A $(CHR)_3$-flat trans-Sasakian manifold

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Abstract. In [4] M. Prvanovic considered several curvaturelike tensors defined for Hermitian manifolds. Developing her ideas in [3], we defined in an almost contact Riemannian manifold another new curvaturelike tensor field, which is called a contact holomorphic Riemannian curvature tensor or briefly $(CHR)_3$-curvature tensor. Then, we mainly researched $(CHR)_3$-curvature tensor in a Sasakian manifold. Also we proved, that a conformally $(CHR)_3$-flat Sasakian manifold does not exist.

In the present paper, we consider this tensor field in a trans-Sasakian manifold. We calculate the $(CHR)_3$-curvature tensor in a trans-Sasakian manifold. Also, the $(CHR)_3$-Ricci tensor $\rho_3$ and the $(CHR)_3$-scalar curvature $\tau_3$ in a trans-Sasakian manifold have been obtained.

Moreover, we define the notion of the $(CHR)_3$-flatness in an almost contact Riemannian manifold. Then, we consider this notion in a trans-Sasakian manifold and determine the curvature tensor, the Ricci tensor and the scalar curvature. We proved that a $(CHR)_3$-flat trans-Sasakian manifold is a generalized $\eta$-Einstein manifold.

Finally, we obtain the expression of the curvature tensor with respect to the Riemannian metric $g$ of a trans-Sasakian manifold, if the latter is $(CHR)_3$-flat.

Keywords: $(CHR)_3$-curvature tensor, trans-Sasakian manifold, $(CHR)_3$-flat almost contact Riemannian manifold, (generalized) $\eta$-Einstein manifold.

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1. ALMOST CONTACT RIEMANNIAN MANIFOLDS

A real \((2n+1)\)-dimensional differentiable Riemannian manifold \((M^{2n+1}, g)\) is said to be an almost contact Riemannian manifold if it has a \((1, 1)\)-tensor \(\varphi\) and a 1-form \(\eta\) which satisfy

\[
\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1, \quad (1.1)
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (1.2)
\]

for any \(Y, X \in TM^{2n+1}\), where \(\xi\) is defined by

\[
g(\xi, X) = \eta(X)
\]

and \(TM^{2n+1}\) is the tangent bundle of \(M^{2n+1}\).

From (1.1), the vector field \(\xi\) is unit and we call this vector field the structure vector field of the almost contact Riemannian manifold. Next, in an almost contact Riemannian manifold \(M^{2n+1}\) we define a 2-form \(F\) by

\[
F(X, Y) = g(\varphi X, Y) \quad (1.3)
\]

for all \(X, Y \in TM^{2n+1}\). Then the 2-form \(F\) is skew-symmetric and we call this tensor field the fundamental 2-form of this almost contact Riemannian manifold. Hereafter, we write the same \(\varphi\) instead of \(F\).

An almost contact manifold \(M^{2n+1}\) is called trans-Sasakian if the fundamental form \(\varphi\) satisfies

\[
(\nabla_X \varphi)(Y, Z) = \alpha\{g(X, Y)\eta(Z) - g(X, Z)\eta(Y)\} +
+ \beta\{\varphi(X, Y)\eta(Z) - \varphi(X, Z)\eta(Y)\}, \quad (1.4)
\]

for certain smooth functions \(\alpha\) and \(\beta\) on \(M^{2n+1}\) and for all tangent vectors \(X, Y, Z \in TM^{2n+1}\), where \(\nabla\) means the covariant differentiation with respect to \(g\). In that case we will say that a trans-Sasakian structure is of type \((\alpha, \beta)\) or of an \((\alpha, \beta)\)-type, [5].

**Remark 1.1.** A \((-1, 0)\)-type (resp. \((0, 1)\)-type) trans-Sasakian manifold is a Sasakian (resp. a Kenmotsu) manifold.
In a trans-Sasakian manifold of \((\alpha,\beta)\)-type, we know, \([5]\), that

\[
\nabla_X\xi = -\alpha\varphi X + \beta\{X - \eta(X)\xi\},
\]

\[
(\nabla_X\eta)(Y) = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y),
\]

\[
R(X, Y, Z, \xi) = (X\alpha)g(\varphi Y, Z) - (Y\alpha)g(\varphi X, Z)
\]

\[
\quad - (X\beta)g(\varphi Y, \varphi Z) + (Y\beta)g(\varphi X, \varphi Z)
\]

\[
\quad + (\alpha^2 - \beta^2)A(X, Y, Z) - 2\alpha\beta A(X, Y, \varphi Z),
\]

\[
\rho(X, \xi) = \{2n(\alpha^2 - \beta^2) - (\xi\beta)\}\eta(X) - \alpha(\varphi X) - (2n-1)(X\beta)
\]

for any \(X, Y \in TM^{2n+1}\), where \(\rho\) is the Ricci tensor with respect to \(g\) and \(A(X, Y, Z)\) is defined as

\[
A(X, Y, Z) = g(Z, Y)\eta(X) - g(Z, X)\eta(Y)
\]

for any \(X, Y, Z \in TM^{2n+1}\).

The following equations (1.7), (1.8), (1.9), (1.10) and (1.11) are very useful for calculations of the \((CHR)\)-curvature tensor in a trans-Sasakian manifold.

By virtue of (1.4) and the Bianci identity, we have

\[-R(X, Y, Z, \varphi W) + R(X, Y, W, \varphi Z) =
\]

\[
\quad = (X\alpha)A(Z, W, Y) - (Y\alpha)A(Z, W, X) +
\]

\[
\quad + (X\beta)A(Z, W, \varphi Y) - (Y\beta)A(Z, W, \varphi X) +
\]

\[
\quad + (\alpha^2 - \beta^2)\{g(Y, W)g(\varphi X, Z) - g(Y, Z)g(\varphi X, W) -
\]

\[
\quad - g(X, W)g(\varphi Y, Z) + g(X, Z)g(\varphi Y, W)\} +
\]

\[
\quad + 2\alpha\beta\{g(\varphi X, W)g(\varphi Y, Z) - g(\varphi Y, W)g(\varphi X, Z) +
\]

\[
\quad + g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\}
\]

for any \(X, Y, Z, W \in TM^{2n+1}\). By virtue of (1.6), we can easily obtain

\[R(\varphi X, Y, Z, \varphi W) + R(\varphi X, Y, \varphi Z, W) =
\]

\[
\quad = -(\varphi X\alpha)A(Z, W, Y) + (Y\alpha)B(Z, W, X) +
\]

\[
\quad + (\varphi X\beta)A(Z, W, Y) - (Y\beta)A(Z, W, X) +
\]

\[
\quad + (\alpha^2 - \beta^2)\{g(\varphi X, Z)g(\varphi Y, W) - g(\varphi X, W)g(\varphi Y, Z)
\]

\[
\quad + g(X, W)g(Y, Z) - g(X, Z)g(Y, W)
\]

\[
\quad + g(Y, W)\eta(X)\eta(Z) - g(Y, Z)\eta(X)\eta(W)\}
\]

\[
\quad + 2\alpha\beta\{g(X, W)g(\varphi Y, Z) - g(X, Z)g(\varphi Y, W)
\]

\[
\quad + g(Y, Z)g(\varphi X, W) - g(Y, W)g(\varphi X, Z)
\]

\[
\quad + g(\varphi Y, W)\eta(X)\eta(Z) - g(\varphi Y, Z)\eta(X)\eta(W),
\]

\[
\quad (1.8)
\]
and

\[
R(X, Y, \varphi Z, \varphi W) = R(X, Y, Z, W) + \\
+ (X\alpha)A(Z, W, \varphi Y) - (Y\alpha)A(Z, W, \varphi X) \\
- (X\beta)A(Z, W, Y) + (Y\beta)A(Z, W, X) \\
+ (\alpha^2 - \beta^2)\{g(Y, W)g(X, Z) - g(X, W)g(Y, Z) \\
- g(\varphi X, Z)g(\varphi Y, W) + g(\varphi X, W)g(\varphi Y, Z)\} \\
+ 2\alpha\beta\{g(\varphi Y, W)g(X, Z) - g(\varphi X, W)g(Y, Z) \\
- g(\varphi Y, Z)g(X, W) + g(\varphi X, Z)g(Y, W)\}
\]

(1.9)

for any \(X, Y, Z, W \in TM^{2n+1}\).

Moreover, we have from the above equation

\[
R(\varphi X, \varphi Y, \varphi Z, \varphi W) = R(X, Y, Z, W) + \\
+ (Z\alpha)A(X, Y, \varphi W) - (W\alpha)A(X, Y, \varphi Z) \\
- (Z\beta)A(X, Y, W) + (W\beta)A(X, Y, Z) \\
- (\varphi X\alpha)A(Z, W, Y) + (\varphi Y\alpha)A(Z, W, X) \\
- (\varphi X\beta)A(Z, W, \varphi Y) + (\varphi Y\beta)A(Z, W, \varphi X) \\
+ (\alpha^2 - \beta^2)\{A(X, Y, W)\eta(Z) - A(X, Y, Z)\eta(W)\} \\
+ 2\alpha\beta\left[2\{g(X, W)g(\varphi Y, Z) - g(X, Z)g(\varphi Y, W) \\
+ g(Y, W)g(\varphi X, Z) - g(Y, Z)g(\varphi X, W)\} \\
- A(X, Y, \varphi W)\eta(Z) + A(X, Y, \varphi Z)\eta(W)\right]
\]

(1.10)

for any \(X, Y, Z, W \in TM^{2n+1}\).

By virtue of \(R(X, Y, Z, W) = R(Z, W, X, Y)\) for any \(X, Y, Z, W \in TM^{2n+1}\), we have from (1.10)

\[
\{(X\alpha) + (\varphi X\beta)\}A(Z, W, \varphi Y) - \{(Y\alpha) + (\varphi Y\beta)\}A(Z, W, \varphi X) \\
- \{(Z\alpha) + (\varphi Z\beta)\}A(X, Y, \varphi W) + \{(W\alpha) + (\varphi W\beta)\}A(X, Y, \varphi Z) \\
- \{(X\beta) - (\varphi X\alpha)\}A(Z, W, Y) + \{(Y\beta) - (\varphi Y\alpha)\}A(Z, W, X) \\
+ \{(Z\beta) - (\varphi Z\alpha)\}A(X, Y, W) - \{(W\beta) - (\varphi W\alpha)\}A(X, Y, Z) \\
= 4\alpha\beta\left[2\{g(X, W)g(\varphi Y, Z) - g(X, Z)g(\varphi Y, W) \\
+ g(Y, Z)g(\varphi X, W) - g(Y, W)g(\varphi X, Z)\} \\
+ A(Z, W, \varphi Y)\eta(X) - A(Z, W, \varphi X)\eta(Y)\right]
\]

(1.11)

for any \(X, Y, Z, W \in TM^{2n+1}\). Thus we have
Proposition 1.2. In an \((\alpha, \beta)\)-type trans-Sasakian manifold \(M^{2n+1}\), the functions \(\alpha\) and \(\beta\) satisfy (1.11).

2. \((CHR)\)_3-curvature tensor in a trans-Sasakian manifold

In this section, we consider the \((CHR)\)_3-curvature tensor in a trans-Sasakian manifold.

The \((CHR)\)_3-curvature tensor in an almost contact Riemannian manifold is defined by

\[
16(CHR)_3(X, Y, Z, W) =
3\{R(X, Y, Z, W) + R(\varphi X, \varphi Y, Z, W) \\
+ R(X, Y, \varphi Z, \varphi W) + R(\varphi X, \varphi Y, \varphi Z, \varphi W)\}
- R(X, Z, \varphi W, \varphi Y) - R(\varphi X, \varphi Z, W, Y) \\
- R(X, W, \varphi Y, \varphi Z) - R(\varphi X, \varphi W, Y, Z) \\
+ R(\varphi X, Z, \varphi W, Y) + R(X, \varphi Z, W, \varphi Y) \\
+ R(\varphi X, W, Y, \varphi Z) + R(X, \varphi W, \varphi Y, Z) \\
+ \eta(X)P(Z, W, Y) - \eta(Y)P(Z, W, X) \\
+ \eta(Z)P(X, Y, W) - \eta(W)P(X, Y, Z) \\
+ \eta(X)\eta(W)Q(Y, Z) - \eta(X)\eta(Z)Q(Y, W) \\
+ \eta(Y)\eta(Z)Q(W, X) - \eta(Y)\eta(W)Q(Z, X),
\]

where we put

\[
P(X, Y, Z) = 3\{R(X, Y, Z, \xi) + R(\varphi X, \varphi Y, Z, \xi)\} + R(\varphi X, \varphi Z, Y, \xi) \\
+ R(\varphi Z, \varphi Y, X, \xi) - R(X, \varphi Z, \varphi Y, \xi) - R(\varphi Z, Y, \varphi X, \xi).
\]

and

\[
Q(X, Y) = 3R(\xi, X, Y, \xi) - R(\xi, \varphi X, \varphi Y, \xi).
\]

for any \(X, Y, Z \in TM^{2n+1}\), [3]. We call this tensor field a contact holomorphic Riemannian curvature tensor or briefly \((CHR)\)_3-curvature tensor in an almost contact Riemannian manifold. Hereafter, we assume that all vector fields are elements of \(TM^{2n+1}\).

Now, to calculate \((CHR)\)_3-curvature tensor in a trans-Sasakian manifold \(M^{2n+1}\), we separate this tensor field as the following 5-parts:

(I) \(R(X, Y, Z, W) + R(\varphi X, \varphi Y, Z, W)+
+ R(X, Y, \varphi Z, \varphi W) + R(\varphi X, \varphi Y, \varphi Z, \varphi W),\)

(II) \(R(X, Z, \varphi W, \varphi Y) + R(\varphi X, \varphi Z, W, \varphi Y)+
+ R(X, W, \varphi Y, \varphi Z) + R(\varphi X, \varphi W, Y, Z),\)
(III) \[ R(\varphi X, Z, \varphi W, Y) + R(X, \varphi Z, W, \varphi Y) + \\
+ R(\varphi X, W, Y, \varphi Z) + R(X, \varphi W, \varphi Y, Z), \]

(IV) \[ \eta(X)P(Z, W, Y) - \eta(Y)P(Z, W, X) + \\
+ \eta(Z)P(X, Y, W) - \eta(W)P(X, Y, Z), \]

(V) \[ \eta(X)\eta(W)Q(Y, Z) - \eta(X)\eta(Z)Q(Y, W) + \\
+ \eta(Y)\eta(Z)Q(W, X) - \eta(Y)\eta(W)Q(Z, X). \]

Then we know that

\[ 16(CHR)_3(X, Y, Z, W) = 3(I) - (II) + (III) + (IV) + (V). \]

Using (1.9) and (1.10) we get that

(I) \[ = 4R(X, Y, Z, W) + \\
+ \{(X\alpha) - (\varphi X\beta)\}A(Z, W, \varphi Y) - \{(Y\alpha) - (\varphi Y\beta)\}A(Z, W, \varphi X) \\
+ 2(Z\alpha)A(X, Y, \varphi W) - 2(W\alpha)A(X, Y, \varphi Z) - \\
- \{(X\beta) + (\varphi X\alpha)\}A(Z, W, Y) + \{(Y\beta) + (\varphi Y\alpha)\}A(Z, W, X) \\
- 2(Z\beta)A(X, Y, W) + 2(W\beta)A(X, Y, Z) \\
+ (\alpha^2 - \beta^2)\left[ 2\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + \\
+ g(\varphi X, W)g(\varphi Y, Z) - g(\varphi X, Z)g(\varphi Y, W) \} \\
+ A(Z, W, Y)\eta(X) - A(Z, W, X)\eta(Y) \right] \]

\[ + 2\alpha\beta\left[ 2\{g(Y, Z)g(\varphi X, W) + g(X, Z)g(\varphi Y, W) \\
+ g(X, W)g(\varphi Y, Z) - g(Y, W)g(\varphi X, Z) \} \\
+ A(X, Y, \varphi Z)\eta(W) - A(X, Y, \varphi W)\eta(Z) \right]. \]

Using (1.9), we obtain
\[-(\text{II}) = 2R(X, Y, Z, W) + (X\alpha)\{A(Z, Y, \varphi W) - A(W, Y, \varphi Z)\} \\
- (Y\alpha)\{A(Z, X, \varphi W) - A(W, X, \varphi Z)\} \\
+ (Z\alpha)\{A(X, W, \varphi Y) - A(Y, W, \varphi X)\} \\
- (W\alpha)\{A(X, Z, \varphi Y) - A(Y, Z, \varphi X)\} \]
\[-(X\beta)\{A(Z, Y, W) - A(W, Y, Z)\} \\
+ (Y\beta)\{A(Z, X, W) - A(W, X, Z)\} \] (2.5)
\[+ (Z\beta)\{A(X, W, Y) - A(Y, W, X)\} \\
+ (W\beta)\{A(X, Z, Y) - A(Y, Z, X)\} \]
\[+ 2(\alpha^2 - \beta^2)\left\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \\
- g(\varphi X, W)g(\varphi Y, Z) + g(\varphi X, Z)g(\varphi Y, W) \\
- 2g(\varphi X, Y)g(\varphi Z, W)\right\}. \]

(III) is separated as \((A) + (B)\), where we put
\[(A) = R(\varphi X, Z, \varphi W, Y) + R(\varphi X, W, Y, \varphi Z) \\
= -\{R(\varphi X, \varphi W, Y, Z) + R(\varphi X, \varphi Z, W, Y)\} \\
- \{R(\varphi X, Y, Z, \varphi W) + R(\varphi X, Y, \varphi Z, W)\}, \]
\[(B) = R(X, \varphi Z, W, \varphi Y) + R(X, \varphi W, \varphi Y, Z). \]

By virtue of (1.9), we obtain
\[\begin{align*}
- \{R(\varphi X, \varphi W, Y, Z) + R(\varphi X, \varphi Z, W, Y)\} & = R(Z, Y, Z, W) + \\
+ (Y\alpha)\{A(X, Z, \varphi W) - A(X, W, \varphi Z)\} + \\
+ (Z\alpha)A(X, W, \varphi Y) - (W\alpha)A(X, Z, \varphi Y) + \\
+ (Y\beta)\{A(X, W, Z) - A(X, Z, W)\} - \\
- (Z\beta)A(X, W, Y) + (W\beta)A(X, Z, Y) \\
+ (\alpha^2 - \beta^2)\left\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \\
+ g(\varphi X, Z)g(\varphi Y, W) - g(\varphi X, W)g(\varphi Y, Z) \\
- 2g(\varphi X, Y)g(\varphi Z, W)\right\} \end{align*} \] (2.6)
\[+ 2\alpha\beta\left\{g(X, Z)g(\varphi Y, W) - g(Y, W)g(\varphi X, Z) \\
+ g(Y, Z)g(\varphi X, W) - g(X, W)g(\varphi Y, Z) \\
- 2g(X, Y)g(\varphi Z, W)\right\}. \]

Thus we have from (2.5) and (2.6)
\[(A) = R(X, Y, Z, W) + \]
\[+ (\varphi X_\alpha) A(Z, W, Y) - 2(Y_\alpha)g(\varphi Z, W)\eta(X) \]
\[+ (Z_\alpha)A(X, W, \varphi Y) - (W_\alpha)A(X, Z, \varphi Y) \]
\[+ (\varphi X_\beta) A(Z, W, \varphi Y) + 2(Y_\beta)A(Z, W, X) \]
\[+ (Z_\beta)A(X, W, Y) + (W_\beta)A(X, Z, Y) \]
\[+ (\alpha^2 - \beta^2)\left[2\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\} \right. \]
\[- A(Z, W, Y)\eta(X) \]
\[+ 2\alpha\beta\left[2\{g(X, Z)g(Y, W) - g(X, W)g(\varphi Y, Z) \]
\[- g(X, Y)g(\varphi Z, W)\} - A(Z, W, \varphi Y)\eta(Y) \right]. \]

Since, (B) is the equation which change X \iff Y and Z \iff W in (A), we have
\[(B) = R(X, Y, Z, W) + (\varphi Y_\alpha) A(W, Z, X) - 2(X_\alpha)g(\varphi W, Z)\eta(Y) \]
\[+ (W_\alpha)A(Y, Z, \varphi X) - (Z_\alpha)A(Y, W, \varphi X) \]
\[+ (\varphi Y_\beta)A(W, Z, \varphi X) + 2(X_\beta)A(W, Z, Y) \]
\[- (W_\beta)A(Y, Z, X) + (Z_\beta)A(Y, W, X) \]
\[+ (\alpha^2 - \beta^2)\left[2\{g(Y, W)g(X, Z) - g(Y, Z)g(X, W)\} - A(W, Z, X)\eta(Y) \right] \]
\[+ 2\alpha\beta\left[2\{g(Y, W)g(\varphi X, Z) - g(Y, Z)g(\varphi X, W) \]
\[- g(X, Y)g(\varphi Z, W)\} - A(W, Z, \varphi X)\eta(Y) \right]. \]

By virtue of the above two equations, we obtain
\[(III) = 2R(X, Y, Z, W) + (\varphi X_\alpha) A(Z, W, Y) - (\varphi Y_\alpha)A(W, Z, X) \]
\[+ 2(X_\alpha)g(\varphi Z, W)\eta(Y) - 2(Y_\alpha)g(\varphi Z, W)\eta(X) \]
\[- (Z_\alpha)\{A(X, Y, \varphi W) - 2g(\varphi X, Y)\eta(W)\} \]
\[+ (W_\alpha)\{A(X, Y, \varphi Z) - 2g(\varphi X, Y)\eta(Z)\} \]
\[+ (\varphi X_\beta) A(Z, W, \varphi Y) - (\varphi Y_\beta)A(Z, W, \varphi X) \]
\[- 2(X_\beta)A(Z, W, Y) + 2(Y_\beta)A(Z, W, X) \]
\[- (Z_\beta)A(X, Y, W) + (W_\beta)A(X, Y, Z) \]
On trans-Sasakian manifolds

\[ + (\alpha^2 - \beta^2) \left[ 4 \{ g(X, Z)g(Y, W) - g(X, W)g(Y, Z) - g(\varphi X, Y)g(\varphi Z, W) \} - A(Z, W, Y)\eta(X) + A(Z, W, X)\eta(Y) \right] \]

\[ + 2\alpha\beta \left[ 2 \{ g(X, Z)g(\varphi Y, W) - g(X, W)g(\varphi Y, Z) + g(Y, W)g(\varphi X, Z) - g(Y, Z)g(\varphi X, W) \} - A(Z, W, \varphi X)\eta(Y) + A(Z, W, \varphi X)\eta(Y) \right]. \]

Next, to calculate (IV) in a trans-Sasakian manifold, we have to get \( P(X, Y, Z) \) which defined by (2.2) in a trans-Sasakian manifold. By virtue of (1.5) we obtain that

\[ P(X, Y, Z) = 4 \{ (X\alpha)g(\varphi Y, Z) - (Y\alpha)g(\varphi X, Z) \]  
\[ - 2 \{ (X\beta)g(\varphi Y, \varphi Z) - (Y\beta)g(\varphi X, \varphi Z) \} 
\[ - 4 \{ (\varphi X\alpha)g(\varphi Y, \varphi Z) - (\varphi Y\alpha)g(\varphi X, \varphi Z) \} 
\[ - 2 \{ (\varphi X\beta)g(\varphi Y, \varphi Z) - (\varphi Y\beta)g(\varphi X, \varphi Z) \} 
\[ + 4(\varphi Z\beta)g(\varphi X, Y) + 2(\alpha^2 - \beta^2)A(X, Y, Z) - 8\alpha\beta A(X, Y, \varphi Z). \]

Using the above equation, we get

\[ (IV) = 2 \{ (2X\alpha) - (\varphi X\beta) \} A(Z, W, \varphi Y) - 2 \{ (2Y\alpha) - (\varphi Y\beta) \} A(Z, W, \varphi X) \]
\[ + 2 \{ (2Z\alpha) - (\varphi Z\beta) \} A(X, Y, \varphi W) - 2 \{ (2W\alpha) - (\varphi W\beta) \} A(X, Y, \varphi Z) \]
\[ - 2 \{ (2\varphi X\alpha) + (X\beta) \} A(Z, W, Y) + 2 \{ (2\varphi Y\alpha) + (Y\beta) \} A(Z, W, X) \]
\[ - 2 \{ (2\varphi Z\alpha) + (Z\beta) \} A(X, Y, W) + 2 \{ (2\varphi W\alpha) + (W\beta) \} A(X, Y, Z) \]
\[ + 4 \{ (\varphi Y\beta)g(\varphi Z, W)\eta(X) - (\varphi X\beta)g(\varphi Z, W)\eta(Y) \]
\[ + (\varphi W\beta)g(\varphi X, Y)\eta(Z) - (\varphi Z\beta)g(\varphi X, Y)\eta(W) \}
\[ + 4(\alpha^2 - \beta^2) \{ A(Z, W, Y)\eta(X) - A(Z, W, X)\eta(Y) \}. \]

Finally, we calculate (V) in a trans-Sasakian manifold.

By virtue of (1.5)\(_3\), we have

\[ R(\xi, X, Y, \xi) = \{ (\xi\alpha) + 2\alpha\beta \} g(\varphi X, Y) + \]
\[ + \{ (\alpha^2 - \beta^2) - (\xi\beta) \} g(\varphi X, \varphi Y). \quad (2.7) \]

In (2.7), the left hand side is symmetric with respect to \( X \) and \( Y \). So we have
Proposition 2.1. In a trans-Sasakian manifold, the condition
\[(\xi\alpha) + 2\alpha\beta = 0\]
holds.

Thus, (2.7) is written as
\[R(\xi, X, Y, \xi) = \{ (\alpha^2 - \beta^2) - (\xi\beta) \} g(\varphi X, \varphi Y). \quad (2.8)\]

By virtue of (2.8), we can easily obtain that
\[R(\xi, X, Y, \xi) = R(\xi, \varphi X, \varphi Y, \xi).\]

Thus we have from the above equation
\[Q(X, Y) = 2\{ (\alpha^2 - \beta^2) - (\xi\beta) \} g(\varphi X, \varphi Y). \quad (2.9)\]

Thus we have from (2.9)
\[(V) = -2\{ (\alpha^2 - \beta^2) - (\xi\beta) \} \{ A(Z, W, Y)\eta(X) - A(Z, W, X)\eta(Y) \} .\]

By virtue of (1.11), (I), (II), (III), (IV) and (V), the \((CHR)_3\)-curvature tensor in a trans-Sasakian manifold is written as follows:
\[16(CHR)_3(X, Y, Z, W) = 16R(X, Y, Z, W) \]
\[+ (X\alpha)\{ 7A(Z, W, \varphi Y) + 4g(\varphi Z, W)\eta(Y) \} \]
\[- (Y\alpha)\{ 7A(Z, W, \varphi X) + 4g(\varphi Z, W)\eta(X) \} \]
\[+ (Z\alpha)\{ 7A(X, Y, \varphi W) + 4g(\varphi X, Y)\eta(W) \} \]
\[- (W\alpha)\{ 7A(X, Y, \varphi Z) + 4g(\varphi X, Y)\eta(Z) \} \]
\[- 5\{ (\varphi X\alpha)A(Z, W, Y) - (\varphi Y\alpha)A(Z, W, X) \]
\[+ (\varphi Z\alpha)A(X, Y, W) - (\varphi W\alpha)A(X, Y, Z) \} \]
\[- 9\{ (X\beta)A(Z, W, Y) - (Y\beta)A(Z, W, X) \]
\[+ (Z\beta)A(X, Y, W) - (W\beta)A(X, Y, Z) \} \]
\[- (\varphi X\beta)\{ 3A(Z, W, \varphi Y) + 4g(\varphi Z, W)\eta(Y) \} \]
\[+ (\varphi Y\beta)\{ 3A(Z, W, \varphi X) + 4g(\varphi Z, W)\eta(X) \} \]
\[- (\varphi Z\beta)\{ 3A(X, Y, \varphi W) + 4g(\varphi X, Y)\eta(W) \} \]
\[+ (\varphi W\beta)\{ 3A(X, Y, \varphi Z) + 4g(\varphi X, Y)\eta(Z) \} \]
\[+ (\alpha^2 - \beta^2)\left[ 12\{ g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \} \right. \]
\[+ 4\{ g(\varphi X, W)g(\varphi Y, Z) - g(\varphi X, Z)g(\varphi Y, W) \]
\[\left. - 2g(\varphi X, Y)g(\varphi Z, W) \right] \]
\[+ 2(\xi\beta)\{ A(Z, W, Y)\eta(X) - A(Z, W, X)\eta(Y) \}. \quad (2.10)\]
From the above equation, we can easily obtain the \((CHR)_3\)-Ricci tensor \(\rho_3\) and the \((CHR)_3\)-scalar curvature \(\tau_3\) as
\[
8\rho_3(X,Y) = 8\rho(X,Y) + (5n+3)[\{(\varphi X)\alpha\} \eta(Y) + \{(\varphi Y)\alpha\} \eta(X)] + (9n-1)[(X\beta)\eta(Y) + (Y\beta)\eta(X)]
- 4(\alpha^2 - \beta^2)\left\{ (3n-1)g(X,Y) + (n+1)\eta(X)\eta(Y) \right\}
+ 2(\xi\beta)\{4g(X,Y) - (n+3)\eta(X)\eta(Y)\}.
\]  
\] (2.11)

and
\[
\tau_3 = \tau - (3n+1)n(\alpha^2 - \beta^2) + 4n(\xi\beta),
\] (2.12)

where \(\tau\) denotes the scalar curvature with respect to \(g\).

By virtue of (1.5)_4 and (2.11), we easily have
\[
8\rho_3(X,\xi) = 5(n-1)[(\varphi X)\alpha - 7(n-1)((X\beta) - (\xi\beta)\eta(X))] \left\{ \right\}. \tag{2.13}
\]

3. \((CHR)_3\)-FLAT TRANS-SASAKIAN MANIFOLDS

An almost contact Riemannian manifold is called \((CHR)_3\)-flat if the \((CHR)_3\)-curvature tensor equals to zero on \(M^{2n+1}\).

Let us consider a \((CHR)_3\)-flat trans-Sasakian manifold. Then the left hand side of (2.10) is zero.

Moreover, if the \((CHR)_3\)-curvature tensor is flat, then the \((CHR)_3\)-Ricci tensor and the \((CHR)_3\)-scalar are flat. So, by virtue of (2.11) and (2.12), we respectively have
\[
8\rho(X,Y) + (5n+3)[\{(\varphi X)\alpha\} \eta(Y) + \{(\varphi Y)\alpha\} \eta(X)]
+ (9n-1)[(X\beta)\eta(Y) + (Y\beta)\eta(X)]
- 4(\alpha^2 - \beta^2)\left\{ (3n-1)g(X,Y) + (n+1)\eta(X)\eta(Y) \right\}
+ 2(\xi\beta)\{4g(X,Y) - (n+3)\eta(X)\eta(Y)\} = 0
\]

and
\[
\tau - (3n+1)n(\alpha^2 - \beta^2) + 4n(\xi\beta) = 0. \tag{3.2}
\]

We know from (2.13)
\[
5\{(\varphi X)\alpha\} - 7\{(X\beta) - (\xi\beta)\eta(X)\} = 0. \tag{3.3}
\]

From the above equation, we get
\[
\{(\varphi X)\alpha\} \eta(Y) + \{(\varphi Y)\alpha\} \eta(X) =
+ \frac{7}{5}\{(X\beta)\eta(Y) + (Y\beta)\eta(X)\} - \frac{14}{5}(\xi\beta)\eta(X)\eta(Y). \tag{3.4}
\]
Substituting (3.4) into (3.1), we get
\[
\rho(X, Y) = \frac{(3n-1)(\alpha^2 - \beta^2) - 2(\xi \beta)}{2} g(X, Y) \\
+ \frac{1}{2} \left\{ \frac{2(10n+9)}{5} (\xi \beta) + (n+1)(\alpha^2 - \beta^2) \right\} \eta(X) \eta(Y) \\
- \frac{2(5n+1)}{5} \left\{ d\beta(X) \eta(X) + d\beta(Y) \eta(X) \right\}.
\]
(3.5)

Thus we have

**Theorem 3.1.** A (CHR)$_3$-flat trans-Sasakian manifold is a generalized $\eta$-Einstein manifold.

**Remark 3.2.** The notion of a generalized $\eta$-Einstein manifold is defined by A. A. Shaikh and Y. Matsuyama, [5]. Moreover, M. C. Chaki called this manifold a *generalized quasi-Einstein manifold*, [1], [2].

From the above theorem, we can easily obtain

**Corollary 3.3.** A (CHR)$_3$-flat trans Sasakian manifold is $\eta$-Einstein if and only if the function $\beta$ is constant. Then the Ricci tensor $\rho$ and the scalar curvature $\tau$ with respect to $g$ are written as
\[
\rho(X, Y) = (\alpha^2 - \beta^2) \left\{ \frac{3n-1}{2} g(X, Y) + \frac{n+1}{2} \eta(X) \eta(Y) \right\} \\
\]
(3.6)
and
\[
\tau = -(3n+1)n(\alpha^2 - \beta^2).
\]

By virtue of Remark 1.1 and the above corollary, we get

**Corollary 3.4.** In a (CHR)$_3$-flat Sasakian, resp. Kenmotsu, manifold, the Ricci tensor $\rho$ and the scalar curvature $\tau$ with respect to $g$ satisfy
\[
\rho(X, Y) = \frac{3n-1}{2} g(X, Y) + \frac{n+1}{2} \eta(X) \eta(Y),
\]
resp.
\[
\rho(X, Y) = -\left\{ \frac{3n-1}{2} g(X, Y) + \frac{n+1}{2} \eta(X) \eta(Y) \right\}
\]
and
\[
\tau = -(3n+1)n \quad (\text{resp.} \quad \tau = (3n+1)n).
\]
(3.7)

Now, from (3.3), we obtain
\[
- 5\{ (\varphi X)\alpha \} A(Z, W, Y) + 5\{ (\varphi Y)\alpha \} A(Z, W, X) - \\
- 5\{ (\varphi Z)\alpha \} A(X, Y, W) + 5\{ (\varphi W)\alpha \} A(X, Y, Z) - \\
- 9\left\{ (X\beta) A(Z, W, Y) - (Y\beta) A(Z, W, X) + 
\]

\[
+ (Z\beta)A(X, Y, W) - (W\beta)A(X, Y, Z) \right)
= -16\left\{ (X\beta)A(Z, W, Y) - (Y\beta)A(Z, W, X) + \\
+ (Z\beta)A(X, Y, W) - (W\beta)A(X, Y, Z) \right\} + \\
+ 14(\xi\beta)\{A(Z, W, Y)\eta(X) - A(Z, W, X)\eta(Y)\}.
\]

From (3.3), we get

\[
\{(\varphi X)\beta\} = \frac{5}{7} (X\alpha) + \frac{5}{7}(\xi\alpha)\eta(X).
\] (3.8)

From this, we have

\[
- 3\left\{ \{(\varphi X)\beta\}A(Z, W, \varphi Y) - \{(\varphi Y)\beta\}A(Z, W, \varphi X) + \\
+ \{(\varphi Z)\beta\}A(X, Y, \varphi W) - \{(\varphi W)\beta\}A(X, Y, \varphi Z) \right\} + \\
+ 7\left\{ (X\alpha)A(Z, W, \varphi Y) - (Y\alpha)A(Z, W, \varphi X) + \\
+ (Z\alpha)A(X, Y, \varphi W) - (W\alpha)A(X, Y, \varphi Z) \right\} = \\
= \frac{64}{7}\left\{ (X\alpha)A(Z, W, \varphi Y) - (Y\alpha)A(Z, W, \varphi X) + \\
+ (Z\alpha)A(X, Y, \varphi W) - (W\alpha)A(X, Y, \varphi Z) \right\}.
\]

Next, since we have

\[
4[(X\alpha) - \{(\varphi X)\beta\}]g(\varphi Z, W)\eta(Y) = \frac{48}{7} (X\alpha) - \frac{20}{7}(\xi\alpha)g(\varphi Z, W)\eta(X)\eta(Y),
\]

we obtain

\[
4[(X\alpha) - \{(\varphi X)\beta\}]g(\varphi Z, W)\eta(Y) - \\
- 4[(Y\alpha) - \{(\varphi Y)\beta\}]g(\varphi Z, W)\eta(X) + \\
+ 4[(Z\alpha) - \{(\varphi Z)\beta\}]g(\varphi X, Y)\eta(W) - \\
- 4[(W\alpha) - \{(\varphi W)\beta\}]g(\varphi X, Y)\eta(Z) = \\
= \frac{48}{7}\left\{ (X\alpha)g(\varphi Z, W)\eta(Y) - (Y\alpha)g(\varphi Z, W)\eta(X) + \\
+ (Z\alpha)g(\varphi X, Y)\eta(W) - (W\alpha)g(\varphi X, Y)\eta(Z) \right\}.
\]
Using (3.6), (3.7) and (3.8), the curvature tensor $R$ with respect to $g$ is written as

$$
R(X, Y, Z, W) = \frac{1}{4} \left[ (X\alpha)\{4A(W, Z, \varphi Y) + 3g(\varphi W, Z), \eta(Y)\} 
- (Y\alpha)\{4A(W, Z, \varphi X) + 3g(\varphi W, Z)\eta(X)\} 
+ (Z\alpha)\{4A(Y, X, \varphi W) + 3g(\varphi Y, X)\eta(W)\} 
- (W\alpha)\{4A(Y, X, \varphi Z) + 3g(\varphi Y, X)\eta(Z)\} \right] 
+ (X\beta)A(W, Z, Y) - (Y\beta)A(W, Z, X) 
- (Z\beta)A(Y, X, W) + (W\beta)A(Y, X, Z) 
+ \frac{1}{4}(\alpha^2 - \beta^2) \left[ 3\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} 
- g(\varphi X, W)g(\varphi Y, Z) + g(\varphi X, Z)g(\varphi Y, W) 
+ 2g(\varphi X, Y)g(\varphi Z, W) \right] 
- \frac{1}{4}\{((\alpha^2 - \beta^2) - (\xi\beta))\} \{A(Z, W, Y)\eta(X) - A(Z, W, X)\eta(Y)\}. 
$$

Thus we get

**Theorem 3.5.** If a trans-Sasakian manifold is $(CHR)_3$-flat, the curvature tensor satisfies (3.9).

By virtue of Remark 1.1 and the above theorem, we have

**Corollary 3.6.** In a $(CHR)_3$-flat Sasakian, resp. Kenmotsu, manifold, the curvature tensor $R$ with respect to $g$ are written by

$$
R(X, Y, Z, W) = \frac{1}{4} \left[ 3\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} 
- g(\varphi X, W)g(\varphi Y, Z) + g(\varphi X, Z)g(\varphi Y, W) + 2g(\varphi X, Y)g(\varphi Z, W) \right] 
- \frac{1}{4}\{A(Z, W, Y)\eta(X) - A(Z, W, X)\eta(Y)\}
$$

resp.

$$
R(X, Y, Z, W) = -\frac{1}{4} \left[ 3\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} 
- g(\varphi X, W)g(\varphi Y, Z) + g(\varphi X, Z)g(\varphi Y, W) + 2g(\varphi X, Y)g(\varphi Z, W) \right] 
+ \frac{1}{4}\{A(Z, W, Y)\eta(X) - A(Z, W, X)\eta(Y)\}. 
$$
Remark 3.7. The above corollary shows that a $(CHR)_3$-flat Sasakian (resp. Kenmotsu) manifold is a Sasakian (resp. Kenmotsu) space form with zero holomorphic sectional curvature.

Remark 3.8. Of course, we can get (3.2) and (3.5) from Theorem 3.1 directly.

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