Partitioning of an arbitrary domain into subdomains without branching of inner boundaries

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Abstract. In this paper we consider an approach to the partitioning of a arbitrary domain into connected subdomains without branching of internal boundaries. The Reeb graph is a simplified representation of the topology of the required domain. Algorithms are shown in this paper is a modification of the presented earlier algorithm of the formation of the plane Reeb graph. The algorithm for constructing the volume Reeb graph allows us to determine the internal topology of the domain. A new algorithm of the formation of the partitioning into subdomains without branching of internal boundaries is presented.

1. Reeb graph constructed by surface triangulation
Define a tree of connections between the critical points \( v^c \) and construct the Reeb graph \( R(V, E) \) with edges \( e(v_i, v_j) \in E \) for the surface triangulation \( T \in \mathbb{R}^3 \), consisting of 2-simplexes (Algorithm 1).

We will use the Morse level function \( f \) to determine the critical points \( v^c \) and introduce the parameter \( c(v^c) \) corresponding to the number of edges outgoing from it and taking the value: 1 - one edge; 2 - two edges; 3 - branch; 0 is the auxiliary vertex \( v^a \), which with the nearest critical \( v^a \) forms curvilinear arcs between the critical points in the branching case [1, 2, 3, 4].

In this paper Reeb graph has two definitions, the usual Reeb graph and a volume Reeb graph. Usual Reeb graph constructed with respect to one coordinate axis, based and constructed on surface mesh [5]. Volume Reeb graph is a set of subgraphs, that use two coordinate axis, to perform layers and usual graph. We will define Volume Reeb graph in next subsection.

**Algorithm 1.** Formation of the Reeb graph

| Data | \{T-triangulation, consisting from 2-simplexes \( \sigma \in T \); Morse level function \( f \)\} |
|------|---------------------------------|
| Result | \{Reeb graph \( R_1(V, E) \)\} |
| While | \( m < |V(T)| \) |
| | Computing values of Morse function \( f(v_m) \) for each edge of triangulation \( T \). |
| | \( m \leftarrow m + 1 \) |
| | \* Form layers of function Morse \* |

**IF** \( (Lk^+(v_h); Lk^-(v_h); Lk^\pm(v_h); Lk^\mp(v_h)) \) |

| | \( v^c_h = v_h \) |
| | \( L(v^c_h) = l \), where \( l = \{1,2,3,4,5\} \) - the type of 0-simplex link. |

**Define auxiliary edges.**

**ForEach** \( v^f_i \) |

| | **IF** \( L(v^f_i) = 2 \land L(v^c_{i+1}) = 3 \) |
While\(\sum_{j=1}^{m}|LS_j(\sigma)| < |T|\)\{ 
ForAll\{\nu\}\{ 
\(LS_m(\sigma) = \{\sigma \in Cl(LS(\alpha)): \forall \nu \leq \sigma, f(\nu) \geq \alpha.\}\)
\} 
\} 
ForEach\{\nu_i\}\{ 
Define 2-simplexes \(\sigma\), containing \(v_i\).
ForEach\{\nu_h \in T\}\{ 
Define link of 0-simplex \(v_h\).
\}
\}

Finding edges \(v_i^a, v_{(i+1)}^a\) on surface between \(v_i^c\) and \(v_{(i+1)}^c\).
\(v_{(i+1)}^c = v_i^a \cap v_{(i+2)}^a = v_{(i+1)}^a, v^a \subseteq v^c, L(v_{(i+1)}^c) = 0 \land L(v_{(i+3)}^c) = 0.\)
Renumbering edges and suppose that \(i \rightarrow i + 2.\)
\}
\}

Form and compile complete Reeb graph \(R_1(V,E)\).

The result of the Algorithm 1 is shown on figure 1 (b). To construct nearly exact domain topology, we use precise graph (figure 1 (b)). Figure 1 (a) shows a simplified Reeb graph, based on the surface triangulation, which passes incomplete holes and other features of topology [5].

Figure 1. Reeb graph: (a) simplified; (b) precise.

Algorithm 1 cannot determine the topological characteristics of arbitrary domains, for example, multidirectional holes. One of possible approaches to solution of this problem is to use volume Reeb graph.

2. The volume Reeb graph
Define a tree of connections between the critical points \(v^c\) and construct the Reeb graph \(R(V,E)\) with the edges \(e(v_i, v_j) \in E\) for the triangulation \(T \in R^3\), consisting of 3-simplexes.
We will use the Morse level function \(f\) of different directions in some coordinate planes. Note by 0 the simplex \(v^s\), as the auxiliary point. In contrast to the previously implemented approach to the formation of the Reeb graph, this algorithm uses a volume triangulation of the domain \(T\).

The volume Reeb graph \(R(V,E)\) is represented as a set of subgraphs that are constructed from the mesh (or cells) layers in different directions of traversal of the Morse function. A graph has the notation \(R_0(V,E)\) the remaining subgraphs will have the notation \(R_{n+1}(V,E)\), where \(n\) is the number of inner layers. The surface mesh is used only to form the \(R_0(V,E)\) graph.
Algorithm 2. Formation of volume Reeb graph

Data \{T – triangulation of 3-simplexes \( \sigma \in T \)\}

Result \{Volume Reeb graph \( R(V, E) \)\}

Construct graph with Algorithm 1, and we get main Reeb graph \( R_0(V, E) \).

While \( m < |V(T)| \)\{
  For \( i \leq n \) \{ * number of layers needed
    If \( v^a – \) auxiliary \{
      Making layer \( S_i = \{ \forall v \subseteq T | v \leq z(v^a) \} \) of 3-simplexes.
    \}
  \}
  ForEach \( S_n \)\{
    Form a subgraph \( R_{(n+1)}(V, E) \) from Algorithm 1 separately for each \( S_n \).
    As direction of function Morse \( f \) we will take one of horizontal axis.
  \}
  Form the volume Reeb graph \( R(V, E) \), where \( R_{(0,n)}(V, E) \subseteq R(V, E) \).
  \( m \leftarrow m + 1 \)
 \}

Algorithm 2 allows us to get as a result of its implementation, the volume Reeb graph for arbitrary topologies. Volume Reeb graph we will use as a topology graph of domain to produce partition without branching of inner boundaries.

Figure 2. Volume Reeb graph: (a) graph; (b) graph of mesh.

Volume Reeb graph is presented by multiplicity of subgraphs and main graph. Subgraphs are limited by the surface mesh and the level surface of the Morse function as secant plane. They perform as layers from mesh on figure 2 (b) and figure 3 (b).
Figure 3. Volume Reeb graph: (a) volume graph; (b) graph of layers; (c) volume graph of mesh.

Subgraph of layers represents slices on auxiliary points of graph $R_0(V, E)$, known as main graph (figure 3 (b)). Main graph and subgraph use mutual edges to connect with each other.

3. Domain partitioning by volume Reeb graph

Segmentation of subdomains into smaller fragments, which are blocks that represent split pants-decomposition [6]. After obtaining the volume Reeb graph it is necessary to cut the domain $T$ along the auxiliary vertexes of the graph. The obtained layers $S$, which represent some contour of 2 or 3 simplexes, have some upper bound. For each layer, the Morse function $f$ with a different direction in the coordinate system is applied. The resulting Reeb graph is associated with the main graph obtained in Algorithm 1 and is positioned relative to the auxiliary points. Thus we obtain a volume connected graph which characterize the complete topology of the given domain. Algorithm of the mesh partitioning based on Reeb graph shown in the figure 4-5.

Algorithm 3. Decomposition based on Reeb graph.

| Data | $T$ - triangulation of 3-simplexes, where $\sigma \in T$; $R(V, E)$ – Volume Reeb graph |
|------|----------------------------------------------------------------------------------|
| Result | The set of layers $LS_{(m)}(\sigma)$ |
| Finding auxiliary edges $v^{(a)} = \{ v^{(a)} \subseteq v^c | c(v^c) \equiv 0 \}$ |
| While ($v^m_m < |V(R_0)|$) |
| Finding nearest auxiliary edges $v^c_m$ for each hole |
| Direct secant plains for 4 auxiliary edges $v^c$ of 2 coordinate axis |
| Form layer $LS_{(m)}(\sigma) = \{ \sigma \in Lk(\sigma^+) | \sigma^+ \in LS_{(m)}(\sigma), \sigma \notin Ls_{(m)}(\sigma) \cup LS_{(m-1)}(\sigma) \}$ |
| $m \leftarrow m + 1$ |

The topology of this domain and the graph characterizing it allow us to divide the domain into minimal independent parts. The partitioning algorithm uses the Reeb graph for each layer, which makes it possible to implement a large number of algorithms.

4. Numerical experiments

Computational experiments were carried out on three-dimensional unstructured meshes for multiply connected domains, differing in the topological type, surface geometry and the details of its description by the mesh itself [7]. Earlier in previous works, the computational cost of the algorithm for the Reeb graph creating for the triangulated surface of a three-dimensional domain $O(K^2 \log^2 K)$,
where $K$ is the total number of vertices, triangles and tetrahedrons of the mesh. At the moment, by parallelizing the main cycles and passes through 2 and 3 - simplexes of triangulation, the following estimates of computations are achieved [8]. The parallel version of algorithm was evaluated in $O(K \log^2 \frac{K}{n})$, $n$ is the number of available threads. The computational costs of the partitioning algorithm based on the Reeb graph is $O(L \log^2 L)$, $L$ – is a total sum of edges, triangles and tetrahedrons.

![Figure 4](image1.png)  
**Figure 4.** Decomposition on subdomains: (a) volume graph; (b) partition on graph; (c) subdomains and graph.

Presented partition on figure 4 illustrates decomposition without branching of inner boundaries. Mesh on figure 4 (c) have 6 connected subdomains. Type of surface of each subdomain in topology will be (0,2). So the genus of subdomain will be 0 and 2 adjoining components.

![Figure 5](image2.png)  
**Figure 5.** Partition on subdomains: (a) decomposition based on one function Morse; (b) decomposition on two functions Morse (x and y axis) of volume Reeb graph from Algorithm 3.
On figure 5 presented partition without branching of inner boundaries by one function Morse (figure 5 (a)) and by two functions in different direction (figure 5 (b)). Domain on figure 5 (a) have been partitioned only on three subdomains. These subdomains are multiply connected and genus is more than one. The introduction of an additional direction in the partition algorithm (Algorithm 3) made it possible to obtain seventeen subdomains without branching of inner boundaries.

The algorithms considered in this paper make it possible to automatically determine the features of the topology of various domains. Algorithms showed sufficiently accurate results, which indicates the reliability of their application. The considered partitions, due to the elimination of branching of inner boundaries, reduce the number and simplify the structure of exchanges between computational processes in parallel algorithms of mesh methods. The layer-by-layer ordering of mesh cells in the subdomains of the unstructured mesh excludes conflicts in computing in the shared memory of computational systems with a hybrid architecture [9].

Partitioning into a given number of subdomains can be provided by recursively dividing the obtained subdomains of an unstructured mesh and combining the layers of cells that provide independent access to the data.

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**References**

[1] Korneev V and Langer U 2004 *Encyclopedia of Computational Mechanics* 617–647

[2] Kopysov S P, Kuzmin I M, Nedozhogin N S and Novikov A K 2012 Parallel Algorithms for Constructing and Solving the Schur Complement on Graphics Accelerators *Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki* 154 (3) 202–215

[3] Martynenko S I 2008 Formalization of Computations at Numerical Solution of Boundary Value Problems *Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki* 150 (1) 76–90

[4] Postnikov M M 1971 *Introduction in Morse Theory* (Moscow: Nauka) 568

[5] Zimovnov A V and Mestetskiy L M 2016 On algorithm of curve-skeleton extraction for 3D model based on planar projections *Vestn. TvGU Seriya: Prikladnaya matematika* (3) 67–83

[6] Hajij M, Dey T and Li X 2016 Segmenting a surface mesh into pants using Morse theory *Graphical Models* 88 12–21

[7] Mestetskiy L M 2009 *Continuous morphology of binary images: figures, skeletons and circulars* (Moscow: Fizmatlit) 288

[8] Harvey W, Wang Y and Wenger R 2010 A Randomized O (M log M) Time Algorithm for Computing Reeb Graphs of Arbitrary Simplicial Complexes *USA: SoCG ’10* 267–276

[9] Novikov A K, Piminova N K, Kopysov S P and Sagdeeva Y A 2016 Layer-by-layer ordering in parallel finite composition on shared-memory multiprocessors *IOP Conf. Ser.: Mater. Sci. Eng.* 158 (1)