Detection Theory for Union of Subspaces

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Abstract—The focus of this paper is on detection theory for union of subspaces (UoS). To this end, generalized likelihood ratio tests (GLRTs) are presented for detection of signals conforming to the UoS model and detection of the corresponding “active” subspace. One of the main contributions of this paper is bounds on the performances of these GLRTs in terms of geometry of subspaces under various assumptions on the observation noise. The insights obtained through geometrical interpretation of the GLRTs are also validated through extensive numerical experiments on both synthetic and real-world data.

Index Terms—Adaptive detection, signal detection, subspace detection, subspace geometry, union of subspaces

I. INTRODUCTION

Detection theory has a long history in the signal processing literature. Classical detection theory is often based on the subspace model, in which the signal to be detected is assumed to come from a low-dimensional subspace embedded in a high-dimensional ambient space [1]. However, recently a nonlinear generalization of the subspace model, termed the union of subspaces (UoS) model [2], has gained attention in the literature due to its ability to better model real-world signals. We refer the reader to [3]–[6] for various applications of the UoS model.

In this paper, we revisit the problem of detection of signals under various additive noise models for the case when the signal conforms to the UoS model. Our goal in this regard is derivation of tests for detection of both the signal and the underlying active subspace, and characterization of the performance of these tests in terms of geometry of the subspaces.

A. Prior work

There exists a rich body of literature concerning detection of signals under the subspace model; see, e.g., [7]–[10]. The most well-studied method in this regard is the matched subspace detector [7], which projects the received signal onto the subspace of interest and compares its energy against a threshold. A naive approach to detection under the UoS model would be to treat it as a subspace detection problem by replacing the union with sum and using the resulting subspace within the matched subspace detector. However, such an approach not only results in high false alarm rates (for obvious reasons), but it also does not enable detection of the active subspace. A better alternative is to treat the detection problem as a multiple hypothesis testing problem, as in [6], with each test given by an individual matched subspace detector. We establish in this paper that such an approach will have the same performance as a GLRT for the case of a single active subspace.

Recently, there have been a few works that are directly related to the detection problem under the UoS model [11]–[15]. Among these, [11] studies the problem of signal detection under the compressive sensing framework [16], with the final results involving analysis of a GLRT for a binary hypothesis test. These compressive detection results can be considered a special instance of those for detection under the UoS model, since a sparse signal can be thought of as lying in a union of (exponentially many) subspaces [2]. The nature of these results, however, does not enable understanding of the general detection problem under the UoS model, especially in relation to geometry of the underlying subspaces. Similar to [11], [12] also studies the compressive detection problem, but in the context of radar-based multi-target detection. While the analysis in [12] is based on the use of the LASSO [17] for detection, it too does not offer geometric insights into the general UoS-based detection problem. In [13], the authors extend the original compressive detection framework of [11] to more general settings, but the final results are still couched in terms of the sparsity framework and they fail to bring out the geometric interplay between the different subspaces.

The work that is most closely related to this paper is [14], in which the authors study the signal and the active subspace detection problems under the UoS framework in the context of radar target detection. The (signal and active subspace) detection schemes proposed in [14] are based on multiple hypothesis testing. The analysis in [14] is for the case of colored Gaussian noise with unknown variance but known covariance matrix. Further, since the analysis is in terms of the spectral signatures of targets, it does not help understand the interplay between the detection performance and the geometry of subspaces. Finally, [14] does not investigate invariance properties of the derived test statistics.

Recently, [15] has studied both recovery of a signal conforming to the UoS model and detection of the corresponding active subspace in the presence of a linear sampling operator. This work, however, is fundamentally focused on understanding the role of the sampling operator within the active subspace detection problem. Further, it assumes white Gaussian noise with known variance, does not investigate the related problem of signal detection, and does not focus on the geometry of subspaces as an integral component of the detection problem.

B. Our contributions

Our focus in this paper is on analysis of various GLRTs for the signal and the active subspace detection problems under the UoS model for different noise settings. Our main contribution in this regard is a comprehensive understanding of the two detection problems in terms of characterization of the performance of the derived GLRTs through the probabilities.
of detection, classification, and false alarm, geometry of the underlying subspaces, and invariance properties of the test statistics. One of the key insights of this work is that the probability of correct identification of the active subspace increases with increasing principal angles between subspaces in the union. While this makes intuitive sense, our analysis provides theoretical justification for such an assertion. Further, our work also helps understand the relationship between a binary and a multiple hypothesis testing approach to the signal detection problem under the UoS model. Finally, we provide extensive numerical experiments to highlight the usefulness of our analysis and its superiority to prior works such as [15]. We refer the reader to Table I for a brief comparison of our work with existing literature.

### C. Notation and Organization

We use bold lowercase and bold uppercase letters to denote vectors and matrices, respectively. Given a matrix $\mathbf{A}$, $a_{ij}$ and $A_{i,j}$ denote its $j$-th column and $(i,j)$-th entry, respectively. Further, $\mathbf{A}^{-1}$ and $|\mathbf{A}|$ denotes its inverse (if it exists) and its determinant, respectively. Given a vector $a$, $\|a\|_p$ denotes its $\ell_p$-norm and $|a|$ denotes its elementwise absolute values. Finally, $Q(\cdot)$, $\Gamma(\cdot)$, and $K_0(\cdot)$ denote the Gaussian $Q$ function, the Gamma function, and the modified Bessel function of the second kind with parameter $n$, respectively.

The rest of the paper is organized as follows. In Sec. II, we formulate the signal and the active subspace detection problems under the UoS model. Sec. III presents and analyzes the GLRTs for these two problems under different noise conditions. Sec. IV provides a discussion of the results obtained in Sec. III. Sec. V presents the results of numerical experiments on both synthetic and real-world data, while we conclude the paper in Sec. VI.

### II. Problem Formulation

We study two interrelated detection problems in this paper. The first one, referred to as signal detection, involves deciding between an observation $y \in \mathbb{R}^m$ being just noise or it being an unknown signal $x \in \mathbb{R}^m$ embedded in noise. Mathematically, this can be posed as a binary hypothesis test with the null ($\mathcal{H}_0$) and the alternate ($\mathcal{H}_1$) hypotheses given by:

$$\mathcal{H}_0: \quad y = n;$$
$$\mathcal{H}_1: \quad y = x + n;$$

where $n \in \mathbb{R}^m$ denotes noise that is typically assumed Gaussian. Traditionally, (1) has been studied under the assumption of $x$ belonging to a low-dimensional subspace of $\mathbb{R}^m$ [7]–[10]. In contrast, our focus is on the case of $x$ belonging to a union of low-dimensional subspaces: $x \in \bigcup_{k=1}^{K_0} S_k$, where $S_k \subset \mathbb{R}^m$ denotes a subspace of $\mathbb{R}^m$. We further assume that the subspaces are pairwise disjoint, $S_k \cap S_{k'} = \emptyset$ for $k \neq k'$, and they have the same dimension: $\forall k, \dim(S_k) = n \ll m$.\footnote{One can extend this work to the case of different dimensional subspaces in a straightforward manner at the expense of notational complexity.}

The second problem studied in this paper, which does not arise in classical subspace detection literature, is referred to as active subspace detection. The goal in this problem is to not only detect whether $y$ contains an unknown signal $x$, but also identify the subspace $S_k$ to which $x$ belongs. Mathematically, this can be posed as a multiple hypothesis test with the null ($\mathcal{H}_0$) and the alternate ($\{\mathcal{H}_k\}_{k=1}^{K_0}$) hypotheses given by:

$$\mathcal{H}_0: \quad y = n;$$
$$\mathcal{H}_k: \quad y = x + n, \quad x \in S_k: \quad k = 1, \ldots, K_0. \quad (2)$$

Our goal in this paper is to derive statistical tests for (1) and (2), and provide a rigorous mathematical understanding of the performance of the derived tests. Our analysis is based on the assumption of $x$ being a colored Gaussian noise that is distributed as $\mathcal{N}(0, \sigma^2 \mathbf{R})$ with $\mathbf{R}$ being a full-rank covariance. In particular, we focus on the three cases of (i) known noise statistics, (ii) known variance ($\sigma^2$), but unknown covariance ($\mathbf{R}$), and (iii) unknown variance and covariance. In contrast to prior works [11]–[15], we are specifically interested in characterizing our results in terms of the geometry of the underlying subspaces. This geometry can be described through the principal angles between the subspaces, where the $i$-th principal angle between subspace $S_j$ and $S_k$, denoted by $\varphi_{i,j,k}$, $i = 1, \ldots, n$, is recursively defined as [18]:

$$\varphi_{i,j,k} = \arccos \left( \max_{\mathbf{u}, \mathbf{v}} \left\{ \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|_2 \|\mathbf{v}\|_2} : \mathbf{u} \in S_j, \mathbf{v} \in S_k, \mathbf{u} \perp \mathbf{u}_\ell, \mathbf{v} \perp \mathbf{v}_\ell, \ell = 1, \ldots, i - 1 \right\} \right), \quad (3)$$

where $(\mathbf{u}_\ell, \mathbf{v}_\ell) \in S_j \times S_k$ denote the principal vectors associated with the $\ell$-th principal angle. It is straightforward to see that $0 \leq \varphi_{1,j,k} \leq \varphi_{2,j,k} \leq \cdots \leq \varphi_{n,j,k} \leq \pi/2$.

We conclude by noting that our statistical tests in the following will be expressed in terms of the following ratios.

### Table I

A brief comparison of this work with related prior works in the literature.

| Work   | Framework                  | Gaussian Noise Model                  | Signal Detection | Active Subspace Detection | Impact of Geometry | Invariance |
|--------|----------------------------|---------------------------------------|------------------|---------------------------|-------------------|------------|
| [11], [12], [13] | compressive sensing       | white, w/ known variance             | ✓                | ✓                         | ✓                 | ✓          |
| [14]   | general UoS                | colored, w/ known cov. and unknown var. | ✓                | ✓                         | ✓                 | ✓          |
| [15]   | linear sampling of UoS     | white, w/ known var.                 | ✓                | ✓                         | ✓                 | ✓          |
| This work | general UoS               | colored, w/ known statistics         | ✓                | ✓                         | ✓                 | ✓          |
|         |                            | colored, w/ partially unknown statistics | ✓                | ✓                         | ✓                 | ✓          |
|         |                            | colored, w/ completely unknown statistics | ✓                | ✓                         | ✓                 | ✓          |
for compactness purposes:

\[ T_x(P) = \frac{z^T P z}{z^T z}, \quad T_{xy}(P) = \frac{z^T P z}{\eta}, \]

\[ T_x(P, Q) = \frac{z^T P z}{z^T Q z}, \quad T_{xy}(P) = \frac{z^T P z}{\eta + z^T z}, \]

where \( z \) and \((P, Q)\) denote a vector and matrices of appropriate dimensions, respectively, while \( \eta > 0 \) denotes a constant.

### A. Performance metrics

The performances of the statistical tests proposed in this paper will be characterized in terms of the probabilities of detection \( (P_D) \), classification \( (P_C) \), and false alarm \( (P_{FA}) \). Specifically, let \( P_{H_\ell}(\cdot) = \Pr(\cdot | H_\ell) \), and define the event \( H_\ell = \{ \text{Hypothesis } H_\ell \text{ is accepted} \} \). Then, in the case of signal detection, we have \( P_D = P_{H_1}(H_1) \) and \( P_{FA} = P_{H_0}(H_1) \). In contrast, in the case of active subspace detection, we have \( P_C = \sum_{k=1}^{K_0} P_{H_k}(H_k) \Pr(H_k) \) and \( P_{FA} = P_{H_0}(H_k) \).

We conclude by pointing out that some of our forthcoming discussion will use the shorthand \( P_{S_k}(\cdot) = \Pr(\cdot | x \in S_k) \) and \( \Psi(\eta, \alpha) = \frac{\eta}{\pi^{n/2} \sigma^2} (\eta \alpha)^{(n-1)/2} K_{(n-1)/2} (\eta \alpha) \) for \( \alpha \in \mathbb{R}_+ \) and \( \eta_0 \in (0, 1/2) \). Using this notation, we can also write \( P_D = \sum_{k=1}^{K_0} P_{S_k}(H_k) \Pr(x \in S_k) \).

### III. MAIN RESULTS

In this section, we present statistical tests for both the detection problems under various noise conditions. In addition, we provide bounds on the performance metrics for these tests.

#### A. Known noise statistics

We begin with the assumption that both the noise variance, \( \sigma^2 \), and the covariance, \( \mathbf{R} \), are known. It is trivial to see that \( y | H_0 \sim \mathcal{N}(0, \sigma^2\mathbf{R}) \) for both detection problems. Further, in the case of signal detection, we have \( y | H_1 \sim \mathcal{N}(x, \sigma^2\mathbf{R}) \). In contrast, the observations \( y \) under the \( k \)-th alternate hypothesis in the case of active subspace detection can be expressed as \( y | H_k \sim \mathcal{N}(\mathbf{H}_k \theta_k, \sigma^2) \). Here, \( \mathbf{H}_k \in \mathbb{R}^{n \times n} \) denotes a basis for subspace \( S_k \) and \( \theta_k \in \mathbb{R}^n \) denotes representation coefficients of \( x \) under basis \( \mathbf{H}_k \). Since \( x \) and \( \theta_k \) are unknown for the signal and the active subspace detection problems, respectively, we resort to the generalized likelihood ratio tests (GLRTs) for the two detection problems. Our results in this regard are based on the following definitions: let \( z = \mathbf{R}^{-\frac{1}{2}} y \) denote the whitened observations, \( \mathbf{w} = \mathbf{R}^{-\frac{1}{2}} \mathbf{n} \) denote the whitened noise, \( \mathbf{G}_k = \mathbf{R}^{-\frac{1}{2}} \mathbf{H}_k \), \( k = 1, \ldots, K_0 \), denote the whitened subspace bases, and \( P_{S_k} = \mathbf{G}_k (\mathbf{G}_k^T \mathbf{G}_k)^{-1} \mathbf{G}_k^T \) and \( P_{\bar{S}_k} = \mathbf{I} - P_{S_k} \), respectively, denote the projection matrix for the \( k \)-th whitened subspace and its orthogonal complement.

**Theorem 1.** Let \( \bar{\gamma} > 0 \) denote the test threshold and define \( \hat{k} = \arg \max_k (z^T P_{\bar{S}_k} z) \). The GLRT for the signal detection and the active subspace detection problem is, respectively, given by

\[ T_{z}^{2 \sigma^2} \left( P_{S_k} \right) \overset{H_1}{\gtrless} \bar{\gamma}, \quad T_{z}^{2 \sigma^2} \left( P_{\bar{S}_k} \right) \overset{H_0}{\gtrless} \bar{\gamma}. \]  

The proof of this theorem is given in Appendix A, while its interpretation as well as its relationship to the classical test for subspace detection are provided in Sec. IV. We now characterize the performance of the statistical tests in (4) in terms of bounds on \( P_{FA} \), \( P_D \), and \( P_C \). Note that we have to resort to bounds, as opposed to exact expressions, because of the complicated joint distributions that arise in our context; we refer the reader to Appendix B for further discussion.

**Theorem 2.** The GLRTs in Theorem 1 for the signal and the active subspace detection problems result in probability of false alarm that is upper bounded by:

\[ P_{FA} \leq \min \left\{ 1, \sum_{k=1}^{K_0} \Pr \left( T_{z}^{2 \sigma^2} \left( P_{S_k} \right) > \bar{\gamma} \right) \right\}. \]  

Further, in the case of signal detection, the probability of detection \( P_D = \sum_{k=1}^{K_0} P_{S_k}(H_k) \Pr(x \in S_k) \) can be upper and lower bounded by the fact that

\[ P_{S_k}(H_k) \leq \min \left\{ 1, \sum_{i=1}^{K_0} P_{S_k} \left( T_{z}^{2 \sigma^2} \left( P_{S_i} \right) > \bar{\gamma} \right) \right\}, \]

\[ P_{S_k}(H_k) \geq \sum_{i=1}^{K_0} P_{S_k} \left( T_{z}^{2 \sigma^2} \left( P_{S_i} \right) > \bar{\gamma} \right)^2. \]

Finally, the probability of classification \( P_C \) for active subspace detection can be lower bounded by the fact that

\[ P_{H_k}(H_k) \geq \max \left\{ 0, \sum_{j=1, j \neq k}^{K_0} P_{S_k} \left( T_{z}^{2 \sigma^2} \left( P_{S_j} \right) > \bar{\gamma} \right) \right\}, \]

\[ \sum_{j=1, j \neq k}^{K_0} P_{S_k} \left( T_{z}^{2 \sigma^2} \left( P_{S_j} \right) > \bar{\gamma} \right) - \left( K_0 - 1 \right). \]

The proof of this theorem is given in Appendix B. It is worth noting that probabilities of the form \( P_{S_k}(T_{z}^{2 \sigma^2} \left( P_{S_k} \right) > \bar{\gamma}) \) correspond to tail probabilities of chi-squared random variables, whereas the probabilities \( P_{S_k}(T_{z}^{2 \sigma^2} \left( P_{S_k} \right), P_{S_j} > \bar{\gamma}) \) involve ratios of dependent chi-squared variables whose distributions can be numerically computed.

**Remark 1.** It is noted in Appendix B that (7) can be further lower bounded using [15, Lemma 1] as \( P_{H_k}(H_k) \geq \max \left\{ 0, P_{S_k} \left( T_{z}^{2 \sigma^2} \left( P_{S_k} \right) > \bar{\gamma} \right) - \sum_{j=1, j \neq k}^{K_0} \Psi(\eta_0, \lambda_{j \neq k}) \right\}, \) where \( \lambda_{j \neq k} = z^T P_{\bar{S}_l} z / \sigma^2 \) when \( z \in \bar{S}_k \). This bound, however, depends further on \( \eta_0 \). Numerical experiments reported in Sec. V show the looseness of this bound for the case of \( \eta_0 = 0.25 \), the value advertised in [15].

**Remark 2.** A heuristic approach to detecting signals under the OoS model would be to use the multiple hypothesis tests of [6], where each test is an individual matched subspace detector. The final decision can then be made by taking the union of binary outputs from each matched detector and declaring detection if any one of them has detected a signal. It is straightforward to see however that this final decision rule coincides with the decision rule in (4). Thus, in the event that only one subspace is active, the testing procedure in [6] effectively reduces to a GLRT.
B. Unknown noise covariance

Next, we consider the case of colored noise with unknown covariance matrix $\mathbf{R}$. In this case, we also assume access to $N_0$ noise samples $\mathbf{\xi}_p \sim \mathcal{N}(0, \mathbf{R}), p = 1, \ldots, N_0$ ($N_0 > m$ to obtain a non-singular estimate of $\mathbf{R}$), which is a standard assumption in the detection literature [8]–[10]. As before, we use GLRTs to obtain decision rules for the two detection problems. Our results make use of the following definitions: let $\mathbf{\Sigma} = \frac{1}{N_0} \sum_{p=1}^{N_0} \mathbf{\xi}_p \mathbf{\xi}_p^T$ denote sample covariance of noise samples, $\mathbf{\bar{z}} = \mathbf{\Sigma}^{-\frac{1}{2}} \mathbf{\bar{y}}$ denote the empirically whitened observations, $\mathbf{\bar{w}} = \mathbf{\Sigma}^{-\frac{1}{2}} \mathbf{\bar{z}}$ denote the empirically whitened noise, $\mathbf{G}_k = \mathbf{\Sigma}^{-\frac{1}{2}} \mathbf{H}_k, k = 1, \ldots, K_0$, denote the empirically whitened subspace bases, and $\mathbf{P}_{\mathbf{S}_k} = \mathbf{G}_k (\mathbf{G}_k^T \mathbf{G}_k)^{-1} \mathbf{G}_k^T$ denote the projection matrix for the $k$-th empirically whitened subspace.

**Theorem 3.** Let $\bar{\gamma} > 0$ denote the test threshold and define $k = \arg \max_k (\mathbf{\bar{z}}^T \mathbf{P}_{\mathbf{S}_k} \mathbf{\bar{z}})$. The GLRT for the signal detection and the active subspace detection problem, respectively, given by:

$$T_{\mathbf{\bar{z}}}^{N_0\sigma^2} (\mathbf{P}_{\mathbf{S}_k}) \overset{\mathcal{H}_0}{\gtrless} \bar{\gamma} \quad \text{and} \quad T_{\mathbf{\bar{z}}}^{N_0\sigma^2} (\mathbf{P}_{\mathbf{S}_k}) \overset{\mathcal{H}_0}{\gtrless} \bar{\gamma}. \quad (8)$$

The proof of this theorem is provided in Appendix C, while some discussion on interpretation and relationship to the classical test for subspace detection is provided in Sec. IV. We now characterize the performance of the statistical tests in (8) in terms of bounds on $P_{FA}, P_D,$ and $P_C$.

**Theorem 4.** The GLRTs for the signal and the active subspace detection problems in Theorem 3 result in the probability of false alarm that is upper bounded by:

$$P_{FA} \leq \min \left\{ 1, \sum_{k=1}^{K_0} \Pr \left( T_{\mathbf{\bar{w}}}^{N_0\sigma^2} (\mathbf{P}_{\mathbf{S}_k}) > \bar{\gamma} \right) \right\}. \quad (9)$$

Further, in the case of signal detection, the probability of detection $P_D = \sum_{k=1}^{K_0} P_{S_k}(\mathcal{H}_1) \Pr(x \in \mathbf{S}_k)$ can be upper and lower bounded by the fact that

$$P_{S_k}(\mathcal{H}_1) \leq \min \left\{ 1, \sum_{i=1}^{K_0} P_{S_k} \left( T_{\mathbf{\bar{z}}}^{N_0\sigma^2} (\mathbf{P}_{\mathbf{S}_i}) > \bar{\gamma} \right) \right\}, \quad \text{and}$$

$$P_{S_k}(\mathcal{H}_1) \geq \frac{1}{K_0} \sum_{i=1}^{K_0} \left[ P_{S_k} \left( T_{\mathbf{\bar{z}}}^{N_0\sigma^2} (\mathbf{P}_{\mathbf{S}_i}) > \bar{\gamma} \right) \right]^2. \quad (10)$$

Finally, the probability of classification $P_C$ for active subspace detection can be lower bounded by the fact that

$$P_{H_k}(\mathcal{H}_k) \geq \max \left\{ 0, P_{S_k} \left( T_{\mathbf{\bar{z}}}^{N_0\sigma^2} (\mathbf{P}_{\mathbf{S}_k}) > \bar{\gamma} \right) + \sum_{j=1, j \neq k}^{K_0} P_{S_k} \left( T_{\mathbf{\bar{z}}} (\mathbf{P}_{\mathbf{S}_k} \mathbf{P}_{\mathbf{S}_j}) > 1 \right) - (K_0 - 1) \right\}. \quad (11)$$

The proof of this theorem follows along similar lines as for the proof of Theorem 2 and is omitted due to space constraints. In contrast to Theorem 2, the terms of the form $P_{S_k}(\mathbf{P}_{\mathbf{S}_k} > \bar{\gamma})$ and $P_{S_k}(T_{\mathbf{\bar{z}}} (\mathbf{P}_{\mathbf{S}_k}) \mathbf{P}_{\mathbf{S}_j}) > 1$ involve the probabilities of the ratios of dependent chi-squared variables and have to be computed numerically.

**Remark 3.** One can again further lower bound (11) using [15, Lemma 1] as:

$$P_{H_k}(\mathcal{H}_k) \geq \max \left\{ 0, P_{S_k} \left( T_{\mathbf{\bar{z}}}^{N_0\sigma^2} (\mathbf{P}_{\mathbf{S}_k}) > \bar{\gamma} \right) + K_0 \sum_{j=1}^{K_0} Q \left( \frac{1}{2}(1 - 2\eta_0) \sqrt{\bar{\lambda}_{j\backslash k}} \right) - \sum_{j=1, j \neq k}^{K_0} \Psi(\eta_0, \bar{\lambda}_{j\backslash k}) \right\},$$

where $\bar{\lambda}_{j\backslash k} = \frac{1}{\sigma^2} \mathbf{\bar{z}}^T \mathbf{P}_{\mathbf{S}_j} \mathbf{\bar{z}}$ when $\mathbf{z} \in \mathbf{S}_k$.

C. Unknown noise statistics

We now address adaptive detection in settings where the covariance matrix $\mathbf{R}$ and variance $\sigma^2$ are both unknown. Once again assuming access to $N_0$ noise samples and using the notation of Sec. III-B, the GLRTs lead to the following decision rules.

**Theorem 5.** Let $\bar{\gamma} > 0$ denote the test threshold and define $k = \arg \max_k (\mathbf{\bar{z}}^T \mathbf{P}_{\mathbf{S}_k} \mathbf{\bar{z}})$. The GLRT for the signal detection and the active subspace detection problem, respectively, given by:

$$T_{\mathbf{\bar{z}}} (\mathbf{P}_{\mathbf{S}_k}) \overset{\mathcal{H}_0}{\gtrless} \bar{\gamma} \quad \text{and} \quad T_{\mathbf{\bar{z}}} (\mathbf{P}_{\mathbf{S}_k}) \overset{\mathcal{H}_0}{\gtrless} \bar{\gamma}. \quad (12)$$

The proof of this theorem is given in Appendix D, with corresponding discussion in Sec. IV. The performance of the statistical tests in (12) is given by the following theorem.

**Theorem 6.** The GLRTs for the signal and the active subspace detection problems in Theorem 5 result in the probability of false alarm that is upper bounded by:

$$P_{FA} \leq \min \left\{ 1, \sum_{k=1}^{K_0} \Pr \left( T_{\mathbf{\bar{w}}} (\mathbf{P}_{\mathbf{S}_k}) > \bar{\gamma} \right) \right\}. \quad (13)$$

Further, in the case of signal detection, the probability of detection $P_D = \sum_{k=1}^{K_0} P_{S_k}(\mathcal{H}_1) \Pr(x \in \mathbf{S}_k)$ can be upper and lower bounded by the fact that

$$P_{S_k}(\mathcal{H}_1) \leq \min \left\{ 1, \sum_{i=1}^{K_0} P_{S_k} \left( T_{\mathbf{\bar{z}}} (\mathbf{P}_{\mathbf{S}_i}) > \bar{\gamma} \right) \right\}, \quad \text{and}$$

$$P_{S_k}(\mathcal{H}_1) \geq \sum_{i=1}^{K_0} \left[ P_{S_k} \left( T_{\mathbf{\bar{z}}} (\mathbf{P}_{\mathbf{S}_i}) > \bar{\gamma} \right) \right]^2. \quad (14)$$

Finally, the probability of classification $P_C$ for active subspace detection can be lower bounded by the fact that

$$P_{H_k}(\mathcal{H}_k) \geq \max \left\{ 0, P_{S_k} \left( T_{\mathbf{\bar{z}}} (\mathbf{P}_{\mathbf{S}_k}) > \bar{\gamma} \right) + \sum_{j=1, j \neq k}^{K_0} P_{S_k} \left( T_{\mathbf{\bar{z}}} (\mathbf{P}_{\mathbf{S}_k} \mathbf{P}_{\mathbf{S}_j}) > 1 \right) - (K_0 - 1) \right\}. \quad (15)$$

The proof of this theorem is also similar to the proof of Theorem 2 and is thus omitted. Similar to Theorem 4, the terms of the form $P_{S_k}(T_{\mathbf{\bar{z}}} (\mathbf{P}_{\mathbf{S}_k}) > \bar{\gamma})$ and $P_{S_k}(T_{\mathbf{\bar{z}}} (\mathbf{P}_{\mathbf{S}_k} \mathbf{P}_{\mathbf{S}_j}) > 1)$ need to be computed numerically.
Remark 4. Similar to Remark 3, a looser lower bound can be derived here as well, with the only difference being that $T_k^{N_0\sigma^2}(\tilde{P}_S)$ is replaced by $T_k(\tilde{P}_S)$.

IV. DISCUSSION

In this section we discuss some characteristics of the various test statistics obtained in Sec. III. First, we compare the test statistics for signal detection under the UoS model ((4),(8) and (12)) with their counterparts under the subspace model [7]–[10]. In the subspace observation model, $x$ is assumed to belong to a single subspace, $x = H\theta$, where $H$ contains the subspace bases. The corresponding test statistics for known noise statistics, unknown noise covariance and unknown noise statistics are, respectively, given by:

$$ T_k^{2\sigma^2} (P_S) \gtrless \gamma, T_k^{N_0\sigma^2} (P_S) \gtrless \gamma, \text{ and } T_k(\tilde{P}_S) \gtrless \gamma. \quad (16) $$

At a first glance, the statistics for the UoS model and the subspace model look similar. However, the numerator of the statistics for the subspace model corresponds to the energy of the observed signal after projection onto the relevant subspace. In contrast, since we deal with multiple subspaces, we have to rely on projection onto the subspace that captures the most energy of the observed signal.

A. Signal detection versus active subspace detection

Notice that the test statistics for active subspace detection have forms similar to those for signal detection. The main difference lies in the performance of these statistics when detecting either the signal or the active subspace. The detection performance for active subspace detection is lower than that for signal detection. This is due to the fact that for signal detection, the detector is not concerned with detecting the true subspace from which the observed signal is coming and can afford to confuse one subspace with another as long as it detects the presence of a signal. That is not the case with active subspace detection, where this confusion matters, and thus we observe the loss in performance. This fact was also highlighted by Gini et al. in [14].

B. Invariance properties of the test statistics

We now examine the invariance properties of our test statistics for signal detection. Since our test statistics for active subspace detection are similar to those for signal detection under UoS model, they exhibit similar invariance properties.

From the expressions in (4), (8) and (12), notice that the statistics are invariant to the rotations in $S_k$. This means all rotated versions of the relevant signal (for rotations in $S_k$) will result in same detection performance. Moreover, the statistics also exhibit invariance with respect to the translations in $S_k$ (which is the orthogonal subspace of $S_k$). This implies that any additive interference from $S_k$ is unnoticeable to the detectors since they only measure the energy of $z$ in the subspace $\bar{S}_k$. Additionally, the test statistic for detection in unknown noise statistics (12) is also invariant to the scaling of the observed signals, i.e., scaled versions of a signal will result in same detection performance with this test. This is due to the fact that both numerator and denominator in (12) are quadratic forms of the whitened/empirically whitened observations $z$, without any additive terms.

C. Influence of geometry between whitened subspaces on detection probability

The detection performance of our detector decreases only slightly as the angles between the subspaces increase. This can be seen from an alternate expression for the probability of union of events. For example, the probability of union of two events, $A$ and $B$, can be written as: $P(A \cup B \cup C) = P(A) + P(B) - P(AB) = P(AB') + P(A'B) + P(AB)$ where $A'$, and $B'$ are the complements of the corresponding events. One can thus see that the probability of union of events is directly proportional to the probability of the intersection of events (and their complements). For the case of detection probability, these intersections are $k$-tuples of the form $\bigcap_{j=1}^{k} \{ T_{S_k} > \gamma \}$ (and their complements). When a pair (or more) of subspaces are close to each other, i.e., the principal angles between whitened/empirically whitened subspaces are small, the probability of these $k$-tuples is larger compared to when the subspaces are far apart.

Intuitively, since signal detection problem is not concerned with the detection of the active subspace, confusing a (noisy) signal coming from one subspace as being generated from another subspace does not matter significantly. In fact, this confusion helps the detection task as long as a signal is actually present. Interestingly, when the subspaces are far apart, i.e., principal angles are large, chances of such confusion are less and the probability of detection is slightly decreased.

D. Influence of geometry between whitened subspaces on correct classification probability

We now examine the influence of geometry between whitened subspaces on the probability of correct classification. This analysis in particular sets us apart from other related works such as [11]–[15], as we make the influence of geometry explicit through the principal angles between subspaces. We start with the case of active subspace detection in known noise statistics. The crux of our analysis is given in the following theorem.

**Theorem 7.** When the active subspaces are detected using the test in Theorem 1, the lower bound on the probability of correct classification increases with increasing principal angles between the whitened subspaces.

The proof of this theorem is detailed in Appendix E. The following corollary can also be obtained form Theorem 7.

**Corollary 1.** Suppose the noise is white Gaussian, i.e., $n \sim N(0, \sigma^2 I)$. When the active subspaces are detected using the test in Theorem 1, the lower bound on the probability of correct classification increases with increasing principal angles between the subspaces in the union.

Similarly, in the case of other noise settings (unknown covariance and unknown noise statistics), the probability of
whitening, suffer more attenuation and have a lower \( \parallel \) (and signals) with more energy in lower indices after unscaled of the eigenvectors of the covariance as its columns. Note that the covariance, i.e., \( Q \) bases of the observation space with the eigenvectors of the \( x \) decomposition of \( x \) where \( \lambda \) performs unscaled whitening. We can see that \( \lambda \) is the closer a subspace is to the leading eigenvectors of noise covariance, the lower is its detection probability as it suffers more attenuation during whitening.

correct classification of individual subspaces increases with increasing principal angles between the empirically whitened subspaces. This also follows trivially from Theorem 7.

E. Influence of geometry of colored noise

To characterize the effect of noise geometry on two detection problems, we focus on the terms \( z^T Q S z \) in (4). We can see that \( z^T Q S z = (\bar{x} + w)^T P S (\bar{x} + w) = x^T P S \bar{x} + 2w^T P S \bar{x} w + w^T P S w \), where \( \bar{x} = R^{-\frac{1}{2}} x \). The norm of \( \bar{x} \) can be expressed as:

\[
\| \bar{x} \|_2^2 = x^T Q \Lambda^{-\frac{1}{2}} Q^T x = \| \Lambda^{-\frac{1}{2}} x^Q \|_2^2 = \sum_{i=1}^{m} \frac{(\lambda_i^Q)^2}{\lambda_i} \tag{17}
\]

where \( x^Q = Q^T x \), and \( R = Q A Q^T \) is the eigenvalue decomposition of \( R \). The matrix \( \Lambda \) contains the eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \lambda_m \) on the diagonal and the matrix \( Q \) has the eigenvectors of the covariance as its columns. Note that \( Q^T \) is a rotation matrix that rotates and aligns the canonical bases of the observation space with the eigenvectors of the covariance, i.e., \( Q^T \) performs unscaled whitening. We can see from the last expression in (17) that \( x_i^Q \) for smaller values of \( i \) gets attenuated by a larger \( \lambda_i \) than \( x_i^Q \) for larger values of \( i \) (since \( \lambda_1 \geq \lambda_2 \geq \ldots \lambda_m \)). This implies that subspaces (and signals) with more energy in lower indices after unscaled whitening, suffer more attenuation and have a lower \( \| \bar{x} \|_2 \). Thus, subspaces (signals) closer to the higher-order eigenvectors of the covariance (i.e., eigenvectors corresponding to higher eigenvalues) end up having a lower \( \| \bar{x} \|_2 \).

With slight algebraic manipulations, we can see also that \( \| \bar{x} \|_2 \) (and other terms proportional to it) appears in the numerator of our test statistics. This dictates that for same signal-to-noise ratio (SNR), i.e., \( SNR = \frac{\| \bar{x} \|_2^2}{\sigma^2} \), a lower \( \| \bar{x} \|_2 \) will result in a lower detection probability. Thus we conclude that for the same SNR, subspaces with more energy closer to the higher-order eigenvectors of the covariance have lower detection probability and vice versa. This make intuitive sense: a subspace with more influence of noise (i.e., a subspace that lives closer to the higher-order eigenvectors of the covariance) will have a lower detection rate than a subspace with less influence of noise. A depiction of this observation is shown in Fig. 1.

Since the same quadratic forms appear in the numerator of the test statistics for active subspace detection, we also conclude from this discussion that the subspaces with more energy near the higher-order eigenvectors of the covariance have lower probability of correct classification.

V. Numerical Experiments

In this section, we present numerical experiments to examine the tightness of various bounds derived in this paper and verify the trends of performance metrics with respect to the geometry of the subspaces.

A. Synthetic data

We run Monte-Carlo experiments for signal and active subspace detection problems under different noise settings using synthetic data. Our general procedure for these experiments is as follows: we consider a union of three 2-dimensional subspaces in a 4-dimensional space. The subspaces are structured to highlight the effect of geometry between subspaces. The first and third subspaces are fixed and the angles between them are kept constant. As for the second subspace, we make different realizations of it with increasing principal angles with respect to the first subspace. This process is repeated for different levels of false alarm probabilities and SNR levels. The threshold for each false alarm level is determined numerically. When unmentioned, the false alarm rate is upper bounded at \( 1e^{-1} \) and the SNR is 10 dB. Each experiment is averaged over 10000 trials.

1) Signal detection problem: The receiver operating characteristic (ROC) curve of signal detection for the tests derived in this paper, with their respective upper and lower bounds, is given in Fig. 2. We can see that the lower union bound is much looser compared to the upper union bound and the lower bound derived in the paper. Moreover, Fig. 3 provides a comparison of different noise scenarios, from which we can conclude that the best performance is given under known noise statistics.

Next, Fig. 4 shows the effect of subspace angles on the detection probability. We see that the principal angles between whitened subspaces have indeed minimal effect on the detection probability under known noise settings. A similar behavior can also be seen for detection probability under other noise settings. However, we omit those plots in the interest of space.

To show the influence of the geometry of noise, we consider three 2-dimensional subspaces in a 4-dimensional space and randomly generate a noise covariance matrix. We then add noise to the eigenvectors of the noise covariance matrix and use them as bases for two of our subspaces. Starting from the eigenvectors corresponding to the smallest eigenvalues, we successively pick \( n \) noisy eigenvectors for subspaces \( S_1 \) and \( S_2 \) in the union. The bases of the third subspace \( S_3 \) are
because as we keep increasing the angles between
probability known noise settings. Notice that for subspace
can be seen in Fig. 6 for active subspace detection under
(whitened/empirically whitened) subspaces. This trend
increases with increasing principal angles between
demonstrate that the probability of correct classification
vice versa. This trend can be clearly seen in Fig. 5 for signal
detection under each noise setting.

2) Active subspace detection problem: We now
demonstrate that the probability of correct classification
increases with increasing principal angles between
(whitened/empirically whitened) subspaces. This trend
can be seen in Fig. 6 for active subspace detection under
known noise settings. Notice that for subspace \( S_2 \), the
probability \( P_{S_2}(\mathcal{H}_2) \) first increases then decreases. This is
because as we keep increasing the angles between \( S_1 \) and \( S_2 \),
\( S_2 \) keeps moving closer to \( S_3 \). Since \( S_1 \) and \( S_3 \) are fixed,
the angles that \( S_2 \) collectively makes with \( S_1 \) and \( S_3 \) first
increase and then decrease, resulting in the observed behavior
for \( P_{S_2}(\mathcal{H}_2) \). This insight is verified in Fig. 7 in terms of
the plot of \( \varphi_1^2 + \varphi_3^2 \) as a function of the number of trials.
Similar trends for probabilities are seen under other noise
settings, which are omitted due to space constraints.

Next, we plot the ROC curves for the probability of correct
classification and the various bounds derived under different
noise settings in Fig. 8. We see that the lower bounds derived
from [15] are very loose, compared to our lower bounds. A
comparison of the probability of correct classification under
different noise settings is provided in Fig. 3.

We further show the influence of noise geometry on active
subspace detection. We use the same setup as for signal
detection. We can see from Fig. 9 that subspaces closer to the
higher-order eigenvectors of the noise covariance have lower
detection probability, and vice versa.

3) Other observations: Fig. 10 shows the gap between
detection and classification probabilities for different noise
settings and different SNR levels. We can see that the gap
decreases for higher SNR levels.

We make a final observation by plotting the ROC curves
under various noise settings for different number of noise
samples. From Fig. 11, we see that the gap between probabil-
ities for known noise settings and unknown noise covariance
decreases as the number of noise samples increases. This is
since with increasing number of noise samples, our estimates
of noise statistics get better and we move closer to the regime
of known noise statistics.

B. Real-world datasets

In this subsection, we report results on some real-world
datasets that potentially conform to the UoS model. The first
dataset we consider is the Salinas ‘A’ Scene Hyperspectral
Data [19]. This data was acquired by a 224-band AVIRIS
sensor over Salinas Valley (California). There are six target
classes in the data. We assume each target class is lying
in a different subspace, thus modeling the set of targets as
belonging to a union of subspaces. To obtain the bases for
the subspaces, we randomly select 20 pixels belonging to
each target and use singular value decomposition (SVD) to
get the bases for 10-dimensional target subspaces. For the
Salinas ‘A’ Scene, the ground truth and the detected targets are
shown in Fig. 12. Assuming noise with unknown statistics and
false alarm probability upper bounded at \( 5e^{-4} \), the targets are
classified with the overall probability of correct classification
0.9116.

Next, the face of a subject with varying illumination con-
ditions has been shown to lie near a 9-dimensional subspace
[20]. Thus a set of subjects can be assumed to lie near a union
of subspaces. Using this assumption, for the Yale Database B
[21], we first obtain subspace bases for each subject by using
SVD on 18 randomly selected subject images. With these
bases and assuming unknown noise statistics, we correctly
identify subjects with probability 0.76 while upper bounding
the false alarm rate at \( 1e^{-3} \).

The third dataset in consideration is the Hopkins 155 motion
segmentation dataset [22], which consists of sequences of two
and three motions extracted from several videos. It has been
argued that different motion sequences extracted from tracking
a set of points in a video lie in 3-dimensional subspaces
[22]. We again use SVD on randomly selected sequences to
learn the subspace bases. Using the UoS model with unknown
noise statistics, the probability of correct classification over all sequences comes out to be 0.7664 by upper bounding the false alarm rate at 5e^{-2}.

C. Discussion

The experiments performed in Sec. V-A suggest that even though the bounds we obtain for probabilities of detection and correct classification are loose, they still predict the effect of subspace geometry on these probabilities correctly. In particular, we correctly predict that as the angles between whitened subspaces increase, the probabilities of detection and correct classification get higher, and vice-versa.

The results obtained in Sec. V-B for real-world datasets are not as good as some state-of-the-art algorithms (e.g., see [22]). However, there are certain advantages that our approach enjoys over the state-of-the-art methods. The first advantage is that our detection and classification methods allow control over the false alarm rate, which is not an option for other methods. Secondly, our method can work with just enough data, i.e., we just need enough samples to get good estimates of subspace bases and noise statistics. The third advantage is that our results explicitly cater to different levels of knowledge about the noise statistics and include that information in the detection and classification processes.

VI. CONCLUSION

We introduced GLRTs for signal and active subspace detection under the UoS model. We analyzed the performance of the derived test statistics under various levels of knowledge about noise and explained the effect of colored noise geometry and geometry between subspaces on the detection and classification capabilities of these statistics. This was achieved by obtaining bounds on detection and classification probabilities in terms of the angles between subspaces and the angles that subspaces make with the noise eigenvectors. We also validated the insights of our analysis through Monte-Carlo experiments and experiments with real-world datasets.

APPENDIX A

PROOF OF THEOREM 1

In the case of the signal detection problem, the likelihoods under the two hypotheses are given by:

\[ l_0(y) \propto \exp \left( -\frac{y^T R^{-1} y}{2\sigma^2} \right), \]
Fig. 6. In known noise settings, the probability of correct classification increases with the increasing principal angles between whitened subspaces.

Fig. 7. Sum of minimum principal angles subspace $S_2$ makes with subspace $S_1$ and subspace $S_3$. As $S_2$ moves away from $S_1$, the average of this sum increases initially and then decreases. The effect of this on the probability of classification $P_{S_2}(\hat{H}_2)$ can be seen in Fig. 6.

\[ l_1(y) \propto \exp \left( -\frac{(y-x)^T R^{-1} (y-x)}{2\sigma^2} \right) \]  
(18)

Since $x$ is unknown in (18), we replace it with its maximum likelihood (ML) estimate $\hat{x}$, which is given by $\arg \min_k (y - H_k \theta)^T R^{-1} (y - H_k \theta)$, where $P_{S_k} = H_k (H_k^T R^{-1} H_k)^{-1} H_k^T R^{-1}$ [7]. Consequently, the GLRT for this problem leads to the decision rule

\[ \frac{l_1(y)}{l_0(y)} \begin{cases} \overset{\mathcal{H}_1}{\gtrless_{\hat{k}}} \gamma & \Rightarrow \frac{y^T R^{-1} P_{S_{\hat{k}}} y}{2\sigma^2} \overset{\mathcal{H}_0}{\gtrless_{\bar{\gamma}}} \bar{\gamma}, \end{cases} \]  
(19)

where $\hat{k} = \arg \max_k (y^T R^{-1} P_{S_k} y)$, and $\bar{\gamma} = \log \gamma$ is the threshold used to control the probability of false alarm. Now, with appropriate substitutions, we can rewrite the final decision rule as:

\[ T_{2\sigma^2} \left( P_{S_{\hat{k}}} \right) \overset{\hat{k}}{\gtrless_{\mathcal{H}_0}} \bar{\gamma} \]  
with $\hat{k} = \arg \max_k z^T P_{S_k} z$.

Similarly, the likelihoods under different hypotheses for the active subspace detection problem are given by:

\[ l_0(y) \propto \exp \left( -\frac{y^T R^{-1} y}{2\sigma^2} \right), \]  
and

\[ l_k(y) \propto \exp \left( -\frac{(y-H_k \theta_k)^T R^{-1} (y-H_k \theta_k)}{2\sigma^2} \right), \]  
(20)

where $k = 1, \ldots, K_0$. Replacing the unknown $\theta_k$’s in (20) with their ML estimates $\theta_k = (H_k^T R^{-1} H_k)^{-1} H_k^T R^{-1} y$ [7]
Fig. 8. ROC curves for active subspace detection under the UoS model (labeled UoSD) and the derived bounds. All subfigures show three plots: the true classification probability under UoS, the lower bound on the classification probability computed numerically and the lower bound derived using [15]. Starting from the left, the sub-figures show the ROC curves under known noise statistics, unknown noise covariance and unknown noise statistics.

Fig. 9. Each subfigure shows that the closer a subspace is to the higher-order eigenvectors of the noise covariance, the lower is its classification probability $P_{H_k}(\hat{H}_k)$). The setup here is similar to the one for Fig. 5.

Fig. 10. Gap between the probability of detection and the probability of correct classification under various noise settings. The two rows have SNR levels 10 dB and 5 dB respectively. We can see that higher SNR results in a lower gap.

and comparing the generalized likelihoods lead to the rule

$$\frac{l_k(y)}{l_0(y)} \gtrless \frac{\mathbf{y}^T \mathbf{R}^{-1} \mathbf{P}_{\hat{S}_k} \mathbf{y}}{\mathbf{y}^T \mathbf{R}^{-1} \mathbf{P}_{\hat{S}_0} \mathbf{y}} \gtrless \frac{\gamma}{\gamma}.$$  \hspace{1cm} (21)

Making the same substitutions as before, the final decision rule becomes: $T_{z_k}^2 \left( \mathbf{P}_{\hat{S}_k} \right) \gtrless \gamma$.

APPENDIX B

PROOF OF THEOREM 2

The probability of false alarm in the case of signal detection is given by:

$$P_{FA} = P_{H_0}(\hat{H}_1) = P_{H_0}\left(T_{z_k}^2(\mathbf{P}_{\hat{S}_k}) > \bar{\gamma}\right)$$

$$\equiv Pr\left(T_{z_k}^2(\mathbf{P}_{\hat{S}_k}) > \bar{\gamma}\right) = Pr\left(\bigcup_{k=1}^{K_0} T_{z_k}^2(\mathbf{P}_{\hat{S}_k}) > \bar{\gamma}\right)$$

$$= \sum_{k=1}^{K_0} Pr\left(T_{z_k}^2(\mathbf{P}_{\hat{S}_k}) > \bar{\gamma}\right).$$
Finally since, the null hypotheses for both signal and active 
(22) explicitly since it contains tail probabilities of 
y by the union bound, i.e., 
the quadratic forms in (22) are neither independent nor fall 
distribution of the 
has a centered chi-squared distribution. This means that the 
w that 
91 unknown noise statistics. The targets were detected with the classification 
Fig. 12. This figure shows the ground truth (left) for different classes in 
Salinas A scene and the detected targets (right) using the UoS detector under 
unknown noise statistics. The targets were detected with the classification 
accuracy of 91.16% when upper bounding the false alarm rate at 5e^{-4}. 
\[
\sum_{k<j}^{K_0} \Pr \left\{ T_{w}^{2\sigma^2} (P_{S_j}) > \gamma \right\} \bigg\} + 
\cdots + (-1)^{K_0-1} \Pr \left\{ \bigcap_{k=1}^{K_0} \{ T_{w}^{2\sigma^2} (P_{S_k}) > \gamma \} \right\},
\] 
where (a) follows because y \mid H_0 = n. We cannot evaluate 
(22) explicitly since it contains tail probabilities of k-tuples 
\bigcap_{j=1}^{k} \{ w^T P S_j w > \gamma \}, \ k = 1, \ldots, K_0. In particular, notice 
that w^T P S_j w is a quadratic form of the variable P S_j w and has a centered chi-squared distribution. This means that the distribution of the k-tuple is the joint distribution of k 
dependent chi-squared variables. These distributions exist in the 
literature for either independent quadratic forms or dependent 
quadratic forms under particular settings [23]–[26]. However, the quadratic forms in (22) are neither independent nor fall 
under these settings. We instead resort to upper bounding (22) 
by the union bound, i.e., 
\[
P_{FA} = \Pr \left( \bigcup_{k=1}^{K_0} \{ T_{w}^{2\sigma^2} (P_{S_k}) > \gamma \} \right) \leq \min \left\{ 1, \sum_{k=1}^{K_0} \Pr \left( T_{w}^{2\sigma^2} (P_{S_k}) > \gamma \right) \right\}.
\] 
Finally since, the null hypotheses for both signal and active 
subspace detection problems are the same, they end up having 
the same probability of false alarm. 
Next, for the probability of detection P_D, note that 
\[
P_{S_k} (\hat{H}_1) = P_{S_k} \left( \bigcup_{i=1}^{K_0} \{ T_{z}^{2\sigma^2} (P_{S_i}) > \gamma \} \right) 
= \sum_{i=1}^{K_0} P_{S_k} \left( T_{z}^{2\sigma^2} (P_{S_i}) > \gamma \right) 
- \sum_{i<j} P_{S_k} \left( \{ T_{z}^{2\sigma^2} (P_{S_i}) > \gamma \} \bigcap \{ T_{z}^{2\sigma^2} (P_{S_j}) > \gamma \} \right) 
- \cdots + (-1)^{K_0-1} P_{S_k} \left( \bigcap_{i=1}^{K_0} \{ T_{z}^{2\sigma^2} (P_{S_i}) > \gamma \} \right)
\leq \min \left\{ 1, \sum_{i=1}^{K_0} P_{S_k} \left( T_{z}^{2\sigma^2} (P_{S_i}) > \gamma \right) \right\},
\] 
where (c) is again obtained using the union bound since the 
k-tuples in (b) cannot be expressed in closed form. Further, the lower bound in (6) follows from [27, Theorem 1]. 
Finally for the probability of classification P_C, we have: 
P_{H_k} (\hat{H}_k) = P_{S_k} \left( \{ T_{z}^{2\sigma^2} (P_{S_k}) > \gamma \}, \bigcap_{j=1,j\neq k}^{K_0} \{ T_{z} (P_{S_j}, P_{S_k}) > 1 \} \right).
\] 
Since (25) cannot be evaluated explicitly as it involves dependent 
definite and indefinite quadratic forms, we lower bound it by using the Fréchet inequalities [28]: 
P_{H_k} (\hat{H}_k) \geq \max \left\{ 0, P_{S_k} \left( T_{z}^{2\sigma^2} (P_{S_k}) > \gamma \right) 
+ \sum_{j=1,j\neq k}^{K_0} P_{S_k} \left( T_{z} (P_{S_k}, P_{S_j}) > 1 \right) \right\}.
\] 
We conclude by noting that one could use [15, Lemma 1] 
to further lower bound (26). Specifically, 
P_{S_k} (T_{z} (P_{S_k}, P_{S_j}) > 1) = P_{S_k} \left( z^T P_{S_j}^\perp z - z^T P_{S_j}^\perp z > 0 \right) 
= 1 - P_{S_k} \left( z^T P_{S_j}^\perp z - z^T P_{S_j}^\perp z < 0 \right) 
\geq 1 - Q_1^{\frac{1}{2} (1-2\eta_0)} \sqrt{\lambda_{i\mid k}} - \Psi(n, \lambda_{j\mid k}),
\] 
where \( \lambda_{i\mid k} = \frac{1}{2} z^T P_{S_j}^\perp z \) when \( z \in S_k \). This leads to 
P_{H_k} (\hat{H}_k) \geq \max \left\{ 0, P_{S_k} \left( T_{z}^{2\sigma^2} (P_{S_k}) > \gamma \right) \right\} - \sum_{j \neq k} Q_1^{\frac{1}{2} (1-2\eta_0)} \sqrt{\lambda_{j\mid k}} - \sum_{j \neq k} \Psi(\eta_0, \lambda_{j\mid k}). \]

APPENDIX C
PROOF OF THEOREM 3
The results derived in this appendix closely follow the derivations in [10]. The likelihood of \( \xi_p \) is given by: 
\[
l(\xi_p) = \frac{1}{(2\pi)^{m/2} |\mathbf{R}|} \exp \left\{ -\frac{1}{2} \xi_p^T \mathbf{R}^{-1} \xi_p \right\},
\] 
which is used to get the joint likelihoods under each hypothesis \( H_1 \) and \( H_0 \): \( l_0(y, \Xi) \) and \( l_1(y, \Xi) \), where \( \Xi = \)
[\xi_1, \xi_2, \ldots, \xi_{N_0}]$. From these joint likelihoods, the ML estimate of $R$ under $H_1$ and $H_0$ can be computed as $\hat{R}_1 = \frac{N_0}{N_0 + 1} \Sigma + \frac{(y - x)(y - x)^T}{\sigma^2(N_0 + 1)}$ and $\hat{R}_0 = \hat{R}_1 |_{x = 0}$, respectively.

Now, following the same steps as in the proof of Theorem 1, we can proceed to calculate the final decision rule for signal detection as $T_k z \sim \mathcal{N}(\hat{P}, \hat{\Sigma}) H_1 \overset{H_0}{\sim} \hat{\gamma}$, where $\hat{k} = \arg \max_k (\hat{z}^T \hat{P} \hat{z})$ and $\hat{\gamma} = \log \hat{\gamma}$.

Next, note that the likelihood in (28) combined with the likelihoods in (20) also provide the joint likelihoods under each hypothesis for the active subspace detection problem. With trivial algebraic manipulations, the ML estimates of $R$ in this case can be expressed as:

$$H_0: \hat{R}_0 = \frac{N_0}{N_0 + 1} \Sigma + \frac{yy^T}{\sigma^2(N_0 + 1)},$$

$$H_k: \hat{R}_k = \frac{N_0}{N_0 + 1} \Sigma + \frac{(y - x)(y - x)^T}{\sigma^2(N_0 + 1)}. \quad (29)$$

where $x | H_k = H_k \theta_k$. Using the ML estimates of $R$ and the joint likelihoods, we can calculate the decision rule (similar to the proof of Theorem 1) as $T_k z \sim \mathcal{N}(\hat{P}, \hat{\Sigma}) H_1 \overset{H_0}{\sim} \hat{\gamma}$. \hfill \blacksquare

**APPENDIX D**

**PROOF OF THEOREM 5**

This proof uses derivations from the proof of Theorem 3. The only additional estimate we need is for the variance $\sigma^2$ which can be found from the joint likelihoods with the estimate $\hat{R}$ substituted in them. This results in:

$$\hat{\sigma}^2 | H_1 = \frac{N_0 - m + 1}{N_0 m} (y - x)^T \Sigma^{-1} (y - x),$$

$$\hat{\sigma}^2 | H_0 = \frac{N_0 - m + 1}{N_0 m} y^T \Sigma^{-1} y. \quad (30)$$

$T_k (\hat{P}, \hat{S}_k) \overset{H_1}{\sim} \hat{\gamma}$, where $\hat{k} = \arg \max_k (\hat{z}^T \hat{P} \hat{z})$ and $\hat{\gamma} = \log \hat{\gamma}$.

Similarly, the active subspace detection problem takes the same from as in Theorem 3 with an additional unknown variable $\sigma^2$. However, we can use the previously calculated ML estimates of $\sigma^2$, $R$, and $x$ to arrive at the final decision rule of $T_k (\hat{P}, \hat{S}_k) \overset{H_1}{\sim} \hat{\gamma}$. \hfill \blacksquare

**APPENDIX E**

**PROOF OF THEOREM 7**

To get a better understanding of the parameters that affect the probability of correct classification, we analyze the terms $P_{S_k} (T_k (\hat{P}, \hat{S}_k), \hat{S}_j) > 1$ in (7) since these terms characterize the interactions between the whitened subspaces. Assuming $x \in S_k$, notice that:

$$T_k (\hat{P}, \hat{S}_k) > 1 \Leftrightarrow z^T \hat{P} \hat{S}_k z > z^T \hat{P} \hat{S}_j z$$

$$\Leftrightarrow (\hat{x} + w)^T \hat{P} \hat{S}_k (\hat{x} + w) > (\hat{x} + w)^T \hat{P} \hat{S}_j (\hat{x} + w)$$

$$\Leftrightarrow w^T \hat{P} \hat{S}_k w - w^T \hat{P} \hat{S}_j w > -\hat{x}^T \hat{P} \hat{S}_k \hat{x} - \hat{x}^T \hat{P} \hat{S}_j \hat{x} - 2w^T \hat{x} + 2w^T \hat{P} \hat{S}_j \hat{x}, \quad (31)$$

where $\hat{x} = R^{-1} x$ is the whitened signal. We now focus on the quadratic forms $x^T \hat{P} \hat{S}_k x$ and $w^T \hat{P} \hat{S}_j x$ in (31) because these are the terms where different subspaces interact with each other and that can be expressed in terms of the principal angles between whitened subspaces. Using the derivation provided in Appendix F, we can bound $P_{S_k} (T_k (\hat{P}, \hat{S}_k), \hat{S}_j) > 1$ as:

$$P_{S_k} (w^T \hat{P} \hat{S}_k w - w^T \hat{P} \hat{S}_j w > -\hat{x}^T \hat{P} \hat{S}_k \hat{x} - \hat{x}^T \hat{P} \hat{S}_j \hat{x} - 2w^T \hat{x} + 2w^T \hat{P} \hat{S}_j \hat{x})$$

$$\geq P_{S_k} (\|n\|^2 (\cos^2 \psi_k - \cos^2 \psi_j) >$$

$$-\sum_{i=1}^{n} \theta_{ki}^2 \sin^2 \phi_i \phi_{j} + 2 \sum_{i<p} |\theta_{ki} \theta_{kp}| \cos \phi_i \phi_{j} \cos \phi_p$$

$$+ \|n\|^2 \cos \psi_j \left(\frac{\sum_{i=1}^{n} \theta_{ki}^2 \cos^2 \phi_i \phi_{j}}{2} - \|n\|^2 \cos \psi_j \left(\|\sum_{i=1}^{n} \theta_{ki}^2 \cos^2 \phi_i \phi_{j}\|^2 \right)^{2}\right), \quad (32)$$

where $\phi_i \phi_{j}$ is the angle that $g_k^i$ (i-th basis vector of whitened subspace $\hat{S}_k$, i.e., i-th column of $G_k$) makes with the whitened subspace $\hat{S}_j$, i.e., $\phi_i \phi_{j}$ is the angle between $g_k^i$ and $g_p^j$ (i.e., the angles between the i-th and p-th basis vectors of whitened subspaces $\hat{S}_k$ and $\hat{S}_j$ after projection onto the whitened subspace $\hat{S}_j$) and $\psi_j$ is the angle between $w$ and the whitened subspace $\hat{S}_j$.

This lower bound on $P_{S_k} (T_k (\hat{P}, \hat{S}_k), \hat{S}_j) > 1$ is dependent on the principal angles $\phi_i \phi_{j}$ between the whitened subspaces $\hat{S}_k$ and $\hat{S}_j$. In particular, we can see that as the principal angles $\phi_i \phi_{j}$ increase, the bound on the right hand side of the inequality (a) in (31) becomes smaller. This implies that lower bound on the tail probability in (32) becomes larger as the principal angles increase. This trend holds for all pairs of whitened subspaces $\hat{S}_j$ and $\hat{S}_k$ (for $j, k = 1, \ldots, K_0$ and $j \neq k$). This means that the lower bound for $P_{H_k} (\hat{H}_k)$ in (7) also increases with increasing principal angles between the whitened subspaces.

We conclude by noting that this trend can also be derived from the lower bound expression in Remark 1. The quantities $Q(\cdot)$ and $\Psi(\cdot)$ in that expression are functions of $\lambda_j \lambda_i$ and decrease monotonically as $\lambda_j \lambda_i$ is increased [15]. This means that an increase in $\lambda_j \lambda_i$ will result in an increase in the probability of correct classification. Since $\lambda_j \lambda_i$ can be expressed as $\lambda_j \lambda_i = \frac{1}{\lambda} z^T \hat{P} \hat{S}_k \hat{x} = \frac{1}{\lambda} (z^T \hat{z} - z^T \hat{P} \hat{S}_k \hat{z}) = \frac{1}{\lambda} (z^T \hat{z} - \hat{x}^T \hat{P} \hat{S}_k \hat{x} - 2w^T \hat{P} \hat{S}_k \hat{x} - 2w^T \hat{P} \hat{S}_j \hat{x} - w^T \hat{P} \hat{S}_j \hat{x})$, one can use results from Appendix F to once again argue that as the angles between whitened subspaces increase, the lower bound on $\lambda_j \lambda_i$ increases which in turn results in larger (lower) bound on the probability of correct classification. \hfill \blacksquare

**APPENDIX F**

**PROBABILITY BOUND ON RATIO OF QUADRATIC FORMS**

The outline of our procedure for deriving a lower bound on the probability of the comparison of quadratic forms is as follows: we first express $\hat{x}^T \hat{P} \hat{S}_k \hat{x}$ and $w^T \hat{P} \hat{S}_j \hat{x}$ in terms of the principal angles between whitened subspaces. We then obtain upper bounds on these quadratic forms that depend on
the principal angles. Next we put these upper bounds in the expression for the probability of correct classification of the individual subspaces and finally we derive a lower bound on the probability of correct classification that is dependent on the principal angles between the whitened subspaces.

Let’s consider $\hat{\mathbf{x}}^T \mathbf{P}_{S_j} \hat{\mathbf{x}}$ when $\mathbf{x} \in S_k$:

$$\hat{x}^T P_{S_j} \hat{x} = \|P_{S_j} \hat{x}\|^2 = \|P_{S_j} G_k \theta_k\|^2$$

where $\varphi_i^{(k,j)}$ are as defined in Appendix E. Note that (a) in (33) follows from $\hat{x} = G_k \theta_k$, (b) uses the notation $g_k^{k-j} = P_{S_j} g_i$, and (c) uses the identity $|a + b|^2 = |a|^2 + |b|^2 + 2a b$.

Now, if we assume $g_k^{k-j}$’s to be the unit-norm principal vectors of $S_k$, we can bound (33) as $\hat{x}^T P_{S_j} \hat{x} \leq \sum_{i=1}^n \theta_k^2 \cos^2 \varphi_i^{(k,j)} + 2 \sum_{i<p} \theta_k \theta_{k'} \cos \varphi_i^{(k,j)} \cos \varphi_{i'}^{(k,j)}$. Similarly, we have $w^T P_{S_j} \hat{x} \leq \|w\|^2 \cos \psi_j \left( \sum_{i=1}^n \theta_k^2 \cos^2 \varphi_i^{(k,j)} \right)^{\frac{1}{2}} + \|w\| \cos \psi_j \left( \sum_{i<p} \theta_k \cos \varphi_i^{(k,j)} \theta_{k'} \cos \varphi_{i'}^{(k,j)} \right)^{\frac{1}{2}}$, where we have used the fact that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ and $\psi_j$ is the angle between $w$ and the whitened subspace $S_j$. Substituting these upper bounds in (31) we get:

$$\|w\|^2 (\cos^2 \psi_k - \cos^2 \psi_j) > \sum_{i=1}^n \theta_k^2 \sin^2 \varphi_i^{(k,j)} + 2 \sum_{i<p} \theta_k \theta_{k'} \cos \varphi_i^{(k,j)} \cos \varphi_{i'}^{(k,j)} + \|w\|^2 \cos \psi_j \left( \sum_{i=1}^n \theta_k^2 \cos^2 \varphi_i^{(k,j)} \right)^{\frac{1}{2}} - \|w\| \cos \psi_k \left( \sum_{i=1}^n \theta_k^2 \right)^{\frac{1}{2}} + \|w\|^2 \cos \psi_j \left( \sum_{i<p} \theta_k \theta_{k'} \cos \varphi_i^{(k,j)} \cos \varphi_{i'}^{(k,j)} \right)^{\frac{1}{2}}$$

which can be used to obtain (32).

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