A Note on Power Instability of Linear Discrete-time Systems in Banach Spaces

Ioan-Lucian Popa

Abstract. In this paper we investigate the power instability properties and give necessary and sufficient conditions for the concepts of uniform power instability, power instability and strong power instability for linear discrete-time system $x_{n+1} = A(n)x_n$ in Banach spaces.

AMS Subject Classification (2000). 39A11; 34D05

Keywords. Linear discrete-time systems, uniform power instability, (strong) power instability.

1 Introduction and Preliminaries

The qualitative theory of difference equations is in a process of continuous development in the past decades (see, e.g. [5], [1], [3], [4] and the references therein). An important result in the theory of linear discrete-time systems have been proved by R.K. Przyluski and S. Rolewicz in [7] for the concept of uniform power stability. In our previous paper [6] we obtained characterizations for uniform an nonuniform exponential stability concepts for linear discrete-time systems. Other results, concerning nonuniform exponential stability concepts have been studied by L. Barreira and C. Valls in [2]. Recently, new concepts of instability have been introduced and studied (see [9] for semigroups of operators, [10], [11] for evolution operators, [8] for linear skew-product flows).
In this paper we present the concept of uniform power instability and two nonuniform concepts (power instability and strong power instability) for linear discrete-time systems. Our main objectives are to establish relations between these concepts and to offer generalizations of R.K. Przyluski type theorem for these concepts.

Let us first introduce the notation used in this note. Let $X$ be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded and linear operators from $X$ into itself. The norm on $X$ and in $\mathcal{B}(X)$ will be denoted by $\| \cdot \|$. The set of all positive integers will be denoted by $\mathbb{N}$, $\Delta$ denotes the set of all pairs $(m, n)$ of positive integers satisfying the inequality $m \geq n$. We also denote by $T$ the set of all triplets $(m, n, p)$ of positive integers with $(m, n)$ and $(n, p) \in \Delta$.

We consider the linear discrete-time system:

$$x_{n+1} = A(n)x_n,$$

where $A : \mathbb{N} \to \mathcal{B}(X)$ is a given $\mathcal{B}(X)-$valued sequence. For $(m, n) \in \Delta$ we denote:

$$A^n_m = \begin{cases} A(m) \cdots A(n+1), & m > n \\ I, & m = n \end{cases}.$$  \hspace{1cm} (1.1)

**Definition 1.1.** The linear discrete-time system (A) is said to be uniformly power instable (and denote u.p.is.) if there are some constants $N \geq 1$ and $r \in (0, 1)$ such that:

$$\| A^n_p x \| \leq Nr^{m-n} \| A^p_m x \|, \text{ for all } (m, n, p, x) \in T \times X.$$  

**Definition 1.2.** The linear discrete-time system (A) is said to be:

i) power instable (and denote p.is.) if there are some constants $N \geq 1$, $r \in (0, 1)$ and $s \geq 1$ such that:

$$\| A^n_p x \| \leq Nr^{m-n}s^n \| A^p_m x \|, \text{ for all } (m, n, p, x) \in T \times X.$$  \hspace{1cm} (1.2)

ii) strongly power instable (and denote s.p.is.) if there are some constants $N \geq 1$, $r \in (0, 1)$ and $s \in \left[ 1, \frac{1}{r} \right)$ such that:

$$\| A^n_p x \| \leq Nr^{m-n}s^n \| A^p_m x \|, \text{ for all } (m, n, p, x) \in T \times X.$$  \hspace{1cm} (1.3)
2 Main results

From the previous definitions it follows that \((u.p.is \implies s.p.is. \implies p.is.)\) The next example illustrate the difference between the concepts of (strong) power instability and uniform power instability for linear discrete-time system \((\mathcal{A})\). We remark that the concepts of power instability and strong power instability are much weaker behaviors in comparison with the classical concept of uniform power instability. A principal motivation for weakening the assumption is that almost all variational equations in a finite dimensional spaces have a nonuniform exponential behavior.

Example 2.1. Let \(X = \mathbb{R}\) be a Banach space, \(c > 0\), \((a_n)_n \subset (0, \infty)\) and \((A_n)_n \subset \mathcal{B}(X)\) defined for all \(n \in \mathbb{N}\) by \(A_n = c \cdot a_n I\), where

\[
a_n = \begin{cases} 2^{-n} & \text{if } n = 2k \\ 2^{n+1} & \text{if } n = 2k + 1 \end{cases}
\]

The following statements are true:

i) \((\mathcal{A})\) is not uniformly power instable;

ii) \((\mathcal{A})\) is power instable if and only if \(c > 1\);

iii) \((\mathcal{A})\) is strongly power instable if and only if \(c > 2\).

Let \((m, n, x) \in \Delta \times X\). According to (1.1) we have that:

\[
\mathcal{A}_m^n x = \begin{cases} c^{m-n} a_{mn} x & m > n \\ x & m = n \end{cases},
\]

where

\[
a_{mn} = \begin{cases} 1 & \text{if } m = 2q \text{ and } n = 2p \\ 2^{-n-1} & \text{if } m = 2q \text{ and } n = 2p + 1 \\ 2^{n+1} & \text{if } m = 2q + 1 \text{ and } n = 2p \\ 2^{-m-n} & \text{if } m = 2q + 1 \text{ and } n = 2p + 1 \end{cases}
\]

(i) If we suppose that \((\mathcal{A})\) is u.p.is. then there exist some constants \(N \geq 1\) and \(r \in (0, 1)\) such that

\[
\| x \| \leq N r^{m-n} \| \mathcal{A}_m^n x \| = N (rc)^{m-n} a_{mn} \| x \|,
\]
for all \((m, n, x) \in \Delta \times X\) which is equivalent with

\[
\begin{cases}
\left( \frac{1}{rc} \right)^{m-n} \leq N & \text{if } m = 2q \text{ and } n = 2p \\
\left( \frac{1}{rc} \right)^{m-n} 2^{n+1} \leq N & \text{if } m = 2q \text{ and } n = 2p + 1 \\
\left( \frac{1}{rc} \right)^{m-n} 2^{-m-1} \leq N & \text{if } m = 2q + 1 \text{ and } n = 2p \\
\left( \frac{1}{rc} \right)^{m-n} 2^{-m+n} \leq N & \text{if } m = 2q + 1 \text{ and } n = 2p + 1.
\end{cases}
\]

There are two cases that can be considered at this point. If \(c \in (0, 1]\) then for all \(r \in (0, 1), m = 2q\) and \(n = 2p \in \mathbb{N}\) fixed we have that

\[
\lim_{q \to \infty} \left( \frac{1}{rc} \right)^{2q - 2p} = \infty. \tag{2.1}
\]

If \(c \in (1, \infty)\), \(r \in (0, 1), n = 2p + 1\) and \(m = n + 1\) it follows that

\[
\lim_{p \to \infty} \left( \frac{1}{rc} \right)^{2p+2} = \infty. \tag{2.2}
\]

According to (2.1) and (2.2) we can conclude that \((\mathfrak{A})\) cannot be u.p.is.

(ii) If \((\mathfrak{A})\) is power instable then there exist some constants \(N \geq 1, r \in (0, 1)\) and \(s \geq 1\) such that:

\[
\| x \| \leq N r^{m-n} s^n \| A_m^nx \| = N (rc)^{m-n} s^n a_{mn} \| x \|,
\]

for all \((m, n, x) \in \Delta \times X\) which is equivalent with

\[
\begin{cases}
\left( \frac{1}{rc} \right)^{m-n} s^{-n} \leq N & \text{if } m = 2q \text{ and } n = 2p \\
\left( \frac{1}{rc} \right)^{m-n} s^{-n} 2^{n+1} \leq N & \text{if } m = 2q \text{ and } n = 2p + 1 \\
\left( \frac{1}{rc} \right)^{m-n} s^{-n} 2^{-m-1} \leq N & \text{if } m = 2q + 1 \text{ and } n = 2p \\
\left( \frac{1}{rc} \right)^{m-n} s^{-n} 2^{-m+n} \leq N & \text{if } m = 2q + 1 \text{ and } n = 2p + 1.
\end{cases} \tag{2.3}
\]

If \(\frac{1}{rc} > 1\), then for \(m = 2q\) and \(n = 2\) we obtain:

\[
\lim_{q \to \infty} \left( \frac{1}{rc} \right)^{2q-2} \left( \frac{1}{s} \right)^2 = \infty. \tag{2.4}
\]
If we consider \( \frac{1}{rc} \leq 1 \) then for \( s \geq 2 \) and \( N \geq 2 \) conditions (2.3) are fulfilled for all \((m, n) \in \Delta\).

Reciprocally, we suppose that \( c > 1 \). In this case we consider \( N \geq 2 \), \( r \in (0, \frac{1}{c}] \) and \( s \geq 2 \) thus (2.3) is verified for all \((m, n) \in \Delta\). Hence, for \( c > 1 \), \((A)\) is p.is.

(iii) If we suppose that \((A)\) is s.p.is. then there exist some constants \( N \geq 1 \), \( r \in (0, 1) \) and \( s \in [1, \frac{1}{r}] \) such that for all \((m, n) \in \Delta\) relation (2.3) to be true. In a similar way as we proved (ii) we obtain that \( c \geq \frac{1}{r} > s \geq 2 \). If \( 0 < c \leq 2 \), then for \( m = 2q \) and \( n = 2 \) we obtain (2.4).

Reciprocally, we consider \( c > 2 \). In this case we consider \( N \geq 2 \), \( s = 2 \) and \( r \in \left[ \frac{1}{c}, \frac{1}{2} \right) \) thus (2.3) is verified for all \((m, n) \in \Delta\). Hence, for \( c > 2 \), \((A)\) is s.p.is.

**Theorem 2.1.** The linear discrete-time system \((A)\) is power instable if and only if there exist some constants \( D \geq 1 \), \( d > 1 \) and \( c \geq 1 \) with \( c \in [1, d) \) such that:

\[
\sum_{k=n}^{m} d^{m-k} \| A^n_k x \| \leq Dc^m \| A^m_n x \|, \text{ for all } (m, n, x) \in \Delta \times X. \tag{2.5}
\]

**Proof. Necessity.** According to Definition 1.2 there are some constants \( N \geq 1 \), \( r \in (0, 1) \) and \( s \geq 1 \) such that:

\[
\| A^n_k x \| \leq Nr^{m-k} s^k \| A^m_n x \|, \text{ for all } (m, k, n, x) \in T \times X.
\]

Then for any \( d > \frac{s}{r} \) and all \((n, x) \in N \times X\) we have:

\[
\begin{align*}
\sum_{k=n}^{m} d^{m-k} \| A^n_k x \| & \leq N \| A^m_n x \| \sum_{k=n}^{m} d^{m-k} r^{m-k} s^k = \\
& = N (dr)^m \| A^m_n x \| \sum_{k=n}^{m} \left( \frac{s}{dr} \right)^k \\
& = \frac{Nd}{dr - s} (dr)^m \| A^m_n x \|.
\end{align*}
\]

**Sufficiency.** Using inequality (2.5) for all \((m, n, x) \in \Delta \times X\) we have that:

\[
d^{m-n} \| x \| \leq \sum_{k=n}^{m} d^{m-k} \| A^n_k x \| \leq Dc^m \| A^m_n x \|.
\]
Hence,
\[ \| x \| \leq D c^n \left( \frac{c}{d} \right)^{m-n} \| A^n_{m} x \|. \]

If we consider \( y = A^n_{p} x \) we obtain
\[ \| A^n_{p} y \| \leq D c^n \left( \frac{c}{d} \right)^{m-n} \| A^n_{m} y \|, \]
for all \((m, n, p, y) \in T \times X\) which proves that (2) is p.is. \( \square \)

**Proposition 2.2.** The linear discrete-time system \((\mathfrak{A})\) is strongly power unstable if and only if there exist some constants \( D \geq 1, d > 1 \) and \( c \geq 1 \) with \( 1 \leq 2c < d \) such that:
\[ \sum_{k=n}^{m} d^{m-k} \| A^n_{k} x \| \leq D c^m \| A^n_{m} x \|, \quad \text{for all } (m, n, x) \in \Delta \times X. \tag{2.6} \]

**Proof.** It is similar with the proof of the Theorem 2.1. \( \square \)

**Corollary 2.3.** The linear discrete-time system \((\mathfrak{A})\) is uniformly power unstable if and only if there exist some constants \( D \geq 1, d > 1 \) such that:
\[ \sum_{k=n}^{m} d^{m-k} \| A^n_{k} x \| \leq D \| A^n_{m} x \|, \quad \text{for all } (m, n, x) \in \Delta \times X. \tag{2.7} \]

**Remark 2.1.** Theorem 2.1, Proposition 2.2 and Corollary 2.3 are generalizations of Przyluski type theorems for the concepts of power instability, strongly power instability and uniform power instability.

**References**

[1] **R. P. Agarwal**, *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] **L. Barreira and C. Valls**, Nonuniform exponential contractions & Lyapunov sequences, *J. Differential Equations*, 246, (2009), 4743-4771.

[3] **S. Elaydi**, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, 3rd edition, Springer, New York, NY, USA, 2005.

[4] **M. I. Gil**, *Difference equations in Normed spaces Stability and Oscillation*, Elsevier, Nort-Holland Mathematics-Studies, 2007.

[5] **A. Halanay and D. Wexler**, *Teoria Calitativă a Sistemelor cu Impulsuri*, Editura Academiei R.S.R., București, 1968.
[6] I.-L. Popa, T. Ceauşu, and M. Megan, On exponential stability for linear discrete-time systems in Banach spaces, *Comput. Math. Appl.*, (2012.), doi:10.1016/j.camwa.2012.01.027

[7] K. M. Przyluski and S. Rolewicz, On stability of linear time-varying infinite-dimensional discrete-time systems, *Syst. Control Lett.*, 4, (1984), 307-315.

[8] M. Megan, A. L. Sasu, and B. Sasu, Exponential stability and exponential instability for linear skew-product flows, *Math. Bohem.*, 129, (2004), 225-243.

[9] M. Megan, A. L. Sasu, B. Sasu, and A. Pogan, Exponential stability and unstability of semigroups of linear operators in Banach spaces, *Math. Inequal. Appl.*, 5, (2002), 557-567.

[10] M. Megan, A. L. Sasu, and B. Sasu, Nonuniform exponential instability of evolution operators in Banach spaces, *Glas. Mat. Ser. III*, 36, (2001), 287-295.

[11] M. A. Tomescu, T. Ceauşu, and A. A. Minda, On nonuniform exponential instability of evolution operators in Banach spaces, *An. Univ. Timişoara Ser. Mat.-Inform.*, XLIX(1), (2011), 153-164

Ioan-Lucian Popa
Department of Mathematics,
Faculty of Mathematics and Computer Science,
West University of Timişoara,
V. Pârvan Blvd. No. 4,
300223-Timişoara,
Romania
E-mail: popa@math.uvt.ro

Received: 28.01.2012
Accepted: 20.06.2012