Far from equilibrium energy flow in quantum critical systems

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We investigate far from equilibrium energy transport in strongly coupled quantum critical systems. Combining results from gauge-gravity duality, relativistic hydrodynamics, and quantum field theory, we argue that long-time energy transport occurs via a universal steady-state for any spatial dimensionality. This is described by a boosted thermal state. We determine the transport properties of this emergent steady state, including the average energy flow and its long-time fluctuations.

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Introduction.— In recent years there has been significant interest in the behavior of strongly correlated systems out of equilibrium. Experiments on cold atomic gases have raised questions ranging from the nature of thermalization in one spatial dimension (d = 1) [1–5] to the dynamics of spontaneous symmetry breaking [4–9]. Theoretical attention has focused on the behavior of many body systems following time-dependent protocols such as rapid quenches [2]. A strong motivation is the possibility of establishing universal results for the far from equilibrium response; for a review see Ref. [8].

Considerable insight into the dynamics of quantum systems has been obtained using integrability and field theory techniques. Recent work in d = 1 has established a universal regime of thermal transport when two isolated critical systems are brought into instantaneous thermal contact [9,11]. The predicted steady state energy flow across the interface has been recently observed [12] using time-dependent Density Matrix Renormalization Group (DMRG) at finite temperature [13-15]. Extending such results to higher dimensions is a major challenge and is the motivation for this manuscript. For recent experiments on thermal transport and the thermal expansion of cold atomic gases see Refs. [16,17].

We will use a combination of methods, including gauge-gravity duality [18,19], relativistic hydrodynamics and field theory techniques, to establish universal results for non-equilibrium thermal transport in d > 1. We will focus on relativistic conformal field theories (CFTs) describing quantum critical points with a linear dispersion ε = v|k|; we will generically set v = h = k_B = 1. We argue that thermal contact between strongly coupled quantum critical systems gives rise to a universal homogeneous steady state with a non-vanishing energy flow. Moreover, the energy transport in this far from equilibrium steady state is fully described by a Lorentz boosted thermal distribution. This governs not only the average energy current but also its fluctuations.

Setup.— We consider energy transport in a homogeneous quantum critical system as depicted in Fig. 1. The system is subject to an initial temperature distribution with a step profile. Equivalently, we bring into thermal contact the two semi-infinite halves of the system that are independently thermalized at left and right temperatures, T_L and T_R. A key question is whether a non-trivial current carrying steady state emerges near the interface at late times; see Fig. 1. In particular, is there a steady state energy flow with J_E ≡ ⟨T^{t,x}\rangle_s \neq 0? Here T^{\mu\nu} is the energy-momentum tensor and s denotes the steady state. If so, what is the value of this energy current and what are the fluctuations in this steady state?

As we will argue below, a non-trivial current carrying state exists in all dimensions. This is in spite of the fact that there are no external heat baths to drive a current. Rather, the semi-infinite sub-systems themselves play the role of baths. Although these effective baths become asymptotically far apart at long times, the steady state is expected to carry a current; at a quantum critical point, the energy current is a conserved quantity and the transport should have a ballistic component.

One Dimension.— In d = 1, the steady state of a CFT in the above setup was shown to exist and is described in Refs. [9,10]. In order to generalize these results to higher...
dimensions it is instructive to examine these findings using general field theory considerations. First, in \( d = 1 \), one can show that the steady state is not accompanied by an energy density gradient, but in fact the energy density must be homogeneous: by conservation of \( T^{\mu \nu} \) and stationarity, \( \partial_s (T^{xx}) = -\partial_t (T^{tx}) = 0 \). Tracelessness (scale invariance) yields \( (T^{xx})_s = (T^{tt})_s = 0 \) and so \( (T^{tt})_s \) is homogeneous. Second, conservation of \( T^{\mu \nu} \) and tracelessness in \( d = 1 \) implies that the dynamics may be factorized into left- and right-moving components. With the initial condition of zero current, this gives \( (T^{tx}(x,t)) = F(x-t) - F(x+t) \) and \( (T^{tt}(x,t)) = F(x-t) + F(x+t) \), corresponding to sharp “shock waves” emanating from the interface at unit speed; see Fig. 1. Using the initial thermal form of the energy density on the left and the right, \( F(x) = (c\pi/12)T_L^2 \Theta(-x) + (c\pi/12)T_R^2 \Theta(x) \), where \( c \) is the central charge \([20,21]\). In the long time limit, the steady-state energy current (for instance at \( x = 0 \)) is given by \( (T^{tx})_s = (c\pi/12)(T_L^2 - T_R^2) \), corresponding to the difference of independently “thermalized” left- and right-moving densities. Equivalently, \( (T^{tx})_s = c\Delta T \) where \( \Delta T \equiv T_L - T_R \) and \( g = (\pi^2/36) T_s \) is the quantum of thermal conductance \([22,24]\) with \( T_s = (T_L + T_R)/2 \).

**Gauge-Gravity Duality.**—What is the nature of the steady state in higher dimensions? To answer this question, we first assume that in any dimension a thermalization quench in a critical system results in a completely homogeneous steady state with an energy flow; we will provide a posteriori evidence for this later in the manuscript. In order to resolve the nature of this steady state we employ gauge-gravity duality or holography \([13,19]\). Gauge-gravity duality offers unique opportunities for advancing our understanding of far from equilibrium dynamics. The emergent behavior is encoded in the real time evolution of black holes residing in AdS (Anti-de Sitter) space-time, in one more spatial dimension; see Fig. 2. This has been used to explore thermalization in strongly coupled gauge theories and the dynamics following quenches \([25,33]\). Although technically the simplest form of the correspondence only holds for strongly interacting theories with “large-N matrix” degrees of freedom, holographic results can be viewed as a generalization of Landau-Ginzburg theory that includes Wilsonian scaling. In particular, it allows access to the behavior of conformally invariant theories in \( d > 1 \), where there are very few tractable microscopic theories.

As we are concerned with the transport of energy, we study the simplest holographic theory, which only contains Einstein–Hilbert gravity:

\[
S = \frac{1}{16\pi G_N} \int d^{d+2}x \sqrt{-g} (R - 2\Lambda),
\]

where \( \Lambda = -d(d+1)/2L^2 \) is a negative cosmological constant and \( L \) is the radius of AdS. The model \([1]\) is dual to a strongly coupled CFT in \( d \) spatial dimensions; see Fig. 2. In particular, the metric \( g^{\mu \nu} \) is dual to the energy-momentum tensor \( T^{\mu \nu} \) of the CFT. On the gravitational side, \( L \) should be large in units of Newton’s constant \( L^d/G_N \gg 1 \), in order to use classical gravity. On the dual gauge theory side, \( L^d/G_N \gg 1 \) encodes the large number of degrees of freedom of the CFT.

The homogeneous, stationary nature of the steady state should be reflected in its gravitational dual. In a future publication \([34]\), we will show that the only regular solutions to Einstein’s equations, dual to theories on flat space-time, which encode a homogeneous constant stress tensor are the boosted black branes:

\[
ds^2 = \frac{L^2}{z^2} \left[ \frac{dz^2}{f(z)} - f(z)(dt \cosh \theta - dx \sinh \theta)^2 + (dx \cosh \theta - dt \sinh \theta)^2 + dy^2 \right],
\]

where

\[
f(z) = 1 - \left( \frac{z}{z_0} \right)^{d+1} \quad \text{and} \quad z_0 = d + 1 \frac{4\pi T}{L}.
\]

Here \( \theta \) is the boost parameter corresponding to a boost in the negative \( x \)-direction, \( y_L \) parameterizes the transverse spatial coordinates, \( T \) is the unboosted temperature of the black hole, and \( z_0 \) is the position of the planar horizon. Hence, at least in the large-\( N \) limit, the steady state is described by a Lorentz boosted equilibrium state, after suitable identifications of \( T \) and \( \theta \) in terms of \( T_{L,R} \). We will argue that this also holds without large-\( N \).

Following the rules of the AdS/CFT correspondence, \( \langle T^{\mu \nu} \rangle_s \) is obtained from the metric of the dual gravitational problem. After transformation to Fefferman-Graham coordinates \([35]\):

\[
\langle T_{\mu \nu} \rangle_s = \frac{L^d}{16\pi G_N} \lim_{z \to 0} \left( \frac{d}{dz} \right)^{d+1} \left( \frac{Z^2}{L^d} g_{\mu \nu}(z(Z)) \right),
\]
where $z(Z) = Z/R - (Z/R)^{d+2}/[2(d + 1) z_0^{d+1}]$ with $R = (d l_0^0)^{1/(d-1)}$. This yields the Lorentz boosted stress tensor of a finite temperature CFT

$$\langle T^{\mu\nu} \rangle_s = a_d T^{d+1} (\eta^{\mu\nu} + (d + 1) u^\mu u^\nu), \quad (5)$$

where $\eta^{\mu\nu} = \text{diag}(-1, 1, \cdots, 1)$ is the CFT metric and $u^\mu = (\cosh \theta, \sinh \theta, 0, \ldots, 0)$ is the resulting velocity. The coefficient $a_d \sim L^d/G_N$ characterizes the rest frame energy density of the CFT. \langle T^{\mu\nu} \rangle = da_d T^{d+1}$ which is analogous to the Stefan–Boltzmann law \cite{36}. In general $a_d$ is a measure of the number of degrees of freedom of the CFT, which depends on the details of the theory, including the strength of the coupling, see e.g. Ref. \cite{37}. The result \cite{5} may also be obtained by direct Lorentz transformation and the steady energy current is given by

$$\langle T^{tx} \rangle_s = \frac{1}{2} a_d T^{d+1} (d + 1) \sinh 2 \theta. \quad (6)$$

Here we recall that $\theta$ and $T$ are to be determined in terms of $T_{L,R}$. In $d = 1$, the above picture is readily interpreted. The boosted black hole coordinate corresponds to a state with its left- and right-movers thermally populated at temperatures $T_L = T e^\theta$ and $T_R = T e^{-\theta}$ \cite{38}. Equivalently, $T_L$ and $T_R$ may be regarded as the apparent temperatures arising from the Doppler shift of the Stefan–Boltzmann radiation \cite{39}. Combining $a_1 = L\pi/4G_N$ with the equation $c = 3L/2G_N$, where $c$ is the central charge, one finds $a_1 = c\pi/6$. One thus obtains the established non-equilibrium result $\langle T^{tx} \rangle_s = (c\pi/12)(T_L^2 - T_R^2) \cite{9}$. For recent work examining energy flows dual to boosted black holes in different setups see Refs. \cite{40,41}.

**Fluctuations.**—A key observation is that the boosted state \cite{10} encodes not only the average energy current, but also its fluctuations. This may be illustrated in $d = 1$, without recourse to gauge-gravity duality and the large-$N$ limit. In $d = 1$ the left- and right-movers are independently thermalized, the exact steady state density matrix is given by $\rho_0 = e^{-\beta_L H_L - \beta_R H_R}$, where $H_{\pm} = \sum_k (|k| \pm k)/2$ are the total energies of the right- and left-moving excitations \cite{9} and $\beta_{L,R} = 1/T_{L,R}$. Equivalently, $H_{\pm} = (H \pm P_x)/2$, where $H$ is the Hamiltonian and $P_x$ is the total momentum. Therefore, $\rho_0 = e^{-\beta_+ H - \beta_- P_x}$ where $\beta_{\pm} = (\beta_L \pm \beta_R)/2$. This is equivalent to a boosted thermal state with $\rho = e^{-\beta \cosh \theta H - \beta \sinh \theta P_x}$, where $\beta = \sqrt{\beta_{L,R}}$ is the inverse temperature in the rest-frame and $\theta$ is the boost parameter, given by $e^{2\theta} = \beta_R / \beta_L$. The non-equilibrium steady state in $d = 1$ \cite{9,12} is therefore also obtained by “running past” a thermal state with velocity $(\beta_R - \beta_L)/(\beta_R + \beta_L)$. Crucially, the exact steady state density matrix $\rho_0$ allows one to compute not only the average energy flow, but also the exact generating function of the energy current fluctuations \cite{9,11}.

In general, one is interested in the full probability distribution of the integrated current density on the interface (that is, the total transfer of energy), $J = \int dt dy T^{tx}(x = 0, y_L, t)$, in the steady state. Scaling out the large transverse area and the long time, the cumulants can be expressed in terms of connected correlation functions, $c_n \equiv \langle J^{n-1} T^{tx}(0) \rangle_s$. The generating function $F(z) \equiv \sum_{n=1}^\infty z^n c_n$ is an important quantity, as non-equilibrium steady states are expected to give rise to nontrivial relations amongst cumulants encoded into the “non-equilibrium fluctuation relations” $F(\beta_L - \beta_R - z) = F(z) \cite{33,45}$. Using AdS/CFT, cumulants are in principle computable from Witten diagrams holographically (see \cite{46} for certain quadratic fluctuations). In \cite{34}, we will argue that, the steady state being a boosted thermal state, we can obtain the exact $F(z)$ in any dimension, whenever there is PT symmetry. Building on \cite{11}, we will argue that in this case the extended fluctuation relations (EFR) hold:

$$\frac{dF(z)}{dz} = J_E(\beta_L - z, \beta_R + z). \quad (7)$$

Here, the knowledge of the current as a function of the temperatures fixes $F(z)$. In particular, with parity symmetry $J_E(\beta_L, \beta_R) = -J_E(\beta_R, \beta_L)$, the EFR implies the non-equilibrium fluctuation relations. In $d = 1$ CFTs the EFR was shown to hold in \cite{11}. In this case it is a direct consequence of left- and right-moving factorization. This also allows us to derive this relation directly from thermal partition functions, equivalently from partition functions of the boosted black brane, $\text{Tr} e^{-\beta \cosh \theta H + \beta \sinh \theta P}$. **Higher Dimensions.**—For $d > 1$ there is no holomorphic factorization and thermalization is expected to modify the dynamics compared to $d = 1$. We will show that it still develops a steady state. At long times, this should be described by relativistic hydrodynamics. Indeed, Eq. \cite{7} corresponds to the energy-momentum tensor of a perfect conformal fluid, where $u^\mu$ is the local fluid velocity. The hydrodynamic equations simply express the conservation of energy and momentum, $\partial_\mu (T^{\mu\nu}) = 0$. In a CFT we also have $\langle T^{\mu\nu} \rangle = 0$. Within this framework one may consider the effects of a range of initial conditions that interpolate between asymptotic heat baths at temperatures $T_L$ and $T_R$. At sufficiently large scales all of these will look like the initial conditions of the Riemann problem, $T_L$ for $x < 0$ and $T_R$ for $x > 0$.

A solution consistent with these initial conditions consists of two planar shock waves emanating from the contact region \cite{37}. We will now elucidate the properties of the steady state in the intermediate region. We consider left- and right-moving shocks that are homogeneous in the transverse spatial directions and move at constant speeds $u_L$ and $u_R$ respectively; given the Riemann conditions the resulting solution is unique. Enforcing energy and momentum conservation across the shocks constrains the form of the steady state energy-momentum tensor:

$$\langle T^{tx} \rangle_s = a_d \frac{T^{d+1}_L - T^{d+1}_R}{u_L + u_R}, \quad (8)$$
Invoking the boosted steady state \[5\] gives explicit expressions for \( u_{L,R} \) in terms of \( T_{L,R} \): \[9\]

\[
\begin{align*}
  u_L &= \frac{1}{d} \sqrt{\frac{x + d}{x + d - 1}}, \\
  u_R &= \sqrt{\frac{x + d - 1}{x + d}},
\end{align*}
\]

where \( \chi \equiv (T_L/T_R)^{(d+1)/2} \). The result in the steady state region is a boosted thermal state, with temperature \( T = \sqrt{T_LT_R} \) and boost velocity given by \((\chi - 1)/\sqrt{(x + d)(x + d^{-1})}; \) in \( d = 1 \) this reduces to our previous result. The validity of these findings in \( d > 1 \) is confirmed numerically in Fig. 3.

The shock waves emanating from the contact region are non-linear generalizations of sound waves. In particular, it follows from energy and momentum conservation that the shock speeds satisfy the constraint \( u_L u_R = c_s^2 \), where \( c_s = 1/\sqrt{d} \) is the speed of sound. As a function of \( T_L/T_R \), greater than unity, \( u_R \) interpolates between \( c_s \) and \( v \), whilst \( u_L \) interpolates between \( c_s \) and \( c_s^2/v \), where we have reinstated the microscopic velocity \( v \).

A notable difference in \( d > 1 \) compared to \( d = 1 \) is that the system now diffuses. By scaling, viscous corrections will not change the late time results. This is readily seen in the linear response regime, \( |T_L - T_R| \ll T_L + T_R \), where we can solve the hydrodynamic equations explicitly. One can show that the two “shocks” propagate at the speed of sound, and have a width growing diffusively as \( \sqrt{T} \).

On long length scales this reduces to the sharp shock dynamics discussed above. In linear response the solution is stable, but large shear perturbations may set off a turbulent instability. It would be interesting to investigate this in future work.

Conclusions.— We have established results for the far from equilibrium energy flow in strongly coupled critical systems in \( d > 1 \). We predict the existence of steady state solutions with a universal description for energy transport in terms of a boosted thermal distribution. Although we have focused on CFTs, we expect that non-trivial steady states may emerge under a broader range of conditions, provided energy and momentum are conserved. It would be interesting to verify these results in experiments using cold atomic gases, or in numerical simulations based on matrix product states.

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