GENERALIZED STATISTICAL LIMIT POINTS AND
STATISTICAL CLUSTER POINTS VIA IDEAL

PRASANTA MALIK* AND ARGHA GHOSH*

* Department of Mathematics, The University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India. Email: pmjupm@yahoo.co.in, papanargha@gmail.com

ABSTRACT. In this paper we have extended the notion of statistical limit point as introduced by Fridy [8] to $I$-statistical limit point of sequences of real numbers and studied some basic properties of the set of all $I$-statistical limit points and $I$-statistical cluster points of real sequences.

Key words and phrases: $I$-statistical convergence, $I$-statistical limit point, $I$-statistical cluster point, $I$-asymptotic density, $I$-statistical boundedness.

AMS subject classification (2010): 40A05, 40D25.

1. Introduction:

The usual notion of convergence of real sequences was extended to statistical convergence independently by Fast [12] and Schoenberg [20] based on the notion of natural density of subsets of $\mathbb{N}$, the set of all positive integers. Since then a lot of works has been done in this area (in particular after the seminal works of Salat [18] and Fridy [7]). Following the notion of statistical convergence in [8] Fridy introduced and studied the notions of statistical limit points and statistical cluster points of real sequences. More primary work on this convergence can be found from [1, 2, 3, 9, 17, 21] where other references can be found.

The concept of statistical convergence was further extended to $I$-convergence by Kostyrko et. al. [13] using the notion of ideals of $\mathbb{N}$. Using this notion of ideals the concepts of statistical limit point and statistical cluster point were naturally extended to $I$-limit point and $I$-cluster point respectively by Kostyrko et. al. in [14]. More works in this line can be found in [6, 10, 11] and many others.

Recently in [4] the notion of $I$-statistical convergence and $I$-statistical cluster point of real sequences have been introduced by Das et. al. using ideals of $\mathbb{N}$, which naturally extends the notions of statistical convergence and statistical
cluster point. Further works on such summability method can be found in [5, 15, 19] and many others.

In this paper using the notion of $I$-statistical convergence we extend the concept of statistical limit point to $I$-statistical limit point of sequences of real numbers and study some properties of $I$-statistical limit points and $I$-statistical cluster points of sequences of real numbers. We also study the sets of $I$-statistical limit points and $I$-statistical cluster points of sequences of real numbers and relationship between them.

2. Basic Definitions and Notations

In this section we recall some basic definitions and notations. Throughout the paper $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{R}$ denotes the set of all real numbers and $x$ denotes the sequence $\{x_k\}_{k \in \mathbb{N}}$ of real numbers.

Definition 2.1. [16] A subset $K$ of $\mathbb{N}$ is said to have natural density (or asymptotic density) $d(K)$ if

$$d(K) = \lim_{n \to \infty} \frac{|K(n)|}{n}$$

where $K(n) = \{j \in K : j \leq n\}$ and $|K(n)|$ represents the number of elements in $K(n)$.

If $\{x_k\}_{j \in \mathbb{N}}$ is a subsequence of the sequence $x = \{x_k\}_{k \in \mathbb{N}}$ of real numbers and $A = \{k_j : j \in \mathbb{N}\}$, then we abbreviate $\{x_{k_j}\}_{j \in \mathbb{N}}$ by $\{x\}_A$. In case $d(A) = 0$, $\{x\}_A$ is called a subsequence of natural density zero or a thin subsequence of $x$. On the other hand, $\{x\}_A$ is a non-thin subsequence of $x$ if $d(A)$ does not have natural density zero i.e., if either $d(A)$ is a positive number or $A$ fails to have natural density.

Definition 2.2. [7] Let $x = \{x_k\}_{k \in \mathbb{N}}$ be a sequence of real numbers. Then $x$ is said to be statistically convergent to $\xi$ if for any $\varepsilon > 0$

$$d(\{k : |x_k - \xi| \geq \varepsilon\}) = 0.$$ 

In this case we write $\text{st - lim}_{k \to \infty} x_k = \xi$.

Definition 2.3. [17] A sequence $x = \{x_k\}_{k \in \mathbb{N}}$ is said to be statistically bounded if there exists a compact set $C$ in $\mathbb{R}$ such that

$$d(\{k : k \leq n, x_k \notin C\}) = 0.$$ 

Definition 2.4. [5] A real number $l$ is a statistical limit point of the sequence $x = \{x_k\}_{k \in \mathbb{N}}$ of real numbers, if there exists a nonthin subsequence of $x$ that converges to $l$.

A real number $L$ is an ordinary limit point of a sequence $x$ if there is a subsequence of $x$ that converges to $L$. The set of all ordinary limit points and the set of all statistical limit points of the sequence $x$ are denoted by $L_x$ and $\Lambda_x$ respectively. Clearly $\Lambda_x \subset L_x$. 

Definition 2.5. \[8\] A real number \( y \) is a statistical cluster point of the sequence \( x = \{x_k\}_{k \in \mathbb{N}} \) of real numbers, if for any \( \varepsilon > 0 \) the set \( \{k \in \mathbb{N} : |x_k - y| < \varepsilon\} \) does not have natural density zero.

The set of all statistical cluster points of \( x \) is denoted by \( \Gamma_x \). Clearly \( \Gamma_x \subseteq L_x \).

We now recall definitions of ideal and filter in a non-empty set.

Definition 2.6. \[13\] Let \( X \neq \emptyset \). A class \( I \) of subsets of \( X \) is said to be an ideal in \( X \) provided, \( I \) satisfies the conditions:

(i) \( \emptyset \in I \),
(ii) \( A, B \in I \Rightarrow A \cup B \in I \),
(iii) \( A \in I, B \subseteq A \Rightarrow B \in I \).

An ideal \( I \) in a non-empty set \( X \) is called non-trivial if \( X \notin I \).

Definition 2.7. \[13\] Let \( X \neq \emptyset \). A non-empty class \( F \) of subsets of \( X \) is said to be a filter in \( X \) provided that:

(i) \( \emptyset \notin F \),
(ii) \( A, B \in F \Rightarrow A \cap B \in F \),
(iii) \( A \in F, B \supseteq A \Rightarrow B \in F \).

Definition 2.8. \[13\] Let \( I \) be a non-trivial ideal in a non-empty set \( X \). Then the class \( F(I) = \{M \subseteq X : \exists A \in I \text{ such that } M = X \setminus A\} \) is a filter on \( X \). This filter \( F(I) \) is called the filter associated with \( I \).

A non-trivial ideal \( I \) in \( X(\neq \emptyset) \) is called admissible if \( \{x\} \in I \) for each \( x \in X \).

Throughout the paper we take \( I \) as a non-trivial admissible ideal in \( \mathbb{N} \) unless otherwise mentioned.

Definition 2.9. \[13\] Let \( x = \{x_k\}_{k \in \mathbb{N}} \) be a sequence of real numbers. Then \( x \) is said to be \( I \)-convergent to \( \xi \) if for any \( \varepsilon > 0 \)

\[ \{k : |x_k - \xi| \geq \varepsilon\} \in I. \]

In this case we write \( I-\lim_{k \to \infty} x_k = \xi \).

Definition 2.10. \[6\] A sequence \( x = \{x_k\}_{k \in \mathbb{N}} \) of real number is said to be \( I \)-bounded if there exists a real number \( G > 0 \) such that \( \{k : |x_k| > G\} \in I \).

Definition 2.11. \[4\] Let \( x = \{x_k\}_{k \in \mathbb{N}} \) be a sequence of real numbers. Then \( x \) is said to be \( I \)-statistically convergent to \( \xi \) if for any \( \varepsilon > 0 \) and \( \delta > 0 \)

\[ \{n \in \mathbb{N} : \frac{1}{n} \{k \leq n : |x_k - \xi| \geq \varepsilon\} \geq \delta\} \in I. \]

In this case we write \( I-st-\lim x = \xi \).

3. \( I \)-statistical limit points and \( I \)-statistical cluster points

In this section, following the line of Fridy \[8\] and Pehlivan et. al. \[17\], we introduce the notion of \( I \)-statistical limit point of real sequences and present an \( I \)-statistical analogue of some results in those papers.
Definition 3.1. [5] A subset $K$ of $\mathbb{N}$ is said to have $I$-natural density (or, $I$-asymptotic density) $d_I(K)$ if 

$$d_I(K) = I - \lim_{n \to \infty} \frac{|K(n)|}{n}$$

where $K(n) = \{j \in K : j \leq n\}$ and $|K(n)|$ represents the number of elements in $K(n)$.

Note 3.1. From the above definition, it is clear that, if $d(A) = r, A \subset \mathbb{N}$, then $d_I(A) = r$ for any nontrivial admissible ideal $I$ in $\mathbb{N}$.

In case $d_I(A) = 0$, $\{x\}_A$ is a subsequence of $I$-asymptotic density zero, or an $I$-thin subsequence of $x$. On the other hand, $\{x\}_A$ is an $I$-nonthin subsequence of $x$, if $d_I(A)$ does not have density zero i.e., if either $d_I(A)$ is a positive number or $A$ fails to have $I$-asymptotic density.

Definition 3.2. A real number $l$ is an $I$-statistical limit point of a sequence $x = \{x_k\}_{k \in \mathbb{N}}$ of real numbers, if there exists an $I$-nonthin subsequence of $x$ that converges to $l$. The set of all $I$-statistical limit points of the sequence $x$ is denoted by $\Lambda^S_x(I)$.

Definition 3.3. [5] A real number $y$ is an $I$-statistical cluster point of a sequence $x = \{x_k\}_{k \in \mathbb{N}}$ of real numbers, if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - y| < \varepsilon\}$ does not have $I$-asymptotic density zero. The set of all $I$-statistical cluster points of $x$ is denoted by $\Gamma^S_x(I)$.

Note 3.2. If $I = I_{fin} = \{A \subset \mathbb{N} : |A| < \infty\}$, then the notions of $I$-statistical limit points and $I$-statistical cluster points coincide with the notions of statistical limit points and statistical cluster points respectively.

We first present an $I$-statistical analogous of some results in [5].

Theorem 3.1. Let $x = \{x_k\}_{k \in \mathbb{N}}$ be a sequence of real numbers. Then $\Lambda^S_x(I) \subset \Gamma^S_x(I)$.

Proof. Let $\alpha \in \Lambda^S_x(I)$. So we have a subsequence $\{x_{k_j}\}_{j \in \mathbb{N}}$ of $x$ with $\lim x_{k_j} = \alpha$ and $d_I(K) \neq 0$, where $K = \{k_j : j \in \mathbb{N}\}$. Let $\varepsilon > 0$ be given. Since $\lim x_{k_j} = \alpha$, so $B = \{k_j : |x_{k_j} - \alpha| \geq \varepsilon\}$ is a finite set. Thus

$$\{k \in \mathbb{N} : |x_k - \alpha| < \varepsilon\} \supset \{k_j : j \in \mathbb{N}\} \setminus B$$

Now if $d_I(\{k \in \mathbb{N} : |x_k - \alpha| < \varepsilon\}) = 0$, then we have $d_I(K) = 0$, which is a contradiction. Thus $\alpha$ is an $I$-statistical cluster point of $x$. Since $\alpha \in \Lambda^S_x(I)$ is arbitrary, so $\Lambda^S_x(I) \subset \Gamma^S_x(I)$. \hfill \Box

Note 3.3. The set $\Lambda^S(I)$ of all $I$-statistical limit points of a sequence $x$ may not be equal to the set $\Gamma^S_x(I)$ of all $I$-statistical cluster points of $x$. To show this we cite the following example.
Theorem 3.2. If \( x = \{ x_k \}_{k \in \mathbb{N}} \) and \( y = \{ y_k \}_{k \in \mathbb{N}} \) are sequences of real numbers such that \( d_I(\{ k : x_k \neq y_k \}) = 0 \), then \( \Lambda^S_x(I) = \Lambda^S_y(I) \) and \( \Gamma^S_x(I) = \Gamma^S_y(I) \).

Proof. Let \( \gamma \in \Gamma^S_x(I) \) and \( \varepsilon > 0 \) be given. Then \( \{ k : |x_k - \gamma| < \varepsilon \} \) does not have \( I \)-asymptotic density zero. Let \( A = \{ k : x_k = y_k \} \). Since \( d_I(A) = 1 \) so \( \{ k : |x_k - \gamma| < \varepsilon \} \cap A \) does not have \( I \)-asymptotic density zero. Thus \( \gamma \in \Gamma^S_y(I) \). Since \( \gamma \in \Gamma^S_x(I) \) is arbitrary, so \( \Gamma^S_x(I) \subset \Gamma^S_y(I) \). By symmetry we have \( \Gamma^S_y(I) \subset \Gamma^S_x(I) \). Hence \( \Gamma^S_x(I) = \Gamma^S_y(I) \).

Also let \( \beta \in \Lambda^S_x(I) \). Then \( x \) has an \( I \)-nonthin subsequence \( \{ x_{k_j} \}_{j \in \mathbb{N}} \) that converges to \( \beta \). Let \( K = \{ k_j : j \in \mathbb{N} \} \). Since \( d_I(\{ k_j : x_{k_j} \neq y_{k_j} \}) = 0 \), we have \( d_I(\{ k_j : x_{k_j} = y_{k_j} \}) \neq 0 \). Therefore from the latter set we have an \( I \)-nonthin subsequence \( \{ y_{k_j} \}_K \) of \( \{ y_{k_j} \}_K \) that converges to \( \beta \). Thus \( \beta \in \Lambda^S_y(I) \). Since \( \beta \in \Lambda^S_x(I) \) is arbitrary, so \( \Lambda^S_x(I) \subset \Lambda^S_y(I) \). By symmetry we have \( \Lambda^S_y(I) \subset \Lambda^S_x(I) \). Hence \( \Lambda^S_x(I) = \Lambda^S_y(I) \).

We now investigate some topological properties of the set \( \Gamma^S_x(I) \) of all \( I \)-statistical cluster points of \( x \).

Theorem 3.3. Let \( A \) be a compact set in \( \mathbb{R} \) and \( A \cap \Gamma^S_x(I) = \emptyset \). Then the set \( \{ k \in \mathbb{N} : x_k \in A \} \) has \( I \)-asymptotic density zero.

Proof. Since \( A \cap \Gamma^S_x(I) = \emptyset \), so for any \( \xi \in A \) there is a positive number \( \varepsilon = \varepsilon(\xi) \) such that

\[
d_I(\{ k : |x_k - \xi| < \varepsilon \}) = 0.
\]

Let \( B_{\varepsilon(\xi)}(\xi) = \{ y : |y - \xi| < \varepsilon \} \). Then the set of open sets \( \{ B_{\varepsilon(\xi)}(\xi) : \xi \in A \} \) form an open covers of \( A \). Since \( A \) is a compact set so there is a finite subcover of \( \{ B_{\varepsilon(\xi)}(\xi) : \xi \in A \} \) for \( A \), say \( \{ A_i = B_{\varepsilon(\xi_i)}(\xi_i) : i = 1, 2, \ldots, q \} \). Then \( A \subset \bigcup_{i=1}^q A_i \) and

\[
d_I(\{ k : |x_k - \xi_i| < \varepsilon(\xi_i) \}) = 0 \quad \text{for} \quad i = 1, 2, \ldots, q.
\]

We can write

\[
|\{ k : k \leq n; x_k \in A \}| \leq \sum_{i=1}^q |\{ k : k \leq n; |x_k - \xi_i| < \varepsilon(\xi_i) \}|
\]

and by the property of \( I \)-convergence,

\[
I- \lim_{n \to \infty} \frac{|\{ k : k \leq n; x_k \in A \}|}{n} \leq \sum_{i=1}^q I- \lim_{n \to \infty} \frac{|\{ k : k \leq n; |x_k - \xi_i| < \varepsilon(\xi_i) \}|}{n} = 0.
\]

Which gives \( d_I(\{ k : x_k \in A \}) = 0 \) and this completes the proof.

Note 3.4. If the set \( A \) is not compact then the above result may not be true.
Theorem 3.4. If a sequence \( x = \{x_k\}_{k \in \mathbb{N}} \) has a bounded \( I\)-non-thin subsequence, then the set \( \Gamma^S_x(I) \) is a non-empty closed set.

Proof. Let \( x = \{x_k\}_{q \in \mathbb{N}} \) be a bounded \( I\)-non-thin subsequence of \( x \) and \( A \) be a compact set such that \( x_k \in A \) for each \( q \in \mathbb{N} \). Let \( P = \{k_q : q \in \mathbb{N}\} \). Clearly \( d_I(P) \neq 0 \). Now if \( \Gamma^S_x(I) = \emptyset \), then \( A \cap \Gamma^S_x(I) = \emptyset \) and so by Theorem 3.3 we have

\[
d_I(\{k : x_k \in A\}) = 0.
\]

But

\[
|\{k : k \leq n, k \in P\}| \leq |\{k : k \leq n, x_k \in A\}|
\]

which implies that \( d_I(P) = 0 \), which is a contradiction. So \( \Gamma^S_x(I) \neq \emptyset \).

Now to show that \( \Gamma^S_x(I) \) is closed, let \( \xi \) be a limit point of \( \Gamma^S_x(I) \). Then for every \( \varepsilon > 0 \) we have \( B_\varepsilon(\xi) \cap (\Gamma^S_x(I) \setminus \{\xi\}) \neq \emptyset \). Let \( \beta \in B_\varepsilon(\xi) \cap (\Gamma^S_x(I) \setminus \{\xi\}) \). Now we can choose \( \varepsilon' > 0 \) such that \( B_{\varepsilon'}(\beta) \subset B_\varepsilon(\xi) \). Since \( \beta \in \Gamma^S_x(I) \) so

\[
d_I(\{k : |x_k - \beta| < \varepsilon'\}) \neq 0
\]

\[
\Rightarrow d_I(\{k : |x_k - \xi| < \varepsilon\}) \neq 0.
\]

Hence \( \xi \in \Gamma^S_x(I) \).

Definition 3.4. (a) A sequence \( x = \{x_k\}_{k \in \mathbb{N}} \) of real numbers is said to be \( I\)-statistically bounded above if, there exists \( L \in \mathbb{R} \) such that \( d_I(\{k \in \mathbb{N} : x_k > L\}) = 0 \).

(b) A sequence \( x = \{x_k\}_{k \in \mathbb{N}} \) of real numbers is said to be \( I\)-statistically bounded below if, there exists \( l \in \mathbb{R} \) such that \( d_I(\{k \in \mathbb{N} : x_k < l\}) = 0 \).

Definition 3.5. A sequence \( x = \{x_k\}_{k \in \mathbb{N}} \) of real numbers is said to be \( I\)-statistically bounded if, there exists \( l > 0 \) such that for any \( \delta > 0 \), the set

\[
A = \{n \in \mathbb{N} : \frac{1}{n}|\{k \in \mathbb{N} : k \leq n, |x_k| > l\}| \geq \delta\} \in I
\]

i.e. \( d_I(\{k \in \mathbb{N} : |x_k| > l\}) = 0 \).

Equivalently, \( x = \{x_k\}_{k \in \mathbb{N}} \) is said to be \( I\)-statistically bounded if, there exists a compact set \( C \) in \( \mathbb{R} \) such that for any \( \delta > 0 \), the set \( A = \{n \in \mathbb{N} : \frac{1}{n}|\{k \in \mathbb{N} : k \leq n, x_k \notin C\}| \geq \delta\} \in I \) i.e., \( d_I(\{k \in \mathbb{N} : x_k \notin C\}) = 0 \).

Note 3.5. If \( I = I_{\text{fin}} = \{A \subset \mathbb{N} : |A| < \infty\} \), then the notion of \( I\)-statistical boundedness coincide with the notion of statistical boundedness.

Corollary 3.5. If \( x = \{x_k\}_{k \in \mathbb{N}} \) is \( I\)-statistically bounded. Then the set \( \Gamma^S_x(I) \) is non empty and compact.

.................THEOREM 3.3.........................
Theorem 3.6. Let \( x = \{x_k\}_{k \in \mathbb{N}} \) be an \( I \)-statistically bounded sequence. Then for every \( \varepsilon > 0 \) the set
\[
\{ k \in \mathbb{N} : d(\Gamma^S_x(I), x_k) \geq \varepsilon \}
\]
has \( I \)-asymptotic density zero, where \( d(\Gamma^S_x(I), x_k) = \inf_{y \in \Gamma^S_x(I)} |y - x_k| \) the distance from \( x_k \) to the set \( \Gamma^S_x(I) \).

Proof. Let \( C \) be a compact set such that \( d_I(\{k \in \mathbb{N} : x_k \notin C\}) = 0 \). Then by Corollary 3.5 we have \( \Gamma^S_x(I) \) is non-empty and \( \Gamma^S_x(I) \subseteq C \).

Now if possible let \( d_I(\{k \in \mathbb{N} : d(\Gamma^S_x(I), x_k) \geq \varepsilon' \}) \neq 0 \) for some \( \varepsilon' > 0 \).

Now we define \( B = \{y \in \Gamma^S_x(I) : d(\Gamma^S_x(I), y) < \varepsilon' \} \) and let \( B = C \setminus B \).

Then \( A \) is a compact set which contains an \( I \)-non-thin subsequence of \( x \). Then by Theorem 3.3 \( A \cap \Gamma^S_x(I) \neq \emptyset \), which is absurd, since \( \Gamma^S_x(I) \subseteq B \). Hence
\[
d_I(\{k \in \mathbb{N} : d(\Gamma^S_x(I), x_k) \geq \varepsilon \}) = 0
\]
for every \( \varepsilon > 0 \).

4. Condition APIO

Definition 4.1. (Additive property for \( I \)-asymptotic density zero sets)

The \( I \)-asymptotic density \( d_I \) is said to satisfy \( \text{APIO} \) if, given a collection of sets \( \{A_j\}_{j \in \mathbb{N}} \) in \( \mathbb{N} \) with \( d_I(A_j) = 0 \), for each \( j \in \mathbb{N} \), there exists a collection \( \{B_j\}_{j \in \mathbb{N}} \) in \( \mathbb{N} \) with the properties \( |A_j \Delta B_j| < \infty \) for each \( j \in \mathbb{N} \) and \( d_I(B) = \sum_{j=1}^{\infty} B_j = 0 \).

Theorem 4.1. A sequence \( x = \{x_k\}_{k \in \mathbb{N}} \) of real number is \( I \)-statistically convergent to \( l \) implies there exists a subset \( B \) with \( d_I(B) = 1 \) and \( \lim_{k \to \infty} x_k = l \) if and only if \( d_I \) has the property \( \text{APIO} \).

Theorem 4.2. Let \( I \) be an ideal such that \( d_I \) has the property \( \text{APIO} \), then for any sequence \( x = \{x_k\}_{k \in \mathbb{N}} \) of real numbers there exists a sequence \( y = \{y_k\}_{k \in \mathbb{N}} \) such that \( L_y = \Gamma^S_x(I) \) and the set \( \{k : x_k \neq y_k \} \) has \( I \)-asymptotic density zero.

5. \( I \)-statistical analogus of Completeness Theorems

In this section following the line of Fridy [8], we formulate and prove an \( I \)-statistical analogue of the theorems concerning sequences that are equivalent to the completeness of the real line.

We first consider a sequential version of the least upper bound axiom (in \( \mathbb{R} \)), namely, Monotone sequence Theorem: every monotone increasing sequence of real numbers which is bounded above, is convergent. The following result is an \( I \)-statistical analogue of that Theorem.

Theorem 5.1. Let \( x = \{x_k\}_{k \in \mathbb{N}} \) be a sequence of real numbers and \( M = \{k : x_k \leq x_{k+1}\} \). If \( d_I(M) = 1 \) and \( x \) is bounded above on \( M \), then \( x \) is \( I \)-statistically convergent.
Proof. Since $x$ is bounded above on $M$, so let $l$ be the least upper bound of the range of $\{x_k\}_{k \in M}$. Then we have
(i) $x_k \leq l$, $\forall k \in M$
(ii) for a pre-assigned $\varepsilon > 0$, there exists a natural number $k_0 \in M$ such that $x_{k_0} > l - \varepsilon$.

Now let $k \in M$ and $k > k_0$. Then $l - \varepsilon < x_{k_0} \leq x_k < l + \varepsilon$. Thus $M \cap \{k : k > k_0\} \subset \{k : l - \varepsilon < x_k < l + \varepsilon\}$. Since the set on the left hand side of the inclusion is of $I$-asymptotic density 1, we have $d_I(\{k : l - \varepsilon < x_k < l + \varepsilon\}) = 1$ i.e., $d_I(\{k : |x_k - l| \geq \varepsilon\}) = 0$. Hence $x$ is $I$-statistically convergent to $l$.

Theorem 5.2. Let $x = \{x_k\}_{k \in \mathbb{N}}$ be a sequence of real numbers and $M = \{k : x_k \geq x_{k+1}\}$. If $d_I(M) = 1$ and $x$ is bounded below on $M$, then $x$ is $I$-statistically convergent.

Proof. The proof is similar to that of Theorem 5.1 and so is omitted.

Note 5.1. (a) In the Theorem 5.1 if we replace the criteria that ‘$x$ is bounded above on $M$’ by ‘$x$ is $I$-statistically bounded above on $M$’ then the result still holds. Indeed if $x$ is a $I$-statistically bounded above on $M$, then there exists $l \in \mathbb{R}$ such that $d_I(\{k \in M : x_k > l\}) = 0$ i.e., $d_I(\{k \in M : x_k \leq l\}) = 1$. Let $S = \{k \in M : x_k \leq l\}$ and $l' = \sup\{x_k : k \in S\}$. Then
(i) $x_k \leq l'$ for all $k \in S$
(ii) for any $\varepsilon > 0$, there exists a natural number $k_0 \in S$ such that $x_{k_0} > l' - \varepsilon$.

Then proceeding in a similar way as in Theorem 5.1 we get the result.

(b) Similarly, In the Theorem 5.2 if we replace the criteria that ‘$x$ is bounded below on $M$’ by ‘$x$ is $I$-statistically bounded below on $M$’ then the result still holds.

Another completeness result for $\mathbb{R}$ is the Bolzano-Weierstrass Theorem, which tells us that, every bounded sequence of real numbers has a cluster point. The following result is an $I$-statistical analogue of that result.

Theorem 5.3. Let $I$ be an ideal such that $d_I$ has the property APIO. Let $x = \{x_k\}_{k \in \mathbb{N}}$ be a sequence of real numbers. If $x$ has a bounded $I$-nonthin subsequence, then $x$ has an $I$-statistical cluster point.

Proof. Using Theorem 4.2, we have a sequence $y = \{y_k\}_{k \in \mathbb{N}}$ such that $L_y = \Gamma^S_I(I)$ and $d_I(\{k : x_k = y_k\}) = 1$. Let $\{x_k\}_K$ be the bounded $I$-nonthin subsequence of $x$. Then $d_I(\{k : x_k = y_k\} \cap K) \neq 0$. Thus $y$ has a bounded $I$-nonthin subsequence and hence by Bolzano-Weierstrass Theorem, $L_y \neq \emptyset$. Thus $\Gamma^S_I(I) \neq \emptyset$.

Corollary 5.4. Let $I$ be an ideal such that $d_I$ has the property APIO. If $x$ is a bounded sequence of real numbers, then $x$ has an $I$-statistical cluster point.
The next result is an $I$-statistical analogue of the Heine-Borel Covering Theorem.

**Theorem 5.5.** Let $I$ be an ideal such that $d_I$ has the property APIO. Let $x$ be a bounded sequence of real numbers, then it has an $I$-thin subsequence $\{x\}_B$ such that $\{x_k : k \in \mathbb{N} \setminus B\} \cup I^*_2(I)$ is a compact set.

**Acknowledgment:** The second author is grateful to University Grants Commissions, India for his fellowship funding under UGC-JRF (SRF fellowship) scheme during the preparation of this paper.

**References**

[1] J. Connor, J. Fridy, and J. Kline, Statistically pre-Cauchy Sequences, *Analysis*, 14(1994) 311-317.
[2] J. Connor, R-type summability methods, Cauchy criteria, P-sets and Statistical convergence, *Proc. Amer. Math. Soc.* 115 (1992), 319-327.
[3] J. Connor, The statistical and strong P-Cesaro convergence of sequences, *Analysis*, 8(1988), 47-63.
[4] P. Das, E. Savas, S.Kr. Ghosal, On generalizations of certain summability methods using ideals, *Appl. Math. Lett.*, 24(2011) 1509-1514.
[5] P. Das, E. Savas, On $I$-statistically pre-Cauchy sequences, *Taiwanese J. Math.*,18(1)(2014) 115-126.
[6] K. Demirci: $I$-limit superior and limit inferior, *Math. Commun.* 6(2)(2001), 165-172.
[7] J. A. Fridy, On statistical convergence, *Analysis*, 5(1985), 301-313.
[8] J. A. Fridy, statistical limit points, *Proc. Amer. Math. Soc.* 118(4)(1993), 1187-1192.
[9] J. A. Fridy and C. Orhan, Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc.*,125(1997), 3625-3631.
[10] B. K. Lahiri, P. Das, $I$ and $I^*$-convergence in topological spaces, *Math. Bohemica*, 126(2005), 153-160.
[11] B. K. Lahiri, P. Das, $I$ and $I^*$-convergence of nets, *Real Analysis Exchange*, 33(2)(2007/2008), 431-442.
[12] H. Fast, Sur la convergence statistique. *Colloq. Math* 2(1951) 241-244.
[13] P. Kostyrko, T. Šalát, W. Wilczyński: $I$-convergence, *Real Anal. Exchange* 26(2)(2000/2001), 669-685.
[14] P. Kostyrko, M. macz, T. Šalát, M. Sleziak: $I$-convergence and external $I$-limit points, *Math. Slovaca* 55(4)(2005), 443-454.
[15] M. Mursaleen, D. Deb NATO and D. Rakshit, $I$-Statistical Limit Superior and $I^*$ Statistical Limit Inferior, *Filomat*, 31.7 (2017), 21032108.
[16] I. Niven and H. S. Zuckerman, An introduction to the theorem of numbers, *4th ed., Wiley, New York*, 1980.
[17] S. Pehlivan, A. Guncan and M. A. Mamedov, Statistical cluster points of sequences in finite dimensional spaces, *Czechoslovak Mathematical Journal*, 54 (129)(2004), 95-102.
[18] T. Šalát, On statistically convergent sequences of real numbers, *Math. Slovaca*, 30(1980), 139-150.
[19] E. Savas, P. Das, A generalized statistical convergence via ideals, *Appl. Math. Lett.*, 24(2011) 826-830.
[20] I. J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, 66(1959), 361-375.
[21] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.*, 2(1951) 73-74.