Internal mode mechanism for collective energy transport in extended systems

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We study directed energy transport in homogeneous nonlinear extended systems in the presence of homogeneous ac forces and dissipation. We show that the mechanism responsible for unidirectional motion of topological excitations is the coupling of their internal and translation degrees of freedom. Our results lead to a selection rule for the existence of such motion based on resonances that explains earlier symmetry analysis of this phenomenon. The direction of motion is found to depend both on the initial and the relative phases of the two harmonic drivings, even in the presence of noise.

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One intriguing phenomenon that is receiving much attention recently is net directed motion induced by zero average forces. Originally motivated by stochastic models of biomolecular (brownian) motors [1], deterministic ratchetlike systems [2, 3] are being intensively studied, chiefly because of their many potential technological applications [4]. Many such models consist of one or two particles on a periodic, asymmetric potential and a periodic force (rocking ratchet [1]). Later, the investigation was generalized to systems with many interacting particles, from noisy soliton-bearing systems [2, 6] to other spatially extended (stochastic and deterministic, overdamped and underdamped) systems, both theoretically [5] and from a more applied viewpoint.

Among this class of problems, net transport in homogeneous extended systems driven by homogeneous ac forces is particularly interesting. A paradigmatic example is the ac driven, damped sine-Gordon (sG) equation:

$$\phi_{tt} - \phi_{xx} + \sin(\phi) = -\beta \phi_t + f(t). \quad (1)$$

A symmetry analysis, suggested for one-particle systems in [5] and extended to this problem [2, 10], indicated that a directed energy current appeared if \( f(t) \) broke the symmetry \( f(t) = -f(t + T/2) \), \( T \) being the period of the external driving. One such choice is \( f(t) \equiv \epsilon_1 \sin(\delta t + \delta_0) + \epsilon_2 \sin(m \delta t + \delta_0 + \theta) \) [2, 10] with \( \delta_0 = \pi/2 \), a case for which numerical simulations of the sG equation confirmed the symmetry analysis results. In what follows, we will refer to \( \delta_0 \) as the initial phase and to \( \theta \) as the relative phase. Transport required a nonzero topological charge, implying the existence of sG solitons (kinks) in the system. In this respect, we stress that kink-mediated transport is impossible with only one harmonic for any value of the damping coefficient \( \beta \) [11]. It was argued in [10] that the observed rectification arises from the nonadiabatic excitation of internal kink modes and their interaction with the translational kink motion. This conjecture had no rigorous support; rather, it was based on plots of sG soliton evolution and on the failure of a collective coordinate (CC) approach [12] with one degree of freedom, which assumed that sG solitons behave like rigid particles. An attempt to include the width degree of freedom has been recently presented in [13], where it was concluded that the directed energy current vanishes unless the width of the kink, \( l(t) \), is a dynamical variable. However, this condition is only a necessary one: \( l(t) \) is a dynamical variable in the one-harmonic case but the kink velocity is zero for any value of the damping as already mentioned [11]. Another point not accounted for in [13] is the connection between the internal mode mechanism and the symmetry analysis, which also prohibits motion in other cases where \( l(t) \) is a dynamical variable. Therefore, the reasons for the phenomena observed in [2, 10] remained largely obscure.

In this letter, a different CC approach allows us to identify the mechanism through which the width oscillation drives the kink and its relation with the symmetry conditions. Furthermore, our theory predicts, and numerical simulations of Eq. (11) confirm, that the direction of motion depends on the initial phase of the driving, even in the presence of additive noise. Our CC theory is based on an Ansatz, proposed in [14], for the perturbed kink depending on two CC, \( X(t) \) and \( l(t) \) (respectively, position and width of the kink). It is not difficult to show [14, 15, 16] that the dynamics of these two CC is given by

$$\frac{dP}{dt} = -\beta P - q f(t), \quad (2)$$

$$\dot{l}^2 - 2\dot{l} - 2\beta \dot{l} = \Omega_R l^2 \left[ 1 + \frac{P^2}{M_0^2} \right] - \frac{1}{\alpha}, \quad (3)$$

where the momentum \( P(t) = M_0 \dot{X}(l(t)), \quad \Omega_R = 1/(\sqrt{\alpha} l_0) \) with \( \alpha = \pi^2/12 \) is the so-called Rice’s frequency, and \( M_0 = 8, q = 2\pi \) and \( l_0 = 1 \) are, respectively, the dimensionless kink mass, topological charge and unperturbed width. Equation (2) can be solved exactly,
and in the long time limit \((t \gg \beta^{-1})\) yields

\[ P(t) = -\sqrt{\epsilon} [a_1 \sin(\delta t + \delta_0 - \chi_1) + a_2 \sin(m \delta t + \delta_0 + \theta - \chi_2)], \]

where \(\epsilon\) is merely a rescaling parameter in the perturbation expansion, to be determined later; \(\chi_1 = \arctan(\delta/\beta)\), \(\chi_2 = \arctan(m \delta/\beta)\), \(a_1 = q e_1/\sqrt{\epsilon(\beta^2 + \delta^2)}\), and \(a_2 = q e_2/\sqrt{\epsilon(\beta^2 + m^2 \delta^2)}\). As we are interested in the damped \((\beta \neq 0)\) case and Eq. 43 cannot be solved in that case, we will study it by a perturbative expansion, \(l(t) = l_0 + \epsilon l_1(t) + \epsilon^2 l_2(t) + \cdots\). At order \(O(\epsilon)\), we obtain

\[ \dot{l}_1(t) + \beta \dot{l}_1(t) + \Omega_R^2 l_1(t) = -\Omega_R^2 P(t) l_0/2 \epsilon M_0^2. \tag{4} \]

The key point is that, by substituting the expression of \(P(t)\) into Eq. 44, we see that the equation for \(l_1(t)\) contains harmonics of frequencies \(2\delta, 2m\delta\) and \((m \pm 1)\beta\), i.e.,

\[ \dot{l}_1(t) + \beta l_1(t) + \Omega_R^2 l_1(t) = A_1 + A_2 \cos(2\delta t + 2\delta_0 - 2\chi_1) + A_3 \cos(2m \delta t + 2\delta_0 + 2\theta - 2\chi_2) + A_4 \cos[(m-1)\delta t + \theta - (\chi_2 - \chi_1)] - A_4 \cos[(m+1)\delta t + 2\delta_0 + \theta - (\chi_2 + \chi_1)], \]

where \(A_1 = -A_2 - A_3\), and \(A_2 = \Omega_R a_2^2/4\sqrt{\epsilon} M_0^2\), \(A_3 = \Omega_R a_2^2/4\sqrt{\epsilon} M_0^2\), \(A_4 = -\Omega_R a_1 a_2/2\sqrt{\epsilon} M_0^2\). After transients elapse, we find

\[ l_1(t) = \frac{A_1}{\Omega_R^2} + \frac{A_2 \sin(2\delta t + 2\delta_0 - 2\chi_1 + \theta_2)}{\sqrt{\Omega_R^2 - 4\beta^2 \delta^2}} + \frac{A_3 \sin(2m \delta t + 2\delta_0 + 2\theta - 2\chi_2 + \theta_{2m})}{\sqrt{\Omega_R^2 - 4m^2 \beta^2 \delta^2}} + \frac{A_4 \sin[(m-1)\delta t + \theta - (\chi_2 - \chi_1) + \theta_{m-1}]}{\sqrt{\Omega_R^2 - (m-1)^2 \beta^2 \delta^2}} - \frac{A_4 \sin[(m+1)\delta t + 2\delta_0 + \theta - (\chi_2 + \chi_1) + \theta_{m+1}]}{\sqrt{\Omega_R^2 - (m+1)^2 \beta^2 \delta^2}}, \tag{5} \]

where \(\theta_m = \arctan \left( (\Omega_R^2 - m^2 \beta^2 \delta^2) / m \beta \delta \right)\). A cumbersome but otherwise trivial calculation yields the harmonics contained in \(l_2(t)\), collected in Table I.

Next, we need to compute the average velocity over one period \(T = 2\pi/\delta\). In the CC approach, we use the definition of the momentum and find

\[ \langle \dot{X}(t) \rangle = \frac{1}{T} \int_0^T \frac{P(t) l(t)}{M_0 l_0} \, dt. \tag{6} \]

From Eq. (7) we see that for \(\epsilon\) to be small the prefactor on the rhs has to be much smaller than 1. A definite, verifiable prediction from this asymptotic expression is the existence of a nonzero velocity for \(m = 2\), with a sinusoidal dependence on \(\delta_0\) and \(\theta\). This means that the velocity depends on both the initial and the relative phases; indeed, by letting \(\delta_0 \equiv \delta \theta_0\) in Eq. 11 and changing variables to \(\tau' = t + \delta_0\) it can be immediately seen that an initial phase \(\delta_0\) is equivalent to a relative phase \(\theta' = \theta - (m - 1)\delta_0\) for a kink with its center shifted to \(x_0 + V \tau_0\). The dependence of the velocity on \(\theta\) agrees with (and explains) Eqs. 12, 17, whereas the dependence on \(\delta_0\) is a totally new result. Nevertheless, these analytical results as well as the numerical simulations we present below strongly support the present conclusion.

For the case \(m = 3\), the average velocity is zero at all orders, a result confirmed by direct numerical simulation of the full sG equation as we will see below. The reason can be understood by looking at Table II. For \(m = 3\), the frequencies of the ac force (or the momentum) are odd harmonics \((\delta, 3\delta)\), whereas the width of the kink oscillates only with even harmonics \((2n\delta, n \in \mathbb{N})\). This leads us to our main conclusion, namely the mechanism for the appearance of net motion and the corresponding
are only qualitatively correct unless the integral is nonzero only if \( l(t) \) is in turn inversely proportional to \( l(t) \). This we do by using the Strauss-Vázquez scheme \([18]\), on systems of length \( L = 100, 1000 \), with steps \( \Delta t = 0.01 \), \( \Delta x = 0.1 \), free boundary conditions, and a kink at rest as initial condition. Instead of the perturbative expressions (which are only qualitatively correct unless \( \epsilon << 1 \)), to assess the validity of our theory we numerically integrate Eq. \(3\) and the equation for the velocity obtained from the expression of \( P(t) \), Eq. \(2\), with a fourth-order Runge-Kutta method. Our main results are shown in Figs. 1-4.

TABLE I: Harmonic content of the first contributions to the perturbative expansion of \( l(t) \).

| harmonic | \( l_1 \) | \( l_2 \) |
|---------|----------|----------|
| \( m \) | \( \delta, 2m\delta, (m \pm 1)\delta \) | \( \delta, 4\delta, 2m\delta, 4m\delta, (m \pm 1)\delta \) |
| 2       | \( \delta, 2\delta, 3\delta, 4\delta \) | \( \delta, 2\delta, 3\delta, 4\delta, 5\delta, 6\delta, 7\delta, 8\delta \) |
| 3       | \( 2\delta, 3\delta, 6\delta \) | \( 2\delta, 3\delta, 6\delta, 8\delta, 10\delta, 12\delta \) |

FIG. 1: Dependence of the kink velocity on the initial phase. Parameters are \( \epsilon_1 = \epsilon_2 = 0.2 \), \( \beta = 0.05 \), \( \delta = 0.1 \). Relative phase \( \theta = \pi/2 \): solid line, CC theory; filled circles, simulation results. Relative phase \( \theta = 0 \): dashed line, CC theory; squares, simulation results.

These predictions from the CC approximation must be confirmed by numerical solution of the full partial differential equation \([1]\). This we do by using the Strauss-Vázquez scheme \([18]\), on systems of length of \( L = 100, 1000 \), with steps \( \Delta t = 0.01 \), \( \Delta x = 0.1 \), free boundary conditions, and a kink at rest as initial condition. Instead of the perturbative expressions (which are only qualitatively correct unless \( \epsilon << 1 \)), to assess the validity of our theory we numerically integrate Eq. \(3\) and the equation for the velocity obtained from the expression of \( P(t) \), Eq. \(2\), with a fourth-order Runge-Kutta method. Our main results are shown in Figs. 1-4.

they fully confirm the accuracy, even quantitative, of our approach. Figure 1 exhibits clearly the sinusoidal dependence of the velocity as a function of the initial phase. The dependence on \( \theta \) is also seen as a simple shift when changing from \( \theta = 0 \) to \( \theta = \pi/2 \). The agreement with the CC results is perfect. As a further check of the robustness of this dependence, following \([8]\) we have simulated Eq. \(1\) with an additional additive gaussian white noise term with variance \( D \). While one could in principle think that this noise would suppress the initial phase dependence, Fig. 2 shows that the opposite is the case: the noise enhances the dependence on the initial phase, increasing the maximum values of the velocity while keeping the same general sinusoidal dependence and the location of the zeros. It is tempting to conclude from this plot that the noise, at least if it is not very large \((D << 1)\), assists the process of energy transfer between the width and the translation degrees of freedom, activating it. Finally, Fig. 3 makes it clear that our main result, namely the interpretation of the physics of the problem, is indeed true, by showing the harmonic content of \( l(t) \) for \( m = 2 \) and 3. In this case, the agreement between our CC theory and the full numerical simulation of Eq. \(1\) is indeed impressive, and validates firmly our resonance criterion for net kink motion. It is important to stress that the present theory does not apply to the net motion found for \( m = 3 \) in \([10]\). We have confirmed their result in our simulations, which allowed us to realize that this is an altogether different phenomenon: first, it appears only above a (moderately large, \( \epsilon_i \geq 0.4 \)) threshold amplitude, and second, it is induced by the kink wings, which are highly distorted in the process yielding the CC picture inappropriate (even kink-antikink pairs are created).

In conclusion, we have found that the symmetry conditions set forth in \([8, 14]\) have their physical origin in the mechanism of the directed motion: the indirect action of the force through the coupling of the translational and width degrees of freedom. To make net motion possible, this indirect driving has to resonate with the available frequencies for the width. This interpretation does not contradict the non-existence of internal modes in sG kinks, shown in \([10]\), because external forces can induce,
via excitation of certain phonons, behavior similar to the one expected from an intrinsic internal mode \cite{14,20}. The fact that a force with only one harmonic would not drive the damped sG kink \cite{11}, and that two harmonics are needed to simultaneously excite the width oscillations and induce net motion fits nicely in this picture. On the other hand, this point rises the question as to the generality of our results, in view of the fact that most kink-bearing systems do have internal modes. To answer this question, we have studied the same problem in the framework of the $\phi^4$ model, reaching the same conclusion \cite{17}. Indeed, the intrinsic internal mode of $\phi^4$ kinks makes the phenomenon even more noticeable, making us confident on the wide applicability of this work.

Another important conclusion is the dependence of the velocity on the initial phase $\delta_0$, not mentioned in earlier work \cite{4,14}. We note that this dependence allows much more flexibility in controlling the kink velocity, providing an alternative to the use of the relative phase suggested earlier. On the other hand, this may have important consequences for applications as a way of separating, e.g., fluxons in long Josephson junctions \cite{15}. Interestingly, such superconducting devices provide the best possible laboratory to verify our results. This experimental confirmation is crucial in order to ascertain their applicability. Given the accuracy with which the sG equation describes long Josephson junctions, and the fact that an external force like the one proposed in this and earlier works \cite{4,10} is easy to implement, we hope that the corresponding measurements will soon be carried out. A conclusive, positive verification of our theory would yield the picture we provide here very useful in that and related contexts.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Discrete Fourier Transform of the kink width. Upper panel: $m = 2$; lower panel: $m = 3$. Solid line: amplitude measured in simulations. Dashed line: numerical integration of the CC equations. Parameters are as in Fig. 4 for relative phase $\theta = \pi/2$ and $\delta_0 = -2.5$.}
\end{figure}

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