A New Approach to Inverse Local Times

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Abstract
In 1981 F. Knight published an article with a partial solution to a problem proposed by Ito-McKean [6], [p.217]. In this paper Knight [8] characterized the Levy measures of gap diffusions also known as quasi-diffusions. The proof is very elegant but it uses quite a lot functional analysis, more specifically spectral Krein Theory. We present a new proof of Knight’s Theorem, defined at the beginning of the Introduction as well as the new proof of the same theorem referred as Theorem 2.7 in Section 2.

1 Introduction
In 1981 F. Knight published an article with a partial solution to a problem proposed by Ito-McKean [6]. In this paper Knight [8] characterized the Levy measures of gap diffusions also known as quasi-diffusions. The proof is very elegant but it uses functional analysis, more specifically spectral Krein Theory.

The task of re-proving such result with different techniques was sugested by the late Dr. Martin L. Silverstein to whom I am deeply grateful for his advice and help. The present work provides a constructive proof of the following Knigth’s theorem which allows a better probablistic interpretation of every step of the process.

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Theorem: The Levy measure \( n(dy) \) of a persistent gap diffusion on \([0, \infty)\) reflected at 0 is absolutely continuous with density

\[
n(y) = \int_0^\infty e^{-y x} \mu(dx)
\]

with measure \( \mu(dx) \geq 0 \) on \((0, \infty)\) such that

\[
\int_0^\infty \frac{1}{x(1+x)} \mu(dx) < \infty.
\]

The techniques used are totally different from those previosly used.
In Section 2 we state but rarely prove the material needed for the proof of main result refered as Theorem 2.7. Exact references will be provided in Appendix 3. Section 2 contains the new proof of Knight’s Theorem A as defined at the beginning of the chapter.

2 Main Tools

Firstly, let us define (see Watanabe [11]) what is called a gap-diffusion

**Definition 2.1.** Let \( 0 < l \leq \infty \). A non negative Borel measure \( m(dx) \) on \([0, l]\) is called an inextensible measure on \([0, l]\) if there exist a non-negative Borel measure which is finite on each compact set \( m'(dx) \) on \([0, l]\) such that, by extending \( m'(dx) \) on \([0, l]\) so that \( m'\{l\} = 0 \)

\[
m(dx) = \begin{cases} 
m'(dx) & \text{if } l = \infty \text{ or } m'([0, l]) = \infty \\
m'(dx) + \infty \delta_l(dx) & \text{if } l + m'([0, l]) < \infty.
\end{cases}
\]

where \( \delta_l \) is the unit measure at \( x = l \).

Let \( m(dx) \) an inextensible measure on \([0, l]\) and \( E_m \in [0, \infty] \) be the support of \( m \). Let us assume, fora for simplicity, \( 0 \in E_m \).

Let \( X = (X_t, P_x)_{x \in \mathbb{R}} \) be a one dimensional standard Brownian motion and \( \Phi(t, x), (t \geq 0, x \in \mathbb{R}) \) be its local time at \( x \)

\[
\int_0^l I_A(X_s) \, ds = 2 \int_A \Phi(t, x) \, m(dx), \quad \forall A \subset \mathcal{B}(\mathbb{R}).
\]
Let
\[ \Theta(t) = \int_{[0,l]} \Phi(t,x) m(dx) \]
and \( \Theta^{-1}(t) \) be the inverse function of \( \Theta(t) \) (inverse local time) and \( \hat{X}(t) = X(\Theta^{-1}(t)) \). By the general theory of time change, \( \hat{X}(t) \) defines a Markov process on \( E_m = E_m \cap [0,l] \) whose life time is identified with the first hitting time for \( l \).

**Definition 2.2.** The Markov process \( \hat{X}(t) = X(\Theta^{-1}(t)) \) above mentioned is called the gap-diffusion process corresponding to the inextensible measure \( m(dx) \).

Let us introduce the following convenient notation:

- \( \tau(t,a,w) = \) local time spent at \( a \) up to time \( t \) by trajectory \( w \) starting at \( a \)
- \( \tau(t,a) = \tau(t,a,w) \)
- \( \tau(t) = \tau(t,0) \)
- \( \tau(t,w) = \tau(t,0,w) \)

**Lemma 2.1.** Let \([W,B,P,a] \in E_1 = \{x_0,x_1\}\) be a gap-diffusion with two points state and let \( \{H_0,H_1\} \) be exponential holding times distributions at point \( x_0 \) and \( x_1 \) with rates \( \{a_0,a_1\} \). Thus
\[
P(H_i > t) = \int_t^\infty h_i(l) \, dl = e^{-a_it}, \quad i = 0,1; \quad a_i > 0.
\]

Define \( c(t) = \) \{number of jumps from \( x_0 \) to \( x_1 \) before time \( t \)\}. Then
\[
P_0(\tau^{-1}(t,w) - t \in ds) = \sum_{k=0}^{\infty} P_0(\tau^{-1}(t,w) - t \in ds | c(t) = k) \quad (1)
\]
with
\[
P_0(c(t) = k) = P_0(H_0^1 + H_0^2 + \ldots + H_0^{k+1} \geq t) - P_0(H_0^1 + H_0^2 + \ldots + H_0^k \geq t)
\]
\[
= \frac{e^{-a_0t}(a_0t)^k}{k!}, \quad k \geq 1
\]
and $H_0$ are such that $H_0 = H_0, \forall 1 \leq i \leq n$. Therefore
\[
P_0(\tau^{-1}(t) - t \in ds) = \sum_{k=0}^{\infty} \frac{e^{-a_0 t}(a_0 t)^k}{k!} S_k^*\]
where $S$ is the probability distribution of $\tau^{-1} - t$ conditioned to $c(t) = 1$.

**Proof:** Will be omitted. ♦

In this particular case $S$ is absolutely continuous with density
\[
S_0(x) = a_1 e^{-a_1 x}, \ x \geq 0
\]
but the argument is general and can be applied to a quasi-diffusion with any finite state space. Thus we can conclude that the inverse local time (at 0) for a gap-diffusion with finite state space is a compound Poisson process with the following probability distribution
\[
\tau^{-1}(t) - t = \sum_{k=0}^{\infty} \frac{e^{-a_0 t}(a_0 t)^k}{k!} S_k^*
\]

\[
S(dx) = P_0(\tau^{-1}(t, w) - t \in dx \mid c(t) = 1) = S(x)dx \quad (\text{II})
\]
as (I) in Lemma 2.1

**Lemma 2.2.** Let $[W, B, P, a \in \{0, 1, 2, \ldots, n\}]$ be a gap-diffusion with finite state space. Let $n(ds)$ be its Levy measure, then
\[
S(x)dx = n(dx) \quad ♦
\]

**Proof:** By Theorem 3.3 we have
\[
E_0[e^{-z\tau^{-1}(t)}] = e^{-t[m_0 z + \int_0^\infty (1-e^{-zx})n(dx)]}
\]
and by Fubini’s Theorem
\[
E_0[e^{-z\tau^{-1}(t)}] = \sum_{k=0}^{\infty} \frac{e^{-a_0 t}(a_0 t)^k}{k!} (T(z))^k = e^{-a_0 t}e^{a_0 T(z)}
\]
where \( T(z) = \int_0^\infty e^{-zx}S(x) \, dx \) and \( S(x) \) as in (II) above. Therefore
\[
m_0z + \int_0^\infty (1 - e^{-zx})n(dx) = a_0 - a_0T(z), \quad z \geq 0
\]
Differentiating with respect to \( z \) in both sides
\[
-a_0T'(z) = m_0 + \int_0^\infty xe^{-zx}n(dx)
\]
\[
-a_0 \frac{d}{dz} \int_0^\infty e^{-zx}S(x) \, dx = m_0 + \int_0^\infty xe^{-zx}n(dx)
\]
\[
a_0 \int_0^\infty xe^{-zx}S(x) \, dx = m_0 \int_0^\infty xe^{-zx}n(dx)
\]
Therefore
\[
S(x) \, dx = n(dx) + m_0\delta(x)
\]
but since \( n(dx) \) is a measure on \((0, \infty)\)
\[
S(x) \, dx = n(dx)
\]
completing the proof of the lemma. ♦

Our next task is to study in more detail the probability distribution \( S(x) \) as in formula (II) for gap-diffusions with \( n \)-points state space.

**Notation :** We mean by (i) the expression (ii)
- (i) \( \frac{a_i}{b_1+x} \odot \frac{a_i}{b_2+x} \), \( a_i \neq 0, \ i = 1, 2 \).
- (ii) \( \frac{a_1}{b_1+x-\{b_2+x\}} \), \( a_i \neq 0, \ i = 1, 2 \).

**Lemma 2.3.** Let \([W, \mathcal{B}, P, a \in E_2 = \{x_0, x_1, x_2\}]\) be a quasi-diffusion with 3 points state space and let \( \{H_0, H_1, H_2\} \) be the exponential holding times at the points \( \{x_0, x_1, x_2\} \) with rates \( \{a_0, a_1, a_2\} \) respectively. Then
\[
L(S)(x) = \frac{a_1}{2(x+a_1)} \odot \frac{a_1a_2}{x+a_2}
\]
where \( S(x) \) is as in (II) ♦
**Proof**: As in Lemma (2.1) let $c(t)$ be the number of jumps made the particle from $x_0$ to $x_1$ up to time $t$.

Once the particle jumps from point $x_0$ it can do any of the following possible movements before it returns to point $x_0$ again. We are only interested in the case $c(t) = 1$.

**Possible Trajectories**

- **Case (1)**:  $< x_1 >$
- **Case (2)**:  $< x_1, x_2, x_1 >$
- **Case (3)**:  $< x_1, x_2, x_1, x_2, x_1 >$

$\ldots \ldots$

Case (1) means that the particle jumps to point $x_1$ and then jump to point $x_0$.

Case (2) particle jumps to point $x_1$, then to point $x_2$ back to point $x_1$ and then returns to point $x_0$.

Same for Case (3) etc., etc.

Let $p, q$ be such that $p + q = 1$, $p, q \geq 0$ representing the probability of jumping to the right and probability of jumping to the left of any given point of the state space. We are going to assume works for general $p = q = \frac{1}{2}$ in the following computations but the same argument works for general $p$ and $q$.

Thus for $p = q = \frac{1}{2}$

$$S(x) = \frac{h_1(x)}{2} + \frac{(h_1)^2 \ast h_2}{4} + \frac{(h_1)^3 \ast (h_2)^2}{8} + \ldots$$

or $p \neq q$

$$S(x) = qh_1(x) + pq \left( (h_1)^2 \ast h_2 \right) + p^2 q \left( (h_1)^3 \ast (h_2)^2 \right) + \ldots$$

$$L(S)(x) = \frac{w_1(x)}{2} \left( 1 + \frac{w_1(x)w_2(x)}{2} + \left( \frac{w_1(x)w_2(x)}{2} \right)^2 + \ldots \right)$$

$$= \frac{w_1(x)}{2} \left( \frac{1}{1 - \frac{w_1(x)w_2(x)}{2}} \right)$$
where
\[ h_i = a_i e^{-a_i x}, \quad i = 0, 1, 2 \]
and
\[ w_i(x) = L(h_i)(x) = \frac{a_i}{x + a_i}, \quad i = 1, 2 \]
Thus, for \( p = q = \frac{1}{2} \)
\[ L(S)(x) = \frac{a_1}{2(x + a_1)(1 - \frac{a_1 a_2}{2(x + a_1) x + a_2})} \]
or \( p \neq q \)
\[ L(S)(x) = \frac{q a_1}{(x + a_1)(1 - \frac{p a_1 a_2}{(x + a_1) x + a_2})} \]
Therefore
\[ L(S)(x) = \frac{a_1}{2(x + a_1) - \frac{a_1 a_2}{x + a_2}} = \frac{a_1}{2(x - a_1)} \ominus \frac{a_1 a_2}{x - a_2} \]
or
\[ L(S)(x) = \frac{q a_1}{(x - a_1)} \ominus \frac{p a_1 a_2}{x - a_2} \]
completing the proof of the lemma. ♦

**Lemma 2.4.** Let \([W, B, P, a \in E_3 = \{x_0, x_1, x_2, x_3\}]\) be a quasi-diffusion with 3 points state space and let \(\{H_0, H_1, H_2, H_3\}\) be the exponential holding times at the points \(\{x_0, x_1, x_2, x_3\}\) with rates \(\{a_0, a_1, a_2, a_3\}\) respectively. Then
\[ L(S)(x) = \frac{a_1}{2(x + a_1)} \ominus \frac{a_1 a_2}{2(x + a_2)} \ominus \frac{a_2 a_3}{(x + a_3)} \]
where \(S(x)\) is as in (II) ♦

**Proof:** Same argument as in Lemma 2.3 we obtain
\[ L(S) = \frac{w_1}{2} \left( \frac{1}{1 - \frac{w_2}{g_2}} \right) \]
where
\[ g_2(x) = \frac{a_2}{2(x + a_2) - \frac{a_2 a_3}{x + a_3}} \]

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and

\[ w_1(x) = \frac{a_1}{x + a_1}. \]

Therefore

\[
L(S)(x) = \frac{a_1}{2(x + a_1) \left( 1 - \frac{a_1}{2(x + a_1)} \left[ \frac{a_2}{2(x + a_2)} \right] \right)}
\]

\[
= \frac{a_1}{2(x + a_1) - \frac{a_1 a_2}{2(x + a_2)}}
\]

\[
= \frac{a_1}{2(x + a_1)} \oplus \frac{a_1 a_2}{2(x + a_2)} \oplus \frac{a_2 a_3}{2(x + a_3)}
\]

establishing the lemma. ♦

**Lemma 2.5.** Let \([W, B, P_a, a \in E_N = \{x_0, x_1, x_2, ..., x_N\}]\) be a quasi-diffusion with \((N + 1)\) points state space and let \(\{H_0, H_1, H_2, ..., H_N\}\) be the exponential holding times at the points \(\{x_0, x_1, x_2, ..., x_N\}\) with rates \(\{a_0, a_1, a_2, ..., a_N\}\) respectively. Let us assume that

\[ P(H_i > t) = \int_t^\infty h_i(s) \, ds = e^{-a_i t}, \quad a_i > 0, \quad i = 0, 1, ..., N. \]

Then

\[
L(S)(x) = \frac{a_1}{2(x + a_1)} \oplus \frac{a_1 a_2}{2(x + a_2)} \oplus \ldots \oplus \frac{a_{N-1} a_N}{(x + a_N)}
\]

where

\[ S(x) = P_0(\tau^{-1}(t, w) - t \in dx | C(t) = 1) \]

with \(\tau^{-1}(t, w)\) being the inverse local time for a given finite state space quasi-diffusion. ♦

**Proof:** We will use an inductive argument over \(N\). For

- \(n = 1 \rightarrow E_1 = \{x_0, x_1,\} \) is proved in Lemma 2.1
- \(n = 3 \rightarrow E_3 = \{x_0, x_1, x_2, x_3\} \) is proved in Lemma 2.3
- \(n = 4 \rightarrow E_4 = \{x_0, x_1, x_2, x_3, x_4\} \) is proved in Lemma 2.4

Suppose true for \((N - 1)\), we shall prove for \(N\). We know by previous lemma thanks

\[
L(S)(x) = \frac{w_1}{2} \frac{1}{1 - \frac{w_1}{2} g_{N-1}}
\]

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where
\[ w_1(x) = \frac{a_1}{x + a_1} \]
and
\[ g_{N-1}(x) = \frac{a_2}{2(x + a_2)} - \left( \frac{a_2}{2(x + a_2)} \odot \cdots \odot \frac{a_{N-1}a_N}{(x + a_N)} \right) \]

Thus
\[ L(S)(x) = \frac{a_1}{2(x + a_1)} \left( 1 - \frac{a_1}{2(x - a_1)} \left[ \frac{a_2}{2(x + a_2)} \odot \cdots \odot \frac{a_{N-1}a_N}{(x + a_N)} \right] \right) \]
\[ = \frac{a_1}{2(x + a_1)} \odot \frac{a_1a_2}{2(x + a_2)} \odot \frac{a_{N-1}a_N}{(x + a_N)} \]
completing the proof of the theorem. ♦

Let \( B^+ = [W; B; P_a; a \in \mathbb{R}^+ = ] \) be a reflected Brownian motion. Let \( D_k = [W^k; B^k; P^k_a; b \in E^k = b^k_0, b^k_1, \ldots, b^k_{n_k}, b^k_0 = 0], \ k = 1, 2, \ldots \) be a quasi-diffusion with finite state space \( E^k \), Levy measure \( n_k(dx) \) and speed measure \( m_k(dx) \) and let \( D = [W, B, P_a, a \in \mathbb{R}^+] \) be a quasi-diffusion with Levy measure \( m(dx) \).

**Theorem 2.6.** If for every \( D_k \), \( k = 1, 2, \ldots \) as above we have that the Levy measure \( n_k(dx) \) of a persistent gap-diffusion (quasi-diffusion) with finite state space \( E^k \), Levy measure \( n_k(dx) \) and speed measure \( m_k(dx) \) and let \( D = [W, B, P_a, a \in \mathbb{R}^+] \) be a quasi-diffusion with Levy measure \( m(dx) \).

Theorem 2.6. If for every \( D_k \), \( k = 1, 2, \ldots \) as above we have that the Levy measure \( n_k(dx) \) of a persistent gap-diffusion (quasi-diffusion) with finite state space \( E^k \), Levy measure \( n_k(dx) \) and speed measure \( m_k(dx) \) and let \( D = [W, B, P_a, a \in \mathbb{R}^+] \) be a quasi-diffusion with Levy measure \( m(dx) \).

\[ n_k(dy) = \int_0^\infty e^{-yx} \mu_k(dx) dy \quad (*) \]

with measure \( \mu_k(dx) \geq 0 \) on \((0, \infty)\) such that
\[ \int_0^\infty \frac{\mu_k(dx)}{x(1 + x)} < \infty, \quad (**) \]

Then the Levy measure \( n(dy) \) of a persistent quasi-diffusion on \([0, \infty)\) reflected at 0 has the representation \((*)\) satisfying condition \((**)\).

**Proof:** Let \( \tau(t, w) \) be the local time spent at 0 by \( D_k \) and \( \tau(t, w) \) be the time spent at 0 by the reflecting Brownian motion. Let \( \{m_k\}_{k=1}^\infty \) be the
speed measure with finite support converging monotonically a.e. to \( m(dx) \). By Theorem 3.1 we know that changing the time of a Brownian motion we can obtain \( D_k, \ k = 1, 2, \ldots \). By [Note A, (A.1)] we have

\[
\dot{\tau}(t, w)(t, \psi, w) = f_k(\tau^{-1}(t, \psi, w)) = \int_0^\infty (\tau(\tau^{-1}(t, \psi, w), w, y)) m_k(dy)
\]

where

\[
f_k(t) = f_k(t, w) = \int_0^\infty \tau(t, w, y) m_k(dy).
\]

Then

\[
\lim_{n \to \infty} \dot{\tau}_n(t, \psi, w) = \dot{\tau}(t, \psi, w)
\]

\[
\dot{\tau}_n(t) = \dot{\tau}(t), \text{ monotonically in distribution}
\]

\[
\lim_{n \to \infty} E_0\{e^{-z\dot{\tau}_n(t)}\} = E_0\{e^{-z\dot{\tau}(t)}\}, \ z \geq 0 \ (6.1)
\]

where above limit is monotone.

By Theorem 3.3 we have that for every \( z > 0 \)

\[
\lim_{k \to \infty} \int_0^\infty (1 - e^{-zy}) n_k(dy) = \int_0^\infty (1 - e^{-zy}) n(dy)
\]

but, since for every \( z > 0 \)

\[
\int_0^\infty (1 - e^{-zy}) n(dy) = \int_0^z \left( \int_0^\infty -ye^{-sy} n(dy) \right) ds
\]

we have for every positive \( z \)

\[
\lim_{k \to \infty} \int_0^z \left( \int_0^\infty -ye^{-sy} n_k(dy) \right) ds = \int_0^z \left( \int_0^\infty -ye^{-sy} n(dy) \right) ds
\]

\[
\int_0^z \left( - \lim_{k \to \infty} \int_0^\infty ye^{-sy} n_k(dy) \right) ds = \int_0^z \left( \int_0^\infty ye^{-sy} n(dy) \right) ds
\]

Therefore, by Theorem 3.4 we know that

\[
\lim_{k \to \infty} n_k(dx) = n(dx), \quad (6.2)
\]

vaguely with the dense set of convergence being equal to \( \mathbb{R}^+ = [0, \infty) \).

Therefore by Lemma 2.2 and Lemma 2.5

\[
n_k(dx) = S_k(x) \ dx
\]
where
\[ S_k(x) = \int_0^\infty e^{-xs} d\Phi_k(s), \quad k = 1, 2, \ldots. \]

By (6.2) and Lemma 2.2 it is easy to see that \( n(dx) = S(x) \, dx \).

We claim that
\[ S(x) = \int_0^\infty e^{-xs} d\Phi(s) \]

Due to (6.2), the fact that \( \{S_k(x)\}_k \) are completely monotonic and the vague convergence, it is true that the following follows:

\[
\lim_{n \to \infty} \int_0^\infty \int_0^\infty e^{-xs} \, d\Phi_n(s) \, dx = \int_0^\infty S(x) \, dx, \quad z > 0
\]

\[
\lim_{n \to \infty} \left( \int_0^\infty e^{-xs} \, dx \right) d\Phi_n(s) = \int_0^\infty S(x) \, dx, \quad z > 0
\]

\[
\lim_{n \to \infty} \int_0^\infty \frac{1}{s} e^{-zs} \, d\Phi_n(s) = \int_0^\infty S(x) \, dx, \quad z > 0
\]

By Theorem 3.4, we can conclude that exists \( d\Phi \) so that

\[
\int_0^\infty S(x) \, dx = \int_0^\infty e^{-zx} \, d\Phi(x) = \int_0^\infty \frac{e^{-zs}}{x} [x \, d\Phi(x)] = \\
= \int_0^\infty \left( \int_0^\infty e^{-xy} \, dy \right) [x \, d\Phi(x)] = \\
= \int_0^\infty \left( \int_0^\infty e^{-xy} [x \, d\Phi(x)] \right) \, dy, \quad z > 0.
\]

Therefore

\[ S(x) = \int_0^\infty e^{-xy} \, d\phi(y), \quad \text{for almost all } x > 0 \]

where

\[ d\phi(y) = y \, d\Phi(y) \]

completing in this way the proof of the theorem. ♦

**Theorem 2.7.** The Levy measure \( n(dy) \) of a persistent gap diffusion on \([0, \infty)\) reflected at 0 is absolutely continuous with density

\[ n(y) = \int_0^\infty e^{-y\mu} \mu(dx) \]
with measure $\mu(dx) \geq 0$ on $(0, \infty)$ such that

$$\int_0^\infty \frac{1}{x(1+x)} \mu(dx) < \infty.$$  

**Proof:** By Theorem 2.6 it suffices to prove the theorem for finite state space quasi-diffusions. By Lemma 2.5 and Theorem 3.10 plus the adequate interpretation in terms of the probabilistic model we get

$$L(S)(z) = \int_0^\infty \frac{d\Phi(t)}{(z + t)}, \quad z > 0$$

where $\Phi(t)$ is a real valued positive step function with discontinuities on $[0, \infty)$ and $S$ as in Lemma 2.2.

By Theorem 3.3

$$L(S) = L\{\int_0^\infty e^{-tu} d\Phi(u)\}$$

and uniqueness of Laplace transform

$$S(x) = \int_0^\infty e^{-xu} d\Phi(u)$$

$$n(dx) = \left( \int_0^\infty e^{-xu} d\Phi(u) \right) dx.$$

Applying Theorem 3.3 E.3 we know that

$$\int_0^\infty (1 - e^{-s}) n(ds) < \infty = \int_0^\infty (1 - e^{-s}) \left( \int_0^\infty e^{-su} d\Phi(u) \right) ds < \infty$$

$$= \int_0^\infty \left( \int_0^\infty e^{-su}(1 - e^{-s}) ds \right) d\Phi(u) < \infty$$

$$= \int_0^\infty \frac{1}{u} - \frac{1}{u+1} d\phi(u) < \infty$$

$$= \int_0^\infty \frac{1}{u(1+u)} d\phi(u) < \infty.$$

completing the proof of the theorem. ♦
3 Appendix

NOTE A: Brownian Motion.

Let us assume that \( Q = [0, \infty) \), \( m(dx) \) is a measure on \( Q \) that is finite on compact subintervals and positive on neighborhoods of \( 0 \). Let \( B^+ \) denote the standard Brownian motion (\( B^+ = |B(t)| \)). Let

\[
\tau^+ = \frac{1}{2} \int_0^t I_{[0,x]}(B^+(s)) \, ds
\]

denote the local time of \( B^+ \) (continuous in \((t,x)\) \( P_0 \) a.s. by Trotter’s theorem).

**Theorem 3.1.** [\[3\], p.55], [\[10\]]. Let \( \tau(t) \) the right continuous inverse of the additive functional

\[
A(t) = \int_0^\infty \tau^+(t,x) m(dx)
\]

with \( \tau(t) = \infty \) for \( t \geq A(\infty) \). Then the process \( X_t = B^+(\tau(t)) \) with the usual generated \( \sigma \)-fields and translation operators \( \Theta_{\tau(t)} \) defines a Hunt process on closure (support) of \( m(dx) \) for the probabilities \( P_x \) of \( B^+ \). It is called the gap diffusion on \([0, \infty)\) (quassi diffusion) with natural scale and speed measure \( m(dx) \).

**Remark 3.2.** [\[5\]]. We know that a solution of \( \tau'(X(t), \psi) = t \) is

\[
X(t) = (\tau')^{-1}(t, \psi) \text{ for every } \psi
\]

If \( X(t) = f(\tau^{-1}(t, \psi)) \) it is a solution too:

\[
\tau'(f(X(t), \psi)) = \tau'(f^{-1}(f(\tau^{-1}(t, \psi))))(\psi) = t.
\]

Showing that

\[
\tau'^{-1}(t, \psi) = f(\tau^{-1}(t, \psi)), \quad (A.1)
\]

where

\[
f(t) = \int_Q \tau(t, \psi) m(d\psi).
\]

**Theorem 3.3.** [\[7\], p.30, Note 1].

If \( \{Y(t), t \geq 0\} \) is a real valued process with stationary independent and nonnegative increments, \( Y(0) = 0 \) and right continuous paths defined on some probability space \((\Omega^*, \epsilon, P)\) then

\[
E[e^{-\lambda Y(t)}] = e^{-t g(\lambda)}; \quad \lambda < 0, \quad (A.2)
\]
where
\[
g(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t}) n(dt), \quad (A.3)
\]
\[
b \geq 0, \quad n(dt) \geq 0, \quad \int_0^\infty (1 - e^{-t}) n(dt) < \infty, \quad (A.4)
\]
n(dt) is the so called Levy measure of the process.

NOTE B: Laplace Transform

Theorem 3.4. [12], Vol.II, p.433.

For n=1,2,... let \( U_n \) be a measure with Laplace transform \( w_n \). If \( w_n(\lambda) \to w(\lambda) \) for \( \lambda > a \), then \( w \) is the Laplace Transform of a measure \( U \) and \( U_n \to U \). Conversely, \( U_n \to U \) and the sequence \( \{w_n(a)\} \) is bounded, then \( w_n(\lambda) \to w(\lambda) \) for every \( \lambda > a \). Recall that the sequence \( \{U_n\}_{n=1}^\infty \) of measures is said to converge to \( U \) if and only if \( U_n(I) \to U(I) < \infty \) for every finite interval of continuity.

Remark 3.5. [12], p.334.

If the integral
\[
f(s) = \int_0^\infty \frac{d\alpha(t)}{s + t}
\]
converges, then
\[
f(s) = \int_0^\infty e^{-st} \phi(t) \, dt, \quad s > 0
\]
where
\[
\phi(t) = \int_0^\infty e^{-tu} \, d\alpha(u), \quad t > 0
\]
NOTE C: Continued Fractions

Remark 3.6. [7], p.17.
A continued fraction is an order pair \( \langle \{a_n\}; \{b_n\}; \{f_n\} \rangle \) where \( a_1, a_2, \ldots \) and \( b_0, b_1, \ldots \) are complex numbers and all \( a_n \neq 0 \) and where \( \{f_n\} \) is a sequence in the extended complex plane defined as follows:

\[
f_n = S_n(0), \quad n = 0, 1, 2, \ldots \quad (C.1)
\]

where

\[
S_0(w) = s_0(w), \quad S_n(w)S_{n-1}(s(n(w)); \quad n = 1, 2, \ldots \\
s_0(w) = b_0 + w, \quad s_n(w) = \frac{a_n}{b_n + w}; \quad n = 1, 2, \ldots
\]

\( f_n \) is called the \( n \)-th-approximant. If \( \{a_n\} \) and \( \{b_n\} \) are infinite sequences then it is called non terminating continued fraction, otherwise is called finite or terminating continued fraction. Corresponding to each continued fraction there are sequences complex \( \{A_n\} \) and \( \{B_n\} \) defined by the system of second order linear difference equations.

\[
A_{-1} = b_0, \quad B_{-1} = 0, \quad B_0 = 1
\]

\[
A_n = b_nA_{n-1} + a_nA_{n-2}, \quad n = 1, 2, \ldots \quad (C.2)
\]

\[
B_n = b_nB_{n-1} + a_nB_{n-2}, \quad n = 1, 2, \ldots
\]

It can be shown by an induction argument that eqnarray

\[
f_n = \frac{A_n}{B_n}, \quad n = 0, 1, 2, \ldots
\]

\[
f_n = b_0 + \frac{a_1}{b_1} \oplus \frac{a_2}{b_2} \oplus \ldots \oplus \frac{a_n}{b_n}
\]

Theorem 3.7. [9]; [12], p.41.
The following relation holds for any three consecutive orthogonal polynomials

\[
K_n(x) = (A_nx + B_n)K_{n-1}(x) - C_n(x), \quad (C.3)
\]

Here \( A_n, B_n \) and \( C_n \) are constants, \( A_n > 0 \) and \( C_n > 0 \). If the highest coefficient of \( K_n(x) \) is denoted by \( a_n \) we have

\[
A_n = \frac{a_n}{a_{n-1}}, \quad C_n = \frac{A_n}{A_{n-1}} = \frac{a_n a_{n-2}}{(a_{n-1})^2}
\]

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Now let \( \{K_n(x)\} \) be the orthonormal set of polynomials associated with the distribution \( d\psi(t) \) on \([0, \infty)\). The recurrence formula (C.3) then suggests the consideration of the continued fraction

\[
\frac{1}{A_1 x + B_1} \oplus \frac{C_2}{A_2 x + B_2} \oplus \frac{C_3}{A_3 x + B_3} \oplus \ldots \quad \text{(C.4)}
\]

with \( \frac{K_n(x)}{L_n(x)} \) as the \( n \)th approximant \( n = 1, 2, 3, \ldots \). ●

**Theorem 3.8.** [7], p.254.
If \( \{L_n\}_{n=1}^\infty \) is a sequence of polynomials satisfying a system of three term recurrence relation of the form

\[
L_0(x) = 1, \quad L_1(x) = l_1 + x, \quad L_n(x)(l_n + x)L_n - k_nL_{n-2}(x) \quad \text{(C.5)}
\]

with \( k_n > 0, \ l_n \) real for \( n = 1, 2, \ldots \). Then there exists a \( \psi(t) \) real valued, bounded monotone, non decreasing with infinitely many points of increase such that \( \{L_n\}_{n=1}^\infty \) is a system of orthogonal polynomials with respect to \( \psi(t) \).

Given a J-fraction as in Theorem 3.7 (C.4) with the \( n \)th-approximant

\[
f_n = \frac{K_n(x)}{L_n(x)}
\]

we know that \( \{L_n(x)\}_{n=1}^\infty \) satisfies relation Theorem 3.8 (C.5).

**Theorem 3.9.** [9]; [12], p.54.
The approximants \( \frac{K_n(x)}{L_n(x)} \) of the J-continued fraction are determined by the formula

\[
K_n(x) = c \int_a^b \frac{L_n(x) - L_n(t) - (l_n + x)\{L_{n-1}(x) - L_{n-1}(t)\}}{(x-t)} - \frac{k_n\{L_{n-2}(x) - L_{n-2}(t)\}}{(x-t)} d\psi(t), \quad n = 0, 1, \ldots \quad \text{(C.6)}
\]

**Proof:** Notice that (C.6) holds also for \( n = 0 \) and \( n = 1 \). For \( n \geq 2 \) we have

\[
K_n(x) = \int_a^b \frac{-L_n(t) + (l_n + x)L_{n-1}(t) - k_nL_{n-2}(t)}{(x-t)} d\psi(t)
\]

\[
= \int_a^b \frac{-L_n(t) + ((l_n + x) + (l_n + t) - (l_n + t))L_{n-1}(t) - k_nL_{n-2}(t)}{(x-t)} d\psi(t)
\]

\[
= \int_a^b L_{n-1}(t) \frac{(x-t)}{(x-t)} d\psi(t).
\]
establishing the theorem.

**Remark 3.10.** [3],p.56; [3],p.212-238; [3],p.336-337.

For a real J-fraction

\[
\frac{k_1}{l_1 + z} \oplus \frac{k_2}{l_2 + z} \oplus \frac{k_3}{l_3 + z} \oplus ..... \quad (C.7)
\]

with \(k_n > 0\) and \(l_n\) real for \(n = 1, 2, 3, \ldots\), let \(K_n(z)\) and \(L_n(z)\) be the \(n\)th-approximant numerator and denominator respectively. Then the zeros \(\{x_k\}_{n=1}^n\) of \(L_n(z)\) are real and distinct and \(\frac{K_n(z)}{L_n(z)}\) has a partial fraction decomposition

\[
\frac{K_n(z)}{L_n(z)} = C \sum_{k=1}^n \frac{\lambda_{nk}}{(z - x_k)} \quad \text{so that } \lambda_{nk} > 0; \quad k = 0, 1, 2 \ldots \quad (C.8)
\]

Thus

\[
\frac{K_n(z)}{L_n(z)} = \int_{-\infty}^{\infty} \frac{d\Phi_n(t)}{(z - t)}
\]

where \(\Phi(t)\) is a step function.

**Proof:** By Lagrange interpolation arguments and since all the zeros of an orthogonal polynomial are distinct and real we have

\[
K_n(x) = C \sum_{k=1}^n K_n(x_k) \frac{L_n(x)}{L_n'(x)(x - x_k)}
\]

By Theorem (3.3)

\[
\frac{K_n(x_k)}{L_n'(x_k)} = \frac{1}{L_n'(x_k)} \int \frac{L_n(t)}{(t - x_k)} d\phi(t) = \
\int \frac{L_n(t)}{L_n'(x_k)(t - x_k)} = \lambda_k, \quad k = 1, 2, 3, \ldots, n
\]

\[
K_n(x) = C \sum_{k=1}^n \lambda_k \frac{L_n(x)}{(x - x_k)}
\]

\[
\frac{K_n(x)}{L_n(x)} = C \sum_{k=1}^n \frac{\lambda_k}{(x - x_k)} \quad (C.9)
\]

where the \(\{\lambda_k\}\) are called Christofell numbers; \(\lambda_k > 0, \quad k = 1, 2, \ldots\)

Now ordering \(x_k\) according to size and defining the step function \(\Phi_n(t)\) by
\( \Phi_n(t) = \begin{cases} 
0 & \text{if } -\infty < t \leq x_1 \\
\sum_{k=1}^{m} \lambda_k & \text{if } x_m < t \leq x_{m+1}, \quad m = 1, 2, ..., n, \\
k_1 & \text{if } x_n < t \leq \infty.
\end{cases} \quad (C.10) \)

we obtain

\[
\frac{K_n(z)}{L_n(z)} = \int \frac{d\phi(t)}{(z-t)}, \quad \text{for every } n. \quad \diamond
\]

**Remark 3.11.** [12]. A necessary and sufficient condition for a rational fraction to have the form

\[
\frac{f_1}{f_0} = \sum_{p=1}^{n} \frac{L_p}{(z-x_p)}
\]

where the \( x_p \) are real and distinct and the \( L_p \) positive, is that \( f_1/f_0 \) have a continued J-fraction expansion as in Remark \[3.6 \] (C.1) in which \( a_i \) are positive and \( b_i \) are real.

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