Scaling asymptotics of spectral Wigner functions

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Abstract

We prove that smooth Wigner–Weyl spectral sums at an energy level $E$ exhibit Airy scaling asymptotics across the classical energy surface $\Sigma_E$. This was proved earlier by the authors for the isotropic harmonic oscillator and the proof is extended in this article to all quantum Hamiltonians $-\hbar^2 \Delta + V$ where $V$ is a confining potential with at most quadratic growth at infinity. The main tools are the Herman–Kluk initial value parametrix for the propagator and the Chester–Friedman–Ursell normal form for complex phases with a one-dimensional cubic degeneracy. This gives a rigorous account of Airy scaling asymptotics of spectral Wigner distributions of Berry, Ozorio de Almeida and other physicists.

Keywords: Airy, Wigner–Weyl, spectral, asymptotics

This article is thus concerned with spectral Wigner functions for a Schrödinger operator

$$\hat{H}_\hbar = -\frac{\hbar^2}{2} \Delta_{\mathbb{R}^d} + V(x),$$

(1)

where $V \in C^\infty(\mathbb{R}^d)$ is a real-valued potential that satisfies $V(x) \to \infty$ as $\|x\| \to \infty$, and is at most of quadratic growth at infinity,

$$\forall \gamma \in \mathbb{N}^d \text{ s.t. } |\gamma| \geq 2 \quad \exists C_\gamma > 0, \quad \sup_x |\partial_\gamma^x V(x)| < C_\gamma.$$  

(2)

Let $\{\psi_j(\hbar)\}$ be a complete orthonormal basis for $L^2(\mathbb{R}^d, dx)$ consisting of eigenfunctions of $\hat{H}_\hbar$:

$$\hat{H}_\hbar \psi_j(\hbar, x) = E_j(\hbar) \psi_j(\hbar, x),$$

where $E_j(\hbar)$ is the eigenvalue corresponding to $\psi_j(\hbar)$. 

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and let
\[ W_{\psi_j,\psi_j}(x, \xi) = (2\pi \hbar)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \psi_j^\ast(h, x + \frac{v}{2}) \psi_j(h, x - \frac{v}{2}) e^{-\sqrt{\frac{\hbar}{2}} v \xi} \frac{dv}{(2\pi \hbar)^d} \]  
be the Wigner functions of the individual eigenstates. The Weyl–Wigner spectral functions for an energy interval \( I_E = [E - a, E + b] \) centered at \( E \) are defined by,
\[ W_{hJ}(x, \xi) := \sum_{\beta E_j \in I_E} W_{\psi_j,\psi_j}(x, \xi). \]

In the recent articles [HZ20, HZb], the authors obtained scaling asymptotics of Airy type for (4) around the energy surface
\[ \Sigma_E = \{(x, \xi) : H(x, \xi) = E\}, \quad H(x, \xi) = \|\xi\|^2 + V(x) \]
in the special case of the isotropic Harmonic oscillator where \( V(x) = \|x\|^2 \). The asymptotics involve two types of localization: (i) spectral localization to the interval \( I_E \) and (ii) phase space localization, where \((x, \xi)\) is localized either near \( \Sigma_E \) or at some prescribed distance from it, either in the energy ball,
\[ B_E = \{(x, \xi) : H(x, \xi) \leq E\} \]
or outside the energy surface. The purpose of this article is to present one generalization of the Airy scaling results to all Schrödinger operators (1) for which \( H \) is strictly convex, hence
\[ B_E \text{ is strictly convex} \]
and to relate the scaling results to statements in Berry’s articles [Ber89, Ber91] as well as to the related articles [O98, dRO] of Ozorio de Almeida (see also [TL]).

The result we generalize here is the following [HZ20, theorem 1.4]. Let \( E_N(h) = \hbar(N + \frac{3}{4}) \) be the \( N \)th distinct eigenvalue of the isotropic harmonic oscillator (see section 9 for notation and background) and define \( h = h_N(E) := \frac{\hbar(N + \frac{3}{4})}{E} \), so that \( E_N(h) = E \). Let \( \Pi_{h_N(h)} \) be the orthogonal projection onto the corresponding eigenspace \( V_N(h) \) and let \( W_{h_N(h)}(x, \xi) \) be its Wigner distribution (in the notation of (4) this means we choose \( a = b = 0 \)). Then, with \( s = \frac{h_N(h) \xi^2}{E} \), and where for simplicity we assume that \( s = \frac{h_N(h) \xi^2}{E} \in (0, 1] \), it follows from [HZ20, (50)] that,
\[ W_{h_N(h)}(x, \xi) = \frac{2}{(2\pi \hbar)^d} \left[ \hbar_E^{1/3} \text{Ai}\left( \hbar_E^{-2/3} B^2(s)\right) \left( 1 + O\left( (s - 1)^{2/3}\right) \right) + \varepsilon_1(h_E, s) \right], \]
where we have set
\[ B(s) = i(3\beta(s)/2)^{1/3}, \quad \beta(s) := \frac{1}{2} \left[ \cos^{-1} \sqrt{s} - \sqrt{s - s^2} \right]. \]
Note that in [HZ20, (50)] we use the large parameter \( \nu = \frac{2}{\hbar_E} \) and that the argument of the Airy function there is \( \nu^2 B^2(s) \); this introduces some additional universal constants in the asymptotics. Note also that since we have restricted for simplicity to \( s \leq 1 \), the formula (8) for \( B(s) \) is pure imaginary, so that its square is negative and the Airy function is evaluated in its oscillatory region. In section 9 we interpret the formulae geometrically and simplify (7) when \( (x, \xi) \) is very close to \( \Sigma_E \).

The present article generalizes the scaling results (7) for the more general Hamiltonians (1). There are both sharp and smooth versions of the scaling results, depending on whether we sum
over an energy interval as in (4) or whether we use a smooth cutoff function \( f \) to define general smoothed Weyl–Wigner sums,

\[
W_{h,f,E}(x, \xi) := \sum_{j=1}^{\infty} f(h^{-\delta}(E - E_j)) W_{\psi_j}(x, \xi),
\]

where \( f \in \mathcal{S}(\mathbb{R}) \) is a Schwartz function with \( \hat{f} \in C_0^\infty(\mathbb{R}) \). In this article we only consider \( \delta = 1 \) and we concentrate on the behavior of (9) in a thin \( h^{1/3} \) neighborhood of \( \Sigma_E \). In section 4 we review other scaling results in [HZ20, HZb] where \( \delta = 0, \frac{2}{3} \), i.e. where we sum over larger spectral intervals.

To state our first result about \( W_{h,f,1,E}(x, \xi) \), we need some notation. Let us denote the Hamilton vector field of \( H : T^*\mathbb{R}^d \to \mathbb{R} \) by \( \Xi_H \), and the Hamiltonian flow of \( \Xi_H \) by \( \Phi_t \). The classical trajectory with initial data \((q, p)\) is denoted by

\[
\Phi_t(q, p) = (q_t, p_t),
\]

and we will write

\[
M_t(q, p) := D_{q,p} \Phi_t(q, p)
\]

for its Jacobian. It was pointed out in [Ber89, O98] that the asymptotics of \( W_{h,f,1,E}(x, \xi) \) for \((x, \xi) \in B_E \) sufficiently close to \( \Sigma_E \) is governed in a sense that we will describe by the midpoint map, \( \Theta^*: \mathbb{R} \times \Sigma_E \to B_E \) given by

\[
\Theta^*(t, q, p) = \Theta_t(q, p) := \frac{1}{2}(I + \Phi'(q, p)) = \left( \frac{q + q_t}{2}, \frac{p + p_t}{2} \right).
\]

The assumption (6) ensures that (12) takes its values in \( B_E \). The asymptotics of \( W_{h,f,1,E}(x, \xi) \) depend on the solutions \((t, q, p)\) of the equations

\[
\Theta'(q, p) = (x, \xi), \quad (q, p) \in \Sigma_E, \quad (x, \xi) \in B_E,
\]

with \((E, x, \xi)\) are fixed, which we will see in proposition 3.1 below arise as critical point equations for a certain phase function related to \( W_{h,f,1,E} \). An important point is that there is an obvious symmetry in the equation (13), namely

\[
\Theta'(q, p) = \Theta^{-1}(\Phi'(q, p)),
\]

which fixes the Hamiltonian arc from \((q, p)\) to \((q_t, p_t)\) but reverses the endpoints. It is proved in section 7.4 (see corollary 3.4) that under assumption (6), there exists \( \varepsilon > 0 \) so that for \((x, \xi)\) in an \( \varepsilon \)-tube around \( \Sigma_E \) there exist unique \((t, q, p)\) up to the symmetry (14) satisfying the midpoint equation,

\[
\Theta(t, q, p) = (x, \xi).
\]
Further, fix \( f \) and denote by \( \gamma \) the endpoints of \( \gamma \) joining \((x, \xi)\) to points of the Hamilton orbit, the integral (18) equals the oriented area \( \int_\beta \omega \).
where $\omega = dp \wedge dq$ is the standard symplectic form of $T^* \mathbb{R}^d$. In section 9 we show that $\rho$ in (17) equals $-B^2(s)$ up to universal constants in (7) in the case of the isotropic harmonic oscillator.

The sign conventions in the argument of the Airy function is discussed below proposition 4.1. It is clearly consistent with (7), i.e. for $H(x, \xi) \leq E$ the argument of the Airy function is negative and therefore the asymptotics are oscillatory.

Further, let us remark that theorem 0.2 concerns the behavior of the smoothed spectral Wigner functions $W_{h,1/6}$ evaluated phase space points $(x, \xi)$ that are $\bar{h}^{2/3}$-close to the energy surface $\Sigma_E$. For such points, we prove that the leading order Airy behavior of $W_{h,1/6}(x, \xi)$ in theorem 0.2 comes from the nearly unique trajectory $\gamma_{x,\xi}$ of the classical Hamilton flow satisfying the mid-point rule relative to $(x, \xi)$ whose length goes to zero as $\hbar \to 0$. As explained above, this ultra-short trajectory is unique up to a time reversal symmetry. This degeneracy yields a universal fold singularity of the Lagrangian underlying the smoothed spectral Wigner function (see section 7.4 for more details).

Next, we remark that the result of theorem 0.2 is stated and a proof is sketched in [O98, TL]. Our motivation for presenting a rigorous proof is to generalize our earlier result (7) and to relate these special asymptotics to other types of asymptotics expansions for Wigner functions. The argument of the Airy function and the leading coefficient and argument of the Airy function agree with the calculations in [O98, TL]. To connect the notations, in those articles, the leading coefficient is given by $A_0(x, \xi, E)\det(1 + M)^{-\frac{3}{2}}$. Also, $\delta_0^{3/2}$ is the argument of Airy in [TL]. In our notation, which follows that of [Hol], the Airy argument is $\rho$ and $\rho^{1/3} = (\rho^{3/2})^\circ = (\delta_0)^{1/6}$.

Further, our proof of theorem 0.2 relies on a stationary phase with fold singularities (see [Hol, page 236]) from which we see that when $\rho(x, \xi) \approx 0$ theorem 0.2 gives an asymptotic expansion in powers of $\hbar^{1/3}$ that remains valid in any region of $T^* \mathbb{R}^d$ in which $\hbar^{-\frac{2}{3}} \rho(x, \xi)$ stays bounded. However, due to the exponential decay of the Airy functions on the positive axis, $\rho(x, \xi)$ becomes very small when $\hbar^{-\frac{2}{3}} \rho(x, \xi)$ is positive, and stationary phase asymptotics given below in proposition 0.5 then become valid.

The Airy scaling asymptotics of $W_{h,1/6}(x, \xi)$ for $(x, \xi)$ in the boundary layer around $\Sigma_E$ (i.e. $\hbar^{1/3}$ close to $\Sigma_E$) are due to a fold singularity in the map (12). More precisely, in section 7.4, it is explained that the Airy asymptotics are due to the fact that the relevant Lagrangian submanifold $\Lambda_E \subset T^*(T^* \mathbb{R}^d)$ in the second cotangent bundle (defined by (57)) has a fold singularity around $\Sigma_E$, i.e. $\Sigma_E$ is a caustic for the Wigner function (see section 7.4). The proof of theorem 0.2 is based on the use of the Herman–Kluk propagator [HK] and on the Airy asymptotics results of Chester–Friedman–Ursell [CFU] as given in [Hol, section 7] and [GSt]. Theorem 0.2 is a rigorous version of the result stated in [O98, (7.21)]. It does not seem to appear in [Ber89].

Finally, we conjecture that when the smooth test function $f$ is replaced by the indicator function of the spectral interval, then the sharp Wigner–Weyl sum has the asymptotics,

$$\sum_{\rho \in \{E_f(h) - E\} \leq Ch} W_{\rho, \psi_f}(x, \xi) = \hbar^{-d} \left( \hbar^{\frac{d}{2}} \text{Ai} \left( -\hbar^{-\frac{2}{3}} \rho(x, \xi) \right) \sum_{\nu=0}^{\infty} u_{0,\nu}(x, \xi) \hbar^{\nu} \right) + O\left( \hbar^{-d+\frac{2}{3}} \right).$$

Classically such asymptotics for spectral intervals are obtained from the smoothed results (17) by applying cosine Tauberian theorems. But Tauberian theorems make the hypothesis that the terms of the sums are non-negative, whereas it is a well-known and important phenomenon that Wigner functions are almost never globally positive. This raises the question whether $W_{\rho, \psi_f}(x, \xi) \geq 0$ for $(x, \xi)$ in an $\hbar^{2/3}$ tube around $\Sigma_E$ when the eigenvalue of $\psi_f$ satisfies $|E_f(h) - E| \leq Ch$. This is plausible, since asymptotically the Airy function is only evaluated
where it is positive; moreover, in a weak sense, the Wigner functions tend to a delta function on $\Sigma_E$. But this is far from sufficient to prove positivity. A possible source of counter-examples where $W_{\psi,\psi_0}(x,\xi)$ can be negative at some points of $\Sigma_E$ is the Wigner distribution of product eigenstates of the isotropic harmonic oscillator,

$$\varphi_{\alpha,h}(x) = h^{-d/4} p_{\alpha}(x) e^{-x^2/2h},$$

(20)

where $\alpha = (\alpha_1, \ldots, \alpha_d) \geq (0, \ldots, 0)$ is a $d-$dimensional multi-index and $p_{\alpha}(x)$ is the product $\prod_{j=1}^d p_{\alpha_j}(x_j)$ of the Hermite polynomials $p_k$ (of degree $k$) in one variable. There are similar products for generic oscillators, where the multiplicity of each eigenvalue equals one. The Wigner distribution is the product of those of the factors and are given by products in the variables $(x_1, \ldots, x_d)$ of one-dimensional Laguerre polynomials of the variables $x_j^\perp$. The product is positive if and only if the number of negative factors is even. It seems non-obvious whether one can construct products with an odd number of negative factors when $|E_j(h) - E| \leq C h$ and $\sum_{j=1}^d x_j^2 = E$. Another source of counter-examples could come from Wigner distributions of coherent states along elliptic periodic orbits. In the case of the isotropic harmonic oscillator, the Wigner distribution is positive near the periodic orbit on $\Sigma_E$ but appears to be negative at some points on $\Sigma_E$ away from the orbit [Lo]. As far as we know, the study of negativity sets of Wigner distributions of eigenfunctions under the above constraints has never been studied.

There are many further scaling asymptotics results for the smoothed and sharp Weyl–Wigner sums (4)–(9). In section 4 we review some model results for the isotropic harmonic oscillator,

1. Pointwise semi-classical asymptotics of Wigner transform of the propagator

As a interesting and useful warm-up to the proof of theorem 0.2, we first give a new proof of what is essentially a well-known result from the physics literature (e.g. [Ber89]) giving pointwise semi-classical asymptotics for the Wigner transformation of the Schrödinger operator $\hat{H}_h$. Before stating the precise result in proposition 0.3, we recall some notation. First, the semi-classical Wigner transform is defined to be the unitary operator

$$\mathcal{W}_h : L^2(\mathbb{R}^d \times \mathbb{R}^d) \to L^2(T^*\mathbb{R}^d),$$

taking Hilbert–Schmidt kernels $K_h \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ to their Wigner distributions. The semi-classical Wigner transform extends to temperate (i.e. Schwartz) distributions $K_h \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ in the dual space of Schwartz space by,

$$\mathcal{W}_h(K_h)(x, \xi) := \int_{\mathbb{R}^d} K_h\left(x + \frac{v}{2}, x - \frac{v}{2}\right) e^{-\frac{i}{\hbar} \xi \cdot v} \frac{dv}{(2\pi \hbar)^d}. $$

(21)

In particular, each Wigner function $W_{\psi,\psi_0}$ appearing in theorem 0.2 can be written as the Wigner transform

$$W_{\psi,\psi_0}(x, \xi) = \mathcal{W}_h(\Pi_{\psi,h})(x, \xi),$$

of the kernel

$$\Pi_{\psi,h}(x,y) := \psi(h,x)\psi(h,y)$$

where it is positive; moreover, in a weak sense, the Wigner functions tend to a delta function on $\Sigma_E$. But this is far from sufficient to prove positivity. A possible source of counter-examples
of the rank one projection onto the state $\psi_h(\cdot)$. One advantage of viewing Wigner functions in this way is that, while Wigner functions $W_{\psi_j,\psi_i}$ are quadratic in $\psi_j$, the Wigner transform $W_h(K_h)$ is linear in $K_h$. We will make use of this when proving theorem 0.2. But first we consider the ‘propagator’ $U_h(t) = e^{-\frac{i}{\hbar}H}$, i.e. the solution operator of the Cauchy problem for the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \phi = \hat{H}_h \phi.$$  

More precisely, we will consider the Schwartz kernel $U_h(t, x, y) = e^{-\frac{i}{\hbar}H}(t, x, y) = \sum_{j \geq 0} e^{-\frac{i}{\hbar}E_j \gamma_j(t)} \Pi_{j, h}(x, y) \tag{22}$

of $U_h(t)$ and its Wigner transform

$$U_{\hat{h}}(t, x, \xi) := W(U_{\hat{h}}(t, \cdot))(x, \xi) = \int e^{-\frac{i}{\hbar^{2n}}q v} U_h(t, x, x - \frac{v}{2}, x - \frac{v}{2}) \frac{dv}{(2\pi\hbar)^d}. \tag{23}$$

Following Berry [Ber89], our first task will be to obtain, for fixed $(t, x, \xi)$, semi-classical pointwise asymptotics for $U_{\hat{h}}(t, x, \xi)$. Specifically, we prove the following formula for pointwise semi-classical asymptotics of the Wigner function of the propagator, stated in a heuristic way by Berry [Ber89, (21)] (see also [TL] and [O98]).

**Proposition 0.3.** Let $\hat{H}_h$ satisfy (2). Let $(t, x, \xi)$ be fixed and consider the set of points $(q_j, p_j) \in T^*\mathbb{R}^d$ and the Hamiltonian arcs $\gamma_j = \gamma_{p_j, t, x, \xi}$ such that $\gamma_{p_j, t, x, \xi}(0) = (q_j, p_j)$ and

$$x = \frac{1}{2} (\gamma_{p_j, t, x, \xi}(0) + \gamma_{p_j, t, x, \xi}(t)), \quad \xi = \frac{1}{2} (\gamma_{p_j, t, x, \xi}(0) + \gamma_{p_j, t, x, \xi}(t)). \tag{24}$$

Consider the Jacobian

$$M_j(t, x, \xi) := D_{q,p} \Psi(q, p)|_{(q,p) = (\gamma_{p_j, t, x, \xi}, p_j)}$$

of the Hamilton flow $\Psi'$ of the endpoint $\gamma_{p_j, t, x, \xi}(t) = \Psi'(q_j, p_j)$ with respect to initial point $(q, p)$, and assume that

$$\det(I + M_j(x, \xi)) \neq 0, \quad \forall \ j.$$  

Then, (23) admits the pointwise semi-classical asymptotics,

$$U_{\hat{h}}(t, x, \xi) = 2^d \sum_j \frac{\exp\left(i \frac{S_j}{\hbar} + \eta_j\right)}{\det^{1/2} (I + M_j(x, \xi))} + O(\hbar),$$

where $S_j$ is the classical action and $\eta_j$ is the Maslov index associated to the path $\gamma_{p_j, t, x, \xi}$ (section 5).

This proposition is essentially well known in the physics literature (e.g. [Ber89] and [O98]). We prove it in a new way using the Herman–Kluk parametrix for the propagator (section 6.1). The basic idea, explained in section 6, of the proof of proposition 0.3 is to use the Herman–Kluk parametrix [R10] for the kernel of the propagator $U_{\hat{h}}(t, x, y)$ to obtain a parametrix for its Wigner distribution of the form

$$U_h(t, x, \xi) \sim \int_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar}\Psi(q, p, t, x, \xi)} a_h(t, q, p) dq dp. \tag{25}$$
where $\Psi$ is an explicit complex-valued phase function depending on the underlying classical Hamiltonian flow (see (37)) and $a_\hbar$ is a polyhomogeneous symbol (see (35)). This parametrix is valid for long times due to the sub-quadratic assumption (6) on our Hamiltonians. A straight forward stationary phase argument, detailed in section 6, yields proposition 0.3.

2. Outline of the proof of theorem 0.2

We will deduce theorem 0.2 by analyzing the following relation between $W_{\hbar, f, 1, E}$ and the Wigner function of the propagator:

$$W_{\hbar, f, 1, E}(x, \xi) = \int_{\mathbb{R}} \hat{f}(t) e^{itE/\hbar} U_\hbar(t, x, \xi) \frac{dt}{2\pi},$$

which is a consequence of Fourier inversion and the linearity of the Wigner transform. As discussed in detail in [HZ20, HZb], $U_\hbar(t, x, y)$ is not locally $L^1$ but is nonetheless well-defined as a tempered distribution in the sense that integrals of the form

$$W_{\hbar, f}(x, \xi) := \int_{\mathbb{R}} \hat{f}(t) U_\hbar(t, x, \xi) dt$$

are well-defined for $f \in \mathcal{S}(\mathbb{R})$. It therefore has a well-defined distributional Wigner transform, making (26) a valid expression. To analyze (26), just as in the derivation of proposition 0.3, we start with the Herman–Kluk [HK, R10] parametrix to obtain a parametrix for the Wigner transform of the propagator as in (25). In combination with (26) this yields an oscillatory integral representation

$$W_{\hbar, f, 1, E}(x, \xi) \sim \int_{\mathbb{R}^4} e^{i\hat{\Theta}(q, p, v, (x, \xi)) + itE} a_\hbar(t, q, p) \frac{dq dp dv}{(2\pi \hbar)^d} \frac{dt}{2\pi}.$$ (28)

To obtain the asymptotic expansion in proposition 0.3 we apply stationary phase. The key point is that the critical points with respect to $p, q, v$ are non-degenerate (as they were in the proof of proposition 0.3) but that the critical points with respect to $t$ are degenerate with a fold singularity at the only critical point in the support to $f$, namely at $t = 0$.

After integrating out the $p, q, v$ variables, to obtain asymptotics for $W_{\hbar, f, 1, E}(x, \xi)$ we must integrate in $t$. Theorem 0.2 follows by applying the Chester–Friedman–Ursell (or Malgrange preparation theorem) to the phase of the $dt$ integral resulting from proposition 0.3. In section 8, we use the fold singularity at $t = 0$ to complete the proof of theorem 0.2. Leveraging proposition 0.5 and proposition 0.3, we will see in proposition 3.1 that $W_{\hbar, f, 1, E}(x, \xi)$ is a semi-classical Fourier integral kernel with a complex phase. The critical points in $(t, q, p, v)$ at which the imaginary part of the phase vanishes (and hence the phase integrand is not exponentially small in $\hbar$) are all solutions of

$$\Theta'(q, p) = (x, \xi), \quad H(q, p) = E.$$

The singularities of (28) are determined geometrically by the following result, which is proved at the same time as proposition 3.3 below.

**Proposition 0.4.** The map $\Theta'$ defined in (12) has a fold singularity along $\{0\} \times \Sigma_E$. That is, (12) fixes the diagonal when $t = 0$ and the kernel of its derivative is spanned by the vector field $\frac{\partial}{\partial t}$.

Proposition 0.4 shows that the Lagrangian generated by the phase in (28) has a fold singularity for critical points at $t = 0$. To analyze it, we rely on variants of the Malgrange preparation
3. Contributions to smooth Wigner functions coming from non-degenerate critical times

Although we do not use it in the proof of theorem 0.2, we state a second known result on the asymptotics of (9) for which the stationary phase method applies. We only use it to make comparisons with the Airy asymptotics as \((x, \xi)\) moves away from the fold singularity. In the next proposition we retain the notation of proposition 0.3.

**Proposition 0.5.** Let \(H_2\) satisfy (2). Fix \((E, x, \xi)\) with \((x, \xi) \in B_E\) and consider all solutions \((t_j, q_j, p_j)\) of (24) for which \((q_j, p_j) \in \Sigma_E\) and \(t_j \in \text{supp}\ f\). Assume that \(0 \notin \text{supp}(f)\) and that \(\frac{d}{dE} \det(1 + M_j(x, t_j(E))) \neq 0\) for all \(j\). Then the smoothed spectral Wigner function (9) admits the pointwise semi-classical asymptotics,

\[
W_{h,f,1,E}(x, \xi) := \frac{2^{d+1}}{\sqrt{2\pi h}} \sum_j f(t_j) \left| \frac{d}{dE} \det(1 + M_j(q_j, p_j, t_j(E))) \right|^{1/2} \times \cos \left( \frac{S_j(x, \xi, E)}{h} + m_j \right) + O(h^{1/2}).
\]

Here, \(S_j(x, \xi, E)\) is the action along the trajectory \(\gamma_{j,x,\xi}\).

Under the assumption (6), there are no solutions \((t_j, q_j, p_j)\) unless \((x, \xi) \in B_E\). Moreover, as explained in [Ber89], if the trajectory through \((q, p)\) is periodic of period \(T\), and if \((t, q, p)\) is a solution of (24), then \((t + T, q, p)\) is another solution, but the non-degeneracy condition is not satisfied. Finally, we remark that theorem 0.2 gives the leading order contribution to the smoothed spectral Wigner functions \(W_{h,f,1,E}(x, \xi)\) at phasespace points \((x, \xi)\) that are \(h^{2/3}\) close to the energy surface \(\Sigma_E\). In contrast, proposition 0.5 concerns the asymptotics of \(W_{h,f,1,E}(x, \xi)\) at a fixed phase space point \((x, \xi)\) inside the energy ball \(B_E\) and gives a version of the results of Almeida [O98] and Berry [B77] concerning scar contributions to the spectral Wigner functions coming from macroscopic Hamilton orbits satisfying the mid-point rule. The key to focusing on these longer trajectories is that proposition 0.5 explicitly considers test functions \(f\) whose Fourier transform do not contain 0 in their support.

4. Further scaling results

The results of Berry [Ber89] pertain mainly to the contribution of periodic orbits to the Wigner spectral asymptotics. It is evident from (13) that if \((q, p)\) is a periodic point of period \(T\) then one gets further solutions by replacing \((q, p)\) by \(\Phi^T(q, p)\). The fold singularity along \(\Sigma_E\) at \(t = 0\) also occurs at time \(t = T\) at the periodic points of \(\Sigma_E\) of period \(T\). The order of the asymptotics is higher than it would be for non-degenerate critical points, and Berry therefore referred to the caustic enhancement of such periodic orbits as ‘scars’. The methods of this article extend to the periodic orbit case with few modifications but for the sake of brevity we do not include them here.
In [HZb], further scaling asymptotics are proved in the case of the isotropic harmonic oscillator. The essential difference to theorem 0.2 is that much larger spectral intervals are assumed. We do not generalize these results to general Hamiltonians satisfying (2) or (6) in this article, but it is very likely that generalizations do exist. We briefly review these additional scaling results.

Instead of spectral intervals of width \( h \) we consider intervals of width \( h^{2/3} \), e.g. \( [E - \lambda, h^{2/3}, E + \lambda, h^{2/3}] \). In the smoothing asymptotics we let \( \delta = \frac{1}{h} \) and consider

\[
W_{h^2/3E,h}(x,x) := \sum_{j=1}^{\infty} f_{h^2/3E,h}(E_j) \hat{W}_{h^2/3E,h}(x,x).
\]

We define the rescaled variable \( u = u(x,\xi) \) at the energy surface \( \Sigma_E \) by

\[
H(x,\xi) = E + u(h/2E)^{2/3}.
\]

We then prove that,

\[
W_{2/3E,h}(x,\xi) = \frac{(2\pi)^d}{4} C_E \int_{-\lambda_+}^{\lambda_+} \text{Ai}\left(\frac{u}{E} + \lambda C_E\right) d\lambda + O(h^{-d+1/3-\delta}), \quad C_E = (E/4)^{1/3}.
\]

Instead of getting the value of the Airy function at the scaled parameter \( u \), we get an integral over the values due to the larger spectral interval.

Furthermore, in the even larger interval \( I_h = [0,E] \), for any \( \varepsilon > 0 \), we obtained the bulk asymptotics,

\[
W_{h,0,E}(x,\xi) = \frac{(2\pi)^d}{4} \int_{-\lambda_+}^{\lambda_+} \text{Ai}\left(\frac{u}{E} + \lambda\right) d\lambda + O(h^{1/3-\varepsilon}|u|^{1/2}) + O(|u|^{5/2} h^{2/3-\varepsilon}),
\]

where the \( O \)-symbol depends only on \( d, \varepsilon \).

The articles [Ber89, O98] use the special (Lorentzian) test function \( f_{\varepsilon}(E - E_j(h)) = \delta_{\varepsilon}(E - E_j(h)) \), with various choices of \( \varepsilon \) stated in [Ber89, pages 220–221] as ranging from the mean level spacing of order \( h^0 \) and the semi-classical scaling \( \frac{h}{\text{typ}} \) where \( T_{\text{typ}} \) is the length of the shortest periodic orbit on \( \Sigma_E \). The same test functions and energy scales are used in [O98, section 7]. The mathematical techniques of this article (along with other spectral asymptotics articles in the mathematics literature) are not valid below the semi-classical scale, and in particular do not give scaling results on the length scale \( \delta = h^\nu \) with \( \nu > 1 \); they do apply on the scale \( \frac{h}{\text{typ}} \) and we prove theorem 0.2 by using a special case of the Malgrange preparation theorem due to Chester–Friedman–Ursell [CPU] to put the phase into normal form; the relevant theorems are explained in detail in [HoI, chapter 7] and in [GST, page 439, page 444]. Such asymptotics are used to determine the asymptotics of oscillatory integrals whose phase functions exhibit fold singularities, such as occur in diffraction theory. In section 7.4 we identify the relevant folding map.

Besides linking the somewhat heuristic asymptotics calculations of [Ber89, O98] to the mathematical literature, in particular making more precise the scale of the asymptotics, a novelty of our presentation is to use the Herman–Kluk propagator as discussed by Robert [R10] to construct a ‘parametrix’ for (9).
5. Background on classical mechanics

In this section, we recall some basic results from classical mechanics and set notation. With \( \Phi^t \) the Hamiltonian flow defined in (10), we continue to denote its derivative at \((q, p)\) by

\[
D_{q,p} \Phi^t := M_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix},
\]

where

\[
A_t = \partial_q q_t, \quad B_t = \partial_p q_t, \quad C_t = \partial_q p_t, \quad D_t = \partial_p p_t,
\]

with rows corresponding to components and columns to derivatives:

\[
(\partial q_{j} q_{t})_k = (A_t)_{k,j}.
\]

Since \( \Phi^t \) is a Hamiltonian flow, we have

\[
\Omega = M_t^T \Omega M_t,
\]

where \( \Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \) is the symplectic form [F] and we therefore have

\[
A_t C_t = C_t^T A_t, \quad A_t^T D_t - C_t^T B_t = \text{Id}, \quad B_t D_t = D_t B_t.
\]

The action along the \( \Phi^t \) orbit with initial value \((q, p)\) is defined by,

\[
S(t, q, p) := \int_0^t (\dot{q}_s \cdot p_s - H(q_s, p_s)) ds
\]

We only deal with autonomous Hamiltonians, for which \( H(s, q_s, p_s) = H(q_s, p_s) \) and then \( H(q_s, p_s) \) is constant along Hamilton orbits and the second term is \( tH(q, p) \). We will have occasion to use the following elementary result.

**Lemma 1.1.** We have

\[
\begin{aligned}
\partial_q S(t, q, p) &= (\partial_q q_t) \cdot p_t - p, \\
\partial_p S(t, q, p) &= (\partial_p q_t) \cdot p_t, \\
\partial_t S(t, q, p) &= \dot{q}_t \cdot p_t - H(q_t, p_t).
\end{aligned}
\]

**Proof.** Using the equations of motion \( \partial_p H = \dot{q}_t \) and \( \partial_q H = -p_t \), we get

\[
\partial_q S(t, q, p) = \partial_q \left( \int_0^t \dot{q}_s \cdot p_s - H(q_s, p_s) ds \right)
\]

\[
= \int_0^t (\partial_q \dot{q}_s \cdot p_s + \dot{q}_s \cdot \partial_q p_s - \partial_q H(q_s, p_s) \partial_q q_s - \partial_p H(q_s, p_s) \partial_q p_s) ds
\]

\[
= \int_0^t \langle \partial_q \dot{q}_s, p_s \rangle + \langle \dot{q}_s, \partial_q p_s \rangle + \langle \dot{p}_s, \partial_q q_s \rangle - \langle \dot{q}_s, \partial_p p_s \rangle ds
\]

\[
= \int_0^t \langle \partial_q \dot{q}_s, p_s \rangle + \langle \dot{p}_s, \partial_q q_s \rangle ds = \int_0^t \partial_t \langle \partial_q q_s, p_s \rangle ds = \langle \partial_q q_t, p_t \rangle |_0^t.
\]
proving the first statement. Similarly,
\[
\partial_p S(t, q, p) = \partial_p \left( \int_0^t \dot{q}_s \cdot p_s - H(q_s, p_s) \, ds \right) \\
= \int_0^t \left( \partial_p \dot{q}_s \cdot p_s + \dot{q}_s \cdot \partial_p p_s - \partial_q H(q_s, p_s) \partial_p q_s - \partial_p H(q_s, p_s) \partial_p p_s \right) \, ds \\
= \int_0^t \left( \partial_p \dot{q}_s \cdot p_s + \dot{q}_s \cdot \partial_p p_s + \langle \dot{p}_s, \partial_p q_s \rangle - \langle \dot{q}_s, \partial_p p_s \rangle \right) \, ds \\
= \int_0^t \langle \partial_p \dot{q}_s \cdot p_s + \langle \dot{p}_s, \partial_p q_s \rangle \rangle \, ds = \int_0^t \frac{d}{ds} \langle p_s, \partial_p q_s \rangle \, ds = \langle p_t, \partial_p q_t \rangle |_{t=0},
\]
proving the second statement. The third statement is obvious. □

6. Herman–Kluk parametrix and the proof of proposition 0.3

In this section, we prove proposition 0.3. For this, we recall in section 6.1 the Herman–Kluk parametrix for the propagator. Then we use this parameterix in conjunction with stationary phase to complete the proof of proposition 0.3 in section 6.2. We then provide in sections 6.3 and 6.4 two conceptual remarks on the proof that will be useful for guiding our subsequent proof of theorem 0.2.

6.1. Herman–Kluk parametrix for the Wigner transform of the propagator

As explained in [R10, RS] (see [R10, (1.7)]), for subquadratic Hamiltonians satisfying 2, one can construct a long-time parameterix
\[
U_{\hbar}(t, x, y) \sim \int_{\mathbb{R}^{2d}} e^{i \frac{\Psi_{HK}(t, q, p, x, y)}{\hbar}} a_{\hbar}(t, q, p) \, dq \, dp
\]
for the propagator $U_{\hbar}(t, x, y)$ (see (22)) with the complex Herman–Kluk phase,
\[
\Psi_{HK}(t, q, p, x, y) := S(t, q, p) + p(x - q_t) - p(y - q) + \frac{i}{2} \left( |x - q_t|^2 + |y - q|^2 \right).
\]

Here, $S(t, q, p)$ is the action given by (32). The phase $\Psi_{HK}$ is called a positive complex phase since its imaginary part is positive. Moreover, $a_{\hbar} \sim \sum_j h^a_j(q, p)$ is a polyhomogeneous symbol with
\[
a_{0} = (\det(A_t + D_t + i(B_t - C_t)))^{1/2} \exp(-iH(q, p)),
\]
where we have written as in (30) section 5
\[
A_t = \frac{\partial q_t}{\partial q}, \quad B_t = \frac{\partial q_t}{\partial p}, \quad C_t = \frac{\partial p_t}{\partial q}, \quad D_t = \frac{\partial p_t}{\partial p}.
\]
The parameterix is valid for all times $t < \log(1/\hbar)$ and the symbol estimates are uniform over compact intervals in $t$. The parameterix (33) for the propagator $U_{\hbar}(t, x, y)$ reveals that its Wigner transform $U_{\hbar}(t, x, \xi)$ is an oscillatory integral with positive complex phase. Namely,
\[
U_{\hbar}(t, x, \xi) \sim (2\pi \hbar)^{-d} \int_{\mathbb{R}^{2d}} e^{i \frac{\Psi_{HK}(t, q, p, x, y)}{\hbar}} a_{\hbar}(t, q, p) \, dq \, dp \, dv \frac{dv}{(2\pi \hbar)^d}.
\]
where
\[
\Psi(t, x, \xi; q, p, v) := S_t + p\left(x + \frac{v}{2} - q_t\right) - p\left(x - \frac{v}{2} - q\right) - v\xi
\]
\[
+ \frac{i}{2}\left(\left|x + \frac{v}{2} - q_t\right|^2 + \left|x - \frac{v}{2} - q\right|^2\right).
\]

The integral is of course exponentially small in \(\hbar\) unless the imaginary part vanishes, i.e.
\[
\text{Im } \Psi = \left|x + \frac{v}{2} - q_t\right|^2 + \left|x - \frac{v}{2} - q\right|^2 = 0.
\]

6.2. Proof of proposition 0.3

We are now ready to complete the proof of proposition 0.3. In the notation of section 5, the main technical result is the following.

**Proposition 2.1.** The Wigner function \(\mathcal{U}_\hbar(t, x, \xi)\) of the propagator is a semi-classical Fourier integral kernel with a positive complex phase. The dominant critical points (in \((q, p, v)\)) at which (38) holds are solutions of
\[
x = \frac{q + q_t}{2}, \quad \xi = \frac{p + p_t}{2}, \quad v = q_t - q.
\]

Moreover, there exist \(\varepsilon, \varepsilon_0 > 0\) such that for \((x, \xi)\) in an \(\varepsilon\)-neighborhood of \(\Sigma_E\) and \(t \in (-\varepsilon_0, \varepsilon_0)\), critical points are uniformly non-degenerate. The critical value of the phase at a critical point \((q, p, v)\) is given by:
\[
\Psi_c(t, x, \xi) := \int_{\gamma_{t,x,\xi}} (pdq - Hds) - \langle(q_t - q), \xi\rangle,
\]
where \(\gamma_{t,x,\xi}\) was defined in (16) as the unique Hamiltonian arc (up to the involution (14)) with \(|t|\) small on \(\Sigma_E\) such that \((x, \xi) = \frac{1}{2}(\gamma_{t,x,\xi}(0) + \gamma_{t,x,\xi}(t))\).

Define the action,
\[
S(t, x, \xi) = \int_{\gamma_{t,x,\xi}} (pdq - Hds)
\]
along the path \(\gamma_{t,x,\xi}\).

**Proof of proposition 2.1.** As mentioned above, the imaginary part of the phase is strictly positive unless
\[
x + \frac{v}{2} - q_t = x - \frac{v}{2} - q = 0.
\]

Hence, any critical points where these conditions are satisfied will provide the dominant contribution to the Wigner function. It is immediate from (41) that at the dominant critical points,
\[
x = \frac{q + q_t}{2}, \quad v = q_t - q.
\]

In addition, the equation \(d_v \Psi = 0\) implies that
$$\xi = \frac{p + p_i}{2}.$$ 

To complete the proof of proposition 0.3, we fix $t > 0$, consider the full $dq dp dv$ integral (36), and apply the method of stationary phase for positive complex phase functions (cf [HoI, theorem 7.7.5]). For this, we need to compute the Hessian of (37) at the dominant critical points. We use the notation (30), which we repeat for the reader’s convenience,

$$A_t(q, p) = \partial_{q} q_t(q, p), \quad B_t(q, p) = \partial_{p} q_t(q, p), \quad C_t(q, p) = \partial_{q} p_t(q, p),$$
$$D_t(q, p) = \partial_{p} p_t(q, p).$$

The key result is the following computation.

**Lemma 2.2.** Fix $t, x, \xi$ and suppose $(q_c, p_c, v_c)$ is a dominant critical point of $\Psi(q, p, v; t, x, \xi)$ (37) in the variables $(q, p, v)$ for which $\partial_{p} q_t \neq 0$. Then

$$\text{det}(\text{Hess}(\Psi))(q_c, p_c, v_c)$$

is given by,

$$(-1)^d \cdot \text{det}(1 + M_t(q_c, p_c)) \text{det}(A_t(q_c, p_c) + D_t(q_c, p_c) + i(B_t(q_c, p_c) - C_t(q_c, p_c))).$$

**Proof.** We claim that at the critical point,

$$\partial_{pp} \Psi = B_t^T (-D_t + iB_t),$$
$$\partial_{pq} \Psi = B_t^T (-C_t + iA_t),$$
$$\partial_{pq} \Psi = \frac{1}{2} (\text{Id} + D_t - iB_t)^T,$$
$$\partial_{qs} \Psi = i \text{Id},$$
$$\partial_{q} \Psi = -A_t^T C_t + i (A_t^T A_t + \text{Id}),$$
$$\partial_{q} \Psi = \frac{1}{2} (C_t + i(-A_t + \text{Id})).$$

We summarize the key points of the calculations. *A priori*, the Hessian involves second derivatives in $(q, p)$ of $(q_t, p_t, \dot{q}_t, \dot{p}_t)$ but in fact these cancel at a dominant critical point. First,

$$\partial_{p} \Psi = \partial_{p} p_t \left( x + \frac{v}{2} - q_t \right) - \left( x_j - \frac{v_j}{2} - q_j \right) + i \left( -\partial_{p} q_t \left( x + \frac{v}{2} - q_t \right) \right), \quad (43)$$

and at a critical point,

$$\partial_{p} \Psi = -\partial_{p} p_t \partial_{p} q_t + i (\partial_{p} q_t \partial_{p} q_t) = (-\partial_{p} (p_t + i\partial_{p} q_t) \partial_{p} q_t = \left( (-D_t + iB_t)^T B_t \right)_{k,l}.$$ 

Also,

$$\partial_{q} \Psi = \partial_{q} p_t \left( x + \frac{v}{2} - q_t \right) - i \left( \partial_{q} q_t \left( x + \frac{v}{2} - q_t \right) + \left( x_j - \frac{v_j}{2} - q_j \right) \right), \quad (44)$$

and at a critical point,

$$\partial_{p} \Psi = -\partial_{q} p_t \partial_{p} q_t + i (\partial_{q} q_t \partial_{p} q_t).$$
Starting from (43), we have
\[
\partial_{q,p_j} \Psi = \frac{1}{2} \left( (\partial_{q,p_j})_k + \delta_{jk} - i(\partial_{q,p_j})_k \right) = \frac{1}{2} (\text{Id} + D_t - iB_t)_{k,j}.
\]

Again starting from (43), we have
\[
\partial_{p_j} \Psi = \left( -\partial_{p_j} p_i + i\partial_{p_j} q_i \right) \dot{q}_i = [\dot{q}_i(-D_t + iB_t)]_j.
\]

Starting from (44), we have
\[
\partial_{q,q_j} \Psi = \partial_{q_j} p_i \partial_{q_i} q_j + i(\partial_{q_j} q_i \partial_{q_k} q_i + \delta_{jk}) = \left[ A_t^T C_t + iA_t^T A_t + \text{Id} \right]_{k,j}.
\]

Finally, from (44) we obtain
\[
\partial_{v,q_j} \Psi = \frac{1}{2} \left( (\partial_{q,q_j})_k + i\left( -\partial_{q,q_j} q_i + \delta_{j,k} \right) \right).
\]

The Hessian of \( \Psi \) at \((q_c, p_c, v_c)\) is therefore
\[
\begin{pmatrix}
\frac{i}{2} \text{Id} & \frac{1}{2} (C_t + i(-A_t + \text{Id})) & \frac{1}{2} (D_t + \text{Id} - iB_t) \\
\frac{1}{2} (C_t^T + i(-A_t + \text{Id})) & -A_t C_t + i(\tilde{A}_t^2 + \text{Id}) & -C_t^T B_t + i\tilde{A}_t B_t \\
\frac{1}{2} (D_t + \text{Id} - iB_t^T) & -B_t^T C_t + iB_t^T A_t & B_t^T (-D_t + iB_t) \\
\end{pmatrix},
\]

as claimed. We now calculate the determinant of the Hessian. Multiplying the first row on the left by \( A_t \) and adding the result to the second row and then multiplying the first row by \( B_t^T \) on the left and adding the result to the third row shows that the determinant of \( \text{Hess}(\Psi)(q_c, p_c, v_c) \) is the same as the determinant of
\[
\begin{pmatrix}
\frac{i}{2} \text{Id} & \frac{1}{2} (C_t + i(-A_t + \text{Id})) & \frac{1}{2} (\tilde{D}_t - iB_t) \\
\frac{1}{2} (C_t^T + i\tilde{A}_t) & -i\tilde{A}_t & \tilde{A}_t \\
\frac{1}{2} (\tilde{D}_t + iB_t^T) & iB_t^T & B_t^T \\
\end{pmatrix},
\]

where for any matrix \( K \), we write \( \tilde{K} = \text{Id} + K \). Next, multiplying the first column by 2 and subtracting it from the second column, we find that the determinant of \( \text{Hess}(\Psi)(q_c, p_c, v_c) \) is the same as the determinant of
\[
\begin{pmatrix}
\frac{i}{2} \text{Id} & \frac{1}{2} (C_t - i\tilde{A}_t) & \frac{1}{2} (\tilde{D}_t - iB_t) \\
\frac{1}{2} (C_t^T + i\tilde{A}_t) & -C_t^T & \tilde{A}_t \\
\frac{1}{2} (\tilde{D}_t + iB_t^T) & -\tilde{D}_t & B_t^T \\
\end{pmatrix}.
\]
Next, by dividing the first row and column by $\sqrt{2/i}$, we find that the determinant of Hess($\Psi(q_c, p_c, v_c)$) is the same as the determinant of

$$
\begin{pmatrix}
\text{Id} & \frac{1}{\sqrt{2i}}(C_i - i\tilde{A}_i) & \frac{1}{\sqrt{2i}}(\tilde{D}_i - iB_i) \\
\frac{1}{\sqrt{2i}}(C_i^T + i\tilde{A}_i) & -C_i^T & \tilde{A}_i \\
\frac{1}{\sqrt{2i}}(\tilde{D}_i + iB_i^T) & -\tilde{D}_i & B_i^T
\end{pmatrix}
$$

times (i/2)^d. Next, writing $\Omega = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$, note that

$$
\begin{pmatrix}
-C_i^T & \tilde{A}_i \\
-\tilde{D}_i & B_i^T
\end{pmatrix} = \tilde{M}_i^T \Omega.
$$

Further, observe that

$$
\begin{pmatrix}
C_i - i\tilde{A}_i, \\
\tilde{D}_i - iB_i
\end{pmatrix} = (-i\text{Id} \text{ Id})\tilde{M}
$$

and that

$$
\begin{pmatrix}
C_i^T + i\tilde{A}_i \\
\tilde{D}_i + iB_i^T
\end{pmatrix} = \tilde{M}_i^T \begin{pmatrix} i\text{Id} \\ \text{Id} \end{pmatrix}.
$$

Hence, the determinant of Hess($\Psi(q_c, p_c, v_c)$) is the same as $(i/2)^d$ times the determinant of

$$
\begin{pmatrix}
\text{Id} & 0 \\
0 & \tilde{M}_i^T
\end{pmatrix}
\begin{pmatrix}
\text{Id} & \frac{1}{\sqrt{2i}}(-i\text{Id} \text{ Id})\tilde{M} \\
\frac{1}{\sqrt{2i}}(i\text{Id} \text{ Id}) & \Omega
\end{pmatrix}.
$$

Therefore, the determinant of Hess($\Psi(q_c, p_c, v_c)$) equals $(2/i)^d$ times $\text{det}(\tilde{M})$ times the determinant of

$$
\begin{pmatrix}
\text{Id} & \frac{1}{\sqrt{2i}}(-i\text{Id} \text{ Id})\tilde{M} \\
\frac{1}{\sqrt{2i}}(i\text{Id} \text{ Id}) & \Omega
\end{pmatrix}.
$$

Take the transpose of this matrix and using the Schur complement formula, we find that the determinant of this matrix is the same as the determinant of

$$
\text{Id} - \frac{1}{2i} \left[ (i\text{Id} \text{ Id})\Omega i\text{Id} \text{ Id} \right] = \frac{1}{2i} \left[ (-\text{Id} \text{ Id})\Omega i\text{Id} \text{ Id} \right],
$$

where we have used that

$$
\frac{1}{2i} \left[ (i\text{Id} \text{ Id})\Omega (-\text{Id} \text{ Id}) \right] = 2i\text{Id}.
$$
Finally, since
\[
(-\text{Id} \ i \ \text{Id})\Omega M^T (-\text{Id} \ i \ \text{Id})^T = A_t + D_t + i(B_t^T - C_t^T)
\]
and the determinant is invariant under transposing the matrix, we find that
\[
\det(\text{Hess}(\Psi)(q_c, p_c, v_c)) = 2^{2d} \det(\text{Id} + M_t) \cdot \det(A_t + D_t + i(B_t - C_t)), \quad (45)
\]
as claimed. □

To complete the proof of proposition 0.3, we note that when \((x, \xi)\) and \(t = 0\), we have \(M_t = I\) and obviously the modulus of (45) is a positive constant. Hence the same is true for \((x, \xi)\) in a tubular neighborhood of some positive radius around \(\Sigma_E\). Thus, proposition 0.3 follows by stationary phase for positive complex phases [HoI]. □

6.3. Remark: integrating out \(dv\) first

We remark that a somewhat simpler analysis is possible for the proof of proposition 0.3 by first integrating in \(dv\), to get a reduced oscillatory integral in \(dq dp\). This is natural since the \(dv\) integral is essentially a Fourier transform of a Gaussian, slightly distorted by the non-constant amplitude. In particular the Hessian \(d_v^2\Psi\) at the critical point is non-degenerate. Eliminating \(dv\) we obtain the reduced integral,
\[
\mathcal{U}_h(t, x, \xi) = \frac{1}{(2\pi \hbar)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i \Psi_c(h, q, p, t)} A(h, q, p, t) dq dp
\]
with reduced phase,
\[
\Psi_c(t, x, \xi; q, p) := S_t(q, p) + (p_t - p) \left( x - \frac{q_t + q}{2} \right) - \langle q_t - q, \xi \rangle + i \left( \left| x - \left( \frac{q_t + q}{2} \right) \right|^2 \right).
\]

Here, we did not use the equation \((x - \frac{q_t + q}{2}) = 0\). The amplitude is that of the original \(dq dp dv\) integral evaluated at \(v = q_t - q\) and multiplied by the constant Hessian determinant in \(v\). We need to further apply stationary phase for complex phase functions in \(dq dp\) to obtain this as a critical point equation. If we do that, we obtain the critical point equation for the real part of (47) simplify to,
\[
\begin{cases}
  x = \frac{q_t + q}{2}, \\
  d_q S_t - \frac{1}{2}(p_t - p)(I - d_q q_t) - \langle d_q q_t, \xi \rangle + \xi = 0, \\
  d_p S_t - \frac{1}{2}(p_t - p)d_p q_t - \langle d_p q_t, \xi \rangle = 0
\end{cases}
\]
and by lemma 1.1 reduce further to,
\[
\begin{cases}
  \left( \frac{1}{2}(p_t + p) - \xi \right)[-I + d_q q_t] = 0, \\
  = \left( \frac{1}{2}(p_t + p) - \xi \right)d_p q_t = 0,
\end{cases}
\]
which (again) imply $\frac{1}{2}(p_t + p) = \xi$. At a dominant critical point, the real part of the Hessian is given by,

$$\text{Re Hess}_q^p \Psi_{c, \text{crit}} = \begin{pmatrix}
A_tB_t - \frac{1}{2}C_t(I + A_t) & [A_tD_t - \frac{1}{2}B_tC_t]I^T \\
A_tD_t - \frac{1}{2}B_tC_t & B_tD_t - \frac{1}{2}D_tC_t + \frac{1}{2}C_t
\end{pmatrix}|_{\varphi'(q,p) = (x,\xi)}.$$

and the imaginary part of the Hessian is given by,

$$\text{Im Hess} = \begin{pmatrix}
\frac{1}{4}[2I + 2(\partial_q q_t, \partial_q q_t) + 2\partial_q q_t] & 2(\partial_p q_t, \partial_q q_t) + 2\partial_q q_t \\
2(\partial_p q_t, \partial_q q_t) + 2\partial_q q_t & \frac{1}{2}(\partial_p q_t, \partial_p q_t)
\end{pmatrix}.$$

We leave this approach at this point since we have already calculated the full Hessian determinant.

### 6.4. Remark: interpretation of proof of proposition 0.3 by Lagrangian submanifolds

For fixed $(t,x,\xi)$, the critical set of the phase (37) in the sense of [HoIII] is therefore

$$C_{W_{\bar{h}}} = \{(x,\xi,q,p,v): x = \frac{q + q_t}{2}, \xi = \frac{p + p_t}{2}, v = q_t - q\}.$$

Using lemma 1.1, we find that the associated Lagrangian submanifold of $T^*(\mathbb{R} \times T^*\mathbb{R}^d)$ is given by,

$$\Lambda = \{(x,\xi,q,p,v): (x,\xi,q,p,v) \in C_{W_{\bar{h}}(x,\xi)}\}$$

$$= \left\{(x,\xi,q,p,v): \frac{q + q_t}{2}, \xi = \frac{p + p_t}{2}\right\}.$$  \hspace{1cm} (48)

We also consider the space-time Lagrangian

$$\Gamma = \left\{(t,x,\xi,q,p,v): (t,x,\xi,q,p,v) \in C_{W_{\bar{h}}(x,\xi)}\right\} \subset T^*(\mathbb{R} \times T^*\mathbb{R}^d)$$

$$= \left\{(t,\varphi'(q,p),x,\xi,q,p,v): \frac{q + q_t}{2}, \xi = \frac{p + p_t}{2}\right\}.$$  \hspace{1.5cm} (49)

We have the natural projection,

$$\pi: \Gamma \rightarrow \mathbb{R} \times T^*\mathbb{R}^d,$$

given by

$$\left(t,\varphi'(q,p) + p_t \left(x + \frac{q_t - q}{2} - q_t\right),x,\xi,q_t,q_p,v\right) \mapsto (t,\xi).$$
Since $v = q_t - q$ is uniquely determined once $(q, p)$ are determined, the fiber of (50) is the set of solutions of

$$\pi^{-1}(t, x, \xi) = \left\{ (q, p, v) : x = \frac{q + q_t}{2}, \quad \xi = \frac{p + p_t}{2}, \quad v = q_t - q \right\}$$

as defined in (12). By the inverse function theorem, the solution is locally unique if $D_{(q, p)}\Theta'$ does not have $-1$ as an eigenvalue. As an example where it is non-unique, we could let $(x, \xi) = (0, 0)$ and consider ‘anti-podal’ times when $\Theta'(q, p) = -(q, p)$. For instance, in the case of the isotropic oscillator such a time is $t = \pi$. An oscillator on $\mathbb{R}^d$ with $k \leq d$ equal frequencies provides an example where the fiber is not discrete but has dimension $d - k$.

7. Proof of theorem 0.2: the $t$ integral and energy asymptotics

In this section we extend the analysis from the previous section, which involved computing the integral (36) in $v, q, p$ to computing pointwise asymptotics for the Wigner transform $\hat{U}_h(t, x, \xi)$ of the propagator, to include also the $t$ integral (see e.g. (9))

$$W_{h, f, 1, E}(x, \xi) \sim \int_{\mathbb{R}^d} e^{i\hat{\Theta}(q, p; x, \xi) + iE} a_h(t, q, p) \frac{dq \, dp \, dv}{(2\pi \hbar)^d} \, dt$$

(51)

to determine the asymptotics of the Wigner distribution of the smoothed spectral projector.

As in [O98, TL], it is the $t$-integral which introduces a degeneracy in the phase and a fold singularity in the associated Lagrangian submanifold (see sections 7.3 and 7.4 for the geometric analysis of folds). In the next section 8, we put the phase into Malgrange normal form, or more correctly a stronger version of it for cubic phases due to Chester–Friedman–Ursell [CFU] and Levinson [L61].

We consider two approaches to evaluating (51): (i) integrate in $(t, q, p, v)$ at once; (ii) integrate in $(q, p, v)$ first, reduce to a one dimensional integral in $dt$ and then determine the asymptotics of this integral. The advantage of (i), which we carry out in section 7.1, is that the critical point equations are considerably simpler to solve. However, the phase is complex-valued and the standard normal forms results need to be modified to apply to it. The advantage of (ii), which we carry out in section 7.2, is that we can directly apply the asymptotics results from the literature [CFU, HoI, GST]. But the critical point equations become more complicated.

7.1. The $dt \, dq \, dp \, dv$ integral

We extend propositions 2.1 and 2.2 by including the additional integration in $t$.

**Proposition 3.1.** Let $f \in \mathcal{S}(\mathbb{R})$. The Wigner function

$$W_{h, E}(x, \xi) = \int_{\mathbb{R}^d} \hat{\rho}(t) W_{h_f}(x, \xi) e^{i\frac{\pi}{\hbar} \frac{d^2}{d\xi^2}} dq \, dp \, dv \, dt$$

of the smoothed spectral function is a semi-classical Fourier integral kernel with a complex phase. The critical points in $(t, q, p, v)$ at which the imaginary part of the phase

$$\tilde{\Psi}_E(t, q, p, v; x, \xi) := \Psi(t, q, p, v, x, \xi) + tE$$

vanishes are all solutions of

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We now rewrite (39) in an advantageous form,

\[
\Psi(t, x, \xi) := \int_{\gamma_{t, x, \xi}} (p \, dq) - \int_{\gamma_{t, x, \xi}} H \, ds = \int_{\tilde{\gamma}_{t, x, \xi}} \omega - \int_{\tilde{\gamma}_{t, x, \xi}} H \, ds,
\]  

(52)

where \( \tilde{\gamma}_{t, x, \xi} \) is the completed contour obtained by adding in the chordal link to obtain the oriented closed curve connecting the endpoints of \( \gamma_{t, x, \xi} \) by the chord \( \alpha_{t, x, \xi}(s) = (1 - s)\gamma_{t, x, \xi}(0) + s\gamma_{t, x, \xi}(1) \) (see (18)). Note that
\[
\int_{\alpha_{t,\xi}} p\, dq = \int_0^1 ((1-s)p_{t,\xi}(0) + sp_{t,\xi}(t))d(1-s)q_{t,\xi}(0) + sq_{t,\xi}(t)
\]
\[
= \left( \int_0^1 ((1-s)p_{t,\xi}(0) + sp_{t,\xi}(t))\, dt \right) \cdot \left( q_{t,\xi}(t) - q_{t,\xi}(0) \right)
\]
\[
= (\xi, q_{t,\xi}(t) - q_{t,\xi}(0)).
\]

As mentioned below (18), \(\tilde{\gamma}_{t,\xi}\) bounds the two-dimensional surface \(\beta_{t,\xi}\) consisting of line segments joining \((x, \xi)\) to points of the Hamilton orbit, the integral (18) equals the oriented area \(\int_{\beta_{t,\xi}} \omega\) where \(\omega = dp \wedge dq\) is the standard symplectic form of \(T^* \mathbb{R}^d\).

**Proposition 3.2.** The Wigner function \(W_{n,E}(x, \xi) = \int_{\mathbb{R}} \tilde{\rho}(t)W_{n,E}(x, \xi)e^{2\pi i E t^2} \, dt\) of the smoothed spectral function is a semi-classical Fourier integral kernel with real phase, 
\[
\Psi_E(t, x, \xi) := \int_{\beta_{t,\xi}} \omega - tH(\gamma_{t,\xi}(0)) + tE,
\]
and with amplitude \(|\det(1 + M(q, p, t)(E))|^{1/2}\), where \(q, p, t(E)\) are as in propositions 0.3 and 0.5. For \((x, \xi)\) near \(\Sigma_E\), the critical point in \(t\) of the phase occur at times \(t\) such that there exist \((q, p)\) in \(\Sigma_E\) for which \((x, \xi) = \Theta(q, p).\) If \(t_{\pm}(E, x, \xi)\) are the times \(t\) such that \(\Theta(q, p) = (x, \xi)\) has a solution, then at a critical point, \(\frac{\partial}{\partial t}\Psi_E(t) = -\partial H(\gamma_{t,\xi}(0))(J_{t,\xi}(0)) = (\frac{\partial T}{\partial \psi})^{-1}E\).

**Proof.** The equation (53) is the same as (52). We start with a general principle: let \(f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) be a smooth function, and consider the critical point equation \(dq_{t,\xi}f(x,y) = 0.\) Let \((a, b)\) be a critical point such that the partial Hessian \((\frac{\partial^2}{\partial y^2}f)(a, b)\) is non-degenerate. Then by the implicit function theorem, there exists an open set \(U \subset \mathbb{R}^n\) containing \(a\) and a unique differentiable function \(g: U \to \mathbb{R}^m\) such that \(g(a) = b\) and \(d_x f(x, g(x)) = 0\) identically in \(U.\) Moreover, the critical point equations \(d_q f(x, g(x)) = 0\) and \(d_q f(x, g(x)) = 0\) are equivalent in \(U\) to \(d_{x,\xi}f(x, y) = 0.\) Indeed, \(\frac{\partial}{\partial y} f(x, g(x)) = \frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial g}(x, g(x)) \frac{\partial g}{\partial x}.\) The second term is zero in \(U\) by definition of \(g(x),\) and then the first term must be zero when the left side is zero.

We apply this to the case where \(n = 1, m = 3d\) and \(f(t, q, p, v)\) of the phase (37). It follows from proposition 2.1 that the non-degeneracy condition is satisfied as long as \(D\Theta^\tau\) is non-singular, i.e. \(D\Theta^\tau\) does not have \(-1\) as an eigenvalue. We then get locally defined branches \(g: \mathbb{R} \to \mathbb{R}^{3d}, g(t) = (q(t), p(t), v(t))\) such that \(\det_{t,\xi,p,v}(t)(q(t), p(t), v(t)) = 0.\) The phase of the additional \(dt\) integral is \(\Psi(t, q(t), p(t), v(t)) + tE.\) By the general principle, its critical points are the same as for those of \(d_{t,\xi,p,v}\Psi = 0\) and by proposition 3.1 these are given by the equations stated there.

**7.2. Calculation of \(\frac{d}{dt}\Psi_E(t): Reynolds formula.** A direct proof without using implicit functions can be obtained from the Reynolds transport formula for the first term of (52),
\[
\frac{d}{dt} \int_{\beta_{t,\xi}} \omega = \int_{\beta_{t,\xi}} J_{t,\xi}(0) \, \omega.
\]
Here, \(J_{t,\xi}(0)\) is the variational (Jacobi) vector field along \(\beta_{t,\xi} = \tilde{\gamma}_{t,\xi}\) obtained by varying \(\gamma_{t,\xi}\) and the chord \(\alpha_{t,\xi}: J_{t,\xi}\) is determined by \(\gamma_{t,\xi}(0),\) since \(\Phi^*\gamma_{t,\xi}(0) = (J_{t,\xi}(s)).\) Hence,
\[
J_{t,\xi}(s) := \frac{d}{dt} \Phi^*\gamma_{t,\xi}(0) = D\gamma_{t,\xi}(0)\Phi^* \left( \frac{d}{dt}\gamma_{t,\xi}(0) \right).
\]
J\textsubscript{t,ξ}(s) is a Jacobi field along γ\textsubscript{t,ξ}(s), i.e. the variational vector field obtained by varying a curve of Hamilton orbits. It is not a Jacobi field on α\textsubscript{t,ξ} since these are not extremals, but for brevity we continue to call it a Jacobi field. We compute the integral by writing

\[ \int_{\gamma_{t,\xi}} t_{t,\xi,\omega} = \int_0^t \omega(J_{t,\xi}(s),\xi) ds = \int_0^t dH(J_{t,\xi}(s)) \]

\[ = \int_0^t (\Phi') dH(J_{t,\xi}(0)) ds = t dH(J_{t,\xi}(0)). \quad (54) \]

On the other hand, since \( \frac{d}{ds} \alpha_{t,\xi}(s) = \frac{d}{ds}((1-s)\gamma_{t,\xi}(0) + s\gamma_{t,\xi}(t)) = \gamma_{t,\xi}(t) - \gamma_{t,\xi}(0) \), and since along α\textsubscript{t,ξ}, J\textsubscript{t,ξ} = (1 - s)J\textsubscript{t,ξ}(0) + sJ\textsubscript{t,ξ}(t), we have

\[ \int_{\alpha_{t,\xi}} t_{t,\xi,\omega} = \int_0^1 \omega(J_{t,\xi}(s),\alpha'_{t,\xi}(s)) ds \]

\[ = \omega\left( \frac{1}{2}(J_{t,\xi}(0) + J_{t,\xi}(t)),\gamma_{t,\xi}(t) - \gamma_{t,\xi}(0) \right) = 0, \]

since

\[ \frac{1}{2}(J_{t,\xi}(0) + J_{t,\xi}(t)) = \frac{1}{2} \frac{d}{dt}(\gamma_{t,\xi}(0) + \gamma_{t,\xi}(t)) = \frac{d}{dt}(x, \xi) = 0. \]

That leaves the second term of (52),

\[ \frac{d}{dt} \int_{\gamma_{t,\xi}} H ds = \frac{d}{dt} H(\gamma_{t,\xi}(0)) = H(\gamma_{t,\xi}(0)) + t dH(J_{t,\xi}(0)). \quad (55) \]

It follows from (54) and (55) that

\[ \frac{\partial}{\partial t} \Psi_E = -H(\gamma_{t,\xi}(0)) + E, \]

proving proposition 3.2.

7.2.2. Calculation of \( \Psi_E^T(\xi) \). We differentiate \( \frac{d}{dt} \Psi_E = -H(\gamma_{t,\xi}(0)) + E \) once more to get

\[ \frac{\partial^2}{\partial t^2} \Psi_E = -\frac{d}{dt} H(\gamma_{t,\xi}(0)) = -dH(\gamma_{t,\xi}(0)) (J_{t,\xi}(0)). \]

We claim that if \( t(E, x, \xi) \) is implicitly one of the solutions of \( \Theta'(q, p) = (x, \xi) \) with \( (q, p) = \gamma_{t,\xi}(0) \in \Sigma_k \) then

\[ \left( \frac{dr(E, x, \xi)}{dE} \right)^{-1} = -dH(\gamma_{t,\xi}(0))(J_{t,\xi}(0)) |_{p=\Theta(q, \xi)}. \quad (56) \]

This follows because \( t(E, x, \xi) \) may be defined as the locally unique pair of solutions near \( t = 0 \) of the equation \( H((\Theta')^{-1}(x, \xi)) = E \). Thus,

\[ \frac{d}{dE} H((\Theta')^{-1}(x, \xi)) = -dH((\Theta')^{-1}(x, \xi)) \frac{d}{dt} ((\Theta')^{-1}(x, \xi)) \cdot \frac{dt}{dE} = 1. \]
Here, as above, \((\Theta')^{-1}\) is multi-valued, so that \((\Theta')^{-1}(x, \xi)\) is a set but near \(\Sigma_E\) it is doublevalued and we mean that the equations hold for either branch. One such branch is given by \((\Theta')^{-1}(x, \xi) = \gamma_{1,\xi}(0)\) (the other being \(\gamma_{2,\xi}(t)\) and \(\frac{d}{dt}(\Theta')^{-1}(x, \xi) = J_{1,\xi}(0)\) (resp. \(J_{2,\xi}(t)\)). This proves (56). Regarding the amplitude, after eliminating \((q, p, v)\) in Lemma 2.2, the amplitude

\[
a_0 = (\det(A_t + D_t + i(B_t - C_t)))^{1/2} \exp(-itH(q, p))
\]

gets multiplied by \((\det(\text{Hess}(\Psi))(q_c, p_c, v_c))^{-1/2}\) where the Hessian is,

\[
(-1)^d \cdot \det(1 + M_t(q_c, p_c)) \det(A_t(q_c, p_c) + D_t(q_c, p_c) + i(B_t(q_c, p_c) - C_t(q_c, p_c))).
\]

This cancels all factors of the amplitude except for \(\det(I + M_t)\). This concludes the proof of proposition 3.2. 

\[\square\]

7.3. Background on folds

In the next section 7.4 we discuss fold singularities of Lagrangian submanifolds associated to the Wigner spectral function. In preparation, we review the definitions and properties of folding maps \(f : X \rightarrow Y\) for general manifolds. References include [G, HoIII].

We follow the exposition in [G, appendix 1, page 109]. Let \(f : X \rightarrow Y\) be a smooth map of \(n\)-dimensional manifolds, and let \(S \subset X\) be a hypersurface (codimension one submanifold) of \(X\). Let \(dV_f\) be a volume form on \(Y\).

Then \(f : X \rightarrow Y\) is a folding map with \(S\) as the fold locus if the following are satisfied,

- \(S\) is set of critical points of \(f\), i.e. \(D_s f : T_s X \rightarrow T_{f(s)}Y\) fails to be surjective;
- At every \(s \in S\), the kernel \(\ker D_s f\) is one-dimensional and is transversal to \(T_s S\);
- \(f^*dV_f\) vanishes to first order at each \(s \in S\), i.e. \(\det D_s f\) vanishes to first order on \(S\).

In this case, one can find coordinates \(x_1, \ldots, x_n\) on \(X\) and coordinates \(y_1, \ldots, y_n\) on \(Y\) so that \(S = \{x_n = 0\}\) and \(f(S) = \{y_n = 0\}\) and so that \(f^*y_j = x_j\) for \(j \leq n - 1\) and \(f^*y_n = x_n^2\).

Associated to the folding map is a canonical involution \(\sigma : V \rightarrow V\) in a neighborhood \(V\) of \(S\) in \(X\), fixing \(S\) and having the local form \(\sigma(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, -x_n)\).

7.4. Analysis of the fold singularity as a singularity of the projection of the Lagrangian

The time integral is the semi-classical Fourier transform of the \(dq \, dp \, dv\) integral, and the associated Lagrangian submanifold generated by the phase is the classical analogue of the Fourier transform, \((t, \tau) \rightarrow (-t, t)\) applied to (48) with \(\tau = E\), namely

\[
\Lambda_E = \{(x, \xi, d_{x, \xi}\Psi_E) : d_{q, p, v}\Psi_E = 0\} \subset T^*(T^*\mathbb{R}^d)
\]

\[
\left\{(x, \xi, p - q) : x = \frac{q + q_0}{2}, \xi = \frac{p + p_0}{2}, H(q, p) = E \right\}
\]

\[
\left\{(x, \xi, -2(x - q), 2(\xi - p)) : x = \frac{q + q_0}{2}, \xi = \frac{p + p_0}{2}, H(q, p) = E \right\}.
\]

(57)

Note that the change of the Lagrangian submanifold \(\Lambda_t \rightarrow \Lambda_E\) of (48) under the \(dt\) integral corresponds to the symplectic map \((t, \tau) \rightarrow (-\tau, t)\) in (49). The phase \(\Psi_E\) induces the Lagrangian immersion,
\( \tilde{\psi}_E : (t, q, p, x, \xi) \in C_{\tilde{\psi}_E} \rightarrow \left( \frac{q + q_i}{2}, \frac{p + p_i}{2}, q - q_i, p - p_i \right) \)

\( = (x, \xi, -2(x - q), 2(\xi - p)) \),

which is closely related to the map (12).

The set of \( \lambda \in \Lambda_{E, t, i} \) on which the natural projection

\( \pi : \Lambda_E \rightarrow T^* \mathbb{R}^d \) (59)

is a submersion called the regular set and its complement where \( D_\lambda \pi \) is singular is called the (Maslov) singular cycle. The image of the Maslov singular cycle under \( \pi \) is called the ‘caustic’.

When \( \Sigma_E \) is strictly convex, the image is \( B_E \) and there is an apparent fold over its boundary \( \Sigma_E \).

We prove this folding statement along with the closely related proposition 0.4.

**Proposition 3.3.**

- (i) If \( \Sigma_E \) is convex, the map (59) \( \pi : \Lambda_E \rightarrow B_E \) has a fold singularity along the zero section \( \Sigma_E \times \{0\} \subset T^* (\mathbb{R}^d) \).
- (ii) The map \( \tilde{\Theta} : \mathbb{R} \times \Sigma_E \rightarrow B_E \) with \( \tilde{\Theta}(t, q, p) = \tilde{\Theta}(q, p) \) (12) fixes the diagonal when \( t = 0 \) and the kernel of its derivative is \( 2 \frac{\partial}{\partial x} - \Xi_H \), i.e. \( D\tilde{\Theta}(2 \frac{\partial}{\partial x} - \Xi_H)|_{t=0} = 0 \).

This is the folding result relevant to the asymptotics of the smoothed Weyl–Wigner asymptotics of theorem 0.2. First we prove proposition 0.4.

**Proof.** We first relate (59) and (12) through the diagram,

\[ \begin{array}{c}
\tilde{\psi}_E : C_{\tilde{\psi}_E} \rightarrow \Lambda_E \\
\pi \downarrow \\
\Theta : \mathbb{R} \times \Sigma_E \rightarrow B_E,
\end{array} \]

where in each case \( \pi \) denotes the natural projection to the base. This shows that \( C_{\tilde{\psi}_E} \) is the fiber-product \( \Lambda_E \times_{B_E} (\mathbb{R} \times \Sigma_E) \) with respect to \( B_E \), i.e. pairs of points in the product space which get mapped to the same point in \( B_E \). This description guides the analysis of the fibers of the maps in the diagram.

It is evident that the projection \( \pi : C_{\tilde{\psi}_E} \rightarrow (\mathbb{R} \times \Sigma_E) \) is a diffeomorphism, since \( (t, q, p, x, \xi) \in C_{\tilde{\psi}_E} \) occurs only when \( (x, \xi) = \tilde{\Theta}(q, p) \), so \( (x, \xi) \) is redundant. From the formula in (58) we see that \((q, p)\) are uniquely determined by the point \((x, \xi, -2(x - q), 2(\xi - p)) \in \Lambda_E\); since \( t \) is defined by \( q_i = 2x - q, p_i = 2\xi - p \), the point \((t, q, p, x, \xi) \in C_{\tilde{\psi}_E} \) is uniquely determined. Hence, also \( \tilde{\psi}_E \) is a diffeomorphism. It follows that the fold singularities of \( \pi : \Lambda_E \rightarrow B_E \) and \( \Psi : \mathbb{R} \times \Sigma_E \rightarrow B_E \) are equivalent.

The fiber of the projection (59) is,

\[ \pi^{-1}(x, \xi) = \left\{ (x, \xi, q - q_i, p - p_i) : H(q, p) = E, (x, \xi) = \left( \frac{q + q_i}{2}, \frac{p + p_i}{2} \right) \right\} = \left\{ ((x, \xi), (q - q_i, p - p_i)) : (q, p) \in (\Theta)^{-1}(x, \xi) \cap \Sigma_E \right\}. \]
It is evident that the inverse image of the boundary under (59) is the zero section \(0_{\Gamma(\mathbb{R};\mathbb{R})}\) over \(\Sigma_E\). It is the fold locus of the map \(\Theta: \mathbb{R} \times \Sigma_E\) at \(t = 0\) and therefore of the natural projection, \(\pi: \Lambda_E \to T^*\mathbb{R}^d\). It is the fixed point set of the involutions \(t, q, p \to (-t, \Phi(q, p))\). Reversing the endpoints of the chord whose midpoint is \((x, \xi)\).

To verify that \(\Theta\) has a fold singularity along \(\{0\} \times \Sigma_E\), i.e. satisfies the conditions for a fold in section 7.3. Consider the curves,

\[
\alpha(s) = \alpha_{q, p}(s) := (2s, \Phi^{-1}(q, p)) : \mathbb{R} \to \mathbb{R} \times \Sigma_E.
\]

Then \(\alpha(0) = (0, q, p)\), \(\alpha'(0) = \frac{2}{m} \Xi_H\) and (referring to (12))

\[
\tilde{\Theta}(\alpha(s)) = \frac{1}{2}(\Phi^{-1}(q, p) + \Phi^{-1}(\Phi^{-1}(q, p))) = \frac{1}{2}(\Phi^{-1}(q, p) + \Phi^{-1}(\Phi^{-1}(q, p)) - \Xi_H(q, p) + \Xi_H(q, p) = 0
\]

we see that \(\tilde{\Theta}\) has a fold along the points \(\{0\} \times \Sigma_E\).

**Corollary 3.4.** If \(\Sigma_E\) is convex, there exists \(\varepsilon > 0\) so that the map \(\Theta\) is 2–1 on \((-\varepsilon, 0) \cup (0, \varepsilon) \times \Sigma_E\), i.e. if \(|t| \leq \varepsilon\), then \((x, \xi)\) lies in \(\Sigma_E\) and \(d((x, \xi), \Sigma_E) < \varepsilon\), then there is a unique Hamiltonian arc of time \(t\) (up to orientation reversal) for which \((x, \xi)\) is the chordal midpoint.

The statement follows from the local description of fold singularities in section 7.3. As noted above, the canonical involution is orientation reversal, \(\sigma(t, q, p) = (-t, \Theta(t, q, p))\), i.e. \(\Theta(t, q, p) = \frac{1}{2}(\Phi(q, p) + \Phi(q, p)) = \Theta^{-1}(\Phi(q, p)) = \frac{1}{2}(\Phi(q, p) + \Phi^{-1}(\Phi(q, p))\).

On the other hand, if \(\Sigma_E\) is convex, then there always exists at least one closed orbit of the Hamiltonian flow on \(\Sigma_E\) if \(H\) is convex [Rab, W]. If \((q, p)\) is a periodic point of period \(T\), then \(\nu_{\Psi}(0, q, p) = \nu_{\Psi}(T, q, p)\). In this case, \(\nu_{\Psi}(x, \xi) = \{((kT + q, p)\} \) has a discrete inverse image. But it is also possible to have an inverse image of positive dimension, for instance if \(V(x) = ||x||^2\) and \((x, \xi) = (0, 0)\) for instance if periodic orbits come in positive dimensional families.

**Remark:** the phase function \(\Psi_E\) of (53) is equivalent to the phase function \(\tilde{\Psi}_E\) in the sense of [HoIII, GST], namely they both parametrize the same Lagrangian.

### 8. Proof of theorem 0.2

The proof of theorem 0.2 is based on a special case of the Malgrange preparation theorem which was proved earlier by Chester–Friedman–Ursell [CFU]. The Malgrange preparation theorem asserts that if \(f(t, x)\) is a smooth function of \((t, x) \in \mathbb{R} \times \mathbb{R}^n\), and if \(k\) is the positive integer so that

\[
f(0, 0) = 0, \quad \frac{\partial}{\partial t}f(0, 0) = 0, \ldots, \frac{\partial^{k-1}}{\partial t^{k-1}}f(0, 0) = 0, \quad \frac{\partial^k}{\partial t^k}f(0, 0) \neq 0, \quad (60)
\]

then there exists a smooth function \(c(t, x)\) which is non-vanishing near \((0, 0)\) and smooth functions \(a_j(x), j = 1, \ldots, k - 1\) so that

\[
f(t, x) = c(t, x)(t^k + a_{k-1}(x)t^{k-1} + \cdots + a_0(x)).
\]
We only apply this theorem when $k = 3$ and use a stronger form from [CFU, L61] and also proved in detail in [Hol, theorem 7.5.13] (see also [GSt, page 444]). With no loss of generality assume that $\frac{\partial^3}{\partial t^3}(0,0) > 0$. Then, there exists a $C^\infty$ function $T = T(t,x)$ near $(0,0)$ with $T(0,0) = 0$, $\frac{\partial^3}{\partial t^3}(0,0) > 0$ and $C^\infty$ functions $\rho, \mu$ near $(0,0)$ such that

$$ f(t,x) = \frac{T^3}{3} + a(x)T + \mu(x), \quad (61) $$

with $a(0) = 0$, $\mu(0) = f(0,0)$.

The detailed statement of the result is given in [Hol, theorem 7.7.19].

**Proposition 4.1.** Let $f(t,x)$ be real valued on $\mathbb{R}^{n+1}$ satisfying (60) with $k = 3$. Then there exist $C^\infty$ functions $u_{0,\nu}, u_{1,\nu}$ such that, in the notation above,

$$ \int u(t,x)e^{i\tau f(t,x)}dt \sim e^{i\tau^3} \left( \tau^{-\frac{3}{2}} Ai \left( \tau \frac{3}{2} u(x) \right) \sum_{\nu=0}^{\infty} u_{0,\nu}(x) \tau^{-\nu} \right) $$

$$ + e^{i\tau^3} \left( \tau^{-\frac{3}{2}} Ai' \left( \tau \frac{3}{2} u(x) \right) \sum_{\nu=0}^{\infty} u_{1,\nu}(x) \tau^{-\nu} \right). $$

Remark: note that $\tau^3 + a(x) = 0$, so that $a(x) < 0$. In [GSt, page 444]), the normal form is stated in the form, $\frac{L^3}{4} - \rho(x)T + \mu(x)$. Hence, in what follows, $a(x) = -\rho(x)$.

We apply the result to the oscillatory integral $W_{h,E,p}(x, \xi) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho(t) W_h(x, \xi) e^{i\tau} dt dq dp dv dt$ and to the one-dimensional reduction in section 7.2.

For the full $\mathbb{R}^{d+1}$ dimensional integral, we use

**Proposition 4.2.** Let $f(y,x)$ be complex valued function with positive imaginary part on $\mathbb{R}^{n+m}$. Assume that $\text{Im} f(0,0) = 0$, that $\text{Hess}(\text{Im} f)(0,0)$ is non-degenerate, and that

- (i) $\text{Re} f_1'(0,0) = 0$.
- (ii) $\text{Rank} \text{Re} f_\lambda''(0,0) = n - 1$.
- (iii) $(X, \partial / \partial x) \text{Re} f(0,0) \neq 0$ if $0 \neq X \in \text{ker} \text{Re} f_\lambda''(0,0)$.
- (iv) $f_1(x,0)$ is real valued.

Then there exist $C^\infty$ functions $\rho(x), \mu(x)$ near $(0,0)$ such that $\rho(0) = 0$, $\mu(0) = f(0)$ and such that, for any $u \in C^\infty_0$ with $\text{Supp} u$ sufficiently close to $(0,0)$, there exist $C^\infty$ functions $u_{0,\nu}, u_{1,\nu}$ such that,

$$ \int u(y,x)e^{i\tau f(y,x)}dx \sim e^{i\tau^3} \left( \tau^{-\frac{3}{2}} Ai \left( -\tau \frac{3}{2} \rho(y) \right) \sum_{\nu=0}^{\infty} u_{0,\nu}(y) \tau^{-\nu} \right) $$

$$ + e^{i\tau^3} \left( \tau^{-\frac{3}{2}} Ai' \left( -\tau \frac{3}{2} \rho(y) \right) \sum_{\nu=0}^{\infty} u_{1,\nu}(y) \tau^{-\nu} \right). $$

Moreover, if $\partial_1 u(x,y) = 0$, then $u_{1,\nu}(y) = 0$.
The proof is similar to that of [Hol, theorem 7.7.18] and also in the reduction to one integral. One integrates first in \((x_2, \ldots, x_n)\) where stationary phase applies, then applies proposition 4.1 to the remaining one dimensional integral. For the sake of completeness, we provide more details as in [Hol, theorem 7.7.19], although it explicitly assumed there that the phase is real-valued. We label the coordinates so that \(x_1 = t\) and \((q, p, v) = (x_2, \ldots, x_n)\) with \(n = 3d + 1\). Let \(f = \text{Re} \Psi\). Define \(X_j(x_1, y)\) for \(j = 2, \ldots, n\) so that the equations \(\frac{\partial X_j}{\partial y} = 0\) determine \(x_j\) as functions \(x_j = X_j(x_1, y)\). Change variables in the \(dx_2, \ldots, dx_n\) integral to \(x'_j = x_j - X_j\). Then the critical points of the \(dx_2, \ldots, dx_n\) integral become \(x_j = 0, j = 2, \ldots, n\). By applying stationary phase with a positive complex phase to this integral, we obtain an asymptotic of the integrand with fixed \(x_1\) of the form,

\[\tau^{-\frac{2d+1}{3}} e^{i\tau f(x_1, 0)} U(x_1, y, 0).\]

Here we use that the imaginary part \(\text{Im} \Psi\) of the phase vanishes at the critical set.

To apply the results to our integrals, it is necessary to show that the hypotheses of proposition 4.1 hold and then to calculate the coefficients \(\rho, \mu\) in (61) and the principal coefficients \(u_{0,0}, u_{1,0}\) in proposition 4.1. The following general calculation is due to [CPU, section 3]; see also [GSt, (6.7), page 440]. By (61), and the convention that \(\rho = -a\), we may express \(\varphi(t, x) = T^2 - \rho T + \mu\). For each critical point \(d_\varphi(t, x)\) there exist two roots \(t_{\pm}(x)\). By \(\varphi_\pm(t_{\pm}, x)\) we denote the critical value of the phase at each critical point. We then have,

**Lemma 4.3.** In the notation above,

\[\begin{align*}
\rho^{3/2}(x) &= \frac{3}{4}(\varphi_-(t_+, x) - \varphi_+(t_-, x)) \\
\mu(x) &= \frac{1}{2}(\varphi_-(t_-, x) + \varphi_+(t_+, x)).
\end{align*}\]

We review the proof, since we need to do the calculations.

**Proof.** We assume that \(\varphi(t, x) = T^3 - \rho(x) T + \mu(x)\) is a smooth expression (here we write \(a = -\rho\)). The critical point equation in \(t\) (viewing \(x\) as a parameter) is

\[\frac{\partial}{\partial t} \varphi(t, x) = \frac{\partial}{\partial T} \varphi(T, x) \frac{\partial T}{\partial t} = \frac{\partial T}{\partial t} \left(T^2 - \rho(x)\right) = 0.\]

Hence, since \(\frac{\partial T}{\partial t} \neq 0\) for \(t, x\) sufficiently close to 0, we have \(T^2 = \rho(x)\) on the critical point set, and there exist two roots \(T_{\pm} = \pm \sqrt{\rho(x)}\) when \(T \neq 0\). It follows that \(\frac{\partial T}{\partial t} \neq 0\) when \(T \neq 0\). The critical values of the phase at the two critical points are given by,

\[\varphi_+(t_{\pm}, x) = \frac{T^3_{\pm}}{3} - \rho(x) T_{\pm} + \mu(x) = \pm \frac{\rho^{3/2}}{3} - \rho(x)(\pm \rho^{1/2}) + \mu(x) = \mp \frac{2}{3} \rho^{3/2} + \mu(x).\]

Thus,

\[\varphi_+(t_{-}, x) = -\frac{2}{3} \rho^{3/2}(x) + \mu(x), \quad \varphi_+(t_{+}, x) = +\frac{2}{3} \rho^{3/2}(x) + \mu(x).\]

\[\square\]
8.1. Completion of the proof of theorem 0.2

8.1.1. Calculation of $\Psi^*_E(t_c)$ and of $\Psi^*_E(t_0)$

**Proof.** When $t = 0$ and $(x, \xi) \in \Sigma_E$, we have

$$\frac{\partial^2}{\partial t^2} \Psi_E = -\frac{d}{dt} H(\gamma_{t,\xi}(0)) = -dH(\gamma_{t,\xi}(0)(J_{t,\xi}(0))|_{t=0},(x,\xi)) \in \Sigma_E.$$

In computing the time derivatives we may assume $(x, \xi) \in \Sigma_E$. But $H(\gamma_{t,\xi}(0))$ has a local minimum when $t = 0$ in that case. Indeed, for any $t$ such that $(x, \xi) = \frac{1}{2}(\gamma_{t,\xi}(0) + \gamma_{t,\xi}(t))$, it is necessary that $H(\gamma_{t,\xi}(0)) \geq H(x, \xi) = E$. It follows also that $\frac{d}{dt}(\Theta^{-1}(x, \xi)|_{t=0} \in T\Sigma_E$, and that

$$dH(\gamma_{t,\xi}(0))|_{t=0} = 0, \ (x, \xi) \in \Sigma_E. \quad (62)$$

We now show that

$$\frac{\partial^3}{\partial t^3} \Psi_E|_{t=0} = -\frac{d^2}{dt^2} H(\gamma_{t,\xi}(0))|_{t=0} \neq 0.$$

We have $\Theta'(\gamma_{t,\xi}(0)) = (x, \xi)$ and $\frac{d}{dt} \Theta' = \frac{1}{2} \frac{d}{dt} \psi' = \frac{1}{2} \Xi_H$. Then, $0 = \frac{d}{dt} \Theta'(\gamma_{t,\xi}(0)) = \frac{1}{2} \Xi_H(x, \xi) + D\Theta \frac{d}{dt} \gamma_{t,\xi}|_{t=0}$. If the second term is zero, we would get $\Xi_H(x, \xi) = 0$, a contradiction. In fact, $D\Theta \frac{d}{dt}|_{t=0} = Id$ and we get $J_{t,\xi}(0) = -\frac{1}{2} \Xi_H(x, \xi)$. But then, by strict convexity of $H$,

$$\frac{d^2}{dt^2} H(\gamma_{t,\xi}(0))|_{t=0} = \text{Hess}_{\gamma_{t,\xi}} H(J_{0,\xi}(0), J_{0,\xi}(0)) > 0.$$

Note that the other term $dH(\gamma_{t,\xi}(0)) \frac{d^2}{dt^2} \gamma_{t,\xi}(0) = 0$ by (62). This completes the proof that the conditions (60) are satisfied.

We may therefore apply proposition 4.1 to obtain an asymptotic expansion of the form stated in theorem 0.2. The remaining step is to use lemma 4.3 to calculate the parameters $\rho, \mu, \omega(0, 0), \omega(1, 0)$.

We therefore fix $(x, \xi) \in B_E$ sufficiently close to $\Sigma_E$ and define $(t, q, p) \in (-\varepsilon_0, \varepsilon_0) \times \Sigma_E$ as above by $\Psi(q, p) = (x, \xi)$. Further, we denote by $t_s(E, x, \xi)$ the two roots of the critical point equation for the phase $\Psi_E(t, x, \xi)$ (see definition 0.1).

**Lemma 4.4.** If $(x, \xi)$ is $\varepsilon_0$-close to $\Sigma_E$, then there exist exactly two critical times $t = t_s(E, x, \xi) \in (-\varepsilon_0, \varepsilon_0)$ (see proposition 2.1) and they are of the form $t_s = \pm t$ where $\Theta'(q, p) = (x, \xi), H(q, p) = E$. Moreover, the action changes sign under the reversal of $\gamma_{t,\xi}$.

Indeed, for such $(x, \xi)$ there is a unique Hamiltonian arc of time $t$ with $|t| \leq \varepsilon$ (up to the symmetry (14)), for which $(x, \xi)$ is the midpoint of the chord with endpoints $(q, p), (q, p)$. The two critical times points $(t, q, p)$ correspond to $\gamma_{t,\xi}$ and its orientation reversal. Reversing the orientation means to change $\gamma_{t,\xi}(s) \rightarrow \gamma_{t,\xi}^*(s) := \gamma_{t,\xi}(t - s)$, so that $\frac{d}{ds} \gamma_{t,\xi}^*(s) := (\frac{d}{ds} \gamma_{t,\xi}(t - s))$. We write $\gamma_{t,\xi}(s) = (q_{t,\xi}(s), p_{t,\xi}(s))$ and $\gamma_{t,\xi}^*(s) = (q_{t,\xi}^*(s), p_{t,\xi}^*(s)) = (q_{t,\xi}(t - s), p_{t,\xi}(t - s))$ and show that

$$\int_0^t p_{t,\xi}(s) \frac{d}{ds} q_{t,\xi}(s) ds = -\int_0^t p_{t,\xi}^*(s) \frac{d}{ds} q_{t,\xi}^*(s) ds.$$
Indeed, the right side equals
\[
\int_0^t p_{t,x,\xi}(t-s)\frac{d}{ds} q_{t,x,\xi}(t-s)ds = -\int_0^t p_{t,x,\xi}(t-s)\dot{q}_{t,x,\xi}(t-s)ds
\]
\[
= \int_0^t p_{t,x,\xi}(s)\dot{q}_{t,x,\xi}(s)ds = -\int_0^t p_{t,x,\xi}(s)\dot{q}_{t,x,\xi}(s)ds.
\]

8.1.2. Calculation of \( \rho \). By the first equation of lemma 4.3, we need to calculate the ‘odd’ part of the phase, defined by
\[
\frac{1}{2}(\Psi_E(t_+(E,x,\xi);x,\xi)-\Psi_E(t_-(E,x,\xi);x,\xi))
\]
\[
= \frac{1}{2}(S_{t_+(E,x,\xi)} - S_{t_-(E,x,\xi)}) - \left\langle \frac{1}{2}(q_{t_+(E,x,\xi)} - q_{t_-(E,x,\xi)}), \xi \right\rangle
\]
\[
+ (t_+(E,x,\xi) - t_-(E,x,\xi))E,
\]
where \( S_t(q, p) = \int_{\gamma_{t,q,p}} (p \, dq - H \, ds) \) is the action integral along the phase space trajectory \( \gamma_{t,q,p} \) with initial value \( (q, p) \). The ‘odd’ part of the action integral is
\[
\frac{1}{2}\left( \int_{t_0}^{t_+(E,x,\xi)} p \, dq - \int_{t_0}^{t_-(E,x,\xi)} p \, dq \right) = \frac{1}{2} \int_{t_-(E,x,\xi)}^{t_+(E,x,\xi)} p \, dq,
\]
where the integral is taken along the Hamilton path with endpoints \((q_{t_-}, p_{t_-}), (q_{t_+}, p_{t_+})\) where \( t_\pm = t_{\pm}(E,x,\xi) \). Since \( H(q_{t_+}, p_{t_+}) = H(q, p) \), the second term of \( S_t \) combines with \( E \) to produce \( (t_+(E,x,\xi) - t_-(E,x,\xi))(E - H(q, p)) = 0 \) at the critical time. One also has,
\[
\left\langle \frac{1}{2}(q_{t_+(E,x,\xi)} - q_{t_-(E,x,\xi)}), \xi \right\rangle = \frac{1}{2} \int_{t_-(E,x,\xi)}^{t_+(E,x,\xi)} \xi \cdot dq.
\]

Hence,
\[
\frac{4}{n} \rho^{\beta/2} = \frac{1}{2} \int_{t_-(E,x,\xi)}^{t_+(E,x,\xi)} (p - \xi) \cdot dq = \int_{\gamma_{t,\xi}} \omega,
\]
where in the second inequality we use the observation noted in (18) and in the proof of proposition 3.2 that this action integral is the oriented area of the surface bounded by the oriented closed curve \( \gamma_{t,\xi} \) consisting of the Hamilton arc followed by the chord.

8.1.3. Calculation that \( \mu = 0 \). We claim that \( \mu = 0 \). This follows from lemma 4.4, since the two critical times correspond to the unique Hamilton orbit and its orientation reversal. The same is true of \( \langle q_t - q, \xi \rangle \), which is clearly odd under orientation reversal.

8.1.4. Calculation of \( u_{0,0} \). We claim that
\[
u_{0,0} = \sqrt{\pi} \rho^{1/4} \left| \frac{d}{dt_j} \det \left[ M_j(x, t_j(E)) \right] \right|^{1/2}.
\]
Hence, Jacobi’s formulashowsthat
\[
\partial \left( \frac{g_0}{t^{2/3}} \right) Ai\left( -\frac{\tau^{2/3}}{3} \right) + \frac{g_1}{t^{5/3}} Ai\left( -\frac{\tau^{2/3}}{3} \right)
\]
to the stationary phase of the same integral for \( \tau^{2/3} \geq 0 \), given by
\[
\frac{\pi^{-1/3} e^{i\pi\sigma}}{\sqrt{\pi t^{2/3} \rho^{1/2}}} \left( g_0 \cos \left( \frac{2\tau \rho^{2/3}}{3} - \frac{\pi}{4} \right) \right) - g_1 \rho^{1/2} \sin \left( \frac{2\tau \rho^{2/3}}{3} - \frac{\pi}{4} \right).
\]

To see this, we first note that a direct computationyields
\[ u_{1,0}(\omega, \xi) = \text{Ai}(t, \xi) \]
and is givenindetail [GST, page 459]. We wishtomatchthe Airy asymptotic expansion [GST, page 442, (6.9)],
\[
\text{Airy } \quad g_0 \iff \text{StPh } \frac{1}{\sqrt{\pi}} \rho^{-1/4} g_0.
\]
Note that \( \tau^{-1/3} \left( \tau^{2/3} \right)^{1/4} = \tau^{-1/3} \tau^{-1/6} = \tau^{-2}. \) The stationary phase calculation uses that,
\[
\text{Ai}(t) \simeq \frac{1}{\sqrt{\pi t^{1/4}}} \cos \left( \frac{2\tau^{3/2}}{3} - \frac{\pi}{4} \right).
\]
For stationary phase points very close to \( \Sigma_E \) one gets by proposition 0.5 the amplitude
\[
\frac{2^{d+1}}{\sqrt{2\pi h}} \sum_j e^{-ct_j/h} A_j(x, E) \cos \left( \frac{S_j(x, E)}{h} + m_j \right),
\]
and applying this to the critical points \((t, q, \rho)\) corresponding to \((x, \xi)\) close to \( \Sigma_E \) gives that
\[
g_0 = \left| \frac{dt}{dE} \det(1 + M(x, t_j(E)))^{-1} \right|^{1/2},
\]
proving (64) and concluding the proof of theorem 0.2.

8.1.5. Vanishing of \( u_{1,0} \) Finally, to complete the proof of theorem 0.2 we must check that \( u_{1,0}(\omega, \xi) = 0 \). To see this, we recall from proposition 3.2 that the amplitude in the one variable \( t \) oscillatory integral representation for \( W_{1,1}(\sum_J, \xi) \) equals
\[
\det(I + M(q_c, p_c; x, \xi)).
\]

Thus, due to the final statement in proposition 4.2, we must check that
\[
\frac{\partial}{\partial t} \mid_{t=0} \det(I + M(q_c, p_c; x, \xi)) = 0.
\]

To see this, we first note that a direct computation yields
\[
M_0 = I, \quad \frac{d}{dr} \mid_{r=0} M_r = \begin{pmatrix} 0 & I \\ -\text{Hess } V(q_0) & 0 \end{pmatrix}.
\]

Hence, Jacobi’s formula shows that \( \frac{\partial}{\partial t} \mid_{t=0} \det(I + M(q_c, p_c; x, \xi)) \) equals
\[
\det(I + M_0(q_c, p_c; x, \xi)) \text{tr} \left( M_0(q_c, p_c; x, \xi)^{-1} \frac{d}{dt} \mid_{t=0} M(q_c, p_c; x, \xi) \right) = 0.
\]

This completes the proof. \( \square \)
9. Example: isotropic harmonic oscillator

In this section, we evaluate all of the objects above in the simplest case of the isotropic harmonic oscillator, \( H(q, p) = \frac{1}{2}(\|p\|^2 + \|q\|^2) \) and check the consistency of (7) and (17).

The first simplifying feature is that the Hamilton flow \( \Phi^t \), resp. the midpoint map \( \Theta^t = \frac{1}{2}(I + \Phi^t) \) are linear; they are given respectively by,

\[
M_t = \Phi^t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},
\]

\[
\Theta^{-t} = \begin{pmatrix} 2 & 2(1 + \cos t) \\ 2 & \sin t \end{pmatrix} \begin{pmatrix} \frac{1 + \cos t}{\sin t} \\ 1 + \cos t \end{pmatrix},
\]

i.e. \((q_t, p_t) = (\cos tq + \sin tp, -\sin tq + \cos tp)\) and on the critical point set of the \((q, p, v)\) integral,

\[
\begin{pmatrix} q \\ p \end{pmatrix} = \Theta^{-t}(x, \xi) = \begin{pmatrix} 1 + \cos t & \sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 + \cos t \\ 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}.
\]

Given \((x, \xi)\) we can solve for the critical time \( t = t(x, \xi, E) \) at which there exists \((q, p) \in \Sigma_E\) for which \((x, \xi)\) is the midpoint of the Hamilton orbit of time \( t \) starting at \((q, p)\). Namely,

\[
\cos \frac{t}{2} = \frac{\sqrt{H(x, \xi)}}{\sqrt{H(q, p)}} = \sqrt{r}, \quad (H(q, p) = E),
\]

where in the second equality we use the notation \( \frac{H(x, \xi)}{E} = (0, 1) \in (8) \). There are two solutions \( t = t_{\pm}(E, x, \xi) \) with \( t_- = -t_+ \).

9.1. Simple geometric formulae for spheres

Let \( C_R \) be the circle of radius \( R \) around the origin. Since the Hamiltonian orbits of an isotropic oscillator are great circles of the sphere of radius \( \sqrt{2E} \), we may regard them as circular arcs on \( C_{\sqrt{2E}} \). The travel time \( t \) of the Hamiltonian flow \( \Phi^t \) on this arc has length \( Rt = \sqrt{2E} t \).

The chord between endpoints of the Hamilton arc of time \( t \) has length \( \frac{1}{2}\sqrt{R^2 - r^2} \) where \( r \) is the distance from the origin to the midpoint of the chord. Since \((x, \xi)\) is the midpoint of the chord, \( r = \sqrt{2H(x, \xi)} \) and the length of the chord is \( \frac{1}{2}\sqrt{2E - 2H(x, \xi)} \). The area of the circular sector bounded by the Hamilton arc and the chord is given by,

\[
A = R^2 \cos^{-1} \left( \frac{r}{R} \right) - rR \sqrt{1 - \frac{r^2}{R^2}} = (2E)\cos^{-1} \sqrt{\frac{H(x, \xi)}{E}}
- 2\sqrt{H(x, \xi)E} \sqrt{1 - \frac{H(x, \xi)}{E}}.
\]

In the notation of (8) with \( \sqrt{s} = \frac{\xi}{R} \), and comparing with (63),

\[
A = R^2 \left( \cos^{-1} \sqrt{s} - \sqrt{s} \sqrt{1 - s} \right) = \frac{4}{3}R^{3/2}.
\]

On the other hand, \( \beta(s) = \frac{1}{2}[\cos^{-1} \sqrt{s} - \sqrt{s - s^2}] \), so \( A = 2R^2 \beta \) and \( B^2 = -[(2R^2)^{-1} A]^2 \).
We recall that \( h = h(E) := \frac{E}{N^2} \) and that the argument of the Airy function in (7) is \( h^{3/2} B^2(s) \).
We cannot and do not use the same convention for a general Hamiltonian, so that the factor
\[(2R^2)^{-1} \Rightarrow E^{1/4} \text{ accounts for the change in the Airy argument between } h_E^{-2/3} B^2(s) \text{ in (7) and } h_E^{-1/3} \rho(x, \xi) \text{ (17).}\]

### 9.2. Simplified asymptotics

For \((x, \xi) \in T^* \mathbb{R}^d\) define the rescaled variable \(u = u(x, \xi)\) centered at the energy surface \(\Sigma_E\) by
\[
\left(\|x\|^2 + \|\xi\|^2\right)/2 = E + u(\hbar/2E)^{2/3}.
\]

It is proved in [HZ20] that, for \(|u| < \hbar^{-1/3}\),
\[
W_{h,E}(x, \xi) = \begin{cases} 
\frac{2}{(2\pi\hbar)^{d/2}} \left(\frac{\hbar}{2E}\right)^{1/3} \left(\text{Ai}(u/E) + O\left((1 + |u|)^{1/4}u^{2/3}\right)\right), & u < 0 \\
\frac{2}{(2\pi\hbar)^{d/2}} \left(\frac{\hbar}{2E}\right)^{1/3} \text{Ai}(u/E) \left(1 + O\left((1 + |u|)^{1/4}u^{2/3}\right)\right), & u > 0.
\end{cases}
\]

The simplified asymptotics follow by Taylor expansion of \(\beta\) in (8) and of the amplitude \(u_{0,0}\) denoted in [HZ20] by
\[
\alpha_0(s) = \begin{cases} 
\frac{\sqrt{2|B(s)|}}{(1 - s)^{1/8}3^{3/4}}, & 0 < s < 1 \\
\frac{\sqrt{2B(s)}}{(s - 1)^{1/8}3^{3/4}}, & s \geq 1
\end{cases}
\]

Namely, the Taylor expansions are,
\[
B^2(1 + t) = 2^{-2/3} t(1 + O(t)), \quad \alpha_0(1 + t) = 2^{1/3}(1 + O(t)).
\]

These estimates yield
\[
\nu^{2/3} B^2(s) = \left(\frac{2}{\hbar_E}\right)^{2/3} B^2(1 + (u/E)\hbar_E^{2/3}) = \frac{u}{E} + O\left(|u|^2 \hbar_E^{2/3}\right),
\]
which leads in [HZ20] to,
\[
\nu^{-1/3} \text{Ai}(\nu^{2/3} B^2(s) \alpha_0(s)) = \hbar_E^{1/3} \left[\text{Ai}(u/E) + O((1 + |u|)^{1/4}u^{2/3})\right]
\]
when \(u < 0\) and
\[
\nu^{-1/3} \text{Ai}(\nu^{2/3} B^2(s) \alpha_0(s)) = \hbar_E^{1/3} \text{Ai}(u/E) \left[1 + O((1 + |u|)^{1/4}u^{2/3})\right]
\]
when \(u > 0\).

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Data availability statement

No new data were created or analysed in this study.

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