A FOOTNOTE TO A THEOREM OF HALÁSZ

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ABSTRACT. We study multiplicative functions $f$ satisfying $|f(n)| \leq 1$ for all $n$, the associated
Dirichlet series $F(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$, and the summatory function $S_f(x) := \sum_{n \leq x} f(n)$. Up to
a possible trivial contribution from the numbers $f(2^k)$, $F(s)$ may have at most one zero or one
pole on the one-line, in a sense made precise by Halász. We estimate $\log F(s)$ away from any
such point and show that if $F(s)$ has a zero on the one-line in the sense of Halász, then $|S_f(x)| \leq (x/\log x) \exp(c\sqrt{\log \log x})$ for all $c > 0$ when $x$ is large enough. This bound is best possible.

Halász obtained in [3, 4] some fundamental results on the mean values of multiplicative func-
tions $f$ subject to the restriction $|f(n)| \leq 1$ for all nonnegative integers $n$. We denote this class of
functions by $\mathcal{M}$ and set $S_f(x) := \sum_{n \leq x} f(n)$ and $F(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$, where the latter series converges absolutely for $\sigma := \text{Re } s > 1$. Following Montgomery [6], we have the following.

Halász’s theorem. Suppose that $f$ belongs to $\mathcal{M}$. Then for every real $t$ with at most one excep-
tion, we have

\begin{equation}
F(\sigma + it) = o\left(\frac{1}{\sigma - 1}\right), \quad \sigma \downarrow 1.
\end{equation}

If there exists an exceptional $t = t_0$ for which (1) does not hold, then

\begin{equation}
F(\sigma + it_0) \approx \frac{1}{\sigma - 1}, \quad 1 < \sigma \leq 2.
\end{equation}

Moreover, the following three assertions are equivalent:

(i) $S_f(x) = o(x), \quad x \to \infty$;
(ii) For every real $t$, $F(\sigma + it) = o(1/(\sigma - 1))$ when $\sigma \downarrow 1$;
(iii) For every real $t$, we have

$$\sum_p \frac{1 - \text{Re}(f(p)p^{-il})}{p} = +\infty \quad \text{or} \quad f(2^k) = -2^{ikt} \quad \text{for all } k \geq 1.$$  

The three equivalent assertions (i), (ii), (iii) give a more precise statement about the case
$S_f(x) = o(x)$ than what is found in the usual “textbook version” of Halász’s theorem; see for example [8 Sect. 4.3]. All the statements above can still be extracted from Satz 1’ of [3]. The

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second alternative in item (iii) accounts for a trivial reason for having \( F(\sigma + it) = o(1/(\sigma - 1)) \) when \( \sigma \searrow 1 \), namely the existence of \( t \) such that

\[
\sum_{k \geq 0} \frac{f(2^k)}{2^{k(\sigma + it)}} = \frac{2^\sigma - 2}{2^\sigma - 1}.
\]

In our first theorem, we exclude this possibility by considering the subclass \( M_2 \) of \( M \) consisting of \( f \) for which \( f(2^k) = 0 \) for every \( k \geq 1 \).

We may think of the exceptional case \( t = t_0 \) in Halász’s theorem as the assertion that \( F(s) \) has a “simple pole” at the point \( s = 1 + it_0 \). Following [7, Thm. 2.1], we find it natural to treat such “poles” on equal terms with possible “zeros” on the line \( \sigma = 1 \). This allows us to incorporate the following consequence of the prime number theorem in the first part of the theorem: if there is such a “zero” or a “pole”, there can be no other point of the same kind. This version of Halász’s result also comes with a precise estimate:

**Theorem 1.** Suppose that \( f \) belongs to \( M_2 \). Then for every real \( t \) with at most one exception,

\[
\lim_{\sigma \searrow 1} \frac{|F(\sigma + it)|^{\varepsilon}}{\varepsilon - 1} = +\infty
\]

for both \( \varepsilon = -1 \) and \( \varepsilon = 1 \). In fact, if there exists a pair \( (\varepsilon, t) = (\varepsilon_0, t_0) \in \{-1, 1\} \times \mathbb{R} \) for which (3) does not hold, then for \( 1 < \sigma \leq 3/2 \),

\[
|F(\sigma + it_0)|^{\varepsilon_0} = (\sigma - 1)^{\varepsilon_0}
\]

and

\[
\varepsilon_0 \log F(\sigma + it) + \log \zeta(\sigma + it - it_0) = o\left(\sqrt{\log \frac{1}{\sigma - 1}}\right),
\]

uniformly for all real \( t \) when \( \sigma \searrow 1 \).

As far as the mean values of \( f \) are concerned, the bound in (4) is of no interest when \( \varepsilon_0 = -1 \). What matters is then only the behavior of \( F(\sigma + it_0) \) when \( \sigma \searrow 1 \), and we will in particular have that \( |S_f(x)|/x \) tends to a positive limit; see [2] for precise information about the relation between \( F(\sigma + it_0) \) and the mean values \( S_f(x)/x \) in the case \( \varepsilon_0 = -1 \). However, when \( \varepsilon_0 = 1 \), the estimate in (4) yields a sharp improvement of the bound in item (i) of Halász’s theorem.

**Theorem 2.** Suppose that \( f \) belongs to \( M \). If there exists a real \( t_0 \) such that

\[
\sum_p \frac{1 + \text{Re}(f(p)p^{-it_0})}{p} < \infty,
\]

then

\[
\limsup_{x \to \infty} \frac{|S_f(x)| \log x}{x \exp(c \sqrt{\log \log x})} = 0
\]

for every constant \( c > 0 \). Conversely, if \( \kappa : [3, \infty) \to \mathbb{R}^+ \) satisfies \( \kappa(x) = o(\sqrt{\log \log x}) \) when \( x \to \infty \), then there exists an \( f \) in \( M \) such that (5) holds for \( t_0 = 0 \) and

\[
\limsup_{x \to \infty} \frac{|S_f(x)| \log x}{x \exp(\kappa(x))} = \infty.
\]
We obtain (6) as an immediate consequence of (4) and a celebrated elucidation of item (i) of Halász’s theorem, expressed in terms of the size of \(|F(s)|\) close to the 1-line. This result also stems from work of Halász [3–4]; see Montgomery’s paper [5], Tenenbaum’s book [8, Sec. III.4.3], or the recent paper [11]. We will therefore give below only the proof of the second part of Theorem 2.

Before proving our two theorems, we establish the following lemma.

**Lemma 1.** Let \( f(p) \) be a sequence of numbers satisfying \(|f(p)| \leq 1\). Suppose that there exist \( \varepsilon_0 \) in \([-1, 1]\) and a real number \( t_0 \) such that

\[
\sum_{p} \frac{1 + \varepsilon_0 \Re \left( f(p) p^{-i t_0} \right)}{p} < \infty.
\]

Then

\[
\varepsilon_0 \sum_{p} \frac{f(p)}{p^s} + \log \zeta(s - i t_0) = o \left( \sqrt{\log \frac{1}{\varepsilon_0 - 1}} \right)
\]

uniformly for \( s = \sigma + i t, \ \sigma \searrow 1, \) and real \( t \).

**Proof of Lemma** Let \( (s = \sigma + i t, \ \sigma \searrow 1, \) and real \( t \)

Our initial assumption is that (8) holds for either \( \varepsilon_0 = -1 \) or \( \varepsilon_0 = 1 \). Writing \( \varepsilon_0 f(p) p^{-i t_0} := -|f(p)| e^{i \theta_p} \) with \(-\pi < \theta_p \leq \pi\), we see that

\[
1 + \varepsilon_0 \Re \left( f(p) p^{-i t_0} \right) = 1 - |f(p)| + |f(p)|(1 - \cos \theta_p) \geq |f(p)|(1 - \cos \theta_p) \geq \frac{|f(p)|}{2\pi} \theta_p^2,
\]

so that (8) implies that

\[
\sum_{p} \frac{|f(p)| \theta_p^2}{p} < \infty.
\]

We may now write

\[
\varepsilon_0 \sum_{p} \frac{f(p)}{p^s} = \sum_{p} \frac{\Re \left( \varepsilon_0 f(p) p^{-i t_0} \right)}{p^{s-i t_0}} + i \sum_{p} \frac{\Im \left( \varepsilon_0 f(p) p^{-i t_0} \right)}{p^{s-i t_0}}
\]

\[
= -\sum_{p} \frac{1}{p^{s-i t_0}} - i \sum_{p} \frac{|f(p)| \sin \theta_p}{p^{s-i t_0}} + O(1)
\]

\[
= -\log \zeta(s - i t_0) - i \sum_{p} \frac{|f(p)| \sin \theta_p}{p^{s-i t_0}} + O(1),
\]

\[(10)\]

\footnote{To this end, we use the classical fact that \( 1/\zeta(\sigma + i t) \ll \log(|t|+ 2) \) holds uniformly for \( \sigma \geq 1 \) and real \( t \).}
which holds uniformly for \( \sigma > 1 \). By Mertens’s theorem for the sum \( \sum_{p \leq x} 1/p \), the Cauchy–Schwarz inequality, and (9),

\[
\sum_p \frac{|f(p)\sin \theta_p|}{p^\sigma} \leq \log \log \frac{1}{\sigma - 1} + O(1) + \sum_{p \geq 1/(\sigma - 1)} \frac{|f(p)\sin \theta_p|}{p^\sigma}
\]

\[
\leq \log \log \frac{1}{\sigma - 1} + O(1) + \left( \sum_p p^{1-2\sigma} \right)^{1/2} \left( \sum_{p \geq 1/(\sigma - 1)} \frac{|f(p)|\theta_p^2}{p} \right)^{1/2}
\]

\[
= o\left( \sqrt{\log \frac{1}{\sigma - 1}} \right)
\]

when \( \sigma \downarrow 1 \). Plugging this estimate into (10), we obtain the desired bound. \( \square \)

**Proof of Theorem 1.** Since \( \zeta(s) \) has a simple pole at \( s = 1 \), is otherwise analytic, and has no zero on \( \sigma = 1 \), the first part of Theorem 1 is an immediate consequence of the second part. To prove the latter assertion, we assume that

\[
|F(\sigma + it_0)|_{t_0} \neq \zeta(\sigma) |F(s)|_{t_0} \exp \left\{ \sum_p 1 + \epsilon_0 \Re \left( \frac{f(p)p^{-it_0}}{p^\sigma} \right) \right\}
\]

does not tend to \( +\infty \) when \( \sigma \downarrow 1 \) for some pair \((\epsilon_0, t_0)\) in \((-1, 1) \times \mathbb{R}\). By assumption, \( f \) is in \( \mathcal{M}_2 \), whence

\[
F(s) := \prod_{p} \sum_{k=0}^\infty \frac{f(p)^k}{p^{ks}}.
\]

For \( p \geq 3 \), we have \( |\sum_{k=1}^\infty f(p^k)p^{-ks}| \leq 1/2 \). We may therefore infer that

\[
\log F(s) = \sum_p f(p)p^{-s} + O(1)
\]

and hence

\[
\log |F(s)| = \sum_p \Re \left( \frac{f(p)p^{-s}}{p^\sigma} \right) + O(1)
\]

when \( \sigma > 1 \). It follows from this and the fact that \( \zeta(s) \) has a simple pole at \( s = 1 \) that

\[
\frac{|F(s)|_{t_0}}{\sigma - 1} = \zeta(\sigma) |F(s)|_{t_0} \exp \left\{ \sum_p 1 + \epsilon_0 \Re \left( \frac{f(p)p^{-it_0}}{p^\sigma} \right) \right\}
\]

when \( 1 < \sigma \leq 3/2 \). By monotone convergence,

\[
\lim_{\sigma \downarrow 1} \sum_p \frac{1 + \epsilon_0 \Re \left( \frac{f(p)p^{-it_0}}{p^\sigma} \right)}{p^\sigma} = \sum_p \frac{1 + \epsilon_0 \Re \left( f(p)p^{-it_0} \right)}{p}.
\]

By assumption, this limit is not \( +\infty \), and hence we may apply Lemma 1 to conclude. \( \square \)

**Proof of the second part of Theorem 2.** We will assume that every \( f \) in \( \mathcal{M} \) for which (5) holds with \( t_0 = 0 \), satisfies

\[
|S_f(x)| \ll \frac{x}{\log x} \exp \left( \kappa(x) \right),
\]

and show that this leads to a contradiction.
We may clearly assume that \( \kappa(x) \) is a continuous function. It is also plain that \( \kappa(x) \) may be assumed to be nondecreasing and that \( \kappa(x)/\sqrt{\log \log x} \) may be taken to be a nonincreasing function. Indeed, if \( \kappa(x) \) failed to be nondecreasing, then we could use instead \( \kappa_0(x) := \max_{3 \leq y \leq x} \kappa(y) \); should moreover \( \kappa_0(x)/\sqrt{\log \log x} \) fail to be nonincreasing, then we could replace it by

\[
\kappa_1(x) := \sqrt{\log \log x} \max_{y \leq x} \frac{\kappa_0(y)}{\sqrt{\log \log y}},
\]

which would still be a nondecreasing function being \( o(\sqrt{\log \log x}) \) when \( x \to \infty \).

By partial summation, we have for every \( 1 < \sigma \leq 3/2 \) and say \( |t| \leq 1 \),

\[
|F(\sigma + it)| \leq 1 + 2 \int_3^\infty |S(f(y))| y^{-\sigma-1} \, dy \ll \int_3^\infty \frac{e^{\kappa(y)}}{y^\sigma \log y} \, dy,
\]

\[
\leq \exp \left( \kappa(e^{\frac{1}{\sigma-1}}) \right) \int_3^{e^{1/(\sigma-1)}} \frac{dy}{y \log y} + \int_{e^{1/(\sigma-1)}}^\infty \frac{e^{-\log \log y + \kappa(y)}}{y^\sigma \log y} \, dy.
\]

Since \( \kappa(y)/\sqrt{\log \log y} \) is a nonincreasing function, the function \( \log \log y - \kappa(y) \) is eventually increasing, whence the above computation leads to the bound

\[
|F(\sigma + it)| \ll \exp \left( \kappa(e^{\frac{1}{\sigma-1}}) \right) \left( \log \frac{1}{\sigma-1} + 1/e \right).
\]

We may write this more succinctly as

\[
|F(\sigma + it)| \leq \exp \left( \alpha \left( e^{\frac{1}{\sigma-1}} \right) \sqrt{\frac{1}{\log \frac{1}{\sigma-1}}} \right),
\]

where \( \alpha : [3, \infty) \to (0, \infty) \) is a nonincreasing function satisfying \( \alpha(x) \to 0 \) when \( x \to \infty \).

We now choose a sequence of positive numbers \( x_j \), growing so rapidly that \( x_j^{\log x_j} < x_{j+1} \) for every \( j \geq 1 \) and the sequence

\[
a_j := \sqrt{\alpha(x_j^{\log x_j})}
\]

is in \( \ell^2 \). We then set

\[
\theta_p := \begin{cases} \frac{a_j}{\sqrt{\log \log p}}, & x_j \leq p < x_j^{\log x_j}, \quad -\text{Re}(ip^{-i}) \geq 1/2 \\ 0, & \text{otherwise}. \end{cases}
\]

We find that

\[
\sum_p \frac{|	heta_p|^2}{p} \leq \sum_{j=1}^{\infty} \frac{a_j^2}{\log \log x_j} \sum_{p \leq x_j} \frac{1}{p} \ll \sum_{j=1}^{\infty} a_j^2,
\]

where we in the last step used Mertens’s theorem for the sum \( \sum_{p \leq x} 1/p \). Hence \( \sum_p |	heta_p|^2 / p < \infty \) by our choice of the sequence \( a_j \). Setting \( f(p) := -e^{i\theta_p} \) and using Taylor’s theorem to write

\[
f(p) = -1 - i\theta_p + O(\theta_p^2),
\]
we infer from this that
\[
\text{Re} \sum_p f(p) p^{-s} = -\text{Re} \log \zeta(s) - \text{Re} i \sum_p \theta_p p^{-s} + O(1).
\]

It does not matter how we define \( f \) for higher prime powers, but for definiteness, let us require that \( f \) be completely multiplicative. Setting \( \sigma = 1 + 1/(\log x_j)^2 \) and \( t = 1 \), we then get
\[
\text{Re} \log F(1 + 1/(\log x_j)^2 + i) = -\text{Re} i \sum_p \theta_p p^{-1-1/(\log x_j)^2-i} + O(1)
\]
\[
\gg \frac{a_j}{\sqrt{\log \log x_j}} \sum_{x_j \leq p \leq x_j} \frac{1}{p} + O(1) \gg a_j \sqrt{\log \log x_j}.
\]

But choosing the same \( \sigma = 1 + 1/(\log x_j)^2 \) and \( t = 1 \) in (12), we reach the bound
\[
\sqrt{\alpha(x_j^{\log x_j})} \gg 1,
\]
contradicting that \( \alpha(x) \searrow 0 \) when \( x \to \infty \), which, as observed above, is a consequence of our assumption that (11) holds for all \( f \) in \( \mathcal{M} \) for which (5) is true. \( \square \)

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