Abstract

The classical model of q-damped oscillator is introduced and solved in terms of Jackson q-exponential function for three different cases, under-damped, over-damped and the critical one. It is shown that in all three cases solution is oscillating in time but is unbounded and non-periodic. By q-periodic function modulation, the self-similar microstructure of the solution for small time intervals is derived. In the critical case with degenerate roots, the second linearly independent solution is obtained as a limiting case of two infinitesimally close roots. It appears as standard derivative of q-exponential and is rewritten in terms of the q-logarithmic function. We extend our result by constructing n linearly independent set of solutions to a generic constant coefficient q-difference equation degree N with n degenerate roots.

1 Introduction

The damped harmonic oscillator as simplest classical model of motion with dissipation, corresponds to friction force proportional to velocity of motion. It appears in many physical problems from quantum theory to inflating universe models. The quantum damped oscillator as one of the simplest quantum system displaying the energy dissipation, has been studied to understand dissipation in quantum theory [1]. Most popular models are, the Bateman-Feshbach-Tikochinsky oscillator as a closed system with two degrees of freedom and the Caldirola-Kanai oscillator as an open system with one degree of freedom and time dependent mass. Due to complicated character of the friction force, several modifications of the damping term were proposed as fractional derivative, time-delay or finite difference derivative etc. The goal of the present paper is to study classical q-extended damped harmonic oscillator, where standard time derivatives are replaced by Jackson q-derivatives [2]. The q-extension of harmonic oscillator and its solution in form of basic trigonometric functions [4] was
The q-extended heat equation and corresponding Burgers equation with q-shock solitons were studied in [5]. In the limit $q \to 1$, the q-deformed model reduces to the standard damped oscillator model. We construct solution in terms of Jackson q-exponential function and find q-periodic modulation of the solution with self-similar properties. Special attention is paid for degenerate roots case. And results are generalized for arbitrary order constant coefficient q-ordinary difference equation. This gives background for further possible quantization of corresponding model.

2 Damped Oscillator

In reality a spring never oscillates forever, since frictional forces will diminish the amplitude of oscillation until the rest. In many situations the frictional force is proportional to the velocity of the mass as follows $f_r = -\gamma v$, where $\gamma > 0$ is the damping constant. Therefore, by adding this frictional force we have the following equation for a spring

$$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0. \quad (1)$$

Solution of this equation in the form $x(t) = e^{\lambda t}$ leads to the characteristic equation

$$m\lambda^2 + \gamma\lambda + k = 0 \quad (2)$$

with two roots

$$\lambda_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m}, \quad \lambda_2 = \frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m}.$$ 

Then according to value of damping constant we have three cases:

**i - Under-damping Case:** When $\gamma^2 < 4mk$, which means that friction is sufficiently weak, we have two complex conjugate roots

$$\lambda_{1,2} = -\frac{\gamma}{2m} \pm i\omega, \quad (3)$$

where $\omega \equiv \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$. Then the general solution of (1) is

$$x(t) = e^{-\frac{\gamma t}{2m}} (A \cos \omega t + B \sin \omega t). \quad (4)$$

If $\gamma = 0$, there is no decay and the spring oscillates forever. If $\gamma$ is big, the amplitude of oscillations decays very fast (the exponential decay).

**ii - Over-damping Case:** When $\gamma^2 > 4mk$, which means that friction is sufficiently strong, both roots are real, this why the solution decays exponentially

$$x(t) = Ae^{-\frac{\gamma + \sqrt{\gamma^2 - 4mk}}{2m} t} + Be^{-\frac{\gamma - \sqrt{\gamma^2 - 4mk}}{2m} t}. \quad (5)$$
This case is called as over-damping because there is no any oscillation.

iii - Critical Case: For $\gamma^2 = 4mk$, we have two degenerate roots

$$\lambda_1 = \lambda_2 = -\frac{\gamma}{2m},$$

then the general solution is

$$x(t) = Ae^{-\frac{\gamma}{2m}t} + Bte^{-\frac{\gamma}{2m}t}. \quad (6)$$

3 q-Harmonic Oscillator

Here we introduce the $q$-Harmonic oscillator. Equation of $q$-deformed classical harmonic oscillator is

$$D_q^2 x(t) + \omega^2 x(t) = 0, \quad (7)$$

where the $q$-derivative is definite as [2],

$$D_q x(t) = \frac{x(qt) - x(t)}{(q - 1)t}. \quad (8)$$

Using the power series method (or the $q$-exponential form $x(t) = e_q(\lambda t)$), we find the general solution of $q$-Harmonic Oscillator in the following form [3]

$$x(t) = A(t) \cos_q \omega t + B(t) \sin_q \omega t, \quad (9)$$

where

$$D_q A(t) = D_q B(t) = 0,$$

means $A(t), B(t)$ in general are $q$-periodic functions, and particularly could be arbitrary constants. Here the Jackson $q$-exponential function is definite as

$$e_q(t) \sum_{n=1}^{\infty} \frac{t^n}{[n]_q!}, \quad (10)$$

and

$$e_q(it) = \cos_q t + i \sin_q t, \quad (11)$$

where $[n]_q = 1 + q + ... + q^{n-1}$. For $q > 1$ this function is entire analytic function, so we restrict consideration by this case only.

In Figure 1 we plot particular $\cos_q t$ solution of $q$-deformed classical harmonic oscillator. In contrast to standard $\sin t$ and $\cos t$ functions, $\sin_q t$ and $\cos_q t$ functions [3], are not bounded and also have no periodicity. In Figure 2 we plot modulation of the same solution with $q$-periodic function $A(t) = \sin \left( \frac{2\pi}{\ln_q t} \right) \cos_q t$, which gives micro oscillations to the solution.
We define equation for \( q \)-analogue of damped oscillator in the form
\[
D_q^2 x(t) + \Gamma D_q x(t) + \omega^2 x(t) = 0,
\]
where
\[
\omega \equiv \sqrt{\frac{k}{m}}, \quad \Gamma \equiv \frac{\gamma}{m}.
\]
By substituting \( x(t) = e_q(\lambda t) \) into equation (12), we obtain
\[
e_q(\lambda t) \left[ \lambda^2 + \Gamma \lambda + \omega^2 \right] = 0.
\]
For \( q > 1 \), \( e_q(\lambda t) \) is an entire function defined for any \( t \), this why it has an infinite set of zeros (no poles). Then, we can choose
\[
\lambda^2 + \Gamma \lambda + \omega^2 = 0.
\]
The roots of this characteristic equation are
\[
\lambda_{1,2} = -\frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega^2}.
\]

4.1 Under-Damping Case
For \( \Gamma^2 < 4\omega^2 \), we have two complex conjugate roots
\[
\lambda_1 = -\frac{\Gamma}{2} + i\Omega, \quad \lambda_2 = -\frac{\Gamma}{2} - i\Omega,
\]
where
\[
\Omega \equiv \sqrt{\omega^2 - \frac{\Gamma^2}{4}}.
\]
Then the general solution of equation (12) is
\[
x(t) = Ae_q \left[ \left( -\frac{\Gamma}{2} + i\Omega \right) t \right] + Be_q \left[ \left( -\frac{\Gamma}{2} - i\Omega \right) t \right]
\]
(14)

In Figure 3 and Figure 4 we plot particular solutions with constant \((A = B = 1)\) and with \(q\)-Periodic modulation, respectively.

![Figure 3: Under-damping case \(A = B = 1\)](chart)

4.2 Over-Damping Case
For \( \Gamma^2 > 4\omega^2 \), we have two distinct real roots \( \lambda_{1,2} \) and solution is
\[
x(t) = A(t)e_q \left[ \left( -\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} - \omega^2} \right) t \right] + B(t)e_q \left[ \left( -\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} - \omega^2} \right) t \right],
\]
(15)
where \( A(t), B(t) \) are \( q \)-periodic functions (or could be arbitrary constants).

In Figure 5 and Figure 6 we plot particular solutions with constant \( A = B = 1 \) and with \( q \)-Periodic modulation, respectively.

### 4.3 Critical Case

For \( \Gamma^2 = 4\omega^2 \), we have degenerate roots \( \lambda_{1,2} = -\frac{E}{2} \). The first obvious solution is \( e_q(-\omega t) \). However if we try the second linearly independent solution in the usual form \( te_q(-\omega t) \), it doesn’t work. This why we follow the next method:

We suppose that the system is very close to the critical case so that \( \frac{\Gamma}{2} = \omega + \epsilon \), where \( \epsilon \ll 1 \). Then the roots of characteristic equation are

\[
\lambda_1 = -\omega + \sqrt{2\omega\epsilon}, \quad \lambda_2 = -\omega - \sqrt{2\omega\epsilon},
\]  

(16)
and the solution is

\[ x(t) = A e_q \left( (-\omega + \sqrt{2\omega \epsilon})t \right) + B e_q \left( (-\omega - \sqrt{2\omega \epsilon})t \right) \]  

(17)

Expanding this solution in terms of \( \epsilon \),

\[
x(t) = A \sum_{n=0}^{\infty} \left( \frac{(-\omega)^n + n \sqrt{2\omega \epsilon} (-\omega)^{n-1} + \ldots}{[n]!} \right) t^n \\
+ B \sum_{n=0}^{\infty} \left( \frac{(-\omega)^n - n \sqrt{2\omega \epsilon} (-\omega)^{n-1} + \ldots}{[n]!} \right) t^n \\
= (A + B) \sum_{n=0}^{\infty} \frac{(-\omega)^n}{[n]!} t^n + (A - B) \sqrt{2\omega \epsilon} \sum_{n=1}^{\infty} \frac{n}{[n]!} (-\omega)^{n-1} t^n + \ldots \\
= (A + B) x_1(t) + (B - A) \sqrt{\frac{2\epsilon}{\omega}} x_2(t) + \ldots, 
\]

(18)

in zero approximation we get the first solution

\[ x_1(t) = e_q(-\omega t). \]  

(19)

In the linear approximation we obtain the second solution in the form

\[ x_2(t) = t \frac{d}{dt} e_q(-\omega t). \]  

(20)

In order to prove that solutions \( x_1(t) = e_q(-\omega t) \) and \( x_2(t) = t \frac{d}{dt} e_q(-\omega t) \) are linearly independent, we check the \( q \)-Wronskian :

\[
W_q = \begin{vmatrix}
  e_q(-\omega t) & t \frac{d}{dt} e_q(-\omega t) \\
  D_q(e_q(-\omega t)) & D_q \left( t \frac{d}{dt} e_q(-\omega t) \right)
\end{vmatrix}
\]
Here we show that the term in parenthesis is not identically zero. For $q > 1$, by using the infinite product representation of $e_q(x)$ (21), we get

$$e_q(-\omega t) = \prod_{n=0}^{\infty} \left( 1 - \left(1 - \frac{1}{q} \right) \frac{1}{q^n \omega t} \right), \quad (22)$$

or

$$t \frac{d}{dt} \ln e_q(-\omega t) = \sum_{n=0}^{\infty} -\omega \left(1 - \frac{1}{q} \right) \frac{1}{q^n \omega t} \frac{1}{q^n},$$

or

$$t \frac{d}{dt} e_q(-\omega t) = Ae_q(-\omega t),$$

where

$$A \equiv \sum_{n=0}^{\infty} -\omega \left(1 - \frac{1}{q} \right) \frac{1}{q^n \omega t}. \quad (23)$$

Expanding the denominator, we have

$$A = \sum_{n=0}^{\infty} \left(1 - \frac{1}{q} \right) \frac{1}{q^n \omega t} \sum_{l=0}^{\infty} \frac{1}{q^n} (\omega t)^l = -\sum_{l=0}^{\infty} \frac{1}{q^n} (\omega t)^l \sum_{l=0}^{\infty} \frac{1}{q^n},$$

or

$$t \frac{d}{dt} e_q(-\omega t) = Ae_q(-\omega t),$$

where $|t| < \frac{\omega}{q}$. We know that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - ... = -\sum_{l=1}^{\infty} \frac{x^l}{l}$$

and the $q$-analogue of this expression is given as [6]

$$\ln_q(1-x) = -\sum_{l=1}^{\infty} \frac{x^l}{[l]_q}. \quad (24)$$

Then we rewrite

$$A = -\frac{1}{1-q} \ln_q \left(1 - \left(1 - \frac{1}{q} \right) \omega t \right). \quad (25)$$
The second solution $x_2(t)$ can also be rewritten in terms of $q$-logarithmic function

$$x_2(t) = t \frac{d}{dt} e_q(-\omega t) = \frac{1}{q-1} \ln_q \left( 1 - \left( 1 - \frac{1}{q} \right) \omega t \right) e_q(-\omega t), \quad (26)$$

where $|t| < \frac{q}{2}$. Finally, the $q$-Wronskian is not vanish

$$W_q = -\omega (e_q(-\omega t))^2 \left( 1 - \frac{1}{q-1} \ln_q \left( 1 - \left( 1 - \frac{1}{q} \right) \omega t \right) \right) \neq 0,$$

since the term

$$1 - \frac{1}{q-1} \ln_q \left( 1 - \left( 1 - \frac{1}{q} \right) \omega t \right)$$

couldn’t be identically zero.

We can also rewrite this in terms of $q$-logarithm, which instead of linear in $t$ term for $q = 1$ case, now includes infinite set of arbitrary powers of $t$,

$$x_2(t) = \frac{1}{1-q} \sum_{l=1}^{\infty} \frac{(1 - \frac{1}{q})^l \omega^l t^l}{[l]} e_q(-\omega t) = -\frac{1}{1-q} \ln_q \left( 1 - \left( 1 - \frac{1}{q} \right) \omega t \right) e_q(-\omega t). \quad (27)$$

It is easy to check that for $q \rightarrow 1$ our solution reduces to the standard second solution $te^{-\omega t}$.

Combining the above results we find the general solution in the degenerate case as

$$x(t) = Ae_q(-\omega t) + B t \frac{d}{dt} e_q(-\omega t). \quad (28)$$

In Figure 7 and Figure 8 we plot particular solutions with constant ($A = B = 1$) and with $q$-Periodic modulation, respectively.

In Figures 9 and 10 we plot solution with $q$-Periodic function modulation at different small scales. Comparing these figures we find very close similarity, this why $q$-periodic function modulation leads to the self-similarity property of the solution.

5 Degenerate Roots for Equation Degree N

$q$-Damped oscillator considered in first section is an example of constant coefficient $q$-difference equation of degree two. The problem of radiation damping leads to a constant coefficient equation of degree three \cite{1}. Here we consider generic constant coefficient $q$-difference equation of degree $N$. Then the result for degenerate roots obtained in previous section can be generalized to this equation of an arbitrary order.
The constant coefficients $q$-difference equation of order $N$ is

$$
\sum_{k=0}^{N} a_k D^k x(t) = 0,
$$

(29)

where $a_k$ are constants (or $q$-periodic functions). By substitution

$$
x = e_q(\lambda t)
$$

(30)

we get the characteristic equation

$$
\sum_{k=1}^{N} a_k \lambda^k = 0.
$$

It has $N$ roots. Suppose $(\lambda_1, \lambda_2, ..., \lambda_N)$ are distinct numbers. Then, the general
solution of (29) is found in the form

\[ x(t) = \sum_{k=1}^{N} c_k e_{q}(\lambda_k t). \]  

(31)

In case, when we have \( n \)-degenerate roots

\[ (D + \omega)^n x = 0, \]

by substituting (30), characteristic equation is found as

\[ (\lambda + \omega)^n = 0. \]

Then the linearly independent solutions for these degenerate roots we can obtain in the following form:

\[ x_1(t) = e_{q}(-\omega t), \]
\[ x_2(t) = t \frac{d}{dt} e_{q}(-\omega t) = \frac{1}{q - 1} \ln_q \left( 1 - (1 - \frac{1}{q})\omega t \right) e_{q}(-\omega t), \]
\[ x_3(t) = \left( \frac{d}{dt} \right)^2 e_{q}(-\omega t) \]

or by using the commutation relation \( [t, \frac{d}{dt}] = -1 \), up to linearly dependent solution, it can be written as

\[ x_3(t) = t^2 \frac{d^2}{dt^2} e_{q}(-\omega t) \]
\[ \ldots \]
\[ x_n(t) = t^{n-1} \frac{d^{n-1}}{dt^{n-1}} e_{q}(-\omega t). \]

For construction solutions with degenerate roots we need following propositions:
Figure 10: Self-similar micro structure at scale 0.05

**Proposition 5.0.1** We have following commutation relation

\[
\left[ dt \frac{d}{dt}, D \right] = -D
\]  \hspace{1cm} (32)

implies

\[
t \frac{d}{dt} D = D \left( t \frac{d}{dt} - 1 \right)
\]

**Proof 5.0.2** By definition of \(D_q\) operator the commutation relation can be found as follows

\[
\left[ dt \frac{d}{dt}, D \right] f = \frac{d}{dt} D f - D t \frac{d}{dt} f
\]

\[
= \frac{d}{dt} \left( \frac{f(qt) - f(t)}{q - 1} \right) - D \left( t \frac{df}{dt} \right)
\]

\[
= t \left( \frac{(q - 1)t (q f'(qt) - f'(t)) - (q - 1) (f(qt) - f(t))}{(q - 1)^2 t^2} \right) - \frac{q t \frac{df}{dt}(qt)}{(q - 1)t} - t f'(t)
\]

\[
= -D f
\]

which implies

\[
\left[ dt \frac{d}{dt}, D \right] = -D.
\]

**Proposition 5.0.3**

\[
t \frac{d}{dt} D^n = D^n \left( t \frac{d}{dt} - n \right)
\]  \hspace{1cm} (34)

**Proof 5.0.4** By using mathematical induction:
For \(n = 1\), from the above commutation relation it is easy to see.
Suppose it is true for \( n \): \( t \frac{d}{dt} D^n = D^n (t \frac{d}{dt} - n) \)
And we should show that it is true for \( n + 1 \)

\[
\begin{align*}
t \frac{d}{dt} D^{n+1} &= t \frac{d}{dt} D^n \quad \text{\( D^n \)} \\
&= D^n (t \frac{d}{dt} - 1) - nD^n + 1 \left( t \frac{d}{dt} - (n + 1) \right) \\
&= D^n D (t \frac{d}{dt} - 1) - n D^{n+1} = D^{n+1} \left( t \frac{d}{dt} - (n + 1) \right) 
\end{align*}
\] (35)

**Proposition 5.0.5** We have more general relation in the following form

\[
t \frac{d}{dt} (\omega + D)^n = (\omega + D)^n t \frac{d}{dt} - n(\omega + D)^{n-1} D.
\] (36)

**Proof 5.0.6** By using mathematical induction:

For \( n = 1 \),

\[
\begin{align*}
t \frac{d}{dt} (\omega + D) &= t \frac{d}{dt} \omega + t \frac{d}{dt} D \\
&= t \frac{d}{dt} \omega + D \left( t \frac{d}{dt} - 1 \right) \\
&= (\omega + D) t \frac{d}{dt} - D \\
\end{align*}
\]

Suppose this relation is true for \( n \):

\[
t \frac{d}{dt} (\omega + D)^n = (\omega + D)^n t \frac{d}{dt} - n(\omega + D)^{n-1} D.
\]

Now we prove that it is true for \( n + 1 \)

\[
\begin{align*}
t \frac{d}{dt} (\omega + D)^{n+1} &= t \frac{d}{dt} (\omega + D)^n (\omega + D) \\
&= (\omega + D)^n \left( t \frac{d}{dt} - n(\omega + D)^{n-1} D \right) \quad \omega + D \\
&= (\omega + D)^n \left( (\omega + D) t \frac{d}{dt} - D \right) - n(\omega + D)^n D \\
&= (\omega + D)^{n+1} t \frac{d}{dt} - (n + 1)(\omega + D)^n D \\
\end{align*}
\]

Using the operator identity (36) we can show that if \( x_0 \) is solution of

\[
(D + \omega) x_0 = 0 \Rightarrow (D + \omega)^2 x_0 = 0 \Rightarrow ... \Rightarrow (D + \omega)^n x_0 = 0.
\]

\[
\begin{align*}
t \frac{d}{dt} (D + \omega)^n x_0 &= 0 \\
(D + \omega)^n t \frac{d}{dt} x_0 - n(D + \omega)^{n-1} x_0 &= 0 \\
(\omega + D)^{n-1} x_0 &= 0 \Rightarrow (\omega + D)^n x_1 = 0,
\end{align*}
\]

13
where
\[ x_1 \equiv t \frac{d}{dt} x_0. \]

Then,
\[ x_1 = t \frac{d}{dt} x_0 \]
is solution of
\[ (D + \omega)^2 x_1 = 0, \]
and as follows
\[ (D + \omega)^n x_1 = 0, \]
e.t.c. And then,
\[ x_{n-1} = t \frac{d^{n-1}}{dt^{n-1}} x_0 \]
is solution of
\[ (D + \omega)^n x_{n-1} = 0. \]
It provides us with \( n \) linearly independent solutions \( x_0, x_1, \ldots, x_{n-1} \) of \( N \)-degree equation with \( n \)-degenerate roots
\[ (D + \omega)^n x = 0. \]

6 Conclusions

In conclusion we like to mention relation of our q-damped oscillator problem with nonlinear q-difference problem. By following substitution
\[ y(t) = \frac{D_q x(t)}{x(t)} \]
equation (37) leads to nonlinear the q-Riccati equation
\[ D_q y(t) + y(qt) y(t) + \Gamma y(t) + \omega^2 = 0, \]
so that every solution of the first one produces a solution of the second one. This means that (32) gives linearization of the nonlinear q-difference equation (38). Similar situation we encounter in the q-Burgers equation linearized by q-Cole-Hopf transformation in terms of the q-heat equation [5].

Finally some comments about degenerate limit of our system. If in addition to our q-damped oscillator model (32) we consider standard oscillator with q-derivative friction
\[ m \frac{d^2}{dt^2} x(t) + \gamma D_q x(t) + k x(t) = 0, \]
then in the case of strong damping \( \gamma >> m \), or the small mass, both equations reduce to the first order q-difference equation
\[ \gamma D_q x(t) + k x(t) = 0. \]
Under the reduction, the two-dimensional phase space for the second order system turns into a one-dimensional one for the first order system. The first order system is called the "degenerate system"[7]. An arbitrary initial value problem in general does not apply to the degenerate system[8]. For the second order system at $t=0$ we can attach an arbitrary value for coordinate $x$ and related velocity $\dot{x}$ or $D_qx$. However, we can describe the same physical system by[10] only after some time interval and, moreover, in such a case $\dot{x}$ or $D_qx$ cannot be arbitrary since $D_qx$ is completely determined by given coordinate $x$, according to[10]. When the mass $m$ is going to zero, transition from a state incompatible with[10] to a compatible one is very fast. Acceleration at the initial time is very high (the related velocity is changing very fast). The transition to the massless limit can be well approximated by the discontinuous jumping condition like in[7]: the energy of the system cannot be changed by a jump. The jumping condition implies that under the jumping the coordinates of the system remain invariant and only the velocities can be changed. Details of this study we present in our future work.

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