Second-Order Corrections to the Power Spectrum in the Slow-Roll Expansion with a Time-Dependent Sound Speed

Hao Wei 1,2*, Rong-Gen Cai 1,3† and Anzhong Wang 3‡

1 Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing 100080, China
2 The Graduate School of the Chinese Academy of Sciences
3 CASPER, Department of Physics, Baylor University, Waco, TX76798-7316, USA

Abstract

We extend Green’s function method developed by Stewart and Gong to calculate the power spectrum of density perturbation in the case with a time-dependent sound speed, and explicitly give the power spectrum and spectral index up to second-order corrections in the slow-roll expansion. The case of tachyon inflation is included as a special case.

* e-mail address: haowei@itp.ac.cn
† e-mail address: cairg@itp.ac.cn
‡ e-mail address: anzhong_wang@baylor.edu
1 Introduction

It is generally believed that the curvature perturbation produced during inflation [1] is the origin of the inhomogeneities necessary for large scale structure formation. The power spectrum of these perturbations is predicted to be approximately scale invariant by most of the inflation scenarios [2], and this is confirmed by many recent cosmic microwave background observations and galaxy surveys [3]. The observation results from WMAP [4], SDSS [5] and other experiments give more accurate measurement of the power spectrum and the spectral index than ever before. Thus, it is very important to calculate the power spectrum precisely to match the observation results fully and to distinguish different inflation models.

Many works have been done on the power spectrum for the density perturbations produced during inflation [6]. For example, the power spectrum and spectral index of density fluctuation are calculated up to the first order of slow-roll parameters in [7]. In [8] Stewart and Gong further set up a formalism that can be used to calculate the power spectrum of the curvature perturbations up to arbitrary order in the slow-roll expansion, and calculate the power spectrum and spectral index up to second-order corrections. Recently, Myung et al. [9] have calculated the second-order corrections in the brane inflation model and in the noncommutative space-time inflation model, respectively.

However, we note that in most calculations made before, the sound speed \( C_s^2 \) is assumed to be constant. Of course, it is true in the inflation models derived by a scalar field of canonical kinetic term, where \( C_s^2 = 1 \). But, as we will see in the following, the sound speed is time-dependent in the general case. For instance, in the case of tachyon inflation (see for example [12, 13] and references therein), which has received a lot of attention in recent years, \( C_s^2 = 1 - \frac{2}{3} \epsilon \), where \( \epsilon \equiv - \dot{H}/H^2 \) is a first-order slow-roll parameter and \( H \) is the Hubble parameter, a dot denotes the derivative with respect to cosmic time \( t \). If one only considers approximations up to the first-order corrections, because \( \dot{\epsilon} \) is a second-order small parameter, \( \epsilon \) and then \( C_s^2 \) can be treated as constant approximately. But when one attempts to calculate second-order corrections, one has to consider the time-dependence of these quantities. As a result, the Mukhanov equation governing quantum fluctuations produced during inflation becomes more difficult to solve than the case with a constant sound speed.

In this paper, we have extended Green’s function method developed by Stewart and Gong [8] and obtained the second-order corrections in the slow-roll expansion to the power spectrum in the case with a time-dependent sound speed. In particular, the case of tachyon inflation is included. Note that the formalism can be used to calculate the power spectrum of the curvature perturbation up to arbitrary order in the slow-roll expansion. We use the units \( \hbar = c = 8\pi G = 1 \) throughout this paper.

2 Preliminaries

2.1 Basics

Our starting point is the following effective action during inflation [10],

\[
S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2}R + p(\varphi, X) \right]
\]  

(2.1)

where \( g \) is the determinant of the metric, \( R \) is the Ricci scalar, \( \varphi \) denotes a scalar field and

\[
X = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi,
\]  

(2.2)
is its kinetic term. The Lagrangian $p$ plays the role of pressure and the energy density is given by

$$\rho = 2X p,_{,X} - p.$$  \hfill(2.3) 

Here $p,_{X}$ denotes the partial derivative of $p$ with respect to $X$. In this paper, we consider a flat universe for simplicity, namely $K = 0$. The background metric is

$$ds^2 = dt^2 - a^2(t)dx^2 = a^2(\eta) \left( d\eta^2 - dx^2 \right),$$  \hfill(2.4) 

where $\eta$ is the conformal time defined by $d\eta = dt/a$. The action for scalar perturbation can be written down as \[10\]

$$S = \frac{1}{2} \int \left[ w'^2 + C_s^2 w \nabla^2 w + \frac{z''}{z} w^2 \right] d\eta d^3x,$$  \hfill(2.5) 

where a prime denotes the derivative with respect to the conformal time $\eta$, the canonical quantization variable $w = z\zeta$, $\zeta$ is the curvature perturbation, and $z$ is defined by

$$z \equiv \frac{a(\rho + p)^{1/2}}{C_s H},$$  \hfill(2.6) 

and the sound speed by

$$C_s^2 \equiv \frac{p,_{X}}{\rho,_{X}}.$$  \hfill(2.7) 

Using the Fourier transform of $w$, from \[10\] one can obtain the equation of motion for mode $w_k$, i.e. the well-known Mukhanov equation \[10\]

$$w''_k + \left( C_s^2 k^2 - \frac{z''}{z} \right) w_k = 0,$$  \hfill(2.8) 

where $k$ is the wave-number for the mode $w_k$. Furthermore, from Eq. \[2.6\] and the dynamical equation of the Hubble parameter

$$\dot{H} = -\frac{1}{2}(\rho + p),$$  \hfill(2.9) 

we can express $z$ by the slow-roll parameter $\epsilon \equiv -\dot{H}/H^2$, which is a first-order slow-roll parameter, as

$$z = \sqrt{2} \frac{a}{C_s} \epsilon^{1/2}.$$  \hfill(2.10) 

\subsection*{2.2 Sound speed}

Obviously, the sound speed $C_s$ defined by Eq. \[2.7\] is generally time-dependant for a general Lagrangian $p(\varphi, X)$. In the slow-roll phase where $X$ is a small quantity, compared to the potential of the scalar field $\varphi$. As a result, a generic Lagrangian is expected to admit an expansion of the form \[11\]

$$p(\varphi, X) = V(\varphi) + D(\varphi)X + B(\varphi)X^2 + \cdots.$$  \hfill(2.11) 

From Eqs. \[2.11\], \[2.3\] and \[2.7\], one has the sound speed

$$C_s^2 = 1 - \frac{4BX}{D} + 24 \left( \frac{BX}{D} \right)^2,$$  \hfill(2.12) 

up to the term of $X$ squared. Substituting Eqs. \[2.11\] and \[2.8\] into Eq. \[2.9\], we get

$$2BX^2 + DX - H^2 \epsilon = 0.$$  \hfill(2.13)
This can be considered as a quadratic equation of $X$ and the solution satisfying the convergence condition $DX \gg BX^2$ is

$$X = \frac{H^2}{D} \epsilon - \frac{2BH^4}{D^2} \epsilon^2,$$

(2.14)

up to the second-order correction of slow-roll parameter. Then, one can expand the sound speed in terms of the slow-roll parameter as

$$C^2_s = 1 - \frac{4BH^2}{D^2} \epsilon + \frac{8B^2H^4}{D^3} \left(1 + \frac{3}{D}\right) \epsilon^2.$$  

(2.15)

Because the coefficients $B$ and $D$ are dependent of $\varphi$, and in turn dependent of time, it is still very difficult to deal with Eq. (2.8). From now on, we restrict ourselves to a class of special case where the sound speed has the form

$$C^2_s = 1 - c_1 \epsilon - c_2 \epsilon^2.$$  

(2.16)

Here $c_1$ and $c_2$ are two constants of order of unity. This form of sound speed includes some well-known examples. For instance, in the case of canonical scalar field one has $c_1 = c_2 = 0$ while $c_1 = 2/3, c_2 = 0$ for the case of tachyon inflation [12, 13].

In fact, by using the formalism developed in section 3, we can consider a class of more general sound speed as

$$C^2_s = 1 - c_1 \epsilon_m - c_2 \epsilon_p \epsilon_q,$$

where $\epsilon_i$ is defined by Eq. (2.17) and $i = m, p, q$ are some positive integers. We will give a simple discussion on this in the appendix. But here, we will concentrate ourselves on the case with the sound speed given by Eq. (2.16).

### 2.3 Slow-roll parameters

In this paper, we introduce a set of slow-roll parameters expressed by Hubble parameter and its derivatives with respect to time as

$$\epsilon_n \equiv \frac{(-1)^n}{H} \frac{d^n H}{dt^n} \left/ \frac{d^{n-1} H}{dt^{n-1}} \right.,$$

(2.17)

namely, one has

$$\epsilon_1 = \epsilon \equiv - \frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\ddot{H}}{HH}, \quad \epsilon_3 = - \frac{\dot{H}}{HH}, \ldots$$

(2.18)

evaluated at sound horizon crossing [10], i.e. $aH = C_s k$. Note that, different from other slow-roll parameters used for example in [8, 2], these parameters defined in (2.17) are all first-order small parameters, namely satisfying $|\epsilon_n| < \xi$ for some small perturbation parameter $\xi$. The higher order slow-roll parameters are products of the first-order parameters $\epsilon_n$. In addition, it is easy to show

$$\frac{1}{H} \dot{\epsilon}_n = \epsilon_n \left[ \epsilon_1 + (-1)^{n+1} (\epsilon_n + \epsilon_{n+1}) \right],$$

(2.19)

which will be used extensively below.

Note that for the case of canonical scalar field, the slow-roll parameters used in Ref. [8] are

$$\epsilon_1 \equiv - \frac{\dot{H}}{H^2} = \frac{1}{2} \left( \frac{\dot{\phi}}{H} \right)^2, \quad \delta_n \equiv \frac{1}{H^n \phi} \frac{d^{n+1} \phi}{dt^{n+1}},$$

and these have the following relations to the slow-roll parameters defined in (2.17)

$$\epsilon_1 = \epsilon_1, \quad \delta_1 = \frac{1}{2} \epsilon_2, \quad \delta_2 = \frac{1}{4} \epsilon_2^2 - \frac{1}{2} \epsilon_2 \epsilon_3, \quad \delta_3 = \frac{3}{8} \epsilon_2^3 + \frac{3}{4} \epsilon_2 \epsilon_3^2 - \frac{1}{2} \epsilon_2 \epsilon_3 \epsilon_4, \ldots$$

(2.20)
3 The calculations

3.1 Slow-roll expansion and the formalism of calculations

Our main task is to solve the Mukhanov equation Eq. (2.8). Introducing new variables \( y = \sqrt{2k\omega_k} \) and \( x = -k\eta \), we can recast the Mukhanov equation as

\[
\frac{d^2y}{dx^2} + \left( C_s^2 \frac{x^2}{z} \right) y = 0.
\] (3.1)

Because \( C_s^2 \) is time-dependent, the problem is more complicated than the case with constant \( C_s \). In order to obtain the solution of (3.1) up to second order corrections in the slow-roll expansion, we develop a formalism, in which the key point is to use Green’s function method developed in Ref. [8] twice. The first step is to decompose the term \( \frac{1}{z} \frac{d^2z}{dx^2} = \frac{z''}{k^2z} \) into two parts and Eq. (3.1) then can be rewritten as

\[
\frac{d^2y}{dx^2} + \left( C_s^2 - 2 \frac{x^2}{z} \right) y = \frac{1}{x^2} g(\ln x) y,
\] (3.2)

where \( g(\ln x) \) is defined by

\[
g(\ln x) = \frac{x^2 z''}{k^2 z} - 2.
\] (3.3)

We can justify this decomposition as follows. From Eq. (2.10) we have

\[
\frac{z''}{z} = \frac{a''}{a} + 2 \frac{a' \epsilon_1}{\epsilon_1} - 2 \frac{a'}{a} \left( \frac{C'_s}{C_s} \right) - \frac{1}{4} \left( \frac{C'_s}{C_s} \right)^2 + \frac{1}{2} \left( \frac{C''_s}{C_s} \right) - \frac{1}{2} \left( \frac{C''_s}{C_s} \right) - 2 \left( \frac{C'_s}{C_s} \right)^2.
\] (3.4)

Note that

\[
\frac{a''}{a} = a^2H^2(2 - \epsilon_1),
\] (3.5)

and

\[
x = -k\eta = -k \int \frac{dt}{a} = -k \int \frac{da}{a^2H} \approx \frac{k}{aH} (1 + \epsilon_1 + 3\epsilon_2^2 + \epsilon_1\epsilon_2).
\] (3.6)

Thus, for any \( C_s^2 \), the leading term of \( \frac{1}{z} \frac{d^2z}{dx^2} \) is \( \frac{2}{x^2} \), which comes from \( \frac{a''}{a} \). Therefore, the decomposition made above is always valid. Furthermore, we can expand

\[
g(\ln x) = \sum_{n=0}^{\infty} \frac{g_{n+1}}{n!} (\ln x)^n,
\] (3.7)

where \( g_n \) is of order \( n \) in the slow-roll expansion, namely \( |g_n| < \xi^n \). This expansion is useful for \( \exp(-1/\xi) \ll x \ll \exp(1/\xi) \). From Eqs. (3.3)-(3.7), (2.10) and (2.19), up to second-order corrections, it is easy to find

\[
g_2 = \left. \frac{d g_2}{d \ln x} \right|_{x=1} \approx \left. \left( -12\epsilon_1^2 - \frac{15}{2} \epsilon_1 \epsilon_2 + \frac{3}{2} \epsilon_2^2 + \frac{3}{2} \epsilon_2 \epsilon_3 \right) \right|_{aH=C_s,k},
\] (3.8)

\[
g_1 = g \big|_{x=1} \approx \left[ 6\epsilon_1 + (20 + 3c_1)\epsilon_2^2 + \frac{3}{2} \epsilon_2 + (9 + 3\epsilon_1 c_1)\epsilon_1 \epsilon_2 - \frac{1}{4} \epsilon_2^2 - \frac{1}{2} \epsilon_2 \epsilon_3 \right] \big|_{aH=C_s,k}.
\] (3.9)

Note that here \( \ln x \big|_{aH=C_s,k} \) is a first-order small quantity in the slow-roll expansion.

The boundary condition of Eq. (3.1) or Eq. (3.2) required to get the power spectrum is

\[
y \to \left\{ \begin{array}{ll}
\lim_{x \to \infty} y_0(x) & \text{as } x \to \infty, \\
\sqrt{2kA_kz} & \text{as } x \to 0,
\end{array} \right.
\] (3.10)

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where \( \ddot{y}_0(x) \) is a solution of equation
\[
\frac{d^2 y}{dx^2} + \left( C_s^2 - \frac{2}{x^2} \right) y = 0. \tag{3.11}
\]

Note that \( C_s^2 \) is time-dependent, and then \( x \)-dependant. To solve Eq. (3.11), it is necessary to express \( C_s^2 \) as a function of \( x \). Since \( \epsilon_1 = -\dot{H}/H^2 \) is time-dependant and therefore is a function of \( x \), too. From Eqs. (2.10) and (3.6), up to second-order, we have
\[
\frac{de_1}{dx} \simeq -x^{-1}\epsilon_1(2\epsilon_1 + \epsilon_2). \tag{3.12}
\]

Here \( \epsilon_1(2\epsilon_1 + \epsilon_2) \) can be treated as a constant approximately because its derivative with respect to time is a third-order small quantity, which can be ignored. Therefore \( \epsilon_1 \) should be of the form of \( const. + const. \ln x \), up to second-order corrections. In fact, we can expand \( \epsilon_1 \) as
\[
\epsilon_1 = \sum_{n=0}^{\infty} \frac{s_{n+1}}{n!} (\ln x)^n, \tag{3.13}
\]
where \( s_n \) is of order \( n \) in the slow-roll expansion, namely \( |s_n| < \xi^n \). This expansion is useful for \( \exp(-1/\xi) \ll x \ll \exp(1/\xi) \). It is then easy to obtain
\[
s_2 = \frac{de_1}{d\ln x} \bigg|_{x=1} \simeq -\epsilon_1(2\epsilon_1 + \epsilon_2)\big|_{aH=C_s k}, \tag{3.14}
\]
\[
s_1 = \epsilon_1 \big|_{x=1} \simeq \epsilon_1\big|_{aH=C_s k} \tag{3.15}
\]

Substituting Eqs. (3.12) into Eq. (2.16), up to second-order, we can recast the sound speed as
\[
C_s^2 = C_{sc}^2 - c_1 s_2 \ln x, \tag{3.16}
\]
in which
\[
C_{sc}^2 = 1 - c_1 s_1 - c_2 s_1^2. \tag{3.17}
\]

Then, Eq. (3.11) can be approximately rewritten as
\[
\frac{d^2 y}{dx^2} + \left( C_{sc}^2 - \frac{2}{x^2} \right) y = (c_1 s_2 \ln x)y. \tag{3.18}
\]

Using Green’s function approach, Equation (3.17) can be perturbatively solved order by order. The solution of Eq. (3.11) or (3.17) is
\[
\ddot{y}_0(x) = y_0(x) + \frac{i c_1}{2C_{sc}} \int_x^1 du \, s_2 \ln u \, \ddot{y}_0(u)[y_0^*(u)y_0(x) - y_0^*(x)y_0(u)], \tag{3.19}
\]
where
\[
y_0(x) = \left( 1 + \frac{i}{C_{sc} x} \right) e^{iC_{sc} x}, \tag{3.20}
\]
satisfying
\[
\frac{d^2 y_0}{dx^2} + \left( C_{sc}^2 - \frac{2}{x^2} \right) y_0 = 0. \tag{3.21}
\]

Ignoring the 4th- and higher order corrections, from (3.19) we get
\[
\ddot{y}_0(x) = y_0(x) + \frac{i c_1}{2C_{sc}} \int_x^1 du \, s_2 \ln u \, y_0(u)[y_0^*(u)y_0(x) - y_0^*(x)y_0(u)] \equiv y_0(x) + y_s(x). \tag{3.22}
\]
Now, we are able to solve Eq. \ref{eq:3.22} by using Green’s function approach once again. The solution is

\[
y(x) \simeq \tilde{y}_0(x) + \frac{i}{2C_{sc}} \int_x^\infty du \frac{1}{u^2} g(\ln u) y(u) \left[ \tilde{y}_0^*(u) \tilde{y}_0(x) - \tilde{y}_0^*(x) \tilde{y}_0(u) \right]
\]

\[
\simeq \tilde{y}_0(x) + \frac{i}{2C_{sc}} \int_x^\infty du \frac{1}{u^2} g(\ln u) y(u) \left[ y_0^*(u) y_0(x) - y_0^*(x) y_0(u) \right],
\]

(3.22)

in which we have ignored the third and higher order corrections. Using Eq. \ref{eq:3.22} twice, we have the perturbative solution up to second-order corrections

\[
y(x) = y_0(x) + y_s(x) + y_1(x) + y_{21}(x) + y_{22}(x),
\]

(3.23)

where \(y_s(x)\) comes from Eq. \ref{eq:3.21}, which is a second-order correction to \(y(x)\), and given by

\[
y_s(x) = \frac{i c_1}{2C_{sc}} \int_x^1 du s_2 \ln u y_0(u) [y_0^*(u) y_0(x) - y_0^*(x) y_0(u)]
\]

\[
= \frac{i c_1 s_2}{2C_{sc}} \left\{ y_0(x) \left[ -1 - x \ln x + x + \frac{1}{C_{sc}} (x^{-1} \ln x - 1 + x^{-1}) \right] - y_0^*(x) \left[ \frac{i}{2C_{sc}} \right] \right. 
\]

\[
e^{2i C_{sc} x} \ln x - \frac{3 i}{2C_{sc}} \int_x^1 du u^{-1} e^{2i C_{sc} u} - \frac{1}{C_{sc}} \left( x^{-1} e^{2i C_{sc} x} (\ln x + 1) - e^{2i C_{sc} x} \right) \right\}.
\]

(3.24)

\(y_1(x)\) is a first-order correction to \(y(x)\),

\[
y_1(x) = \frac{i g_1}{2C_{sc}} \int_x^\infty du u^{-2} y_0(u) [y_0^*(u) y_0(x) - y_0^*(x) y_0(u)]
\]

\[
= -\frac{1}{3} g_1 \left[ \frac{2 i}{3C_{sc}} x^{-1} e^{i C_{sc} x} + y_0(x) \int_x^\infty du u^{-1} e^{2i C_{sc} u} \right],
\]

(3.25)

and the second-order corrections \(y_{21}(x)\) and \(y_{22}(x)\) are, respectively, given by

\[
y_{21}(x) = \frac{i g_1}{2C_{sc}} \int_x^\infty du u^{-2} y_1(u) [y_0^*(u) y_0(x) - y_0^*(x) y_0(u)]
\]

\[
= -\frac{i g_1^2}{9} \left[ \frac{2}{3C_{sc}} x^{-1} e^{i C_{sc} x} + \left( -\frac{5}{3C_{sc}} x^{-1} + \frac{i}{3} \right) \int_x^\infty du u^{-1} e^{2i C_{sc} u} 
\]

\[
+i y_0(x) \int_x^\infty du u^{-1} e^{-2i C_{sc} u} \int_u^\infty dv v^{-1} e^{2i C_{sc} v} \right\},
\]

(3.26)

\[
y_{22}(x) = \frac{i g_2}{2C_{sc}} \int_x^\infty du u^{-2} \ln u y_0(u) [y_0^*(u) y_0(x) - y_0^*(x) y_0(u)]
\]

\[
= \frac{i g_2}{3} \left[ \frac{1}{C_{sc}} x^{-1} \left( \frac{8}{3} + 2 \ln x \right) e^{i C_{sc} x} + \frac{7 i}{3} y_0^*(x) \int_x^\infty du u^{-1} e^{2i C_{sc} u} 
\]

\[
+i y_0^*(x) \int_x^\infty du u^{-1} \ln u e^{2i C_{sc} u} \right\}.
\]

(3.27)

Note that in writing \(y_{22}(x)\) we have ignored a term

\[
\frac{i g_2 y_0(x)}{6 C_{sc}^2} x^{-3} \left( 1 - \frac{1}{C_{sc}} \right) \left( \ln x + \frac{1}{3} \right),
\]

since it is a third-order correction. In addition, the perturbative solution Eq. \ref{eq:3.26} should be accurate for \(\exp(1/\xi) \gg x \gg \exp(-1/\xi)\).
3.2 The power spectrum

Having obtained the solution $y(x)$ of the Mukhanov equation \((3.28)\), we are in a position to calculate the power spectrum of density fluctuation during inflation. Taking the limit $x \to 0$ while keeping $\xi \ln(1/x)$ fixed and small, after some tedious calculations, we obtain the asymptotic forms for $y_0(x)$, $y_s(x)$, $y_1(x)$, $y_{21}(x)$ and $y_{22}(x)$. They are

$$y_0(x) \to \frac{i}{C_{sc}} x^{-1},$$

$$y_s(x) \to \frac{3i C_1 s_2}{4C_{sc}^3} x^{-1} \left[ \frac{4}{3} - \frac{2i}{3 C_{sc}} - \frac{2i C_2}{3} + \frac{2i e^{2i C_{sc}}}{3 C_{sc}} + \alpha - 2 - Ei(2i C_{sc}) - \frac{i \pi}{2} - \ln C_{sc} \right],$$

$$y_1(x) \to \frac{ig_1}{3 C_{sc}} x^{-1} \left[ \alpha + \frac{i \pi}{2} - \ln C_{sc} \right] - \frac{ig_1}{3 C_{sc}} x^{-1} \ln x,$$

$$y_{21}(x) \to \frac{ig_2}{18 C_{sc}} \left[ \alpha^2 - \frac{2}{3} \alpha - 4 + \frac{\pi^2}{4} + (\ln C_{sc})^2 + \left( \frac{2}{3} - 2 \alpha \right) \ln C_{sc} + i \pi \left( \alpha - \frac{1}{3} - \ln C_{sc} \right) \right] x^{-1},$$

$$\frac{-ig_2}{9 C_{sc}} x^{-1} \ln x - \frac{ig_2}{6 C_{sc}} x^{-1} (\ln x)^2,$$

and

$$y_{22}(x) \to \frac{ig_2}{6 C_{sc}} \left[ \alpha^2 + \frac{2}{3} \alpha + \frac{\pi^2}{12} + i \pi \left( \alpha + \frac{1}{3} - \ln C_{sc} \right) - \frac{2}{3} \left( \alpha + \frac{1}{3} \right) \ln C_{sc} + (\ln C_{sc})^2 \right] x^{-1}$$

$$- \frac{ig_2}{9 C_{sc}} x^{-1} \ln x - \frac{ig_2}{6 C_{sc}} x^{-1} (\ln x)^2,$$

respectively, where $\alpha \equiv 2 - \ln 2 - \gamma$, $\gamma \simeq 0.5772156649$ is the Euler-Mascheroni constant, and $Ei(z) \equiv \int_{-\infty}^{z} \frac{e^\chi}{\chi} d\chi$ is the exponential integral function. Especially, $Ei(2i) \simeq 0.4229808288 + i 3.1762093036$.

The exact asymptotic form of $y(x)$ in the limit $x \to 0$ is given by Eq. \((3.10)\). By comparing \((3.10)\) with the sum of asymptotic forms given by \((3.28)-(3.32)\), we can obtain the coefficient $A_k$. To this end, we need to expand $z$ first,

$$xz = \sum_{n=0}^{\infty} \frac{f_n}{n!} (\ln x)^n,$$

where $f_n/f_0$ is of order $n$ in the slow-roll expansion. This expansion is useful for $\exp(-1/\xi) \ll x \ll \exp(1/\xi)$. By using Eqs. \((2.10)-(3.6)\), \((2.16)\) and \((2.19)\), up to second-order, we obtain

$$f_2 = \frac{d^2(xz)}{(d \ln x)^2} \bigg|_{x=1} \simeq \frac{k}{H} \sqrt{\frac{\epsilon_1}{2}} \left( 16 \epsilon_1^2 + 9 \epsilon_1 \epsilon_2 - \frac{1}{2} \epsilon_2^2 - \epsilon_2 \epsilon_3 \right) |_{aH=C_{k}} ,$$

$$f_1 = \frac{d(xz)}{d \ln x} \bigg|_{x=1} \simeq \frac{k}{H} \sqrt{\frac{\epsilon_1}{2}} \left( -4 \epsilon_1 - \epsilon_2 - 12 \epsilon_2^2 - 4 \epsilon_1 \epsilon_3 - 4 \epsilon_1^2 \epsilon_2 - \frac{3}{2} \epsilon_1 \epsilon_2 \epsilon_3 \right) |_{aH=C_{k}},$$

$$f_0 \simeq \frac{k}{H} \sqrt{\frac{\epsilon_1}{2}} \left[ 2 + (2 + c_1) \epsilon_1 + \left( 10 + 3 c_1 + \frac{3}{4} c_1^2 + c_2 \right) \epsilon_2^2 + \left( 3 + \frac{1}{2} c_1 \right) \epsilon_1 \epsilon_2 \right] |_{aH=C_{k}} .$$

Then, up to second-order corrections, the asymptotic form \((3.10)\) for $y(x)$ in the limit $x \to 0$ can be expressed as

$$y(x) \to \sqrt{2k} A_k f_0 x^{-1} + \sqrt{2k} A_k f_1 x^{-1} \ln x + \frac{1}{2} \sqrt{2k} A_k f_2 x^{-1} (\ln x)^2 .$$

(3.37)
Collecting Eqs. (3.28)-(3.32) together also gives an asymptotic form for \( y(x) \) up to second-order corrections. Note that

\[
C_{sc} \simeq 1 + \mathcal{O}(\xi) + \mathcal{O}(\xi^2), \quad \frac{1}{C_{sc}} \simeq 1 + \mathcal{O}(\xi) + \mathcal{O}(\xi^2), \quad \ln C_{sc} \simeq \mathcal{O}(\xi) + \mathcal{O}(\xi^2).
\]

We can therefore throw away the third and higher order corrections in Eqs. (3.28)-(3.32) coming from \( C_{sc} \). Comparing the result with Eq. (3.37), the coefficient of \( x^{-1} \) will give \( A_k \) up to second-order corrections. The coefficients of \( x^{-1} \ln x \) and \( x^{-1}(\ln x)^2 \) simply give the consistent asymptotic behavior, namely proportional to \( x \). We finally arrive at

\[
A_k = \frac{i}{\sqrt{2k}} \left\{ \frac{1}{C_{sc}} + \frac{c_1 s_2}{4} \left[ -2 - 4i + 3\alpha + \beta_1 + i\beta_2 - \frac{3i\pi}{2} \right] \right.
\]

\[
+ \frac{g_1}{3C_{sc}} \left[ \alpha - \ln C_{sc} + \frac{i\pi}{2} \right] + \frac{g_1^2}{18} \left[ \alpha^2 - \frac{2}{3}\alpha - 4 + \frac{\pi^2}{4} + i\pi \left( \alpha - \frac{1}{3} \right) \right] \right.
\]

\[
+ \frac{g_2}{6} \left[ \alpha^2 - \frac{2}{3}\alpha - \frac{\pi^2}{12} + i\pi \left( \alpha + \frac{1}{3} \right) \right] \right\}, \tag{3.38}
\]

where \( \beta_1 + i\beta_2 \equiv \lim_{x \to 0} \frac{\psi_x}{x} = -0.5496523673 + i 8.6963342377 \).

The power spectrum is defined by

\[
P_s(k) = \frac{H^2}{2\pi^2} \left( \text{lim}_{x \to 0} \left| \frac{w_k}{z} \right|^2 \right) = \frac{k^3}{2\pi^2} |A_k|^2. \tag{3.39}
\]

Substituting Eqs. (3.38), (3.40), (3.41), (3.42) and (3.43) into the above formula, we obtain

\[
P_s(k) = \frac{H^2}{8\pi^2\epsilon_1} \left\{ 1 + (4\alpha - 2)\epsilon_1 + \alpha\epsilon_2 + \left[ 4\alpha^2 - 23 + \frac{7\pi^2}{3} + (2 - \alpha - \beta_1)\epsilon_1 \right] \epsilon_3^2 \right.
\]

\[
+ \left[ \frac{3}{2}\alpha^2 + \alpha - 11 + \frac{29\pi^2}{24} + \frac{1}{2}(2 - \alpha - \beta_1)\epsilon_1 \right] \epsilon_2 \epsilon_3 \right.
\]

\[
+ \left( \alpha^2 - 1 + \frac{\pi^2}{12} \right) \epsilon_2^2 + \frac{1}{2} \left( \alpha^2 - \frac{\pi^2}{12} \right) \epsilon_2 \epsilon_3 \right\}, \tag{3.40}
\]

where the right hand side should be evaluated at \( aH = C_s k \). The spectral index is defined by

\[
n_s(k) - 1 \equiv \frac{d \ln P_s(k)}{d \ln k}. \tag{3.41}
\]

It is easy to get the result in present case,

\[
n_s(k) = 1 - 4\epsilon_1 - \epsilon_2 + 8(\alpha - 1)\epsilon_1^2 - \alpha\epsilon_2^2 + (5\alpha - 3)\epsilon_1\epsilon_2 - \alpha\epsilon_2\epsilon_3
\]

\[
+ 4 \left[ -4\alpha^2 + 10\alpha - 27 + \frac{7\pi^2}{3} + (3 - \alpha - \beta_1)\epsilon_1 \right] \epsilon_3^2 + \left( 2 - \alpha^2 - \frac{\pi^2}{6} \right) \epsilon_2^3
\]

\[
+ \frac{1}{2} \left[ -31\alpha^2 + 60\alpha - 172 + \frac{199\pi^2}{12} + (20 - 7\alpha - 7\beta_1)\epsilon_1 \right] \epsilon_2^2
\]

\[
+ \left[ 2 - \frac{3}{2}\alpha^2 - \frac{\pi^2}{8} \right] \epsilon_2^2 \epsilon_3 + \left( \alpha^2 - \alpha - 2 + \frac{\pi^2}{6} + \frac{1}{2}\epsilon_1 \right) \epsilon_1 \epsilon_2
\]

\[
+ \frac{1}{2} \left[ 7\alpha^2 - 8\alpha + 22 - \frac{31\pi^2}{12} + (\alpha + \beta_1 - 2)\epsilon_1 \right] \epsilon_1 \epsilon_2 \epsilon_3 + \frac{1}{2} \left( \alpha^2 - \frac{\pi^2}{12} \right) \epsilon_2 \epsilon_3 \epsilon_4, \tag{3.42}
\]

where also the right hand side should be evaluated at \( aH = C_s k \).
4 Some special cases

4.1 The case of canonical scalar field

In this case, $C_s^2 = 1$, namely $c_1 = c_2 = 0$. Substituting into Eqs. (3.40) and (3.42), we will obtain the power spectrum and spectral index for the case of canonical scalar field. It is interesting to compare our results with the one obtained in Ref. [8]. Having considered the relations between the slow-roll parameters in Ref. [8] and in this paper given Eq. (2.20), and for the case of canonical scalar field,

$$\epsilon_1 = -\frac{\dot{H}}{H^2} = \frac{1}{2} \left( \frac{\dot{\phi}}{H} \right)^2,$$

it is easy to see our results are completely the same as those of Ref. [8]. This gives a challenging check of our formalism.

4.2 The case of tachyon inflation

In this case [12, 13], $C_s^2 = 1 - \dot{T}^2 = 1 - \frac{2}{3} c_1$, namely $c_1 = \frac{2}{3}$ and $c_2 = 0$. Substituting these into Eqs. (3.40) and (3.42), we obtain the power spectrum and spectral index for tachyon case. Note that the first order corrections have been calculated, for example, in [13]. The result of the second-order corrections to the power spectrum and spectral index for the case of tachyon inflation is new. This is an concrete example that the sound speed is time-dependant. When one attempts to calculate the second and higher order corrections, $C_s^2$ can not be taken as a constant approximately. Thus, the formalism presented in this paper is applicable in this case.

4.3 The first-order corrections

When one considers only first-order corrections, the slow-roll parameters can be regarded approximately as some constants. From Eqs. (3.40) and (3.42), for any $C_s^2 = 1 - c_1 \epsilon_1 - c_2 \epsilon_1^2$, we have

$$P_s(k) \simeq \frac{H^2}{8\pi^2 \epsilon_1} [1 + (4\alpha - 2) \epsilon_1 + \alpha \epsilon_2], \quad (4.1)$$

and

$$n_s(k) \simeq 1 - 4 \epsilon_1 - \epsilon_2 + 8(\alpha - 1) \epsilon_1^2 - \alpha \epsilon_1^2 + (5\alpha - 3) \epsilon_1 \epsilon_2 - \alpha \epsilon_2 \epsilon_3, \quad (4.2)$$

where again the right hand side should be evaluated at $aH = C_s k$. Note that in these expressions, the coefficients $c_1$ and $c_2$ do not appear. This implies that up to the first order corrections, the time-dependence of sound speed does not have any effect on the power spectrum of density perturbation and spectral index.

In summary, we have extended Green’s function method developed by Stewart and Gong to calculate the power spectrum and its spectral index of density fluctuation produced during inflation in the case with a time-dependent sound speed, up to second order corrections in the slow-roll expansion. The result for the tachyon inflation is included as a special case. We have noted that up to the first order corrections, there are no effects of the time-dependence of sound speed on the power spectrum and spectral index.
5 Appendix

As mentioned at the end of section 2.2, we can deal with a class of more general sound speed as

\[ C_s^2 = 1 - c_1 \epsilon_m - c_2 \epsilon_p \epsilon_q, \]  

where \( m, p \) and \( q \) are some positive integers. Similar to the case of \( \epsilon_1 \), from Eqs. (2.19) and (3.6), we have

\[ \frac{d\epsilon_m}{dx} \simeq -x^{-1} \epsilon_m \left[ \epsilon_1 + (-1)^{m+1}(\epsilon_m + \epsilon_{m+1}) \right] \]  

for any \( \epsilon_m \). And \( \epsilon_m \left[ \epsilon_1 + (-1)^{m+1}(\epsilon_m + \epsilon_{m+1}) \right] \) can be treated as a constant approximately because its derivative with respect to time is a third-order small quantity which can be ignored. Thus integrating Eq. (5.2) yields

\[ \epsilon_m = s_{m1} + s_{m2} \ln x, \]  

up to second-order, where

\[ s_{m1} \simeq \epsilon_m \big|_{\alpha H = C_s k}, \quad s_{m2} \simeq -\epsilon_m \left[ \epsilon_1 + (-1)^{m+1}(\epsilon_m + \epsilon_{m+1}) \right] \big|_{\alpha H = C_s k}. \]  

We then can recast the sound speed as

\[ C_s^2 = C_{sc}^2 - c_1 s_{m2} \ln x, \]  

up to second-order, where

\[ C_{sc}^2 = 1 - c_1 s_{m1} - c_2 s_{p1} s_{q1}. \]  

The formalism developed in section 3 is still valid for this \( C_{sc}^2 \) because the calculations keep the forms of \( C_{sc}, s_n, g_n \) and \( f_n \) unchanged. The only difference is to replace the \( s_2 \) in \( A_k \) by \( s_{m2} \). Because the variable \( z \) is dependent of \( C_s \) as in Eq. (2.10), we must calculate the slow-roll expansions of \( z \) and \( g(\ln x) = \frac{x^2 z''}{k^2 z} - 2 \) accordingly. Once having the new \( f_0, g_1 \) and \( g_2 \), and using the new \( C_{sc}^2 \) given in Eq. (5.6) and \( s_{m1}, s_{m2} \) in Eq. (5.4), we then can get the corresponding \( A_k \). Thus the final power spectrum and spectral index are in hand. The final results are very involved and their expressions are long in length, we do not present them here.

Acknowledgments

We thank Profs. Y.S. Myung and Y.Z. Zhang for useful discussions. We are also grateful to Hong-Sheng Zhang, Qi Guo, Zong-Kuan Guo, Da-Wei Pang, Xun Su, Ren Wei, and Ding-Fang Zeng for discussions and help. One of the authors (RGC) would like to express his gratitude to the Physics Department, Baylor University for its hospitality. This work was supported by Baylor University, a grant from Chinese Academy of Sciences, a grant from NSFC, China (No. 13325525), and a grant from the Ministry of Science and Technology of China (No. TG1999075401).

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