Research Article

Strategies To Evaluate The Riemann Zeta Function

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Abstract

This paper continues a series of investigations on converging representations for the Riemann Zeta function. We generalize some identities which involve Riemann’s zeta function, and moreover we give new series and integrals for the zeta function. The results originate from attempts to extend the zeta function by classical means on the complex plane. This is particularly of interest for representations which converge rapidly in a given area of the complex plane, or for the purpose to make error bounds available.

Keywords: Riemann Zeta function, q-series, Euler-MacLaurin summation, Bernoulli Number and Polynomial, Riemann hypothesis.

1. Introduction and Definitions

1.1. Extensions To The Complex Plane

The Riemann Zeta function is classically defined as

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s},$$

which is a convergent representation for $\text{Re } s > 1$. An immediate extension to $\text{Re } s > 0$ is given by the variant

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i^s},$$

which results from separating summands of the form $\frac{1}{2^i}$ and $\frac{1}{2^{i+1}}$. This alternating series converges for $\text{Re } s > 0$ and reveals that $\zeta$ has a pole with residue 1 at $s = 1$, as $\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} = \ln 2$.

Probably less well-known is the similar representation

$$\zeta(s) = \frac{1}{1 - 3^{1-s}} \sum_{i=1}^{\infty} \left( \frac{1}{(3i-2)^s} - \frac{1}{(3i-1)^s} + \frac{2}{3i^s} \right),$$

which converges as well for $\text{Re } s > -1$.

Other interesting representations, as they seem to not appear in the literature and which extend the zeta function even further, include

- $\left(1 - \frac{1}{2^s} - \frac{2}{3^s}\right) \zeta(s) = 1 + \sum_{i=1}^{\infty} \left( \frac{1}{(4i-1)^s} - \frac{2}{(4i)^s} + \frac{1}{(4i+1)^s} \right)$ for $\text{Re } s > -1$,
- $\left(1 - \frac{5}{3^s} + \frac{5}{6^s} - \frac{1}{6^s}\right) \zeta(s) = \sum_{i=0}^{\infty} \left( \frac{1}{(6i+1)^s} - \frac{4}{(6i+2)^s} + \frac{6}{(6i+3)^s} - \frac{4}{(6i+4)^s} + \frac{1}{(6i+5)^s} \right)$ and
- $\left(4 - \frac{5}{2^s} - \frac{4}{3^s} - \frac{1}{6^s}\right) \zeta(s) = 4 - \frac{1}{2^s} - \sum_{i=1}^{\infty} \left( \frac{1}{(6i-2)^s} - \frac{4}{(6i-1)^s} + \frac{6}{(6i)^s} - \frac{4}{(6i+1)^s} + \frac{1}{(6i+2)^s} \right)$ for $\text{Re } s > -3$.

They are all verified in a similar way as the initial formula.
1.2. Euler-Maclaurin

To further extend the Zeta function to complex arguments with even smaller real part one may apply Euler-Maclaurin’s summation formula, which states that

\[
\sum_{i=1}^{n} f(i) = \int_{1}^{n} f(x) \, dx + \sum_{j=1}^{\Delta} B_j \left( f^{(j)}(n) - f^{(j)}(1) \right) - (-1)^{\Delta} \int_{n}^{\infty} f^{(\Delta)}(x) \frac{B_{\Delta}(x)}{\Delta!} \, dx;
\]

\( f \) is a function with (continuous) derivatives \( f^{(j)} \) up to order \( \Delta \), \( \{x\} = x - \lfloor x \rfloor \) is the fractional part of \( x \), \( B_i \) the Bernoulli number and \( B_i(x) = \sum_{j=0}^{i} \binom{i}{j} B_j x^j \) the Bernoulli polynomial (cf. [AS64]).

The statement may be applied to (1), giving thus after some rearrangements the identity

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{j=0}^{\Delta} \frac{B_j}{n^{s+j-1}} \left( \frac{1}{1} - \frac{1}{s-j} \right) + (-1)^{\Delta} \frac{1}{\Delta!} \int_{n}^{\infty} B_\Delta(x) \frac{1}{x^{s+\Delta}} \, dx \quad \text{(2)}
\]

\( \zeta \) is analytic in the entire complex plane except for \( s = 1 \), where is a pole with residue 1. The argument, however, can be refined for even more explicit error bounds and rates (cf. [CO92]). But as a general paster the expression converge, provided that \( \Re s > 1 - \Delta \).

**Remark 1.** Identity (2) is frequently given in the form

\[
\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s} - \sum_{j=0}^{\Delta} \frac{B_j}{n^{s+j-1}} \left( \frac{1}{1} - \frac{1}{s-j} \right) + (-1)^{\Delta} \frac{1}{\Delta!} \int_{n}^{\infty} B_\Delta(x) \frac{1}{x^{s+\Delta}} \, dx,
\]

which is equivalent because \( B_{2(j+1)} = 0 \) (except for \( B_1 = -\frac{1}{2} \)) and \( \binom{j-x-1}{j} = -(-1)^{j-x-1} \).

**Remark 2.** As \( B_\Delta \) is a polynomial, \( B_\Delta \) is uniformly bounded for \( x \in \mathbb{R} \), and it even holds that \( |B_\Delta(x)| \leq \frac{\Delta!}{(1-2^{-x})^{2\pi}} \) for \( x \in [0, 1] \) (cf. [Leh40]). This allows to conclude that

\[
\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s} - \sum_{j=0}^{\Delta} \frac{B_j}{n^{s+j-1}} \left( \frac{1}{1} - \frac{1}{s-j} \right) + O \left( \frac{1}{n^{s+\Delta-1}} \right),
\]

and whence, \( \zeta \) is analytic in the entire complex plane except for \( s = 1 \), where is a pole with residue 1. The argument, however, can be refined for even more explicit error bounds and rates (cf. [CO92]). But as a general paster the expression converge, provided that \( \Re s > 1 - \Delta \).

**Remark 3.** Notice as well that \( \binom{j}{\Delta} = 0 \) for \( s \) a negative integer and \( \Delta \) big enough, which enables to recover

\[
\zeta(-n) = -\frac{B_{n+1}}{n+1}
\]

for \( n \in \{1, 2, \ldots\} \).

**Remark 4.** The partial series is frequently denoted

\[
H_n(s) := \sum_{i=1}^{n} \frac{1}{i^s},
\]

which is the generalized harmonic number of order \( n \) of \( s \).

**Remark 5.** For future reference recall that

\[
H_n(-k) = \sum_{i=1}^{n} i^k = \frac{B_{k+1}(n+1) - B_{k+1}}{k+1}
\]

in the notation just introduced for natural numbers \( k \in \{0, 1, 2, \ldots\} \) – this is sometimes referred to as Faulhaber’s formula.

\footnote{Note again that \( B_{2(j+1)} = 0 \), so the order of convergence in the next statement actually is \( O \left( \frac{1}{n^{s+\Delta}} \right) \), whenever \( \Delta \) is even.}
2. Results And Extension To Various Directions

It turns out that a lot of different relations for the Riemann Zeta function again involve the expression $\sum_{j=0}^{\infty} B_j \frac{(s+j)}{s+j}$. This is natural, as we have seen that this expression describes the asymptotic behavior of $\sum_{n=1}^{\infty} \frac{1}{n^s}$.

So does the following extension of an identity which is sometimes referred to as Stark’s formula; an extension to Hurwitz Zeta function is known under the name Stark-Keiper formula:

**Theorem 6** (Generalization of Stark’s formula). For $s \neq 1$ and any $\Delta \in \{1, 2, \ldots \}$,

$$\zeta(s) = 1 + \frac{1}{s-1} \sum_{k=1}^{\Delta-1} \left( \frac{s+k-2}{k} \right) \frac{1}{s+i-1} \sum_{i=1}^{\infty} \frac{1}{i^s}$$

**Remark 7.** The latter series converges for all $s \neq 1$. As $\zeta(s+i)$ ($i \geq \Delta$) are easily available even by means of formula (1) provided that $\text{Re} s > 1 - \Delta$, the statement makes $\zeta(s)$ available by different choices of $\Delta$ for $\text{Re} s \leq 1$.

The particular statement for $\Delta = 1$ (sometimes called Stark’s formula, although it seems it was published first by Landau) is (for example) contained in [Tit86].

**Proof.** Notice first that

$$1 - \frac{1}{s-1} \sum_{i=1}^{\Delta-1} \left( \frac{s+i-1}{i+1} \right) = \left( 1 - \frac{1}{s-1} \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^s} \right)$$

$$= 1 - \frac{1}{s-1} \sum_{n=2}^{\infty} \frac{1}{n^s} + \frac{1}{1^s} = 1 - \frac{1}{s-1} (1 + (1-s) (\zeta(s) - 1))$$

which is the statement for $\Delta = 1$.

The general result follows by induction on $\Delta$: To this end assume that (5) holds true and replace $s$ in (6) by $s + \Delta$. Subtracting appropriate multiples of (6) from (5) to eliminate $\zeta(s+\Delta)$ reveals the assertion for $\Delta + 1$, where additionally the identity

$$\sum_{k=0}^{\Delta} \binom{\Delta+1}{k} B_k = 0$$

($\Delta \geq 1$) is involved.

**Theorem 8.** For $\Delta \in \{1, 2, \ldots \}$ and $s \neq 1$ the relation

$$\zeta(s) = \sum_{k=0}^{\Delta} \binom{s+k-1}{k} \frac{1}{2^s} \sum_{j=1}^{\Delta-1} \sum_{i=1}^{\Delta-1} \binom{k}{j} \frac{1}{j+1}$$

holds true.

**Remark 9.** The latter series converges for all $s \neq 1$. As above, $\zeta(s+i)$ is easily available by means of (1) for $\text{Re} s > 1 - \Delta$, the statement whence again makes $\zeta(s)$ accessible for arbitrary $\text{Re} s \leq 1$.

**Proof.** The statement for $\Delta = 1$ is

$$\zeta(s) = \frac{1}{2^s} \sum_{k=1}^{\infty} \binom{s+k-1}{k} \frac{1}{2^k}$$

(8)
and sometimes called Ramaswami’s formula. This is verified straight forward, as

\[ \sum_{k=1}^{\infty} \binom{s+k-1}{k} \zeta(s+k) \frac{1}{2^k} = \sum_{i=1}^{\infty} \frac{1}{i^s} \sum_{k=1}^{\infty} \binom{s+k-1}{k} \left( \frac{1}{2^i} \right)^k \]

\[ = \sum_{i=1}^{\infty} \frac{1}{i^s} \left( \left( \frac{1}{2i} \right)^s - 1 \right) = \sum_{i=1}^{\infty} \frac{1}{i^s} \left( \left( \frac{2i}{2i-1} \right)^s - 1 \right) \]

\[ = 2^s \sum_{i=1}^{\infty} \left( \frac{1}{2i-1} \right)^s - \sum_{i=1}^{\infty} \frac{1}{i^s} = 2^s (1 - 2^{-s}) \zeta(s) - \zeta(s) \]

\[ = (2^s - 2) \zeta(s). \]

As for the general statement one may replace \( s \) in (8) by \( s + \Delta \) and substitute this for the first summand in (7). This burns down then to the identity

\[ \sum_{j=0}^{\Delta-1} B_j \frac{1}{2^{s-2^j}} \sum_{i=0}^{\Delta-1} \binom{\Delta}{i} \frac{1}{i+1} = \sum_{j=0}^{\Delta} B_j \frac{\binom{\Delta+1}{j}}{2^{s-2^j} \Delta + 1} \]

which holds true because

\[ \sum_{j=0}^{\Delta-1} B_j \frac{1}{2^{s-2^j}} \sum_{i=0}^{\Delta-1} \binom{\Delta}{i} \frac{1}{i+1} = \sum_{j=0}^{\Delta} B_j \frac{\binom{\Delta+1}{j}}{2^{s-2^j} \Delta + 1} \]

\[ \Rightarrow \sum_{j=0}^{\Delta} B_j \frac{\binom{\Delta+1}{j}}{2^{s-2^j} \Delta + 1} = (\Delta + 1) B_\Delta. \]

**Remark 10.** Equation (8) is similar to other representations found in [Sri03].

The ingredients from the previous results can be processed together to give a representation of (2) which does not involve integrals: It is the compelling advantage of the following representation over (2) and (3) to replace the integral by a usual sum. As the error term can be evaluated precisely, this makes \( \zeta \) available without computing the limit and letting \( n \to \infty \) in the respective sum.

**Theorem 11.** For \( n \in \{1, 3, \ldots\}, p \in \{2, 3, \ldots\} \) and \( \Delta \in \{0, 1, 2, \ldots\} \) the representation

\[ \zeta(s) = \sum_{j=1}^{n} \frac{1}{j^s} + \sum_{k=0}^{\Delta-1} \binom{s+k-2}{k} \frac{B_k}{s-1} n^{s+k-1} + \]

\[ + \frac{1}{(s-1)n^s} \sum_{i=0}^{\Delta} \binom{s+i-1}{i+1} \sum_{j=0}^{i} \frac{1}{j!} \sum_{p=0}^{i-j} \frac{B_j (n(p-1))^{i-j}}{p^s - p^{i-j}} \]

holds true, which represents a converging series at least for \( \text{Re } s > 1 - \Delta \).

**Remark 12.** Notice that (9) even holds for \( n = 1. \)
Remark 13. The order of summation must not be changed in (9): As formally
\[
\sum_{i=\Delta}^{\Delta+1} \frac{\binom{i+1}{i}}{(i+1)(n^p)} \sum_{j=\Delta}^{i} \binom{i+1}{j} B_j \frac{(n(p-1))^{i+1-j}}{p^j - p^{i-j}} = \sum_{j=\Delta}^{\Delta+1} \frac{B_j}{p^j - p^{i-j}} \sum_{i=\Delta}^{\infty} \binom{i+1}{j} \frac{(n(p-1))^{i+1-j}}{(i+1)(n^p)} \frac{(n(p-1))^{i+1-j}}{p^j - p^{i-j}} = \sum_{j=\Delta}^{\Delta+1} \frac{B_j}{n^{s+1}} \frac{(n(p-1))^{i+1-j}}{s-1},
\]
which characterizes the asymptotic behavior, but this series does not converge.

Remark 14. For \( p \to 1 \) and (formally) interchanging this limit with the sum the (non-converging) identity
\[
\zeta(s) = \sum_{j=1}^{n-1} \frac{1}{j^s} + \sum_{k=0}^{\infty} \frac{\binom{k}{2k}}{s-1} \frac{B_k}{n^{s+1}}
\]
is obtained; and for \( p \to \infty \), the same argument gives
\[
\zeta(s) = \sum_{j=1}^{n-1} \frac{1}{j^s} + \sum_{k=0}^{\Delta} \frac{\binom{k}{2k}}{s-1} \frac{B_k}{n^{s+1}}.
\]

Proof. As for the proof notice that
\[
\sum_{k=0}^{\Delta+1} \frac{\binom{k}{2k}}{s-1} \frac{B_k}{n^{s+1}} = \frac{1}{n^s} \sum_{j=\Delta}^{\infty} \frac{\binom{i+1}{i}}{(i+1)(n^p)} \sum_{j=\Delta}^{i} \binom{i+1}{j} B_j \frac{(n(p-1))^{i+1-j}}{p^j - p^{i-j}} = \frac{1}{n^s} \sum_{j=\Delta}^{\infty} \frac{B_j}{(n(p-1))^{i+1-j}} \frac{1}{p^j - p^{i-j}} \sum_{i=\Delta}^{\infty} \binom{i+1}{j} \left( 1 - \frac{1}{p} \right)^i = (\ast).
\]

Next we have for \( p > 1 \) the geometric series
\[
\frac{1}{n^s} \frac{(n(p-1))^{i+1-j}}{p^j - p^{i-j}} = \frac{(p-1)^{i+1-j}}{n^{s+1-j} p^{s+1-i}} = \frac{1}{1 - \frac{1}{n^i (p-1)p^{s+1-i}}} = \frac{(n(p-1)p^i)^{i+1-j}}{(n(p-1)p^i)^{s+1-i}},
\]
and thus
\[
(\ast) = \sum_{k=0}^{\Delta+1} \frac{\binom{k}{2k}}{s-1} \frac{B_k}{n^{s+1}} - \sum_{j=\Delta}^{\infty} \frac{B_j}{(n(p-1))^{i+1-j}} \frac{1}{p^j - p^{i-j}} \sum_{i=\Delta}^{\infty} \binom{i+1}{j} \left( 1 - \frac{1}{p} \right)^i + \sum_{i=\Delta}^{\infty} \binom{i+1}{j} \sum_{k=0}^{\Delta} \frac{1}{(n(p-1)p^i)^{j+i+1}} \frac{(n(p-1)p^i)^{i+1-j}}{(n(p-1)p^i)^{s+1-i}}.
\]
The inner series has an explicit evaluation, as
\[
\sum_{i=\Delta}^{\infty} \binom{i+1}{j} \left( 1 - \frac{1}{p} \right)^i = (1-p)^{1-j} (p^i - p^{i-1}) \frac{\binom{j}{i}}{s-1}.
\]
For this and the fact that \( B_{2j+1} = 0 \) (except for \( B_1 = -\frac{1}{2} \)) the first two sums collapse to \( \frac{1}{n^s} \), and thus

\[
(*) = \frac{1}{n^s} + \sum_{k=0}^\infty \left( \frac{1}{n^{p^k+1}} \right)^s (\frac{1}{n^{p^{k+1}}})^{s-1} \cdot \sum_{j=0}^\infty \frac{1}{j!} \sum_{i=0}^\infty \binom{s-1}{i} B_i (n (p-1) p^j)^{i+1-j}.
\]

Next observe that the inner sum is just

\[
\frac{1}{i+1} \sum_{j=0}^i \binom{i+1}{j} B_j (n (p-1) p^j)^{i+1-j} = \sum_{j=0}^\infty j^j
\]

(cf. (4)), thus

\[
(*) = \frac{1}{n^s} + \sum_{k=0}^\infty \left( \frac{1}{n^{p^k+1}} \right)^s \sum_{j=0}^\infty \binom{s-1}{j} B_j (n (p-1) p^j)^{j+1-j} = \sum_{j=0}^\infty j^j.
\]

In this situation we may evaluate the power series and rearrange the resulting terms as

\[
(*) = \frac{1}{n^s} + \sum_{k=0}^\infty \left( \frac{1}{n^{p^k+1}} \right)^s \sum_{j=0}^\infty \binom{s-1}{j} B_j (n (p-1) p^j)^{j+1-j} = \sum_{j=0}^\infty j^j.
\]

which finally completes the proof.

To analyze convergence it should be mentioned that

\[
\frac{1}{n^s} \sum_{j=0}^\infty \frac{(s-1+i)}{i+1 (n p)^j} \sum_{j=0}^\infty \binom{i+1}{j} B_j (n (p-1) p^j)^{i+1-j} = \frac{1}{n^s} \sum_{j=0}^\infty \binom{s-1+i}{i} B_i (n (p-1) p^j)^{j+1-j}.
\]

Recall from (2) and (4) that \( \sum_{j=1}^{\Delta-1} j^j = O \left( \frac{n^{\Delta-1}}{\Delta!} \right) \), and further that

\[
\sum_{j=1}^{\Delta-1} j^j = \frac{1}{1+i} \sum_{j=0}^{\Delta-1} \binom{i+1}{j} B_j \cdot n^{i+1-j} = O \left( \frac{n^{i-1}}{\Delta!} \right).
\]
defines an analytic function for all values of \( s \) for all derivatives including the function \( \tilde{\theta} \) and it is an interesting observation that this is the continuous Fourier transform of \( x \) from by parts twice for \( s \). It deduces immediately from

\[
\sum_{j=0}^{n(p-1)p^{i+1}} \left( \frac{1}{p} \right)^{i+1-j} = \mathcal{O} \left( \left( \frac{n(p-1)p^{i+1}}{p^{i+1}} \right)^{s} \right)
\]

\[
= \mathcal{O} \left( \left( \frac{n(p-1)p^{i+1}}{p^{i+1}} \right)^{i+1-\Delta} \right)
\]

which shows that the series converge at least for \( \text{Re } s > 1 - \Delta \).

Whence convergence can be described by

\[
\sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \text{ being Jacobi’s elliptic theta function (we choose } q = e^{-\pi i t} \text{ here for convenience; [Edw74] uses } \Psi \text{ for } \Psi(t) = \frac{1}{2} (\theta_3(0; it) - 1) = \sum_{n=1}^{\infty} e^{-\pi n \tau} \text{). The latter representation (10) deduces immediately from }
\]

\[
\Gamma \left( \frac{s}{2} \right) = n \pi^{\frac{s}{2}} \cdot \int_{0}^{\infty} t^{-\frac{1}{2}} e^{-n^2 \pi t} dt
\]

and holds true for \( \text{Re } s > 1 \). In order to get a formula for all complex numbers \( s \) one may apply integration by parts twice for

\[
s(s-1) \zeta(s) \frac{\Gamma(s)}{\pi^s} = \int_{0}^{\infty} t^{-\frac{1}{2}} \tilde{\theta}(t) dt,
\]

where

\[
\tilde{\theta}(t) = 2t^2 \theta''(t) + 3t \theta'(t) = \sum_{n=1}^{\infty} \left( 4 \left( \frac{n^2 \pi}{t} \right)^2 - 6n^2 \pi \right) e^{-n^2 \pi t}.
\]

It can be proved directly that \( \tilde{\theta} \) inherits the property

\[
\tilde{\theta}(t) = \frac{1}{\sqrt{t}} \tilde{\theta} \left( \frac{1}{t} \right)
\]

from \( \theta \), but additionally

\[
0 = \lim_{t \to 0^+} \tilde{\theta}^{(k)}(x) = \lim_{x \to 0} \tilde{\theta}^{(k)}(x)
\]

for all derivatives including the function \( \tilde{\theta} \) itself (\( k = 0, 1, \ldots \)) holds true, such that the next integral (11) defines an analytic function for all values of \( x \in \mathbb{C} \).

The symmetric version (for \( e^{\frac{1}{2}} \tilde{\theta}(e^{x}) = e^{\frac{1}{2}} \tilde{\theta}(e^{-x}) \)) again by (12) of (11) reads

\[
\int_{-\infty}^{\infty} e^{\frac{1}{2} x} \tilde{\theta}(e^{x}) dx,
\]

and it is an interesting observation that this is the continuous Fourier transform of \( x \mapsto e^{\frac{1}{2}} \tilde{\theta}(e^{x}) \):

\[
\int_{-\infty}^{\infty} e^{\frac{1}{2} x} \tilde{\theta}(e^{x}) dx = \sqrt{2\pi} \cdot \mathcal{F} \left( e^{\frac{1}{2}} \tilde{\theta}(e^{x}) \right) \left( \frac{s}{2} \left( s - \frac{1}{2} \right) \right),
\]

\[
= \sqrt{2\pi} \cdot \mathcal{F} \left( e^{\frac{1}{2}} \tilde{\theta}(e^{x}) \right) \left( \frac{s}{2} \left( s - \frac{1}{2} \right) \right),
\]

\[
= \sqrt{2\pi} \cdot \mathcal{F} \left( e^{\frac{1}{2}} \tilde{\theta}(e^{x}) \right) \left( \frac{s}{2} \left( s - \frac{1}{2} \right) \right),
\]
where $F(f)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) \, dx$ is the Fourier transform of $f$.

The property (12) may be used to easily deduce Riemann’s functional equation. The function decays rather quickly to 0, for $e^{\frac{i}{2} \theta(x)} \sim 4\pi^{2} e^{\frac{i}{4} \pi \epsilon^{-1}}$ as $x$ tends to infinity.

A good proxy of (11) in terms of a polynomial in $s$ with convergence as $n \rightarrow \infty$ is given by the asymptotic properties of the Hermite polynomial $H_n$ (cf. [Pol27]) and Laguerre polynomial $L_n$,

$$(-1)^n \cdot n! \int_{-\infty}^{\infty} H_{2n}\left(x, \frac{s-\frac{1}{2}}{4i \sqrt{n}}\right) e^{\frac{i}{2} \theta(x)} \, dx =$$

$$= \frac{1}{(n^{2})} \int_{-\infty}^{\infty} L_{n}(-\frac{1}{2}) \left(\frac{x^{2}}{n} - \frac{s-\frac{1}{2}}{4i}\right)^{2} \cdot e^{\frac{i}{2} \theta(x)} \, dx$$

$$= \frac{1}{(n^{2})} \int_{-\infty}^{\infty} L_{n}(-\frac{1}{2}) \left(\frac{x^{2}}{16n} - \frac{s-\frac{1}{2}}{i}\right)^{2} \cdot e^{\frac{i}{2} \theta(x)} \, dx$$

$$\xrightarrow{n \rightarrow \infty} s(s-1) \zeta(s) \cdot \frac{\Gamma(\frac{1}{2})}{\pi^{\frac{1}{2}}}.$$

The symmetric integrand is maintained by the following form in terms of complete Elliptic Integrals of the first and second kind.

**Theorem 15.** For any $s \in \mathbb{C}$

$$s(s-1) \zeta(s) \frac{\Gamma(\frac{1}{2})}{\pi^{\frac{1}{2}}} = \int_{0}^{1} \left(\frac{K(1-m)}{K(m)}\right)^{(s-\frac{1}{2})} U(m) \, dm,$$

where $U$ is the symmetric function

$$U(m) = \left(\frac{K(m)K(1-m)}{\pi \sqrt{2m(1-m)}}\right)^{\frac{1}{2}} \begin{pmatrix} 3(1-m) \cdot K(m) \cdot E(1-m) \\ +3m \cdot E(m) \cdot K(1-m) \\ -3 \cdot E(m) \cdot E(1-m) \\ (1-m) \cdot K(m) \cdot K(1-m) \end{pmatrix}.$$  

Here, $K(m) = \int_{0}^{\frac{\pi}{2}} \frac{dv}{\sqrt{1-m \sin^{2}v}} = \frac{\pi}{2} {}_{2}F_{1}\left(\frac{1}{2}, \frac{1}{2}, 1, m\right)$ is the complete elliptic integral of the first kind, and $E(m) = \int_{0}^{\frac{\pi}{2}} \sqrt{1-m \sin^{2}x} \, dx = \frac{\pi}{2} {}_{2}F_{1}\left(\frac{1}{2}, -\frac{1}{2}, 1, m\right)$ the complete elliptic integral of the second kind, which describes the circumference of the ellipse with eccentricity $\sqrt{m}$.

We shall use the abbreviations $K = K(m), K' = K(1-m), E = E(m)$ and $E' = E(1-m)$ in the sequel. A power series analysis of $U$ gives that

$$U(m) = \frac{1}{4 \sqrt{\pi}} \left(\ln \frac{16}{m} - \frac{3}{2} \right) \left(\ln \frac{16}{m}\right)^{\frac{1}{2}} + o(1)$$

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\(^{2}\)Elliptic integrals are often defined with the parameter $m = k^{2}$ instead; in the present context, for symmetry, it is more convenient to use the parameter $m$. 

8
and
\[
\frac{K(1-m)}{K(m)} = \frac{1}{\pi} \ln \frac{16}{m} + o(1).
\]
As
\[
\int_0^1 \left( \ln \frac{16}{m} \right)^\alpha \, dm = 16 \cdot \Gamma \left( \alpha + 1, \ln \frac{16}{x} \right) \approx x \left( \ln \frac{16}{x} \right)^\alpha \left( 1 + \frac{\alpha}{\ln \frac{16}{x}} + \ldots \right)
\]
is integrable for any \( \alpha \) with a result expressed by the upper incomplete Gamma function \( \Gamma \). It is thus evident that the representation (15) converges for all values of \( s \).

**Proof.** For the nome \( q(m) := e^{-\pi K(1-m)/K(m)} \) the q-series identity
\[
1 + 2 \sum_{n=1} q(m)^n = \sqrt{\frac{2}{\pi}} K(m)
\]
holds true. Observe that
\[
\frac{d}{dm} \log q(m) = -\pi \frac{d}{dm} \frac{K(1-m)}{K(m)} = -\pi \frac{EK' + KE' - KK'}{2m (1-m) K^2} = \frac{\pi^2}{4m (1-m) K^2}
\]
due to the well-known derivatives for \( \frac{d}{dm} K = \frac{E-(1-m)K}{2m(1-m)} \) and \( \frac{d}{dm} E = \frac{E-K}{2m} \) and Legendre’s relation (cf. [EMOT53] or [GR00]).

Taking the derivative of the formula (16) reveals the closed form
\[
\sum_{n=1} q(m)^n (n^2 \pi^2) = \sqrt{\frac{2}{\pi}} K(m) \frac{K^2}{2\pi} (E - (1-m) K)
\]
after simplification, and proceeding in the same way further
\[
\sum_{n=1} q(m)^n (n^2 \pi^2)^2 = \sqrt{\frac{2}{\pi}} K(m) \frac{K^2}{2\pi} \left( 3E^2 - 6(1-m)E + (3-m)(1-m)K^2 \right).
\]

Combining the latter equalities and after further simplifications involving Legendre’s relation again one obtains
\[
t(m)^{-\frac{1}{2}} \cdot \tilde{\theta}(t(m)) \cdot t'(m) = U(m),
\]
where \( t(m) := -\frac{1}{\pi} \log q(m) = \frac{K(1-m)}{K(m)} \). Whence
\[
s(s-1) \zeta(s) \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} = \int_0^\infty t^{s-1} \tilde{\theta}(t) \, dt
\]
\[
= \int_0^1 t(m)^{-\frac{1}{2}} \tilde{\theta}(t(m)) \cdot t'(m) \, dm = \int_0^1 t(m)^{-\frac{1}{2}} t(m)^{-\frac{1}{2}} \tilde{\theta}(t(m)) \cdot t'(m) \, dm
\]
\[
= \int_0^1 \left( \frac{K(1-m)}{K(m)} \right)^{\frac{3}{2}} \frac{1}{U(m)} \, dm,
\]
which is the desired assertion. \( \square \)

The results found can be combined in the following way.

Recall the inverse elliptic nome function \( q^{-1}(q) := \frac{\theta_1(q)}{\theta_3(q)} \) (cf. [WW62, Chapter XXI, p. 486])\(^3\); with this choice \( x^{-1}(x) = q^{-1}(e^{-\pi x}) \) is the inverse function of \( x(m) := \log \frac{K(m)}{K(1-m)} \).

---

\(^3\)This is called the problem of inversion, cf. [EMOT53, volume II, p. 362].
This can be used to rewrite (13) in different variants as

\[
\frac{\Gamma\left(\frac{1}{2}\right)}{\pi^\frac{3}{2}} = \int_0^\infty e^\frac{1}{2}(t^2 s)^{\frac{1}{2}} U\left(x^{-1}(s)\right) x'\left(x^{-1}(s)\right) dx
\]

\[
= \int_0^\infty e^\frac{1}{2}(t^2 s)^{\frac{1}{2}} \cdot \hat{U}\left(t \left(\frac{\partial_x^2}{\theta_3(e^{-x})}\right)\right) dt
\]

\[
= \int_0^1 \left( -\frac{\log q}{\pi} \right)^{\frac{1}{2}} \frac{1}{q^\frac{1}{2}} \hat{U}\left(\frac{\theta_3^2(q)}{\theta_3^2(q)}\right) dq
\]

where

\[
\hat{U} (m) = \frac{4(K(m)K(1 - m))^{\frac{1}{2}}}{\pi^2 \sqrt{2\pi}} \begin{pmatrix}
3 (1 - m) \cdot K(m) \cdot E(1 - m) \\
+3m \cdot E(m) \cdot K(1 - m) \\
-3 \cdot E(m) \cdot E(1 - m) \\
-m (1 - m) \cdot K(m) \cdot K(1 - m)
\end{pmatrix}.
\]

Introducing \(B(m) := \frac{E(1-m)K}{m}\) we have \(\frac{1}{\sqrt{1 - m \sin^2 \phi}} = \hat{K}(\frac{1}{2}, \frac{1}{2}, 2, m)\) this function rewrites as

\[
\hat{U} (m) = \sqrt{8\pi} \frac{m (1 - m)}{\pi^3} (K'K)^{\frac{1}{2}} (2K'K - 3B'B).
\]

Comparing the identity with (13) we finally obtain that \(t^\frac{1}{2} \hat{\theta}(t) = \hat{U}\left(\frac{\theta_2(e^{-x})}{\theta_3(e^{-x})}\right) = \hat{U}\left(\frac{\theta_4(e^{-x})}{\pi^2 (e^{-x})}\right)\), or

\[
e^\frac{1}{2} \hat{\theta}(e^x) = \hat{U}\left(\frac{\theta_2(e^{-x})}{\theta_3(e^{-x})}\right) = \hat{U}\left(q^{-\frac{1}{2}} (e^{-x})\right)
\]

by symmetry (12). The inversion moreover has the well-known (cf. [EMOT53, Volume II, p. 362], correcting the misprints there) expansion \(\frac{\theta_4(q)}{\theta_6(q)} = \prod_{n=0}^{\infty} (1 - q^{2n+1})^2\) as an infinite product.

The power series expansion

\[
s(s - 1) \zeta(s) \frac{\Gamma\left(\frac{1}{2}\right)}{\pi^\frac{3}{2}} = \sum_{k=0}^{\infty} \frac{(s - \frac{1}{2})^{2k}}{(2k)!4^k} \int_0^\infty x^{2k} \cdot e^\frac{1}{2}(t^2 s)^{\frac{1}{2}} \cdot \hat{U}\left(t \left(\frac{\partial_x^2}{\theta_3(e^{-x})}\right)\right) dx
\]

for even and positive coefficients then is just a by-product. The convergence of (14), however, is faster.

4. Summary And Acknowledgment

4.1. Summary

Various algorithms to evaluate the Riemann \(\zeta\) function start with the Euler-Maclaurin summation formula, as it properly describes the asymptotics of the series representation. The closed forms available for the error bound typically involve integrals. The method proposed here generalizes other methods and gives the precise error bound as sum.

4.2. Acknowledgment

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The plots have been produced by use of Mathematica.
5. References

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