Some spectral properties of uniform hypergraphs

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Abstract

For a $k$-uniform hypergraph $H$, we obtain some trace formulas for the Laplacian tensor of $H$, which imply that $\sum_{s=1}^{k} d_{s}^{s}$ is determined by the Laplacian spectrum of $H$, where $d_{1}, \ldots, d_{n}$ is the degree sequence of $H$. Using trace formulas for the Laplacian tensor, we obtain expressions for some coefficients of the Laplacian polynomial of a regular hypergraph. We give some spectral characterizations of odd-bipartite hypergraphs, and give a partial answer to a question posed by Shao et al \cite{17}. We also give some spectral properties of power hypergraphs, and show that a conjecture posed by Hu et al \cite{7} holds under certain conditions.

Keywords: Hypergraph eigenvalue, Adjacency tensor, Laplacian tensor, Signless Laplacian tensor, Power hypergraph

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1. Introduction

Recently, the research on spectral theory of hypergraphs has attracted extensive attention [1,5-8,11,13,14,16-18]. We first introduce some necessary concepts and notations. For a positive integer $n$, let $[n] = \{1, \ldots, n\}$. An order $k$ dimension $n$ tensor $A = (a_{i_{1}\cdots i_{k}}) \in \mathbb{C}^{n\times\cdots\times n}$ is a multidimensional array with $n^{k}$ entries, where $i_{j} \in [n]$, $j = 1, \ldots, k$. We sometimes write $a_{i_{1}\cdots i_{k}}$ as $a_{i_{1}\alpha}$, where $\alpha = i_{2}\cdots i_{k}$. When $k = 1$, $A$ is a column vector of dimension $n$. When $k = 2$, $A$ is an $n \times n$ matrix. The unit tensor of

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order $k \geq 2$ and dimension $n$ is a diagonal tensor $\mathcal{I}_n = (\delta_{i_1\ldots i_k})$ such that $\delta_{i_1\ldots i_k} = 1$ if $i_1 = i_2 = \cdots = i_k$, and $\delta_{i_1\ldots i_k} = 0$ otherwise. In \cite{15}, Shao defined the following product of tensors, which is a generalization of the matrix multiplication.

**Definition 1.1.** \cite{15} Let $A$ and $B$ be order $m \geq 2$ and order $k \geq 1$, dimension $n$ tensors, respectively. The product $AB$ is the following tensor $C$ of order $(m-1)(k-1) + 1$ and dimension $n$ with entries

$$c_{i\alpha_1\ldots\alpha_{m-1}} = \sum_{i_2,\ldots,i_m \in [n]} a_{i_1i_2\ldots i_m} b_{i_2\alpha_1} \cdots b_{i_m\alpha_{m-1}} \quad (i \in [n], \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}).$$

Let $A$ be an order $k \geq 2$ dimension $n$ tensor, and let $x = (x_1, \ldots, x_n)^T$. From Definition 1.1 the product $Ax$ is a vector in $\mathbb{C}^n$ whose $i$-th component is (see Example 1.1 in \cite{15})

$$(Ax)_i = \sum_{i_2,\ldots,i_k \in [n]} a_{i_1i_2\ldots i_k} x_{i_2} \cdots x_{i_k}.$$ 

The concept of tensor eigenvalues was posed in \cite{9, 12}. If there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x^{[k-1]}$, then $\lambda$ is called an eigenvalue of $A$, $x$ is an eigenvector of $\lambda$, where $x^{[k-1]} = (x_1^{k-1}, \ldots, x_n^{k-1})^T$. The determinant of $A$, denoted by $\det(A)$, is the resultant of the system of polynomials $f_i(x_1, \ldots, x_n) = (Ax)_i \quad (i = 1, \ldots, n)$. The characteristic polynomial of $A$ is defined as $\phi_A(\lambda) = \det(\lambda I_n - A)$, where $I_n$ is the unit tensor of order $k$ and dimension $n$. It is known that eigenvalues of $A$ are exactly roots of $\phi_A(\lambda)$ \cite{12}. The multiset of roots of $\phi_A(\lambda)$ (counting multiplicities) is the spectrum of $A$, denoted by $\sigma(A)$. The maximal modulus of eigenvalues of $A$ is called the spectral radius of $A$, denoted by $\rho(A)$. More details on eigenvalues and characteristic polynomials of tensors can be found in \cite{4, 12}.

A hypergraph $H$ is called $k$-uniform if each edge of $H$ contains exactly $k$ distinct vertices. Let $V(H)$ and $E(H)$ denote the vertex set and the edge set of $H$, respectively. In \cite{13}, Qi defined the Laplacian and the signless Laplacian tensor of a uniform hypergraph as follows.

**Definition 1.2.** \cite{7, 13} The adjacency tensor of a $k$-uniform hypergraph $H$, denoted by $A_H$, is an order $k$ dimension $|V(H)|$ tensor with entries

$$a_{i_1i_2\ldots i_k} = \begin{cases} 
\frac{1}{(k-1)!} & \text{if } i_1i_2\ldots i_k \in E(H), \\
0 & \text{otherwise.}
\end{cases}$$
Let $D_H$ be an order $k$ dimension $|V(H)|$ diagonal tensor whose diagonal entries are vertex degrees of $H$. The tensors $L_H = D_H - A_H$ and $Q_H = D_H + A_H$ are the Laplacian tensor and the signless Laplacian tensor of $H$, respectively. Eigenvalues of $A_H$, $L_H$ and $Q_H$ are called eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of $H$, respectively. Characteristic polynomials of $L_H$ and $Q_H$ are called Laplacian polynomial and signless Laplacian polynomial of $H$, respectively.

This paper is organized as follows. In Section 2, we give some trace formulas for the Laplacian tensor of a uniform hypergraph, and obtain expressions for some coefficients of the Laplacian polynomial of a regular hypergraph. In Section 3, we give some spectral characterizations of odd-bipartite hypergraphs. In Section 4, we give some spectral properties of power hypergraphs.

2. Laplacian spectra and degree sequence of hypergraphs

Traces of tensors are very useful in the study of spectral theory of tensors. The $d$-th order trace of an order $k \geq 2$ dimension $n$ tensor $T = (t_{i_1 \cdots i_k})$ is defined as \cite{1, 4, 10}

$$Tr_d(T) = (k-1)^{n-1} \sum_{d_1 + \cdots + d_n = d} \prod_{i=1}^{n} \frac{1}{(d_i(k-1))!} \left( \sum_{y \in [n]^{k-1}} t_{iy} \frac{\partial}{\partial a_{iy}} \right)^{d_i} \text{tr}(A^{d(k-1)}),$$

where $A = (a_{ij})$ is an $n \times n$ auxiliary matrix, and $\frac{\partial}{\partial a_{iy}} = \frac{\partial}{\partial a_{i_2y}} \cdots \frac{\partial}{\partial a_{i_ky}}$ if $y = i_2 \cdots i_k$. The codegree $d$ coefficient of the characteristic polynomial of $\mathcal{T}$ can be expressed in terms of $Tr_1(\mathcal{T}), \ldots, Tr_d(\mathcal{T})$ (see \cite[Theorem 6.3]{4}). It is also known that $Tr_t(\mathcal{T}) = \sum_{\lambda \in \sigma(\mathcal{T})} \lambda^t$ for any $t \in [n(k-1)^{n-1}]$ (see \cite[Theorem 6.10]{4}). Hence $Tr_d(\mathcal{T})$ is an important invariant in the spectral theory of tensors.

Shao et al \cite{16} give a graph theoretical formula for $Tr_d(\mathcal{T})$. In order to describe this formula, we introduce some notations in \cite{16}. For an integer $d > 0$, we define

$$\mathcal{F}_d = \{(i_1\alpha_1, \ldots, i_d\alpha_d)|1 \leq i_1 \leq \cdots \leq i_d \leq n; \alpha_1, \ldots, \alpha_d \in [n]^{k-1}\}.$$ 

For $F = (i_1\alpha_1, \ldots, i_d\alpha_d) \in \mathcal{F}_d$ and an order $k \geq 2$ dimension $n$ tensor $\mathcal{T} = (t_{i_1 \cdots i_k})$, we write $\pi_F(\mathcal{T}) = \prod_{j=1}^{d} t_{ij\alpha_j}$. Let $p_i(F)$ be the total number of times that the index $i$ appears in $F$. If $p_i(F)$ is a multiple of $k$ for any $i \in [n]$, then $F$ is called $k$-valent.
Definition 2.1. \[16\] Let \( F = (i_1\alpha_1, \ldots, i_d\alpha_d) \in \mathcal{F}_d \), where \( i_j\alpha_j \in [n]^k \), \( j = 1, \ldots, d \). Then

(1) Let \( E(F) = \bigcup_{j=1}^d E_j(F) \), where \( E_j(F) \) is the arc multi-set \( E_j(F) = \{ (i_j, v_1), \ldots, (i_j, v_{k-1}) \} \)

if \( \alpha_j = v_1 \cdots v_{k-1} \).

(2) Let \( b(F) \) be the product of the factorials of the multiplicities of all the arcs of \( E(F) \).

(3) Let \( c(F) \) be the product of the factorials of the outdegrees of all the vertices in the arc multi-set \( E(F) \).

(4) Let \( W(F) \) be the set of all closed walks \( W \) with the arc multi-set \( E(F) \).

Shao et al give a graph theoretical formula for \( Tr_d(T) \) as follows (see equation (3.5) in [16]).

Lemma 2.2. \[16\] Let \( T = (T_{i_1 \cdots i_k}) \) be an order \( k \geq 2 \) dimension \( n \) tensor. Then

\[
Tr_d(T) = (k - 1)^{n-1} \sum_{F \in \mathcal{F}_d} \frac{b(F)}{c(F)} \pi_F(T)|W(F)|,
\]

where \( \mathcal{F}_d' = \{ F | F \in \mathcal{F}_d, F \text{ is } k\text{-valent} \} \).

For a \( k \)-uniform hypergraph \( H \), Cooper and Dutle [1] proved that \( Tr_d(A_H) = 0 \) for \( d \in [k-1] \). We give some trace formulas for the Laplacian (signless Laplacian) tensor of uniform hypergraphs as follows.

Theorem 2.3. Let \( H \) be a \( k \)-uniform hypergraph with degree sequence \( d_1, \ldots, d_n \). Then

\[
Tr_t(L_H) = Tr_t(Q_H) = (k - 1)^{n-1} \sum_{i=1}^n d_i^t, \ t = 1, \ldots, k - 1,
\]

\[
Tr_k(L_H) = (-1)^k k^{k-1} (k - 1)^{n-k} |E(H)| + (k - 1)^{n-1} \sum_{i=1}^n d_i^k,
\]

\[
Tr_k(Q_H) = k^{k-1} (k - 1)^{n-k} |E(H)| + (k - 1)^{n-1} \sum_{i=1}^n d_i^k.
\]
Proof. By Lemma 2.2, we have

\[ Tr_t(\mathcal{L}_H) = (k - 1)^{n-1} \sum_{F \in F'_t} \frac{b(F)}{c(F)} \pi_F(\mathcal{L}_H)|W(F)|, \tag{1} \]

where \( F'_t = \{ F | F \in F_t, F \text{ is k-valent} \} \). For \( F = (i_1\alpha_1, \ldots, i_t\alpha_t) \in F_t \), if \( \pi_F(\mathcal{L}_H) = \prod_{j=1}^{t}(\mathcal{L}_H)_{i_j\alpha_j} \neq 0 \), then \( i_j\alpha_j = i_ji_j \cdots i_j \in [n]^k \) or \( i_j\alpha_j \in E(H) \) for any \( 1 \leq j \leq t \).

Let \( F \in F_t \) satisfies \( \pi_F(\mathcal{L}_H) \neq 0 \). If \( t < k \), then \( F \) is k-valent if and only if \( F = (i_1i_1 \cdots i_1, \ldots, i_ti_t \cdots i_t) \). In this case, \( |W(F)| \neq 0 \) if and only if \( i_1 = \cdots = i_t \). Let \( F_i = (i_i \cdots i_i, \ldots, i_i \cdots i_i) \in F'_t (t < k) \). From Eq. (1) and Definition 2.1, we have

\[ Tr_t(\mathcal{L}_H) = (k - 1)^{n-1} \sum_{i=1}^{n} \frac{b(F_i)}{c(F_i)} \pi_{F_i}(\mathcal{L}_H)|W(F_i)| \]

\[ = (k - 1)^{n-1} \sum_{i=1}^{n} \frac{(t(k-1))!}{(t(k-1))!} d_i^t = (k - 1)^{n-1} \sum_{i=1}^{n} d_i^t. \]

Similar with the above procedure, we can also get \( Tr_t(\mathcal{Q}_H) = (k - 1)^{n-1} \sum_{i=1}^{n} d_i^t, t = 1, \ldots, k - 1 \).

Let \( F \in F_k \) satisfies \( \pi_F(\mathcal{L}_H) \neq 0 \). Then \( F \) is k-valent and \( |W(F)| \neq 0 \) if and only if \( F = (i_i \cdots i_i, \ldots, i_i \cdots i_i) \) or \( F = (i_1\alpha_1, \ldots, i_k\alpha_k) \), where \( i_1\alpha_1, \ldots, i_k\alpha_k \) correspond to the same edge \( i_1i_2 \cdots i_k \in E(H) \). Let \( F_i = (i_i \cdots i_i, \ldots, i_i \cdots i_i) \in F'_k \). From Eq. (1) and Definition 2.1, we have

\[ Tr_k(\mathcal{L}_H) = (-1)^k Tr_k(\mathcal{A}_H) + (k - 1)^{n-1} \sum_{i=1}^{n} \frac{b(F_i)}{c(F_i)} \pi_{F_i}(\mathcal{L}_H)|W(F_i)| \]

\[ = (-1)^k Tr_k(\mathcal{A}_H) + (k - 1)^{n-1} \sum_{i=1}^{n} \frac{(k(k-1))!}{(k(k-1))!} d_i^k \]

\[ = (-1)^k Tr_k(\mathcal{A}_H) + (k - 1)^{n-1} \sum_{i=1}^{n} d_i^k. \]

From the proof of [1, Theorem 3.15], we have \( Tr_k(\mathcal{A}_H) = k^{k-1}(k-1)^{n-k}|E(H)| \).

Hence

\[ Tr_k(\mathcal{L}_H) = (-1)^k k^{k-1}(k-1)^{n-k}|E(H)| + (k - 1)^{n-1} \sum_{i=1}^{n} d_i^k. \]
Similar with the above procedure, we can also get

\[ T_{rk}(Q_H) = k^{k-1}(k-1)^{n-k}|E(H)| + (k-1)^{n-1} \sum_{i=1}^{n} d_i^k. \]

\[ \square \]

**Remark.** Note that traces of a tensor are determined by its spectrum [3, Theorem 6.3]. For a \( k \)-uniform hypergraph \( H \), by Theorem 2.3, we know that \( \sum_{i=1}^{n} d_i^s \) (\( s = 1, \ldots, k \)) is determined by the Laplacian (signless Laplacian) spectrum of \( H \), where \( d_1, \ldots, d_n \) is the degree sequence of \( H \).

Let \( p_t(M) \) denote the codegree \( t \) coefficient of the characteristic polynomial of a tensor \( M \).

**Lemma 2.4.** Let \( M \) be an order \( k \geq 2 \) dimension \( n \) tensor. Then

\[
\begin{align*}
\frac{t!}{p_t(M)} = & \det \begin{pmatrix}
- Tr_t & Tr_1 & Tr_2 & \cdots & Tr_{t-1} \\
- Tr_{t-1} & t-1 & Tr_1 & \cdots & Tr_{t-2} \\
- Tr_{t-2} & 0 & t-2 & \cdots & \vdots \\
\vdots & \vdots & \iddots & \iddots & Tr_1 \\
- Tr_1 & 0 & \cdots & 0 & 1
\end{pmatrix},
\end{align*}
\]

where \( Tr_t = Tr_t(M), t \in [n(k-1)^{n-1}] \).

**Proof.** From [4, Theorem 6.10], we have

\[
\begin{pmatrix}
t & Tr_1 & Tr_2 & \cdots & Tr_{t-1} \\
0 & t-1 & Tr_1 & \cdots & Tr_{t-2} \\
0 & 0 & t-2 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \iddots & Tr_1 \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
p_t(M) \\
p_{t-1}(M) \\
p_{t-2}(M) \\
p_{t-3}(M) \\
p_1(M)
\end{pmatrix}
= \begin{pmatrix}
- Tr_t \\
- Tr_{t-1} \\
- Tr_{t-2} \\
- Tr_{t-3} \\
- Tr_1
\end{pmatrix}.
\]

We can obtain the expression of \( t!p_t(M) \) from Cramer’s rule. \( \square \)

A uniform hypergraph \( H \) is called \( d\)-regular if each vertex of \( H \) has degree \( d \). The following are some coefficients of the Laplacian (signless Laplacian) polynomial of regular hypergraphs.
Theorem 2.5. Let $H$ be a $d$-regular $k$-uniform hypergraph with $n$ vertices. Then
\[
p_t(L_H) = p_t(Q_H) = (-1)^t d^k \binom{n(k-1)^{n-1}}{t}, \quad t = 1, \ldots, k-1,
\]
\[
p_k(L_H) = (-1)^{k+1} k^{k-3} (k-1)^{n-k} nd + (-1)^k d^k \binom{n(k-1)^{n-1}}{k},
\]
\[
p_k(Q_H) = -k^{k-3} (k-1)^{n-k} nd + (-1)^k d^k \binom{n(k-1)^{n-1}}{k}.
\]

Proof. By Lemma 2.4, we have
\[
t! p_t(L_H) = \det \begin{pmatrix}
-T_{r_t} & T_{r_1} & T_{r_2} & \cdots & T_{r_{t-1}} \\
-T_{r_{t-1}} & t-1 & T_{r_1} & \cdots & T_{r_{t-2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-T_{r_2} & 0 & t-2 & \cdots & T_{r_1} \\
-T_{r_1} & 0 & \cdots & 0 & 1
\end{pmatrix}, \quad (2)
\]
where $T_{r_t} = Tr_t(L_H)$. Since $H$ is $d$-regular, by Theorem 2.3, we have $Tr_i = dTr_{i-1} = n(k-1)^{n-1}d^i, i = 2, \ldots, k-1$. If $t < k$, then by Eq. (2), we have
\[
t! p_t(L_H) = \det \begin{pmatrix}
0 & T_{r_1} & T_{r_2} & \cdots & T_{r_{t-1}} \\
0 & t-1 & T_{r_1} & \cdots & T_{r_{t-2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \vdots & \vdots & \ddots & T_{r_1} \\
d-Tr_1 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]
\[
= \det \begin{pmatrix}
0 & T_{r_1} - (t-1)d & 0 & \cdots & 0 \\
0 & t-1 & \cdots & \cdots & \vdots \\
\vdots & 0 & \ddots & T_{r_1} - 2d & 0 \\
0 & \vdots & \ddots & 2 & T_{r_1} \\
d-Tr_1 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]
\[
= (-1)^t \prod_{i=0}^{t-1} (Tr_i - id).
\]
Since $Tr_1 = n(k-1)^{n-1}d$, we have

$$p_t(L_H) = (-1)^t \prod_{i=0}^{t-1} \frac{(n-1)^{n-1}d-idi}{t!} = (-1)^t d^t \prod_{i=0}^{t-1} \frac{(n-1)^{n-1} - id}{t!} = (-1)^t d^t \binom{n(k-1)^{n-1}}{t}.$$ 

Similar with the above procedure, we can also get

$$p_t(Q_H) = (-1)^t d^t \binom{n(k-1)^{n-1}}{t}, \ t = 1, \ldots, k-1.$$ 

Since $H$ is $d$-regular, by Theorem 2.3 we have $Tr_k = (-1)^k k^{k-2}(k-1)^{n-k}nd + dTr_{k-1}$ and $Tr_i = dTr_{i-1} = n(k-1)^{n-1}d^i, \ i = 2, \ldots, k-1$. From Eq. (2), we have

$$k! p_k(L_H) = \text{det} \begin{pmatrix} (-1)^{k+1} k^{k-2}(k-1)^{n-k}nd & Tr_1 & Tr_2 & \cdots & Tr_{k-1} \\ 0 & k-1 & Tr_1 & \cdots & Tr_{k-2} \\ \vdots & 0 & k-2 & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & Tr_1 \\ d-Tr_1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$= \text{det} \begin{pmatrix} (-1)^{k+1} k^{k-2}(k-1)^{n-k}nd & Tr_1 - (k-1)d & 0 & \cdots & 0 \\ 0 & k-1 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & Tr_1 - 2d & 0 \\ 0 & \vdots & \ddots & 2 & Tr_1 \\ d-Tr_1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$= (-1)^{k+1} k^{k-2}(k-1)^{n-k}(k-1)nd + (-1)^k d^k \prod_{i=0}^{k-1} (n(k-1)^{n-1} - i).$$

$$p_k(L_H) = (-1)^{k+1} k^{k-3}(k-1)^{n-k}nd + (-1)^k d^k \binom{n(k-1)^{n-1}}{k}.$$ 

Similar with the above procedure, we can also get

$$p_k(Q_H) = -k^{k-3}(k-1)^{n-k}nd + (-1)^k d^k \binom{n(k-1)^{n-1}}{k}.$$ 

$\square$
3. Eigenvalues and odd-bipartite hypergraphs

A \( k \)-uniform hypergraph \( H \) is called odd-bipartite, if there exists a proper subset \( V_1 \) of \( V(H) \) such that each edge of \( H \) contains exactly odd number of vertices in \( V_1 \) [6, 17]. Spectral characterizations of odd-bipartite hypergraphs will be investigated in this section. We first give some auxiliary lemmas. The following lemma can be obtained from equation (2.1) in [15].

**Lemma 3.1.** Let \( A = (a_{i_1 \cdots i_k}) \) be an order \( k \geq 2 \) dimension \( n \) tensor, and let \( P = (p_{ij}), Q = (q_{ij}) \) be \( n \times n \) matrices. Then

\[
(PAQ)_{i_1 \cdots i_k} = \sum_{j_1, \ldots, j_k \in [n]} a_{j_1 \cdots j_k} p_{i_1 j_1} q_{j_2 i_2} \cdots q_{j_k i_k}.
\]

**Lemma 3.2.** [6] Let \( H \) be a connected \( k \)-uniform hypergraph. A nonzero vector \( x \) is an eigenvector of \( Q_H \) corresponds to the zero eigenvalue if and only if there exist nonzero \( \gamma \in \mathbb{C} \) and integers \( \alpha_i \) such that \( x_i = \gamma \exp\left(\frac{2\alpha_i \pi}{k} \sqrt{-1}\right) \) for each \( i \in V(H) \), and

\[
\sum_{j \in e} \alpha_j = \sigma_e k + \frac{k}{2}
\]

for some integer \( \sigma_e \) associated with each \( e \in E(H) \).

Weakly irreducible tensors are defined in [3]. It is known that a \( k \)-uniform hypergraph \( H \) is connected if and only if \( A_H \) is weakly irreducible [11].

**Lemma 3.3.** [17, 19] Let \( A \) be an order \( k \) dimension \( n \) nonnegative weakly irreducible tensor. If \( \rho(A) \exp(\theta \sqrt{-1}) \) is an eigenvalue of \( A \), then there exists a diagonal matrix \( U \) with unit diagonal entries such that

\[
A = \exp(-\theta \sqrt{-1}) U^{-(k-1)} A U.
\]

For a tensor \( \mathcal{T} \), let \( H\sigma(\mathcal{T}) = \{ \lambda | \lambda \in \sigma(\mathcal{T}), \lambda \text{ has a real eigenvector} \} \). For a connected \( k \)-uniform hypergraph \( G \), Shao et al [17] proved that

\[
H\sigma(L_G) = H\sigma(Q_G) \implies \sigma(L_G) = \sigma(Q_G).
\]

Shao et al wish to know whether the reverse implication is true. We show that the reverse is true when \( k \) is not divisible by 4.
Theorem 3.4. Let $G$ be a connected $k$-uniform hypergraph, and $k$ is not divisible by 4. Then the following are equivalent:

(1) $k$ is even and $H$ is odd-bipartite.
(2) $H \sigma(L_G) = H \sigma(Q_G)$.
(3) $\sigma(L_G) = \sigma(Q_G)$.
(4) 0 is a signless Laplacian eigenvalue of $G$.

Proof. From [17, Theorem 2.2], we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Since 0 is always an eigenvalue of $L_G$ (see [13]), we have (3) $\Rightarrow$ (4). Next we prove that (4) $\Rightarrow$ (1).

If 0 is an eigenvalue of $Q_G$, then by Lemma 3.2, there exists a vertex labeling $f : V(G) \rightarrow [k]$ such that

$$\sum_{i \in e} f(i) \equiv \frac{k}{2} \pmod{k}$$

for each $e \in E(G)$. Hence $k$ is even. Since $k$ is not divisible by 4, we know that $\frac{k}{2}$ is odd. So $\sum_{i \in e} f(i)$ is odd for each $e \in E(G)$. Let $V_1 = \{u | u \in V(G), f(u) \text{ is odd}\}$. For any $e \in E(G)$, since $\sum_{i \in e} f(i)$ is odd, $e$ contains exactly odd number of vertices in $V_1$. Hence $G$ is odd-bipartite.

When $k = 2$, Theorem 3.4 becomes a classic result in spectral graph theory, i.e., a connected graph $G$ is bipartite if and only if 0 is a signless Laplacian eigenvalue of $G$. It is also well known that a connected graph $G$ is bipartite if and only if $-\rho(A_G)$ is an eigenvalue of $G$. We generalize this result as follows.

Theorem 3.5. Let $H$ be a connected $k$-uniform hypergraph, and $k$ is not divisible by 4. Then the following are equivalent:

(1) $k$ is even and $H$ is odd-bipartite.
(2) $-\rho(A_H)$ is an eigenvalue of $H$.

Proof. From [17, Theorem 2.3], we have (1) $\Rightarrow$ (2). If (2) holds, then by Lemma 3.3 there exists a diagonal matrix $U$ with unit diagonal entries such that $A_H = -U^{-(k-1)}A_H U$. By Lemma 3.1 we have

$$a_{i_1i_2\cdots i_k} = -a_{i_1i_2\cdots i_k}u_{i_1}^{-(k-1)}u_{i_2}\cdots u_{i_k},$$

where $u_{i_j}$ is the diagonal entry of $U$ corresponds to vertex $i_j$ ($j = 1, \ldots, k$). For any edge $e = i_1i_2\cdots i_k \in E(H)$, we get

$$u_{i_1}^{-(k-1)}u_{i_2}\cdots u_{i_k} = -1, \quad u_{i_1}u_{i_2}\cdots u_{i_k} = -u_{i_1}^k.$$
Similarly, we have $u_1 u_2 \cdots u_k = -u_1^k = \cdots = -u_k^k$. Since $u_1, \ldots, u_k$ are unit complex number, there exist integers $\alpha_1, \ldots, \alpha_k$ and $\theta$ such that

$$u_{ij} = \exp\left(\frac{2\pi\alpha_{ij} + \theta}{k}\sqrt{-1}\right), \quad j = 1, \ldots, k.$$ Then

$$u_1 u_2 \cdots u_k = \exp\left(\frac{k\theta + 2\pi \sum_{j=1}^k \alpha_{ij}}{k}\sqrt{-1}\right) = -u_1^k = -\exp\left(\theta\sqrt{-1}\right),$$

$$\exp\left(2\pi \sum_{j=1}^k \alpha_{ij}\sqrt{-1}\right) = -1.$$ Hence $\sum_{j=1}^k \alpha_{ij} \equiv \frac{k}{2} \pmod{k}$, $k$ is even. Since $k$ is not divisible by 4, $\sum_{j=1}^k \alpha_{ij}$ is odd for any edge $e = i_1 i_2 \cdots i_k \in E(H)$. Let $V_1 = \{u \mid u \in V(H), \alpha_u \text{ is odd}\}$. For any $e = i_1 i_2 \cdots i_k \in E(H)$, since $\sum_{j=1}^k \alpha_{ij}$ is odd, $e$ contains exactly odd number of vertices in $V_1$. Hence $H$ is odd-bipartite. 

Let $H$ be a connected $k$-uniform hypergraph. If 0 is an eigenvalue of $Q_H$, then by the proof of Theorem 3.4, we know that there exists a vertex labeling $f : V(H) \to [k]$ such that $\sum_{i \in e} f(i) \equiv \frac{k}{2} \pmod{k}$ for each $e \in E(H)$. We pose the following conjecture.

**Conjecture 3.6.** Let $H$ be a connected $k$-uniform hypergraph. Then the following are equivalent:

1. $k$ is even and $H$ is odd-bipartite.
2. 0 is a signless Laplacian eigenvalue of $H$.
3. $-\rho(A_H)$ is an eigenvalue of $H$.
4. There exists a vertex labeling $f : V(H) \to [k]$ such that $\sum_{i \in e} f(i) \equiv \frac{k}{2} \pmod{k}$ for each $e \in E(H)$.

### 4. Eigenvalues of power hypergraphs

A vertex with degree one is called a core vertex [7]. For a $k$-uniform hypergraph $H$, if $e \in E(H)$ contains core vertices, then we use $H - e$ to denote a $k$-uniform sub-hypergraph of $H$ obtained by deleting the edge $e$ and all core vertices in $e$.

**Theorem 4.1.** Let $H$ be a $k$-uniform hypergraph, and let $e \in E(H)$ be an edge contains at least two core vertices. If $\lambda$ is an eigenvalue of $H - e$, then $\lambda$ is an eigenvalue of $H$. 


Theorem 4.4. Let \( x \) be an eigenvector of the eigenvalue \( \lambda \) of \( H - e \). Let \( y \) be a column vector of dimension \( |V(H)| \) such that \( y_u = x_u \) if \( u \in V(H - e) \), and \( y_u = 0 \) if \( u \in V(H) \) is a core vertex in \( e \). Since \( A_{H-e}x = \lambda x^{[k-1]} \), we have \( A_Hy = \lambda y^{[k-1]} \). So \( \lambda \) is an eigenvalue of \( H \). \( \square \)

In [7], Hu et al defined power hypergraphs as follows.

Definition 4.2. [7] Let \( G \) be an ordinary graph (i.e. 2-uniform hypergraph). For any \( k \geq 3 \), the \( k \)-th power of \( G \), denoted by \( G^k \), is a \( k \)-uniform hypergraph with edge set \( E(G^k) = \{e \cup \{i_{e,1}, \ldots, i_{e,k-2}\} | e \in E(G)\} \), and vertex set \( V(G^k) = V(G) \cup \{i_{e,j} | e \in E(G), j \in [k-2]\} \).

Some examples of power hypergraphs are given in [7, Fig.1]. From Definition 4.2, we know that each edge of a power hypergraph \( G^k \) contains two adjacent vertices in \( V(G) \) and \( k-2 \) core vertices not in \( V(G) \).

If \( H \) is a connected \( k \)-uniform hypergraph, then \( A_H \) and \( Q_H \) are both weakly irreducible [13]. So we obtain the following lemma from [13, Theorem 2.2].

Lemma 4.3. Let \( H \) be a connected \( k \)-uniform hypergraph. If \( \lambda \) is an eigenvalue of \( A_H \) (or \( Q_H \)) with a positive eigenvector, then \( \lambda = \rho(A_H) \) (or \( \lambda = \rho(Q_H) \)).

Theorem 4.4. If \( \lambda \neq 0 \) is an eigenvalue of a graph \( G \), then \( \lambda^\frac{k}{k-1} \) is an eigenvalue of \( G^k \). Moreover, \( \rho(A_{G^k}) = \rho(A_G)^\frac{k}{k-1} \).

Proof. Suppose that \( x \) is an eigenvector of the eigenvalue \( \lambda \neq 0 \) of graph \( G \). Then \( \sum_{j \in N_G(i)} x_j = \lambda x_i \) for any \( i \in V(G) \), where \( N_G(i) \) is the set of all neighbors of \( i \) in \( G \). Let \( y \) be a column vector of dimension \( |V(G^k)| \) such that \( y_u = (x_u)^{\frac{k}{k-1}} \) if \( u \in V(G) \), and \( y_u = (\lambda^{-1}x_i x_j)^{\frac{k}{k-1}} \) if \( u \in V(G^k) \setminus V(G) \) is a core vertex in the edge contains two adjacent vertices \( i, j \in V(G) \). For any \( i \in V(G) \), by \( \sum_{j \in N_G(i)} x_j = \lambda x_i \), we have

\[
(A_{G^k}y)_i = \sum_{j \in N_G(i)} (\lambda^{-1}x_i x_j)^{\frac{k-2}{k}} (x_j)^{\frac{k}{k-1}} = \lambda^{\frac{k}{k-1}} (x_i)^{\frac{2(k-1)}{k}} = \lambda^{\frac{k}{k-1}} (y_i)^{k-1}.
\]

For any \( u \in V(G^k) \setminus V(G) \), we have

\[
(A_{G^k}y)_u = (\lambda^{-1}x_i x_j)^{\frac{k-2}{k}} (x_i)^{\frac{k}{k-1}} = \lambda^{\frac{k}{k-1}} (\lambda^{-1}x_i x_j)^{\frac{k-1}{k}} = \lambda^{\frac{k}{k-1}} (y_u)^{k-1}.
\]

Hence \( \lambda^{\frac{k}{k-1}} \) is an eigenvalue of \( G^k \) with an eigenvector \( y \).
If $G$ is connected and $\lambda = \rho(A_G)$, then we can choose $x$ as a positive eigenvector of $\rho(A_G)$. In this case, $y$ is a positive eigenvector of the eigenvalue $\rho(A_G)^\frac{k}{2}$ of $G^k$. Lemma 4.3 implies that $\rho(A_G^k) = \rho(A_G)^{\frac{k}{2}}$ when $G$ is connected.

If $G$ has $r \geq 2$ components $G_1, \ldots, G_r$, then

$$\rho(A_G^k) = \max\{\rho(A_{G_1}^k), \ldots, \rho(A_{G_r}^k)\} = \max\{\rho(A_{G_1})^\frac{k}{2}, \ldots, \rho(A_{G_r})^\frac{k}{2}\} = \rho(A_G)^{\frac{k}{2}}.$$

We can obtain the following result from Theorem 4.4.

**Corollary 4.5.** For any nontrivial graph $G$, we have $\lim_{k \to \infty} \rho(A_G^k) = 1$. Moreover, $\{\rho(A_G^k)\}$ is a strictly decreasing sequence if $\rho(A_G) > 1$.

The following corollary follows from Theorem 4.1 and 4.4.

**Corollary 4.6.** If $\lambda \neq 0$ is an eigenvalue of any subgraph of a graph $G$, then $\lambda^\frac{k}{2}$ is an eigenvalue of $G^k$ for $k \geq 4$.

Let $P_n$ and $S_n$ be the path and the star of order $n$, respectively. The following result was proved by Li et al [8]. Here we give a different proof.

**Corollary 4.7.** Let $T$ be a tree with $n$ vertices. Then

$$\rho(A_{P_n}^k) \leq \rho(A_{T}^k) \leq \rho(A_{S_n}^k),$$

where the left equality holds if and only if $T = P_n$, and the right equality holds if and only if $T = S_n$.

**Proof.** Among all trees with $n$ vertices, $P_n$ is the unique tree with the smallest adjacency spectral radius, and $S_n$ is the unique tree with the largest adjacency spectral radius [2]. By Theorem 4.4 we have

$$\rho(A_{P_n}^k) \leq \rho(A_{T}^k) \leq \rho(A_{S_n}^k),$$

where the left equality holds if and only if $T = P_n$, and the right equality holds if and only if $T = S_n$. \qed

**Theorem 4.8.** If $\alpha \neq 0$ is an eigenvalue of a $d$-regular graph $G$, then the roots of $(x - d)(x - 1)^{\frac{k-2}{2}} - \alpha = 0$ are signless Laplacian eigenvalues of $G^k$. Moreover, $\rho(Q_{G^k})$ is the largest real root of $(x - d)(x - 1)^{\frac{k-2}{2}} - d = 0$. 

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Theorem 4.10. If \( \alpha \neq 0 \) is an eigenvalue of a \( d \)-regular graph \( G \), then the roots of \( (d-x)(1-x) \frac{k-2}{k} - \alpha = 0 \) are Laplacian eigenvalues of \( G^k \).

Proof. Suppose that \( x \) is an eigenvector of the eigenvalue \( \alpha \neq 0 \) of graph \( G \). Then \( \sum_{j \in N_G(i)} x_j = \alpha x_i \) for any \( i \in V(G) \), where \( N_G(i) \) is the set of all neighbors of \( i \) in \( G \). Let \( \lambda \in \mathbb{C} \) be any number such that \((\lambda-d)(\lambda-1) \frac{k-2}{k} = \alpha \), then \( \lambda \neq 1 \). Let \( y \) be a column vector of dimension \( |V(G^k)| \) such that \( y_u = (x_u) \frac{k}{k-2} \) if \( u \in V(G) \), and \( y_u = (\lambda - 1)^{-\frac{1}{k-2}} (x_i x_j) \frac{k}{k-2} \) if \( u \in V(G^k) \setminus V(G) \) is a core vertex in the edge contains two adjacent vertices \( i, j \in V(G) \). For any \( i \in V(G) \), by \( \sum_{j \in N_G(i)} x_j = \alpha x_i \) and \((\lambda - d)(\lambda - 1) \frac{k-2}{k} = \alpha \), we have
\[
(Q_{G^k} y)_i = d(x_i) \frac{2^{(k-1)}}{k} + \sum_{j \in N_G(i)} (\lambda - 1)^{-\frac{k-2}{2}} (x_i x_j) \frac{k-2}{k} \frac{k}{k-2} (x_j) \frac{k}{k-2} = \lambda(x_i) \frac{2^{(k-1)}}{k} = \lambda(y_i) \frac{k-1}{k}.
\]
For any \( u \in V(G^k) \setminus V(G) \), we have
\[
(Q_{G^k} y)_u = (\lambda - 1)^{-\frac{1}{k-2}} (x_i x_j) \frac{k-2}{k-2} + (\lambda - 1)^{-\frac{k-2}{2}} (x_i x_j) \frac{k-2}{k} \frac{k}{k-2} (x_j) \frac{k}{k-2} = \lambda(\lambda - 1) \frac{k-2}{k} (x_i x_j) \frac{k-2}{k} = \lambda(y_u) \frac{k-1}{k}.
\]
Hence \( \lambda \) is a signless Laplacian eigenvalue of \( G^k \) with an eigenvector \( y \).

If \( G \) is connected and \( \alpha = d = \rho(A_G) \), then we can choose \( x \) as a positive eigenvector of \( \rho(A_G) \). In this case, \( y \) is a positive eigenvector of the signless Laplacian eigenvalue \( \lambda \) of \( G^k \). Lemma 4.3 implies that \( \rho(Q_{G^k}) \) is the largest real root of \((x-d)(x-1) \frac{k-2}{k} - d = 0 \) when \( G \) is connected.

If \( G \) has \( r \geq 2 \) components \( G_1, \ldots, G_r \), then
\[
\rho(Q_{G^k}) = \max \{ \rho(Q_{G_1^k}), \ldots, \rho(Q_{G_r^k}) \}.
\]
Since \( G_1, \ldots, G_r \) are connected \( d \)-regular graphs, we know that \( \rho(Q_{G_1^k}) = \rho(Q_{G_1^k}) = \cdots = \rho(Q_{G_r^k}) \) is equal to the largest real root of \((x-d)(x-1) \frac{k-2}{k} - d = 0 \).

The following corollary follows from Theorem 4.8.

Corollary 4.9. For any \( d \)-regular graph \( G \), we have \( \lim_{k \to \infty} \rho(Q_{G^k}) = d \).
Moreover, \( \rho(Q_{G^k}) \) is a strictly decreasing sequence if \( d > 1 \).

Remark. In [7, Conjecture 4.1], Hu et al conjectured that \( \rho(Q_{G^k}) \) is a strictly decreasing sequence for any graph \( G \) and even \( k \). By Corollary 4.9, this conjecture holds when \( G \) is regular of degree \( d > 1 \).

The proof of the following theorem is similar with that of Theorem 4.8. So we omit it.

Theorem 4.10. If \( \alpha \neq 0 \) is an eigenvalue of a \( d \)-regular graph \( G \), then the roots of \((d-x)(1-x) \frac{k-2}{k} - \alpha = 0 \) are Laplacian eigenvalues of \( G^k \).
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