Diamagnetism of confined Dirac fermions in disordered graphene

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Abstract
The diamagnetism of confined Dirac fermions submitted to a uniform magnetic field in disordered graphene is investigated. The solutions of the energy spectrum are used to discuss the orbital magnetism from a statistical mechanical point of view. More precisely, by the technique of Green’s functions, the self-energy for short- and long-ranged disorders is obtained. This allows us to determine the susceptibility for short- and long-ranged disorders together with confinement. We compare our results with already published work and point out the relevance of these findings to a systematic formulation of the diamagnetism in a confining potential.

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1. Introduction

The magnetism of graphene was first studied as a simple model for three-dimensional graphite [1] where the susceptibility of the disorder-free graphene was calculated within the effective mass approximation. It was found that the system exhibits a large diamagnetism at the Fermi energy $\varepsilon_F = 0$, expressed as a $\delta$-function of $\varepsilon_F$ at the absolute zero temperature. The graphene magnetism was considered again in studies on the graphite intercalation compounds, where the tight-binding model was applied for a wide range of Fermi energies [2–5]. The effects of disorder on graphene under magnetic fields have been examined in early theoretical studies before the experimental discovery of graphene, where the electronic structure [6], the transport properties [6–8] and the de Haas–van Alphen effect [9] were investigated. Recently,
the Shubnikov–de Haas oscillation was studied in disordered graphene [10, 11] and the spectral and transport properties were examined in the presence of lattice defects under the magnetic fields [12].

Very recently, an interesting development on the diamagnetism of disordered graphene was reported by Koshino and Ando [13]. They studied the graphene monolayer orbital magnetism within the effective mass approximation. In models of short- and long-ranged disorders, magnetization was calculated with the self-consistent Born approximation. In the zero-field limit, the susceptibility becomes highly diamagnetic around zero energy, while it has a long tail proportional to the inverse of the Fermi energy. It was demonstrated how the magnetic oscillation vanishes and converges to the susceptibility, on going from a strong-field regime to a zero field. Additionally, the behavior at zero energy was shown to be highly singular.

On the other hand, an exact solution of a related problem, that has been studied at various levels by researchers dealing with different physical issues (see for instance [14, 15]), was given by one of us (AJ) and his collaborators [16], considering a relativistic particle subjected to an external magnetic field as well as to a confining potential. By a similarity transformation, the system can be diagonalized in a simple way. Solving the eigenvalue equation and accounting for the complete space of the eigenfunctions, one can include various cases related to different physical settings. More precisely, from the nature of the problem, it was possible to obtain separate angular and radial solutions. The radial equation leads to an exact relation between the two spinor components. In fact, depending on the range of values of three physical quantities, the full solution space split into eight disconnected subspaces as summarized in table 1.

Motivated by different investigations on Dirac fermions in (2+1)-dimensions, in particular by references [13, 16], we treat the diamagnetism of a confined system in a statistical mechanical way. More precisely, we study the orbital magnetism of Dirac fermions in a uniform magnetic field, disordered graphene and confining potential. In fact, we combine studies reported in [13, 16] to generalize the results of Koshino and Ando [13] about diamagnetism in disordered graphene to the confinement case. This can be done by using the energy spectrum solutions to study the self-energy for both regimes: short- and long-ranged disorders by the technique of Green’s functions. The self-energy allows us to obtain the shape of the density of states that is needed to determine the related thermodynamical quantities and discuss different issues.

Subsequently, we calculate the susceptibility for two regimes and underline what makes the difference with respect to the standard case [13], namely the analysis without confinement ($\kappa = 0$). For the short-ranged disorder, we obtain interesting results in terms of a parameter of confinement and disorder strength, called $C$. In particular, we show that there is a quantum correction to the result obtained by Koshino and Ando [13], which disappears by switching off $\kappa$. Furthermore, we note that the clean limit result without confinement can be obtained by considering $C$ going to zero.

As far as the long-range disorder is concerned, the susceptibility shows an additional second term of the order of $O(C)$ compared to the short-range case. However, this gives a minor effect since $C$ is assumed to be small. When the terms of the order $O(C)^2$ are

| Frequency | $\kappa > l_B^{-2}$ | $\kappa < l_B^{-2}$ |
|-----------|---------------------|---------------------|
| Energy    | $\varepsilon > 0$  | $\varepsilon < 0$  |
| Azimuth   | $k \geq 0$          | $k < 0$             |
neglected, the susceptibility becomes just $1 - 3C$ times as large as in the short-ranged disorder. Accordingly, the integration of the susceptibility over the energy $\varepsilon$ depends weakly on $C$, while in the limit $C \to 0$ we again obtain the susceptibility as a $\delta$-function. Finally, we note that the case $\kappa = 0$ allows us to recover the results and related conclusions proposed in [13].

The paper is organized as follows. In section 2, we review the energy spectrum solutions of the confinement problem needed to deal with different issues. Section 3 is devoted to introduce disordered graphene where we give the corresponding four-component spinors as well as the disorder and confining potentials. In section 4, we use the self-consistent Born approximation to determine the self-energy and the density of states. These will allow us to treat the orbital magnetism by distinguishing between the short- and long-ranged disorders in section 5. Finally, we conclude by discussing the main results and possible extension of our work.

2. Confinement problem

We start by formulating the problem in terms of our approach. In doing so, we introduce a similarity transformation of the Dirac equation in polar coordinates. This will be convenient to handle the exact relationship between spinor components and thus derive the full spectrum accounting for the complete Hilbert space.

2.1. Hamiltonian

The problem of a charged particle moving in a constant magnetic field $\vec{B} = B \hat{z}$ is a 2D problem in the plane normal to the field (the Cartesian $(x, y)$-plane or cylindrical $(r, \theta)$-plane). In the relativistic units, $\hbar = c = 1$, the Dirac equation in $(2+1)$-dimensions for a spinor of charge $e$ and massless in the electromagnetic potential $A_\mu = (A_0, \vec{A})$ reads

$$[i \gamma^\mu (\partial_\mu + i e A_\mu)] \psi = 0, \quad \mu = 0, 1, 2, \quad \gamma^\mu = (\gamma^0, \vec{\gamma})$$

where the summation convention over repeated indices is used. $\gamma^\mu = (\gamma^0, \vec{\gamma})$ are three unimodular square matrices satisfying the anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 G^{\mu\nu}, \quad (2)$$

where $G$ is the metric of Minkowski spacetime, which is equal to diag$(+ - -)$. A minimal irreducible matrix representation that satisfies this relation is given by $\gamma^0 = \sigma_3$, $\vec{\gamma} = i \vec{\sigma}$, where $\{\sigma_i\}_{i=1}^3$ are the $2 \times 2$ Hermitian Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3)$$

Equation (1) can be rewritten as

$$i \frac{\partial}{\partial t} \psi = (-i \vec{\alpha} \cdot \vec{\nabla} + e \vec{a} \cdot \vec{A} + e A_0) \psi, \quad (4)$$

where $\vec{a}$ is the Hermitian matrix $\vec{a} = i \sigma_3 \vec{\sigma}$. We will see below that the symmetry of the problem is preserved even if we introduce an additional coupling to the 2D Dirac-oscillator potential. This coupling is introduced by the substitution $\vec{\nabla} \to \vec{\nabla} + \kappa \sigma_3$ where $\kappa$ is a constant parameter.

For time-independent potentials, the two-component spinor wavefunction $\psi(t, r, \theta)$ is written as

$$\psi(t, r, \theta) = e^{-i\varepsilon t} \psi(r, \theta) \quad \psi(t, r, \theta)$$
and (4) becomes the energy eigenvalue equation \((H - \epsilon)\psi = 0\), where \(\epsilon\) is the relativistic energy. The Dirac Hamiltonian \(H\) is the \(2 \times 2\) matrix operator

\[
H = i \sigma_3 \hat{r} \cdot \hat{r} + i \sigma_3 \hat{\theta} \cdot \hat{\theta} + H_r + i \sigma_3 \hat{r} \cdot \hat{r} + i \sigma_3 \hat{\theta} \cdot \hat{\theta} + H_\theta,
\]

where \((\hat{r}, \hat{\theta})\) are the unit vectors in cylindrical coordinates and

\[
H_r = -i \partial_r + eA_r - ikr \sigma_3,
\]

\[
H_\theta = -\frac{i}{r} \partial_\theta + eA_\theta.
\]

For regular solutions of (4), square integrability (with respect to the measure \(d^2\hat{r} = r \, dr \, d\theta\)) and the boundary conditions require that \(\psi(r, \theta)\) satisfies

\[
\sqrt{r} \psi(r, \theta) \bigg|_{r=0} = 0, \quad \sqrt{r} \psi(r, \theta) \bigg|_{r\to\infty} = 0, \quad \psi(\theta + 2\pi) = \psi.
\]

To simplify the construction of the solution, we look for a local \(2 \times 2\) similarity transformation \(\Lambda_1(r, \theta)\) that maps the cylindrical projection of the Pauli matrices \((\hat{\sigma} \cdot \hat{r}, \hat{\sigma} \cdot \hat{\theta})\) into their canonical Cartesian representation \((\sigma_1, \sigma_2)\), respectively. That means

\[
\Lambda_1 \hat{\sigma} \cdot \hat{r} \Lambda_1^{-1} = \sigma_1, \quad \Lambda_1 \hat{\sigma} \cdot \hat{\theta} \Lambda_1^{-1} = \sigma_2.
\]

A \(2 \times 2\) matrix that satisfies this requirement is

\[
\Lambda_1(r, \theta) = \lambda(r, \theta) e^{i \sigma_3 \theta},
\]

where \(\lambda(r, \theta)\) is a real function and the exponential is a \(2 \times 2\) unitary matrix. The Dirac Hamiltonian (6) gets mapped into

\[
H = \Lambda H \Lambda^{-1} = -\sigma_2 H_r + \sigma_1 H_\theta,
\]

where

\[
H_r = -i \left( \partial_r - \frac{\lambda_r}{\lambda} \right) + ie A_r - ikr \sigma_3,
\]

\[
H_\theta = -\frac{i}{r} \left( \partial_\theta - \frac{\lambda_\theta}{\lambda} - \frac{i}{2} \sigma_3 \right) + eA_\theta
\]

with \(\lambda_k = \partial_k \lambda\). Therefore, the \(2 \times 2\) Dirac Hamiltonian becomes

\[
H = \begin{pmatrix}
0 & \partial_r - \frac{\lambda_r}{\lambda} + \frac{1}{2r} + ie A_r - \frac{1}{r} \left( \partial_\theta - \frac{\lambda_\theta}{\lambda} \right) + eA_\theta \\
-\partial_r + \frac{\lambda_r}{\lambda} - \frac{1}{2r} - ie A_r - \frac{1}{r} \left( \partial_\theta - \frac{\lambda_\theta}{\lambda} \right) + eA_\theta & 0
\end{pmatrix}.
\]

Thus, Hermiticity of (13) requires that

\[
\lambda_{\theta} = 0, \quad \frac{\lambda_r}{\lambda} - \frac{1}{2r} = 0
\]

and fixes the exact form of the modulus of the similarity transformation to be \(\lambda(r, \theta) = \sqrt{r}\). It is interesting to note that \(\lambda^2\) turns out to be the integration measure in 2D cylindrical coordinates. We could have eliminated the \(\lambda\) factor in the definition of \(\Lambda\) in (10) by proposing that the new spinor wavefunction \(\chi\) be replaced by \(\frac{1}{\sqrt{r}}\chi(r, \theta)\). In that case, the transformation matrix \(\Lambda\) becomes simply \(e^{i \sigma_3 \theta}\), which is unitary. However, making the presentation as above

\[\text{Any other choice for the pair of Pauli matrices can be obtained from the present one by a unitary transformation, hence leaving the physics of the problem unaltered.}\]
gave us a good opportunity to show (in a different approach) why it is customary to take the radial component of the wavefunction in 2D cylindrical coordinates to be proportional to \( \sqrt{r} \).

Finally, we obtain the \((2+1)\)-dimensional Dirac equation \((H - \varepsilon) \chi = 0\) for a charged spinor in static electromagnetic potential as

\[
(H - \varepsilon) \chi(r, \theta) = 0
\]

where \( \chi_{\pm} \) are the components of the transformed wavefunction \( |\chi\rangle = \Lambda |\psi\rangle \). This equation will be solved by choosing an appropriate gauge to end up with the full Hilbert space.

2.2. Energy spectrum

Now, we specialize in the case where a constant magnetic field of strength \( B \) is applied at right angles to the \((r, \theta)\)-plane, which is \( \vec{B} = B \hat{z} \). Therefore, the electromagnetic potential has the time and space components

\[
A_0 = 0, \quad \vec{A}(r, \theta) = \frac{1}{2} Br \hat{\theta}.
\]

Consequently, (15) becomes completely separable, and we can write the spinor wavefunction as

\[
\chi_{\pm}(r, \theta) = \phi_{\pm}(r) \tau(\theta).
\]

Thus, the angular component satisfies

\[
-\frac{i}{\sqrt{2\pi}} \frac{d}{d\theta} \xi \tau = \xi \tau,
\]

where \( \xi \) is a real separation constant giving the function

\[
\tau(\theta) = \frac{1}{\sqrt{2\pi}} e^{i\theta}.
\]

On the other hand, the boundary condition \( \psi(\theta + 2\pi) = \psi(\theta) \) requires that \( e^{i2\pi \xi} = e^{-i\pi v} = +1 \) which, in turn, demands that \( e^{i2\pi \xi} = -1 \) giving the following quantum number:

\[
\xi = \frac{1}{2} v, \quad v = \pm 1, \pm 3, \pm 5 \ldots.
\]

Consequently, the Dirac equation for the two-component radial spinor is reduced to

\[
\left( \begin{array}{c}
-\varepsilon - \frac{\xi}{r} + \omega r
\
-\frac{d}{dr} + \frac{\xi}{r} + \omega r
\end{array} \right) \left( \begin{array}{c}
\phi_{+}(r)
\
\phi_{-}(r)
\end{array} \right) = 0,
\]

where the physical constant \( \omega \) is given by \( \omega = l_B^{-2} - \kappa \) and \( l_B \) is the magnetic length \( l_B = \frac{1}{\sqrt{\varepsilon B}} \).

Thus, the presence of the 2D Dirac-oscillator coupling did, in fact, maintain the symmetry of the problem as stated below (4). Moreover, its introduction is equivalent to changing the magnetic field as \( eB \rightarrow eB - 2\kappa \). As a result of the wave equation (20), the two spinor components satisfy the ‘kinetic balance’ relation

\[
\phi_{\pm}(r) = \frac{1}{\varepsilon} \left[ \mp \frac{d}{dr} + \frac{\xi}{r} + \omega r \right] \phi_{\mp}(r),
\]

where \( \varepsilon \neq 0 \). Therefore, the solution of the problem with the top/bottom sign corresponds to the positive/negative energy solution. Using the exact relation (21) to eliminate one component in terms of the other in (20) results in the following Schrödinger-like differential equation for each spinor component:

\[
\left\{ -\frac{d^2}{dr^2} + \frac{\xi (\xi \mp 1)}{r^2} + \omega^2 r^2 + [-\varepsilon^2 + \omega(2\xi \pm 1)] \right\} \phi_{\pm}(r) = 0.
\]
We stress that this equation gives only one radial spinor component. One must choose either the top or bottom sign to obtain the component that corresponds to the positive or negative energy solutions, respectively. The second component is obtained by substituting this into the exact relation (21). Nonetheless, we only need to find one solution (the positive- or negative-energy solution), because the other is obtained by a simple map. For example, the following map takes the positive energy solution into the negative energy solution:

\[ \varepsilon \rightarrow -\varepsilon, \quad v \rightarrow -v, \quad \omega \rightarrow -\omega, \quad \phi_{\pm} \rightarrow \phi_{\mp}, \]  

(23)

which, in fact, is the CPT transformation. Here the charge conjugation C means that \( e \rightarrow -e \) and \( \kappa \rightarrow -\kappa \). It is easy to check that the above map (23) originates from the fact that the Dirac equation (20) is invariant under such transformation. Hence, we just need to solve for positive energies and use the above transformation to obtain the negative energy solutions.

The total spinor wavefunction reads

\[ \psi(r, \theta) = \frac{1}{\sqrt{r}} e^{i\xi \theta} e^{-i\sigma^3 \theta} \phi(r), \]  

(24)

where \( \phi(r) \) has two components, such as

\[ \phi = \left( \phi_+ \phi_- \right). \]  

(25)

Equation (22) looks like the non-relativistic oscillator problem with a certain parameter map of the frequency, angular momentum, and energy. For regular solutions of (22), the bound states will be of the form

\[ \phi_{\pm} \sim z^\mu e^{-z^2/2} L_n^\nu(z), \]  

(26)

where \( L_n^\nu(z) \) is the associated Laguerre polynomial of order \( n = 0, 1, 2, \ldots \) and \( z = \rho^2r^2 \). The constants \( \{ \mu, \nu, \rho \} \) are real and related to the physical parameters \( B, \kappa \) and \( \xi \). Square integrability and the boundary conditions require that \( 2\mu \geq \frac{1}{2} \) and \( \nu > -1 \).

Substituting the ansatz (26) into (22) and using the differential equation for the Laguerre polynomials [17], we obtain four equations. Three of them determine the parameters \( \{ \mu, \nu, \rho \} \) and one determines the energy spectrum. The first three are

\[ 2\mu = \nu + \frac{1}{2}, \quad \rho^2 = |\omega|, \]

\[ \nu = \pm \left\{ \xi - \frac{1}{2}, \quad \varepsilon > 0 \right\} \]

\[ \xi + \frac{1}{2}, \quad \varepsilon < 0. \]  

(27)

For regular solutions of (22), the \( \pm \) sign in the expression for \( \nu \) corresponds to \( \pm \xi > 0 \). The fourth equation gives the following (positive and negative) energy spectra:

\[ \varepsilon_{n,\kappa}^\pm = \pm \sqrt{2|\omega|[2n + 1 \pm s - s' + \xi(s + s')]}, \]  

(28)

where \( s = \text{sgn}(\omega) = \frac{\kappa}{l_B} \) and \( s' = \text{sgn}(\xi) \). The sign of \( \omega \) depends on whether \( \kappa \) is larger or smaller than the magnetic length \( l_B \). To compare our work with frequently used notation in the literature, we can replace the quantum number \( \xi \) by \( k + \frac{1}{2} \), where \( k = 0, \pm 1, \pm 2, \ldots \) and \( \xi \rightarrow -\xi \) implies that \( \kappa \rightarrow -\kappa \). In that case, one may write the positive eigenvalues as

\[ \varepsilon_{n,k}^+ = \sqrt{2|\omega|[2n + 1 + s + k(s + s')]}, \]  

(29)

and the negative ones as

\[ \varepsilon_{n,k}^- = -\sqrt{2|\omega|[2n + 1 + s' + k(s + s')]}, \]  

(30)
where \( s' = +1 \) for \( k = 0 \). It is interesting to note that for \( \xi \omega < 0 \) the spectrum is infinitely degenerate because it is independent of \( \xi \). However, for \( \xi \omega > 0 \) the degeneracy is finite and equal to \( n + k + 1 \). Substituting the wavefunction parameters given by (27) into the ansatz (26) gives for \( \varepsilon > 0 \)

\[
\phi_+(r) = x^{k+\frac{1}{2}} e^{-\frac{1}{2}x^2} \begin{cases} A^{++}_{n,k} L_n^k(x^2), & k \geq 0 \\ A^{-+}_{n,k} x L^{-k-1}_n(x^2), & k < 0, \end{cases}
\]

as well as for \( \varepsilon < 0 \)

\[
\phi_-(r) = x^{k+\frac{1}{2}} e^{-\frac{1}{2}x^2} \begin{cases} A^{+-}_{n,k} x L^{k+1}_n(x^2), & k \geq 0 \\ A^{--}_{n,k} L^{-k-1}_n(x^2), & k < 0, \end{cases}
\]

where \( x = r \sqrt{|\omega|} \) and \( A^{ij}_{n,k} \) are normalization constants that depend on the physical quantities \( l_B \) and \( \kappa \). The lower component \( \phi_-(r) \) is obtained by substituting (31) and (32) into the exact relation (21). Doing so while exploiting the differential and recursion properties of the Laguerre polynomials, we obtain the following for \( \varepsilon > 0 \):

\[
\phi_-(r) = \frac{\sqrt{|\omega|}}{e^{\gamma_{n,k}}} x^{k+\frac{1}{2}} e^{-\frac{1}{2}x^2} \times \begin{cases} A^{++}_{n,k} \left[(s-1)\frac{1}{2}L_n^k(x^2) + 2L^{k+1}_n(x^2)\right], & k \geq 0 \\ A^{+-}_{n,k} \left[(s-1)(n-k)L^{-k-1}_n(x^2) - (s+1)(n+1)L^{k+1}_{n+1}(x^2)\right], & k < 0, \end{cases}
\]

On the other hand, repeating the same calculation for the upper component of the negative energy solution gives the function

\[
\phi_+(r) = \frac{\sqrt{|\omega|}}{e^{\gamma_{n,k}}} x^{k+\frac{1}{2}} e^{-\frac{1}{2}x^2} \times \begin{cases} A^{--}_{n,k} \left[(1+s)(n+k+1)L_n^k(x^2) + (1-s)(n+1)L^{k+1}_n(x^2)\right], & k \geq 0 \\ A^{+-}_{n,k} \left[(1+s)L^{-k-1}_n(x^2) - 2L^{k+1}_n(x^2)\right], & k < 0, \end{cases}
\]

which can also be obtained by applying the \( CPT \) map (23) to (33). Thus, the structure of the whole Hilbert space solution consists of eight disconnected spaces that can be displayed in a tabular form as shown in table 1.

Using the standard definition, we calculate all normalization constants in the above wavefunctions. These are summarized in the table 2, where as stated above \( s = \text{sgn}(|\omega|) = |\omega|/\omega \).

### 3. Disordered sublattices

Our main task is to analyze the diamagnetism of confined Dirac fermions in disordered graphene. In the previous section, we settled the required tools for the confinement; however, we still need to introduce the disorder potential and related matter. For this, we write the eigenspinors as well as the impurity and confining potentials in four components.

Having obtained the energy spectrum solutions corresponding to one sublattice, it is worthwhile to deal with a system of graphene. Such a system is composed of a honeycomb network of carbon atoms where a unit cell contains one atom each from a pair of sublattices, denoted by I and II. It can be described by a \( 4 \times 4 \) matrix Hamiltonian, such as

\[
H = \begin{pmatrix} 0 & a & 0 & 0 \\ a^\dagger & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & a^\dagger & 0 \end{pmatrix}
\]

(35)
represent the envelope functions at I and II sites for point $K$ where in polar coordinates the two operators $a_J$ as well as $\frac{1}{\alpha}$, we introduce the disorder potential characterized by two simple models: short- and making use of some mapping.

Clearly, they act on one component of the eigenspinors $n,k(r, \theta)$.

Energy $\varepsilon > 0$, $k \geq 0$ \[ A_{n,k}^+ = \sqrt{\frac{2\varepsilon + \lambda}{n}} \left[ \left( \frac{2n}{\alpha} \right)^2 (2(n + k + 1) + n(1 - s)) \right]^{-1} \]

Energy $\varepsilon > 0$, $k < 0$ \[ A_{n,k}^- = \sqrt{\frac{2\varepsilon + \lambda}{n}} \left[ \left( \frac{2n}{\alpha} \right)^2 (2(n + 1) + (k + 1)(s - 1)) \right]^{-1} \]

Energy $\varepsilon < 0$, $k \geq 0$ \[ A_{n,k}^+ = \sqrt{\frac{2\varepsilon + \lambda}{n}} \left[ \left( \frac{2n}{\alpha} \right)^2 (2(n + 1) + k(s + 1)) \right]^{-1} \]

Energy $\varepsilon < 0$, $k < 0$ \[ A_{n,k}^- = \sqrt{\frac{2\varepsilon + \lambda}{n}} \left[ \left( \frac{2n}{\alpha} \right)^2 (2(n - k) + n(s + 1)) \right]^{-1} \]

where in polar coordinates the two operators $a$ and $a^\dagger$ take the form

\[ a = \partial_r - \frac{\lambda r}{\alpha} + \frac{1}{2r} + i r A_r - \kappa r - \frac{i}{r} \left( \partial_\theta - \frac{\lambda \theta}{\alpha} \right) + eA_\theta \] \quad (36)

\[ a^\dagger = -\partial_r + \frac{\lambda r}{\alpha} - \frac{1}{2r} - i r A_r - \kappa r - \frac{i}{r} \left( \partial_\theta - \frac{\lambda \theta}{\alpha} \right) + eA_\theta \] \quad (37)

Clearly, they act on one component of the eigenspinors $F_{IK}^I, F_{IK}^K, F_{IK}^{K'}$, $F_{IK}^I, F_{IK}^K$ where $F_{IK}^I, F_{IK}^K$ represent the envelope functions at I and II sites for point $K$, respectively, and $F_{IK}^{K'}, F_{IK}^{K''}$ for $K'$, with $K$ and $K'$ indexing the two valleys.

It is convenient for our task to label the eigenspinors by $\alpha = (j, n, k)$ with the valley index $j = K, K'$, the Landau level index $n = 0, \pm 1, \ldots$ and the wavevector $k$. The eigenspinors for $k \geq 0$ and $\varepsilon > 0$ can easily be deduced from the above analysis. For $K$ we have

\[ F_{n,k}^K(r, \theta) = A_{n,k}^+ |\alpha|^2 e^{i(k+\frac{1}{2})\sigma_x} e^{-\frac{1}{2}\sigma_y x} [\frac{(s-1)\lambda}{\alpha} + 2\alpha L_n^{k+1}(x^2)] \]

as well as

\[ F_{n,k}^{K'}(r, \theta) = A_{n,k}^+ |\alpha|^2 e^{i(k+\frac{1}{2})\sigma_x} e^{-\frac{1}{2}\sigma_y x} [\frac{(s-1)\lambda}{\alpha} + 2\alpha L_n^{k+1}(x^2)] \]

for the point $K'$. Without loss of generality, in the forthcoming analysis we only focus on the case where $k \geq 0$ and $\varepsilon > 0$. Other configurations can be recovered from the first one by making use of some mapping.

To consider disordered graphene and make comparisons with already published work [13], we introduce the disorder potential characterized by two simple models: short- and
long-ranged scatterers [6]. The first is an on-site potential localized at a particular I or II site with a random amplitude. A scatterer on site I at $\vec{R}_I$ is represented as

$$U(\vec{r}) = \begin{pmatrix} 1 & 0 & z_I^* z_I' & 0 \\ 0 & 0 & 0 & 0 \\ z_I z_I' & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} u_i \delta(\vec{r} - \vec{R}_I),$$

and that on site II at $\vec{R}_II$ as

$$U(\vec{r}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & z_{II}^* z_{II}' \\ 0 & 0 & 0 & 0 \\ 0 & z_{II} z_{II}'^* & 0 & 1 \end{pmatrix} u_i \delta(\vec{r} - \vec{R}_{II}).$$

where we introduced $z_X = e^{i\vec{K} \cdot \vec{R}_X}$, $z_X' = e^{i\vec{K}' \cdot \vec{R}_X}$ with $X = I$ and $II$, and $u_i = (\sqrt{3} a^2/2) U_i$ with the on-site energy $U_i$. We assume that the scatterers are equally distributed on I and II sites with density $n_I = n_{II} = n_i/2$ and the mean square amplitude $\langle (u_I^2) \rangle = \langle (u_{II}^2) \rangle = u_i^2$.

Dominant scatterers in graphene are expected to have a potential range larger than the lattice constant for which inter-valley scattering is much smaller than intra-valley scattering. Further, realistic scatterers are likely to have the range comparable to the Fermi wavelength [18–20]. In the following, however, we shall assume scatterers with the potential range smaller than the Fermi wavelength. The reason is that the results are expected [13] to remain qualitatively the same and further that actual calculations are practically possible.

In this long-range model, a scatterer at $\vec{R}$ is expressed by

$$U(\vec{r}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} u_i \delta(\vec{r} - \vec{R}).$$

We assume the scatterer density $n_i$ and the mean square amplitude $u_i^2$. It was shown that the transport properties in the short-ranged disorder and the long-ranged one are qualitatively similar [6–8].

To complete our model, we consider the confining potential as a $4 \times 4$ matrix as well. This is

$$V(\vec{r}) = \begin{pmatrix} \kappa r & 0 & 0 & 0 \\ 0 & -\kappa r & 0 & 0 \\ 0 & 0 & \kappa r & 0 \\ 0 & 0 & 0 & -\kappa r \end{pmatrix}.$$
4.1. Self-energy

To deal with different issues, we introduce the Green function that is related to the self-energy $\Sigma$ via the Dyson equation. It is

$$\langle G_{\alpha\alpha}(\epsilon) \rangle = \Sigma_{\alpha\alpha}'(\epsilon) G_{\alpha}(\epsilon) + G_{\alpha}(\epsilon) \sum_{\alpha'} \Sigma_{\alpha'\alpha'}(\epsilon) \langle G_{\alpha'\alpha}(\epsilon) \rangle,$$  \hspace{1cm} (44)

where the first term contains the matrix elements of the unperturbed Green’s function corresponding to the Hamiltonian considered before, such as

$$G_{\alpha\alpha}^0(\epsilon) = \langle \alpha | \frac{1}{\epsilon - H} | \alpha' \rangle = \delta_{\alpha\alpha}' G_{\alpha}(\epsilon).$$  \hspace{1cm} (45)

To proceed further, we make use of the self-consistent Born approximation for our system. The self-energy of the disorder-averaged Green’s function $\langle G_{\alpha,\alpha}(\epsilon) \rangle$ can be written as

$$\Sigma_{\alpha\alpha}(\epsilon) = \sum_{\alpha_1\alpha_2} \langle U_{\alpha\alpha_1} U_{\alpha_2\alpha}' \rangle \langle G_{\alpha_1\alpha_2}(\epsilon) \rangle,$$  \hspace{1cm} (46)

where the symbol $\langle \cdots \rangle$ represents the average over the impurity configurations.

The above equation can be solved by considering the range of the disorder. Specifically, in the short-ranged model, one can show that the self-energy and the averaged Green’s function are diagonal with respect to $\alpha$ [13]. Furthermore, the self-energy becomes independent of $\alpha$ and leads to

$$\langle G_{\alpha\alpha}(\epsilon) \rangle = \delta_{\alpha\alpha} G_{\alpha}(\epsilon),$$  \hspace{1cm} (47)

where $G_{\alpha}(\epsilon)$ is given by

$$G_{\alpha}(\epsilon) = G(\epsilon, \epsilon_\alpha) \equiv \frac{1}{\epsilon - \epsilon_\alpha - \Sigma(\epsilon)}.$$  \hspace{1cm} (48)

From (46) and (48), it is clear that one has to find the appropriate solution for $\Sigma$. To do so, one needs to introduce respective approximations.

Having described the needed tools, let us see how they can be applied to analyze the basic features of the present system. According to the former analysis, we have two contributions to the total self-energy $\Sigma(\epsilon) \equiv \Sigma_{\text{tot}}(\epsilon)$, namely

$$\Sigma_{\text{tot}}(\epsilon) = \Sigma_{\text{dis}}(\epsilon) + \Sigma_{\text{conf}}(\epsilon),$$  \hspace{1cm} (49)

where $\Sigma_{\text{dis}}$ and $\Sigma_{\text{conf}}$ correspond to the disordered and confining potentials, respectively. In the forthcoming analysis, we separately determine each part. Note that, what makes a difference with respect to the study reported in [13] is the second contribution, and therefore one can see its impact on such study.

As a first step, we have to evaluate the matrix elements of different potentials. For the impurity potential, using the eigenspinors it is straightforward to show that

$$\langle \alpha | U_{\alpha'} \rangle = \sqrt{|\alpha|} u_{\alpha} \delta_{\alpha\alpha'},$$  \hspace{1cm} (50)

where $\alpha$ is the parameter introduced in (20), and the integration is performed over the coordinates $r$ and $\theta$. This can be used together with (46) for the short-ranged potential to straightforwardly obtain

$$\Sigma_{\text{dis}}(\epsilon) = \frac{n_i u_i^2}{2} |\alpha| \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} g(\epsilon_{n,k}) \frac{-\epsilon - \epsilon_{n,k} - \Sigma_{\text{tot}}(\epsilon)}{\epsilon_\alpha - \Sigma_{\text{tot}}(\epsilon)},$$  \hspace{1cm} (51)

where the cutoff function $g(\epsilon)$ is given as

$$g(\epsilon) = \begin{cases} 1, & |\epsilon| < \epsilon_c \\ 0, & \text{otherwise}. \end{cases}$$  \hspace{1cm} (52)
As far as the confining potential is concerned, the corresponding matrix elements can be evaluated to end up with
\[
\langle \alpha | \pm \kappa r | \alpha' \rangle = \sqrt{\omega} \kappa \left[ 1 - \frac{\omega}{\epsilon_{n,k}^+ \epsilon_{n,k}^-} 2s(n + k + 1) \right] \delta_{\alpha \alpha'}.
\] (53)

This leads to the self-energy for the confinement:
\[
\Sigma_{\text{conf}}(\epsilon) = \frac{\kappa^2}{2} |\omega| \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \left[ 1 - \frac{s(n + k + 1)}{2n + (s + 1)(k + 1)} \right] \frac{g(\epsilon_{n,k})}{\epsilon - \epsilon_{n,k} - \Sigma_{\text{tot}}(\epsilon)}.
\] (54)

It is clear that \( \Sigma_{\text{conf}} \) is strongly \( \kappa \)-dependent, which is an expected result because of the confining potential expression (43).

So far, we have obtained the different contributions to the total self-energy. This can be written as
\[
\Sigma_{\text{tot}}(\epsilon) = \frac{\kappa^2}{2} |\omega| \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \left\{ n_i u_i^2 + \kappa^2 \left[ 1 - \frac{s(n + k + 1)}{2n + (s + 1)(k + 1)} \right] \right\} \frac{g(\epsilon_{n,k})}{\epsilon - \epsilon_{n,k} - \Sigma_{\text{tot}}(\epsilon)}.
\] (55)

This expression can be simplified by choosing \( s = 1 \) and requiring the condition \( n + k + 1 \neq 0 \). In this case, (55) reduces to
\[
\Sigma_{\text{tot}}(\epsilon) = C |\omega| \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{g(\epsilon_{n,k})}{\epsilon - \epsilon_{n,k} - \Sigma_{\text{tot}}(\epsilon)},
\] (56)

where \( C \) is a dimensionless parameter defined as
\[
C = n_i u_i^2 + \frac{\kappa^2}{4},
\] (57)

which depends on the set of parameters. Thus, one can fix them to derive specific results and offer different interpretations.

At this stage, one can inspect the above results to underline their basic properties. One way to do so is to look at the case \( \kappa = 0 \), i.e. without confining the system. This simply reduces \( C \) to the so-called disorder strength
\[
C_{|\kappa=0} = n_i u_i^2,
\] (58)

which leads to the self-energy
\[
\Sigma(\epsilon) = \frac{n_i u_i^2}{2} l_B^{-2} \sum_{n=-\infty}^{\infty} \frac{g(\epsilon_n)}{\epsilon - \epsilon_n - \Sigma(\epsilon)},
\] (59)

where the corresponding eigenvalues are
\[
\epsilon_n = l_B^{-2} \text{sgn}(n) \sqrt{|n|}.
\] (60)

This was obtained by studying the Dirac fermions in the magnetic field and disordered graphene; more details can be found in [13]. On the other hand, comparing (57) and (58), \( C \) can be interpreted as a parameter of confinement and disorder strength.

4.2. Density of states

As we claimed before, the density of states is strongly needed and will play a crucial role in the forthcoming analysis. Specifically, it is related to different thermodynamical quantities and therefore allows us to determine them in an appropriate way. This statement will be clarified in the next section.
For later convenience, we consider the density of states used in [12] by dealing with some features of graphene. This is
\[ \rho(\varepsilon) = -\frac{1}{\pi} \sum_{\alpha} \text{Im} G_{\alpha}(\varepsilon + i0). \] (61)
This form can be handled by fixing different conditions. For this, we distinguish between short- and long-ranged disorders. Returning to our results, we have
\[ \rho(\varepsilon) = -\frac{1}{\pi} \frac{2}{C_{[\alpha]}} \text{Im} \Sigma_{\text{tot}}(\varepsilon + i0) \] (62)
for the short-ranged disorder.
Let us recall that the transport properties in the short- and long-ranged disorders are qualitatively similar [13]. In the last case, the self-energy and Green function have off-diagonal matrix elements between \((j,n,k)\) and \((j,-n,k)\). Thus, we obtain
\[ \Sigma_{\alpha,\alpha'}(\varepsilon) = \delta_{j,j'} \delta_{k,k'} [\delta_{n,n'} \Sigma^d(\varepsilon) + \delta_{n,-n'} \Sigma^o(\varepsilon)]. \] (63)
Splitting the self-energy into two parts
\[ \Sigma_{\text{tot}}^\pm \equiv \Sigma_{\text{tot}}^d \pm \Sigma_{\text{tot}}^o, \] (64)
we derive the positive contribution
\[ \Sigma_{\text{tot}}^+(\varepsilon) = C_{[\alpha]} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(\varepsilon - \Sigma_{\text{tot}})(g(\varepsilon_{n,k}))}{(\varepsilon - \Sigma_{\text{tot}})(\varepsilon - \Sigma_{\text{tot}}) - (\varepsilon_{n,k})^2}, \] (65)
as well as the negative one
\[ \Sigma_{\text{tot}}^-(\varepsilon) = C_{[\alpha]} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(\varepsilon - \Sigma_{\text{tot}})(g(\varepsilon_{n,k}))}{(\varepsilon - \Sigma_{\text{tot}})(\varepsilon - \Sigma_{\text{tot}}) - (\varepsilon_{n,k})^2}. \] (66)
where \(C\) has the same form as for the short-range case, i.e. (57). These parts can be used to derive the density of states for the long-range case. More precisely, we obtain
\[ \rho(\varepsilon) = -\frac{1}{\pi} \frac{1}{C_{[\alpha]}} \text{Im} \left[ \Sigma_{\text{tot}}^d(\varepsilon + i0) + \Sigma_{\text{tot}}^o(\varepsilon + i0) \right], \] (67)
which reduces to that obtained in [13] by switching off the confining parameter \(\kappa\). With this, we finish the derivations of the tools needed to tackle different issues. In fact, we will see how the above results can be applied to deal with the diamagnetism of the present system and emphasize what makes the difference with respect to the case without confinement.

5. Thermodynamic properties

Now we show the relevance of the above tools. We focus on the study of the diamagnetism and proceed in the standard way evaluating different physical quantities. More precisely, we determine the magnetization, number of fermions, and susceptibility to describe the physical properties of the system.

5.1. Thermodynamic quantities

We recall useful definitions of different thermodynamic quantities. The magnetization is given by
\[
\mathcal{M} = -\left( \frac{\partial \Omega}{\partial B} \right)_\mu,
\]
(68)
where \( \Omega(T, \mu, B) \) is the thermodynamic potential and \( \mu \) is the chemical potential.

To determine the number of fermions, we can use one of two methods. The first one is based on the definition
\[
\mathcal{N} = -\left( \frac{\partial \Omega}{\partial \mu} \right)_B
\]
(69)
to obtain the Maxwell relation
\[
\left( \frac{\partial \mathcal{M}}{\partial \mu} \right)_B = \left( \frac{\partial \mathcal{N}}{\partial B} \right)_\mu.
\]
(70)

On the other hand, in terms of the density of states \( \rho \), we have
\[
\mathcal{N} = \int_{-\infty}^{\infty} \rho(\varepsilon, B) f(\varepsilon) \, d\varepsilon,
\]
(71)
where the fermionic distribution is \( f(\varepsilon) = 1/(1 + e^{(\varepsilon-\mu)/k_B T}) \).

From the above formulas, one can establish an interesting relation. After a straightforward calculation, we obtain
\[
\mathcal{M} = \int_{-\infty}^{\infty} \rho(\varepsilon, B) f(\varepsilon) \int_{-\infty}^{\varepsilon} d\varepsilon' \frac{\partial \rho(\varepsilon', B)}{\partial B}
\]
in terms of the density of states. Furthermore, one can also obtain the magnetic susceptibility
\[
\chi = \left. \frac{\partial \mathcal{M}}{\partial B} \right|_{B=0}.
\]
(73)
The above quantities will be much more simplified by considering the self-consistent Born approximation and fixing the type of disorder.

5.2. Short-ranged disorder

To calculate different quantities, we specify the nature of disorder. According to (62) and (72), the susceptibility in the self-consistent Born approximation for the short-ranged disorder can be written as
\[
\chi = -\frac{1}{\pi C} \int_{-\infty}^{\infty} d\varepsilon f(\varepsilon) \int_{-\infty}^{\varepsilon} d\varepsilon' \left| \frac{\partial \Sigma_{\text{tot}}(\varepsilon', B)}{\partial B} \right|_{\varepsilon'=\varepsilon, B=0}.
\]
(74)

It is convenient to introduce
\[
\varepsilon = \varepsilon - \Sigma_{\text{tot}},
\]
(75)
which allows us to rewrite
\[
\Sigma_{\text{tot}}(\varepsilon, B) = \Sigma_{\text{tot}}(\varepsilon, B) = \frac{C}{2} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{g(\varepsilon_{n,k})}{X - \varepsilon_{n,k}}.
\]
(76)

According to the above equations, second derivatives are needed. The first-order derivative of \( \Sigma_{\text{tot}} \) with respect to \( B \) gives
\[
\left. \frac{\partial \Sigma_{\text{tot}}(\varepsilon, B)}{\partial B} \right|_{\varepsilon'=\varepsilon, B=0} = \left[ 1 - \left( \frac{\partial \Sigma_{\text{tot}}(X, B)}{\partial X} \right)^{-1} \frac{\partial \Sigma_{\text{tot}}(X, B)}{\partial B} \right],
\]
(77)
which leads to the second as
\[ \frac{\partial^2 \Sigma_{tot}}{\partial B^2} = \left[ 1 - \frac{\partial \Sigma_{tot}}{\partial X} \right]^{-1} \left[ \frac{\partial^2 \Sigma_{tot}}{\partial B^2} - 2 \frac{\partial \Sigma_{tot}}{\partial X \partial B} \left( \frac{\partial \Sigma_{tot}}{\partial B} \right) + \frac{\partial^2 \Sigma_{tot}}{\partial X^2} \left( \frac{\partial \Sigma_{tot}}{\partial B} \right)^2 \right]. \] (78)

One can expand (76) into a series in terms of a function \( h \):
\[ \tilde{\Sigma}_{tot}(X, B) = C \frac{\Delta t}{\Delta} \left[ \frac{1}{2} h(0) + \sum_{n=-\infty, \neq 0}^{\infty} \sum_{k=1}^{\infty} h \left( \frac{2 [n + k + 1] \Delta t}{\Delta} \right) \right], \] (79)
where \( \Delta t = |\omega| = |l_B^{-2} - \kappa| \) and the function has the form
\[ h(t) = \frac{2}{\Delta} g(\sqrt{t}) Xg \left( \sqrt{\frac{t}{\Delta}} \right) X^2 - t. \] (80)

Note that, taking \( \kappa = 0 \) we recover the results obtained in [13]. This tells us that those results have been generalized to the present case, and we will see how they can be interpreted.

To go further, we introduce some relevant assumptions. If the condition \( \text{Im}(X) \gg \sqrt{\Delta t} \) is valid, one can simplify (79) to
\[ C \Delta t \left[ \frac{1}{2} h(0) + \sum_{n=-\infty, \neq 0}^{\infty} \sum_{k=0}^{\infty} h(2[n + k + 1]) \right] \]
\[ = C \int_{0}^{\infty} h(t) \, dt - C \frac{\Delta t}{\Delta} \frac{1}{12} \left[ h''(0) + \frac{1}{2} h''(\infty) \right], \] (81)
where \( \theta(\Delta t)^3 \) is neglected. This leads to
\[ \tilde{\Sigma}_{tot}(X, B) - \tilde{\Sigma}_{tot}(X, 0) = -\frac{C}{24} h''(0)(\Delta t)^2, \] (82)
which can be used to calculate the above derivatives. Otherwise, after neglecting all terms containing a power of \( X \) larger than 3, we obtain
\[ \left. \frac{\partial \Sigma_{tot}}{\partial B} \right|_{B=0} = \left[ 1 - \frac{\partial \tilde{\Sigma}_{tot}}{\partial X} \right]^{-1} \left[ \frac{C}{6} \kappa \epsilon \frac{1}{X^3} \right], \] (83)
\[ \left. \frac{\partial^2 \Sigma_{tot}}{\partial B^2} \right|_{B=0} = \left[ 1 - \frac{\partial \tilde{\Sigma}_{tot}}{\partial X} \right]^{-1} \left[ \frac{C}{6} \epsilon^2 \frac{1}{X^3} \right]. \] (84)

With the help of (77)–(78) and (83)–(84), the susceptibility (74) becomes
\[ \chi = \frac{e^2}{3\pi \kappa} \int_{-\infty}^{\infty} df(x) \text{Im} \int_{-\infty}^{\infty} dX' \left[ 1 - \frac{\partial \tilde{\Sigma}_{tot}}{\partial X'} \right]^{-1} \frac{1}{X'^3} \bigg|_{B=0}. \] (85)

By introducing
\[ dX' = \left[ 1 - \frac{\partial \tilde{\Sigma}_{tot}}{\partial X'} \right] dX', \] (86)

it is not hard to find
\[ \chi = -\frac{e^2}{6\pi \kappa} \int_{-\infty}^{\infty} df(x) \text{Im} \frac{1}{|x - \Sigma_{tot}(\epsilon)|^2} \bigg|_{B=0}. \] (87)

As far as the magnetization is concerned for the short-ranged disorder, one can use (62), (83), (84) and (86). These give
\[ \mathcal{M} = -\frac{e}{6\pi} \int_{-\infty}^{\infty} df(x) \text{Im} \frac{1}{|x - \Sigma_{tot}(\epsilon)|^2}. \] (88)
On the other hand, the number of fermions can also be formulated as

\[ N = -\frac{1}{\pi} \frac{2}{C(\omega)} \int_{-\infty}^{\infty} df(\epsilon) \Im \Sigma_{\text{tot}}(\epsilon, B). \]  

(89)

Now we will see how the above results will be simplified by making use of different considerations. In fact, to underline what makes a difference with respect to other studies, we consider the zero-field limit and determine the corresponding susceptibility. More precisely, in such a limit, (56) can be written as

\[ \Sigma_{\text{tot}}(\epsilon) = C \int_{0}^{\infty} t dt \frac{(\epsilon - \Sigma_{\text{tot}})}{(\epsilon - \Sigma_{\text{tot}})^2 - t^2} - \frac{\kappa^2}{12} C \frac{1}{(\epsilon - \Sigma_{\text{tot}})^3} + \cdots, \]  

(90)

which can be approximated by assuming that \( \epsilon \ll \epsilon_c \). In this situation, we end up with

\[ \Sigma_{\text{tot}}(\epsilon) = -C(\epsilon - \Sigma_{\text{tot}}) \log \left[ -\epsilon^2_c \frac{(\epsilon - \Sigma_{\text{tot}})}{2} \right] - \frac{\kappa^2}{12} C \frac{1}{(\epsilon - \Sigma_{\text{tot}})^3} + \cdots. \]  

(91)

To further simplify the above form, one may consider the condition \( \kappa \ll 1 \) and choose an appropriate branch of the logarithm. Thus, one obtains

\[ \Sigma_{\text{tot}}(\epsilon) = \epsilon - \epsilon \left[ 2Cf_L \left( -\frac{i\epsilon}{2\Gamma C} \right) \right]^{-1}, \]  

(92)

where \( f_L(z) \) is the Lambert C-function (also called Omega function which is the inverse function of \( f(z) = ze^z \)[21]) and \( \Gamma \) is given by

\[ \Gamma = \epsilon_c \exp \left( -\frac{1}{2C} \right). \]  

(93)

In the region where the energy fulfills the constraint \( |\epsilon| \gg \Gamma \), \( \Sigma_{\text{tot}} \) becomes

\[ \Sigma_{\text{tot}}(\epsilon + i0) \approx -2C \log \frac{\epsilon_c}{\epsilon} - i\pi |\epsilon| C. \]  

(94)

Following from this expression, we can derive specific results and offer different discussions. Let us split (94) as

\[ \Sigma_{\text{tot}}(\epsilon + i0) \approx -n_i u_i^2 \left[ 2\epsilon \log \frac{\epsilon_c}{\epsilon} + i\pi |\epsilon| \right] - \frac{\kappa^2}{4} \left[ 2\epsilon \log \frac{\epsilon_c}{\epsilon} + i\pi |\epsilon| \right]. \]  

(95)

The first term is similar to that obtained in [13] and becomes exactly the same if the constraint \( \kappa = 0 \) is taken into account. Accordingly, we can interpret the second as a quantum correction.

To treat the susceptibility for the short-ranged disorder, we distinguish two cases. In the first case, the energy is constrained by the condition \( \epsilon \ll \epsilon_c \). At zero temperature, we evaluate the integral to end up with

\[ \chi(\epsilon_F) = -\frac{\epsilon_c^2}{3\pi} \frac{2C}{\Gamma} F \left( \frac{\epsilon_F}{2\Gamma C} \right), \]  

(96)

where the function \( F(x) \) is given by

\[ F(x) = -\frac{1}{x} \Im \left[ f_L(-ix) + \frac{1}{2} f_L^2(-ix) \right]. \]  

(97)

Without confinement, this result reduces to

\[ \chi(\epsilon_F) \bigg|_{\kappa=0} = -\frac{\epsilon_c^2}{3\pi} \frac{2n_i u_i^2}{\Gamma} F \left( \frac{\epsilon_F}{2n_i u_i^2 \Gamma} \right), \]  

(98)

which is similar to that obtained in [13]. Therefore, (96) is general in the sense that one can change two parameters, i.e. disorder \( u_i \) and confinement \( \kappa \), to offer different interpretations. In particular, we mention that even when the disorder becomes smaller, the peak of (96) does
not become narrower. However, it happens when $C$ becomes smaller. On the other hand, by noting that $F(x)$ has its maximum at $x = 0$ with $F(0) = 1$, we obtain

$$\chi(\varepsilon_F) = -\frac{e^2}{3\pi}\frac{2C}{\Gamma}.$$  \hfill (99)

It is worthwhile to see what happens in the case where $|\varepsilon| \gg \Gamma$. Using (94) the susceptibility becomes

$$\chi(\varepsilon_F) \approx -\frac{e^2}{3}\frac{C}{|\varepsilon_F|}.$$  \hfill (100)

At this stage, we have different comments. First, it is easy to see that if $C$ is constant, then (100) monotonically decreases as $|\varepsilon_F|$ increases. Second, let us write (100) as

$$\chi(\varepsilon_F) \approx -\frac{e^2}{3}\frac{\eta u^2}{|\varepsilon_F|} - \frac{e^2}{12}\frac{\kappa^2}{|\varepsilon_F|},$$  \hfill (101)

which tells us that the first term is due to the disorder and the second is a manifestation of the confinement. Clearly, this can be interpreted as a quantum correction to the first term. According to (101), this conclusion disappears if $\kappa = 0$. Furthermore, in the limit $C \to 0$, the susceptibility becomes a $\delta$-function,

$$\chi(\varepsilon_F) \approx -\frac{e^2}{6}\delta(\varepsilon_F).$$  \hfill (102)

Obviously, this conclusion cannot be reached in the clean limit $u_i \to 0$ as in [13].

### 5.3. Long-ranged disorder

It is worthwhile to ask about the susceptibility for the long-ranged disorder. In a similar way to the short-ranged case, we use (67) and (72) to obtain

$$\chi = -\frac{1}{\kappa C} \int_{-\infty}^{\infty} d\varepsilon f(\varepsilon) \int_{-\infty}^{\varepsilon} d\varepsilon' \lim_{\Gamma \to \infty} \frac{1}{2\partial B^2} \left[ \Sigma^+_{\text{tot}}(\varepsilon', B) + \Sigma^-_{\text{tot}}(\varepsilon', B) \right] \bigg|_{B=0}. $$  \hfill (103)

By analogy to (75), we change the variable to

$$X^\pm = \varepsilon \pm \Sigma^\pm_{\text{tot}}$$  \hfill (104)

and define $\Sigma^\pm_{\text{tot}}(X^+, X^-, B)$ as

$$\tilde{\Sigma}^+_{\text{tot}} = C|\omega| \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{X^+ g(\epsilon_{n,k})}{X^+ X^- - (\epsilon_{n,k})^2};$$  \hfill (105)

$$\tilde{\Sigma}^-_{\text{tot}} = C|\omega| \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{X^- g(\epsilon_{n,k})}{X^+ X^- - (\epsilon_{n,k})^2}.$$  \hfill (106)

The derivatives of $\Sigma_{\text{tot}}$ can be written in terms of $\tilde{\Sigma}_{\text{tot}}$ as

$$\frac{\partial \Sigma_{\text{tot}}}{\partial B} = A_{ij} \frac{\partial \tilde{\Sigma}_{\text{tot}}}{\partial B}$$  \hfill (107)

$$\frac{\partial^2 \Sigma_{\text{tot}}}{\partial B^2} = A_{ij} \left[ \frac{\partial^2 \tilde{\Sigma}_{\text{tot}}}{\partial B^2} - \frac{1}{2} \frac{\partial^2 \tilde{\Sigma}_{\text{tot}}}{\partial X^k \partial B} \frac{\partial \Sigma_{\text{tot}}}{\partial B} + \frac{\partial^2 \tilde{\Sigma}_{\text{tot}}}{\partial X^k \partial X^l} \frac{\partial \Sigma_{\text{tot}}}{\partial B} \frac{\partial \Sigma_{\text{tot}}}{\partial B} \right],$$  \hfill (108)
where $i, j, l = \pm$ and repeated indices indicate summation. The involved matrix elements are given by

$$\mathcal{A}_{ij} \equiv \left( \delta_{ij} + \frac{\partial \tilde{\Sigma}^l}{\partial X_i} \right)^{-1}.$$ (109)

One can calculate the derivatives of $\tilde{\Sigma}^\pm$ at $B = 0$, in a similar way to the short-range case, and then obtain those for $\Sigma^\pm$ using (107) and (108), to obtain

$$\frac{\partial}{\partial B}(\Sigma^{+} + \Sigma^{-}) = (1 + \alpha + 2\beta)^{-1} \kappa \frac{e}{X^3} \left( \frac{C}{6} \right),$$ (110)

$$\frac{\partial^2}{\partial B^2}(\Sigma^{+} + \Sigma^{-}) = (1 + \alpha + 2\beta)^{-1} \kappa^2 \frac{e^2}{X^3} \left( -\frac{C}{6} + \frac{1}{1 - \alpha} - \frac{2\beta}{(1 - \alpha)^2} C^2 - \frac{2}{4} \right),$$ (111)

where $X \equiv \lim_{B \to 0} X^{+} = \lim_{B \to 0} X^{-}$ and the two parameters are

$$\alpha = 2C \int_0^\infty t \, dt \frac{g(t)}{X^2 - t^2},$$ (112)

$$\beta = 2C \int_0^\infty t \, dt \frac{-X^2 g(t)}{(X^2 - t^2)^2}.$$ (113)

We have now derived all ingredients to write the expression for the susceptibility in this case:

$$\chi = -\frac{e^2}{6\pi \kappa} \int_{-\infty}^\infty \! \! df(\varepsilon) \int_{X(-\infty)}^{X(\varepsilon)} \! \! dX' \frac{1}{X^2} \text{Im} \left[ \frac{1}{X^3} \left( -1 + 3C \left( \frac{1}{1 - \alpha} - \frac{\beta'}{(1 - \alpha')^2} \right) \right) \right] \bigg|_{B=0},$$ (114)

where the integration over $\varepsilon'$ has been replaced by

$$dX' = (1 + \alpha' + 2\beta')^{-1} d\varepsilon'.$$ (115)

For the long-ranged disorder in the region $|\varepsilon| \ll \varepsilon_c$, (12) and (13) can be approximated as

$$\alpha \approx -C \log \left( \frac{\varepsilon^2}{X^2} \right),$$ (116)

$$\beta \approx C.$$ (117)

By substituting them in (114) and performing the integration over $X'$, we see that the susceptibility is given by

$$\chi = -\frac{e^2}{6\pi \kappa} \int_{-\infty}^\infty \! \! df(\varepsilon) \int_{X(-\infty)}^{X(\varepsilon)} \! \! dX' \frac{1}{X^2} \text{Im} \left[ \frac{1}{X^3} \left( 1 - \frac{3C}{1 + C \log \left( \frac{-\varepsilon^2}{X^2} \right)} \right) \right] \bigg|_{B=0},$$ (118)

with $X = \varepsilon - \Sigma_{\text{rel}}(\varepsilon)$. This can be rewritten as

$$\chi = -\frac{e^2}{6\pi \kappa} \int_{-\infty}^\infty \! \! df(\varepsilon) \int_{X(-\infty)}^{X(\varepsilon)} \! \! dX' \frac{1}{X^2} \text{Im} \left[ \frac{1}{X^2 \left( 1 + C \log \left( \frac{-\varepsilon^2}{X^2} \right) \right)} \right] \bigg|_{B=0} + \frac{e^2}{2\pi \kappa} C \int_{-\infty}^\infty \! \! df(\varepsilon) \int_{X(-\infty)}^{X(\varepsilon)} \! \! dX' \frac{1}{X^2} \text{Im} \left[ \frac{1}{X^2 \left( 1 + C \log \left( \frac{-\varepsilon^2}{X^2} \right) \right)} \right] \bigg|_{B=0}.$$ (119)

The first term is the short-range contribution (87). The second term can be regarded as a contribution of the order of $\mathcal{O}(C)$, but this gives a minor effect since $C$ is assumed to be small. When $\mathcal{O}(C)^2$ is neglected, the susceptibility becomes just $1 - 3C$ times as large as in the case of short-ranged disorder. Accordingly the integration of $\chi$ over $\varepsilon$ weakly depends on $C$, while in $C \to 0$ we again obtain (102). Finally, note that taking $\kappa = 0$ we end up with the results obtained by Koshino and Ando in [13].
6. Conclusion

This paper was devoted to give a complete solution to the confined Dirac fermion system in the presence of a perpendicular magnetic field. Using a similarity transformation, we have formulated our problem in terms of the polar coordinate representation that allows us to handle easily the exact relationship between spinor components. One spinor component was obtained by solving a second-order differential equation, while the other component was obtained using the exact relationship (21). It resulted in a complete solution space made of eight subspaces, which suggests that it is necessary to include all components of this subspace in the computations of any physical quantity. A failure to do so will result in erroneous conclusions.

Considering the disordered graphene, we formulated our solutions to capture the impurity potential. After providing the necessary tools, we used the Green function technique to determine the self-energy. For this, two cases are discussed, which concern the short- and long-ranged disorders. These allowed us to give a simplified form of the density of states that was used to determine different thermodynamical quantities.

For further studies, we distinguished between two cases. As far as the short-ranged disorder is concerned, we further simplified the self-energy and found a quantum correction to the susceptibility for Dirac fermions in disordered graphene. On the other hand, we noted that $\kappa = 0$ allowed us to recover the results of disordered graphene in the presence of a magnetic field [13]. Furthermore, in the limit $C \to 0$, the susceptibility becomes a $\delta$-function (102).

Subsequently, we treated the long-ranged case and stressed what makes a difference with respect to the former one. By comparing the results obtained in (87) and (118), we noted that there is an extra term of the order of $O(C^2)$, but this gives a minor effect since $C$ is assumed to be small. When $O(C^2)$ is neglected, the susceptibility becomes just $1 - 3C$ times as large as in the short-ranged disorder. Accordingly, the integration of $\chi$ over $\epsilon$ weakly depends on $C$, while in the limit $C \to 0$ we again obtain (102).

This work can be extended to other cases. One may numerically check the above results and compare them with those obtained before. On the other hand, an interesting question arises about what happens for a variable magnetic field, in particular an exponential variation of the field. This question is under investigation.

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