Classification of Finite Irreducible Conformal Modules over $N=2$ Lie Conformal Superalgebras of Block Type

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Abstract
We introduce the $N=2$ Lie conformal superalgebras $\mathfrak{K}(p)$ of Block type, and classify their finite irreducible conformal modules for any nonzero parameter $p$. In particular, we show that such a conformal module admits a nontrivial extension of a finite conformal module $M$ over $K_2$ if $p = -1$ and $M$ has rank $(2 + 2)$, where $K_2$ is an $N = 2$ conformal subalgebra of $\mathfrak{K}(p)$. As a byproduct, we obtain the classification of finite irreducible conformal modules over a series of finite Lie conformal superalgebras $\mathfrak{f}(n)$ for $n \geq 1$. Composition factors of all the involved reducible conformal modules are also determined.

Keywords
Finite conformal module · Lie conformal superalgebra · $N = 2$ conformal superalgebra · Composition factor

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1 Introduction
The present paper is the third in our series of papers on representation theory of infinite Lie conformal superalgebras with Cartan type conformal subalgebras, the first two of which are [15] and [16].

Lie conformal superalgebras, introduced by Kac [10], encode the singular part of the operator product expansion of chiral fields in conformal field theory. During the last two decades, many advances have been made in the theory of finite Lie conformal superalgebras [1–6, 8, 9, 11, 12, 18, 19]. A complete classification of finite simple Lie conformal superalgebras can be found in [8, 9], which includes Cartan type and current type Lie conformal superalgebras. The most physically important Cartan type Lie conformal superalgebras include the Virasoro conformal algebra $K_0$, the Neveu-Schwarz conformal algebra $K_1$, and the $N = 2$ conformal superalgebra $K_2$. For the classification of finite irreducible conformal modules (FICMs) over finite simple Lie conformal superalgebras, see [1–6, 12].
main methods include description of extremal vectors and degenerate modules [1–6] and distinctive matrix realization over the Weyl algebra [12].

The theory of infinite Lie conformal superalgebras, however, is only in the early stages of development. Let \( p \) be a nonzero complex number. In [15], we introduced the infinite Lie conformal algebra \( \mathfrak{B}(p) \), which contains a Virasoro conformal subalgebra \( K_0 \) (denoted \( \mathfrak{Vir} \) in [15]) and has close relation with Lie algebras of Block type [13, 14, 17]. In [16], we constructed the Lie conformal superalgebra \( \mathfrak{S}(p) \) as the super analogue of \( \mathfrak{B}(p) \), which contains a Neveu-Schwarz conformal subalgebra \( K_1 \) (denoted \( \mathfrak{N}\mathfrak{S} \) in [16]). It is well-known that \( K_0 \subset K_1 \subset K_2 \). Motivated by this, in the present paper, we shall construct the Lie conformal superalgebra \( \mathfrak{R}(p) \) such that the following embedding diagram is commutative:

\[
\begin{array}{ccc}
K_0 & \longrightarrow & K_1 \\
\downarrow & & \downarrow \\
\mathfrak{B}(p) & \longrightarrow & \mathfrak{S}(p) \\
\downarrow & & \downarrow \\
\mathfrak{R}(p) & \longrightarrow & \mathfrak{R}(p)
\end{array}
\]

Naturally, we refer to \( \mathfrak{R}(p) \)'s as \( N = 2 \) Lie conformal superalgebras of Block type.

As one can see later, \( \mathfrak{R}(p) = \mathfrak{R}(p)_0 \oplus \mathfrak{R}(p)_1 \), where the even part \( \mathfrak{R}(p)_0 \) has \( \mathbb{C}[\delta] \)-basis \( \{ L_i, J_i | i \in \mathbb{Z}_+ \} \) and the odd part \( \mathfrak{R}(p)_1 \) has \( \mathbb{C}[\delta] \)-basis \( \{ G_i^\pm | i \in \mathbb{Z}_+ \} \), satisfying \( \lambda \)-brackets

\[
\begin{align*}
[L_i, L_j] &= ((i + p)\delta + (i + j + 2p)\lambda)L_{i+j}, & (1) \\
[L_i, J_j] &= ((i + p)\delta + (i + j + p)\lambda)J_{i+j}, & (2) \\
[L_i, G_j^\pm] &= ((i + p)\delta + (i + j + \frac{3}{2}p)\lambda)G_{i+j}^\pm, & (3) \\
[J_i, G_j^\pm] &= \pm G_{i+j}^\pm, & (4) \\
[G_i^+, G_j^-] &= ((2i + p)\delta + (2i + j + p)\lambda)J_{i+j} + 2L_{i+j}. & (5)
\end{align*}
\]

Other \( \lambda \)-brackets are given by the skew-symmetry or vanish. It is worth to highlight some interesting features of \( \mathfrak{R}(p) \). Firstly, each \( \mathfrak{R}(p) \) contains an \( N = 2 \) conformal subalgebra. Setting \( L = \frac{1}{p}L_0, J = J_0, G^\pm = \frac{1}{\sqrt{p}}G_{0}^\pm \), one can check that the subalgebra

\[
K_2 = \mathbb{C}[\delta]L \oplus \mathbb{C}[\delta]J \oplus \mathbb{C}[\delta]G^+ \oplus \mathbb{C}[\delta]G^- \quad (6)
\]

of \( \mathfrak{R}(p) \) is exactly the \( N = 2 \) conformal superalgebra [8] (see also Section 2.2). Secondly, the subalgebra of \( \mathfrak{R}(p) \) with \( \mathbb{C}[\delta] \)-basis \( \{ L_i, G_i = \sqrt{2} (G_i^+ + G_i^-) | i \in \mathbb{Z}_+ \} \) is isomorphic to \( \mathfrak{S}(p) \) [16]. These two features suggest that the above embedding diagram is commutative. Thirdly, there are embedding relations among \( \mathfrak{R}(p) \)'s. For any integer \( n \geq 1 \), \( \mathfrak{R}(p) \) can be embedded into \( \mathfrak{R}(np) \) via \( L_i \mapsto \frac{1}{n} L_{ni}, J_i \mapsto J'_{ni}, G_i^\pm \mapsto \frac{1}{\sqrt{n}} G_{ni}^\pm \). Finally, \( \mathfrak{R}(n) \) contains a series of finite Lie conformal superalgebras as quotient algebras (cf. Eq. 12)

\[
\mathfrak{R}(n) = \mathfrak{R}(n)/\mathfrak{R}(n)_{(n+1)}.
\]

The special cases \( \mathfrak{R}(1) \) and \( \mathfrak{R}(2) \) will be respectively referred to as \( N = 2 \) Heisenberg conformal superalgebra and \( N = 2 \) Schrödinger conformal superalgebra. See Section 2.3 for more details.

Our main goal in this paper is to classify FICMs over \( \mathfrak{R}(p) \). A complete classification of FICMs over \( K_2 \) was achieved in [6]. In particular, the rank of a nontrivial FICM over \( K_2 \) is either \((1 + 1)\) or \((2 + 2)\). Obviously, any conformal module over \( K_2 \subset \mathfrak{R}(p) \) can be trivially extended to a conformal module over \( \mathfrak{R}(p) \). Our main result indicates that a FICM over \( \mathfrak{R}(p) \) admits a nontrivial extension of a finite conformal module \( M \) over \( K_2 \) if and only
if \( p = -1 \) and \( M \) has rank \((2 + 2)\) (see Table 1). As a byproduct of our main result, we also obtain the classification of FICMs over the finite Lie conformal superalgebra \( \mathfrak{k}(n) \) (see Table 2).

Remarkably, any rank \((1 + 1)\) \( K_2 \)-module \( M \) cannot be nontrivially extended to a \( \mathfrak{sl}(p) \)-module, even in case that \( p = -1 \) and \( M \) can be degenerated from a rank \((2 + 2)\) \( K_2 \)-module (see Remark 2). This is essentially different from the extensions from \( K_1 \)-modules to \( \mathfrak{sp}(p) \)-modules [16].

Along the way, we in fact classify all the free conformal modules of ranks \((1 + 1)\) and \((2 + 2)\) over \( \mathfrak{sl}(p) \) and \( \mathfrak{k}(n) \) without the irreducibility assumption, and completely determine their composition factors (see Propositions 3 and 4) and multiplicities. Composition factors for reducible rank \((2 + 2)\) conformal modules \( V_{\Delta, \lambda, \alpha} \) and \( V_{\Delta, \lambda, \alpha, 0} \) with \( \Delta = \pm 2 \Delta \) are more attractive (see Table 3). Here, \( \varsigma_{-\alpha} \) denotes an even one-dimensional trivial module, and \( \tilde{M} \) denotes the same conformal module as \( M \) but with reversed parity. As a corollary, the composition factors of \( K_2 \)-modules of small rank are also obtained (see Remark 3).

The outline of the paper is as follows. In Section 2, after recalling some basic definitions and notations, we construct the main object \( \mathfrak{sl}(p) \), and present certain quotient algebras \( \mathfrak{sl}(p)[n] \) (cf. Eq. 11) of \( \mathfrak{sl}(p) \). In particular, we introduce the \( N = 2 \) Heisenberg and Schrödinger conformal superalgebras. In Section 3, for certain subquotient algebra \( \mathfrak{g} \) (cf. Eq. 13) of the annihilation superalgebra of \( \mathfrak{sl}(p) \), by introducing the row ideals and column ideals (cf. Eqs. 14 and 15), we conceptually determine the dimension of any irreducible \( \mathfrak{g} \)-module. The Schur’s lemma for Lie superalgebras will play an important role. In Section 4, we classify all the free conformal \( \mathfrak{sl}(p) \)-modules of ranks \((1 + 1)\) and \((2 + 2)\) by analytical techniques. Also, for reducible ones, we completely determine their composition factors. In the last section, we complete the classification of FICMs over \( \mathfrak{sl}(p) \) by showing that they must be free of rank \((1 + 1)\) or \((2 + 2)\). As an application, we also obtain the classification of FICMs over \( \mathfrak{k}(n) \) by the feature Eq. 7 of \( \mathfrak{sl}(−n) \).

## 2 Preliminaries

Throughout this paper, the notation \(|a| \in \mathbb{Z}/2\mathbb{Z}\) denotes the parity of an element \( a \) in a super vector space, and \( a \) is always assumed to be homogeneous if \(|a| \) appears in an expression. The angle bracket \( \langle \cdot, \cdot \rangle \) denotes “the Lie conformal superalgebra generated over \( \mathbb{C}[\partial] \) by”. The symbol \( \delta \) denotes the Kronecker delta.

### Table 1 Nontrivial FICMs over \( \mathfrak{sl}(p) \)

| \( \mathfrak{sl}(p) \) | FICMs (up to parity change) | Reference |
|----------------------|-----------------------------|-----------|
| \( p \neq -1 \)      | \( V_{\Delta, \alpha}^{(1)} \), \( V_{\Delta, \alpha}^{(2)} \), \( V_{\Delta, \lambda, \alpha} \) | Theorem 4 |
| \( p = -1 \)         | \( V_{\Delta, \alpha}^{(1)} \), \( V_{\Delta, \alpha}^{(2)} \), \( V_{\Delta, \lambda, \alpha, \beta} \) | Theorem 4 |

### Table 2 Nontrivial FICMs over \( \mathfrak{k}(n) \)

| \( \mathfrak{k}(n) \) | FICMs (up to parity change) | Reference |
|----------------------|-----------------------------|-----------|
| \( n > 1 \)          | \( V_{\Delta, \alpha}^{(1)} \), \( V_{\Delta, \alpha}^{(2)} \), \( V_{\Delta, \lambda, \alpha} \) | Corollary 2 |
| \( n = 1 \)          | \( V_{\Delta, \alpha}^{(1)} \), \( V_{\Delta, \alpha}^{(2)} \), \( V_{\Delta, \lambda, \alpha, \beta} \) | Corollary 2 |
### 2.1 Basic Definitions

Let us first recall some basic definitions, see [8, 10, 16] for more details.

**Definition 1** A Lie conformal superalgebra $R = R₀ ⊕ R₁$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R ⊗ R → \mathbb{C}[\lambda] ⊗ R$, $a ⊗ b → [a, b]$ called $\lambda$-bracket, and satisfying the following axioms ($a$, $b$, $c ∈ R$):

- (conformal sesquilinearity) $[\partial a, b] = -\lambda[a, b]$, $[a, \partial b] = (\partial + \lambda)[a, b]$,
- (skew-symmetry) $[a, b] = -(-1)^{|a||b|}[b, -\lambda - a]$,
- (Jacobi identity) $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$.

Let $R$ be a Lie conformal superalgebra. We call $R$ finite if it is finitely generated over $\mathbb{C}[\partial]$; $\mathbb{Z}$-graded if $R = ⊕_{i ∈ \mathbb{Z}} R_i$, where $R_i$ is a $\mathbb{C}[\partial]$-submodule and $[R_i, R_j] ⊂ R_{i+j}[\lambda]$ for $i, j ∈ \mathbb{Z}$.

**Definition 2** A conformal module $M = M₀ ⊕ M₁$ over a Lie conformal superalgebra $R$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R ⊗ M → \mathbb{C}[\lambda] ⊗ M$, $a ⊗ v → a_\lambda v$ called $\lambda$-action, such that ($a$, $b ∈ R$, $v ∈ M$)

- $(\partial a)_\lambda v = -\lambda a_\lambda v$, $a_\lambda (\partial v) = (\partial + \lambda)a_\lambda v$,
- $[a_\lambda b]_{\lambda + \mu} v = a_\lambda (b_\mu v) - (-1)^{|a||b|}b_\mu (a_\lambda v)$.

Let $M = M₀ ⊕ M₁$ be a conformal $R$-module. We call $M$ finite if it is finitely generated over $\mathbb{C}[\partial]$. As $\mathbb{C}[\partial]$-modules, if $M₀$ has rank $m$ and $M₁$ has rank $n$, we say that $M$ has rank $(m + n)$, denoted by rank($M$) = $(m + n)$. In case $R$ is $\mathbb{Z}$-graded, we call $M$ $\mathbb{Z}$-graded if $M = ⊕_{i ∈ \mathbb{Z}} M_i$, where $M_i$ is a $\mathbb{C}[\partial]$-submodule and $R_i M_j ⊂ M_{i+j}[\lambda]$ for $i, j ∈ \mathbb{Z}$. Furthermore, if each $M_i$ is freely generated by one element over $\mathbb{C}[\partial]$, we call $M$ a $\mathbb{Z}$-graded free intermediate series module.

Clearly, for any fixed $α ∈ \mathbb{C}$, the $\mathbb{C}[\partial]$-module $c_α$ with $\partial c_α = α c_α$, $a_\lambda c_α = 0$ for $a ∈ R$, is a conformal $R$-module, which will be referred to as the even (resp., odd) one-dimensional trivial module if $|c_α| = 0$ (resp., $|c_α| = 1$).

### Table 3 Composition factors

| $V_{Δ,α}/V_{Δ,α,0}$ | Composition factors | $V_{Δ,α}/V_{Δ,α,0}$ | Composition factors |
|----------------------|---------------------|----------------------|---------------------|
| $Δ ≠ 0, -\frac{1}{2}$ | $V^{(1)}_{\Delta,α}, V^{(1)}_{\Delta+\frac{1}{2},α}$ | $Δ ≠ 0, -\frac{1}{2}$ | $V^{(2)}_{\Delta,α}, V^{(2)}_{\Delta+\frac{1}{2},α}$ |
| $Δ = 0$ | $V^{(1)}_{\frac{1}{2},α}, V^{(2)}_{\frac{3}{2},α} \otimes c_{-α}$ | $Δ = 0$ | $V^{(2)}_{\frac{1}{2},α}, V^{(2)}_{-\frac{1}{2},α} \otimes c_{c_{-α}}$ |
| $Δ = -\frac{1}{2}$ | $V^{(1)}_{1/2,α}, V^{(2)}_{-1/2,α} \otimes c_{c_{-α}}$ | $Δ = -\frac{1}{2}$ | $V^{(1)}_{1/2,α}, V^{(2)}_{-1/2,α} \otimes c_{c_{-α}}$ |
Definition 3 The annihilation superalgebra $\mathcal{A}(R)$ of a Lie conformal superalgebra $R$ is a Lie superalgebra with $\mathbb{C}$-basis $\{a_n \mid a \in R, n \in \mathbb{Z}_+\}$ and relations

$$(\lambda a)_n = \lambda a_n, \quad (a + b)_n = a_n + b_n, \quad (\partial a)_n = -na_{n-1},$$

$$[a_m, b_n] = \sum_{k \in \mathbb{Z}_+} \binom{m}{k} (a(k)b)_{m+n-k},$$

(8)

where $a(k)b$ is called the $k$-product, given by the following inversion formula:

$$[a \lambda b] = \sum_{k \in \mathbb{Z}_+} \lambda^{(k)} a(k)b \quad \text{with} \quad \lambda^{(k)} = \frac{\lambda^k}{k!}.$$  

Here, the parity $|a_n|$ of $a_n \in \mathcal{A}(R)$ is the same as $|a|$ for any $a \in R$ and $n \in \mathbb{Z}_+$. Note that $\mathcal{A}(R)$ admits a derivation $T$ given by $T(a_n) = -na_{n-1}$ for any $a_n \in \mathcal{A}(R)$. The extended annihilation superalgebra $\mathcal{A}(R)^e$ of a Lie conformal superalgebra $R$ is defined by $\mathcal{A}(R)^e = \mathbb{C}T \ltimes \mathcal{A}(R)$ with $[T, a_n] = -na_{n-1}$. The representation theory of $R$ is controlled by the representation theory of $\mathcal{A}(R)^e$ in the following sense:

Proposition 1 A conformal module $M$ over a Lie conformal superalgebra $R$ is the same as a module over the Lie superalgebra $\mathcal{A}(R)^e$ satisfying $a_n v = 0$ for $a \in R, v \in M, n \gg 0$.

2.2 Construction of $\mathfrak{R}(p)$

Recall that [8] the $N = 2$ conformal superalgebra $K_2$ is a Cartan $K$ type Lie conformal superalgebra, which has $\mathbb{C}[\partial]$-basis $\{L, J, G^\pm\}$ with $|L| = |J| = 0$ and $|G^\pm| = 1$, satisfying

$$[L \lambda L] = (\partial + 2\lambda)L,$$

$$[L \lambda J] = (\partial + \lambda)J,$$

$$[L \lambda G^\pm] = (\partial + \frac{3}{2}\lambda)G^\pm,$$

$$[J \lambda G^\pm] = \pm G^\pm,$$

$$[G^+ \lambda G^-] = (\partial + 2\lambda)J + 2L.$$  

The Lie conformal algebra $\mathfrak{B}(p)$, introduced in [15], is an $\mathbb{Z}$-graded Lie conformal algebra, which has $\mathbb{C}[\partial]$-basis $\{L_i \mid i \in \mathbb{Z}_+\}$, satisfying $\lambda$-brackets

$$[L_i \lambda L_j] = ((i + p)\partial + (i + j + 2p)\lambda)L_{i+j}.$$  

Motivated by an important class of $\mathbb{Z}$-graded free intermediate series $\mathfrak{B}(p)$-module structure [16]

$$M(a) = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[\partial]m_i \quad \text{with} \quad \lambda\text{-action} \quad L_i \lambda m_j = ((i + p)\partial + (i + j + a)\lambda)m_{i+j},$$

and the conformal structure of $K_2$, let us consider a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module

$$R := R \left( \phi^{(k)}_{i,j} | k = 1, 2, 3, 4 \right) = R_0 \oplus R_1,$$
where the even part $R_0$ has $\mathbb{C}[\partial]$-basis $\{L_i, J_i \mid i \in \mathbb{Z}_+\}$ and the odd part $R_1$ has $\mathbb{C}[\partial]$-basis $\{G_i^+ \mid i \in \mathbb{Z}_+\}$, satisfying (1)–(3) (we hope that $K_2$ is exactly the zeroth graded component of $\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module $R$), and

$$[J_{i, j} \ G_j^+] = \phi_{i, j}^{(1)}(\partial, \lambda)G_{i+j}^+, \quad [J_{i, j} \ G_j^-] = \phi_{i, j}^{(2)}(\partial, \lambda)G_{i+j}^-,$$

$$[G_i^+ \ G_j^-] = \phi_{i, j}^{(3)}(\partial, \lambda)J_{i+j} + \phi_{i, j}^{(4)}(\partial, \lambda)L_{i+j}, \quad [X_i \ X_j] = 0 \text{ for } X = J, G^\pm.$$

Here, $\phi_{i, j}^{(k)}(\partial, \lambda) \in \mathbb{C}[\partial, \lambda], k = 1, 2, 3, 4$, referred to as the structure polynomials, satisfy $\phi_{i, j}^{(k)}(\partial, \lambda) \neq 0$ for some $i, j \in \mathbb{Z}_+$.

**Lemma 1** The $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module $R$ becomes a Lie conformal superalgebra if and only if $\phi_{i, j}^{(1)}(\partial, \lambda) = -\phi_{i, j}^{(2)}(\partial, \lambda) = a$, $\phi_{i, j}^{(3)}(\partial, \lambda) = \frac{c}{a}((i + \frac{1}{2}j)\partial + (i + j + p)\lambda)$, $\phi_{i, j}^{(4)}(\partial, \lambda) = c$ for all $i, j \in \mathbb{Z}_+$, where $a$ and $c$ are constants. Up to isomorphism, we may assume that $a = 1$, $c = 2$, and the resulting algebra is exactly $\mathfrak{c}(p)$.

**Proof** The sufficiency is straightforward, so let us prove the necessity. Let $R$ be a Lie conformal superalgebra. We need to determine the structure polynomials $\phi_{i, j}^{(k)}(\partial, \lambda), k = 1, 2, 3, 4$. Roughly speaking, our strategy is to apply the Jacobi identity for certain triples, and then to determine $\phi_{i, j}^{(k)}(\partial, \lambda)$ by using the substitutability of variables, the arbitrariness of subscripts and some analytical techniques.

Using the Jacobi identity for triple $(L_0, G_i^+, G_j^-)$ and matching the coefficients of $J_{i+j}$ and $L_{i+j}$, we obtain

$$(p\partial + (i + j + p)\lambda)\phi_{i, j}^{(3)}(\partial + \lambda, \mu) = \left( i + \frac{1}{2}p \right) \lambda - p\mu \phi_{i, j}^{(3)}(\partial, \lambda + \mu) + (p\partial + \mu) + \left( j + \frac{3}{2}p\lambda \right) \phi_{i, j}^{(3)}(\partial, \mu),$$

$$(p\partial + (i + j + 2p)\lambda)\phi_{i, j}^{(4)}(\partial + \lambda, \mu) = \left( i + \frac{1}{2}p \right) \lambda - p\mu \phi_{i, j}^{(4)}(\partial, \lambda + \mu) + (p\partial + \mu) + \left( j + \frac{3}{2}p\lambda \right) \phi_{i, j}^{(4)}(\partial, \mu),$$

where $i, j \in \mathbb{Z}_+$. Taking $\mu = 0$, we have

$$\left( i + \frac{1}{2}p \right) \phi_{i, j}^{(3)}(\partial, \lambda) = p\partial \frac{\phi_{i, j}^{(3)}(\partial + \lambda, 0) - \phi_{i, j}^{(3)}(\partial, 0)}{\lambda} + (i + j + p)\phi_{i, j}^{(3)}(\partial + \lambda, 0) - \left( j + \frac{3}{2}p \right) \phi_{i, j}^{(3)}(\partial, 0),$$

$$\left( i + \frac{1}{2}p \right) \phi_{i, j}^{(4)}(\partial, \lambda) = p\partial \frac{\phi_{i, j}^{(4)}(\partial + \lambda, 0) - \phi_{i, j}^{(4)}(\partial, 0)}{\lambda} + (i + j + 2p)\phi_{i, j}^{(4)}(\partial + \lambda, 0) - \left( j + \frac{3}{2}p \right) \phi_{i, j}^{(4)}(\partial, 0).$$

Taking $\lambda \to 0$, we obtain

$$\frac{d}{d\partial} (\phi_{i, j}^{(3)}(\partial, 0)) = \phi_{i, j}^{(3)}(\partial, 0), \quad \frac{d}{d\partial} (\phi_{i, j}^{(4)}(\partial, 0)) = 0. \quad (9)$$
The second formula in Eq. 9 implies that \( \phi^{(4)}_{i,j}(\partial, 0) = c_{i,j} \in \mathbb{C} \). Processing with the same procedure for triple \((L_i, G^+_0, G^-_j)\) and matching the coefficients of \(L_{i+j}\), we obtain

\[
\lambda \left( \frac{1}{2} p \phi^{(4)}_{i,j}(\partial, \lambda) + (i + j + \frac{3}{2} p)\lambda \right) c_{0,i+j} - (i + j + 2p) c_{0,j} = (i + p) \partial (c_{0,j} - c_{0,i+j}).
\]  

(10)

By the arbitrariness of \(i\) and \(j\), we have \(\lambda (c_{0,j} - c_{0,i+j})\), and so \(c_{0,j} = c\) for all \(j \in \mathbb{Z}_+\), where \(c\) is a constant. Substituting this back to Eq. 10, we see that \(\phi^{(4)}_{i,j}(\partial, \lambda) = c\) for all \(i, j \in \mathbb{Z}_+\).

To save space, we proceed with an outline of the characterization of the other structure polynomials; the details are omitted. For \(\phi^{(k)}_{i,j}(\partial, \lambda)\), \(k = 1, 2\), by considering triple \((L_m, J_i, G^\pm_j)\), one can first show that \(\phi^{(k)}_{i,j}(\partial, \lambda) = a^{(k)}\) for all \(i, j \in \mathbb{Z}_+\), where \(a^{(k)}\) is a constant. Then, by triple \((G^+_0, J_i, G^-_j)\) one can derive that \(a^{(1)} = -a^{(2)}\) (simply denote \(a^{(1)}\) by \(a\)) and obtain a preliminary form of \(\phi^{(3)}_{i,j}(\partial, \lambda)\). Using this preliminary form, the first formula in Eq. 9, and triple \((G^+_0, G^+_0, G^-_j)\), one can prove that \(\phi^{(3)}_{i,j}(\partial, \lambda)\) must have the required form.

For the last statement, we only need to note that the map \(L_i \mapsto L_i, J_i \mapsto \frac{1}{a} J_i, G^\pm_i \mapsto \sqrt{\frac{2}{c}} G^\pm_i\) defines an automorphism of \(R\). This completes the proof. \(\square\)

### 2.3 Quotient Algebras of \(\mathfrak{R}(p)\)

One of attractive features of \(\mathfrak{R}(p)\), as we mentioned in the Introduction, is that one can obtain new interesting finite Lie conformal superalgebras from \(\mathfrak{R}(p)\). Let \(n\) be a positive integer. Consider a subalgebra \(\mathfrak{R}(p)_{(n)}\) of \(\mathfrak{R}(p)\) defined by

\[
\mathfrak{R}(p)_{(n)} = \{ L_i, J_i, G^\pm_i | i \geq n \}.
\]

Note that \(\mathfrak{R}(p)_{(n)}\) is in fact a Lie conformal superalgebra ideal of \(\mathfrak{R}(p)\). For \(n \in \mathbb{Z}_+\), define a quotient algebra \(\mathfrak{R}(p)_{(n)}\) of \(\mathfrak{R}(p)\) by

\[
\mathfrak{R}(p)_{(n)} = \mathfrak{R}(p)/\mathfrak{R}(p)_{(n+1)}.
\]

(11)

Clearly, \(\mathfrak{R}(p)_{(0)} \cong K_2\). In addition, the special cases \(p = -n\) with \(n \in \mathbb{Z}_{\geq 1}\) supply a series of new finite non-simple Lie conformal superalgebras:

\[
\mathfrak{t}(n) = \mathfrak{R}(-n)_{(n)} = \mathfrak{R}(-n)/\mathfrak{R}(-n)_{(n+1)}.
\]

(12)

Let us say more about the conformal structures of \(\mathfrak{t}(1)\) and \(\mathfrak{t}(2)\).

**Example 1** By definition Eq. 12, one can check that the subalgebra \(\langle \tilde{L}_0, \tilde{J}_0, \tilde{G}_0^\pm \rangle\) of \(\mathfrak{t}(1)\) is isomorphic to the \(N = 2\) conformal superalgebra \(K_2\). While the subalgebra \(\langle \tilde{L}_i, \tilde{G}_i^+ + \tilde{G}_i^- | i = 0, 1 \rangle\) of \(\mathfrak{t}(1)\) is isomorphic to the Heisenberg-Neveu-Schwarz conformal algebra [16]. Following our previous name rule, we refer to \(\mathfrak{t}(1)\) as \(N = 2\) Heisenberg conformal superalgebra.

**Example 2** Similarly, the subalgebra \(\langle \tilde{L}_0, \tilde{J}_0, \tilde{G}_0^\pm \rangle\) of \(\mathfrak{t}(2)\) is also isomorphic to the \(N = 2\) conformal superalgebra \(K_2\). While the subalgebra \(\langle \tilde{L}_i, \tilde{G}_i^+ + \tilde{G}_i^- | i = 0, 1, 2 \rangle\) of \(\mathfrak{t}(2)\) is isomorphic to the Schrödinger-Neveu-Schwarz conformal algebra [16]. Following our previous name rule, we refer to \(\mathfrak{t}(2)\) as \(N = 2\) Schrödinger conformal superalgebra.
3 Annihilation Superalgebra and Related Representations

In this section, we construct a subquotient algebra $g$ (see Eq. 13 below) of the annihilation superalgebra $A(\mathcal{R}(p))$ of $\mathcal{R}(p)$. Then, by introducing the so-called row ideals and column ideals of $g$, we determine the dimension of any nontrivial finite-dimensional irreducible module over $g$.

3.1 Annihilation Superalgebra $A(\mathcal{R}(p))$

Let us first give the Lie superalgebra structure on the annihilation superalgebra $A(\mathcal{R}(p))$.

Lemma 2 We have

$$A(\mathcal{R}(p)) \cong \text{span}_{\mathbb{C}} \{ L_{i,m}, J_{i,n}, G_{i,t}^\pm \mid i, n \in \mathbb{Z}_+, m \in \mathbb{Z}_{\geq -1}, t \in \frac{1}{2} + \mathbb{Z}_{\geq -1} \}.$$ 

with super-commutation relations

$$[L_{i,m}, L_{j,n}] = ((j + p)(m + 1) - (i + p)(n + 1))L_{i+j,m+n},$$

$$[L_{i,m}, J_{j,n}] = (j(m + 1) - (i + p)n)J_{i+j,m+n},$$

$$[L_{i,m}, G_{j,n}^\pm] = ((j + \frac{p}{2})(m + 1) - (i + p)(n + 1))G_{i+j,m+n}^\pm,$$

$$[J_{i,m}, G_{j,n}^\pm] = \pm G_{i+j,m+n}^\pm,$$

$$[G_{i,m}^+, G_{j,n}^-] = (2j + p)
\left(\frac{m + \frac{1}{2}}{2} - (2i + p)\left(\frac{n + \frac{1}{2}}{2}\right)\right) J_{i+j,m+n} + 2L_{i+j,m+n}.$$

Proof By the inversion formula in Definition 3, one can first transfer the $\lambda$-brackets of $\mathcal{R}(p)$ to $k$-products. Then, by Eq. 8 and making the shift $L_{i,m} = (L_i)_{m+1}$, $J_{i,n} = (J_i)_n$, $G_{i,t}^\pm = (G_i^\pm)_{t+\frac{1}{2}}$ for $i, n \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq -1}$, $t \in \frac{1}{2} + \mathbb{Z}_{\geq -1}$, one can obtain the super-commutation relations of $A(\mathcal{R}(p))$. \hfill $\square$

3.2 Subquotient Algebras of $A(\mathcal{R}(p))$

Next, we construct a class of subquotient algebras of $A(\mathcal{R}(p))$. Let $k, N \in \mathbb{Z}_+$ be two non-negative integers. Consider a subalgebra $A(\mathcal{R}(p))_+$ of $A(\mathcal{R}(p))$:

$$\mathcal{A}(\mathcal{R}(p))_+ = \text{span}_{\mathbb{C}} \{ L_{i,m}, J_{i,n}, G_{i,t}^\pm \in A(\mathcal{R}(p)) \mid i, m \in \mathbb{Z}_+, n \in \mathbb{Z}_{\geq 1}, t \in \frac{1}{2} + \mathbb{Z}_+ \}.$$ 

This Lie superalgebra contains an ideal $\mathcal{I}(k, N)$:

$$\mathcal{I}(k, N) = \text{span}_{\mathbb{C}} \left\{ L_{i,m}, J_{i,n}, G_{i,t}^\pm \in \mathcal{A}(\mathcal{R}(p))_+ \mid i > k, m > N, n > N + 1, t > N + \frac{1}{2} \right\}.$$ 

Define $g(k, N)$ by

$$g(k, N) := \mathcal{A}(\mathcal{R}(p))_+ / \mathcal{I}(k, N).$$

We shall determine the dimension of any finite-dimensional irreducible module over $g(k, N)$. To this end, we introduce two types of ideals of $g(k, N)$:

- **row ideals** $\tau(k, N)$ defined by

$$\tau(k, N) = \text{span}_{\mathbb{C}} \{ L_{k,n}, J_{k,n+1}, G_{k,n+\frac{1}{2}}^\pm \in g(k, N) \mid n \leq N \}.$$ (14)
• column ideals \( c(k, N) \) defined by
\[
c(k, N) = \text{span}_\mathbb{C}\{\bar{L}_{i,N}, \bar{J}_{i,N+1}, \bar{G}^\pm_{i,N+\frac{1}{2}} \in \mathfrak{g}(k, N) \mid i \leq k\}. \tag{15}
\]

### 3.3 Irreducible Representations of \( \mathfrak{g}(k, N) \)

Let \( V = V_0 \oplus V_1 \) be a nontrivial finite-dimensional irreducible module over \( \mathfrak{g}(k, N) \). If \( V_0 \) has dimension \( m \) and \( V_1 \) has dimension \( n \), we say that \( V \) has dimension \( (m|n) \). The main result in this section is the following.

**Theorem 1** The dimension of \( V \) is either \( (1|1) \) or \( (2|2) \).

Let us first introduce some auxiliary sets:
\[
\Omega = \{(j, n) \mid \bar{L}_{j,n}, \bar{J}_{j,n+1}, \bar{G}^\pm_{j,n+\frac{1}{2}} \in \mathfrak{g}(k, N)\} \setminus \{(0, 0)\},
\]
\[
\Omega(L) = \{(j, n) \in \Omega \mid j - pn = 0\}, \tag{16}
\]
\[
\Omega(J) = \{(j, n) \in \Omega \mid j - p(n + 1) = 0\}, \tag{17}
\]
\[
\Omega(G) = \{(j, n) \in \Omega \mid j - p(n + \frac{1}{2}) = 0\}. \tag{18}
\]

The following facts can be easily checked.

**Proposition 2** Suppose \( k, N \geq 1 \). We have
\[
(1) \quad \Omega(L) \neq \emptyset \iff \Omega(J) \neq \emptyset, \text{ and if } \Omega(L) \neq \emptyset, \text{ then } p > 0;
\]
\[
(2) \quad \Omega(G) \neq \emptyset \iff p \in 2\mathbb{Z}_{\geq 1}, \text{ and if } \Omega(G) \neq \emptyset, \text{ then } \Omega(L) \neq \emptyset.
\]

**Lemma 3** If \( \Omega(L) = \Omega(J) = \Omega(G) = \emptyset \), then \( \dim V = (1|1) \).

**Proof** Consider the following decomposition of \( \mathfrak{g}(k, N) \):
\[
\mathfrak{g}(k, N) = \mathbb{C}\bar{L}_{0,0} + \bar{\mathfrak{g}}(k, N), \quad \text{where } \bar{\mathfrak{g}}(k, N) = \mathfrak{g}(k, N) \setminus \mathbb{C}\bar{L}_{0,0}.
\]
Clearly, \( \bar{\mathfrak{g}}(k, N) \) is a nilpotent ideal of \( \mathfrak{g}(k, N) \). Since \( \Omega(L) = \Omega(J) = \Omega(G) = \emptyset \), one can see that \( \bar{\mathfrak{g}}(k, N) \) is a completely reducible \( \mathbb{C}\bar{L}_{0,0} \)-module with no trivial summand. By ([4, 5], Lemma 1), \( \bar{\mathfrak{g}}(k, N) \) acts trivially on \( V \). Hence, \( V \) is simply a finite-dimensional \( \mathbb{C}\bar{L}_{0,0} \)-module, and so \( \dim V = (1|1) \).

**Lemma 4** Suppose \( k, N \geq 1 \) and \( \Omega(L) \neq \emptyset, \Omega(G) = \emptyset \). Let
\[
j_L = \max\{j \mid (j, n) \in \Omega(L)\}, \quad n_L = \max\{n \mid (j, n) \in \Omega(L)\},
\]
\[
j_J = \max\{j \mid (j, n) \in \Omega(J)\}, \quad n_J = \max\{n \mid (j, n) \in \Omega(J)\}.
\]

(1) If \( j_J < k \), then the row ideal \( \mathfrak{c}(k, N) \) acts trivially on \( V \);  
(2) If \( j_J = k, j_L < k \), then the row ideal \( \mathfrak{c}(k, N) \) acts trivially on \( V \);  
(3) If \( j_J = j_L = k, n_L < N \), then the column ideal \( \mathfrak{c}(k, N) \) acts trivially on \( V \);  
(4) If \( j_J = j_L = k, n_L = N \), then the column ideal \( \mathfrak{c}(k, N) \) acts trivially on \( V \).

**Proof** Note first that \( j_L \leq j_J \) and (1)–(4) cover all possible cases.
(1) Assume that $\tau(k, N)$ acts non-trivially on $V$. We have $V = \tau(k, N)V$ by the irreducibility of $V$. Consider the action of $\bar{L}_{0,0}$ on $\tau(k, N)$,

$$[\bar{L}_{0,0}, \bar{L}_{k,n}] = b_1 \bar{L}_{k,n}, \text{ where } b_1 = k - pn \neq 0 \text{ (since } k > j_f \geq j_L).$$

(19) $[\bar{L}_{0,0}, \bar{J}_{k,n+1}] = b_2 \bar{J}_{k,n+1}, \text{ where } b_2 = k - p(n + 1) \neq 0 \text{ (since } k > j_f).$

(20) $[\bar{L}_{0,0}, \bar{G}^\pm_{k,n+\frac{1}{2}}] = b_3 \bar{G}^\pm_{k,n+\frac{1}{2}}, \text{ where } b_3 = k - p(n + \frac{1}{2}) \neq 0 \text{ (since } \Omega(G) = \emptyset).$

(21) Hence, $\bar{L}_{k,n}, \bar{J}_{k,n+1}, \bar{G}^\pm_{k,n+\frac{1}{2}} \in \tau(k, N)$ act nilpotently on $V$. Since $\tau(k, N)$ is abelian, $\tau(k, N)$ acts nilpotently on $V$, which contradicts to $V = \tau(k, N)V$.

(2) In this case, we have $n_J = N$. Assume that $\tau(k, N)$ acts non-trivially on $V$. We have $V = \tau(k, N)V$ by the irreducibility of $V$. Consider the decomposition of $\tau(k, N)$:

$$\tau(k, N) = \bar{C} \bar{J}_{k,N+1} + \bar{t}(k, N), \text{ where } \bar{t}(k, N) = \tau(k, N) \setminus \bar{C} \bar{J}_{k,N+1}.$$ 

Note that $\bar{J}_{k,N+1}$ is an even central element of $g(k, N)$. We may assume that the action of $\bar{J}_{k,N+1}$ is a scalar $c$. In addition, we have

$$[\bar{L}_{0,0}, \bar{J}_{0,N+1}] = -(N + 1)(k + p) \bar{J}_{k,N+1}.$$ 

(22) Consider the actions of both sides of Eq. 22 on $V$, and compare the traces of the matrices with respect to a basis of $V$. The right hand side is $-c(N + 1)(k + p) \dim V$, while the left hand side is zero, since the corresponding matrix has form $AB - BA$. Hence, $c = 0$ (note that $k + p > 0$ by Proposition 2(1)), and so $V = \bar{t}(k, N)V$. Considering the action of $\bar{L}_{0,0}$ on $\bar{t}(k, N)$, we still have (19)–(21). The only difference is the reason for $b_2 \neq 0$, which should be replaced by $n < N = n_J$. Hence, all elements in $\bar{t}(k, N)$ act nilpotently on $V$. Since $\bar{t}(k, N)$ is abelian, $\bar{t}(k, N)$ acts nilpotently on $V$, which contradicts to $V = \bar{t}(k, N)V$.

(3) In this case, we have $n_J = n_L - 1$. Assume that $c(k, N)$ acts non-trivially on $V$. We have $V = c(k, N)V$ by the irreducibility of $V$. Consider the action of $\bar{L}_{0,0}$ on $c(k, N)$,

$$[\bar{L}_{0,0}, \bar{L}_{i,N}] = b_4 \bar{L}_{i,N}, \text{ where } b_4 = i - pN \neq 0 \text{ (since } N > n_L).$$

(23) $[\bar{L}_{0,0}, \bar{J}_{i,N+1}] = b_5 \bar{J}_{i,N+1}, \text{ where } b_5 = i - p(N + 1) \neq 0 \text{ (since } N > n_L = n_J).$

(24) $[\bar{L}_{0,0}, \bar{G}^\pm_{i,N+\frac{1}{2}}] = b_6 \bar{G}^\pm_{i,N+\frac{1}{2}}, \text{ where } b_6 = i - p(N + \frac{1}{2}) \neq 0 \text{ (since } \Omega(G) = \emptyset).$

(25) Hence, $\bar{L}_{i,N}, \bar{J}_{i,N+1}, \bar{G}^\pm_{i,N+\frac{1}{2}} \in c(k, N)$ act nilpotently on $V$. Since $c(k, N)$ is abelian, $c(k, N)$ acts nilpotently on $V$, which contradicts to $V = c(k, N)V$.

(4) In this case, we have $n_J = N - 1$. Assume that $c(k, N)$ acts non-trivially on $V$. Consider the decomposition of $c(k, N)$:

$$c(k, N) = c_L(k, N) + c_{JG}(k, N), \text{ where } c_L(k, N) = \text{span}_C \{\bar{L}_{i,N} \in g(k, N) | i \leq k\},$$

$$c_{JG}(k, N) = \text{span}_C \{\bar{J}_{i,N+1}, \bar{G}^\pm_{i,N+\frac{1}{2}} \in g(k, N) | i \leq k\}.$$ 

One can first show that the action of $c_{JG}(k, N)$ on $V$ is trivial. In fact, if this is not true, since $c_{JG}(k, N)$ is an ideal of $g(k, N)$, then we have $V = c_{JG}(k, N)V$ by the irreducibility of $V$. Consider further the decomposition of $c_{JG}(k, N)$:

$$c_{JG}(k, N) = \bar{C} \bar{J}_{k,N+1} + \tilde{c}_{JG}(k, N), \text{ where } \tilde{c}_{JG}(k, N) = c_{JG}(k, N) \setminus \bar{C} \bar{J}_{k,N+1}.$$ 

As in (2), by the comparing traces technique, we must have that the action of $\bar{J}_{k,N+1}$ is trivial, and so $V = \tilde{c}_{JG}(k, N)V$. Considering the action of $\bar{L}_{0,0}$ on $\tilde{c}_{JG}(k, N)$, we still
have (24) and (25). Since $\hat{\epsilon}_{JG}(k, N)$ is abelian, $\hat{\epsilon}_{JG}(k, N)$ acts nilpotently on $V$, which contradicts to $V = \hat{\epsilon}_{JG}(k, N)V$.

Now, by the above assumption, we have $V = \epsilon_L(k, N)V$. Similarly, consider the decomposition of $\epsilon_L(k, N)$:

$$\epsilon_L(k, N) = C\bar{L}_{k,N} + \hat{\epsilon}_L(k, N),$$

where $\hat{\epsilon}_L(k, N) = \epsilon_L(k, N)\backslash C\bar{L}_{k,N}$.

Note that $\bar{L}_{k,N}$ is an even central element of $g(k, N)$. Here keep in mind that the actions of $J_{k,N+1}$, $\bar{G}_{k,N+\frac{1}{2}}^\pm \in \epsilon_{JG}(k, N)$ on $V$ are trivial. Consider the actions of both sides of the following equation on $V$:

$$[\bar{L}_{k,0}, \bar{L}_{0,\nu}] = -((N + 1)k + N\nu)\bar{L}_{k,N}.$$

Again, by the comparing traces technique, one can show that the action of $\bar{L}_{k,N}$ is trivial. Hence, $V = \hat{\epsilon}_L(k, N)V$. Considering the action of $\bar{L}_{0,0}$ on $\hat{\epsilon}_L(k, N)$, we still have (23). The only difference is the reason for $b_4 \neq 0$, which should be replaced by $i < k = j_L$. Hence, all elements in $\hat{\epsilon}_L(k, N)$ act nilpotently on $V$. Since $\hat{\epsilon}_L(k, N)$ is abelian, $\hat{\epsilon}_L(k, N)$ acts nilpotently on $V$, which contradicts to $V = \hat{\epsilon}_L(k, N)V$.

**Lemma 5** Suppose $k, N \geq 1$ and $\Omega(G) \neq \emptyset$. Use the same notations $j_L$, $j_J$, $n_L$, $n_J$ as in Lemma 4. Let further

$$j_G = \max\{j \mid (j, n) \in \Omega(G)\}, \quad n_G = \max\{n \mid (j, n) \in \Omega(G)\}.$$

1. If $\max\{j_G, j_J\} < k$, then the row ideal $\tau(k, N)$ acts trivially on $V$;
2. If $j_G < j_J = k$, $j_L < k$, then the row ideal $\tau(k, N)$ acts trivially on $V$;
3. If $j_G < j_J = j_L = k$, $n_L < N$, then the column ideal $\epsilon(k, N)$ acts trivially on $V$;
4. If $j_G < j_J = j_L = k$, $n_L = N$, then the column ideal $\epsilon(k, N)$ acts trivially on $V$;
5. If $j_J < j_G = k$, $n_G < N$, then the column ideal $\epsilon(k, N)$ acts trivially on $V$;
6. If $j_J < j_G = k$, $n_G = N$, then the row ideal $\tau(k, N)$ acts trivially on $V$.

**Proof** Note first that $j_J \neq j_G$, and so (1)–(6) cover all possible cases.

The conclusions (1)–(4) can be respectively proved in a similar way as Lemma 4(1)–(4). The differences lie in the reasons for $b_3 \neq 0$ and $b_6 \neq 0$, which should be replaced by $k > j_G$.

For (5), we have $j_L = j_J (< k), n_L = n_G (< N)$, and $n_J = n_G - 1$. Assume that $\epsilon(k, N)$ acts non-trivially on $V$. We have $V = \epsilon(k, N)V$ by the irreducibility of $V$. Considering the action of $L_{0,0}$ on $\epsilon(k, N)$, we still have (23)–(25). The only difference is the reason for $b_6 \neq 0$, which should be replaced by $N > n_G$. The remaining arguments are the same as in Lemma 4(3).

For (6), we have $k = p(N + \frac{1}{2}) > pN \geq 2N$ (recall that $p \in 2\mathbb{Z}_{\geq 1}$ by Proposition 2(2)). Assume that $\tau(k, N)$ acts non-trivially on $V$. Consider the decomposition of $\tau(k, N)$:

$$\tau(k, N) = \text{span}_C\{J_{k,N+1}, \bar{G}_{k,N+\frac{1}{2}}^\pm\} + \hat{\tau}(k, N),$$

where

$$\hat{\tau}(k, N) = \tau(k, N)\backslash \text{span}_C\{J_{k,N+1}, \bar{G}_{k,N+\frac{1}{2}}^\pm\}.$$

First, recall that $J_{k,N+1}$ is an even central element of $g(k, N)$. As in Lemma 4(2), we must have that the action of $J_{k,N+1}$ is trivial. Then, by relations

$$\left[\bar{G}_{k,N+\frac{1}{2}}^+, \bar{G}_{0,\frac{1}{2}}^-, \bar{G}_{k,N+\frac{1}{2}}^-, \bar{G}_{0,\frac{1}{2}}^+\right] = -(N + 1)pJ_{k,N+1}, \quad \left[\bar{G}_{k,N+\frac{1}{2}}^-, \bar{G}_{0,\frac{1}{2}}^+, \bar{G}_{k,N+\frac{1}{2}}^+, \bar{G}_{0,\frac{1}{2}}^-\right] = (N + 1)p\bar{J}_{k,N+1},$$

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where \( c \neq 0, d = \dim V_0 = \dim V_1 \), and \( I_d \) denotes the identity matrix of order \( d \). We may assume that the matrices of the actions of \( \tilde{L}_{k,0} \) and \( \tilde{G}_{0,N+\frac{1}{2}}^+ \) on \( V \) with respect to the above basis (adjust the even and odd parts of basis if necessary) respectively have forms

\[
\begin{pmatrix}
A_1 & \cdots & A_n \\
B_1 & \cdots & B_n
\end{pmatrix}
\]

where \( A_1 = A_2, B_1, B_2 \) are matrices of order \( d \). Comparing the traces of both sides of the first equation in Eq. 26, we obtain

\[
(A_1 - B_1)A_2 - B_1A_2 = -bcdI_d, \quad A_2B_2 - B_2A_1 = bcI_d.
\]

We obtain

\[
\begin{pmatrix}
A_1 & \\
B_1
\end{pmatrix}
\begin{pmatrix}
A_2 \\
B_2
\end{pmatrix} = \begin{pmatrix}
A_1A_2 - B_1A_2 \\
B_1A_2 - B_2A_1
\end{pmatrix} = \begin{pmatrix}
bcdI_d \\
bcI_d
\end{pmatrix}.
\]

Now, we can give the proof of Theorem 1.

**Proof of Theorem 1** If \( k = 0 \), then \( g(k, N) = g(0, N) = A(\mathfrak{h}(p))_{+}/I(0, N) \). It is equivalent to consider the problem for \( \mathfrak{h}(p)/\mathfrak{h}(p)_{(1)} = \mathfrak{h}(p)_{[0]} \). By results in [6], we must have that \( \dim V = (1|1) \) or \( (2|2) \).

Next, we assume that \( k \geq 1 \) and \( N = 0 \). Note that the notations \( \Omega(J), \Omega(G) \) (cf. Eqs. 17 and 18) and \( \mathfrak{t}(k, 0) \) (cf. Eq. 14) still make sense. We claim that the action of \( \mathfrak{t}(k, 0) \) on \( V \) is trivial.

**Case 1**: \( p \neq \mathbb{Z}_+ \). In this case, we have \( \Omega(J) = \Omega(G) = \emptyset \). One can prove the claim as in Lemma 4(1).

**Case 2**: \( p \in 1 + 2\mathbb{Z}_+ \). In this case, we have \( \Omega(G) = \emptyset \). If \( j_j < k \), then \( \Omega(J) = \emptyset \). One can prove the claim as in Lemma 4(1). If \( j_j = k \), then \( \Omega(J) \neq \emptyset \). One can prove the claim as in Lemma 4(2).

**Case 3**: \( p \in 2\mathbb{Z}_{\geq 1} \). Note that \( j_j \neq j_G \) if \( \max\{j_G, j_j\} < k \), then \( \Omega(J) = \Omega(G) = \emptyset \). One can prove the claim as in Lemma 4(1). If \( j_j = k \), then \( \Omega(J) \neq \emptyset \). One can prove the claim as in Lemma 4(2). If \( j_G = k \), then \( \Omega(G) \neq \emptyset \). One can prove the claim as in Lemma 5(6).

The claim implies that \( V \) is simply an irreducible module over \( g(k-1, 0) \). By induction on \( k \), we must have that \( \dim V = (1|1) \) or \( (2|2) \).

At last, we assume that \( k, N \geq 1 \). Note that if the row ideal \( \mathfrak{r}(k, N) \) (resp., column ideal \( \mathfrak{c}(k, N) \)) of \( g(k, N) \) acts trivially on \( V \), then \( V \) can be viewed as an irreducible module over \( g(k-1, N) \) (resp., \( g(k, N-1) \)). By simultaneous induction on \( k \) and \( N \), using Lemmas 3–5, we must have that \( \dim V = (1|1) \) or \( (2|2) \).
4 Free Conformal $\mathcal{R}(p)$-Modules of Small Rank

In this section, we classify all the free conformal modules of ranks $(1 + 1)$ and $(2 + 2)$ over $\mathcal{R}(p)$ by using the classification result of those over $K_2$. For reducible ones, we completely determine their composition factors.

4.1 Finite Conformal Modules over $K_2$

Let us first list some finite conformal modules over $K_2 = \langle L, J, G^\pm \rangle$. For the convenience of later use, we choose $\{L_0 = pL, J_0 = J, G^\pm_0 = \sqrt{p}G^\pm \}$ as a $\mathbb{C}[\partial]$-basis of $K_2$ (cf. Eq. 6).

1. Trivial $K_2$-modules
   
   For any $\alpha \in \mathbb{C}$, the even one-dimensional trivial $K_2$-module $\zeta_\alpha$ satisfy $L_0 \lambda c_\alpha = J_0 \lambda c_\alpha = G^\pm_0 \lambda c_\alpha = 0$ and $\partial c_\alpha = \alpha c_\alpha$, where $|c_\alpha| = 0$.

2. $K_2$-modules of rank $(1 + 1)$
   
   For any $\lambda, \alpha \in \mathbb{C}$, there is a conformal $K_2$-module $K^{(1)}_{\lambda,\alpha}$ of rank $(1 + 1)$ with $\mathbb{C}[\partial]$-basis $\{v_0, v_1\}$ and $\lambda$-actions
   \[
   \begin{aligned}
   &L_0 \lambda v_0 = p(\partial + \Delta \lambda + \alpha)v_0, \\
   &J_0 \lambda v_0 = -2\Delta v_0, \\
   &G^+_0 \lambda v_0 = \sqrt{p}v_1, \\
   &G^-_0 \lambda v_0 = 0,
   
   &L_0 \lambda v_1 = p(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v_1, \\
   &J_0 \lambda v_1 = (2\Delta - 1)v_1, \\
   &G^+_0 \lambda v_1 = 2\sqrt{p}(\partial + 2\Delta \lambda + \alpha)v_0, \\
   &G^-_0 \lambda v_1 = 0.
   \end{aligned}
   \]
   
   There is another conformal $K_2$-module $K^{(2)}_{\Delta,\alpha}$ of rank $(1 + 1)$ with $\mathbb{C}[\partial]$-basis $\{v_0, v_1\}$ and $\lambda$-actions
   \[
   \begin{aligned}
   &L_0 \lambda v_0^{(1)} = p(\partial + \Delta \lambda + \alpha)v_0^{(1)}, \\
   &J_0 \lambda v_0^{(1)} = -2\Delta v_0^{(1)}, \\
   &G^+_0 \lambda v_0^{(1)} = \sqrt{p}v_1^{(1)}, \\
   &G^-_0 \lambda v_0^{(1)} = 0, \\
   &L_0 \lambda v_1^{(1)} = p(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v_1^{(1)}, \\
   &J_0 \lambda v_1^{(1)} = (2\Delta - 1)v_1^{(1)}, \\
   &G^+_0 \lambda v_1^{(1)} = 2\sqrt{p}(\partial + 2\Delta \lambda + \alpha)v_0^{(1)}, \\
   &G^-_0 \lambda v_1^{(1)} = 0.
   \end{aligned}
   \]

3. $K_2$-modules of rank $(2 + 2)$
   
   For any $\Delta, \lambda, \alpha \in \mathbb{C}$, there is a conformal $K_2$-module $K_{\Delta,\lambda,\alpha}$ of rank $(2 + 2)$ with $\mathbb{C}[\partial]$-basis $\{v_0^{(1)}, v_0^{(2)}, v_1^{(1)}, v_1^{(2)}\}$ and $\lambda$-actions
   \[
   \begin{aligned}
   &L_0 \lambda v_0^{(1)} = p(\partial + \Delta \lambda + \alpha)v_0^{(1)}, \\
   &L_0 \lambda v_0^{(2)} = p(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v_0^{(2)}, \\
   &J_0 \lambda v_0^{(1)} = \Lambda v_0^{(1)}, \\
   &J_0 \lambda v_0^{(2)} = \Lambda v_0^{(2)} + (2\Delta + \lambda)\lambda v_0^{(1)}, \\
   &J_0 \lambda v_1^{(1)} = (\Lambda + 1)v_1^{(1)}, \\
   &J_0 \lambda v_1^{(2)} = (\Lambda - 1)v_1^{(2)}, \\
   &G^+_0 \lambda v_0^{(1)} = \sqrt{p}v_1^{(1)}, \\
   &G^+_0 \lambda v_0^{(2)} = -\sqrt{p}(2\Delta + \lambda)\lambda v_0^{(1)}, \\
   &G^+_0 \lambda v_1^{(1)} = \sqrt{p}(\partial + (2\Delta - \Lambda)\lambda + 2\alpha)v_0^{(1)}, \\
   &G^+_0 \lambda v_1^{(2)} = -\sqrt{p}(2\partial + (2\Delta - \Lambda)\lambda + 2\alpha)v_0^{(2)}, \\
   &G^-_0 \lambda v_0^{(1)} = \sqrt{p}v_1^{(2)}, \\
   &G^-_0 \lambda v_0^{(2)} = \sqrt{p}(2\partial + (2\Delta - \Lambda + 2)\lambda + 2\alpha)v_1^{(2)}, \\
   &G^-_0 \lambda v_1^{(1)} = \sqrt{p}(2\partial + (2\Delta - \Lambda + 2)\lambda + 2\alpha)v_1^{(2)}, \\
   &G^-_0 \lambda v_1^{(2)} = 0.
   \end{aligned}
   \]
Lemma 6 Let \( V \) be a finite irreducible conformal module over \( K_2 \). Up to parity change, \( V \) is isomorphic to \( \xi_\alpha, K^{(1)}_{\Delta,\alpha}, K^{(2)}_{\Delta,\alpha} \) with \( \Delta \neq 0 \), or \( K_{\Delta,\lambda,\alpha} \) with \( 2\Delta \pm \lambda \neq 0 \) listed above.

Remark 1 The above classification result of FICMs over \( K_2 \) was given in [6]. By using a lemma in [4, 5] (see Lemma 11) or the arguments in [7, Remark 8.3], we have that

(1) up to parity change, any conformal \( K_2 \)-module of rank \((1 + 1)\) has form \( K^{(1)}_{\Delta,\alpha} \) or \( K^{(2)}_{\Delta,\alpha} \);

(2) up to parity change, any conformal \( K_2 \)-module of rank \((2 + 2)\) has form \( K_{\Delta,\lambda,\alpha} \).

Furthermore, \( K^{(1)}_{\Delta,\alpha} \) and \( K^{(2)}_{\Delta,\alpha} \) are irreducible if and only if \( \Delta \neq 0 \), while \( K_{\Delta,\lambda,\alpha} \) is irreducible if and only if \( 2\Delta \pm \lambda \neq 0 \). The composition factors of reducible \( K_2 \)-modules \( K^{(1)}_{0,\alpha}, K^{(2)}_{0,\alpha}, K_{\Delta,\pm 2\Delta,\alpha} \) will be given in Remark 3.

4.2 \( \mathfrak{h}(p) \)-Modules of Rank \((1 + 1)\)

In this subsection, we classify all the free conformal \( \mathfrak{h}(p) \)-modules of rank \((1 + 1)\). Obviously, for any \( \Delta, \alpha \in \mathbb{C} \), the conformal \( K_2 \)-modules \( K^{(1)}_{\Delta,\alpha} \) and \( K^{(2)}_{\Delta,\alpha} \) can be trivially extended to conformal \( \mathfrak{h}(p) \)-modules \( V^{(1)}_{\Delta,\alpha} \) and \( V^{(2)}_{\Delta,\alpha} \):

(I-1) \( V^{(1)}_{\Delta,\alpha} = \mathbb{C}[\partial]v_0 \oplus \mathbb{C}[\partial]v_1 \) with Eq. 27 and \( L_i \, \partial \, v_s = J_i \, \partial \, v_s = G^\pm_i \, \partial \, v_s = 0 \), \( i \geq 1, s \in \mathbb{Z}/2\mathbb{Z} \);

(I-2) \( V^{(2)}_{\Delta,\alpha} = \mathbb{C}[\partial]v_0 \oplus \mathbb{C}[\partial]v_1 \) with Eq. 28 and \( L_i \, \partial \, v_s = J_i \, \partial \, v_s = G^\pm_i \, \partial \, v_s = 0 \), \( i \geq 1, s \in \mathbb{Z}/2\mathbb{Z} \).

Lemma 7 Let \( M \) be a nontrivial finite conformal module over \( \mathfrak{h}(p) \). Then the \( \lambda \)-actions of \( L_1, J_1, G^\pm_1 \in \mathfrak{h}(p) \) on \( M \) are trivial for all \( i \gg 0 \). In particular, a finite conformal module over \( \mathfrak{h}(p) \) is simply a finite conformal module over \( \mathfrak{h}(p)_{\langle n \rangle} \) for some big enough integer \( n \), where \( \mathfrak{h}(p)_{\langle n \rangle} \) is defined by Eq. 11.

Proof Since \( M \) can be viewed as a conformal module over the conformal subalgebra \( \langle L_i \mid i \in \mathbb{Z}_+ \rangle \) of \( \mathfrak{h}(p) \), it follows from [15] that the \( \lambda \)-action of \( L_i \) on \( M \) is trivial for all \( i \gg 0 \). Furthermore, from Eqs. 2 and 3 we see that the \( \lambda \)-actions of \( J_i, G^\pm_i \) on \( M \) are also trivial for all \( i \gg 0 \). (Here, we would like to take this opportunity to point out that [16, Theorem 4.2] for \( \mathfrak{g}(p) \) can be also concisely proved by using the above observation.) \[ \square \]

Theorem 2 Let \( M \) be a nontrivial free conformal module of rank \((1 + 1)\) over \( \mathfrak{h}(p) \). Then, up to parity change, \( M \cong V^{(1)}_{\Delta,\alpha} \) or \( V^{(2)}_{\Delta,\alpha} \) for some \( \Delta, \alpha \in \mathbb{C} \).

Proof Let \( M = \mathbb{C}[\partial]v_0 \oplus \mathbb{C}[\partial]v_1 \). By regarding \( M \) as a conformal module over \( K_2 \), we see that, up to parity change, the \( \lambda \)-actions of \( L_0, J_0, G^\pm_0 \) have forms Eqs. 27 or 28, where \( \Delta, \alpha \in \mathbb{C} \). By Lemma 7, \( L_i \, \partial \, v_s = J_i \, \partial \, v_s = G^\pm_i \, \partial \, v_s = 0 \) for \( i \gg 0, s \in \mathbb{Z}/2\mathbb{Z} \). Note that \( \mathfrak{h}(p) \) is \( \mathbb{Z} \)-graded in the sense that \( \mathfrak{h}(p) = \oplus_{i \in \mathbb{Z}} \mathfrak{h}(p)_i \), where \( \mathfrak{h}(p)_i = \mathbb{C}[\partial]L_i \oplus \mathbb{C}[\partial]J_i \oplus \mathbb{C}[\partial]G^+_i \oplus \mathbb{C}[\partial]G^-_i \). Assume that \( k \in \mathbb{Z}_+ \) is the largest integer such that the action of \( \mathfrak{h}(p)_k \) on \( M \) is nontrivial.
If $k = 0$, then $M$ is simply a conformal $K_2$-module. By Remark 1(1), the conclusion holds.

Next, we consider the case $k > 0$. By assumption, we can suppose

$$L_k, v_s = a_s(\partial, \lambda)v_s, \quad J_k, v_s = b_s(\partial, \lambda)v_s, \quad G_k^{\pm}(\partial, \lambda)v_s = c_s^{\pm}(\partial, \lambda)v_{s+1},$$

where $a_s(\partial, \lambda), b_s(\partial, \lambda), c_s^{\pm}(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$ for $s \in \mathbb{Z}/2\mathbb{Z}$, and at least one of them is nonzero. Considering the action of $[L_k L_k]_{\lambda+\mu} = 0$ on $v_s$, we obtain

$$a_s(\partial, \lambda)a_s(\partial + \lambda, \mu) = a_s(\partial, \mu)a_s(\partial + \mu, \lambda), \quad s \in \mathbb{Z}/2\mathbb{Z}.$$  

Comparing the coefficients of $\lambda$, we see that $a_s(\partial, \lambda)$ is independent of the variable $\partial$, and so we can denote $a_s(\lambda) = a_s(\partial, \lambda)$. Then, considering the action of the operator $[L_0 L_k]_{\lambda+\mu} = ((k + p)\lambda - p\mu)L_k \lambda + \mu$ on $v_s$, we obtain

$$(p\mu - (k + p)\lambda)a_s(\lambda + \mu) = p\mu a_s(\mu), \quad s \in \mathbb{Z}/2\mathbb{Z}. \quad (30)$$

If $k \neq -p$, then $k + p \neq 0$. By Eq. 30 with $\mu = 0$, we obtain $a_s(\lambda) = 0$. Hence, the action of $L_k$ on $M$ is trivial. Furthermore, by relations

$$(k + p)(\partial + \lambda)J_k = [L_k J_0],$$

$$((k + p)\partial + \left(k + \frac{3}{2}p\right)\lambda)G_k^{\pm} = [L_k G_0^{\pm}],$$

we see that the actions of $J_k$ and $G_k^{\pm}$ on $M$ are also trivial, a contradiction.

If $k = -p$, then $p$ is a negative integer. By Eq. 30, we see that $a_s(\lambda)$ is independent of the variable $\lambda$, and so we can denote $a_s = a_s(\lambda), s \in \mathbb{Z}/2\mathbb{Z}$.

If $p \leq -2$, then $k \geq 2$. We claim that the action of $\check{\mathfrak{R}}(p)_{k-1}$ on $M$ is trivial. As in [16, Thenrem 5.1], one can first prove that the action of $L_{k-1}$ on $M$ is trivial. Then, by relations

$$-(\partial + \lambda)J_{k-1} = [L_{k-1} J_0],$$

$$-(\partial + \left(1 - \frac{1}{2}p\right)\lambda)G_{k-1}^{\pm} = [L_{k-1} G_0^{\pm}],$$

we see that the actions of $J_{k-1}$ and $G_{k-1}^{\pm}$ on $M$ are also trivial, and thus the claim holds.

At last, note that in case $p \leq -2$, the $\mathbb{C}[\partial]$-basis of $\check{\mathfrak{R}}(p)_{k}$ can be generated by those of $\check{\mathfrak{R}}(p)_{k-1}$ and $L_1$. Hence, the action of $\check{\mathfrak{R}}(p)_{k}$ on $M$ is trivial, a contradiction.

If $p = -1$, then $k = 1$. If $M \cong K_{\Delta,a}^{(1)}$ as a conformal $K_2$-module, then the actions of $L_0, J_0$ and $G_0^{\pm}$ have the form of Eq. 27. Applying the operator $[L_1 G_0^{\pm}]_{\lambda+\mu} = -\frac{1}{2}\lambda G_1^{\pm} \lambda + \mu$ on $v_0$, we obtain

$$\sqrt{-1}(a_0 - a_1)v_1 = \frac{1}{2}\lambda c_0^{\pm}(\partial, \lambda + \mu)v_1,$$

which implies that $a_0 = a_1$ and $c_0^{\pm}(\partial, \lambda) = 0$. Then, applying the above operator on $v_1$, we obtain

$$0 = -\frac{1}{2}\lambda c_1^{\pm}(\partial, \lambda + \mu)v_0,$$

which implies that $c_1^{\pm}(\partial, \lambda) = 0$. Denote $\beta = a_0 = a_1$. Applying the operator $[L_1 G_0^{-} G_0^{\pm}]_{\lambda+\mu} = -\frac{1}{2}\lambda G_1^{-} \lambda + \mu$ on $v_0$ and $v_1$, respectively, we obtain

$$0 = -\frac{1}{2}\lambda c_1^{-}(\partial, \lambda + \mu)v_1, \quad 2\sqrt{-1}\beta \lambda v_0 = -\frac{1}{2}\lambda c_1^{-}(\partial, \lambda + \mu)v_0.$$
which imply that $c_0^{-} (\partial, \lambda) = 0$ and $c_1^{-} (\partial, \lambda) = -4\sqrt{-1} \beta$. Applying the operator 
$[G_0^+ \lambda G_1^-] \lambda + \mu = (-\partial J_1 + 2L_1) \lambda + \mu$ on $v_s$, we obtain

$$4\beta = (\lambda + \mu)b_s(\partial, \lambda + \mu) + 2\beta, \ s \in \mathbb{Z}/2\mathbb{Z},$$

which imply that $\beta = 0$ and $b_s(\partial, \lambda) = 0$ for $s \in \mathbb{Z}/2\mathbb{Z}$. Hence, $M \cong V_{\Delta,\alpha}^{(1)}$. Similarly, if $M \cong K_{\Delta,\alpha}^{(2)}$ as a conformal $K_2$-module, then one can show that $M \cong V_{\Delta,\alpha}^{(2)}$. This completes the proof.

**Remark 2** In [16], we showed that a rank $(1 + 1)$ $K_1$-module can be nontrivially extended to a $\mathcal{G}(p)$-module if $p = -1$. However, by Theorem 2, we see that any rank $(1 + 1) K_2$-module $M$ cannot be nontrivially extended to a $\mathcal{R}(p)$-module, even in case that $p = -1$ and $M$ can be degenerated from a rank $(2 + 2)$ $K_2$-module (note that a $K_2$-module $K_{\Delta,\Lambda,\alpha}$ will degenerate to a $K_2$-module $K_{\Delta,\alpha}^{(1)}$ if we set $v_0^{(2)} = v_1^{(2)} = 0$ and $\Lambda = -2\Delta$). We shall see in the next subsection that a rank $(2 + 2)$ $K_2$-module can be nontrivially extended to a $\mathcal{R}(p)$-module if $p = -1$.

### 4.3 $\mathcal{R}(p)$-Modules of Rank $(2 + 2)$

In this subsection, we classify all the free conformal $\mathcal{R}(p)$-modules of rank $(2 + 2)$. As above, for any $\Delta, \Lambda, \alpha, \beta \in \mathbb{C}$, the conformal $K_2$-module $K_{\Delta,\Lambda,\alpha}$ can be trivially extended to a conformal $\mathcal{R}(p)$-module $V_{\Delta,\Lambda,\alpha}$, while for $\mathcal{R}(-1)$, the conformal $K_2$-module $K_{\Delta,\Lambda,\alpha}$ can be extended to a conformal $\mathcal{R}(p)$-module $V_{\Delta,\Lambda,\alpha,\beta}$, which is a nontrivial extension of $K_{\Delta,\Lambda,\alpha}$ if $\beta \neq 0$.

(II-1) $V_{\Delta,\Lambda,\alpha} = \mathbb{C}[\partial]v_0^{(1)} \oplus \mathbb{C}[\partial]v_0^{(2)} \oplus \mathbb{C}[\partial]v_1^{(1)} \oplus \mathbb{C}[\partial]v_1^{(2)}$ with Eq. 29 and $L_{i,\lambda} v_s^{(\ell)} = J_{i,\lambda} v_s^{(\ell)} = G_1^\pm v_s^{(\ell)} = 0$, $i \geq 1$, $s \in \mathbb{Z}/2\mathbb{Z}$, $\ell = 1, 2$;

(II-2) $V_{\Delta,\Lambda,\alpha,\beta} = \mathbb{C}[\partial]v_0^{(1)} \oplus \mathbb{C}[\partial]v_0^{(2)} \oplus \mathbb{C}[\partial]v_1^{(1)} \oplus \mathbb{C}[\partial]v_1^{(2)}$ with Eq. 29 and

$$
\begin{align*}
L_{1,\lambda} v_0^{(1)} &= -\sqrt{-1} \beta v_0^{(1)}, & L_{1,\lambda} v_1^{(1)} &= -\sqrt{-1} \beta v_1^{(1)}, \\
L_{1,\lambda} v_0^{(2)} &= -\sqrt{-1} \beta (v_0^{(2)} + \lambda v_0^{(1)}), & L_{1,\lambda} v_1^{(2)} &= -\sqrt{-1} \beta v_1^{(1)}, \\
J_{1,\lambda} v_0^{(1)} &= 0, & J_{1,\lambda} v_1^{(1)} &= 0, \\
J_{1,\lambda} v_0^{(2)} &= \sqrt{-1} \beta v_0^{(1)}, & J_{1,\lambda} v_1^{(2)} &= 0, \\
G_1^+ \lambda v_0^{(1)} &= 0, & G_1^+ \lambda v_1^{(1)} &= 0, \\
G_1^+ \lambda v_0^{(2)} &= \beta v_1^{(1)}, & G_1^+ \lambda v_1^{(2)} &= -\beta v_1^{(1)}, \\
G_1^- \lambda v_0^{(1)} &= 0, & G_1^- \lambda v_1^{(1)} &= 0, \\
G_1^- \lambda v_0^{(2)} &= -\beta v_1^{(2)}, & G_1^- \lambda v_1^{(2)} &= 0, \\
X_{1,\lambda} v_s^{(\ell)} &= 0, & X_{1,\lambda} v_s^{(\ell)} &= 0,
\end{align*}
$$

where $X = L, J, G^\pm, i \geq 2, s \in \mathbb{Z}/2\mathbb{Z}$, $\ell = 1, 2$.

**Theorem 3** Let $M$ be a nontrivial free conformal module of rank $(2 + 2)$ over $\mathcal{R}(p)$.

1. If $p \neq -1$, then, up to parity change, $M \cong V_{\Delta,\alpha}^{(1)}$ for some $\Delta, \Lambda, \alpha \in \mathbb{C}$.
2. If $p = -1$, then, up to parity change, $M \cong V_{\Delta,\alpha,\beta}^{(2)}$ for some $\Delta, \Lambda, \alpha, \beta \in \mathbb{C}$.
Proof Suppose $M$ is a nontrivial free conformal $\mathcal{R}(p)$-module of rank $(2 + 2)$. Write $M = M_0 \oplus M_1$, where $M_1 = \mathbb{C}[\Delta]v^{(1)}_s \oplus \mathbb{C}[\Delta]v^{(2)}_s$, $s \in \mathbb{Z}/2\mathbb{Z}$. By regarding $M$ as a conformal module over $K_2$, we see that, up to parity change, the $\lambda$-actions of $L_0$, $J_0$, $G^\pm_0$ have forms Eq. 29, where $\Delta, \Lambda, \alpha \in \mathbb{C}$. By Lemma 7, $L_i \lambda v_s = J_i \lambda v_s = G^\pm_i \lambda v_s = 0$ for $i \gg 0$, $s \in \mathbb{Z}/2\mathbb{Z}$. Assume that $k \in \mathbb{Z}_+$ is the largest integer such that the action of $\mathcal{R}(p)_k$ on $M$ is nontrivial.

If $k = 0$, then $M$ is simply a conformal $K_2$-module. By Remark 1(2), Theorem 3 holds. More precisely, up to parity change,

$$M \cong \begin{cases} V_{\Delta, \Lambda, \alpha}, & \text{if } p \neq -1; \\ V_{\Delta, \Lambda, \alpha, 0}, & \text{if } p = -1. \end{cases}$$

Next, we consider the case $k > 0$. By assumption, we can suppose

$$L_k \lambda v_s^{(t)} = a_s^{(t)}(\partial, \lambda)v_s^{(1)} + d_s^{(t)}(\partial, \lambda)v_s^{(2)},$$

$$J_k \lambda v_s^{(t)} = b_s^{(t)}(\partial, \lambda)v_s^{(1)} + e_s^{(t)}(\partial, \lambda)v_s^{(2)},$$

$$G^\pm_k \lambda v_s^{(t)} = c_s^{\pm(t)}(\partial, \lambda)v_s^{(1)}_{s+1} + f_s^{\pm(t)}(\partial, \lambda)v_s^{(2)}_{s+1},$$

where $x_s^{(t)}(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$ for $x = a, b, c^\pm, d, e, f^\pm, s \in \mathbb{Z}/2\mathbb{Z}$, $t = 1, 2$, and at least one of them is nonzero. Let us first determine the actions of $G^+_k$ and $J_k$ on $M$ through three lemmas.

Lemma 8 There exist $\beta, \gamma \in \mathbb{C}$ such that

(1) $G^+_k \lambda v_1^{(1)} = 0$, $G^+_k \lambda v_0^{(1)} = \delta^*_k v_1^{(1)}$, where $\delta^*_k = \delta_{k, p}\delta_{\Delta, 2}\lambda + 1\gamma(\partial + (\Lambda + 1)\lambda + \alpha)$;

(2) $G^+_k \lambda v_0^{(2)} = \delta_{k+p, 0}\beta v_1^{(1)}$, $G^+_k \lambda v_1^{(2)} = -\delta_{k+p, 0}\beta v_0^{(1)}$.

Proof (1) Considering the action of $[J_0 \lambda G^+_k]_{\lambda + \mu} = G^+_k \lambda + \mu$ on $v_0^{(1)}$, we obtain

$$\begin{align*}
(\Lambda + 1)c_0^{+(1)}(\partial + \lambda, \mu) - \Lambda c_0^{+(1)}(\partial, \mu) &= c_0^{+(1)}(\partial, \lambda + \mu), \\
(\Lambda - 1)f_0^{+(1)}(\partial + \lambda, \mu) - \Lambda f_0^{+(1)}(\partial, \mu) &= f_0^{+(1)}(\partial, \lambda + \mu).
\end{align*}$$

(32) (33)

Taking $\lambda = 0$ in Eq. 33, we obtain $f_0^{+(1)}(\partial, \mu) = 0$. Then, considering the action of $[G^+_0 \lambda G^+_k]_{\lambda + \mu} = 0$ on $v_0^{(1)}$, we obtain $c_0^{+(1)}(\partial, \mu) = f_0^{+(1)}(\partial, \mu) = 0$. Namely, $G^+_k \lambda v_0^{(1)} = 0$. Furthermore, applying $[L_0 \lambda G^+_k]_{\lambda + \mu} = ((k + 1)\frac{1}{2}p)\lambda - p\mu)G^+_k \lambda + \mu$ on $v_0^{(1)}$, we obtain

$$p \left( \partial + \left( \Delta + \frac{1}{2} \right) \lambda + \alpha \right) c_0^{+(1)}(\partial + \lambda, \mu) - p(\partial + \mu + \Delta\lambda + \alpha)c_0^{+(1)}(\partial, \mu)$$

$$= \left( p \left( \frac{k}{2} + \frac{1}{2}p \right) \lambda - p\mu \right) c_0^{+(1)}(\partial, \lambda + \mu).$$

Taking $\mu = 0$, we obtain

$$p(\partial + \Delta\lambda + \alpha) \frac{c_0^{+(1)}(\partial + \lambda, 0) - c_0^{+(1)}(\partial, 0)}{\lambda} + \frac{p}{2}c_0^{+(1)}(\partial + \lambda, 0) = (k + \frac{1}{2}p)c_0^{+(1)}(\partial, \lambda).$$

(34)
Taking $\lambda \to 0$, we obtain

$$p(\partial + \alpha) \frac{d}{d\partial} c_{0}^{+(1)}(\partial, 0) = kc_{0}^{+(1)}(\partial, 0). \tag{35}$$

Let $\deg c_{0}^{+(1)}(\partial, 0) = n$. The solution to Eq. 35 is $k = pn$ and $c_{0}^{+(1)}(\partial, 0) = \gamma_n(\partial + \alpha)^n$, where $\gamma_n \in \mathbb{C}$ and $\gamma_n \neq 0$ if $n \geq 1$. By Eq. 32 with $\mu = 0$ and Eq. 34, we have

$$(\partial + \lambda + \alpha)^n \left(\left(\left(n + \frac{1}{2}\right)\Lambda + n - \Delta\right) \lambda - \partial - \alpha\right) \gamma_n = (\partial + \alpha)^n \left(\left(n + \frac{1}{2}\right)\Lambda - \Delta\right) \lambda - \partial - \alpha \right) \gamma_n.$$

If $n \geq 2$, by comparing the coefficients of $\lambda^{n+1}$ in the above equation, we must have $(n+\frac{1}{2})\Lambda+n-\Delta = 0$. Then, by comparing the coefficients of $\lambda^n$, we obtain $-(\partial + \alpha)\gamma_n = 0$, a contradiction. If $n = 1$, then $k = p$. By comparing the coefficients of $\lambda^2$ in the above equation, we have $\frac{3}{2}\Lambda + 1 - \Delta = 0$. Then, by Eq. 32 with $\mu = 0$, we obtain $c_{0}^{+(1)}(\partial, \lambda) = \gamma_1(\partial + (\Lambda + 1)\lambda + \alpha)$. If $n = 0$, using Eq. 35 and then Eq. 32 with $\mu = 0$, we obtain $c_{0}^{+(1)}(\partial, \lambda) = 0$. Namely, we have

$$G_{k, \lambda} v_{0}^{(1)} = \begin{cases} \gamma_1(\partial + (\Lambda + 1)\lambda + \alpha)v_{1}^{(1)}, & \text{if } k = p, \Delta = \frac{3}{2}\Lambda + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Denote $\delta_{k, \lambda}^* = \delta_{k, p} \delta_{\Delta, \frac{3}{2}\Lambda + 1} \gamma(\partial + (\Lambda + 1)\lambda + \alpha)$, where $\gamma = \gamma_1$. Then, we have $G_{k, \lambda} v_{0}^{(1)} = \delta_{k, \lambda}^* v_{1}^{(1)}$.

(2) Applying $[J_{0, \lambda} G_{k, \lambda}^+]_{\lambda, p} = G_{k, \lambda, \mu}^+$ on $v_{0}^{(2)}$, we obtain

$$(\Lambda + 1)c_{0}^{+(2)}(\partial + \lambda, \mu) - \Lambda c_{0}^{+(2)}(\partial, \mu) = c_{0}^{+(2)}(\partial, \lambda + \mu) - (2\Delta + \Lambda)\lambda \delta_{\mu}^*.$$

(37)

Taking $\lambda = 0$ in Eq. 37, we obtain $f_{0}^{+(2)}(\partial, \mu) = 0$. Then, similar to Eq. 34, by considering the operator $[L_{0, \lambda} G_{k, \lambda}^+]_{\lambda, p} = ((k + \frac{1}{2}p)\lambda - p\mu)G_{k, \lambda, \mu}^+$ on $v_{0}^{(2)}$, we obtain

$$p(\partial + (\Lambda + 1)\lambda + \alpha) \frac{c_{0}^{+(2)}(\partial + \lambda, 0) - c_{0}^{+(2)}(\partial, 0)}{\lambda} - \frac{p}{2} c_{0}^{+(2)}(\partial + \lambda, 0)$$

$$= (k + \frac{1}{2}p)c_{0}^{+(2)}(\partial, \lambda) + p \left(\Delta + \frac{\Lambda}{2}\right) \lambda \delta_{\mu}^*.$$ \tag{38}

Taking $\lambda \to 0$, we obtain

$$p(\partial + \alpha) \frac{d}{d\partial} c_{0}^{+(2)}(\partial, 0) = (k + p)c_{0}^{+(2)}(\partial, 0). \tag{39}$$

Following the discussion in (1), by Eqs. 36, 38 and 39, one can show that $c_{0}^{+(2)}(\partial, \lambda) = \delta_{k+p, 0, 0, \beta}$ for some $\beta \in \mathbb{C}$. Namely, the action of $G_{k, \lambda}^+$ on $v_{0}^{(2)}$ has the required form. Furthermore, considering the action of $[G_{0, \lambda}^+ G_{k, \lambda}^+]_{\lambda, p, \mu} = 0$ on $v_{0}^{(2)}$, we obtain

$$c_{1}^{+(2)}(\partial + \lambda, \mu) - (2\Delta + \Lambda)\lambda(f_{1}^{+(2)}(\partial + \lambda, \mu) - \delta_{\mu}^*) + \delta_{k+p, 0}\beta = 0.$$ \tag{40}

Taking $\lambda = 0$, we see that $c_{1}^{+(2)}(\partial, \mu) = -\delta_{k+p, 0}\beta$. As above, by considering the actions of operators $[J_{0, \lambda} G_{k, \lambda}^+]_{\lambda, p} = G_{k, \lambda, \mu}^+$ and $[L_{0, \lambda} G_{k, \lambda}^+]_{\lambda, p} = ((k + \frac{1}{2}p)\lambda - p\mu)G_{k, \lambda, \mu}^+$ on
\(v^{(2)}_1\), one can show that \(f^{(2)}_1(\partial, \lambda) = 0\). Hence, the action of \(G^+_k\) on \(v^{(2)}_1\) has the required form. 

**Lemma 9** Let \(\gamma\) be as in Lemma 8. We have \(\gamma = 0\), and \(J_k \lambda \ v^{(1)}_0 = J_k \lambda \ v^{(1)}_1 = 0\).

**Proof** We first show that \(J_k \lambda \ v^{(1)}_0 = 0\). Considering the action of \([J_k \lambda \ G^+_0]_{\lambda + \mu} = G^+_k \lambda + \mu\) on \(v^{(1)}_0\), by Lemma 8(1), we obtain \(e^{(1)}_1(\partial, \lambda) = 0\) and

\[
b^{(1)}_1(\partial, \lambda) - b^{(1)}_0(\partial + \mu, \lambda) + (2\Delta + \Lambda)\lambda e^{(1)}_0(\partial + \mu, \lambda) = \frac{1}{\sqrt{p}} \delta^\mu_{\lambda + \mu}. \tag{40}
\]

Then, considering the action of \([J_k \lambda J_k]_{\lambda + \mu} = 0\) on \(v^{(1)}_1\), we obtain

\[
b^{(1)}_1(\partial, \lambda)b^{(1)}_1(\partial + \mu, \mu) = b^{(1)}_1(\partial, \mu)b^{(1)}_1(\partial + \mu, \lambda).
\]

Comparing the coefficients of \(\lambda\) in the above equations, we see that \(b^{(1)}_1(\partial, \lambda)\) is independent of the variable \(\partial\), and so we can denote \(b^{(1)}_1(\lambda) = b^{(1)}(\partial, \lambda)\). Furthermore, applying \([L_0 \lambda J_k]_{\lambda + \mu} = (k\lambda - p\mu)J_{k\lambda + \mu}\) on \(v^{(1)}_1\), we obtain \(-p\mu b^{(1)}_0(\mu) = (k\lambda - p\mu)b^{(1)}_1(\lambda + \mu)\).

Taking \(\mu = 0\), we see that \(b^{(1)}_0(\partial, \lambda) = b^{(1)}_0(\lambda) = 0\). Hence, \(J_k \lambda \ v^{(1)}_1 = 0\).

Now, taking \(\mu = 0\) and \(\lambda = 0\) in Eq. \(40\), respectively, we obtain

\[
b^{(1)}_0(\partial, \lambda) = (2\Delta + \Lambda)\lambda e^{(1)}_0(\partial, \lambda) - \frac{1}{\sqrt{p}} \delta^\mu_{\lambda + \mu}, \tag{41}
\]

\[
b^{(1)}_0(\partial + \mu, 0) = -\frac{1}{\sqrt{p}} \delta^\mu_{\lambda + \mu}. \tag{42}
\]

Considering the action of \([J_0 \lambda J_k]_{\lambda + \mu} = 0\) on \(v^{(1)}_0\) and equating the coefficients of \(v^{(1)}_0\), we obtain

\[
\Lambda b^{(1)}_0(\partial + \mu, \mu) - \Lambda b^{(1)}_0(\partial, \mu) + (2\Delta + \Lambda)\lambda e^{(1)}_0(\partial + \mu, \mu) = 0. \tag{43}
\]

Next, we show that \(\gamma = 0\), or equivalently \(\delta^\mu_{\lambda + \mu} = 0\). Note first that from the definition of \(\delta^\mu_{\lambda + \mu}\), we only need to consider the case \(k = p\) and \(\Delta = \frac{3}{2} \Delta + 1\). By Eq. 42, we see that \(\delta^\mu_{\lambda + \mu}\) is a polynomial on \(\partial + \mu\). Recall again the definition of \(\delta^\mu_{\lambda + \mu}\), we may further assume that \(\Lambda = 0\), and so \(\Delta = 1\). Then, by Eq. 43, we have \(e^{(1)}_0(\partial, \mu) = 0\). On one hand, this, together with Eq. 41, implies that \(b^{(1)}_0(\partial, \lambda) = -\frac{1}{\sqrt{p}} \delta^\mu_{\lambda + \mu} = -\frac{1}{\sqrt{p}} \gamma(\partial + \lambda + \alpha)\). On the other hand, applying \([J_k \lambda J_k]_{\lambda + \mu} = 0\) on \(v^{(1)}_0\), as above, we must have that \(b^{(1)}_0(\partial, \lambda)\) is independent of the variable \(\partial\). Hence, \(\gamma = 0\).

Applying \([L_0 \lambda J_k]_{\lambda + \mu} = (k\lambda - p\mu)J_{k\lambda + \mu}\) on \(v^{(1)}_0\), and then taking \(\mu = 0\), we obtain

\[
p(\partial + \Delta\lambda + \alpha)\frac{b^{(1)}_0(\partial + \lambda, 0) - b^{(1)}_0(\partial, 0)}{\lambda} + p(\Delta + \frac{\Lambda}{2})\lambda e^{(1)}_0(\partial + \lambda, 0) = kb^{(1)}_0(\partial, \lambda), \tag{44}
\]

\[
p(\partial + \Delta\lambda + \alpha)\frac{e^{(1)}_0(\partial + \lambda, 0) - e^{(1)}_0(\partial, 0)}{\lambda} + pe^{(1)}_0(\partial + \lambda, 0) = ke^{(1)}_0(\partial, \lambda). \tag{45}
\]
Taking $\lambda \to 0$ in Eq. 45, we obtain
\[ p(\partial + \alpha) \frac{d}{d\partial} e_0^{(1)}(\partial, 0) = (k - p)e_0^{(1)}(\partial, 0). \] (46)

At last, we show that $J_{k,\lambda}v_0^{(1)} = 0$ in two cases. First, we consider the case $2\Delta + \Lambda \neq 0$. On one hand, by Eq. 41 (recall that $\delta^*_\lambda = 0$), we have $b_0^{(1)}(\partial, \lambda) = (2\Delta + \Lambda)\lambda e_0^{(1)}(\partial, \lambda)$. In particular, $b_0^{(1)}(\partial, 0) = 0$. On the other hand, by Eq. 44, we have $b_0^{(1)}(\partial, \lambda) = \frac{p}{\Delta}(\Delta + \Lambda)\lambda e_0^{(1)}(\partial + \lambda, 0)$. Hence, we have $\frac{p}{\Delta}\lambda e_0^{(1)}(\partial + \lambda, 0) = e_0^{(1)}(\partial, \lambda)$. If $p \neq 2k$, by comparing the coefficients of $\partial$ on both sides, we must have that $e_0^{(1)}(\partial, \lambda) = 0$, and thus $b_0^{(1)}(\partial, \lambda) = 0$. Namely, $J_{k,\lambda}v_0^{(1)} = 0$. If $p = 2k$, by Eq. 46, we must have $e_0^{(1)}(\partial, 0) = 0$. Then, by Eq. 45 and then Eq. 41, we also have that $J_{k,\lambda}v_0^{(1)} = 0$.

Next, we suppose $2\Delta + \Lambda = 0$. First, by Eq. 41, we have $b_0^{(1)}(\partial, \lambda) = 0$. So, we only need to show $e_0^{(1)}(\partial, \lambda) = 0$. If $k = -p$, by Eq. 46, we must have $e_0^{(1)}(\partial, 0) = 0$. As above, by Eq. 45 and then Eq. 41, we have $J_{k,\lambda}v_0^{(1)} = 0$. If $k \neq -p$, then by Lemma 8 and $\delta^*_\lambda = 0$, we see that the action of $G_k^+$ on $M$ is trivial. Applying the operator $[G_k^+, G_0^-]_{\lambda+\mu} = (p\lambda - (2k + p)\mu)J_{k,\lambda+\mu} + 2L_k J_{\lambda+\mu}$ on $v_0^{(1)}$, we see that $L_k J_{\lambda+\mu}v_0^{(1)} = -\frac{1}{2}(p\lambda - (2k + p)\mu)J_{k,\lambda+\mu}v_0^{(1)}$. In particular, taking $\mu = 0$, we have $L_k v_0^{(1)} = -\frac{1}{2}p\lambda J_{k,\lambda}v_0^{(1)}$. Then, applying $[L_k, J_0]_{\lambda+\mu} = -(k + p)\mu J_{k,\lambda+\mu}$ on $v_0^{(1)}$, we obtain
\[
\frac{p}{\Delta}\lambda(e_0^{(1)}(\partial + \mu, \lambda) - e_0^{(1)}(\partial, \lambda)) = -(k + p)\mu e_0^{(1)}(\partial, \lambda + \mu).
\]
Taking $\lambda = 0$, we see that $e_0^{(1)}(\partial, \mu) = 0$. This completes the proof. \hfill \Box

**Lemma 10** Let $\beta$ be as in Lemma 8. We have $J_{k,\lambda}v_1^{(2)} = 0, J_{k,\lambda}v_2^{(2)} = -\delta_{k+p,0}e_0^{(1)}v_0^{(1)}$.

**Proof** We first show that $J_{k,\lambda}v_2^{(2)} = -\delta_{k+p,0}e_0^{(1)}v_0^{(1)}$. Considering the action of $[J_{k,\lambda} G_0^+]_{\lambda+\mu} = G_k^+ J_{k,\lambda+\mu}$ on $v_0^{(2)}$, by Lemma 8(2) and Lemma 9, we obtain
\[
b_0^{(2)}(\partial + \mu, \lambda) + (2\Delta + \Lambda)\lambda e_0^{(2)}(\partial + \mu, \lambda) = -\delta_{k+p,0}\frac{\beta}{\sqrt{p}}.
\]
Taking $\lambda = \mu = 0$, we have $b_0^{(2)}(\partial, 0) = -\delta_{k+p,0}\frac{\beta}{\sqrt{p}}$. Considering the action of $[J_0, J_k]_{\lambda+\mu} = 0$ on $v_0^{(2)}$ and equating the coefficients of $v_0^{(1)}$, by Lemma 9, we obtain
\[
\Lambda b_0^{(2)}(\partial + \lambda, \mu) - \Lambda b_0^{(2)}(\partial, \mu) + (2\Delta + \Lambda)\lambda e_0^{(1)}(\partial + \lambda, \mu) = 0.
\] (47)
Applying $[L_0, J_k]_{\lambda+\mu} = (k\lambda - p\mu)J_{k,\lambda+\mu}$ on $v_0^{(2)}$, and then taking $\mu = 0$, we obtain (recall that we have shown $b_0^{(2)}(\partial, 0) = -\delta_{k+p,0}\frac{\beta}{\sqrt{p}}$)
\[
-p\left(\Delta + \frac{\Lambda}{2}\right)\lambda e_0^{(2)}(\partial + \lambda, 0) + \delta_{k+p,0}\sqrt{p}\beta = kb_0^{(2)}(\partial, \lambda),
\] (48)
\[
p(\partial + (\Delta + 1)\lambda + \alpha)\frac{e_0^{(2)}(\partial + \lambda, 0) - e_0^{(2)}(\partial, 0)}{\lambda} = ke_0^{(2)}(\partial, \lambda).
\] (49)
Taking $\lambda \to 0$ in Eq. 49, we obtain
\[ p(\partial + \alpha) \frac{d}{d\theta} e_0^{(2)}(\partial, 0) = ke_0^{(2)}(\partial, 0). \] (50)

By Eq. 47 with $\mu = 0$, we have $(2\Delta + \Lambda)\lambda e_0^{(1)}(\partial + \lambda, 0) = 0$. This, together with Eq. 48, implies that $b_0^{(2)}(\partial, \lambda) = \delta_k p_0 0 \frac{\sqrt{p}}{k} = -\delta_k p_0 0 \frac{p}{\sqrt{p}}$. Let deg $e_0^{(2)}(\partial, 0) = n$. Assume that $n \geq 1$. By Eq. 50, we must have $k = pn$, which implies $k + p = p(n + 1) \neq 0$. Hence, $b_0^{(2)}(\partial, \lambda) = 0$. Then, applying $[J_k, J_k]_{\lambda + \mu} = 0$ on $v_0^{(2)}$, as in Lemma 9, we must have that $e_0^{(2)}(\partial, \lambda)$ is independent of the variable $\partial$. In particular, $e_0^{(2)}(\partial, 0)$ is independent of the variable $\partial$, which contradicts to the assumption deg $e_0^{(2)}(\partial, 0) = n \geq 1$. Thus, $n = 0$. By Eq. 50, we have $e_0^{(2)}(\partial, 0) = 0$. Furthermore, by Eq. 49, we have $e_0^{(2)}(\partial, \lambda) = 0$. Hence, the action of $J_k$ on $v_0^{(2)}$ has the required form.

Continuation of the proof of Theorem 3 If $k \neq -p$, then $k + p \neq 0$. By Lemmas 8–10, we see that the actions of $G_k^+$ and $J_k$ on $M$ are trivial. Then, by relations $[J_k, G_k^+] = -G_k^-$ and $[G_0^+, G_k^-] = (\partial + 2(k + p)\lambda)J_k + 2L_k$, we see that the actions of $G_k^+$ and $L_k$ on $M$ are also trivial. This contradicts to the assumption that the action of $\mathfrak{h}(p)k$ is nontrivial.

Next, we assume that $k = -p$ (and thus $p$ is a negative integer). Considering the action of $[J_k, G_0^-]_{\lambda + \mu} = -G_k^-_{\lambda + \mu}$ on $M$, by Lemmas 9 and 10, we obtain
\[ G_k^- \lambda v_0^{(2)} = G_k^- \lambda v_0^{(2)} = 0, \quad G_k^- \lambda v_0^{(2)} = -\beta v_0^{(2)}, \quad G_k^- \lambda v_0^{(2)} = -\beta v_0^{(2)}. \] (51)

Then, applying $[G_0^+, G_k^-]_{\lambda + \mu} = -p(\lambda + \mu)J_k_{\lambda + \mu} + 2L_k_{\lambda + \mu}$ on $M$, by Lemmas 9 and 10, we obtain
\[ L_k \lambda v_0^{\ell} = -\frac{\sqrt{p}}{2} \beta v_0^{\ell} + \delta_{s,0} \delta_{\ell,2} \lambda v_0^{(1)}, \quad s \in \mathbb{Z}/2\mathbb{Z}, \quad \ell = 1, 2. \] (52)

If $p \leq -2$, then $k \geq 2$. Following the arguments for case $k + p \neq 0$, one can show that the action of $\mathfrak{h}(p)k_{-1}$ on $M$ is trivial. Then, by relations
\[ [L_1, L_{k-1}] = ((1 + p)\partial + p\lambda)L_k, \]
\[ [L_1, J_{k-1}] = (1 + p)\partial J_k, \]
\[ [L_1, G_{k-1}^\pm] = ((1 + p)\partial + \frac{1}{2}p\lambda)G_k^\pm, \]
we see that the action of $\mathfrak{h}(p)k$ on $M$ is also trivial, a contradiction. Hence, $p = -1$, and by Lemmas 8–10, Eqs. 51 and 52, we have Eq. 31. This completes the proof.

4.4 Composition Factors

Let $M$ be a conformal module over $\mathfrak{h}(p)$. In this subsection, we use $\tilde{M}$ to denote the same module with reversed parity. Next, we determine the composition factors of all the free conformal $\mathfrak{h}(p)$-modules of small rank obtained in Theorems 2 and 3. In fact, for rank
(1 + 1) \( \mathcal{R}(p) \)-modules in Theorem 2, one can easily give their simplicities and composition factors.

**Proposition 3** Let \( M \) be a conformal \( \mathcal{R}(p) \)-module in Theorem 2.

1. If \( M \cong V_{\Delta, \alpha}^{(1)} \), then \( M \) is simple if and only if \( \Delta \neq 0 \). Furthermore, \( V_{0, \alpha}^{(1)} \) contains a unique nontrivial submodule \( \mathbb{C}[\delta](\delta + \alpha)v_0 \oplus \mathbb{C}[\delta](\frac{1}{2}v_1) \cong V_{\frac{1}{2}, \alpha}^{(2)} \), and the quotient is an even trivial module \( \zeta_{\frac{1}{2}} \).

2. If \( M \cong V_{\Delta, \alpha}^{(2)} \), then \( M \) is simple if and only if \( \Delta \neq 0 \). Furthermore, \( V_{0, \alpha}^{(2)} \) contains a unique nontrivial submodule \( \mathbb{C}[\delta](\delta + \alpha)v_0 \oplus \mathbb{C}[\delta](\frac{1}{2}v_1) \cong V_{\frac{1}{2}, \alpha}^{(1)} \), and the quotient is an even trivial module \( \zeta_{-\alpha} \).

In addition, the above conclusions still hold if we reverse the parity of all \( \mathcal{R}(p) \)-modules.

Nevertheless, for rank \((2+2)\) \( \mathcal{R}(p) \)-modules in Theorem 3, the situation is not so obvious.

**Proposition 4** Let \( M \) be a conformal \( \mathcal{R}(p) \)-module in Theorem 3.

1. If \( p \neq -1 \) and \( M \cong V_{\Delta, \Lambda, \alpha} \), then \( M \) is simple if and only if \( 2\Delta + \Lambda \neq 0 \). Furthermore,
   - \( V_{\Delta, -2\Delta, \alpha} \) contains a submodule isomorphic to \( \overline{V_{\Delta+1, \alpha}}^{(1)} \), and the quotient is isomorphic to \( V_{\Delta, \alpha}^{(1)} \).
   - \( V_{\Delta, 2\Delta, \alpha} \) contains a submodule isomorphic to \( \overline{V_{\Delta-1, \alpha}}^{(2)} \), and the quotient is isomorphic to \( V_{\Delta, \alpha}^{(2)} \).

2. If \( p = -1 \) and \( M \cong V_{\Delta, \Lambda, \alpha, \beta} \), then \( M \) is simple if and only if \( (2\Delta + \Lambda, \beta) \neq (0, 0) \). Furthermore,
   - \( V_{\Delta, -2\Delta, \alpha, 0} \) contains a submodule isomorphic to \( \overline{V_{\Delta+1, \alpha}}^{(1)} \), and the quotient is isomorphic to \( V_{\Delta, \alpha}^{(1)} \).
   - \( V_{\Delta, 2\Delta, \alpha, 0} \) contains a submodule isomorphic to \( \overline{V_{\Delta-1, \alpha}}^{(2)} \), and the quotient is isomorphic to \( V_{\Delta, \alpha}^{(2)} \).

In addition, the above conclusions still hold if we reverse the parity of all \( \mathcal{R}(p) \)-modules. All composition factors of \( M \) have multiplicity one. The composition factors of \( V_{\Delta, \pm 2\Delta, \alpha} \) and \( V_{\Delta, \pm 2\Delta, \alpha, 0} \) are listed in Table 3 in the Introduction.

**Proof** (1) First, recall that \( V_{\Delta, \Lambda, \alpha} \) is a trivial extension of the \( K_2 \)-module \( K_{\Delta, \Lambda, \alpha} \). By Remark 1, \( M \) is simple if and only if \( 2\Delta + \Lambda \neq 0 \).

If \( 2\Delta + \Lambda = 0 \), then \( M \cong V_{\Delta, -2\Delta, \alpha} \). Let \( v_0 = v_0^{(2)} \), \( v_1 = v_1^{(2)} \). By Eq. 29 with \( \Lambda = -2\Delta \), it is easy to observe that \( M_1 = \mathbb{C}[\delta]v_0 \oplus \mathbb{C}[\delta]v_1 \) is a submodule isomorphic to \( \overline{V_{\Delta-1, \alpha}}^{(2)} \), and \( M/M_1 \cong V_{\Delta, \alpha}^{(1)} \). By the first conclusion of Proposition 3(1), if \( \Delta \neq 0 \), \(-\frac{1}{2}\), then both \( M_1 \)
and \( M/M_1 \) are irreducible. If \( \Delta = 0 \), then \( M_1 \cong V^{(1)}_{\frac{1}{2},\alpha} \) is irreducible, while \( M/M_1 \cong V^{(1)}_{0,\alpha} \) has composition factors \( \overline{V}^{(2)}_{\frac{1}{2},\alpha} \) and \( \mathbb{C} c_{-\alpha} \) by the second conclusion of Proposition 3(1). If \( \Delta = -\frac{1}{2} \), then \( M/M_1 \cong V^{(1)}_{-\frac{1}{2},\alpha} \) is irreducible, while \( M_1 \cong \overline{V}^{(1)}_{0,\alpha} \) has composition factors \( V^{(2)}_{\frac{1}{2},\alpha} \) and \( \mathbb{C} c_{-\alpha} \) by the parity reverse version of Proposition 3(1).

If \( 2\Delta - \Lambda = 0 \), then \( M \cong V_{\Delta,\pm2\Delta,\alpha} \). Although any trivial combination of \( v_0^{(1)}, v_0^{(2)}, v_1^{(1)}, v_1^{(2)} \) is not a rank \( (1 + 1) \) submodule of \( \mathfrak{R}(p) \), we have a feeling that there should be certain symmetry between cases \( 2\Delta - \Lambda = 0 \) and \( 2\Delta + \Lambda = 0 \). So, let us consider the structure of \( V^{(2)}_{\Delta + \frac{1}{2},\alpha} \) (cf. Eq. 28):

\[
\begin{align*}
L_{0,\lambda} v_1 &= p(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v_1, \\
J_{0,\lambda} v_1 &= (2\Delta + 1)v_1, \\
G_{0,\lambda}^+ v_1 &= 0, \\
G_{0,\lambda}^- v_1 &= \sqrt{p}v_0, \\
X_i \lambda v_s &= 0,
\end{align*}
\]

where \( X = L, J, G^\pm, i \geq 1, s \in \mathbb{Z}/2\mathbb{Z} \). Comparing the above with Eq. 29 with \( \Lambda = 2\Delta \) (especially the action of \( G_{0}^- \)), we naturally set \( v_0 = 2(\partial + \alpha)v_0^{(1)} - v_0^{(2)}, v_1 = v_1^{(1)} \). Then, one can check that \( M_2 = \mathbb{C}[\partial]v_0 \oplus \mathbb{C}[\partial]v_1 \) is a submodule isomorphic to \( V^{(2)}_{\Delta + \frac{1}{2},\alpha} \) and \( M/M_2 \cong V^{(2)}_{\Delta,\alpha} \). As in case \( 2\Delta + \Lambda = 0 \), the composition factors of \( M \) can be determined by considering the simplicities of \( M_2 \) and \( M/M_2 \) by Proposition 3(2) or its parity reverse version.

(2) If \( \beta = 0 \), then the arguments are the same as those in (1). We only need to note that if \( \beta \neq 0 \), then \( M \) is irreducible. If this is not true, we may assume that \( M' \) is a submodule of \( M \). If \( M' \) has rank \( (2 + 2) \), then \( M/M' \) is a trivial module. One can easily derive a contradiction from Eq. 31. If \( M' \) has rank \( (1 + 1) \), then \( M/M' \) also has rank \( (1 + 1) \). This is also impossible, since the action of \( \mathfrak{R}(p)_1 \) on any rank \( (1 + 1) \) module is trivial by Theorem 2.

Remark 3 Recall that \( K_2 \cong \mathfrak{R}(p)_{(0)} \) (cf. Eq. 11) can be viewed as a quotient algebra of \( \mathfrak{R}(p) \). Hence, the classification of composition factors of reducible \( \mathfrak{R}(p) \)-modules \( V^{(1)}_{0,\alpha}, V^{(2)}_{0,\alpha}, V_{\Delta,\pm2\Delta,\alpha} \) in Proposition 3 and Proposition 4(1), is also true for reducible \( K_2 \)-modules \( K^{(1)}_{0,\alpha}, K^{(2)}_{0,\alpha}, K_{\Delta,\pm2\Delta,\alpha} \) (one only need to replace the symbols \( V \)’s by \( K \)’s in all the involved modules).

5 Classification Theorems

5.1 Main Result

Our main result is the following.

Theorem 4 Let \( M \) be a nontrivial FICM over \( \mathfrak{R}(p) \).
Lemma 11 Let $\mathcal{L}$ be a Lie superalgebra with a descending sequence of subspaces $\mathcal{L} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \ldots$ and an element $T$ satisfying $[T, \mathcal{L}_n] = \mathcal{L}_{n-1}$ for $n \geq 1$. Let $V$ be an $\mathcal{L}$-module and let $V_n = \{v \in V \mid \mathcal{L}_n v = 0\}, \ n \in \mathbb{Z}_+$. Suppose that $V_n \neq 0$ for $n \gg 0$, and that the minimal $N \in \mathbb{Z}_+$ for which $V_N \neq 0$ is positive. Then $\mathbb{C}[T]V_N = \mathbb{C}[T] \otimes_{\mathbb{C}} V_N$. In particular, $V_N$ is finite-dimensional if $V$ is a finitely generated $\mathbb{C}[T]$-module.

Lemma 12 The conformal $\mathfrak{R}(p)$-module $M$ must be of free rank $(1+1)$ or $(2+2)$.

Proof Based on technical results prepared in Sections 3 and 4 (especially Theorem 1), one can safely generalize the arguments in our previous papers [15, 16] to the case here. For completeness, we still write down the details.

Note first that, by Lemma 7, the $\lambda$-actions of $L_i$, $J_i$, $G_i^{\pm}$ on $M$ are trivial for $i \gg 0$. Suppose that $k \in \mathbb{Z}_+$ is the largest integer such that the $\lambda$-action of $\mathfrak{R}(p)_k$ (with $\mathbb{C}[\partial]$-basis $(L_k, J_k, G_k^{\pm})$) on $M$ is nontrivial. Then $M$ is simply a nontrivial FICM over $\mathfrak{R}(p)_k$, where $\mathfrak{R}(p)_k$ is defined by Eq. 11. Furthermore, by Proposition 1, $M$ can be viewed as a module over the Lie superalgebra $\mathcal{L} = \mathcal{A}(\mathfrak{R}(p)_k)^\mathfrak{e}$ satisfying

$$\tilde{L}_{i,m} v = \tilde{J}_{i,n} v = \tilde{G}_{i,t}^{\pm} v = 0 \quad \text{for} \quad v \in M, \ 0 \leq i \leq k, \ 0 \ll m, n \in \mathbb{Z}, \ \frac{1}{2} \ll t \ll \frac{1}{2} + \mathbb{Z}. \ (53)$$

For $z \in \mathbb{Z}_+$, let

$$\mathcal{L}_z = \text{span}_{\mathbb{C}} \{\tilde{L}_{i,m}, \tilde{J}_{i,n}, \tilde{G}_{i,t}^{\pm} \in \mathcal{L}\mid 0 \leq i \leq k, \ z-1 \leq m \in \mathbb{Z}, z \leq n \in \mathbb{Z}, z-\frac{1}{2} \leq t \ll \frac{1}{2} + \mathbb{Z}\}.$$

Then $\mathcal{L}_0 = \mathcal{A}(\mathfrak{R}(p)_k)$ and $\mathcal{L} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \ldots$. By the definition of extended annihilation superalgebra, we see that the element $T \in \mathcal{L}$ satisfies $[T, \mathcal{L}_z] = \mathcal{L}_{z-1}$ for $z \geq 1$. Let

$$M_z = \{v \in M \mid \mathcal{L}_z v = 0\}, \ z \in \mathbb{Z}_+.$$

By Eq. 53, $M_z \neq 0$ for $z \gg 0$. Assume that $N \in \mathbb{Z}_+$ is the smallest integer such that $M_N \neq \emptyset$.

Similar to our previous results for $\mathfrak{B}(p)$ [15] and $\mathfrak{S}(p)$ [16], the case $N = 0$ is impossible.

Next, consider the case $N \geq 1$. By the definition of extended annihilation superalgebra and the shift used in the proof of Lemma 2, we have that $T - \frac{1}{p} \tilde{L}_{0,-1} \in \mathcal{L}$ is an even central
element, and so $T - \frac{1}{p} \tilde{L}_{0,-1}$ acts on $M$ as a scalar. Therefore, $\mathcal{L}_0$ acts irreducibly on $M$. Furthermore, since

$$\tilde{L}_{i,-1} = \frac{1}{p} [\tilde{L}_{i,0}, \tilde{L}_{0,-1}], \quad \tilde{J}_{i,0} = \frac{1}{p} [\tilde{J}_{i,1}, \tilde{L}_{0,-1}], \quad \tilde{G}_{i,-\frac{1}{2}} = \frac{1}{p} [\tilde{G}_{i,\frac{1}{2}}, \tilde{L}_{0,-1}],$$

we see that the action of $\mathcal{L}_0$ is determined by $\mathcal{L}_1$ and $\tilde{L}_{0,-1}$ (or equivalently, determined by $\mathcal{L}_1$ and $T$). Note that $M_N$ is $\mathcal{L}_1$-invariant. By the irreducibility of $M$ and Lemma 11, we see that $M = \mathbb{C}[T] \otimes \mathbb{C} M_N$ and $M_N$ is a nontrivial irreducible finite-dimensional $\mathcal{L}_1$-module.

If $N = 1$, by definition we see that $M_1$ is in fact a trivial $\mathcal{L}_1$-module, a contradiction.

If $N \geq 2$, by definition we see that $M_N$ can be viewed as a $\mathcal{L}_1/\mathcal{L}_N$-module. Note that $\mathcal{L}_1/\mathcal{L}_N \cong \mathfrak{g}(k, N - 2)$. By Theorem 1, we must have that the dimension of $M_N$ is either $(1|1)$ or $(2|2)$. Equivalently, as a conformal $\mathfrak{g}(p)$-module, $M$ is free of rank $(1 + 1)$ or $(2 + 2)$.

5.2 Applications

Recall definition Eq. 12 of $\mathfrak{f}(n)$, we have $\mathfrak{f}(n) = \langle \tilde{L}_i, \tilde{J}_i, \tilde{G}^\pm_i | 0 \leq i \leq n \rangle$ with the following nontrivial $\lambda$-brackets ($i + j \leq n$):

$$[\tilde{L}_i, \tilde{J}_j] = ((i - n)\lambda + (i + j - 2n)\lambda) \tilde{L}_{i+j},$$

$$[\tilde{L}_i, \tilde{G}^+_j] = ((i - n)\lambda + (i + j - n)\lambda) \tilde{G}^+_j,$$

$$[\tilde{J}_i, \tilde{G}^+_j] = \pm \tilde{G}^+_j,$$

$$[\tilde{G}^+_i, \tilde{J}_j] = ((2i - n)\lambda + 2(i + j - n)\lambda) \tilde{J}_{i+j} + 2 \tilde{L}_{i+j}.$$

Clearly, the following two $\mathbb{C}[\partial]$-modules are rank $(1 + 1)$ conformal modules over $\mathfrak{f}(n)$ (here, we adopt the same notations as in (I-1) and (I-2) for $\mathfrak{g}(p)$, see Section 4.2).

(i-1) $V^{(1)}_{\Delta, \alpha} = \mathbb{C}[\partial] v_0 \oplus \mathbb{C}[\partial] v_1$ with

$$\begin{cases}
\tilde{L}_0 v_0 = -n(\theta + \Delta \lambda + \alpha) v_0, & \tilde{L}_0 v_1 = -n(\theta + (\Delta + \frac{1}{2}) \lambda + \alpha) v_1, \\
\tilde{J}_0 v_0 = -2\Delta v_0, & \tilde{J}_0 v_1 = (1 - 2\Delta) v_1, \\
\tilde{G}^+_0 v_0 = -\sqrt{-n} v_1, & \tilde{G}^+_0 v_1 = 0, \\
\tilde{G}^-_0 v_0 = 0, & \tilde{G}^-_0 v_1 = 2\sqrt{n}(\theta + 2\Delta \lambda + \alpha) v_0, \\
\tilde{X}_i v_s = 0, & 1 \leq i \leq n, s \in \mathbb{Z}/2\mathbb{Z}.
\end{cases}$$

where $X = L, J, G^\pm$ and $\Delta, \alpha \in \mathbb{C}$;

(ii-2) $V^{(2)}_{\Delta, \alpha} = \mathbb{C}[\partial] v_0 \oplus \mathbb{C}[\partial] v_1$ with

$$\begin{cases}
\tilde{L}_0 v_0 = -n(\theta + \Delta \lambda + \alpha) v_0, & \tilde{L}_0 v_1 = -n(\theta + (\Delta + \frac{1}{2}) \lambda + \alpha) v_1, \\
\tilde{J}_0 v_0 = 2\Delta v_0, & \tilde{J}_0 v_1 = (2\Delta - 1) v_1, \\
\tilde{G}^+_0 v_0 = 0, & \tilde{G}^+_0 v_1 = 2\sqrt{-n}(\theta + 2\Delta \lambda + \alpha) v_0, \\
\tilde{G}^-_0 v_0 = -\sqrt{-n} v_1, & \tilde{G}^-_0 v_1 = 0, \\
\tilde{X}_i v_s = 0, & 1 \leq i \leq n, s \in \mathbb{Z}/2\mathbb{Z}.
\end{cases}$$

where $X = L, J, G^\pm$ and $\Delta, \alpha \in \mathbb{C}$;
The following $V_{\Delta,\Lambda,\alpha}$ is a rank $(2+2)$ conformal module over $\mathfrak{l}(n)$. One can check that the following $V_{\Delta,\Lambda,\alpha,\beta}$ is a more general rank $(2+2)$ conformal module over $\mathfrak{l}(1)$. (here, we adopt the same notations as in (II-1) for $\mathfrak{h}(n)$ and (II-2) for $\mathfrak{h}(1)$, see Section 4.3.)

(ii-1) \[ V_{\Delta,\Lambda,\alpha} = \mathbb{C}[\partial]v_{\partial}^{(1)} \oplus \mathbb{C}[\partial]v_{\partial}^{(2)} \oplus \mathbb{C}[\partial]v_{\partial}^{(1)} \oplus \mathbb{C}[\partial]v_{\partial}^{(2)} \]

\[ (\ell) \]

\[ \begin{align*}
\tilde{L}_{\lambda} v_{\partial}^{(1)} &= - \sqrt{\frac{\lambda}{n}} v_{\partial}^{(1)}, \\
\tilde{L}_{\lambda} v_{\partial}^{(2)} &= - \sqrt{\frac{\lambda}{n}} v_{\partial}^{(2)}, \\
\tilde{J}_{\lambda} v_{\partial}^{(1)} &= 0, \\
\tilde{J}_{\lambda} v_{\partial}^{(2)} &= 0, \\
\tilde{G}_{\lambda} v_{\partial}^{(1)} &= 0, \\
\tilde{G}_{\lambda} v_{\partial}^{(2)} &= 0,
\end{align*} \]

\[(\ell)\]

where $X = L, J, G^\pm$ and $\Delta, \Lambda, \alpha \in \mathbb{C}$;

(ii-2) \[ V_{\Delta,\Lambda,\alpha,\beta} = \mathbb{C}[\partial]v_{\partial}^{(1)} \oplus \mathbb{C}[\partial]v_{\partial}^{(2)} \oplus \mathbb{C}[\partial]v_{\partial}^{(1)} \oplus \mathbb{C}[\partial]v_{\partial}^{(2)} \] with Eq. 56 and

\[ (\ell) \]

\[ \begin{align*}
\tilde{L}_{\lambda} v_{\partial}^{(1)} &= - \sqrt{\frac{\lambda}{n}} v_{\partial}^{(1)}, \\
\tilde{L}_{\lambda} v_{\partial}^{(2)} &= - \sqrt{\frac{\lambda}{n}} v_{\partial}^{(2)}, \\
\tilde{J}_{\lambda} v_{\partial}^{(1)} &= 0, \\
\tilde{J}_{\lambda} v_{\partial}^{(2)} &= 0, \\
\tilde{G}_{\lambda} v_{\partial}^{(1)} &= 0, \\
\tilde{G}_{\lambda} v_{\partial}^{(2)} &= 0,
\end{align*} \]

\[(\ell)\]

where $\Delta, \Lambda, \alpha, \beta \in \mathbb{C}$.

Since $\mathfrak{l}(n)$ is a quotient algebra of $\mathfrak{h}(n)$ (cf. Eqs. 7 and 12), by Theorems 2 and 3, Propositions 3 and 4, we have

**Corollary 1** Let $M$ be a nontrivial free conformal module of rank $(1+1)$ or $(2+2)$ over $\mathfrak{l}(n)$. 

1. If $n > 1$, then, up to parity change, $M$ is isomorphic to $V_{\Delta,\alpha}^{(1)}$ or $V_{\Delta,\alpha}^{(2)}$ for some $\Delta, \alpha \in \mathbb{C}$, or $V_{\Delta,\Lambda,\alpha}$ for some $\Delta, \Lambda, \alpha \in \mathbb{C}$.

2. If $n = 1$, then, up to parity change, $M$ is isomorphic to $V_{\Delta,\alpha}^{(1)}$ or $V_{\Delta,\alpha}^{(2)}$ for some $\Delta, \alpha \in \mathbb{C}$, or $V_{\Delta,\Lambda,\alpha,\beta}$ for some $\Delta, \Lambda, \alpha, \beta \in \mathbb{C}$.

Furthermore, for the above modules we have the same assertions on simplicities and composition factors as those for $\mathfrak{h}(n)$-modules in Propositions 3 and 4.

Furthermore, by Theorem 4, we have that
Corollary 2 The irreducible modules in Corollary 1 exhaust all nontrivial finite irreducible conformal modules over $\mathfrak{f}(n)$.

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Declarations

Conflicts of Interest/Competing Interests Not applicable.

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