Angular $N$-point spectra and cosmic variance on the light-cone

Ermis Mitsou,1 Jaiyul Yoo,1,2 Ruth Durrer,3 Fulvio Scaccabarozzi,1 and Vittorio Tansella3

1 Center for Theoretical Astrophysics and Cosmology, Institute for Computational Science
University of Zürich, Winterthurerstrasse 190, CH-8057, Zürich, Switzerland
2 Physics Institute, University of Zürich, Winterthurerstrasse 190, CH-8057, Zürich, Switzerland
3 Département de Physique Théorique and Center for Astroparticle Physics, Université de Genève,
24 quai Ansermet, CH–1211 Genève 4, Switzerland
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Abstract
We develop the general theory of the angular $N$-point spectra and derive the cosmic variance on the light cone. While the angular bispectrum and the trispectrum are well developed in literature, these higher point angular spectra in general are only shown to be the solutions under the symmetry requirements, rather than uniquely constructed from the $N$-point orthonormal harmonic components, rendering it difficult to go beyond $N = 4$. Here we extend the Wigner 3-$j$ symbols of triangles to construct the multilateral Wigner symbols of polygons and compute the angular $N$-point spectra directly from cosmological observables. We apply the Ergodic hypothesis to cosmological observations on a single light-cone and derive the cosmic variance of the angular $N$-point spectra.
1 Introduction

Cosmological observables such as the cosmic microwave background (CMB) temperature anisotropies are measured in the observer rest frame in terms of their angular position in the observer’s sky. The angular statistics such as angular correlation functions and their power spectra, therefore, provide the most natural way to characterize the cosmological observables and probe the initial conditions in the early Universe. In standard inflationary models, quantum fluctuations in the vacuum are stretched beyond the horizon scale during the period of inflationary expansion, and they “freeze” as classical fluctuations, providing the seeds for the subsequent nonlinear evolution such as galaxy formation. In this standard picture, cosmological observables following the initial conditions are characterized by Gaussian fluctuations, and hence the two-point statistics completely captures all the cosmological information (see, e.g., [1, 2]).

In any inflationary model beyond the standard model, the initial fluctuations, however, deviate from the perfect Gaussianity due to non-trivial interactions of extra fields with the inflaton field, and this deviation, called primordial non-Gaussianity, provides the smoking gun to falsify standard single-field inflationary models. Since the primordial non-Gaussianity manifests itself in higher (connected) $N$-point spectra such as the bispectrum and the trispectrum, the angular $N$-point spectra provide a direct way to test Gaussianity of initial conditions, and they received large attention in literature [3–6] (see, e.g., [7, 8]). Moreover, these higher-order statistics are not only imprinted from primordial non-Gaussianity, but also generated by the late-time nonlinear evolution of structure. Our own Galaxy often contributes non-Gaussianity to measurements of cosmological observables. Therefore, $N$-point spectra beyond the power spectrum play crucial roles in probing the initial conditions and the late-time evolution as well as providing a consistency check for systematic errors.

The angular bispectrum and trispectrum are well developed in literature [9–17], following the lead in [18]. However, there exist no general discussions of the angular $N$-point spectra with $N > 4$. We suspect that this absence is due to the two complicating factors, one in practice and one in theory, not necessarily from the lack of theoretical and/or observational motivations. The difficulty
to measure the angular $N$-point spectra in practice is natural, but not insurmountable. The difficulty in theory arises because there is no simple way to construct the angular $N$-point spectra directly out of cosmological observables. In the pioneering work [18], the angular $N$-point spectra are developed by imposing rotation and parity invariance and finally demanding the orthonormality condition. For instance, the angular bispectrum of a cosmological observable $O(\hat{n})$ can be written [18] as

$$\langle O_{l_1m_1} O_{l_2m_2} O_{l_3m_3} \rangle = \sum_{m_1'm_2'm_3'} \langle O_{l_1m_1'} O_{l_2m_2'} O_{l_3m_3'} \rangle \times D_{l_1,m_1m_1'} D_{l_2,m_2m_2'} D_{l_3,m_3m_3'}, \quad (1.1)$$

from the considerations of the rotational invariance, where $D_{lmm'}$ is the Wigner matrix and the cosmological observable is harmonically decomposed as $O(\hat{n}) = \tilde{O}_{lm} Y_{lm}(\hat{n})$. A further manipulation is made by using the addition of angular momentum for the Wigner matrix to reduce the number of Wigner matrices

$$\langle O_{l_1m_1} O_{l_2m_2} O_{l_3m_3} \rangle = \sum_{m_1'm_2'm_3'} \langle O_{l_1m_1'} O_{l_2m_2'} O_{l_3m_3'} \rangle \times \sum_{LMM'} (2L+1)(-1)^{M+M'} \times \left( \begin{array}{ccc} l_1 & l_2 & L \\ m_1 & m_2 & -M \end{array} \right) \left( \begin{array}{ccc} l_1' & l_2' & L' \\ m_1' & m_2' & -M' \end{array} \right) D_{L,M,M'} D_{l_3,m_3m_3'}, \quad (1.2)$$

and to demand the orthonormality relation of the two Wigner matrices to arrive at the final expression for the reduced angular bispectrum $B_{l_1l_2l_3}$:

$$\langle O_{l_1m_1} O_{l_2m_2} O_{l_3m_3} \rangle = \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) B_{l_1l_2l_3}, \quad (1.3)$$

This procedure becomes quickly complicated for $N \geq 4$.

Here we solve this issue by directly constructing the “multilateral” Wigner symbols, a generalization of the Wigner 3-$j$ symbols of the triangles in $N$-point polygons. Using the multilateral Wigner symbols, the orthonormal harmonic components can be readily constructed out of the product of spherical harmonics, and the angular $N$-point spectra are just the coefficients of the cosmological $N$-point functions in the harmonic basis.

This paper is organized as follows: In section 2, we begin by discussing the observer frame, in which the observer establishes local coordinates for observations and in which cosmological observables are measured. We also discuss the symmetry associated with the observer frame and the corresponding transformations of the observables. In section 3, we develop the general theory of the observed angular $N$-point spectra, generalizing previous work [18] in the literature. This is our main result. In section 4, we construct the theoretical correlation functions to describe the observed correlation function, and we apply the Ergodic hypothesis to the light-cone observations to establish the relation between the (theoretical) ensemble average and the (observed) geometrical average on the observer sky. Given this theoretical and observational framework, we derive the cosmic variance limit for angular $N$-point spectra in section 5. We compute the connected angular $N$-point spectra and derive the covariance for the two-point spectrum as a worked example in section 6. A common mistake (and complication) in literature is discussed in section 7, regarding observable and their dimensionless fluctuations. We conclude in section 8.

2 Preliminary considerations: Angle, redshift, and observables

A cosmological observable associated with some localized source (galaxy, supernovae, etc.) is a function of two space-time points $O(x_o; x_s)$, the “observer” point $x_o$ and the “source” point $x_s$, that
are constrained to lie on a common light-like geodesic, with \( x_s \) in the past of \( x_o \). Thus, for a given observer at \( x_o \), the set of possible source points \( x_s \) forms the past light-cone of \( x_o \). The \( x_o \) and \( x_s \) are not observables, as they are ambiguous due to the freedom of performing coordinate transformations. The only physical information is the relation between observables, so one must parametrize the past-light cone at \( x_o \) in terms of observables. One of them is the incoming photon direction in the sky, \( n \), leading to an angular parametrization of the space-time manifold. For the radial parametrization there are several choices, such as the observed redshift \( z \) or the luminosity/angular distances \( D_{L,A} \). Here we will consider the former, which is also the most widely used and model independent. Thus, the observer at \( x_o \) parametrizes her light-cone in terms of the observables \( z \) and \( n \), and the physical information lies in the relation between \( \mathcal{O} \) and \((z,n)\) that is the function \( \mathcal{O} (x_o; z,n) \).

The quantities \( z \) and \( n \) are defined with respect to the observer rest-frame, i.e. a tetrad \( e_a \) at \( x_o \)

\[
g_a (e_a, e_b) \equiv \eta_{ab} ,
\]

whose time-component \( e_0 \) is the observer’s 4-velocity, and the source 4-velocity \( u_s \) satisfying \( g_s(u_s, u_s) \equiv -1 \). More precisely, if \( k \) denotes the momentum 4-vector of the photon, we have

\[
1 + z := \frac{g(u_s, k)}{g(e_0, k)} , \quad n_i := -\frac{g(e_i, k)}{\sqrt{g(e_j, k) g(e_j, k)}} ,
\]

where \( i \in \{1, 2, 3\} \) is the spatial part of the 4-dimensional index \( a \in \{0, 1, 2, 3\} \). Note that \( n_i \) is the observed angular direction, not the propagation direction. These quantities are therefore invariant under coordinate transformations in the spacetime manifold, as any observable should, since a measurement cannot depend on how we parametrize space-time. The observer tetrad (2.1) is defined only up to a Lorentz transformation \( e_a \rightarrow \Lambda_a^b e_b \), but the boosts alter the observer 4-velocity, so the only ambiguity is the orientation of the spatial frame \( e_i \rightarrow R_i^j e_j \), leading to an SO(3) ambiguity for \( n \)

\[
n \rightarrow R^{-1}n .
\]

As for the observable \( \mathcal{O} \), the theoretical expression must also be a scalar under coordinate transformations [19]

\[
\tilde{\mathcal{O}} (\tilde{x}_o; z,n) \equiv \mathcal{O} (x_o; z,n) ,
\]

but it can be a tensor with respect to the Lorentz index \( a \) of the observer tetrad. For instance, \( \omega \) and \( n \) are components of the Lorentz vector \( k_a \) at \( x_o \)

\[
k_a = -\omega (1, n_i) ,
\]

which is why they transform non-trivially under Lorentz transformations of \( e_a \). Since the boost part here is fixed by the definite observer 4-velocity, we only have to deal with the SO(3) ambiguity (2.3).

To that end, we express \( n \) in terms of observed angles \((\vartheta, \varphi)\)

\[
n \equiv \sin \vartheta \cos \varphi e_1 + \sin \vartheta \sin \varphi e_2 + \cos \vartheta e_3 ,
\]

so that \( \mathcal{O} \equiv \mathcal{O} (x_o; z, \vartheta, \varphi) \) becomes a function on the unit sphere \( S \). The latter is a 2-dimensional manifold with coordinates \( \vartheta^A \in \{\vartheta, \varphi\} \) and with the admissible coordinate transformations being the ones induced by the rotation (2.3).² The generic observable will therefore be a tensor field on that

²Note that this manifold is defined with respect to the tetrad basis of the tangent space at \( x_o \), so these angles have nothing to do with some angular parametrization of the space-time manifold, i.e. they are not space-time coordinates.
manifold $\mathcal{O}_{A_1...A_n}$. However, in two dimensions any tensor can be reduced to scalars and pseudo-scalars. For instance, a vector can be decomposed into a scalar $v$ and a pseudo-scalar $\tilde{v}$ via

$$V_A = \nabla_A v + \varepsilon_A^B \nabla_B \tilde{v},$$

while a tensor can be decomposed into two scalars $T, t$ and two pseudo-scalars $\tilde{T}, \tilde{t}$

$$T_{AB} = s_{AB} T + \varepsilon_{AB} \tilde{T} + \left[ \nabla_A \nabla_B - \frac{1}{2} s_{AB} \nabla^2 \right] t + \varepsilon_{(C} \nabla_{B)} \nabla_C \tilde{t},$$

where $s_{AB}, \varepsilon_{AB}$ and $\nabla_A$ are the metric, volume form and covariant derivative on the 2-sphere $S$. Further decomposing these (pseudo-)scalars into spherical harmonics leads to the decomposition of $\mathcal{O}_{A_1...A_n}$ into spin-weighted spherical harmonics. To avoid the introduction of the latter, which complicates unnecessarily the formalism, here we will assume that our observables are the (pseudo-) scalars of the above decomposition (up to possible Laplacians). For instance, in the case of the CMB polarization tensor $P_{AB}$ we would directly work with its “electric” and “magnetic” components $\nabla_A \nabla_B P_{AB}$ and $\varepsilon_{(C} \nabla_{B)} \nabla_C P_{AB}$, respectively.

Finally, another important observable is the incoming photon’s frequency

$$\omega := -g(e_0, k_o),$$

so the most general parametrization of light-based observables is a spectral distribution $\mathcal{O}(x_o; z, n, \omega)$. In the case of observables associated with diffuse sources (e.g. CMB), i.e. that do not have a particular emission moment and therefore no associated redshift, then we simply have $\mathcal{O}(x_o; n, \omega)$. For the sake of simplicity, here we will assume that $\omega$ is either fixed, or that it is integrated over with some given spectral distribution, as in the case of the CMB temperature for instance. We will therefore work with observable functions of the form $\mathcal{O}(x_o; z, n)$, but the inclusion of $\omega$ is straightforward, it will basically enter our equations exactly as the redshift dependence.

3 General theory of observed angular $N$-point spectra

Here we develop a general formalism to compute the observed angular $N$-point spectra and show that it reduces to the well-known expressions for $N \leq 4$ (see, e.g., [18]).

The observer only has access to a single light-cone $x_o$. Given the SO(3) ambiguity of the observer’s spatial frame, or equivalently of $n$ (2.3), the physical information available to the observer are all the SO(3)-invariant functions one can build out of $\mathcal{O}(x_o; z, n)$. The building blocks for such functions are the average of the products $\prod_{k=1}^N \mathcal{O}(x_o; z_k, n_k)$ over all possible common rotations of the directions $n_k$, i.e. the average over the SO(3) group

$$\frac{\int dR \prod_{k=1}^N \mathcal{O}(x_o; z_k, R^{-1} n_k)}{\int dR},$$

where $dR$ is the Haar measure on SO(3) and is invariant under group multiplication. Thanks to this, if one rotates the $n_k$ in (3.1) with the same rotation $R'$, then this can be reabsorbed in the dummy variable $R$ by the redefinition $R \rightarrow R'R$, thus leaving the average invariant. To turn (3.1) into a well-defined integral, we consider an arbitrary orthonormal reference frame $e_i$ and parametrize $R$ as

$$R(\alpha, \beta, \gamma) := R_3(\alpha) R_2(\beta) R_3(\gamma),$$

$$\frac{\int dR \prod_{k=1}^N \mathcal{O}(x_o; z_k, R^{-1} n_k)}{\int dR},$$

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$$R(\alpha, \beta, \gamma) := R_3(\alpha) R_2(\beta) R_3(\gamma),$$

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where \( R_i(\theta) \) denotes the matrix corresponding to a rotation around \( e_i \) with angle \( \theta \), so that \( \alpha, \beta, \gamma \) are the Euler angles
\[
\alpha \in [0, 2\pi] , \quad \beta \in [0, \pi] , \quad \gamma \in [0, 2\pi] . \tag{3.3}
\]
In particular, the inverse matrix is simply
\[
R^{-1}(\alpha, \beta, \gamma) \equiv R(-\gamma, -\beta, -\alpha) . \tag{3.4}
\]
The Haar measure \( dR \) on \( \text{SO}(3) \) now reads
\[
dR(\alpha, \beta, \gamma) \equiv \sin \beta d\alpha d\beta d\gamma , \tag{3.5}
\]
so the average over \( \text{SO}(3) \) of some function \( f(n_1, \ldots, n_N) \) is given by
\[
\langle f(n_1, \ldots, n_N) \rangle_{\text{SO}(3)} := \frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^\pi \sin \beta d\beta \int_0^{2\pi} d\gamma f \left( R^{-1}(\alpha, \beta, \gamma) n_1, \ldots, R^{-1}(\alpha, \beta, \gamma) n_N \right) , \tag{3.6}
\]
and thus, the \( N \)-point observational correlation functions (OCF) are defined by
\[
G^{\text{ob}}(x_o; \{z_k, n_k\}_{k=1}^N) := \left\langle \prod_{k=1}^N \mathcal{O}(x_o; z_k, n_k) \right\rangle_{\text{SO}(3)} . \tag{3.7}
\]
In particular, the \( N = 1 \) case reduces to the average over the sphere
\[
G^{\text{ob}}(x_o; z) \equiv \frac{1}{4\pi} \int d\Omega \mathcal{O}(x_o; z, n) , \tag{3.8}
\]
which is shown by picking \( e_3 = n \). For \( N > 1 \), three out of the \( 2N \) angles in \( G^{\text{ob}} \) are redundant, since we are free to rotate at will, or equivalently, to choose the reference frame \( \{e_i\}_{i=1}^3 \) arbitrarily. For instance, one can pick (assuming that \( n_1 \) and \( n_2 \) are not parallel)
\[
e_3 = n_1 , \quad e_2 = \frac{n_2 - (n_1 \cdot n_2) n_1}{\sqrt{1 - (n_1 \cdot n_2)^2}} , \quad e_1 = \frac{n_1 \times n_2}{\sqrt{1 - (n_1 \cdot n_2)^2}} , \tag{3.9}
\]
thus leaving us with a dependence on the \( 2N - 3 \) angles \( \vartheta_2, \ldots, \vartheta_N \) and \( \varphi_3, \ldots, \varphi_N \) that parametrize \( \{n_k\}_{k=2}^N \) in the \( e_i \) basis.

Now note that the \( N \)-point OCF is a redshift-dependent function on
\[
\mathbb{S}^N := \mathbb{S} \times \cdots \times \mathbb{S} , \quad \text{\( N \) times} \tag{3.10}
\]
that is symmetric under a common rotation of these \( N \) spheres. It is therefore natural to decompose them in a basis of \( \text{SO}(3) \)-symmetric functions on \( \mathbb{S}^N \). To that end, we start by decomposing the observable in terms of spherical harmonics
\[
\mathcal{O}(x_o; z, n) \equiv \mathcal{O}_{lm}(x_o; z) Y_{lm}(n) , \tag{3.11}
\]
where the summation over \( l, m \) indices will be kept implicit for notational simplicity. In what follows, we will encounter both dummy and free \( l, m \) indices, so their nature will be inferable by looking at both sides of the equation. The \( m \) indices will always be clearly associated to some \( l \) value and
therefore run from \(-l\) to \(l\), while the \(l\) indices run from \(s\) to \(\infty\), where \(s\) is the spin of the observable under consideration.\(^3\) Also, we choose to work with the less conventional normalization of spherical harmonics

\[
\frac{1}{4\pi} \int d\Omega \ Y_{lm}(n) \ Y_{l'm'}^*(n) \equiv \delta_{ll'} \delta_{mm'}, \tag{3.12}
\]

i.e. the one that is unit-normed under spherical average, since it is the natural one in the present context. The \(N\)-point OCF (3.7) now reads\(^4\)

\[
G^{\text{ob}} \left( x_o; \{ z_k, n_k \} \right)_{k=1}^{N} \equiv \left[ \prod_{k=1}^{N} O_{l_k m_k}(x_o; z_k) \right] \frac{1}{8\pi^2} \int dR \prod_{k=1}^{N} Y_{l_k m_k}(R^{-1} n_k), \tag{3.13}
\]

where the Haar measure \(dR\) is given in (3.5). We can then extract the \(R\)-dependence out of the spherical harmonics by using their transformation property under rotations

\[
Y_{lm} \left( R^{-1} (\alpha, \beta, \gamma) \right) n \equiv Y_{lm}(n) \ D_{l,m'm}(\alpha, \beta, \gamma), \tag{3.14}
\]

where the \(D_l\) are the Wigner matrices forming the \((2l+1)\)-dimensional irreducible representation of \(\text{SO}(3)\)

\[
D_l(R) \ D_l(R') \equiv D_l(R R'), \quad D_l(R) \ D_l^\dagger(R) \equiv I. \tag{3.15}
\]

We thus have

\[
G^{\text{ob}} \left( x_o; \{ z_k, n_k \} \right)_{k=1}^{N} \equiv I_{l_1...l_N}^{m_1...m_N, m_1...m_N} \prod_{k=1}^{N} O_{l_k m_k}(x_o; z_k) \ Y_{l_k m_k}(n_k), \tag{3.16}
\]

where the multipole coefficients are defined as

\[
I_{m_1...m_N, m_1...m_N}^{l_1...l_N} := \frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{\pi} \sin \beta \ d\beta \int_0^{2\pi} d\gamma \prod_{k=1}^{N} D_{l_k m_k}(\alpha, \beta, \gamma). \tag{3.17}
\]

In the \(N = 2\) case, we can use the identity

\[
D^*_{l,m'm'} \equiv (-1)^{m+m'} D_{l,-m-m'}, \tag{3.18}
\]

and the orthonormality relation

\[
\frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{\pi} \sin \beta \ d\beta \int_0^{2\pi} d\gamma \ D_{l_1,m_1m_1}(\alpha, \beta, \gamma) \ D^*_{l_2,m_2m_2}(\alpha, \beta, \gamma) = \frac{1}{2l_1+1} \delta_{l_1 l_2} \delta_{m_1 m_2} \delta_{m_1' m_2'}, \tag{3.19}
\]

to obtain

\[
I_{m_1' m_2', m_1 m_2}^{l_1 l_2} = \frac{(-1)^{m_1+m_1'}}{2l_1+1} \delta_{l_1 l_2} \delta_{m_1 m_2} \delta_{m_1' m_2'}, \tag{3.20}
\]

and thus

\[
G^{\text{ob}} \left( x_o; \{ z_1, z_2 \}, n_1, n_2 \right) \equiv G^{\text{ob}}_l \left( x_o; \{ z_1, z_2 \}, (2l+1) \ P_l(n_1 \cdot n_2) \right), \tag{3.21}
\]

\(^3\)For instance, we have \(s = 0\) for CMB temperature maps, while \(s = 2\) for the maps of the electric and magnetic components of the polarization field.

\(^4\)In literature, the \(N\)-point OCFs are often denoted as \(\xi_N\) (or its Greek variants). Here we use \(G\) instead of \(\xi\) to emphasize that \(G\) is constructed directly out of the observables without separating the background and the perturbation contributions, while \(\xi\) is constructed from the dimensionless fluctuations after scaling the background quantity. This point and its impacts are further discussed in section 7.
where the $P_l$ are the Legendre polynomials and

$$G_{l}^{\text{ob}}(x_0; z_1, z_2) := \frac{O_{l,m}(x_0; z_1) O_{l,m}^*(x_0; z_2)}{2l + 1},$$  \hspace{1cm} (3.22)$$

are the “harmonic” components of the 2-point OCF.\footnote{Due to our normalization convention for spherical harmonics, the harmonic components $G^N_l$ of the $N$-point OCF are related to the standard $N$-point OCF $G^N_{\text{std}}$ as $G^N_{\text{std}} = (4\pi)^N/2 G^N_l$.} Indeed, the only $SO(3)$ invariant quantity one can form out of two directions $n_1$ and $n_2$ is their scalar product $n_1 \cdot n_2 \in [-1, 1]$, and the Legendre polynomials form a basis for functions on that interval. The inverse of (3.21) is

$$G_{l}^{\text{ob}}(x_0; z_1, z_2) \equiv \int \frac{d\Omega_1}{4\pi} \frac{d\Omega_2}{4\pi} P_l(n_1 \cdot n_2) G_{l}^{\text{ob}}(x_0; z_1, z_2, n_1, n_2),$$  \hspace{1cm} (3.23)$$

For $N > 2$, we need to use iteratively the “Clebsch-Gordan” composition rule

$$D_{l_1,m_1 m_1'} D_{l_2,m_2 m_2'} = \binom{l_1 \ l_2 \ L}{m_1 \ m_2 \ -M} \binom{l_1' \ l_2' \ L'}{m_1' \ m_2' \ -M'} (2L + 1) (-1)^{M + M'} D_{l,M,M'},$$  \hspace{1cm} (3.24)$$

in order to reduce the product in (3.17) down to a single pair, in which case we can again use (3.18) and (3.19). The result is

$$I_{m_1' \ m_2' \ m_1 \ m_2}^{l_1 \ldots l_N} \equiv W_{m_1' \ m_2'}^{l_1 \ldots l_N} W_{m_1 \ m_2}^{l_1 \ldots l_N},$$  \hspace{1cm} (3.25)$$

where the “multilateral” Wigner symbols

$$W_{m_1 \ldots m_N}^{l_1 \ldots l_N} = \binom{l_1 \ l_2 \ L}{m_1 \ m_2 \ -M} \prod_{k=1}^{N-4} (-1)^{M_k} \sqrt{2L_k + 1} \begin{bmatrix} L_k & l_{k+2} & l_{k+1} \\ M_k & m_{k+2} & -M_{k+1} \end{bmatrix} \times (-1)^{M_{N-3}} \sqrt{2L_{N-3} + 1} \begin{bmatrix} L_{N-3} & l_{N-1} & l_N \\ M_{N-3} & m_{N-1} & m_N \end{bmatrix}.$$  \hspace{1cm} (3.26)$$

Note that for $N = 3$ this simply reduces to the Wigner $3 - j$ symbol

$$W_{m_1 m_2 m_3}^{l_1 l_2 l_3} \equiv \binom{l_1 \ l_2 \ l_3}{m_1 \ m_2 \ m_3},$$  \hspace{1cm} (3.27)$$

which is proportional to the “triangle delta”\footnote{Note that the more common definition is that $\{l_1 \ l_2 \ l_3\}$ is one if and only if $l_1 \in \{|l_2 - l_3|, \ldots, l_2 + l_3\}$, which is due to the relation between the $3 - j$ symbols and the Clebsch-Gordan coefficients.}

$$\{l_1 \ l_2 \ l_3\} := \begin{cases} 1 \text{ if there exists a triangle with lengths } l_1, l_2, l_3 \\ 0 \text{ otherwise} \end{cases}.$$  \hspace{1cm} (3.28)$$

Because of this, the symbols defined in (3.26) are proportional to

$$\{l_1 \ldots l_N | L_1 \ldots L_{N-3}\} = \{l_1 \ l_2 \ L_1\} \prod_{k=1}^{N-4} \{L_k \ l_{k+2} \ L_{k+1}\} \{L_{N-3} \ l_{N-1} \ l_N\}.$$  \hspace{1cm} (3.29)$$

Generalizing the picture laid out in [18] for the $N = 4$ case, this quantity can be interpreted as a “multilateral delta” in the following way. Being a product of $N - 2$ triangle deltas, it is non-zero only if all of the involved integers $l_1, \ldots, l_k$ and $L_1, \ldots, L_{N-3}$ form their respective triangles. Moreover,
with respect to the triangle ordering in (3.29), one of the edges of two neighboring triangles must have the same length $L_k$, so we can picture the triangles as being stuck along their common edges. The resulting shape is therefore a multilateral with $N$ edges of lengths $l_1, \ldots, l_N$, while the $L_1, \ldots, L_{N-3}$ integers correspond to the lengths of the $N-3$ diagonals connected to the vertex where the $l_1$ and $l_N$ edges join (see figure 1). Thus, the multilateral delta is not unity for any set of $l_1, \ldots, l_N$ that can form a multilateral, but only for those whose diagonals also have integer length. We can therefore refer to the symbols $W$ defined in (3.26) as the “multilateral” Wigner symbols, in which case the $3-j$ symbols would be the “triangular” ones.

Let us now come back to the equation of interest (3.16) which, given (3.25), yields

$$G_{\text{rob}}^{m_1, m_2, m_3} (x_o; \{z_k, n_k\}) \equiv C_{l_1 \ldots l_N}^{m_1, m_2, m_3} (x_o; \{z_k\}) Y_{l_1 \ldots l_N}^{m_1, m_2, m_3} (\{n_k\}) ,$$

where

$$C_{l_1 \ldots l_N}^{m_1, m_2, m_3} (x_o; \{z_k\}) := W_{m_1 \ldots m_N}^{l_1 \ldots l_N} \prod_{k=1}^{N} \mathcal{O}_{l_k m_k}(x_o; z_k) ,$$

are the “harmonic” components of the $N$-point OCF and

$$Y_{l_1 \ldots l_N}^{m_1, m_2, m_3} (\{n_k\}) := W_{m_1 \ldots m_N}^{l_1 \ldots l_N} \prod_{k=1}^{N} Y_{l_k m_k} (n_k) ,$$

form a basis for $\text{SO}(3)$-invariant functions on $S^N$. Using the following relation of the $3-j$ symbols

$$\sum_{m_1, m_2, m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3' \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \{l_1 \ l_2 \ l_3\} \delta_{l_3 l_3'},$$

Figure 1. The multilateral illustration for the case $N = 7$.
we obtain
\[ \sum_{m_1, \ldots, m_N} W_{m_1 \ldots m_N}^{l_1 \ldots l_N|L_1 \ldots L_{N-3}} \equiv \{l_1 \ldots l_N|L_1 \ldots L_{N-3}\} \prod_{k=1}^{N-3} \delta^{L_k L_k'}, \tag{3.34} \]
and therefore the orthonormality relation for our basis
\[ \int \left( \prod_{k=1}^{N} \frac{d\Omega_k}{4\pi} \right) Y_{l_1 \ldots l_N|L_1 \ldots L_{N-3}} Y_{l_1' \ldots l_N'|L_1' \ldots L_{N-3}'} \equiv \{l_1 \ldots l_N|L_1 \ldots L_{N-3}\} \prod_{k=1}^{N} \delta^{l_k l_k'} \prod_{k=1}^{N-3} \delta^{L_k L_k'}, \tag{3.35} \]
which, in particular, allows us to invert (3.30)
\[ G^{ob}_{l_1 \ldots l_N|L_1 \ldots L_{N-3}} (x_o; \{z_k\})_{k=1}^{N} \equiv \int \left( \prod_{k=1}^{N} \frac{d\Omega_k}{4\pi} \right) Y^*_{l_1 \ldots l_N|L_1 \ldots L_{N-3}} (\{n_k\})_{k=1}^{N} G^{ob}_{l_1 \ldots l_N|L_1 \ldots L_{N-3}} (x_o; \{z_k, n_k\})_{k=1}^{N}. \tag{3.36} \]
Note how these quantities are explicitly SO(3)-invariant, since they only depend on total angular momentum numbers \(l_k\) and \(L_k\), making a total of \(2N - 3\) numbers, which is the number of independent angles present in the corresponding correlation functions. In particular, for the \(N = 3, 4\) cases, we recover the known results [18]
\[ G^{ob}_{l_1 l_2 l_3} (x_o; z_1, z_2, z_3) \equiv \left( \frac{l_1 l_2 l_3}{m_1 m_2 m_3} \right)^3 \prod_{k=1}^{3} O_{l_k m_k} (x_o; z_k), \tag{3.37} \]
\[ G^{ob}_{l_1 l_2 l_3 l_4} (x_o; z_1, z_2, z_3, z_4) \equiv (-1)^M \sqrt{2L + 1} \left( \frac{l_1 l_2 L}{m_1 m_2 M} \right) \left( \frac{l_3 l_4}{m_3 m_4} \right)^4 \prod_{k=1}^{4} O_{l_k m_k} (x_o; z_k). \]
Finally, note that for \(N > 3\) the above decomposition is not unique [18], a fact which is most easily understood using the multilateral picture shown in Figure 1. Indeed, for \(N > 3\) there is an ambiguity in choosing the vertex with respect to which the diagonals are drawn, which is equivalent to the fact that the first and last triangles in (3.29) contain a single diagonal instead of two. One could therefore choose any pair of \(l_k\) to be the ones that are distinguished in this way, which would then lead to a different ordering of the \(l_k\)’s, a different interpretation of the \(L_k\)’s in the multilateral Wigner symbols and therefore to a different \(G^{ob}_{l_1 \ldots l_N|L_1 \ldots L_{N-3}}\). As shown in [18] for the \(N = 4\) case, however, all these alternative versions are related to each other by linear combinations involving \(6 - j\) Wigner symbols.

4 Theoretical correlation functions and Ergodic hypothesis

Let us now turn our attention to the theorist which works with tensor fields on the full space-time manifold \(\mathcal{M}\), which we collectively denote by \(\Phi(x)\). The observable \(O(x_o; z, n)\) is an integro-differential functional of these fields
\[ O(x_o; z, n) \equiv O(x_o; z, n)[\Phi], \tag{4.1} \]
typically involving integrals over the line-of-sight of the fields and their derivatives. Within cosmological perturbation theory, one splits \(\Phi\) into a homogeneous and isotropic “background” solution \(\bar{\Phi}\) and a fluctuation \(\phi\)
\[ \Phi = \bar{\Phi} + \phi, \tag{4.2} \]
where the latter is subject to a gauge ambiguity. This is the manifestation of coordinate transformations in this framework, since the background is kept fixed under the transformation. The observable $O(x_o; z, n)$ should, by definition, be independent of the choice of gauge, so one can fix the latter. Moreover, one can use any constraint equations (e.g. the Poisson equation) to reduce the number of fields down to those carrying degrees of freedom, which we denote by $\phi_a(x)$, i.e. the “a” index includes both space-time and internal indices. Thus, the $\phi_a(t_0, \vec{x})$ data at a given time $t_0$ uniquely determine the ones at any other time $t$

$$\phi_a(t, \vec{x}) \equiv \phi_a(t, \vec{x})[\phi_a(t_0, \vec{y})].$$

(4.3)

Here it is assumed that the $\phi_a$ also contain the momenta/velocities, for fields obeying second-order equations in time. Next, in order to compare with observations, one needs to consider a statistical ensemble of solutions. The $\phi_a$ are therefore promoted to stochastic fields with a corresponding probability distribution functional (pdf) associating a probability density to each field solution $\phi_a(x)$. Since the latter are completely determined by their configuration at some reference time $\phi_a(t_0, \vec{x})$, it suffices to define that pdf on these field configurations $P \equiv P[\phi(t_0)]$ (dropping the index $a$ and the position $\vec{x}$ for notational simplicity). One can then define the moments, i.e. the statistical averages of field products

$$\langle \phi_{a_1}(t_0, \vec{x}_1) \ldots \phi_{a_n}(t_0, \vec{x}_n) \rangle_P := \int D\phi(t_0) P[\phi(t_0)] \phi_{a_1}(t_0, \vec{x}_1) \ldots \phi_{a_n}(t_0, \vec{x}_n),$$

(4.4)

which completely determine the functional $P$, and with these one can define the field correlation functions (FCF)

$$F_{a_1 \ldots a_n}(x_1, \ldots, x_n) := \langle \phi_{a_1}(t_1, \vec{x}_1) \ldots \phi_{a_n}(t_n, \vec{x}_n) \rangle_P$$

$$\equiv \langle \phi_{a_1}(t_1, \vec{x}_1)[\phi(t_0)] \ldots \phi_{a_n}(t_n, \vec{x}_n)[\phi(t_0)] \rangle_P,$$

(4.5)

thanks to the linearity of the averaging operation (even though $\phi_{a_k}(t_k, \vec{x}_k)[\phi(t_0)]$ is in general not a linear functional of $\phi(t_0)$). In particular, $P$ is chosen such that

$$\langle \phi_a(x) \rangle_P \equiv 0,$$

(4.6)

which can be alternatively stated as

$$\langle \Phi(x) \rangle_P \equiv \bar{\Phi}(x).$$

(4.7)

With the above definitions one can now perform statistical averages of arbitrary functionals of the $\phi_a(x)$. In particular, the theoretical correlation functions (TCF) of the observables are defined by

$$G^{\text{th}}(x_o; \{z_k, n_k\}_{k=1}^N) := \left\langle \prod_{k=1}^N O(x_o; z_k, n_k)[\Phi] \right\rangle_P,$$

(4.8)

which are therefore ultimately a functional of $\bar{\Phi}$ and the FCFs (4.5). Now note that, although strict homogeneity and isotropy are lost as soon as $\phi \neq 0$, these notions can be reintroduced at the statistical level by imposing the corresponding symmetries on $P[\phi(t_0)]$. Thus, we require that, if the two configurations $\phi_a(t_0)$ and $\phi'_a(t_0)$ are related by an isometry of the background geometry, then $P[\phi(t_0)] = P[\phi'(t_0)]$. As a result, the FCFs are invariant under isometries.

We now wish to relate the observational and theoretical $N$-point functions of observables (3.7) and (4.8). We start by considering the set of 3-dimensional field configurations $\phi_a(t_0)$ over which we sum when performing the ensemble average in (4.4). Because of the invariance of $P[\phi(t_0)]$
under the isometry group, this ensemble can be partitioned into equivalence classes, where two field configurations are deemed equivalent if they can be related by an isometry. For the sake of simplicity, let us consider here the case of flat background space with the trivial Cartesian coordinates, so that the isometries form the Euclidean group \( \mathbb{E} := \text{SO}(3) \ltimes \mathbb{R}^3 \). Any element of a given equivalence class can be described as an isometry of some fixed representative \( \hat{\phi}_a(t_0) \)

\[
\phi_{a,R,c}(t_0, \vec{x}) = M_a^b(R) \hat{\phi}_b(t_0, R\vec{x} + \vec{c}) ,
\]

where \( R \) is a rotation matrix, \( \vec{c} \) a translation vector and \( M_a^b \) is the matrix that rotates tensor indices in \( a \). We can therefore split the functional integration in (4.4) into an integral over the elements of a given class followed by an integral over all possible classes. By the latter we mean an integral over suitably chosen representatives \( \hat{\phi}_a(t_0) \) such that the corresponding functional integral is well-defined.

\( \text{7} \) Since the pdf is constant over all representatives of a given class, the integral in (4.4) becomes

\[
\langle \phi_{a_1}(t_0, \vec{x}_1) \ldots \phi_{a_n}(t_0, \vec{x}_n) \rangle_P = \frac{\int D\hat{\phi}(t_0) P[\hat{\phi}(t_0)] \langle \hat{\phi}_{a_1}(t_0, \vec{x}_1) \ldots \hat{\phi}_{a_n}(t_0, \vec{x}_n) \rangle_E}{\int D\hat{\phi}(t_0) P[\hat{\phi}(t_0)]} ,
\]

where we now integrate only over the set of representatives, and

\[
\langle X[\phi(t_0)] \rangle_E := \int \frac{d^3c}{V} \int \frac{dR}{8\pi^2} X[M_a^b(R) \phi_b(t_0, R\vec{x} + \vec{c})] .
\]

is the average over the Euclidean group action over the field configurations, \( V \) is the total volume and \( dR \) is the SO(3) Haar measure. As one may expect, the ensemble average therefore contains a purely geometric average over the symmetry group of the pdf.

Apart from subsets of measure zero, the configurations \( \hat{\phi}(t_0) \) appearing in the integral (4.10) have a rich spatial dependence. In particular, they are non-periodic functions which therefore probe a large variety of local field profiles for large enough \( V \). At the \( V \to \infty \) limit, which we consider here, the ergodic hypothesis states that this probing is thorough enough to make the Euclidean average in (4.10) independent of the configuration \( \hat{\phi}(t_0) \). As a result, that average factorizes out of the integral, thus yielding

\[
\langle \phi_{a_1}(t_0, \vec{x}_1) \ldots \phi_{a_n}(t_0, \vec{x}_n) \rangle_P \overset{\text{erg.}}{=} \left\langle \phi_{a_1}(t_0, \vec{x}_1) \ldots \phi_{a_n}(t_0, \vec{x}_n) \right\rangle_E ,
\]

where now on the right-hand side it is a random field configuration that is considered. We can then generalize this manipulation straightforwardly to the case of the FCFs (4.5), since the Euclidean group action is independent of the time variable \( t \), and therefore also to the case of the TCF (4.8)

\[
G^{4\text{th}}(x_o; \{z_k, n_k\}_{k=1}^N) \overset{\text{erg.}}{=} \left\langle \prod_{k=1}^N \mathcal{O}(x_o; z_k, n_k) [\Phi] \right\rangle_E
\]

\[
= \int \frac{d^3c}{V} \int \frac{dR}{8\pi^2} \prod_{k=1}^N \mathcal{O}(x_o; z_k, n_k) \left[ M_a^b(R) \phi_b(t, R\vec{x} + \vec{c}) \right] .
\]

where now we act with \( \mathbb{E} \) directly on the 4-dimensional fields \( \phi_a(x) \) instead of the \( \phi_a(t_0, \vec{x}) \). Note also that we have not specified the \( \bar{\Phi} \) dependence in (4.13) for simplicity. The above expression

\( \text{7} \) The existence of such a splitting of the integration is a non-trivial mathematical assertion, whose proof, if possible, would go beyond the scope of this paper.
allows us to make contact with the OCFs (3.7). Let us first redefine the dummy variable $\vec{c} \rightarrow \vec{c} - R\vec{x}_o$ in (4.13) and let us also define the notation for shifted fields
\[
\phi_{a,\vec{c}}(t, \vec{x}) := \phi_a(t, \vec{x} + \vec{c}) ,
\]
(4.14)
to get
\[
G^{\text{th}} \left( x_o; \{ z_k, n_k \}_{k=1}^N \right) \equiv \int \frac{d^3c}{V} \int \frac{dR}{8\pi^2} \prod_{k=1}^N O(x_o; z_k, n_k) \left[ M^b_a(R) \phi_{a,\vec{c}}(t, R(\vec{x} - \vec{x}_o)) \right] .
\]
(4.15)
We next observe that, for a given value of $\vec{c}$, the fields $\phi_{a,\vec{c}}$ are rotated by $R^{-1}$ around $\vec{x}_o$ in (4.15). But rotating the fields around $\vec{x}_o$ is tantamount to rotating the observed angles $n$ in the opposite direction, so the SO(3) average on the fields translates into an OCF
\[
G^{\text{th}} \left( x_o; \{ z_k, n_k \}_{k=1}^N \right) \equiv \frac{1}{V} \int d^3 x_o \ G^{\text{ob}}(t_o, x_o; \{ z_k, n_k \}_{k=1}^N) ,
\]
(4.17)
i.e. the average over the action of the Euclidean group on the fields $\phi_a$ is equivalent to averaging over all observer reference frames, as in $G^{\text{ob}}$, but also over all observer positions. This is simply because we have imposed both statistical isotropy and homogeneity. In particular, we see that $G^{\text{th}}$ actually only depends on the observer time $t_o$, contrary to $G^{\text{ob}}$ which also depends on $\vec{x}_o$. For instance, the CMB maps one would obtain from another viewpoint of the universe $\vec{x}_o'$, would be different from the ones we observe on earth today, while the theoretical 2-point correlation function or the power spectrum which we calculate are the same for all vantage points. From now on we will drop the $t_o$ dependence for notational simplicity.

Since the TCFs $G^{\text{th}}$ are SO(3)-invariant functions on $\mathbb{S}^N$ too, we can also decompose them in the basis (3.32) and obtain the harmonic components using (3.36). Since this manipulation is linear, and so is (4.17), the ergodic hypothesis in terms of the harmonic components simply reads
\[
G^{\text{th}}_{l_1,...,l_N|l_1,...,l_{N-3}}(\{ z_k \}_{k=1}^N) \equiv \frac{1}{V} \int d^3 x_o \ G^{\text{ob}}_{l_1,...,l_N|l_1,...,l_{N-3}}(\vec{x}_o; \{ z_k \}_{k=1}^N) .
\]
(4.18)

5 Cosmic variance on the light cone

Since the information from several $\vec{x}_o$ is not observationally available, the best thing one can do in practice in order to compare theory and observation is to use the following approximation to (4.17)
\[
G^{\text{th}} \left( \{ z_k, n_k \}_{k=1}^N \right) \approx G^{\text{ob}}(x_o; \{ z_k, n_k \}_{k=1}^N) .
\]
(5.1)
In the literature $G_{\text{ob}}$ is called an unbiased estimator for $G^{\text{th}}$. To estimate the associated error, one usually considers the statistical covariance matrix

$$\text{Cov}_{\text{stat}}(\alpha_N; \alpha_M') \equiv \left\langle \left[ G_{\text{ob}}(x_o; \alpha_N) - G^{\text{th}}(\alpha_N) \right] \left[ G_{\text{ob}}(x_o; \alpha_M') - G^{\text{th}}(\alpha_M') \right] \right\rangle_p,$$

where $\alpha_N$ collectively denotes the set $\{z_k, n_k\}_{k=1}^N$ for notational simplicity. Here it is understood that the fields inside $G_{\text{ob}}$ are promoted to stochastic ones with distribution $P[\phi(t_0)]$, so that

$$\left\langle G_{\text{ob}}(x_o; \alpha_N) \right\rangle_p = G^{\text{th}}(\alpha_N).$$

(5.3)

The absolute “1-sigma” error of the estimator (5.1) is then

$$\Sigma(\alpha_N) := \sqrt{\text{Cov}_{\text{stat}}(\alpha_N, \alpha_N)}.$$

(5.4)

On the other hand, one can also define a spatial covariance matrix

$$\text{Cov}_{\text{spat}}(\alpha_N; \alpha_M') := \frac{1}{V} \int d^3x_o \left[ G_{\text{ob}}(x_o; \alpha_N) - G^{\text{th}}(\alpha_N) \right] \left[ G_{\text{ob}}(x_o; \alpha_M') - G^{\text{th}}(\alpha_M') \right]$$

erg. $\frac{1}{V} \int d^3x_o G_{\text{ob}}(x_o; \alpha_N) G_{\text{ob}}(x_o; \alpha_M') - G^{\text{th}}(\alpha_N) G^{\text{th}}(\alpha_M'),$

(5.5)

since (with the ergodic hypothesis) $G^{\text{th}}(\alpha_N)$ appears as the spatial average of $G_{\text{ob}}(x_o; \alpha_N)$ over $x_o$ in (4.17). Now note that a product of two OCFs can be expressed as a partial SO(3) average of a single OCF

$$G_{\text{ob}}(x_o; \alpha_N) G_{\text{ob}}(x_o; \alpha_M') \equiv \frac{1}{(8\pi^2)^2} \int dR \int dR' \prod_{k=1}^N \mathcal{O}(x_o; z_k, R^{-1}n_k) \prod_{l=1}^M \mathcal{O}(x_o; z'_l, R'^{-1}n'_l)$$

$$R \rightarrow R' \equiv \frac{1}{(8\pi^2)^2} \int dR' \prod_{k=1}^N \mathcal{O}(x_o; z_k, R'^{-1}n_k) \prod_{l=1}^M \mathcal{O}(x_o; z'_l, R'^{-1}n'_l)$$

$$= \frac{1}{8\pi^2} \int dR' G_{\text{ob}}(x_o; \{z_k, R^{-1}n_k\}_{k=1}^N, \{z'_l, n'_l\}_{l=1}^M),$$

(5.6)

inserting this in (5.2) and (5.5) and using (4.17) and (5.3) we find that both covariances are equal

$$\text{Cov}_{\text{spat.}} \equiv \text{Cov}_{\text{stat.}} =: \text{Cov},$$

(5.7)

and that

$$\text{Cov}(\alpha_N; \alpha_M') \equiv \frac{1}{8\pi^2} \int dRG^{\text{th}}(\{z_k, R^{-1}n_k\}_{k=1}^N, \{z'_l, n'_l\}_{l=1}^M) - G^{\text{th}}(\alpha_N) G^{\text{th}}(\alpha_M').$$

(5.8)

The equality (5.7) is just another consequence of the ergodic hypothesis. It implies that the fundamental statistical uncertainty (5.4), usually known as “cosmic variance”, is actually the error due to the fact that we observe a single realization of the universe and from a single viewpoint $\bar{x}_o$. Indeed, if either of these two conditions were dropped, then there would be no cosmic variance. On one hand, if we had simultaneous access to the data of a single realization from all possible $\bar{x}_o$, then the ergodic hypothesis (4.17) would allow us to match the theoretical predictions exactly.\(^8\) On the other hand, if we could observe all possible universe realizations, even from a single viewpoint $\bar{x}_o$, then we would be able to compute directly the theoretical $N$-point functions, which are independent of $\bar{x}_o$. In the first case we are technically setting $\text{Cov}_{\text{spat.}} \rightarrow 0$, whereas in the second one it is rather $\text{Cov}_{\text{stat.}} \rightarrow 0$.

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\(^8\)Here we neglect the fact that this information would also require a time $\sim V^{1/3}$ to get collected by a main observer and therefore analyzed.
6 Connected correlation functions and angular N-point spectra

The linear relations (4.17) or (4.18) will not hold in general for the connected correlation functions, because the latter are non-linear combinations of the full correlation functions. For instance, in the case of the connected 2-point functions we have

\[
C^{\text{th}}(z_1, z_2, n_1, n_2) := C^{\text{th}}(z_1, z_2, n_1, n_2) - G^{\text{th}}(z_1) G^{\text{th}}(z_2)
\]

\[
= \frac{1}{V} \int d^3 x_o \, C^{\text{ob}}(x_o; z_1, z_2, n_1, n_2) - G^{\text{th}}(z_1) G^{\text{th}}(z_2)
\]

\[
\equiv \frac{1}{V} \int d^3 x_o \left[ C^{\text{ob}}(x_o; z_1, z_2, n_1, n_2) + G^{\text{ob}}(x_o; z_1) G^{\text{ob}}(x_o; z_2) \right] - G^{\text{th}}(z_1) G^{\text{th}}(z_2)
\]

\[
\equiv \frac{1}{V} \int d^3 x_o \, C^{\text{ob}}(x_o; z_1, z_2, n_1, n_2) + \text{Cov}(z_1; z_2),
\]

(6.1)

where \( \text{Cov}(z_1; z_2) \) is the covariance matrix of the 1-point function with itself (see Eq. (5.5)). Let us now investigate this difference more closely. We start by defining the observable fluctuations, in the observational and theoretical cases, as the deviation of the observable from the respective 1-point functions

\[
\Delta^{\text{ob}} \mathcal{O}(x_o; z, n) := \mathcal{O}(x_o; z, n) - G^{\text{ob}}(x_o; z), \quad \Delta^{\text{th}} \mathcal{O}(x_o; z, n) := \mathcal{O}(x_o; z, n) - G^{\text{th}}(z),
\]

(6.2)

or, in terms of the harmonic components,

\[
\Delta^{\text{ob}} \mathcal{O}_{lm}(x_o; z) := \mathcal{O}_{lm}(x_o; z) - \delta^0_l \delta^0_m G^{\text{ob}}(x_o; z), \quad \Delta^{\text{th}} \mathcal{O}_{lm}(x_o; z) := \mathcal{O}_{lm}(x_o; z) - \delta^0_l \delta^0_m G^{\text{th}}(z).
\]

(6.3)

We note in particular that, by construction, the observational monopole is identically zero, while the theoretical one captures precisely the difference between the two 1-point functions

\[
\Delta^{\text{ob}} \mathcal{O}_{00}(x_o; z) \equiv G^{\text{ob}}(x_o, z) - G^{\text{ob}}(x_o, z) \equiv 0, \quad \Delta^{\text{th}} \mathcal{O}_{00}(x_o; z) \equiv G^{\text{ob}}(x_o, z) - G^{\text{th}}(z) \neq 0.
\]

(6.4)

The corresponding connected 2,3,4-point functions can then be defined by

\[
C^*(z_1, z_2, n_1, n_2) := \langle \Delta^* \mathcal{O}(z_1, n_1) \Delta^* \mathcal{O}(z_2, n_2) \rangle^* , \quad (6.5)
\]

\[
B^*(z_1, z_2, n_1, n_2, n_3) := \langle \Delta^* \mathcal{O}(z_1, n_1) \Delta^* \mathcal{O}(z_2, n_2) \Delta^* \mathcal{O}(z_3, n_3) \rangle^* , \quad (6.6)
\]

\[
T^*(z_1, z_2, z_3, z_4, n_1, n_2, n_3, n_4) := \langle \Delta^* \mathcal{O}(z_1, n_1) \Delta^* \mathcal{O}(z_2, n_2) \Delta^* \mathcal{O}(z_3, n_3) \Delta^* \mathcal{O}(z_4, n_4) \rangle^* \quad (6.7)
\]

\[
- \langle \Delta^* \mathcal{O}(z_1, n_1) \Delta^* \mathcal{O}(z_2, n_2) \rangle^* \langle \Delta^* \mathcal{O}(z_3, n_3) \Delta^* \mathcal{O}(z_4, n_4) \rangle^*,
\]

\[
- \langle \Delta^* \mathcal{O}(z_1, n_1) \Delta^* \mathcal{O}(z_3, n_3) \rangle^* \langle \Delta^* \mathcal{O}(z_2, n_2) \Delta^* \mathcal{O}(z_4, n_4) \rangle^*,
\]

\[
- \langle \Delta^* \mathcal{O}(z_1, n_1) \Delta^* \mathcal{O}(z_4, n_4) \rangle^* \langle \Delta^* \mathcal{O}(z_2, n_2) \Delta^* \mathcal{O}(z_3, n_3) \rangle^* ,
\]

where the star is respectively “ob” and “th”, the corresponding averages are respectively \( \langle \ldots \rangle_{\text{SO}(3)} \) and \( \langle \ldots \rangle_P \) and we have omitted the \( \vec{x}_o \) dependencies for notational simplicity. To get the harmonic components we can proceed as in the previous section, or simply use (3.36). In the observational
case, the (absolute) power spectrum, bispectrum and trispectrum read
\[
C_l^{\text{ob}}(\vec{x}_o; z_1, z_2) \equiv \frac{\Delta^{\text{ob}}O_{lm}(\vec{x}_o; z_1) \Delta^{\text{ob}}O^{*}_{lm}(\vec{x}_o; z_2)}{2l + 1},
\]
(6.8)
\[
B_{l_1l_2l_3}^{\text{ob}}(\vec{x}_o; z_1, z_2, z_3) \equiv \left( \frac{l_1 l_2 l_3}{m_1 m_2 m_3} \right)^3 \prod_{k=1}^{3} \Delta^{\text{ob}}O_{lk_mk}(\vec{x}_o; z_k),
\]
(6.9)
\[
T_{l_1l_2l_3l_4|L}^{\text{ob}}(\vec{x}_o; z_1, z_2, z_3, z_4) \equiv (-1)^{M}\sqrt{2L+1} \left( \frac{l_1 l_2 L}{m_1 m_2 - M} \right) \left( \frac{L l_3 l_4}{M m_3 m_4} \right) \prod_{k=1}^{4} \Delta^{\text{ob}}O_{lk_mk}(\vec{x}_o; z_k)
- (-1)^{l_1+l_3} \sqrt{(2L+1)(2L+3)} \delta_{l_1l_2} \delta_{l_3l_4} \delta_{l_00}C_{l_1}^{\text{ob}}(\vec{x}_o; z_1, z_2) C_{l_3}^{\text{ob}}(\vec{x}_o; z_3, z_4)
- (-1)^{l_1+l_2+L} \sqrt{2L+1} \left\{ l_1 l_2 \right\} \delta_{l_1l_4} \delta_{l_2l_3} C_{l_1}^{\text{ob}}(\vec{x}_o; z_1, z_3) C_{l_2}^{\text{ob}}(\vec{x}_o; z_2, z_4)
- \sqrt{2L+1} \left\{ l_1 l_2 \right\} \delta_{l_1l_4} \delta_{l_2l_3} C_{l_1}^{\text{ob}}(\vec{x}_o; z_1, z_4) C_{l_2}^{\text{ob}}(\vec{x}_o; z_2, z_3),
\]
(6.10)
where we have used (3.33),
\[
\sum_{m} (-1)^{m} \frac{(L l l)}{0 m_{-m}} = \sqrt{2l+1}(-1)^{l} \delta^{L_{0}},
\]
(6.11)
and the $3 - j$ symbol symmetries. The theoretical case is then found by simply replacing the label “ob” → “th” and including the averages over $\vec{x}_o$
\[
C_l^{\text{th}}(z_1, z_2) \equiv \frac{1}{V} \int d^3x_o \frac{\Delta^{\text{th}}O_{lm}(\vec{x}_o; z_1) \Delta^{\text{th}}O^{*}_{lm}(\vec{x}_o; z_2)}{2l + 1},
\]
(6.12)
\[
B_{l_1l_2l_3}^{\text{th}}(z_1, z_2, z_3) \equiv \left( \frac{l_1 l_2 l_3}{m_1 m_2 m_3} \right) \frac{1}{V} \int d^3x_o \prod_{k=1}^{3} \Delta^{\text{th}}O_{lk_mk}(\vec{x}_o; z_k),
\]
(6.13)
\[
T_{l_1l_2l_3l_4|L}^{\text{th}}(z_1, z_2, z_3, z_4) \equiv (-1)^{M}\sqrt{2L+1} \left( \frac{l_1 l_2 L}{m_1 m_2 - M} \right) \left( \frac{L l_3 l_4}{M m_3 m_4} \right) \prod_{k=1}^{4} \Delta^{\text{th}}O_{lk_mk}(\vec{x}_o; z_k)
- (-1)^{l_1+l_3} \sqrt{(2L+1)(2L+3)} \delta_{l_1l_2} \delta_{l_3l_4} \delta_{l_00}C_{l_1}^{\text{th}}(z_1, z_2) C_{l_3}^{\text{th}}(z_3, z_4)
- (-1)^{l_1+l_2+L} \sqrt{2L+1} \left\{ l_1 l_2 \right\} \delta_{l_1l_4} \delta_{l_2l_3} C_{l_1}^{\text{th}}(z_1, z_3) C_{l_2}^{\text{th}}(z_2, z_4)
- \sqrt{2L+1} \left\{ l_1 l_2 \right\} \delta_{l_1l_4} \delta_{l_2l_3} C_{l_1}^{\text{th}}(z_1, z_4) C_{l_2}^{\text{th}}(z_2, z_3),
\]
(6.14)
Because of (6.4), the observational spectra (6.8), (6.9) and (6.10) are identically zero whenever at least one of their $l_k$ entries is zero, i.e. they have vanishing “monopoles” by construction. Thus, the physical information contained in the spectra starts at $l > 0$. In contrast, this is not the case for the theoretical spectra (6.12), (6.13) and (6.14). Coming back to the case of the connected 2-point correlation function (6.1), it is also obvious that the difference occurs precisely at the level of the monopole $l = 0$, since the extra term has no angular dependence. The corresponding relation in harmonic space can be found either by using (3.36) on (6.1), or by simply using (6.3) to express

---

Note that (6.10) agrees with Eqs. (19) and (20) of [18], as the normalization convention is different: $T_{l_1l_2l_3l_4|L} \equiv Q_{l_3l_4}(L)/\sqrt{2L+1}$. This implies that the harmonic components in [18] are normalized as in our case.
\( \Delta^\text{th} \mathcal{O}_{lm} \) in terms of \( \Delta^\text{ob} \mathcal{O}_{lm} \) in (6.12). The result is

\[
C_l^{\text{th}}(z_1, z_2) \overset{\text{erg}}{=} \frac{1}{V} \int d^3x_o C_l^{\text{th}}(x_o; z_1, z_2) + \delta_{ll} \text{Cov}(z_1; z_2). \tag{6.15}
\]

Thus, the relation (4.18) between theory and observation holds only for \( l > 0 \). The case \( l = 0 \) of (6.15) tells us that the theoretical power spectrum monopole \( C_0^{\text{th}}(z_1, z_2) \), which has no observational analogue since \( C_0^{\text{ob}}(z_1, z_2) \equiv 0 \), actually amounts to the information of the covariance matrix of the 1-point function \( \text{Cov}(z_1; z_2) \). Therefore, the absolute 1-sigma cosmic variance associated with the 1-point function approximation (5.4)

\[
C_l^{\text{ob}}(x_o; z) \approx C_l^{\text{th}}(z) \pm \Sigma(z), \tag{6.16}
\]

is simply the square root of the theoretical power spectrum monopole at equal redshift

\[
\Sigma(z) := \sqrt{\text{Cov}(z; z)} \equiv \sqrt{C^0_{l0}(z, z)}. \tag{6.17}
\]

This picture generalizes to the case of higher \( N \), as one can check by expressing \( \Delta^\text{th} \mathcal{O}_{lm} \) in terms of \( \Delta^\text{ob} \mathcal{O}_{lm} \) in (6.13) and (6.14) for instance. One finds that Eq. (4.18) holds only up to monopole terms that compensate the fact that the observational spectrum has identically zero monopoles. Moreover, these extra terms can be related to the covariance matrix and higher order analogues (skewness, kurtosis, etc.) of lower-\( N \) spectra.

Let us now compute the covariance matrix of the power spectrum

\[
\text{Cov}_{ll'}(z_1, z_2, z_3, z_4) := \frac{1}{V} \int d^3x_o \left[ C_l^{\text{ob}}(x_o; z_1, z_2) - C_l^{\text{th}}(z_1, z_2) \right] \left[ C_{l'}^{\text{ob}}(x_o; z_3, z_4) - C_{l'}^{\text{th}}(z_3, z_4) \right] \overset{\text{erg}}{=} \frac{1}{V} \int d^3x_o C_l^{\text{ob}}(x_o; z_1, z_2) C_{l'}^{\text{ob}}(x_o; z_3, z_4) - C_l^{\text{th}}(z_1, z_2) C_{l'}^{\text{th}}(z_3, z_4) + \text{mon.}, \tag{6.18}
\]

where the extra monopole terms, i.e. proportional to \( \delta_{ll} \) and/or \( \delta_{ll'} \), arise already at this level because of (6.15) and will be neglected in what follows. To compute the above quantity, we can simply consider (6.14) with \( l_1 = l_2 = l \), \( l_3 = l_4 = l' \) and \( L = 0 \), i.e. a squeezed quadrilateral, and reconstruct and isolate the covariance matrix in the resulting expression. Using

\[
\binom{ l & l & 0 \\ m & -m & 0 } \equiv \sqrt{2l + 1} (-1)^{l-m}, \tag{6.19}
\]

we find

\[
\text{Cov}_{ll'}(z_1, z_2, z_3, z_4) \equiv \frac{(-1)^{l+l'}}{\sqrt{(2l + 1)(2l' + 1)}} \left[ \delta_{ll'} \left[ C_l^{\text{th}}(z_1, z_3) C_l^{\text{th}}(z_2, z_4) + C_l^{\text{th}}(z_1, z_4) C_l^{\text{th}}(z_2, z_3) \right] \\
+ T_{lll'l'}^{\text{th}}(z_1, z_2, z_3, z_4) \right] + \text{mon.} \tag{6.20}
\]

In particular, the cosmic variance of the power spectrum, i.e. the absolute 1-sigma error in the approximation (5.4)

\[
C_l^{\text{ob}}(x_o; z_1, z_2) \approx C_l^{\text{th}}(z_1, z_2) \pm \Sigma_l(z_1, z_2), \tag{6.21}
\]

is

\[
\Sigma_l^2(z_1, z_2) := \text{Cov}_{ll}(z_1, z_2, z_1, z_2) \overset{\text{erg}}{=} \frac{1}{2l + 1} \left[ C_l^{\text{th}}(z_1, z_2) \right]^2 + C_l^{\text{th}}(z_1, z_1) C_l^{\text{th}}(z_2, z_2) + T_{lll'l'}^{\text{th}}(z_1, z_2, z_1, z_2) \right] + \text{mon.}. \tag{6.22}
\]
For Gaussian statistics $T = 0$ and equal redshifts $z_1 = z_2$, we recover the well-known result

$$\tilde{\Sigma}_l(z, z)|_{\mathrm{Gauss.}} = \sqrt{\frac{2}{2l + 1}} C_{l}^{\mathrm{th}} (z, z).$$ \hspace{1cm} (6.23)

### 7 Mind the monopole when using relative fluctuations

Until now we have considered the “absolute” fluctuations $\Delta^{\mathrm{ob}} O$ and $\Delta^{\mathrm{th}} O$, but in practice the most convenient ones to work with are the relative ones

$$\delta^{\mathrm{ob}} O(\vec{x}_o; z, n) := \frac{\Delta^{\mathrm{ob}} O(\vec{x}_o; z, n)}{G^{\mathrm{ob}}(\vec{x}_o; z)}; \quad \delta^{\mathrm{th}} O(\vec{x}_o; z, n) := \frac{\Delta^{\mathrm{th}} O(\vec{x}_o; z, n)}{G^{\mathrm{th}}(z)}.$$ \hspace{1cm} (7.1)

This introduces a non-linear difference between the observational and theoretical definitions and will affect the covariance matrices of the corresponding spectra. Indeed, consider for instance the case of the relative power spectra

$$\tilde{C}_l^{\mathrm{ob}}(\vec{x}_o; z_1, z_2) := \frac{\delta^{\mathrm{ob}} O_{lm}(\vec{x}_o; z_1) \delta^{\mathrm{ob}} O_{lm}^*(\vec{x}_o; z_2)}{2l + 1} \equiv \frac{C_l^{\mathrm{ob}}(\vec{x}_o; z_1, z_2)}{G^{\mathrm{ob}}(\vec{x}_o; z_1) G^{\mathrm{ob}}(\vec{x}_o; z_2)},$$ \hspace{1cm} (7.2)

$$\tilde{C}_l^{\mathrm{th}}(z_1, z_2) \equiv \frac{1}{V} \int d^3 x \delta^{\mathrm{th}} O_{lm}(\vec{x}_o; z_1) \delta^{\mathrm{th}} O_{lm}^*(\vec{x}_o; z_2) \equiv \frac{C_l^{\mathrm{th}}(z_1, z_2)}{G^{\mathrm{th}}(z_1) G^{\mathrm{th}}(z_2)},$$ \hspace{1cm} (7.3)

which are denoted using tilded letters to distinguish them from the absolute ones $C_l^{\mathrm{ob}}$ and $C_l^{\mathrm{th}}$. The corresponding covariance matrix is given by

$$\tilde{\mathrm{Cov}}^{ll'}(z_1, z_2, z_3, z_4) := \frac{1}{V} \int d^3 x \left[ \tilde{C}_l^{\mathrm{ob}}(\vec{x}_o; z_1, z_2) - \tilde{C}_l^{\mathrm{th}}(z_1, z_2) \right] \left[ \tilde{C}_l^{\mathrm{ob}}(\vec{x}_o; z_3, z_4) - \tilde{C}_l^{\mathrm{th}}(z_3, z_4) \right], \hspace{1cm} (7.4)

and one would naively expect this to be simply

$$\tilde{\mathrm{Cov}}^{ll'}(z_1, z_2, z_3, z_4) = \frac{\mathrm{Cov}^{ll'}(z_1, z_2, z_3, z_4)}{G^{\mathrm{th}}(z_1) G^{\mathrm{th}}(z_2) G^{\mathrm{th}}(z_3) G^{\mathrm{th}}(z_4)},$$ \hspace{1cm} (7.5)

where $\mathrm{Cov}^{ll'}$ is the covariance matrix of the absolute power spectrum given in (6.22). In particular, one would then infer the analogue of (6.23) for the corresponding cosmic variance in the equal redshift Gaussian case

$$\tilde{\Sigma}_l(z, z)|_{\mathrm{Gauss.}} = \sqrt{\frac{2}{2l + 1}} \tilde{C}_l^{\mathrm{th}} (z, z).$$ \hspace{1cm} (7.6)

However, as we will now show, Eqs. (7.5) and (7.6) are actually only approximate, because in order to obtain them one must wrongly assume that $G^{\mathrm{ob}}(\vec{x}_o; z) = G^{\mathrm{th}}(z)$ or, according to (6.4), that the monopole of the observable is zero at $\vec{x}_o$

$$\delta^{\mathrm{th}} O_{00}(\vec{x}_o, z) = 0.$$ \hspace{1cm} (7.7)

Indeed, neglecting $l, l' = 0$ terms

$$\mathrm{Cov}^{ll'}(z_1, z_2, z_3, z_4) \equiv \frac{1}{V} \int d^3 x \left[ C_l^{\mathrm{ob}}(\vec{x}_o; z_1, z_2) \right]^2 - \left[ C_l^{\mathrm{th}}(z_1, z_2) \right]^2 \hspace{1cm} (7.8)

\equiv \frac{1}{V} \int d^3 x G^{\mathrm{ob}}(\vec{x}_o; z_1) G^{\mathrm{ob}}(\vec{x}_o; z_2) G^{\mathrm{th}}(\vec{x}_o; z_3) G^{\mathrm{th}}(\vec{x}_o; z_4) \tilde{C}_l^{\mathrm{ob}}(\vec{x}_o; z_1, z_2) \tilde{C}_l^{\mathrm{ob}}(\vec{x}_o; z_3, z_4)

- G^{\mathrm{th}}(z_1) G^{\mathrm{th}}(z_2) G^{\mathrm{th}}(z_3) G^{\mathrm{th}}(z_4) \tilde{C}_l^{\mathrm{th}}(z_1, z_2) \tilde{C}_l^{\mathrm{th}}(z_3, z_4)

\equiv G^{\mathrm{th}}(z_1) G^{\mathrm{th}}(z_2) G^{\mathrm{th}}(z_3) G^{\mathrm{th}}(z_4) \left[ \tilde{\mathrm{Cov}}^{ll'}(z_1, z_2, z_3, z_4) + R^{ll'}(z_1, z_2, z_3, z_4) \right],$$

\hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} 17
where the reminder term $\tilde{R}_{ll'}$ depends on the monopole $\delta^1\mathcal{O}_{00}(x_o; z_k)$ that comes from converting the $G^{ab}(x_o, z_k)$ into $G^{ab}(z_k)$. It is therefore given by a combination of monopoles of the theoretical power spectrum, bispectrum and trispectrum which, unlike the extra terms described in the previous section, contribute to all $l$ values, not just to $l = 0$. Thus, it is important to know whether one compares absolute quantities or relative ones, because the corresponding covariance matrices are different and therefore so will be the results of the corresponding Fisher forecasts.

8 Discussion and Summary

In this paper we have developed the general theory of the angular $N$-point spectra and clarified the cosmic variance of the observed angular $N$-point spectra due to the fact that our observations are confined to a single light-cone. Previous work [9–17] on the angular $N$-point spectra draw heavily on the pioneering work [18], in which the angular bispectrum and the trispectrum are derived. However, these spectra are derived by imposing the rotational invariance and parity, in a way that it becomes quickly difficult to derive the equation for the spectra for $N > 4$. In contrast, we have constructed for the first time the “multilateral” Wigner symbols by generalizing the Wigner 3-$j$ symbols of triangles to polygons to compute the orthonormal harmonic basis of the $N$-point spectra, such that any $N$-point spectra can be mechanically computed in terms of the multilateral Wigner symbols.

To test cosmological models (or any other data on the sphere with is statistically isotropic) against the observed angular $N$-point spectra, we have computed the theoretical $N$-point spectra upon ensemble average and separated the field configurations into different equivalence classes, where two field configurations in the same class are related by an isometry. Applying the Ergodic hypothesis, we can then relate the average over the Euclidean group to the ensemble average. The average over the Euclidean group $E = SO(3) \ltimes \mathbb{R}^3$ is, however, not quite the observed average on the sky, as we can only perform the (angle) average over $SO(3)$, not over the translations (of the observer positions). This physical limitation of the single light-cone observations gives rise to cosmic variance, or the variance of the observable quantities around their theoretical ensemble average. Our new formulation nicely complements the standard perspective (see, e.g., [20, 21]), in which cosmological observables such as the CMB temperature anisotropies are harmonically decomposed and there exist only $(2l + 1)$ number of independent modes.

A direct limitation from these single light-cone observations is that we do not have any access to the background value of the observable quantities. The best way to estimate the background value is to perform the angle average of the observables, and this estimate includes not only the background value, but also the monopole perturbation. Consequently, there is no observed monopole in observations, but the real monopole contribution fluctuates at each observation point in the Universe, setting the cosmic variance limit to our estimate of the background value. For example, observations of the cosmic microwave background temperatures yield the observed CMB temperature [22] with no observed monopole. However, this observation is not the fundamental cosmological parameter ($T_\gamma$ or $\omega_\gamma$), and it is cosmic-variance limited to the monopole power spectrum ($\sim 10^{-5}$) [23].

As a worked example, we have computed the cosmic variance of the angular power spectrum and shown that the standard formula is recovered under the assumption of Gaussianity. Finally, we caution that due to the difference in the observed mean and the background, the dimensionless fluctuation constructed from the observables suffers from the unnecessary nonlinearity due to the monopole contribution in the denominator, which should be avoided by introducing an additional cosmological parameter for the background [23]. The cosmic variance limit from single light-cone observations translates into the fundamental floor in the cosmological parameter constraints derivable
from observations. A systematic study requires a detailed analytical and numerical investigation of
the cosmological information contents on the light cone [24].

Higher-order $N$-point angular spectra provide the best way to probe the non-Gaussianity of the
initial conditions. The bispectrum and the trispectrum are well studied and measured [10–13, 17]
in CMB observations, whittling down the parameter spaces of the inflationary models. There exists,
however, beyond-the-standard inflationary models [6, 25] with negligible bispectrum and trispectrum,
but with significant enhancements in higher $N$-point spectra ($N > 4$). Our new theoretical framework
to construct the angular $N$-point spectra will play a crucial role in systematically exploring the non-
Gaussianities of the initial conditions in observation and theory.

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