambda_2 \), we get

\[ \| f_1 \|_{M_1^{\alpha, \lambda_1}} = \left( \int_{\mathbb{R}^n} \left| x \right|^{-\alpha+n(\lambda_1+\lambda_2)} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/q_1} \]

\[ = 2^{-(\alpha_1-\lambda_1)} \| S_n \left( 2^{(\alpha_1-\lambda_1)q_1-1} \right) \|_{\mathcal{L}_1^{\alpha_1-\lambda_1}}^{1/q_1} \]

\[ \| f_1 \|_{M_1^{\alpha, \lambda_1}} \]

\[ = 2^{\lambda_1} \left( \frac{1}{2^{\lambda_1} p_1 - 1} \right)^{1/p_1} \| S_n \left( 2^{(\alpha_1-\lambda_1)q_1-1} \right) \|_{\mathcal{L}_1^{\alpha_1-\lambda_1}}^{1/q_1} \] \[ \| f_2 \|_{M_1^{\alpha, \lambda_2}} \]

\[ = 2^{\lambda_2} \left( \frac{1}{2^{\lambda_2} p_2 - 1} \right)^{1/p_2} \| S_n \left( 2^{(\alpha_2-\lambda_2)q_1-1} \right) \|_{\mathcal{L}_1^{\alpha_2-\lambda_2}}^{1/q_2} \]

\[ = 2^{\lambda_1} \left( \frac{1}{2^{\lambda_1} p_1 - 1} \right)^{1/p_1} \| S_n \left( 2^{(\alpha_1-\lambda_1)q_1-1} \right) \|_{\mathcal{L}_1^{\alpha_1-\lambda_1}}^{1/q_1} \] \[ \| f_2 \|_{M_1^{\alpha, \lambda_2}} \]

\[ = 2^{\lambda_2} \left( \frac{1}{2^{\lambda_2} p_2 - 1} \right)^{1/p_2} \| S_n \left( 2^{(\alpha_2-\lambda_2)q_1-1} \right) \|_{\mathcal{L}_1^{\alpha_2-\lambda_2}}^{1/q_2} \]

where \( S_n = n^n/\Gamma(1+n/2) \). Thus

\[ \| f_2 \|_{M_1^{\alpha, \lambda_2}} \]

\[ = 2^{\lambda_2} \left( \frac{1}{2^{\lambda_2} p_2 - 1} \right)^{1/p_2} \| S_n \left( 2^{(\alpha_2-\lambda_2)q_1-1} \right) \|_{\mathcal{L}_1^{\alpha_2-\lambda_2}}^{1/q_2} \]

It is similar to obtain

\[ \| f_2 \|_{M_1^{\alpha, \lambda_2}} \]

\[ = 2^{\lambda_1} \left( \frac{1}{2^{\lambda_1} p_1 - 1} \right)^{1/p_1} \| S_n \left( 2^{(\alpha_1-\lambda_1)q_1-1} \right) \|_{\mathcal{L}_1^{\alpha_1-\lambda_1}}^{1/q_1} \]

\[ \times \omega(t_1, t_2) dt_1 dt_2 \]

By simple computation, we get

\[ \| \mathcal{H}_\omega^2 (f_1, f_2) \|_{\mathcal{L}_1^{\alpha, \lambda}}^p \]

\[ = \left( \int_{\mathbb{R}^n} \left| \mathcal{H}_\omega^2 (f_1, f_2) (x) \right|^p \chi_k (x) dx \right)^{1/p} \]

\[ = \left( \int_{\mathbb{R}^n} \left| x \right|^{-(\alpha+n/2)} \chi_k (x) \right)^{1/p} \]

\[ \times \left( \int_0^1 \left| x \right|^{-(\alpha+n/2)} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p} \]

\[ \times \omega(t_1, t_2) dt_1 dt_2 \]

\[ \leq \left( \int_{\mathbb{R}^n} \left| x \right|^{-(\alpha+n/2)} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p} \]

\[ \times \left( \int_0^1 \left| x \right|^{-(\alpha+n/2)} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p} \]

\[ \times \omega(t_1, t_2) dt_1 dt_2 \]

\[ \leq \left( \int_{\mathbb{R}^n} \left| x \right|^{-(\alpha+n/2)} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p} \]

\[ \times \left( \int_0^1 \left| x \right|^{-(\alpha+n/2)} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p} \]

\[ \times \omega(t_1, t_2) dt_1 dt_2 \]

\[ \leq \left( \int_{\mathbb{R}^n} \left| x \right|^{-(\alpha+n/2)} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p} \]

\[ \times \left( \int_0^1 \left| x \right|^{-(\alpha+n/2)} \omega(t_1, t_2) dt_1 dt_2 \right)^{1/p} \]

\[ \times \omega(t_1, t_2) dt_1 dt_2 \]

It means that \( \mathcal{H}_\omega^2 \) is bounded from \( M_\mathcal{K}_{\alpha_1, \lambda_1}^{\alpha, \lambda} \times \cdots \times M_\mathcal{K}_{\alpha_n, \lambda_n}^{\alpha, \lambda} \) to \( M_\mathcal{K}_{p, q}^{\alpha, \lambda} \).
Thus, we have

\[ 2^{-\alpha (\lambda kp)|S_n(2^{(\alpha - \lambda)q}) - 1|^{p/q}} \]

\[ \times \left( \int_0^1 t_1^{-(\alpha + n/q_1 - \lambda_1)} t_2^{-(\alpha + n/q_2 - \lambda_2)} \omega(t_1, t_2) dt_1 dt_2 \right) = \left( \int_0^1 t_1^{n/q_1} t_2^{n/q_2} \omega(t_1, t_2) dt_1 dt_2 \right)^{\frac{1}{p'}}. \]

Since \( \alpha_1 = \alpha_2 = (1/2) \alpha \), \( p_1 = p_2 = 2p \), \( q_1 = q_2 = 2q \), and \( \lambda_1 = \lambda_2 = (1/2) \lambda \), we have

\[ \left\| H^2 \right\|_{L^2(\mathbb{R}^n)} = \sup_{k \in \mathbb{Z}} 2^{-k \lambda}(\sum_{k = -\infty}^{k_n} 2^{kp}) \left\| H^2 \right\|_{L^2(\mathbb{R}^n)}^{\frac{1}{p'}} \]

\[ = \sup_{k \in \mathbb{Z}} 2^{-k \lambda}(\sum_{k = -\infty}^{k_n} 2^{kp}) \left( \int_0^1 t_1^{-(\alpha + n/q_1 - \lambda_1)} t_2^{-(\alpha + n/q_2 - \lambda_2)} \omega(t_1, t_2) dt_1 dt_2 \right)^{\frac{1}{p'}}. \]

So

\[ \left\| H^2 \right\|_{L^2(\mathbb{R}^n)} = (S_n \ln 2)^{1/q} \left( \int_0^1 t_1^{n/q_1} t_2^{n/q_2} \omega(t_1, t_2) dt_1 dt_2 \right)^{\frac{1}{p'}}. \]

Thus, we get

(50)

Then, we get

(51)

(52)

(53)

Since \( \alpha_1 = \lambda_1 \) and \( \alpha_2 = \lambda_2 \), we get \( \alpha = \lambda \). Thus

(54)

Thus,

(55)

(56)

This tells us that (39) also holds in this case.
Furthermore, when one of $\alpha_1 = \lambda_1$, $\alpha_2 = \lambda_2$ holds, we suppose $\alpha_1 = \lambda_1$ but $\alpha_2 \neq \lambda_2$ here. From the above computation, we have

$$
\left\| f_1 \right\|_{MK^{\alpha_1, \lambda_1}_{p_1, q_1}} = 2^{1/2} \left( \frac{1}{2^{1/p_1} - 1} \right)^{1/p_1} (S_n \ln 2)^{1/q_1},
$$

$$
\left\| f_2 \right\|_{MK^{\alpha_2, \lambda_2}_{p_2, q_2}} = 2^{1/2} \left( \frac{1}{2^{1/p_2} - 1} \right)^{1/p_2} \left| \frac{S_n (2(\alpha_2 - \lambda_2)q_2 - 1)}{(\alpha_2 - \lambda_2)q_2} \right|^{1/q_2}.
$$

(58)

It gives

$$
\mathcal{H}_w^2 (f_1, f_2) (x) = \left| x \right|^{-(\alpha_2 + n/q - \lambda_2)} \int_{t_1}^{1} \int_{t_2}^{1} \omega (t_1, t_2) dt_1 dt_2.
$$

(59)

So, we have

$$
\left\| \mathcal{H}_w^2 (f_1, f_2) \right\|_{MK^{\alpha_1, \lambda_1}_{p_1, q_1}} = \sup_{k \in Z} 2^{k\alpha_1} \left( \sum_{k=-\infty}^{\infty} 2^{k \rho} \left\| \mathcal{H}_w^2 (f_1, f_2) \chi_k \right\|_{L^1 (\mathbb{R}^n)} \right)^{1/p}.
$$

$$
= \sup_{k \in Z} 2^{k\alpha_1} \left( \sum_{k=-\infty}^{\infty} 2^{k \rho} \left| \frac{S_n (2(\alpha_2 - \lambda_2)q_2 - 1)}{(\alpha_2 - \lambda_2)q_2} \right|^{1/q_2} \right)^{1/p}
$$

$$
\times \left\{ \int_{t_1}^{1} \int_{t_2}^{1} \omega (t_1, t_2) dt_1 dt_2 \right\}.
$$

(60)

Thus, we have

$$
\left\| \mathcal{H}_w^2 (f_1, f_2) \right\|_{MK^{\alpha_1, \lambda_1}_{p_1, q_1}} \geq \int_0^{\infty} \left( \int_0^\infty \omega (t_1, t_2) dt_1 dt_2 \right) \frac{1}{\left| \frac{S_n (2(\alpha_2 - \lambda_2)q_2 - 1)}{(\alpha_2 - \lambda_2)q_2} \right|^{1/q_2}}.
$$

(62)

The above inequality gives

$$
\left\| \mathcal{H}_w^2 (f_1, f_2) \right\|_{MK^{\alpha_1, \lambda_1}_{p_1, q_1}} \geq \int_0^{\infty} \left( \int_0^\infty \omega (t_1, t_2) dt_1 dt_2 \right) \frac{1}{\left| \frac{S_n (2(\alpha_2 - \lambda_2)q_2 - 1)}{(\alpha_2 - \lambda_2)q_2} \right|^{1/q_2}}.
$$

(63)

Since $\mathcal{H}_w^2$ is bounded from $MK^{\alpha_1, \lambda_1}_{p_1, q_1}$ to $MK^{\alpha_1, \lambda_1}_{p_1, q_1}$ and $\mathcal{H}_w^2$ is bounded from $MK^{\alpha_1, \lambda_1}_{p_1, q_1}$ to $MK^{\alpha_1, \lambda_1}_{p_1, q_1}$, we know that (39) holds and

$$
\left\| \mathcal{H}_w^2 (f_1, f_2) \right\|_{MK^{\alpha_1, \lambda_1}_{p_1, q_1}} \geq \int_0^{\infty} \left( \int_0^\infty \omega (t_1, t_2) dt_1 dt_2 \right) \frac{1}{\left| \frac{S_n (2(\alpha_2 - \lambda_2)q_2 - 1)}{(\alpha_2 - \lambda_2)q_2} \right|^{1/q_2}}.
$$

(64)

The proof of Theorem 9 is complete.

\section*{Conflict of Interests}

The authors declare that there is no conflict of interests regarding the publication of this paper.

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