Flux-Splitting Schemes for Parabolic Problems

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Received January 18, 2012

Abstract—Splitting with respect to space variables can be used in solving boundary value problems for second-order parabolic equations. Classical alternating direction methods and locally one-dimensional schemes could be examples of this approach. For problems with rapidly varying coefficients, a convenient tool is the use of fluxes (directional derivatives) as independent variables. The original equation is written as a system in which not only the desired solution but also directional derivatives (fluxes) are unknowns. In this paper, locally one-dimensional additional schemes (splitting schemes) for second-order parabolic equations are examined. By writing the original equation in flux variables, certain two-level locally one-dimensional schemes are derived. The unconditional stability of locally one-dimensional flux schemes of the first and second approximation order with respect to time is proved.

DOI: 10.1134/S0965542512080106

Keywords: Cauchy problem, second-order parabolic equation, operator-difference schemes, splitting schemes.

In applied mathematical simulation of time-dependent processes, boundary value problems for second-order parabolic equations play the role of basic problems. To solve such problems numerically, one uses finite difference, finite volume, or finite element methods for space discretization and (usually) simplest two- or three-level schemes for time discretization [1–4].

Suppose that the coefficients of a problem vary rapidly. It is then convenient to use fluxes (that is, directional derivatives) as independent variables. The original equation is written as a system in which not only the desired solution but also directional derivatives (fluxes) are unknowns. The idea of introducing flux variables goes back to papers [5, 6] in which stationary and nonstationary one-dimensional problems were treated. In [7, 8], flux variables were applied to designing iterative methods for two-dimensional parabolic problems with rapidly varying coefficients. In the numerical solution of applied problems, the introduction of flux variables can be directly related to the use of mixed and hybrid finite elements (see [9, 10]).

To solve numerically boundary value problems for second-order multidimensional parabolic equations, one uses splitting schemes for space variables, in particular, classical alternating direction methods and locally one-dimensional schemes (see [11–13]). Splitting schemes for parabolic equations written in stream variables are of undoubted interest. Schemes of this type were examined in [14, 15], and locally one-dimensional predictor-corrector schemes were proposed for problems with stream variables. Potentials of splitting schemes were demonstrated by solving certain model problems. Unfortunately, stability was proved only in the case of commuting operators (that is, equations with constant coefficients), which is of little interest from a practical viewpoint.

In this paper, we examine locally one-dimensional additional schemes (splitting schemes) for second-order parabolic equations. Both conventional and flux variables are used. A detailed discussion is given to the grid problem in a rectangle. The transition to three-dimensional problems is of only technical nature. The main result of the paper is as follows: by writing the original equation in flux variables, we construct unconditionally stable two-level locally one-dimensional schemes of the first and second approximation order with respect to time.

1. DIFFERENTIAL PROBLEM

As a typical example, we consider the boundary value problem for a second-order parabolic equation. Assume that, in a bounded domain Ω, the unknown function $u(x, t)$, where $x = (x_1, x_2, \ldots, x_m)$,
satisfies the equation

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \left( k(x) \frac{\partial u}{\partial x_\alpha} \right) = f(x,t), \quad x \in \Omega, \quad 0 < t \leq T. \tag{1}$$

Here, $k \leq k(x) \leq \bar{k}$, $x \in \Omega$, and $k > 0$. We supplement Eq. (1) with Dirichlet homogeneous boundary conditions

$$u(x,t) = 0, \quad x \in \partial \Omega, \quad 0 < t \leq T. \tag{2}$$

In addition, we pose the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{3}$$

Often, the flux formulation of problem (1)–(3) is of interest. Define the new vector variable $q$ (flux) such that

$$q + k \text{grad} u = 0. \tag{4}$$

Then Eq. (1) can be written as

$$\frac{\partial u}{\partial t} + \text{div} q = f, \quad x \in \Omega, \quad 0 < t \leq T. \tag{5}$$

The system (4), (5) is used in computational applications if, for instance, mixed and hybrid finite elements are involved (see [9, 10]).

Since we are going to construct locally one-dimensional schemes (locally one-dimensional splitting schemes), the components of the vector $q = (q_1, q_2, \ldots, q_m)$ are used rather than this vector itself. Consequently, instead of (4), we set

$$q_\alpha + k \frac{\partial u}{\partial x_\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{6}$$

and rewrite Eq. (6) in the form

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{m} \frac{\partial q_\alpha}{\partial x_\alpha} = f(x,t), \quad x \in \Omega, \quad 0 < t \leq T. \tag{7}$$

Let $\mathcal{H} = L_2(\Omega)$. On the set of functions satisfying boundary conditions (2), the gradient and divergence operators obey the relation

$$\int_{\Omega} \text{div} \varphi dx + \int_{\Omega} \text{grad} \varphi dx = 0,$$

that is, in the corresponding spaces,

$$\text{div} = -\text{grad}^*. \tag{8}$$

This permits us to write Eq. (5) as

$$\frac{\partial u}{\partial t} - \text{grad}^* q = f, \quad 0 < t \leq T. \tag{9}$$

An operator formulation with adjoint operators similar to (4), (8) can also be given to system (6), (7). It is convenient to perform symmetrization by inserting the coefficient $k(x)$ in the corresponding differential operators.

We define differential operators along the individual directions $x_\alpha$, $\alpha = 1, \ldots, m$ using the relations

$$sA_\alpha u = -k^{1/2}(x) \frac{\partial u}{\partial x_\alpha}, \quad \alpha = 1, \ldots, m. \tag{10}$$

For their adjoint operators and functions in $\mathcal{H} = L_2(\Omega)$ satisfying condition (2), we have

$$sA_\alpha^* v = \frac{\partial (k^{1/2}(x)v)}{\partial x_\alpha}, \quad \alpha = 1, \ldots, m. \tag{11}$$
For the elliptic operator of problem (1)–(3), relations (9) and (10) imply that

$$- \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_{\alpha}} \left( k(x) \frac{\partial u}{\partial x_{\alpha}} \right) = \sum_{\alpha=1}^{m} \mathcal{A}_{\alpha}^{*} \mathcal{A}_{\alpha} u.$$ 

This makes possible to write problem (1)–(3) as the Cauchy problem for the operator-differential equation

$$\frac{du}{dt} + \sum_{\alpha=1}^{m} \mathcal{A}_{\alpha}^{*} \mathcal{A}_{\alpha} u = f(t), \quad 0 < t \leq T, \quad u(0) = u^{0}. \tag{11}$$

By introducing new variables, Eq. (11) can be written as a system of equations. By analogy with (6), we can set, for instance,

$$g_{\alpha} - \mathcal{A}_{\alpha} u = 0, \quad \alpha = 1, \ldots, m. \tag{13}$$

Equation (11) takes the form

$$\frac{du}{dt} + \sum_{\alpha=1}^{m} \mathcal{A}_{\alpha}^{*} g_{\alpha} = f(t), \quad 0 < t \leq T. \tag{14}$$

From Eq. (11), we obtain system (13), (14) involving adjoint operators. Constructions of this type are typical for many applied problems.

Equation (11) can be considered as an implication of system (13), (14). Another option is to eliminate the unknown $$u$$ from this system. By differentiating (13) and substituting (14), we obtain

$$\frac{dg_{\alpha}}{dt} + \mathcal{A}_{\alpha} \sum_{\beta=1}^{m} \mathcal{A}_{\beta}^{*} g_{\beta} = \mathcal{A}_{\alpha} f(t), \quad \alpha = 1, \ldots, m, \quad 0 < t \leq T. \tag{15}$$

We supplement this system by the initial conditions

$$g_{\alpha}(0) = \mathcal{A}_{\alpha} u^{0}, \quad \alpha = 1, \ldots, m, \tag{16}$$

which follow from (12) and (13).

When the Cauchy problem for system (13), (14) is solved numerically, one can use either the single Eq. (11) or system (13). Equations in the system are interrelated, and, instead of a single equation, we have $$m$$ equations. With regard to this fact, the computer implementation based on solving problem (11), (12) seems much preferable over the one using problem (15), (16).

2. SPACE DISCRETIZATION

We illustrate the general discussion by an example of a model two-dimensional parabolic problem posed in the rectangle

$$\Omega = \{ \mathbf{x} | x = (x_{1}, x_{2}), 0 < x_{\alpha} < l_{\alpha}, \alpha = 1,2 \}.$$ 

As a basic grid in $$\Omega$$, we take the uniform rectangular grid

$$\bar{\Omega} = \{ \mathbf{x} | x = (x_{1}, x_{2}), x_{\alpha} = i_{\alpha} h_{\alpha}, i_{\alpha} = 0,1, \ldots, N_{\alpha}, N_{\alpha} h_{\alpha} = l_{\alpha} \}.$$ 

Let $$\omega$$ be the set of its interior nodes (thus, $$\bar{\omega} = \omega \cup \partial \omega$$). The components of the fluxes are prescribed on the corresponding edges of this grid (see figure). Define

$$\bar{\omega}_{1} = \{ x_{1} = (i_{1} + 0.5) h_{1}, i_{1} = 0, 1, \ldots, N_{1} - 1, x_{2} = i_{2} h_{2}, i_{2} = 0, 1, \ldots, N_{2} \},$$

$$\bar{\omega}_{2} = \{ x_{1} = i_{1} h_{1}, i_{1} = 0, 1, \ldots, N_{1}, x_{2} = (i_{2} + 0.5) h_{2}, i_{2} = 0, 1, \ldots, N_{2} - 1 \}$$

and set $$\bar{\omega}_{3} = \partial \omega_{3}, \partial \omega_{3}, \alpha = 1, 2.$$ 

For grid functions obeying the condition $$\gamma(x) = 0$$ for $$x \in \partial \omega$$, we define the Hilbert space $$H = L_{2}(\omega)$$ in the conventional way so that the inner product and the norm are given by the relations

$$(\gamma, w) = \sum_{x \in \bar{\omega}} \gamma(x) w(x) h_{1} h_{2}, \quad ||\gamma|| = \sqrt{(\gamma, \gamma)}. \tag{1/2}$$
In a similar way, for grid functions that are defined on the grids $\omega_\alpha, \alpha = 1, 2$ and vanish on $\partial \omega_\alpha$, we define the Hilbert spaces $H_\alpha, \alpha = 1, 2$ in which
\[(y, w)_\alpha = \sum_{x \in \omega_\alpha} y(x)w(x)h_1h_2, \quad \|y\|_\alpha = (y, y)_\alpha^{1/2}, \quad \alpha = 1, 2.\]

Let us construct grid counterparts to the differential operators $\mathcal{A}_\alpha$ and $\mathcal{A}_\alpha^*$, $\alpha = 1, 2$, defined by formulas (9) and (10). Space derivatives are approximated by the conventional central differences. We set
\[(A_1y)(x) = -k^{1/2}(x)\frac{y(x_1 + 0.5h_1, x_2) - y(x_1 - 0.5h_1, x_2)}{h_1}, \quad x \in \omega_1, \]
thus, $A_1 : H \rightarrow H_1$. The operator $A_2 : H \rightarrow H_2$ is defined similarly:
\[(A_2y)(x) = -k^{1/2}(x)\frac{y(x_1, x_2 + 0.5h_2) - y(x_1, x_2 - 0.5h_2)}{h_2}, \quad x \in \omega_2.\]

This construction implies that
\[A_\alpha u = \mathcal{A}_\alpha u + O(h_\alpha^2), \quad \alpha = 1, 2.\]

By direct calculations, we verify that the adjoint operators $A_\alpha^* : H_\alpha \rightarrow H, \alpha = 1, 2$, can be represented as
\[(A_1^*y)(x) = k^{1/2}(x_1 + 0.5h_1, x_2)y(x_1 + 0.5h_1, x_2) - k^{1/2}(x_1 - 0.5h_1, x_2)y(x_1 - 0.5h_1, x_2), \quad x \in \omega, \]
\[(A_2^*y)(x) = k^{1/2}(x_1, x_2 + 0.5h_2)y(x_1, x_2 + 0.5h_2) - k^{1/2}(x_1, x_2 - 0.5h_2)y(x_1, x_2 - 0.5h_2), \quad x \in \omega.\]

For sufficiently smooth functions $u$, it holds that
\[A_\alpha^* u = A_\alpha^* u + O(h_\alpha^2), \quad \alpha = 1, 2.\]

Define the grid operator
\[D = \sum_{\alpha=1}^2 D_\alpha, \quad D_\alpha = A_\alpha^* A_\alpha, \quad \alpha = 1, 2.\]
Taking into account relations (17), (18), (20), and (21), we obtain the following representations for the grid operators \( D_\alpha : H \to H, \alpha = 1, 2 \):

\[
(D_1y)(x) = -\frac{1}{h_1}
\left(k(x_1 + 0.5h_1, x_2)\frac{y(x_1 + h_1, x_2) - y(x)}{h_1}
- k(x_1 - 0.5h_1, x_2)\frac{y(x_1 - h_1, x_2)}{h_1}\right), \quad x \in \omega,
\]

\[
(D_2y)(x) = -\frac{1}{h_2}
\left(k(x_1, x_2 + 0.5h_2)\frac{y(x_1, x_2 + h_2) - y(x)}{h_2}
- k(x_1, x_2 - 0.5h_2)\frac{y(x_1, x_2 - h_2)}{h_2}\right), \quad x \in \omega.
\]

By analogy with (19) and (22), we have

\[
D_\alpha u = \mathcal{D}_\alpha u + O(h_\alpha^2), \quad \alpha = 1, 2,
\]

for sufficiently smooth coefficients \( k \) and functions \( u \) (see [1, 16]). Moreover, in the space \( H \) of grid functions, it holds that

\[
D_\alpha = D_\alpha^h, \quad k\delta_\alpha E \leq D_\alpha \leq k\Delta_\alpha E,
\]

\[
\delta_\alpha = \frac{4}{h_\alpha^2}\sin^2\frac{\pi h_\alpha}{2h_\alpha}, \quad \Delta_\alpha = \frac{4}{h_\alpha^2}\cos^2\frac{\pi h_\alpha}{2h_\alpha}, \quad \alpha = 1, 2,
\]

where \( E \) is the identity operator in \( H \).

As a result of space discretization, we replace problem (1)–(3) by the Cauchy problem for the operator-differential equation

\[
\frac{du}{dt} + Du = f(x, t), \quad x \in \omega, \quad 0 < t \leq T,
\]

\[
u(x, 0) = u^0(x), \quad x \in \omega.
\]

(We preserve the notation for the solution.) With system (25), (26), we associate the system of the operator-differential equations

\[
g_\alpha - A_\alpha u = 0, \quad x \in \omega_\alpha, \quad \alpha = 1, 2,
\]

\[
\frac{du}{dt} + \sum_{\alpha=1}^{2} A_\alpha^h g_\alpha = f(x, t), \quad x \in \omega, \quad 0 < t \leq T,
\]

which is supplemented by initial conditions (26).

From (15) and (16), we derive

\[
\frac{dg_\alpha}{dt} + A_\alpha \sum_{\beta=1}^{2} A_\beta^h g_\beta = A_\alpha f, \quad x \in \omega_\alpha, \quad \alpha = 1, 2, \quad 0 < t \leq T,
\]

\[
g_\alpha(0) = A_\alpha u^0, \quad x \in \omega_\alpha, \quad \alpha = 1, 2.
\]

Taking into account relations (19), (22), and (25), we conclude that the above finite difference problems are second-order space approximations of the corresponding differential problems.

3. DIFFERENCE SCHEMES FOR THE PARABOLIC EQUATION

We give a separate discussion of conventional two-level schemes and locally one-dimensional splitting schemes for parabolic equation (24), (25) and the flux system of equations (29). They correspond to two different methods of the partial elimination of unknowns in system (27), (28).
Let $\tau$ be the step of a uniform time grid, and let $y^n = y(t^n)$, $\sigma^n = n\tau$ ($n = 0, 1, \ldots, N$), and $N\tau = T$. Equation (25) is approximated by the weighted two-level scheme

$$\frac{y^{n+1} - y^n}{\tau} + D(\sigma)y^{n+\frac{1}{2}} + (1 - \sigma)y^n = \varphi^n, \quad n = 0, 1, \ldots, N - 1, \tag{31}$$

where, for instance, $\varphi^n = f(\sigma y^{n+\frac{1}{2}} + (1 - \sigma)y^n)$. Taking (26) into account, we supplement the operator-difference Eq. (31) by the initial condition

$$y^0 = u^0. \tag{32}$$

For the difference scheme (31), (32), the truncation error is $O(|h|^2 + \tau^2 + (\sigma - 0.5)\tau)$, where $|h|^2 = h_1^2 + h_2^2$.

Here is a typical result concerning the stability of weighted difference schemes for a first-order evolutionary equation (see [1, 17, 18]). Difference scheme (23), (31), (32) is unconditionally stable for $\sigma \geq 0.5$; moreover, at each step, the difference solution satisfies the estimate

$$\|y^{n+1}\| \leq \|y^n\| + \tau \|\varphi^n\|, \quad n = 0, 1, \ldots, N - 1,$n

which implies in a conditional way the desired stability estimate

$$\|y^{n+1}\| \leq \|y^0\| + \sum_{k=0}^{n} \tau \|\varphi^k\|.$$n

Unconditionally stable operator-difference schemes (31) for parabolic Eq. (1) are not always convenient from the computational viewpoint. Let us consider conventional additive (splitting) schemes for problem (23), (25), (26). For these schemes, the transition to a new time level requires the solution of simpler problems; namely, the individual operators $D_\alpha = A_{\alpha}^nA_\alpha$ ($\alpha = 1, 2$) must be inverted rather than their sum (that is, the operator $D$). Taking into account the nature of the above operators $A_\alpha^n$ and $A_\alpha$ ($\alpha = 1, 2$), we call such schemes locally one-dimensional.

In the construction of additive schemes, the two-component splitting is specially set out. In the case under consideration (see (23)), we can be guided by operator analogs of the classical alternating direction methods. To solve problem (23), (25), (26) numerically, we use the factorized scheme

$$(E + \sigma D_1)(E + \sigma D_2)\frac{y^{n+1} - y^n}{\tau} + Dy^n = \varphi^n, \tag{33}$$

where $E$ is the identity operator in $H$. Scheme (33) corresponds to the application of the Douglas–Rechford method (see [19]) if $\sigma = 1$ and the Peaceman–Rechford method (see [20]) if $\sigma = 1/2$.

It is well known (see [1, 13]) that the factorized operator-difference scheme (32), (33) as applied to problem (23), (25), (26) is unconditionally stable for $\sigma \geq 1/2$ and, at each step, the difference solution satisfies the estimate

$$\|(E + \sigma D_2)y^{n+1}\|^2 \leq \|(E + \sigma D_2)y^n\|^2 + \tau \|(E + \sigma D_1)^{-1}\varphi^n\|^2. \tag{34}$$

Based on the a priori estimate (34), one can show that the difference solution converges to the exact one to an accuracy of $O(|h|^2 + \tau^2 + (\sigma - 0.5)\tau)$.

Additive difference schemes with splittings into three and more pairwise noncommuting operators are constructed on the basis of the concept of full approximation (see [1, 13]). For problem (23), (25), (26), the conventional scheme of componentwise splitting (locally one-dimensional scheme) has the form

$$\frac{y^{n+\alpha/2} - y^{n+\alpha/2}}{\tau} + D(\sigma_\alpha)y^{n+\alpha/2} + (1 - \sigma_\alpha)y^n(1 - \sigma_\alpha)y^n = \varphi_\alpha^n, \tag{35}$$

where $\alpha = 1, 2$, $n = 0, 1, \ldots, N - 1$, and the right-hand sides satisfy the relation

$$\varphi^n = \sum_{\alpha=1}^{2} \varphi_\alpha^n.$$
For \( \sigma_\alpha \geq 1/2 \), the componentwise splitting scheme (35) is unconditionally stable. The convergence analysis of schemes with componentwise splitting is based on a special representation of the right-hand sides \( \varphi^{n}_\alpha \) (\( \alpha = 1, 2 \)) such that
\[
\varphi^{n}_\alpha = \varphi^{n}_\alpha + \tilde{\varphi}^{n}_\alpha, \quad \alpha = 1, 2, \quad \sum_{\alpha=1}^{2} \tilde{\varphi}^{n}_\alpha = 0.
\]
(36)
This form of the right-hand side is of fundamental importance in the error analysis of an additive scheme. It can be shown that, for \( 2 \geq \sigma_\alpha \geq 1/2 \) (\( \alpha = 1, ..., p - 1 \)), the difference solution of problem (32), (35), (36) satisfies the a priori estimate
\[
\|\varphi^{n+1}_\alpha\| \leq \|\varphi^{n}_\alpha\| + \tau \sum_{\alpha=1}^{2} \left\|\tilde{\varphi}^{n}_\alpha\right\| + \tau \left\|D_\alpha \sum_{\beta=\alpha}^{2} \tilde{\varphi}^{n}_\beta\right\|.
\]
(37)

If the splitting is based on an approximation of the transition operator, then the corresponding componentwise splitting scheme can be interpreted as a full approximation scheme. For this case, a priori stability estimates are given in [21].

The stability estimate (37) underlies the error analysis of componentwise splitting schemes. The problem for the error of the difference solution is stated in form (35), (36). An important point is that, in the componentwise splitting schemes under discussion, as well as in additively-averaged schemes (see [13]), the stability estimates depend heavily on splitting (36). In the general case of the splitting into three and more components, additive scheme (32), (35), (36) yields a first order approximation with respect to time for any \( \sigma \). However, in the case of two-component splitting, this scheme has the second order if \( \sigma = 1/2 \) (see [17]).

An important advantage of the factorized schemes over the componentwise splitting schemes are that the former are pseudo-time evolution schemes. For problems with a constant right-hand side, the solution to a time-dependent problem converges to that of the corresponding stationary problem as \( t \to \infty \). In pseudo-time evolution schemes, this property holds for the grid solution (after time-discretization). Schemes of this type are used for both time-dependent and stationary problems. They can be a basis for the construction of iterative methods for solving stationary problems.

4. DIFFERENCE SCHEMES FOR THE STREAM SYSTEM OF EQUATIONS

It is convenient to interpret system (29) as a single evolutionary equation for the vector \( g = \{g_1, g_2\} \):
\[
\frac{dg}{dt} + Ag = r(t), \quad 0 < t \leq T.
\]
(38)
Here, \( r = \{A_1 f, A_2 f\} \). We supplement Eq. (38) with the initial condition
\[
g(0) = s,
\]
(39)
where \( s = \{s_1, s_2\} \) and \( s_\alpha = g_\alpha(0) = A_\alpha u^0 \) (\( \alpha = 1, 2 \)). For the operator matrix \( A \), we have the representation
\[
A = \begin{pmatrix}
A_1 A^*_1 & A_1 A^*_2 \\
A_2 A^*_1 & A_2 A^*_2
\end{pmatrix}.
\]
(40)
On the direct sum \( H = H_1 \oplus H_2 \), we define
\[
(u, v) = \sum_{\alpha=1}^{2} (u_\alpha, v_\alpha)_\alpha, \quad \|u\|^2 = \sum_{\alpha=1}^{2} \|u_\alpha\|^2.
\]
It is straightforward to verify that the operator \( A \) defined by (40) is self-adjoint and nonnegative in \( H \) (that is, \( A = A^* \geq 0 \)). This implies that the solution to problem (38), (39) satisfies the a priori estimate
\[
\|g(t)\| \leq \|s\| + \int_{0}^{t} \|r(\theta)\| d\theta,
\]
(41)
which expresses the stability with respect to the initial data and the right-hand side.
To solve problem (29), (30) numerically, we use the weighted two-level scheme

\[
\frac{y^{n+1}_a - y^n_a}{\tau} + A_a \sum_{\beta=1}^2 A^{\beta}_b (\sigma y^{n+1}_b + (1 - \sigma) y^n_b) = \phi^n_\alpha, \quad n = 0, 1, \ldots, N - 1,
\]

(42)

where \( \phi^n_\alpha = A_\alpha f(\sigma y^{n+1}_\alpha + (1 - \sigma) y^n_\alpha) \). In matrix form, scheme (42), (43) can be written as

\[
\frac{y^{n+1}_a - y^n_a}{\tau} + A(\sigma y^{n+1}_a + (1 - \sigma) y^n_a) = \phi^n_a, \quad n = 0, 1, \ldots, N - 1,
\]

(44)

\[
y^0_a = s_\alpha, \quad \alpha = 1, 2,
\]

(43)

To derive stability conditions for the weighted scheme (44), (45), we rewrite (44) in the form

\[
B \frac{y^{n+1}_a - y^n_a}{\tau} + A \frac{y^{n+1}_a + y^n_a}{2} = \phi^n_a, \quad n = 0, 1, \ldots, N - 1,
\]

(46)

where the operator \( B \) is given by the formula

\[
B = E + \sigma \tau A - \frac{\tau}{2} A,
\]

and \( E \) is the identity operator in \( \mathbf{H} \). If \( \sigma \geq 1/2 \) (which is the standard restriction on the weight), then \( B = B^* \geq E \).

Take the inner product in \( \mathbf{H} \) of (46) and \( \tau (y^{n+1}_a + y^n_a) \). Since the operator \( A \) is nonnegative, this yields

\[
\|y^{n+1}_a\|_\mathbf{H}^2 - \|y^n_a\|_\mathbf{H}^2 \leq \tau (\phi^n_a, (y^{n+1}_a + y^n_a)).
\]

Using the relations

\[
(\phi^n_a, (y^{n+1}_a + y^n_a)) \leq \|\phi^n_a\| \left( \|y^{n+1}_a\| + \|y^n_a\| \right),
\]

\[
\|y^{n+1}_a\| + \|y^n_a\| \leq \|y^{n+1}_a\| + \|y^n_a\|,
\]

\[
\|y^{n+1}_a\|_\mathbf{H}^2 - \|y^n_a\|_\mathbf{H}^2 = (\|y^{n+1}_a\| - \|y^n_a\|) (\|y^{n+1}_a\| + \|y^n_a\|),
\]

we arrive at the grid analog of the a priori estimate (41):

\[
\|y^{n+1}_a\|_\mathbf{H} \leq \|y^n_a\|_\mathbf{H} + \tau \|\phi^n_a\|, \quad n = 0, 1, \ldots, N - 1.
\]

(47)

Thus, the weighted scheme (44), (45), as well as scheme (42), (43), is unconditionally stable for \( \sigma \geq 1/2 \). For \( \sigma = 1/2 \), the convergence to the solution to problem (29), (30) is quadratic in time, while, for other values of \( \sigma \), the convergence is linear.

The computer implementation of scheme (42), (43) involves solving the system of grid equations

\[
y_1^{n+1} + \sigma \tau A_1^* y_1^{n+1} + \sigma \tau A_2^* y_2^{n+1} = \chi_1^n,
\]

\[
y_2^{n+1} + \sigma \tau A_1^* y_1^{n+1} + \sigma \tau A_2^* y_2^{n+1} = \chi_2^n.
\]

The equations in this system are strongly interrelated through the operators \( A_1^* \) and \( A_2^* \). Computational complexity of this approach is higher than that of the implementation of difference scheme (31), (32) for the parabolic problem.

Additive schemes for problem (23), (29), (30) lead in a natural way to simpler problems at each new time level

\[
y_1^{n+1} + \sigma \tau A_1^* y_1^{n+1} = \chi_1^n,
\]

\[
y_2^{n+1} + \sigma \tau A_2^* y_2^{n+1} = \chi_2^n.
\]

Here, the individual components of the solution are not connected with each other.
Starting from the weighted scheme (42), (43), we construct a locally one-dimensional scheme in which only the diagonal part of the operator \( A \) is used at the new time level. Formula (42) is replaced by the relations

\[
\frac{y_1^{n+1} - y_1^n}{\tau} + A_1 A_1^n (\sigma y_1^{n+1} + (1 - \sigma) y_1^n) + A_1 A_2^n y_2^n = \varphi_1^n,
\]

\[
\frac{y_2^{n+1} - y_2^n}{\tau} + A_2 A_1^n y_1^n + A_2 A_2^n (\sigma y_2^{n+1} + (1 - \sigma) y_2^n) = \varphi_2^n.
\]

Using the vector notation introduced above, we can write this scheme in form (46) with

\[
B = E + \sigma \tau D - \frac{\tau}{2} A.
\]  

(48)

For the operator \( D \), we adopt the representation

\[
D = \begin{pmatrix}
A_1 A_1^n & 0 \\
0 & A_2 A_2^n
\end{pmatrix}.
\]  

(49)

The analysis of the operator-difference scheme (45), (46), (48), (49) is performed in the same way as for scheme (44), (45). Estimate (47) is valid if the self-adjoint operator \( B \) is positive. The analysis is based on the inequality

\[
(Ay, y) = \sum_{\alpha=1}^{2} A_\alpha y_\alpha y_\alpha \leq 2 \sum_{\alpha=1}^{2} (A_\alpha y_\alpha, A_\alpha y_\alpha) = 2(Dy, y).
\]

From this, we conclude that \( B = B^* \geq E \) if \( \sigma \geq 1 \).

With such weights, scheme (45), (46), (48), (49) is unconditionally stable; moreover, the difference solution satisfies the stability estimate (47) with respect to the initial data and the right-hand side. This scheme yields an approximation of the first order with respect to time. Its basic potential advantage is that, at each new time level, the components of the approximate solution can be calculated asynchronously. This feature may be important if, for instance, the calculations are organized in a parallel mode.

Another class of additive schemes for solving problem (38), (39) involves the triangular splitting of the self-adjoint matrix operator \( A \), where

\[
A = A_1 + A_2, \quad A_1^* = A_2.
\]  

(50)

In view of (40), we have

\[
A_1 = \begin{pmatrix}
\frac{1}{2} A_1 A_1^n & 0 \\
A_2 A_2^n & \frac{1}{2} A_2 A_2^n
\end{pmatrix}, \quad A = \begin{pmatrix}
\frac{1}{2} A_1 A_1^n & A_1 A_2^n \\
0 & \frac{1}{2} A_2 A_2^n
\end{pmatrix}.
\]  

(51)

Similar additive representations of the operator were used for solving other vector problems (e.g., see [22, 23]). The underlying idea is attributed to the alternating-triangular method due to A.A. Samarskii (see [24]).

For the numerical solution of problem (38), (39), we use the alternating-triangular method

\[
(E + \sigma \tau A_1)(E + \sigma \tau A_2)\frac{y^{n+1} - y^n}{\tau} + Ay^n = \varphi^n, \quad n = 0, 1, \ldots, N - 1.
\]  

(52)

For \( \sigma = 1/2 \), difference equation (52) is a second-order approximation (with respect to \( \tau \)) of the operator-differential equation (38), while, for \( \sigma \neq 1/2 \), it is only a first-order approximation.

Scheme (52) can be written in form (46) with

\[
B = (E + \sigma \tau A_1)(E + \sigma \tau A_2) - \frac{\tau}{2} A.
\]

(53)

It is straightforward to verify that, for \( \sigma \geq 1/2 \),

\[
B = B^* \geq E.
\]
This implies that, for $\sigma \geq 1/2$, the factorized operator-difference scheme (45), (50)–(52) is unconditionally stable; moreover, the difference solution satisfies the stability estimate (47) with respect to the initial data and the right-hand side.

The computational implementation of factorized scheme (52) requires that the operators $E + \sigma \tau A_1$ and $E + \sigma \tau A_2$ be inverted. Taking into account the structure of $A_1$ and $A_2$, we must sequentially solve the grid problems

$$y_1^{n+1/2} + \frac{\tau}{2} A_1 A_1^{n+1/2} y_1^n = \chi_1^n,$$

$$y_2^{n+1/2} + \sigma \tau A_2 A_2^{n+1/2} y_1^n + \frac{\tau}{2} A_2 A_2^{n+1/2} y_2^n = \chi_2^n,$$

$$y_2^{n+1} + \frac{\tau}{2} A_2 A_2^{n+1} y_2^n = \chi_2^{n+1},$$

$$y_1^{n+1} + \frac{\tau}{2} A_1 A_1^{n+1} y_1^n + \sigma \tau A_1 A_2^{n+1} y_2^n = \chi_1^{n+1}.$$

The factorized operator-difference scheme (50)–(52) belongs to the class of pseudo-time evolution schemes. Suppose that it is used for solving the Cauchy problem for system (38), (39), (29), (30) (this involves the calculation of the components $g_{1\alpha}$, $\alpha = 1, 2$). Then the computational complexity of this approach is comparable with that of solving problem (25), (26) for a single component.

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