Maximum principle for viscosity solutions on Riemannian manifolds

Shige Peng∗and Detang Zhou†

October 3, 2008

Abstract. In this work we consider viscosity solutions to second order partial differential equations on Riemannian manifolds. We prove maximum principles for solutions to Dirichlet problem on a compact Riemannian manifold with boundary. Using a different method, we generalize maximum principles of Omori and Yau to a viscosity version.

Key words: viscosity solution, maximum principle, Riemannian manifold

1 Introduction

The theory of viscosity solutions on \( \mathbb{R}^n \) has been an important area in analysis since the concept was introduced in the early 1980’s by Michael Crandall and Pierre-Louis Lions. As a generalization of the classical concept of what is meant by a “solution” to a partial differential equation, it has been found that the viscosity solution is the natural solution concept in many applications of PDE’s, including for example first order equations arising in optimal control (the Hamilton-Jacobi-Bellman equation), differential games (the Isaacs equation) or front evolution problems, as well as second-order equations such as the ones arising in stochastic optimal control or stochastic differential games(see [CIL] and references therein). It is a natural question to ask how to generalize the theory to problems on Riemannian manifolds. Some special cases have been discussed in comparison theory for Riemannian distance function and reduced distance function( see section 9.4 and 9.5 in [CCG] and references therein). Up to now, little is known for general second order partial differential equations in the references. Recently Azagra, Ferrera and Sanz [AFS]

∗This author thanks the partial support from The National Basic Research Program of China (973 Program) grant No. 2007CB814900.
†Partially supported by CNPq and FAPERJ of Brazil.
published a paper in which among the other results they obtained a Hessian estimate for distance functions and generalized some of results in [CIL] to compact Riemannian manifold with some curvature conditions.

We will study the maximum principles for viscosity solutions to second order partial differential equation of the form

$$F(x, u, Du, D^2u) = 0$$

where \( u : M \to \mathbb{R} \) is a function and \( M \) is a compact Riemannian manifold with boundary or a complete Riemannian manifold. It is straightforward to generalize the concepts of viscosity subsolution and supersolution to any Riemannian manifold. As is well known the classical maximum principle for Dirichlet problem on the domains in Euclidean space can be easily generalized to any compact Riemannian manifold without restrictions on curvature. If we follow the method used in [CIL], we see that even for the hyperbolic space the situation the proof does not go directly. The reason is that on \( \mathbb{R}^n \), one need to compute the square of distance function \( \frac{1}{2}|x - y|^2 \) and as a function on \( \mathbb{R}^n \times \mathbb{R}^n \), its Hessian is

$$
\begin{pmatrix}
I_n & -I_n \\
-I_n & I_n
\end{pmatrix}.
$$

As for the Riemannian manifold, the square of distance function \( \frac{1}{2}d(x, y)^2 \) is much more complicated. It may be non-differentiable at some points and even on hyperbolic space \( \mathbb{H}^n(-1) \) of constant curvature \(-1\), it is smooth and its Hessian can be written as

$$
\begin{pmatrix}
1 & -1 \\
I_{n-1}d\coth d & -\frac{d}{\sinh d}I_{n-1} \\
-1 & 1 \\
-\frac{d}{\sinh d}I_{n-1} & I_{n-1}d\coth d
\end{pmatrix},
$$

where \( I_{n-1} = \text{diag}(1, 1, \cdots, 1) \) is \((n - 1) \times (n - 1)\) unit matrix. We will use \( I \) instead of \( I_n \) when the dimension is obvious.

When \( M \) is compact Riemannian manifold with boundary, we consider the Dirichlet problem. From the proof in section 3 in [CIL], we can deal with problems in a sufficiently small neighborhood of the limit point where the function \( \frac{1}{2}d(x, y)^2 \) is smooth. But to prove the comparison theorem we need to estimate the Hessian of \( \frac{1}{2}d(x, y)^2 \). In section 4, we prove a sharp Hessian estimate for \( \frac{1}{2}d(x, y)^2 \) for manifold with sectional curvature bounded below by a constant and a sharp estimate of Laplacian type for \( \frac{1}{2}d(x, y)^2 \) for manifolds with Ricci curvature bounded below a constant. According to these estimates we need to modify the proof in [CIL] to obtain the desired comparison theorem which does not require a curvature restriction. More precisely, we prove
Theorem 1.1. (see Theorem 4.3) Let $M$ be a compact Riemannian manifold with or without boundary $\partial M$, $F \in C(\mathcal{F}(M), \mathbb{R})$ a proper function satisfying

\begin{equation}
\beta (r - s) \leq F(x,r,p,X) - F(x,s,p,X) \quad \text{for } r \geq s,
\end{equation}

for some positive constant $\beta$ and the following condition (H):

there exists a function $\omega : [0, \infty] \rightarrow [0, \infty]$ satisfying $\omega(0^+) = 0$ such that

\begin{equation}
F(y,r,\delta \iota(\gamma'(0)), X_1) - F(x,r,\delta \iota(\gamma'(l)), X_2) \leq \omega(\delta d(x,y)^2 + d(x,y))
\end{equation}

for $X_1 \in S^2T^*_x M$ and $X_2 \in S^2T^*_y M$, satisfying $X_1 \geq X_2 \circ P_{\gamma}(l)$. Here $\iota$ is the dual map between the tangent and cotangent bundles and $\delta$ is a positive constant. Here $S^2T^*M$ is the bundle of symmetric covariant tensors over $M$.

Let $u_1 \in USC(\bar{M})$ and $u_2 \in LSC(\bar{M})$ be a subsolution and supersolution of $F = 0$ respectively. Then $u_1 - u_2$ cannot achieve a positive local maximum at any interior point. In particular if $M$ is a compact Riemannian manifold with boundary $\partial M$ and if $u_1 \leq u_2$ on $\partial M$, then $u_1 \leq u_2$ on $\bar{M}$.

When $M$ is a complete Riemannian manifold without boundary. Both the results and proofs are more interesting to us. We have not found a similar result in the Euclidean case. The new obstruction is that we may not have a limit point when $M$ is noncompact. We first generalize the maximum principle of Omori and Yau (see [O] and [Y]) to a viscosity-type. As is known, Yau’s maximum principle ([Y]) is stated as the following:

Theorem 1.2. Let $M$ be a complete Riemannian manifold with Ricci curvature bounded below by a constant. Let $u : M \rightarrow \mathbb{R}$ is a $C^2$ function with $\inf u > -\infty$, then for any $\varepsilon > 0$, there exists a point $x_\varepsilon \in M$ such that

\begin{align*}
&u(x_\varepsilon) < \inf u + \varepsilon, \\
&|\nabla u|(x_\varepsilon) < \varepsilon, \\
&\Delta u(x_\varepsilon) > -\varepsilon.
\end{align*}

Its proof uses the gradient estimate of the distance function to a fixed point (see [CY]) and for a viscosity solution we do not require the function is differentiable and we cannot use the gradient estimate. Besides overcoming the difficulties appeared in compact cases, we used a new penalty function to the corresponding maximum principles of Omori and Yau for viscosity solutions. Even for $C^2$ function, we provide a new proof of Omori and Yau’s theorems. We would like remark that such a viscosity version is also new for $\mathbb{R}^n$. We prove the following (see section 2 for the definitions of $\bar{J}^{2,+} u(x)$ and $\bar{J}^{2,-} u(x)$.)
**Theorem 1.3.** (see Theorem 5.1) Let \( M \) be a complete Riemannian manifold with sectional curvature bounded below by a constant \(-\kappa^2\). Let \( u \in \text{USC}(M) \), \( v \in \text{LSC}(M) \) be two functions satisfying
\[
\mu_0 := \sup_{x \in M} [u(x) - v(x)] < +\infty.
\]
Assume that \( u \) and \( v \) are bounded from above and below respectively and there exists a function \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \omega(0) = \omega(0^+) = 0 \) such that
\[
(4) \quad u(x) - u(y) \leq \omega(d(x, y)).
\]
Then for each \( \varepsilon > 0 \), there exist \( x_\varepsilon, y_\varepsilon \in M \), such that \( (p_\varepsilon, X_\varepsilon) \in \bar{J}^{2,+}u(x_\varepsilon) \), \( (q_\varepsilon, Y_\varepsilon) \in \bar{J}^{2,-}v(y_\varepsilon) \), such that
\[
u(x_\varepsilon) - v(y_\varepsilon) \geq \mu_0 - \varepsilon,
\]
and such that
\[
d(x_\varepsilon, y_\varepsilon) < \varepsilon, \quad |p_\varepsilon - q_\varepsilon \circ P_\gamma(l)| < \varepsilon, \quad X_\varepsilon \leq Y_\varepsilon \circ P_\gamma(l) + \varepsilon P_\gamma(l),
\]
where \( l = d(x_\varepsilon, y_\varepsilon) \) and \( P_\gamma(l) \) is the parallel transport along the shortest geodesic connecting \( x_\varepsilon \) and \( y_\varepsilon \).

**Theorem 1.4.** (see Theorem 5.2) Let \( M \) be a complete Riemannian manifold with Ricci curvature bounded below by a constant \(-(n-1)\kappa^2\). Let \( u \in \text{USC}(M) \), \( v \in \text{LSC}(M) \) be two functions satisfying
\[
\mu_0 := \sup_{x \in M} [u(x) - v(x)] < +\infty.
\]
Assume that \( u \) and \( v \) are bounded from above and below respectively and there exists a function \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \omega(0) = \omega(0^+) = 0 \) such that
\[
(6) \quad u(x) - u(y) \leq \omega(d(x, y)).
\]
Then for each \( \varepsilon > 0 \), there exist \( x_\varepsilon, y_\varepsilon \in M \), \( (p_\varepsilon, X_\varepsilon) \in \bar{J}^{2,+}u(x_\varepsilon) \), and \( (q_\varepsilon, Y_\varepsilon) \in \bar{J}^{2,-}v(y_\varepsilon) \), such that
\[
u(x_\varepsilon) - v(y_\varepsilon) \geq \mu_0 - \varepsilon,
\]
and such that
\[
d(x_\varepsilon, y_\varepsilon) < \varepsilon, \quad |p_\varepsilon - q_\varepsilon \circ P_\gamma(l)| < \varepsilon, \quad \text{tr}X_\varepsilon \leq \text{tr}Y_\varepsilon + \varepsilon,
\]
where \( l = d(x_\varepsilon, y_\varepsilon) \) and \( P_\gamma(l) \) is the parallel transport along the shortest geodesic connecting \( x_\varepsilon \) and \( y_\varepsilon \).
One can see that if we take \( u \) as a constant function in above theorems and \( v \) is a \( C^2 \) function we can obtain the estimates about Hessian and Laplacian respectively. Therefore the above theorems generalizes Omori and Yau’s maximum principles. It is well known that Yau’s theorem has been applied to many geometrical problems and many functions in geometry problems are naturally non-differentiable.

We can follow the text of Users’ Guide by Crandall, Ishii and Lions [CIL] to obtain the corresponding results about existence, uniqueness of partial differential equations. We hope to apply the results to study more equations as well as some geometrical problems in the future.

The rest of this paper is organized as follows. In section 2 we give some basic notations and definitions for viscosity solutions. In section 3, we prove the comparison theorems for distance function. The sections 4 and 5 deal with maximum principles for viscosity solutions on compact and complete Riemannian manifolds respectively. Finally we consider the parabolic equations in section 6.

Acknowledgements. We wish to thank Xuehong Zhu who carefully read the entire manuscript and gave us helpful comments and suggestions.

2 Preliminaries

Let \( M \) denote a Riemannian manifold. \( TM \) and \( T^*M \) are tangent and cotangent bundles respectively. \( S^2T^*M \) is the bundle of symmetric covariant tensors over \( M \). Denote by \( F(M) \) the bundle product of \( M \times \mathbb{R}, T^*M \) and \( S^2T^*M \).

Given a function \( f \in C^2(M, \mathbb{R}) \), the Hessian of \( f \) is defined as

\[
D^2f(X,Y) = \langle \nabla X \nabla f, Y \rangle = X(Yf) - (\nabla_X Y)f
\]

where \( X, Y \) are vector fields on \( M \) and \( \nabla \) is the Riemannian connection on \( M \), hence \( D^2f \in S^2T^*M \). In this paper, we abuse the notations to write \( A = D^2f \) for a matrix \( A \) which means \( \langle AX, Y \rangle = D^2f(X, Y) \) for all \( X, Y \in TM \). For a differentiable map \( \phi \) between two Riemannian manifolds \( N \) and \( M \) and a function \( f \in C^2(M, \mathbb{R}) \), we have

\[
D^2f \circ \phi(\xi, \eta) = \xi(\eta(f \circ \phi)) - (\nabla^N_\xi \eta)(f \circ \phi),
\]

where \( \xi, \eta \) are vector fields on \( N \) and \( \nabla^N \) is the Riemannian connection on \( N \). As a special case, for any fixed point \( x_0 \in M \), we take \( N \) as \( T_{x_0}M \) and \( \phi \) as the exponential map \( \exp : T_{x_0}M \to M \), then in this normal coordinates the (7) implies, at point \( O \),

\[
D^2f \circ \phi(\xi, \eta) = \xi(\eta(f \circ \phi)).
\]

Besides this, it is well known that the differential of the exponential map \( d\exp \) at \( O \) gives an isomorphism between \( T_0(T_{x_0}M) \) and \( T_{x_0}M \).
The viscosity solution theory applies to certain partial differential equations of the form \( F(x, u, Du, D^2u) = 0 \) where \( F : \mathcal{F}(M) \to \mathbb{R} \) is a continuous function. As in [CIL], we require \( F \) to satisfy some fundamental monotonicity conditions called proper which are made up of the two conditions

\[
F(x, r, p, X) \leq F(x, s, p, X) \quad \text{whenever} \quad r \leq s;
\]

and

\[
F(x, r, p, X) \leq F(x, r, p, Y) \quad \text{whenever} \quad Y \leq X;
\]

where \( (x, r, p, X) \in \mathcal{F}(M) \) and \( (x, s, p, Y) \in \mathcal{F}(M) \).

For any function \( u : M \to \mathbb{R} \), we define

**Definition 2.1.**

\[
J^{2,+}_2(u(x)) = \{(p, X) \in T^*_x M \times S^2T^*_x M, \quad \text{satisfies (12).}\}
\]

where (12) is

\[
u(x) \leq u(x_0) + p(\exp_{x_0}^{-1} x) + \frac{1}{2}X(\exp_{x_0}^{-1} x, \exp_{x_0}^{-1} x) + o(|\exp_{x_0}^{-1} x|^2), \quad \text{as} \quad x \to x_0,
\]

where \( \exp_{x_0} : T_{x_0} M \to M \) is the exponential map. And \( J^{2,-}_2(u(x_0)) \) is defined as

\[
J^{2,-}_2(u(x_0)) = \{(p, X) \in T^*_x M \times S^2T^*_x M \text{ such that } (-p, -X) \in J^{2,+}(-u)(x_0)\}.
\]

**Remark 2.2.** (12) is equivalent to that the function \( \bar{u} \) on \( T_{x_0} M \) defined by \( \bar{u}(y) = u(\exp_{x_0} y) \) satisfies

\[
\bar{u}(y) \leq \bar{u}(0) + \langle p, y \rangle + \frac{1}{2}X(y, y) + o(|y|^2), \quad \text{as} \quad y \to 0.
\]

So we can identify \( T_0(T_{x_0} M) \) with \( T_{x_0} M \) such that \( (p, X) \in J^{2,+}_2(u(x_0)) \) if and only if \( (p, X) \in J^{2,+}_2(\bar{u}(0)) \).

We also use the following notations.

- \( \text{USC}(M) = \{\text{upper semicontinuous functions on } M\} \),
- \( \text{LSC}(M) = \{\text{lower semicontinuous functions on } M\} \).

**Definition 2.3.** A viscosity subsolution of \( F = 0 \) on \( M \) is a function \( u \in \text{USC}(M) \) such that

\[
F(x, u, p, X) \leq 0 \quad \text{for all} \quad x \in M \quad \text{and} \quad (p, X) \in J^{2,+}_2(u(x)).
\]
A viscosity supersolution of $F = 0$ on $M$ is a function $u \in \text{LSC}(M)$ such that
\begin{equation}
F(x,u,p,X) \geq 0 \quad \text{for all } x \in M \text{ and } (p,X) \in J^{2,-}u(x).
\end{equation}

$u$ is a viscosity solution of $F = 0$ on $M$ if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ on $M$.

Following [CIL], we can similarly define $\bar{J}^{2,+}u(x)$, $\bar{J}^{2,-}u(x)$ as
\begin{equation}
\bar{J}^{2,+}u(x) = \left\{ (p,X) \in T_{x_0}^*M \times S^2T_{x_0}^*M, \text{ such that } (x_0, u(x_0), p, X) \text{ is a limit point of } \right. \\
\left. (x_k, u(x_k), p_k, X_k) \in J^{2,+}u(x_k) \text{ in the topology of } \mathcal{F}(M), \right\}
\end{equation}
and
\begin{equation}
\bar{J}^{2,-}u(x) = \left\{ (p,X) \in T_{x_0}^*M \times S^2T_{x_0}^*M, \text{ such that } (x_0, u(x_0), p, X) \text{ is a limit point of } \right. \\
\left. (x_k, u(x_k), p_k, X_k) \in J^{2,-}u(x_k) \text{ in the topology of } \mathcal{F}(M). \right\}
\end{equation}

3 **Hessian-type comparison Theorem for $d(x, y)^2$**

One of the key points to generalize the maximum principle to a complete Riemannian manifold is a new Hessian comparison theorem for Riemannian manifold. This is essentially an application of second variational formula for the arclength.

Let $M$ be a Riemannian manifold. We define the function of square of the distance function as $\varphi : M \times M \to \mathbb{R}$ as
\begin{equation}
\varphi(x,y) = d(x,y)^2.
\end{equation}
It is well known that this function is smooth when $d(x,y)$ is small. We will prove

**Theorem 3.1.** Let $M$ be a connected Riemannian manifold with sectional curvature bounded below by $-\kappa^2$. Given two points $x, y \in M$ with $d(x,y) < \min\{i(x), i(y)\}$, $\gamma : [0,l] \to M$ is the unique geodesic of unit speed with $\gamma(0) = x$ and $\gamma(l) = y$. Denote by $P_\gamma(t) : T_{\gamma(0)}M \to T_{\gamma(t)} M$ the parallel transport along $\gamma$. Then any two vectors $V_1$ and $V_2$ satisfying $\langle V_1, \gamma'(0) \rangle = \langle V_2, \gamma'(l) \rangle = 0$, the Hessian of the square of distance function $\varphi$ on $M \times M$ satisfies
\begin{equation}
D^2\varphi((V_1, V_2), (V_1, V_2)) \leq 2l\kappa[\coth \kappa l\langle V_1, V_1 \rangle + \coth \kappa l\langle V_2, V_2 \rangle] \\
- 2l\kappa\left[\frac{2}{\sinh \kappa l}\langle V_2, P_\gamma(l)V_1 \rangle\right],
\end{equation}

Particularly,
\begin{equation}
D^2\varphi((V_1, P_\gamma(l)V_1), (V_1, P_\gamma(l)V_1)) \leq 4|V_1|^2\kappa l \tanh \frac{\kappa l}{2}.
\end{equation}
Before proving the theorem we give some remarks.

**Remark 3.2.** When \( M \) is the hyperbolic space \( \mathbb{H}^n(−\kappa^2) \), we can also write the \( D^2\varphi \) on the subspace \( \{\gamma(0)\}^\perp \times \{\gamma(l)\}^\perp \subset T_xM \times T_yM \) as

\[
2l\kappa \begin{pmatrix}
\coth \kappa l & -\frac{1}{\sinh \kappa l}
\frac{1}{\sinh \kappa l} & I \coth \kappa l
\end{pmatrix}.
\]

**Remark 3.3.** The estimates in the theorem is sharp in sense that all inequalities becomes equalities for space forms. The theorem improves significantly Proposition 3.3 in [AFS] also part (1) of Proposition 3.1.

**Remark 3.4.** Note that we allow \( \kappa \) to be an imaginary and in case that the curvature is bounded below by a positive constant, we can get a corresponding estimate.

**Remark 3.5.** We also prove a version similar to Laplacian comparison theorem when we have a Ricci curvature lower bound. We will see that it is useful in applications.

**Theorem 3.6.** Let \( M \) be a connected Riemannian manifold. Given two points \( x, y \in M \) with \( d(x, y) < \min\{i(x), i(y)\} \), \( \gamma : [0, l] \to M \) is the unique geodesic of unit speed with \( \gamma(0) = x \) and \( \gamma(l) = y \), then for any two vectors \( V_1 \in T_xM, V_2 \in T_yM \) \( D^2\varphi((V_1, V_2), (V_1, V_2)) \) satisfying \( \langle V_1, \gamma'(0) \rangle = \langle V_2, \gamma'(l) \rangle = 0 \)

\[
(18) \quad D^2\varphi((V_1, V_2), (V_1, V_2)) = 2l \int_0^l [\langle \nabla_{\gamma'} J, J \rangle - \langle R(\gamma', J)\gamma', J \rangle] dt,
\]

where \( J \) is the Jacobi field along \( \gamma \) with \( J(0) = V_1 \) and \( J(l) = V_2 \).

Recall that given a \( C^2 \) curve \( \gamma : [a, b] \to M \). A variation \( \eta \) of \( \gamma \) is a \( C^2 \) mapping \( \eta : [a, b] \times (-\epsilon_0, \epsilon_0) \to M \) for some \( \epsilon_0 > 0 \), for which \( \gamma(t) = \eta(t, 0) \) for all \( t \in [a, b] \). We write \( \partial_t \eta, \partial_\epsilon \eta \) for \( \eta_*(\partial_t), \eta_*(\partial_\epsilon) \) respectively, and denote differentiation of vector field along \( \eta \) with respect to \( \partial_t, \partial_\epsilon \) by \( \nabla_t, \nabla_\epsilon \) respectively. Then the length of \( \eta_\epsilon \), namely

\[
L(\epsilon) = \int_a^b |\partial_\epsilon \eta(t, \epsilon)| dt,
\]

is differentiable and

\[
(19) \quad \frac{dL}{d\epsilon} = \langle \partial_\epsilon \eta, \partial_\epsilon \eta/|\partial_\epsilon \eta| \rangle b^b_a - \int_a^b \langle \partial_\epsilon \eta, \nabla_t (\partial_\epsilon \eta/|\partial_\epsilon \eta|) \rangle dt.
\]

In particular, if \( \eta \) is parameterized with respect to arc length, and we set \( V(t) = (\partial_\epsilon \eta)(t, 0) \), then for the first derivative of \( L \) we have

\[
(20) \quad \frac{dL}{d\epsilon}(0) = \langle \partial_\epsilon \eta, \gamma' \rangle b^b_a - \int_a^b \langle \partial_\epsilon \eta, \nabla_t \gamma' \rangle dt.
\]
and for the second derivative of $L$ we have
\[
\frac{d^2L}{de^2}(0) = \langle \nabla_e \partial_e \eta |_{e=0}, \gamma' \rangle|_a^b - \\
\int_a^b \left[ |\nabla_t V|^2 + \langle R(\gamma', V) \gamma', V \rangle - \langle \nabla_e \partial_e \eta, \nabla_t \gamma' \rangle - \langle \gamma', \nabla_t V \rangle^2 \right] dt.
\] (21)

Let $x, y$ be two non conjugate points connected by a unique minimizing geodesic $\gamma : [0, l] \to M$ with $l = d(x, y) \leq \min\{i(x), i(y)\}$. Given two vectors $V_1 \in T_x M, V_2 \in T_y M$, there exist a unique Jacobi field $J : [0, l] \to TM$ such that $J(0) = V_1$ and $J(l) = V_2$. Thus we can construct a variation $\eta$ of $\gamma$, $\eta : [0, l] \times (-\epsilon_0, \epsilon_0) \to M$ for some $\epsilon_0 > 0$, for which $\gamma(t) = \eta(t, 0)$ for all $t \in [0, l]$ and $\partial_t \eta(0, 0) = V_1$ and $\partial_t \eta(l, 0) = V_2$. We know from the properties of Jacobi field that $\eta(\cdot, \epsilon)$ is a geodesic for each $\epsilon \in (-\epsilon_0, \epsilon_0)$. So
\[
L(\epsilon) = \int_a^b |\partial_t \eta(t, \epsilon)| dt = d(\eta(0, \epsilon), \eta(l, \epsilon)).
\] (22)

Thus, by the first variational formula, we can calculate the differential of $\varphi$,
\[
D\varphi(V_1, V_2) = 2l[\langle V_2, \gamma'(l) \rangle - \langle V_1, \gamma'(0) \rangle].
\] (23)

To compute the second differential, we only need to compute $D^2\varphi((V_1, V_2), (V_1, V_2))$ for $V_1$ and $V_2$ satisfying $\langle V_1, \gamma'(0) \rangle = \langle V_2, \gamma'(l) \rangle = 0$. In this case, $J$ are normal Jacobi fields. Note that from the construction of variation from the Jacobi field, both $\eta(0, \cdot)$ and $\eta(l, \cdot)$ are geodesics and we are working in a convex neighborhood. So
\[
D^2\varphi((V_1, V_2), (V_1, V_2)) = 2l \frac{d^2L}{de^2}(0)
\]
\[
= 2l \int_0^l [||\nabla_t J||^2 - \langle R(\gamma', J) \gamma', J \rangle] dt.
\] (24)

**PROOF.** Since the Jacobi fields minimize the index form $I(X, X)$ with the same boundary conditions $X(0) = V_1$ and $X(l) = V_2$, where
\[
I(X, X) = \int_0^l [||\nabla_{\gamma'} X||^2 - \langle R(\gamma', X) \gamma', X \rangle] dt.
\] (25)

We can obtain vector fields $V_1(t)$ and $V_2(t)$ by parallel transport of $V_1$ and $V_2$ respectively. Choose
\[
X(t) = (\cosh \kappa t - \coth \kappa l \sinh \kappa t) V_1(t) + \frac{\sinh \kappa t}{\sinh \kappa l} V_2(t).
\]
Then
\[ I(X, X) = \langle \nabla_{\gamma}X, X \rangle_0^t - \int_0^t \left[ \langle \nabla_{\gamma}X, X \rangle + \langle R(\gamma', X)\gamma', X \rangle \right] dt \]
\[ \leq \langle \nabla_{\gamma'}X, X \rangle_0^t \]
\[ = \langle \kappa(\sinh \kappa l - \coth \kappa l \cosh \kappa l)V_1(l), V_2 \rangle + \langle \kappa \coth \kappa lV_2, V_2 \rangle - \langle -\kappa \coth \kappa lV_1 + \frac{\kappa}{\sinh \kappa l}V_2(0), V_1 \rangle \]
\[ = \kappa \coth \kappa l\langle V_1, V_1 \rangle + \kappa \coth \kappa l\langle V_2, V_2 \rangle - \frac{2\kappa}{\sinh \kappa l}\langle V_2(0), V_1 \rangle. \]

The theorem follows from
\[ D^2\varphi((V_1, V_2), (V_1, V_2)) \leq 2I(X, X). \]
when \( V_2 = P_\gamma(l)V_1 \), we have
\[ D^2\varphi((V_1, P_\gamma(l)V_1), (V_1, P_\gamma(l)V_1)) \leq 2\kappa \left[ \coth \kappa l\langle V_1, V_1 \rangle + \kappa \coth \kappa l\langle P_\gamma(l)V_1, P_\gamma(l)V_1 \rangle \right] \]
\[ - 2\kappa \left[ \frac{2}{\sinh \kappa l} \langle P_\gamma(l)V_1, P_\gamma(l)V_1 \rangle \right] \]
\[ = 4\kappa \tanh \frac{\kappa l}{2}\|V_1\|^2. \]

\[ \square \]

**Theorem 3.7.** Let \( M \) be a Riemannian manifold with Ricci curvature bounded below by \(-(n - 1)\kappa^2\). Given two points \( x, y \in M \) with \( d(x, y) < \min\{i(x), i(y)\} \), \( \gamma : [0, l] \to M \) is the unique geodesic of unit speed with \( \gamma(0) = x \) and \( \gamma(l) = y \). For any normal base at \( x \), \( \{\gamma'(0), e_2, \ldots, e_n\} \).

\[ \sum_{i=2}^{n-1} D^2\varphi((e_i, P_\gamma(l)e_i), (e_i, P_\gamma(l)e_i)) \leq 4(n - 1)\kappa l \tanh \frac{\kappa l}{2}. \]

**Proof.** Similar to the above proof, we use the property that the Jacobi field minimize the index form \( I(X, X) \) with the same boundary conditions \( X(0) = V_1 \) and \( X(l) = V_2 \). We can obtain vector fields \( e_i(t) \) by parallel transport of \( e_i \). For any function \( \psi \in C^2[0, l] \) with \( \psi(0) = \psi(l) = 1 \), we choose
\[ X_i(t) = \psi(t)e_i(t). \]

Then
\[ D^2\varphi((e_i, P_\gamma(l)e_i), (e_i, P_\gamma(l)e_i)) \leq 2l \int_0^l \left[ \|\nabla_{\gamma}X_i\|^2 - \langle R(\gamma', X_i)\gamma', X_i \rangle \right] dt \]
\[ = 2l \int_0^l \|\psi'\|^2 - \psi^2 K(\gamma', e_i) \| dt, \]
where $K(X, Y)$ is the sectional curvature of the plane spanned by vectors $X, Y$. Then we have
\[
\sum_{i=2}^{n} D^2\varphi((e_i, P_\gamma(l)e_i), (e_i, P_\gamma(l)e_i)) \leq 2l \int_0^l [(n-1)|\psi'|^2 - \psi^2\text{Ric}(\gamma', \gamma')] dt
\]
\[
\leq 2(n-1)l \int_0^l [||\psi'||^2 + \kappa^2\psi^2] dt.
\]
where Ric denotes the Ricci curvature. Let
\[
\psi(t) = \cosh \kappa t + \frac{1 - \cosh \kappa l}{\sinh \kappa l} \sinh \kappa t.
\]
The claimed result follows from a direct computation.

\[\square\]

4 The maximum principle for viscosity solutions

4.1 Dirichlet problem case

The same proof of Lemma 3.1 in [CIL] gives the following lemma.

**Lemma 4.1.** Let $M$ be a compact Riemannian manifold with or without boundary, $u \in \text{USC}(M)$ $v \in \text{LSC}(M)$ and
\[
(29) \quad \mu_\alpha = \sup\{u(x) - v(y) - \alpha \frac{1}{2}d(x, y)^2, x, y \in M\}
\]
for $\alpha > 0$. Assume that $\mu_\alpha < +\infty$ for large $\alpha$ and $(x_\alpha, y_\alpha)$ satisfy
\[
(30) \quad \lim_{\alpha \to \infty} [\mu_\alpha - (u(x_\alpha) - v(y_\alpha) - \alpha \frac{1}{2}d(x_\alpha, y_\alpha)^2)] = 0.
\]
Then the following holds:

\[
(31) \quad \begin{cases} 
(i) & \lim_{\alpha \to \infty} \alpha d(x_\alpha, y_\alpha)^2 = 0, \text{ and} \\
(ii) & \lim_{\alpha \to \infty} \mu_\alpha = u(x_0) - v(x_0) = \sup(u(x) - v(x)), \\
\end{cases}
\]
where $x_0 = \lim_{\alpha \to \infty} x_\alpha$.

From Remark 2.2 we can deduce from Theorem 3.2 in [CIL] the following lemma.

**Lemma 4.2.** Let $M_1, M_2$ be Riemannian manifolds with or without boundary, $u_1 \in \text{USC}(M_1)$, $u_2 \in \text{LSC}(M_2)$ and $\varphi \in C^2(M_1 \times M_2)$. Suppose $(\hat{x}, \hat{y}) \in M_1 \times M_2$ is a local maximum of $u_1(x) - u_2(y) - \varphi(x, y)$. Then for any $\varepsilon > 0$ there exist $X_1 \in ST^*_\hat{x}M_1$ and $X_2 \in S^2T^*\hat{y}M_2$ such that
\[
(D_{\hat{x}}\varphi(\hat{x}, \hat{y}), X_i) \in \tilde{J}^{2,+} u_i(\hat{x}_i), \text{ for } i = 1, 2,
\]
and the block diagonal matrix satisfies

\[
- \left( \frac{1}{\epsilon} + \|A\| \right) I \leq \begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \leq A + \epsilon A^2
\]

where \( A = D^2 \varphi(\hat{x}, \hat{y}) \in S^2 T^* (M_1 \times M_2) \) and

\[
\|A\| = \sup \{|A(\xi, \xi)|, \xi \in T_{(\hat{x}, \hat{y})} M_1 \times M_2, |\xi| = 1\}.
\]

Now we are in a position to prove our comparison theorem.

**Theorem 4.3.** Let \( M \) be a compact Riemannian manifold with or without boundary \( \partial M \), \( F(M) \in C(F, \mathbb{R}) \) a proper function satisfying

\[
\beta(r - s) \leq F(x, r, p, X) - F(x, s, p, X) \text{ for } r \geq s,
\]

for some positive constant \( \beta \) and the following condition (H):

there exists a function \( \omega : [0, \infty) \to [0, \infty] \) satisfying \( \omega(0+) = 0 \) such that

\[
F(y, r, -\alpha \iota(\gamma'(l)), X_2) - F(x, r, -\alpha \iota(\gamma'(0)), X_1) \leq \omega(\alpha d(x, y)^2 + d(x, y))
\]

for \( X_1 \in ST^*_x M \) and \( X_2 \in S^2 T^*_y M \), satisfying \( X_1 \leq X_2 \circ P_\gamma(l) \). Here \( \iota \) is the dual map between the tangent and cotangent bundles.

Let \( u_1 \in \text{USC}(\bar{M}) \) and \( u_2 \in \text{LSC}(\bar{M}) \) be a subsolution and supersolution of \( F = 0 \) respectively. Then \( u_1 - u_2 \) cannot achieve a positive local maximum at any interior point. In particular if \( M \) is a compact Riemannian manifold with boundary \( \partial M \) and if \( u_1 \leq u_2 \) on \( \partial M \), then \( u_1 \leq u_2 \) on \( \bar{M} \).

**Proof.** For the sake of a contradiction we assume that \( 2\delta := \sup_{x \in M} \{u_1(x) - u_2(x)\} \) > 0. Then for \( \alpha \) large,

\[
0 < 2\delta \leq \mu_{\alpha} := \sup \{u_1(x) - u_2(y) - \frac{\alpha}{2} d(x, y)^2, x, y \in M\} < +\infty.
\]

There exists \((x_\alpha, y_\alpha)\) such that

\[
\mu_{\alpha} - (u_1(x_\alpha) - u_2(y_\alpha) - \frac{\alpha}{2} d(x_\alpha, y_\alpha)^2) = 0.
\]

Then the following holds:

\[
\begin{aligned}
(i) & \quad \lim_{\alpha \to \infty} \alpha d(x_\alpha, y_\alpha)^2 = 0, \text{ and } \\
(ii) & \quad \lim_{\alpha \to \infty} \mu_{\alpha} = u_1(x_0) - u_2(x_0) = 2\delta,
\end{aligned}
\]

where \( x_0 = \lim_{\alpha \to \infty} x_\alpha \).
We can suppose that there exists a convex neighborhood \( D \) of \( x_0 \) such that \( x_\alpha \in D \) and \( y_\alpha \in D \) for all \( \alpha \) large. So there exists a constant \( \kappa \) such that the sectional curvature of \( M \) is bounded below by a constant \(-\kappa^2\). Without loss of generality, we assume the diameter \( L(D) \) of \( D \) is small such that \( \cosh \kappa L(D) \leq 2 \). Let \( \gamma_\alpha : [0, l_\alpha] \to D \) be the unique normal geodesic joining \( x_\alpha \) and \( y_\alpha \), where \( l_\alpha = d(x_\alpha, y_\alpha) \). In this case

\[
\nabla \varphi = (-2l\gamma'(0), 2l\gamma'(l_\alpha)).
\]

We can decompose \( T_x M \times T_y M \) as \((\gamma'(0) \mathbb{R} \oplus \{\gamma'(0)\} \mathbb{L}) \times (\gamma'(0) \mathbb{R} \oplus \{\gamma'(l_\alpha)\} \mathbb{L})\). According to this decomposition and from Theorem 3.1 we know \( A_\alpha := D^2(\frac{\kappa}{2}d^2) \) satisfies

\[
A_\alpha \leq \alpha \begin{pmatrix}
1 & -1 \\
\frac{l_\alpha}{\sinh \kappa l_\alpha} I & -1 \\
-\frac{l_\alpha}{\sinh \kappa l_\alpha} I & \frac{l_\alpha}{\sinh \kappa l_\alpha} I \\
\end{pmatrix},
\]

where \( I \) is \((n-1) \times (n-1)\) unit matrix. Then

\[
\|A_\alpha\| \leq \alpha \frac{\kappa l_\alpha}{\sinh \kappa l_\alpha} (\cosh \kappa l_\alpha + 1) \leq 3\alpha,
\]

Here we have assumed that \( \kappa l_\alpha \) is not big. For any \( \varepsilon > 0 \) there exist \( X_{1\alpha} \in S^2T^*_x M \) and \( X_{2\alpha} \in S^2T^*_y M \) such that the block diagonal matrix satisfies

\[
(36) \quad - \left( \frac{1}{\varepsilon} + \|A_\alpha\| \right) I \leq \begin{pmatrix} X_{1\alpha} & 0 \\ 0 & -X_{2\alpha} \end{pmatrix} \leq A_\alpha + \varepsilon A_\alpha^2.
\]

Choosing \( \varepsilon = \alpha \frac{1}{(\cosh \kappa L(D) + 1)} \) we have the following

\[
(37) \quad -4\alpha I \leq \begin{pmatrix} X_{1\alpha} & 0 \\ 0 & -X_{2\alpha} \end{pmatrix} \leq 4\alpha \begin{pmatrix} 1 & -1 \\
\frac{l_\alpha}{\sinh \kappa l_\alpha} I & -\frac{l_\alpha}{\sinh \kappa l_\alpha} I \\
-\frac{l_\alpha}{\sinh \kappa l_\alpha} I & \frac{l_\alpha}{\sinh \kappa l_\alpha} I \\
\end{pmatrix},
\]

Let us denote \( l \) the dual map between the tangent and cotangent bundles. We have proved that for a sufficiently large \( \alpha \) at a maximum \((x_\alpha, y_\alpha)\) of \( u_1(x) - u_2(y) - \frac{\alpha}{2}d(x_\alpha, y_\alpha)^2 \) there exist \((-\alpha l_\alpha l_\gamma'(0), X_{1\alpha}) \in J^{2+}_M (u_1(x_\alpha)) \) and \((-\alpha l_\alpha l_\gamma'(l_\alpha), X_{2\alpha}) \in J^{2-}_M (u_2(y_\alpha)) \) such that \( X_{2\alpha} \geq X_{1\alpha} \circ P_\gamma(l) - 4\alpha l_\alpha^2 \kappa^2 Id \) holds.

Since \( u_1 \) and \( u_2 \) are subsolution and supersolution we have

\[
(38) \quad F(x_\alpha, u_1(x_\alpha), -\alpha l_\alpha l_\gamma'(0), X_{1\alpha}) \leq 0 \leq F(y_\alpha, u_2(y_\alpha), -\alpha l_\alpha l_\gamma'(l_\alpha), X_{2\alpha}).
\]
Then for $\alpha$ sufficiently large

$$0 < \beta \delta < \beta (u_1(x) - u_2(y)) - \frac{\alpha}{2} d^2(x_a, y_a)$$

$$\leq F(x_a, u_1(x), -\alpha l_a(t(\gamma'_a(0)), X_1 \circ P_\gamma(0))) - F(x_a, u_2(y), -\alpha l_a(t(\gamma'_a(0)), X_1 \circ P_\gamma(0))) + F(y_a, u_2(y), -\alpha l_a(t(\gamma'_a(0)), X_2 \circ P_\gamma(0))) + F(y_a, u_2(y), -\alpha l_a(t(\gamma'_a(0)), X_2 \circ P_\gamma(0)))$$

$$\leq \omega(\alpha d(x_a, y_a)^2 + d(x_a, y_a)) - \frac{\beta \alpha}{2} d^2(x_a, y_a) +$$

$$- F(y_a, u_2(y), -\alpha l_a(t(\gamma'_a(0)), X_2 \circ P_\gamma(0))) + F(y_a, u_2(y), -\alpha l_a(t(\gamma'_a(0)), X_2 \circ P_\gamma(0))) +$$

$$\leq \omega(\alpha d(x_a, y_a)^2 + d(x_a, y_a)) - \frac{\beta \alpha}{2} d^2(x_a, y_a) +$$

$$- F(y_a, u_2(y), -\alpha l_a(t(\gamma'_a(0)), X_2 \circ P_\gamma(0))) + F(y_a, u_2(y), -\alpha l_a(t(\gamma'_a(0)), X_2 \circ P_\gamma(0)))$$

Let $\alpha \to +\infty$. Since $F$ is continuous in $X$, the left hand side goes to zero and we have arrived at a contradiction.

5  Viscosity maximum principle on complete Riemannian manifolds

Maximum principles for $C^2$ functions on complete Riemannian manifolds were already done by Omori [O] and Yau [Y], but our approach and the spirit of the results are quite different. In [Y], Yau developed gradient estimate for $C^2$ functions satisfying some inequalities and proved a maximum principle which is well known and has many important applications in geometry. In this section we will generalize both Omori and Yau’s maximum principle to non-differentiable functions. The approach is quite different. Even for $C^2$ function we present here a new proof for Omori-Yau’s maximum principle.

**Theorem 5.1.** Let $M$ be a complete Riemannian manifold with sectional curvature bounded below by a constant $-\kappa^2$. Let $u \in \text{USC}(M)$, and $v \in \text{LSC}(M)$ be two functions satisfying

$$\mu_0 := \sup_{x \in M} [u(x) - v(x)] < +\infty.$$  

Assume that $u$ and $v$ are bounded from above and below respectively and there exists a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = \omega(0+) = 0$ such that

$$u(x) - u(y) \leq \omega(d(x, y)).$$
Then for each $\varepsilon > 0$, there exist $x_\varepsilon, y_\varepsilon \in M$, such that $(p_\varepsilon, X_\varepsilon) \in \overline{J}^2 + u(x_\varepsilon)$, $(q_\varepsilon, Y_\varepsilon) \in \overline{J}^2 - v(y_\varepsilon)$, such that

$$u(x_\varepsilon) - v(y_\varepsilon) \geq \mu_0 - \varepsilon,$$

and such that

$$d(x_\varepsilon, y_\varepsilon) < \varepsilon, \quad |p_\varepsilon - q_\varepsilon \circ P_{\gamma}(l)| < \varepsilon, \quad X_\varepsilon \leq Y_\varepsilon \circ P_{\gamma}(l) + \varepsilon P_{\gamma}(l),$$

where $l = d(x_\varepsilon, y_\varepsilon)$ and $P_{\gamma}(l)$ is the parallel transport along the shortest geodesic connecting $x_\varepsilon$ and $y_\varepsilon$.

**Proof.** We divide the proof into two parts.

**Part 1.** Without loss of generality, we assume that $\mu_0 > 0$. Otherwise we replace $u$ by $u - \mu_0 + 1$. For each $\alpha > 0$, we take $\hat{x}_\alpha \in M$ such that

$$u(\hat{x}_\alpha) - v(\hat{x}_\alpha) + \omega(\sqrt{\frac{\mu_0}{\alpha}}) \geq \mu_0.$$

Let $\lambda_\alpha = -\frac{1}{\ln \omega(\sqrt{\frac{\mu_0}{\alpha}})}$. We consider the following maximization

$$\sigma_\alpha = \sup_{x, y \in M} [u(x) - v(y) - \frac{\alpha}{2}d(x,y)^2 - \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha,x)^2 - \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha,y)^2]$$

Taking $\alpha$ large if necessary, we have $(x_\alpha, y_\alpha)$ satisfying

$$\sigma_\alpha = u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}d(x_\alpha, y_\alpha)^2 - \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha, x_\alpha)^2 - \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha, y_\alpha)^2$$

Let

$$\sigma_\alpha^0 = \sup_{x, y \in M} [u(x) - v(x) - \lambda_\alpha d(\hat{x}_\alpha, x)^2]$$

It is straightforward to see that

$$\sigma_\alpha \geq \sigma_\alpha^0 \geq u(\hat{x}_\alpha) - v(\hat{x}_\alpha) \geq \mu_0 - \omega(\sqrt{\frac{\mu_0}{\alpha}})$$

and it follows from the boundedness of $u$ and $v$ that $\sigma_\alpha$ is bounded from below by some constant $\sigma_* \leq \sigma_\alpha^0$. Then there exists a constant $C$ such that

$$\frac{\alpha}{2}d(x_\alpha, y_\alpha)^2 \leq C$$

and

$$\frac{\alpha}{2}d(x_\alpha, y_\alpha)^2 + \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha, x_\alpha)^2 + \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha, y_\alpha)^2 = u(x_\alpha) - v(y_\alpha) - \sigma_\alpha \leq u(x_\alpha) - u(y_\alpha) + u(y_\alpha) - v(y_\alpha) - \mu_0 + \omega(\sqrt{\frac{\mu_0}{\alpha}})$$

$$\leq \omega(d(x_\alpha, y_\alpha)) + u(y_\alpha) - v(y_\alpha) - \mu_0 + \omega(\sqrt{\frac{\mu_0}{\alpha}})$$

$$\leq \omega(d(x_\alpha, y_\alpha)) + \omega(\sqrt{\frac{\mu_0}{\alpha}}) \to 0.$$
So (44) can be improved as \( \frac{\alpha}{2}d(x, y)^2 \leq \frac{\mu_0}{\alpha} \). Using the same process of (45), we have

\[
\frac{\alpha}{2}d(x, y)^2 + \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha, x)^2 + \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha, y)^2 \leq 2\omega\left(\sqrt{2}\frac{\mu_0}{\alpha}\right).
\]

Thus, when \(\alpha \to \infty\),

\[
d(x, y)^2 \leq \frac{4\omega\left(\sqrt{2}\frac{\mu_0}{\alpha}\right)}{\alpha} \to 0,
\]

\[
d(x, \hat{x}_\alpha)^2 + d(y, \hat{x}_\alpha)^2 \leq \frac{4\omega\left(\sqrt{2}\frac{\mu_0}{\alpha}\right)}{\lambda_\alpha} \to 0.
\]

**Part 2.** We apply now Theorem 3.2 in [CIL] to \(\varphi_\alpha(x, y) = \frac{\alpha}{2}d(x, y)^2 + \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha, x)^2 + \frac{\lambda_\alpha}{2}d(\hat{x}_\alpha, y)^2\). We have for any \(\delta > 0\) there exist \(X_\alpha \in ST^*_x M\) and \(Y_\alpha \in ST^*_y M\) such that

\[
(D_x\varphi_\alpha(x, y), X_\alpha) \in J^{2, u}(x, \alpha), \quad (-D_y\varphi_\alpha(x, y), Y_\alpha) \in J^{2, v}(y, \alpha),
\]

and the block diagonal matrix satisfies

\[
-\left(\frac{1}{\delta} + \|A_\alpha\|\right) I \leq \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \leq A_\alpha + \delta A_\alpha^2,
\]

where \(A_\alpha = D^2\varphi_\alpha(x, y) \in S^2T^*(M \times M)\) and

\[
\|A_\alpha\| = \sup\{|A_\alpha(\xi, \xi)|, \xi \in T(x, y) M \times M, |\xi| = 1\}.
\]

We denote

\[
P_\alpha = \frac{\alpha}{2}D^2d(x, y)^2, \quad Q_\alpha = \frac{\lambda_\alpha}{2}D^2[d(\hat{x}_\alpha, x)^2 + d(\hat{x}_\alpha, y)^2].
\]

Hence \(A_\alpha = P_\alpha + Q_\alpha\). Since \(\lim_{\alpha \to +\infty} x_\alpha = \lim_{\alpha \to +\infty} y_\alpha = \lim_{\alpha \to +\infty} \hat{x}_\alpha\), we can assume \(\alpha\) large enough so that all \(x_\alpha\) and \(y_\alpha\) are in a small convex neighborhood \(D_\alpha\) of \(\hat{x}_\alpha\). Let \(\gamma\) be the minimal geodesic connecting \(x_\alpha\) and \(y_\alpha\). We decompose \(T_{x_\alpha} M \times T_{y_\alpha} M\) as \((\gamma'(0) \mathbb{R} \oplus \{\gamma'(0)\}^\perp) \times (\gamma'(0) \mathbb{R} \oplus \{\gamma'(0)\}^\perp)\). According this decomposition and from Theorem 3.1, we know \(P_\alpha\) satisfies

\[
P_\alpha \leq \alpha\begin{pmatrix} 1 & -\frac{t_\alpha}{\sinh \kappa l_\alpha} I \\ \frac{t_\alpha}{\sinh \kappa l_\alpha} I & 1 \end{pmatrix},
\]

where \(I\) is \((n - 1) \times (n - 1)\) unit matrix. Then

\[
\|P_\alpha\| \leq \alpha\frac{\kappa l_\alpha}{\sinh \kappa l_\alpha}(\cosh \kappa l_\alpha + 1).
\]
We decompose $T_{x_0}M \times T_{y_0}M$ as $(\nabla d(\hat{x}_a, \cdot)\mathbb{R} \oplus \{\nabla d(\hat{x}_a, \cdot)\}^\perp) \times (\nabla d(\hat{x}_a, \cdot)\mathbb{R} \oplus \{\nabla d(\hat{x}_a, \cdot)\}^\perp)$. According this decomposition, the classical Hessian comparison Theorem implies that $Q_\alpha$ satisfies

$$Q_\alpha \leq \lambda_\alpha \begin{pmatrix} 1 & Il_\alpha \coth \kappa l_\alpha \\ Il_\alpha \coth \kappa l_\alpha & 1 \end{pmatrix}.$$  

Since the $d(x_\alpha, y_\alpha)$, $d(\hat{x}_\alpha, x_\alpha)$, and $d(\hat{x}_\alpha, y_\alpha)$ tend to zero as $\alpha \to +\infty$, we can assume the diameter of the convex neighborhood $D_\alpha$ is small so that both $\|P_\alpha\|$ and $\|Q_\alpha\|$ are bounded by $4\alpha$ and $2\lambda_\alpha$ respectively.

So for any $V \in T_{x_\alpha}M$ and $V \perp \gamma'$,

$$\langle X_\alpha V, V \rangle - \langle Y_\alpha P_\gamma(l_\alpha)V, P_\gamma(l_\alpha)V \rangle \leq (P_\alpha + Q_\alpha + \delta P_\alpha^2 + \delta Q_\alpha^2 + \delta P_\alpha Q_\alpha + Q_\alpha P_\alpha)(V, P_\gamma(l_\alpha)V)^2 \leq P_\alpha(V, P_\gamma(l_\alpha)V)^2 + \|Q_\alpha\| + 2\delta(\|P_\alpha^2\| + \|Q_\alpha^2\|)(V, P_\gamma(l_\alpha)V)^2.$$

Since $\|P_\alpha\| \leq 4\alpha$, $\|Q_\alpha\| \leq 2\lambda_\alpha$, we have

$$\langle X_\alpha V, V \rangle - \langle Y_\alpha P_\gamma(l_\alpha)V, P_\gamma(l_\alpha)V \rangle \leq 2\kappa^2 \alpha l_\alpha^2 |V|^2 + 4\lambda_\alpha |V|^2 + 16\delta(16\alpha^2 + 4\lambda_\alpha^2)|V|^2.$$

For any $\varepsilon > 0$, we can choose $\delta = \frac{\varepsilon}{64(4\alpha^2 + \lambda_\alpha^2)}$. Since $\alpha l_\alpha^2 \to 0$ and , then there exists $\alpha_1 > 0$ such that when $\alpha > \alpha_1$

$$\langle X_\alpha V, V \rangle - \langle Y_\alpha P_\gamma(l_\alpha)V, P_\gamma(l_\alpha)V \rangle \leq \frac{\varepsilon}{2} |V|^2.$$

For any $V \in T_{x_\alpha}M$, we have

$$D_x \varphi_\alpha(x_\alpha, y_\alpha)(V) = \alpha \lambda_\alpha \langle -\gamma'(0), V \rangle + \lambda_\alpha d(\hat{x}_\alpha, x_\alpha) \langle \nabla d(\hat{x}_\alpha, x_\alpha), V \rangle$$
$$D_y \varphi_\alpha(x_\alpha, y_\alpha)(P_\alpha(l_\alpha)V) = \alpha \lambda_\alpha \langle \gamma'(l_\alpha), P_\alpha(l_\alpha)V \rangle + \lambda_\alpha d(\hat{x}_\alpha, y_\alpha) \langle \nabla d(\hat{x}_\alpha, y_\alpha), V \rangle,$$

then we have

$$D_x \varphi_\alpha(x_\alpha, y_\alpha)(V) + D_y \varphi_\alpha(x_\alpha, y_\alpha)(P_\alpha(l_\alpha)V) = \lambda_\alpha d(\hat{x}_\alpha, x_\alpha) \langle \nabla d(\hat{x}_\alpha, x_\alpha), V \rangle + \lambda_\alpha d(\hat{x}_\alpha, y_\alpha) \langle \nabla d(\hat{x}_\alpha, y_\alpha), V \rangle,$$

Since $\lim_{\alpha \to +\infty} \lambda_\alpha = 0$ and $\lim_{\alpha \to +\infty} d(\hat{x}_\alpha, x_\alpha) = 0$, then there exists $\alpha_2 > 0$ such that when $\alpha > \alpha_2$, 

$$|D_x \varphi_\alpha(x_\alpha, y_\alpha) + D_y \varphi_\alpha(x_\alpha, y_\alpha) \circ P_\alpha(l_\alpha)| < \varepsilon.$$ 

There exists $\alpha_3 > 0$ such that when $\alpha > \alpha_3$, $\omega(\sqrt{\frac{\varepsilon}{\alpha}}) < \varepsilon$. Therefore

$$u(x_\alpha) - v(y_\alpha) = \sigma_\alpha + \frac{\alpha}{2} d(x_\alpha, y_\alpha)^2 + \frac{\lambda_\alpha}{2} d(\hat{x}_\alpha, x_\alpha)^2 + \frac{\lambda_\alpha}{2} d(\hat{x}_\alpha, y_\alpha)^2 \geq \mu_0 - \omega(\sqrt{\frac{\mu_0}{\alpha}}) > \mu_0 - \varepsilon.$$

Finally we choose $\bar{\alpha} = \max\{\alpha_1, \alpha_2, \alpha_3\}$ such that all the inequalities in the theorem are satisfied. The proof is complete. \qed
Theorem 5.2. Let $M$ be a complete Riemannian manifold with Ricci curvature bounded below by a constant $-(n-1)\kappa^2$. Let $u \in \text{USC}(M)$, $v \in \text{LSC}(M)$ be two functions satisfying

\[(49) \mu_0 := \sup_{x \in M} |u(x) - v(x)| < +\infty.\]

Assume that $u$ and $v$ are bounded from above and below respectively and there exists a function $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\omega(0) = \omega(0^+) = 0$ such that

\[(50) u(x) - u(y) \leq \omega(d(x,y)).\]

Then for each $\varepsilon > 0$, there exist $x_\varepsilon, y_\varepsilon \in M$, $(p_\varepsilon, X_\varepsilon) \in \bar{J}^{2,+}u(x_\varepsilon)$, $(q_\varepsilon, Y_\varepsilon) \in \bar{J}^{2,-}v(y_\varepsilon)$, such that

\[u(x_\varepsilon) - v(y_\varepsilon) \geq \mu_0 - \varepsilon,\]

and such that

\[d(x_\varepsilon, y_\varepsilon) < \varepsilon, \quad |p_\varepsilon - q_\varepsilon \circ P_\gamma(l)| < \varepsilon, \quad \text{tr}X_\varepsilon \leq \text{tr}Y_\varepsilon + \varepsilon,\]

where $l = d(x_\varepsilon, y_\varepsilon)$ and $P_\gamma(l)$ is the parallel transport along the shortest geodesic connecting $x_\varepsilon$ and $y_\varepsilon$.

Proof. Part 1 is the same as the above theorem. We start our proof from the Part 2. Let $D_\alpha$ be the convex neighborhood of $\hat{x}_\alpha$ chosen in the proof of the last theorem and $-\kappa_\alpha^2$ is the lower bound of the sectional curvature in $D_\alpha$. We can assume the diameter of $D_\alpha$ is small such that both $\|P_\alpha\|$ and $\|Q_\alpha\|$ are bounded by $2\alpha$ and $2\lambda\alpha$ respectively. By Theorem 3.8, we have, for any orthonormal base $\{e_1, e_2, \cdots, e_n\}$ at $x_\alpha$ with $e_1 = \gamma'(0)$,

\[\sum_{i=1}^{n} \langle X_\alpha e_i, e_i \rangle - \sum_{i=1}^{n} \langle Y_\alpha P_\gamma(l_\alpha) e_i, P_\gamma(l_\alpha) e_i \rangle \leq 2(n-1)\kappa\alpha l_\alpha \frac{\sinh \frac{2\alpha l_\alpha}{2}}{\cosh \frac{2\alpha l_\alpha}{2}} + 4(n-1)\delta(\alpha^2 + \lambda_\alpha^2).\]

Here we may change the base if necessary since we are computing the traces of $X_\alpha$ and $Y_\alpha$ respectively. Therefore we have $\alpha_1 > 0$, such that when $\alpha > \alpha_1$,

\[\text{tr}X_\alpha - \text{tr}Y_\alpha \leq \frac{\varepsilon}{2} < \varepsilon.\]

The rest of the proof is the same as that of the preceding theorem.

As a corollary we have the famous Yau’s maximum principle.

Corollary 5.3. Let $M$ be a complete Riemannian manifold with Ricci curvature bounded below by a constant $-(n-1)\kappa^2$ and $f$ a $C^2$ function on $M$ bounded from below. Then for any $\varepsilon > 0$ there exists a point $x_\varepsilon \in M$ such that

\[f(x_\varepsilon) \leq \inf f + \varepsilon, \quad |\nabla f|(x_\varepsilon) < \varepsilon, \quad \Delta f(x_\varepsilon) > -\varepsilon.\]

Proof. Let $u = \inf f$ and $v = f$. $\omega$ can be chosen to be constant function. It is straightforward to verify that all conditions in Theorem are satisfied.
6 The maximum principle for parabolic PDE

For any function \( u : [0,T] \times M \rightarrow \mathbb{R} \), we define

**Definition 6.1.** We define
\[
P_M^{2+}u(t_0, x_0) = \{ (a, p, X) \in \mathbb{R} \times T^{*}_{x_0} M \times S^{2}T^{*}_{x_0} M, \text{ satisfies (51).} \}
\]
where (51) is
\[
u(t, x) \leq u(t_0, x_0) + a(t - t_0) + (p(\exp_{x_0}^{-1} x) + \frac{1}{2} X(\exp_{x_0}^{-1} x, \exp_{x_0}^{-1} x) + o(|x - x_0|^2), \text{ as } x \rightarrow x_0, t \rightarrow t_0.
\]

We also set \( P_M^{2-}u(t_0, x_0) = -P_M^{2+}(-u)(t_0, x_0) \).

Correspondingly, we set
\[
P_M^{2+}u(t_0, x_0) = \left\{ (a, p, X) \in \mathbb{R} \times T^{*}_{x_0} M \times S^{2}T^{*}_{x_0} M, \text{ such that } (t_0, x_0, u(t_0, x_0), a, p, X) \right\}
\]
\[\text{is a limit point of } (t_k, x_k, u(t_k, x_k), a_k, p_k, X_k), (a_k, p_k, X_k) \in P_M^{2+}u(t_k, x_k) \right\}
\]
as well as \( P_M^{2-}u(t_0, x_0) = -P_M^{2+}(-u)(t_0, x_0) \).

**Definition 6.2.** A viscosity subsolution of \( \partial_t u + F = 0 \) on \((0, T) \times M\) is a function \( u \in \text{USC}((0, T) \times M) \) such that
\[
a + F(x, u, p, X) \leq 0
\]
for all \((t, x) \in (0, T) \times M\) and \((a, p, X) \in P_M^{2+}u(t, x)\). A viscosity supersolution of \( \partial_t u + F = 0 \) on \((0, T) \times M\) is a function \( u \in \text{LSC}((0, T) \times M) \) such that
\[
a + F(x, u, p, X) \geq 0
\]
for all \((t, x) \in (0, T) \times M\) and \((a, p, X) \in P_M^{2-}u(t, x)\). \( u \) is a viscosity solution of \( \partial_t u + F = 0 \) on \((0, T) \times M\) if it is both a viscosity subsolution and a viscosity supersolution of \( \partial_t u + F = 0 \).

**Lemma 6.3.** Let \( u_i \in \text{USC}((0, T) \times M_i), i = 1, 2 \) and let \( \varphi \in C^{1,2}((0, T) \times M_1 \times M_2) \). Suppose that \( \hat{t} \in (0, T) \) and \( \hat{x}_1 \in M_1, \hat{x}_2 \in M_2 \) satisfy:
\[
u_1(\hat{t}, \hat{x}_1) + u_2(\hat{t}, \hat{x}_2) - \varphi(\hat{t}, \hat{x}_1, \hat{x}_2) \geq u_1(t, x_1) + u_2(t, x_2) - \varphi(t, x_1, x_2)
\]
for \( t \in (0, T) \) and \( x_1 \in M_1, x_2 \in M_2 \). Assume that there exists an \( r > 0 \) such that for every \( K > 0 \) there exists a \( C \) such that for \( i = 1, 2 \): there are \( b_1, b_2 \in \mathbb{R} \) and \( X_i \in S^{2}T^{*}_{x_i} M_i \) such that
\[
b_i \leq C \text{ whenever } (b_i, q_i, X_i) \in P_{M_i}^{2+}u(t, x_i), \text{ d}(x_i, \hat{x}_i) + |t - \hat{t}| \leq r \text{ and } |u_i(t, x_i)| + |q_i| + \|X_i\| \leq K.
\]
Then for each $\varepsilon > 0$, there exist $X_i \in ST_x^*M_i$ such that

(i) $(b_i, D_{x_i}\varphi(\hat{t}, \hat{x}_1, \hat{x}_2), X_i) \in \bar{P}^{2, +}_{M_i}u(\hat{t}, \hat{x}_i)$, for $i = 1, 2$
(ii) $-(\frac{1}{\varepsilon} + \|A\|) I \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq A + \varepsilon A^2$
(iii) $b_1 + b_2 = \partial_t\varphi(\hat{t}, \hat{x}_1, \hat{x}_2)$

where $A = D^2_{(x_1, x_2)}\varphi(\hat{t}, \hat{x}_1, \hat{x}_2)$.

Now we are in a position to assert our comparison theorem for parabolic situation.

**Theorem 6.4.** Let $M$ be a compact Riemannian manifold with boundary $\partial M$, $F \in C([0, +\infty) \times \mathcal{F}(M), \mathbb{R})$ be a proper function such that there exists a function $\omega : [0, \infty] \to [0, \infty]$ satisfying (35). Let $u \in USC([0, T] \times \bar{M})$ and $v \in LSC([0, T] \times \bar{M})$ be a subsolution and supersolution of

$$\partial_t u + F = 0$$

respectively such that $u \leq v$ on $[0, T] \times \partial M$ and $u|_{t=0} \leq v|_{t=0}$ on $M$. Then $u \leq v$ on $[0, T] \times \bar{M}$.

**Proof.** We first observe that for each $\varepsilon > 0$, $\tilde{u} = u - \varepsilon/(T - t)$ satisfies

$$\partial_t \tilde{u} + F(t, x, \tilde{u} + \varepsilon/(T - t), D\tilde{u}, D^2\tilde{u}) + \frac{\varepsilon}{(T - t)^2} \leq 0.$$

We thus only need to prove $\tilde{u} \leq v$. In fact it suffices to prove the comparison theorem for the subsolution $u$ satisfying

(i) $\partial_t u + F(t, x, u + \varepsilon/(T - t), Du, D^2 u) + \frac{\varepsilon}{(T - t)^2} \leq 0$
(ii) $\lim_{t \uparrow T} u(t, x) = -\infty$, uniformly on $M$.

We can see that $u, -v$ be bounded above. For the sake of a contradiction we assume that

$$\delta := \sup_{(t, x) \in [0, T] \times M} \{u(t, x) - v(t, x)\} > 0.$$

Then for $\alpha$ large,

$$0 < \delta \leq \mu_\alpha := \sup_{t \in [0, T], x, y \in M} \{u(t, x) - v(t, y) - \alpha \frac{x}{2} d(x, y)^2\} < +\infty.$$

There exists $(t_\alpha, x_\alpha, y_\alpha)$ such that

$$\lim_{\alpha \to \infty} [\mu_\alpha - (u(t_\alpha, x_\alpha) - v(t_\alpha, y_\alpha) - \alpha \frac{x}{2} d(x_\alpha, y_\alpha)^2)] = 0.$$
Then the following holds:

\[
\begin{align*}
(i) & \lim_{\alpha \to \infty} \alpha d(x_\alpha, y_\alpha)^2 = 0, \quad \text{and} \\
(ii) & \lim_{\alpha \to \infty} \mu_\alpha = u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}) = \delta, \\
\end{align*}
\]

where \((\hat{t}, \hat{x}) = \lim_{\alpha \to \infty} (t_\alpha, x_\alpha)\).

Let \((t_\alpha, x_\alpha, y_\alpha)\) be a maximum point of \(u(t, x) - v(t, y) - (\alpha/2)d(x_\alpha, y_\alpha)^2\) over \([0, T] \times \bar{M} \times \bar{M}\) for \(\alpha > 0\). Such a maximum exists in view of the assumed bounded above on \(u, -v\), the compactness of \(\Omega\), and (52)(ii). The purpose of the term \((\alpha/2)d(x, y)^2\) is as the elliptic case. Set

\[M_\alpha = u(t_\alpha, x_\alpha) - v(t_\alpha, y_\alpha) - \frac{\alpha}{2}d(x_\alpha, y_\alpha)^2.\]

By (53), \(M_\alpha \geq \delta\). If \(t_\alpha = 0\), we have

\[0 < \delta \leq M_\alpha \leq \sup_{x, y \in \bar{M}} [u(0, x) - u(0, y) - \frac{\alpha}{2}d(x, y)^2].\]

But the right hand side tends to zero as \(\alpha \to \infty\), so when \(\alpha\) is large we have \(t_\alpha > 0\). Similarly, since \(u \leq v\) on \([0, T] \times \partial M\) we have \(x_\alpha, y_\alpha \in \Omega\).

We now apply Lemma 6.3 at \((t_\alpha, x_\alpha, y_\alpha)\): there are \(a, b \in \mathbb{R}\) and \(X \in S^2T^*_x M, Y \in ST^*_y M\) such that

\[(a, D_x \varphi(t_\alpha, x_\alpha, y_\alpha), X) \in \bar{P}^2_M u(t_\alpha, x_\alpha), \quad (b, -D_y \varphi(t_\alpha, x_\alpha, y_\alpha), Y) \in \bar{P}^2_M v(t_\alpha, y_\alpha),\]

\[a = b\]

and

\[- \left( \frac{1}{\varepsilon} + \|A\| \right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varepsilon A^2.\]

The relation

\[a + F(t_\alpha, x_\alpha, u(t_\alpha + \varepsilon/(T - t_\alpha), x_\alpha), \alpha t(\gamma_\alpha'(0)), X) \leq -c, \]

\[b + F(t_\alpha, y_\alpha, v(t_\alpha, y_\alpha), \alpha t(\gamma_\alpha'(l_\alpha)), Y) \geq 0.\]

We thus have

\[c \leq -a - F(t_\alpha, x_\alpha, u(t_\alpha, x_\alpha), -l_\alpha \alpha t(\gamma_\alpha'(0)), X_\alpha)\]

\[\leq F(t_\alpha, y_\alpha, v(t_\alpha, y_\alpha), -l_\alpha \alpha t(\gamma_\alpha'(l_\alpha)), Y_\alpha) - F(t_\alpha, x_\alpha, u(t_\alpha, x_\alpha), -l_\alpha \alpha t(\gamma_\alpha'(0)), X_\alpha)\]

\[\leq \omega(d(x_\alpha, y_\alpha)^2 + d(x_\alpha, y_\alpha)).\]

Let \(\alpha \to \infty\) we have arrived at a contradiction.
References

[AFS] D. Azagra, J. Ferrera, B. Sanz, Viscosity solutions to second order partial differential equations I, J. Differential Equations 245(2008) 307–336.

[CY] S.Y. Cheng and S.T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. 28 (1975) 333–354.

[CCG] B. Chow, S. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, p. Lu, F. Luo, L. Ni, Ricci flow: techniques and applications, Part I: geometric aspects. Mathematical Surveys and Monographs, 135, AMS, Providence, RJ, 2007.

[CIL] M.G. Crandall, H. Ishii, P.L. Lions, User’s guide to viscosity solutions of second order partial differential equations., Bull. Amer Math. Soc. 27(1992), 1–67.

[O] H. Omori, Isometric immersions of Riemannian manifolds, J. Math. Soc. Japan 19 (1967), 205-214. MR 35:6101

[Y] S.T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. 28 (1975), 201-228. MR 55:4042

Shige Peng
School of Mathematics
Shandong University
Jinan, Shandong 250100
China
e-mail: peng@sdu.edu.cn

Detang Zhou
Insitituto de Matemática
Universidade Federal Fluminense- UFF
Centro, Niterói, RJ 24020-140
Brazil
e-mail: zhou@impa.br