Quantitative de Jong theorems in any dimension
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Abstract
We develop a new quantitative approach to a multidimensional version of the well-known de Jong’s central limit theorem under optimal conditions, stating that a sequence of Hoeffding degenerate U-statistics whose fourth cumulants converge to zero satisfies a CLT, as soon as a Lindeberg-Feller type condition is verified. Our approach allows one to deduce explicit (and presumably optimal) Wasserstein bounds in the case of general U-statistics of arbitrary order \(d \geq 1\). One of our main findings is that, for vectors of U-statistics satisfying de Jong’s conditions and whose covariances admit a limit, componentwise convergence systematically implies joint convergence to Gaussian: this is the first instance in which such a phenomenon is described outside the frameworks of homogeneous chaoses and of diffusive Markov semigroups.

Keywords: quantitative CLTs; de Jong’s Theorem; exchangeable pairs; Hoeffding decomposition; degenerate U-statistics; multidimensional convergence; Stein’s method.

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1 Introduction, framework and main results
1.1 Overview
Let \( \{W_n : n \geq 1\} \) be a sequence of unit variance U-statistics of order \(d \geq 1\) (not necessarily symmetric) with underlying independent data \(X_1, \ldots, X_n\), that are degenerate in the sense of Hoeffding (see Section 1.2 for formal definitions) and have a finite fourth moment. In the landmark paper [12] (see also [11]), P. de Jong proved the following remarkable fact, valid as \(n \to \infty\): if \(E[W_n^4] \to 3\) and a Lindeberg-Feller-type condition is verified, then \(W_n\) converges in distribution towards a standard Gaussian random variable \(Z\) (note that \(3 = E[Z^4]\)). This surprising result represents a drastic simplification of the method of moments and cumulants (see e.g. [41, Section A.3]), which should be

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Quantitative de Jong theorems in any dimension

contrasted with the ‘typical’ non-central asymptotic behaviour of degenerate $U$-statistics
of a fixed order $d \geq 2$ and with a fixed kernel — see e.g. [22], [62], [60], [15] or [27, Ch.
11]; it also provides a general explanation of the ubiquitous emergence of the Gaussian
distribution in geometric models where counting statistics can be naturally represented
in terms of degenerate $U$-statistics, see e.g. [26, 48, 54].

One should notice that de Jong’s central limit theorem (CLT) is a one dimensional
qualitative statement: in particular, it does not provide any meaningful information about
the rate of convergence of the law of $W_n$ towards the target Gaussian distribution. Our
aim in this paper is to use Stein’s method of exchangeable pairs, as originally developed
in Stein’s monograph [63], in order to prove new quantitative and multidimensional
versions of de Jong’s central limit theorem under minimal conditions, in the setting
of degenerate and non-symmetric $U$-statistics that do not necessarily have the form
of homogeneous sums. In particular, we are interested in characterizing the joint
convergence of those vectors of degenerate $U$-statistics, whose components verify one-
dimensional CLTs.

One of the main motivations for pursuing our goal is that the findings of [11] have
anticipated a modern and very fruitful direction of research, where tools of infinite-
dimensional calculus are used in order to deduce fourth moment theorems in the spirit of
de Jong (but, crucially, without the use of Lindeberg-Feller-type conditions) for random
variables belonging to the homogeneous chaos of some general random field. The
best-known results in this area gravitate around the main discovery of [46] (as well as its
multidimensional extension [51]), where it is proved that a sequence of normalized random
variables $\{Y_n : n \geq 1\}$, belonging to a fixed Wiener chaos of a Gaussian field,
verifies a central limit theorem (CLT) if and only if $E[Y^4_n] \to 3$. The combined use of
Malliavin calculus and Stein’s method has consequently allowed one to deduce strong
quantitative versions with explicit Berry-Esseen bounds of these results (see [40, 41]),
and it is therefore a natural question to ask whether the original CLT by de Jong can be
endowed with explicit bounds, that are comparable with those available in a Gaussian
setting.

The reader can consult the constantly updated webpage

https://sites.google.com/site/malliavinstein/home

for an overview of the emerging domain of research connected to [40, 41, 46, 51].
Among the many notable ramifications of the results of [40, 46] to which our findings
should be compared, we quote: [32, 44, 55] for results involving homogeneous sums
in the Rademacher (also called Walsh) chaos, [17, 34, 35, 49, 52, 57, 61] for the analysis
of Poissonized $U$-statistics living in the Wiener chaos associated with a Poisson
measure, [1, 5, 30, 45] for fourth moment theorems involving homogeneous sums in
a non-commutative setting, and [2, 6, 36] for results in the setting of chaotic random
variables associated with a diffusive Markov semigroup. Central and non-central quanti-
tative versions of de Jong’s results in the case of fully symmetric Poissonized $U$-statistics
can be found in [17, 20, 50].

Two sets of references are particularly relevant for the present work:

(a) In reference [43] (see also [53]) de Jong’s CLT in the special case of homogeneous
sums was studied in the framework of the powerful theory of universality and
influence functions initiated in [39]. In particular, explicit bounds were obtained
for vectors of homogeneous sums satisfying a CLT.

(b) In the already quoted reference [51], the following striking phenomenon was dis-
covered. For $r \geq 2$, let $Y^m = (Y_1^m, \ldots, Y_r^m)$, $m \geq 1$, be a sequence of random vectors
whose components live in a fixed Wiener chaos, and assume that the covariance
matrix of $Y^m$ converges to some $\Sigma \geq 0$ and that each component $Y_i^m$ verifies a
CLT; then, $Y^m$ converges in distribution towards a Gaussian vector with covariance $\Sigma$, that is: for vectors of random variables living in a fixed chaos, componentwise
convergence to Gaussian, systematically implies joint convergence. As explained
e.g. in [41, Chapter 6], such a phenomenon serves as a key stepping stone in order to deduce Gaussian approximations for general functionals of Gaussian fields.
Since then, this result has been extended (at least, partially), to the framework of
the homogeneous chaos associated with a Poisson measure (see [4, 52]), to general
vectors of homogeneous sums (see [43, Section 7] and [42]), to the free probability
setting (see [45]), as well as to the framework of Markov chaoses (see [6]).

The achievements of the present paper are twofolds:

(1) On the one hand, we will obtain a general quantitative version of the
one-dimensional de Jong CLT, displaying explicit bounds on the 1-Wasserstein distance. As
anticipated, we will do that in the full general setting of degenerate $U$-statistics
that do not necessarily have the form of homogeneous sums, and that are not
necessarily symmetric. In particular, this extends the CLTs for homogeneous sums
proved in [43], as well as the results for Poissonized and symmetric $U$-statistics
proved in [17, 34].

(2) On the other hand, we will deduce (quantitative) multidimensional versions of de
Jong theorems, showing that the crucial phenomenon observed in [51] (see the
discussion at Point (b) above) basically extends to the framework of degenerate
$U$-statistics. Our main theorems on the matter show that the case of $U$-statistics
of the same order must take into account at least one cumulant of order four —
thus echoing recent results from [6]. Our forthcoming Theorem 1.7 marks the first
instance in which the phenomenon observed in [51] is described in full generality,
outside the frameworks of homogeneous chaoses, and of the chaoses associated
with a diffusive Markov semigroup.

We will now describe our setting and our main results in more detail.

1.2 Main results, 1: univariate normal approximations

Let us fix the following setup and notation, which we essentially adopt from [12]. We
refer the reader to the classical references [25, 31, 29, 62, 64], as well as to the more
recent works [18, 19, 33, 47], for an introduction to degenerate $U$-statistics, Hoeffding
decompositions and their use in stochastic analysis.

Let $(\Omega, F, P)$ be a probability space and for an integer $n \geq 1$ let $X_1, \ldots, X_n$ be
independent random elements on this space assuming values in the respective measurable
spaces $(E_1, \mathcal{E}_1), \ldots, (E_n, \mathcal{E}_n)$. Further, assume that

$$f : \prod_{j=1}^n E_j \to \mathbb{R} \text{ is } \bigotimes_{j=1}^n \mathcal{E}_j - B(\mathbb{R}) \cdot \text{measurable}$$

and that

$$W := f(X_1, \ldots, X_n) \in L^4(P)$$

satisfies

$$\mathbb{E}[W] = 0 \quad \text{and} \quad \mathbb{E}[W^2] = 1. \quad (1.1)$$

We write

$$[n] := \{1, \ldots, n\}$$

and for $J \subseteq [n]$ we also write

$$\mathcal{F}_J := \sigma(X_j, j \in J).$$
We write
\[ W = \sum_{J \subseteq [n]} W_J \] (1.2)
to indicate the Hoeffding decomposition of \( W \). Note that this means that, for each \( J \subseteq [n] \), \( W_J \) is \( \mathcal{F}_J \)-measurable and that
\[ \mathbb{E}[W_J \mid \mathcal{F}_K] = 0, \] (1.3)
whenever \( J \nsubseteq K \). It is well-known that \( W \) admits a Hoeffding decomposition of the type (1.2), as long as \( W \in L^1(\mathbb{P}) \) and that it is almost surely unique and given by
\[ W_J = \sum_{L \subseteq J} (-1)^{|J| - |L|} \mathbb{E}[W \mid \mathcal{F}_L], \quad J \subseteq [n]. \] (1.4)

We can thus write
\[ W_J = f_J(X_j, j \in J) \] (1.5)
for some measurable functions
\[ f_J : \prod_{j \in J} E_j \to \mathbb{R}, \quad J \subseteq [n]. \]

Let us also define
\[ \sigma_J^2 := \text{Var}(W_J), \quad J \subseteq [n]. \]

One major assumption in what follows will be that, for some fixed integer \( d \in [n] \), \( W \) is a degenerate \( U \)-statistic of order \( d \) (or \( d \)-degenerate \( U \)-statistic), i.e. that the Hoeffding decomposition (1.2) has the form
\[ W = \sum_{J \in \mathcal{D}_d} W_J, \] (1.6)
where
\[ \mathcal{D}_d := \{ J \subseteq [n] : |J| = d \} \]
denotes the collection of all \( \binom{n}{d} \) \( d \)-subsets of \([n]\). Equivalently, we have \( W_K = 0 \) whenever \( K \subseteq [n] \) is such that \( |K| \neq d \). Hence, we have
\[ W = f(X_1, \ldots, X_n) = \sum_{J \in \mathcal{D}_d} f_J(X_j, j \in J). \]

The next lemma lists important properties of the Hoeffding decomposition of \( W \) which will be used without further mention.

**Lemma 1.1.** Let the above notation and definitions prevail. Then, one has the following properties:
1) Whenever \( J, K_1, \ldots, K_s \in \mathcal{D}_d \) are such that
\[ J \nsubseteq \bigcup_{i=1}^s K_i =: K \]
and \( g(W_{K_1}, \ldots, W_{K_s}) \) is square-integrable, then
\[ \mathbb{E}[W_J g(W_{K_1}, \ldots, W_{K_s})] = 0. \]

In particular, \( W_J, J \in \mathcal{D}_d \), are uncorrelated.
2) For all \( J, K \in \mathcal{D}_d \) such that \( J \neq K \) we have \( \mathbb{E}[W_J \mid \mathcal{F}_K] = 0 \).
3) For all \( J, K \in \mathcal{D}_d \) we have \( \mathbb{E}[W_J] = 0 \) and \( \mathbb{E}[W_J W_K] = \delta_{J,K} \sigma_J^2 \).
4) \( \sum_{J \in \mathcal{D}_d} \sigma_J^2 = 1 \).
5) Whenever \( J_1, \ldots, J_r, K_1, \ldots, K_s \in \mathcal{D}_d \) are such that
Wasserstein distance between the distributions of $Lip(1)$ where \( d \) is satisfied and let \( (1.1) \) holds that $E[W_J | \mathcal{F}_K] = 0$, as $J \not\subseteq K$ and, hence, by conditioning we have $$E[W_J g(W_{K_1}, \ldots, W_K)] = E[g(W_{K_1}, \ldots, W_K)E[W_J | \mathcal{F}_K]] = 0.$$ Now, Point 2) follows since it is a special case of Point 1) and also Point 3) and Point 4) are immediately implied by Point 1), in view of assumption (1.1). Finally, Point 5) follows from independence as well as the disjoint block theorem. 

Let us furthermore define the quantity
$$g^2 := g_n^2 := \max_{1 \leq j \leq n} \sum_{K \subseteq D_j, i \in K} \sigma_K^2 \tag{1.7}$$

The next result corresponds to de Jong’s celebrated (qualitative) CLT discussed in Section 1.1.

**Theorem 1.2** (See [12]). Fix $d \geq 1$, and let \( \{n_m : m \geq 1\} \) be a sequence of integers diverging to infinity. Let \( \{W_m : m \geq 1\} \) be a sequence of unit variance degenerate $U$-statistics of order $d$, such that each $W_m$ is a function of the vector of independent variables $(X_1^{(m)}, \ldots, X_n^{(m)})$. Then, as $m \to \infty$, if $E[W_m^4] \to 3$ and $g_n^2 \to 0$, one has that $W_m$ converges in distribution towards a standard Gaussian random variable.

Note that the condition $\lim_{m \to \infty} g_n^2 = 0$ guarantees that, as $m \to \infty$, the influence of each of the random variables $(X_1^{(m)}, \ldots, X_n^{(m)})$ on the total variance of $W_m$ is negligible. In fact, in the case $d = 1$ it reduces to the classical Lindeberg-Feller condition
$$\lim_{m \to \infty} \max_{1 \leq j \leq n_m} \sigma_j^2 = 0,$$
from the Lindeberg-Feller CLT (see e.g. Theorem 5.12 in [28]). Here, we wrote $\sigma_j^2$ for $\sigma_{(j)}^2$. Our first main statement provides an explicit bound in the Wasserstein distance $d_{\text{Wass}}$ for Theorem 1.2. We recall that, given two integrable random variables $X$ and $Y$, the Wasserstein distance between the distributions of $X$ and $Y$ is given by the quantity
$$d_{\text{Wass}}(X, Y) = \sup_{h \in \text{Lip}(1)} |E[h(X)] - E[h(Y)]|,$$
where Lip(1) stands for the class of 1-Lipschitz functions.

**Theorem 1.3.** As before, let $W \in L^4(\mathbb{P})$ be a degenerate $U$-statistic of order $d$ such that (1.1) is satisfied and let $Z \sim N(0, 1)$ be a standard normal random variable. Then, it holds that
$$d_{\text{Wass}}(W, Z) \leq \sqrt{\frac{2}{\pi}} \left( E[W^4] - 3 + \kappa_d g_n^2 \right)^{1/2} + \frac{2\sqrt{2}}{3} \left( 2E[W^4] - 3 \right) + 3\kappa_d g_n^2 \right)^{1/2} \leq \left( \sqrt{\frac{2}{\pi}} + \frac{4\sqrt{2}}{3} \right) \sqrt{|E[W^4] - 3|} + \sqrt{\kappa_d \left( \sqrt{\frac{2}{\pi}} + \frac{2\sqrt{2}}{\sqrt{3}} \right) g_n},$$
where $\kappa_d$ is a finite constant which only depends on $d.$
Quantitative de Jong theorems in any dimension

Recall that a degenerate $U$-statistic $W$ of order $d$ as given by (1.6) is called symmetric, if, additionally, the measurable spaces $(E_1, \mathcal{E}_1), \ldots, (E_n, \mathcal{E}_n)$ all coincide, the random variables $X_1, \ldots, X_n$ are i.i.d. and if there is a measurable kernel $g : E_1^d \to \mathbb{R}$ such that $f_J = g$ for all $J \in \mathcal{D}_d$. In this special situation, the relations

$$1 = \text{Var}(W) = \sum_{J \in \mathcal{D}_d} \mathbb{E}[g^2(X_j, j \in J)] = \binom{n}{d} \mathbb{E}[g^2(X_1, \ldots, X_d)]$$

and

$$g^2_n = \sum_{J \in \mathcal{D}_d \colon \mathbb{1}_i \notin J} \mathbb{E}[g^2(X_j, j \in J)] = \binom{n-1}{d-1} \mathbb{E}[g^2(X_1, \ldots, X_d)]$$

imply that

$$g^2_n = \frac{d}{n}.$$

Hence, we arrive at the following corollary of Theorem 1.3.

**Corollary 1.4.** Let $W \in L^4(\mathbb{P})$ be a normalized, degenerate and symmetric $U$-statistic of order $d$ and let $Z \sim N(0, 1)$ be a standard normal random variable. Then,

$$d_{\text{Wass}}(W, Z) \leq \left(\sqrt{\frac{2}{\pi}} + \frac{4}{3}\right) \sqrt{|\mathbb{E}[W^4]| - 3} + \frac{4d_{\text{Wass}}(g_4, \mathbb{P})}{\sqrt{8n}} \left(\sqrt{\frac{2}{\pi}} + \frac{2\sqrt{2}}{\sqrt{3}}\right).$$

In particular, under the assumptions of Theorem 1.2, a sequence $\{W_m : m \geq 1\}$ of degenerate and symmetric $U$-statistics of a fixed order $d$ converges in distribution to $Z \sim N(0, 1)$, whenever $\lim_{m \to \infty} \mathbb{E}[W^4_m] = 3$.

**Remark 1.5.** (a) The previous Theorem 1.3 is a complete quantitative counterpart to de Jong’s Theorem 1.2. The constant $\kappa_d$ appearing in the bound is given by $\kappa_d = C_d + 2d$, where $C_d$ is a combinatorial constant defined in Equation (4.5) below.

(b) In the context of multilinear forms in independent and standardized real-valued random variables $(X_i)_{i \in \mathbb{N}}$ considered in [43], the authors had to assume that the uniform moment condition $\sup_{i \in \mathbb{N}} \mathbb{E}[X_i^4] < \infty$ is satisfied. It is easy to check that, for homogeneous sums, this condition is in fact equivalent to the hypercontractivity condition

$$\sup_{n \in \mathbb{N}} D_n < \infty \quad \text{where} \quad D_n := \max_{J \in \mathcal{D}_d} \frac{\mathbb{E}[W^4_J]}{\sigma_J^4}.$$  

Interestingly, this condition was also assumed in the monograph [11] by de Jong who was only able to dispense with it in the later paper [12]. Note further that the bounds for multilinear forms in independent random variables with arbitrary distributions derived in [43] are stated in terms of three times differentiable test functions whose first three derivatives are uniformly bounded by a constant. Hence, our Theorem 1.3 is not only more general than the corresponding result from [43] as far as the class of random functionals dealt with is concerned but is also stated in terms of much less smooth test functions.

(c) It should be mentioned that the original proof of Theorem 1.2 in [12] applies a quantitative martingale CLT from [24] and, by carefully revising its proof, one would be able to derive a bound on the rate of convergence. This issue is also briefly addressed in the introduction of the monograph [11] but not pursued any further. The resulting rate, however, would be of a much worse order than the rate provided by Theorem 1.3. Roughly, the power $1/2$ appearing in our statements would have to be systematically replaced by the power $1/5$. Furthermore, as was shown in [23] by means of an example, the Berry-Esseen bound for martingales from [24] cannot in general be improved with respect to the rate of convergence. Consequently, the techniques used by de Jong are not capable of providing sharp error bounds for
his qualitative statement. Note that the phenomenon of generally sharp bounds on
the rate of convergence for martingale CLTs which reduce to sub-optimal bounds in
particular situations was already discovered in the paper [3]. We also stress that,
unlike our work, references [11, 12] do not contain any multidimensional statements.

Finally, we would like to mention that the paper [58] also deals with bounds on the
normal approximation of so-called degenerate weighted $U$-statistics of order $d = 2$, which
have the form

$$ U = \sum_{1 \leq i < j \leq n} w_{i,j} \psi(X_i, X_j) $$

for some vector $X = (X_1, \ldots, X_n)$ of i.i.d. random variables, some symmetric, degenerate
kernel $\psi$ and with nonnegative weights $w_{i,j}$, $1 \leq i < j \leq n$. Note that the class of
weighted $U$-statistics is strictly included in our framework, since we can define the
degenerate kernel $f_{\{i,j\}}$ corresponding to the subset $\{i,j\} \in D_2$ by $f_{\{i,j\}} = w_{i,j}\psi$, leading
to the Hoeffding components $W_{\{i,j\}} = w_{i,j}\psi(X_i, X_j)$, $1 \leq i < j \leq n$. This, of course, also
holds for arbitrary positive integers $d$. Note that, in contrast to our work, the bounds
given in [58] are expressed in terms of quantities which are related explicitly to the
kernel $\psi$ and to the weights $w_{i,j}$ rather than in terms of the fourth cumulant of $U$ and,
hence, cannot be immediately compared to ours.

1.3 Main results, II: multivariate normal approximations

In this subsection we state a new approximation theorem for the distribution of
vectors of degenerate, non-symmetric $U$-statistics by a suitable multivariate normal
distribution. In particular, we show that an analog of de Jong’s theorem 1.2 holds in any
dimension, see Theorem 1.7. Note that, in the multivariate case, even this qualitative
result relating the asymptotic normality of the vector of degenerate, non-symmetric
$U$-statistics to fourth moment conditions is completely novel.

As before, let $X_1, \ldots, X_n$ be the underlying sequence of independent random vari-
ables, let $r \in \mathbb{N}$ and for $1 \leq i \leq r$ let $W(i)$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ which is
measurable with respect to $\mathcal{F}_i = \sigma(X_1, \ldots, X_i)$ and whose Hoeffding decomposition is
given by

$$ W(i) = \sum_{J \in \mathcal{D}_{p_i}} W_J(i) $$

for some $p_i \in \mathbb{N}$, i.e. $W(i)$ is a degenerate $U$-statistic of order $p_i$. Without loss of
generality, we can assume that $p_i \leq p_k$ whenever $1 \leq i < k \leq r$. Thus, there is an
$s \in \{1, \ldots, r\}$, positive integers $r_1, \ldots, r_s$ with $1 \leq r_1 < r_2 < \ldots < r_s = r$ and integers
$1 \leq q_1 < q_2 < \ldots < q_s$ such that

$$ p_i = q_l \quad \text{for all} \quad i \in \{r_l - 1 + 1, \ldots, r_l\} \quad \text{and all} \quad l = 1, \ldots, s, $$

where we set $r_0 := 0$. We define

$$ W := (W(1), \ldots, W(r))^T $$

and assume that each $W(i) \in L^4(\mathbb{P})$ with

$$ \mathbb{E}[W(i)] = 0 \quad \text{and} \quad \text{Var}(W(i)) = \mathbb{E}[W(i)^2] = \sum_{J \in \mathcal{D}_{p_i}} \mathbb{E}[W_{J}(i)^2] = 1, \quad 1 \leq i \leq r. $$

We also let

$$ v_{i,k} := \text{Cov}(W(i), W(k)) = \mathbb{E}[W(i)W(k)], \quad 1 \leq i < k \leq r, $$

and

$$ \mathbb{V} = \mathbb{V}(W) := \text{Cov}(W) = (v_{i,k})_{1 \leq i, k \leq r}. $$
Quantitative de Jong theorems in any dimension

Note that $v_{i,i} = 1$ for $i = 1, \ldots, r$ and $|v_{i,k}| \leq 1$ for $1 \leq i, k \leq r$, by the Cauchy-Schwarz inequality. Note also that $v_{i,k} = 0$ unless $p_i = p_k$. Hence, $V$ is a block diagonal matrix. Throughout this section we denote by

$$Z = (Z(1), \ldots, Z(r))^T \sim N_r(0, V)$$

a centered Gaussian vector with covariance matrix $V$. For $1 \leq k \leq r$ and $J \in \mathcal{D}_{p_k}$ we define

$$\sigma_J(k)^2 := \text{Var}(W_J(k)) = E[W_J(k)^2] \quad \text{and} \quad \sigma_J^2 := \max_{1 \leq j \leq n} \sum_{J \in \mathcal{D}_{p_k}} \sigma_J(k)^2.$$ 

Before stating our multivariate normal approximation theorem, we have to introduce some more notation: For a vector $x = (x_1, \ldots, x_r)^T \in \mathbb{R}^r$ we denote by $\|x\|_2$ its Euclidean norm and for a matrix $A \in \mathbb{R}^{r \times r}$ we denote by $\|A\|_{\text{op}}$ the operator norm induced by the Euclidean norm, i.e.,

$$\|A\|_{\text{op}} := \sup \{\|Ax\|_2 : \|x\|_2 = 1 \}.$$ 

More generally, for a $k$-multilinear form $\psi : (\mathbb{R}^r)^k \to \mathbb{R}$, $k \in \mathbb{N}$, we define the operator norm

$$\|\psi\|_{\text{op}} := \sup \{\|\psi(u_1, \ldots, u_k)\| : u_j \in \mathbb{R}^r, \|u_j\|_2 = 1, j = 1, \ldots, k\}.$$ 

Recall that for a function $h : \mathbb{R}^r \to \mathbb{R}$, its minimum Lipschitz constant $M_1(h)$ is given by

$$M_1(h) := \sup_{x, y \neq y} \frac{|h(x) - h(y)|}{\|x - y\|_2} \in [0, \infty) \cup \{\infty\}. $$

If $h$ is differentiable, then $M_1(h) = \sup_{x \in \mathbb{R}^r} \|Dh(x)\|_{\text{op}}$. More generally, for $k \geq 1$ and a $(k - 1)$-times differentiable function $h : \mathbb{R}^r \to \mathbb{R}$ let

$$M_k(h) := \sup_{x, y \neq y} \frac{\|D^{k-1}h(x) - D^{k-1}h(y)\|_{\text{op}}}{\|x - y\|_2},$$

viewing the $(k - 1)$-th derivative $D^{k-1}h$ of $h$ at any point $x$ as a $(k - 1)$-multilinear form. Then, if $h$ is actually $k$-times differentiable, we have $M_k(h) = \sup_{x \in \mathbb{R}^r} \|D^k h(x)\|_{\text{op}}$. Having in mind this identity, we define $M_0(h) := \|h\|_{\infty}$.

Recall that, for two matrices $A, B \in \mathbb{R}^{r \times r}$, their Hilbert-Schmidt inner product is defined by

$$\langle A, B \rangle_{\text{H.S.}} := \text{Tr}(AB^T) = \text{Tr}(BA^T) = \text{Tr}(B^TA) = \sum_{i,j=1}^r a_{ij}b_{ij}.$$ 

Thus, $\langle \cdot, \cdot \rangle_{\text{H.S.}}$ is just the standard inner product on $\mathbb{R}^{r \times r} \cong \mathbb{R}^{r^2}$. The corresponding Hilbert-Schmidt norm will be denoted by $\|\cdot\|_{\text{H.S.}}$. With this notion at hand, following [8] and [38], for $k = 2$ we finally define

$$\tilde{M}_2(h) := \sup_{x \in \mathbb{R}^r} \|\text{Hess } h(x)\|_{\text{H.S.}},$$

where $\text{Hess } h$ is the Hessian matrix corresponding to $h$.

**Theorem 1.6.** There exist finite constants $C_{q_l}, 1 \leq l \leq s$, only depending on $q_l$ as well as finite constants $C_{i,k}, 1 \leq i, k \leq r$, depending on $i$ and $k$ only through $p_i$ and $p_k$ such that, with the definition

$$A := 4 \sum_{l=1}^s q_{l}^2 \frac{q_{l}}{q_{l}} \sum_{i,k=r_{i-1}+1}^{r_{i}} \left( E[W(i)^2W(k)^2] - E[Z(i)^2Z(k)^2] \right)$$

EJP 22 (2017), paper 2. http://www.imstat.org/ejp/
Quantitative de Jong theorems in any dimension

and under the above assumptions, we have the following bounds:

(i) For any \( h \in C^3(\mathbb{R}^r) \) such that \( E[|h(W)|] < \infty \) and \( E[|h(Z)|] < \infty \),

\[
|E[h(W)] - E[h(Z)]| \leq \frac{1}{4q_1} M_2(h) \sqrt{A} + \frac{\sqrt{2\pi}}{9} M_3(h) \sum_{l=1}^{s} \frac{q_l}{q_1} \sum_{i=r_1}^{r_1} \left( 2(E[W(i)^4] - 3) + 3(C_{q_l} + 2q_l) \varrho_{n,i}^2 \right)^{1/2}.
\]

(ii) If \( V \) is in addition positive definite, then for each \( h \in C^2(\mathbb{R}^r) \) such that \( E[|h(W)|] < \infty \) and \( E[|h(Z)|] < \infty \),

\[
|E[h(W)] - E[h(Z)]| \leq \frac{1}{\sqrt{2\pi q_1}} M_1(h) \|V^{-1/2}\|_{\text{op}} \sqrt{A} + \frac{\sqrt{\pi}}{6} M_2(h) \|V^{-1/2}\|_{\text{op}} \sum_{l=1}^{s} \frac{q_l}{q_1} \sum_{i=r_1}^{r_1} \left( 2(E[W(i)^4] - 3) + 3(C_{q_l} + 2q_l) \varrho_{n,i}^2 \right)^{1/2}.
\]

Fix \( r \in \mathbb{N} \). Since the class of all compactly supported, three times differentiable functions \( h \) on \( \mathbb{R}^r \) is convergence-determining, from Theorem 1.6 (i) we obtain the following statement, which is a new multidimensional extension of Theorem 1.2.

**Theorem 1.7.** Fix \( r \geq 2 \), as well as integers \( p_1, \ldots, p_r \), and let \( n_m \to \infty \), as \( m \to \infty \).

Let \( W_m := (W_m(1), \ldots, W_m(r))^T, m \geq 1 \), be a sequence of random vectors such that each \( W_m(k) \) is a centered, unit variance degenerate \( U \)-statistic of order \( p_k \), whose argument is the vector of independent random elements \((X^{(m)}_1, \ldots, X^{(m)}_{r_1})\). Furthermore, let \( \Sigma \in \mathbb{R}^{r \times r} \) be a positive semi-definite matrix with \( \Sigma(j,j) = 1 \) for \( j = 1, \ldots, r \) and denote by \( N = (N(1), \ldots, N(r))^T \sim N_r(0, \Sigma) \) a centered Gaussian vector with covariance matrix \( \Sigma \). Assume the following:

(i) The covariance matrix of \( W_m \) converges to \( \Sigma \);
(ii) As \( m \to \infty \), \( \varrho_{n,m,k}^2 \to 0 \), for every \( k = 1, \ldots, r \);
(iii) As \( m \to \infty \), \( E[W_m(k)^4] \to 3 \), for every \( k = 1, \ldots, r \);
(iv) If \( j \neq k \) but \( p_j = p_k \) then, as \( m \to \infty \), \( E[W_m(j)^2W_m(k)^2] \to E[N(j)^2N(k)^2] = 1 + (\Sigma(j,k))^2 \).

Then, as \( m \to \infty \), \( W_m \) converges in distribution to \( N \).

In the framework of the normal approximation of vectors of eigenfunctions of diffusive Markov semigroups, a condition similar to (iv) in the above statement has been recently introduced and applied in [6]. The rest of the paper is organized as follows: Section 2 contains the proof of our one-dimensional result, Section 3 focusses on our multidimensional statements, whereas Section 4 contains the detailed proofs of several technical lemmas.
2 Proof of the one-dimensional theorem

In this section we give a detailed proof of Theorem 1.3. First we review Stein’s method of exchangeable pairs for univariate normal approximation.

2.1 Stein’s method of exchangeable pairs

The exchangeable pairs approach within Stein’s method dates back to Stein’s celebrated monograph [63]. Recall that a pair \((X, X')\) of random elements on a common probability space is called exchangeable, if

\[(X, X') \overset{D}{=} (X', X).\]

In [63] C. Stein extensively illustrated the fact that a given normalized random variable \(W\) is close in distribution to \(Z \sim N(0,1)\), whenever one can construct another random variable \(W'\) on the same space such that: (i) \(W'\) is ‘close’ to \(W\) in some proper, quantifiable sense, (ii) the pair \((W, W')\) is exchangeable, (iii) the linear regression property

\[E[(W' - W) \mid W] = -\lambda W\]  \hspace{1cm} (2.1)

is satisfied for some small \(\lambda > 0\), and (iv) the conditional second moment of \(W' - W\) given \(W\) is close to its mean, the constant \(2\lambda\), in the \(L^1\) metric. For a precise statement see Theorem 2.1 below.

The range of examples to which this method can be applied was considerably extended by the work [58] by Rinott and Rotar, who proved bounds on the distance to normality under the condition that the linear regression property is only approximately satisfied, i.e. that there is some negligible remainder term \(R\) such that

\[\frac{1}{\lambda} E[(W' - W) \mid G] = -W + R\]  \hspace{1cm} (2.2)

is satisfied, where \(G\) is a sub-\(\sigma\)-field of \(F\) such that \(\sigma(W) \subseteq G\). The method of exchangeable pairs has been generalized to other absolutely continuous distributions, like the exponential (7 and [21]), the multivariate normal (8, [56] and [38]) and the Beta distribution [14]. It has also been developed for general classes of one-dimensional absolutely continuous distributions in [9], [16] and [14]. As was observed in [59], in the case of one-dimensional distributional approximation one may in general relax the exchangeability condition to the assumption that \(W\) and \(W'\) be identically distributed.

In this article we focus on the exchangeable pairs method in the context of one- and multidimensional normal approximation. The following result is a variant of Theorem 1, Lecture 3 in [63] (see also Theorem 4.9 in [10]). It slightly improves on these result with respect to the constants appearing in the bound and is also stated in terms of identically distributed random variables \(W, W'\) as opposed to exchangeable ones as well as for general sub-\(\sigma\)-fields \(G\) of \(F\) with \(\sigma(W) \subseteq G\). The proof is standard and therefore omitted from the paper. Moreover, the result is a direct consequence of Proposition 3.19 in [14] together with the best known bounds on the first two derivatives of the solution to the standard normal Stein equation for Lipschitz test functions (see e.g. Lemma 2.4 in [10]).

**Theorem 2.1.** Let \((W, W')\) be a pair of identically distributed, square-integrable random variables on \((\Omega, \mathcal{F}, P)\) such that, for some \(\lambda > 0\), (2.1) holds. Furthermore, let \(G\) be a sub-\(\sigma\)-field of \(F\) with \(\sigma(W) \subseteq G\). Then, we have the bound

\[d_{Wass}(W, Z) \leq \sqrt{\frac{2}{\pi}} \sqrt{\text{Var}\left(\frac{1}{2\lambda} E\left[(W' - W)^2 \mid G\right]\right)} + \frac{1}{3\lambda} E|W' - W|^3.\]  \hspace{1cm} (2.3)

For the proof of Theorem 1.3 we will need the following new auxiliary result about exchangeable pairs satisfying identity (2.1) which might be of independent interest.
Lemma 2.2. Let \((W, W')\) be an exchangeable pair of real-valued random variables in \(L^4(\mathbb{P})\) such that, for some \(\lambda > 0\), (2.1) is satisfied and let \(\mathcal{G}\) be a sub-\(\sigma\)-field of \(\mathcal{F}\) with \(\sigma(W) \subseteq \mathcal{G}\). Then,

\[
\frac{1}{4\lambda} \mathbb{E}[(W' - W)^4] = 3\mathbb{E}[(W' - W)^2 | \mathcal{G}] - \mathbb{E}[W^4].
\]

Proof. By exchangeability of \((W, W')\) we have

\[
\frac{1}{2} \mathbb{E}[(W' - W)^4] = \mathbb{E}[W(W - W')^3] = \mathbb{E}[W^4 - 3W^3W' + 3(WW')^2 - W(W')^3]
\]

\[
= \mathbb{E}[W^4] + 3\mathbb{E}[(WW')^2] - 4\mathbb{E}[W^3W'].
\]

(2.4)

Also, by (2.2)

\[
\mathbb{E}[W^3W'] = \mathbb{E}[W^3 \mathbb{E}[W' | \mathcal{G}]] = (1 - \lambda)\mathbb{E}[W^4]
\]

(2.5)

and

\[
\mathbb{E}[(WW')^2] = \mathbb{E}[(W' - W + W)^2 | \mathcal{G}]
\]

\[
= \mathbb{E}[W^2\mathbb{E}[(W' - W)^2 + 2W(W' - W) + W^2 | \mathcal{G}]]
\]

\[
= \mathbb{E}[W^4] - 2\lambda\mathbb{E}[W^4] + \mathbb{E}[W^2\mathbb{E}[(W' - W)^2 | \mathcal{G}]]
\]

\[
= (1 - 2\lambda)\mathbb{E}[W^4] + \mathbb{E}[W^2\mathbb{E}[(W' - W)^2 | \mathcal{G}]].
\]

(2.6)

Thus, from (2.4), (2.5) and (2.6) we obtain that

\[
\frac{1}{2} \mathbb{E}[(W' - W)^4] = \left(1 + 3(1 - 2\lambda) - 4(1 - \lambda)\right)\mathbb{E}[W^4] + 3\mathbb{E}[W^2\mathbb{E}[(W' - W)^2 | \mathcal{G}]]
\]

\[
= 3\mathbb{E}[W^2\mathbb{E}[(W' - W)^2 | \mathcal{G}]] - 2\lambda\mathbb{E}[W^4],
\]

proving the lemma.

\[\square\]

2.2 Proof of Theorem 1.3

Let \(W \in L^4(\mathbb{P})\) be as in Theorem 1.3 such that its Hoeffding decomposition is given by (1.6). We are going to apply Theorem 2.1 to the \(\sigma\)-field \(\mathcal{G} = \sigma(X_1, \ldots, X_n)\) and to the exchangeable pair \((W, W')\) which is constructed as follows: Let \(Y := (Y_j)_{1 \leq j \leq n}\) be an independent copy of \(X := (X_j)_{1 \leq j \leq n}\) and let \(\alpha\) be uniformly distributed on \(\{1, \ldots, n\}\) such that \(X, Y\) and \(\alpha\) are jointly independent. Letting, for \(j = 1, \ldots, n\),

\[
X'_j := \begin{cases} Y_j, & \text{if } \alpha = j \\ X_j, & \text{if } \alpha \neq j \end{cases}
\]

and

\[
X' := (X'_1, \ldots, X'_n)
\]

it is easy to see that the pair \((X, X')\) is exchangeable. Finally, as exchangeability is preserved under functions, defining

\[
W' := f(X'_1, \ldots, X'_n) = \sum_{j=1}^n 1_{\{\alpha = j\}} \left( \sum_{J \in D_j: \text{j.g.} J} \sum_{j \in J} W_j^{(J)} \right)
\]

\[
= \sum_{J \in D_j: \text{g.o.} J} W_j + \sum_{J \in D_j: \text{j.g.} J} W_j^{(\alpha)},
\]

also the pair \((W, W')\) is exchangeable. Here, for \(J = \{j_1, \ldots, j_d\} \in D_d\) with \(1 \leq j_1 < j_2 < \ldots < j_d \leq n\) and \(j = j_k \in J\), we write
Quantitative de Jong theorems in any dimension

\[ W_j^{(i)} := f_j(X_{j_1}, \ldots, X_{j_{k-1}}, Y_{j_k}, X_{j_{k+1}}, \ldots, X_{j_n}), \]

where the kernel \( f_j \) is given by (1.5). We now show that the pair \((W, W')\) satisfies Stein’s linear regression property (2.1) exactly with coefficient \( \lambda = d/n \).

**Lemma 2.3.** With the above definitions, we have

\[ \mathbb{E}[W' - W \mid W] = \mathbb{E}[W' - W \mid X] = -\frac{d}{n} W. \]

**Proof.** It suffices to prove the second equality. Note that

\[ W' - W = \sum_{j=1}^{n} \mathbb{1}_{\{\alpha = j\}} \sum_{J \in \mathcal{D}_d; j \in J} (W_j^{(j)} - W_j) = \sum_{J \in \mathcal{D}_d; j \in J} (W_j^{(\alpha)} - W_j). \]

Hence, by independence,

\[ \mathbb{E}[W' - W \mid X] = \frac{1}{n} \sum_{j=1}^{n} \sum_{J \in \mathcal{D}_d; j \in J} \left( \mathbb{E}[W_j^{(j)} \mid X] - W_j \right) \]

\[ = \frac{1}{n} \sum_{j=1}^{n} \sum_{J \in \mathcal{D}_d; j \in J} \left( \mathbb{E}[W_j^{(j)} \mid X_i, i \in J \setminus \{j\}] - W_j \right) \]

\[ = \frac{1}{n} \sum_{j=1}^{n} \sum_{J \in \mathcal{D}_d; j \in J} \left( \mathbb{E}[W_j \mid \mathcal{F}_{J \setminus \{j\}}] - W_j \right) \]

\[ = -\frac{1}{n} \sum_{j=1}^{n} \sum_{J \in \mathcal{D}_d; j \in J} W_j = -\frac{1}{n} \sum_{J \in \mathcal{D}_d; j \in J} \sum_{j \in J} W_j \]

\[ = -\frac{d}{n} \sum_{J \in \mathcal{D}_d} W_j = -\frac{d}{n} W. \]

Here, we have used the defining property of the Hoeffding decomposition to obtain the fourth equality. \( \square \)

We would like to mention that the same construction of the exchangeable pair \((W, W')\) was used in [58] in the situation of weighted \( U \)-statistics. They also noted the validity of (2.1) with \( \lambda = d/n \) in the special case of completely degenerate weighted \( U \)-statistics of order \( d \).

In order to apply (2.3), by Lemma 2.3, we thus have to compute an upper bound on the variance of \( \frac{1}{d^2} \mathbb{E}[(W' - W)^2 \mid X] \). This is done by finding the Hoeffding decomposition of this quantity in terms of the Hoeffding decomposition of \( W^2 \) for which we will now find a new convenient expression. More generally, we derive a formula for the Hoeffding decomposition of the product of two degenerate \( U \)-statistics, which will also be needed for the proof of Theorem 1.6.

Assume that \( 1 \leq p, q \leq n \) and that \( W \) and \( V \) are square-integrable \( p \)- and \( q \)-degenerate \( U \)-statistics with respect to the same underlying sequence \( X \), respectively, with Hoeffding decompositions

\[ W = \sum_{J \in \mathcal{D}_d} W_J \quad \text{and} \quad V = \sum_{K \in \mathcal{D}_d} V_K. \]  (2.7)

The product \( U := VW \) in general is not a degenerate \( U \)-statistic, but it clearly has a Hoeffding decomposition of the form

\[ U = \sum_{M \subseteq [n]} U_M = \sum_{M \subseteq [n]; |M| \leq p+q} U_M. \]  (2.8)

The following simple observation will be crucial for the computation of the Hoeffding decompositions of both \( VW \) and of the quantity \( \frac{1}{d^2} \mathbb{E}[(W' - W)^2 \mid X] \).
Quantitative de Jong theorems in any dimension

Lemma 2.4. If $L \subseteq [n]$ is such that $J \Delta K = (J \setminus K) \cup (K \setminus J) \not\subseteq L$, then

$$\mathbb{E}[W_j V_K \mid F_L] = 0.$$  

Proof. Assume e.g. that $(J \setminus K) \setminus L = J \setminus (K \cup L) \not\subseteq \emptyset$. Then,

$$\mathbb{E}[W_j V_K \mid F_L] = \mathbb{E}[V_K \mathbb{E}[W_j \mid F_K \cup L] \mid F_L] = \mathbb{E}[V_K \cdot 0 \mid F_L] = 0,$$

as $\mathbb{E}[W_j \mid F_K \cup L] = 0$ because $J \not\subseteq K \cup L$. \qed

Lemma 2.5. Let $J \in D_p$ and $K \in D_q$, respectively.

(a) The Hoeffding decomposition of $W_j V_K$ is given by

$$W_j V_K = \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq M \subseteq J \cup K} (-1)^{|M| - |L|} \mathbb{E}[W_j V_K \mid F_L]. \quad (2.9)$$

(b) If $j \in J \cap K$, then we have the Hoeffding decomposition

$$\mathbb{E}[W_j V_K \mid F_{(J\cup K) \setminus \{j\}}] = \sum_{M \subseteq [n]} \sum_{L \subseteq [n]} (-1)^{|M| - |L|} \mathbb{E}[W_j V_K \mid F_L]. \quad (2.10)$$

Proof. The claim of (a) follows immediately from Lemma 2.4 and from the general formula for the Hoeffding decomposition of an $F_{J \cup K}$-measurable random variable $T$ which is given by

$$T = \sum_{M \subseteq [n]} \left( \sum_{L \subseteq [n]} (-1)^{|M| - |L|} \mathbb{E}[T \mid F_L] \right).$$

The claim of (b) follows similarly upon observing that, for $L \subseteq (J \cup K) \setminus \{j\}$ we have

$$\mathbb{E}\left[ \mathbb{E}[W_j V_K \mid F_{(J\cup K) \setminus \{j\}}] \mid F_L \right] = \mathbb{E}[W_j V_K \mid F_L]. \quad \Box$$

The next result which might be of independent interest plays a similar role as the product formula for two multiple Wiener-Itô integrals (see e.g. [41]).

Theorem 2.6 (Product formula for degenerate $U$-statistics). Let $1 \leq p, q \leq n$ and let $W, V \in L^2(\mathbb{P})$ be $p$- and $q$-degenerate $U$-statistics, respectively, with respective Hoeffding decompositions given by (2.7). Then, the Hoeffding decomposition (2.8) of $U := VW$ is given by the following formula:

$$VW = \sum_{M \subseteq [n]} \left( \sum_{J \in D_p, K \in D_q} \sum_{J \Delta K \subseteq M \subseteq J \cup K} (-1)^{|M| - |L|} \mathbb{E}[W_j V_K \mid F_L] \right) = \sum_{M \subseteq [n]} \left( \sum_{L \subseteq M} (-1)^{|M| - |L|} \sum_{J \in D_p, K \in D_q} \mathbb{E}[W_j V_K \mid F_L] \right),$$

i.e. for $M \subseteq [n]$ with $|M| \leq p + q$ we have

$$U_M = \sum_{J \in D_p, K \in D_q} \sum_{J \Delta K \subseteq M \subseteq J \cup K} (-1)^{|M| - |L|} \mathbb{E}[W_j V_K \mid F_L]$$

$$= \sum_{L \subseteq M} (-1)^{|M| - |L|} \sum_{J \in D_p, K \in D_q} \mathbb{E}[W_j V_K \mid F_L].$$
Quantitative de Jong theorems in any dimension

Proof. By the linearity of the Hoeffding decomposition and since we have
\[ VW = \sum_{J \in D_p, K \in D_q} W_J V_K, \]
it suffices to collect the terms resulting from the Hoeffding decompositions of the summands \( W_J V_K \) in a suitable way. By Lemma 2.5 (a) we have
\[
VW = \sum_{J \in D_p, K \in D_q} W_J V_K = \sum_{J \in D_p, K \in D_q} \sum_{M \subseteq [n]: |M| \leq p+q} \sum_{L \subseteq M: J \setminus K \subseteq L} \sum_{J \Delta L \subseteq M} (-1)^{|M| - |L|} E[W_J V_K | F_L]
\]
\[ = \sum_{M \subseteq [n]: |M| \leq p+q} \left( \sum_{L \subseteq M} (-1)^{|M| - |L|} \sum_{J \in D_p, K \in D_q: J \Delta L \subseteq M} E[W_J V_K | F_L] \right). \]

Now we are in the position to express the Hoeffding decomposition of \( \frac{n}{2d} E[(W' - W)^2 | X] \) in terms of that of \( W^2 \). Since we prove a more general result, Lemma 3.3 below, we do not give its proof, here.

Lemma 2.7. Let \( W^2 = \sum_{|M| \leq 2d} U_M \) be the Hoeffding decomposition of \( W^2 \). Then, we have the Hoeffding decomposition
\[
\frac{n}{2d} E[(W' - W)^2 | X] = \sum_{M \subseteq [n]: |M| \leq 2d - 1} a_M U_M,
\]
with
\[ a_M = 1 - \frac{|M|}{2d} \in [0, 1] \text{ for each } M \subseteq [n] \text{ with } |M| \leq 2d. \]

Before we proceed, let us, following [11] and [12], introduce the following important classes of quadruples \((J_1, J_2, J_3, J_4) \in D^4_d\). We call an element \( j \in J_1 \cup J_2 \cup J_3 \cup J_4 \) a free index, if it appears in \( J_i \) for exactly one \( i \in \{1, 2, 3, 4\} \). Note that this implies that
\[
E[W_{J_1} W_{J_2} W_{J_3} W_{J_4}] = 0 \quad (2.11)
\]
by Lemma 1.1 4). We say that \((J_1, J_2, J_3, J_4) \) is bifold, if each element in the union \( J_1 \cup J_2 \cup J_3 \cup J_4 \) appears in \( J_i \) for exactly two values of \( i \in \{1, 2, 3, 4\} \), i.e. if
\[ 1_{J_1} + 1_{J_2} + 1_{J_3} + 1_{J_4} = 2 \cdot 1_{J_1 \cup J_2 \cup J_3 \cup J_4} . \]

Let us denote by \( B = B_d \) the set of all bifold quadruples. Among the bifold quadruples, the most important ones are given by the subclass \( S_0 \) which is defined by
\[ S_0 = \left\{ (J, K, L, M) \in D^4_d : J \cap K = L \cap M = \emptyset, \emptyset \nsubseteq J \cap L = J \setminus (J \cap M) \nsubseteq J \right\} . \]

Further, we denote by \( T = T_d \) the set of all quadruples \((J_1, J_2, J_3, J_4) \in D^4_d\) that are neither bifold nor have a free index. This just means that
\[ 1_{J_1} + 1_{J_2} + 1_{J_3} + 1_{J_4} \geq 2 \cdot 1_{J_1 \cup J_2 \cup J_3 \cup J_4} \]
and there exists at least one \( j \in [n] \) such that
\[ 1_{J_1}(j) + 1_{J_2}(j) + 1_{J_3}(j) + 1_{J_4}(j) \geq 3, \]
i.e. each element of the union \( J_1 \cup J_2 \cup J_3 \cup J_4 \) appears in \( J_i \) for at least two values of \( i \in \{1, 2, 3, 4\} \) and there is an element of the union \( J_1 \cup J_2 \cup J_3 \cup J_4 \) that appears in \( J_i \) for at least three values of \( i \in \{1, 2, 3, 4\} \).
Following [12] let us define the quantities
\[
S_0 := \sum_{J,K,L,M \in D_d : J \cap K = \emptyset = L \cap M, \emptyset \subseteq J \cap L \subseteq J, \emptyset \subseteq J \cap M \subseteq J} E[W_J W_K W_L W_M],
\]
as well as
\[
\tau := \tau_d := \sum_{(J,K,L,M) \in T} \sigma_J \sigma_K \sigma_L \sigma_M.
\]
Note that the last identity in the definition of \(S_0\) is true by virtue of (2.11). The following result is Proposition 5 (b) of [12]. We will prove a more general version stated as Proposition 3.5 to deal with the multivariate case.

**Proposition 2.8.** We have
\[
S_0 \geq -\tau.
\]

Recall the definition of the Lindeberg-Feller quantity \(\rho = \rho_n\) given in (1.7). Next, we state a substantial improvement of Lemma B in [12]. Indeed, there the upper bound on \(\tau\) is of order \(\rho\) as compared to the order \(\rho^2\) which we obtain. Its proof is deferred to Section 4.

**Proposition 2.9.** For each \(d \in \mathbb{N}\) there is a finite constant \(C_d\) which is independent of \(n\) such that
\[
\tau = \sum_{(J,K,L,M) \in T} \sigma_J \sigma_K \sigma_L \sigma_M \leq C_d \rho^2.
\]
Furthermore, we can let \(C_2 = 13\).

The next two lemmas will be very useful for what follows.

**Lemma 2.10.** Again, let \(W^2 = \sum_{|M| \leq 2d} U_M\) denote the Hoeffding decomposition of \(W^2\). Then, we have the bound
\[
\sum_{M \subseteq [n] : |M| \leq 2d-1} \text{Var}(U_M) \leq E[W^4] - 3 + \kappa_d \rho^2,
\]
where \(\kappa_d = C_d + 2d\) and \(C_d\) is the constant from Proposition 2.9.

**Proof.** We have
\[
\sum_{M \subseteq [n] : |M| \leq 2d-1} \text{Var}(U_M) = \text{Var}(W^2) - \sum_{M \subseteq [n] : |M| = 2d} \text{Var}(U_M)
= E[W^4] - 1 - \sum_{M \subseteq [n] : |M| = 2d} \text{Var}(U_M)
= E[W^4] - 2 - \sum_{M \subseteq [n] : |M| = 2d} E[U_M^2]
= E[W^4] - 3 + \left(2 - \sum_{J,K,L,M \in D_d : J \cap K = \emptyset = L \cap M} E[W_J W_K W_L W_M]\right) \tag{2.12}
\]
For the last equality we have used the fact that for \(|M| = 2d\) we have
\[
U_M = \sum_{J,K \in D_d : J \cap K = \emptyset, J, K = M} W_J W_K.
\]

Also, we can write

EJP 22 (2017), paper 2.  
http://www.imstat.org/ejp/
Quantitative de Jong theorems in any dimension

\[ \sum_{J, K, L, M \in D_d \cap K = \emptyset = L \cap M} \sigma_J^2 \sigma_K^2 + S_0 \]

where we have used Lemma 4.1 to obtain the last inequality. Thus, from (2.12) and Propositions 2.8 and 2.9 we conclude that

\[ \sum_{M \subseteq \{n\} : |M| \leq 2d - 1} \text{Var}(U_M) = \text{Var}(W^2) - \sum_{M \subseteq \{n\} : |M| = 2d} \text{Var}(U_M) \]

\[ \leq E[W^4] - 3 + 2d \rho^2 - S_0 \]

\[ \leq E[W^4] - 3 + (2d + C_d) \rho^2 \]

\[ = E[W^4] - 3 + \kappa_d \rho^2 , \]

which proves the claim.

Now we are able to bound the first term on the right hand side of (2.3).

**Lemma 2.11.** For the above constructed exchangeable pair we have

\[ \frac{n}{2d} E[(W' - W)^2 \mid X] \leq E[W^4] - 3 + \kappa_d \rho^2 . \]  

**Proof.** Using the orthogonality of the summands within the Hoeffding decomposition as well as \( a_M \in [0, 1], |M| \leq 2d - 1 \), from Lemma 2.7 we obtain that

\[ \text{Var}\left( \frac{n}{2d} E[(W' - W)^2 \mid X] \right) = \text{Var}\left( \sum_{M \subseteq \{n\} : |M| \leq 2d - 1} a_M U_M \right) = \sum_{M \subseteq \{n\} : |M| \leq 2d - 1} a_M^2 \text{Var}(U_M) \]

\[ \leq \sum_{M \subseteq \{n\} : |M| \leq 2d - 1} \text{Var}(U_M) \]

\[ \leq E[W^4] - 3 + \kappa_d \rho^2 , \]

where the final inequality is by Lemma 2.10.

Now we proceed to bounding the second error term appearing in the bound (2.3) from Theorem 2.1. The next lemma will be crucial for doing this.

**Lemma 2.12.** For the above constructed exchangeable pair we have the bound

\[ \frac{n}{4d} E[(W' - W)^4] \leq 2(E[W^4] - 3) + 3 \kappa_d \rho^2 . \]

**Proof.** From Lemmas 2.3, 2.2 and 2.7 we obtain that

\[ \frac{n}{4d} E[(W' - W)^4] = 3 E\left[ W^2 \frac{n}{2d} E[(W' - W)^2 \mid X] \right] - E[W^4] \]

\[ = 3 \sum_{M, N \subseteq \{n\} : |M|, |N| \leq 2d} a_M a_N E[U_M U_N] - E[W^4] , \]
where we recall that $a_M = 1 - \frac{|M|}{2d} \in [0, 1]$, for all $M \subseteq [n]$ such that $|M| \leq 2d$. Noting that $a_\emptyset = 1$, $a_M = 0$ whenever $|M| = 2d$, and using the orthogonality of the Hoeffding decomposition yield

$$n \frac{a}{4d} E[(W' - W)^4] = 3a_\emptyset U_\emptyset^2 - E[W^4] + 3 \sum_{M \subseteq [n]; 1 \leq |M| \leq 2d-1} a_M \text{ Var}(U_M)$$

$$\leq 3 - E[W^4] + 3 \sum_{M \subseteq [n]; 1 \leq |M| \leq 2d-1} \text{ Var}(U_M)$$

$$\leq 3 - E[W^4] + 3(E[W^4] - 3 + \kappa d \theta^2)$$

$$= 2(E[W^4] - 3) + 3 \kappa d \theta^2,$$

where we have used Lemma 2.10 to obtain the last inequality.

From the fact that

$$\frac{2d}{n} = 2\lambda = E[(W' - W)^2]$$

and using the Cauchy-Schwarz inequality we obtain

$$\frac{1}{3\lambda} E[|W' - W|^3] \leq \frac{n}{3d} E[(W' - W)^2]^{1/2} (E[W' - W^4])^{1/2}$$

$$= \frac{2\sqrt{2}}{3} \left( \frac{n}{4d} E[(W' - W)^4] \right)^{1/2}. \quad (2.15)$$

Hence, by virtue of Lemma 2.12 we have

$$\frac{1}{3\lambda} E[|W' - W|^3] \leq \frac{2\sqrt{2}}{3} \left( 2(E[W^4] - 3) + 3 \kappa d \theta^2 \right)^{1/2}. \quad (2.16)$$

Theorem 1.3 now follows from (2.3), Lemma 2.11 and from (2.16).

3 Proof of the multidimensional theorem

3.1 Stein’s method of exchangeable pairs for multivariate normal approximation

Although the exchangeable pairs coupling lies at the heart of univariate normal approximation by Stein’s method, it was only in 2008 in [8] that the problem of developing an analogous technique in the multivariate setting was finally attacked. In their work, for a given random vector

$$W = (W(1), \ldots, W(r))^T,$$

the authors assume the existence of another random vector

$$W' = (W'(1), \ldots, W'(r))^T,$$

defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $W'$ has the same distribution as $W$ and such that the linear regression property

$$E[W' - W | W] = -\lambda W$$

is satisfied for some positive constant $\lambda$. Under these assumptions the authors prove several theorems which bound the distance from $W$ to a standard normal random vector in terms of the pair $(W, W')$. 

EJP 22 (2017), paper 2.  
http://www.imstat.org/ejp/
In [56] the authors motivate and investigate the more general linear regression property
\[
E[W' - W \mid G] = -\Lambda W + R, \tag{3.1}
\]
where now \( \Lambda \) is an invertible non-random \( r \times r \) matrix, \( G \subseteq \mathcal{F} \) is a sub-\( \sigma \)-field of \( \mathcal{F} \) such that \( \sigma(W) \subseteq G \) and \( R = (R(1), \ldots, R(r))^T \) is a small remainder term. However, in contrast to [8] and to the univariate situation presented in Subsection 2.1, in [56] the full strength of the exchangeability of the vector \((W, W')\) is needed. Finally, in [38] the two approaches from [8] and [56] are combined, allowing for the more general linear regression property from [56] and using sharper coordinate-free bounds on the solution to the Stein equation similar to those derived in [8]. The following result, quoted from [13], is a version of Theorem 3 in [38] but with better constants.

**Theorem 3.1.** Let \((W, W')\) be an exchangeable pair of \( \mathbb{R}^r \)-valued \( L^2(\mathbb{P}) \) random vectors defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( G \subseteq \mathcal{F} \) be a sub-\( \sigma \)-field of \( \mathcal{F} \) such that \( \sigma(W) \subseteq G \). Suppose there exist a non-random invertible matrix \( \Lambda \in \mathbb{R}^{r \times r} \), a non-random positive semidefinite matrix \( \Sigma \), a \( G \)-measurable random vector \( R \) and a \( G \)-measurable random matrix \( S \) such that (3.1) and
\[
E[(W' - W)(W' - W)^T \mid G] = 2\Lambda \Sigma + S \tag{3.2}
\]
hold true. Finally, denote by \( Z \) a centered \( r \)-dimensional Gaussian vector with covariance matrix \( \Sigma \).

(a) For any \( h \in C^3(\mathbb{R}^r) \) such that \( E[|h(W)|] < \infty \) and \( E[|h(Z)|] < \infty \),
\[
|E[h(W)] - E[h(Z)]| \leq \|\Lambda^{-1}\|_{\text{op}} \left( M_1(h) E[\|R\|_2] + \frac{1}{4} M_2(h) E[\|S\|_{\text{H.S.}}] \right) + \frac{1}{18} M_3(h) E[\|W' - W\|_2^2]
\]
\[
\leq \|\Lambda^{-1}\|_{\text{op}} \left( M_1(h) E[\|R\|_2] + \frac{\sqrt{r}}{4} M_2(h) E[\|S\|_{\text{H.S.}}] \right) + \frac{1}{18} M_3(h) E[\|W' - W\|_2^2].
\]

(b) If \( \Sigma \) is actually positive definite, then for each \( h \in C^3(\mathbb{R}^r) \) such that \( E[|h(W)|] < \infty \) and \( E[|h(Z)|] < \infty \) we have
\[
|E[h(W)] - E[h(Z)]| \leq M_1(h) \|\Lambda^{-1}\|_{\text{op}} \left( E[\|R\|_2] + \frac{\|\Sigma^{-1/2}\|_{\text{op}} E[\|S\|_{\text{H.S.}}]}{\sqrt{2\pi}} \right) + \frac{\sqrt{2\pi}}{24} M_2(h) \|\Lambda^{-1}\|_{\text{op}} \|\Sigma^{-1/2}\|_{\text{op}} E[\|W' - W\|_2^2].
\]

### 3.2 Proof of Theorem 1.6

Recall the notation and assumptions from Subsection 1.3. Starting from the random vector \( W = (W(1), \ldots, W(r))^T \) we will construct another vector
\[
W' := (W'(1), \ldots, W'(r))^T
\]
such that \((W, W')\) is an exchangeable pair in the following way: For each \( 1 \leq i \leq r \) we construct \( W'(i) \) in the same way as we did in the one-dimensional situation treated in Subsection 2.2 and from the same independent copy \( Y = (Y_1, \ldots, Y_n) \) of \( X = (X_1, \ldots, X_n) \) and the same \( \alpha \) which is independent of \((X, Y)\) and uniformly distributed on \([n] \). We will apply Theorem 3.1 with \( \Sigma = V \) and \( G = \sigma(X_1, \ldots, X_n) \).
Lemma 3.2. With the above definitions and notation we have
\[ E[W' - W \mid X] = -\Lambda W, \]
where the matrix \( \Lambda \) is given by \( \Lambda = \text{diag}\left( \frac{p_1}{n}, \ldots, \frac{p_r}{n} \right) \).

Proof. This follows immediately from Lemma 2.3.

Hence, we obtain that
\[ \| \Lambda^{-1} \|_{op} = \max_{i=1, \ldots, r} \frac{n}{p_i} = \frac{n}{p_i}. \] (3.3)

Let us define the random matrix \( S = (S_{i,k})_{1 \leq i, k \leq r} \) by the relation
\[ E[(W' - W)(W' - W)^T \mid X] = 2\Lambda V + S. \] (3.4)

From Lemma 3.2 and the fact that \( v_{i,k} = 0 \) unless \( p_i = p_k \) we easily conclude that \( S \) is symmetric. Also, using exchangeability, it is readily checked that
\[ E[S] = E[(W' - W)(W' - W)^T] - 2\Lambda V = 0. \] (3.5)

Lemma 3.3. Let \( 1 \leq i \leq k \leq r \) and let
\[ W(i)W(k) = \sum_{M \subseteq [n]: |M| \leq p_i + p_k} U_M(i, k) \]
be the Hoeffding decomposition of \( W(i)W(k) \). Then, we have the Hoeffding decomposition
\[ nE[(W'(i) - W(i))(W'(k) - W(k)) \mid X] = \sum_{M \subseteq [n]: |M| \leq p_i + p_k - 1} a_M(i, k)U_M(i, k), \]
where
\[ a_M(i, k) = p_i + p_k - |M|. \]

Proof. First note that we have the representation
\[ (W'(i) - W(i))(W'(k) - W(k)) \]
\[ = \sum_{j=1}^n 1_{\{a=j\}} \sum_{J \in D_{p_j}, K \in D_{p_k}: j \in J \cap K} (W_j'(i) - W_j(i))(W_K'(k) - W_K(k)) \]
\[ = \sum_{j=1}^n 1_{\{a=j\}} \sum_{J \in D_{p_j}, K \in D_{p_k}: j \in J \cap K} \left( W_j'(i)W_K'(k) + W_j(i)W_K(k) - W_j(i)W_K'(k) - W_j'(i)W_K(k) \right) \]
which implies that
\[ nE[(W'(i) - W(i))(W'(k) - W(k)) \mid X] \]
\[ = \sum_{j=1}^n \sum_{J \in D_{p_j}, K \in D_{p_k}: j \in J \cap K} \left( E[W_j(i)W_K(k) \mid \mathcal{F}(J \cup K) \setminus \{j\}] + W_j(i)W_K(k) \right) \]
\[ = \sum_{J \in D_{p_j}, K \in D_{p_k}: J \cap K \neq \emptyset} |J \cap K|W_j(i)W_K(k) + \sum_{j=1}^n \sum_{J \in D_{p_j}, K \in D_{p_k}: j \not\in J \cap K} E[W_j(i)W_K(k) \mid \mathcal{F}(J \cup K) \setminus \{j\}] \]
Using Lemma 2.5 (b) we have

\[
\sum_{j=1}^{n} \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_{(J \cup K) \setminus \{j\}} \right]
\]

\[
= \sum_{j=1}^{n} \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq \mathcal{M}_{J \cup K} \setminus \{j\}} (-1)^{|M|-|L|} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_L \right]
\]

\[
= \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} \sum_{J \cap K \neq \emptyset} \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq \mathcal{M}_{J \cup K} \setminus \{j\}} (-1)^{|M|-|L|} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_L \right]
\]

\[
= \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} \sum_{J \cap K \neq \emptyset} \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq \mathcal{M}_{J \cup K} \setminus \{j\}} (-1)^{|M|-|L|} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_L \right]
\]

\[
\sum_{j=1}^{n} \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq \mathcal{M}_{J \cup K} \setminus \{j\}} (-1)^{|M|-|L|} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_L \right]
\]

\[
(3.6)
\]

Note that for the third equality we have used the crucial fact that

\[
(J \cup K) \setminus M = (J \cap K) \setminus M, \quad \text{whenever } J \Delta K \subseteq M
\]

which implies that

\[
|J \cap K| \setminus M = |J \cup K| - |M| \quad \text{for } J \Delta K \subseteq M \subseteq J \cup K.
\]

Also, from Lemma 2.5 (a) we obtain that

\[
\sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} |J \cap K| \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq \mathcal{M}_{J \cup K} \setminus \{j\}} (-1)^{|M|-|L|} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_L \right]
\]

\[
= \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} |J \cap K| \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq \mathcal{M}_{J \cup K} \setminus \{j\}} (-1)^{|M|-|L|} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_L \right]
\]

\[
= \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} |J \cap K| \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq \mathcal{M}_{J \cup K} \setminus \{j\}} (-1)^{|M|-|L|} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_L \right]
\]

\[
(3.7)
\]

Combining (3.6) and (3.7) we thus have

\[
n \mathbb{E} \left[ (W'(i) - W(i))(W'(k) - W(k)) \mid X \right]
\]

\[
= \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} \sum_{J \cap K \neq \emptyset} \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq \mathcal{M}_{J \cup K} \setminus \{j\}} (-1)^{|M|-|L|} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_L \right]
\]

\[
= \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} \sum_{J \cap K \neq \emptyset} \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq \mathcal{M}_{J \cup K} \setminus \{j\}} (-1)^{|M|-|L|} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_L \right]
\]

\[
= \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p} \sum_{J \cap K \neq \emptyset} \sum_{M \subseteq [n]} \sum_{J \Delta K \subseteq \mathcal{M}_{J \cup K} \setminus \{j\}} (-1)^{|M|-|L|} E \left[ W_J(i) W_K(k) \mid \mathcal{F}_L \right]
\]

\[
EJP 22 (2017), paper 2. Page 20/35
http://www.imstat.org/ejp/

\[
\]
Quantitative de Jong theorems in any dimension

\[ \sum_{M \subseteq [n]: |M| \leq p_i + p_k - 1} (p_i + p_k - |M|) \sum_{J \subseteq \mathcal{D}_{p_i}, K \subseteq \mathcal{D}_{p_k}: J \cup K \subseteq M} \sum_{L \subseteq [n]: |L| \leq p_k} (-1)^{|M|-|L|} \mathbb{E}[W_J(i)W_K(k) | J_L] \]

\[ = \sum_{M \subseteq [n]: |M| \leq p_i + p_k - 1} (p_i + p_k - |M|) \mathcal{U}_M(i, k), \]

as claimed. □

Since \( S \) is centered, from (3.4) and Lemma 3.3 we obtain that

\[ n^2 \mathbb{E}[S_{i,k}^2] = \text{Var}(nS_{i,k}) = \mathbb{V} \left( n \mathbb{E} \left[ (W'(i) - W(i))(W'(k) - W(k)) | X \right] \right) \]

\[ = \sum_{M \subseteq [n]: |M| \leq p_i + p_k - 1} (p_i + p_k - |M|)^2 \text{Var}(U_M(i, k)) \]

\[ \leq (p_i + p_k)^2 \sum_{M \subseteq [n]: |M| \leq p_i + p_k - 1} \text{Var}(U_M(i, k)) \]

\[ = (p_i + p_k)^2 \left( \text{Var}(W(i)W(k)) - \sum_{M \subseteq [n]: |M| = p_i + p_k} \mathbb{E}[U_M(i, k)]^2 \right) \]

\[ = (p_i + p_k)^2 \left( \text{Var}(W(i)W(k)) - \sum_{J,L \subseteq \mathcal{D}_{p_i}, K \subseteq \mathcal{D}_{p_k}: J \cap K = \emptyset} \mathbb{E}[W_J(i)W_K(k)W_L(i)W_M(k)] \right). \] (3.8)

For \( 1 \leq i \leq k \leq r \) define

\[ S_0(i, k) := \sum_{J,L \subseteq \mathcal{D}_{p_i}, K \subseteq \mathcal{D}_{p_k}: J \cap K = \emptyset} \mathbb{E}[W_J(i)W_K(k)W_L(i)W_M(k)]. \]

If \( p_i < p_k \), then from (3.8) we have that

\[ n^2 \mathbb{E}[S_{i,k}^2] \leq (p_i + p_k)^2 \left( \text{Var}(W(i)W(k)) - \sum_{J \subseteq \mathcal{D}_{p_i}, K \subseteq \mathcal{D}_{p_k}: J \cap K = \emptyset} \sigma_J(i)^2 \sigma_K(k)^2 - S_0(i, k) \right). \] (3.9)

Lemma 4.1 immediately yields that

\[ \sum_{J \subseteq \mathcal{D}_{p_i}, K \subseteq \mathcal{D}_{p_k}: J \cap K = \emptyset} \sigma_J(i)^2 \sigma_K(k)^2 = 1 - \sum_{J \subseteq \mathcal{D}_{p_i}, K \subseteq \mathcal{D}_{p_k}: J \cap K \neq \emptyset} \sigma_J(i)^2 \sigma_K(k)^2 \]

\[ \geq 1 - \min \left( p_i \sigma_{n,k}^2, p_k \sigma_{n,i}^2 \right). \] (3.10)

If \( p_i = p_k \), then we obtain that

\[ n^2 \mathbb{E}[S_{i,k}^2] \leq 4p_i^2 \left( \text{Var}(W(i)W(k)) - \sum_{J,K \subseteq \mathcal{D}_{p_i}: J \cap K = \emptyset} \sigma_J(i)^2 \sigma_K(k)^2 \right) \]

\[ - \sum_{J,K \subseteq \mathcal{D}_{p_i}: J \cap K = \emptyset} \mathbb{E}[W_J(i)W_J(k)] \mathbb{E}[W_K(i)W_K(k)] - S_0(i, k) \]. \] (3.11)

Similarly to Lemma 4.1 we obtain for \( p_i = p_k \) that
That, if, on the other hand,

\[ \sum_{J,K \in D_p_i; \ J \cap K \neq \emptyset} \left| \sum_{j \in J} \sum_{k \in K} \mathrm{E}[W_J(i)W_J(k)] \right| \]

\[ \leq \sum_{J \in D_p_i} \left| \sum_{j \in J} \sum_{k \in K} \sigma_J(i)^2 \right|^{1/2} \left( \sum_{K \in D_p_k} \sigma_K(k)^2 \right)^{1/2} \]

\[ \leq p_i \varrho_{n,k} \varrho_{n,i} \sum_{J \in D_p_i} \sum_{K \in D_p_k} \sigma_J(i) \sigma_J(k) \]

\[ \leq p_i \varrho_{n,k} \varrho_{n,i} \left( \sum_{J \in D_p_i} \sigma_J(i)^2 \right)^{1/2} \left( \sum_{K \in D_p_k} \sigma_K(k)^2 \right)^{1/2} \]

\[ = p_i \varrho_{n,k} \varrho_{n,i}. \]

Hence, if \( p_i = p_k \) we have that

\[ \sum_{J,K \in D_p_i; \ J \cap K = \emptyset} \mathrm{E}[W_J(i)W_J(k)]\mathrm{E}[W_K(i)W_K(k)] \]

\[ = \left( \sum_{J \in D_p_i} \mathrm{E}[W_J(i)W_J(k)] \right)^2 - \sum_{J,K \in D_p_i; \ J \cap K \neq \emptyset} \mathrm{E}[W_J(i)W_J(k)]\mathrm{E}[W_K(i)W_K(k)] \]

\[ = v_{i,k}^2 - \sum_{J,K \in D_p_i; \ J \cap K \neq \emptyset} \mathrm{E}[W_J(i)W_J(k)]\mathrm{E}[W_K(i)W_K(k)] \]

\[ \geq v_{i,k}^2 - p_i \varrho_{n,k} \varrho_{n,i}. \]  \hfill (3.12)

Note that we can write

\[ \mathrm{Var}(W(i)W(k)) = \mathrm{E}[W(i)^2W(k)^2] - \left( \mathrm{E}[W(i)W(k)] \right)^2 \]

\[ = \mathrm{Cov}(W(i)^2, W(k)^2) + \mathrm{E}[W(i)^2]\mathrm{E}[W(k)^2] - v_{i,k}^2 \]

\[ = \mathrm{Cov}(W(i)^2, W(k)^2) + 1 - v_{i,k}^2. \]  \hfill (3.13)

Hence, if \( p_i < p_k \), then, since \( v_{i,k} = 0 \), from (3.9), (3.10) and (3.13) we see that

\[ n^2 \mathrm{E}[S_{i,k}^2] \leq (p_i + p_k)^2 \left( \mathrm{Cov}(W(i)^2, W(k)^2) + \min(p_i \varrho_{n,k}^2, p_k \varrho_{n,i}^2) - S_0(i, k) \right). \]  \hfill (3.14)

If, on the other hand, \( p_i = p_k \), then from (3.11), (3.10), (3.13) and (3.12) we conclude that

\[ n^2 \mathrm{E}[S_{i,k}^2] \leq 4p_i^2 \left( \mathrm{Cov}(W(i)^2, W(k)^2) - 2v_{i,k}^2 + p_i \min(\varrho_{n,k}^2, \varrho_{n,i}^2) \right. \]

\[ + p_i \varrho_{n,k} \varrho_{n,i} - S_0(i, k) \]

\[ = 4p_i^2 \left( \mathrm{E}[W(i)^2W(k)^2] - 1 - 2v_{i,k}^2 + p_i \min(\varrho_{n,k}^2, \varrho_{n,i}^2) \right. \]

\[ + p_i \varrho_{n,k} \varrho_{n,i} - S_0(i, k) \]

\[ = 4p_i^2 \left( \mathrm{E}[W(i)^2W(k)^2] - \mathrm{E}[Z(i)^2Z(k)^2] + p_i \min(\varrho_{n,k}^2, \varrho_{n,i}^2) \right. \]

\[ + p_i \varrho_{n,k} \varrho_{n,i} - S_0(i, k) \].  \hfill (3.15)
Quantitative de Jong theorems in any dimension

For the last identity we have used the elementarily verifiable fact that
\[ E[Z(i)^2 Z(k)^2] = 1 + 2n^2 \]
for all \( 1 \leq i, k \leq r \).

For \( p_i < p_k \), by the orthogonality of the Hoeffding decomposition and by the Cauchy-Schwarz inequality we have that
\[
\text{Cov}(W(i)^2, W(k)^2) = \sum_{M,N \subseteq [n]; |M| \leq 2p_i, |N| \leq 2p_k} E[U_M(i, i)U_N(k, k)] - E[W(i)^2]E[W(k)^2] \\
= \sum_{M \subseteq [n]; 1 \leq |M| \leq 2p_i} E[U_M(i, i)U_M(k, k)] \\
\leq \sum_{M \subseteq [n]; 1 \leq |M| \leq 2p_i} \left( E[U_M(i, i)^2] \right)^{1/2} \left( E[U_M(k, k)^2] \right)^{1/2} \\
\leq \left( \sum_{M \subseteq [n]; 1 \leq |M| \leq 2p_i} \text{Var}(U_M(i, i)) \right)^{1/2} \left( \sum_{M \subseteq [n]; 1 \leq |M| \leq 2p_i} \text{Var}(U_M(k, k)) \right)^{1/2} \\
= \left( E[W(i)^4] - 1 \right)^{1/2} \left( \sum_{M \subseteq [n]; 1 \leq |M| \leq 2p_i} \text{Var}(U_M(k, k)) \right)^{1/2}.
\]
(3.16)

Since \( p_i < p_k \), by means of (3.16) we can further bound
\[
\text{Cov}(W(i)^2, W(k)^2) \leq \left( E[W(i)^4] - 1 \right)^{1/2} \left( \sum_{M \subseteq [n]; 1 \leq |M| \leq 2p_k - 1} \text{Var}(U_M(k, k)) \right)^{1/2} \\
= \left( E[W(i)^4] - 1 \right)^{1/2} \left( \text{Var}(W(k)^2) - \sum_{M \subseteq [n]; |M| = 2p_k} E[U_M(k, k)^2] \right)^{1/2} \\
\leq \left( E[W(i)^4] - 1 \right)^{1/2} \left( E[W(k)^4] - 3 + (2p_k + C_{p_k}) \varrho^2 \right)^{1/2},
\]
(3.17)
where the final inequality is true by (2.13).

From (3.14) and (3.17) and from (3.15), respectively, we thus obtain the following result.

**Lemma 3.4.** Let \( 1 \leq i \leq k \leq r \).

(i) If \( p_i < p_k \), then
\[
n^2 E[S_{i,k}^2] \leq (p_i + p_k)^2 \left[ \left( E[W(i)^4] - 1 \right)^{1/2} \left( E[W(k)^4] - 3 + (2p_k + C_{p_k}) \varrho^2 \right)^{1/2} \\
+ \min \left( p_i \varrho^2, p_k \varrho^2 \right) - S_0(i, k) \right].
\]

EJP 22 (2017), paper 2.
Quantitative de Jong theorems in any dimension

(ii) If \( p_i = p_k \), then

\[
2 n^2 E[S_{i,k}^2] \leq 4 p_i^2 \left( E[W(i)^2 W(k)^2] - E[Z(i)^2 Z(k)^2] \right) + p_i \min \left( \varrho_{n,k}, \varrho_{n,i} \right)
\]

\[
+ p_i \varrho_{n,k} \varrho_{n,i} - S_0(i, k) \right).
\]

It remains to bound the quantities \( S_0(i, k), 1 \leq i \leq k \leq r \). The concepts of free indices and bifold quadruples from Subsection 2.2 generalize in the obvious way to quadruples \((J_1, J_2, J_3, J_4) \in D_{p_1} \times D_{p_2} \times D_{p_3} \times D_{p_4} =: D_{p_1}^4 \). We denote by \( B_{i,k} \) the collection of all bifold quadruples in \( D_{p_1}^4 \). Also, we denote by \( T_{i,k} \) the set of quadruples \((J_1, J_2, J_3, J_4) \in D_{p_1}^4 \) which are neither bifold nor have a free index, i.e. which satisfy

\[
1 \leq J_1 + J_2 + J_3 + J_4 \geq 21_{J_1 \cup J_2 \cup J_3 \cup J_4}
\]

and there is an \( i, j \in [n] \) such that

\[
1 \leq J_1(i) + J_2(j) + J_3(i) + J_4(j) \geq 3.
\]

With these definitions, for \( 1 \leq i \leq k \leq r \), we define

\[
\tau_{i,k} := \sum_{(J,K,L,M) \in T_{i,k}} \sigma_j(i) \sigma_K(k) \sigma_L(i) \sigma_M(k).
\]

The next result is a generalization of Proposition 2.8.

**Proposition 3.5.** With these definitions, for \( 1 \leq i \leq k \leq r \), we have

\[
S_0(i, k) \geq -\tau_{i,k}.
\]

The proof is postponed to Section 4. It remains to obtain a bound on the quantities \( \tau_{i,k} \) in terms of \( \varrho_{n,i}^2 \) and \( \varrho_{n,k}^2 \). This is provided by the following result which generalizes Proposition 2.9. An outline of the main elements of the proof is given in Section 4.

**Proposition 3.6.** For each \( 1 \leq i, k \leq r \), there exists a finite constant \( C_{i,k} \) which depends on \( i \) and \( k \) only through \( p_i \) and \( p_k \) and which is independent of \( n \) such that

\[
\tau_{i,k} = \sum_{(J,K,L,M) \in T_{i,k}} \sigma_j(i) \sigma_K(k) \sigma_L(i) \sigma_M(k) \leq C_{i,k} \max \left( \varrho_{n,i}^2, \varrho_{n,k}^2 \right).
\]

Furthermore, we have \( C_{i,k} = C_{k,i} \).

Combining Propositions 3.5 and 3.6, we thus obtain that

\[
S_0(i, k) \geq -C_{i,k} \max \left( \varrho_{n,i}^2, \varrho_{n,k}^2 \right)
\]

for all \( 1 \leq i, k \leq r \).

Observe that, using (3.3) and the symmetry of \( S \), we can bound

\[
\lambda^{-1} ||\lambda||_E[||S||_{H.S.}] \leq \frac{1}{p_1} \left( \frac{1}{p_1} \left( \sum_{i,k=1}^{r} \sum_{l,m=1}^{s} n^2 E[S_{i,k}^2] \right) \right)^{1/2}.
\]

(3.18)

Now, using Lemma 3.4 we have

\[
\sum_{i,k=1}^{r} n^2 E[S_{i,k}^2] \leq \sum_{i,k=1}^{r} \sum_{l,m=1}^{s} \sum_{l,m=1}^{s} n^2 E[S_{i,k}^2]
\]

\[
= \sum_{i,k=1}^{r} \sum_{l,m=1}^{s} n^2 E[S_{i,k}^2] + 2 \sum_{1 \leq m \leq s} \sum_{l,m=1}^{s} \sum_{l,m=1}^{s} n^2 E[S_{i,k}^2]
\]

http://www.imstat.org/ejp/
Quantitative de Jong theorems in any dimension

\[ \sum_{i=1}^{s} 4q_i^2 \sum_{i,k=r_{i-1}+1}^{r_i} \left( E[W(i)^2W(k)^2] - E[Z(i)^2Z(k)^2] + q_i \min(\theta_{n,k}^2, \theta_{n,i}^2) + q_i \min(\theta_{n,k}^2, \theta_{n,i}^2) + C_{i,k} \max(\theta_{n,k}^2, \theta_{n,i}^2) \right) \]

\[ + 2 \sum_{1 \leq i < m \leq s} (q_i + q_m)^2 \sum_{i=r_{i-1}+1}^{r_i} \sum_{k=r_{m-1}+1}^{r_m} \left[ \left( E[W(i)^4] - 1 \right)^{1/2} \left( E[W(k)^4] - 3 + (2q_m + C_{q_m}) \theta_{n,k}^2 \right)^{1/2} \right. \]

\[ + \min(\theta_{n,k}^2, \theta_{n,i}^2) + C_{i,k} \max(\theta_{n,k}^2, \theta_{n,i}^2) \right] = A. \quad (3.20) \]

Here, the constants \( C_{q_m} \) are defined by Proposition 2.9.

Note that from Lemma 2.12 applied to the exchangeable pair \((W(i), W'(i))\) we have

\[ \frac{n}{p_i} E|W'(i) - W(i)|^4 \leq 2\left( E[W(i)^4] - 3 + 3(C_{p_i} + 2p_i) \theta_{n,i}^2 \right). \quad (3.21) \]

Using (3.3) as well as Jensen’s inequality, we obtain

\[ \|A^{-1}\|_{op} E[\|W' - W\|^2] = \frac{n}{p_i} E[\|W' - W\|^2] \]

\[ = \frac{n}{p_i} \left( \frac{1}{t} \sum_{i=1}^{r} E|W'(i) - W(i)|^2 \right)^{3/2} \]

\[ \leq \frac{n}{p_i} \left( \frac{1}{t} \sum_{i=1}^{r} E|W'(i) - W(i)|^3 \right)^{1/2} \]

\[ \leq \frac{n}{p_i} \left( \frac{1}{t} \sum_{i=1}^{r} E|W'(i) - W(i)| \right)^{1/2}. \]

Thus, by (3.21) we have

\[ \|A^{-1}\|_{op} E[\|W' - W\|^2] \]

\[ \leq \frac{n}{p_i} \left( \frac{1}{t} \sum_{i=1}^{r} \left( E|W'(i) - W(i)|^2 \right)^{1/2} \left( E|W'(i) - W(i)|^4 \right)^{1/2} \right)^{1/2} \]

\[ = 2\sqrt{2} \sum_{i=1}^{r} \frac{p_i}{p_i} \left( \frac{n}{4p_i} E|W'(i) - W(i)|^4 \right)^{1/2} \]

\[ \leq 2\sqrt{2} \sum_{i=1}^{r} \frac{p_i}{p_i} \left( 2\left( E[W(i)^4] - 3 + 3(C_{q_i} + 2q_i) \theta_{n,i}^2 \right) \right)^{1/2} \]

\[ = 2\sqrt{2} \sum_{i=1}^{r} \frac{q_i}{q_i} \sum_{i=r_{i-1}+1}^{r_i} \left( 2\left( E[W(i)^4] - 3 + 3(C_{q_i} + 2q_i) \theta_{n,i}^2 \right) \right)^{1/2}. \quad (3.22) \]

Theorem 1.6 now follows from Theorem 3.1 and from the respective bounds (3.19), (3.20) and (3.22).

4 Proofs of several technical results

Lemma 4.1. In the situation of Section 3, for all \( 1 \leq i \leq k \leq r \) we have

\[ \sum_{J \in \mathcal{T}_{p_i}, K \in \mathcal{T}_{p_k}} \sigma_J(i)^2 \sigma_K(k)^2 \leq \min \left( p_i \theta_{n,k}^2, p_k \theta_{n,i}^2 \right). \]
Quantitative de Jong theorems in any dimension

Proof. Note that we have
\[ \sigma_{J}(i)^2 \sigma_{K}(k)^2 = \sum_{J \in \mathcal{D}_p, K \in \mathcal{D}_p: J \cap K \neq \emptyset} \sigma_{J}(i)^2 \sum_{K \in \mathcal{D}_p: J \cap K \neq \emptyset} \sigma_{K}(k)^2 \leq \sum_{J \in \mathcal{D}_p} \sigma_{J}(i)^2 \sum_{K \in \mathcal{D}_p: J \cap K \neq \emptyset} \sum_{J \in J} \sigma_{K}(k)^2 \leq p_i \sigma_k^2 \sum_{J \in \mathcal{D}_p} \sigma_{J}(i)^2 = p_i \sigma_k^2. \]

The claim follows by symmetry. \(\square\)

**Lemma 4.2** (Generalization of Lemma 4 of [12]). Let \((J, K, L, M) \in \mathcal{B}_{i,k}\) be a bifold quadruple. Then, in the situation of Section 3 we have
\[ E[W_{J}W_{K}W_{L}W_{M}] = E[E[W_{J}W_{K} | \mathcal{F}_{J \Delta K}] E[W_{L}W_{M} | \mathcal{F}_{L \Delta M}]. \]

**Proof.** We repeat the short proof from [12]. By independence, we have
\[ E[W_{J}W_{K}W_{L}W_{M}] = E[E[W_{J}W_{K} | \mathcal{F}_{L \cup M}] E[W_{L}W_{M} | \mathcal{F}_{L \cup M}]. \]
\[ = E[E[W_{J}W_{K} | \mathcal{F}_{J \cup K \cap (L \cup M)}] E[W_{L}W_{M} | \mathcal{F}_{J \cup K \cap (L \cup M)}]. \]

Now, the claim follows from the fact that for a bifold quadruple \((J, K, L, M)\) the identity \((J \cup K) \cap (L \cup M) = J \Delta K = L \Delta M\) holds true. \(\square\)

**Lemma 4.3** (Generalization of Lemma 3 of [12]). In the situation of Section 3, for \(1 \leq i \leq k \leq r, J \in \mathcal{D}_p, K \in \mathcal{D}_p\) we have
\[ E\left[ \left( E[W_{J}W_{K} | \mathcal{F}_{J \Delta K}] \right)^2 \right] \leq \sigma_{K}^2(i) \sigma_{K}^2. \]

**Proof.** Again, we imitate the proof given in [12]. Using first the conditional version of the Cauchy-Schwarz inequality and then twice the independence of the underlying random variables \(X_1, \ldots, X_n\) we obtain
\[ E\left[ \left( E[W_{J}W_{K} | \mathcal{F}_{J \Delta K}] \right)^2 \right] \leq E\left[ E[W_{J}W_{K} | \mathcal{F}_{J \Delta K}] E[W_{K}^2 | \mathcal{F}_{J \Delta K}] \right] \]
\[ = E\left[ E[W_{J}^2 | \mathcal{F}_{J \Delta K}] E[W_{K} | \mathcal{F}_{J \Delta K}] \right] \]
\[ = \sigma_{K}^2(i) \sigma_{K}^2(k). \]

**Proof of Proposition 3.5.** We generalize the argument used in the proof of Proposition 5 (b) of [12]. For \(l = 1, \ldots, p_l - 1\) we have
\[ 0 \leq \sum_{C \subseteq \{n\}: |C| = p_l + p_k - 2l} \sum_{B, B' \subseteq C: |B| = p_l - 1, |B'| = p_k - 1} E \left( \sum_{J \in \mathcal{D}_p, M \in \mathcal{D}_p: J \cap M = B, M \setminus J = B'} E[W_{J}(i)W_{M}(k) | \mathcal{F}_{J \Delta M}] \right)^2. \]
Quantitative de Jong theorems in any dimension

\[ = \sum_{C \subseteq [n]} \sum_{B, B' \subseteq C: |C| = p_i + p_k - 2l, |B| = p_i - l, |B'| = p_k - l} E\left[ E\left[ W_J(i)W_M(k) \mid \mathcal{F}_{J\Delta M} \right] E\left[ W_L(i)W_K(k) \mid \mathcal{F}_{L\Delta K} \right] \right] \]

\[ + \sum_{(J,K,L,M) \in \mathcal{T}_{i,k}: J \setminus M = L \setminus K} E\left[ E\left[ W_J(i)W_M(k) \mid \mathcal{F}_{J\Delta M} \right] E\left[ W_L(i)W_K(k) \mid \mathcal{F}_{L\Delta K} \right] \right] \]

\[ \vdash: S_0(i, k, l) = S_0(i, k) \]

where we have used Lemma 4.2 to obtain the second equality. Note that

\[ \sum_{l=1}^{p_i - 1} S_0(i, k, l) = S_0(i, k) \]

because for a bifold quadruple \((J, K, L, M)\) the identity \(J \setminus M = L \setminus K\) implies that \(J \cap K = L \cap M = \emptyset\) and because we have

\[ S_0(i, k) = \sum_{(J,K,L,M) \in \mathcal{B}_{i,k}: J \setminus M = L \setminus K, \|J\| = \|L\|} E\left[ W_J(i)W_K(k) \mid \mathcal{F}_{J\Delta M} \right] E\left[ W_L(i)W_K(k) \mid \mathcal{F}_{L\Delta K} \right] \]

Further, by the Cauchy-Schwarz inequality and by Lemma 4.3, we have

\[ \left( \sum_{l=1}^{p_i - 1} R_l \right) \leq \sum_{l=1}^{p_i - 1} \left( E\left[ \left( E\left[ W_J(i)W_M(k) \mid \mathcal{F}_{J\Delta M} \right]^2 \right) \right] \right)^{1/2} \]

\[ \leq \sum_{(J,K,L,M) \in \mathcal{T}_{i,k}: J \setminus M = L \setminus K} \left( E\left[ \left( E[ W_L(i)W_K(k) \mid \mathcal{F}_{L\Delta K}] \right)^2 \right] \right)^{1/2} \]

\[ \leq \sum_{(J,K,L,M) \in \mathcal{T}_{i,k}} \sigma_J(i)\sigma_K(k)\sigma_L(i)\sigma_M(k) = \tau_{i,k} \]

Thus, the claim follows.

**Proof of Proposition 2.9.** In order to prove Proposition 2.9 let us review the following concepts and notation, introduced in [11]. For a quadruple \((J_1, J_2, J_3, J_4) \in \mathcal{D}_d^4\) write

\[ I := J_1 \cup J_2 \cup J_3 \cup J_4 = \{i_1, \ldots, i_r\} \quad \text{with} \quad 1 \leq i_1 < i_2 < \ldots < i_r \leq n \]

and define the shadow \((J'_1, J'_2, J'_3, J'_4)\) of \((J_1, J_2, J_3, J_4)\) by

\[ J'_l := \{a \in \{1, \ldots, r\} : i_a \in J_l\}, \quad 1 \leq l \leq 4. \]

Note that since we have the equivalence

\[ a \in J'_l \iff i_a \in J_l \]

the sets \(J'_l\) satisfy obvious relations like

\[ |J'_l| = |J_l| = d, \quad |J'_l \cap J'_m| = |J_l \cap J_m| \quad \text{etc.} \]

and that a quadruple \((J_1, J_2, J_3, J_4) \in \mathcal{D}_d^4\) is completely determined by its shadow and by \(I = \bigcup_{l=1}^{d} J_l\). Note also that if \((J_1, J_2, J_3, J_4) \in \mathcal{T}_d\), then we have

EJP 22 (2017), paper 2.  
http://www.imstat.org/ejp/  
Page 27/35
Quantitative de Jong theorems in any dimension

\[ J_1^r \cup J_2^r \cup J_3^r \cup J_4^r = \{1, \ldots, r\} \]

for some \( r \in \{d, d+1, \ldots, 2d-1\} \) and \( J_i \cap J_k \neq \emptyset \) for all \( i, k = 1, 2, 3, 4 \). Indeed, if, for instance, \( J_1 \cap J_2 \) were empty and \( j_0 \in J_i \) for at least three values of \( i \in \{1, 2, 3, 4\} \), then necessarily \( j_0 \in J_3 \cap J_4 \) implying \( |J_3 \cup J_4| \leq 2d - 1 \). Hence, \( J_1 \cup J_2 \not\subset J_3 \cup J_4 \) because \( |J_1 \cup J_2| = 2d \) by disjointness. Thus, \((J_1, J_2, J_3, J_4)\) has a free index and, hence, cannot be in \( \mathcal{T}_d \). By the above observation (4.1), this immediately implies that also \( J_i^r \cap J_k^r \neq \emptyset \) for all \( i, k = 1, 2, 3, 4 \).

In general, we call a quadruple of sets \( F = (F_1, F_2, F_3, F_4) \) a shadow (a \( d \)-shadow) if there is an \( r \in \{d, d+1, \ldots, 2d-1\} \) such that \( F := F_1 \cup F_2 \cup F_3 \cup F_4 = \{1, \ldots, r\} \) and \( |F_1| = d \) for \( l = 1, 2, 3, 4 \). We call \( r \) the size of the shadow \( F \). We say that the shadow \( F \) is induced by the quadruple \((J_1, J_2, J_3, J_4) \in \mathcal{D}_d^4\), if \( F = (J_1^r, J_2^r, J_3^r, J_4^r) \). We write \( F(J_1, J_2, J_3, J_4) \) for the shadow induced by \((J_1, J_2, J_3, J_4) \). If \( F' = (F_1', F_2', F_3', F_4') \) is another \( d \)-shadow with \( F' := F_1' \cup F_2' \cup F_3' \cup F_4' = \{1, \ldots, r'\} \), then we say that \( F \) and \( F' \) are equivalent and write \( F \sim F' \), if \( r = r' \) and there is a permutation \( \sigma \in \mathcal{S}_r \) such that

\[ F_1' = \sigma(F_1) \quad \text{for} \ l = 1, 2, 3, 4. \quad (4.2) \]

We denote the latter fact by \( F' = F_\sigma \). This clearly defines an equivalence relation on the set of \( d \)-shadows and we denote by \( [F]_\sim \) the equivalence class of \( F \). We further denote by \( \gamma(F) \) the number of permutations \( \sigma \in \mathcal{S}_r \) that leave \( F \) fixed in the sense that

\[ \sigma(F_l) = F_l \quad \text{for all} \ l = 1, 2, 3, 4. \quad (4.3) \]

The set of these permutations is just the stabilizer of \( F \) with respect to the natural action of \( \mathcal{S}_r \) on the set of \( d \)-shadows of size \( r \). Note that, for \( F' \sim F \), we have \( \gamma(F) = \gamma(F') \) and that \( \gamma(F) \) also gives the number of permutations \( \sigma \) such that (4.2) holds. Let us define the function \( \gamma : [n]^d \to \mathbb{R} \) by

\[ g(j_1, \ldots, j_d) := \begin{cases} \sigma_{\{j_1, \ldots, j_d\}} & \text{if } \{j_1, \ldots, j_d\} = d \\ 0 & \text{otherwise}. \end{cases} \]

Then, \( g \) is a symmetric function vanishing on the complement \( \Delta_c = [n]^d \setminus \Delta \) of

\[ \Delta := \Delta_d^{(n)} := \{ (j_1, \ldots, j_d) \in [n]^d : j_i \neq j_m \text{ whenever } l \neq m \}. \]

Further, for a shadow \( F = (F_1, F_2, F_3, F_4) \) which is induced by some quadruple \((J_1, J_2, J_3, J_4) \in \mathcal{D}_d^4\) and with \( F := F_1 \cup F_2 \cup F_3 \cup F_4 = \{1, \ldots, r\} \) and \( \pi_{F_l} \) being the natural projection \( [n]^F \to [n]^{F_l} \) given by \( (j_a)_{a \in F} \mapsto (j_a)_{a \in F_l} \), define \( G_{F_l} : [n]^F \to \mathbb{R} \) by \( G_{F_l} := g \circ \pi_{F_l} \). Here, we tacitly identify \([n]^d \) with \([n]^{F_l} \) and \([n]^d \) with \([n]^{F} \).

**Lemma 4.4.** Let \( F = (F_1, F_2, F_3, F_4) \) be a \( d \)-shadow of size \( r \) which is induced by some quadruple \((J_1, J_2, J_3, J_4) \in \mathcal{T} \). Then, we have the bound

\[
\sum_{(J,K,L,M) \in \mathcal{T}} \varepsilon(J,K,L,M) \sigma_{J} \sigma_{K} \sigma_{L} \sigma_{M} \leq \frac{d!(d-1)!}{\gamma(F)} \epsilon_n^2.
\]

**Proof.** For ease of notation, in this proof we use bold letters \( a \) to denote tuples \( a = (a_1, \ldots, a_s) \in [n]^s \), where \( s \) is some natural number. Also, for two such tuples \( a = (a_1, \ldots, a_s) \in [n]^s \) and \( b = (b_1, \ldots, b_t) \in [n]^t \) we write \( a \cap b \neq \emptyset \) if there are indices \( 1 \leq i \leq s \) and \( 1 \leq j \leq t \) such that \( a_i = b_j \), i.e. if

\[ \{a_1, \ldots, a_s\} \cap \{b_1, \ldots, b_t\} \neq \emptyset. \]
Quantitative de Jong theorems in any dimension

We begin the proof with the remark that

\[
\sum_{(i_1, \ldots, i_r) \in [n]^r} G_F(i_1, \ldots, i_r) G_F(i_1, \ldots, i_r) G_F(i_1, \ldots, i_r) G_F(i_1, \ldots, i_r) \geq \gamma(F) \sum_{(J,K,L,M) \in T: F(J,K,L,M) \in [F]_\ast} \sigma_J \sigma_K \sigma_L \sigma_M.
\]

This follows from

\[
\sum_{(i_1, \ldots, i_r) \in [n]^r} \prod_{l=1}^{1} G_F(i_1, \ldots, i_r) \geq \sum_{(i_1, \ldots, i_r) \in [n]^r} \prod_{l=1}^{4} G_F(i_1, \ldots, i_r)
\]

\[
= \sum_{1 \leq j_1 < \ldots < j_r \leq n} \sum_{\sigma \in S_r} \prod_{l=1}^{4} g(\pi_{\sigma(l)}(j_1, \ldots, j_r))
\]

\[
= \sum_{1 \leq j_1 < \ldots < j_r \leq n} \sum_{\sigma \in S_r} \prod_{l=1}^{4} g(\pi_{\sigma(l)}(j_1, \ldots, j_r))
\]

\[
= \gamma(F) \sum_{F' \in [F]_\ast} \sum_{1 \leq j_1 < \ldots < j_r \leq n} \prod_{l=1}^{4} G_F(i_1, \ldots, i_r)
\]

\[
= \gamma(F) \sum_{F' \in [F]_\ast} \sum_{\sigma \in (J,K,L,M) \in T: F(J,K,L,M) = [F]_\ast} \sigma_J \sigma_K \sigma_L \sigma_M.
\]

Here, we used the notation \([n]^r\) for the set of all tuples \((i_1, \ldots, i_r) \in [n]^r\) such that \(i_j \neq i_k\) whenever \(j \neq k\). Hence, it suffices to show that we always have the bound

\[
\sum_{(i_1, \ldots, i_r) \in [n]^r} \prod_{l=1}^{4} G_F(i_1, \ldots, i_r) \leq d!(d-1)!g_n^2
\]

if \(F\) is as in the statement of the Lemma.

We first treat the simple cases that either two or all of the sets \(F_l, l = 1, 2, 3, 4\), are equal. Note that the case of exactly three equal sets is vacuous for a quadruple in \(T\). Assume first that e.g. \(F_3 \neq F_1 = F_2 \neq F_4\). It might be that also \(F_3 = F_1\) but this is immaterial. Then, we have

\[
\sum_{(i_1, \ldots, i_r) \in [n]^r} G_F(i_1, \ldots, i_r) G_F(i_1, \ldots, i_r) G_F(i_1, \ldots, i_r) G_F(i_1, \ldots, i_r)
\]

\[
= \sum_{(i_1, \ldots, i_r) \in [n]^r} G_{F_1}^2(i_1, \ldots, i_r) G_{F_2}^2(i_1, \ldots, i_r) G_{F_3}(i_1, \ldots, i_r)
\]

\[
= \sum_{(i_1, \ldots, i_r) \in [n]^r} g(j)^2 \sum_{k \in [n]^{r_1}} G_{F_1}(j,k) G_{F_2}(j,k)
\]

\[
\leq \sum_{j \in [n]^{r_1}} g(j)^2 \left( \sum_{k \in [n]^{r_1}} G_{F_3}^2(j,k) \right)^{1/2} \left( \sum_{k \in [n]^{r_1}} G_{F_4}^2(j,k) \right)^{1/2}
\]

\[
= \sum_{j \in [n]^{r_1}} g(j)^2 \left( \sum_{l \in [n]^{r_3}: l \neq j} g(l)^2 \right)^{1/2} \left( \sum_{m \in [n]^{r_4}: m \neq j} g(m)^2 \right)^{1/2}
\]
Thus, it remains to bound the quantity $EJP$ pairwise different. Then, using the Cauchy-Schwarz inequality, we can bound this case as well as by the definition of $g_n^2$. If $F_1 = F_2 = F_3 = F_4$, then we have $r = d$ and

$$
\sum_{(i_1, \ldots, i_r) \in [n]^r} G_{F_1}(i_1, \ldots, i_r) G_{F_2}(i_1, \ldots, i_r) G_{F_3}(i_1, \ldots, i_r) G_{F_4}(i_1, \ldots, i_r)
= \sum_{(j_1, \ldots, j_d) \in [n]^d} g(j_1, \ldots, j_d)^4
\leq \sum_{j_1=1}^{n} \max_{(j_2, \ldots, j_d) \in [n]^{d-1}} g(j_1, \ldots, j_d)^2 \sum_{(k_2, \ldots, k_d) \in [n]^{d-1}} g(j_1, k_2, \ldots, k_d)^2
\leq (d-1)!^2 g_n^2 \sum_{j_1=1}^{n} \max_{(j_2, \ldots, j_d) \in [n]^{d-1}} g(j_1, \ldots, j_d)^2
\leq (d-1)!^2 g_n^2 \sum_{(j_1, \ldots, j_d) \in [n]^d} g(j_1, \ldots, j_d)^2
= d![(d-1)!^2 g_n^2].
$$

For the remainder of this proof we may thus assume that the sets $F_l$, $l = 1, 2, 3, 4$, are pairwise different. Then, using the Cauchy-Schwarz inequality, we can bound

$$
\sum_{(i_1, \ldots, i_r) \in [n]^r} G_{F_1}(i_1, \ldots, i_r) G_{F_2}(i_1, \ldots, i_r) G_{F_3}(i_1, \ldots, i_r) G_{F_4}(i_1, \ldots, i_r)
= \sum_{j \in [n]^{r_1}} g(j) \sum_{k \in [n]^{r_1 \setminus r_1}} G_{F_1}(j, k) G_{F_2}(j, k) G_{F_3}(j, k)
\leq \left( \sum_{j \in [n]^{r_1}} g(j)^2 \right)^{1/2} \left( \sum_{k \in [n]^{r_1 \setminus r_1}} \left( \sum_{j \in [n]^{r_1}} G_{F_1}(j, k) G_{F_2}(j, k) G_{F_3}(j, k) \right)^2 \right)^{1/2}
= \sqrt{d!} \left( \sum_{j \in [n]^{r_1}} \left( \sum_{k \in [n]^{r_1 \setminus r_1}} G_{F_1}(j, k) G_{F_2}(j, k) G_{F_3}(j, k) \right)^2 \right)^{1/2}
\leq (4.4)
$$

Thus, it remains to bound the quantity

$$
A := \sum_{j \in [n]^{r_1}} \left( \sum_{k \in [n]^{r_1 \setminus r_1}} G_{F_1}(j, k) G_{F_2}(j, k) G_{F_3}(j, k) \right)^2.
$$

Let us distinguish the following cases.

1) Each element in $F = F_1 \cup F_2 \cup F_3 \cup F_4$ appears in at least three of the sets $F_1, F_2, F_3, F_4$. This implies that

$$
F \setminus F_k = F_j \setminus F_k \quad \text{for all distinct } j, k \in \{2, 3, 4\}.
$$

Then, using Cauchy-Schwarz, we can bound

$$
A \leq \sum_{j \in [n]^{r_1}} \left( \sum_{k \in [n]^{r_1 \setminus r_1}} G_{F_2}^2(j, k) \sum_{k \in [n]^{r_1 \setminus r_1}} G_{F_3}^2(j, k) G_{F_4}^2(j, k) \right)
= \sum_{j \in [n]^{r_1}} \left( \sum_{l \in [n]^{r_1 \setminus r_1} : l \neq j} g(l)^2 \sum_{k \in [n]^{r_1 \setminus r_1}} G_{F_2}^2(j, k) G_{F_4}^2(j, k) \right).
$$
Quantitative de Jong theorems in any dimension

\[
\leq (d - 1)! g_n^2 \sum_{j \in [n] \setminus F} \sum_{k \in [n] \setminus F} G_F^2(j, k) G_{F_i}(j, k)
\]

\[
= (d - 1)! g_n^2 \sum_{k \in [n] \setminus F} \left( \sum_{a \in [n] \setminus F} g(a, k) \right)^2 G_{F_i}(j, a, k, l) G_{F_i}(j, a, k, l)
\]

\[
\leq (d - 1)! g_n^2 \left( \sum_{j \in [n] \setminus F} \sum_{a \in [n] \setminus F} g(a, k) \right)^2
\]

\[
\cdot \left( \sum_{j \in [n] \setminus F} \sum_{a_1 \in [n] \setminus F} G_{F_i}(j, a_1, k, l) G_{F_i}(j, a_1, k, l) \right)^2
\]

\[
= (d - 1)! g_n^2 \sum_{j \in [n] \setminus F} \sum_{a_1 \in [n] \setminus F} \sum_{a_2 \in [n] \setminus F} G_{F_i}(j, a_1, a_2, k, l) G_{F_i}(j, a_1, a_2, k, l)
\]

\[
= (d - 1)! g_n^2 \sum_{j \in [n] \setminus F} \sum_{a_1 \in [n] \setminus F} \sum_{a_2 \in [n] \setminus F} \left( \sum_{l \in [n] \setminus F} G_{F_i}(j, a_1, a_2, k, l) G_{F_i}(j, a_1, a_2, k, l) \right)^2
\]

Note that we have used the fact that

\[
(F_4 \cap F_1) \setminus F_3 = F \setminus F_3 \neq \emptyset
\]

to obtain the last inequality.

2) There is an element \( j_0 \in F = F_1 \cup F_2 \cup F_3 \cup F_4 \) which is contained in exactly two of the sets \( F_1, F_2, F_3, F_4 \). We may assume that \( j_0 \in F_1 \). We claim that then there are distinct indices \( j, k \in \{2, 3, 4\} \) such that

\[
F_1 \not\subset F_j \cup F_k.
\]

Indeed, we have

\[
(F_2 \cup F_3) \cap (F_2 \cup F_4) \cap (F_3 \cup F_4) = (F_2 \cap F_3) \cup (F_2 \cap F_4) \cup (F_3 \cap F_4)
\]

and, hence, \( j_0 \) cannot be contained in the set on the right hand side. Thus, let us assume that \( F_1 \not\subset F_3 \cup F_4 \). We obtain that

\[
A \leq \sum_{j \in [n] \setminus F} \sum_{a_1 \in [n] \setminus F} \left( \sum_{a_2 \in [n] \setminus F} \left( \sum_{k \in [n] \setminus F} G_{F_i}(j, a_1, k, l) G_{F_i}(j, a_2, k, l) \right)^2 \right)
\]

where

\[
a_2^* \in [n] \setminus F_1 \cup F_2 \setminus (F_3 \cup F_4)
\]

is arbitrary but fixed. Now note that due to the fact that \( F \) is induced by some quadruple in \( \mathcal{T} \) we have

EJP 22 (2017), paper 2. http://www.imstat.org/ejp/
Quantitative de Jong theorems in any dimension

\[ F_3 \cup F_4 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1) \cup [(F_3 \cup F_4) \cap F_1 \cap F_2] \cup [(F_3 \cup F_4) \setminus (F_1 \cap F_2)], \]

where the union on the right hand side is disjoint. Thus, the last bound becomes

\[
A \leq (d-1)! g_n^2 \sum_{m \in [n]|(F_1 \cup F_4) \cap (F_1 \cup F_2)} \sum_{p \in [n]|(F_3 \cup F_4) \cap (F_1 \cup F_2)} G_{F_2}(m, l) \\
+ (d-1)! g_n^2 \sum_{a \in [n]|(F_1 \cap F_4) \cap (F_1 \cup F_2)} \sum_{b \in [n]|(F_3 \cup F_4) \cap (F_1 \cup F_2)} \sum_{l \in [n]|(F_3 \cup F_4) \setminus (F_1 \cup F_2)} g(a, b, l)^2 \\
+ \sum_{c \in [n]|(F_1 \cup F_4) \cap (F_1 \cup F_2)} \sum_{p \in [n]|(F_3 \cup F_4) \setminus (F_1 \cup F_2)} g(a, c, p)^2 \\
\leq ((d-1)!)^2 g_n^4 \sum_{j \in [n]} j^2 \\
= d!((d-1)!)^2 g_n^4 
\]

\[
End of the proof of Proposition (2.9).\]

Let \( F_1, \ldots, F_s \) be a complete system of pairwise non-equivalent \( d \)-shadows which are induced by quadruples \((J, K, L, M) \in \mathcal{T}\). Then, clearly, \( s \) is independent of \( n \) and by Lemma 4.4 we have

\[
\tau = \sum_{(J, K, L, M) \in \mathcal{T}} \sigma_{J} \sigma_{K} \sigma_{L} \sigma_{M} = \sum_{j=1}^{s} \sum_{(J, K, L, M) \in \mathcal{T}} \sigma_{j} \sigma_{K} \sigma_{L} \sigma_{M} \\
\leq \left(d!((d-1)!)^2 \gamma(F_j)^{-1}\right) g_n^2 
\]

so that we can let

\[
C_d := d!((d-1)!)^2 \sum_{j=1}^{s} \gamma(F_j)^{-1} \tag{4.5} 
\]

which is independent of \( n \).

\[
\text{Remark 4.5. Using the fact that the equivalence class of a shadow} \ F = (F_1, F_2, F_3, F_4) \text{ is determined by the cardinalities of all finite intersections of the sets} \ F_1, F_2, F_3, F_4, \text{ one can get an upper bound on the number} \ s \text{ of all equivalence classes of shadows induced by quadruples in} \ \mathcal{T}. \text{ Using that} \ \gamma(F) \geq 1 \text{ immediately gives a crude bound on} \ C_d. \text{ It is not difficult to verify that} \ C_d = 13 \text{ by distinguishing all possible cases. Furthermore, by some clever combinatorial argument, it might be possible to compute sharp bounds on} \ C_d \text{ starting from (4.5). This would be of great interest for deriving limit theorems in situations where} \ d = d_n \to \infty \text{ with} \ n. \text{ We leave this as an interesting problem for possible future work.}
\]

\[
\text{Idea of the proof of Proposition 3.6. The proof of Proposition 2.9 can be easily generalized to the present situation by introducing the concept of a} \ \text{(} p_i, p_k) \text{-shadow corresponding to a quadruple} \ (J_1, J_2, J_3, J_4) \in \mathcal{D}_{i,k}^+ \text{ and following exactly the same lines of the proof. We have, however, refrained from giving the proof in this more general situation for mainly two reasons. Firstly, the proof of Proposition 2.9 already involves a lot of notation and introducing even more of it might make the argument less transparent. Secondly, and more importantly, the precise dependence of the constant} \ C_{i,k} \text{ on} \ p_i \text{ and} \ p_k \text{ would be more complicated and less explicit than the formula given by (4.5) which}
\]

EJP 22 (2017), paper 2. http://www.imstat.org/ejp/
can be exactly evaluated for small values of \( d \) and, as mentioned in Remark 4.5, might be suitably bounded for general \( d \).

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