The excitations of the symplectic integrable models and their applications

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Abstract

The Bethe ansatz equations of the fundamental $Sp(2N)$ integrable model are solved by a peculiar configuration of roots leading us to determine the nature of the excitations. They consist of $N$ elementary generalized spinons and $N-1$ composite excitations made by special convolutions between the spinons. This fact is essential to determine the low-energy behaviour which is argued to be described in terms of $2N$ Majorana fermions. Our results have practical applications to spin-orbital systems and also shed new light to the connection between integrable models and Wess-Zumino-Witten field theories.

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The study of quantum one-dimensional integrable models has turned out to be a fruitful venture since the seminal work of Bethe in 1931 [1]. Over the years, solvable models have been extremely useful in many subfields of physics, providing us a rich laboratory in which new theoretical insights and non-perturbative methods can readily be tested. Recent progress in the experimental study of low-dimensional materials, e.g. spin ladders and carbon nanotubes [2], has been an additional source of motivation to investigate one-dimensional exactly solvable models.

The basic concept of quantum integrability is the $S$-matrix which represents either the factorized scattering of particles of $(1+1)$ quantum field theories or the statistical weights of integrable two-dimensional lattice models. It turns out that the symmetry of the $S$-matrix plays a fundamental role in the theory and classification of integrable systems [3, 4]. Of particular interest is the $Sp(2N)$ symmetry which preserves bilinear antisymmetric metrics, typical of systems with $N$ component Dirac fermions. Even though the Bethe ansatz solution of the integrable $Sp(2N)$ models has long been known [5, 6], basic properties such as the nature of the elementary excitations and the low-energy behaviour have not yet been determined.

The purpose of this paper is to unveil the physical content of the fundamental (vector representation) $Sp(2N)$ solvable magnet. We argue that the low-energy properties are given in terms of $2N$ Majorana fermions due to the presence of special low-lying excitations in the spectrum. There exists at least two immediate applications of this result. First, it provides us a unique counterexample to the conjecture that integrable models based on the vector representation of Lie algebras should be lattice realizations of Wess-Zumino-Witten (WZW) conformal theories [1, 8, 9]. In fact, we predict $c = N$ for the fundamental integrable $Sp(2N)$ model while the central charge of the $Sp(2N)$ WZW theory is $c = N(2N + 1)/(N + 2)$. Next, our study is of utility to one-dimensional systems with coupled spin and orbital degrees of freedom such as the spin-orbital [10, 11, 12] and spin-tube [13] models. More precisely, we recall that the effective spin-isospin Hamiltonian describing these systems may be written in
the form [10, 14]

\[
H_{SO}(J_0, J_1, J_2) = \sum_{i=1}^{L} \sum_{\alpha=0}^{2} J_{\alpha} P^{(\alpha)}_{i,i+1}
\]  

(1)

where \(J_{\alpha}\) are superexchange constants and \(P^{(\alpha)}_{i,i+1}\) denote the respective projections on the singlet, triplet and doublet spin-isospin states, see ref.[14] for details. By writing these projectors in terms of two commuting sets of Pauli matrices it is not difficult to identify that the integrable \(Sp(4)\) spin chain [17] corresponds to the point \(J_0/J_1 = J_0/J_2 = 1/3\). This point is interesting because it corresponds to both anisotropic and asymmetric spin-1/2 couplings [18], being closer to represent the properties of realistic materials [19, 20] than the integrable \(SU(4)\) case \(J_0 = J_1 = J_2\) [16]. This then provides us a rare opportunity to determine exactly the nature of the excitations in a relevant spin-orbital model.

In the context of statistical mechanics the integrable \(Sp(2N)\) model is a multistate vertex system defined on the square lattice whose bonds variables take \(2N\) possible values. For each type of configuration of four bonds \(a,b,c,d\) meeting at a vertex we associate a Boltzmann weight factor \(S^{cd}_{ab}(\lambda)\) where \(\lambda\) is the spectral parameter. Compatibility between integrability and the \(Sp(2N)\) invariance (“ice-type” restriction) leads us to the following amplitudes [4, 6]

\[
S^{cd}_{ab}(\lambda) = \delta_{a,d}\delta_{c,b} + \lambda\delta_{a,c}\delta_{b,d} - \frac{\lambda}{\lambda + N + 1} \epsilon_a \epsilon_c \delta_{a,b} \delta_{c,d}
\]

(2)

where \(\bar{a} = 2N + 1 - a\), \(\epsilon_a = 1\) for \(1 \leq a \leq N\) and \(\epsilon_a = -1\) for \(N + 1 \leq a \leq 2N\).

For any integrable vertex model one can associate a local spin chain commuting with the corresponding transfer-matrix whose matrix elements are given by ordered product of \(L\) factors \(S^{cd}_{ab}(\lambda)\). As usual, the Hamiltonian is proportional to the logarithmic derivative of the transfer matrix at the regular point \(\lambda = 0\), and in this case the expression is

\[
H_{Sp(2N)} = \sum_{i=1}^{L} \left[ \delta_{a,d}\delta_{c,b} - \frac{1}{N+1} \epsilon_a \epsilon_c \delta_{a,b} \delta_{c,d} \right] e^{(i)}_{ac} \otimes e^{(i+1)}_{bd}
\]

(3)

where \(e^{(i)}_{ab}\) are the elementary matrix \([e_{ab}]_{l,k} = \delta_{a,l}\delta_{b,k}\) acting on site \(i\). We observe that the spectrum of \(H_{SO}(J_1/3, J_1, J_1)\) matches that of \(J_1 H_{Sp(4)} - J_1 L\). The \(Sp(2N)\) Hamiltonian (3) is solvable by the Bethe ansatz [3, 6] and its eigenvalues \(E(L)\) can be parametrized in terms of a
set of variables $\lambda_j^{(a)}$, $j = 1, \cdots, m_a$ and $a = 1, \cdots, N$, satisfying the following Bethe equations

$$
\begin{split}
\left[ \lambda_j^{(a)} - \frac{\delta_{a,1}}{\eta_a} \right]^L & = \prod_{b=1}^{N} \prod_{k=1, \ k \neq j}^{m_b} \frac{\lambda_j^{(a)} - \lambda_k^{(b)} - i \frac{C_{a,b}}{\eta_a}}{\lambda_j^{(a)} + \frac{\delta_{a,1}}{\eta_a}}
\end{split}
$$

(4)

and the eigenvalues are given by

$$
E(L) = - \sum_{i=1}^{m_1} \frac{1}{\left[ \lambda_i^{(1)} \right]^2 + 1/4} + L
$$

(5)

where $C_{ab}$ is the Cartan matrix and $\eta_a$ is the normalized length of the $a$-th root of the $Sp(2N)$ algebra.

We start our study considering first the $Sp(4)$ model motivated by its direct relevance to the physics of spin-orbital systems. In fact, this is the simplest symplectic invariant system since $N = 1$ is equivalent to the isotropic six-vertex model. Later on we will show that the technicalities entering in the analysis of the $Sp(4)$ model can be easily generalized to include arbitrary $N > 2$. Essential to our study is to determine the configurations of roots that describe the absolute ground state and the elementary excitations. This can be done by solving the Bethe equations (4,5) for some values of $L$ and comparing it with the exact diagonalization of the $Sp(4)$ spin chain. Its spectrum is parametrized by two set of variables $\lambda_j^{(1,2)}$ and we found that the ground state and the low-lying excitations are characterized by strings with different lengths, namely

$$
\lambda_j^{(1)} = \begin{cases} 
\xi_j^{(1)} \\
\xi_j^{(3)} \pm i \left[ \frac{1}{2} + O(e^{-\gamma L}) \right]
\end{cases}
$$

$$
\lambda_j^{(2)} = \xi_j^{(2)}
$$

(6)

with $\gamma$ positive and $\xi_j^{(\alpha)}$ $\alpha = 1, 2, 3$ are real numbers.

The important point here is to observe that the rapidities $\lambda_j^{(1)}$ are described in terms of two independent type of strings, i.e 1-strings and 2-strings. This feature should be contrasted to the behaviour of the Bethe ansatz roots of other fundamental integrable systems such as $SU(N)$ and $O(N)$ models [16, 8]. In fact, for the latter systems each $a$-th root $\lambda_j^{(a)}$ characterizing the infinite volume properties has a unique string length. To explore the consequences of the peculiar string configuration (6) we substitute it in the Bethe ansatz (4) and we obtain the
following effective equations for the variables $\xi_j^{\alpha}$

$$L \left[ \psi_{1/2}(\xi_j^{\alpha}) \delta_{\alpha,1} + \psi_1(\xi_j^{\alpha}) \delta_{\alpha,3} \right] = 2\pi Q_j^{\alpha} + \sum_{\beta=1}^{3} \sum_{k=1}^{N_\beta} \phi_{\alpha\beta}(\xi_j^{\alpha} - \xi_j^{(\beta)})$$

(7)

where $\psi_a(x) = 2 \arctan(x/a)$, $N_\alpha$ is the number of roots $\xi_j^{(\alpha)}$, $Q_j^{(\alpha)} = -(N_\alpha - 1)/2 + j - 1$ with $j = 1, \cdots, N_\alpha$ and the matrix elements $\phi_{\alpha\beta}(x)$ are given by

$$\phi_{\alpha\beta}(x) = \begin{pmatrix} \psi_1(x) & -\psi_1(x) & \psi_{1/2}(x) + \psi_{3/2}(x) \\ -\psi_1(x) & \psi_2(x) & -\psi_{1/2}(x) - \psi_{3/2}(x) \\ \psi_{1/2}(x) + \psi_{3/2}(x) & -\psi_{1/2}(x) - \psi_{3/2}(x) & 2\psi_1(x) + \psi_2(x) \end{pmatrix}$$

(8)

The ground state for even $L$ consists of a sea of 1-strings and 2-strings with $N_1 = N_2 = 2N_3 = L/2$. For large $L$, the roots $\xi_j^{(\alpha)}$ are densely packed into its density distribution $\sigma(\xi_j^{(\alpha)}) = 1/L(\xi_j^{(\alpha)} + 1 - \xi_j^{(\alpha)})$ and the relations (7) in the $L \to \infty$ limit become integral equations for such densities. These integral equations are solved by elementary Fourier techniques and we find that

$$\sigma^{(1)}(x) = \frac{1}{2 \cosh(\pi x)}, \quad \sigma^{(2)}(x) = \frac{1}{6 \cosh(\pi x/3)}$$

(9)

and

$$\sigma^{(3)}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sigma^{(1)}(y) \sigma^{(2)}(x-y) dy$$

$$= \frac{2}{3\sqrt{3}} \frac{\sinh(2\pi x/3) - x}{6 \sinh(\pi x)}$$

(10)

where we have emphasized the remarkable fact that $\sigma^{(3)}(x)$ is exactly the convolution of the densities $\sigma^{(1)}(x)$ and $\sigma^{(2)}(x)$. Recall that function $\sigma^{(\alpha)}(x)$ is related to the continuous probability densities of finding the rapidity $\xi^{(\alpha)}$ with a given value $x$. We may therefore interpret $\sigma^{(3)}(x)$ as the probability for the sum of two independent events with probability $\sigma^{(1)}(y)$ at an arbitrary value $y$ and $\sigma^{(2)}(x-y)$ at the complementary value $x-y$.

We now have the basic ingredients to investigate the thermodynamic limit properties. The ground state energy per site $e_\infty$ is calculated by using Eqs.(5,9,10) after replacing the sum by an integral. The final result is

$$e_\infty = -2 \left[ \frac{2 \ln(2)}{3} + \frac{\pi}{9\sqrt{3}} - \frac{1}{3} \right]$$

(11)
The low-lying excitations are obtained by inserting holes in the density distribution of $\xi_j^{(\alpha)}$ which means the removal of certain quantum numbers $Q_j^{(\alpha)}$ of the Bethe equations (7). The necessary manipulations of these equations are standard \[14, 16\] and we find that the energy $\varepsilon^{(\alpha)}(\xi)$ and the momentum $p^{(\alpha)}(\xi)$ of one hole excitation in the sea of $\xi_j^{(\alpha)}$, measured from the ground state, have the form

$$\varepsilon^{(\alpha)}(\xi) = \pi \sigma^{(\alpha)}(\xi), \quad p^{(\alpha)}(\xi) = \int_{\xi}^{+\infty} \varepsilon^{(\alpha)}(x)dx$$

where $\xi$ is the $\alpha$-hole rapidity. For the first two excitations one can easily eliminate the variable $\xi$ leading us to the following dispersion relations

$$\varepsilon^{(1)}(p) = \frac{\pi}{2} \sin(2p), \quad \varepsilon^{(2)}(p) = \frac{\pi}{6} \sin(2p)$$

implying that these excitations are gapless and that their low-energy limits $\varepsilon^{(\alpha)}(p) \sim v^{(\alpha)}p$ are governed by distinct sound velocities, i.e $v^{(1)} = \pi$ and $v^{(2)} = \pi/3$. The contribution to the total spin of each of these excitations is $\frac{1}{2}$, and therefore we shall interpret them as spinons propagating with different velocities.

Similar computation for the third excitation leads us to transcendental equations and an analytical expression for the dispersion relation is hard to be obtained. However, it is possible to study the low-energy behaviour of such excitation by expanding the density $\sigma^{(3)}(\lambda)$ in powers of $e^{-\lambda}$. We see that low-momenta regime is dominated by both sound velocities $v^{(1)}$ and $v^{(2)}$ and strictly in the $p \to 0$ limit the lowest one prevails. This massless excitation turns out to be a spinless mode whose speed of sound is $v^{(3)} = \pi/3$. At this point we note that recently the compound $NaV_2O_5$ has been modeled by an anisotropic/asymmetric spin-orbital model \[19\]. Remarkably, the three-particle continuum found above is in accordance with the excitation spectrum proposed in ref.\[19\] to explain the optical properties of this material.

Next we would like to identify the underlying conformal field theory which describes the low-energy limit of the integrable $Sp(4)$ model. This can be investigated by analyzing the behaviour of the finite size spectrum of the corresponding spin chain. For a conformally
invariant theory, the ground state energy $E_0(L)$ in a finite lattice of size $L$ behaves as \cite{21, 22}

\begin{equation}
\frac{E_0(L)}{L} = e_{\infty} - \frac{\pi}{6L^2} \sum_{a=1}^{3} v^{(a)} c^{(a)}
\end{equation}

where $c^{(a)}$ is the central charge associated to the $a$-th excitation. Similarly, the conformal
dimension $X_i^{(a)}$ of the operator corresponding to the excited state $E_i(L)$ is proportional to the
finite size gap

\begin{equation}
\frac{E_i(L)}{L} - \frac{E_0(L)}{L} = \frac{2\pi}{L^2} \sum_{a=1}^{3} v^{(a)} X_i^{(a)}
\end{equation}

Within the string hypothesis (6), the finite size corrections to the eigenspectrum can be
evaluated analytically by applying the root density method \cite{23} to the “string” Bethe equations (7). For instance, we find that each $a$-th massless excitation is associated with a central charge $c^{(a)} = 1$. However, the string assumptions gives us only an idea of the true finite size behaviour, because the complex part of the roots may contribute to the term $1/L^2$ as well. In order to obtain the correct finite size properties, we solve numerically the original Bethe ansatz equations (4) up to $L = 36$. In table (1) we exhibit such numerical results for the effective
central charge $c_{ef} = \sum_{i=1}^{3} \frac{v^{(i)}}{v^{(3)}} c^{(i)}$ and the conformal dimension $X_{ef} = \sum_{i=1}^{3} \frac{v^{(i)}}{v^{(3)}} X_i^{(i)}$ associated with the lowest spin-wave excitation over the ground state. Clearly, the imaginary parts of the roots $\lambda_j^{(1)}$ have conspired together and canceled the $1/L^2$ correction proportional to the third mode. This is the first example we know that complex strings can produce negative contribution to the central charge. It is therefore tempting to think the third excitation as a composite state of two elementary spinons and that it does not contribute to the low-energy limit. The same phenomenon happens to the excited states and this allows us to interpret the operator content of the lowest excitation as $X_1^{(1)} = X_1^{(2)} = 1/4$. Putting these information together, we see that the underlying conformal field theory is likely to be represented in terms of four Majorana fermions rather than given by a $Sp(4)$ WZW theory.

Let us now turn to the problem of extending our results for general $N > 2$. The Bethe
ansatz equations are parametrized by $N$ different types of roots, and it turns out that the
first $N - 1$ roots are given by both 1-strings and 2-strings while the last one behaves like
1-strings. If we characterize the center of the strings by $\xi_j^{(\alpha)}$, $\alpha = 1, \cdots, 2N - 1$, we find that
the corresponding density distributions are

\[
\sigma^{(\alpha)}(x) = \begin{cases} 
\frac{1}{N} \frac{\sin(\pi \alpha / N)}{\cosh(2\pi x / N) - \cos(\pi \alpha / N)}, \quad \alpha = 1, \cdots, N-1 \\
\frac{1}{2(N+1)} \frac{1}{\cosh(\pi x / (N+1))}, \quad \alpha = N 
\end{cases}
\]  \hspace{1cm} (16)

while for \( \alpha = N+1, \cdots, 2N-1 \) they are given by the convolution

\[
\sigma^{(\alpha)}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sigma^{(2N-\alpha)}(y) \sigma^{(N)}(x-y) dy.
\]

The corresponding excitations are gapless, consisting of \( N \) generalized spinons whose speed of sound are \( v^{(\alpha)} = \frac{2\pi}{N} \) for \( \alpha = 1, \cdots, N-1 \) and \( v^{(N)} = \frac{\pi}{N+1} \). In addition, we have \( N-1 \) composite modes made by the convolution between the first \( N-1 \) spinons with the \( N \)-th excitation. For \( N > 2 \) numerical results for large \( L \) become difficult to be obtained since the number of roots to be determined grow rapidly with both \( N \) and \( L \). However, for \( N = 3 \) and small \( L \sim 18 \) our numerical analysis is consistent with the fact the only modes contributing to the low-energy properties are the spinons, each one with \( c = 1 \). All these results seem to be strong evidence that the continuum limit of such \( Sp(2N) \) integrable models can indeed be described in terms of \( 2N \) Majorana fermions.

In conclusion, we have studied the excitation spectrum of the simplest integrable \( Sp(2N) \) spin chain. Contrary to the common belief this system is not the lattice realization of the \( Sp(2N) \) WZW conformal theory. Our study indicates that the nature of the excitations in spin-orbital systems can be rather involving. In fact, the isotropic point \( J_0 = J_1 = J_2 \) is known to have three basic excitations [16], being the lattice realization of a \( SU(4) \) WZW field theory [4]. However, the anisotropic point \( J_0/J_1 = J_0/J_2 = 1/3 \) has only two independent excitations and one composite mode that do not contribute to the low-energy limit, and it is described by a \( c = 2 \) conformal field theory. This work prompts us to ask some questions that may open up new interesting avenues. What is the nature of the excitations of the spin-orbital model (1) in the crossover regime \( 1/3 \leq J_0/J_1 = J_0/J_2 \leq 1 \)? What is the mechanism that made one of the excitations to become a composite state? What is the integrable lattice \( Sp(2N) \) model whose continuum limit corresponds to the \( Sp(2N) \) WZW theory?
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| $L$ | $c_{ef}$     | $X_{ef}$  |
|-----|-------------|----------|
| 12  | 4.074 654  | 1.000 985 |
| 16  | 4.046 720  | 1.004 993 |
| 20  | 4.033 092  | 1.006 011 |
| 24  | 4.025 305  | 1.006 017 |
| 28  | 4.020 376  | 1.005 692 |
| 32  | 4.017 024  | 1.005 272 |
| 36  | 4.014 620  | 1.004 845 |
| Extr.| 4.001 (±2) | 1.006 (±2) |

Table 1: Finite size sequences and the extrapolations of the effective central charge and the lowest conformal dimension of the $Sp(4)$ spin chain.