PARACONSISTENT INTUITIONISTIC FUZZY RELATIONAL DATA MODEL

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In this paper, we present a generalization of the relational data model based on paraconsistent intuitionistic fuzzy sets. Our data model is capable of manipulating incomplete as well as inconsistent information. Fuzzy relation or intuitionistic fuzzy relation can only handle incomplete information. Associated with each relation are two membership functions one is called truth-membership function $T$ which keeps track of the extent to which we believe the tuple is in the relation, another is called false-membership function which keeps track of the extent to which we believe that it is not in the relation. A paraconsistent intuitionistic fuzzy relation is inconsistent if there exists one tuple $a$ such that $T(a) + F(a) > 1$. In order to handle inconsistent situation, we propose an operator called split to transform inconsistent paraconsistent intuitionistic fuzzy relations into pseudo-consistent paraconsistent intuitionistic fuzzy relations and do the set-theoretic and relation-theoretic operations on them and finally use another operator called combine to transform the result back to paraconsistent intuitionistic fuzzy relation. For this model, we define algebraic operators that are generalisations of the usual operators such as union, selection, join on fuzzy relations. Our data model can underlie any database and knowledge-base management system that deals with incomplete and inconsistent information.

Keywords: Paraconsistent intuitionistic fuzzy relations, fuzzy relations, paraconsistent relations, generalized relational algebra, inconsistent information

1. Introduction

Relational data model was proposed by Ted Codd’s pioneering paper. Since then, relational database systems have been extensively studied and a lot of commercial relational database systems are currently available. This data model usually takes care of only well-defined and unambiguous data. However, imperfect information is ubiquitous – almost all the information that we have about the real world is not certain, complete and precise. Imperfect information can be classified as: incompleteness, imprecision, uncertainty, inconsistency. Incompleteness arises from the absence of a value, imprecision from the existence of a value which cannot be measured with suitable precision, uncertainty from the fact that a person has
given a subjective opinion about the truth of a fact which he/she does not know for certain, and inconsistency from the fact that there are two or more conflicting values for a variable.

In order to represent and manipulate various forms of incomplete information in relational databases, several extensions of the classical relational model have been proposed. In some of these extensions, a variety of "null values" have been introduced to model unknown or not-applicable data values. Attempts have also been made to generalize operators of relational algebra to manipulate such extended data models. The fuzzy set theory and fuzzy logic proposed by Zadeh provide a requisite mathematical framework for dealing with incomplete and imprecise information. Later on, the concept of interval-valued fuzzy sets was proposed to capture the fuzziness of grade of membership itself. In 1986, Atanassov introduced the intuitionistic fuzzy set which is a generalization of fuzzy set and provably equivalent to interval-valued fuzzy set. The intuitionistic fuzzy sets consider both truth-membership $T$ and false-membership $F$ with $T(a), F(a) \in [0, 1]$ and $T(a) + F(a) \leq 1$. Because of the restriction, the fuzzy set, interval-valued fuzzy set and intuitionistic fuzzy set cannot handle inconsistent information. Some authors have studied relational databases in the light of fuzzy set theory with an objective to accommodate a wider range of real-world requirements and to provide closer man-machine interactions. Probability, possibility and Dempster-Shafer theory have been proposed to deal with uncertainty. Possibility theory is built upon the idea of a fuzzy restriction. That means a variable could only take its value from some fuzzy set of values and any value within that set is a possible value for the variable. Because values have different degrees of membership in the set, they are possible to different degrees. Prade and Testemale initially suggested using possibility theory to deal with incomplete and uncertain information in database. Their work is extended to cover multivalued attributes. Wong proposes a method that quantifies the uncertainty in a database using probabilities. His method may be the simplest one which attaches a probability to every member of a relation, and to use these values to provide the probability that a particular value is the correct answer to a particular query. Carvallo and Pittarelli also use probability theory to model uncertainty in relational databases systems. Their method augmented projection and join operations with probability measures.

However, unlike incomplete, imprecise and uncertain information, inconsistent information has not enjoyed enough research attention. In fact, inconsistent information exists in a lot of applications. For example, in data warehousing application, inconsistency will appear when trying to integrate the data from many different sources. Another example is that in the expert system, there exist facts which are inconsistent with each other. Generally, two basic approaches have been followed in solving the inconsistency problem in knowledge bases: belief revision and para-consistent logic. The goal of the first approach is to make an inconsistent theory consistent, either by revising it or by representing it by a consistent semantics. On
the other hand, the paraconsistent approach allows reasoning in the presence of inconsistency, and contradictory information can be derived or introduced without trivialization \cite{26}. Bagai and Sunderraman \cite{27} proposed a paraconsistent relational data model to deal with incomplete and inconsistent information. This data model is based on paraconsistent logics which were studied in detail by de Costa \cite{28} and Belnap \cite{29}.

In this paper, we present a new relational data model – paraconsistent intuitionistic fuzzy relational data model (PIFRDM). Our model is based on the paraconsistent intuitionistic fuzzy set theory which is an extension of intuitionistic fuzzy set theory \cite{30} and is capable of manipulating incomplete as well as inconsistent information. We use both truth-membership function grade $\alpha$ and false-membership function grade $\beta$ to denote the status of a tuple of a certain relation with $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 2$. PIFRDM is the generalization of fuzzy relational data model (FRDM). That is, when $\alpha + \beta = 1$, paraconsistent intuitionistic fuzzy relation is the ordinary fuzzy relation. This model is distinct with paraconsistent relational data model (PRDM), in fact it can be easily shown that PRDM is a special case of PIFRDM. That is when $\alpha, \beta = 0$ or 1, paraconsistent intuitionistic fuzzy relation is just paraconsistent relation. We can use Figure 1 to express the relationship among FRDM, PRDM and PIFRDM.

We introduce paraconsistent intuitionistic fuzzy relations, which are the fundamental mathematical structures underlying our model. These structures are strictly more general than classical fuzzy relations and intuitionistic fuzzy relations (interval-valued fuzzy relations), in that for any fuzzy relation or intuitionistic fuzzy relation (interval-valued fuzzy relation) there is a paraconsistent intuitionistic fuzzy relation with the same information content, but not vice versa. The claim is also true for the relationship between paraconsistent intuitionistic fuzzy relations and
paraconsistent relations. We define algebraic operators over paraconsistent intuitionistic fuzzy relations that extend the standard operators such as selection, join, union over fuzzy relations.

There are many potential applications of our new data model. Here are some examples:

(a) Web mining. Essentially the data and documents on the Web are heterogeneous, inconsistency is unavoidable. Using the presentation and reasoning method of our data model, it is easier to capture imperfect information on the Web which will provide more potentially value-added information.

(b) Bioinformatics. There is a proliferation of data sources. Each research group and each new experimental technique seems to generate yet another source of valuable data. But these data can be incomplete and imprecise and even inconsistent. We could not simply throw away one data in favor of other data. So how to represent and extract useful information from these data will be a challenge problem.

(c) Decision Support System. In decision support system, we need to combine the database with the knowledge base. There will be a lot of uncertain and inconsistent information, so we need an efficient data model to capture these information and reasoning with these information.

The paper is organized as follow. Section 2 of the paper deals with some of the basic definitions and concepts of fuzzy relations and operations. Section 3 introduces paraconsistent intuitionistic fuzzy relations and two notions of generalising the fuzzy relational operators such as union, join, projection for these relations. Section 4 presents some actual generalised algebraic operators for paraconsistent intuitionistic fuzzy relations. These operators can be used for specifying queries for database systems built on such relations. Section 5 gives an illustrative application of these operators. Finally, Section 6 contains some concluding remarks and directions for future work.

2. Fuzzy Relations and Operations

In this section, we present the essential concepts of a fuzzy relational database. Fuzzy relations associate a value between 0 and 1 with every tuple representing the degree of membership of the tuple in the relation. We also present several useful query operators on fuzzy relations.

Let a relation scheme (or just scheme) $\Sigma$ be a finite set of attribute names, where for any attribute name $A \in \Sigma$, $\text{dom}(A)$ is a non-empty domain of values for $A$. A tuple on $\Sigma$ is any map $t : \Sigma \rightarrow \bigcup_{A \in \Sigma} \text{dom}(A)$, such that $t(A) \in \text{dom}(A)$, for each $A \in \Sigma$. Let $\tau(\Sigma)$ denote the set of all tuples on $\Sigma$.

**Definition 1.** A fuzzy relation on scheme $\Sigma$ is any map $R : \tau(\Sigma) \rightarrow [0, 1]$. We let $\mathcal{F}(\Sigma)$ be the set of all fuzzy relations on $\Sigma$. $\square$
If $\Sigma$ and $\Delta$ are relation schemes such that $\Delta \subseteq \Sigma$, then for any tuple $t \in \tau(\Delta)$, we let $t^\Sigma$ denote the set $\{t' \in \tau(\Sigma) \mid t'(A) = t(A), \text{ for all } A \in \Delta\}$ of all extensions of $t$. We extend this notion for any $T \subseteq \tau(\Delta)$ by defining $T^\Sigma = \bigcup_{t \in T} t^\Sigma$.

2.1. Set-theoretic operations on Fuzzy relations

Definition 2. Union: Let $R$ and $S$ be fuzzy relations on scheme $\Sigma$. Then, $R \cup S$ is a fuzzy relation on scheme $\Sigma$ given by

$$(R \cup S)(t) = \max\{R(t), S(t)\}, \text{ for any } t \in \tau(\Sigma).$$

Definition 3. Complement: Let $R$ be a fuzzy relation on scheme $\Sigma$. Then, $-R$ is a fuzzy relation on scheme $\Sigma$ given by

$$(-R)(t) = 1 - R(t), \text{ for any } t \in \tau(\Sigma).$$

Definition 4. Intersection: Let $R$ and $S$ be fuzzy relations on scheme $\Sigma$. Then, $R \cap S$ is a fuzzy relation on scheme $\Sigma$ given by

$$(R \cap S)(t) = \min\{R(t), S(t)\}, \text{ for any } t \in \tau(\Sigma).$$

Definition 5. Difference: Let $R$ and $S$ be fuzzy relations on scheme $\Sigma$. Then, $R - S$ is a fuzzy relation on scheme $\Sigma$ given by

$$(R - S)(t) = \min\{R(t), 1 - S(t)\}, \text{ for any } t \in \tau(\Sigma).$$

2.2. Relation-theoretic operations on Fuzzy relations

Definition 6. Let $R$ and $S$ be fuzzy relations on schemes $\Sigma$ and $\Delta$, respectively. Then, the natural join (or just join) of $R$ and $S$, denoted $R \bowtie S$, is a fuzzy relation on scheme $\Sigma \cup \Delta$, given by

$$(R \bowtie S)(t) = \min\{R(\pi_\Sigma(t)), S(\pi_\Delta(t))\}, \text{ for any } t \in \tau(\Sigma \cup \Delta).$$

Definition 7. Let $R$ be a fuzzy relation on scheme $\Sigma$ and let $\Delta \subseteq \Sigma$. Then, the projection of $R$ onto $\Delta$, denoted by $\Pi_\Delta(R)$ is a fuzzy relation on scheme $\Delta$ given by

$$(\Pi_\Delta(R))(t) = \max\{R(u) \mid u \in t^\Sigma\}, \text{ for any } t \in \tau(\Delta).$$

Definition 8. Let $R$ be a fuzzy relation on scheme $\Sigma$, and let $F$ be any logic formula involving attribute names in $\Sigma$, constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\lor$ and $\land$. Then, the selection of $R$ by $F$, denoted $\sigma_F(R)$, is a fuzzy relation on scheme $\Sigma$, given by

$$(\sigma_F(R))(t) = \begin{cases} R(t) & \text{if } t \in \sigma_F(\tau(\Sigma)) \\ 0 & \text{Otherwise} \end{cases}$$

where $\sigma_F$ is the usual selection of tuples satisfying $F$. □
3. Paraconsistent Intuitionistic Fuzzy Relations

In this section, we generalize fuzzy relations in such a manner that we are now able to assign a measure of belief and a measure of doubt to each tuple. We shall refer to these generalized fuzzy relations as paraconsistent intuitionistic fuzzy relations. So, a tuple in a paraconsistent intuitionistic fuzzy relation is assigned a measure \((\alpha, \beta)\), \(0 \leq \alpha, \beta \leq 1\). \(\alpha\) will be referred to as the belief factor and \(\beta\) will be referred to as the doubt factor. The interpretation of this measure is that we believe with confidence \(\alpha\) and doubt with confidence \(\beta\) that the tuple is in the relation. The belief and doubt confidence factors for a tuple need not add to exactly 1. This allows for incompleteness and inconsistency to be represented. If the belief and doubt factors add up to less than 1, we have incomplete information regarding the tuple’s status in the relation and if the belief and doubt factors add up to more than 1, we have inconsistent information regarding the tuple’s status in the relation.

In contrast to fuzzy relations where the grade of membership of a tuple is fixed, paraconsistent intuitionistic fuzzy relations bound the grade of membership of a tuple to a subinterval \([\alpha, 1 - \beta]\) for the case \(\alpha + \beta \leq 1\).

The operators on fuzzy relations can also be generalised for paraconsistent intuitionistic fuzzy relations. However, any such generalisation of operators should maintain the belief system intuition behind paraconsistent intuitionistic fuzzy relations.

This section also develops two different notions of operator generalisations.

We now formalize the notion of a paraconsistent intuitionistic fuzzy relation.

Recall that \(\tau(\Sigma)\) denotes the set of all tuples on any scheme \(\Sigma\).

**Definition 9.** A paraconsistent intuitionistic fuzzy relation \(R\) on scheme \(\Sigma\) is any map

\[
R : \tau(\Sigma) \rightarrow [0, 1] \times [0, 1].
\]

For any \(t \in \tau(\Sigma)\), we shall denote \(R(t) = (R(t)^+, R(t)^{-})\), where \(R(t)^{+}\) is the belief factor assigned to \(t\) by \(R\) and \(R(t)^{-}\) is the doubt factor assigned to \(t\) by \(R\). We let \(\mathcal{V}(\Sigma)\) be the set of all paraconsistent intuitionistic fuzzy relations on \(\Sigma\). □

**Definition 10.** A paraconsistent intuitionistic fuzzy relation \(R\) on scheme \(\Sigma\) is consistent if \(R(t)^{+} + R(t)^{-} \leq 1\), for all \(t \in \tau(\Sigma)\). We let \(\mathcal{C}(\Sigma)\) be the set of all consistent paraconsistent intuitionistic fuzzy relations on \(\Sigma\). Moreover, \(R\) is said to be complete if \(R(t)^{+} + R(t)^{-} \geq 1\), for all \(t \in \tau(\Sigma)\). If \(R\) is both consistent and complete, i.e. \(R(t)^{+} + R(t)^{-} = 1\), for all \(t \in \tau(\Sigma)\), then it is a total paraconsistent intuitionistic fuzzy relation, and we let \(\mathcal{T}(\Sigma)\) be the set of all total paraconsistent intuitionistic fuzzy relations on \(\Sigma\). \(R\) is said to be pseudo-consistent if \(\max(R(t_i)^{+}) + \max(R(t_i)^{-}) > 1, R(t_i)^{+} + R(t_i)^{-} = 1\), for some \(t_i \in \tau(\Sigma)\), these \(t_i\)’s have the same values on \(\Sigma\) with different belief factor and doubt factor and for all the other \(t \in \tau(\Sigma), R(t)^{+} + R(t)^{-} \leq 1\). We let \(\mathcal{P}(\Sigma)\) be the set of all pseudo-consistent paraconsistent intuitionistic fuzzy relations on \(\Sigma\). □
Note that pseudo-consistent paraconsistent intuitionistic fuzzy relation is a subclass of consistent paraconsistent intuitionistic fuzzy relation.

It should be observed that total paraconsistent intuitionistic fuzzy relations are essentially fuzzy relations where the uncertainty in the grade of membership is eliminated. We make this relationship explicit by defining a one-one correspondence \( \lambda_\Sigma : T(\Sigma) \rightarrow F(\Sigma) \), given by \( \lambda_\Sigma(R)(t) = R(t)^+ \), for all \( t \in \tau(\Sigma) \). This correspondence is used frequently in the following discussion.

**Operator Generalisations**

It is easily seen that paraconsistent intuitionistic fuzzy relations are a generalisation of fuzzy relations, in that for each fuzzy relation there is a paraconsistent intuitionistic fuzzy relation with the same information content, but not \textit{vice versa}. It is thus natural to think of generalising the operations on fuzzy relations such as union, join, projection etc. to paraconsistent intuitionistic fuzzy relations. However, any such generalisation should be intuitive with respect to the belief system model of paraconsistent intuitionistic fuzzy relations. We now construct a framework for operators on both kinds of relations and introduce two different notions of the generalisation relationship among their operators.

An \( n \)-ary operator on fuzzy relations with signature \( \langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle \) is a function \( \Theta : F(\Sigma_1) \times \cdots \times F(\Sigma_n) \rightarrow F(\Sigma_{n+1}) \), where \( \Sigma_1, \ldots, \Sigma_{n+1} \) are any schemes. Similarly, an \( n \)-ary operator on paraconsistent intuitionistic fuzzy relations with signature \( \langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle \) is a function \( \Psi : V(\Sigma_1) \times \cdots \times V(\Sigma_n) \rightarrow V(\Sigma_{n+1}) \).

**Definition 11.** An operator \( \Psi \) on paraconsistent intuitionistic fuzzy relations with signature \( \langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle \) is \textit{totality preserving} if for any total paraconsistent intuitionistic fuzzy relations \( R_1, \ldots, R_n \) on schemes \( \Sigma_1, \ldots, \Sigma_n \), respectively, we have

\[
\Psi(R_1, \ldots, R_n) = \Theta(\lambda_\Sigma_1(R_1), \ldots, \lambda_\Sigma_n(R_n)).
\]

**Definition 12.** A totality preserving operator \( \Psi \) on paraconsistent intuitionistic fuzzy relations with signature \( \langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle \) is a \textit{weak generalisation} of an operator \( \Theta \) on fuzzy relations with the same signature, if for any total paraconsistent intuitionistic fuzzy relations \( R_1, \ldots, R_n \) on schemes \( \Sigma_1, \ldots, \Sigma_n \), respectively, we have

\[
\lambda_{\Sigma_{n+1}}(\Psi(R_1, \ldots, R_n)) = \Theta(\lambda_{\Sigma_1}(R_1), \ldots, \lambda_{\Sigma_n}(R_n)).
\]

The above definition essentially requires \( \Psi \) to coincide with \( \Theta \) on total paraconsistent intuitionistic fuzzy relations (which are in one-one correspondence with the fuzzy relations). In general, there may be many operators on paraconsistent intuitionistic fuzzy relations that are weak generalisations of a given operator \( \Theta \) on fuzzy relations. The behavior of the weak generalisations of \( \Theta \) on even just the consistent paraconsistent intuitionistic fuzzy relations may in general vary. We require
a stronger notion of operator generalisation under which, at least when restricted to consistent intuitionistic fuzzy relations, the behavior of all the generalised operators is the same. Before we can develop such a notion, we need that of ‘representations’ of a paraconsistent intuitionistic fuzzy relation.

We associate with a consistent paraconsistent intuitionistic fuzzy relation $R$ the set of all (fuzzy relations corresponding to) total paraconsistent intuitionistic fuzzy relations obtainable from $R$ by filling in the gaps between the belief and doubt factors for each tuple. Let the map $\text{reps}_\Sigma : C(\Sigma) \to 2^{F(\Sigma)}$ be given by

$$\text{reps}_\Sigma(R) = \{ Q \in F(\Sigma) \mid \bigwedge_{t_i \in \tau(\Sigma)} (R(t_i)^+ \leq Q(t_i) \leq 1 - R(t_i)^-) \}.$$ 

The set $\text{reps}_\Sigma(R)$ contains all fuzzy relations that are ‘completions’ of the consistent paraconsistent intuitionistic fuzzy relation $R$. Observe that $\text{reps}_\Sigma$ is defined only for consistent paraconsistent intuitionistic fuzzy relations and produces sets of fuzzy relations. Then we have following observation.

**Proposition 1.** For any consistent paraconsistent intuitionistic fuzzy relation $R$ on scheme $\Sigma$, $\text{reps}_\Sigma(R)$ is the singleton $\{\lambda_\Sigma(R)\}$ iff $R$ is total. $\square$

**Proof.** It is clear from the definition of consistent and total paraconsistent intuitionistic fuzzy relations and from the definition of $\text{reps}$ operation. $\square$

We now need to extend operators on fuzzy relations to sets of fuzzy relations. For any operator $\Theta : F(\Sigma_1) \times \cdots \times F(\Sigma_n) \to F(\Sigma_{n+1})$ on fuzzy relations, we let $S(\Theta) : 2^{F(\Sigma_1)} \times \cdots \times 2^{F(\Sigma_n)} \to 2^{F(\Sigma_{n+1})}$ be a map on sets of fuzzy relations defined as follows. For any sets $M_1, \ldots, M_n$ of fuzzy relations on schemes $\Sigma_1, \ldots, \Sigma_n$, respectively,

$$S(\Theta)(M_1, \ldots, M_n) = \{ \Theta(R_1, \ldots, R_n) \mid R_i \in M_i, \text{ for all } i, 1 \leq i \leq n \}.$$ 

In other words, $S(\Theta)(M_1, \ldots, M_n)$ is the set of $\Theta$-images of all tuples in the cartesian product $M_1 \times \cdots \times M_n$. We are now ready to lead up to a stronger notion of operator generalisation.

**Definition 13.** An operator $\Psi$ on paraconsistent intuitionistic fuzzy relations with signature $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ is consistency preserving if for any consistent paraconsistent intuitionistic fuzzy relations $R_1, \ldots, R_n$ on schemes $\Sigma_1, \ldots, \Sigma_n$, respectively, $\Psi(R_1, \ldots, R_n)$ is also consistent. $\square$

**Definition 14.** A consistency preserving operator $\Psi$ on paraconsistent intuitionistic fuzzy relations with signature $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ is a strong generalisation of an operator $\Theta$ on fuzzy relations with the same signature, if for any consistent paraconsistent intuitionistic fuzzy relations $R_1, \ldots, R_n$ on schemes $\Sigma_1, \ldots, \Sigma_n$, respectively, we have

$$\text{reps}_{\Sigma_{n+1}}(\Psi(R_1, \ldots, R_n)) = S(\Theta)(\text{reps}_{\Sigma_1}(R_1), \ldots, \text{reps}_{\Sigma_n}(R_n)). \square$$
Given an operator $\Theta$ on fuzzy relations, the behavior of a weak generalisation of $\Theta$ is ‘controlled’ only over the total paraconsistent intuitionistic fuzzy relations. On the other hand, the behavior of a strong generalisation is ‘controlled’ over all consistent paraconsistent intuitionistic fuzzy relations. This itself suggests that strong generalisation is a stronger notion than weak generalisation. The following proposition makes this precise.

**Proposition 2.** If $\Psi$ is a strong generalisation of $\Theta$, then $\Psi$ is also a weak generalisation of $\Theta$. □

**Proof.** Let $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ be the signature of $\Psi$ and $\Theta$, and let $R_1, \ldots, R_n$ be any total paraconsistent intuitionistic fuzzy relations on schemes $\Sigma_1, \ldots, \Sigma_n$, respectively. Since all total relations are consistent, and $\Psi$ is a strong generalisation of $\Theta$, we have that

$$\text{reps}_{\Sigma_{n+1}}(\Psi(R_1, \ldots, R_n)) = S(\Theta)(\text{reps}_{\Sigma_1}(R_1), \ldots, \text{reps}_{\Sigma_n}(R_n)),$$

Proposition 1 gives us that for each $i$, $1 \leq i \leq n$, $\text{reps}_{\Sigma_i}(R_i)$ is the singleton set $\{\lambda_{\Sigma_i}(R_i)\}$. Therefore, $S(\Theta)(\text{reps}_{\Sigma_1}(R_1), \ldots, \text{reps}_{\Sigma_n}(R_n))$ is just the singleton set:

$$\{\Theta(\lambda_{\Sigma_1}(R_1), \ldots, \lambda_{\Sigma_n}(R_n))\}.$$

Here, $\Psi(R_1, \ldots, R_n)$ is total, and $\lambda_{\Sigma_{n+1}}(\Psi(R_1, \ldots, R_n)) = \Theta(\lambda_{\Sigma_1}(R_1), \ldots, \lambda_{\Sigma_n}(R_n))$, i.e. $\Psi$ is a weak generalisation of $\Theta$. □

Though there may be many strong generalisations of an operator on fuzzy relations, they all behave the same when restricted to consistent paraconsistent intuitionistic fuzzy relations. In the next section, we propose strong generalisations for the usual operators on fuzzy relations. The proposed generalised operators on paraconsistent intuitionistic fuzzy relations correspond to the belief system intuition behind paraconsistent intuitionistic fuzzy relations.

First we will introduce two special operators on paraconsistent intuitionistic fuzzy relations called split and combine to transform inconsistent paraconsistent intuitionistic fuzzy relations into pseudo-consistent paraconsistent intuitionistic fuzzy relations and transform pseudo-consistent paraconsistent intuitionistic fuzzy relations into inconsistent paraconsistent intuitionistic fuzzy relations.

**Definition 15. (Split)** Let $R$ be paraconsistent intuitionistic fuzzy relations on scheme $\Sigma$. Then, $\triangle(R) = \{t_i \in R|(R(t_i)^+ + R(t_i)^-) > 1) \land (\triangle(R)(t_i)^+ = R(t_i)^+ \land \triangle(R)(t_i)^- = 1 - R(t_i)^+ \lor \triangle(R)(t_i)^- = 1 - R(t_i)^- \land \triangle(R)(t_i)^+ = R(t_i)^- \lor (R(t_i)^+ + R(t_i)^- \leq 1) \land (\triangle(R)(t_i)^+ = R(t_i)^+ \land \triangle(R)(t_i)^- = R(t_i)^-)\}$. □

It is obvious that $\triangle(R)$ is pseudo-consistent if $R$ is inconsistent.

**Definition 16. (Combine)** Let $R$ be paraconsistent intuitionistic fuzzy relations on scheme $\Sigma$. Then, $\nabla(R) = \{t_i \in R|(\forall i)(\nabla(R)(t_i)^+ = \max(R(t_i)^+) \land (\nabla(R)(t_i)^- = \max(R(t_i)^-))\}$
It is obvious that $\nabla(R)$ is inconsistent if $R$ is pseudo-consistent.

Note that strong generalization defined above only holds for consistent or pseudo-consistent paraconsistent intuitionistic fuzzy relations. For any arbitrary paraconsistent intuitionistic fuzzy relations, we should first use split operation to transform them into non inconsistent paraconsistent intuitionistic fuzzy relations and apply the set-theoretic and relation-theoretic operations on them and finally use combine operation to transform the result into arbitrary paraconsistent intuitionistic fuzzy relation. For the simplification of notation, the following generalized algebra is defined under such assumption.

4. Generalized Algebra on Paraconsistent Intuitionistic Fuzzy Relations

In this section, we present one strong generalisation each for the fuzzy relation operators such as union, join, projection. To reflect generalisation, a hat is placed over a fuzzy relation operator to obtain the corresponding paraconsistent intuitionistic fuzzy relation operator. For example, $\hat{\bowtie}$ denotes the natural join among fuzzy relations, and $\hat{\bowtie}$ denotes natural join on paraconsistent intuitionistic fuzzy relations. These generalized operators maintain the belief system intuition behind paraconsistent intuitionistic fuzzy relations.

Set-Theoretic Operators

We first generalize the two fundamental set-theoretic operators, union and complement.

Definition 17. Let $R$ and $S$ be paraconsistent intuitionistic fuzzy relations on scheme $\Sigma$. Then,

(a) the union of $R$ and $S$, denoted $R \hat{\cup} S$, is a paraconsistent intuitionistic fuzzy relation on scheme $\Sigma$, given by

$$(R \hat{\cup} S)(t) = \langle \max\{R(t)^+, S(t)^+\}, \min\{R(t)^-, S(t)^-\} \rangle,$$

for any $t \in \tau(\Sigma)$;

(b) the complement of $R$, denoted $\hat{-} R$, is a paraconsistent intuitionistic fuzzy relation on scheme $\Sigma$, given by

$$(\hat{-} R)(t) = \langle R(t)^-, R(t)^+ \rangle,$$

for any $t \in \tau(\Sigma)$.

An intuitive appreciation of the union operator can be obtained as follows: Given a tuple $t$, since we believed that it is present in the relation $R$ with confidence $R(t)^+$ and that it is present in the relation $S$ with confidence $S(t)^+$, we can now believe that the tuple $t$ is present in the “either-$R$-or-$S$” relation with confidence which is equal to the larger of $R(t)^+$ and $S(t)^+$. Using the same logic, we can now believe in the absence of the tuple $t$ from the “either-$R$-or-$S$” relation with confidence which...
is equal to the smaller (because $t$ must be absent from both $R$ and $S$ for it to be absent from the union) of $R(t)^-$ and $S(t)^-$. The definition of complement and of all the other operators on paraconsistent intuitionistic fuzzy relations defined later can (and should) be understood in the same way.

**Proposition 3.** The operators $\hat{\cup}$ and unary $\sim$ on paraconsistent intuitionistic fuzzy relations are strong generalisations of the operators $\cup$ and unary $-$ on fuzzy relations.

**Proof.** Let $R$ and $S$ be consistent paraconsistent intuitionistic fuzzy relations on scheme $\Sigma$. Then $\text{reps}_\Sigma(R \hat{\cup} S)$ is the set
\[
\{ Q \mid \bigwedge_{t_i \in \tau(\Sigma)} (\max\{R(t_i)^+, S(t_i)^+\} \leq Q(t_i) \leq 1 - \min\{R(t_i)^-, S(t_i)^-\}) \}
\]
This set is the same as the set
\[
\{ r \cup s \mid \bigwedge_{t_i \in \tau(\Sigma)} (R(t_i)^+ \leq r(t_i) \leq 1 - R(t_i)^-), \bigwedge_{t_i \in \tau(\Sigma)} (S(t_i)^+ \leq s(t_i) \leq 1 - S(t_i)^-) \}
\]
which is $S(\bigcup)(\text{reps}_\Sigma(R), \text{reps}_\Sigma(S))$. Such a result for unary $\sim$ can also be shown similarly.

For sake of completeness, we define the following two related set-theoretic operators:

**Definition 18.** Let $R$ and $S$ be paraconsistent intuitionistic fuzzy relations on scheme $\Sigma$. Then,

(a) the intersection of $R$ and $S$, denoted $R \hat{\cap} S$, is a paraconsistent intuitionistic fuzzy relation on scheme $\Sigma$, given by
\[
(R \hat{\cap} S)(t) = \langle \min\{R(t)^+, S(t)^+\}, \max\{R(t)^-, S(t)^-\} \rangle, \text{ for any } t \in \tau(\Sigma);
\]
(b) the difference of $R$ and $S$, denoted $R \hat{\setminus} S$, is a paraconsistent intuitionistic fuzzy relation on scheme $\Sigma$, given by
\[
(R \hat{\setminus} S)(t) = \langle \min\{R(t)^+, S(t)^-\}, \max\{R(t)^-, S(t)^+\} \rangle, \text{ for any } t \in \tau(\Sigma);
\]

The following proposition relates the intersection and difference operators in terms of the more fundamental set-theoretic operators union and complement.

**Proposition 4.** For any paraconsistent intuitionistic fuzzy relations $R$ and $S$ on the same scheme, we have
\[
R \hat{\cap} S = \neg(\neg R \hat{\cup} \neg S), \text{ and } \\
R \hat{\setminus} S = \neg(\neg R \hat{\cap} S).
\]
The second part of the result can be shown similarly.

\begin{proof}
By definition, 
\[ \hat{R}(t) = \langle R(t)^-, R(t)^+ \rangle \]
\[ \hat{S}(t) = \langle S(t)^-, S(t)^+ \rangle \]
and \[ \langle \hat{R} \cup \hat{S} \rangle(t) = \langle \max(R(t)^-, S(t)^-), \min(R(t)^+, S(t)^+) \rangle \]
so, \[ \langle \hat{R} \cap \hat{S} \rangle(t) = \langle \min(R(t)^+, S(t)^+), \max(R(t)^-, S(t)^-) \rangle \]
\[ = R \cap S(t). \]
\end{proof}

Relation-Theoretic Operators

We now define some relation-theoretic algebraic operators on paraconsistent intuitionistic fuzzy relations.

**Definition 19.** Let \( R \) and \( S \) be paraconsistent intuitionistic fuzzy relations on schemes \( \Sigma \) and \( \Delta \), respectively. Then, the *natural join* (further for short called *join*) of \( R \) and \( S \), denoted \( R \Join S \), is a paraconsistent intuitionistic fuzzy relation on scheme \( \Sigma \cup \Delta \), given by

\[ (R \Join S)(t) = \{ \min\{R(\pi_\Sigma(t))^+, S(\pi_\Delta(t))^+\}, \max\{R(\pi_\Sigma(t))^-, S(\pi_\Delta(t))^+\} \}, \]

where \( \pi \) is the usual projection of a tuple.

It is instructive to observe that, similar to the intersection operator, the minimum of the belief factors and the maximum of the doubt factors are used in the definition of the join operation.

**Proposition 5.** \( \Join \) is a strong generalisation of \( \sqcap \).

\begin{proof}
Let \( R \) and \( S \) be consistent paraconsistent intuitionistic fuzzy relations on schemes \( \Sigma \) and \( \Delta \), respectively. Then \( \text{reps}_{\Sigma \cup \Delta}(R \Join S) \) is the set \( \{ Q \in \mathcal{F}(\Sigma \cup \Delta) \mid \bigwedge_{t \in \tau(\Sigma \cup \Delta)}(\min\{R_{\pi_\Sigma}(t_i)^+, S_{\pi_\Delta}(t_i)^+\} \leq Q(t_i) \leq 1 - \max\{R_{\pi_\Sigma}(t_i)^-, S_{\pi_\Delta}(t_i)^-\}) \} \) and \( S(\Join)(\text{reps}_\Sigma(R), \text{reps}_\Delta(S)) = \{ r \Join s \mid r \in \text{reps}_\Sigma(R), s \in \text{reps}_\Delta(S) \} \).

Let \( Q \in \text{reps}_{\Sigma \cup \Delta}(R \Join S) \). Then \( \pi_\Sigma(Q) \in \text{reps}_\Sigma(R) \), where \( \pi_\Sigma \) is the usual projection over \( \Sigma \) of fuzzy relations. Similarly, \( \pi_\Delta(Q) \in \text{reps}_\Delta(S) \). Therefore, \( Q \in S(\Join)(\text{reps}_\Sigma(R), \text{reps}_\Delta(S)) \).

Let \( Q \in S(\Join)(\text{reps}_\Sigma(R), \text{reps}_\Delta(S)) \). Then \( Q(t_i) \geq \min\{R_{\pi_\Sigma}(t_i)^+, S_{\pi_\Delta}(t_i)^+\} \) and \( Q(t_i) \leq \min\{1 - R_{\pi_\Sigma}(t_i)^-, 1 - S_{\pi_\Delta}(t_i)^-\} = 1 - \max\{R_{\pi_\Sigma}(t_i)^-, S_{\pi_\Delta}(t_i)^-\} \), for any \( t_i \in \tau(\Sigma \cup \Delta) \), because \( R \) and \( S \) are consistent. Therefore, \( Q \in \text{reps}_{\Sigma \cup \Delta}(R \Join S) \).

We now present the projection operator.
Definition 20. Let $R$ be a paraconsistent intuitionistic fuzzy relation on scheme $\Sigma$, and $\Delta \subseteq \Sigma$. Then, the projection of $R$ onto $\Delta$, denoted $\hat{\pi}_\Delta(R)$, is a paraconsistent intuitionistic fuzzy relation on scheme $\Delta$, given by

$$(\hat{\pi}_\Delta(R))(t) = \langle \max\{R(u)^+|u \in t^\Sigma\}, \min\{R(u)^-|u \in t^\Sigma\} \rangle.$$  

The belief factor of a tuple in the projection is the maximum of the belief factors of all of the tuple’s extensions onto the scheme of the input paraconsistent intuitionistic fuzzy relation. Moreover, the doubt factor of a tuple in the projection is the minimum of the doubt factors of all of the tuple’s extensions onto the scheme of the input paraconsistent intuitionistic fuzzy relation.

We present the selection operator next.

Definition 21. Let $R$ be a paraconsistent intuitionistic fuzzy relation on scheme $\Sigma$, and let $F$ be any logic formula involving attribute names in $\Sigma$, constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\lor$ and $\land$. Then, the selection of $R$ by $F$, denoted $\hat{\sigma}_F(R)$, is a paraconsistent intuitionistic fuzzy relation on scheme $\Sigma$, given by

$$(\hat{\sigma}_F(R))(t) = \langle \alpha, \beta \rangle,$$

where

$$\alpha = \begin{cases} R(t)^+ & \text{if } t \in \sigma_F(\tau(\Sigma)) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta = \begin{cases} R(t)^- & \text{if } t \in \sigma_F(\tau(\Sigma)) \\ 1 & \text{otherwise} \end{cases}$$

where $\sigma_F$ is the usual selection of tuples satisfying $F$ from ordinary relations. □

If a tuple satisfies the selection criterion, it’s belief and doubt factors are the same in the selection as in the input paraconsistent intuitionistic fuzzy relation. In the case where the tuple does not satisfy the selection criterion, its belief factor is set to 0 and the doubt factor is set to 1 in the selection.

Proposition 6. The operators $\hat{\pi}$ and $\hat{\sigma}$ are strong generalisations of $\pi$ and $\sigma$, respectively.

Proof. Similar to that of Proposition 5. □

Example 1. Relation schemes are sets of attribute names, but in this example we treat them as ordered sequences of attribute names (which can be obtained through permutation of attribute names), so tuples can be viewed as the usual lists of values. Let $\{a, b, c\}$ be a common domain for all attribute names, and let $R$ and $S$ be the following paraconsistent intuitionistic fuzzy relations on schemes $\langle X, Y \rangle$ and $\langle Y, Z \rangle$ respectively.
For other tuples which are not in the paraconsistent intuitionistic fuzzy relations $R(t)$ and $S(t)$, their $\langle \alpha, \beta \rangle = \langle 0, 0 \rangle$ which means no any information available. Because $R$ and $S$ are inconsistent, we first use split operation to transform them into pseudo-consistent and apply the relation-theoretic operations on them and transform the result back to arbitrary paraconsistent intuitionistic fuzzy set using combine operation. Then, $T_1 = \triangledown(\Delta(R) \bowtie \Delta(S))$ is a paraconsistent intuitionistic fuzzy relation on scheme $\langle X, Y, Z \rangle$ and $T_2 = \triangledown(\delta_{X \rightarrow Z}(\Delta(T_1)))$ and $T_3 = \sigma_{X \rightarrow Z}(T_2)$ are paraconsistent intuitionistic fuzzy relations on scheme $\langle X, Z \rangle$. $T_1$, $T_2$ and $T_3$ are shown below:

| $t$   | $R(t)$ | $t$   | $S(t)$ |
|------|--------|------|--------|
| $(a, a)$ | $(0, 1)$ | $(a, c)$ | $(1, 0)$ |
| $(a, b)$ | $(0, 1)$ | $(b, a)$ | $(1, 1)$ |
| $(a, c)$ | $(0, 1)$ | $(c, b)$ | $(0, 1)$ |
| $(b, b)$ | $(1, 0)$ | $(b, c)$ | $(1, 0)$ |
| $(b, c)$ | $(1, 0)$ | $(c, b)$ | $(1, 0)$ |
| $(c, b)$ | $(1, 0)$ | $(c, c)$ | $(0, 1)$ |

5. An Application

Consider the target recognition example presented in 31. Here, an autonomous vehicle needs to identify objects in a hostile environment such as a military battlefield. The autonomous vehicle is equipped with a number of sensors which are used to collect data, such as speed and size of the objects (tanks) in the battlefield. Associated with each sensor, we have a set of rules that describe the type of the object based on the properties detected by the sensor.
Let us assume that the autonomous vehicle is equipped with three sensors resulting in data collected about radar readings, of the tanks, their gun characteristics and their speeds. What follows is a set of rules that associate the type of object with various observations.

**Radar Readings:**

- Reading $r_1$ indicates that the object is a T-72 tank with belief factor 0.80 and doubt factor 0.15.
- Reading $r_2$ indicates that the object is a T-60 tank with belief factor 0.70 and doubt factor 0.20.
- Reading $r_3$ indicates that the object is not a T-72 tank with belief factor 0.95 and doubt factor 0.05.
- Reading $r_4$ indicates that the object is a T-80 tank with belief factor 0.85 and doubt factor 0.10.

**Gun Characteristics:**

- Characteristic $c_1$ indicates that the object is a T-60 tank with belief factor 0.80 and doubt factor 0.20.
- Characteristic $c_2$ indicates that the object is not a T-80 tank with belief factor 0.90 and doubt factor 0.05.
- Characteristic $c_3$ indicates that the object is a T-72 tank with belief factor 0.85 and doubt factor 0.10.

**Speed Characteristics:**

- Low speed indicates that the object is a T-60 tank with belief factor 0.80 and doubt factor 0.15.
- High speed indicates that the object is not a T-72 tank with belief factor 0.85 and doubt factor 0.15.
- High speed indicates that the object is not a T-80 tank with belief factor 0.95 and doubt factor 0.05.
- Medium speed indicates that the object is not a T-80 tank with belief factor 0.80 and doubt factor 0.10.

These rules can be captured in the following three paraconsistent intuitionistic fuzzy relations:

| Reading | Object | Confidence Factors |
|---------|--------|-------------------|
| $r_1$   | T-72   | (0.80, 0.15)      |
| $r_2$   | T-60   | (0.70, 0.20)      |
| $r_3$   | T-72   | (0.05, 0.95)      |
| $r_4$   | T-80   | (0.85, 0.10)      |
The autonomous vehicle uses the sensors to make observations about the different objects and then uses the rules to determine the type of each object in the battlefield. It is quite possible that two different sensors may identify the same object as of different types, thereby introducing inconsistencies.

Let us now consider three objects \( o_1, o_2 \) and \( o_3 \) which need to be identified by the autonomous vehicle. Let us assume the following observations made by the three sensors about the three objects. Once again, we assume certainty factors (maybe derived from the accuracy of the sensors) are associated with each observation.

Given these observations and the rules, we can use the following algebraic expression
to identify the three objects:

\[
\pi_{\text{Object-id, Object}}(\text{RadarData} \bowtie \bowtie \text{RadarRules}) \cap \\
\pi_{\text{Object-id, Object}}(\text{GunData} \bowtie \bowtie \text{GunRules}) \cap \\
\pi_{\text{Object-id, Object}}(\text{SpeedData} \bowtie \bowtie \text{SpeedRules})
\]

The intuition behind the intersection is that we would like to capture the common (intersecting) information among the three sensor data. Evaluating this expression, we get the following paraconsistent relation:

| Object-id | Object | Confidence Factors |
|-----------|--------|-------------------|
| o_1       | T-72   | (0.05, 0.0)       |
| o_2       | T-80   | (0.0, 0.05)       |
| o_3       | T-80   | (0.05, 0.0)       |

It is clear from the result that by the given information, we could not infer any useful information that is we could not decide the status of objects \(o_1, o_2\) and \(o_3\).

6. Conclusions and Future Work

We have presented a generalization of fuzzy relations, intuitionistic fuzzy relations (interval-valued fuzzy relations) and paraconsistent relations, called paraconsistent intuitionistic fuzzy relations, in which we allow the representation of confidence (belief and doubt) factors with each tuple. The algebra on fuzzy relations is appropriately generalized to manipulate paraconsistent intuitionistic fuzzy relations.

Various possibilities exist for further study in this area. Recently, there has been some work in extending logic programs to involve quantitative paraconsistency. Paraconsistent logic programs were introduced in \(^{32}\) and probabilistic logic programs in \(^{33}\). Paraconsistent logic programs allow negative atoms to appear in the head of clauses (thereby resulting in the possibility of dealing with inconsistency), and probabilistic logic programs associate confidence measures with literals and with entire clauses. The semantics of these extensions of logic programs have already been presented, but implementation strategies to answer queries have not been discussed. We propose to use the model introduced in this paper in computing the semantics of these extensions of logic programs. Exploring application areas is another important thrust of our research.

We developed two notions of generalising operators on fuzzy relations for paraconsistent intuitionistic fuzzy relations. Of these, the stronger notion guarantees that any generalised operator is “well-behaved” for paraconsistent intuitionistic fuzzy relation operands that contain consistent information.

For some well-known operators on fuzzy relations, such as union, join, projection, we introduced generalised operators on paraconsistent intuitionistic fuzzy relations. These generalised operators maintain the belief system intuition behind paraconsistent intuitionistic fuzzy relations, and are shown to be “well-behaved” in the sense mentioned above.
Our data model can be used to represent relational information that may be incomplete and inconsistent. As usual, the algebraic operators can be used to construct queries to any database systems for retrieving vague information.

References
1. Codd, E.: A relational model for large shared data banks. Communications of the ACM 13 (1970) 377–387
2. Elmasri, Navathe: Fundamentals of Database Systems. Third edn. Addison–Wesley, New York (2000)
3. Silberschatz, A., Korth, H.F., Sudarshan, S.: Database System Concepts. Third edn. McGraw–Hill, Boston (1996)
4. Parsons, S.: Current approaches to handling imperfect information in data and knowledge bases. IEEE Trans. Knowledge and Data Engineering 3 (1996) 353–372
5. Biskup, J.: A foundation of codd’s relational maybe–operations. ACM Trans. Database Syst. 8 4 (1983) 608–636
6. Brodie, M.L., Mylopoulos, J., Schmidt, J.W.: On the development of data models. On Conceptual Modelling (1984) 19–47
7. Codd, E.: Extending the database relational model to capture more meaning. ACM Transactions of Database Systems 4 (1979) 397–434
8. Lipski, W.: On semantic issues connected with incomplete information databases. ACM Trans. Database Syst. 4 3 (1979) 262–296
9. Lipski, W.: On databases with incomplete information. Journal of the Association for Computing Machinery 28 (1981) 41–70
10. Maier, D.: The Theory of Relational Databases. Computer Science Press, Rockville, Maryland (1983)
11. Zadeh, L.A.: Fuzzy sets. Inf. Control 8 (1965) 338–353
12. Turksen, I.: Interval valued fuzzy sets based on normal forms. Fuzzy Sets and Systems 20 (1986) 191–210
13. Atanassov, K.: Intuitionistic fuzzy sets. Fuzzy Sets and Systems 20 (1986) 87–96
14. Anvari, M., Rose, G.F.: Fuzzy relational databases. In: Proceedings of the 1st International Conference on Fuzzy Information Processing (Kuauai, Hawaii), CRC Press (1984)
15. Baldwin, J.F.: A fuzzy relational inference language for expert systems. In: Proceedings of the 13th IEEE International Symposium on Multivalued Logic (Kyoto, Japan). (1983) 416–423
16. Buckles, B.P., Petry, F.E.: A fuzzy representation for relational databases. Fuzzy Sets Syst. 7 (1982) 213–226
17. Chang, S.K., Ke, J.S.: Database skeleton and its application to fuzzy query translation. IEEE Trans. Softw. Eng. SE–4 (1978) 31–43
18. Kacprzyk, J., Ziolkowski, A.: Database queries with fuzzy linguistic quantifiers. IEEE Trans. Syst. Man Cybern. SMC–16 3 (1986) 474–479
19. Prade, H.: Lipskiš approach to incomplete information databases restated and generalised in the setting of zadeh’s possibility theory. Inf. Syst. 9 1 (1984) 27–42
20. Raju, K.V.S.V.N., Majumdar, A.K.: Fuzzy functional dependencies and lossless join decomposition of fuzzy relational database systems. ACM Trans. on Database Syst. 13 2 (1988) 129–166
21. Zadeh, L.A.: Fuzzy sets as the basis for a theory of possibility. Fuzzy Sets and Systems 1 (1978) 1–27
22. Prade, H., Testemale, C.: Generalizing database relational algebra for the treatment
of incomplete or uncertain information and vague queries. Information Sciences 34 (1984) 115–143
23. Prade, H., Testemale, C.: Representation of soft constraints and fuzzy attribute values by means of possibility distributions in databases. Analysis of Fuzzy Information, Volume II, Artificial Intelligence and Decision Systems (1987) 213–229
24. Wong, E.: A statistical approach to incomplete information in database systems. ACM Trans. on Database Systems 7 (1982) 470–488
25. Cavallo, R., Pottarelli, M.: The theory of probabilistic databases. In: Proceedings of the 13th Very Large Database Conference. (1987) 71–81
26. de Amo, S., Carnielli, W., Marcos, J.: A logical framework for integrating inconsistent information in multiple databases. In: Proc. PolIKS’02, LNCS 2284. (2002) 67–84
27. Bagai, R., Sunderraman, R.: A paraconsistent relational data model. International Journal of Computer Mathematics 55 (1995)
28. Costa, N.C.A.D.: On the theory of inconsistent formal systems. Notre Dame Journal of Formal Logic 15 (1977) 621–630
29. Belnap, N.D.: A useful four-valued logic. In Eppstein, G., Dunn, J.M., eds.: Modern Uses of Many-valued Logic. Reidel, Dordrecht (1977) 8–37
30. Lung Gau, W., Buehrer, D.J.: Vague sets. IEEE Trans. On Syst. Man Cybern. SMC–23 2 (1993) 610–614
31. Subrahmanian, V.S.: Amalgamating knowledge bases. ACM Transactions on Database Systems 19 (1994) 291–331
32. Blair, H.A., Subrahmanian, V.S.: Paraconsistent logic programming. Theoretical Computer Science 68 (1989) 135–154
33. Ng, R., Subrahmanian, V.: Probabilistic logic programming. Information and Computation 101 (1992) 150–201