Quanta without quantization

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Abstract

The dimensional properties of fields in classical general relativity lead to a tangent tower structure which gives rise directly to quantum mechanical and quantum field theory structures without quantization.

We derive all of the fundamental elements of quantum mechanics from the tangent tower structure, including fundamental commutation relations, a Hilbert space of pure and mixed states, measurable expectation values, Schrödinger time evolution, “collapse” of a state and the probability interpretation.

The most central elements of string theory also follow, including an operator valued mode expansion like that in string theory as well as the Virasoro algebra with central charges.

Introduction  The detailed structure of quantum systems follows by fully developing the scaling structure of spacetime. Thus quantum systems - despite their Hilbert space of states, operator-valued observables, interfering complex quantities, and probabilities - are rendered in terms of classical spacetime variables which simultaneously form a Lie algebra of operators on the space of conformal weights.

This remarkable result follows from the tangent tower structure implicit in general relativity. We develop this structure, cite a central theorem, then examine some properties of tensors over the tangent tower. Finally, we apply these ideas to quantum mechanics and string theory, showing how the core elements of both arise classically.
The weight tower and weight maps Consider a spacetime \((M, g)\) with
dynamical fields, \(\{\Phi_A \mid A \in A\}\) of various conformal weights and
tensorial types. Under a global change of units by \(\lambda\), let
\[
\Phi_A \rightarrow (\lambda)^{w_A} \Phi_A
\]
where \(w_A\) is called the conformal weight of \(\Phi_A\). Normally, physicists simply
insert these scale factors by hand when needed, without mentioning the im-
licit mathematical structures their use requires, but these structures turn
out to be interesting and important.

The tower structure begins with the set of conformal weights,
\(W \equiv \{w_A \mid A \in A\}\), which must be closed under addition of any two different elements.
Possible sets include the reals \(R\), the rationals \(Q\), the integers \(J\), and the
finite set \(\{0, 1, -1\}\). For most physical problems we can choose the unit-
weight objects to give \(W = J\).

Next, define the equivalence relation
\[
\Phi_A \cong \Phi_B \text{ if } w_A = w_B \text{ and } \exists \eta \text{ such that } \Phi_A = \eta \Phi_B
\]
which partitions the tangent space \(TM\) into a tower of projective Minkowski
spaces, \(PM\), one copy for each \(n \in J = W\). This partition enlarges the
linear transformation group of the tangent space into the direct product of
the Lorentz group and the group of weight maps. While the Lorentz trans-
formations have their usual effect, general weight maps act on conformal
weight. To see that the group is a direct product, consider the product of
an \(n\)-weight scalar field with an \(m\)-weight vector field. Since the linear trans-
formations preserve the Lorentz inner product between arbitrarily weighted
vectors, the resulting \((n + m)\)-weight vector field remains parallel to the orig-
inal \(m\)-weight vector field under Lorentz transformation. Therefore, Lorentz
transformations map different weight vectors in the same way.

We now investigate weight maps. Just as the only measurable Lorentz
objects are scalars, the only measurable tangent tower quantities are zero
weight scalars\(^1\). Therefore, to readily form zero weight scalars, we classify
weight maps and tensors over \(W\) by their conformal weights.

The generator of global scale changes, \(D\), determines the weights of fields
according to
\[
D \Phi_A = w_A \Phi_A
\]
\(^1\)Eg., we may measure the dimensionless ratio of the length of a table to the length of
a meter stick.
\(^2\)In conformal geometries, \(D\) generates dilations.
while the generators $M_\alpha$ of definite-weight maps satisfy
\[ [D, M_\alpha] = n_\alpha M_\alpha \]
where $n_\alpha \in J$. Then $M_\alpha$ maps $n$-weight fields to $(n+n_\alpha)$-weight fields, making the construction of 0-weight quantities straightforward.

The following theorem now holds [1]:

**Theorem.** Let $V$ be a maximally non-commuting Lie algebra consisting of exactly one weight map of each conformal weight. Then $V$ is the Virasoro algebra with central charge,
\[ [M_{(m)}, M_{(n)}] = (m - n)M_{(m+n)} + cm(m^2 - 1)\delta^0_{m+n}1 \]

The lengthy proof relies on explicit construction through a series of inductive arguments. It is highly significant to note that the real-projective tangent tower produces the same central charge for $V$ as unitarily-projective string theory, despite its classical character. This is the first concrete evidence that some phenomena widely regarded as “quantum” can be understood from a classical standpoint.

**Tensors over the weight tower** Having understood the algebraic character of definite-weight weight maps, we next look at the tensors they act on. Since Lorentz transformations decouple from weight maps, the Lorentz and conformal ranks are independent. Thus
\[ T_{n_1 \ldots n_s}^{a_1 \ldots a_r} : (PM)^r \otimes J^s \rightarrow R \]
is a typical rank $(r,s)$ tensor. Weight maps act linearly on each label $n_i$.

Particularly relevant to quantum systems are $(0,1)$ tensors. With $\eta_k$ a $k$-weight scalar, $D\eta_k = k\eta_k$, a general $(0,1)$ tensor is an indefinite-weight linear combination
\[ \Phi = \sum_{k=-\infty}^{\infty} \phi_k \eta_k \]
We immediately see the need for some convergence criterion. Imposing a norm provides such a criterion, and with a norm these objects form a Hilbert space, $H$. The simplest norm uses a continuous representation for the $(0,1)$ tensors defined by
\[ \Phi(x) \equiv \sum_{n=-\infty}^{\infty} \phi_n e^{inx}. \]
Clearly, $\Phi(x)$ exists only under appropriate convergence conditions, provided by including as the $(0,1)$ tensors those vectors satisfying

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(x)\bar{\Phi}(x) < \infty \quad (9)$$

Using the product

$$(MN)(x,y) = \int_{-\pi}^{\pi} M(x,z)N(z,y)dz \quad (10)$$

we straightforwardly find the representations

$$D(x,y) = -i \frac{\partial}{\partial x} \delta(x-y) + c1 \quad (11)$$

for $D$ and

$$M_{(k)}(x,y) = e^{ikx}(-i\partial_x - k)\delta(x-y) \quad (12)$$

for the $k$-weight Virasoro operator. In eq.(11), $D(x,y)$ requires the central distribution $c$ to cancel surface terms from the product integral. Such terms are an artifact of the continuous representation.

**Physical effects of the tangent tower**  Now we describe physical effects of the tangent tower. We replace the usual $(r,0)$ tensors of quantum mechanics and quantum field theory by $(r,1)$ or $(r,2)$ tensors. In the remaining two sections we do this in detail for quantum mechanics, then briefly for string theory.

**Conformal weight and quantum mechanics**  The tangent tower underpinning of all axiomatic features of quantum mechanics now follows immediately: commutation relations of operator-valued position and momentum vectors, a Hilbert space of pure and mixed states, measurable expectation values, Schrödinger time evolution, “collapse” of a state and the probability interpretation.

First, consider canonical commutators. Since $w_x = 1$ and $w_{p/h} = -1$, canonical coordinates lie in the subalgebra determined by $W_0 = \{0,1,-1\} \subset W$. We temporarily consider operators in this subspace, replacing symplectic coordinates $Q^A = (q^i, \pi_j)$ by weight-map-valued 6-vectors $\hat{Q}^A = (\hat{q}^i, \hat{\pi}_j) \in T_{nm}^a$ forming the Lie algebra

$$[\hat{Q}^A, \hat{Q}^B] = c^{AB}1 \quad (13)$$
where projective representation allows arbitrary central charges $c^{AB}$. The only invariant antisymmetric symplectic tensor is the symplectic 2-form

$$\Omega^{AB} = \begin{pmatrix} 0 & -\delta^i_j \\ \delta^i_j & 0 \end{pmatrix}$$

(14)

so we set

$$[\hat{Q}^A, \hat{Q}^B] = \Omega^{AB}$$

(15)

Furthermore, the natural symplectic metric

$$K^{AB} = \begin{pmatrix} 0 & \delta^i_j \\ \delta^i_j & 0 \end{pmatrix}$$

(16)

may be diagonalized by a symplectic transformation to new variables $\hat{R}^A = (\hat{X}^i, \hat{P}^j)$ such that

$$\tilde{K}^{AB} = \begin{pmatrix} \delta_{ij} & -\delta^{ij} \\ \delta^{ij} & \delta_{ij} \end{pmatrix}$$

(17)

or equivalently $\hat{R}^{tA} = (\hat{X}^i, i\hat{P}^j)$ with

$$\tilde{K}^{tAB} = \begin{pmatrix} \delta_{ij} & \delta^{ij} \\ \delta^{ij} & \delta_{ij} \end{pmatrix}$$

(18)

The zero signature of the symplectic metric requires an imaginary unit in relating $\hat{R}^{tA}$ to $\hat{P}^j$ because the physical $\hat{X}^i$ and $\hat{P}^j$ have the same metric, $+\delta_{ij}$. The projective algebra of $\hat{X}^i$ and $\hat{P}^j$ is the canonical one

$$[\hat{X}^i, \hat{P}^j] = i\delta^{ij}$$

(19)

$$[\hat{X}^i, \hat{X}^j] = [\hat{P}^i, \hat{P}^j] = 0$$

(20)

Thus, “quantum” $(\hat{X}^i, \hat{P}^j)$ commutators follow from classical scaling considerations by replacing classical variables with weight tower operators in the \{0, 1, −1\} subalgebra, and using the natural symplectic structure.

Other quantum structures follow easily. Thus, definite weight scalars replace pure quantum states, while indefinite weight objects such as $\Phi(x) \in H$ replace mixed states. The construction of expectation values is simply a rule to generate a 0-weight scalar. The rule works by matching elements of
$H$ with their complex conjugates but the definition of $\Phi(x)$ above translates this into the manifestly 0-weight sum

\[
\langle \Phi, \Psi \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(x) \overline{\Psi(x)} dx
\]

(21)
\[
= \sum \phi_n \psi_{-n}
\]

(22)

Finally, consider the time evolution of $\Phi(x)$. In standard quantum mechanics, states evolve \textit{continuously} via the Schrödinger equation and \textit{discontinuously} during measurement. Conformal properties account for both processes.

\textbf{Continuous evolution} of weighted quantities is described by parallel transport, where the metric-compatible covariant derivative $\nabla$ must be augmented by the $(1,2)$ gauge vector $W_\mu$ of the \textit{full} tangent tower symmetry. With the weight-map-valued 1-form $W = W_\mu dx^\mu$ exact, the geometry remains Riemannian, but having $W$, we can write expressions which are manifestly scale-invariant, even locally. Parallel transport of a $(0,1)$ tensor obeys

\[
u^\mu \nabla_\mu \Phi(x'; x) = -u^\mu W_\mu \Phi(x'; x)
\]

(23)

where $u^\mu \nabla_\mu$ reduces to $\frac{d}{d\tau}$ in flat spacetime while $u^\mu W_\mu \Phi$ gives the action of a weight map $H \equiv u^\mu W_\mu$ on the weight superposition $\Phi(x'; x)$. Identifying $H$ as the Hamiltonian operator [2], eq.(22) becomes the Schrödinger equation. The free-particle form for $H$ is the zero-weight quantity

\[
H = \left( \frac{d\hat{X}_j}{d\tau} \right) \left( i\hat{P}_j \right)
\]

(24)

so the imaginary unit enters correctly.

\textbf{Discrete evolution} and the probability interpretation are natural outcomes of measuring \textit{properties} with definite conformal weight. Such measurement involves projecting a superposition $\Phi(x)$ into an object with, eg., units of length (giving $\phi_1$), inverse area (giving $\phi_{-2}$), etc. What we actually measure are dimensionless ratios, $\frac{\phi_1}{L}$, $L^2 \phi_{-2}$, etc., with $L$ some standard unit of length.

Therefore the rules of quantum mechanics have natural classical interpretations in terms of the scale-invariance properties of spacetime.
Conformal weight and quantum field theory Just as field theory emerges as the limiting case of multiparticle dynamics, and quantum field theory emerges as a blend of quantum mechanics and special relativity [3], we can imagine retracing the preceding arguments to derive the principles of quantum field theory. After all, the quantum state is a field, even in quantum mechanics. Thus, we may anticipate a classical tangent tower interpretation of quantum field theory. As striking as this conjecture seems, we now demonstrate a more striking claim: the tangent tower contains the essential elements of string theory [4].

Minor manipulations of a definite weight $(1, 2)$ tensor $\alpha^\mu_{(k)}(x, y)$ found by writing the $\delta$-function in $D$ as

$$\delta(x - y) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(x-y)}$$

lead to

$$\alpha^\mu_{(k)}(x, y) = (2\pi)^{-1} i k x \sum_m \left[ \alpha^\mu_m e^{im(x-y)} + \tilde{\alpha}^\mu_m e^{im(x+y)} \right]$$

with the 0-weight $(1, 2)$ tensors given by the Fourier series

$$\alpha^\mu(x, y) = (2\pi)^{-1} \sum_m \left[ \alpha^\mu_m e^{im(x-y)} + \tilde{\alpha}^\mu_m e^{im(x+y)} \right]$$

This expression has the form of a string mode expansion, with each mode being a Hilbert space operator. Viewing the two modes as left and right moving waveforms, gives the abstract $(x, y)$ space a Lorentz norm. Under mild assumptions [5], time-oriented embeddings of the abstract space into spacetime exist, and have the properties of string world sheets.

Nothing is more central to the theory of quantized string than these mode operators. Through the Virasoro algebra they determine the physical states. Mode operators give a representation of the Poincaré generators thereby classifying those states by mass and spin. Ultimately this gives a gauge theory of a massless spin-2 mode governed at lowest order by the Einstein-Hilbert action. Additionally, the Virasoro central charge determines the critical dimension of spacetime. The existence of heterotic and supersymmetric representations of the symmetry eliminates tachyons, guarantees fermions and provides a large internal nonabelian gauge symmetry.

All of these string properties may now be studied as properties of $(1, 2)$ tensors on the conformal tangent tower of spacetime.
References

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