HOMOTOPICALLY NON-TRIVIAL MAPS WITH SMALL K-DILATION

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Abstract. We construct homotopically non-trivial maps from $S^m$ to $S^n$ with arbitrarily small 3-dilation for certain pairs $(m, n)$. The simplest example is the case $m = 4, n = 3$, and there are other pairs with arbitrarily large values of both $m$ and $n$. We show that a homotopy class in $\pi_7(S^4)$ can be represented by maps with arbitrarily small 4-dilation if and only if the class is torsion.

The k-dilation of a map measures how much the map stretches k-dimensional volumes. If $f$ is a $C^1$ map between Riemannian manifolds, we say that the k-dilation of $f$ is at most $D$ if $f$ maps each k-dimensional submanifold of the domain with volume $V$ to an image with volume at most $DV$. We get the same k-dilation whether we consider all submanifolds or whether we consider only small disks, and so the k-dilation can also be defined in terms of the first derivative $df$. Recall that $\Lambda^k df$, the k-fold exterior product of the derivative $df$, maps $\Lambda^k T M$ to $\Lambda^k T N$. If $f$ is $C^1$, the k-dilation of $f$ is equal to the supremal value of the norm $|\Lambda^k df|$.

In this paper, we examine to what extent a bound on the k-dilation of a map controls the homotopy type of the map. A beautiful result of this kind was recently obtained by Tsui and Wang in [7].

Theorem. (Tsui and Wang) Let $f$ be a $C^1$ map from $S^m$ to $S^n$, where $m \geq 2$. If the 2-dilation of $f$ is less than 1, then $f$ is nullhomotopic.

The main result of this paper shows that the situation is very different for 3-dilation.

Theorem 1. For each $n$, there are infinitely many $m$ so that the following holds: there are homotopically non-trivial maps from $S^m$ to $S^n$ with arbitrarily small 3-dilation.

This result partly answers a question raised by Gromov in [5] (page 231). Gromov asked for which values of $k$, $q$, $m$, and $n$ is a map $f : S^m \to S^n$ with a sufficiently small norm $|\Lambda^k df|_{L^q}$ necessarily null-homotopic.

We make the following definition. A homotopy class $a$ in $\pi_m(S^n)$ lies in $V_k \pi_m(S^n)$ if there are maps in the homotopy class $a$ with arbitrarily small k-dilation. We will prove that $V_k \pi_m(S^n)$ is a subgroup of $\pi_m(S^n)$ and that $V_k \pi_m(S^n) \subset V_{k+1} \pi_m(S^n)$. Therefore, $V_k \pi_m(S^n)$ defines a filtration of $\pi_m(S^n)$. More generally, we will define a filtration $V_k \pi_m(X)$ for any space $X$.

Our methods give some partial information about the filtration $V_k \pi_m(S^n)$. The information is most interesting for the filtration $V_k \pi_7(S^4)$. Recall that $\pi_7(S^4)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_{12}$.

Theorem 2. The group $V_4 \pi_7(S^4)$ is the torsion subgroup of $\pi_7(S^4)$. It is a proper, non-zero subgroup.
The paper is organized as follows. In the first section, we prove the first theorem. In the second section, we summarize some lower bounds for k-dilation that appear in the literature. In the third section, we define \( V_k \pi_m(X) \), and prove its basic algebraic properties. In the fourth section, we look at some examples, including \( V_k \pi_7(S^4) \).

This paper is based on a section of my thesis. I would like to thank my thesis advisor, Tom Mrowka, for his help and support.

1. Homotopically non-trivial maps with small k-dilation

This section contains the main result of the paper. It gives a construction for maps between spheres with very small k-dilation which are homotopic to high order suspensions.

**Proposition 1.** Fix a homotopy class \( a \) in \( \pi_m(S^n) \) and then consider its p-fold suspension \( \Sigma^p a \) in \( \pi_{m+p}(S^{n+p}) \), and let \( k \) be an integer greater than \( n + (n/m)p \). Then there are maps from \( S^{m+p} \) to \( S^{n+p} \) in the homotopy class \( \Sigma^p a \) with arbitrarily small k-dilation.

**Proof.** Let \( f_1 \) be a map in the homotopy class \( a \) from \([0, 1]^m\) to the unit n-sphere, taking the boundary of the domain to the basepoint of \( S^n \). Let \( f_2 \) be a degree 1 map from \([0, 1]^p\) to the unit p-sphere, taking the boundary of the domain to the basepoint of \( S^p \). We can assume both maps are \( C^1 \), and we pick a number \( L \) which is bigger than the Lipshitz constant of either map.

Inside of the unit \((m+p)\)-sphere, we can quasi-isometrically embed a rectangle \( R \) with dimensions \([0, \epsilon]^m \times [0, \epsilon^{-m/p}]^p\). (The quasi-isometric constant does not depend on \( \epsilon \).) Now we construct a map \( F \) from \( R \) to \( S^n \times S^p \). The map \( F \) is a direct product of a map \( F_1 \) from \([0, \epsilon]^m\) to \( S^n \) and a map \( F_2 \) from \([0, \epsilon^{-m/p}]^p\) to \( S^p \). The map \( F_1 \) is just a dilation from \([0, \epsilon]^m\) to the unit cube, composed with the map \( f_1 \). The map \( F_2 \) is just a dilation from \([0, \epsilon^{-m/p}]^p\) to \([0, 1]^p\), composed with the map \( f_2 \).

When \( k \) is bigger than \( n \), the k-dilation of \( F \) is less than \( (L \epsilon^{-1})^n (L \epsilon^{m/p})^{k-n} \). Expanding this expression gives \( L^k \epsilon^{-n+(m/p)k-(m/p)n} \). The important part of the expression is the power of \( \epsilon \), which is equal to \((m/p)(k-n-(n/m)p)\). We have assumed that \( k \) is greater than \( n + (n/m)p \), and so the exponent of \( \epsilon \) is positive. For \( \epsilon \) sufficiently small, the k-dilation of \( F \) is arbitrarily small.

The map \( F \) takes the boundary of \( R \) to \( S^n \cup S^p \). We compose \( F \) with a smash map, which is a degree 1 map from \( S^n \times S^p \) to \( S^{n+p} \), taking \( S^n \cup S^p \) to the base point. The result is a map from \( R \) to \( S^{n+p} \) which takes the boundary of \( R \) to the basepoint. We can easily extend this map to all of \( S^{n+p} \) by mapping the complement of \( R \) to the basepoint of \( S^{n+p} \). The resulting map is homotopic to \( \Sigma^p(a) \), and it has arbitrarily small k-dilation.

We can apply our proposition to the non-trivial homotopy class in \( \pi_4(S^3) \), which is represented by the suspension of the Hopf fibration. In this case, we are considering a 1-fold suspension of a map from \( S^3 \) to \( S^2 \). Therefore \( p = 1, m = 3 \), and \( n = 2 \). Since \( 3 > 2 + (2/3)1 \), this homotopy class can be realized by maps from \( S^4 \) to \( S^3 \) with arbitrarily small 3-dilation.

We now turn to the proof of our main theorem, which gives infinitely many examples of non-trivial homotopy classes that can be realized with arbitrarily small 3-dilation. The proof of this theorem uses some deep results in algebraic topology.
to guarantee that certain high-order suspensions are homotopically non-trivial. I would like to thank Haynes Miller for helping me to find the relevant results in the topology literature.

**Theorem 1.** For every \( N \geq 2 \), there are infinitely many \( M \) so that there are homotopically non-trivial maps from \( S^M \) to \( S^N \) with arbitrarily small 3-dilation.

**Proof.** When \( i = 8j + 1 \), the homotopy group \( \pi_i(SO) \) is equal to \( \mathbb{Z}_2 \). The stable J-homomorphism maps \( \pi_i(SO) \) to the \( i \)-th stable stem of the homotopy groups of spheres. The image of the J-homomorphism was studied by Adams in [1]. When \( i = 8j + 1 \), he proved that the stable J-homomorphism is injective. Its image is a copy of \( \mathbb{Z}_2 \). This image contains a non-trivial element in \( \pi_{i+n}(S^n) \), for large \( n \).

It turns out that this non-trivial element is the suspension of a class in \( \pi_{i+2}(S^2) \). This statement is made clearly in the introduction to the paper [3], and the proof appears in the older paper [2]. For each \( i = 8j + 1 \) greater than \( 2N - 6 \), let \( a_i \) be a homotopy class in \( \pi_{i+2}(S^2) \) whose suspension is the non-trivial element in the image \( J(\pi_i(SO)) \). In particular, the \( p \)-fold suspension \( \Sigma^p a_i \) is non-trivial for every \( p \) and every \( i \).

Now let \( N \) be any integer at least 2. For each \( i = 8j + 1 \) greater than \( 2N - 6 \), consider the class \( \Sigma^{N-2}a_i \) in \( \pi_{i+N}(S^N) \). We let \( M = N+i \). Each of these homotopy classes is non-trivial. In the language of Proposition 1, we have \( p = N-2, m = i+2, \) and \( n = 2 \). The condition that \( i \) is greater than \( 2N - 6 \) exactly guarantees that \( 3 > n + (n/m)p \). Therefore, each of these homotopy classes can be realized with arbitrarily small 3-dilation.

This proof constructs homotopically non-trivial maps from \( S^M \) to \( S^N \) with arbitrarily small 3-dilation for many pairs \( (M, N) \). For example, when \( N = 3 \), we can take \( M = 4, 12, 20, 28, 36 \), and so on; and when \( N = 4 \), we can take \( M = 13, 21, 29, 37 \), and so on.

2. Lower bounds for k-dilation

In this section, we survey several lower bounds for the k-dilation of mappings. We begin by recalling the theorem of Tsui and Wang mentioned in the introduction.

**Theorem.** (Tsui and Wang, [7]) Let \( f \) be a \( C^1 \) map from \( S^m \) to \( S^n \), where \( m \geq 2 \). If the 2-dilation of \( f \) is less than 1, then \( f \) is nullhomotopic.

The proof by Tsui and Wang uses the mean curvature flow to deform the graph of the map \( f \) as a submanifold of \( S^m \times S^n \). They prove that the mean curvature flow converges to the graph of a constant function and that at each time \( t \) the flowed submanifold is the graph of a map \( f_t \). Therefore, \( f_t \) provides a homotopy from \( t \) to a constant map.

Earlier, Gromov had proved a slightly weaker theorem in the same spirit. He proved that for each \( m \) and \( n \), there exists a number \( \epsilon(m, n) > 0 \), so that any \( C^1 \) map from \( S^m \) to \( S^n \) with 2-dilation less than \( \epsilon(m, n) \) is null-homotopic. Gromov’s proof is completely different from the proof of Tsui and Wang. He approaches the map from \( S^m \) to \( S^n \) as a family of maps from \( S^2 \) to \( S^n \). If the 2-dilation is less than \( \epsilon \), then each image of \( S^2 \) has area less than \( 4\pi \epsilon \). Gromov uses the Riemann mapping theorem to change coordinates on the domain so that each map in the family has Dirichlet energy less than \( C\epsilon \). Then he uses the borderline Sobolev inequality, which bounds the BMO norm of each map in the family by \( C\epsilon \). Using the bound on the BMO norm, he constructs a homotopy from the initial family of
maps to a family of constant maps. This argument is sketched on page 179 of the long essay [5].

Gromov’s estimates are less sharp than those of Tsui and Wang, but he is able to push them through with much weaker hypotheses. For example, Gromov proves that if \( f \) maps \( S^m \) to \( S^n \) with \( |A^2 df|_{L^{m-1+i}} < \epsilon(m, n, \delta) \) then \( f \) is null-homotopic. Gromov’s method also applies to more manifolds. He proves that for any two compact simply connected Riemannian manifolds \( M \) and \( N \) there is a constant \( \epsilon(M, N) \) so that any map \( f : M \to N \) with 2-dilation less than \( \epsilon(M, N) \) is null-homotopic. (Both these results are special cases of the Corollary on page 230 of [5].)

Lower bounds for the k-dilation with \( k \) greater than 2 are much rarer. We begin with a very elementary example. If a \( C^1 \) map \( f \) from \( S^n \) to \( S^n \) is homotopically non-trivial, then it has n-dilation at least 1. This result follows because a map with n-dilation equal to 1 – \( \epsilon \) has an image with volume at most \( (1 - \epsilon)Vol(S^n) \). Such a map is not surjective, and so it is null-homotopic.

Gromov also proved lower bounds for the k-dilation of maps with non-zero Hopf invariant. The following argument appears in [4] on pages 358-359.

**Theorem.** (Gromov) Let \( f \) be a map from \( S^{4n-1} \) to \( S^{2n} \) with 2n-dilation \( D \). Then the norm of the Hopf invariant of \( f \) is bounded by \( C(n)D^2 \). Since the Hopf invariant is an integer, any map with non-zero Hopf invariant has 2n-dilation at least \( C(n)^{-1/2} \).

**Proof.** Let \( \omega \) be a 2n-form on \( S^{2n} \) with \( \int \omega = 1 \). The pullback \( f^*(\omega) \) is a closed 2n-form on \( S^{4n-1} \). Since \( H^{2n}(S^{4n-1}) = 0 \), this form is exact. We let \( Pf^*(\omega) \) denote any primitive of \( f^*(\omega) \). Then the Hopf invariant of \( f \) is equal to \( \int_{S^{4n-1}} Pf^*(\omega) \wedge f^*(\omega) \).

During this proof, we let \( C \) denote a constant depending on \( n \) that may change from line to line. We can assume that \( |\omega| < C \) at every point of \( S^{2n} \). The norm of \( f^*(\omega) \) is bounded by \( CD \) pointwise. Therefore, the \( L^2 \) norm of \( f^*(\omega) \) is bounded by \( C D \). Using Hodge theory, we can choose \( Pf^*(\omega) \) to be perpendicular to all of the exact (2n-1)-forms. For this choice, the \( L^2 \) norm of \( Pf^*(\omega) \) is bounded by \( \lambda^{-1/2}CD \), where \( \lambda \) is the smallest eigenvalue of the Laplacian on exact (2n)-forms. The eigenvalue \( \lambda \) is greater than zero and depends only on \( n \). Finally, the norm of the Hopf invariant is bounded by \( \|f^*(\omega)\|_{L^2} \cdot \|Pf^*(\omega)\|_{L^2} \), which is bounded by \( C(n)D^2 \).

The method above can be generalized to many non-torsion homotopy classes. For more information, see Gromov’s discussion in [5], pages 220-223.

The methods outlined here leave many open questions. If \( a \) is a torsion homotopy class in some homotopy group \( \pi_m(S^n) \), I don’t know any way to lower bound the 3-dilation of maps in the homotopy class \( a \). For example, the 100-fold suspension of the Hopf map is a non-trivial element in \( \pi_{103}(S^{102}) \). According to Proposition 1, it can be realized by maps with arbitrarily small 69-dilation. It would be interesting to know whether it can be realized by maps with arbitrarily small 3-dilation.

3. A Filtration on Homotopy Groups

In this section, we define a filtration \( V_k \pi_m(X) \) which measures which homotopy classes in \( \pi_m(X) \) can be realized with arbitrarily small k-dilation. We derive some
easy formal properties of this filtration. In the next section, we will calculate it in some examples and show that it can be non-trivial.

As a first step, we define $V_k \pi_m(W)$ when W is a finite simplicial complex. We put a piecewise Riemannian metric on W by equipping each simplex with the metric of an equilateral simplex in Euclidean space with edges of length 1. We then define $V_k \pi_m(W)$ to be the set of homotopy classes in $\pi_m(W)$ that can be realized by maps $S^m \to W$ with arbitrarily small k-dilation.

The set $V_k \pi_m(W)$ is in fact a subgroup of $\pi_m(W)$. If $a$ lies in $V_k \pi_m(W)$, then let $f_i$ be a sequence of maps from $S^m$ to W in the homotopy class $a$ with k-dilation tending to zero. Let $I$ be a reflection, mapping $S^m$ to itself with degree $-1$, and taking the basepoint of $S^m$ to itself. Then the maps $f_i \circ I$ have k-dilations tending to zero and lie in the homotopy class $-a$. Therefore $-a$ lies in $V_k \pi_m(W)$. Next, suppose that $a$ and $b$ lie in $V_k \pi_m(W)$. Again, let $f_i$ be a sequence of (pointed) maps in the class $a$ with k-dilation tending to zero, and let $g_i$ be a sequence of (pointed) maps in the homotopy class $b$ with k-dilation tending to zero. Let I be a map from $S^m$ to $S^m \vee S^m$ with degree $(1,1)$. Let $h_i$ be the map from $S^m \vee S^m$ to W whose restriction to the first copy of $S^m$ is equal to $f_i$ and whose restriction to the second copy of $S^m$ is equal to $g_i$. Then the sequence $h_i \circ I$ has k-dilation tending to zero. Each map in the sequence lies in the homotopy class $a+b$. So $a+b$ lies in $V_k \pi_m(W)$.

The sets $V_k \pi_m(W)$ are nested, with $V_k \pi_m(W) \subset V_{k+1} \pi_m(W)$. To see this, we express the k-dilation of a map in terms of the singular values of its derivative. Let $f$ be a piecewise $C^1$ map from $S^m$ to W. Suppose that the singular values of $df$ at a point are equal to $0 \leq s_1 \leq \ldots \leq s_m$. Then the norm $|\Lambda^k df|$ at that point is equal to $s_{m-k+1} \ldots s_m$. Therefore, $|\Lambda^{k+1} df| \leq |\Lambda^k df| \frac{1}{s_{m-k}}$. If a map f has k-dilation D, then the $(k+1)$-dilation of f is at most $D \frac{1}{s_{m-k}}$. In particular, if $f_i$ is a sequence of maps with k-dilation tending to zero, then the $(k+1)$-dilation of $f_i$ also tends to zero. Therefore, $V_k \pi_m(W) \subset V_{k+1} \pi_m(W)$. Any map from $S^m$ has $(m+1)$-dilation zero, and so $V_{m+1} \pi_m(W)$ is always the whole homotopy group $\pi_m(W)$.

The last paragraph shows that the sequence of groups $V_k \pi_m(W)$ is a filtration of $\pi_m(W)$.

$$0 = V_1 \pi_m(W) \subset V_2 \pi_m(W) \subset \ldots \subset V_m \pi_m(W) \subset V_{m+1} \pi_m(W) = \pi_m(W).$$

The filtration $V_k$ behaves naturally under mappings in the following sense. If $F : W \to V$ is a continuous pointed mapping between finite simplicial complexes, then $F_* : \pi_m(W) \to \pi_m(V)$ takes $V_k \pi_m(W)$ into $V_k \pi_m(V)$. To prove this, first approximate F by a PL map with some finite Lipshitz constant L. Let $a$ be a class in $V_k \pi_m(W)$, realized by mappings $f_i : S^m \to W$ with k-dilation tending to zero. The map $F \circ f_i$ from $S^m$ to V has k-dilation less than $L^k$ times the k-dilation of $f_i$, so the the sequence $F \circ f_i$ has k-dilation tending to zero. Each map $F \circ f_i$ lies in the homotopy class $F_*(a)$. Therefore, $F_*(a)$ lies in $V_k \pi_m(V)$.

Now we define $V_k \pi_m(X)$ for an arbitrary space X. A homotopy class $a$ in $\pi_m(X)$ belongs to $V_k \pi_m(X)$ if there is a finite simplicial complex W and a map $F : W \to X$ so that $a$ lies in the image $F_*(V_k \pi_m(W))$. In case X is a finite simplicial complex, this definition agrees with our first definition because of the mapping property of $V_k$ proved in the last paragraph. As above, $V_k \pi_m(X)$ defines a filtration of $\pi_m(X)$. Also, if $f : X \to Y$ is any continuous map of spaces, then $f_*$ maps $V_k \pi_m(X)$ into $V_k \pi_m(Y)$. From this mapping property, it follows that the filtration $V_k \pi_m(X)$ is a homotopy invariant.
We can rephrase the main theorems of this paper in the language of the filtration $V_k \pi_m(S^n)$. The theorem of Tsui and Wang implies that $V_2 \pi_m(S^n) = 0$ for $m \geq 2$. Theorem 1 says that for each $n \geq 2$, the group $V_3 \pi_m(S^n)$ is non-zero for infinitely many $m$. Theorem 2 says that $V_4 \pi_7(S^4)$ is exactly the torsion subgroup of $\pi_7(S^4)$, which is isomorphic to $\mathbb{Z}_{12}$.

### 4. Examples of the $V_k$ Filtration

In this section, we use the tools from sections 1 and 2 to compute the filtration $V_k \pi_m(S^n)$ for a few small values of $m$ and $n$. For other values of $m$ and $n$, we obtain some partial information about the filtration. The most interesting example is a computation of $V_4 \pi_7(S^4)$. We use the lists of homotopy groups of spheres and the suspension maps between them given in [6] on pages 39-42. According to the theorem of Tsui and Wang, $V_2 \pi_m(S^n)$ is zero in all the examples below, so we will only discuss $V_k$ for $k \geq 3$.

#### The filtration of $\pi_m(S^2)$

Any element of $\pi_m(S^2)$ lies in $V_3 \pi_m(S^2)$, because any smooth map from $S^m$ to $S^2$ has 3-dilation equal to zero.

#### The filtration of $\pi_4(S^3)$

The group $\pi_4(S^3)$ is isomorphic to $\mathbb{Z}_2$, and the non-trivial element is given by the suspension of the Hopf fibration from $S^3$ to $S^2$. Applying Proposition 1, this non-trivial element lies in $V_5 \pi_4(S^3)$.

#### The filtration of $\pi_5(S^3)$

The group $\pi_5(S^3)$ is isomorphic to $\mathbb{Z}_2$, and the non-trivial element is the suspension of a class in $\pi_4(S^2)$. Applying Proposition 1, this non-trivial element lies in $V_5 \pi_5(S^3)$.

#### The filtration of $\pi_6(S^3)$

The group $\pi_6(S^3)$ is isomorphic to $\mathbb{Z}_{12}$. One non-trivial element is the suspension of a class in $\pi_5(S^2)$. Applying Proposition 1, this non-trivial element lies in $V_5 \pi_6(S^3)$. I don’t know whether the other non-trivial elements lie in $V_5 \pi_6(S^3)$.

#### The filtration of $\pi_7(S^4)$

The group $\pi_7(S^4)$ is isomorphic to $\mathbb{Z}_2$, and the non-trivial element is given by the double suspension of the Hopf fibration from $S^3$ to $S^2$. Applying Proposition 1, this non-trivial element lies in $V_4 \pi_7(S^4)$. I don’t know whether it lies in $V_3 \pi_7(S^4)$.

#### The filtration of $\pi_8(S^4)$

The group $\pi_8(S^4)$ is isomorphic to $\mathbb{Z}_2$, and the non-trivial element is the double suspension of a class in $\pi_7(S^2)$. Applying Proposition 1, this non-trivial element lies in $V_4 \pi_8(S^4)$. I don’t know whether it lies in $V_3 \pi_8(S^4)$.

#### The filtration of $\pi_9(S^4)$

The group $\pi_9(S^4)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_{12}$. The Hopf invariant gives a map $H : \pi_7(S^4) \to \mathbb{Z}$. In section 2, following Gromov, we proved that $V_4 \pi_7(S^4)$ lies in the kernel of $H$. The kernel of $H$ is isomorphic to $\mathbb{Z}_{12}$. The suspension map is an isomorphism from $\pi_0(S^3)$ onto the kernel of $H$, as follows from the long exact sequence of the Hopf fibration $S^3 \to S^7 \to S^4$. (This fact is stated in [6] on page 2.) Therefore, every element in the kernel of $H$ is a suspension. Applying Proposition 1, we see that the kernel of $H$ lies in $V_4 \pi_7(S^4)$. Therefore, $V_4 \pi_7(S^4)$ is exactly the kernel of $H$. This proves the second theorem in the introduction.
In addition, one non-trivial element in $\pi_7(S^4)$ is a double suspension of an element in $\pi_5(S^2)$. Applying Proposition 1 to this element, we see that it lies in $V_5\pi_7(S^4)$. I don’t know whether the other torsion elements of $\pi_7(S^4)$ lie in $V_3\pi_7(S^4)$.

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