Propagating spinors on a tetrahedral spacetime lattice

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We derive a discrete path integral for massless fermions on a hypercubic spacetime lattice with null faces. The amplitude for a path with \( N \) steps and \( B \) bends is \( \pm (1/2)^N (i/\sqrt{3})^B \).

In this article, we discuss a four-dimensional variant of ‘Feynman’s Checkerboard’ problem—a construction of the propagator for the Dirac equation in 1+1 dimensions via a sum-over-paths method, where the paths traverse a null lattice. In that problem, the amplitude for a given zig-zag path is just \( (i\epsilon m)^R \), where \( \epsilon \) is the time duration of a lattice step, \( m \) is the particle mass, and \( R \) is the number of direction reversals (and we use units with \( \hbar = c = 1 \)). If the mass vanishes the particle moves only to the right or the left with the speed of light, and these two motions correspond to the two chiralities for a Dirac spinor in 1+1 dimensions.

Feynman’s checkerboard path integral is striking in its simplicity, and intriguing because it accounts for relativistic propagation and Dirac matrices with nothing more than a simple factor of \( i \) associated with a geometric property—a bend—in a piecewise lightlike path. It hints at the possibility that a simple, discrete dynamics could underly the usual continuum description of relativistic quantum field theory. A longstanding question has been whether or not a path integral could be found in \( 3+1 \) dimensions preserving these surprising features. Here we show that one can indeed come very close.

Most work on lattice formulations of spinor propagation is directed at lattice field theory calculations, and thus involves Grassmann variables and “path” integrals over field configurations in a spacetime of Euclidean signature \( \mathbb{R}^4 \). Here instead we seek a bosonic (i.e. non-Grassmanian) formulation in terms of a sum over particle paths on a Minkowski signature lattice. Such formulations have been studied previously in \( \mathbb{R}^3 \) and references therein. What is new here is the interesting structure of the lattice employed, and the fact that it allows for a particularly simple rule for the amplitudes.

We focus on the massless case, which in \( 3+1 \) dimensions is far more interesting than in \( 1+1 \) dimensions. The decoupled right and left chiralities are then described by two-component spinors satisfying the Weyl equation,

\[
\sigma^\mu \partial_\mu \Psi = 0,
\]

where \( \sigma^\mu = (1, \pm \vec{\sigma}) \) with \( \vec{\sigma} \) the Pauli matrices. (The + sign corresponds to right handed spinors.)

The lattice we use here is topologically hypercubical but “tilted on its corner”. A quantum network based on this lattice was called “hyperdiamond” and previously studied by Finkelstein and collaborators \( \cite{6} \) (see also \( \cite{7} \)). We use this name here for the lattice itself.

A diagonal of the hypercube defines our time axis. The edges from one vertex lie in the direction of the four spacetime vectors

\[
n_i^\mu = (1, \alpha \hat{n}_i).
\]  

The four spatial unit vectors \( \hat{n}_i \) point to the vertices of a tetrahedron, and the step speed \( \alpha \) is for the moment unspecified. The \( \hat{n}_i \) sum to zero, and the inner product or angle between any distinct two is the same. Hence for \( i \neq j \) we have \( \hat{n}_i \cdot \hat{n}_j = -1/3 \), and the angle is equal to \( \cos^{-1}(-1/3) \approx 109^\circ \).

The spatial lattice at one time is a face-centered cubic (fcc) lattice. A way to see this is to begin with the tetrahedron of points that lies at one time step to the future of a given spacetime point \( p \). The four dimensional lattice has translation symmetries that map any point to any other point, and the spatial lattice at one time must share this property. Hence it can be grown from this tetrahedral seed by translation along the edges of the tetrahedron, which produces the fcc lattice shown in Fig. 1.

![FIG. 1: Face-centered cubic lattice of points at one time step. The tetrahedron (dotted lines) is comprised of the four points reached from the center of the small cube in one time step (dashed lines). The continuum sphere of light is enclosed by and tangent to the tetrahedron. The distance from the center to a tetrahedron vertex is three times the radius of the sphere. The step length \( a \) and cube edge length \( L \) are shown.](image)

Evolving the spatial lattice one time step to the future
amounts to shifting it along the displacement from the center of one tetrahedron to one of its vertices, yielding a distinct but equivalent spatial lattice. After four such steps the original spatial lattice is recovered.

It might seem natural to choose the step speed $\alpha = 1$, so that the links $n_i^\mu$ of the hyperdiamond would be null as envisaged in (4). However, in this case the lattice propagator would fail to converge at all in the continuum limit. The reason is that such a spacetime lattice violates the well-known “Courant condition” for stability: the discrete region of causal influence must contain the continuum one.

To marginally satisfy the Courant condition, the polyhedral cone formed by the four hypersurfaces spanned by three of the $n_i^\mu$’s must be tangent to the continuum light cone. That is, the faces of the hypercube must be null. Thus each of these hyperplanes must contain one and only one null direction. By symmetry this null direction must coincide with the sum of the three link vectors, e.g. $N^\mu = (3, \alpha(\hat{n}_1 + \hat{n}_2 + \hat{n}_3))$. The Minkowski norm of $N^\mu$ is $9 - \alpha^2$, hence if it is to be null we must choose $\alpha = 3$. Moreover, $N^\mu n_{1\mu} = 3 - (\alpha^2/3)$, so if $\alpha = 3$ the null vector $N^\mu$ is orthogonal to all vectors in the hyperplane, confirming that the hyperplane is indeed null.

Now consider the tetrahedral quartet of unit vectors $\hat{n}_i$. The sums $\sum_i \hat{n}_i^a \hat{n}_i^b$ and $\sum_i \hat{n}_i^a \hat{n}_i^b$ (with ‘$a’$ denoting the ‘$a$’ component of $\hat{n}_i$) are invariant under the symmetries of the tetrahedron, hence the first sum must vanish and the second sum must be proportional to the Euclidean metric $\delta^{ab}$. Since the trace is equal to four this yields the relation

$$\sum_i \hat{n}_i^a \hat{n}_i^b = \frac{4}{3} \delta^{ab}. \tag{3}$$

Using these identities and the definition (2) of the 4-vectors $n_i^\mu$, the matrix 4-vector $\sigma^\mu = (1, \sigma)$ can be expressed as

$$\sigma^\mu = \frac{1}{2} \sum_i \frac{1}{2} \left(1 + \frac{3}{\alpha} \hat{n}_i \cdot \vec{\sigma}\right) n_i^\mu. \tag{4}$$

In the special case $\alpha = 3$, for which the polyhedral light cone consists of null hyperplanes, this becomes just

$$\sigma^\mu = \frac{1}{2} \sum_i P_i n_i^\mu, \tag{5}$$

where

$$P_i = \frac{1}{2} (1 + \hat{n}_i \cdot \vec{\sigma}) \tag{6}$$

is the projector for spin up in the direction $\hat{n}_i$. Using the identity (5) the Weyl equation (1) for right-handed spinors takes the form

$$\frac{1}{2} \sum_i P_i n_i^\mu \partial_\mu \Psi = 0. \tag{7}$$

We consider the hyperdiamond lattice with step vectors $\epsilon n_i^\mu$ scaled by the step size $\epsilon$. With the partial derivatives replaced by finite differences,

$$\epsilon n_i^\mu \partial_\mu \Psi(x) \cong \Psi(x) - \Psi(x - \epsilon n_i), \tag{8}$$

the Weyl equation (7) yields the one-step evolution prescription for determining $\Psi(x)$ on the lattice from the values $\Psi(x - \epsilon n_i)$ at the immediately preceding points,

$$\Psi(x) = \frac{1}{2} \sum_i P_i \Psi(x - \epsilon n_i). \tag{9}$$

The finite difference equation (9) is different from any typically used in lattice field theory calculations. For a plane wave solution of the form $\Psi(x^\mu) = \exp(ik^\mu x^\mu)\Psi_0$ (with $\Psi_0$ a constant spinor) it implies the dispersion relation

$$\Psi_0 = \frac{1}{2} \sum_i P_i e^{-i k^\mu n_i^\mu} \Psi_0. \tag{10}$$

Expanding in $\epsilon$ and using equation (5), we find at first order in $\epsilon$ the standard Weyl equation for momentum eigenstates, $\sigma^\mu k_\mu \Psi_0 = 0$. The solutions obey the relativistic dispersion relation, $k_\mu k^\mu = 0$, and the corresponding zero frequency solution has vanishing wave vector.

Fermion doubling does not occur. That is, there are no additional zero-frequency solutions to Eq. (10), as we argue later. The Nielsen-Ninomiya theorem (2) is presumably evaded since this finite difference equation does not satisfy the hermiticity condition. The attendant lack of unitarity would be a problem for a lattice field theory application, but it does not present a problem for our purpose, which is just to extract a representation of the propagator as the continuum limit of a sum over paths on the lattice.

Iterating Eq. (10) backwards in time, we see that the retarded propagator between two points can be written as a sum over multi-step paths involving moves in the directions $\hat{n}_{i_2} \ldots \hat{n}_{i_N}$ at step speed $3c$, with the amplitude for such a path given by the operator

$$\frac{1}{2^N} P_{i_N} \ldots P_{i_1}. \tag{11}$$

The propagator is then the sum of these operators over all paths that connect the points. In terms of the unit eigenspinor $|i\rangle$ of $P_i = |i\rangle\langle i|$ the amplitude (11) becomes

$$\frac{1}{2^N} |i_{N}\rangle \langle i_N| i_{N-1} \langle i_{N-1}| \ldots \langle i_2| i_2\rangle \langle i_1|. \tag{12}$$

Note that for fixed initial and final points the number of each of the four step types is fixed, so one sums only over the order in which the steps are taken.

The amplitude (12) is independent of the choice of phases for the spinors, so we are free to adjust those phases to produce a particularly nice result. The unit
spinor corresponding to a unit vector with spherical angles \((\theta, \phi)\) is \((\cos(\theta/2), \sin(\theta/2) \exp(i\phi))\) times an arbitrary overall phase. The spinors \(|i\rangle\), which correspond to the four unit vectors pointing to the vertices of a tetrahedron, can be taken as

\[
|1\rangle = e^{i\psi_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad |3\rangle = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{6}} \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}, \quad |4\rangle = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{3}} \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix} \tag{13}
\]

If we choose \(\psi_1 = -\pi/2\) and \(\psi_2 = \psi_3 = \psi_4 = 0\), then the inner products between the various spinors become identical up to a sign that depends on the order:

\[
\langle 1|2, 3, 4\rangle = \langle 2|3\rangle = \langle 3|4\rangle = \langle 4|2\rangle = \frac{i}{\sqrt{3}} \tag{14}
\]

Thus the amplitude for an \(N\) step path with \(B\) bends is

\[
\pm \frac{1}{2^N} \left( \frac{i}{\sqrt{3}} \right)^B. \tag{15}
\]

To eliminate the operator \(|i_N\rangle\langle i_1|\) from the amplitude \(12\), we considered here the matrix element of the propagator with respect to fixed initial and final spinors \(|i_0\rangle\) and \(|i_{N+1}\rangle\) selected from the set \(13\). These specify directions of arrival at the initial point and departure from the final point, and the amplitude \(15\) includes any contributions from initial and final bends. Unfortunately we have not been able to find a simple way to specify the overall sign other than by reference to the positive two-step orders given in \(14\).

For left handed Weyl spinors the amplitude for a step is given by the orthogonal spin projector relative to the right handed case. Hence we can use the charge conjugates of the four spinors \(13\), with the result that the imaginary unit \(i\) in the amplitude \(15\) is replaced by \(-i\). The effect of a mass \(m\) can be included by allowing for chirality flips between right and left handed spinor propagation at each time step, with an associated amplitude factor \(icm\) \(F\), as on Feynman’s checkboard.

We now evaluate the sum over paths for the lattice propagator \(K_\epsilon(\Delta x)\) for a spacetime displacement \(\Delta x\) and demonstrate that it reproduces the continuum propagator in the limit \(\epsilon \to 0\).

The lattice displacement \(\Delta x\) can be expanded in terms of the four basis vectors:

\[
\Delta x^\mu = \sum_j \Delta x^j n^\mu_j, \tag{16}
\]

hence the displacement is determined by a unique set of four integers \(N^j = \Delta x^j/\epsilon\). The constraint that a path connects the two points of interest can be incorporated as four Kronecker deltas, which we express in a Fourier representation:

\[
\prod_j \delta(N^j, \Delta x^j/\epsilon) = \int_{-\pi}^{\pi} \frac{d\theta_j}{(2\pi)^j} e^{-i\sum_j \theta_j(N^j - \Delta x^j/\epsilon)}. \tag{17}
\]

The lattice propagator is given by

\[
K_\epsilon(\Delta x) = \sum_{N=0}^{\infty} \int_{-\pi}^{\pi} \frac{d\theta_j}{(2\pi)^j} e^{-i\sum_j \theta_j(N^j - \Delta x^j/\epsilon)} |A(\theta)|^N, \tag{18}
\]

where

\[
A(\theta) = \frac{1}{2} \sum_j P_j e^{i\theta_j}. \tag{19}
\]

The sum over \(N\) of \(|A(\theta)|^N\) produces every possible sequence of projection operators, each with the appropriate exponential factor encoding the number of steps in each direction. When the integrals over \(\theta_j\) are carried out, only those step sequences that produce the displacement \(\Delta x^\mu\) will survive. In particular, only the value of \(N\) equal to the total number of steps contributes.

As the step size \(\epsilon\) goes to zero, the number of steps \(N\) for a fixed time interval goes to infinity as \(\Delta t/\epsilon\). Convergence thus requires that the norm \(\|A(\theta)\|\) (i.e., the maximum norm of \(A(\theta)\) acting on a unit spinor) be less than or equal to unity. It is shown in the appendix that \(\|A(\theta)\| < 1\) except when at least three of the \(\theta_j\) coincide.

Thus, as \(N\) becomes larger, \(|A(\theta)|^N\) converges to zero pointwise except at these degenerate \(\theta_j\) values. (Had we assumed an arbitrary step-speed \(\alpha\), convergence would have required \(\alpha \geq 3\).)

When three \(\theta_j\) coincide \(A(\theta)\) takes the form \(A(\theta) = \frac{1}{2}(e^{i\theta_1} + e^{i\theta_2})P_3 + e^{i\theta_2}P_1\), where \(P\) is the projector orthogonal to \(P_3\). It is easily seen that the action of \(A(\theta)\) then decreases the norm of any spinor except an eigenspinor of \(P_3\). In the exceptional case where all four \(\theta_j\) coincide \(A(\theta)\) is just \(e^{i\theta}\) times the identity. This leads to no difficulty, as the integral over \(\theta = \sum_j \theta_j\) in \(18\) just produces a Kroneker delta which sets \(N\) equal to the total number of steps. Fermion doubling would occur if Eq. \(10\) had an extra solution with zero frequency. This would be an eigenvector of \(A(\theta)\) with unit eigenvalue, where \(\theta_i = -\epsilon k_i n_i^\mu\) and \(\sum_i \theta_i = 0\). Inspection of the above degenerate form of \(A(\theta)\) reveals that such an eigenvector exists only if all \(\theta_i\) vanish, i.e., if \(k_\mu = 0\).

We next introduce the new variables \(k_j := \theta_j/\epsilon\), in terms of which \(18\) takes the form

\[
K_\epsilon(\Delta x) = \epsilon \sum_{N=0}^{\infty} \int_{-\pi/\epsilon}^{\pi/\epsilon} \frac{d^4k}{(2\pi)^4} e^{-i\sum_j k_j \Delta x^j} |A(\epsilon k)|^N. \tag{20}
\]

Taking the limit \(\epsilon \to 0\), with the time interval \(\Delta t = N\epsilon\) fixed, we have

\[
[A(\epsilon k)]^N = \left[ \sum_j \frac{1}{2} P_j e^{i\epsilon k_j} \right]^N \tag{21}
\]

\[= e^{iN\epsilon \sum_j k_j P_j / 2} + O(\epsilon^2). \tag{22}
\]

Moreover the limits of integration in \(20\) approach \(\pm \infty\),
hence (20) yields

\[ K_{\epsilon \to 0}(\Delta x) = \epsilon^4 \sum_{N=0}^{\infty} \int_{-\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} e^{-ik_\mu \Delta x^\mu} e^{iN\epsilon \sum_j k_j P_j/2} \]  

(23)

The components \( \Delta x^\mu \) are defined in (16) relative to the tetrahedral basis of 4-vectors \( n_i^\mu \). The quantities \( k_j \) can be viewed as components of a co-vector in the dual basis, and the corresponding components in an arbitrary basis are denoted \( k_\mu \),

\[ k_j = k_\mu n_j^\mu. \]  

(24)

Hence we have \( \sum_j k_j \Delta x^j = k_\mu \Delta x^\mu \). Substituting (24) for \( k_j \), and using (16), the sum in the last exponent of (23) becomes \( k_\mu \sigma^\mu \). Changing integration variables from \( k_j \) to \( k_\mu \) in (23) gives rise to a Jacobian \( |\partial k_j/\partial k_\mu| = |n_j^\mu| \), which can be computed using an explicit form of the tetrahedron of unit 3-vectors, yielding

\[ d^4 k_j = 48\sqrt{3}d^4 k_\mu. \]  

(25)

The final step in taking the limit is to replace the discrete variable \( N \) by a continuous one \( s = N\epsilon \), in terms of which the sum \( \sum_N \) becomes \( \int ds/\epsilon \). With this replacement, and the change of variables from \( k_j \) to \( k_\mu \), (23) becomes

\[ K_{\epsilon \to 0}(\Delta x) = 48\sqrt{3}\epsilon^3 \int_0^{\infty} ds \int_{-\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} e^{-ik_\mu \Delta x^\mu} e^{i\epsilon k_\mu \sigma^\mu} \]  

(26)

Except for the peculiar factor in front, this is just the continuum retarded propagator, albeit in a perhaps slightly unfamiliar form. The integral over \( k_0 \) produces a Dirac delta function \( \delta(s - \Delta x^0) \), after which the \( s \)-integral sets \( s \) equal to \( \Delta x^0 \) (assuming \( \Delta x^0 > 0 \)), yielding the retarded propagator in the more common guise of a three-dimensional Fourier transform. Had we kept the subleading terms of order \( N\epsilon^2 \) (= \( se \)) in (23) the convergence factor for the integration limit \( s \to \infty \) would have been supplied much as in (3).

It remains to account for the prefactor \( 48\sqrt{3}\epsilon^3 \). We computed the propagator to go between two points on the hyperlattice. In the continuum, the amplitude to arrive at one point starting from another point is zero, since only by integrating over a finite region should a nonzero amplitude arise. The prefactor is none other than the volume per point in the lattice, with the step length \( \epsilon \) equal to 3\( \epsilon \). Hence what we have actually obtained is the continuum propagator integrated over the volume associated with the initial lattice point.

We have shown that a very simple discrete path integral converges in the continuum limit to the retarded propagator for the Weyl equation. Although the underlying lattice is not Lorentz invariant, that symmetry is recovered by the propagator in the continuum limit.

Another symmetry not possessed by the discrete propagator is unitarity. This is not because discreteness and unitarity are necessarily in conflict. Indeed, as shown in (4), one can write a unitary discrete evolution rule on a body centered cubic lattice (or on an alternating pair of simple cubic lattices) whose continuum limit is the Weyl equation. (Interestingly, unitarity and locality were shown there to imply the Weyl equation.) Consider an initial state that is non-vanishing only at one lattice point, with normalized spin state \(|\psi\rangle\). At the next time step according to (4) it has support at the four corners of a tetrahedron, with the amplitudes \( \frac{1}{2} |P_i| |\psi\rangle \). The norm of the state after one step is then \( |\langle \psi | \sum_i \frac{1}{2} P_i |\psi\rangle| = \frac{1}{4} \), i.e. it has decreased by a factor of two, violating unitarity.

It is not just the norm change that violates unitarity. Also, the evolutions of orthogonal states do not remain orthogonal. Two points at one time have either one or no common points in their one-step future. In the former case, the one-step evolution of two orthogonal states concentrated on the two initial points are clearly not orthogonal, because they overlap in just one point which will make the unique non-zero (since \( P_i P_j \neq 0 \)) contribution to the inner product. The evolutions therefore have “more overlap than they should”, which presumably counteracts the loss of norm of each individual evolution in such a way that the continuum limit is unitary.

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APPENDIX

In this appendix we prove that the norm of the matrix \( A \) defined in Eq. (16) is less than unity unless at least three of the \( \theta_i \) coincide, in which case the norm is unity. The proof is due to Eli Hawkins.

Let \(|\psi\rangle\) be any unit spinor. The squared norm of \( A|\psi\rangle \) is \( ||A|\psi\rangle||^2 = \langle \psi | A^\dagger A |\psi\rangle \), whose maximum is the larger eigenvalue of \( A^\dagger A \). This value defines the squared norm \( ||A||^2 \).

Using the definition of the spin projection operators \( P_i \) and the inner products of the unit vectors \( \hat{n}_i \) (1 if \( i = j \) and \(-\frac{1}{2} \) if \( i \neq j \)) we find

\[ \text{tr}(A^\dagger A) = 1 + \frac{1}{4} \sum_{(ij)} \cos(\theta_i - \theta_j) \]  

(A.1)

where the sum is over the 6 choices of \( \{i, j\} \subset \{1, 2, 3, 4\} \). This trace is at most 2, and therefore the smaller eigenvalue of \( A^\dagger A \) is less than 1 unless \( A^\dagger A = 1 \). Hence

\[ \Phi := \det (A^\dagger A - 1) \]  

(A.2)

has the same sign as \( 1 - ||A|| \).
The matrix $A^\dagger A$ is linear in terms of $e^{i(\theta_i - \theta_j)}$, therefore $\Phi$ is quadratic. Because $\Phi$ is invariant under all permutations of the $\theta$'s, it can be written as a quadratic function of the cosines $\cos(\theta_i - \theta_j)$. Because $\Phi$ vanishes when the $\theta$'s are all equal, it is convenient to write it in terms of the cosines minus 1. It thus takes the form,

$$\Phi = a \sum_{(ij)} (1 - \cos[\theta_i - \theta_j]) + b \sum_{(ij)} (1 - \cos[\theta_i - \theta_j])^2 + c \sum_{(ij)(kl)} (1 - \cos[\theta_i - \theta_j]) (1 - \cos[\theta_k - \theta_l]).$$

(A.3)

The last sum is over the 3 partitions of $\{1, 2, 3, 4\}$ into pairs. By setting $\theta_1 = \theta_2 = \theta_3$ in this expression, we see that $a = b = 0$. To determine the value of $c$, consider the case that $\theta_1 = \theta_2 = 0$ and $\theta_3 = \theta_4 = \pi$. Then $A$ is the hermitian matrix $\frac{1}{2}(\hat{n}_1 + \hat{n}_2) \cdot \vec{\sigma}$, which has eigenvalues $\pm 1/\sqrt{3}$, hence $\Phi = 4/9$. The last sum in (A.3) is 8, so $c = 1/18$.

The determinant $\Phi$ is thus given by

$$\Phi = \frac{1}{18} \sum_{(ij)(kl)} (1 - \cos[\theta_i - \theta_j]) (1 - \cos[\theta_k - \theta_l]).$$

(A.4)

This satisfies $\Phi \geq 0$, therefore $\|A\| \leq 1$. Each term is non-negative, therefore $\Phi = 0$ only if every term vanishes. This occurs only if at least three of the $\theta$'s are equal.

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