The flux of noncommutative $U(1)$ instanton through the fuzzy spheres

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March 27, 2022

Abstract

From the ADHM construction on noncommutative $\mathbb{R}^4_\theta$ vector space we investigate different $U(1)$ instanton solutions tied by partial isometry transformations. We recast these solutions under a form of vector fields in noncommutative $\mathbb{R}^3_\theta$ vector space which makes possible the calculus of their fluxes through fuzzy spheres. For this end, we establish the noncommutative analog of Gauss theorem from which we show that the flux of the $U(1)$ instantons through fuzzy spheres does not depend on the radius of these spheres and it is invariant under partial isometry transformations.

PACS NUMBER: 11.10.Nx, 11.15.Tk.

Introduction

Noncommutative geometry is a generalization of the usual differential geometry in the sense that the usual description of manifolds by their corresponding algebra of functions is reformulated by using noncommutive algebras which are considered as algebras on noncommutative spaces [1, 2, 3]. In physics one can hope that noncommutative geometry gives alternatives to solve many problems such as renormalization of quantum field theories where the fuzzy spheres are used in the regularization scheme [4], quantization of gravity [5], superstring and M-theory [6, 7, 8] and quantum Hall effect [9, 10].

In the several last years a great variety of works in field theories on noncommutative geometry have been developed. In particular the Yang-Mills gauge theories which are emerged from certain low energy limit of string theory [11, 12] or from M theory compactification [7, 8]. In the most part of this development are treated some non perturbative aspects of noncommutative gauge theories especially to describe noncommutative instantons and their topological charge [13, 14, 15, 16, 17, 18, 19, 20].

In this work we mainly treat the invariance of the one topological charge under partial isometry transformations of $U(1)$ instantons on noncommutative
\( \mathbb{R}^4_\theta \) and establish the analog of Gauss theorem in noncommutative \( \mathbb{R}^4_\theta \) space from which we show that the flux of \( U(1) \) instantons through fuzzy spheres does not depend on their radius (does not depend on the Hilbert space representations of different fuzzy sphere algebras).

We begin this paper by recalling in section 1 some properties of noncommutative \( \mathbb{R}^4_\theta \) space and review briefly in section 2 the ADHM (Atiyah-Drinfeld-Hitchin-Manin) construction of instantons on this noncommutative space [22, 23, 24, 13].

In the third section, explicit solutions deduced from partial isometries are investigated. We show that these partial isometries acts as noncommutative \( U(1) \) gauge transformations leaving invariant the instanton number.

In section 4 we recast the \( U(1) \)-one-instantons in terms of operator algebra over \( \mathbb{R}^3_\theta \). This will leads us to view the strength field of the \( U(1) \) instanton as a vector field in \( \mathbb{R}^3_\theta \). Finally we establish in the last section a noncommutative analog to the Gauss theorem to calculate the flux of the \( U(1) \) instanton fields through fuzzy spheres then we show its invariance under partial isometry transformations.

1 Noncommutative \( \mathbb{R}^4_\theta \)

In this section we consider \( \mathbb{R}^4_\theta \) noncommutative space which can be described by the complex coordinates

\[
\hat{z}_1 = \hat{x}_2 + i\hat{x}_1, \quad \hat{z}_2 = \hat{x}_4 + i\hat{x}_3,
\]
satisfying the commutation relations : 

\[
[\hat{z}_\alpha, \hat{z}_\beta] = -2\delta_{\alpha,\beta}\theta, \quad [\hat{z}_\alpha, \hat{z}_\beta] = 0, \quad (\alpha, \beta = 1, 2)
\]

The Hilbert space representation \( \mathcal{H} \), on which the operators \( \hat{z}_\alpha, \hat{\bar{z}}_\alpha \) act, is spanned by the basis

\[
|n_1, n_2\rangle = \frac{\hat{z}_{1}/\sqrt{2\theta}}{\sqrt{n_1!}} |0, 0\rangle, \quad \hat{\bar{z}}_\alpha |0, 0\rangle = 0
\]

where the states \( |n_1, n_2\rangle \) are orthonormalized

\[
\langle n_1, n_2 |m_1, m_2\rangle = \delta_{n_1, m_1}\delta_{n_2, m_2}.
\]

\( \hat{z}_\alpha \) and \( \hat{\bar{z}}_\alpha \) act on theses states as:

\[
\hat{z}_1 |n_1, n_2\rangle = \sqrt{2\theta (n_1 + 1)} |n_1 + 1, n_2\rangle, \quad \hat{\bar{z}}_2 |n_1, n_2\rangle = \sqrt{2\theta (n_2 + 1)} |n_1, n_2 + 1\rangle,
\]
\[
\hat{z}_1 |n_1, n_2\rangle = \sqrt{2\theta n_1} |n_1 - 1, n_2\rangle, \quad \hat{\bar{z}}_2 |n_1, n_2\rangle = \sqrt{2\theta n_2} |n_1, n_2 - 1\rangle.
\]
The operator algebra over $\mathbb{R}^4_{\theta}$, denoted by $\hat{A}^4_{\theta}$, can also be described by the algebra $A^4_{\theta}$ of c-number functions $f(z, \overline{z})$ endowed with the normal ordering star product [25]

$$f(z, \overline{z}) \star g(z, \overline{z}) = e^{2\theta \partial_{\overline{z}} \partial_{z}'} f(z, \overline{z}) g(z', \overline{z'})|_{z'=z, \overline{z'}=\overline{z}}.$$  

The correspondence between the operators and the c-number functions is given by:

$$\hat{f}(\hat{z}_\alpha, \hat{z}_\alpha) :\leftrightarrow f(z_\alpha, \overline{z}_\alpha) = \langle \overline{z} | :\hat{f}(\hat{z}_\alpha, \hat{z}_\alpha) : | z \rangle,$$  

where

$$|z\rangle = \exp\left(-\frac{z_\alpha \hat{z}_\alpha}{4\theta}\right) \exp\left(\frac{\overline{z}_\alpha \hat{z}_\alpha}{2\theta}\right) |0, 0\rangle$$

is the coherent state satisfying $\hat{z}_\alpha |z\rangle = z_\alpha |z\rangle$ and $\langle \overline{z} | \overline{z} \rangle = 1$.

The derivatives $\partial_\alpha = \frac{\partial}{\partial z_\alpha}$ and $\partial_{\overline{z}_\alpha} = \frac{\partial}{\partial \overline{z}_\alpha}$ satisfy Leibnitz rules with respect to this star product [2]. In the operator description, these derivatives are defined as

$$\hat{\partial}_\alpha (\cdot) = \frac{1}{2\theta} [\hat{z}_\alpha, \cdot], \quad \hat{\partial}_{\overline{z}_\alpha} (\cdot) = -\frac{1}{2\theta} [\hat{\overline{z}}_\alpha, \cdot].$$  

For the integration, the correspondence between the operator description and the c-number function is given by

$$\int d^4x f \leftrightarrow (2\pi)^2 Tr_\mathcal{H}(\hat{f})$$

where the trace of the operator is over the Hilbert space representation $\mathcal{H}$.

## 2 Instantons and ADHM construction

Instantons are localized finite-action non-perturbative solutions for the Euclidean equations of motion of Yang-Mills gauge theories. In this section we will recall the basic algorithm of ADHM construction [13] to get instanton solutions in noncommutative space $\mathbb{R}^4_{\theta}$ which are just deformed versions of the commutative one [22, 23, 24].

The different steps of the ADHM construction for $U(N)$ $k$-instantons can be summarized as follows

1. Solve the deformed ADHM equations

$$\begin{bmatrix} B_1, B_1^\dagger \end{bmatrix} + \begin{bmatrix} B_2, B_2^\dagger \end{bmatrix} + \begin{bmatrix} II^\dagger - J^\dagger J \end{bmatrix} = 4\theta id_k.$$

where $B_1, B_2 \in End \ (C^k), I \in Hom \ (C^N, C^k)$ and $J \in Hom \ (C^k, C^N)$

2. Define Dirac operator $D_z : (C^k \oplus C^k \oplus C^N) \otimes \hat{A}^4_{\theta} \rightarrow (C^k \oplus C^k) \otimes \hat{A}^4_{\theta}$

$$D_z = \begin{pmatrix} B_2 - \hat{z}_2 & B_1 - \hat{z}_1 & I \\ -(B_1^\dagger - \hat{z}_1) & B_2 - \hat{z}_2 & J^\dagger \end{pmatrix}. \quad (6)$$
3. Look for all the $N$ orthonormalized solutions $\Psi^a : \mathbb{A}^4 \rightarrow (C^k \oplus C^k \oplus C^N) \otimes \hat{A}_\theta$ (the zero-modes) to the equation

$$D_z \Psi^a = 0, \quad \Psi^a \Psi^b = \delta^{ab}.$$

4. Construct the $U(N)$ gauge field

$$\hat{A} = \Psi^i d\Psi.$$

The field strength of the gauge field $\hat{A}$ is given by $\hat{F} = d\hat{A} + \hat{A}^2$ and the topological number is defined by

$$K = -\frac{1}{16\pi^2} (2\pi\theta)^2 \text{Tr} \left( \hat{F} \right)^2$$

where the trace is taken both on the group indices (for the general case of the $U(N)$ gauge group) and on the Hilbert space $\mathcal{H}$. The formula (7) is the noncommutative version of the second Chern character defined by:

$$K = -\frac{1}{16\pi^2} \int d^4x \text{Tr}_{U(N)} \left( F_{\mu\nu} \tilde{F}^{\mu\nu} \right)$$

where $\tilde{F}^{\mu\nu} = *F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F^{\rho\sigma}$ is the dual of the strength field.

3 $U(1)$-one-Instanton and partial isometries

We will now concentrate on the partial isometry transformations of the known example of $U(1)$-one-instanton solution on $\mathbb{R}^4$ (see [11]). For the $U(1)$-one-instanton solutions on $\mathbb{R}^4$, $k = N = 1$, thus the matrices $B_{1,2}, I, J$ become complex numbers. Inserting this in the ADHM equations we obtain $II^\dagger - JJ^\dagger J = 4\theta$ and $IJ = 0$, we take $I = 2\sqrt{\theta}$, $J = 0$. The parameters $B_{1,2}$ are interpreted as the position of the instanton. Due to the translational invariance of $\mathbb{R}^4$ we can consider the case where the instanton is localized at the origin i.e. $B_{1,2} = 0$. In this case the Dirac operator reads

$$D_z = \begin{pmatrix} -\tilde{z}_2 & -\tilde{z}_1 & 2\sqrt{\theta} \\ \tilde{z}_1 & -\tilde{z}_2 & 0 \end{pmatrix}.$$ (8)

Finely we look for the solution $\Psi$ of the Dirac equation

$$D_z \Psi = 0$$

whose solution is given by:

$$\psi_0 = \begin{pmatrix} 2\sqrt{\theta} & 2\sqrt{\theta} & 1 \\ \tilde{z}_2 & \tilde{z}_1 & \sqrt{\tilde{z}_2(\tilde{z}_2 + 4\theta)} \end{pmatrix}.$$ (10)
where \( \hat{F} = \hat{z}_1 \hat{z}_2 + \hat{z}_2 \hat{z}_1 \). We can see that this solution is not normalized as said in the ADHM data, because of the operator \( \frac{1}{\sqrt{\hat{z}_1 \hat{z}_2 + 4\theta}} \) which indefinite on the vacuum state \( |0, 0\rangle \). However this solution is well normalized in the subspace where the state \( |0, 0\rangle \) is projected out. The subspace \( \mathcal{H} - \{0, 0\} \) will be denoted by \( \mathcal{H}_{00} \).

The gauge field is directly calculated in \( \mathcal{H}_{00} \) by

\[
\hat{A}_{00} = \psi_0 \frac{1}{2\theta} \left[ \frac{1}{\sqrt{\hat{z}_1 \hat{z}_2 + 4\theta}} \right] \, dz^i - \frac{1}{2\theta} \left[ \frac{1}{\sqrt{\hat{z}_1 \hat{z}_2 + 4\theta}} \right] \, d\bar{z}^i. \tag{11}
\]

By using the relations \( \hat{z}_a \hat{f} (\hat{z}) = \hat{f} (\hat{z}_1 - 2\theta) \hat{z}_a, \hat{z}_a \hat{f} (\hat{z}) = \hat{f} (\hat{z} + 2\theta) \hat{z}_a \) and the solution (10) we may compute explicitly (11) to get:

\[
\hat{A}_{00} = \frac{1}{2\theta} (\frac{\hat{z}_1 \hat{z}_2 + 6\theta}{(\hat{z}_1 + 2\theta)(\hat{z}_2 + 4\theta)})^\dagger - 1) \hat{z}_a dz^a - h.c. \tag{12}
\]

leading to a strength field \( \hat{F}_{00} = d\hat{A}_{00} + \hat{A}_{00} \cdot \hat{A}_{00} \) of the form

\[
\hat{F}_{00} = -\frac{8\theta}{\hat{z}_1 \hat{z}_2 + 2\theta}(\hat{z}_1 \hat{z}_2) d\hat{z}_1 d\hat{z}_2 - \frac{8\theta}{\hat{z}_1 \hat{z}_2 + 2\theta}(\hat{z}_1 \hat{z}_2) \, d\hat{z}_2 d\hat{z}_1
\]

\[
-\frac{4\theta}{\hat{z}_2 \hat{z}_1 + 2\theta}(\hat{z}_1 \hat{z}_2) \, d\hat{z}_2 d\hat{z}_1 + \frac{4\theta}{\hat{z}_1 \hat{z}_2 + 2\theta}(\hat{z}_1 \hat{z}_2) \, d\hat{z}_2 d\hat{z}_1 \tag{13}
\]

which exhibits the anti-self duality conditions:

\[ \hat{F}_{00z_1 z_2} = -\hat{F}_{00z_2 z_1}, \quad \hat{F}_{00z_1 z_2} = 0. \]

The topological charge of the \( U(1) \) instanton expressed like in relation (12) by using

\[
(F_{00})^2 = \left[ 16 \left( \hat{F}_{00z_1 z_2} \hat{F}_{00z_2 z_1} - \frac{1}{2} \left( \hat{F}_{00z_1 z_2} \hat{F}_{00z_2 z_1} \hat{F}_{00z_1 z_2} \hat{F}_{00z_2 z_1} \right) \right) \right]
\]

\[
= \frac{(16\theta)^2}{\hat{z}_1 \hat{z}_2 + 2\theta}(\hat{z}_1 \hat{z}_2 + 4\theta) \tag{14}
\]

to get (13)

\[
\mathcal{K} = \frac{\theta^2}{4} Tr_{\mathcal{H}_{00}} \left( \frac{(16\theta)^2}{\hat{z}_1 \hat{z}_2 + 2\theta}(\hat{z}_1 \hat{z}_2 + 4\theta) \right) = \sum_{n \neq 0} \frac{4}{(n_1 + n_2)(n_1 + n_2 + 1)(n_1 + n_2 + 2)}
\]

\[
= \frac{4}{n(n + 1)(n + 2)} = 1.
\]

where we denote the ordered couple \((n_1, n_2)\) by \( \vec{n} \) and the sum \( n_1 + n_2 \) by \( n \).

The investigation of the transformed solution by partial isomery is deduced from the fact that the solution
\( \bar{\psi}_0 = \begin{pmatrix} 2\sqrt{\theta} \hat{z}_2 \\ 2\sqrt{\theta} \hat{z}_1 \end{pmatrix} \)

of (4) belong to a right module. Then we may also consider others solutions of the form

\[ \psi = \bar{\psi}_0 \hat{z}_0^K \hat{z}_2^L \ (K \geq 0, L \geq 0). \]

The normalization of these solutions gives

\[ \psi = \psi_0 U = \begin{pmatrix} 2\sqrt{\theta} \hat{z}_2 \\ 2\sqrt{\theta} \hat{z}_1 \end{pmatrix} \frac{1}{\sqrt{\hat{z}_2(\hat{z}_2 + 4\theta)}} \left( \frac{\hat{z}_1}{\sqrt{\hat{z}_1^2}} \right)^K \left( \frac{\hat{z}_2}{\sqrt{\hat{z}_2^2}} \right)^L \]

(15)

where \( U \) is given by

\[ U = \left( \frac{\hat{z}_1}{\sqrt{\hat{z}_1^2}} \right)^K \left( \frac{\hat{z}_2}{\sqrt{\hat{z}_2^2}} \right)^L \].

(16)

One can check that (15) can be rewritten as

\[ \psi = \bar{\psi}_0 \hat{z}_0^K \hat{z}_2^L N(\hat{z}, \hat{z})^{-\frac{1}{2}} \]

(17)

where

\[ N(\hat{z}, \hat{z}) = (\hat{z}_2 - 2\theta(K+L))(\hat{z}_2 - 2\theta(K+L-2)) \prod_{k=1}^K (\hat{z}_1 - 2\theta(k-1)) \prod_{l=1}^L (\hat{z}_2 - 2\theta(l-1)). \]

We will restrict ourselves, for the same reason explained above, on the Hilbert subspace states \( \mathcal{H}_{KL} \) on which the solution is well defined and well normalized i.e., the eigenstates of \( N(\hat{z}, \hat{z}) \) whose the eigenvalues do not vanish. These nonsingular states are \( |n_1, n_2\rangle \) such that \( n_1 \geq K, n_2 \geq L \), and \( \langle n_1, n_2\rangle \neq (K, L) \). This basis span the subspace \( \mathcal{H}_{KL} = P_{KL} \mathcal{H} \) where the projector \( P_{KL} \) reads

\[ P_{KL} = 1 - \sum_{n_1=0}^{K} \sum_{n_2=0}^{L} |n_1, n_2\rangle \langle n_1, n_2| - \sum_{n_1=0}^{K} \sum_{n_2=L+1}^{\infty} |n_1, n_2\rangle \langle n_1, n_2| - \sum_{n_1=K+1}^{\infty} \sum_{n_2=0}^{L} |n_1, n_2\rangle \langle n_1, n_2| . \]

Before we study this solution, we investigate some properties of \( U \) transformations which we will see below that they act like noncommutative \( U(1) \) gauge transformations. They are not unitary but satisfy the properties of partial
isometries.\(^1\)

\[
UU^\dagger = id_H, \quad U^\dagger U = P_{KL}, \quad P_{KL}U^\dagger = U^\dagger, \quad UP_{KL} = U.
\]  

(18)

In \(\mathcal{H}_{KL}\), the relations \(^1\) read

\[
UU^\dagger = U^\dagger U = id_{\mathcal{H}_{KL}}.
\]  

(19)

Under \(U\), the gauge field transforms in this subspace as

\[
\hat{A}_{KL} = \psi^\dagger d\psi = U^\dagger \psi_0^\dagger d(\psi_0)U + U^\dagger \psi_0^\dagger \psi_0 dU
\]  

(20)

where we have used the fact that in \(\mathcal{H}_{KL}\) we have \(U^\dagger \psi_0^\dagger \psi_0 = U^\dagger\). We can see from the relation \(^2\) that \(U\) acts (from \(\mathcal{H}_{00}\) to \(\mathcal{H}_{KL}\)) like noncommutative \(U(1)\)-gauge transformations act on the gauge fields.

Now the strength field in \(\mathcal{H}_{KL}\) is defined as usual by:

\[
\hat{F}_{KL} = d\hat{A}_{KL} + \hat{A}_{KL} \hat{A}_{KL}.
\]  

(21)

Using \(^2\), \(dU^\dagger U = -U^\dagger dU\) and \(dUU^\dagger = -dU^\dagger U\) in \(\mathcal{H}_{KL}\), we get

\[
\hat{F}_{KL} = U^\dagger \hat{F}_{00} U
\]  

(22)

which is the gauge transform of the strength field \(\hat{F}_{00}\) (in \(\mathcal{H}_{00}\)) in the subspace \(\mathcal{H}_{KL}\). The instanton number must be unchanged by the \(U\)-transformations, this is what we will see in the calculus of

\[
K = -\frac{1}{16\pi^2} (2\pi\theta)^2 Tr_{\mathcal{H}_{KL}} \left( \hat{F}_{KL} \right)^2 = -\frac{1}{16\pi^2} (2\pi\theta)^2 Tr_{\mathcal{H}_{KL}} \left( U^\dagger \left( \hat{F}_{00} \right)^2 U \right).
\]  

(23)

In fact, by using \(^2\) we get

\[
\left( \hat{F}_{KL} \right)^2 = U^\dagger \left( \hat{F}_{00} \right)^2 U = -U^\dagger \frac{(16\theta)^2}{\left( \hat{z} \right)^2 + 4\theta} U
\]  

(24)

The transformation \(U\) correspond to a multiplication by a phase factor in the classical limit \(\theta \rightarrow 0\). In this limit \(z_i, \bar{z}_i\) become usual complexe coordinates, then if we put \(z_i = r_i e^{i\varphi_i}\) (polar coordinates \(r_i, \varphi_i\)) it is obvious to have \(\left( \frac{1}{\sqrt{r_i^2 + (\bar{z}_i)^2}} \right)^j J = (e^{i\varphi_i})^j\), thus the transformation take the usual form \(U = e^{i\Phi}\), where \(\Phi = K\varphi_1 + L\varphi_2\).
Inserting (24) in (23), we obtain

\[ \mathcal{K} = 4 \sum_{n_1 \geq K \atop n_2 \geq L} \frac{1}{(n - (K + L))(n - (K + L - 1))^2(n - (K + L - 2))} \]

\[ = 4 \sum_{\pi \neq (0,0)} \frac{1}{n(n + 1)^2(n + 2)} = 1. \]

which shows the invariance of the instanton number partial isometries.

4 Reduction to the vector space \( \mathbb{R}_\theta^3 \)

In this section we will calculate the flux of the \( U(1) \)-one instanton strength field through the fuzzy sphere. This investigation is justified by the fact that the \( U(1) \) instanton components (13) can be written in terms of the \( \mathbb{R}_\theta^3 \) vector space coordinates \( \hat{x}^i \) as a three vector \( \vec{F} \) whose components are defined in \( \mathcal{H}_{00} \) as:

\[ \hat{F}_{0021} = \vec{F}^3 = \frac{-\theta \hat{x}^3}{\hat{x}^0(\hat{x}^0 + \theta)(\hat{x}^0 + 2\theta)}, \]

\[ \hat{F}_{0012} = \vec{F}^+ = \frac{-\theta}{\hat{x}^0(\hat{x}^0 + \theta)(\hat{x}^0 + 2\theta)} (\hat{x}^3 + i \hat{x}^2) = F^1 + i F^2, \]

\[ \hat{F}_{0021} = \vec{F}^- = \frac{-\theta}{\hat{x}^0(\hat{x}^0 + \theta)(\hat{x}^0 + 2\theta)} (\hat{x}^3 - i \hat{x}^2) = F^1 - i F^2 \]

with \( \hat{x}^0 = \frac{1}{2} \hat{z}^2 \). The coordinates on \( \mathbb{R}_\theta^3 \) are defined from those on \( \mathbb{R}_\theta^4 \) by (Hopf fibration)

\[ \hat{x}^i = \frac{1}{2} \hat{z}^0 \tau^i_{\alpha \beta} \hat{z}_\beta, \quad i = 1, 2, 3; \quad \alpha, \beta = 1, 2 \]

(26)

where \( \tau^i \) are the three Pauli matrices. The coordinates \( \hat{x}^i \) generate a subalgebra \( \hat{A}_\theta^3 \subset \hat{A}_\theta^4 \) satisfying the commutation rules

\[ [\hat{x}^i, \hat{x}^j] = i2\theta \epsilon^{ijk} \hat{x}^k, \quad [\hat{x}^0, \hat{x}^i] = 0 \]

(27)

and the relation

\[ \hat{x}^i \cdot \hat{x}^i = \hat{x}^0 (\hat{x}^0 + 2\theta) \]

where the repeated indices are summed over. It is convenient to use, for the \( \hat{A}_\theta^3 \) algebra the Schwinger basis of the Hilbert space

\[ |J, m\rangle = \begin{pmatrix} \hat{z}_1 \sqrt{28} \\ \hat{z}_2 \sqrt{28} \end{pmatrix}^{J+m} \begin{pmatrix} \hat{z}_1 \sqrt{28} \\ \hat{z}_2 \sqrt{28} \end{pmatrix}^{J-m} |0, 0\rangle \]

where \( J = 0, \frac{1}{2}, 1, \ldots \infty \) and \( m \) runs by integer steps over the range \( -J \leq m \leq J \).
The algebra $\hat{A}_b^0$ can also be described by the algebra $A_3^0$ of c-number function $f(x)$ of $x^i = \frac{1}{2} z_{ij} x^i y^j$ and $x^i \cdot x^i = (x^0)^2$ endowed with the star product of 21

\[
f(x) \star g(x) = e^{\theta(x^0 \delta^{ij} + i e^{ij} x^i)} \frac{\partial}{\partial x^0} \frac{\partial}{\partial y^i} f(x) g(y) |_{y^i = x^i}.
\]

deduced from the star product of $A_3^0$ by restriction.

In terms of elements of the algebra $A_3^0$ we can decompose the strength field components 25 as

\[
F^i_j = - \sum_{j=\frac{1}{2}}^{\infty} x^0 \star (x^0 + \theta) \star (2x^0 + 2\theta) \star x^i \star P_J (x^0)
\]

\[
= - \frac{1}{4 \theta^2} \sum_{j=\frac{1}{2}}^{\infty} \frac{1}{J(J+1)(2J+1)} x^i \star P_J (x^0)
\]

\[
= - \frac{1}{4 \theta^2} \sum_{j=\frac{1}{2}}^{\infty} \frac{1}{J(J+1)(2J+1)} x^i P_j - \sum_{j=\frac{1}{2}}^{\infty} F^i_j
\]

where we have used $x^0 \star P_J (x^0) = 2 \theta J P_J (x^0)$ to define $(x^0)^{-1} \star P_J (x^0) = \frac{1}{2 \theta J} P_J (x^0)$ for $J \neq 0$ and $x^i \star P_J (x^0) = x^i P_j - \frac{1}{2} \theta^2 x^j P_J (x^0)$.

The corresponding operator subalgebra $\hat{S}_{\theta,j}^2$ generated by $\hat{x}^i , \hat{P}_j$ satisfies the commutation relation rules on the fuzzy sphere

\[
\left[ \hat{x}^i , \hat{P}_j \right] = i2 \theta e^{ijk} \hat{x}^k , \quad [\hat{x}^0 , \hat{x}^i] = 0.
\]

and

\[
\hat{x}^i \cdot \hat{x}^i = \hat{x}^0 (\hat{x}^0 + 2 \theta) = 4 \theta^2 J(J+1) \hat{P}_j
\]

The subalgebra $\hat{S}_{\theta,j}^2$ is realized in the Hilbert subspace $\mathcal{H}_J \subset \mathcal{H}$ spanned by the basis $|J,m\rangle$, $-J \leq m \leq J$.

5 Flux through the Fuzzy sphere

The form $(\hat{F} \approx \frac{\vec{F}}{(x^0)^2})$ of the strength field 25 permits us to view it as the noncommutative analog of the classical Dirac magnetic monopole field (or by duality as the static coulombian electric field). So it is interesting to calculate
the flux of this noncommutative vector field through a fuzzy sphere. To perform the calculus of the flux of the vector fields through fuzzy spheres, we generalize the technics of the Gauss theorem to the noncommutative space by using the integration measures on $\mathbb{R}_o^3$ and on the fuzzy sphere investigated in [21].

For this end we start with the derivatives with respect to $z_\alpha$ and $\tau_\alpha$ of a scalar field $\varphi(x) \in A_o^3$ giving components $\partial_\alpha \varphi(x)$, $\partial_\alpha \varphi(x) \in A_o^3$ of a quadri-vector given as

$$\partial_\alpha \varphi(x) = \partial_\alpha (x^i) \partial_i \varphi(x) = \frac{1}{2} \tau_{\alpha\beta} \varphi_\beta \partial_\alpha \varphi(x), \quad \partial_\alpha \varphi(x) = \partial_\alpha (x^i) \partial_i \varphi(x) = \frac{1}{2} \tau_{\beta\alpha} \varphi_\beta \partial_\alpha \varphi(x)$$

(30)

where $\partial_i = \frac{\partial}{\partial x^i}$. These relations are similar to the transformations of the covariant components of a quadri-vector under a coordinate transformations. In our case the transformation $z_\alpha, \tau_\alpha \rightarrow x^i = \frac{1}{2} z_\alpha \tau_{\alpha\beta} \tau_\beta$ is a surjection from $\mathbb{R}_o^4$ to $\mathbb{R}_o^3$ which is not invertible. In what follows, we only consider quadri-vectors $V$ of components $V_\alpha, V_\alpha \in A_o^3$ verifying the same transformations like in (30)

$$V_\alpha(z, \tau) = \frac{1}{2} \tau_{\alpha\beta} \varphi_\beta V_\beta(x), \quad V_\alpha(z, \tau) = \frac{1}{2} \tau_{\beta\alpha} \varphi_\beta V_\beta(x).$$

(31)

where $V^i(x) \in A_o^3$ are the components of a three vector $\overrightarrow{V}(x)$ of $\mathbb{R}_o^3$. The quadri-divergence of (31) is given in terms of three-divergence of $\overrightarrow{V}(x)$ as

$$\partial_\alpha V_\alpha + \partial_\alpha \overrightarrow{V}_\alpha = x^0 \partial_\alpha V_\alpha \in A_o^3$$

(32)

where we have used $Tr(\tau) = 0$ and the relation $\frac{1}{4} \tau_{\alpha\beta} \tau_{\rho\sigma} \tau_{\gamma\delta} = \frac{1}{2} (x^0 \delta_{ij} + i e^{ijk} x^k)$. The integration of the above divergence over $\mathbb{R}_o^3$ reduces to the integration over $\mathbb{R}_o^4$. In fact for a coordinate system of the form

$$z_1 = R \cos \theta e^{i \frac{\varphi}{2}} , \quad z_2 = R \sin \theta e^{i \frac{\psi}{2}}$$

the measure on $\mathbb{R}_o^4$ reads

$$dz_1 dz_2 dz_3 = R^2 \left( \frac{R^2}{2} \right) \sin \theta d\theta d\varphi d\psi = x^0 dx^0 d\Omega d\psi$$

where $\frac{R^2}{2} = \tau^2 = x^0$. Since the elements of $A_o^3 \subset A_o^4$ do not depend on the angle $\psi$, their integration over $\mathbb{R}_o^4$ factorize into an integration over $\mathbb{R}_o^3$ as $\int x^0 dx^0 d\Omega A_o^3$ times an integration along the angle $\psi$, $\int d\psi = \pi$. Then the measure over the $\mathbb{R}_o^4$ is given by $d^3 x = x^0 dx^0 d\Omega$ which differs from the usual $d^3 x$ by factors of $\frac{1}{\pi}$. The integration of (32) in $\mathbb{R}_o^3$ is given by

$$\int \frac{dz_1 dz_2 dz_3}{\pi} (\partial_\alpha V_\alpha + \partial_\alpha V_\alpha) = \int d^3 x x^0 \partial_\alpha V_\alpha = \int d^3 x \partial_\alpha V_\alpha,$$

(33)

This relation suggests that for the integration over $\mathbb{R}_o^3$, the correspondence between the operator description and the c-number functions is given by

$$\int \frac{d^3 x}{x^0} f(x) \leftrightarrow \frac{(2\pi \theta)^2}{\pi} Tr \hat{\mathbf{K}} \left( \hat{\mathbf{f}}(\hat{\mathbf{x}}) \right)$$

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where \( f(x) \in \mathcal{A}_d^4 \) and \( \hat{f}(\hat{x}) \in \mathcal{A}_d^{\hat{4}} \). Now if we take \( V^i = F^i \), the divergence\(^2 \partial_i F^i \) reads

\[
\partial_i F^i = -\frac{1}{4\theta^2} \sum_{J=0}^{\infty} \left( \frac{2P_{J-\frac{1}{2}}(x^0)}{(J)(2J+1)} - \frac{2P_J(x^0)}{(J+1)(2J+1)} \right)
\]

Its integration over a volume in \( \mathbb{R}^3_\theta \) bounded by two fuzzy spheres of radii \( J_2 > J_1 \) respectively \( (J_1 \neq 0) \) reads

\[
\Phi = \int d^3x \sum_{J=J_1}^{J_2} \partial_i F^i = -\frac{1}{4\theta^2} \int d^3x \sum_{J=J_1}^{J_2} \left( \frac{2P_{J-\frac{1}{2}}(x^0)}{(J)(2J+1)} - \frac{2P_J(x^0)}{(J+1)(2J+1)} \right)
\]

The contribution of the first term for \( J + \frac{1}{2} \) automatically cancel those of the second term of \( J \) to leave the contributions at the boundaries as

\[
\Phi = -\frac{4\pi}{4\theta^2} \int (x^0)^2 dx^0 \sum_{J=J_1}^{J_2} \left( \frac{2P_{J-\frac{1}{2}}(x^0)}{(J)(2J+1)} - \frac{2P_J(x^0)}{(J+1)(2J+1)} \right) = 0 \quad (34)
\]

where we have used \( \int d^3x P_J(x^0) = 4\pi\theta^3(2J+2)(2J+1) \). The formula \( (34) \) shows clearly that the contributions to the divergence of the vector field \( \vec{F} \) to the integral come from the boundaries only i.e. the two fuzzy spheres of radii \( J_1 \) and \( J_2 \). This suggests that each of the two terms in \( (34) \) represent the flux (of magnitude \( -4\pi\theta \) ) through the fuzzy sphere \( J_{1,2} \) independently from the radii \( J_i \). The opposite signs represent the directions of the fluxes; entering or outgoing flux in the volume bounded by the two spheres of radii \( J_{1,2} \). The fact that the result of the integral is zero shows the absence of a charge between the spheres of radii \( J_{1,2} \neq 0 \) which express the absence of singularities of \( \vec{F} \) in this integration volume. In what follows we will see that the singularity of the \( U(1) \) instanton field \( \vec{F} \) at the origin is in fact due to a charge.

Each term in \( (34) \) can be represented by a flux of \( F_j \) through a fuzzy sphere of radius \( J \) as

\[
-\frac{4\pi}{4\theta^2} \int (x^0)^2 dx^0 \frac{2P_{J-\frac{1}{2}}(x^0)}{J(2J+1)} = -\frac{4\pi}{4\theta^2} \int x^0 dx^0 \frac{2x^0 P_{J-\frac{1}{2}}(x^0)}{J(2J+1)}
\]

\[
= \frac{4\pi}{4\theta} \int x^0 dx^0 \frac{1}{4\theta} \frac{8\theta^2 J(J+1) P_J(x^0)}{(J+1)(2J+1)^2}
\]

\[
= \frac{1}{2\theta} \int \frac{d^3x}{x^0} (x^i * F^i_j + F_j^i * x^i) = -4\pi\theta
\]

where we have used \( x^0 P_{J-\frac{1}{2}}(x^0) = 2J\theta P_J(x^0) \), \( x^i * x^i * P_J(x^0) = x^0 * (x^0 + 2\theta) * P_J(x^0) = 4\theta^2 J(J+1) P_J(x^0) \) and \( \int x^0 dx^0 P_J(x^0) = \theta^3 (2J+1) \).

The third term of the right hand side of \( (35) \) represents the analog of the classical

\(^2\)Since the derivative \( \partial_i \) is not a proper derivative with respect to the star-product \( [21] \), we perform the calculus of the star-products then use the derivatives \( \partial_i \) on the result which is a c-number function of \( x^i \).
flux of $F^j_i$ through the fuzzy sphere of radius $J$. In what follows we will establish the noncommutative analog of the Gauss theorem for any vector $\vec{V}(x) \in \mathcal{A}_0^3$ to justify the form of the third term of the right hand side of (35). For this end we calculate the flux in the operator formalism. In terms of operators, the integration (33) over a volume bounded by a fuzzy sphere of radius $J$ is represented by a trace over the Hilbert space representation $\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_J$ as

$$\Phi = \frac{(2\pi\theta)^2}{\pi} \sum_{j=0}^{J} \sum_{m=-J}^{J} \langle j, m | \frac{1}{2\theta} \left[ \hat{z}_\alpha, \hat{V}_\alpha \right] - \frac{1}{2\theta} \left[ \hat{z}_\alpha, \hat{V}_\alpha \right] | j, m \rangle$$

(36)

$$= 2\pi\theta \sum_{m=-J}^{J} \langle J, m | \hat{z}_\alpha \hat{V}_\alpha + \hat{V}_\alpha \hat{z}_\alpha | J, m \rangle$$

(37)

here the same mechanism of cancellation occurs for $j < J$ to leave only the contribution of the boundary term which is a trace over $\mathcal{H}_J$.

$$\hat{z}_\alpha \hat{V}_\alpha + \hat{V}_\alpha \hat{z}_\alpha$$

can now be translated in terms of c-number function as:

$$\hat{z}_\alpha \hat{V}_\alpha + \hat{V}_\alpha \hat{z}_\alpha = \hat{z}_\alpha V_\alpha + (2\theta) \frac{\partial}{\partial z_\alpha} V_\alpha + V_\alpha z_\alpha + (2\theta) \frac{\partial}{\partial z_\alpha} V_\alpha$$

(38)

By using (31) and (32) we can rewrite (38) in terms of star-product in $\mathcal{A}_0^3$ as:

$$x^i V_i + V_i x^i + x^0 \partial_i V_i = x^i * V_i + V_i * x^i$$

which corresponds in terms of operators to $\hat{x}^i \hat{V}_i + \hat{V}_i \hat{x}^i \in \mathcal{A}_0^3$. Then the right hand side of (37) can be rewritten as

$$\Phi = 2\pi\theta \sum_{m=-J}^{J} \langle J, m | \hat{x}^i \hat{V}_i + \hat{V}_i \hat{x}^i | J, m \rangle = 2\pi\theta Tr_{\mathcal{H}_J} \left( \hat{x}^i \hat{V}_i + \hat{V}_i \hat{x}^i \right)$$

(39)

$$= 2\pi\theta Tr_{\mathcal{H}} \left( \hat{x}_J \hat{V}_i^J + \hat{V}_i^J \hat{x}_J \right)$$

(40)

which is the flux of the vector field $\hat{V}$ through the fuzzy sphere of radius $J$. $
\hat{x}_J = \hat{P}_J \hat{x}_J \hat{P}_J$ and $\hat{V}_i^J = \hat{P}_J \hat{V}_i^J \hat{P}_J$ belong to the fuzzy sphere subalgebra $\mathcal{A}_J^3 \subset \mathcal{A}_0^3$.

Note that we can use the correspondence (3) to translate (31) in terms of operators as

$$\hat{V}_\alpha \left( \hat{x}, \hat{\tau} \right) = \hat{V}_i \left( \hat{x} \right) \frac{1}{2} \tau_{\alpha \beta} \hat{\tau}^{\beta}, \quad \hat{V}_\alpha \left( \hat{x}, \hat{\tau} \right) = \frac{1}{2} \hat{z}_\beta \tau_{\beta} \hat{V}_i \left( \hat{x} \right)$$

Then from (37), we get directly (40) by using the commutation relations of the coordinates $\hat{z}_\alpha$ and $\hat{\tau}_\alpha$, $Tr(\tau) = 0$ and (26).

By combining (33) with (40) translated in term of integral of $x^j_i * V^j_i + V^j_i * x^j_i$ over a fuzzy sphere of radius $J$, we can establish the noncommutative analog of Gauss theorem for any vector field $\vec{V} = \sum_{j=0}^{\infty} P_j * \vec{V} * P_j = \sum_{j=0}^{\infty} \vec{V}^j_j \in \mathcal{A}_0^3$ as
\[ \int d^3x \sum_{J=0}^{J_1} \partial_i V_i^J = \frac{1}{2\theta} \int \frac{d^3x}{x^0} (x_{J_1}^i \ast V_i^J + V_i^{J_1} \ast x_J^i) \]

where the left hand side is the integral of the divergence of \( \vec{V} \) over a volume in \( \mathbb{R}^3_\theta \) bounded by a fuzzy sphere of radius \( J_1 \) and the right hand side is a surface term which is the integral over the fuzzy sphere of radius \( J_1 \) which express the flux of the vector field \( \vec{V} \in \mathcal{A}^3_\theta \) through this sphere.

Applying the above formalism to the case of our strength field \( \hat{F}^i \), we get from (40) the flux through the fuzzy sphere as

\[ \Phi = 2\pi \theta Tr_H \left( \hat{x}^i \hat{F}^i + \hat{F}^i \hat{x}^i \right) = 2\pi \theta Tr_H \left( \frac{-2\hat{x}_{J_1} \hat{P}_J}{4\theta^2 J (J+1)(2J+1)} \right) \]

\[ = 4\pi \theta Tr_H \left( \frac{-J(J+1)}{J(J+1)(2J+1)} \right) = -4\pi \theta \sum_{m=-J}^{J} \frac{1}{(2J+1)} = -4\pi \theta. \]

This result shows that the Flux \( \Phi \) is independent from the choice of the representation \( J \) i.e. it is independent from the choice of the sphere radius \( \Phi = -4\pi \theta \forall J \) the radius of the fuzzy sphere. (41)

This is similar to the classical Gauss theorem which states that the total charge calculated by the flux of a coulombian or Dirac monopole field through a closed surface is independent from the shape of this closed surface.

Thus (41) is the noncommutative analog of the Gauss theorem on the noncommutative space \( \mathbb{R}^3_\theta \).

This result shows also that the quantity \( \Phi \) represents a source of the field \( \hat{\vec{F}} \) which we can be interpreted as a magnetic charge. Thus the \( U(1) \)-one-instanton on \( \mathbb{R}^4_\theta \) gives rise to a magnetic charge at the origin of the noncommutative space \( \mathbb{R}^3_\theta \).

5.1 \( \mathcal{H}_{KL} \)-Instanton solution Flux through the fuzzy sphere and gauge invariance

In this section we try to see in what sense the magnetic flux is invariant under partial isometries. First we rewrite the transformation \( U \) in the Schwinger basis as

\[ U = \sum_{J=0}^{\infty} \sum_{m=-J}^{J} |J, m\rangle \langle J + P, m + Q| = \sum_{J=0}^{\infty} \sum_{m=-J}^{J} |J - P, m - Q\rangle \langle J, m| \]

where \( P = \frac{1}{2} (K + L) \), \( Q = \frac{1}{2} (K - L) \). Then the components of the transformed field is given by

\[ \hat{\vec{F}}_{KL} = U \hat{\vec{F}}_{00} U = \hat{\vec{F}}_u = -\frac{\theta}{x_u \left( \frac{x_0^0}{x_u^0} + 2\theta \right) \left( \frac{2x_0^0}{x_u^0} + 2\theta \right)} \hat{\vec{F}}_u \]

(43)
where

\[ \hat{x}_u^i = U^\dagger \hat{x}_u^i U, \text{ and } \hat{x}_u^0 = U^\dagger \hat{x}_u^0 U. \]  \hspace{1cm} (44)

These new coordinates \( \hat{x}_u^i \) satisfy the same commutation relations as the old ones

\[ [\hat{x}_u^i, \hat{x}_u^j] = i2\theta \epsilon^{ijk} \hat{x}_u^k, \quad [\hat{x}_u^0, \hat{x}_u^i] = 0. \]

and

\[ \hat{x}_u^i \hat{x}_u^i = \hat{x}_u^0 (\hat{x}_u^0 + 2\theta). \]

Note that in this case the formalism is expressed in the subalgebra \( \hat{A}_{3\theta,u}^3 \subset U^\dagger \hat{A}_{3\theta}^3 U \subset \hat{A}_{4\theta}^4 \). The formula (44) is similar to the transformations of the angular momentum in quantum mechanics \(^3\). These transformations are concretely given by

\[ \hat{x}_u^3 = \sum_{J=P}^\infty \sum_{m=-(J-P)+Q}^{J-P+Q} 2\theta \theta (m - Q) |J, m\rangle \langle J, m|, \]

\[ \hat{x}_u^+ = \sum_{J=P}^\infty \sum_{m=-(J-P)+Q}^{J-P+Q} 2\theta \sqrt{(J - P) (J - P + 1) - (m - Q) (m - Q + 1)} |J, m + 1\rangle \langle J, m|, \]

\[ \hat{x}_u^- = \sum_{J=P}^\infty \sum_{m=-(J-P)+Q}^{J-P+Q} 2\theta \sqrt{(J - P) (J - P + 1) - (m - Q) (m - Q - 1)} |J, m - 1\rangle \langle J, m|, \]

\[ \hat{x}_u^0 = \sum_{J=P}^\infty \sum_{m=-(J-P)+Q}^{J-P+Q} 2\theta (J - P) |J, m\rangle \langle J, m|. \]

As in the above section, we can use the projector on the fuzzy sphere \( \hat{P}_J \) with \( J = J' + P \), to define the map (surjection): \( H_J \rightarrow H_{J'} \)

\[ U_J = U \hat{P}_J = \sum_{m=-(J-P)+Q}^{J-P+Q} |J - P, m - Q\rangle \langle J, m| \]

satisfying

\[ U_J^\dagger U_J = \sum_{m=-(J-P)+Q}^{J-P+Q} |J, m\rangle \langle J, m|. \]

which is a projector on a part of states of the fuzzy sphere of radius \( J \) and

\[^3\]The difference between the quantum mechanical gauge transformation and the isometry we discuss in our present paper is that unitary gauge transformations in quantum mechanics just rotate the states \( |Jm\rangle \), whereas in our case the isometry (which is a gauge transformation) generate a shift of \( J \) then a rotation of the same basis vectors \( |Jm\rangle \).
\[ U_J U_J^\dagger = \sum_{m=-(j-P)+Q}^{j-P+Q} |j - P, m - Q\rangle \langle j - P, m - Q| = P_J' \]

is a projector on the fuzzy sphere of radius \( J' \). Under \( U_J \), the coordinates \( \hat{x}^i \) transform as

\[ \hat{x}^i_{u, J} = U_J^\dagger \hat{x}^i U_J = \hat{P}_J \hat{x}^i \hat{P}_J. \]

They satisfy the commutation rules on the fuzzy sphere

\[ [\hat{x}^i_{u, J}, \hat{x}^j_{u, J}] = i2\theta \epsilon^{ijk} \hat{x}^k_{u, J}, \quad [\hat{x}^0_{u, J}, \hat{x}^i_{u, J}] = 0. \]

and

\[ \hat{x}^i_{u, J} \hat{x}^i_{u, J} = \hat{x}^0_{u, J}(\hat{x}^0_{u, J} + 2\theta), \]

where \( \hat{x}^0_{u, J} = U_J^\dagger \hat{x}^0 U_J \).

The transformed coordinates generate the subalgebra \( \hat{S}^2_{u, J} \) which is an algebra describing a truncated fuzzy sphere realized in the Hilbert subspace \( \mathcal{H}_{u, J} \subset \mathcal{H}_J \) spanned by the basis \( |J, m - Q\rangle \) with \( J = J' + P \) and \( J' \leq m \leq J' \).

In this truncated space \( \mathcal{H}_{u, J} \) the coordinates \( \hat{x}^i_{u, J} \) act as

\[ \hat{x}^3_{u, J} |J, m\rangle = 2\theta (m - Q) |J, m\rangle, \]
\[ \hat{x}^+_{u, J} |J, m\rangle = 2\theta \sqrt{(J - P)(J - P + 1) - (m - Q)(m - Q + 1)} |J, m + 1\rangle, \]
\[ \hat{x}^-_{u, J} |J, m\rangle = 2\theta \sqrt{(J - P)(J - P + 1) - (m - Q)(m - Q - 1)} |J, m - 1\rangle \]

which show that the highest state is \( |J, J - P + Q\rangle \langle \hat{x}^+_{u, J} |J, J - P + Q\rangle = 0 \)
and the lower state is \( |J, -(J - P) + Q\rangle \langle \hat{x}^-_{u, J} |J, -(J - P) + Q\rangle = 0 \).

Now this formalism is expressed in the subalgebra \( \hat{S}^2_{u, J} \subset \hat{A}^3_{\theta, u} \subset \hat{A}^3_{\theta} \subset \hat{A}^4_{\theta} \).

The calculus of the flux is now given by a trace over \( \mathcal{H}_{u, J} \).

\[ \Phi = 2\pi \theta Tr_{\mathcal{H}_{u, J}} \left( \hat{x}^i_{u, J} \hat{F}^i_{u, J} + \hat{F}^i_{u, J} \hat{x}^i_{u, J} \right) = -2\pi \theta Tr_{\mathcal{H}_{u, J}} \frac{2\theta \hat{x}^i_{u, J} \hat{x}^i_{u, J}}{\hat{x}^0_{u, J}(\hat{x}^0_{u, J} + 2\theta)(2\hat{x}^0_{u, J} + 2\theta)} \]

\[ = \frac{-2(2\pi \theta)}{(2J - 2P^2 + 1)} \sum_{m=-(J-P)+Q}^{J-P+Q} 1 = -4\pi \theta \]

Thus we obtain the same result as in (41) and also in this gauge the flux of the transformed strength field is independent from the choice of the radius i.e. independent from the Hilbert space representation \( \mathcal{H}_{u, J} \).

\[ \Phi = -4\pi \theta \quad \forall J > P \text{ the radius of the fuzzy sphere, } \forall K, L \geq 0 \]

or \( P = 0, \frac{1}{2}, 1, \ldots \infty \) and \( Q \) runs by integer step over \( -P \leq Q \leq P \).
Discussion and conclusion

In this paper we have investigated $U(1)$-one-instanton solutions with the intension of studying the relation between the instanton number and partial isometries. This allows us to see that the partial isometry transformations which act like $U(1)$ gauge transformations leave invariant the instanton number.

The second result, obtained in this paper, is that the description of the $U(1)$-one-instanton solution in terms of $\hat{A}_3$ algebra, gives rise to an object behaving like Dirac magnetic monopole field. Its flux through the fuzzy spheres is independent from the choice of the radius of these ones. This fact can be seen as the noncommutative analog of the Gauss theorem for the coulombian forces in the classical case. Furthermore we have seen that the flux is invariant under the partial isometry transformations of the $U(1)$ instanton field, provided that the coordinates on $\mathbb{R}_3^\theta$ transform as $\hat{x}_i = U^\dagger \hat{x}_i U$. This transformation preserve the commutation rules of the coordinates algebra on $\mathbb{R}_3^\theta$, and confirm the mixture between gauge theory and geometrical transformations acting on the base space, which, in our case, rotate the fuzzy sphere and shifts the radius.

Finally we hope to comment the fact that the flux $\Phi$ can be related to the quantization of the magnetic charge; This can be done by remarking that the action of the $\mathbb{R}_3^\theta$ coordinates is the same as the quantum mechanical angular momentum. Thus we have in one part:

\[
J^3 |J, m\rangle = m \hbar |J, m\rangle
\]
\[
J^\pm |J, m\rangle = \hbar \sqrt{J (J + 1) - m (m \pm 1)} |J, m \pm 1\rangle
\]

and in the other part:

\[
\hat{x}^3 |J, m\rangle = m (2\theta) |J, m\rangle
\]
\[
\hat{x}^\pm |J, m\rangle = (2\theta) \sqrt{J (J + 1) - m (m \pm 1)} |J, m \pm 1\rangle
\]

So we can identify $\hbar$ with $(2\theta)$, then the flux will be given by $\Phi = -2\pi \hbar$, which correspond to a monopole magnetic charge $-1$.

As further directions one can generalize the statements above on noncommutative multi-$U(1)$-instanton solutions on $\mathbb{R}_3^\theta$ by searching for general solutions to calculate their flux through fuzzy spheres to hope to find higher monopole charges.

Acknowledgment

I would like to thank M. Dubois-Violette for hepful discussion.

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