THE DUPONT HOMOTOPY FORMULA AND STELLAR SUBDIVISION

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Abstract. The Dupont homotopy, a classical construction in the algebraic topology of triangulated smooth manifolds, has been revived in the last decade in the construction of an effective field theory where it appears as a propagator. In this paper, we ask and answer a question of relevance to the renormalization group of this theory: is Dupont’s construction compatible with stellar subdivision?

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1. Introduction

Whitney realized (see the monograph [8]) that for any triangulated manifold $M$, there is a cochain map $W : C^\bullet(M) \to \Omega^\bullet(M)$ that is a section (a right inverse) to the integration map $I : \Omega^\bullet(M) \to C^\bullet(M)$. Because, by the de Rham isomorphism theorem, $I$ induces an isomorphism on cohomology, $W$ also induces an isomorphism on cohomology, so the image of $W$, the space of Whitney forms, generates the de Rham cohomology of $M$.

Several decades later, Dupont (see [2]) proved the stronger result that there is a deformation retraction of $\Omega^\bullet(M)$ onto $C^\bullet(M)$. That is, he showed there exists a homotopy $s$ between 1 and $WI$. Dupont was interested in the study of characteristic classes and constructing a universal Chern-Weil homomorphism taking an invariant polynomial on the Lie algebra of a Lie group $G$ to a cohomology class of the classifying space $BG$. Dupont used his homotopy to relate the de Rham complex of a simplicial manifold to the simplicial cochain complex of its geometric realization, the classifying space $BG$ being the simplicial manifold of interest.
More recently, Getzler \[3\] has made use of the Dupont homotopy in his study of nilpotent Lie algebras and more generally nilpotent \(L_\infty\)-algebras. He uses the Dupont homotopy to construct a space \(\gamma(g)\) that is homeomorphic to \(BG\) for \(g\) a nilpotent Lie algebra. Since \(\gamma(g)\) exists for any nilpotent \(L_\infty\)-algebra \(g\), it can be thought of as a generalized notion of classifying space.

In a different direction, Mnev \[7\] used the Dupont homotopy as a propagator for BF theory on triangulated manifolds. In his paper, the effective action, a functional on the space of Lie algebra valued simplicial cochains, is calculated for a variety of familiar topological spaces, and explicit combinatorial formulas are written down. The paper also begins the study of the gluing of Dupont homotopies. In a subsequent paper, Cattaneo, Mnev, and Reshetikhin \[1\] treat the more general setting of a cellular complex where one is required to make a noncanonical choice of a deformation retraction of \(\Omega^\bullet(M)\) onto \(C^\bullet(M)\), the space of cellular cochains. They construct the effective BF action on cellular manifolds with boundary and show that the construction is compatible with gluing of cobordisms.

A natural question to ask is whether the Dupont homotopy is compatible with stellar subdivision. By a theorem of Alexander (see Lickorish \[6\] for a modern proof), any two triangulations of a manifold with a common refinement are related by a sequence of stellar subdivisions and stellar weldings (inverse stellar subdivisions). An elementary problem in algebraic topology might be to show that the simplicial chains of any two triangulations (at least with common refinement) are homotopy equivalent. To do so, it would suffice to be able to exhibit, for a simplicial complex \(M\), a deformation retraction from the simplicial chains on a stellar subdivision \(*M\) onto the simplicial chains of \(M\). With such a deformation retraction in hand, by dualizing we in particular have the ability to include simplicial cochains on \(M\) as a subspace of simplicial cochains of its stellar subdivision \(*M\). We then ask the question, if \(M\) is a triangulated manifold, whether the Whitney forms on \(M\) are a subspace of the Whitney forms on \(*M\).

In the realm of quantum field theory on a lattice, the renormalization group should relate the physics on the lattice to the physics on any refinement of the lattice. By “integrating out” the additional degrees of freedom of the fields on the refined lattice, one recovers the physics depending on the fields on the original lattice. This is the renormalization group picture pioneered by Kadanoff and Wilson (see \[4\], Chapter 3). In the setting of simplicial BF theory the renormalization group should be interpreted as a compatibility between the effective actions associated to different triangulations, or by Alexander’s theorem, just \(M\) and its stellar subdivision \(*M\). This compatibility statement reduces to a statement about propagators: The sum of the propagator used to construct the effective action for \(*M\) (the Dupont homotopy for \(*M\)) and the propagator used to integrate out the additional degree of freedom for fields on \(*M\) (which is related to the stellar subdivision homotopy) is equal to the propagator used to construct the effective action for \(M\) (the Dupont homotopy for \(M\)). The main theorem in this paper, Theorem \[8\] gives a slightly weaker version of this statement, that is nevertheless sufficient for effective field theory.

As a broad outline of the paper, in Section \[2\] we define the integration map, the Whitney map, and the Dupont homotopy and show that for a triangulated manifold \(M\) these maps together specify a deformation retraction of \(\Omega^\bullet(M)\) onto \(C^\bullet(M)\). The exposition essentially follows the classic monograph by Dupont \[2\]
and the paper of Getzler \[3\]. Any original contributions (new proofs, correcting of errors) are noted along the way. In Section 3 we construct the Dupont homotopy for cubical chains following the tensor product construction for deformation retractions. Some of these formulas can be found in Mnev’s paper \[7\], but we derive additional properties.

In Section 4, we define a deformation retraction from simplicial (co)chains on any stellar subdivision of the \(n\)-simplex onto simplicial (co)chains on the \(n\)-simplex. The construction immediately generalizes to a triangulated manifold. In Section 5, we define the notion of cubical stellar subdivision. Employing the tensor product construction, we find a deformation retraction from cubical (co)chains on any cubical stellar subdivision of the \(n\)-cube onto the cubical (co)chains on the \(n\)-cube and more generally on any cubulated manifold \(M\).

Section 6 lays out the main results of the paper. Suppose \(M\) a triangulated manifold and \(\ast_\sigma M\) is its stellar subdivision at a simplex \(\sigma\). The Dupont deformation retraction of \(\Omega^\bullet(\ast_\sigma M)\) onto \(C^\bullet(\ast_\sigma M)\) can be composed with the stellar subdivision deformation retraction of \(C^\bullet(\ast_\sigma M)\) onto \(C^\bullet(M)\). In the main theorem of the paper, Theorem 8, we find that this composed deformation retraction is homotopic to the Dupont deformation retraction of \(\Omega^\bullet(M)\) onto \(C^\bullet(M)\). More precisely, we find the composed inclusion map is equal to Whitney map, the composed projection map is equal to the integration map, and the composed homotopy and the Dupont homotopy are cohomologous. In Section 7, we examine the cubical case of this compatibility result.

Lastly, in Sections 8 and 9 we take a different approach to constructing the stellar subdivision deformation retraction. In Section 8 we recall the elementary collapse deformation retraction (whose formula can be found in \[1\]). In Section 9 we show that a stellar subdivision can be constructed by a sequence of elementary expansions (inverse elementary collapses) followed by elementary collapses. For stellar subdivision at a \(k\)-simplex, there are \(k + 1\) such sequences. For each such sequence, composing the zigzag of elementary collapse deformation retractions is still in fact a deformation retraction. We prove that the average of these \(k + 1\) deformation retractions gives rise to a deformation retraction that is equal to our stellar subdivision deformation retraction.

### 2. Dupont Homotopy Formula

#### 2.1. Definitions.

It is important to clarify that the image of the Whitney map will not be in smooth forms. The space of smooth forms needs to be suitably extended to contain the image. The natural extension, which we shall always denote by \(\Omega^\bullet(M)\) whenever \(M\) is a triangulated manifold, is the space of piecewise smooth forms. More precisely, we specify a smooth form \(\omega_T\) for each simplex \(T\) with the compatibility condition that the pullback of \(\omega_T\) to a subsimplex \(T'\) of \(T\) is \(\omega_{T'}\).

Let

\[ \Delta^n = [e_0, \ldots, e_n] = \{ t_0 e_0 + \cdots + t_n e_n : \sum_i t_i = 1 \} \subset \mathbb{R}^{n+1} \]

denote the \(n\)-simplex. For any simplicial chain \(\alpha \in C_\bullet(\Delta^n)\), let \(\hat{\alpha} \in C^\bullet(\Delta^n)\) be its dual simplicial cochain.
Definition 1. Define the Whitney form $\omega_{i_0, \ldots, i_p} = p! \omega_{i_0, \ldots, i_p}$, a $p$-form on $\Delta^n$, where

$$\omega_{i_0, \ldots, i_p} = \sum_l (-1)^l t_{i_l} dt_{i_0} \ldots \hat{dt}_{i_l} \ldots dt_{i_n}.$$ 

for each subsimplex $[i_0, \ldots, i_p]$ in $\Delta^n$. Then define the Whitney map $W : C^\bullet(\Delta^n) \to \Omega^\bullet(\Delta^n)$ by

$$W(\hat{\alpha}) = \omega_{i_0, \ldots, i_p} = p! \omega_{i_0, \ldots, i_p}$$

for $\alpha = [i_0, \ldots, i_p]$. This is a cochain map

$$dW(\hat{\alpha}) = p! \sum_l (-1)^l dt_{i_l} dt_{i_0} \ldots \hat{dt}_{i_l} \ldots dt_{i_p} = (p + 1)! dt_{i_0} \ldots dt_{i_p},$$

where we have used the fact that

$$\sum_k \omega_{k, i_0, \ldots, i_p} = \sum_k \omega_{k, i_0, \ldots, i_p} = W(d\hat{\alpha})$$

Recall the integration map

$$I = \sum_{p=0}^n \sum_{i_0 < \cdots < i_p} [i_0, \ldots, i_p] \int_{[i_0, \ldots, i_p]}$$

is also a cochain map as a consequence of Stokes' Theorem.

We shall verify that $\int_{[i_0, \ldots, i_p]} \omega_{i_0, \ldots, i_p} = \frac{1}{p!}$ in the next section. This implies that

$$JW([i_0, \ldots, i_p] \omega_{i_0, \ldots, i_p}) = [i_0, \ldots, i_p] p! \int_{[i_0, \ldots, i_p]} \omega_{i_0, \ldots, i_p} = [i_0, \ldots, i_p].$$

That is, $JW = 1$. Dupont discovered that while $WI \neq 1$, there is a homotopy between 1 and WI.

The Dupont homotopy is expressed in terms of Whitney forms and degree $-1$ maps $h^i$ where $i$ ranges from 0 to $n$. We define the map

$$\phi^i : [0, 1] \times \Delta^k \to \Delta^k$$

by

$$\phi^i (s, \sum_j t_j e_j) = (1 - s) \sum_j t_j e_j + se_i = \sum_j ((1 - s)t_j + s\delta_{ij}) e_j.$$ 

Note that $\phi^i$ is the contraction of the simplex onto a single vertex. Now let $h^i = \pi_\ast (\phi^i)\ast$ where $\pi_\ast$ is integration along the fiber $[0, 1]$.

Definition 2. The Dupont homotopy is given by the formula

$$s = -\sum_{k=0}^{n-1} \sum_{i_0 < \cdots < i_k} \omega_{i_0, \ldots, i_k} h^{i_k} \cdots h^{i_0}$$
In the next section, where in particular we derive the basic properties of \( h^i \), we will show that
\[
(-1)^p \varepsilon^h h^{ip-1} \cdots h^{i0}(\omega_{i0}, \ldots, i_p) = \frac{1}{p!},
\]
where \( \varepsilon^h : \Omega^\bullet(\Delta^n) \to \Omega^0(\Delta^n) \to \mathbb{R} \) is evaluation at the vertex \( e_k \).

We have also claimed that
\[
\int_{[i_0, \ldots, i_p]} \omega_{i_0}, \ldots, i_p = \frac{1}{p!}.
\]

This motivates a more general statement: for any \( \omega \in \Omega^\bullet(\Delta^n) \),
\[
\int_{[i_0, \ldots, i_p]} \omega = (-1)^p \varepsilon^h h^{ip-1} \cdots h^{i0}(\omega).
\]
We shall prove this as a lemma in Section 2.3.

With \( I, s \) and \( W \) written in terms of \( h^i \) and \( \omega_{i_0}, \ldots, i_p \), the formula \( ds + sd = 1 - WI \) seems plausible. We shall prove this along with the properties \( s^2 = 0 \), \( sW = 0 \), and \( Is = 0 \) in Section 2.3. In summary:

**Theorems 1 & 2.** The Dupont homotopy gives a (special) deformation retraction of the differential forms \( \Omega^\bullet(\Delta^n) \) on the \( n \)-simplex onto the simplicial cochains \( C^\bullet(\Delta^n) \).

\[
(C^\bullet(\Delta^n), d) \xrightarrow{I} (\Omega^\bullet(\Delta^n), d) \xrightarrow{s}
\]

where \( W \) is the Whitney map, \( I \) is the integration map, and \( s \) is the Dupont homotopy.

We shall prove Theorems 1 & 2 in Section 2.3 after having thoroughly developed the properties of the Dupont homotopy. The proof we shall give of Theorems 1 & 2 differs somewhat from the one in Getzler’s paper [3]. Getzler [3] establishes that \( s \) is a gauge by direct calculation, meaning the condition \( s^2 = 0 \). We also give direct arguments for the conditions \( Is = 0 \) and \( sW = 0 \). Furthermore, our approach to the proof of Lemma 1 is perhaps more illustrative. We also correct an error in the definition of \( \phi^i \) as well as subsequent formulas in which the error is carried through.

### 2.2. Basic Properties

We compute in coordinates
\[
\begin{align*}
 h^i(fdt_{i_0} \cdots dt_{i_p}) &= \pi_\ast [f(\phi^i(s, t)) d((1-s)t_{i_0}) \cdots d((1-s)t_{i_i} + s) \cdots d((1-s)t_{i_p})] \\
 &= \pi_\ast \left[ f(\phi^i(s, t))(1-s)^p \sum_j (-1)^j (\delta_{i, i_j} - t_{i_j}) ds dt_{i_0} \cdots \widehat{dt_{i_j}} \cdots dt_{i_p} \right] \\
 &= \left( \int_0^1 (1-s)^p f(\phi^i(s, t)) ds \right) \sum_j (-1)^j (\delta_{i, i_j} - t_{i_j}) dt_{i_0} \cdots \widehat{dt_{i_j}} \cdots dt_{i_p}
\end{align*}
\]

Introduce the vector fields \( E_i \) on \( \Delta^n \) given by
\[
E_i = \sum_j (\delta_{i, j} - t_j) \partial_j.
\]
A priori, these are only vector fields on $\mathbb{R}^{n+1}$. However,

$$E_i \left( \sum_k t_k - 1 \right) = 1 - \sum_k t_k$$

so $E_i$ preserves the ideal generated by $\sum_k t_k - 1$, which implies that $E_i$ descends to a vector field on $\Delta^n$.

The relation $(\phi^i_s)^*(\delta_{i,j} - t_j) = (1 - s)(\delta_{i,j} - t_j)$ implies that

$$(\phi^i_s)^*E_i (f dt_{i_0} \cdots dt_{i_p}) = f(\phi^i(s, t))(1 - s)^{p+1} \sum_j (-1)^j(t_{i_j} - \delta_{i,i_j}) dt_{i_0} \cdots \hat{dt}_{i_j} \cdots dt_{i_p}$$

and thus:

**Proposition 1.**

$$h^i(f dt_{i_0} \cdots dt_{i_p}) = \int_0^1 \frac{ds}{1 - s} (\phi^i_s)^*E_i (f dt_{i_0} \cdots dt_{i_p}).$$

We would now like to show that $(-1)^{p \varepsilon_i} h^{i_{p-1}} \cdots h^{i_0} \omega_{i_0, \ldots, i_p} = 1/p!$.

**Proposition 2.**

$$i^*E_i(\omega_{i_0, \ldots, i_p}) = (-1)^{i+1} \omega_{i_0, \ldots, \hat{i}_i, \ldots, i_p},$$

and therefore

$$h^i(\omega_{i_0, \ldots, i_p}) = \frac{(-1)^{i+1}}{p} \omega_{i_0, \ldots, \hat{i}_i, \ldots, i_p}$$

if $i = i_l$ for some $l$ and $h^i(\omega_{i_0, \ldots, i_p}) = 0$ otherwise.

**Proof.** Let $i_{\Delta^n} : \Delta^n \to \mathbb{R}^{n+1}$ denote the inclusion map. From the definitions,

$$i^*E_i(\omega_{i_0, \ldots, i_p}) = i^*_\Delta \left( \sum_j t_{i_j} \delta_{i,j} dt_{i_0} \cdots dt_{i_p} \right)$$

$$= i^*_\Delta \left( \sum_{j,k} t_{i_j} (\delta_{i,k} - t_{i_k}) \partial_{i,k} \partial_{i,j} (dt_{i_0} \cdots dt_{i_p}) \right)$$

$$= -i^*_\Delta \left( \sum_j t_{i_j} \partial_{i,j} (dt_{i_0} \cdots dt_{i_p}) \right)$$

which implies that

$$(\phi^i_s)^*i^*E_i(\omega_{i_0, \ldots, i_p}) = -(1 - s)^p i^*_\Delta \left( \sum_j t_{i_j} \partial_{i,j} (dt_{i_0} \cdots dt_{i_p}) \right)$$

$$= (-1)^{i+1} (1 - s)^p \omega_{i_0, \ldots, \hat{i}_i, \ldots, i_p}.$$ 

Thus, if $i = i_l$ for some $l$

$$h^i(\omega_{i_0, \ldots, i_p}) = \frac{(-1)^{i+1}}{p} \omega_{i_0, \ldots, \hat{i}_i, \ldots, i_p}.$$

Otherwise, $h^i(\omega_{i_0, \ldots, i_p}) = 0$. \qed
In conclusion, we see that \((-1)^p \varepsilon^{i_p} h^{i_{p-1}} \ldots h^{i_0}(\omega_{i_0, \ldots, i_p}) = 1/p!\). We would also like to show that \(\int_{[i_0, \ldots, i_p]} \omega_{i_0, \ldots, i_p} = 1/p!\). To calculate the integral, we could pull back \(\omega_{i_0, \ldots, i_p}\) using a parametrization \([0, 1]^p \to [i_0, \ldots, i_p] \subset \Delta^n\), i.e. a smooth map which restricts on \((0, 1)^p\) to a diffeomorphism onto the interior of \([i_0, \ldots, i_p]\).

Let \(t_p : [0, 1]^p \to [0, 1]^p \times \Delta^n\) be the product of the identity map and the inclusion of the vertex \(e_{i_p}\). A natural candidate for the parametrization \([0, 1]^p \to [i_0, \ldots, i_p]\) is then

\[
F_{i_0, \ldots, i_p} = \phi^{i_0} \circ (\text{id}_{[0, 1]} \times \phi^{i_1} \circ \cdots \circ (\text{id}_{[0, 1]} \times \phi^{i_{p-1}}) \circ t_1^{i_p}
\]

Note that for each \(k \in \{0, \ldots, p\}\) the image of \(\phi^{i_k}|_{[0, 1] \times [i_{k+1}, \ldots, i_p]}\) is equal to \([i_k, \ldots, i_p]\). Therefore the image of \(F_{i_0, \ldots, i_p}\) is equal to \([i_0, \ldots, i_p]\). It turns out that \(F_{i_0, \ldots, i_p}\) is orientation preserving for \(p\) even and orientation reversing for \(p\) odd. Let \(R_p : [0, 1]^p \to [0, 1]^p\) be defined by \(R_p(s_0, \ldots, s_{p-1}) = (1 - s_0, \ldots, 1 - s_{p-1})\).

**Proposition 3.** The restriction of \(F_{i_0, \ldots, i_p} \circ R_p\) to \((0, 1)^p\) is an orientation preserving diffeomorphism onto its image.

**Proof.** Let \(G_{i_0, \ldots, i_p} = F_{i_0, \ldots, i_p} \circ R_p\). In coordinates,

\[
G_{i_0, \ldots, i_p}(s_0, \ldots, s_{p-1}) = (1 - s_0)e_{i_0} + \cdots + s_0 \ldots s_{p-2}(1 - s_{p-1})e_{i_{p-1}} + s_0 \ldots s_{p-1}e_{i_p}
\]

Due to the inverse function theorem, it suffices to show that the differential of \(G_{i_0, \ldots, i_p}\) is injective on \((0, 1)^p\). But

\[
\frac{G_{i_0, \ldots, i_p}}{\partial s_k} = 0 \quad \text{if} \quad k > j, \quad \text{and} \quad \frac{G_{i_0, \ldots, i_p}}{\partial s_k} = -\prod_{l=0}^{k-1} s_l
\]

showing that differential has rank \(p\) on \((0, 1)^p\) and is therefore injective. We claim that the determinant of the matrix \((1_{p \times 1} DG)\) is positive on \((0, 1)^p\) which implies that \(G_{i_0, \ldots, i_p}\) is orientation preserving. The top row of this matrix only has only two nonzero entries. Therefore, it is sufficient to prove that the two corresponding terms in the cofactor expansion are positive on \((0, 1)^p\), which can be shown by induction on \(p\).

We can now calculate that \(\int_{[i_0, \ldots, i_p]} \omega_{i_0, \ldots, i_p} = 1/p!\) directly or use the identification:

**Lemma 1.** For any \(p\)-form \(\omega\) on \(\Delta^n\)

\[
\int_{[i_0, \ldots, i_p]} \omega = (-1)^p \varepsilon^{i_p} h^{i_{p-1}} \ldots h^{i_0}(\omega)
\]

**Proof.** Because

\[
\varepsilon^{i_p} h^{i_{p-1}} \ldots h^{i_0}\omega = \varepsilon^{i_p} \int_{[0, 1]^p} (\text{id}_{[0, 1]^{p-1}} \phi^{i_{p-1}})^* \circ \cdots \circ (\text{id}_{[0, 1]} \times \phi^{i_1}) \circ (\phi^{i_0})^* \omega = \int_{[0, 1]^p} F^*_{i_0, \ldots, i_p} \omega = (-1)^p \int_{[0, 1]^p} G^*_{i_0, \ldots, i_p} \omega
\]

the result follows from the fact that \(G_{i_0, \ldots, i_p}\) is a parametrization of \([i_0, \ldots, i_p]\). □

The proof of this lemma in Getzler's paper [3], is by arguing by induction on \(p\) using the formula \(dh^i + h^i d = 1 - \varepsilon^i\) and assuming that \(\omega\) is exact for \(p > 0\). This is a less direct approach and it seems to us that it only proves the result for closed forms.
2.3. **Proof of Main Theorems.** Note that the proof of the formula \( dh^i + h^i d = \varepsilon^i - 1 \) is a consequence of the relation \( d \Delta_k \pi_* = -\pi_* d \Delta_k \) and the fundamental theorem of calculus

\[
\pi_* d_{[0,1]}(\phi^i)^* \omega = (\phi^i_1)^* \omega - (\phi^i_0)^* \omega \\
= \varepsilon^i \omega - \omega.
\]

**Theorem 1.** The Dupont homotopy \( s \) is a deformation retraction.

**Proof.** Following Getzler \[3\], we compute

\[
[d, s] = -\sum_{k=0}^{n-1} \sum_{i_0 < \cdots < i_k \neq \{i_0, \ldots, i_k\}} \mathfrak{w}_{i_0 \ldots i_k} h^{i_k} \ldots h^{i_0} \\
- \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \sum_{i_0 < \cdots < i_k} \mathfrak{w}_{i_0 \ldots i_k} h^{i_k} \ldots [d, h^j i] \ldots h^{i_0}
\]

Using the formula \( dh^i + h^i d = \varepsilon^i - 1 \), the second term becomes

\[
\text{id} + \sum_{k=1}^{n} \sum_{j=0}^{k} (-1)^j \sum_{i_0 < \cdots < i_k} \mathfrak{w}_{i_0 \ldots i_k} h^{i_k} \ldots [h^j i] \ldots h^{i_0} - WI.
\]

The middle term of the three terms above is equal to

\[
\sum_{k=0}^{n-1} \sum_{i_0 < \cdots < i_k \neq \{i_0, \ldots, i_k\}} \mathfrak{w}_{i_0 \ldots i_k} h^{i_k} \ldots h^{i_0}.
\]

We conclude that \([d, s] = 1 - WI. \]

To show that \( s^2 = 0 \), we shall need the identity

**Lemma 2.** If \( i \notin \{i_0, \ldots, i_k\} \), then

\[
h^i \omega_{i_0, \ldots, i_k} = (-1)^k \omega_{i_0, \ldots, i_k} h^i - (k+1) h^i \omega_{i_0, \ldots, i_k, i} h^i
\]

which implies that

\[
h^i \mathfrak{w}_{i_0, \ldots, i_k} = (-1)^k \mathfrak{w}_{i_0, \ldots, i_k} h^i - h^i \mathfrak{w}_{i_0, \ldots, i_k, i} h^i
\]

**Proof.** Here we distinguish between \( h^i(\omega) \), \( h^i \) applied to \( \omega \), and \( h^i \omega \), \( h^i \) composed with multiplication by \( \omega \). Getzler \[3\] observes that we have

\[
h^i \omega_{i_0, \ldots, i_k} = \int_0^1 \frac{ds}{1-s} (\phi^i_s)^* tE_i \omega_{i_0, \ldots, i_k}
\]

\[
= (-1)^k \omega_{i_0, \ldots, i_k} \int_0^1 ds (1-s)^k (\phi^i_s)^* tE_i
\]

And the other hand,

\[
h^i \omega_{i_0, \ldots, i_k, i} h^i = \int_0^1 \int_0^1 \frac{dsds'}{(1-s)(1-s')} (\phi^i_s)^* tE_i \omega_{i_0, \ldots, i_k, i} (\phi^i_{s'})^* tE_i
\]

\[
= (-1)^k \omega_{i_0, \ldots, i_k} \int_0^1 \int_0^1 \frac{dsds'(1-s)^k}{(1-s')} (\phi^i_s)^* (\phi^i_{s'})^* tE_i
\]
Note that $\phi_{1-s} \circ \phi_s = \phi_{1-s'}$. Upon making the change of variables from $(s, s')$ to $(w, s')$ where $w = ss'$, we see that
\[ \int_0^1 \int_0^1 \frac{dsdw}{(1-s')^k} (\phi_{i-s})^*(\phi_{i-s'})^* = \int_0^1 \int_0^1 \frac{dsdw}{s'} (\phi_{1-s})^*(\phi_{1-s'})^* = \frac{1}{k+1} \int_0^1 ds \frac{ds}{1-s} (\phi_{i-s})^* - \frac{1}{k+1} \int_0^1 ds (1-s)^k (\phi_{i-s'})^* \]

establishing the lemma.

\textbf{Theorem 2.} The Dupont homotopy $s$ is a special deformation retraction.

\textbf{Proof.} Using Lemma 2,

\[-h_{ik} \ldots h_{io} s = \sum_{l=0}^{n-1} \sum_{0 \prec \cdots \prec j_l \prec f \{j_0, \ldots, j_l\}} h_{ik} \ldots h_{il} h_{i0} \overline{w}_{j_0, \ldots, j_l} h_{j_l} \ldots h_{j_0} + \sum_{l=0}^{n-1} \sum_{0 \prec \cdots \prec j_l \prec f \{j_0, \ldots, j_l\}} h_{ik} \ldots h_{il} ((-1)^l h_{i0} \overline{w}_{j_0, \ldots, j_l} - h_{il} h_{i0} \overline{w}_{j_0, \ldots, j_l, j_l}) h_{i0} h_{j_l} \ldots h_{j_0} \]

immediately implying that
\[ s^2 = \sum_{k,l=0}^{n-1} (-1)^{(l+1)k} \sum_{0 \prec \cdots \prec j_k, \cdots \prec j_l \prec f \{j_0, \ldots, j_l\}} \overline{w}_{i_0, \ldots, i_k} \overline{w}_{j_0, \ldots, j_l} h_{i_k} \ldots h_{i_0} h_{j_l} \ldots h_{j_0}. \]

But $h^i h_j + h^j h_i = 0$ for $i \neq j$, so we can interchange the $i$ and $j$ indices in a term to and pick up a factor of $(-1)^{(l+1)(k+1)+lk}$. The overall sign for an interchanged term is $(-1)^{(l+1)k}((-1)^{(l+1)(k+1)+lk} = -(-1)^{(l+1)k}$, so we get pairwise cancellation, implying that $s^2 = 0$.

It remains to show firstly that $Is = 0$, which is a consequence of

\[ \varepsilon^{ik} h_{ik-1} \ldots h_{io} s = -\varepsilon^{ik} h_{ik-1} \ldots h_{io} h_{ik} \]

and the identity $\varepsilon^j h^j = 0$.

And lastly, $sW = 0$ because
\[ s(\overline{w}_{j_0, \ldots, j_l}) = -\sum_{k=0}^{n-1} \sum_{0 \prec \cdots \prec j_k} \overline{w}_{i_0, \ldots, i_k} h_{i_k} \ldots h_{i_0} (\overline{w}_{j_0, \ldots, j_l}) \]

But for all $k$,
\[ \sum_{0 \prec \cdots \prec i_k} \overline{w}_{i_0, \ldots, i_k} h_{i_k} \ldots h_{i_0} (\overline{w}_{j_0, \ldots, j_l}) = 0 \]
because
\[ \omega_{i_0, \ldots, i_k} h^{i_k} \ldots h^{i_0} (\omega_{j_0, \ldots, j_l}) = i^*_\Delta (i_{E} dt_{i_0} \ldots dt_{i_k}) i^*_\Delta (i_{E} t_{\partial_{i_k}} \ldots t_{\partial_{i_0}} dt_{j_0} \ldots dt_{j_l}) \]
\[ = i^*_\Delta \sum_{i,j=0}^{l} t_i t_j (i_{\partial_{i}} dt_{i_0} \ldots dt_{i_k}) (i_{\partial_{j}} t_{\partial_{i_k}} \ldots t_{\partial_{i_0}} dt_{j_0} \ldots dt_{j_l}) \]
where \( E = \sum_{i} t_i \partial_{i} \). The terms in the above sum are zero unless \( \{i_0, \ldots, i_k\} \subset \{j_0, \ldots, j_l\} \), \( i \in \{i_0, \ldots, i_k\} \), and \( j \in \{j_0, \ldots, j_l\} \) \( \setminus \{i_0, \ldots, i_k\} \). So we can cancel the term indexed by \( i_0 < \cdots < i_k \) with its pair: the term indexed by \( i'_0 < \cdots < i'_k \) which is given by removing \( i \) from \( \{i_0, \ldots, i_k\} \) and adding \( j \) to it. \( \square \)

2.4. Globalizing the Construction. The general statement for a triangulated manifold follows as a consequence of Theorems 1 & 2 as we now show:

**Corollary 1.** The Dupont homotopy \( s \) is a well-defined (special) deformation retraction of the differential forms \( \Omega^\bullet(M) \) on a triangulated manifold \( M \) onto the simplicial cochains \( C^\bullet(M) \).

\[ (C^\bullet(M), d) \xrightarrow{I} (\Omega^\bullet(M), d) \xrightarrow{\epsilon} s \]

where \( W \) is the Whitney map and \( I \) is the integration map, which are also well-defined.

**Proof.** Firstly, we need that \( W, I \) and \( s \) are equivariant under the action of the symmetric group. This means that the maps do not depend on the ordering that we choose for vertices. This is necessary because a triangulation of a manifold does not come with an ordering of the vertices of its simplices. For \( W \) and \( I \) this follows directly from the definition. Turning to \( s \) for a general permutation \( \sigma \in S_n \) let \( \tau_\sigma : \Delta^n \to \Delta^n \) be the induced map \( \tau_\sigma(t_0, \ldots, t_n) = (t_{\sigma(0)}, \ldots, t_{\sigma(n)}) \). We have

\[ \tau_\sigma s = - \sum_{k=0}^{n-1} \sum_{i_0 < \cdots < i_k} \tau_\sigma^* \om_{i_0} \cdots i_k \tau_\sigma^* h^{i_k} \ldots h^{i_0} \]
\[ = - \sum_{k=0}^{n-1} \sum_{i_0 < \cdots < i_k} \om_{\sigma(i_0)} \cdots \sigma(i_k) h^{\sigma(i_k)} \ldots h^{\sigma(i_0)} \tau_\sigma^* \]
\[ = s \tau_\sigma^* . \]

Here we have used the fact that \( \tau_\sigma^* h^i = h^\sigma(i) \tau_\sigma^* \) which follows from the identity \( \tau_\sigma \circ \phi^\sigma(i) = \phi^\sigma \circ (\id_{[0,1]} \times \tau_\sigma) \).

It now suffices to show that \( W, I \) and \( s \) commute with pullback by the face maps \( \epsilon_i : \Delta^{n-1} \to \Delta^n \) for \( i = 0, \ldots, n \). For \( W \) and \( I \) this follows directly from the definition. For \( s \) we use that

\[ \epsilon_i \circ \phi^j = \begin{cases} \phi^j \circ (\id_{[0,1]} \times \epsilon_i) & \text{if } i > j \\ \phi^{i+1} \circ (\id_{[0,1]} \times \epsilon_i) & \text{if } i \leq j \end{cases} \]

This implies that

\[ h^j(\epsilon_i)^* = \begin{cases} (\epsilon_i)^* h^j & \text{if } i > j \\ (\epsilon_i)^* h^{j+1} & \text{if } i \leq j \end{cases} \]
and therefore
\[
(\epsilon_i)^* s = - \sum_{k=0}^{n-1} \sum_{\substack{i_0 < \cdots < i_k \\
i \notin \{i_0, \ldots, i_k\}}} (\epsilon_i)^* (\omega_{i_0, \ldots, i_k}) (\epsilon_i)^* h^{i_k} \cdots h^{i_0} = - \sum_{k=0}^{n-1} \sum_{\substack{i_0 < \cdots < i_k \\
i < i_{j+1} < \cdots < i_k \cdots < i_{j+1}}} \omega_{i_0, \ldots, i_j, i_{j+1}-1, \ldots, i_{k-1}} h^{i_k-1} \cdots h^{i_{j+1}-1} \cdots h^{i_0} (\epsilon_i)^* = s(\epsilon_i)^*
\]

\[\square\]

3. CUBICAL DUPONT HOMOTOPY FORMULA

The Dupont homotopy formula for cubical forms can be constructed from the Dupont homotopy formula on the 1-simplex \([0, 1]\) through the tensor product construction.

Let \(t\) be the natural coordinate on \(\Delta^1 = [0, 1]\). The degree 0 Whitney forms on \(\Delta^1\) are \(\omega_0 = 1 - t\) and \(\omega_1 = t\). The degree 1 Whitney form is \(\omega_{01} = (1 - t)dt - t(1 - t) = dt\).

Given a form \(\omega = f(t) + g(t) dt\) on \(\Delta^1 = [0, 1]\),
\[
IW(\omega) = \omega_0 I_0 \omega + \omega_1 I_1 \omega + \omega_{01} I_{0,1} \omega = (1 - t)f(0) + tf(1) + \int_0^1 g(t) dt
\]
and the Dupont homotopy is given by
\[
s(\omega) = -\omega_0 h^0(\omega) - \omega_1 h^1(\omega)
= (1 - t) t \int_0^1 g((1 - s)t) ds + t(t - 1) \int_0^1 g((1 - s)t + s) ds
= (1 - t) \int_0^t g(u) du - t \int_t^1 g(u) du
= \int_0^t g(u) du - t \int_0^1 g(u) du
\]

Let us recall the tensor construction. Note that \(C^\bullet(\square^n) = C^\bullet(\Delta^1) \otimes \cdots \otimes C^\bullet(\Delta^1)\) and \(\Omega^\bullet(\square^n) = \Omega^\bullet(\Delta^1) \otimes \cdots \otimes \Omega^\bullet(\Delta^1)\) where \(\otimes\) denotes the completed projective tensor product. Due to the continuity of \(I : \Omega^\bullet(\Delta^1) \rightarrow C^\bullet(\Delta^1)\), \(WI : \Omega^\bullet(\Delta^1) \rightarrow \Omega^\bullet(\Delta^1)\) and \(s : \Omega^\bullet(\Delta^1) \rightarrow \Omega^\bullet(\Delta^1)\), the following definitions make sense:

**Definition 3.** We define the integration map \(I : \Omega^\bullet(\square^n) \rightarrow C^\bullet(\square^n)\) by \(I = I \otimes \cdots \otimes I\) and the cubical Whitney map \(W : C^\bullet(\square^n) \rightarrow \Omega^\bullet(\square^n)\) by \(W = W \otimes \cdots \otimes W\). Define
\[
s_0 = \sum_{j=1}^n 1 \otimes \cdots \otimes 1 \otimes s \otimes WI \otimes \cdots \otimes WI
\]
and define the cubical Dupont homotopy $s$ as the symmetrization of $s_0$. That is if $\tau_\sigma : \Omega^*(\square^n) \to \Omega^*(\square^n)$ is the induced linear map coming from the permutation $\sigma \in S_n$, we have
\[
s = \frac{1}{n!} \sum_{\sigma \in S_n} \tau_\sigma \circ s_0 \circ \tau_\sigma^{-1}
\]
\[
= \frac{1}{n!} \sum_{\epsilon} C_{[\epsilon], n} \sum_{j=1}^{n} (WI)^{\epsilon_1} \otimes \cdots \otimes (WI)^{\epsilon_j-1} \otimes s \otimes (WI)^{\epsilon_j} \otimes \cdots \otimes (WI)^{\epsilon_{n-1}}
\]
\[
= \frac{1}{n!} \sum_{\epsilon} C_{[\epsilon], n} \psi_\epsilon
\]
where $C_{[\epsilon], n} = |\epsilon|!(n - 1 - |\epsilon|)!$ and the outer sum is over $\epsilon_k = 0, 1$.

**Theorem 3.** The cubical Dupont homotopy $s$ is a (special) deformation retraction of the differential forms $\Omega^*(\square^n)$ on the $n$-cube onto the cubical cochains $C^*(\square^n)$.

\[
(C^*(\square^n), d) \xrightarrow{\frac{1}{W}} (\Omega^*(\square^n), d) \xrightarrow{s}
\]

where $W$ is the Whitney map and $I$ is the integration map.

**Proof.** The theorem also holds replacing $s$ with $s_0$. The reason for working with $s$ rather than $s_0$ is to be able to pass to cubulated manifolds where there is no fixed identification of the $n$-cube as an ordered product of 1-simplices.

Because $d$ and $WI$ commute with $\tau_\sigma$ for any permutation $\sigma \in S_n$, to show that $s$ is a deformation retraction, it suffices to show that $s_0$ is a deformation retraction. But
\[
d s_0 + s_0 d = \sum_{j=1}^{n} 1^{\otimes (j-1)} \otimes (ds + sd) \otimes (WI)^{\otimes (n-j)}
\]
\[
= 1 - WI
\]
It is clear that $sW = 0$ and $Is = 0$. Lastly $s^2 = 0$ follows from the relation $\psi_\epsilon \psi_\epsilon' = \psi_{\epsilon \epsilon'} \psi_{\epsilon'}$.

**Corollary 2.** The cubical Dupont homotopy formula gives a (special) deformation retraction of the piecewise smooth differential forms $\Omega^*(M)$ on a cubulated manifold $M$ onto the cubical cochains $C^*(M)$.

\[
(C^*(M), d) \xrightarrow{I} (\Omega^*(M), d) \xrightarrow{s}
\]

where $W$ is the Whitney map, $I$ is the integration map, and $s$ is the Dupont homotopy.

### 4. Stellar Subdivision Formula

We begin with the statement in one dimension for simplicity. Let $*\Delta^1$ denote the stellar subdivision of the 1-simplex $\Delta^1$. That is $*\Delta^1$ is the simplicial complex with vertices $e_*, e_0, e_1$ and edges $[e_*, e_0]$ and $[e_*, e_1]$. We visualize $e_*$ as lying at the barycenter of the 1-simplex $[e_0, e_1]$ of $\Delta^1$.

There is a natural inclusion map $i_* : C_*(\Delta^1) \to C_*(\star\Delta^1)$ defined by $e_0 \mapsto e_0$, $e_1 \mapsto e_1$ and $[e_0, e_1] \mapsto [e_0, e_*] + [e_*, e_1]$, which is a chain map. There is a natural projection map $p_* : C_*(\star\Delta^1) \to C_*(\Delta^1)$ defined by $e_* \mapsto \frac{1}{2}(e_0 + e_1)$, $e_0 \mapsto e_0$, $e_1 \mapsto e_1$. 

$e_1 \mapsto e_1$ and $[e_0, e_*] \mapsto \frac{1}{2}[e_0, e_1]$ and $[e_*, e_1] \mapsto \frac{1}{2}[e_0, e_1]$, which is also a chain map. We have $p_*i_* = 1$ and we would like to find a homotopy $a_*$ between the identity $1$ and $i_*p_*$. We define $a_*$ by $e_* \mapsto \frac{1}{2}(e_0, e_* - [e_*, e_1]), e_0 \mapsto 0,$ and $e_1 \mapsto 0$. Then $\partial a_* + a_* \partial = 1 - i_*p_*$. Furthermore, we have $a_*i_* = 0$, $p_*a_* = 0$, and $a_*^2 = 0$.

The dual deformation retraction in one dimension has inclusion $i^*: C^*({\Delta^1}) \to C^*(\ast\Delta^1)$ defined by $e_0 \mapsto \tilde{e}_0 + \frac{1}{2}\tilde{e}_1$, $\tilde{e}_1 \mapsto \tilde{e}_1 + \frac{1}{2}\tilde{e}_2$, $[e_0, e_1] \mapsto \frac{1}{2}(e_0 + e_1 + [e_0, e_1])$. It has projection $p^*: C^*(\ast\Delta^1) \to C^*({\Delta^1})$ defined by $e_0 \mapsto e_0$, $\tilde{e}_1 \mapsto -\tilde{e}_1$, $e_* \mapsto 0$, $[e_0, e_*] \mapsto [e_0, e_1]$ and $[e_*, e_1] \mapsto [e_0, e_1]$. Lastly, the homotopy $a^*$ defined by $[e_0, e_*] \mapsto \frac{1}{2}e_*$ and $[e_*, e_1] \mapsto -\frac{1}{2}e_*$. Generalizing now to the $n$-simplex:

**Definition 4.** For $k \leq n$ and $0 \leq i_0 < \cdots < i_k \leq n$, we define the **stellar subdivision** $\ast_{i_0, \ldots, i_k} \Delta^n$ for $k \leq n$. This is a simplicial complex having vertex $e_*$ as well as vertices $e_{i_0}, \ldots, e_{i_k}$. We allow all simplices $[e_{i_0}, \ldots, e_{i_k}]$ as well as $[e_*, e_{i_0}, \ldots, e_{i_k}]$ where $[e_{i_0}, \ldots, e_{i_k}] \not\subset [e_{i_0}, \ldots, e_{i_k}]$. In other words, for $I = \{i_0, \ldots, i_k\}$ and $J = \{j_0, \ldots, j_l\}$ with $I \not\subset J$, we include all simplices of the form $[e_{i_0}, \ldots, e_{i_k}]$ and $[e_*, e_{i_0}, \ldots, e_{i_k}]$. When $k = n$ we shall simply write $\ast\Delta^n$.

**Definition 5.** We define the **stellar subdivision inclusion map** $i_* : C^*({\Delta^n}) \to C^*(\ast_{i_0, \ldots, i_k} \Delta^n)$ by $i_*[e_{i_0}, \ldots, e_{i_k}] =$

$$
\begin{cases}
[e_{i_0}, \ldots, e_{i_k}] & \text{for } [e_{i_0}, \ldots, e_{i_k}] \not\supset [e_{i_0}, \ldots, e_{i_k}] \\
\sum_{j_i \in I} (-1)^i [e_*, e_{i_0}, \ldots, e_{i_k}] & \text{for } [e_{i_0}, \ldots, e_{i_k}] \supset [e_{i_0}, \ldots, e_{i_k}]
\end{cases}
$$

We quickly verify that $i_*$ is a chain map since for $[e_{i_0}, \ldots, e_{i_k}] \supset [e_{i_0}, \ldots, e_{i_k}]$

$$
\partial i_*[e_{i_0}, \ldots, e_{i_k}] = i_* \sum_{j_i \in I} (-1)^i [e_{i_0}, \ldots, e_{i_k}] \\
+ i_* \sum_{j_i \not\in I} (-1)^i [e_{i_0}, \ldots, e_{i_k}] \\
= i_* \partial [e_{i_0}, \ldots, e_{i_k}]
$$

**Definition 6.** Define the **stellar subdivision projection map** $p_* : C^*(\ast_{i_0, \ldots, i_k} \Delta^n) \to C^*(\Delta^n)$ for $J \not\supset I$, by

$$
p_*[e_{i_0}, \ldots, e_{i_k}] = [e_{i_0}, \ldots, e_{i_k}]
$$

and

$$
p_*[e_*, e_{i_0}, \ldots, e_{i_k}] = \frac{1}{k + 1} \sum_{\alpha \in I \setminus J} [e_{\alpha}, e_{i_0}, \ldots, e_{i_k}]
$$

This is a chain map because

$$
\partial p_*[e_*, e_{i_0}, \ldots, e_{i_k}] = \frac{|I \setminus J|}{k + 1} [e_{i_0}, \ldots, e_{i_k}] - \frac{1}{k + 1} \sum_{\alpha \in I \setminus J} \sum_{i=0}^l (-1)^i [e_*, e_{i_0}, \ldots, e_{i_k}, e_i] \\
= [e_{i_0}, \ldots, e_{i_k}] - \frac{1}{k + 1} \sum_{i=0}^l \sum_{\alpha \in I \setminus (J \cup \{i\})} (-1)^i [e_*, e_{i_0}, \ldots, e_i, e_{i_k}]
$$

$$
p_*[e_*, e_{i_0}, \ldots, e_{i_k}] = p_* \partial [e_*, e_{i_0}, \ldots, e_{i_k}]
$$
Theorem 4. Define the stellar subdivision homotopy by $a_* : C^*(\ast_{i_0,\ldots,i_k} \Delta^n) \to C^{\ast+1}(\ast_{i_0,\ldots,i_k} \Delta^n)$ for $J \not\supset I$ by

$$a_*[e_{j_0}, \ldots, e_{j_l}] = 0$$

and for $\sigma = [e_{j_0}, \ldots, e_{j_l}]$ with $I \setminus J = \{i_m\}$,

$$i_*p_*[e_{j_0}, \ldots, e_{j_l}] = \frac{1}{k+1}[e_{j_0}, \ldots, e_{j_l}] - \frac{1}{k+1} \sum_{j_i \in I} (-1)^l [e_{i_0}, e_{j_0}, \ldots, e_{j_l}].$$

and lastly, for $\sigma = [e_{j_0}, \ldots, e_{j_l}]$ with $|I \setminus J| \geq 2$, we have

$$i_*p_*\sigma = \frac{1}{k+1} \sum_{\alpha \in I \setminus J} [e_{\alpha}, e_{j_0}, \ldots, e_{j_l}]$$

We now compute for $I \setminus J = \{i_m\}$

$$(\partial a_* + a_* \partial)([e_{j_0}, \ldots, e_{j_l}]) = - \sum_{i=0}^{l} (-1)^i a_*[e_{j_0}, \ldots, e_{j_{i-1}}, e_{j_{i+1}}, \ldots, e_{j_l}]$$

$$= \frac{k}{k+1}[e_{j_0}, \ldots, e_{j_l}]$$

$$+ \frac{1}{k+1} \sum_{j_i \in I} (-1)^l [e_{i_0}, e_{j_0}, \ldots, e_{j_l}].$$

For $|I \setminus J| \geq 2$, we have

$$(\partial a_* + a_* \partial)([e_{j_0}, \ldots, e_{j_l}]) = - \sum_{i=0}^{l} (-1)^i a_*[e_{j_0}, \ldots, e_{j_{i-1}}, e_{j_{i+1}}, \ldots, e_{j_l}]$$

$$- \frac{1}{k+1} \sum_{\alpha \in I \setminus J} \partial[e_{\alpha}, e_{j_0}, \ldots, e_{j_l}]$$

$$= (1 - i_*p_*)([e_{j_0}, \ldots, e_{j_l}]).$$

It is clear from the definitions that $a_*i_* = 0$. For $|I \setminus J| \geq 2$, we have

$$p_*a_*[e_{j_0}, \ldots, e_{j_l}] = - \frac{1}{(k+1)^2} \sum_{\alpha \in I \setminus (J \cup \{\alpha\})} \sum_{\alpha' \in I \setminus J} [e_{\alpha}, e_{\alpha'}, e_{j_0}, \ldots, e_{j_l}] = 0$$

and if $|I \setminus J| \geq 3$,

$$a_*^2[e_{j_0}, \ldots, e_{j_l}] = \frac{1}{(k+1)^2} \sum_{\alpha \in I \setminus (J \cup \{\alpha\})} \sum_{\alpha' \in I \setminus J} [e_{\alpha}, e_{\alpha'}, e_{j_0}, \ldots, e_{j_l}] = 0$$
Definition 8. We define the stellar subdivision inclusion map on cochains by
\[ i^*[e_{j_0}, \ldots, e_{j_I}] = [e_{j_0}, \ldots, e_{j_I}] + \frac{1}{k+1} \sum_{j_i \in J} (-1)^i [e_{\ast}, e_{j_0}, \ldots, e_{j_I}, \ldots, e_{j_J}] \]
for \( J \nsubseteq I \) and
\[ i^*[e_{j_0}, \ldots, e_{j_J}] = \frac{1}{k+1} \sum_{j_i \in I} (-1)^i [e_{\ast}, e_{j_0}, \ldots, e_{j_J}, \ldots, e_{j_I}] \]
for \( J \supset I \).

Definition 9. We define the stellar subdivision projection map on cochains for \( J \nsubseteq I \) by
\[ p^*[e_{j_0}, \ldots, e_{j_I}] = [e_{j_0}, \ldots, e_{j_I}] \]
and
\[ p^*[e_{\ast}, e_{j_0}, \ldots, e_{j_I}] = \begin{cases} [e_{i_m}, e_{j_0}, \ldots, e_{j_I}] & \text{if } I \setminus J = \{i_m\} \\ 0 & \text{otherwise} \end{cases} \]

Definition 10. We define the stellar subdivision homotopy on cochains for \( J \nsubseteq I \) by
\[ a^*[e_{j_0}, \ldots, e_{j_I}] = 0 \]
and
\[ a^*[e_{\ast}, e_{j_0}, \ldots, e_{j_I}] = -\frac{1}{k+1} \sum_{j_i \in I} (-1)^i [e_{\ast}, e_{j_0}, \ldots, e_{j_I}, \ldots, e_{j_J}] \]

Theorem 5. The stellar subdivision induces a special deformation retraction on cochains
\[(C^\ast(\Delta^n), d) \xrightarrow{p^*} (C^\ast(*_{i_0,\ldots,i_k} \Delta^n), d) \xleftarrow{a^*} \]
which is dual to the stellar subdivision deformation retraction on chains.

5. Cubical Stellar Subdivision Formula

We define the cubical stellar subdivision \(*_{i_1,\ldots,i_k} \Delta^n\) for \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \),
to be the product \( I_1 \times \cdots \times I_n \) where \( I_i = \ast \Delta^1 \) if \( i \in \{i_1, \ldots, i_k\} \) and \( I_i = \Delta^1 \) otherwise. To simplify notation, we shall always assume in what follows that \( i_j = j \)
so that
\[ *_{i_1,\ldots,i_k} \Delta^n = *_{1,\ldots,k} \Delta^n = (\ast \Delta^1)^{\times k} \times (\Delta^1)^{\times (n-k)}. \]
The formulas for the cubical stellar subdivision deformation retraction that we will present can be recovered for general \( \{i_1, \ldots, i_k\} \) by composing with an appropriate permutation.

Definition 11. Define \( i_{\ast} = i_{\ast}^{\otimes k} \otimes 1^{\otimes (n-k)}, p_{\ast} = p_{\ast}^{\otimes k} \otimes 1^{\otimes (n-k)} \) and
\[ a_{\ast} = \frac{1}{k!} \sum_{\sigma \in S_k} \tau_{\sigma \times 1^{n-k}} \circ \left( \sum_{j=1}^{n} 1^{\otimes (j-1)} \otimes a_{\ast} \otimes (i_{\ast} p_{\ast})^{\otimes (k-j)} \otimes 1^{\otimes (n-k)} \right) \circ \tau_{\sigma \times 1^{n-k}}^{-1} \]
respectively the cubical stellar subdivision inclusion, projection and homotopy on cubical chains. Define \( i^* = (i^*)^\otimes k \otimes 1^\otimes (n-k) \), \( p^* = (p^*)^\otimes k \otimes 1^\otimes (n-k) \) and
\[
a^* = \frac{1}{k!} \sum_{\sigma \in S_k} \tau_{\sigma \times 1^{n-k}} \circ \left( \sum_{j=1}^{n} 1^\otimes (j-1) \otimes a^* \otimes (i^* p^*)^\otimes (k-j) \otimes 1^\otimes (n-k) \right) \circ \tau_{1^{n-k}}^{-1}
\]
respectively the cubical stellar subdivision inclusion, projection and homotopy on cubical cochains.

**Theorem 6.** Cubical stellar subdivision induces a special deformation retraction on cubical chains
\[
(C_\bullet(\Box^n), \partial) \xrightarrow{p^*} (C_\bullet(*_{1,...,k}\Box^n), \partial) \xleftarrow{a^*}
\]
whose dual is special deformation retraction on cubical cochains
\[
(C^\bullet(\Box^n), d) \xrightarrow{p^*} (C^\bullet(*_{1,...,k}\Box^n), d) \xleftarrow{a^*}
\]
Lastly, it is worth mentioning that cubical stellar subdivision on a cubulated manifold has a different flavor than simplicial stellar subdivision on a triangulated manifold. Simplicial stellar subdivision is a local operation that exists for any choice of a simplex in the triangulation. However, for cubical stellar subdivision, for one must specify a collection of \( k \)-cubes in the cubulation such that any \( n \)-cube in the cubulation has closure containing exactly \( 2^{n-k} \) \( k \)-cubes from the collection that are opposite faces of the \( n \)-cube.

6. **Compatibility of the DHF and Stellar Subdivision**

We begin with \( n = 1 \) where the formulas and proofs are much simpler. The Dupont homotopy formula gives a deformation retraction
\[
(C^\bullet(*\Delta^1), d) \xrightarrow{j^*} (\Omega^\bullet(*\Delta^1), d) \xleftarrow{\ast}
\]
Explicitly, the Whitney forms are \( \omega_* = 2t \chi_{[0,1/2]} + (2 - 2t) \chi_{[1/2,1]} \), \( \omega_0 = (1 - 2t) \chi_{[0,1/2]} \), \( \omega_1 = (2t - 1) \chi_{[1/2,1]} \), \( \omega_{0*} = \chi_{[0,1/2]} dt \), and \( \omega_{1*} = \chi_{[1/2,1]} dt \). The Dupont homotopy is given by
\[
\hat{s}(g(t) dt) = \left[ \int_0^t g(u) du - 2t \int_0^{1/2} g(u) du \right] \chi_{[0,1/2]} + \left[ \int_{1/2}^t g(u) du - 2(t - 1/2) \int_{1/2}^1 g(u) du \right] \chi_{[1/2,1]}
\]
Here \( \chi_{[a,b]} \) is the characteristic function for the interval \([a, b]\).

**Theorem 7.** The Dupont deformation retraction on \(*\Delta^1\) can be composed with the stellar subdivision deformation retraction to get a new deformation retraction
\[
(C^\bullet(\Delta^1), d) \xrightarrow{p^* j^*} (\Omega^\bullet(*\Delta^1), d) \xleftarrow{\ast + \hat{w}_a \ast}
\]
This is equal to the Dupont deformation retraction on $\Delta^1$

$$(C^{\bullet}(\Delta^1), d) \xrightarrow{\text{I}} (\Omega^{\bullet}(\Delta^1), d) \hookrightarrow (\Omega^{\bullet}(\ast \Delta^1), d) \xleftarrow{\text{s}}$$

where we identify $\Omega^{\bullet}(\Delta^1)$ as a subspace and extend $s$.

**Proof.** We have

$$p^* \dot{I}(f(t)) = p^*(f(0)\hat{e}_0 + f(1)\hat{e}_1)$$

$$= f(0)\hat{e}_0 + f(1)\hat{e}_1$$

$$= I(f(t))$$

and

$$p^* \dot{I}(g(t) \, dt) = p^* \left( \int_0^{1/2} g(t) \, dt + \int_{1/2}^1 g(t) \, dt \right)$$

$$= \int_0^1 g(t) \, dt$$

$$= I(g(t) \, dt)$$

That is $p^* \dot{I} = I$.

Secondly, $Wi^*([e_0, e_1]) = \frac{1}{2}W([e_0, e_*] + [e_*, e_1]) = dt$,

$$\hat{W}i^*(\hat{e}_0) = \omega_0 + \frac{1}{2}\omega_* = 1 - t = W(\hat{e}_0)$$

and

$$\hat{W}i^*(\hat{e}_1) = \frac{1}{2}\omega_* + \omega_2 = t = W(\hat{e}_1)$$

That is, $\hat{W}i^* = W$.

Lastly,

$$\hat{W}a^* \dot{I}(g(t) \, dt) = \hat{W} \left( \frac{\dot{e}_*}{2} \int_0^{1/2} g(t) \, dt - \frac{\dot{e}_*}{2} \int_{1/2}^1 g(t) \, dt \right)$$

$$= (t \chi[0,1/2] + (1 - t) \chi[1/2,1]) \left( \int_0^{1/2} g(t) \, dt - \int_{1/2}^1 g(t) \, dt \right)$$

and thus $\hat{W}a^* \dot{I} + s = s$. \hfill \Box

In general, the simplicial complex $\ast_{i_0, \ldots, i_k} \Delta^n$ has $k + 1$ top dimensional simplices $[e_*, e_0, \ldots, e_m, \ldots, e_n]$ for $m = 0, \ldots, k$. For each top dimensional simplex in $\ast_{i_0, \ldots, i_k} \Delta^n$, there are barycentric coordinates which we would like to relate to barycentric coordinates on $\Delta^n$.

Writing this down explicitly, a point $t_0e_0 + \cdots + t_ne_n$ in $\Delta^n$ is in the $m$-th top dimensional simplex of $\ast_{i_0, \ldots, i_k}(\Delta^n)$ if it is a convex combination

$$t'_ie_* + t'_ie_0 + \cdots + t'_{i_{m-1}}e_{i_{m-1}} + t'_{i_m+1}e_{i_{m+1}} + \cdots + t'_ne_n$$

where $e_* = (e_{i_0} + \cdots + e_{i_k})/(k + 1)$. Or equivalently, if

$$t_{i_m} \leq 1/(k + 1) \quad \text{and} \quad t_i \geq t_{i_m} \quad \text{for all} \quad i \in I \setminus \{i_m\}.$$
The change of barycentric coordinates is given by
\[ t'_s = (k+1)t_{i_m} \quad \text{and} \quad t'_i = t_i - t_{i_m} \quad \text{and} \quad t'_j = t_j \]
for \( i \in I \setminus \{i_m\} \) and for \( j \notin I \).

**Proposition 4.** On \([e_*, e_0, \ldots, \hat{e}_{i_m}, \ldots, e_n]\), the Whitney forms for \( J \neq i_m \) are denoted \( \overline{\omega}_{j_0, \ldots, j_i} \) and \( \underline{\omega}_{*, j_0, \ldots, j_i} \). These are related to the restriction of the Whitney forms on \( \Delta^n \) by
\[ \overline{\omega}_{j_0, \ldots, j_i} = \overline{\omega}_{j_0, \ldots, j_i} - \sum_{j_i \in I} (-1)^i \overline{\omega}_{i_m, j_0, \ldots, \hat{j}_i, \ldots, j_i} \]
and
\[ \underline{\omega}_{*, j_0, \ldots, j_i} = (k+1)\overline{\omega}_{i_m, j_0, \ldots, j_i} \]

**Proof.** We note the identities on \([e_*, e_0, \ldots, \hat{e}_{i_m}, \ldots, e_n]\)
\[ dt'_{j_0} \ldots dt'_{j_i} = dt_{j_0} \ldots dt_{j_{i_m}} \sum_{j_i \in I} dt_{j_0} \ldots \hat{dt}_{j_{i_m}} \ldots dt_{j_i} \]
and
\[ dt'_s \omega'_{j_0, \ldots, j_i} = (k+1)dt_{i_m} \omega_{j_0, \ldots, j_i} - (k+1)t_{i_m} dt_{j_0} \sum_{j_i \in I} dt_{j_0} \ldots \hat{dt}_{j_{i_m}} \ldots dt_{j_i} \]
The Whitney forms on \([e_*, e_0, \ldots, \hat{e}_{i_m}, \ldots, e_n]\) are thus given by
\[ \overline{\omega}_{j_0, \ldots, j_i} = \overline{\omega}_{j_0, \ldots, j_{i_m}} - l! \sum_{j_i \in I} (-1)^i \left( t_{i_m} dt_{j_0} \ldots \hat{dt}_{j_{i_m}} \ldots dt_{j_i} - dt_{i_m} \omega_{j_0, \ldots, \hat{j}_i, \ldots, j_i} \right) \]
\[ = \overline{\omega}_{j_0, \ldots, j_{i_m}} - \sum_{j_i \in I} (-1)^i \overline{\omega}_{i_m, j_0, \ldots, \hat{j}_i, \ldots, j_i} \]
and
\[ \underline{\omega}_{*, j_0, \ldots, j_i} = (l+1)! \left( t'_s dt'_{j_0} \ldots dt'_{j_i} - dt'_s \omega'_{j_0, \ldots, j_i} \right) \]
\[ = (l+1)! \left( k+1 \right) \left( t_{i_m} dt_{j_0} \ldots dt_{j_i} - dt_{i_m} \omega_{j_0, \ldots, j_i} \right) \]
\[ = (k+1)\overline{\omega}_{i_m, j_0, \ldots, j_i} \]
\[ \square \]

The Dupont homotopy on \(*_{i_0, \ldots, i_k} \Delta^n\) is a deformation retraction
\[ (C^*(*_{i_0, \ldots, i_k} \Delta^n), d) \xrightarrow{i} (\Omega^*(*_{i_0, \ldots, i_k} \Delta^n), d) \]

**Theorem 8.** Composing with the stellar subdivision deformation retraction gives a new deformation retraction
\[ (C^*(\Delta^n), d) \xrightarrow{\overline{\text{stellar}} \, \bar{i}} (\Omega^*(*_{i_0, \ldots, i_k} \Delta^n), d) \]
This is homotopic to Dupont deformation retraction on \( \Delta^n \),
\[ (C^*(\Delta^n), d) \xrightarrow{i} (\Omega^*(\Delta^n), d) \leftrightarrow (\Omega^*(*_{i_0, \ldots, i_k} \Delta^n), d) \]
where we identify $\Omega^*(\Delta^n)$ as a subspace and extend $s$. That is, $\hat{W}i^* = W$, $p^*\hat{I} = I$ and $\hat{s} + \hat{W}a^*\hat{I} - s = dQ - Qd$ for some $Q$.

Proof. We have

$$p^*\hat{I} = p^* \left[ \sum_{j_0 < \cdots < j_l} \int_{[j_0, \ldots, j_l]} + \sum_{j_0 < \cdots < j_l} [\ast, j_0, \ldots, j_l] \int_{[\ast, j_0, \ldots, j_l]} \right]$$

$$= \sum_{j_0 < \cdots < j_l} \int_{[j_0, \ldots, j_l]} + \sum_{j_0 < \cdots < j_l} [\ast, j_0, \ldots, j_l] \int_{[\ast, j_0, \ldots, j_l]}$$

$$= I$$

Secondly, for $J \supset I$, we have on $[e_*, e_0, \ldots, e_i, \ldots, e_n]$ for each $i_m = j_0 \in I$

$$\hat{W}i^*[\hat{e}_{j_0}, \ldots, e_{j_l}] = \frac{1}{k + 1} \sum_{j_i \in I} (-1)^i \hat{\omega}_{*, j_0, \ldots, \check{j}_i, \ldots, j_l} = \frac{(-1)^b}{k + 1} \hat{\omega}_{*, j_0, \ldots, \check{j}_l, \ldots, j_l}$$

$$= (-1)^b \hat{\omega}_{j_0, j_0, \ldots, \check{j}_l, \ldots, j_l} = \hat{\omega}_{j_0, \ldots, j_l} = W[e_{j_0}, \ldots, e_{j_l}]$$

For $J \not\supset I$, we have on $[e_*, e_0, \ldots, e_i, \ldots, e_n]$ for each $i_m \in I \setminus J$

$$\hat{W}i^*[e_{j_0}, \ldots, e_{j_l}] = \hat{\omega}_{j_0, \ldots, j_l} + \frac{1}{k + 1} \sum_{j_i \in I} (-1)^i \hat{\omega}_{i_m, j_0, \ldots, \check{j}_i, \ldots, j_l}$$

$$= \hat{\omega}_{j_0, \ldots, j_l} - \sum_{j_i \in I} (-1)^i \hat{\omega}_{i_m, j_0, \ldots, \check{j}_i, \ldots, j_l} + \sum_{j_i \in I} (-1)^i \hat{\omega}_{i_m, j_0, \ldots, \check{j}_i, \ldots, j_l}$$

$$= \hat{\omega}_{j_0, \ldots, j_l} = W[e_{j_0}, \ldots, e_{j_l}]$$

For $J \not\supset I$, we have on $[e_*, e_0, \ldots, e_i, \ldots, e_n]$ for each $j_i \in I$

$$\hat{W}i^*[e_{j_0}, \ldots, e_{j_l}] = \hat{\omega}_{j_0, \ldots, j_l} + \frac{1}{k + 1} \sum_{j_i \in I} (-1)^i \hat{\omega}_{*, j_0, \ldots, \check{j}_i, \ldots, j_l}$$

$$= \frac{(-1)^b}{k + 1} \hat{\omega}_{*, j_0, \ldots, \check{j}_l, \ldots, j_l}$$

$$= \hat{\omega}_{j_0, \ldots, j_l} = W[e_{j_0}, \ldots, e_{j_l}]$$

Solely based on the identities $\hat{W}i^* = W$ and $p^*\hat{I} = I$, it follows that $\hat{W}a^*\hat{I} + \hat{s} - s$ is closed. More strongly, in fact, $\hat{W}a^*\hat{I} + \hat{s} - s$ is exact. This is because $\Delta^n$ is contractible, so the homomorphism complex $\text{Hom}^*(\Omega(\Delta^n), \Omega(\Delta^n))$ defined by

$$\text{Hom}^i(\Omega(\Delta^n), \Omega(\Delta^n)) = \oplus_{j=0}^n \text{Hom}(\Omega^j(\Delta^n), \Omega^{j+i}(\Delta^n))$$

has cohomology

$$H^i(\text{Hom}(\Omega(\Delta^n), \Omega(\Delta^n))) \cong \begin{cases} \mathbb{K} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Therefore a closed degree $-1$ linear endomorphism of $\Omega^*(\Delta^n)$ must be exact. □
Turning now to the Dupont homotopy, we compute on \([e_*, e_0, \ldots, e_{i_m}, \ldots, e_n]\) that
\[
\phi^* \left( s, \sum_j t_j e_j \right) = (1-s)t'_* e_* + (1-s) \sum_{j \neq i_m} t'_j e_j + se_*
\]
\[
= \sum_j (1-s)t_j e_j + \frac{1}{k+1} \sum_{i \in I} se_i
\]
Also, \(\phi^* = \phi^*\), for \(\alpha \neq i_m\) and therefore \(h^\alpha = h^\alpha\), for \(\alpha \neq i_m\). For consistency, we also make a change of notation dropping the prime for \(h^{\star}\) and writing instead \(h^*\).

The Dupont homotopy thus is given by
\[
\hat{s} = -\sum_{l=0}^{n-1} \sum_{j_0 < \ldots < j_l} \overline{w}_j \frac{h^{j_l} \ldots h^{j_0}}{l!}
- \sum_{l=-1}^{n-2} \sum_{j_0 < \ldots < j_l} \overline{w}_j R_{j_0, \ldots, j_l} h^{j_l} \ldots h^{j_0}
\]
For \(l = -1\) we are simply defining the inner sum to be equal to \(\overline{w}_* h^*\)

**Theorem 9.**
\[
\hat{W} a^* I + \hat{s} - s = \sum_{l=-1}^{n-2} \sum_{j_0 < \ldots < j_l} \overline{w}_l R_{j_0, \ldots, j_l} h^{j_l} \ldots h^{j_0}
\]
where
\[
R_{j_0, \ldots, j_l} = (-1)^{l+1} \left( 1 - \varepsilon^* \right) \sum_{i_l \in I \setminus J} h^{i_l} - (k+1)h^* \right) h^{j_l} \ldots h^{j_0}
\]
Here \(\varepsilon^*\) is evaluation at \(e_*\), the barycenter of \([i_0, \ldots, i_k]\).

**Proof.** On \([e_*, e_0, \ldots, e_{i_m}, \ldots, e_n]\) the Dupont homotopy restricts to
\[
\hat{s} = -\sum_{l=0}^{n-1} \sum_{j_0 < \ldots < j_l} \overline{w}_j \frac{h^{j_l} \ldots h^{j_0}}{l!}
- \sum_{l=-1}^{n-2} \sum_{j_0 < \ldots < j_l} \overline{w}_j \frac{h^{j_l} \ldots h^{j_0} h^*}{l!}
\]
\[
= -\sum_{l=0}^{n-1} \sum_{j_0 < \ldots < j_l} \overline{w}_j \frac{h^{j_l} \ldots h^{j_0}}{l!}
+ \sum_{l=-1}^{n-2} \sum_{j_0 < \ldots < j_l} \overline{w}_l R_{j_0, \ldots, j_l} h^{j_l} \ldots h^{j_0} \left( \sum_{i_l \in I \setminus \{i_m\} \cup J} h^{i_l} - (k+1)h^* \right)
\]
We have
\[
\hat{W} a^* I = -\frac{1}{k+1} \sum_{l=0}^{n-1} \sum_{j_0 < \ldots < j_l} \sum_{j_l \in I} (-1)^l \overline{w}_j R_{j_0, \ldots, j_l} h^{j_l} \ldots h^{j_0} \int_{[*, j_0, \ldots, j_l]} \]
On $[e_*, e_0, \ldots, e_{\widehat{m}}, \ldots, e_n]$, this reduces to

$$W a^* \dot{I} = -\frac{1}{k + 1} \sum_{l=0}^{n-1} \sum_{j_0 < \cdots < j_l \in I} \sum_{i_m \notin J} (-1)^l \mathcal{W}_{*,j_0,\ldots,j_l} \int_{[*,j_0,\ldots,j_l]}$$

$$-\frac{1}{k + 1} \sum_{l=0}^{n-1} \sum_{j_0 < \cdots < j_l} (-1)^l \mathcal{W}_{*,j_0,\ldots,j_l} \int_{[*,j_0,\ldots,j_l]}$$

$$= -\sum_{l=-1}^{n-2} \sum_{j_0 < \cdots < j_l} \mathcal{W}_{i_m, j_0, \ldots, j_l} \sum_{\alpha \in I \setminus \{J, \widehat{J} \}} \int_{[*, i_m, j_0, \ldots, j_l]}$$

$$= \sum_{l=-1}^{n-2} \sum_{j_0 < \cdots < j_l} \mathcal{W}_{i_m, j_0, \ldots, j_l} \int_{[*, i_m, j_0, \ldots, j_l]}$$

Therefore

$$\dot{W} a^* \dot{I} = -\sum_{l=-1}^{n-2} \sum_{j_0 < \cdots < j_l} \mathcal{W}_{i_m, j_0, \ldots, j_l} \sum_{\alpha \in I \setminus J} \int_{[*, i_m, j_0, \ldots, j_l]}$$

Collecting the results gives

$$R_{j_0, \ldots, j_l} = -\sum_{\alpha \in I \setminus J} \int_{[*, i_m, j_0, \ldots, j_l]}$$

$$+ h^j \ldots h^0 \left( \sum_{i_m \in I \setminus \{J, \widehat{J} \}} h^{i_m} - (k + 1) h^* + h^{i_m} \right).$$

Bringing $h^j \ldots h^0$ to the right gives the desired result.

$$R_{j_0, \ldots, j_l} = (-1)^{l+1} \left( -\sum_{i_m \in I \setminus J} \varepsilon^* h^{i_m} + \sum_{i_m \in I \setminus J} h^{i_m} - (k + 1) h^* \right) h^j \ldots h^0$$

As a sanity check, we can verify that $R_0(\omega) = 0$ for all $\omega$, when $n = 1$. That is,

$$(1 - \varepsilon^*) \sum_{i_m \in I} h^{i_m}(\omega) = (k + 1) h^*(\omega) = 0,$$

which for $n = 1$ becomes the formula

$$(1 - \varepsilon^*)(h^0 + h^1)(g(t) \, dt) = (1 - \varepsilon^*) \left[ \int_t^0 g(u) \, du + \int_t^1 g(u) \, du \right]$$

$$= 2 \int_t^{1/2} g(u) \, du = 2 h^*(g(t) \, dt)$$

as expected.

For general $n$, however, it is not true that $\dot{W} a^* \dot{I} + \dot{s} - s = 0$. The best we can do is give an explicit formula for the homotopy $Q$. 

\[\square\]
Theorem 10. Let

\[ Q = - \sum_{i=-1}^{n-2} \sum_{j_0, \ldots, j_l} \sum_{i_{m} \in I \setminus J} \omega_{i_{m}, j_0, \ldots, j_l} h^{i_{m}}h^{j_0}h^{j_1} \ldots h^{j_l} \]

Then \( \mathcal{W}a^*I + \mathcal{s} - s = dQ - Qd \).

Proof. The contraction homotopy to the barycenter \( \phi^* \) induces the homotopy \( h^* \) of cochain maps satisfying

\[ dh^* + h^*d = \varepsilon^* - 1 \]

Given a homomorphism \( G \in \text{Hom}^i(\Omega^*(\Delta^n), \Omega^*(\Delta^n)) \), let

\[ H(G) = -h^*G\varepsilon^* - (-1)^{|G|}Gh^* \]

and let

\[ E(G) = \varepsilon^*G\varepsilon^* \]

Then

\[ DH + HD = 1 - E \]

where \( D \) is the action of \( d \) on linear maps; that is, by graded commutator, \( D(G) = [d, G] \)

Since \( \mathcal{W}a^*I + \mathcal{s} - s \) is closed under \( D \) and \( E(\mathcal{W}a^*I + \mathcal{s} - s) = 0 \),

\[ \mathcal{W}a^*I + \mathcal{s} - s = DH(\mathcal{W}a^*I + \mathcal{s} - s) \]

Therefore,

\[ Q = H(\mathcal{W}a^*I + \mathcal{s} - s) = (\mathcal{W}a^*I + \mathcal{s} - s)h^* \]

But

\[ R_{j_0, \ldots, j_l}h^* = - \sum_{i_{n} \in I \setminus J} h^{i_{n}}h^{j_0}h^{j_1} \ldots h^{j_l} \]

Notice that in the above proof, we could have also chosen \( H'(G) = -h^*G - (-1)^{|G|}\varepsilon^*Gh^* \), for which the formula \( DH' + H'D = 1 - E \) also holds. Because

\[ \varepsilon^*(\mathcal{W}a^*I + \mathcal{s} - s) = \frac{1}{k+1} \left( \sum_{i_{n} \in I \setminus J} \varepsilon^*h^{i_{n}} + \sum_{i_{n} \in I \setminus J} \varepsilon^*h^{i_{n}} - (k+1)\varepsilon^*h^* \right) = 0 \]

\[ \mathcal{W}a^*I + \mathcal{s} - s = DH'(\mathcal{W}a^*I + \mathcal{s} - s) = -D(h^*(\mathcal{W}a^*I + \mathcal{s} - s)) \]

Then \( Q' = -h^*(\mathcal{W}a^*I + \mathcal{s} - s) \) is another valid choice of homotopy. Curiously, we now have the relation \( Q - Q' = [\mathcal{W}a^*I + \mathcal{s} - s, h^*] \) is closed and therefore exact.
7. Compatibility of the Cubical DHF and Cubical Stellar Subdivision

The Dupont homotopy on $\star_{1,\ldots,k}\square^n$ is a deformation retraction

$$(C^\bullet(\star_{1,\ldots,k}\square^n), d) \overset{\Phi}{\longrightarrow} (\Omega^\bullet(\star_{1,\ldots,k}\square^n), d) \overset{\cdot}{\longrightarrow} s$$

where $\Phi = (I)^{\otimes k} \otimes I^{\otimes (n-k)} = \Phi_k \otimes I_{n-k}$, $\Phi = (W)_{\otimes k} \otimes W^{\otimes (n-k)} = \Phi_k \otimes W_{n-k}$

and

$$s = \frac{1}{n!} \sum \epsilon C_{\epsilon|n} \psi_{\epsilon}$$

where $C_{\epsilon|n} = |\epsilon|!(n-1-|\epsilon|)!$ and the outer sum is over $\epsilon_k = 0, 1$ and where

$$\psi_{\epsilon} = \sum_{j=1}^{k} (W^n)^{j_1} \otimes \cdots \otimes (W^n)^{j_{k-1}} \otimes (W^n)^{j_k} \otimes \cdots \otimes (W^n)^{j_n}$$

Theorem 11. Composing with the stellar subdivision deformation retraction gives a new deformation retraction

$$(C^\bullet(\square^n), d) \overset{p \cdot \Phi}{\longrightarrow} (\Omega^\bullet(\star_{1,\ldots,k}\square^n), d) \overset{\cdot}{\longrightarrow} s + \Phi^* \alpha \cdot i$$

This is homotopic to the cubical Dupont deformation retraction

$$(C^\bullet(\square^n), d) \overset{I}{\longrightarrow} (\Omega^\bullet(\star_{1,\ldots,k}\square^n), d) \overset{\cdot}{\longrightarrow} s$$

That is, $p \cdot \Phi = I$, $\Phi^* = W$, and there exists $Q$ such that $\Phi^* \alpha \cdot i + s - dQ - Qd$.

Proof. The identities $p \cdot \Phi = I$ and $\Phi^* = W$ follow from the identities $p \cdot \Phi = I$ and $\Phi^* = W$ on $\Delta^1$ that were proved in Theorem 7.

We shall try to write down the homotopy $Q$ explicitly only in the more symmetric case $k = n$ for which the formulas are easier to handle. For $k = n$,

$$\Phi^* \alpha \cdot i + s - dQ - Qd = \sum_{\sigma \in S_n} \tau_{\sigma} \circ \left( \sum_{j=1}^{n} (W^n)^{\otimes (j-1)} \otimes (W^n)^{\otimes (n-j)} \right) \circ \tau_{\sigma}^{-1}$$

$$+ \sum_{\sigma \in S_n} \tau_{\sigma} \circ \left( \sum_{j=1}^{n} (W^n)^{\otimes (j-1)} \otimes s \otimes (W^n)^{\otimes (n-j)} \right) \circ \tau_{\sigma}^{-1}$$

$$- \sum_{\sigma \in S_n} \tau_{\sigma} \circ \left( \sum_{j=1}^{n} (W^n)^{\otimes (j-1)} \otimes s \otimes (W^n)^{\otimes (n-j)} \right) \circ \tau_{\sigma}^{-1}$$
and thus

\[
\dot{W}a^* \dot{I} + \dot{s} - s = \sum_{\sigma \in S_n} \tau_\sigma \circ \left( \sum_{j=1}^n (\dot{W}I - 1) \otimes (j-1) \otimes s \otimes (WI) \otimes (n-j) \right) \circ \tau^{-1}_\sigma + \sum_{\sigma \in S_n} \tau_\sigma \circ \left( \sum_{j=1}^n (\dot{W}I) \otimes (j-1) \otimes \dot{s} \otimes (1 - WI) \otimes (n-j) \right) \circ \tau^{-1}_\sigma
\]

The symmetric tensor construction applied to \(h^*\) gives a homotopy \(h^*,\) satisfying

\[d h^* + h^* d = \epsilon^* - 1\]

Then

\[Q = (\dot{W}a^* \dot{I} + \dot{s} - s) h^*\]

satisfies \(\dot{W}a^* \dot{I} + \dot{s} - s = d Q - Q d.\) The formula can be simplified using the identities \(W I \epsilon^* = \epsilon^*\) and \(\dot{W}I \epsilon^* = \epsilon^*,\) but we shall not write it down any more explicitly.

\[\square\]

8. ELEMENTARY EXPANSION AND COLLAPSE

Let \(Y\) be a simplicial complex containing a \(k\)-simplex \(\sigma\) and a \((k-1)\)-simplex \(\sigma'\) such that \(\sigma\) is the only \(k\)-simplex whose boundary contains \(\sigma'.\) Let \(X \subset Y\) be the subcomplex obtained by removing the pair \(\sigma, \sigma'\) from \(Y.\) Then one calls \(X\) an elementary collapse of \(Y\) and \(Y\) an elementary expansion of \(X.\)

To write down simplicial chains, we choose an orientation for each simplex. Suppose that \(\partial \sigma = \sum \varepsilon_\tau \tau\) where \(\varepsilon_\tau = \pm 1.\) There is a natural projection \(p_\downarrow: C_*(Y) \rightarrow C_*(X)\) sending \(\sigma' \mapsto \sigma' - \varepsilon_{\sigma'} \partial \sigma\) and \(\sigma \mapsto 0.

**Proposition 5.** There is an elementary collapse deformation retraction

\[C_*(X) \xrightarrow{i_\downarrow} C_*(Y) \xleftarrow{a_\downarrow} C_*(X)\]

where \(i_\downarrow\) is the natural inclusion, \(p_\downarrow\) is as above, and \(a_\downarrow(\sigma') = \varepsilon_{\sigma'} \sigma\) and \(a_\downarrow(\tau) = 0\) for \(\tau \neq \sigma'.\)

**Proof.** We verify that \(p_\downarrow\) is a chain map by computing \(\partial p_\downarrow(\sigma') = \partial (\sigma' - \varepsilon_{\sigma'} \partial \sigma) = p_\downarrow \partial (\sigma')\) and

\[
\varepsilon_{\sigma'} p_\downarrow \partial (\sigma) = p_\downarrow (\sigma') + p_\downarrow (\varepsilon_{\sigma'} \partial (\sigma) - \sigma')
\]

\[= \sigma' - \varepsilon_{\sigma'} \partial (\sigma) + \varepsilon_{\sigma} \partial (\sigma) - \sigma' = 0 = \partial p_\downarrow (\sigma).\]

Lastly, we verify that \(\partial a_\downarrow + a_\downarrow \partial = 1 - i_\downarrow p_\downarrow.\) We have

\[\partial a_\downarrow (\sigma) + a_\downarrow (\partial (\sigma)) = a_\downarrow (\partial (\sigma)) = \sigma = \sigma - i_\downarrow p_\downarrow (\sigma)\]

and

\[\partial a_\downarrow (\sigma') + a_\downarrow (\partial (\sigma')) = \partial a_\downarrow (\sigma') = \varepsilon_{\sigma'} \partial (\sigma) = \sigma' - (\sigma' - \varepsilon_{\sigma'} \partial (\sigma)) = \sigma' - i_\downarrow p_\downarrow (\sigma').\]

\[\square\]
Let
\[ i^i(\hat{\tau}) = \begin{cases} \hat{\tau} - \varepsilon_{\sigma'}\hat{\tau}' & \text{if } \tau \text{ is in } \partial\sigma - \sigma' \\ \hat{\tau} & \text{otherwise} \end{cases} \]
\[ p^i(\hat{\tau}) = \begin{cases} \hat{\tau} & \text{if } \tau \text{ is in } X \\ 0 & \text{if } \tau = \sigma, \sigma' \end{cases} \]
and lastly,
\[ a^i(\hat{\tau}) = \begin{cases} \varepsilon_{\sigma'}\hat{\tau}' & \text{if } \tau = \sigma \\ 0 & \text{otherwise} \end{cases} \]

**Proposition 6.** There is an elementary collapse deformation retraction on cochains
\[ C^\bullet(X) \xrightarrow{p^i} C^\bullet(Y) \xleftarrow{a^i} \]
where \( i^i, p^i \) and \( a^i \) are as defined above. This is dual to the deformation retraction defined in Proposition 5.

9. Stellar Subdivision from Elementary Expansions and Collapses

A stellar subdivision of the \( n \)-simplex can be constructed as a sequence of elementary expansions and elementary collapses. For example for the 1-simplex, there are two such sequences:

\[ \begin{array}{cccc}
  & & & \\
  & & \downarrow & \\
  & & & \\
  & \downarrow & & \\
  & & & \\
\end{array} \]

\[ \begin{array}{cccc}
  & & & \\
  & & \downarrow & \\
  & & & \\
  \downarrow & & & \\
  & & & \\
\end{array} \]

We start by identifying \( \Delta^1 = [e_0, e_1] \) in \( \Delta^2 = [e_0, e_*, e_1] \). All simplices that we consider in the sequence of elementary expansions and collapses shall be given the induced orientation from \( \Delta^2 \). We shall indicate the simplex \( \sigma \) in the subscript of the maps in the elementary collapse deformation retractions, and use primes to indicate the subsimplex \( \sigma' \subset \sigma \).

We consider the first sequence of elementary expansions. The first inclusion map is
\[ i^{[e_0, e_1]}(\hat{\tau}) = \begin{cases} \hat{\tau} & \text{if } \tau = e_0 \\ \varepsilon_{e_0} + \varepsilon_* & \text{if } \tau = e_* \end{cases} \]
and the second inclusion map is
\[ i^{[e_0, e_*, e_1]}(\hat{\tau}) = \begin{cases} [e_0, e_*] & \text{if } \tau = [e_0, e_*] \\ [e_0, e_1] & \text{if } \tau = [e_0, e_1] \\ \hat{\tau} & \text{otherwise} \end{cases} \]
and their composition is the inclusion map
\[ i^{[e_0, e_0, e_1]}_{1} (\hat{\tau}) = \begin{cases} [e_0, e_1] + [e_0, e_1] & \text{if } \tau = [e_0, e_1] \\ \hat{\tau} & \text{if } \tau = e_0 \\ \hat{\tau} & \text{otherwise} \end{cases} \]
the projection map

\[ p_1^1(\tilde{\tau}) = p^{[e_0, e_i]} p^{[e_0, e_i]}(\tilde{\tau}) = \begin{cases} \tilde{\tau} & \text{if } \tau \in \Delta^1 \\ 0 & \text{otherwise} \end{cases} \]

and the homotopy

\[ a_1^1(\tilde{\tau}) = (a^{[e_0, e_i]} + i^{[e_0, e_i]} a^{[e_0, e_i]} p^{[e_0, e_i]})(\tilde{\tau}) = \begin{cases} [e_*, e_1] & \text{if } \tau = [e_0, e_*, e_1] \\ [e_*, e_*] & \text{if } \tau = [e_0, e_*] \\ 0 & \text{otherwise} \end{cases} \]

We then apply the elementary collapse to get

\[ i_1(\tilde{\tau}) = p^{[e_0, e_i]} i_1^4(\tilde{\tau}) = \begin{cases} [e_*, e_1] & \text{if } \tau = [e_0, e_1] \\ e_0 + e_* & \text{if } \tau = e_0 \\ \tilde{\tau} & \text{otherwise} \end{cases} \]

and

\[ \hat{i}_1(\tilde{\tau}) = p_1^4 i^{[e_0, e_i]}(\tilde{\tau}) = \begin{cases} [e_0, e_1] & \text{if } \tau = [e_0, e_*] \\ [e_0, e_1] & \text{if } \tau = [e_*, e_1] \\ 0 & \text{if } \tau = e_* \\ \tilde{\tau} & \text{otherwise} \end{cases} \]

and lastly

\[ \hat{a}_1(\tilde{\tau}) = p^{[e_0, e_*]} a_1^4(\tilde{\tau}) = \begin{cases} e_* & \text{if } \tau = [e_0, e_*] \\ 0 & \text{otherwise}. \end{cases} \]

For the second sequence of elementary expansions: The inclusion map is

\[ i_2^1(\tilde{\tau}) = i^{[e_0, e_i]} i_1^4(\tilde{\tau}) = \begin{cases} [e_0, e_1] + [e_0, e_*] & \text{if } \tau = [e_0, e_1] \\ e_1 + e_* & \text{if } \tau = e_1 \\ \tilde{\tau} & \text{otherwise} \end{cases} \]

the projection map is

\[ p_2^1(\tilde{\tau}) = p^{[e_*, e_1]} p^{[e_*, e_1]}(\tilde{\tau}) = \begin{cases} \tilde{\tau} & \text{if } \tau \in \Delta^1 \\ 0 & \text{otherwise} \end{cases} \]

and the homotopy is

\[ a_2^1(\tilde{\tau}) = (a^{[e_0, e_*]} + i^{[e_0, e_*]} a^{[e_0, e_*]} p^{[e_0, e_*]})(\tilde{\tau}) = \begin{cases} [e_0, e_*] & \text{if } \tau = [e_0, e_*, e_1] \\ -e_* & \text{if } \tau = [e_*, e_1] \\ 0 & \text{otherwise} \end{cases} \]

We then apply the elementary collapse to get

\[ i_2(\tilde{\tau}) = p^{[e_0, e_*]} i_2^4(\tilde{\tau}) = \begin{cases} [e_0, e_*] & \text{if } \tau = [e_0, e_1] \\ e_1 + e_* & \text{if } \tau = e_1 \\ \tilde{\tau} & \text{otherwise} \end{cases} \]
and
\[
\hat{p}_2(\hat{\tau}) = p_2^{l_1, l_0, e_*}(\hat{\tau}) = \begin{cases} 
[e_0, e_1] & \text{if } \tau = [e_0, e_*] \\
[e_0, e_1] & \text{if } \tau = [e_*, e_1] \\
0 & \text{if } \tau = e_* \\
\hat{\tau} & \text{otherwise}
\end{cases}
\]
and lastly
\[
\hat{a}_2(\hat{\tau}) = a_2^{l_1, l_0, e_*}(\hat{\tau}) = \begin{cases} 
-\epsilon_* & \text{if } \tau = [e_* , e_1] \\
0 & \text{otherwise}
\end{cases}
\]
Let
\[
\hat{i}(\hat{\tau}) = \frac{1}{2}(\hat{\tau} + \hat{\tau})(\hat{\tau}) = \begin{cases} 
\frac{1}{2}([e_0, e_*] + [e_* , e_1]) & \text{if } \tau = [e_0, e_1] \\
\frac{1}{2}c_0 + \epsilon_* & \text{if } \tau = e_0 \\
\frac{1}{2}c_1 + \epsilon_* & \text{if } \tau = e_1 \\
\hat{\tau} & \text{otherwise},
\end{cases}
\]
let \( \hat{p} = \hat{p}_1 = \hat{p}_2 \) and let
\[
\hat{a}(\hat{\tau}) = \frac{1}{2}(\hat{a}_1 + \hat{a}_2)(\hat{\tau}) = \begin{cases} 
\frac{1}{2}c_1 & \text{if } \tau = [e_0, e_1] \\
-\frac{1}{2}c_1 & \text{if } \tau = [e_1, e_2] \\
0 & \text{otherwise}
\end{cases}
\]
Then we have \( d\hat{a} + \hat{a}d = 1 - \hat{i}p \), and remarkably we still have the property \( \hat{p}^* = 1 \), despite only composing a zigzag of deformations retractions. More remarkably, this deformation retraction is equal to the stellar subdivision deformation retraction on \( \Delta^1 \) that we defined at the beginning of Section 4. That is, \( \hat{i} = 0, \hat{p} = p^* \) and \( \hat{h} = h^* \).

More generally, for \( \star \Delta^n \), the stellar subdivision of the \( n \)-simplex, we shall give \( n + 1 \) different sequences of elementary expansions followed by an elementary collapse. One such sequence is illustrated in the following picture

Once again, we begin by embedding \( \Delta^1 \) in \( \Delta^{n+1} \) by identifying \( \Delta^{n+1} \) with \([e_*, e_0, \ldots, e_n]\). We shall again use primes to indicate the subsimplex \( \sigma' \subset \sigma \) in each elementary expansion. The sequence of elementary expansion begins by choosing a vertex \( e_j \) from \( e_0, \ldots, e_n \) and adding the edge \([e'_j, e_j]\). Then for each vertices \( e_k \) in \( e_0, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n \) we add the 2-simplex \([e'_k, e_j, e_k]\). Then for each pair of vertices \( e_k, e_k \) in \( e_0, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n \), we add the 3-simplex \([e'_k, e_j, e_k, e_k]\). We continue inductively and end up adding an \( l \)-simplex for each of the \( \binom{n}{l-1} \) choices of \( l - 1 \) vertices.

Every simplex in the simplicial complex \( \Delta^{n+1} \) will be added by this procedure. Clearly, any simplex containing both \( e_* \) and \( e_j \) will be added. Any simplex not containing \( e_* \) is already present from \( \Delta^n \). A simplex \([e_*, e_k, \ldots, e_k]\), which does not contain \( e_j \), is added with \([e_0, e_j, e_k, \ldots, e_k]\) in elementary expansion.
For each elementary expansion the inclusion map is given by

\[
\left[ e_{*}, e_{j}, e_{k_{1}}, \ldots, e_{k_{p}} \right] = \begin{cases} 
\left[ e_{*}, e_{j}, e_{k_{1}}, \ldots, e_{k_{p}} \right] & \text{if } \tau = [e_{*}, e_{j}, e_{k_{1}}, \ldots, e_{k_{p}}] \\
\left[ e_{j}, e_{k_{1}}, \ldots, e_{k_{p}} \right] & \text{if } \tau = [e_{j}, e_{k_{1}}, \ldots, e_{k_{p}}] \\
\tau & \text{otherwise}
\end{cases}
\]

the projection map is given by

\[
p_{\left[ e_{*}, e_{j}, e_{k_{1}}, \ldots, e_{k_{p}} \right]} (\tau) = \begin{cases} 
0 & \text{if } \tau = [e_{*}, e_{j}, e_{k_{1}}, \ldots, e_{k_{p}}] \\
0 & \text{if } \tau = [e_{*}, e_{k_{1}}, \ldots, e_{k_{p}}] \\
\tau & \text{otherwise}
\end{cases}
\]

and the homotopy is given by

\[
a_{\left[ e_{*}, e_{j}, e_{k_{1}}, \ldots, e_{k_{p}} \right]} (\tau) = \begin{cases} 
-\left[ e_{*}, e_{k_{1}}, \ldots, e_{k_{p}} \right] & \text{if } \tau = [e_{*}, e_{j}, e_{k_{1}}, \ldots, e_{k_{p}}] \\
0 & \text{otherwise}
\end{cases}
\]

The inclusion map for the sequence of elementary expansions is thus

\[
i_{j}^{\left[ e_{*} \right]} (\tau) = \begin{cases} 
\left[ e_{j}, e_{k_{1}}, \ldots, e_{k_{p}} \right] & \text{if } \tau = [e_{j}, e_{k_{1}}, \ldots, e_{k_{p}}] \text{ for } p = 0, \ldots, n \\
\tau & \text{otherwise}
\end{cases}
\]

the projection map is thus

\[
p_{j}^{\left[ e_{*} \right]} (\tau) = \begin{cases} 
0 & \text{if } \tau = [e_{*}, e_{j}, e_{k_{1}}, \ldots, e_{k_{p}}] \text{ for } p = 0, \ldots, n - 1 \\
0 & \text{if } \tau = [e_{*}, e_{k_{1}}, \ldots, e_{k_{p}}] \text{ for } p = 0, \ldots, n \\
\tau & \text{otherwise}
\end{cases}
\]

and the homotopy is thus

\[
a_{j}^{\left[ e_{*} \right]} (\tau) = \begin{cases} 
-\left[ e_{*}, e_{k_{1}}, \ldots, e_{k_{p}} \right] & \text{if } \tau = [e_{*}, e_{j}, e_{k_{1}}, \ldots, e_{k_{p}}] \\
0 & \text{otherwise}
\end{cases}
\]

Composing with the elementary collapse \([e_{*}, e_{0}', \ldots, e_{n}']\) gives

\[
i_{\left[ e_{*} \right]} (\tau) = p_{\left[ e_{*}, e_{0}', \ldots, e_{n}' \right]} i_{j}^{\left[ e_{*} \right]} (\tau) = \begin{cases} 
(-1)^{j} [e_{*}, e_{0}', \ldots, \tilde{e}_{j}, \ldots, e_{n}] & \text{if } \tau = [e_{0}, \ldots, e_{n}] \\
\left[ e_{j}, e_{k_{1}}, \ldots, e_{k_{p}} \right] & \text{if } \tau = [e_{j}, e_{k_{1}}, \ldots, e_{k_{p}}] \text{ for } p = 0, \ldots, n - 1 \\
\tau & \text{otherwise}
\end{cases}
\]

and because

\[
i_{\left[ e_{*} \right]}^{\left[ e_{0}', \ldots, e_{n}' \right]} (\tau) = \begin{cases} 
\left[ e_{*}, e_{0}', \ldots, \tilde{e}_{l}, \ldots, e_{n} \right] & \text{if } \tau = [e_{*}, e_{0}', \ldots, \tilde{e}_{l}, \ldots, e_{n}] \\
(-1)^{j} [e_{0}, \ldots, e_{n}] & \text{otherwise}
\end{cases}
\]
we have
\[ \hat{\psi}_j(\hat{\tau}) = p_j^j |_{[e_*, e_0', \ldots, e_n']} |_{\hat{\tau}} = \begin{cases} (-1)^l |e_0, \ldots, e_n| & \text{if } \tau = [e_*, e_0, \ldots, \hat{e}_l, \ldots, e_n] \text{ for } l = 0, \ldots, n \\ \hat{\tau} & \text{otherwise} \end{cases} \]
and lastly
\[ \hat{\alpha}_j(\hat{\tau}) = p_j^j |_{[e_*, e_0', \ldots, e_n']} |_{\hat{\tau}} = \begin{cases} -[e_*, e_{k_1}, \ldots, e_{k_p}] & \text{if } \tau = [e_*, e_j, e_{k_1}, \ldots, e_{k_p}] \text{ for } p = 0, \ldots, n - 1 \\ 0 & \text{otherwise} \end{cases} \]

Let \( \hat{i} = \frac{1}{n+1} \sum_j \hat{i}_j \), let \( \hat{p} = \hat{p}_j \) and let \( \hat{a} = \frac{1}{n+1} \sum_j \hat{a}_j \). Because we have composed a zigzag of deformation retractions, we have \( d\hat{a} + \hat{a}d = 1 - \hat{i}p \). Remarkably it is still the case the \( \hat{p}^* = 1 \). Furthermore,

**Theorem 12.** The deformation retraction

\[ C^\bullet(\Delta^n) \xleftarrow{\hat{p}} C^\bullet(\ast\Delta^n) \xrightarrow{\hat{a}} \]

is equal to the stellar subdivision on cochains deformation retraction of Theorem 5. That is \( i = i^* \), \( p = p^* \) and \( a = a^* \).

For simplicity, we have chosen in this section to focus on the construction for the stellar subdivision \( \ast\Delta^n = \ast_1 \ldots n \Delta^n \).

We indicate how one would proceed in constructing the more general stellar subdivision \( \ast_{i_0, \ldots, i_k} \Delta^n \). For each \( i_\alpha \in \{i_0, \ldots, i_k\} \), we follow the same sequence of elementary expansions followed by an elementary collapse as specified above. Each of these \( k + 1 \) sequences constructs \( \ast\Delta^n \) from \( \Delta^n \). We succeed each sequence with addition elementary collapses of all the simplices \([e_*, e_{j_0}', \ldots, e_{j_\alpha}']\) containing \([e_*, e_{i_0}, \ldots, e_{i_k}]\) starting in dimension \( n - 1 \) and moving to lower dimensions. Let \( \hat{i}_\alpha, \hat{p}_\alpha, \text{ and } \hat{a}_\alpha \) be the inclusion, projection, and homotopy respectively that results from composing the specified sequence of elementary expansions followed by elementary collapses. Let \( \hat{i} = \frac{1}{k+1} \sum_\alpha \hat{i}_\alpha \), let \( \hat{p} = \hat{p}_\alpha \), and let \( \hat{a} = \frac{1}{k+1} \sum_\alpha \hat{a}_\alpha \). In conclusion, we claim that these maps form a deformation retraction that is equal to the stellar subdivision deformation retraction that was constructed in Theorem 5.

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