Equivalence groupoids of classes of linear ordinary differential equations and their group classification

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Abstract. Admissible point transformations of classes of rth order linear ordinary differential equations (in particular, the whole class of such equations and its subclasses of equations in the rational form, the Laguerre–Forsyth form, the first and second Arnold forms) are exhaustively described. Using these results, the group classification of such equations is carried out within the algebraic approach in three different ways.

1. Introduction
The study of Lie symmetries of ordinary differential equations (ODEs) has a long history, and “Lie theory” was just started as a systematical and elegant approach to integration of various classes of ODEs. The first results on possible dimensions of the maximal Lie invariance algebras of ODEs of any fixed order were obtained by Sophus Lie, see, e.g., [1, pp 294–301] and a modern treatment in [2, Section 2]. Namely, Sophus Lie proved that the dimension of the maximal Lie invariance algebra of an rth order ODE is infinite for r = 1, not greater than 8 for r = 2 and not greater than r + 4 for r ⩾ 3. He also showed that each ODE of order r = 1 is similar with respect to a point transformation to the elementary equation x' = 0 and that, for equations of order r ⩾ 2, the maximal dimension of invariance algebras is reached for equations that are reduced by point transformations to the elementary equation x(r) = 0; cf. [3, Theorem 14] and [4, Theorems 6.39 and 6.43].

In spite of the fact that transformational and, in particular, symmetry properties of linear ODEs were intensively investigated (see, e.g., detailed reviews [5–7] and textbooks [4,8–10]), in the present paper we consider them from another side, describing the whole set of admissible transformations between such equations. This creates a basis for the group classification of linear ODEs within the framework of the algebraic approach, which is quite effective for solving group classification problems for both ordinary and partial differential equations; see, e.g., [11–15] and references therein. Previously, in [16, 17], the group classification of linear ODEs was carried out within the framework of the classical infinitesimal approach, which led to cumbersome calculations. Although the infinitesimal approach is the most commonly used in group analysis of differential equations, it is efficient only for classes of simple structure. In [4, pp 217–218] the...
solution of the group classification problem of linear ODEs was related to Wilczynski’s result [18] on relative invariants of the Laguerre–Forsyth form of these equations. The similar problem on the classification of linear ODEs up to contact transformations as well as the associated equivalence problem were considered in detail in [19–21].

The main purpose of the present paper is to carry out the complete group classification of the class $L$ of $r$th order ($r \geq 3$) linear ODEs in more elegant algebraic ways, using subalgebra analysis of the equivalence algebra associated with $L$. This properly works since the class $L$ is (pointwise) normalized (in the usual sense), i.e., transformations from its (usual) point equivalence (pseudo)group $G^\sim$ generate all admissible point transformations between equations from $L$. The group classification of the class of second-order linear ODEs is trivial since its equivalence group acts transitively. Note that the equivalence group $G^\sim$ of the class $L$ with $r \geq 2$ was first found by Stäckel [22].

The set of admissible transformations of any class of differential equations possesses the groupoid structure and is called the equivalence groupoid of this class [14, 15]. See, e.g., [13–15, 26] for the definition of normalized classes and other related notions. So, we can say that the equivalence groupoid $G^\sim$ of the class $L$ with $r \geq 3$ is generated by its equivalence group $G^\sim$.

In Section 2 we begin the study of the class $L$ of $r$th order ($r \geq 2$) linear ODEs with the description of its equivalence groupoid in terms of its equivalence group and normalization. As a rule, we present only the components of equivalence transformations that correspond to the dependent and independent variables. One can gauge arbitrary elements of the class $L$ by parameterized families of transformations from $G^\sim$, which induces mappings of the class $L$ onto its subclasses. Two such gauges for arbitrary elements related to the subleading-order derivatives are well known. They result in the rational form with the first subleading coefficient being equal to zero (the subclass $L_1$) and Laguerre–Forsyth form with the first two subleading coefficients being equal to zero (the subclass $L_2$). It appears that for $r \geq 3$ both the subclasses $L_1$ and $L_2$ are also normalized with respect to their equivalence groups. Then we study two gauges for arbitrary elements related to the lowest-order derivatives, which give the first and second Arnold forms. The corresponding subclasses are even not semi-normalized and hence these gauges are not convenient for symmetry analysis. Having the chain of nested normalized classes $L \supset L_1 \supset L_2$ for $r \geq 3$ and the associated chain of classes of homogeneous equations, which are peculiarly semi-normalized, we can classify Lie symmetries of $r$th order linear ODEs within the algebraic approach in three different ways, which is done in Section 3. In the final section we summarize results of the paper and discuss their connection with possible approaches to solving group classification problems for classes of systems of linear differential equations.

2. Equivalence groupoids of classes of linear ODEs

Consider the class $L$ of $r$th order linear ODEs, which have the form

$$x^{(r)} + a_{r-1}(t)x^{(r-1)} + \cdots + a_1(t)x^{(1)} + a_0(t)x = b(t),$$

where $a_{r-1}, \ldots, a_1, a_0$ and $b$ are arbitrary smooth functions of $t$, $x = x(t)$ is the unknown function, $x^{(k)} = d^k x/dt^k$, $k = 1, \ldots, r$, $r \geq 2$. Below we also use the notation $x' = dx/dt$

1 There exist other names for this notion, e.g., “structure invariance group” [10]. The attribute “usual” and the prefix “pseudo-” are usually omitted for usual equivalence pseudogroups. We will also say “normalization” without attributes in the case of pointwise normalization in the usual sense.

2 An admissible (point) transformation of a class of differential equations is a triple of the form $(E, \hat{E}, T)$. Here $E$ and $\hat{E}$ are equations from the class or, equivalently, the corresponding values of the arbitrary elements parameterizing the class. The transformational part $T$ of the admissible transformation is a point transformation mapping the equation $E$ to the equation $\hat{E}$.

3 Recall also the contribution by Halphen [23], Laguerre [24], Forsyth [25] and Wilczynski [18] in the study of point transformations between linear ODEs.
and $x'' = d^2x/dt^2$ for the first and second derivatives, respectively. The subscripts $t$ and $x$ denote differentiation with respect to the corresponding variables. We assume that all variables, functions and other values are either real or complex, i.e., the underlying field $\mathbb{F}$ is either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, respectively. We work within the local approach.

2.1. General class

There are two different cases for the structure of the equivalence groupoid of the entire class $\mathcal{L}$ depending on the value $r$, namely $r = 2$ and $r \geq 3$. We begin with the case $r = 2$.

**Proposition 1.** The equivalence group $G^\sim$ of the class $\mathcal{L}$ with $r = 2$ consists of the transformations whose projections to the variable space$^4$ have the form

$$\tilde{t} = T(t), \quad \tilde{x} = X_1(t)x + X_0(t),$$

(2)

where $T$, $X_1$ and $X_0$ are arbitrary smooth functions of $t$ with $T_t X_1 \neq 0$.

**Proof.** Suppose that a point transformation $\mathcal{T}$ of the general form

$$\tilde{t} = T(t,x), \quad \tilde{x} = X(t,x),$$

(3)

where the Jacobian $J = |\partial(T,X)/\partial(t,x)|$ does not vanish, $J \neq 0$, connects two fixed second-order linear ODEs $\mathcal{E}$ and $\tilde{\mathcal{E}}$. We substitute the expressions for the new variables (which are with tildes) and the corresponding derivatives in terms of the old variables (which are without tildes) into $\tilde{\mathcal{E}}$. The equality obtained should be identically satisfied on solutions of the equation $\mathcal{E}$. Therefore, additionally substituting the expression for $x''$ implied by $\mathcal{E}$, we can split the equality with respect to the derivative $x'$. Collecting the coefficients of $(x')^3$, we obtain the equation

$$X_{xx}T_x - X_xT_{xx} + \tilde{a}_1 X_x T_x^2 + (\tilde{a}_0 X - \tilde{b})T_x^3 = 0.$$

As $\tilde{a}_1$, $\tilde{a}_0$ and $\tilde{b}$ are the only arbitrary elements involved in this equation and we study the equivalence group, we can vary the arbitrary elements and hence split with respect to them. Hence $T_x = 0$, i.e., $T = T(t)$. Then the terms with $(x')^2$ give the equation $T_t X_{xx} = 0$. As the condition $J \neq 0$ reduces to the inequality $T_t X_x \neq 0$, we have $X = X_1(t)x + X_0(t)$ with $X_1 \neq 0$. The other determining equations, which are derived by the additional splitting with respect to $x'$ and $x$, define the transformation components for the arbitrary element as functions of the variables and the arbitrary elements. \hfill \Box

**Proposition 2.** The equivalence groupoid of the class $\mathcal{L}$ of second-order linear ODEs is generated by compositions of transformations from the equivalence group $G^\sim$ of this class with transformations from the point symmetry group of the equation $x'' = 0$. Therefore, the class $\mathcal{L}$ is semi-normalized but not normalized.

**Proof.** The free particle equation $x'' = 0$ admits point symmetry transformations that are truly fractional linear with respect to $x$ or whose components for $t$ depend on $x$. Each of these properties is not consistent with the transformation form (2). This is why there exists admissible transformations in the class $\mathcal{L}$ that are not generated by its equivalence transformations, i.e., this class is not normalized.

$^4$ Recall that for all equivalence transformations we present only the components corresponding to the variables $t$ and $x$ since each pair of these components completely determine the corresponding equivalence transformation. As the form (2) is the most general for relevant equivalence transformations, the transformation components for arbitrary elements can be derived using Faà di Bruno’s formula and the general Leibniz rule, and thus they are quite cumbersome.
It is commonly known that any second-order linear ODE \( E \) is locally reduced to the equation \( x'' = 0 \) by an equivalence transformation, so-called Arnold transformation,

\[
\tilde{t} = \frac{\varphi_2(t)}{\varphi_1(t)}, \quad \tilde{x} = \frac{x - \varphi_0(t)}{\varphi_1(t)},
\]

where \( \varphi_0 \) is a particular solution of the equation \( E \), \( \varphi_1 \) and \( \varphi_2 \) are linearly independent solutions of the corresponding homogeneous equation, see, e.g., [27, p 43] or [28]. In other words, the class \( \mathcal{L} \) is a single orbit of its equivalence group \( G^\sim \). Any class with this property is semi-normalized. We show this in detail. Consider two fixed equations \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) from the class \( \mathcal{L} \) with \( r = 2 \) and a point transformation \( \mathcal{T} \) linking these equations. Let \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) be the projections of elements of \( G^\sim \) that map the equations \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), respectively, to \( x'' = 0 \). Then the transformation \( \mathcal{T}_0 := \mathcal{T}_2 \mathcal{T}_1^{-1} \) belongs to the point symmetry group of the equation \( x'' = 0 \). This implies the representation \( \mathcal{T} = \mathcal{T}_2^{-1} \mathcal{T}_1 \mathcal{T}_1 \). Roughly speaking, \( \mathcal{T} \) is the composition of the equivalence transformations \( \mathcal{T}_1 \) and \( \mathcal{T}_2^{-1} \) and the symmetry transformation \( \mathcal{T}_0 \) of the equation \( x'' = 0 \). It is obvious that any transformation possessing such a representation maps the equation \( \mathcal{E}_1 \) to the equation \( \mathcal{E}_2 \). The above representation can be rewritten as \( \mathcal{T} = \mathcal{T} \mathcal{T} \), where \( \mathcal{T} = \mathcal{T}_2^{-1} \mathcal{T}_1 \) is an equivalence transformation of the class \( \mathcal{L} \) and \( \mathcal{T} = \mathcal{T}_1^{-1} \mathcal{T}_0 \mathcal{T}_1 \) is a symmetry transformation of the equation \( \mathcal{E}_1 \), which means that the class \( \mathcal{L} \) is semi-normalized.

**Remark 3.** The equivalence group \( G^\sim \) induces all fiber-preserving admissible transformations of the class \( \mathcal{L} \) with \( r = 2 \). This directly follows from the fact that imposing the constraint \( T_x = 0 \) for admissible transformations also implies the condition \( X_{xx} = 0 \).

In what follows we consider the class \( \mathcal{L} \) with \( r \geq 3 \). Although the projections of transformations from the equivalence group to the variable space in the case \( r \geq 3 \) coincide with that in the case \( r = 2 \), the corresponding equivalence groupoids have different structures.

**Proposition 4.** The equivalence group \( G^\sim \) of the class \( \mathcal{L} \), where \( r \geq 3 \), consists of the transformations whose projections to the variable space have the form (2). This group generates the entire equivalence groupoid of the class \( \mathcal{L} \), i.e., the class \( \mathcal{L} \) is normalized.

**Proof.** In order to study admissible transformations in the class \( \mathcal{L} \), we consider a pair of equations from this class, namely an equation \( \mathcal{E} \) of the form (1) and an equation \( \tilde{\mathcal{E}} \) of the same form, where all variables, derivatives and arbitrary elements are with tildes, and assume that these equations are connected by a point transformation \( \mathcal{T} \) of the general form (3). At first we express derivatives with tildes in terms of the variables without tildes,

\[
\tilde{x}^{(k)} = \left( \frac{1}{DT} D \right)^k X,
\]

where \( D = \partial_t + x' \partial_x^1 + x'' \partial_x^2 + \cdots \) is the total derivative operator with respect to the variable \( t \). After substituting the expressions for the variables and derivatives with tildes into \( \tilde{\mathcal{E}} \), we derive an equation in the variables without tildes. It should be an identity on the manifold determined by \( \mathcal{E} \) in the \( r \)th order jet space with the independent variable \( t \) and the dependent variable \( x \). The coefficient of \( x^{(r)x^{(r-1)}} \) in this equation equals

\[
- \frac{J}{(DT)^{r+2}} T_x \left( 3 + \frac{(r-2)(r+3)}{2} \right) = 0,
\]

and hence \( T_x = 0 \), i.e., the function \( T \) does not depend on the variable \( x \), \( T = T(t) \). The nondegeneracy condition \( J \neq 0 \) is simplified to \( T_x X_x \neq 0 \). Taking into account the condition \( T_x = 0 \), we collect coefficients of \( x^{(r)x^{(r-1)}} \), which gives \( rT_x^{-r} X_{xx} = 0 \). This equation implies that
\(X_{xx} = 0\), i.e., \(X\) is a linear function of \(x\), \(X = X_1(t)x + X_0(t)\). Therefore, the transformation \(T\) has the form (2). The other determining equations, which are obtained by splitting with respect to derivatives of \(x\) after substituting for \(x^{(r)}\) in view of \(E\), establish the relation between arbitrary elements of the initial and the transformed equations.

The transformation \(T\) maps any equation from the class \(L\) to another equation from the same class, and its prolongation to the arbitrary elements \(a_{r-1}, \ldots, a_0\) and \(b\), which is given by the above relation, is a point transformation in the joint space of the variables and the arbitrary elements. Hence such prolongations of the transformations of the form (2) constitute the (usual) equivalence group \(G^\sim\).

Consider the corresponding subclass \(\hat{L}\) of \(r\)th order \((r \geq 2)\) linear homogeneous ODEs, which is singled out from the class \(L\) by the constraint \(b = 0\). The arbitrary element \(b\) can be gauge to zero by equivalence transformations. Namely, the class \(L\) is mapped to its subclass \(\hat{L}\) by a family of point transformations with \(T = t\), \(X^1 = 1\) and \(X^0\) being a particular solution of the initial equation, and thus these transformations are parameterized by \(b\).

Corollary 5. The equivalence group \(\hat{G}^\sim\) of the subclass \(\hat{L}\) is obtained from the equivalence group \(G^\sim\) of the class \(L\) by setting \(X_0 = 0\) and neglecting the transformation component for \(b\).

Corollary 6. The subclass \(\hat{L}\) with \(r = 2\) is a single orbit of the elementary equation \(x'' = 0\) under the action of the equivalence group \(\hat{G}^\sim\). Hence the subclass \(\hat{L}\) is semi-normalized but not normalized.

Corollary 7. Given any equation \(E\) from the subclass \(\hat{L}\) with \(r \geq 3\), a point transformation maps \(E\) to another equation from the same subclass if and only this transformation has the form (2), where the ratio \(X_0/X_1\) is a solution of \(E\).

In other words, the transformational part of any admissible transformation within the class \(\hat{L}\) with \(r \geq 3\) can be represented as the composition of a linear superposition symmetry transformation of the initial equation with the projection of an element of the equivalence group \(\hat{G}^\sim\) to the variable space. For all equations from \(\hat{L}\), the associated groups of linear superposition symmetry transformations are of the same structure. In particular, they are commutative and \(r\)-dimensional. Therefore, although the class of \(\hat{L}\) is not normalized, it is semi-normalized in a quite specific way, which is a particular case of so-called uniform semi-normalization.\(^5\) For short, in similar situations we will say that a class is uniformly semi-normalized with respect to linear superposition of solutions.

2.2. Rational form

Using parameterized families of transformations from the equivalence group \(G^\sim\), we can gauge arbitrary elements of the class \(L\). For example, we can set \(a_{r-1} = 0\). This gauge can be realized by the parameterized family of projections of equivalence transformations to the \((t, x)\)-space

\[
\tilde{t} = t, \quad \tilde{x} = \exp\left(\frac{1}{r} \int a_{r-1}(t)dt\right) x,
\]

which maps the class \(L\) onto the subclass \(L_1\) of equations in the rational form

\[
x^{(r)} + a_{r-2}(t)x^{(r-2)} + \cdots + a_1(t)x' + a_0(t)x = b(t),
\]

where we omitted tildes over the variables and the arbitrary elements. This form was used in [16, 17] for the group classification of linear ODEs within the framework of the infinitesimal approach.

\(^5\) Similar properties are known for classes of homogeneous linear PDEs whose corresponding classes of (in general, inhomogeneous) linear PDEs are normalized [11].
**Proposition 8.** The equivalence group $G_1^\sim$ of the subclass $L_1$ consists of the transformations whose projections to the variable space have the form

$$\tilde{t} = T(t), \quad \tilde{x} = C(T(t))^{\frac{1}{1-\hat{r}}} x + X_0(t),$$

where $T$ and $X_0$ are arbitrary smooth functions of $t$ with $T_t \neq 0$, and $C$ is an arbitrary nonzero constant. The subclass $L_1$ with $r = 2$ is semi-normalized. If $r \geq 3$, then the group $G_1^\sim$ generates the equivalence groupoid of this subclass, i.e., it is normalized.

**Proof.** In the case $r = 2$ we follow the proof of Proposition 1 and derive the form (2) for equivalence transformations of the subclass $L_1$. Then further collecting coefficients of $x'$ and $x$ gives the equation

$$\frac{X_1}{T_t} \left( \frac{X_{1,t}}{X_1} - \frac{1}{2} \frac{T_{tt}}{T_t} \right) = 0,$$

which is integrated to the relation $X_1 = CT_t^{\frac{1}{1-\hat{r}}}$ with an arbitrary nonzero constant $C$, as well as the equivalence transformation components for the arbitrary elements $a_0$ and $b$ of the subclass $L_1$ with $r = 2$. The semi-normalization of this subclass is proved in the same way as Proposition 2.

In the case $r \geq 3$ we describe the entire equivalence groupoid. Suppose that an equation $E$ of the form (6) and an equation $\tilde{E}$ of the same form, where all variables, derivatives and arbitrary elements are with tildes, are connected by a point transformation $T$. In view of Proposition 4 this transformation has the form (2). We express the variables with tildes and the corresponding derivatives in terms of the variables and derivatives without tildes, substitute the expressions obtained into $\tilde{E}$ and collect terms containing the derivative $x^{(r-1)}$, which gives

$$r \frac{X_1}{T_t} \left( \frac{X_{1,t}}{X_1} - \frac{r-1}{2} \frac{T_{tt}}{T_t} \right) x^{(r-1)}.$$

The coefficient of $x^{(r-1)}$ vanishes only if $X_1 = CT_t^{\frac{1}{1-\hat{r}}}$, where $C$ is an arbitrary nonzero constant. Therefore, the point transformation $T$ has the form (7). Analogously to the end of the proof of Proposition 4, the transformations of this form when prolonged to the arbitrary elements $a_{r-2}, \ldots, a_0$ and $b$ constitute the equivalence group $G_1^\sim$ of the subclass $L_1$.

The status and properties of the corresponding subclass $\hat{L}_1$ of homogeneous equations within the class $L_1$ are the same as those of the subclass $\hat{E}$ within the class $E$.

**Corollary 9.** The equivalence group $\widehat{G}_1^\sim$ of the subclass $\hat{L}_1$ is derived from the equivalence group $G_1^\sim$ of the class $L_1$ by setting $X_0 = 0$ and neglecting the transformation component for $b$.

The equivalence groupoid of the subclass $\hat{L}_1$ is exhaustively described by the following two assertions depending on the value of $r$.

**Corollary 10.** If $r = 2$, then the subclass $\hat{L}_1$ is semi-normalized since it is a single orbit of the elementary equation $x'' = 0$ under the action of the group $\widehat{G}_1^\sim$.

**Corollary 11.** For each equation $E$ from the subclass $\hat{L}_1$ with $r \geq 3$, a point transformation is the transformational part of an admissible transformation in $\hat{L}_1$ with $E$ as source if and only if it is of the form (7), where the product $T_t^{\frac{1}{1-\hat{r}}} X_0$ is a solution of $E$. This means that this subclass is uniformly semi-normalized with respect to linear superposition of solutions.

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6 For even $r$, the power of $T_t$ is half-integer and hence the absolute value of $T_t$ should be substituted instead of $T_t$ in the real case or a branch of square root should be fixed in the complex case.
2.3. Laguerre–Forsyth form

Transformations from \( G_1^- \) are parameterized by an arbitrary function \( T = T(t) \) with \( T_t \neq 0 \). Hence we can set \( a_{r-2} = 0 \) in the equation (6) by a transformation from the group \( G_1^- \), where the parameter-function \( T \) is a solution of the equation

\[
T_{tt}T_t - \frac{3}{2}T_t^2 + \frac{12}{r(r^2 - 1)}a_{r-2}T_t^4 = 0.
\]

Thus, a family of such equivalence transformations parameterized by the arbitrary element \( a_{r-2} \) maps the subclass \( \mathcal{L}_1 \) onto the narrower subclass \( \mathcal{L}_2 \) of equations in the Laguerre–Forsyth form

\[
x^{(r)} + a_{r-3}(t)x^{(r-3)} + \cdots + a_1(t)x' + a_0(t)x = b(t). \tag{8}
\]

Note that, in contrast to the transformation (5), the above map does not preserve the corresponding subclass of linear ODEs with constant coefficients.

**Proposition 12.** The equivalence group \( G_2^- \) of the subclass \( \mathcal{L}_2 \) with \( r \geq 2 \) consists of the transformations whose projections to the variable space have the form

\[
\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{C}{(\gamma t + \delta)^{r-1}}x + X_0(t), \tag{9}
\]

where \( \alpha, \beta, \gamma, \delta \) and \( C \) are arbitrary constants with \( \alpha \delta - \beta \gamma \neq 0 \) and \( C \neq 0 \) that are defined up to obvious rescaling (so, only four constants among them are essential), and \( X_0 \) is an arbitrary smooth function of \( t \). The subclass \( \mathcal{L}_2 \) is semi-normalized in the case \( r = 2 \). If \( r \geq 3 \), then the group \( G_2^- \) generates the equivalence groupoid of this subclass, i.e., it is normalized.

**Proof.** We should again consider the cases \( r = 2 \) and \( r \geq 3 \) separately.

For \( r = 2 \) we repeat the proof of Proposition 1 and the corresponding part of the proof of Proposition 8 and derive the form (2) for equivalence transformations of the subclass \( \mathcal{L}_2 \). Then further collecting coefficients of \( x \) gives the equation

\[
\frac{T_{tt}}{T_t} - \frac{3}{2} \left(\frac{T_t}{T_t}\right)^2 = 0. \tag{10}
\]

In other words, the Schwarzian derivative of the function \( T \) vanishes, i.e., the function \( T \) is fractional linear,

\[
T(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \tag{11}
\]

where \( \alpha, \beta, \gamma, \delta \) are arbitrary constants with \( \alpha \delta - \beta \gamma \neq 0 \) that are defined up to nonvanishing constant multiplier. The rest of terms in the determining equation results in the equivalence transformation component for the arbitrary element \( b \) of the subclass \( \mathcal{L}_2 \) with \( r = 2 \). The semi-normalization of this subclass is proved in the same way as Proposition 2.

In order to describe the entire equivalence groupoid the subclass \( \mathcal{L}_2 \) in the case \( r \geq 3 \), we suppose that a point transformation \( \mathcal{T} \) links equations \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) from the class \( \mathcal{L}_2 \). (We assume that all the values in the equation \( \tilde{\mathcal{E}} \) are with tildes.) Thus, \( \mathcal{T} \) has the form (7). We express the derivatives of \( \tilde{x} \) with respect to \( \tilde{t} \) in terms of \( (t, x) \), substitute these expressions into the equation \( \tilde{\mathcal{E}} \) and then substitute the expression for \( x^{(r-2)} \) implied by \( \mathcal{E} \). Collecting the coefficients of \( x^{(r-2)} \) gives the equation (10), i.e., the function \( T \) is of the form (11). We substitute the expression for \( T \) into (7) and obtain transformations that map any equation from the subclass \( \mathcal{L}_2 \) to another equation from the same subclass, and the new arbitrary elements functionally depend on the old variables and the old arbitrary elements. Therefore, prolongations of these transformations to the arbitrary elements \( a_{r-3}, \ldots, a_0 \) and \( b \) constitute the equivalence group \( G_2^- \) of the subclass \( \mathcal{L}_2 \).
Denote by $\widehat{\mathcal{L}}_2$ the subclass of homogeneous equations within the class $\mathcal{L}_2$. The case $r = 2$ is singular for transformational properties of $\widehat{\mathcal{L}}_2$ since then the only element of this subclass is the free particle equation $x'' = 0$. This implies that the subclass $\widehat{\mathcal{L}}_2$ is normalized, and its equivalence group coincides with the point symmetry group of the equation $x'' = 0$, which consists of fractional linear transformations in the space of $(t, x)$. The case $r \geq 3$ for $\widehat{\mathcal{L}}_2$ is similar to ones for $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}_1$.

**Corollary 13.** If $r \geq 3$, the equivalence group $\widehat{\mathcal{G}}_2$ of the subclass $\widehat{\mathcal{L}}_2$ is derived from the equivalence group $\mathcal{G}^*_2$ of the class $\mathcal{L}_2$ by setting $X_0 = 0$ and neglecting the transformation component for $b$. The subclass $\widehat{\mathcal{L}}_2$ is uniformly semi-normalized with respect to linear superposition of solutions. More precisely, the equivalence groupoid of this subclass is constituted by triples of the form $(\mathcal{E}, \mathcal{E}, T)$, where the equation-source $\mathcal{E}$ runs through the entire subclass $\widehat{\mathcal{L}}_2$, the transformational part $T$ is of the form (9) with the product $(\gamma t + \delta)^r X_0$ being an arbitrary solution of $\mathcal{E}$, and the equation-target is defined by $\mathcal{E} = T(\mathcal{E})$.

**Remark 14.** In fact, the equivalence and admissible transformations for the classes $\widehat{\mathcal{L}}$, $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ were known in the literature for a long time due to Stäckel, Laguerre, Forsyth et al.; see, e.g., [10, Section 4.1] and [18, Chapter I and § II.4]. At the same time, we rigorously formulate these results and explicitly describe the associated equivalence groupoids by proving that these classes are normalized if $r \geq 3$ and semi-normalized if $r = 2$.

### 2.4. First Arnold Form

The above gauges of arbitrary elements of the class $\mathcal{L}$, which are commonly used, concern the subleading coefficients $a_{r-1}$ and $a_{r-2}$, but this is not a unique possibility. Instead of $a_{r-1}$ and $a_{r-2}$ one can gauge the lowest coefficients $a_0$ and $a_1$. We can set $a_0 = 0$ in any equation from the class $\mathcal{L}$ by an Arnold transformation

$$\tilde{t} = t, \quad \tilde{x} = \frac{x}{\varphi_1(t)},$$

where $\varphi_1$ is a nonzero solution of the corresponding homogeneous equation. As a result, we obtain the subclass $\mathcal{A}_1$ of the class $\mathcal{L}$ that is constituted by the equations of the form

$$x^{(r)} + a_{r-1}(t)x^{(r-1)} + \cdots + a_1(t)x^{(1)} = b(t). \quad (12)$$

Following the (eponymous) Arnold Principle,\(^\dagger\) we call this form the first Arnold form.

**Proposition 15.** The equivalence groupoid of the class $\mathcal{A}_1$, where $r \geq 3$, is constituted by the admissible transformations whose equations-sources exhaust the whole class $\mathcal{A}_1$ and whose transformational parts are of the form

$$\tilde{t} = T(t), \quad \tilde{x} = \frac{x}{\psi_1(t)} + X_0(t), \quad (13)$$

where $T$ and $X_0$ are arbitrary smooth functions of $t$ with $T_t \neq 0$, and $\psi_1 = \psi_1(t)$ is a nonzero solution of the homogeneous equation associated with the corresponding equation-source.

**Proof.** Let $T$ be a point transformation between equations $\mathcal{E}$ and $\mathcal{E}$ of the form (12). Then the transformation $T$ is of the general form (2). As the equation $\mathcal{E}$ belongs to the class $\mathcal{A}_1$, the function $\tilde{\psi}_1 \equiv 1$ is a solution of the corresponding homogeneous equation. Hence $\psi_1 = 1/X_1$ is a solution of the homogeneous equation associated with $\mathcal{E}$. Therefore, the transformation $T$ is of the form (13).

\(^\dagger\) The Arnold Principle states that if a notion bears a personal name, then this name is not the name of the discoverer. The Berry Principle extends the Arnold Principle by stating the following: the Arnold Principle is applicable to itself.
Remark 16. In the case $r = 2$, the same assertion is true for the fiber-preserving part of the equivalence groupoid of the class $\mathcal{A}_1$; cf. Remark 3.

Proposition 17. The equivalence group $G^\sim_{\mathcal{A}_1}$ of the class $\mathcal{A}_1$, where $r \geq 2$, consists of the transformations whose projections to the variable space have the form

$$\tilde{t} = T(t), \quad \tilde{x} = Cx + X_0(t),$$

where $T$ and $X_0$ are arbitrary smooth functions of $t$ with $T_1 \neq 0$ and $C$ is an arbitrary nonzero constant.

Proof. The projection of any transformation from the group $G^\sim_{\mathcal{A}_1}$ to the variable space is of the form (13). For $r \geq 3$, this is an obvious consequence of Proposition 15. In the case $r = 2$, similarly to the proof of Proposition 1, we can first show that projections of equivalence transformations to the variable space are fiber-preserving and then take into account Remark 16.

Note that constant functions are solutions of any homogeneous equation from the class $\mathcal{A}_1$. Therefore, the group $G^\sim_{\mathcal{A}_1}$ contains the transformations whose restrictions on the space of $(t, x)$ have the form (14). Moreover, only these transformations are in the group $G^\sim_{\mathcal{A}_1}$. Indeed, consider a transformation $\mathcal{T}$ of the form (13) with $\psi_1 \neq \text{const}$. Then there exists a homogeneous equation from $\mathcal{A}_1$ that is not satisfied by the function $\psi_1$. Hence the coefficient $\tilde{a}_0$ of the corresponding transformed equation is nonzero. This means that the transformation $\mathcal{T}$ is not a projection of an element of the group $G^\sim_{\mathcal{A}_1}$.

By $\hat{\mathcal{A}}_1$ we denote the subclass consisting of homogeneous equations of the form (12), i.e., singled out from the class $\mathcal{A}_1$ by the constraint $b = 0$.

Corollary 18. The equivalence group $\hat{G}^\sim_{\mathcal{A}_1}$ of the subclass $\hat{\mathcal{A}}_1$ with $r \geq 2$ is obtained from the group $G^\sim_{\mathcal{A}_1}$ by setting $X_0 = 0$ and neglecting the transformation component for the arbitrary element $b$. A point transformation $\mathcal{T}$ relates two equations from this subclass if and only if it has the form (13), where the product $\psi_1 X_0$ is a solution of the corresponding initial equation.

Corollary 19. The classes $\mathcal{A}_1$ and $\hat{\mathcal{A}}_1$ with $r \geq 3$ are not semi-normalized.

Proof. There exists an equation $\mathcal{E}$ in $\hat{\mathcal{A}}_1 \subset \mathcal{A}_1$ whose point symmetry group consists only of transformations related to the linearity and the homogeneity of $\mathcal{E}$.

Remark 20. Analogously to Proposition 2, the classes $\mathcal{A}_1$ and $\hat{\mathcal{A}}_1$ with $r = 2$ are semi-normalized since are orbits of the elementary equation $x'' = 0$ with respect to the equivalence groups $G^\sim_{\mathcal{A}_1}$ and $\hat{G}^\sim_{\mathcal{A}_1}$, respectively.

A common fact is that similar equations have similar point symmetry groups. Moreover, any admissible transformation in the class $\mathcal{L}$ maps point symmetries associated to the linearity and homogeneity of the corresponding initial equation to symmetries of the same kind that are admitted by the target equation. Hence it suffices to find an equation $\hat{\mathcal{E}}$ with trivial point symmetries in the class $\mathcal{L}_2$. In view of Corollary 13, possible point transformations of $\hat{\mathcal{E}}$ within the class $\mathcal{L}_2$ are exhausted, up to point symmetries associated to the linearity and the homogeneity of $\hat{\mathcal{E}}$, by the transformations of the form (9), where $C = 1$ and $X'' = 0$. Equations that are not invariant with respect to any of such transformations exist even among equations from $\mathcal{L}_2$ with polynomial coefficients.
2.5. Second Arnold form

Using a transformation of the form (4) with \( \varphi_0 = 0 \), we can set additionally \( a_1 = 0 \) in any equation from the class \( A_1 \). In this way the class \( A_1 \) (as well as the entire class \( L \)) is mapped onto its subclass \( A_2 \) that consists of the equations of the form

\[
x^{(r)} + a_{r-1}(t)x^{(r-1)} + \cdots + a_2(t)x^{(2)} = b(t),
\]

(15)
called the second Arnold form.

**Proposition 21.** The equivalence groupoid of the subclass \( A_2 \), where \( r \geq 3 \), is constituted by the admissible transformations whose equations-sources exhaust the whole class \( A_2 \) and whose transformational parts are of the form

\[
\tilde{t} = \frac{\psi_2(t)}{\psi_1(t)}, \quad \tilde{x} = \frac{x}{\psi_1(t)} + X_0(t),
\]

(16)

where \( \psi_1 = \psi_1(t) \) and \( \psi_2 = \psi_2(t) \) are arbitrary linearly independent solutions of the homogeneous equation associated with the corresponding equation-source, and \( X_0 \) is an arbitrary smooth function of \( t \).

**Proof.** Suppose that a point transformation \( T \) connects two equations \( E \) and \( \tilde{E} \) from the class \( A_2 \). Then the transformation \( T \) has the form (13). Note that the function \( \tilde{\psi}_2 = \tilde{t} \) is a solution of the homogeneous equation associated with \( \tilde{E} \). Hence the function \( \psi_2 = \psi_1 T \) is a solution of the homogeneous equation corresponding to \( E \), i.e., \( T = \psi_2 / \psi_1 \). As a result, the transformation \( T \) is of the form (16). \( \square \)

**Proposition 22.** The equivalence group \( G_{A_2}^\sim \) of the subclass \( A_2 \), where \( r \geq 3 \), consists of the transformations whose projections to the variable space have the form

\[
\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{x}{\gamma t + \delta} + X_0(t),
\]

(17)

where \( \alpha, \beta, \gamma \) and \( \delta \) are arbitrary constants with \( \alpha \delta - \beta \gamma \neq 0 \).

**Proof.** Any transformation from the group \( G_{A_2}^\sim \) generates a family of elements from the equivalence groupoid of the subclass \( A_2 \), and hence its projection to the variable space has the form (16). Since all linear functions of \( \tilde{t} \) are solutions of any homogeneous equation from the subclass \( A_2 \), then the group \( G_{A_2}^\sim \) contains the transformations whose restrictions on the variable space are of the form (17). We prove that there are no other transformations in the group \( G_{A_2}^\sim \), i.e., a transformation \( T \) of the form (16) does not induce an equivalence transformation of the class \( A_2 \) if at least one of the parameter-functions \( \psi_1 \) or \( \psi_2 \) is a nonlinear function of \( t \). First consider the case where \( \psi_1 \) is nonlinear. We take a homogeneous equation \( E \) from the subclass \( A_2 \) that is not satisfied by \( \psi_1 \). Then the corresponding transformed equation \( \tilde{E} \) possesses no constant solutions and hence its coefficient \( \tilde{a}_0 \) is nonzero. This means that the equation \( \tilde{E} \) does not belong to the subclass \( A_2 \), which gives the necessary statement. Now consider the complementary case, namely, where the function \( \psi_1 \) is linear and the function \( \psi_2 \) is nonlinear. Then there is a homogeneous equation \( E \) of the form (15) that is not satisfied by \( \psi_2 \). As \( \psi_1 \) is a solution of the equation \( E \), the coefficient \( \tilde{a}_0 \) is equal to zero in the corresponding transformed equation \( \tilde{E} \). Moreover, the function \( \tilde{\psi}_2 \equiv \tilde{t} \) is not a solution of the equation \( \tilde{E} \), so its coefficient \( \tilde{a}_1 \) is nonzero. Therefore, the equation \( \tilde{E} \) is not contained in the subclass \( A_2 \), which completes the proof. \( \square \)

Transformational properties of the subclass \( \tilde{A}_2 \) of homogeneous equations from \( A_2 \) with \( r \geq 3 \) are similar to ones of the class \( \tilde{A}_1 \).
**Corollary 23.** The equivalence group $\hat{G}_{A_2}$ of the subclass $\hat{A}_2$ with $r \geq 3$ is obtained from the group $G_{A_2}$ by setting $X_0 = 0$ and neglecting the transformation component for the arbitrary element $b$. A point transformation relates two equations from this subclass if and only if it has the form (16), where the product $\psi_1 X_0$ is a solution of the corresponding initial equation.

**Corollary 24.** The classes $A_2$ and $\hat{A}_2$ with $r \geq 3$ are not semi-normalized.

**Proof.** Following the proof of Corollary 19, consider an admissible transformation $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{T})$ in the class $\mathcal{A}_2 \subset \hat{A}_2$, where the point symmetry group of $\mathcal{E}_1$ consists only of transformations related to the linearity and the homogeneity of $\mathcal{E}_1$, in the representation (16) for $\mathcal{T}$ the solution $\psi_1$ of $\mathcal{E}_1$ is not affine in $t$, $\psi_1'' \neq 0$, and $\mathcal{E}_2 = \mathcal{T}(\mathcal{E}_1)$. At the same time, the compositions of symmetry transformations of $\mathcal{E}_1$ with projections of transformations from $G_{\mathcal{A}_2}$ (resp. $\hat{G}_{\mathcal{A}_2}$) to the variable space are at most of the form (17). Therefore, the admissible transformation $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{T})$ is not generated by one of the above compositions.

**Remark 25.** For $r = 2$, the classes $\mathcal{A}_2$ and $\hat{A}_2$ coincides with the classes $\mathcal{L}_2$ and $\hat{\mathcal{L}}_2$, respectively.

### 3. Group classification

Any inhomogeneous linear ODE can be reduced to the corresponding homogeneous equation by a transformation from the equivalence group $G^\ast$. The class $\mathcal{L}$ is normalized and its subclass $\hat{\mathcal{L}}$ of homogeneous equations is semi-normalized. Therefore, for solving the group classification problem for the class $\mathcal{L}$ it suffices to solve the similar problem for the subclass $\hat{\mathcal{L}}$.

As remarked earlier, any second-order linear ODE can be reduced to the elementary equation $x'' = 0$ (see, e.g., [29]), which admits the eight-dimensional Lie invariance algebra

$$\langle \partial_t, \partial_x, t \partial_t, x \partial_t, t \partial_x, x \partial_x, t x \partial_t + x^2 \partial_x, t^2 \partial_t + t x \partial_x \rangle.$$ 

This gives the exhaustive group classification of second-order linear ODEs.

Let $r \geq 3$. It is also a classical result by Sophus Lie [1, pp 296–298] that the dimension of Lie invariance algebras for $r$th order ODEs with $r \geq 3$ is not greater than $r + 4$. Much later this result was partially reproved in [16,17] only for linear ODEs.

Consider an $r$th order linear homogeneous ODE $\mathcal{E}$. Due to linearity leading to the linear superposition principle, this equation admits the $r$-dimensional abelian Lie invariance algebra $\mathfrak{g}^\mathcal{E}_0$ spanned by the vector fields

$$\varphi_1(t) \partial_x, \varphi_2(t) \partial_x, \ldots, \varphi_r(t) \partial_x,$$

where the functions $\varphi_i = \varphi_i(t)$, $i = 1, \ldots, r$, form a fundamental set of solutions of the equation $\mathcal{E}$. By virtue of homogeneity, the equation $\mathcal{E}$ also admits the one-parameter symmetry group of scale transformations generated by the vector field $x \partial_x$. Therefore, each equation $\mathcal{E}$ from the subclass $\hat{\mathcal{L}}$ admits the $(r+1)$-dimensional Lie invariance algebra

$$\mathfrak{g}^\mathcal{E}_0 = \langle x \partial_x, \varphi_1(t) \partial_x, \varphi_2(t) \partial_x, \ldots, \varphi_r(t) \partial_x \rangle.$$ 

Corollary 7 implies that any Lie symmetry operator $Q$ of the equation $\mathcal{E}$ is of the general form

$$Q = \tau(t) \partial_t + (\xi_1(t) x + \xi_0(t)) \partial_x,$$

where $\tau$, $\xi_1$ and $\xi_0$ are smooth functions of $t$, and $\xi_0$ is additionally a solution of $\mathcal{E}$. Hence the maximal Lie invariance algebra $\mathfrak{g}^\mathcal{E}$ of $\mathcal{E}$ contains the subalgebra $\mathfrak{g}^\mathcal{E}_0$ as an ideal.

Now we carry out the group classification of linear ODEs in three different ways, which respectively involve the Laguerre–Forsyth form (8), the rational form (6) and Lie’s classification of realizations of finite-dimensional Lie algebras in the space of two variables.
3.1. The first way: Laquerre–Forsyth form

The group classification of the class $\hat{\mathcal{L}}$ can be further reduced to the group classification of its subclass $\hat{\mathcal{L}}_2$ of $r$th order homogeneous linear ODEs in the Laquerre–Forsyth form, which is singled out from $\hat{\mathcal{L}}$ by the constraint $a_{r-1} = a_{r-2} = 0$. Indeed, both the arbitrary elements $a_{r-1}$ and $a_{r-2}$ can be gauged to zero by a family of point transformations, which are parameterized by these arbitrary elements and are associated with equivalence transformations. Moreover, equations from the class $\hat{\mathcal{L}}$ are $\hat{G}^\sim$-equivalent if and only if their images in the class $\hat{\mathcal{L}}_2$ are $G^\sim_2$-equivalent.

In view of Corollary 13, the class $\hat{\mathcal{L}}_2$ is not normalized. At the same time, it is uniformly semi-normalized with respect to linear superposition of solutions (i.e., with respect to the symmetry groups of its equations related to linear superposition of their solutions). This suffices for using an advanced version of the algebraic method to group classification of the class $\hat{\mathcal{L}}_2$. The equivalence group $\hat{G}^\sim_2$ of $\hat{\mathcal{L}}_2$ consists of the transformations whose projections to the variable space are of the form (9) with $X_0 = 0$. The scalings of $x$ constitute the kernel group $\hat{G}^\sim_2$ of $\hat{\mathcal{L}}_2$ since they are only common point symmetry transformations for all equations from the class $\hat{\mathcal{L}}_2$. By trivial prolongation to the arbitrary elements, the group $\hat{G}^\sim_2$ is embedded into $\hat{G}^\sim$ as a normal subgroup. The factor group $\hat{G}^\sim_2/\hat{G}^\sim$ can be identified with the subgroup $H$ of $\hat{G}^\sim_2$ singled out by the constraint $C = 1$, where additionally the expression $\alpha \delta - \beta \gamma$ is equal to 1 or $\pm 1$ in the complex or real case, respectively. Up to point symmetry transformations associated with the linear superposition principle and homogeneity, i.e., related to the Lie algebra $\mathfrak{g}_0^2$, all admissible transformations of the equation $\mathcal{E}$ within $\hat{\mathcal{L}}_2$ are exhausted by the projections of elements from $H$ to the variable space. The subgroup $H$ is isomorphic to the projective general linear group $\text{PGL}(2, \mathbb{F})$ of fractional linear transformations of $t$.

Therefore, all possible Lie symmetry extensions within the class $\hat{\mathcal{L}}_2$ are necessarily associated with subgroups of $\text{PGL}(2, \mathbb{F})$. In infinitesimal terms, the maximal Lie invariance algebra of the equation $\mathcal{E}$ is a semidirect sum of the algebra $\mathfrak{g}_0^2$ and a subalgebra of the realization of $\mathfrak{sl}(2, \mathbb{F})$ spanned by the vector fields

$$\mathcal{P} = \partial_x, \quad \mathcal{D} = t \partial_t + \frac{1}{2} (r - 1) x \partial_x, \quad \mathcal{K} = t^2 \partial_t + (r - 1) t x \partial_x.$$  

Subalgebras of $\mathfrak{sl}(2, \mathbb{F})$ are well known (see, e.g., [30] or the appendix in the arXiv version of [31]). Up to internal automorphisms of $\mathfrak{sl}(2, \mathbb{F})$, a complete list of subalgebras of $\mathfrak{sl}(2, \mathbb{F})$ is exhausted by the zero subalgebra $\{0\}$, the one-dimensional subalgebras $\langle \mathcal{P} \rangle$, $\langle \mathcal{D} \rangle$ and, only for $\mathbb{F} = \mathbb{R}$, $\langle \mathcal{P} + \mathcal{K} \rangle$, the two-dimensional subalgebra $\langle \mathcal{P}, \mathcal{D} \rangle$ and the entire realization $\langle \mathcal{P}, \mathcal{D}, \mathcal{K} \rangle$. The equivalence of subalgebras of $\mathfrak{sl}(2, \mathbb{F})$ well agrees with the similarity of equations within the class $\hat{\mathcal{L}}_2$.

The zero subalgebra $\{0\}$ corresponds to the general case with no extension.

For one-dimensional extensions of algebras of the form (19) by the subalgebras $\langle \mathcal{P} \rangle$, $\langle \mathcal{D} \rangle$ and $\langle \mathcal{P} + \mathcal{K} \rangle$, the corresponding equations from the class $\hat{\mathcal{L}}_2$ respectively take the forms

$$x^{(r)} + c_{r-3} x^{(r-3)} + \cdots + c_1 x' + c_0 x = 0, \quad x^{(r)} + c_{r-3} t^{-3} x^{(r-3)} + \cdots + c_1 t^{-r+1} x' + c_0 t^{-r} x = 0, \quad x^{(r)} + q_{r-3}(t) x^{(r-3)} + \cdots + q_1(t) x' + q_0(t) x = 0,$$

(20) (21) (22)

9 The subalgebra $\langle \mathcal{P} + \mathcal{K} \rangle$ is equivalent to $\langle \mathcal{P} \rangle$ if $\mathbb{F} = \mathbb{C}$. This is why the part of the consideration corresponding to the subalgebra $\langle \mathcal{P} + \mathcal{K} \rangle$ can be then just neglected in the complex case.
where \( c_0, \ldots, c_{r-3} \) are arbitrary constants,

\[
q_m(t) = \frac{c_m}{(1 + t^2)^r} - \frac{(m + 1)(r - m - 1)}{(1 + t^2)^{r - m}} \int (1 + t^2)^{r - m - 1} q_{m+1}(t)dt, \quad m = r - 4, \ldots, 0,
\]

and the integral denotes a fixed antiderivative. However, by the point transformations\(^\text{10}\)

\[
\tilde{t} = \ln t, \quad \tilde{x} = xt^{-{(r-1)/2}} \quad \text{and} \quad \tilde{t} = \arctan t, \quad \tilde{x} = x(1 + t^2)^{-(r-1)/2}
\]

the maximal Lie invariance algebras of equations of the form (21) (known as the Euler–Cauchy equation, or just Euler’s equation) and the form (22) are reduced to the algebras looking as

\[
\langle \partial_{\tilde{t}}, \tilde{x} \partial_{\tilde{x}}, \tilde{\varphi}_1(\tilde{t}) \partial_{\tilde{x}}, \tilde{\varphi}_2(\tilde{t}) \partial_{\tilde{x}}, \ldots, \tilde{\varphi}_r(\tilde{t}) \partial_{\tilde{x}} \rangle.
\]

Moreover, the above transformations map these equations to constant-coefficient maps from the class (6), where \( a_{r-2} = -\frac{1}{2} r(r^2 - 1) \) for (21) and \( a_{r-2} = \frac{1}{2} r(r^2 - 1) \) for (22). Additionally scaling \( t \), we can set \( a_{r-2} = -1 \) and \( a_{r-2} = 1 \), respectively. Thus, any equation from \( \tilde{L}_2 \) that admits an \((r+2)\)-dimensional Lie invariance algebra is equivalent to a homogeneous equation with constant coefficients from the class (6), in which \( a_{r-2} = 0, a_{r-2} = -1 \) and \( a_{r-2} = 1 \) for (20), (21) and (22), respectively.

If an equation from \( \tilde{L}_2 \) possesses the \((r+3)\)-dimensional Lie invariance algebra

\[
\langle \partial_{\tilde{t}}, t \partial_{\tilde{t}} + \frac{1}{2}(r - 1)x \partial_x, \ x \partial_x, \ \varphi_1(t) \partial_x, \ \varphi_2(t) \partial_x, \ldots, \ \varphi_r(t) \partial_x \rangle,
\]

then it has the form \( x^{(r)} = 0 \) and hence its maximal Lie invariance algebra is \((r+4)\)-dimensional,

\[
\langle \partial_{\tilde{t}}, t \partial_{\tilde{t}} + \frac{1}{2}(r - 1)x \partial_x, \ t^2 \partial_{\tilde{t}} + (r - 1)tx \partial_x, \ x \partial_x, \ \varphi_1(t) \partial_x, \ \varphi_2(t) \partial_x, \ldots, \ \varphi_r(t) \partial_x \rangle.
\]

As the functions \( \varphi_1, \ldots, \varphi_r \) form a fundamental set of solutions of the elementary equation, we can choose \( \varphi_i = t^{i-1}, i = 1, \ldots, r \). Thus, there is no \( r \)th order linear ODE whose maximal Lie invariance algebra is \((r+3)\)-dimensional. Moreover, if such an equation admits an \((r+4)\)-dimensional Lie invariance algebra, then it is similar to the elementary equation \( x^{(r)} = 0 \) with respect to a point transformation of the form (2).

### 3.2. The second way: rational form

Consider now the subclass \( \tilde{L}_1 \) of homogeneous ODEs of the rational form (6). Again, the group classification of the class \( \tilde{L} \) reduces to that of the subclass \( \tilde{L}_1 \) since the gauge \( a_{r-1} = 0 \) singling out the subclass \( \tilde{L}_1 \) from \( \tilde{L} \) is realized by a family of equivalence transformations. Moreover, the subclass \( \tilde{L}_1 \) is uniformly semi-normalized with respect to linear superposition of solutions.

Each equation \( \tilde{E} \) from \( \tilde{L}_1 \) possesses the \((r+1)\)-dimensional Lie invariance algebra \( \tilde{g}^E_0 \), which is an ideal of the maximal Lie invariance algebra \( \tilde{g}^E \) of \( \tilde{E} \). At the same time, in contrast to the Laguerre–Forsyth form, the equivalence group \( \tilde{G}^*_1 \) of \( \tilde{L}_1 \) is parameterized by an arbitrary function \( T = T(t) \). Corollary 11 implies that the algebra \( \tilde{g}^E \) is contained in the algebra \( \langle R(\tau) \rangle + \tilde{g}^E_0 \), where plus denotes the sum of vector spaces,

\[
R(\tau) = \tau(t) \partial_t + \frac{1}{2}(r - 1)\tau(t)x \partial_x,
\]

\(^{10}\)For the first transformation, the absolute value of \( t \) should be substituted instead of \( t \) in the real case or branches of the \( \ln \) and power functions should be fixed in the complex case.
and the parameter $\tau$ runs through the set of smooth functions of $t$. It is easy to see that $[\mathcal{R}(\tau^1), \mathcal{R}(\tau^2)] = \mathcal{R}(\tau^1 \tau^2 - \tau^2 \tau^1)$. Hence $\mathfrak{g}_{\hat{t}}^\mathcal{E} := \langle \mathcal{R}(\tau) \rangle \cap \mathfrak{g}^\mathcal{E}$ is a (finite-dimensional) subalgebra of $\mathfrak{g}^\mathcal{E}$. Each vector field $\mathcal{R}(\tau)$ is completely defined by its projection $\text{pr}_t \mathcal{R}(\tau)$ to the space of the variable $t$, $\text{pr}_t \mathcal{R}(\tau) = \tau(t) \partial_t$. In other words, the algebras $\mathfrak{g}_{\hat{t}}^\mathcal{E}$ and $\text{pr}_t \mathfrak{g}_{\hat{t}}^\mathcal{E}$ are isomorphic. Moreover, the corresponding projection of the equivalence group of the subclass $\hat{L}_1$ coincides with the group of all local diffeomorphisms in the space of the variable $t$.

As a result, the group classification of the class $\hat{L}_1$ reduces to the classification of realizations of finite-dimensional Lie algebras by vector fields in the space of the single variable $t$. The latter classification is well known and was done by Sophus Lie himself. A complete list of inequivalent Lie symmetry extensions within the class $L_1$. Here $\mathcal{P} = \mathcal{R}(1)$, $\mathcal{D} = \mathcal{R}(t)$ and $\mathcal{K} = \mathcal{R}(t^2)$ are the same operators as those in the first way. If the equation $\mathcal{E}$ admits the two-dimensional extension $\langle \mathcal{P}, \mathcal{D} \rangle$, then it coincides with the elementary equation $x^{(r)} = 0$, which possesses the three-dimensional extension $\langle \mathcal{P}, \mathcal{D}, \mathcal{K} \rangle$. This is why the two-dimensional extension is improper. Finally, we have three inequivalent cases of Lie symmetry extensions in the class $L$, which are

- the general case with no extension,
- general constant-coefficient equations admitting the one-dimensional extension $\langle \mathcal{P} \rangle$, and
- the elementary equation $x^{(r)} = 0$ possessing the three-dimensional extension $\langle \mathcal{P}, \mathcal{D}, \mathcal{K} \rangle$.

### 3.3. The third way: general form

The group classification of the entire class $\hat{L}$ can also be obtained directly from Lie’s classification of realizations of finite-dimensional Lie algebras by vector fields in the spaces of two real or complex variables [29, 32]. A modern treatment of these results on realizations was presented, e.g., in [3, 4, 33, 34]. In order to solve the group classification problem for the class $\hat{L}$, from Lie’s list of realizations we select candidates for the maximal invariance algebras of equations from $\hat{L}$. All candidates should satisfy the following obvious properties, which are preserved by point transformations:

- The maximal Lie invariance algebra $\mathfrak{g}^\mathcal{E}$ of each $r$th order linear ODE $\mathcal{E}$ $(r \geq 3)$ contains the $(r+1)$-dimensional almost abelian ideal $\mathfrak{g}_0^\mathcal{E}$. More precisely, the ideal $\mathfrak{g}_0^\mathcal{E}$ is the semidirect sum of an $r$-dimensional abelian ideal $\mathfrak{g}_a^\mathcal{E}$ of the whole algebra $\mathfrak{g}^\mathcal{E}$ and the linear span of one more vector field whose adjoint action on the ideal $\mathfrak{g}_a^\mathcal{E}$ is the identity operator.
- Moreover, the ideal $\mathfrak{g}_0^\mathcal{E}$ is an intransitive Lie algebra of vector fields, $\text{rank} \mathfrak{g}_0^\mathcal{E} = 1$.

The above properties are satisfied by realization families 21, 23, 26 and 28 from [33, Table 1] (or realization families 3.2, 1.6, 1.9 and 1.11 from [4, pp 472–473], or realization families 49, 51, 54 and 56 from [34, Table 1.1], respectively). As in the previous two ways, among $r$th order linear ODEs, only the elementary equation $x^{(r)} = 0$ admits the third realization. At the same time, this equation also possesses the fourth realization, which is of greater dimension than the third one. This is why the third realization should be neglected.

The equivalence within the chosen families of realizations well conforms with the point equivalence of linear ODEs. Indeed, given two such realizations that are equivalent with respect to a point transformation $\mathcal{T}$ and whose $(r+1)$-dimensional almost abelian ideals are of the form (19), the transformation $\mathcal{T}$ should have the form (2), where $X_0/X_1$ is a linear combinations of the parameters-functions $\varphi_1, \ldots, \varphi_r$ of the initial realization. The proof of this claim is based on two facts. The first fact is trivial: The mapping generated by a point transformation between realizations of a Lie algebra by vector fields is a Lie algebra isomorphism and, in particular, establishes a bijection between the corresponding nilradicals. The second fact is that nilradicals of appropriate realizations are spanned by vector fields $\varphi_1(t) \partial_x, \ldots, \varphi_r(t) \partial_x$, where $\varphi_i = \varphi_i(t)$,
i = 1, ..., r, are linearly independent functions. This fact is obvious for realizations reducing to the forms (19) and (24). Suppose that it is not the case for a realization reducing to the form (23), where tildes over all values are omitted. Then for some constant \( \nu \) the corresponding nilradical includes the vector field \( \partial_t + \nu \partial_x \), and commutation relations within the nilradical imply the existence of a constant nilpotent matrix \((\mu_{ij})\) such that \( \varphi'_i - \nu \varphi_i = \mu_{i1} \varphi_1 + \cdots + \mu_{ir} \varphi_r \). The constant \( \nu \) can be set to zero by a point transformation \( t = t, \ x = e^{\nu t} x \). As the functions \( \varphi_i \) are linearly independent, up to their linear combining the matrix \((\mu_{ij})\) can be assumed to coincide with the \( r \times r \) nilpotent Jordan block, and hence we can set \( \varphi_i = t^{i-1} \). At the same time, the only equation that belongs to the class \( \hat{L} \) and is invariant with respect to the algebra \( \langle \partial_t, t \partial_t, \ldots, t^{r-1} \partial_t \rangle \) is the elementary equation \( x'' = 0 \), but the maximal Lie invariance algebra of this equation is of higher dimension.

As a result, using the algebraic method of group classification we have reproved the following assertion in three different ways (see, e.g., [4,10,16,17,21]):

**Proposition 26.** The dimension of the maximal Lie invariance algebra \( \mathfrak{g}^E \) of an \( r \)th order (\( r \geq 3 \)) linear ODE \( E \) takes a value from \( \{ r+1, r+2, r+4 \} \). If \( \dim \mathfrak{g}^E \geq r + 2 \), then the equation \( E \) is similar to a linear ODE with constant coefficients. In the case \( \dim \mathfrak{g}^E = r + 4 \) the equation \( E \) is reduced by a point transformation of the form (2) to the elementary equation \( x^{(r)} = 0 \).

4. Conclusion

In this paper we exhaustively describe the equivalence groupoid of the class \( L \) of \( r \)th order linear ODEs as well as equivalence groupoids of its subclasses \( L_1, L_2, A_1 \) and \( A_2 \) associated with the rational, the Laguerre–Forsyth, the first and second Arnold forms, i.e., the classes of equations of the form (1), (6), (8), (12) and (15), respectively. The corresponding classes \( \hat{L}, \hat{L}_1, \hat{L}_2, \hat{A}_1 \) and \( \hat{A}_2 \) of homogeneous equations are also studied from the point of view of admissible transformations.

The case \( r = 2 \) is singular for all the above classes. Each of the classes \( L, \hat{L}, L_1, \hat{L}_1, L_2 = A_2, A_1 \) and \( \hat{A}_1 \) with \( r = 2 \) is an orbit of the elementary equation \( x'' = 0 \) with respect to the equivalence group of this class. This is why its equivalence groupoid is generated by the compositions of transformations from its equivalence group with transformations from the point symmetry group of the equation \( x'' = 0 \). Hence the class is semi-normalized, cf. the proof of Proposition 2. The class \( \hat{L}_2 = \hat{A}_2 \) is constituted by the single equation \( x'' = 0 \) and, thus, is normalized.

For \( r \geq 3 \), the equivalence groupoids of the classes \( L, L_1 \) and \( L_2 \) are generated by the corresponding (usual) equivalence groups. In other words, each of these classes is normalized, see Propositions 4, 8 and 12. The associated subclasses of homogeneous equations are uniformly semi-normalized with respect to the linear superposition symmetry groups of their equations. This allows us to classify Lie symmetries of linear ODEs using the algebraic tools in three different ways. The purpose of the presentation of various ways for carrying out the known classification is to demonstrate advantages and disadvantages of each of them, which is important, e.g., to effectively apply the algebraic approach to systems of linear ODEs. Thus, the classification based on the Laguerre–Forsyth form, which is associated with a maximal gauge of arbitrary elements, is just reduced to the classification of subalgebras of the algebra \( \mathfrak{sl}(2, E) \), which is finite-dimensional (more precisely, three-dimensional). The use of the rational form leads to involving the classification of all possible realizations of finite-dimensional Lie algebras on the line. At the same time, the single classification case of constant-coefficient equations in the rational form is split in the Laguerre–Forsyth form into three (resp. two) cases over the real (resp. complex) field, and two (resp. one) of them are related to variable-coefficient equations.
If we neglect the possibility of gauging arbitrary elements and consider general linear ODEs, we need to classify specific realizations of specific Lie algebras in the space of two variables.

The structure of the equivalence groupoids of the classes $A_1$, $\tilde{A}_1$, $A_2$ and $\tilde{A}_2$ associated with the Arnold forms, where $r \geq 3$, is more complicated since these classes are even not semi-normalized. This is why they are not usable for the group classification of the class $\mathcal{L}$, although these are the forms that are involved in reduction of order of linear ODEs.

In contrast to single linear ODEs, results concerning group properties of normal systems of second-order linear ODEs are very far from to be completed, not to mention general systems of linear ODEs; see a more detailed discussion in [35]. Only recently the group classification of systems of second-order linear ODEs with commuting constant-coefficient matrices was considered for various particular cases of the number of equations (two, three or four) and of the structure of the coefficient matrices in a series of papers [36–39] and was then exhaustively solved in [35]. In spite of a number of publications on the subject, the group classification of systems of linear second-order ODEs with noncommuting constant-coefficient matrices or with general nonconstant coefficients was carried out only for the cases of two and three equations [40–42]. The consideration of a greater number of equations or equations of higher and different orders within the framework of the classical infinitesimal approach requires cumbersome computations. Hence there is a demand for the development of new more powerful algebraic and geometric tools, which, for instance, involve a deep investigation of associated equivalence groupoids and other related algebraic structures.

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