Optimized dynamical decoupling sequences in protecting two-qubit states

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Abstract

Aperiodic dynamical decoupling (DD) sequences of \( \pi \) pulses are of great interest to decoherence control and have been recently extended from single-qubit to two-qubit systems. If the environmental noise power spectrum is made available, then one may further optimize aperiodic DD sequences to reach higher efficiency of decoherence suppression than known universal schemes. This possibility is investigated in this work for the protection of two-qubit states, using an exactly solvable pure dephasing model including both local and nonlocal noise. The performance of optimized DD sequences in protecting two-qubit states is compared with that achieved by nested Uhrig’s DD (nested-UDD) sequences, for several different types of noise spectrum. Except for the cases with noise spectrum decaying slowly in the high-frequency regime, optimized DD sequences with tens of control pulses can perform orders of magnitude better than that of nested-UDD. A two-qubit system with highly unbalanced local noise is also examined to shed more light on a recent experiment. Possible experiments that may be motivated by this work are discussed.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In one way or another, any two-level system (qubit) is coupled to some environmental degrees of freedom. This inevitable system–environment coupling leads to decoherence. To protect quantum coherence as a key resource for quantum technologies, many schemes have been proposed to dynamically eliminate the unwanted qubit–environment coupling [1–5].

Analogous to the spin-echo technique widely adopted in nuclear-magnetic-resonance studies [6], various dynamical decoupling (DD) sequences of instantaneous control pulses [2, 7–9] have been extensively studied. In particular, due to Uhrig’s DD (UDD) sequence [8, 9], research activities focusing on DD have surged recently [10–15]. In addition to its high efficiency in theory, the power of UDD lies in its universality [16, 17]. Indeed, the working mechanism of UDD does not rely on detailed assumptions about the system–environment coupling or about the environment. Nevertheless, if the actual form of the noise spectrum of the environment is available, then a locally optimized DD (LODD) [10] sequence based on the noise spectrum can outperform UDD, with the pulse locations optimized according to an exact decoherence function [18–20]. The reason for the success of optimization is simple. In suppressing the pure-dephasing of a qubit, a UDD sequence minimizes a decoherence filter function (defined later) in the neighbourhood of zero frequency but gradually becomes less effective for large frequencies. As such, if the noise spectrum of the environment is known, then its actual behaviour at appreciably nonzero frequencies makes room for further optimization of DD sequences. This optimization approach is somewhat in the same spirit of the continuous DD approach [3, 4], insofar as both attempt to make the full use of the noise spectrum.

Extension of DD to two-qubit (or even multi-qubit) systems is crucial towards efficient protection of quantum
entanglement. For a known initial state of a two-qubit system, an extended UDD sequence is found by involving nonlocal control operators, with its theoretical performance essentially identical with that in one-qubit UDD cases [21]. For general situations, nested-UDD sequences, initially proposed for suppressing both dephasing and relaxation in one-qubit systems [22], are advocated to protect two-qubit states in a universal manner, i.e., without knowing the details of the system–environment coupling or of the noise spectrum [23, 24]. For example, it was shown that to lock an unknown superposition state of two known basis states to the $|N\rangle$, three layers of UDD sequences and hence about $N^3$ control pulses in total will be needed [21]. If the initial state is totally unknown, then to reach the same level of decoherence suppression one needs four layers of UDD sequences and hence about $N^4$ control pulses [23, 24]. One important question then arises. That is, if the noise spectrum of the environment of a two-qubit system is available, then can we significantly improve entanglement protection by further optimizing the locations of the instantaneous control pulses as what was done in [10, 18] for single-qubit systems? If yes, then the required number of pulses can be much less and more understandings of entanglement protection might emerge. Using an approach extended from the above-mentioned LODD for single-qubit systems, this question is answered here via a pure dephasing model of an open two-qubit system. The performance of optimized DD sequences in preserving two-qubit states is compared with that achieved by nested-UDD for several different types of noise spectrum. Except for the Lorentzian type of noise spectrum, it is found that optimized DD sequences can protect two-qubit states orders of magnitude better than nested-UDD. As a result, on the one hand nested-UDD can be seen as a powerful and a general-purpose scheme for protecting two-qubit states, and on the other hand the optimized DD approach can be seen as system-specific DD schemes with even better performance. In addition, to shed more light on a recent experiment of entanglement protection via DD [25], a two-qubit system that shares important noise features with the experiment is studied.

Our plan for this paper is as follows. In the next section, we discuss a model that describes pure dephasing processes of a two-qubit system in the presence of instantaneous $\pi$ pulses. Based on exact expressions of decoherence effects, we elaborate in section 3 our optimization procedure to find the optimized pulse locations. In section 4, the performance of the optimized two-qubit DD sequence is illustrated in four subsections treating different types of noise spectrum. Section 5 concludes this work and proposes two types of experiments.

2. Pure dephasing model of two-qubit systems

In general, decoherence as a complicated process involves both population relaxation and dephasing. Yet, if we increase the strength of an external polarization field such that all the energy level splittings are sufficiently large, then the relaxation can be made negligible within a time scale of interest and dephasing becomes the only source of decoherence. Under such a pure dephasing assumption, the environmental noise may be modelled as classical random fields causing random phase shifts, a valid treatment in many dephasing environments like spin bath in solid-state systems [10, 12, 13] and background noise in superconducting qubits [26]. Here we follow the methodology proposed in [9, 26]. We can then write the pure-dephasing Hamiltonian of an open two-qubit system as

$$H = f_1(t)\sigma_z + f_2(t)\sigma_z + f_3(t)\sigma_z\sigma_z,$$

where $f_i(t)$ are assumed to be independent random noise variables with a Gaussian distribution. We further assume

$$\left\langle f_i(t)\right\rangle = 0,$$

$$\left\langle f_i(t_1)f_i(t_2)\right\rangle = g_i(t_1 - t_2),$$

with the correlation function $g_i(t)$ being an even function. The $f_i(t)$ term in the Hamiltonian depicts the noise seen by the first (second) qubit alone, whereas the $f_3(t)$ term reflects how noise may jointly impact the two qubits. In the following, we loosely call the $f_3(t)$ noise term as a term of nonlocal noise. If the two qubits are far apart, then this nonlocal noise term should be very small and can be neglected. As a consequence the decoherence control problem is expected to share important features with single-qubit DD [27] (note that $f_1(t)$ and $f_2(t)$ are assumed to be independent here). However, our concern in this study is two nearby qubits interacting with each other, and therefore the nonlocal noise term should be included to account for random fluctuations in the qubit–qubit mutual interaction (for example, due to the fluctuations in the qubit–qubit distance caused by lattice vibrations, or in the context of superconducting qubits [28], due to the fluctuations in a third device that is responsible for qubit–qubit coupling). Certainly, in a more realistic environment, the $f_3(t)$ term might be correlated with the local noise terms. Such kinds of potential correlations are neglected in our model. From a different perspective, one may regard the first qubit as a part of the environment of the second qubit, and then the $f_3(t)$ term models how the dynamics of one qubit might change the environment experienced by the other qubit. Throughout we assume dimensionless units.

Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be the eigenstates of $\sigma_z$; then, a general two-qubit pure state at time zero can be written as

$$\left|\Psi(0)\right\rangle = \alpha |\downarrow\downarrow\rangle + \beta |\downarrow\uparrow\rangle + \gamma |\uparrow\downarrow\rangle + \eta |\uparrow\uparrow\rangle,$$

where the upward or downward arrows in each of the four components represent the spin states of the two qubits. The Hamiltonian in equation (1) then gives rise to the following state vector at time $t$:

$$\left|\Psi(t)\right\rangle = \alpha e^{-iF_1(t)\sigma_z} + \beta e^{-iF_2(t)\sigma_z} + \gamma e^{-iF_3(t)\sigma_z} + \eta e^{-iF_3(t)\sigma_z}$$

$$\left|\Psi(t)\right\rangle = \alpha e^{-iF_1(t)\sigma_z + F_3(t)\sigma_z + F_3(t)\sigma_z} \left|\Psi(0)\right\rangle + \beta e^{-iF_2(t)\sigma_z + F_3(t)\sigma_z} \left|\Psi(0)\right\rangle,$$

$$\left|\Psi(t)\right\rangle = \alpha e^{-iF_1(t)\sigma_z + F_3(t)\sigma_z + F_3(t)\sigma_z} \left|\Psi(0)\right\rangle + \beta e^{-iF_2(t)\sigma_z + F_3(t)\sigma_z} \left|\Psi(0)\right\rangle + \gamma e^{-iF_3(t)\sigma_z + F_3(t)\sigma_z} \left|\Psi(0)\right\rangle + \eta e^{-iF_3(t)\sigma_z + F_3(t)\sigma_z} \left|\Psi(0)\right\rangle,$$

$$\left|\Psi(t)\right\rangle = \alpha e^{-iF_1(t)\sigma_z + F_3(t)\sigma_z + F_3(t)\sigma_z} \left|\Psi(0)\right\rangle + \beta e^{-iF_2(t)\sigma_z + F_3(t)\sigma_z} \left|\Psi(0)\right\rangle + \gamma e^{-iF_3(t)\sigma_z + F_3(t)\sigma_z} \left|\Psi(0)\right\rangle + \eta e^{-iF_3(t)\sigma_z + F_3(t)\sigma_z} \left|\Psi(0)\right\rangle,$$

where $F_i(t) \equiv \int_0^t f_i(t') dt'$. To analyse the density matrix $\rho(t) = \left|\Psi(t)\right\rangle \langle\Psi(t)\right|$ averaged over noise histories (denoted $\langle\cdot\rangle$), we further define four basis states $|0\rangle = \left|\downarrow\downarrow\right\rangle, |1\rangle = \left|\downarrow\uparrow\right\rangle, |2\rangle = \left|\uparrow\downarrow\right\rangle, |3\rangle = \left|\uparrow\uparrow\right\rangle$. Then all averaged density matrix elements (in the absence of DD control pulses) can

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Y Pan et al
be easily worked out. For example, the mean value of \( \rho_0(t) \), denoted \( \bar{\rho}_0(t) \), can be expressed as

\[
\bar{\rho}_0(t) = \alpha^* \beta e^{-\Delta F_3(t)} + \beta^* \gamma e^{-\Delta F_2(t)} + \gamma^* \alpha e^{-\Delta F_1(t)}
\]

Here, in obtaining the last equality we have used the relation \((\bar{f}_2 f_3)^2 = 0 \) as well as the Gaussian nature of the noise. Similar expressions can be obtained for all other density matrix elements.

In our pure-dephasing model, the error operator for the first (second) qubit is \( \sigma_1 \) (\( \sigma_2 \)), whose detrimental effect can be suppressed by a local control operator \( \sigma_2 \) (\( \sigma_1 \)). As seen from \([23, 24]\), a nested-UDD scheme with two layers of \( \sigma_1 \) and \( \sigma_2 \) pulses can suppress the dephasing to the Nth order, with about \( N^2 \) pulses in total. To see if we can further improve the performance by optimizing the pulse locations, let us now consider the following scenario: \( n \) pulses of \( \pi \) rotation along the \( x \)-axis are applied to the first qubit, with the pulse locations given by \( t_1, t_2, \ldots, t_n \), whereas \( m \) analogous pulses are applied to the second qubit at times \( t_1', t_2', \ldots, t_m' \). As a result, totally \( n+m \) pulses are applied to the two-qubit system. We arrange the \( n+m \) pulse timings in increasing order and denote them by \( t_1, t_2, \ldots, t_{n+m} \), with \( t_j < t_j \). At each of such instants, either \( \sigma_1 \) or \( \sigma_2 \) operator switches its sign. At the same time, the operator \( \sigma_1 \), \( \sigma_2 \) changes its sign \( n+m \) times at these instants. This motivates us to define three switch functions \([9]\):

\[
s_1(t') = (-1)^k, \quad t_k < t' \leq t_{k+1}, \quad k = 0, 1, \ldots, n,
\]

\[
s_2(t') = (-1)^k, \quad t_k < t' < t_{k+1}, \quad k = 0, 1, \ldots, m,
\]

\[
s_3(t') = (-1)^k, \quad t_k < t' < t_{k+1}, \quad k = 0, 1, \ldots, n+m,
\]

with \( t_0 = 0 \) and \( t_{n+m} = t_{n+m}' = t \). For times outside the domain \([0, t] \) these switch functions are defined to be zero. The influence of the DD pulses can then be expressed in a rather compact form. Still taking the decay of \( \rho_0(t) \) as an example, we obtain

\[
\bar{\rho}_0(t) = \alpha^* \beta e^{-\Delta F_3(t)} + \beta^* \gamma e^{-\Delta F_2(t)} + \gamma^* \alpha e^{-\Delta F_1(t)}
\]

To proceed we next define three filter functions \([9]\) from the Fourier transform of \( s_i(t) \), i.e.

\[
\int_{-\infty}^{\infty} s_1(t') e^{i\omega t'} dt' = \frac{i}{\omega} y_m(\omega t'),
\]

\[
\int_{-\infty}^{\infty} s_2(t') e^{i\omega t'} dt' = \frac{i}{\omega} y_m(\omega t'),
\]

\[
\int_{-\infty}^{\infty} s_3(t') e^{i\omega t'} dt' = \frac{i}{\omega} y_{n+m}(\omega t').
\]

Here, the filter function with \( M \) (which is \( n, m \) or \( n+m \)) pulses is defined as

\[
y_M(\omega t) = 1 + (-1)^{M+1} e^{i\omega t} + 2 \sum_{j=1}^{M} (-1)^j e^{i\omega t_j},
\]
assuming the statistical independence of \( f_1(t) \) and \( f_2(t) \). If \( f_1(t) \) and \( f_2(t) \) has nonzero correlations (noise under this correlated situation may also be called nonlocal noise, which is much different from our case here), then many of the density matrix elements cannot be evaluated analytically.

3. Optimization procedure

Next we aim to optimize the pulse locations to keep \( \tilde{\rho}(t) \) as close as possible to the initial state. Taking the trace fidelity \( C(t) = \text{Tr}[\tilde{\rho}(t)\rho(0)] \) as a measure of DD performance, it is straightforward to carry out the optimization if the initial state is known. However, in many cases, a two-qubit state to be protected or stored is unknown, and as such the average fidelity for all possible initial states can be of more interest. Averaging \( C(t) \) over all initially pure states and using the fact that \( |\alpha|^2 = |\beta|^2 = |\gamma|^2 = |\eta|^2 = \frac{1}{2} \), we arrive at

\[
\tilde{C}(t) = \frac{1}{4} + \frac{1}{4} (e^{-\Gamma_1 t/2} + e^{-\Gamma_2 t/2} + e^{-\Gamma_3 t/2} + e^{-\Gamma_4 t/2}).
\]  

(18)

The optimization then becomes the minimization of \( \Phi(t) = 4|1 - C(t)| \). The function \( \Phi(t) \), called the performance function below, is given by

\[
\Phi(t) = 3 - (e^{-\Gamma_1 t} + e^{-\Gamma_2 t} + e^{-\Gamma_3 t} + e^{-\Gamma_4 t}).
\]  

(19)

Clearly \( 0 \leq \Phi(t) \leq 3 \), and a smaller value of \( \Phi(t) \) indicates a higher degree of decoherence suppression. Therefore, \( \Phi(t) \) can also be regarded an error function.

To minimize the value of the performance function, we first consider a total of \( n + m \) pulses, among which \( m \) pulses are applied to the second qubit. Assuming that the \( n + m \) pulses are applied at the timings

\[
\delta_1 < \delta_2 < \delta_3 < \cdots < \delta_{n+m},
\]

one first needs to pick out \( m \) pulse locations to apply the \( \sigma_z \) control operator. The number of such choices is given by the combination \( C_{n+m}^m \). Then the analytic form of the three filter functions \( y_n(\omega) \), \( y_m(\omega) \) and \( y_{n+m}(\omega) \) can be fixed. We next minimize the performance function \( \Phi(t) \) by optimizing the \( (n + m) \) pulse locations using the line-search algorithm.

In every iteration, the algorithm searches for the minimum along the direction set by a gradient. We use \( n+m \) equally spacing pulses and a two-layer nested-UDD sequence as our initial guess. Once the result converges, we obtain the locally optimized locations for \( n+m \) control pulses in a particular \( (n+m, m) \) scheme. To find the optimal pulse locations, we test all \( C_{n+m}^m \) possible choices of pulse allocations for the second qubit. Furthermore, by scanning the value of \( m \) from 1 to the total number of pulses, we can find an optimal pulse-number partition for a fixed total number of pulses.

4. Performance of optimized sequences

In the following, for convenience we set \( t = 1 \) when comparing cases of different noise spectra. Effects of varying \( t \) can be understood as the result of a rescaling of the strength and shape of the noise spectrum [20]. To appreciate that the issue of two-qubit decoherence suppression is significantly different from a one-qubit problem, let us first illustrate the influence of the nonlocal noise term \( f_3(t)\sigma_z\sigma_z \) on the performance of DD with \( n + m = 8 \). Computational examples are shown in table 1. It is seen that in the absence of nonlocal noise, the optimized performance (error) is of the order of \( 10^{-10} \), and the optimized DD pulses are applied at

\[
[0.10 \quad 0.10 \quad 0.35 \quad 0.35 \quad 0.65 \quad 0.65 \quad 0.90 \quad 0.90],
\]

which is simply two optimized 4-pulse sequences simultaneously applied to the two qubits. However, this DD sequence cannot suppress any nonlocal noise because every pair of such simultaneous pulses will keep the sign of \( \sigma_z \), unchanged, and the associated decay exponent \( \Gamma_3 \) would be the same as that without DD, which is given by

\[
\Gamma_3 = 4 \int_0^\infty S_j(\omega) \sin^2(\omega/2) \frac{\sin^2(\omega/2)}{\omega^2} d\omega.
\]  

(20)

Interestingly, as shown in table 1, even with a very weak nonlocal noise added, e.g. \( S_j(\omega) = 0.1 \omega \Theta(0.1 - \omega) \), where \( \Theta(\cdot) \) is the Heaviside step function, the minimal error that can be achieved by an optimized DD sequence is already increased by five orders of magnitude! This clearly addresses the importance of taking nonlocal noise into consideration for decoherence suppression. With this understanding we are now ready to examine the optimization of DD in two-qubit systems.

4.1. Spectrum with hard cutoff

Previous work showed that in single-qubit cases, an optimized DD sequence can greatly outperform UDD for an ohmic noise spectrum with hard cutoff [10, 18–20]. It is observed that this finding also holds in our two-qubit dephasing model. Here, we set the local noise spectrum of the two qubits to be the same, and then vary the intensity of the nonlocal noise spectrum, from twice as much as the local noise spectrum to a relatively weak one. As seen from table 2, the optimized DD sequence is in general much better than nested-UDD. In particular, for a 15-pulse sequence \( (n + m = 15) \), the optimization improves the performance by many orders of magnitude as compared with a two-layer nested-UDD(3) (here nested-UDD(5) means that 5 pulses applied to the second qubit in the outside layer, and within each pulse interval a k-pulse UDD sequence is applied to the first qubit). As a matter of fact, by comparing table 2 and

| Nonlocal spectrum \( S_j(\omega) \) | Pulse location | Performance |
|---------------------------------|---------------|-------------|
| \( 2\omega \Theta(2 - \omega) \) | 2,4,6,8 | 4.59 \times 10^{-5} |
| \( 0.5 \omega \Theta(0.5 - \omega) \) | 2,4,6,8 | 4.59 \times 10^{-5} |
| \( 0.1 \omega \Theta(0.1 - \omega) \) | 2,4,6,8 | 4.43 \times 10^{-5} |
| \( \frac{2\omega}{\pi^2} \) (infinite cutoff) | 2,4,6,8 | 1.67 \times 10^{-3} |
| No nonlocal noise | 2,4,6,8 | 4.08 \times 10^{-10} |

| Pulse location | Performance |
|---------------|-------------|
| 2,4,6,8 | 4.59 \times 10^{-5} |
| 2,4,6,8 | 4.59 \times 10^{-5} |
| 2,4,6,8 | 4.43 \times 10^{-5} |
| 2,4,6,8 | 1.67 \times 10^{-3} |
| 2,4,6,8 | 4.08 \times 10^{-10} |
Table 2. Performance of optimized two-qubit DD sequences as compared with nested-UDD for a few examples of local and nonlocal noise spectrum. Performance is optimized for each combination $C^m_{an}$ taking into account all possibilities of allocating $m$ pulses to the second qubit. In the column of pulse location, a listed integer $j$ means that the $j$th pulse is a $\sigma_{xj}$ pulse (i.e. applied to the second qubit) during the first half of the dynamics. The second half of the pulse sequence is symmetric to the first half. The ohmic noise spectrum with hard cutoff is considered.

| Pulse combination | Pulse location | Performance |
|-------------------|----------------|-------------|
| $S_1 = \omega \Theta(1 - \omega)$, $S_2 = \omega \Theta(1 - \omega)$, $S_3 = 2\omega \Theta(2 - \omega)$ | 3 | $7.32 \times 10^{-4}$ |
| Nested-UDD(2) | 3 | $8.66 \times 10^{-5}$ |
| $C^4_2$ | 2.4 | $4.59 \times 10^{-5}$ |
| Nested-UDD(3) | 4.8 | $2.45 \times 10^{-6}$ |
| $C^4_{15}$ | 4.8 | $3.04 \times 10^{-7}$ |
| Nested-UDD(3) | 2.5, 8 | $6.14 \times 10^{-9}$ |
| $C^4_{15}$ | 1.3, 5, 7, 8 | $1.17 \times 10^{-10}$ |
| $S_1 = \omega \Theta(1 - \omega)$, $S_2 = \omega \Theta(1 - \omega)$, $S_3 = 0.5\omega \Theta(0.5 - \omega)$ | 3 | $3.26 \times 10^{-4}$ |
| Nested-UDD(2) | 3 | $8.14 \times 10^{-5}$ |
| $C^4_2$ | 2.4 | $4.59 \times 10^{-5}$ |
| Nested-UDD(3) | 4.8 | $1.66 \times 10^{-6}$ |
| $C^4_{15}$ | 3.8 | $1.88 \times 10^{-7}$ |
| Nested-UDD(3) | 3.5, 8 | $7.06 \times 10^{-11}$ |
| $C^4_{15}$ | 1.3, 4, 6, 8 | $6.26 \times 10^{-10}$ |

Table 3. Same as in table 2, but for ohmic noise spectrum with hard cutoff at larger frequencies and for $1/f$ spectrum with hard cutoff.

| Pulse combination | Pulse location | Performance |
|-------------------|----------------|-------------|
| $S_1 = \omega \Theta(5 - \omega)$, $S_2 = \omega \Theta(5 - \omega)$, $S_3 = \omega \Theta(3 - \omega)$ | 3 | $1.55$ |
| Nested-UDD(2) | 3 | $0.80$ |
| $C^4_6$ | 2.4 | $0.54$ |
| Nested-UDD(3) | 4.8 | $0.36$ |
| $C^4_{15}$ | 3.8 | $6.63 \times 10^{-2}$ |
| $C^4_{15}$ | 2.4, 6, 8 | $1.48 \times 10^{-6}$ |
| $S_1 = \frac{1}{m} \Theta(10 - \omega)$, $S_2 = \frac{1}{m} \Theta(10 - \omega)$, $S_3 = \frac{1}{m} \Theta(5 - \omega)$ | 3 | $0.61$ |
| Nested-UDD(2) | 3 | $0.60$ |
| $C^4_6$ | 2.4 | $0.41$ |
| Nested-UDD(3) | 4.8 | $0.32$ |
| $C^4_{15}$ | 4.8 | $0.22$ |
| $C^4_{15}$ | 2.4, 6, 8 | $9.96 \times 10^{-5}$ |

Figure 1. Comparison of specific pulse locations between nested-UDD and our optimized two-qubit DD sequences. $S_1 = \omega \Theta(5 - \omega)$, $S_2 = \omega \Theta(5 - \omega)$ and $S_3 = \omega \Theta(3 - \omega)$. The left panel is for $n + m = 8$, and the right panel is for $n + m = 15$. The grey (red) lines stand for timings of the pulses applied to the second qubit.

In single-qubit cases [9, 18–20], LODD is more powerful for a noise spectrum with a larger cutoff. This motivated us to investigate some representative cases with a larger frequency cutoff. Here we consider ohmic and $1/f$ spectrum in table 3 (with cutoff). Here the $1/f$-type noise is of interest because it has been observed in many solid-state implementations, especially in superconducting qubits [26, 29, 30]. As seen from table 3, in both cases the two-qubit dephasing can be greatly suppressed. Interestingly, although the ohmic and $1/f$ cases represent drastically different noise spectrum, in these two cases the optimized 15-pulse two-qubit DD sequences give similar performance that is five or six orders of magnitude better than nested-UDD(3). Clearly then, the improvement afforded by optimization is sensitive to the cutoff frequency value, but rather insensitive to the shape of the noise spectrum before the cutoff. Similar improvements have been observed previously in single qubit LODD [9, 18].

In figure 1 we show the optimized pulse locations in the case of $S_1 = \omega \Theta(5 - \omega)$, $S_2 = \omega \Theta(5 - \omega)$ and $S_3 = \omega \Theta(3 - \omega)$. In the case of $n + m = 8$ with $m = 2$, the locations of all $\sigma_{xj}$ pulses in the optimized DD sequence differ remarkably from that in nested-UDD(2). In the case of $n + m = 15$ with $m = 3$, the first $\sigma_{x1}$ pulse is the third
pulse in the optimized two-qubit sequence but the fourth pulse in nested-UDD(3). These observations might be relevant to future experiments.

4.2. Spectrum with a soft cutoff

The noise spectrum with a soft cutoff only gradually decays to zero. The associated task of decoherence suppression is more challenging. References [9, 18, 19, 26] showed that for single-qubit dephasing caused by soft-cutoff noise, UDD can only give a performance analogous to the Carr–Purcell–Meiboom–Gill (CPMG) sequence [31]. Mathematically, this can be understood from the decay exponents $\Gamma_1$ determined by an integral of a filter function multiplied by a noise spectrum. Because the noise spectrum is not rapidly vanishing in the high frequency regime, the behaviour of a filter function in the high frequency regime may be important and as a result the optimization becomes less effective.

Here we study the optimization of DD using various noise spectrum combinations, involving both super-ohmic ($\omega^p, p > 1$) spectrum and Lorentzian spectrum with soft cutoff. The results are presented in table 4. In the first super-ohmic case with an exponential soft cutoff, the improvement of optimized DD sequence over nested-UDD is still magnificent (note that a non-optimized DD sequence might even speed up the decoherence process [32]). Indeed, the exponential cutoff can be regarded as an intermediate case between a truly slow local noise spectrum and Lorentzian spectrum with soft cutoff.

| Pulse combination | Pulse location | Performance |
|-------------------|----------------|-------------|
| $S_1 = \omega_0 e^{-\omega t}$, $S_2 = \omega_0 e^{-\omega t}$, $S_3 = \omega_0 e^{-\omega t}$ | 3 | $5.31 \times 10^{-3}$ |
| Nested-UDD(2) | 2.4 | $1.04 \times 10^{-3}$ |
| $S_1 = \omega_0 \Theta(1 - \omega)$, $S_2 = \omega_0 \Theta(1 - \omega)$, $S_3 = \frac{\omega_0}{\omega^2 + \omega_1^2}$ | 4.8 | $1.44 \times 10^{-4}$ |
| Nested-UDD(3) | 2.58 | $5.25 \times 10^{-5}$ |

4.3. Asymmetric local spectrum

So far local noise spectrum is assumed to be Lorentzian. For convenience, the noise spectrum is assumed to be a constant subject to a hard cutoff, i.e. $S_1(\omega) = 10(10 - \omega)$, $S_2(\omega) = 0.1\Theta(0.1 - \omega)$ and $S_3(\omega) = 0.05\Theta(0.05 - \Omega)$. For these noise parameters, the dimensionless dephasing rate $\Gamma_1$ is highly limited, then it is beneficial to control only one of the two qubits. The optimization results are shown in table 5. At least two interesting observations can be made. First, in the case of $n + m = 4$, assigning some control pulses to the second qubit may bring down the performance. Hence, if the local noise spectrum is strongly unbalanced and if $n + m$ is highly limited, then it is beneficial to control only one of the two qubits. This is consistent with the experimental implementation in [25]. Second, when the minimized total error approaches the error caused by the free decay of the second qubit, solely applying $\sigma_z$ pulses can no longer improve the DD performance. As shown in table 5, the DD performance without any $\sigma_z$ pulses is bounded by about $10^{-2}$. However, if now two $\sigma_z$ pulses are applied, then the DD performance can be improved by many orders of magnitude (e.g. in the 12-pulse case). Note also that for fixed $n + m$, increasing $m$ too much undermines the performance. Again, this is because the

| Pulse combination | Pulse location | Performance |
|-------------------|----------------|-------------|
| $S_1 = 10\Theta(10 - \omega)$, $S_2 = 0.1\Theta(0.1 - \omega)$, $S_3 = 0.05\Theta(0.05 - \Omega)$ | No pulse | 1.3 |
| $C_7^*$ | 2 | 2.00 |
| $C_7^*$ | No pulse | $2.00 \times 10^{-2}$ |
| $C_7^*$ | 3 | $7.64 \times 10^{-3}$ |
| $C_7^*$ | 1.3 | 1.30 |
| $C_7^*$ | 1.2,4 | 2.00 |
| $C_7^*$ | No pulse | $1.99 \times 10^{-2}$ |
| $C_7^*$ | 4 | $1.57 \times 10^{-7}$ |
| $C_7^*$ | 3.5 | $6.25 \times 10^{-6}$ |
| $C_7^*$ | 2.4,6 | $7.45 \times 10^{-3}$ |
| $C_7^*$ | 1.3,4,6 | 1.29 |
the optimization algorithm becomes much slower in each run and may fail to give a converged result. For example, for \( n + m = 24 \), the converged pulse locations might be inappropriate, i.e. violating the initial ordering of the pulse locations. Worse still, for \( n + m = 24 \), we have to run the algorithm 924 times for the single combination \( C^2_{12} \) (12 pulses for the second qubit). Hence computationally it is prohibitively expensive if hundreds of pulses need to be optimized. Second, as the pulse number increases, the optimized result becomes more sensitive to the initial guess. Mathematically it is also a challenging question to find the global minimum in a high-dimensional parameter space. For \( n + m = 24 \), we have considered two initial guesses, namely, an equally spaced DD sequence and a nested-UDD(4) sequence. The associated results are shown in table 6. It is seen that for the same noise spectrum, the performance is only slightly better than that for \( n + m = 15 \). It is hence possible that our result as a locally optimized result is still far away from the globally optimized result. More effective optimization algorithms together with more initial guesses may be the ultimate solution.

It should also be noted that the total pulse number is physically limited. Realistic pulses are imperfect and an ideal instantaneous \( \pi \) rotation is never possible [33]. As a result, the pulse-to-pulse errors may accumulate with the increase of the pulse number [34, 35] and hence more pulses do not necessarily lead to better performance. Furthermore, constraint on the minimal pulse interval might place another intrinsic limit on the pulse number we could consider [36]. For fixed total evolution time \( t \), such a constraint imposes an upper bound for the pulse number. For a varying \( t \), the optimized performance in practice will be connected with the available pulsing rate as well as the spectral bandwidth [37, 38].

5. Conclusions

Using a pure dephasing model incorporating both local and nonlocal Gaussian noise, we have studied how the DD protection of two-qubit states can be better achieved by optimizing the pulse locations as well as the partition of the pulse numbers allocated to each qubit. Compared with nested-UDD as a general-purpose DD scheme for two-qubit systems, our optimization procedure may improve the performance
by many orders of magnitude, using only a few tens of instantaneous $\pi$ pulses. This makes it possible to use much less pulses to obtain a given fidelity of decoherence control in two-qubit systems. The price of this performance gain is the required knowledge of the noise spectrum. In addition, it is also seen that if the noise spectrum decays to zero slowly (e.g. a Lorentzian shape), then even our optimized DD may not perform very well and it is only slightly better than nested-UDD.

The results here also help to gain more insights into the issue of DD protection in two-qubit or multi-qubit systems. The importance of fighting against the nonlocal noise in two-qubit decoherence control (or more generally, the importance of taking into account the impact of the dynamics of one qubit on the environment of the other qubit) becomes clearer. Without the nonlocal noise, our optimized DD sequence simply degenerates into two single-qubit optimized DD sequences. In the presence of nonlocal noise, the optimized two-qubit DD sequence is quite different from two optimized single-qubit DD sequences, because the pulse numbers and pulse locations for each qubit should be adjusted (in a somewhat subtle manner) based on both nonlocal and local noise. If the local noise spectrum of the two qubits is highly unbalanced, then most pulses should be applied to one of the two qubits to defeat its rapid dephasing. It is these specific strategies, which are not exploited by the rather universal nested-UDD scheme, that make optimization possible. Note also that in nested-UDD schemes, the number of control pulses or the control order may be different for different control layers. But as seen from one optimization result in section 4.3, the performance of such a type of asymmetric-nested-UDD is still far from being optimized.

There have been wide experimental interests in both fundamental aspects of decoherence and the DD approach to decoherence suppression. Two types of experiments may be motivated by this work. First, for two-qubit systems with a known noise spectrum (local and nonlocal), it is of immediate interest to apply optimized DD sequences to extend the lifetime of entangled states, hopefully with better efficiency. Second, one may experimentally study the nonlocal noise of a two-qubit system by the use of single-qubit DD. That is, if two single-qubit DD sequences are applied on top of each other, then (assuming that the local noise of the two qubits are independent) only the nonlocal noise is not suppressed and hence its impact on two-qubit dephasing may be directly observed. Such types of experiments are of importance to the design of optimized two-qubit DD sequences and to the understanding of decoherence mechanism in two-qubit systems embedded in a solid-state environment.

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