QUASI PROJECTIVE DIMENSION FOR COMPLEXES

TIRDAD SHARIF

Abstract. In this note, we extend the quasi-projective dimension of finite (that is, finitely generated) modules to homologically finite complexes, and we investigate some of homological properties of this dimension.

1. Introduction

Throughout, all rings are commutative and Noetherian. In [1] Avramov, and in [5] Avramov, Gasharov and Peeva defined and studied the complexity, the complete intersection dimension, and the quasi projective dimension of finite modules.

Let \( D_f^f(R) \) be the category of homologically finite \( R \)-complexes, and \( X \in D_f^f(R) \).

The complexity and the complete intersection dimension of \( X \), denoted by \( cX_R \) and \( CI-dim_R X \), respectively, were defined and studied by Sather-Wagstaff [14]. In this work, we introduce the quasi-projective dimension, as a refinement of the projective dimension, for homologically finite complexes and verify some of it’s homological properties analogous to those holding for modules.

Let \( Y, Z \in D_f^f(R) \). In our main result, Theorem 3.5, as an application of the Intersection Theorem for the quasi-projective dimension, Proposition 3.2(b), and the depth formula for the complete intersection dimension, Proposition 3.4, a lower bound and an upper bound for \(-\sup R\hom_R(Z, (X \otimes_R Y))\) with respect to \( CI-dim_R X \), \( cx_R(X) \) and \(-\sup R\hom_R(Z, Y)\) is determined, when \( \sup (X \otimes_R Y) < \infty \) and \( CI-dim_R X < \infty \).

This result is as an extension of a grade inequality in [16, (3.3)] to complexes; see Example 3.8 about this inequality.

2. Homology theory of complexes

In this paper, definitions and results are formulated within the framework of the derived category of complexes. The reader is referred to [3, 6, 9, 10] for details of the following brief summary. Let \( X \) be a complex of \( R \)-modules and \( R \)-homomorphisms.

For an integer \( n \), the \( n \)-th shift or suspension of \( X \) is the complex \( \Sigma^n X \) with \((\Sigma^n X)_\ell = X_{\ell - n}\) and \( \partial^{\Sigma^n X}_\ell = (-1)^n \partial^X_\ell \) for each \( \ell \). The \( n \)-th cokernel of \( X \) is \( C_n^X = \text{cokernel} \partial^X_{n+1} \). The supremum and the infimum of a complex \( X \), denoted by \( sup(X) \) and \( inf(X) \), respectively, are defined by the supremum and the infimum of \( \{ i | H_i(X) \neq 0 \} \) and let \( amp(X) = sup(X) - inf(X) \). The symbol \( D(R) \) denotes the derived category of \( R \)-complexes. The full subcategories \( D_n(R), D_{<}(R), D_{>}(R) \) and \( D_0(R) \) of \( D(R) \) consist of \( R \)-complexes \( X \) while \( H_\ell(X) = 0 \), for respectively \( \ell \gg 0 \), \( \ell \ll 0 \), \( |\ell| \gg 0 \) and \( \ell \neq 0 \). By \( D^f \) we denote the full subcategory of complexes with all homology modules are finite, called homologically degreewise finite complexes.
A complex $X$ is called homologically finite, if it is homologically both bounded and degreewise finite. The right derived functor of the homomorphism functor of $R$-complexes and the left derived functor of the tensor product of $R$-complexes are denoted by $\mathbf{R}\text{Hom}_R(-, -)$ and $- \otimes^L_R -$, respectively. A homology isomorphism is a morphism $\alpha : X \to Y$ such that $H(\alpha)$ is an isomorphism; homology isomorphisms are marked by the sign $\simeq$, while $\cong$ is used for isomorphisms. The equivalence relation generated by the homology isomorphisms is also denoted by $\simeq$.

2.1. Let $X$ and $Y$ be in $D_+(R)$, then there is an inequality

$$\inf(X \otimes_R^L Y) \geq \inf X + \inf Y.$$  

Equality holds if $i = \inf X$ and $j = \inf Y$ are finite and $H_i(X) \otimes_R H_j(Y) \neq 0$.

2.2. The support of a complex $X$, $\text{Supp}(X)$, consists of all $p \in \text{Spec}(R)$ such that the $R_p$-complex $X_p$ is not homologically trivial.

2.3. Let $R$ be a ring. If $(R, m, k)$ is local, depth of a complex $X \in D_-(R)$ is defined as the following

$$\text{depth}_R X = - \sup \mathbf{R}\text{Hom}_R(k, X).$$

(a) Let $X \in D^f_0(R)$, then the following inequality holds

$$\text{depth}_R X \leq \text{depth}_{R_p} X_p + \dim R/p.$$  

The dimension of a complex $X \in D_+(R)$ is defined by the following formula

$$\dim_R X = \sup\{\dim R/p - \inf X_p | p \in \text{Supp} X\}. $$

(b) Let $X \in D^f_+(R)$, then the following inequality holds

$$\dim_{R_p} X_p + \dim R/p \leq \dim_R X.$$

(c) Let $Y \in D^f_+(R)$ and $X \in D_-(R)$, then the next equality holds

$$- \sup \mathbf{R}\text{Hom}_R(Y, X) = \inf \{\text{depth}_{R_p} X_p + \inf Y_p | p \in \text{Supp} X \cap \text{Supp} Y\}.$$  

2.4. For a ring $R$, let $(-)^* = \text{Hom}_R(-, R)$. A finite $R$-module $M$ is totally reflexive over $R$ if $M$ is reflexive and $\text{Ext}_R^i(M, R) = 0 = \text{Ext}_R^i(M^*, R)$ for all $i > 0$. Let $X \in D^f_0(R)$. A G-resolution of $X$ is a complex $G \simeq X$, such that each $G_i$ is totally reflexive over $R$. The Gorenstein dimension of $X$ is

$$\text{G-dim}_R(X) = \inf\{\sup\{i | G_i \neq 0\} | G \text{ is a G-resolution of } X\}.$$  

3. Quasi-projective dimension for complexes.

In this section, all rings are local. We introduce the quasi-projective dimension, as a refinement of the projective dimension for homologically finite complexes. At first, we bring some notions and definitions.

3.1. Let $R$ be a ring, a codimension $c$, quasi-deformation of $R$ is a diagram of local homomorphisms $R \to R' \leftarrow Q$ such that the first map is flat and the second map is surjective with kernel generated by a $Q$-sequence of length $c$. The complete intersection dimension and the complexity of a complex $X \in D^f_0(R)$ are defined analogous to those of modules as the following

$$\text{CI-dim}_R X = \inf\{pd_Q X' - pd_Q R'| R \to R' \leftarrow Q \text{ is a quasi-deformation}\},$$
where $X' = X \otimes_R R'$, and
\[
\text{cx}_R(X) = \inf\{d \in \mathbb{N}_0 | \beta_n(X) \leq \gamma n^{d-1} \text{ for some } \gamma \in \mathbb{R}\}
\]
in which, $\beta_n(X) = \dim_k H_n(X \otimes_R^L k)$ is the $n$-th Betti number of $X$.

Now we define the quasi-projective dimension of $X \in \mathcal{D}_0^f(R)$ similar to that of modules as the following
\[
\text{qpd}_R X = \inf\{\text{pd}_Q X'| R \to R' \leftarrow Q \text{ is a quasi-deformation}\}.
\]

In the first part of the following proposition, we extend \[5\] (5.11) to complexes. In the second part, we show that in the Intersection Theorem, \[9\] (18.5), we can replace the projective dimension with that of the quasi-projective dimension. Note that by \[9\] (16.22) it is an extension of \[16\] (3.1) to complexes.

**Proposition 3.2.** Let $X \in \mathcal{D}_0^f(R)$ with CI-$\dim_R X < \infty$ and $Y \in \mathcal{D}_0^f(R)$. Then
\begin{enumerate}[(a)]
  \item $\text{qpd}_R X = \text{CI-$\dim_R X + \text{cx}_R(X)$}.$
  \item $\dim_R Y \leq \text{dim}_R(X \otimes_R^L Y) + \text{qpd}_R X$.
\end{enumerate}

**Proof.** (a) From \[14\] (3.10) it follows that $\text{cx}_R(X) < \infty$. Thus $\text{CI-$\dim_R X + \text{cx}_R(X)$}$ and $\text{qpd}_R X$ are finite. Let $P$ be a projective resolution of $X$. We only need to show that $\text{qpd}_R P = \text{CI-$\dim_R P + \text{cx}_R(P)$}$. Take $n \geq \sup X$, and let $C_n^P$ be the $n$-th cokernel of $P$. By \[14\] (2.12) and \[14\] (3.7) we get $\text{cx}_R(X) = \text{cx}_R(C_n^P)$ and $\text{CI-$\dim_R C_n^P < \infty$}$, respectively. Hence by \[5\] (5.10) we can choose a quasi-deformation $R \to R' \leftarrow Q$ of codimension equal to $\text{cx}_R(X)$ such that $\text{pd}_Q C_n^P < \infty$, where $P' = P \otimes_R R'$. Let $P_\leq n$ and $P_\geq n$ be the hard left and the hard right truncations of $P$, respectively. The following sequence of complexes is exact
\[
0 \to P_\leq n \to P \to P_\geq n \to 0. \tag{3.2.1}
\]
Since $R'$ is faithful flat as an $R$-module, the following sequence of $R'$-complexes is also exact
\[
0 \to P'_\leq n \to P' \to P'_\geq n \to 0.
\]
It is clear that $\text{pd}_{R'} P'_\leq n < \infty$. Since $\text{pd}_Q R' < \infty$, thus $\text{pd}_Q P'_\leq n - 1 < \infty$. On the other hand, $P'_\geq n \simeq \Sigma^n C_n^P$ and $\text{pd}_Q C_n^{P''} < \infty$, thus we have $\text{pd}_Q P'_\geq n < \infty$. Now from the above exact sequence we find that $\text{pd}_Q P' < \infty$, and this implies that $\text{qpd}_R P \leq \text{pd}_Q P'$. From \[14\] (3.3) we have the following (in)equalities
\[
\text{qpd}_R P - \text{CI-$\dim_R P$} \leq \text{pd}_Q P' - \text{G-$\dim_R P$} = \text{G-$\dim_R P' + \text{cx}_R(P) - \text{G-$\dim_R P$} = \text{cx}_R(P),}
\]
where the equalities hold by \[6\] (2.3.10), \[6\] (2.3.12), and \[14\] (5.11), respectively. On the other hand, there is a quasi-deformation of codimension $c$, $R \to R' \leftarrow Q'$ such that $\text{qpd}_R P = \text{pd}_Q P''$, where $P'' = P \otimes_R R''$. Therefore from (3.2.1) we get $\text{pd}_Q C_n^{P''} < \infty$, and from \[5\] (5.9) it follows that $\text{cx}_R P = \text{cx}_R(C_n^P) \leq c$. We have the following equalities
\[ \text{qpd}_R P - \text{CI-dim}_R P = \text{pd}_{Q'} P'' - \text{G-dim}_R P \]
\[ = \text{G-dim}_Q P'' - \text{G-dim}_R P \]
\[ = \text{G-dim}_R P'' + c - \text{G-dim}_R P \]
\[ = c \geq cx_R(C^P_n) = cx_R(P) \]

in which, we have used [6] (2.3.10]), [6] (2.3.12]), and [7] (5.11]) again. Now assertion holds.

(b) Let \( \rho : A \to B \) be a surjective homomorphism of rings and let \( X, Z \in \mathcal{D}_b^f(B) \). Then it is easy to see that, \( \dim_A(X \otimes^L_Z Z) = \dim_B(X \otimes^L_B Z) \). Now let \( \text{qpd}_R X = \text{pd}_Q X' \), when \( R \to R' \leftarrow Q \) is a quasi deformation. From [9] (18.5) we get

\[ \dim_Q Z' \leq \dim_Q(X' \otimes^L_Q Z') + \text{pd}_Q X'. \]  \( (3.2.2) \)

We have \( \dim_Q Z' = \dim_R Z' \) and \( \dim_Q(X' \otimes^L_Q Z') = \dim_R(X' \otimes^L_R Z') \), because \( Q \to R' \) is surjective. By the associativity of the derived tensor product we get \( X' \otimes^L_R Z' = (X \otimes^L_R Z)' \). Since \( R' \) is a flat \( R \)-algebra, from [4] (2.1) we have the following equalities

\[ \dim_R Z' = \dim_R Z + \dim_R R'/m R', \]

and

\[ \dim_{R'}(X \otimes^L_R Z)' = \dim_{R'}(X \otimes^L_R Z) + \dim_R R'/m R'. \]

By combining (3.2.2) with the above equalities, we get the following inequality

\[ \dim Z \leq \text{qpd}_R X + \dim_R(X \otimes^L_R Z). \]

Now let \( Y \in \mathcal{D}_b^f(R) \). Since \( H_m(Y) \) is a finite \( R \)-module, from the above inequality it follows that

\[ \dim_R H_m(Y) \leq \text{qpd}_R X + \dim_R(X \otimes^L_R H_m(Y)). \]  \( (3.2.3) \)

Let \( P \) be a projective resolution of \( X \). Thus \( P \simeq X \) and \( P_i = 0 \), for \( i < 0 \). By applying [9] (16.24.b)] we get

\[ \dim_R(P \otimes_R Y) = \sup\{\dim_R(P \otimes_R H_m(Y)) - m|m \in \mathbb{Z}\}. \]

Therefore

\[ \dim_R(X \otimes^L_R Y) = \sup\{\dim_R(X \otimes^L_R H_m(Y)) - m|m \in \mathbb{Z}\}. \]

Now from (3.2.3) assertion holds. \qed

**Remark 3.3.** Let \( \mathcal{F}(R) \) be the category of \( R \)-modules of finite flat dimension. The *large restricted flat dimension* of \( X \in \mathcal{D}_+(R) \), \( \text{Rfd}_R X \), was introduced and studied by Christensen, Foxby and Frankild in [8], and is defined as the following

\[ \text{Rfd}_R X = \sup\{\sup(T \otimes^L_R X)|T \in \mathcal{F}(R)\}. \]

It is shown that \( \text{Rfd} \) is a refinement of the flat dimension, [8] (2.5)]. By [15] (3.6), it is easy to see that \( \text{Rfd}_R X \) is also a refinement of \( \text{CI-dim}_R X \), for \( X \in \mathcal{D}_b^f(R) \).

Using these facts, we can prove the first part of the above proposition, without any using of the Gorenstein dimension. However, by [9] (12,13)], [11] (2.6)], and [14] (3.3)], we can also prove it, without any using of the above dimensions, see the proof of [5] (5.11)].
The next proposition is the depth formula for the complete intersection dimension of complexes, immediately follows from [13 (3.3)] and [14 (3.3)].

**Proposition 3.4.** Let CI-$\dim_R X < \infty$ and $Y \in D^f_\omega(R)$, if $\lambda = \sup(X \otimes^L_R Y) < \infty$, then
\[
\text{depth}_R(X \otimes^L_R Y) = \text{depth}_R Y - \text{CI-$\dim_R X$}.
\]

Now we are in the position of proving our main result.

**Theorem 3.5.** Let $X, Y, Z \in D^f_\omega(R)$, if CI-$\dim_R X < \infty$ and $\sup(X \otimes^L_R Y) < \infty$, then
\[
\begin{align*}
(a) & \quad - \sup R \hom_R(Z, Y) - \text{CI-$\dim_R X$} \leq - \sup R \hom_R(Z, (X \otimes^L_R Y)). \\
(b) & \quad - \sup R \hom_R(Z, (X \otimes^L_R Y)) \leq \text{amp} Y + \text{cx}_R(X) - \sup R \hom_R(Z, Y) - \inf X.
\end{align*}
\]

**Proof.** (a) From 2.3(c) it follows that there is $p \in \text{Supp}_R X \cap \text{Supp}_R Y \cap \text{Supp}_R Z$ such that
\[
- \sup R \hom_R(Z, (X \otimes^L_R Y)) = \text{depth}_{R_p}(X_p \otimes^L_{R_p} Y_p) + \inf Z_p.
\]

Thus we have
\[
- \sup R \hom_R(Z, (X \otimes^L_R Y)) = \text{depth}_{R_p} Y_p - \text{CI-$\dim_{R_p} X_p$} + \inf Z_p \geq \text{depth}_{R_p} Y_p - \text{CI-$\dim_R X$} + \inf Z_p \geq - \sup R \hom_R(Z, Y) - \text{CI-$\dim_R X$}
\]
in which, the equality holds by Proposition 3.4 and the inequalities hold by [14 (3.4)] and 2.3(c), respectively.

(b) By 2.3(c) for some $p \in \text{Supp}_R Z \cap \text{Supp}_R Y$, we get $- \sup R \hom_R(Z, Y) = \text{depth}_{R_p} Y_p + \inf Z_p$. Now let $W = (R/p \otimes^L_{R} X \otimes^L_{R} Y)$. Since $X \otimes^L_{R} Y \in D^f_\omega(R)$, thus we can choose a prime ideal $q$ of $R$ which is minimal in $\text{Supp}_R W$. Therefore $p \subseteq q$ and $q \in \text{Supp}_R X \cap \text{Supp}_R Y$. On the one hand, since $p \in \text{Supp}_R Z$, thus $q \in \text{Supp}_R Z$. Let $V = (R/p \otimes^L_{R} Y)$. Since $V_q \in D^f_\omega(R_q)$ and $X_q \in D^f_\omega(R_q)$, from Proposition 3.2(b) it follows that
\[
\text{dim}_{R_q} V_q \leq \text{qpd}_{R_q} X_q + \text{dim}_{R_q} W_q.
\]

Using 2.1 and the definition of dimension of complexes, it is straightforward to verify that $\text{dim}_{R_q} V_q = - \inf Y_p$ and $\text{dim}_{R_q} W_q = - \inf X_q - \inf Y_q$.

By 2.3(a) and 2.3(b), we have the following inequalities, respectively.
\[
\text{dim}_{R_q} V_q \geq \text{dim}_{R_q} Y_p + \text{dim}_{R_q/pR_q}.
\]

Therefore the following inequality holds
\[
- \inf Y_p + \text{depth}_{R_q} Y_q - \text{depth}_{R_p} Y_p \leq \text{qpd}_{R_q} X_q - \inf X_q - \inf Y_q.
\]

Proposition 3.4 yields an equality
\[
\text{depth}_{R_q} Y_q = \text{depth}_{R_q}(X_q \otimes^L_{R_q} Y_q) + \text{CI-$\dim_{R_q} X_q$}.
\]
Now from Proposition 3.2(a) we get the following inequality
\[ \text{depth}_{R_q}(X_q \otimes_{R_q} Y_q) + \inf X_q + \inf Y_q - \inf Z_q \leq \text{cx}_{R_q}(X_q) + \text{depth}_{R_p} Y_p + \inf Z_p. \]
It is easy to see that \( \text{cx}_{R_q}(X_q) \leq \text{cx}(X) \), and \( \inf Z_q \leq \inf Z_p \). Thus from (2.3(c)) we have
\[ - \sup R \text{Hom}_R(Z, X \otimes_R Y) + \inf X_q + \inf Y_q - \inf Y_p \leq \text{cx}(X) - \sup R \text{Hom}_R(Z, Y). \]
Therefore
\[ - \sup R \text{Hom}_R(Z, X \otimes_R Y) \leq \text{amp} Y + \text{cx}(X) - \sup R \text{Hom}_R(Z, Y) - \inf X. \]
Now assertion holds. \( \square \)

Let \( M \) and \( N \) be \( R \)-modules. Then
\[ \text{grade}_R(M, N) = \inf \{ \ell \mid \text{Ext}^\ell_R(M, N) \neq 0 \}. \]
If \( \text{Ext}^i_R(M, N) = 0 \), for all \( i \), then \( \text{grade}_R(M, N) = \infty \). We say that \( M \) and \( N \) are Tor-independent, if \( \text{Tor}^i_R(M, N) = 0 \), for \( i > 0 \).

The following grade inequality, see [16] (3.3)), is an immediate corollary of the above theorem.

**Corollary 3.6.** Let \( M, N \) and \( L \) be finite \( R \)-modules with CI-dim \( R, N < \infty \) such that \( M \) and \( N \) are Tor-independent \( R \)-modules. Then
\[ \text{grade}(L, M) - \text{CI-dim}_R N \leq \text{grade}(L, M \otimes_R N) \leq \text{grade}(L, M) + \text{cx}_R(N) \]

In Example 3.8, it is shown that \( \text{grade}(L, M \otimes_R N) \) may be arbitrary greater than \( \text{grade}(L, M) - \text{CI-dim}_R N \). This example also shows that in the above inequality, the term \( \text{cx}_R(N) \) is strongly necessary.

**3.7.** Let \( k \) be a field and let \( R_t \) for \( t = 1, 2 \) be \( k \)-algebras, and suppose that \( M_t \) is an \( R_t \)-module. Let \( R = R_1 \otimes_k R_2, \ N = M_1 \otimes_k R_2, \) and \( M = M_2 \otimes_k R_1 \). It is easy to see that \( \text{Tor}^i_R(M, N) = 0 \), for \( i > 0 \), see [12] (4.2)], and \( N \cong M_1 \otimes R_1, R, M \cong M_2 \otimes R_2 R \).

**Example 3.8.** We use the above notations. Take an arbitrary integer \( n \geq 1 \). Let \( Q_1 = k[[X_j, Y_j]], B_1 = (X_j, Y_j), \) for \( 1 \leq j \leq n \) and \( Q_2 = k[[X_j, Y_j]], B_2 = (X_j, Y_j), \) for \( n < j \leq 2n \). It is clear that the local rings \( R_t = Q_t/B_t, \) for \( t = 1, 2 \), are non-regular complete intersections of codimension \( n \). Let \( M_1 = R_t/(Y_i)_{R_1} \) and \( M_2 = R_2/(Y_j)_{R_2}, \) for \( 1 \leq i \leq n \) and \( n < j \leq 2n \), respectively. We have \( \text{depth} R_t = \text{depth}_{R_t} M_t = n, \) for \( t = 1, 2 \), and the Auslander-Buchsbaum formula for the complete intersection dimension yields CI-dim \( R_t, M_t = 0, \) for \( t = 1, 2 \).

Let \( L_1 = R_t/(X_i)_{R_1}, \) for \( 1 \leq i \leq n \). Theorem 5.2(b) and 9 (16.22)] yield \( \text{dim}_{R_t} L_1 \leq \text{cx}_{R_t}(M_t) + \text{dim}_{R_t}(L_t \otimes_{R_t} M_t). \) Since \( L_1 \otimes_{R_t} M_t \cong k \) and \( \text{dim}_{R_t} L_1 = n, \) thus \( \text{cx}_{R_t}(M_t) \geq n, \) and from [2] (8.1.2) we have \( n = \text{cx}_{R_t}(k) \geq \text{cx}_{R_t}(M_t), \) thus \( \text{cx}_{R_t}(M_t) = n. \) It is easy to see that \( R = k[[X_t, Y_t]]/(X_t Y_t) \) and \( M \otimes_R N = R/(Y_t)R, \) for \( 1 \leq \ell \leq 2n. \) By 3.7 \( M \) and \( N \) are Tor-independent \( R \)-modules. Now let \( L = R/(X_t)R, \) for \( 1 \leq \ell \leq 2n.\) Then we get \( \text{grade}_R(L, M) = n, \) and \( \text{grade}_R(L, M \otimes_R N) = 2n. \) The natural local homomorphism \( \varphi_t : R_t \rightarrow R \) is flat. Using 3.7 since \( R \) is complete intersection, from [5] (1.13.1)] and [5] (5.2.3)] we get CI-dim \( R, N = 0 \) and \( \text{cx}_R(N) = n, \) respectively. Therefore the left side of the inequality of Corollary 3.6 is equal to \( n, \) and it’s right side is equal to \( 2n. \)
References

1. L. L. Avramov, *Homological asymptotics of modules over local rings*, in Commutative Algebra, Vol. 15, pp. 3362, MSRI, Berkeley, 1982; Springer-Verlag, New York, 1989.
2. L. L. Avramov, *Infinite free resolutions*. Six lectures on commutative algebra (Bellaterra, 1996), 1–118, Progr. Math. 166, Birkhäuser, Basel, 1998.
3. L. L. Avramov, H. B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra. 71 (1991), 129-155.
4. L. L. Avramov, H. B. Foxby, *Cohen-Macaulay properties of ring homomorphisms*, Adv. Math. 133 (1998), 54-95.
5. L. L. Avramov, V. N. Gasharov, I. V. Peeva, *Complete intersection dimension*, Inst. Hautes Etudes Sci. Publ. Math. 86 (1997), 67–114.
6. L. W. Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, 1747. Springer-Verlag, Berlin, 2000.
7. L. W. Christensen, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. 353 (2001), 1839–1883.
8. L. W. Christensen, H. B. Foxby, and A. Frankild, *Restricted homological dimensions and Cohen-Macaulayness*, J. Algebra, 251 (2002), 479-502.
9. H. B. Foxby, *Hyperhomological algebra and commutative rings*, Københavns Univ. Mat. Inst. Preprint, 1998.
10. H. B. Foxby, *Bounded complexes of flat modules*, J. Pure Appl. Algebra. 15 (1979), 149-172.
11. S. Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. 230 (1999), 545–567.
12. D. A. Jorgensen, *A generalization of the Auslander-Buchsbaum formula*, J. Pure Appl. Algebra. 144 (1999), 145–155.
13. P. Sahandi, T. Sharif, S. Yassemi, *Depth formula via complete intersection flat dimension*, Comm. Algebra 39 (2011), 4002-4013.
14. S. Sather-Wagsta, *Complete intersection dimensions for complexes*, J. Pure Appl. Algebra. 190 (2004), 267–290.
15. T. Sharif, S. Yassemi, *Depth formula, restricted Tor-dimension under base change*, Rocky Mountain. J. Math. 34 (2004), 1131-1146.
16. T. Sharif, S. Yassemi, *Special homological dimensions and intersection theorem*, Math. Scand. 96 (2005), 161–168.

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran Iran.
E-mail address: sharif@ipm.ir