A new Lagrange-Galerkin scheme for solving fluid flow in porous media

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Abstract. We introduce a new Lagrange–Galerkin scheme for computing fluid flow through porous media. The method of volume averaging of the velocity and pressure deviation in the pore is employed to derive the macroscopic mass and momentum conservation in the porous medium in which this technique was established by Hsu and Cheng. We derive the Lagrange–Galerkin scheme by extending the idea of the method of characteristics to overcome the difficulty which comes from the non-homogeneous porosity. The numerical experiment are conducted to verify the performance of the new numerical scheme. The new numerical scheme shows a good agreement with the analytical solution and has 2\textsuperscript{nd} order accuracy both in space and time.

1. Introduction
Numerical simulations of the fluid flow in porous media have recently become helpful in the engineering areas such as in petroleum engineering, environmental engineering, hydrology, and geothermal engineering. Accurate characterization of the fluid flow in porous media is required to successfully design the operations of the engineering projects [1]. As an example in the geothermal engineering, that characterization has essential information to determine the location of the well-bore in the reservoir. By precisely determining its location, the reservoir can generate a significant impact on the steam and water production which implies the increasing supply of electricity.

A geothermal reservoir is a large volume of hot water and steam trapped in subsurface porous and permeable rock structure. Figure 1 is a schematic view of a typical geothermal reservoir system. The dynamical of the fluid in the reservoir is not steady, so we can not model the fluid flow and heat transfer using Darcy law.

Many scientists and researchers are attempting to model the transport processes in porous media. In 1856, Henry Darcy observed water flowing through packed sand at the low porosity and permeability. According to his experiment, he concluded that the pressure gradient is proportional to the fluid velocity in the porous media. In the one-dimensional case, Darcy’s equation is expressed by

\[
-\frac{dp}{dx} = \frac{\mu v}{k},
\]

where \( p \) is the pressure, \( x \) is spatial coordinate, \( v \) is the macroscopic velocity, \( \mu \) is the dynamic viscosity, and \( k \) is the permeability. This relationship is called Darcy’s Law. In the application where the permeability and porosity of the media are small such as in the groundwater and petroleum flows [2,10], Darcy’s low has an excellent performance to describe those phenomena.
However, in the application where the permeability and porosity of the media are significant such as in the geothermal system, Darcy’s law failed to describe it [10]. In 1947, Brinkman improved Darcy’s law by including a nonlocal term (divergence of velocity) which represents the viscous term. This results in Brinkman’s equation.

\[- \frac{dp}{dx} = \frac{\mu v}{k} - \mu \frac{\partial^2 v}{\partial x^2}.\]  

The Brinkman’s equation describes the transport processes in the porous medium more generally than Darcy’s equation. However, it only can be applied in a stationary setting.

Dupuit (1863) and in Forchheimer (1901) empirically found that for high velocities, the pressure drop is proportional to the square of the velocity [3,10]. Using this fact Dupuit and Forchheimer added the second order derivative of the velocity term to represent the microscopic inertial effect. The resulting equation becomes Brinkman–Forchheimer’s equation:

\[- \frac{dp}{dx} = \frac{\mu v}{k} - \mu \frac{\partial^2 v}{\partial x^2} + \beta \rho v^2,\]  

where \( \beta = \frac{E \phi}{\sqrt{k}} \) is the non-Darcy coefficient and \( \rho \) is density of the fluid. This equation is more general than Brinkman’s equation, but again, it only applies to the stationary setting.

In 1967, Whitaker introduced the volume average method to relate the average of spatial derivative to the spatial derivative of the volume average. By using this method, it is possible to transform the microscopic equation into the macroscopic equation of fluid flow through porous media [6]. Hsu and Cheng (1989) applied the volume average in the represented volume to derive the equation for fluid flow through the porous media. In their equation, they represented the drag force with Ergun’s relation [5,7].

In this work, we introduced a new numerical scheme based on the Lagrange-Galerkin method to solve the model propose by Hsu and Cheng numerically. The analytical solution is employed to investigate the performance of the new numerical scheme by determining the order of convergence of the scheme.

2. Governing Equation

Hsu and Cheng [6] derive the equation of Navier–Stokes with the porous media for velocity and pressure based on the method of volume average of the velocity and pressure deviations in the pore scale. Let \( V(x) \equiv V_\alpha(x) \cup V_\beta(x) \subset \mathbb{R}^3 \) and \( |V(x)| = |V_\alpha(x)| + |V_\beta(x)| < \infty, \ |V_\alpha| \) is the

**Figure 1.** Schematic of geothermal reservoir system with production and injection wells installed.

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\[ \frac{dp}{dx} = \frac{\mu v}{k} - \mu \frac{\partial^2 v}{\partial x^2} + \beta \rho v^2, \]
volume occupied by the $\alpha$-phase, $|V_\beta|$ is the volume occupied by the $\beta$-phase, $x \in \Omega$ is the macroscopic coordinate, and $x' \in V_\alpha$ is the microscopic coordinate. The macroscopic intrinsic phase average for the velocity vector $w$ defined by

$$w := \langle v'_\alpha \rangle = \frac{1}{|V_\alpha|} \int_{V_\alpha} v'_\alpha dx', \quad (4)$$

where $v'_\alpha \in \mathbb{R}^3$ is the microscopic velocity vector. Assuming that the total macroscopic source of the system at a point $x$ is equal to the sum of microscopic source of the system at a point $x'$ and the total flux through the surface $A_{\alpha\beta}$ (see fig 2), Whitaker and Slattery [8] obtained the average volume which relates the volume average of the spatial derivative to the spatial derivative of the volume average, i.e.,

$$\nabla \cdot \left[ \frac{1}{|V|} \int_{V_\alpha} v'_\alpha dx' \right] = \frac{1}{|V|} \int_{V_\alpha} \nabla' \cdot v'_\alpha dx' + \frac{1}{|V|} \int_{A_{\alpha\beta}} v'_\alpha \cdot n_{\beta\alpha} ds. \quad (5)$$

To obtain the equation of fluid flow through the porous media, we integrate the Navier-Stokes equation respect to the representative volume in the porous media and divide the resulting expression by $|V|$. This yields

$$\frac{1}{|V|} \int_{V_\alpha} (\nabla' \cdot \nu') dx' = 0, \quad (6)$$

$$\frac{1}{|V|} \int_{V_\alpha} \rho_\alpha \left[ \frac{\partial \nu'}{\partial t} + \nabla' \cdot (\nu' \otimes \nu') \right] dx' = \frac{1}{|V|} \int_{V_\alpha} [-\nabla' p_\alpha + \mu_\alpha \nabla'^2 \nu'] dx'. \quad (7)$$

From (5), we obtain

$$\nabla \cdot u = 0,$$

$$\rho \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla) \frac{u}{\phi} \right] = -\nabla p + \mu \Delta u + B, \quad (8)$$

where $\phi$ is the porosity, $\rho$ is the density of the fluid, $\mu$ is the dynamic viscosity, $p := \phi \bar{p}$ is the macroscopic pressure, $u := \phi \bar{v}$ is the Darcy velocity, and $B(u, \phi)$ is the total drag force per unit volume due to the presence of the solid particle. $B(u, \phi)$ satisfies Ergun’s expression:

$$B = B(u, \phi) = B(u, \phi; \mu, \rho, d_p) \equiv -\frac{\mu \phi u}{K(\phi)} - \rho \frac{F(\phi) \phi u \theta}{\sqrt{K(\phi)}}, \quad (9)$$
where $F : (0, 1] \to (0, \infty)$ and $K : (0, 1] \to (0, \infty]$ are functions defined by
\[ F(\phi) = b/\sqrt{a\phi^3}, \quad K(\phi) = d_p^2\phi^3/a(1 - \phi)^2, \] (10)
which correspond to the Forchheimer constant and the Kozeny–Carman absolute permeability, respectively. The positive constant $d_p$ is a particle diameter, and the values of $a$ and $b$ are empirically given by $a = 150$ and $b = 1.75$ in [7].

3. Statement of the Model
In this section, we introduce a functional framework for the model introduced in the previous section given by equations (8), (9), and (10).

The notation to be used in this paper is as follows. For $d = 2, 3$, let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $\Gamma$ the boundary of $\Omega$, and $T$ a positive constant. $\Gamma$ is divided into three parts, $\Gamma_i$, $i = 0, 1, 2$, which satisfy $\bar{\Gamma} = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ and $\bar{\Gamma}_i \cap \bar{\Gamma}_j = \emptyset$ for all $i \neq j$. We suppose that $\Gamma$ is a Lipschitz boundary, and that, for each $i \in \{0, 1, 2\}$, $\Gamma_i$ is piecewise smooth, where the total number of the smooth boundaries of $\Gamma_i$ is finite. The Lebesgue space on $\Omega$ for $p \in [1, \infty]$ is denoted by $L^p(\Omega)$ and the Sobolev space $W^{1,2}(\Omega)$ is denoted by $H^1(\Omega)$ with the norm
\[ \|u\|_{H^1(\Omega)} := \left( \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right)^{1/2}. \]
The vector- and matrix-valued function spaces corresponding to, e.g., $L^2(\Omega)$ are denoted by $L^2(\Omega)^d$ and $L^2(\Omega)^{d \times d}$, respectively. The inner products in $L^2(\Omega)$, $L^2(\Omega)^d$, and $L^2(\Omega)^{d \times d}$ are all represented by $(\cdot, \cdot)$.

We consider the following problem governed by the Navier–Stokes equations with the non-homogeneous porous media [1]; find $(u, p) : \Omega \times (0, T) \to \mathbb{R}^d \times \mathbb{R}$ such that
\begin{align*}
\rho \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right] + \nabla p &= f + B(u, \phi) \quad \text{in } \Omega \times (0, T), \quad (11a) \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \times (0, T), \quad (11b) \\
u &= g \quad \text{on } \Gamma_0 \times (0, T), \quad (11c) \\
2\mu D(u)n - pn &= 0 \quad \text{on } \Gamma_1 \times (0, T), \quad (11d) \\
u \cdot n &= 0 \quad \text{on } \Gamma_2 \times (0, T), \quad (11e) \\
(2\mu D(u)n - pn) \times n &= 0 \quad \text{on } \Gamma_2 \times (0, T), \quad (11f) \\
u &= u^0 \quad \text{in } \Omega, \quad \text{at } t = 0, \quad (11g)
\end{align*}
where $u$ is the velocity, $p$ is the pressure, $\mu > 0$ is a dynamic viscosity, $u^0 : \Omega \to \mathbb{R}^d$ is a given initial velocity, $f : \Omega \times (0, T) \to \mathbb{R}^d$ is a given external force, $g : \Gamma_0 \times (0, T) \to \mathbb{R}^d$ is a given boundary velocity, $\phi : \Omega \to (0, 1]$ is a given porosity, and $D(u) : \Omega \times (0, T) \to \mathbb{R}^{d \times d}_{\text{sym}}$ is the strain-rate tensor defined by
\[ D(u) := \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right]. \]
Let $W := H^1(\Omega; \mathbb{R}^N)$, $V(\phi) := \{ v = g \text{ on } \Gamma_0, \quad v \cdot n = 0 \text{ on } \Gamma_2 \}$, $V := V(0)$, and $Q := L^2(\Omega)$. We define the bilinear forms $a_0$ on $W \times W$, $b$ on $W \times Q$, and $c_0$ on $W \times W$ as follows:
\[ a_0(u, v) := 2\mu \left( D(u), D(v) \right), \quad b(v, q) := -(\nabla \cdot v, q), \quad c_0(u, v) := \mu \left( \frac{\phi}{K(\phi)} u, v \right). \]
We also define a trilinear form $c_1$ on $W \times W \times L^\infty(\Omega)$ by
\[ c_1(u, |u|, v) := \left( \frac{F(\phi)\phi|u|}{\sqrt{K(\phi)}}, v \right). \]
Then the weak formulation for the problem (11) is to find \((u, p): (0, T) \rightarrow V \times Q\) such that, for \(t \in (0, T),\)
\[
\left( \frac{Dw}{Dt}, v \right) + a_0(u, v) + b(v, p) + b(u, q) + c_0(u, v) + c_1(u, |u|, v) = (f(t), v),
\forall (v, q) \in V \times Q,
\]
with \(u(0) = u^0 \in L^2(\Omega)^d.\)

4. A Lagrange-Galerkin Scheme

In this section, we present a first-order in time Lagrange–Galerkin scheme for problem (11). For \(u\) and \(\phi\) in (12), we introduce the velocity \(w : \Omega \times (0, T) \rightarrow \mathbb{R}\) and the material derivative with respect to \(w\) defined by
\[
w := \frac{u}{\phi}, \quad \frac{D}{Dt} := \frac{\partial}{\partial t} + \phi \frac{\nabla}{\phi}.
\]
Thus, we can rewrite \(\partial u/\partial t + (u \cdot \nabla)(u/\phi)\) as
\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \phi \left[ \frac{\partial w}{\partial t} + (w \cdot \nabla)w \right] = \phi \frac{Dw}{Dt},
\]
which is an essential relationship to develop our new numerical scheme.

Let \(\tau\) be a time increment and \(t^k := k\tau\) for \(k \in \mathbb{N} \cup \{0\}\). For a function \(\psi\) defined in \(\Omega \times (0, T)\) or \(\Gamma_0 \times (0, T)\), we denote \(\psi(\cdot, t^k)\) simply by \(\psi^k\). Let \(X : (0, T) \rightarrow \mathbb{R}^d\) be a solution of the following ordinary differential equation,
\[
X'(t) = w(X(t), t), \quad t \in (0, T),
\]
subject to an initial condition \(X(t^0) = x\). Physically, \(X(t)\) represents the position of a fluid particle with respect to the velocity \(w\) at time \(t\). For a given velocity \(v : \Omega \rightarrow \mathbb{R}^d\), let \(X_1(v, \tau) : \Omega \rightarrow \mathbb{R}^d\) be the mapping defined by
\[
X_1(v, \tau)(x) := x - v(x)\tau,
\]
which is an upwind point of \(x\) with respect to the velocity \(v\) and a time increment \(\tau\). Now, we approximate the material derivative by a first-order approximation in time as following:
\[
\phi(x) \frac{Dw}{Dt}(x, t^k) = \phi(x) \frac{d}{dt} (w(X(t), t)) \big|_{t=t^k} = \phi(x) \left( \frac{w^k(x) - w^{k-1} \circ X_1(w^{k-1}, \tau)(x)}{\tau} \right) + O(\tau)
\]
\[
= \frac{1}{\tau} \left( w^k(x) - \phi(x) \left( \frac{w^{k-1} \circ X_1(w^{k-1}, \tau)}{\tau} \right) \right) + O(\tau),
\]
where the symbol “\(\circ\)” denotes the composition of functions.

Let \(T_h := \{K\}\) be a triangulation of \(\Omega (= \cup_{K \in T_h})\), \(h_K\) a diameter of \(K \in T_h\), and \(h := \max_{K \in T_h} h_K\) the maximum element size. We define the function spaces \(X_h, M_h, V_h\) and \(Q_h\) by
\[
X_h := \left\{ v_h \in C(\bar{\Omega}; \mathbb{R}^d); v_h|_K \in P_2(K; \mathbb{R}^d) \right\}, \quad M_h := \left\{ q_h \in C(\bar{\Omega}); q_h|_K \in P_1(K) \right\},
\]
\[V_h := X_h \cap V, \quad Q_h := M_h \cap Q = M_h,\]
respectively, where \(P_k(K; \mathbb{R}^d)\) is the polynomial space of degree \(k \in \mathbb{N}\) on \(K\), and \(P_1(K) := P_1(K; \mathbb{R}^1)\). Let \(N_T := \lfloor T/\tau \rfloor\) be the total number of time steps.
Suppose \( f \in C([0,T]; L^2(\Omega; \mathbb{R}^d)) \) and \( u^0 \in V \). Let \( u^0_h = u^0 \in V_h \) with \( u^0 \) a given function. Our new Lagrange-Galerkin scheme for solving the problem (12) is to find \( \psi \) where \( \psi = \sin(x) \sin(y) e^{-2t} \), such that, for \( k = 1, \ldots , N_T \),

\[
\frac{u^k_h - \phi u^{k-1}_h}{\tau} X_1(u^{k-1}_h, \tau, v_h) + a_0(u^k_h, v_h) + b(u^k_h, q_h) + b(v_h, p^k_h)
+ c_0(u^k_h, v_h) + c_1(|u^{k-1}|, u^k_h, v_h) = (f^n, v_h) \quad \forall (v_h, q_h) \in V_h \times Q_h,
\]

for \( (k = 0, \ldots , N_T) \) we defined \( w^n_h := \frac{u^n_h}{\phi} \).

5. Numerical Results

To compute all of this simulation we used device Intel core i-7 with RAM 4GB and for typing a code, we are used software FreeFem++ 3.61.

5.1. Order of Convergence

In this section, the two-dimensional test problem is computed by the scheme in (17) in order to check the order of convergence of the scheme. In this computation, we set \( \Omega = (0, \pi) \times (0, \pi) \), with division number \( N = 4, 8, 16, 32, 64, 128 \), \( h = \pi/N, \tau = h^2, N_T = \lceil 1/\tau \rceil, \rho = 1 \text{ gr/cm}^2, \mu = 8.9 \times 10^{-3} \text{ dyn-s/cm}^2, d_p = 1 \times 10^{-2} \text{ cm}, \phi = \frac{2+\sin(2\pi)}{3}, \) and we used FEM \( P2/P1 \). The functions \( f \) and \( w^0 \) are given. The manufactured solution is

\[
u(x,t) = \left( -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right) (x,t), \quad p = \sin(x) \sin(y) e^{-2t},
\]

where \( \psi \) is the stream function, defined by

\[\psi = \sin^3(x) \sin^3(y) e^{-2t}. \]

For the solution \((u_h, p_h)\) of The scheme in (17) we define maximum errors \( E_{r1} \) and \( E_{r2} \) as follows:

\[
E_{r1} := \max_{k=0,\ldots,N_T} \| u^k_h - u^k \|_{H^1(\Omega; \mathbb{R}^2)}, \quad E_{r2} := \max_{k=0,\ldots,N_T} \| p^k_h - p^k \|_{L^2(\Omega)}.
\]

| N   | \( E_{r1} \)    | \( E_{r2} \)    | Slope of \( E_{r1} \) | Slope of \( E_{r2} \) |
|-----|----------------|----------------|------------------------|------------------------|
| 4   | 1.9 \times 10^{-1} | 7.1 \times 10^{-1} | -                      | -                      |
| 8   | 6.3 \times 10^{-2} | 6.4 \times 10^{-2} | 1.6                    | 3.5                    |
| 16  | 1.3 \times 10^{-2} | 5.7 \times 10^{-3} | 2.3                    | 3.5                    |
| 32  | 3.1 \times 10^{-3} | 1.1 \times 10^{-3} | 2.1                    | 2.4                    |
| 64  | 1.0 \times 10^{-3} | 2.4 \times 10^{-4} | 1.6                    | 2.1                    |
| 128 | 3.5 \times 10^{-4} | 7.0 \times 10^{-5} | 1.6                    | 1.8                    |

From the table 1 and the fig 3, we can conclude that the order of convergence of the scheme (17) is two.
5.2. Simulation

In this section, we present the results of our numerical simulation for fluid flow through the porous media. In this simulation, we set $\Omega = (0, 2) \times (0, 1)$, division number $N = 100$, $h = 1/N$, $\tau = h$, $N_T = \lceil 1/\tau^2 \rceil$, $Re = 1000$, $Da = 6.6 \times 10^{-4}$, and we used FEM $P_2/P_1$. We gave function $f = 0$ and $u^0$ defined by

$$u^0 = \begin{pmatrix} 0.005\eta(x_1) \\ 0 \end{pmatrix}, \quad \eta(x_1) = \begin{cases} \cos(\frac{\pi}{2}x_1) & \text{if } (0 \leq x_1 \leq 0.5) \\ 0 & \text{if } (0.5 < x_1 \leq 2) \end{cases} \quad (20)$$

For the porosity $\phi(x)$ we set

$$\phi(x) = \frac{(\beta - \alpha)}{2} \sin(4\pi x_2) + \frac{\beta + \alpha}{2}$$

where $\alpha = 0.15$ and $\beta = 0.65$. To aid the understanding of the problem setting in this simulation, we plot the distribution function of porosity in the computation domain in fig 4:

![Computation domain and pore distribution](image)

The results of the simulation presented in fig 5 until fig 10. The results of the simulation presented in fig 5 until 10. From the figures clearly that the fluid flowing faster in the high porosity.

6. Conclusion

We have presented a new numerical scheme based on the Lagrange-Galerkin method to solve the fluid flow through a porous media, and we have performed some numerical simulations to test the order of convergence. From the work that we have done, we conclude that the new numerical scheme has a good performance to solve the problem. The scheme is second-order accurate both in space and time.
Figure 5. Time evolution of velocity magnitude.

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