Local characterization of polyhedral spaces

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Abstract
We show that a compact length space is polyhedral if a small spherical neighborhood of any point is conic.

1 Introduction

In this note we characterize polyhedral spaces as the spaces where every point has a conic neighborhood. Namely, we prove the following theorem; see Section 2 for all necessary definitions.

1.1. Theorem. A compact length space $X$ is polyhedral if and only if a neighborhood of each point $x \in X$ admits an open isometric embedding to Euclidean cone which sends $x$ to the tip of the cone.

Note that we do not make any assumption on the dimension of the space. If the dimension is finite then the statement admits a simpler proof by induction; this proof is indicated in the last section.

A priori, it might be not clear why the space in the theorem is even homeomorphic to a simplicial complex. This becomes wrong if you remove word “isometric” from the formulation. For example, there are closed 4-dimensional topological manifold which does not admit any triangulation, see [2, 1.6].

The Theorem 1.1 is applied in [4], where it is used to show that an Alexandrov space with the maximal number of extremal points is a quotient of $\mathbb{R}^n$ by a cocompact properly discontinuous isometric action.

About the proof. In the first approximation, the proof can be described as following. We cover $X$ by finite number of spherical conic neighborhood and consider its nerve, say $\mathcal{N}$. Then we map $\mathcal{N}$ barycentrically back to $X$. If we could show that the image of this map cover whole $X$ that would nearly finish the proof. Uninformatively we did not manage to show this statement and have make a walk around; this is the only subtle point in the proof.

Acknowledgment. We would first like to thank Arseniy Akopyan, Vitali Kapovitch, Alexander Lytchak and Dmitri Panov for their help.

2 Definitions

In this section we give the definition of polyhedral space of arbitrary dimension. It seems that these spaces were first considered by Milka in [5]; our definitions are equivalent but shorter.
Metric spaces. The distance between points \( x \) and \( y \) in a metric space \( X \) will be denoted as \(|x - y|\) or \(|x - y|_X\). Open \( \varepsilon \)-ball centered at \( x \) will be denoted as \( B(x, \varepsilon) \); i.e.,
\[
B(x, \varepsilon) = \{ y \in X \mid |x - y| < \varepsilon \}.
\]
If \( B = B(x, \varepsilon) \) and \( \lambda > 0 \) we use notation \( \lambda \cdot B \) as a shortcut for \( B(x, \lambda \cdot \varepsilon) \).

A metric space is called length space if the distance between any two points coincides with the infimum of lengths of curves connecting these points.

A minimizing geodesic between points \( x \) and \( y \) will be denoted by \([xy]\).

Polyhedral spaces. A length space is called polyhedral space if it admits a finite triangulation such that each simplex is (globally) isometric to a simplex in Euclidean space.\(^1\)

Cones and homotheties. Let \( \Sigma \) be a metric space with diameter at most \( \pi \). Consider the topological cone \( K = [0, \infty) \times \Sigma / \sim \) where \((0, x) \sim (0, y)\) for every \( x, y \in \Sigma \). Let us equip \( K \) with the metric defined by the rule of cosines; i.e., for any \( a, b \in [0, r) \) and \( x, y \in \Sigma \) we have
\[
|(a, x) - (b, y)|_K^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos |x - y|_\Sigma.
\]
The obtained space \( K \) will be called Euclidean cone over \( \Sigma \). All the pairs of the type \((0, x)\) correspond to one point in \( K \) which will be called the tip of the cone. A metric space which can be obtained in this way is called Euclidean cone. Equivalently, Euclidean cone can be defined as a metric space \( X \) which admits a one parameter family of homotheties \( m_\lambda : X \to X \) for \( \lambda \geq 0 \) such that for any fixed \( x, y \in X \) there are real numbers \( \zeta, \eta \) and \( \vartheta \) such that \( \zeta, \vartheta \geq 0 \), \( \eta^2 \leq \zeta \cdot \vartheta \) and
\[
|m_\lambda(x) - m_\mu(y)|_X^2 = \zeta \cdot \lambda^2 + 2 \cdot \eta \cdot \lambda \cdot \mu + \vartheta \cdot \mu^2.
\]
for any \( \lambda, \mu \geq 0 \). The point \( m^0(x) \) is the tip of the cone; it is the same point for any \( x \in X \).

Once the family of homotheties is fixed, we can abbreviate \( \lambda \cdot x \) for \( m^\lambda(x) \).

Conic neighborhoods.

2.1. Definition. Let \( X \) be a metric space, \( x \in X \) and \( U \) a neighborhood of \( x \). We say that \( U \) is a conic neighborhood of \( x \) if \( U \) admits an open distance preserving embedding \( \iota : U \to K_x \) into Euclidean cone \( K_x \) which sends \( x \) to the tip of the cone.

If \( x \) has a conic neighborhood then the cone \( K_x \) as in the definition will be called the cone at \( x \). Note that in this case \( K_x \) is unique up to an isometry which sends the tip to the tip. In particular, any conic neighborhood \( U \) of \( x \) admits an open distance preserving embedding \( \iota_U : U \to K_x \) which sends \( x \) to the tip of \( K_x \). Moreover, it is easy to arrange that these embeddings commute with inclusions; i.e., if \( U \) and \( V \) are two conic neighborhoods of \( x \) and \( U \supset V \) then the restriction of \( \iota_U \) to \( V \) coincides with \( \iota_V \). The later justifies that we omit index \( U \) for the embedding \( \iota : U \to K_x \).

\(^1\)Note that according to our definition, the polyhedral space has to be compact.
Assume $x \in X$ has a conic neighborhood and $K_x$ is the cone at $x$. Given a geodesic $[xy]$ in $X$, choose a point $\bar{y} \in [xy]$ sufficiently close to $x$ and set

$$\log[xy] = \frac{|x - y|_X}{|x - \bar{y}|_X}, \epsilon(\bar{y}) \in K_x.$$  

Note that $\log[xy]$ does not depend on the choice of $\bar{y}$.

\section{Preliminary statements}

\subsection{Definition.}
Let $X$ be a metric space and $[px_1],[px_2], \ldots,[px_m]$ are geodesics in $X$. We say that a neighborhood $U$ of $p$ splits in the direction of the geodesics $[px_1], [px_2], \ldots, [px_k]$ if there is an open distance preserving map $i$ from $U$ to the product space $E \times K'$, such that $E$ is a Euclidean space and the inclusion $i(U \cap [px_i]) \subset E \times \{\epsilon'\}$ holds for a fixed $\epsilon' \in K'$ and any $i$.

\subsection{Lemma.}
Let $X$ be a metric space and $p \in X$, $B_i = B(x_i,r_i)$, $i \in \{1, \ldots, k\}$ are conic neighborhoods of $x_i$. Assume $p \in B_i$ for each $i$. Then any conic neighborhood of $p$ splits in the direction of $[px_1], \ldots, [px_k]$.

In the proof we will use the following statement; its proof is left to the reader.

\subsection{Proposition.}
Assume $K$ is a metric space which admits cone structures with different tips $x_1, \ldots, x_k$. Then $K$ is isometric to the product space $E \times K'$, where $E$ is a Euclidean space and $K'$ is a cone with tip $\epsilon'$ and $x_i \in \epsilon' \times E$ for each $i$.

\textit{Proof of Lemma 3.2.} Fix sufficiently small $\epsilon > 0$. For each point $x_i$, consider point $x'_i \in [px_i]$ such that $|p - x'_i| = \epsilon |p - x_i|$. Since $\epsilon$ is sufficiently small, we can assume that $x'_i$ lies in the conic neighborhood of $p$.

Note that for the right choice of parameters close to 1, the composition of homotheties with centers at $x_i$ and $p$ produce a homothety with center at $x'_i$ and these are defined in a fixed conic neighborhood of $p$. These homotheties can be extended to the cone $K_p$ at $p$ and taking their compositions we get the homotheties for all values of parameters with the centers at $x'_i = \log[px'_i] \in K_p$. It remains to apply Proposition 3.3. \hfill \square

From the Lemma 3.2, we get the following corollary.

\subsection{Corollary.}
Let $X$ be a compact length space and $x \in X$. Suppose $B = B(x,r)$ is a conic neighborhood of $x$ which splits in the direction of $[px_1], \ldots, [px_k]$ and $\iota : B \hookrightarrow E \times K'$ be the corresponding embedding. Then the image $\iota(B)$ is a ball of radius $r$ with a center $(\iota(x), \epsilon')$.

In particular, for any point $q \in B$ such that $|q - p|_X = \rho$ and $\iota(q) \in E \times \{\epsilon'\}$ the ball $B(q,r - \rho)$ is a conic neighborhood of $q$.  

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3.5. Lemma. Let $B_i = B(x_i, r_i), i \in \{0, \ldots, k\}$ be balls in the metric space $X$. Assume each $B_i$ forms a conic neighborhood of $x_i$ and $x_i \in B_j$ if $i \leq j$. Then $X$ contains a subset $Q$ which contains all $x_i$ and is isometric to a convex polyhedron.

Moreover the geodesics in $Q$ do not bifurcate; i.e., if geodesic $\gamma: [a, b] \to X$ lies in $Q$ and an other geodesics $\gamma': [a, b] \to X$ coincides with $\gamma$ on some interval then $\gamma' = \gamma$.

Proof. To construct $Q = Q_k$ we apply induction on $k$ and use the cone structures on $B_i$ with the tip at $x_i$ consequently.

The base $k = 0$ is trivial.

By the induction hypothesis, there is a set $Q_{k-1}$ containing all $x_0, \ldots, x_{k-1}$.

Note that $B_k$ is strongly convex; i.e., any minimizing geodesic with ends in $B_k$ lies completely in $B_k$. In particular $Q_{k-1} \cap B_k$ is convex. Since $x_i \in B_k$ for all $i < k$, we may assume that $Q_{k-1} \subset B_k$.

Note that the homothety $m^\lambda_k$ with center $x_k$ and $\lambda \leq 1$ is defined for all points in $B_k$. Set

$$Q_k = \{ m^\lambda_k(x) \mid x \in Q_{k-1} \text{ and } \lambda \leq 1 \}.$$  

Since $Q_{k-1}$ is isometric to a convex polytope, so is $Q_k$.

To show that the geodesic $\gamma: [a, b] \to X$ in $Q$ can not bifurcate, it is sufficient to show that if $a < c < b$ then a neighborhood of $p = \gamma(c)$ splits in the direction of $\gamma$.

Without loss of generality, we may assume that $p$ lies in the intersection $\bigcap_i B_i$; if this is not the case, we can move $p$ in this intersection applying a composition of the homotheties $m^\lambda_k$ for $\lambda_k \geq 1$ and chop the interval around $c$ to keep $\gamma$ in $Q$.

It remains to apply Lemma 3.2. \qed

4 The proof

The proof of Theorem 1.1 is based on the following lemma; its proof is generously left to the reader.

4.1. Lemma. Assume a length space $X$ is covered by finite number of sets such that each finite intersection of these sets is isometric to a convex polytope. Then $X$ is a polyhedral space.

Proof of Theorem 1.1. We need to show the “if” part; the “only if” part is trivial.

Fix a finite cover of $X$ by open balls $B_i = B(x_i, r_i), i \in \{0, \ldots, n\}$ such that for each $i$, the ball $4 \cdot B_i$ is a conic neighborhood of $x_i$.

Given $i \in \{0, \ldots, n\}$ and $z \in X$ set

$$f_i(z) = |x_i - z|^2_X - r_i^2.$$  

Clearly $f_i(z) < 0$ if and only if $z \in B_i$. Set

$$f(z) = \min_i \{f_i(z)\}.$$  

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It follows that \( f(z) < 0 \) for any \( z \in X \).

Consider Voronoi domains \( V_i \) for the functions \( f_i \); i.e.,
\[
V_i = \{ z \in X \mid f_i(z) \leq f_j(z) \text{ for all } j \}.
\]

From above we get that \( V_i \subset B_i \) for each \( i \).

Given a subset \( \sigma \subset \{0, \ldots, n\} \) set
\[
V_\sigma = \bigcap_{i \in \sigma} V_i.
\]

Note that \( V_{(j)} = V_j \) for any \( i \in \{0, \ldots, n\} \).

Let \( \mathcal{N} \) be the nerve of the covering \( \{V_i\} \); i.e., \( \mathcal{N} \) is the abstract simplicial complex with \( \{0, \ldots, n\} \) as the set of vertexes and such that a subset \( \sigma \subset \{0, \ldots, n\} \) forms a simplex in \( \mathcal{N} \) if and only if \( V_\sigma \neq \emptyset \).

Let us fix a simplex \( \sigma \) in \( \mathcal{N} \). While \( \sigma \) is fixed, we may assume without loss of generality that \( \sigma = \{0, \ldots, k\} \) for some \( k \leq n \) and \( r_0 \leq r_1 \leq \ldots \leq r_k \). In particular \( 2 \cdot B_i \ni x_0 \) for each \( i \leq k \).

From above \( V_\sigma \subset B_0 \). Since \( 4 \cdot B_i \) is a conic neighborhood of \( x_i \) and \( 2 \cdot B_i \ni x_0 \) for each \( i \in \sigma \), we can apply Lemma 3.2 for the balls \( 4 \cdot B_0, \ldots, 4 \cdot B_k \). Denote by \( f: 4 \cdot B_0 \leftrightarrow E \times K \) the distance preserving embedding provided by this lemma.

We can assume that the Euclidean factor \( E \) has minimal possible dimension; i.e., the images \( f(B_0 \cap [x_0 x_i]) \) span whole \( E \). In this case the projection of \( f(V_\sigma) \) on \( E \) is a one-point set, say \{z\}. Denote by \( x_\sigma \in B_0 \) the point such that \( f(x_\sigma) = z \). Set \( r_\sigma = r_0 \) and \( B_\sigma = B(x_\sigma, r_\sigma) \). (The point \( x_\sigma \) plays the role of radical center of the collection of balls \( \{B_i\}_{i \in \sigma} \).)

According to Corollary 3.4 the ball \( 3 \cdot B_\sigma \) forms a conic neighborhood of \( x_\sigma \). Clearly \( B_\sigma \ni V_\sigma \) for any simplex \( \sigma \) in \( \mathcal{N} \).

Let \( \sigma \) be a simplex of \( \mathcal{N} \) and \( \varphi, \psi \) be two faces of \( \sigma \); i.e., \( \varphi, \psi \subset \sigma \). Note that \( 3 \cdot B_{\varphi} \ni x_\psi \) if \( r_\varphi \leq r_\psi \). Therefore Lemma 3.5 provides a subset, say \( Q_\sigma \), isometric to a convex polyhedron and contains all \( x_\varphi \) for \( \varphi \subset \sigma \).

It remains to show
(a) \( X = \bigcup_{\sigma} Q_\sigma \), where the union is taken for all the simplices \( \sigma \) in \( \mathcal{N} \).
(b) The intersection of arbitrary collection of \( Q_\sigma \) is isometric to a convex polytope.

Once (a) and (b) are proved, Lemma 4.1 finishes the proof.

Part (b) follows since the geodesics in \( Q_\sigma \) do not bifurcate; see Lemma 3.5.

Given \( p \in X \), set
\[
\sigma(p) = \{ i \in \{0, \ldots, n\} \mid p \in V_i \}.
\]

Note that \( \sigma(p) \) forms a simplex in \( \mathcal{N} \) and \( p \in V_{\sigma(p)} \).

Therefore \( p \in B_{\sigma(p)} \).

Recall that \( B_{\sigma(p)} \) forms a conic neighborhood of \( x_{\sigma(p)} \). If \( p \neq x_{\sigma(p)} \) then moving \( p \) away from \( x_{\sigma(p)} \) in the radial direction keeps the point in \( V_{\sigma(p)} \) till the moment it hits a new Voronoi domain, say \( V_j \) with \( j \notin \sigma(p) \). Denote this end point by \( p' \). In other words, \( p' \) is the point such that
\footnote{It also follows that \( V_i \) forms a strongly convex subset of \( X \); i.e., any minimizing geodesic in \( X \) with ends in \( V_i \) lies completely in \( V_i \) This property is not needed in our proof, but it is used in the alternative proof; see the last section.}
(i) $p$ lies on the geodesic $[x_{\sigma(p)}p]$;
(ii) $p' \in V_i$ for any $i \in \sigma(p)$;
(iii) the distance $|x_{\sigma(p)} - p'|_X$ takes the maximal possible value.

Start with arbitrary point $p$ and consider the recursively defined sequence $p = p_0, p_1, \ldots$ such that $p_{i+1} = p'_i$.

Note that $\sigma(p)$ forms a proper subset of $\sigma(p')$. It follows that the sequence $(p_i)$ terminates after at most $n$ steps; in other words $p_k = x_{\sigma(p_k)}$ for some $k$.

In particular $p_k \in Q_{\sigma(p_k)}$. By construction it follows that $p_i \in Q_{\sigma(p_k)}$ for each $i \leq k$. Hence $p \in Q_{\sigma(p_k)}$; i.e., (a) follows.

5 Final remarks

**Finite dimensional case.** Let $X$ be a compact length space such that each point $x \in X$ admits a conic neighborhood.

Note that from Theorem 1.1, it follows in particular that dimension of $X$ is finite. If we know a priori the dimension (topological or Hausdorff) of $X$ is finite then one can build an easier proof using induction on the dimension which we are about to indicate.

Consider the Voronoi domains $V_i$ as in the beginning of proof of Theorem 1.1. Note that all $V_{\{i,j\}}$ are convex and $\dim V_{\{i,j\}} < \dim X$ if $i \neq j$.

By induction hypothesis we can assume that all $V_{\{i,j\}}$ are polyhedral spaces. Cover each $V_{\{i,j\}}$ by isometric copies of convex polyhedra satisfying Lemma 4.1. Applying the cone construction with center $x_i$ over these copies in $V_{\{i,j\}}$ for all $i \neq j$, we get a covering of $X$ by a finite number of copies of convex polyhedra such that all their finite intersections are isometric to convex polyhedra. It remains to apply Lemma 4.1.

**Spherical and hyperbolic polyhedral spaces.** Analogous characterization holds for spherical and hyperbolic polyhedral spaces. One needs to use spherical and hyperbolic rules of cosine in the definition of cone; after that proof goes without any changes.

**Locally compact case.** One may define polyhedral space as a complete length space which admits a locally finite triangulation such that each simplex is isometric to a simplex in Euclidean space.

In this case a locally compact length space is polyhedral if every point admits a conic neighborhood. The proof is the same.

**One more curvature free result.** Our result is curvature free — we do not make any assumption on the curvature of $X$. Besides our theorem, we are aware about only one statement of that type — the polyhedral analog of Nash–Kuiper theorem. It states that any distance nonexpanding map from $m$-dimensional polyhedral space to the Euclidean $m$-space can be approximated by a piecewise distance preserving map to the Euclidean $m$-space. In full generality this result was proved recently by Akopyan [1], his proof is based on earlier results obtained by Zalgaller [7] and Krat [3]. Akopyan’s proof is sketched in the lecture notes of the second author [6].
References

[1] Akopyan, A. V., *A piecewise linear analogue of Nash–Kuiper theorem*, a preliminary version (in Russian) can be found on www.moebiuscontest.ru

[2] Freedman, M. H., *The topology of four-dimensional manifolds*. J. Differential Geom. 17 (1982), no. 3, 357–453.

[3] Krat, S. *Approximation problems in Length Geometry*, Thesis, 2005

[4] Nina Lebedeva, *Alexandrov spaces with maximal number of extremal points*, arXiv:1111.7253

[5] Milka, A. D. *Multidimensional spaces with polyhedral metric of nonnegative curvature*. I. (Russian) Ukrain. Geometr. Sb. Vyp. 5–6 1968 103–114.

[6] Petrunin, A.; Yashinski, A. *Folding the polyhedral spaces*.

[7] Zalgaller, V. A. *Isometric imbedding of polyhedra*. (Russian) Dokl. Akad. Nauk SSSR 123 1958 599–601.