Linear Multifractional Stable Sheets in the Broad Sense: Existence and Joint Continuity of Local Times

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We introduce the notion of linear multifractional stable sheets in the broad sense (LMSS) with \(\alpha \in (0, 2]\), to include both linear multifractional Brownian sheets (\(\alpha = 2\)) and linear multifractional stable sheets (\(\alpha < 2\)). The purpose of the present paper is to study the existence and joint continuity of the local times of LMSS, and also the local Hölder condition of the local times in the set variable. Among the main results of this paper, Theorem 2.4 provides a sufficient and necessary condition for the existence of local times of LMSS; Theorem 3.1 shows a sufficient condition for the joint continuity of local times; and Theorem 4.1 proves a sharp local Hölder condition for the local times in the set variable. All these theorems improve significantly the existing results for the local times of multifractional Brownian sheets and linear multifractional stable sheets in the literature.

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1. Introduction

The purpose of this paper is to develop a unified framework for improving the results concerning the local times of multifractional Brownian sheets [12], linear fractional stable sheets [1], and linear multifractional stable sheets [16]. We will use the notion of linear multifractional stable sheets in the broad sense (LMSS), where the stability index is \(\alpha\), which controls the tail-heaviness of the distributions, is ranged in \((0, 2]\). As a consequence of the present paper, some results obtained in [1, 4, 5, 7, 12, 16, 19, 22] are extended to the setting of LMSS and improved significantly. Below we describe the main contributions of this paper:

(i). A sufficient condition for the existence of local times of multifractional Brownian sheets was given in [12], which was extended to linear multifractional stable sheets in [16]. However, their conditions are not optimal. In particular, no necessary condition had been proved for the existence of local times of multifractional Brownian sheets or linear multifractional stable sheets in the literature. We fill this gap by proving in Theorem 2.4 a sufficient and necessary condition for the existence of the local times of LMSS. This solves completely the problem on the existence of local times for multifractional Brownian sheets in [12] and linear multifractional stable sheets in [16].

(ii). In Theorem 3.1, we provide a sufficient condition for the joint continuity of the local times of LMSS, which is significantly weaker than the conditions proved in [12] for multifractional Brownian sheets and in [16] for linear multifractional stable sheets. We remark that [16] makes crucial use of the arguments in [22], which rely on the local nondeterminism property and the assumption of \(\alpha \in (1, 2]\). Our Theorem 3.1 holds for all \(\alpha \in (0, 2]\) and its proof builds upon an extension
of the direct approach in [2] for linear fractional stable sheets that can provide more precise information on the upper bound for the moments of local times than those in [16, 22].

(iii). We prove a local Hölder condition for the local times of LMSS, see Theorem 4.1. This theorem is useful for studying the local Hausdorff dimension and exact Hausdorff measure of the level sets of LMSS [5, 20]. This latter problem goes beyond the scope of the present paper and we plan to study it in a subsequent paper.

(iv). Through proving the aforementioned theorems we have extended and improved several results in the literature. These include Lemma 3.2, Lemma 3.3 and Remark 3.4, which may have their own interests.

Throughout this paper, if not specified, we adopt the following notations and assumptions:

- For $0 < \epsilon < T$, let $I := [\epsilon, T]^N$. For every $u, v \in \mathbb{R}^N$ such that $u_l \leq v_l$ ($l = 1, \ldots, N$), $[u, v]$ denotes the closed rectangle defined by $[u, v] := \prod_{l=1}^N [u_l, v_l]$.
- For $x, y \in \mathbb{R}$, define $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$.
- Denote by $\| \cdot \|$ the Euclidean norm.

Before we introduce LMSS, let us first define the linear fractional stable sheet in the broad sense (LFSS). The phrase “broad sense” refers to the fact that we allow cases of $X$ where:

Definition 1.1 (Linear fractional stable sheet in the broad sense). For any $\alpha \in (0, 2]$, any vectorial exponent $H = (h_1, \ldots, h_N) \in (0, 1)^N$, a real-valued linear fractional stable sheet in the broad sense $X_H^0 = \{ X_0^H(u), u \in \mathbb{R}_+^N \}$ is defined via the following integral representation:

$$X_0^H(u) := \int_{\mathbb{R}_{+}^N} g^H(u, v) M_\alpha \, (dv), \quad \text{for all } u \in \mathbb{R}_+^N := [0, +\infty)^N,$$

(1.1)

where:

- For $\alpha \in (0, 2)$, $M_\alpha$ denotes a rotationally invariant $\alpha$-stable random measure on $\mathbb{R}^N$ with Lebesgue control measure. When $\alpha = 2$, $M_\alpha$ stands for the standard Gaussian measure (or Gaussian white noise). See Samorodnitsky and Taqqu [15] for the definition and properties of the integral in (1.1).
- The kernel function $g^H : \mathbb{R}_{+}^N \times \mathbb{R}^N \to \mathbb{R}_+$ is defined as

$$g^H(u, v) := c_H \prod_{l=1}^N \left[ (u_l - v_l)^{h_l-1/\alpha} - (-v_l)^{h_l-1/\alpha} \right],$$

(1.2)

where $a_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$ and the normalizing constant $c_H > 0$ is chosen such that $\|X_0^H(1)\|_\alpha = 1$ (see the forthcoming (1.6) for the definition of $\| \cdot \|_\alpha$).

For any integer $d \geq 1$, an $(N, d)$-LFSS is defined by

$$X^H = \{ X^H(u), u \in \mathbb{R}_+^N \} := \{ (X_1^H(u), \ldots, X_d^H(u)), u \in \mathbb{R}_+^N \},$$

where $X_1^H, \ldots, X_d^H$ are $d$ independent copies of $X_0^H$.

Remark 1.2. When $\alpha = 2$, $X^H$ is known as a fractional Brownian sheet. When $h_1 = \ldots = h_N = 1/\alpha$, $X^H$ becomes the ordinary stable sheet, studied in [9].

Several authors have studied the sample path properties of the LFSS $X^H$. For example, Ayache et al. [2, 3] considered the local and asymptotic properties of the paths of the LFSS with $\alpha < 2$. Ayache
et al. [1] proved the existence and joint continuity of the local times of LFSS with \( \alpha < 2 \), subject to some conditions on \( d \) and \( H \). Xiao and Zhang [23] and Ayache et al. [4] studied the existence and joint continuity of the local times of fractional Brownian sheets (LFSS with \( \alpha = 2 \)), respectively. Xiao [22] proved that LFSS has the property of sectorial local nondeterminism and applied this property to study the regularity properties of the local times of LFSS. The results show that the regularity and fractal properties of LFSS \( X^H \) are determined by the stability index \( \alpha \) and the constant vector \( H = (h_1, \ldots, h_N) \). In particular, many random sets generated by \( X^H \) such as its trajectories and level sets are “monofractals”.

In order to construct more flexible stochastic models with varying local regularity and fractal properties, Stevo and Taqqu [17, 18] introduced linear multifractional stable motion and studied its stochastic and sample properties. Recently Shen et al. [16] obtained a sufficient condition for the existence of local times of the linear multifractional stable sheets with \( \alpha < 2 \). By extending the definition of linear multifractional stable motion in [16–18] to the random field setting in Definition 1.1, we define the linear multifractional stable sheet in the broad sense (LMSS) as follows.

**Definition 1.3** (Linear multifractional stable sheet in the broad sense). Let \( \alpha \in (0, 2] \) and let \( H(u) = (h_1(u), \ldots, h_N(u)) \), \( u \in \mathbb{R}^N_+ \), be a deterministic function such that

\[
0 < h_l(u) < 1 \quad \text{for all } u \in \mathbb{R}^N_+ \text{ and all } l \in \{1, \ldots, N\}.
\]

A real-valued linear multifractional stable sheet in the broad sense with Hurst functional index \( H(\bullet) \) is defined by

\[
X_0^{H(u)}(u) := \int_{\mathbb{R}^N} g^{H(u)}(u,v) M_\alpha(\text{d}v), \quad u \in \mathbb{R}^N_+,
\]

where \( M_\alpha \) is given in Definition 1.1 and, for any \( u, v \in \mathbb{R}^N_+ \), \( g^{H(u)}(u,v) \) is defined as in (1.2) with \( h_l \) replaced by \( h_l(u) \) and with its normalizing constant \( c_H(1) \) chosen to satisfy \( \|X_0^{H(1)}(1)\|_\alpha = 1 \).

For \( d \geq 1 \), an \((N, d)\)-LMSS is defined to be

\[
X^{H(\bullet)} = \{X^{H(u)}(u), u \in \mathbb{R}^N_+ \} := \left\{(X_1^{H(u)}(u), \ldots, X_d^{H(u)}(u)), u \in \mathbb{R}^N_+ \right\},
\]

where \( X_1^{H(\bullet)}, \ldots, X_d^{H(\bullet)} \) are \( d \) independent copies of \( X_0^{H(\bullet)} \).

**Remark 1.4.**

- Recall from [13, 15, 22] that, for \( \alpha \in (0, 2] \), if \( \{\tilde{X}(u), u \in \mathbb{R}^N_+ \} \) is a symmetric \( \alpha \)-stable random field having the following integral representation:

\[
\tilde{X}(u) := \int_{\mathbb{R}^N} g(u,v) M_\alpha(\text{d}v), \quad \text{for all } u \in \mathbb{R}^N_+,
\]

then for all \( a_1, \ldots, a_n \in \mathbb{R}, u^1, \ldots, u^n \in \mathbb{R}^N_+ \), the characteristic function of the joint distribution of \( (\tilde{X}(u^1), \ldots, \tilde{X}(u^n)) \) is given by

\[
\mathbb{E}\left(e^{i\sum_{j=1}^n a_j \tilde{X}(u^j)}\right) = e^{-\|\sum_{j=1}^n a_j \tilde{X}(u^j)\|_\alpha},
\]

where

\[
\|\sum_{j=1}^n a_j \tilde{X}(u^j)\|_\alpha := \int_{\mathbb{R}^N} \left|\sum_{j=1}^n a_j g(u^j, v)\right|^\alpha \text{d}v.
\]

It is worth noting that \( \|\bullet\|_\alpha \) defines an \( L^\alpha \)-norm only when \( \alpha \geq 1 \); when \( \alpha \in (0, 1) \), \( \|\bullet\|_\alpha \) does not satisfy the triangle inequality and it is called an \( L^\alpha \)-quasinorm.
• In particular when \( \tilde{X} = X_0^H(\bullet) \), the characteristic function of \( (X_0^H(u^1)(u^1), \ldots, X_0^H(u^n)(u^n)) \) is given by

\[
\mathbb{E}(e^{i \sum_{j=1}^n a_j X_0^H(u^j)(u^j)}) = e^{-\| \sum_{j=1}^n a_j X_0^H(u^j)(u^j) \|^\alpha_{\alpha}}
\]

(1.7)

for all \( a_1, \ldots, a_n \in \mathbb{R} \) and \( u^1, \ldots, u^n \in \mathbb{R}^N_+ \), where

\[
\| \sum_{j=1}^n a_j X_0^H(u^j)(u^j) \|^\alpha \equiv \int_{\mathbb{R}^N} \left| \sum_{j=1}^n a_j g^H(u^j)(u^j) \right|^{\alpha} \, dv.
\]

When \( \alpha = 2 \), \( X^H(\bullet) \) becomes a multifractional Brownian sheet [12]. Since the normalizing factor in \( g^H(\bullet) \) is chosen such that \( \| X_0^H(1) \|^2 = 1 \), we obtain that: for \( \alpha = 2 \),

\[
\| \sum_{j=1}^n a_j X_0^H(u^j)(u^j) \|^2 = \frac{1}{2} \var \left( \sum_{j=1}^n a_j g^H(u^j)(u^j) \right).
\]

• When \( H(\bullet) \equiv H \) (constant), \( X^H(\bullet) \) becomes the LFSS with Hurst index \( H \) in Definition 1.1 (see [1–3]).

Next we recall the notion of local times as in [10]. For more information on local times of Gaussian and stable processes or random fields we refer the readers to [1, 6–8, 12–14, 22] and the references therein.

Let \( Y : \mathbb{R}^N \rightarrow \mathbb{R}^d \) be a (deterministic or random) Borel vector field and let \( \lambda_N \) be the Lebesgue measure on \( \mathbb{R}^N \). For any Borel set \( I \subseteq \mathbb{R}^N \), the occupation measure of \( Y \) on \( I \) is the Borel measure on \( \mathbb{R}^d \) defined by

\[
\mu_I(\bullet) := \lambda_N \{ t \in I : Y(t) \in \bullet \}.
\]

(1.8)

**Definition 1.5** (Local time). If \( \mu_I \) is absolutely continuous with respect to the Lebesgue measure \( \lambda_d \) in \( \mathbb{R}^d \), \( Y \) is said to have local times on \( I \) and its local time \( L(\bullet, I) \) is defined to be the Radon-Nikodým derivative of \( \mu_I \) with respect to \( \lambda_d \), that is,

\[
L(x, I) := \frac{d \mu_I}{d \lambda_d}(x), \quad \text{for a.e. } x \in \mathbb{R}^d,
\]

where \( x \) and \( I \) are called the space variable and time variable, respectively.

Heuristically speaking, for any Borel set \( A \subseteq \mathbb{R}^d \), \( \mu_I(A) \) measures the amount of "time" that \( Y \) spends in \( A \) during the time period \( I \); \( L(x, I) \) measures the amount of "time" that \( Y \) spends at \( x \) during \( I \). Sometimes, we write \( L(x, t) \) in place of \( L(x, [0, t]) \).

The following remark contains two consequences of Definition 1.5. We will make use of the second observation in the proofs of the main results, Theorems 2.4 and 3.1.

**Remark 1.6.** Notice that if \( Y \) has local times on \( I \), for any Borel set \( J \subseteq I \), \( L(x, J) \) also exists. On the other hand, if \( I_1, \ldots, I_n \) is an arbitrary partition of \( I \) and \( Y \) has local times on \( I_i \) \( (i = 1, \ldots, n) \), \( Y \) admits local times \( L(x, I) \) on \( I \) and \( L(x, I) = \sum_{i=1}^n L(x, I_i) \) a.e..

**Definition 1.7** (Joint continuity of local time). Let \( Y : \mathbb{R}^N \rightarrow \mathbb{R}^d \) be a random field and let \( I = [a, b] \subseteq \mathbb{R}^N \). If the local time of \( Y \), \( L(x, [a, t]) \), is an almost surely continuous function of \( (x, t) \in \mathbb{R}^d \times [a, b] \), we say that \( Y \) has a joint locally continuous local time on \( I \).
In order to study the existence and joint continuity of the local times of LMSS, we introduce some assumptions on the smoothness of the Hurst functional index $H(\bullet)$, and assume they hold throughout the rest of the paper.

$\mathcal{H}_1$: For $l = 1, \ldots, N$, there are constants $0 < m_l < M_l < 1$ such that $m_l \leq h_l(u) \leq M_l$ for all $u \in \mathbb{R}_+^N$.

$\mathcal{H}_2$: Let $I \subseteq \mathbb{R}_+^N$ be a compact set. There is a constant $c = c(I) > 0$ such that for $l = 1, \ldots, N$,

$$|h_l(u) - h_l(v)| \leq c\rho(u,v),$$

for all $u, v \in I$,

where $\rho(u, v)$ is the metric in $\mathbb{R}^N$ defined by

$$\rho(u, v) := \sum_{l=1}^N \min \left\{|u_l - v_l|^{m_l}, |u_l - v_l|^{M_l}\right\},$$

for all $u, v \in \mathbb{R}^N$,

where $(m_1, \ldots, m_N)$ and $(M_1, \ldots, M_N) \in (0, 1)^N$ are given in $\mathcal{H}_1$.

Notice that Condition $\mathcal{H}_2$ implies that the Hurst functional index $H(\bullet)$ is continuous. Moreover, when $|u - v| \leq 1$ we have $\rho(u, v) = \sum_{l=1}^N |u_l - v_l|^{M_l}$. Therefore, the metric $\rho$ coincides locally with the metric $\rho_K$ introduced in [12].

### 2. Existence of the local times

This section is devoted to studying the existence of local times of the $(N, d)$-LMSS $\{X_t^{H(u)}(u)\}_{u \in \mathbb{R}_+^N}$. As the first main result, Theorem 2.4 derives a necessary and sufficient condition for the existence of local times. The key idea to its proof is: first, observe that the existence of local times is equivalent to

$$\int_I \int_I \|X_t^{H(u)}(u) - X_t^{H(v)}(v)\|^{-d} \, du \, dv < +\infty. \quad (2.1)$$

Next, by using the fact that $\|X_t^{H(u)}(u) - X_t^{H(v)}(v)\|_{\alpha}$ is compatible with $\sum_{l=1}^N |u_l - v_l|^{h_l(u)}$ for all $u, v \in I$ (see Lemma 2.2 below), we obtain that (2.1) is equivalent to

$$\int_I \int_I \left(\sum_{l=1}^N |u_l - v_l|^{h_l(u)}\right)^{-d} \, du \, dv < +\infty. \quad (2.2)$$

Finally, by applying Lemma 2.3 below to an argument by induction on $N$, we derive the necessary and sufficient condition for (2.2) to hold.

Before stating Theorem 2.4, we give some preliminary results that are useful in its proof. Recall the following elementary inequalities.

**Lemma 2.1.** For any $\alpha > 0$, $x_1, \ldots, x_N \in \mathbb{R}$,

$$\left| \sum_{l=1}^N x_l \right|^\alpha \leq \begin{cases} N^{\alpha-1} \sum_{l=1}^N |x_l|^\alpha, & \text{if } \alpha \geq 1; \\ \sum_{l=1}^N |x_l|^\alpha, & \text{if } 0 < \alpha < 1. \end{cases} = (N^{\alpha-1} + 1) \sum_{l=1}^N |x_l|^\alpha. \quad (2.3)$$

**Proof.** The inequalities in (2.3) hold for $\alpha \geq 1$ thanks to Jensen’s inequality; they hold for $\alpha \in (0, 1)$ due to the triangle inequality $\left| \sum_{l=1}^N x_l \right|^\alpha \leq (\sum_{l=1}^N |x_l|)^\alpha$ and the fact that the mapping $x \mapsto (x + a)^\alpha - x^\alpha - a^\alpha$ with $a > 0$ is decreasing over $x \in \mathbb{R}_+$. \qed

The lemma below describes an approximation of the increments of $X_0^{H(\bullet)}$ in the $L^\alpha$-(quasi)norm.
Lemma 2.2. Let $X_0^H(\bullet)$ be an $(N,1)$-LMSS and let $0 < \epsilon < T$ be two constants such that $|T - \epsilon|$ is sufficiently small. Then there exist constants $0 < c_{2,1} \leq c_{2,2}$ such that for all $u, v \in I = [\epsilon, T]^N$,
\[
c_{2,1} \sum_{l=1}^{N} |u_l - v_l|^{a_l(\hat{u})} \leq \|X_0^H(u) - X_0^H(v)\|_\alpha \leq c_{2,2} \sum_{l=1}^{N} |u_l - v_l|^{a_l(\hat{u})},
\]
for any $\hat{u} := (\hat{u}_1, \ldots, \hat{u}_N) \in \prod_{l=1}^{N} [u_l \wedge v_l, u_l \vee v_l]$.

**Proof.** First, it was proved in [12, Lemma 2.2] for $\alpha = 2$ and in [16, Lemma 3.2] for $0 < \alpha < 2$, that for $|T - \epsilon|$ being sufficiently small, there exist constants $0 < c_{2,3} \leq c_{2,4}$ such that for all $u, v \in I = [\epsilon, T]^N$,
\[
c_{2,3} \left( \sum_{l=1}^{N} |u_l - v_l|^{a_l(\hat{u})} \right)^{1/\alpha} \leq \|X_0^H(u) - X_0^H(v)\|_\alpha \leq c_{2,4} \left( \sum_{l=1}^{N} |u_l - v_l|^{a_l(\hat{u})} \right)^{1/\alpha},
\]
for any $\hat{u} := (\hat{u}_1, \ldots, \hat{u}_N) \in \prod_{l=1}^{N} [u_l \wedge v_l, u_l \vee v_l]$. Next using Lemma 2.1 yields
\[
(N^{\alpha-1} \wedge 1)^{-1/\alpha} \sum_{l=1}^{N} |u_l - v_l|^{a_l(\hat{u})} \leq \left( \sum_{l=1}^{N} |u_l - v_l|^{a_l(\hat{u})} \right)^{1/\alpha} \leq (N^{1/\alpha-1} \wedge 1) \sum_{l=1}^{N} |u_l - v_l|^{a_l(\hat{u})}. \tag{2.6}
\]
Finally (2.4) follows from (2.5) and (2.6). \hfill \Box

The following lemma is an extension of [21, Lemma 8.6].

**Lemma 2.3.** Let $\alpha > 0, \beta > 0$ and $0 \leq a < b$ be constants. Then for all $A > 0$ and $t_0 \in [a, b]$,
\[
\int_a^b (A + |t - t_0|^\alpha)^{-\beta} \, dt \asymp \begin{cases} 
A^{-(\beta - 1/\alpha)} & \text{if } \alpha \beta > 1, \\
\log((1 + (b - t_0)A^{-1/\alpha})(1 + (t_0 - a)A^{-1/\alpha})) & \text{if } \alpha \beta = 1, \\
1 & \text{if } \alpha \beta < 1
\end{cases} \tag{2.7}
\]
Here and below, for two positive real-valued functions $f$ and $B$ defined on a set $D$, $f \asymp B$ means that there exist $c_{2,5}, c_{2,6} > 0$ such that $c_{2,5}B(x) \leq f(x) \leq c_{2,6}B(x)$ for all $x \in D$.

**Proof.** On one hand, from [21, Lemma 8.6] we see that for any given constants $\alpha > 0, \beta > 0$,
\[
\int_0^b (A + t^\alpha)^{-\beta} \, dt \asymp \begin{cases} 
A^{-(\beta - 1/\alpha)} & \text{if } \alpha \beta > 1, \\
\log(1 + bA^{-1/\alpha}) & \text{if } \alpha \beta = 1, \\
1 & \text{if } \alpha \beta < 1
\end{cases} \tag{2.8}
\]
On the other hand, by the change of variable $u = t - t_0$, we obtain
\[
\int_a^b (A + |t - t_0|^\alpha)^{-\beta} \, dt = \int_0^{t_0 - a} (A + u^\alpha)^{-\beta} \, du + \int_0^{b - t_0} (A + u^\alpha)^{-\beta} \, du. \tag{2.9}
\]
Since $t_0 - a, b - t_0 \geq 0$, applying (2.8) to the right-hand side of (2.9) yields (2.7). \hfill \Box

As our first main result, Theorem 2.4 below provides a sufficient and necessary condition $C$ for the existence of the local times of LMSS. The condition $C$ significantly improves the sufficient conditions in [12] for multifractional Brownian sheets and in [16] for linear multifractional stable sheets with $0 < \alpha < 2$. 
Theorem 2.4. Assume $\alpha \in (0, 2]$. Let $X^H(\bullet)$ be an $(N, d)$-LMSS with Hurst functional index $H(u) = (h_1(u), \ldots, h_N(u))$ and let $I = [\epsilon, T]^N$ with $0 < \epsilon < T$. $X^H(\bullet)$ admits an $L^2(\lambda_d)$-integrable local time $L(\bullet, I)$ almost surely if and only if the following condition $C$ holds:

$$C : \quad d \leq \inf_{v \in I} \sum_{l=1}^{N} \frac{1}{h_l(v)} \quad \text{and} \quad \int_{I} \left( \sum_{l=1}^{N} \frac{1}{h_l(v)} - d \right)^{-1} dv < \infty. \quad (2.10)$$

**Proof.** By Remark 1.6, we first assume that $|T - \epsilon|$ is sufficiently small so that Lemma 2.2 is applicable. Denote by $\mu_I$ the occupation measure of $X^H(\bullet)$ on $I$ (see (1.8)). The Fourier transform of $\mu_I$ is

$$\hat{\mu}_I(\xi) = \int_{I} e^{i\langle \xi, X^H(u) \rangle} du. \quad (2.11)$$

Define

$$\mathcal{J}(I) := \mathbb{E} \int_{\mathbb{R}^d} |\hat{\mu}_I(\xi)|^2 d\xi. \quad (2.12)$$

Plugging (2.11) into (2.12) and applying Fubini’s theorem, we get

$$\mathcal{J}(I) = \int_{I} \int_{\mathbb{R}^d} \mathbb{E} \exp(i\langle \xi, X^H(u) - X^H(v) \rangle) d\xi du dv. \quad (2.13)$$

According to [10, Theorem 21.9], Theorem 2.4 is equivalent to: $\mathcal{J}(I) < \infty$ if and only if (2.10) holds. It follows from (2.13), (1.5) and the following equation: for any constants $a > 0$, $b \geq 0$, and $A > 0$,

$$\int_{-\infty}^{+\infty} |x|^b e^{-|x|^a} dx = 2^{1-b} a^{b-a} A^{-b/a}, \quad (2.14)$$

that

$$\mathcal{J}(I) = 2^d \left( \Gamma\left( \frac{1}{\alpha} + 1 \right) \right)^d \int_{I} \int_{I} \|X^H_1(u) - X^H_1(v)\|^{-d}_\alpha du dv. \quad (2.15)$$

Since $|T - \epsilon|$ is sufficiently small, by Lemma 2.2, there exist constants $c_{2.1}, c_{2.2} > 0$ such that for every $u, v \in I$ with $u \neq v$,

$$(c_{2.2} \sum_{l=1}^{N} |u_l - v_l| h_l(v))^{-d} \leq \|X^H_1(u) - X^H_1(v)\|^{-d}_\alpha \leq (c_{2.1} \sum_{l=1}^{N} |u_l - v_l| h_l(v))^{-d}. \quad (2.16)$$

It follows from (2.15) and (2.16) that

$$c_{2.7} \int_{I} \int_{I} \left( \sum_{l=1}^{N} |u_l - v_l| h_l(v) \right)^{-d} du dv \leq \mathcal{J}(I) \leq c_{2.8} \int_{I} \int_{I} \left( \sum_{l=1}^{N} |u_l - v_l| h_l(v) \right)^{-d} du dv,$$

where $c_{2.7} = (2\Gamma(1/\alpha + 1)c_{2.2})^{-d}$ and $c_{2.8} = (2\Gamma(1/\alpha + 1)c_{2.1})^{-d}$. Therefore to prove the theorem it suffices to verify

$$\int_{I} \int_{I} \left( \sum_{l=1}^{N} |u_l - v_l| h_l(v) \right)^{-d} dv du < \infty \quad \text{if and only if (2.10) holds.} \quad (2.17)$$

To this end, we will prove a more general result: for any function $\theta(\bullet)$ continuous over $I$,

$$\int_{I} \int_{I} \left( \sum_{l=1}^{N} |u_l - v_l| h_l(v) \right)^{-\theta(v)} du dv < \infty \iff \theta(v) \leq \sum_{l=1}^{N} \frac{1}{h_l(v)} \quad \text{for } v \in I$$

and

$$\int_{I} \left( \sum_{l=1}^{N} \frac{1}{h_l(v)} - \theta(v) \right)^{-1} dv < \infty. \quad (2.18)$$
Before we prove the aforementioned claim, let us fix some notations. For \( m = 1, \ldots, N \), denote by
\[
I_m := [\epsilon, T]^m, \quad \mathfrak{u}_m := (u_1, \ldots, u_m).
\]

Notice that \( I_N = I \). For \( \mathfrak{u}_m \in I_m, v \in I_N \), let
\[
A_m(\mathfrak{u}_m, v) := \sum_{l=1}^{m} |u_l - v_l|h_l(v), \quad J_m, \theta(v)(v) := \int_{I_m} (A_m(\mathfrak{u}_m, v))^{-\theta(v)} \, d\mathfrak{u}_m \quad \text{and} \quad J_m := \int_{I} J_m, \theta(v)(v) \, dv.
\]

Then the left-hand side integral in (2.18) is \( J_N \).

We now prove the following statement by using induction: for any \( m = 1, \ldots, N \),
\[
J_m < \infty \iff \theta(v) \leq \sum_{l=1}^{m} \frac{1}{h_l(v)} \text{ for } v \in I \quad \text{and} \quad \int_{I} \left( \sum_{l=1}^{m} \frac{1}{h_l(v)} - \theta(v) \right)^{-1} \, dv < \infty. \quad (2.19)
\]

Since the integral \( \int_{\{v: \theta(v) \leq 0\}} J_m, \theta(v)(v) \, dv \leq c_{2.9} < \infty \) for \( m = 1, \ldots, N \), the set \( \{ v: \theta(v) \leq 0 \} \) does not affect the statement.

**Step 1:** Consider first the case \( m = 1 \).

Notice that in order for \( J_1 < \infty \), we necessarily have \( \theta(v) \leq 1/h_1(v) \) for all \( v \in I \). This is because if \( \theta(v) > 1/h_1(v) \) for some \( v \in I \), then by the continuity of \( h_1 \) there exists a vector \( \delta \in (0, \infty)^N \) with equal-valued coordinates such that \( \theta(u)h_1(u) > 1 \) for all \( u \in I \cap [v - \delta, v + \delta] \). As a result,
\[
J_1 \geq \int_{I([v - \delta, v + \delta])} J_1, \theta(u)(u) \, du = \int_{I([v - \delta, v + \delta])} \int_{\mathfrak{u}} h_1(v) \, du = \int_{I([v - \delta, v + \delta])} \int_{\mathfrak{u}} h_1(v) \, du = \infty,
\]
where \( u_1 \) denotes the first coordinate of \( u \). Hence we may assume \( \theta(v) \leq 1/h_1(v) \) for all \( v \in I \). Subject to this constraint, we can write \( I = \mathcal{V} \cup \mathcal{V}_0 \), with
\[
\mathcal{V} := \{ v \in I : \theta(v)h_1(v) < 1 \} \quad \text{and} \quad \mathcal{V}_0 := \{ v \in I : \theta(v)h_1(v) = 1 \}.
\]

Then two cases follow.

**Case 1:** \( \mathcal{V} \) is dense in \( I \), i.e., \( \overline{\mathcal{V}} = I \).

Since the Lebesgue measures of the open sets \( \mathcal{V} \) and \( I \) are equal, we have
\[
\int_{I} J_1, \theta(v)(v) \, dv = \int_{\mathcal{V}} J_1, \theta(v)(v) \, dv \quad \text{and} \quad \int_{I} \left( \frac{1}{h_1(v)} - \theta(v) \right)^{-1} \, dv = \int_{\mathcal{V}} \left( \frac{1}{h_1(v)} - \theta(v) \right)^{-1} \, dv.
\]

For all \( v \in \mathcal{V} \), we can write
\[
J_1, \theta(v)(v) = \frac{(v_1 - \epsilon)^{1-\theta(v)} h_1(v) + (T - v_1)^{1-\theta(v)} h_1(v)}{1 - \theta(v) h_1(v)}.
\]

Using (2.21) and the fact that \( h_1(v) \in (m_1, M_1) \) for all \( v \in \mathcal{V} \), there exist \( c_{2.10}, c_{2.11} > 0 \) such that for all \( v \in \mathcal{V} \),
\[
c_{2.10} \left( \frac{1}{h_1(v)} - \theta(v) \right)^{-1} \leq J_1, \theta(v)(v) \leq c_{2.11} \left( \frac{1}{h_1(v)} - \theta(v) \right)^{-1}.
\]

It follows from (2.20) and (2.22) that
\[
c_{2.10} \int_{I} \left( \frac{1}{h_1(v)} - \theta(v) \right)^{-1} \, dv \leq \int_{I} J_1, \theta(v)(v) \, dv \leq c_{2.11} \int_{I} \left( \frac{1}{h_1(v)} - \theta(v) \right)^{-1} \, dv.
\]

Therefore in Case 1,
\[
J_1 < \infty \iff \theta(v) \leq \frac{1}{h_1(v)} \text{ for } v \in I \text{ and } \int_{I} \left( \frac{1}{h_1(v)} - \theta(v) \right)^{-1} \, dv < \infty.
\]

**Case 2:** \( \mathcal{V} \) is not dense in \( I \).
In this case, \( \mathcal{V}_0 \) is a closed and non-empty set. Let us first show the interior \( \mathcal{V}_0 \neq \emptyset \). If \( \mathcal{V}_0 = \emptyset \), then for every \( \nu_0 \in \mathcal{V}_0 \), there is a sequence \( \{v_k\}_{k \geq 1} \) in \( \mathcal{V} \) such that \( v_k \to \nu_0 \) as \( k \to \infty \). Therefore \( \nu_0 \in \mathcal{V} \). This implies that \( \mathcal{V}_0 \subseteq \mathcal{V} \) and \( \mathcal{V} \) is dense in \( I \), which is a contradiction.

Now, since \( \mathcal{V}_0 \neq \emptyset \), there exist \( \nu_0 \in \mathcal{V}_0 \) and \( \delta \in (0, \infty)^N \) with equal-valued coordinates such that \( I \cap [\nu_0 - \delta, \nu_0 + \delta] \subseteq \mathcal{V}_0 \). Consequently,

\[
\int_I J_1, \theta(v)(v) \, dv \geq \int_{I \cap [\nu_0 - \delta, \nu_0 + \delta]} \int_{\varepsilon}^T |\varepsilon - v_1|^{-\theta(v)h_1(v)} \, d\varepsilon \, dv = \infty.
\]

Therefore \( J_1 = \infty \) in Case 2.

Combining Cases 1 and 2, we obtain that \( J_1 < \infty \) implies

\[
\theta(v) = \frac{1}{h_1(v)} \quad \text{for } v \in I \quad \text{and} \quad \int_I \left( \sum_{l=1}^n \frac{1}{h_l(v)} - \theta(v) \right)^{-1} \, dv < \infty
\]

and \( \mathcal{V} \) is dense in \( I \). This proves the necessity part. In the other direction, we see that by using similar argument in Case 2, (2.23) implies that \( \mathcal{V} \) should be dense in \( I \). Then it follows from Case 1 that \( J_1 < \infty \). Therefore, we have shown that \( J_1 < \infty \) if and only if (2.23) is satisfied.

**Step 2:** Assume that for some \( n \in \{1, \ldots, N-1\} \),

\[
J_n = \int_I J_n, \theta(v)(v) \, dv < \infty \quad \iff \quad \theta(v) \leq \frac{1}{h_1(v)} \quad \text{for } v \in I \quad \text{and} \quad \int_I \left( \sum_{l=1}^n \frac{1}{h_l(v)} - \theta(v) \right)^{-1} \, dv < \infty
\]

Now we consider \( J_{n+1} \). By applying (2.7), we have for any \( v \in I \),

\[
J_{n+1}, \theta(v)(v) = \int_{\nu_n+1}^{\nu_{n+1}} (A_{n+1}(\nu_{n+1}, v))^{-\theta(v)} \, d\nu_{n+1}
\]

\[
= \int_I \left\{ \int_{\varepsilon}^T (A_n(\nu_n, v) + |\nu_{n+1} - v_n|^{h_{n+1}(v)} - \theta(v)) \, d\nu_{n+1} \right\} \, d\nu_n
\]

\[
= \begin{cases} 
J_n \left( A_n(\nu_n, v) \right)^{-(\theta(v) - 1/h_{n+1}(v))} \, d\nu_n & \text{if } \theta(v)h_{n+1}(v) > 1, \\
J_n \log \left( \left( 1 + (T - v_n + 1) \right) A_n(\nu_n, v) \right)^{-1/h_{n+1}(v)} \, d\nu_n & \text{if } \theta(v)h_{n+1}(v) = 1, \\
J_n, 1 \, d\nu_n & \text{if } \theta(v)h_{n+1}(v) < 1,
\end{cases}
\]

\[
= \begin{cases} 
J_n, \theta(v) - 1/h_{n+1}(v) \, d\nu_n & \text{if } \theta(v)h_{n+1}(v) > 1, \\
J_n \log \left( \left( 1 + (T - v_n + 1) \right) A_n(\nu_n, v) \right)^{-1/h_{n+1}(v)} \, d\nu_n & \text{if } \theta(v)h_{n+1}(v) = 1, \\
(T - \varepsilon)^n & \text{if } \theta(v)h_{n+1}(v) < 1.
\end{cases}
\]

From (2.25) we see that \( \int_{\{\theta(v)h_{n+1}(v) \leq 1\}} J_{n+1}, \theta(v)(v) \, dv < \infty \). Hence \( J_{n+1} < \infty \) if and only if

\[
\int_{\{\theta(v)h_{n+1}(v) > 1\}} J_{n+1}, \theta(v)(v) \, dv < \infty.
\]

By (2.25) and the remark below (2.19), we see that (2.26) is equivalent to

\[
\int_I J_{n}, \theta(v) - 1/h_{n+1}(v) \, dv < \infty.
\]

Replacing \( \theta(v) \) in the induction hypothesis (2.24) with \( \theta(v) - 1/h_{n+1}(v) \) yields that, (2.27) holds if and only if

\[
\theta(v) - 1/h_{n+1}(v) \leq \sum_{l=1}^n \frac{1}{h_l(v)} \quad \text{for } v \in I \quad \text{and} \quad \int_I \left( \sum_{l=1}^n \frac{1}{h_l(v)} - \left( \theta(v) - \frac{1}{h_{n+1}(v)} \right) \right)^{-1} \, dv < \infty.
\]
We conclude that (2.19) and thus (2.18) are proved. Taking \( \theta(\bullet) \equiv d \) in (2.18) yields (2.17). Theorem 2.4 is proved for \( |T - \epsilon| > 0 \) being sufficiently small.

Finally we consider an arbitrary \( I = [\epsilon, T]^{\mathbb{N}} \) and let \( I_1, \ldots, I_P \) be an arbitrary partition (rectangles) of \( I \) such that the size of each \( I_i \) is sufficiently small. According to Remark 1.6 and the fact that

\[
d \leq \inf_{v \in I_i} \sum_{l=1}^{N} \frac{1}{h_l(v)} \quad \text{and} \quad \int_{I_i} \left( \sum_{l=1}^{N} \frac{1}{h_l(v)} - d \right)^{-1} \, dv < \infty, \quad \text{for all } i = 1, \ldots, P,
\]

is equivalent to

\[
d \leq \inf_{v \in I} \sum_{l=1}^{N} \frac{1}{h_l(v)} \quad \text{and} \quad \int_{I} \left( \sum_{l=1}^{N} \frac{1}{h_l(v)} - d \right)^{-1} \, dv < \infty.
\]

Hence, Theorem 2.4 holds for arbitrary \( I \). The proof is complete.

**Remark 2.5.** It is easy to see that \( C \) is equivalent to either the following condition \( C_1 \) or \( C_2 \) holds:

\[
C_1: \quad d < \inf_{v \in I} \sum_{l=1}^{N} \frac{1}{h_l(v)}.
\]

\[
C_2: \quad d = \inf_{v \in I} \sum_{l=1}^{N} \frac{1}{h_l(v)} \quad \text{and} \quad \int_{I} \left( \sum_{l=1}^{N} \frac{1}{h_l(v)} - d \right)^{-1} \, dv < \infty.
\]

The integral constraint in \( C_2 \) is some requirement on the convergence rate for the function \( v \mapsto \sum_{l=1}^{N} 1/h_l(v) \) to approach its infimum on \( I \). It requires the function \( v \mapsto \sum_{l=1}^{N} 1/h_l(v) \) to be “rough enough” around its minimizers in \( I \). We can see that linear fractional stable sheets do not satisfy \( C_2 \). As a result our Theorem 2.4 includes [1, Theorem 2.2] as a particular case. In Table 1 below we compare our results to the literature ones in more detail. From the table we see that our Theorem 2.4 improves the sufficient conditions in [12, Corollary 3.2] for multifractional Brownian sheets and in [16, Theorem 3.1] for linear multifractional stable sheets (\( \alpha < 2 \)).

**Table 1.** Summary of conditions for the existence of local times of \((N, d)\)-LSS.

| Reference | \((N, d)\)-LSS Type | \( \alpha \) | Condition Type | Condition |
|-----------|---------------------|-------------|----------------|-----------|
| Xiao and Zhang (2002) [23, Theorem 3.6] | Fractional Brownian sheets | 2 | Sufficient | \( C_1 \) |
| Appleche, Roueff and Xiao (2007) [1, Theorem 2.2] | Linear fractional stable sheets | \((0, 2]\) | Suf. & nec. | \( C_1 \) |
| Meerschaert, Wu and Xiao (2008) [12, Corollary 3.2] | Multifractional Brownian sheets | 2 | Sufficient | \( C_1 \) |
| Shen, Yu and Li (2020) [16, Theorem 3.1] | Linear multifractional stable sheets | \((0, 2]\) | Sufficient | \( d < \sum_{l=1}^{N} 1/\sup_{v \in I} h_l(v) \) |
| Ding, Peng and Xiao (2022), Theorem 2.4 | LMSS | \((0, 2]\) | Suf. & nec. | \( C_1 \) or \( C_2 \) |

Below we provide a simple example to illustrate how the conditions for the existence of local times to be derived. Consider an \((N, d)\)-LSS with

\[
\alpha \in (0, 2], \quad N = 1 \quad \text{and} \quad h_1(v) = \frac{1}{m} - (v - q)^k \quad \text{for} \quad v \in I = \left[ q, \frac{1}{m} \right],
\]

where the integer \( m \geq 2 \) and the real numbers \( q \geq 0, \; k > 0 \) are chosen to satisfy \( h_1(1/m) > 0 \).

If \( k \in (0, 1) \), from [12, Corollary 3.2] and [16, Theorem 3.1] we know that \( d < \inf_{v \in I} 1/h_1(v) = m \) is a sufficient condition for the existence of local times on \( I \). As an improvement, Theorem 2.4 yields that \( d \leq m \) is a sufficient and necessary condition, because in this case, either \( C_1 \) or \( C_2 \) is satisfied: we have either \( d < m \) or \( d = m \) with

\[
\int_{I} \left( \frac{1}{h_1(v)} - d \right)^{-1} \, dv = \int_{q}^{1/m} \left( \frac{1}{h_1(v)} - \frac{1}{h_1(q)} \right)^{-1} \, dv = \frac{1}{m} \int_{q}^{1/m} \left( \frac{1}{m(v - q)^k} - 1 \right) \, dv < \infty.
\]
If \( k \geq 1 \), it is easy to see that \( C_2 \) can not hold, therefore by Theorem 2.4, the sufficient and necessary condition becomes \( C_1 : d < m \).

3. Joint continuity of the local times

In this section we obtain that the assumption  \( C_1 \) in Remark 2.5 is also a sufficient condition for the joint continuity of the local times of \((N,d)\)-LMSS, which is significantly weaker than the ones in [12, Theorem 3.4] and [16, Theorem 3.2] for multifractional Brownian sheets and linear multifractional stable sheets, respectively. The main result is stated below.

**Theorem 3.1.** Assume \( \alpha \in (0, 2] \). Let \( X^{H(\bullet)} \) be an \((N,d)\)-LMSS. It has a jointly continuous local time on \( I := [\epsilon, T]^N \), provided \( C_1 : d < \inf_{v \in I} \sum_{i=1}^N 1/h_i(v) \).

The proof of Theorem 3.1 will be based on a multiparameter version of Kolmogorov’s continuity theorem (cf. [11]) and estimates on the higher-order moments of the local times of \( X^{H(\bullet)} \) (see Lemmas 3.6 and 3.9). The proofs of Lemmas 3.6 and 3.9 are technical, as they require a careful control of the upper and lower bounds for the weighted sum of the elements in \( X_0^{H(\bullet)} \) in the \( L^\alpha \)-(quasi)norm.

Similarly to [12, 16, 22], we decompose \( X_0^{H(\bullet)} \) into sum of independent multifractional sheets \( Y_1, Y_2, \) and \( Z_l, l = 1, \ldots, N \) (see (3.3)) and control their bounds separately. The new idea in this paper is that, instead of using the property of local nondeterminism, we extend the direct approach in [2] for linear fractional stable sheets to LMSS which allows us to derive more precise information on the upper bound for the moments of local times than those in [12, 16, 22].

Denote by

\[
g_l(u_l, v_l) := c_{H(1)}^{l/N} \left( (u_l - v_l)^{h_l(u) - 1/\alpha} - (-u_l)^{h_l(u) - 1/\alpha} \right).
\]

In terms of (1.3) and (3.1), for any \( u \in \mathbb{R}^N_+ \), we can write

\[
X_0^{H(u)}(u) = \int_{(-\infty,u[0,u]} \prod_{l=1}^N g_l(u_l, v_l) M_{\alpha}(dv) + Y^{H(u)}(u),
\]

where \( Y^{H(u)}(u) := \int_{[0,u]} \prod_{l=1}^N g_l(u_l, v_l) M_{\alpha}(dv) \). For any \( u = (u_1, \ldots, u_N) \in [\epsilon, T]^N \) and \( l = 1, \ldots, N \), denote by \( R_l^1(u) := [0, \epsilon] \) and \( R_l^2(u) := (\epsilon, u_l] \). Hence the rectangle \([0, u]\) can be decomposed into the union of disjoint sub-rectangles:

\[
[0, u] = \bigcup_{i_1, \ldots, i_N \in \{1, 2\}} \prod_{l=1}^N R_l^{i_l}(u) \cup \bigcup_{i_1, \ldots, i_N \in \{1, 2\}} \prod_{l=1}^N R_l^{i_l}(u) \cup \bigcup_{i_1, \ldots, i_N \in \{1, 2\}} \prod_{l=1}^N R_l^{i_l}(u) \]

\[
= [0, \epsilon]^N \cup \bigcup_{l=1}^N Q_l(u) \cup Q(u),
\]

where \( Q_l(u) := R_1^1(u) \times \ldots \times R_{l-1}^1(u) \times R_l^2(u) \times R_{l+1}^1(u) \times \ldots \times R_N^1(u) \) and \( Q(u) \) is the union of \( 2^N - N - 1 \) disjoint sub-rectangles:

\[
Q(u) := \bigcup_{i_1, \ldots, i_N \in \{1, 2\}} \prod_{l=1}^N R_l^{i_l}(u).
\]
Thus, we can write
\[ Y^H(u)(u) = Y_1(u) + \sum_{l=1}^N Z_l(u) + Y_2(u), \] (3.2)
where
\[ Y_1(u) := \int_{[0,t]^N} g^H(u,v)M_\alpha(\mathrm{d}v); \quad Y_2(u) := \int_{Q(t)} g^H(u,v)M_\alpha(\mathrm{d}v); \]
\[ Z_l(u) := \int_{Q_l(u)} g^H(u,v)M_\alpha(\mathrm{d}v), \text{ for } l = 1,\ldots,N, \] (3.3)
with \( g^H(u,v) \) being defined in (1.2). We claim that the random fields \( Y_1, Y_2, \) and \( Z_l (1 \leq l \leq N) \) are independent since they are defined over disjoint sets. This together with (3.2) leads to the following result: for \( a_j \in \mathbb{R}, u^j \in I \ (j = 1,\ldots,n), \)
\[ \left\| \sum_{j=1}^n a_j X^0_\alpha(u^j)(\omega) \right\|^\alpha \geq \left\| \sum_{j=1}^n a_j Y^H(u^j)(\omega) \right\|^\alpha \geq \sum_{l=1}^N \left\| \sum_{j=1}^n a_j Z_l(u^j) \right\|^\alpha. \] (3.4)
Thanks to (3.4), the random fields \( Z_l, l = 1,\ldots,N \) play a key role in studying the joint continuity of local times of \( X^H(\bullet). \)

Lemma 3.2 below is an extension of (2.14) to multivariate integral and it is used to derive the forthcoming Lemma 3.3. Its proof is given in Appendix A.

**Lemma 3.2.** Let \( \alpha \in (0,2], n \geq 1, b_1,\ldots,b_n \geq 0 \) and let the upper triangle matrix \((a_{i,j})_{i,j=1,\ldots,n}\) satisfy \( a_{i,i} \neq 0 \) for \( i = 1,\ldots,n \) and \( a_{i,j} = 0 \) for \( j < i \). Then the following inequality holds:
\[ \int_{\mathbb{R}^n} \left( \prod_{i=1}^n |x_i|^{b_i} \right) \frac{1}{\sum_{j=1}^n |a_{i,j}|x_j}^\alpha \mathrm{d}x_1 \ldots \mathrm{d}x_n \leq c_{3,1}(n) \left( \prod_{i=1}^n a_{i,i} \right)^{-1} \prod_{i=1}^n \sum_{j=1}^n |a_{i,j}|^{b_i}, \] where \((a_{i,j})_{i,j=1,\ldots,n}\) is the inverse matrix of \((a_{i,j})_{i,j=1,\ldots,n}\) and
\[ c_{3,1}(n) = \left( \prod_{i=1}^n (a_i^{b_i-1} \vee 1) \right) \left( \frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha}{\alpha})} \right)^n \left( \sup_{1 \leq j_m \leq n \in \{1,\ldots,n\}} 1 + \frac{\sum_{i=1}^n b_i}{\alpha} \right)^n. \] (3.5)

By applying Lemma 3.2, a crucial inequality related to the weighted sum of \( Z_l(u^j), j = 1,\ldots,n \) in the \( L^\alpha \)-quasinorm is obtained in (3.6) in Lemma 3.3 below. This inequality is essential for estimating high-order moments of the local times of LMSS in Lemma 3.6. For multifractional Brownian sheets (\( \alpha = 2 \)), a result similar to (3.6) was proved in [12, Equations (3.25) and (3.29)]. For linear fractional stable sheets with \( 1 < \alpha < 2 \), it follows from the proof of [22, Equation (4.37)]. A similar inequality was also obtained in [16, Equations (3.33) and (3.34)] for linear multifractional stable sheets. But the argument in [16] makes use of the notions of “metric projection” and “orthogonality” in [22], which relies on the assumption of \( 1 < \alpha < 2 \). Moreover, because the dependence of \( C \) on \( n \) in [16, Equation (3.22)] is not described, their Lemma 3.6 is not strong enough for proving Theorem 3.3 in [16]. Our proof of the inequality (3.6) in Lemma 3.3 is based on an extension of the direct approach in [2] and provides more precise information on the constant \( c_{3,2}(n) \) in (B.17). As a special case of Lemma 3.3, we derive in Remark 3.4 that the constant in (3.7) is of the form \( c_{3,3}^n \). This is crucial for proving the local Hölder condition for the local times in Section 4. We provide the proof of Lemma 3.3 in Appendix B.
Lemma 3.3. Assume $\alpha \in (0, 2]$. For any $n \geq 1$, $b_1, \ldots, b_n \geq 0$, $l \in \{1, \ldots, N\}$, and $u^j \in [\varepsilon, T]^N$ ($j = 1, \ldots, n$) with $0 = u^0_1 \leq u^1_1 \leq \ldots \leq u^n_1$, we have:

$$
\int_{\mathbb{R}^n} \left( \prod_{i=1}^n |x_i|^{b_i} \right) e^{-\|x_i^nZ_{l}(u^j)\|^\alpha} \, dx_1 \ldots dx_n \leq c_{3,2}(n) \left( \prod_{j=1}^n (u^j_1 - u^j_1 - 1)^{-h_l(u^j)}(1 + \sum_{i=1}^n b_i) \right), \tag{3.6}
$$

where $c_{3,2}(n) > 0$ does not depend on $u^j$, $j = 1, \ldots, n$ and its expression is given in (B.17).

Remark 3.4. If $b_1 = \ldots = b_n = 0$, we use the convention $\prod_{i \in \emptyset}(\bullet) \equiv 1$ to observe

$$
\prod_{i \in \{1, \ldots, n\}} \sum_{j \neq 0} |u_{i,j}|^{b_i} = 1
$$

and $c_{3,1}(n) = ((2/\alpha)\Gamma(1/\alpha))^n$ in (3.5). Then $c_{3,2}(n) = c_{3,3}^{n}$ in (B.17) for some $c_{3,3} > 0$ independent of $n$. Hence (3.6) becomes

$$
\int_{\mathbb{R}^n} e^{-\|x_i^nZ_{l}(u^j)\|^\alpha} \, dx \leq c_{3,3}^{n} \prod_{j=1}^n |u^j_1 - u^j_1 - 1|^{-h_l(u^j)}. \tag{3.7}
$$

Note that (3.7) has been obtained in [2] for LFSS with $\alpha \geq 1$. Our result extends it to LMSS with $\alpha \in (0, 2]$. We will make use of (3.7) in the proofs of Lemma 3.6 and Theorem 4.1 (see Section 4).

The lemma below is also used in the proof of Lemmas 3.6 and 3.9; it can be found in [12, Lemma 2.10] or [4, Lemma 3.4].

Lemma 3.5. Let $(\vartheta_1, \ldots, \vartheta_N) \in (0, 1)^N$. For any $q \in [0, \sum_{i=1}^N \vartheta_i^{-1})$, let $\tau \in \{1, \ldots, N\}$ be the unique integer such that $\sum_{i=1}^{\tau q-1} \vartheta_i \leq q < \sum_{i=1}^{\tau q} \vartheta_i$, with the convention that $\sum_{i=1}^{0} \vartheta_i := 0$. Then there exists a positive constant $\Delta_\tau \leq 1$, depending only on $(\vartheta_1, \ldots, \vartheta_N)$, such that for every $\Delta \in (0, \Delta_\tau)$, we can find real numbers $p_1, \ldots, p_\tau \geq 1$ satisfying:

$$
\sum_{i=1}^\tau \frac{1}{p_i} = 1, \quad \frac{\vartheta q}{p_i} < 1 \quad \text{for all } i = 1, \ldots, \tau \quad \text{and} \quad (1 - \Delta) \sum_{i=1}^\tau \frac{\vartheta q}{p_i} \leq \vartheta q + \tau - \sum_{i=1}^\tau \frac{\vartheta q}{p_i}. \tag{3.8}
$$

Moreover, let $\alpha_{\tau} := \sum_{i=1}^\tau 1/\vartheta_i - q$, then for any $\kappa \in (0, \alpha_{\tau}/(2\tau))$, there is $l_0 \in \{1, \ldots, \tau\}$ such that

$$
\vartheta_{l_0} \left( \frac{q}{p_{l_0}} + 2\kappa \right) < 1. \tag{3.9}
$$

We apply Remark 3.4 in the proof of Lemma 3.6 and apply Lemma 3.3 in the proof of Lemma 3.9 below. Lemma 3.6 improves [12, Lemma 3.5], through obtaining a smaller upper bound for the $n$th moment of the local times of LMSS under a weaker condition. This upper bound is useful for proving a sharp local H"{o}lder condition on the local times (Theorem 4.1 in Section 4) and for studying fractal properties of the level sets of LMSS.

Lemma 3.6. Assume $\alpha \in (0, 2]$ and $C_1 : d < \inf_{v \in I} \sum_{i=1}^N 1/h_l(v)$. Denote by $\mathcal{S}(N)$ the group of permutations of $\{1, \ldots, N\}$. For each $\sigma \in \mathcal{S}(N)$, let

$$
\sigma(H(v)) := \{ h_{\sigma(1)}(v), \ldots, h_{\sigma(N)}(v) \}, \quad v \in I. \tag{3.10}
$$

Also denote by

$$
\gamma(H(v)) := \min \left\{ m \in \{1, \ldots, N\} : d < \frac{1}{\sum_{i=1}^m 1/h_i(v)} \right\}. \tag{3.11}
$$
Then, for every integer \( n \geq 1 \), \( x \in \mathbb{R}^d \), and \( I_{a, \delta} = \prod_{l=1}^{N} [a_l, a_l + \delta] \subseteq I \) with \( \delta \in (0, 1] \) sufficiently small, we have

\[
E[L(x, I_{a, \delta})^n] \leq c_{3.4}(n) \delta^{3n},
\]

(3.12)

where \( c_{3.4}(n) \) is given in (3.30) and \( \overline{\beta} = \sup_{v \in I_{a, \delta}, \sigma \in \mathcal{S}(N)} \beta(\sigma(H(v))) \) with

\[
\beta(\sigma(H(v))) := N - \gamma(\sigma(H(v))) + h(\gamma(\sigma(H(v))))(\sum_{l=1}^{\infty} \frac{1}{h_{\sigma(l)}(v)} - d).
\]

(3.13)

**Proof.** By [10, Equation (25.5)], we have: for all \( x, y \in \mathbb{R}^d \), all Borel sets \( J \subseteq I \), and all integers \( n \geq 1 \),

\[
E[L(x, J)^n] = (2\pi)^{-nd} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \sum_{j=1}^{n} (v^j - x^j)^2} E\left[ e^{i \sum_{j=1}^{n} (v^j, X^H(u^j) - 0)} \right] dv du,
\]

(3.14)

where \( \bar{v} := (v^1, \ldots, v^n) \), \( \bar{u} := (u^1, \ldots, u^n) \) and \( v^j := (v_{1j}, \ldots, v_{dj}) \in \mathbb{R}^d \), \( u^j := (u_{1j}, \ldots, u_{dj}) \in I \) for each \( j = 1, \ldots, n \). By (3.14), the fact that the coordinate processes \( X^H(u^1), \ldots, X^H(u^n) \) are independent and identically distributed, and (1.7), we have

\[
E[L(x, I_{a, \delta})^n] \leq (2\pi)^{-nd} \int_{I_{a, \delta}}^d \prod_{k=1}^{d} Q_k(\mathbf{v}) d\mathbf{v},
\]

(3.15)

where

\[
Q_k(\mathbf{v}) := \int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_{j=1}^{n} v^j_k Z_l(u^j)||_\alpha^\alpha} d\mathbf{v}_k, \quad \text{with} \quad \mathbf{v}_k := (v_{1k}, \ldots, v_{nk}) \in \mathbb{R}^n.
\]

Let \( u^* \in I_{a, \delta} \) be fixed, but arbitrary. Since \( d < \inf_{v \in I} \sum_{l=1}^{N} 1/h_l(v) \leq \inf_{v \in I_{a, \delta}} \sum_{l=1}^{N} 1/h_l(v) \), (3.11) guarantees that the choice of \( \gamma(H(u^*)) \) is unique. Observe that, by (3.4) and the fact that \( \gamma(H(u^*)) \leq N \), we have for \( k \in \{1, \ldots, d\} \),

\[
Q_k(\mathbf{v}) \leq \int_{\mathbb{R}^n} e^{-\sum_{l=1}^{N} v_{lk} Z_l(u^l)||_\alpha^\alpha} d\mathbf{v}_k \leq \int_{\mathbb{R}^n} e^{-\sum_{l=1}^{N} \gamma(H(u^*)) v_{lk} Z_l(u^l)||_\alpha^\alpha} d\mathbf{v}_k
\]

(3.16)

By (3.16) and the generalized Hölder’s inequality we get

\[
Q_k(\mathbf{v}) \leq \prod_{l=1}^{d} \left[ \int_{\mathbb{R}^n} e^{-\sum_{j=1}^{n} v_{lj} \pi_l(u^j)||_\alpha^\alpha} d\mathbf{v}_k \right]^{1/p_l} \leq \prod_{l=1}^{d} \left[ \int_{\mathbb{R}^n} e^{-p_l \sum_{j=1}^{n} (u_{lj}^{\pi_l(j)} - u_{lj}^{\pi_l(j-1)}) - h_l(u^*^{\pi_l(j)})/p_l} \right]^{1/p_l}
\]

(3.17)

where \( p_1, \ldots, p_n(\gamma(\pi^*)) \geq 1 \) satisfy \( \sum_{l=1}^{N} \gamma(H(u^*))/p_l = 1 \). They are chosen as in Lemma 3.5 with \( q = d \) and \( \theta_l = h_l(u^*) \). Applying (3.7) to (3.17) yields

\[
Q_k(\mathbf{v}) \leq c_{3.5}^{n} \prod_{l=1}^{N} \prod_{j=1}^{\max_{m \in \{1, \ldots, N\}} (c_{3.5} \theta_l^{1/\alpha})^{1/p_l}} \left( u_{lj}^{\pi_l(j)} - u_{lj}^{\pi_l(j-1)} - h_l(u^{\pi_l(j)})/p_l \right)
\]

(3.18)

where \( c_{3.5} = \max_{m \in \{1, \ldots, N\}} \prod_{l=1}^{N} \left( c_{3.3} \theta_l^{1/\alpha} \right)^{1/p_l} \) and, for each \( l \in \{1, \ldots, N\} \), \( \pi_l \in \mathcal{S}(n) \) satisfies

\[
a_l \leq u_{lj}^{\pi_l(1)} \leq \cdots \leq u_{lj}^{\pi_l(n)} \leq a_l + \delta \quad \text{and} \quad \pi_l(0) = 0.
\]

For each \( l = 1, \ldots, N \), define

\[
\Pi_l := \{(u_{lj}^{1}, \ldots, u_{lj}^{n}) \in [a_l, a_l + \delta]^n : a_l \leq u_{lj}^{\pi_l(1)} \leq \cdots \leq u_{lj}^{\pi_l(n)} \leq a_l + \delta \}.
\]

(3.19)
Due to $\mathcal{H}_2$ and the fact that $|u_i^l - u_i^l| \leq \delta \leq 1$ for all $u_i^l, u_i^l \in [a_l, a_l + \delta]$, we have: for $u^* \in I_{a,\delta}$,

\[(u_i^l - u_i^l)^{\gamma_l(u_i^l)} - h_l(u_i^l) \leq (u_i^l - u_i^l)^{-h_l(u_i^l)} - h_l(u_i^l) \leq (u_i^l - u_i^l)^{(h_l(u_i^l) + c_0(\delta))} \quad (3.20)\]

where

\[c_0(\delta) := c \sum_{l=1}^{N_d} d_{hl} > 0, \quad (3.21)\]

with $c > 0$ being the constant in $\mathcal{H}_2$. Combining (3.15) - (3.21), we derive:

\[
E[L(x, I_{a,\delta})^n] \leq c_{3,6}^n \sum_{\pi_1, \ldots, \pi_N \in S(n)} \int_{\Pi_1 \times \cdots \times \Pi_N} \left( \prod_{l=1}^{n} \prod_{j=1}^{\gamma_l(H(u^*))} (u_i^l - u_i^l)^{\gamma_l(u_i^l)} - dh_l(u_i^l) \right)^{p_l} d\pi
\]

\[
\leq c_{3,6}^n \delta^n (N - \gamma(H(u^*))) \times \prod_{\pi_1, \ldots, \pi_N(H(u^*)) \in S(n)} \left( \prod_{l=1}^{n} \prod_{j=1}^{\gamma_l(H(u^*))} (u_i^l - u_i^l)^{\gamma_l(H(u^*))} - dh_l(u_i^l) \right)^{p_l} d\pi, \quad (3.22)\]

where $c_{3,6} = (2\pi)^{-\frac{d}{3}}d_{3,5}^d$. Next consider the following integral in (3.22):

\[
I_{\pi_l} := \int_{\Pi_l} \prod_{j=1}^{n} (u_i^l - u_i^l)^{\gamma_l(u_i^l)} - dh_l(u_i^l) \left( \frac{d(h_l(u_i^l) + c_0(\delta))}{p_l} \right)^{p_l} d\pi_l. \quad (3.23)\]

Recall that, by Lemma 3.5, the real numbers $p_1, \ldots, p_{\gamma_l(H(u^*))}$ chosen in (3.17) also satisfy $\frac{d(h_l(u_i^l) + c_0(\delta))}{p_l} < 1$ for $l = 1, \ldots, \gamma_l(H(u^*))$, we can then choose $c_0(\delta)$ small enough such that

\[
\frac{d(h_l(u_i^l) + c_0(\delta))}{p_l} < 1 \quad \text{for all } l = 1, \ldots, \gamma_l(H(u^*)), \quad (3.24)\]

Then apply [12, Lemma 2.11] (or [4, Lemma 3.6]) to (3.23) to obtain: there is a constant $c_{3,7}(l, u^*) > 0$ depending only on $l$ and $u^*$ (continuously) such that

\[
I_{\pi_l} \leq c_{3,7}^n(l, u^*)(n) \delta^n \left( \frac{d(h_l(u_i^l) + c_0(\delta))}{p_l} \right)^{p_l} \delta^n (N - \frac{1}{\delta})^{\gamma_l(H(u^*))} \frac{d(h_l(u_i^l) + c_0(\delta))}{p_l}, \quad (3.25)\]

Combining (3.22) and (3.25) yields

\[
E[L(x, I_{a,\delta})^n] \leq c_{3,8}(n)(n) \left(1 - \frac{1}{\delta}\right) \sum_{l=1}^{\gamma_l(H(u^*))} \frac{d(h_l(u_i^l) + c_0(\delta))}{p_l} \delta^n (N - \frac{1}{\delta})^{\gamma_l(H(u^*))} \frac{d(h_l(u_i^l) + c_0(\delta))}{p_l}, \quad (3.26)\]

where

\[
c_{3,8}(n) = c_{3,7}^n 2^{N^m} \sup_{u \in I} \left\{ \prod_{l=1}^{m} c_{3,7}^n(l, u)(n) \delta^{-n} \frac{d(h_l(u_i^l) + c_0(\delta))}{p_l} \right\}, \quad (3.27)\]
does not depend on $\delta$. It is worth noting that the above $\sup_{\delta \in [0,1]} \delta^{-c_0/\delta} < \infty$, thanks to the fact that $\lim_{\delta \to 0} \delta^{-c_0} = 1$. Applying Lemma 3.5 with $\Delta = n^{-1}, q = d, \vartheta_l = 1$, we obtain

\[
\left(1 - \frac{1}{n}\right) \sum_{l=1}^{\Delta} \frac{b_l(u^*)}{p_l} \leq h_\gamma(H(u^*)) (u^*) d + \gamma(H(u^*)) - \sum_{l=1}^{\Delta} \frac{b_l(H(u^*)) (u^*)}{h_l(u^*)}.
\] (3.28)

Therefore, (3.26) together with (3.28) yields

\[
\mathbb{E}[L(x, I_{a, \delta})^n] \leq c_{3,8}(n) (n!)^{(N-\beta(H(u^*)))} \delta^{n \beta(H(u^*))} \leq c_{3,4}(n) \delta^{n \beta(H(u^*))},
\] (3.29)

where $\beta(\bullet)$ is defined in (3.13) and

\[
c_{3,4}(n) = c_{3,8}(n) \sup_{u \in I} \left\{ (n!)^{N-\beta(H(u))} \right\}.
\] (3.30)

In (3.29), since $u^*$ can be arbitrarily chosen in $I_{a, \delta}$, and the $h_l(\bullet)$’s in $H(\bullet)$ can be arbitrarily ordered, taking the infimum over $u^* \in I_{a, \delta}$ and $\sigma \in \mathcal{S}(N)$ on both hands sides of (3.29) leads to (3.12). Therefore, Lemma 3.6 is proved.

**Remark 3.7.** For each fixed $n \geq 1$, if we let $c_0(\delta) \leq 1/n$ in Lemma 3.6, we obtain $c_{3,8}(n) \leq c_{3,9}^n$ in (3.27) for some $c_{3,9} > 0$, thanks to Stirling’s formula. As a result, Lemma 3.6 becomes

\[
\mathbb{E}[L(x, I_{a, \delta})^n] \leq c_{3,9}^n \sup_{u \in I} \left\{ (n!)^{N-\beta(H(u))} \right\} \delta^{n \beta(H(u^*))}.
\] (3.31)

This observation will be used in the proof of Theorem 4.1. Now we compare the moment estimates in (3.12) and (3.29) with those in [4, 12, 16]. When either (i) LMSS is reduced to LFSS or (ii) $H_\gamma(H(u^*)) (u^*) = \sup_v \in \mathcal{I}_{a, \delta} H_\gamma(H(v)) (v)$, we can replace $c_0(\delta)$ in (3.20) by 0. As a consequence, $c_{3,8}(n) \leq c_{3,9}^n$. Hence (3.29) includes [4, Equation (3.38)] for fractional Brownian sheets and [12, Equation (3.16)] for multifractional Brownian sheets where $H_\gamma(H(u^*)) (u^*)$ is replaced with $\sup_v \in \mathcal{I}_{a, \delta} H_\gamma(H(v)) (v)$ as special cases. However, a stronger condition $d < \sum_{l=1}^{N} 1/\sup_v \in \mathcal{I}_{a, \delta} h_l(v)$ than that in Lemma 3.6 was assumed in [12]. In [16, Equation (3.22)], a result similar to Lemma 3.6 was also proved for LMSS, but the dependence of $C$ on $n$ was not described there. As a consequence, the estimate (3.22) in [16] was not strong enough for proving the claimed Theorem 3.3 in [16]. With (3.30) our Lemma 3.6 fills this gap, which is important for proving the local Hölder condition in Theorem 4.1, where for each $n \geq 1$, we will take $c_0(\delta) \geq 2^{1/n} \leq 1/n$.

**Remark 3.8.** For the proof of Theorem 3.1, we will make use of the multiparameter version of Kolmogorov’s continuity theorem (cf. [11]) and only moments of the local times of large but fixed order $n$ will be needed. It is sufficient to use the following simpler variant of Lemma 3.6. In the last inequality of (3.16), we replace $\gamma(H(u^*))$ with $N$ and apply the generalized Hölder’s inequality to the second inequality in (3.17) with $N$ positive numbers $p_1, \ldots, p_N$ defined by $p_l := \sum_{l=1}^{N} h_l(u^*) / h_p(u^*), l = 1, \ldots, N$. Then following the same proof in Lemma 3.6, we have, for every interval $I_{a, \delta} = \prod_{l=1}^{N} \{ a_l, a_l + \delta \} \subseteq I$, (3.26) becomes

\[
\mathbb{E}[L(x, I_{a, \delta})^n] \leq c_{3,10}(n) \sum_{l=1}^{N} \frac{b_l(u^*)}{p_l} \prod_{l=1}^{N} \delta^{n(1 - b_l(u^*) / p_l)} = c_{3,10}(n) (n!)^{N \nu} \left( \lambda_N(I_{a, \delta}) \right)^{n(1 - \nu)},
\] (3.31)

where $\nu := d / (\sum_{l=1}^{N} 1 / h_l(u^*))$, $c_{3,10}(n) > 0$ does not depend on $\delta$.

Lemma 3.9 below is another key step leading to Theorem 3.1. We provide its proof in Appendix C.
Lemma 3.9. Let $X^H(\bullet)$ be an $(N, d)$-LMSS and assume $d < \inf_{v \in I} \sum_{l=1}^{N} 1/h_l(v)$. Then for any integer $n \geq 1$, there exists $c_{3.11}(n) > 0$ such that for any subintervals $I_a, \delta = \prod_{l=1}^{N} [a_l, a_l + \delta] \subseteq I$ with $\delta \in (0, 1)$ small enough, any $x, y \in \mathbb{R}^d$ with $|x - y| \leq 1$,

$$E[|L(x, I_a, \delta) - L(y, I_a, \delta)|^n] \leq c_{3.11}(n) \inf_{v \in I_a, \delta} \sigma(S(N)) \left\{|x - y|^{n \kappa_n(\sigma(H(v)))} \delta^n (\beta(\sigma(H(v))) - (n-1)) h_\gamma(\sigma(H(v))) \kappa_n(\sigma(H(v)))\right\},$$

(3.32)

where $\sigma(H(v)), \gamma(H(v)), \beta(H(v))$ are defined in (3.10), (3.11), and (3.13), respectively; for each $v \in I_a$, $\kappa_n(H(v))$ (depending on $n$) is some real number satisfying

$$n \kappa_n(H(v)) \in \left(0, 1, \frac{\alpha(H(v))}{2\gamma(H(v))}\right) \text{ with } \alpha(H(v)) := \sum_{l=1}^{N} \frac{1}{h_l(v)} - d.$$

(3.33)

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $I = [\epsilon, T]^N$ and $d < \inf_{v \in I} \sum_{l=1}^{N} 1/h_l(v)$. It follows from Lemma 3.9 and the multiparameter version of Kolmogorov’s continuity theorem in [11] that for every $I_a, \delta \subseteq I$, the LMSS $\{X^H(u)(u)\}_{u \in \mathbb{R}^N}$ has almost surely a local time $L(x, I_a, \delta)$ that is continuous for all $x \in \mathbb{R}^d$.

To prove the joint continuity, observe from Lemma 2.1 that for all $x, y \in \mathbb{R}^d$ and $s, t \in I$ such that $|s - t| > 0$ small enough, we have for $n \geq 1$,

$$E[|L(x, I_a, s, a) - L(y, I_a, t, a)|^n] \leq 2^{n-1} \left(E[|L(x, I_a, s, a) - L(x, I_a, t, a)|^n + E[|L(x, I_a, t, a) - L(y, I_a, t, a)|^n].

(3.34)

The term $L(x, I_a, s, a) - L(x, I_a, t, a)$ in (3.34) can be rewritten as a sum of a finite number (which only depends on $N$) of terms of $L(x, I_j)$, where each $I_j$ is a closed subinterval of $I$ satisfying $\lambda_N(I_j) \leq c|s - t|$, with $c > 0$ not depending on $s, t$. Then for $|s - t|$ small enough, we apply (3.31) to bound it as $E[|L(x, I_j)|^n] \leq c_{3.12}(j, n) (\lambda_N(I_j))^{n \nu_1} \leq c_{3.13}(j, n)|s - t|^{n \nu_1}$, where $c_{3.12}(j, n), c_{3.13}(j, n) > 0$ are constants that do not depend on the edge lengths of $I_j$ and $\nu_1 \in (0, 1)$. Hence the first term in (3.34) can be bounded as $\mathbb{E}[|L(x, I_a, s, a) - L(x, I_a, t, a)|^n] \leq c_{3.14}(n)|s - t|^{n \nu_2}$. On the other hand, the difference $L(x, I_a, s, a) - L(y, I_a, t, a)$ in (3.34) can be rewritten as a sum of a finite number of terms of $L(x, I_j) - L(y, I_j)$, where each $I_j$ is a closed subinterval of $I_a$ satisfying $\lambda_N(I_j)$ is small enough. Then each term can be bounded by Lemma 3.9 as $E[|L(x, I_j) - L(y, I_j)|^n] \leq c_{3.15}(n)|x - y|^{n \nu_2}$, where $\nu_2 \in (0, 1)$. Therefore, there exist $\nu_1 \in (0, 1)$ and $c_{3.16}(n) > 0$ such that (3.34) yields $E[|L(x, I_a, s, a) - L(y, I_a, t, a)|^n] \leq c_{3.16}(n)(|x - y| + |s - t|)^{n \nu}$. Again by the multiparameter version of Kolmogorov’s continuity theorem, the joint continuity of the local times on $I$ holds. The proof is complete.

4. Local Hölder condition for the local times

For any fixed $x \in \mathbb{R}^d$, let $L(x, \bullet)$ be the local time of the $(N, d)$-LMSS $\{X^H(u)(u)\}_{u \in \mathbb{R}^N}$ at $x$. When the local time is jointly continuous, $L(x, \bullet)$ can be extended to be a measure supported by the level set $\Gamma_x = \{u \in \mathbb{R}_+^N : X^H(u)(u) = x\}$. Hence, the following theorem on the local oscillation of $L(x, \bullet)$ is useful for studying the fractal properties of $\Gamma_x$. See, e.g., [10, 12, 19, 21]. Compared with [12, Theorem 4.3] for multifractional Brownian sheets, the condition of our Theorem 4.1 is sharper, which can be applied to derive more precise information on the Hausdorff measure of $\Gamma_x$. A similar result for linear multifractional stable sheets was stated in [16, Theorem 3.3] and it was claimed that it would follow from their Lemma 3.6. As we mentioned earlier, because the dependence of $C$ on $n$ in [16, Equation
Based on \((\text{cupation density formula): for any Borel function} \ g\)

\[ L(x, U(t, r)) = \frac{1}{\varphi(r)} \text{ for } L(x, \bullet) \text{-almost all } t \in I, \]

(4.1)

where \(U(t, r)\) is the open ball in \(I\) with center \(t \in I\) and radius \(r > 0\), and the scaling function \(\varphi(r) := r^{\beta(H(t))} (\log(r^{-1}))^{N-\beta(H(t))}\), for \(0 < r < e^{-1}\), with \(\beta(H(t))\) being defined in (3.13).

**Proof.** For every integer \(k > 0\), define the random measure \(L_k(x, \bullet)\) on the Borel subset \(C\) of \(I\) to be

\[ L_k(x, C) := (2\pi)^{-d} \int_C e^{-i\langle x, \xi \rangle} \frac{1}{\varphi(r)} \, d\xi. \]

(4.2)

According to [10, Theorem 6.4], the local times have a measurable modification that satisfies the occupation density formula: for any Borel function \(g(t, x) \geq 0\) on \((t, x) \in I \times \mathbb{R}^d\),

\[ \int_I g(t, X(t)) \, dt = \int_{\mathbb{R}^d} \int_I g(t, x) L(x, dt) \, dx. \]

(4.3)

Based on (4.3), we can obtain

\[ \int_C e^{i\langle x, \xi \rangle} \, d\xi = \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} L(x, C) \, dx. \]

(4.4)

Since the right hand-side of (4.4) is the characteristic function of a random variable with density \(L(x, C)\), by the inversion theorem we can derive

\[ L(x, C) = (2\pi)^{-d} \int_C \frac{1}{\varphi(r)} \, d\xi. \]

Now by the continuity of the mapping \(y \mapsto L(y, C)\), we have \(L_k(x, C) \overset{a.s.}{\underset{k \rightarrow \infty}{\longrightarrow}} L(x, C)\) for every Borel set \(C \subseteq I\). Define \(f_n(t) := L(x, U(t, 2^{-m}))\), \(m \geq 1\). From the proof of Theorem 3.1, we can see that almost surely the functions \(f_n\)'s are continuous and bounded. Hence, by the Lebesgue’s dominated convergence theorem, for all integers \(m, n \geq 1\),

\[ \int_I (f_n(t))^m L_k(x, dt) \overset{a.s.}{\underset{k \rightarrow \infty}{\longrightarrow}} \int_I (f_n(t))^n L(x, dt). \]

(4.5)

It results from (4.2), (4.5) and the proof of Proposition 3.1 in [14] that for each integer \(n \geq 1\),

\[ \mathbb{E} \int_I (f_n(t))^n L(x, dt) \]

\[ = \frac{1}{(2\pi)^{(n+1)d}} \int_{\mathbb{R}^{n+1, 2-m}} e^{i \sum_{j=1}^{n+1} \langle x, u_j \rangle} \mathbb{E} e^{i \sum_{j=1}^{n+1} \langle u_j, X(t) \rangle} \, d\nu \]

\[ \leq \frac{1}{(2\pi)^{(n+1)d}} \int_{\mathbb{R}^{n+1, 2-m}} \prod_{k=1}^d Q_k(\varphi) \, d\varphi, \]

(4.6)

where \(Q_k(\varphi) := \int_{\mathbb{R}^{n+1}} e^{-\| \sum_{j=1}^{n+1} u_k X_j^H(s_j) \|_2^2} \, du_k\), with \(\varphi := (s^1, \ldots, s^{n+1}) \in \mathbb{R}^{(n+1)N}\) and \(\varphi := (u^1, \ldots, u^{n+1}) \in \mathbb{R}^{(n+1)d}\). In the following, we provide an upper bound of the right-hand side of (4.6) for sufficiently large \(m\), by modifying the proof of (3.12) in Lemma 3.6. For consistency, we use the same notations as in the proof of Lemma 3.6.
For \( l = 1, \ldots, N \), denote \( U_l(s^{n+1}, 2^{-m}) := [s_l^{n+1} - 2^{-m}, s_l^{n+1} + 2^{-m}] \) as the projection of \( U(s^{n+1}, 2^{-m}) \) onto the \( l \)-th dimension. For each \( l = 1, \ldots, N \) and each permutation \( \pi_l \in S(n+1) \), define

\[
\Pi_l := \left\{ (s_1^{\pi_l}, \ldots, s_l^{\pi_l}) \in (U_l(s^{n+1}, 2^{-m}))^n \times [\epsilon, T] : s_l^{\pi_l(1)} \leq \ldots \leq s_l^{\pi_l(n+1)} \right\} \neq \emptyset,
\]

with the convention that \( s_l^{\pi_l(0)} = s_l^0 := 0 \). For each \( l = 1, \ldots, N \), let \( j_n \in \{1, \ldots, n+1\} \) be the unique integer such that \( \pi_l(j_n) = n + 1 \), we then define

\[
\Pi_l^- := \left\{ (s_1^{\pi_l(1)}, \ldots, s_l^{\pi_l(j_n-1)}) \in (U_l(s^{n+1}, 2^{-m}))^{j_n-1} : s_l^{\pi_l(1)} \leq \ldots \leq s_l^{\pi_l(j_n-1)} \leq s_l^{n+1} \right\},
\]

\[
\Pi_l^+ := \left\{ (s_1^{\pi_l(j_n)}, \ldots, s_l^{\pi_l(n+1)}) \in (U_l(s^{n+1}, 2^{-m}))^{n-j_n+1} : s_l^{n+1} \leq s_l^{\pi_l(j_n)} \leq \ldots \leq s_l^{\pi_l(n+1)} \right\}.
\]

It results from (3.16), (3.17), and Remark 3.4 that

\[
Q_k(\mathcal{F}) \leq c_{4,2}^n \prod_{l=1}^{n+1} \prod_{j=1}^{l-1} (s_l^{\pi_l(j)} - s_l^{\pi_l(j-1)})^{-h_l(s^{\pi_l(j)})/p_l(H(s^{n+1}))},
\]

where \( c_{4,2} > 0 \) does not depend on \( n, \mathcal{F}, \gamma(H(s^{n+1})) \) and \( p_1(H(s^{n+1})), \ldots, p_{\gamma(H(s^{n+1}))}(H(s^{n+1})) \geq 1 \) satisfy \( \sum_{l=1}^{\gamma(H(s^{n+1}))} 1/p_l(H(s^{n+1})) = 1 \). Combining (4.6) - (4.9), we have

\[
\mathbb{E} \int_t [f_m(t)]^n L(x, dt)
\]

\[
\leq c_{4,3}^n \sum_{\pi_1, \ldots, \pi_N \in S(n)} \prod_{l=1}^{n+1} \prod_{j=1}^{l-1} (s_l^{\pi_l(j)} - s_l^{\pi_l(j-1)})^{-h_l(s^{\pi_l(j)})/p_l(H(s^{n+1}))} ds_l^{\pi_l(1)} \ldots ds_l^{\pi_l(j_n-1)}
\]

\[
\leq c_{4,3}^n \sum_{\pi_1, \ldots, \pi_N \in S(n)} \int_{[\epsilon, T]^n} (2^{n-mN})^{N-\gamma(H(s^{n+1}))} \prod_{l=1}^{n+1} (s_l^{\pi_l(j)} - s_l^{\pi_l(j-1)})^{-h_l(s^{\pi_l(j)})/p_l(H(s^{n+1}))} ds_l^{\pi_l(1)} \ldots ds_l^{\pi_l(j_n-1)}
\]

\[
\times \left\{ \int_{\Pi_1^-} \prod_{j=1}^{j_n-1} (s_l^{\pi_l(j)} - s_l^{\pi_l(j-1)})^{-h_l(s^{\pi_l(j)})/p_l(H(s^{n+1}))} ds_l^{\pi_l(1)} \ldots ds_l^{\pi_l(j_n-1)} \right\} ds_l^{j_n+1} \ldots ds_l^{n+1},
\]

where \( c_{4,3} = (c_{4,2}/(2\pi))^{d} \). Similar to (3.22) - (3.25), for sufficiently large \( m \) and letting \( \delta(m) = 2^{1-m} \), we obtain

\[
\int_{\Pi_1^+} \prod_{j=j_n+1}^{n+1} (s_l^{\pi_l(j)} - s_l^{\pi_l(j-1)})^{-h_l(s^{\pi_l(j)})/p_l(H(s^{n+1}))} ds_l^{\pi_l(1)} \ldots ds_l^{\pi_l(n+1)}
\]

\[
\leq c_{4,4}^{n-j_n+1}(l)((n-j_n+1)!) \frac{d_{s_l^{\pi_l(n+1)}+\epsilon_0(\delta(m))}}{p_l(H(s^{n+1}))} - 1
\]

\[
\times (2^{-m})^{(n-j_n+1)(1-(1-1/(n-j_n+1))d_{s_l^{\pi_l(n+1)}+\epsilon_0(\delta(m)))})
\]
and
\[
\int_{\Pi_1^n} \prod_{j=1}^n (s_l^{\pi_l(j)} - s_l^{\pi_l(j-1)}) \frac{d\nu_l(s_l^{\pi_l(j)})}{\nu_l(H^{\pi_l(j)+1})} \, ds_l^{\pi_l(1)} \ldots ds_l^{\pi_l(n)} \\
\leq c_{4.5}^{-1}(l) (\log n)^{-1} \frac{(\log n + c_0(\delta(m)))}{\nu_l(H^{\pi_l(n)+1})}.
\]  
(4.12)

We then use the bounds in (4.11) and (4.12) and the mean value theorem to obtain
\[
\int_{[\epsilon, T]^N} \prod_{l=1}^n \left\{ \int_{\Pi_1^{n+1}} \prod_{j=1}^{n+1} (s_l^{\pi_l(j)} - s_l^{\pi_l(j-1)}) \frac{d\nu_l(s_l^{\pi_l(j)})}{\nu_l(H^{\pi_l(j)+1})} \, ds_l^{\pi_l(1)} \ldots ds_l^{\pi_l(n+1)} \right\} \, ds_l^{n+1} \ldots ds_l^{n+1} \\
\leq (T - \epsilon)^N \prod_{l=1}^n \left\{ \frac{\gamma(H(u^*))}{c_{4.6}(l)(n!)} \sum_{i=1}^n \frac{d\nu_l(s_l^{\pi_l(i)})}{\nu_l(H^{\pi_l(i)}))} \, ds_l^{\pi_l(1)} \ldots ds_l^{\pi_l(n+1)} \right\}.
\]  
(4.13)

where \( u^* \in I \) is some element depending on \( m, n \).

We now take \( n = \lfloor \log m \rfloor \), where \( \lfloor \cdot \rfloor \) denotes the integer part. With this choice, the terms depending on \( c_0(\delta(m)) \) in the right-hand side of (4.13) could be upper bounded by a constant which does not depend on \( m \):
\[
c_{4.7} = \sup_{m \geq 1} \left\{ \left( n! 2^{mn} \right) c_0(\delta(m)) \right\} < +\infty.
\]  
(4.14)

The above supremum exists, thanks to the fact that
\[
\lim_{m \to +\infty} (n! 2^{mn}) c_0(\delta(m)) = \lim_{m \to +\infty} (n! 2^{en} n^{2(1-e^n)} = 1.
\]

It follows from (4.10), (4.13), (4.14) and the similar arguments to (3.25) - (3.28) that
\[
\mathbb{E} \int_I (f_m(t))^n L(x, \, dt) \leq c_{4.8}^n (n!)^{N-\beta(H(u^*))} 2^{-mn \delta(H(u^*))},
\]  
(4.15)

where
\[
c_{4.8} = \max \left\{ 2^{2Nc_{4.3}c_{4.7}c_{4.8}} \sup_{k \in \{1, \ldots, N\}} \left\{ \prod_{l=1}^k c_{4.6}(l) \right\}, 1 \right\}
\]
does not depend on \( n \). We again point out that obtaining the scaling constant \( c_{4.8}^n \) in (4.15) is crucial for deriving the value of \( \tau \) below.

Let \( \tau > 0 \) be a constant, the value of which will be determined later. We consider the random set
\[
I_m = \{ t \in I : f_m(t) \geq \tau \phi_{u^*}(2^{-m}) \}.
\]
Denote by \( \mu_{\omega} \) the restriction of the random measure \( L(x, \bullet) \) to \( I \), that is, \( \mu_{\omega}(E) = L(x, E \cap I) \) for all Borel set \( E \subseteq \mathbb{R}_+^N \). Since \( n = \lfloor \log m \rfloor \), following the same approach in [12, proof of Theorem 4.3] and
applying the crucial inequality (4.15) and Stirling’s formula, we have

\[ E \mu_\omega(I_m) \leq \mathbb{E} \int L(x, dt) \leq \frac{c_{4,8}(\tau e^{2m})^{N-\beta(H(2^m))}2^{-mn(1-H(2^m))}}{\tau^{N-\beta(H(2^m))}2^{-mn(1-H(2^m))}} \leq m^{-2}, \]

provided \( \tau > 0 \) is chosen large enough, say, \( \tau \geq c_{4,9} := c_{4,8}e^2 \). This implies

\[ \mathbb{E} \left[ \sum_{m=1}^{\infty} \mu_\omega(I_m) \right] \leq \sum_{m=1}^{\infty} m^{-2} < +\infty. \]

Therefore by the Borel-Cantelli lemma, with probability 1 for \( \mu_\omega \)-almost all \( t \in I \),

\[ \limsup_{m \to \infty} \frac{L(x, U(t, 2^{-m}))}{\varphi_t(2^{-m})} \leq c_{4,9}. \]

Thanks to the continuity of \( h_t(\bullet) \) (l = 1, ..., N), it can be verified that

\[ \gamma(H(u^*)) \xrightarrow{u^* \to t} \gamma(H(t)) \quad \text{and} \quad \beta(H(u^*)) \xrightarrow{u^* \to t} \beta(H(t)). \]

Since \( \gamma(H(u^*)) \) and \( \gamma(H(t)) \) are integer-valued, we have \( \gamma(H(u^*)) = \gamma(H(t)) \) for all \( m \) large enough. By this and Condition \( H_2 \), one can verify that there exists a constant \( c_{4,10} > 0 \) such that \( \varphi_u(2^{-m}) \leq c_{4,10} \varphi_t(2^{-m}) \) for all \( m > 0 \). Therefore,

\[ \limsup_{m \to \infty} \frac{L(x, U(t, 2^{-m}))}{\varphi_t(2^{-m})} \leq c_{4,10} \limsup_{m \to \infty} \frac{L(x, U(t, 2^{-m}))}{\varphi_u(2^{-m})} \leq c_{4,11}, \]

where \( c_{4,11} = c_{4,9}c_{4,10} \). Hence, for any \( r > 0 \) small enough, there exists an integer \( m \) such that \( 2^{-m} \leq r < 2^{-m+1} \) and since \( \varphi_t(\bullet) \) is increasing in the neighborhood of 0, we have

\[ \limsup_{m \to \infty} \frac{L(x, U(t, r))}{\varphi_t(r)} \leq \limsup_{m \to \infty} \frac{L(x, U(t, 2^{-m+1}))}{\varphi_t(2^{-m})} \leq c_{4,11} \sup_{m \geq 1} \left\{ \frac{\varphi(t, 2^{-m+1})}{\varphi_t(2^{-m})} \right\} < +\infty. \]

This proves (4.1). \( \square \)

**Appendix A: Proof of Lemma 3.2**

For \( i = 1, \ldots, n \), define \( y_i := \sum_{j=1}^{n} a_{i,j}x_j \). Setting \( a_{i,j} = 0 \) if \( i > j \). Since \( (a_{i,j})_{i,j=1,\ldots,n} \) is an upper triangle matrix, we thus obtain, for \( i = 1, \ldots, n, x_i = \sum_{j=1}^{n} u_{i,j}y_j \), for some \( u_{i,j} \in \mathbb{R}, j = 1, \ldots, n \), which only depend on \( a_{i,j} \)'s and satisfy \( u_{i,i} = a_{i,i}^{-1}, u_{i,j} = 0 \) for \( i > j \). Therefore we can write:

\[ \int_{\mathbb{R}^n} \left( \prod_{i=1}^{n} |x_i|^{b_i} \right) e^{-\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}x_j} \prod_{i=1}^{n} \left| \prod_{j=1}^{n} u_{i,j}y_j \right| e^{-\sum_{i=1}^{n} |y_i|^{b_i}} \, dy, \]

where \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \). Using the inequality (2.3) and the multinomial formula, we obtain

\[ \prod_{i=1}^{n} \left| \sum_{j=1}^{n} u_{i,j}y_j \right|^{b_i} = \prod_{i \in \{1, \ldots, n\}} \left| \sum_{j=1}^{n} u_{i,j}y_j \right|^{b_i} \leq \prod_{i \in \{1, \ldots, n\}} \left( \sum_{b_{i,j} \neq 0} n_{b_{i,j}} \sum_{j=1}^{n} |u_{i,j}y_j|^{b_i} \right) = \left( \prod_{i=1}^{n} (n_{b_{i,j}} - 1 \lor 1) \right) \sum_{j_1, \ldots, j_n \in \{1, \ldots, n\}} \prod_{i \in \{1, \ldots, n\}} |u_{i,j_i}y_j_i|^{b_i}. \]
Therefore by using (A.1), (A.2) and (2.14) iteratively,
\[
\int_{\mathbb{R}^n} \left( \prod_{i=1}^n |x_i|^{b_i} \right) e^{-\sum_{i=1}^n \left| \sum_{j=i}^n a_{i,j} x_j \right|^\alpha} dx \leq \left( \prod_{i=1}^n (n^{b_i-\varepsilon} + 1) \right) \left( \prod_{i=1}^n a_{i,i} \right)^{-1} \\
\times \sum_{j_1,\ldots,j_n \in \{1,\ldots,n\}} \left( \prod_{i \in \{1,\ldots,n\}} |u_{i,j_i}|^{b_i} \right) \int_{\mathbb{R}^n} \left( \prod_{i \in \{1,\ldots,n\}} |y_{j_i}|^{b_i} \right) e^{-\sum_{i=1}^n |y_i|^{\alpha}} dy
\]
\[
\leq c_{3,1}(n) \left( \prod_{i=1}^n (n^{b_i-\varepsilon} + 1) \right) \left( \prod_{i=1}^n a_{i,i} \right)^{-1} \sum_{j_1,\ldots,j_n \in \{1,\ldots,n\}} |u_{i,j_i}|^{b_i} = c_{3,1}(n) \left( \prod_{i=1}^n a_{i,i} \right)^{-1} \prod_{i \in \{1,\ldots,n\}} \sum_{j_1,\ldots,j_n \in \{1,\ldots,n\}} |u_{i,j_i}|^{b_i},
\]
where
\[
c_{3,1}(n) = \left( \prod_{i=1}^n (n^{b_i-\varepsilon} + 1) \right) \left( \frac{\alpha}{\alpha} \right)^n \left( \sup_{j_1,\ldots,j_n \in \{1,\ldots,n\}} \left( 1 + \frac{\sum_{i=1}^n b_i}{\alpha} \right) \right)^n.
\]

Lemma 3.2 is proved.

Appendix B: Proof of Lemma 3.3

The proof is based on an extension of the direct approach in [2] for linear fractional stable sheets. For \( n \geq 1 \), by the definition of \( Z_l(\bullet) \) we can write \( \sum_{j=1}^n Z_l(u^j) \) as sum of independent components and obtain the following:
\[
\left\| \sum_{j=1}^n x_j Z_l(u^j) \right\|_{\alpha}^\alpha = \sum_{j=1}^n \int_{Q_l(u^j) \setminus Q_l(u^{j-1})} \left| \sum_{j=1}^n x_j g^{H(u^j)}(u^j, r) \right|^{\alpha} dr,
\]
where \( Q_l(u^j) := 0 \) and \( Q_l(u^j) \setminus Q_l(u^{j-1}) = [0, \varepsilon]^l \times (u_l^{j-1}, u_l^j) \times [0, \varepsilon]^{N-l} \). Using the definition of \( g_l \) in (3.1), for every \( i \in \{1, \ldots, n\} \),
\[
\int_{Q_l(u^j) \setminus Q_l(u^{j-1})} \left| \sum_{j=1}^n x_j g^{H(u^j)}(u^j, r) \right|^{\alpha} dr = c_{Q_l}(1) \int_{Q_l(u^j) \setminus Q_l(u^{j-1})} \left| \sum_{j=1}^n x_j \prod_{p=1}^N (u_p - r_p)^{\alpha} \left( u_p^{j-1} - u_p^j \right) \right|^{\alpha} dr,
\]
where \( r = (r_1, \ldots, r_N) \). Applying the following change of variables to (B.2):
\[
r_l \rightarrow u_l^{j-1} + (u_l^j - u_l^{j-1})(1 - r_l),
\]
we obtain
\[
\int_{Q_l(u^j) \setminus Q_l(u^{j-1})} \left| \sum_{j=1}^n x_j \prod_{p=1}^N (u_p - r_p)^{\alpha} \left( u_p^{j-1} - u_p^j \right) \right|^{\alpha} dr = \int_{S_l(1)} |F(u^j, x, r)|^{\alpha} dr,
\]
where
\[
S_l(1) := \{ r \in [0, +\infty)^N : 0 \leq r_p \leq \varepsilon \text{ if } p \neq l, 0 < r_l \leq 1 \}
\]
and
\[
F(u^j, x, r) := \sum_{j=1}^n x_j \left( u_l^j - u_l^{j-1} \right)^{\alpha} \left( u_l^{j-1} - u_l^j \right)^{\alpha} \left( u_l^{j-1} - u_l^j \right)^{\alpha} \prod_{p \neq l} (u_p - r_p)^{\alpha - \alpha}.
\]
Below we distinguish with 2 cases: $1 \leq \alpha \leq 2$ and $0 < \alpha < 1$.

If $\alpha \in [1, 2]$, it follows from (B.3), H"older’s inequality and (B.5) that
\[
\int_{S_1(1)} |F(u^i, x, r)|^\alpha \, dr \geq c_{5.1} \int_{S_1(1)} |F(u^i, x, r)| \, d\alpha = c_{5.1} \left| \sum_{j=1}^n \theta_{i,j} x_j \right|^\alpha, \tag{B.6}
\]
where $c_{5.1} = e^{(N-1)(1-\alpha)}$ and
\[
\theta_{i,j} = \left( u_i^j - u_i^{j-1} \right)^{1/\alpha} \int_{S_1(1)} \left( u_i^j - u_i^{j-1} - (u_i^j - u_i^{j-1})(1-r_j) \right)^{h_i(u^j)-1/\alpha} \prod_{p \neq l} (u_p^j - r_p)^{h_p(u^j)-1/\alpha} \, dr. \tag{B.7}
\]
Combining (B.1), (B.2), (B.3), and (B.6) we obtain
\[
\left\| \sum_{j=1}^n x_j Z_l(u^j) \right\|_H^\alpha \geq c_{5.1} c_H^\alpha \left( \sum_{i=1}^n \left| \sum_{j=1}^n \theta_{i,j} x_j \right|^\alpha \right)^{-1/\alpha}, \tag{B.8}
\]
(B.8) together with Lemma 3.2 and (B.7) yields:
\[
\int_{\mathbb{R}^n} \left( \prod_{i=1}^n |x_i|^{b_i} \right) e^{-\sum_{j=1}^n x_j Z_l(u^j)} \, dx \leq \int_{\mathbb{R}^n} \left( \prod_{i=1}^n |x_i|^{b_i} \right) e^{-c_{5.1} c_H^\alpha \sum_{i=1}^n \left| \sum_{j=1}^n \theta_{i,j} x_j \right|^\alpha} \, dx \leq \left( c_{5.1} c_H^\alpha \right)^{-\left( \sum_{i=1}^n b_i + n \right)/\alpha} c_{3.1}(n) \left( \prod_{i=1}^n \theta_{i,i} \right) \sum_{j_1, \ldots, j_n \in \{1, \ldots, n\}} \prod_{i \in \{1, \ldots, n\}} \prod_{b_i \neq 0} |\xi_{i,j_i}|^{b_i}, \tag{B.9}
\]
where $c_{3.1}(n)$ is given in (3.5); $(\xi_{i,j})_{i,j=1,\ldots,n}$ is the inverse matrix of $(\theta_{i,j})_{i,j=1,\ldots,n}$. Note that each $\xi_{i,j}$ has the representation
\[
\xi_{i,j} = p_{i,j} ((\theta_{i,j})_{i,j=1,\ldots,n}) \prod_{i=1}^n \theta_{i,i}^{-1}, \tag{B.10}
\]
where $p_{i,j} ((\theta_{i,j})_{i,j=1,\ldots,n})$ denotes the $(i,j)$-element of the adjugate of the matrix $(\theta_{i,j})_{i,j=1,\ldots,n}$ thus it is a polynomial of $u^1, \ldots, u^n$. It follows from (B.9) and (B.10) that
\[
\int_{\mathbb{R}^n} \left( \prod_{i=1}^n |x_i|^{b_i} \right) e^{-\sum_{j=1}^n x_j Z_l(u^j)} \, dx \leq \left( c_{5.1} c_H^\alpha \right)^{-\left( \sum_{i=1}^n b_i + n \right)/\alpha} c_{3.1}(n) \left( \prod_{i=1}^n \theta_{i,i} \right) \sum_{j_1, \ldots, j_n \in \{1, \ldots, n\}} \prod_{i \in \{1, \ldots, n\}} \prod_{b_i \neq 0} |p_{i,j} ((\theta_{i,j})_{i,j=1,\ldots,n})|^{b_i} \leq c_{5.2}(n) \prod_{i=1}^n |u_i^j - u_i^{j-1} - h_i(u^j)(1 + \sum_{j=1}^n b_j)|, \tag{B.11}
\]
where
\[
c_{5.2}(n) = \left( c_{5.1} c_H^\alpha \right)^{-\left( \sum_{i=1}^n b_i + n \right)/\alpha} c_{3.1}(n) \times \left\{ \inf_{u \in [e, T]^N} \left\{ \int_{S_1(1)} \left( u_i^j(u) - 1/\alpha \right) \prod_{p \neq l} (u_p^j - r_p)^{h_p(u) - 1/\alpha} \, dr \right\} \right\}^{-1} \times \sup_{u^1, \ldots, u^n \in [e, T]^N} \left\{ \prod_{i \in \{1, \ldots, n\}} \sum_{j=1}^n |p_{i,j} ((\theta_{i,j})_{i,j=1,\ldots,n})|^{b_i} \right\}. \tag{B.12}
\]
Next we consider the case for $0 < \alpha < 1$. Since the function $\phi(r) = e^{-\beta r}$ with $\beta > 0$ is convex on $[0, +\infty)$, by using Jensen’s inequality we have
\[
e^{-c_{H(1)}^n} \sum_{i=1}^{n} \int_{S_i(1)} |F(u', x, r)|^\alpha \, dr \leq \frac{1}{e^{N-1}} \int_{S_i(1)} e^{-c_{H(1)}^n} \sum_{i=1}^{n} |F(u', x, r)|^\alpha \, dr.
\] (B.13)
It results from (B.1), (B.3), (B.13) and Fubini’s theorem that
\[
\int_{\mathbb{R}^n} \left( \prod_{j=1}^{n} |x_j|^{b_j} \right) e^{-\| \sum_{j=1}^{n} x_j Z_i(u) \|_n^\alpha} \, dx 
\leq \frac{1}{e^{N-1}} \int_{S_i(1)} \int_{\mathbb{R}^n} \left( \prod_{j=1}^{n} |x_j|^{b_j} \right) e^{-c_{H(1)}^n} \sum_{i=1}^{n} |\sum_{j=1}^{n} \eta_{i,j}(r)|^\alpha x \, dx dr,
\]
where
\[
\eta_{i,j}(r) = (u_i^j - u_{i-1}^j)^{1/\alpha}(u_i^j - u_{i-1}^j - (u_i^j - u_{i-1}^j)(1 - r_t))b_i(u_t)^{1-\alpha} \prod_{p \neq l} (u_p^j - r_p)b_p(u_t)^{-1/\alpha},
\] (B.14)
Then similar to the way to obtain (B.11), applying again Lemma 3.2 we get
\[
\int_{\mathbb{R}^n} \left( \prod_{j=1}^{n} |x_j|^{b_j} \right) e^{-\| \sum_{j=1}^{n} x_j Z_i(u) \|_n^\alpha} \, dx \leq c_{5,3}(n) \prod_{i=1}^{n} |u_i^j - u_{i-1}^j|^{-b_i(u_t)}(1 + \sum_{j=1}^{n} b_t),
\] (B.15)
where
\[
c_{5,3}(n) = e^{(1-N)} \\left( e^{(N-1)} c_{H(1)}^n \right)^{-\left( \sum_{j=1}^{n} b_j + n \right)/\alpha} c_{3,1}(n)
\times \left( \sup_{u \in [r, T]^N} \left\{ \int_{S_i(1)} 3^{1/\alpha - b_i(u)} \prod_{p \neq l} (u_p - r_p)^{1/\alpha - b_p(u)} \, dr \right\} \right)^{1 + \sum_{j=1}^{n} b_j}
\times \sup_{u^1, \ldots, u^n \in [r, T]^N} \left\{ \int_{S_i(1)} \prod_{i \in \{1, \ldots, n\}} \sum_{j=1 \ b_j \neq 0}^{n} |\eta_{i,j}((\eta_{i,j}(r)))_{i,j=1,\ldots,n}|^{b_i} \, dr \right\},
\] (B.16)
where each $\eta_{i,j}((\eta_{i,j}(r)))_{i,j=1,\ldots,n}$ is the $(i, j)$-element of the adjugate of the matrix $((\eta_{i,j}(r)))_{i,j=1,\ldots,n}$.

Finally Lemma 3.3 follows from (B.11) and (B.15), with
\[
c_{3,2}(n) = c_{5,2}(n) \lor c_{5,3}(n).
\] (B.17)

**Appendix C: Proof of Lemma 3.9**

We first point out that, in order to show (3.32) holds for all integer $n \geq 1$, it suffices to prove that it holds for even integers $n \geq 2$, thanks to the Cauchy-Schwarz inequality. Therefore in the following we assume $n$ is an even integer.

By [10, Equation (25.7)], we have: for all $x, y \in \mathbb{R}^d$, Borel sets $J \subseteq I$, and all even integer $n \geq 2$,
\[
E[(L(x, J) - L(y, J))^n] = (2n)^{-n} \int_{J_n} \int_{J_{\mathbb{R}^d}} \prod_{j=1}^{n} (e^{-i(i^j \cdot x)} - e^{-i(i^j \cdot y)}) E\left[e^{i \sum_{j=1}^{n} (i^j \cdot X^H(i^j)(u'))} \right] \, d\nu \, d\bar{u}.
\] (C.1)
Pick any $u^* \in I_{n, \delta}$ and let $\gamma(H(u^*)) \in \{1, \ldots, N\}$ be the unique integer satisfying (3.11). Let $\kappa_n(H(u^*))$ be the real number satisfying (3.33). By the elementary inequality
\[
|e^{ix} - 1| \leq 2^{1 - \kappa_n(H(u^*))}|x|^{\kappa_n(H(u^*))}, \quad \text{for all } x \in \mathbb{R}
\]
and the triangle-type inequalities in (2.1), we have for all $v_1, \ldots, v_n, x, y \in \mathbb{R}^d$,
\[
\prod_{j=1}^n |e^{-i(v^j, x)} - e^{-i(v^j, y)}| \leq 2n(1 - \kappa_n(H(u^*))) |x - y|^n \kappa_n(H(u^*)) \sum_{j \in \{1, \ldots, n\}} \prod_{k \in \{1, \ldots, d\}} n |v^j_k| \kappa_n(H(u^*)).
\]
(C.2)

The inequalities (3.4) and the fact that $\gamma(H(u^*)) \leq N$ yield
\[
\left\| \sum_{j=1}^n \eta_j \times \prod_{k=1}^d \left( \prod_{j=1}^n |v^j_k| \kappa_n(H(u^*)) \right) \right\|_{\alpha} \geq \sum_{l=1}^N \left\| \sum_{j=1}^n \eta_j Z_l(u^j) \right\|_{\alpha} \geq \sum_{l=1}^N \left\| \sum_{j=1}^n \eta_j Z_l(u^j) \right\|_{\alpha}^\alpha.
\]
(C.3)

Since $n \geq 2$ is even, the left-hand side of (C.1) is nonnegative. Combining (C.1), (C.2) and (C.3), and using the independence of $X_k^{(H(\bullet)), k = 1, \ldots, d}$, we have
\[
\mathbb{E}[|L(x, I_a, \delta) - L(y, I_a, \delta)|^n] \leq (2\pi)^{-n/d} 2^n (1 - \kappa_n(H(u^*))) |x - y|^n \kappa_n(H(u^*))
\]
\[
\times \sum_{j \in \{1, \ldots, n\}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \prod_{k=1}^d \left( \prod_{j=1}^n |v^j_k| \kappa_n(H(u^*)) \right) \right) \prod_{k=1}^d \left( \sum_{l=1}^N \eta_j Z_l(u^j) \right) \prod_{k=1}^d d\eta_k.
\]
\[
\leq |x - y|^n \kappa_n(H(u^*)) \sum_{j \in \{1, \ldots, n\}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \prod_{k=1}^d \left( \prod_{j=1}^n |v^j_k| \kappa_n(H(u^*)) \right) \eta_j(k_j) \right) \prod_{k=1}^d d\eta_k.
\]
(C.4)

where $\eta_k(u) := \begin{cases} 1 & \text{if } u = k, \\ 0 & \text{if } u \neq k. \end{cases}$ Now take $\Delta = n^{-1}, q = d, \delta_l = h_l(u^*)$ for $l = 1, \ldots, N$ in Lemma 3.5 and let $p_1, \ldots, p_\gamma(H(u^*))$ satisfy (3.8). Observe that since $n \kappa_n(H(u^*)) \in (0, \alpha(H(u^*)) / 2\gamma(H(u^*))$, it follows from (3.9) that there exists $l_0 \in \{1, \ldots, \gamma(H(u^*))\}$ (depending on $\kappa_n(H(u^*))$) such that
\[
h_{l_0}(u^*) \left( \frac{d}{p_{l_0}} + 2n \kappa_n(H(u^*)) \right) < 1.
\]
(C.7)

Combining (C.4) with the generalized Hölder’s inequality, we obtain
\[
\mathbb{E}[|L(x, I_a, \delta) - L(y, I_a, \delta)|^n] \leq |x - y|^n \kappa_n(H(u^*))
\]
\[
\times \sum_{j \in \{1, \ldots, n\}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \prod_{k=1}^d \left( \prod_{j=1}^n |v^j_k| \kappa_n(H(u^*)) \right) \eta_j(k_j) \right) \prod_{k=1}^d \left( \sum_{l=1}^N \eta_l Z_l(u^j) \right) \prod_{k=1}^d d\eta_k, \quad \text{(C.6)}
\]

where
\[
\mathcal{M}_{l_0, k_1, \ldots, k_n}(\overline{v}) := \int_{\mathbb{R}^d} \left( \prod_{j=1}^n |v^j_k| \kappa_n(H(u^*)) \eta_l(k_j) \right) \prod_{l \neq l_0} \mathcal{M}_{l, k}(\overline{v}) \prod_{k=1}^d d\eta_k
\]
(C.7)

and
\[
\mathcal{M}_{l, k}(\overline{v}) := \int_{\mathbb{R}^d} e^{-p_{l_0} \left( \sum_{j=1}^n |v^j_k Z_l(u^j) \right) \| \overline{v} \|_{\alpha}^\alpha d\eta_k.
\]

Next we provide upper bounds of $\mathcal{M}_{l_0, k_1, \ldots, k_n}(\overline{v})$ and $\mathcal{M}_{l, k}(\overline{v})$, respectively.
Upper bound of $\mathcal{M}_{l_0,k_1,\ldots,k_n}(\pi)$:
Taking $b_i = \kappa_n(H(u^*))\eta_k(k_i)p_{l_0}$, $j = 1,\ldots,n$ in Lemma 3.3 and using (3.20), we derive that, for $\delta > 0$ small enough, there is a constant $c_{5,4}(l_0, n) > 0$ such that
\[
\mathcal{M}_{l_0,k_1,\ldots,k_n}(\pi) \leq c_{5,4}(l_0, n) \prod_{j=1}^{n} (u_{l_0}^{\pi_0(j)} - u_{l_0}^{\pi_0(j-1)}) - (h_{l_0}(u^*) + c_0(\delta))(1 + \kappa_n(H(u^*))p_{l_0} \sum_{l=1}^{n} \eta_k(k_i)),
\]
where $c_0(\delta)$ is given in (3.21).

Upper bound of $\mathcal{M}_{l,k}(\pi)$:
Similarly, applying Remark 3.4 and (3.20), we easily obtain, for each $l \in \{1,\ldots,\gamma(H(u^*))\}\{l_0\}$, there is $c_{5,5}(l) > 0$ such that
\[
\mathcal{M}_{l,k}(\pi) \leq c_{5,5}^2(l) \prod_{j=1}^{n} (u_l^{\pi_l(j)} - u_l^{\pi_l(j-1)}) -(h_l(u^*) + c_0(\delta))\eta_k(k_i).
\]

For $l = 1,\ldots,N$, let $\pi_l$ be defined as in (3.19). Now combining (C.6), (C.8), (C.9) and using the fact that $\sum_{l=1}^{d} \sum_{i=1}^{n} \eta_k(k_i) = n$ for $k_i = 1,\ldots,d$, we obtain
\[
\mathbb{E}[\langle L(x, I_{a,\delta}) - L(y, I_{a,\delta}) \rangle^n] \leq c_{5,6}(n)|x-y|^n\mathcal{E}(H(u^*))
\]
\[\times \left\{ \prod_{l=1}^{i} \prod_{l \neq l_0} \left( \int_{\Pi_{l_0}}^{\Pi_{l_l}} (u_l^{\pi_l(j)} - u_l^{\pi_l(j-1)}) - (h_l(u^*) + c_0(\delta))d\eta_k(k_i) \right) \right\} \delta^n(N-\gamma(H(u^*))),
\]
where
\[
c_{5,6}(n) = \sup_{l_0, m \in \{1,\ldots,N\}} c_{5,4}(l_0, n)_{l=1}^{d/p_{l_0}} \prod_{l=1, l \neq l_0}^{m} c_{5,5}^2(l)_{nd/p_l}.
\]

Since (C.5) holds, we are able to choose $\delta \in (0,1]$ small enough so that the following inequality also holds:
\[
\left( h_{l_0}(u^*) + c_0(\delta) \right) \left( \frac{d}{p_{l_0}} + 2n\kappa_n(H(u^*)) \right) < 1.
\]

Thanks to (3.24) and (11), the integrals in (C.10) are finite. Then similar to the derivation of (3.25),
\[
\mathbb{E}[\langle L(x, I_{a,\delta}) - L(y, I_{a,\delta}) \rangle^n] \leq c_{3,11}(n)|x-y|^n\mathcal{E}(H(u^*))\delta^n(N-(1-1/n)\sum_{l=1}^{n} \gamma(H(u^*))) dh_l(u^*)/p_l - (1-nh_{l_0}(u^*)n\kappa_n(H(u^*))),
\]
where
\[
c_{3,11}(n) = c_{5,6}(n)_{d^n} \sup_{m \in \{1,\ldots,N\}} \sup_{u \in \mathcal{U}} \left\{ \frac{(n)!}{(h_m(u) + c_0(\delta))n\kappa_n(H(u))} \right\}
\times \prod_{l=1}^{m} c_{5,5}(l, u)_{(n)!} dh_l(u^*)/p_l \xi^{-n(1-1/n)h_{l_0}(u^*)}(d\eta_k(k_i)+c_0(\delta)n\kappa_n(H(u^*))).
Applying Lemma 3.5 with $\Delta = n^{-1}$, $q = d$, $\vartheta_1 = h_1(u^*)$, we obtain

$$
\left(1 - \frac{1}{n}\right)^{\gamma(H(u^*))} \sum_{l=1}^{\gamma(H(u^*))} \frac{h_i(u^*)}{\pi_l} \leq h_\gamma(H(u^*)) h_i(h(u^*)) + \gamma(H(u^*)) - \sum_{l=1}^{\gamma(H(u^*))} \frac{h_i(h(u^*))}{h_i(u^*)}.
$$

W.l.o.g., we can assume that $0 < h_1(u^*) \leq \ldots \leq h_{N}(u^*) < 1$. Therefore, (C.12) yields

$$
E[(L(x, I_{a,\delta}) - L(y, I_{a,\delta}))^n] \leq C_{3,11}(n) x - y |^{\gamma n} H(u^*) \delta^n (\beta(H(u^*)) - (n-1) h_\gamma(H(u^*)) h_i(H(u^*))).
$$

Since the choice of $u^*$ in (C.13) is arbitrary in $I_{a,\delta}$ and the order of coordinates in $H(\bullet)$ can be arbitrary, taking the infimum over $u^* \in I_{a,\delta}$ and $\sigma \in S(N)$ on both hand sides of (C.13) leads to (3.32). Lemma 3.9 is proved.

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References

[1] Ayache, A., Roueff, F. and Xiao, Y. (2007). Joint continuity of the local times of linear fractional stable sheets. Comptes Rendus Mathématique 344 635–640. https://doi.org/10.1016/j.crma.2007.03.028 MR2334075
[2] Ayache, A., Roueff, F. and Xiao, Y. (2007). Local and asymptotic properties of linear fractional stable sheets. Comptes Rendus Mathématique 344 389–394. https://doi.org/10.1016/j.crma.2007.01.017 MR2310675
[3] Ayache, A., Roueff, F. and Xiao, Y. (2009). Linear fractional stable sheets: wavelet expansion and sample path properties. Stochastic Processes and their Applications 119 1168–1197. https://doi.org/10.1016/j.spa.2008.06.004 MR2508569
[4] Ayache, A., Wu, D. and Xiao, Y. (2008). Joint continuity of the local times of fractional Brownian sheets. Annales de l’Institut Henri Poincaré, Probabilités et Statistiques 44 727–748. https://doi.org/10.1214/07-AIHP131 MR2446295
[5] Ayache, A. and Xiao, Y. (2005). Asymptotic properties and Hausdorff dimensions of fractional Brownian sheets. Journal of Fourier Analysis and Applications 11 407–439. https://doi.org/10.1007/s00041-005-4048-3 MR2169474
[6] Berman, S. M. (1970). Gaussian processes with stationary increments: Local times and sample function properties. The Annals of Mathematical Statistics 41 1260–1272. https://doi.org/10.1214/aoms/1177696901 MR0272035
[7] Boufoussi, B., Dozzi, M. and Guerbaz, R. (2006). On the local time of the multifractional Brownian motion. Stochastics 78 33–49. https://doi.org/10.1080/17442500600578073 MR2219711
[8] Dozzi, M. (2003). Occupation density and sample path properties of $N$-parameter processes. In Topics in Spatial Stochastic Processes (Martina Franca, 2001). Lecture Notes in Mathematics, 1802 127–166. Springer, Berlin https://doi.org/10.1007/978-3-540-36259-3_4 MR1975519
[9] EHM, W. (1981). Sample function properties of multi-parameter stable processes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **56** 195–228. https://doi.org/10.1007/BF00535741. MR0618272

[10] GEMAN, D. and HOROWITZ, J. (1980). Occupation densities. *The Annals of Probability* **8** 1–67. https://doi.org/10.1214/aop/1176994824. MR0556414

[11] KHOSNEVISAN, D. (2002). *Multiparameter Processes: An Introduction to Random Fields*. Springer, New York https://doi.org/10.1007/b97363.

[12] MEERSCHAERT, M., WU, D. and XIAO, Y. (2008). Local times of multifractional Brownian sheets. *Bernoulli* **14** 865–898. https://doi.org/10.3150/08-BEJ126. MR2537815

[13] NOLAN, J. P. (1989). Local nondeterminism and local times for stable processes. *Probability Theory and Related Fields* **82** 387–410. https://doi.org/10.1007/BF00339994. MR1001520

[14] PITT, L. D. (1978). Local times for Gaussian vector fields. *Indiana University Mathematics Journal* **27** 309–330. https://doi.org/10.1512/iumj.1978.27.27024. MR471055

[15] SAMORODNITSKY, G. and TAQQU, M. S. (1994). *Stable Non-Gaussian Random Processes*. Chapman & Hall, New York. MR1280932

[16] SHEN, G., YU, Q. and LI, Y. (2020). Local times of linear multifractional stable sheets. *Applied Mathematics. A Journal of Chinese Universities. Ser. B* **35** 1–15. https://doi.org/10.1134/s1001541420010031. MR4078814

[17] STOEV, S. and TAQQU, M. S. (2004). Stochastic properties of the linear multifractional stable motion. *Advances in Applied Probability* **36** 1085–1115. https://doi.org/10.1239/aap/1103662959. MR2119856

[18] STOEV, S. and TAQQU, M. S. (2005). Path properties of the linear multifractional stable motion. *Fractals* **13** 157–178. https://doi.org/10.1142/S0218348X05002775. MR2151096

[19] XIAO, Y. (1997). Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. *Probability Theory and Related Fields* **109** 129–157. https://doi.org/10.1007/s004400050128. MR1469923

[20] XIAO, Y. (1999). The Hausdorff dimension of the level sets of stable processes in random scenery. *Acta Universitatis Szegedensis. Acta Scientiarum Mathematicarum* **65** 385–395. MR1702175

[21] XIAO, Y. (2009). Sample Path Properties of Anisotropic Gaussian Random Fields. In *A Minicourse on Stochastic Partial Differential Equations. Lecture Notes in Mathematics*, **1962** 145–212. Springer, Berlin https://doi.org/10.1007/978-3-540-85994-9_5. MR2508776

[22] XIAO, Y. (2011). Properties of strong local nondeterminism and local times of stable random fields. In *Seminar on Stochastic Analysis, Random Fields and Applications VI*. *Progress in Probability*, **63** 279–308. Birkhäuser/Springer Basel AG, Basel https://doi.org/10.1007/978-3-0348-0021-1_18. MR2857032

[23] XIAO, Y. and ZHANG, T. (2002). Local times of fractional Brownian sheets. *Probability Theory and Related Fields* **124** 204–226. https://doi.org/10.1007/s004400200210. MR1936017