The calculation of the thermodynamic quantities of the Bardeen black hole

Jingyun Man        Hongbo Cheng*
Department of Physics, East China University of Science and Technology,
Shanghai 200237, China
The Shanghai Key Laboratory of Astrophysics, Shanghai 200234, China

Abstract

In this work we research on the thermodynamical properties of the Bardeen black holes. We compute the series of new thermodynamic quantities such as local temperature, heat capacity, off-shell free energy of this kind of black hole in detail. We further analyze the thermodynamical characteristics of the Bardeen black hole by varying its charge $q$ to check the existence and stability of the black hole.

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*E-mail address: hbcheng@ecust.edu.cn
I. Introduction

The thermodynamics of various kinds of black holes which are space regions that nothing can escape from become a focus recently. More than thirty years ago, Bekenstein predicted that black holes should have finite temperature which is not equal to zero and further the entropy of a black hole is proportional to its surface area [1-3]. Hawking et. al. researched on the quantum mechanics of scalar particles around a black hole to reveal that the black hole has a thermal radiation with the temperature due to its surface gravity [4-6]. A lot of notes on the phase transitions of black holes in the frame of semiclassical gravity were listed in Ref. [7]. The thermodynamic characteristics of modified Schwarzschild black hole and quantum-corrected Schwarzschild black hole have been investigated [8-10]. The thermodynamic behaviours including phase transition in Born-Infeld-anti-de Sitter black holes were probed in virtue of various ways [11, 12].

A kind of black hole whose distinct feature is that its spacetime is regular without a singularity but a horizon was put forward by Bardeen [13]. Some other regular black holes were also discussed [14, 15]. More efforts have been made for the Bardeen black holes. A Bardeen black hole and some similar regular black holes as exact solutions to a model of nonlinear electrodynamics coupled to Einstein gravity was studied [16-19]. Eiroa and Sendra examined the gravitational lensing of the Bardeen black holes [20]. The geodesic motion of the test particles around the regular black hole was also studied [21]. The gravitational and electromagnetic stability of this kind of black holes have been explored [22]. The quasinormal modes of the scalar field perturbations of the Bardeen black hole is discussed [23].

It is necessary to scrutinize the thermodynamic properties of the Bardeen black holes. The quantum corrections to some thermodynamical quantities such as temperature and entropy for a Bardeen black hole were evaluated within a quantum tunneling approach over semiclassical approximation [24]. In the noncommutative spacetime the temperature, entropy, singularity, horizon and mass function of the charged regular black hole are examined [25]. In Ref. [24] and [25], only the quantum and noncommutative corrections to the Hawking temperature and entropy of the Bardeen black hole were revealed, and the further research on the thermodynamic characteristics of this kind of black hole have not been performed. We wish to find the other thermodynamical characteristics of the Bardeen black hole. We plan to derive and calculate the local temperature, heat capacity and off-shell free energy of the Bardeen black hole in order to check the existence and stability of the black hole. We certainly wonder how the thermodynamical quantities related to the magnetic charge of the black hole. The results and conclusions will be listed finally.

II. The thermodynamics quantities Bardeen black holes

The regular static-charged black hole named as Bardeen black hole is the gravitational field of a magnetic monopole arising from nonlinear electrodynamics [13, 16]. According to the Einstein equations and field equations from the proposed action involving the nonlinear electrodynamical term, the metric of the so-called Bardeen black hole as a static spherically solution for the equations is
given as,

\[ ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \]  \tag{1} \]

where

\[ f(r) = 1 - \frac{2Mr}{(r^2 + q^2)^{\frac{3}{2}}} \]  \tag{2} \]

Here \( q \) and \( M \) are the magnetic charge and the mass of the magnetic monopole respectively. As \( q = 0 \), the spacetime structure of Bardeen black hole certainly recover to be the Schwarzschild metric.

Asymptotically, the metric function \( f(r) \) can be approximated as,

\[ f(r) \approx 1 - \frac{2M}{r} + \frac{3Mq^2}{r^3} + O\left(\frac{1}{r^5}\right) \]  \tag{3} \]

which is different from Reissner-Nordstrom metric. According to Figure 1, if the horizons exist, the inner one as Killing horizon and the outer one as event horizon, the charge \( q \) has to be smaller than \( \frac{4\sqrt{3}}{9}M \). When \( q = \frac{4\sqrt{3}}{9}M \), the two horizons meet. When \( q > \frac{4\sqrt{3}}{9}M \), there is no horizon for \( f(r) \).

The event horizon of metric (1) is located at,

\[ r_H = \left[ \frac{1}{3}(4M^2 - 3q^2) + \frac{\sqrt{2}}{3}(32M^6 - 72M^4q^2 + 27M^2q^4 + 3\sqrt{3}\sqrt{27M^4q^8 - 16M^6q^6})^\frac{1}{2} \right. \]

\[ - \left. \frac{24M^2q^2 - 16M^4}{3\sqrt{2}(32M^6 - 72M^4q^2 + 27M^2q^4 + 3\sqrt{3}\sqrt{27M^4q^8 - 16M^6q^6})^\frac{1}{2}} \right]^{\frac{1}{2}} \]  \tag{4} \]

The relation between the total mass and the event horizon also results from metric (1),

\[ M = \frac{(r_H^2 + q^2)^{\frac{3}{2}}}{2r_H^2} \]  \tag{5} \]

then the minimal mass \( M_0 \) reads,

\[ M_0 = \frac{3\sqrt{3}}{4}q \]  \tag{6} \]

which has something to do with the magnetic charge. The total mass as a function of event horizon and charge is depicted in Figure 2. The Hawking temperature can be calculated as,

\[ T_H(q) = \frac{1}{4\pi} \left[ \sqrt{\left(-g^{tt}g^{rr}g_{\theta\theta}\right)} \right]_{r=r_H} \]

\[ = \frac{1}{4\pi r_H^2} \left( \frac{r_H^2 - 2q^2}{r_H^2 + q^2} \right) \]  \tag{7} \]

where the prime stands for the derivative with respect to the radial coordinate \( r \). According to Ref. [26], the local temperature is given by,
The temperature associated with the black hole radius and charge is plotted in Figure 3. It is clear that the shape of temperature is similar to those of the noncommutative Schwarzschild black hole and the Reissner-Nordstrom black hole [8] although the expressions are different. There is a zero for the local temperature \( T_{\text{loc}}(q) |_{r_H = r_0} = 0 \) and here \( r_0 = \sqrt{2}q \). Minimizing the local temperature (8) with respect to the black hole radius \( r_H \) like \( \left( \frac{\partial T_{\text{loc}}}{\partial r_H} \right)_r = 0 \), we find the following equation,

\[
(r_H^2 - 2q^2)^2 + \left[ 1 - \frac{r^2}{r_H^2} \left( \frac{r_H^2 + q^2}{r^2 + q^2} \right)^3 \right] (18r_H^2q^2 - 3r_H^4) = 0
\]

whose real roots are \( r_1 \) and \( r_2 \). The numerical calculation shows that the smaller radius \( r_1 \) becomes larger while the larger one \( r_2 \) gets smaller with increasing charge \( q \). The local temperature has two extrema denoted as \( T_i = T |_{r_H = r_i} \) with \( i = 1, 2 \), where \( r_1 \) and \( r_2 \) are the positions of the two extrema respectively. We choose charge \( q = 0.5 \) to obtain \( T_1 = 0.0411 \) and \( T_2 = 0.0204 \) with \( r_2 = 6.594 \). It is also interesting that the higher temperature \( T_1 \) decreases remarkably and the lower one \( T_2 \) just increase a little when the charge \( q \) becomes larger. There is one small black hole for \( 0 < T < T_2 \) and one large black hole for \( T > T_1 \). When the local temperature is intermediate like \( T_2 < T < T_1 \), there will exist three black holes.

According to Bekenstein’s approach, the entropy is proportional to the area of event horizon and denoted as [1-3],

\[
S = \frac{A}{4} = \pi r_H^2
\]

leading \( dS = 2\pi r_H dr_H \).

We combine the local temperature (8), entropy (10) and the first law of thermodynamics \( dE_{\text{loc}} = T_{\text{loc}} dS \) to find that,

\[
dE_{\text{loc}} = \frac{1}{2} \frac{r_H^2 - 2q^2}{r_H^2 + q^2} \left( \frac{1}{\sqrt{1 - \frac{r^2}{r_H^2} \left( \frac{r_H^2 + q^2}{r^2 + q^2} \right)^2}} \right) dr_H
\]

In order to check the stability of the Bardeen black hole, we should derive the black hole’s capacity as,

\[
C(q) = \left( \frac{\partial E_{\text{loc}}}{\partial T_{\text{loc}}} \right)_r
\]
The curves of the heat capacity of Bardeen black hole is exhibited in Figure 4. The heat capacity is positive for \( r_0 < r_H < r_1 \) and \( r_H > r_2 \), which means that the small and the large Bardeen black hole can survive for long time. Within the region \( r_1 < r_H < r_2 \), the black hole heat capacity is negative, showing that the intermediate black holes decay quickly. As mentioned above, the positions \( r_1 \) and \( r_2 \) approach each other as the black hole charge becomes stronger. There are two zeros for the heat capacity denoted as \( C(q) |_{r_H=r_0,r} = 0 \). It should be pointed out that the stability of the black holes can be displayed in their off-shell free energy.

The off-shell free energy is defined as,

\[
F_{\text{off}} = E_{\text{loc}} - TS
\]

where \( T \) is an arbitrary temperature. According to Eq. (10) and (11), the off-shell free energy of Bardeen black holes can be represented in the integral form as,

\[
F_{\text{off}}(q) = \frac{1}{2} \int_{r_0}^{r_H} \frac{x^2 - 2q^2}{x^2 + q^2} \sqrt{1 - \frac{x^2}{r_H^2} (\frac{x^2}{x^2 + q^2})^2} \, dx - \pi r_H^2 T
\]

It is evident that the off-shell free energy expression is not explicit, but the relation between the off-shell free energy and the event horizon for several temperatures within the region \( r_H \leq r \) can be shown graphically. How the off-shell free energy of Bardeen black holes changes with the horizon under several temperatures is exhibited in Figure 5. When \( T < T_2 \), the curves of off-shell free energy have only one minimum respectively, which representing a small stable black hole. As the temperature rises to the value between \( T_2 \) and \( T_1 \), three black holes will emerge. Among the three black holes, the small one and the large one are stable while the intermediate black hole will be unstable, corresponding to two minimum and one maxima of the off-shell free energy, which is exhibited in Figure 6. If the temperature is sufficiently high like \( T > T_1 \), there will be only one minimum for the free energy at the position that the horizon is larger, and certainly one black hole which is large and stable will appear.

III. Discussion

In this paper we explore the existence and the stability of the Bardeen black hole in a new direction. We calculate the thermodynamic quantities such as local temperature, heat capacity, off-shell free energy for black hole to show how the quantities change with the horizon \( r_H \) and the charge \( q \). In general, one small black hole will generate when the temperature is not too low. With the higher and higher temperature there will be three black holes. For the sufficiently high temperature, one larger black hole remains. According to the heat capacity, only when the event horizon is between \( r_0 \) and \( r_1 \) or larger than \( r_2 \), the Bardeen black holes can be stable because of
the positive parts of the quantity. Within the region that the size of the Bardeen black hole is between $r_1$ and $r_2$, the heat capacity is negative, corresponding to an unstable black hole. It can be checked as a requirement for off-shell free energy owing to its extrema. It is obvious that only when the temperature is higher than $T_2$ but smaller than $T_1$ the stable small and large black holes corresponding to two minima of the off-shell free energy survive for long time while the middle one due to the maximum of the free energy will decay quickly. For $T > T_1$, one large Bardeen black hole exists stably. It should be pointed out that the influence from the charge of black hole on the black hole size and the critical temperatures is manifest and distinct.

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Figure 1: The relation between $f(r)$ and $r$ for magnetic charge $q = 0, 0.5, 0.7698$ respectively.
Figure 2: The total mass of Bardeen black hole as a function of event horizon for magnetic charge $q = 0, 0.5, 0.7698$ respectively.
Figure 3: The dependence of local temperature on the event horizon for magnetic charge $q = 0, 0.5, 0.7698$ respectively.
Figure 4: The solid and dotted curves of the dependence of the local temperatures on the horizon with $8\pi G\eta^2 \approx 10^{-5}$, $r = 10$ and $\psi_0 = 0.02$ for the Schwarzschild black hole with a global monopole or an $f(R)$ global monopole respectively.
Figure 5: The solid, dotted and dashed curves of the off-shell free energy of the Bardeen black hole as a function of the horizon with $q = 0.5$ under the temperature $T = 0.015, 0.028, 0.06$ respectively.
Figure 6: The dependence of the off-shell free energy of the Bardeen black hole on the horizon with $q = 0.5$ and $r = 10$ under the temperature 0.028.