Investigation of the Motion of Viscous Fluid between Rotating Disks

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Abstract. The issue of the motion of fluid between two rotating disks often arises in practice and it has been little investigated yet. The practical instances of this issue are designing distributors of disk-type fuel pumps, lubrication of smooth disk surfaces during rotation etc. The present article studies the process of the leakage of fluid between disks when the fluid is supplied to the disks under a certain pressure P. The objective of the research is to obtain mathematical models for determining the amount of the fluid leaking at the pin and the amount of the fluid leaking out. It is shown that the leakage of the fluid between the disks can be described using the Navier-Stokes equation system. The mathematical models for determining the amount of the fluid leaking at the pin and the amount of the fluid leaking out have been obtained.

1. Introduction
The physical model of the flow of fluid between disks is shown in Figure 1.

Figure 1. The scheme of the flow of fluid between two rotating disks and the adopted coordinate system:
 h is the spacing between the disks; P is the pressure under which the fluid is supplied into the
spacing between the discs; \( R \) is the radius of a disk; \( R'_{0} \) is the radius of the pin; \( r_{0} \) is the radius of the fluid inlet port.

2. Relevance

Let us determine the amount of the fluid flowing to the center of the disk per unit of time. For this we will use a vector equation of the motion of viscous incompressible fluid (the Navier–Stokes equation). In the projections on the coordinate axes this equation is represented by a system of equations (1) [1,2] and a continuity equation (2):

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = F_x - \frac{1}{\rho} \frac{\partial P}{\partial x} + \psi \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = F_y - \frac{1}{\rho} \frac{\partial P}{\partial y} + \psi \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = F_z - \frac{1}{\rho} \frac{\partial P}{\partial z} + \psi \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\]

where \( t \) is time, sec; \( u,v,w \) are the flow velocity components in the \( x,y \) and \( z \) directions, respectively, \( m/s; F_x, F_y, F_z \) are the body forces in the \( x,y \) and \( z \) directions, respectively, \( N; \rho \) is the density, \( kg/m^3; P \) is the pressure, \( MPa; \psi \) the bracketed expressions are the operational definitions of velocities in the directions of the corresponding axes; \( \nu \) is the kinematic viscosity of the fluid, \( m^2/s \).

3. Theoretical research

At a steady rotation speed the process is stationary and the time derivatives of the velocities components along the axes are equal [3]:

\[
\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0
\]

Hence, the initial conditions for solving the system (1) are not required.

The boundary conditions are as follows:

\[
\text{at } z=0; \quad u = 0, \quad v = 0.
\]

(3)

These conditions indicate that the molecules of fluid, which are situated closely to the lower disk, are immobile (i.e. the condition of the complete adhesion of particles),

\[
\text{at } z = h; \quad w = 0, \quad v_r = \frac{\pi n r}{30}
\]

where \( v_r \) and \( v_t \) are the tangent and radial components of the velocity vector.

Taking into account that \( u = -v_r \sin \phi = -v_r \frac{y}{r}, \quad v = v_r \cos \phi = v_r \frac{x}{r}, \) we have: at \( z=h:\)

\[
u = -\frac{\pi n y}{30}, \quad v = -\frac{\pi n x}{30}.
\]

(4)

These conditions also indicate that the particles that are closely situated to the upper disk will rotate along with it.

The periodical pressure \( P \) acts at the points \( r = r_0 \) of the disk. As the calculations, that take into account the periodical pressure, are very complicated we will average out the pressure and presume that constant pressure \( P_0 \) acts at every point. Such substitution will not significantly affect the result, as our task is to calculate the permeability only (the need for such averaging is obvious at high rotation rates of the disk).

Obviously, the radial pressure is \( P_r = P \cos \phi \) and the total pressure along the radius is:
\[
P_r = \int_{\pi/2}^{\pi} P \cos \varphi d\varphi = 2P.
\]

Hence, the average pressure is:
\[
P_{cp} = \frac{2P}{\pi r_0}.
\]

Therefore, we have another two conditions:
\[
\begin{align*}
&\text{at } r = r_0 \quad P = P_{cp} = \frac{2P}{\pi r_0}, \\
&\text{at } r = r_0' \quad P = P_\infty,
\end{align*}
\]

where \( r = r_0' \) is the radius of the pin.

In order to solve the system we can use the fact that the Reynolds number is small and the fluid flows almost in a concentric circle. Thus, we can neglect the inertial forces in the left hand side of the system (1). It should be noted that such a presumption will only increase the permeability in the direction of the pin, because the inertial forces act in the opposite direction.

\[
\begin{align*}
\frac{\partial P}{\partial x} &= \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\
\frac{\partial P}{\partial y} &= \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\
\frac{\partial P}{\partial z} &= \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{align*}
\]

Obviously, \( w = 0 \).

Further, taking into the consideration that both \( h \) and the boundary conditions (3) and (4) are small, the variation of \( u \) along the \( z \)-axis will go much faster than the variation of \( u \) along the \( x \)- and \( y \)-axes:
\[
\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial z} \land \frac{\partial u}{\partial y} \ll \frac{\partial u}{\partial z}.
\]

Hence:
\[
\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial z^2} \land \frac{\partial^2 u}{\partial y^2} \ll \frac{\partial^2 u}{\partial z^2}.
\]

This can also be written as:
\[
\frac{\partial^2 v}{\partial x^2} \ll \frac{\partial^2 v}{\partial z^2} \land \frac{\partial^2 v}{\partial y^2} \ll \frac{\partial^2 v}{\partial z^2}.
\]

In this case the equation (7) becomes:
\[ \frac{\partial P}{\partial x} = \mu \left( \frac{\partial^2 u}{\partial z^2} \right), \]
\[ \frac{\partial P}{\partial y} = \mu \left( \frac{\partial^2 v}{\partial z^2} \right). \]

The third equation of this system shows that \( P \) depends only on \( x \) and \( y \), i.e \( P = f(x, y) \). Thus, the first and second equations of the system (8) can be easily integrated along the \( z \)-axis.

\[ \mu u = \frac{\partial P}{\partial x} \frac{z^2}{2} + c_1 z + c_2, \]
\[ \mu v = \frac{\partial P}{\partial y} \frac{z^2}{2} + c_3 z + c_4, \]

where \( c_1, c_2, c_3, c_4 \) are the functions of \( x \) and \( y \).

We can find them from the conditions (3) and (4). We obtain: \( c_2 = c_4 = 0 \);

\[ c_1 = -\left( \mu \frac{\pi ny}{30h} + \frac{\partial P}{\partial x} \frac{h}{2} \right), \]
\[ c_3 = \left( \mu \frac{\pi nx}{30h} - \frac{\partial P}{\partial y} \frac{h}{2} \right). \]

Hence,

\[ u = \frac{1}{2\mu} \frac{\partial P}{\partial x} \cdot z(z-h) - \frac{\pi ny}{30h} \cdot z, \]
\[ u = \frac{1}{2\mu} \frac{\partial P}{\partial y} \cdot z(z-h) - \frac{\pi nx}{30h} \cdot z. \]

We now need to determine \( \frac{\partial P}{\partial x} \) and \( \frac{\partial P}{\partial y} \).

Plugging (9) into the last equation of the system (8), we find:

\[ \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0. \]

It is very important that if the pressure \( P \) satisfies the equation (10), the formulae (9) will give the accurate solution of the system (7) because in this case the neglected terms are equal to zero. So:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left( \frac{1}{2\mu} \frac{\partial^2 P}{\partial x} \cdot z(z-h) + \frac{1}{2\mu} \frac{\partial^2 P}{\partial y} \cdot z(z-h) \right) = \frac{1}{2\mu} z(z-h) \frac{\partial}{\partial x} \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) = 0 \]

or:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left[ \frac{1}{2\mu} z(z-h) \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) \right] = 0 \]

Let us solve the equation (11) under the conditions (5) and (6). After transiting to polar coordinates the equation (11) can be written:

\[ \frac{\partial^2 P}{\partial z^2} + \frac{1}{r} \frac{\partial P}{\partial r} + \frac{1}{r^2} \frac{\partial^2 P}{\partial \phi^2} = 0, \]
The pressure $P$ in the direction of the coordinate $\varphi$ doesn’t depend on the angle, thus:

$$\frac{\partial^2 P}{\partial \varphi^2} = 0 .$$  \hspace{1cm} (13)

Taking into account (13) the equation (12) can be rearranged to give:

$$\frac{\partial^2 P}{\partial z^2} + \frac{1}{r} \frac{\partial P}{\partial r} = 0 .$$  \hspace{1cm} (14)

Solving the equation (14) we can obtain:

$$P = c \ln r + c_0 ,$$

where $c$ and $c_0$ are determined from the conditions (5) and (6):

$$c = \frac{P_{cp} - P_a}{\ln r_0 - \ln r}_0 , \hspace{0.5cm} c_0 = \frac{P_{cp} - P_a}{\ln r_0 - \ln r}_0 \cdot \ln r_0 .$$

Taking into account the expressions for the constants $c$ and $c_0$:

$$P = \frac{P_{cp} - P_a}{\ln r_0 - \ln r}_0 (\ln r_0 - \ln r_0') + P_{cp}$$

And hence,

$$\frac{\partial P}{\partial x} = \frac{P_{cp} - P_a}{\ln r_0 - \ln r}_0 \cdot \frac{x}{r^2} ,$$  \hspace{1cm} (16)

$$\frac{\partial P}{\partial y} = \frac{P_{cp} - P_a}{\ln r_0 - \ln r}_0 \cdot \frac{y}{r^2} .$$

From (10) and (16) we have:

$$v_r = u \cos \varphi + v \sin \varphi = \frac{1}{2\mu} \left( \frac{P_{cp} - P_a}{\ln r_0 - \ln r}_0 (z - h) \right) .$$  \hspace{1cm} (17)

The amount of the fluid leaking at the pin per unit of time (Figure 2):

$$dQ = ds \cdot v \cdot r'_0 ,$$

but $ds = 2\pi r'_0 dz$, therefore $dQ = 2\pi r'_0 \cdot v \cdot r'_0 dz$. Then from (17):

$$Q = \int_0^h 2\pi r'_0 \cdot v \cdot r'_0 dz = \frac{\pi}{6\mu} \frac{P_{cp} - P_a}{\ln r_0 - \ln r}_0 h^3 .$$  \hspace{1cm} (18)

**Figure 2.** Determination of the leakage of fluid at the pin.
In the same manner, we can obtain the formula for determining the amount of the fluid that flows out:

\[ Q = \frac{\pi \cdot (P_{cp} - P_a) \cdot h^3}{6\mu \ln R - \ln r_0}, \]  

(19)

Where \( R \) is the radius of the disk, m; \( P_a \) is the external pressure, MPa.

The formulae (18) and (19) show that the main parameters that influence the leakage of the fluid between the disks are the following: the dynamic viscosity \( \mu \), the differential pressure between the inlet port and the environmental pressure, the size of the spacing \( h \) between the disks and the difference of the logarithms of the corresponding radiuses. Also, there is a linear dependence between the amount of the fluid that leaked out and the differential pressure. The intensity of the variation of the leakage amount is determined by the tilt angle:

\[ \alpha = \frac{\pi \cdot h^3}{6\mu(\ln r_0 - \ln r'_0)}. \]

The leakage rate depending on the difference of the logarithms of the radiiuses of the inlet ports and the pin is more intense when the differential pressure is constant and it is changing in accordance with the hyperbolic law:

\[ Q = \frac{A}{(\ln r_0 - \ln r'_0)}, \]

where \( A = \frac{\pi \cdot (P_{cp} - P_a) \cdot h^3}{6\mu} \).

The fluid leakage between the disks depending on the spacing size occurs as a cubic dependence:

\[ Q = bh^3, \]

where \( b = \frac{\pi \cdot (P_{cp} - P_a)}{6\mu \ln R - \ln r_0} \).

The dependence of the amount of the fluid leaking out between the disks on the viscosity is due to the hyperbolic law:

\[ Q = \frac{c}{\mu}, \]

where

\[ c = \frac{\pi \cdot (P_{cp} - P_a) \cdot h^3}{6 \ln R - \ln r_0}. \]

4. Research result
Figure 3. The dependence of the water leakage at the pin on the differential pressure and the difference between the radiuses of the inlet post and the pin.

\[ \ln r_0 - \ln r_0' : 1 - 0.223; 2 - 0.405; 3 - 0.560; 4 - 0.693; 5 - 0.811; 0.916. \]

Figure 4. The dependence of the water leakage in the disks on differential pressure and the difference between the radiuses of the inlet port and the pin.

\[ \ln R - \ln r_0' : 1 - 1.099; 2 - 1.253; 3 - 1.386; 4 - 1.504; 5 - 1.609; 6 - 1.705. \]

Figure 5. The dependence of the water leakage at the pin on the differential pressure and the spacing between the \( h \) (\( \mu m \)): 1 - 5 \( \mu m \); 2 - 10 \( \mu m \); 3 - 15 \( \mu m \); 4 - 20 \( \mu m \); 5 - 25 \( \mu m \).
5. Conclusions
1. Solution of the Navier–Stokes equation system and a continuity equation allows obtaining an equation that governs the process of fluid leakage between disks.
2. The equations obtained allow determining the factors and the degree of their influence on the process of fluid leakage between disks.
3. The formulae (18) and (19) show that the amount of the leaking fluid is in direct proportion to the differential pressure at the disk inlet port, the spacing between the disks $h$ raised to the third power and is in inverse proportion to the difference of the logarithms of the external radiuses of disks.

6. References
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