Prescribed-Time Synchronization of Multiweighted and Directed Complex Networks
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Abstract—In this note, we study the prescribed-time (PT) synchronization of the multiweighted and directed complex networks (MWDCNs). Unlike finite-time and fixed-time synchronization, the time for synchronization can be prescribed as needed, which is independent of initial values and parameters, such as coupling matrix. At first, we define a class of regulatory functions whose improper integrals diverge, and it is proved to play a key role in PT stability for dynamical systems. Many special functions established previously for PT stability can be included in this class. Then, we apply this new result on MWDCNs as an application. The synchronization error at the PT is discussed carefully, so PT synchronization can be reached. The network topology can be directed and disconnected, which means that the outer coupling matrices (OCMs) can be asymmetric and not connected. The relationships between nodes are allowed to be cooperative or competitive, so elements in OCMs and inner coupling matrices (ICMs) can be positive or negative. We use the rearranging variables’ order technique to combine ICMs and OCMs together to get the sum matrices, which can make a bridge between multiweighted and single-weighted networks. Finally, simulations are presented to illustrate its effectiveness.

Index Terms—Complex networks, improper integral, multiweighted and directed, prescribed-time (PT) synchronization, regulatory functions.

I. INTRODUCTION

A complex dynamical network can be regarded as a graph with sundry nodes, complex structure, and divers connections. Over the past years, many scholars have conducted research on its dynamics, such as consensus [1], synchronization [2], and control [3], which means that every node in the network reaches the same state after a duration of time.

The settling time of synchronization is a key index to evaluate the network efficiency, which relies on the development of corresponding stability theories. This time means that synchronization should be achieved within it, which helps to estimate the completion time of tasks. By now, this performance index has become a hot topic [4], [5], [6], [7], [8], [9]. One type is the finite-time stability, which is an effective way to accelerate speed [4]. It has advantages of good disturbance rejection and robustness against uncertainties, but the settling time heavily depends on the initial values of the network. The other type is the fixed-time stability first proposed in [5], where the settling time can be independent of initial values. Finite-time and fixed-time stability/synchronization was further investigated in [6] and [7] by using the inverse function method.

The settling time of fixed-time stability actually depends on system parameters, such as control gain. Thus, the settling time for finite-time and fixed-time stabilities cannot be prescribed arbitrarily. However, many practical applications, such as robotic manipulators [10] and permanent magnet synchronous motor system [11], need this requirement. The key difficulty lies in how to design a suitable controller to obtain prescribed-time (PT) stability. A new concept called “prescribed convergence time” was first proposed in [12], but it was often larger than the actual convergence time. To overcome this, Sanchez-Tones et al. [13] put forward “PT stability.” It has been studied by means of backstepping method [14], [15], [16], impulsive regulation [17], linear time-varying inequality-based approach [24], and other methods to design a dynamic state feedback controller. These control laws are often bounded and time varying. For example, Espitia and Perruquetti [14] used a backstepping method to transform and select appropriately the PT stability target system for controllable linear time-invariant systems with input delay while using the polynomial method for the case without delay. A nonscaling backstepping design scheme was proposed in [15] and [16] for nonlinear systems. For affine dynamical systems with uncertain input noise, He et al. [17] applied the proposed impulsive regulation to regulate the settling time to fit the prescribed terminal time; then, the systems could be stabilized. Wang et al. [18] proposed the concept of “practical PT stability” and developed an adaptive fault-tolerant controller. Shakouri and Assadian [19] defined “triangular stability” and proved that a PT controller could achieve it. Similar to [20], Tran and Yucelen [21] established generalized time transformation functions and executed a control algorithm for the stability analysis on perturbed systems, whereas time transformation was also applied on multiaagent systems [22]. Linear state and observer-based output feedbacks were designed for PT stabilization in [23], Zhou and Zhang [24] used a linear time-varying inequality-based and backstepping approach to investigate PT stability, and proposed a unified framework of finite-time, fixed-time, and PT stability.

Along with the development of these PT stability theories, PT consensus [25], [26], [27], PT synchronization [28], [29], [30], and control [31], [32], [33], [34], [35], [36], [37] are investigated. For consensus problem, Wang et al. [25] presented a novel distributed protocol upon a new scaling function for multiaagent systems. Based on a similar scaling function, Guo and Liang [26] designed the PT bipartite consensus protocol by using the time scale theory. Second-order multiaagent systems were also investigated in [27]. For PT synchronization, an event-triggered controller with a time-varying control gain was developed in [28] and a smooth controller was designed in [29] for PT cluster synchronization. Shao et al. [30] designed smooth controllers to achieve synchronization on Luo’s network topology. For PT cluster synchronization, Shao et al. [9]—Senior Member, IEEE—by using the inverse function and per-
networks. For stochastic nonlinear strict-feedback systems, Li and Krstic considered the PT mean-square stabilization in [15] and PT output-feedback control in [31]. Instead of the fractional-power state feedback, the authors in [32], [33], [34], [35], and [36] built a time-varying PT controller based on the regular state feedback. Most of them only considered specific forms of the regulatory functions, while Hua et al. [37] considered a more general adjustment function recently by using the improper integral.

From the aforementioned papers, although PT synchronization has been studied from lots of aspects, most of them only discussed networks with a single weight. Nodes may have many kinds of connections in practice, for instance, one can travel from one point to another point by railway, highway, ship, airplane, etc. Yao et al. [38] considered the synchronization of fractional-order multilevel complex networks, and Wang et al. [39] investigated the \( H_{\infty} \) synchronization. For multilevel and directed complex networks (MDWCNs), the design of a Lyapunov function is a difficulty, and Liu [40] proposed two useful techniques to deal with this problem, whose routes were also adopted and applied in synchronization of the coupled reaction–diffusion neural networks with both state coupling and spatial coupling [41], [42].

In this note, we further investigate the PT stability under a general function, especially the value of derivation at the PT; moreover, the PT synchronization of multilevel complex networks with or without pinning control is also addressed. The main contributions of this article are listed as follows.

1) A general form of time-varying (regulatory) function is first defined. PT stability is proved strictly with improper integral and Taylor expansion. The time-varying functions are more general than those in previous papers [29], [30], [32], [33], [34], [35], [36]. Moreover, the derivative at the PT is discussed, which represents the feasibility of the designed control strategy.

2) We apply the obtained results on the PT synchronization problem for MDWCNs. In contrast to [38] and [39], where outer coupling matrices (OCMs) are required to be symmetric, we relax this requirement to be asymmetric and these OCMs are not necessarily strongly connected or even not connected. In addition to considering the cooperative relationship between nodes, we also consider the competitive relationship, which makes its application scope larger than in [40].

3) Compared with many existing studies on finite-time and fixed-time synchronization, our work on PT synchronization is more challenging. The settling time is fully independent of initial states and network parameters. Rearranging variables’ order technique (ROT) is applied to obtain the sum matrices by combining inner coupling matrices (ICMs) and OCMs.

The rest of this article is organized as follows. In Section II, a new class of regulatory functions are defined with improper integral, and two theorems about PT stability are first established. In Section III, PT synchronization for MDWCNs is investigated based on the new PT stability results and ROT. Three numerical simulations are given in Section IV. Finally, Section V concludes this article.

II. PT STABILITY FOR DYNAMICAL SYSTEMS

A careful investigation of PT stability for dynamical systems should be first presented.

Definition 1 (see [43]): For the following dynamical system:

\[
\dot{x}(t) = F(x(t), T)
\]  

(1)

where \( x(t) \in \mathbb{R}^n, F(\cdot) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n \) with \( F(0, T) = 0 \) is continuous, and \( T \) is the user-defined settling time. The origin \( x(t) = 0 \) of system (1) is said to achieve PT stability if it is

1) stable, if for any \( \epsilon > 0 \), there always exists \( \delta(\epsilon) \) such that for any \( x(0) \) that satisfies \( ||x(0)|| < \delta(\epsilon) \) and for all \( t > 0 \), the trajectories of (1) satisfy \( ||x(t)|| < \epsilon \).

2) PT attractive, if there exists an attractive domain of the origin such that for any \( x(0) \) that belongs to it and each solution \( x(t) \) exists with \( \lim_{t \to T^-} ||x(t)|| = 0 \).

Definition 2: For a PT \( T > 0 \), a regulatory function \( C(t), t \in [0, T] \) is denoted as \( C(t) \in \text{PTF}(T) \), if the following hold:

1) \( C(t) \in C^\infty[0, T] \), i.e., it is a smooth function;

2) \( C(t) \geq 0 \), and \( C(t) = 0 \) if and only if \( t = T \);

3) the following improper integral diverges:

\[ \int_0^T C(t)^{-1} dt = +\infty. \]  

(2)

Next, we present some important results for PT stability.

Theorem 1: For a continuous and nonnegative real function \( V(t) \) defined in the neighborhood of the origin, suppose

\[ \dot{V}(t) = -\delta C(t)^{-1}V(t), t \in [0, T] \]  

(3)

where the scalar \( \delta > 0 \), and the function \( C(t) \in \text{PTF}(T) \), then PT stability can be realized.

Proof: First, we compare PT stability with exponential stability. Since \( C(t) \) is continuous in interval \([0, T]\), denote \( \varpi = \max_{t \in [0, T]} C(t) \), we have \( 0 \leq C(t) \leq \varpi \), thus

\[ \dot{V}(t) = -\delta C(t)^{-1}V(t) \leq -\delta \varpi^{-1}V(t). \]

Obviously, exponential stability can be ensured.

Second, with the variable separation method, (3) becomes

\[ V(t)^{-1}dV(t) = -\delta C(t)^{-1}dt. \]  

(4)

By integration, the abovementioned differential equation would become

\[ \ln V(t) - \ln V(0) = -\delta \int_0^T C(t)^{-1} dt \]  

(5)

so if PT stability is realized \( (V(T) = \lim_{t \to T^-} V(t) = 0) \), one necessary condition is (2). Especially, if \( C(t) = (T - t)^\ell \), then (2) holds with \( \ell \geq 1 \), and in this case, \( C(t) \in \text{PTF}(T) \). On the other hand, by solving (5), we have \( V(t) = V(0)e^{-\delta \int_0^t C(s)^{-1}ds} \). Hence, (2) can deduce \( V(T) = 0 \). Therefore, (2) is a necessary and sufficient condition for \( V(T) = 0 \).

Third, when \( t \to T^- \), \( C(t) \to 0 \), and \( V(t) \to 0 \), then the right-hand side of (3) may become infinite, which has no sense in real applications. Therefore, it is important to discuss \( \dot{V}(T) \), which represents the feasibility of the control, and strictly according to Definition 1, we should also check \( \dot{V}(T) = 0 \).

We consider the polynomial forms of \( C(t) \) to start. Denote \( \Phi(T) = C(t)^{-1}V(t) \). If \( C(t) = T - t \), then

\[ \Phi(T) = \lim_{T \to T^-} \Phi(T) = \lim_{T \to T^-} V(0)(T - t)^{-(T - t)^{\delta - 1}} \]

therefore, if \( \delta < 1 \), then \( \Phi(T) \) would be infinite; if \( \delta = 1 \), then \( \Phi(T) \) would be a nonzero constant; and if \( \delta > 1 \), then \( \Phi(T) = 0 \).

On the other hand, if \( C(t) = (T - t)^\ell, \ell > 1 \), by letting \( q = (T - t)^{-1} \), when \( t \to T^- \), then \( q \to +\infty \); thus, we have

\[ \Phi(T) = V(0)e^{\frac{1}{q^{1-q}}} \lim_{q \to +\infty} \frac{q^q e^q}{(T - t)^{\ell q^{1-q}}} = 0. \]  

(6)

Now, for a general function \( C(t) \in \text{PTF}(T) \), suppose there exists \( k^* \geq 1 \), such that the \( k^* \)th order derivative \( C^{(k^*)}(T) \) satisfies \( C^{(k^*)}(T) \neq 0 \).
0, while $C^{(k)}(T) = 0, 0 \leq k < k^*$, so its Taylor expansion at $t = T$ would be

$$C(t) = \frac{C^{(k^*)}(T)}{k^*!} (t-T)^{k^*} + \frac{C^{(k^*+1)}(a)}{(k^*+1)!} (t-T)^{k^*+1}$$

$$= L_1(T-t)^{k^*} + L_2(T-t)^{k^*+1}$$

where $a$ is in a small left neighborhood $(T-\theta, T)$, $L_1 = (-1)^k C^{(k)}(T)/k^*!$, and $L_2 = (-1)^{k^*+1} C^{(k^*+1)}(a)/(k^*+1)!$. According to the fact that $C(t) > 0, t \neq T$, one has $L_1 > 0$.

**Case 1:** If $L_2 = 0$, then $C(t)$ is a polynomial expression, which has been proved previously.

**Case 2:** If $L_2 > 0$, let $\theta \leq 0.5L_1L_2^{-1}$, then $L_2(T-t)^{k^*+1} \leq 0.5L_1(T-t)^{k^*}$; thus, $L_1(T-t)^{k^*} \leq C(t) \leq 1.5L_1(T-t)^{k^*}$.

$$\Phi(T) = \lim_{t \to T} \frac{V(t)}{C(t)} = \lim_{t \to T} \frac{V(0)e^{-\delta \int_0^T C(s)^{-1} ds}}{C(t)}$$

$$\leq \lim_{t \to T} \frac{V(0)e^{-\delta \int_0^T (1.5L_1(T-s)^{k^*})^{-1} ds}}{0.5L_1(T-t)^{k^*}}.$$ 

With the process similar to (6), we can obtain $\Phi(T) = 0$.

**Case 3:** If $L_2 < 0$, let $\theta \leq 0.5L_1|L_2|^{-1}$, then $L_2(T-t)^{k^*+1} \geq -0.5L_1(T-t)^{k^*}$; thus, $0.5L_1(T-t)^{k^*} \leq C(t) \leq L_1(T-t)^{k^*}$. We have

$$\Phi(T) \leq \lim_{t \to T} \frac{V(0)e^{-\delta \int_0^T (1.5L_1(T-s)^{k^*})^{-1} ds}}{0.5L_1(T-t)^{k^*}}.$$ 

This case is similar to Case 2, we can also get that $\Phi(T) = 0$.

In all, PT stability can be obtained with $V(T) = 0$. Moreover, in almost all the cases, $V(T) = -\delta \Phi(T) = 0$, except for some special functions, such as $C(t) = t - \delta$ and $\delta \leq 1$.

**Remark 1:** Hua et al. [37] first put forward a PT adjustment function (named $\mu(t)$) with an improper integral method to solve the PT stability problem. According to its definition, $C(t)$ can also be called a PT adjustment function. However, our ideas have some different aspects in solving PT stability. Similar to the inverse function idea proposed in [6] and [7], we use a variable separation method in (4) to derive the divergence requirement of the improper integral. Moreover, we not only prove that $V(T) = 0$, but also prove the important requirement that $V(T) = 0$.

An example for the dynamics of (3) with $C(t) = (1-t)^{\ell}$ is shown in Fig. 1 with $\ell = 1, 2, 3$, $\delta = 0.5, 1, 2$, and $V(0) = 15$. For fixed $\ell$, the larger the $\delta$ is, the higher the convergence speed is (but the settling times are the same, $T = 1$); see the red and blue lines. On the other hand, for fixed $\delta$, the larger the $\ell$ is, the higher the convergence speed is; see the three solid lines.

From Fig. 1, we further confirm that in order to achieve PT stability, $\ell$ should be no less than 1 in this example. Then, for $\delta, \delta > 1$ when $\ell = 1$ and $\delta > 0$ when $\ell > 1$.

In lots of previous papers that investigate PT stability, they have constructed many valid forms of $C(t)$. Here, we summarize them as two categories: one contains exponential functions, whereas the other contains power functions. For the former, in [23], $C(t) = (e^{\alpha T} - e^{-\alpha T})/(e^{\alpha T} - 1), \alpha > 0$, so

$$\int_0^T \frac{dt}{C(t)} = \int_0^T \frac{e^{\alpha T} - 1}{e^{\alpha T} - e^{-\alpha T}} dt = \frac{e^{\alpha T} - 1}{e^{\alpha T} - e^{-\alpha T}} \int_0^T \frac{1}{C(t)^{-1}} dt$$

$$= \frac{e^{\alpha T} - 1}{ae^{\alpha T}} \left[ \log \frac{e^{\alpha T}}{1 - e^{\alpha T}} - \log \frac{e^{-\alpha T}}{1 - e^{-\alpha T}} \right] = +\infty$$

and in [18], $C(t) = (e^{\alpha(T-t)} - 1)/(ae^{\alpha(T-t)}), \alpha > 0$, which can be integrated similarly as mentioned previously except the coefficient. For the latter, there are many forms of $(T-t)^{k}$, for example, $\ell = 1$ in [29], [30], [32], [33], and [34], $\ell \geq 1$ in [19], $\ell = 2$ in [27], $\ell \geq 2$ in [31] and [35], and $\ell = 4$ in [36]. They all satisfy the requirements in Definition 2, so they are also in the class PTF($T$).

In this note, we construct a new form by ourselves

$$C(t) = \left( T-t - \frac{3}{4\widehat{t}} (T-t)^2 \right)^2.$$  

(7)

For this $C(t)$, it is not decreasing on $[0, T]$, but has the maximum value at $t = T/3$. Since $\int_0^T \frac{dt}{C(t)}$ diverges, and

$$\lim_{t \to T} \frac{C(t)}{(T-t)^2} = \lim_{t \to T} \left[ 1 - \frac{3}{4\widehat{t}} (T-t)^2 \right] = 1$$

thus condition (2) holds. The new form (7) can be classified into the second category. Interested readers can also construct more new forms of $C(t) \in$ PTF($T$).

**Remark 2:** In fact, a more general model can be set up

$$\dot{V}(t) = -\delta C(t)^{-1} V^{\dagger}(t), \ V(t) \geq 0, \ C(t) \in$ PTF($T$).  

(8)

When $p = 1$, it becomes (3). Otherwise

$$V(a)^{-p+1} - V(0)^{-p+1} = (p-1)\delta \int_0^T C(t)^{-1} dt.$$ 

Therefore, if $p > 1$, then the improper integral (2) should diverge, and PT stability can be obtained with the similar analysis as the abovementioned theorem. On the other hand, if $0 < p < 1$, then integral (2) should converge, for example, $C(t)$ is a constant, then it becomes the finite-time stability like [6]. Therefore, we still can consider the finite-time or fixed-time stability over PT interval, and in this case, an improper integral may become a normal integral.

Next, we consider the PT stability for a more complicated but more useful model.

**Theorem 2:** For the nonnegative continuous function $V(t)$

$$\dot{V}(t) = \delta_1 V^{\dagger}(t) - \delta_2 C(t)^{-1} V(t), \ \delta_1 \geq 0, \ \delta_2 > 0$$  

(9)

Fig. 1. Different dynamics of $V(t)$ under different $\ell$ and $\delta$. One should notice: when $\ell = 1$, if $\delta \leq 1$, see the first and the second red lines, the derivative of $V(t)$ at time $T$ would be infinity or nonzero constant, whereas if $\ell > 1$, then no matter what $\delta$ is, the derivative of $V(t)$ at time $T$ would be zero.
where \( p > 0 \), \( C(t) \in \text{PTF}(T) \), and PT stability can be achieved with \( \delta_2 > (\delta_1 + 1)\max_{t \in [0,T]} C(t) \), for \( p \geq 1 \), or \( \delta_2 > C^{(1)}(T)/(1-p) \), for \( 0 < p < 1 \).

**Proof:** First, if \( p = 1 \), the abovementioned model is written as

\[
\dot{V}(t) = (\delta_1 - \delta_2 C(t)^{-1}) V(t) = -\delta_2 C^{(1)}(t) V(t)
\]

where \( C^{(1)}(t) = \partial C(t)/\partial t \). Since \( \delta_2 > (\delta_1 + 1)\max_{t \in [0,T]} C(t) \), \( C^{(1)}(t) > 0, t \neq T \), and \( C(T) = 0 \). Moreover, with the fact that

\[
\lim_{t \to T^-} C(t) = \lim_{t \to T^-} \delta_2 = \frac{\delta_2}{\delta_1 - \delta_2 C(T)} = 1
\]

and \( C(t) \in \text{PTF}(T) \), we derive that \( C^{(1)}(t) \in \text{PTF}(T) \) too, so this model has already been investigated in Theorem 1.

Second, we consider the case \( p > 1 \). If we want to ensure that \( V(0) \) is the largest value in \( [0,T] \) for function \( V(t) \), then \( \dot{V}(t) < 0 \) is sufficient. Thus, if \( \delta_2 > (\delta_1 + 1)\max_{t \in [0,T]} C(t) \), then \( \delta_2 C(t)^{-1} > (\delta_1 + 1) V^{-1}(t) \). And \( V(t) \) would decrease definitely as follows:

\[
-\delta_2 C(t)^{-1} V(t) \leq \dot{V}(t) = (\delta_1 - \delta_2 C(t)^{-1}) V(t) \leq (\delta_1 - \delta_2 C(t)^{-1}) V(t).
\]

This inequality can be discussed similarly by (10) for the right-hand side, Theorem 1 for the left-hand side and the comparison theorem, so PT stability will also be achieved at time \( T \).

Third, we consider \( 0 < p < 1 \). Let \( V(t) = V^{1-p}(t) \), then

\[
\dot{V}(t) = \delta_1(1-p) - \delta_2(1-p) C(t)^{-1} V(t)
\]

thus, by solving this equation, we have

\[
V^{1-p}(t) = e^{\delta_2(1-p) \int_0^t C(s)^{-1} ds} \int_0^t V^{1-p}(s) e^{-\delta_2(1-p) \int_0^t C(s)^{-1} ds} ds.
\]

By using the L’Hospital rule, we have

\[
V^{1-p}(T) = \lim_{t \to T^-} \frac{\int_0^T (1-p) \delta_1 e^{\delta_2 (1-p) \int_0^t C(s)^{-1} ds}}{e^{\delta_2 (1-p) \int_0^T C(s)^{-1} ds}} = \lim_{t \to T^-} \frac{(1-p) \delta_1 e^{\delta_2 (1-p) \int_0^t C(s)^{-1} ds}}{\delta_2 (1-p) C(T)^{-1} e^{\delta_2 (1-p) \int_0^T C(s)^{-1} ds}} = \lim_{t \to T^-} \delta_1 C(T)/\delta_2 = 0
\]

and

\[
\lim_{t \to T^-} C(t)^{-1} V^{1-p}(t) = \frac{(1-p) \delta_1}{C^{(1)}(T) + (1-p) \delta_2}
\]

which is a positive constant when \( \delta_2 > -C^{(1)}(T)/(1-p) \).

Therefore, \( V(T) = 0 \) and

\[
\dot{V}(T) = -\delta_2 \lim_{t \to T^-} C(t)^{-1} V(t) = -\delta_2 \lim_{t \to T^-} (C(t)^{-1} V(t)) V^{1-p}(t) = 0.
\]

In all, no matter the value of \( p \), when \( \delta_2 \) is large enough, system (9) can converge exactly at time \( T \) (neither before nor after this time), so PT stability can be finally realized.

**Remark 3:** For \( p = 0 \), (9) can be expressed as the form (12), since condition (14) holds, \( V(T) \neq 0 \), i.e., PT stability cannot hold for this case. This fact means that for a heterogeneous network analysis, PT synchronization may not hold, which can be regarded as a disadvantage, since finite-time and fixed-time synchronization protocols are both robust to disturbances. How to overcome this flaw would be our future work.

### III. PT SYNCHRONIZATION AND CONTROL FOR MWDCNs

In this section, we will apply our obtained results on the synchronization and control problems for MWDCNs.

**Definition 3** (see [22]): An irreducible matrix \( M = (M_{ij}) \in R^{N \times N} \) is denoted as \( M \in \mathcal{SC} \), if

\[
\left\{ \begin{array}{ll}
M_{ij} \geq 0, M_{ij} = -\sum_{j=1}^N M_{ij} & \forall i \neq j \\
\Re \{\lambda(M)\} < 0
\end{array} \right.
\]

where \( \Re \{\cdot\} \geq 0 \) means that the real parts of eigenvalues are all negative except an eigenvalue 0 with the right eigenvector \( 1_N = (1, 1, \ldots, 1)^T \) and multiplicity 1, notation \( \mathcal{T}^N \) means the transpose of a vector or a matrix.

### A. PT Synchronization Without Pinning Control

In this section, to realize PT synchronization only by mutual coupling, we design the network model as

\[
x_i(t) = f(x_i(t)) + \eta/\sqrt{C(t)} \sum_{i=1}^W \sum_{j=1}^N M_{ij} \Gamma^{uw} x_j(t)
\]

where \( x_i(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n \) is the state of the ith node, \( i \in \Omega = \{1, 2, \ldots, N\} \), \( W > 1 \) means the multiple (single) network topologies, \( \eta > 0 \) is a parameter, \( C(t) \in \text{PTF}(T) \) is the regulatory function, and \( \eta/C(t) \) can be regarded as the adaptive coupling strength. \( \Gamma = (\Gamma_{ij}^{uw})_{N \times N} \) stands for asymmetric OCMMs with zero-row-sum, i.e., \( \sum_{j=1}^N \Gamma_{ij}^{uw} = 0 \forall i \). Symmetric ICMMs \( \Gamma^{uw} = (\gamma_{ij}^{uw}) \in R^{w \times n} \); \( f(\cdot) : R^n \rightarrow R^n \) is a continuous function, and there exists \( H_f > 0 \), such that for all \( x, \bar{x} \in R^n \)

\[
(x - \bar{x})^T f(x) - f(\bar{x}) \leq H_f (x - \bar{x})^T (x - \bar{x}) \leq 0.
\]

**Definition 4:** Network (15) is said to realize PT synchronization if \( \lim_{t \to T^-} \|x_i(t) - x_j(t)\|_2 = 0 \), and \( \lim_{t \to T^-} \|\dot{x}_i(t) - \dot{x}_j(t)\|_2 = 0 \), for all \( i, j \in \Omega \).

The synchronization problem is investigated by considering each dimension of the state of all nodes separately, which is called ROT in [40]. Therefore, we consider the dynamics of \( x_i(t) \)

\[
\dot{x}_i(t) = f(x_i(t)) + \eta/\sqrt{C(t)} \sum_{i=1}^W \sum_{j=1}^N M_{ij} \left( \gamma_{ij}^{uw} x_j(t) + \sum_{d \neq i} \gamma_{ij}^{ud} x_j(t) \right)
\]

where \( f(x_i(t)) \) is the ith dimension of \( f(x_i(t)) \), \( d = 1, 2, \ldots, n \). For any dimension \( d \), we denote

\[
X_i^{[d]}(t) = (x_1^{[d]}(t), x_2^{[d]}(t), \ldots, x_N^{[d]}(t))^T
\]

\[
F_i^{[d]}(t) = (f(x_i(t))^{[d]}, f(x_2(t))^{[d]}, \ldots, f(x_N(t))^{[d]})^T
\]

then

\[
\dot{X}_i^{[d]}(t) = F_i^{[d]}(t) + \eta/\sqrt{C(t)} \left( M_i^{[d]} X_i^{[d]}(t) + \sum_{e \neq d} M_i^{[de]} X_i^{[e]}(t) \right)
\]

where the abovementioned sum (union) matrices are denoted as

\[
M^{[d]} = \sum_{w=1}^W \gamma_{ij}^{uw} M^{[uw]}, \quad d, e = 1, 2, \ldots, n.
\]
Since all $M^u$ are zero-row-sum matrices, $M^{[dd]}$ and $M^{[de]}$ defined in (19) are also zero-row-sum matrices. Furthermore, we have the following assumption.

**Assumption 1:** We assume $M^{[dd]} \in SC, d = 1, 2, \ldots, n$.

Notice that we just restate the above-mentioned assumption, so there is no requirement on single OCM $M^u$ and ICM $\Gamma^w$, elements $M^u_{ij}$ and $\gamma^w_{ij}$ can be positive (cooperative relationship), negative (competitive relationship), or zero. The network topology can be directed and even disconnected. Hence, this condition is more general and has wider applications than [40].

Based on the results in [2], under Assumption 1, the normalized left eigenvector (NLEVec) corresponding to eigenvalue 0 of $M^{[dd]}$ exists, i.e., $(\psi^{[dd]}_d)^T M^{[dd]} = 0$, where $\psi^{[dd]} = (\psi^{[dd]}_1, \ldots, \psi^{[dd]}_N)^T \in \mathbb{R}^N$, with $\sum_{i=1}^N \psi^{[dd]}_i = 1$ and $\psi^{[dd]}_i > 0, i \in \Omega$. Define $\Psi^{[dd]} = diag(\psi^{[dd]}_i)$, then symmetric matrices $I_{\psi} = diag(0, I_{\psi}[2], \ldots, I_{\psi}[m])$ (20)

$$M = \begin{bmatrix}
M^{[1]} & M^{[1n]} \\
\vdots & \ddots & \vdots \\
M^{[m]} & \ldots & M^{[mn]}
\end{bmatrix}$$ (21)

**Theorem 3:** For the MWDCN (15) with $C(t) \in PTF(T)$, suppose Assumption 1 holds, if the following matrix

$$\mathbb{M} = (I_{\psi} M + M^{T} I_{\psi})/2$$ (22)

is negative definite in the transverse space $TS = \{S = (S^1, \ldots, S^n) \in R^{1 \times N} | S^i \in R^{1 \times N}, i = 1, 2, \ldots, n, S^i \cdot (1, 1, \ldots, 1) = 0\}$, then $\mathbb{M}$ is positive definite. Then the pinning control is defined as

$$\Gamma = (\psi^{[de]}_1, \ldots, \psi^{[de]}_N)$$ (23)

where $\lambda_{\psi}(\cdot)$ signifies the second largest eigenvalue in the whole space, which can also be called Fiedler [1].

**Proof:** By using NLEVec $\psi^{[de]}$, we can define dummy synchronization targets as $x^d_{\psi}(t) = \sum_{i=1}^N \psi^{[de]}_i x^d_i(t)$. Thus, the PT synchronization in Definition 4 is equivalent to prove that $lim_{t \to \infty} \|x^d_{\psi}(t) - x^d_{\psi}(t)\| = 0$, and $lim_{t \to \infty} \|x^d(t) - x^d(t)\| = 0$.

Define the Lyapunov function

$$V(t) = 0.5 \sum_{i=1}^N \sum_{d=1}^n X^{[de]}_i(t) X^{[de]}_i(t)$$ (24)

Its derivative satisfies

$$\dot{V}(t) = \sum_{d=1}^n X^{[de]}_i(t)^T \dot{I}_{\psi,d} X^{[de]}_i(t)$$

$$\leq \sum_{d=1}^n X^{[de]}_i(t)^T \dot{I}_{\psi,d} X^{[de]}_i(t) + \eta C(t) - I_{\psi,d} M^{[de]} X^{[ci]}(t)$$

$$\leq 2H^v_i V(t) + \eta C(t) - I_{\psi,d} M^{[de]} X^{[ci]}(t)$$

$$\leq 2H^v_i V(t) + 2\gamma_2(\mathbb{M}) \eta C(t) - I_{\psi,d} M^{[de]} X^{[ci]}(t)$$ (25)

where $X^{[ci]}(t) = (X^{[ci]}_1(t), \ldots, X^{[ci]}_n(t))^T$.

Form (25) is the same as that in Theorem 2 if we set $\delta_1 = 2H^v_i$ and $\delta_2 = -2\gamma_2(\mathbb{M}) \eta$. According to Theorem 2, if condition (23) holds, PT stability for $V(t)$ holds, i.e., $V(t)$ would converge to zero when $t \to T^*$ and the derivative of $V(T)$ is also zero. Therefore, from the concrete form of $V(t)$ defined in (24), PT synchronization is realized globally.

**Remark 4:** If the function $f(\cdot)$ in (15) does not satisfy the condition (16) but with strong nonlinearities [44], such as

$$(x_i - x_j)^T f(x_i) - f(x_j))$$

$$\leq \frac{2H^v_i}{\eta}(x_i - x_j)^T (x_i - x_j) \geq 0$$

then we can also apply the models in Section II to investigate PT synchronization. For example, suppose $f(\cdot)$ satisfies

$$(x_i - x_j)^T f(x_i) - f(x_j)$$

then (25) can be changed to the following form:

$$\dot{V}(t) \leq 2H^v_i V(t) + 2\lambda_2(\mathbb{M}) \eta C(t) - I_{\psi,d} M^{[de]} X^{[ci]}(t)$$ (26)

Hence, according to Theorem 2 with $p = 2$, PT synchronization can be also realized with

$$\eta \geq \frac{1}{2} \lambda_{\psi}(\mathbb{M})$$ (26)

Recalling the prerequisite that matrix $\mathbb{M}$ defined in (22) should be negative definite in the transverse space, we know that this condition cannot be ensured for any matrix [40]; therefore, in the following, we will consider a special case: all ICMs are diagonal, i.e., $\Gamma^w = diag(\gamma^w_{11}, \ldots, \gamma^w_{nn})$. In this case, sum matrices defined in (19) satisfy:

$$M^{[de]} = 0 \text{ for any } d \neq c, \text{ and matrix } M^{[de]} \text{ defined in } (21) \text{ would be }$$

$$M^{[de]} = diag(M^{[1]}_1, \ldots, M^{[n]}_n) \text{. According to [2], matrices}$$

$$\psi^{[de]} X^{[de]}_i(t) = \left(\psi^{[de]}_1 X^{[de]}_1(t), \ldots, \psi^{[de]}_n X^{[de]}_n(t)\right)$$ (27)

are symmetric and negative definite in TS, which means that matrix $\mathbb{M}$ defined in (22) is surely negative definite in TS.

**Corollary 1:** For the MWDCN (15) with all diagonal ICMs $\Gamma^w$, suppose $C(t) \in PTF(T)$ and Assumption 1 holds, then PT synchronization can be realized under condition (23).

**B. PT Synchronization With Pinning Control**

In this section, we will consider the pinning control problem, whose network model can be described as

$$\dot{x}_i(t) = f(x_i(t)) + \eta C(t) \sum_{j=1}^N \sum_{w=1}^W M^u_{ij} \Gamma^w x_j(t) + u_i$$ (28)

where the meanings of parameters are the same as those in model (15), the pinning control is defined as

$$u_1 = -\eta \kappa C(t)^{-1} \Gamma^1 x_1(t) - x_0(t), \quad \kappa > 0$$

and $u_i = 0, i \in \Omega - \{1\}$, symmetric matrix $\Gamma = (\gamma_{ij})$ can be independent of ICMs $\Gamma^w$, and $x_0(t)$ is the control target satisfying

$$\dot{x}_0(t) = f(x_0(t))$$ (29)

**Remark 5:** According to Assumption 1, $M^{[de]}$ is irreducible, so the corresponding digraph is strongly connected [2]. We can choose any node as the pinned node since each node influences each other directly or indirectly [3].

**Definition 5:** A complex network under pinning control (28) is said to realize PT synchronization if $lim_{t \to \infty} \|x_i(t) - x_0(t)\| = 0, \text{ and } lim_{t \to \infty} \|\dot{x}_i(t) - \dot{x}_0(t)\| = 0, i \in \Omega$. 

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For any dimension $d = 1, 2, \ldots, n$, denote
\begin{equation}
Y^{[d]}(t) = (x_1^d(t) - x_0^d(t), \ldots, x_n^d(t) - x_0^d(t))^T
\end{equation}
\begin{equation}
\hat{F}^{[d]}(t) = (f(x_1^d) - f(x_0^d), \ldots, f(x_n^d) - f(x_0^d))^T
\end{equation}
then
\begin{equation}
Y^{[d]}(t) = \hat{F}^{[d]}(t) + \frac{\eta}{C(t)} \left( M^{[d]e}Y^{[e]}(t) + \sum_{e \neq d} M^{[d]e}Y^{[e]}(t) \right).
\end{equation}
The abovementioned (union) matrices are denoted as
\begin{equation}
M^{[d]e} = M^{[d]e} - \text{diag}(\gamma_{de}, 0, \ldots, 0)
\end{equation}
where $M^{[d]e}$ is defined in (19).

According to [3], under Assumption 1, the NLEVec $\psi^{[d]}$ of $M^{[d]}$ exists, which can make the new matrices
\begin{equation}
\psi^{[d]} \hat{M}^{[d]} = (\psi^{[d]} \hat{Y}^{[d]} + (\hat{M}^{[d]}\psi^{[d]})/2
\end{equation}
be symmetric and negative definite, where $\psi^{[d]} = \text{diag}(\psi^{[d]})$.

Theorem 4: For the MWDCNs (28) with $C(t) \in \text{PFT}(T)$, suppose Assumption 1 holds, if the following matrix:
\begin{equation}
\hat{M} = (\psi \hat{M} + \hat{M}^T \psi)/2
\end{equation}
is negative definite, where $\Psi = \text{diag}(\psi^{[1]}, \ldots, \psi^{[n]})$, and
\begin{equation}
\hat{M} = \begin{pmatrix}
M^{[1]} & \cdots & M^{[n]} \\
\vdots & \ddots & \vdots \\
M^{[n]} & \cdots & M^{[1]} 
\end{pmatrix}
\end{equation}
then PT synchronization under pinning control is realized with
\begin{equation}
\eta > (H_f + 0.5) \max_{t \in [0, T]} C(t)/\lambda_{\max}(\hat{M})
\end{equation}
where $\lambda_{\max}(\cdot)$ signifies the largest eigenvalue.

Proof: PT synchronization in Definition 5 is equivalent to proving that $\lim_{t \to T^-} \|x_i^d(t) - x_0^d(t)\| = 0$ and $\lim_{t \to T^-} \|\hat{x}_i^d(t) - \hat{x}_0^d(t)\| = 0$. Therefore, we choose the Lyapunov function as
\begin{equation}
V(t) = 0.5 \sum_{i=1}^N \sum_{d=1}^n \psi^{[d]}(x_i^d(t) - x_0^d(t))^2
\end{equation}
\begin{equation}
= 0.5 \sum_{d=1}^n Y^{[d]}(t)^T \psi^{[d]} Y^{[d]}(t).
\end{equation}
Its derivative satisfies
\begin{equation}
\dot{V}(t) = \sum_{d=1}^n Y^{[d]}(t)^T \dot{\psi}^{[d]} Y^{[d]}(t)
\end{equation}
\begin{equation}
\leq 2H_f V(t) + \eta C(t)^{-1} Y(t)^T \hat{M} Y(t)
\end{equation}
\begin{equation}
\leq 2H_f V(t) + 2\lambda_{\max}(\hat{M}) \eta C(t)^{-1} V(t)
\end{equation}
where $Y(t) = (Y^{[1]}(t)^T, \ldots, Y^{[n]}(t)^T)^T$.
With the same arguments as those in Theorem 3, we can also get the PT synchronization globally under (36).

Similarly, when all ICMs are diagonal, i.e., $\Gamma^w = \text{diag}(\gamma_{1w}, \ldots, \gamma_{nw})$ and $\Gamma = \text{diag}(\gamma_{11}, \ldots, \gamma_{nn})$, $M^{[d]} = 0, d \neq e$; therefore, $\hat{M} = \text{diag}(\hat{M}^{[1]}, \ldots, \hat{M}^{[n]})$, according to (33), matrix $\hat{M}$ would be negative definite surely.

Corollary 2: For the MWDCN (28) with all diagonal ICMs $\Gamma^w$, suppose $C(t) \in \text{PFT}(T)$ and Assumption 1 holds, then PT synchronization can be realized under condition (36).

Remark 6: Different from finite-time or fixed-time synchronization, which can be achieved by using nonlinear coupling/control functions, the previously designed PT synchronization protocol only uses linear coupling/control functions, and the multiple coupling matrices can be asymmetric, so it is simpler in real applications, whereas the main difference from classical synchronization [2] is the design of coupling strength.

IV. NUMERICAL SIMULATION

Simulation 1: Consider an MWDCN (15) with three nodes and four weights, and the function $f(\cdot)$ is described
\begin{equation}
f(x_i(t)) = -x_i(t) + \begin{pmatrix}
-1.25 & -3.2 & -3.2 \\
-3.2 & 1.1 & -4.4 \\
-3.2 & 4.4 & 1
\end{pmatrix} \begin{pmatrix}
h(x_1^d(t)) \\
h(x_2^d(t)) \\
h(x_3^d(t))
\end{pmatrix}
\end{equation}
where $x_i(t) = (x_1^1(t), x_2^1(t), x_3^1(t))^T$ and $h(x_i(t)) = (|x_i^1(t) + 1| - |x_i^1(t) - 1|)/2, d = 1, 2, 3$. Hence, $H_f = 5.4704$.

OCMs are chosen as
\begin{equation}
M^{[1]} = \begin{pmatrix}
-3 & 3 & 0 \\
0 & 0 & 0 \\
3 & 0 & -3
\end{pmatrix},
M^{[2]} = \begin{pmatrix}
0 & -6 & 6 \\
-3 & 0 & -3
\end{pmatrix},
M^{[3]} = \begin{pmatrix}
2 & 2 & 0 \\
0 & -5 & 5 \\
4 & 0 & -4
\end{pmatrix}.
\end{equation}

Obviously, $M^{[1]}$, $M^{[2]}$, and $M^{[3]}$ mean that the network topologies are not strongly connected, and nodes can have competitive relationships. OCMs are chosen as $\Gamma^1 = \Gamma^2 = \text{diag}(5, 7, 6), \Gamma^3 = \text{diag}(6, -1, 1), \Gamma^4 = \text{diag}(5, 7, 6)$.

According to (19), the sum matrices are
\begin{equation}
M^{[1]}[1] = \begin{pmatrix}
-21 & 9 & 12 \\
0 & -72 & 72 \\
90 & 0 & -90
\end{pmatrix},
M^{[2]}[2] = \begin{pmatrix}
-27 & 17 & 10 \\
0 & -55 & 55 \\
46 & 0 & -46
\end{pmatrix}
\end{equation}
\begin{equation}
M^{[3]}[3] = \begin{pmatrix}
-30 & 16 & 14 \\
0 & -71 & 71 \\
68 & 0 & -68
\end{pmatrix}.
\end{equation}

Therefore, Assumption 1 holds, and the corresponding NLEVec are: $\psi^{[1]} = \frac{1}{\sqrt{2}} (120, 15, 28)^T$, $\psi^{[2]} = \frac{1}{\sqrt{2}} (2530, 782, 1485)^T$, and $\psi^{[3]} = \frac{1}{\sqrt{2}} (2414, 544, 1065)^T, \lambda_{\hat{M}}(\hat{M}) = -9.9387$. Once we choose the function $C(t) \in \text{PFT}(T)$, and according to (23) in Corollary 1, if $\eta > (H_f + 0.5) \max_{t \in [0, T]} C(t)/\lambda_{\max}(\hat{M})$, then PT synchronization can be realized. Set the PT $T = 3$, if $C(t) = (3 - t)^2$, then $\eta > 5.4$, and if $C(t)$ is defined in (7), then $\max_{t \in [0, T]} C(t) = 1$, and $\eta > 0.6$.

Choose the initial values as: $x_1(0) = (1, 1.5, 2)^T$, $x_2(0) = (2, 5, 3.5)^T$, and $x_3(0) = (4, 4.5, 5)^T$. The index $E_1(t) = \|x_2(t) - x_1(t)\|^2 + \|x_2(t) - x_3(t)\|^2$ is used to denote the synchronization error between nodes. Fig. 2 shows the dynamics of $E_1(t)$ with $\eta = 0.35$ under different $C(t)$.

Simulation 2: Next, we consider PT synchronization under pinning control for a MWDCN (28), where the parameters are the same as defined previously, and the pinning control is added on the first node with $\kappa = 1$ and $\Gamma = \text{diag}(11, 13, 15)$. Therefore, according to (32), the new sum matrices are
\begin{equation}
\hat{M}^{[1]}[1] = \begin{pmatrix}
-32 & 9 & 12 \\
0 & -72 & 72 \\
90 & 0 & -90
\end{pmatrix},
\hat{M}^{[2]}[2] = \begin{pmatrix}
-40 & 17 & 10 \\
0 & -55 & 55 \\
46 & 0 & -46
\end{pmatrix}.
\end{equation}
Dynamics of \( E_1(t) \) under \( \eta = 0.35 \), which shows that PT synchronization can be realized. Compared with our constructed form in (7) and [29], [30], and [36], our function makes the error converge faster for this case.

\[
\hat{M}_{[33]} = \begin{pmatrix} -45 & 16 & 14 \\ 0 & -71 & 71 \\ 68 & 0 & -68 \end{pmatrix}.
\]

Hence, \( \lambda_{\text{max}}(\hat{M}) = -1.8241 \). Set \( T = 3 \). Therefore, once we choose the function \( C(t) \in \text{PTF}(3) \), for example, \( C(t) \) is defined in (7), then according to condition (36) in Corollary 2, if \( \eta > (H_t + 0.5) \max_{t \in [0, T]} |C(t)|/\lambda_{\text{max}}(\hat{M}) = 3.273 \), then PT synchronization can be realized. Denote the synchronization error as \( E_2(t) = \sqrt{\sum_{i=1}^{3} \| x_i(t) - x_0(t) \|^2 } \), where the target \( x_0(t) \) in (29) starts from \( x_0(0) = (1, 1, 1)^T \). Fig. 3 shows the dynamics of \( E_2(t) \) with \( \eta = 1 \) under different \( C(t) \).

Simulation 3: Furthermore, we consider a complex network with 100 nodes, whose intrinsic dynamic is described by \( \dot{x}_i(t) = (x_i(t))^2, x_i(t) \in R \), which has a strong nonlinearity. The initial values are chosen as \( x_i(0) = 0.1 \cdot i, i = 1, \ldots, 100 \). As for the coupling matrices, one coupling matrix is chosen as fully connected case, i.e., \( M^i_{ij} = 1, i \neq j \), and the other coupling matrix is chosen as the chain case: \( M^i_{i+1} = 1, i = 1, \ldots, 99 \). The pinning controllers are added to the first 10 nodes, i.e., \( \kappa = 100 \). Set \( T = 3 \), and \( C(t) \) is defined in (7). Similar to condition (26) in Remark 4, we can also calculate the lower bound of \( \eta \). The final target is set to be zero, Fig. 4 shows the state of each node, so PT synchronization under pinning control is realized with \( \eta = 0.8 \).

V. CONCLUSION

In this note [45], we first study the PT stability, whose settling time is independent of initial values and system parameters. We define a class of regulatory functions, which have the divergency property for improper integral, and prove that for these functions, PT stability can be achieved by two useful theorems. The regulatory functions in many previous papers are included in our class. Based on these results, we investigate the PT synchronization for MWDCNs with or without pinning control. Both the cooperative and competitive relationships between nodes are allowed. Symmetric ICs can be diagonal or nondiagonal. We use RIT to consider the PT synchronization for each dimension separately. At last, three numerical simulations are given to verify the theoretical results.

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