1 Introduction

Suppose that $X$ is an orbifold. In general, $K_X$ is an orbifold vector bundle or a $Q$-divisor only. When the $X$ is so called Gorenstein, $K_X$ is a bundle or a divisor. For Gorenstein orbifold, a resolution $\pi : Y \to X$ is called a crepant resolution if $\pi^* K_X = K_Y$. Here, ”crepant” can be viewed as a minimality condition with respect to canonical bundle. Crepant resolution always exists when dimension is two or three. A nice way to construct it is to use Hilbert scheme of points. However, the crepant resolution in dimension three is not unique. Different crepant resolutions are connected by flops. When the dimension is bigger than four, the crepant resolution does not always exist. It is an extremely interesting problem in algebraic geometry to find out when it does exist. One of famous example is Hilbert scheme of points of algebraic surfaces, which is a crepant resolution of the symmetry product of an algebraic surface. When orbifold string theory was first constructed over global quotient [DHVW], one of the first invariants from it is orbifold Euler characteristic. It was conjectured that the orbifold Euler characteristic is the same as the Euler characteristic of its crepant resolution. This fits well with McKay correspondence in algebro-geometry. It had been the main attraction before the current development. By the work of Batyrev and others [B1] [DL], this conjecture has been extended and solved for orbifold Hodge number of Gorenstein global quotients. Very recently, it was solved in the complete generalities by Lupercio-Poddar [LP] and Yusuda [Y].

It has long been an interesting problem to compute cohomology of Hilbert scheme of points of algebraic surfaces. A great deal of works have been done on this topic. Among all the progress, a less known but crucial ingredient is the link to the orbifold cohomology of the symmetry product. Suppose that $M$ is an algebraic surface. We use $M^{[n]}$ to denote the Hilbert scheme of points of length $n$ of $M$. In his thesis [G], Göttsche computed the generating function of Euler number $\sum_{n=1}^{\infty} \chi(M^{[n]})q^n$ and showed that it has a surprising modularity. In 1994, in order to explain its modularity, Vafa-Witten [VW] computed $\mathcal{H} = \oplus_n H^*(M^{[n]}/S_n, \mathbb{C})$. Motivated by orbifold conformal field theory, they directly wrote $\mathcal{H}$ as a ”Fock space” or a representation of Heisenberg algebra. Then, the generating function of Euler characteristic is interpreted as the correlation function of an elliptic curve. Therefore, it should be invariant under modular transformations of the elliptic curve. This shows that the space of cohomology itself has more structure. Orbifold string theory conjecture predicates that $\oplus_n H^*(M^{[n]}, \mathbb{C})$ should also admit a representation of Heisenberg algebra. This conjecture was verified by a beautiful work of Nakajima [N] and others. One of theme of this short note is that the orbifold $M^n/S_n$ will continue to play a crucial role to compute ring structure of $H^*(M^{[n]}, \mathbb{C})$.

During last two years, there was a surge of activities to study mathematics of the orbifold string theory, which author called the stringy orbifolds. A few curious physical concepts such as the orbifold Euler-Hodge numbers of global quotients found their places in a much broader and deeper theory. For example, a new cohomology (orbifold cohomology) was constructed [CR1]. The growth was so explosive that the author believes that there is an emerging new subject of mathematics. He learned in graduate school that the test of the relevance of a new theory has been the progress it
made from old problems. Therefore, it is particularly significant to revisit the problem of computing the ring structure of $M[n]$ and a crepant resolution in general. A lot of information is known for additive structure of $H^*(M[n], \mathbb{C})$. The ring structure of $M[n]$ is quite subtle and more interesting. Partial results has been obtained by Fantechi-G"ottsche [FG], Ellingsrud-Stromme [ES1] [ES2], Beauvill [Bea], Mark [Mar]. Based on an important observation by Frenkel and Wang [FW], Lehn-Sorger [LS1] determined the cohomology ring of $(\mathbb{C}^2)^n$. At the same time, the author was computing orbifold cohomology $(\mathbb{C}^2)^n/S_n$ and the result from both calculations matches perfectly. Based on physical motivation and the strong evidence from $(\mathbb{C}^2)^n/S_n$. The author proposed a conjecture in the case of a hyperkahler resolution.

**Cohomological Hyperkahler Resolution Conjecture:** Suppose that $\pi : Y \rightarrow X$ is a hyperkahler resolution. Then, the ordinary cohomology ring of $Y$ is isomorphic to the orbifold cohomology ring of $X$.

In the case of Hilbert scheme points of surfaces, the Cohomological Hyperkahler Resolution Conjecture (CHRC) implies that $K3^n$,$(T^4)^n$ have isomorphic cohomology ring as the orbifold cohomology rings of $K3^n/S_n$,$(T^4)^n/S_n$. The later was proved recently by the beautiful works of Lehn-Sorge [LS2], Fantechi-G"ottsche [FG2] and Uribe [U]. It should be mentioned that Fantechi-G"ottsche-Uribe’s work computed the orbifold ring structure of $X^n/S_n$ for an arbitrary complex manifold $X$ which may or may not be $K3,T^4$. There is a curious phenomenon that over rational number Lehn-Sorge, Fantanch-G"ottsche and Uribe showed that one must modify the ring structure of the orbifold cohomology by a sign in order to match to the cohomology of Hilbert scheme. However, Qin-Wang observed that such a sign modification is unnecessary over complex number [QW]. All the conjectures stated in this article (in fact any conjecture motivated by physics) are the statements over complex number. The ring structure of $X[n]$ for a general algebraic surface $X$ is still unknown.

It is easy to check that CHRC is false if we drop the hyperkahler condition. One of main purposes of this article is to propose a conjecture for the arbitrary crepant resolution.

As mentioned previously, the crepant resolutions are not unique. The different crepant resolutions are connected by "K-equivalence" [W]. Two smooth (or Gorenstein orbifolds) complex manifolds $X,Y$ are $K$-equivalent iff there is a common resolution $\phi, \psi : Z \rightarrow X,Y$ such that $\phi^*K_X = \psi^*K_Y$. Batyrev-Wang [B], [W] showed that two $K$-equivalent projective manifolds have the same betti number. It is natural to ask if they have the same ring structures. This question is obviously related to CHRC. Suppose that CHRC holds for non-hyperkahler resolutions. It implies that different resolutions ($K$-equivalent) have the same ring structures. Unfortunately, they usually have different ring structures, and hence CHRC fails in general. It is easy to check this in case of three dimensional flops. A key idea to remedy the situation is to include the quantum corrections. The author proposed

**Quantum Minimal Model Conjecture:** Two $K$-equivalent projective manifolds have the same quantum cohomologies.

Li and the author proved Quantum Minimal Model conjecture in complex dimension three. In higher dimensions, it seems to be a difficult problem. In many ways, Quantum Minimal Model Conjecture unveils the deep relation between the quantum cohomology and the birational geometry [R2]. However, it is a formidable task to master the quantum cohomology machinery for any non-experts. In this article, we proposed another conjecture focusing on the cohomology instead of the quantum cohomology. As mentioned before, the cohomology ring structures are not isomorphic for $K$-equivalent manifolds. Therefore, some quantum information must be included. Our new con-
jecture requires a minimal set of quantum information involving the GW-invariants of exceptional rational curve.

Finally, the motivation behind the conjectures should be described here. Let’s first go back to the motivation of CHRC. Naively, physics indicates that the orbifold quantum cohomology of $X$ should be "equivalent" to the quantum cohomology of $Y$. It is not clear how to formulate the precise meaning of the "equivalence". However, for the hyperkahler resolutions, there are no quantum corrections and the quantum cohomology is just cohomology. All the difficulties to formulate the "equivalence" of the quantum information disappear. We should just get an isomorphism between cohomologies. This is the reasoning behind Cohomological Hyperkahler Resolution Conjecture. For a general crepant resolution, quantum corrections do appear. Mathematically, it means that the cup product of a crepant resolution is the orbifold cup product of the orbifold plus some quantum corrections. A further study shows that the quantum corrections come from the GW-invariants of exceptional rational curves only. However, when we try to count these quantum corrections, we encounter a serious problem. The quantum corrections appear to be an infinite series of the GW-invariants corresponding to the multiple degree of exceptional rational curves. In the quantum cohomology, we insert a quantum variable $q$ to keep track of this degree. Intuitively, we should set $q = 1$. By checking a few examples, one can find that the quantum corrections diverge at $q = 1$ and some kind of "renormalization" is necessary. If we believe that a solution can be found in the other value of $q$, it is instructive to find the possible value of $q$. When we match the quantum cohomologies under 3-dimensional flops, there is a change of quantum variable $q \rightarrow \frac{1}{q}$. If there is an uniform way to set the value of $q$, said $q = \lambda$. Then, we must have $\lambda = \frac{1}{\lambda}$. Hence, $\lambda^2 = 1$. Namely, the other choice is $q = -1$. In a different context, there was beautiful works by P. Aspinwall \[A\] and E. Wendland \[W\] in physics concerning the conformal theory on $K3$ \[W\]. It suggests that after the quantization, the value of B-field "shifts" to "$q = -1$"!

This short article is organized as follows. In the section two, we will formulate our conjectures. In the section three, we will verify our conjectures using several examples. Special thanks goes to P. Aspinwall and E. Witten for bringing me the attention of \[W\] and K. Wendland for a wonderful talk on Workshop on Mathematical Aspect of Orbifold String Theory to explain \[W\]. Finally, I would like to thank Wei-Ping Li, Zhenbo Qin for interesting discussions.

## 2 Conjectures

Suppose that $\pi : Y \to X$ is one crepant resolution of Gorenstein orbifold $X$. Then, $\pi$ is a Mori contraction and the homology classes of rational curves $\pi$ contracted are generated by so called extremal rays. Let $A_1, \cdots, A_k$ be an integral basis of extremal rays. We call $\pi$ non-degenerate if $A_1, \cdots, A_k$ are linearly independent. For example, the Hilbert-Chow map $\pi : M^{[n]} \to M^n/S_n$ satisfies this hypothesis. Then, the homology class of any effective curve being contracted can be written as $A = \sum_i a_i A_i$ for $a_i \geq 0$. For each $A_i$, we assign a formal variable $q_i$. Then, $A$ corresponds to $q_1^{a_1} \cdots q_k^{a_k}$. We define a 3-point function

$$< \alpha, \beta, \gamma >_{qc} (q_1, \cdots q_k) = \sum_{a_1, \cdots, a_k} \Psi^X_A(\alpha, \beta, \gamma) q_1^{a_1} \cdots q_k^{a_k},$$

where $\Psi^X_A(\alpha, \beta, \gamma)$ is Gromov-Witten invariant and $qc$ stands for the quantum correction. We view $< \alpha, \beta, \gamma >_{qc} (q_1, \cdots, q_k)$ as analytic function of $q_1, \cdots q_k$ and set $q_i = -1$ and let

$$< \alpha, \beta, \gamma >_{qc} = < \alpha, \beta, \gamma >_{qc} (-1, \cdots, -1).$$
We define a quantum corrected triple intersection

\[ <\alpha,\beta,\gamma>_\pi = <\alpha,\beta,\gamma> + <\alpha,\beta,\gamma>_{qc}, \]

where \( <\alpha,\beta,\gamma> = \int_X \alpha \cup \beta \cup \gamma \) is the ordinary triple intersection. Then we define the quantum corrected cup product \( \alpha \cup_\pi \beta \) by the equation

\[ <\alpha \cup_\pi \beta,\gamma> = <\alpha,\beta,\gamma>, \]

for arbitrary \( \gamma \). Another way to understand \( \alpha \cup_\pi \beta \) is as following. Define a product \( \alpha \ast_{qc} \beta \) by the equation

\[ <\alpha \ast_{qc} \beta,\gamma> = <\alpha,\beta,\gamma>_{qc} \]

for arbitrary \( \gamma \in H^*(Y,\mathbb{C}) \). Then, the quantum corrected product is the ordinary cup product corrected by \( \alpha \ast_{qc} \beta \). Namely,

\[ (2.3) \quad \alpha \cup_\pi \beta = \alpha \cup \beta + \alpha \ast_{qc} \beta. \]

We denote the new quantum corrected cohomology ring as \( H^*_\pi(Y,\mathbb{C}) \).

**Cohomological Crepant Resolution Conjecture:** Suppose that \( \pi \) is non-degenerate and hence \( H^*_\pi(Y,\mathbb{C}) \) is well-defined. Then, \( H^*_\pi(Y,\mathbb{C}) \) is the ring isomorphic to orbifold cohomology ring \( H^*_{orb}(X,\mathbb{C}) \).

Recently, Li-Qin-Wang [LQW3] proved a striking theorem that the cohomology ring of \( M^{[n]} \) is universal in the sense that it depends only on homotopy type of \( M \) and \( K_X \). Combined with their result, CCRC yields

**Conjecture:** For \( M^{[n]} \), \( \ast \) product depends only on \( K_M \).

It suggests an interesting way to calculate \( \ast \) product by first finding a universal formula (depending only on \( K_M \)) and calculating a special example such as \( \mathbb{P}^2 \) to determine the coefficient.

Next, we formulate a closely related conjecture for \( K \)-equivalent manifolds.

Suppose that \( X, X' \) are \( K \)-equivalent and \( \pi : X \to X' \) is the birational map. Again, exceptional rational curves makes sense. Suppose that \( \pi \) is nondegenerate. Then, we go through the previous construction to define ring \( H^*_\pi(X,\mathbb{C}) \).

**Cohomological Minimal Model Conjecture:** Suppose that \( \pi, \pi^{-1} \) are nondegenerate. Then, \( H^*_\pi(X,\mathbb{C}) \) is the ring isomorphic to \( H^*_\pi^{-1}(X',\mathbb{C}) \)

Here \( \pi^{-1} : X' \to X \) is the inverse birational transformation of \( \pi \). When \( X, X' \) are the different crepant resolutions of the same orbifolds, Cohomological minimal model conjecture follows from Cohomological crepant resolution conjecture. However, it is well-known that most of \( K \)-equivalent manifolds are not crepant resolution of orbifolds. Cohomological minimal model conjecture can be generalized to orbifold provided that the quantum corrections are defined using orbifold Gromov-Witten invariants introduced by Chen-Ruan [CR2].

**Remark 3.4:** (1) The author does not know how to define quantum corrected cohomology if \( \pi \) is not nondegenerate. (2) All the conjectures in this section should be understood as the conjectures up to certain slight modifications (see next sections).
3 Verification of Conjectures

Example 3.1: Suppose that $\Sigma$ is one Riemann surface of genus $\geq 2$ and $E \to \Sigma$ is a rank two bundle such that $C_1(E) = 2g - 2$. Then, $E$ is an example of local Calabi-Yau manifold. Let $\tau$ be the involution acting on $E$ as the multiplication of $-1$. $X = E/\tau$ is a Calabi-Yau orbifold. Let $\tilde{E}$ be the blow-up of $E$ along $\Sigma$. The action of $\tau$ extends over $\tilde{E}$. Let $Y = \tilde{E}/\tau$. The projection $\pi: Y \to X$ is a crepant resolution of $X$. Let’s verify Cohomological Crepant Resolution Conjecture in this case. For simplicity, we consider the even cohomology only. Moreover, it is easy to compute triple intersections $\langle \alpha, \beta, \gamma \rangle > 0$ for $\alpha, \beta, \gamma \in H^2$.

Note that $X$ is homotopic equivalent to $\Sigma$. Therefore, the nontwisted sector contributes one generator to $H^2_{\text{orb}}(X, \mathbb{C})$. Let $\alpha$ be the generator with the integral one on $\Sigma$. Since $\Sigma$ has local group $\mathbb{Z}_2$, it generates a twisted sector with degree shifting number 1. It contributes to a generator $\beta$ to $H^2_{\text{orb}}(X, \mathbb{C})$, where $\beta$ represents the constant function 1 on the twisted sector. It is easy to compute

\[
\langle \alpha, \alpha, \alpha \rangle > 0, \langle \alpha, \alpha, \beta \rangle > 0, \langle \alpha, \beta, \beta \rangle = \frac{1}{2}, \langle \beta, \beta, \beta \rangle > 0.
\]

Let’s compute $H^2(Y, \mathbb{C})$. Let $\alpha' = \pi^* \alpha$. The exceptional divisor $S$ is a ruled surface of $\Sigma$. Let $\beta'$ be its Poincaré dual. It is clear that

\[
\langle \alpha', \alpha', \alpha' \rangle > 0, \langle \alpha', \alpha', \beta' \rangle > 0.
\]

Nonzero ones are $\langle \alpha', \beta', \beta' \rangle = K_S[C] = -2, \langle \beta', \beta', \beta' \rangle = K_S^2 = 8(1 - g)$ (Lemma 3.2), where $K_S$ is the canonical bundle of $S$ and $C$ is the fiber of $S$.

Next, we compute the correction term. By Wilson [W] (Lemma 3.3), a small complex deformation can deform $S$ into $2(g - 1)$ many $\mathcal{O}(-1) + \mathcal{O}(-1)$ curves. Wilson’s argument is completely local and works for this case. It is well-known that

\[
\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{\text{qc}} (g) = 2(g - 1)\gamma_1([C])\gamma_2([C])\gamma_3([C]) \sum_{i=1}^3 q^i = 2(g - 1)\gamma_1([C])\gamma_2([C])\gamma_3([C]) \frac{q}{1 - q}.
\]

Hence,

\[
\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{\text{qc}} = (1 - g)\gamma_1([C])\gamma_2([C])\gamma_3([C]).
\]

Moreover, $\alpha'[C] = 0, \beta'[C] = K_S[C] = -2$. Then, we obtain that

\[
\langle \alpha', \alpha', \alpha' \rangle_{\text{qc}} = \langle \alpha', \alpha', \beta' \rangle_{\text{qc}} = \langle \alpha', \beta', \beta' \rangle_{\text{qc}} = 0.
\]

\[
\langle \beta', \beta', \beta' \rangle_{\text{qc}} = -8(1 - g).
\]

After corrected by $\langle \ldots \rangle_{\text{qc}}$, $\langle \beta, \beta, \beta \rangle > \langle \beta', \beta', \beta' \rangle > \pi$ match perfectly. But there is still a discrepancy between $\langle \alpha, \beta, \beta \rangle > \langle \alpha', \beta', \beta' \rangle > \pi$ which is the reminisce of discrepancy for surface quotient singularities (See Remark). We don’t know a canonical way to construct a homomorphism between orbifold cohomology and cohomology of its crepant resolution. Nevertheless, up to a sign, the map $\alpha \to \alpha', \beta \to 2\beta'$ gives an ring isomorphism. □

Example 3.2: Next, we use the work of Li-Qin [LQ] to verify Cohomological Crepant Resolution Conjecture for $M^{[2]}$. To simplify the formula, we assume that $M$ is simply connected.
It is easy to compute the orbifold cohomology $H^*_{orb}(X,\mathbb{C})$ for $X = M^2/\mathbb{Z}_2$. The nontwisted sector can be identified with invariant cohomology of $M^2$. Let $h_i \in H^2(M,\mathbb{C})$ be a basis and $H \in H^4(M,\mathbb{C})$ be Poincare dual to a point. Then, the cohomology of the nontwisted sectors are generated by $1, 1 \otimes h_i + h_i \otimes 1, 1 \otimes H + H \otimes 1, h_i \otimes h_j + h_j \otimes h_i, h_i \otimes H + H \otimes h_i, H \otimes H$. The twisted sector is diffeomorphic to $M$ with degree shifting number 1. We use $\bar{1}, \bar{h}_i, \bar{H}$ to denote the generators. They are of degrees 2, 4, 6. By the definition, triple intersections

$$<\text{twisted sector}, \text{nontwisted sector}, \text{nontwisted sector}>=0,$$

$$<\text{twisted sector}, \text{twisted sector}, \text{twisted sector}>=0.$$  

Following is the table of nonzero triple intersections involving classes from the twisted sector

$$<\bar{1}, \bar{1}, 1 \otimes H + H \otimes 1>=1, <\bar{1}, \bar{1}, h_i \otimes h_j + h_j \otimes h_i>=<h_i, h_j>,$$

(3.7) $$<\bar{1}, \bar{h}_i, h_j \otimes 1 + h_j \otimes 1>=<h_i, h_j>.$$  

Next, we review the construction of $Y = M^{[2]}$. Let $\tilde{M}^2$ be the blow-up of $M^2$ along the diagonal. Then, $\mathbb{Z}_2$ action extends to $\tilde{M}^2$. Then, $Y = M^2/\mathbb{Z}_2$. It is clear that we should map the classes from nontwisted sector to its pull-back $\pi: Y \to X$. We use the same notation to denote them. The exceptional divisor $E$ of Hilbert-Chow map $\pi: Y \to X$ is a $P^1$-bundle over $M$. Let $\bar{1}, \bar{h}_i, \bar{H}$ be the Poincare dual to $E$, $p^{-1}(PD(h_i))$ and fiber $[C]$, where $p: E \to M$ is the projection.

Notes that $\bar{1}|_E = 2\mathcal{E}$, where $\mathcal{E}$ is the tautological divisor of $P^1$-bundle $E \to M$. It is clear that $E = P(N_{\Delta(X)|X^2})$, where $\Delta(X) \subset X^2$ is the diagonal. Hence,

(3.8) $$<\bar{1}, \bar{1}, \bar{h}_i> = 4\mathcal{E}^2|_{p^{-1}(PD(h_i))} = 4C_1(N_{\Delta(X)|X^2})\mathcal{E}|_{p^{-1}(PD(h_i))} = -4 <C_1(X), h_i>.$$  

(3.9) $$<\bar{1}, \bar{1}, 1 \otimes H + H \otimes 1>= 2\mathcal{E}(1 \otimes H + H \otimes 1)(E) = 4\mathcal{E}(C) = -4.$$  

(3.10) $$<\bar{1}, \bar{1}, h_i \otimes h_j + h_j \otimes h_i> = -4 <h_i, h_j>.$$  

(3.11) $$<\bar{1}, \bar{h}_i, 1 \otimes h_j + h_j \otimes 1> = -4 <h_i, h_j>.$$  

Others are zero.

The quantum corrections have been computed by Li-Qin [LQ] (Proposition 3.021). The only nonzero terms are

$$<\bar{1}, \bar{1}, \bar{h}_i>_{qc}(q) = \sum_{d=1}^{\infty} \frac{1}{d^2} (d[C])^2 \Psi_{\frac{d}{d[C]}}(\bar{h}) q^d$$

(3.12) $$= \sum_{d=1}^{\infty} \frac{d^2 (2<\bar{K}_X, h_i>) q^d}{d^2}.$$  

Hence,

(3.13) $$<\bar{1}, \bar{1}, \bar{h}_i>_{qc} = 4 <\bar{K}_X, h_i> = 4 <C_1(X), h_i>$$

 cancelling $<\bar{1}, \bar{1}, \bar{h}_i>$.  

It is clear that the map $\bar{1} \to 21, \bar{h} \to 2\bar{h}, \bar{H} \to \bar{H}$ is a ring isomorphism. $\square$

Next, we give two examples to verify Cohomological Minimal Model Conjecture (CMMC).
Example 3.4: The first example is the flop in dimension three. This case has been worked out in great detail by Li-Ruan [LR]. For example, they proved a theorem that quantum cohomology rings are isomorphic under the change of the variable $q \rightarrow \frac{1}{q}$. Notes that if we set $q = -1$, $\frac{1}{q} = -1$. We set other quantum variables zero. Then, the quantum product becomes the quantum corrected product $\alpha \cup_{\pi} \beta$. Hence, CMMC follows from Li-Ruan’s theorem. However, It should be pointed out that one can directly verify CMMC without using Li-Ruan’s theorem. In fact, it is an much easier calculation.

Example 3.5: There is a beautiful four dimensional birational transformation called Mukai transform as follows. Let $P^2 \subset X^4$ with $N_{P^2|X^4} = T^*P^2$. Then, one can blow up $P^2$. The exceptional divisor of the blow up is a hypersurface of $P^2 \times P^2$ with the bidegree $(1, 1)$. Then, one can blow down in another direction to obtain $X'$. $X, X'$ are $K$-equivalent. In his Ph.D thesis [Z], Wanchuan Zhang showed that the quantum corrections $<\alpha, \beta, \gamma >_{qc}$ are trivial, and cohomologies of $X, X'$ are isomorphic.

4 Remarks

In the computation of orbifold cohomology of symmetry product and its relation to that of Hilbert scheme of points, there are two issues arisen. It was showed in the work of Lehn-Sorger, Fantechi-Göttsche and Uribe that one has to add a sign in the definition of orbifold product in order to match that of Hilbert scheme of points over rational number. This sign was described as follows.

Recall the definition of orbifold cup product

\begin{equation}
\alpha \cup_{\text{orb}} \beta = \sum_{(h_1, h_2) \in T_2, h_i \in (g_i)} (\alpha \cup_{\text{orb}} \beta)_{(h_1, h_2)},
\end{equation}

where $(\alpha \cup_{\text{orb}} \beta)_{(h_1, h_2)} \in H^*(X(h_1, h_2), \mathbb{C})$ is defined by the relation

\begin{equation}
< (\alpha \cup_{\text{orb}} \beta)_{(h_1, h_2)}, \gamma >_{\text{orb}} = \int_{X(h_1, h_2)} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge e_A(E(g_i)).
\end{equation}

for $\gamma \in H_c^*(X_h(h_1, h_2), \mathbb{C})$. Then, we add a sign to each term.

\begin{equation}
\alpha \cup_{\text{orb}} \beta = \sum_{(h_1, h_2) \in T_2, h_i \in (g_i)} (-1)^{\epsilon(h_1, h_2)} (\alpha \cup_{\text{orb}} \beta)_{(h_1, h_2)},
\end{equation}

where

\begin{equation}
\epsilon(h_1, h_2) = \frac{1}{2}(\iota(h_1) + \iota(h_2) - \iota(h_1 h_2)).
\end{equation}

Since

\begin{equation}
\epsilon(h_1, h_2) + \epsilon(h_1 h_2, h_3) = \frac{1}{2}(\iota(h_1) + \iota(h_2) + \iota(h_3) - \iota(h_1 h_2 h_3)) = \epsilon(h_1, h_2 h_3) + \epsilon(h_2, h_3),
\end{equation}

such a sign modification does not affect the associativity of orbifold cohomology.

However, Qin-Wang [QW] observed that the orbifold cohomology modified by such a sign is isomorphic to original orbifold cohomology over complex number by an explicit isomorphism

\begin{equation}
\alpha \rightarrow (-1)^{\frac{\iota(g)}{2}} \alpha
\end{equation}
for $\alpha \in H^*(X_g, C)$. $\epsilon(h_1, h_2)$ is often an integer (for example symmetric product) while $\frac{\iota(g)}{2}$ is just a fraction. Hence, $(-1)^{\frac{\iota(g)}{2}}$ is a complex number only.

Another issue is the example of the crepant resolution of surface singularities $C^2/\Gamma$. As Fantechi-Göttche [FG2] pointed out, the Poincare paring of $H^2_{\text{orb}}(C^2/\Gamma, C)$ is indefinite while the Poincare paring of its crepant resolution is negative definite. There is an easy way to fix this case (suggested to this author by Witten). We view the involution $I : H^*(X_g, C) \rightarrow H^*(X_g, C)$ as a "complex conjugation". Then, we define a "hermitian inner product"

$$<< \alpha, \beta >> = < \alpha, I^*(\beta)>.$$  

If we use this "hermitian" inner product, the intersection paring is positive definite again. The above process has its conformal theory origin (see [NW]). It is attempting to perform this modification on $H^*(X_g, C) \oplus H^*(X_{g-1}, C)$ whenever $\iota(g) = \iota(g-1)$. The author does not know if it will affect the associativity of orbifold cohomology.

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