ARITHMETIC AND ANALYSIS OF THE SERIES \[ \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n} \]

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To the memory of our friend Carlos Berenstein

Abstract. In this paper we connect a celebrated theorem of Nyman and Beurling on the equivalence between the Riemann hypothesis and the density of some functional space in \( L^2(0, 1) \) to a trigonometric series considered first by Hardy and Littlewood. We highlight some of its curious analytical and arithmetical properties.

1. Introduction

The main purpose of this work is to bring to light a new relationship between two facets of Riemann’s zeta function: On the one hand a functional analysis approach to the Riemann hypothesis due to Nyman and Beurling, and on the other hand a trigonometric series first studied by Hardy and Littlewood [16], and then followed by Flett [15], Segal [25] and Delange [13]. The trigonometric series in question is

\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{x}{n} \right). \]

It differs from the finite sum \( \sum_{n \leq x} \frac{1}{n} \sin \left( \frac{x}{n} \right) \), as \( x \) tends to \( \infty \), by

\[ \sum_{n>1} \frac{1}{n} \sin \left( \frac{x}{n} \right) = O \left( \sum_{n>x} \frac{x}{n^2} \right) = O(1). \]

Hardy and Littlewood proved [16] that, as \( x \) tends to \( \infty \),

\[ f(x) = O \left( (\log x)^{\frac{3}{4}} (\log \log x)^{\frac{3}{4} + \epsilon} \right) \]

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and that
\[
f(x) = \Omega \left( (\log \log x)^{\frac{3}{4}} \right)
\]
from the fact that for \( x \geq 5 \), the number of \( n \leq x \) whose prime divisors are equivalent to 1 modulo 4 is \( C \frac{\log \log x}{\log x} \), where \( C \) is a constant. Delange \cite{13} showed that \( f(x) \) is not bounded on the real line only from the following result on the reciprocals of primes in arithmetic progressions
\[
\sum_{p \text{ prime}, \ p \equiv 1(\text{mod} \ 4)} \frac{1}{p} = \infty
\]
and obtained the \( \Omega \)-result of Hardy and Littlewood just because
\[
\sum_{p \text{ prime} \leq x, \ p \equiv 1(\text{mod} \ 4)} \frac{1}{p} = \frac{1}{2} \log \log x + c + o(1).
\]

This trigonometric series, despite its simplicity, has many similarities with the Riemann zeta function \cite{15} and deep relation to the divisor functions through the sawtooth function
\[
\{t\} = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin 2m\pi t}{m} = \begin{cases} t - \lfloor t \rfloor - \frac{1}{2} & \text{if } t \neq \lfloor t \rfloor \\ 0 & \text{if } t = \lfloor t \rfloor. \end{cases}
\]

For \( s \in \mathbb{C} \) we define
\[
\sigma^{s}(n) = \sum_{d|n} d^{s}, \quad \sigma_{s}(n) = \sum_{d|n} d^{-s}
\]
so that \( n^{s} \sigma_{s}(n) = \sigma^{s}(n) \). For example if we define
\[
S_{1}(x) = \sum_{n \leq x} \sigma_{1}(n), \quad S^{1}(x) = \sum_{n \leq x} \sigma^{1}(n)
\]
and
\[
\rho(x) = \sum_{n \leq x} \frac{1}{n} \left\{ \frac{x}{n} \right\} = \sum_{n \leq x} \frac{1}{n} \left( \frac{x}{n} - \left\lfloor \frac{x}{n} \right\rfloor - \frac{1}{2} \right)
\]
then the divisors and the fractional parts functions are related by
\[
S_{1}(x) = \sum_{n \leq x} \frac{1}{n} \left\lfloor \frac{x}{n} \right\rfloor = x \sum_{n \leq x} \frac{1}{n^2} - \rho(x)
\]
\[
= \frac{\pi^{2}}{2} x - \frac{1}{2} \log x - \rho(x) + O(1).
\]
Similarly \cite{32} (p.70):
\[
S^{1}(x) = \frac{\pi^{2}}{12} x^2 - x \rho(x) + O(x).
\]
We will see (3.5) with \( f(2\pi x) = \sin x \) an integral representation of the partial sums of \( f(x) \), using the sawtooth function.

2. NYMAN-BEURLING CRITERION FOR THE RIEMANN HYPOTHESIS

2.0.1. Nyman-Beurling theorem. For \( x > 0 \), let \( \rho(x) \) be the fractional part of \( x \) so that \( \rho(x) = x - \lfloor x \rfloor \). To each \( 0 < \theta \leq 1 \) we associate the function \( \rho_\theta(x) = \rho(\theta x) \).

Then \( 0 \leq \rho_\theta(x) \leq 1 \) and \( \rho_\theta(x) = \frac{\theta}{x} \) if \( \theta < x \). We introduce, as in [6], [14], [3], [4], [5], [22], [29] and the more recent book [23]

\[
\mathcal{M} = \left\{ f, f(x) = \sum_{n=1}^{N} a_n \rho(\theta_n x), a_n \in \mathbb{R}, \theta \in (0, 1], \sum_{n=1}^{N} a_n \theta_n = 0, N \geq 1 \right\}
\]

Each function in \( \mathcal{M} \) has at most a countable set of points of discontinuity, and is identically zero for \( x > 0 \).

**Theorem 2.1 (Nyman-Beurling).** Let \( 1 < p \leq \infty \). The subspace \( \mathcal{M} \) is dense in the Banach space \( L^p(0, 1) \) if and only if the Riemann zeta function \( \zeta(s) \) has no zero in the right half plane \( \text{Re}s > 1 \).

The fundamental relations in the proof of this theorem are

\[
\int_{0}^{1} \rho(\theta x) x^{s-1} dx = -\frac{\theta}{1-s} - \theta^s \frac{\zeta(s)}{s}, \quad \text{Res} > 1
\]

which is just a variant of the classical representation

\[
\zeta(s) = \frac{s}{s-1} - s \int_{0}^{\infty} \frac{u - \lfloor u \rfloor}{(u+1)^{s+1}} du
\]

It follows from (2.1) that for \( f(x) \in \mathcal{M} \)

\[
\int_{0}^{1} f(x) x^{s-1} dx = -\frac{\zeta(s)}{s} \sum_{k=1}^{N} a_k \theta_k^s.
\]

The study of the function \( f(x) \) is intimately linked to that of following function

\[
\{t\} = \begin{cases} 
  t - \lfloor t \rfloor - \frac{1}{2} & \text{if } t \neq \lfloor t \rfloor \\
  0 & \text{if } t = \lfloor t \rfloor.
\end{cases}
\]

We have the formal Fourier series expansion [11], [12]

\[
\sum_{n=1}^{\infty} \frac{a_n}{n} \{n\theta\} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{A_n}{n} \{\sin 2\pi n\theta\}
\]
where

\[ A_n = \sum_{d|n} a_d. \]

Davenport considered the cases of

\[ a_n = \mu(n); \quad a_n = \lambda(n); \quad a_n = \Lambda(n); \quad a_{n^2} = \mu(n), a_n = 0, \quad n \neq m^2. \]

These arithmetical functions have their usual number-theoretic meanings. For example if \( \omega(n) \) is the number of distinct prime factors of \( n \) or in other terms \( \omega(n) = \sum_{b|n} 1 \)

and \( \omega(1) = 0 \), then the Möbius function \( \mu(n) \) is defined by

\[
\mu(n) = \begin{cases} 
0 & \text{if } n \text{ is divisible by a perfect square } > 1 \\
(-1)^{\omega(n)} & \text{otherwise}
\end{cases}
\]

and the Von Mangoldt function \( \Lambda(n) \) is defined by

\[
\Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^\alpha \text{ for a prime } p \text{ and some } \alpha \in \mathbb{N} \\
0 & \text{otherwise}
\end{cases}
\]

In the case of the Möbius function \( a_n = \mu_n \), Davenport uses Vinogradov’s method, a refinement of Weyl’s method on estimating trigonometric sums, to prove that for any fixed \( h \)

\[ \sum_{n \leq y} \mu(n) e^{2\pi i n x} = O(y(\log y)^{-h}) \]

uniformly in \( x \in \mathbb{R}/\mathbb{Z} \). The implied constants are not effective. There have been several results justifying (2.3) for other particular sequences \( (a_n) \). The most general problem is considered in [17].

It should be noted that the Davenport or Hardy Littlewood estimates admit a common analysis. For the convenience of the reader we gather together a few classical results on exponential sums. Let \( I \) be an interval of length at most \( N \geq 1 \) and let \( f : I \to \mathbb{R} \) be a smooth function satisfying the estimates \( x \in I, 2 \leq N \ll T, j \geq 1 \)

\[ |f^{(j)}(x)| = \exp \left( O(j^2) \right) \frac{T}{N^j} \]

then with \( f(x) = e^{x} \)

1. Van der Corput estimate: For any natural number \( k \geq 2 \), we have

\[ \frac{1}{N} \sum_{n \in I} e(f(n)) = O \left( \frac{T}{N^k} \frac{1}{\log \frac{1}{2}(2 + T)} \right) \]
ARITHMETIC AND ANALYSIS OF THE SERIES $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n}$

(2) Vinogradov estimate: For some absolute constant $c > 0$,

$$\frac{1}{N} \sum_{n \in I} e(f(n)) \ll N^{-\frac{c}{4}}$$

2.0.2. The functions $[x]$, $\rho(x)$ and $\{x\}$. The Hardy-Littlewood-Flett function $f(x)$ is related, in many ways, to the three functions $[x]$, $\rho(x)$ and $\{x\}$. The floor function $[x]$ is related to the divisor function $d(n) = 1 \ast 1(n)$, the multiplicative square convolution product of the constant function $1$, through the Dirichlet hyperbola method. More generally if $g, h$ are two multiplicative functions and $f = g \ast h$. The Dirichlet hyperbola method is just the evaluation of a sum in two different ways:

$$\sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{ab = n} g(a)h(b) = \sum_{a \leq \sqrt{x}} \sum_{b \leq \frac{x}{a}} g(a)h(b) + \sum_{b \leq \sqrt{x}} \sum_{a \leq \frac{x}{b}} g(a)h(b) - \sum_{a \leq \sqrt{x}} \sum_{b \leq \sqrt{x}} g(a)h(b).$$

If $g = h$, then

$$\sum_{n \leq x} f(n) = 2 \sum_{a \leq \sqrt{x}} \sum_{b \leq \frac{x}{a}} g(a)h(b) - \left( \sum_{a \leq \sqrt{x}} g(a) \right)^2.$$

As an application we have the estimate [28] (p. 262) for the divisor function $d = 1 \ast 1$:

$$d(x) = x \log x + (2\gamma - 1)x + O(x^{\frac{1}{4}}).$$

The importance of the functions $\{x\}$ and $\rho(x)$ lies in the integral representations of the Riemann zeta-function:

$$\zeta(s) = -s \int_{0}^{\infty} \frac{\{x\} - \frac{1}{x^{s+1}}}{dx} = -s \int_{0}^{\infty} \frac{\rho(x)}{x^{s+1}} dx$$

valid for $-1 < \text{Re} s < 0$. Making the change of variable $x = \frac{1}{u}$ and applying Mellin inversion formula gives

$$\rho\left(\frac{1}{u}\right) = -\frac{1}{2i\pi} \int_{c-i \infty}^{c+i \infty} \frac{\zeta(s)}{s} u^{-s} ds.$$

For later use, we give some details on the case considered by Davenport in (2.3). From (2.4) we obtain for $-1 < c < 0$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \rho(nx) = -\frac{1}{2i\pi} \int_{c-i \infty}^{c+i \infty} \frac{\zeta(s)}{s \zeta(1-s)} x^{s} ds.$$
By the functional equation of the Riemann \(\zeta\)-function and the functional equation of the \(\Gamma\)-function we obtain for \(0 < a < 1\)
\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \rho(nx) = -\frac{1}{2i\pi^2} \int_{a-i \infty}^{a+i \infty} \Gamma(s) \sin\left(\frac{1}{2} \pi s\right)(2\pi x)^{-s} ds = -\frac{1}{\pi} \sin(2\pi x).
\]

Using the classical equivalent formulation of the Prime Number Theorem that \(\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0\) we obtain Davenport’s relation

\[(2.7)\]
\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \{nx\} = -\frac{1}{\pi} \sin(2\pi x)
\]

where the convergence is uniform by Davenport estimate \((2.4)\). We will need two important properties of the function \(\{x\}\):

Kubert identity:

\[(2.8)\]
\[
\sum_{l \mod m} \left\{ x + \frac{l}{m} \right\} = \{mx\}
\]

Franel formula:

\[(2.9)\]
\[
\int_{0}^{1} \{ax\} \{bx\} = \frac{\text{lcm}(a, b)}{12ab}
\]

Kubert identity and Franel’s formula are interesting features shared by many functions. Let \(B_r(x)\) be the Bernoulli polynomial defined by
\[
\frac{te^{tx}}{e^t - 1} = \sum_{r=0}^{\infty} B_r(x)t^r, \quad |t| < 2\pi,
\]
so that
\[
B_1(x) = x - \frac{1}{2}, \quad 2!B_2(x) = x^2 - x + \frac{1}{6}, \cdots
\]
If \(r \geq 2\) is even then for \(0 \leq x \leq 1\)
\[
(2\pi)^r B_r(x) = (-1)^{\frac{r}{2} - 1} \sum_{l=1}^{\infty} \frac{2 \cos(2l \pi x)}{l^r}
\]
with absolute convergence of the series. The Hurwitz zeta function \(\zeta(s, x)\) is defined for \(\text{Re}\ s > 1\) by
\[
\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(x + n)^s}.
\]
Then \(B_r(x)\) and \(\zeta(s, x)\) both satisfy the functional equation \([21]\)
\[
f(x) + f(x + \frac{1}{k}) + \cdots + f(x + \frac{k-1}{k}) = f^{(k)}(kx),
\]
where \( f^{(k)} = k^{1-n} \) if \( f(x) = B_n(x) \) and \( f^{(k)} = k^s \) if \( f(x) = \zeta(s, x) \). Furthermore
\[
\int_0^1 B_r(ax) B_r(bx) \, dx = (-1)^{r-1} \frac{B_{2r}}{(2r)!} \left( \frac{(a, b)}{[a, b]} \right)^r
\]
and for \( \text{Re} s > \frac{1}{2} \)
\[
\int_0^1 \zeta(1-s, ax) \zeta(1-s, bx) \, dx = \frac{2\Gamma^2(s)\zeta(2s)}{(2\pi)^2s} \left( \frac{(a, b)}{[a, b]} \right)^s.
\]

Similarly to (2.2) we have
\[
\zeta(s, w) = \frac{1}{(s-1)w^{s-1}} + \frac{1}{w^s} - s \int_0^\infty \frac{u - \lfloor u \rfloor}{(u + w)^{s+1}} \, du,
\]
the function \( \zeta(s, w) - \frac{1}{(s-1)w^{s-1}} \) being analytic in \( \{ \text{Re} s > 0 \} \). In the next section we use two summation formulas

If \( F \) is an antiderivative of \( f \), then, formally
\[
(2.10) \quad \int_0^1 \rho(\theta t)f(t) \, dt = \int_0^\theta \frac{f(t)}{t} \, dt + \sum_{p=1}^\infty n \left( F\left( \frac{\theta}{n} \right) - F\left( \frac{\theta}{n + 1} \right) \right)
\]
and if \( \mu \) is the Möbius function and if \( 0 < \theta, x \leq 1 \), we have, pointwise
\[
\sum_{n=1}^\infty \mu(n) \left\{ \rho \left( \frac{\theta}{nx} \right) - \frac{1}{n} \rho \left( \frac{\theta}{x} \right) \right\} = -\chi_{[0,\theta]}(x)
\]

3. From Beurling’s theorem to Hardy-Littlewood-Flett function \( \hat{f}(x) \)

3.1. The emergence of Franel integral type. To show that the constant function \( 1 \in \mathcal{M} \) one has, as in [1], to minimize the norms in \( L^2([0, 1]) \)
\[
(3.1) \quad \| 1 + \sum_{j=1}^N a_j \rho \left( \frac{\alpha_j}{x} \right) \|
\]
which brings back to the evaluation of integrals of Franel type, computed in [1]:
\[
J(\beta) = \int_0^1 \rho \left( \frac{1}{x} \right) \rho \left( \frac{\beta}{x} \right) \, dx, \quad \beta \in [0, 1].
\]
To show that the function \( \sin x \in \mathcal{M} \) one has, this time, to minimize the norms
\[
(3.2) \quad \| \sin x + \sum_{j=1}^N a_j \rho \left( \frac{\alpha_j}{x} \right) \|
Using (2.7) the minimization problem reduces to evaluation of the scalar products in $L^2(0, 1)$

$$(\{\frac{\theta}{x}\} | \sqrt{2} \sin(n\pi x)) = \sqrt{2} \int_0^1 \{\frac{\theta}{x}\} \sin(n\pi x) dx = -\pi \sqrt{2} \sum_{j \geq 1} \frac{\mu_j}{j} \int_0^1 \{\frac{\theta}{x}\} \{jn\} dx.$$ 

and then to the evaluation of $\int_0^1 \{\frac{a}{x}\} \{bx\} dx$, another kind of integrals of Franel type. We compute these integrals in the case $a = m, b = n$.

### 3.2. The second kind of Franel type integrals

$I_{n,m} = \int_0^1 \{nx\} \{\frac{m}{x}\} dx$, $n, m \in \mathbb{N}^*$. The values of the integrals $I_{n,m}$ are given by the following

**Theorem 3.1.** For positive integers $m, n$, the modified Franel integrals are given by

$$I_{n,m} = \frac{n}{m} + m \log m + m(n - 1) \log(mn) - m(\log((n - 1)!))$$

$$- \frac{n(n - 1)}{2} - \frac{nm^2}{2} (\zeta(2) - \sum_{1 \leq j \leq m} (1 - \frac{m}{j})) + \sum_{1 \leq k \leq n, mn \geq jk} (1 - \frac{mn}{jk}).$$

Let us first give few examples:

$I_{(2,1)} = \frac{5}{2} - \log(2) - \zeta(2)$; $I_{(3,1)} = \frac{25}{6} + \log(2) - 2 \log(3) - \frac{3}{2} \zeta(2)$

$I_{(4,1)} = \frac{35}{6} - 5 \log(2) + \log(3) - 2 \zeta(2)$; $I_{(5,1)} = \frac{35}{6} - 5 \log(2) + \log(3) - 2 \zeta(2)$

$I_{(1,2)} = \frac{7}{2} - 2 \zeta(2)$; $I_{(1,3)} = \frac{61}{8} - \frac{9}{2} \zeta(2)$

$I_{(1,4)} = \frac{5989}{288} - \frac{25}{2} \zeta(2)$; $I_{(2,2)} = \frac{49}{6} - 2 \log(2) - 4 \zeta(2)$

$I_{(2,3)} = \frac{171}{10} - 3 \log(2) - 9 \zeta(2)$; $I_{(2,4)} = \frac{18469}{630} - 4 \log(2) - 16 \zeta(2)$

$I_{(2,5)} = \frac{15059}{336} - 5 \log(2) - 25 \zeta(2)$; $I_{(3,2)} = \frac{196}{15} + 2 \log(2) - 4 \log(3) - 6 \zeta(2)$

We observe that in all these examples the factor $\zeta(2) = \frac{\pi^2}{6}$ is present.

For the proof we consider the two functions defined on $[0, +\infty[$

$$f(x) = f_n(x) = x \chi_{[0,1]}(x) \{nx\}, \quad g(x) = \{x\} \chi_{[1, +\infty]}(x)$$

and their multiplicative convolution multiplicative

$$(f * g)(a) = \int_0^{+\infty} f(x) g(\frac{a}{x}) \frac{dx}{x}, \quad (f * g)(m) = I_{n,m}.$$ 

We split the computations in several steps. A natural method is to use first the Mellin transform with its property $\mathcal{M}(f * g)(s) = \mathcal{M}(f)(s) \mathcal{M}(g)(s)$, followed by an
inversion. The main idea is the following easily proved decomposition formula, valid for suitable $f(x)$:

$$
\int_0^1 \rho_\theta(x)dx = \theta \int_0^1 f(x)dx - \sum_{n=1}^{\infty} n \int_{\frac{\theta}{n+1}}^{\frac{\theta}{n}} (x - n) f(x)dx + \int_\theta^1 \frac{\theta}{x} f(x)dx
$$
or, equivalently

$$
\rho_\theta(x) = \sum_{n=1}^{\infty} \left( \frac{\theta}{x} - n \right) \chi_{\frac{\theta}{n+1}, \frac{\theta}{n}}(x) + \frac{\theta}{x} \chi_{[\theta,1]}
$$

where $\chi_A$ is the characteristic function of the set $A$.

3.2.1. Computations of different integrals.

(1) Computation of $F(s) = M(f)(s)$ For $\sigma = \text{Re}(s) > -2$ we have

$$
F(s) = \int_0^1 \{nx\} x^s dx
= \sum_{0 \leq k \leq n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (nx - k)x^s dx
= n \int_0^1 x^{s+1} dx - \sum_{1 \leq k \leq n-1} k \int_{\frac{k}{n}}^{\frac{k+1}{n}} x^s dx
= \frac{n}{s+2} - \frac{1}{(s+1)n^{s+1}} \sum_{1 \leq k \leq n-1} k((k+1)^{s+1} - k^{s+1})
= \frac{n}{s+2} - \frac{1}{(s+1)n^{s+1}} \left\{ n^{s+2} - (1 + 2^{s+1} + 3^{s+1} + \cdots n^{s+1}) \right\}
$$

(2) Computation of $G(s) = M(g)(s)$ For $-2 < \sigma = \text{Res} < -1$ on a

$$
G(s) = \int_1^{+\infty} \{x\}^{s-1} dx
= \sum_{k \geq 1} \int_k^{k+1} (x - k)x^{s-1} dx
= \int_1^{+\infty} x^s dx - \sum_{k \geq 1} k \int_k^{k+1} x^{s-1} dx
= \int_1^{+\infty} x^s dx - \frac{1}{s} \sum_{k \geq 1} k((k+1)^s - k^s)
= \frac{1}{s+1} - \frac{\zeta(-s)}{s} \quad \sigma < -1
$$
Hence for $-2 < c < -1$ we can write

$$I_{n,m} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left( 1 - \frac{\zeta(-s)}{s} \right) \left( \frac{n}{s + 2} - \frac{1}{(s + 1)n^{s+1}}(n^{s+1} - (1 + 2^{s+1} + 3^{s+1} + \cdots (n-1)^{s+1}) \right) \frac{ds}{m^s}.$$ 

and, by changing $s$ in $-s$, we get for $1 < c < 2$

$$I_{n,m} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left( \frac{1}{1 - s} + \frac{\zeta(s)}{s} \right) \left( \frac{n}{2 - s} - \frac{1}{(1-s)n^{1-s}}(n^{1-s} - (1 + 2^{1-s} + 3^{1-s} + \cdots (n-1)^{1-s}) \right) \frac{ds}{m^{-s}}.$$ 

By expanding we find:

$$I_{n,m} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{n \cdot m^s}{(1 - s)(2 - s)} ds - \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{m^s}{(1 - s)^2 n^{1-s}}(n^{1-s} - (1 + 2^{1-s} + 3^{1-s} + \cdots (n-1)^{1-s}) ds$$

$$+ \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s)}{s} \left( \frac{n}{2 - s} - \frac{1}{(1-s)n^{1-s}}(n^{1-s} - (1 + 2^{1-s} + \cdots + (n-1)^{1-s}) \right) m^s ds$$

We treat the last type by developing the $\zeta$ function in Dirichlet series. We will treat each type of integrals appearing separately. Then we proceed to the necessary groupings in order to conclude.

In the following we write $\int (c)$ instead of $\int c-i\infty$, with $1 < c < 2$.

(3) Computation of $\frac{n}{2i\pi} \int (c) \frac{m^s}{(1 - s)(2 - s)} ds$. We set $f(x) = -\frac{1}{x}$ for $0 < x \leq 1$

and $f(x) = -\frac{1}{x^2}$ for $x > 1$. Its Mellin transform is $\frac{1}{(1-s)(2-s)}$ for $1 < \sigma < 2$. We obtain $\frac{n}{m}$ for $m \geq 1$.

(4) Computation of $-\frac{1}{2i\pi} \int (c) \frac{m^s}{(1 - s)^2} ds$. We take $f(x) = \log x$ pour $0 < x < 1$

et $0$ pour $x \geq 1$. Its Mellin transform is $-\frac{1}{(s-1)^2}$ for $\sigma > 1$. Here we obtain $m \log m$ for $m \geq 1$.

(5) Computation of $\frac{k}{n} \int (c) \frac{(mn)^s ds}{(1 - s)^2 k^s}$. As before we find $m \log \left( \frac{mn}{k} \right)$ if $mn \geq k$

and zero $0$ if $mn < k$.

(6) Computation of $\frac{n}{2i\pi} \int (c) \frac{m^s ds}{s(2 - s)j^s}$, $j \geq 1$. We take $f(x) = -\frac{1}{2}$ for $0 < x \leq 1$

and $f(x) = -\frac{1}{2x^2}$ for $x > 1$. We get $-\frac{n}{2}$ if $j \leq m$ and $-\frac{nm^2}{2j^2}$ if $j > m$. 
ARITHMETIC AND ANALYSIS OF THE SERIES $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n}$

(7) Computation of $-\frac{1}{2i\pi} \int_{(c)} \frac{m^d s}{s(1-s)^j} ds$, $j \geq 1$. Here we take $f(x) = 1 - \frac{1}{x}$ if $0 < x \leq 1$ and $0$ for $x > 1$. We obtain $1 - \frac{m}{j}$ if $m \geq j$ and $0$ otherwise.

(8) Computation of $\frac{k}{n} \frac{1}{2i\pi} \int_{(c)} \frac{(nm)^s}{s(1-s)(jk)^s}$. Here we obtain $1 - \frac{nm}{jk}$ if $mn \geq jk$ and $0$ otherwise.

By putting together these partial results we end the proof of Theorem (3.1).

3.3. **Second approach $\{\theta_x\}$**. The most interesting approach for the evaluation of the integral $\int_{0}^{1} \{\theta_x\} \sin(n\pi t) dt$ is to use (2.10):

$$
\int_{0}^{1} \{\theta_x\} \sin(n\pi t) dt = \int_{0}^{\theta} \{\theta_x\} \sin(n\pi t) dt + \theta \int_{\theta}^{1} \frac{1}{t} \sin(n\pi t) dt.
$$

Moreover

$$
\int_{0}^{\theta} \{\theta_x\} \sin(n\pi t) dt = \sum_{p \geq 1} \int_{\theta/p+1}^{\theta} \sin(n\pi t) \left(\frac{\theta}{t} - p\right) dt
$$

$$
= \theta \sum_{p \geq 1} \int_{\theta/p+1}^{\theta} \frac{\sin(n\pi t)}{t} dt - \sum_{p \geq 1} p \int_{\theta/p+1}^{\theta} \sin(n\pi t) dt
$$

$$
= \theta \int_{0}^{\theta} \frac{\sin(n\pi t)}{t} dt + \frac{1}{n\pi} \sum_{p \geq 1} p \left(\cos \frac{n\pi \theta}{p} - \cos \frac{n\pi \theta}{p+1}\right)
$$

Hence

$$
a_n = \sqrt{2} \left(\theta \int_{0}^{1} \frac{\sin(n\pi t)}{t} dt + \frac{1}{n\pi} \sum_{p \geq 1} p \left(\cos \frac{n\pi \theta}{p} - \cos \frac{n\pi \theta}{p+1}\right)\right).
$$

Seeking for the coefficient corresponding to $f(x) = \sum_{1 \leq \nu \leq N} c_{\nu} \{\theta_x\}$ the first integral does not matter since $\sum_{1 \leq \nu \leq N} c_{\nu} \theta_{\nu} = 0$. It remains to compute

$$
A = \sum_{p \geq 1} p \left(\cos \frac{n\pi \theta}{p} - \cos \frac{n\pi \theta}{p+1}\right).
$$

Let

$$
A_N = \sum_{1 \leq \nu \leq N} p \left(\cos \frac{n\pi \theta}{p} - \cos \frac{n\pi \theta}{p+1}\right)
$$
for \( x > 0 \).
We have by a partial summation
\[
A_N = \cos \frac{x}{1} + \cdots + \cos \frac{x}{N} - N \cos \frac{x}{N + 1}
\]
\[
= (\cos \frac{x}{1} - 1) + \cdots + (\cos \frac{x}{N} - 1) + N(1 - \cos \frac{x}{N + 1})
\]
\[
= -2 \sum_{n=1}^{N} \sin^2 \frac{x}{n} + 2N \sin^2 \frac{x}{N + 1}
\]

Hence
\[
(3.3) \quad \lim_{N \to +\infty} A_N = -2 \sum_{n=1}^{\infty} \sin^2 \frac{x}{n}
\]
and finally
\[
a_n = \sqrt{2} \left( \frac{\theta}{\pi} \int_0^1 \frac{\sin(\pi nt)}{t} dt - \frac{1}{n\pi} \sum_{n=1}^{\infty} \sin^2 \frac{x}{n} \right).
\]

We essentially used the following relation
\[
(3.4) \quad \rho(\frac{\theta}{t}) = \sum_{n=1}^{\infty} (\frac{\theta}{t} - n))\chi(\frac{\theta}{t+n}, \frac{\theta}{t}) + \frac{\theta}{t} \chi(\theta, 1)
\]
where \( \chi(a,b) \) is the characteristic function of \((a,b)\). In particular if \( f \) is a periodic function, of period one, such that \( \int_0^1 |f(x)| dx < \infty \), then
\[
\int_0^1 \rho(\frac{\theta}{t})f(t)dt = \sum_{n=1}^{\infty} \int_{\frac{\theta}{n+1}}^{\frac{2\theta}{n+1}} (\frac{\theta}{t} - n))f(t) dt + \int_{\theta}^{1} \frac{\theta}{t} f(t) dt.
\]

We adapt an interesting method, due to Delange \[13\], and use a result of Saffari and Vaughan\[24\]. First we introduce for \( 0 < \alpha \leq 1 \)
\[
c_\alpha(u) = \begin{cases} 
1 & \text{if } u - \lfloor u \rfloor = \rho(u) < \alpha \\
0 & \text{otherwise}
\end{cases}
\]
Furthermore for \( x > 0, y > 1 \) let
\[
\vartheta_{x,y}(u) = \frac{1}{\log y} \sum_{n \leq y} \frac{1}{n} c_\alpha \left( \frac{u}{n} \right).
\]
According to \[24\] we have
Lemma 3.1. We have the estimate
\[ \vartheta_{x,y}(u) = u + O \left( (\log x)^{\frac{2}{3}}(\log y)^{-1} \right), \]
the O is uniform in u.

If \( f \) is continuously differentiable function on \([0,1]\)
\[ f(2\pi x/n) = -2\pi \int_0^1 f'(2\pi u) du = -2\pi \int_0^1 c_u(x/n) f'(2\pi u) du. \]

Hence
\[ \sum_{n \leq x} \frac{1}{n} f(2\pi x/n) = -2\pi (\log x) \int_0^1 \vartheta_{x,x}(u) f'(2\pi u). \]

Since
\[
\int_0^1 \vartheta_{x,x}(u) f'(2\pi u) du = \int_0^1 f'(2\pi u) du + \int_0^1 (\vartheta_{x,x}(u) - u) f'(2\pi u) du
\]
\[ = \int_0^1 (\vartheta_{x,x}(u) - u) f'(2\pi u) du. \]

From the lemma (3.1) we get, since \( f' \) is bounded on \((0,1)\)
\[ \sum_{n \leq x} \frac{1}{n} f(2\pi x/n) = O (\log x)^{\frac{2}{3}}. \]

A natural example is to consider a Dirichlet character modulo \( N, \chi \). In this case
\[ \sum_{n \leq x} \frac{\chi(n)}{n} \sin(2\pi x/n) = O (\log x)^{\frac{2}{3}}. \]

We shall not try to give sufficient conditions to justify the process here. The main interest of the remark is that it suggests a method of dealing with various other sums than \( f(x) \).

4. Almost periodicity

The goal of this section is to show, by elementary methods, that the Hardy-Littelwood-Flett function \( f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n} \) is not bounded on the real line. First we recall two fundamental results on Bohr-almost periodic functions (p.39, 44, and 58).
Theorem 4.1 (The Mean value theorem). For every almost periodic function \( f(x) \), there exists a mean value
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(x) \, dx = M\{f(x)\}
\]
and
\[
\lim_{T \to +\infty} \frac{1}{T} \int_a^{a+T} f(x) \, dx = M\{f(x)\}
\]
uniformly with respect to \( a \). In particular if \( f \) is an odd almost periodic function, then its mean \( M\{f(x)\} \) is zero.

Theorem 4.2 (The antiderivative theorem). The integral \( F(x) = \int_0^x f(t) \, dt \) of an almost-periodic function \( f(x) \) is almost-periodic if and only if it is bounded.

Let
\[
\mathfrak{F}(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{x}{n}.
\]
The series defining \( \mathfrak{F}(x) \) is uniformly convergent on the real line. The partial sums
\[
\mathfrak{F}_n(x) = \sum_{p=1}^{n} \frac{1}{p^2} \cos \frac{x}{p}
\]
are almost periodic \([8]\) (Corollary, p.38), and then \( \mathfrak{F}(x) \) is also almost periodic \([8]\) (Theorem IV, p.38). It is interesting to note that \( \mathfrak{F}_n \) is periodic of period \( p_n = \text{lcm}(1, 2, \cdots, n) = e^{\psi(n)} \), with \( \psi(x) \) the Chebyshev function, given by \( \psi(x) = \sum_{p \leq x} \Lambda(p) \), where \( \Lambda(n) \) is the Mangoldt function.

The prime number theorem asserts that \( p_n = e^{n(1+o(1))} \) as \( n \to \infty \) \([26]\) (p.261). Actually \( p_n \leq 3^n \).

Lemma 4.1. We have
\[
\lim_{x \to +\infty} \frac{1}{x} \sum_{n=1}^{\infty} \sin^2 \frac{x}{n} = \frac{\pi}{2}
\]
Let \( x > 0 \) and \( n_x = \left\lfloor \frac{2x}{\pi} \right\rfloor \). The function \( h : x \to \sin^2 \frac{1}{x} \), being bounded on \([0, \frac{\pi}{2}]\) and continuous on each \([\alpha, \frac{\pi}{2}]\), is Riemann-integrable on \([0, \frac{\pi}{2}]\), so by considering Riemann sums:
\[
(4.1) \quad \lim_{x \to +\infty} \frac{1}{n_x} \sum_{n=1}^{n_x} \sin^2 \frac{x}{n} = \int_0^{\frac{2}{\pi}} \sin^2 \frac{1}{t} \, dt = \int_{\frac{\pi}{2}}^{\infty} \frac{\sin^2 u}{u^2} \, du
\]
For \( x > 0 \) the function of \( g(t) = \sin^2 \frac{x}{t} \) is decreasing on \( \left( \frac{2x}{\pi}, +\infty \right) \) and thus

\[
\sum_{n=n_x+1}^{\infty} \sin^2 \frac{x}{n} - \int_{\frac{2x}{\pi}}^{\infty} \sin^2 \frac{x}{t} dt \leq 1.
\]

Since \( \int_{\frac{2x}{\pi}}^{\infty} \sin^2 \frac{x}{t} dt = x \int_{0}^{\frac{\pi}{2}} \frac{\sin^2 u}{u^2} du \) we deduce the lemma from (4.8) (4.9) and the relations

\[
\int_{0}^{\infty} \frac{\sin^2 u}{u^2} du = \int_{0}^{\infty} \frac{\sin u}{u} du = \frac{\pi}{2}.
\]

**Corollary 4.2.** The function \( f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n} \) is not bounded on the real line.

**Proof.** Assume that \( f(x) \) is bounded on \( \mathbb{R} \) then it would be almost periodic by the antiderivative theorem (4.2) and the remark that \( f'(x) = \mathcal{F}(x) \). Since \( f(x) \) is odd its mean is zero. This is in contradiction with the limit \( \frac{\pi}{2} \) given by the lemma (4.1). \( \square \)

**Remark 4.3.** The same analysis applies to the series \( \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \sin \left( \frac{x}{n} \right) \), \( \chi \) being a Dirichlet character modulo \( N \).

We will need to consider some Bessel functions. We recall that for \( \text{Re} s > 0 \) the \( \Gamma \)-function is

\[
\Gamma(s) = \int_{0}^{\infty} u^{s-1} e^{-u} du.
\]

By Fubini’s theorem

\[
\Gamma^2(s) = \int_{0}^{\infty} \int_{0}^{\infty} (uv)^{s-1} e^{-(u+v)} dudv = \int_{0}^{\infty} u^{s-1} \xi_0(u) du,
\]

where

\[
\xi_0(u) = \int_{1}^{\infty} \frac{2e^{2t\sqrt{u}}}{\sqrt{t^2 - 1}} dt
\]

More generally the iterated integrals \([30], [31]\):

\[
\xi_1(x) = \int_{x}^{\infty} \xi_0(t) dt, \cdots, \xi_k(x) = \int_{x}^{\infty} \xi_{k-1}(t) dt.
\]

satisfy the differential equation of Bessel type:

\[
x \frac{d^2 \xi_k(x)}{dx^2} + (1 - k) \frac{d \xi_k(x)}{dx} - \xi_k(x) = 0
\]
The ordinary Bessel function of order $\nu$ is
\[ J_{\nu}(z) = \sum_{m=0}^{\infty} (-1)^m \frac{(z/2)^{2m+\nu}}{m! \Gamma(m+1+\nu)}, \quad I_{\nu}(z) = i^{-\nu} J_{\nu}(iz), \quad |z| < \infty. \]

The $K$-Bessel function of order $\nu$, for $\nu$ not an integer, is:
\[ K_{\nu}(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_{\nu}(z)}{\sin \pi \nu}. \]
When $\nu$ is an integer we take the limiting value. It could be also defined by
\[(4.3) \quad K_{\nu}(z) = \frac{1}{2} \int_{0}^{\infty} t^{\nu-1} e^{-z/2(t+1/t)} \, dt, \quad \text{Re} \nu \geq 0. \]

The Mellin transform of the $J_{0}$-Bessel function is:
\[ \int_{0}^{\infty} J_{0}(\sqrt{x}) x^{s-1} \, dx = 4^s \frac{\Gamma(s)}{\Gamma(1-s)}. \]

We will need two Mellin transforms, due essentially to Voronoi:
\[ \int_{0}^{\infty} x^{s-1} K_{0}(4\pi \sqrt{x}) \, dx = \frac{1}{2} (2\pi)^{-2s} \Gamma^2(s), \]
\[ \int_{0}^{\infty} x^{s-1} Y_{0}(4\pi \sqrt{x}) \, dx = -\frac{1}{\pi} (2\pi)^{-2s} \cos(\pi s) \Gamma^2(s). \]

4.0.1. Summations formulas and beyond. Various classical summation formulas, as Poisson summation formula, Voronoi summation formula or Hardy-Ramanujan summation formula can all be given an unified formulation. The following Generalized Poisson summation formula is proved in [9].

Theorem 4.3. Let $a = a(n)$ be an arithmetic function with moderate growth. We define the Dirichlet series
\[ L(a, s) = \sum_{n=1}^{\infty} a(n) n^{-s}, \quad \text{Res} > 1 \]
and we suppose that $L(a, s)$ has an analytic continuation to $\mathbb{C}$ with only a possible pole at $s = 1$. We suppose also that there are positive constants $A, a_1, \cdots, a_g$ such that with the $\Gamma$-factors
\[ \gamma(s) = A^s \prod_{j=1}^{g} \Gamma(a_j s) \]
$L(a, s)$ satisfies the functional equation
\[ \gamma(s)L(a, s) = \gamma(1-s)L(a, 1-s). \]
Furthermore for \( f \in S(\mathbb{R}) \), the Schwartz space, we define a very special Hankel’s transform:

\[
g(x) = \int_0^\infty f(y)K(xy)dy, \quad \text{with} \quad K(x) = \int_{\text{Res} = \frac{3}{2}} \frac{\gamma(s)}{\gamma(1-s)}x^{-s}ds.
\]

Then,

\[
\sum_{n=1}^\infty a(n)f(n) = f(0)L(a,0) + \text{Res}_{s=1} \mathcal{M}(f)(s)L(a,s) + \sum_{n=1}^\infty a(n)g(n).
\]

**4.0.2. 2 classical choices.**

1. For \( a(n) = 1 \) we have \( L(a,s) = \zeta(s) \) and

\[
\gamma(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \quad K(x) = 2\cos(2\pi x).
\]

We recover Poisson summation formula for even functions in \( f(x) \in S(\mathbb{R}) \):

\[
\sum_{n=1}^\infty f(n) = -\frac{1}{2}f(0) + \int_0^\infty f(x)dx + 2\sum_{n=1}^\infty \int_0^\infty f(y)\cos(2\pi ny)dy.
\]

2. If \( a(n) = d(n) \) we have \( L(d,s) = \zeta^2(s) \) and

\[
\gamma(s) = \pi^{-s}\Gamma\left(\frac{s}{2}\right)^2, \quad \frac{\gamma(s)}{\gamma(1-s)} = (2\pi)^{-2s}(2 + 2\cos\pi s)\Gamma(s)^2
\]

and

\[
K(x) = 4K_0(4\pi \sqrt{x}) - 4Y_0(4\pi \sqrt{x}).
\]

We recover Voronoi summation formula

\[
\sum_{n=1}^\infty f(n)d(n) = \frac{1}{4}f(0) + \int_0^\infty f(x)(2\gamma + \log x) \, dx + \sum_{n=1}^\infty d(n) \int_0^\infty f(y) \left(4K_0\left(4\pi(ny)^{\frac{1}{2}}\right) - 2\pi Y_0\left(4\pi(ny)^{\frac{1}{2}}\right)\right) \, dy
\]

As a consequence we have Koshliakovs formula valid for \( a > 0 \):

\[
\sqrt{a} \left(\gamma - \log\left(\frac{4\pi}{a}\right) + 4 \sum_{n=1}^\infty d(n)K_0(2\pi an)\right)
\]

\[
= \frac{1}{\sqrt{a}} \left(\gamma - \log(4\pi a) + 4 \sum_{n=1}^\infty d(n)K_0\left(\frac{2\pi n}{a}\right)\right).
\]

This formula was proved by Ramanujan about ten years earlier (He did not appeal to Voronois formula) and by many authors later.
4.1. Another function of Hardy and Littlewood. Hardy and Littlewood gave in [16] (p.269) the following relation

\[ F(z) = \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - e^{-z/n} \right) = 2 \log z + 2\gamma \]

\[ -2 \sum_{n=1}^{\infty} \left\{ K_0 \left( \sqrt{2n\pi i}z \right) + K_0 \left( \sqrt{-2n\pi i}z \right) \right\} \]

where \( \text{Re}z > 0 \), \( \gamma \) is Euler’s constant

For \( |z| < 1 \):

\[ \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - e^{-z/n} \right) = -\sum_{n=1}^{\infty} \zeta(n+1) \frac{(-z)^n}{n!}. \]

An immediate consequence of this expansion is obtained by taking real and imaginary parts with \( z = ix, x \in \mathbb{R}, |x| < 1 \):

\[ \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - \cos \frac{x}{n} \right) = 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin^2 \frac{x}{2n} = -\sum_{k \geq 0} \zeta(4k+1) \frac{x^{4k}}{(4k)!} + \sum_{k \geq 0} \zeta(4k+3) \frac{x^{4k+2}}{(4k+2)!} \]

\[ = \sum_{k \geq 0} (-1)^{k-1} \zeta(2k+1) \frac{x^{2k}}{(2k)!} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n} = \sum_{k \geq 0} \zeta(4k+2) \frac{x^{4k+1}}{(4k+1)!} - \sum_{k \geq 0} \zeta(4k+4) \frac{x^{4k+3}}{(4k+3)!} \]

More generally we define the series

\[ G_\nu(z) = \sum_{n>\text{Re}\nu+1} \zeta(n-\nu) \frac{(-z)^n}{n!} \]

which has a Mellin-Barnes type integral representation when \( x > 0, c \) is fixed with \( \text{Re}\nu + 1 < c < \text{Re}\nu + 2 \):

\[ G_\nu(z) = \frac{1}{2i\pi} \int_{(c)} \Gamma(-s) \zeta(s-\nu)x^s ds. \]

The proof of the main equality results from the deformation of the contour of integration and the fact that the pair

\[ x^\nu K_\nu(x), \quad 2^{s+\nu-2} \Gamma(s/2) \Gamma(s/2 + \nu), \quad \text{Res} > \text{max}(0, -2\text{Re}\nu) \]

is a pair of Mellin transforms. The details are given in [18], [19]. The series (4.5) has many remarkable properties. It may be differentiated term by
term to get $\mathcal{G}(-x)$ where $\mathcal{G}(x)$ is the function defined in [26] (p.243):

$$\mathcal{G}(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{\frac{x}{n^2}}.$$  

(4.7)

The following formula is mentioned in [27]

$$\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \mathcal{G}(2\pi nk) = \sum_{d|n} \frac{1}{d^2}.$$  

4.2. Laplace transform of $a\sqrt{t}J_{1}(a\sqrt{t})$, $a > 0$, $t > 0$, another approach to Segal’s formula. In [25] (formula (12)) Segal proves the following result

Theorem 4.4. If $g(z) := \sum_{k \geq 1} (1 - \cos \frac{z}{k})$ then

$$g(z) = \frac{\pi z}{2} - \frac{1}{2} + \frac{1}{4} \sum_{k \geq 1} \frac{2\sqrt{2k \pi \sqrt{z}}}{k} J_{1}(2\sqrt{2k \pi \sqrt{z}}).$$

This formula is interesting compared to (4.5), as we have for real $z$, $g(z) = \text{Re} \mathcal{G}(iz)$. The proof given [25] uses a rather elaborated tools such the three Bessel functions $J_1$, $J_2$, $J_3$, the functional equation of the Riemann $\zeta$-function etc. We give here a simpler proof.

The Laplace transform $\mathcal{L}(a\sqrt{t}J_{1}(a\sqrt{t}))(p)$ is given by

$$\mathcal{L}(a\sqrt{t}J_{1}(a\sqrt{t}))(p) = a \int_{0}^{+\infty} \sqrt{t}J_{1}(a\sqrt{t}) e^{-tp} dt, \quad \text{Rep} > 0.$$  

We set $u^2 = t$, then

$$\mathcal{L}(a\sqrt{t}J_{1}(a\sqrt{t}))(p) = 2a \int_{0}^{+\infty} J_{1}(au) e^{-pa^2 u^2} du, \quad \text{Rep} > 0.$$  

According to [33] (page 394, formula(4)) we have for with $|\text{Arg} p| < \frac{\pi}{4}$

$$\int_{0}^{+\infty} J_{\nu}(au) e^{-p^2 u^2} u^{\nu+1} du = \frac{a^{\nu}}{(2p^2)^{\nu+1}} e^{-\frac{a^2}{4p^2}}.$$  

Replacing $p$ by $\sqrt{p}$ with $|\text{Arg} p| < \frac{\pi}{2}$ and taking $\nu = 1$ we obtain

$$\int_{0}^{+\infty} J_{1}(au) e^{-pa^2 u^2} du = \frac{a}{4p^2} e^{-\frac{a^2}{4p^2}}.$$  

Hence

$$\mathcal{L}(a\sqrt{t}J_{1}(a\sqrt{t}))(p) = \frac{a^2}{2p^2} e^{-\frac{a^2}{4p^2}} \quad \text{Rep} > 0.$$
Note that $a\sqrt{t}J_1(a\sqrt{t})$ is not in $L^2([0, +\infty[)$ since its Laplace transform is not bounded in the $L^2$-norm on the lines $\text{Re} p = c > 0$. With $a = 2\sqrt{2k\pi}$ we get

$$\mathcal{L}(2\sqrt{2k\pi}tJ_1(2\sqrt{2k\pi}t))(p) = \frac{4k\pi}{p^2}e^{-\frac{2k\pi}{p}}.$$ 

As we have

$$\mathcal{L}\left(\frac{\pi t - 1}{2}\right)(p) = \frac{\pi}{2p^2} - \frac{1}{2p},$$

and

$$g(t) = \frac{\pi t - 1}{2} + \frac{1}{4}\sum_{k \geq 1} \frac{2\sqrt{2k\pi}t}{k} J_1(2\sqrt{2k\pi}t)$$

the Laplace transform of the sum in $g(t)$ is:

$$\frac{\pi}{p^2} \sum_{k \geq 1} (e^{-\frac{2k\pi}{p}})^k$$

which converges in $\text{Re} p > 0$. Hence, by continuity, the Laplace transform of $g(t)$ is:

$$\frac{\pi}{2p^2} - \frac{1}{p} + \frac{\pi}{p^2} \frac{e^{-\frac{2\pi}{p}}}{1 - e^{-\frac{2\pi}{p}}} = \frac{\pi}{2p^2} - \frac{1}{p} + \frac{\pi}{p^2} \frac{1}{e^{\frac{2\pi}{p}} - 1}.$$ 

This shows the desired equality by using the well known partial fractions decomposition:

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k \geq 1} \frac{2z}{z^2 + 4k^2\pi^2}, \quad z \in \mathbb{C} \setminus 2i\pi\mathbb{Z},$$

where we have to set $z = \frac{1}{2\pi p}$.

### 4.3. Some Mellin transforms and the cube of theta functions.

It has been noticed in [15] (p.14) that the function

$$R(t) = \sum_{n \leq t} \frac{1}{n} e^{\frac{it}{n}}$$

is very similar to $\zeta(1 + it)$ in its asymptotic behaviour as $t \to +\infty$. This could suggest a link between this function and the function $\vartheta_3(q) = \sum_{n \in \mathbb{Z}} q^{n^2}$. In this section, following a suggestion of Crandall [10] we would like to briefly show, by considering Mellin transforms, an unexpected link to the third power of the (fourth)
Jacobi theta function $\vartheta_4(q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}$, $|q| < 1$. We define two functions

\begin{align*}
\bar{\chi}(s, t) &= \sum_{n=1}^{\infty} \frac{e^{-\frac{t}{n}}}{n^s}, \\
\chi(s, t) &= \sum_{n=1}^{\infty} (-1)^n \frac{e^{-\frac{t}{n}}}{n^s}.
\end{align*}

These two functions are defined for $x \in \mathbb{C}$ and $\text{Re} s > 1$ for $\bar{\chi}(s, t)$, $\text{Re} s > 0$ for $\chi(s, t)$. They are related by

$$\chi(s, t) = \frac{1}{2^{s-1}} \bar{\chi}\left(\frac{s}{2}, t\right) - \bar{\chi}(s, t).$$

We have

$$\int_0^\infty x^{s-1} \chi(s, t)^2 \, dx = \Gamma(s) \sum_{n,m=1}^{\infty} (\frac{1}{n+m})^s = \Gamma(s) (\zeta(s-1) - \zeta(s))$$

and

$$\int_0^\infty x^{s-1} \bar{\chi}(s, t)^2 \, dx = \Gamma(s) \sum_{n,m=1}^{\infty} (-1)^{n+m} (\frac{1}{n+m})^s = \Gamma(s) (\eta(s-1) - \eta(s)),$$

where

$$\eta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

is the Dirichlet $\eta$-function. Furthermore

$$\frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \bar{\chi}(s, t)^3 \, dx = \sum_{p,q,r=1}^{\infty} \frac{1}{(pq + qr + rs)^s}$$

and for $\chi(s, t)$ we have a more interesting result

$$\frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \chi(s, t)^3 \, dx = \sum_{p,q,r=1}^{\infty} (-1)^{p+q+r} (\frac{1}{pq + qr + rp})^s.$$

We have the following lemma due to Andrews [2] (p.124):

**Lemma 4.4.** For $|q| < 1$

$$\vartheta_4^3(q) = \sum_{n \in \mathbb{Z}} (-1)^n r_3(n) q^n = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1 + q^n} - 2 \sum_{n=1}^{\infty} q^{n^2-j^2} (1 - q^n)(-1)^j \sum_{\substack{j \leq n \atop |j| \leq n}} \frac{1 + q^n}{1 + q^j},$$

where $r_3(n)$ is the number of representations of $n$ as sum of three squares. According to Fermat an integer is a sum of three squares if and only if it is not of form
$4^n(8m + 7)$. There are some gaps in the expansion in power series of the left hand side of (4.11). Similarly to (4.10) we have

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\vartheta_3^2(q) - 1) \, dt = \sum_{p,q,r \in \mathbb{Z}} \frac{(-1)^{p+q+r}}{(p^2 + q^2 + r^2)^s}$$

A link between (4.14) and (4.17) is given by Crandall [10], by expanding the right hand side in (4.15) in power series, in the form of a relation between two Epstein zeta functions associated with the two not equivalent ternary forms

$$q_1(u,v,w) = u^2 + v^2 + w^2, \quad q_2(u,v,w) = uv + vw + wu$$

in the form

$$\sum_{p,q,r \in \mathbb{Z}} \frac{(-1)^{p+q+r}}{(p^2 + q^2 + r^2)^s} = -6(1 - 2^{-s})^2 \zeta_2(s) - 4 \sum_{p,q,r=1}^{\infty} \frac{(-1)^{p+q+r}}{(pq + qr + rp)^s}$$

Next we establish a functional equation

Theorem 4.5. For $t > 0$ the function $\chi(\frac{1}{2}, t)$ satisfies the following functional equation

$$\chi(\frac{1}{2}, t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-t/n} = \sqrt{i} \sum_{\mathcal{O}} e^{-\gamma \sqrt{2\pi n t}/\sqrt{n}},$$

with $\gamma = 1 - i$ and $\mathcal{O}$ is the set of odd integers.

We could also seek for a result similar to (4.14) for

$$\tilde{\chi}(s, t) = \sum_{n=1}^{\infty} \frac{1}{n^s} e^{-t/n}, \quad \text{Re}s > 1.$$ 

The relevance of this function lies in its relation to a Hardy-Littlewood-Flett like function:

$$\text{Im}\tilde{\chi}(1, -it) = \text{Im} \sum_{n=1}^{\infty} \frac{1}{n} e^{it/n} = \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{t}{n} \right),$$

In order to prove (4.14) we consider, for a fixed $t > 0$, the function

$$f(x) = e^{i\pi x} e^{-t/x} \sqrt{x}, \quad x > 0$$

extended to the origin by $f(0) = 0$ and to $\mathbb{R}$ as an even function. The obtained function is $C^\infty$ on the real line to which we apply the Poisson summation formula (4.4) to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} e^{-t/n} = \int_0^\infty e^{i\pi x} e^{-t/x} \sqrt{x} \, dx + 2 \sum_{n=1}^{\infty} \int_0^\infty e^{i\pi x} e^{-t/x} \sqrt{x} \cos(2\pi nx) \, dx.$$
Remark 4.1. The function $y \mapsto e^{\frac{-i}{\sqrt{y}}} \cos(2\pi ny)$ is continuous on $[0, \infty]$ and decreases to 0 at infinity, hence the proper integrals $\int_0^\infty f(y) \cos(2\pi ny) \, dy, \ n \geq 0$ are convergent.

We compute $\mathcal{F}(f)(n)$ as follows:

$$\mathcal{F}(f)(n) = \int_0^\infty e^{i\pi y} e^{\frac{-i}{\sqrt{y}}} \cos(2\pi ny) \, dy$$

(4.17) $$= \frac{1}{2} \left\{ \int_0^\infty e^{(i\pi+2i\pi n)y-t/y} y^{-1/2} \, dy + \int_0^\infty e^{(i\pi-2i\pi n)y-t/y} y^{-1/2} \, dy \right\}.$$

We recall the modified Bessel function (4.3), written in the form

$$\int_0^\infty w^{\nu-1} e^{-w-a/w} \, dw = 2 \left( \frac{1}{a} \right)^{\nu/2} K_\nu \left( 2\sqrt{a} \right)$$

that we use in the form

(4.18) $$\int_0^\infty w^{\nu-1} e^{-bw-a/w} \, dw = 2 \left( \frac{a}{b} \right)^{\nu/2} K_\nu \left( 2\sqrt{ab} \right)$$

Actually for (4.17) we need only the simplest case of

(4.19) $$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

The first integral in the left hand side of (4.17), with

$$a = t, \quad -b = i\pi + 2i\pi n = i\pi(2n + 1)$$

is equal to

(4.20) $$2 \left( \frac{t}{-i\pi(2n + 1)} \right)^{1/4} K_{1/2} \left( 2\sqrt{t (-i\pi(2n + 1))} \right).$$

The second integral with

$$a = t, \quad -b = i\pi - 2i\pi n = i\pi(-2n + 1)$$

is equal to

(4.21) $$2 \left( \frac{t}{-i\pi(-2n + 1)} \right)^{1/4} K_{1/2} \left( 2\sqrt{t (-i\pi(-2n + 1))} \right).$$

By using (4.19) we see that (4.17) is the sum of

$$\left( \frac{t}{-i\pi(2n + 1)} \right)^{1/4} \sqrt{\frac{\pi}{2.2\sqrt{t (-i\pi(2n + 1))}}} e^{-2\sqrt{t (-i\pi(2n + 1))}}$$
and
\[
\left(\frac{t}{-i\pi(-2n+1)}\right)^{1/4} \sqrt{\frac{\pi}{2.2\sqrt{t(-i\pi(-2n+1))}}} e^{-2\sqrt{t(-i\pi(-2n+1))}}.
\]

As in [10] we denote by \(\gamma = 1 - i\), \(d = \pm 2n + 1\), \(n \in \mathbb{N}^*\) with \(\sqrt{|d|} = i\sqrt{|d|}\). Then \(d\) describes \(\mathcal{O} \setminus \{1\}\) and
\[
\left(\frac{t}{-i\pi d}\right)^{1/4} \sqrt{\frac{\pi}{2.2\sqrt{t(-i\pi d)}}} e^{-2\sqrt{-it\pi d}} = \frac{1}{2} \sqrt{i} \frac{e^{-\gamma\sqrt{t\pi d}}}{\sqrt{d}}.
\]

Hence
\[
(4.22) \quad 2 \sum_{n=1}^{\infty} \int_{0}^{\infty} f(y) \cos(2\pi ny) dy = \sqrt{i} \sum_{d \in \mathcal{O}, d \neq 1} e^{-\gamma\sqrt{t\pi d}} \frac{e^{-\gamma\sqrt{t\pi d}}}{\sqrt{d}}.
\]

For the remaining term in (4.16) we use (4.20), with \(n = 0\) to obtain
\[
\int_{0}^{\infty} f(x) dx = \mathcal{F}(f)(0) = \sqrt{i} e^{-\gamma\sqrt{2\pi t}}
\]
which together with (4.22) gives finally (4.14):
\[
\chi(1/2, t) = \sqrt{i} \sum_{\mathcal{O}} e^{-\gamma\sqrt{2\pi dt}} \frac{e^{-\gamma\sqrt{2\pi dt}}}{\sqrt{d}}.
\]

In close analogy to Jacobi’s transformation of Theta functions (4.14) appears as a convergence acceleration of a slowly convergent series.

Incidentally \(\chi(1/2; t^2/4)\) is a Fourier transform of a function of the Schwartz class.

Indeed let \(g(x) = \frac{1}{1 + e^{2x}}\), the reciprocity formulas are
\[
\hat{g}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1 + e^{2x}} e^{itx} dx, \quad g(x) = \int_{\mathbb{R}} \hat{g}(t) e^{-itx} dt
\]
with
\[
\hat{g}(t) = \frac{1}{2\pi} \sum_{n > 0} (-1)^n \int_{\mathbb{R}} e^{itx} e^{-nx^2} dx
\]
\[
= \frac{1}{2\sqrt{\pi}} \sum_{n > 0} \frac{(-1)^n}{\sqrt{n}} e^{-\frac{t^2}{4n}} = \frac{1}{2\sqrt{\pi}} \chi(\frac{1}{2}; \frac{t^2}{4}).
\]
Remark 4.2. The convolution of three functions $f, g, h \in \mathcal{S}(\mathbb{R})$ is, as well known,

\[
(f \ast (g \ast h))(x) = \int_{\mathbb{R}} f(y)(g \ast h)(x-y)dy
= \int_{\mathbb{R}} f(y) \left( \int_{\mathbb{R}} g(z)h(x-y-z) \right) dy
= \int \int_{\mathbb{R} \times \mathbb{R}} f(y)g(z)h(x-y-z)dzdy.
\]

With $f(x) = g(x) = h(x) = \frac{1}{1 + e^{x^2}}$ we have

\[
(f \ast (g \ast h))(x) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{dt \, du}{(1 + e^{y^2})(1 + e^{z^2})(1 + e^{(x-y-z)^2})}
\]

so that for the Fourier transform

\[
\hat{f} \ast \hat{g} \ast \hat{h} (t) = (2\pi)^2 \hat{g}^3(t) = (2\pi)^2 \frac{1}{8\pi \sqrt{\pi}} \chi^3 \left( \frac{1}{2}; \frac{t^2}{4} \right) = \frac{\sqrt{\pi}}{2} \chi^3 \left( \frac{1}{2}; \frac{t^2}{4} \right)
\]

or

\[
f \ast g \ast h(x) = \frac{\sqrt{\pi}}{2} \int_{\mathbb{R}} \chi^3 \left( \frac{1}{2}; \frac{t^2}{4} \right)e^{-itx} dt.
\]

Evaluating at $x = 0$ we obtain

\[
\int_{\mathbb{R}^2} \frac{dy \, dz}{(1 + e^{y^2})(1 + e^{z^2})(1 + e^{(x-y-z)^2})} = \int_{\mathbb{R}^2} \frac{dy \, dz}{(1 + e^{y^2})(1 + e^{z^2})(1 + e^{(x-y-z)^2})}
= \frac{\sqrt{\pi}}{2} \int_{\mathbb{R}} \chi^3 \left( \frac{1}{2}; \frac{t^2}{4} \right) dt = \sqrt{\pi} \int_0^{\infty} \frac{1}{\sqrt{u}} \chi^3 \left( \frac{1}{2}; u \right) du.
\]

From (4.10), with $s = \frac{1}{2}$ we have (Compare with [10])

\[
\frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{1}{\sqrt{x}} \chi(s, x)^3 dx = \sum_{p, q, r=1}^{\infty} \frac{(-1)^{p+q+r}}{(pq + qr + rp)^{\frac{3}{2}}}.
\]

We end this study by using an interesting integral representation due to Mellin [20] (p. 22-23):

\[
\frac{\Gamma(s)}{(w_0 + w_1 + \cdots + w_p)^s} = \frac{1}{(2i\pi)^p} \int_{\kappa_1-i\infty}^{\kappa_1+i\infty} \cdots \int_{\kappa_p-i\infty}^{\kappa_p+i\infty} \Gamma(s - z_1 \cdots z_p)
\times \Gamma(z_1) \cdots \Gamma(z_p)
\times \frac{\Gamma(z_1) \cdots \Gamma(z_p)}{w_1^z \cdots w_p^z} dz_1 \cdots dz_p.
\]

$\kappa_\nu > 0, \nu = 1, \cdots, p; \ Res \kappa_1 + \cdots + \kappa_p > 0.$
In the case of $p = 2$ we obtain at once

\[
\sum_{p,q,r \geq 1} \frac{(-1)^{p+q+r}}{(pq + qr + rp)^s} = -\frac{1}{4\pi^2} \int_{\kappa_1-i\infty}^{\kappa_1+i\infty} \int_{\kappa_2-i\infty}^{\kappa_2+i\infty} \Gamma(s-u_1u_2)\Gamma(u_1)\Gamma(u_2)
\]

\[
= -\frac{1}{4\pi^2} \int_{\kappa_1-i\infty}^{\kappa_1+i\infty} \int_{\kappa_2-i\infty}^{\kappa_2+i\infty} \Gamma(s-u_1u_2)\Gamma(u_1)\Gamma(u_2)K(s;u_1,u_2)\,du_1du_2,
\]

where

\[
K(s;u_1,u_2) = \eta(s+u_1-u_1u_2)\eta(s+u_2-u_1u_2)\eta(u_1+u_2).
\]

This representation of the Epstein zeta function of the ternary form $q_2(u,v,w) = uv + vw + wu$ in terms of the Dirichlet $\eta$-function and similar other representations can shed some light on their analytic continuation.

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