Width of spherical convex bodies

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Abstract. For every hemisphere \( K \) supporting a convex body \( C \) on the sphere \( S^d \) we define the width of \( C \) determined by \( K \). We show that it is a continuous function of the position of \( K \). We prove that the diameter of every convex body \( C \subset S^d \) equals the maximum of the widths of \( C \) provided the diameter of \( C \) is at most \( \pi/2 \). In a natural way, we define spherical bodies of constant width. We also consider the thickness \( \Delta(C) \) of \( C \), i.e., the minimum width of \( C \). A convex body \( R \subset S^d \) is said to be reduced if \( \Delta(Z) < \Delta(R) \) for every convex body \( Z \) properly contained in \( R \). For instance, bodies of constant width on \( S^d \) and regular spherical odd-gons of thickness at most \( \pi/2 \) on \( S^2 \) are reduced. We prove that every reduced smooth spherical convex body is of constant width.

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1. Introduction

Let \( S^d \) be the unit sphere in the \((d + 1)\)-dimensional Euclidean space \( E^{d+1} \), where \( d \geq 2 \). By a great circle of \( S^d \) we mean the intersection of \( S^d \) with any two-dimensional subspace of \( E^{d+1} \). The common part of the sphere \( S^d \) with any hyper-subspace of \( E^{d+1} \) is called a \((d - 1)\)-dimensional great sphere of \( S^d \). In particular, for \( S^2 \) the \((d - 1)\)-dimensional great spheres are great circles. By a pair of antipodes of \( S^d \) we mean any pair of points of intersection of \( S^d \) with a straight line through the origin of \( E^{d+1} \). Observe that if two different points are not antipodes, there is exactly one great circle containing them.

If two different points \( a, b \in S^d \) are not antipodes, by the arc \( ab \) connecting them we mean the shorter part of the great circle containing \( a \) and \( b \). By the spherical distance \( |ab| \), or shortly distance, of these points we understand the length of the arc connecting them. Moreover, we put \( \pi \), if the points are antipodes and 0 if the points coincide.
By a spherical ball of radius \( \rho \in (0, \pi/2] \), or shorter a ball, we mean the set of points of \( S^d \) having distance at most \( \rho \) from a fixed point, called the center of this ball. An open ball is the set of points of \( S^d \) having distance smaller than \( \rho \) from a point. Balls on \( S^2 \) are called disks. Spherical balls of radius \( \pi/2 \) are called hemispheres. In other words, by a hemisphere of \( S^d \) we mean the common part of \( S^d \) with any closed half-space of \( E^{d+1} \). We denote by \( H(m) \) the hemisphere whose center is \( m \). Two hemispheres whose centers are antipodes are called opposite hemispheres. By an open hemisphere we mean the set of points having distance less than \( \pi/2 \) from a fixed point.

By a spherical \((d-1)\)-dimensional ball of radius \( \rho \in (0, \pi/2] \) we mean the set of points of a \((d-1)\)-dimensional great sphere of \( S^d \) which are at distance at most \( \rho \), from a fixed point, called the center of this ball. The \((d-1)\)-dimensional balls of radius \( \pi/2 \) are called \((d-1)\)-dimensional hemispheres. If \( d = 2 \), we call them semicircles.

We say that a set \( C \subset S^d \) is convex if it does not contain any pair of antipodes and if together with every two points it contains the whole arc connecting them. By a convex body on \( S^d \) we mean a closed convex set with non-empty interior. Observe that a set \( C \subset S^d \) is a convex body if and only if it is contained in an open hemisphere and is an intersection of hemispheres. For a short survey of definitions of convexity on \( S^d \) we refer to Sect. 9.1 of [1]. The literature concerning this subject is very large. For instance see [2,3] and [4].

Clearly, the intersection of every family of convex sets is also convex. Thus for every set \( Q \subset S^d \) contained in an open hemisphere of \( S^d \) there exists the unique smallest convex set containing \( Q \). It is called the convex hull of \( Q \) and it is denoted by \( \text{conv}(Q) \).

**Lemma 1.** If \( Q \subset S^d \) is a closed subset of an open hemisphere, then \( \text{conv}(Q) \) is also closed.

This lemma follows by applying an analogous theorem for compact sets in \( E^{d+1} \).

If a \((d-1)\)-dimensional great sphere \( G \) of \( S^d \) has a common point \( t \) with a convex body \( C \subset S^d \) and if its intersection with the interior of \( C \) is empty, we say that \( G \) is a supporting \((d-1)\)-dimensional great sphere of \( C \) passing through \( t \). We also say that \( G \) supports \( C \) at \( t \). If \( H \) is the hemisphere bounded by \( G \) and containing \( C \), we say that \( H \) supports \( C \) at \( t \). If at every boundary point of a convex body \( C \subset S^d \) exactly one hemisphere supports \( C \), we say that the body is smooth.

By the well known fact that a set \( C \subset S^d \) is convex if and only if the cone generated by it in \( E^{d+1} \) is convex and from the classic separation theorem in Euclidean space we obtain the following analogous fact for \( S^d \).
Lemma 2. Every two convex bodies on the sphere $S^d$ with empty intersection of their interiors are subsets of some two opposite hemispheres.

Let $P \subset S^d$ be a convex body. Let $Q \subset S^d$ be a convex body or a hemisphere. We say that $P$ touches $Q$ from outside if $P \cap Q \neq \emptyset$ and $\text{int}(P) \cap \text{int}(Q) = \emptyset$. We say that $P$ touches $Q$ from inside if $P \subset Q$ and $\text{bd}(P) \cap \text{bd}(Q) \neq \emptyset$. In both cases, points of $\text{bd}(P) \cap \text{bd}(Q)$ are called points of touching.

The convex hull $V$ of $k \geq 3$ points on $S^2$ such that none of them belongs to the convex hull of the remaining points is called a spherical convex $k$-gon. The mentioned points are called the vertices of $V$. We write $V = v_1v_2 \ldots v_k$ provided $v_1, v_2, \ldots, v_k$ are successive vertices of $V$ when we go around $V$ on the boundary of $V$. In particular, when we take $k \geq 3$ successive points in a spherical circle of radius less than $\frac{\pi}{2}$ on $S^2$ with equal distances of every two successive points, we obtain a regular spherical $k$-gon.

2. Lunes

If hemispheres $G$ and $H$ of $S^d$ are different and not opposite, then $L = G \cap H$ is called a lune of $S^d$. This notion is considered in many books and papers, for lunes on $S^2$ see e.g. [5], p. 18. The $(d-1)$-dimensional hemispheres bounding the lune $L$ and contained in $G$ and $H$, respectively, are denoted by $G/H$ and $H/G$.

Claim 1. Every pair of different points $a$, $b$ which are not antipodes determines exactly one lune $L$ such that $a$, $b$ are the centers of the $(d-1)$-dimensional hemispheres bounding $L$.

Proof. If $|ab| \leq \frac{\pi}{2}$, then on the great circle containing the arc $ab$ we find points $p$ and $q$ such that $a \in pb$, $b \in aq$, $|pb| = |qa| = \frac{\pi}{2}$. If $|ab| > \frac{\pi}{2}$, then on the great circle containing the arc $ab$ we find points $p$ and $q$ such that $q \in pb$, $p \in aq$, $|pb| = |qa| = \frac{\pi}{2}$. The lune $L = H(p) \cap H(q)$ is the one that we are looking for. 

Since every lune $L$ determines exactly one pair of centers of the $(d-1)$-dimensional hemispheres bounding $L$, from Claim 1 we see that there is a one-to-one correspondence between lunes and pairs of points (different and not antipodes) of $S^d$.

Clearly, $(G/H) \cup (H/G)$ is the boundary of the lune $G \cap H$. In particular, every lune of $S^2$ is bounded by two different semicircles. Denote by $c_{G/H}, c_{H/G}$ the centers of $G/H$ and $H/G$, respectively. Points of $(G/H) \cap (H/G)$ are called corners of the lune $G \cap H$. Of course, $r \in (G/H) \cup (H/G)$ is a corner of $G \cap H$ if and only if $r$ is equidistant from $c_{G/H}$ and $c_{H/G}$. In particular, every lune on $S^2$ has exactly two corners. They are antipodes.
By the thickness $\Delta(L)$ of a lune $L = G \cap H \subset S^d$ we mean the spherical distance of the centers of the $(d - 1)$-dimensional hemispheres $G/H$ and $H/G$ bounding $L$. Observe that it equals each of the non-oriented angles $\angle c_{G/H}c_{H/G}$, where $r$ is any corner of $L$.

We omit the simple proof of the following lemma.

**Lemma 3.** Let $H$ and $G$ be different and not opposite hemispheres. Consider the lune $L = H \cap G$. Let $x \neq c_{G/H}$ belong to $G/H$. If $\Delta(L) < \frac{\pi}{2}$, we have $|xc_{H/G}| > |c_{G/H}c_{H/G}|$. If $\Delta(L) = \frac{\pi}{2}$, we have $|xc_{H/G}| = |c_{G/H}c_{H/G}|$. If $\Delta(L) > \frac{\pi}{2}$, we have $|xc_{H/G}| < |c_{G/H}c_{H/G}|$.

For a convex body $C \subset S^d$ they matter lunes containing it, and in particular such lunes for which both $(d - 1)$-dimensional hemispheres bounding the lune have non-empty intersection with $C$. We say that a lune passes through a boundary point $p$ of a convex body $C \subset S^d$ if the lune contains $C$ and if the boundary of the lune contains $p$. If the centers of both $(d - 1)$-dimensional hemispheres bounding a lune belong to $C$, then we call such a lune an orthogonally supporting lune of $C$.

By applying the classical Blaschke selection theorem (e.g., see [6], p. 64) in $E^{d+1}$ we easily obtain its spherical analogue and also the following lemma.

**Lemma 4.** From every sequence of lunes on $S^d$ we may select a subsequence of lunes convergent to a lune.

### 3. Width and thickness of a spherical convex body

For every hemisphere $K$ supporting a convex body $C \subset S^d$ we are looking for hemispheres $K^*$ supporting $C$ such that the lunes $K \cap K^*$ are of the minimum thickness, i.e., which are the “narrowest” lunes of the form $K \cap K'$ over all hemispheres $K'$ supporting $C$. By compactness arguments we immediately see that at least one such hemisphere $K^*$ exists, and thus at least one corresponding lune $K \cap K^*$ exists. Denote by width$_K(C)$ its thickness and we call it the width of $C$ determined by $K$. This notion of width of $C \subset S^d$ is an analogue of the notion of width of a convex body of $E^d$. How to find the width of $C$ determined by a given hemisphere $K$? Theorem 1 presented below and its proof present a procedure for establishing width$_K(C)$.

First let us prove a lemma needed for the proof of Theorem 1.

**Lemma 5.** Let $G$ and $H$ be different and not opposite hemispheres, and let $g$ denote the center of $G$. If $g \notin \text{bd}(H)$, then by $B$ denote the ball with center $g$ which touches $H$ (from inside or outside) and by $t$ the point of touching. If $g \in \text{bd}(H)$, we put $t = g$. We claim that $t$ is always at the center of the $(d - 1)$-dimensional hemisphere $H/G$. 
Proof. If $g \notin \text{bd}(H)$ consider any two corners $r_1$ and $r_2$ of the lune $G \cap H$. Look at the triangles $gr_1$ and $gr_2$. Below we explain that they have three equal elements. They have the length of the common side $gt$. Since $g$ is the center of $G$, we have $|gr_1| = \frac{\pi}{2} = |gr_2|$. By the orthogonality of $gt$ and $H$ it follows that $\angle gr_1 = \frac{\pi}{2} = \angle gr_2$. Since these three elements are equal, we have $|tr_1| = |tr_2|$ (by the way, they are both equal to $\frac{\pi}{2}$ since $|r_1r_2| = \pi$). When $g \in \text{bd}(H)$, we have $|gr_1| = |gr_2|$; still $r_1,r_2$ belong to the boundary of $H$. Thus $t$ is always the center of the hemisphere $H/G$. \hfill $\square$

**Theorem 1.** Let $K$ be a hemisphere which supports a convex body $C \subset S^d$. Denote by $k$ the center of $K$.

I. If $k \notin C$, then there exists a unique hemisphere $K^*$ supporting $C$ such that the lune $L = K \cap K^*$ contains $C$ and has thickness $\text{width}_K(C)$. This hemisphere supports $C$ at the point $t$ at which the largest ball $B$ with center $k$ touches $C$ from outside. We have $\Delta(K \cap K^*) = \frac{\pi}{2} - \rho_B$, where $\rho_B$ denotes the radius of $B$.

II. If $k \in \text{bd}(C)$, then there exists at least one hemisphere $K^*$ supporting $C$ such that $L = K \cap K^*$ is a lune containing $C$ which has thickness $\text{width}_K(C)$. This hemisphere supports $C$ at $t = k$. We have $\Delta(K \cap K^*) = \frac{\pi}{2}$.

III. If $k \in \text{int}(C)$, then there exists at least one hemisphere $K^*$ supporting $C$ such that $L = K \cap K^*$ is a lune containing $C$ which has thickness $\text{width}_K(C)$. Every such $K^*$ supports $C$ at exactly one point $t \in \text{bd}(C) \cap B$, where $B$ denotes the largest ball with center $k$ contained in $C$, and for every such $t$ this hemisphere $K^*$, denoted $K^*_t$, is unique. For every $t$ we have $\Delta(K \cap K^*_t) = \frac{\pi}{2} + \rho_B$, where $\rho_B$ denotes the radius of $B$.

**Proof.** Figures 1 and 2 illustrate this theorem and its proof. They show the orthogonal look to the hemisphere $K$ from outside.

Part I.

Since $C$ is a convex body and $B$ is a ball, we see that $B$ touches $C$ from outside and the point of touching is unique. Denote it by $t$ (see Fig. 1). By Lemma 2, the bodies $C$ and $B$ are in some two opposite hemispheres. What is more, since $B$ is a ball touching $C$ from outside, this pair of hemispheres is unique. Denote by $K^*_t$ the one which contains $C$. We intend to show that $K^*_t$ is nothing else but the promised $K^*$.

Denote by $k^*$ the center of $K^*_t$. Since $k$ is also the center of $B$ and since $B$ and $K^*_t$ touch from outside at $t$, we have $t = kk^*$. From Lemma 5 we see that $t$ is the center of the $(d-1)$-dimensional hemisphere $K^*_t/K$. Analogously, from this lemma we conclude that the common point $u$ of $kk^*$ and the boundary of $K$ is the center of $K/K^*_t$. Since $t$ and $u$ are centers of the $(d-1)$-dimensional hemispheres bounding the lune $K \cap K^*_t$, we have $|tu| = \Delta(K \cap K^*_t)$. This and $|kt| + |tu| = |ku| = \frac{\pi}{2}$ imply $\Delta(K \cap K^*_t) = \frac{\pi}{2} - \rho_B$. 

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Proof. If $g \notin \text{bd}(H)$ consider any two corners $r_1$ and $r_2$ of the lune $G \cap H$. Look at the triangles $gr_1$ and $gr_2$. Below we explain that they have three equal elements. They have the length of the common side $gt$. Since $g$ is the center of $G$, we have $|gr_1| = \frac{\pi}{2} = |gr_2|$. By the orthogonality of $gt$ and $H$ it follows that $\angle gr_1 = \frac{\pi}{2} = \angle gr_2$. Since these three elements are equal, we have $|tr_1| = |tr_2|$ (by the way, they are both equal to $\frac{\pi}{2}$ since $|r_1r_2| = \pi$). When $g \in \text{bd}(H)$, we have $|gr_1| = |gr_2|$; still $r_1,r_2$ belong to the boundary of $H$. Thus $t$ is always the center of the hemisphere $H/G$. \hfill $\square$

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II. If $k \in \text{bd}(C)$, then there exists at least one hemisphere $K^*$ supporting $C$ such that $L = K \cap K^*$ is a lune containing $C$ which has thickness $\text{width}_K(C)$. This hemisphere supports $C$ at $t = k$. We have $\Delta(K \cap K^*) = \frac{\pi}{2}$.

III. If $k \in \text{int}(C)$, then there exists at least one hemisphere $K^*$ supporting $C$ such that $L = K \cap K^*$ is a lune containing $C$ which has thickness $\text{width}_K(C)$. Every such $K^*$ supports $C$ at exactly one point $t \in \text{bd}(C) \cap B$, where $B$ denotes the largest ball with center $k$ contained in $C$, and for every such $t$ this hemisphere $K^*$, denoted $K^*_t$, is unique. For every $t$ we have $\Delta(K \cap K^*_t) = \frac{\pi}{2} + \rho_B$, where $\rho_B$ denotes the radius of $B$.

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If we assume that there exists a hemisphere $M \supset C$ with $\Delta(K \cap M) < \frac{\pi}{2} - \rho_B$, then the lune $K \cap M$ must be disjoint with $B$, and hence it does not contain $C$. A contradiction. Thus $K \cap K_t^*$ is a narrowest lune of the form $K \cap N$ containing $C$. It is the unique lune of this form by the uniqueness of $t$ and $K_t^*$ explained at the beginning of the proof of Part I.
Part II.
Clearly, there is at least one hemisphere $K^*$ supporting $C$ at $k$. Of course, $\Delta(K \cap K^*) = \frac{\pi}{2}$. By Lemma 5 we see that $k$ is the center of $K^*/K$.

Part III.
Take the largest ball $B \subset C$ with center $k$. Clearly, there is at least one boundary point $t$ of $C$ which is also a boundary point of $B$ (see Fig. 2). We find a hemisphere $K_t^*$ which supports $C$ at $t$. Of course, it also supports $B$ and thus, for given $t$, it is unique.

For every $t$ there is a unique point $u \in K/K_t^*$ such that $k \in tu$. This, $|ku| = \frac{\pi}{2}$ and $|kt| = \rho_B$ imply $|tu| = \frac{\pi}{2} + \rho_B$. Hence the facts, resulting from Lemma 5, that $t$ is the center of $K_t^*/K$ and that $u$ is the center of $K/K_t^*$ give $\Delta(K \cap K_t^*) = \frac{\pi}{2} + \rho_B$.

If we assume that there exists a hemisphere $M \supset C$ such that the lune $K \cap M$ is narrower than $\frac{\pi}{2} + \rho_B$, then this lune does not contain $B$, and hence it does not contain $C$ either. A contradiction. Thus the narrowest lunes of the form $K \cap N$ containing $C$ are of the form $K \cap K_t^*$. □

Let us point out that in Part I, so if the center $k$ of $K$ does not belong to $C$, the lune $K \cap K^*$ is unique. In Part II this narrowest lune $K \cap K^*$ containing $C$ is sometimes unique and sometimes not. This depends on the point $k = t$ of $C$ which belongs to the boundary of $B$. In Part III for any given point $t$ of touching $C$ by $B$ from inside (we may have one, or finitely many, or infinitely many such points $t$), the lune $K \cap K_t^*$ is unique.

For instance, if $C \subset S^2$ is a regular spherical triangle of sides $\frac{\pi}{2}$ and the circle bounding a hemisphere $K$ contains a side of this triangle, then $K \cap K^*$ is not unique. Namely, as $K^*$ we may take any hemisphere containing $C$, whose boundary contains this vertex of $C$ which does not belong to $K$. The thickness of every such lune $K \cap K^*$ equals $\frac{\pi}{2}$. If $C$ is a regular spherical triangle of sides over $\frac{\pi}{2}$ and the boundary of $K$ contains a side of this triangle, then $K \cap K^*$ is not unique either. This time the boundary of $K^*$ contains a side of $C$ different from the side which is in $K$. So we have exactly two positions of $K^*$.

Here are two corollaries from Theorem 1 (for the second we also apply Lemma 5).

Corollary 1. If $k \notin C$, then $\text{width}_{K}(C) = \frac{\pi}{2} - \rho_B$. If $k \in \text{bd}(C)$, we have $\text{width}_{K}(C) = \frac{\pi}{2}$. If $k \in \text{int}(C)$, then $\text{width}_{K}(C) = \frac{\pi}{2} + \rho_B$.

Corollary 2. The point $t$ of support in Theorem 1 is the center of the $(d - 1)$-dimensional hemisphere $K^*/K$.

We define the thickness $\Delta(C)$ of a convex body $C \subset S^d$ as follows:

$$\Delta(C) = \text{inf}\{\text{width}_{K}(C); K \text{ is a supporting hemisphere of } C\}.$$ 

Compactness arguments show that the infimum is realized. As a consequence, $\Delta(C) = \min\{\text{width}_{K}(C); K \text{ is a supporting hemisphere of } C\}$. By the
definitions of width and thickness we conclude that the thickness of every convex body \( C \subset S^d \) is equal to the minimum thickness of a lune containing \( C \).

At this moment observe that our definition of width \( K(C) \) has an advantage, when applied to find the thickness of a convex body \( C \subset S^d \). Namely, it is sufficient to find the minimum of the values of width \( K(C) \) over all hemispheres \( K \) supporting \( C \). Theorem 1 helps to establish every width \( K(C) \).

Example 1. Applying Theorem 1 we easily find the thickness of any regular triangle \( T_\alpha \) of angles \( \alpha \). Formulas of spherical trigonometry imply that

\[
\Delta(T_\alpha) = \arccos \frac{\cos \alpha}{\sin \frac{\alpha}{2}} \quad \text{for} \quad \alpha < \frac{\pi}{2}.
\]

If \( \alpha \geq \frac{\pi}{2} \) (but, of course, \( \alpha < \frac{2}{3}\pi \)), then

\[
\Delta(T_\alpha) = \alpha.
\]

In both cases \( \Delta(T_\alpha) \) is realized for width \( K(T_\alpha) \), where \( K \) is a hemisphere whose bounding semicircle contains a side of \( T_\alpha \). In the first case \( T_\alpha \) is symmetric with respect to the arc \( A \) connecting the centers of \( K/K^* \) and \( K^*/K \), while in the second \( T_\alpha \) is symmetric with respect to the arc passing through the middle of \( A \) and having endpoints at the corners of the lune \( K \cap K^* \). For \( \alpha = \frac{\pi}{2} \) there are infinitely many positions in \( K^* \).

Theorem 2. As the position of the \((d - 1)\)-dimensional supporting hemisphere of a convex body \( C \subset S^d \) changes, the width of \( C \) determined by this hemisphere changes continuously.

**Proof.** We keep the notation of Theorem 1. Of course, the positions of \( k \) and thus of \( B \) depend continuously on \( K \). Hence \( \frac{\pi}{2} - \rho_B \) and \( \frac{\pi}{2} + \rho_B \) change continuously. This and Corollary 1 imply the thesis of our theorem. It does not matter here that for a fixed \( K \) sometimes the lunes \( K \cap K^* \) are not unique; still they are all of equal. \( \square \)

Claim 2. Consider a convex body \( C \subset S^d \) and any lune \( L \) of thickness \( \Delta(C) \) containing \( C \). Both centers of the \((d - 1)\)-dimensional hemispheres bounding \( L \) belong to \( C \).

This claim results immediately from Corollary 2 applied twice: for each of the two \((d - 1)\)-dimensional hemispheres bounding \( L \).

If for every hemisphere supporting a convex body \( C \subset S^d \) the width of \( C \) determined by \( K \) is the same, we say that \( C \) is a body of constant width. In particular, spherical balls of radius smaller than \( \frac{\pi}{2} \) are bodies of constant width.

Also every spherical Reuleaux odd-gon is a convex body of constant width. Recall this notion. Take a regular spherical \( k \)-gon \( v_1 v_2 \ldots v_k \subset S^2 \), where \( k \geq 3 \) is odd. Clearly, all the distances \( |v_i v_{i+k-1}| \) and \( |v_i v_{i+k+1}| \) for \( i = 1, \ldots, k \) are equal (the indices are taken modulo \( k \)). Denote them by \( \delta \). Assume that \( \delta \leq \frac{\pi}{2} \). Let \( B_i \), where \( i = 1, \ldots, k \), be the disk with center \( v_i \) and radius \( \delta \). The set \( B_1 \cap \cdots \cap B_k \) is just a spherical Reuleaux \( k \)-gon.

By the definition of width and by Claim 2, if \( C \subset S^d \) is a body of constant width, then every supporting hemisphere \( G \) of \( C \) determines a supporting
hemisphere $H$ of $C$ for which $G \cap H$ is a lune such that the centers of $G/H$ and $H/G$ belong to the boundary of $C$. Is the opposite true? More precisely, is a convex body $C \subset S^d$ of constant width provided every supporting hemisphere $G$ of $C$ determines at least one hemisphere $H$ supporting $C$ such that $G \cap H$ is a lune with the centers of $G/H$ and $H/G$ in the boundary of $C$?

4. Diameter

By the *diameter* $\text{diam}(C)$ of a set $C \subset S^d$ we mean the supremum of the spherical distances between pairs of points of $C$. Clearly, if $C$ is closed, the diameter of $C$ is realized for at least one pair of points of $C$.

Claim 3. Let $\text{diam}(C) \leq \frac{\pi}{2}$ for a convex body $C \subset S^d$ and assume that $\text{diam}(C) = |ab|$ for some points $a, b \in C$. Denote by $L$ the lune such that $a$ and $b$ are the centers of the $(d-1)$-dimensional hemispheres bounding $L$. We have $C \subset L$.

Proof. We apply Claim 1. Let us keep the notation of its proof. Since $|ab| = \text{diam}(C)$, for every $x \in C$ we have $|ax| \leq |ab|$. Moreover $\text{diam}(C) \leq \frac{\pi}{2}$ implies $a \in pb$. Hence $|px| \leq |pa| + |ax| \leq |pa| + |ab| = |pb| = \frac{\pi}{2}$. Thus $C \subset H(p)$. Similarly, $C \subset H(q)$. Consequently, $C \subset H(p) \cap H(q) = L$. □

Remark 1. In general, Claim 3 does not hold true without the assumption that $\text{diam}(C) \leq \frac{\pi}{2}$. A simple counterexample is the triangle $T = abc$ with $|ab| = \frac{2}{3}\pi \approx 2.0944$, $|bc| = \frac{\pi}{6} \approx 0.5236$ and $\angle abc = 95^\circ$. From the Al Battani formulas, also called law of cosines for sides, (see, e.g., [5], p. 45), we get $|ac| \approx 2.0609$. Consequently, $|ab| = \frac{2}{3}\pi$ is the diameter of $T$. Since $\angle abc = 95^\circ$, the lune with centers $a$ and $b$ of the semicircles bounding it does not contain $c$. Still its thickness is $\frac{2}{3}\pi$. Thus this lune does not contain $T$.

Compactness arguments lead to the conclusion that for every convex body $C \subset S^d$ the supremum of $\text{width}_H(C)$ over all hemispheres $H$ supporting $C$ is realized for a supporting hemisphere of $C$, that is, we may take here the maximum instead of supremum.

The following theorem is an analog of the classic theorem for Euclidean space.

Theorem 3. Let $\text{diam}(C) \leq \frac{\pi}{2}$ for a convex body $C \subset S^d$. We have

$$\max\{\text{width}_K(C); K \text{ is a supporting hemisphere of } C\} = \text{diam}(C).$$

Proof. Let $K$ be an arbitrary hemisphere supporting $C$ and let $s \in C$ be a point of support by $K$ (see Fig. 1). Take $k, t, u$ and $K^*$ like in Parts I and II of Theorem 1. By Lemma 3 we have $|st| \geq |ut|$. Hence $\text{diam}(C) \geq |st| \geq |ut| = \text{width}_K(C)$. This and the assumption that $K$ is an arbitrary hemisphere
supporting \( C \) imply that \( \text{diam}(C) \) is at least the maximum of \( \text{width}_K(C) \) over all supporting hemispheres \( K \) of \( C \).

Let \( a, b \in C \) be such that \( |ab| = \text{diam}(C) \). Take the lune \( L \) from Claim 3, i.e., \( H(p) \cap H(q) \) like in its proof. Thus \( \text{diam}(C) \) equals the thickness of \( L \), i.e., \( \text{width}_{H(p)}(C) \). Hence \( \text{diam}(C) \) is at most the maximum of \( \text{width}_K(C) \) over all supporting hemispheres \( K \) of \( C \).

The following example shows that Theorem 3 requires the assumption \( \text{diam}(C) \leq \frac{\pi}{2} \).

**Example 2.** Let \( T \) be an isosceles triangle with base of length \( \lambda \) close to 0 and the height perpendicular to it of length \( \mu \in \left( \frac{\pi}{2}, \pi \right) \). Denote by \( w \) the center of the base and by \( v \) the opposite vertex of \( T \). Lemma 3 implies that \( wv \) is the diametrical segment of \( T \). Take the hemisphere \( K \) supporting \( T \) at \( w \). Denote by \( k \) the center of \( K \). Clearly, \( k \in wv \), so \( k \) is in the interior of \( T \). Let \( \rho \) be the radius of the largest disk \( B \) with center \( k \) contained in \( T \). The radius \( \rho \) of \( B \) is arbitrarily close to 0, as \( \lambda \) is sufficiently small. Applying Part III of Theorem 1 we conclude that the width of \( T \) determined by \( K \) is \( \frac{\pi}{2} + \rho \). Hence it may be arbitrarily close to \( \frac{\pi}{2} \), as \( \lambda \) is sufficiently small. On the other hand, the diameter \( |wv| \) of \( T \) may be arbitrarily close to \( \pi \), as \( \mu \) is sufficiently close to \( \pi \).

**Proposition 1.** Let \( \text{diam}(C) > \frac{\pi}{2} \) for a convex body \( C \subset S^d \). We have

\[
\max\{\text{width}_K(C); K \text{ is a supporting hemisphere of } C\} \leq \text{diam}(C).
\]

**Proof.** Let \( K \) be an arbitrary hemisphere supporting \( C \) and let \( s \in C \) be a point of support by \( K \) (see Fig. 2). Take \( k, t \) and \( K^* \) like in Parts I–III of Theorem 1.

If \( k \notin \text{int}(C) \), so if we apply Parts I and II of Theorem 1, we repeat the consideration of the first paragraph of the proof of Theorem 3 which gives \( \text{width}_K(C) \leq \text{diam}(C) \).

Assume that \( k \in \text{int}(C) \), so that we apply Part III of Theorem 1. Clearly, \( |sk| = \frac{\pi}{2} \). Take the largest ball \( B \) with center \( k \) contained in \( C \). Denote by \( \rho \) its radius. By Part III we have \( \text{width}_K(C) = \frac{\pi}{2} + \rho \). Provide the great circle through \( s \) and \( k \). It intersects the boundary of \( B \) at two points. Denote by \( z \) this from these two points for which \( k \in sz \). We have \( |sz| = |sk| + |kz| = \frac{\pi}{2} + \rho \), which, by Part III, equals \( \text{width}_K(C) \). This and \( |sz| \leq \text{diam}(C) \) lead to the conclusion that \( \text{width}_K(C) \leq \text{diam}(C) \).

Since \( K \) is an arbitrary hemisphere supporting \( C \), we get the thesis.

5. Reduced bodies

In analogy to the definition of reduced bodies in Euclidean space \( E^d \) introduced in [7] (see also [8–10] and [11]), we define reduced convex bodies on \( S^d \). We
say that a convex body \( R \subset S^d \) is reduced if \( \Delta(Z) < \Delta(R) \) for every convex body \( Z \subset R \) different from \( R \).

By our definition of bodies of constant width on \( S^d \) we see that they are reduced bodies. In particular, every Reuleaux polygon on \( S^2 \) is a reduced body.

It is easy to show that all regular odd-gons on \( S^2 \) of thickness at most \( \frac{\pi}{2} \) are reduced bodies. The assumption that the thickness is at most \( \frac{\pi}{2} \) matters here. For instance take the regular triangle \( T_\alpha \) of angles \( \alpha > \frac{\pi}{2} \) (see Example 1). Take the hemisphere \( K \) whose boundary contains a side of \( T_\alpha \) and apply Part III of Theorem 1. The corresponding ball \( B \subset T_\alpha \) touches \( T_\alpha \) from inside at exactly two points \( t_1, t_2 \). Cutting off a part of \( T_\alpha \) by the shorter arc of the boundary of \( B \) between \( t_1 \) and \( t_2 \) we obtain a convex body \( Z \subset T_\alpha \). We have \( \Delta(Z) = \Delta(T_\alpha) \), which implies that \( T_\alpha \) is not reduced.

Dissect a disk on \( S^2 \) by two orthogonal great circles through its center. The four obtained parts are called quarters of disks. In particular, the triangle of sides and angles \( \frac{\pi}{2} \) is a quarter of a disk. It is easy to see that every quarter of a disk is a reduced body and that the thickness of it is equal to the radius of the original disk. More generally, each of the \( 2^d \) parts of a spherical ball on \( S^d \) dissected by \( d \) pairwise orthogonal great \((d-1)\)-dimensional spheres through the center of this ball is a reduced body of \( S^d \). We call it the \( \frac{1}{2^d} \)-th part of a ball. Clearly, its thickness is equal to the radius of the above ball.

We say that \( e \) is an extreme point of a convex body \( C \subset S^d \) provided the set \( C \setminus \{e\} \) is convex. From the analogue of the Krein–Milman theorem for convex cones (e.g., see [12]) its analogue for spherical convex bodies follows: every convex body \( C \subset S^d \) is the convex hull of its extreme points. This and the fact that the common part of any closed convex body \( C \subset S^d \) with any of its supporting \((d-1)\)-dimensional great sphere is a closed convex set imply the following lemma.

**Lemma 6.** The boundary of every supporting hemisphere of a convex body \( C \subset S^d \) passes through an extreme point of \( C \).

**Theorem 4.** Through every extreme point \( e \) of a reduced body \( R \subset S^d \) a lune \( L \supset R \) of thickness \( \Delta(R) \) passes with \( e \) as the center of one of the two \((d-1)\)-dimensional hemispheres bounding \( L \).

**Proof.** Let \( B_i \) be the open ball of radius \( \Delta(R)/i \) centered at \( e \) and let \( R_i = \text{conv}(R \setminus B_i) \) for \( i = 2, 3, \ldots \). By Lemma 1 every \( R_i \) is a convex body. Moreover, since \( e \) is an extreme point of \( R \), \( R_i \) is a proper subset of \( R \). So, since \( R \) is reduced, \( \Delta(R_i) < \Delta(R) \). By the definition of thickness of a convex body, \( R_i \) is contained in a lune \( L_i \) of thickness \( \Delta(R_i) \).

From Lemma 4 we conclude that there exists a subsequence of the sequence \( L_2, L_3, \ldots \) converging to a lune \( L \). Since \( R_i \subset L_i \) for \( i = 2, 3, \ldots \), we obtain that \( R \subset L \). Since \( \Delta(L_i) = \Delta(R_i) < \Delta(R) \) for every \( i \), we get \( \Delta(L) \leq \Delta(R) \). This and \( R \subset L \) imply \( \Delta(L) = \Delta(R) \).
Let $m_i, m'_i$ be the centers of the $(d - 1)$-dimensional hemispheres $H_i, H'_i$ bounding $L_i$. We claim that at least one of these two centers, say $m_i$, belongs to the closure of $R \setminus R_i$. The reason is that in the opposite case, there would be a neighborhood $N_i$ of $m_i$ such that $N_i \cap R_i = N_i \cap R$, which would imply that $H_i$ supports $R$ at $m_i$. Moreover, $H'_i$ supports $R$ at $m'_i$. Hence $\Delta(R) = \Delta(L_i) = \Delta(R_i)$, in contradiction with $\Delta(R_i) < \Delta(R)$.

Since $m_i \in R \setminus R_i$ for $i = 2, 3, \ldots$, we see that the sequence of points $m_2, m_3, \ldots$ tends to $e$. Consequently, $e$ is the center of a $(d - 1)$-dimensional hemisphere bounding $L$.

Remark 2. Besides the lune from Theorem 4, sometimes we have additional lunes $L' \supset R$ of thickness $\Delta(R)$ through $e$ for which $e$ is not in the middle of a $(d - 1)$-dimensional hemisphere bounding $L'$. This happens, for instance, when $R$ is a spherical regular triangle $T_\alpha$ with $\alpha \leq \pi/2$.

Theorem 4 leads to the following questions. Is it true that through every boundary point $p$ of a reduced body $R \subset S^2$ a lune $L \supset R$ of thickness $\Delta(R)$ passes? A consequence would be that every reduced body $R \subset S^2$ is an intersection of lunes of thickness $\Delta(R)$. Is a stronger version of the preceding question true, namely, that there is always such a lune $L$ with $p$ at the center of one of the two $(d - 1)$-dimensional hemispheres bounding $L$?

There are more questions on spherical reduced bodies. For instance, which properties of reduced bodies in $E^d$, and especially in $E^2$ (see [9] and [10]), may be reformulated and proved for reduced bodies on $S^d$? Are there reduced spherical polytopes on $S^d$, where $d \geq 3$, different from $1/2\pi$-th part of a ball? Or at least a spherical simplex different from $1/2\pi$ of $S^d$ (comp. [11]).

By Theorem 4 we obtain the following spherical analog of a theorem from [8], see also Corollary 1 in [9] and [10].

**Theorem 5.** Every smooth reduced body on $S^d$ is of constant width.

**Proof.** Let $R \subset S^d$ be a smooth reduced body. Take any supporting hemisphere $K$ of $R$. By Lemma 6 the boundary of $K$ contains an extreme point $e$ of $R$. Since $R$ is smooth, $K$ is the unique supporting hemisphere of $R$ at $e$. Moreover, from Theorem 4 we see that through $e$ a lune $L \supset R$ of thickness $\Delta(R)$ passes. Thus $L = K \cap K^*$ and hence width$_K(R) = \Delta(R)$. This and the arbitrariness of $K$ imply the thesis of our theorem. 

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