SELF-SIMILARITY IN GENERAL RELATIVITY

B. J. Carr
Astronomy Unit
Queen Mary and Westfield College
University of London
London, England

and

A. A. Coley
Department of Mathematics
Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
Canada B3H 3J5

Abstract. The different kinds of self-similarity in general relativity are discussed, with special emphasis on similarity of the “first” kind, corresponding to spacetimes admitting a homothetic vector. We then survey the various classes of self-similar solutions to Einstein’s field equations and the different mathematical approaches used in studying them. We focus mainly on spatially homogenous and spherically symmetric self-similar solutions, emphasizing their possible roles as asymptotic states for more general models. Perfect fluid spherically symmetric similarity solutions have recently been completely classified, and we discuss various astrophysical and cosmological applications of such solutions. Finally we consider more general types of self-similar models.

1. Introduction

The purpose of this review is to summarize recent developments in the study of self-similar solutions in general relativity and to discuss recent applications of these solutions. It thus combines a mathematical and physical approach to the subject. In these introductory remarks, we will first discuss the issues involved in rather broad terms, this serving to delineate the general scope of the review. We will then present some technical mathematical background to elucidate the nature of self-similarity. Finally, as preparation for the more
detailed discussion in later sections, we will present an overview of the different types of solutions.

A. Overview.

Forms of self-similarity

In Newtonian hydrodynamics self-similar solutions occur when the physical quantities depend on functions of $x/l(t)$, where $x$ and $t$ are independent space and time variables and $l$ is a time-dependent scale. This means that the spatial distribution of the characteristics of motion remains similar to itself at all times during the motion and that all dimensional constants entering the initial and boundary conditions vanish or become infinite (Barenblatt and Zeldovich 1972). When the form of the self-similar asymptotics can be obtained from dimensional considerations, the solutions are referred to as self-similar solutions of the first kind (Barenblatt and Zeldovich 1972). Examples of these appear in the study of strong explosions (Sedov 1946 and 1967, Taylor 1950) and thermal waves (Zeldovich and Kompaneets 1950, Barenblatt 1952, Zeldovich and Raizer 1963). Otherwise, the solutions are referred to as self-similar solutions of the more general second kind. Self-similar solutions also describe the “intermediate-asymptotic” behaviour of solutions in the region in which they no longer depend on the details of the initial and/or boundary conditions but in which the system may still be far from equilibrium.

In general relativity the concept of self-similarity is perhaps less straightforward, since in principle there are various ways of generalizing the Newtonian concept and also a covariant characterization is required. First, it is important to distinguish between different types of self-similarity. The existence of self-similar solutions of the first kind is related to conservation laws and to the invariance of the problem with respect to the group of similarity transformations of quantities with independent dimensions. This can be characterized within general relativity by the existence of a homothetic vector. In this case, one assumes a certain regularity of the limiting process in passing from the original non-self-similar regime to the self-similar regime. However, in general such a passage need not be regular, so the expressions for the self-similar variables are not determined from dimen-
sional analysis alone. Solutions are then called self-similar solutions of the second kind. As in the Newtonian context, a characteristic of these solutions is that they contain dimensional constants which are not determined from the conservation laws but can be found by matching the self-similar solutions with the non-self-similar solutions whose asymptotes they represent (Barenblatt and Zeldovich 1972). Most of this review will be concerned with self-similarity of the first kind but we shall consider more general kinds of self-similarity in Section 5. In particular, the important example of kinematic self-similarity (Carter and Henriksen 1989, Coley 1997a) will be reviewed.

Second, in general relativity one must distinguish between geometrical and physical self-similarity. Geometrical similarity is a property of the spacetime metric, whereas physical similarity is a property of the matter fields. As discussed in Section 1B, these need not be equivalent and the relationship between them also depends on the nature of the matter. In much of this review we will be focussing on perfect fluid solutions admitting a homothetic vector and in this case geometrical self-similarity implies physical self-similarity. However, some of the discussion will pertain to more general fluids.

Third, it is important to distinguish between continuous and discrete self-similarity. For example, in the spherically symmetric case, the continuous kind involves a similarity variable, $\zeta$, with all dimensionless quantities $\Psi(\zeta)$ being invariant under any coordinate transformation for which $\zeta$ is constant. A discrete self-similarity is then one in which all dimensionless variables $\Psi$ repeat themselves on some spacetime scale: this condition can usually be written as $\Psi(\tau, \zeta) = \Psi(\tau - n\Delta, \zeta)$ for some constant $\Delta$, where $n$ is an integer and $\tau$ is another variable. Since one recovers continuous self-similarity in the limit $\Delta \to 0$, continuous self-similarity can be regarded as a special case of discrete self-similarity and is much easier to deal with mathematically. Although we are mainly concerned with continuous self-similarity in this review, discrete self-similarity is of great interest in its own right and, as we will see in Section 4C, a focus of considerable interest in the context of critical phenomena.

Relevance of self-similarity

There are two important reasons for studying self-similar solutions of the Einstein field
equations (EFEs). First, the assumption of self-similarity reduces the mathematical complexity of the governing differential equations, often leading in problems of physical interest to the reduction of partial differential equations (PDEs) to ordinary differential equations (ODEs). This makes such solutions easier to study mathematically. Indeed self-similarity in the broadest (Lie) sense refers to an invariance which allows such a reduction.

Second, self-similar solutions play an important role in describing the asymptotic properties of more general models. This is discussed in detail in Section 2B for spatially homogeneous models and the same idea may apply in some spherically symmetric contexts. For example, the expansion of the Universe from the big bang and the collapse of a star to a singularity might both exhibit self-similarity in some form since it might be expected that the initial conditions would be “forgotten” in some sense. In the cosmological context, the suggestion that fluctuations might naturally evolve from complex initial conditions via the Einstein equations to self-similar form has been termed the “similarity hypothesis” (Carr 1993). This certainly does not apply in all circumstances but it may do so whenever one has non-linear perturbations and non-zero pressure. One of our aims here is to discuss under what circumstances the similarity hypothesis might hold. As a first step in this direction, we discuss the stability of self-similar solutions to non-self-similar perturbations in Section 4D.

Classes of self-similar solutions

As discussed in Section 1C, there are many different classes of self-similar solutions. In particular, there are self-similar spatially homogeneous models, which will be reviewed in Section 2, and self-similar spherically symmetric models, which have now been classified completely (Carr and Coley 1998a) and will be reviewed in Section 3. There are other exact homothetic models, including for example self-similar $G_2$ models and plane-symmetric models, although these are discussed in less detail.

These different types of solutions tend to attract different types of mathematical analysis. Recent studies of the spatially homogenous models often use a dynamical systems approach. This is because the governing ODEs reduce to an autonomous system and this approach facilitates the qualitative analysis of the models. As discussed in Section 3A, some
studies of the spherically symmetric models have also used a dynamical systems approach. However, because of the mathematical simplicity involved in this case, one can often write the solutions explicitly, as emphasized in Section 3C, and this may offer a more physically intuitive approach.

Self-similar models can also be analyzed using either a tetrad or coordinate approach, a variety of preferred gauges (e.g., coordinate systems), and a number of natural variables. In the spherically symmetric case, in particular, one can use three possible coordinate approaches. The first one uses “comoving” coordinates and was the one pioneered by Cahill and Taub (1973) and then followed by Carr and Henriksen and coworkers. The second approach, followed by Bogoyavlenski and coworkers, uses “homothetic” coordinates, in which the homothetic vector is along either the time or space axis. In this case, the equations can be reduced to that of a dynamical system and one can exploit results derived from the study of hypersurface homogeneous models. A third approach uses “Schwarzschild” coordinates and was included in the analysis of Ori and Piran (1990). In Section 3 we will mainly emphasize the first approach.

Applications of self-similarity

Besides their intrinsic mathematical interest, there are many applications of similarity solutions in astrophysics and cosmology. The astrophysical applications include gravitational collapse and the occurrence of naked singularities (Section 4A). Indeed, most of the examples of naked singularities in the literature involve self-similar solutions. The cosmological applications include features of gravitational clustering and cosmic voids (Section 4B). These features are particularly relevant because they allow the similarity hypothesis to be tested observationally (Section 4D). A distinctively relativistic application includes the crucial role of self-similar solutions in critical phenomena (Section 4C), surely one of the most exciting developments in general relativity in recent years.

We hope that the discussion in Section 4 will prove useful in drawing connections between different areas of research. We do not pretend that our selection of applications is complete - nearly all of them are drawn from the spherically symmetric context - but it should be broad enough to give a taste of the subject. Also we will be discussing topics which involve
different areas of expertise, so we will attempt to avoid too many technicalities. Note that we shall restrict our attention to general relativity in this review, although there are many applications of self-similar solutions in other theories of gravity.

B. Mathematical Background.

In this review we shall be particularly concerned with the case in which the source of the gravitational field is a perfect fluid; i.e., the energy-momentum tensor is given by

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab}, \quad (1.1)$$

where $u^a$ is the normalized fluid 4-velocity, $\mu$ is the density and $p$ is the pressure. Unless stated otherwise, a linear barotropic equation of state of the form

$$p = \alpha\mu \quad (1.2)$$

will be assumed, where the constant $\alpha$ satisfies $0 \leq \alpha \leq 1$ for ordinary matter and $-1 \leq \alpha < -1/3$ for models that undergo inflation. Causality requires $-1 \leq \alpha \leq 1$.

Cahill and Taub (1971) were the first to study perfect fluid similarity solutions in general relativity. They did so in the cosmological context under the assumption of spherical symmetry. They assumed that all dependent variables are functions of a single dimensionless combination of $r$ and $t$ (i.e., the solution is invariant under the transformation $\tilde{t} = at$, $\tilde{r} = ar$ for any constant $a$) and that the model contains no other dimensional constants. This corresponds to the existence of a similarity of the first kind and they showed that it can be invariantly formulated in terms of the existence of a homothetic vector.

For a general spacetime a proper homothetic vector (HV) is a vector field $\xi$ which satisfies

$$\mathcal{L}_\xi g_{\mu\nu} = 2g_{\mu\nu}, \quad (1.3)$$

where $g_{\mu\nu}$ is the metric and $\mathcal{L}$ denotes Lie differentiation along $\xi$. An arbitrary constant on the right-hand-side of (1.3) has been rescaled to unity. If this constant is zero, i.e., $\mathcal{L}_\xi g_{\mu\nu} = 0$, then $\xi$ is a Killing vector (KV). A homothetic motion or homothety captures the geometric notion of “invariance under scale transformations”. From (1.3) it follows that

$$\mathcal{L}_\xi R^a_{\ bcd} = 0, \quad (1.4)$$
and hence

\[ \mathcal{L}_\xi R_{ab} = 0, \quad \mathcal{L}_\xi G_{ab} = 0. \tag{1.5a,b} \]

A vector field \( \xi \) that satisfies equation (1.4) is called a curvature collineation, one that satisfies equation (1.5a) is called a Ricci collineation, and one that satisfies equation (1.5b) is called a matter collineation.

For a perfect fluid, it follows from equation (1.3) and the EFEs that the physical quantities transform according to

\[ \mathcal{L}_\xi u^a = -u^a, \tag{1.6} \]

and

\[ \mathcal{L}_\xi \mu = -2\mu, \quad \mathcal{L}_\xi p = -2p, \tag{1.7} \]

where

\[ \mathcal{L}_\xi T_{ab} = 0 \tag{1.8} \]

(Cahill and Taub 1971, Eardley 1974). Perfect fluid spacetimes admitting a HV within general relativity have been comprehensively studied by Eardley (1974). In such spacetimes equations (1.6) and (1.7) imply that all physical quantities transform according to their respective dimensions, so “geometrical” and “physical” self-similarity coincide. However, this need not always be the case, and it is unfortunate that a rather misleading terminology has been introduced (Eardley 1974) which equates self-similarity with the existence of a homothety and which refers to the homothetic group as the similarity group.

The question of whether the matter field exhibits the same symmetries as the geometry within general relativity is called the symmetry “inheritance” problem. If the source is not a perfect fluid, then the spacetime symmetries need not be inherited by the matter (Coley and Tupper 1989), so a homothety is a purely geometric property of a spacetime rather than a self-similarity. In this case, it is only through the EFEs that properties of the matter like similarity can be inferred. On the other hand, if the matter fields exhibit self-similarity, then the EFEs place restrictions on the geometry. For example, if the self-similarity is of the first kind (i.e., resulting from dimensional considerations), then \( \mathcal{L}_\xi T_{ab} = 0 \) implies

\[ \mathcal{L}_\xi G_{ab} = 0, \tag{1.9} \]
in which case $\xi$ is a matter collineation (Kramer et al. 1980, Carot et al. 1994). Although a HV satisfies equation (1.9), a matter collineation is not necessarily a HV. Indeed, equation (1.9) need not imply equation (1.5a), which in turn need not imply equation (1.4); i.e., neither a curvature collineation nor even a Ricci collineation need be a HV. The general problem of determining the constraints on the form of the metric from an equation like (1.9) has been termed the “inverse” symmetry inheritance problem, and the study of matter collineations in which $\mathcal{L}_\xi C^a_{bcd} \neq 0$ (otherwise $\xi$ is necessarily a HV) was recently undertaken by Carot et al. (1994).

C. Spacetimes Admitting a Homothetic Vector.

The differential geometric properties of HVs were studied by Yano (1955). The totality of HVs on a spacetime form a Lie algebra $H_n$ of dimension $n$ which (if $H_n$ is non-trivial) contains an $(n - 1)$ dimensional subalgebra of KVs, $G_{n-1}$. Except when the spacetime is conformal to a “generalized plane-wave” spacetime, it follows that if the orbits of $H_n$ are $r$-dimensional, then the orbits of $G_{n-1}$ are $(r - 1)$-dimensional (Eardley 1974). If, in addition, the spacetime is not conformally flat, then it is conformally related to a spacetime for which the Lie algebra $H_n$ is the Lie algebra of KVs (Defrise-Carter 1975). In the trivial case of a (locally) flat spacetime, the dimension of the homothetic algebra is eleven and that of its associated Killing subalgebra is ten. The orbits of the homothetic group and the isometry group can coincide only if they are four-dimensional or three-dimensional and null, the resulting spacetime is consequently either locally flat or is a special type of “generalized plane-wave” spacetime (cf. Hall and Steele 1990).

Vacuum spacetimes admitting a HV were studied by McIntosh (1975), who showed that a non-flat vacuum spacetime can only admit a non-trivial HV if that HV is neither null nor hypersurface-orthogonal. He also showed that a perfect fluid spacetime cannot admit a non-trivial HV which is orthogonal to the fluid 4-velocity unless $p = \mu$. If a spacetime admits a non-trivial HV and there is an equation of state of the form $p = p(\mu)$, then necessarily $p = \alpha \mu$ (Cahill and Taub 1971, Wainwright 1985); i.e., equation (1.2) results from equation (1.7). We note that homothetic initial data is preserved by the EFEs (Eardley 1974). In the case of radiation (with $p = \frac{1}{3}\mu$), $T \equiv T^{ab}g_{ab} = 0$ and the existence of a HV implies the
existence of a conserved current. In general, if we define $P^a = T^{ab}\xi_b$, energy-momentum conservation implies $P^a;_a = T$. For radiation, $P^a = \frac{4}{3}(4u^a w^b \xi^b + \xi^a)$ and so $P^a;_a = 0$.

In addition to flat Minkowski spacetime, all FRW models admit a timelike HV in the special case of matter with $p = -\frac{1}{3} \mu$ (Eardley 1974). Otherwise, only the flat model admits a HV, and this occurs for all $p = \alpha \mu$ models in which the scale function has power-law dependence on time (Maartens and Maharaj 1986).

There are many exact self-similar spatially homogeneous and spherically symmetric solutions, and these will be reviewed in Sections 2 and 3, respectively. In addition, there are a number of exact homothetic $G_2$ models. In these solutions there are two commuting spacelike KVs acting orthogonally transitively, which together with the HVs form a three-dimensional homothety group, $H_3$. Exact solutions have been found for the cases in which the orbits of $H_3$ are spacelike (Eardley 1974, Luminet 1978, Chao 1981, Carot and Sintes 1997), timelike (Hewitt and Wainwright 1990, Hewitt, Wainwright and Goode, 1988, Hewitt, Wainwright and Glaum 1991, Uggla 1992, Carot and Sintes 1997), and null (Carot and Sintes 1997), although the emphasis in the timelike case has primarily been on the qualitative analysis of the models (cf. Wainwright and Ellis 1996). Recently, Haager and Mars (1998) have analyzed algebraically general, non-diagonal $G_2$ self-similar tangent dust models (which are tangent in the sense that the fluid flow is tangent to the orbits of the $H_3$). Homothetic cylindrically symmetric perfect fluid solutions also exist, but to our knowledge these only occur as special cases of known (more general) abelian $G_2$ solutions. Abelian $G_2$ models in which one of the commuting KVs is timelike (i.e., the stationary axisymmetric case) are also of astrophysical interest (Kramer et al. 1980). Finally, exact self-similar solutions have been found for plane symmetric spacetimes (Taub 1972, Shikin 1979, Chao 1981, Foglizzo and Henriksen 1993), hyperbolically symmetric spacetimes (Chao 1981), Weyl spacetimes (Godfrey 1972), and diagonal hypersurface homogeneous spacetimes (Uggla et al. 1995). Carot and Sintes (1997) have recently studied spacetimes admitting an $H_3$, in which the two (spacelike) KVs are not necessarily orthogonally transitive nor commuting (and in which the perfect fluid does not necessarily admit of a barotropic equation of state), or an $H_4$. This extends and unifies previous work cited above.
2. SELF-SIMILAR MODELS AS ASYMPTOTIC STATES OF MORE GENERAL MODELS

Self-similar models are often related to the asymptotic states of more general models (Hsu and Wainwright 1986). In particular, self-similar models play an important role in the asymptotic properties of spatially homogeneous models, spherically symmetric models, $G_2$ models and silent universe models (Bruni et al. 1995). In this section, we will focus on spatially homogeneous models, which have been discussed in Ellis and MacCallum (1969) and Kramer et al. (1980), and $G_2$ models, which have been discussed in Kramer et al. (1980). The terminology used follows that of these references. For the definitions of any technical terms in dynamical system theory used below, reference should be made to any modern textbook or Wainwright and Ellis (1997; WE). We note that the self-similar Bianchi models discussed below are transitively self-similar, in the sense that the orbits of the $H_4$ are the whole spacetime, while the self-similar $G_2$ and spherically symmetric models are not transitively self-similar (unless they admit additional symmetry). However, the three exact power-law self-similar spherically symmetric solutions discussed in Section 3C are transitively self-similar.

A. Spatially Homogeneous Models.

Many people have studied self-similar spatially homogeneous models, both as exact solutions and in the context of qualitative analyses (see WE and Coley 1997b and references therein). Exact spatially homogeneous solutions were first displayed in early papers; however, it was not until after 1985 that many of them were recognized by Wainwright (1985) and Rosquist and Jantzen (1985) as being self-similar [although Eardley (1974) first pointed out that some simple Bianchi models are self-similar and appears to have been the first to have introduced the notion of asymptotic self-similarity in cosmology]. The complete set of self-similar orthogonal spatially homogeneous perfect fluid and vacuum solutions were given by Hsu and Wainwright (1986) and they have also been reviewed in WE. Kantowski-Sachs models were studied by Collins (1977). Exact self-similar solutions were given by Burd and Coley (1994) and Coley and van den Hoogen (1994a, 1995) for imperfect fluid sources and by Feinstein and Ibañez (1993), Ibañez et al. (1993) and Coley et al. (1997) for scalar field models with an exponential potential.
Spatially homogeneous models have attracted considerable attention since the governing equations reduce to a relatively simple finite-dimensional dynamical system, thereby enabling the models to be studied by standard qualitative techniques. Planar systems were initially analyzed by Collins (1971, 1974) and a comprehensive study of general Bianchi models was made by Bogoyavlenski and Novikov (1973) and Bogoyavlenski (1985) and more recently (using automorphism variables and Hamiltonian techniques) by Jantzen and Rosquist (Jantzen 1984, Rosquist 1984, Jantzen and Rosquist 1986, Rosquist and Jantzen 1988, Rosquist et al. 1990). Perhaps the most illuminating approach has been that of Wainwright and collaborators (Hsu and Wainwright 1986, Wainwright and Hsu 1989, Hewitt and Wainwright 1993), in which the more physically or geometrically natural expansion-normalized (dimensionless) configuration variables are used. In this case, the physically admissible states typically lie within a bounded region, the dynamical system remains analytic both in the physical region and its boundaries, and the asymptotic states typically lie on the boundary represented by exact physical solutions rather than having singular behaviour. We note that the physically admissible states do not lie in a bounded region for Bianchi models of types VII$_0$, VIII and IX; see WE for details.

Wainwright utilizes the orthonormal frame method (Ellis and MacCallum 1969) and introduces expansion-normalized (commutation function) variables and a new “dimensionless” time variable to study spatially homogeneous perfect fluid models satisfying $p = \alpha\mu$. The equations governing the models form an $N$-dimensional system of coupled autonomous ODEs. When the ODEs are written in expansion-normalized variables, they admit a symmetry which allows the equation for the time evolution of the expansion $\theta$ (the Raychaudhuri equation) to decouple. The reduced $N - 1$-dimensional dynamical system is then studied. At all of the singular points of the reduced system, $\dot{\theta}$ is proportional to $\theta^2$ and hence all such points correspond to transitively self-similar cosmological models (Hsu and Wainwright 1986). This is why the self-similar models play an important role in describing the asymptotic dynamics of the Bianchi models.

For orthogonal Bianchi models of class A, the resulting reduced state space is five-dimensional (Wainwright and Hsu 1989). Orthogonal Bianchi cosmologies of class B were studied by Hewitt and Wainwright (1993) and are governed by a five-dimensional system
of analytic ODEs with constraints. In further work, imperfect fluid Bianchi models were studied under the assumption that all physical quantities satisfy “dimensionless equations of state”, thereby ensuring that the singular points of the resulting reduced dynamical system are represented by exact self-similar solutions (Coley and van den Hoogen 1994a and b). Models satisfying the linear Eckart theory of irreversible thermodynamics were studied by Burd and Coley (1993) and Coley and van den Hoogen (1994a), those satisfying the truncated causal theory of Israel-Stewart by Coley and van den Hoogen (1995), and those satisfying the full (i.e., non-truncated) relativistic Israel-Stewart theory by Coley et al. (1996). The singular points of the reduced dynamical system for scalar field Bianchi cosmological models with an exponential potential again correspond to exact self-similar solutions; such models have been studied by Ibañez et al. (1995), van den Hoogen et al. (1997) and Coley et al. (1997).

It is interesting to ask under what circumstances the singular points correspond to (and hence the asymptotic properties of Bianchi models can be represented by) self-similar models. This depends critically on the equation of state. For perfect fluid models with \( p/\mu \) asymptotically constant, it is plausible that self-similarity of the asymptotic limits is preserved (Wainwright and Hsu 1989) and this was indeed proved for a class of two-fluid models (Coley and Wainwright 1992). However, this property is not robust, and self-similarity of the asymptotic limits is broken for perfect fluid models with more complicated equations of state or for imperfect fluid models that do not have “dimensionless equations of state” (Coley and van den Hoogen 1994b). It is also broken for sources consisting of a homogeneous scalar field with a non-exponential potential (cf. Ibanez et al. 1995) or if the strong energy condition is violated (e.g., if there is a cosmological constant).

B. Self-Similar Models as Asymptotic States of Bianchi Models.

We now discuss the primary role of exact self-similar models in describing the asymptotic states of Bianchi models, again assuming \( p = \alpha \mu \) with \( 0 \leq \alpha \leq 1 \). We will summarize the work of Wainwright and Hsu (1989) and Hewitt and Wainwright (1993), who studied the asymptotic states of orthogonal spatially homogeneous models in terms of attractors of the associated dynamical system for class \( A \) and class \( B \) models, respectively. Due to the
existence of monotone functions, it is known that there are no periodic or recurrent orbits in class A models. Although “typical” results can be proved in a number of Bianchi type B cases, these are not “generic” due to the lack of knowledge of appropriate monotone functions. In particular, there are no sources or sinks in the Bianchi invariant sets $B_0^\pm$ (VIII) or $B^\pm$ (IX).

• A large class of orthogonal spatially homogeneous models (including all class B models) are asymptotically self-similar at the initial singularity and are approximated by exact perfect fluid or vacuum self-similar power law models. Examples include self-similar Kasner vacuum models or self-similar locally rotationally symmetric (class III) Bianchi type II perfect fluid models (Collins and Stewart 1971; see also Collins 1971 and Doreshkevich et al. 1973).

However, this behaviour is not generic; general orthogonal models of Bianchi type IX and VIII have an oscillatory behaviour with chaotic-like characteristics, with the matter density becoming dynamically negligible as one follows the evolution into the past towards the initial singularity. Ma and Wainwright (1994) show that the orbits of the associated cosmological dynamical system are negatively asymptotic to a lower two-dimensional attractor. This is the union of three ellipsoids in $\mathbb{R}^5$ consisting of the Kasner ring joined by Taub separatrices; the orbits spend most of the time near the Kasner vacuum equilibrium points. Clearly the self-similar Kasner models play a primary role in the asymptotic behaviour of these models.

• Exact self-similar power law models can also approximate general Bianchi models at intermediate stages of their evolution (e.g., radiation Bianchi VII$_h$ models; Doreshkevich et al., 1973). Of special interest are those models which can be approximated by an isotropic solution at an intermediate stage of their evolution (e.g., those models whose orbits spend a period of time near to a flat Friedmann equilibrium point).

This last point is of particular importance in relating Bianchi models to the real universe, and is discussed further in general terms in WE (see, especially, Chapter 15) and specifically for Bianchi VII$_h$ models in Wainwright et al. (1998). In particular, the flat Friedmann equilibrium point is universal in that it is contained in the state space of each Bianchi type. Isotropic intermediate behaviour has also been found in tilted Bianchi V models (Hewitt
and Wainwright 1992), and it appears that many tilted models have isotropic intermediate behaviour (see WE).

- Self-similar solutions can describe the behaviour of Bianchi models at late times (i.e., as $t \to \infty$). Examples include self-similar flat space and self-similar homogeneous vacuum plane waves (Collins 1971, Wainwright 1985).

All models expand indefinitely except for the Bianchi IX models. The question of which Bianchi models can isotropize was addressed in the famous paper by Collins and Hawking (1973), in which it was shown that, for physically reasonable matter, the set of homogeneous initial data that give rise to models that isotropize asymptotically to the future is of zero measure in the space of all homogeneous initial data (see also Barrow and Tipler 1986, and WE).

All vacuum models of Bianchi (B) types IV, V, VI$_h$ and (especially) VII$_h$ are asymptotic to plane wave states to the future. Type V models tend to the Milne form of flat spacetime (Hewitt and Wainwright 1993). Typically, and perhaps generically (Hewitt and Wainwright 1993), non-vacuum models are asymptotic in the future to either plane-wave vacuum solutions (Doroshkevich et al. 1973, Siklos 1981) or non-vacuum Collins type VI$_h$ solutions (Collins 1971).

Bianchi (A) models of types VII$_o$ (non-vacuum) and VIII expand indefinitely but are found to have oscillatory (though non-chaotic) behaviour in the Weyl curvature (Wainwright, unpublished). Bianchi type IX models obey the “closed universe recollapse” conjecture (Lin and Wald 1989). All orbits in the Bianchi invariant sets $B^\pm_{\alpha}(VII_0) \ (\Omega > 0), B^\pm_{\alpha}(VIII)$ and $B^\pm(IX)$ are positively departing; in order to analyse the future asymptotic states of such models, it is necessary to compactify phase-space. The description of these models in terms of conventional expansion-normalized variables is only valid up to the point of maximum expansion (where $\theta = 0$), although recently Wainwright has introduced more appropriate variables which are valid for all values of $\theta$ (WE).

In summary, due to the non-existence of periodic, recurrent and homoclinic orbits in the Bianchi state space (deduced from the existence of monotone functions), the dynamical
behaviour of Bianchi models is dominated by equilibrium points and heteroclinic sequences (or heteroclinic cycles contained in the Mixmaster attractor for class A models). This is why self-similar models, which correspond to equilibrium points, play a dominant role in the dynamics of Bianchi cosmological models. These issues are further discussed in WE. In particular, one can generalize the above analysis to the exceptional Bianchi $VI_{-1/9}$ models, to two-fluid models (Coley and Wainwright 1992), and to inflationary models with $-1 \leq \alpha < -1/3$ (cf. the cosmic no-hair theorems; Wald 1983). Tilted Bianchi models and models with more general sources than a perfect fluid (including, for example, scalar fields, imperfect fluids and magnetic fields) are also discussed in WE (see, especially, Chapter 8 by Hewitt, Uggla and Wainwright). Self-similar spatially homogeneous massless scalar field models, which are formally equivalent to stiff ($\alpha = 1$) perfect fluid models, have also been discussed by Coley and Wainwright (1998).

C. $G_2$ Models.

Inhomogeneous perfect fluid $p = \alpha \mu$ cosmological models admitting two commuting spacelike KVs acting orthogonally transitively — the so-called $G_2$ cosmologies — have been studied by Hewitt and Wainwright (1990) with a view to describing their asymptotic behaviour near the big bang and at late times. In particular, they showed that the EFEs can be written as an autonomous system of first-order, quasi-linear (formally hyperbolic) PDEs (without constraints) in terms of two independent dimensionless variables, the state space being an infinite-dimensional function space. By defining the dynamical equilibrium states in terms of an appropriately invariantly defined time derivative of the state vector being zero, Hewitt and Wainwright (1990) prove that these states correspond to cosmological models that are self-similar (but not necessarily spatially homogeneous). In this case, the EFEs reduce to a system of autonomous ODEs (with spatial dependence) that can be studied qualitatively by normal techniques; the spatially homogeneous subcases have been studied previously (see above).

Hewitt and Wainwright (1990) conjecture that the dynamical equilibrium states may describe the asymptotic or intermediate dynamical behaviour of the orthogonally transitive $G_2$ models. In particular, they show that in the subclass of separable diagonal $G_2$
cosmologies (in which the two KVs are hypersurface orthogonal), the models do indeed asymptote towards the dynamical equilibrium states. Thus the models in this special subclass are asymptotically self-similar. In these models, the orbits of the three-dimensional homothetic group are timelike, the velocity vector being tangent to the group orbits, so the time evolution is completely determined and the spatial structure is governed by a two-dimensional plane autonomous system of ODEs. The qualitative properties of the diagonal self-similar $G_2$ cosmologies have been studied by Hewitt et al. (1988, 1991).

It remains to be determined to what extent a typical $G_2$ cosmology which expands indefinitely from an initial singularity is asymptotically self-similar into the past and the future. Since the flat Friedmann model is a saddle point of the governing $G_2$ evolution equations, intermediate isotropization will occur for a subset of models, but the size of this subset of $G_2$ models is unclear. These issues are discussed further in WE.

3. Spherically Symmetric Models

The most extensive literature on self-similarity involves spherically symmetric models since these obviously afford the greatest mathematical simplification and have a number of important applications. The first work focussed on dust ($\alpha = 0$) solutions since, in this case, the solutions can often be expressed analytically and are just a special subclass of the more general spherically symmetric Tolman-Bondi solutions (Tolman 1934, Bondi 1947, Bonnor 1956); see, for example, Gurovich (1967), Dyer (1979), Chao (1981), Maharaj (1988) and, more recently, Joshi and Darivedi (1993, Sintes (1996) and Carr and Coley (1998a). More extensive references can be found in Kramer et al. (1980) and Krasinski (1997). Self-similar dust solutions have played an important historical role in the subject but here we will mainly focus on the more general case in which the fluid has an equation of state $p = \alpha \mu$ with $0 < \alpha < 1$. It has recently been claimed that such models can be classified completely, which makes our discussion particularly timely. Much of the analysis will also be applicable in the dust ($\alpha = 0$) and stiff fluid ($\alpha = 1$) limits but it should be cautioned that not all features of the solutions will carry over in these special cases. In some contexts, we will consider negative pressure models. Here we require $-1 < \alpha < 0$ but it should be noted that $\alpha = -1$ and $\alpha = -1/3$ are also special cases.
A. Different Mathematical Approaches.

Due to the existence of several preferred geometric structures in self-similar spherically symmetric perfect fluid models, a number of different approaches (i.e., coordinate systems) may be used in studying them (Bogoyavlenski 1985). In particular, one can use “comoving” coordinates, “homothetic” coordinates or “Schwarzschild” coordinates. All of these approaches are complementary and which is most suitable depends on what type of problem is being studied. The relationship between the various approaches, and the precise coordinate transformations between them, can be found in a number of sources (see Section IV.3 in Bogoyavlenski 1985, Ori and Piran 1990, Appendix C of Carr and Coley 1998a, Appendices B of Goliath et al. 1998a and 1998b, and Carr et al. 1998).

In the comoving approach, pioneered by Cahill and Taub (1973) and employed by Carr and Henriksen and coworkers and Ori and Piran (1990), the coordinates are adapted to the fluid four-velocity vector. We shall primarily adopt this approach here since it affords the best physical insights and is the most convenient one with which to discuss the solutions explicitly. Even within this approach, different authors use different notation, so it is sometimes difficult to relate their results; in the discussion below we will primarily use the notation of Cahill and Taub. Recently Carr and Coley (1998a) presented a comprehensive and unified analysis of spherically symmetric self-similar perfect fluid models using the comoving approach, relating many of the results obtained earlier by Ori and Piran (1990) and Foglizzo and Henriksen (1993).

In the homothetic approach, used by Bogoyavlenski and coworkers and adopted more recently by Brady (1994) and Goliath et al. (1998a and 1998b), the coordinates are adapted to the homothetic vector and a “conformally static” metric is employed. In this case, the governing equations reduce to an autonomous system of ODEs and hence dynamical systems theory can be exploited to study them. The results of the dynamical systems analysis complement and, in some cases, provide more rigorous demonstrations of the results obtained in the comoving approach. However, in the homothetic approach spacetime must be covered by several coordinate patches, one in which the HV is spacelike and one in which it is timelike. These regions must then be joined by a surface in which the HV is null.
and this surface is associated with important physics. Bogoyavlenski (1985) studied the spacelike and timelike cases simultaneously and continuously matched the two regions to obtain the behaviour of solutions crossing the null surface; however, it should be noted that Bogoyavlenski changed to comoving coordinates explicitly in order to describe the physics of the associated solutions.

Recently Goliath et al. (1998a and 1998b) have reinvestigated self-similar, spherically symmetric perfect fluid of models using the homothetic approach. They introduce dimensionless variables, so that the number of equations in the coupled system of autonomous differential equations is reduced, with the resulting reduced phase-space being compact and regular. In this way the similarities with the equations governing hypersurface orthogonal models, and in particular spatially homogeneous models (WE, Nilsson and Ugglas 1997), can be exploited. In their approach, all equilibrium points are hyperbolic, in contrast to the earlier work in which Bogoyavlenski used non-compact variables (which resulted in parts of phase-space being “crushed”). The spatially self-similar case was studied by Goliath et al. (1998a) and the timelike case, which contains the more interesting physics (e.g., shocks and sound-waves), was studied by Goliath et al. (1998b).

The Schwarzschild approach was adopted by Ori and Piran (1990) and more recently by Maison (1995). In order to obtain physically reasonable models, spacetimes are often required to be asymptotically flat. Since asymptotically flat spacetimes are not self-similar, one therefore needs to match a self-similar interior region to a non-self-similar exterior solution. This is usually taken to be Schwarzschild, in which case Schwarzschild coordinates are better suited to finding global solutions. This approach is also suitable for solving the equations of motion for (radial) null geodesics, enabling the causal structure of spacetime to be studied. Consequently it was used by Ori and Piran (1990) since one of their primary goals was to study naked singularities and test the cosmic censorship hypothesis (Section 4A). However, the Schwarzschild coordinates break down at $t = 0$. Null coordinates can also be used to analyse the global structure (Henriksen and Patel 1991) and these are often used in the study of spherically symmetric scalar fields (Choptuik 1993, Gundlach 1997).

B. General Features of Similarity Solutions.
Throughout the subsequent discussion we will use comoving coordinates, so the metric in the spherically symmetric situation can be written in the form

\[ ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + R^2 d\Omega^2, \quad d\Omega^2 \equiv d\theta^2 + \sin^2 \theta \, d\phi^2 \]  

(3.1)

where \( \nu, \lambda \) and \( R \) are functions of \( r \) and \( t \). The equations have a “first integral” \( m(r,t) \) which can be interpreted as the mass within comoving radius \( r \) at time \( t \). There is also a dimensionless quantity \( E(r,t) \) which represents the total energy per unit mass for the shell with comoving coordinate \( r \). Unless \( p = 0 \), both these quantities decrease with increasing \( t \) because of the work done by the pressure.

Spherically symmetric homothetic solutions were first investigated by Cahill and Taub (1971), who showed that by a suitable coordinate transformation they can be put into a form in which all dimensionless quantities such as \( \nu, \lambda, E \) and 

\[ S \equiv \frac{R}{r}, \quad M \equiv \frac{m}{R}, \quad P \equiv pR^2, \quad W \equiv \mu R^2 \]  

(3.2)

are functions only of the dimensionless similarity variable \( z \equiv r/t \). The homothetic vector in these coordinates is

\[ \xi^a \frac{\partial}{\partial x^a} = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}. \]  

(3.3)

Values of \( z \) for which \( M = 1/2 \) correspond to a black hole or cosmological apparent horizon since the congruence of outgoing null geodesics have zero divergence. Another important quantity is the function

\[ V(z) = e^{\lambda - \nu} z, \]  

(3.4)

which represents the velocity of the fluid relative to spheres of constant \( z \). These spheres contract relative to the fluid for \( z < 0 \) and expand for \( z > 0 \). The homothetic vector is timelike for \( V < 1 \) and spacelike for \( V > 1 \). Special significance is attached to values of \( z \) for which \( |V| = \sqrt{\alpha} \) and \( |V| = 1 \). The first corresponds to a sonic point (where the pressure and density gradients are not uniquely determined), the second to a Cauchy horizon (either a black hole event horizon or a cosmological particle horizon).

We have seen that the only barotropic equation of state compatible with the similarity ansatz is one of the form \( p = \alpha \mu \). It is convenient to introduce a dimensionless function \( x(z) \) defined by

\[ x(z) \equiv \left( 4\pi \mu r^2 \right)^{-\alpha/(1+\alpha)}. \]  

(3.5)
The conservation equations $T^\mu_{\nu;\nu} = 0$ can then be integrated to give
\[ e^\nu = \beta x^{2\alpha/(1+\alpha)}, \quad e^{-\lambda} = \gamma x^{-1/\alpha} S^2 \] (3.6)
where $\beta$ and $\gamma$ are integration constants. The remaining field equations reduce to a set of ODEs for $x$ and $S$ in terms of the similarity variable [see eqns (2.11) to (2.15) of Carr and Yahil (1990)]. These specify integral curves in the three-dimensional $(x, S, \dot{S})$ space (where a dot denotes $zd/dz$). For a given equation of state parameter $\alpha$, there is therefore a two-parameter family of spherically symmetric similarity solutions.

In $(x, S, \dot{S})$ space the sonic condition $|V| = \sqrt{\alpha}$ specifies a two-dimensional surface. Where a curve intersects this surface, the equations do not uniquely determine $\dot{x}$ (since the coefficient of $\dot{x}$ disappears in one of them), so there can be a number of different solutions passing through the same point. However, only integral curves which pass through a line $Q$ on the sonic surface are “regular” in the sense that $\dot{x}$ is finite and they can be extended beyond there. The equations permit just two values of $\dot{x}$ at each point of $Q$ and there will be two associated values of $\dot{V}$. On some parts of $Q$ these values will be complex (corresponding to a “focal” point), so the solution will still be unphysical. Otherwise both values of $\dot{V}$ will be real and at least one of them will be positive. If both values of $\dot{V}$ are positive (corresponding to a “nodal” point), the smaller one is associated with a 1-parameter family of solutions, while the larger one is associated with an isolated solution. If one of the values of $\dot{V}$ is negative (corresponding to a “saddle” point), both values are associated with isolated solutions. This behaviour has been analysed in detail by Bogoyavlenski (1977), Bicknell and Henriksen (1978), Carr and Yahil (1990) and Ori and Piran (1990).

On each side of the sonic point, $\dot{x}$ may have either of the two values. If one chooses different values for $\dot{x}$, there will be a discontinuity in the pressure gradient. If one chooses the same value, there may still be a discontinuity in the higher derivatives of $x$. Only the isolated solution and a single member of the one-parameter family of solutions are analytic. This contrasts with the case of a shock, where $x$ is itself discontinuous (Cahill and Taub 1971, Bogoyavlenski 1985, Anile et al. 1987, Moschetti 1987). One can show that the part of $Q$ containing nodes, for which there is a one-parameter family of solutions, corresponds to two ranges of values for $z$. One range ($z_1 < z < z_2$) lies to the left of the Friedmann sonic point $z_F$ and includes the static sonic point $z_S$. The other range ($z > z_3$) includes
the Friedmann sonic point $z_F$. The values of $z_1$, $z_2$ and $z_3$ depend on $\alpha$. The ranges for $\alpha = 1/3$ are indicated in Figure (1); in this case, $z_2 = z_S$ and $z_3 = z_F$.

Carr and Coley (1998a; CC) have classified the $p = \alpha \mu$ spherically symmetric similarity solutions completely. The key steps in their analysis are: (1) a complete analysis of the dust solutions, since this provides a qualitative understanding of certain features of the general solutions with pressure; (2) an elucidation of the link between the $z > 0$ and $z < 0$ solutions; (3) a proof that, at large and small values of $|z|$, all similarity solutions must have an asymptotic form in which $x$ and $S$ have a power-law dependence on $z$; and (4) a demonstration that there are only three such power-law solutions (apart from the trivial flat solution).

C. Exact Power-Law Similarity Solutions.

We first discuss the power-law models explicitly since they play a central role in what follows. We will assume $z > 0$ but the equations below can be easily extended to the $z < 0$ regime by replacing $z$ by $|z|$ and reversing the sign of $V$.

- The $k = 0$ Friedmann solution. For this one can choose $\beta$ and $\gamma$ in equation (3.6) such that

$$x = z^{-2 \alpha/(1 + \alpha)}, \quad S = z^{-2/[3(1 + \alpha)]}$$

and then

$$\mu = \frac{1}{4\pi l^2}, \quad V = \left(\frac{1 + 3 \alpha}{\sqrt{6}}\right) z^{(1 + 3 \alpha)/[3(1 + \alpha)]},$$

One can put the metric in a more familiar form by making a transformation of the radial coordinate.

- A self-similar Kantowski-Sachs (KS) model. For each $\alpha$ there is a unique self-similar KS solution and this can be put in the form

$$S = S_* z^{-1}, \quad x = x_* z^{-2 \alpha/(1 + \alpha)}$$

where $x_*$ and $S_*$ are constants determined by $\alpha$. One can take $\beta$ and $\gamma$ to have the same values as in the Friedmann solution for $\alpha < 0$ and $i$ times those values for $\alpha > 0$. One then
has
\[ \mu t^2 = \left( \frac{1}{3|\alpha|} \right)^{(1+\alpha)/(\alpha-1)}, \quad V = -\frac{(1-\alpha)(1+3\alpha)^2}{2\sqrt{6\alpha}} \left( \frac{1}{3|\alpha|} \right)^{-2\alpha/(1-\alpha)} z^{(1+3\alpha)/(1+\alpha)} \] (3.10)
and one can again put the metric in a familiar form with a radial coordinate transformation. However, it should be stressed that only solutions with \( \alpha < -1/3 \) are physical: for \( 0 < \alpha < 1 \), \( V \) is negative and this corresponds to tachyonic solutions (i.e., the t coordinate is spacelike and the r coordinate is timelike); for \( -1/3 < \alpha < 0 \), \( V \) is positive but \( S \) is imaginary. The mass is also negative for \( -1/3 < \alpha < 0 \). For \( \alpha < -1/3 \), the mass is positive and the coordinates play their usual roles. Although such solutions have negative pressure and violate the strong energy condition, they may be relevant in the early Universe.

- A self-similar static solution. In this case

\[ x = x_o, \quad S = S_o \] (3.11)

where the constants \( x_o \) and \( S_o \) are determined by \( \alpha \), so there is just one static solution for each equation of state. The other interesting functions are

\[ \mu = x_o^{-(1+\alpha)/\alpha} \rho^{-2}, \quad V = x_o^{-(1-\alpha)/2\alpha} z^{(1-\alpha)/(1+\alpha)}. \] (3.12)

The metric can be put in an explicitly static form under an appropriate change of variables. Misner and Zapolsky (1964) first found this solution (also see Oppenheimer and Volkoff 1939) but did not appreciate its self-similarity; it has subsequently been studied by Henriksen and Wesson (1978) and Carr and Yahil (1990). Note that there is a naked singularity at the origin.

There is an interesting connection between the static and KS solutions: if one interchanges the \( r \) and \( t \) coordinates in the static metric and also changes the equation of state parameter to

\[ \alpha' = -\frac{\alpha}{1+2\alpha}, \] (3.13)

one obtains the KS metric. For a static solution with a normal equation of state \((1 > \alpha > 0)\), \( \alpha' \) must lie in the range \(-1/3 \) to \( 0 \), so negative pressure (negative mass) KS solutions can also be interpreted as positive pressure (positive mass) static solutions.
The forms of $V(z)$ for the Friedmann solution, KS and static solutions in the $\alpha = 1/3$ case is shown in Figure (1). The full family of similarity models comprises solutions asymptotic to these at large and small $|z|$. To study their asymptotic behaviour, one introduces functions $A(z)$ and $B(z)$ defined by

$$x \equiv x_i e^A, \quad S \equiv x_i e^B,$$

where $x_i$ is given by equation (3.7) in the Friedmann case, equation (3.9) in the KS case and equation (3.11) in the static case. The ODEs for $x$ and $S$ then become ODEs for $A$ and $B$. The solutions in each family can be specified by the values of $A$ and $B$ as $|z| \to \infty$ (denoted by $A_\infty$ and $B_\infty$) and their values as $|z| \to 0$ (denoted by $A_0$ and $B_0$), although these values may not be independent. The form of the full family of solutions in the $\alpha = 1/3$ case is summarized in Figure (2).

**D. Asymptotically Friedmann Solutions.**

In the supersonic (large $z$) regime, asymptotically Friedmann similarity solutions are described by the single parameter $B_\infty$ since $A_\infty = 0$. The solutions are overdense relative to the Friedmann solution for $B_\infty < 0$ and underdense for $B_\infty > 0$. If $B_\infty$ is sufficiently negative, $V$ reaches a minimum value and then rises again to infinity as $z$ decreases. Such solutions contain black holes and were originally studied because there was interest in whether black holes could grow at the same rate as the particle horizon. Carr and Hawking (1974) showed that such solutions exist for radiation ($\alpha = 1/3$) and dust ($\alpha = 0$) but only if the universe is asymptotically rather than exactly Friedmann ($B_\infty \neq 0$); i.e., black holes formed through purely local processes in the early Universe cannot grow as fast as the particle horizon. Carr (1976) and Bicknell and Henriksen (1978a) then extended this result to a general $0 < \alpha < 1$ fluid. Lin et al. (1976) claimed that there is a similarity solution in an exact Friedmann universe for the special case of a stiff fluid ($\alpha = 1$) but Bicknell and Henriksen (1978b) showed that this requires the inflowing material to turn into a null fluid at the event horizon. In fact, for fixed $\alpha$, it is now known that all subsonic solutions which can be attached to an exact Friedmann model via a sound wave are non-physical: as one goes inward from the sound-wave they either enter a negative mass regime or reach another sonic point at which the pressure diverges (Bicknell and Henriksen 1978a).
It is likely that asymptotically Friedmann solutions which contain black holes are supersonic everywhere (in the sense that $V$ never falls below $1/\sqrt{\alpha}$), although this has not been rigorously proved. However, all the solutions with $B_\infty$ exceeding some critical negative value $B_\infty^{crit}$ reach the sonic surface and those which do so with $z_1 < z < z_2$ or $z > z_3$ may be attached to the origin by subsonic solutions. The latter are also described by a single parameter and this can be taken to be $A_0$, solutions being overdense relative to the Friedmann solution for $A_0 < 0$ and underdense for $A_0 > 0$. These transonic solutions represent density fluctuations in a flat Friedmann model which grow at the same rate as the particle horizon (Carr and Yahil 1990). While there is a continuum of regular underdense solutions, regular overdense solutions only occur in successive and very narrow bands (with just one solution per band being analytic at the sonic point). The overdense solutions also exhibit oscillations in the subsonic region, with the number of oscillations identifying the band. The existence of these bands was first pointed out by Bogovalenski (1985) and also studied by Ori and Piran (1990). The band structure arises even in the Newtonian situation (Whitworth and Summers 1985). The higher bands are all nearly static near the sonic point but they deviate from the static solution as one goes towards the origin.

The forms of $S(z)$ and $V(z)$ in the general $\alpha$ asymptotically Friedmann solutions are indicated in Figure (3). The curves are here parametrized by the “asymptotic energy” parameter ($E_\infty$), which is related to $B_\infty$ by

$$E_\infty = \frac{1}{2}(e^{6B_\infty} - 1).$$

(3.15)

This is a more convenient parameter if one wishes to relate the asymptotically Friedmann solutions to the other ones discussed below. The $z > 0$ solutions correspond to models which start from an initial Big Bang singularity at $z = \infty$ ($t = 0$) and then either expand to infinity as $z \rightarrow 0$ ($t \rightarrow \infty$) for $E_\infty > E_{crit}$ or recollapse to a black hole at some positive value of $z$ for $E_\infty < E_{crit}$. Here $E_{crit}$ is related to $B_\infty^{crit}$ by equation (3.15). In the dust case, $E_{crit} = 0$ because there is no pressure to stop the collapse if the energy is negative. However, if there is pressure, $E_{crit} < 0$ and we have seen that there are overdense solutions without black holes which expand to infinity providing $E_\infty$ lies in narrow bands between $E_{crit}$ and 0; outside these bands the solutions are irregular at the sonic point. The underdense solutions (with $E_\infty > 0$) are all regular at the sonic point. The $E_\infty < E_{crit}$
solutions contain a black hole event horizon and a cosmological particle horizon for values of $E_\infty$ exceeding another critical value $E_\ast$ (corresponding to $V_{\text{min}} = 1$). Note that the mass function $M$ has a minimum below $1/2$, whatever the value of $E_\infty$, so there is always a black hole and cosmological apparent horizon.

The analysis is trivially extended to the $z < 0$ regime since the solutions are symmetric in $z$, as illustrated in Figure (3). Since $r$ is always taken to be positive, the $z < 0$ solutions are the time-reverse of the $z > 0$ ones. Thus the $E_\infty > E_{\text{crit}}$ models collapse from an infinitely dispersed initial state to a big crunch singularity as $z$ decreases from 0 to $-\infty$ (i.e. as $t$ increases from $-\infty$ to 0), while the $E_\infty < E_{\text{crit}}$ models emerge from a white hole and are never infinitely dispersed.

E. Asymptotically Kantowski-Sachs Solutions.

In this case, the asymptotic behaviour depends on the value of $\alpha$. For $0 < \alpha < 1$, the solutions can be characterized by a single parameter and this can be taken to be $A_\infty$ in the supersonic regime and $A_0$ in the subsonic regime. However, there are only isolated solutions at a sonic point, so solutions which hit the sonic surface are unlikely to be regular there. In any case, these solutions - like the KS similarity solution itself - are presumably unphysical. For $-1 < \alpha < -1/3$, there is no sonic point and the solutions can be characterized by either $A_0$ or $A_\infty$. Note that the asymptotic energy in all solutions is $E_\infty = -1/2$.

The asymptotically KS models were studied by Carr and Koutras (1993), who integrated the equations numerically for $\alpha = 1/3$ and $\alpha = -1/2$. Figure (2) shows the form of $V(z)$ in the former case; although the meaning of these solutions is unclear (since they are unphysical), they are still of mathematical interest in that they serve to fill in the $V(z)$ solution space. The solutions with $-1 < \alpha < -1/3$ may be relevant in the early Universe due to inflation or particle production effects. In particular, they may be related to the growth of $p > 0$ bubbles formed at a phase transition in a $p < 0$ cosmological background (Wesson 1986). For example, Henriksen et al. (1983) have shown that a bubble in a de Sitter background can be modelled by a KS solution (although this involves similarity of the second kind since the de Sitter model contains a scale–see Section 5B). Generalized
negative-pressure KS similarity solutions (in which the equation of state is different from
\( p = \alpha \mu \) and which can be interpreted as a mixture of false vacuum and dust) have been
studied by Ponce de Leon (1988) and Wesson (1989).

A rather peculiar feature of the asymptotically KS solutions is that the mass can go
negative. Indeed, this is a general feature of similarity solutions and can occur even for
\( V > 0 \) (e.g., when one has shell-crossing). This may seem unphysical but - in the context of
the big bang model - Miller (1976) has given a possible interpretation in terms of “lagging”
cores. She gives an explicit example of an \( \alpha = 1 \) self-similar solution for which the mass
goes negative. In the \( \alpha = 1/3 \) case (but only in this case), one can show that there is a
well-defined curve in the \( V(z) \) diagrams where \( M = 0 \) and this is shown in Figure (1). This
curve has two parts: the upper part (with \( V > 0 \)) is relevant for asymptotically Friedmann
solutions, while the lower part (with \( V < 0 \)) is relevant for asymptotically KS solutions.
\( M \) is negative in between the two parts and this region includes the KS solution itself (as
expected).

F. Asymptotically Static Solutions.

For \( 0 < \alpha < 1 \), the asymptotically static solutions are described by two parameters at
large values of \( z \). These can be taken to be \( A_\infty \) and \( B_\infty \), which in this case can be chosen
independently. This also determines the asymptotic energy parameter

\[
E_\infty = \frac{(1 + \alpha)^2}{2(1 + 6\alpha + \alpha^2)} e^{6B_\infty - 2A_\infty / \alpha} - \frac{1}{2}.
\]

Such solutions are of particular interest because they represent the most general asymptotic
behaviour. As discussed in Sections 4A and 4B, they are also associated with the formation
of naked singularities and the occurrence of critical phenomena in gravitational collapse.
However, describing these solutions as “asymptotically static” at large \( z \) is rather misleading
because one can show that the velocity of the fluid relative to the constant \( R \) surfaces,

\[
V_R = \frac{V S}{S + \dot{S}} ,
\]

is generally non-zero as \( z \to \infty \). Indeed, the asymptotic value of \( V_R \) can also be expressed
in terms of \( A_\infty \) and \( B_\infty \), and this is zero only for a one-parameter subfamily of solutions.
This agrees with the description of Foglizzo and Henriksen (1993), who term such solutions “symmetric”. CC describe the more general solutions as asymptotically “quasi-static” since they still have $\dot{S}$ and $\dot{x}$ tending to 0 at infinity. There are no asymptotically quasi-static solutions at the origin since all solutions must be either exactly static, asymptotically Friedmann or asymptotically KS at small $z$.

A proper understanding of these solutions requires that one allows for both positive and negative values of $z$. This is because the solutions necessarily span both regimes, as illustrated by the form of $S(z)$ in Figure (4). This shows that the introduction of the second parameter (which CC term D) has two important consequences. First, although one still has expanding and collapsing solutions for $D > 0$, the Big Bang singularity occurs at $z = -1/D$ rather than at $z = \infty$ (i.e., before $t = 0$), while the Big Crunch singularity occurs at $z = +1/D$ (i.e., after $t = 0$). Second, the solution is asymptotic to a quasi-static model rather than the Friedmann model as $|z| \to \infty$. On the other hand, the second parameter has relatively little effect in the subsonic regime, so one can still use the results of the asymptotically Friedmann analysis here (at least qualitatively). In particular, the model can only collapse from infinity if $E_\infty$ is positive or lies in discrete bands if negative. Otherwise it must expand from an initial white hole singularity at some negative value of $z$ (just as in the collapsing asymptotically Friedmann models) before collapsing to the black hole singularity at $z = 1/D$. The expanding solutions are just the time reverse of the collapsing solutions.

The form of $V(z)$ in the $D > 0$ collapsing solutions is also illustrated in Figure (3). The solutions start with $V = 0$ at $z = 0$ and then, as $z$ decreases, pass through a Cauchy horizon (where $V = -1$) and then a sonic point (where $V = -\sqrt{\alpha}$) before tending to the quasi-static form at $z = -\infty$. They then jump to $z = +\infty$ and enter the $z > 0$ regime. As $z$ further decreases, $V$ first reaches a minimum and then diverges to infinity when it encounters the Big Crunch singularity at $z_S = 1/D$. The minimum of $V$ may be either above or below 1, depending on the values of $E_\infty$ and $D$, but (as discussed in Section 4B) one necessarily has a naked singularity in the latter case. The singularity forms with zero mass at $t = 0$, but its mass $m_S = (M_S)z_S t$ then grows as $t$ . As pointed out by Ori and Piran (1990), such solutions have an analogue in Newtonian theory (Larson 1969, Penston
1969). If the minimum of $V$ is below $\sqrt{\alpha}$, there is probably no solution since it would need to pass through two further sonic points. At a given value of $t$, the form of $\mu t^2$ specifies the density profile. If $z < 0$, this corresponds to an isothermal static distribution ($\mu \propto r^{-2}$) for $|z| >> 1/D$ with a uniform core for $|z| << 1/D$. If $z > 0$, it again corresponds to an "isothermal" static distribution for $z >> 1/D$ but with a density singularity at the origin.

Some of the two-parameter family of similarity solutions with pressure have been studied numerically by Foglizzo and Henriksen (1993), although they only focus on the collapsing ones. They confirm that the solutions are described by two parameters at large $|z|$ and by one parameter at small $|z|$. They also identify the expected behaviour at the sonic point. In their phase-space analysis, the orbits corresponding to the overdense solutions converge on and then spiral around the static solution for a while before heading to the origin. This corresponds to the oscillations found by Carr and Yahil (1990) and Ori and Piran (1990).

4. Applications of Spherically Symmetric Similarity Solutions

A. Gravitational Collapse and Naked Singularities.

One of the major goals of classical general relativity in recent years has been the study and testing of the cosmic censorship hypothesis. This asserts, in very general terms, that singularities which develop from regular initial conditions have no causal influence on spacetime (Penrose 1969, Israel 1984). Until recently most possible counter-examples to the cosmic censorship hypothesis have been restricted to spherically symmetric spacetimes which involve shell-crossing or shell-focussing, such solutions being globally naked for a suitable choice of initial data (Eardley and Smarr 1979, Lake 1992).

Self-similarity is very relevant to this issue because most of the known examples of shell-focussing singularities involve exact homothetic solutions (Eardley et al. 1986, Zannias 1991, Lake 1992). Indeed, it has been shown that a large subclass of self-similar solutions have a central singularity from which null geodesics emerge to infinity (Henriksen and Patel 1991) and it has been argued that one might generally expect a naked singularity to have a horizon structure similar to that of the global homothetic solution (Lake 1992).
The occurrence of naked singularities in spherically symmetric, perfect fluid, self-similar collapse has been studied by Ori and Piran (1987, 1990), Waugh and Lake (1988, 1989), Lake and Zannias (1990), Henriksen and Patel (1991), and Foglizzo and Henriksen (1993) [see also Ref. 2 in Lake 1992]. Their occurrence in the spherically symmetric collapse of a self-similar massless scalar field has been studied by Brady (1995).

Most of the early work focussed on dust solutions of the Tolman-Bondi class, including in particular the analytical work of Eardley and Smarr (1979) and Christodolou (1984). Ori and Piran (1990; OP) extended this work by studying spherically symmetric homothetic with pressure. For reasonable equations of state it might be expected that pressure gradients would prevent the formation of shell-crossing singularities (the situation is less clear for shell-focussing singularities). However, OP proved the existence of a “significant” class of perfect fluid self-similar solutions with a globally-naked central singularity. They explicitly studied the causal nature of these solutions by analysing the equations of motion for the radial null geodesics, thereby demonstrating that the null geodesics emerge to infinity. OP noted that these perfect fluid (non-dust) solutions might constitute the strongest known counter-example to cosmic censorship, although they have not been shown to be stable and consequently may not contradict the formulation of cosmic censorship due to Penrose (1969). Clearly it is important to study the stability of perfect fluid spherically symmetric homothetic solutions with naked singularities with respect to non-self-similar perturbations (and eventually nonspherical perturbations). We discuss this further in Section 4D.

Foglizzo and Henriksen (1993; FH) extend OP’s analysis of the gravitational collapse of homothetic perfect fluid gas spheres with \( p = \alpha \mu \) for all \( \alpha \) between 0 and 1, partially utilizing the powerful dynamical systems approach of Bogoyavlenski (1985). They show that the set of globally \textit{analytic} naked solutions is discrete but finite (and even empty for large values of \( \alpha \)) and they confirm that the number of oscillations in the flow is a good index, with the approach to the “static” solution being recovered as this index grows. FH discuss how the initial part of the “precursor” singularity (OP, Lake 1992), which is the only component which can become naked, is formed from initially inwardly directed trajectories.

Recent work, both theoretical (Barrabas et al. 1991) and numerical (Shapiro and Teukol-
sky 1992), has shown that naked singularities may also arise in non-spherical collapse. In
the latter work, the axisymmetric collapse of a prolate cluster of collisionless matter to a
singular “spindle state” was studied. However, these solutions are not self-similar. FH also
considered planar homothetic collapse and found that, in this case, the singularity is never
naked.

B. Critical Phenomena.

One of the most exciting developments in general relativity in recent years has been
the discovery of critical phenomena in gravitational collapse. This first arose in studying
the gravitational collapse of a spherically symmetric massless (minimally coupled) scalar
field (Choptuik 1993). If one considers a family of imploding scalar wave packets whose
strength is characterized by a continuous parameter \( l \), one finds that the final outcome is
either gravitational collapse for \( l > l^* \) or dispersal, leaving behind a regular spacetime, for
\( l < l^* \). For \((l - l^*)/l^* \) positive and small, the final black hole mass obeys a scaling law

\[
M_{BH} = C(l - l^*)^{\eta}
\]

where \( C \) is a family-dependent parameter and \( \eta = 0.37 \) is family-independent. Initial data
with \( l = l^* \) leads to a critical solution which exhibits “echoing”. This is a discrete self-
similarity (DSS) in which all dimensionless variables \( \Psi \) repeat themselves on ever-decreasing
spacetime scales: \( \Psi(t, r) = \Psi(e^{-n\Delta t}, e^{-n\Delta r}) \) where \( n \) is a positive integer and \( \Delta = 3.44 \).
Near-critical initial data first evolves towards the critical solution, showing some echoing on
small space scales, but then rapidly evolve away from it to either form a black hole \((l > l^*)\)
or disperse \((l < l^*)\).

The structure of the critical solution has been studied by Gundlach (1995). He claims
that the solution is unique providing the metric is regular at the origin \((r = 0)\) and analytic
across the past null-cone of \( r = t = 0 \). (The null-cone is also a sonic surface since the speed
of sound is the speed of light for a scalar field.) The solution has a naked singularity at
\( r = t = 0 \). Gundlach also shows that the critical solution is unstable: spherically symmetric
perturbations about it contain a single growing mode. A similar picture has emerged from
the numerical analysis of spherically symmetric field collapse for a non-minimally coupled
scalar field and for a self-interacting scalar field (Choptuik 1994). It also applies in the case of vacuum, axisymmetric gravitational wave collapse (Abraham and Evans 1993).

Obtaining analytical solutions with a DSS is difficult, so attempts have been made to elucidate critical phenomena by studying spherically symmetric solutions which possess continuous self-similarity (CSS; e.g., admit a homothetic vector). There is evidence that CSS is a good approximation in the near-critical regime, so mathematical simplification is not the only motivation for these studies. Spherically symmetric homothetic spacetimes containing radiation (i.e., $\alpha = 1/3$) have been investigated by Evans and Coleman (1994). They study models containing ingoing Gaussian wave packets of radiation numerically and find analogous non-linear behaviour to the scalar case. The scaling law even has the same exponent $\eta$, although this appears to be a coincidence since the exponent is different for other equations of state (Maison 1995, Koike et al. 1995). They also obtain a homothetic critical solution which qualitatively resembles the scalar DSS critical solution; in particular, it has a curvature singularity at $t = r = 0$ but is regular at $r = 0$ and the sonic point. As in the scalar case, they find that the critical solution is an intermediate attractor: as the critical point is approached, the evolution of the fluid and gravitational field develops a self-similar region (given by the exact critical solution) near the centre of collapse. However, only a precisely critical model is described by this solution everywhere. Although Evans and Coleman claim that their solution possesses a similarity of the second kind, Carr and Henriksen (1998) show that the solution is actually of the first kind.

Maison (1995) and Koike et al. (1995) have extended Evans and Coleman’s work to the more general $p = \alpha \mu$ case. By considering spherically symmetric non-self-similar perturbations to the critical solution, Maison also manages to obtain the scaling behaviour indicated by equation (4.1) analytically. This analysis cannot be applied directly to the scalar field case, even though a massless scalar field can formally be described as a stiff ($\alpha = 1$) perfect fluid whenever the gradient of the scalar field is timelike. However, spherically symmetric self-similar spacetimes with a massless scalar field have been investigated by a number of authors (Brady 1995, Koike et al. 1995, Hod and Piran 1997, Frolov 1997). The self-similar solution of Roberts (1989) is also relevant in this context. This describes the implosion of scalar radiation from past null infinity. The solution is described by a single parameter: it
collapses to a black hole when this parameter is positive, disperses to future null infinity leaving behind Minkowski spacetime when it is negative, and exhibits a null singularity when it is zero. Although this is reminiscent of the usual critical behaviour, the critical solution is not an intermediate attractor since nearby solutions do not evolve towards it.

Continuous self-similarity also arises in the collapse of a complex scalar field (Hirschmann and Eardley 1995a, 1995b) or the axion/dilation field found in string theory (Hamadè et al. 1995). However, in these cases, the CSS solutions are unstable and the universal critical attractor is DSS (Hamadè et al. 1995). Gundlach (1995, 1997) has proposed running detailed collapse calculations for more complicated matter models, such as the continuous one-parameter family of $p = \alpha \mu$ perfect fluid models. In one parameter region the critical solution might be discretely self-similar, while in another it might be continuously self-similar. Parameter values may even exist in which two equally strong attractors could coexist, perhaps leading to new interesting non-linear behaviour. For example, Choptuik and Liebling (1996) have studied a massless scalar field in Brans-Dicke gravity (which is equivalent to two scalar fields with a particular coupling in general relativity) and observed a transition from continuous to discrete self-similarity in the intermediate attractor as the Brans-Dicke parameter is varied and Choptuik et al. (1996) have investigated Einstein-Yang-Mills collapse and again found discrete self-similarity at the blackhole threshold as well as another region of parameter space where the intermediate attractor is the $n = 1$ static Bartnik-McKinnon solution. In addition, Brady et al. (1998) have studied massive scalar field collapse.

It is clearly important to relate these studies of homothetic solutions to the earlier ones described in Section 3 and to identify the critical solutions among the complete family described in Carr and Coley (1998a). The overdense asymptotically Friedmann solutions already exhibit some of the features of the critical solution in that they are nearly static inside the sonic point and exhibit oscillations. They are also regular at the origin and at the sonic point. However, they cannot be identified with the critical solution itself since they do not contain a naked singularity at the origin. Nor can the static solution itself be so identified since it has a naked singularity at $r = 0$ for all $t$, whereas the critical solution only has a singularity at the origin for $t = 0$. 

To identify the critical solution, one needs to consider the full two-parameter family of spherically symmetric similarity solutions. This has been discussed by Carr and Henriksen (1998), who argue that the critical solution should be identified with the collapsing solution for which \( V_{\text{min}} = 1 \) since various global studies of naked singularities in these solutions (see references in Section 4A) have shown that the condition \( V_{\text{min}} = 1 \) heralds the appearance of a naked singularity at the space-time origin. We have seen that the family of analytic, homothetic solutions that contain (initially massless) black holes and naked singularities at the space-time origin is a discrete one parameter set. These solutions can be characterized by the number of oscillations they contain in the subsonic region \( n_{\alpha} \). As \( n_{\alpha} \) increases, the minimum value \( V_{\text{min}} \) obtained by \( V \) in the region \( z > 0 \) (where the singularity lies) decreases. Thus there is always a first \( n_{\alpha} \) for which \( V_{\text{min}} < 1 \). This value, \( n_{\alpha}^{*} \), say, then labels the threshold for the formation of massless black holes and naked singularities. FH give \( n_{\alpha}^{*} = 1 \) for \( \alpha = 1/16 \), 4 for \( \alpha = 1/3 \) and 6 for \( \alpha = 9/16 \). They surmise that \( n_{\alpha}^{*} \rightarrow \infty \) as \( \alpha \rightarrow 1 \) but do not prove this assertion. For sufficiently small \( \alpha \), \( n_{\alpha}^{*} = 0 \) (OP).

Unfortunately, there is a problem with this simple criterion. First, the discreteness in the family of analytic solutions means that none of them is likely to have \( V_{\text{min}} = 1 \) precisely. This suggests that the critical solution is likely to be \( C^{1} \) rather than analytic at the sonic point and it is not clear from the Evans-Coleman paper whether this is the case. Second, all the known critical solutions have a single oscillation in the subsonic regime. It is possible that single-oscillation solutions can have \( V_{\text{min}} = 1 \) if one allows non-analyticity at the sonic point but that would suggest that the critical solution may not be unique. The precise identification of the critical solution therefore remains uncertain and is the subject of further work (Carr et al. 1998).

C. Self-Similar Voids.

A few years ago measurements of the Hubble constant \( H_{0} \) obtained through studying Cepheid variables in galaxies in the Virgo and Leo clusters gave values of around 80 \( \text{km s}^{-1}\text{Mpc}^{-1} \) (Pierce et al. 1994, Freedman et al. 1994). In the standard Big Bang model without a cosmological constant this made it hard to reconcile the age of the Universe with the ages of globular clusters (at least 12 Gyr). More recent estimates of \( H_{0} \) using Cepheids
yield values closer to 70 (Freedman 1997), but there is still an age problem. However, it must be stressed that these large values of $H_0$ are all obtained within the relatively local distance of 100 Mpc, which is much less than the horizon size of order 10 Gpc. Observations based on the Sunyaev-Zeldovich effect for clusters (Birkinshaw and Hughes 1994) and the time delay in gravitational lensed quasars (Rhee 1991, Roberts et al. 1991) at much larger distances give lower values for the Hubble constant, which would be compatible with the ages of globular clusters.

Several people have pointed out that the apparent discrepancy between the local and distant values of the Hubble constant can be reconciled if we live in a region of the Universe for which the local density is considerably less than its global value (Moffat and Tatarski 1992 and 1994, Nakao et al. 1995, Shi et al. 1996, Maartens et al. 1997). This could also explain why the local density parameter (e.g., the density inferred from the analysis of Virgocentric flow) is less than the global value that would be required by inflation. Such a region will be described as a “void” even though it is not completely empty. This suggestion might not seem too radical since we already know that the Universe contains large-scale voids (Geller and Huchra 1989), as well as large-scale flows (Lauer and Postman 1994). However, to resolve the age problem, we need the local void to extend to at least 100 Mpc (so that it includes the Coma cluster, which is assumed to have negligible deviation from the Hubble flow in the Cepheid estimates of $H_0$) and this is much larger than the typical void.

In analysing this proposal, one needs to assume a particular model for the void. Since the similarity hypothesis (discussed in Section 4D) suggests that cosmological density perturbations may inevitably evolve to self-similar form, it is natural to model the voids by the sort of underdense asymptotically Friedmann solutions discussed in Section 3C. Indeed Newtonian studies already support this suggestion. These show that voids evolve towards self-similar form at late times, with most of the matter piling up onto a surrounding shell (Hoffman et al. 1983, Hausman et al. 1983, Bertschinger 1985). This applies whether the void is produced by a cosmic explosion (Schwartz et al. 1975, Ikeuchi et al. 1983) or merely evolves from the primordial density perturbations. Bertschinger (1985) has applied this idea to explain giant cosmic voids but finds that one needs more than the linear
fluctuations which arise in the standard hierarchical clustering scenario.

Although the studies cited above were all Newtonian, voids have also been studied in the relativistic context; for extensive references see Sato (1984) and Krasinski (1997). Indeed, a relativistic treatment is obligatory for voids whose radius is non-negligible compared to the particle horizon size. Many such treatments use the Tolman-Bondi solution to model the void as an underdense sphere embedded in an Einstein-de Sitter and then determine the ratio of the global and local Hubble parameters (Wu et al. 1995, Suto et al. 1995). However, if primordial fluctuations arise from an inflationary phase, it is also natural to consider fluctuations which are only asymptotically rather than exactly Friedmann. This has motivated Carr and Whinnett (1997) to model cosmic voids by the underdense self-similar asymptotically Friedmann solutions discussed in Section 3C. Related solutions have also been discussed by Tomita (1995, 1997a, 1997b).

We have seen that the precise form of such self-similar voids depends upon the equation of state. After the time of decoupling at around $10^5$ yrs, the Universe can be treated as pressureless “dust” ($\alpha = 0$). In this case, there is no sonic point and the underdense self-similar solutions can be analysed analytically as a special case of the Tolman-Bondi solutions. Carr and Whinnett express the various Hubble and density parameter profiles in terms of the (negative) energy parameter $E_\infty$. Although they find that the local values of these parameters may indeed differ considerably from their global values, they also note that the origin of the self-similar dust solution is non-regular in that the circumference function is non-zero in the limit $r \to 0$ unless $E_\infty = 0$. Thus the coordinate origin is an expanding two-sphere and the solution must be patched onto a non-self-similar solution at the centre. This produces anomalous behaviour in the $r$-dependence of the Hubble parameter, which is in contradiction with the observational data.

Although the self-similar dust solution is not a viable model for a void in the real Universe, the similarity hypothesis is not really expected to apply in the dust situation since it probably requires the effects of pressure. It is therefore more natural to assume that a void only tends to self-similar form in the radiation-dominated ($\alpha = 1/3$) era before decoupling. Since the special conditions required for self-similar evolution in the radiation
era are incompatible with self-similar evolution after decoupling, this suggests that one should merely use the self-similar radiation solution to set the initial conditions for the non-self-similar Tolman-Bondi evolution in the dust era. More precisely, the similarity solutions specify the forms of $R$, $m$ and $E$ as functions of $r$ along the decoupling hypersurface and the Tolman-Bondi equations then give the evolution of $R(t, r)$ for each shell of constant $r$. From this one can calculate the various Hubble parameters at any given epoch.

Carr and Whinnnett find models that are in agreement with the observational data and clearly show a variation of the Hubble parameter with distance. However, these models have a drawback. The strength of the initial radiation perturbation is determined by the single parameter $A_0$, which fixes both the density contrast and the size of the void relative to the particle horizon. To obtain a void that is large enough to contain the Coma cluster at the present epoch it is necessary to choose a value for $A_0$ which implies that the density contrast at decoupling exceeds the mean perturbations allowed by the data from the COBE satellite. In addition, at the current epoch, the void has a local density parameter which is much lower than the observed value. One can select a smaller value of $A_0$ to produce the required density contrast but, in this case, the void radius is too small.

D. The Similarity Hypothesis.

The “similarity hypothesis” proposes that, in a variety of physical situations, solutions may naturally evolve to self-similar form even if they start out more complicated. We have already mentioned several examples of this. For example, we noted in Section 1 that self-similar asymptotics can be obtained from dimensional considerations in a wide range of contexts in fluid dynamics (Barenblatt and Zeldovich 1972) and we saw in Section 2 that self-similar solutions act as asymptotic states in the context of spatially homogeneous cosmological models. We also noted in Section 2C the conjecture of Hewitt and Wainwright (1990) that self-similar solutions may describe the asymptotic or intermediate dynamical behaviour of orthogonally transitive $G_2$ models. It is even possible that the hypothesis extends to self-similarity of the second kind. For example, the “cosmic no-hair theorem” asserts that all cosmological models asymptote to the de Sitter solution in the presence of a cosmological constant (cf. Wald 1993). de Sitter spacetime is not homothetic but it is
self-similar in that it admits a kinematic self-similarity of the zeroth kind (Coley 1997a; see also Section 5A).

In this section, we will consider the plausibility of this hypothesis in the spherically symmetric context. There are a variety of astrophysical and cosmological situations (both Newtonian and relativistic) in which spherically symmetric solutions seem to evolve to self-similar form. For example, an explosion in a homogeneous background produces fluctuations which may be very complicated initially but which tend to be described more and more closely by a spherically symmetric similarity solution as time evolves (Sedov 1967). We saw in Section 4C that this applies even if the explosion occurs in an expanding cosmological background and it may indeed be a general feature of voids, whatever their origin. Overdense regions in the hierarchically clustering scenario in Newtonian cosmology may also tend to self-similar form (Quinn et al. 1986, Frenk et al. 1988) due to non-linear effects (Gunn and Gott 1972, Gunn 1977, Fillmore and Goldreich 1984, Bertschinger and Watts 1988, Syer and White 1997). In the non-cosmological context, a gravitationally bound cloud collapsing from an initially uniform static configuration may also evolve to self-similar form (Penston 1969, Larson 1969). This is understood theoretically, at least in a Newtonian context, as arising from such processes as virialization, shell-crossing and violent relaxation (Lynden-Bell 1967). In addition, it is known that in general relativity all static, spherically symmetric perfect fluid solutions with $p = \alpha \mu$ are asymptotically self-similar (Collins 1985, Goliath et al. 1998b).

These considerations led Carr (1993) to propose the “cosmological similarity hypothesis”. This states says that, under certain circumstances (e.g., non-zero pressure, high non-linearity, shell-crossing, processes analogous to virialization), cosmological solutions will naturally evolve to a spherically symmetric self-similar form, whatever the nature of the primordial fluctuations. We saw an application of this in Section 4C. Another application involves the studies of hierarchical cosmological models (Wesson 1979, 1981, 1982). In principle, this proposal is directly testable using numerical studies of spherically symmetric perturbations of Friedmann models with pressure (cf. Frauendorfer and Schmidt 1993), but this has not yet been done.
Presumably a necessary (but not sufficient) condition for the similarity hypothesis to be valid is that spherically symmetric similarity solutions (or at least some subset of them) be stable to non-self-similar perturbations. As a first step to studying this, Carr and Coley (1998b) have therefore investigated the stability of spherically symmetric similarity solutions within the more general class of spherically symmetric solutions. Following Cahill and Taub (1971), they express all functions in terms of the similarity variable \( z = r/t \) and the radial coordinate \( r \) and regard these as the independent variables rather than \( r \) and \( t \). They also assume that the perturbations in \( S \) and \( x \) (defined in Section 3A) can be expressed in the form

\[
S(z, r) = S_0(z)[1 + S_1(r)], \quad x(z, r) = x_0(z)[1 + x_1(r)]
\]

(4.3)

where a subscript 0 indicates the form of the function in the exact self-similar case and a subscript 1 indicates the fractional perturbation in that function (taken to be small; i.e., \( S_1 << 1 \) and \( x_1 << 1 \)). The perturbation equations for \( S_1(r) \) and \( x_1(r) \) can then be expressed as second order differential equations in \( r \) and Carr and Coley test whether a particular similarity solution is stable by examining, for example, whether the perturbation terms grow or decay at large values of \( r \). It might seem more natural to examine whether the solution grows or decays with time. However, the \( t \) evolution is entirely contained within the \( z \) evolution (viz. \( \partial f/\partial t = -t^{-1}\dot{f} \)), so if one wrote the perturbation equations in terms of \( t \) and \( z \) (instead of \( r \) and \( z \)), one would get exactly equivalent results.

Carr and Coley (1998b) come to the following conclusions:

- The asymptotically Friedmann solutions are stable providing \( \alpha > -1/3 \). This directly relates to the issue of whether density perturbations naturally evolve to self-similar form. Of course, it does not prove the validity of the similarity hypothesis since only small perturbations of self-similar models are considered. Neither do they consider non-spherical perturbations, for which the equations would be even more complicated. However, it does seem to be a rather general property of perturbations in an expanding Universe that they tend to sphericity at late times.

- As already shown by Ori and Piran (1988), transonic similarity solutions which are not analytic at the sonic point are unstable. This relates to what they term the “kink”
instability: non-analytic solutions either develop shocks or are driven towards analytic ones. This also applies in the Newtonian context (Whitworth and Summers 1985).

- The asymptotically Kantowski-Sachs solutions are only stable for \(-1 < \alpha < -1/3\), corresponding precisely to the range in which the solutions are physical. This may relate to the formation of bubbles in an inflationary scenario and hence to the instability of the inflationary phase itself.

- The stability of the asymptotically quasi-static solutions is still undetermined but is clearly of great physical interest. In particular, the stability of the ones containing naked singularities presumably bears upon the cosmic censorship hypothesis (discussed in Section 4A), while the stability of the critical solutions must relate to the results of previous studies (discussed in Section 4B) which show that the critical solutions are unstable to a single mode.

5. Generalized Self-Similarity

Self-similarity in the broadest sense refers to the situation in which a system is not restricted to be invariant under the relevant group action but merely appropriately rescaled. The basic condition for a manifold vector field \(\xi\) to be a self-similar generator is that there exist constants \(d_i\) such that, for each independent physical field \(\Phi^i\),

\[
\mathcal{L}_\xi \Phi^i = d_i \Phi^i. \tag{5.1}
\]

For each \(i\), \(d_i\) is a constant which (formally) is the scalar product of the dimensionality covector of \(\Phi^i\) with respect to the rescaling algebra and some rescaling algebra vector (Carter and Henriksen 1991).

In the Newtonian case, the physical fields consist of \(\mu\), \(p\) and \(\phi\) (the Newtonian gravitational potential), and \(\xi\) is the generator of a self-similarity with respect to a three-dimensional rescaling algebra vector (time, length and mass). Since \(\mu\), \(p\) and \(\phi\) are scalar fields, \(\mathcal{L}_\xi\) denotes their directional derivative. The ordinary continuity and dynamical evolution equations are preserved by the action of this three-parameter rescaling group.
Unfortunately, in general the self-similarity will not survive when additional laws governing $p$ (the equation of state) and $\phi$ (Poisson’s equation) complete the system of equations (Carter and Henriksen 1991). In order for the system to remain invariant, further restrictions are imposed. The effective invariance of the Poisson equation restricts one to a two-dimensional subalgebra. Further restrictions then arise from the imposition of an equation of state, except in the pressure-free case. For example, the special polytropic case $p = p_0 \mu^\gamma$ (where $\gamma$ is the polytropic index) effects reduction to a one-parameter rescaling subalgebra, resulting in a specific “self-similar” index. It is through such an index that a formal basis can be provided for classifying self-similarity as first class or, more generally, second class (Barenblatt and Zeldovich 1972).

These Newtonian ideas have been adapted to the relativistic context by Carter and Henriksen (1989; CH). Clearly, the characterizing equations must be generalized to tensorial counterparts of the covariant form

$$\mathcal{L}_\xi \Phi^i_A = d_i \Phi^i_A,$$  \hspace{1cm} (5.2)

where the fields $\Phi_A$ can be scalar (e.g., $\mu$), vectorial (e.g., $u_a$) or tensorial (e.g., $g_{ab}$). In particular, in general relativity the gravitational potential $\phi$ is replaced by the metric tensor $g_{ab}$ and an appropriate definition of “geometrical” self-similarity is necessary. In the seminal work by Cahill and Taub (1971), the simplest generalization was effected, whereby the metric itself satisfies an equation of the form (5.2); in this case, $\xi$ is a homothetic vector and this corresponds to Zeldovich’s similarity of the first kind.

However, in general relativity it is not the energy-momentum tensor itself that must satisfy (5.2); rather each of the physical fields making up the energy-momentum tensor must separately satisfy an equation of this form. For a fluid characterized by a timelike congruence $u_a$, the energy-momentum tensor can be uniquely decomposed with respect to $u_a$ (Ellis 1971), each component having a physical interpretation in terms of the energy, pressure, heat flow and anisotropic stress as measured by an observer comoving with the fluid, and each separately satisfying an equation of the form (5.2). In the same way, if the metric can be uniquely, physically and covariantly decomposed, then the homothetic condition can be replaced by the condition that each component must satisfy (5.2). For a
fluid, the metric can be uniquely decomposed, through the projection tensor

\[ h_{ab} = g_{ab} + u_a u_b . \]  

(5.3)

This represents the projection of the metric into the 3-spaces orthogonal to \( u^a \) (i.e., into the rest frame of the comoving observers). If \( u_a \) is irrotational, these 3-spaces are surface-forming, the decomposition is global and \( h_{ab} \) represents the intrinsic metric of these 3-spaces. The projection tensor is the first fundamental form of the hypersurfaces orthogonal to \( u^a \). It can be regarded as the relativistic counterpart of the Newtonian metric tensor, when the flow-independent \( u_a \) is defined as the relativistic counterpart of the preferred (irrotational) Newtonian time covector \(-t, a\) (CH).

A. Kinematic Self-Similarity.

By using arguments similar to these and, more importantly, comparing with self-similarity in a continuous Newtonian medium, CH have introduced the covariant notion of *kinematic self-similarity* in the context of relativistic fluid mechanics. A kinematic self-similarity vector \( \xi \) satisfies the conditions

\[ \mathcal{L}_\xi u_a = \overline{\alpha} u_a, \]  

(5.4)

where \( \overline{\alpha} \) is a constant and

\[ \mathcal{L}_\xi h_{ab} = 2h_{ab} . \]  

(5.5)

\( \xi \) has been normalized so that the constant in (5.5) has been set to unity. Evidently, in the case \( \overline{\alpha} = 1 \), it follows that \( \xi \) is a HV (Cahill and Taub 1971), corresponding to self-similarity of the first kind (Barenblatt and Zeldovich 1972). CH then argue that the case \( \overline{\alpha} \neq 1 \) is the natural relativistic counterpart of self-similarity of the more general second kind, while \( (\overline{\alpha} = 0) \) corresponds to self-similarity of the zeroth kind.

The parameter \( \overline{\alpha} \) is the constant proportionality factor between the rates of dilation of the length-scales and time-scales. When \( \overline{\alpha} \neq 1 \) (i.e., when \( \xi \) is not a HV), the relative rescaling of space and time under \( \xi \) are not the same. When \( \overline{\alpha} = 0 \), there is space dilation without time amplification. The parameter \( \overline{\alpha} \) defined by (5.4) is equivalent to the self-similar index which arises in the Newtonian case under the usual normalization \( (u_a t, a = 1) \).
All of the physical fields must satisfy equations of the form (5.2). In the case of a perfect fluid, in addition to the “kinematic” self-similarity condition (5.4) and the “geometric” self-similarity condition (5.5), the physical energy and pressure must therefore satisfy the conditions

\[ \mathcal{L}_\xi \mu = a\mu, \quad \mathcal{L}_\xi p = b p, \]  

(5.6)

where \( a \) and \( b \) are constants. In the exceptional pressure-free case we have

\[ \mathcal{L}_\xi T_{ab} = (2\alpha + a)T_{ab}. \]  

(5.7)

If \( p = p(\mu) \), then one necessarily has a polytropic equation of state, \( p = p_0\mu^\gamma \), and equations (5.6) imply that \( b = a\gamma \). Kinematic self-similarities in perfect fluid spacetimes have been extensively studied in Coley (1997a). In particular, a set of “integrability” conditions for the existence of a proper kinematic self-similarity in such spacetimes was derived; these integrability conditions constitute a set of further constraints arising from the compatibility of the EFEs and equations (5.4)–(5.6).

**B. Examples.**

In the *spherically symmetric* case, CH have shown that there exist comoving coordinates in which the self-similar generator is

\[ \xi^a \frac{\partial}{\partial x^a} = (\overline{\alpha}t + \overline{\beta}) \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}, \]  

(5.9)

and the metric is given by (3.1), where \( \nu, \lambda \) and \( S = R/r \) again depend only on the self-similarity coordinate \( z \). The metric is manifestly of the same form as in Cahill and Taub (1971) and the resulting governing differential equations do indeed reduce to a system of ODEs.

For self-similarity of the *first* kind (the homothetic case), \( \overline{\alpha} = 1 \), \( \overline{\beta} \) can be rescaled to zero, and \( z = r/t \) as usual. In the less-studied *zeroth* case, \( \overline{\alpha} = 0 \), \( \overline{\beta} \) can be rescaled to unity and \( z = re^{-t} \). Examples of this case are provided by the solution of Henriksen, Emslie and Wesson (1983), in which a dimensional constant (and hence a fundamental scale) is introduced via the cosmological constant, and the solution of Alexander et al. (1989), which
represents self-similar perturbations of a de Sitter universe. These solutions also relate to the Kantowski-Sachs models with $\alpha = -\frac{1}{3}$ studied in Section 3C.

In the general case with $\overline{\alpha} \neq 0$ or 1 and $\overline{\beta}$ rescaled to zero, corresponding to self-similarity of the second kind, the self-similarity coordinate is given by

$$z = r(\overline{\alpha}t)^{-1/\overline{\alpha}}.$$  \hfill (5.10)

An important example of this is provided by a class of zero-pressure perfect fluid models (i.e., dust models in which $u^a$ is geodesic) first studied by Lynden-Bell and Lemos (1988) and described in detail by Henriksen (1989) and CH. This class of models has recently been generalized to the non-zero-pressure case as follows (Benoit and Coley 1998a; BC). Using the similarity variables defined above and the same notation as in Section 3, the metric for $S + \dot{S} \neq 0$ is given by

$$ds^2 = -dt^2 + (S + \dot{S})^2dr^2 + r^2S^2d\Omega^2,$$  \hfill (5.11)

where $S(z)$ is a solution of the differential equation

$$2\ddot{S}S + \dot{S}^2 + 2\alpha\dot{S}S + k = 0,$$  \hfill (5.12)

and $k$ is a dimensional constant. This assumes that $\dot{S} \neq 0$ (i.e., the spacetime is not static) and that $\overline{\alpha}$ is neither 0 nor 1 (i.e., $\xi$ is not a homothetic vector). The comoving perfect fluid is described by

$$\mu = W(\xi)t^{-2}, \quad p = \frac{1}{4}k\overline{\alpha}^{-4}t^{-2}.$$  \hfill (5.13)

where $W(\xi)$ is a function of $\xi$ given by BC. [This solution is similar to the spherically symmetric generalization of the Kantowski-Sachs model discussed by Wesson (1989) in which $S + \dot{S} = 0$.] The dust solution of the Tolman family, described in CH, is obtained when $k = 0$. We note that in the solution above there is a dimensional constant appearing in the pressure, a property that is characteristic of self-similarity of the second kind (Barenblatt and Zeldovich 1972). This class of perfect fluid solutions does not admit any homothetic vectors. It has been generalized to the anisotropic fluid case by Benoit and Coley (1998b).

There is another case of potential interest in which the parameter $\overline{\alpha}$ occurring in equation (5.4) approaches infinity, and we could refer to this case as kinematic self-similarity of the
“infinite” kind. This case could be covariantly defined by equation (5.4) with $\alpha$ normalized and equation (5.5) with the right-hand side set to zero (i.e., $\mathcal{L}_\xi h_{ab} = 0$). Consequently $\xi$ represents a generalized “rigid motion”. This case has been investigated by Benoit et al. (1998).

Recently, kinematic self-similarity in perfect fluid spacetimes has been studied in the spherically symmetric case (Benoit and Coley 1998a), the plane symmetric and hyperbolic cases (Benoit et al. 1998) and the locally rotationally symmetric case (Sintes 1997). Note that there is some evidence that kinematic self-similar spherically symmetric solutions asymptote towards exact solutions that admit a HV. For example, if $p = \alpha \mu$, these models necessarily reduce to either the exact flat Friedmann model or the static model, both of which admit a HV. In general, kinematic self-similar models do not admit an equation of state. However, all kinematic self-similar spherically symmetric models in which the energy density and the pressure separately satisfy the “physical” self-similar conditions (5.6) have been shown to asymptote to an exact solution that admits a HV (BC). It is therefore possible that the exact Friedmann, Kantowski-Sachs and static homothetic solutions play the same role in describing the asymptotic behaviour of generalized self-similar solutions as they did for the homothetic models discussed in Section 3.

Another generalization of homothety is so-called partial homothety (Tomita 1981; see also Tomita and Jantzen 1983) and this corresponds to an intrinsic symmetry (Collins 1979). Ponce de Leon (1993) has studied partial homotheties in spherically symmetric fluid models and attempted to relate the existence of such a symmetry to the notion of generalized self-similarity. However, Ponce de Leon’s approach is not covariant: even in the spherically symmetric situation there are ambiguities in the shear-free case. In addition, the matter fields do not satisfy equations like (5.2) [i.e., like (5.6)], so the models are not “physically” self-similar (Ponce de Leon 1988, 1989).

6. Final Remarks

We conclude with some general remarks about outstanding problems and areas of the subject which are likely to see exciting developments in the next few years.
• We saw in Section 5B that the asymptotic analysis of the spherically symmetric homothetic models discussed in Section 3 might be applicable more generally. Most of this review has focussed on perfect fluids with an equation of state of the form $p = \alpha \mu$, so our analysis does not cover more general self-similar perfect fluids (Anile et al. 1987) or anisotropic fluids (Herrera and Ponce de Léon 1985a,b), even though these may be of physical interest. However, there are clearly ways of extending the analysis. For example, it can be shown that a two-perfect-fluid model, in which the two (necessarily comoving) fluids each have an equation of state of the form $p_i = \alpha_i \mu_i \ (i = 1, 2)$, is formally equivalent to a model with a single perfect fluid that does not have an equation of state. It is plausible that perfect fluid models for which $p/\mu$ is asymptotically constant have the same asymptotic behaviour as the $p = \alpha \mu$ perfect fluid models discussed here. This is indeed the case for the self-similar two-perfect-fluid model if each of the two fluids separately satisfy the conservation equations (Carr and Coley 1998a). In addition, an anisotropic fluid solution in which the fluid has the form of a perfect fluid asymptotically might also display these properties. For example, a source consisting of a perfect fluid plus an electromagnetic field (satisfying the Einstein-Maxwell equations) can be formally equivalent to an anisotropic fluid source.

• It is clear that a full understanding of the relationship between critical phenomena and self-similarity will yield important insights, even though the precise relationship between continuous self-similar solutions and critical phenomena remains controversial (Carr and Henriksen 1998, Carr et al. 1998). In particular, an existence proof for the critical solutions, which so far have only been constructed numerically, may be possible (Brady 1995, Gundlach 1997). It is still not clear why the critical solution is sometimes associated with a discrete self-similarity and whether this type of similarity is more generic than its continuous counterpart. Doubtless resolving these issues will require a deeper understanding of discrete self-similarity in general.

• Much work remains to be done in understanding the status of the similarity hypothesis, especially in the spherically symmetric context. In particular, the conditions under which this hypothesis is likely to hold have still not been identified. On the other hand, this is a problem which is ideally suited for both numerical studies and empirical cosmological tests, so one can expect progress. Further studies of the stability of the self-similar solutions will
likely yield new insights.

ACKNOWLEDGMENTS

We would like to thank J. Carot, M. Choptuik, M. Goliath, C. Gundlach, R. Henriksen, C. Hewitt, K. Lake, M. Mars, U. Nilsson, A. Sintes, C. Uggla, J. Wainwright and the referee for helpful comments. AAC would like to acknowledge the Natural Sciences and Engineering Research Council for financial support. BJC would like to thank the Department of Mathematics, Statistics and Computing Science at Dalhousie University for hospitality while this work was carried out.
REFERENCES

A. M. Abrahams and C. R. Evans, 1993, Phys. Rev. Lett. 70, 2980.

D. Alexander, R. M. Green and A. G. Emslie, 1989, MNRAS, 237, 93.

A. M. Anile, G. Moschetti and O. I. Bogoyavlenski, 1987, J. Math. Phys. 28, 2942.

G. I. Barenblatt, 1952, Prikl. Mat. Mekh. 16, 67.

G. I. Barenblatt and Ya B. Zeldovich, 1972, Ann. Rev. Fluid Mech. 4, 285.

J. D. Barrow and F. J. Tipler, 1986, The Anthropic Principle (Oxford University Press, Oxford).

C. Barrabes, W. Israel and P. S. Letelier, 1991, Phys. Lett. A 160, 41.

P. M. Benoit and A. A. Coley, 1998a, Class. Quantum Grav.

P. M. Benoit and A. A. Coley, 1998b, J. Math. Phys.

P. M. Benoit, A. A. Coley and A-M. Sintes, 1998, preprint.

E. Bertschinger, 1985, Ap. J. 268, 17.

E. Bertschinger and P. N. Watts, 1984, Ap. J. 328, 23.

G. V. Bicknell and R. N. Henriksen, 1978a, Ap. J. 219, 1043.

G. V. Bicknell and R. N. Henriksen, 1978b, Ap. J. 225, 237.

G. V. Bicknell and R. N. Henriksen, 1979, Ap. J. 232, 670.

M. Birkinshaw and J. P. Hughes, 1994, Ap. J. 420, 33.

O. I. Bogoyavlenski, 1977, Sov. Phys. JETP 46, 634.

O. I. Bogoyavlenski, 1985, Methods in the Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamics (Springer-Verlag).

O. I. Bogoyavlenski and S. P. Novikov, 1973, Sov. Phys.-JETP 37, 747.

H. Bondi, 1947, MNRAS 107, 410.

W. B. Bonnor, 1956, Z. Astrophys. 39, 143.

P. R. Brady, 1995, Phys. Rev. D 51, 4168 .

P. R. Brady, C. M. Chambers and S. M. C. V. Concalves, 1998, Phys. Rev. D.

M. Bruni, S. Matarrase and O. Pantano, 1995, Ap. J. 445, 958.

A. B. Burd and A. A. Coley, 1994, Class. Quantum Grav. 11, 83.

A. H. Cahill and M. E. Taub, 1971, Comm. Math. Phys. 21, 1.

J. Carot and A. M. Sintes, 1997, Class Quantum Grav. 14, 1183.

J. Carot, J. da Costa and E. G. L. R. Vaz, 1994, J. Math. Phys. 35, 4832.
B. J. Carr, 1976, Ph.D. thesis (Cambridge).
B. J. Carr, 1993, preprint prepared for but omitted from *The Origin of Structure in the Universe*, ed. E. Gunzig and P. Nardone (Kluwer).
B. J. Carr, 1997, in *Proceedings of the Seventh Canadian Conference on General Relativity and Relativistic Astrophysics*, ed. D. Hobill (Calgary Press).
B. J. Carr and S. W. Hawking, 1974, MNRAS 168, 399.
B. J. Carr and A. Yahil, 1990, Ap. J. 360, 330.
B. J. Carr and A. Koutras, 1992, Ap. J. 405, 34.
B. J. Carr and A. A. Coley, 1998a, Phys. Rev. D.
B. J. Carr and A. A. Coley, 1998b, preprint.
B. J. Carr and R. N. Henriksen, 1998, preprint.
B. J. Carr and A. Whinnett, 1997, MNRAS.
B. J. Carr, A. A. Coley, M. Goliath, U. S. Nilsson and C. Uggla, 1998, preprint.
B. Carter and R. N. Henriksen, 1989, Ann. Physique Supp. 14, 47.
B. Carter and R. N. Henriksen, 1991, J. Math. Phys. 32, 2580.
W. Z. Chao, 1981, Gen. Rel. Grav. 13, 625.
M. W. Choptuik, 1993, Phys. Rev. Lett. 70, 9.
M. W. Choptuik, 1994, in *Deterministic Chaos in General Relativity*, ed. D. Hobill et al. (Plenum, New York).
M. W. Choptuik and S. Liebling, 1996, Phys. Rev. Lett. 77, 1424.
M. W. Choptuik, T. Chmaj and P. Bizon, 1996, Phys. Rev. Lett. 77, 424.
D. Christodoulou, 1984, Commun. Math. Phys. 93, 171.
A. A. Coley, 1997a, Class. Quantum Grav. 14, 87.
A. A. Coley, 1997b, in *Proceedings of the Sixth Canadian Conference on General Relativity and Relativistic Astrophysics*, eds. S. Braham, J. Gegenberg and R. McKellar, Fields Institute Communications Series (AMS), Volume 15, p.19 (Providence, RI).
A. A. Coley and B. O. J. Tupper, 1983, Ap. J. 271, 1.
A. A. Coley and B. O. J. Tupper, 1989, J. Math. Phys. 30, 2616.
A. A. Coley and B. O. J. Tupper, 1990, Class. Quantum Grav. 7, 1961.
A. A. Coley and J. Wainwright, 1992, Class. Quantum Grav. 9, 651
A. A. Coley and R. J. van den Hoogen, 1994a, J. Math. Phys. 35, 4117.
A. A. Coley and R. J. van den Hoogen, 1994b, in *Deterministic Chaos in General
Relativity, ed. D. Hobill et al. (Plenum, New York).

A. A. Coley and R. J. van den Hoogen, 1995, Class. Quantum Grav., to appear.
A. A. Coley and J. Wainwright, 1998, preprint.
A. A. Coley, R. J. van den Hoogen and R. Maartens, 1996, Phys. Rev. D. 54, 1393.
A. A. Coley, J. Ibañez and R.J. van den Hoogen, 1997, J. Math. Phys. 38, 5256.
C. B. Collins, 1971, Comm. Math. Phys. 23, 137.
C. B. Collins, 1974, Comm. Math. Phys. 39, 131.
C. B. Collins, 1977, J. Math. Phys. 18, 2116.
C. B. Collins, 1979, Gen. Rel. Grav. 10, 925.
C. B. Collins, 1985, J. Math. Phys. 26, 2268.
C. B. Collins and J. M. Stewart, 1971, MNRAS 153, 419.
C. B. Collins and S. W. Hawking, 1973, Ap. J. 180, 317.
C. B. Collins and J. M. Lang, 1987, Class. Quantum Grav. 4, 61.
L. Defrise-Carter, 1975, Comm. Math. Phys. 40, 273.
A. G. Doroshkevich, V. N. Lukash and I. D. Novikov, 1973, Sov. Phys.-JETP 37, 739.
C. C. Dyer, 1979, MNRAS 189, 189.
C. Eckart, 1940, Phys. Rev. 58, 919.
D. M. Eardley, 1974, Comm. Math. Phys. 37, 287.
D. M. Eardley and L. Smarr, 1979, Phys. Rev. D 19, 2239.
D. M. Eardley, J. Isenberg, J. Marsden and V. Moncrief, 1986, Comm. Math. Phys. 106, 137.
G. F. R. Ellis, 1971, Relativistic Cosmology, in General Relativity and Cosmology, XLVII Corso, Varenna, Italy (1969), ed R. Sachs (Academic, New York).
G. F. R. Ellis and M. A. H. MacCallum, 1969, Comm. Math. Phys. 12, 108.
C. R. Evans and J. S. Coleman, 1994, Phys. Rev. Lett. 72, 1782.
A. Feinstein and J. Ibañez, 1993, Class. Quantum Grav. 10, 93.
J. A. Fillmore and P. Goldreich, 1984, Ap. J. 281, 1.
T. Foglizzo and R. N. Henriksen, 1993, Phys. Rev. D. 48, 4645.
J. Frauendiener and B. G. Schmidt, 1993, Gen. Rel. Grav. 25, 373.
W. L. Freedman et al., 1994, Nature 371, 757.
W. L. Freedman, 1997, preprint.
C. S. Frenk et al., 1988, Ap. J. 351, 10.
A. V. Frolov, 1997, preprint.
M. J. Geller and J. P. Huchra, 1989, Science 246, 897.
B. B. Godfrey, 1972, Gen. Rel. Grav. 3, 3.
M. Goliath, U. S. Nilsson and C. Uggla, 1998a, Class Quantum Grav. 15, 167.
M. Goliath, U. S. Nilsson and C. Uggla, 1998b, Class Quantum Grav.
C. Gundlach, 1995, Phys. Rev. Lett. 75, 3214.
C. Gundlach, 1997, Phys. Rev. D. 55, 695.
J. E. Gunn, 1977, Ap. J. 218, 592.
J. E. Gunn and J. R. Gott, 1972, Ap. J. 176, 1.
V. T. Gurovich, 1967, Sov. Phys. Doklady 11, 569.
G. Haager and M. Mars, 1998, Class. Quantum Grav.
G.S. Hall and D. Steele, 1990, Gen. Rel. Grav. 22, 457.
G. S. Hall, D. J. Low and J. R. Pulham, 1994, J. Math. Phys. 35, 5930.
R. S. Hamadé, J. H. Horne and J. M. Stewart, 1996, Class. Quantum Grav. 13, 2241.
M. A. Hausman et al., 1983, Ap. J. 270, 351.
R. N. Henriksen, 1989, MNRAS 240, 917.
R. N. Henriksen and P. S. Wesson, 1978a, Ap. Sp. Sci. 53, 429.
R. N. Henriksen and P. S. Wesson, 1978b, Ap. Sp. Sci. 53, 445.
R. N. Henriksen, A. G. Emslie and P. S. Wesson, 1983, Phys. Rev. D 27, 1219.
R. N. Henriksen and K. Patel, 1991, Gen. Rel. Grav. 23, 527.
L. Herrera and J. Ponce de Léon, 1985a, J. Math. Phys. 26, 2302.
L. Herrera and J. Ponce de Léon, 1985b, J. Math. Phys. 27, 2987.
C. G. Hewitt and J. Wainwright, 1990, Class. Quantum Grav. 7, 2295.
C. G. Hewitt and J. Wainwright, 1992, Phys. Rev. D 46, 4242.
C. G. Hewitt and J. Wainwright, 1993, Class. Quantum Grav. 10, 99.
C. G. Hewitt, J. Wainwright and S. W. Goode, 1988, Class. Quantum Grav. 5, 1313.
C. G. Hewitt, J. Wainwright and M. Glaum, 1991, Class. Quantum Grav. 8, 1505.
E. W. Hirschmann and D. M. Eardley, 1995a, Phys. Rev. D 51, 4198.
E. W. Hirschmann and D. M. Eardley, 1995b, Phys. Rev. D 52, 5850.
S. Hod and T. Piran, 1997, Phys. Rev. D. 55, R440.
Y. Hoffman and J. Shaham, 1985, Ap. J. 297, 16.
R. J. van den Hoogen, 1995, Ph.D. Thesis (Dalhousie University).
R. J. van den Hoogen, A. A. Coley and J. Ibañez, 1997, Phys. Rev. D. 55, 1.
L. Hsu and J. Wainwright, 1986, Class. Quantum Grav. 3, 1105.
J. Ibañez, R. J. van den Hoogen and A. A. Coley, 1995, Phys. Rev. D 51, 928.
S. Ikeuchi, K. Tomisaka and J. P. Ostriker, 1983, Ap. J. 265, 583.
W. Israel, 1984, Found. Phys. 14, 1049.
K. C. Jacobs, 1968, Ap. J. 153, 661.
R. T. Jantzen, 1984, Cosmology of the Early Universe, ed R. Ruffini and L. Fang (World
Scientific, Singapore).
R. T. Jantzen and K. Rosquist, 1986, Class. Quantum Grav. 3, 281.
R. Kantowski and R. Sachs, 1966, J. Math. Phys. 7, 443.
T. Koike, T. Hara and S. Adachi, 1995, Phys. Rev. Lett. 74, 5170.
A. Koutras, 1992, Ph.D. thesis (Queen Mary and Westfield College).
D. Kramer, H. Stephani, M. A. H. MacCallum and E. Herlt, 1980, Exact Solutions of
Einstein’s Field Equations (Cambridge University Press, Cambridge).
A. Krasinzie, 1997, Physics in an Inhomogeneous Universe (Cambridge University
Press, Cambridge).
K. Lake, 1992, Phys. Rev. Lett. 68, 3129.
K. Lake and T. Zannias, 1990, Phys. Rev. D 41, 3866.
R. B. Larson, 1969, MNRAS 145, 271.
T. R. Lauer and M. Postman, 1994, Ap. J. 425, 418.
D. N. C. Lin, B. J. Carr and S. D. M. Fall, 1978, MNRAS 177, 151.
X. Lin and R. M. Wald, 1989, Phys. Rev. D 40, 3280.
J. P. Luminet, 1978, Gen. Rel. Grav. 9, 673.
D. Lynden-Bell, 1967, MNRAS 136, 101.
D. Lynden-Bell and J. P. S. Lemos, 1988, MNRAS 233, 197.
P. K-H. Ma and J. Wainwright, 1994 in Deterministic Chaos in General Relativity, ed.
D. Hobill et al. (Plenum, New York).
R. Maartens and S. D. Maharaj, 1986, Class. Quantum Grav. 3, 1005.
R. Maartens, N. P. Humphreys, D. R. Matravers and W. R. Stoeger, 1996, preprint.
S. D. Maharaj, 1988, J. Math. Phys. 29, 1443.
D. Maison, 1996, Phys. Lett. B. 366, 82.
C. B. G. McIntosh, 1975, Gen. Rel. Grav. 7, 199.
B. D. Miller, 1976, Ap. J. 208, 275.
C. W. Misner and H. S. Zapolsky, 1964, Phys. Rev. Lett. 12, 635.
J. W. Moffat and D. C. Tatarski, 1992, Phys. Rev. D. 45, 3512.
J. W. Moffat and D. C. Tatarski, 1995, Ap. J. 353, 17.
G. Moschetti, 1987, Gen. Rel. Grav. 19, 155.
K. Nakao et al., 1995, Ap. J. 453, 541.
U. S. Nilsson and C. Uggla, 1997, Class. Quantum Grav. 14, 1965.
A. Ori and T. Piran, 1987, Phys. Rev. Lett. 59, 2137.
A. Ori and T. Piran, 1988, Mon. Not. R. Astron. Soc. 234, 821.
A. Ori and T. Piran, 1990, Phys. Rev. D. 42, 1068.
R. Penrose, 1969, Nuovo Cim. 1, 252.
M. V. Penston, 1969, MNRAS 144, 449.
M. J. Pierce et al., 1994, Nature 371, 211.
J. Ponce de Leon, 1988, J. Math. Phys. 29, 2479.
J. Ponce de Leon, 1990, J. Math. Phys. 31, 371.
J. Ponce de Leon, 1991, MNRAS 250, 69.
J. Ponce de Leon, 1993, Gen. Rel. Grav. 25, 865.
P. J. Quinn et al., 1986, Nature 322, 329.
G. F. R. N. Rhee, 1991, Nature 350, 211.
M. D. Roberts, 1989, Gen. Rel. Grav. 21, 907.
D. H. Roberts et al., 1991, Nature 352, 43.
K. Rosquist, 1984, Class. Quantum Grav. 1, 81.
K. Rosquist and R. T. Jantzen, 1988, Phys. Rep. 166, 189.
K. Rosquist, C. Uggla and R. T. Jantzen, 1990, Class. Quantum Grav. 7, 625.
H. Sato, 1984, in General Relativity and Gravitation, p289, ed. B. Bertotti et al. (Reidel, Dordrecht).
J. Schwartz, J. P. Ostriker and A. Yahil, 1975, Ap. J. 202, 1.
L. I. Sedov, 1946, Prikl. Mat. Mekh. 10, 241.
L. I. Sedov, 1967, Similarity and Dimensional Methods in Mechanics (New York, Academic).
S. L. Shapiro and S. A. Teukolsky, 1992, Phys. Rev. D 45, 2006.
X. Shi, L. M. Widrow and L. J. Dursi, 1996, MNRAS 281, 565 (1996).
I. S. Shikin, 1979, Gen. Rel. Grav. 11, 433.
S. T. C. Siklos, 1981, J. Phys. A 14, 395.
A. M. Sintes, 1996, Ph.D thesis (University of the Balearic Islands).
A. M. Sintes, 1997, preprint, “KSS LRS models”.
Y. Suto et al., 1995, Prog. Theor. Phys. 93, 839.
Syer and White, 1997.
A. H. Taub, 1972, General Relativity, papers in honour of J. L. Synge, ed. L.O’Raifeartaigh
(Oxford University Press, London).
A. H. Taub, 1973, Comm. Math. Phys. 29, 79.
G. I. Taylor, 1950, Proc. Roy. Soc. London A201, 175.
K. S. Thorne, 1967, Ap. J. 148, 51.
R. C. Tolman, 1934, Proc. Nat. Acad. Sci. 20, 169.
K. Tomita, 1981, Prog. Theor. Phys. 66, 2025.
K. Tomita, 1995, Ap. J. 451, 1.
K. Tomita, 1997a, Phys. Rev. D. 56, 3341.
K. Tomita, 1997b, Gen. Rel. Grav. 29, 815.
K. Tomita and R.T. Jantzen, 1983, Prog. Theor. Phys. 70, 886.
C. Uggla, 1992, Class. Quantum Grav. 9, 2287.
C. Uggla, R. T. Jantzen and K. Rosquist, 1995, Phys. Rev. D 51, 5522.
J. Wainwright, 1985, in Galaxies, Axisymmetric Systems and Relativity, ed. M. Mac-Callum (Cambridge University Press, Cambridge).
J. Wainwright and L. Hsu, 1989, Class. Quantum Grav. 6, 1409.
J. Wainwright and G. F. R. Ellis, 1997, Dynamical systems in cosmology (Cambridge University Press, Cambridge).
J. Wainwright, W. C. W. Ince and B. J. Marshman, 1979, Gen. Rel. Grav. 10, 259.
J. Wainwright, A. A. Coley, G. F. R. Ellis and M. Hancock, 1998, Class. Quantum Grav. 15, 331.
R. M. Wald, 1983, Phys. Rev. D. 28, 2118.
B. Waugh and K. Lake, 1988, Phys. Rev. D 38, 1315.
B. Waugh and K. Lake, 1989, Phys. Rev. D 40, 2137.
P. S. Wesson, 1979, Ap. J. 228, 647.
P. S. Wesson, 1981, Phys. Rev. D 23, 2137.
P. S. Wesson, 1982, Ap. J. 259, 20.

P. S. Wesson, 1989, Ap. J. 336, 58.

A. Whinnett, B. J. Carr and A. A. Coley (1998). In preparation.

A. Whitworth and D. Summers, 1985, MNRAS, 214, 1.

X. Wu et al., 1995, preprint.

K. Yano, 1955, The Theory of Lie Derivatives (North-Holland, Amsterdam).

T. Zannias, 1991, Phys. Rev. D 44, 2397.

Ya. B. Zeldovich and A. S. Kompaneets, 1950, Collection Dedicated to Joffe 61, ed. P.I. Lukirsky (Izd. Akad. Nauk SSSR, Moscow).

Ya. B. Zeldovich and Yu. P. Raizer, 1963, Physics of Shock Waves and High Temperature Phenomena (New York, Academic).

Ya. B. Zeldovich and I. D. Novikov, 1967, Soviet Astr. AJ. 10, 602.
Figures

FIGURE (1). This shows the form of \( V(z) \) for the exact Friedmann, static and (non-physical) Kantowski-Sachs solutions in the \( \alpha = 1/3 \) case. Also shown are the curves corresponding to \( M = 0 \) (solid), the sonic lines \( |V| = 1/\sqrt{\alpha} \) (broken) and the range of values of \( z \) (bold) in which curves can cross the sonic line regularly.

FIGURE (2). This shows the form of the scale factor \( S(z) \) and the velocity function \( V(z) \) for the full family of spherically symmetric similarity solutions with \( \alpha = 1/3 \). The exact Friedmann, Kantowski-Sachs and static solutions are indicated by the bold lines. Also shown (for different values of \( E_\infty \)) are the asymptotically Friedmann solutions and (for different values of \( E_\infty \) and \( D \)) the asymptotically quasi-static solutions. The latter contain a naked singularity when the minimum of \( V \) is below 1. The negative \( V \) region is occupied by the asymptotically Kantowski-Sachs solutions, though these may not be physical since the mass is negative. Solutions which are irregular at the sonic point are shown by broken lines. The dotted curve corresponds to a negative mass solution.

FIGURE (3). This shows the forms of the scale factor \( S(z) \) and the velocity function \( V(z) \) for the asymptotically Friedmann solutions with different values of \( E_\infty \). The \( z > 0 \) (\( z < 0 \)) solutions contain black (white) holes for \( E_\ast < E_\infty < E_{\text{crit}} \).

FIGURE (4). This shows the forms of the scale factor \( S(z) \) and the velocity function \( V(z) \) for the asymptotically quasi-static solutions with different values of \( E_\infty \) and \( D \). The solutions contain a naked singularity for \( E_\ast(D) < E_\infty(D) < E_{\text{crit}}(D) \).