A scheme for determining fundamental interactions and the universality principle

Zhongmin Qian*

Summary. The Standard Model of particle physics was established based on the equivalence principle and gauge invariance. The Lagrangians were built upon experimental data demonstrating the violation of discrete symmetries together with ideas of spontaneous symmetry breaking. The features of vector and axial-vector interactions and chirality principle are manually added on in order to fit in the observation data, rather than explained by the model as one may hope. Here we develop a theory of interactions based on the Dirac algebra together with the exterior operations the wedge product and the star operator on the space-time, and develop a scheme for determining all possible interaction formulations. In our scheme, the chirality transformation and the left and right handedness come up naturally. A simple postulate about the universality of physical interactions based on the space-time causal structure yields the attracting features mentioned above as consequences.

1 Introduction

The Standard Model (see for example [14, 30, 36, 37, 13]) of the strong, electromagnetic and weak interactions of particles is an $SU(3) \times SU(2) \times U(1)$ gauge theory, which was built upon the equivalence principle [10, 16], the gauge invariance principle [38, 40, 34, 37, 28], and ideas of spontaneous breaking of local gauge symmetries [11, 15] together with the Higgs mechanism [17, 15]. The gauge group uniquely determines the number of gauge bosons mediating interactions with matter fields via co-variant derivatives. Lagrangians of the Standard Model have been assembled by using experiment data, guided mainly by the universality of weak interactions, the $V$-$A$ current-current theory of weak interactions (the chirality principle) [12, 33, 23] based on the discoveries of the parity violation [20, 39] and the $CP$ violation [6, 1, 2]. The equivalence principle demands the invariance of laws of Nature under transformations which preserve the gravitational field and the causal structure [16]. In the context of special relativity, the equivalence principle can be made more precise: mathematical theories describing laws of Nature should be invariant under any change of coordinate systems, and invariant under proper and orthochonous Lorentz transformations and translations of the space-time. In particular, the parity or the time inversion conservation, which was once taken as granted due to the correspondence principle, has been abandoned. It is not widely appreciated that the gravitational field $(g_{\mu\nu})$ defines the duality of various fundamental fields and its significance in the understanding of parity and time inversion symmetries. It is the causal structure of the space-time which is the origin of violations of discrete symmetries.

Fermi made use of the Dirac matrices to propose a model for weak interactions, since then the Dirac algebra has become the main ingredient in the construction of Lagrangians within the framework of gauge theories. The chirality matrix $\gamma^5$ and the helicity defined in terms of the left and right hand projections $\frac{1}{2}(1 \pm \gamma^5)$ have been appeared manually in the Standard Model. It is a mystery why $\gamma^5$ and $\frac{1}{2}(1 \pm \gamma^5)$ play so fundamental roles in the high energy physics, and it remains to reveal their meanings. In this article we consider the Dirac algebra combining with the exterior calculus [9, 4] on the space-time, and demonstrate that the chirality principle and the concept of helicity are better understood in terms of the Hodge star operator * [9, 4]. We develop a general theory of interactions which allows us to determine Lagrangians which obey the fundamental principles of equivalence and the gauge invariance. The violation of discrete symmetries, and the universal theory of interactions and the chirality follow as consequences of a simple postulate based on
the space-time structure rather than added into theoretical models manually.

2 Universality of interactions

Let \((g_{\mu \nu})\) denote the Minkowski metric and \(\Omega\) be the volume form \(-dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\). In quantum field theory, only two kinds of fields (and their corresponding quantized fields), tensor fields and Dirac’s (fermion) fields, are needed. The tensor fields transfer under any diffeomorphism \(F\) of the space-time by the differentiation \(F\), (see [4] for a definition). The exterior algebra over the space-time is equipped with the wedge product \(\wedge\) and exterior differentiation \(d\).

The operations \(\wedge\) and \(d\) are defined on the existence of coordinate system, and no physical processes are involved. The star operator \(*\) is however determined by the gravitational field \((g_{\mu \nu})\), which takes a \(p\)-form \(\alpha = (\alpha_1, \ldots, \alpha_p)\) to a \(4-p\) form \(*\alpha = ((\alpha_1, \ldots, \alpha_p))\). \(*\) is linear operating on the exterior algebra and \(** = -1\), which is introduced to determine the adjoint of \(d\), \(\delta = -d*\). The \(*\) operator is determined by the following rules: \(\star 1 \to \Omega,\ dx^0 \to dx^1 \wedge dx^2 \wedge dx^3,\ dx^1 \to dx^0 \wedge dx^2 \wedge dx^3,\ dx^2 \to \cdots\). The exterior differentiation \(d\) doesn’t obey the gauge invariance principle, so that it must be replaced by gauge theories. In a gauge theory, interactions are mediated by bosons (which are differential forms of degree one on the space-time), which appear as connection forms of covariant derivatives. Suppose the gauge group is a simple Lie group \(G\) with its Lie algebra \(\mathfrak{g}\). Choose a normal basis \(\{T_a\}\) of \(\mathfrak{g}\) (in physics literature, \(\iota_a = iT_a\) are used), so that every element in the connected component of \(G\) at the identity can be expressed as \(\exp(e^{a}T_a)\) for some reals \(e^a\). Consider a \(g\)-valued connection \(D = d + \omega\) on the space-time, where \(\omega = A^0T_a\) and \(A^0 = A^0_\mu dx^\mu\) are bosons, real valued differential forms on space-time. If \(\Phi = (\Phi_\mu)\) is a family of fermion fields, then \(D\Phi = [D_\mu \Phi] dx^\mu\), where \(D_\mu \Phi = \partial_\mu \Phi + A^0_\mu T_\alpha \Phi\). Recall that we can only integrate differential forms of degree 4, i.e. top forms, on the space-time, hence Lagrangians which have to obey both the equivalence principle and the gauge invariance, must be constructed from Lorentz invariant differential forms of degree 4 on the space-time. On the other hand, Lorentz invariant top forms which also verify the gauge invariance must be constructed via the bosons \(A^aT_a\), \(\gamma^\mu dx^\mu\) together with spinors \(\Phi\) and \(\Psi\). The bosons \(A^a\) define a \(g\)-valued differential form of degree two \(\sum_{\mu < \nu} F^a_{\mu \nu} T_a dx^\mu \wedge dx^\nu\), the curvature form. There are only two possible forms of degree 4 one can build from \(F\) alone, that is, \(F \wedge F\) (which is the gauge Lagrangian) and \(F \wedge \Phi\). The top form \(F \wedge F\) is however a Chern form and therefore it has definite integration, from which it might be possible to construct useful Lorentz and gauge invariant quantities.

The gamma matrices were originally introduced by Dirac [7, 8] to derive the relativistic wave equation that \(\gamma^\mu (i\partial_\mu - eA_\mu)\psi = m\psi\), where \(\gamma = (\gamma^\mu)\) is a unitary representation of Dirac matrices [25, 24] so that \(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^\mu\nu\) and \(\gamma^\mu = i\gamma^\mu\). \{\gamma^\mu\} generates the Dirac algebra of 16 elements, which give rise to transformation rules for spinors determined by von Neumann [35] who classified elements of the Dirac algebra into 5 classes (see Messiah [24], page 896, Table XX.1.). The Dirac algebra was used by Lee and Yang [21], Schwinger [32], Salam [29], Salam and Ward [31], Sudarshan and Marshak [33], Feynman and Gellmann [12] to generalize Fermi’s theory of weak interactions into the V-A current-current theory [19]. The Dirac algebra has thus become the standard tool (see [28, 37, 13] for example) and the main ingredient in the construction of effective Lagrangians.

Our new idea is to make use of the Dirac algebra through the exterior calculus [9]. Let \(\gamma^\mu = g_{\mu \nu} \gamma^\nu\) and define a matrix-valued differential form (of degree one) \(\gamma^\nu = g_{\nu \mu} dx^\mu\) which assigns the transformation rule as a tensor field for \(\gamma^\nu\) under any coordinate change of the space-time (and therefore under every Lorentz transformation in particular). By using \(\wedge\), one can generate further three differential forms of various degrees, namely the “tensor” \(\mathcal{F} = \frac{1}{2} \gamma^\nu \wedge \gamma^\nu\), the “axial vector” \(\mathcal{A} = \frac{2}{3} \gamma^\nu \wedge \gamma^\nu\) and the “pseudo scalar” \(\mathcal{S} = \frac{4}{3} \gamma^\nu \wedge \gamma^\nu \wedge \gamma^\nu\), which are the fundamental forms to construct Lorentz invariant Lagrangians. Applying the star operator \(*\) to these fundamental forms to generate differential forms \(*1 = \Omega\), \(*\gamma^\nu = i\gamma^\nu \mathcal{F}\), \(*\mathcal{F} = i\gamma^\nu \mathcal{A}\) and \(*\mathcal{A} = i\gamma^\nu \mathcal{S}\), where \(\mathcal{F} = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma_5\). Therefore \(i\gamma^\nu\) plays a role of the Hodge duality (which is determined by the grav-
It is not widely recognized that Lorentz invariance may not always coincide. In fact $S_A [\Upsilon \Psi] = S(A)|\Upsilon \Psi|$ and $S_A [\Upsilon^\dagger \Psi] = \Lambda$, while $S_A [\Upsilon^\dagger \Psi] = \det (A) S(\Lambda) |\Upsilon \Psi|$ for $\Upsilon = 1, \Upsilon, \Upsilon^\dagger$. If $\det (A) = -1$, then the spin action $S_A$ on the field $\Upsilon^\dagger \Psi$ differs from the natural action $\Lambda$, by this we mean the parity violation (and always to the maximum). It follows that the possible Lorentz (quadratic) invariants are $\Upsilon^\dagger \Psi$, $\Upsilon^2 \Psi$. For two pairs of spinors, in addition to quadratic products, the only Lorentz invariants are $[\Upsilon^\dagger \Phi]_\mu \gamma_\nu [\Upsilon \gamma_5 \Psi]_\mu [\Upsilon \gamma^\nu \Phi]_\mu$ and $[\Upsilon^\dagger \Phi]_\mu \gamma_\nu [\Upsilon \gamma_5 \Psi]_\mu [\Upsilon \gamma^\nu \Phi]_\nu$, also the following couplings which will not appear in any pair production: $[\Upsilon^\dagger \Phi]_\mu \gamma_\nu [\Upsilon \gamma_5 \Psi]_\mu [\Upsilon \gamma^\nu \Phi]_\nu$ and $[\Upsilon^\dagger \Phi]_\mu \gamma_\nu [\Upsilon \gamma_5 \Psi]_\nu [\Upsilon \gamma^\nu \Phi]_\mu$ (where summations run over $\mu < \nu$).

It remains to explain the mystery why the operators $1 \pm \gamma^5$ enter into the construction of the standard model? To answer this question, we observe that there are exactly two differential forms of degree one, namely $\Upsilon = \gamma_\mu dx^\mu$ and $-i \mathcal{A} = \gamma_\mu dx^\mu$. Nature may select the simplest possible mechanism to mediate interactions among the most fundamental particles to reveal her beauty of laws. With this faith, and the observation that all interactions appear in gauge theories are mediated and coupled with fields and the differential form $\Upsilon$ of degree one via $\Lambda$ and $\ast$, one naturally asks the question why $\Upsilon$? To preserve the Lorentz invariance, we can start with a general differential form of degree one and we would like to derive natural conditions which must be independent of any interactions or fields, however ensure the simplicity of laws of Nature. The most general differential form of degree one can be written as $\mathcal{F} = \lambda_1 \Upsilon - \lambda_2 i \ast \mathcal{A}$. We seek for an $\mathcal{F}$ which generates a simpler algebra as possible (by using $\Lambda$ and $\ast$). To this end we calculate that

$$\mathcal{F} \wedge \mathcal{F} = \left( \lambda_1^2 - \lambda_2^2 \right) \sum_{\mu < \nu} \gamma_\mu \gamma_\nu dx^\mu \wedge dx^\nu, \quad (2)$$

$$\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} = 3 \left( \lambda_1^2 - \lambda_2^2 \right) \left( \lambda_1 + \lambda_2 \gamma^5 \right) \sum_{\sigma < \mu < \nu} \gamma_\sigma \gamma_\mu \gamma_\nu dx^\sigma \wedge dx^\mu \wedge dx^\nu \quad (3)$$

and

$$\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} = -12 \left( \lambda_1^2 - \lambda_2^2 \right)^2 i \gamma^5 \Omega. \quad (5)$$

It follows that the simplest algebra generated by $\mathcal{F}$ is achieved when $\mathcal{F} \wedge \mathcal{F} = 0$, i.e. the differential
one form $\mathcal{F}$ is self-dual. This self-duality of $\mathcal{F}$ is equivalent to that $\lambda_1 = \pm \lambda_2$, which leads to two possible differential forms (up to a multiple constant) $\mathcal{L} = \frac{1}{2} (1 + \gamma^5) \gamma_\mu dx^\mu$ and $\mathcal{R} = \frac{1}{2} (1 - \gamma^5) \gamma_\mu dx^\mu$, and in this way the projections $1 \pm \gamma^5$ appear naturally. One may work out the algebra generated by $\mathcal{L}$ and $\mathcal{R}$ too through the exterior multiplication $\wedge$ and the star operator $\star$. The results are $\mathcal{L} \wedge \mathcal{R} = (1 + \gamma^5) \mathcal{F}$, $\mathcal{R} \wedge \mathcal{L} = (1 - \gamma^5) \mathcal{F}$, and $\mathcal{R} \wedge \mathcal{L} \wedge \mathcal{R} = 3(1 - \gamma^5) \mathcal{A}$. It follows that there are only 6 (instead of 10 for Dirac algebra) non-trivial differential forms one may constructed from $\mathcal{L}$ and $\mathcal{R}$, two for each degree. The six elements are self-dual in the sense that $\star (\mathcal{L} \wedge \mathcal{R}) = i \mathcal{L} \wedge \mathcal{R}$, $\star (\mathcal{R} \wedge \mathcal{L}) = -i \mathcal{R} \wedge \mathcal{L}$, $\star \mathcal{L} = \frac{1}{3!} \mathcal{L} \wedge \mathcal{R} \wedge \mathcal{L}$ and $\star \mathcal{R} = -\frac{1}{3!} \mathcal{R} \wedge \mathcal{L} \wedge \mathcal{R}$. In particular, the differential form $\mathcal{L}$, in contrast with the case of $\gamma^5$, only generates one more differential form (of degree three) $\star \mathcal{L}$. Observing the amazing duality, one can not stop to formulate the following proposition.

Postulate of Universality. The weak interactions are mediated through $\mathcal{L}$ (or equivalently $\mathcal{R}$) only.

Under this version of the universality principle, the only coupling allowed under this postulate is the top form $[\Phi \mathcal{L} \Psi] \wedge [\Phi \star \mathcal{L} \Psi]$, which has the density (up to a constant) $[\Phi (1 + \gamma^5) \gamma_\mu \Psi] [\Phi (1 + \gamma^5) \gamma_\mu \Psi]$. Therefore the universal $V-A$ theory for weak interactions follows. There is only one possible coupling with bosons and spinors one can build, namely $\Phi \star \mathcal{L} \wedge D \Psi$ which gives the interaction $\Phi (1 + \gamma^5) [\gamma_\mu D_\mu \Psi]$ up to a constant. This is exactly the chirality principle. While, there is no non-trivial two form generated by $\mathcal{L}$ alone, therefore, a kinetic term involving $\Phi \mathcal{L} \Psi \wedge \Phi \star \mathcal{L} \Psi$, which can be embedded into Lagrangians for interactions.

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