Local distinction, quadratic base change and automorphic induction for $\text{GL}_n$

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Abstract

Behind this sophisticated title hides an elementary exercise on Clifford theory for index two subgroups and self-dual/conjugate-dual representations. When applied to semi-simple representations of the Weil-Deligne group $W_F'$ of a non Archimedean local field $F$, and further translated in terms of representations of $\text{GL}_n(F)$ via the local Langlands correspondence when $F$ has characteristic zero, it yields various statements concerning the behaviour of different types of distinction under quadratic base change and automorphic induction. When $F$ has residual characteristic different from 2, combining one of the simple results that we obtain with the triviality of conjugate-orthogonal root numbers ([GGP12]), we recover without using the LLC a result of Serre on the parity of the Artin conductor of orthogonal representations of $W_F'$ ([Ser71]). On the other hand we discuss its parity for symplectic representations using the LLC and the Prasad and Takloo-Bighash conjecture.

Introduction

Let $E/F$ be a separable quadratic extension of non Archimedean local fields. Then thanks to the known local Langlands correspondence for $\text{GL}_n(E)$ and $\text{GL}_n(F)$, one has a base change map $\text{BC}_E^F$ from the set of isomorphism classes of irreducible representations of $\text{GL}_n(F)$ to that of $\text{GL}_n(E)$, and an automorphic induction map $\text{AI}_E^F$ from the set or isomorphism classes of irreducible representations of $\text{GL}_n(E)$ to that of $\text{GL}_n(F)$. A typical statement proved in this note (for $F$ of characteristic zero) is that if $\pi$ is a generic unitary representation of $\text{GL}_n(F)$ with orthogonal Langlands parameter (orthogonal in short), then $\text{BC}_E^F(\pi)$ is orthogonal and $\text{GL}_n(F)$-distinguished, and that the converse holds if $\pi$ is a discrete series (see Corollary 3.1 for the general statement). Corollary 3.1 is itself a translation via the LLC of our main result which concerns representations of the Weil-Deligne group of $F$ (Proposition 3.1). Another lucky application of Proposition 3.1 is that the result of [Ser71] on the parity of Artin conductors of representations of the Weil-Deligne group of $F$ is a consequence of that in [Del76] on root numbers of orthogonal representations, when $F$ has odd residual characteristic, as we show in Corollary 4.1. We also discuss its parity for symplectic representations using the LLC and the Prasad and Takloo-Bighash conjecture in Corollary 4.2.

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1 Notation, definitions and basic facts about self-dual and conjugate-dual representations

For $K$ a non Archimedean local field we denote by $W_K$ the Weil group of $K$ (see [Tat79]), and by $W'_K = W_K \times \text{SL}_2(\mathbb{C})$ the Weil-Deligne group of $K$. By a representation of $W_K$ we mean a finite dimensional representation smooth complex representation of $W_K$. By a representation of $W'_K$ we mean a representation which is a direct sum of representations of the form $\phi \otimes S$, where $\phi$ is an irreducible representation of $W_K$ and $S$ is an irreducible algebraic representation of $\text{SL}_2(\mathbb{C})$. We sometimes abbreviate $^*\phi$ is a representation of $W'_K$ as $^*\phi \in \text{Rep}(W'_K)$. We denote by $^*\phi^\vee \in \text{Rep}(W'_K)$ the dual of $\phi \in \text{Rep}(W'_K)$.

For the following facts on self-dual and conjugate-dual representations of $W'_K$, we refer to [GGP12, Section 3]. We recall that representation $\phi$ of $W'_K$ is self-dual if and only if there exists $\phi \times \phi$ a $W'_K$-invariant bilinear form $B$ which is non degenerate: we will say that $B$ is $W'_K$-bilinear (which in particular means non degenerate). If moreover $B$ is alternate, we say that $B$ is $(W'_K,-1)$-bilinear in which case we say that $\phi$ is symplectic or $(-1)$-self-dual, whereas if $B$ is symmetric, and we say that $B$ is $(W'_K,1)$-bilinear in which case we say that $\phi$ is orthogonal or $1$-self-dual. If $\phi$ is irreducible and self-dual, then there is up to nonzero scaling a unique $W'_K$-bilinear form on $\phi \times \phi$, which is either $(W'_K,-1)$-bilinear or $(W'_K,1)$-bilinear, but not both.

Now suppose that $L/K$ is a separable quadratic extension so that $W_L$ has index two in $W_K$, and fix $s \in W_K - W_L$. For $\phi$ a representation of $W_L$, we denote by $^*\phi^s$ the representation of $W'_L$ defined as $^*\phi^s := \phi(s \cdot s^{-1})$. We say that $\phi$ is $L/K$-dual or conjugate-dual if $^*\phi^s \cong ^*\phi^\vee$. The representation $^*\phi^s \in \text{Rep}(W'_L)$ is conjugate-dual if and only if there is on $\phi \times \phi$ a non-degenerate bilinear form $B$ such that

$$B(w.x, sws^{-1}.y) = B(x,y)$$

for all $(w,x,y)$ in $W'_L \times \phi \times \phi$. We say that such a bilinear form $B$ is $L/K$-bilinear (this in particular means non degenerate). If moreover there is $\varepsilon \in \{\pm 1\}$ such that $B$ satisfies

$$B(x, s^2.y) = \varepsilon B(y,x)$$

for all $(x,y)$ in $\phi \times \phi$ we say that $B$ is $(L/K,\varepsilon)$-bilinear, in which case we say that $\phi$ is $(L/K,\varepsilon)$-dual or conjugate-symplectic if $\varepsilon = -1$ and conjugate-orthogonal if $\varepsilon = 1$. All the definitions above do not depend on the choice of $s$. When $\phi$ is $L/K$-dual and also irreducible, then there is up to nonzero scaling a unique $L/K$-bilinear form on $\phi \times \phi$, which is either $(L/K,-1)$-bilinear or $(L/K,1)$-bilinear, but not both.

2 Preliminary results

2.1 Clifford-Mackey theory for index two subgroups

We refer to [CSST10, Section 3] for the following standard results.

Theorem 2.1. Let $G$ be a finite group and $H$ be a finite subgroup of index 2, and let $\eta : G \rightarrow \{\pm 1\}$ be the nontrivial character of $G$ trivial on $H$.

- For $\phi$ a (finite dimensional complex) representation of $H$ which is irreducible, the representation $\text{Ind}_H^G(\phi)$ is irreducible if and only $\phi^* \neq \phi$, which is also equivalent to the fact that $\phi$ does not extend to $G$. If it is reducible then $\phi$ extends to $G$, and if $\phi$ is such an extension, then $\eta \otimes \phi$ is the only other extension different from $\phi$, and $\text{Ind}_H^G(\phi) \cong \phi \otimes \eta \otimes \phi$. 

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Proposition 2.1. An irreducible representation \( \phi' \) of \( G \) restricts to \( H \) either irreducibly, or breaks into two irreducible pieces, and the second case occurs if and only \( \phi' \cong \eta \otimes \phi' \), which is also equivalent to \( \phi' = \text{Ind}_H^G(\phi) \) for \( \phi \) and irreducible representation of \( H \) such that \( \phi'' \neq \phi \).

For \( E/F \) a separable quadratic extension of non Archimedean local fields, we denote by \( \eta_{E/F} : W_F' \rightarrow \{ \pm 1 \} \) the nontrivial character of \( W_F' \) trivial on \( W_K' \). We also recall as a consequence of [BH04, 28.6] that if \( \phi_K \) is an irreducible representation of \( W_K' \) for \( K \) local and non Archimedean, then there is a character \( \chi \) of \( W_K' \) such that \( \chi \otimes \phi_K \) is irreducible, we call it a LLC representation of \( W_K' \), which in turn defines a character \( \chi \otimes \phi_K \). This observation allows one to extend Theorem 2.1 to the following situation.

Corollary 2.1. Let \( E/F \) be a separable quadratic extension of non Archimedean local fields, and fix \( s \in W_F - W_E \).

- For \( \phi_E \in \text{Rep}(W_E') \) be an irreducible representation, the representation \( \text{Ind}_{W_E}^{W_F}(\phi_E) \) is irreducible if and only \( \phi_E \neq \phi_E \), which is also equivalent to the fact that \( \phi_E \) does not extend to \( W_F' \). If it is reducible then \( \phi_E \) extends to \( W_F' \), and if \( \phi_F \) is such an extension, then \( \eta_{E/F} \otimes \phi_F \) is the only other extension different from \( \phi_F \), and \( \text{Ind}_{W_E}^{W_F}(\phi_E) \simeq \phi_F \otimes \eta_{E/F} \otimes \phi_F \).

- An irreducible representation \( \phi_F \) of \( W_F' \) restricts to \( W_E' \) either irreducibly, or breaks into two irreducible pieces, and the second case occurs if and only \( \phi_F \simeq \eta_{E/F} \otimes \phi_F \), which is also equivalent to \( \phi_F \simeq \text{Ind}_{W_E}^{W_F}(\phi_E) \) for \( \phi_E \) and irreducible representation of \( W_E' \) such that \( \phi_E \neq \phi_E \).

We will tacitly use the above corollary from now on.

2.2 Distinction and LLC for \( \text{GL}_n \)

Let \( F \) be a non Archimedean local field, we denote by LLC the local Langlands correspondence ([LRS93, HT02, Henn04]). For any \( n \geq 1 \), it restricts as a bijection from the set of isomorphism classes of \( n \)-dimensional representations of \( W_F' \) to that of (smooth and complex) irreducible representations of \( \text{GL}_n(F) \). If \( E/F \) is a quadratic extension, and \( \pi = \text{LLC}(\phi_F) \) for \( \phi \) a representation of \( W_F' \), we set \( \text{BC}^{\text{LLC}}_E(\pi) = \text{LLC}(\text{Res}_{W_E}^{W_F}(\phi)) \) (the quadratic base change of \( \pi \)), whereas if \( \tau = \text{LLC}(\phi_E) \) for \( \phi \) a representation of \( W_E' \), we set \( \text{AI}^{\text{LLC}}_E(\tau) = \text{LLC}(\text{Ind}_{W_E}^{W_F}(\phi_E)) \) (the quadratic automorphic induction of \( \tau \)). For \( \pi \) a representation of \( \text{GL}_n(F) \), we denote by \( \pi' \) its dual. If \( \pi \) is irreducible, we call it a discrete series representation if it has a matrix coefficient \( c \) such that \( |\chi \otimes \psi|^2 \) is integrable on \( \text{GL}_n(F)/F^*I_n \) (with respect to any Haar measure on the group \( \text{GL}_n(F)/F^*I_n \)) for some character \( \chi \) of \( \text{GL}_n(F) \). A representation \( \phi \) of \( W_F' \) is irreducible if and only if LLC(\( \phi \)) is a discrete series.

Let \( N_n(F) \) be the subgroup of \( \text{GL}_n(F) \) of upper triangular unipotent matrices, and let \( \psi \) be a non trivial character of \( F \), which in turn defines a character \( \tilde{\psi} : u \mapsto \psi(u_{1,2} + \cdots + u_{n-1,n}) \) of \( N_n(F) \). We say that an irreducible representation \( \pi \) of \( \text{GL}_n(F) \) is generic if \( \text{Hom}_{N_n(F)}(\pi, \tilde{\psi}) \neq \{0\} \) and this does not depend on the choice of \( \psi \). Genericity can be read on the Langlands parameter from [Zel80, Theorem 9.7] (one way to state it is that LLC(\( \phi \)) is generic if and only if the adjoint L factor of \( \phi \) is holomorphic at \( s = 1 \)). From this one easily deduces the direct implications of the following proposition, the converse implications being special cases of [MS20, Theorem 9.1].

Proposition 2.1. Let \( \pi \) be an irreducible representation of \( \text{GL}_n(F) \). If \( \text{BC}^{\text{LLC}}_E(\pi) \) is generic, then \( \pi \) is generic, and conversely if \( \pi \) is generic unitary, then \( \text{BC}^{\text{LLC}}_E(\pi) \) is generic (unitary).
Let $\tau$ be an irreducible representation of $\text{GL}_n(F)$. If $\text{Art}_F(\tau)$ is generic, then $\tau$ is generic, and conversely if $\tau$ is generic unitary, then $\text{Art}_F(\tau)$ is generic (unitary).

We denote by $\text{GL}_n(F)$ the double cover of $\text{GL}_n(F)$ defined for example in [Kap17, Section 2.1]. Following [Kap17] we call a map $\gamma : F^* \to C^*$ a pseudo-character if it satisfies $\gamma(xy) = \gamma(x)\gamma(y)(x, y)_{F^*}$ for all $x$ and $y$ in $F^*$, where $( , )_2$ is the Hilbert symbol of $F^*$. For $\gamma$ a pseudo-character of $F^*$ we denote by $\theta_\gamma$ the Kazhdan-Patterson exceptional representation of $\text{GL}_n(F)$ denoted by $\theta_{1,\gamma}$ in [Kap17, Section 2.5]. We say that an irreducible representation $\pi$ of $\text{GL}_n(F)$ is $F$-distinguished if there exist pseudo-characters $\gamma$ and $\gamma'$ of $F^*$ such that $\text{Hom}_{\text{GL}_n(F)}(\theta_{1,\gamma} \otimes \theta_{1,\gamma'}, \pi') \neq \{0\}$.

When $n$ is even, we denote by $S_n(F)$ the Shalika subgroup of $\text{GL}_n(F)$ consisting of matrices of the form $s(g, x) = \text{diag}(g, g) \left( \begin{array}{cc} I_{n/2} & x \\ & I_{n/2} \end{array} \right)$ for $g \in \text{GL}_{n/2}(F)$ and $x \in M_{n/2}(F)$, and for $\psi$ a non trivial character of $F$, we denote by $\Psi$ the character of $S_n(F)$ defined by $\Psi(s(g, x)) = \psi(\text{tr}(x))$.

We say that an irreducible representation $\pi$ of $\text{GL}_n(F)$ is $F$-distinguished if $n$ is even and $\text{Hom}_{S_n(F)}(\pi, \Psi) \neq \{0\}$. This does not depend on the choice of $\psi$.

Finally if $E/F$ is quadratic separable, identifying $\eta_{E/F}$ to the character of $F^*$ trivial on $N_{E/F}(E^*)$ via local class field theory, we say that an irreducible representation $\tau$ of $\text{GL}_n(E)$ is $1_{E/F}$-distinguished if $\text{Hom}_{\text{GL}_n(E)}(\tau, 1) \neq \{0\}$ and $\eta_{E/F}$-distinguished if $\text{Hom}_{\text{GL}_n(E)}(\tau, \eta_{E/F} \circ \det) \neq \{0\}$.

The following theorem follows from [Hen10], [Kab04], [AKT04], [AR05], [Mat11], [KR12], [Jo20], [Mat17], [Yam17], [Kap17]. Parts of it are known to hold when $F$ is of positive characteristic and odd residual characteristic ([AKM+21, Appendix A]).

**Theorem 2.2.** Suppose that $F$ has characteristic zero.

- Let $\pi = \text{LLC}(\phi_F)$ be a generic representation of $\text{GL}_n(F)$, then $\phi_F$ is symplectic if and only $\pi$ is $\Psi_F$-distinguished, whereas $\phi_F$ is orthogonal if and only if $\pi$ is $\Theta_F$-distinguished.

- Let $\tau = \text{LLC}(\phi_E)$ be a generic representation of $\text{GL}_n(E)$, then $\phi_E$ is conjugate-symplectic if and only $\tau$ is $\eta_{E/F}$-distinguished, whereas $\phi_E$ is conjugate-orthogonal if and only if $\tau$ is $1_{E/F}$-distinguished.

### 2.3 A reminder on epsilon factors

Let $K' / K$ be a finite separable extension of non Archimedean local fields. We denote by $\varpi_K$ a uniformizer of $K$ and by $P_K$ the maximal ideal of the ring of integers $O_K$ of $K$. If $\psi$ is a non trivial character of $K$, we denote by $\psi_{K'}$ the character $\psi \circ \text{tr}_{K'/K}$. We call the conductor of $\psi$ and write $d(\psi)$ for the smallest integer $d$ such that $\psi$ is trivial on $P_K^d$. When $K' / K$ is unramified, it follows from [Wei74, Chapter 8, Corollary 3] that

$$d(\psi_{K'}) = d(\psi).$$

Similarly if $\chi$ is a character of $W'_K$, identified by local class field theory with a character of $K^*$, we call the Artin conductor of $\chi$ the integer $a(\chi)$ equal to zero if $\chi$ is unramified, or equal to the smallest integer $a$ such that $\chi$ is trivial on $1 + P_K^a$ if $\chi$ is ramified. More generally one can define the Artin conductor $a(\phi)$ (which is an integer) of any representation $\phi$ of $W'_K$, see [Lat79, 3.4.5]
when \( \phi \) is a representation of \( W_K \) and \[GR10\] Section 2.2, (10)] in general. The Artin conductor is additive:

\[
a(\phi \oplus \phi') = a(\phi) + a(\phi')
\]

for \( \phi \) and \( \phi' \) in \( \text{Rep}(W'_K) \). If \( \phi \) is a representation of \( W'_K \), and \( \psi \) is a non trivial character of \( K \), we refer to \[Tat79\, 3.6.4\] and \[BH06\, 31.3\] or \[GR10\, Section 2.2\] for the definition of the root number \( \epsilon(1/2, \phi, \psi) \). One then defines the Langlands \( \lambda \)-constant:

\[
\lambda(K'/K, \psi) = \frac{\epsilon(1/2, \text{Ind}_{W'_K}^W(K'_K, \psi))}{\epsilon(1/2, 1_{W'_K}, \psi_{K'})}.
\]

For \( a \in K^* \), we set \( \psi_a = \psi(a \cdot) \). These constants enjoy the following list of properties, which we will freely use later in the paper.

1. \( \epsilon(1/2, \phi \oplus \phi', \psi) = \epsilon(1/2, \phi, \psi)\epsilon(1/2, \phi', \psi) \) where \( \phi' \) is another representation of \( W'_K \) (\[Tat79\, (3.4.2)]).

2. \( \epsilon(1/2, \phi, \psi_a) = \det(\phi(a))\epsilon(1/2, \phi, \psi) \) (\[Tat79\, (3.6.6)]).

3. \( \epsilon(1/2, \phi, \psi)^2 = \det(\phi)(-1) \) when \( \phi \) is self-dual (\[GR10\, Section 2.3, (11)]).

4. If \( d(\psi) = 0 \) and \( \mu \) is an unramified character of \( K^* \), it follows from \[GR10\, Section 2.3, (9)] that:

\[
\epsilon(1/2, \mu \otimes \phi, \psi) = \mu(\omega^a(\phi))\epsilon(1/2, \phi, \psi).
\]

5. If \( K'/K \) is quadratic with \( K \) of characteristic not 2, \( \delta \in \ker(\text{tr}_{K'/K}) - \{0\} \), and \( \phi \) is a \( K'/K \)-orthogonal representation of \( W'_K \), then by \[GGP12\, Proposition 5.2\] (generalizing \[FQ73\, Theorem 3\]):

\[
\epsilon(1/2, \phi, \psi_{K'}) = \det(\phi)(\delta).
\]

6. If \( \phi_{K'} \) is an \( r \)-dimensional representation of \( W'_K \), then

\[
\epsilon(1/2, \text{Ind}_{W'_K}^W(\phi_{K'}), \psi) = \lambda(K'/K, \psi)^r \epsilon(1/2, \phi_{K'}, \psi_{K'})
\]

(\[BH06\, (30.4.2)]). When applied to a \( K'/K \) quadratic and \( \phi_{K'} = \text{Res}_{W'_K}^W(\phi) \) for \( \phi \) a representation of \( W'_K \), one gets

\[
\epsilon(1/2, \phi, \psi)\epsilon(1/2, \eta_{K'/K} \otimes \phi, \psi) = \lambda(K'/K, \psi)^r \epsilon(1/2, \text{Res}_{W'_K}^W(\phi), \psi_{K'})
\]

7. If \( K'/K \) is unramified with \([K'/K] = n\):

\[
\lambda(K'/K, \psi) = (-1)^{d(\psi)(n-1)}
\]

(for example \[Moy86\] and \[2\], together with Equation (2)). In particular if \( d(\psi) = 0 \) then

\[
\lambda(K'/K, \psi) = 1.
\]
3 Distinction, base change, and automorphic induction

From now on $E/F$ is a separable quadratic extension of non Archimedean local fields. Our main result is the following proposition, and we notice that half of its first point is \cite[Lemma 3.5. (i)]{GGP12}.

Proposition 3.1. 1. Let $\phi_E$ be a semi-simple representation of $W'_E$ which is either $\varepsilon$-self-dual or $(E/F, \varepsilon)$-dual, then $\text{Ind}_{W'_E}^{W_F}(\phi_E)$ is $\varepsilon$-self-dual.

2. Conversely if $\phi_E$ is irreducible and $\text{Ind}_{W'_E}^{W_F}(\phi_E)$ is $\varepsilon$-self-dual:

(a) if $\text{Ind}_{W'_E}^{W_F}(\phi_E)$ is irreducible, i.e. $\phi_E^\ast \neq \phi_E$, then either $\phi_E$ is $\varepsilon$-self-dual or $(E/F, \varepsilon)$-dual, but not both together,

(b) if $\text{Ind}_{W'_E}^{W_F}(\phi_E)$ is reducible, i.e. $\phi_E^\ast \simeq \phi_E$, then $\phi_E$ is both $\varepsilon$-self-dual and $(E/F, \varepsilon)$-dual.

3. Let $\phi_F$ be a semi-simple representation of $W'_F$ which is $\varepsilon$-self-dual, then $\text{Res}_{W'_E}^{W'_F}(\phi_F)$ is $\varepsilon$-self-dual and $(E/F, \varepsilon)$-dual.

4. Conversely, if $\phi_F$ is irreducible and $\text{Res}_{W'_E}^{W'_F}(\phi_F)$ is $\varepsilon$-self-dual then $\phi_F$ is also $\varepsilon$-self-dual.

Proof. 1. First suppose that $B_E$ is a $(E/F, \varepsilon)$-bilinear form on $\phi_E$. Write an element $v$ (resp. $v'$) in $\text{Ind}_{W'_E}^{W_F}(\phi_E)$ under the form $v = x + s^{-1}y$ (resp. $v' = x' + s^{-1}y'$) for $x$, $x'$, $y$, $y'$ in $\phi_E$, and set

$$B_F(v, v') = B_E(x, y') + \varepsilon B_E(x', y).$$

Then $B_F$ is $W'_E$-invariant because $B_E$ is $(W'_E, \varepsilon)$-conjugate (it is non-degenerate because so is $B_E$). Finally

$$B_F(s.v, s.v') = B_E(y, s^2.x') + \varepsilon B_E(y', s^2.x) = \varepsilon B_E(x', y) + B_E(x, y') = B_F(v, v').$$

Similarly if $B_E$ is $(W'_E, \varepsilon)$-bilinear, then one checks that

$$B_F(x + s^{-1}y, x' + s^{-1}.y') = B_E(x, x') + B_E(y, y')$$

defines a $(W'_E, \varepsilon)$-bilinear form on $\phi_F$.

2. Suppose that $\phi_E$ is irreducible and that $\text{Ind}_{W'_E}^{W_F}(\phi_E)$ is $\varepsilon$-self-dual with $(W'_F, \varepsilon)$-bilinear form $B_F$.

(a) If $\phi_E^\ast \neq \phi_E$, because $\text{Ind}_{W'_E}^{W_F}(\phi_E)$ is self-dual then either $\phi_E$ is self-dual, or $\phi_E^\ast \simeq \phi_E^\ast$ but not both together. In the first case, say that $\phi_E$ is $\varepsilon'$-self-dual, then so is $\text{Ind}_{W'_E}^{W_F}(\phi_E)$ by \cite{1} but then $\varepsilon' = \varepsilon$ by irreducibility of $\text{Ind}_{W'_E}^{W_F}(\phi_E)$. If $\phi_E^\ast \simeq \phi_E^\ast$ we conclude in a similar manner.

(b) If $\phi_E^\ast \simeq \phi_E$ then $\text{Ind}_{W'_E}^{W_F}(\phi_E) \simeq \phi \otimes \eta_{E/F} \otimes \phi$ for $\phi$ extending $\phi_E$, and $\phi \neq \eta_{E/F} \otimes \phi$. Because $\phi \neq \eta_{E/F} \otimes \phi$ there are two disjoint cases. The first is when $\phi$ is self-dual, in which case $\phi \neq \eta_{E/F} \otimes \phi$ and $B_F$ restricts non trivially to $\phi \otimes \phi$ and $\eta_{E/F} \otimes \phi \otimes \eta_{E/F} \otimes \phi$. Then $\phi_E$ is $\varepsilon$-dual and $(\varepsilon, s)$-dual by \cite{3}. Otherwise $\phi^\ast \simeq \eta_{E/F} \otimes \phi$ and $B_F$ is zero on
Corollary 3.1. \( GL \) its extension to of the results recalled in Section 2.2. For this we denote by \( \sigma \). We suppose that \( \phi \) 3. Let \( B_F \) be a \((W_F, \varepsilon)\)-bilinear form on \( \phi_F \), then it remains a \((W_F, \varepsilon)\)-bilinear on \( \text{Res}^W_{W_E}(\phi_F) \), and on the other hand \( B_E(x, y) = B_F(x, s^{-1}y) \) is an \((E/F, \varepsilon)\)-bilinear form on \( \text{Res}^W_{W_E}(\phi_F) \).

4. We suppose that \( \phi_F \) is irreducible and that \( \text{Res}^W_{W_E}(\phi_F) \) is \( \varepsilon \)-self-dual and also \((E/F, \varepsilon)\)-dual. There are two cases to consider.

First if \( \text{Res}^W_{W_E}(\phi_F) \) is irreducible, then denote by \( B_E \) the \((W', \varepsilon)\)-bilinear form on \( \text{Res}^W_{W_E}(\phi_F) \). Now set \( D_E(x, y) = B_E(x, s^{-1}y) \) for \( x, y \in \text{Res}^W_{W_E}(\phi_F) \). Clearly \( D_E \) is \( E/F \)-bilinear, but by irreducibility \( \text{Res}^W_{W_E}(\phi_F) \) affords at most one such form up to scalar, hence \( D_E \) must be \((E/F, \varepsilon)\)-bilinear. This implies that for \( x \) and \( y \) in \( \text{Res}^W_{W_E}(\phi_F) \) one has \( B_E(s.x, s.y) = D_E(s.x, s^2.y) = \varepsilon D_E(y, s.x) = \varepsilon B_E(y, x) = B_E(x, y) \).

All in all, when \( \text{Res}^W_{W_E}(\phi_F) \) is irreducible we deduce that \( B_E \) is in fact \( W_F \)-invariant hence that \( \phi_F \) is \( \varepsilon \)-self-dual.

It remains to treat the case where \( \text{Res}^W_{W_E}(\phi_F) \) is reducible. In this case it is of the form \( \phi_E \otimes s^{-1}.\phi_E \) where \( \phi_E \) is an irreducible of \( W_E \) such that \( \phi_E \not\equiv \phi_E \) and \( \phi_F = \text{Ind}^W_{W_E}(\phi_F) \).

First because \( \text{Res}^W_{W_E}(\phi_F) \) is \( \varepsilon \)-self-dual, then the \((W', \varepsilon)\)-bilinear form \( B_E \) on \( \text{Res}^W_{W_E}(\phi_F) \) either induces an isomorphism \( \phi_E \equiv \phi_E \) or \( \phi_E \equiv s^{-1}.\phi_E \) for \( B_E \). Similarly the \((E/F, \varepsilon)\)-bilinear form \( C_E \) on \( \text{Res}^W_{W_E}(\phi_F) \) either induces an isomorphism \( \phi_E \equiv \phi_E \) or \( \phi_E \equiv s^{-1}.\phi_E \) for \( C_E \). Suppose that \( B_E \) induces an isomorphism \( \phi_E \equiv \phi_E \), then one must have \( \phi_E \equiv s^{-1}.\phi_E \) for \( C_E \) because \( \phi_E \not\equiv \phi_E \equiv s^{-1}.\phi_E \). This implies that \( C_E \) induces an \((E/F, \varepsilon)\)-bilinear form on \( \phi_F \) and by point 1 we deduce that \( \phi_F \) is \( \varepsilon \)-self-dual. On the other hand if \( \phi_F \equiv s^{-1}.\phi_F \) for \( B_E \) then \( B_E \) induces an \((W', \varepsilon)\)-bilinear form on \( \phi_F \) and \( \phi_F \) is \( \varepsilon \)-self-dual again by point 1.

\[
\phi \times \phi \text{ and } \eta_{E/F} \otimes \phi \times \eta_{E/F} \otimes \phi. \text{ In this case there is up to scaling a unique } W'_F \text{-invariant bilinear form on } \text{Ind}^W_{W_E}(\phi_E), \text{ namely } B_F. \text{ Because } \phi_E \equiv \phi_E \text{ (by restricting the relation } \phi \equiv \eta_{E/F} \otimes \phi \text{ to } W'_E), \text{ } \phi_E \text{ must be } \varepsilon'-\text{self-dual, hence } \text{Ind}^W_{W_E}(\phi_E) \text{ as well by 1 but then we have } \varepsilon' = \varepsilon \text{ by multiplicity one of } W'_F \text{-invariant bilinear form on } \text{Ind}^W_{W_E}(\phi_E). \text{ Moreover because } \phi'_E = \phi_E \text{ the parameter } \phi_E \text{ is also } (\varepsilon'', s)-\text{self-dual and by 1 again we deduce that } \phi'' = \varepsilon. \]

3. Let \( B_F \) be a \((W', \varepsilon)\)-bilinear form on \( \phi_F \), then it remains a \((W', \varepsilon)\)-bilinear on \( \text{Res}^W_{W_E}(\phi_F) \), and on the other hand \( B_E(x, y) = B_F(x, s^{-1}y) \) is an \((E/F, \varepsilon)\)-bilinear form on \( \text{Res}^W_{W_E}(\phi_F) \).

Supposing that \( F \) has characteristic zero, we translate Proposition 2.1 via the LLC, in view of the results recalled in Section 2.2. For this we denote by \( \sigma \) the Galois conjugation of \( E/F \) and its extension to \( \text{GL}_n(E) \), et set \( \tau^2 = \tau \sigma \sigma \) for any representation of \( \text{GL}_n(E) \).

Corollary 3.1. 1. Let \( \tau \) be an irreducible representation of \( \text{GL}_n(E) \) such that \( \Lambda(E)(\tau) \) is generic \((\text{for example } \tau \text{ generic unitary}) \). If \( \tau \) is either \( \Theta_E \text{-distinguished or } 1_{E/F} \text{-distinguished, then } \Lambda(E)(\tau) \) is \( \Theta_F \text{-distinguished, whereas if } \tau \) is either \( \Psi_E \text{-distinguished or } \eta_{E/F} \text{-distinguished, then } \Lambda(E)(\tau) \) is \( \Theta_F \text{-distinguished. } \)
2. Conversely if $\tau$ is a discrete series representation $\text{GL}_n(E)$.

(a) Suppose that $\text{AI}_E^F(\tau)$ is $\Psi_F$-distinguished:
\begin{itemize}
  \item[i.] if $\text{AI}_E^F(\tau)$ is a discrete series, i.e. if $\tau^\sigma \neq \tau$, then either $\tau$ is $\Psi_E$-distinguished or $\eta_{E/F}$-distinguished, but not both together,
  \item[ii.] if $\text{AI}_E^F(\tau)$ is not a discrete series, i.e. $\tau^\sigma \equiv \tau$, then $\tau$ is both $\Psi_E$-distinguished and $\eta_{E/F}$-distinguished.
\end{itemize}

(b) Suppose that $\text{AI}_E^F(\tau)$ is $\Theta_F$-distinguished:
\begin{itemize}
  \item[i.] if $\text{AI}_E^F(\tau)$ is a discrete series, i.e. $\tau^\sigma \neq \tau$, then either $\tau$ is $\Theta_F$-distinguished or $1_{E/F}$-distinguished, but not both together,
  \item[ii.] if $\text{AI}_E^F(\tau)$ is reducible, i.e. $\tau^\sigma \equiv \tau$, then $\tau$ is both $\Theta_E$-distinguished and $1_{E/F}$-distinguished.
\end{itemize}

3. Let $\pi$ be an irreducible representation of $\text{GL}_n(F)$ such that $BC_E^F(\pi)$ is generic (for example $\pi$ generic unitary). If $\pi$ is $\Theta_F$-distinguished, then $BC_E^F(\pi)$ is $\Theta_E$-distinguished and $1_{E/F}$-distinguished, whereas if $\pi$ is $\Psi_F$-distinguished, then $BC_E^F(\pi)$ is $\Psi_E$-distinguished and $\eta_{E/F}$-distinguished.

4. Conversely suppose that $\pi$ is a discrete series. If $BC_E^F(\pi)$ is $\Theta_E$-distinguished and $1_{E/F}$-distinguished, then $\pi$ is $\Theta_F$-distinguished, whereas if $BC_E^F(\pi)$ is $\Psi_E$-distinguished and $\eta_{E/F}$-distinguished, then $\pi$ is $\Psi_F$-distinguished.

4 Parity of the Artin conductor of self-dual representations

In this section $F$ is again a non Archimedean local field. First, using [GGP12, Proposition 5.2] (which is itself a quick but non trivial consequence of a difficult result of Deligne [Del76] on root numbers of orthogonal representations), we quickly recover in odd residual characteristic from Proposition 3.1 and Section 2.3 the following result due to Serre [Ser71] (the result in question also holds in even residual characteristic by [Ser71]). In other words we show that the result of [Del76] implies that of [Ser71] for non Archimedean local fields of odd residual characteristic.

**Corollary 4.1** (of Proposition 3.1 and Ser71). Let $\phi$ be an orthogonal representation of $W_F^\sigma$. We have the following congruence of Artin conductors: $a(\phi) = a(\det(\phi))[2]$.

**Proof.** As we said the result is true for $F$ of any residual characteristic, and we recover it in this proof for $F$ of residual characteristic different from $2$. Let $E$ be the unramified quadratic extension of $F$, and take $\psi$ a character of $F$ of conductor zero. We have according to Section 2.3 Points 6 and 7

$$\epsilon(1/2, \text{Res}_{\psi_E} W_{\phi}^\sigma(\phi)) = \epsilon(1/2, \phi, \psi)\epsilon(1/2, \eta_{E/F} \otimes \phi, \psi).$$

(2)

Now denoting by $q$ the residual cardinality of $F$, let $u$ be an element of order $q^{1/2}$ in $E^*$, so that $\delta := u^{(q+1)/2}$ does not belong to $F$ but $\Delta := \delta^2$ belongs to $F$. Note that $\Delta$ is a generator the copy of $O_F^*/1 + P_F$ inside $F^*$. Then $\epsilon(1/2, \text{Res}_{\psi_E} W_{\phi}^\sigma(\phi), \psi_E^{-1}(\phi)) = 1$ by Proposition 3.1 and Section 2.3 Point 5 hence

$$\epsilon(1/2, \text{Res}_{\psi_E} W_{\phi}^\sigma(\phi), \psi_E) = \det(\text{Res}_{\psi_E} W_{\phi}^\sigma(\phi)(\delta)) = \det(\phi)(N_{E/F}(\delta)) = \det(\phi)(\Delta)$$
thanks to Section 2.3 Point 2. Now observe that \( \det(\phi) \) is quadratic as \( \phi \) is self-dual, but because \( q \) is odd it is trivial on \( 1 + P_F \), hence it has conductor 0 or 1, and it is of conductor zero if and only if \( \det(\phi)(\Delta) = 1 \), hence \( \det(\phi)(\Delta) = (-1)^{\alpha(\det(\phi))} \), so

\[
\epsilon(1/2, \text{Res}_{W'_E}^W(\phi), \psi_E) = (-1)^{\alpha(\det(\phi))} \det(\phi)(-1).
\]

Now \( \epsilon(1/2, \eta_{E/F} \otimes \phi, \psi) = (-1)^{\alpha(\phi)} \epsilon(1/2, \phi, \psi) \) thanks to Section 2.3 Point 1 hence Section 2.3 Point 3 implies the following:

\[
\epsilon(1/2, \phi, \psi) \epsilon(1/2, \eta_{E/F} \otimes \phi, \psi) = (-1)^{\alpha(\phi)} \epsilon(1/2, \phi, \psi)^2 = (-1)^{\alpha(\phi)} \det(\phi)(-1).
\]

The result now follows from Equation (2). \( \square \)

One can legitimately ask about the parity of the Artin conductor of symplectic representations of \( W'_E \). The answer seems much more complicated, and one way to adress it is via the LLC, using the so called Prasad and Takloo-Bighash conjecture, which is now a theorem when \( F \) has characteristic zero and residual characteristic different from 2 ([Xue21], [Sé20], [Suz21], [SX20]). To this end we recall that for \( E/F \) a separable quadratic extension, then \( \mathcal{M}_F(2n) \) embeds uniquely up to \( \text{GL}_{2n}(F) \)-conjugacy into \( \mathcal{M}_E(F) \) as an \( F \)-subalgebra by the Skolem-Noether theorem. We fix such an embedding, which in turn gives rise to an embedding of \( \text{GL}_n(E) \) into \( \text{GL}_{2n}(F) \). We then say that an irreducible representation \( \pi \) of \( \text{GL}_{2n}(F) \) is \( 1^{E/F} \)-distinguished if and only if \( \text{Hom}_{\text{GL}_n(E)}(\pi, 1) \neq \{0\} \). We recall the following theorem, which is a consequence of one part of the Prasad and Takloo-Bighash conjecture.

**Theorem 4.1** ([Xue21], [Sé20], [SX20]). Suppose that \( F \) has characteristic zero and residual characteristic different from 2. If \( \phi \) is an irreducible symplectic representation of \( W'_{E/F} \) of dimension \( 2n \), then

\[
\epsilon(1/2, \phi \otimes \text{Ind}_{W'_{E/F}}^W(\phi)) = \eta_{E/F}(-1)^n
\]

if LLC(\( \phi \)) is \( 1^{E/F} \)-distinguished and

\[
\epsilon(1/2, \phi \otimes \text{Ind}_{W'_{E/F}}^W(\phi)) = -\eta_{E/F}(-1)^n
\]

otherwise.

**Remark 4.1.** In the statement above, as the determinant of a symplectic representation is equal to 1, we suppressed the dependence of the root number \( \epsilon(1/2, \phi \otimes \text{Ind}_{W'_{E/F}}^W(\phi), \psi) \) on the non-trivial additive character \( \psi \) of \( F \).

As an immediate corollary we obtain the following result on the parity of Artin conductors of symplectic representations.

**Corollary 4.2.** Suppose that \( F \) has characteristic zero and residual characteristic different from 2, denote by \( E \) the unramified quadratic extension of \( F \), and let \( \phi \) be an irreducible symplectic representation of \( W'_{E/F} \) of dimension \( 2n \). Then \( \alpha(\phi) \) is even if and only if LLC(\( \phi \)) is \( 1^{E/F} \)-distinguished.

**Proof.** It easily follows, along the lines of the proof of Corollary 4.1 from Theorem 4.1 noting that \( \eta_{E/F}(-1) = 1 \). \( \square \)
Remark 4.2. A general symplectic representation $\phi$ of $W'_F$ being a direct sum of the form $\oplus_{i=1}^r \phi_i \oplus \phi_j$ for $\phi_i$ irreducible symplectic and $\phi_j$ irreducible, we deduce the parity of $a(\phi)$ from Corollary 4.2 and such a decomposition. Namely, by Corollary 4.1 $a(\phi_j \oplus \phi_j) \equiv 0 \mod 2$. Hence setting $\epsilon_i \in \{\pm 1\}$ being equal to 1 if and only if LLC($\phi_i$) is $1^E/F$-distinguished, we deduce by additivity of the Artin conductor that $(-1)^{a(\phi)} = \prod_{i=1}^r \epsilon_i$.

Remark 4.3. Looking at it from another angle, one sees that a symplectic discrete series representation of $GL_{2n}(F)$ is $1^E/F$-distinguished ($E/F$ unramified) if and only if it has even conductor.

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