Anomalous Transport Phenomena in $p_x + ip_y$ Superconductors

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Spontaneous breaking of time-reversal symmetry in superconductors with the $p_x + ip_y$ symmetry for a class of effects which are analogous to the anomalous Hall effect in ferromagnets. These effects exist below the critical temperature, $T < T_c$. We develop a kinetic theory of such effects. In particular, we consider anomalous Hall thermal conductivity, the polar Kerr effect, the anomalous Hall effect, and the anomalous photo- and acousto-galvanic effects.

Introduction: One of the leading candidates for $p$-wave pairing in electronic systems is Sr$_2$RuO$_4$. There are numerous pieces of experimental evidence that the superconducting state of Sr$_2$RuO$_4$ has odd parity, breaks time reversal symmetry and is spin triplet [1–6]. An order parameter consistent with these experiments is given by the chiral $p$-wave state [7] which is an analog of $^3$He-$A$. It has the form $\Delta_{\alpha\beta}(p) \sim p_x \pm ip_y$ where $\Delta_{\alpha\beta}(p)$ is the Fourier transform of $\Delta_{\alpha\beta}(r - r')$. However, the observation of power laws in specific heat [8] and NMR [9], the absence of electric currents along edges [10], and the absence of a split transition in the presence of an in-plane magnetic field [11] are inconsistent with the theoretically expected properties of a simple chiral superconductor. Consideration of additional experimental manifestations of spontaneous breaking of time-reversal symmetry in $p_x + ip_y$ superconductors may clarify the nature of superconducting state in Sr$_2$RuO$_4$.

Due to spontaneous breaking of time-reversal symmetry, $p_x + ip_y$ superconductors must exhibit anomalous transport phenomena similar to those which exist in metallic ferromagnets (see for example reviews Refs. [14] and [15]). Below we generalize this approach to superconductors without time reversal symmetry, which exhibit anomalous transport phenomena. In the clean regime, $l \gg \xi$, and at sufficiently low frequencies, $\omega \ll |\Delta|$, where $|\Delta|$ is the modulus of the order parameter, the quasiparticle dynamics can be described by the Boltzmann kinetic equation for quasiparticles

$$\frac{\partial n_p(r, t)}{\partial t} + \frac{\partial \tilde{\epsilon}_p}{\partial r} \frac{\partial n_p}{\partial r} - \frac{\partial \tilde{\epsilon}_p}{\partial \Phi} \frac{\partial n_p}{\partial \Phi} = I_{st}, \tag{1}$$

where

$$\tilde{\epsilon}_p = \epsilon_p + v \cdot p_s, \quad \epsilon_p = \sqrt{\tilde{\epsilon}_p^2 + |\Delta|^2}, \tag{2a}$$

$$\tilde{\epsilon}_p = \epsilon_p + \Phi + \frac{p_s^2}{2m}, \quad \xi_p = \frac{p_s^2}{2m} - \epsilon_F. \tag{2b}$$

In Eq. (2) $m$ is the electron mass, while $p_s$ and $\Phi$, are given by

$$p_s = \frac{\hbar}{2} \nabla \chi - \frac{e}{c} A, \quad \Phi = \frac{\hbar}{2} \partial_t \chi + e \phi, \tag{3}$$

where $\chi$ is the order parameter phase, and $\phi$ and $A$ are the scalar and vector potentials. From Eq. (3) one obtains the equation for the acceleration of the condensate

$$\partial_t p_s = e E + \nabla \Phi. \tag{4}$$

Equations (1)-(3) should be supplemented by the expression for the current density,

$$j = \frac{eN}{m} p_s + e \int d\Gamma \nabla n_p, \tag{5}$$

and by the charge neutrality condition,

$$\nu \Phi = \int d\Gamma \frac{\tilde{\epsilon}_p}{\tilde{\epsilon}_p} n_p, \tag{6}$$

that relates the gauge invariant scalar potential and the odd in $\xi$ part of the quasiparticle distribution function,
the self-consistency equation for the order parameter. Here \( d^* = V d^3 p / (2\pi\hbar)^3 \) (\( V \) is the volume of the sample) and \( v = d \xi / dp \).

We work in linear response to external perturbations, and neglect corrections to equilibrium value of \(|\Delta|\). The collision integral \( I_{st} = I_{st}^{(e)} + I_{st}^{(c)} \) in Eq. (1) describes both elastic and inelastic scattering. We will assume that \( \tau \gg \tau_s \), where \( \tau_s \) and \( \tau \) are inelastic and elastic mean free time respectively. Therefore the main contribution to the aforementioned anomalous effects comes from elastic scattering, which is described by the collision integral

\[
I_{st} = \int (W_{pp'} n_{p'} - W_{p'p} n_p) \delta (\epsilon_p - \epsilon_{p'}) d\Gamma'. \tag{7}
\]

Skew scattering of quasiparticles corresponds to the part of scattering probability in Eq. (7) that is associated with breaking of time reversal symmetry, \( \delta W_{pp'} = W_{pp'} - W_{p'p} = 0 \). Thus, all the aforementioned effects are proportional to \( \delta W_{pp'} \). Skew scattering arises beyond the lowest Born approximation for the scattering amplitude. Below we consider point-like impurities. In the normal state such impurities scatter electrons only in the \( s \)-wave channel and do not cause skew scattering. Therefore in the superconducting state skew scattering of quasiparticles is entirely due to the breaking of time reversal symmetry by the \( p_x + ip_y \) order parameter. The elastic scattering probability for quasiparticles with energy \( \epsilon \) can be characterized by \( \xi \equiv \xi_p, \xi' \equiv \xi_{p'} = \pm \xi \) and the asimuthal angles \( \varphi, \varphi' \), which define the direction of \( p \) and \( p' \) in the \( xy \)-plane. For simplicity, we assume cylindrical Fermi surface and obtain for the scattering probability (see appendix for details).

\[
W_{pp'} = W_0 + W_1 \left[ 1 - \cos (\varphi - \varphi' + 2\delta_s) \right]. \tag{8}
\]

Here \( \delta_s \) is the energy-dependent scattering phase shift. It is related to the \( s \)-wave scattering phase shift \( \delta_n \) in the normal state by

\[
\delta_s = \arctan \frac{\delta_n \epsilon}{\sqrt{\epsilon^2 - \Delta^2}}. \tag{9}
\]

We assume weak impurities, for which \( \delta_n \approx \tan \delta_n = -\pi \nu V_0 \) is small. Here \( \nu \) is the density of states on the Fermi level and \( V_0 \) is the impurity pseudo-potential [16]. In this case \( W_0 \) and \( W_1 \) are given by

\[
W_0(\xi, \xi') = \frac{\zeta(\epsilon)}{2\nu \tau} \left( \frac{\xi + \xi'}{2\epsilon^2} \right)^2, \tag{10a}
\]

\[
W_1(\xi, \xi') = \frac{\zeta(\epsilon)}{2\nu \tau} \frac{\Delta^2}{\epsilon^2}. \tag{10b}
\]

Here \( \tau^{-1} = 2\pi n_i V_0 / c_n \), with \( n_i \) being the impurity density, is the elastic scattering rate in the normal state. The coefficient \( \zeta(\epsilon) = (\epsilon^2 - \Delta^2) / [\epsilon^2 (1 + \delta_n^2) - \Delta^2] \) represents the enhancement factor of the quasiparticle scattering cross-section over the normal state value. The first term in Eq. (8), \( W_0 \) given by Eq. (10a) has the same structure as in \( s \)-wave superconductors. It describes scattering only within the same (particle-like, \( \xi > 0 \), or hole-like, \( \xi < 0 \) branch and does not lead to branch imbalance relaxation. The second term, \( W_1 \) in Eq. (8) is absent in \( s \)-wave superconductors. It leads to both skew scattering and scattering between branches of quasiparticle spectrum with different signs of \( \xi \). The skew scattering cross-section, described by the \( \sin(\varphi - \varphi') \sin \delta_s \) term in Eq. (8), is energy-dependent. It follows from Eqs. (8), (9), and (10b) that it changes sign when impurity potential \( V_0 \) changes from repulsive to attractive.

Below we consider linear response to several external perturbations and look for the quasiparticle distribution function in the form \( n_p = n_p(0) + n_p^{(1)} \), where \( n_p(0) \) is a locally equilibrium Fermi distribution, and \( n_p^{(1)} \) describes the deviation from equilibrium. Noting that the collision integral (7) is nullified by an arbitrary function \( n_p^{(0)}(\epsilon_p) \) we write the linearized Boltzmann equation in the form

\[
S(p) = \int d\Gamma' W_{pp'} (n_p^{(0)} - n_p^{(1)}) \delta(\epsilon_p - \epsilon_{p'}), \tag{11}
\]

where the specific form of the source \( S(p) \) depends on the type of perturbation.

**Anomalous Hall thermal conductivity:** We consider the Hall component of the thermal conductivity \( \kappa_{xy} \), which describes the heat flux perpendicular to the direction (\( x \)-axis) of the temperature gradient. In this case the source term in Eq. (11) has the form

\[
S(p) = -\frac{\xi}{T} \nabla T \frac{\partial n_p^{(0)}}{\partial \epsilon}. \tag{12}
\]

The expression for the heat flux is

\[
j^Q = \int d\Gamma' \epsilon_p \frac{\partial n_p^{(1)}}{\partial \epsilon}. \tag{13}
\]

Note that \( \partial \epsilon_p / \partial p = \epsilon / c \) is the group velocity of the quasiparticles while \( v \) is the bare velocity as in a normal metal, \( |v| = v_F \). The solution of Eqs. (11), (13) has the form

\[
n_p^{(1)} = -\frac{\xi}{T} v_F \nabla T \frac{\partial n_p^{(0)}}{\partial \epsilon} \left[ \alpha_s(\epsilon) \sin \varphi + \alpha_c(\epsilon) \cos \varphi \right]. \tag{14}
\]

The Hall component of the thermal conductivity tensor, \( \kappa_{xy} \), is determined by \( \alpha_s(\epsilon) \) in the above expression, which is given by \( \alpha_s(\epsilon) = a(\epsilon) / [b(\epsilon) + a^2(\epsilon)] \), with

\[
a(\epsilon) = \frac{\zeta(\epsilon)}{2\tau} \frac{\Delta^2}{2c|\xi|} \sin 2\delta_s, \tag{14a}
\]

\[
b(\epsilon) = \frac{|\xi|}{\epsilon \tau} + \frac{\zeta(\epsilon)}{2\tau} \frac{\Delta^2}{2c|\xi|} (\cos 2\delta_s + 2). \tag{14b}
\]

For weak impurities, \( |\delta_n| \ll 1 \), we obtain, close to \( T_c \)

\[
\kappa_{xy} = 3\kappa \left( \frac{\Delta}{\pi T} \right)^2 \delta_n, \tag{15}
\]
where $\kappa = \pi^2 \nu TD / 3$ (with $D = v_F^2 \tau / 2$ being the diffusion constant) is the normal state thermal conductivity.

**Polar Kerr effect:** Next we consider a linearly polarized electromagnetic wave at normal incidence to the $xy$ surface of $p_x + ip_y$ superconductor. The reflected wave is elliptically polarized with the major axis rotated with respect to the incident one by the polar Kerr angle [17]

$$\theta_k = \frac{\left(1 - n^2 + \kappa^2\right) \Delta \kappa + 2 n \kappa \Delta n}{(1 - n^2 + \kappa^2)^2 + (2 n \kappa)^2},$$  

(16)

where $n$ and $\kappa$ are, respectively, the real and imaginary part of the refraction index and

$$\Delta n + i \Delta \kappa = -\frac{4 \pi (n - i \kappa) \sigma_{xy}}{n^2 + \kappa^2},$$  

(17)

where $\sigma_{xy}$ is the complex ac conductivity.

In this case the electric field is uniform in the direction parallel to the surface of the sample, $\Phi = 0$, and the value of $p_x(t)$ is given by [14]

$$\sigma_{xx} \approx \sigma_D + \frac{i N_s(T)}{\omega},$$  

(18)

where $N_s(T)$ is the temperature dependent superfluid density and $\sigma_D = e^2 \nu D$ is the Drude conductivity. In contrast to the thermal conductivity consideration, in the present case $n(0) = 1/\exp[p_x(t) + v \cdot p_y(t)/T + 1]$ gives a nonvanishing contribution to the current response (via $N_s(T)$) because the superfluid momentum depends on the electric field. The Kerr angle $\theta_k$ is determined by the value of the Hall component of conductivity $\sigma_{xy}$.

To find $\sigma_{xy}$ we seek the solution of Eq. (1) in the form

$$n = n^{(0)}(\epsilon_p/T) + n^{(1)}_p.$$  

The source in Eq. (11) becomes

$$S(p) = -i \omega n^{(1)}_p - e \nu \cdot E \frac{\partial n^{(0)}_p}{\partial \epsilon},$$  

(19)

where the external electric field $E$ is along $x$-direction. The nonequilibrium distribution $n^{(1)}_p$ has the form

$$n^{(1)}_p = -e E v_F \frac{\partial n^{(0)}_p}{\partial \epsilon} \left[ \beta_x(\epsilon) \sin \varphi + \beta_y(\epsilon) \cos \varphi \right].$$  

(20)

The Hall conductivity depends only on the function $\beta_x(\epsilon)$, which is given by $\beta_x(\epsilon) = a(\epsilon) / \{|b(\epsilon) - i \omega| + \frac{1}{2} a^2(\epsilon)\}$, with $a(\epsilon)$ and $b(\epsilon)$ being defined in Eq. (14). Substituting Eq. (20) into Eq. (5) we obtain the Hall conductivity in the weak impurity limit, $|\pi e V_0| \ll 1$, in the form

$$\sigma_{xy}(\omega) = \sigma_D \frac{\Delta}{2T} \delta_n \int_0^\infty \frac{dx}{x^2 + 1} \frac{\cos^2(\sqrt{x^2 + 1} \Delta / 2T)}{(-i \omega \tau x \sqrt{x^2 + 1} + x^2 + 3/4)^2},$$  

(21)

where $x = |\Delta| / \Delta$. At temperature close to $T_c$ and at low frequencies, $\omega \tau \ll 1$, this expression yields

$$\sigma_{xy} = \frac{7 \pi}{12 \sqrt{3}} \delta_n \frac{\Delta}{T} \sigma_D.$$  

(22)

This result was derived assuming $p_x + i p_y$ state the Hall conductivity $\sigma_{xy}$ has opposite sign. It also changes sign if the impurity potential $V_0$ changes from repulsive, $\delta_n < 0$, to attractive, $\delta_n > 0$, in agreement with Ref. [18]. Note that our result for the low frequency Hall conductivity, Eq. (22), is proportional to the elastic mean free time $\tau$ and to the density of quasiparticles.

There is another contribution to $\sigma_{xy}$ associated with the existence of the transverse component of the superfluid velocity $v_y \sim p_x$, which is proportional to the condensate acceleration in the $x$-direction. It may not be obtained within the present formalism that is based on the Boltzmann kinetic equation for the quasiparticles. At $T \sim T_c$ this contribution is smaller than the quasiparticle contribution, Eq. (22). However at $T \ll T_c$ when the quasiparticle contribution becomes exponentially small in Eq. (21) it becomes the dominant contribution. The requirement for this contribution to exist is violation of Galilean invariance in the system. Thus it should exist in any crystalline superconductors with $p_x + i p_y$ symmetry [19, 20]. It can also be caused by electron-impurity scattering. In this case this contribution is inversely proportional to the electron mean free time [18].

**Hall effect for normal current injection:** Let us now consider a normal metal/$p_x + i p_y$-superconductor junction, through which a steady current is flowing. At $T \ll \Delta$ this situation was considered in Ref. [21]. In this regime conversion of normal current to supercurrent is mediated by multiple Andreev reflections. Here we work near the critical temperature and consider a setup, in which the normal current is injected into the superconductor in the $x$-direction, as shown in the inset in Fig. 1. In this case the conversion of quasiparticle current to the supercurrent occurs in the superconductor. Just as in the case of $s$-wave superconductor, near $T_c$, the electric field penetrates into superconductor to a large distance $L_Q \gg l$, which is determined by the relaxation of imbalance between the populations of quasiparticles in electron-like, $\xi > 0$, and hole-like, $\xi < 0$ branches of the spectrum (see for example Ref. [14] and references therein). The new feature of normal current injection that appears in $p_x + i p_y$ superconductors is that skew scattering of quasiparticles generates nonequilibrium current that is perpendicular to the electric field. Another aspect is that, in contrast to $s$-wave superconductors, impurity scattering leads to branch imbalance relaxation even if the magnitude of the order parameter $|\Delta|$ is isotropic in the Fermi surface. Below we assume that the inelastic scattering rate is smaller than $1/\tau$ and thus impurity scattering gives the dominant contribution to branch imbalance relaxation.

In linear response we write the quasiparticle distribution function in the superconductor in the form
where \( j \) is the superconductor. For the latter satisfies the diffusion equation with relaxation distribution function averaged over the momentum directions. The latter satisfies the diffusion equation with energy relaxation for both the inelastic mean free path in the normal metal, \( l \), and the branch imbalance relaxation length \( \xi \) in the superconductor. For \( l_c \gg \xi \), the boundary condition is

\[
\bar{n}_a(\xi,0) = \text{sign}(\xi) \frac{eE_x(0)\xi \Delta}{4T \cosh^2(\xi/2T)},
\]

where \( E_x(0) \) is the electric field in the normal metal generating the steady current. Here we used the fact that in the stationary case \( E = -\nabla \Phi/e \), which follows from Eq. (4). Using this relation and substituting Eqs. (27), (26) into Eqs. (6) and (24) we obtain the spatial distributions of the electric field \( E_x(x) \) and the Hall current \( j_y(x) \) in the superconductor,

\[
E_x(x) = E_x(0)F_0 \left( \frac{x}{\xi} \right),
\]

\[
j_y(x) = \sigma_D E_x(0)\delta_n \left( \frac{\Delta}{T} \right)^2 F_{-2} \left( \frac{x}{\xi} \right),
\]

where \( \langle L_Q \rangle = 2\ln2(Tl/\Delta) \) and the functions \( F_n \), are defined as

\[
F_n(x) = \int_0^\infty dy \frac{y^n}{\cosh^2(y)} \exp \left( -\ln2 \frac{x}{y} \right),
\]

and are plotted in Fig. 1. The spatial distributions of the Hall current \( j_y(x) \) and the electric field \( E_x(x) \) are drastically different, and cannot be related by a local Hall conductivity \( \sigma_{xy} \). At relatively short distances, \( l \ll x \ll \langle L_Q \rangle \), we see from Eq. (29) that \( j_y(x) \propto \sigma_D E_x(0)\delta_n \left( \frac{\Delta}{T} \right)^2 \langle L_Q \rangle /x \), so that the Hall current is

\[
I_y = \int dx j_y(x) \approx \sigma_D E_x(0)\delta_n \left( \frac{\Delta}{T} \right)^2 \langle L_Q \rangle \ln \frac{\langle L_Q \rangle}{l}.
\]

**FIG. 1.** Plot of the functions \( F_{-2}(x) \) (solid line) and \( F_0(x) \) (dashed line) in Eq. (30). The inset shows a schematic setup of the normal current injection experiment. Electric current is injected into the superconductor \( S \) from the normal metal \( N \) along the \( x \)-axis. Skew scattering of quasiparticles generates an anomalous Hall current in the \( y \)-direction.

Anomalous photo- and acousto-galvanic effects: When an electromagnetic or an acoustic wave propagates through a conductor it generates an anisotropic in momentum \( \mathbf{p} \) distribution function. The induced current density is proportional to the rate of the the momentum transfer from the wave to the electron system [15],

\[
J_x = I \alpha_{xx}.
\]

Here \( I \) is the rate of momentum density transfer due to the wave adsorption, and \( x \) is the direction of the wave propagation. In \( p_x + ip_y \) superconductors an anomalous current in the \( y \)-direction is generated. Considerations similar to those leading to Eq. (22) near \( T_c \) yield

\[
\alpha_{xy} \sim \alpha_{xx} \frac{\Delta}{T} \delta_n.
\]

Finally, we note that all anomalous transport phenomena discussed above are driven by the underlying symmetry of the superconducting state. Therefore they should exist in any superconductor whose order parameter breaks time reversal symmetry, see for example [22–26]. Although our consideration focused on \( p_x + ip_y \) materials we believe our approach is applicable to other superconductors with broken time-reversal invariance.
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Appendix: Derivation of the scattering probability

In this appendix we obtain the scattering probability $W_{pp'}$ in Eq. (8) for simplicity we consider point-like impurities whose scattering matrix elements are independent of momentum, $V_{pp'} = V_0$. Skew scattering appears beyond the lowest Born approximation as a result of a particular structure of the order parameter in $p_x + i p_y$ superconductors. We note that even weak impurities, $|\nu V_0| \ll 1$, lead to the existence of bound states in $p_x + i p_y$ superconductors with binding energies $E_b \sim |\nu V_0|^2$ [27]. Therefore low energy quasiparticles undergo resonant scattering.

The scattering properties of a single impurity are described by the $T$-matrix. In a superconductor, for each initial and final momenta $p'$ and $p$ the $T$-matrix acquires an additional $2 \times 2$ structure in the Nambu-Gorkov space. We denote these $2 \times 2$ matrices by $T_{pp'}$. The quasiparticle scattering amplitude is given by the on-shell matrix element $T_{pp'} \equiv \langle p | T | p' \rangle$, with $|p\rangle$ being the two component quasiparticle wave function. Using the Fermi Golden rule the scattering probability can be expressed as

$$W_{pp'} = 2\pi n_i |\tilde{T}_{pp'}|^2$$

where $n_i$ is the impurity concentration. The $T$-matrix obeys the Lipmann-Schwinger equation

$$\tilde{T}_{pp'} = \hat{V}_{pp'} + \sum_{pp''} \hat{V}_{pp''} \hat{G}_0(p'') \tilde{T}_{pp''},$$

For point-like impurities, $\hat{V}_{pp'} = V_0 \hat{\gamma}_3$ (with $\hat{\gamma}_3$ being the Pauli matrices in the Nambu-Gorkov space), the $T$-matrix depends only on the energy $E$. Thus Eq. (A.2) simplifies to

$$\tilde{T}(E) = V_0 \hat{\gamma}_3 \sum_p \hat{G}_0(p) \tilde{T}(E)$$

The order parameter of $p_x + i p_y$ superconductor can be expressed as $\Delta(p_x + i p_y)/p_F = \Delta \exp(i \varphi_p)$ where $\varphi_p$ is the momentum-dependent phase such that $\cos \varphi_p = p_x/p_F$ and $\sin \varphi_p = p_y/p_F$. This leads to the BCS Hamiltonian

$$\hat{H}(p) = \left( \begin{array}{cc} \xi_p & \Delta e^{i\varphi_p} \\ \Delta e^{-i\varphi_p} & -\xi_p \end{array} \right),$$

and the Green function $\hat{G}_0(p) = \left( E - \hat{H}(p) \right)^{-1}$,

$$\hat{G}_0(p) = \frac{1}{E^2 - \xi_p^2} \left( \begin{array}{cc} E + \xi_p & \Delta e^{i\varphi_p} \\ \Delta e^{-i\varphi_p} & E - \xi_p \end{array} \right).$$

Replacing $\sum_p$ with $V \int d^3p/(2\pi\hbar)^3 \to \nu \int d\nu_p \int d\varphi_p/2\pi$, where $V$ is the volume of the sample and $\nu$ is the density of state at Fermi surface, and using Eq. (A.3) and Eq. (A.4), we obtain for the $T$-matrix,

$$\tilde{T}(E) = \left( \begin{array}{cc} \frac{V_0}{1 + (\nu V_0)^2 E^2/(E^2 - \Delta^2)} & 0 \\ 0 & \frac{-V_0}{1 + (\nu V_0)^2 E^2/(E^2 - \Delta^2)} \end{array} \right).$$

The pole of the $T$-matrix describes the subgap bound state [27] with the binding energy $E_b = \Delta/\sqrt{1 + (\nu V_0)^2}$. For the energy in the continuum, $|E| > \Delta$, we obtain the $T$-matrix in the form

$$\tilde{T}(E) = \left( \begin{array}{cc} f(E)e^{i\delta_E} & 0 \\ 0 & -f(E)e^{-i\delta_E} \end{array} \right)$$

where $\delta_E$ is given by Eq. (9) and

$$f(E) = \frac{\sqrt{V_0}}{\sqrt{1 + (\nu V_0)^2 E^2/(E^2 - \Delta^2)}}.$$

The quasiparticle scattering amplitude is given by the on-shell matrix elements between two quasiparticle states with energy $\epsilon = \sqrt{\Delta^2 + \xi_p^2} = \sqrt{\Delta^2 + \xi_{p'}^2}$,

$$\tilde{T}_{pp'} = \langle p | \tilde{T}(\epsilon) | p' \rangle.$$  

In a $p_x + i p_y$ superconductor the Bogoliubov amplitudes carry a momentum-dependent phase $\varphi_{p'}$,

$$|p\rangle = (u_p e^{-i\varphi_p}, v_p),$$

where $u_p$ and $v_p$ have the standard BCS form and are independent of the momentum direction,

$$u_p = \sqrt{(1 + \xi_p/\xi_p)/2}, \quad v_p = \sqrt{(1 - \xi_p/\xi_p)/2}.$$  

Substituting Eqs. (A.9) and (A.10) into (A.8) we get

$$\left| \tilde{T}_{pp'} \right|^2 = f^2(\epsilon) \left\{ \left( u_p u_{p'} - v_p v_{p'} \right)^2 + 2u_p u_{p'} v_p v_{p'} \times \left[ 1 - \cos(2\delta_e + \varphi_p - \varphi_{p'}) \right] \right\}$$

$$= \frac{1}{2} f^2(\epsilon) \left[ 1 + \frac{\xi_p \xi_{p'} - \Delta^2}{\xi_p^2} \right] + \frac{f^2(\epsilon) \Delta^2}{2\xi_p^2} \left[ 1 - \cos (2\delta_e + \varphi_p - \varphi_{p'}) \right].$$

Substituting this result into Eq. (A.1), and recalling that for weak impurities, $|\nu V_0| \ll 1$, the scattering rate in the normal state is $\tau^{-1} = 2\pi n_i \nu V_0^2$ we obtain the scattering probability in the form of Eq. (8) and Eq. (10).
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