High Energy Scattering in Perturbative Quantum Gravity at Next to Leading Power

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We consider the relativistic scattering of unequal-mass scalar particles through graviton exchange in the small-angle high-energy regime and calculate the scattering amplitude up to the next-to-leading eikonal correction, including gravitational effects of the same order. We find that these power corrections are consistent with the exponentiation of a corrected phase in impact parameter space, once all possible graviton exchanges are summed. The leading power correction is suppressed by a single power of the ratio of momentum transfer to the energy of the light particle in the rest frame of the heavy particle, independent of the heavy particle mass. For large enough heavy-particle mass, the saddle point for the impact parameter is modified compared to the leading order by a multiple of the Schwarzschild radius determined by the mass of the heavy particle, independent of the energy of the light particle.

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I. INTRODUCTION

There has been a renewed interest in perturbative quantum gravity in part due to recent developments illuminating a relationship between gravity and gauge theory amplitudes [1, 2] and also because of its relevance in high energy, small angle scattering [3–8]. The long distance regime of this theory is particularly interesting as quantum gravity has a simpler infrared structure than gauge theories. For instance, there is a cancellation of collinear divergences in the former theory but not the latter [9–11] and the logarithmically divergent soft amplitudes have a ladder like structure.

Since the infrared behavior of perturbative quantum gravity is tractable, it is important to look for observables for which long distance effects are particularly important. A fundamentally infrared dominated physical quantity of note in perturbative quantum gravity is the eikonal phase that has been calculated for small-angle high-energy scattering processes. An eloquent overview of the eikonal regime for this type of scattering is given in a set of lectures by Giddings [8].

This process has been shown to be largely independent of short distance physics. Giddings, Gross and Maharana [6] argued that the scattering of strings may be replaced by graviton contributions alone with short distance physics playing no significant role. As reviewed in [8] the leading eikonal approximation is justified for high energy, small angle scattering of equal-mass particles, because the impact parameter as determined from the saddle point of the loop integrations is larger than the typical gravitational radius $2GE$ by a factor of $s/t$.

Thus non-local string effects are actually subdominant to higher loop gravitational processes for a large range of impact parameters. In a slightly different kinematic regime, to be discussed later, we find the first subleading contributions coming from field theory corrections to the eikonal phase.

The eikonal approximation provides the leading contribution, which exponentiates in impact parameter space. Large impact parameters dominate the scattering in this regime. The next-to-eikonal contribution to small-angle high-energy gravitational scattering has to our knowledge not previously been calculated, although there has been much interesting progress on this topic for non-abelian gauge theories [12, 13]. In [14] the effective vertices at the next-to-eikonal level for gravity are given. Our specific calculations will also make contact with the extensive computations of Ref. [15]. A number of our results will involve special limits of the diagrammatic calculations outlined there.

In this paper, we consider the near forward scattering of unequal-mass scalar particles, and compute power corrections to the eikonal approximation in $\Delta/E_P$, where $\Delta$ is the momentum transfer and $E_P$ the energy of a light projectile that undergoes small-angle scattering by a heavy target, nearly at rest. We will find potentially substantial corrections, linear in this ratio, associated with both the next-to-eikonal expansion and, at the same level, corrections due to the nonlinear gravitational interactions.

In section II A we will review the calculation for the leading eikonal phase to establish our conventions. We proceed to calculating the next-to-eikonal phase in section II B. This is done by expanding the scalar propagator to next-to-eikonal order as well as allowing seagull type vertices in the diagram. We find that part of this contribution exponentiates, and calculate the correction to the saddlepoint at this order. We conclude with a summary and brief discussion of our results. Certain technical calculations are relegated to an appendix.

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II. RELATIVISTIC SCATTERING

It was shown by ‘t Hooft [3] that the scattering of ultra-relativistic particles could be reliably studied by considering graviton exchange in perturbative quantum gravity. In the rest frame of one of the particles, the gravitational field of the rapidly moving particle is that of a gravitational shock wave, which can be described by the Aichelburg-Sexl metric [18]. The particle at rest is a quantum particle whose dynamics can be described by the solutions of Klein-Gordon equation in the Aichelburg-Sexl background. This was ‘t Hooft’s approach [3] and the results can also be obtained by summing a class of Feynman diagrams in the eikonal approximation [8, 19]. In this section we will demonstrate how this latter approach can be extended to the next-to-eikonal level.

We will investigate the small-angle gravitational scattering of an ultra-relativistic light scalar particle of energy $E_\phi$ off of a very heavy scalar particle. The mass of the heavy particle, $M_\sigma$, is bigger than $E_\phi$ and both are much larger than the transferred momentum. This approximation is made not only because it simplifies the calculations needed to determine the next-to-eikonal corrections. It will also lead to power corrections of the form $\sigma \tau$ is much larger than the corrections that characterize the scattering of equal-mass particles.

We first show the exponentiation of the amplitude in the leading eikonal approximation in the kinematic regime described above. Then we consider the next-to-leading eikonal corrections. To make our kinematics explicit, throughout this paper we will work in the frame where $\Delta = \phi$ is the scalar-scalar-graviton vertex, and $\kappa^2 = 32\pi G$.

We will use the fact that in the large $M_\sigma$ limit

$$L_{\mu\nu\alpha\beta} \tau^{\mu\nu}(q,q', M_\sigma) = 2M_\sigma^2 (2\delta_{\sigma\alpha}\delta_{\beta\beta} - \eta_{\alpha\beta}) + \mathcal{O}(M_\sigma^4).$$

To leading power in $M_\sigma$, the right hand side of (5) is independent of the momentum flowing through the vertex so we will always be free to use this identity for any vertex on a heavy line.

In the ultra-relativistic limit $M_\phi = 0$ and $p^0 = |p| \equiv E_\phi$, so,

$$(2\delta_{\sigma\alpha}\delta_{\beta\beta} - \eta_{\alpha\beta}) \tau^{\alpha\beta}(p,p', M_\phi) = 4E_\phi^2 + \mathcal{O}(\Delta^2).$$

Corrections are down by two powers of $E_\phi$. Note that in the $\Delta^0 = 0$ frame $p^0 = p^0$, and (6) is valid whenever the $p^0 = p^0$ condition holds.

Let us first work with the simplest case where only a single graviton is exchanged (Fig. 1). The matrix element corresponding to this diagram can be written as

$$iM_1^0 = \left(-\frac{i\kappa}{2}\right)^2 \frac{1}{2\Delta^2 + i\epsilon} L_{\mu\nu\alpha\beta} \tau^{\alpha\beta}(p,p', M_\phi) \tau^{\mu\nu}(q,q', M_\sigma),$$

where

$$L_{\mu\nu\alpha\beta} = \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}$$

is the numerator of the de Donder gauge graviton propagator,

$$\tau^{\mu\nu}(q,q', M_\sigma) = q^\mu q'^\nu + q^\nu q'^\mu - \eta^{\mu\nu}((q \cdot q') - M_\sigma^2)$$

is the scalar-scalar-graviton vertex, and $\kappa^2 = 32\pi G$.

We emphasize that the terms we have ignored at leading discussion, we will not need to consider corrections arising from such numerator factors. This is one of the primary benefits of working in this particular kinematic regime.

Using (5) and (6), we have

$$iM_1^0 = \left(-\frac{i\kappa}{2}\right)^2 \frac{1}{2(2M_\sigma^2)(4E_\phi^2)} \frac{1}{\Delta^2 + i\epsilon}.$$  

FIG. 1: A scattering process with a single graviton exchanged. The heavy scalar is the solid line and the light scalar is the dotted line.

1. One Graviton Exchange

Let us first consider this process in the eikonal limit to establish our conventions. In this limit it is well known that only the ladder and crossed-ladder type diagrams contribute. It will be instructive to consider the cases where one and two gravitons are exchanged between the scalars before moving to the general $n$-graviton case.

A. Eikonal Phase

We will investigate the small-angle gravitational scattering of an ultra-relativistic light scalar particle of energy $E_\phi$ off of a very heavy scalar particle. The mass of the heavy particle, $M_\sigma$, is bigger than $E_\phi$ and both are much larger than the transferred momentum. This approximation is made not only because it simplifies the calculations needed to determine the next-to-eikonal corrections. It will also lead to power corrections of the form $\sigma \tau$, which in terms of invariants is $\sqrt{(\Delta^2 - \Delta_0^2)}M_\sigma/(s - M_\sigma^2)$, much larger than the corrections that characterize the scattering of equal-mass particles.

We first show the exponentiation of the amplitude in the leading eikonal approximation in the kinematic regime described above. Then we consider the next-to-leading eikonal corrections. To make our kinematics explicit, throughout this paper we will be working in the large center of mass energy and small angle scattering regime for the scattering,

$$p + q \rightarrow p' + q', \quad p^2 = p'^2 = 0, \quad q^2 = q'^2 = M_\sigma^2 \quad (1)$$

so we will take the incoming and outgoing momentum $p$ and $p'$ of the light scalar particle $\phi$ to be much larger than the momentum transfer between the two scalars, $\Delta = p' - p$. We will take $q$ and $q'$ to be the incoming and outgoing momenta of the heavy scalar $\sigma$. We will also work in the frame where $\Delta^0 = 0$ and in the de Donder gauge. We will always work to leading power in $M_\sigma$, and seek the first power corrections in $E_\phi$.
FIG. 2: A two-graviton ladder diagram. The heavy scalar is the solid line and the light scalar is the dotted line.

FIG. 3: A two-graviton crossed ladder diagram. The heavy scalar is the solid line and the light scalar is the dotted line.

2. Two Graviton Exchange

Let us now consider the scattering process where two gravitons are exchanged. In this case the matrix element becomes

\[
i M_2^0 = \left(-\frac{ik}{2}\right)^4\frac{i}{2}(i)^2(2\pi)^{-4}(2M_\sigma^2)^2 \times \int d^4k_1 d^4k_2 \delta^4(k_1 + k_2 + \Delta) \times \tau^{\alpha_1\beta_1}(p, p - k_1, M_\phi) \tau^{\alpha_2\beta_2}(p - k_1, p', M_\phi) \times 2\delta_{\alpha_1\beta_1} - \eta_{\alpha_1\beta_1}2\delta_{\alpha_2\beta_2} - \eta_{\alpha_2\beta_2} \frac{1}{k_1^2 + i\epsilon} \times \frac{1}{k_2^2 + i\epsilon} \frac{1}{(q + k_1)^2 - M_\sigma^2 + i\epsilon} \frac{1}{(q + k_2)^2 - M_\sigma^2 + i\epsilon},
\]  

where we have made use of (5). Note that the sum in the last line of (8) corresponds to summing over the ladder (Fig. 2) and crossed ladder (Fig. 3) diagrams. In the large $M_\sigma$ limit and the $\Delta^0 = 0$ frame, we have,

\[
\delta(k_1^0 + k_2^0 + \Delta^0) \times \left[ \frac{1}{(q + k_1)^2 - M_\sigma^2 + i\epsilon} + \frac{1}{(q + k_2)^2 - M_\sigma^2 + i\epsilon} \right] = (2M_\sigma)^{-1}\delta(k_1^0 + k_2^0) \left[ \frac{1}{k_1^2 + i\epsilon} + \frac{1}{k_2^2 + i\epsilon} \right].
\]  

This expression, can be simplified by use of the identities (see [20] and references therein), which we will have several occasions to use below, on both the heavy and light scalar lines.

\[
\sum_{\text{Perms of } \omega_i} \delta(\omega_1 + ... + \omega_n) \frac{1}{\omega_1 + i\epsilon} ... \frac{1}{\omega_1 + ... + \omega_{n-1} + i\epsilon} = \delta(\omega_1 + ... + \omega_n) \sum_{\omega_j} \frac{1}{\omega_j + i\epsilon} = (-2\pi i)^n \delta(\omega_1) ... \delta(\omega_n).
\]  

In (8), $\omega_i = k_i^0$, $n = 2$. After applying (10) we have,

\[
i M_2^0 = \left(-\frac{ik}{2}\right)^4\frac{i}{2}(i)^2(2\pi)^{-4}M_\sigma(-2\pi i) \times \int d^4k_1 d^4k_2 \delta(k_1^0)\delta(k_2^0)\delta^3(k_1 + k_2 + \Delta) \times \tau^{\alpha_1\beta_1}(p, p - k_1, M_\phi)\tau^{\alpha_2\beta_2}(p - k_1, p', M_\phi) \times \frac{2\delta_{\alpha_1\beta_1} - \eta_{\alpha_1\beta_1}2\delta_{\alpha_2\beta_2} - \eta_{\alpha_2\beta_2}}{k_1^2 + i\epsilon} \times \frac{1}{k_2^2 + i\epsilon} \frac{1}{(p - k_1)^2 + i\epsilon}.
\]  

Note that the delta functions over $k_1^0$ and $k_2^0$ guarantee that $p^0 = (p^0 - k_1^0) = p^0$ so we are free to use the light scalar vertex identity, (6). After applying this identity $M_2^0$ becomes

\[
i M_2^0 = \left(-\frac{ik}{2}\right)^4\frac{i}{2}(i)^2(2\pi)^{-4}M_\sigma(-2\pi i)(4E_\phi^2)^2 \times \int d^4k_1 d^4k_2 \delta(k_1^0)\delta(k_2^0)\delta^3(k_1 + k_2 + \Delta) \times \frac{1}{k_1^2 + i\epsilon} \frac{1}{k_2^2 + i\epsilon} \frac{1}{(p - k_1)^2 + i\epsilon}.
\]  

3. n-Graviton Exchange

Let us now consider the form of the matrix element when $n$ gravitons are exchanged. In this case, the matrix element is

\[
i M_n^0 = \left(-\frac{ik}{2}\right)^{2n}\frac{i}{2}(i)^{2n-2}(2\pi)^{-4n}(2M_\sigma^2)^n \times \int d^4k_1 ... d^4k_n \delta^4(k_1 + ... + k_n + \Delta) \times \prod_{i=1}^{n} \left[ \tau^{\alpha_i\beta_i}(p - K_{i-1}, p - K_i, M_\phi) \delta_{\alpha_i\beta_i} - \eta_{\alpha_i\beta_i} \right] \times \prod_{i=1}^{n-1} \frac{1}{(p - K_i)^2 + i\epsilon} \times \sum_{\text{perms of } k_i} \left[ \frac{1}{(q + k_1)^2 - M_\sigma^2 + i\epsilon} \right],
\]  

where
where \( k_i \) are the graviton momenta and

\[
K_i \equiv \sum_j i \cdot k_j .
\]  

(14)

The tensors \( \tau_i^{\alpha_i \beta_i} (p - K_{i-1}, p - K_i, M_\phi) \) are the \( n \) scalar-graviton vertices on the light scalar line. We have made use of the heavy scalar vertex approximation, Eq. (5). As we saw in the two graviton case, the summation over all permutations of the ordering of the graviton momenta onto the heavy line generates all the diagrams.

Just as in the two graviton case, we can simplify (13) by applying the identities (10) and (6) in succession. The result is

\[
i \mathcal{M}_n^0 = \frac{-i \kappa}{2} \frac{2n}{(2\pi)^n} \int d^4 k_1 \ldots d^4 k_n \delta(k_1^0) \ldots \delta(k_n^0) \delta^3(k_1 + \ldots + k_n + \Delta)
\]

\[
\times \prod_{i=1}^n \left( \frac{1}{k_i^2 + i \epsilon} \right) \left( \frac{1}{(p - K_i)^2 + i \epsilon} \right) \left[ \frac{1}{2p \cdot K_i - K_i^2 + i \epsilon} \right] .
\]  

(15)

The remaining scalar propagators can be expanded as

\[
\frac{1}{2p \cdot K_i - K_i^2 + i \epsilon} \approx \frac{1}{2E_\phi} \left( \frac{1}{K_i^2 + i \epsilon} + \frac{K_i^2}{2E_\phi (K_i^2 + i \epsilon)^2} \right) .
\]  

(16)

where we have taken \( p \) to be in the \( z \)-direction. For the leading eikonal phase we will only need to keep the first term in the expansion. It will be useful to symmetrize across the \( n \) graviton momenta so that we have

\[
i \mathcal{M}_n^0 = -4i(2\pi)^3 E_\phi M_\sigma \frac{1}{n!} \left( \frac{-\kappa^2 M_\sigma E_\phi}{4(2\pi)^3} \right)^n
\]

\[
\times \int d^3 k_1 \ldots d^3 k_n \delta^3(k_1 + \ldots + k_n + \Delta)
\]

\[
\times \prod_{i=1}^n \left[ \frac{1}{k_i^2 + i \epsilon} \right] \sum_{\text{perms of } k_i} \left( \frac{1}{k_i^2 + i \epsilon} \right) .
\]  

(17)

Again, in the frame \( \Delta^0 = 0, p^0 = p^{00} \equiv E_\phi = |p| \), which we take nearly in the \( z \)-direction. In fact, we may choose \( \Delta^z = 0 \) by taking \( p^z = p^{zz} = \sqrt{E_\phi^2 - \Delta^2}/4 \) so that \( p^z = E_\phi \) up to corrections of relative order \( E_\phi^{-2} \), which we may neglect. Thus, we can again apply (10), this time for the \( k^z \) components, to get

\[
i \mathcal{M}_n^0 = 4(2\pi)^2 E_\phi M_\sigma \frac{1}{n!} \left( \frac{\kappa^2 M_\sigma E_\phi}{4(2\pi)^3} \right)^n
\]

\[
\times \delta(k_1^z) \ldots \delta(k_n^z) \delta^2(k_1^+ + \ldots + k_n^+ + \Delta^+) \prod_{i=1}^n \left( \frac{1}{(k_i^+)^2} \right) .
\]  

(18)

Now let us Fourier transform into (transverse) impact parameter space,

\[
i \tilde{\mathcal{M}}_n^0 (b^+) = \int \frac{d^2 \Delta}{(2\pi)^2} e^{ib^+ \cdot \Delta} \mathcal{M}_n^0 (\Delta^+)
\]

\[
= 4(E_\phi M_\sigma) \frac{1}{n!} \left( \frac{\kappa^2 M_\sigma E_\phi}{4(2\pi)^3} \right)^n \int d^2 k_1^+ \ldots d^2 k_n^+
\]

\[
\times \prod_{i=1}^n \left[ \frac{1}{(k_i^+)^2} e^{-ib^+ k_i^+} \right] .
\]  

(19)

After summing over all \( n \) we have,

\[
i \tilde{\mathcal{M}}_n^0 = 2(s - M_\phi^2) e^{i\chi_0 - 1} ,
\]  

(20)

where

\[
\chi_0 (b^+) = \frac{\kappa^2 M_\sigma E_\phi}{4(2\pi)^3} \int d^2 k^+ \frac{1}{(k^+)^2} e^{-ib^+ k^+}
\]

\[
= 4GM_\phi E_\phi \left[ \frac{1}{d - 4} - \log b^+ + \text{terms independent of } b^+ \right] .
\]  

(21)

and where we have evaluated the integral in \( d \) dimensions and omitted finite terms independent of \( b \), as reviewed in the appendix. The inverse Fourier transform back to momentum space is dominated by the stationary phase point at

\[
b^+ \sim 4GM_\phi E_\phi \frac{\Delta^+}{\Delta^z} = 2 R_s E_\phi \frac{\Delta^+}{\Delta^z} ,
\]  

(22)

where \( R_s \) is the Schwarzschild radius of the heavy particle and \( \Delta^z \sim t \). Thus in the ultrarelativistic regime of small momentum transfer, this process is dominated by impact parameters larger than the Schwarzschild radius of the target particle. We note, however, that because of the asymmetry between the masses of the two scalar particles, the point of stationary phase of the inverse transform occurs at a multiple of the Schwarzschild radius that is set by the ratio of the momentum transfer to the energy, not to the heavy particle mass. Therefore, at small but finite angles, the scattering is dominated by impact parameters of order \( R_s \) divided by the scattering angle, rather than by the ratio of the momentum transfer divided by the total center of mass energy. In this situation, non-linear gravitational effects and corrections to the eikonal approximation are of comparable size. We will return the transformation to impact parameter below, when interpreting next-to-eikonal corrections, neglecting any contributions that are concentrated at \( b^+ = 0 \).
amplitude in Eq. (15) is then,

\[ i\mathcal{M}_2^a = -2i(2\pi)^3M_\sigma \left(-\frac{\kappa^2 M_\sigma E_\phi^2}{2(2\pi)^3}\right)^2 (2E_\phi)^{-2} \times \int d^3k_1 d^3k_2 \delta^3(k_1 + k_2 + \Delta) \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{k_1^2}{(k_1^2 + i\epsilon)^2}. \] (23)

If we symmetrize in the graviton momenta we have

\[ i\mathcal{M}_2^{1a} = -2i(2\pi)^3M_\sigma \left(-\frac{\kappa^2 M_\sigma E_\phi^2}{2(2\pi)^3}\right)^2 (2E_\phi)^{-2} \frac{1}{2} \times \int d^3k_1 d^3k_2 \delta^3(k_1 + k_2 + \Delta) \times \frac{1}{k_1^2} \frac{1}{k_2^2} \left[ \frac{k_1^2}{(k_1^2 + i\epsilon)^2} + \frac{k_2^2}{(k_2^2 + i\epsilon)^2} \right]. \] (24)

As expected, this expression is suppressed by a power of \( E_\phi \) compared to the leading power eikonal approximation. Dimensional analysis also shows that it behaves as \( 1/|\Delta| \) rather than the \( 1/|\Delta|^2 \), as at leading power. We will give an explicit analysis of these integrals in the context of \( n \) graviton exchange below.

Let us now consider the seagull interaction, Fig. 4, which we denote as

\[ i\mathcal{M}_2^{\text{seg}} = \left(-\frac{i\kappa}{2}\right)^2 (i\kappa^2)^2(i)(2\pi)^{-4}(2M_\sigma^2)^2(\frac{1}{2}) \times \int d^4k_1 d^4k_2 \delta^4(k_1 + k_2 + \Delta)\tau^{\alpha_1\beta_1\alpha_2\beta_2}(p,p') \times 2\delta_{\alpha_0\beta_1} \delta_{\alpha_0\beta_2} - \eta_{\alpha_0\beta_1} \delta_{\alpha_0\beta_2} - \eta_{\alpha_0\beta_2} \delta_{\alpha_0\beta_1} \times \frac{1}{k_1^2 + i\epsilon} \frac{1}{k_2^2 + i\epsilon} \left[ \frac{(q + k_1)^2 - M_\sigma^2 + i\epsilon}{(q + k_2)^2 - M_\sigma^2 + i\epsilon} \right]. \] (25)

We have again symmetrized graviton momenta and used the heavy scalar vertex approximation, Eq. (5). The explicit scalar-graviton seagull vertex, \( \tau^{\alpha_1\beta_1\alpha_2\beta_2}(p,p') \) in this expression, can be found in the convenient appendices of Refs. [15] and [17].

We now use the identities (9) and (10) to simplify Eq. (25),

\[ i\mathcal{M}_2^{\text{seg}} = \left(-\frac{i\kappa}{2}\right)^2 (i\kappa^2)^2(i)(2\pi)^{-4}(2M_\sigma^2)^2(\frac{1}{2}) \times (2M_\sigma)^{-1}(-2\pi i) \int d^4k_1 d^4k_2 \delta^4(k_1^0) \delta^4(k_2^0) \times \delta^4(k_1 + k_2 + \Delta)\tau^{\alpha_1\beta_1\alpha_2\beta_2}(p,p') \times 2\delta_{\alpha_0\beta_1} \delta_{\alpha_0\beta_2} - \eta_{\alpha_0\beta_1} \delta_{\alpha_0\beta_2} - \eta_{\alpha_0\beta_2} \delta_{\alpha_0\beta_1} \times \frac{1}{k_1^2 + i\epsilon} \frac{1}{k_2^2 + i\epsilon} \left[ \frac{(q + k_1)^2 - M_\sigma^2 + i\epsilon}{(q + k_2)^2 - M_\sigma^2 + i\epsilon} \right]. \] (26)

The numerator factors for the seagull diagram are readily evaluated, and give

\[ (2\delta_{\alpha_0\beta_1} \delta_{\alpha_0\beta_2} - \eta_{\alpha_0\beta_1} \delta_{\alpha_0\beta_2} - \eta_{\alpha_0\beta_2} \delta_{\alpha_0\beta_1})\tau^{\alpha_1\beta_1\alpha_2\beta_2}(p,p') = (4p^0 p^0 - p \cdot p') \approx 4E_\phi^2. \] (27)
where we have used that fact that \( p \cdot p' \) is subleading by two powers of \( E_\phi \) compared to \( p^0 p'^0 \) in the ultrarelativistic limit.

Organizing its factors in the same manner as for the seagull, the graviton triangle diagram, Fig. 5 is given by

\[
iM_2^{tri} = \left( \frac{i \kappa}{2} \right)^4 \left( \frac{i}{2} \right)^3 (2\pi)^{-4} (2M_\sigma^2)^2 \frac{1}{2} \times (2M_\sigma)^{-1} (-2\pi i) \int d^4k_1 d^4k_2 \delta(k_1^0) \delta(k_2^0) \times \delta^0(k_1 + k_2 + \Delta) \tau^{\mu \nu} (p, p') L_{\mu \nu \sigma \lambda} \frac{E_\phi}{E} \times \frac{2 \delta_{\alpha_1 \beta_1} \delta_{\alpha_2 \beta_2} - \eta_{\alpha_1 \beta_1} \eta_{\alpha_2 \beta_2}}{k_1^2 + \i \epsilon} \times \tau^{\alpha_1 \beta_1 \alpha_2 \beta_2 \eta \lambda} (k_1, k_2),
\]

where \( \tau^{\alpha_1 \beta_1 \alpha_2 \beta_2 \eta \lambda} (k_1, k_2) \) is the three graviton vertex, the moderately lengthy expression for which may be found in [15, 17].

The relevant numerator factors for the triangle diagram are found by a straightforward calculation, and give

\[
(2\delta_{\alpha_1 \beta_1} \delta_{\alpha_2 \beta_2} - \eta_{\alpha_1 \beta_1} \eta_{\alpha_2 \beta_2}) \times \tau^{\mu \nu} (p, p') L_{\mu \nu \sigma \lambda} \tau^{\alpha_1 \beta_1 \alpha_2 \beta_2 \eta \lambda} (k_1, k_2) = 8 E_\phi^2 (k_1^2 + k_2^2) + \cdots.
\]

Terms not explicitly included are either suppressed by the ratio \( p \cdot p' / E_\phi^2 \) or are proportional to \( k_i^2, i = 1, 2 \). The former are negligible to first non-leading power, while the second give integrals that vanish in dimensional regularization.

Applying Eqs. (26)-(29), we find for the sum of the seagull and triangle diagrams,

\[
iM_2^{1b} (\Delta) = 8iM_\sigma (2\pi)^3 \left( \frac{-\kappa^2 M_\sigma E_\phi}{4(2\pi)^3} \right)^2 \frac{1}{2} \times \int d^3k_1 d^3k_2 \delta^3(k_1 + k_2 + \Delta) \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{\Delta^2 + k_1^2} \times \frac{1}{\Delta^2 + k_2^2} = 2(S - M_\sigma^2) i \chi_2^b (\Delta^\perp),
\]

where in the second equality we isolate the tree-level prefactor. This correction to the amplitude in (30) has the same power behavior in \( E_\phi \) and \( \Delta \) as the lowest-order power corrections given above in Eq. (24). The dimensionless factor that multiplies the prefactor is

\[
i\chi_2^b (\Delta^\perp) = i \left( \frac{\kappa^4 M_\sigma^2 E_\phi}{16(2\pi)^3} \right) \times \int d^3k_1 d^3k_2 \delta^3(k_1 + k_2 + \Delta) \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{\Delta^2 + k_1^2} \frac{1}{\Delta^2 + k_2^2}.
\]

These integrals are not difficult to evaluate, using dimensional regularization in terms of which the relevant integrals are finite [15, 16]. The result is,

\[
i\chi_2^b (\Delta^\perp) = i \left( \frac{\kappa^4 M_\sigma^2 E_\phi}{16(2\pi)^3} \right) \left( \frac{15\pi^3}{16} \right) \frac{1}{\Delta^\perp}.
\]

For use below, we transform \( \chi_2^b \) to impact parameter space,

\[
i\chi_2^b (b^\perp) = i \int \frac{d^2 \Delta}{(2\pi)^2} e^{ib^\perp \cdot \Delta} = i \left( \frac{\kappa^3 M_\sigma^2 E_\phi}{256\pi} \right) \left( \frac{15}{16} \right) \frac{1}{b^\perp}.
\]

We will encounter this contribution to the imaginary part in our discussion of \( n \)-graviton exchange in the next subsection.

Finally, we note that the diagrams that are mirror reflections of those in (Fig. 4) and (Fig. 5) (i.e., those with the two distinct matter-graviton vertices on the light line) are suppressed in the limit of a heavy sigma particle. This simplifies the combinatorics of the next section and is a practical advantage of this particular kinematic limit.

2. \( n \)-Graviton Exchange

We now consider the \( n \)-graviton case. Let us first start by considering the contribution from expanding the light scalar propagators. From (15) and (16) we see that

\[
iM_2^{1\alpha} (\Delta) = -2i(2\pi)^3 M_\sigma \left( \frac{-\kappa^2 M_\sigma E_\phi^2}{2(2\pi)^3} \right)^n (2E_\phi)^{-n} \times \int d^3k_1 \ldots d^3k_n \delta^3(k_1 + \ldots + k_n + \Delta) \prod_{i=1}^n \frac{1}{k_i^2} \times \prod_{i=1}^{n-1} \frac{1}{K_i^2 + \i \epsilon} \sum_{i=1}^{n-1} \frac{1}{K_i^2 + \i \epsilon},
\]

where the sum at the end corresponds to replacing each of the \( n - 1 \) scalar propagators with the second order term from (16) one time. In these expressions, the identity (10) that reduces all graviton energies to zero has already been employed. The expression is not symmetric in the
The second equality, we organize the terms into two groups, labelled \( J_m \) and \( J_{\alpha\beta} \): those that have \( k_m^\perp \) in the numerator and those that have \( k_{\alpha} \cdot k_{\beta} \) in the numerator, respectively. We have also absorbed into the \( J \) one of the second equalities of Eq. (35), and is proportional to

\[
iM^{1a}_n = -\frac{2i}{n!}(2\pi)^3 M_\sigma \left( -\frac{\kappa^2 M_\sigma E_\phi}{2(2\pi)^3} \right)^n (2E_\phi)^{-n}
\times \int d^3k_1 \ldots d^3k_n \delta^3(\mathbf{k}_1 + \ldots + \mathbf{k}_n + \Delta) \prod_{i=1}^n \frac{1}{k_i^2}
\times \sum_{\text{Perms over } k_i} \left[ \prod_{i=1}^{n-1} \frac{1}{K_i^2 + i\epsilon} \sum_{j=1}^{n-1} \frac{K_j^2}{K_j^2 + i\epsilon} \right]
= \frac{2i}{n!}(2\pi)^3 M_\sigma \left( -\frac{\kappa^2 M_\sigma E_\phi}{4(2\pi)^3} \right)^n
\times \prod_{i=1}^n \int d^2k_i^\perp \delta^2 \left( \sum_{j=1}^n k_j^\perp + \Delta^\perp \right)
\times \left( \sum_{m=1}^n J_m + \sum_{\alpha\beta} J_{\alpha\beta} \right), \tag{35}\]

In the second equality, we organize the terms into two groups, labelled \( J_m \) and \( J_{\alpha\beta} \): those that have \( k_m^\perp \) in the numerator and those that have \( k_{\alpha} \cdot k_{\beta} \) in the numerator, respectively. We have also absorbed into the \( J \) one of the second equalities of Eq. (35), and is proportional to

\[
\int \frac{d^2\Delta}{(2\pi)^2} e^{ib^\perp \cdot \Delta^\perp} \prod_{i=1}^n \int d^2k_i^\perp e^{-ib^\perp \cdot k_i^\perp} \frac{dk_i^z}{k_i^2}
\times \left( k_m^\perp \frac{\partial}{\partial k_m^\perp} \right) \prod_{j=1}^{n-1} \frac{1}{K_j^2 + i\epsilon}. \tag{36}\]

We see immediately that the factor of \( k_m^\perp \) from the eikonal expansion cancels all explicit \( k_i^2 \)-dependence aside from the exponential, which then gives \( \delta^2(\mathbf{b}^\perp) \). In effect, the remaining interactions, which are treated in eikonal approximation, are forced to zero impact parameter.  \(^2\) As discussed above, we will not consider these contributions further.

We next analyze the factors including non-diagonal numerator momentum products in Eq. (35), which will contribute for all impact parameters. We begin by isolating the coefficients of each such momentum factor,

\[
J_{\alpha\beta} = \prod_{i=1}^n \int dk_i^z \cdot k_{\alpha} I_{\alpha\beta}, \tag{37}\]

where

\[
I_{\alpha\beta} = \sum_{\text{Perms over } k_i} \left[ \prod_{i=1}^{n-1} \frac{1}{K_i^2 + i\epsilon} \delta \left( \sum_{j=1}^n k_j^2 \right) \right]
\times \sum_{K_i \in \kappa(\alpha, \beta)} \frac{1}{K_i^2 + i\epsilon}. \tag{38}\]

Here, \( \kappa(\alpha, \beta) \) is the set of denominators \( K_i \) that include both \( k_\alpha \) and \( k_\beta \).

We now derive a variation of the standard eikonal identity applicable to such sums, with squares of denominators involving pairs of momenta, \( k_\alpha \) and \( k_\beta \). Such denominators occur in two sets of diagrams, those in which \( k_\alpha \) appears alone and \( k_\beta \) only with \( k_\alpha \), and those in which the roles of \( k_\alpha \) and \( k_\beta \) are reversed. We may think of these as diagrams in which the graviton with momentum \( k_\alpha \) is emitted by the light scalar line before \( k_\beta \), and the other way around. This suggests that we rewrite \( I_{\alpha\beta} \) as integrals over “times”, \( x_i \), \( i = 1 \ldots n \). In any permutation of the momenta we can always order \( x_n \geq x_{n-1} \ldots \geq x_1 \), where \( \alpha, \beta \neq n \). Because the function \( I_{\alpha\beta} \) is completely symmetric in the permutations that relate these orderings, the sum over all its diagrams can be written in terms of integrals over the \( x_i \)’s as a sum over the ‘latest time’ \( x_n \), with free integrals up to \( x_i = x_n \) for all \( i < n \),

\[
I_{\alpha\beta} = -\sum_{n \neq \alpha, \beta} \int_{-\infty}^{x_n} \frac{dx_n}{2\pi} e^{ix_n k_n^z}
\times (-i)^{n-1} \left[ \frac{\partial}{\partial k_n^z} \prod_{i=1}^{n-1} \int_{-\infty}^{x_n} dx_i e^{-i(k_i^z + i\epsilon)x_i} \theta(x_\beta - x_\alpha) + \frac{\partial}{\partial k_\alpha^z} \prod_{i=1}^{n-1} \int_{-\infty}^{x_n} dx_i e^{-i(k_i^z + i\epsilon)x_i} \theta(x_\alpha - x_\beta) \right]. \tag{39}\]

where the sum for fixed \( \alpha \) and \( \beta \) is over choices of momentum \( k_n \) that do not appear in any denominators. For fixed \( k_n \), the unrestricted integrals over the \( x_i \)’s generate every term in \( I_{\alpha\beta} \), with each such term corresponding to one ordering of the \( x_i \)’s, including \( x_\alpha \) and \( x_\beta \). The derivatives then double each denominator in the set \( \kappa(\alpha, \beta) \). For any fixed \( k_n \), the \( x_n \) integral provides the \( k^2 \) delta function for all of the orderings. We can do all the integrals (39) trivially, thus combining \((n-1)!\) diagrams for each choice of \( k_n \), and then take the derivatives. The

---

\(^2\) In fact, the coefficient of \( \delta^2(\mathbf{f}) \) is real and infrared divergent in this case.
In the second equality we have inserted unity in the form of an integral of a new momentum, $k'^z$, which is fixed by a delta function. In effect, on the light scalar ($p$) line, the two graviton momenta $k_α^z$ and $k_β^z$ have been replaced by a single momentum $k'^z$, which is set to zero along with other single graviton momenta. The $k_α^z$ and $k_β^z$ integrals remain, but can be treated as a separate loop integral. In this form, we can employ the eikonal identities of Eq. (10), which will set $k'^z$ to zero, along with all other $k_γ^z$ carried by the light scalar line. It will be important, however, to do the $k_α$ and $k_β$ integrals first at fixed $k'^z$, before using relations like Eq. (10), because the new integral is not guaranteed to commute with such distribution identities. With this in mind, we apply Eq. (10), but introduce a notation that implies that this algebraic distribution identity is applied only after the $k_α$ and $k_β$ integrals,

\[
I_{α, β} = \sum_{k_α ≠ α, β} δ \left( \sum_{j=1}^{n} k_j^z \right) \prod_{i=1 \atop i ≠ α, β}^{n-1} \frac{1}{k_i^z + iε} \times \frac{1}{k_α^z + ε} \frac{1}{k_β^z + iε} \times \int d k'^z \delta (k'^z - k_α^z - k_β^z) \frac{1}{k'^z + iε} \frac{1}{k_α^z + iε} \frac{1}{k_β^z + iε}.
\]

(40)

where the term in square brackets will be inserted in the integrals over $k_α$ and $k_β$, and the limit taken. We now insert our result into the expression for $J_{α, β}$, (37), and then to $i\mathcal{M}_n^α$, Eq. (35), and take the transform to impact parameter space. This gives the first non-leading power as a product of leading exponents, times an integral over $k_α$ and $k_β$,

\[
i\tilde{\mathcal{M}}_n^α \left( b^+ \right) = -\frac{2}{(n-2)!} \frac{M_α}{2π} \left( \frac{κ^2 M_σ E_φ}{4(2π)^2} \right)^n \times \int d^2k_α^⊥ d^2k_β^⊥ e^{-ik_α^⊥ \cdot b^+} \delta \left( k'^z - k_α^z - k_β^z \right) \frac{k_α^⊥ \cdot k_β^⊥}{|k_α^⊥ + iε||k_β^⊥ + iε|} - \frac{2}{(n-2)!} (s - M_α^2) (i\tilde{χ}_0 \left( b^+ \right))^n - 2(i\tilde{χ}_2 \left( b^+ \right))
\]

where $\tilde{χ}_0$ is given in Eq. (21) and where we define,

\[
i\tilde{χ}_2 \left( b^+ \right) = i\frac{κ^4 M_σ^2 E_φ}{32(2π)^5} \lim_{k'^z → 0} \int d^2k_α^⊥ d^2k_β^⊥ e^{-ik_α^⊥ \cdot b^+} \delta \left( k'^z - k_α^z - k_β^z \right) \frac{k_α^⊥ \cdot k_β^⊥ + k_α^z k_β^z}{|k_α^⊥ + iε||k_β^⊥ + iε|}.
\]

(43)

In deriving Eq. (42) we have used the fact that every choice of $α$, $β$ gives the same contribution so we can drop the sum and just multiply by $n(n-1)$. Eq. (43), like the leading power phase, is purely imaginary. We evaluate the integral in the appendix, and will combine its result with the correction due to the gravitational corrections of Figs. 4 and 5, to which we now turn.

Let consider the next-to-eikonal contributions arising from a combination of multiple gravitons with a single seagull at the light scalar line or graviton triangle at the heavy line. We have already computed the lowest-order examples in momentum space, resulting in Eq. (30). We wish to embed these diagrams into diagrams with $n - 2$ additional gluon exchanged, treated at leading power. In each such diagram, there will be $n$ vertices on the heavy scalar line and $n - 1$ on the light scalar line. As in the examples above, the use of identity (10) will simplify the answers, and as in (42), we will find a factorization between powers of the leading-power phase, and a new imaginary power correction, which by analogy to Eq. (43) we will label $i\tilde{χ}_b^2$. 

\[
i\overline{\mathcal{M}}_n^α \left( b^+ \right) = -\frac{2}{(n-2)!} \frac{M_α}{2π} \left( \frac{κ^2 M_σ E_φ}{4(2π)^2} \right)^n \times \int d^2k_α^⊥ d^2k_β^⊥ e^{-ik_α^⊥ \cdot b^+} \delta \left( k'^z - k_α^z - k_β^z \right) \frac{k_α^⊥ \cdot k_β^⊥ + k_α^z k_β^z}{|k_α^⊥ + iε||k_β^⊥ + iε|}.
\]

(41)
C. Multigraviton exchange beyond ladders

Consider first the sum of $n$-graviton exchange diagrams with a single seagull, which can be written as

\[ i\mathcal{M}_n^{1b}(\Delta) = \frac{(2\pi)^4}{E_\phi^2} \left( \frac{\kappa^2 M_g E_\phi^2}{(2\pi)^4} \right)^n \frac{1}{2} \times \int d^4k_1 \cdots d^4k_n \delta^4(k_1 + \cdots + k_n + \Delta) \]

\[ \times \prod_{i=1}^n \frac{1}{k_i^2} \sum_{j=1}^{n-1} \prod_{i \neq j} \frac{1}{(p - K_i)^2 + i\epsilon} \]

\[ \times \sum_{\text{perms of } k_i} \frac{1}{(q + k_1)^2 - M_g^2 + i\epsilon} \]

\[ \times \cdots (q + k_1 + \cdots + k_{n-1})^2 - M_g^2 + i\epsilon \]

(44)

The products and sum in the third line generate all possible places where the seagull vertex can be placed on the light scalar line, and the permutations in the third line generate all of the possible diagrams for each placement of this vertex. Note there is an over counting because exchanging the order of the seagull legs does not result in distinct diagrams, so we divide by 2. A similar manipulation can be carried out for the three-gluon triangle diagram.

The integrals of Eq. (44) and the corresponding expression where a three-gluon triangle replaces the seagull diagram, include the same sum of permutations over $n - 1$ heavy particle ($q$) propagators, with the same overall delta function ensuring zero energy transfer. Thus we can apply (10) in the ultrarelativistic limit as we did for the purely eikonal contribution. On the light scalar line, there are now $n - 1$ vertices and $n - 2$ propagators. Of these, $n - 2$ vertices connect to single gravitons, and at a single vertex to the sum of the seagull (Fig. 4) and graviton triangle diagrams (Fig. 5). In the notation of Eq. (44), the latter vertex is in position $j$, $1 \leq j \leq n - 1$, and carries momentum $k_j \equiv k_j + k_{j+1}$ out of the light scalar line. Here, momenta $k_j$ and $k_{j+1}$ play the role that momenta $k_n$ and $k_3$ played in our analysis of $n$-single graviton exchange above.

The application of Eq. (10) now factorizes the dependence of all of all single-graviton momenta on the heavy scalar line, setting all graviton energies to zero, and we find

\[ i\mathcal{M}_n^{1b} = \left( \frac{-\kappa^2 M_g E_\phi}{4(2\pi)^3} \right)^{n-2} \times \sum_{j=1}^{n-1} \int d^3k_j' i\mathcal{M}_2^{1b}(k_j') \]

\[ \times \prod_{i=1}^{n-1} \int \frac{d^3k_i}{k_i^2} \delta^3(\sum_{i=1}^{n} (j)k_i + \Delta) \]

\[ \times \prod_{i=1}^{n-1} \frac{1}{K_i^2 + i\epsilon} . \]

(45)

where we absorb a three-dimensional delta function that sets $k_j' = k_j + k_{j+1}$ into the function, $\mathcal{M}_n^{1b}(k_j')$, which then becomes the same function of momentum transfer as in Eq. (30) above, including its prefactor. The superscript $(j)$ refers to this change, and indicates that in the sums and products $k_j$ and $k_{j+1}$ are combined to a single term, $k_j'$.

We have in Eq. (44) all placements of vertex $j$, which carries momentum $k_j'$. We may sum over all permutations of the $n - 2$ remaining vertices on the p line, which are all indistinguishable, and must be compensated for by an overall factor of $1/(n - 2)!$. We thus have

\[ i\mathcal{M}_n^{1b} = \frac{1}{(n - 2)!} \left( \frac{-\kappa^2 M_g E_\phi}{4(2\pi)^3} \right)^{n-2} \times \int d^3k_j' i\mathcal{M}_2^{1b}(k_j') \]

\[ \times \prod_{i=1}^{n-1} \int \frac{d^3k_i}{k_i^2} \delta^3(\sum_{i=1}^{n} (j)k_i + \Delta) \]

\[ \times \delta(\sum_{i=1}^{n} (j)k_i) \prod_{k_i} \text{perms of } \frac{1}{K_i^2 + i\epsilon} . \]

(46)

For the set of momenta $k_1, \ldots, k_{j-1}, k_j', k_{j+2}, \ldots, k_n$, the final line is in precisely the form necessary to apply the identity (10), so that we can set all of their $z$ components to zero,

\[ i\mathcal{M}_n^{1b}(\Delta) = \frac{1}{(n - 2)!} \left( \frac{\kappa^2 M_g E_\phi}{4(2\pi)^2} \right)^{n-2} \times \int d^3k_j' i\mathcal{M}_2^{1b}(k_j') \]

\[ \times \prod_{i=1}^{n-1} \int \frac{d^2k_i}{k_i^2} \delta^2(\sum_{i=1}^{n} (j)k_i + \Delta) . \]

(47)

This expression is ready to be transformed to impact parameter space, which gives

\[ i\tilde{\mathcal{M}}_n^{1b}(b) = 2(s - M_g^2)^{n-2} \int_{(n-2)!} i\chi_2(b) , \]

(48)
where \( \hat{M}^{1b} \) is the transform of the same combination of seagull and vertex diagrams given in Eq. (30), and where \( \chi_0^b \) is given in momentum and impact parameter space by Eqs. (32) and (33), respectively.

Summing over all \( n \) in \( i\hat{M}^0 + i\hat{M}^1 \), and We can now combine the leading order result of Eq. (20) with the first non leading powers from expanding light scalar denominators (42) and adding gravitational corrections (48), to find,

\[
i\hat{M}^{0+1}(\mathbf{b}^\perp) = 2(s - M_b^2)(1 + i\chi_2(\mathbf{b}^\perp))[e^{i\chi_0(\mathbf{b}^\perp)} - 1]
\]

where power corrections are included in the imaginary term \( i\chi_2 \), with one contribution from expanding denominators and another from the seagull and graviton triangle, Eq. (33) for \( \chi_2^b \) and (A20), respectively,

\[
\tilde{\chi}_2(\mathbf{b}^\perp) = \tilde{\chi}_2^b(\mathbf{b}^\perp) + \tilde{\chi}_2^g(\mathbf{b}^\perp) = 8(GM_b)^2E_\phi \left(-1 + \frac{15\pi}{32}\right) \frac{1}{\mathbf{b}^\perp}.
\]

These are our basic results. The power correction has left the exponentiation of leading corrections unchanged, and indeed, is consistent with the exponentiation of the correction itself, since it is a pure phase. The terms found from this exponentiation would be further suppressed by powers, however, so that we will not discuss them further here.

We now see how the power correction affects the amplitude by transforming back to momentum space. We start by rewriting (49) as

\[
i\hat{M}^{0+1} \approx 2(s - M_b^2)[e^{i(\chi_0 - i\ln[1 + i\chi_2])} - 1],
\]

to find the correction to the stationary phase point (22) due to \( \chi_2 \),

\[
\hat{M}^{0+1}(\mathbf{\Delta}^\perp) = 2(s - M_b^2) \int d^2\mathbf{b}^\perp e^{-i\mathbf{\Delta}^\perp \cdot \mathbf{b}^\perp} [e^{i(\chi_0 - i\ln[1 + i\chi_2])} - 1].
\]

As usual, the condition \( \frac{\partial}{\partial\mathbf{b}^\perp} \text{Phase} = 0 \) determines the new saddle point. In terms of \( R_s = 2GM_\phi \), this condition reduces to

\[
\mathbf{\Delta}^\perp(\mathbf{b}^\perp)^2 - 2R_sE_\phi\mathbf{b}^\perp + (\frac{15\pi}{32} - 1)2R_s^2E_\phi = 0.
\]

The relevant solution, at large impact parameter, is

\[
\mathbf{b}^\perp \sim R_s \left(\frac{2E_\phi}{\Delta} - 1\right) \left(\frac{15\pi}{32} - 1\right).
\]

The first term on the right is the leading order saddle point for the impact parameter, and the second term is the correction due to the next to leading power eikonal phase. While the leading order term shows that the impact parameter differs from the Schwarzschild radius of the heavy particle by a large factor of \( E_\phi/\Delta^\perp \), no such enhancement is present for the correction. Thus as expected, the leading order result suggests that the scattering process is dominated by large values of the impact parameter for near-forward scattering. Nevertheless, the power correction to the leading eikonal result shifts by a finite factor times the Schwarzschild radius. Corrections are proportional to the ratio of momentum transfer to projectile energy, and are independent of the heavy particle mass. This suggests that for the system under study, small angle scattering is a transition region, where corrections to eikonal propagation and gravitational self-interactions are of comparable sizes. In this light it may be interesting to look at yet higher powers in the our expansion about the eikonal approximation.

### III. CONCLUSIONS

In this paper we have calculated the next-to-eikonal contribution to the gravitational scattering amplitude for the near-forward scattering of a light scalar by a much heavier scalar particle. Working at leading power in the heavy particle mass, we expanded the light particle propagator to next-to-eikonal power and included gravitational interactions of comparable size, finding power corrections suppressed by a single power of \( \Delta/E_\phi \) with respect to the leading eikonal term. Corrections are pure phases, leaving leading-power exponentiation unaffected, and are consistent with the exponentiation of the power corrections themselves. The next-to-eikonal and gravitational corrections cause the saddle point of the impact parameter to shift in magnitude by an amount comparable to the Schwarzschild radius of the heavy particle. Our calculations are of relevance for the small angle scattering of a light particle from a black hole. The power corrections we find are potentially bigger than in the case of scattering of equal mass particles.

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### Appendix A: Useful Integrals

In this appendix we evaluate the integrals used in the main text.
1. \( \chi_0 \) Integral

Here we evaluate the integral that is needed in (21),

\[
I_0 = \int \frac{d^2k_{\perp}}{(k_{\perp}^2)^2} e^{-ib^\perp \cdot k^\perp} = \int dk_{\perp} d\theta e^{-ik_{\perp} \cdot b \cos \theta} k^\perp_\perp^{d-5},
\]

(A1)

where in the last line we make the number of dimensions \( d \) general. In terms of Bessel functions we can write this as

\[
I_0 = 2\pi \int dk_{\perp} k^\perp_{\perp} J_0(k_{\perp} b) .
\]

(A2)

Note that since

\[
\int_0^\infty dyy^m J_0(y) = 2^{m} \frac{\Gamma[(1 + m)/2]}{\Gamma((1 - m)/2)},
\]

(A3)

for \(-1 < m < \frac{1}{2}\), we have

\[
I_0 = 2\pi b^{4-d} \frac{1}{\Gamma(6-d)/2}.  \]

(A4)

If we expand \( \Gamma((n-4)/2) \) about \( n = 4 \), we get

\[
I_0 \approx 2\pi \frac{1}{d-4} - \log b, \]

(A5)

Taylor expanding the non pole part and only keeping the finite part dependent on \( b \),

\[
I_0 \approx 2\pi \frac{1}{d-4} - \log b, \]

(A6)

which is used in Eq. (21).

2. \( \chi_2 \) Integral

The integral associated with \( \chi_2 \), Eq. (43), is

\[
i\chi_2 = \frac{\kappa^4 M_1^2 E_\phi}{32(2\pi)^5} I_2^a
\]

\[
I_2^a = \lim_{k^z \to 0} \int \frac{dk^x}{k^x} \frac{dk^y}{k^y} e^{-i(k_{\perp}^x + k_{\perp}^y) \cdot b^\perp} \]

\[
\times \delta (k^z - k^x_{\alpha} - k^y_{\beta}) \frac{k^x_{\alpha} \cdot k^y_{\beta} + k^x_{\alpha} k^y_{\beta}}{|k^x_{\alpha} + i\epsilon||k^y_{\beta} + i\epsilon|}. \quad (A7)
\]

We begin at fixed \( z \) components, and integrate over the transverse momenta, using

\[
\int \frac{d^2k_{\perp}^+}{k^2} e^{-ib^\perp \cdot k^\perp +} = 2\pi K_0 (b^\perp |k^\perp|). \quad (A8)
\]

for any real \( k^\perp \). For the transverse components \( i = x, y \) we then have

\[
\int d^2k_{\perp}^+ \frac{k^i}{k^2} e^{-ib^\perp \cdot k^\perp +} = 2\pi i \frac{\partial}{\partial b^i} K_0 (b^\perp |k^\perp|)
\]

\[
= 2\pi i \frac{b^i}{b^\perp} \frac{\partial}{\partial b^\perp} K_0 (b^\perp |k^\perp|)
\]

\[
= 2\pi i \frac{b^i}{b^\perp} |k_{\perp}^\perp| K_0^\prime (b^\perp |k^\perp|). \quad (A9)
\]

Substituting these results in (A7) now gives

\[
I_2^a = -\frac{(2\pi)^2}{b^\perp} \lim_{k^z \to 0} \int \frac{dk^x_{\alpha} d\theta^\perp_{\beta}}{k^z_{\alpha}} \delta (k^z - k^x_{\alpha} - k^y_{\beta}) \]

\[
\times [K_0^\prime (|y_{\alpha}^\perp|) K_0^\prime (|y_{\beta}^\perp|) \sigma(y_{\alpha}^\perp y_{\beta}^\perp) - K_0 (|y_{\alpha}^\perp|) K_0 (|y_{\beta}^\perp|) ] . \quad (A10)
\]

where \( \sigma(\pm x) = \pm 1 \) for \( x > 0 \) arises from canceling the poles in \( k^z \) with absolute values. The arguments of the Bessel functions are simplified by rescaling the integration variables with \( b^\perp \),

\[
I_2^a = -\frac{(2\pi)^2}{b^\perp} \lim_{k^z \to 0} \int \frac{dy^x_{\alpha} dy^y_{\beta}}{k^z_{\alpha}} \delta (k^z b^\perp - y^x_{\alpha} - y^y_{\beta}) \]

\[
\times [K_0^\prime (|y_{\alpha}^\perp|) K_0^\prime (|y_{\beta}^\perp|) \sigma(y_{\alpha}^\perp y_{\beta}^\perp) - K_0 (|y_{\alpha}^\perp|) K_0 (|y_{\beta}^\perp|) ] . \quad (A11)
\]

To proceed, we use the following representation of the delta function,

\[
\delta (k^z b^\perp - y^x_{\alpha} - y^y_{\beta}) = \frac{1}{2\pi} \lim_{\delta \to 0} \int_{-\infty}^{\infty} da e^{i(y_{\alpha}^\perp y_{\beta}^\perp - k^z b^\perp + i\delta) a} + \int_{-\infty}^{\infty} da e^{i(y_{\alpha}^\perp y_{\beta}^\perp - k^z b^\perp - i\delta) a}. \quad (A12)
\]

Consider first the contribution from the \( a > 0 \) term, leaving the limits \( k^z \to 0 \) and \( \delta \to 0 \) implicit,

\[
I_{2A}^z = \frac{2\pi i}{b^\perp} \int_0^\infty da e^{i(y^x_{\alpha} y^y_{\beta} - k^z b^\perp + i\delta) a} \]

\[
\times [K_0^\prime (|y_{\alpha}^\perp|) K_0^\prime (|y_{\beta}^\perp|) \sigma(y_{\alpha}^\perp y_{\beta}^\perp) - K_0 (|y_{\alpha}^\perp|) K_0 (|y_{\beta}^\perp|) ] . \quad (A13)
\]

We treat the two terms in brackets separately, labeling them \( I_{2A}^z \) and \( I_{2B}^z \). For the first, with the derivatives of the Bessel function, we have

\[
I_{2A}^z = \frac{i2\pi i}{b^\perp} \int_0^\infty da e^{i(y^x_{\alpha} y^y_{\beta} - k^z b^\perp + i\delta) a} \]

\[
\times [K_0^\prime (|y_{\alpha}^\perp|) K_0^\prime (|y_{\beta}^\perp|) \sigma(y_{\alpha}^\perp y_{\beta}^\perp) - K_0 (|y_{\alpha}^\perp|) K_0 (|y_{\beta}^\perp|) ] . \quad (A14)
\]

\[
= \frac{2\pi i}{b^\perp} \int_0^\infty da e^{-iak^z b^\perp} \left[ \int_{-\infty}^{\infty} dy e^{i\alpha y^x_{\alpha} y^y_{\beta} - k^z b^\perp + i\delta) a} \right] ^2 . \quad (A15)
\]
The $y$ integrals can now be carried out by dividing the integration region,

\[
\int_{-\infty}^{\infty} dy \ e^{iya} K'_0(|y|) \sigma(y) \\
= \int_{0}^{\infty} dy \ e^{iya} K'_0(|y|) - \int_{-\infty}^{0} dy \ e^{iya} K'_0(|y|)
\]

\[
= 2i \int_{\epsilon}^{\infty} dy \ \sin(ya) K'_0(y) + \int_{-\epsilon}^{\epsilon} dy \ K'_0(y),
\]

(A15)

where we have split off an infinitesimal region $(-\epsilon, \epsilon)$ in the integral around the origin. For consistency in the integration region, $y = 0$ must be defined near the origin by the replacement $K'_0(y) \approx -\frac{1}{y} \rightarrow -\frac{1}{y + \epsilon}$. Then, using the principal value prescription near $y = 0$ and taking the limits $\delta \to 0$ and $k^2 \to 0$, we find that

\[
\int_{-\infty}^{\infty} dy \ e^{iya} K'_0(|y|) \sigma(y) = i\pi \left(1 - \frac{a}{\sqrt{a^2 + 1}}\right)
\]

(A16)

Inserting this result into (A14), we find for the first term

\[
I^a_{2A} = -\frac{i(2\pi)^3}{4b^4} \int_{0}^{\infty} da \left(1 - \frac{a}{\sqrt{a^2 + 1}}\right)^2
\]

\[
= \frac{i(2\pi)^3}{4b^4} \left(\frac{\pi}{2} - 2\right).
\]

(A17)

We note now that the integral $I^a_B$ would have been infrared divergent if not for the use of the ordered limits in $k^2$ and $\delta$ to account for the pole structure. The contribution from $a < 0$ gives the same result ($I^a_{2A} = I^a_{2A}^+$) so that the full contribution from the differentiated Bessel functions is

\[
I^a_{2A} = \frac{i(2\pi)^3}{2b^4} \left(\frac{\pi}{2} - 2\right).
\]

(A18)

For the second term in $I^a_B$, Eq. (A7), with no derivatives on $K_0$, the limits can be imposed inside the integrands, and we can do the $a$ integral all at once,

\[
I^a_{2B} = \frac{2\pi i}{b^4} \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} dy \ e^{i(y_2^a + y_2^b)a} \left(y_2^0\right) K_0(|y_2^a|) \\
= \frac{2\pi i}{b^4} \int_{-\infty}^{\infty} da \left(\int_{-\infty}^{\infty} dy \ e^{iya} K_0(|y|)\right)^2
\]

\[
= \frac{i(2\pi)^3}{4b^4} \int_{-\infty}^{\infty} da \frac{1}{a^2 + 1}
\]

\[
= \frac{i(2\pi)^3}{2b^4} \left(\frac{\pi}{2}\right).
\]

(A19)

and the full expression for $\chi^a_2$ is

\[
i\chi^a_2 (b^\perp) = \frac{\kappa^4 M_2^b E_0}{32(2\pi)^8} (I^a_{2A} - I^a_{2B})
\]

\[
= -\frac{i}{b^4} \frac{\kappa^4 M_2^b E_0}{128\pi^2},
\]

(A20)

which is used in Eq. (50).