DARBOUX TRANSFORMATIONS OF JACOBI MATRICES AND PADÉ APPROXIMATION

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Abstract. Let $J$ be a monic Jacobi matrix associated with the Cauchy transform $F$ of a probability measure. We construct a pair of the lower and upper triangular block matrices $L$ and $U$ such that $J = LU$ and the matrix $\tilde{J}_C = UL$ is a monic generalized Jacobi matrix associated with the function $\tilde{F}_C(\lambda) = \lambda F(\lambda) + 1$. It turns out that the Christoffel transformation $\tilde{J}_C$ of a bounded monic Jacobi matrix $J$ can be unbounded. This phenomenon is shown to be related to the effect of accumulating at $\infty$ of the poles of the Padé approximants of the function $\tilde{F}_C$ although $\tilde{F}_C$ is holomorphic at $\infty$.

The case of the $UL$-factorization of $J$ is considered as well.

1. Introduction

Let $\sigma$ be a probability measure with an infinite support contained in $\mathbb{R}$. Also, assume that all the moments of $\sigma$, i.e.

$$\int_{\mathbb{R}} t^j d\sigma(t), \quad j \in \mathbb{Z}_+, \quad \text{are finite.}$$

It is well known that the sequence $\{P_j\}_{j=0}^{\infty}$ of monic polynomials orthogonal with respect to $\sigma$ satisfies a three-term recurrence relation \[1\]

$$\lambda P_j(\lambda) = b_j P_j(\lambda) + c_j P_{j-1}(\lambda), \quad j \in \mathbb{Z}_+$$

with the initial conditions

$$P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1,$$

where $b_j \in \mathbb{R}$ and $c_j > 0, j \in \mathbb{Z}_+$. Recall that the polynomials $P_j$ are also called the polynomials of the first kind corresponding to $\sigma$. Besides the polynomials of the first kind, one can also associate with $\sigma$ polynomials $Q_j$ of the second kind as solutions of \[1.1\] with the following initial conditions

$$Q_{-1}(\lambda) = -1, \quad Q_0(\lambda) = 0.$$

Clearly, the relation \[1.1\] can be rewritten as follows

$$Jp(\lambda) = \lambda p(\lambda),$$
where \( p = (P_0, P_1, P_2, \ldots)^\top \) and \( J \) is a semi-infinite tridiagonal matrix of the form

\[
J = \begin{pmatrix}
  b_0 & 1 & & & \\
  c_0 & b_1 & 1 & & \\
         & c_1 & b_2 & \ddots & \\
         &       & \ddots & \ddots & \\
         &       &       & \cdots & \cdots \\
\end{pmatrix}
\]

The matrix \( J \) is said to be \textit{the monic Jacobi matrix associated with} \( \sigma \).

As is known \cite{6}, a monic Jacobi matrix \( J \) admits the unique \( LU \)-factorization \( J = LU \) with the bidiagonal matrices \( L \) and \( U \) having the forms

\[
L = \begin{pmatrix}
  1 & 0 & & & \\
  l_1 & 1 & 0 & & \\
          & l_2 & 1 & \ddots & \\
          &       & \ddots & \ddots & \\
          &       &       & \cdots & \cdots \\
\end{pmatrix}, \quad U = \begin{pmatrix}
  u_1 & 1 & & & \\
         & u_2 & 1 & & \\
         &       & 0 & \ddots & \\
         &       & u_3 & \ddots & \\
         &       &       & \cdots & \cdots \\
\end{pmatrix}
\]

if and only if the condition

\[
P_j(0) \neq 0, \quad j \in \mathbb{Z}_+
\]

is satisfied. \textit{The Christoffel transformation of} \( J \) is defined as follows

\[
J = LU \rightarrow J_C := UL.
\]

The matrix \( J_C \) is the tridiagonal matrix associated to the measure \( t \sigma(t) \) \cite{6}.

On the other hand, if \( s_{-1} \) is a real number such that

\[
Q_j(0) - s_{-1}P_j(0) \neq 0, \quad j \in \mathbb{Z}_+, \tag{1.5}
\]

then the monic Jacobi matrix \( J \) admits the \( UL \)-factorization

\[
J = UL
\]

where \( U \) and \( L \) are defined by (1.3). The matrix

\[
J_G := LU
\]

is clearly a tridiagonal matrix and it is called \textit{the Geronimus transformation of} \( J \) \textit{with the parameter} \( s_{-1} \). Notice that the \( LU \)-factorization of a monic Jacobi matrix is unique while the \( UL \)-factorization depends on the free parameter \( s_{-1} \). Both the Christoffel and Geronimus transformation are also called \textit{the Darboux transformations} \cite{6}. The study of discrete Darboux transformations was originated in \cite{25} and has been further developed in \cite{15}, \cite{29}, \cite{30}. Also, note that discrete Darboux transformations are applied in bispectral problems \cite{16}, Toda lattices \cite{20} and numerical linear algebra \cite{19}. We refer the reader to \cite{6} for more details.

In the present paper we extend the Darboux transformations beyond the conditions (1.4) and (1.5). Namely, for an arbitrary monic Jacobi matrix \( J \) a pair of lower and upper triangular block matrices \( L \) and \( U \) is constructed such that \( J = LU \) and the matrix

\[
J_C = UL
\]

is a monic generalized Jacobi matrix in the sense of \cite{11}, i.e. \( J_C \) is a tridiagonal block matrix with blocks specified below in Definition 2.4.
Similarly, for any monic Jacobi matrix $J$ and the free parameter $s_{-1}$ the $UL$-factorization $J = \Omega \mathcal{L}$ with the lower and upper triangular block matrices $\mathcal{L}$ and $\Omega$ is found. In this case the Geronimus transform

$$\mathcal{J}_G = \Omega \mathcal{L}$$

is also a tridiagonal block matrix (a monic generalized Jacobi matrix).

Recall that the monic Jacobi matrix $J$ can be also associated with the Nevanlinna function (see [1])

$$F(\lambda) = \int_{\mathbb{R}} \frac{d\sigma(t)}{t - \lambda}.$$ 

Analogously, one can associate the generalized Jacobi matrices $\mathcal{J}_C$ and $\mathcal{J}_G$ with the functions $\mathcal{F}_C$ and $\mathcal{F}_G$, respectively (see [11]). We show that for the case in question they have the following forms

$$\mathcal{F}_C(\lambda) = \lambda F(\lambda) + 1, \quad \mathcal{F}_G(\lambda) = -\frac{s_{-1}}{\lambda} + \frac{F(\lambda)}{\lambda},$$

where $s_{-1}$ is a free parameter.

It turns out that the Christoffel transformation $\mathcal{J}_C$ of a bounded monic Jacobi matrix can be unbounded, that is, the entries of $\mathcal{J}_C$ are not necessarily bounded. In Section 6 we relate this phenomenon to the effect of accumulating at $\infty$ of the poles of the Padé approximants to $\mathcal{F}_C$. In fact, we show that the $LU$-factorization of $J$ is bounded in the sense that $L$ and $U$ are bounded if and only if the diagonal Padé approximants to the corresponding function $\mathcal{F}_C$ converge to $\mathcal{F}_C$ locally uniformly outside of a finite interval of the real line. A similar situation takes place in the case of the Geronimus transformation.

The paper is organized as follows. In Section 2 we recall some basic facts and definitions. In particular, the definition of monic generalized Jacobi matrices is given. The Darboux transformations for arbitrary Jacobi matrices as well as for monic generalized Jacobi matrices are presented in Sections 3 and 4. Next section deals with triangular factorizations of arbitrary symmetric Jacobi matrices. In Section 6, we restrict our consideration to the case of probability measures supported on $[-1, 1]$ and some criteria for the locally uniform convergence, outside of a finite interval, of the diagonal Padé approximants to the functions $\mathcal{F}_C$ and $\mathcal{F}_G$ in terms of the $LU$- and $UL$-factorizations are obtained. Finally, in Section 7 we provide the reader with some concrete examples.

2. Preliminaries

2.1. Classes $\mathbf{D}^+_{-\infty}$ and $\mathbf{D}^-_{-\infty}$. Let $\mathbf{N}$ be the class of all Nevanlinna functions which map the upper half plane $\mathbb{C}_+$ into the upper half plane $\mathbb{C}_+$. We will say (cf. [17]) that a Nevanlinna function $F \in \mathbf{N}$ belongs to the class $\mathbf{N}_{-2n}$ if for some numbers $s_0, \ldots, s_{2n} \in \mathbb{R}$ the following asymptotic expansion holds true

$$F(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_{2n}}{\lambda^{2n+1}} + o \left( \frac{1}{\lambda^{2n+1}} \right), \quad \lambda \to \infty,$$

where $\lambda \to \infty$ means that $\lambda$ tends to $\infty$ non-tangentially, i.e. inside the sector $\varepsilon < \arg \lambda < \pi - \varepsilon$ for some $\varepsilon > 0$. Let us set

$$\mathbf{N}_{-\infty} := \bigcap_{n \geq 0} \mathbf{N}_{-2n}.$$
Recall [1] that \( F \in \mathbb{N}_{-\infty} \) if and only if it admits the following representation

\[
F(\lambda) = \int_{\mathbb{R}} \frac{d\sigma(t)}{t - \lambda},
\]

where \( \sigma \) is a bounded measure supported on the real line and \( s_j = \int_{\mathbb{R}} \nu d\sigma(t) \), \( j \in \mathbb{Z}_+ \). It will be sometimes convenient to use the following notation

\[
(2.2) \quad F(\lambda) \sim -\sum_{j=0}^{\infty} \frac{s_j}{\lambda^{j+1}}, \quad \lambda \to \infty
\]

to denote the validity of (2.1) for all \( n \in \mathbb{Z}_+ \). The Jacobi matrix \( J \) associated with the probability measure \( \sigma/s_0 \) will be also called the Jacobi matrix associated with \( F \in \mathbb{N}_{-\infty} \).

**Definition 2.1.** Let us say that a function \( \tilde{\mathcal{G}} \) meromorphic in \( \mathbb{C}_+ \) belongs to the class \( \mathbb{D}^+_{\infty} \) if it admits the asymptotic expansion

\[
(2.3) \quad \tilde{\mathcal{G}}(\lambda) \sim -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_n}{\lambda^{n+1}} - \cdots, \quad \lambda \to \infty,
\]

with some \( s_j \in \mathbb{R} \) \( (j \in \mathbb{Z}_+) \) and \( \lambda \tilde{\mathcal{G}}(\lambda) + s_0 \in \mathbb{N}_{-\infty} \).

**Definition 2.2.** Let us say that a function \( \tilde{\mathcal{G}} \) meromorphic in \( \mathbb{C}_+ \) belongs to the class \( \mathbb{D}^-_{\infty} \) if there exists a number \( s_{-1} \in \mathbb{R} \setminus \{0\} \) such that

\[
-\frac{s_{-1}}{\lambda} + \frac{\tilde{\mathcal{G}}(\lambda)}{\lambda} \in \mathbb{N}_{-\infty}.
\]

In what follows we use the Gothic script for all the notations associated with the \( \mathbb{D}^\pm_{\infty} \)-functions and the Roman script for the \( \mathbb{N}_{-\infty} \)-functions to avoid confusion.

2.2. **The Schur transform of the \( \mathbb{D}^\pm_{\infty} \)-functions.** Let \( \tilde{\mathcal{G}} \) be a nonrational function from \( \mathbb{D}^\pm_{\infty} \) having an asymptotic expansion of the form (2.3). Define the set \( \mathcal{N}(s) \) of normal indices of the sequence \( s = \{s_j\}_{j=0}^{\infty} \) by

\[
(2.4) \quad \mathcal{N}(s) = \{n_j : \det(s_{i+k})_{i,k=0}^{n_j-1} \neq 0, \quad j = 1, 2, \ldots \}.
\]

We will say that the function \( \tilde{\mathcal{G}} \in \mathbb{D}^\pm_{\infty} \) is normalized if the first nontrivial coefficient in its asymptotic expansion (2.3) has modulus 1, i.e. \( |s_{n_1-1}| = 1 \).

The following statement can be found in [1] (Theorem 4.8).

**Theorem 2.3.** Let \( \tilde{\mathcal{G}} \) be a nonrational normalized function of the class \( \mathbb{D}^\pm_{\infty} \), let the sequence \( \mathcal{S} = \{s_j\}_{j=0}^{\infty} \) be defined by the asymptotic expansion (2.3), and let \( \mathcal{N}(s) = \{n_i\}_{i=1}^{\infty} \) be a set of normal indices of \( s \). Then \( \tilde{\mathcal{G}} \) admits the expansion into the following \( P \)-fraction

\[
(2.5) \quad -\frac{\epsilon_0}{p_0(\lambda)} - \frac{\epsilon_0 \epsilon_1 b_1^2}{p_1(\lambda)} - \cdots - \frac{\epsilon_{N-1} \epsilon_N b_N^{2N-1}}{p_N(\lambda)} - \cdots,
\]

where \( p_i \) are monic polynomials of degree \( \ell_i := n_{i+1} - n_i \) \( (\leq 2) \), \( \epsilon_i = \pm 1 \), \( b_i > 0 \), \( i \in \mathbb{Z}_+ \), and \( n_0 = 0 \).

For the convenience of the reader we sketch the proof. Actually, the proof is based on the following step-by-step Schur process [1] (see also [2], [7]). If \( s_0 \neq 0 \) then \( n_1 = 1 \) and the polynomial \( p_0 \) is defined as follows

\[
p_0(\lambda) = \lambda - \frac{s_1}{s_0} = \frac{1}{s_0} \begin{vmatrix} s_0 & s_1 \\ 1 & \lambda \end{vmatrix}.
\]
If $s_0 = 0$ then $s_1 \neq 0$, $n_1 = 1$, and

$$p_0(\lambda) = \frac{1}{\det(s_{i+j})_{i,j=0}^{1}} \begin{vmatrix} 0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ 1 & \lambda & \lambda^2 \end{vmatrix}.$$ 

In both cases the Schur transform $\hat{F}$ of $F \in D_{-\infty}$ is defined by the equality

$$(2.6) \quad -\frac{1}{\hat{\delta}(\lambda)} = \epsilon_0 p_0(\lambda) + b_0^2 \hat{\delta}(\lambda),$$

where $\epsilon_0 = s_{n_1-1}$ and $b_0$ is chosen in such a way that $\hat{F}$ is normalized. Also, it is easy to see that $\hat{F} \in D_{-\infty}$ (see [11] for details). Further, considering the asymptotic expansion of $F_1 := \hat{F}$

$$\hat{\delta}(\lambda) \sim -\frac{s_0^{(1)}}{\lambda} - \frac{s_1^{(1)}}{\lambda^2} - \cdots - \frac{s_{2n}^{(1)}}{\lambda^{2n+1}} - \cdots, \quad \lambda \to \infty,$$

one can construct the number $\epsilon_1 = \pm 1$, the monic polynomial $p_1$, and the function $\hat{F}_2 := \hat{F}_1$. Continuing this procedure leads us to (2.5).

2.3. Generalized Jacobi matrices associated with the $D_{-\infty}$-functions. The $P$-fraction (2.5) enables us to associate a generalized Jacobi matrix with the function $F \in D_{-\infty}$ (see [8, 9, 10]).

**Definition 2.4.** Let $F$ be a nonrational normalized $D_{-\infty}$-function having the $P$-fraction expansion (2.5). Then a monic generalized Jacobi matrix associated with the function $F \in D_{-\infty}$ is the tridiagonal block matrix

$$(2.7) \quad \hat{J} = \begin{pmatrix} \mathcal{B}_0 & \mathcal{D}_0 & \mathcal{C}_0 & \mathcal{B}_1 & \mathcal{D}_1 & \mathcal{C}_1 & \mathcal{B}_2 & \cdots \\ \mathcal{C}_0 & \mathcal{B}_1 & \mathcal{D}_1 & \mathcal{C}_1 & \mathcal{B}_2 & \cdots \\ \mathcal{C}_1 & \mathcal{B}_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where the diagonal entries have the following form

$$\mathcal{B}_j = \begin{cases} -p_0^{(j)}, & \text{if } \epsilon_j = 1; \\ 0 & 1 \end{cases} \begin{pmatrix} 0 & -p_0^{(j)} \\ 1 & -p_1^{(j)} \end{pmatrix},$$

the super-diagonal is built up with $\mathfrak{t}_j \times \mathfrak{t}_{j+1}$ matrices

$$\mathcal{D}_j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \epsilon_j & 0 \\ 0 & \epsilon_j \end{pmatrix},$$

and the sub-diagonal consists of $\mathfrak{t}_{j+1} \times \mathfrak{t}_j$ matrices

$$\mathcal{C}_j = \begin{pmatrix} 0 & \epsilon_j \\ \epsilon_j & 0 \end{pmatrix}, \begin{pmatrix} 0 & \epsilon_j \\ \epsilon_j & 0 \end{pmatrix}.$$ 

here $c_j = \epsilon_j \epsilon_{j+1} b_j^2$.

**Remark 2.5.** Actually, one can associate a monic generalized Jacobi matrix $\hat{J}$ with an arbitrary nonrational function $\hat{F} \in D_{-\infty}$. Namely, every function $F \in D_{-\infty}$ can be normalized as follows $F_{nor} = F/[s_{n_1-1}]$ and the monic generalized Jacobi matrix associated with $F_{nor}$ will be also called associated with $F$ (see Remark 2.5).
Setting $c_{-1} = \epsilon_0$, let us define polynomials of the first kind $P_j(\lambda)$, $j \in \mathbb{Z}_+$, as solutions $y_j = P_j(\lambda)$ of the following system:
\begin{equation}
\epsilon_{j-1} y_{j-1} - p_j(\lambda) y_j + y_{j+1} = 0, \quad j \in \mathbb{Z}_+,
\end{equation}
with the initial conditions
\begin{equation}
y_{-1} = 0, \quad y_0 = 1.
\end{equation}
Similarly, the polynomials of the second kind $\Omega_j(\lambda)$, $j \in \mathbb{Z}_+$, are defined as solutions $u_j = \Omega_j(\lambda)$ of the system\footnote{See (2.9)} with the initial conditions
\begin{equation}
y_{-1} = -1, \quad y_0 = 0.
\end{equation}
It follows from (2.8) that $p_j$ is a monic polynomial of degree $n_j$ and $\Omega_j$ is a polynomial of degree $n_j - t_0$ with the leading coefficient $\epsilon_0$. The equations\footnote{See (2.8)} coincide with the three-term recurrence relations associated with $P$-fractions\footnote{See (2.8)}. The following statement is immediate from (2.8).

**Proposition 2.6.** \footnote{See (2.8)} Polynomials $P_j$ and $P_{j+1}$ ($\Omega_j$ and $\Omega_{j+1}$) are coprime.

Recall (see\footnote{See (2.8)}) that the polynomials $P_j$ can be found by the formulas
\begin{equation}
P_j(\lambda) = \frac{1}{d_j} \det \left( \begin{array}{ccc} s_0 & s_1 & \cdots & s_{n_j} \\
 s_{n_j-1} & s_{n_j} & \cdots & s_{2n_j-1} \\
 1 & \lambda & \cdots & \lambda^{n_j} \end{array} \right),
\end{equation}
and hence
\begin{equation}
P_j(0) = \frac{(-1)^{n_j}}{d_j} \det(s_{i+k+1})_{i,k=0}^{n_j-1}, \quad \Omega_j(0) = \frac{(-1)^{n_j}}{d_j} \det \left( \begin{array}{ccc} 0 & s_0 & \cdots & s_{n_j-1} \\
 s_{n_j} & s_1 & \cdots & s_{n_j} \\
 \cdots & \cdots & \cdots & \cdots \\
 s_{2n_j-1} & s_{n_j+1} & \cdots & s_{2n_j-1} \end{array} \right),
\end{equation}
since $\Omega_j(\lambda) = \mathcal{C}_t \left( \frac{P_j(\lambda) - \mathcal{C}_t(i)}{\lambda - t} \right)$, where $\mathcal{C}_t$ is a linear functional such that
\begin{equation}
\mathcal{C}_t(i^j) = \delta_j, \quad j \in \mathbb{Z}_+.
\end{equation}

Introducing for arbitrary $j \in \mathbb{Z}_+$ the shortened matrices
\begin{equation}
\mathcal{J}_{[0,j]} = \begin{pmatrix} \mathcal{C}_0 & \mathcal{B}_0 \\
 \mathcal{C}_1 & \mathcal{B}_1 \\
 \vdots & \ddots \\
 \mathcal{C}_{j-1} & \mathcal{B}_{j-1} \\
 \mathcal{C}_j & \mathcal{B}_j \end{pmatrix}, \quad \mathcal{J}_{[1,j]} = \begin{pmatrix} \mathcal{C}_1 & \mathcal{B}_1 \\
 \mathcal{C}_2 & \mathcal{B}_2 \\
 \vdots & \ddots \\
 \mathcal{C}_{j-1} & \mathcal{B}_{j-1} \\
 \mathcal{C}_j & \mathcal{B}_j \end{pmatrix},
\end{equation}
one can obtain the following connection between the polynomials of the first and second kind $P_j$, $\Omega_j$ and the shortened Jacobi matrices $\mathcal{J}_{[0,j]}$ and $\mathcal{J}_{[1,j]}$ (for the classical case see\footnote{See (2.8)} Section 7.1.2))
\begin{equation}
P_j(\lambda) = \det(\lambda - \mathcal{J}_{[0,j]-1}), \quad \Omega_j(\lambda) = \epsilon_0 \det(\lambda - \mathcal{J}_{[1,j]-1}).
\end{equation}

**Remark 2.7.** Let us define an infinite matrix $G$ by the equality
\begin{equation}
G = \text{diag}(G_0, \ldots, G_n, \ldots), \quad G_j = \begin{cases} \epsilon_j, & \text{if } t_j = 1; \\
 \epsilon_j \begin{pmatrix} 0 & 1 \\
 1 & -p_1(j) \end{pmatrix}, & \text{if } t_j = 2,
\end{cases}
\end{equation}
and let $\ell^2_{[0,\infty)}(G)$ be the space of $\ell^2$-vectors with the inner product
\begin{equation}
(x, y) = (Gx, y)_{\ell^2_{[0,\infty)}}, \quad x, y \in \ell^2_{[0,\infty)}.
\end{equation}
The inner product (2.16) is indefinite, if either $k_j > 1$ for some $j \in \mathbb{Z}_+$, or at least one $\varepsilon_j$ is equal to $-1$. The space $\ell^2_{[0,\infty)}(G)$ is equivalent to a Krein space (see [3]) if both $G$ and $G^{-1}$ are bounded in $\ell^2_{[0,\infty)}$. If $k_j = \varepsilon_j = 1$ for all $j$ big enough, then $\ell^2_{[0,\infty)}(G)$ is a Pontryagin space.

The $m$-function of the shortened matrix $J_{[0,j-1]}$ is defined by
\begin{equation}
m_{[0,j-1]}(\lambda) = ([J_{[0,j-1]} - \lambda]^{-1}e_0, e_0),
\end{equation}
where $e_0 = (1, 0, 0, \ldots, 0)^\top \in \mathbb{C}^n$. Due to (2.14) it is calculated by
\begin{equation}
m_{[0,j-1]}(\lambda) = -\varepsilon_0 \frac{\det(\lambda - J_{[1,j-1]})}{\det(\lambda - J_{[0,j-1]})} = -\frac{\Omega_j(\lambda)}{\Psi_j(\lambda)}.
\end{equation}

**Remark 2.8.** Let us emphasize that the polynomials $\Psi_j$ and $\Omega_j$ have no common zeros (see [9] Proposition 2.7) and due to (2.18) the set of holomorphy of $m_{[0,j-1]}$ coincides with the resolvent set of $J_{[0,j-1]}$.

It was shown in [9] that $m_{[0,j-1]}$ has the following property
\begin{equation}
\mathfrak{F}(\lambda) - m_{[0,j-1]}(\lambda) = O\left(\frac{1}{\lambda^{2n_j + 1}}\right), \quad \lambda \to \infty.
\end{equation}
The latter means that one can reconstruct the sequence $\{s_k\}_{k=0}^\infty$ by the monic generalized Jacobi matrix $J$. Namely, it follows from (2.19) that for any $k \in \mathbb{Z}_+$
\begin{equation}
s_k = \left([J_{[0,j-1]}]^k\right) e_0, e_0, \quad \text{for all } j \text{ such that } n_j \geq \frac{k}{2}.
\end{equation}

### 3. The Christoffel transformation and its inverse

From now on we always assume that $F \in \mathbb{N}_{-\infty}$ (or $\mathfrak{F} \in \mathbb{D}_{-\infty}$) is a nonrational function. Then there is a monic Jacobi (or generalized Jacobi) matrix associated with $F$ (or $\mathfrak{F}$, respectively).

#### 3.1. LU-factorizations of Jacobi matrices

In the following proposition we introduce a block LU-factorization of an arbitrary Jacobi matrix.

**Proposition 3.1.** Let $J$ be a monic Jacobi matrix associated with $F \in \mathbb{N}_{-\infty}$ with the asymptotic expansion (2.1), let $\{n_j\}_{j=1}^\infty$ be the set of normal indices of the sequence $s = \{s_j\}_{j=0}^\infty := \{s_{j+1}\}_{j=0}^\infty$ and let $t_j := n_{j+1} - n_j$, $j \in \mathbb{Z}_+$, where $n_0 = 0$. Then $J$ admits the following factorization
\begin{equation}
J = LU,
\end{equation}
where $L$ and $U$ are block lower and upper triangular matrices having the forms
\begin{equation}
\begin{pmatrix}
I_{t_0} & 0 \\
L_1 & I_{t_1} & 0 \\
& L_2 & I_{t_2} & \ddots \\
& & & \ddots & \ddots
\end{pmatrix}, \quad
\begin{pmatrix}
U_0 & D_0 \\
0 & U_1 & D_1 \\
& 0 & U_2 & \ddots \\
& & & \ddots & \ddots
\end{pmatrix}
\end{equation}
in which the sub-diagonal of $L$ consists of $\mathfrak{f}_{j+1} \times \mathfrak{f}_j$ matrices

$$L_{j+1} = \begin{pmatrix} l_{j+1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} l_{j+1} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} l_{j+1} & 0 \end{pmatrix}, \quad (l_{j+1}),$$

the diagonal entries of $U$ are of the form

$$U_j = \begin{cases} u_0^{(j)}, & \text{if } \mathfrak{f}_j = 1; \\ 0 & 1 \\ u_0^{(j)} & u_1^{(j)} \end{cases}$$

and the super-diagonal of $U$ is built up with $\mathfrak{f}_j \times \mathfrak{f}_{j+1}$ matrices

$$D_j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (1).$$

Moreover, the factorization (3.1) is unique and the following relations hold true

$$u_0^{(j)} = -\frac{P_{n_j+1}(0)}{P_{n_j}(0)}, \quad u_1^{(j)} = b_{n_j+1}, \quad j \in \mathbb{Z}_+.$$

**Proof.** It is natural to consider the following relation

$$LU = \begin{pmatrix} U_0 & D_0 \\ L_1U_0 & L_1D_0 + U_1 \\ & \vdots \end{pmatrix} = \begin{pmatrix} b_0 & 1 & 1 \\ c_0 & b_1 & 1 \\ & \vdots & \ddots \end{pmatrix},$$

which enables us to find the entries of $L$ and $U$. By the definition, there are two possibilities for the matrix $U_0$. If $\mathfrak{f}_0 = 1$ then

$$u_0^{(0)} = b_0 = -\frac{P_1(0)}{P_0(0)} = -\frac{P_{n_1}(0)}{P_{n_0}(0)}.$$

In case $\mathfrak{f}_0 = 2$ we have

$$\begin{pmatrix} 0 \\ u_0^{(0)} \\ u_1^{(0)} \end{pmatrix} = \begin{pmatrix} b_0 \\ c_0 \\ b_1 \end{pmatrix},$$

and, hence, one obtains the equalities

$$u_0^{(0)} = c_0, \quad u_1^{(0)} = b_1.$$

Since $\mathfrak{f}_0 = 2$ then $s_1 = 0$, $P_1(0) = 0$ and (6.1) yields

$$u_0^{(0)} = c_0 = -\frac{P_2(0)}{P_0(0)} = -\frac{P_{n_1}(0)}{P_{n_0}(0)}.$$

The equality $b_0 = 0$ follows from the relation $b_0 = \frac{a_0}{s_0}$ (see [1]).

Now, assume that

$$u_0^{(j-1)} = -\frac{P_{n_j}(0)}{P_{n_{j-1}}(0)}.$$

We will consider 4 cases.

**Case 1.** $\mathfrak{f}_j = \mathfrak{f}_{j-1} = 2$, $j \in \mathbb{N}$. In this case one obtains the following relations

$$L_jU_{j-1} = \begin{pmatrix} 0 & l_j \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_{n_{j-1}} \\ 0 & 0 \end{pmatrix}.$$


Consequently, by taking into account the equalities and formula (3.12) follows from (1.1) and the fact that
\[ (3.13) \]
we get that
\[ (3.12) \]
\[ l_j = c_{nj-1}, \quad u_1^{(j)} = b_{nj+1}, \quad u_0^{(j)} = c_{nj}. \]

It follows from \( \xi_j = \xi_{j-1} = 2 \) that \( \det(s_{n+1})_{j=0}^{n+1} = 0 \). Hence by (3.12) \( P_{nj+1}(0) = P_{nj-1}(0) = 0 \) and then (1.1) yields \( b_{nj} = 0 \). Moreover, from (1.1) one gets also
\[ (3.9) \]
\[ u_0^{(j)} = c_{nj} = \frac{P_{nj+2}(\lambda) + (b_{nj+1} - \lambda)P_{nj+1}(\lambda)}{P_{nj}(\lambda)} \bigg|_{\lambda = 0} = -\frac{P_{nj+1}(0)}{P_{nj}(0)}. \]

Case 2. \( \xi_j = 2, \xi_{j-1} = 1, j \in \mathbb{N} \). In this case we have
\[ L_j U_{j-1} = \begin{pmatrix} l_j u_0^{(j)} \\ 0 \end{pmatrix} = \begin{pmatrix} c_{nj-1} \\ 0 \end{pmatrix}, \]
\[ L_j D_{j-1} + U_j = \begin{pmatrix} l_j \\ u_0^{(j)} \\ u_1^{(j)} \end{pmatrix} = \begin{pmatrix} b_{nj} \\ c_{nj} \\ b_{nj+1} \end{pmatrix}. \]

Since \( P_{nj+1}(0) = 0 \), one concludes as in (3.9) that
\[ (3.10) \]
\[ u_0^{(j)} = c_{nj+1}, \quad u_1^{(j)} = c_{nj} = -\frac{P_{nj+1}(0)}{P_{nj}(0)}. \]

In order to see that the relations
\[ (3.11) \]
\[ l_j = b_{nj}, \quad l_j u_0^{(j-1)} = c_{nj-1} \]
hold true simultaneously, let us set \( l_j := b_{nj} \). Thus, according to the assumption (3.7), the second relation in (3.11) can be rewritten as follows
\[ (3.12) \]
\[ b_{nj} P_{nj}(0) + c_{nj-1} P_{nj-1}(0) = 0, \]
and formula (3.12) follows from (1.1) and the fact that \( P_{nj+1}(0) = 0 \).

Case 3. \( \xi_j = 1, \xi_{j-1} = 2, j \in \mathbb{N} \). The identities \( \xi_j = 1 \) and \( \xi_{j-1} = 2 \) yield
\[ L_j U_{j-1} = (0 \quad l_j) = (0 \quad c_{nj-1}), \quad L_j D_{j-1} + U_j = u_0^{(j)} = b_{nj}. \]

Consequently, by taking into account the equalities \( P_{nj-1}(0) = 0 \) and \( n_{j+1} = n_j + 1 \) we get that
\[ (3.13) \]
\[ l_j = c_{nj-1}, \quad u_0^{(j)} = b_{nj} = \frac{P_{nj+1}(0)}{P_{nj}(0)}. \]

Case 4. \( \xi_j = 1, \xi_{j-1} = 1, j \in \mathbb{N} \). This case reduces to the following relations
\[ (3.14) \]
\[ L_j U_{j-1} = l_j u_0^{(j)} = c_{nj-1}, \quad L_j D_{j-1} + U_j = l_j + u_0^{(j)} = b_{nj}. \]

Eliminating \( l_j \) from the first equation in (3.14) and substituting it to the second one, we get
\[ (3.15) \]
\[ u_0^{(j)} = \frac{b_{nj} P_{nj}(0) + c_{nj-1} P_{nj-1}(0)}{P_{nj}(0)} = -\frac{P_{nj+1}(0)}{P_{nj}(0)}. \]

\[ \square \]

**Theorem 3.2.** Let \( J \) be a monic Jacobi matrix associated with \( F \in \mathbb{N}_{-\infty} \) and let \( J = LU \) be its LU factorization. Then the matrix \( \lambda J = \lambda L U \) is the monic generalized Jacobi matrix associated with \( \lambda F(\lambda) + s_0 \in \mathbb{D}_{-\infty} \).
Proof. The fact that $\mathcal{J} = UL$ is a monic generalized Jacobi matrix can be easily verified by straightforward calculations.

We can assume that the function $F$ is normalized and let $F$ have the asymptotic expansion (2.2). Then the function $\lambda F(\lambda) + 1$ has the asymptotic expansion

$$
\lambda F(\lambda) + 1 \sim -\frac{s_1}{\lambda} - \frac{s_2}{\lambda^2} - \ldots \quad (\lambda \to \infty)
$$

and is not necessarily normalized. Since according to (3.6) we have

$$
e_0 u_0^{(0)} = -\epsilon_0 p_{n_0}(0) \begin{cases}
e_0 s_1, & \text{if } t_0 = 1; \\
e_0 s_2, & \text{if } t_0 = 2
\end{cases} = \begin{cases}s_1, & \text{if } t_0 = 1; \\
s_2, & \text{if } t_0 = 2\end{cases}
$$

the function $\mathfrak{F}_{nor} = (F(\lambda) + 1)/\epsilon_0 u_0^{(0)}$ is normalized.

Further, observe that

(3.16) \hspace{1cm} L_{[0,j]}^\top e_0 = e_0, \quad G_{[0,j]} e_0 = \frac{\epsilon_0}{u_0^{(0)}} U_{[0,j]} e_0, \quad j \in \mathbb{Z}_+,

where the shortened matrices $L_{[0,j]}$, $U_{[0,j]}$, and $G_{[0,j]}$ are defined analogously to (2.13).

Clearly, for $j$ big enough

$$s_n = \left(e_0, J_{[0,n_{j+1}]}\right), \quad n \in \mathbb{Z}_+,
$$

where $J_{[0,n_{j+1}]}$ is the leading principal submatrix of $J$ constructed from the first $n_{j+1}$ rows and columns of $J$. Thus, we have

$$s_n = \left(e_0, J_{[0,n_{j+1}]}\right) = \left(e_0, L_{[0,j]} U_{[0,j]} \ldots L_{[0,j]} U_{[0,j]} e_0\right) =
$$

$$= \left(L_{[0,j]}^\top e_0, U_{[0,j]} L_{[0,j]} \ldots U_{[0,j]} L_{[0,j]} e_0\right), \quad n \in \mathbb{N}.
$$

Further, using (3.16) gives the equality

$$s_n = \epsilon_0 u_0^{(0)} \left(e_0, U_{[0,j]} L_{[0,j]}\right)^{n-1} G e_0.
$$

Hence for sufficiently large $j$ one calculates the moments

(3.17) \hspace{1cm} \left(\mathfrak{F}_{[0,j]}\right)^{n-1} e_0, e_0 = \frac{s_n}{\epsilon_0 u_0^{(0)}},

which coincide with the coefficients in the asymptotic expansion of $\mathfrak{F}_{nor}$. In view of (2.20) this means that the generalized Jacobi matrix $\mathfrak{J}$ is associated with the function $\lambda F(\lambda) + s_0 \in D_{\infty}$. \hfill $\square$

The transform $J = LU \mapsto \mathfrak{J} = UL$ is called the Christoffel transform of the Jacobi matrix $J$.

Corollary 3.3. Let $J$ be a monic Jacobi matrix associated with $F \in \mathbb{N}_{-\infty}$ with the asymptotic expansion (2.1). Assume that $P_j(0) \neq 0$ for all $j \in \mathbb{N}$. Then $n_j = j$ for $j \in \mathbb{N}$ and

(i) The matrix $J$ admits the LU-factorization (3.1), where $L$ and $U$ are lower and upper triangular matrices having the forms (3.2) with

(3.18) \hspace{1cm} D_j = 1, \quad U_j = -\frac{P_{j+1}(0)}{P_j(0)}, \quad L_{j+1} = b_{j+1} - U_{j+1}, \quad j \in \mathbb{Z}_+.

(ii) The matrix $\mathcal{J} = UL$ is the monic tridiagonal generalized Jacobi matrix associated with $\lambda F(\lambda) + s_0 \in \mathcal{D}_{-\infty}$.

The statement of this corollary is contained in [6], where, in fact, more general monic three-diagonal matrices of the form (1.2) were considered (their matrices are associated with a class of quasi-definite linear functionals which includes finite measures on the real line as a subclass).

3.2. UL-factorizations of generalized Jacobi matrices. Here we present a block UL-factorization of monic generalized Jacobi matrices.

**Proposition 3.4.** Let $\mathcal{J}$ be a monic generalized Jacobi matrix associated with $\mathcal{J} \in \mathcal{D}_{-\infty}$, let $\{n_j\}_{j=0}^{\infty}$ be the set of normal indices of the sequence $s = \{s_j\}_{j=0}^{\infty}$ defined by the asymptotic expansion (2.3) of the function $\mathcal{J}$, $\xi_j := n_{j+1} - n_j$, $j \in \mathbb{Z}_+$, where $n_0 = 0$, let $s_{-1} \in \mathbb{R}$ be the same as in Definition 2.2, let $\mathcal{J}_j(\lambda)$ and $\Omega_j(\lambda)$ be polynomials of the first and the second kind associated with the sequence $s = \{s_j\}_{j=0}^{\infty}$, and let the polynomials $\mathcal{J}_j$ be defined by

$$\mathcal{J}_j(\lambda) = \mathcal{J}_j(\lambda) - \frac{1}{s_{-1}}\Omega_j(\lambda).$$

Then:

(i) $\mathcal{J}_j(0) \neq 0$ for $j \in \mathbb{Z}_+$;

(ii) the GJM $\mathcal{J}$ admits the following factorization

$$\mathcal{J} = UL$$

where $U$ and $L$ are block lower and upper triangular matrices having the form (3.2) and (3.3). Moreover, the following relations hold true

$$u_1^{(j)} = -p_1^{(j)}, \quad l_{j+1} = -\frac{\mathcal{J}_{j+1}(0)}{\mathcal{J}_j(0)}, \quad u_0^{(0)} = \frac{1}{s_{-1}}, \quad u_0^{(j+1)} = \frac{c_j}{l_{j+1}}.$$

**Proof.** (i) Notice first that the sequence $\{s_j\}_{j=0}^{\infty} = \{s_{j-1}\}_{j=0}^{\infty}$ satisfies the asymptotic expansion (2.3) of the function $F(\lambda) = \frac{\mathcal{J}(\lambda) - s_{-1}}{\lambda} \in \mathcal{N}_{-\infty}$. Then

$$\vartheta_k := \det \begin{pmatrix} s_{-1} & s_0 & \cdots & s_k \\ s_0 & s_1 & \cdots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_k & s_{k+1} & \cdots & s_{2k+1} \end{pmatrix} \neq 0, \quad k \in \mathbb{Z}_+,$$

by Definition 2.2 and the property of Nevanlinna functions. In particular, one gets from (2.2) the relation

$$\vartheta_{n_j-1} = s_{-1} \det \begin{pmatrix} s_1 & \cdots & s_{n_j} \\ \vdots & \ddots & \vdots \\ s_{n_j} & \cdots & s_{2n_j-1} \end{pmatrix} + \det \begin{pmatrix} 0 & s_0 & \cdots & s_{n_j-1} \\ s_0 & s_1 & \cdots & s_{n_j} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n_j-1} & s_{n_j+1} & \cdots & s_{2n_j-1} \end{pmatrix},$$

$$= (-1)^{j+1} (s_{-1}\mathcal{J}_j(0) - \Omega_j(0)) \det(s_{i+k}_{i,k=0}^{n_j-1}) \neq 0,$$

which shows that $\mathcal{J}_j(0) \neq 0$ for $j \in \mathbb{Z}_+$.

(ii) Further, the equality $\mathcal{J} = UL$ yields

$$u_1^{(j)} = -p_1^{(j)}, \quad u_0^{(j)} + l_{j+1} = -p_0^{(j)}, \quad u_0^{(j+1)}l_{j+1} = c_j, \quad j \in \mathbb{Z}_+.$$
Setting $u_0^{(0)} := 1/s_{-1}$, we see that
\[ l_1 = -p_0^{(0)} - \frac{1}{s_{-1}} = -\hat{\mathcal{P}}_1(0)/\mathcal{P}_0(0). \]

Next, the second formula in (3.24) follows by induction with the help of the second and third equations in (3.21), and the fact that the polynomials $\hat{\mathcal{P}}_j$ satisfy (2.8).

**Remark 3.5.** It should be noted that, actually, both the matrices in the UL-decomposition (3.19) depend on $s_{-1}$, that is, $U = U(s_{-1})$ and $L = L(s_{-1})$.

**Theorem 3.6.** Let $J$ be a monic generalized Jacobi matrix associated with $\mathfrak{J} \in \mathbb{D}^+_{-\infty}$, let $s_{-1} \in \mathbb{R}$ be the same as in Definition 2.2, and let $J = UL$ be its UL factorization of the form (3.19), (3.2)–(3.3). Then the matrix $J = LU$ is the monic Jacobi matrix associated with $\frac{\mathfrak{J}(\lambda)}{\lambda} - \frac{s_{-1}}{\lambda} \in \mathbb{N}_{-\infty}$.

**Proof.** The fact that $J$ is a classical Jacobi matrix can be easily verified by straightforward calculations. The rest of the proof can be done by reversing the reasoning given in the proof of Theorem 3.2. \qed

### 4. The Geronimus Transformation and Its Inverse

#### 4.1. LU-factorizations of Generalized Jacobi Matrices

**Proposition 4.1.** Let $J$ be a monic generalized Jacobi matrix associated with $\mathfrak{J} \in \mathbb{D}^+_{-\infty}$, and let $\ell_j := n_{j+1} - n_j$, $j \in \mathbb{Z}_+$, where $n_0 = 0$ and $\{n_j\}_{j=1}^{\infty}$ is the set of normal indices of the sequence $s = \{s_j\}_{j=1}^{\infty}$ defined by (2.3) and let $\mathcal{P}_j(\lambda)$ be polynomials of the first kind associated with the sequence $s = \{s_j\}_{j=1}^{\infty}$. Then $\mathcal{P}_j(0) \neq 0$ for all $j \in \mathbb{Z}_+$ and the GJM $J$ admits the following factorization
\[ J = \mathcal{L}\mathcal{U}, \]
where $\mathcal{L}$ and $\mathcal{U}$ are block lower and upper triangular matrices having the forms
\[ \mathcal{L} = \begin{pmatrix} E_0 & 0 & \cdots \\ L_1 & E_1 & \cdots \\ & \ddots & \ddots \\ L_2 & E_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} U_0 & D_0 & \cdots \\ 0 & U_1 & \cdots \\ & \ddots & \ddots \\ 0 & \ddots & \ddots \end{pmatrix} \]
in which the sub-diagonal of $\mathcal{L}$ consists of $\ell_j \times \ell_{j-1}$ matrices
\[ \mathcal{L}_j = \begin{pmatrix} 0 & 0 \\ 0 & l_j \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ l_j \end{pmatrix}, \quad \begin{pmatrix} l_j \end{pmatrix}, \quad \ell_j, \]
$D_j$ are of the form (3.5) and $U_j$, $E_j$ are $\ell_j \times \ell_j$ matrices
\[ U_j = \begin{cases} 0, & \text{if } \ell_j = 1; \\ 1, & \text{if } \ell_j = 1; \end{cases}, \quad E_j = \begin{cases} 0, & \text{if } \ell_j = 2; \\ 1, & \text{if } \ell_j = 2; \end{cases} \]
Moreover, the following relations hold true
\[ u_j = -\frac{\mathcal{P}_{j+1}(0)}{\mathcal{P}_j(0)}, \quad \ell_{j+1} = \frac{\ell_j}{u_j}, \quad c_j = -p_1^{(j)}, \quad j \in \mathbb{Z}_+. \]
Proof. First, notice that it follows from formula (2.11) that
\[ P_j(0) = \frac{\det(s_i+k+1)^n_{i,k=0}}{\det(s_i+1)^n_{i,k=0}} \neq 0, \]
since \( \mathcal{F} \in D^\pm_{-\infty} \) and \( s_i := s_i+1 \) \((i \in \mathbb{Z}_+)\) are moments of the Nevanlinna function \( \lambda \mathcal{F}(\lambda) + s_0 \in \mathbb{N}_{-\infty} \).

Consider the product LU of the matrices L and U
\[
\begin{pmatrix}
  L_0 & \mathcal{C}_0 \mathcal{D}_0 \\
  L_1 & L_1 \mathcal{D}_0 + \mathcal{C}_1 \mathcal{U}_1 \\
  L_2 & L_2 \mathcal{D}_1 + \mathcal{C}_2 \mathcal{U}_2 \\
  \vdots & \vdots
\end{pmatrix}
= \begin{pmatrix}
  \mathcal{B}_0 & \mathcal{D}_0 \\
  \mathcal{C}_0 & \mathcal{B}_1 & \mathcal{D}_1 \\
  \mathcal{C}_1 & \mathcal{B}_2 & \ddots
\end{pmatrix},
\]
Comparing it with the matrix \( \mathcal{J} \) in (2.7) one finds the entries of \( L \) and \( U \).

Next, we will prove formula (4.5).

If \( k_0 = 1 \) then the equality \( E_0 U_0 = B_0 \) yields
\[ u_0 = -P_0(0) = \frac{P_1(0)}{P_0(0)}. \]

In the case \( k_0 = 2 \) we have
\[ E_0 U_0 = \begin{pmatrix} 0 & 1 \\ u_0 & e_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p_j(0) & -p_j(1) \end{pmatrix}, \]
and, again, we obtain the equalities
\[ u_0 = -p_0(0) = \frac{P_1(0)}{P_0(0)}; \quad e_0 = -p_1(0). \]

Now, assume that
\[ u_{j-1} = \frac{P_j(0)}{P_{j-1}(0)}, \quad j \in \mathbb{N}. \]

We will analyze four cases.

Case 1. \( k_j = k_{j-1} = 2, j \in \mathbb{N}. \) In this case one gets the following relations
\[ L_j U_{j-1} = \begin{pmatrix} 0 & 0 \\ l_j u_{j-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ e_{j-1} & 0 \end{pmatrix}, \]
\[ L_j D_{j-1} + E_j U_j = \begin{pmatrix} 0 & 1 \\ l_j + u_j & e_j \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p_j(0) & -p_j(1) \end{pmatrix}, \]

Hence one obtains
\[ l_j = \frac{e_{j-1}}{u_{j-1}}, \quad e_j = -p_j(1), \]
\[ u_j = -(p_j(0) + l_j) = -p_j(0) \frac{P_j(0) - e_{j-1} P_{j-1}(0)}{P_j(0)} = \frac{P_{j+1}(0)}{P_j(0)}. \]
Case 2. Let \( k_j = 2, k_{j-1} = 1, j \in \mathbb{N} \). Then (4.9) holds true and (4.8) takes the form

\[
L_j U_{j-1} = \begin{pmatrix} 0 \\ l_j u_{j-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \epsilon_{j-1} \end{pmatrix},
\]

Hence one obtains (4.10) and (4.11).

Case 3. Let \( k_j = 1, k_{j-1} = 2, j \in \mathbb{N} \). Then

\[
(4.12) L_j U_{j-1} = \begin{pmatrix} l_j u_{j-1} \\ 0 \end{pmatrix} = \begin{pmatrix} \epsilon_{j-1} \\ 0 \end{pmatrix},
\]

\[
(4.13) L_j D_{j-1} + E_j U_j = l_j + u_j = -p^{(j)}_0.
\]

Case 4. In the case \( k_j = 1, k_{j-1} = 1, j \in \mathbb{N} \) the equality (4.13) holds true and (4.12) takes the form

\[
l_j u_{j-1} = \epsilon_{j-1}.
\]

In both cases the calculations in (4.11) are still in force.

\[\square\]

**Theorem 4.2.** Let \( \tilde{J} \) be a monic generalized Jacobi matrix associated with \( \tilde{\mathcal{F}} \in \mathcal{D}^+_{-\infty} \) and let \( \mathcal{J} = \mathcal{LU} \) be its LU-factorization of the form (4.1) - (4.5). Then the matrix \( J = UL \) is the monic Jacobi matrix associated with \( \lambda \mathcal{F}(\lambda) + s_0 \in \mathbb{N}_{-\infty} \).

**Proof.** The fact that \( J = UL \) is a monic Jacobi matrix follows by straightforward calculations. Next, notice that

\[
(4.14) \quad \mathcal{L}_{[0,j]}^T e_0 = e_0, \quad \mathcal{U}_{[0,j]} G_{[0,j]} e_0 = \alpha e_0, \quad j \in \mathbb{Z}_+,
\]

where

\[
\alpha = \begin{cases} 
\epsilon_0 u_0, & \text{if } \epsilon_0 = 1; \\
\epsilon_0, & \text{if } \epsilon_0 = 2
\end{cases}
\]

and the shortened matrices \( \mathcal{L}_{[0,j]}, \mathcal{U}_{[0,j]}, \) and \( G_{[0,j]} \) are defined analogously to (2.13). We can assume, without loss of generality, that the function \( \tilde{\mathcal{F}} \) is normalized. Then it follows from (4.6) that \( \alpha = s_1 \) both for \( \epsilon_0 = 1 \) and \( \epsilon_0 = 2 \).

It follows from (2.20) and (4.1) that for \( j \) big enough one has

\[
s_k = e_0, J_{[0,j-1]} G_{[0,j-1]} e_0 = \left( e_0, \mathcal{L}_{[0,j]}^T \mathcal{U}_{[0,j]} G_{[0,j]} e_0 \right) = \left( e_0, \underbrace{\mathcal{L}_{[0,j]}^T \mathcal{U}_{[0,j]} G_{[0,j]} e_0}_{k \text{ times}} \right) = \left( e_0, \underbrace{\mathcal{L}_{[0,j]}^T \mathcal{U}_{[0,j]} G_{[0,j]} e_0}_{k-1 \text{ times}} \right), \quad k \in \mathbb{N}.
\]

Further, using (4.14) we get

\[
s_k = \alpha \left( e_0, \left( \mathcal{U}_{[0,j]} \mathcal{L}_{[0,j]} \right)^{k-1} e_0 \right),
\]

which for sufficiently large \( j \) can be rewritten as follows

\[
\left( e_0, J_{[0,n+j-1]} e_0 \right) = \frac{s_k}{s_1}, \quad k \in \mathbb{N}.
\]

This implies that the Jacobi matrix \( J \) is associated with the normalized function \( F(\lambda) = \frac{\lambda \tilde{\mathcal{F}}(\lambda) + s_0}{s_1} \in \mathbb{N}_{-\infty} \) and, thus, also with \( \lambda \tilde{\mathcal{F}}(\lambda) + s_0 \). \[\square\]
4.2. UL-factorizations of Jacobi matrices.

**Proposition 4.3.** Let $J$ be a monic Jacobi matrix associated with $F \in \mathbb{N}_-\infty$ which has the asymptotic expansion (4.1) $s_{-1} \in \mathbb{R}$, and let $P_j(\lambda)$ and $Q_j(\lambda)$ be polynomials of the first and the second kind, respectively, associated with the sequence $s = \{s_j\}_{j=0}^{\infty}$. Then:

(i) the normal indices $n_j$ ($j \in \mathbb{N}$) of the sequence $s = \{s_{j-1}\}_{j=0}^{\infty}$ can be characterized by the conditions

$$\hat{P}_{n_j-1}(0) \neq 0, \quad j \in \mathbb{N},$$

where $\hat{P}_j(\lambda) := Q_j(\lambda) - s_{-1} P_j(\lambda)$;

(ii) the Jacobi matrix $J$ admits the following factorization

$$(4.15) \quad J = \mathcal{U} \mathcal{L},$$

where $\mathcal{L}$ and $\mathcal{U}$ are block lower and upper triangular matrices having the form (4.12)-(4.14). Moreover, the following relations hold true

$$(4.16) \quad l_j = -\frac{\hat{P}_{n_j+1-1}(0)}{\hat{P}_{n_j-1}(0)}, \quad j \in \mathbb{Z}_+.$$

**Proof.** (i) The normal indices $n_j$ ($j \in \mathbb{N}$) of the sequence $s = \{s_{j-1}\}_{j=0}^{\infty}$ can be characterized by the conditions

$$d_j := \det \begin{pmatrix} s_{-1} & s_0 & \cdots & s_{n_j-2} \\ s_0 & s_1 & \cdots & s_{n_j-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n_j-2} & s_{n_j-1} & \cdots & s_{2n_j-3} \end{pmatrix} \neq 0, \quad j \in \mathbb{N}.$$

Now the first statement follows by the equalities

$$d_j = s_{-1} \det \begin{pmatrix} s_1 & \cdots & s_{n_j-1} \\ \vdots & \ddots & \vdots \\ s_{n_j-2} & \cdots & s_{2n_j-3} \end{pmatrix} + \det \begin{pmatrix} 0 & s_0 & \cdots & s_{n_j-2} \\ s_0 & s_1 & \cdots & s_{n_j-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n_j-1} & s_{n_j+1} & \cdots & s_{2n_j-3} \end{pmatrix},$$

$$= (-1)^n_j (s_{-1} P_{n_j-1}(0) - Q_{n_j-1}(0)) \det(s_{i+k}^{\infty}_{n_j\infty} - Q_{n_j-1}(0)) \neq 0,$$

(ii) Comparing the matrix

$$\mathcal{U} \mathcal{L} = \begin{pmatrix} \mathcal{U}_0 \mathcal{E}_0 + \mathcal{D}_0 \mathcal{L}_1 & \mathcal{D}_0 \mathcal{E}_1 \\ \mathcal{U}_1 \mathcal{L}_1 & \mathcal{U}_1 \mathcal{E}_1 + \mathcal{D}_1 \mathcal{L}_2 & \mathcal{D}_1 \mathcal{E}_2 \\ \vdots & \vdots & \ddots \end{pmatrix},$$

with the matrix $\mathcal{J}$ in (1.2) one finds the entries of $\mathcal{L}$ and $\mathcal{U}$. Let us consider four cases.

**Case 1.** $\mathfrak{e}_0 = \mathfrak{e}_1 = 2$. In this case $s_{-1} = \hat{P}_0(0) = 0$, $\hat{P}_1(\lambda) \equiv 1$ and $\hat{P}_2(\lambda) = Q_2(\lambda) = \lambda - b_1$.

Since $b_1 = -\hat{P}_2(0) = 0$ the equation

$$(4.17) \quad \mathcal{U}_0 \mathcal{E}_0 + \mathcal{D}_0 \mathcal{L}_1 = \begin{pmatrix} \mathfrak{e}_0 & 1 \\ \mathfrak{u}_0 & 0 \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ \mathfrak{e}_1 & b_1 \end{pmatrix},$$
is solvable and it follows from
\[(4.18) \quad U_1 L_1 = \begin{pmatrix} 0 & I_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}\]
that
\[I_1 = c_2 = -\frac{\hat{P}_2(0)}{P_1(0)}\]

**Case 2.** $k_0 = 2, k_1 = 1$. In this case $s_{-1} = \hat{P}_0(0) = 0$. It follows from the equation
\[\begin{align*}
U_0 E_0 + D_0 L_1 &= \begin{pmatrix} e_0 & u_0 \\ 0 & I_1 \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ c_1 & b_1 \end{pmatrix},
\end{align*}\]
that
\[I_1 = b_1 = -\frac{\hat{P}_2(0)}{P_1(0)}\]

**Case 3.** In the case $k_0 = 1, k_1 = 2$ one gets $\hat{P}_0(0) \neq 0, \hat{P}_1(0) = 0$, and the equations take the form
\[\begin{align*}
U_0 E_0 + D_0 L_1 &= u_0 = b_0, \\
U_1 L_1 &= \begin{pmatrix} I_1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}.
\end{align*}\]
Hence
\[I_1 = c_1 = -\frac{\hat{P}_2(0)}{P_0(0)}\]

**Case 4.** In the case $k_0 = k_1 = 1$ one has $s_{-1} = \hat{P}_0(0) \neq 0$, and the equation
\[U_0 E_0 + D_0 L_1 = u_0 + I_1 = b_0,
\]
contains a free parameter $u_0$. Setting $u_0 = -\frac{P_0(0)}{s_{-1}}$ one obtains
\[I_1 = b_0 - u_0 = -\frac{\hat{P}_1(0)}{P_0(0)}\]

Assume now that (4.19) is satisfied for some $j \in \mathbb{N}$ and consider again four cases.  

1) $j = j - 1 = 2$. These conditions can be rewritten as follows
\[\hat{P}_{n_j-1}(0) \neq 0, \quad \hat{P}_{n_j+1}(0) \neq 0, \quad \hat{P}_{n_j}(0) = 0, \quad \hat{P}_{n_j+2}(0) = 0.
\]
This implies $b_{n_j+1} = 0$ and hence the equations
\[(4.19) \quad U_j E_j + D_j L_{j+1} = \begin{pmatrix} e_j & 1 \\ u_j & 0 \end{pmatrix} = \begin{pmatrix} b_{n_j} & 1 \\ c_{n_j} & b_{n_j+1} \end{pmatrix}, \]
\[U_1 L_1 = \begin{pmatrix} 0 & I_{j+1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_{n_j+1} \\ 0 & 0 \end{pmatrix} \]
are solvable and
\[I_{j+1} = c_{n_j+1} = -\frac{\hat{P}_{n_{j+2}-1}(0)}{\hat{P}_{n_{j+1}-1}(0)}\]

2) $j = 2, \ j = 1$. In this case
\[\hat{P}_{n_j-1}(0) \neq 0, \quad \hat{P}_{n_j}(0) = 0, \quad \hat{P}_{n_{j+1}}(0) \neq 0, \quad \hat{P}_{n_{j+2}}(0) \neq 0.
\]
It follows from the equation
\[
U_j E_j + D_j L_{j+1} = \begin{pmatrix} e_j & 1 \\ u_j & l_{j+1} \end{pmatrix} = \begin{pmatrix} b_{n_j} & 1 \\ c_{n_j} & b_{n_j+1} \end{pmatrix},
\]
that
\[
(4.21) \quad l_{j+1} = b_{n_j+1} = -\frac{\hat{P}_{n_j+2-1}(0)}{\hat{P}_{n_j+1-1}(0)}.
\]

3) In the case \(k_{j-1} = 1, k_j = 2\) one obtains
\[
\hat{P}_{n_j-1}(0) \neq 0, \quad \hat{P}_{n_j}(0) \neq 0, \quad \hat{P}_{n_j+1}(0) = 0, \quad \hat{P}_{n_j+2}(0) \neq 0.
\]
It follows that
\[
U_j E_j + D_j L_{j+1} = u_j = b_{n_j}.
\]
Hence
\[
(4.22) \quad l_{j+1} = c_{n_j+1} = -\frac{\hat{P}_{n_j+2-1}(0)}{\hat{P}_{n_j+1-1}(0)}.
\]

4) In the case \(k_{j-1} = k_j = 1\) one has
\[
\hat{P}_{n_j-1}(0) \neq 0, \quad \hat{P}_{n_j}(0) \neq 0, \quad \hat{P}_{n_j+1}(0) \neq 0,
\]
and the equations
\[
(4.23) \quad U_j E_j + D_j L_{j+1} = u_j l_j = c_{n_j-1},
\]
yield
\[
(4.24) \quad l_{j+1} = b_{n_j} - u_j = b_{n_j} - \frac{c_{n_j-1}}{l_j} = \frac{b_{n_j} \hat{P}_{n_j}(0) + c_{n_j-1} \hat{P}_{n_j-1}(0)}{\hat{P}_{n_j}(0)} = -\frac{\hat{P}_{n_j+2-1}(0)}{\hat{P}_{n_j+1-1}(0)}.
\]

This completes the proof of \((4.16)\). \(\square\)

Remark 4.4. It should be noted that, actually, both the matrices in the \(UL\)-decomposition \((4.15)\) depend on \(s_{-1}\), that is, \(U = \mathfrak{U}(s_{-1})\) and \(L = \mathfrak{L}(s_{-1})\).

Theorem 4.5. Let \(J\) be a monic Jacobi matrix associated with \(F \in \mathbb{N}_{-\infty}, s_{-1} \in \mathbb{R}\), and let \(J = \mathfrak{U}L\) be the corresponding \(UL\) factorization of \(J\) of the form \((4.2) - (4.3)\).

Then \(\tilde{J} = L\mathfrak{U}\) is the monic generalized Jacobi matrix associated with \(F(\lambda) = \frac{\lambda - s_{-1}}{\lambda} \in \mathbb{D}_{0, -\infty}\).

Proof. The fact that \(\tilde{J}\) is the monic generalized Jacobi matrix can be easily verified by straightforward calculations. The rest of the proof can be done by reversing the reasoning given in the proof of Theorem 4.2. \(\square\)

The transform \(J = \mathfrak{U}L \mapsto \tilde{J} = L\mathfrak{U}\) in Theorem 4.5 is called the Geronimus transform of the Jacobi matrix \(J\) with the parameter \(s_{-1}\).
4.3. Jacobi matrices associated with generalized Stieltjes functions. Let \( \kappa \) be a nonnegative integer. Remind that a function \( F \), meromorphic in \( \mathbb{C} \setminus \mathbb{R} \), is said to belong to the class \( N_\kappa \) if the domain of holomorphy \( \rho(F) \) of the function \( F \) is symmetric with respect to \( \mathbb{R} \), \( F(\overline{\lambda}) = \overline{F(\lambda)} \) for \( \lambda \in \rho(F) \), and the kernel
\[
\begin{cases}
N_F(\lambda, \omega) = \frac{F(\lambda) - F(\omega)}{\lambda - \omega}, & \lambda, \omega \in \rho(F); \\
N_F(\lambda, X) = F'(\lambda), & \lambda \in \rho(F)
\end{cases}
\]
has \( \kappa \) negative squares on \( \rho(F) \). The last statement means that for every \( n \in \mathbb{N} \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \rho(F) \), \( n \times n \) matrix \( (N_F(\lambda_i, \lambda_j))_{i,j=1}^n \) has at most \( \kappa \) negative eigenvalues (with account of multiplicities) and for some choice of \( n, \lambda_1, \lambda_2, \ldots, \lambda_n \) it has exactly \( \kappa \) negative eigenvalues (see [21]). Clearly, \( N_0 = \mathbb{N} \).

**Definition 4.6.** Let us say that a function \( F \) holomorphic in \( \mathbb{C}_+ \) belongs to the generalized Stieltjes class \( S^{\pm \kappa} \) if \( F(\lambda) \in \mathbb{N} \) and \( \lambda^{\pm 1}F(\lambda) \in \mathbb{N}_\kappa \). Let us set
\[
S^{\pm \kappa}_\infty = S^{\kappa} \cap \mathbb{N}_-\infty.
\]
According to [13] Theorem 2.4 any function \( F \in S^{\kappa}_\infty \) admits the integral representation
\[
F(\lambda) = \sum_{j=1}^{p} A_j \frac{1}{t_j - \lambda} + \int_0^\infty \frac{d\sigma(t)}{t - \lambda},
\]
where \( A_j > 0 \) (\( j = 1, \ldots, p \)), \( p = \kappa \) and \( \sigma \) is a finite measure on \( [0, +\infty) \), such that
\[
\int_0^\infty t^{2n}d\sigma(t) < \infty \text{ for all } n \in \mathbb{N}.
\]

Similarly, any function \( F \in S^{-\kappa}_\infty \) admits the integral representation [12], where \( A_j > 0 \) (\( j = 1, \ldots, p \)), \( \sigma \) is a finite measure on \( [0, +\infty) \), which satisfies (4.26) and
\[
p = \begin{cases} \kappa - 1, & \text{if } 0 < F(0^-) \leq \infty; \\ \kappa, & \text{if } F(0^-) \leq 0. \end{cases}
\]
This implies, in particular, that every function \( F \in S^{-\kappa}_\infty \) belongs either to \( S^{\kappa}_\infty \) or to \( S^{-\kappa+1}_\infty \).

**Corollary 4.7.** Let \( F \in S^{\kappa}_\infty \) have the asymptotic expansion (2.1), let \( J \) be a monic Jacobi matrix associated with \( F \), let \( \{n_j\}_{j=1}^{\infty} \) be the set of normal indices of the sequence \( s = \{s_{j+1}\}_{j=0}^{\infty} \), and let \( J = LU \) be its LU-factorization of the form (3.1)-(3.2). Then:

(i) the sequence \( t_j := n_{j+1} - n_j, \ j \in \mathbb{Z}_+ \) is stabilized, i.e. there is \( N > 0 \) such that \( t_j = 1 \) for \( j \geq N \);
(ii) the matrix \( \mathfrak{J} = UL \) is the monic generalized Jacobi matrix of the form
\[
\mathfrak{J} = \begin{pmatrix} \mathfrak{J}_{[0,N]} & \mathfrak{J}_{12} \\ \mathfrak{J}_{21} & \mathfrak{J}_{[N+1,\infty]} \end{pmatrix}, \quad \mathfrak{J}_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathfrak{J}_{21} = \begin{pmatrix} 0 \\ \mathbb{C}^N \end{pmatrix},
\]
\( \mathfrak{J}_{[N+1,\infty]} \) is a monic Jacobi matrix, \( \mathfrak{J}_{[0,N]} \) is a monic generalized Jacobi matrix of the form (2.7);
(iii) the matrix \( \mathfrak{J} = UL \) is associated with \( \lambda F(\lambda) + s_0 \in D^-\infty \).
5. Generalized Cholesky decomposition

Recall that the Cholesky decomposition is a decomposition of a symmetric positive-definite real matrix $A$ as the product of a lower triangular matrix $L$ and its transpose $L^\top$, that is $A = LL^\top$. The decomposition can be also rewritten in the form $A = L\Lambda L^\top$, where $\Lambda$ is a diagonal matrix and all the diagonal entries of $L$ are equal to 1. In this section, we give a generalization of the latter decomposition to an arbitrary symmetric Jacobi matrix $J_s$.

At first, note that every monic Jacobi matrix $J_s$ can be reduced to a symmetric Jacobi matrix via the following transformation

\begin{equation}
J_s = \Psi^{-1} J \Psi,
\end{equation}

where $\Psi = \text{diag}(1, \sqrt{c_0}, \sqrt{c_0 c_1}, \sqrt{c_0 c_1 c_2}, \ldots)$; here the square root is chosen to be positive. In fact, every symmetric Jacobi matrix can be represented in such a way. So, now we can reformulate Proposition 3.1 in an appropriate way.

**Proposition 5.1.** Let $J_s$ be a symmetric Jacobi matrix associated with $F \in \mathbb{N}_{-\infty}$ and let $\ell_j := n_{j+1} - n_j$, $j \in \mathbb{Z}_+$, where $n_0 = 0$ and $\{n_j\}_{j=1}^\infty$ is the set of normal indices of the sequence $s = \{s_j\}_{j=0}^\infty$ defined by the asymptotic expansion (2.3) of the function $\lambda F(\lambda) + s_0$. Then $J_s$ admits the following generalized Cholesky decomposition

\begin{equation}
J_s = L\Lambda L^\top,
\end{equation}

where $L$ is a block lower triangular matrix having the form

\[
L = \begin{pmatrix}
I_{\ell_0} & 0 & 0 \\
\hat{L}_1 & I_{\ell_1} & 0 \\
\hat{L}_2 & I_{\ell_2} & \ddots \\
& \ddots & \ddots
\end{pmatrix},
\]

in which the sub-diagonal of $L$ consists of $\ell_{j+1} \times \ell_j$ matrices

\[
\hat{L}_{j+1} = \begin{pmatrix}
\hat{\ell}_{j+1} & 0 \\
0 & \hat{\ell}_j
\end{pmatrix}, \quad \begin{pmatrix}
\hat{\ell}_{j+1} & 0 \\
0 & \hat{\ell}_j
\end{pmatrix}, \quad \begin{pmatrix}
\hat{\ell}_{j+1} & 0 \\
0 & \hat{\ell}_j
\end{pmatrix}, \quad \begin{pmatrix}
\hat{\ell}_{j+1} & 0 \\
0 & \hat{\ell}_j
\end{pmatrix},
\]

and $\Lambda = \text{diag}(\Lambda_0, \Lambda_1, \ldots)$ is a block diagonal matrix with the entries

\[
\Lambda_j = \begin{cases}
\Lambda_0^{(j)} & \text{if } \ell_j = 1; \\
0 & \Lambda_0^{(j)} \Lambda_1^{(j)} & \text{if } \ell_j = 2,
\end{cases}
\]

Moreover, the factorization (5.2) is unique.

**Proof.** Making use of (3.1) and (5.1) gives

\begin{equation}
J_s = \Psi^{-1} LU \Psi.
\end{equation}
Further, let $\Psi = \text{diag} (\Psi_0, \Psi_1, \Psi_2, \ldots)$ be a partition corresponding to the one of $L$. Then (6.3) can be rewritten as follows

\begin{equation}
\left( \begin{array}{cccc}
\sqrt{c_0} & b_0 & \sqrt{c_1} & b_1 \\
\sqrt{c_1} & b_2 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \end{array} \right) = \left( \begin{array}{cccc}
\hat{\Lambda}_0 & \hat{D}_0 & \hat{L}_1 \hat{D}_0 + \hat{\Lambda}_1 & \hat{D}_1 \\
\hat{L}_1 \hat{\Lambda}_0 & \hat{L}_1 \hat{D}_0 + \hat{\Lambda}_1 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \end{array} \right),
\end{equation}

where

$$\hat{\Lambda}_j = \Psi_j^{-1} U_j\Psi_j, \quad \hat{D}_j = \Psi_j^{-1} D_j\Psi_j, \quad \hat{L}_j = \Psi_{j+1}^{-1} L_{j+1} \Psi_j, \quad j \in \mathbb{Z}_+.$$ \hfill $\Box$

### 6. Convergence of Padé approximants for $D_{\pm\infty}^*$ functions

In this section we restrict our consideration to the case of probability measures supported on $[-1, 1]$. As a consequence, the corresponding monic classical Jacobi matrices will be bounded. To specify the results of this section, let us recall the notion of Padé approximants. Suppose we are given a formal power series $F(\lambda) = -\sum_{j=0}^{\infty} \frac{s_j}{\lambda^{j+1}}$ with $s_j \in \mathbb{R}$. Let $L, M$ be positive integers. Then an $[L/M]$ Padé approximant for $F$ is defined as a ratio

$$F^{[L/M]}(\lambda) = \frac{A^{[L/M]}(\lambda)}{B^{[L/M]}(\lambda)}$$

of polynomials $A^{[L/M]}$, $B^{[L/M]}$ of formal degree $L$ and $M$, respectively, such that $B^{[L/M]}(0) \neq 0$ and

\begin{equation}
\sum_{j=0}^{L+M-1} \frac{s_j}{\lambda^{j+1}} + F^{[L/M]}(\lambda) = O \left( \frac{1}{\lambda^{L+M+1}} \right), \quad \lambda \to \infty.
\end{equation}

More information about Padé approximants can be found in [4]. In what follows $\hat{\mathbb{C}}$ is considered as $\mathbb{C} \cup \{\infty\}$ equipped with the spherical metric.

### 6.1. Padé approximants for $D_{-\infty}^*$ functions

In a previous work we proved the following result.

**Theorem 6.1 ([4]).** Let $\sigma$ be a finite nonnegative measure on $E = [-1, \alpha] \cup [\beta, 1]$, $0 \in [\alpha, \beta]$, and let

\begin{equation}
\mathfrak{F}(\lambda) = \int_E \frac{t \sigma(t)}{t - \lambda}.
\end{equation}

Then $\mathfrak{F} \in D_{-\infty}^*$ and the sequence $\{\mathfrak{F}^{[n]/[n]}\}_{n=0}^{\infty}$ converges to $\mathfrak{F}$ locally uniformly in $\hat{\mathbb{C}} \setminus \{[-1, \alpha] \cup [\beta, 1+\varepsilon]\}$ for some $\varepsilon > 0$ if and only if the condition

\begin{equation}
\sup_{j \in \mathbb{Z}_+} \left| \frac{P_{n_j+1}(0)}{P_{n_j}(0)} \right| < \infty
\end{equation}

is fulfilled for the polynomials $P_j$ orthogonal with respect to $\sigma$. 
Remark 6.2. In [11, Section 5.3] we gave some sufficient conditions for the measure \( \sigma \) to possess the property (6.3).

This result can be interpreted in terms of the Darboux transformations as follows.

Theorem 6.3. Let \( \mathfrak{F} \in D_{-\infty} \) have the form (6.2). Then the following statements are equivalent:

(i) for the number \( s_{-1} := \int E \sigma(t) \), the operator \( U = U(s_{-1}) \) in the factorization (3.19) of \( \mathfrak{F} \) corresponding to \( \mathfrak{F} \) is bounded in \( \ell^2_{[0, \infty)} \);

(ii) the operator \( U \) in the factorization (6.1) of \( J \) corresponding to the function

\[
F(\lambda) = \int_E \frac{d\sigma(t)}{t - \lambda}
\]

is bounded in \( \ell^2_{[0, \infty)} \);

(iii) the sequence \( \{\mathfrak{F}^{[n,j]/n_j}\}_{j=0}^{\infty} \) of diagonal Padé approximants converges to \( \mathfrak{F} \) locally uniformly in \( \hat{C} \setminus ([-1 - \varepsilon, \alpha] \cup [\beta, 1 + \varepsilon]) \) for some \( \varepsilon \geq 0 \).

Proof. Clearly, the first statement is equivalent to the second one by construction. So, it remains to prove the equivalence of (ii) and (iii). First, suppose that (iii) holds true and, therefore, from Theorem 6.1 we see that (6.3) is fulfilled. Then, according to (3.8), (3.10), (3.13), and (3.15), the operator \( U \) is bounded. The implication from (ii) to (iii) follows from formulas (3.13), (3.15) and Theorem 6.1. \( \square \)

Remark 6.4. Under the condition of Theorem 6.3, from (3.8), (3.10), (3.11), (3.13), and (3.14) we see that the operators \( U \) and \( L \), given by (3.1), are bounded or unbounded simultaneously.

In some cases, we can specialize Theorem 6.3.

Corollary 6.5. Let \( \mathfrak{F} \) admit the following representation

\[
\mathfrak{F}(\lambda) = \int_0^1 t d\sigma(t) + \sum_{j=1}^{p} a_j t_j, \quad t_j \leq 0, \quad a_j \geq 0 \quad \text{for} \quad j = 1, \ldots, p.
\]

Then the following statements hold true:

(i) the sequence \( \{\mathfrak{F}^{[n,j]/n_j}\}_{j=0}^{\infty} \) of diagonal Padé approximants converges to \( \mathfrak{F} \) locally uniformly in \( \hat{C} \setminus ([0, 1] \cup \{t_k\}_{k=1}^{p}) \);

(ii) the polynomials \( P_j \) orthogonal with respect to \( \sigma + \sum_{j=1}^{p} a_j \Theta(\cdot - t_j) \) possess the following property

\[
\sup_{j \in \mathbb{Z}_+} \left| \frac{P_{n_j+1}(0)}{P_{n_j}(0)} \right| < \infty
\]

(here \( \Theta \) denotes the Heaviside step function);

(iii) for the number

\[
s_{-1} := \int_0^1 d\sigma(t) + \sum_{j=1}^{p} a_j,
\]
the operator $U = U(s_{-1})$ in the factorization (3.19) of $\tilde{\mathcal{J}}$ corresponding to $\tilde{\mathfrak{a}}$ is bounded in $l^2_{[0, \infty)}$;

(iv) the operator $U$ in the factorization (3.1) of $J$ corresponding to the function

\begin{equation}
F(\lambda) = \int_0^1 \frac{d\sigma(t)}{t - \lambda} + \sum_{j=1}^p \frac{a_j}{t_j - \lambda}
\end{equation}

is bounded in $l^2_{[0, \infty)}$.

Proof. Clearly, $\tilde{\mathfrak{a}} \in D_{-\infty}$. In fact, statement (i) for the class in question was proved in \cite{27}. Statement (ii) is an immediate consequence of statement (i) and Theorem 6.1. The rest follows from Theorem 6.3. \hfill \Box

Remark 6.6. As was mentioned in Subsection 4.3 the function $F$ in (6.5) belongs to the class $S^{-p}$.

6.2. Padé approximants for $D_{-\infty}^+$ functions. In this section we present convergence results for diagonal Padé approximants to $D_{-\infty}^+$ functions.

By re-examining \cite[Theorem 4.16]{10}, \cite[Theorem 3.3]{11} and \cite[Theorem 5.5]{11}, we arrive at the following more general result.

Proposition 6.7. Let

\begin{equation}
F(\lambda) = \int_{-1}^1 \frac{d\sigma(t)}{t - \lambda}
\end{equation}

Then for any sequence $\{\tau_j\}_{j=1}^\infty$ of real numbers the following relation holds true

\begin{equation}
\hat{F}^{|n_j/n_j|}(\lambda) := -\frac{Q_{n_j}(\lambda)}{P_{n_j}(\lambda)} + \tau_j \frac{Q_{n_j-1}(\lambda)}{P_{n_j-1}(\lambda)} - \sum_{i=0}^{2n_j-2} \frac{s_i}{\lambda^{i+1}} + O\left(\frac{1}{\lambda^{2n_j}}\right), \quad \lambda \to \infty,
\end{equation}

where $P_j$ and $Q_j$ are polynomials of the first and second kind, respectively, corresponding to $\sigma$. Moreover, the sequence of the modified Padé approximants $\hat{F}^{|n_j/n_j|}$ converges to $F$ locally uniformly in $\hat{\mathcal{G}} \setminus ([-1 + \varepsilon, 1 - \varepsilon])$ for some $\varepsilon \geq 0$ if and only if the condition

\begin{equation}
\sup_{j \in \mathbb{N}} |\tau_j| < \infty
\end{equation}

is satisfied.

Proof. The relation (6.6) is an appropriate reformulation of the results of \cite[Section I.4.2]{11}. Next, notice that the symmetric Jacobi matrix $J$ associated with $\sigma$ is a bounded linear operator in $l^2_{[0, \infty)}$. Furthermore, using the classical versions of formulas (2.14), (2.17), and (2.18), the modified Padé approximant can be rewritten as follows

\begin{equation}
\hat{F}^{|n_j/n_j|}(\lambda) = -\frac{Q_{n_j}(\lambda)}{P_{n_j}(\lambda)} + \tau_j \frac{Q_{n_j-1}(\lambda)}{P_{n_j-1}(\lambda)} - \frac{\det(\lambda - J^{|n_j|}_{[1,n_j-1]})}{\det(\lambda - J^{|n_j|}_{[0,n_j-1]})} \left( (J^{|n_j|}_{[0,n_j-1]} - \lambda)^{-1} e, e \right),
\end{equation}

is bounded in $l^2_{[0, \infty)}$.\hfill \Box
where \( \mathcal{J}_{[0,n_j-1]}^{(\tau_j)} \) is a rank-one perturbation of \( \mathcal{J}_{[0,n_j-1]} \) of the form

\[
\mathcal{J}_{[0,n_j-1]}^{(\tau_j)} = \begin{pmatrix}
  b_0 & \sqrt{c_0} & & \\
  \sqrt{c_0} & \ddots & & \\
  & \ddots & \ddots & \\
  & & \sqrt{c_{n_j-2}} & b_{n_j-1} + \tau_j
\end{pmatrix} = \mathcal{J}_{[0,n_j-1]} + \text{diag} \{0, \ldots, 0, \tau_j\}
\]

(cf. \cite{28} Proposition 5.8). Now, let us suppose that \( \mathcal{J}_{[0,n_j-1]} \) is satisfied. Then one obtains

\[
\|\mathcal{J}_{[0,n_j-1]}^{(\tau_j)} - \lambda\|^{-1} \leq \frac{1}{|\lambda| - \|\mathcal{J}_{[0,n_j-1]}^{(\tau_j)}\|} (|\lambda| > \|\mathcal{J}_{[0,n_j-1]}\|)
\]

and \( \mathcal{J}_{[0,n_j-1]}^{(\tau_j)} \) that

\[
|\mathcal{F}^{[n_j/n_j]}(\lambda) - (\mathcal{J}_{[0,n_j-1]}^{(\tau_j)} - \lambda)^{-1} e, e\|_2 \leq \frac{1}{|\lambda| - 1 - \varepsilon}.
\]

for \( |\lambda| > 1 + \varepsilon \). The pointwise convergence \( \mathcal{F}^{[n_j/n_j]}(\lambda) \) to \( F(\lambda) \) for \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) follows from the proof of \cite{28} Proposition 5.3]. Finally, it remains to apply the Vitali theorem.

Let us prove the necessity by proving the contrary statement. So, suppose that

\[
\sup_{j \in \mathbb{N}} |\tau_j| = \infty.
\]

Further, note that the poles of the modified Padé approximant \( \mathcal{F}^{[n_j/n_j]} \) coincide with eigenvalues of the matrix \( \mathcal{J}_{[0,n_j-1]}^{(\tau_j)} \). Taking into account that \( \mathcal{J}_{[0,n_j-1]}^{(\tau_j)} \) is a self-adjoint matrix, one obtains

\[
|\lambda_{\max}(\mathcal{J}_{[0,n_j-1]}^{(\tau_j)})| = \|\mathcal{J}_{[0,n_j-1]}^{(\tau_j)}\| \geq |(\mathcal{J}_{[0,n_j-1]}^{(\tau_j)} e_{n_j-1}, e_{n_j-1})| = |b_{n_j-1} + \tau_j|,
\]

where \( \lambda_{\max}(\mathcal{J}_{[0,n_j-1]}^{(\tau_j)}) \) is the eigenvalue of the matrix \( \mathcal{J}_{[0,n_j-1]}^{(\tau_j)} \) with the largest absolute value. Since the sequence \( \{b_k\}_{k=0}^{\infty} \) is bounded, we have that infinity is an accumulation point of the set of all poles of the modified Padé approximants \( \mathcal{F}^{[n_j/n_j]} \).

\[\square\]

**Theorem 6.8.** Let \( \sigma \) be a finite nonnegative measure on \([-1, 1]\) and let

\[
\mathfrak{F}(\lambda) = \frac{1}{\lambda} \int_{-1}^{1} \frac{d\sigma(t)}{t - \lambda} - \frac{s_{-1}}{\lambda},
\]

where \( s_{-1} \) is a fixed real number. Then \( \mathfrak{F} \in \mathbb{D}^{\pm} \) and the sequence \( \{\mathfrak{F}^{[n_j/n_j]}\}_{j=0}^{\infty} \) converges to \( \mathfrak{F} \) locally uniformly in \( \mathbb{C} \setminus ([-1 - \varepsilon, 1 + \varepsilon]) \) for some \( \varepsilon \geq 0 \) if and only if the condition

\[
\sup_{j \in \mathbb{Z}^+} \left| \frac{Q_{n_j}(0) - s_{-1} P_{n_j}(0)}{Q_{n_j-1}(0) - s_{-1} P_{n_j-1}(0)} \right| < \infty
\]
is fulfilled for the polynomials $P_j$ and $Q_j$ of the first and second kind corresponding to $\sigma$.

**Proof.** Observe that by setting

$$
\tau_j = \frac{Q_{n_j}(0) - s_{-1}P_{n_j}(0)}{Q_{n_j-1}(0) - s_{-1}P_{n_j-1}(0)}
$$

we have that $\mathfrak{F}^{[n_j/n_j]}(0) = s_{-1}$. Thus, due to (6.4) and (6.6) one has the following representation of the diagonal Padé approximants to $\mathfrak{F}$

$$
\mathfrak{F}^{[n_j/n_j]}(\lambda) = \frac{1}{\lambda} \left( \frac{Q_{n_j}(\lambda) + \tau_j Q_{n_j-1}(\lambda)}{P_{n_j}(\lambda) + \tau_j P_{n_j-1}(\lambda)} - s_{-1} \right).
$$

Now the statement is an immediate consequence of Proposition 6.7.

The above result can be reformulated in terms of the Darboux transformations as follows.

**Theorem 6.9.** Let $\mathfrak{F} \in D^{+}_{-\infty}$ have the form (6.2). Then the following statements are equivalent:

(i) the operator $L$ in the factorization (4.11) of $\mathfrak{F}$ corresponding to $\mathfrak{F}$ is bounded in $L^2_{[0, \infty)}$;

(ii) the operator $L = \Sigma(s_{-1})$ in the factorization (4.15) of $J$ with the parameter $s_{-1} = \varepsilon_0$ corresponding to the function

$$
F(\lambda) = \int_E \frac{d\sigma(t)}{t - \lambda}
$$

is bounded in $L^2_{[0, \infty)}$;

(iii) the sequence $\{\mathfrak{F}^{[n_j/n_j]}\}_{n_j=0}^{\infty}$ of diagonal Padé approximants converges to $\mathfrak{F}$ locally uniformly in $\hat{C} \setminus ([-1 - \varepsilon, 1 + \varepsilon])$ for some $\varepsilon \geq 0$.

**Proof.** Obviously, the first statement is equivalent to the second one by construction. So, the equivalence of (ii) and (iii) proves the theorem. First, suppose that (iii) holds true and, therefore, from Theorem 6.8 we see that (6.11) is fulfilled. Then, according to (4.20), (4.21), (4.22), and (4.24), the operator $L$ is bounded. The implication from (ii) to (iii) follows from formulas (4.21), (4.24) and Theorem 6.8.

In some cases, we can specialize Theorem 6.9.

**Corollary 6.10.** Let $\mathfrak{F}$ admit the following representation

$$
\mathfrak{F}(\lambda) = \frac{1}{\lambda} \int_0^1 \frac{d\sigma(t)}{t - \lambda} + \frac{1}{\lambda} \sum_{j=1}^{p} \frac{a_j}{t_j - \lambda} - \frac{s_{-1}}{\lambda},
$$

where $\sigma$ is a finite nonnegative measure on $[0, 1]$, $t_j < 0$, $a_j \geq 0$ for $j = 1, \ldots, p$, and $s_{-1}$ is a fixed real number. Then the following statements hold true:

(i) the sequence $\{\mathfrak{F}^{[n_j/n_j]}\}_{n_j=0}^{\infty}$ of diagonal Padé approximants converges to $\mathfrak{F}$ locally uniformly in $\hat{C} \setminus ([0, 1] \cup \{t_k\}_{k+1})$;
(ii) the polynomials $P_j$ and $Q_j$ corresponding to $\sigma + \sum_{j=1}^{p} a_j \Theta(\cdot - t_j)$ possess the following property
\[
\sup_{j \in \mathbb{Z}^+} \left| \frac{Q_{n_j}(0) - s_{-1} P_{n_j}(0)}{Q_{n_j-1}(0) - s_{-1} P_{n_j-1}(0)} \right| < \infty;
\]
(iii) the operator $\varLambda$ in the factorization (4.1) of $\mathfrak{J}$ corresponding to $\mathfrak{F}$ is bounded in $\ell^2_{[0,\infty)}$;
(iv) the operator $L = L(s_{-1})$ in the factorization (4.15) of $J$ with the parameter $s_{-1} = s_0$ corresponding to the function
\[
F(\lambda) = \int_{-1}^{1} \frac{d\sigma(t)}{t - \lambda} + \sum_{j=1}^{p} \frac{a_j}{t_j - \lambda}
\]
is bounded in $\ell^2_{[0,\infty)}$.

**Proof.** Clearly, $\mathfrak{F} \in D_{-}\infty$. Actually, statement (i) for the class in question was proved in [10]. Statement (ii) is an immediate consequence of statement (i) and Theorem 6.8. The rest follows from Theorem 6.9. \qed

**Remark 6.11.** 1) In fact, in [10] we proved the convergence results for the Padé approximants to $N_\kappa$-functions having the asymptotic expansion (2.1).

2) As follows from (4.25), (4.27) the function $F$ in (6.13) belongs to the class $S_{-\kappa}$, where $\kappa = p$, if $F(0^-) = \int_{-1}^{1} \frac{d\sigma(t)}{t} + \sum_{j=1}^{p} \frac{a_j}{t_j} \leq 0$, and $\kappa = p+1$, if $0 < F(0^-) \leq \infty$.

7. **Examples**

7.1. **The appearance of the block structure.** In this subsection we give a family of Jacobi matrices such that the Christoffel transform of these matrices leads to block tridiagonal matrices.

Let us consider a sequence $\{P_j\}_{j=0}^{\infty}$ of monic orthogonal polynomials satisfying the following three term recurrence relation
\[
\lambda P_j(\lambda) = P_{j+1}(\lambda) + c_{j-1} P_{j-1}(\lambda), \quad j \in \mathbb{Z}^+,
\]
where $c_j > 0$ and $P_{-1}(\lambda) = 0$, $P_0(\lambda) = 1$. For example, if $c_j = 1/4$, $j \in \mathbb{Z}^+$, then $P_j$ are Chebyshev polynomials of second kind. Clearly, the polynomials $P_j$ are symmetric, i.e.
\[
P_j(-\lambda) = (-1)^j P_j(\lambda), \quad j \in \mathbb{Z}^+.
\]
Besides, the corresponding monic Jacobi matrix has the following form
\[
J = \begin{pmatrix}
0 & 1 \\
c_0 & 0 & 1 \\
& c_1 & 0 & \ddots \\
& & \ddots & \ddots
\end{pmatrix}
\]
and is associated with the Markov function
\[
F(\lambda) = \int_{-1}^{1} \frac{d\sigma(t)}{t - \lambda},
\]
where $d\sigma$ is a symmetric measure.

It follows from (7.1) that $P_{2j+1}(0) = 0$ and $P_{2j}(0) \neq 0$ for every $j \in \mathbb{Z}^+$. The latter fact immediately implies that $n_j = 2j$, $j \in \mathbb{Z}^+$, and, thus, $\xi_j = 2$ for $j \in \mathbb{Z}^+$. 
Now, Proposition\textsuperscript{3.1} gives the $LU$-factorization $J = LU$, where $L$ and $U$ are block lower and upper triangular matrices having the forms

$$L = \begin{pmatrix} I_{t_0} & 0 & 0 \\ L_1 & I_{t_1} & 0 \\ & \ddots & \ddots \end{pmatrix}, \quad U = \begin{pmatrix} U_0 & D_0 & D_1 \\ 0 & U_1 & D_1 \\ & \ddots & \ddots \end{pmatrix},$$

which entries are as follows

$$L_{j+1} = \begin{pmatrix} c_{2j+1} & 0 \\ 0 & 0 \end{pmatrix}, \quad U_j = \begin{pmatrix} 0 & 1 \\ c_{2j} & 0 \end{pmatrix}, \quad D_j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad j \in \mathbb{Z}_+.$$

Multiplying $L$ and $U$ the other way around, we arrive at the monic generalized Jacobi matrix

$$\tilde{J} = UL = \begin{pmatrix} \mathcal{B}_0 & \mathcal{D}_0 \\ \mathcal{C}_0 & \mathcal{B}_1 & \mathcal{D}_1 \\ & \mathcal{C}_1 & \mathcal{B}_2 & \ddots \end{pmatrix},$$

where the entries have the following form

$$\mathcal{B}_j = \begin{pmatrix} 0 & 1 \\ c_{2j} & 0 \end{pmatrix}, \quad \mathcal{C}_j = \begin{pmatrix} 0 & 0 \\ c_{j+1}c_{j+2} & 0 \end{pmatrix}, \quad \mathcal{D}_j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad j \in \mathbb{Z}_+.$$

Finally, observe that $F$ can be represented as follows

$$F(\lambda) = \lambda \int_0^1 \frac{d\rho(t)}{t - \lambda^2} \in \mathbb{N}.$$

Thus, due to Theorem\textsuperscript{3.2} we have that $\tilde{J}$ is associated with the function

$$\lambda F(\lambda) + 1 = \lambda^2 \int_0^1 \frac{d\rho(t)}{t - \lambda^2} + 1 = \int_0^1 \frac{td\rho(t)}{t - \lambda^2} \in \mathcal{D}_{-\infty}.$$

7.2. \textbf{Unboundedness of the Christoffel transform.} Here we give an example of a bounded Jacobi matrix such that its Christoffel transform is an unbounded matrix.

Consider the following monic 2-periodic Jacobi matrix (see \cite[Example 1]{11})

$$J = \begin{pmatrix} a_0 & 1 \\ 1 & a_1 & 1 \\ & 1 & a_2 & \ddots \end{pmatrix}, \quad a_n = \frac{(-1)^n + 1}{2}, \quad n \in \mathbb{Z}_+,$$

associated with the function

$$F(\lambda) = -\frac{1}{2} + \frac{1}{2} \sqrt{\frac{\lambda^2 - \lambda - 4}{\lambda - 1}},$$

where the branch is chosen so that $F(\lambda) \to 0$ as $\lambda \to \infty$ (see \cite{11}).
Let us find the $LU$-factorization of $J$, that is, let us represent $J$ as follows

$$J = LU = \begin{pmatrix} u_1 & 1 & 1 \\ u_1l_1 & u_2 + l_1 & 1 \\ u_2l_2 & u_3 + l_2 & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ \end{pmatrix}.$$ 

Clearly, we have the following relations

$$u_1 = 1, \quad u_kl_k = 1, \quad u_{2k} + l_{2k-1} = 0, \quad u_{2k+1} + l_{2k} = 1, \quad k \in \mathbb{N}.$$ 

Next, by induction we have that

$$u_1 = 1, \quad l_1 = 1, \quad u_2 = -1, \quad l_2 = -1,$$

$$u_{2k+1} = k + 1, \quad u_{2k+2} = -\frac{1}{k+1}, \quad l_{2k+1} = \frac{1}{k+1}, \quad l_{2k+2} = -(k+1), \quad k \in \mathbb{N}.$$ 

Therefore, the operators $L$ and $U$ in $LU$-factorization of $J$ are unbounded. Moreover, the Christoffel transformation

$$J_C = UL = \begin{pmatrix} u_1 + l_1 & 1 \\ u_2l_1 & u_2 + l_2 & 1 \\ u_3l_2 & u_3 + l_3 & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \\ \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -2 & 1 \\ -2 & 5/2 & \ddots & \ddots \\ \end{pmatrix}$$

is also an unbounded monic generalized Jacobi matrix associated with the function

$$\lambda F(\lambda) + 1 = -\frac{\lambda^2}{2} + 1 + \frac{\lambda}{2} \sqrt{\frac{\lambda(\lambda^2 - \lambda - 4)}{\lambda - 1}} \in D_{-\infty}.$$ 

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