COMPLEXITY FOR EXTENDED DYNAMICAL SYSTEMS

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ABSTRACT. We consider dynamical systems for which the spatial extension plays an important role. For these systems, the notions of attractor, e-entropy and topological entropy per unit time and volume have been introduced previously. In this paper we use the notion of Kolmogorov complexity to introduce, for extended dynamical systems, a notion of complexity per unit time and volume which plays the same role as the metric entropy for classical dynamical systems. We introduce this notion as an almost sure limit on orbits of the system. Moreover we prove a kind of variational principle for this complexity.

1. Introduction

Dynamical systems are called “extended” when the spatial extension plays an important role. They occur for example, in nonlinear partial differential equations of parabolic or hyperbolic type when the size of the domain is much larger than the typical size of the structures developed by the solutions. As in Statistical Mechanics, one can try to use the infinite volume limit as an approximation.

For several classes of such systems, motivated by physical models, it has been shown that one can define the semi flow of evolution in unbounded domains acting on bounded functions with some regularity (see for example \[3\], \[14\], \[19\], \[6\]). This is particularly convenient when studying traveling solutions or waves, since one would not like to fix some particular boundary conditions which restrict the nature of the solution (for example fixing a particular spatial period). Once the dynamics has been defined in unbounded domain, one can ask for a notion of attractor. Such a notion was introduced by Feireisl (see \[13\] and \[20\]) by observing the system in bounded windows and inferring the result for the unbounded domain. When the evolution equation does not depend explicitly on space (homogeneous system), the attractor is translation invariant and often non compact of infinite dimension. However, if restricted to a finite window it is often a compact set. A situation which occurs in several examples is that the functions in

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the attractor are analytic and bounded in a strip around the real domain (see for example [4], [25]). Compactness follows in bounded (real) regions when using $C^k$ norms for example. For such systems with non compact translation invariant infinite dimensional attractors, one can try to define extensive quantities as in statistical mechanics. A notion of dimension per unit volume can be defined from the $\epsilon$-entropy per unit volume of Kolmogorov (see [17]) where it was used in particular to quantify the fact that some function spaces are larger than others. Looking for example at an attractor composed of functions analytic in a strip and of infinite dimension, since an analytic function is completely determined by its data in a finite domain, the dimension observed in any finite window will always be infinite. To avoid this uninteresting result, one first fixes a precision $\epsilon > 0$. One then counts for example the minimal number $N^\epsilon_{\Lambda}$ of balls of radius $\epsilon$ needed to cover the attractor in the finite window $\Lambda$. The next step is to prove that $H(\epsilon) = \lim_{|\Lambda| \to \infty} \log_2 N^\epsilon_{\Lambda}/|\Lambda|$ exists, and then to consider the quantity $H(\epsilon)/\log_2 2^{-1}$ for small $\epsilon$. As mentioned above, the order in which the limits in $\Lambda$ and $\epsilon$ are taken is important. If for fixed $\Lambda$ one lets first $\epsilon$ tend to zero, the result is in general infinite, while in the other order, one can get finite results. These ideas were applied to the attractors of various extended systems (see [7], [12] and [5]). These ideas can also be adapted to give a definition of the topological entropy per unit volume (see [8], [9], [5] and [27]). One first fixes a finite precision, considers the maximal number $N^\epsilon_{\Lambda}(T)$ of different trajectories one can observe in a finite window $\Lambda$ on the time interval $[0,T]$ at this given precision. One then considers the limits

$$h_{top} = \lim_{\epsilon \to 0} \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \lim_{T \to \infty} \frac{\log_2 N^\epsilon_{\Lambda}(T)}{T}.$$  

Here again the order of the limits is crucial, otherwise one may get an infinite quantity.

Regarding similar approaches, angular limits have been proposed in [22] for cellular automata, and for entropies along subspaces we refer to [21] and references therein. In [8], a similar definition was proposed for the metric entropy per unit time and volume, however this definition involves several limits which are up to now not known to exist.

In order to circumvent this difficulty we deal in the present paper with the Kolmogorov complexity. For dynamical systems on a compact phase space with an ergodic invariant measure, it is known that the complexity per unit time of a typical trajectory is equal to the metric entropy (see [2], [26]). A first advantage of the complexity per unit time is that it can be defined for individual trajectories with initial conditions on a full measure set. We will also see below that the complexity satisfies some useful subadditivity properties allowing to define a complexity per unit time and unit volume. The strategy is the same as for the topological entropy. We first fix a precision $\epsilon$. We then consider the complexity per unit time of a coding of these trajectories in the window $\Lambda$ using a covering by balls of radius at
most $\epsilon$. We then show that this quantity grows like the volume and define a complexity per unit time and unit volume at a fixed precision, finally letting the precision become infinite.

We will deal in the present paper with systems satisfying some hypothesis inspired by the known results on extended systems (for example, reaction-diffusion equations, models of convection etc., see [4]). In particular we will not assume that the attractor is compact but that it is translation invariant. We will also assume a space time invariant ergodic measure is given. In other words, our results apply to $\mathbb{R} \times \mathbb{R}$ actions satisfying the hypothesis given below. In particular, we assume that the semi-flow on the function space is a flow when restricted to the attractor of the system. This follows for example from the analyticity in time of the solutions of the evolution equations. The procedure described above differs with the more standard approach to the space time entropy which uses boxes of roughly the same size in the space and time direction. It is however more natural from the point of view of the definition of the attractors of such systems.

In order to open the possibility of using other type of complexities, we have tried to isolate the properties we need without reference to a particular example, although the Kolmogorov complexity satisfies all the requirements. In order to simplify the proofs, we only discuss the case of one space dimension, although most results extend easily to higher dimension.

The paper is organised as follows. In section 2 we first state the required hypothesis on the dynamical system and on the complexity, and show that these hypothesis are satisfied by Kolmogorov complexity. We then state the main results. In section 3 we prove that under these hypothesis one can define a complexity per unit time and unit volume. This is done following the scheme briefly mentioned above of fixing first a finite precision and removing it only at the end. In section 4 we prove a variational principle which shows that in the concrete examples of extended systems studied up to now, the complexity we have defined is finite. In fact, we show that computing the supremum of the complexity for functions in the supports of the invariant measures of the system, one obtains the topological entropy defined in [5].

2. Settings and results

Let $F$ be a set of real functions defined on $\mathbb{R}$ and consider the following actions on $F$: the space translation

$$\mathbb{R} \ni y \mapsto (\zeta_y u)(x) := u(x + y)$$

and a flow of time evolution $\varphi_t : F \to F$ defined for $t \in \mathbb{R}$. We assume that the two actions commute.

We assume that the set $F$ is endowed with a translation invariant metric $d$ and that there exists a probability measure $\mu$ on $F$, such that $\mu$ is invariant and ergodic with respect to the $(\zeta, \varphi)$ action.
We make the following assumptions on the set $F$ and the flow $\varphi$. Let us assume that for any interval $\Lambda \subset \mathbb{R}$ the set
\[
F|_{\Lambda} := \{ g : \Lambda \to \mathbb{R} : \exists f \in F \text{ with } f|_{\Lambda} \equiv g \}
\]
is endowed with a metric $d|_{\Lambda}$ such that $(F,d)$ is the projective limit of $(F|_{\Lambda},d|_{\Lambda})$ as $|\Lambda| \to \infty$. If for example $F \subset C^0_R(\mathbb{R})$, the set of real bounded continuous functions on $\mathbb{R}$, and $d$ is the sup-norm, then for every $\Lambda \subset \mathbb{R}$ we have $d|_{\Lambda}(g_1,g_2) = \sup_{x \in \Lambda} |g_1(x) - g_2(x)|$. We assume that for any interval $\Lambda \subset \mathbb{R}$ we have
\[
(A1) \quad |\Lambda| < \infty \implies F|_{\Lambda} \text{ is pre-compact}
\]
By assumption $(A1)$, for any $\epsilon > 0$ and $|\Lambda| < \infty$ we can define the set
\[
(2.1) \quad C^e_{\Lambda} = \{ \text{finite open coverings of } F|_{\Lambda} \text{ with balls of radius } < \epsilon \}
\]
and we denote by $U^e_\Lambda$ an element of $C^e_{\Lambda}$. Fixed two finite intervals $\Lambda_1$ and $\Lambda_2$ with disjoint interiors let $\Lambda$ be the union $\Lambda := \Lambda_1 \cup \Lambda_2$, then we assume that
\[
(A2) \quad \text{there exists an integer } q \text{ depending only on the metric } d \text{ such that, for any } U^e_{\Lambda_1} \in C^e_{\Lambda_1} \text{ and } U^e_{\Lambda_2} \in C^e_{\Lambda_2} \text{ and two balls } B_1 \in U^e_{\Lambda_1} \text{ and } B_2 \in U^e_{\Lambda_2}, \text{ either the intersection } B_1 \cap B_2 \text{ is empty or can be covered by } q \text{ balls of a covering } U^e_{\Lambda} \in C^e_{\Lambda}.
\]
The last assumption on the system is about the separation speed of two nearby functions with time. We assume that there are constants $\gamma > 0$, $\Gamma > 1$ and $C > 0$ such that, for any $|\Lambda| < \infty$ and any $\epsilon > 0$ satisfying $\text{diam}(\Lambda) > 2C\epsilon^{-1}$ and for any initial conditions $f_1$ and $f_2$ in $F$ such that $d|_{\Lambda}(f_1,f_2) < \epsilon$, we have
\[
(A3) \quad d|_{\Lambda \setminus \{d(x,\partial\Lambda) < C\epsilon^{-1}(t+1)\}}(\varphi_t(f_1),\varphi_t(f_2)) < \Gamma e^{\gamma t}\epsilon
\]
for any $t \in (0,C^{-1}\text{diam}(\Lambda)\epsilon)$ (cfr. [8]).

Under these assumptions a notion of topological entropy for the flow $\varphi$ has been defined in [8]. Let
\[
(2.2) \quad N^e_{\Lambda}(T) := \max \{ \text{card}(S^e_{\Lambda}(T)) : S^e_{\Lambda}(T) \text{ is made of } (\Lambda,T,\epsilon)\text{-distinguishable orbits} \}
\]
where we say that $f$ and $g$ in $F|_{\Lambda}$ have $(\Lambda,T,\epsilon)$-indistinguishable orbits up to time $T$ and with resolution $\epsilon$ if
\[
d|_{\Lambda}(\varphi_t(f),\varphi_t(g)) < \epsilon \quad \forall \ t \in (0,T)
\]
In [8], [9] and [27] it is proved that
\[
(2.3) \quad h_{\text{top}} := \lim_{\epsilon \downarrow 0} \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \lim_{T \to \infty} \frac{\log_2 N^e_{\Lambda}(T)}{T}
\]
exists and is finite under some additional assumptions, in fact it is bounded by $\gamma D_{\text{up}}$, where $D_{\text{up}}$ is called the upper local dimension per unit length of the set $F$ in [8] or capacity per unit length in [17].

The aim of this paper is to introduce a measure of the complexity of the action of the flow $\varphi$ on $F$ which would be the analogous of the metric entropy.
for dynamical systems. To this aim we need to define a notion of complexity. Our definition is inspired by the notion of Kolmogorov complexity ([18]).

Let \( \mathcal{A}^* \) be the set of finite words on a finite alphabet \( \mathcal{A} \), and for a word \( s \) let us denote by \( |s| \) its length. We say that \( K : \mathcal{A}^* \to \mathbb{R}^+ \), defined for any alphabet \( \mathcal{A} \), is a “good” complexity function if it satisfies the following hypothesis \((\text{H1})-\text{(H4)}\).

The first hypothesis is about the behaviour of the complexity function on sub-words and a sub-additivity property. Let \( s = uv \) be the concatenation of two words \( u \) and \( v \), then

\[
(\text{H1.a}) \quad K(u) \leq K(s) + \log_2 |u| + \text{const}
\]

for a constant independent on \( s \) and \( u \). Moreover let us assume that there exists a function \( h : \mathbb{N} \to \mathbb{R}^+ \) satisfying \( \lim_{n \to \infty} \frac{h(n)}{n} = 0 \) such that

\[
(\text{H1.b}) \quad K(s) \leq K(u) + K(v) + h(|u|) + h(|v|)
\]

Let now \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be two different alphabets, with \( r_i := \text{card}(\mathcal{A}_i) \). Moreover let \( \tilde{\mathcal{A}} \) be an alphabet with \( \text{card}(\tilde{\mathcal{A}}) = qr_1r_2 \) for some integer number \( q \geq 1 \), and we assume that there exists a surjective map \( \pi : \tilde{\mathcal{A}} \to \mathcal{A}_1 \times \mathcal{A}_2 \), with coordinate maps \( \pi_1 \) and \( \pi_2 \) on \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), respectively. Let \( s \in \tilde{\mathcal{A}}^* \) and \( \pi_i(s) \in \mathcal{A}_i^* \) be its projections. Then

\[
(\text{H2.a}) \quad K(\pi_i(s)) \leq K(s) + \text{const}
\]

\[
(\text{H2.b}) \quad K(s) \leq K(\pi_1(s)) + K(\pi_2(s)) + |s| \log_2 q + \text{const}
\]

where the constants are independent on \( s \).

The third hypothesis is an estimate on \( K \) that comes from observations by Shannon for his definition of information content ([24]). Let \( E \subset \mathcal{A}^* \times \mathbb{N} \) be a recursively enumerable set (for a definition see for example [18]), and for any \( n \in \mathbb{N} \) let \( L_n := \{ s \in \mathcal{A}^* : (s, n) \in E \} \) be a set of finite cardinality. Then we assume that for all \( n \in \mathbb{N} \) it holds

\[
(\text{H3}) \quad K(s) \leq \log_2 (\text{card}(L_n)) + \log_2 n + \text{const} \quad \forall s \in L_n
\]

where the constant only depends on the set \( E \).

Finally we ask for a relation between the bound of the complexity on a set of words and the cardinality of this set. We assume that

\[
(\text{H4}) \quad \text{card} \{ s \in \mathcal{A}^* : K(s) < c \} \leq 2^c \quad \forall c \in \mathbb{R}
\]

By using a “good” complexity function let us now define the complexity of the flow \( \varphi_t \).

Consider a fixed probability measure \( \mu \) which is invariant and ergodic for the action of \( (\zeta, \varphi) \). For a given \( \epsilon > 0 \) and an interval \( \Lambda \subset \mathbb{R} \) with \( |\Lambda| < \infty \), we consider on \( \mathcal{F}|\Lambda \) the set of coverings \( \mathcal{C}_\Lambda^\epsilon \). We will use such coverings to code the orbit of a function \( f \in \mathcal{F} \) under \( \varphi \). To this aim, we introduce a time step \( \tau > 0 \) and consider the orbits \( (f, \varphi_\tau(f), \varphi_{2\tau}(f), \ldots) \). By the method of symbolic dynamics we can associate to an orbit \( (\varphi_{j\tau}(f))_{j=0}^{n-1} \) a set of \( n \)-long
words $\psi(f, n, U^\Lambda_n)$ on a finite alphabet $A = A(U^\Lambda_n) = \{1, \ldots, \text{card}(U^\Lambda_n)\}$. If we denote $U^\Lambda_n := \{U_1, \ldots, U_{\text{card}(U^\Lambda_n)}\}$, we define

$$
\psi(f, n, U^\Lambda_n) := \{\omega^{n-1} \in A(U^\Lambda_n) : \varphi_j(f) \in U_{\omega_j} \forall j = 0, \ldots, n - 1\}
$$

In the same way we can define in the general case $\psi(\varphi_m\tau(f), n - m, U^\Lambda_n)$ as the set of possible symbolic representations of the orbit $(\varphi_j(f))_{j=m}^{n-1}$. At this point we can use a complexity function $K$ to define

$$
K(f, \tau, U^\Lambda_n, m, n) := \min \{K(\omega^{n-1}) : \omega^{n-1} \in \psi(\varphi_m\tau(f), n - m, U^\Lambda_n)\}
$$

To simplify notations, for $m = 0$ we will write

$$
K(f, \tau, U^\Lambda_n, n) := \min \{K(\omega^{n-1}) : \omega^{n-1} \in \psi(f, n, U^\Lambda_n)\}
$$

We can define the asymptotic linear rate of increase in $n$ by

$$
K(f, \tau, U^\Lambda_n) := \lim_{n \to \infty} \frac{K(f, \tau, U^\Lambda_n, n)}{n}
$$

To get rid of the dependence on the covering we define

$$
K(f, \tau, \epsilon, \Lambda) := \inf \{K(f, \tau, U^\Lambda_n) : U^\Lambda_n \in C_\epsilon\}
$$

The next step will be to study the asymptotic rate of increase in $|\Lambda|$. We restrict ourselves to a class of intervals defined as follows.

**Definition 2.1.** A sequence of sets $\Lambda = \{\Lambda_k\}_k$ is called *admissible* if $\Lambda_k = [a_k, b_k]$ for two sequences $\{a_k\}_k$ and $\{b_k\}_k$ satisfying $a_k < b_k$ for all $k \geq 1$ and

$$
\lim_{k \to \infty} (b_k - a_k) = +\infty
$$

$$
\liminf_{k \to \infty} \frac{b_k - a_k}{\max\{a_k, 0\}} > 0
$$

$$
\liminf_{k \to \infty} \frac{b_k - a_k}{\min\{b_k, 0\}} > 0
$$

Intuitively, this definition says that these sequences do not move too fast to the left or to the right.

If $\Lambda$ is an admissible sequence of sets, let us define

$$
K_\mu(f, \tau, \epsilon) := \lim_{k \to \infty} \frac{K(f, \tau, \epsilon, \Lambda_k)}{|\Lambda_k|}
$$

Given these definitions, we will prove that

**Theorem 2.2.** For a given ergodic probability measure $\mu$, if the complexity function $K$ satisfies (H1) and (H2), the limits in (2.6) and (2.11) exist almost surely and $K(f, \tau, \epsilon)$ is almost surely equal to a constant $K_\mu(\tau, \epsilon)$.
not depending on the admissible sequence \( \Lambda \) of sets. Moreover the function \( K_\mu(\tau, \epsilon) \) is not decreasing in \( \epsilon \), hence the limit
\[
K_\mu(\tau) := \lim_{\epsilon \to 0} K_\mu(\tau, \epsilon)
\]
extists and moreover there exists a constant \( K_\mu \) such that for all \( \tau > 0 \)
\[
\frac{K_\mu(\tau)}{\tau} = K_\mu
\]

**Theorem 2.3.** If the complexity function satisfies also (H3) and (H4), then
\[
\sup \{ K_\mu : \mu \text{ invariant probability measures} \} = h_{top}
\]
where \( h_{top} \) is defined in (2.3).

Before giving the proofs of these theorems, we recall that for a finite word \( s \in \{0, 1\}^* \), the Kolmogorov complexity or Algorithmic Information Content of \( s \) is defined as
\[
C(s) := \min \{|w| : w \in \{0, 1\}^*, U(w) = s\}
\]
where \(|\cdot|\) denotes the length of a word, and \( U \) is a universal Turing machine. For more details we refer to [18].

**Theorem 2.4.** The Kolmogorov complexity satisfies hypotheses (H1)-(H4).

**Proof.** We recall that the translation of a finite word from the binary alphabet to any other finite alphabet \( A \) requires only a constant amount of information content not dependent on the word. Hence we assume that these constants are included in the hypotheses (H1)-(H3).

Hypothesis (H1) and (H2) follow from [18], equation (2.2) and arguments used in [18], section 2.1.2.

Hypothesis (H3) is a corollary of Theorem 2.1.3 in [18].

Hypothesis (H4) is Theorem 2.2.1 in [18].

\( \square \)

3. **Proof of Theorem 2.2**

Let us consider any fixed probability measure \( \mu \) which is invariant and ergodic for the action of \( (\zeta, \varphi) \).

The first part of the proof relies on the application of arguments related to the sub-additivity property to define the quantities in (2.6) and (2.11).

Let \( X = (X_{m,n}) \) be a family of real random variables with indexes \( m,n \in \mathbb{N} \). We recall that \( X \) is almost subadditive if there exists a family of random variables \( U = (U_j) \), with \( j \in \mathbb{N} \), defined on the same probability space of \( X \) such that
\[
X_{m,n} \leq \sum_{i=1}^{k-1} (X_{j_i,j_{i+1}} + U_{j_{i+1} - j_i})
\]
for all \( 1 \leq m < n \) and all partitions \( m = j_1 < j_2 < \cdots < j_k = n \). In [23] the following result is proved
Theorem 3.1. Let $X$ and $U$ be jointly stationary and let $X$ be almost subadditive with respect to $U$. Assume that $X^+_{0,1} \in L^1_+$ and that there exists an increasing sequence of integers $(m_k)_k$ with $m_1 \geq 1$ such that

\[(3.2) \quad \lim_{k \to \infty} \inf \frac{X_{0,n+m_k}}{n+m_k} \geq \lim_{k \to \infty} \frac{X_{0,m_k}}{m_k} \quad \text{almost surely} \]

for all $n \geq 1$ and

\[(3.3) \quad \lim_{k \to \infty} \frac{U_{m_k}}{m_k} = 0 \quad \text{almost surely} \]

Then

\[\lim_{k \to \infty} \frac{X_{0,m_k}}{m_k} = \bar{x} \quad \text{exists almost surely} \]

with $-\infty \leq \bar{x} < \infty$ almost surely.

We first apply this theorem to $K(f, \tau, U^\epsilon, n)$ as defined in (2.5), identifying $X_{m,n}$ with $K(f, \tau, U^\epsilon, m, n)$. Then, since for all $\omega \in \mathcal{A}^N$ it holds

\[K(f, \tau, U^\epsilon, 1) = \min \{K(\omega_0) : \omega_0 \in \psi(f, 1, U^\epsilon)\} \in L^1\]

Moreover by (H1.a) we have

\[K(f, \tau, U^\epsilon, k) \leq K(f, \tau, U^\epsilon, n+k) + \log_2 k + \text{const}\]

for all $n \geq 1$ and all $f \in \mathcal{F}$, hence condition (3.2) of the previous theorem is satisfied with $m_k = k$. We now show the sub-additivity property with respect to a family of random variables.

Lemma 3.2. For any fixed $\tau > 0$, $\epsilon > 0$, $|\Lambda| < \infty$ and $U^\epsilon \in \mathcal{C}_\Lambda$, the family $(K(f, \tau, U^\epsilon, m, n))_{m,n}$ is almost subadditive with respect to the family of functions $h = h(j)$ defined in (H1.b).

Proof. Without loss of generality we can assume $m = 0$ because of stationarity. Let us fix a function $f \in \mathcal{F}$. For all $\omega_0^{n-1} \in \psi(f, n, U^\epsilon)$ by (H1.b) we have

\[K(\omega_0^{n-1}) \leq \sum_{i=1}^{k-1} (K(\omega_{j_i}^{j_{i+1}}) + h(j_{i+1} - j_i))\]

for any partition $0 = j_1 < j_2 < \cdots < j_{k-1} = n - 1$. Fixed any such partition, let $\omega_{j_i}^{j_{i+1}}, i = 1, \ldots, k-1, j_i$, be a collection of finite words such that

\[K(f, \tau, U^\epsilon, j_i, j_{i+1}) = K(\omega_{j_i}^{j_{i+1}})\]

Then, if we denote by $\tilde{\omega}_0^{n-1}$ the concatenation

\[\tilde{\omega}_0^{n-1} := \tilde{\omega}_{j_1}^{j_2} \tilde{\omega}_{j_2}^{j_3} \cdots \tilde{\omega}_{j_{k-1}}^{j_k}\]

it holds

\[\sum_{i=1}^{k-1} (K(\omega_{j_i}^{j_{i+1}}) + h(j_{i+1} - j_i)) \geq K(\tilde{\omega}_0^{n-1}) \geq K(f, \tau, U^\epsilon, n)\]
hence the sub-additivity property is proved. □

Since condition (3.3) is verified by the function \( h(n) \) and the probability measure \( \mu \) is invariant, we can apply Theorem 3.1 to obtain that the limit \( K(f, \tau, U^c_{\Lambda}) \) exists and is finite \( \mu \) almost surely. Let us denote by \( Y^{r}_{\epsilon, \Lambda} \subset \mathcal{F} \) a set with \( \mu((Y^{r}_{\epsilon, \Lambda})^c) = 0 \) on which the limit exists. Then we define

\[
K(f, \tau, U^c_{\Lambda}) := \begin{cases} \\
\lim_{n \to \infty} \frac{K(f, \tau, U^c_{\Lambda}, n)}{n} & \text{if } f \in Y^{r}_{\epsilon, \Lambda} \\
\infty & \text{otherwise}
\end{cases}
\]

(3.4)

We can then prove

**Lemma 3.3.** There exists a set \( Y^{r}_{\epsilon, \Lambda} \subset \mathcal{F} \) with \( \mu((Y^{r}_{\epsilon, \Lambda})^c) = 0 \) such that

\[
K(f, \tau, \epsilon, \Lambda) := \inf \{ K(f, \tau, U^c_{\Lambda}) : U^c_{\Lambda} \in C^c_{\Lambda} \}
\]

is well defined and finite for all \( f \in Y^{r}_{\epsilon, \Lambda} \). Moreover there exists a sequence \( \{ \hat{V}_s \}_s \) of coverings in \( C^c_{\Lambda} \) such that

\[
\lim_{s \to \infty} K(f, \tau, \hat{V}_s) = K(f, \tau, \epsilon, \Lambda) \quad \forall f \in Y^{r}_{\epsilon, \Lambda}
\]

(3.5)

and the sequence \( \{ K(f, \tau, \hat{V}_s) \}_s \) is non-increasing for all \( f \in Y^{r}_{\epsilon, \Lambda} \).

**Proof.** We first restrict to a countable set of coverings \( D^c_{\Lambda} \subset C^c_{\Lambda} \). Let \( G = \{ g_j \} \subset \mathcal{F} \) be a countable set dense in \( \mathcal{F}|\Lambda \), and define \( D^c_{\epsilon} \) as the set of finite open coverings

\[
D^c_{\epsilon} := \{ V^c_{\Lambda} \subset C^c_{\Lambda} : \text{the centers are in } G \text{ and radii are rational} \}
\]

Restricting to coverings \( V^c_{\Lambda} \in D^c_{\epsilon} \) we can define

\[
\bar{K}(f, \tau, \epsilon, \Lambda) := \inf \{ K(f, \tau, V^c_{\Lambda}) : V^c_{\Lambda} \in D^c_{\epsilon} \}
\]

(3.7)

on the set \( Y^{r}_{\epsilon, \Lambda} \subset \mathcal{F} \) with \( \mu((Y^{r}_{\epsilon, \Lambda})^c) = 0 \) defined by

\[
Y^{r}_{\epsilon, \Lambda} := \bigcap_{V^c_{\Lambda} \in D^c_{\epsilon}} Y^{r}_{\epsilon, \Lambda}
\]

We now show that \( \bar{K}(f, \tau, \epsilon, \Lambda) \) is equal to \( K(f, \tau, \epsilon, \Lambda) \) on \( Y^{r}_{\epsilon, \Lambda} \). To this aim it is enough to prove that for any \( U^c_{\Lambda} \in C^c_{\Lambda} \) there exists \( V^c_{\Lambda} \in D^c_{\epsilon} \) such that

\[
K(f, \tau, V^c_{\Lambda}) \leq K(f, \tau, U^c_{\Lambda})
\]

(3.8)

Indeed from this and (3.3), on \( Y^{r}_{\epsilon, \Lambda} \) we have that

\[
\bar{K}(f, \tau, \epsilon, \Lambda) \leq K(f, \tau, \epsilon, \Lambda)
\]

and the other inequality is obtained by using \( D^c_{\epsilon} \subset C^c_{\Lambda} \).

Let us now prove (3.3). Let \( U^c_{\Lambda} = \{ U_1, \ldots, U_c \} \) be a covering in \( C^c_{\Lambda} \) with \( c = \text{card}(\mathcal{U}^c_{\Lambda}) \), and define \( r = \max \{ r(U_j) : j = 1, \ldots, c \} < \epsilon \) where \( r(U_j) \) is the radius of the ball \( B_j \). Then by density of the set of functions \( G \) in \( \mathcal{F}|\Lambda \), we can find a covering \( V^c_{\Lambda} \in D^c_{\epsilon} \) with balls \( \{ V_j \}_{j=1,\ldots,c} \) such that \( U_j \subset V_j \) for all \( j = 1, \ldots, c \). Indeed it is enough to choose balls \( V_j \) with centres in
functions of the set $G$ at distances less than $\frac{\varepsilon}{2}$ from the centres of the balls $U_j$.

For this choice of coverings for all $n \geq 1$ it holds
\[
\psi(f, n, U_\Lambda^t) \subset \psi(f, n, V_\Lambda^t)
\]
hence from (2.5) and (3.4) it follows that for all $f \in Y^{\tau, \Lambda}$
\[
K(f, \tau, U_\Lambda^t) = \lim_{n \to \infty} K(f, \tau, U_\Lambda^t, n) = K(f, \tau, U_\Lambda^t)
\]
We now prove the second part of the assertion. Let us consider an enumeration of the coverings in $D_\Lambda^\varepsilon = \{V_j\}_{j}$, then we define
\[
\tilde{V}_s := \bigwedge_{1 \leq j \leq s} V_j
\]
where, for two finite open coverings $U$ and $V$, by $U \wedge V$ we denote the finite open covering which contains all the balls of $U$ and $V$. By definition, it is clear that $V_s \in D_\Lambda^\varepsilon$ for all $s \geq 1$. Moreover, since $\tilde{V}_s$ contains all the balls of the coverings $V_1, \ldots, V_s$, we have that, modulo a renumbering of the balls of $\tilde{V}_s$, $\psi(f, n, V_j) \subset \psi(f, n, \tilde{V}_s)$ for all $j = 1, \ldots, s$ and all $f \in \mathcal{F}$. Hence for all $j = 1, \ldots, s$
\[
K(f, \tau, \tilde{V}_s, n) \leq K(f, \tau, V_j, n) + \text{const} \quad \forall f \in \mathcal{F}
\]
where the constant is independent on the length $n$ of the symbolic words. Dividing by $n$ and taking the limit as $n \to \infty$, we obtain for all $f \in \mathcal{F}$
\[
\bar{K}(f, \tau, \varepsilon, \Lambda) \leq \liminf_{s \to \infty} K(f, \tau, \tilde{V}_s) \leq \limsup_{s \to \infty} K(f, \tau, \tilde{V}_s) \leq \bar{K}(f, \tau, \varepsilon, \Lambda)
\]
where the first two inequalities come from the definition of upper and lower limit. Hence (3.5) is proved.

By the same argument as above, it is immediate to verify that for all $f \in \mathcal{F}$ the sequence $\{K(f, \tau, \tilde{V}_s)\}_s$ is non-increasing. Hence the lemma is proved. \hfill $\square$

The next step is to show the existence of the limit in (2.11) for an admissible sequence of intervals to define $K(f, \tau, \varepsilon)$. We need the following general lemma

**Lemma 3.4.** Let $T : (X, \nu) \to (X, \nu)$ be a measure preserving invertible transformation of a probability space $(X, \nu)$. Let $\vartheta$ and $\xi$ be two real functions on $X$ in the space $L^1(X, \nu)$ and let $\xi(x) \geq 0$ for all $x \in X$. Then there exists a set $Y \subset X$ with $\nu(Y^c) = 0$ such that for any sequences $\{a_k\}_k$ and $\{b_k\}_k$ of integers satisfying conditions (2.8)-(2.10) we have
\[
(3.9) \quad \bar{\vartheta}(x) := \lim_{k \to \infty} \frac{1}{b_k - a_k} \sum_{j = a_k}^{b_k - 1} \vartheta(T^j(x))
\]
exists, is finite for all \( x \in Y \) and it is in \( L^1(X, \nu) \). Moreover it satisfies

\[
\int_X \bar{\vartheta}(x) d\nu = \int_X \vartheta(x) d\nu
\]

For the function \( \xi \) we have

\[
\lim_{k \to \infty} \frac{\xi(T^{b_k}(x))}{b_k - a_k} = 0
\]

for all \( x \in Y \).

This result is in the spirit of results in [16] and [15], where it is proved that we cannot ask for weaker conditions on the sequences \( \{a_k\}_k \) and \( \{b_k\}_k \). However we could not relate directly our lemma to their results, hence in the appendix we give a proof.

We will use this lemma for the space translation action to show that there exists a set \( Y_\tau^\epsilon \subset F \), with \( \mu(Y_\tau^\epsilon) = 0 \), such that the limit along any admissible sequence of intervals \( \Lambda = \{\Lambda_k\} \)

\[
\lim_{k \to \infty} \frac{K(f, \tau, \epsilon, \Lambda_k)}{|\Lambda_k|}
\]

exists and is finite for all \( f \in Y_\tau^\epsilon \). The ergodicity of the measure \( \mu \) will imply that this limit is independent on \( f \) and it is a constant \( K_\mu(\tau, \epsilon) \). Moreover from the proof it will follow that the limit does not depend on the admissible sequence \( \Lambda \) of sets, indeed it will be given by (3.13).

We will study separately the superior and the inferior limits. For the superior limit we use the functions \( K(f, \tau, \epsilon, [0,1]) \) for \( p \in \mathbb{N} \). We first prove that we can apply Theorem 3.1 to this sequence of functions. We start by verifying that \( K(f, \tau, \epsilon, [0,1]) \in L^1 \). For any \( \mathcal{V}_{[0,1]}^\epsilon \in \mathcal{D}_{[0,1]}^\epsilon \) and the associated alphabet \( \mathcal{A} \), we can write

\[
K(f, \tau, \mathcal{V}_{[0,1]}^\epsilon, n) \leq n \max_{\alpha \in \mathcal{A}} K(\alpha) + \text{const}
\]

hence for all \( f \in Y_{[0,1]} \)

\[
K(f, \tau, \epsilon, [0,1]) \leq \inf \left\{ \max_{\alpha \in \mathcal{A}(\mathcal{V}_{[0,1]}^\epsilon)} K(\alpha) : \mathcal{V}_{[0,1]}^\epsilon \in \mathcal{D}_{[0,1]}^\epsilon \right\} + \text{const}
\]

Assumption \((A1)\) implies that \( K(f, \tau, \epsilon, [0,1]) \in L^1 \). Note that the bound depends only on the length of the interval \( \Lambda = [0,1] \).

Let \( \Lambda_1 \) and \( \Lambda_2 \) be two fixed intervals with disjoint interiors and denote their union \( \Lambda := \Lambda_1 \cup \Lambda_2 \), let \( \mathcal{C}_\Lambda^\epsilon \subset \mathcal{C}_\Lambda^\epsilon \) be the set of coverings of \( \mathcal{F}|_{\Lambda} \) built as in \((A2)\) by two coverings \( \mathcal{U}_{\Lambda_1}^\epsilon \subset \mathcal{C}_\Lambda^\epsilon \) and \( \mathcal{U}_{\Lambda_2}^\epsilon \subset \mathcal{C}_\Lambda^\epsilon \). For any \( \mathcal{U}_\Lambda^\epsilon \subset \mathcal{C}_\Lambda^\epsilon \) we can write, by using \((H2.a)\)

\[
\frac{K(f, \tau, \pi_1(\mathcal{U}_\Lambda^\epsilon), n)}{n |\Lambda_1|} \leq \frac{K(f, \tau, \mathcal{U}_\Lambda^\epsilon, n)}{n |\Lambda|} \frac{|\Lambda|}{|\Lambda_1|} + \frac{\text{const}}{n |\Lambda_1|}
\]

where \( \pi_1(\mathcal{U}_\Lambda^\epsilon) \) denotes the “projection” of the covering \( \mathcal{U}_\Lambda^\epsilon \) onto \( \mathcal{C}_{\Lambda_1}^\epsilon \).
Applying this argument with \( \Lambda_1 = [0, p] \) and \( \Lambda_2 = [p, m + p] \) for all \( m \geq 1 \), and by taking the limit as \( n \to \infty \) and the infimum limit on \( |\Lambda_1| = p \to \infty \), we obtain condition (3.2) for any fixed \( |\Lambda_2| = m \).

**Lemma 3.5.** For any fixed \( \epsilon > 0 \) and \( \tau > 0 \), the family \( \{K(f, \tau, \epsilon, [0, p])\}_{p \in \mathbb{N}} \) is almost subadditive with respect to the constant function \( u(|\Lambda|) \equiv \log_2 q \), where \( q \) is the constant defined in (A2).

**Proof.** Let us consider two disjoint intervals \( \Lambda_1 = [a, b] \) and \( \Lambda_2 = [b, c] \) and the union \( \Lambda = [a, c] \). Let us fix a function \( f \in \mathcal{F} \) and let \( \omega_0^{n-1} \in \psi(f, n, \mathcal{U}_A^n) \) for a covering \( \mathcal{U}_A^n \in \mathcal{C}_A^n \). By (H2.b) we have

\[
K(\omega_0^{n-1}) \leq K(\pi_1(\omega_0^{n-1})) + K(\pi_2(\omega_0^{n-1})) + n \log_2 q + \text{const}
\]

since the map \( \pi = (\pi_1, \pi_2) \) is surjective. Then by the same argument as in Lemma 3.2 we have for all \( f \in \mathcal{Y}_{\mathcal{U}_A^n} \cap \mathcal{Y}_{\mathcal{U}_{A_1}} \cap \mathcal{Y}_{\mathcal{U}_{A_2}} \)

\[
K(f, \tau, \mathcal{U}_A^n, n) \leq K(f, \tau, \mathcal{U}_{A_1}^n, n) + K(f, \tau, \mathcal{U}_{A_2}^n, n) + n \log_2 q + \text{const}
\]

for \( \mathcal{U}_{A_i} = \pi_i(\mathcal{U}_A^n) \). Then we divide by \( n \) and take the limit as \( n \to \infty \). These limits exist as proved above, and we get

\[
K(f, \tau, \mathcal{U}_{A_1}^n) \leq K(f, \tau, \mathcal{U}_{A_1}^n) + K(f, \tau, \mathcal{U}_{A_2}^n) + \log_2 q
\]

for all coverings \( \mathcal{U}_{A_i} \in \mathcal{C}_{A_i}^{n+1} \) and the special covering \( \mathcal{U}_{A_i}^n \in \mathcal{C}_A^n \) built from the two. For any fixed \( \delta > 0 \) let us choose two coverings \( \mathcal{U}_{A_i}^n \) satisfying

\[
K(f, \tau, \mathcal{U}_{A_i}^n) \leq K(f, \tau, \epsilon, \Lambda_i) + \delta
\]

then

\[
K(f, \tau, \epsilon, \Lambda_1) + K(f, \tau, \epsilon, \Lambda_2) + 2\delta + \log_2 q \geq K(f, \tau, \mathcal{U}_{A_1}^n) \geq K(f, \tau, \epsilon, \Lambda)
\]

and sub-additivity is proved since it holds for all \( \delta > 0 \). \( \square \)

Since \( u(|\Lambda|) = \log_2 q \) obviously satisfies condition (3.3), we have that there exist a set \( \mathcal{Y}_\epsilon^\tau \subset \mathcal{F} \) with \( \mu((\mathcal{Y}_\epsilon^\tau)^c) = 0 \) on which \( K(f, \tau, \epsilon, [0, p]) \) is defined for all \( p \in \mathbb{N} \), and the limit

\[
\lim_{p \to \infty} \frac{K(f, \tau, \epsilon, [0, p])}{p} =: K_\mu(f, \tau, \epsilon)
\]

exists, is finite almost surely and it is in \( L^1(\mathcal{F}, \mu) \). Moreover the limit holds also in \( L^1 \) and we denote

\[
K_\mu(\tau, \epsilon) := \int_\mathcal{F} K_\mu(f, \tau, \epsilon) \, d\mu
\]

We remark that if the measure \( \mu \) is ergodic then \( K_\mu(f, \tau, \epsilon) \) is almost surely constant and equal to \( K_\mu(\tau, \epsilon) \).

Following the notation of Lemma 3.4 we denote

\[
\tilde{K}(f, \tau, \epsilon, [0, p]) := \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} K(\zeta_{jp} f, \tau, \epsilon, [0, p])
\]
where it exists. Then we prove the following lemma

**Lemma 3.6.** For any fixed $\tau$ and $\epsilon$, there exists a set $\bar{Y}_\epsilon^\tau$ with $\mu((\bar{Y}_\epsilon^\tau)^c) = 0$ such that for all $f \in \bar{Y}_\epsilon^\tau$ and any admissible sequence of intervals $\Lambda = \{\Lambda_k\}$, it holds

$$\limsup_{k \to \infty} \frac{K(f, \tau, \epsilon, \Lambda_k)}{|\Lambda_k|} \leq \liminf_{p \to \infty} \frac{\bar{K}(f, \tau, \epsilon, [0, p])}{p}$$

If moreover the measure $\mu$ is ergodic then

$$\limsup_{k \to \infty} \frac{K(f, \tau, \epsilon, \Lambda_k)}{|\Lambda_k|} \leq K_\mu(\tau, \epsilon)$$

**Proof.** Let us consider an admissible sequence of intervals with integer boundary points. For a fixed integer $p \in \mathbb{N}$, we can use the sub-additivity property (H2.b) as in Lemma 3.5 to show that for all $f \in \tilde{Y}_{\tau, \epsilon}$ we have, by setting $\lfloor \tilde{b}_k p \rfloor =: \tilde{b}_k$ and $\lfloor \tilde{a}_k p \rfloor + 1 =: \tilde{a}_k$,

$$K(f, [a_k, b_k]) \leq \sum_{j=\tilde{a}_k}^{\tilde{b}_k - 1} (K(f, [jp, (j + 1)p]) + \log_2 q) + K(f, [a_k, \tilde{a}_k p]) + K(f, [\tilde{b}_k p, b_k]) + 2 \log_2 q$$

where the dependence on $\tau$ and $\epsilon$ has been ignored to simplify notations. First of all, by repeating the same argument we used to prove (3.12), we can prove that there exists a constant depending only on $p$, see remark after (3.12), that is a bound from above for $K(f, [a_k, \tilde{a}_k p])$ and $K(f, [\tilde{b}_k p, b_k])$ for all $f \in \bar{Y}_\epsilon^\tau$. Moreover we can write

$$K(f, [jp, (j + 1)p]) = K(\zeta_p f, [0, p])$$

hence

$$\frac{K(f, [a_k, b_k])}{b_k - a_k} \leq \frac{1}{b_k - a_k} \sum_{j=\tilde{a}_k}^{\tilde{b}_k - 1} (K(\zeta_p f, [0, p]) + \log_2 q) + \frac{\text{const}}{b_k - a_k}$$

We now apply Lemma 3.4 to the action of the space translation $\zeta$ and with $K(f, [0, p])$ having the role of the $L^1$ function $\vartheta$. Let $Y_p \subset \mathcal{F}$ be the full measure set given for $K(f, [0, p])$ by Lemma 3.4 then we conclude by (3.9) that for all $f \in \bar{Y}_\epsilon^\tau := \tilde{Y}_\epsilon^\tau \cap (\cap_p Y_p)$, we have

$$\limsup_{k \to \infty} \frac{K(f, [a_k, b_k])}{b_k - a_k} \leq \frac{\bar{K}(f, [0, p])}{p} + \frac{\log_2 q}{p}$$

for all $p \in \mathbb{N}$. Hence we obtain the first part of the assertion.

The second part follows by first applying Lemma 3.4 to an ergodic measure, from which we get that for all $p \in \mathbb{N}$

$$\bar{K}(f, [0, p]) = \int_X K(f, [0, p]) \, d\mu$$

almost surely. Then we use (3.13) and (3.14) to conclude.
The result for the sequences \( \{a_k\}_k \) and \( \{b_k\}_k \) follows by writing
\[
K(f, [a_k, b_k]) \leq K(f, [a_k, [a_k] + 1]) + K(f, [[a_k] + 1, [b_k]]) +
+ K(f, [[b_k], b_k]) + 3 \log_2 q
\]
and reducing to the above argument. \( \square \)

To prove a similar result for the inferior limit we use the following result proved in [11], Theorem 3.7 \([11]\). Let \( T : (X, \nu) \to (X, \nu) \) be a measure preserving invertible transformation of a probability space \( (X, \nu) \). Let \( \{\beta_n\}_n \) be a sequence of integrable real functions on \( X \) such that
\[
\text{(3.16)} \quad \inf_n \frac{1}{n} \int_X \beta_n(x) \, d\nu(x) > -\infty
\]
and for all \( n, k \)
\[
\beta_{n+k}(x) - \beta_n(x) - \beta_k(T^n(x)) \leq h_k(T^n(x))
\]
for a sequence of functions \( \{h_k\}_k \) satisfying \( h_k \geq 0 \) and \( \int_X h_k \, d\nu \leq \text{const} \).

Then there exists a function \( \vartheta \in L^1(X, \nu) \) such that
\[
\text{(3.18)} \quad \int_X \vartheta(x) \, d\nu = \lim_{n \to \infty} \frac{1}{n} \int_X \beta_n(x) \, d\nu
\]
and a function \( \xi \in L^1(X, \nu) \) such that \( \xi \geq 0 \) and
\[
\text{(3.19)} \quad \sum_{j=0}^{n-1} \vartheta(T^j(x)) \leq \beta_n(x) + \xi(T^n(x))
\]
for almost all \( x \in X \) and all \( n \in \mathbb{N} \).

**Lemma 3.8.** Under the hypothesis of ergodicity for the probability measure \( \mu \) on \( \mathcal{F} \), for any fixed \( \tau \) and \( \epsilon \), there exists a set \( Y^\tau_\epsilon \) with \( \mu(Y^\tau_\epsilon^c) = 0 \) such that for all \( f \in Y^\tau_\epsilon \) and any admissible sequence of intervals \( \Lambda = \{\Lambda_k\} \), it holds
\[
\liminf_{k \to \infty} \frac{K(f, \tau, \epsilon, \Lambda_k)}{|\Lambda_k|} \geq K_\mu(\tau, \epsilon)
\]

**Proof.** Let us consider first the case of sequences \( \{a_k\} \) and \( \{b_k\} \) of integers. We apply Theorem 3.7 to the sequence \( \beta_p(f) := K(f, \tau, \epsilon, [0, p]) \) for \( f \in Y^\tau_\epsilon \) and \( p \in \mathbb{N} \). Indeed condition (3.16) is easily verified since \( K(f, [0, p]) \) is non-negative\(^2\), and condition (3.17) is the subadditive property we proved in Lemma 3.5. Hence there exists a full measure set \( Y(\xi, \vartheta) \subset \mathcal{F} \) on which (3.19) is verified for two given functions \( \vartheta \) and \( \xi \). Moreover from (3.13) and (3.14), for an ergodic measure \( \mu \) we have that
\[
\int_\mathcal{F} \vartheta(f) \, d\mu = K_\mu(\tau, \epsilon)
\]

\(^2\)To simplify notations we neglect the dependence on \( \tau \) and \( \epsilon \).
Now let us define the set
\[ Y_\zeta(\xi, \vartheta) := \bigcap_{n \in \mathbb{Z}} \\{ \zeta_n f : f \in Y(\xi, \vartheta) \} \]

since \( \mu \) is \( \zeta \)-invariant it holds \( \mu((Y_\zeta(\xi, \vartheta))^c) = 0 \). On \( Y_\zeta(\xi, \vartheta) \) we can write
\[ K(f, [a_k, b_k]) = K(\zeta_{a_k} f, [0, b_k - a_k]) \]
and, using (3.19),
\[ \sum_{j=0}^{b_k-a_k-1} \vartheta(\zeta_j + a_k f) \leq K(f, [a_k, b_k]) + \xi(\zeta_{b_k} f) \]
At this point we apply Lemma 3.4 to \( \vartheta \) and \( \xi \), and we obtain using the ergodicity of the measure \( \mu \)
\[ K_\mu(\tau, \epsilon) = \lim_{k \to \infty} \frac{1}{b_k - a_k} \sum_{j=a_k}^{b_k-1} \vartheta(\zeta_j(f)) \leq \lim_{k \to \infty} \inf \frac{K(f, [a_k, b_k])}{b_k - a_k} \]
for almost all \( f \in \mathcal{F} \). Let us call this full measure set \( Y_\tau^\epsilon \).

As in Lemma 3.6 the general case for real sequences \( \{a_k\} \) and \( \{b_k\} \) follows by
\[ K(f, [a_k, b_k]) \geq K(f, [\lfloor a_k \rfloor, \lfloor b_k \rfloor + 1]) - K(f, [\lfloor a_k \rfloor, a_k]) - K(f, [b_k, \lfloor b_k \rfloor + 1]) - 3 \log_2 q \]
The lemma is proved.

Putting together Lemmas 3.6 and 3.8 we have that, for any fixed \( \tau \) and \( \epsilon \), there exists a set \( Y_\epsilon^\tau := \bar{Y}_\epsilon^\tau \cap Y_\epsilon^\tau \subset \mathcal{F} \) with \( \mu((Y_\epsilon^\tau)^c) = 0 \), such that for all \( f \in Y_\epsilon^\tau \) the limit
\[ \lim_{k \to \infty} \frac{K(f, \tau, \epsilon, |A_k|)}{|A_k|} = K_\mu(f, \tau, \epsilon) \]
exists and is finite for all admissible sequence of intervals \( \Lambda = \{A_k\} \). Moreover, for an ergodic measure \( \mu \), for all \( f \in Y_\epsilon^\tau \) and any admissible sequence of intervals \( \Lambda \), it is equal to the constant \( K_\mu(\tau, \epsilon) \) defined in (3.14). Hence the limit in (2.11) exists.

To finish the proof of Theorem 2.2 we first have to prove that \( K_\mu(\tau, \epsilon) \) is non decreasing in \( \epsilon \). Let \( \epsilon_1 < \epsilon_2 \), then according to the above arguments, we can define \( K(f, \tau, \epsilon_1, \Lambda) \) and \( K(f, \tau, \epsilon_2, \Lambda) \) for all \( |\Lambda| < \infty \) as in (2.7) and all \( f \in Y_\epsilon^\tau \). Moreover, since \( C_{\Lambda_1}^{\epsilon_1} \subset C_{\Lambda_2}^{\epsilon_2} \) we have \( Y_{\epsilon_1, \Lambda} \subset Y_{\epsilon_2, \Lambda} \) and
\[ K(f, \tau, \epsilon_2, \Lambda) \leq K(f, \tau, \epsilon_1, \Lambda) \]
Restricting to \( \Lambda = [a_k, b_k] \) with \( a_k < b_k \) for all \( k \geq 1 \) and satisfying (2.8) and (2.10), dividing by \( |\Lambda| = (b_k - a_k) \) and taking the limit as in (2.11) gives
\[ K_\mu(\tau, \epsilon_2) \leq K_\mu(\tau, \epsilon_1) < \infty \]
on the set \( Y^\tau_{\epsilon_k} \subset Y^\tau_{\epsilon_2} \). Let us now consider a monotonically vanishing sequence \( \{\epsilon_k\}_k \subset \mathbb{Q}^+ \) and define
\[
Y^\tau := \bigcap_{k \geq 1} Y^\tau_{\epsilon_k}
\]
Then \( \mu((Y^\tau)^c) = 0 \) and \( K_\mu(\tau, \epsilon) \) is finite on \( Y^\tau \) for all \( \epsilon > 0 \) and non-decreasing on \( \epsilon \). Hence we can define
\[
K_\mu(\tau) = \lim_{\epsilon \to 0} K_\mu(\tau, \epsilon)
\]
on the full measure set of functions \( Y^\tau \). In Theorem 2.3 we prove that it is finite for complexity functions satisfying also \( (H3) \) and \( (H4) \).

Now it remains to prove that \( K_\mu(\tau) \) does not depend on \( \tau \). We will use assumption \( (A3) \).

**Lemma 3.9.** The full measure set \( Y := Y^\tau \) does not depend on \( \tau \), and there exists a constant \( \bar{K}_\mu \) such that \( \frac{K_\mu(\tau)}{\tau} = \bar{K}_\mu \) on \( Y \) for all \( \tau > 0 \).

**Proof.** Let us fix constants \( \gamma, \Gamma, C \) as in \( (A3) \), an interval \( |\Lambda| < \infty \), \( \epsilon > 0 \) and a time step \( \tau \in (0, C^{-1} \text{diam}(\Lambda) \epsilon) \). Then for all \( \tau' > 0 \) we denote \( \epsilon' := \Gamma e^{\gamma \tau} \epsilon \)
\[
\Lambda' := \Lambda \setminus \{d(x, \partial \Lambda) < C\epsilon^{-1}(\tau + 1)\}
\]
Moreover, given a covering \( \mathcal{U}_n' \subset \mathcal{C}'_\Lambda \), we consider the covering \( \bar{\mathcal{U}}_n' \subset \mathcal{C}'_\Lambda \), which has balls with the same centres as those in \( \mathcal{U}_n \) and radius increased by a factor \( \Gamma e^{\gamma \tau} \). By assumption \( (A3) \), for any function \( f \in F \), we can build a symbolic orbit \( \bar{\omega}_n' \subset \psi(f, n', \bar{\mathcal{U}}_n') \) by using the information contained in a symbolic orbit \( \omega_n' \subset \psi(f, n, \mathcal{U}_n) \), with \( n = n' \frac{\tau'}{\tau} \) by defining
\[
\bar{\omega}_j' = \omega_j \quad \text{with} \quad j = \begin{cases} 
\left\lfloor j' + \left( \frac{\tau'}{\tau} - 1 \right) \right\rfloor & \text{if } \tau' > \tau \\
\left\lfloor j' \frac{\tau}{\tau'} \right\rfloor & \text{if } \tau' < \tau
\end{cases}
\]
hence
\[
K(\bar{\omega}_n'^{-1}) \leq K(\omega_n'^{-1}) + n' \frac{\tau'}{\tau} + \text{const}
\]
with a constant dependent only on the complexity function, and the term \( n' \frac{\tau'}{\tau} \) that contains the information we need each time that the difference between \( j \) and \( j' \) increases of one unit.

Let now \( \bar{\omega}_n'^{-1} \) be the symbolic orbit on which the minimum for the function \( K(f, \tau, \mathcal{U}_n') \) is attained. Then by (3.20) we have
\[
K(f, \tau', \bar{\mathcal{U}}_n', n') \leq K(\bar{\omega}_n'^{-1}) \leq K(f, \tau, \mathcal{U}_n, n) + n' \frac{\tau'}{\tau} + \text{const}
\]
and dividing by \( n \) and taking the limit for \( n \to \infty \) we have
\[
\frac{K(f, \tau', \bar{\mathcal{U}}_n')}{\tau'} \leq \frac{K(f, \tau, \mathcal{U}_n)}{\tau} + \frac{1}{\tau}
\]
for all \( f \in Y^\tau \cap Y^{\tau'} \). At this point, let us fix a \( \delta > 0 \) and let \( U^\delta_{\Lambda} \) be a covering such that
\[
K(f, \tau, U^\delta_{\Lambda}) < K(f, \tau, \epsilon, \Lambda) + \delta
\]
then using the induced covering \( \bar{U}^\delta_{\Lambda} \), and (3.21) we have
\[
(3.22) \quad \frac{K(f, \tau', \epsilon', \Lambda')}{\tau'|\Lambda'|} \leq \frac{K(f, \tau', \bar{U}^\delta_{\Lambda})}{\tau'|\Lambda'|} < \frac{K(f, \tau, \epsilon, \Lambda) + \delta + 1}{\tau|\Lambda|}
\]
for all \( \Lambda = [a_k, b_k] \) as in (2.8)-(2.10). Then the limit for \( |\Lambda| \to \infty \) gives
\[
\frac{K_{\mu}(\tau', \epsilon')}{\tau'} < \frac{K_{\mu}(\tau)}{\tau} < \infty
\]
on \( Y^\tau \cap Y^{\tau'} \), where we have used \( \frac{|\Lambda'|}{|\Lambda|} \to 1 \) as \( |\Lambda| \to \infty \), and we have suppressed the dependence on the function \( f \) because of the ergodicity of the measure \( \mu \).

The final step is the limit for \( \epsilon \), and since \( \epsilon' \to 0 \) as \( \epsilon \to 0 \) we have
\[
\frac{K_{\mu}(\tau', \epsilon')}{\tau'} < \frac{K_{\mu}(\tau)}{\tau} < \infty
\]
on \( Y^\tau \cap Y^{\tau'} = Y^\tau \).

Repeating the argument interchanging the roles of \( \tau \) and \( \tau' \), the lemma is proved.

Hence Theorem 2.2 is proved.

4. Proof of Theorem 2.3

Since we proved that \( K_\mu(\tau) = K_\mu \) for all \( \tau > 0 \), in this proof we can fix \( \tau = 1 \) for simplicity of notation and drop it from formulas.

The first inequality
\[
(4.1) \quad \sup \{ K_\mu : \mu \text{ invariant probability measures} \} \leq h_{\text{top}}
\]
follows by showing that for any \( (\zeta, \varphi) \)-invariant probability measure \( \mu \), and for any fixed \( \epsilon > 0 \) and any finite interval \( \Lambda \) it holds
\[
(4.2) \quad \int_{\mathcal{F}} K(f, \epsilon, \Lambda) \, d\mu \leq \lim_{T \to \infty} \frac{\log_2 (N^\epsilon/4_{\Lambda}(T))}{T}
\]
where the right hand side is defined as in (2.8). Indeed (4.2) implies (4.1) just dividing by \( |\Lambda| \), and using the \( L^1 \) convergence proved in Theorem 2.2 for the limits as \( |\Lambda| \to \infty \) and \( \epsilon \to 0 \).

To prove (4.1), since for any \( f \in \mathcal{F} \) it holds \( K(f, \epsilon, \Lambda) \leq K(f, U^\epsilon_{\Lambda}) \) for any covering \( U^\epsilon_{\Lambda} \in \mathcal{C}_\Lambda^\epsilon \), it is enough to prove the following lemma

Lemma 4.1. There exists a covering \( \bar{U}^\epsilon_{\Lambda} \in \mathcal{C}_\Lambda^\epsilon \) such that
\[
(4.3) \quad \int_{\mathcal{F}} K(f, \bar{U}^\epsilon_{\Lambda}) \, d\mu \leq \lim_{n \to \infty} \frac{\log_2 (N^\epsilon/4_{\Lambda}(n))}{n}
\]
Proof. Let us consider a covering $U_\Lambda^c \in C_\Lambda$ with balls of radius $\frac{\varepsilon}{2}$, and a new covering $\bar{U}_\Lambda^c$ with balls with the same centres as before and radius $\frac{\varepsilon}{3}$.

We recall that the Lebesgue number lemma states that for a finite open covering of a compact metric space, there is a finite number $\gamma > 0$ such that the $\gamma$-neighbourhood of any point is contained in at least one open set of the covering. The number $\gamma$ is called the Lebesgue number of the covering.

Since $F|\Lambda$ is a metric compact set, the covering $\bar{U}_\Lambda^c$ has a finite Lebesgue number, and by its construction we conclude that its Lebesgue number is not less than $\frac{\varepsilon}{4}$.

The idea of the proof is to use $(H3)$, hence we need to count all possible symbolic orbits. For all $n$ let us consider a minimal $\left(n, \frac{\varepsilon}{4}\right)$-spanning set $\Sigma(n, \frac{\varepsilon}{4})$ for $F|\Lambda$, that is for any $f \in F$ there exists $g \in \Sigma(n, \frac{\varepsilon}{4})$ such that

$$d|\Lambda(\varphi_k(f), \varphi_k(g)) \leq \frac{\varepsilon}{4} \quad \forall \; k = 0, \ldots, n-1$$

For any $g \in \Sigma(n, \frac{\varepsilon}{4})$ let us denote by $R_g \subset F$ the set of functions $f \in F$ which are $\frac{\varepsilon}{4}$-spanned by $g$. We can make $\{R_g\}_g$ a partition of $F$ just by choosing for each $f \in F$ only one function spanning it.

For any function $g \in \Sigma(n, \frac{\varepsilon}{4})$ we can consider the sequence of balls $\{B(\varphi_k(g), \frac{\varepsilon}{4})\}_k$ with $k = 0, \ldots, n-1$. At the same time, by our result on the Lebesgue number of $\bar{U}_\Lambda^c$, we can associate to each such sequence of balls a symbolic orbit $\omega_0^{n-1}(g) \in \psi(g, n, \bar{U}_\Lambda^c)$. Then we have

$$\int_F \frac{K(f, \bar{U}_\Lambda^c, n)}{n} \, d\mu \leq \frac{1}{n} \sum_{g \in \Sigma(n, \frac{\varepsilon}{4})} K(\omega_0^{n-1}(g)) \mu(R_g) \tag{4.4}$$

Let us now consider for the alphabet $A = \{1, \ldots, \text{card}(\bar{U}_\Lambda^c)\}$, the set

$$E := \bigcup_{n \in \mathbb{N}} \bigcup_{g \in \Sigma(n, \frac{\varepsilon}{4})} (\omega_0^{n-1}(g), n) \subset A^* \times \mathbb{N}$$

It is a recursively enumerable set, hence we can apply hypothesis $(H3)$ getting

$$K(\omega_0^{n-1}(g)) \leq \log_2 \left(\sigma \left(n, \frac{\varepsilon}{4}\right)\right) + \log_2 n + \text{const} \tag{4.5}$$

where $\sigma \left(n, \frac{\varepsilon}{4}\right) := \text{card}(\Sigma \left(n, \frac{\varepsilon}{4}\right))$. Since this estimate is uniform on $g$, applying it to (4.4) we get

$$\int_F \frac{K(f, \bar{U}_\Lambda^c, n)}{n} \, d\mu \leq \frac{\left(\log_2 \left(\sigma \left(n, \frac{\varepsilon}{4}\right)\right) + \log_2 n + \text{const}\right)}{n} \tag{4.6}$$

since $\sum_g \mu(R_g) = 1$.

To finish the proof of the lemma we use the inequality

$$\sigma \left(n, \frac{\varepsilon}{4}\right) \leq N_\Lambda^{\varepsilon/4}(n)$$

which is well known in ergodic theory, see for example [10].
Let us now prove the other inequality. For any fixed \( \epsilon > 0 \), from the definition of topological entropy (2.3) we define

\[
(4.7) \quad h_\Lambda(\epsilon) := \lim_{n \to \infty} \frac{\log_2 N_\Lambda^\epsilon(n)}{n}
\]

\[
(4.8) \quad h(\epsilon) := \lim_{|\Lambda| \to \infty} \frac{h_\Lambda(\epsilon)}{|\Lambda|}
\]

The quantity \( h(\epsilon) \) is non-decreasing in \( \epsilon \) and its limit as \( \epsilon \to 0 \) is \( h_{\text{top}} \). With respect to the quantities defined above, given any fixed \( \delta > 0 \) there exist \( \epsilon_0 > 0 \), \( \lambda_0(\epsilon) > 0 \) and \( n_0(\epsilon, \Lambda) > 0 \) such that

\[
(4.9) \quad h_{\text{top}} - \delta < h(\epsilon) < h_{\text{top}} \quad \forall \ \epsilon < \epsilon_0
\]

\[
(4.10) \quad (h(\epsilon) - \delta)|\Lambda| < h_\Lambda(\epsilon) < (h(\epsilon) + \delta)|\Lambda| \quad \forall \ |\Lambda| > \lambda_0(\epsilon)
\]

\[
(4.11) \quad 2^{n(h_\Lambda(\epsilon) - \delta|\Lambda|)} < N_\Lambda^\epsilon(n) < 2^{n(h_\Lambda(\epsilon) + \delta|\Lambda|)} \quad \forall \ n > n_0(\epsilon, \Lambda)
\]

We first state a lemma we need in the following

**Lemma 4.2.** Let us consider a fixed \( \epsilon > 0 \) and a finite interval \( \Lambda \). If \( \{\rho_j\}_j \) is a sequence of probability measure on \( \mathcal{F} \), invariant with respect to the time evolution \( \varphi_1 \), then there exists a sub-sequence \( \{\rho_{j_h}\}_h \) such that \( \rho_{j_h} \) is weakly convergent to a probability measure \( \rho \), invariant with respect to \( \varphi_1 \), and

\[
\limsup_{h \to \infty} \int_{\mathcal{F}} K(f, \epsilon, \Lambda) \ d\rho_{j_h} \leq \int_{\mathcal{F}} K(f, \epsilon, \Lambda) \ d\rho
\]

**Proof.** The existence of the \( \varphi_1 \)-invariant probability measure \( \rho \) follows by the compactness of the space \( \mathcal{F}|\Lambda \). For simplicity of notations, let us assume that \( \{\rho_j\}_j \) is weakly convergent to \( \rho \).

By the monotonicity of the sequence \( \{K(f, \tilde{V}_s)\}_s \) proved in Lemma we have

\[
(4.12) \quad \int_{\mathcal{F}} K(f, \epsilon, \Lambda) \ d\rho_j = \lim_{s \to \infty} \int_{\mathcal{F}} K(f, \tilde{V}_s) \ d\rho_j
\]

for all \( j \in \mathbb{N} \) and also for the \( \varphi_1 \)-invariant measure \( \rho \). By the sub-additive ergodic theorem in \( L^1 \) it holds

\[
(4.13) \quad \int_{\mathcal{F}} K(f, \tilde{V}_s) \ d\rho_j = \lim_{n \to \infty} \int_{\mathcal{F}} K(f, \tilde{V}_s, n) \ d\rho_j = \inf_{n \to \infty} \int_{\mathcal{F}} K(f, \tilde{V}_s, n) \ d\rho_j
\]

\(^3\)We recall that we consider fixed \( \tau = 1 \).
for all $s \in \mathbb{N}$. Moreover, for each fixed $n \in \mathbb{N}$, the function $f \mapsto K(f, \tilde{V}_s, n)$ is upper semi-continuous, hence using weak convergence of $\rho_j$ we get for all $s \in \mathbb{N}$ (see for example [1])

\[\inf_{n \to \infty} \int_{\mathcal{F}} \frac{K(f, \tilde{V}_s, n)}{n} \, dp_j \leq \inf_{n \to \infty} \int_{\mathcal{F}} \frac{K(f, \tilde{V}_s, n)}{n} \, dp = \int_{\mathcal{F}} K(f, \tilde{V}_s) \, dp\]

where for the last equality we have used (4.13) for the $\varphi_1$-invariant measure $\rho$. The assertion follows by putting together (4.13) and (4.14), and by applying (4.12) to both sides. □

**Lemma 4.3.** Given any $\delta > 0$, there exists $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$ there exists $\lambda_0(\epsilon) > 0$ such that for any finite interval $|\Lambda| > \lambda_0(\epsilon)$ it holds: there exists a probability measure $\mu^\epsilon_\Lambda$ on $\mathcal{F}$, invariant with respect to the time evolution $\varphi_1$, such that

\[\frac{1}{|\Lambda|} \int_{\mathcal{F}} K(f, \epsilon/2, \Lambda) \, d\mu^\epsilon_\Lambda \geq h_{\text{top}} - 4\delta\]

**Proof.** For any fixed $\delta > 0$ let us consider $\epsilon_0 > 0$ as defined for (4.9), and let us fix $\epsilon < \epsilon_0$. Referring to (2.2), for any finite interval $\Lambda$ let us denote by $S^\epsilon_\Lambda(n) := \{f_j\}_{j=1}^{N^\epsilon_\Lambda(n)}$ the functions of a maximal set of $(\Lambda, n, \epsilon)$-different orbits. On this set we consider the sequence of probability measures on $\mathcal{F}$ given by

\[\nu^\epsilon_\Lambda, n := \frac{1}{N^\epsilon_\Lambda(n)} \sum_{j=1}^{N^\epsilon_\Lambda(n)} \delta_{f_j}\]

where $\delta$ denotes the usual Dirac mass. Hence by definition of the set $S^\epsilon_\Lambda(n)$, for any open covering $U^\epsilon/2_\Lambda \in C^\epsilon/2$ we have

\[\frac{1}{|\Lambda|} \int_{\mathcal{F}} K(f, U^\epsilon/2_\Lambda, n) \, d\nu^\epsilon_\Lambda, n = \frac{1}{|\Lambda|} \frac{1}{N^\epsilon_\Lambda(n)} \sum_{j=1}^{N^\epsilon_\Lambda(n)} K(f_j, U^\epsilon/2_\Lambda, n)\]

We now use hypothesis (H4). For any given $\delta > 0$ we have

\[\text{card} \left\{ f \in S^\epsilon_\Lambda(n) : K(f, U^\epsilon/2_\Lambda, n) < n|\Lambda|(h(\epsilon) - 3\delta) \right\} \leq 2^{n|\Lambda|(h(\epsilon) - 3\delta)}\]

Putting together (4.16) and (4.17) we obtain

\[\frac{1}{|\Lambda|} \int_{\mathcal{F}} K(f, U^\epsilon/2_\Lambda, n) \, d\nu^\epsilon_\Lambda, n \geq \left( N^\epsilon_\Lambda(n) - 2^{n|\Lambda|(h(\epsilon) - 3\delta)} \right) \frac{(h(\epsilon) - 3\delta)}{N^\epsilon_\Lambda(n)}\]

For any fixed $m \in \mathbb{N}$ let us write $n = tm + r$ with $0 \leq r < m$. Using the sub-additivity property for the family of functions $(K(f, U^\epsilon/2_\Lambda, m, n))_{m,n}$
proved in Lemma \(3.2\) we write for any \(f \in \mathcal{F}\)
(4.19)
\[
K(f, U_{\lambda}^{\ell/2}, n) \leq \sum_{j=0}^{t-2} K(\varphi_i(f), U_{\lambda}^{\ell/2}, jm, (j+1)m) + \\
+ K(f, U_{\lambda}^{\ell/2}, l) + K(f, U_{\lambda}^{\ell/2}, (t-1)m + l, n) + \theta h(m)
\]
for all \(l = 0, \ldots, m - 1\), where \(h(\cdot)\) is the function defined in (H1.b). Moreover for all \(j = 0, \ldots, t - 2\) and all \(l = 0, \ldots, m - 1\), it holds
\[
\int_{\mathcal{F}} K(\varphi_i(f), U_{\lambda}^{\ell/2}, jm, (j+1)m) \, d\nu^{\ell}_n = \int_{\mathcal{F}} K(f, U_{\lambda}^{\ell/2}, m) \, d(\varphi_i jm \nu^{\ell}_n)
\]
and for all \(f \in \mathcal{F}\) the uniform estimate
\[
K(f, U_{\lambda}^{\ell/2}, l) + K(f, U_{\lambda}^{\ell/2}, (t-1)m + l, n) \leq 2m \log_2(\text{card}(U_{\lambda}^{\ell/2})) + \text{const}
\]
holds. Hence letting
\[
\mu^{\ell}_n := \frac{1}{m} \sum_{l=0}^{m-1} \left( \frac{1}{(t-1)l} \sum_{j=0}^{t-2} \varphi_{i+jm} \nu^{\ell}_n \right) = \frac{1}{(t-1)m} \sum_{i=0}^{(t-1)m-1} \varphi_i \nu^{\ell}_n
\]
we obtain
(4.20)
\[
\int_{\mathcal{F}} K(f, U_{\lambda}^{\ell/2}, n) \, d\nu^{\ell}_n \leq \frac{(t-1)m}{tm + r} \int_{\mathcal{F}} K(f, U_{\lambda}^{\ell/2}, m) \, d\mu^{\ell}_n + \frac{\theta h(m) + o(n)}{tm + r}
\]
Let \(\mu^{\ell}_n\) be an accumulation point for the sequence of probability measures \(\{\mu^{\ell}_n\}_n\) given by Lemma \(1.2\). It follows that \(\mu^{\ell}_n\) is a probability measure on \(\mathcal{F}\) which is invariant for the time action \(\varphi_1\). Using \(4.20\) and the upper semi-continuity of the function \(f \mapsto K(f, U_{\lambda}^{\ell/2}, m)\) for all \(m \in \mathbb{N}\), we have
(4.21)
\[
\limsup_{n \to \infty} \int_{\mathcal{F}} K(f, U_{\lambda}^{\ell/2}, n) \, d\nu^{\ell}_n \leq \int_{\mathcal{F}} K(f, U_{\lambda}^{\ell/2}, m) \, d\mu^{\ell}_n + \frac{h(m)}{m}
\]
for all \(m \in \mathbb{N}\).

Let now \(\lambda_0(\varepsilon) > 0\) and \(n_0(\varepsilon, \Lambda) > 0\) be defined as in (4.10) and (4.11). Then from (4.10) we have that if \(|\Lambda| > \lambda_0\) and \(n > n_0\) it holds
\[
\frac{1}{|\Lambda|} \int_{\mathcal{F}} K(f, U_{\lambda}^{\ell/2}, n) \, d\nu^{\ell}_n \geq \left( 1 - 2^{-n|\Lambda|\delta} \right) (h(\varepsilon) - 3\delta)
\]
Hence for all \(m \in \mathbb{N}\), if \(\varepsilon < \varepsilon_0\) and \(|\Lambda| > \lambda_0(\varepsilon)\) we have
(4.22)
\[
\int_{\mathcal{F}} K(f, U_{\lambda}^{\ell/2}, m) \, d\mu^{\ell}_n + \frac{h(m)}{m} \geq (h_{\text{top}} - 4\delta)|\Lambda|
\]
for all coverings \(U_{\lambda}^{\ell/2} \in \mathcal{U}_{\lambda}^{\ell/2}\), where we have used (4.9). By the sub-additive ergodic theorem in \(L^1\) we have that
\[
\lim_{m \to \infty} \int_{\mathcal{F}} K(f, U_{\lambda}^{\ell/2}, m) \, d\mu^{\ell}_n = \int_{\mathcal{F}} K(f, U_{\lambda}^{\ell/2}) \, d\mu^{\ell}_n
\]
where $K(f, U_\Lambda^{\epsilon/2})$ is defined as in (3.1). Hence, since $h(m) = o(m)$, from (4.22) we have that given any $\delta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$ there exists $\lambda_0(\epsilon) > 0$ such that for any $|\Lambda| > \lambda_0$ it holds

\[
\int_K K(f, U_\Lambda^{\epsilon/2}) \ d\mu_\lambda^\epsilon \geq (h_{top} - 4\delta)|\Lambda|
\]

for any finite open covering $U_\Lambda^{\epsilon/2} \in C_\Lambda^{\epsilon/2}$. To finish the proof of the lemma, we use the sequence of coverings $\left\{\tilde{V}_s\right\}_s$ defined in Lemma 3.3 to obtain

\[
\int_K K(f, \epsilon/2, \Lambda) \ d\mu_\lambda^\epsilon = \lim_{s \to \infty} \int_K K(f, \tilde{V}_s) \ d\mu_\lambda^\epsilon \geq (h_{top} - 4\delta)|\Lambda|
\]

where the last inequality is given by (4.23). \hfill \Box

**Lemma 4.4.** For any given $\delta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$ it holds: there exists a probability measure $\rho^\epsilon$ on $F$, invariant with respect to the $(\zeta_1, \varphi_1)$ action such that

\[
K_{\rho^\epsilon}(\epsilon/2) := \int_K K(f, \epsilon/2) \ d\rho^\epsilon \geq h_{top} - 4\delta
\]

**Proof.** In Theorem 2.2 we proved that $K_{\rho^\epsilon}(\epsilon)$, as defined in (3.14), is not dependent on the admissible sequence $\Lambda$ of intervals. Hence we will restrict to the family of intervals $\Lambda = \left\{[0, p]\right\}$, for $p \in \mathbb{N}$. For a fixed $q \in \mathbb{N}$, writing $p = tq + r$ with $0 \leq r < q$ and using the sub-additivity property proved in Lemma 3.5 we have for all $f \in F$

\[
K(f, \epsilon/2, [0, p]) \leq \sum_{j=0}^{t-2} K(\zeta(f), \epsilon/2, [jq, (j+1)q]) + K(f, \epsilon/2, [0, l]) + t \log_2 q
\]

for all $l = 0, \ldots, q - 1$, where $K(f, \epsilon/2, [0, l])$ and $K(f, \epsilon/2, [(t-1)q + l, p])$ are $O(q)$ as shown in (3.12). Fixed an $\epsilon < \epsilon_0$, for all $p > \lambda_0(\epsilon)$, where $\lambda_0(\epsilon)$ is given as in (4.10) and in Lemma 4.3 we denote by $\mu_\rho^\epsilon$ the $\varphi_1$-invariant probability measure associated to $[0, p]$. For all $p > \lambda_0$ we write using (4.25)

\[
\int_K K(f, \epsilon/2, [0, p]) \ d\mu_\rho^\epsilon \leq \sum_{j=0}^{t-2} \frac{K(\zeta(f), \epsilon/2, [jq, (j+1)q])}{p} \ d\mu_\rho^\epsilon + \frac{t \log_2 q + o(p)}{p}
\]

for all $l = 0, \ldots, q - 1$. Moreover for all $j = 0, \ldots, t-2$ and all $l = 0, \ldots, q - 1$ it holds

\[
\int_K K(\zeta(f), \epsilon/2, [jq, (j+1)q]) \ d\mu_\rho^\epsilon = \int_K K(f, \epsilon/2, [0, q]) \ d(\zeta_\rho^{\epsilon+j} \mu_\rho^\epsilon)
\]

hence we define

\[
\rho_\rho := \sum_{l=0}^{q-1} \frac{1}{q} \sum_{j=0}^{t-2} \zeta_\rho^{\epsilon+j} \mu_\rho^\epsilon = \frac{1}{(t-1)q} \sum_{i=0}^{(t-1)q-1} \zeta_\rho^{\epsilon} \mu_\rho^\epsilon
\]
and obtain, using Lemma 4.3 and (4.26)

\[ h_{\text{top}} - 4\delta \leq \frac{(t - 1)q}{tq + r} \int_{\mathcal{F}} \frac{K(f, \tau, [0, q])}{q} \, dp + \frac{t \log q + o(p)}{tq + r} \]

We now apply Lemma 4.2 to the sequence of measures \( \{ \rho_p \} \), and we obtain a probability measure \( \rho' \), invariant with respect to the \( (\zeta_1, \varphi_1) \) action, which satisfies

\[ \int_{\mathcal{F}} \frac{K(f, \tau, [0, q])}{q} \, dp + \frac{\log q}{q} \geq h_{\text{top}} - 4\delta \]

for all \( q \in \mathbb{N} \). Letting \( q \to \infty \) we have

\[ K_{\rho'}(\epsilon/2) \geq h_{\text{top}} - 4\delta \]

by using the definition of \( K_{\rho'}(\epsilon/2) \) as given in (3.13) and (3.14).

\[ \Box \]

In Theorem 2.2 we have proved that \( K_\mu(\epsilon) \) is non-decreasing in \( \epsilon \) for any probability invariant measure \( \mu \). Hence, from Lemma 4.4, we obtain that for all \( \epsilon < \epsilon_0(\delta) \) it holds

\[ K_{\rho'}(\epsilon) \geq h_{\text{top}} - 4\delta \]

where we recall that \( \rho' \) are probability measure invariant with respect to the \( (\zeta_1, \varphi_1) \) action. To finish the proof of the theorem, we only need to construct a probability measure satisfying (4.32), which is invariant with respect to space translation and time evolution for all \( (x, t) \in \mathbb{R} \times \mathbb{R} \).

Let us choose a fixed \( \epsilon < \epsilon_0 \) and denote \( \rho := \rho' \). The probability measure

\[ \nu := \int_0^1 \int_0^1 \zeta_x(\varphi_{x+t}^t, \rho) \, dt \, dx \]

is invariant with respect to space translation and time evolution for all \( (x, t) \in \mathbb{R} \times \mathbb{R} \) by definition. We now prove

**Lemma 4.5.** The probability measure \( \nu \) defined in (4.33) satisfies

\[ K_\nu \geq h_{\text{top}} - 4\delta \]

**Proof.** It is enough to prove that for \( \epsilon' \) small enough it holds

\[ K_{\nu}(\epsilon') \geq h_{\text{top}} - 4\delta \]

For all \( p \in \mathbb{N} \) let us write

\[ \int_{\mathcal{F}} \frac{K(f, \epsilon', [0, p])}{p} \, d\nu = \int_0^1 \int_0^1 \left( \int_{\mathcal{F}} \frac{K(f, \epsilon', [0, p])}{p} \, d(\zeta_x(\varphi_{x+t}^t, \rho)) \right) \, dt \, dx \]

and for the moment consider \( (x, t) \) fixed. The first step is to write

\[ \int_{\mathcal{F}} K(f, \epsilon', [0, p]) \, d(\zeta_x(\varphi_{x+t}^t, \rho)) = \int_{\mathcal{F}} K(\zeta_x(f), \epsilon', [0, p]) \, d(\varphi_{x+t}^t, \rho) = \]

\[ \int_{\mathcal{F}} K(f, \epsilon', [x, p + x]) \, d(\varphi_{x+t}^t, \rho) \]
By using the sub-additivity property proved in Lemma 3.5, we write
\[ \int_{\mathcal{F}} K(f, \varepsilon', [0, p + 1]) \, d(\varphi^*, \rho) \leq \int_{\mathcal{F}} K(f, \varepsilon', [x, p + x]) \, d(\varphi^*, \rho) + o(p) \]
where the term \( o(p) \) contains the constant \( 2 \log_2 q \), and the integral of the terms \( K(f, \varepsilon', [0, x]) \) and \( K(f, \varepsilon', [p+x, p+1]) \) which are bounded as in (3.12). Hence
\[ \lim_{p \to \infty} \frac{\int_{\mathcal{F}} K(f, \varepsilon', [0, p + 1]) \, d(\varphi^*, \rho)}{p} \leq \lim_{p \to \infty} \frac{\int_{\mathcal{F}} K(f, \varepsilon', [x, p + x]) \, d(\varphi^*, \rho)}{p} \]
where on the left hand side we know that the limit exists because \( \varphi^*, \rho \) is \((\zeta_1, \varphi_1)\)-invariant. We now want to estimate the left hand side. Let us start by writing
\[ \int_{\mathcal{F}} K(f, \varepsilon', [0, p]) \, d(\varphi^*, \rho) = \int_{\mathcal{F}} K(\varphi_-(f), \varepsilon', [0, p]) \, d \rho = \]
\[ = \lim_{s \to \infty} \lim_{n \to \infty} \int_{\mathcal{F}} \frac{K(\varphi_-(f), \tilde{V}_s, n)}{n} \, d \rho \]
where we used the sequence of open coverings \{\tilde{V}_s\} in \( \mathcal{C}'_{[0,p]} \) defined in Lemma 3.5. By definition of \( K(\varphi_-(f), \tilde{V}_s, n) \) we look at the complexity of the symbolic words in \( \psi(\varphi_-(f), n, \tilde{V}_s) \) (see (2.5)). Hence we have
\[ K(\varphi_-(f), \tilde{V}_s, n) = K(f, \varphi(f), \tilde{V}_s, n) \]
where \( \varphi(f) \) is a covering of \( \mathcal{F}_{[0, p]} \). Indeed, for \( g \in \mathcal{F} \) there exists \( V_j \in \tilde{V}_s \) such that \( \varphi_-(g) \subseteq V_j \), because \( \varphi \) is invertible. Hence \( g \in \varphi(f)(V_j) \subseteq \varphi_1(\tilde{V}_s) \).
Moreover, by assumption (A3), we have that there are constants \( \gamma > 0 \), \( \Gamma > 1 \) and \( C > 0 \) such that for \( p > 2C(\varepsilon')^{-1} \) if \( d_{[0, p]}(f_1, f_2) < \varepsilon' \) then
\[ d_{[2C(\varepsilon')^{-1}, p-2C(\varepsilon')^{-1}]}(f_1, f_2) < \Gamma \varepsilon \varepsilon' \]
This implies that for any \( t \in [0, 1] \), the covering \( \varphi(f) \) is a covering of \( \mathcal{F}_{[2C(\varepsilon')^{-1}, p-2C(\varepsilon')^{-1}]} \) and each of its set is contained in a ball of radius \( \eta = \Gamma \varepsilon \varepsilon' \). Hence there exists a covering \( \tilde{U}_s \in \mathcal{C}'_{[2C(\varepsilon')^{-1}, p-2C(\varepsilon')^{-1}]} \) such that
\[ K(f, \varphi_1(\tilde{V}_s), n) \geq K(f, \tilde{U}_s, n) \]
for all \( f \in \mathcal{F} \) and all \( n \in \mathbb{N} \). By using (1.39) and (1.40) we get for all \( s \in \mathbb{N} \)
\[ \lim_{n \to \infty} \int_{\mathcal{F}} \frac{K(\varphi(f), \tilde{V}_s, n)}{n} \, d \rho \geq \lim_{n \to \infty} \int_{\mathcal{F}} \frac{K(f, \tilde{U}_s, n)}{n} \, d \rho \geq \]
\[ \int_{\mathcal{F}} K(f, \eta, [2C(\varepsilon')^{-1}, p-2C(\varepsilon')^{-1}]) \, d \rho \]
hence
\[
\lim_{p \to \infty} \int_{\mathcal{F}} \frac{K(f, \varepsilon', [0, p+1])}{p} \, d\nu = K'_{\rho}(\eta)
\]

(4.41)

\[
\geq \lim_{p \to \infty} \int_{\mathcal{F}} \frac{p+1-4C'(\varepsilon')^{-1}}{p} \frac{K(f, \eta, [0, p+1-2C'(\varepsilon')^{-1}])}{p+1-4C'(\varepsilon')^{-1}} \, d\rho = K_{\rho}(\eta)
\]

At this point, we would put together (4.37), (4.38) and (4.41), and use Lemma 3.6 to get
\[
K_{\rho}(\varepsilon') \geq K_{\rho}(\eta)
\]

The assertion would then follow by choosing $\varepsilon'$ small enough to have $\eta < \frac{\varepsilon}{2}$ and use (4.32). The only problem to this argument is that we have tacitly assumed that it is possible to exchange the order of limit in $p$ and integrations in $(x, t)$ in (4.36). However, since by Lemma 3.5 we have
\[
\frac{K(f, \varepsilon', [0, p])}{p} \leq K(f, \varepsilon', [0, 1]) + \log_2 q \leq \text{const}
\]
for all $f \in \mathcal{F}$, we can integrate with respect to $\zeta^*(\varphi^\tau, \rho)$ and apply Lebesgue dominated convergence theorem. 

Hence Theorem 2.3 is proved.

**APPENDIX A. PROOF OF LEMMA 3.4**

Let us denote as usual
\[
(S_n \vartheta)(x) := \sum_{j=0}^{n-1} \vartheta(T^j(x))
\]

(A.1)

From Birkhoff ergodic theorem we have that there exists a set $Y_1 \subset X$ with $\nu(Y_1^c) = 0$ such that for any diverging sequence $(n_k)_k \subset \mathbb{Z}$ it holds
\[
\bar{\vartheta}(x) := \lim_{k \to \infty} \frac{(S_{n_k} \vartheta)(x)}{|n_k|} = \lim_{k \to \infty} \frac{1}{|n_k|} \sum_{j=0}^{n_k-1} \vartheta(T^j(x))
\]

(A.2) exists, is finite for all $x \in Y_1$ and it is in $L^1(X, \nu)$. Moreover $\bar{\vartheta}$ satisfies (3.10). Let $Y_1$ be such that the same holds for $|\vartheta|$.

To prove (3.10), given any sequence of integers $(a_k)_k$ and $(b_k)_k$ as in the hypothesis, we write for all $x \in Y_1$
\[
\frac{1}{b_k-a_k} \sum_{j=a_k}^{b_k-1} \vartheta(T^j(x)) = \frac{b_k}{b_k-a_k} \frac{(S_{b_k} \vartheta)(x)}{b_k} - \frac{a_k}{b_k-a_k} \frac{(S_{a_k} \vartheta)(x)}{a_k}
\]

(A.3) and to study the convergence of (A.3) we divide the indices $k \in \mathbb{N}$ into four sets:
\[
I_1 := \left\{ k \in \mathbb{N} : |a_k| \geq \sqrt{b_k-a_k} \quad |b_k| \geq \sqrt{b_k-a_k} \right\}
\]
\[
I_2 := \left\{ k \in \mathbb{N} : |a_k| \geq \sqrt{b_k-a_k} \quad |b_k| < \sqrt{b_k-a_k} \right\}
\]
We remark that \( I \) and analogously for the other two possible combinations, \((-+)\) and \((-(-))\).

First of all we can neglect \( I_4 \) since it contains only a finite number of indices by (2.8). Moreover we introduce for \( i = 1, 2, 3 \) the notation
\[
I_i^{(+)} := \{ k \in \mathbb{N} : a_k \geq 0 \ ; \ b_k \geq 0 \}
\]
and analogously for the other two possible combinations, \((-+)\) and \((-(-))\).

We remark that \( I_2 = I_2^{(-)} \cup I_2^{(-(-))} \) and \( I_3 = I_3^{(+)} \cup I_3^{(-(-))} \).

Let us consider (A.3) for the indices \( k \in I_1^{(+)} \). By (2.8) and (A.2), for any given \( x \in Y_1 \) and any fixed \( \eta > 0 \) there exists \( k_0 \in \mathbb{N} \) such that for all \( k \geq k_0 \) we have
\[
\left| \frac{S_{a_k} \vartheta(x)}{b_k} - \bar{\vartheta}(x) \right| < \eta
\]
\[
\left| \frac{S_{a_2} \vartheta(x)}{a_k} - \bar{\vartheta}(x) \right| < \eta
\]
Also by (2.9) we can assume that for \( k \geq k_0 \) we have
\[
a_k \leq \lim_{k \to \infty} \sup_{k \geq k_0} \frac{a_k}{b_k - a_k} + \eta = \frac{1}{l_a} + \eta \quad (A.4)
\]
\[
b_k \leq 1 + \lim_{k \to \infty} \sup_{k \geq k_0} \frac{a_k}{b_k - a_k} + \eta = 1 + \frac{1}{l_a} + \eta \quad (A.5)
\]
Applying these inequalities to (A.3) we have that for all \( k \geq k_0 \)
\[
\left| \frac{1}{b_k - a_k} \sum_{j=k}^{b_k - 1} \tilde{\vartheta}(T^j(x)) - \tilde{\vartheta}(x) \right| < \eta \left( 1 + \frac{1}{l_a} + \eta \right) \left( \frac{1}{l_a} + \eta \right)
\]
This proves (3.9) for all sequences \( a_k \) and \( b_k \) with \( k \in I_1^{(+)} \). The same argument applies to \( k \) in \( I_2^{(-(-))} \) and \( I_1^{(-(-))} \) by writing the right hand side of (A.3) respectively as
\[
\frac{b_k}{a_k^2 + |a_k|^2} \frac{(S_{a_k} \vartheta(x))}{b_k} + \frac{|a_k|}{|a_k| + |a_k|^2} \frac{(S_{a_k} \vartheta(x))}{|a_k|} \quad (A.6)
\]
\[
\frac{|a_k|}{|a_k| - |b_k|} \frac{(S_{a_k} \vartheta(x))}{|a_k|} - \frac{|b_k|}{|a_k| - |b_k|} \frac{(S_{b_k} \vartheta(x))}{|b_k|} \quad (A.7)
\]
and using conditions (2.8)–(2.10).

Let us consider now \( k \in I_2^{(-(+))} \). First of all it holds
\[
0 \leq \lim_{k \to \infty} \frac{b_k}{b_k + |a_k|} < \lim_{k \to \infty} \frac{1}{\sqrt{b_k + |a_k|}} = 0 \quad (A.8)
\]
hence
\[
\lim_{k \to \infty} \frac{|a_k|}{b_k + |a_k|} = 1 \quad (A.9)
\]
Moreover we can apply (A.2) to \( \frac{(S_{a_k} \vartheta)(x)}{|a_k|} \) and, since \( b_k < \sqrt{b_k + |a_k|} \), it holds

\[
\limsup_{k \to \infty} \frac{|(S_{b_k} \vartheta)(x)|}{b_k + |a_k|} \leq \limsup_{k \to \infty} \frac{1}{\sqrt{b_k + |a_k|}} \frac{(S_{\sqrt{b_k + |a_k|}} \vartheta)(x)}{\sqrt{b_k + |a_k|}} = 0
\]

Hence, using (A.6) and applying (A.10) to the first term and (A.9) and (A.2) to the second term, we prove (3.9) for \( k \in I_2^{(+)2} \).

The same arguments apply also to \( I_2^{(-)2} \), and to \( I_3 \) by interchanging the role of \( a_k \) and \( b_k \).

The proof of (3.11) follows by a similar argument. First we show that there exists a set \( Y_2 \subset X \) with \( \nu(Y_2^c) = 0 \) such that for any diverging sequence of integers \( \{n_k\} \) it holds

\[
\lim_{k \to \infty} \frac{\xi(T^{n_k}(x))}{|n_k|} = 0
\]

for all \( x \in Y_2 \). Since \( \xi \in L^1 \) and \( \xi \geq 0 \), and since the transformation \( T \) is measure preserving, for all \( \eta > 0 \) it holds

\[
\sum_{k=1}^{\infty} \nu \{ \xi(T^k(x)) > k\eta \} = \sum_{k=1}^{\infty} \nu \{ \xi(x) > k\eta \} = \\
= \frac{1}{\eta} \sum_{k=1}^{\infty} k\eta \nu \{(k+1)\eta \geq \xi(x) > k\eta \} = \frac{1}{\eta} \sum_{k=1}^{\infty} \int_{\{(k+1)\eta \geq \xi(x) > k\eta \}} \xi(x) \, d\nu < \\
< \frac{1}{\eta} \int_X \xi(x) \, d\nu < \infty
\]

hence from the Borel-Cantelli lemma it follows that the measure of the set on which \( \xi(T^k(x)) > \eta \) infinitely often is zero. Let us moreover assume that the function \( \xi(x) \) satisfies Birkhoff theorem (condition (A.2)) for all \( x \in Y_2 \).

We now use (A.11) as we used (A.2) before. If \( |b_k| \geq \sqrt{b_k - a_k} \) then by (2.8) \( b_k \to \infty \), hence we can apply (A.11) to \( b_k \). Hence by using (A.5), we have that for all \( x \in Y_2 \) and for any given \( \eta > 0 \) there exists \( k_0(x) \) such that for all \( k \geq k_0(x) \) it holds

\[
\frac{\xi(T^{b_k}(x))}{b_k - a_k} \leq \eta \left( 1 + \frac{1}{l_a} + \eta \right)
\]

hence (3.11) holds in \( Y_2 \) for these indices.

If instead \( |b_k| < \sqrt{b_k - a_k} \), we can apply (A.2) by writing

\[
\limsup_{k \to \infty} \frac{\xi(T^{b_k}(x))}{b_k - a_k} \leq \limsup_{k \to \infty} \frac{1}{\sqrt{b_k - a_k}} \frac{(S_{\sqrt{b_k - a_k}} \xi)(x)}{\sqrt{b_k - a_k}} = 0
\]

using \( \xi \geq 0 \). Hence (3.11) holds in \( Y_2 \) also in this case. This finishes the proof of the lemma by choosing \( Y := Y_1 \cap Y_2 \).
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