Adiabatic Perturbations in Homologous Conventional Polytropic Core Collapses of a Spherical Star

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Abstract
We perform a non-radial adiabatic perturbation analysis on homologous conventional polytropic stellar core collapses. The core collapse features a polytropic exponent $\Gamma = 4/3$ relativistic gas under self-gravity of spherical symmetry while three-dimensional perturbations involve an adiabatic exponent $\gamma$ with $\gamma \neq \Gamma$ such that the Brunt-Väisälä buoyancy frequency $N$ does not vanish. With proper boundary conditions, we derive eigenvalues and eigenfunctions for different modes of oscillations. In reference to stellar oscillations and earlier results, we examine behaviours of different modes and the criterion for instabilities. The acoustic $p$-modes and surface $f$-modes remain stable. For $\gamma < \Gamma$, convective instabilities appear as unstable internal gravity $g^-$-modes. For $\gamma > \Gamma$, sufficiently low-order internal gravity $g^+$-modes are stable, whereas sufficiently high-order $g^-$-modes, which would have been stable in a static star, become unstable during self-similar core collapses. For supernova explosions, physical consequences of such inevitable $g$-mode instabilities are speculated.

Key words: hydrodynamics — instabilities — stars: neutron — stars: oscillations (including pulsations) — supernovae: general — waves

1 INTRODUCTION
Stability properties of core collapses in massive progenitors stars before supernova (SN) explosions (Goldreich & Weber 1980 – GW hereafter; Goldreich et al. 1996; Lai 2000; Lai & Goldreich 2000; Blondin et al. 2003; Murphy et al. 2004; Burrows et al. 2006, 2007; Lou & Cao 2008; Cao & Lou 2009) have come to focus after three decades, because of the realization after numerous unsuccessful one-dimensional SN simulations that the breakdown of spherical symmetry inevitably occurs and plays a key role in SN explosions (e.g. Burrows 2000). Numerical simulations (e.g. Bruenn 1989a, b) indicate that pre-SN stellar core collapses may be approximately described by a homologous process, first analyzed by GW for a conventional polytropic equation of state (EoS) $P = \kappa \rho^\Gamma$ where $P$ and $\rho$ are the pressure and density and both $\kappa$ and $\Gamma = 4/3$ are constant. For perturbations obeying identical EoS of the background core collapse, GW explored linear stability properties of such stellar collapses and concluded that these collapses are stable. Yahil (1983) extended homologous collapses to polytropic exponent $\Gamma < 4/3$, noting the presence of an outer supersonic envelope besides the inner core collapse. Lai (2000) performed perturbation analysis to these extended solutions and claimed that perturbations are stable for $\Gamma > 1.09$.

Lou & Cao (2008) substantially extended the $\Gamma = 4/3$ self-similar solutions, including those of GW, using a general polytropic EoS with a temporally and radially variable $\kappa$ being conserved along streamlines. We obtained a broad family of homologous core collapses allowing $\kappa$ as an arbitrary function of the independent self-similar variable.

The specific entropy of an ideal or perfect gas is

$$s = \frac{k_B}{\gamma - 1} \ln \left( \frac{P}{\rho^\gamma} \right) + \text{constant},$$

where $k_B$ is the Boltzmann constant and $\gamma$ is the adiabatic exponent with $\gamma \neq \Gamma$. Here $\Gamma$ determines the structure and evolution of core collapses while $\gamma$ controls specific entropy perturbation properties. The earlier isentropic assumption requires $\gamma = \Gamma$ (e.g. GW). Numerical simulations (e.g. Bethe et al. 1979; Bruenn 1985, 1989a, 1989b; Woosley et al. 1993, 2002) for the structure of massive stars and SN explosions support variable radial distributions of specific entropy. Thus the model of Lou & Cao (2008) allows an arbitrary radial profile of specific entropy. Using this general polytropic model, we conducted a non-radial adiabatic perturbation analysis and classified different perturbation...
modes parallel to stellar oscillations (see Unno et al. 1979 and Cao & Lou 2009) and concluded that in addition to internal gravity g-modes for convective instabilities, sufficiently high-order g-modes also become unstable.

Cao & Lou (2009) demonstrated that the criterion of using the sign of the Brunt-Väisälä buoyancy frequency squared $N^2$ (see definition 13) for the existence of g-modes remains valid in self-similar dynamic core collapses. In a conventional polytropic process, isentropic perturbations with $\gamma = \Gamma$ make all g-modes disappear, i.e. $N^2 = 0$. Thus previous perturbation analyses (GW; Lai 2000; Lai & Goldreich 2000) considered only the stability of acoustic modes. In addition to p-modes and f-modes, Cao & Lou (2009) found g-mode instabilities in stellar core collapses and speculated possible consequences for neutron star kicks etc.

From the existence criterion of $N^2 \neq 0$ for g-modes, we realize that in a conventional polytropic core collapse, non-radial adiabatic perturbations with $\gamma \neq \Gamma$ should allow g-modes. Such nonisentropic assumption has been invoked decades ago (e.g. Ledoux 1965; Unno et al. 1979) and applied to helioseismology and white dwarf oscillations (e.g. Shibahashi et al. 1988). For star formation, McKee & Holliman (1999) also studied nonisentropic radial perturbations in molecular clouds. They used a thermal free energy to discuss cloud stability and gave critical cloud masses for locally and globally adiabatic models. These early results motivate us to perform such a perturbation analysis for stellar core collapses and study the existence and stability of g-modes in addition to other modes. The main thrust of this Letter is to show such g-mode instabilities and speculate consequences for stellar core collapses and SN explosions.

## 2 PERTURBATIONS IN CORE COLLAPSES

The governing equations include conservations of mass and momentum, Poisson equation for self-gravity and EoS. The gas is ideal and the background is a $\Gamma = 4/3$ conventional polytrope while the adiabatic perturbation with $\gamma \neq \Gamma$ conserves the specific entropy along streamlines. The background collapse is spherically symmetric and as in GW, we introduce a dimensionless independent variable $x = r/a(t)$ where $a(t) = \rho_c^{-1/3}[(\pi/\Gamma)]^{1/2}$ is the Jeans length with a central mass density $\rho_c$ and $G$ is the gravitational constant. Physical variables are cast into the following forms where the first term of each variable is the dynamic background and the last term is the first order-perturbation:

\[
\begin{align*}
  u &= \hat{u}(t)x + \frac{a(t)}{t_{ff}}\psi_1(x, \theta, \phi)\tau(t), \\
  \rho &= \left(\frac{\kappa}{\pi G}\right)^{3/2}a(t)^{-3}f^3(x)[1 + \beta(x, \theta, \phi)\tau(t)], \\
  P &= \left(\frac{\kappa}{\pi G}\right)^2a(t)^{-4}f^4(x)[1 + \beta(x, \theta, \phi)\tau(t)], \\
  \Phi &= \frac{4}{3}\left(\frac{\kappa}{\pi G}\right)^{1/2}a(t)^{-1}[\psi(x) + \psi_1(x, \theta, \phi)\tau(t)].
\end{align*}
\]  

Here the time-dependent temporal factor $\tau(t)$ is

\[
\tau(t) = \exp\left(p\int_{t_0}^t t_{ff}'\,dt'\right),
\]

where the free-fall timescale $t_{ff}$ is defined by

\[
t_{ff} = \left[\frac{4}{3}\pi G\rho_c(t)\right]^{-1/2}
\]

and $p$ is a dimensionless ‘frequency’. We do not use stream function for velocity perturbations to allow for vorticities. The dynamic collapse background is that of GW. Defining a dimensionless collapse parameter $\lambda$ (Yahil 1983) as

\[
\frac{1}{\lambda} = \frac{8\pi G\rho_0}{3\nu_0^2}(\tau_{r_0} = \left(\frac{u_{ff}}{u_0}\right)^2)_{r=0},
\]

where the subscript 0 indicates background variables and $u_{ff}$ is the free-fall velocity, we obtain the background density profile determined by the equation for $f(x)$, namely

\[
\frac{1}{x^2\frac{df(x)}{dx}} + f^3 = \lambda,
\]

with boundary conditions $f(0) = 1$ and $f'(0) = 0$. Other background variables are then readily determined. Solutions of eq. 9 are sensible for core collapses of massive stars when density vanishes at a moving boundary $x_0$. For this purpose, the variable $\hat{a}$ should be negative and the time reversal operation must be taken in the governing equations. We assume adiabatic perturbations with the exponent $\gamma$ different from $\Gamma = 4/3$. In our model, we define $v_1 = pw_1$ and $m = p[p + (\lambda/2)^{1/2}]$ for convenient analysis. After the linearization of hydrodynamic equations including conservations of mass and momentum, Poisson equation for self-gravity and the adiabatic EoS, the angular variation factors of $w_1$, $\psi_1$, $f_1$ and $\psi_1$ can be separated out involving the spherical harmonics $Y_{lm}(\theta, \phi)$. We further let $w_1$ take the consistent form of

\[
w_1 = e_r w_r(x, t)Y_{lm} + w_1(x, t)\hat{\nabla}_\perp Y_{lm} + \hat{\nabla}_\perp \times (w_{rot}Y_{lm}e_r),
\]

where the modified transverse gradient operator is

\[
\hat{\nabla}_\perp = e_\theta \frac{\partial}{\partial \theta} + e_\phi \frac{1}{\sin \theta} \frac{\partial}{\partial \phi}.
\]

As $w_{rot}$ decouples from other perturbation equations, we suppress it in the following analysis. Eliminating $\beta_1$ and $f_1$, we obtain a set of ordinary differential equations (ODEs):

\[
\frac{f}{x^2}\frac{d}{dx}\left(x^2\frac{dw_r}{dx}\right) - \frac{l(l+1)f}{x}w_r + \frac{4}{\gamma}w_r\frac{df}{dx} = -\frac{4}{\gamma^2}(mxw_r + \psi_1) = 0,
\]

\[
\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\psi_1}{dx}\right) - \frac{l(l+1)}{x^2}\psi_1 = 3f^2\left(\frac{4}{\gamma} - 3\right)w_r\frac{df}{dx} = -\frac{4}{\gamma^2}f^2(mxw_r + \psi_1),
\]

\[
mxw_r - m\frac{d}{dx}(xw_r) = \frac{(3\gamma - 4)}{\gamma f}(mxw_1 + \psi_1) + \frac{3(4 - 3\gamma)}{\gamma f}\left(\frac{df}{dx}\right)^2w_r,
\]

where the same notation is used for the $x-$dependent factor of $v_1$. ODES (12) - (14) for non-radial perturbations reduce to those of GW by setting $\gamma = \Gamma = 4/3$ and introducing a stream function for velocity perturbations.
Regular boundary conditions are imposed, namely
\[
\begin{align*}
\psi_1 & \propto x^1, \quad w_r = lw_1, \quad \text{for } x \to 0^+ \\
\psi_1 & \propto x^{-(l+1)}, \quad 3w_r df/dx - m x w_r = \psi_1, \quad \text{for } x = x_b
\end{align*}
\]
The last of the outer boundary conditions requires a zero Lagrangian pressure. With these boundary conditions, we can prove the orthogonality of eigenfunctions, i.e. \( \int f^3 w^{(k)} \cdot w^{(l)} \, dr = 0 \) where superscripts \((i)\) and \((j)\) indicate eigenfunctions of different eigenvalues. Meanwhile, the variational principle (Chandrasekhar 1964) can be formulated for this eigenvalue problem.

To solve this eigenvalue problem, we use the following numerical procedure. After solving the background physical variables by using the fourth-order explicit Runge-Kutta scheme, we discretize ODEs \((12) - (14)\) with a proper mesh. Then the inverse iteration method (e.g. Wilkinson 1965) is applied to determine eigenvalues and eigenfunctions. To enhance the efficiency of the inverse iteration method, a relaxation scheme is also implemented.

3 RESULTS OF OUR MODEL ANALYSIS

We first test our numerical code to solve \( \lambda = 4/3 \) isentropic cases and confirm the results of Cao & Lou (2009) to guarantee the correctness and reliability of the code. This check also confirms the errors of \( p \)-mode eigenvalues in GW.

The spectra of eigenvalues versus the mode degree \( l \) contain characteristic features of oscillations. A typical spectrum of \( \lambda > 4/3 \) modes is shown in Fig. 1. In these cases, all eigenvalues \( \lambda \) are negative. We use two dashed lines to separate three distinct classes of modes: \( p \)-modes for \( l \geq 0 \), a unique branch of \( f \)-modes for \( l \geq 2 \) and \( g^{-}\)-modes for \( l \geq 1 \) (see Cowling 1941 for mode classifications of stellar oscillations and Unno et al. 1979 for properties of different modes.). The \( p \)-modes are pressure-driven and gravity-modified acoustic oscillations. The \( f \)-modes are essentially trapped surface Lamb waves (Lamb 1932; Unno et al. 1979; Lou 1990, 1991) and decay exponentially inwards. The \( g^{-} \)-modes are driven by buoyancy and are trapped deep inside the core. The \( g^+ \)-modes are one subclass of \( g \)-modes with \( \lambda > 4/3 \). For a given \( l \), the absolute value of eigenvalue \( |\lambda| \) decreases as the radial order increases. In that limit, eigenvalue \( \lambda \) approaches 0. There is no \( l = 0 \) \( g^+ \)-mode.

The other subclass \( g^{-} \)-modes exist for \( \lambda < 4/3 \) with similar characteristics of \( g^{-} \)-modes, except that the eigenvalues \( \lambda \) are positive and decrease towards zero as the radial order increases. By numerical explorations, we find similar behaviours of each perturbation mode in the core collapsing background as those in stellar oscillations. As \( p \)-modes have been studied by GW for \( \lambda = \Gamma = 4/3 \), we emphatically show \( g^+ \)-mode eigenfunctions and their eigenvalues in Fig. 2.

Between \( p \)-modes and \( g^+ \)-modes exists the unique branch of \( f \)-modes for \( l \geq 2 \). Eigenvalues of \( f \)-modes separate the \( p \)- and \( g \)-modes. Such \( f \)-modes are characterized by eigenfunctions of density and radial velocity perturbations without nodes. They are acoustic in nature and relate to the surface Lamb waves (Lamb 1932; Lou 1990, 1991).

In stellar oscillations, the existence criterion for both types of \( g \)-modes depends on the square of the Brunt-Väisälä buoyancy frequency \( N^2 \) defined by
\[
N^2 = -\frac{1}{\rho} \frac{dP}{dr} \left( \frac{d \ln \rho}{dr} - \frac{1}{\gamma} \frac{d \ln P}{dr} \right),
\]
where \( P(r) \) and \( \rho(r) \) are for a hydrostatic equilibrium, and \( \gamma \) is the adiabatic exponent of perturbation. Since our numerical explorations reveal that if an eigenvalue \( \lambda \) with its eigenfunction can be obtained for \( \lambda = 0 \) for the hydrostatic limit, its counterpart can also be determined for \( \lambda > 0 \) in a continuous manner. Thus, the existence criterion for \( g \)-modes in stellar oscillations can be applied in our analysis by using
partial derivatives instead of derivatives with respect to the radius. For $\gamma < 4/3$ ($\Lambda^2 < 0$), $g^-$-modes appear, whereas for $\gamma > 4/3$ ($\Lambda^2 > 0$), $g^+$-modes manifest. For $\gamma = \Gamma = 4/3$ ($\Lambda^2 = 0$), both $g$-modes are suppressed.

We have readily computed the $p$-mode eigenfunctions with $\gamma \neq \Gamma = 4/3$ and they appear qualitatively similar to those shown in GW and in Cao & Lou (2009).

We demonstrate mathematically that a $g^+$-mode does not change to a $g^-$-mode or vice versa as $\lambda$ increases from $0$ to $\lambda_M$, i.e. for $\gamma \neq 4/3$, there is no $m = 0$ eigenvalue for all $0 < \lambda < \lambda_M$. By setting $m = 0$ in ODEs (12) - (14) and after straightforward manipulations, we derive

$$\int_V \left[ \|\nabla (\psi_1 Y_m)\|^2 + 3 f^2 \psi_1^2 Y_m^2 \right] d^3 r = 0 ,$$

which requires $\psi_1 = 0$. Thus no non-trivial eigenfunctions can be found. Consequently, the eigenvalue curve $m(\lambda)$ for each perturbation mode does not intersect the $m = 0$ line. We show the variation trends of eigenvalues $m$ as $\lambda$ increases from $0$ to $\lambda_M$ in Fig. 3 for typical eigenvalues $m$ of the three lowest $p$-modes and $g$-modes for $l = 1$, 2 and of $f$-mode for $l = 2$ as functions of $\lambda$. Fig. 3 shows that eigenvalues $m$ larger than $25\lambda_M/8$ approach this limiting value as $\lambda \rightarrow \lambda_M$ (see GW). Fig. 4 illustrates the variation of eigenvalues $m$ for the first two lowest orders of $p$-modes and $g^+$-modes as well as that of $f$-modes all for $\lambda = 0.002$ and $l = 2$ as the adiabatic exponent $\gamma$ varies from 1.34 to 1.66.

A key question for perturbation analysis is the stability of a core collapse. The inequality $\Lambda^2 < 0$ for stellar perturbations is the Schwarzschild criterion for convection. In our dynamic core collapse background, we emphasize that this inequality is only sufficient but not necessary for instability. The temporal factor $\tau(t)$ for $\lambda > 0$ bears a power law $t^\xi$ with $\xi = -1/6 \pm [1/36 + 2m/(9\lambda)]^{1/2}$. For a time reversal operation, GW showed that the compression of a collapse is responsible for an amplification of $t^{-1/6}$. Thus the determinant under the square root in $\xi$ decides the mode stability. For $m > -\lambda/8$ so that $\xi$ has two real roots, the mode is unstable for one of the two roots. For $m < -\lambda/8$ so that $\xi$ has a pair of complex conjugate roots, a perturbation oscillates stably in addition to the compression amplification of core collapse. By this new criterion for instability, we find that all $g^-$-modes and sufficiently high-order $g^+$-modes are unstable. It can be shown by a local analysis that the definition of $N$ remains valid for dynamical core collapses, such that $g^-$-modes lead to convective instabilities. It also announces that $g^+$-mode instabilities are uniquely associated with self-similar dynamic core collapse.

We compare our results with earlier analyses (GW; Lai 2000; Cao & Lou 2009). First, in contrast to oscillations in a static star, we examine non-radial adiabatic perturbations in a self-similar conventional polytropic core collapse. The adiabatic exponent $\gamma$ differs from the background $\Gamma = 4/3$, whereas earlier stability analyses of the dynamic background are restricted to $\gamma = \Gamma = 4/3$. Secondly, for a conventional polytropic core collapse, we classify different adiabatic perturbation modes, including $p$-modes, $g$-modes and $f$-modes. GW and Lai (2000) studied only $p$-mode perturbations as their isentropic EoS makes $g$-modes vanish. Cao & Lou (2009) obtained $g$-modes by a general polytropic EoS with a variable specific entropy distribution, even though the perturbation EoS remains the same as the background EoS. In comparison, our model here describes non-radial adiabatic perturbations with $\gamma \neq \Gamma = 4/3$. This leads to $N \neq 0$ so that $g$-modes may exist. We indeed confirm this by numerical explorations. For various efficiencies of heat transport and radiative losses, the case of $\gamma = \Gamma$ should be a special and rare situation. Therefore $g$-modes should exist in stellar core collapses in general.

For core-collapse SN explosions, such $g$-mode instabilities should bear physical consequences. These instabilities complement those revealed by Cao & Lou (2009). They occur during the pre-SN core collapse stage and should also influence the formation of proto-neutron stars and subsequent emergence and evolution of rebound shocks. At least, such instabilities lead to early breakdown of spherical sym-
metry before the emergence of rebound shocks (e.g. Lou & Wang 2006, 2007; Wang & Lou 2008; Hu & Lou 2009). So far, most proposed instabilities occur either in a massive progenitor star prior to the core collapse (e.g. Goldreich et al. 1996) or after the core rebound (e.g. Blondin et al. 2003; Burrows et al. 2006, 2007). Such instabilities may affect the formation, motion and evolution of a proto-neutron star or a pulsar as speculated by Cao & Lou (2009). Moreover, our perturbation analysis serves to link perturbations in a massive progenitor star and those after the core rebound. For example, the ‘ε-mechanism’ (e.g. Goldreich et al. 1996) may provide seed perturbations during a core collapse while such perturbations during a core collapse may stimulate those after the core rebound.

We emphasize that while p−modes, f−modes and low-order g−modes are stable, if radiative losses and diffusive processes are involved instead of the adiabatic approximation, these modes might become overstable.

4 SUMMARY AND CONCLUSIONS

In this Letter, we examine 3D adiabatic perturbations in a self-similar conventional polytropic collapsing core with γ ≠ Γ = 4/3. The gas is non-isentropic in the sense that the adiabatic index γ of perturbations differs from that of the background Γ = 4/3. As γ determines the specific entropy perturbation conserved along streamlines, the non-isentropic process actually involves a nonzero buoyancy frequency N giving rise to g−mode perturbations. In comparison, perturbation analysis of Cao & Lou (2009) emphasizes a general polytropic core collapse with a variable specific entropy (Lou & Cao 2008) and 3D adiabatic perturbations of γ = Γ = 4/3.

By imposing proper boundary conditions, we solve the perturbation eigenvalue problem and derive distinct modes: acoustic p−modes, surface f−modes, internal gravity g− and g+−modes which are classified by their series of eigenvalues m and eigenfunctions in reference to stellar oscillations. In parallel, g−modes involve two types, viz. g−− and g−+−modes; their existence depends on the square of the Brunt-Väisälä buoyancy frequency N^2: the former requires N^2 > 0 while the latter needs N^2 < 0. For adiabatic perturbations, g+−modes correspond to γ > 4/3 while g−−modes occur for γ < 4/3.

Stability properties of these modes in stellar core collapses are examined. The instability criterion shifts from m > 0 in a static Lane-Emden polytropic sphere of λ = 0 to m > −λ/8 for 0 < λ < λM core collapses. Consequently, p−modes, f−modes and sufficiently low-order g−− modes oscillate stably. The g−− modes are unstable leading to convections. Sufficiently high-order g+−modes which would have been stable in a static polytrope now become unstable for 0 < λ < λM GW dynamic core collapse. The specific radial order that g+−modes become unstable depends on the parameter pair of λ and γ.

We speculate that such inevitable g−mode instabilities may offer valuable clues to SN simulations and that the formation, motion and evolution of proton-neutron stars can be nontrivially influenced (Cao & Lou 2009). These perturbations again lead to instabilities before the core rebound and they serve as seeds of later fluctuations. Their excitations can come from oscillations of the progenitor star before the onset of core collapse. According to our numerical exploration and typical stellar parameters with γ > Γ = 4/3, unstable g+−modes appear with fairly high radial orders. This implies that the central mass blob can be quite small and a SN might even break the core into pieces without forming a NS.

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