IDENTITIES OF EULERIAN AND ORDERED STIRLING NUMBERS OVER A MULTISET

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Abstract. By considering Eulerian numbers and ordered Stirling numbers of the second and third kinds over a multiset, we generalize identities of Eulerian numbers and Stirling numbers of the second and third kinds and provide q-analogs of these generalizations. Using these generalizations, we also compute Eulerian numbers and ordered Stirling numbers of the second kind over a multiset.

1. Introduction

Let \( \binom{d}{i} \) be an Eulerian number, which is the number of permutations of \( \{1, 2, \ldots, d\} \) with \( i \) descents. A well-known identity of Eulerian numbers is Worpitzky’s identity \[1\]

\begin{equation}
\left( \begin{array}{c}
\frac{x^d}{d} \\
\end{array} \right) = \sum_{i=0}^{d-1} \binom{d}{i} \left( \frac{x - 1 - i + d}{d} \right),
\end{equation}

and the following Carlitz’s identity \[2, 3\] is a q-analog of this identity

\begin{equation}
\left[ \begin{array}{c}
x^d \\
1
\end{array} \right]_q = \sum_{i=1}^{d} A_{d,i}(q) \left[ \begin{array}{c}
x - 1 + i \\
d
\end{array} \right]_q
\end{equation}

where \( \left[ \begin{array}{c}
x \\
m
\end{array} \right]_q = \prod_{i=1}^{m} (1 - q^{x-m+i})/(1 - q^i) \) and \( A_{d,i}(q) \) is a polynomial of \( q \).

Eulerian numbers are defined over an ordinary set, thus one way to generalize identities \[1\] and \[2\] is to consider Eulerian numbers over a multiset. By considering Eulerian numbers over a multiset, we may be able to generalize identities \[1\] and \[2\].

We can apply such an idea to identities of Stirling numbers of the second and third kinds. Let \( \{ \begin{array}{c}
\frac{d}{k} \\
\end{array} \} \) be a Stirling number of the second kind, which is the number of partitions of a \( d \) element set into \( k \) nonempty sets. String numbers of the second kind satisfy the following polynomial identity

\begin{equation}
x^d = \sum_{k=1}^{d} k! \left\{ \begin{array}{c}
d \\
k
\end{array} \right\} \left( \begin{array}{c}
x \\
k
\end{array} \right)
\end{equation}
A Stirling number of the third kind \([4]\) (or a Lah number), denoted by \(\lfloor \frac{d}{d} \rfloor\), is the number of ways to partition a \(d\) element set into \(k\) nonempty linearly ordered subsets \([5, 6]\). For example, the partitions of \(\{1, 2, 3\}\) into 2 nonempty linearly ordered sets are \(\{(1), (2, 3)\}\), \(\{(1), (3, 2)\}\), \(\{(2), (1, 3)\}\), \(\{(2), (3, 1)\}\), \(\{(3), (1, 2)\}\), and \(\{(3), (2, 1)\}\). Lah \([7]\) defined Stirling numbers of the third kind by the following polynomial identity

\[
d! \left( \frac{x - 1 + d}{d} \right) = \sum_{k=1}^{d} k! \left\lfloor \frac{d}{k} \right\rfloor \left( \begin{array}{c} x \\ k \end{array} \right).
\]

By considering Stirling numbers of the second and third kinds over a multiset, we may also be able to generalize identities \([3]\) and \([4]\) and provide \(q\)-analogs of these generalizations.

Motivated by these ideas, we will generalize identities \([1]\), \([3]\), and \([4]\) and provide \(q\)-analogs of these generalizations. In particular, we generalize identities \([3]\) and \([4]\) and obtain \(q\)-analogs of these generalizations by considering ordered Stirling numbers of the second kind \(k!\left\{ \frac{d}{k} \right\}\) and third kind \(k! \left\lfloor \frac{d}{k} \right\rfloor\) over a multiset, rather than ordinary Stirling numbers of the second and third kinds. Using generalizations of identities \([1]\) and \([3]\), we also compute Eulerian numbers and ordered Stirling numbers of the second kind over a multiset.

The gist of our idea is the following. For a sequence of finite sets of lattice points \(S_0, S_1, S_2, \ldots\), we compute the numbers of elements in \(S_n\) by two different ways and obtain a polynomial identity. To obtain a \(q\)-analog of this identity, we compute the following generating function

\[
\sum_{(x_1, x_2, \ldots, x_d) \in S_n} q^{x_1 + x_2 + \cdots + x_d}
\]

by two different ways.

2. A Triangulation of the Product of Simplexes

In this section, we introduce a triangulation of the product of simplexes. This triangulation will be the main tool to generalize identities \([1]\), \([3]\), and \([4]\) and provide \(q\)-analogs of these generalizations.
For a positive integer \( n \), let \([n] = \{1, 2, \ldots, n\}\) and \([n]_0 = \{0, 1, \ldots, n\}\). We denote a point in \( \mathbb{R}^d \) by \( \mathbf{x} = (x_1, x_2, \ldots, x_d) \). We denote the zero vector of \( \mathbb{R}^d \) by \( \mathbf{e}_{d,0} \) and for each \( i \in [d] \) we denote the \( i \)th unit vector of \( \mathbb{R}^d \) by \( \mathbf{e}_{d,i} = (0, \ldots, 0, 1, 0, \ldots, 0) \) (with 1 in the \( i \)th coordinate). We define \( \alpha^d_i \) to be a \( d \)-dimensional simplex whose vertexes are \( \mathbf{e}_{d,0}, \mathbf{e}_{d,0} + \mathbf{e}_{d,1}, \ldots, \mathbf{e}_{d,0} + \mathbf{e}_{d,1} + \cdots + \mathbf{e}_{d,d} \). Note that \( \alpha^d_0 \) is the set of points \( \mathbf{x} \) such that \( 1 \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0 \).

Let \( d = d_1 + d_2 + \cdots + d_l \) be a sum of nonnegative integers. For each \( j \in [l] \) writing a point of \( \mathbb{R}^{d_j} \) by \( \mathbf{x}_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,d_j}) \), we also denote a point of \( \mathbb{R}^d = \prod_{j=1}^l \mathbb{R}^{d_j} \) by \( \mathbf{x} = \prod_{j=1}^l \mathbf{x}_j \). Letting \( \{ (j_h, i_h) \mid h \in [d] \} = \bigcup_{j=1}^l \{ j \} \times [d_j] \), for a point \( \mathbf{x} \) in \( \mathbb{R}^d \) we always assume that

1. \( x_{j_1,i_1} \geq x_{j_2,i_2} \geq \cdots \geq x_{j_d,i_d} \),
2. if \( x_{j_h,i_h} = x_{j_{h+1},i_{h+1}} \) then either \( j_h < j_{h+1} \) or \( (j_h, i_h) = (j_{h+1}, i_{h+1} - 1) \). Let \( \mathbf{d} = (d_1, d_2, \ldots, d_l) \). We denote the product of simplexes \( \alpha^d_0, \alpha^d_0, \ldots, \alpha_{d_l}^d \) by \( \alpha^d = \prod_{j=1}^l \alpha^d_j \) and the vertexes of \( \alpha^d \) by \( \mathbf{e}_{1,2,\ldots,i_l} = \prod_{j=1}^l (\mathbf{e}_{d_j,0} + \mathbf{e}_{d_j,1} + \cdots + \mathbf{e}_{d_j,d_j}) \) where \((i_1, i_2, \ldots, i_l)\) ranges over the set \( \prod_{j=1}^l [d_j] \).

Let \( S(\mathbf{d}) = S(d_1, d_2, \ldots, d_l) \) be a multiset such that for each \( j \in [l] \) the number of \( j \) in \( S(\mathbf{d}) \) is \( d_j \). We define \( S(\mathbf{d}) \) to be the permutation set of \( S(\mathbf{d}) \) and denote a permutation of \( S(\mathbf{d}) \) by \( \sigma = [\sigma_1, \sigma_2, \ldots, \sigma_d] \). For a permutation \( \sigma \) of \( S(\mathbf{d}) \) we define \( \alpha^d(\sigma) \) to be a \( d \)-simplex with the vertexes \( \mathbf{e}_{i_{h,1},i_{h,2},\ldots,i_{h,l}} \) for \( h \in [d]_0 \) such that \((i_{0,0}, i_{0,2}, \ldots, i_{0,l}) = \mathbf{e}_{l,0} \) and \((i_{h,1}, i_{h,2}, \ldots, i_{h,l}) = \sum_{a=1}^{h} \mathbf{e}_{l_a} \sigma_a \) for \( h \in [d] \). By definition, \( \alpha^d(\sigma) \) is the set of points \( \mathbf{x} \) such that

1. \( x_{\sigma_1,i_1} \geq x_{\sigma_2,i_2} \geq \cdots \geq x_{\sigma_d,i_d} \),
2. if \( x_{\sigma_h,i_h} = x_{\sigma_{h+1},i_{h+1}} \) then either \( \sigma_h < \sigma_{h+1} \) or \( (\sigma_h, i_h) = (\sigma_{h+1}, i_{h+1} - 1) \).

Let \( \mathbf{x} \) be a point of \( \alpha^d \). Letting \( x_{j_0,i_0} = 1 \) and \( x_{j_{d+1},i_{d+1}} = 0 \), we can represent \( \mathbf{x} \) by the following sum

\[
\mathbf{x} = \sum_{k=0}^{d} (x_{j_k,i_k} - x_{j_{k+1},i_{k+1}}) \mathbf{e}_{i_{k+1},i_{k+2},\ldots,i_{d}}
\]

where \((i_{0,1}, i_{0,2}, \ldots, i_{0,l}) = \mathbf{e}_{l,0} \) and \((i_{h,1}, i_{h,2}, \ldots, i_{h,l}) = \sum_{a=1}^{h} \mathbf{e}_{l_a} j_a \) for \( h \in [d] \). This implies that \( \mathbf{x} \) is a convex sum of the vertexes of \( \alpha^d([j_1, j_2, \ldots, j_d]) \). Since this is true for every point of \( \alpha^d \), it follows that \( \alpha^d \) is the union of the \( d \)-simplexes \( \alpha^d(\sigma) \) for \( \sigma \in S(\mathbf{d}) \). Moreover, the set composed of the
d-simplexes $\alpha^d(\sigma)$ for $\sigma \in \mathcal{S}(d)$ and their faces is a triangulation of $\alpha^d$ by definition. We denote this triangulation by $T_{\alpha^d}$.

3. Eulerian numbers over a multiset

Let $R$ be a subset of $\mathbb{R}^d$. For a nonnegative integer $n$ we denote $nR = \{nr \mid r \in R\}$ and we define $Z(R)$ to be the set of lattice points in $R$. Recall that a lattice point is a point with nonnegative integers entries. For a point $x$ let $q^x = q_{x_1}^1 + q_{x_2}^2 + \cdots + q_{x_d}^d$. To generalize identity (1), we will compute the number of elements in $Z(n\alpha^d)$ by two different ways and to obtain a $q$-analog of this generalization, which is a generalization of identity (2), we will compute $f_1(q) = \sum_{x \in Z(n\alpha^d)} q^x$ by two different ways.

For a permutation $\sigma$ of $S^d$ let $D(\sigma)$ be the descent set of $\sigma$, that is, the set of indexes $h$ such that $\sigma_h > \sigma_{h+1}$. We define $A(\sigma)$ to be a subset of $\alpha^d$ composed of points $x$ such that $x_{\sigma_h,i_h} > x_{\sigma_h+1,i_{h+1}}$ for all $h$ where $h$ ranges over $D(\sigma)$. Then for each point $x$ in $\alpha^d$ there is a unique permutation $\sigma$ such that $A(\sigma)$ contains $x$. Therefore $\alpha^d$ is the disjoint union of the sets $A(\sigma)$ for $\sigma \in \mathcal{S}(d)$. We call this disjoint union of $\alpha^d$ the first decomposition of $\alpha^d$. Stanley [8] obtained such a decomposition when $d_1 = d_2 = \cdots = d_l = 1$, that is, $\alpha^d$ is an $l$-dimensional hypercube.

The set $Z(n\alpha^d)$ is the product of $Z(n\alpha^{d_1})$, $Z(n\alpha^{d_2})$, \ldots, $Z(n\alpha^{d_l})$ and for each $j \in [l]$ the number of elements in the set $Z(n\alpha^{d_j})$ is $\binom{n+d_j}{d_j}$, thus

$$|Z(n\alpha^d)| = \prod_{j=1}^l |Z(n\alpha^{d_j})| = \prod_{j=1}^l \binom{n+d_j}{d_j}.$$ 

A point $x$ is an element of $Z(nA(\sigma))$ if and only $x$ is a lattice point such that

1. $n \geq x_{\sigma_1,i_1} \geq x_{\sigma_2,i_2} \geq \cdots \geq x_{\sigma_d,i_d} \geq 0,$
2. $x_{\sigma_h,i_h} \geq x_{\sigma_{h+1},i_{h+1}} + 1$ whenever $h$ is a descent of $\sigma$.

Thus the number of elements in $Z(nA(\sigma))$ is $\binom{n-|D(\sigma)|+d}{d}$. Therefore if we denote by $\langle \binom{id}{i} \rangle$ the number of permutations of $S(d)$ with $i$ descents, then
by the first decomposition of \( \alpha^d \) we obtain

\[
|Z(n\alpha^d)| = |Z(\bigoplus_{\sigma\in S(d)} nA(\sigma))| = \sum_{\sigma\in S(d)} |Z(nA(\sigma))| = \sum_{\sigma\in S(d)} \left( n - |D(\sigma)| + d \right) \binom{n + d - 1}{i} \binom{n - i + d}{d}.
\]

As a result, we obtain the following identity

\[
(5) \quad \prod_{j=1}^{l} \binom{n + d_j}{d_j} = \sum_{i=0}^{d-1} \binom{d}{i} \binom{n - i + d}{d}.
\]

Note that Kim and Lee [Polytope numbers and their properties, arXiv:1206.0511] derived identity (5) by using the concept of polytope numbers. Since identity (5) is true for every nonnegative integer \( n \), it follows that

\[
(6) \quad \prod_{j=1}^{l} \binom{x + d_j}{d_j} = \sum_{i=0}^{d-1} \binom{d}{i} \binom{x - i + d}{d}.
\]

**Theorem 3.1.** Eulerain numbers over a multiset \( S(d) \) satisfies

\[
\prod_{j=1}^{l} \binom{x + d_j}{d_j} = \sum_{i=0}^{d-1} \binom{d}{i} \binom{x - i + d}{d}.
\]

When \( d_1 = d_2 = \cdots = d_l = 1 \), identity (6) becomes identity (1). Thus identity (6) is a generalization of Worpitzky’s identity [1].

Using identity (6), we can obtain Eulerian numbers over \( S(d) \) in the following way. Let \( M_1 \) be a \( d \times d \) matrix defined by

\[
M_1 = \begin{bmatrix}
\binom{d}{d} & 0 & \cdots & 0 \\
\binom{d+1}{d} & \binom{d}{d} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
\binom{2d-1}{d} & \binom{2d-2}{d} & \cdots & \binom{d}{d}
\end{bmatrix}.
\]

Then from identity (5) we can obtain the following matrix identity

\[
\begin{bmatrix}
\prod_{j=1}^{l} \binom{d_j}{d_j} \\
\prod_{j=1}^{l} \binom{1+d_j}{d_j} \\
\vdots \\
\prod_{j=1}^{l} \binom{d-1+d_j}{d_j}
\end{bmatrix} = \begin{bmatrix}
\binom{d}{d} & 0 & \cdots & 0 \\
\binom{d+1}{d} & \binom{d}{d} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
\binom{2d-1}{d} & \binom{2d-2}{d} & \cdots & \binom{d}{d}
\end{bmatrix} \begin{bmatrix}
\binom{d}{0} \\
\binom{d}{1} \\
\vdots \\
\binom{d}{d-1}
\end{bmatrix}.
\]
Since the inverse of $M_1$ is

$$M_1^{-1} = \begin{bmatrix} (-1)^0 \binom{d+1}{0} & 0 & \cdots & 0 \\ (-1)^1 \binom{d+1}{1} & (-1)^0 \binom{d+1}{0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{d-1} \binom{d+1}{d-1} & (-1)^{d-2} \binom{d+1}{d-2} & \cdots & (-1)^0 \binom{d+1}{0} \end{bmatrix}$$

by the Gaussian elimination, it follows that

$$\langle d \rangle_i = \sum_{h=0}^{i} (-1)^{i-h} \binom{d+1}{i-h} \prod_{j=1}^{l} \binom{h + d_j}{d_j}.$$ 

**Theorem 3.2.** Eulerian numbers over a multiset $S(d)$ are given by

$$\langle d \rangle_i = \sum_{h=0}^{i} (-1)^{i-h} \binom{d+1}{i-h} \prod_{j=1}^{l} \binom{h + d_j}{d_j}.$$ 

Now we consider $f_1(q)$. By definition, $x_j$ is in $\mathbb{Z}(n\alpha_{d_j}^{d_j})$ if and only if $x_j$ is a lattice point such that $n \geq x_{j,1} \geq x_{j,2} \geq \cdots \geq x_{j,d_j} \geq 0$, thus the generating function $\sum_{x_j \in \mathbb{Z}(n\alpha_{d_j}^{d_j})} q^{x_j}$ is a $q$-binomial coefficient $\left[ \binom{n + d_j}{d_j} \right]_q$. Since $\mathbb{Z}(n\alpha^d) = \prod_{j=1}^{l} \mathbb{Z}(n\alpha_{d_j}^{d_j})$ and $q^x = \prod_{j=1}^{l} q^{x_j}$, it follows that

$$f_1(q) = \prod_{j=1}^{l} \sum_{x_j \in \mathbb{Z}(n\alpha_{d_j}^{d_j})} q^{x_j} = \prod_{j=1}^{l} \left[ \binom{n + d_j}{d_j} \right]_q.$$ 

For a permutation $\sigma$ of $S(d)$ we define the major index of $\sigma$ to be $maj(\sigma) = \sum_{j \in D(\sigma)} j$. A lattice point $x$ is in $\mathbb{Z}(nA(\sigma))$ if and only if

$$\begin{cases} x_{\sigma_h,i_h} \geq x_{\sigma_{h+1},i_{h+1}} & \text{for } h \in [d+1] \setminus D(\sigma) \\ x_{\sigma_h,i_h} \geq x_{\sigma_{h+1},i_{h+1}} + 1 & \text{for } h \in D(\sigma) \end{cases}.$$

Therefore

$$\sum_{x \in \mathbb{Z}(nA(\sigma))} q^x = \sum_{x \in \mathbb{Z}(nA(\sigma))} \prod_{k=1}^{d} q^{x_{\sigma_k,i_k}} = q^{maj(\sigma)} \left[ \binom{n - |D(\sigma)| + d}{d} \right]_q.$$
Since $\mathbb{Z}(n\alpha^d)$ is the disjoint union of the sets $\mathbb{Z}(nA(\sigma))$ for $\sigma \in \mathcal{S}(d)$, we obtain

$$f_1(q) = \sum_{\sigma \in \mathcal{S}(d)} \sum_{x \in \mathbb{Z}(nA(\sigma))} q^x$$

$$= \sum_{\sigma \in \mathcal{S}(d)} q^{maj(\sigma)} \left[ \frac{n - |D(\sigma)| + d}{d} \right]_q$$

$$= \sum_{i=1}^{d} A_{d,i}(q) \left[ \frac{n + i}{d} \right]_q$$

(8)

where $A_{d,i}(q) = \sum_{\sigma \in \mathcal{S}(d)} q^{maj(\sigma)}$. Therefore combining identities (7) and (8), we obtain

$$\prod_{j=1}^{l} \left[ \frac{n + d_j}{d_j} \right]_q = \sum_{i=1}^{d} A_{d,i}(q) \left[ \frac{n + i}{d} \right]_q.$$

(9)

Identity (9) is true for every nonnegative integer $n$, thus by the substitution $n \rightarrow x$ we obtain the following $q$-analog of identity (6)

$$\prod_{j=1}^{l} \left[ \frac{x + d_j}{d_j} \right]_q = \sum_{i=1}^{d} A_{d,i}(q) \left[ \frac{x + i}{d} \right]_q.$$

(10)

**Theorem 3.3.** A $q$-analog of identity (6) is

$$\prod_{j=1}^{l} \left[ \frac{x + d_j}{d_j} \right]_q = \sum_{i=1}^{d} A_{d,i}(q) \left[ \frac{x + i}{d} \right]_q.$$

Note that identity (10) is a generalization of Carlitz’s identity (2).

### 4. Ordered Stirling numbers of the second kind over a multiset

Similar to the case of Eulerian numbers over a multiset, to generalize identity (3), we will compute the number of elements in $\mathbb{Z}(n\alpha^d)$ by two different ways and to obtain a $q$-analog of this generalization, we will compute $\sum_{x \in \mathbb{Z}(n\alpha^d)} q^x$ by two different ways.

For two vectors $\mathbf{v} = (v_1, v_2, \ldots, v_l)$ and $\mathbf{v}' = (v'_1, v'_2, \ldots, v'_l)$, we define $\mathbf{v} \leq \mathbf{v}'$ if $v_i \leq v'_i$ for all $i \in [l]$. Let $T(\alpha^d)$ be the set of simplexes in $T_{a,d}$ that contains $e_{0,0,\ldots,0}$ and $e_{d_1,d_2,\ldots,d_l}$. For a $k$-dimensional simplex $\alpha^k$ ($k \in [d]$) in
$\mathcal{I}(\alpha^d)$ let \( \{e_{i_1, i_2, \ldots, i_t} \mid h \in [k]_0 \} \) be the vertex set of \( \alpha^k \) where

\[
\begin{aligned}
(i_{0,1}, i_{0,2}, \ldots, i_{0,l}) &= 0 \\
(i_{h,1}, i_{h,2}, \ldots, i_{h,t}) &< (i_{h+1,1}, i_{h+1,2}, \ldots, i_{h+1,t}) \quad \text{for } h \in [k-1]_0 \\
(i_{k,1}, i_{k,2}, \ldots, i_{k,l}) &= (d_1, d_2, \ldots, d_l).
\end{aligned}
\]

(11)

We define \( I(\alpha^k) \) to be the set of convex sums \( x = \sum_{h=0}^{k} c_h e_{i_{h,1}, i_{h,2}, \ldots, i_{h,t}} \) of the vertexes of \( \alpha^k \) such that \( c_h > 0 \) for \( h \in [k-1] \). This is implies that if \( x \) is a point of \( I(\alpha^k) \), then there are \( k-1 \) numbers \( h_1, h_2, \ldots, h_{k-1} \) in \([d]\) such that

\[
\begin{aligned}
x_{j_1,i_1} = \cdots = x_{j_{h_1},i_{h_1}} > \\
x_{j_{h_1+1},i_{h_1+1}} = \cdots = x_{j_{h_2},i_{h_2}} > \\
\vdots \\
x_{j_{h_{k-1}+1},i_{h_{k-1}+1}} = \cdots = x_{j_d,i_d}
\end{aligned}
\]

(12)

Let \( x \) be a point of \( \alpha^d \). Suppose that \( x \) satisfies condition (12). Then a simplex \( \alpha^k \) in \( \mathcal{I}(\alpha^d) \) with the vertex set \( \{e_{i_1, i_2, \ldots, i_t} \mid a \in [k]_0 \} \) satisfying

\[
\begin{aligned}
(i_{0,1}, i_{0,2}, \ldots, i_{0,l}) &= 0 \\
(i_{a,1}, i_{a,2}, \ldots, i_{a,t}) &= \sum_{j=1}^{h_a} e_{i,j} \quad \text{for } a \in [k-1] \\
(i_{k,1}, i_{k,2}, \ldots, i_{k,l}) &= (d_1, d_2, \ldots, d_l)
\end{aligned}
\]

is the unique simplex such that \( I(\alpha^k) \) contains \( x \). Since \( I(\alpha^k) \) is a subset of \( \alpha^d \), it follows that \( \alpha^d \) is the disjoint union of the sets \( I(\alpha^k) \) for \( \alpha^k \in \mathcal{I}(\alpha^d) \).

We call this disjoint union of \( \alpha^d \) the second decomposition of \( \alpha^d \).

Let \( \alpha^k \) be a \( k \)-dimensional simplex in \( \mathcal{I}(\alpha^d) \) with the vertex set \( \{e_{i_{h,1}, i_{h,2}, \ldots, i_{h,t}} \mid h \in [k]_0 \} \) that satisfies condition (11). We define the major index of \( \alpha^k \) to be \( \text{maj}(\alpha^k) = \sum_{h=1}^{k-1} \sum_{j=1}^{i_{h,j}} i_{h,j} \). For each \( h \in [k] \) if we denote \( S_h = S(i_{h,1}, i_{h,2}, \ldots, i_{h,t}) \), then \( (S_1, S_2, \ldots, S_k) \) is an ordered partition of \( S(d) \) into \( k \) nonempty multisets. Therefore the number of \( k \)-dimensional simplexes in \( \mathcal{I}(\alpha^d) \) is the number of ordered partitions of \( S(d) \) into \( k \) nonempty multisets. We call this number an ordered Stirling number of the second kind and denote it by \( \{d\}_k^1 \). Note that if \( d_1 = d_2 = \cdots = d_l = 1 \), then \( \{d\}_k^1 = k!{\binom{l}{k}} \).
By definition, the number of elements in the set $\mathbb{Z}(nI(\alpha^k))$ is $\binom{n+1}{k}$, thus by the second decomposition of $\alpha^d$ it follows that

$$\prod_{j=1}^{l} \binom{n+d_j}{d_j} = \prod_{j=1}^{l} |\mathbb{Z}(n\alpha^{d_j})| = |\mathbb{Z}(n\alpha^d)|$$

$$= |\mathbb{Z}(\biguplus_{k=1}^{d} \alpha^k I(\alpha^k))|$$

$$= \sum_{k=1}^{d} \sum_{\alpha^k \in I(\alpha^d)} |\mathbb{Z}(nI(\alpha^k))|$$

$$= \sum_{k=1}^{d} \left\{ \binom{d}{k} \right\}_O \binom{n+1}{k}.$$

As a result, by the substitution $n \to x$ we obtain

$$\prod_{j=1}^{l} \binom{x+d_j}{d_j} = \sum_{k=1}^{d} \left\{ \binom{d}{k} \right\}_O \binom{x+1}{k}.$$

**Theorem 4.1.** Ordered Stirling numbers of the second kind over $S(d)$ satisfies

$$\prod_{j=1}^{l} \binom{x+d_j}{d_j} = \sum_{k=1}^{d} \left\{ \binom{d}{k} \right\}_O \binom{x+1}{k}.$$

Similar to the case of Eulerian numbers over a multiset, we can obtain ordered Stirling numbers of the second kind over a multiset as follows. Let $M_2$ be a $d \times d$ matrix defined by

$$M_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \binom{1}{2} & \binom{2}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{d}{1} & \binom{d}{2} & \cdots & \binom{d}{d} \end{bmatrix},$$

Then we can rewrite identity (13) in the following form

$$\prod_{j=1}^{l} \begin{bmatrix} \binom{d_j}{1} \\ \binom{1+d_j}{1} \\ \vdots \\ \binom{d-1+d_j}{1} \\ \binom{d}{1} \end{bmatrix} = \prod_{j=1}^{l} \begin{bmatrix} \binom{d}{1} \\ \binom{d}{2} \\ \vdots \\ \binom{d}{d} \end{bmatrix}_O \begin{bmatrix} \binom{d}{1} \\ \binom{d}{2} \\ \vdots \\ \binom{d}{d} \end{bmatrix}_O.$$
Since by the Gaussian elimination the inverse of $M_2$ is

$$M_2^{-1} = \begin{pmatrix} (-1)^0 \binom{1}{0} & 0 & \cdots & 0 \\ (-1)^1 \binom{2}{1} & (-1)^0 \binom{2}{0} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ (-1)^{d-1} \binom{d}{d-1} & (-1)^{d-2} \binom{d}{d-2} & \cdots & (-1)^0 \binom{d}{0} \end{pmatrix},$$

it follows that

$$\{d \atop k\} = \sum_{h=0}^{k-1} (-1)^{k-1-h} \binom{k}{h} \prod_{j=1}^{l} \binom{h + d_j}{d_j}.$$

**Theorem 4.2.** Stirling numbers of the second kind over $\overline{S(d)}$ are given by

$$\{d \atop k\} = \sum_{h=0}^{k-1} (-1)^{k-1-h} \binom{k}{h} \prod_{j=1}^{l} \binom{h + d_j}{d_j}.$$  

For each $\alpha^k \in I(\alpha^d)$ the sum of $q^x$ for $x \in \mathbb{Z}(nI(\alpha^k))$ is $q^{\text{maj}(\alpha^k)} \left[ n + 1 \atop k \right]_q$. Thus it follows that

$$f_1(q) = \sum_{k=1}^{d} \sum_{\alpha^k \in I(\alpha^d)} \sum_{x \in \mathbb{Z}(nI(\alpha^k))} q^x$$

$$= \sum_{k=1}^{d} \sum_{\alpha^k \in I(\alpha^d)} q^{\text{maj}(\alpha^k)} \left[ n + 1 \atop k \right]_q$$

$$= \sum_{k=1}^{d} B_{d,k}(q) \left[ n + 1 \atop k \right]_q$$

where $B_{d,k}(q) = \sum_{\alpha^k \in I(\alpha^d)} q^{\text{maj}(\alpha^k)}$. In addition, $f_1(q) = \prod_{j=1}^{l} \left[ \frac{n + d_j}{d_j} \right]_q$, therefore we obtain

$$\prod_{j=1}^{l} \left[ \frac{n + d_j}{d_j} \right]_q = \sum_{k=1}^{d} B_{d,k}(q) \left[ n + 1 \atop k \right]_q.$$  

As a result, by the substitution $n \to x$ we get the following $q$-analog of identity (13)

$$\prod_{j=1}^{l} \left[ \frac{x + d_j}{d_j} \right]_q = \sum_{k=1}^{d} B_{d,k}(q) \left[ x + 1 \atop k \right]_q.$$  

**Theorem 4.3.** A $q$-analog of identity (13) is

$$\prod_{j=1}^{l} \left[ \frac{x + d_j}{d_j} \right]_q = \sum_{k=1}^{d} B_{d,k}(q) \left[ x + 1 \atop k \right]_q.$$
IDENTITIES OF EULERIAN AND ORDERED STIRLING NUMBERS OVER A MULTISET

Note that identity (14) is a generalization of the following identity (11)
\[
\left[ \frac{x^d}{1} \right]_q = \sum_{k=0}^{d} q^k (k-1)^a_{d,k}(q) \left[ \frac{x^d}{k} \right]_q
\]
where \( a_{d,k}(q) \) is a polynomial of \( q \).

5. ORDERED STIRLING NUMBERS OF THE THIRD KIND OVER A MULTISET

To derive a multiset version of identity (11), we will compute \( \sum_{\sigma \in \mathcal{E}(d)} |Z(n\alpha^d(\sigma))| \)
by two different ways and to obtain a \( q \)-analog of this identity, we will compute \( f_2(q) = \sum_{\sigma \in \mathcal{E}(d)} \sum_{x \in Z(n\alpha^d(\sigma))} q^{\sigma(15)} \) by two different ways.

Let \( x \) be a point in \( Z(n\alpha^d) \). Since \( \alpha^d \) is the disjoint union of the sets \( I(\alpha^k) \) for \( \alpha^k \in I(\alpha^d) \), there is a unique simplex \( \alpha^k \) in \( I(\alpha^d) \) such that \( Z(nI(\alpha^k)) \) contains \( x \). If \( \alpha^d(\sigma) \) is a \( d \)-dimensional simplex in \( I(\alpha^d) \) that contains \( \alpha^k \), then we can represent \( (\alpha^d(\sigma), \alpha^k) \) by a \( k \)-tuple of the form
\[
(\sigma_1, \ldots, \sigma_{i_1}), (\sigma_{i_1+1}, \ldots, \sigma_{i_2}), \ldots, (\sigma_{i_{k-1}+1}, \ldots, \sigma_d),
\]
By a simple computation, the number of such \( k \)-tuples is \( \binom{d}{d_1, d_2, \ldots, d_l} \frac{(d-1)!}{(k-1)!} \).

Note that the vertex set of \( \alpha^k \) is \( \{e_{i_1, i_2, \ldots, i_{d_k}} | h \in [k]_0 \} \) where
\[
\begin{cases}
(i_{0,1}, i_{0,2}, \ldots, i_{0,l}) = 0 \\
(i_{h,1}, i_{h,2}, \ldots, i_{h,l}) = \sum_{i=1}^{i_h} e_{i, j} \text{ for } h \in [k-1] \\
(i_{k,1}, i_{k,2}, \ldots, i_{k,l}) = (d_1, d_2, \ldots, d_l)
\end{cases}
\]
Therefore if we denote by \( \binom{d}{k}_O \) the number of \( k \)-tuples of form (13), which is called an ordered Stirling number of the third kind (or an ordered Lah number) over a multiset, then we obtain
\[
\sum_{\sigma \in \mathcal{E}(d)} |Z(n\alpha^d(\sigma))| = \sum_{\sigma \in \mathcal{E}(d)} \sum_{k=1}^{d} \sum_{\alpha^k \subseteq \alpha^d(\sigma)} |Z(nI(\alpha^k))| = \sum_{k=1}^{d} \binom{d}{k}_O \binom{n+1}{k}.
\]
In addition, \( \sum_{\sigma \in \mathcal{E}(d)} |Z(n\alpha^d(\sigma))| = \binom{d}{d_1, d_2, \ldots, d_l} \binom{n+d}{d} \), thus it follows that
\[
\binom{d}{d_1, d_2, \ldots, d_l} \binom{n+d}{d} = \sum_{k=1}^{d} \binom{d}{k}_O \binom{n+1}{k}.
\]
As a result, we get a multiset version of identity (14)
\[
\binom{d}{d_1, d_2, \ldots, d_l} (x + d) = \sum_{k=1}^{d} \binom{d}{k} (x + 1)^k.
\]

**Theorem 5.1.** Ordered Stirling numbers of the third kind over \(S(d)\) satisfy
\[
\binom{d}{d_1, d_2, \ldots, d_l} (x + d) = \sum_{k=1}^{d} \binom{d}{k} (x + 1)^k.
\]

Now we consider \(f_2(q)\). The number of permutations of \(S(d)\) is \(d_1, d_2, \ldots, d_l\) and \(\sum_{x \in Z(n \alpha d(\sigma))} q^x = \binom{n + d}{d}_q\), thus it follows that
\[
f_2(q) = \binom{d}{d_1, d_2, \ldots, d_l} \left[ \binom{n + d}{d}_q \right].
\]

In addition, by the second decomposition of \(\alpha d\) we can obtain
\[
f_2(q) = \sum_{\sigma \in \Theta(d)} \sum_{x \in Z(n \alpha d(\sigma))} q^x
\]
\[
= \sum_{\sigma \in \Theta(d)} \sum_{k=1}^{d} \sum_{\alpha^k \in Z(n \alpha d(\sigma))} \sum_{x \in Z(n I(\alpha^k))} q^x
\]
\[
= \sum_{k=1}^{d} \sum_{\sigma \in \Theta(d)} \sum_{\alpha^k \in Z(n \alpha d(\sigma))} q^{maj(\alpha^k)} \left[ \binom{n + 1}{k}_q \right]
\]
\[
= \sum_{k=1}^{d} C_{d,k}(q) \left[ \binom{n + 1}{k}_q \right]
\]
(18)

where \(C_{d,k}(q) = \sum_{\sigma \in \Theta(d)} \sum_{\alpha^k \in Z(n \alpha d(\sigma))} q^{maj(\alpha^k)}\). Therefore by combining identities (17) and (18) we obtain
\[
\binom{d}{d_1, d_2, \ldots, d_l} \left[ \binom{n + d}{d}_q \right] = \sum_{k=1}^{d} C_{d,k}(q) \left[ \binom{n + 1}{k}_q \right].
\]

As a result, we get the following \(q\)-analog of identity (16)
\[
\binom{d}{d_1, d_2, \ldots, d_l} \left[ \binom{x + d}{d}_q \right] = \sum_{k=1}^{d} C_{d,k}(q) \left[ \binom{x + 1}{k}_q \right].
\]

**Theorem 5.2.** A \(q\)-analog of identity (16) is
\[
\binom{d}{d_1, d_2, \ldots, d_l} \left[ \binom{x + d}{d}_q \right] = \sum_{k=1}^{d} C_{d,k}(q) \left[ \binom{x + 1}{k}_q \right].
\]
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