ON THE RATIONALITY OF THE MODULI OF HIGHER SPIN CURVES IN LOW GENUS

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ABSTRACT: The global geometry of the moduli spaces of higher spin curves and their birational classification is largely unknown for \( g \geq 2 \) and \( r > 2 \). Using quite related geometric constructions, we almost complete the picture of the known results in genus \( g \leq 4 \) showing the rationality of the moduli spaces of even and odd 4-spin curves of genus 3, of odd spin curves of genus 4 and of 3-spin curves of genus 4.

Key words: Rationality, Higher spin curves, Higher theta-characteristics, Low genus.

Mathematics Subject Classification (2010): 14H10, 14H45, 14E05, 14E08.

1. Introduction

Let \( C \) be a smooth, irreducible complex projective curve of genus \( g \), a theta characteristic on \( C \) is a square root \( \eta \) of the canonical sheaf \( \omega_C \). By definition a pair \((C,\eta)\) is a spin curve. It is said to be even or odd according to the parity of \( h^0(\eta) \). Starting from Cornalba’s paper \[CS9\], the moduli space \( S_g \) of spin curves of genus \( g \) and its compactifications became object of systematic investigations. As is well known \( S_g \) is split in two irreducible connected components \( S^+_g \) and \( S^-_g \). They respectively correspond to moduli of even and odd spin curves. The Kodaira dimension of \( S^+_g \) is completely known, as well as several facts about rationality or unirationality in low genus. The picture is as follows for even or odd spin curves:

- \( S^+_g \) is uniruled for \( g \leq 7 \),
- \( S^+_8 \) has Kodaira dimension zero,
- \( S^+_g \) is of general type for \( g \geq 9 \),
- \( S^-_g \) is uniruled for \( g \leq 11 \),
- \( S^-_g \) is of general type for \( g \geq 12 \).

Moreover the unirationality of \( S^-_g \) and \( S^+_g \) has been proved respectively for \( g \leq 8 \) and \( g \leq 6 \). Concerning the rationality problem, \( S^+_g \) is classically known to be rational for \( g \leq 3 \), while the rationality of \( S^+_g \) is a recent result. For more details on the above picture see \[Dol10\], \[F10\], \[FV10\], \[FV12\], \[TZ09\], \[V13\].

Higher spin curves generalize spin curves. By definition a higher spin curve of genus \( g \) and order \( r \) is a pair \((C,\eta)\) such that \( \eta^{\otimes r} \cong \omega_C \). The moduli spaces of these pairs are denoted by \( S^{1/r}_g \). They were constructed by Jarvis in \[J98\] and then studied by several authors, see for instance \[CCC07\], \[Ch08\], \[J01\].

Supported by PRIN Project 2010-11 'Geometria delle varietà algebriche' of MIUR and by GNSAGA group of INdAM.
Concerning the irreducibility of these spaces, it is useful to recall how they behave: \( S_{g/r}^1 \) is irreducible if \( r \) is odd and \( g \geq 2 \), while \( S_{g/r}^1 \) is split in two irreducible connected components if \( r \) is even and \( g \geq 2 \) \cite{[00]}. They are distinguished by the condition that \( \eta \otimes r^2 \) is an even or odd theta characteristic. However, with the exception of the case of genus 1, the global geometry of \( S_{g/r}^1 \) appears largely unknown for \( r > 2 \).

From another side a natural, elementary, remark is that for every curve \( C \) the canonical sheaf \( \omega_C \) not only admits square roots, but the roots of order \( g - 2 \) and \( g - 1 \) as well. Restricting to \( g - 1 \) roots, they form configurations of line bundles of degree two which are worth of being studied.

For \( r = g - 1 \) the forgetful map \( f : S_{g/(g-1)}^1 \rightarrow \mathcal{M}_g \) has degree \((g-1)^2\). Since this grows up very fast, it is seems natural to expect that \( S_{g/(g-1)}^1 \) becomes of general type after very few exceptions. About this, assume that \( g - 1 \) is even so that \( \eta \otimes (g-1)/2 \) is a theta characteristic. Then every irreducible component of \( S_{g/(g-1)}^1 \) dominates \( S_{g}^+ \) or \( S_{g}^- \) if \( g \) is odd, via the assignment \((C, \eta) \rightarrow (C, \eta \otimes (g-1)/2)\).

Therefore, in view of the picture on moduli of spin curves, there exist irreducible components of \( S_{g/(g-1)}^1 \) of non negative Kodaira dimension as soon as \( g \geq 8 \). In this frame the first unknown case of low genus to be considered is the genus 4 case. Somehow surprisingly this is still an exception. We prove in this note that

**Theorem 1.1.** The moduli space of 3-spin curves of genus 4 is rational.

Let \((C, \eta)\) be a general spin curve of genus 4 and order 3. The starting point for proving the theorem is the remark that giving \((C, \eta)\) is equivalent to give the unique effective divisor \( t \in |\eta^\otimes 2| \). Furthermore, let \( C \) be canonically embedded in \( \mathbb{P}^3 \), then \( 3t \) is the complete intersection of two quadrics and a cubic surface. We show that the GIT-quotient \( Q \) of the family of these complete intersections is rational and that there is a natural birational map between \( Q \) and \( S_{1/3}^1 \).

Adding up this result to the known picture we obtain a list of cases of genus \( g \leq 4 \) where the rationality of \( S_{g/r}^1 \) is confirmed. Here is the complementary list of unknown cases for \( g \leq 4 \):

- Moduli of 4-spin curves of genus 3.
- Moduli of odd spin curves of genus 4.
- Moduli of 6-spin curves of genus 4.

In particular it seems that the case of odd spin curves of genus 4 was not considered in the literature. Notice also that \( S_{g/(2g-2)}^1 \) splits into the union of two components: the moduli of pairs \((C, \eta)\) such that \( \eta \otimes g-1 \) is an even theta characteristic and the complementary component. We will denote them respectively by

\[ S_{g/(2g-2)}^1, S_{g/(2g-2)}^- \]

We will say that \((C, \eta)\) is an even (odd) \( r \)-spin curve if \( \eta \otimes r^2 \) is an even (odd) theta characteristic. In the final part of this paper we almost complete the picture of the known results in genus \( g \leq 4 \). Building on quite related geometric constructions and methods, we prove the following theorems.

**Theorem 1.2.** The moduli space of odd spin curves of genus 4 is rational.

**Theorem 1.3.** The moduli spaces of 4-spin curves of genus 3 are rational.
We have not found evidence to the uniruledness of $S_g^{1/r}$ in the only two missing cases in genus $g \leq 4$, namely for $S_4^{1/6^+}$ and $S_4^{1/6^-}$. The same lack of evidence appears for further very low values of $g$, say $g \leq 7$ and $r \geq 3$. Already for these cases, it could be interesting to apply some recent results on the structure of the Picard group of the Deligne-Mumford compactification of $S_g^{1/r}$ to obtain informations on the Kodaira dimension of these spaces, (cfr. for instance [P13] and [RW12]).

2. Third roots on genus 4 curves

Let $(C, \eta)$ be a spin curve of genus $g$ and order $r$. We will assume that $C$ is canonically embedded in $\mathbb{P}^{g-1}$.

Putting $k = \lfloor \frac{g - 1}{\deg \eta} \rfloor + 1$, we have $h^0(\eta^\otimes k) \geq 1$ by Riemann-Roch. This implies that each effective divisor $t \in |\eta^\otimes k|$ satisfies the condition $rt = C \cdot F$, where $F$ is a hypersurface of degree $k$. If $\deg \eta$ divides $g$ then $\deg t = g$ and we expect that $t$ is isolated, which is equivalent to $h^1(\eta^\otimes k) = 0$.

Let us focus on the case $g = 4$ and $r = 3$. In this situation $C \subset \mathbb{P}^3$ is a genus 4 curve of degree 6 and $t$ is a divisor in the linear system $|\eta^\otimes 2|$. Then $3t$ is a bicanonical divisor and there exists a quadric surface $S$ such that

$$3t = C \cdot S.$$  

Lemma 2.1. Let $C$ be a general curve of genus 4, then $h^0(\eta) = 0$ for every spin curve $(C, \eta)$ of order 3.

Proof. We can assume that $C = Q \cap F$, where $Q$ is a fixed, smooth quadric and $F$ a cubic surface. Now assume $h^0(\eta) = 1$ for some cubic root $\eta$ of $\omega_C$. Then there exist points $x, y \in C$ such that $x + y \in |\eta|$ and $3x + 3y = C \cdot H$, where $H \in |\mathcal{O}_Q(1)|$. Let $F$ be the family of complete intersections $3x' + 3y' = C' \cdot H'$, where $H' \in |\mathcal{O}_Q(1)|$ and $C' \in |\mathcal{O}_Q(3)|$ is smooth. It is easy to see that the action of $\text{Aut} \ Q$ on $F$ has finitely many orbits. On the other hand, since $3x + 3y$ is a complete intersection, it follows $\dim |I_{3x+3y}(C)| = 8$, where $I_{3x+3y}$ is the ideal sheaf of $3x + 3y$. But then, since the moduli space of $C$ is 9-dimensional, $C$ is not general: a contradiction. □

From now on our spin curve $(C, \eta)$ will be sufficiently general. In particular we fix the following assumptions:

Assumption 2.1.

- $C$ is a complete intersection in $\mathbb{P}^3$ of a smooth quadric $Q$ and a cubic $F$,
- for each $x \in C$ one has $h^0(\mathcal{O}_C(3x)) = 1$,
- $h^0(\eta) = 0$ so that $h^0(\eta^\otimes 2) = 1$.

The second condition is just equivalent to say that the two $g^3_4$’s on $C$ have simple ramification. The third one is satisfied iff the unique effective divisor $t \in |\eta^\otimes 2|$ is not contained in any plane.

It is clear that the locus of moduli of pairs $(C, \eta)$ satisfying these assumptions is a dense open subset of $S_4^{1/3}$. It is also clear from the previous remarks that the scheme $3t$ is a complete intersection, namely

$$3t = F \cdot Q \cdot S,$$

where $S$ is a quadric. This defines a second curve, that we denote from now on as

$$E := Q \cdot S.$$
We point out that $E$ is uniquely defined by $(C, \eta)$. $E$ is a quartic curve of arithmetic genus one. We will denote by $I_{at}$ the ideal sheaf in $Q$ of the divisor at $C$. Let $o \in t$ be a closed point, we can fix local parameters $x, y$ at $o$ so that $y$ is a local equation of $C$ and $x$ restricts to a local parameter in $O_{C,o}$. Then $I_{at}$ is generated at $o$ by $x^{am}$ and $y$, where $m$ is the multiplicity of $t$ at $o$. We observe that $3t$ is a 0-dimensional scheme of length 12, embedded in the smooth curve $C$.

Now assume for simplicity that $E$ is smooth. Since $3t = E \cdot C$, it follows that $3t$, a divisor in $E$, belongs to $|O_E(3)|$. Let us define

$$\epsilon(1) := O_E(t).$$

Since we are assuming that $h^0(O_C(t)) = 1$, we know that then $t$ is not contained in a plane. Hence the line bundle $\epsilon$ is non trivial. On the other hand we have $3t \in |O_E(3)|$ so that $\epsilon^{\otimes 3} \cong O_E$. It follows that

**Lemma 2.2.** The line bundle $\epsilon$ is a non trivial 3-torsion element of $\text{Pic}^0 E$.

Actually the condition that $E$ be smooth is satisfied as soon as the the pair $(C, \eta)$ is sufficiently general. This is proven in the next theorem, where some useful conditions, satisfied by a general pair $(C, \eta)$, are summarized.

**Theorem 2.3.** On a dense open set $U \subset S_4^{1/3}$ every point is the moduli point of a spin curve $(C, \eta)$ such that:

1) $(C, \eta)$ is general as in assumption 2.1,
2) $E$ is a smooth quartic elliptic curve,
3) $t$ is a smooth divisor of $E$,
4) $t \in |\epsilon(1)|$, where $\epsilon$ is a non trivial third root of $O_E$.

**Proof.** We use the irreducibility of $S_g^{1/r}$ when $r$ is odd and $g \geq 2$, [00]. The space $S_4^{1/3}$ is irreducible, so that every non-empty open subset of it is dense. Conditions 1), 2), 3) and 4) are open on families of triples $(C, \eta, E)$ and hence they define open subsets of $S_4^{1/3}$. We already know that the open set defined by 1) is not empty. Therefore, to prove the theorem, it suffices to produce one pair $(C, \eta)$ satisfying 2), 3), 4). We start from a smooth elliptic quartic $E$. We have $E = Q \cdot S \subset P^3$, where $Q, S$ are smooth quadrics. Let $\epsilon \in \text{Pic}^0 E$ be a non trivial element such that $\epsilon^{\otimes 3} \cong O_E$. Since $\epsilon(1)$ is very ample, a general $t \in |\epsilon(1)|$ is smooth and not contained in a plane. Note that $3t \in |O_E(3)|$. Then, since $E$ is projectively normal, we have

$$3t = Q \cdot S \cdot F$$

where $F$ is a cubic surface. Let $I_{3t}$ be the ideal sheaf of $3t$ in $Q$, then we have $h^0(I_{3t}(3)) = 5$. Moreover the base locus of $|I_{3t}(3)|$ is $3t$. Hence, by Bertini theorem, a general $C \in |I_{3t}(3)|$ is smooth along $C - t$. To prove that a general $C$ is smooth along $t$ it suffices to produce one element with this property. This is the case for $E + L$, where $L$ is a general plane section. Let $C \in |I_{3t}(3)|$ be smooth and let $\eta := O_C(1 - t)$. Then $(C, \eta)$ is a spin curve of order 3 satisfying 2), 3), 4). \qed

3. Projective bundles related to $S_4^{1/3}$

Let $(C, \eta)$ be a general spin curve of order 3 and genus 4. We keep the previous conventions, so that $C$ is canonically embedded in $P^3$ as $Q \cap F$. 

It follows from the above theorem that the moduli point \([C, \eta]\) uniquely defines, up to isomorphisms, a triple \((E, \epsilon, t)\) such that \(E\) is a smooth quartic elliptic curve in \(\mathbb{P}^3\) and \(\epsilon\) is a non trivial third root of \(\mathcal{O}_E\).

Moreover \(t\) is a smooth element of \(|\epsilon(1)|\) and \(3t\) is a complete intersection

\[3t = C \cdot E = F \cdot Q \cdot S \subset \mathbb{P}^3,\]

where \(S\) is a quadric. As a divisor in \(C\), \(t\) is the the unique element of \(|\eta^{\otimes 2}|\). In order to prove the rationality of \(S_{4}^{1/3}\) our strategy is as follows. We consider the moduli space of elliptic curves \(E\) endowed with a non trivial 3-torsion element of \(\text{Pic}^0 E\), namely

\[\mathcal{R}_{1,3} := \{[E, \epsilon] \mid g(E) = 1, \quad \epsilon \not\in \mathcal{O}_E, \quad \epsilon^{\otimes 3} \cong \mathcal{O}_E\}.\]

Over it we have the moduli space \(\mathcal{P}_{1,4}\) of triples \((E, \epsilon, H)\) such that \(H \in \text{Pic}^4 E\). This can be also defined via the Cartesian square

\[
\begin{array}{ccc}
\mathcal{P}_{4,1} & \longrightarrow & \mathcal{P} \text{i}c_{4,1} \\
\downarrow & & \downarrow \\
\mathcal{R}_{1,3} & \longrightarrow & \mathcal{M}_1.
\end{array}
\]

As usual, \(\mathcal{P} \text{i}c_{4,1}\) denotes the universal Picard variety, that is, the moduli space of pairs \((H, E)\) such that \(E\) is an elliptic curve and \(H \in \text{Pic}^4 E\).

The space \(\mathcal{P}_{4,1}\) is a rational surface. Proving its unirationality, so that the rationality follows, is easy. Starting from \(\mathcal{P}_{4,1}\) we construct a suitable “tower”

\[
P_c \xrightarrow{\epsilon} P_b \xrightarrow{b} P_a \xrightarrow{a} \mathcal{P}_{4,1}
\]

of projective bundles \(a, b, c\). Clearly, as a “tower” of projective bundles over a rational base, \(P\) is rational. Let \(\psi : S_{4}^{1/3} \to \mathcal{P}_{4,1}\) be the rational map defined as follows: \(\psi([C, \eta]) := [E, \epsilon]\). Then we will show that \(\psi\) factors through a natural birational map between \(S_{4}^{1/3}\) and \(P_c\), so proving that \(S_{4}^{1/3}\) is rational. In the next subsections we produce the projective bundles which are needed.

3.1. The ambient bundle \(\mathbb{P}\). Let us start with the universal elliptic curve over \(\mathcal{M}_1\) and its pull-back \(\mathcal{E} \to \mathcal{R}_{1,3}\). As is well known there exists a Poincaré bundle \(\mathcal{P}\) on the fibre product \(\mathcal{P}_{4,1} \times_U \mathcal{E}\), where \(U \subset \mathcal{R}_{1,3}\) is a suitable dense open set. In particular the restriction of \(\mathcal{P}\) to the fibre at \([E, \epsilon, H]\) of the projection map

\[\alpha : \mathcal{P}_{4,1} \times_U \mathcal{E} \to \mathcal{P}_{4,1}\]

is given by \(\mathcal{P} \otimes \mathcal{O}_{\mathcal{P}_{4,1}} \otimes \mathcal{E} \cong H\). Note that \((\alpha_{\mathcal{P}})[E, \epsilon, H] = H^0(H)\) has constant dimension 4. Let \(\mathcal{H} := \alpha_{\mathcal{P}}\); then, by Grauert’s theorem, \(\mathcal{H}\) is a vector bundle of rank 4 over \(\mathcal{P}_{4,1}\). We define the the ambient bundle \(\mathbb{P}\) as follows:

\[\mathbb{P} := \mathcal{P} \mathcal{H}^*.
\]

Its structure map will be denoted as \(p : \mathbb{P} \to \mathcal{P}_{4,1}\). It is a \(\mathbb{P}^2\)-bundle over \(\mathcal{P}_{4,1}\). In particular, the tautological bundle \(\mathcal{O}_{\mathbb{P}}(1)\) defines an embedding

\[\mathcal{P}_{4,1} \times_U \mathcal{E} \subset \mathbb{P}.
\]

At \(x := [E, \epsilon, H]\) this is the embedding \(E \subset \mathbb{P}_{x} = \mathbb{P} H^0(H)^*\) defined by \(H\).
3.2. The bundle of quadrics $a : \mathbb{P}_a \to \mathcal{P}_{a,1}$. Let us consider the map 
\[ \mu : \text{Sym}^2 \mathcal{H} \to \alpha_*(\mathcal{P} \otimes 2) \]
of vector bundles on $\mathcal{P}_{a,1}$. At $x := [E, \epsilon, H]$ we have $\alpha_*(\mathcal{P} \otimes 2)_x = H^0(H \otimes 2)$ and 
\[ \mu_x : \text{Sym}^2 H^0(H) \to H^0(H \otimes 2) \]
is the multiplication map. Putting $Q := \ker \mu$ and $\mathbb{P}_a := \mathbb{P}Q$, we denote as 
\[ a : \mathbb{P}_a \to \mathcal{P}_{a,1} \]
the structure map. The bundle $a$ is a $\mathbb{P}^1$-bundle and the fibre $\mathbb{P}_x$ parametrizes the quadrics containing the tautological embedding $E \subset \mathbb{P}_x$ defined by $H$.

3.3. The $\mathbb{P}^3$-bundle $b : \mathbb{P}_b \to \mathbb{P}_a$. At first we define the $\mathbb{P}^3$-bundle 
\[ e : \mathbb{P}_c \to \mathcal{P}_{4,1}. \]
Its fibre $\mathbb{P}_{c,x}$ will be $[\epsilon \otimes H]$ at $x := [E, \epsilon, H]$. On $\mathcal{P}_{a,1} \times U \mathcal{E}$ we fix a vector bundle $\mathcal{N}$ whose restriction to the fibre of $\alpha : \mathcal{P}_{a,1} \times U \mathcal{E} \to \mathcal{P}_{a,1}$ at $x$ is 
\[ \mathcal{N} \otimes \mathcal{O}_{\alpha^*x} \cong \epsilon. \]
The construction of $\mathcal{N}$ is standard: let $\beta : \mathcal{P}_{a,1} \times U \mathcal{E} \to \mathcal{R}_{1,3} \times U \mathcal{E}$ be the natural map. Then we define $\mathcal{N} := \beta^* \mathcal{L}$, where $\mathcal{L}$ is a Poincaré bundle on $\mathcal{R}_{1,3} \times U \mathcal{E}$. Note that $\mathcal{L}$ restricted to the fibre at $[E, \epsilon]$ of the projection $\gamma : \mathcal{R}_{1,3} \times U \mathcal{E} \to \mathcal{R}_{1,3}$ is the line bundle $\epsilon$. We consider the tensor product $\mathcal{H} \otimes \mathcal{N}$ and finally $\alpha_*(\mathcal{H} \otimes \mathcal{N})$. The latter is a rank 4 vector bundle with fibre $H^0(\mathcal{H} \otimes \epsilon)$ at $x$. We define 
\[ \mathbb{P}_b := a^* \mathbb{P}\alpha_*(\mathcal{H} \otimes \epsilon). \]
The bundle $\mathbb{P}_b$ is a $\mathbb{P}^3$-bundle over $\mathbb{P}_a$. The fibre at $x$ of the map $a \circ b : \mathbb{P}_b \to \mathcal{P}_{a,1}$ is the Segre product $[\epsilon \otimes H] \times [I_3(2)]$, where $I_3$ is the ideal sheaf of the embedding $E \subset \mathbb{P}_x$.

3.4. The $\mathbb{P}^4$-bundle $c : \mathbb{P}_c \to \mathbb{P}_b$. In the fibre product $\mathbb{P}_b \times \mathcal{P}_{a,1}$ we define the following subvarieties 
\[ t \subset E \subset Q \subset \mathbb{P}_b \times \mathcal{P}_{a,1}. \]
Let $o \in \mathbb{P}_b \times U \mathbb{P}$, then $o$ defines a pair $(x, z)$ where $z \in \mathbb{P}_x$ and $x := a \circ b(o) = [E, \epsilon, H]$. Moreover, the point $o$ is an element $t \in [\epsilon \otimes H]$ of the fibre of $\mathbb{P}_b$ at $b(o)$. Finally $b(o)$ is an element $Q \in [I_3(2)]$, where $I_3$ is the ideal sheaf of the tautological embedding $E \subset \mathbb{P}_x$. Clearly we have $t \subset E \subset Q$.

The conditions $z \in t$, $z \in E$, $z \in Q$ respectively define the closed sets $t$, $E$, $Q$. In particular $E$ is a natural embedding of $\mathcal{P}_{1,4} \times U \mathcal{E}$ in $\mathbb{P}_b \times \mathcal{P}_{a,1}$ and $t$ is a Weil divisor in $E$. Let us consider the standard exact sequence 
\[ 0 \to I_{3t} \to \mathcal{O}_Q \to \mathcal{O}_{3t} \to 0 \]
where $I_{3t}$ is the ideal sheaf of $t$ in $Q$. We pull-back the line bundle $\mathcal{O}_Q(3)$ to the fibre product $\mathbb{P}_b \times \mathcal{P}_{a,1}$ and tensor the above exact sequence by it. The resulting exact sequence is denoted in the following way: 
\[ 0 \to I_{3t}(3) \to \mathcal{O}_Q(3) \to \mathcal{O}_{3t}(3) \to 0. \]
Let $\beta : \mathbb{P}_b \otimes \mathbb{P} \to \mathbb{P}_b$ be the projection onto $\mathbb{P}_b$. Then we apply the push-down functor $\beta_*$ to this new exact sequence. We obtain the exact sequence 
\[ 0 \to \beta_* I_{3t}(3) \to \beta_* \mathcal{O}_Q(3) \to \beta_* \mathcal{O}_{3t}(3) \to R^1 \beta_* I_{3t}(3) = 0. \]
Here the sheaf $R^1\beta_*\mathcal{I}_{3t}(3)$ is zero because at any point $p = (t, Q, [E, \epsilon, H]) \in \mathbb{P}_b$ its fibre is $H^1(\mathcal{I}_{3t/Q}(3)) = 0$. Notice also that the sheaf $\mathcal{F} := \beta_*\mathcal{I}_{3t}(3)$ is a rank 5 vector bundle with fibre $H^0(\mathcal{I}_{3t/Q}(3))$ at the same point $p$. Finally we define

$$P_c := \mathbb{P}\mathcal{F}.$$  

We denote the structure map of this $\mathbb{P}^4$-bundle as $c : P_c \to \mathbb{P}_b$. The fibre of $c$ at $p$ is the linear system of cubic sections $C$ of $Q$ containing the scheme $3t \subset E$. Notice that a smooth $C$ is a canonical curve of genus 4 endowed with the order 3 spin structure $\eta := \omega_C(-t)$.

4. The rationality of $S_{4}^{1/3}$

Let $\mathcal{I}_{2t/\mathbb{P}^3}$ be the ideal sheaf of $2t \subset C \subset \mathbb{P}^3$. Notice also that

**Lemma 4.1.** $|\mathcal{I}_{2t/\mathbb{P}^3}(2)|$ is a pencil of quadrics with base locus $E$.

**Proof.** Observe that $\omega_{C}^{\otimes 2}(-2t) \cong \eta^{\otimes 2}$. Moreover, this is also the sheaf $\mathcal{I}_{2t/C}(2)$. Consider the standard exact sequence of ideal sheaves

$$0 \to \mathcal{I}_{C/\mathbb{P}^3}(2) \to \mathcal{I}_{2t/\mathbb{P}^3}(2) \to \eta^{\otimes 2} \to 0.$$  

Since we have $h^0(\mathcal{I}_{C/\mathbb{P}^3}(2)) = h^0(\eta^{\otimes 2}) = 1$, the statement follows. \[\square\]

Due to the latter construction there exists a natural moduli map

$$\phi : P_c \to S_{4}^{1/3}$$  

which sends a point $z = (C, t, Q, [E, \epsilon, H]) \in P_c$ to the point

$$\phi(z) := (C, \eta),$$  

with $\eta = \omega_C(-t)$. Clearly $\phi$ is defined at $z$ iff $C$ is smooth. Since $P_c$ is rational we can finally deduce the rationality of $S_{4}^{1/3}$, stated in the Introduction. We show that

**Theorem 4.2.** The map $\phi : P_c \to S_{4}^{1/3}$ is birational, so that $S_{4}^{1/3}$ is rational.

**Proof.** At first we show that the map $\phi$ is dominant. Starting with a general point $[C, \eta] \in S_{4}^{1/3}$ it is possible to reconstruct a point $z = (C, t, Q, [E, \epsilon, H]) \in P_c$ such that $\phi(z) = [C, \eta]$. Indeed $t$ is the unique element of $|\eta^{\otimes 2}|$. Then, from the canonical embedding $C \subset \mathbb{P}^3$, we reconstruct $E$ as the smooth base locus of the pencil of quadrics $|\mathcal{I}_{2t}(2)|$, considered above. Then we have $H := O_E(1)$ and $\epsilon := H(-t)$. The quadric $Q$ is the unique quadric of $|\mathcal{I}_{2t/\mathbb{P}^3}(2)|$ containing $C$. It is clear that $[C, \eta] = \phi(z)$, with $z = (C, t, Q, [E, \epsilon, H])$. Conversely the inverse map of $\phi$ is well-defined too. Starting from a general $[C, \eta]$ the point $z$ is indeed uniquely reconstructed as above. Hence $\phi^{-1}$ is well defined and $\phi$ is birational. \[\square\]

In the next sections we prove the other rationality results announced in the Introduction.
5. The rationality of $S_4^-$

We start from an odd spin curve $(C,\eta)$ of genus 4. As in the previous sections, $C$ will be sufficiently general. Thus, passing to its canonical model, we have

$$C \subset Q \subset \mathbb{P}^3,$$

where $Q = \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric and $C$ has bidegree $(3, 3)$ in it. Since $\eta$ is odd, there exists a unique $d \in |\eta|$ and we have

$$2d = L \cdot C,$$

where $L$ is a plane section of $Q$ and a conic tritangent to $C$. The condition that both $d$ and $L$ be smooth clearly defines an open set $U \subset S_4^-$. Furthermore it is easily seen that $U \neq \emptyset$. Then, since $S_4^-$ is irreducible, the next lemma follows.

**Lemma 5.1.** For a general $C$ both the divisor $d$ and the conic $L$ are smooth.

Let $o_1, o_2, o_3$ be the three points of $d$. They are not collinear because $h^0(\eta) = 1$. Hence we can fix projective coordinates $(x_0 : x_1) \times (y_0 : y_1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ so that

$$o_1 = (1 : 0) \times (1 : 0), \quad o_2 = (0 : 1) \times (0 : 1), \quad o_3 = (1 : 1) \times (1 : 1).$$

In particular we can assume that these points are in the diagonal

$$L := \{x_0y_1 - x_1y_0 = 0\}$$

of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathcal{I}_{2d}$ be the ideal sheaf of $2d$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and let

$$I := H^0(\mathcal{I}_{2d}(3, 3)).$$

We consider the 9-dimensional linear system $\mathcal{I}I$. This is endowed with the map

$$m : \mathcal{I}I \to S_4^-$$

defined as follows. Let $C \in \mathcal{I}I$ be smooth, then $m(C) := [C, \eta]$, where $\eta := \mathcal{O}_C(o_1 + o_2 + o_3)$. It is clear from the construction that $m$ is dominant. Let

$$G \subset \text{Aut} \mathbb{P}^1 \times \mathbb{P}^1$$

be the stabilizer of the set $\{o_1, o_2, o_3\}$. We have:

**Lemma 5.2.** Assume $C_1, C_2 \in \mathcal{I}I$ are smooth. Then $m(C_1) = m(C_2)$ if and only if $C_2 = \alpha(C_1)$ for some $\alpha \in G$.

**Proof.** Let $m(C_i) = [C_i, \eta_i], \ i = 1, 2$. If $m(C_1) = m(C_2)$ there exists a birational map $\alpha : C_2 \to C_1$. Since $\mathcal{O}_{C_i}(1, 1) \cong \omega_{C_i}$, it follows that $\alpha$ induces an isomorphism $a^* : H^0(\mathcal{O}_{C_1}(1, 1)) \to H^0(\mathcal{O}_{C_2}(1, 1))$. This implies that $a$ is induced by some $\alpha \in \text{Aut} \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, the condition $m(C_1) = m(C_2)$ also implies that $a^*\mathcal{O}_{C_2}(o_1 + o_2 + o_3) \cong \mathcal{O}_{C_1}(o_1 + o_2 + o_3)$. Hence $\alpha \in G$. The converse is obvious. \(\square\)

Now observe that $G$ acts, in the natural way, on $\mathcal{I}I$ and that $m : \mathcal{I}I \to S_4^-$ is dominant. Then, as an immediate consequence of the previous lemma, we have

**Corollary 5.3.** $S_4^-$ is birational to the quotient $\mathcal{I}I/G$.

Thus the rationality of $S_4^-$ follows if $\mathcal{I}I/G$ is rational. In order to prove this, we preliminarily describe the group $G$ and its action on $\mathcal{I}I$. We recall that the natural inclusion $\text{Aut} \mathbb{P}^1 \times \text{Aut} \mathbb{P}^1 \subset \text{Aut} \mathbb{P}^1 \times \mathbb{P}^1$ induces the exact sequence

$$0 \to \text{Aut} \mathbb{P}^1 \times \text{Aut} \mathbb{P}^1 \to \text{Aut} \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{Z}_2 \to 0,$$
where $\mathbb{Z}_2$ is generated by the class of the projective involution
\[ \iota : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \]
exchanging the factors. From the above exact sequence we have the exact sequence
\[ 0 \to G_3 \to G \to \mathbb{Z}_2 \to 0. \]
Here $G_3$ denotes the stabilizer of the set $O := \{o_1, o_2, o_3\}$ in $\text{Aut} \mathbb{P}^1 \times \text{Aut} \mathbb{P}^1$. Since $O$ is a subset of the diagonal $L$, $L$ itself is fixed by $G_3$. In particular it follows that $G_3$ is the diagonal embedding in $\text{Aut} \mathbb{P}^1 \times \text{Aut} \mathbb{P}^1$ of the stabilizer of $\{o_1, o_2, o_3\}$ in $\text{Aut} L$. As is very well known, this is a copy of the symmetric group $S_3$.

Now we proceed to an elementary and explicit description of the $G$-invariant subspaces of $\mathbb{P} I$. From it the rationality of $\mathbb{P} I / G$ will follow. We fix the notation $l := x_0y_1 - x_1y_0$ for the equation of the diagonal $L$. Let
\[ R = \oplus_{a, b \in \mathbb{Z}} R_{a, b} \]
be the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$, where $R_{a, b}$ is the vector space of forms of bidegree $a, b$. We can assume that $\iota^* : R \to R$ is the involution such that $\iota^* x_i = y_i, i = 0, 1$. On the other hand let
\[ h_1 := x_0(y_1 - y_0) + y_0(x_1 - x_0), \ h_2 := x_1(y_1 - y_0) + y_1(x_0 - x_1), \ h_3 := x_0y_1 + x_1y_0, \]
so that $\{l, h_1, h_2, h_3\}$ is a basis of $R_{1, 1}$. We can also assume that, for each $\sigma \in G_3$, the map $\sigma^*: R \to R$ is such that $\sigma^* l = l$ and $\sigma^*$ permutes the elements of the set $\{h_1, h_2, h_3\}$. Then we observe that the eigenspaces of $\iota^*: R_{1, 1} \to R_{1, 1}$ are
\[ R_{1, 1}^+ = l, \ R_{1, 1}^- = h_1, h_2, h_3 \]
This implies that
\[ R_{1, 1} = l \oplus h_1 + h_2 + h_3 \oplus h_1 - h_3, h_2 - h_3 \]
where all the summands are $G$-invariant. Considering the multiplication map
\[ \mu : \text{Sym}^2 R_{1, 1} \to R_{2, 2} \]
one can check that
\[ \text{Ker} \mu = l^2 - (h_1 - h_3)(h_2 - h_3). \]
Then, putting $h := h_1 + h_2 + h_3$ and $h_{ij} := h_i - h_j$, it is easy to deduce that the eigenspaces of $\iota^*: R_{2, 2} \to R_{2, 2}$ decompose as follows:
\[ R_{2, 2}^+ = h^2 \oplus h_{13}, \ hh_{23} \oplus h_{13}^2, \ h_{23}^2 \oplus h_{13} h_{23} \]
and
\[ R_{2, 2}^- = lh \oplus lh_{13}, \ lh_{23}, \]
where each summand appearing above is $G$-invariant. Finally, we consider the vector space $I$ and observe that, taking the multiplication by $l$, we have an injection
\[ l < l > \cap R_{2, 2} \to I. \]
Its image $lR_{2, 2} \subset I$ is a subspace codimension one. Moreover we have
\[ lR_{2, 2}^+ \subseteq I^-, \ lR_{2, 2}^- \subseteq I^+, \]
where $I^+, I^-$ are the eigenspaces of $\iota^*: I \to I$. Let us consider
\[ c = x_0x_1(x_0 - x_1) + y_0y_1(y_0 - y_1). \]
Notice that $c \in I$ and that $\text{div}(c)$ is $G$-invariant. Indeed, $\text{div}(c)$ is the union of the six lines in the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ passing through the points $o_1, o_2, o_3$. Notice also that $c$ is not in $lR_{Q,2}$, in particular $I = \langle c \rangle \oplus lR_{Q,2}$. Notice also that $\iota^*c = c$.

Summing all the previous remarks up, we can finally describe the eigenspaces of $\iota^*: I \to I$ and their decompositions as a direct sum of $G$-invariant summands.

**Lemma 5.4.** Let $I^+, I^-$ be the eigenspaces of $\iota^*: I \to I$, then we have

- $I^+ = \langle c \rangle \oplus \langle l^2h > \oplus \langle l^2h_{13}, l^2h_{23} >$,
- $I^- = \langle lh^2 > \oplus \langle lh_{13}, lh_{23} > \oplus \langle lh_{13}^2, lh_{23}^2 > \oplus \langle lh_{13}h_{23} >$,

where each summand is an irreducible representation of $G$.

Now it is straightforward to conclude. For instance let us consider

$$B := \mathbb{P}I^+ \times \mathbb{P}I^-$$

and then the variety

$$\mathbb{P} := \{(x, p) \in \mathbb{P}I \times B \mid x \in \mathbb{P}_p \subset \mathbb{P}I \times B,$$

where $p := (p^+, p^-) \in \mathbb{P}I^+ \times \mathbb{P}I^-$ and $\mathbb{P}_p$ denotes the line joining $p^+$ and $p^-$. The variety $\mathbb{P}$ is endowed with its two natural projections

$$\mathbb{P}I \overset{\beta}{\leftarrow} \mathbb{P} \overset{\alpha}{\twoheadrightarrow} B.$$

Note that $\beta: \mathbb{P} \to \mathbb{P}I$ is birational, since there exists a unique line $\mathbb{P}_p$ passing through a point in $\mathbb{P}I - (\mathbb{P}I^+ \cup \mathbb{P}I^-)$. Moreover

$$\alpha: \mathbb{P} \to B$$

is a $\mathbb{P}^1$-bundle structure with fibre $\mathbb{P}_p$ at the point $p = (p^+, p^-) \in B$. It is also clear that the action of $G$ on $\mathbb{P}I$ induces an action of $G$ on $\mathbb{P}$ and that

$$\mathbb{P}I/G \cong \mathbb{P}/G.$$ 

More precisely, the map $\iota^*$ acts as the identity on $B$, since its two factors are projectivized eigenspaces of $\iota^*$. Moreover each fibre $\mathbb{P}_p$ of $\alpha$ is $\iota^*$-invariant. Indeed $\iota^*/\mathbb{P}_p$ is a projective involution with fixed points $p^+$, $p^-$ on the line $\mathbb{P}_p$.

Note that the induced action of $G_3$ on $B$ is faithful, since the 2-dimensional summands of $I^\pm$ are standard representations of $S_3$. Furthermore $G_3$ acts linearly on the fibres of $\alpha: \mathbb{P} \to B$.

Indeed consider any $\phi \in G_3$ and any $p = (p^+, p^-) \in B$. Then $\phi(\mathbb{P}_p)$ is the line $\mathbb{P}_{\phi(p)}$, where $\phi(p) = (\phi(p^+), \phi(p^-))$. In particular the map $\phi/\mathbb{P}_p: \mathbb{P}_p \to \mathbb{P}_{\phi(p)}$ is a projective isomorphism. Let

$$\overline{\mathbb{P}} := \mathbb{P}/G_3.$$ 

Then the latter remarks imply that $\alpha: \mathbb{P} \to B$ descends to a $\mathbb{P}^1$-bundle

$$\overline{\alpha}: \overline{\mathbb{P}} \to B/G_3$$

over a non empty open set $U \subset B/G_3$. Now let us consider $\iota \in G$ and the involution $\iota : \mathbb{P} \to \mathbb{P}$ due to the action of $G$ on $\mathbb{P}$. It is clear from the previous construction that $\iota$ descends to an involution

$$\overline{\iota}: \overline{\mathbb{P}} \to \overline{\mathbb{P}},$$

which is fixing each fibre of $\overline{\alpha}$ and acts linearly on it. Passing to the quotient

$$\hat{\mathbb{P}} := \overline{\mathbb{P}}/\langle \overline{\iota} \rangle,$$

it follows that $\overline{\alpha}$ induces a $\mathbb{P}^1$-bundle structure $\hat{\alpha}: \hat{\mathbb{P}} \to B/G_3$. 

Remark 5.1. Actually $\hat{\alpha}$ has two natural sections $s^\pm : B/G_3 \to \hat{P}$. They are defined as follows: let $\overline{\sigma} \in B/G$ be the orbit of $p = (p^+, p^-) \in B$. Then the fixed points of $\overline{\tau} : \overline{P} \to \overline{P}^\pm$ are the orbits $\overline{P}^\pm$ of $p^+, p^-$. Passing to the quotient by $\overline{\tau}$ they define two distinguished points $\hat{p}^+, \hat{p}^- \in \hat{P}$, by definition $\hat{p}^\pm = s^\pm(\overline{\sigma})$.

Theorem 5.5. The quotient $\mathbb{P}/G$ is rational.

Proof. Since $\mathbb{P}/G \cong \hat{P}$ and $\hat{\alpha} : \hat{P} \to B/G_3$ is a $\mathbb{P}^1$-bundle, the preceding remarks imply that $\mathbb{P}/G \cong B/G_3 \times \mathbb{P}^1$. Hence it remains to show the rationality of $B/G_3$. This is now straightforward: we have $B = PI^+ \times PI^-$ and $G_3$ acts linearly on both factors. Considering $B$ as the trivial projective bundle over $PI^+$, it follows that $B/G_3$ is a $\mathbb{P}^2$-bundle over $PI^+/G_3$. The rationality of $PI^+/G_3$ is a standard property. Since $PI^+ = \mathbb{P}^1$, it is easily proven considering the decomposition of $I^+$ as a sum of irreducible representations of $G_3$. Hence $B/G_3$ is rational. \(\square\)

We have already proved that $S^4_1$ is birational to $\mathbb{P}/G$. Hence it follows:

Corollary 5.6. The moduli space of odd spin curves of genus 4 is rational.

6. The rationality of $S^{4-4}_3$

The rationality result to be proven in this section naturally relies on the geometry of odd spin curves of genus 4 considered above. To see this relation let us fix from now on a general curve $C$ of genus three and two distinct points $n_1, n_2 \in C$. As is well known, the line bundle $\omega_C(n_1 + n_2)$ defines a morphism

$$\phi : C \to C_n \subset Q \subset \mathbb{P}^3$$

where $C_n := \phi(C)$ and $Q := \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric. $\phi$ is an embedding on $C - \{n_1, n_2\}$. Moreover $C_n$ is a curve of bidegree $(3, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with exactly one node $n := \phi(n_1) = \phi(n_2)$, see [GL86]. The condition that $\mathcal{O}_C(n_1 + n_2)$ is a theta characteristic is reflected by the projective model $C_n$ as follows:

Lemma 6.1. Let $R_1$ and $R_2$ be the two lines of $Q$ passing through the node $n$. Then the following conditions are equivalent:

- $R_1, R_2$ are tangent to the branches of $n$, that is $\phi^* R_i = 2n_i + n_j$ for $i \neq j$.
- $\mathcal{O}_C(n_1 + n_2)$ is a theta characteristic.

We omit for brevity the standard proof of this lemma. Assume now that $[C, \eta]$ is a general point of $S^{4-4}_3$ so that $\eta^{\otimes 2}$ is an odd theta characteristic on $C$. This is equivalent to say that there exist two distinct points $n_1, n_2 \subset C$ such that

$$\mathcal{O}_C(n_1 + n_2) \cong \eta^{\otimes 2} \quad \text{and} \quad \eta^{\otimes 6} \cong \omega_C(n_1 + n_2) \cong \mathcal{O}_C(3n_1 + 3n_2).$$

Considering the morphism $\phi$ defined by $\eta^{\otimes 6}$, we have as above that its image

$$C_n \subset Q \subset \mathbb{P}^3$$

is a curve with exactly one node $n = \phi(n_1) = \phi(n_2)$ and no other singular point. Now we observe that the linear system $|\eta^{\otimes 6}|$ contains the two distinct elements:

- $3n_1 + 3n_2$, where $n_1 + n_2 \in |\eta^{\otimes 2}|$,
- $2o_1 + 2o_2 + 2o_3$, where $o_1 + o_2 + o_3 \in |\eta^{\otimes 3}|$.

Lemma 6.2. One has $h^0(\eta) = 0$, so that $h^0(\mathcal{O}_C(o_1 + o_2 + o_3)) = 1$. 


Proof. If \( h^0(\eta) \geq 1 \) then \( \eta \cong \mathcal{O}_C(p) \) for some point \( p \in C \). But then \( 4p \in |\omega_C| \), which is impossible on a general \( C \) of genus 3. Now observe that \( \omega_C(-o_1-o_2-o_3) \cong \eta \). Since \( h^0(\eta) = 0 \) it follows \( h^0(\mathcal{O}_C(o_1+o_2+o_3)) = 1 \) by Riemann-Roch.

**Lemma 6.3.** The points \( o_1, o_2, o_3 \) are distinct and \( \{o_1 o_2 o_3\} \cap \{n_1 n_2\} = \emptyset \). Moreover one has \( 2o_1 + 2o_2 + 2o_3 = L \cdot C_n \) where \( L \in |\mathcal{O}_Q(1)| \) is smooth.

Proof. It suffices to produce one pair \((C, \eta)\) satisfying the statement. Fix in \( \mathbb{P}^2 \) five general points \( o_1, o_2, o_3, n_1, n_2 \) and let \( L \) be the conic through them. Consider the linear system \( \Sigma \) of all quartics \( C \) which are tangent to \( L \) at \( o_1, o_2, o_3 \) and tangent to the line \( <n_1, n_2> \) at \( n_1, n_2 \). It is easy to check that the general \( C \in \Sigma \) is smooth. Let \( \eta = \mathcal{O}_C(o_1 + o_2 + o_3 - n_1 - n_2) \), then \((C, \eta)\) satisfies the statement.

**Remark 6.1.** As above let \( \mathcal{O}_C(n_1 + n_2) \) be an odd theta characteristic and let \( C_n \subset \mathbb{P}^3 \) be the image of the map defined by \( |\omega_C(n_1 + n_2)| \). It follows from the previous discussion that there exists a bijection between the set of square roots \( \eta \) of \( \mathcal{O}_C(n_1+n_2) \) and the set of tritangent planes \( P \) to \( C_n - \{n\} \). This bijection associates to \( P \) the line bundle \( \eta = \mathcal{O}_C(o_1 + o_2 + o_3 - n_1 - n_2) \), where \( P \cdot C_n = 2o_1 + 2o_2 + 2o_3 \).

To prove the rationality result of this section we proceed as in the previous one. We fix coordinates \((x_0 : x_1) \times (y_0 : y_1)\) on \( Q \) so that \( o_1 = (1:0:1:0), o_2 = (0:1:0:1) \) and \( o_3 = (1:1,1:1) \). Then we observe that the diagonal \( L = \{x_0 y_1 - x_1 y_0\} \) is tritangent to the the previous curve \( C_n \) at \( o_1, o_2, o_3 \) and that \( n \in Q - L \). Keeping the notations of the previous section we consider the linear system \( \mathbb{P}^1 \). \( C_n \) is in the family of the singular elements of \( \mathbb{P}^1 \). Let

\[
U := Q - L,
\]

for each \( n \in U \) we consider the 4-dimensional linear system

\[
\mathbb{F}_n
\]

of all curves \( D \) of bidegree \((3,3)\) such that:

1. \( 2o_1 + 2o_2 + 2o_3 \subset L \cdot D \),
2. \( D \) has multiplicity \( \geq 2 \) at \( n \),
3. \( R_i \cdot D = 3n \) for \( i = 1, 2 \), where \( R_1 \) and \( R_2 \) are the lines of \( Q \) through \( n \).

Condition (1) implies the inclusion \( \mathbb{F}_n \subset \mathbb{P}^1 \). We consider the incidence correspondence

\[
\mathbb{F} := \{(D,n) \in \mathbb{P}^1 \times U \mid D \in \mathbb{F}_n\}
\]

together with its two projection maps

\[
\mathbb{P}^1 \xleftarrow{\pi_1} \mathbb{F} \xrightarrow{\pi_2} U
\]

Note that \( \mathbb{F} \) is a \( \mathbb{P}^1 \)-bundle via the map \( \pi_2 : \mathbb{F} \rightarrow U \). On the other hand the closure of \( \pi_1(\mathbb{F}) \) is the locus of singular elements of \( \mathbb{P}^1 \). Now we define a rational map

\[
m : \mathbb{F} \rightarrow S^{1/4}_{\mathbb{G}}
\]

as follows. Let \( C_n \in \mathbb{F}_n \) be nodal with exactly one node \( n \), so that its normalization \( \nu : C \rightarrow C_n \) is of genus 3. Defining \( \eta := \nu^* \mathcal{O}_C(o_1 + o_2 + o_3 - \nu^* n) \), one has by definition

\[
m(C_n) := [C, \eta].
\]

Note that the group \( \mathbb{G} \), defined as in the previous section, acts on \( \mathbb{F} \) in the natural way. The action of \( \alpha \in \mathbb{G} \) on \( \mathbb{F} \) is the isomorphism \( f_\alpha : \mathbb{F} \rightarrow \mathbb{F} \) sending \((D,n) \in \mathbb{F} \)
to $(\alpha(D), \alpha(n))$. The proof of the next lemma is completely analogous to the proof of Lemma 6.2 and hence we omit it. The corollary is immediate.

**Lemma 6.4.** Let $D_1, D_2 \in \mathbb{F}$. Then $m(D_1) = m(D_2)$ iff there exists $\alpha \in G$ such that $\alpha(D_1) = \alpha(D_2)$.

**Corollary 6.5.** The quotient $\mathbb{F}/G$ is birational to $S^{1/4-}_3$.

Finally we can deduce that

**Theorem 6.6.** $S^{1/4-}_3$ is rational.

**Proof.** It is easy to see, and it follows from the analysis of the previous section on the action of $G$ on $\mathbb{P}^1$, that the action of $G$ on $F$ is faithful and linear between the fibres of $\mathbb{F}$. Hence the $\mathbb{P}^1$-bundle $\pi_2 : \mathbb{F} \to U$ descends to a $\mathbb{P}^1$-bundle $\mathbb{F} \to U/G$, which is just $\mathbb{F}/G$. But $U/G$ is rational, since it is a unirational surface, therefore $\mathbb{F} = \mathbb{F}/G$ is rational. Then, by the previous corollary, $S^{1/4-}_3$ is rational. □

7. The rationality of $S^{1/4+}_3$

Let us recall that, for any smooth curve $C$ and any divisor $e$ of degree two on it, the line bundle $\omega_C(e)$ is very ample iff $h^0(\mathcal{O}_C(e)) = 0$. Let $C$ be a general curve of genus 3 and let $\eta$ be any 4-th root of $\omega_C$. Then $\eta \otimes 2$ is an even theta characteristic. Hence we assume that $\eta \otimes 2$ is odd in the previous section.

From now on we assume that $[C, \eta]$ is in $S^{1/4+}$, so that $h^0(\eta \otimes 2) = 0$. Then the line bundle $\omega_C \otimes \eta \otimes 2$ is very ample and moreover it defines an embedding of $C$ in $\mathbb{P}^3$ as a projectively normal curve whose ideal is generated by cubics, see [Dol10, §6.3]. Obviously no quadric contains $C$ and we cannot argue as in the previous section. Though the beautiful geometry of cubic surfaces through $C$ can be used, it is simpler to consider the canonical model of $C$. Hence we assume that $C$ is embedded in $\mathbb{P}^2$ as a general plane quartic.

**Lemma 7.1.**

1. One has $h^0(\eta \otimes 3) = 1$. Moreover, the unique divisor of $|\eta \otimes 3|$ is supported on three distinct points $o_1, o_2, o_3$.

2. There exists exactly one cubic $E$ such that $C \cdot E = 4(o_1 + o_2 + o_3)$ and $E$ is smooth.

**Proof.** We have $h^0(\eta \otimes 3) \geq 2$ iff $h^0(\omega_C \otimes \eta^{- \otimes 3}) = 1$. This implies that $C$ has a Weierstrass point $p$ such that $4p \in |\omega_C|$. But then $C$ is not a general curve. To complete the proof of (1) and to prove (2) it suffices to construct a pair $(C, \eta)$ with the required properties. Starting from a smooth cubic $E$ this construction is standard: adapt the argument analogous to the one of the proof of Theorem 2.3.

Furthermore, let $H := \mathcal{O}_E(1)$ and, as above, $4(o_1 + o_2 + o_3) = E \cdot C$. Let $\epsilon := H(-o_1 - o_2 - o_3)$. Clearly $\epsilon$ is a 4-th root of $\mathcal{O}_E$. Moreover:

**Lemma 7.2.** The line bundle $\epsilon \otimes 2$ is not trivial.

**Proof.** If $\epsilon \otimes 2$ were trivial then $2o_1 + 2o_2 + 2o_3 = B \cdot E$, where $B$ is a conic. This would imply that $h^0(\eta \otimes 2) = h^0(\mathcal{O}_C(B - 2o_1 - 2o_2 - 2o_3)) = 1$, which is against our assumption that $\eta \otimes 2$ is an even theta. □
Moving \( \varrho_1 + \varrho_2 + \varrho_3 \) in \( [H \otimes \epsilon^{-1}] \), we can see that a general \( d \in [H \otimes \epsilon^{-1}] \) defines a linear system of genus 3 spin curves \( (D, \eta_D) \) of order 4, such that \( \eta_D^{\otimes 2} \) is an even theta characteristic on \( D \). Indeed, let \( \mathcal{I}_{4d} \) be the ideal sheaf of \( 4d \subset E \). Then
\[
|\mathcal{I}_{4d}(4)|
\]
is a 3-dimensional linear system of plane quartics \( D \) such that the line bundle \( \eta_D := \omega_D(-d) \) satisfies the previous requirements.

Since the previous curve \( C \) was general in moduli, the construction implies that a dense open set of \( S_3^{1/4+} \) is filled up by points \( [D, \eta_D] \) realized as above. We now use the previous remarks to prove that \( S_3^{1/4+} \) is birational to a suitable tower of projective bundles over a rational modular curve.

Let \( \mathcal{T} \) be the moduli space of abelian curves, polarized by a degree 3 polarization and endowed with a 4-torsion point whose square is not trivial. We can think of \( \mathcal{T} \) as a rational curve.

\[ \text{Proposition 7.3.} \quad \mathcal{T} \text{ is a rational curve.} \]

**Proof.** Observe that, on a smooth plane cubic \( E \), a 4-th root of \( O_E \) is a line bundle \( \tau := O_E(t-o) \) such that \( o, t \in E \) and moreover
\begin{enumerate}[(i)]  
\item \( 3o \in |O_E(1)| \),
\item \( 4t + q + r \in |O_E(2)| \),
\item \( q, r \in E \) and \( q + r \in |2o| \).
\end{enumerate}

Indeed these conditions are just equivalent to say that \( 4t \sim 4o \). Notice also that they are fulfilled iff there exists a conic \( B \) such that \( B \cdot E = 4t + q + r \) and \( q + r \sim 2o \).

Furthermore, it is easy to see that either \( \tau^{\otimes 2} \) is not trivial and \( B \) is smooth or \( B \) is a double line and \( B \cdot E = 2(2t + o) \). Assuming the former case we consider the plane cubic \( A + B \), where \( A \) is the flex tangent to \( E \) at \( o \). Let \( P \) be the pencil of cubics generated by \( E \) and \( A + B \), then its base locus is the 0-dimensional scheme \( 4t + q + r + 3o \subset E \). Let \( F \in P \) be smooth, then \( F \) is endowed with the line bundles \( \tau_F := O_F(t-o) \) and \( H_F := O_F(1) \). Hence there exists a rational map \( m : P \to \mathcal{T} \) defined as follows: \( m(F) = [F, H_F, t] \). This map clearly dominates \( \mathcal{T} \) and hence \( \mathcal{T} \) is a rational curve.

\[ \square \]

The space \( \mathcal{T} \) is a finite cover of the moduli space \( \mathcal{A}(3) \) of abelian curves endowed with a degree 3 polarization. An example of such a cover is the forgetful map
\[ f : \mathcal{T} \to \mathcal{A}(3) \]
sending \( [E, H, t] \) to \( [E, H] \). Let \( \mathcal{E} \) be the universal family of abelian curves over \( \mathcal{A}(3) \). Over a suitable open set of \( \mathcal{A}(3) \times \mathcal{M}_1 \), \( \mathcal{E} \) we fix a Poincaré line bundle \( \mathcal{P} \), whose restriction to the curve \( [E, H] \times E \) is the line bundle \( H \). We consider the map
\[ f \times \text{id}_{\mathcal{E}} : \mathcal{T} \times \mathcal{M}_1, \mathcal{E} \to \mathcal{A}(3) \times \mathcal{M}_1, \mathcal{E} \]
and the pull-back
\[ \mathcal{P} := (f \times \text{id}_{\mathcal{E}})^* \mathcal{P} \]
of \( \mathcal{P} \) over the surface
\[ \hat{\mathcal{E}} := \mathcal{T} \times \mathcal{M}_1, \mathcal{E}. \]
Let $u : \tilde{E} \to T$ be the elliptic fibration defined by projection onto $T$. We have two natural sections
\[ s_0, s_1 : T \to \tilde{E} \]
of $u$ which are so defined: $s_1([E, H, t]) = t \in E$ and $s_0([E, H, t]) = o \in E$. The section $s_0$ is just induced by the zero section of $E \to A_1(3)$. Let $D_0 := s_0(T)$ and $D_1 := s_1(T)$. Over a dense open set of $T$ we can finally define the $\mathbb{P}^2$-bundles:
\begin{itemize}
  \item $T := \mathbb{P}(u_*\overline{\mathcal{P}} \otimes \mathcal{O}_{\mathcal{E}}(D_1 - D_0))$,
  \item $\mathbb{P} := \mathbb{P}(u_*\overline{\mathcal{P}}^*)$.
\end{itemize}

The fibre of $T$ at the point $[E, H, t]$ is the linear system $|H(t - o)|$, while the fibre of $\mathbb{P}$ at the same point is $\mathbb{P}H^0(H)^*$. Now we consider the tautological embedding
\[ \tilde{E} \subset \mathbb{P}. \]

At the point $e := [E, H, t]$ the fibre of $\tilde{E}$ is $E$ and the tautological embedding restricts to the embedding $E \subset \mathbb{P}_e = \mathbb{P}H^0(H)^*$, defined by $H$. Then we consider the fibre product
\[ F := T \times_T \mathbb{P} \]
and the incidence correspondence $Z \subset F$ parametrizing the points $[E, H, t; d, x] \in T \times_T \mathbb{P}$ such that
\begin{itemize}
  \item $x \in d \subset E \subset \mathbb{P}H^0(H)^*$,
  \item $d \in |H(t - o)|$.
\end{itemize}

Let $\pi_1 : F \to T$ and $\pi_2 : F \to \mathbb{P}$ be the projection maps; we have the embeddings
\[ Z \subset \pi_1^*\tilde{E} \subset F \]
where $Z$ is a divisor in $\pi_1^*\tilde{E}$ and the latter, up to shrinking its base, is a smooth family of elliptic curves. Such a family contains the divisor $4Z$ and this is a subscheme of $F$. Let $J$ be its ideal sheaf; then our construction yields the projective bundle
\[ Q := \mathbb{P}_{\pi_1^*}(J \otimes \pi_2^*\mathcal{O}_p(4)) \]
over $T$. Let $p := [E, H, t, d]$ be a general point of $T$, then the fibre of $Q$ at $p$ is
\[ Q_p = |\mathcal{I}_{4d}(4)|, \]
where $\mathcal{I}_{4d}$ is the ideal sheaf of $4d$ in $E \subset \mathbb{P}H^0(H)^*$. In particular, by Grauert’s theorem, $Q$ is a $\mathbb{P}^1$-bundle over $T$. Furthermore, the bundle $T$ is rational, since it is a projective bundle over the rational curve $T$, and hence also $Q$ is rational.

The conclusion is now quite clear: we can construct a birational map
\[ m : Q \to S^{1/4}_3 \]
defined as follows. Let $p \in T$ be a general point as above and consider a general element $D \in Q_p = |\mathcal{I}_{4d}(4)|$. By definition, the map $m$ sends it to the point $[D, \omega_D(-d)]$ of $S^{1/4}_3$.

Furthermore $m$ is generically invertible. Indeed let $(C, \eta)$ be as above, then there exists a unique $E$ such that $E \cdot C = d$ and $\eta \cong \omega_C(-d)$. Let $\tilde{E} = \text{Pic}^0 E$, then $\tilde{t} := H(-d)$ is a 4-torsion point of $\tilde{E}$ and $\tilde{H} := \mathcal{O}_{\tilde{E}}(3\tilde{t})$ is the degree 3 polarization defined by $\tilde{\eta} := \mathcal{O}_{\tilde{E}} \subset \tilde{E}$. Let $d = x_1 + x_2 + x_3$ and $\tilde{d} = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3$, where $\tilde{x}_i := H(-3x_i) \in \tilde{E}$. Then $m^{-1}$ is the rational map sending a general $(C, \eta)$ to the point $[\tilde{E}, \tilde{H}, \tilde{t}, \tilde{d}]$ of $Q$. We omit some further details. Now $S^{1/4}_3$ is irreducible and $\dim S^{1/4}_3 = \dim Q$. Hence $m$ is birational and it follows that
Theorem 7.4. \( S_{1/4}^{1/4} \) is rational.

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