COMPOSITION OPERATORS ON SOBOLEV SPACES AND WEIGHTED MODULI INEQUALITIES

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In this paper, we study connections between composition operators on Sobolev spaces and mappings defined by $p$-moduli inequalities ($p$-capacity inequalities). We prove that weighted moduli inequalities lead to composition operators on corresponding Sobolev spaces and conversely, that composition operators on Sobolev spaces imply weighted moduli inequalities.

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1. INTRODUCTION

In this paper, we study connections between composition operators on Sobolev spaces and mappings defined by weighted $p$-moduli inequalities of curves families or corresponding weighted $p$-capacity inequalities. In the case $p = n$, the mappings defined by conformal moduli inequalities are quasiconformal mappings [1, 34] and the conformal modules method is one of the basic methods in the geometric theory of quasiconformal mappings [34].

The topological mappings defined by $p$-capacity inequalities were firstly studied in [5]. In this article [5], the notion of $p$-distortion of mappings was introduced and the Lipschitz properties of such mappings was proved in the case $n - 1 < p < \infty$, $p \neq n$. The topological mappings defined by $(p, q)$-capacity inequalities were studied in [30] in connections with composition operators on Sobolev spaces. In [30, 40] the weak differentiability of inverse mappings and their Hölder continuity properties were proved in the case $n - 1 < q < p < \infty$. Continuous mappings satisfying $(p, q)$-capacity inequalities were considered in [32]. In this work [32], Liouville type theorems were proved and removability properties of singular sets were considered. In the recent paper [37] (see also [31, 36]), mappings which satisfy weighted $(p, q)$-capacity inequalities were considered in connection with problems of the geometric theory of composition operators on Sobolev spaces [30, 41, 42].
Significant contributions to the theory of mappings defined by moduli inequalities belong to the Donetsk geometric mapping theory school. In particular, for mappings satisfying to weighted Poletsky’s type inequalities for $p$-modulus of families of curves, their differentiability almost everywhere and the local integrability of partial derivatives were established \[28\]. Some problems concerning the local behavior of such mappings and the problem of removability of an isolated singular point were investigated in \[6, 7, 27\]. In particular, mappings with a distortion of the modulus of order $n - 1 < p < n$ have no essential singular points, that fundamentally distinguishes them from analytic functions and mappings with bounded distortion, see \[6, \text{Theorem 1.1}\]. Let us note that some results concerning of weighted inequalities with respect to the $p$-modulus for such classes of mappings were obtained in \[29\]. It should be noted that the similar theory of mappings was developing independently in the context of directly weighted $p$-moduli, that can be found in works \[2, 3\].

Composition operators on Sobolev spaces arise in the geometric analysis of Sobolev spaces \[9, 13, 14\] and are closely connected with the quasiconformal Reshetnyak problem \[38\]. In series of works \[30, 35, 40, 41, 42\] and \[10\] was founded the geometric theory of composition operators on Sobolev spaces. This theory has significant applications in the spectral theory of elliptic equations, see, for example, \[11, 16, 17\].

The composition operators on Sobolev spaces are generated by weak $(p, q)$-quasiconformal mappings \[10, 30, 40\] and allow characterization in the terms of capacity inequalities. Hence, there is a connection between the geometric theory of composition operators on Sobolev spaces and the theory of mappings defined moduli inequalities. $Q$-homeomorphisms \[22\] were considered in connection with composition operators on Sobolev spaces in \[25, 26\].

In the present work, we give connection between mappings defined by weighted moduli inequalities and weak $(p, q)$-quasiconformal mappings (mappings that generate composition operators on Sobolev spaces). Namely, we prove the following statement:

Let a homeomorphism $\varphi: \Omega \to \tilde{\Omega}$ satisfy to the $p$-modulus inequality

$$M_p(\varphi \Gamma) \leq \int_\Omega Q(x) \cdot \rho^p(x)dx, \ n - 1 < p < \infty,$$

with a non-negative function $Q \in L_1(\Omega)$. Then $\varphi$ generates the bounded composition operator

$$\varphi^*: L_{p'}^1(\tilde{\Omega}) \to L_1^1(\Omega), \ p' = p/(p - n + 1).$$
The inverse assertion states:
Let a homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \) generate a bounded composition operator
\[
\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_{n-1}(\Omega), \quad n - 1 < p < \infty.
\]
Suppose that the mapping \( \varphi \) satisfies Luzin’s \( N \)-property. Then
\[
M_{p'}(\varphi \Gamma) \leq \int_{\Omega} Q(x) \cdot \rho^{p'}(x) dx, \quad p' = \frac{p}{p - n + 1},
\]
with a non-negative function \( Q \in L_1(\Omega) \).

The suggested methods are based on the geometric theory of composition operators on Sobolev spaces and moduli inequalities.

2. SOBOLEV SPACES AND COMPOSITION OPERATORS

2.1. Sobolev spaces

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \). The Sobolev space \( W^1_p(\Omega) \), \( 1 \leq p \leq \infty \), is defined\[23\] as a Banach space of locally integrable weakly differentiable functions \( f : \Omega \to \mathbb{R} \) equipped with the following norm:
\[
\| f \|_{W^1_p(\Omega)} = \| f \|_{L^p(\Omega)} + \| \nabla f \|_{L^p(\Omega)},
\]
where \( \nabla f \) is the weak gradient of the function \( f \), i.e., \( \nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \).

The homogeneous seminormed Sobolev space \( L^1_p(\Omega) \), \( 1 \leq p \leq \infty \), is defined as a space of locally integrable weakly differentiable functions \( f : \Omega \to \mathbb{R} \) equipped with the following seminorm:
\[
\| f \|_{L^1_p(\Omega)} = \| \nabla f \|_{L^p(\Omega)}.
\]

In the Sobolev spaces theory, a crucial role belongs to capacity as an outer (capacitary) measure. This measure regulates some natural properties (for example, convergence properties) of corresponding Sobolev spaces\[21\] \[23\]. In accordance to this approach, elements of Sobolev spaces \( W^1_p(\Omega) \) are equivalence classes up to a set of \( p \)-capacity zero\[24\).

Recall the definition of the capacity\[12\] \[21\] \[23\]. Suppose \( \Omega \) is an open set in \( \mathbb{R}^n \) and \( F \subset \Omega \) is a compact set. The \( p \)-capacity of \( F \) with respect to \( \Omega \) is defined by
\[
\text{cap}_p(F; \Omega) = \inf \{ \| \nabla f \|_{L^p(\Omega)}^p \},
\]
where the infimum is taken over all functions \( f \in C_0(\Omega) \cap L^1_p(\Omega) \) such that \( f \geq 1 \) on \( F \) and which are called admissible functions for the compact set \( F \subset \Omega \). If \( U \subset \Omega \) is an open set, we define
\[
\text{cap}_p(U; \Omega) = \sup \{ \text{cap}_p(F; \Omega) : F \subset U, \ F \text{ is compact} \}.
\]
In the case of an arbitrary set \( E \subset \Omega \), we define the inner \( p \)-capacity

\[
\underline{\cap}_p(E; \Omega) = \sup \{ \cap_p(F; \Omega) : F \subset E \subset \Omega, \text{ } F \text{ is compact} \},
\]

and the outer \( p \)-capacity

\[
\overline{\cap}_p(E; \Omega) = \inf \{ \cap_p(U; \Omega) : E \subset U \subset \Omega, \text{ } U \text{ is open} \}.
\]

A set \( E \subset \Omega \) is called \( p \)-capacity measurable, if \( \underline{\cap}_p(E; \Omega) = \overline{\cap}_p(E; \Omega) \). Let \( E \subset \Omega \) be a \( p \)-capacity measurable set. The value

\[
\cap_p(E; \Omega) = \underline{\cap}_p(E; \Omega) = \overline{\cap}_p(E; \Omega)
\]

is called the \( p \)-capacity measure of the set \( E \subset \Omega \). Borel subsets \( E \subset \Omega \) are \( p \)-capacity measurable \[21\].

The mapping \( \varphi : \Omega \to \mathbb{R}^n \) belongs to the Sobolev space \( W_{1,\text{loc}}^1(\Omega) \), if its coordinate functions belongs to \( W_{1,\text{loc}}^1(\Omega) \). In this case, the formal Jacobi matrix \( D\varphi(x) \) and its determinant (Jacobian) \( J(x, \varphi) \) are well defined at almost all points \( x \in \Omega \). The norm \( |D\varphi(x)| \) is the operator norm of \( D\varphi(x) \),

Let us recall the change of variable formula in the Lebesgue integral \[4, 20\]. Suppose a homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \) is such that there exists a collection of closed sets \( A_k \subset A_{k+1} \subset \Omega, \text{ } k = 1, 2, \ldots, \), for which restrictions \( \varphi|_{A_k} \) are Lipschitz mappings on the sets \( A_k \) and

\[
\left| \Omega \setminus \bigcup_{k=1}^{\infty} A_k \right| = 0.
\]

Then there exists a Borel set \( S \subset \Omega, \text{ } |S| = 0, \) such that on the set \( \Omega \setminus S \) the homeomorphism \( \varphi \) has the Luzin \( N \)-property (the image of a set of measure zero has measure zero) and the change of variable formula

\[
(1) \quad \int_E f \circ \varphi(x) |J(x, \varphi)| \, dx = \int_{\tilde{\Omega} \setminus \varphi(S)} f(y) \, dy
\]

holds for every measurable set \( E \subset \Omega \) and every non-negative measurable function \( f : \tilde{\Omega} \to \mathbb{R} \).

Note, that Sobolev homeomorphisms of the class \( W_{1,\text{loc}}^1(\Omega) \) satisfy the conditions of the change of variable formula \[20\] and, therefore, for Sobolev homeomorphisms the change of variable formula \[1\] holds.

If the mapping \( \varphi \) possesses the Luzin \( N \)-property, then \( |\varphi(S)| = 0 \) and the second integral can be rewritten as the integral on \( \tilde{\Omega} \). Note, that Sobolev homeomorphisms of the class \( L_p^1(\Omega), \text{ } p \geq n, \) possess the Luzin \( N \)-property \[39\].
2.2. Composition operators and regularity of inverse mappings

Let $\Omega$ and $\tilde{\Omega}$ be domains in the Euclidean space $\mathbb{R}^n$. Then a homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ generates a bounded composition operator

$$\varphi^* : L_p^1(\tilde{\Omega}) \to L_q^1(\Omega), \ 1 \leq q \leq p \leq \infty,$$

by the composition rule $\varphi^*(f) = f \circ \varphi$, if for any function $f \in L_p^1(\tilde{\Omega})$, the composition $\varphi^*(f) \in L_q^1(\Omega)$ defined quasi-everywhere in $\Omega$ and there exists a constant $K_{p,q}(\varphi; \Omega) < \infty$ such that

$$\|\varphi^*(f) \|_{L_q^1(\Omega)} \leq K_{p,q}(\varphi; \Omega) \|f \|_{L_p^1(\tilde{\Omega})}.$$

Recall that the $p$-dilatation $[5]$ of a Sobolev mapping $\varphi : \Omega \to \tilde{\Omega}$ at a point $x \in \Omega$ defined as

$$K_p(x) = \inf\{k(x) : |D\varphi(x)| \leq k(x)|J(x, \varphi)|^{1/p}\}.$$

The following theorem gives a characterization of composition operators in terms of integral characteristics of mappings of finite distortion. Recall that a weakly differentiable mapping $\varphi : \Omega \to \tilde{\Omega}$ between two domains $\Omega$ and $\tilde{\Omega}$ generates a bounded composition operator

$$\varphi^* : L_p^1(\tilde{\Omega}) \to L_q^1(\Omega), \ 1 \leq q \leq p \leq \infty,$$

if and only if $\varphi \in W_{q,\text{loc}}^1(\Omega)$, has finite distortion, and

$$K_{p,q}(\varphi; \Omega) := \|K_p \|_{L_\kappa(\Omega)} \| < \infty, \ 1/q - 1/p = 1/\kappa \ (\kappa = \infty, \ if \ p = q).$$

The norm of the operator $\varphi^*$ is estimated as $\|\varphi^*\| \leq K_{p,q}(\varphi; \Omega)$.

This theorem, in the case $p = q = n$, was proved in $[38]$, the case $p = q > n$ was proved in $[35]$ (see also $[10]$). The general case $q \leq p < \infty$ was proved in $[30]$ (see also $[40]$) and the limit case $p = \infty$ was considered in $[15]$.

Let us recall, that homeomorphisms $\varphi : \Omega \to \tilde{\Omega}$ which satisfy conditions of Theorem 2.1 are called weak $(p, q)$-quasiconformal mappings $[10, 40]$.

In the case of weak $(p, q)$-quasiconformal mappings, the following composition duality property was introduced in $[30]$ (the detailed proof can be found in $[18]$):

**Theorem 2.2.** Let the homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ between two domains $\Omega$ and $\tilde{\Omega}$ generate a bounded composition operator

$$\varphi^* : L_p^1(\tilde{\Omega}) \to L_q^1(\Omega), \ n - 1 < q \leq p < \infty.$$
Then the inverse mapping \( \varphi^{-1} : \tilde{\Omega} \to \Omega \) generates a bounded composition operator

\[
(\varphi^{-1})^* : L^1_{q'}(\Omega) \to L^1_{p'}(\tilde{\Omega}),
\]

where \( p' = p/(p-n+1) \), \( q' = q/(q-n+1) \). In the case \( n = 2 \), this theorem is correct for \( 1 \leq q \leq p < \infty \), the inverse assertion is also correct and \( p'' = p'/p' - 1 = p \), \( q'' = q'/(q' - 1) = q \).

The limit case of this theorem \( p = \infty \) and \( q = n - 1 \) was considered in [15] in the frameworks of the weak inverse theorem for Sobolev spaces.

In the present work, we consider the case \( q = n - 1 < p < \infty \).

**Theorem 2.3.** Let the homeomorphism \( \varphi : \Omega \to \tilde{\Omega} \) between two domains \( \Omega \) and \( \tilde{\Omega} \) possess the Luzin \( N \)-property and generate a bounded composition operator

\[
\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_{n-1}(\Omega), \quad n - 1 < p < \infty.
\]

Then the inverse mapping \( \varphi^{-1} : \tilde{\Omega} \to \Omega \) generates a bounded composition operator

\[
(\varphi^{-1})^* : L^1_\infty(\Omega) \to L^1_{p'}(\tilde{\Omega}),
\]

where \( p' = p/(p-n+1) \). In the case \( n = 2 \) the inverse assertion is also correct and \( p'' = p'/p' - 1 = p \).

**Proof.** Since \( \varphi : \Omega \to \tilde{\Omega} \) generates a bounded composition operator

\[
\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_{n-1}(\Omega), \quad n - 1 < p < \infty,
\]

then by [30] the mapping \( \varphi \in W^1_{n-1,\text{loc}}(\Omega) \) and has a finite distortion. Because \( \varphi \) possesses the Luzin \( N \)-property, then the inverse mapping belongs to \( W^1_{1,\text{loc}}(\tilde{\Omega}) \) and is a mapping of finite distortion [15]. Hence

\[
|D\varphi^{-1}(y)| \leq \frac{|D\varphi(x)|^{n-1}}{|J(x, \varphi)|},
\]

for almost all \( x \in \Omega \setminus (S \cup Z) \), \( y = \varphi(x) \in \tilde{\Omega} \setminus \varphi(S \cup Z) \), and \( |D\varphi^{-1}(y)| = 0 \) for almost all \( y \in \varphi(S) \), where \( Z = \{x \in \Omega : J(x, \varphi) = 0\} \) and \( S \) is the singular set in the change of variables formula [11]. Because the measure of \( S \) is zero and the mapping \( \varphi \) has \( N \)-property then the measure of \( \varphi(S) \) is also zero.
Therefore,
\[
\int_{\tilde{\Omega} \setminus \varphi(S \cup Z)} |D\varphi^{-1}(y)|^{p'} \, dy = \int_{\tilde{\Omega} \setminus \varphi(S \cup Z)} \left( \frac{|D\varphi^{-1}(y)|^{n-1}}{|J(\varphi^{-1}(y), \varphi)|} \right)^{p'} \, dy
\]
\[
= \int_{\Omega \setminus (S \cup Z)} \left( \frac{|D\varphi(x)|^{n-1}}{|J(x, \varphi)|} \right)^{p'} \, dx
\]
\[
= \int_{\Omega} \left( \frac{|D\varphi(x)|^{p}}{|J(x, \varphi)|} \right)^{\frac{n-1}{p-(n-1)}} \, dx < \infty,
\]
by Theorem 2.1. Hence [15], \( \varphi^{-1} : \tilde{\Omega} \to \Omega \) generates a bounded composition operator
\[
(\varphi^{-1})^* : L^1_{\infty}(\Omega) \to L^{1}_{p'}(\tilde{\Omega}),
\]
where \( p' = p/(p - n + 1) \). \( \square \)

3. COMPOSITION OPERATORS AND MODULI INEQUALITIES

3.1. Modulus and capacity

Let \( \Gamma \) be a family of curves in \( \mathbb{R}^n \). Denote by \( \text{adm}(\Gamma) \) the set of Borel functions (admissible functions) \( \rho : \mathbb{R}^n \to [0, \infty] \) such that the inequality
\[
\int_{\gamma} \rho \, ds \geq 1
\]
holds for locally rectifiable curves \( \gamma \in \Gamma \).

Let \( \Gamma \) be a family of curves in \( \overline{\mathbb{R}}^n \), where \( \overline{\mathbb{R}}^n \) is a one point compactification of the Euclidean space \( \mathbb{R}^n \). The quantity
\[
M_p(\Gamma) = \inf_{\mathbb{R}^n} \int \rho^p \, dx
\]
is called the \( p \)-module of the family of curves \( \Gamma \) [22]. The infimum is taken over all admissible functions \( \rho \in \text{adm}(\Gamma) \).

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and \( F_0, F_1 \) be disjoint non-empty compact sets in the closure of \( \Omega \). Let \( M_p(\Gamma(F_0, F_1; \Omega)) \) denote the moduli of a family of curves which connect \( F_0 \) and \( F_1 \) in \( \Omega \). Then [22]
\[
M_p(\Gamma(F_0, F_1; \Omega)) = \text{cap}_p(F_0, F_1; \Omega),
\]
where \( \text{cap}_p(F_0, F_1; \Omega) \) is the \( p \)-capacity of the condenser \( (F_0, F_1; \Omega) \) [23].
Suppose that a homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ between two domains $\Omega$ and $\tilde{\Omega}$ satisfies the moduli inequality

$$M_p(\varphi \Gamma) \leq \int_{\Omega} Q(x) \cdot \rho^p(x) dx$$

with a non-negative measurable function $Q$ for every family $\Gamma$ of rectifiable curves in $\Omega$ and every admissible function $\rho$ for $\Gamma$. Such homeomorphisms are called $Q$-homeomorphisms.

The next section is devoted to connections between $Q$-homeomorphisms and the composition operators in the case $Q \in L_1(\Omega)$.

### 3.2. Composition operators and $Q$-homeomorphisms

Firstly, we define two dilatation functions for Sobolev mappings of finite distortion $\varphi : \Omega \to \tilde{\Omega}$.

The outer $p$-dilatation is the following quantity

$$K_p^O(x, \varphi) = \begin{cases} \frac{|D\varphi(x)|^p}{|J(x, \varphi)|^p}, & J(x, \varphi) \neq 0, \\ 0, & J(x, \varphi) = 0. \end{cases}$$

The inner $p$-dilatation is the following quantity

$$K_p^I(x, \varphi) = \begin{cases} \frac{|J(x, \varphi)|^p}{l(D\varphi(x))^p}, & J(x, \varphi) \neq 0, \\ 0, & J(x, \varphi) = 0, \end{cases}$$

where $l(D\varphi(x)) = \min_{h=1} |D\varphi(x) \cdot h|$ for almost all $x \in \Omega$.

**Theorem 3.1.** Let the homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ satisfy the moduli inequality

$$M_p(\varphi \Gamma) \leq \int_{\Omega} Q(x) \cdot \rho^p(x) dx, \quad n - 1 < p < \infty,$$

with a non-negative function $Q \in L_1(\Omega)$. Then $\varphi$ generates the bounded composition operator

$$\varphi^* : L_p^1(\tilde{\Omega}) \to L_1^p(\Omega), \quad p' = p/(p - n + 1).$$

**Proof.** Because $\varphi$ satisfies the moduli inequality with $Q \in L_1(\Omega)$, then by [28] the mapping $\varphi \in ACL(\Omega)$, has finite distortion and the inequality

$$|D\varphi(x)|^p \leq C(n, p)|J(x, \varphi)|^{p-n+1}Q^{n-1}(x)$$
holds for almost all $x \in \Omega$. Hence
\[
\left( \frac{|D\varphi(x)|^{\frac{p}{p-n+1}}}{|J(x, \varphi)|} \right)^{\frac{p-n+1}{n-1}} \leq C(n, p)Q(x) \text{ for almost all } x \in \Omega.
\]

Since $Q \in L_1(\Omega)$, we have
\[
\int_{\Omega} \left( \frac{|D\varphi(x)|^{\frac{p}{p-n+1}}}{|J(x, \varphi)|} \right)^{\frac{p-n+1}{n-1}} dx \leq C(n, p) \int_{\Omega} Q(x) \, dx < \infty.
\]

Then, by Theorem 2.1 the mapping $\varphi$ generates the bounded composition operator $\varphi^* : L^1_{p'}(\Omega) \to L^1_q(\Omega)$, where $p' = p/(p-n+1)$ and the number $q' = 1$ is the solution of the equation
\[
\frac{q'}{p' - q'} = \frac{p - n + 1}{n - 1}.
\]

In [8], the integrability of Jacobians of open discrete mappings with controlled $p$-modulus was considered. As a consequence of Theorem 3.1, we obtain that homeomorphisms which satisfy the weighted $p$-moduli inequality [4] posses measure distortion properties as weak $(p', 1)$-quasiconformal mappings [40, 41].

Now, using the composition duality property in the case of planar domains $\Omega, \tilde{\Omega} \subset \mathbb{R}^2$, we obtain:

**Theorem 3.2.** Let the homeomorphism $\varphi : \Omega \to \tilde{\Omega}$, $\Omega, \tilde{\Omega} \subset \mathbb{R}^2$, satisfy the moduli inequality
\[
M_p(\varphi \Gamma) \leq \int_{\Omega} Q(x) \cdot \rho^p(x) \, dx, \quad 1 \leq p < \infty,
\]
with a non-negative function $Q \in L_1(\Omega)$. Then, the inverse mapping $\varphi^{-1} : \tilde{\Omega} \to \Omega$ generates the bounded composition operator
\[
(\varphi^{-1})^* : L^1_p(\Omega) \to L^1_p(\tilde{\Omega}).
\]
In particular, $\varphi^{-1} \in L^1_p(\tilde{\Omega})$.

Now, we prove the inverse property.

**Theorem 3.3.** Let the homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ generate the bounded composition operator
\[
\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_n(\Omega), \quad n - 1 < p < \infty.
\]
Suppose that the mapping $\varphi$ satisfies Luzin’s $N$-property. Then $\varphi$ is a $Q$-homeomorphism with respect to $p'$-modulus with $Q(x) = K^f_{p'}(x, \varphi) \in L_1(\Omega)$, where $p' = \frac{p}{p-n+1}$.
Proof. On the first step, we prove that under conditions of the theorem the inner distortion function \( K^I_{p'}(x, \varphi) \in L_1(\Omega) \). Since \( \varphi \) generates the bounded composition operator

\[
\varphi^* : L^1_p(\widetilde{\Omega}) \to L^1_{n-1}(\Omega), \quad n - 1 < p < \infty.
\]

then, by Theorem \[2.3\] the inverse mapping \( \varphi^{-1} : \widetilde{\Omega} \to \Omega \) generates a bounded composition operator

\[
(\varphi^{-1})^* : L^1_p(\Omega) \to L^1_{p'}(\widetilde{\Omega}), \quad p' = \frac{p}{p - n + 1}
\]

and belongs to the Sobolev space \( L^1_{p'}(\widetilde{\Omega}) \) \[15\].

Because the inverse mapping is a mapping of finite distortion, then by \[19\]

\[
\int_{\Omega} \frac{|J(x, \varphi)|}{l(D\varphi(x))^{p'}} \, dx = \int_{\Omega \setminus Z} \frac{|J(x, \varphi)|}{l(D\varphi(x))^{p'}} \, dx = \int_{\Omega \setminus Z} |D\varphi^{-1}(\varphi(x))|^{p'} |J(x, \varphi)| \, dx
\]

\[
= \int_{\widetilde{\Omega}} |D\varphi^{-1}(y)|^{p'} \, dy < \infty.
\]

Now, by the definition of the \( Q \)-homeomorphism with respect to \( p' \)-modulus, we have to show that for every family \( \Gamma \) of curves in \( \Omega \) and every \( \rho \in \text{adm}(\Gamma) \)

\[
M_{p'}(\varphi \Gamma) \leq \int_{\Omega} K^I_{p'}(x, \varphi) \rho^{p'}(x) \, dx.
\]

First, note that by Theorem \[2.1\] \( \varphi \in W^1_{n-1,\text{loc}}(\Omega) \). Also, \( \varphi^{-1} \in W^1_{p',\text{loc}}(\widetilde{\Omega}) \) by Theorem \[2.2\]. It implies, that \( \varphi^{-1} \in \text{ACL}^p_{\text{loc}}(\widetilde{\Omega}) \), is differentiable a.e. (see \[33\] Lemma 3]).

By Fuglede’s theorem (see \[34\], p. 95), if \( \widetilde{\Gamma} \) is the family of all curves \( \gamma \in \varphi \Gamma \) for which \( \varphi^{-1} \) is absolutely continuous on all closed subcurves of \( \gamma \), then \( M_{p'}(\varphi \Gamma) = M_{p'}(\widetilde{\Gamma}) \). Then, for given \( \rho \in \text{adm}(\Gamma) \), one consider

\[
\widetilde{\rho}(y) = \begin{cases} \rho(\varphi^{-1}(y))|D\varphi^{-1}(y)|, & y \in \widetilde{\Omega}, \\ 0, & \text{otherwise.} \end{cases}
\]

Then, for \( \widetilde{\gamma} \in \widetilde{\Gamma} \)

\[
\int_{\widetilde{\gamma}} \widetilde{\rho} \, ds \geq \int_{\varphi^{-1} \circ \widetilde{\gamma}} \rho \, ds \geq 1,
\]

and consequently \( \widetilde{\rho} \in \text{adm}(\widetilde{\Gamma}) \).
We denote by $Z_0$ the set of all points $y \in \tilde{\Omega}$, where $J_{\varphi^{-1}}(y) = 0$. By change of variable formula (see [4, Theorem 3.2.5]), we obtain that

$$M_{p'}(\varphi \Gamma) = M_{p'}(\tilde{\Gamma}) \leq \int_{\tilde{\Omega}} \rho^{p'} \, dy$$

$$= \int_{\tilde{\Omega}} \rho^{p'}(\varphi^{-1}(y))|D\varphi^{-1}(y)|^{p'} \, dy = \int_{\tilde{\Omega} \setminus Z_0} \frac{\rho^{p'}(\varphi^{-1}(y))}{l(D\varphi(\varphi^{-1}(y)))} \, dy$$

$$= \int_{\tilde{\Omega}} \rho^{p'}(\varphi^{-1}(y))K_{p'}^{I}(\varphi^{-1}(y), \varphi)J_{\varphi^{-1}}(y) \, dy \leq \int_{\Omega} K_{p'}^{I}(x, \varphi)\rho^{p'}(x) \, dx$$

which completes the proof. \qed

In the planar case $\Omega, \tilde{\Omega} \subset \mathbb{R}^2$, we have the following theorem:

**Theorem 3.4.** Let $\varphi : \Omega \to \tilde{\Omega}$ be a homeomorphism of the domains $\Omega, \tilde{\Omega} \subset \mathbb{R}^2$. Then $\varphi$ satisfies the moduli inequality

(6) \quad $M_p(\varphi \Gamma) \leq \int_{\Omega} Q(x) \cdot \rho^p(x) \, dx$, \quad $1 < p < \infty$,

with a non-negative function $Q \in L_1(\Omega)$, if and only if $\varphi$ generates the bounded composition operator

$$\varphi^* : L_{p'}^1(\tilde{\Omega}) \to L_1^1(\Omega)$, \quad p' = p/(p - 1)$.$

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