Zeros of Jones Polynomials for Families of Knots and Links

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Abstract

We calculate Jones polynomials $V_L(t)$ for several families of alternating knots and links by computing the Tutte polynomials $T(G, x, y)$ for the associated graphs $G$ and then obtaining $V_L(t)$ as a special case of the Tutte polynomial. For each of these families we determine the zeros of the Jones polynomial, including the accumulation set in the limit of infinitely many crossings. A discussion is also given of the calculation of Jones polynomials for non-alternating links.
1 Introduction

Knots and links are of interest in mathematics and because of their connection to Yang-Baxter relations and solvable models [11]-[12]. Here a link is defined as an embedding of $n_c$ circles $S^1$ (each forming a component of the link) in $\mathbb{R}^3$, and a knot is the special case $n_c = 1$ of a link involving one such circle. We shall generally use the notation $L$ and $K$ for links and knots and $L$ when referring to both. A longstanding problem in mathematics has been to reduce the classification of knots and links to an algebraic formulation. Various polynomials have been constructed in an effort to achieve an algebraic classification of knots. Here we shall focus on the Jones polynomial $V_L(t)$ [10, 11]. Technically, this is not, in general, a polynomial, since it can involve non-integral powers of its variable $t$ and, even in cases where it only involves integral powers, these may be negative. However, $V_L(t)$ can always be written as a polynomial in $t$ multiplied by a monomial prefactor. With this qualification, we shall follow common usage in referring to $V_L(t)$ as a polynomial. Lists of knots and links are given with standard ordering and labelling in, e.g., [1], and we follow the labelling conventions in this reference. The usual presentation of knots and links involves a projection from the embedding space $\mathbb{R}^3$ to $\mathbb{R}^2$ with resultant apparent crossings of the curves on the one or more embedded $S^1$’s. According to standard labelling, $5_2 \equiv 5_{12}$ denotes the second knot with five crossings in the list, $6_2$ denotes the second link containing three intertwined components with six crossings, etc. We recall that an alternating link is defined as one in which, as one travels along each of the $n_c$ embedded circles $S^1$, one traverses a crossing in an overlying manner, then underlying, then overlying, and so forth.

Here we shall calculate Jones polynomials for certain families of knots and, more generally, links with arbitrarily many crossings and analyze certain properties in the limit of infinitely many crossings. An interesting pioneering study of this type has been carried out recently by Wu and Wang [13]. We shall concentrate on alternating links but shall also briefly discuss the non-alternating case. Let us denote the member of a generic family as $L_r$, meaning that this has $r$ crossings. Other information about the links will be indicated as necessary below but is suppressed here. We introduce the notation for the formal limit $\lim_{r \to \infty} L_r = \{L\}$. It is also useful to investigate the zeros of the Jones polynomial for classes of links or knots in the complex $t$ plane and their accumulation set $\mathcal{B}$ in the limit where the number $r$ of crossings goes to infinity.

There are several motivations for this study. It is worthwhile to have new exact calculations of Jones polynomials for various families of links; furthermore, the fact that these apply for arbitrarily many crossings enables one to investigate the limit where the number of crossings goes to infinity. As noted above, knots and links are also relevant to physics
because of their relation to Yang-Baxter equations. These equations are conditions for (i) transfer matrices of exactly soluble models in statistical mechanics, and (ii) equivalently, $S$-matrix elements of relativistic particle scattering in one space and one time dimension, so that these $S$ matrix elements factorize, i.e., the $S$ matrix an $n \to n$ scattering process factorizes into a product of $S$ matrix elements for simple $2 \to 2$ scatterings, without particle production. Indeed, there are well-known connections between Yang-Baxter relations, the braid group, and Temperley-Lieb algebras (e.g., [4]-[6]). As we shall discuss next, there are also useful relations between the Jones polynomial and the Tutte polynomial, or equivalently, the partition function of the Potts model, a valuable model of phase transitions and critical phenomena.

2 Some General Results

2.1 Connection between Jones and Tutte Polynomials

Our method for calculating the Jones polynomials $V_L(t)$ for a given family of alternating links is to compute the Tutte polynomial $T(G, x, y)$ for a certain associated graph $G$ and then use the fact that $V_L(t)$ can be obtained as a special case of the Tutte polynomial for $x = t$ and $y = -1/t$. To explain this method, we first discuss some background. For a given knot, draw the associated shaded-region diagram. There are two such diagrams. To see this, consider a given crossing, say in the form of an $\times$; in one shading, the regions to the left and right of the $\times$ are light and those above and below the $\times$ are dark. In the other shading, the regions to the left and right are dark while those above and below are light. These two possible shadings are labelled as plus and minus with the following convention: as one exits a given crossing on the curve that is overlying, if the region on the left (right) is shaded, assign $+$ ($-$) to this crossing. For a given knot or link $L$, the numbers of the $\pm$ crossings are denoted $r_{\pm}(L)$. It is easily seen that for an alternating knot, if one crossing on the shaded-region diagram is $+$, then all are, and similarly if one crossing is $-$, then all are. We denote the corresponding shaded region diagrams as $D_+(L)$ and $D_-(L)$. We denote the numbers of dark ($d$) and light ($\ell$) regions in the shaded region diagram $n_d(D_{\pm})$ and $n_{\ell}(D_{\pm})$. Clearly, $n_d(D_+) = n_{\ell}(D_-)$ and $n_{\ell}(D_+) = n_d(D_-)$. From these shaded-region diagrams $D_{\pm}(L)$, one then constructs the associated graphs $G_+(L)$ and $G_-(L)$ by assigning vertices to the shaded regions and edges (bonds) connecting these vertices. We recall that a graph $G = (V, E)$ is defined by its vertex and edge sets $V$ and $E$. Strictly speaking, the objects that we are dealing with here are pseudographs, since they can have multiple edges and loops, but we shall continue to call them simply graphs. Let the number of vertices and
edges of the graph $G$ be denoted $n(G) = |V|$ and $e(G) = |E|$. Clearly, the total number of crossings is equal to the number of edges of the associated graph:

$$r(L) = e(G_+(L)) = e(G_-(L))$$

and the respective numbers of dark (light) regions in the $D_+$ ($D_-$) region diagrams is equal to the number of vertices in the associated graphs:

$$n_d(D_+) = n(G_+(L)), \quad n_d(D_-) = n(G_-(L)).$$

Furthermore,

$$G_+(L) = [G_-(L)]^*$$

i.e., $G_-(L)$ is the (planar) dual of $G_+(L)$. (Here, from a planar graph $G$, the (planar) dual $G^*$ is constructed by assigning to each face of $G$ a vertex and connecting these vertices by edges traversing the edges of $G$.) Another standard definition that we will need is the writhe of the knot or link $L$, denoted $w(L)$. For this we first define another type of sign $s = \pm$ for each crossing in a link. For each circle on the link, we consider an orientation, which can be regarded as a direction of motion along the circle. Next, we represent a given crossing as two arrows, one pointing to the upper left and one to the upper right. If the lower-right to upper-left arrow is over (under) the lower-left to upper-right arrow, the sign $s$ is $-$ ($+)$.

Denote the total number of crossings with $s = + (s = -)$ as $N_{s+}$ ($N_{s-}$), respectively. Then the writhe is

$$w(L) = (+1)N_{s+} + (-1)N_{s-}$$

To each knot or link $K$ or $L$, there corresponds another, which we label as $R(K)$ and $R(L)$ obtained by reversing all crossings, i.e. replacing an over-under crossing by an under-over crossing, and vice versa. A basic property of the Jones polynomial is that

$$V_{L}(t) = V_{R(L)}(t^{-1}).$$

We next proceed to the relation between the Jones polynomial $V_{L}(t)$ and the Tutte polynomial of the associated graph $G_+(L)$ which applies for the case in which $L$ is an alternating link or knot. The Tutte polynomial $T(G, x, y)$ of a graph $G$ (which may include multiple edges and loops) is

$$T(G, x, y) = \sum_{G' \subseteq G} (x - 1)^{k(G') - k(G)}(y - 1)^{c(G')},$$

where $G'$ is a spanning subgraph of $G$, i.e., $G'$ has the same vertex set and a subset of the edge set, of $G$: $G' = (V, E')$ with $E' \subseteq E$. In eq. (2.1.6), $k(G')$, $c(G') = |E'|$, and $c(G')$. 

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denote the number of components, edges, and (linearly independent) circuits of \( G' \), with the usual relation
\[
c(G') = e(G') + k(G') - n(G')
\]
(2.1.7)
where, as above, \( n(G') \) is the number of vertices of \( G' \). The Tutte polynomials of a planar graph \( G \) and its (planar) dual \( G^* \) are related according to
\[
T(G, x, y) = T(G^*, y, x).
\]
(2.1.8)
The relation between the Jones polynomial \( V_L(t) \) and the Tutte polynomial of the associated graph \( G_+(L) \) for an alternating link or knot is then
\[
V_L(t) = (-1)^w t^{(n_L(G_+)-n_d(G_+)+3w)/4} T(G_+(L), -t, -1/t).
\]
(2.1.9)
Note that for a link involving an odd number \( n_c \) of components (circles) and hence, in particular, for a knot, \( n_L(G_+)-n_d(G_+)+3w \) is always a multiple of 4, so that \( V_L(t) \) is a multinomial, i.e., Laurent polynomial, in \( t \). For a link consisting of an even number \( n_c \geq 2 \) of components \( n_L(G_+)-n_d(G_+)+3w = 2 \mod 4 \), so that \( V_L(t) \) is of the form \( V_L(t) = t^q \text{Poly}_L(t) \), where \( q \) is a positive or negative half-odd-integer.

2.2 Some Further Properties
A basic property of the Jones polynomial \( V_L(t) \) for a link \( L \) is that
\[
V_L(e^{2\pi i/3}) = (-1)^{n_c}
\]
(2.2.1)
where, as before, \( n_c \) denotes the number of \( S^1 \)'s in the link. Since in general, \( V_L(t) \) (aside from its prefactor) may be a high-order polynomial in \( t \), this suggests that \( V_L(t) \) may have a complex-conjugate pair of zeros near to \( t = e^{\pm 2\pi i/3} \). Indeed, for the special case \( n_c = 1 \), i.e., a knot, one has
\[
V_K(t) = 1 - (1 - t)(1 - t^3) W_K(t)
\]
(2.2.2)
where \( W_K(t) \) is a Laurent polynomial in \( t \). As shown in [13], it follows that \( V_K(t) \) has a complex-conjugate pair of zeros that approach the points
\[
t = e^{\pm 2\pi i/3}
\]
(2.2.3)
as \( r(K) \to \infty \).

Two other basic properties concern the highest and lowest powers of \( t \) in \( V_L(t) \) for an alternating knot or link. To determine these, we first derive the values of the respective maximal degrees of the Tutte polynomial in its variables \( x \) and \( y \):
\[
\text{max} \left( \text{deg}_x [T(G, x, y)] \right) = n(G) - 1
\]
(2.2.4)
and
\[ \max(\deg_y[T(G, x, y)]) = c(G). \] (2.2.5)

To prove (2.2.4) we maximize the power \( k(G') - k(G) \) in the definition (2.1.9). This maximum power is realized for \( G' = (V, \emptyset) \), i.e., the case where the spanning subgraph has no edges. For the case of interest here, where \( G \) itself is connected, this power then takes on the maximal value \( n(G) - 1 \). To prove (2.2.5), we want to maximize the number of circuits in \( G' \). Clearly, \( c(G') \leq c(G) \), so we achieve this maximum number of circuits by taking \( G' = G \).

Then, combining (2.2.4) and (2.2.5) with (2.1.9), we have, for an alternating link (indicating also the sign of \( t \))
\[ \max(\deg[V_L(t)]) = (-1)^{w(L) + n(G_+(L)) - 1} \left( n(G_+(L)) - 1 + n(L) - n_d(G_+(L)) + 3w(L) \right)/4 \] (2.2.6)
\[ \min(\deg[V_L(t)]) = (-1)^{w(L) - c(G_+(L)) - 1} \left( c(G_+(L)) + n(L) - n_d(G_+(L)) + 3w(L) \right)/4 \] (2.2.7)

From these results together with the relation (2.1.7), it follows that
\[ \max(\deg[V_L(t)]) - \min(\deg[V_L(t)]) = e(G_+(L)) \]
\[ = r(L). \] (2.2.8)

We can simplify the sign in Eq. (2.2.6). Since \((-1)^{w(L)} = (-1)^{r(L)} = (-1)^{e(G_+(L))}\), we have
\[ (-1)^{w(L) + n(G_+(L)) - 1} = (-1)^{n_d(D_-(L)) - 1} \] (2.2.9)
which makes use of the Euler relation for a planar graph \( n(G) - e(G) + f(G) = 2 \), where \( f(G) = v(G^*) \) is the number of faces of \( G \), which is also equal to the number of vertices of the planar dual, \( G^* \).

### 2.3 Limit of Links with Infinitely Many Crossings

We next define a function
\[ U(\{L\}, t) = \lim_{n(G_+(L)) \to \infty} \left( (-1)^{n_d(D_-(L)) - 1} V_L(t) \right)^{1/n(G_+(L))}. \] (2.3.1)

With the multiplicative factor incorporated as indicated, the coefficient of the maximal power of \( t \) is positive and hence the resultant function is real and positive for large real positive \( t \). Consequently, among the \( n \) different \( 1/n \)’th roots, one can, with no ambiguity, pick the real positive one, and this is understood in eq. (2.3.1). The largest region in the complex \( t \) plane to which one can analytically continue the function \( U(\{L\}, t) \) we shall call region \( R_1 \).
We remark that there is a subtlety in the definition (2.3.1) due to the fact that at certain special points $t = t_s$ one can encounter the noncommutativity

$$\lim_{t \to t_s} \lim_{n \to \infty} [V_L(t)]^{1/n} \neq \lim_{n \to \infty} \lim_{t \to t_s} [V_L(t)]^{1/n}$$

(2.3.2)

where $n \equiv n(G_s(L))$. Denote $U_{tn}$ and $U_{nt}$ as the functions defined by the different order of limits on the left and right-hand sides of (2.3.2). Here we make the choice

$$U(\{L\}, t) \equiv U_{tn}(\{L\}, t)$$

(2.3.3)

since this has the advantage of removing certain isolated discontinuities that are present in $U_{nt}$.

The definition (2.3.1) is analogous to the definition of the exponent of the reduced free energy $f$ in statistical mechanics according to

$$e^f = \lim_{n \to \infty} Z^{1/n}$$

(2.3.4)

where $Z$ is the partition function. Again, in this definition one implicitly takes the real positive $1/n$‘th root, and there is no ambiguity since for physical values of temperature and external magnetic field, $Z$ is real and positive. Another analogy is the ground state degeneracy $W$ per site for the Potts antiferromagnet on a graph $G$, which can be obtained from the chromatic polynomial $P(G, q)$, counting the number of ways of coloring the vertices of a graph $G$ subject to the constraint that no two adjacent vertices have the same color.

Denoting the formal limit as $n \to \infty$ of a family of graphs $G$ as $\{G\}$, we define

$$W(\{G\}, q) = \lim_{n \to \infty} P(G, q)^{1/n}.$$  

(2.3.5)

In [18] it was noted that at certain special points $q_s$ (typically $q_s = 0, 1, ..., \chi(G)$, where $\chi(G)$ is the chromatic number of the graph $G$, i.e. the minimum number of colors necessary for a proper vertex coloring of $G$), one has the noncommutativity of limits

$$\lim_{q \to q_s} \lim_{n \to \infty} P(G, q)^{1/n} \neq \lim_{n \to \infty} \lim_{q \to q_s} P(G, q)^{1/n}$$

(2.3.6)

and hence it is necessary to specify the order of the limits in the definition of $W(\{G\}, q_s)$ [18]. Denoting $W_{qn}$ and $W_{nq}$ as the functions defined by the different order of limits on the left and right-hand sides of (2.3.6), the choice $W \equiv W_{qn}$ was made in [18] and subsequent our works since this has the advantage of removing certain isolated discontinuities that are present in $W_{nq}$. For a given family of graphs $G$, and for sufficiently large real positive $q$, $P(G, q)$ is real and positive, so again one can choose, without any ambiguity, the real positive $1/n$‘th root.
in (2.3.3). In [18] the largest region in the complex $q$ plane to which one can analytically continue the function $W(\{G\}, q)$ from the real positive axis for large $q$ was denoted as $R_1$. Just as $W(\{G\}, q)$ is analytic in certain regions of the complex $q$ plane, $U(\{L\}, t)$ is analytic in certain regions of the complex $t$ plane. And just as was true for $W$, in regions other than $R_1$, $V_L(t)$ can be negative or complex, hence there is no canonical choice of the phase to take in evaluating the $1/n$’th root in (2.3.1), and only the magnitude $|U(\{L\}, t)|$ can be determined unambiguously.

The continuous locus where $U(\{L\}, t)$ is nonanalytic, called $B$, arises as the continuous accumulation set of the zeros of $V_L(t)$. A pioneering discussion of this type of accumulation of zeros to form non-analytic boundaries for a model in statistical physics was given by Yang and Lee [19, 20] for the zeros of the Ising model partition function as a function of the complex variable $\mu = e^{-2\beta H}$, where $\beta = (k_B T)^{-1}$, $T$ is the temperature, and $H$ is the external magnetic field. A second example is the zeros of a spin model partition function in the complex variable $a^K$, where $K = \beta J$ and $J$ is a spin-spin exchange coupling [21]. A more complicated example is provided by the study of the zeros and their accumulation set in the $\mathbb{C}^2$ space defined by $(a, \mu)$ [22, 23].

2.4 Connection with Potts Model Partition Function

It is useful to observe that the Tutte polynomial of a graph is essentially equivalent to the partition function of the Potts model on this graph (for a review of the Potts model, see [24]). We recall that on $G$ (often taken to be a regular lattice in physics) at temperature $T$ the Potts model is defined by the partition function

$$Z(G, q, v) = \sum_{\{\sigma_i\}} e^{-\beta H}$$

with the (zero-field) Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}$$

where $\sigma_i = 1, ..., q$ are the spin variables on each vertex $i \in G$; $\beta = (k_B T)^{-1}$; and $\langle ij \rangle$ denotes pairs of adjacent vertices, and again with $K = \beta J$,

$$v = e^K - 1. \quad (2.4.3)$$

This function $Z(G, q, v)$ can be written as the sum [25]

$$Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} v^{e(G')}$$
\[
Z(G) = \sum_{r=k(G)}^{n(G)} \sum_{s=0}^{e(G)} z_{rs} q^r v^s .
\] (2.4.4)

This formula allows one to generalize the definition of the Potts model from \(q \in \mathbb{Z}_+\) to \(q \in \mathbb{C}\) and from the real physical range \(-1 \leq v < \infty\) to \(v \in \mathbb{C}\). Now let

\[
x = 1 + \frac{q}{v}
\] (2.4.5)

and

\[
y = v + 1
\] (2.4.6)

so that

\[
q = (x - 1)(y - 1) .
\] (2.4.7)

Then the Tutte polynomial and Potts model are related according to

\[
Z(G, q, v) = (x - 1)^{k(G)}(y - 1)^{n(G)} T(G, x, y) .
\] (2.4.8)

Another relation concerns the chromatic polynomial \(P(G, q)\), which counts the number of ways of coloring the vertices of \(G\) subject to the condition that no two adjacent vertices have the same color [26, 27]. This is identical to the partition function of the Potts antiferromagnet at \(T = 0\), i.e., \(v = -1\):

\[
P(G, q) = Z(G, q, -1)
\] (2.4.9)

or, equivalently,

\[
P(G, q) = (-q)^{k(G)}(-1)^{n(G)} T(G, 1 - q, 0) .
\] (2.4.10)

Thus, via the connection to the Tutte polynomial or Potts model partition function, one sees that the study of the zeros of the Jones polynomial is related to studies of complex-temperature zeros of Ising and Potts models [21], [30]-[43], to studies of zeros in complex \(q\) of chromatic polynomials (e.g., [44]-[62], [18]), and the studies of zeros of Potts model partition functions in the \(C^2\) space spanned by \((q, v)\) [69]-[70], [56].

It will be convenient to recall some definitions from graph theory. The circuit graph \(C_n\) is the graph of \(n\) vertices connected with edges to form a circle. The (planar) dual of this graph is called the fat link, \(FL_n\), and consists of two vertices with \(n\) edges connecting them. The join \(G + H\) of two graphs \(G\) and \(H\) is the graph formed by connecting each vertex of \(G\) to all of the vertices of \(H\). The wheel graph \((Wh)_n\) is the join of a single vertex with the circuit graph \(C_{n-1}\). Note that the wheel graph is self-dual: \((Wh)_n^* = (Wh)_n\). The degree of
a vertex is the number of edges that connect to it. A homeomorphic expansion (= inflation) of a graph $G$ is obtained by inserting one or more degree-2 vertices to edges of $G$. Given a graph, the homeomorphic expansion of this graph is another graph. The inverse operation, homeomorphic reduction, consists in removing one or more degree-2 vertices from a graph. This can take one out of the class of proper graphs by resulting in a multigraph; for example, the homeomorphic reduction of $C_3$ is the multigraph $C_2$.

3 A Structural Theorem for $V_L(t)$

We next derive a basic theorem which will be important for our further analysis. It was proved in [69] that the Tutte polynomial for a recursive graph consisting of $m$ repetitions of some basic subgraph can be written in the form

$$T(G, x, y) = \sum_{j=1}^{N_{T, G, \lambda}} c_{G,j}(\lambda_{G,j})^m$$

(3.1)

where the coefficients $c_{G,j}$ and the terms $\lambda_{G,j}$ depend on the type of graph but not on $m$. It follows from this and the relation (2.1.9) for alternating links that the Jones polynomial for links $L$ whose associated graphs $G_{\pm}(L)$ are recursive has the same type of form,

$$V_L(t) = \sum_{j=1}^{N_{L, \lambda}} c_{L,j}(\lambda_{L,j})^m$$

(3.2)

where

$$N_{L, \lambda} \leq N_{T, G_{\pm}(L), \lambda}.$$  

(3.3)

The inequality is used here because when one sets $x = -t$ and $y = -1/t$, some $\lambda_{G,j}(x, y)$’s may become identically equal or may vanish. There is one case that we have considered where $N_{T, \lambda} = 10$ but when one sets $x = 1/y = -t$, two of the $\lambda_{T,j}$’s become equal, so that $N_{L, \lambda} = 9$. However, for all of the families discussed here, (3.3) is realized as an equality. A term $\lambda_{L,j}$ is dominant if its magnitude $|\lambda_{L,j}|$ is greater than the magnitudes of other terms $\lambda_{L,j'}$ occurring in (3.2). This definition is motivated by the fact that in a given region of the complex $t$ plane, as $m$ gets large, the $\lambda_{L,j}$ with largest magnitude dominates the sum to a greater and greater extent. Thus, the continuous locus $B$ where $U_L(t)$ is nonanalytic occurs where there is a switching between dominant terms $\lambda_{L,j}$ as one moves throughout the complex $t$ plane. This locus, i.e. the set of curves and possible line segments, arises as the continuous accumulation set of the zeros of $V_L(t)$. This is evident also in the work of [13]. There are also discrete points on the accumulation set, at $t = e^{\pm 2\pi i/3}$ [13].
4 The Family of Knots $A_n$ with $G_+(A_n) = D1C_{n-1}$

Let us define the (multi)graph $D1C_n$ as the result of doubling one edge (denoted as $D1$) of the circuit graph $C_n$. Note that $D1C_2 = FL_3$. These multigraphs are associated with a certain family of knots. We have found this association by observing a number of special cases and then generalizing this correspondence. Specifically, we observe that

$$G_+(31) = D1C_2$$  \hspace{1cm} (4.1)

$$G_+(41) = D1C_3$$  \hspace{1cm} (4.2)

$$G_+(52) = D1C_4$$  \hspace{1cm} (4.3)

and

$$G_+(61) = D1C_5.$$  \hspace{1cm} (4.4)

We generalize this to a family of knots with $r$ crossings, denoted $A_n$ with the property that

$$G_+(A_n) = D1C_{n-1}$$  \hspace{1cm} (4.5)

Thus, $A_3 = 31$, $A_4 = 41$, $A_5 = 52$, $A_6 = 61$, etc. As an example, we show in Fig. 1 a picture of the knot $A_9$, which is $9_2$ in the standard list. Higher members of this family are obtained by adding more twists to the bottom part of the knot, keeping the entwinement in the upper central part the same.

The knot $A_n$ has writhe

$$w(A_n) = \begin{cases} 
-n & \text{if } n \text{ is odd} \\
4 - n & \text{if } n \text{ is even} 
\end{cases}.$$  \hspace{1cm} (4.6)

The shaded diagram $D_+$ corresponding to this knot has

$$n_d(D_+(A_n)) = n - 1$$  \hspace{1cm} (4.7)

dark regions and

$$n_l(D_+(A_n)) = 3$$  \hspace{1cm} (4.8)

light regions.

We calculate the Jones polynomial for this family of knots by first calculating the Tutte polynomial for the graph $D1C_n$. We find

$$T(D1C_n, x, y) = (1 + y) \left[ y + \sum_{j=1}^{n-2} x^j \right] + x^{n-1}$$

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Using the relation (2.1.9), we then have

\[ V_{A_n}(t) = \frac{t^k}{1+t} \left[ (1+t^{-2}) + (1-t^{-1})(1+t+t^{-1})(-t)^{1-n} \right] \]  

(4.10)

where

\[ k = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
3 & \text{if } n \text{ is even} 
\end{cases} \]  

(4.11)

This is of the form (3.2) with \( N_{L,\lambda} = 2 \), the terms

\[ \lambda_{A_n,1} = 1 \]  

(4.12)

\[ \lambda_{A_n,2} = -t^{-1} \]  

(4.13)

and the corresponding coefficients

\[ c_{A_n,1} = \frac{t^k(1+t^{-2})}{1+t} \]  

(4.14)

\[ c_{A_n,2} = \frac{t^k(1-t)(1+t+t^{-1})}{1+t} \]  

(4.15)
Our formula (4.10) contains as special cases the known results for

\[ V_{A_3}(t) = t^{-4}(-1 + t + t^3) \]  \hspace{1cm} (4.16)

\[ V_{A_4}(t) = t^{-2}(1 - t + t^2 - t^3 + t^4) \]  \hspace{1cm} (4.17)

and so forth for higher \( A_n \).

Figure 2: Singular locus \( B \) for the \( n \to \infty \) limit of the family \( A_n \). For comparison, zeros of the Jones polynomial \( V_{A_n}(t) \) are shown for \( n = 50 \).

In the limit \( n \to \infty \), the continuous accumulation set \( B \) of the zeros of \( V_{A_n}(t) \) is the solution of the degeneracy equation \( |t| = 1 \), i.e., the unit circle in the complex \( t \) plane. We find that the density of the zeros on this circle is uniform. Evidently, this locus separates the \( t \) plane into two regions. Note that for this family, the discrete points given in eq. (2.2.3) lie on the continuous accumulation set. We denote the region exterior to this circle, i.e., \( |t| > 1 \) as \( R_1 \) and the interior of the circle, \( |t| < 1 \) as region \( R_2 \). In region \( R_1 \), \( \lambda_{A_n,1} \) is dominant and

\[ U(\{A\}, t) = 1 \quad \text{for} \quad t \in R_1. \] \hspace{1cm} (4.18)
Thus, as $t \to \infty$, $U$ is a constant. For other families of links, we find that $U$ can increase in magnitude without bound as $t \to \infty$. In region $R_2$, $\lambda_{A_n,2}$ is dominant and

$$|U(A, t)| = \frac{1}{|t|} \quad \text{for} \quad t \in R_2.$$  \hspace{1cm} (4.19)

$|U(A, t)|$ diverges like $1/|t|$ as $t \to 0$. Indeed, we find that for each of the families of links that we have considered, $U$ diverges as $t \to 0$ (with the power depending on the family).

5 **Families of Knots and Links $B_n$ with $G_+(B_n) = (Wh)_n$**

We observe that a number of knots and links yield wheel graphs as their associated graphs. Specifically, we find that

$$G_+(4_1) = (Wh)_3 = D1C_3 \hspace{1cm} (5.1)$$

$$G_+(6_2^3) = (Wh)_4 \hspace{1cm} (5.2)$$

and

$$G_+(8_{18}) = (Wh)_5 \hspace{1cm} (5.3)$$

A picture of the knot $8_{18}$ is shown in Fig. 3. We generalize this to the family $B_n$ of links with $G_+(B_n) = (Wh)_n$. The links in this family always have an even number of crossings, $r(B_n)$. This family has

$$n(G_+(B_n)) = n = \frac{r(B_n)}{2} + 1 \hspace{1cm} (5.4)$$

$$n_d(D_+(B_n)) = n \ell(D_+(B_n)) \hspace{1cm} (5.5)$$

and

$$w(B_n) = 0. \hspace{1cm} (5.6)$$

For the Tutte polynomial of the wheel graph $(Wh)_n$ we find, in the notation of (3.1), $N_\lambda = 3$ and, with $m = n - 1$,

$$\lambda_{Wh,1} = 1 \hspace{1cm} (5.7)$$

$$\lambda_{Wh,(2,3)} = \frac{1}{2} \left[1 + x + y \pm \left[(1 + x + y)^2 - 4xy\right]^{1/2}\right] \hspace{1cm} (5.8)$$

$$c_{Wh,1} = q - 2 = xy - x - y - 1 \hspace{1cm} (5.9)$$

and

$$c_{Wh,j} = 1 \quad \text{for} \quad j = 2, 3 \hspace{1cm} (5.10)$$

so that

$$T((Wh)_n, x, y) = xy - x - y - 1 + (\lambda_{Wh,2})^{n-1} + (\lambda_{Wh,3})^{n-1}. \hspace{1cm} (5.11)$$
Figure 3: The knot $8_{18}$, as an example of the family $B_n$ for $n = 5$.

Using the relation (2.1.9), we have

$$V_{B_n}(t) = t + t^{-1} + (\lambda_{W,h,2})^{n-1} + (\lambda_{W,h,3})^{n-1}$$

(5.12)

where

$$\lambda_{W,h,(2,3)} = \frac{1}{2} \left[ 1 - t - t^{-1} \pm \left( -1 - 2(t + t^{-1}) + t^2 + t^{-2} \right)^{1/2} \right].$$

(5.13)

Note the symmetry

$$V_{B_n}(t) = V_{B_n}(t^{-1})$$

(5.14)

which follows from the self-dual property of the wheel graphs and the symmetry (2.1.8) of the Tutte polynomial. Our general formula subsumes the following special cases for $B_3 = 4_1$, $B_5 = 8_{18}$:

$$V_{B_3} = t^{-2}(1 - t + t^2 - t^3 + t^4)$$

(5.15)

$$V_{B_5} = t^{-4}(1 - 4t + 6t^2 - 7t^3 + 9t^4 - 7t^5 + 6t^6 - 4t^7 + t^8).$$

(5.16)

The locus $\mathcal{B}$ is determined by the degeneracy of leading $\lambda$’s, in this case simply by $|\lambda_{W,h,2}| = |\lambda_{W,h,3}|$ and consists of the union of an arc of the unit circle with a line segment on the positive real axis as shown in Fig. 4:

$$\mathcal{B}: \quad \{ t = e^{i\theta}, \quad -\frac{2\pi}{3} \leq \theta \leq \frac{2\pi}{3} \} \cup \{ \frac{3 - \sqrt{5}}{2} \leq t \leq \frac{3 + \sqrt{5}}{2} \}. \quad (5.17)$$
Figure 4: Singular locus $\mathcal{B}$ for the $n \to \infty$ limit of the family $B_n$. For comparison, zeros of the Jones polynomial $V_{B_n}(t)$ are shown for $n = 42$. 
Numerically, \((3 - \sqrt{5})/2 = 0.381966\ldots\) and \((3 + \sqrt{5})/2 = 2.61803\ldots\) The four endpoints of \(\mathcal{B}\) occur where the numerator of the function in the square root, \(1 - 2t - t^2 - 2t^3 + t^4 = (t^2 + t + 1)(t^2 - 3t + 1)\) vanishes. Note that the endpoints of the circular arc occur precisely at the special points in eq. (2.2.3). Evidently, the locus \(\mathcal{B}\) does not separate the \(t\) plane into different regions. On the positive real axis for \(t > (3 + \sqrt{5})/2\), the dominant term is \(\lambda_{W,n,3}\) and hence
\[
U(\{B\}, t) = -\lambda_{W,n,3} .
\] (5.18)
This expression applies throughout the complex \(t\) plane, taking account of the fact that the sign of the square root reverses when one passes across the branch cut associated with this square root, which is the same here as the locus \(\mathcal{B}\). Thus, in the vicinity of the origin, the dominant term is \(\lambda_{W,n,3}\) and \(U(\{B\}, t)\) diverges like \(t^{-1}\) as \(t \to 0\). As \(t \to \infty\), \(U\) again diverges, like \(t\).

6 Family of Links \(E_n\) with \(G_+(E_n) = H_{3,n}\)

In earlier work on continuous accumulation sets \(\mathcal{B}\) of zeros of chromatic polynomials as the number of vertices of a graph in some family goes to infinity, a certain family of graphs called “hammock” graphs were studied \[55\] and it was shown that the locus \(\mathcal{B}\) for these graphs was noncompact in the \(q\) plane, i.e. passed through the origin of the \(1/q\) plane. Thus, the zeros of the chromatic polynomials for these graphs have magnitudes that are unbounded as one takes the number of vertices to infinity \[53, 54\]. The hammock graph \(H_{k,\ell}\) is defined as follows: start with two end vertices and connect these together by \(\ell\) “ropes”, each consisting of \(k\) vertices, counting the end vertices. The dual of the graph \(H_{3,\ell}\) is a multigraph which we denote \(DC_{\ell}\):
\[
(H_{3,\ell})^* = DC_{\ell} .
\] (6.1)
Here, \(DC_{\ell}\) is the multigraph obtained by starting with a circuit graph and doubling each edge. Note that \(DC_2 = FL_4\) and \(FL_4 = C_4 = H_{3,2}\). We calculate the correspondences
\[
G_+(4^2_1) = DC_2 , \ i.e., \ G_-(4^2_1) = H_{3,2} \] (6.2)
\[
G_+(6^3_1) = H_{3,3} , \ i.e., \ G_-(6^3_1) = DC_3 \] (6.3)
\[
G_+(8^4_1) = H_{3,4} , \ i.e., \ G_-(8^4_1) = DC_4 . \] (6.4)
We denote \(E_2 = R(4^2_1)\), \(E_3 = 6^3_1\), and \(E_4 = 8^4_1\) and generalize this to the family of links \(E_n\) with the property
\[
G_+(E_n) = H_{3,n} , \ i.e., \ G_-(E_n) = DC_n . \] (6.5)
The links in this family always have an even number \( r(E_n) \) of crossings. We note that the number of vertices in the associated graph, \( |V(G_+(E_n))| \), is given by

\[
|V(G_+(E_n))| = \frac{r(E_n)}{2} + 2 = n + 2 .
\]

(6.6)

In Fig. 5 we show a picture of the knot \( 8^4_1 \) as an example of this family.

![Figure 5: The knot 8^4_1 as an example of the family \( E_n \) for \( n = 4 \).](image)

The family \( E_n \) has

\[
n_d(D_+(E_n)) = n + 2 \quad \text{(6.7)}
\]

\[
n_t(D_+(E_n)) = n . \quad \text{(6.8)}
\]

For a link containing \( n_c \geq 2 \) intertwined components (in contrast to a knot), the writhe depends on the orientations chosen for these components (circles) and we follow the convention that for a given link \( L \), \([L]_1\) refers to the choice of orientations with the smallest value of the writhe, \([L]_2\) to the next smallest, and so forth. With an appropriate convention, we have

\[
w([E_n]_1) = -2n . \quad \text{(6.9)}
\]

For the Tutte polynomial for the hammock graph \( H_{3,n} \) we find \( N_{T,\lambda} = 2 \) and

\[
T(H_{3,n}, x, y) = \frac{1}{(y-1)} \left[ (xy - x - y)(1 + x)^n + (x + y)^n \right] .
\]

(6.10)
Using the relation (2.1.9), we find
\[ V_{E_n}(t) = \frac{-t^{1/2}}{1+t} \left[ (1+t+t^{-1}) \left( \frac{1-t}{t^{3/2}} \right)^n + (-1)^n \left( \frac{t+t^{-1}}{t^{3/2}} \right)^n \right]. \] (6.11)

This is of the form (3.2) with the terms
\[ \lambda_{E_n,1} = \frac{1}{\sqrt{t}}(-1 + t^{-1}) \] (6.12)
\[ \lambda_{E_n,2} = -\frac{1}{\sqrt{t}}(1 + t^{-2}) \] (6.13)
and corresponding coefficients
\[ c_{E_n,1} = -\frac{t^{1/2}(1+t+t^{-1})}{(1+t)} \] (6.14)
\[ c_{E_n,2} = -\frac{t^{1/2}}{1+t}. \] (6.15)

This yields the special cases
\[ V_{\left[R(\Delta^2)\right]_1}(t) = t^{-9/2}(-1 - t^2 + t^3 - t^4) \] (6.16)
\[ V_{\left[6\Delta^2\right]_1}(t) = t^{-7}(1 - t + 3t^2 - t^3 + 3t^4 - 2t^5 + t^6). \] (6.17)

The continuous locus \( B \) is determined by the degeneracy of magnitudes \(| -1 + t^{-1}| = |1 + t^{-2}|\). Let us introduce polar coordinates
\[ t = re^{i\theta} \] (6.18)

The locus \( B \) is described by the equation
\[ -1 + \frac{1}{r^2} + 2r \cos \theta + 2 \cos(2\theta) = 0 \] (6.19)
i.e.,
\[ \cos \theta = \frac{1}{4} \left[ -r + \left( r^2 + 12 - \frac{4}{r^2} \right)^{1/2} \right]. \] (6.20)

This curve passes through \( t = -1 \), crosses the imaginary axis at \( t = \pm \frac{i}{\sqrt{3}} \), passes into the \( Re(t) > 0 \) half-plane, then bends back toward the imaginary axis and extends upward and downward to \( \pm i\infty \). Asymptotically,
\[ \cos \theta = \frac{3}{2r} + O\left( \frac{1}{r^3} \right) \quad \text{as} \quad r \to \infty. \] (6.21)
A portion of this curve is shown in Fig. 6. Thus, the locus $B$ is noncompact in the $t$ plane, and this family of knots shows that the zeros of Jones polynomials have magnitudes that are not necessarily bounded. In Fig. 6 we show $B$ in the $t$ plane together with zeros calculated for a link with $n = 42$ crossings. These zeros lie close to the asymptotic locus $B$ but, as expected for finite $n$, do not continue to track near it as it extends arbitrary far from the origin. This is similar to the behavior that was found for the comparison of the location of chromatic zeros and their asymptotic accumulation set for families of graphs where this set was unbounded [48, 18, 53, 54, 55, 56]. Another noteworthy feature of the locus $B$ for this family is that it does not cross the positive real $t$ axis, thereby showing that this is not a necessary property of such a locus. Evidently, the locus $B$ divides the $t$ plane into two different regions. For this family the special discrete points on the accumulation set at the values given in eq. (2.2.3) are separate from the continuous accumulation set $B$.

In the respective regions $R_1$ and $R_2$ to the right and left of $B$, $\lambda_{E_n,2}$ and $\lambda_{E_n,1}$ are dominant, so that

$$U(\{E\}, t) = -\lambda_{E_n,2} \quad \text{for} \quad t \in R_1$$

$$|U(\{E\}, t)| = |\lambda_{E_n,1}| \quad \text{for} \quad t \in R_2$$

$U(\{E\}, t)$ diverges like $t^{-5/2}$ as $t \to 0$ and vanishes like $t^{-1/2}$ as $t \to \infty$.

In the $s = 1/t$ plane, the image of $B$ is compact, forming a closed curve passing through $s = -1$ and $s = 0$. Using the relation (2.1.5), it follows that the image of $B$ in the $s$ plane is identical to the locus $B$ for the family $R(\{E\})$ in the $t$ plane. We show this in Fig. 7.

7 Families of Knots $F_n$ with $G(F_n) = HW_n$

We define the graph $HW_n$ as the graph obtained by starting with the wheel graph $(Wh)_m$ and performing a single homeomorphic expansion on each edge forming a spoke of the wheel, where, as a result,

$$n = 2m + 1$$

is necessarily odd. We observe the correspondences $G_+(6_1) = HW_5$ and generalize this to a family of knots $F_n$ with

$$G(F_n) = HW_n.$$  \hspace{1cm} (7.2)

Thus, the next member of the family beyond $F_5 = 6_1$ is $F_7 = 9_{41}$, and so forth for higher members. An example of this family is shown in Fig. 8.

The knot $F_n$ has writhe

$$w(F_n) = -\frac{(n - 1)}{2}. \hspace{1cm} (7.3)$$
Figure 6: Singular locus $\mathcal{B}$ for the $n \to \infty$ limit of the family $E_n$. For comparison, zeros of the Jones polynomial $V_{E_n}(t)$ are shown for $n = 42$. 
Figure 7: Singular locus $\mathcal{B}$ for the $n \to \infty$ limit of the family $R(E_n)$ in the $t$ plane, or equivalently of the family $E_n$ in the $1/t$ plane. For comparison, zeros of the Jones polynomial $V_{R(E_n)}(t)$ are shown for $n = 42$.

Figure 8: The knot $9_{41}$ as an example of the family $F_n$ for $n = 7$. 

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The shaded diagram $D_+$ corresponding to this knot has

$$n_d(D_+(F_n)) = n$$  \hspace{1cm} (7.4)$$
dark regions and

$$n_l(D_+(F_n)) = \frac{n + 1}{2}$$  \hspace{1cm} (7.5)$$
light regions.

We calculate the Jones polynomial for this family of knots by first calculating the Tutte polynomial for the graph $HW_n$. We find

$$T(HW_n, x, y) = (xy - x - y - 1)(x + 1)^m + \lambda_{HW,1}^m + \lambda_{HW,2}^m$$  \hspace{1cm} (7.6)$$
where $m = (n - 1)/2$ as given by eq. (7.1) and

$$\lambda_{HW,(1,2)} = \frac{1}{2} \left[ 1 + 2x + x^2 + y \pm \left[ (1 + 2x + x^2 + y)^2 - 4x(x + 1)(x + y) \right]^{1/2} \right]$$  \hspace{1cm} (7.7)$$
Using the relation (2.1.9), we then have

$$V_{F_n}(t) = 1 + (t + t^{-1}) (1 - t^{-1})^m + (t^{-2} - t^{-1} + 1 - t)^m$$  \hspace{1cm} (7.8)$$
This is of the form (3.2) with $N_{L,\lambda} = 3$, with the terms

$$\lambda_{F_n,1} = 1$$  \hspace{1cm} (7.9)$$
$$\lambda_{F_n,2} = 1 - t^{-1}$$  \hspace{1cm} (7.10)$$
$$\lambda_{F_n,3} = t^{-2} - t^{-1} + 1 - t$$  \hspace{1cm} (7.11)$$
and the corresponding coefficients

$$c_{F_n,1} = 1$$  \hspace{1cm} (7.12)$$
$$c_{F_n,2} = t + t^{-1}$$  \hspace{1cm} (7.13)$$
and

$$c_{F_n,3} = 1 .$$  \hspace{1cm} (7.14)$$
Our formula (7.8) contains as special cases the known results for

$$V_{F_5}(t) = t^{-4}(1 - t + t^2 - 2t^3 + 2t^4 - t^5 + t^6)$$  \hspace{1cm} (7.15)$$
$$V_{F_7}(t) = t^{-6}(1 - 3t + 5t^2 - 7t^3 + 8t^4 - 8t^5 + 8t^6 - 5t^7 + 3t^8 - t^9)$$  \hspace{1cm} (7.16)$$
and so forth for higher $F_n$.  

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From these results, we have then calculated the zeros of $V_{F_n}(t)$ and the singular locus $\mathcal{B}$, which is shown in Fig. [2.3]. This locus divides the $t$ plane into six regions, of which two are self-conjugate and the other four comprise two complex-conjugate pairs. The two special points (2.2.3) lie on two of the ovals comprising the locus $\mathcal{B}$. The right-hand oval is the solution to the degeneracy equation $|\lambda_{F_n,1}| = |\lambda_{F_n,3}|$; in the region $R_1$ outside of this oval, $\lambda_{F_n,3}$ is dominant, while inside the oval, in a region that we denote as $R_2$, $\lambda_{F_n,1}$ is dominant. The large upper and lower ovals are, apart from the small bubble regions, the solution to the equation $|\lambda_{F_n,2}| = |\lambda_{F_n,3}|$ and in the interior of these large ovals, which we call regions $R_3$ and $R_3^*$, $\lambda_{F_n,2}$ is dominant. Finally, in the small bubble phases on the lower (upper) right part of the upper (lower) oval, which we denote $R_4$ and $R_4^*$, $\lambda_{F_n,1}$ is dominant. We thus have

$$U(\{F\}, t) = (-\lambda_{F_n,3})^{1/2} \quad \text{for} \quad t \in R_1$$

$$|U(\{F\}, t)| = |\lambda_{F_n,1}|^{1/2} = 1 \quad \text{for} \quad t \in R_2, R_4, R_4^*$$

$$|U(\{F\}, t)| = |\lambda_{F_n,2}|^{1/2} \quad \text{for} \quad t \in R_3, R_3^*$$

$|U(\{F\}, t)|$ diverges like $|t|^{-1}$ as $t \to 0$. The locus $\mathcal{B}$ intersects the real axis at the two points 0.6823278... and 1.754877... (solutions of the cubic degeneracy equations given above). As $t \to \infty$, $U$ diverges like $t^{1/2}$. Thus, the behavior of $U$ for large positive real $t$ varies widely among the families of links that we have studied, ranging from vanishing, to being a constant, to diverging.

8 $V_L(t)$ for non-alternating links

For a non-alternating knot or link, one does not have a simple relation between the Jones polynomial $V_L(t)$ and the Tutte polynomial as given in eq. (2.1.9). In this case, there are two types of edges, $e_{\pm}(G)$, in the associated graph instead of just one type of edge as in the case of an alternating link. In general, the Jones polynomial for any link can be expressed in term of Potts model partition function $[3]$ as

$$V_L(t) = q^{-1/2}(-t^{-3/4})^{-w}Z(W_+, W_-)$$

where

$$W_\pm(a, b) = A_\pm(1 + v_\pm \delta_{ab})$$

and

$$v_\pm = e^{K_\pm} - 1.$$
Figure 9: Singular locus $\mathcal{B}$ for the $n \to \infty$ limit of the family $F_n$. For comparison, zeros of the Jones polynomial $V_{F_n}(t)$ are shown for $n = 63$. 
The definition of $Z(W_+, W_-)$ related to the usual $Z(G, q, v_\pm)$ by

$$Z(W_+, W_-) = A_+^{e_+(G)} A_-^{e_-(G)} q^{-n(G)/2} Z(G, q, v_\pm) \quad (8.4)$$

where $n(G)$ is the total vertex number of the associated graph. To satisfy all three Reidemeister moves, we have to set $A_\pm = t^{\pm 1/4}$, $v_\pm = -(1 + t^\mp)$, and $q = t + 2 + 1/t$. For alternating links, all the edges are either $+$ type or $-$ type. Here we have $K_+ = -K_-$, which means that we have both positive and negative spin-spin interaction with the same magnitude. The relation between $v_+$ and $v_-$ is that

$$v_\pm = -v_{\mp} \quad (8.5)$$

Therefore, we have to modify eq. (2.4.4) to be

$$Z(G, q, v_-) = \sum_{G' \subseteq G} q^{k(G')} v_-^{e_-(G')} \left( \frac{-1}{1 + v_-} \right)^{e_+(G')} \quad (8.6)$$

or

$$Z(G, q, v_+) = \sum_{G' \subseteq G} q^{k(G')} v_+^{e_+(G')} \left( \frac{-1}{1 + v_+} \right)^{e_-(G')} \quad (8.7)$$

where $e_\pm(G')$ is the number of plus (minus) type edges in $G'$. Equivalently, we have

$$T(G_{na}, x, y) = \sum_{G' \subseteq G} (x-1)^{k(G')-k(G)} (y-1)^{c(G')} \left( \frac{-1}{y} \right)^{e'(G')} \quad (8.8)$$

where $e'(G')$ is the number of plus type edges in $G'$ if $y = -t$, and the number of minus type edges in $G'$ if $y = -1/t$. Without loss of generality, consider the associated graph $G$ such that the majority of its edges are plus type. If all vertices of $G$ are connected by at least one of the plus edges, which can be achieved most of the time by performing the type III Reidemeister move, one can rewrite eq. (8.8) as

$$T(G_{na}, x, y) = \sum_{\{e_-\}} \Delta T(G_{\{e_-\}}, x, y) \left( \frac{-1}{y} \right)^{e_-} \quad (8.9)$$

where we sum all possible minus type edge set $\{e_-\}$, and $\Delta T(G_{\{e_-\}}, x, y)$ is the difference of Tutte polynomials for the graph with and without a particular edge set $\{e_-\}$. Combining eqs. (2.4.8), (8.4), and (8.4), we have

$$V(t)_{na} = (-t^{3/4}) w(t^{(2-2n(G)+e_+(G)-e_-(G))}/4) T(G_{na}, -t, -1/t) = (-t^{3/4}) w(t^{(-2+2n(G)+e_+(G)-e_-(G))}/4) T(G_{na}, -1/t, -t) \quad (8.10)$$
Another way to deal with non-alternating knots or links is even simpler and makes use of the skein relation for the Jones polynomial,

\[
\frac{1}{t} V_{L_+}(t) - t V_{L_-}(t) = \left[ \sqrt{t} - \frac{1}{\sqrt{t}} \right] V_{L_0}(t)
\]

(8.11)

where the associated graph for \( L_0 \) corresponds to (i) the graph with a contraction of the edge considered if the definition of ± crossings is the same for the oriented links and that for the shading, and (ii) the graph with a deletion operation performed on this edge if these two definitions are different. The Jones polynomial can be obtained by using this relation to flip the sign of the crossings until all the graphs become alternating, after which one can apply eq. (2.1.9).

9 A Remark on Computational Complexity

The question of the computational complexity associated with determining basic properties of knots and links, i.e., roughly speaking, the number of steps needed to obtain this information, is a subject of considerable interest in pure and applied mathematics and computer science (e.g., [9]). Recently, for example, an upper bound has been given for the number of Reidemeister moves (RM) needed to transform the unknot to a circle with no crossings, where the unknot is defined as the trivial knot [72]. This bound is of the form \( N_{RM} \leq 2^{c_1 r} \), where, as before, \( r \) is the number of crossings and \( c_1 \) is a certain (large) constant. We would like to point out here that for the families of knots and links for which we have calculated the Jones polynomial \( V_L(t) \), our method, using the intermediate step of computing the Tutte polynomial of the associated graph \( G_+(L) \), requires a polynomial, rather than exponential number of steps. Specifically, we use the standard iterative use of the deletion-contraction theorem for the Tutte polynomial. This difference in computational complexity is quite similar to the situation for the Potts model partition function: a formal evaluation of the sum over states for this model on an arbitrary graph involves exponentially many steps since there are exponentially many spin configurations. However, by means of the transfer matrix method or the related iterated deletion-contraction theorem for the equivalent Tutte polynomial, one can, for specific families of graphs, carry out the evaluation in a polynomial number of steps, since the partition function is just the trace of a power of the transfer matrix.

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