INVERSE PROBLEMS UNDER SARMANOV DEPENDENCE STRUCTURE

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Abstract. Consider a sequence \( \{(X_i, Y_i)\} \) of independent and identically distributed random vectors, with joint distribution bivariate Sarmanov. This is a natural set-up for discrete time financial risk models with insurance risks. Of particular interest are the infinite time ruin probabilities

\[
P[\sup_{n \geq 1} \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j > x].
\]

When the \( Y_i \)'s are assumed to have lighter tails than the \( X_i \)'s, we investigate sufficient conditions that ensure each \( X_i \) has a regularly varying tail, given that the ruin probability is regularly varying. This is an inverse problem to the more traditional analysis of the ruin probabilities based on the tails of the \( X_i \)'s. We impose moment-conditions as well as non-vanishing Mellin transform assumptions on the \( Y_i \)'s in order to achieve the desired results. But our analysis departs from the more conventional assumption of independence between the sequences \( \{X_i\} \) and \( \{Y_i\} \), instead assuming each \( (X_i, Y_i) \) to be jointly distributed as bivariate Sarmanov, a fairly broad class of bivariate distributions.

1. Introduction

In this article we consider a discrete-time risk model with insurance and financial risks. We refer the reader to Paulsen (2008) which describes the history and evolution of this model in detail. The survey discusses the pertinent integro-differential equations, asymptotics results and bounds on the ruin probability. It encompasses both continuous and discrete time risk model theories. The model we are concerned with constitutes the insurance risk \( X_n \) and the financial risk or the stochastic discount factor \( Y_n \) at time \( n \). The stochastic discount value of aggregate net losses up to time \( n \) in this set-up is given by

\[
S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j.
\]

In this set-up the finite time ruin probabilities are defined as

\[
\Psi(x, n) = P[\max_{1 \leq k \leq n} S_k > x],
\]

and the infinite time ruin probability is given by

\[
\Psi(x) = P[\sup_{n \geq 1} S_n > x].
\]

In general we derive the behaviour of \( \Psi(x, n) \) and \( \Psi(x) \) from the tail of \( X_1 \). However, the focus of this paper will be to study the inverse problem, i.e. the tail of \( X_1 \) given the behaviour of \( \Psi(x, n) \) and \( \Psi(x) \).

In the risk-model set-up described above, each \( X_i \) is generally assumed to follow a regularly varying distribution. Resnick (1987) has studied this class of distributions extensively, and its uses in applied probability are detailed in Bingham and Teugels (1987). Its applications in stochastic recurrence equations have been studied in Basrak et al. (2002) and Denisov and Zwart (2007) among others. Nyrhinen (2012) and Yang and Wang (2013) have investigated the myriad applications of regularly varying distributions in the above-mentioned risk-model problems. In particular, Yang and Wang (2013) considers a set up where the random vectors \( (X_n, Y_n) \) are i.i.d. jointly distributed as bivariate Sarmanov, as in Definition 1.3.
Recall that a function $f$ is said to be regularly varying at $\infty$ with index $\beta$ if $f(xy) \sim y^\beta f(x)$ as $x \to \infty$. Similarly, a random variable $X$ is said to have a regularly varying tail of index $-\alpha$, $\alpha > 0$, denoted by $X \in RV_{-\alpha}$, if its tail distribution $F$ satisfies, for all $y > 0$,

$$F(xy) \sim y^{-\alpha}F(x), \quad x \to \infty.$$ 

Here and henceforth, for two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ as $x \to \infty$ to mean that $\lim_{x \to \infty} a(x)/b(x) = 1$. When $X \in RV_{-\alpha}$ and $Y$ is independent of $X$, satisfying $E[Y^{n+\epsilon}] < \infty$ for some $\epsilon > 0$, Breiman (1965) shows that $XY$ will also be regularly varying with $P[XY > x] \sim E[Y^\alpha]P[X > x]$ as $x \to \infty$. The inverse problem corresponding to this same set-up has been studied by Jacobsen et al. (2009). When $XY$ is regularly varying, they propose sufficient conditions for $X$ to be regularly varying. Resnick and Willekens (1991) extended the product result of Breiman to finite and infinite sums. Hazra and Maulik (2012) explored the inverse problems corresponding to the finite and infinite sums under the Resnick-Willekens conditions. In Section 2 we are interested in the inverse problem for products, but with the classically studied independence structure between $X$ and $Y$ replaced by the bivariate Sarmanov distribution as in Definition 1.3. Yang and Wang obtained sufficient conditions for regular variation of $\Psi(x, n)$ and $\Psi(x)$ when each $X_n$ is regularly varying. Maulik and Podder extended their results for conditions similar to those proposed by Denisov and Zwart (2007). In Sections 3 and 4 we shall be interested in analyzing the inverse problem for sums – finite and infinite cases respectively, imposing Breiman-like moment conditions on the $Y_i$’s. In section 5 we show the necessity of non-vanishing Mellin transform of the appropriate measure. We are motivated by the example given in Jacobsen et al. (2009) which we adapt appropriately for our set-up. In our study, each of the i.i.d. random vectors $(X_i, Y_i)$ follows a bivariate Sarmanov distribution as in Definition 1.3. Given that $\Psi(x, n)$ or $\Psi(x) \in RV_{-\alpha}$, we shall investigate sufficient conditions to ensure that $X_n \in RV_{-\alpha}$ for each $n$, under the above mentioned independence assumptions. In this context, we also refer to Damek et al. (2014) for discussions on inverse problems for regular variations in multivariate cases and where regular variation is not restricted to one direction or quadrant. They discuss inverse problems for the convolution of two multivariate random measures, assuming independence between them. They also focus on the inverse problems for weighted sums of multivariate regularly varying measures, but with the weights being non-random matrices. Our work in this paper departs both from their independence assumption as well as deterministic weights.

We give a brief description of the results from Jacobsen et al. (2009), as our results are vitally based on these. Given a probability measure $\nu$ and a $\sigma$-finite measure $\rho$ on $(0, \infty)$, we define a new measure on $(0, \infty)$ by

$$(\nu \oplus \rho)(B) = \int_0^\infty \nu(x^{-1}B)\rho(dx), \quad B \text{ a Borel set on } (0, \infty).$$

Following Jacobsen et al., we call this the multiplicative convolution of $\nu$ and $\rho$, since in the case where $\nu$ and $\rho$ are probability measures, $\nu \oplus \rho$ gives the law of the product of two independent random variables with marginals $\nu$ and $\rho$.

We now provide a paraphrased version of Theorem 2.3 of Jacobsen et al. (2009) which inspires our main result in Section 2.

**Theorem 1.1.** Let $\rho$ be a non-zero $\sigma$-finite measure such that, for some $\alpha > 0$,

$$\int_0^\infty y^{\alpha-\delta} \lor y^{\alpha+\delta} \rho(dy) < \infty, \quad \text{for some } \delta > 0. \quad (1.4)$$

and the non-vanishing Mellin transform condition holds, i.e.,

$$\int_0^\infty y^{\alpha+i\theta} \rho(dy) \neq 0 \quad \text{for all } \theta \in \mathbb{R}. \quad (1.5)$$

If $\nu \oplus \rho \in RV_{-\alpha}$, then $\nu \in RV_{-\alpha}$ as well, with

$$\lim_{x \to \infty} \frac{(\nu \oplus \rho)(x, \infty)}{\nu(x, \infty)} = \int_0^\infty y^\alpha \rho(dy).$$
Remark 1.2. In their original result, Jacobsen et al. allowed \( \nu \) to be a \( \sigma \)-finite measure, for which they required an additional integrability assumption. It is not needed when \( \nu \) is a probability measure.

We relax the independence between \( \nu \) and \( \rho \) as considered by Jacobsen et al. in defining the product convolution \( \ast \). We extend our dependence structure to the much broader class of bivariate Sarmanov distributions, which is defined as follows.

Definition 1.3. The pair of random variables \((X, Y)\) is said to follow a bivariate Sarmanov distribution, if

\[
P(X \in dx, Y \in dy) = (1 + \theta \phi_1(x)\phi_2(y))F(dx)G(dy), \quad x \in \mathbb{R}, y \geq 0,
\]

where the kernels \( \phi_1 \) and \( \phi_2 \) are two real valued functions and the parameter \( \theta \) is a real constant satisfying

\[
E\{\phi_1(X)\} = E\{\phi_2(Y)\} = 0
\]

and

\[
1 + \theta \phi_1(x)\phi_2(y) \geq 0, \quad x \in D_X, y \in D_Y,
\]

where \( D_X \subset \mathbb{R} \) and \( D_Y \subset \mathbb{R}^+ \) are the supports of \( X \) and \( Y \), with marginals \( F \) and \( G \) respectively.

This class of bivariate distributions is quite wide, covering a large number of well-known copulas such as the Farlie-Gumbel-Morgenstern (FGM) copula, which is recovered by taking \( \phi_1(x) = 1 - 2F(x) \) and \( \phi_2(y) = 1 - 2G(y) \). We refer the reader to [Lee (1996)] for further discussion. A bivariate Sarmanov distribution is called proper if \( \theta \neq 0 \) and none of \( \phi_1 \) and \( \phi_2 \) vanishes identically.

As has been discussed above, Yang and Wang studied this class of distributions. They additionally assumed

\[
\lim_{x \to \infty} \phi_1(x) = d_1. \tag{1.6}
\]

Yang and Wang made the crucial observation that the bivariate Sarmanov dependence is not very far from independence. If \((X, Y)\) is bivariate Sarmanov, then asymptotically, the product \( XY \) has the same tail distribution as the product \( X Y^*_\theta \) where \( X \) and \( Y^*_\theta \) are independent and \( Y^*_\theta \) is obtained through a change of measure performed on the distribution of \( Y \). It has the distribution function \( G^*_\theta \) with

\[
G^*_\theta(dy) = P[Y^*_\theta \in dy] = (1 + \theta d_1 \phi_2(y))G(dy). \tag{1.7}
\]

To state the result formally, we first need to define the class of dominatedly-tail-varying distributions. A random variable \( X \) with distribution function \( F \) is called dominatedly-tail-varying, denoted by \( X \in \mathcal{D} \) or \( F \in \mathcal{D} \), if for all \( 0 < y < 1 \),

\[
\limsup_{x \to \infty} \frac{\mathcal{F}(xy)}{F(x)} < \infty. \tag{1.8}
\]

Lemma 3.1 of [Yang and Wang (2013)] shows the weak dependence of the bivariate Sarmanov distribution, but we shall need a less generalized version of it, stated as follows.

Theorem 1.4. Assume that \((X, Y)\) follows a bivariate Sarmanov distribution and \( \|1.6\) holds. Let \( X^* \) and \( Y^* \) be two independent random variables identically distributed as \( X \) and \( Y \) respectively, i.e. having marginals \( F \) and \( G \) respectively. Let \( \mathcal{F}(x) = P[X^*Y^* > x] \). If now \( H^* \in \mathcal{D} \) and \( G(x) = o(\mathcal{F}(x)) \), then

\[
P[XY > x] \sim P[X^*Y^*_\theta > x], \tag{1.9}
\]

where \( X^*, Y^*_\theta \) mutually independent and \( Y^*_\theta \sim G^*_\theta \) as defined in \( \|1.7\).

Yang and Wang also noted the following property of bivariate Sarmanov which will also be important for establishing our results.

Lemma 1.5. Assume that \((X, Y)\) follows a proper bivariate Sarmanov distribution. Then there exists two positive constants \( b_1 \) and \( b_2 \) such that \( |\phi_1(x)| \leq b_1 \) for all \( x \in D_X \) and \( |\phi_2(y)| \leq b_2 \) for all \( y \in D_Y \).
2. Inverse problem for product

We now state our result concerning the tail of one of the multiplicands from the regularly varying tail of the product of two random variables.

**Theorem 2.1.** Suppose the pair of random variables \((X, Y)\) jointly follow bivariate Sarmanov distribution, as defined in Definition 1.3, with \(\lim_{x \to \infty} \phi_1(x) = d_1\). We also assume that \(F \in D\) and \(\mathcal{G}(x) = o(F(x))\). Suppose \(XY \in RV_{-\alpha}\) for some \(\alpha > 0\). If now we have \(E[Y^{\alpha + \epsilon}] < \infty\) for some \(\epsilon > 0\) and for all \(\beta \in \mathbb{R}\),

\[
E[Y^{\alpha + \beta}] + \theta d_1 E[\phi_2(Y)Y^{\alpha + \beta}] \neq 0,
\]

then \(X \in RV_{-\alpha}\) and \(P[XY > x] \sim \{E[Y^{\alpha}] + \theta d_1 E[\phi_2(Y)Y^{\alpha}]\} P[X > x]\).

We shall assume without loss of generality that \(\epsilon \in (0, \alpha)\).

**Proof of Theorem 2.1.** Let \(X^*, Y^*\) be mutually independent copies of \(X\) and \(Y\), with marginals \(F\) and \(G\) respectively. We define \(H^*\) by \(H^*(x) = P[X^*Y^* > x]\). Since \(F \in D\), from Theorem 3.3 of Cline and Samorodnitsky (1994), we conclude that \(H^* \in D\) as well.

Choosing a suitable \(a\) such that \(\mathcal{G}(a) > 0\), we have

\[
H^*(x) \geq \frac{F(x)}{a} \mathcal{G}(a).
\]

Therefore, using \(F \in D\) and \(\mathcal{G}(x) = o(F(x))\), we have

\[
\limsup_{x \to \infty} \frac{\mathcal{G}(x)}{H^*(x)} \leq \limsup_{x \to \infty} \frac{\mathcal{G}(x)}{F(x)} \limsup_{x \to \infty} \frac{F(x)}{x} \limsup_{x \to \infty} \frac{F(x/a)}{H^*(x)} = 0.
\]

Thus we have established that \(\mathcal{G}(x) = o(H^*(x))\). Recall the twisted version \(Y^*_\theta\) of \(Y\) defined in (1.7). Then by Theorem 1.4 we know that

\[
P[XY > x] \sim \{E[Y^{\alpha}] + \theta d_1 E[\phi_2(Y)Y^{\alpha}]\} P[X > x] \implies X^*Y^*_\theta \in RV_{-\alpha}.
\]

Using Lemma 1.6 we get

\[
E[Y^*_{\theta}^{\alpha + \epsilon}] \leq (1 + |\theta d_1| b_2) E[Y^{\alpha + \epsilon}] < \infty.
\]

As defined in (1.7), if \(G_\theta\) denotes the marginal of \(Y^*_\theta\), then

\[
\int_0^\infty y^{\alpha - \epsilon} \vee y^{\alpha + \epsilon} G_\theta(dy) = \int_0^1 y^{\alpha - \epsilon} G_\theta(dy) + \int_1^\infty y^{\alpha + \epsilon} G_\theta(dy) \leq 1 + E[Y^*_{\theta}^{\alpha + \epsilon}] < \infty.
\]

By (2.1), for all \(\alpha \in \mathbb{R}\), we have \(\int_0^\infty y^{\alpha + \epsilon} G_\theta(dy) \neq 0\). We are now able to conclude, from Theorem 1.1, that \(X^*\) and hence \(X\) is in \(RV_{-\alpha}\). The final result follows using Breiman (1965)’s theorem.

\[
\square
\]

3. Inverse problem for finite sum

We start with the same set-up as described in Yang and Wang (2013). Let \(\{(X_i, Y_i)\}\) be a sequence of i.i.d. random vectors, with the generic vector \((X, Y)\) jointly having bivariate Sarmanov distribution, as in Definition 1.3. Recall that \(\Psi(x, n)\) is the finite time ruin probability defined as \(\Psi(x, n) = P\left[\max_{1 \leq k \leq n} S_k > x\right]\) where \(S_n\) is as in (1.4). We provide sufficient conditions under which \(\Psi(x, n) \in RV_{-\alpha}\) implies \(X \in RV_{-\alpha}\). To this end, we state and prove the following important lemma.

**Lemma 3.1.** Let \(\{(X_i, Y_i)\}\) be a sequence of i.i.d. random vectors, with the generic vector \((X, Y)\) jointly bivariate Sarmanov distribution, as in Definition 1.3. We assume that \(F \in D\) and \(\mathcal{G}(x) = o(F(x))\). Denoting \(P[XY > x]\) by \(\mathcal{H}(x)\), we assume that for every \(v > 0\), the quantity

\[
\mathcal{H}(v) = \sup_{x > 0} \frac{\mathcal{H}(x/v)}{\mathcal{H}(x)}
\]

(3.1)
satisfies the condition
\[ \int_0^\infty \bar{H}(v)G(dv) < \infty. \]

Then we can conclude that
\[ P[X_1Y_1 > x, X_2Y_2 > x] = o(P[X_1 > x]). \quad (3.3) \]

Proof. Conditioning on \( Y_1 \), and noting that \((X_2, Y_2)\) independent of \((X_1, Y_1)\), we get
\[ \frac{P[X_1Y_1 > x, X_2Y_2Y_1 > x]}{P[X_1Y_1 > x]} = \frac{\int_0^\infty P[X_1 > x/v|Y_1 = v] \bar{H}(x/v)G(dv)}{\bar{H}(x)}. \quad (3.4) \]

Using the bivariate dependence structure between \( X_1 \) and \( Y_1 \), and Lemma 3.2, we have
\[ P[X_1 > x/v|Y_1 = v] \leq \int_0^\infty (1 + |\theta||\phi_1(u)||\phi_2(v)|)F(du) \leq (1 + |\theta|b_1b_2)\bar{F}(x/v). \quad (3.5) \]

Observe that \( \bar{H}(v) - (\bar{F}(x/v)\bar{H}(x/v))/\bar{H}(x) \geq 0 \), where \( \bar{H} \) is as in (3.1). Applying Fatou’s lemma, we now get
\[ \int_0^\infty \bar{H}(v)G(dv) - \limsup_{x \to \infty} \int_0^\infty \frac{\bar{F}(x/v)\bar{H}(x/v)}{\bar{H}(x)}G(dv) \geq \int_0^\infty \left[ \bar{H}(v) - \limsup_{x \to \infty} \frac{\bar{F}(x/v)\bar{H}(x/v)}{\bar{H}(x)} \right]G(dv) \]
\[ = \int_0^\infty \bar{H}(v)G(dv), \quad (3.6) \]

where the last equality follows from the assumption of (3.2). From (3.6) we conclude that
\[ \limsup_{x \to \infty} \int_0^\infty \frac{\bar{F}(x/v)\bar{H}(x/v)}{\bar{H}(x)}G(dv) = 0. \quad (3.7) \]

Combining (3.4), (3.5) and (3.7), we get the desired result. \( \square \)

The next lemma is stated under the same conditions as Lemma 3.1. It additionally assumes Breiman-like moment condition on the random variables \( Y_i \), and that the tail of the product \( X_1Y_1 \) is bounded above by a regularly varying function of index \( -\alpha \). Under these additional assumptions, the next lemma shows negligibility of joint tails of higher products with respect to the general dominator, which is a regularly varying function.

**Lemma 3.2.** Let \( \{(X_i, Y_i)\} \) be a sequence of i.i.d. random vectors, with the generic vector \( (X, Y) \) jointly bivariate Sarmanov distribution, as in Definition 1.3 and satisfying \( \lim_{x \to \infty} \phi_1(x) = d_1 \). We assume that \( F \in \mathcal{D} \) and \( \bar{G}(x) = o(\bar{F}(x)) \). Denoting \( P[XY > x] \) by \( \bar{H}(x) \), and \( \bar{H} \) as in (3.1), we assume that the condition (3.2) holds. Suppose \( X \) is nonnegative and
\[ P[XY > x] \leq W(x), \quad \text{for all } x > 0, \quad (3.8) \]

where \( W \) is a bounded regularly varying function with index \( -\alpha \). Finally, we assume that \( E[Y^{\alpha+\epsilon}] < \infty \) for some \( \epsilon > 0 \). Then, for \( 1 \leq s < t \leq n \),
\[ P \left[ X_s \prod_{i=1}^s Y_i > x, X_t \prod_{i=1}^t Y_i > x \right] = o(W(x)), \quad \text{as } x \to \infty. \quad (3.9) \]

Proof. Let \( Z_s = X_sY_s, \ Z_t = X_tY_tY_s, \ \Theta_s = \prod_{j=1}^{s-1} Y_j \) and \( \Theta_t = \prod_{1 \leq j \leq t-1, j \neq s} Y_j \). Observe that \( (Z_s, Z_t) \) is independent of \( (\Theta_s, \Theta_t) \). Let \( G_{s,t} \) be the joint distribution of \( (\Theta_s, \Theta_t) \), and \( G_s, G_t \) the respective marginals.

We condition on \( (\Theta_s, \Theta_t) \) to get
\[ P \left[ X_s \prod_{i=1}^s Y_i > x, X_t \prod_{i=1}^t Y_i > x \right] \]
\[ = \int_0^\infty \int_0^\infty P[Z_s > x/u, Z_t > x/v] G_{s,t}(du, dv) \]
\[ \leq \int_{u > v} P[Z_s > x/u, Z_t > x/u] G_{s,t}(du, dv) \]
Theorem 3.3. Let \( x \) which goes to zero as \( x \to 0 \).

From Lemma 4.3 of Hazra and Maulik (2012), we know that, for any \( 1 < \alpha < 2 \), we have

\[
\lim_{x \to \infty} \frac{W(x/u)}{W(x)} = \begin{cases} 
M_{u^{-\alpha}} & \text{if } u < 1 \\
M_{u^{1+\alpha}} & \text{if } u \geq x/x_0.
\end{cases}
\]

(3.11)

For \( u \leq x/x_0 \), we bound the integrand of (3.10) by \( M(1 + u^{1+\alpha}) \).

Using Lemma 3.1 and applying dominated convergence theorem, we get

\[
\lim_{x \to \infty} \int_{0}^{x/x_0} \frac{P[X_1Y_1 > x/u, X_2Y_2 > x/u]}{P[X_1Y_1 > x/u]} W(x/u) \frac{W(x/u)}{W(x)} (G_s + G_t)(du) = 0.
\]

For \( u > x/x_0 \), the integral of (3.10) is bounded above by

\[
\sup_{y > 0} W(y) \cdot \left( P[\Theta > x/x_0] + P[\Theta > x/x_0] \right) \leq \sup_{y > 0} W(y) \cdot \alpha^+ \cdot \epsilon x^+ \cdot W(x)
\]

which goes to zero as \( x \to \infty \).

With the aid of Lemmas 3.1 and 3.2, we are now able to state and prove the main result of this section.

Theorem 3.3. Let \( \{(X_i, Y_i)\} \) be a sequence of i.i.d. random vectors, with the generic vector \( (X, Y) \) jointly bivariate Sarmanov distribution, as in Definition 1.1, and satisfying \( \lim_{x \to \infty} \phi_1(x) = d_1 \). We assume that \( F \in \mathcal{D} \) and \( G(x) = o(F(x)) \). Denoting \( P[X > x] \) by \( H(x) \), and \( \bar{F} \) as in (1.1), we assume that the condition of (3.2) holds. Suppose \( X \) is nonnegative and \( S_n \in RV_{-\alpha} \), where \( S_n \) is as defined in (1.1). We assume that \( E[Y^{1+\beta}] < \infty \) for some \( \epsilon > 0 \). For all \( \beta \in \mathbb{R} \),

\[
E[Y^{1+\beta}] + \theta d_1 E[\phi_2(Y)Y^{1+\beta}] \neq 0,
\]

(3.12)

and

\[
\sum_{k=0}^{n-1} \{E[Y^{1+\beta}]\}^k \neq 0.
\]

(3.13)

Then each \( X_i \in RV_{-\alpha} \) and

\[
P[S_n > x] \sim \frac{(1 - E[Y^{1+\beta}]^n)(E[Y^{1+\beta}] + \theta d_1 E[\phi_2(Y)Y^{1+\beta}])}{(1 - E[Y^{1+\beta}])} \bar{F}(x), \quad \text{as } x \to \infty.
\]

(3.14)

Proof. From Lemma 4.3 of Hazra and Maulik (2012), we know that, for any \( 1/2 < \delta < 1 \),

\[
P[S_n > x] \geq \sum_{i=1}^{n} P[X_i \prod_{j=1}^{i} Y_j > x] - \sum_{1 \leq s \neq t \leq n} P[X_s \prod_{j=1}^{s} Y_j > x, X_t \prod_{j=1}^{t} Y_j > x],
\]

(3.15)

and

\[
P[S_n > x] \leq \sum_{i=1}^{n} P[X_i \prod_{j=1}^{i} Y_j > \delta x] + \sum_{1 \leq s \neq t \leq n} P[X_s \prod_{j=1}^{s} Y_j > \frac{1-\delta}{n-1} x, X_t \prod_{j=1}^{t} Y_j > \frac{1-\delta}{n-1} x].
\]

(3.16)
From (3.15) and Lemma 3.2, we have
\[
\limsup_{x \to \infty} \frac{\sum_{i=1}^{n} P\left[ X_{i} \prod_{j=1}^{i} Y_{j} > x \right]}{P[S_n > x]} \leq 1. \tag{3.17}
\]
From (3.16) we have
\[
\liminf_{x \to \infty} \frac{\sum_{i=1}^{n} P\left[ X_{i} \prod_{j=1}^{i} Y_{j} > x \right]}{P[S_n > x]} \geq \liminf_{x \to \infty} \frac{P[S_n > x]}{P[S_n > \delta x]}
- \sum_{1 \leq s \neq t \leq n} \limsup_{x \to \infty} \frac{P\left[ X_{s} \prod_{j=1}^{s} Y_{j} > \frac{1}{n+\epsilon} x, X_{t} \prod_{j=1}^{t} Y_{j} > \frac{1}{n+\epsilon} x \right]}{P[S_n > \delta x]} \tag{3.18}
\]
and the right side of (3.18) equals \(\delta^\alpha\), using the regular variation of \(S_n\) and Lemma 3.2. Hence letting \(\delta \to 1\), we have
\[
P[S_n > x] \sim \sum_{k=1}^{n} P[X_k \prod_{j=1}^{k} Y_{j} > x] \quad \text{as} \quad x \to \infty. \tag{3.19}
\]
Let \(\nu\) be the law induced by each \(X_i Y_i\), and \(\rho\) be the law given by
\[
\rho(B) = \sum_{k=1}^{n} P\left[ \prod_{j=1}^{k-1} Y_{j} \in B \right], \quad B \text{ a Borel set in } (0, \infty),
\]
where the empty product is defined as 1. Since \(E[Y^{\alpha+\epsilon}] < \infty\), \(\rho\) is a finite measure satisfying the moment condition (1.4). Due to (3.13), it also satisfies the non-vanishing Mellin transform condition (1.5). The multiplicative convolution \(\nu \ast \rho\) is the distribution on the right side of (3.19), hence in \(RV_{-\alpha}\). From Theorem 1.1 we conclude that \(\nu\) and hence \(X_1 Y_1 \in RV_{-\alpha}\). Finally we invoke Theorem 2.1 to conclude that \(X_1 \in RV_{-\alpha}\).

\[\square\]

4. Inverse problem for infinite sum

We start with the same set-up as described in Section 3. We are now interested in sufficient conditions that ensure \(X \in RV_{-\alpha}\) given that the infinite time ruin probability \(\Psi(x) = P\left[ \sup_{n \geq 1} S_n > x \right] \in RV_{-\alpha}\), where \(S_n\) as in (1.4). We shall state and prove two lemmas before the main result of this section. In this section, we shall additionally assume that \(E[Y^{\alpha+\epsilon}] < 1\) for some \(\epsilon \in (0, \alpha)\). This is required for finiteness of the geometric sum of the expectations. The next lemma shows the negligibility of the tail of the tail sums as well as the tail sum of the tails with respect to the dominator \(W\).

**Lemma 4.1.** We start with the same set-up as in Lemma 3.2, but additionally assume that \(E[Y^{\alpha+\epsilon}] < 1\) for some \(\epsilon \in (0, \alpha)\). Recall that \(W \in RV_{-\alpha}\) is a bounded function with \(P[XY > x] \leq W(x)\) for all \(x > 0\). Then
\[
\lim_{m \to \infty} \limsup_{x \to \infty} \frac{P\left[ \sum_{t=m+1}^{\infty} X_t \prod_{j=1}^{t} Y_{j} > x \right]}{W(x)} = 0, \tag{4.1}
\]
and
\[
\lim_{m \to \infty} \limsup_{x \to \infty} \frac{\sum_{t=m+1}^{\infty} P\left[ X_t \prod_{j=1}^{t} Y_{j} > x \right]}{W(x)} = 0. \tag{4.2}
\]
Proof. Using the notion of one large jump, we split the numerator on the left side of (4.1) and bound it as follows.

\[
P \left[ \sum_{t=m+1}^{\infty} X_t \prod_{j=1}^{t} Y_j > x \right] \leq \sum_{t=m+1}^{\infty} P \left[ X_t \prod_{j=1}^{t} Y_j > x \right] + P \left[ \sum_{t=m+1}^{\infty} X_t \prod_{j=1}^{t} Y_j 1_{[X_t \prod_{j=1}^{t} Y_j \leq x]} > x \right].
\] (4.3)

If \( G_t \) denotes the distribution of \( \Theta_t = \prod_{j=1}^{t-1} Y_j \) then we have

\[
\sum_{t=m+1}^{\infty} \int_{0}^{\infty} W(x/u)G_t(du).
\] (4.4)

For any \( \gamma > \alpha \), from Karamata’s theorem, we can find \( M(\gamma) > 0 \) such that

\[
E \left[ \{X_tY_t\}^\gamma 1_{[X_tY_t \leq z]} \right] = \gamma \int_{0}^{x} u^{\gamma-1} P[X_tY_t > u]du \leq M(\gamma)W(x)x^\gamma.
\] (4.5)

We now bound the second term on the right side of (4.3) separately for \( \alpha < 1 \) and \( \alpha \geq 1 \). For \( \alpha < 1 \), using Markov’s inequality, and (4.3) for \( \gamma = 1 \), we have

\[
P \left[ \sum_{t=m+1}^{\infty} X_t \prod_{j=1}^{t} Y_j 1_{[X_t \prod_{j=1}^{t} Y_j \leq x]} > x \right] \leq \sum_{t=m+1}^{\infty} \int_{0}^{\infty} (x/v)^{-1}E \left[ X_tY_t 1_{[X_tY_t \leq x/v]} \right] G_t(du)
\]

\[
\leq M(1) \sum_{t=m+1}^{\infty} \int_{0}^{\infty} W(x/v)G_t(du).
\] (4.6)

For \( \alpha \geq 1 \), we use Markov’s inequality, Minkowski’s inequality, and (4.3) for \( \gamma = \alpha + \epsilon \) to get the bound

\[
P \left[ \sum_{t=m+1}^{\infty} X_t \prod_{j=1}^{t} Y_j 1_{[X_t \prod_{j=1}^{t} Y_j \leq x]} > x \right] \leq M(\alpha + \epsilon) \left\{ \sum_{t=m+1}^{\infty} \left( \int_{0}^{\infty} W(x/v)G_t(du) \right)^{\alpha + \epsilon} \right\} \rightarrow \infty.
\] (4.7)

From (4.4), (4.6) and (4.7), it suffices to show that

\[
\lim_{m \to \infty} \limsup_{x \to \infty} \sum_{t=m+1}^{\infty} \int_{0}^{\infty} W(x/v)W(x)G_t(du) = 0, \text{ when } \alpha < 1,
\] (4.8)

\[
\lim_{m \to \infty} \limsup_{x \to \infty} \sum_{t=m+1}^{\infty} \left( \int_{0}^{\infty} W(x/v)W(x)G_t(du) \right) \frac{x^{\alpha + \epsilon}}{W(x)} = 0, \text{ when } \alpha \geq 1.
\] (4.9)

We split the integral in (4.8) over three intervals: \((0, 1), (1, x/x_0)\) and \((x/x_0, \infty)\), where \( x_0 \) is as in (3.11). Then the integral over \((0, 1)\) is bounded by \( ME \left[ \Theta_t^{\alpha - \epsilon} \right] \), which is further bounded by \( M \left\{ E \left[ Y^{\alpha + \epsilon} \right] \right\} \frac{t-1}{\alpha + \epsilon} \), using Potter’s bounds (as in (3.11)) and Jensen’s inequality. The integral over \((1, x/x_0)\) is bounded by \( M \left\{ E \left[ Y^{\alpha + \epsilon} \right] \right\} t-1 \) again by Potter’s bounds.

Because \( W \) is bounded, the integral over \((x/x_0, \infty)\) is bounded as follows:

\[
\int_{x/x_0}^{\infty} \frac{W(x/v)}{W(x)}G_t(du) \leq \sup_{y > 0} W(y) \cdot \frac{P[\Theta_t > x/x_0]}{W(x)} \leq \sup_{y > 0} W(y) \cdot \frac{x_0^{\alpha + \epsilon} \left\{ E \left[ Y^{\alpha + \epsilon} \right] \right\} t-1}{x^{\alpha + \epsilon} W(x)}.
\]

Thus the final bound becomes, for a suitably large \( M_0 \),

\[
\int_{0}^{\infty} \frac{W(x/v)}{W(x)}G_t(du) \leq M_0 \left\{ E \left[ Y^{\alpha + \epsilon} \right] \right\} \frac{(\alpha + \epsilon)(t-1)}{(\alpha + \epsilon)^{t-1}} + \frac{E \left[ Y^{\alpha + \epsilon} \right]}{x^{\alpha + \epsilon}} \int_{0}^{\infty} W(x/v)G_t(du).
\] (4.10)

For \( \alpha \geq 1 \), from (4.10), we get the bound

\[
\left( \int_{0}^{\infty} \frac{W(x/v)}{W(x)}G_t(du) \right) \frac{x^{\alpha + \epsilon}}{W(x)} \leq M_0 \left\{ E \left[ Y^{\alpha + \epsilon} \right] \right\} \frac{(\alpha + \epsilon)(t-1)}{(\alpha + \epsilon)^{t-1}} + \frac{E \left[ Y^{\alpha + \epsilon} \right]}{x^{\alpha + \epsilon}} \frac{t-1}{xW(x^{\alpha + \epsilon})}.
\] (4.11)
Because $W$ is bounded, the denominator $xW(x)^{1/\alpha} \to \infty$, and using the fact that $E[Y^{\alpha+\varepsilon}] < 1$, we get the final desired results of (4.1) and (4.2).

We shall need one final lemma in order to show that, under the set-up described in Lemma 4.1, but now with $S = \sum_{i=1}^{\infty} X_i \prod_{j=1}^{t} Y_j \in RV_{-\alpha}$, the tail of $S$ is going to be asymptotically like the sum of the tails of the individual summands in $S$.

**Lemma 4.2.** Consider the exact same set-up as in Lemma 4.1, but now consider $W(x) = P[S > x]$ where $S = \sum_{i=1}^{\infty} X_i \prod_{j=1}^{t} Y_j \in RV_{-\alpha}$. Then

$$P[S > x] \sim \sum_{i=1}^{\infty} P\left[ X_i \prod_{j=1}^{t} Y_j > x \right], \quad \text{as } x \to \infty. \quad (4.12)$$

**Proof.** From Lemma 4.1 for all $n \in \mathbb{N}$, we get

$$\lim_{n \to \infty} \limsup_{x \to \infty} \frac{P\left[ \sum_{i=n+1}^{\infty} X_i \prod_{j=1}^{t} Y_j > x \right]}{P[S > x]} = 0, \quad (4.13)$$

$$\lim_{n \to \infty} \limsup_{x \to \infty} \sum_{i=n+1}^{\infty} \frac{P\left[ X_i \prod_{j=1}^{t} Y_j > x \right]}{P[S > x]} = 0, \quad (4.14)$$

and from Lemma 3.2 for all $s \neq t$, we have

$$\lim_{x \to 0} \frac{P\left[ X_s \prod_{j=1}^{t} Y_j > x, X_t \prod_{j=1}^{t} Y_j > x \right]}{P[S > x]} = 0. \quad (4.15)$$

Recall $S_n$ as defined in (1.1). For any $\delta > 0$,

$$P[S > (1 + \delta)x] \leq P[S_n > x] + P\left[ \sum_{t=n+1}^{\infty} X_t \prod_{j=1}^{t} Y_j > \delta x \right]. \quad (4.16)$$

From (4.13) and because $S \in RV_{-\alpha}$, we get

$$\lim_{n \to \infty} \liminf_{x \to \infty} \frac{P[S_n > x]}{P[S > x]} \geq \lim_{x \to \infty} \liminf_{n \to \infty} \frac{P[S > (1 + \delta)x]}{P[S > x]} - \lim_{n \to \infty} \limsup_{x \to \infty} \frac{P\left[ \sum_{t=n+1}^{\infty} X_t \prod_{j=1}^{t} Y_j > \delta x \right]}{P[S > x]} = (1 + \delta)^{-\alpha},$$

so that by letting $\delta \to 0$, we get the lower bound

$$\lim_{n \to \infty} \liminf_{x \to \infty} \frac{P[S_n > x]}{P[S > x]} \geq 1. \quad (4.17)$$

But we trivially also have $P[S_n > x] \leq P[S > x]$, hence we conclude

$$\lim_{n \to \infty} \limsup_{x \to \infty} \frac{P[S_n > x]}{P[S > x]} \leq 1. \quad (4.18)$$

We invoke the inequalities from Lemma 4.3 of Hazra and Maulik (2012), as in Lemma 3.3 and consider (4.15) and (4.16). From (4.13), (4.14) and (4.16), we have

$$\lim_{n \to \infty} \limsup_{x \to \infty} \frac{\sum_{t=1}^{n} P\left[ X_t \prod_{j=1}^{t} Y_j > x \right]}{P[S > x]} \leq 1.$$

Finally, from (4.16), (4.17) and again (4.15), and using the regular variation of $S$, we get

$$\lim_{n \to \infty} \liminf_{x \to \infty} \frac{\sum_{t=1}^{n} P\left[ X_t \prod_{j=1}^{t} Y_j > x \right]}{P[S > x]} \geq 1.$$

Combining these with (4.14) we get the final result. \qed
We finally come to the main result of this section, which infers about the tail behavior of each $X_i$ from the regularly varying tail of $S = \sum_{i=1}^{\infty} X_i \prod_{j=1}^{i} Y_j$.

**Theorem 4.3.** Consider the exact same set-up as in Lemma 4.2. Additionally, we assume that for all $\beta \in \mathbb{R}$, \((3.12)\) holds and
\[
\sum_{k=0}^{\infty} \{ E[Y^{\alpha+i\beta}] \}^k \neq 0. \tag{4.19}
\]
Then we conclude that each $X_i \in RV_{-\alpha}$ and
\[
P[S > x] \sim \frac{E[Y^\alpha] + \theta d_1 E[\phi_2(Y) Y^\alpha]}{1 - E[Y^\alpha]} F(x) \quad \text{as } x \to \infty.
\tag{4.20}
\]

**Proof.** The proof is similar to that of Theorem 3.3 with $\rho$ defined as
\[
\rho(B) = \sum_{k=1}^{\infty} P \left( \prod_{j=1}^{k-1} Y_j \in B \right), \quad \text{B a Borel set in } (0, \infty). \]

\[\square\]

5. **Necessity of the non-vanishing Mellin transform condition**

Each of Theorems 3.3, 4.3 and 4.4 has non-vanishing Mellin transform condition(s) imposed on the sequence $\{Y_i\}$ of random variables, similar to the condition \((1.5)\) in Theorem 1.1. In this section we shall show that such a condition cannot be relaxed for proving our results. This is similar to the assertion made by Jacobsen et al. (2009) in Theorem 2.3. They show that if \((1.5)\) does not hold for some $\beta$, then the $\sigma$-finite measure $\nu$ without a regularly varying tail can be found such that $\nu \oplus \rho$ is regularly varying. The construction of the counterexample in Theorem 5.2 is inspired by Jacobsen et al.

To this end, recall the class of dominatedly tail varying distributions given in \((1.8)\). We start with the following useful remark.

**Remark 5.1.** From Foss et al. (2013), we know that, if $F$ and $G$ are distribution functions with $G \in RV_{-\alpha}$ and for some constants $0 < c_1 < c_2 < \infty$,
\[
c_1 G(x) \leq F(x) \leq c_2 G(x)
\]
then $F \in \mathcal{D}$.

**Theorem 5.2.** Let $G$ be a distribution function on $(0, \infty)$. We are given two bounded functions $\phi_1$ and $\phi_2$ on $(0, \infty)$, and $\theta \in \mathbb{R}$ such that:

(i) $\phi_1$ takes both positive and negative values,
(ii) $\lim_{x \to \infty} \phi_1(x) = d_1 \in \mathbb{R}$ exists,
(iii) for all $x > 0, y > 0$ we have $1 + \theta \phi_1(x) \phi_2(y) \geq 0$,
(iv) $\int_0^{\infty} \phi_2(y) G(dy) = 0$.

For some $\alpha > 0, \epsilon > 0$ and $\beta_0 \in \mathbb{R}$, assume that $\int_{0}^{\infty} y^{\alpha+i\beta_0} G(dy) < \infty$ and
\[
\int_{0}^{\infty} y^{\alpha+i\beta_0} G(dy) + \theta d_1 \int_{0}^{\infty} \phi_2(y) y^{\alpha+i\beta_0} G(dy) = 0. \tag{5.1}
\]
If $Y \sim G$, then there exists $X$, with not regularly varying tail, such that $(X, Y)$ is jointly distributed as bivariate Sarmanov, as defined in Definition 1.3 with kernel functions $\phi_1, \phi_2,$ and constant $\theta$, with $XY$ having regularly varying tail with index $-\alpha$.

**Proof.** Find $Y$ such that its marginal is $G$, then consider its twisted version $Y^\theta$ as defined in \((1.7)\), with marginal $G_\theta$ given by $G_\theta(dy) = (1 + \theta d_1 \phi_2(y)) G(dy)$. From the condition of \((5.1)\), we find that
\[
E[Y^\theta_{\alpha+i\beta_0}] = 0.
\]
Since \(\phi_2\) is bounded, let \(|\phi_2(y)| \leq b_2\) for all \(y > 0\), for some finite \(b_2\). Then
\[
E[Y_0^{\alpha + \epsilon}] \leq (1 + |\theta||d_1|b_2) \int_0^\infty y^{\alpha + \epsilon} G(dy) < \infty.
\]
(5.2)
This shows that the moment condition of (5.4),
\[
\int_0^\infty y^{\alpha - \epsilon} \vee y^{\alpha + \epsilon} G_\theta(dy) < \infty,
\]
holds. We adopt the idea of Theorem 2.1 of Jacobsen et al. (2009) to define two distribution functions \(F_\alpha\) and \(\bar{F}\) as follows:
\[
F_\alpha(dx) = \alpha x^{-(\alpha + 1)} dx, \quad \text{for all } x > 1,
\]
which means \(F \in RV^{-\alpha}\), and
\[
\bar{F}(dx) = g(x)F_\alpha(dx), \quad x > 1,
\]
where \(g(x) = 1 + a \cos(\beta_0 \log x) + b \sin(\beta_0 \log x)\) for some constants \(a > 0, b > 0\) with \(0 < a + b \leq 1\). Then again, from the conclusion of Theorem 2.1 of Jacobsen et al. (2009), we have \(\bar{F} \oplus G_\theta \sim \bar{F}_\alpha \oplus G_\theta \in RV^{-\alpha}\).

But from Theorem 2.3 of Jacobsen et al. (2009), we know that \(\bar{F}\) does not have a regularly varying tail.

We now have to tweak \(\bar{F}\) to get our desired \(F\), so that all conditions of Theorem 5.2 hold. Choosing \(c > 1\) so that \(\bar{F}(c) < 1\), we define a new distribution \(F^{(1)}\) as follows:
\[
F^{(1)}(y) = \begin{cases} 
\bar{F}(y) & \text{for } y > c, \\
\bar{F}(y) & \text{for } 1 < y \leq c, \\
1 & \text{for } y \leq 1.
\end{cases}
\]

We shall now show that \(\bar{F}^{(1)} \oplus G_\theta \sim \bar{F} \oplus G_\theta\). Observe that
\[
\bar{F}^{(1)} \oplus G_\theta(x) = \int_{(0,x/c)} + \int_{[x/c,x)} + \int_{(x,\infty)} \bar{F}^{(1)}(\frac{x}{u}) G_\theta(du)
= \bar{F} \oplus G_\theta(x) - \int_{x/c}^{\infty} \bar{F}(\frac{x}{u}) G_\theta(du) + \bar{F}(c)G_\theta([x/c,x)) + G_\theta([x, \infty)).
\]
(5.3)
We deal with the second term of the sum in (5.3) first. Let \(\|G_\theta\|_\alpha\) denote the integral \(\int_0^\infty y^\alpha G_\theta(dy)\). From the definitions of \(F_\alpha\) and \(\bar{F}\), and the fact that \(\bar{F} \oplus G_\theta \sim \bar{F}_\alpha \oplus G_\theta\), we get:
\[
\lim_{x \to \infty} \frac{\int_{x/c}^{\infty} \bar{F}(\frac{x}{u}) G_\theta(du)}{\bar{F} \oplus G_\theta(x)} = \lim_{x \to \infty} \frac{\int_{x/c}^{\infty} \bar{F}(\frac{x}{u}) G_\theta(du)}{\bar{F}_\alpha \oplus G_\theta(x)}
= \lim_{x \to \infty} \int_{x/c}^{\infty} \frac{g(y)F_\alpha(dy)G_\theta(du)}{x^{-\alpha} \|G_\theta\|_\alpha}
\leq \lim_{x \to \infty} (1 + a + b) \int_{x/c}^{\infty} \frac{\frac{\bar{F}(\frac{x}{u}) G_\theta(du)}{x^{-\alpha} \|G_\theta\|_\alpha}}
= \lim_{x \to \infty} (1 + a + b) \int_{x/c}^{\infty} u^\alpha G_\theta(du).
\]
(5.4)
From (5.2) we conclude that \(\int_0^\infty y^\alpha G_\theta(dy) < \infty\), so that applying dominated convergence, the numerator on the right side of (5.3) goes to 0 as \(x \to \infty\). We now consider the last two summands on the right side of (5.3).
\[
\lim_{x \to \infty} \frac{\bar{F}(c)G_\theta([x/c,x)) + G_\theta([x, \infty))}{\bar{F} \oplus G_\theta(x)} \leq \lim_{x \to \infty} \frac{\bar{F}(c)G_\theta(x/c) + G_\theta(x)}{\bar{F}_\alpha \oplus G_\theta(x)}
\]
Immediate for (i). For (ii), we consider India.

From the definitions of $F^{(1)}$ and $\tilde{F}$ in terms of $F_\alpha$, and Remark $5.4$, it is immediate that $F^{(1)} \in D$. As $\tilde{F}$ and $F^{(1)}$ eventually have the same tail, $F^{(1)}$ cannot be regularly varying. The last step in this proof is to adjust $F^{(1)}$ slightly to get the final desired distribution $F$ so that $\int_0^\infty \phi_1(x) F(dx) = 0$.

For this purpose, we define $\hat{\phi}_1$ as $\hat{\phi}_1(x) = \int_x^\infty \phi_1(x) F^{(1)}(dx)$, $x > 0$. Because $\phi_1$ is bounded, $\hat{\phi}_1$ is continuous on $(1, \infty)$. We now subdivide into three cases:

(i) If $\hat{\phi}_1$ takes both positive and negative values on $(1, \infty)$, by intermediate value theorem, we find $x_0 > 1$ such that $\hat{\phi}_1(x_0) = \int_{x_0}^\infty \phi_1(x) F^{(1)}(dx) = 0$. Then we define $F$ as $F(x) = \tilde{F}(x)/F^{(1)}(x_0)$ for $x \geq x_0$.

(ii) Suppose $\phi_1$ takes only strictly positive values on $(0, \infty)$. Because $\phi_1$ takes both positive and negative values, we find $x_1 > 0$ and $c_1 > 0$ such that $\phi_1(x_1) = -c_1$. Let $\hat{\phi}_1(1) = \int_1^\infty \phi_1(x) F^{(1)}(dx) = c_0 > 0$. We define the probability measure $\mu$ as follows:

$$\mu(B) = \frac{\mu^{(1)}[B \cap [1, \infty]] + c_0 \delta_{x_1}(B)}{\mu^{(1)}(1, \infty) + c_1}, \quad B \text{ a Borel set on } (0, \infty),$$

where $\mu^{(1)}$ is the law induced by $F^{(1)}$. Then we take $F$ to be the distribution function for $\mu$.

(iii) Suppose $\phi_1$ takes only strictly negative values on $(0, \infty)$. Again, we can find $x_2 > 0$ and $c_2 > 0$ such that $\phi_1(x_1) = c_2$. Let $\hat{\phi}_1(1) = \int_1^\infty \phi_1(x) F^{(1)}(dx) = -c_0 < 0$. We define the probability measure $\mu$ as follows:

$$\mu(B) = \frac{\mu^{(1)}[B \cap [1, \infty]] + c_0 \delta_{x_1}(B)}{\mu^{(1)}(1, \infty) + c_2}, \quad B \text{ a Borel set on } (0, \infty).$$

Then we take $F$ to be the distribution function for $\mu$.

We claim that for a suitable constant $\kappa$, $F \oplus G_\theta \sim \kappa F^{(1)} \oplus G_\theta$, which gives $F \oplus G_\theta \in RV_{-\alpha}$. This is immediate for (i). For (ii), we consider

$$F \oplus G_\theta(x) = \frac{1}{\mu^{(1)}(1, \infty) + c_1} \left[ \int_0^x \frac{F^{(1)}(y)}{u} G_\theta(du) + \frac{F^{(1)}(1)G_\theta([x, \infty)) + c_0 G_\theta \left[ \frac{x}{x_1}, \infty \right]}{c_1} \right].$$

We now deal with the second term in the sum on the right side of $5.5$ the same way as the second term in the sum of $5.4$, and the sum of the last two terms in the same way as the last two terms of $5.3$. From the definition of $F$ in terms of $F^{(1)}$ and hence $\tilde{F}$, and because $\tilde{F}$ is not regularly varying, we conclude that $F$ also not regularly varying. Case (iii) is dealt with similarly. This completes the proof.

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