Online Boosting for Multilabel Ranking with Top-k Feedback

Daniel T. Zhang
Facebook Infrastructure Engineering
dtzhang@fb.com

Young Hun Jung
University of Michigan
yhjung@umich.edu

Ambuj Tewari
University of Michigan
tewaria@umich.edu

November 7, 2019

Abstract

We present online boosting algorithms for multilabel ranking with top-k feedback, where the learner only receives information about the top k items from the ranking it provides. We propose a novel surrogate loss function and unbiased estimator, allowing weak learners to update themselves with limited information. Using these techniques we adapt full information multilabel ranking algorithms (Jung and Tewari, 2018) to the top-k feedback setting and provide theoretical performance bounds which closely match the bounds of their full information counterparts, with the cost of increased sample complexity. The experimental results also verify these claims.

1. INTRODUCTION

The classical theory of boosting is an impressive algorithmic and theoretical achievement (see Schapire and Freund (2012) for an authoritative treatment). However, for the most part it assumes that learning occurs with a batch of data that is already collected and that ground truth labels are fully observed by the learning algorithm. Modern “big data” applications require us to go beyond these assumptions in a number of ways. First, large volumes of available data mean that online algorithms (Shalev-Shwartz 2012; Hazan 2016) are needed to process them effectively. Second, in many applications such as text categorization, multimedia (e.g., images and videos) annotation, bioinformatics, and cheminformatics, the ground truth may not be just a single label but a set of labels (Zhang and Zhou 2013; Gibaja and Ventura 2015). Third, in a multilabel setting, a common design decision (Tsoumakas et al. 2011) is to have the learner output a ranking of the labels. Fourth, human annotators may not have the patience to go down the full ranking to give us the ground truth label set. Therefore, the learner may have to deal with partial feedback. A very natural partial feedback is top-k feedback (Chaudhuri and Tewari 2017) where the annotator only provides ground truth for the top-k ranked labels. Theory and algorithms for online, multilabel boosting with top-k feedback have thus far been missing. Our goal in this paper is to fill this gap in the literature.

Existing literature has dealt with some of the challenges mentioned above. For example, recent work has developed the theory of online boosting for single label problems such as binary classification (Beygelzimer et al. 2015) and multiclass classification (Jung et al. 2017). This was followed by an extension to the complex label setting of multilabel ranking (Jung and Tewari 2018). All of these works were in the full information setting where ground truth labels are fully revealed to the learner. Zhang et al. (2018) recently extended the theory of multiclass boosting to the bandit setting where the learner only observes whether the (single) label it predicted is correct or not. However, none of the available extensions of classical boosting has all of the following three desired attributes at once, namely online updates, multilabel rankings, and the ability to learn from only top-k feedback.

Note that top-k feedback is not bandit feedback. Unlike the bandit multiclass setting, the learner does not even get to compute its own loss! Thus, a key challenge in our setting is to use the structure of the loss to design estimators that can produce unbiased estimates of the loss from only top-k feedback. This intricate interplay between loss functions and partial feedback does not occur in previous work on online boosting.

Specifically, we extend the full information algorithms of Jung and Tewari (2018) to the top-k feedback setting for the multilabel ranking problems. Our algorithms randomize their predictions and construct novel unbiased loss estimates. In this way, we can still let our weak learners update themselves even with partial feedback. Interestingly, the performance bounds of the proposed algorithms match the bounds of their full information counterparts with increased sample complexities. That is, even with the top-k feedback, one can eventually obtain the same level of accuracy provided sufficient data. We also run our algorithms on the same data sets that are investigated by Jung and Tewari (2018), and obtain results supporting our theoretical findings. Our empirical results also verify that a larger k (i.e., more information to the learner) does decrease the sample complexity of the algorithms.
2. PRELIMINARIES

The space of all possible labels is denoted by \([m] = \{1, \cdots, m\}\) for some positive integer \(m\), which we assume is known to the learner. For a technical reason, we assume \(m \geq 5\). We denote the indicator function as \(\mathbb{I}(\cdot)\), the \(i\)th standard basis vector as \(e_i\), and the zero vector as \(0\). Let \(D_m = \{e_i| i \in [m]\}\). We denote a ranking as an ordered tuple. For example, a ranking \(r = (2, 1, 3)\) ranks label 2 the highest and label 3 the lowest. Given a ranking \(r\), we let \(T^k(r)\) return an unordered set of the \(k\) top ranked elements. For example, \(T^2((2, 1, 3)) = \{1, 2\}\).

We will frequently use a score vector \(s \in \mathbb{R}^m\) to denote a ranking. We convert it to a ranking using the function \(\sigma(s)\), which orders the members of \([m]\) according to the score \(s[i]\). We break ties by preferring smaller labels, for example 3 is preferred over 4 if votes are even. This makes the mapping \(\sigma(s)\) injective. For example, \(\sigma(3, 7, 5, 3) = (2, 3, 4, 1)\). When it is clear from the context, we will use a score vector and the corresponding ranking interchangeably.

2.1. Problem Setting

We first describe the multilabel ranking (MLR) problem with top-\(k\) feedback. At each timestep \(t = 1, \cdots, T\), the relevant labels are \(R_t \subseteq [m]\), and the irrelevant labels are \(\overline{R}_t\). An adversary chooses a labelled example \((x_t, R_t) \in X \times 2^{|m|}\) (where \(X\) is some domain) and sends \(x_t\) to the learner. As we are interested in the MLR setting, the learner then produces an \(m\)-dimensional score vector \(y_t\), and sends this result to the adversary. In the full information setting, the learner then observes \(R_t\) and suffers a loss \(L(y_t, R_t)\) which will be defined later. In the top-\(k\) feedback setting, however, it only observes whether \(l\) is in \(R_t\) for each label \(l \in T^k(\sigma(y_t))\). That is to say, if \(k = m\), then it becomes the full information problem, and smaller \(k\) implies less information. For a technical reason, we assume \(k \geq 3\). This feedback occurs naturally in applications such as ads placement and information retrieval, where the end user of the system has limited feedback capabilities. In such scenarios, \(R_t\) may be the set of ads or documents which the user finds relevant, and \([m]\) may be the total set of documents. The user only gives feedback (e.g., by clicking relevant ads) for a few documents placed on top by the algorithm. The learner’s end goal is still to minimize the loss \(L(y_t, R_t)\). It might not be able to compute the exact value of the loss because \(R_t\) is unknown. We want to emphasize that even the size of relevant labels \(|R_t|\) is not revealed.

To tackle this problem we use the online multilabel boosting setup of [Jung and Tewari 2018]. In this setting, the learner is constructed from \(N\) online weak learners, \(WL^1, \cdots, WL^N\) plus a booster which manages the weak learners. Each weak learner predicts a probability distribution across all possible labels, which we write as \(h_t^i \in \mathbb{R}^m\). Previous work has shown that this weak learner restriction encompasses a variety of prediction formats including binary predictions, multiclass single-label predictions, and multiclass multilabel predictions [Jung and Tewari 2018].

In the MLR version of boosting, each round starts when the booster receives \(x_t\). It shares this with all the weak learners and then aggregates their predictions into the final score vector \(y_t\). Once the booster receives its feedback, it constructs a cost vector \(c_t^i \in \mathbb{R}^m\) for \(WL^i\) so that the weak learners incur loss \(c_t^i \cdot h_t^i\), where \(h_t^i\) is the \(i\)th weak learner’s prediction at time \(t\). Each weak learner’s goal is to adjust itself over time to produce \(h_t^i\) that minimizes its loss. The goal of the booster is to generate cost vectors which encourage the weak learners to cooperate in creating better \(y_t\). It should be noted that despite the top-\(k\) feedback, our weak learners get full-information feedback, which means the entire cost vector is revealed to them. Constructing a complete cost vector with partial feedback is one of the main challenges in this problem.

2.2. Estimating a Loss Function

Because of top-\(k\) feedback, we require methods to estimate loss functions dependent on labels outside of the top-\(k\) labels from our score vector \(y_t\). One common way of dealing with partial feedback is to introduce randomized predictions and construct an unbiased estimator of the loss using the known distribution of the prediction. This way, we can obtain a randomized loss function for our learner to use. Thus, we propose a novel unbiased estimator to randomize arbitrary \(y_t\). This estimator requires some structure within the loss function it is estimating.

We require that loss to be writable as a sum of functions which only require as input the scores and relevance of two particular labels, each containing one relevant and irrelevant label. In particular, our loss must have the form

\[
L(s, R) = \sum_{a \in R} \sum_{b \notin R} f(s[a], s[b]) =: \sum_{a, b \in [m]} f^{a,b}(s),
\]

where \(f^{a,b}(s) = \mathbb{I}(a \in R) \mathbb{I}(b \notin R) f(s[a], s[b])\).

Here \(s\) is an arbitrary score vector in \(\mathbb{R}^m\), and \(f\) is a given function. We call this property pairwise decomposability. This decomposability allows us to individually estimate each \(f^{a,b}\) and thus \(L\).

In fact, various valid MLR loss functions are pairwise decomposable. An example is the unweighted rank loss

\[
L^{\text{rank}}(s, R_t) = \sum_{a \in R_t} \sum_{b \notin R_t} \mathbb{I}(s[a] \leq s[b]),
\]

which has various surrogates, including the following.
We note that given any two distinct labels \( a \) and \( b \), the random ranking from the previously described procedure will be filled in by the two boosting algorithms. This process is more complicated than simply using a random ranking with probability \( \rho \), but with our method, \( \tilde{r}_i \) stays closer to \( r_t \), which would be favorable provided \( r_t \) has a small loss. Figure 1 presents an example of this exploration step.

In case the loss is a function of score vector instead of rankings, we can get a random score \( \tilde{r}_i \) with probability \( \rho \) we use \( r_t \) as our final ranking. Otherwise, with probability \( \rho \), we choose \( A \) two elements, denoted by \( A \), from \( T^k(\tilde{r}_i) \) and two elements, denoted by \( B \), from \( T^k(\tilde{r}_i)^c \), the set of labels which have rank lower than \( k \). Then, we take the higher ranked labels from \( A \) and \( B \) and swap them, and do the same for the lower ranked labels, producing our final ranking. This process is more complicated than simply using a random ranking with probability \( \rho \). However, our method, \( \tilde{r}_i \) stays closer to \( r_t \), which would be favorable provided \( r_t \) has a small loss. Figure 1 presents an example of this exploration step. In case the loss is a function of score vector instead of rankings, we can get a random score \( \tilde{r}_i \) with probability \( \rho \) we use \( r_t \) as our final ranking. Otherwise, with probability \( \rho \), we choose \( A \) two elements, denoted by \( A \), from \( T^k(\tilde{r}_i) \) and two elements, denoted by \( B \), from \( T^k(\tilde{r}_i)^c \), the set of labels which have rank lower than \( k \). Then, we take the higher ranked labels from \( A \) and \( B \) and swap them, and do the same for the lower ranked labels, producing our final ranking. This process is more complicated than simply using a random ranking with probability \( \rho \). However, our method, \( \tilde{r}_i \) stays closer to \( r_t \), which would be favorable provided \( r_t \) has a small loss. Figure 1 presents an example of this exploration step.

We now present our unbiased estimator. Let \( \tilde{r}_i \) be the random ranking from the previously described procedure, and let \( s \) be an arbitrary score vector in \( \mathbb{R}^m \). We note that given any two distinct labels \( a \) and \( b \), \( \Pr[a, b \in T^k(\tilde{r}_i)] > 0 \). Since being in the top-\( k \) provides the learner with full information regarding the relevance and scores of the labels, we have this unbiased estimator.

Using importance sampling

\[
\hat{L}(s, R_t) = \sum_{a, b \in [m]} \frac{\mathbb{I}(a, b \in T^k(\tilde{r}_i)) \Pr[a, b \in T^k(\tilde{r}_i)]}{f^{n,b}(s)}.
\]

We prove that this is an unbiased estimator in Lemma 5 in the appendix. Our algorithms will use this unbiased estimator to estimate certain surrogate functions which we construct to be pairwise decomposable.

One useful quality of this estimator is what we call \( b \)-boundedness. We say a random vector \( Y \) is \( b \)-bounded if almost surely, \( \| Y - EY \|_\infty \leq b \). This definition also applies to scalar random variables, in which case the infinity norm becomes the absolute value. In Lemma 6 in the appendix, we prove that if the pairwise functions are bounded by some \( z \), then any such unbiased estimator like in Eq. 1 is bounded with a constant that is \( O(\frac{e^{m^2}}{z^2}) \).

Now suppose that the cost vector \( c^t_i \) (to be fed to the \( i \)-th weak learner at time \( t \)) requires full knowledge of \( R_t \) to compute. If each of its entries is a function that is pairwise decomposable, we can use the same unbiased estimation strategy to obtain random cost vectors \( \tilde{c}^t_i \) that are in expectation equal to \( c^t_i \).

3. ALGORITHMS

We introduce two different online boosting algorithms along with their performance bounds. Our bounds rely on the number and quality of the weak learners, so we define the edge of a weak learner. Our first algorithm assumes every weak learner has a positive edge \( \gamma \), while our second algorithm uses an edge measured adaptively.

These edges have a close relationship to each other and are also closely related to the full information edges defined by Jung and Tewari [2018], allowing us to show that our theoretical error bounds closely match theirs. Our first algorithm uses this edge information to achieve an exponentially decreasing error bound with respect to the number and quality of weak learners. For our second algorithm we use empirical edges to bound the loss, and allow adaptive weighting of weak learners. This makes it more practical while sacrificing exponentially decreasing bounds.

3.1. Algorithm Template

We describe the template which our two boosting algorithms share. It does not specify certain steps, which will be filled in by the two boosting algorithms. Also, in our template we do not restrict weak learners in any way except that each WL predicts a distribution \( h^t_i \) over \([m]\), receives a full cost vector \( c^t_i \in \mathbb{R}^m \), and suffers the loss \( c^t_i \cdot h^t_i \) according to its prediction.

The booster keeps updating the learner weights \( \alpha^t_i \) and constructs \( N \) experts, where the \( j \)-th expert is the...
using an adaptive distribution. The booster then uses more weight on each label in $\gamma$, the cost vectors tainted by some noise. and with the weak learners only observing versions of their loss better than a randomly guessing competitor, 3.2.1 Ranking Weak Learning Condition learners used, up to a constant factor.

Our first algorithm, TopkBBM\footnote{Boost-By-Majority for ranking with top-$k$ feedback}, assumes the ranking weak learning condition and is optimal, meaning it matches the asymptotic loss bounds of an optimal full information boosting algorithm in the number of weak learners used, up to a constant factor.

3.2. An Optimal Algorithm

Our first algorithm, TopkBBM\footnote{Boost-By-Majority for ranking with top-$k$ feedback}, assumes the ranking weak learning condition and is optimal, meaning it matches the asymptotic loss bounds of an optimal full information boosting algorithm in the number of weak learners used, up to a constant factor.

\begin{algorithm}
1: \textbf{Input:} Exploration rate $\rho$ and a loss $L(\cdot, \cdot)$ where $L$ is pairwise decomposable
2: \textbf{Initialize:} WL weights $\alpha_i^t$ for $i \in [N]
3: \textbf{for} t = 1, \ldots, T \textbf{do}
4: \hspace{1em} Receive example $x_t$
5: \hspace{1em} Obtain distribution $h_i^t$ from WL$i$ for $i \in [N]
6: \hspace{1em} Compute experts $s_i^t = \sum_{j=1}^{j} \alpha_i^t h_i^j$ for $j \in [N]
7: \hspace{1em} Select an index $i_t \in [N]
8: \hspace{1em} Obtain $\tilde{y}_t$ from $\sigma(\tilde{y}_t)$ using $s_i^t$ and the random process defined in Section 2.2
9: \hspace{1em} Booster suffers loss $L(\tilde{y}_t, R_t)$, but this is not shown to the booster
10: \hspace{1em} For each $l \in T^k(\tilde{y}_t)$, receive feedback $\mathbb{I}(l \in R_t)$
11: \hspace{1em} Compute cost vectors $c_i^t$ for each $i \in [N]
12: \hspace{1em} Weak learners suffer loss $\tilde{c}_i^t \cdot h_i^t$ and update internal parameters
13: \hspace{1em} Set $\alpha_{i+1}^t$ for all $i \in [N]
14: \hspace{1em} Update booster’s parameters, if any
15: \textbf{end for}
\end{algorithm}

weighted cumulative votes from the first $j$ weak learners $s_i^t := \sum_{j=1}^{j} \alpha_i^t h_i^j \in \mathbb{R}^m$. The booster chooses an expert index $i_t \in [N]$ at each round $t$ to use. The first algorithm fixes $i_t$ to be $N$, while the second one draws it randomly using an adaptive distribution. The booster then uses $s_i^t$ to compute its final random prediction $\tilde{y}_t$. After obtaining feedback, the booster computes random cost vectors $c_i^t$ for each weak learner and lets them update parameters.

3.2.1 Ranking Weak Learning Condition

The ranking weak learning condition states that within the cost vector framework, weak learners can minimize their loss better than a randomly guessing competitor, so long as the cost vectors satisfy certain conditions, and with the weak learners only observing versions of the cost vectors tainted by some noise.

We define the randomly guessing competitor at time $t$ as $u_{R_t}$, which is an almost uniform distribution placing $\gamma$ more weight on each label in $R_t$. In particular, for a any label $l$ we define it as

$$u_{R_t}^\gamma, l = \begin{cases} 
\frac{1 - |R_t| \gamma}{m} + \gamma & \text{if } l \in R_t \\
\frac{1 - |R_t| \gamma}{m} & \text{if } l \notin R_t
\end{cases}$$

The intuition is that if a weak learner predicts a label using $u_{R_t}^\gamma$ at each round, then its accuracy would be better than random guessing by at least an edge of $\gamma$.

Given $R_t$, we specify the set of possible cost vectors as

$$\mathcal{C}(R_t) = \{c \in [0, 1]^m \mid \min_{l \in R_t} c[l] = 0, \max_{l \notin R_t} c[l] = 1, \max_{l \in R_t} c[l] \leq \min_{j \notin R_t} c[j]\}.$$ 

We also allow a sample weight $w_t \in [0, 1]$ to be multiplied by this cost vector. This feasible set of cost vectors is equivalent to those used in the full information setting studied by Jung and Tewari (2018).

As in the full information setting, at each round we allow the adversary to choose an arbitrary cost vector from $\mathcal{C}(R_t)$ and its weight for the learner. In our top-$k$ feedback setting, we further permit the adversary to generate random cost vectors and weights, so long as in expectation each random cost vector is in $\mathcal{C}(R_t)$.

We now introduce our top-$k$ feedback weak learning condition, presented beside the full information online weak learning condition from [Jung and Tewari (2018)], to show their similarity.

Definition 1 (OnlineWLC). For parameters $\gamma, \delta$, and $S$, a pair of a learner and adversary satisfies OnlineWLC($\gamma, \delta, S$) if for any $T$, with probability $1 - \delta$, the learner can generate predictions that satisfy

$$\sum_{t=1}^{T} w_t c_t \cdot h_t \leq \sum_{t=1}^{T} w_t c_t \cdot u_{R_t}^\gamma + S.$$ 

Definition 2 (Top-kWLC). For parameters $\gamma, \delta, b$, and $S$, a pair of a learner and adversary satisfies Top-kWLC($\gamma, \delta, S, b$) if for any $T$, with probability $1 - \delta$, the learner can generate predictions that satisfy

$$\sum_{t=1}^{T} w_t c_t \cdot h_t \leq \sum_{t=1}^{T} w_t c_t \cdot u_{R_t}^\gamma + S,$$

while only observing random cost vectors $\tilde{c}_t$, where all $\tilde{c}_t$ are $b$-bounded and $\mathbb{E}\tilde{c}_t = w_t c_t$.

In these definitions, $S$ is called the excess loss. The weak learning conditions only differ by the introduction of random noise. In Top-kWLC if the variance of each $\tilde{c}_t$ is 0, then $\tilde{c}_t = c_t$, and we recover the full information weak learning condition. For a positive $b$, the definition of $b$-boundedness implies $\frac{1}{b}$ is 1-bounded. From this, we can infer $S = \mathcal{O}(b)$ in the top-$k$ setting.

3.2.2 TopkBBM Details

We require that our loss function $L(s, R_t)$ be pairwise decomposable, and that each of its pairwise function $f$ has three properties, which we now describe.
Algorithm 2 TopkBBM

2: Initialize: WL weights $\alpha^i_1 = 1$ for $i \in [N]$
7: Set $i_t = N$
11: Compute cost vectors $\hat{c}_i^t$ for each $i$ using Eq. 3
13: Set $\alpha^i_{t+1} = 1$
14: No booster parameters to update

We record important qualities of this surrogate potential as a proposition, with the proof in Appendix A.2.

**Proposition 1.** $\Phi^N_t(s)$ is proper and convex, and for any $R$, $\gamma$, $N$, and $s$, we have $\Upsilon^N_t(s) \leq \Phi^N_t(s)$.

We also stress that $\Phi^N_t$ is pairwise decomposable into each of its smaller potential functions.

Returning to the algorithm, we assume that weak learners satisfy Top-$4\text{WLC}(\gamma, \delta, S, b)$. Our goal is to set $c^t[i] = \Phi^N_t(s^i_1 + c_t)$. Because $\Phi$ is pairwise-decomposable, we can create an unbiased estimator of it using the technique in Section 2.2 as

$$\hat{\Phi}^N_t(s) = \sum_{a \in R_t} \sum_{b \not\in R_t} \mathbb{I}(a, b \in T^k(\hat{r}_t)) \Pr[a, b \in T^k(\hat{r}_t)] \Lambda^a, b, N_t(s).$$

Because $\Lambda^a, b, N_t$ is simply a potential function using $f_{a, b}$, any upper bound on $f_{a, b}$ also upper bound $\Lambda^a, b, N_t$. Then we can use Lemma 5 to claim $\hat{\Phi}^N_t$ is $b$-bounded.

Thus, we can create unbiased estimates of $c^t[i]$ as

$$\hat{c}^t[i] = \hat{\Phi}^N_t(s^i_1 + c_t).$$  (3)

The rest of the algorithm is straightforward. We set $\alpha^i_t = 1$ for all $i \in [N]$, and select the best expert to be $i_t = N$. This means that we always take an equal-weighted vote from all the weak learners. Intuitively, the booster wants to use all weak learners because they are all guaranteed to do better than random guessing in the long run, and weigh them equally because all weak learners have the same edge $\gamma$. Lastly, given the last expert $s^N_t$, our algorithm predicts using the same random process described in Section 2.2, so that we can use the unbiased estimator.

### 3.2.3 TopkBBM Loss Bound

We can theoretically guarantee the performance of TopkBBM on any proper and pairwise decomposable loss function. In our theorem, we bound $L(s^N_t, R_t)$ instead of the true, randomized loss $L(\hat{y}_t, R_t)$ because without extra guarantees, we cannot say anything about how random predictions will affect the loss. We provide corollaries later for specific losses which we will bound $L(\hat{y}_t, R_t)$. The following theorem holds for any pairwise decomposable loss functions whose pairwise losses satisfy the three qualities listed earlier. The proof appears in Appendix A.3.

**Theorem 2** (TopkBBM, General Loss Bound). For any $T, N$ satisfying $\delta \ll \frac{1}{N}$, the total loss incurred by TopkBBM satisfies the following inequality with probability at least $1 - N\delta$

$$\sum_{t=1}^{T} L(s^N_t, R_t) \leq \Phi^N_t(0)T + O(\frac{2m^2 - k^2}{\rho}zN),$$

where $z$ is the maximum possible value that any $f_{a, b}$ can output, and $O$ suppresses dependence on $\log \frac{1}{\delta}$. 

Note that there is no single canonical loss in the MLR setting unlike the classification setting where the 0-1 loss is quite standard. Still, the weighted rank loss comes close to being canonical since it is often used in practice and is implemented in standard MLR libraries (Tsoumakas et al., 2011). It has also been analyzed theoretically in previous work on ranking (e.g., see Cheng et al. (2010) and Gao and Zhou (2011)).

We note that this loss is not convex, and thus not pairwise decomposable. Thus we use the unweighted hinge loss as a surrogate. Since the unweighted hinge loss upper bounds the rank loss, Theorem 2 can be used to bound it. This allows us to present the following corollary, whose proof can be found in Appendix A.3.

**Corollary 3** (TopkBBM, Rank Loss Bound). For any $T$ and $N \ll \frac{1}{\delta}$, TopkBBM’s randomized predictions $\tilde{y}_t$ satisfy the following bound on the rank loss with probability at least $1 - \delta$:

$$
\sum_{t=1}^{T} L_t^\text{rank}(\tilde{y}_t) \leq \frac{m^2}{4}(N + 1) \exp(-\frac{\gamma^2 N}{2})T + 2\rho m T + \mathcal{O}(\frac{m^2 - k^2}{\rho} N^2 \sqrt{T}).
$$

We can optimize $\rho \propto N \sqrt{\frac{2m^2 - k^2}{m}} T^{-\frac{1}{4}}$ so that the first term in the bound becomes the asymptotic average loss bound. We can compare it to the asymptotic error bounds in Jung and Tewari (2018) by multiplying the full information algorithm loss bounds by $\frac{m^2}{m}$, which is the maximum value of the rank loss normalization constant. Let $s_i^t$ be the score vectors produced by the full information algorithm. Then we have that

$$
\sum_{t=1}^{T} L_t^\text{rank}(s_i^t) \leq \frac{m^2}{4}(N + 1) \exp(-\frac{\gamma^2 N}{2})T + \frac{m^2}{2} N S.
$$

We see that the asymptotic losses, after optimizing $\rho$, are identical, so that the cost of top-$k$ feedback appears only in the excess loss. Furthermore, since the optimal algorithm in Jung and Tewari (2018) is optimal in the number of weak learners it requires to achieve some asymptotic loss, TopkBBM is also optimal in this regard since the problem it faces is only harder because of partial information.

### 3.3. An Adaptive Algorithm

While TopkBBM is theoretically sound, it has a number of drawbacks in real world applications. Firstly, it is difficult to actually measure $\gamma$ for a particular weak learner, and usually the weak learners will not all have the same edge. Secondly, potential functions often do not have closed form definitions, and thus require expensive random walks to compute. To address these issues, we propose an adaptive algorithm, TopkAdaptive, modifying Ada.OLMR from Jung and Tewari (2018) so that it can use top-$k$ feedback.

#### 3.3.1 Logistic Loss and Empirical Edges

Like other adaptive boosting algorithms, we require a surrogate loss. We take the logistic loss for multilabel ranking from Ada.OLMR, but ignore its normalization (as that would require knowledge of $|R_t|$)

$$
L^\text{log}(s, R_t) := \sum_{a \in R_t, b \notin R_t} \log(1 + \exp(s[a] - s[b])).
$$

This loss is proper and convex. As in Ada.OLMR, the booster’s prediction is still graded using the (unweighted) rank loss. This surrogate loss only plays a role in optimizing parameters.

Similarly to $\Phi$, we create an unbiased estimator $\hat{L}^\text{log}(s, R_t)$, of the logistic loss as

$$
\sum_{a \in R_t, b \notin R_t} \mathbb{I}(a, b \in T^k(\hat{r}_t)) \log(1 + \exp(s[a] - s[b])).
$$

Our goal is to set $c_i^t := \nabla \hat{L}^\text{log}(s_i^{t-1})$. However, because we cannot always evaluate the logistic loss, we use $\hat{L}^\text{log}(\cdot)$ instead to make random cost vectors which in expectation are the desired cost vectors:

$$
\hat{c}_i^t = \nabla \hat{L}^\text{log}(s_i^{t-1}). \tag{4}
$$

Even though the algorithm is adaptive, we still need an empirical measure of the weak learner’s predictive powers for performance bounds. As in Ada.OLMR, we use the following empirical edge of WL$^t$:

$$
g_{\text{WL}} = \sum_{t=1}^{T} \frac{c_i^t h_i^t}{\|w_i^t\|^2} \frac{1}{1 + \exp(s_i^{t-1}[a] - s_i^{t-1}[b])}, \tag{5}
$$

where $w_i^t$ is the definition of the weight of a cost vector taken from Jung and Tewari (2018). A useful remark is that if a weak learner satisfies Top-$k$WL with edge $\gamma_i$, then for large $T$ it should have an empirical edge $\gamma_i \geq \frac{\gamma}{2}$ with high probability.

Having a similar edge as Ada.OLMR allows us to precisely evaluate the cost of bandit feedback. We can check that since $\sum_t |c_i^t[t]| = w_i^t[t]$, $\gamma_i \in [-1, 1]$. It is apparent the empirical edge is not visible to the learner, since it requires the expected cost vector to compute. This is fine because this value is only useful in proving the loss bound, and is not used by the algorithm.

#### 3.3.2 TopkAdaptive Details

We now go into the details of TopkAdaptive. The choice of cost vectors $\hat{c}_i^t$ is discussed in the previous section. As this is an adaptive algorithm, we want to choose the weak learner’s weights $\alpha_i^t$ at each round. We
Algorithm 3 TopkAdaptive

2: Initialize: WL weights \( \alpha_i^1 = 0 \) and expert weights \( \nu_i^1 = 1 \) for \( i \in [N] \)
7: Select an index \( i_t \in [N] \) with \( \Pr[ i_t = i ] \propto \nu_i^t \)
11: Compute cost vectors \( \hat{s}_t^i \) for each \( i \) using Eq. 4
13: Set \( \alpha_{i+1}^t = \frac{8pT}{(2m^2-k^2)\sqrt{T}} \)
14: Set \( \nu_{i+1}^t = \nu_i^t \cdot \exp(-L_t(s_i^t)) \)

would like to choose them to minimize the cumulative logistic loss

\[
\sum_t g_t^i(\alpha_i^t) \quad \text{where} \quad g_t^i(\alpha_i^t) = \hat{L}_t^\log(s_i^{t-1} + \alpha_i^t e_i^t)
\]

with only the unbiased estimate

\[
\sum_t \hat{g}_t^i(\alpha_i^t) \quad \text{where} \quad \hat{g}_t^i(\alpha_i^t) = \hat{L}_t^\log(s_i^{t-1} + \alpha_i^t e_i^t)
\]

at each time step available to the booster.

Since the logistic loss is convex, we can use our partial feedback to run stochastic gradient descent (SGD). To apply SGD, besides convexity we require that the feasible space be compact, so we let \( F = [-2, 2] \). To stay in the feasible space, we use the projection function \( \Pi(\cdot) = \max\{-2, \min\{\cdot, 2\}\} \) in our update rule as \( \alpha_{i+1}^t = \Pi(\alpha_i^t - \eta_t \hat{g}_t^i(\alpha_i^t)) \), where \( \eta_t \) is the learning rate. We bound the loss from SGD and show it provides a regret within \( O\left(\frac{m^2-k^2}{\rho}\sqrt{T}\right) \). The details are in the proof in Appendix A.3.

We cannot prove that the last expert is the best because our weak learners do not adhere to any weak learning condition. Instead, we prove that at least one expert is reliable. Our algorithm uses the Hedge algorithm from Freund and Schapire (1997) to select the best expert from the ones available, taking as input for the \( i \)th expert the unweighted rank loss of the \( i \)th expert, which we define as

\[
\hat{L}_t^\rank(s_i^t, R_t) = \sum_{a,b \in [m]} \mathbb{I}(a, b \in T^k(\tilde{r}_t)) \frac{\Pr[a, b \in T^k(\tilde{r}_t)]}{\Pr[\tilde{r}_t]} \mathbb{I}(s[a] \geq s[b]).
\]

Because the exploration rate \( \rho \) controls the variance of the loss estimate, we can combine the analysis of the Hedge algorithm with a concentration inequality to obtain a similar result.

3.3.3 TopkAdaptive Loss Bound

We now bound the cumulative rank loss of TopkAdaptive using the weak learner’s empirical edges. The proof appears in Appendix A.3.

Theorem 4 (TopkAdaptive, Rank Loss Bound). For any \( T, N \) satisfying \( \delta \ll \frac{1}{T} \), the cumulative rank loss of TopkAdaptive, \( \sum_{t=1}^T \hat{L}_t^\rank(\tilde{y}_t, R_t) \), satisfies the following bound with probability at least \( 1 - (N + 4)\delta \):

\[
\frac{2m^2}{\sum_i |\gamma_i|} T + 2\rho m T + \tilde{O}\left(\frac{(2m^2 - k^2)N\sqrt{T}}{\rho \sum_i |\gamma_i|}\right),
\]

where \( \tilde{O} \) suppresses dependence on \( \log \frac{1}{\delta} \).

By optimizing \( \rho \propto \sqrt{\frac{(2m^2-k^2)N}{m \sum_i |\gamma_i|}} T^{-\frac{1}{2}} \), we get the first term of the bound as the asymptotic average loss bound. To compare it with the adaptive algorithm in Jung and Tewari (2018), we again multiply the bound by \( \frac{m}{T} \) to account for the normalization constant. Let \( s_i^t \) be the scores of the full information adaptive algorithm at time \( t \). Then we have

\[
\sum_t L_t^\rank(s_i^t) \leq \frac{2m^2}{\sum_i |\gamma_i|} T + \tilde{O}(\frac{N^2m^2}{\sum_i |\gamma_i|}).
\]

Which matches the asymptotic loss in TopkAdaptive, after optimizing for \( \rho \). Thus, the cost of top-\( k \) feedback is again only present in the excess loss.

4. EXPERIMENTS

We compare various boosting algorithms on benchmark data sets using publicly available codes. The models we use are our own, TopkBBM (TopOpt) and TopAdaptive (TopAda), along with OnlineBMR (FullOpt) and Ada.OLMR (FullAda) by Jung and Tewari (2018), the full information algorithms we compared our theoretical results to. All of these boosters use multilabel weak learners.

We examine several data sets from the UCI data repository from Tsoumakas et al. (2011) that have been used to evaluate the full information algorithms. We follow the preprocessing steps from Jung and Tewari (2018) to ensure consistent comparisons, and the data set details and statistics appear in Table 3 in the appendix. However, because the top-\( k \) feedback algorithms require more data to converge, we loop over the training set a number of times before evaluating on the testing data set. We consider the number of loops a hyper-parameter. However, it never exceeds 20. The other hyper-parameters we optimize are the number of weak learners \( N \), and the the edge \( \gamma \) for TopOpt. Experiment details can be found in Appendix 3.

4.1. Asymptotic Performance

Since for the rank loss, the theoretical asymptotic error bounds of the proposed algorithms match their full information counterparts, we first compare the models empirical asymptotic performance. For the full information algorithms, we looped the training set once and then ran the test set, while for the top-\( k \) algorithms, we looped them as described in the previous subsection. In these tests, we set \( k = 3 \). The selected hyper-parameters, including number of loops, appear in Table
Table 1: Average weighted rank loss on the test set

| Data    | m  | TopOpt | TopAda | FullOpt | FullAda |
|---------|----|--------|--------|---------|---------|
| Emotions| 6  | 0.19   | 0.23   | 0.17    | 0.16    |
| Scene   | 6  | 0.11   | 0.13   | 0.07    | 0.09    |
| Yeast   | 14 | 0.22   | 0.22   | 0.18    | 0.19    |
| Mediamill|101| 0.07   | 0.08   | –       | 0.05    |
| M-reduced|101| 0.11   | 0.11   | 0.06    | 0.06    |

In the appendix. Each table entry is the result of 10 runs averaged together.

In Table 1 we see that in each data set, the full information algorithms outperform their top-k feedback counterparts, but that the gap is quite small. The largest gap is between TopAda and FullAda on M-reduced at 0.05. In part, this is due to differences in weighting between the unweighted and weighted rank loss. When the number of relevant labels per example is constant, the weighted and unweighted rank loss are exactly proportional because the rank loss weight is constant, but when the number of labels varies, this rank loss weight will change. This leads to a discrepancy in the goals the full information algorithms and our algorithms are boosting towards. Overall, however, especially after accounting for exploration, the smallness of this gap shows our algorithms are learning as effectively as their full information counterparts.

Another factor is the number of loops run. For the data sets with smaller $m$, the number of loops multiplied by $k$ implies that our algorithms could have observed each label multiple times in its training. For example, with $m = 6$ and $k = 3$, theoretically in two loops of the training set, our algorithms could have observed them in their entirety. However, in the Mediamill and M-reduced data sets, our algorithms manage comparable asymptotic performance, while at best, given $m$ and $k$ they could only have possibly observed 60 labels, or about 60% training labels. This shows they are capable of making inferences even with partial information.

4.2. Effects of Varying Observability

To show the empirical effects of top-k feedback on model convergence and asymptotic loss, we repeat our experiments with the Yeast data set, keeping the same hyper-parameters, but increasing $k$. Figure 2 plots the weighted rank loss averaged over every 100 consecutive rounds for TopkBBM models with various $k$. Each line in the figure is itself averaged from 10 runs, with the same hyper-parameters in Table 2. Clearly as $k$ increases, the number of rounds TopOpt requires to converge decreases. Despite this, by the 8000th round of feedback, the three lines in figure 2 have closed tightly on each other, supporting our theory that the cost of changing $k$ is only taken by the excess loss.

Figure 2: Learning curves of TopkBBM run on the Yeast dataset, with varying $k$.

Acknowledgements

Part of this work occurred while DZ was an undergraduate student at the University of Michigan where he was partially supported by the NSF RTG grant DMS-1646108. YJ and AT acknowledge the support of NSF CAREER grant IIS-1452099.

References

Alina Beygelzimer, Satyen Kale, and Haipeng Luo. Optimal and adaptive algorithms for online boosting. In *International Conference on Machine Learning*, pages 2323–2331, 2015.

Nicolo Cesa-Bianchi and Gabor Lugosi. *Prediction, learning, and games*. Cambridge university press, 2006.

Sougata Chaudhuri and Ambuj Tewari. Online learning to rank with top-k feedback. *The Journal of Machine Learning Research*, 18(1):3599–3648, 2017.

Weiwei Cheng, Eyke Hüllermeier, and Krzysztof J Dembczynski. Bayes optimal multilabel classification via probabilistic classifier chains. In *Proceedings of the 27th international conference on machine learning (ICML-10)*, pages 279–286, 2010.

Yoav Freund and Robert E Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. *Journal of computer and system sciences*, 55(1):119–139, 1997.

Wei Gao and Zhi-Hua Zhou. On the consistency of multi-label learning. In *Proceedings of the 24th annual conference on learning theory*, pages 341–358, 2011.

Eva Gibaja and Sebastián Ventura. A tutorial on multilabel learning. *ACM Computing Surveys (CSUR)*, 47(3):52, 2015.
Elad Hazan. Introduction to online convex optimization. *Foundations and Trends in Optimization*, 2(3-4):157–325, 2016.

Young Hun Jung and Ambuj Tewari. Online boosting algorithms for multi-label ranking. In *Proceedings of the 21st International Conference on Artificial Intelligence and Statistics*, volume 84 of *Proceedings of Machine Learning Research*, pages 279–287, 2018.

Young Hun Jung, Jack Goetz, and Ambuj Tewari. Online multiclass boosting. In *Advances in Neural Information Processing Systems 30*, pages 920–929, 2017.

Indraneel Mukherjee and Robert E Schapire. A theory of multiclass boosting. *Journal of Machine Learning Research*, 14(Feb):437–497, 2013.

Robert E Schapire and Yoav Freund. *Boosting: Foundations and Algorithms*. MIT Press, 2012.

Shai Shalev-Shwartz. Online learning and online convex optimization. *Foundations and Trends in Machine Learning*, 4(2):107–194, 2012.

Grigorios Tsoumakas, Eleftherios Spyromitros-Xioufis, Jozef Vilcek, and Ioannis Vlahavas. Mulan: A java library for multi-label learning. *Journal of Machine Learning Research*, 12(Jul):2411–2414, 2011.

Daniel Zhang, Young Hun Jung, and Ambuj Tewari. Online multiclass boosting with bandit feedback. *arXiv preprint arXiv:1810.05290*, 2018.

Min-Ling Zhang and Zhi-Hua Zhou. A review on multilabel learning algorithms. *IEEE Transactions on Knowledge and Data Engineering*, 26(8):1819–1837, 2013.

Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the 20th International Conference on Machine Learning (ICML-03)*, pages 928–936, 2003.
A. DETAILED PROOFS

We provide proofs that are skipped in the main body.

A.1. Proofs for the Unbiased Estimator

Lemma 5. Suppose that we have $k \geq 2$, and a random ranking $\tilde{r}$ such that for any $a, b \in [m]$, $\Pr[a, b \in T_k(\tilde{r})] > 0$. Let $g(s) = \sum_{a \in R_t} \sum_{b \notin R_t} f_{a,b}(s)$. Then the expectation of Eq. 1 becomes $g(s)$.

Proof. We first write out the unbiased estimator that Eq. 1 provides of $g$

$$\hat{g}(s) = \sum_{a \in R_t} \sum_{b \notin R_t} \frac{\mathbb{I}(a, b \in T_k(\tilde{r}))}{\Pr[a, b \in T_k(\tilde{r})]} f_{a,b}(s).$$

Then we rewrite the expectation of $\hat{g}$ by moving it inside the sum.

$$\mathbb{E}_s[\hat{g}(s, R_t)] = \sum_{a \in R_t} \sum_{b \notin R_t} \mathbb{E}_s \left[ \frac{\mathbb{I}(a, b \in T_k(\tilde{r}))}{\Pr[a, b \in T_k(\tilde{r})]} f_{a,b}(s) \right]$$

$$= \sum_{a \in R_t} \sum_{b \notin R_t} \Pr[a, b \in T_k(\tilde{r})] \frac{f_{a,b}(s)}{\Pr[a, b \in T_k(\tilde{r})]}$$

$$= g(s)$$

where the middle equality is due to the expectation inside the summation being zero unless $a, b \in T_k(\tilde{r})$. $\square$

Lemma 6. Suppose our loss function $L$ is pairwise decomposable, $\rho < 0.25$, and thus has an unbiased estimator $\hat{L}$ like in Eq. 1. If there exists $z$ such that $f_{a,b}(s) \leq z$ for all feasible $s$ available to the booster, then we have

$$|\hat{L}(s, R_t) - L(s, R_t)| = O(z \frac{2m^2 - k^2}{\rho})$$

almost surely.

Proof. We record again the definition of $\hat{L}$ from Eq. 1.

$$\hat{L}(s, R_t) = \sum_{a, b \in [m]} \frac{\mathbb{I}(a, b \in T_k(\tilde{r}_t))}{\Pr[a, b \in T_k(\tilde{r}_t)]} f_{a,b}(s) = \sum_{a \in R_t} \sum_{b \notin R_t} \frac{\mathbb{I}(a, b \in T_k(\tilde{r}_t))}{\Pr[a, b \in T_k(\tilde{r}_t)]} f(s[a], s[b]).$$

We first bound the case where our estimator underestimates the true loss. In this case, the worst scenario would be all the pairwise loss functions which we activate evaluate to 0, while all other functions evaluate to $z$. In these cases, we are bounded by $O(z(m^2 - k^2)).$

We now bound the cases where our estimator overestimates the true loss. We proceed by bounding the difference in our estimate from the expectation on a case-by-case basis, and then counting the number of each case that can arise in a worst-case scenario. We recall that $\hat{L}$ has implicit parameter $\rho$ that is used to generate $\tilde{r}$, as described in Section 2.2. Let $V = T^k(r_t)$ and $\tilde{V} = T^k(\tilde{r}_t)$. If we decide not to explore, and $V = \tilde{V}$, then our worst case scenario where we overestimate is when all the functions which we don’t activate are 0, and all the functions which we do activate are $z$. In this case, we are bounded by $O(k^2(\frac{1}{1-\rho} - 1)) = O(k^2 \rho)$. To see that this is in $O(\frac{2m^2 - k^2}{\rho})$, we note that $O(k^2 \rho + \frac{m^2 - k^2}{\rho}) = O(\frac{m^2 - k^2}{\rho}) = O(\frac{2m^2 - k^2}{\rho})$.

Now we consider the case where we decide to explore. In this case, because we overestimate all of our activated pairwise functions, our worst case scenario is where all the pairwise functions we activate evaluate to $z$, and all other pairwise functions evaluate to 0. Firstly, suppose $a, b \in V$. Then we have $\Pr[a, b \in V] \geq 1 - \rho$, which implies $\frac{1}{\Pr[a, b \in T_k(\tilde{r}_t)]} \leq \frac{4}{3}$. In expectation, we expect this weight to be 1, so we can upper bound the deviation of each pair by $\frac{1}{3}$, and we know there must be fewer than $k^2$ of these pairs.

Secondly, suppose exactly one of $a$ and $b$ is in $V$. Without loss of generality, let us assume $a \in V$ and $b \notin V$. In this case, for both labels to be in $V$, the algorithm must decide to explore, with probability $\rho$. Then it must select $b$ to be in the set of $d$ taken from outside $\tilde{V}$, and it must not select $a$ from $V$. From this, we obtain

$$\Pr[a, b \in \tilde{V}] = \rho \cdot \frac{2}{m-k} \cdot \frac{k-2}{k}.$$
Thus in this case, our bound on the difference between estimator and expectation is \( \frac{1}{\Pr[a,b \in \tilde{V}] - 1} = \mathcal{O}(\frac{m-k}{\rho}) \). We again count that the maximum number of such pairs that could appear is \( 2(k-2) \), because there are \( k-2 \) such pairs for each label transplanted from outside of the top-\( k \).

Lastly, suppose \( a,b \not\in V \). They must be chosen to be moved up when the algorithm decides to explore with probability \( \rho \). Therefore, we have

\[
\Pr[a,b \in \tilde{V}] = \rho \left( \frac{1}{\binom{m-k}{2}} \right) = \rho \left( \frac{2}{(m-k)(m-k-1)} \right).
\]

Thus in this case, \( \frac{1}{\Pr[a,b \in \tilde{V}]} \leq \left( \frac{m-k}{2\rho} \right)^2 \). There must be only one of these labels present, because we only ever choose two labels from outside of the top-\( k \) to swap. Now, to produce our final bound, we multiply the weight produced from each case by the number of times it can occur, to obtain the sum

\[
G = z \left[ \frac{1}{3} k^2 + 2(k-2) \frac{k(m-k)}{2(k-2)\rho} + \frac{(m-k)^2}{2 \rho} \right]
\]

\[
= z \left[ \frac{1}{3} k^2 + \frac{k(m-k)}{\rho} + \frac{(m-k)^2}{2 \rho} \right]
\]

\[
= z \left[ \frac{1}{3} k^2 + \frac{1}{2} \frac{m^2 - k^2}{\rho} \right]
\]

\[
\leq z \frac{m^2 - k^2 + \frac{1}{6} k^2}{2 \rho} \leq z \frac{2m^2 - k^2}{\rho} = \mathcal{O}(z \frac{2m^2 - k^2}{\rho})
\]

where the first inequality results from \( \rho \leq 0.25 \).

\[\Box\]

### A.2. Proofs for the Optimal Algorithm

For our optimal algorithm proofs, we require an important lemma comparing biased uniform distributions and our surrogate potential function distributions.

**Lemma 7.** Let \( a,b \in [m] \) and \( f^{a,b}(-) \) be a pairwise function which satisfies the three properties stated in Section 3.2.2. Then for any set of relevant labels \( R \subset [m] \), we have

\[
\mathbb{E}_{e_l \sim u_\gamma^a}[f^{a,b}(s + e_l)] \leq \mathbb{E}_{e_l \sim u^a_R}[f^{a,b}(s + e_l)].
\]

where \( u_\gamma^a \) and \( u^a_R \) are the biased uniform distributions, placing \( \gamma \) more weight on members of \( R \) and the label \( a \) respectively.

**Proof.** Recall \( f^{a,b}(s) = \mathbb{I}(a \in R)\mathbb{I}(b \not\in R) f(s[a],s[b]) \). Hence, if \( a \not\in R \) or \( b \in R \), then \( f^{a,b} \) becomes a zero function, and the inequality trivially holds.

Suppose \( a \in R \) and \( b \not\in R \). By definition of \( u_\gamma^a \) and \( u^a_R \), we have

\[
u_\gamma^a[a] - \nu_\gamma^b[b] = \nu^a_R[a] - \nu^a_R[b] = \gamma,
\]

from which we can deduce

\[
u_\gamma^a[a] - \nu^a_R[a] = \nu_\gamma^b[b] - \nu^a_R[b] =: \Delta > 0 \quad \text{and} \quad \sum_{l \in [m] \setminus \{a,b\}} u_\gamma^a[l] - \nu^a_R[l] = -2\Delta.
\]

Furthermore, observe that if \( l \not\in \{a,b\} \), then \( f^{a,b}(s) = f^{a,b}(s + e_l) \). From this, we can infer

\[
\mathbb{E}_{e_l \sim u_\gamma^a}[f^{a,b}(s + e_l)] - \mathbb{E}_{e_l \sim u^a_R}[f^{a,b}(s + e_l)] = \sum_{l \in [m]} (u_\gamma^a[l] - \nu^a_R[l]) f^{a,b}(s + e_l)
\]

\[
= \Delta \cdot \left( f^{a,b}(s + e_a) + f^{a,b}(s + e_b) - 2f^{a,b}(s) \right).
\]

Using the uncrossability and convexity of \( f \), we can show

\[
f^{a,b}(s + e_a) + f^{a,b}(s + e_b) \geq 2f^{a,b}(s + \frac{1}{2}(e_a + e_b)) \geq 2f^{a,b}(s),
\]

which finishes the proof.

\[\Box\]
A.2.1 Proof of Proposition 1

**Proof.** We note that $\Phi$ is the sum of many pairwise potential functions, each of which uses a convex and proper pairwise function. By nature of potential functions, these smaller potentials must be proper and convex thus $\Phi$ must be as well.

To prove the upper bound, we first expand $\Upsilon$ and $\Phi$ as the sum over pairs of potential functions:

$$
\Upsilon_t^N(s) = \sum_{a \in R_t} \sum_{b \notin R_t} \varphi_{a,b}^N(s, t^a, b)
$$

$$
\Phi_t^N(s) = \sum_{a \in R_t} \sum_{b \in R_t} \Lambda_t^{a,b,N}(s).
$$

Lemma 7 implies

$$
\varphi_{a,b}^{1,\gamma}(s, t^a, b) \leq \Lambda_t^{a,b,1}(s).
$$

We also note that $\Lambda_t^{a,b,N}(s)$ is also proper, uncrossable, and convex in its scores, and thus

$$
\varphi_{a,b}^{2,\gamma}(s, t^a, b) \leq \Lambda_t^{a,b,2}(s).
$$

We can repeat this process recursively to obtain for any $N$

$$
\varphi_{a,b}^{N,\gamma}(s, t^a, b) \leq \Lambda_t^{a,b,N}(s)
$$

and finally sum across all pairs in $R_t \times R_t$ to obtain the desired inequality.

\[ \square \]

A.3. Proof of Theorem 2

**Proof.** We first show a recurrence relation involving $\Lambda$.

$$
\Lambda_t^{a,b,N-i}(s_t) = \varphi_{a,b}^{N-i}(s_t, t^a, b)
$$

$$
= \mathbb{E}_{e_i \sim u_{R_t}} \varphi_{a,b}^{N-i-1}(s_t^i + e_i, t^a, b)
$$

$$
= \mathbb{E}_{e_i \sim u_{R_t}} \Lambda_t^{a,b,N-i-1}(s_t^i + e_i),
$$

where the inequality holds from Lemma 7. Then, summing across all pairs of elements, we have that

$$
\Phi_t^{N-i}(s_t^i) = \sum_{a \in R_t} \sum_{b \notin R_t} \Lambda_t^{a,b,N-i}(s_t^i)
$$

$$
\geq \mathbb{E}_{e_i \sim u_{R_t}} \left[ \sum_{a \in R_t} \sum_{b \notin R_t} \Lambda_t^{a,b,N-i-1}(s_t^i + e_i) \right]
$$

$$
= c_t^i \cdot u_{R_t}^\gamma
$$

$$
= c_t^i \cdot (u_{R_t}^\gamma - h_t^i) + c_t^i \cdot h_t^i
$$

$$
\geq c_t^i \cdot (u_{R_t}^\gamma - h_t^i) + \Phi_t^{N-i-1}(s_t^{i+1})
$$

where the last inequality holds due to the convexity of $\Phi$ and Jensen’s inequality.

Then summing over $t$, we have that

$$
\sum_t \Phi_t^{N-i}(s_t) \geq \sum_t c_t^i \cdot (u_{R_t}^\gamma - h_t^i) + \Phi_t^{N-i-1}(s_t^{i+1}).
$$

and with probability at least $1 - \delta$, the Top-$\delta$WLC($\gamma, \delta, S$) holds, and we have that

$$
\sum_t \Phi_t^{N-i}(s_t) - \Phi_t^{N-i-1}(s_t^{i+1}) \geq -S.
$$

Then summing across all $N$, by the telescoping rule we have that

$$
\sum_t \Phi_t^{N}(0) + NS \geq \sum_t \Phi_t^{0}(s_t^N) = \sum_t L(s_t^N, R_t).
$$

Simplifying and noting that $S = \tilde{O}(\frac{2m^2 - k^2}{n} \cdot 2)$ proves the theorem.

\[ \square \]
A.4. Proof of Corollary 3

We first prove a lemma regarding potential functions and the hinge loss. It modifies Lemma 8 by Jung and Tewari (2018).

Lemma 8. With the unweighted hinge loss as the loss function, we have that

\[ \Phi_t^N(0) \leq (N + 1) \frac{m^2}{4} \exp(-\frac{\gamma^2 N}{2}). \]

Proof. For convenience, we drop the subscript \( t \) from this proof. Recalling that \( \Phi^N \) is the sum of many \( \Lambda^{a,b,N} \), where \( a \in R_i, b \notin R_i \), we bound each \( \Lambda \) on its own first. Let \( X^N \) be the result of the random walk from the potential function \( \Lambda \) is defined with, sampling \( N \) times from \( u_0^2 \) and adding the result to 0. Then we can rewrite \( \Lambda \) as

\[ \Lambda^{a,b,N}(0) = \mathbb{E}_{X^N} \left[ \max\{0, X^N[b] - X^N[a]\} \right] \]

\[ = \sum_{n=0}^N \Pr[X^N[b] - X^N[a] \geq n] \]

\[ \leq (N + 1) \Pr[X^N[b] - X^N[a] \geq 0]. \]

Next, we define the probabilities \( p := u_0^2[a], q := u_0^2[b] \), and we interpret \( \Pr[X^N[b] - X^N[a] \geq 0] \) using a game, where with probability \( p \) we draw \(-1\), while with probability \( q \) we draw \(1\). Then \( \Pr[X^N[b] - X^N[a] \geq 0] \) equals the probability that the summation of \( N \) i.i.d. random numbers are non-negative. Thus we can apply Hoeffding’s inequality to obtain

\[ \Pr[X^N[b] - X^N[a] \geq 0] \leq \exp(-\frac{\gamma^2 N}{2}) \]

and plugging back into Eq. 6 to obtain

\[ \Lambda^{a,b,N}(0) \leq (N + 1) \exp(-\frac{\gamma^2 N}{2}). \]

Lastly, we recall that \( \Phi_t^N \) sums over every pair of relevant and irrelevant label using \( \Lambda \). This requires us to note that the maximum possible number of such pairs is \( \frac{m^2}{4} \), when \( \frac{m}{2} \) of the labels are relevant. Multiplying \( \Lambda^{a,b,N}(0) \) by the maximum possible number of pairs provides our bound.

Now we are ready to prove Corollary 3.

Proof. Firstly, we note that if we decide to explore, then the number of additional pairs that could be dis-ordered is no greater than \( 2m \). To see this, we observe the worst case scenario for exploration, where the the top ranked label is relevant but is being swapped with the lowest ranked label, which is irrelevant. Immediately this provides our first newly disordered pair. Let \( M \) be the set of labels not ranked first or last, in other words, all the other labels. For every irrelevant label in \( M \), another pair associated with the formerly top-ranked and relevant label will become incorrect. Then, for every relevant label in \( M \), another pair associated with the formerly lowest-ranked and relevant label will become incorrect after swapping. Thus for every member of \( M \), an additional pair is disordered by the swap, which is upper bounded by \( m \). Thus, the total number of pairs disordered is \( |M| + 1 = m - 1 < m \). Multiplying by the number of swaps provides our upper bound. Noting that the worst case scenario upper bounds the expected loss from exploration, we have the inequality

\[ \sum_{t=1}^{T} \mathbb{E}_{\tilde{y}_t}[L^\text{rk}(\tilde{y}_t, R_t)] - L^\text{rk}(s_t^N, R_t) \leq 2pmT. \]

Then, we note that the random additional unweighted rank loss we obtain from exploration is bounded each round by \( m^2 \) because this upper bounds the unweighted rank loss. We apply concentration inequalities and rearrange, so that with probability at least \( 1 - \delta \), an upper bound on the deviation of our true exploration loss from the expected exploration loss is

\[ \sum_{t=1}^{T} L^\text{rk}(\tilde{y}_t, R_t) - \mathbb{E}_{\tilde{y}_t}[L^\text{rk}(\tilde{y}_t, R_t)] \leq m^2 \sqrt{T \log \frac{1}{\delta}}. \]
Combing these inequalities and plugging into Theorem 2, we have

$$\sum_t L_{\text{rk}}(\hat{y}_t, R_t) \leq \sum_t L_{\text{rk}}(s_t^N, R_t) + 2\rho mT + m^2 \sqrt{T \log \frac{1}{\delta}}$$

$$\leq \Phi_t^N(0)T + \tilde{O}\left(\frac{2m^2 - k^2}{\rho}N^2\right) + 2\rho mT + m^2 \sqrt{T \log \frac{1}{\delta}}$$

where in obtaining the expression inside $\tilde{O}$ we use that each pairwise function in the unweighed hinge loss is upper bounded by $N$.

Finally, using Lemma 8 we can produce our final bound.

$$\sum_t L_{\text{rk}}(\hat{y}_t, R_t) \leq \frac{m^2}{4} (N + 1) \exp(-\frac{\gamma^2 N}{2}T) + \rho dmT + \tilde{O}\left(\frac{2m^2 - k^2}{\rho}N^2 \sqrt{T}\right)$$

$$\leq \frac{m^2}{4} (N + 1) \exp(-\frac{\gamma^2 N}{2}T) + \rho dmT + \tilde{O}\left(\frac{2m^2 - k^2}{\rho}N^2 \sqrt{T}\right).$$

\[\square\]

A.5. Proof of Theorem 4

Proof. We start by defining the unbiased estimate and true unweighted rank loss suffered by the $i$th expert as

$$\hat{M}_i = \sum_t \hat{L}_{\text{rk}}(s_t^i, R_t)$$

$$M_i = \sum_t L_{\text{rk}}(s_t^i, R_t).$$

Recall that our unbiased estimator for unweighted rank loss is bounded by $\tilde{O}(\frac{2m^2 - k^2}{\rho})$. We let $\hat{M}_0 = \sum_t \hat{L}(0, R_t)$ and $M_0 = \sum_t L_{\text{rk}}(0, R_t)$. If we write $i^* = \arg\min_i M_i$, then we have by the concentration inequality, with probability at least $1 - \delta$ that

$$\min_i \hat{M}_i \leq \hat{M}_{i^*} \leq \min_i M_i + \tilde{O}\left(\frac{2m^2 - k^2}{\rho}T \log \frac{1}{\delta}\right) = \min_i M_i + \tilde{O}\left(\frac{2m^2 - k^2}{\rho}\sqrt{T}\right).$$

Because the booster chooses an expert through the Hedge algorithm and because we feed the hedge algorithm $\hat{M}_i$ each round, a standard analysis as in Corollary 2.3 of Cesa-Bianchi and Lugosi (2006) gives that

$$\sum_t L_{\text{rk}}(\hat{y}_t, R_t) \leq \sum_t \hat{L}_{\text{rk}}(\hat{y}_t, R_t) + \tilde{O}\left(\frac{2m^2 - k^2}{\rho}\sqrt{T}\right) \leq 2 \min_i M_i + 2 \log N + \tilde{O}\left(\frac{2m^2 - k^2}{\rho}\sqrt{T}\right).$$

where the inequality between $L_{\text{rk}}$ and $\hat{L}_{\text{rk}}$ is again using concentration inequalities, and with another probability at least $1 - \delta$.

Now we check that $\frac{1}{1 + \exp(a - b)} \leq \frac{1}{2} \mathbb{I}(a \leq b)$, so that

$$w^i[t] \geq \frac{1}{2} L_{\text{rk}}(s_{i-1}^i, R_t)$$

and $\|w^i\|_1 \geq \frac{M_{i-1}}{2}$

with $w^i[t]$ defined as in Eq. 5.

Now we let $\Delta_i$ denote the difference between cumulative logist loss between two experts

$$\Delta_i = \sum_t L_{\log}(s_t^i, R_t) - L_{\log}(s_{i-1}^i, R_t)$$

$$= \sum_t L_{\log}(s_{i-1}^i + \alpha_t h_t^i, R_t) - L_{\log}(s_{i-1}^i, R_t).$$

Then a standard analysis of stochastic gradient descent (Zinkevich, 2003) provides that, with probability at least $1 - \delta$

$$\Delta_i \leq \min_{\alpha \in [-2, 2]} \sum_t \left[ L_{\log}(s_{i-1}^i + \alpha h_t^i, R_t) - L_{\log}(s_{i-1}^i, R_t) \right] + \tilde{O}\left(\frac{2m^2 - k^2}{\rho}\sqrt{T}\right)$$

(9)
We now record a useful inequality
\[ \log(1 + e^{s+\alpha}) - \log(1 + e^s) = \log(1 + e^{\alpha} - 1) \]
Using this we can expand
\[
\sum_t L^\log(s_t^{i-1} + \alpha h_t^i, R_t) - L^\log(s_t^{i-1}, R_t) = \sum_t \sum_{a \in R_t, b \notin R_t} \log \left( 1 + \frac{\exp(s_t^{i-1}[b] - s_t^{i-1}[a] + \alpha(h_t^i[b] - h_t^i[a]))}{1 + \exp(s_t^{i-1}[b] - s_t^{i-1}[a])} \right)
\leq \sum_t \sum_{a \in R_t, b \notin R_t} \exp(\alpha(h_t^i[b] - h_t^i[a]) - 1) \frac{1 + \exp(s_t^{i-1}[b] - s_t^{i-1}[a])}{1 + \exp(s_t^{i-1}[b] - s_t^{i-1}[a])} := f(\alpha)
\]
We also rewrite \(\|w^i\|_1\) and \(\gamma_i\) as the following:
\[
\|w^i\|_1 = \sum_t \sum_{a \in R_t, b \notin R_t} \frac{1}{1 + \exp(s_t^{i-1}[b] - s_t^{i-1}[a])}
\gamma_i = \sum_t \sum_{a \in R_t, b \notin R_t} \frac{h_t^i[a] - h_t^i[b]}{\|w^i\|_1 1 + \exp(s_t^{i-1}[b] - s_t^{i-1}[a])}.
\]
For ease of notation, let \(j\) be an index which loops over all possible combinations in \([T] \times R_t \times R_t^c\), and let \(a_j\) and \(b_j\) be the following terms:
\[
a_j = \frac{1}{\|w^i\|_1 1 + \exp(s_t^{i-1}[b] - s_t^{i-1}[a])}
\]
\[
b_j = h_t^i[a] - h_t^i[b].
\]
Then from Eq. \[10\] we have that \(\sum_j a_j = 1\) and \(\sum_j a_j b_j = \gamma_i\). Then, we can express \(f(\alpha)\) using \(a_j\) and \(b_j\) as
\[
\frac{f(\alpha)}{\|w^i\|_1} = \sum_j a_j \exp(\alpha b_j - 1)
\leq \exp(-\alpha \sum_j a_j b_j) - 1
= \exp(-\alpha \gamma_i) - 1
\]
where the inequality holds by Jensen’s. From this, we can deduce that
\[
\min_{\alpha \in [-2, 2]} f(\alpha) \leq -\frac{\gamma_i}{2} \|w^i\|_1.
\]
Then combining equations \[8\, 9\] and \[11\] we have
\[
\Delta_i \leq -\frac{|\gamma_i|}{4} M_{i-1} + \tilde{O}(\frac{2m^2 - k^2}{\rho} \sqrt{T}).
\]
Summing over \(i\), we get by telescoping rule
\[
\sum_t L^\log(s_t^N, R_t) - L^\log(0, R_t) \leq \frac{1}{4} \sum_{i=1}^{N} |\gamma_i| M_{i-1} + \tilde{O}(\frac{(2m^2 - k^2)N}{\rho} \sqrt{T})
\leq \frac{1}{4} \sum_{i=1}^{N} |\gamma_i| \min_i M_i + \tilde{O}(\frac{2m^2 - k^2}{\rho} N \sqrt{T}).
\]
Then, solving for \(\min_i M_i\), we obtain
\[
\min_i M_i \leq \frac{4L^\log(0, R_t)}{\sum_k |\gamma_i|} + \tilde{O}(\frac{(2m^2 - k^2)N}{\rho \sum_k |\gamma_i|} \sqrt{T})
\leq \frac{m^2 \log 2}{\sum_k |\gamma_i|} T + \tilde{O}(\frac{(2m^2 - k^2)N}{\rho \sum_k |\gamma_i|} \sqrt{T}),
\]
where we use that in $L^\log(0, R_t)$, each pairwise function must evaluate to log 2, and there can be at most $\frac{m^2}{T}$ pairwise functions. Plugging this result into Eq. 7 we have

$$\sum_t L^{\text{lnk}}(\hat{y}_t, R_t) \leq \frac{2m^2 \log 2}{\sum_i |\gamma_i|} T + \hat{O}(\frac{(2m^2 - k^2)N}{\rho \sum_i |\gamma_i|} \sqrt{T}).$$

Finally, since we predict with $\tilde{y}_t$ instead of $\hat{y}_t$, we must add in the loss from exploration. We follow the same steps as in the proof of Corollary 3 to obtain

$$\sum_t L^{\text{lnk}}(\tilde{y}_t, R_t) \leq \frac{2m^2 \log 2}{\sum_i |\gamma_i|} T + \rho dm T + \hat{O}(\frac{(2m^2 - k^2)N}{\rho \sum_i |\gamma_i|} \sqrt{T}).$$

\[\square\]

A.6. Additional Proofs

**Lemma 9.** The logistic and hinge unweighed are pairwise decomposable into functions which are proper, convex, and uncrossable.

**Proof.** We first note that for any $a \in \mathbb{R}, b \notin \mathbb{R}$, we can write the logistic and hinge losses as

$$L^{\text{log}}(s, R) = \sum_{a \in \mathbb{R}} \sum_{b \notin \mathbb{R}} \log(1 + \exp(s[b] - s[a])), \text{ and}$$

$$L^{\text{hinge}}(s, R) = \sum_{a \in \mathbb{R}} \sum_{b \notin \mathbb{R}} \max\{0, 1 + s[b] - s[a]\}.$$ 

In both cases, adding a constant scalar to the scores for both $a$ and $b$ will cancel itself out, thus satisfying the uncrossability. Also, the fact that the functions inside the summations are proper and convex finishes the proof. \[\square\]

B. EXPERIMENT DETAILS

| Dataset   | # Weak Learners | $\rho$   | # Loops |
|-----------|----------------|----------|---------|
| Emotions  | 50/30          | 0.04/0.04| 20/10   |
| Scene     | 50/50          | 0.02/0.04| 10/10   |
| Yeast     | 30/60          | 0.04/0.04| 10/10   |
| Mediamill | 10/10          | 0.02/0.06| 20/20   |
| M-reduced | 20/60          | 0.04/0.06| 20/20   |

| Dataset   | # train | # test | dim | $m$ | min | mean | max |
|-----------|---------|--------|-----|-----|-----|------|-----|
| Emotions  | 391     | 202    | 72  | 6   | 1   | 1.87 | 3   |
| Scene     | 1211    | 1196   | 294 | 6   | 1   | 1.07 | 3   |
| Yeast     | 1500    | 917    | 103 | 14  | 1   | 4.24 | 11  |
| Mediamill | 30993   | 12914  | 120 | 101 | 0   | 4.38 | 18  |
| M-reduced | 1500    | 500    | 120 | 101 | 0   | 4.39 | 13  |

We set loops to be either 10 or 20, optimize $N$ with granularity down to multiples of 10, and the edge $\gamma$ for TopOpt from the set $\{0.1, 0.2, 0.3, 0.4\}$. In constructing our randomized exploration, we first swap the scores of labels in $s$, as described in Section 2.2.

In our experiments, one additional heuristic we added to the adaptive algorithm is to clip the gradient estimates that were of magnitude greater than 1.0, down to either 1.0 or $-1.0$ based on their sign. We found this improved convergence of SGD.