LETTER TO THE EDITOR

Intermediate asymptotics of the Kerr quasinormal spectrum

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Abstract

We study \textit{analytically} the quasinormal mode spectrum of near-extremal (rotating) Kerr black holes. We find an analytic expression for these black-hole resonances in terms of the black-hole physical parameters: its Bekenstein–Hawking temperature, $T_{BH}$, and its horizon’s angular velocity, $\Omega$. The expression is valid in the intermediate asymptotic regime $1 \ll |\omega| \ll T_{BH}^{-1}$.

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Gravitational waves emitted by a perturbed black hole are dominated by ‘quasinormal ringing’, damped oscillations with a \textit{discrete} spectrum (see e.g., [1] for a detailed review). At late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell [2–5]. Being the characteristic ‘sound’ of the black hole itself, these free oscillations are of great importance from the astrophysical point of view. They allow a direct way of identifying the spacetime parameters (especially, the mass and angular momentum of the black hole). This has motivated a flurry of activity with the aim of computing the spectrum of oscillations.

The ringing frequencies are located in the complex frequency plane characterized by $\text{Im} \omega < 0$. It turns out that for a Schwarzschild black hole, for a given angular harmonic index $l$ there exist an infinite number of quasinormal modes, for $n = 0, 1, 2, \ldots$, characterizing oscillations with decreasing relaxation times (increasing imaginary part) [6, 7]. On the other hand, the real part of the Schwarzschild black-hole resonances approaches an asymptotic constant value.

The quasinormal modes (QNMs) have been the subject of much recent attention (see e.g., [8–67] and references therein), with the hope that these classical frequencies may shed some light on the elusive theory of quantum gravity. These recent studies are motivated by an earlier work [68], in which the application of \textit{Bohr’s correspondence principle} to the black-hole resonances was suggested in order to determine the value of the fundamental area unit in a quantum theory of gravity. The use of black-hole QNM frequencies to fix the value of the Immirzi parameter in canonical quantum gravity [69, 70] was later suggested. These
proposals [68] have motivated a flurry of research attempting to calculate the asymptotic ringing frequencies of various types of black holes.

Leaver [6] was the first to address the problem of computing the highly damped (asymptotic) ringing frequencies of the Schwarzschild black hole. Nollert [71] found numerically (see also [72]) that the asymptotic (large \( \text{Im} \, \omega \)) behaviour of the ringing frequencies of a Schwarzschild black hole is given by (we normalize \( G = c = 2M = 1 \))

\[
\omega_n = 0.0874247 \frac{i}{2} \left( n + \frac{1}{2} \right). \tag{1}
\]

On the basis of Bohr’s correspondence principle, it was suggested [68] that this asymptotic (real) value actually equals \( \ln(3)/(4\pi) \). This identity was further motivated by a heuristic picture, which was based on thermodynamical and statistical physics arguments [68, 73, 74]. An analytical proof of this equality was later given in [9]. This was followed by an analytical calculation of the asymptotic QNM frequencies of the (charged) Reissner–Nordström (RN) black hole [11].

It should be emphasized, however, that less is known about the corresponding QNM spectrum of the generic (rotating) Kerr black hole, which is the most interesting from a physical point of view. Former studies of the Kerr asymptotic spectrum [6, 16, 30, 31, 75, 76] used numerical tools, and there are in fact no analytical results for the asymptotic Kerr QNM frequencies. In this work we provide analytical formulae for the intermediate asymptotic behaviour of the Kerr QNM spectrum. This is done by using a similarity between the intermediate QNMs of the Kerr black hole and the known asymptotic spectrum of the RN black hole.

The dynamics of black-hole perturbations is governed by the Regge–Wheeler equation [77] in the case of a Schwarzschild black hole, and by the Teukolsky equation [78] for the Kerr black hole. The black-hole QNMs correspond to solutions of the wave equations with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the event horizon [79]. Such boundary conditions single out a discrete set of resonances \( \{ \omega_n \} \) (assuming a time dependence of the form \( e^{-i\omega t} \)). The solution to the radial Teukolsky equation may be expressed as [6] (assuming an azimuthal dependence of the form \( e^{im\phi} \))

\[
R_{lm} = e^{i\omega r} (r - r_-)^{-1-s+i\sigma r} (r - r_+)^{-s-i\sigma r} \sum_{n=0}^{\infty} d_n \left( \frac{r - r_+}{r - r_-} \right)^n, \tag{2}
\]

where \( r_\pm = M \pm (M^2 - a^2)^{1/2} \) are the black-hole (event and inner) horizons, \( a \equiv J/M \) is the black-hole angular momentum per unit mass, and \( \sigma \equiv (\omega r - ma)/(r_+ - r_-) \). The field spin-weight parameter \( s \) takes the values 0, -1, and -2, respectively, for scalar, electromagnetic and gravitational fields.

The sequence of expansion coefficients \( \{ d_n : n = 1, 2, \ldots \} \) is determined by a recurrence relation of the form [6]

\[
\alpha_n d_{n+1} + \beta_n d_n + \gamma_n d_{n-1} = 0, \tag{3}
\]

with initial conditions \( d_0 = 1 \) and \( \alpha_0 d_1 + \beta_0 d_0 = 0 \). The recursion coefficients \( \alpha_n, \beta_n, \) and \( \gamma_n \) are given in [6]. The quasinormal frequencies are determined by the requirement that the series in equation (2) is convergent, that is \( \sum d_n \) exists and is finite [6].

We find that the physical content of the recursion coefficients becomes clear when they are expressed in terms of the black-hole physical parameters: the Bekenstein–Hawking temperature \( T_{BH} = (r_+ - r_-)/A \), and the horizon’s angular velocity \( \Omega = 4\pi a/A \), where \( A = 4\pi (r_+^2 + a^2) \) is the black-hole surface area. The recursion coefficients obtain a surprisingly
simple form in terms of these physical quantities,

\[ \alpha_n = (n + 1)(n + 1 - s - 2i\beta_+\hat{\omega}), \quad (4) \]

\[ \beta_n = -2(n + \frac{1}{2} - 2i\beta_+\hat{\omega})(n + \frac{1}{2} - 2i\omega_r) - (a\omega)^2 + 2ma\omega - s - \frac{1}{2} - A_{lm}, \quad (5) \]

and

\[ \gamma_n = (n - 2i\omega)(n + s - 2i\beta_+\hat{\omega}), \quad (6) \]

where \( \beta_+ \equiv (4\pi T_{\text{BH}})^{-1} \) is the black-hole inverse temperature, \( \hat{\omega} \equiv \omega - m\Omega \), and the angular separation constants, \( A_{lm}(a\omega) \), are given by an independent recurrence relation \([6]\). To our best knowledge, this compact formulation of the recursion coefficients in terms of the black-hole physical parameters has not been done so far.

The Teukolsky equation also describes the propagation of scalar \((s = 0)\) waves in the RN spacetime (one should simply replace \( r_\pm = M \pm (M^2 - a^2)^{1/2} \) by \( r_\pm = M \pm (M^2 - Q^2)^{1/2} \), and take \( a = 0 \) elsewhere). The spectrum of the RN black hole may be found using a recursion relation of form (3). The \( \{\alpha_n\} \) and \( \{\gamma_n\} \) coefficients of the RN perturbations have exactly the same form as in the Kerr case (where for the RN black hole \( \hat{\omega} = \omega \)). In addition, \( \beta_{\text{RN}}^n = \beta_{\text{Kerr}} + (a\omega)^2 - 2ma\omega \).\(^3\) In the intermediate asymptotic range, defined as \( 1 \ll |\omega| \ll T_{\text{BH}}^{-1} \), \( \beta_n \) is dominated by its first term in equation (5). This term is at least of order \( |\omega|/T_{\text{BH}} \), and is therefore much larger in magnitude than the \((a\omega)^2\) and \( A_{lm} \) terms in this (intermediate asymptotic) regime. Hence, in this limit one finds that the \( \{\beta_n\} \) coefficients of the Kerr black hole coincide with those of the RN black hole (in addition to the coincidence of the \( \{\alpha_n\} \) and \( \{\gamma_n\} \) terms).

Thus, the intermediate asymptotic \((1 \ll |\omega| \ll T_{\text{BH}}^{-1})\) quasinormal frequencies of the near extremal Kerr black hole are related to the asymptotic frequencies of the RN black hole. The asymptotic spectrum of the RN black hole is determined, for \( \text{Re } \omega > 0 \), by the equation \([11]\)^4

\[ 2e^{-4\pi\beta_+\omega} + 3e^{-4\pi\omega} = -1. \quad (7) \]

The preceding discussion indicates that an expression similar to equation (7) \(^5\) must hold true for the intermediate asymptotic QNMs of the Kerr black hole. Equation (7) suggests that the spectrum depends on the combinations \( \beta_+\omega \) and \( \omega \) appearing in equations (4)–(6), but does not depend explicitly on \( \omega_r \). Under this assumption, one finds that for a Kerr black hole\(^5\)

\[ 2e^{-4\pi\beta_+\omega} + 3e^{-4\pi\omega} = -1. \quad (8) \]

In the extremal limit \( \beta_+ \rightarrow \infty \), thus yielding

\[ \omega \rightarrow m\Omega + T_{\text{BH}} \ln 2 - i2\pi T_{\text{BH}}(n + \frac{1}{2}) \quad (9) \]

for \( m > 0 \), and

\[ \omega \rightarrow \frac{\ln 3}{4\pi} - \frac{i}{2} \left(n + \frac{1}{2}\right) \quad (10) \]

for \( m \leq 0 \). For \( m = 0 \) there is an additional branch in the cases where \( \beta_+ \) can be written as \( \beta_+ = 2k/(2n + 1) \), with \( k \) and \( n \) natural numbers. In this case, one finds

\[ \omega \rightarrow -\frac{i}{2} \left(n + \frac{1}{2}\right). \quad (11) \]

\(^3\) For \( |a\omega| \gg 1 \), the angular wavefunctions that satisfy the angular Teukolsky equation approach the prolate spheroidal wavefunctions, implying that \( A_{lm} = O(a\omega) \) \([80, 81, 31]\). Thus, \( A_{lm} \) is subdominant to the \((a\omega)^2\) term in \( \beta_{\text{Kerr}}^n \). For the RN case, \( A_{lm} = il(l + 1) - s(s + 1) \).

\(^4\) The final result of \([11]\) for the RN spectrum can be written as equation (7) \(^5\) by simply multiplying both sides of equation (6) in \([11]\) by \( e^{-4\pi\omega} \).

\(^5\) This relation is valid for \( \text{Re } \omega > 0 \). The frequencies with \( \text{Re } \omega < 0 \) may be determined from the well-known symmetry of the Kerr spacetime \([6]\): if \( \omega_{n,m} \) is a QNM frequency, then \(-\omega_{-n,-m}^*\) is also a solution.
These analytical results are consistent with previous numerical analyses [6, 76], which have begun to probe the intermediate asymptotic regime. In particular, for \( m > 0 \) the numerical results suggest that \( \text{Re} \omega \to m \Omega_1 \) in the extremal limit, and that in this limit the spacing between frequencies is \( \Delta \omega \simeq i 2 \pi T_{\text{BH}} \) [13], in accord with the analytical result of equation (9). For \( m < 0 \), it was found numerically that \( \Delta \omega \simeq i / 2 \) in the extremal limit, in agreement with equation (10). Two comments are in place. First, note that extrapolations of equation (7) from the RN to the Kerr black hole that differ from equation (8) will not agree with the numerical results. Second, the numerical results hold true for gravitational and electromagnetic perturbations, suggesting that equations (8)–(11) are valid for these perturbation fields as well.

In summary, we have studied analytically the QNM spectrum of nearly extremal, rotating Kerr black holes. It was found that the intermediate asymptotic resonances can be expressed in terms of the black-hole physical parameters: its temperature \( T_{\text{BH}} \), and its horizon’s angular velocity \( \Omega_1 \), see equations (9)–(11). Our results may provide an important link towards an analytical calculation of the asymptotic spectrum, which is of great interest, both classically and quantum mechanically.

Acknowledgments

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References

[1] For an excellent review and a detailed list of references, see Nollert H P 1999 Class. Quantum Grav. 16 R159
[2] Press W H 1971 Astrophys. J. 170 L105
[3] de la Cruz V, Chase J E and Israel W 1970 Phys. Rev. Lett. 24 423
[4] Vishveshwara C V 1970 Nature 227 936
[5] Davis M, Ruffini R, Press W H and Price R H 1971 Phys. Rev. Lett. 27 1466
[6] Leaver E W 1985 Proc. R. Soc. A 402 285
[7] Bachelot A and Motet-Bachelot A 1993 Ann. Inst. H. Poincaré 59 3
[8] Kunstatter G 2003 Phys. Rev. Lett. 90 161301
[9] Motl L 2003 Adv. Theor. Math. Phys. 6 1135
[10] Corichi A 2003 Phys. Rev. D 67 087502
[11] Motl L and Neitzke A 2003 Adv. Theor. Math. Phys. 7 307
[12] Cardoso V and Lemos J P S 2003 Phys. Rev. D 67 084020
[13] Hod S 2003 Phys. Rev. D 67 081501
[14] Kaul R K and Rama S K 2003 Phys. Rev. D 68 024001
[15] Abdalla E, Castello-Branco K H C and Lima-Santos A 2003 Mod. Phys. Lett. A 18 1435
[16] Berti E and Kokkotas K D 2003 Phys. Rev. D 68 044027
[17] Cho H T 2003 Phys. Rev. D 68 024003
[18] van den Brink A M 2004 J. Math. Phys. 45 327
[19] Glampedakis K and Andersson N 2003 Class. Quantum Grav. 20 3441
[20] Neitzke A 2003 Preprint hep-th/0304080
[21] Molina C 2003 Phys. Rev. D 68 064007
[22] Polychronakos A P 2004 Phys. Rev. D 69 044010
[23] van den Brink A M 2003 Phys. Rev. D 68 047501
[24] Xue L H, Shen Z X, Wang B and Su R K 2004 Mod. Phys. Lett. A 19 239
[25] Cardoso V, Konoplya R and Lemos J P S 2003 Phys. Rev. D 68 044024
[26] Birmingham D, Carlip S and Chen Y 2003 Class. Quantum Grav. 20 L239
[27] Swain J 2003 Int. J. Mod. Phys. D 12 1729
[28] Padmanabhan T and Patel A 2003 Preprint hep-th/0305165
[29] Birmingham D 2003 Phys. Lett. B 569 199
[30] Berti E, Cardoso V, Kokkotas K D and Onozawa H 2003 Phys. Rev. D 68 124018
[31] Berti E, Cardoso V and Yoshida S 2004 Phys. Rev. D 69 124018
[32] Andersson N and Howls C J 2004 Class. Quantum Grav. 21 1623
[33] Oppenheimer J 2004 Phys. Rev. D 69 044012
[34] Yoshida S and Futamase T 2004 Phys. Rev. D 69 064025
[35] Musiri S and Siopsis G 2003 Class. Quantum Grav. 20 L285
  Musiri S and Siopsis G 2004 Phys. Lett. B 579 25
[36] Konoplya R A 2003 Phys. Rev. D 68 124017
  Konoplya R A 2004 Phys. Rev. D 70 047503
  Konoplya R A 2005 Phys. Rev. D 71 024038
[37] Ling Y and Zhang H 2003 Phys. Rev. D 68 101501
[38] Cardoso V, Lemos J P S and Yoshida S 2004 Phys. Rev. D 69 044004
[39] Medvev A J M, Martin D and Visser M 2004 Class. Quantum Grav. 21 1393
  Medvev A J M, Martin D and Visser M 2004 Class. Quantum Grav. 21 2393
[40] Padmanabham T 2004 Class. Quantum Grav. 21 L1
[41] Birmingham D and Carlip S 2004 Phys. Rev. Lett. 92 111302
[42] Choudhury T R and Padmanabham T 2004 Phys. Rev. D 69 064033
[43] Setare M R 2004 Phys. Rev. D 69 044016
[44] Cardoso V, Lemos J P S and Yoshida S 2003 J. High Energy Phys. JHEP12(2003)41
[45] Zhang H, Cao Z, Gong X and Zhou W 2004 Class. Quantum Grav. 21 917
[46] Jing J 2004 Phys. Rev. D 69 084009
[47] Gour G and Suneeta V 2004 Class. Quantum Grav. 21 3405
[48] Vanzo L and Zerbini S 2004 Phys. Rev. D 70 044030
[49] Musiri S and Siopsis G 2004 Phys. Lett. B 579 25
[50] Tamaki T and Nomura H 2004 Phys. Rev. D 69 044014
[51] Kiefer C 2004 Class. Quantum Grav. 21 L123
[52] Ohashi A and Sakagami M 2004 Class. Quantum Grav. 21 3973
[53] Kiefer C, Kunstatter G and Medved A J M 2004 Class. Quantum Grav. 21 5317
[54] Cardoso V, Lemos J P S and Yoshida S 2003 J. High Energy Phys. JHEP12(2003)41
[55] Konoplya R A 2004 Phys. Rev. D 70 044004
[56] Jing J 2004 Phys. Rev. D 69 084046
[57] Vanzo L and Zerbini S 2004 Phys. Rev. D 70 044030
[58] Das S and Shankaranarayanan S 2005 Class. Quantum Grav. 22 L7
[59] Shu F W and Shen Y G 2004 Phys. Rev. D 70 084046
[60] Castello-Branco K H C, Konoplya R A and Zhidenko A 2005 Phys. Rev. D 71 047502
[61] Natario J and Schiappa R 2004 Preprint hep-th/0411267
[62] Fernando S and Holbrook C 2005 Preprint hep-th/0501138
[63] Shu F W and Shen Y G 2005 Preprint gr-qc/0501098
[64] Jing J and Pan Q 2005 Preprint gr-qc/0502011
[65] Jing J 2005 Preprint gr-qc/0502023
[66] Giannatto M and Moss I G 2005 Preprint gr-qc/0502010
[67] Berti E and Kokkotas K D 2005 Preprint gr-qc/0502065
[68] Hod S 1998 Phys. Rev. Lett. 81 4293
[69] Dreyer O 2003 Phys. Rev. Lett. 90 081301
[70] Ashtekar A, Bauz J C, Corichi A and Krasnov K 1998 Phys. Rev. Lett. 80 904
[71] Collot H P 1993 Phys. Rev. D 47 5253
[72] Andersson N 1993 Class. Quantum Grav. 10 L61
[73] Mukhanov V 1986 JETP Lett. 44 63
[74] Bekenstein J D 1973 Phys. Rev. D 7 2333
[75] Detweiler S 1980 Astrophys. J. 239 292
[76] Onozawa H 1997 Phys. Rev. D 55 3593
[77] Regge T and Wheeler J A 1974 Phys. Rev. 108 1063
[78] Teukolsky S A 1972 Phys. Rev. Lett. 29 1114
[79] Teukolsky S A 1973 Astrophys. J. 185 635
[80] Abramowitz M 1965 Handbook of Mathematical Functions (New York: Dover)
[81] Flammer C 1957 Spheroidal Wave Functions (Stanford, CA: Stanford University Press)