Maximal Independent Sets in Polygonal Cacti

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Abstract

Counting the number of maximal independent sets of graphs was started over 50 years ago by Erdős and Mooser. The problem has been continuously studied with a number of variations. Interestingly, when the maximal condition of an independent set is removed, such the concept presents one of topological indices in molecular graphs, the so called Merrifield-Simmons index. In this paper, we applied the concept of bivariate generating function to establish the recurrence relations of the numbers of maximal independent sets of regular \( n \)-gonal cacti when \( 3 \leq n \leq 6 \). By the ideas on meromorphic functions and the growth of power series coefficients, the asymptotic behaviors through simple functions of these recurrence relations have been established.

Keywords: Independent Domination; Independence Polynomial; Cactus Graphs; Asymptotic Behavior

AMS subject classification: 05C69; 05A15; 05C30; 05C31; 05C92; 30E15

1 Introduction

Let \( G = (V(G), E(G)) \) be a graph with the vertex set \( V(G) \) and the edge set \( E(G) \). The order of \( G \) is \( |V(G)| \). For a vertex \( v \in V(G) \), the neighbor set of \( v \) in \( G \), \( N_G(v) \), is the set \( \{u | uv \in E(G)\} \). The degree \( \deg_G(v) \) of a vertex \( v \) in \( G \) is \( |N_G(v)| \). A cycle of length \( n \) is denoted by \( C_n \). For a graph \( G \) containing a vertex \( v \), we say that \( v \) is a cut vertex of \( G \) if \( G - v \) has more components than \( G \). A maximal subgraph that does not have a cut vertex is called a block. A block \( B \) is an end block if \( B \) has exactly one cut vertex of \( G \). For a natural number \( n \geq 3 \), a regular \( n \)-gonal cactus is a graph that has exactly two end blocks and all the blocks are cycles of the same length. A regular \( n \)-gonal cactus is said to be ortho if the two cut vertices of \( G \) that belong to a non-end block are adjacent. A regular \( n \)-gonal cactus
$G$ is said to be *meta* if the two cut vertices of $G$ that belong to a non-end block are a pair of vertices at distance two of the polygon. Further, a regular $n$-gonal cactus $G$ is said to be *para* if the two cut vertices of $G$ that belong to a non-end block a pair of vertices at distance three of the polygon.

A vertex subset $I$ of $V(G)$ is *independent* if any pair of vertices in $I$ are not adjacent in $G$. An independent set $I$ is *maximal* if $I \cup \{v\}$ is no longer independent for any vertex $v \in V(G) \setminus I$. A vertex subset $K$ of $V(G)$ is *clique* if any pair of vertices in $K$ are adjacent in $G$. A clique $K$ is *maximal* if $K \cup \{v\}$ is not a clique for any vertex $v \in V(G) \setminus I$. Clearly, if $\alpha(n)$ is the number of maximal independent sets of all graphs of order $n$ and $\omega(n)$ is the number of maximal cliques of all graphs of order $n$, then

$$\alpha(n) = \omega(n).$$

The study of counting the largest number of maximal independent sets, or cliques, was started over fifty years ago when Erdős and Mooser rose the question how large $\alpha(n)$ can be and what is the graphs that satisfy this upper bound. This problem was solved by Moon and Mooser [37] in 1965. However, the extremal graphs in [37] are not connected. Hence, it was asked further by Wilf [47], who found the largest number of maximal independent sets of trees, what is the largest number of maximal independent sets of all connected graphs of order $n$. This problem was solved by Griggs et al. [21].

Although the value $\alpha(n)$ for graphs, connected graphs or trees had been found, a number of graph theorists still have paid attention to count the number of maximal independent sets. The study is to give some condition to the graphs or is to focus on some particular classes of graphs. For examples, counting the number of maximal independent sets of graphs with given the number of cycles see [28, 29, 30, 41], counting the number of maximal independent sets of comparability graphs (the graphs that are constructed from partial order sets) see [13], counting the number of maximal independent sets of unlabelled cycle of length $n$ see [14], counting the number of maximal independent sets of random graphs see [13, 15] and for example of studies on maximal independent sets see [28, 29].

In 1952, Husimi [27] generalized the cluster and irreducible integrals from the book of Statistical Mechanics by Mayer and Mayer [32] to solve problems in the Theory of Condensation. Later the same year, Uhlenbeck [46] pointed out that Husimi’s integrals can be interpreted by graphs. The such graphs were called Husimi trees, the graphs whose each edge is in at most one cycle. Husimi trees have been attracted much attention as they can be applied to explain many of condensation phenomena, for example of the studies see [24, 26, 40]. The Husimi trees were known in graph theory literature as cacti after Harary and Palmer [25] published their classical book on graph enumeration in 1973 which was 20 years since it was first introduced.

For graphs that are applied in chemistry, Merrifield and Simmons [33, 34, 35, 36] found relationship between physical properties of hydrocarbons and topological indices of graphs representing their molecular structures such as the number of independent sets, connected sets, irredundant sets and maximal independent sets (kernel). In particular, in [33], Merrifield and Simmons observed that the numbers of independent sets of graphs representing Alkanes
have inverse variations to the boiling points and heat of formations of these compounds. By these observations, the number of independent sets of graphs representing molecular structures is known to be Merrifield-Simmons index and has been studied by a number of graph theorists, see \cite{10,11,38} for example. For more studies on topological indices of molecular graphs see \cite{6,7,8,9,20,22,23,31,39,43,44,45} for example.

In this paper, we applied the concept of bivariate generating functions, which have been used in \cite{10,11,12}, to establish recurrence relations of the number of maximal independent sets of ortho, meta and para $n$-gonal cacti when $3 \leq n \leq 6$. Further, the asymptotic behaviors through simple functions of these recurrence relations has been established by the ideas on meromorphic functions and the growth of power series coefficients which were introduced in \cite{48}.

\section{Main Results}

In this section, we state all our main results as detailed in each of the following subsections. All the results will be summarized in Tables 1 and 2 at the end of this section.

\subsection{Triangular Cacti}

Let $T(n)$ be a triangular cactus of $n$ triangles as detailed in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{triangular_cactus.png}
\caption{The triangular cactus $T(n)$ of $n$ triangles}
\end{figure}

\textbf{Theorem 1} Let $t(n,k)$ be the number of maximal independent sets containing $k$ vertices of $T(n)$. If $T(x,y)$ is a bi-variate generating function of $t(n,k)$, then

$$T(x,y) = \frac{1 + 2xy + x^2y^2}{1 - xy - x^2y}.$$ 

\textbf{Theorem 2} Let $t(n)$ be the number of maximal independent sets of $T(n)$. Then, $t(1) = 3, t(2) = 5$ and

$$t(n) = t(n-1) + t(n-2)$$

for $n \geq 3$. 

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From Theorem 2 we obtain the exact formula of \( t(n) \) as follows.

**Theorem 3** If \( t(n) \) is the number of maximal independent sets of \( T(n) \). Then,

\[
t(n) = 0.10559(-0.61803)^n + 1.8944(1.61803)^n.
\]

Further, by considering the principal part of the expansion of the generating function around the singularity, we can gain the growth rate of the recurrence relation in Theorem 2 that:

**Theorem 4** If \( t(n) \) is the number of maximal independent sets of \( T(n) \). Then,

\[
t(n) \approx 1.1708 \frac{n}{0.61803}.
\]

For the last formula of \( t(n) \) concerning the average size of a maximal independent set of \( T(n) \), we find the first order partial derivative of \( T(x, y) \) from Theorem 3 (the method can be found in [11, 48] as well). By letting \( y = 1 \) in \( \frac{\partial T(x, y)}{\partial y} \), we will obtain the recurrence relation of the summation of sizes of all maximal independent sets of \( T(n) \). By finding its asymptotic formula and dividing by the approximation of \( t(n) \) from Theorem 4 we have the following theorem.

**Theorem 5** If \( \overline{t}(n) \) is the average size of a maximal independent set of \( T(n) \). Then,

\[
\overline{t}(n) \approx 0.72361n + 0.33475.
\]

### 2.2 Diamond Cacti

Let \( D(n) \) be the diamond (meta-rectangular) cactus of \( n \) diamonds as detailed in Figure 2.

![Figure 2: The diamond cactus \( D(n) \) of \( n \) diamonds](image)

**Theorem 6** Let \( d(n, k) \) be the number of maximal independent sets of \( D(n) \) containing \( k \) vertices. If \( D(x, y) \) is a bi-variate generating function of \( d(n, k) \), then

\[
D(x, y) = \frac{1 + xy^2 - xy - x^2y^4 - x^3y^3 + 2x^2y^3}{1 - xy - x^2y^3 - x^3y^3}.
\]
**Theorem 7** Let $d(n)$ be the number of maximal independent sets of $D(n)$. Then, $d(1) = 2, d(2) = 4, d(3) = 7$ and

$$d(n) = 2d(n - 1) - d(n - 2) + d(n - 3)$$

for $n \geq 4$.

Next, we give a brief argument to find the exact formula of $d(n)$ from Theorem 7. By replacing $d(n - i)$ with $r^{n-i}$ for all $i \in \{0, 1, 2, 3\}$ and divide by $r^{n-3}$ throughout the equation, we have

$$r^3 - 2r^2 + r - 1 = 0$$

which we obtain complex roots $0.12256 + 0.74486i$ and $0.12256 - 0.74486i$. Then, the polar form is considered together with Euler’s formula to cancel the imaginary parts of the solutions. We obtain the exact formula of $d(n)$ as follows.

**Theorem 8** If $d(n)$ is the number of maximal independent sets of $D(n)$. Then,

$$d(n) = 1.2672(1.7549)^n - 0.27965\cos(1.4077n - 0.765).$$

By similar arguments as Theorem 8 we can approximate the recurrence relation in Theorem 7 that:

**Theorem 9** If $d(n)$ is the number of maximal independent sets of $D(n)$. Then,

$$d(n) \approx \frac{0.6213}{0.5698^{n+1}}.$$

By similar arguments as Theorem 9 we obtain the following theorem.

**Theorem 10** If $\overline{d}(n)$ is the average size of a maximal independent set of $D(n)$. Then,

$$\overline{d}(n) \approx 1.2345n + 0.89458.$$

### 2.3 Square Cacti

Let $S(n)$ be the square (ortho-rectangular) cactus of $n$ squares as detailed in Figure 3.

**Theorem 11** Let $s(n, k)$ be the number of maximal independent sets of $S(n)$ containing $k$ vertices. If $S(x, y)$ is a bi-variate generating function of $s(n, k)$, then

$$S(x, y) = \frac{1 - 2xy + 2xy^2 - 2x^2y^2 + x^2y^3 + x^2y^4}{1 - 2xy + x^2y^2 - x^2y^3}.$$
Theorem 12 Let $s(n)$ be the number of maximal independent sets of $S(n)$. Then $s(1) = 2$ and, for $n \geq 2$, we have that
\[ s(n) = 2s(n - 1). \]

Namely, $s(n) = 2^n$.

By finding the summation of sizes of all maximal independent sets with similar arguments as Theorem 5 and divided by $2^n$, we obtain the following theorem.

Theorem 13 If $\bar{s}(n)$ is the average size of a maximal independent set of $S(n)$. Then,
\[ \bar{s}(n) \approx 1.25n + 0.75. \]

2.4 Pentagonal Cacti

Let $P(n)$ be the (ortho) pentagonal cactus of $n$ pentagons as illustrated in Figure 4.
Theorem 14 Let $p(n,k)$ be the number of maximal independent sets of $P(n)$ containing $k$ vertices. If $P(x,y)$ is a bi-variate generating function of $p(n,k)$, then
\[
P(x,y) = \frac{1 + 4x^3y^5 - 4x^3y^4 + 4x^2y^4 - x^2y^3 + 4xy^2}{1 - 5x^2y^3 - xy^2 - 4x^3y^4}.
\]

Theorem 15 Let $p(n)$ be the number of maximal independent sets of $P(n)$. Then $p(1) = 5, p(2) = 13, p(3) = 42$ and for $n \geq 4$, we have that
\[
p(n) = p(n - 1) + 5p(n - 2) + 4p(n - 3).
\]

From Theorem 15 we obtain the exact formula of $p(n)$ by similar arguments of Theorem 8 as follows.

Theorem 16 If $p(n)$ is the number of maximal independent sets of $P(n)$. Then,
\[
p(n) = 1.4492(3.0606)^n - 0.56449 \cos(3.5897n - 0.428).
\]

The recurrence relation in Theorem 15 can be approximated as:

Theorem 17 If $p(n)$ is the number of maximal independent sets of $P(n)$. Then,
\[
p(n) \approx \frac{0.4735}{0.3267^{n+1}}.
\]

The average size of a maximal independent set is detailed as follows.

Theorem 18 If $\overline{p}(n)$ is the average size of a maximal independent set of $P(n)$. Then,
\[
\overline{p}(n) \approx 1.5516n + 0.55731.
\]

2.5 Meta-Pentagonal Cacti

We let $M(n)$ be the meta-pentagonal cactus of $n$ pentagons as shown in Figure 5.

![Figure 5: The meta-pentagonal cactus $M(n)$ of $n$ pentagons.](image-url)
**Theorem 19** Let \( m(n, k) \) be the number of maximal independent sets of \( M(n) \) containing \( k \) vertices. If \( M(x, y) \) is a bi-variate generating function of \( m(n, k) \), then
\[
M(x, y) = \frac{-11x^2y^4 + 5xy^2 + 2x^3y^6 + 2x^3y^5 - x^2y^3}{1 - 4xy^2 - xy + 3x^2y^3 + 4x^2y^4 - 2x^3y^5} + 1.
\]

**Theorem 20** Let \( m(n) \) be the number of maximal independent sets of \( M(n) \). Then, \( m(1) = 5, m(2) = 13 \) for \( n \geq 3 \), we have that
\[
m(n) = 3m(n-1) - m(n-2).
\]

From Theorem 20, we obtain the exact formula of \( m(n) \) as follows.

**Theorem 21** If \( m(n) \) is the number of maximal independent sets of \( M(n) \). Then,
\[
m(n) = 0.10573(0.38197)^n + 1.89443(2.61803)^n.
\]

The recurrence relation in Theorem 20 can be approximated as:

**Theorem 22** If \( m(n) \) is the number of maximal independent sets of \( M(n) \). Then,
\[
m(n) \approx \frac{0.7236}{0.382^{n+1}}.
\]

The average size of a maximal independent set is detailed as follows.

**Theorem 23** If \( \overline{m}(n) \) is the average size of a maximal independent set of \( M(n) \). Then,
\[
\overline{m}(n) \approx 1.7236n + 0.24033.
\]

### 2.6 Meta-Hexagonal Cacti

We may let the meta-hexagonal cactus \( H(n) \) of \( n \) hexagons be the graph in Figure 6.

![Figure 6: The meta-hexagonal cactus \( H(n) \) of \( n \) hexagons.](image)
Theorem 24 Let $h(n, k)$ be the number of maximal independent sets of $H(n)$ containing $k$ vertices. If $H(x, y)$ is a bi-variate generating function of $h(n, k)$, then

$$H(x, y) = \frac{1 - 2x^2y^3 + xy^2 - x^3y^5 + x^2y^5 + 5x^2y^4 + xy^3 - x^2y^6 + x^3y^7}{1 - 3x^2y^3 - xy^3 - 2xy^2 - x^3y^5 + x^2y^5 + x^2y^4 - x^3y^6}.$$ 

Theorem 25 Let $h(n)$ be the number of maximal independent sets of $H(n)$. Then $h(1) = 5, h(2) = 19, h(3) = 64$ and, for $n \geq 4$, we have that

$$h(n) = 3h(n - 1) + h(n - 2) + 2h(n - 3).$$

From Theorem 25, we obtain the exact formula of $h(n)$ by similar arguments as Theorem 8 as follows.

Theorem 26 If $h(n)$ is the number of maximal independent sets of $H(n)$. Then,

$$h(n) = 1.5499(3.4567^n - 0.65668 \cos(1.8757n - 0.88031)).$$

The asymptotic analysis provides that:

Theorem 27 If $h(n)$ is the number of maximal independent sets of $H(n)$. Then,

$$h(n) \approx \frac{0.4484}{0.2893^{n+1}}.$$ 

The average size of a maximal independent set is detailed as follows.

Theorem 28 If $\overline{h}(n)$ is the average size of a maximal independent set of $H(n)$. Then,

$$\overline{h}(n) \approx 1.9409n + 0.56686.$$ 

2.7 Para-Hexagonal Cacti

Let $G(n)$ be the para-hexagonal cactus of $n$ hexagons as shown in Figure 7.

Figure 7: The para-hexagonal cactus $G(n)$ of $n$ hexagons.
Theorem 29 Let \( g(n, k) \) be the number of maximal independent sets of \( G(n) \) containing \( k \) vertices. If \( G(x, y) \) is a bi-variate generating function of \( g(n, k) \), then
\[
G(x, y) = \frac{-2x^2y^5 - 3x^2y^4 - x^3y^5 + x^2y^6 + x^3y^6 - xy^2 + 4xy^3 - xy + 1}{1 - xy - xy^2 + 4x^2y^3 - 3xy^2 + x^2y^4 - x^3y^5 - x^2y^5}.
\]

Theorem 30 Let \( g(n) \) be the number of maximal independent sets of \( G(n) \). Then \( g(1) = 5, g(2) = 19, g(3) = 76 \) and, for \( n \geq 4 \), we have that
\[
g(n) = 5g(n - 1) - 4g(n - 2) + g(n - 3).
\]

From Theorem 30, we obtain the exact formula of \( g(n) \) by similar arguments as Theorem 8 as follows.

Theorem 31 If \( g(n) \) is the number of maximal independent sets of \( G(n) \). Then,
\[
g(n) = 1.115(4.0796)^n - 1.3688 \cos(0.3777n + 1.5292).
\]

We can approximate the recurrence relation in Theorem 30 that:

Theorem 32 If \( g(n) \) is the number of maximal independent sets of \( G(n) \). Then,
\[
g(n) \approx \frac{0.2733}{0.2451^n + 1}.
\]

The average size of a maximal independent set is detailed as follows.

Theorem 33 If \( \overline{g}(n) \) is the average size of a maximal independent set of \( G(n) \). Then,
\[
\overline{g}(n) \approx 2.9143n + 0.88506.
\]

2.8 Ortho-Hexagonal Cacti

The ortho-hexagonal cactus \( Q(n) \) of \( n \) hexagons is shown in Figure 8.
Theorem 34 Let $q(n, k)$ be the number of maximal independent sets of $Q(n)$ containing $k$ vertices. If $Q(x, y)$ is a bi-variate generating function of $q(n, k)$, then

$$Q(x, y) = \frac{x^2y^6 + 2xy^3 + 1}{1 - x^2y^5 - x^2y^4 - x^2y^3 - 3xy^2}.$$ 

Theorem 35 Let $q(n)$ be the number of maximal independent sets of $Q(n)$. Then $q(1) = 5, q(2) = 19$ and, for $n \geq 3$, we have that

$$q(n) = 3q(n - 1) + 3q(n - 2).$$

From Theorem 35, we obtain the exact formula of $q(n)$ as follows.

Theorem 36 If $q(n)$ is the number of maximal independent sets of $Q(n)$. Then,

$$q(n) = 0.01201(-0.79129)^n + 1.32132(3.79129)^n.$$ 

In a similar fashion, we can approximate the recurrence relation in Theorem 35 that:

Theorem 37 If $q(n)$ is the number of maximal independent sets of $Q(n)$. Then,

$$q(n) \approx \frac{0.3485}{0.2638^{n+1}}.$$ 

The average size of a maximal independent set is detailed as follows.
**Theorem 38** If $\overline{q}(n)$ is the average size of a maximal independent set of $Q(n)$. Then,

$$\overline{q}(n) \approx 2n + 0.41742.$$ 

Finally, we summarize all our results in this paper in Tables 1 and 2 as follows. In Table 1, the entries of the column G.F. are obtained by letting $y = 1$ in each bivariate generating function.

| Cacti | Bivariate Generating Functions | G.F. | Recurrence Relations |
|-------|--------------------------------|------|---------------------|
| $T(n)$ | $\frac{1+2xy+x^2y^2}{1-xy-x^2y}$ | $\frac{1+2x+x^2}{1-x-x^2}$ | $t(n) = t(n-1) + t(n-2)$  
$t(1) = 3, t(2) = 5, t(3) = 8$ |
| $D(n)$ | $\frac{1+xy^2-xy-x^2y^4+x^3y^3+2x^2y^3}{1-xy-x^2y^2+x^2y^3-x^3y^4}$ | $\frac{1+x^2+x^3}{1-2x-x^2}$ | $d(n) = 2d(n-1) - d(n-2) + d(n-3)$  
$d(1) = 2, d(2) = 4, d(3) = 7, d(4) = 12$ |
| $S(n)$ | $\frac{1-2xy+2x^2y^2-2x^2y^3+x^2y^4+x^2y^5}{1-2xy+x^2y^2-x^3y^3}$ | $\frac{1}{1-2x}$ | $s(n) = 2s(n-1)$  
$s(1) = 2$ |
| $P(n)$ | $\frac{1+4x^3y^4-4x^3y^4+4x^2y^5-2x^2y^5+4xy^7}{1-5x^2y^3-4x^3y^4}$ | $\frac{3x^2+4x+1}{-4x^3-5x^2-x+1}$ | $p(n) = p(n-1) + 5p(n-2) + 4p(n-3)$  
$p(1) = 5, p(2) = 13, p(3) = 42, p(4) = 127$ |
| $M(n)$ | $\frac{-11x^2y^4+5xy^5+2x^2y^6+2x^2y^6-x^2y^4+x^2y^5}{1-4xy-x^2y+3x^2y^2+4x^2y^3-2x^2y^4}$ | $\frac{x^2+2x+1}{x^2-3x+1}$ | $m(n) = 3m(n-1) - m(n-2)$  
$m(1) = 5, m(2) = 13, m(3) = 34, m(4) = 89$ |
| $H(n)$ | $\frac{1-2x^2y^3+xy^2-x^3y^3+2x^2y^4+5x^2y^4+x^3y^6-x^2y^7}{1-3x^2y^3-xy^2-2x^2y^4-2x^2y^5-x^3y^6-x^3y^7}$ | $\frac{3x^2+2x+1}{-2x^3-x^2-3x+1}$ | $h(n) = 3h(n-1) + h(n-2) + 2h(n-3)$  
$h(1) = 5, h(2) = 19, h(3) = 64, h(4) = 221$ |
| $G(n)$ | $\frac{-2x^2y^6-3x^2y^3-x^3y^5-x^2y^4+x^3y^6-x^2y^7-xy^2}{1-xy-xy^2+4x^2y^3-3xy^4+4x^2y^3-xy^2}$ | $\frac{1-2x^2}{1-5x^2-4x^2-x^2}$ | $g(n) = 5g(n-1) - 4g(n-2) + g(n-3)$  
$g(1) = 5, g(2) = 19, g(3) = 76, g(4) = 309$ |
| $Q(n)$ | $\frac{x^2y^2+2xy^3+1}{1-x^2y^2-y^2-x^2y^3-3xy^2}$ | $\frac{x^2+2x+1}{1-3x-3x^2}$ | $q(n) = 3q(n-1) + 3q(n-2)$  
$q(1) = 5, q(2) = 19, q(3) = 72, q(4) = 273$ |

Table 1: Bivariate generating functions, generating functions and recurrence relations of n-gonal cacti.

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Table 2: Asymptotic formulae, exact formulae and average size of a maximal independent set of the cacti.

| Cacti | Exact Formulae | Asymptotic Formulae | Average Size |
|-------|----------------|---------------------|--------------|
| \( T(n) \) | \( 0.10559(-0.61803)^n + 1.8944(1.61803)^n \) | \( \frac{1.17082}{(0.61803)^{n^{++}}} \) | \( 0.72361n + 0.33475 \) |
| \( D(n) \) | \( 1.2672(1.7549)^n - 0.27965 \cos(1.4077n - 0.765) \) | \( \frac{0.62126}{(0.59884)^{n^{++}}} \) | \( 1.2345n + 0.89458 \) |
| \( S(n) \) | \( 2^n \) | \( 2^n \) | \( 1.25n + 0.75 \) |
| \( P(n) \) | \( 1.4492(3.0606)^n - 0.56449 \cos(3.5897n - 0.428) \) | \( \frac{0.47351}{(0.32673)^{n^{++}}} \) | \( 1.5516n + 0.55731 \) |
| \( M(n) \) | \( 0.10573(0.38197)^n + 1.89443(2.61803)^n \) | \( \frac{0.72361}{(0.38196)^{n^{++}}} \) | \( 1.7236n + 0.24033 \) |
| \( H(n) \) | \( 1.5499(3.4567)^n - 0.65668 \cos(1.8757n - 0.88031) \) | \( \frac{0.44836}{(0.28330)^{n^{++}}} \) | \( 1.9409n + 0.56686 \) |
| \( G(n) \) | \( 1.115(4.0796)^n - 1.3688 \cos(0.3777n + 1.5292) \) | \( \frac{0.27331}{(0.24512)^{n^{++}}} \) | \( 2.9143n + 0.88506 \) |
| \( Q(n) \) | \( 0.01201(-0.79129)^n + 1.32132(3.79129)^n \) | \( \frac{0.34851}{(0.26376)^{n^{++}}} \) | \( 2n + 0.41742 \) |

3 Proofs

3.1 Triangular Cacti

First, we name all the vertices of \( T(n) \) as detailed in Figure 9. Then, we recall that

\[ t(n) = \text{the number of all maximal independent sets of } T(n) \]

and

\[ t(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } T(n). \]

Thus,

\[ t(n) = \sum_{k \geq 0} t(n, k). \]
Further, we let

\[ T(x) = \sum_{n \geq 0} t(n)x^n \]

be the generating function of \( t(n) \) and we let

\[ T(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} t(n, k)x^n y^k \]

be the bi-variate generating function of \( t(n, k) \). It is worth noting that, when \( y = 1 \), we have

\[ T(x, 1) = \sum_{n \geq 0} \left( \sum_{k \geq 0} t(n, k)(1)^k \right)x^n = \sum_{n \geq 0} t(n)x^n = T(x). \]  

(2)

Next, we let \( \mathcal{T}(n) \) be constructed from \( T(n) \) by joining one vertex to one vertex of degree two of the \( n^{th} \) triangle. The graph \( \mathcal{T}(n) \) is illustrated by Figure 10.

Then, we let

\[ \overline{t}(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } \mathcal{T}(n). \]

and

\[ \mathcal{T}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \overline{t}(n, k)x^n y^k \]
be the bi-variate generating function of $\overline{T}(n, k)$.

We are ready to prove Theorem 1.

**Proof of Theorem 1.** First, we will establish a recurrence relation of $t(n, k)$. Let $x_1^n$ be the vertex of degree 4 of the $n^{th}$ triangle of $T_n$ and let $D$ be a maximal independent set of $T(n)$ containing $k$ vertices. We distinguish two cases.

**Case 1:** $x_1^n \in D$.

Thus, the two vertices of degree two $x_2^n, x_3^n$ are not in $D$. Further, $x_2^{n-1}, x_3^{n-1} \notin D$. Removing $x_1^n, x_2^n, x_3^n, x_2^{n-1}, x_3^{n-1}$ from $T(n)$ results in $\overline{T}(n-3)$. Thus $D = D' \cup \{x_1^n\}$ where $D'$ is a maximal independent set of $\overline{T}(n-3)$ containing $k-1$ vertices. Clearly, there are $\overline{T}(n-3, k-1)$ possibilities of $D'$ yielding that there are $\overline{T}(n-3, k-1)$ possibilities of $D$.

**Case 2:** $x_1^n \notin D$.

By maximality of $D$, either $x_2^n \in D$ or $x_3^n \in D$. In each case, $|D \cap \{x_1^n, x_2^n, x_3^n\}| = 1$. Removing $x_1^n, x_2^n, x_3^n$ from $D$ results in $\overline{T}(n-2)$. Thus, $D = D' \cup \{w\}$ where $w \in \{x_2^n, x_3^n\}$ and $D'$ is a maximal independent set containing $k-1$ vertices of $\overline{T}(n-2, k-1)$. Therefore, for each $w \in \{x_2^n, x_3^n\}$, there are $\overline{T}(n-2, k-1)$ possibilities of $D'$ yielding that there are totally $2\overline{T}(n-2, k-1)$ possibilities of $D$.

From Cases 1 and 2, we have that

$$t(n, k) = \overline{T}(n-3, k-1) + 2\overline{T}(n-2, k-1)$$ (3)

For $n \geq 3$ and $k \geq 1$, we multiply $x^n y^k$ throughout (3) and sum over all $x^n y^k$. Thus, we have that

$$\sum_{n \geq 3} \sum_{k \geq 1} t(n, k) x^n y^k = \sum_{n \geq 3} \sum_{k \geq 1} \overline{T}(n-3, k-1) x^n y^k + 2 \sum_{n \geq 3} \sum_{k \geq 1} \overline{T}(n-2, k-1) x^n y^k.$$ (4)

We first consider the term $\sum_{n \geq 3} \sum_{k \geq 1} t(n, k) x^n y^k$. It can be checked that

$t(0, 0) = 1, t(0, 1) = 0$ and $t(0, k) = 0$ for all $k \geq 2$

$t(1, 0) = 0, t(1, 1) = 3$ and $t(1, k) = 0$ for all $k \geq 2$

$t(2, 0) = 0, t(2, 1) = 1, t(2, 2) = 4$ and $t(2, k) = 0$ for all $k \geq 3$.

Thus,

$$\sum_{n \geq 3} \sum_{k \geq 1} t(n, k) x^n y^k = (\sum_{n \geq 3} \sum_{k \geq 1} t(n, k) x^n y^k + t(0, 0) + t(1, 1) xy + t(2, 1) x^2 y + t(2, 2) x^2 y^2)$$

$$- t(0, 0) - t(1, 1) xy - t(2, 1) x^2 y - t(2, 2) x^2 y^2$$

$$= T(x, y) - t(0, 0) - t(1, 1) xy - t(2, 1) x^2 y - t(2, 2) x^2 y^2$$

$$= T(x, y) - 1 - 3xy - x^2 y - 4x^2 y^2.$$ (5)
Now, we consider the term $\sum_{n \geq 3} \sum_{k \geq 1} \overline{T}(n - 3, k - 1)x^n y^k$. Clearly,

$$
\sum_{n \geq 3} \sum_{k \geq 1} \overline{T}(n - 3, k - 1)x^n y^k = x^3 y \sum_{n \geq 3} \sum_{k \geq 1} \overline{T}(n - 3, k - 1)x^{n-3} y^{k-1}
$$

$$
= x^3 y \sum_{n \geq 0} \sum_{k \geq 0} \overline{T}(n, k)x^n y^k
$$

$$
= x^3 y \overline{T}(x, y).
$$

(6)

Finally, we consider the term $2 \sum_{n \geq 3} \sum_{k \geq 1} \overline{T}(n - 2, k - 1)x^n y^k$. It can be check that

$$
\overline{T}(0, 0) = 0, \overline{T}(0, 1) = 2 \text{ and } \overline{T}(0, k) = 0 \text{ for all } k \geq 2.
$$

Thus,

$$
2 \sum_{n \geq 3} \sum_{k \geq 1} \overline{T}(n - 2, k - 1)x^n y^k = 2x^2 y \sum_{n \geq 3} \sum_{k \geq 1} \overline{T}(n - 2, k - 1)x^{n-2} y^{k-1}
$$

$$
= 2x^2 y \sum_{n \geq 3} \sum_{k \geq 1} \overline{T}(n, k)x^n y^k
$$

$$
= 2x^2 y (\sum_{n \geq 1} \sum_{k \geq 0} \overline{T}(n, k)x^n y^k + \overline{T}(0, 1)y - \overline{T}(0, 1)y)
$$

$$
= 2x^2 y (\sum_{n \geq 0} \sum_{k \geq 0} \overline{T}(n, k)x^n y^k - 2y)
$$

$$
= 2x^2 y \overline{T}(x, y) - 4x^2 y^2.
$$

(7)

Plugging (5), (6) and (7) to (4), we have

$$
T(x, y) - 1 - 3xy - x^2 y - 4x^2 y^2 = x^3 y \overline{T}(x, y) + 2x^2 y \overline{T}(x, y) - 4x^2 y^2
$$

which can be solved that

$$
T(x, y) = 1 + 3xy + x^2 y + (x^3 y + 2x^2 y) \overline{T}(x, y).
$$

(8)

Next, we will establish the recurrence relation of $\overline{T}(n, k)$. For the graph $\overline{T}(n)$, we let $x_{n+1}$ be the vertex of degree one who is adjacent to $x_0^n$ of $T_n$. Further, we let $D$ be a maximal independent set of $\overline{T}(n)$ containing $k$ vertices. There are 2 cases.

**Case 1:** $x_{n+1} \in D$

Thus $x_3^n \notin D$. Removing $x_3^n, x_{n+1}$ from $\overline{T}(n)$ results in $\overline{T}(n - 1)$. Thus, $D = D' \cup \{x_{n+1}\}$ where $D'$ is a maximal independent set of $\overline{T}(n - 1)$ containing $k - 1$ vertices. Therefore, there are $\overline{T}(n - 1, k - 1)$ possibilities of $D'$ yielding that there are $\overline{T}(n - 1, k - 1)$ possibilities of $D$. 16
Case 2: $x_{n+1} \notin D$

By maximality of $D$, $x_3^n \in D$. Hence, $x_1^n, x_2^n, x_3^n, x_{n+1}$ do not belong to $D$. Removing $x_1^n, x_2^n, x_3^n, x_{n+1}$ from $T(n)$ results in $T(n-2)$. Thus, $D = D' \cup \{x_3^n\}$ where $D'$ is a maximal independent set of $T(n-2)$ containing $k-1$ vertices. Therefore, there are $\tilde{T}(n-2, k-1)$ possibilities of $D'$ yielding that there are $\tilde{T}(n-2, k-1)$ possibilities of $D$.

From Cases 1 and 2, we have that

$$\tilde{T}(n, k) = \tilde{T}(n-1, k-1) + \tilde{T}(n-2, k-1).$$  \hfill (9)

For $n \geq 2$ and $k \geq 1$, we multiply $x^n y^k$ throughout (9) and sum over all $x^n y^k$. Thus,

$$\sum_{n \geq 2} \sum_{k \geq 1} \tilde{T}(n, k) x^n y^k = \sum_{n \geq 2} \sum_{k \geq 1} \tilde{T}(n-1, k-1) x^n y^k + \sum_{n \geq 2} \sum_{k \geq 1} \tilde{T}(n-2, k-1) x^n y^k. \hfill (10)$$

We first consider the term $\sum_{n \geq 2} \sum_{k \geq 1} \tilde{T}(n, k) x^n y^k$. It can be checked that

$$\tilde{T}(0, 0) = 0, \tilde{T}(0, 1) = 2 \text{ and } \tilde{T}(0, k) = 0 \text{ for all } k \geq 2$$

$$\tilde{T}(1, 0) = 0, \tilde{T}(1, 1) = 1, \tilde{T}(1, 2) = 2 \text{ and } \tilde{T}(0, k) = 0 \text{ for all } k \geq 3.$$

Thus,

$$\sum_{n \geq 2} \sum_{k \geq 1} \tilde{T}(n, k) x^n y^k = (\sum_{n \geq 2} \sum_{k \geq 1} \tilde{T}(n, k) x^n y^k + \tilde{T}(0, 1) y + \tilde{T}(1, 1) xy + \tilde{T}(1, 2) xy^2)$$

$$- \tilde{T}(0, 1) y - \tilde{T}(1, 1) xy - \tilde{T}(1, 2) xy^2$$

$$= \sum_{n \geq 2} \sum_{k \geq 1} \tilde{T}(n, k) x^n y^k - 2y - xy - 2xy^2$$

$$= T(x, y) - 2y - xy - 2xy^2 \hfill (11)$$

We next consider the term $\sum_{n \geq 2} \sum_{k \geq 1} \tilde{T}(n-1, k-1) x^n y^k$. Recall that

$$\tilde{T}(0, 0) = 0, \tilde{T}(0, 1) = 2 \text{ and } \tilde{T}(0, k) = 0 \text{ for all } k \geq 2.$$

Thus,

$$\sum_{n \geq 2} \sum_{k \geq 1} \tilde{T}(n-1, k-1) x^n y^k = xy \sum_{n \geq 2} \sum_{k \geq 1} \tilde{T}(n-1, k-1) x^{n-1} y^{k-1}$$

$$= xy \sum_{n \geq 1} \sum_{k \geq 0} \tilde{T}(n, k) x^n y^k$$

$$= xy \left( \sum_{n \geq 1} \sum_{k \geq 0} \tilde{T}(n, k) x^n y^k + \tilde{T}(0, 1) y - \tilde{T}(0, 1)y \right)$$

$$= xy \left( \sum_{n \geq 0} \sum_{k \geq 0} \tilde{T}(n, k) x^n y^k - 2y \right)$$

$$= xy T(x, y) - 2xy^2 \hfill (12)$$

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Finally, we consider the term \( \sum_{n \geq 2} \sum_{k \geq 1} t(n - 2, k - 1)x^n y^k \). Clearly,

\[
\sum_{n \geq 2} \sum_{k \geq 1} t(n - 2, k - 1)x^n y^k = x^2 y \sum_{n \geq 2} \sum_{k \geq 1} t(n - 2, k - 1)x^{n-2} y^{k-1}
\]

\[
= x^2 y \sum_{n \geq 2} \sum_{k \geq 1} t(n) x^n y^k 
\]

\[
= x^2 y T(x, y). 
\] (13)

Plugging (11), (12) and (13) to (10), we have

\[
T(x, y) - 2y - xy - x^2 y^2 = xy T(x, y) - 2xy^2 + x^2 y T(x, y)
\]

which can be solved that

\[
T(x, y) = \frac{2y + xy}{1 - xy - x^2 y}. 
\] (14)

By plugging (14) to (8), we have

\[
T(x, y) = 1 + 2xy + x^2 y^2 
\]

as required. \( \square \)

Next, we will prove Theorem 2.

Proof of Theorem 2 From the equation of Theorem 1 by letting \( y = 1 \) and (2), we have that

\[
\sum_{n \geq 0} t(n)x^n = T(x, 1)
\]

\[
= 1 + 2x + x^2 
\]

\[
= \frac{1 - x - x^2}{1 - x - x^2} 
\] (16)

which can be solved that

\[
x^2 + 2x + 1 = (1 - x - x^2) \sum_{n \geq 0} t(n)x^n
\]

\[
= \sum_{n \geq 0} t(n)x^n - \sum_{n \geq 0} t(n)x^{n+1} - \sum_{n \geq 0} t(n)x^{n+2}
\]

\[
= \sum_{n \geq 0} t(n)x^n - \sum_{n \geq 1} t(n - 1)x^n - \sum_{n \geq 2} t(n - 2)x^n
\]

\[
= t(0) + t(1)x - t(0)x + \sum_{n \geq 2} (t(n) - t(n - 1) - t(n - 2))x^n. 
\] (17)
Because the order of the polynomial on the left hand side of (17) is two, the coefficients of $x^n$ for all $n \geq 3$ must be 0. Thus, $t(n) - t(n - 1) - t(n - 2) = 0$ implying that

$$t(n) = t(n - 1) + t(n - 2).$$

(18)

This completes the proof. $\square$

### 3.2 Diamond Cacti

First, we may name all the vertices of $D(n)$, the Diamond cacti of $n$ Diamonds, as detailed in Figure 11.

![Diagram of a labelled diamond cactus of $n$ diamonds](image)

Figure 11: A labelled diamond cactus of $n$ diamonds

Then, we recall that

$$d(n) = \text{the number of all maximal independent sets of } D(n)$$

and

$$d(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } d(n).$$

Thus,

$$d(n) = \sum_{k \geq 0} d(n, k).$$

(19)

Further, we let

$$D(x) = \sum_{n \geq 0} d(n)x^n$$

be the generating function of $d(n)$ and we let

$$D(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} d(n, k)x^ny^k$$
be the bi-variate generating function of $d(n, k)$. It is worth noting that, when $y = 1$, we have

$$D(x, 1) = \sum_{n \geq 0} \left( \sum_{k \geq 0} d(n, k)(1)^k \right)x^n = \sum_{n \geq 0} d(n)x^n = D(x). \quad (20)$$

Next, we let $\overline{D}(n)$ be constructed from $D(n)$ by joining two vertices to a vertex at distance two from the cut vertex of the $n^{th}$ diamond.

![Figure 12: The graph $\overline{D}(n)$](image)

Then, we let

$$\overline{d}(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } \overline{D}(n).$$

and let

$$\overline{D}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \overline{d}(n, k)x^ny^k$$

be the bi-variate generating function of $\overline{d}(n, k)$.

Now, we are ready to prove Theorem 6.

**Proof of Theorem 6** First, we will establish a recurrence relation of $d(n, k)$. Let $D$ be a maximal independent set of $D(n)$ containing $k$ vertices. We distinguish two cases.

**Case 1**: $x_3^n \in D$.
Because $D$ is independent, $x_2^n, x_4^n \notin D$. Removing $x_2^n, x_3^n, x_4^n$ form $D(n)$ results in $D(n - 1)$. Thus, $D = D' \cup \{x_3^n\}$ where $D'$ is a maximal independent set of $D(n - 1)$ containing $k - 1$ vertices. There are $d(n - 1, k - 1)$ possibilities of $D'$ giving that there are $d(n - 1, k - 1)$ possibilities of $D$.

**Case 2**: $x_3^n \notin D$.
By maximality of $D$, $x_2^n \in D$ or $x_4^n \in D$. In any case, $x_1^n \notin D$ where $x_2^n, x_4^n \in D$. Removing $x_1^n, x_2^n, x_3^n, x_4^n$ from $D$ results in $\overline{D}(n - 2)$. Thus, $D = D' \cup \{x_2^n, x_4^n\}$ where $D'$ is a maximal independent set of $D(n - 2)$ containing $k - 1$ vertices.

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set of $\overline{D}(n-2)$. There are $\overline{d}(n-2, k-2)$ possibilities of $D'$ yielding that there are $\overline{d}(n-2, k-2)$ possibilities of $D$.

From Cases 1 and 2, we have that

$$d(n, k) = d(n - 1, k - 1) + \overline{d}(n - 2, k - 2)$$  \hspace{1cm} (21)

For $n \geq 2$ and $k \geq 2$, we multiply $x^n y^k$ throughout (21) and sum over all $x^n y^k$. Thus, we have that

$$\sum_{n\geq 2} \sum_{k\geq 2} d(n, k) x^n y^k = \sum_{n\geq 2} \sum_{k\geq 2} d(n - 1, k - 1) x^n y^k + \sum_{n\geq 2} \sum_{k\geq 2} \overline{d}(n - 2, k - 2) x^n y^k$$  \hspace{1cm} (22)

We first consider the term $\sum_{n\geq 2} \sum_{k\geq 2} d(n, k) x^n y^k$. It can be checked that

- $d(0, 0) = 1, d(0, 1) = 0, d(0, 2) = 0$, and $d(0, k) = 0$ for all $k \geq 2$
- $d(1, 0) = 0, d(1, 0) = 0, d(1, 2) = 2$, and $d(0, k) = 0$ for all $k \geq 2$
- and $d(n, k) = 0$ for all $n \geq 2, k \leq 1$.

Clearly,

$$\sum_{n\geq 2} \sum_{k\geq 2} d(n, k) x^n y^k = \sum_{n\geq 2} \sum_{k\geq 2} d(n, k) x^n y^k + d(0, 0) + d(1, 2) xy^2 - d(0, 0) - d(1, 2) xy^2$$

$$= \sum_{n\geq 0} \sum_{k\geq 0} t(n, k) x^n y^k - 1 - 2 xy^2$$

$$= D(x, y) - 1 - 2 xy^2$$  \hspace{1cm} (23)

Now, we consider the term $\sum_{n\geq 2} \sum_{k\geq 2} d(n - 1, k - 1) x^n y^k$. Clearly,

$$\sum_{n\geq 2} \sum_{k\geq 2} d(n - 1, k - 1) x^n y^k = xy \left( \sum_{n\geq 2} \sum_{k\geq 2} d(n - 1, k - 1) x^{n-1} y^{k-1} \right)$$

$$= xy \left( \sum_{n\geq 2} \sum_{k\geq 2} d(n - 1, k - 1) x^{n-1} y^{k-1} \right) + d(0, 0) - d(0, 0))$$

$$= xy \left( \sum_{n\geq 0} \sum_{k\geq 0} d(n - 1, k - 1) x^{n-1} y^{k-1} - 1 \right)$$

$$= xyD(x, y) - xy$$  \hspace{1cm} (24)

Finally, we consider the term $\sum_{n\geq 2} \sum_{k\geq 2} d(n - 2, k - 2) x^n y^k$. It can be check that,
\[
\sum_{n \geq 1} \sum_{k \geq 2} d(n-2,k-2)x^n y^k = x^2 y^2 \sum_{n \geq 2} \sum_{k \geq 2} d(n-2,k-2)x^{n-2} y^{k-2} \\
= x^2 y^2 \sum_{n \geq 0} \sum_{k \geq 0} d(n,k)x^n y^k \\
= x^2 y^2 D(x,y).
\]  

(25)

Plugging (23), (24) and (25) to (22), we have

\[
D(x,y) - 1 - 2xy^2 = xyD(x,y) - xy + x^2 y^2 D(x,y) 
\]

which can be solved that

\[
D(x,y) = 1 + 2xy^2 + xyD(x,y) - xy + x^2 y^2 D(x,y).
\]

(26)

Next, we will establish the recurrence relation of \(\overline{d}(n,k)\). For the graph \(\overline{D}(n)\), we let \(D\) be a maximal independent set of \(\overline{D}(n)\) containing \(k\) vertices. There are 2 cases.

Case 1: \(x_3^n \in D\)

Thus \(x_2^n, x_1^n, x_{n+1}, x'_{n+1} \notin D\). Removing \(x_2^n, x_1^n, x_{n+1}, x'_{n+1}\) from \(\overline{D}(n)\) results in \(\overline{D}(n-1)\). Thus, \(D = D' \cup \{x_3^n\}\) where \(D'\) is a maximal independent set of \(\overline{D}(n-1)\) containing \(k-1\) vertices. Therefore, there are \(d(n-1,k-1)\) possibilities of \(D'\) yielding that there are \(d(n-1,k-1)\) possibilities of \(D\).

Case 2: \(x_3^n \notin D\)

By maximality of \(D\), \(x_{n+1}, x'_{n+1} \in D\). Removing \(x_3^n, x_{n+1}, x'_{n+1}\) from \(\overline{D}(n)\) results in \(\overline{D}(n-1)\). Thus, \(D = D' \cup \{x_{n+1}, x'_{n+1}\}\) where \(D'\) is a maximal independent set of \(\overline{D}(n-1)\) containing \(k-2\) vertices. Therefore, there are \(\overline{d}(n-1,k-2)\) possibilities of \(D'\) yielding that there are \(\overline{d}(n-1,k-2)\) possibilities of \(D\).

From Cases 1 and 2, we have that

\[
\overline{d}(n,k) = d(n-1,k-1) + \overline{d}(n-1,k-2).
\]

(27)

For \(n \geq 1\) and \(k \geq 2\), we multiply \(x^n y^k\) throughout (27) and sum over all \(x^n y^k\). Thus, we have that

\[
\sum_{n \geq 1} \sum_{k \geq 2} \overline{d}(n,k)x^n y^k = \sum_{n \geq 1} \sum_{k \geq 2} d(n-1,k-1)x^n y^k + \sum_{n \geq 1} \sum_{k \geq 2} \overline{d}(n-1,k-2)x^n y^k
\]

(28)

We first consider the term \(\sum_{n \geq 1} \sum_{k \geq 2} \overline{d}(n,k)x^n y^k\). It can be checked that

\[
\overline{d}(0,0) = 0, \overline{d}(0,1) = 1, \overline{d}(0,2) = 1 \text{ and } \overline{d}(0,k) = 0 \text{ for all } k \geq 3
\]
\[\sum \sum d(n, k)x^n y^k = \sum \sum d(n, k)x^n y^k \]
\[+ \bar{d}(0, 1)y + \bar{d}(0, 2)y^2 - \bar{d}(0, 1)y - \bar{d}(0, 2)y^2 \]
\[= \sum \sum d(n, k)x^n y^k - y - y^2 \]
\[= \bar{D}(x, y) - y - y^2 \quad (29)\]

We next consider the term \(\sum \sum d(n - 1, k - 1)x^n y^k\). Recall that
\[d(0, 0) = 1, \text{ and } d(n, 0) = 0 \text{ for all } n \geq 1.\]

\[\sum \sum d(n - 1, k - 1)x^n y^k = xy\left(\sum \sum d(n - 1, k - 1)x^{n-1}y^{k-1}\right)\]
\[= xy\left(\sum \sum d(n, k)x^n y^k\right)\]
\[= xy\left(\sum \sum d(n, k)x^n y^k + d(0, 0) - d(0, 0)\right)\]
\[= xy\left(\sum \sum d(n, k)x^n y^k - 1\right)\]
\[= xyD(x, y) - xy \quad (30)\]

Finally, we consider the term \(\sum \sum \bar{d}(n - 1, k - 2)x^n y^k\). Clearly,
\[\sum \sum \bar{d}(n - 1, k - 2)x^n y^k = xy^2\left(\sum \sum \bar{d}(n - 1, k - 2)x^{n-1}y^{k-2}\right)\]
\[= xy^2\left(\sum \sum \bar{d}(n, k)x^n y^k\right)\]
\[= xy^2\bar{D}(x, y) \quad (31)\]

Plugging (29), (30) and (31) to (28), we have
\[\bar{D}(x, y) - y - y^2 = xyD(x, y) - xy + xy^2\bar{D}(x, y).\]

which can be solved that
\[ D(x, y) = y + y^2 + xyD(x, y) - xy + xy^2D(x, y). \] (32)

By plugging (26) to (32), we have

\[ D(x, y) = \frac{1 + xy^2 - xy - x^2y^4 - x^3y^3 + 2x^2y^3}{1 - xy - xy^2 + x^2y^3 - x^3y^3}. \]

as required. This proves Theorem 6. ☐

Proof of Theorem 7 By Theorem 6 with \( y = 1 \), we have that

\[ \sum_{n \geq 0} d(n)x^n = D(x, 1) \]

\[ = \frac{1 + x^2 - x^3}{1 - 2x + x^2 - x^3}. \] (33)

which can be solved that

\[ 1 + x^2 - x^3 = (1 - 2x + x^2 - x^3) \sum_{n \geq 0} d(n)x^n \]

\[ = \sum_{n \geq 0} d(n)x^n - \sum_{n \geq 0} 2d(n)x^{n+1} + \sum_{n \geq 0} d(n)x^{n+2} - \sum_{n \geq 0} d(n)x^{n+3} \]

\[ = \sum_{n \geq 0} d(n)x^n - \sum_{n \geq 1} 2d(n - 1)x^n + \sum_{n \geq 2} d(n - 2)x^n - \sum_{n \geq 3} 2d(n - 3)x^n \]

\[ = d(0) + d(1)x + d(2)x^2 + d(3)x^3 - 2d(0)x - 2d(1)x^2 - 2d(2)x^3 + d(0)x^2 \]

\[ + d(1)x^3 - 2d(0)x^3 + \sum_{n \geq 4} (d(n) - 2d(n) - d(n - 2) - d(n - 3))x^n \] (34)

Because the order of the polynomial on the left hand side of (34) is three, the coefficients of \( x^n \) for all \( n \geq 4 \) must be 0. Thus, \( d(n) - 2d(n - 1) + d(n - 2) - d(n - 3) = 0 \) implying that

\[ d(n) = 2d(n - 1) - d(n - 2) + d(n - 3). \]

This proves Theorem 7. ☐

3.3 Square Cacti

First we may name all the vertices of \( S(n) \), a square cacti of \( n \) squares, as detailed in Figure 13.

Then, we let

\[ s(n) = \text{the number of all maximal independent sets of } S(n) \]

and

\[ s(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } S(n). \]
Thus,

\[
s(n) = \sum_{k \geq 0} s(n, k)
\]

Further, we let

\[
S(x) = \sum_{n \geq 0} s(n)x^n
\]

be the generating function of \( s(n) \) and we let

\[
S(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} s(n, k)x^ny^k
\]

be the bi-variate generating function of \( s(n, k) \). It is worth noting that, when \( y = 1 \), we have

\[
S(x, 1) = \sum_{n \geq 0} (\sum_{k \geq 0} s(n, k)(1)^k)x^n = \sum_{n \geq 0} s(n)x^n = S(x).
\] (35)

Next, we let \( \overline{S}(n) \) be constructed from \( S(n) \) by joining an end vertex of a path of length one to a vertex at distance one from the cut vertex of the \( n^{th} \) square. The graph \( \overline{S}(n) \) is shown in Figure 14.

Then, we let

\[
\overline{s}(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } \overline{S}(n).
\]

and let

\[
\overline{S}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \overline{s}(n, k)x^ny^k
\]

be the bi-variate generating function of \( \overline{s}(n, k) \).
Figure 14: The graph $\overline{S}(n)$.

Now, we are ready to prove Theorem 11.

Proof of Theorem 11 First, we will establish a recurrence relation of $s(n, k)$. Let $D$ be a maximal independent set of $S(n)$ containing $k$ vertices. We distinguish two cases.

Case 1: $x_2^n \in D$.
Because $D$ is independent, $x_1^n, x_3^n \notin D$. Removing $x_1^n, x_2^n, x_3^n$ form $D(n)$ results in $S(n-1)$. Thus, $D = D' \cup \{x_2^n\}$ where $D'$ is a maximal independent set of $S(n-1)$ containing $k-1$ vertices. There are $s(n-1, k-1)$ possibilities of $D'$ giving that there are $s(n-1, k-1)$ possibilities of $D$.

Case 2: $x_2^n \notin D$.
By maximality of $D$, $x_1^n \in D$ or $x_3^n \in D$. We may assume that $|D \cap \{x_1^n, x_3^n\}| = 1$. Suppose without loss of generality that $x_1^n \in D$, then $x_2^n \notin D$. Because $D$ is independent, $x_3^n \notin D$. Removing $x_1^n, x_2^n, x_3^n$ from $D$ results in $\overline{S}(n-2)$. Thus, $D = D' \cup \{x_1^n, x_3^n\}$ where $D'$ is a maximal independent set of $\overline{S}(n-2)$ containing $k-2$ vertices. There are $\overline{s}(n-2, k-2)$ possibilities of $D'$ yielding that there are $\overline{s}(n-2, k-2)$ possibilities of $D$.

From Cases 1 and 2, we have that

$$s(n, k) = s(n-1, k-1) + \overline{s}(n-2, k-2)$$

(36)

For $n \geq 2$ and $k \geq 2$, we multiply $x^n y^k$ throughout (36) and sum over all $x^n y^k$. Thus, we have that

$$\sum_{n \geq 2} \sum_{k \geq 2} s(n, k) x^n y^k = \sum_{n \geq 2} \sum_{k \geq 2} s(n-1, k-1) x^n y^k + \sum_{n \geq 2} \sum_{k \geq 2} \overline{s}(n-2, k-2) x^n y^k$$

(37)

We first consider the term $\sum_{n \geq 2} \sum_{k \geq 2} s(n, k) x^n y^k$. It can be checked that

$s(0, 0) = 1, s(0, 1) = 0$ and $s(0, k) = 0$ for all $k \geq 2$

$s(1, 0) = 0, s(1, 1) = 0, s(1, 2) = 2$ and $s(0, k) = 0$ for all $k \geq 3$
\[ s(n,k) = 0 \text{ for all } n \geq 2, k \leq 1. \]

Thus,
\[
\sum_{n \geq 2} \sum_{k \geq 2} s(n,k) x^n y^k = \sum_{n \geq 2} \sum_{k \geq 2} s(n,k) x^n y^k + s(0,0) + s(1,2)xy^2 - s(0,0) - s(1,2)xy^2 \\
= \sum_{n \geq 0} \sum_{k \geq 0} s(n,k) x^n y^k - 1 - 2xy^2 \\
= S(x,y) - 1 - 2xy^2
\tag{38}
\]

We next consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} s(n-1,k-1)x^n y^k \). Recall that
\[
\sum_{n \geq 2} \sum_{k \geq 2} s(n-1,k-1)x^n y^k = xy(\sum_{n \geq 2} \sum_{k \geq 2} s(n,k)x^n y^k + s(0,0) - s(0,0)) \\
= xy(\sum_{n \geq 2} \sum_{k \geq 2} s(n,k)x^n y^k - 1) \\
= xyS(x,y) - xy 
\tag{39}
\]

Finally, we consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} s(n-2,k-2)x^n y^k \). Clearly,
\[
\sum_{n \geq 2} \sum_{k \geq 2} s(n-2,k-2)x^n y^k = x^2y^2(\sum_{n \geq 2} \sum_{k \geq 2} s(n,k)x^n y^k) \\
= x^2y^2(\sum_{n \geq 2} \sum_{k \geq 2} s(n,k)x^n y^k) \\
= x^2y^2S(x,y) 
\tag{40}
\]

Plugging (38), (39) and (40) to (37), we have
\[
S(x,y) - 1 - 2xy^2 = xyS(x,y) - xy + x^2y^2S(x,y) \\
\]
which can be solved that
\[
S(x,y) = 1 + 2xy^2 + xyS(x,y) - xy + x^2y^2S(x,y). 
\tag{41}
\]

Next, we will establish the recurrence relation of \( \overline{s}(n,k) \). For the graph \( \overline{S}(n) \), we let \( D \) be a maximal independent set of \( \overline{S}(n) \) containing \( k \) vertices. There are 2 cases.
Case 1: \( x'_{n+1} \in D \)

Thus \( x_{n+1} \notin D \). Removing \( x_{n+1}, x'_{n+1} \) from \( S(n) \) results in \( S(n+1) \) containing \( k-1 \) vertices. Therefore, \( D^1 = D' \cup \{x'_{n+1}\} \) where \( D' \) is a maximal independent set of \( S(n) \) containing \( k-1 \) vertices. Therefore, there are \( s(n, k-1) \) possibilities of \( D' \) yielding that there are \( s(n, k-1) \) possibilities of \( D \).

Case 2: \( x'_{n+1} \notin D \)

By maximality of \( D \), \( x_{n+1} \in D \). Because \( D \) is independent, \( x_n \notin D \). Removing \( x_n, x_{n+1}, x'_{n+1} \) from \( S(n) \) results in \( S(n-1) \). Thus, \( D = D' \cup \{x_{n+1}\} \) where \( D' \) is a maximal independent set of \( S(n-1) \) containing \( k-1 \) vertices. Therefore, there are \( s(n-1, k-1) \) possibilities of \( D' \) yielding that there are \( s(n-1, k-1) \) possibilities of \( D \).

From Cases 1 and 2, we have that

\[
\begin{align*}
\bar{s}(n, k) &= s(n, k-1) + s(n-1, k-1).
\end{align*}
\] (42)

For \( n \geq 1 \) and \( k \geq 1 \), we multiply \( x^n y^k \) throughout (42) and sum over all \( x^n y^k \). Thus, we have that

\[
\sum_{n \geq 1} \sum_{k \geq 1} \bar{s}(n, k) x^n y^k = \sum_{n \geq 1} \sum_{k \geq 1} \bar{s}(n, k-1) x^n y^k + s(0,1) + s(0,2) \bar{s}(x,y) - s(0,1)y - s(0,2)y^2
\] (43)

We first consider the term \( \sum_{n \geq 1} \sum_{k \geq 1} \bar{s}(n, k) x^n y^k \).

Next, it can be checked that

\[
\begin{align*}
\bar{s}(0,0) &= 0, \bar{s}(0,1) = 1, \bar{s}(0,2) = 1 \text{ and } \bar{s}(0,k) = 0 \text{ for all } k \geq 3,
\end{align*}
\]

\[
\begin{align*}
\bar{s}(1,0) &= 0, \bar{s}(1,1) = 0, \bar{s}(1,2) = 1 \text{ and } \bar{s}(1,k) = 0 \text{ for all } k \geq 3.
\end{align*}
\]

Thus,

\[
\sum_{n \geq 1} \sum_{k \geq 1} \bar{s}(n, k) x^n y^k = \sum_{n \geq 1} \sum_{k \geq 1} \bar{s}(n, k) x^n y^k
\]

\[
+ s(0,1) + s(0,2) \bar{s}(x,y) - s(0,1)y - s(0,2)y^2
\]

\[
= \bar{s}(x,y) - y - y^2
\] (44)

Now, we consider the term \( \sum_{n \geq 1} \sum_{k \geq 1} s(n, k-1) x^n y^k \). Clearly,

\[
\begin{align*}
s(0,0) &= 1, s(0,1) = 0 \text{ and } s(0,k) = 0 \text{ for all } k \geq 2.
\end{align*}
\]
\[
\sum_{n \geq 1} \sum_{k \geq 1} s(n, k-1)x^n y^k = y\left(\sum_{n \geq 1} \sum_{k \geq 1} s(n, k-1)x^n y^{k-1} + s(0, 0) - s(0, 0)\right)
\]
\[= y \sum_{n \geq 0} \sum_{k \geq 0} s(n, k)x^n y^k - y = yS(x, y) - y \]  

(45)

Finally, we consider the term \(\sum_{n \geq 1} \sum_{k \geq 1} \bar{s}(n - 1, k - 1)x^n y^k\). It can be check that,

\[
\sum_{n \geq 1} \sum_{k \geq 1} \bar{s}(n - 1, k - 1)x^n y^k = xy\left(\sum_{n \geq 1} \sum_{k \geq 1} \bar{s}(n - 1, k - 1)x^{n-1} y^{k-1}\right)
\]
\[= xy\left(\sum_{n \geq 0} \sum_{k \geq 0} \bar{s}(n, k)x^n y^k\right) = xy\bar{S}(x, y) \]  

(46)

Plugging (44), (45) and (46) to (43), we have

\[S(x, y) - y - y^2 = yS(x, y) - y + xy\bar{S}(x, y).\]

This together with (41) yield

\[S(x, y) = \frac{1 - 2xy + 2xy^2 - 2x^2y^3 + x^2y^2 + x^2y^4}{1 - 2xy + x^2y^2 - x^2y^3}\]

as required. This proves Theorem 11.\(\square\)

Proof of Theorem 12 By Theorem 11 with \(y = 1\), we have that

\[\sum_{n \geq 0} s(n)x^n = (x, 1) = \frac{1}{1 - 2x}\]

(47)

which can be solved that
1 = (1 − 2x) \sum_{n \geq 0} s(n)x^n

= \sum_{n \geq 0} s(n)x^n - \sum_{n \geq 0} 2s(n)x^{n+1}

= \sum_{n \geq 0} s(n)x^n - \sum_{n \geq 1} 2s(n-1)x^n

= s(0) + \sum_{n \geq 1} (s(n) - 2s(n-1))x^n \tag{48}

Because the order of the polynomial on the left hand side of (48) is zero, the coefficients of \(x^n\) for all \(n \geq 1\) must be 0. Thus, \(s(n) - 2s(n-1) = 0\) implying that

\[ s(n) = 2s(n-1). \tag{49} \]

This proves Theorem 12 □

### 3.4 Pentagonal Cacti

First, we name all the vertices of \(P(n)\) as shown in Figure 15.

![Figure 15: A labelled pentagonal cactus of \(n\) pentagons.](image)

Then, recall that

\[ p(n) = \text{the number of all maximal independent sets of } P(n) \]

and

\[ p(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } p(n). \]
Thus,

\[ p(n) = \sum_{k \geq 0} p(n, k). \]  \hspace{1cm} (50)

Further, we let

\[ P(x) = \sum_{n \geq 0} p(n)x^n \]

be the generating function of \( p(n) \) and we let

\[ P(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} p(n, k)x^ny^k \]

be the bi-variate generating function of \( p(n, k) \). It is worth noting that, when \( y = 1 \), we have

\[ P(x, 1) = \sum_{n \geq 0} \left( \sum_{k \geq 0} p(n, k)(1)^k \right)x^n = \sum_{n \geq 0} p(n)x^n = P(x). \]  \hspace{1cm} (51)

Next, we let \( P(n) \) be constructed from \( p(n) \) by joining an end vertex of a path of length two to a vertex at distance one from the cut vertex of the \( n^{th} \) pentagon. The graph \( P(n) \) is shown in Figure 16.

Then, we let

\[ \overline{p}(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } P(n). \]

and let

\[ \overline{P}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \overline{p}(n, k)x^ny^k \]
be the bi-variate generating function of $\overline{p}(n,k)$. Now, we are ready to prove Theorem [14].

**Proof of Theorem [14]** Let $D$ be a maximal independent set of $P(n)$ containing $k$ vertices. We distinguish 2 cases.

**Case 1:** $x_1^n \in D$

Thus, $x_2^n, x_5^n \notin D$. We further consider 2 subcases.

**Subcase 1.1:** $x_2^n \notin D$

Thus, removing $x_1^n, ..., x_5^n$ from $P(n)$ results in $\overline{p}(n-2)$. Thus, $D = D' \cup \{x_2^n, x_4^n\}$ where $D'$ is a maximal independent set of $\overline{p}(n-2)$ containing $k-2$ vertices. Therefore, there are $\overline{p}(n-2, k-2)$ possibilities of $D'$ yielding that there are $\overline{p}(n-2, k-2)$ possibilities of $D$.

**Subcase 1.2:** $x_5^n \notin D$

By maximality of $D$, $x_1^n \in D$. Thus, $x_2^n, x_4^n \notin D$. By maximality of $D$, either $x_3^n \in D$ or $x_3^n \notin D$. Removing all vertices of the $n^{th}$ and $(n-1)^{th}$ pentagons results in $\overline{p}(n-3)$. Thus, for any $u \in \{x_2^n, x_4^n\}$, we have that $D = D' \cup \{u, x_1^n, x_3^n\} D'$ is a maximal independent set of $\overline{p}(n-3)$ containing $k-3$ vertices. Therefore, there are $\overline{p}(n-3, k-3)$ possibilities of $D'$ yielding that there are $2\overline{p}(n-3, k-3)$ possibilities of $D$.

**Case 2:** $x_1^n \notin D$

We further have the following 3 subcases.

**Subcase 2.1:** $n_3^n \notin D$ and $x_2^n \in D$.

Thus, $x_2^n \in D$. Removing all the vertices of the $n^{th}$ pentagon results in $\overline{p}(n-2)$. We have that there are $\overline{p}(n-2, k-2)$ possibilities of $D$.

**Subcase 2.2:** $n_3^n \in D$ and $x_5^n \notin D$.

Thus, $x_5^n \notin D$. By the maximality of $D$, $x_1^n \in D$ and this implies that either $x_2^n \in D$ or $x_3^n \in D$. Removing all the vertices of the $(n-1)^{th}$ and $n^{th}$ pentagons results in $\overline{p}(n-3)$. We have that there are $2\overline{p}(n-3, k-3)$ possibilities of $D$.

**Subcase 2.3:** $n_3^n \in D$ and $x_2^n \in D$.

Thus, $x_2^n \in D$. Removing all the vertices of the $n^{th}$ pentagon results in $\overline{p}(n-2)$. We have that there are $\overline{p}(n-2, k-2)$ possibilities of $D$.

From, all the cases, we have that

$$ p(n, k) = 3\overline{p}(n-2, k-2) + 4\overline{p}(n-3, k-3) \tag{52} $$

For $n \geq 3$ and $k \geq 3$, we multiply $x^n y^k$ throughout [52] and sum over all $x^n y^k$. Thus, we have that

$$ \sum_{n \geq 3} \sum_{k \geq 3} p(n, k) x^n y^k = \sum_{n \geq 3} \sum_{k \geq 3} 3\overline{p}(n-2, k-2) x^n y^k + \sum_{n \geq 3} \sum_{k \geq 3} 4\overline{p}(n-3, k-3) x^n y^k \tag{53} $$

We first consider the term $\sum_{n \geq 3} \sum_{k \geq 3} p(n, k) x^n y^k$. It can be checked that

$$ p(0,0) = 1, \text{ and } p(0,k) = 0 \text{ for all } k \geq 1 $$

$$ p(1,0) = 0, p(1,1) = 0, p(1,2) = 5, \text{ and } p(1,k) = 0 \text{ for all } k \geq 3 $$

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\[ p(2, 0) = p(2, 1) = p(2, 2) = 0, p(2, 3) = 4, p(2, 4) = 9, \text{ and } p(2, k) = 0 \text{ for all } k \geq 5. \]

Thus,
\[
\sum_{n \geq 3} \sum_{k \geq 3} p(n, k) x^n y^k = \sum_{n \geq 3} \sum_{k \geq 3} p(n, k) x^n y^k + p(0, 0) + p(1, 2)x^2y^2 + p(2, 3)x^2y^3 + p(2, 4)x^2y^4
- p(0, 0) - p(1, 2)x^2y^2 - p(2, 3)x^2y^3 - p(2, 4)x^2y^4
= \sum_{n \geq 0} \sum_{k \geq 0} p(n, k) x^n y^k - 5x^2y^2 - 4x^2y^3 - 9x^2y^4
= P(x, y) - 1 - 5x^2y^2 - 4x^2y^3 - 9x^2y^4. \tag{54}
\]

Now, we consider the term \( \sum_{n \geq 3} \sum_{k \geq 3} 3\overline{p}(n - 2, k - 2)x^n y^k \). Clearly,
\[
\overline{p}(0, 0) = 0, \overline{p}(0, 1) = 0, \overline{p}(0, 2) = 3 \text{ and } \overline{p}(0, k) = 0 \text{ for all } k \geq 3
\]
\[
\overline{p}(1, 0) = 0, \overline{p}(1, 1) = 0, \overline{p}(1, 2) = 0, \overline{p}(1, 3) = 7, \overline{p}(1, 4) = 3, \text{ and } \overline{p}(1, k) = 0 \text{ for all } k \geq 5
\]
\[
\overline{p}(n, 0) = 0 \text{ for all } n \geq 0.
\]

Thus,
\[
\sum_{n \geq 3} \sum_{k \geq 3} 3\overline{p}(n - 2, k - 2)x^n y^k = 3x^2y^2 \sum_{n \geq 3} \sum_{k \geq 3} \overline{p}(n - 2, k - 2)x^{n-2}y^{k-2}
= 3x^2y^2(\sum_{n \geq 1} \sum_{k \geq 1} \overline{p}(n, k)x^n y^k + \overline{p}(0, 2)y^2 - \overline{p}(0, 2)y^2)
= 3x^2y^2(\sum_{n \geq 0} \sum_{k \geq 0} \overline{p}(n, k)x^n y^k - 3y^2)
= 3x^2y^2P(x, y) - 9x^2y^4. \tag{55}
\]

Finally, we consider the term \( \sum_{n \geq 3} \sum_{k \geq 3} 4\overline{p}(n - 3, k - 3)x^n y^k \). It can be check that,
\[
\sum_{n \geq 3} \sum_{k \geq 3} 4\overline{p}(n - 3, k - 3)x^n y^k = 4x^3y^3 \sum_{n \geq 3} \sum_{k \geq 3} \overline{p}(n - 3, k - 3)x^{n-3}y^{k-3}
= 4x^3y^3 \sum_{n \geq 0} \sum_{k \geq 0} \overline{p}(n, k)x^n y^k
= 4x^3y^3P(x, y). \tag{56}
\]

Plugging (54), (55) and (56) to (53), we have
\[
P(x, y) - 1 - 5x^2y^2 - 4x^2y^3 - 9x^2y^4 = 3x^2y^2P(x, y) - 9x^2y^4 + 4x^3y^3P(x, y)
\]
which can be solved that
Next, we will find a recurrence relation of $\overline{\nu}(n, k)$. Let $D$ be a maximal independent set of $\overline{P}(n)$ containing $k$ vertices. We distinguish 2 cases.

**Case 1:** $x_3^{n+1} \in D$

Thus, $x_2^{n+1}, x_4^{n+1} \notin D$. Removing $x_2^{n+1}, x_3^{n+1}, x_4^{n+1}$ from $\overline{P}(n)$ results in $P(n)$. Thus, there are $p(n, k-1)$ possibilities of $D$.

**Case 2:** $x_3^{n+1} \notin D$

By maximality of $D$, $x_3^{n+1} \in D$. We further distinguish 2 subcases.

**Case 2.1:** $x_2^{n+1} \notin D$

By maximality of $D$, $x_2^{n} \in D$. Thus, $x_2^{n}, x_1^{n} \notin D$. By maximality of $D$, either $x_2^{n} \in D$ or $x_3^{n} \in D$. Removing all vertices of the $n^{th}$ pentagon and $x_2^{n+1}, x_3^{n+1}, x_4^{n+1}$ from $\overline{P}(n)$ results in $P(n-2)$. Thus, for any $u \in \{x_2^{n}, x_3^{n}\}$, we have that $D = D' \cup \{u, x_4^{n+1}, x_3^{n}\}$ where $D'$ is a maximal independent set of $\overline{P}(n-2)$ containing $k-3$ vertices. Therefore, there are $\overline{\nu}(n-2, k-3)$ possibilities of $D'$ yielding that there are $2\overline{\nu}(n-2, k-3)$ possibilities of $D$.

**Case 2.2:** $x_2^{n+1} \in D$

Thus, $x_2^{n} \notin D$. Removing $x_2^{n}, x_3^{n+1}, x_4^{n+1}$ from $\overline{P}(n)$ results in $P(n-1)$. There are $\overline{\nu}(n-1, k-2)$ possibilities of $D$.

From all the cases, we have that

$$\overline{\nu}(n, k) = p(n, k-1) + 2\overline{\nu}(n-2, k-3) + \overline{\nu}(n-1, k-2).$$  \hfill (58)

For $n \geq 2, k \geq 3$, we multiply $x^{n}y^{k}$ throughout (58) and sum over all $n, k$. We have that

$$\sum_{n \geq 2} \sum_{k \geq 3} \overline{\nu}(n, k) = \sum_{n \geq 2} \sum_{k \geq 3} p(n, k-1) + \sum_{n \geq 2} \sum_{k \geq 3} 2\overline{\nu}(n-2, k-3) + \sum_{n \geq 2} \sum_{k \geq 3} \overline{\nu}(n-1, k-2).$$  \hfill (59)

We first consider the term $\sum_{n \geq 2} \sum_{k \geq 3} \overline{\nu}(n, k)x^{n}y^{k}$. It can be checked that,

$$\overline{\nu}(0, 0) = 0, \overline{\nu}(0, 1) = 0, \overline{\nu}(0, 2) = 3$$
$$\overline{\nu}(1, 0) = 0, \overline{\nu}(1, 1) = 0, \overline{\nu}(1, 2) = 0, \overline{\nu}(1, 3) = 7, \overline{\nu}(1, 4) = 3,$$
$$\overline{\nu}(2, 0) = 0, \overline{\nu}(2, 1) = 0, \overline{\nu}(2, 2) = 0,$$
$$\overline{\nu}(n, k) = 0 \text{ for all } n \geq 2, k \leq 2.$$
\[
\sum_{n \geq 2} \sum_{k \geq 3} \overline{p}(n, k) x^ny^k = \sum_{n \geq 2} \sum_{k \geq 3} \overline{p}(n, k) x^ny^k + \overline{p}(0, 2)y^2 + \overline{p}(1, 3)xy^3 + \overline{p}(1, 4)xy^4
\]
\[
- \overline{p}(0, 2)y^2 - \overline{p}(1, 3)xy^3 - \overline{p}(1, 4)xy^4
\]
\[
= \sum_{n \geq 0} \sum_{k \geq 0} \overline{p}(n, k) x^ny^k - 3y^2 - 7xy^3 - 3xy^4
\]
\[
= \mathcal{P}(x, y) - 3y^2 - 7xy^3 - 3xy^4
\]

We next consider the term \( \sum_{n \geq 2} \sum_{k \geq 3} p(n, k - 1)x^n y^k \). Recall that

\[
\sum_{n \geq 2} \sum_{k \geq 3} p(n, k - 1)x^n y^k = y(\sum_{n \geq 2} \sum_{k \geq 3} p(n, k - 1)x^n y^{k-1})
\]
\[
= y(\sum_{n \geq 2} \sum_{k \geq 2} p(n, k)x^ny^k)
\]
\[
= y(\sum_{n \geq 2} \sum_{k \geq 2} p(n, k)x^ny^k + p(0, 0) + p(1, 2)xy^2 - p(0, 0) - p(1, 2)xy^2)
\]
\[
= y(\sum_{n \geq 0} \sum_{k \geq 0} p(n, k)x^ny^k - 1 - 5xy^2)
\]
\[
= y\mathcal{P}(x, y) - y - 5xy^2
\]

We next consider the term \( \sum_{n \geq 2} \sum_{k \geq 3} 2\mathcal{P}(n - 2, k - 3)x^n y^k \). It can be checked that,

\[
\sum_{n \geq 2} \sum_{k \geq 3} 2\mathcal{P}(n - 2, k - 3)x^n y^k = 2x^2y^3(\sum_{n \geq 2} \sum_{k \geq 3} \mathcal{P}(n - 2, k - 3)x^{n-2}y^{k-3})
\]
\[
= 2x^2y^3(\sum_{n \geq 0} \sum_{k \geq 0} \mathcal{P}(n, k)x^ny^k)
\]
\[
= 2x^2y^3\mathcal{P}(x, y)
\]

Finally, we consider the term \( \sum_{n \geq 2} \sum_{k \geq 3} \mathcal{P}(n - 1, k - 2)x^n y^k \). Clearly,

\[
\sum_{n \geq 2} \sum_{k \geq 3} \mathcal{P}(n - 1, k - 2)x^n y^k = xy^2(\sum_{n \geq 2} \sum_{k \geq 3} \mathcal{P}(n - 1, k - 2)x^{n-1}y^{k-2})
\]
\[
= xy^2(\sum_{n \geq 1} \sum_{k \geq 1} \mathcal{P}(n, k)x^ny^k)
\]
\[
= xy^2(\sum_{n \geq 1} \sum_{k \geq 1} \mathcal{P}(n, k)x^ny^k + \mathcal{P}(0, 2)y^2 - \mathcal{P}(0, 2)y^2)
\]
\[
= xy^2(\sum_{n \geq 0} \sum_{k \geq 0} \mathcal{P}(n, k)x^ny^k - 3y^2)
\]
\[
= xy^2\mathcal{P}(x, y) - 3xy^4
\]
Plugging (60), (61), (62) and (63) to (59), we have

\[
\overline{P}(x, y) - 3y^2 - 7y^3 - 3x^4 = yP(x, y) - y - 5xy^3 + 2x^2y^2\overline{P}(x, y) + xy^2\overline{P}(x, y) - 3x^4.
\]

which can be solved that

\[
\overline{P}(x, y) = 3y^2 + 7xy^3 + yP(x, y) - y - 5xy^3 + 2x^2y^2\overline{P}(x, y) + xy^2\overline{P}(x, y).
\]

(64)

By (57) and (64), we have

\[
P(x, y) = 1 + 4x^3y^5 - 4x^3y^4 + 4x^2y^4 - x^2y^3 + 4xy^2
\]

\[
\frac{1 - 5xy^3 - xy^2 - 4x^3y^4}{1 - 5x^2y^3 - xy^2 - 4x^3y^4}
\]

as required. \(\Box\)

We will prove Theorem 15.

Proof of Theorem 15 By Theorem 14 with \(y = 1\), we have that

\[
\sum_{n \geq 0} p(n)x^n = P(x, 1)
\]

\[
= \frac{3x^2 + 4x + 1}{-4x^3 - 5x^2 - x + 1}
\]

(65)

which can be solved that

\[
3x^2 + 4x + 1 = (-4x^3 - 5x^2 - x + 1) \sum_{n \geq 0} p(n)x^n
\]

\[
= \sum_{n \geq 0} p(n)x^n - \sum_{n \geq 0} 5p(n)x^{n+2} - \sum_{n \geq 0} p(n)x^{n+1} - \sum_{n \geq 0} 4p(n)x^{n+3}
\]

\[
= \sum_{n \geq 0} p(n)x^n - \sum_{n \geq 2} 5p(n-2)x^n - \sum_{n \geq 1} p(n-1)x^n - \sum_{n \geq 3} 4p(n-3)x^n
\]

\[
= p(0) + p(1)x + p(2)x^2 - 5p(0)x^2 - p(0)x - p(1)x^2
\]

\[
+ \sum_{n \geq 3} (p(n) - 5p(n-2) - p(n-1) - 4p(n-3))x^n
\]

(66)

Because the order of the polynomial on the left hand side of (66) is two, the coefficients of \(x^n\) for all \(n \geq 3\) must be 0. Thus, \(p(n) - 5p(n-2) - p(n-1) - 4p(n-3) = 0\) implying that

\[
p(n) = p(n-1) + 5p(n-2) + 4p(n-3)
\]
3.5 Meta-Pentagonal Cacti

First, we many name all the vertices of $M(n)$ as shown in Figure 17.

![Figure 17: A labelled meta-pentagonal cactus of $n$ pentagons.](image)

Then, we recall that:

$m(n) =$ the number of all maximal independent sets of $M(n)$

and

$M(n, k) =$ the number of maximal independent sets containing $k$ vertices of $M(n)$.

Thus,

$$m(n) = \sum_{k \geq 0} m(n, k). \quad (67)$$

Further, we let

$$M(x) = \sum_{n \geq 0} m(n)x^n$$

be the generating function of $m(n)$ and we let

$$M(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} m(n, k)x^ny^k$$

be the bi-variate generating function of $m(n, k)$. It is worth noting that, when $y = 1$, we have

$$M(x, 1) = \sum_{n \geq 0} \left( \sum_{k \geq 0} m(n, k)(1)^k \right)x^n = \sum_{n \geq 0} m(n)x^n = M(x). \quad (68)$$

Next, we let $\overline{M}(n)$ be constructed from $M(n)$ by joining one vertex to a vertex at distance two from the cut vertex of the $n^{th}$ pentagon. The graph $\overline{M}(n)$ is illustrated by Figure 18. Further, we define
the graph $\tilde{M}$ from $M(n)$ by identifying a vertex of degree two of a path of length three to a vertex at distance two from the cut vertex of the $n^{th}$ pentagon. The graph $M(n)$ is shown in Figure 19.

![Figure 18: The graph $M(n)$.](image18)

Then, we let

$\overline{m}(n, k) =$ the number of maximal independent sets containing $k$ vertices of $\overline{M}(n)$,

$\tilde{m}(n, k) =$ the number of maximal independent sets containing $k$ vertices of $\tilde{M}(n)$,

and let

$$\overline{M}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \overline{m}(n, k) x^n y^k$$
\[
\tilde{M}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \tilde{m}(n, k) x^n y^k.
\]

Now, we are ready to prove Theorem 19.

**Proof of Theorem 19**

Let \( D \) be a maximal independent set of \( M(n) \) containing \( k \) vertices. We distinguish 2 cases.

**Case 1:** \( x_3^n \in D \)

Thus, \( x_3^n, x_4^n, x_5^n \notin D \). Removing \( x_3^n, x_4^n, x_5^n \) from \( M(n) \) result in \( \overline{M}(n - 1) \). There are \( \overline{m}(n - 1, k - 1) \) possibilities of \( D \).

**Case 2:** \( x_3^n \notin D \)

We further have the following 3 subcases.

**Subcase 2.1:** \( n_3^n \notin D \) and \( x_5^n \in D \).

Thus, \( x_1^n \notin D \) and \( x_2^n \in D \). Removing all the vertices of the \( n^{th} \) pentagon results in \( \tilde{M}(n - 2) \). We have that there are \( \tilde{m}(n - 2, k - 2) \) possibilities of \( D \).

**Subcase 2.2:** \( n_3^n \in D \) and \( x_5^n \notin D \).

Thus, \( x_2^n \notin D \). By the maximality of \( D \), \( x_1^n \in D \) and this implies that \( x_2^{n-1}, x_4^{n-1} \notin D \). Removing all the vertices of the \( n^{th} \) pentagon and \( x_2^{n-1}, x_4^{n-1} \) results in \( \overline{M}(n - 2) \). We have that there are \( \overline{m}(n - 2, k - 2) \) possibilities of \( D \).

**Subcase 2.3:** \( n_3^n \in D \) and \( x_5^n \in D \).

Thus, \( x_2^n \in D \). Removing all the vertices of the \( n^{th} \) pentagon results in \( \tilde{M}(n - 2) \). We have that there are \( \tilde{m}(n - 2, k - 2) \) possibilities of \( D \).

From all the cases, we have that

\[
m(n, k) = \overline{m}(n - 1, k - 1) + 2\tilde{m}(n - 2, k - 2) + \overline{m}(n - 2, k - 2) \tag{69}
\]

For \( n \geq 2 \) and \( k \geq 2 \), we multiply \( x^n y^k \) throughout (69) and sum over all \( x^n y^k \). Thus, we have that

\[
\sum_{n \geq 2} \sum_{k \geq 2} m(n, k) x^n y^k = \sum_{n \geq 2} \sum_{k \geq 2} \overline{m}(n - 1, k - 1) x^n y^k + 2 \sum_{n \geq 2} \sum_{k \geq 2} \tilde{m}(n - 2, k - 2) x^n y^k + \sum_{n \geq 2} \sum_{k \geq 2} \overline{m}(n - 2, k - 2) x^n y^k. \tag{70}
\]

We first consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} m(n, k) x^n y^k \). It can be checked that

\[
m(0, 0) = 1, m(0, 1) = 0, m(0, 2) = 0, \text{ and } m(0, k) = 0 \text{ for all } k \geq 3
\]
\[
m(1, 0) = 0, m(1, 0) = 0, m(1, 2) = 5, \text{ and } m(0, k) = 0 \text{ for all } k \geq 3
\]
\[ \sum_{n \geq 2} \sum_{k \geq 2} m(n, k)x^ny^k = \sum_{n \geq 2} \sum_{k \geq 2} m(n, k)x^ny^k + m(0, 0) + m(1, 2)xy^2 - m(0, 0) - m(1, 2)xy^2 \\
= \sum_{n \geq 0} \sum_{k \geq 0} m(n, k)x^ny^k - 1 - 5xy^2 \\
= M(x, y) - 1 - 5xy^2 \tag{71} \]

Now, we consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} \mathfrak{m}(n - 1, k - 1)x^ny^k \). Clearly,

\( \mathfrak{m}(0, 0) = 0, \mathfrak{m}(0, 1) = 2 \) and \( \mathfrak{m}(0, k) = 0 \) for all \( k \geq 2 \)

\( \mathfrak{m}(1, 0) = 0, \mathfrak{m}(1, 1) = 0 \) and \( \mathfrak{m}(1, k) = 0 \) for all \( k \geq 2 \)

\[ \sum_{n \geq 2} \sum_{k \geq 2} \mathfrak{m}(n - 1, k - 1)x^ny^k = xy\left(\sum_{n \geq 2} \sum_{k \geq 2} \mathfrak{m}(n - 1, k - 1)x^{n-1}y^{k-1}\right) \\
= xy\left(\sum_{n \geq 2} \sum_{k \geq 2} \mathfrak{m}(n - 1, k - 1)x^{n-1}y^{k-1} + \mathfrak{m}(0, 1)y - \mathfrak{m}(0, 1)y\right) \\
= xy\left(\sum_{n \geq 0} \sum_{k \geq 0} \mathfrak{m}(n, k)x^ny^k - 2y\right) \\
= xyM(x, y) - 2xy^2. \tag{72} \]

Next, we consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} 2\mathfrak{m}(n - 2, k - 2)x^ny^k \). It can be check that,

\[ \sum_{n \geq 2} \sum_{k \geq 2} 2\mathfrak{m}(n - 2, k - 2)x^ny^k = 2x^2y^2\left(\sum_{n \geq 2} \sum_{k \geq 2} \mathfrak{m}(n - 2, k - 2)x^{n-2}y^{k-2}\right) \\
= 2x^2y^2\left(\sum_{n \geq 0} \sum_{k \geq 0} \mathfrak{m}(n, k)x^ny^k\right) \\
= 2x^2y^2\mathfrak{M}(x, y). \tag{73} \]

Finally, we consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} \mathfrak{m}(n - 2, k - 2)x^ny^k \). It can be check that,

\[ \sum_{n \geq 2} \sum_{k \geq 2} \mathfrak{m}(n - 2, k - 2)x^ny^k = x^2y^2\left(\sum_{n \geq 2} \sum_{k \geq 2} \mathfrak{m}(n - 2, k - 2)x^{n-2}y^{k-2}\right) \\
= x^2y^2\left(\sum_{n \geq 0} \sum_{k \geq 0} \mathfrak{m}(n, k)x^ny^k\right) \\
= x^2y^2\mathfrak{M}(x, y). \tag{74} \]

Plugging (71), (72), (73) and (74) to (70), we have

\[ M(x, y) - 1 - 5xy^2 = xy\mathfrak{M}(x, y) - 2xy^2 + 2x^2y^2\mathfrak{M}(x, y) + x^2y^2\mathfrak{M}(x, y) \]

40
which can be solved that

\[ M(x, y) = 1 + 5xy^2 + xyM(x, y) - 2xy^2 + 2x^2y^2\hat{M}(x, y) + x^2y^2\tilde{M}(x, y). \] (75)

Next, we will find the recurrence relation of \( m(n, k) \). Let \( D \) be a maximal independent set of \( M(n) \) containing \( k \) vertices. We distinguish 2 cases.

**Case 1:** \( x_2^{n+1} \in D \)

Thus, \( x_3^n \notin D \). Removing \( x_3^n, x_2^{n+1} \) from \( M(n) \) results in \( \hat{M}(n-1) \). There are \( \hat{m}(n-1, k-1) \) possibilities of \( D \).

**Case 2:** \( x_2^{n+1} \notin D \)

By the maximality of \( D \), \( x_3^n \in D \). Thus, \( x_2^n, x_4^n \notin D \). Removing the vertices \( x_2^n, x_3^n, x_4^n, x_2^{n+1} \) from \( M(n) \) results in \( \hat{M}(n-1) \). There are \( \hat{m}(n-1, k-1) \) possibilities of \( D \).

From the two cases, we have that

\[ m(n, k) = \hat{m}(n-1, k-1) + m(n-1, k-1) \] (76)

For \( n \geq 1 \) and \( k \geq 1 \), we multiply \( x^ny^k \) throughout (76) and sum over all \( x^ny^k \). Thus, we have that

\[ \sum_{n \geq 1} \sum_{k \geq 1} m(n, k)x^ny^k = \sum_{n \geq 1} \sum_{k \geq 1} \hat{m}(n-1, k-1)x^ny^k + \sum_{n \geq 1} \sum_{k \geq 1} m(n-1, k-1)x^ny^k \] (77)

We first consider the term \( \sum_{n \geq 1} \sum_{k \geq 1} m(n, k)x^ny^k \). It can be checked that,

\[ m(0, 0) = 0, m(0, 1) = 2 \text{ and } m(0, k) = 0 \text{ for all } k \geq 2 \]
\[ m(1, 0) = 0, m(1, 1) = 0 \text{ and } m(1, k) = 0 \text{ for all } k \geq 2 \]

\[ \sum_{n \geq 1} \sum_{k \geq 1} m(n, k)x^ny^k = \sum_{n \geq 1} \sum_{k \geq 1} \hat{m}(n-1, k-1)x^ny^k + m(0, 1)y - m(0, 1)y \\
= \sum_{n \geq 0} \sum_{k \geq 0} m(n, k)x^ny^k - 2y \\
= \hat{M}(x, y) - 2y \] (78)

We next consider the term \( \sum_{n \geq 1} \sum_{k \geq 1} \hat{m}(n-1, k-1)x^ny^k \). It can be checked that,
\[
\sum_{n \geq 1} \sum_{k \geq 1} \tilde{m}(n-1,k-1)x^n y^k = xy\left(\sum_{n \geq 1} \sum_{k \geq 1} \tilde{m}(n-1,k-1)x^{n-1} y^{k-1}\right)
\]
\[= xy\left(\sum_{n \geq 0} \sum_{k \geq 0} \tilde{m}(n,k)x^n y^k\right)
= xy\tilde{M}(x,y)
\] (79)

Finally, we next consider the term \(\sum_{n \geq 1} \sum_{k \geq 1} m(n-1,k-1)x^n y^k\). Clearly

\[
\sum_{n \geq 1} \sum_{k \geq 1} m(n-1,k-1)x^n y^k = xy\left(\sum_{n \geq 1} \sum_{k \geq 1} m(n-1,k-1)x^{n-1} y^{k-1}\right)
\]
\[= xy\left(\sum_{n \geq 0} \sum_{k \geq 0} m(n,k)x^n y^k\right)
= xyM(x,y)
\] (80)

Plugging (78), (79) and (80) to (77) we have

\[
M(x,y) - 2y = xy\tilde{M}(x,y) + xyM(x,y).
\]

which can be solved that

\[
M(x,y) = 2y + xy\tilde{M}(x,y) + xyM(x,y).
\] (81)

Finally, we will find the recurrence relation of \(\tilde{m}(n,k)\). Let \(D\) be a maximal independent set of \(\tilde{M}(n)\) containing \(k\) vertices. We distinguish 2 cases.

**Case 1:** \(x_2^{n+1} \in D\)

Thus, \(x_2^n, x_1^{n+1}, x_3^{n+1}, x_4^{n+1} \notin D\). By the maximality of \(D\), \(x_4^{n+1} \in D\). Removing \(x_2^n, x_3^n, x_1^{n+1}, \ldots, x_4^{n+1}\) from \(\tilde{M}(n)\) results in \(\tilde{M}(n-1)\). There are \(\tilde{m}(n-1,k-2)\) possibilities of \(D\).

**Case 2:** \(x_3^{n+1} \notin D\)

By the maximality of \(D\), \(x_1^{n+1} \in D\). Further, \(x_3^{n+1} \in D\) or \(x_4^{n+1} \in D\). Removing all the vertices \(x_1^{n+1}, \ldots, x_4^{n+1}\) results in \(\tilde{M}(n-1)\). We have that there are \(2\tilde{m}(n-1,k-2)\) possibilities of \(D\).

From all the cases, we have that

\[
\tilde{m}(n,k) = \tilde{m}(n-1,k-2) + 2\tilde{m}(n-1,k-2)
\] (82)

For \(n \geq 1\) and \(k \geq 2\), we multiply \(x^n y^k\) throughout (82) and sum over all \(x^n y^k\). Thus, we have that

\[
\sum_{n \geq 1} \sum_{k \geq 2} \tilde{m}(n,k)x^n y^k = \sum_{n \geq 1} \sum_{k \geq 2} \tilde{m}(n-1,k-2)x^n y^k + \sum_{n \geq 1} \sum_{k \geq 2} 2\tilde{m}(n-1,k-2)x^n y^k
\] (83)
We first consider the term \( \sum_{n \geq 1} \sum_{k \geq 2} \tilde{m}(n,k)x^n y^k \). It can be checked that,

\[
\begin{align*}
\tilde{m}(0,0) &= 0, \quad \tilde{m}(0,1) = 0, \quad \tilde{m}(0,2) = 3 \text{ and } \tilde{m}(0,k) = 0 \text{ for all } k \geq 3 \\
\tilde{m}(1,0) &= 0, \quad \tilde{m}(1,1) = 0, \quad \tilde{m}(0,2) = 0 \text{ and } \tilde{m}(0,k) = 0 \text{ for all } k \geq 3
\end{align*}
\]

\[
\begin{align*}
\sum_{n \geq 1} \sum_{k \geq 2} \tilde{m}(n,k)x^n y^k &= \sum_{n \geq 1} \sum_{k \geq 2} \tilde{m}(n,k)x^n y^k + \tilde{m}(0,2)y^2 - \tilde{m}(0,2)y^2 \\
&= \sum_{n \geq 0} \sum_{k \geq 0} \tilde{m}(n,k)x^n y^k - 3y^2 \\
&= \tilde{M}(x,y) - 3y^2 \quad (84)
\end{align*}
\]

Now, we consider the term \( \sum_{n \geq 1} \sum_{k \geq 2} m(n-1,k-2)x^n y^k \). Clearly,

\[
\begin{align*}
\sum_{n \geq 1} \sum_{k \geq 2} m(n-1,k-2)x^n y^k &= xy^2(\sum_{n \geq 1} \sum_{k \geq 2} m(n-1,k-2)x^{n-1} y^{k-2}) \\
&= xy^2(\sum_{n \geq 0} \sum_{k \geq 0} m(n,k)x^n y^k) \\
&= xy^2\tilde{M}(x,y) \quad (85)
\end{align*}
\]

Finally, we consider the term \( \sum_{n \geq 1} \sum_{k \geq 2} 2\tilde{m}(n-1,k-2)x^n y^k \). It can be checked that,

\[
\begin{align*}
\sum_{n \geq 1} \sum_{k \geq 2} 2\tilde{m}(n-1,k-2)x^n y^k &= 2xy^2(\sum_{n \geq 1} \sum_{k \geq 2} \tilde{m}(n-1,k-2)x^{n-1} y^{k-2}) \\
&= 2xy^2(\sum_{n \geq 0} \sum_{k \geq 0} \tilde{m}(n,k)x^n y^k) \\
&= 2xy^2\tilde{M}(x,y) \quad (86)
\end{align*}
\]

Plugging (84), (85), and (86) to (83), we have

\[
\tilde{M}(x,y) - 3y^2 = xy^2\tilde{M}(x,y) + 2xy^2\tilde{M}(x,y)
\]

which can be solved that

\[
\tilde{M}(x,y) = 3y^2 + xy^2\tilde{M}(x,y) + 2xy^2\tilde{M}(x,y) \quad (87)
\]
By plugging (75) to (81) to (87), we have

\[ M(x, y) = \frac{-11x^2y^4 + 5xy^2 + 2x^3y^6 + 2x^3y^8 - x^2y^9 + 1}{1 - 4xy^2 - xy + 3x^2y^4 + 4x^2y^5 - 2x^3y^6} \]

as required. □

Now, we are ready to prove Theorem 20

**Proof of Theorem 20** By Theorem 19 with \( y = 1 \), we have that

\[ \sum_{n \geq 0} m(n)x^n = M(x, 1) = \frac{1 - 5x^2 + 2x^3}{1 - 5x + 7x^2 - 2x^3} = \frac{-x^2 + 2x + 1}{x^2 - 3x + 1} \quad (88) \]

which can be solved that

\[ -x^2 + 2x + 1 = (x^2 - 3x + 1) \sum_{n \geq 0} m(n)x^n \]

\[ = \sum_{n \geq 0} m(n)x^{n+2} - 3 \sum_{n \geq 0} m(n)x^{n+1} + \sum_{n \geq 0} n(n)x^n \]

\[ = \sum_{n \geq 2} m(n-2)x^n - 3 \sum_{n \geq 1} m(n-1)x^n + \sum_{n \geq 0} m(n)x^n \]

\[ = m(0) - 3m(0)x + m(1)x + \sum_{n \geq 2} (m(n-2) - 3m(n-1) + m(n))x^n \quad (89) \]

Because the order of the polynomial on the left hand side of (89) is two, the coefficients of \( x^n \) for all \( n \geq 3 \) must be 0. Thus, \( m(n) - 3m(n-1) + m(n-2) = 0 \) implying that

\[ m(n) = 3m(n-1) - m(n-2). \]

### 3.6 Meta-Hexagonal Cacti

We first name all the vertices of \( H(n) \) as shown in Figure 20.
Figure 20: The graphs $H(n)$ whose all vertices are labelled.

Then, we recall that

$$h(n) = \text{the number of all maximal independent sets of } H(n)$$

and

$$H(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } H(n).$$

Thus,

$$h(n) = \sum_{k \geq 0} h(n, k). \quad (90)$$

Further, we let

$$H(x) = \sum_{n \geq 0} h(n)x^n$$

be the generating function of $h(n)$ and we let

$$H(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} h(n, k)x^ny^k$$

be the bi-variate generating function of $h(n, k)$. It is worth noting that, when $y = 1$, we have

$$H(x, 1) = \sum_{n \geq 0} (\sum_{k \geq 0} h(n, k)(1)^k)x^n = \sum_{n \geq 0} h(n)x^n = H(x). \quad (91)$$

Next, we let $\overline{H}(n)$ be constructed from $H(n)$ by joining two vertices to a vertex at distance two from
the cut vertex of the $n^{th}$ hexagon. Further, we let $\overline{H}(n)$ be constructed from $\overline{H}(n)$ by joining an end
vertex of a path of length one to a vertex of degree one of $\overline{H}(n)$. The graphs $\overline{H}(n)$ and $\tilde{H}(n)$ are shown in Figures 21 and 22 respectively.

![Graph H(n)](image1)

**Figure 21:** The graph $\overline{H}(n)$.

![Graph H(n)](image2)

**Figure 22:** The graph $\tilde{H}(n)$.

Then, we let

$\overline{n}(n,k) = \text{the number of maximal independent sets containing } k \text{ vertices of } \overline{H}(n)$,
\( \hat{h}(n, k) = \) the number of maximal independent sets containing \( k \) vertices of \( \tilde{H}(n) \),

and let

\[
\overline{\mathcal{H}}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \mathcal{H}(n, k)x^n y^k, \\
\hat{H}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \hat{h}(n, k)x^n y^k.
\]

Now, we are ready to prove Theorem 24.

**Proof of Theorem 24**

Let \( D \) be a maximal independent set of \( H(n) \) containing \( k \) vertices. We distinguish 2 cases.

**Case 1:** \( x_3^n \in D \)

Thus, \( x_3^n, x_4^n, x_5^n \) do not belong to \( H(n) \). Removing the vertices of the \( n^{th} \) hexagon results in \( \tilde{H}(n-1) \). There are \( \hat{h}(n-1, k-1) \) possibilities of \( D \).

**Case 2:** \( x_3^n \notin D \)

We further have the following 3 subcases.

**Subcase 2.1:** \( n_3^n \notin D \) and \( x_3^n \in D \).

Thus, \( x_3^n \notin D \) and \( x_6^n \in D \). This implies that \( x_1^n \notin D \). Removing all the vertices of the \( n^{th} \) hexagon results in \( \tilde{H}(n-2) \). We have that there are \( \hat{h}(n-2, k-2) \) possibilities of \( D \).

**Subcase 2.2:** \( n_3^n \in D \) and \( x_6^n \notin D \).

Thus, \( x_2^n \notin D \) and \( x_6^n \in D \). This implies that \( x_1^n \notin D \). Removing all the vertices of the \( n^{th} \) hexagon results in \( \tilde{H}(n-2) \). We have that there are \( \hat{h}(n-2, k-2) \) possibilities of \( D \).

**Subcase 2.3:** \( n_3^n \in D \) and \( x_6^n \in D \).

Thus, \( x_2^n, x_6^n \notin D \). Removing the vertices \( x_2^n, ..., x_6^n \) results in \( H(n-1) \). We have that there are \( h(n-1, k-2) \) possibilities of \( D \).

From the two cases, we have that

\[
h(n, k) = \overline{h}(n-1, k-1) + 2\hat{h}(n-2, k-2) + h(n-1, k-2).
\]

For \( n \geq 2 \) and \( k \geq 2 \), we multiply \( x^n y^k \) throughout (69) and sum over all \( x^n y^k \). Thus, we have that

\[
\sum_{n \geq 2} \sum_{k \geq 2} h(n, k)x^n y^k = \sum_{n \geq 2} \sum_{k \geq 2} \overline{h}(n-1, k-1)x^n y^k + 2 \sum_{n \geq 2} \sum_{k \geq 2} \hat{h}(n-2, k-2)x^n y^k \\
+ \sum_{n \geq 2} \sum_{k \geq 2} h(n-1, k-2)x^n y^k. 
\]

(93)

We first consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} h(n, k)x^n y^k \). It can be checked that

\[
h(0, 0) = 1, h(0, 1) = 0, h(0, 2) = 0, \text{ and } h(0, k) = 0 \text{ for all } k \geq 3
\]
Next, we consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} h(n, k)x^ny^k \). Thus,

\[
\sum_{n \geq 2} \sum_{k \geq 2} h(n, k)x^ny^k = \sum_{n \geq 2} \sum_{k \geq 2} h(n, k)x^ny^k + h(0, 0) + h(1, 2)xy^2 + h(1, 3)xy^3 - h(0, 0)
- h(1, 2)xy^2 - h(1, 3)xy^3
= \sum_{n \geq 0} \sum_{k \geq 0} h(n, k)x^ny^k - 3xy^2 - 2xy^3
= H(x, y) - 1 - 3xy^2 - 2xy^3
\]

(94)

Now, we consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} \bar{h}(n - 1, k - 1)x^ny^k \). Clearly,

\[
\bar{h}(0, 0) = 0, \bar{h}(0, 1) = 1, \bar{h}(0, 2) = 1 \quad \text{and} \quad \bar{h}(0, k) = 0 \quad \text{for all} \quad k \geq 2
\]

\[
\bar{h}(1, 0) = 0, \bar{h}(1, 1) = 0 \quad \text{and} \quad \bar{h}(1, k) = 0 \quad \text{for all} \quad k \geq 2
\]

\[
\sum_{n \geq 2} \sum_{k \geq 2} \bar{h}(n - 1, k - 1)x^ny^k = xy(\sum_{n \geq 2} \sum_{k \geq 2} \bar{h}(n - 1, k - 1)x^{n-1}y^{k-1})
= xy(\sum_{n \geq 2} \sum_{k \geq 2} \bar{h}(n - 1, k - 1)x^{n-1}y^{k-1} + \bar{h}(0, 1)y + \bar{h}(0, 2)y^2
- \bar{h}(0, 1)y - \bar{h}(0, 2)y^2)
= xy(\sum_{n \geq 0} \sum_{k \geq 0} \bar{h}(n, k)x^n y^k - y^2)
= xyH(x, y) - xy^2 - xy^3.
\]

(95)

Next, we consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} 2\tilde{h}(n - 2, k - 2)x^ny^k \). It can be checked that,

\[
\tilde{h}(0, 0) = 0, \tilde{h}(0, 1) = 0, \tilde{h}(0, 2) = 3, \tilde{h}(0, 3) = 1 \quad \text{and} \quad \tilde{h}(0, k) = 0 \quad \text{for all} \quad k \geq 4
\]

\[
\tilde{h}(1, 0) = 0, \tilde{h}(1, 1) = 0, \tilde{h}(1, 2) = 0, \tilde{h}(1, 3) = 2, \tilde{h}(1, 4) = 5, \tilde{h}(1, 5) = 4, \tilde{h}(1, 6) = 1 \quad \text{and} \quad \tilde{h}(1, k) = 0 \quad \text{for}
\quad \text{all} \quad k \geq 7
\]

Thus,

\[
\sum_{n \geq 2} \sum_{k \geq 2} 2\tilde{h}(n - 2, k - 2)x^ny^k = 2x^2y^2(\sum_{n \geq 2} \sum_{k \geq 2} \tilde{h}(n - 2, k - 2)x^{n-2}y^{k-2})
= 2x^2y^2(\sum_{n \geq 0} \sum_{k \geq 0} \tilde{h}(n, k)x^n y^k
= 2x^2y^2 H(x, y).
\]

(96)
Finally, we consider the term $\sum_{n \geq 2} \sum_{k \geq 2} h(n-1, k-2)x^ny^k$.

$$
\sum_{n \geq 2} \sum_{k \geq 2} h(n-1, k-2)x^ny^k = xy^2(\sum_{n \geq 2} \sum_{k \geq 2} h(n-1, k-2)x^{n-1}y^{k-2})
$$

$$
= xy^2(\sum_{n \geq 1} \sum_{k \geq 0} h(n, k)x^ny^k + h(0, 0) - h(0, 0))
$$

$$
= xy^2H(x, y) - xy^2. \hspace{1cm} (97)
$$

Plugging (94), (95), (96) and (97) to (93), we have

$$
H(x, y) - 1 - 3xy^2 - 2xy^3 = xy\overline{H}(x, y) - xy^2 - xy^3 + 2x^2y^2\tilde{H}(x, y) + xy^2H(x, y) - xy^2.
$$

which can be solved that

$$
H(x, y) = 1 + xy^2 + xy^3 + xy\overline{H}(x, y) + 2x^2y^2\tilde{H}(x, y) + xy^2H(x, y). \hspace{1cm} (98)
$$

Next, we will find the recurrence relation of $\overline{h}(n, k)$. Let $D$ be a maximal independent set of $\overline{H}(n)$ containing $k$ vertices. We distinguish 2 cases.

Case 1: $x_5^n \in D$

Thus, $x_2^n, x_4^n, x_2^{n+1}, x_6^{n+1} \notin D$. We further distinguish 2 subcases.

Subcase 1.1: $x_5^n \in D$

Thus, $x_5^n \notin D$. Removing $x_2^n, ..., x_6^n$ and $x_2^{n+1}, x_6^{n+1}$ from $\overline{H}(n)$ results in $H(n - 1)$. There are $h(n - 1, k - 2)$ possibilities of $D$.

Subcase 1.2: $x_5^n \notin D$

By the maximality of $D$, $x_5^n \in D$. Thus, $x_5^n \notin D$. Removing all vertices of the $n^{th}$ hexagon and $x_2^{n+1}, x_6^{n+1}$ from $\overline{H}(n)$ result in $\tilde{H}(n - 2)$. There are $\tilde{h}(n - 2, k - 2)$ possibilities of $D$.

Case 2: $x_5^n \notin D$

By the maximality of $D$, $x_2^{n+1}, x_6^{n+1} \in D$. Removing the vertices $x_3^n, x_2^{n+1}, x_6^{n+1}$ from $\overline{H}(n)$ result in $\tilde{H}(n - 1)$. There are $\tilde{h}(n - 1, k - 2)$ possibilities of $D$.

From the two cases, we have that

$$
\overline{h}(n, k) = h(n - 1, k - 2) + \tilde{h}(n - 2, k - 2) + \tilde{h}(n - 1, k - 2) \hspace{1cm} (99)
$$

For $n \geq 2, k \geq 2$, we multiply $x^ny^k$ throughout (99) and sum over all $n$ and $k$. We have that

$$
\sum_{n \geq 2} \sum_{k \geq 2} \overline{h}(n, k) = \sum_{n \geq 2} \sum_{k \geq 2} h(n - 1, k - 2) + \sum_{n \geq 2} \sum_{k \geq 2} \tilde{h}(n - 2, k - 2) + \sum_{n \geq 2} \sum_{k \geq 2} \tilde{h}(n - 1, k - 2). \hspace{1cm} (100)
$$

We first consider the term $\sum_{n \geq 2} \sum_{k \geq 2} \overline{h}(n, k)x^ny^k$. It can be checked that,
\( \vec{h}(0, 0) = 0, \vec{h}(0, 1) = 1, \vec{h}(0, 2) = 1 \) and \( \vec{h}(0, k) = 0 \) for all \( k \geq 3 \)

\( \vec{h}(1, 0) = 0, \vec{h}(1, 1) = 0, \vec{h}(1, 2) = 1, \vec{h}(1, 3) = 1, \vec{h}(1, 4) = 3, \vec{h}(1, 5) = 1 \) and \( \vec{h}(1, k) = 0 \) for all \( k \geq 6 \)

\[
\sum_{n \geq 2} \sum_{k \geq 2} \vec{h}(n, k) x^n y^k = \sum_{n \geq 2} \sum_{k \geq 2} \vec{h}(n, k) x^n y^k + \vec{h}(0, 1) y + \vec{h}(0, 2) y^2 + \vec{h}(1, 2) x y^2 \\
+ \vec{h}(1, 3) x y^3 + \vec{h}(1, 4) x y^4 + \vec{h}(1, 5) x y^5 - \vec{h}(0, 1) y - \vec{h}(0, 2) y^2 \\
- \vec{h}(1, 2) x y^2 - \vec{h}(1, 3) x y^3 - \vec{h}(1, 4) x y^4 - \vec{h}(1, 5) x y^5 \\
= \sum_{n \geq 0} \sum_{k \geq 0} \vec{h}(n, k) x^n y^k - y - y^2 - x y^2 - x^3 y - 3 x^4 y - x y^5 \\
= \vec{H}(x, y) - y - y^2 - x y^2 - x^3 y - 3 x^4 y - x y^5 \quad (101)
\]

We next consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} h(n - 1, k - 2) x^n y^k \), clearly

\[
\sum_{n \geq 2} \sum_{k \geq 2} h(n - 1, k - 2) x^n y^k = xy^2 (\sum_{n \geq 2} \sum_{k \geq 2} h(n - 1, k - 2) x^{n-1} y^{k-2} + h(0, 0) - h(0, 0)) \\
= xy^2 (\sum_{n \geq 1} \sum_{k \geq 0} h(n, k) x^n y^k - 1) \\
= xy^2 H(x, y) - xy^2 \quad (102)
\]

We next consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} \tilde{h}(n - 2, k - 2) x^n y^k \).

\[
\sum_{n \geq 2} \sum_{k \geq 2} \tilde{h}(n - 2, k - 2) x^n y^k = x^2 y^2 (\sum_{n \geq 2} \sum_{k \geq 2} \tilde{h}(n - 2, k - 2) x^{n-2} y^{k-2}) \\
= x^2 y^2 \vec{H}(x, y) \quad (103)
\]

Finally, We next consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} \tilde{h}(n - 1, k - 2) x^n y^k \), Clearly

\[
\sum_{n \geq 2} \sum_{k \geq 2} \tilde{h}(n - 1, k - 2) x^n y^k = xy^2 (\sum_{n \geq 2} \sum_{k \geq 2} \tilde{h}(n - 1, k - 2) x^{n-1} y^{k-2} + \tilde{h}(0, 2) + \tilde{h}(0, 3)) \\
- \tilde{h}(0, 2) - \tilde{h}(0, 3) \\
= xy^2 (\sum_{n \geq 0} \sum_{k \geq 0} \tilde{h}(n, k) x^n y^k - 3 y^2 - y^3) \\
= xy^2 \tilde{H}(x, y) - 3 x y^4 - x y^5 \quad (104)
\]

Plugging (101), (102), (103) and (104) to (100), we have

\[
\vec{H}(x, y) - y - y^2 - x y^2 - x^3 y - 3 x^4 y - x y^5 \\
= xy^2 H(x, y) - x y^2 + x^2 y^2 \vec{H}(x, y) + xy^2 \tilde{H}(x, y) - 3 x y^4 - x y^5. \quad (105)
\]
which can be solved that
\[ \mathbf{F}(x, y) = xy^2 H(x, y) + (x^2 y^2 + xy^2)\tilde{H}(x, y) + y + y^2 + xy^3. \]
(106)

Next, we will find the recurrence relation of \( \tilde{h}(n, k) \). Let \( D \) be a maximal independent set of \( \tilde{H}(n) \) containing \( k \) vertices. We distinguish 2 cases.

**Case 1: \( x_3^n \in D \)**

Thus, \( x_2^n, x_4^n, x_2^{n+1}, x_6^n \notin D \). Clearly, either \( x_3^{n+1} \in D \) or \( x_4^{n+1} \in D \). Further, we distinguish 2 subcases.

**Subcase 1.1: \( x_3^n \in D \)**

Thus, \( x_2^n \notin D \). Removing \( x_2^n, \ldots, x_6^n \) and \( x_2^{n+1}, x_3^{n+1}, x_4^{n+1}, x_6^{n+1} \) from \( \tilde{H}(n) \) result in \( H(n-1) \). Since either \( x_3^{n+1} \) or \( x_4^{n+1} \) is in \( D \), there are \( 2h(n-1, k-3) \) possibilities of \( D \).

**Subcase 1.2: \( x_3^n \notin D \)**

By the maximality of \( D \), \( x_3^n \in D \). Thus, \( x_1^n \notin D \). Removing all vertices of the \( n \)th hexagon and \( x_2^{n+1}, x_3^{n+1}, x_4^{n+1}, x_6^{n+1} \) from \( \tilde{H}(n) \) result in \( H(n-2) \). Since either \( x_3^{n+1} \) or \( x_4^{n+1} \) is in \( D \), there are \( 2h(n-2, k-3) \) possibilities of \( D \).

**Case 2: \( x_3^n \notin D \)**

By the maximality of \( D \), \( x_6^{n+1} \in D \). We further distinguish 2 subcases.

**Subcase 2.1: \( x_2^{n+1} \in D \)**

Thus, \( x_2^{n+1} \notin D \). By maximality of \( D \), \( x_3^{n+1} \in D \). Removing the vertices \( x_3^n, x_2^{n+1}, x_3^{n+1}, x_4^{n+1}, x_6^{n+1} \) from \( \tilde{H}(n) \) result in \( H(n-1) \). There are \( h(n-1, k-3) \) possibilities of \( D \).

**Subcase 2.2: \( x_2^{n+1} \notin D \)**

By the maximality of \( D \), \( x_3^{n+1} \in D \). Thus, \( x_3^{n+1} \notin D \). Removing the vertices \( x_3^n, x_2^{n+1}, x_3^{n+1}, x_4^{n+1}, x_6^{n+1} \) from \( \tilde{H}(n) \) result in \( H(n-1) \). There are \( h(n-1, k-2) \) possibilities of \( D \).

From all the cases, we have that
\[ \tilde{h}(n, k) = 2h(n-1, k-3) + 2\tilde{h}(n-2, k-3) + \tilde{h}(n-1, k-3) + \tilde{h}(n-1, k-2) \]
(107)

For \( n \geq 2, k \geq 3 \), we multiply \( x^n y^k \) throughout (107) and sum over all \( n, k \). We have that
\[
\sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n, k) = \sum_{n \geq 2} \sum_{k \geq 3} 2h(n-1, k-3) + \sum_{n \geq 2} \sum_{k \geq 3} 2\tilde{h}(n-2, k-3) \\
+ \sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n-1, k-3) + \sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n-1, k-2). 
\]
(108)

We first consider the term \( \sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n, k) x^n y^k \). Recall that,
\[
\tilde{h}(0, 0) = 0, \tilde{h}(0, 1) = 0, \tilde{h}(0, 2) = 3, \tilde{h}(0, 3) = 1 \text{ and } \tilde{h}(0, k) = 0 \text{ for all } k \geq 4
\]
\[
\tilde{h}(1, 0) = 0, \tilde{h}(1, 1) = 0, \tilde{h}(1, 2) = 0, \tilde{h}(1, 3) = 2, \tilde{h}(1, 4) = 5, \tilde{h}(1, 5) = 4, \tilde{h}(1, 6) = 1 \text{ and } \tilde{h}(1, k) = 0 \text{ for all } k \geq 7.
\]
Thus,
\[
\sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n, k)x^ny^k = \sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n, k)x^ny^k + \tilde{h}(0, 2)y^2 + \tilde{h}(0, 3)y^3 + \tilde{h}(1, 3)xy^3 \\
+ \tilde{h}(1, 4)xy^4 + \tilde{h}(1, 5)xy^5 + \tilde{h}(1, 6)xy^6 - \tilde{h}(0, 2)y^2 - \tilde{h}(0, 3)y^3 \\
- \tilde{h}(1, 3)xy^3 - \tilde{h}(1, 4)xy^4 - \tilde{h}(1, 5)xy^5 - \tilde{h}(1, 6)xy^6
\]
\[
= \sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n, k)x^ny^k - 3y^2 - y^3 - 2xy^3 - 5xy^4 - 4xy^5 - xy^6
\]  
(109)

We next consider the term \(\sum_{n \geq 2} \sum_{k \geq 3} 2h(n - 1, k - 3)x^ny^k\).

\[
\sum_{n \geq 2} \sum_{k \geq 3} 2h(n - 1, k - 3)x^ny^k = 2xy^3(\sum_{n \geq 2} \sum_{k \geq 3} h(n - 1, k - 3)x^{n-1}y^{k-3} + h(0, 0) - h(0, 0))
\]
\[
= 2xy^3(\sum_{n \geq 2} \sum_{k \geq 0} h(n, k)x^ny^k - 1)
\]
\[
= 2xy^3\tilde{H}(x, y) - 2xy^3
\]  
(110)

Now, we consider the term \(\sum_{n \geq 2} \sum_{k \geq 3} 2\tilde{h}(n - 2, k - 3)x^ny^k\).

\[
\sum_{n \geq 2} \sum_{k \geq 3} 2\tilde{h}(n - 2, k - 3)x^ny^k = 2x^2y^3(\sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n - 2, k - 3)x^{n-2}y^{k-3})
\]
\[
= 2x^2y^3\tilde{H}(x, y)
\]  
(111)

Next, we consider the term \(\sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n - 1, k - 3)x^ny^k\).

\[
\sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n - 1, k - 3)x^ny^k = xy^3(\sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n - 1, k - 3)x^{n-1}y^{k-3}
\]
\[
+ \tilde{h}(0, 2)y^2 + \tilde{h}(0, 3)y^3 - \tilde{h}(0, 2)y^2 - \tilde{h}(0, 3)y^3
\]
\[
= xy^3(\sum_{n \geq 0} \sum_{k \geq 0} \tilde{h}(n, k)x^ny^k - 3y^2 - y^3)
\]
\[
= xy^3\tilde{H}(x, y) - 3xy^5 - xy^6
\]  
(112)

Finally, we consider the term \(\sum_{n \geq 2} \sum_{k \geq 3} \tilde{h}(n - 1, k - 2)x^ny^k\). It can be checked that,
\[
\sum_{n \geq 2 \ k \geq 3} \hat{h}(n-1, k-2) x^n y^k = xy^2 \left( \sum_{n \geq 2 \ k \geq 3} \hat{h}(n-1, k-2) x^{n-1} y^{k-2} + \hat{h}(0, 2) y^3 + \hat{h}(0, 3) y^3 - \hat{h}(0, 2) y^2 - \hat{h}(0, 3) y^3 \right) \\
= xy^2 \left( \sum_{n \geq 0 \ k \geq 0} \hat{h}(n, k) x^n y^k - 3y^2 - y^3 \right) \\
= xy^2 \tilde{H}(x, y) - 3xy^4 - xy^5 \tag{113}
\]

Plugging (109) , (110) , (111) , (112) , and (113) to (108), we have

\[
\tilde{H}(x, y) - 3y^2 - y^3 - 2xy^3 - 5xy^4 - 4xy^5 - xy^6 = 2xy^3 \tilde{H}(x, y) - 2xy^3 + 2x^2 y^3 \tilde{H}(x, y) + xy^3 \tilde{H}(x, y) - 3xy^5 - xy^6 + xy^2 \tilde{H}(x, y) - 3xy^4 - xy^5 \tag{114}
\]

which can be solved that

\[
\tilde{H}(x, y) = \frac{1}{1 - 2x^2 y^3 - xy^3 - xy^5} (2xy^3 \tilde{H}(x, y) + 3y^2 + y^3 + 2xy^4) \tag{115}
\]

By plugging (98) to (100) to (115) , we have

\[
H(x, y) = \frac{1 - 2x^2 y^3 + xy^2 - x^3 y^5 + x^2 y^5 + 5x^2 y^4 + xy^3 - x^2 y^6 + x^3 y^7}{1 - 3x^2 y^3 - xy^3 - 2xy^2 - x^3 y^5 + x^2 y^5 + x^2 y^4 - x^3 y^6} \tag{116}
\]

as required. \( \square \)

Now, we are ready to prove Theorem 25. By Theorem 24 with \( y = 1 \), we have that

**Proof of Theorem 25**

\[
\sum_{n \geq 0} h(n)x^n = H(x, 1) = \frac{3x^2 + 2x + 1}{-2x^3 - x^2 - 3x + 1} \tag{117}
\]

which can be solved that
\[3x^2 + 2x + 1 = (-2x^3 - x^2 - 3x + 1) \sum_{n \geq 0} h(n)x^n\]
\[= -2 \sum_{n \geq 0} h(n)x^{n+3} - \sum_{n \geq 0} h(n)x^{n+2} - 3 \sum_{n \geq 0} h(n)x^{n+1} + \sum_{n \geq 0} h(n)x^n\]
\[= -2 \sum_{n \geq 3} h(n-3)x^n - \sum_{n \geq 2} h(n-2)x^n - 3 \sum_{n \geq 1} h(n-1)x^n + \sum_{n \geq 0} h(n)x^n\]
\[= h(0) + h(1)x + h(2)x^2 - 3h(0)x - 3h(1)x^2 - h(0)x^2\]
\[+ \sum_{n \geq 3} (-2h(n-3) - 3h(n-1) - h(n-2) + h(n))x^n\]  
(118)

Because the order of the polynomial on the left hand side of (118) is two, the coefficients of \(x^n\) for all \(n \geq 3\) must be 0. Thus, \(h(n) - 3h(n-1) - h(n-2) - 2h(n-3) = 0\) implying that

\[h(n) = 3h(n-1) + h(n-2) + 2h(n-3)\]

and this completes the proof. \(\Box\)

### 3.7 Para-Hexagonal Cacti

First, we may name all the vertices of \(G(n)\) as shown in Figure 23.

![Figure 23: The para-hexagonal cactus \(G(n)\) of \(n\) hexagons.](image)

Then, we recall that

\(g(n) = \) the number of all maximal independent sets of \(G(n)\)

and

\(G(n, k) = \) the number of maximal independent sets containing \(k\) vertices of \(G(n)\).

Thus,

\[G(n) = \sum_{k \geq 0} g(n, k).\]  
(119)
Further, we let
\[ G(x) = \sum_{n \geq 0} g(n) x^n \]
be the generating function of \( g(n) \) and we let
\[ G(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} g(n, k) x^n y^k \]
be the bi-variate generating function of \( g(n, k) \). It is worth noting that, when \( y = 1 \), we have
\[ G(x, 1) = \sum_{n \geq 0} \left( \sum_{k \geq 0} g(n, k)(1)^k \right) x^n = \sum_{n \geq 0} g(n) x^n = G(x). \quad (120) \]

Next, we let \( \overline{G}(n) \) be constructed from \( G(n) \) by joining two vertices to a vertex at distance three from the cut vertex of the \( n^{th} \) hexagon. Further, we let \( \tilde{G}(n) \) be constructed from \( \overline{G}(n) \) by joining two vertices to the end vertices of \( \overline{G}(n) \), one vertex each. The graphs \( \overline{G}(n) \) and \( \tilde{G}(n) \) are shown in Figures 24 and 25 respectively.

![Figure 24: The graph \( \overline{G}(n) \).](image)

![Figure 25: The graph \( \tilde{G}(n) \).](image)

Then, we let
\[ \gamma(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } \overline{G}(n), \]
\[ \tilde{\gamma}(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } \tilde{G}(n), \]
and let
\[ G(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} g(n, k) x^n y^k \]
\[ \tilde{G}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \tilde{g}(n, k) x^n y^k \]

Now, we are ready to prove Theorem 29.

**Proof of Theorem 29**

Let \( D \) be a maximal independent set of \( G(n) \) containing \( k \) vertices. We distinguish 2 cases.

**Case 1:** \( x_3^n \in D \)

Thus, \( x_3^n, x_4^n, x_5^n \notin D \). Removing the vertices \( x_3^n, x_4^n, x_5^n \) from \( G(n) \) results in \( G(n-1) \). There are \( \bar{g}(n-1, k-1) \) possibilities of \( D \).

**Case 2:** \( x_3^n \notin D \)

We further have the following 3 subcases.

**Subcase 2.1:** \( n_3^n \notin D \) and \( x_5^n \in D \).

Thus, \( x_2^n \notin D \) and \( x_6^n \in D \). This implies that \( x_1^n \notin D \). Removing all the vertices of the \( n^{th} \) hexagon results in \( G(n-2) \). We have that there are \( \bar{g}(n-2, k-2) \) possibilities of \( D \).

**Subcase 2.2:** \( n_3^n \in D \) and \( x_5^n \notin D \).

Thus, \( x_2^n \notin D \) and \( x_6^n \in D \). This implies that \( x_1^n \notin D \). Removing all the vertices of the \( n^{th} \) hexagon results in \( \tilde{G}(n-2) \). We have that there are \( \bar{g}(n-2, k-2) \) possibilities of \( D \).

**Subcase 2.3:** \( n_3^n \in D \) and \( x_5^n \in D \).

Thus, \( x_2^n, x_6^n \notin D \). Removing the vertices \( x_2^n, ..., x_6^n \) results in \( G(n-1) \). We have that there are \( g(n-1, k-2) \) possibilities of \( D \).

From the two cases, we have that
\[ g(n, k) = \bar{g}(n-1, k-1) + 2\bar{g}(n-2, k-2) + g(n-1, k-2) \] (121)

For \( n \geq 2 \) and \( k \geq 2 \), we multiply \( x^n y^k \) throughout (121) and sum over all \( x^n y^k \). Thus, we have that
\[ \sum_{n \geq 2} \sum_{k \geq 2} g(n, k) x^n y^k = \sum_{n \geq 2} \sum_{k \geq 2} \bar{g}(n-1, k-1) x^n y^k + 2 \sum_{n \geq 2} \sum_{k \geq 2} \bar{g}(n-2, k-2) x^n y^k \\
+ \sum_{n \geq 2} \sum_{k \geq 2} g(n-1, k-2) x^n y^k \] (122)

We first consider the term \( \sum_{n \geq 2} \sum_{k \geq 2} g(n, k) x^n y^k \). It can be checked that
\[ g(0, 0) = 1, g(0, 1) = 0, g(0, 2) = 0, \text{ and } g(0, k) = 0 \text{ for all } k \geq 3 \]
\[ g(1, 0) = 0, g(1, 1) = 0, g(1, 2) = 3, g(1, 3) = 2, \text{ and } g(1, k) = 0 \text{ for all } k \geq 4 \]
\[
\sum_{n \geq 2} \sum_{k \geq 2} g(n, k)x^ny^k = \sum_{n \geq 2} \sum_{k \geq 2} g(n, k)x^ny^k + g(0, 0) + g(1, 2)xy^2 + g(1, 3)xy^3 - g(0, 0) - g(1, 2)xy^2 - g(1, 3)xy^3 = \sum_{n \geq 2} \sum_{k \geq 2} g(n, k)x^ny^k - 3xy^2 - 2xy^3 = G(x, y) - 3xy^2 - 2xy^3
\]

(123)

Now, we consider the term \(\sum_{n \geq 2} \sum_{k \geq 2} \mathcal{F}(n - 1, k - 1)x^ny^k\). Clearly,

\[
\mathcal{F}(0, 0) = 0, \mathcal{F}(0, 1) = 1, \mathcal{F}(0, 2) = 1 \text{ and } \mathcal{F}(0, k) = 0 \text{ for all } k \geq 2
\]

\[
\sum_{n \geq 2} \sum_{k \geq 2} \mathcal{F}(n - 1, k - 1)x^ny^k = xy \left( \sum_{n \geq 2} \sum_{k \geq 2} \mathcal{F}(n - 1, k - 1)x^{n-1}y^{k-1} \right)
\]

\[
= xy \left( \sum_{n \geq 2} \sum_{k \geq 2} \mathcal{F}(n - 1, k - 1)x^{n-1}y^{k-1} + \mathcal{F}(0, 1)y + \mathcal{F}(0, 2)y^2 - \mathcal{F}(0, 1)y - \mathcal{F}(0, 2)y^2 \right)
\]

\[
= xy \left( \sum_{n \geq 2} \sum_{k \geq 2} \mathcal{F}(n, k)x^ny^k - y - y^2 \right)
\]

\[
= xyG(x, y) - xy^2 - xy^3.
\]

(124)

Next, we consider the term \(\sum_{n \geq 2} \sum_{k \geq 2} 2\mathcal{G}(n - 2, k - 2)x^ny^k\). It can be check that,

\[
\sum_{n \geq 2} \sum_{k \geq 2} 2\mathcal{G}(n - 2, k - 2)x^ny^k = 2x^2y^2 \left( \sum_{n \geq 2} \sum_{k \geq 2} \mathcal{G}(n - 2, k - 2)x^{n-2}y^{k-2} \right)
\]

\[
= 2x^2y^2 \left( \sum_{n \geq 0} \sum_{k \geq 0} \mathcal{G}(n, k)x^ny^k \right)
\]

\[
= 2x^2y^2 G(x, y).
\]

(125)

Finally, we consider the term \(\sum_{n \geq 2} \sum_{k \geq 2} g(n - 1, k - 2)x^ny^k\). It can be checked that,

\[
\sum_{n \geq 2} \sum_{k \geq 2} g(n - 1, k - 2)x^ny^k = xy ^2 \left( \sum_{n \geq 2} \sum_{k \geq 2} g(n - 1, k - 2)x^{n-1}y^{k-2} \right)
\]

\[
= xy^2 \left( \sum_{n \geq 2} \sum_{k \geq 2} g(n - 1, k - 2)x^{n-1}y^{k-2} + g(0, 0) - g(0, 0) \right)
\]

\[
= xy^2 \left( \sum_{n \geq 2} \sum_{k \geq 0} g(n, k)x^ny^k - 1 \right)
\]

\[
= xy^2 G(x, y) - xy^2
\]

(126)
Plugging \((123), (124), (125)\) and \((126)\) to \((122)\), we have

\[
G(x, y) - 1 - 3xy^2 - 2xy^3 = xyG(x, y) - xy^2 - xy^3 + 2x^2y^2G(x, y) + xy^2G(x, y) - xy^2
\]

which can be solved that

\[
G(x, y) = 1 + 3xy^2 + 2xy^3 + xyG(x, y) - xy^2 - xy^3 + 2x^2y^2G(x, y) + xy^2G(x, y) - xy^2.
\] (127)

Next, we will find the recurrence relation of \(G(n)\). Let \(D\) be a maximal independent set of \(G(n)\) containing \(k\) vertices. We distinguish 2 cases.

**Case 1:** \(x_3^n \in D\)

Thus, \(x_3^n, x_5^n, x_6^{n+1} \notin D\). Removing \(x_3^n, x_4^n, x_5^n, x_6^{n+1}\) from \(G(n)\) results in \(G(n - 1)\). There are \(g(n - 1, k - 1)\) possibilities of \(D\).

**Case 2:** \(x_3^n \notin D\)

By the maximality of \(D\), \(x_4^n, x_6^{n+1} \in D\). Removing \(x_4^n, x_5^n, x_6^{n+1}\) from \(G(n)\) results in \(\tilde{G}(n - 1)\). There are \(\tilde{g}(n - 1, k - 2)\) possibilities of \(D\).

From the two cases, we have that

\[
\mathcal{G}(n, k) = \mathcal{G}(n - 1, k - 1) + \tilde{g}(n - 1, k - 2) \quad (128)
\]

For \(n \geq 1\) and \(k \geq 2\), we multiply \(x^n y^k\) throughout \((128)\) and sum over all \(x^n y^k\). Thus, we have that

\[
\sum_{n \geq 1} \sum_{k \geq 2} \mathcal{G}(n, k) x^n y^k = \sum_{n \geq 1} \sum_{k \geq 2} \mathcal{G}(n - 1, k - 1) x^n y^k + \sum_{n \geq 1} \sum_{k \geq 2} \tilde{g}(n - 1, k - 2) x^n y^k
\] (129)

We first consider the term \(\sum_{n \geq 1} \sum_{k \geq 2} \mathcal{G}(n, k) x^n y^k\). It can be checked that,

\[
\mathcal{G}(0, 0) = 0, \mathcal{G}(0, 1) = 1, \mathcal{G}(0, 2) = 1 \text{ and } \mathcal{G}(0, k) = 0 \text{ for all } k \geq 3
\]

\[
\sum_{n \geq 1} \sum_{k \geq 2} \mathcal{G}(n, k) x^n y^k = \sum_{n \geq 1} \sum_{k \geq 2} \mathcal{G}(n, k) x^n y^k + \mathcal{G}(0, 1)y + \mathcal{G}(0, 2)y^2 - \mathcal{G}(0, 1)y - \mathcal{G}(0, 2)y^2
\]

\[
= \sum_{n \geq 0} \sum_{k \geq 0} \mathcal{G}(n, k) x^n y^k - y - y^2
\]

\[
= \mathcal{G}(x, y) - y - y^2
\] (130)

We next consider the term \(\sum_{n \geq 1} \sum_{k \geq 2} \mathcal{G}(n - 1, k - 1) x^n y^k\), clearly
\[
\sum_{n \geq 1} \sum_{k \geq 2} g(n-1, k-1)x^n y^k = xy \left( \sum_{n \geq 1} \sum_{k \geq 2} g(n-1, k-1)x^n y^{k-1} \right)
\]
\[
= xy \left( \sum_{n \geq 0} \sum_{k \geq 1} g(n, k)x^n y^k \right)
\]
\[
= xy \left( \sum_{n \geq 0} \sum_{k \geq 0} g(n, k)x^n y^k \right)
\]
\[
= xy \tilde{G}(x, y)
\]

Finally, we consider the term \( \sum_{n \geq 1} \sum_{k \geq 2} \tilde{g}(n-1, k-2)x^n y^k \). It can be checked that,

\[
\sum_{n \geq 1} \sum_{k \geq 2} \tilde{g}(n-1, k-2)x^n y^k = xy^2 \left( \sum_{n \geq 1} \sum_{k \geq 2} \tilde{g}(n-1, k-2)x^{n-1} y^{k-2} \right)
\]
\[
= xy^2 \tilde{G}(x, y)
\]  

(132)

Plugging (130), (131) and (132) to (129), we have

\[
\tilde{G}(x, y) - y - y^2 = xy \tilde{G}(x, y) + xy^2 \tilde{G}(x, y).
\]

(133)

which can be solved that

\[
\tilde{G}(x, y) = y + y^2 + xy \tilde{G}(x, y) + xy^2 \tilde{G}(x, y).
\]

(134)

Next, we will find the recurrence relation of \( \tilde{G} \). Let \( D \) be a maximal independent set of \( \tilde{G}(n) \) containing \( k \) vertices. We distinguish 4 cases.

Case 1: \( x_3^{n+1} \in D \) and \( x_6^{n+1} \in D \)

Thus, \( x_2^{n+1}, x_6^{n+1} \notin D \). Removing \( x_2^{n+1}, x_3^{n+1}, x_5^{n+1}, x_6^{n+1} \) from \( \tilde{G}(n) \) results in \( G(n) \). There are \( g(n, k-2) \) possibilities of \( D \).

Case 2: \( x_3^{n+1} \notin D \) and \( x_6^{n+1} \notin D \)

Thus, \( x_2^{n+1} \in D \) but \( x_6^{n+1} \notin D \). Since \( D \) is independent, \( x_1^{n+1} \notin D \). Removing \( x_1^{n+1}, x_2^{n+1}, x_3^{n+1}, x_5^{n+1}, x_6^{n+1} \) from \( \tilde{G}(n) \) results in \( \tilde{G}(n-1) \). There are \( \tilde{g}(n-1, k-2) \) possibilities of \( D \).

Case 3: \( x_3^{n+1} \in D \) and \( x_5^{n+1} \notin D \)

By similar arguments in Case 2, there are \( \tilde{g}(n-1, k-2) \) possibilities of \( D \).

Case 4: \( x_3^{n+1} \notin D \) and \( x_5^{n+1} \notin D \)

Thus, \( x_2^{n+1}, x_6^{n+1} \in D \). Since \( D \) is independent, \( x_1^{n+1} \notin D \). Removing \( x_1^{n+1}, x_2^{n+1}, x_3^{n+1}, x_5^{n+1}, x_6^{n+1} \) from \( \tilde{G}(n) \) results in \( \tilde{G}(n-1) \). There are \( \tilde{g}(n-1, k-2) \) possibilities of \( D \).

From the three cases, we have that

\[
\tilde{g}(n, k) = g(n, k-2) + 3\tilde{g}(n-1, k-2).
\]

(135)
For \( n \geq 1 \) and \( k \geq 2 \), we multiply \( x^n y^k \) throughout (135) and sum over all \( x^n y^k \). Thus, we have that

\[
\sum_{n \geq 1} \sum_{k \geq 2} \tilde{g}(n, k) x^n y^k = \sum_{n \geq 1} \sum_{k \geq 2} g(n, k - 2) x^n y^k + \sum_{n \geq 1} \sum_{k \geq 2} 3\tilde{g}(n - 1, k - 2) x^n y^k. \tag{136}
\]

We first consider the term \( \sum_{n \geq 1} \sum_{k \geq 2} \tilde{g}(n, k) x^n y^k \). It can be checked that,

\[
\tilde{g}(0, 0) = 0, \tilde{g}(0, 1) = 0, \tilde{g}(0, 2) = 3, \tilde{g}(0, 3) = 1 \text{ and } \tilde{g}(0, k) = 0 \text{ for all } k \geq 4.
\]

\[
\sum_{n \geq 1} \sum_{k \geq 2} \tilde{g}(n, k) x^n y^k = \sum_{n \geq 1} \sum_{k \geq 2} \tilde{g}(n, k) x^n y^k + \tilde{g}(0, 2) y^2 + \tilde{g}(0, 3) y^3 - \tilde{g}(0, 2) y^2 - \tilde{g}(0, 3) y^3 \\
= \sum_{n \geq 0} \sum_{k \geq 0} \tilde{g}(n, k) x^n y^k - 3y^2 - y^3 \\
= \tilde{G}(x, y) - 3y^2 - y^3 \tag{137}
\]

We next consider the term \( \sum_{n \geq 1} \sum_{k \geq 2} g(n, k - 2) x^n y^k \), clearly

\[
\sum_{n \geq 1} \sum_{k \geq 2} g(n, k - 2) x^n y^k = y^2 \left( \sum_{n \geq 1} \sum_{k \geq 2} g(n, k - 2) x^n y^{k-2} + g(0, 0) - g(0, 0) \right) \\
= y^2 \left( \sum_{n \geq 0} \sum_{k \geq 0} g(n, k) x^n y^k - 1 \right) \\
= y^2 G(x, y) - y^2 \tag{138}
\]

Finally, we consider the term \( \sum_{n \geq 1} \sum_{k \geq 2} 3\tilde{g}(n - 1, k - 2) x^n y^k \). It can be checked that,

\[
\sum_{n \geq 1} \sum_{k \geq 2} 3\tilde{g}(n - 1, k - 2) x^n y^k = 3xy^2 \sum_{n \geq 1} \sum_{k \geq 2} \tilde{g}(n - 1, k - 2) x^{n-1} y^{k-2} \\
= 3xy^2 \left( \sum_{n \geq 0} \sum_{k \geq 0} \tilde{g}(n, k) x^n y^k \right) \\
= 3xy^2 \tilde{G}(x, y) \tag{139}
\]

Plugging (137), (138) and (139) to (136), we have

\[
\tilde{G}(x, y) - 3y^2 - y^3 = y^2 G(x, y) - y^2 + 3xy^2 \tilde{G}(x, y) \tag{140}
\]
which can be solved that

\[ \tilde{G}(x, y) = 3y^2 + y^3 + y^2G(x, y) - y^2 + 3xy^2\tilde{G}(x, y) \]  \hspace{1cm} (141)

By plugging (127) to (134) to (141), we have

\[ G(x, y) = \frac{x^3y^6 - x^3y^5 + x^2y^6 - 2x^2y^5 - 3x^2y^4 + 2x^2y^3 + 2xy^3 - xy^2 - xy + 1}{1 - xy - 4xy^2 + 4x^2y^3 - x^2y^5 + x^2y^4 - x^3y^5} , \]

as required. \( \Box \)

We will prove Theorem 30.

Proof of Theorem 30. By Theorem 29 with \( y = 1 \), we have that

\[ \sum_{n \geq 0} g(n)x^n = G(x, 1) \]

\[ = \frac{1 - 2x^2}{1 - 5x + 4x^2 - x^3} \]  \hspace{1cm} (142)

which can be solved that

\[ 1 - 2x^2 = (1 - 5x + 4x^2 - x^3) \sum_{n \geq 0} g(n)x^n \]

\[ = - \sum_{n \geq 0} g(n)x^{n+3} + 4 \sum_{n \geq 0} g(n)x^{n+2} - 5 \sum_{n \geq 0} g(n)x^{n+1} + \sum_{n \geq 0} g(n)x^n \]

\[ = - \sum_{n \geq 3} g(n-3)x^n + 4 \sum_{n \geq 2} g(n-2)x^n - 5 \sum_{n \geq 1} g(n-1)x^n + \sum_{n \geq 0} g(n)x^n \]

\[ = g(0) + g(1)x + g(2)x^2 - 5g(0)x - 5g(1)x^2 + 4g(0)x^2 \]

\[ + \sum_{n \geq 3} (-g(n-3) + 4g(n-2) - 5g(n-1) + g(n))x^n \]  \hspace{1cm} (143)

Because the order of the polynomial on the left hand side of (143) is two, the coefficients of \( x^n \) for all \( n \geq 3 \) must be 0. Thus, \( g(n) - 5g(n-1) + 4g(n-2) - g(n-3) = 0 \) implying that

\[ g(n) = 5g(n-1) - 4g(n-2) + g(n-3) \]

and this proves Theorem 30. \( \Box \)

### 3.8 Ortho-Hexagonal Cacti

First, we may name all the vertices of the ortho-hexagonal cactus of \( n \) hexagons as shown by Figure 26.
Then, we recall that

\[ q(n) = \text{the number of all maximal independent sets of } Q(n) \]

and

\[ q(n, k) = \text{the number of maximal independent sets containing } k \text{ vertices of } Q(n). \]

Thus,

\[ Q(n) = \sum_{k \geq 0} q(n, k). \tag{144} \]

Further, we let

\[ Q(x) = \sum_{n \geq 0} q(n) x^n \]

be the generating function of \( q(n) \) and we let

\[ Q(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} q(n, k) x^n y^k \]

be the bi-variate generating function of \( q(n, k) \). It is worth noting that, when \( y = 1 \), we have

\[ Q(x, 1) = \sum_{n \geq 0} (\sum_{k \geq 0} q(n, k)(1)^k) x^n = \sum_{n \geq 0} q(n) x^n = Q(x). \tag{145} \]

Next, we let \( \overline{Q}(n) \) be constructed from \( Q(n) \) by joining two vertices to a vertex at distance one from the cut vertex of the \( n^{th} \) hexagons. Further, we let \( \overline{Q}(n) \) be constructed from \( Q(n) \) by joining an end

Figure 26: The graph \( Q(n) \) with labelled vertices.
vertex of a path of length three to a vertex at distance one from the cut vertex of the $n^{th}$ hexagons. The graphs $\overline{Q}(n)$ and $\tilde{Q}(n)$ are shown in Figures 27 and 28, respectively.

![Figure 27: The graph $\overline{Q}(n)$](image)

![Figure 28: The graph $\tilde{Q}(n)$.](image)

Then, we let

$\overline{Q}(n, k) =$ the number of maximal independent sets containing $k$ vertices of $\overline{Q}(n)$,

$\tilde{Q}(n, k) =$ the number of maximal independent sets containing $k$ vertices of $\tilde{Q}(n)$
and let
\[
\overline{Q}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \overline{q}(n, k)x^n y^k
\]
\[
\tilde{Q}(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} \tilde{q}(n, k)x^n y^k
\]
be the bi-variate generating functions of \(q(n, k)\) and \(\tilde{q}(n, k)\), respectively. Now, we are ready to prove Theorem 34.

**Proof of Theorem 34**

Let \(D\) be a maximal independent set of \(Q(n)\) containing \(k\) vertices. We distinguish 2 cases.

**Case 1:** \(x_3^n \in D\)

Thus, \(x_3^n, x_5^n \notin D\). Removing the vertices \(x_3^n, x_4^n, x_5^n\) from \(Q(n)\) result in \(\overline{Q}(n-1)\). There are \(\overline{q}(n-1, k-1)\) possibilities of \(D\).

**Case 2:** \(x_3^n \notin D\)

We further have the following 3 subcases.

**Subcase 2.1:** \(n_3^n \notin D\) and \(x_5^n \in D\).

Thus, \(x_3^n \notin D\) and \(x_4^n \in D\). This implies that \(x_1^n \notin D\). Removing all the vertices of the \(n^{th}\) hexagon results in \(\tilde{Q}(n-2)\). We have that there are \(\tilde{q}(n-2, k-2)\) possibilities of \(D\).

**Subcase 2.2:** \(n_3^n \in D\) and \(x_5^n \notin D\).

Thus, \(x_3^n \notin D\) and \(x_4^n \in D\). This implies that \(x_1^n \in D\). Removing all the vertices of the \(n^{th}\) hexagon results in \(\tilde{Q}(n-2)\). We have that there are \(\tilde{q}(n-2, k-2)\) possibilities of \(D\).

**Subcase 2.3:** \(n_3^n \in D\) and \(x_5^n \in D\).

Thus, \(x_3^n \in D\), \(x_5^n \notin D\). Removing the vertices \(x_2^n, ..., x_6^n\) results in \(Q(n-1)\). We have that there are \(q(n-1, k-2)\) possibilities of \(D\).

From, the two cases, we have that
\[
q(n, k) = \overline{q}(n-1, k-1) + 2\tilde{q}(n-2, k-2) + q(n-1, k-2) \tag{146}
\]

For \(n \geq 2\) and \(k \geq 2\), we multiply \(x^n y^k\) throughout (146) and sum over all \(x^n y^k\). Thus, we have that
\[
\sum_{n \geq 2} \sum_{k \geq 2} q(n, k)x^n y^k = \sum_{n \geq 2} \sum_{k \geq 2} \overline{q}(n-1, k-1)x^n y^k + 2\sum_{n \geq 2} \sum_{k \geq 2} \tilde{q}(n-2, k-2)x^n y^k + \sum_{n \geq 2} \sum_{k \geq 2} q(n-1, k-2)x^n y^k. \tag{147}
\]

We first consider the term \(\sum_{n \geq 2} \sum_{k \geq 2} q(n, k)x^n y^k\). It can be checked that
\[
q(0, 0) = 1, q(0, k) = 0 \text{ for all } k \geq 1
\]
\[
q(1, 0) = 0, q(1, 1) = 0, q(1, 2) = 3, q(1, 3) = 2, \text{ and } q(1, k) = 0 \text{ for all } k \geq 4
\]
\[
\sum_{n \geq 2} \sum_{k \geq 2} q(n, k)x^ny^k = \sum_{n \geq 2} \sum_{k \geq 2} q(n, k)x^ny^k + q(0, 0) + q(1, 2)xy^2 + q(1, 3)xy^3 - q(0, 0)
- q(1, 2)xy^2 - q(1, 3)xy^3
= \sum_{n \geq 0} \sum_{k \geq 0} q(n, k)x^ny^k - 3xy^2 - 2xy^3
= Q(x, y) - 3xy^2 - 2xy^3
\] (148)

Now, we consider the term \(\sum_{n \geq 2} \sum_{k \geq 2} \mathcal{Q}(n - 1, k - 1)x^ny^k\). Clearly,

\[
\mathcal{Q}(0, 0) = 0, \mathcal{Q}(0, 1) = 1, \mathcal{Q}(0, 2) = 1\\text{and } \mathcal{Q}(0, k) = 0 \text{ for all } k \geq 3
\]

\[
\sum_{n \geq 2} \sum_{k \geq 2} \mathcal{Q}(n - 1, k - 1)x^ny^k = xy(\sum_{n \geq 2} \sum_{k \geq 2} \mathcal{Q}(n - 1, k - 1)x^{n-1}y^{k-1})
= xy(\sum_{n \geq 2} \sum_{k \geq 2} \mathcal{Q}(n - 1, k - 1)x^{n-1}y^{k-1} + \mathcal{Q}(0, 1)y + \mathcal{Q}(0, 2)y^2
- \mathcal{Q}(0, 1)y - \mathcal{Q}(0, 2)y^2)
= xy(\sum_{n \geq 0} \sum_{k \geq 0} \mathcal{Q}(n, k)x^ny^k - y^2)
= xyQ(x, y) - xy^2 - xy^3.
\] (149)

Next, we consider the term \(\sum_{n \geq 2} \sum_{k \geq 2} 2\mathcal{Q}(n - 2, k - 2)x^ny^k\). It can be check that,

\[
\sum_{n \geq 2} \sum_{k \geq 2} 2\mathcal{Q}(n - 2, k - 2)x^ny^k = 2x^2y^2(\sum_{n \geq 2} \sum_{k \geq 2} \mathcal{Q}(n - 2, k - 2)x^{n-2}y^{k-2})
= 2x^2y^2\sum_{n \geq 0} \sum_{k \geq 0} \mathcal{Q}(n, k)x^ny^k
= 2x^2y^2Q(x, y).
\] (150)

Finally, we consider the term \(\sum_{n \geq 2} \sum_{k \geq 2} q(n - 1, k - 2)x^ny^k\). It can be checked that,

\[
\sum_{n \geq 2} \sum_{k \geq 2} q(n - 1, k - 2)x^ny^k = xy^2(\sum_{n \geq 2} \sum_{k \geq 2} q(n - 1, k - 2)x^{n-1}y^{k-2})
= xy^2(\sum_{n \geq 2} \sum_{k \geq 2} q(n - 1, k - 2)x^{n-1}y^{k-2} + q(0, 0) - q(0, 0))
= xy^2(\sum_{n \geq 0} \sum_{k \geq 0} q(n, k)x^ny^k - 1)
= xy^2Q(x, y) - xy^2
\] (151)

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Plugging (148), (149), (150) and (151) to (147), we have
\[
Q(x, y) - 1 - 3xy^2 - 2xy^3 = xyQ(x, y) - xy^2 - xy^3 + 2x^2y^2\tilde{Q}(x, y) + xy^2Q(x, y) - xy^2
\]
which can be solved that
\[
Q(x, y) = 1 + xy^2 + xy^3 + xyQ(x, y) + 2x^2y^2\tilde{Q}(x, y) + xy^2Q(x, y).
\]  

Next, we will find the recurrence relation of \(\mathcal{Q}(n)\). Let \(D\) be a maximal independent set of \(\mathcal{Q}(n)\) containing \(k\) vertices. We distinguish 2 cases.

Case 1: \(x_6^n \in D\)
Thus, \(x_3^n, x_6^n, x_2^{n+1}, x_6^{n+1} \notin D\). We further distinguish 2 cases.

Case 1.1: \(x_2^n, x_4^n \in D\)
Removing the vertices of the \(n^{th}\) hexagon and \(x_2^{n+1}, x_6^{n+1}\) from \(\mathcal{Q}(n)\) results in \(\tilde{Q}(n-2)\). There are \(\tilde{q}(n-2, k-3)\) possibilities of \(D\).

Case 1.2: \(x_3^n \in D\)
Thus, \(x_2^n, x_4^n \notin D\). Removing the vertices of the \(n^{th}\) hexagon and \(x_2^{n+1}, x_6^{n+1}\) from \(\mathcal{Q}(n)\) results in \(\tilde{Q}(n-1)\). There are \(\tilde{q}(n-1, k-2)\) possibilities of \(D\).

Case 2: \(x_6^n \notin D\)
By the maximality of \(D\), \(x_2^{n+1}, x_6^{n+1} \in D\). Removing \(x_6^n, x_2^{n+1}, x_6^{n+1}\) from \(\mathcal{Q}(n)\) results in \(\tilde{Q}(n-1)\). There are \(\tilde{q}(n-1, k-2)\) possibilities of \(D\).

From all the cases, we have that
\[
\mathcal{Q}(n, k) = \tilde{q}(n-2, k-3) + \tilde{q}(n-1, k-2) + \tilde{q}(n-2, k-2)
\]  

For \(n \geq 2\) and \(k \geq 3\), we multiply \(x^ny^k\) throughout (146) and sum over all \(x^ny^k\). Thus, we have that
\[
\sum_{n \geq 2} \sum_{k \geq 3} \mathcal{Q}(n, k) x^ny^k = \sum_{n \geq 2} \sum_{k \geq 3} \tilde{q}(n-2, k-3)x^ny^k + \sum_{n \geq 2} \sum_{k \geq 3} \tilde{q}(n-1, k-2)x^ny^k + \sum_{n \geq 2} \sum_{k \geq 3} \tilde{q}(n-2, k-2)x^ny^k
\]  

We first consider the term \(\sum_{n \geq 2} \sum_{k \geq 3} \mathcal{Q}(n, k) x^ny^k\). It can be checked that,

\[
\mathcal{Q}(0, 0) = 0, \mathcal{Q}(0, 1) = 1, \mathcal{Q}(0, 2) = 1 \text{ and } \mathcal{Q}(0, k) = 0 \text{ for all } k \geq 3
\]

\[
\mathcal{Q}(1, 0) = 0, \mathcal{Q}(1, 1) = 0, \mathcal{Q}(1, 2) = 1, \mathcal{Q}(1, 3) = 1
\]

\[
\mathcal{Q}(1, 4) = 3, \mathcal{Q}(1, 5) = 1 \text{ and } \mathcal{Q}(1, k) = 0 \text{ for all } k \geq 6
\]
\[
\sum_{n \geq 2} \sum_{k \geq 3} q(n, k)x^ny^k = \sum_{n \geq 2} \sum_{k \geq 3} q(n, k)x^ny^k + \sum_{n \geq 1} \sum_{k \geq 3} q(0, k)x^ny^k + q(0, 1)xy + q(0, 2)y^2 + q(1, 2)xy^2 + q(1, 3)xy^3
\]
\[
+ q(1, 4)xy^4 + q(1, 5)xy^5 - q(0, 1)y - q(0, 2)y^2
\]
\[
- q(1, 2)xy^2 - q(1, 3)xy^3 - q(1, 4)xy^4 - q(1, 5)xy^5
\]
\[
= \sum_{n \geq 0} \sum_{k \geq 0} q(n, k)x^ny^k - y - y^2 - xy^2 - xy^3 - 3xy^4 - xy^5
\]
\[
= \bar{Q}(x, y) - y - y^2 - xy^2 - xy^3 - 3xy^4 - xy^5
\] (156)

It can be checked that,
\[
\bar{q}(0, 0) = 0, \bar{q}(0, 1) = 0, \bar{q}(0, 2) = 3, \bar{q}(0, 3) = 1 \text{ and } \bar{q}(0, k) = 0 \text{ for all } k \geq 4
\]
\[
\bar{q}(1, 0) = 0, \bar{q}(1, 1) = 0, \bar{q}(1, 2) = 0, \bar{q}(1, 3) = 1
\]
\[
\bar{q}(1, 4) = 10, \bar{q}(1, 5) = 4 \text{ and } \bar{q}(1, k) = 0 \text{ for all } k \geq 6
\]

We next consider the term \( \sum_{n \geq 2} \sum_{k \geq 3} \bar{q}(n - 2, k - 3)x^ny^k \), clearly
\[
\sum_{n \geq 2} \sum_{k \geq 3} \bar{q}(n - 2, k - 3)x^ny^k = x^2y^3(\sum_{n \geq 2} \sum_{k \geq 3} \bar{q}(n - 2, k - 3)x^{n-2}y^{k-3})
\]
\[
= x^2y^3\bar{Q}(x, y)
\] (157)

We next consider the term \( \sum_{n \geq 2} \sum_{k \geq 3} \bar{q}(n - 1, k - 2)x^ny^k \), clearly
\[
\sum_{n \geq 2} \sum_{k \geq 3} \bar{q}(n - 1, k - 2)x^ny^k = xy^2(\sum_{n \geq 2} \sum_{k \geq 3} \bar{q}(n - 1, k - 2)x^{n-1}y^{k-2} - \bar{q}(0, 2)y^2 - \bar{q}(0, 3)y^3)
\]
\[
= xy^2\bar{Q}(x, y) - 3xy^4 - xy^5
\] (158)

Finally, we consider the term \( \sum_{n \geq 2} \sum_{k \geq 3} \bar{q}(n - 2, k - 2)x^ny^k \).
\[
\sum_{n \geq 2} \sum_{k \geq 3} \bar{q}(n - 2, k - 2)x^ny^k = x^2y^2(\sum_{n \geq 2} \sum_{k \geq 3} \bar{q}(n - 2, k - 2)x^{n-2}y^{k-2})
\]
\[
= x^2y^2\bar{Q}(x, y)
\] (159)

Plugging (156), (157), (158) and (159) to (155), we have
\[
\bar{Q}(x, y) - y - y^2 - xy^2 - xy^3 - 3xy^4 - xy^5 = x^2y^3\bar{Q}(x, y) + xy^2\bar{Q}(x, y) - 3xy^4 - xy^5
\]
\[
+ x^2y^2\bar{Q}(x, y)
\] (160)
which can be solved that

\[
\tilde{Q}(x, y) = y + y^2 + xy^2 + xy^3 + (x^2y^3 + xy^2 + x^3y^2)\tilde{Q}(x, y)
\]  \hspace{1cm} (161)

Next, we will find the recurrence relation of \( \tilde{Q}(n) \). Let \( D \) be a maximal independent set of \( \tilde{Q}(n) \) containing \( k \) vertices. We distinguish 2 cases.

**Case 1:** \( x_5^{n+1} \in D \)

Thus, \( x_4^{n+1} \notin D \). We further distinguish 2 cases.

**Case 1.1:** \( x_3^{n+1} \in D \)

Thus, \( x_3^{n} \notin D \). Removing the vertices \( x_2^{n+1}, \ldots, x_5^{n+1} \) from \( \tilde{Q}(n) \) results in \( Q(n) \). There are \( q(n, k - 2) \) possibilities of \( D \).

**Case 1.2:** \( x_3^{n+1} \notin D \)

By maximality of \( D \), \( x_2^{n+1} \in D \). Thus, \( x_3^{n} \notin D \). Removing the vertices \( x_2^{n}, x_3^{n}, x_4^{n+1} \) from \( \tilde{Q}(n) \) results in \( Q(n - 1) \). There are \( q(n - 2, k - 2) \) possibilities of \( D \).

**Case 2:** \( x_5^{n+1} \notin D \)

By the maximality of \( D \), \( x_4^{n+1} \in D \). Thus, \( x_5^{n+1} \notin D \). We further distinguish 2 cases.

**Case 2.1:** \( x_3^{n+1} \in D \)

Thus, \( x_3^{n} \notin D \). Removing \( x_2^{n}, x_3^{n}, x_4^{n+1} \) from \( \tilde{Q}(n) \) results in \( \tilde{Q}(n - 1) \). There are \( q(n - 1, k - 2) \) possibilities of \( D \).

**Case 2.2:** \( x_3^{n+1} \notin D \)

By the maximality of \( D \), \( x_4^{n-1} \in D \). Thus, \( x_5^{n} \notin D \).

**Case 2.2.1:** \( x_2^{n}, x_3^{n} \in D \)

Removing the vertices of the \( n^{th} \) hexagon and \( x_2^{n+1}, \ldots, x_6^{n+1} \) from \( \tilde{Q}(n) \) results in \( \tilde{Q}(n - 2) \). There are \( q(n - 2, k - 4) \) possibilities of \( D \).

**Case 2.2.2:** \( x_3^{n} \in D \)

Thus, \( x_2^{n}, x_4^{n} \notin D \). Removing the vertices of the \( n^{th} \) hexagon and \( x_2^{n+1}, \ldots, x_6^{n+1} \) from \( \tilde{Q}(n) \) results in \( \tilde{Q}(n - 2) \). There are \( q(n - 2, k - 3) \) possibilities of \( D \).

From all the cases, we have that

\[
\tilde{q}(n, k) = q(n, k - 2) + 2\tilde{q}(n - 1, k - 2) + \tilde{q}(n - 2, k - 3) + \tilde{q}(n - 2, k - 4)
\]  \hspace{1cm} (162)

For \( n \geq 2 \) and \( k \geq 4 \), we multiply \( x^n y^k \) throughout \( (162) \) and sum over all \( x^n y^k \). Thus, we have that

\[
\sum_{n \geq 2} \sum_{k \geq 4} \tilde{q}(n, k) x^n y^k = \sum_{n \geq 2} \sum_{k \geq 1} q(n, k - 2) x^n y^k + \sum_{n \geq 2} \sum_{k \geq 4} 2\tilde{q}(n - 1, k - 2) x^n y^k + \sum_{n \geq 2} \sum_{k \geq 4} \tilde{q}(n - 2, k - 3) x^n y^k + \sum_{n \geq 2} \sum_{k \geq 4} \tilde{q}(n - 2, k - 4) x^n y^k
\]  \hspace{1cm} (163)
We first consider the term $\sum \sum_{n \geq 2 \ k \geq 4} \tilde{q}(n,k)x^n y^k$.

$$\sum \sum_{n \geq 2 \ k \geq 4} \tilde{q}(n,k)x^n y^k = \sum \sum_{n \geq 2 \ k \geq 4} \tilde{q}(n,k)x^n y^k + \tilde{q}(0,2)y^2 + \tilde{q}(0,3)y^3$$

$$+ \tilde{q}(1,3)xy^3 + \tilde{q}(1,4)xy^4 + \tilde{q}(1,5)xy^5$$

$$- \tilde{q}(0,2)y^2 - \tilde{q}(0,3)y^3 - \tilde{q}(1,3)xy^3 - \tilde{q}(1,4)xy^4 - \tilde{q}(1,5)xy^5$$

$$= \sum \sum_{n \geq 0 \ k \geq 0} \tilde{q}(n,k)x^n y^k - 3y^2 - y^3 - 10xy^4 - 4xy^5$$

$$= Q(x,y) - 3y^2 - y^3 - 10xy^4 - 4xy^5 \quad (164)$$

We next consider the term $\sum \sum_{n \geq 2 \ k \geq 4} q(n,k-2)x^n y^k$, clearly

$$\sum \sum_{n \geq 2 \ k \geq 4} q(n,k-2)x^n y^k = y^2(\sum \sum_{n \geq 2 \ k \geq 4} q(n,k-2)x^n y^{k-2} + q(0,0) + q(1,2)xy^2 + q(1,3)xy^3$$

$$- q(0,0) - (1,2)xy^2 - q(1,3)xy^3)$$

$$= y^2(\sum \sum_{n \geq 0 \ k \geq 0} q(n,k)x^n y^k - 1 - 3xy^2 - 2xy^3)$$

$$= y^2 Q(x,y) - y^2 - 3xy^2 - 2xy^3 \quad (165)$$

We next consider the term $\sum \sum_{n \geq 2 \ k \geq 4} 2\tilde{q}(n-1,k-2)x^n y^k$, clearly

$$\sum \sum_{n \geq 2 \ k \geq 4} 2\tilde{q}(n-1,k-2)x^n y^k = 2xy^2(\sum \sum_{n \geq 2 \ k \geq 4} \tilde{q}(n-1,k-2)x^{n-1} y^{k-2}$$

$$+ \tilde{q}(0,2)y^2 + \tilde{q}(0,3)y^3 - \tilde{q}(0,2)y^2 - \tilde{q}(0,3)y^3)$$

$$= 2xy^2(\sum \sum_{n \geq 0 \ k \geq 0} \tilde{q}(n,k)x^n y^k - 3y^2 - y^3)$$

$$= 2xy^2 \tilde{Q}(x,y) - 6xy^4 - 2xy^5 \quad (166)$$

We next consider the term $\sum \sum_{n \geq 2 \ k \geq 4} \tilde{q}(n-2,k-3)x^n y^k$.

$$\sum \sum_{n \geq 2 \ k \geq 4} \tilde{q}(n-2,k-3)x^n y^k = x^2y^3(\sum \sum_{n \geq 2 \ k \geq 4} \tilde{q}(n-2,k-3)x^{n-2} y^{k-3})$$

$$= x^2y^3 \tilde{Q}(x,y) \quad (167)$$

Finally, we consider the term $\sum \sum_{n \geq 2 \ k \geq 4} \tilde{q}(n-2,k-4)x^n y^k$.
\[
\sum_{n \geq 2} \sum_{k \geq 4} \tilde{q}(n-2, k-4)x^n y^k = x^2 y^4 \left( \sum_{n \geq 2} \sum_{k \geq 4} \tilde{q}(n-2, k-4)x^{n-2} y^{k-4} \right)
\]
\[
= x^2 y^4 \tilde{Q}(x, y)
\]  

(168)

Plugging (164), (165), (166), (167) and (168) to (163), we have

\[
\tilde{Q}(x, y) - 3y^2 - y^3 - xy^3 - 10xy^4 - 4xy^5 = y^2 Q(x, y) - y^2 - 3xy^4 - 2xy^5
\]
\[
+ 2xy^2 \tilde{Q}(x, y) - 6xy^4 - 2xy^5
\]
\[
+ x^2 y^3 \tilde{Q}(x, y) + x^2 y^4 \tilde{Q}(x, y)
\]  

(169)

which can be solved that

\[
\tilde{Q}(x, y) = \frac{1}{1 - 2xy^2 - x^2 y^3 - x^2 y^4} \left( 2y^2 + y^3 + y^2 Q(x, y) + xy^3 + xy^4 \right)
\]  

(170)

By plugging (153) to (161) to (170), we have

\[
Q(x, y) = \frac{x^2 y^6 + 2xy^3 + 1}{1 - 2xy^5 - x^2 y^4 - x^2 y^3 - 3xy^2}
\]  

(171)

as required. \( \square \)

We will prove Theorem 35

Proof of Theorem 35 By Theorem 34 with \( y = 1 \), we have that

\[
\sum_{n \geq 0} q(n) x^n = Q(x, 1)
\]
\[
= \frac{x^2 + 2x + 1}{1 - 3x - 3x^2}
\]  

(172)

which can be solved that

\[
x^2 + 2x + 1 = (1 - 3x - 3x^2) \sum_{n \geq 0} q(n) x^n
\]
\[
= \sum_{n \geq 0} q(n) x^n - 3 \sum_{n \geq 0} q(n) x^{n+1} - 3 \sum_{n \geq 0} q(n) x^{n+2}
\]
\[
= \sum_{n \geq 0} q(n) x^n - 3 \sum_{n \geq 1} q(n-1) x^n - 3 \sum_{n \geq 2} q(n-2) x^n
\]
\[
= q(0) + q(1)x - 3q(0)x + \sum_{n \geq 2} (q(n) - 3q(n-1) - 3q(n-2)) x^n.
\]  

(173)
Because the order of the polynomial on the left hand side of (173) is two, the coefficients of $x^n$ for all $n \geq 3$ must be 0. Thus, $q(n) - 3q(n-1) - 3q(n-2) = 0$ implying that

$$q(n) = 3q(n-1) + 3q(n-2).$$

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