CHARACTERIZATIONS OF FREE QUASICONFORMALITY IN METRIC SPACES

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Abstract. In this paper, we investigate the concept of (dimension) free quasiconformality in metric spaces. We establish three results demonstrating that this concept is useful in a very general metric setting. First, we show several sufficient conditions for a homeomorphism to be fully semisolid in suitable metric spaces. These conditions indicate that the quasihyperbolic metrics are quasi-invariant under several different kinds of mappings, for example, quasisymmetric mappings, weakly quasisymmetric mappings etc. One of these sufficient conditions is a generalization of the main result, Theorem 1.6, in [15]. Second, as the main result of this paper, we prove that, in suitable Boman metric spaces, all the sufficient conditions obtained for full semisolidity are also necessary, and then, as a direct corollary, we obtain six alternative characterizations for free quasiconformality of a homeomorphism. Finally, as an application of our main result, we prove that the composition of two locally weakly quasisymmetric mappings in a large class of metric spaces is locally quasisymmetric, and also it is quasiconformal.

1. Introduction and main results

The quasihyperbolic metric (briefly, QH metric) was introduced by Gehring and his students Palka and Osgood in the 1970’s [10][11] in the setting of Euclidean spaces \( \mathbb{R}^n \) \((n \geq 2)\). Since its first appearance, the quasihyperbolic metric has become an important tool in the geometric function theory of Euclidean spaces.

From late 1980’s onwards, Väisälä has developed the theory of (dimension) free quasiconformal mappings (briefly, free theory) in Banach spaces [29][30][31][32][34], which is based on the quasihyperbolic metric. The main advantage of this approach over generalizations based on the conformal modulus (see [12] and references therein) is that it does not make use of volume integrals, which allows one to study the quasiconformality of homeomorphisms in infinite dimensional Banach spaces. In the free theory, Väisälä has mainly studied the relationships between quasiconformal mappings and quasisymmetric mappings, as well as properties of the quasihyperbolic metric and various classes of domains. The importance of the quasihyperbolic metric

2000 Mathematics Subject Classification. Primary: 30C65, 30F45; Secondary: 30C20.

Key words and phrases. Quasihyperbolic metric; quasiconformality; free quasiconformality; quasisymmetry; weak quasisymmetry; full semisolidity; relativity; ring property; Boman chain condition; quasiconvex metric space.

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The research was partly supported by NSF of China (No. 11101138).
in this setting arises from the distortion inequality in Definition 1.3 (cf. the Schwarz-Pick type result given in [10]). This line of research has recently attracted substantial interest in the research community (see e.g. [16, 21, 22, 35]).

The class of quasisymmetric mappings on the real axis were first introduced by Beurling and Ahlfors [1], who found a way to obtain a quasiconformal extension of a quasisymmetric self-mapping of the real axis to a self-mapping of the upper half-plane. This concept was later generalized by Tukia and Väisälä to study quasisymmetric mappings between metric spaces [24]. In 1998, Heinonen and Koskela [14] proved a remarkable result, showing that these two concepts, quasiconformality and quasisymmetry, are quantitatively equivalent in a large class of metric spaces, which includes Euclidean spaces. In Banach spaces, Väisälä also proved the quantitative equivalence among free quasiconformality, quasisymmetry and weak quasisymmetry. See [34, Theorem 7.15].

Our study is motivated by Väisälä’s theory of freely quasiconformal mappings and other related maps in the setup of Banach spaces, as well as the work of Heinonen and others in the metric setting. A reference of particular importance here is the very recent study in [15] dealing with quasisymmetric mappings in metric spaces. In particular, the series of related papers of Väisälä has emphasized very much on the properties considered in our study, such as characterizations of semisolidity or full semisolidness of homeomorphisms in Banach spaces (see [29, 30, 31, 32, 34]). The main goal of this paper is to merge the above approaches by finding characterizations of full semisolidity and free quasiconformality of homeomorphisms in suitable metric spaces.

We begin with some basic definitions. Throughout this paper, we always assume that $X$ and $Y$ are metric spaces. Following notations and terminology of [13, 14, 15, 26, 34], we begin with the definitions of quasiconformality and quasisymmetry.

**Definition 1.1.** A homeomorphism $f$ from $X$ to $Y$ is said to be

1. quasiconformal if there is a constant $H < \infty$ such that

\[
\limsup_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} \leq H
\]

for all $x \in X$;

2. quasisymmetric if there is a constant $H < \infty$ such that

\[
\frac{L_f(x,r)}{l_f(x,r)} \leq H
\]

for all $x \in X$ and all $r > 0$,

where $L_f(x,r) = \sup_{|y-x| \leq r} \{|f(y) - f(x)|\}$ and $l_f(x,r) = \inf_{|y-x| \geq r} \{|f(y) - f(x)|\}$.

Here and in what follows, we always use $|x - y|$ to denote the distance between $x$ and $y$.

**Definition 1.2.** A homeomorphism $f$ from $X$ to $Y$ is said to be

1. $\eta$-quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

\[
|x - a| \leq t|x - b| \implies |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)|
\]
for each $t > 0$ and for each triplet $x, a, b$ of points in $X$;
(2) weakly $H$-quasisymmetric if
$$|x - a| \leq |x - b| \text{ implies } |f(x) - f(a)| \leq H|f(x) - f(b)|$$
for each triplet $x, a, b$ of points in $X$.

**Remark 1.1.** The following observations follow immediately from Definitions 1.1 and 1.2.

1. The quasisymmetry implies the quasiconformality;
2. A homeomorphism $f$ from $X$ to $Y$ is quasisymmetric with coefficient $H$ defined by (1.2) if and only if it is weakly $H$-quasisymmetric;
3. The $\eta$-quasisymmetry implies the weak $H$-quasisymmetry, where $H = \eta(1)$. In general, the converse is not true (cf. [34, Theorem 8.5]). See [17] for the related discussions.

**Definition 1.3.** Let $G \subset X$ and $G' \subset Y$ be two domains (open and connected), and let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism. We say that a homeomorphism $f : G \rightarrow G'$ is

1. $\varphi$-semisolid if $k_{G'}(f(x), f(y)) \leq \varphi(k_G(x, y))$ for all $x, y \in G$;
2. $\varphi$-solid if both $f$ and $f^{-1}$ are $\varphi$-semisolid;
3. fully $\varphi$-semisolid (resp. fully $\varphi$-solid) if $f$ is $\varphi$-semisolid (resp. $\varphi$-solid) in every proper subdomain of $G$. Fully $\varphi$-solid mappings are also called freely $\varphi$-quasiconformal mappings, or briefly, $\varphi$-FQC mappings.

In [10], Gehring and Osgood proved that every $K$-quasiconformal mapping in any domain $G \subset \mathbb{R}^n$ is a $\varphi$-semisolid mapping, where $\varphi(t) = c \max \{t, t^{1/(1-n)}\}$ and $c = c(K, n)$ which means that the constant $c$ depends only on the coefficient $K$ of quasiconformality and the dimension $n$ of the Euclidean space $\mathbb{R}^n$, and thus every $K$-quasiconformal mapping in $G$ is $\varphi$-solid since $f^{-1}$ is also $K$-quasiconformal. Hence every $K$-quasiconformal mapping in any domain $G \subset \mathbb{R}^n$ is $\varphi$-FQC. In [25, Theorem 6.12], Tukia and Väisälä proved that the converse is also true. This implies that $f$ is $K$-quasiconformal if and only if it is $\varphi$-FQC, where $K$ and $\varphi$ depend on each other and $n$, from which, we see that FQC mappings are natural generalizations of quasiconformal mappings in the setting of metric spaces. But the implication from conformality to quasisymmetry of a homeomorphism $f : G \rightarrow G'$ still remains open when $G$ is a proper domain in $X$ (including the case $X = \mathbb{R}^n$).

For the generalization of convexity, we introduce the following definition.

**Definition 1.4.** For $c \geq 1$, a metric space $X$ is $c$-quasiconvex if each pair of points $x, y \in X$ can be joined by an curve $\gamma$ with length $\ell(\gamma) \leq c|x - y|$.

We remark that $X$ is convex if and only if $c = 1$, and obviously, if $X$ is $c_1$-quasiconvex, then it must be $c_2$-quasiconvex for any $c_2 \geq c_1$.

For a homeomorphism $f$ from domain $G \subset X$ to domain $G' \subset Y$, for the sake of convenience, we introduce the following conventions:
“$P_1$” means that $f$ is fully $\varphi$-semisolid;
“$P_2$” means that $f$ is fully $(\theta; t_0)$-relative;
“$P_3$” means that $f$ is fully $C$-coarsely $M$-Lipschitz in the $QH$ metric;
“$P_4$” means that $f$ has the $(M; \alpha; \beta)$-ring property;
“$P_5$” means that $f$ is fully $\theta$-relative;
“$P_6$” means that $f$ is $q$-locally $\eta$-quasisymmetric for some $0 < q < 1$;
“$P_7$” means that $f$ is $q$-locally weakly $H$-quasisymmetric for some $0 < q < 1$.

Other concepts and notations appearing in the above statements will be introduced in the later sections.

By studying full semisolidity of homeomorphisms in Banach spaces, Väisälä obtained several characterizations of FQC mappings (see, e.g., [34, Theorem 7.9]). The main purpose of this paper is to establish the quantitative implications, in particular, the quantitative equivalence, among $P_1, \ldots, P_7$. This can be regarded as an analogue of [34, Theorem 7.9 and Corollary 7.12] in the setting of metric spaces. It should be noted that Huang and Liu have started the study of this topic in [15]. As the main result in their paper, Huang and Liu proved that every weakly quasisymmetric mapping in quasiconvex and complete metric spaces is semisolid ([15, Theorem 1.6]). The phrase “quantitative implication” should be understood as follows. For example, a condition $A$ with data $\nu$ implies the condition $B$ with data $\mu$ so that $\mu$ depends only on $\nu$ and other given quantities, then we say that $A$ implies $A'$ quantitatively. If $A'$ also implies $A$ quantitatively, then we say that $A$ and $A'$ are quantitatively equivalent.

Our first result is about the quantitative equivalence among $P_1, \ldots, P_4$.

**Theorem 1.** Suppose $X$ is a $c$-quasiconvex metric space and $Y$ is a $c'$-quasiconvex metric space and both $G \subseteq X$ and $G' \subseteq Y$ are domains. If $f : G \to G'$ is a homeomorphism, then $f$ satisfies the following quantitative equivalence:

$$P_1 \iff P_2 \iff P_3 \iff P_4.$$ 

With the extra assumptions “completeness” on the metric space $X$ and “being non-point-cut” on $G$ in Theorem 1, we get quantitative implications among $P_1, \ldots, P_7$.

**Theorem 2.** Suppose $X$ is a $c$-quasiconvex and complete metric space, $Y$ is a $c'$-quasiconvex metric space, and both $G \subseteq X$ and $G' \subseteq Y$ are domains. If $f : G \to G'$ is a homeomorphism, then $f$ satisfies the following quantitative implications:

$$P_6 \Rightarrow P_7 \Rightarrow P_4 \Leftrightarrow P_3 \Leftrightarrow P_2 \Leftrightarrow P_1.$$ 

Furthermore, if $G$ is non-point-cut, then $f$ satisfies one more quantitative implication, that is,

$$P_5 \Rightarrow P_6.$$ 

Here $G$ is said to be non-point-cut if for any $x \in G$, the set $G \setminus \{x\}$ is a subdomain of $G$.

**Remark 1.2.** (1) By Remarks 1.1 and 4.1, we see that the implication $P_7 \Rightarrow P_1$ in Theorem 2 is a generalization of the main result, Theorem 1.6, in [15].
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(2) If \( f : G \to G' \) is a homeomorphism, and \( G \) and \( G' \) are domains in \( X \) and \( Y \), respectively, then \( G \) is non-point-cut if and only if \( G' \) is non-point-cut.

In [15], the authors proposed a problem that whether the converse of [15, Theorem 1.6] is true or not. Our next purpose is to consider this problem. In our case, this problem becomes whether the implication \( P_1 \Rightarrow P_7 \), which, combining with Theorem 2, implies the quantitative equivalence among \( P_1, \ldots, P_7 \). In this way, we get six alternative characterizations of full semisolidity, and also as a direct corollary, we give a partial answer to the related problem. Our result is as follows.

Theorem 3. Suppose \( X \) is a \( c \)-quasiconvex, complete, \( \kappa \)-finite and Boman metric space, \( Y \) is a \( c' \)-quasiconvex metric space, and suppose \( G \subsetneq X \) and \( G' \subsetneq Y \) are domains. If \( f : G \to G' \) is a homeomorphism, and if \( G \) is non-point-cut, then \( f \) satisfies the following quantitative equivalences:

\[
P_1 \Leftrightarrow P_2 \Leftrightarrow P_3 \Leftrightarrow P_4 \Leftrightarrow P_5 \Leftrightarrow P_6 \Leftrightarrow P_7.
\]

As a direct corollary of Theorem 3, we obtain the following six alternative characterizations for free quasiconformality.

Corollary 1. Suppose that \( X \) and \( Y \) are quasiconvex, complete, \( \kappa \)-finite and Boman metric spaces, that \( G \subsetneq X \) and \( G' \subsetneq Y \) are domains, that \( f : G \to G' \) is a homeomorphism, and that \( G \) is non-point-cut. Then the following statements are quantitatively equivalent:

1. \( f \) is \( \varphi \)-FQC;
2. \( 2 \leq j \leq 7 \), both \( f \) and \( f^{-1} \) satisfy the property \( P_j \).

The following relationship between free quasiconformality and quasiconformality also easily follows from Theorem 3 or Corollary 1 together Remark 4.1(2).

Corollary 2. Under the assumptions in Corollary 1, if \( f \) is \( \varphi \)-FQC, then both \( f \) and \( f^{-1} \) are quasiconformal.

It is easy to see that the composition of two \( \varphi \)-quasisymmetric mappings is also a \( \varphi \)-quasisymmetric mapping, see [24, Theorem 2.2]. But, in [24], Tukia and Väisälä have indicated that the composition of two weakly quasisymmetric mappings need not be weakly quasisymmetric. So, it is significant to find conditions under which the composition of two locally weakly quasisymmetric mappings is quasisymmetric or quasiconformal. As an application of Theorem 3, we get the following.

Theorem 4. Suppose that \( X_1 \) is a \( c_1 \)-quasiconvex, complete, \( \kappa \)-finite, Boman metric space, and that \( X_i (i = 1, 2, 3) \) are \( c_i \)-quasiconvex metric spaces. For domains \( G_i \subsetneq X_i \) \((i = 1, 2, 3)\), if \( f : G_1 \to G_2 \) is a \( q_1 \)-locally weakly \( H_1 \)-quasisymmetric mapping, \( g : G_2 \to G_3 \) is a \( q_2 \)-locally weakly \( H_2 \)-quasisymmetric mapping, and if \( G_1 \) is non-point-cut, then

1. the composition \( g \circ f \) is a \( q \)-locally \( \eta \)-quasisymmetric mapping;
2. the composition \( g \circ f \) is also a \( K \)-quasiconformal mapping,

where the constants \( q, \eta \) and \( K \) depend only on \( c_1, c_2, c_3, q_1, q_2, H_1, H_2 \).
The rest of this paper is organized as follows. In Section 2, we introduce some necessary concepts and notations, and obtain several auxiliary and interesting results which will be used later on. We give a proof of Theorem 1 together with the definitions of relativity, ring property and coarsely (or fully coarsely) Lipschitz mapping, in Section 3 and then, in Section 4 we first introduce the definitions of local (or locally weak) quasisymmetry, and then prove Theorem 2. Section 5 is devoted to the introduction of the concepts of Boman chain condition, Boman space and \( \kappa \)-finite property, and the proof of the main result, Theorem 3, in this paper. Finally, Section 6 contains a proof of Theorem 4.

2. QUASIHYPERBOLIC METRIC

We always denote the open (resp. closed) metric ball with center \( x \in X \) and radius \( r > 0 \) by

\[
\mathbb{B}(x, r) = \{ z \in X : |z - x| < r \} \quad \text{(resp. } \overline{\mathbb{B}}(x, r) = \{ z \in X : |z - x| \leq r \}).
\]

and the metric sphere by

\[
\partial \mathbb{B}(x, r) = \{ z \in X : |z - x| = r \}.
\]

For \( \lambda > 0 \),

\[
\lambda \mathbb{B}(x, r) = \{ z \in X : |z - x| < \lambda r \}.
\]

For a set \( A \) in \( X \), we always use \( \partial A \) (resp. \( \overline{A} \)) to denote the boundary (resp. the closure) of \( A \).

By a curve, we mean any continuous function \( \gamma : [a, b] \to X \). The length of \( \gamma \) is denoted by

\[
\ell(\gamma) = \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \right\},
\]

where the supremum is taken over all partitions \( a = t_0 < t_1 < t_2 \ldots < t_n = b \). The curve is rectifiable if \( \ell(\gamma) < \infty \). In particular, if the metric is taken to be the quasihyperbolic metric, the length of \( \gamma \) is denoted by \( \ell_{qh}(\gamma) \).

The length function associated with a rectifiable curve \( \gamma : [a, b] \to X \) is

\[
s_{s}(t) = s_{s}(t) = \gamma(\ell(\gamma)_{[a, b]}).
\]

For any rectifiable curve \( \gamma : [a, b] \to X \), there is a unique curve \( \gamma_s : [0, \ell(\gamma)] \to X \) such that \( \gamma = \gamma_s \circ s_{\gamma} \). Obviously, \( \ell(\gamma_s|_{[0,t]}) = t \) for \( t \in [0, \ell(\gamma)] \). The curve \( \gamma_s \) is called the arclength parametrization of \( \gamma \).

For a rectifiable curve \( \gamma \) in \( X \), the line integral over \( \gamma \) of each Borel function \( \varrho : X \to [0, \infty) \) is

\[
\int_{\gamma} \varrho ds = \int_{0}^{\ell(\gamma)} \varrho \circ \gamma_s(t) dt.
\]

Let \( X \) be a connected metric space. For any open set \( G \) in \( X \), if \( G \neq X \), it follows from [15, Remark 2.2] that \( \partial G \neq \emptyset \). For \( r \in (0, \delta_G(z)) \), the ball \( \mathbb{B}(z, r) \) is not necessarily contained in \( G \), where \( \delta_G(z) \) denotes the distance from \( z \) to \( \partial G \). Thus we need to consider \( G \) as a metric space whose metric is the restriction of the one in \( X \). If \( G \not\subset X \) is a domain, it follows from [15, Observation 2.6] that \( G \) is rectifiably connected. For a metric ball \( \mathbb{B}(z, r) \) with \( z \in G \) and \( 0 < r < \delta_G(z) \), the intersection \( \mathbb{B}(z, r) \cap G \) is not always connected. So we always consider the
component of the intersection $B(z,r) \cap G$ containing the center $z$, which is denoted by $B^G(z,r)$. Similarly, we use $\overline{B}^G(z,r)$ (resp. $\partial B^G(z,r)$) to denote the closure (resp. the boundary) of the component of $B(z,r) \cap G$ containing the center $z$. In particular, for $\lambda > 0$ and $0 < r < \frac{1}{\lambda} \delta_G(x)$, we always use $\lambda B^G(z,r)$ to denote the component $B^G(z,\lambda r)$. Then we have

**Lemma 2.1.** Suppose $X$ is a $c$-quasiconvex metric space and $G \subseteq X$ is a domain. For any rectifiably connected set $D \subset B(z,r)$ with $z \in D \cap G$, if $r \leq \delta_G(z)$, then

$$D \subset \overline{B}^G(z,r) \subset G.$$ 

**Proof.** It follows from [15, Observation 2.6] that $G$ is rectifiably connected. Obviously, it suffices to prove that $D \subset G$. We show this by contradiction. Suppose on the contrary that $D$ is not contained in $G$. Then there exists a point $u \in D \subset B(z,r)$, but $u \notin G$. Let $\gamma$ be a rectifiable curve in $D$ joining the points $z$ and $u$. It follows from [15, Remark 2.2] that $\partial G \neq \emptyset$. Since $u \notin G$, there must exist a point $u_0$ such that $u_0 \in \gamma \cap \partial G$. Hence

$$|z - u_0| \geq \delta_G(z) \geq r > |z - u_0|,$$

which is the desired contradiction. \hfill $\square$

Further, we have the following result.

**Lemma 2.2.** Suppose that $X$ is a $c$-quasiconvex metric space and $G \subseteq X$ is a domain.

1. Suppose that $z \in G$ and $0 < r \leq \frac{2}{c+2} \delta_G(z)$. Then $B(z,r) \subset G$;
2. Suppose that $z \in G$ and $0 < r \leq \frac{1}{c+1} \delta_G(z)$. Then for any $x$ and $y \in B(z,r)$, there must exist a curve $\gamma \subset B^G(z,(c+1)r)$ such that

$$\ell(\gamma) \leq c|x - y|;$$
3. Suppose that $z \in G$ and $0 < r \leq \frac{1}{c+1} \delta_G(z)$. Then

$$B(z,r) \subset B^G(z,(c+1)r).$$

In particular, for any $z \in G$ and $0 < r \leq \frac{1}{c+1} \delta_G(z)$,

$$B(z,r) \subset B^G(z,\delta_G(z)).$$

**Proof.** We prove the first assertion by contradiction. Suppose that there exists some $x \in B(z,r)$ such that $x \notin G$. By the assumption, we see that there is a curve $\beta \subset X$ connecting $z$ and $x$ such that

$$\ell(\beta) \leq c|x - z| < cr.$$ 

Obviously, there is some point $z_0$ such that $z_0 \in \beta \cap \partial G$. Let $w_0$ be the point in $\beta$ such that

$$\ell(\beta|_{z,w_0}) = \ell(\beta|_{w_0,x}),$$
where $\beta_{[z,w_0]}$ denotes the part of $\beta$ with the endpoints $z$ and $w_0$. Then we claim that for $w \in \beta$, 

\[
|w - z| < \frac{c + 2}{2}r.
\]

To show this claim, we consider two cases. For the first case where $w \in \beta_{[z,w_0]}$, we easily see that

\[
|w - z| \leq \frac{1}{2} \ell(\beta) < \frac{c}{2}r.
\]

For the other case where $w \in \beta_{[x,w_0]}$, we easily see that

\[
|w - z| \leq |w - x| + |x - z| < \frac{1}{2} \ell(\beta) + r \leq \frac{c + 2}{2}r.
\]

Hence (2.1) holds, and the claim is proved.

Since $z_0 \in \beta \cap \partial G$, we see from (2.1) that

\[
|z_0 - z| \geq \delta_G(z) \geq \frac{c + 2}{2}r > |z_0 - z|.
\]

This is the desired contradiction, which shows that the assertion (1) in the lemma is true.

To get proofs of the second and the third assertions in the lemma, for any $x$ and $y \in \mathbb{B}(z,r)$, we let $\gamma$ denote a curve in $X$ joining $x$ to $y$ such that

\[
\ell(\gamma) \leq c|x - y| < 2cr.
\]

Now, we check that

\[
(2.2) \quad \gamma \subset \mathbb{B}(z, (c + 1)r) \cap G.
\]

Since for all $w \in \gamma$,

\[
(2.3) \quad |w - z| \leq \min\{|w - x|, |w - y|\} + r \leq \frac{1}{2} \ell(\gamma) + r < (c + 1)r,
\]

which implies that $\gamma \subset \mathbb{B}(z, (c + 1)r)$.

It remains to prove the conclusion $\gamma \subset G$. Again, we show this by contradiction. Suppose that $\gamma$ is not contained in $G$. Since it follows from the assertion (1) that $x$ and $y \in G$, there must be a point $u_0$ which is contained in the intersection $\gamma \cap \partial G$, and so (2.3) implies

\[
\delta_G(z) \leq |u_0 - z| < (c + 1)r \leq \delta_G(z).
\]

This is the desired contradiction. Hence $\gamma \subset \mathbb{B}(z, (c + 1)r) \cap G$.

Obviously, the assertion (3) follows from (2.2) and the arbitrariness of the points $x$ and $y$ in $\mathbb{B}(z,r)$.

It follows from the assertion (3) and (2.2) that (2) holds, and hence the proof of our lemma is complete.

The proof of the following useful result is based on Lemma 2.2.
Lemma 2.3. Suppose $X$ is a $c$-quasiconvex metric space and $G \subset X$ is a domain. For any metric ball $B(z, r)$ with $z \in G$ and $0 < r \leq \delta_G(z)$, let $B = B^G(z, r)$. Then

$$\delta_B(z) = r.$$

Proof. Suppose $\delta_B(z) < r$. Since $\partial B \neq \emptyset$, there must exist $z_0 \in \partial B$ such that

$$\delta_B(z) \leq |z_0 - z| < r,$$

which implies that $\delta_G(z_0) > 0$, and so $z_0 \in G$. Also, we know that there is a sequence $\{x_i\}_{i=1}^{\infty} \subset B$ such that $\lim_{i \to \infty} x_i = z_0$.

Let $r_1 = r - |z_0 - z|$. Obviously, $B(z_0, r_1) \subset B(z, r)$ and

$$\delta_G(z_0) \geq \delta_G(z) - |z_0 - z| \geq r - |z_0 - z| = r_1.$$

Let $r_2 = \frac{1}{c+1} r_1$. Then Lemma 2.2 implies

$$B(z_0, r_2) \subset B^G(z_0, r_1).$$

Since for all sufficiently large $i$, $x_i \in B(z_0, r_2)$, we see that

$$B(z_0, r_2) \subset B^G(z_0, r_1) \subset B.$$

This is the desired contradiction. Hence $\delta_B(z) = r$. \hfill \Box

The quasihyperbolic length of a rectifiable curve or a path $\gamma$ in the metric in a domain $G \subset X$ is the number:

$$\ell_{kG}(\gamma) = \int_{\gamma} \frac{|dz|}{\delta_G(z)}.$$

For any $z_1, z_2$ in $G$, the quasihyperbolic distance $k_G(z_1, z_2)$ between $z_1$ and $z_2$ is defined by

$$k_G(z_1, z_2) = \inf \{ \ell_{kG}(\gamma) \},$$

where the infimum is taken over all rectifiable curves $\gamma$ joining $z_1$ to $z_2$ in $G$.

Gehring and Palka [11] introduced the quasihyperbolic metric of a domain in $\mathbb{R}^n$. For the basic properties of this metric we refer to [10]. Recall that a curve $\gamma$ from $z_1$ to $z_2$ is a quasihyperbolic geodesic if $\ell_{kG}(\gamma) = k_G(z_1, z_2)$. Each subcurve of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between any two points in $E$ exists if the dimension of $X$ is finite, see [10] Lemma 1. This is not true in arbitrary metric spaces (cf. [33, Example 2.9]).

We establish a comparison result between the metrics $|\cdot|$ and $k_G$ in a $c$-quasiconvex metric space. It is a modified version of [15, Theorems 2.7 and 2.8], which is needed in our proofs later on.

Lemma 2.4. Let $X$ be a $c$-quasiconvex metric space and let $G \subset X$ be a domain.

(1) For all $x, y \in G$,

$$|x - y| \leq (e^{k_G(x, y)} - 1) \delta_G(x);$$

(2.4)
(2) Suppose $z \in G$ and $0 < t < 1$. Then for $x, y \in \mathbb{B}^G(z, \frac{1}{2c} \delta_G(z))$, 
\begin{equation}
\frac{c}{c + t} \frac{|x - y|}{\delta_G(z)} \leq k_G(x, y) \leq \frac{c}{1 - \frac{c+1}{2c} t} \frac{|x - y|}{\delta_G(z)};
\end{equation}

(3) Suppose that $x, y \in G$ and either $|x - y| \leq \frac{1}{3c} \delta_G(x)$ or $k_G(x, y) \leq 1$. Then 
\begin{equation}
\frac{1}{2} \frac{|x - y|}{\delta_G(x)} < k_G(x, y) \leq 3 \frac{|x - y|}{\delta_G(x)}.
\end{equation}

Proof. The inequality (2.4) follows from [15, Theorem 2.7]. In the following, we prove (2.5) and (2.6).

Since $X$ is a $c$-quasiconvex metric space, we see from Lemma 2.2 that there is a curve $\gamma$ in $\mathbb{B}^G(z, \frac{c+1}{2c} t\delta_G(z))$ joining $x$ and $y$ with 
$$\ell(\gamma) \leq c|x - y|.$$ 
Since for $w \in \gamma,$ 
$$\delta_G(w) \geq \delta_G(z) - |w - z| \geq \left(1 - \frac{c + 1}{2c} t\right) \delta_G(z),$$ 
we see that 
$$k_G(x, y) \leq \frac{1}{c + t} \frac{|dw|}{\delta_G(x)} \leq \frac{c}{(1 - \frac{c+1}{2c} t) \delta_G(z)} |x - y|.$$ 
This shows that the right-side inequality of (2.5) is true. Now, we give a proof of the left-side inequality of (2.5). It follows from [15, Observation 2.6] that $G$ is rectifiably connected. Hence it suffices to prove that for any rectifiable curve $\alpha \subset G$ joining $x$ and $y$, 
$$\ell_{k_G}(\alpha) \geq \frac{c}{c + t} \frac{|x - y|}{\delta_G(z)}.$$ 
To this end, let $\alpha$ be a rectifiable curve in $G$ joining $x$ and $y$. We divide the proof into two cases. For the first case where $\alpha \subset \mathbb{B}^G(z, \frac{c}{2c} \delta_G(z))$, we see that 
\begin{equation}
\delta_G(w) \leq |w - z| + \delta_G(z) \leq \left(1 + \frac{t}{c}\right) \delta_G(z)
\end{equation}
for all $w \in \alpha$, and hence 
$$\ell_{k_G}(\alpha) = \int_{\alpha} \frac{|dw|}{\delta_G(w)} \geq \frac{c}{c + t} \frac{|x - y|}{\delta_G(z)},$$ 
as required.

For the remaining case, that is, $\alpha \not\subset \mathbb{B}^G(z, \frac{c}{2c} \delta_G(z))$, obviously, $\alpha$ has two sub-curves $\alpha_1$ and $\alpha_2$ in $\mathbb{B}^G(z, \frac{c}{2c} \delta_G(z))$ joining the sets $\partial \mathbb{B}^G(z, \frac{c}{2c} \delta_G(z))$ and $\partial \mathbb{B}^G(z, \frac{c}{2c} \delta_G(z))$. Again, (2.7) implies 
$$\ell_{k_G}(\alpha) \geq \int_{\alpha_1 \cup \alpha_2} \frac{|dw|}{\delta_G(w)} \geq \frac{c}{c + t} \frac{|x - y|}{\delta_G(z)},$$ 
since $\ell(\alpha_i) \geq \frac{t}{2c} \delta_G(z) \geq \frac{|x - y|}{2}$ for $i = 1, 2$. Hence the proof of (2.5) is complete.
To prove (2.6), we first consider the case where $|x - y| \leq \frac{1}{3c} \delta_G(x)$. By taking $t = \frac{2}{3}$ in (2.5), obviously, we have that

$$|x - y| < \frac{c|x - y|}{(c + \frac{2}{3})\delta_G(x)} \leq \frac{3c^2}{2c - 1} \frac{|x - y|}{\delta_G(x)} \leq 3c \frac{|x - y|}{\delta_G(x)}.$$ 

For the remaining case, that is, $|x - y| > \frac{1}{3c} \delta_G(z)$ and $k_G(x, y) \leq 1$, the right-side inequality of (2.6) is obvious. Since for $r \in (0, 1]$, $e^r - 1 < 2r$,

it follows from (2.4) that

$$|x - y| \leq (e^{k_G(x, y)} - 1)\delta_G(x) < 2k_G(x, y)\delta_G(x),$$

which implies the left-side inequality of (2.6) is true too. Hence the proof of (2.6) is complete. □

The following result is on the quasiconvexity of a domain $G$ in $X$ with respect to the quasihyperbolic metric $k_G$.

**Lemma 2.5.** Let $X$ be a $c$-quasiconvex metric space and let $G \subsetneq X$ be a domain.

1. Let $\gamma$ be a rectifiable path in $G$. Then $\ell_{k_G}(\gamma)$ is the length of $\gamma$ in the metric space $(G, k_G)$, i.e., $\ell_{k_G}(\gamma) = \ell_{qh}(\gamma)$;
2. the space $(G, k_G)$ is $\lambda$-quasiconvex for all $\lambda > 1$.

**Proof.** The proof of the statement (1) easily follows from (2.5) and [5, Lemma 2.6], and the second statement easily follows from the first one. □

### 3. The Proof of Theorem 1

We start this section with some necessary definitions.

**Definition 3.1.** Let $G \subsetneq X$ and $G' \subsetneq Y$ be two domains. We say that a homeomorphism $f : G \to G'$ is

1. $C$-coarsely $M$-Lipschitz in the QH metric if there are constants $M > 0$ and $C > 0$ such that

$$k_{G'}(f(x), f(y)) \leq M k_G(x, y) + C$$

for all $x$ and $y$ in $G$;

2. fully $C$-coarsely $M$-Lipschitz in the QH metric if there are constants $M > 0$ and $C > 0$ such that $f$ is $C$-coarsely $M$-Lipschitz in every proper subdomain of $G$.

**Definition 3.2.** A function $f$ is said to be

1. $(\theta; t_0)$-relative if there are $t_0 \in (0, 1]$ and a homeomorphism $\theta : [0, t_0) \to [0, \infty)$ such that

$$\frac{|f(x) - f(y)|}{\delta_G(f(x))} \leq \theta\left(\frac{|x - y|}{\delta_G(x)}\right)$$

for all $x$ and $y$ in $G$.
whenever \(x, y \in G\) with \(|x - y| < t_0\delta_G(x)\); In particular, if \(t_0 = 1\), then \(f\) is called to be \(\theta\)-relative;

(2) fully \((\theta; t_0)\)-relative (resp. fully \(\theta\)-relative) if \(f\) is \((\theta; t_0)\)-relative (resp. \(\theta\)-relative) in every proper subdomain of \(G\).

**Definition 3.3.** A function \(f\) is said to have the \((M; \alpha; \beta)\)-ring property in \(G\) if there are \(1 < \alpha \leq \beta\) and \(M > 0\) such that for any \(B = B^G(z, r)\) with \(z \in G\) and \(\beta r < \delta_G(z)\),

\[
\text{diam}(f(B)) \leq M \text{dist}(f(B), \partial f(\alpha B)),
\]

where \(\text{diam}(U)\) (resp. \(\text{dist}(U, V)\)) means the diameter (resp. the distance) of the set \(U\) (resp. between the two disjoint sets \(U\) and \(V\)).

**Remark 3.1.** It follows from Definition 3.3 that if \(f\) has the \((M; \alpha; \beta)\)-ring property in \(G\), then for any subdomain \(D\) in \(G\), the restriction \(f_D\) of \(f\) in \(D\) also has the \((M; \alpha; \beta)\)-ring property in \(D\).

Now, we are ready to prove Theorem 1. We verify the implications indicated by the following routes.

\[ P_1 \Rightarrow P_3 \Rightarrow P_4 \Rightarrow P_2 \Rightarrow P_1. \]

The proof is based on a refinement of the method due to Väisälä. The proof is given in the following four subsections.

3.1. The implication from \(P_1\) and \(P_3\). If we assume that \(f\) is fully \(\varphi\)-semisolid, then it suffices to show that for any domain \(D \subsetneq G\) and any pair \(x, y \in D\),

\[
k_{f(D)}(f(x), f(y)) \leq M k_D(x, y) + C,
\]

where \(M = M(c, \varphi)\) and \(C = C(c, \varphi)\).

To this end, we first prove the following claim for any fixed \(0 < q < 1\).

**Claim 3.1.1.** If \(x, y \in D\) with \(k_D(x, y) \leq \frac{q}{6c}\), then

\[
k_{f(D)}(f(x), f(y)) \leq \psi(k_D(x, y)),
\]

where \(\psi(t) = \varphi(\frac{\psi}{q} t)\).

It follows from (2.16) that

\[
|x - y| < 2k_D(x, y)\delta_D(x) \leq \frac{q}{3c} \delta_D(x),
\]

which implies \(y \in B(x, \frac{q}{3c} \delta_D(x))\). Set \(B_1 = B^D(x, q\delta_D(x))\). Then Lemma 2.2 implies that \(y \in B_1\). Since by Lemma 2.3,

\[
|x - y| < \frac{q}{3c} \delta_D(x) = \frac{1}{3c} \delta_{B_1}(x),
\]

again, we see from (2.6) that

\[
k_{B_1}(x, y) \leq 3c \frac{|x - y|}{\delta_{B_1}(x)} = 3c \frac{|x - y|}{q \delta_D(x)} \leq \frac{6c}{q} k_D(x, y),
\]

and thus, by the assumption, we get
\[
k_{f(D)}(f(x), f(y)) \leq k_{f(B_1)}(f(x), f(y)) \leq \varphi(k_{B_1}(x, y)) \leq \varphi\left(\frac{6c}{q} k_D(x, y)\right)
= \psi(k_D(x, y)),
\]
where \(\psi(t) = \varphi\left(\frac{6c}{q} t\right)\), which shows that the claim holds.

Since by Lemma 2.5, \((D, k_D)\) is \(\lambda\)-quasiconvex for all \(\lambda > 1\), we see that for all \(x, y \in D\), there is a curve \(\gamma\) in \(D\) joining \(x\) and \(y\) with
\[
\ell_{k_D}(\gamma) \leq \lambda k_D(x, y).
\]
Obviously, there is a unique integer \(n \geq 0\) satisfying
\[
\frac{n}{6c} q < \ell_{k_D}(\gamma) \leq \frac{n + 1}{6c} q.
\]
If \(n = 0\), that is, \(\ell_{k_D}(\gamma) \leq \frac{q}{6c}\), then Claim 3.1.1 shows that
\[
k_{f(D)}(f(x), f(y)) \leq \psi\left(\frac{q}{6c}\right).
\]
Hence (3.1) holds with \(M = 0\) and \(C = \psi\left(\frac{q}{6c}\right)\).

Now, we assume that \(n > 0\). Let \(x = x_0, \ldots, x_{n+1} = y\) be successive points in \(\gamma\) such that
\[
\ell_{k_D}(\gamma|x_{i-1},x_i|) = \frac{q}{6c} \quad \text{for all } 1 \leq i \leq n \quad \text{and} \quad 0 < \ell_{k_D}(\gamma|x_n,x_{n+1}|) \leq \frac{q}{6c}.
\]

![Figure 1. The arc \(\gamma\) in \(D\) and the related points.](image)

Then \(n \leq \frac{6c}{q} \ell_{k_D}(\gamma)\), and it follows from Claim 3.1.1 that for \(i \in \{1, \ldots, n + 1\}\),
\[
k_{f(D)}(f(x_{i-1}), f(x_i)) \leq \psi(k_D(x_{i-1}, x_i)) \leq \psi\left(\frac{q}{6c}\right).
\]
Hence
\[ k_{f(D)}(f(x), f(y)) \leq \sum_{i=1}^{n+1} k_{f(D)}(f(x_{i-1}), f(x_i)) \leq (1 + n)\psi\left(\frac{q}{6c}\right) \]
\[ \leq \left(1 + \frac{6c}{q} \ell_{k_D}(\gamma)\right)\psi\left(\frac{q}{6c}\right) \]
\[ \leq \frac{6c}{q} \lambda \psi\left(\frac{q}{6c}\right)k_D(x, y) + \psi\left(\frac{q}{6c}\right). \]

By taking \( q = \frac{1}{2} \), \( \lambda = 2 \), and letting \( M = 24c\psi\left(\frac{1}{12c}\right) \), \( C = \psi\left(\frac{1}{12c}\right) \), we see that (3.1) is also true. Hence the proof of \( P_1 \Rightarrow P_3 \) is complete.

3.2. The implication from \( P_3 \) to \( P_1 \). Assume \( f \) is fully \( C \)-coarsely \( M \)-Lipschitz in the \( QH \) metric. We shall show that for each pair of constants \( \alpha \) and \( \beta \) with \( 2c < \alpha \leq \beta \), there is a constant \( M_0 \) such that for any \( B = B^G(z, r) \) with \( z \in G \) and \( \beta r < \delta_G(z) \),
\[ \text{diam}(f(B)) \leq M_0 \text{dist}(f(B), \partial f(\alpha B)), \]
where \( M_0 = M_0(c, C, M, \alpha) \).

To find such a constant, for any \( a \in \overline{B} \), we let \( b \in \overline{B} \) be such that
\[ |f(a) - f(b)| \geq \frac{1}{3} \text{diam}(f(B)). \]

Since Lemma 2.3 leads to
\[ a, b \in \overline{B} = \overline{B^D}(z, \frac{1}{\alpha} \delta_D(z)), \]
where \( D = \alpha B \), it follows from (2.5) that
\[ k_D(a, b) \leq \frac{c}{1 - (1 + c)/\alpha} \frac{|a - b|}{\delta_D(z)} \leq \frac{2c}{\alpha - 1 - c}. \]

Hence
\[ \log \left(\frac{|f(a) - f(b)|}{\delta_{f(D)}(f(a))}\right) \leq k_{f(D)}(f(a), f(b)) \leq Mk_D(a, b) + C \leq \frac{2cM}{\alpha - 1 - c} + C = M_1, \]
and so
\[ \text{diam}(f(\overline{B})) \leq 3|f(a) - f(b)| \leq 3e^{M_1}\delta_{f(D)}(f(a)). \]

The arbitrariness of \( a \in \overline{B} \) implies that (3.2) holds with \( M_0 = 3e^{M_1} \).

3.3. The implication from \( P_1 \) to \( P_2 \). Assume \( f \) has the \((M; \alpha; \beta)\)-ring property in \( G \), where \( 1 < \alpha \leq \beta \). For any domain \( D \subsetneq G \), it follows from Remark 3.1 that \( f \) also has the \((M; \alpha; \beta)\)-ring property in \( D \). Let \( t_0 = \frac{1}{2c(2c\alpha+\gamma)\beta} \). To prove that \( f \) is fully \((\theta, t_0)\)-relative, it suffices to prove that \( f \) is \((\theta, t_0)\)-relative in \( D \) for some homeomorphism \( \theta: [0, t_0) \to [0, \infty) \), where \( \theta = \theta_{c, c', M, \alpha, \beta} \) which means that the homeomorphism \( \theta \) depends only on the given constants \( c, c', M, \alpha \) and \( \beta \).

To find such a homeomorphism, we let \( x \) and \( y \in D \) with \( |x - y| = t\delta_D(x) \), where \( t \in (0, t_0) \), and let \( m \) be the largest integer with
\[ 2c(2c\alpha)^m \beta t < 1. \]
Obviously, $m \geq 3$. Set $B_0 = B^D(x, t\delta_D(x))$ and $s_j = (2\alpha)^j$ for $1 \leq j \leq m$. Then $2cs_j\beta t\delta_D(x) \leq 2cs_m\beta t\delta_D(x) < \delta_D(x)$.

Let $z'_0 \in \partial f(D)$ be such that $|z'_0 - f(x)| \leq 2\delta f(D)(f(x))$.

Then there is a curve $\gamma \subset Y$ joining $f(x)$ and $z'_0$ such that $\ell(\gamma) \leq c'|z'_0 - f(x)| \leq 2c'\delta f(D)(f(x))$.

Let $B_j = B^D(x, s_j t\delta_D(x))$. Then it follows that for each $j \leq m$,

$$\text{diam}(f(B_j)) \leq M \text{dist}(f(B_j), \partial f(\alpha B_j)) \leq M \text{dist}(f(B_j), \partial f(2\alpha B_j)),$$

and so

$$\text{dist}(f(B_j), \partial f(D)) \geq \text{dist}(f(B_m), \partial f(D)) \geq \text{dist}(f(B_m), \partial f(2\alpha B_m)) \geq \frac{1}{M} \text{diam}(f(B_m)) > 0,$$

which implies that for each $j \in \{1, 2, \ldots, m\}$, the intersection set $\gamma \cap \partial f(B_j)$ is not empty. Let $y_j \in \gamma \cap \partial f(B_j)$. Then it follows that for each $3 \leq j \leq m$,

$$|y_2 - f(x)| \leq \text{diam}(f(B_{j-1})) \leq M \text{dist}(f(B_{j-1}), \partial f(2\alpha B_{j-1})) = M \text{dist}(f(B_{j-1}), \partial f(B_j)) \leq M \text{dist}(\partial f(B_{j-1}), \partial f(B_j)) \leq M |y_j - y_{j-1}|.$$

Summing over the indices $j$, we obtain that

$$(m - 2)|y_2 - f(x)| \leq M \sum_{j=3}^{m} |y_j - y_{j-1}| < M\ell(\gamma) \leq 2Mc'\delta f(D)(f(x)).$$
Moreover, since \( y \in \overline{B}(x, t_D(x)) \), Lemma 2.2 guarantees that \( y \in B_1 \), and so it follows that
\[
|f(x) - f(y)| \leq \text{diam}(f(B_1)) \leq M \text{dist}(f(B_1), \partial f(B_2)) \leq M|y_2 - f(x)|.
\]
Hence
\[
\left| \frac{f(x) - f(y)}{\delta_{f(D)}(f(x))} \right| \leq \frac{2M^2}{m - 2c'}.
\]
Since \( 2c(2c\alpha)^3\beta t < 1 \leq 2c(2c\alpha)^{m+1}\beta t \), we have
\[
m - 2 \geq \frac{\log(1/(2c\beta t)) - 3\log(2c\alpha)}{\log(2c\alpha)},
\]
which implies that
\[
\frac{|f(x) - f(y)|}{\delta_{f(D)}(f(x))} \leq \frac{2M^2c\log(2c\alpha)}{\log(1/(2c(2c\alpha)^3\beta t))}.
\]
Now, let
\[
\theta(t) = \frac{2M^2c\log(2c\alpha)}{\log(1/(2c(2c\alpha)^3\beta t))}.
\]
Obviously, this \( \theta \) is the desired. Hence \( P_2 \) holds.
\[\square\]

3.4. The implication from \( P_2 \) to \( P_1 \). Assume \( f \) is fully \((\theta; t_0)\)-relative in \( G \). To prove that \( f \) is fully \( \varphi \)-semisolid in \( G \), obviously, we only need to prove that \( f \) is \( \varphi \)-semisolid in any proper subdomain \( D \) of \( G \), where \( \varphi = \varphi_{c', t_0, \theta} \). For this, we choose
\[
t_1 = \frac{1}{2} \min \left\{ 1, t_0, \theta^{-1}\left( \frac{1}{3c'} \right) \right\},
\]
and let \( x \) and \( y \) be any two points in \( D \) with \( k_D(x, y) < t_1 \). Then it follows from (2.6) that
\[
\frac{|x - y|}{\delta_D(x)} \leq 2k_D(x, y) < 2t_1 \leq t_0,
\]
which implies
\[
|f(x) - f(y)| \leq \theta(2t_1)\delta_D(f(x)) \leq \frac{1}{3c'}\delta_D(f(x)).
\]
Again, (2.6) leads to
\[
k_D(f(x), f(y)) \leq 3c'\frac{|f(x) - f(y)|}{\delta_D(f(x))} \leq 3c'\theta\left( \frac{|x - y|}{\delta_D(x)} \right) \leq 3c'\theta(2k_D(x, y)).
\]
Hence, \( f \) is \((\psi; t_1)\)-uniformly continuous in the \( QH \) metric with \( \psi(t) = 3c'\theta(2t) \) (See, for example, [28] or [29] for the definition). Obviously, \( \psi(0) = 0 \). Since \((D, k_D)\) is 2-quasiconvex, we see from [28, Lemma 2.5] or [29, Lemma 3.2] that there is a homeomorphism \( \varphi = \varphi_{t_1, \psi} : [0, \infty) \to [0, \infty) \) such that \( f \) is \( \varphi \)-semisolid in \( D \). Obviously, \( \varphi = \varphi_{c', t_0, \theta} \), and thus \( f \) satisfies \( P_1 \).
\[\square\]
4. The proof of Theorem 2

First, we introduce some necessary definitions.

**Definition 4.1.** Let $G \subseteq X$ and $G' \subseteq Y$ be domains. A homeomorphism $f$ from $G$ to $G'$ is said to be

1. $q$-locally $\eta$-quasisymmetric for some $0 < q < 1$ if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that
   \[ |x - a| \leq t|x - b| \implies |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)| \]
   for each $t > 0$, each triple $x, a, b$ of points in $B^G(z, q\delta_G(z))$ and any $z \in G$;
2. $q$-locally weakly $H$-quasisymmetric for some $0 < q < 1$ if
   \[ |x - a| \leq |x - b| \implies |f(x) - f(a)| \leq H|f(x) - f(b)| \]
   for each triple $x, a, b$ of points in $B^G(z, q\delta_G(z))$ and any $z \in G$.

**Remark 4.1.**
1. The weak $H$-quasisymmetry implies the $q$-locally weak $H$-quasisymmetry for any $0 < q < 1$;
2. The $q$-locally weak $H$-quasisymmetry with $0 < q < 1$ or the weak $H$-quasisymmetry implies the $H$-quasisymmetry.

Now, we are ready for the proof of Theorem 2. To prove the first part of Theorem 2, Theorem 1 implies that we need to verify the following implications:

\[ P_6 \implies P_7 \implies P_4. \]

Since the implication from $P_6 \implies P_7$ is obvious, it suffices to verify the implication $P_7 \implies P_4$.

4.1. The implication from $P_7$ to $P_4$. Assume there is $0 < q < 1$ such that

\[ (4.1) \quad |f(x) - f(a)| \leq H|f(x) - f(b)| \]

for any triple $x, a$ and $b$ of points in $B^G(z, q\delta_G(z))$ with $|x - a| \leq |x - b|$ and any $z \in G$. We shall show that $f$ has the $(M; \alpha; \beta)$-ring property with $M = 2H^2(q + 1)$, $\alpha = 3$, and $\beta = \frac{q^2}{q^2 - q}.

For $w \in G$, $0 < q < 1$, let $K = B^G(w, r)$, where $0 < r < \frac{q^2}{q^2 - q}\delta_G(w)$. Then $K \subset 3K \subset B^G(w, q\delta_G(w))$. We first show two lemmas.

**Lemma 4.1.1.** $\diam(f(K)) \leq 2H^2(q + 1) \text{dist}(f(K), f(\partial(3K)))$.

**Proof.** Obviously, it suffices to show that for all $a, b, y \in K$ and $z \in \partial(3K)$,

\[ |f(a) - f(b)| \leq 2H^2(q + 1)|f(y) - f(z)|. \]

Since for any $u \in \partial K$, $\max\{|a - w|, |b - w|\} \leq |u - w| \leq |z - u|$, we know from (4.1) that

\[ |f(a) - f(b)| \leq |f(a) - f(w)| + |f(b) - f(w)| \leq 2H|f(u) - f(w)| \leq 2H^2|f(z) - f(u)|. \]

Meanwhile, the fact $|y - u| \leq |z - y|$ implies

\[ |f(y) - f(u)| \leq H|f(y) - f(z)|, \]
Obviously, there is $z$ satisfying the sequence. Then there is easily follows from this assertion.

Let $f$ so

We prove this assertion by contradiction. Suppose $w$ is not a Cauchy sequence in $f(3K)$. Since $X$ is complete, it suffices to show that $\{w_j\}_{j=1}^\infty$ is also Cauchy, since this will imply that $\{w_j\}_{j=1}^\infty$ converges to a point $u \in 3K$, and so $f(w_j) \to f(u) \in f(3K)$. Then the conclusion $f(\partial(3K)) = \partial f(3K)$ in the lemma easily follows from this assertion.

We prove this assertion by contradiction. Suppose $\{w_j\}_{j=1}^\infty$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ such that for each positive integer $k$, there is $j(k) > k$ satisfying

$$|w_k - w_{j(k)}| \geq \varepsilon.$$  

Obviously, there is $z_k \in \{w_k, w_{j(k)}\}$ such that

$$|z_k - w_1| \geq \varepsilon/2.\tag{4.2}$$

Let $t = 6r/\varepsilon$. Then we have

$$|w_1 - w_{j(1)}| \leq 6r = t\varepsilon \leq 2t|z_k - w_1|.$$  

For their images under $f$, we have the following estimate.

**Claim 4.1.1.** $|f(w_1) - f(w_{j(1)})| \leq (1 + 2ct)H|f(z_k) - f(w_1)|$.

We divide the proof into two cases. For the first case where $2t \leq 1$, the claim is obvious from (4.1) since $w_1, w_{j(1)}$ and $z_k \in \mathbb{B}^G(w, \frac{t\varepsilon}{2}\delta_G(w)) \subseteq \mathbb{B}^G(w, q\delta_G(w))$.

For the remaining case, that is, $2t > 1$, by Lemma 2.2 there is a curve $\gamma$ in $\mathbb{B}^G(w, \frac{t\varepsilon}{2}q\delta_G(w))$ joining $w_1$ and $w_{j(1)}$ such that

$$\ell(\gamma) \leq c|w_1 - w_{j(1)}|.$$  

Define inductively the successive points $w_1 = a_0, \ldots, a_s = w_{j(1)}$ of $\gamma$ so that for each $i \in \{1, \ldots, s\}$, $a_i$ denotes the last point of $\gamma$ in $\mathbb{B}(a_{i-1}, |z_k - w_1|)$. Obviously, $s \geq 2$. The following upper bound of $s$ is needed in the proof later on. Since for $i \in \{1, \ldots, s-1\}$,

$$|z_k - w_1| = |a_i - a_{i-1}| \leq \ell(\gamma[a_{i-1}, a_i]),$$

we see that

$$(s - 1)|z_k - w_1| \leq \ell(\gamma) \leq c|w_1 - w_{j(1)}| \leq 2ct|z_k - w_1|.$$  

Hence

$$s \leq 1 + 2ct.\tag{4.3}$$
Since all $z_k$ and $a_i$ ($i \in \{0, \ldots, s\}$) are contained in $B^G(z, q\delta_G(z)) \subset B^G(w, q\delta_G(w))$, we know that
\[|f(w_1) - f(w_j(1))| \leq \sum_{i=1}^{s} |f(a_i) - f(a_{i-1})| \leq sH|f(z_k) - f(w_1)| \leq (1+2ct)H|f(z_k) - f(w_1)|.

Hence our claim holds.

Let us proceed with the proof of our lemma. Since
\[|z_k - w_1| \leq 6r = te \leq t|w_k - w_j(k)|,

by replacing $2t$ with $t$, the similar reasoning as in the proof of Claim 4.1.1 shows that
\[|f(z_k) - f(w_1)| \leq (1 + ct)H|f(w_k) - f(w_j(k))|.

Since $\{f(w_j)\}_{j=1}^{\infty}$ is a Cauchy sequence, we see that $|f(w_k) - f(w_j(k))| \rightarrow 0$ as $k \rightarrow \infty$, and so $f(z_k) \rightarrow f(w_1)$. Then it follows from Claim 4.1.1 that $f(w_1) = f(w_j(1))$, which is a contradiction since $f$ is homeomorphic. The lemma is proved. \qed

By Lemmas 4.1.1 and 4.1.2, we see that
\[\text{diam}(f(K)) \leq 2H^2(H+1) \text{dist}(f(K), f(\partial(3K))) \leq 2H^2(H+1) \text{dist}(f(K), \partial f(3K)).

Hence the proof of $P_1 \Rightarrow P_4$ is complete. \qed

Now, we prove the second part of Theorem 2 that is, the implication $P_5 \Rightarrow P_6$, with the extra assumption that $G$ is non-point-cut.

4.2. The implication from $P_5$ to $P_6$. Assume that $f$ is fully $\theta$-relative in $G$. Since $\theta : [0, 1) \rightarrow [0, \infty)$ is homeomorphic, without loss of generality, we assume that
\[M = \theta(2c/(2c + 1)) > 1.

For $x$, $a$, $b \in B = B^G(z, q\delta_G(z))$ with $|x - a| = t|x - b|$, where $t > 0$ and $0 < q < 1$, to prove that $f$ has the property $P_6$, we need a relationship between $|f(a) - f(x)|$ and $|f(x) - f(b)|$. To this end, we divide the discussions into three cases according to the location of the parameter $t$. We first discuss the case where $0 < t \leq \frac{2c}{2c+1}$. In this case, we have

Claim 4.2.1. If $0 < t \leq \frac{2c}{2c+1}$, then for any $0 < q < \frac{1}{3}$,
\[|f(a) - f(x)| \leq \theta(t)|f(x) - f(b)|.

Set $D = G \setminus \{b\}$. Obviously, $f(D) = G' \setminus \{f(b)\}$. By the assumptions, we see that $D$ is a subdomain in $G$, and thus, $f$ is $\theta$-relative in $D$. Since
\[\delta_G(x) \geq \delta_G(z) - |x - z| \geq \delta_G(z) - q\delta_G(z) > 2q\delta_G(z) \geq |x - b|,

we have
\[|x - a| = t|x - b| \leq t\delta_D(x) < \delta_D(x),

and so
\[|f(a) - f(x)| \leq \theta(t)\delta_D(f(x)) \leq \theta(t)|f(x) - f(b)|,

from which Claim 4.2.1 follows.
Claim 4.2.2. If \( \frac{2c}{2c+1} < t \leq 1 \), then for any \( 0 < q \leq \frac{1}{3c+1} \),
\[
|f(a) - f(x)| \leq H|f(x) - f(b)|,
\]
where \( H = H(\theta, c) \).

In this case, by constructing a curve connecting \( x \) and \( a \), and then choosing suitable points on this curve, we shall apply the similar reasoning as in Claim 4.2.1 to any two adjacent points to obtain our desired inequality.

By Lemma 2.2, there is a curve \( \gamma \subset B^G(z, (c + 1)q\delta_G(z)) \) joining \( x \) and \( a \) with
\[
\ell(\gamma) \leq c|x - a|.
\]

Define inductively the successive points \( x = x_0, \ldots, x_k = a \) of \( \gamma \) so that \( x_j \) is the last point of \( \gamma \) in \( B(x_{j-1}, (\frac{2c}{2c+1})^j|x - b|) \). Obviously, \( k \geq 2 \), and for \( 1 \leq j \leq k - 1 \),
\[
|x_{j-1} - x_j| = \left( \frac{2c}{2c+1} \right)^j |x - b|
\]
and
\[
|x_{k-1} - x_k| \leq \left( \frac{2c}{2c+1} \right)^k |x - b|.
\]

Now, we need an upper bound for \( k \). Since for \( 1 \leq j \leq k - 1 \),
\[
l(\gamma_{[x_{j-1}, x_j]}) \geq |x_{j-1} - x_j| = \left( \frac{2c}{2c+1} \right)^j |x - b|,\]
we have
\[
\sum_{j=1}^{k-1} \left( \frac{2c}{2c+1} \right)^j |x - b| \leq l(\gamma) \leq c|x - a| \leq ct|x - b| \leq c|x - b|.
\]
Hence we obtain
\[
k \leq \frac{\log 2}{\log(2c+1) - \log(2c)} + 1 = k_0.
\]

For points \( x_1, x_0 = x \) and \( b \), since \( |x_1 - x_0| = \frac{2c}{2c+1}|x - b| \), the similar method in the proof of (4.4) shows that
\[
|f(x_1) - f(x_0)| \leq M|f(x) - f(b)|.
\]

For points \( x_{j+1}, x_j \) and \( x_{j-1} \), where \( 1 \leq j \leq k - 1 \), we set \( D_j = G \setminus \{x_j\} \). Obviously, \( f(D_j) = G' \setminus \{f(x_j)\} \). Since
\[
\delta_G(x_j) - |x_j - x_{j-1}| \geq \delta_G(x_0) - \sum_{i=0}^{j-1} |x_i - x_{i+1}| - |x_j - x_{j-1}|
\]
\[
\geq \delta_G(z) - |x_0 - z| - \sum_{i=0}^{j-1} |x_i - x_{i+1}| - |x_j - x_{j-1}|
\]
\[
> (1 - (4c + 1)q)\delta_G(z) \geq 0,
\]
Obviously, \( \theta(t) \leq M \).

Second, we consider the case where \( \frac{2c}{2c+1} \leq t \leq 1 \). Then we have
we know that
\[ |x_j - x_{j+1}| \leq \frac{2c}{2c+1} |x_j - x_{j-1}| = \frac{2c}{2c+1} \delta_{D_j}(x_j) < \delta_{D_j}(x_j). \]

Since \( f \) is \( \theta \)-relative in \( D_j \) for each \( 1 \leq j \leq k - 1 \), we have
\[
|f(x_2) - f(x_1)| \leq M \delta_{f(D_0)}(f(x_1)) \leq M|f(x_1) - f(x_0)| \leq M^2|f(x) - f(b)|,
\]
\[
\cdots
\]
\[
|f(x_k) - f(x_{k-1})| \leq M \delta_{f(D_{k-2})}(f(x_{k-1})) \leq M^k|f(x) - f(b)|,
\]
whence
\[
|f(a) - f(x)| \leq \sum_{j=1}^k |f(x_j) - f(x_{j-1})| < kM^k|f(x) - f(b)|.
\]
By (4.5), this yields
\[
|f(x) - f(a)| \leq H|f(x) - f(b)|,
\]
where \( H = k_0M^{k_0} \). The claim is proved.

By Claims 4.2.1 and 4.2.2 in fact, we have proved the following conclusion.

Lemma 4.2.1. \( f \) is weakly \( H \)-quasisymmetric in \( B^G(z, \frac{1}{4c+1} \delta_G(z)) \) with \( H = k_0M^{k_0} \geq 1 \).

Now, we consider the final case, that is, \( t > 1 \).

Claim 4.2.3. For any \( x, a, b \in B = B^G(z, \frac{1}{2c(4c+1)} \delta_G(z)) \) with \( |x - a| = t|x - b| \), if \( t > 1 \), then
\[
|f(a) - f(x)| \leq (1 + ct)H|f(b) - f(x)|.
\]

We shall apply Lemma 4.2.1 to prove this claim. Since \( x, a, b \in B = B^G(z, \frac{1}{2c(4c+1)} \delta_G(z)) \), Lemma 2.2 implies that there is a curve \( \gamma \subset B^G(z, \frac{1}{4c+1} \delta_G(z)) \) joining \( x \) and \( a \) with
\[
\ell(\gamma) \leq c|x - a|.
\]

Define inductively the successive points \( x = u_0, \ldots, u_\varrho = a \) of \( \gamma \) so that for each \( 1 \leq i \leq \varrho \), \( u_i \) denotes the last point of \( \gamma \) in \( B(u_{i-1}, |b - x|) \). Then the similar reasoning as in the proof of (4.3) ensures that
\[
2 \leq \varrho \leq 1 + ct.
\]

Since \( b \) and all \( u_i \) \((i \in \{0, \ldots, \varrho\})\) are contained in \( B^G(z, \frac{1}{4c+1} \delta_G(z)) \), Lemma 4.2.1 guarantees that
\[
|f(a) - f(x)| \leq \sum_{i=1}^\varrho |f(u_i) - f(u_{i-1})| \leq \varrho H|f(x) - f(b)|.
\]
Thus, (4.7) implies
\[
|f(a) - f(x)| \leq (1 + ct)H|f(b) - f(x)|.
\]
The proof of Claim 4.2.3 is complete.
Now, we are ready to finish the proof. Let
\[
\eta(t) = \begin{cases} 
(1 + c) \frac{H}{M} \theta(t), & \text{if } 0 < t \leq \frac{2c}{2c+1}, \\
(1 + c) H, & \text{if } \frac{2c}{2c+1} \leq t \leq 1, \\
H(1 + ct), & \text{if } t > 1.
\end{cases}
\]
Then it follows from (4.4), (4.6) and (4.8) that for any \(x, a, b \in B = B_G(z, \frac{1}{2(4c+1)} \delta_G(z))\) with \(|a - x| = t|b - x|\),
\[
|f(a) - f(x)| \leq \eta(t) |f(b) - f(x)|.
\]
This shows that \(f\) has the property \(P_6\) with \(q = \frac{1}{2(4c+1)}\).

\[\Box\]

5. The proof of Theorem 3

Before the proof of Theorem 3, let us recall some definitions. The first is the so-called “Boman chain condition” which was originally introduced in the setting of homogeneous spaces.

**Definition 5.1.** Suppose \(G\) is a domain in a \(c\)-quasiconvex metric space \(X\).

1. \(G\) is said to satisfy the \((M, \lambda, 4cC_1, C_1)\)-Boman chain condition if there exist positive constants \(M, \lambda > 1, C_1 > 1\) and a family \(\mathcal{F}\) of disjoint metric balls \(B\) such that
   (a) \(G = \bigcup_{B \in \mathcal{F}} C_1B\);
   (b) \(\sum_{B \in \mathcal{F}} \lambda^{4cC_1B(x)} \leq M \chi_G(x)\) for all \(x \in X\);
   (c) There is a so-called “central ball” \(B_* \in \mathcal{F}\) such that
      (i) for each ball \(B \in \mathcal{F}\), there is a positive integer \(k = k(B)\) and a chain of balls \(\{B_j\}_{j=0}^k\) such that \(B_0 = B, B_k = B_*\), and \(C_1B_j \cap C_1B_{j+1}\) contains a metric ball \(D_j\) whose radius is comparable to those of both \(B_j\) and \(B_{j+1}\); and
      (ii) for all \(j = 1, \ldots, k(B)\), \(B_0 \subset \lambda B_j\).

2. \(G\) is said to be Boman if it satisfies the \((M, \lambda, 4cC_1, C_1)\)-Boman chain condition.

We remark that, for our purpose, the constant \(C_2\) in the corresponding version of the definition of “Boman chain condition” in the Euclidean spaces (see, e.g., Definition 2.1 in [6]) is replaced by \(4cC_1\).

In 1982, Boman [3], in his work on \(L^p\)-estimates for elliptic systems, introduced the Boman chain condition. The domains satisfying the Boman chain condition soon appeared in many connections in the study of Hardy-Littlewood type inequality, Sobolev-Poincaré inequality, etc (see, e.g., [2, 4, 6, 7, 8, 9, 18, 19, 20, 23]). In the Carnot-Carathéodory setting, the fact that metric balls satisfy chain condition is implicit in [19], and it is explicitly proved in [20]. The same result is given in a more general setting in [6] and [8], where certain geodesic conditions are assumed. The important observation in connection with certain metric spaces is that metric balls in a metric are Boman domains which allows one to patch up global estimates from local ones. Naturally, we introduce the following definition.
Definition 5.2. A $c$-quasiconvex metric space $X$ is said to be Boman if there are positive constants $M > 0$, $\lambda > 1$ and $C_1 > 1$ such that every metric ball in $X$ is an $(M, \lambda, 4cC_1, C_1)$-Boman domain.

Obviously, all corresponding metric spaces discussed in \cite{4, 8, 19, 20} are Boman.

Definition 5.3. A metric space $X$ is $\kappa$-finite if there is an increasing function $\kappa : [1/2, \infty) \to [1, \infty)$ such that if $a_1, \ldots, a_s$ are points in $B(x, r)$ with $|a_i - a_j| \geq t > 0$ for all $i \neq j$, then $s \leq \kappa(r/t)$.

We remark that if $X$ is $\kappa$-homogeneously totally bounded, briefly, $\kappa$-HTB, then it must be $\kappa$-finite. See \cite{24} Remarks 2.8 or \cite{27} Section 2.8.

5.1. The proof of Theorem 3. Assume that every metric ball in $X$ is $(M, \lambda, 4cC_1, C_1)$-Boman with $M > 0$, $\lambda > 1$, $C_1 > 1$. It follows from Theorem 2 that, to prove Theorem 3, it suffices to verify the implication:

$$P_1 \Rightarrow P_5.$$ 

Assume that $f$ is fully $\varphi$-semisolid and that $D$ is a proper subdomain of $G$. To show the implication $P_1 \Rightarrow P_5$, obviously, it suffices to construct a homeomorphism $\theta : [0, 1) \to [0, \infty)$ such that $f$ is $\theta$-relative in $D$, where $\theta = \theta_{c, \kappa, \lambda, \varphi}$. The procedure of the construction of such a homeomorphism consists of six steps. First, we need to show that for $x, y \in D$ with $|x - y| = t \delta_D(x)$ ($0 < t < 1$), there is a homeomorphism $\theta' : [0, 1) \to [0, \infty)$ such that

$$(5.1) \quad k_D(x, y) \leq \theta'(t),$$

where $\theta' = \theta'_{c, \kappa, \lambda}$. 

To get such a homeomorphism, we need to construct a curve in $D$ connecting $x$ and $y$, which satisfies certain conditions. This construction consists of four steps. The fifth step is to prove (5.1). Based on (5.1), in the last step, we shall complete the proof of the implication $P_1 \Rightarrow P_5$.

Before the statement of the first step, we need some preparation.

Let $\tilde{B}$ denote the metric ball $B(x, \delta_D(x))$. Then $y \in \tilde{B}$. Since $\tilde{B}$ is a Boman domain, Lemma 2.1 implies that $\tilde{B} \subset D$, and also there is a family $\mathcal{F}$ of disjoint metric balls $B$ satisfying the $(M, \lambda, 4cC_1, C_1)$-Boman chain condition. We start with a lemma. In what follows in this section, for convenience, we always use $x(B)$ (resp. $r(B)$) to denote the center (resp. the radius) of the metric ball $B$, i.e., $B = B(x(B), r(B))$.

Lemma 5.1.1. For any ball $B$ in $\mathcal{F}$,

1. $4cC_1B \subset B$;
2. $r(B) \leq \frac{1}{4cC_1} \delta_B(x(B)) \leq \frac{1}{4cC_1} \delta_D(x(B))$; and
3. $r(B) \leq \frac{1}{2cC_1} \delta_D(x)$.

Proof. The statement (1) directly follows from the fact: for any $v \in 4cC_1B$,

$$\sum_{B \in \mathcal{F}} \chi_{4cC_1B}(v) \leq M \chi_B(v).$$
The second statement easily follows from the first one, and the third statement follows from \([1]\) and the obvious fact: For any two metric balls \(B_1 = B(z, r_1)\) and \(B_2 = B(z, r_2)\) in \(X\), if \(B_1 \subset B_2\), then \(r_1 \leq 2r_2\). Hence the proof of the lemma is complete. \(\square\)

We assume that \(B_n\) is a central ball of \(F\). Our first step is to show the existence of some special ball chains in \(F\) as the following shows.

**Step 5.1.1.** Starting from any fixed ball in \(F\), there is a ball chain in \(F\) with the rectified geometric growth rate.

The construction is a modification of Buckley, Koskela and Lu \([6]\). Fix \(B_0 \in F\). By definition, we see that there exists a ball chain \(\{B_j\}_{j=0}^n\) in \(F\), which is from \(B_0\) to \(B_n\). First, we prove a claim.

**Claim 5.1.1.1.** There is some constant \(M_1 = M_1(κ, λ)\) such that

either \(n \leq M_1\) or \(r(B_j) > 2r(B_0)\) for some \(0 < j \leq M_1\).

Since for \(i \neq j \in \{0, \ldots, n\}\), \(B_i \cap B_j = \emptyset\) and \(B_0 \subset λB_j\), we see that

\[
|x(B_j) - x(B_i)| > r(B_j) > \frac{1}{2λ} r(B_0).
\]

Then it follows from the assumption “\(X\) being \(κ\)-finite” that the number of chain balls contained in \((2λ + 3)B_0\) is at most \(M_0 = κ(10λ^2)\). Now, we set \(M_1 = M_0 + 1\) which is the number required. To see this, we assume that \(n > M_1\). Then there must exist some \(j \in \{0, \ldots, M_1\}\) such that \(B_j\) is not contained in \((2λ + 3)B_0\). Since \(B_0 \subset λB_j\), we have

\[
(2λ + 3)r(B_0) - r(B_j) \leq |x(B_0) - x(B_j)| \leq λr(B_j),
\]

which implies the required estimate \(r(B_j) > 2r(B_0)\). The claim is proved.

We proceed with the construction. Based on Claim 5.1.1.1, from \(F\), we shall pick up a ball chain with the rectified geometric growth rate.

If \(r(B_j) \leq 2r(B_0)\) for \(j \in \{0, \ldots, n\}\), then we denote \(B_{j+1}^1 = B_j\) for \(j \in \{0, \ldots, n\}\) and \(m_1 = n\). Obviously, it follows from Claim 5.1.1.1 that \(m_1 \leq M_1\).

Otherwise, there is some \(j \in \{0, \ldots, n\}\) such that \(r(B_j) > 2r(B_0)\). We let \(p_1 = m_1 + 1\) be the minimum \(j\) such that \(r(B_j) > 2r(B_0)\), and let \(B_{j+1}^1 = B_j\) for \(j \in \{0, \ldots, m_1\}\) and \(B_{m_1}^1 = B_{m_1+1}\). Obviously, \(m_1 \leq M_1\). If \(B_{m_1}^1 \neq B_n\), then we see from the definition that there exists a chain \(\{C_j\}_{j=0}^{m_1}\) in \(F\), which is from \(C_0 = B_1^1\) to \(B_{m_1}\).

If \(r(C_j) \leq 2r(B_{m_1}^2)\) for \(j \in \{0, \ldots, n_1\}\), then we denote \(B_{j+1}^2 = C_j\) for \(j \in \{0, \ldots, n_1\}\) and \(m_2 = n_1\). Again, it follows from Claim 5.1.1.1 that \(m_2 \leq M_1\).

Otherwise, there is some \(j\) such that \(r(C_j) > 2r(B_{m_1}^2)\). We let \(p_2 = m_2 + 1\) be the minimum \(j\) such that \(r(C_j) > 2r(B_{m_1}^2)\), and let \(B_{j+1}^2 = C_j\) for \(j \in \{0, \ldots, m_2\}\) and \(B_{m_2+1}^2 = C_{m_2+1}\). Also, \(m_2 \leq M_1\). If \(B_{m_2}^2 \neq B_n\), then we see from the definition that there exists a chain \(\{D_j\}_{j=0}^{m_2}\) in \(F\), which is from \(D_0 = B_1^2\) to \(B_{m_2}\).
Since the metric ball $\tilde{B}$ is bounded, by repeating the above procedure at most finitely many times, we find a subchain

$$\mathcal{F}' = \{B_j^k\}_{1 \leq j \leq m_k+1, 1 \leq k \leq q}$$

of $\mathcal{F}$, which is from $B_1^1$ to $B_s$, i.e., $B_{m_q+1}^q = B_s$, where $q$ is finite and $m_k \leq M_1$ for each $k \in \{1, \ldots, q\}$. We remark that if there is some $k$ such that $m_k = 0$, then $k = q$ and $B_1^q = B_s$.

**Claim 5.1.1.2.** $\mathcal{F}'$ satisfies the following properties.

1. For $k \in \{1, \ldots, q\}$ and $j \in \{1, \ldots, m_k + 1\}$,
   $$|x(B_j^k) - x(B_1^k)| \leq \lambda r(B_j^k);$$
2. If $q > 1$, then for $k \in \{1, \ldots, q-1\}$,
   $$|x(B_k^k) - x(B_{k+1}^k)| \leq \lambda r(B_{k+1}^k);$$
3. If $q > 1$, then for $k \in \{1, \ldots, q-1\}$ and $j \in \{1, \ldots, m_k + 1\}$,
   $$r(B_{k+1}^k) > 2r(B_k^k) \geq r(B_j^k) > \frac{1}{2\lambda} r(B_1^k);$$
4. If $k = q$, then for $j \in \{1, \ldots, m_q + 1\}$,
   $$2r(B_j^q) \geq r(B_j^q) > \frac{1}{2\lambda} r(B_1^q);$$
5. For any two adjacent balls $B_I$ and $B_{II}$ in $\mathcal{F}'$, $C_1B_I \cap C_1B_{II} \neq \emptyset$, and so
   $$|x(B_I) - x(B_{II})| \leq C_1(r(B_I) + r(B_{II}));$$
6. For any ball $B$ in $\mathcal{F}'$,
   $$B_1^1 \subset (3\lambda + 4\lambda^2)B.$$

From the construction of $\mathcal{F}'$, we see that it suffices to prove the item [6]. For any ball $B$ in $\mathcal{F}'$, obviously, there are $k \in \{1, \ldots, q\}$ and $j \in \{1, \ldots, m_k + 1\}$ such that $B = B_j^k$. If $q = 1$, the proof is obvious. If $q > 1$, then for any $k \in \{1, \ldots, q\}$, and $j \in \{1, \ldots, m_k\}$ (Here, without loss of generality, we assume that $m_k > 0$), then it follows from Claim 5.1.1.2[1] and Claim 5.1.1.2[2] that

$$|x(B_j^k) - x(B_1^1)| \leq |x(B_k^k) - x(B_1^1)| + |x(B_j^k) - x(B_1^1)| + \ldots + |x(B_1^1) - x(B_1^1)| \leq \lambda r(B_1^1) + \lambda r(B_1^1) + \ldots + \lambda r(B_1^1) \leq \lambda r(B_1^1) + \lambda r(B_1^1) \left(1 + \frac{1}{2} + \ldots + \left(\frac{1}{2}\right)^{k-2}\right) < \lambda r(B_1^1) + 2\lambda r(B_1^1) \leq (\lambda + 4\lambda^2)r(B_1^1)$$

and

$$r(B_1^1) < \frac{1}{2} r(B_2^2) < \ldots < \frac{1}{2^{k-1}} r(B_k^k) \leq \frac{2\lambda}{2^{k-1}} r(B_1^1).$$

Hence for any $z \in B_1^1$,

$$|x(B_j^k) - z| \leq |x(B_k^k) - x(B_1^1)| + |x(B_j^k) - z| < (3\lambda + 4\lambda^2)r(B_1^1),$$
from which (6) easily follows. □

Remark 5.1. Because the family $\mathcal{F}'$ possesses the properties (3) and (4) in Claim 5.1.1.2, we say that the family $\mathcal{F}'$ is a ball chain from $\mathcal{F}$, which has the rectified geometric growth rate.

Based on Step 5.1.1, we shall construct a curve in $D$ connecting $x(B_*)$ and $x(B)$ for any fixed ball $B$ in $\mathcal{F}$, which is as follows.

**Step 5.1.2.** For any ball $B \in \mathcal{F}$, there exists a curve $\gamma$ joining $x(B)$ and $x(B_*)$ in $D$ such that for any $z \in \gamma$,

\begin{align}
\ell(\gamma_{[x(B),z]}) &\leq \mu \delta_D(z) \\
\ell(\gamma) &\leq \nu \delta_D(x),
\end{align}

where $\mu = \frac{16}{3} \lambda (M_1 + 1)$ and $\nu = 4(M_1 + 1)$.

Fix $B \in \mathcal{F}$. Then, by Step 5.1.1. there exists a ball chain $\mathcal{F}' = \{B_j^k\}$ in $\mathcal{F}$ such that

1. it is from $B_1^1 = B$ to the central ball $B_*$;
2. it satisfies Claim 5.1.1.2.

For any $1 \leq k \leq q$ and $1 \leq j \leq m_k$ (if $m_k > 0$), we connect $x(B_j^k)$ and $x(B_{j+1}^k)$ by a curve $\alpha_{j}^{k} \subset X$ with

$$\ell(\alpha_{j}^{k}) \leq c|x(B_{j+1}^k)|,$$

where $x(B_j^k)$ and $x(B_{j+1}^k)$ denote the centers of two adjacent balls $B_j^k$ and $B_{j+1}^k$, respectively. Thus Claim 5.1.1.2 implies

$$\ell(\alpha_{j}^{k}) \leq cC_1(r(B_j^k) + r(B_{j+1}^k)).$$

**Figure 3.** The curve $\alpha_{j}^{k}$.

For $k \in \{1, \ldots, q - 1\}$ (if $q > 1$), we join $x(B_{m_k+1}^k)$ and $x(B_{1+1}^{k+1})$ by a curve $\alpha_{0}^{k+1} \subset X$ with

$$\ell(\alpha_{0}^{k+1}) \leq c|x(B_{m_k+1}^k) - x(B_{1+1}^{k+1})|.$$
and then, again, Claim 5.1.2(5) shows that
\begin{equation}
\ell(\alpha_0^{k+1}) \leq cC_1 \left( r(B_{m_k+1}^k) + r(B_1^{k+1}) \right).
\end{equation}

Further, we show that these curves are contained in \( \tilde{B} \) as the following claim demonstrates.

**Claim 5.1.2.1.** (1) For any \( 1 \leq k \leq q \) and \( 1 \leq j \leq m_k \) (if \( m_k > 0 \)),
\[ \alpha_j^k \subset \tilde{B} \subset D; \]
(2) For \( k \in \{1, \ldots, q - 1\} \) (if \( q > 1 \)),
\[ \alpha_0^{k+1} \subset \tilde{B} \subset D. \]

Obviously, it is sufficient to prove the first assertion (1) because the proof of (2) easily follows from a similar argument. By Lemma 5.1.1[2] and (5.4), we have that
\[ \ell(\alpha_j^k) \leq cC_1 \left( r(B_j^k) + r(B_{j+1}^k) \right) \leq \frac{1}{4} \left( \delta_{\tilde{B}}(x(B_j^k)) + \delta_{\tilde{B}}(x(B_{j+1}^k)) \right) \]
\[ \leq \frac{1}{2} \max \left\{ \delta_{\tilde{B}}(x(B_j^k)), \delta_{\tilde{B}}(x(B_{j+1}^k)) \right\}, \]
which implies that \( \alpha_j^k \subset \tilde{B} \). Hence the claim is proved.

Let \( \beta^1 = \cup_{j=1}^{m_1} \alpha_j^1 \), and for \( 2 \leq k \leq q \) (if \( q > 1 \)), let \( \beta^k = \cup_{j=0}^{m_k} \alpha_j^k \). Obviously, \( \beta^1 \) (resp. \( \beta^k \)) is a curve joining the centers of the balls \( B_1^1 \) and \( B_{m_1+1}^1 \) (resp. \( B_{m_k+1}^{k-1} \) and \( B_{m_k+1}^k \) for \( 2 \leq k \leq q \)). It follows from Claim 5.1.2.1 that for \( 1 \leq k \leq q \),
\[ \beta^k \subset \tilde{B} \subset D. \]

Further, it follows from (5.4), (5.5) and Claim 5.1.2.1(3) that
\begin{equation}
\ell(\beta^1) = \sum_{j=1}^{m_1} \ell(\alpha_j^1) \leq cC_1 \sum_{j=1}^{m_1} \left( r(B_j^1) + r(B_{j+1}^1) \right) \leq 4cC_1 M_1 r(B_1^1),
\end{equation}
and for $2 \leq k \leq q$,

$$\ell(\beta^k) = \sum_{j=0}^{m_k} \ell(\alpha_j^k)$$

$$\leq c C_1 \sum_{j=1}^{m_k} (r(B^k_j) + r(B^k_{j+1})) + c C_1 r(B^k_{m_k-1}) + r(B^k_1)$$

$$\leq 4 c C_1 \sum_{j=1}^{m_k} r(B^k_j) + 2 c C_1 r(B^k_1)$$

$$\leq 4 c C_1 (M_1 + 1) r(B^k_1).$$

Let $\gamma = \cup_{k=1}^q \beta^k$. Then $\gamma$ is a curve joining $x(B)$ and $x(B_*)$, $\gamma \subset B \subset D$, and (5.6), (5.7) together with Lemma 5.1.1 lead to

$$\ell(\gamma) = \sum_{k=1}^q \ell(\beta^k) \leq 4 c C_1 \sum_{k=1}^q (M_1 + 1) r(B^k_1)$$

$$\leq 4 c C_1 (M_1 + 1) r(B^q_1) \left(1 + \frac{1}{2} + \ldots + \left(\frac{1}{2}\right)^q\right)$$

$$\leq \nu \delta_D(x),$$

where $\nu = 4(M_1 + 1)$. Hence (5.3) holds.

Figure 5. The curve $\gamma$ connecting $x(B)$ and $x(B_*)$.  

To get the inequality (5.2), we need the following lower bound on $\delta_D(z)$ in terms of some $r(B^k_1)$ for every $z \in \gamma$.

**Claim 5.1.2.2.** For any $z \in \gamma$, if $z \in \alpha_j^k$ for $k \in \{1, \ldots, q\}$ and $j \in \{1, \ldots, m_k\}$, or $z \in \alpha_j^{k+1}$ for $k \in \{1, \ldots, q-1\}$, then

$$\delta_D(z) \geq \frac{3 c C_1}{2 \lambda} r(B^k_1).$$

For the first case, i.e., $z \in \alpha_j^k$ for $k \in \{1, \ldots, q\}$ and $j \in \{1, \ldots, m_k\}$, we see from Lemma 5.1.1, (5.4) and Claim 5.1.1.2 that
\[\delta_D(z) \geq \max \{\delta_D(x(B_i^k)) - |x(B_i^k) - z|, \delta_D(x(B_j^k)) - |x(B_j^k) - z|\}\]
\[\geq \frac{1}{2}\left(\delta_D(x(B_i^k)) + \delta_D(x(B_j^k)) - (|x(B_i^k) - z| + |x(B_j^k) - z|)\right)\]
\[\geq \frac{1}{2} \left(4cC_1r(B_i^k) + 4cC_1r(B_j^k) - \ell(a_j^k)\right)\]
\[\geq 3cC_1\left(r(B_i^k) + r(B_j^k)\right)\]
\[\geq \frac{3cC_1}{2\lambda} r(B_i^k).\]

For the remaining case, that is, \(z \in \alpha_{\delta}^{k+1}\) for \(k \in \{1, \ldots, q - 1\}\), again we see from Lemma 5.1.1[5.5] and Claim 5.1.1.2[3] that
\[\delta_D(z) \geq \max \{\delta_D(x(B_i^k)) - |x(B_i^k) - z|, \delta_D(x(B_j^k)) - |x(B_j^k) - z|\}\]
\[\geq \frac{3}{2} cC_1 \left(r(B_i^k) + r(B_j^k)\right)\]
\[\geq \frac{3cC_1}{2\lambda} r(B_i^k).\]

Hence the proof of the claim is complete.

Now, we are ready to finish the proof of Step 5.1.2. For any \(z \in \gamma\), there must exist \(k \in \{1, \ldots, q\}\) such that \(z \in \alpha_j^k\) (1 \(\leq j \leq m_k\)) or \(\alpha_j^k\). Then it follows from (5.6), (5.7), Claims 5.1.1.2(3) and 5.1.2.2 that
\[\ell(\gamma_{[x(B), z]}) \leq \sum_{i=1}^{k} \ell(\beta^i) \leq 4cC_1(M_1 + 1) \sum_{i=1}^{k} r(B_i^k) < 8cC_1(M_1 + 1)r(B_1^k) \leq \mu\delta_D(z),\]
where \(\mu = \frac{16}{3} \lambda(M_1 + 1)\). Hence the proof of Step 5.1.2 is complete.

**Remark 5.1.1.** Both constants \(\mu\) and \(\nu\) in Step 5.1.2 depend only on \(\kappa\) and \(\lambda\).

Further, we shall construct a curve in \(D\) connecting \(x\) and \(x(B)\) for any fixed ball \(B\) in \(F\). This construction is given in the following step.

**Step 5.1.3.** For any ball \(B \in F\), there exists a curve \(\gamma\) joining \(x(B)\) and \(x\) in \(D\) such that for any \(z \in \gamma\),
\[\ell(\gamma_{[x(B), z]}) \leq 12\mu\nu\delta_D(z)\]
and
\[\ell(\gamma) \leq 3\nu\delta_D(x),\]
where \(\mu\) and \(\nu\) are the same as in Step 5.1.2.

Since \(\bigcup_{B \in F} C_1B = \tilde{B}\), there exists a ball \(B_w \in F\) such that \(x \in C_1B_w\), and then Lemma 5.1.1[3] implies
\[(5.8) \quad |x - x(B_w)| \leq C_1r(B_w) \leq \frac{1}{2c}\delta_D(x).\]

By the quasiconvexity of \(X\), there is a curve \(\gamma_w\) in \(X\) joining \(x\) and \(x(B_w)\) with
\[\ell(\gamma_w) \leq c|x - x(B_w)|,\]
and so
\[ \ell(\gamma_w) \leq \frac{1}{2} \delta_D(x). \]  

(5.9)

Obviously, \( \gamma_w \subset D \). We come to determine the position of \( \gamma_w \) more precisely.

**Claim 5.1.3.1.** \( \gamma_w \subset \overline{B}(x, \frac{c+2}{4c} \delta_D(x)) \).

For any \( z \in \gamma_w \), if \( \ell(\gamma_w[x,z]) \geq \frac{1}{2} \ell(\gamma_w) \), then (5.8) shows that
\[ |x - z| \leq |x - x(B_w)| + |x(B_w) - z| \leq \frac{c + 2}{4c} \delta_D(x), \]
and if \( \ell(\gamma_w[x,z]) < \ell(\gamma_w)/2 \), then
\[ |x - z| < \frac{1}{2} \ell(\gamma_w) \leq \frac{1}{4} \delta_D(x). \]

Hence the claim holds.

For any ball \( B \) in \( \mathcal{F} \), we come to construct a curve connecting \( x(B) \) and \( x \) satisfying the needed property. By Step 5.1.2, there exist two curves \( \gamma_1 \) and \( \gamma_2 \) joining \( x(B_*) \) and \( x(B_*') \) and \( x(B_w) \), respectively. Let
\[ \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_w. \]

Then \( \gamma \) is a curve joining \( x(B) \) and \( x \), and \( \gamma \subset D \). In the following, we prove that this \( \gamma \) is the required.

![Figure 6. The curve \( \gamma \) connecting \( x(B) \) and \( x \).](image)

First, it is obvious from Step 5.1.2 and (5.9) that
\[ \ell(\gamma) = \ell(\gamma_1) + \ell(\gamma_2) + \ell(\gamma_w) \leq \nu \delta_D(x) + \nu \delta_D(x) + \frac{1}{2} \delta_D(x) < 3\nu \delta_D(x). \]

(5.10) \[ \ell(\gamma) = \ell(\gamma_1) + \ell(\gamma_2) + \ell(\gamma_w) \leq \nu \delta_D(x) + \nu \delta_D(x) + \frac{1}{2} \delta_D(x) < 3\nu \delta_D(x). \]

To finish the proof, we need to get an estimate on \( \ell(\gamma_{x(B),z}) \) in terms of \( \delta_D(z) \) for any \( z \in \gamma \). To this end, we separate the argument into two possibilities: \( z \in \gamma_1 \) and \( z \in \gamma_2 \cup \gamma_w \).

For the first possibility, that is, \( z \in \gamma_1 \), we obtain from Step 5.1.2 that
\[ \ell(\gamma_{x(B),z}) \leq \mu \delta_D(z). \]

(5.11) \[ \ell(\gamma_{x(B),z}) \leq \mu \delta_D(z). \]

We now consider the other possibility, that is, \( z \in \gamma_2 \cup \gamma_w \). In this case, we first get a comparison result between \( \delta_D(z) \) and \( \delta_D(x) \), which is as follows.

**Claim 5.1.3.2.** If \( z \in \gamma_2 \cup \gamma_w \), then \( \delta_D(x) \leq 4\mu \delta_D(z) \).
We divide the proof into two cases. For the first case where \( z \in B \left( x, 2 + \frac{c}{4c} \delta_D(x) \right) \), we know

\[
\delta_D(z) \geq \left( 1 - \frac{2 + c}{4c} \right) \delta_D(x) = \frac{3c - 2}{4c} \delta_D(x).
\]

For the other case where \( z \not\in B \left( x, 2 + \frac{c}{4c} \delta_D(x) \right) \), it follows from Claim 5.1.3.1 that \( z \in \gamma_2 \), and then (5.8) leads to

\[
|z - x(B_w)| \geq |z - x| - |x - x(B_w)| \geq \left( \frac{2 + c}{4c} - \frac{1}{2c} \right) \delta_D(x) = \frac{1}{4} \delta_D(x),
\]

which, together with Step 5.1.2, implies that

\[
\delta_D(z) \geq \frac{1}{\mu} \ell(\gamma_{[x(B_w),z]}) \geq \frac{1}{\mu} |x(B_w) - z| \geq \frac{1}{4\mu} \delta_D(x).
\]

Hence the claim is proved.

The combination of (5.10) and Claim 5.1.3.2 shows that

\[
(5.12) \quad \ell(\gamma_{[x,B]}(z)) < 3\nu \delta_D(x) \leq 12\mu \nu \delta_D(z).
\]

It follows from (5.10), (5.11) and (5.12) that the proof of Step 5.1.3 is complete.

Now, we are ready to finish the construction of the needed curve in \( D \) connecting \( x \) and \( y \). This construction is given in the following step.

**Step 5.1.4.** There exists a curve \( \gamma \) joining \( y \) and \( x \) in \( D \) such that for any \( z \in \gamma \),

\[
(5.13) \quad \ell(\gamma_{[x,B]}(z)) \leq 24\mu \nu \delta_D(z)
\]

and

\[
(5.14) \quad \ell(\gamma) \leq 4\nu \delta_D(x).
\]

Based on Step 5.1.3, we shall construct the desired curve. It follows from the assumptions that there exists a ball \( B \in \mathcal{F} \) such that \( y \in C_1B \). Then Lemma 5.1.1(2) implies

\[
(5.15) \quad |y - x(B)| \leq C_1r(B) \leq \frac{1}{4c} \delta_D(x(B)).
\]

By Step 5.1.3, there is a curve \( \alpha \) in \( D \) joining \( x(B) \) and \( x \) such that for any \( z \in \alpha \),

\[
\ell(\alpha_{[x(B),z]}) \leq 12\mu \nu \delta_D(z)
\]

and

\[
\ell(\alpha) \leq 3\nu \delta_D(x).
\]

By Lemmas 2.2(2), 5.1.1(1) and 5.1.1(3), together with (5.15), we see that there is a curve \( \beta \) joining \( y \) and \( x(B) \) such that \( \beta \subset 2cC_1B \subset B \subset D \) and

\[
(5.16) \quad \ell(\beta) \leq c|y - x(B)| \leq cC_1r(B) \leq \frac{1}{2} \delta_D(x).
\]

Let \( \gamma = \alpha \cup \beta \). Then \( \gamma \subset D \) is a curve joining \( x \) and \( y \), and

\[
\ell(\gamma) = \ell(\alpha) + \ell(\beta) \leq 3\nu \delta_D(x) + \frac{1}{2} \delta_D(x) < 4\nu \delta_D(x),
\]

which shows that (5.14) is true.
Next, we verify the inequality (5.13). For any \( z \in \gamma \), we consider the following three possibilities:

1. If \( z \in \beta \), then Lemma 5.1.1(2) and (5.16) show that
   \[
   \delta_D(z) \geq \delta_D(x(B)) - |x(B) - z| \geq 3cC_1r(B) \geq 3\ell(\beta) \geq 3\ell(\gamma_{y,z});
   \]

2. If \( z \in \alpha \cap \mathbb{B}(x(B), cC_1r(B)) \), then (5.16) implies
   \[
   \ell(\gamma_{y,z}) = \ell(\beta) + \ell(\alpha_{[x(B),z]}) \leq cC_1r(B) + 12\mu\nu\delta_D(z) \leq \left(\frac{1}{3} + 12\mu\nu\right)\delta_D(z),
   \]
   since by Lemma 5.1.1(2) and (5.16), \( \delta_D(z) \geq \delta_D(x(B)) - |z - x(B)| \geq 3cC_1r(B) \);

3. If \( z \in \alpha \) and \( z \not\in \mathbb{B}(x(B), cC_1r(B)) \), then
   \[
   \ell(\gamma_{y,z}) = \ell(\beta) + \ell(\alpha_{[x(B),z]}) \leq cC_1r(B) + 12\mu\nu\delta_D(z) \leq 24\mu\nu\delta_D(z),
   \]
   since \( 12\mu\nu\delta_D(z) \geq \ell(\alpha_{[x(B),z]}) \geq |z - x(B)| \geq cC_1r(B) \).

In conclusion, (5.13) holds. Hence the proof of Step 5.1.4 is complete.

It is the time for us to complete the proof of (5.1), which is given in the next step. We recall that \( x, y \in D \) and \( |x - y| = t\delta_D(x) \), where \( t \in [0, 1) \).

**Step 5.1.5.** There is a homeomorphism \( \theta' : [0, 1) \to [0, \infty) \) which satisfies (5.1).

To find such a homeomorphism, we divide the proof into two cases. For the first case where \( 0 < t \leq \frac{1}{3\lambda} \), we know from (2.6) that
\[
(5.17)
\]
   \[
   k_D(x, y) \leq 3ct.
   \]

For the remaining case, we have

**Claim 5.1.5.1.** If \( \frac{1}{3\lambda} < t < 1 \), then \( k_D(x, y) \leq \frac{100}{1-t}\mu\nu^2 \).

By Step 5.1.4, there must exist a curve \( \gamma \) in \( D \) joining \( y \) and \( x \) such that for any \( z \in \gamma \),
\[
\ell(\gamma_{[y,z]}) \leq 24\mu\nu\delta_D(z)
\]
and
\[
\ell(\gamma) \leq 4\nu\delta_D(x).
\]
Here, we recall that \( \mu = \frac{16}{3}\lambda(M_1 + 1) \) and \( \nu = 4(M_1 + 1) \), where \( M_1 = \kappa(10\lambda^2) + 1 \).
Let $\gamma : [0, \ell(\gamma)] \to G$ be the arc-length parametrization of $\gamma$ with $\gamma(0) = y$. Define

$$u_0 = \sup \{ u \in [0, \ell(\gamma)] : \gamma_{\cdot|[0,u]} \subseteq \overline{B}(y, \frac{1 - t}{2} \delta_D(x)) \}.$$ 

If $u_0 = \ell(\gamma)$, then $\gamma \subseteq \overline{B}(y, \frac{1 - t}{2} \delta_D(x))$, and thus for $z \in \gamma$,

$$\delta_D(z) \geq \delta_D(y) - |y - z| \geq \delta_D(x) - |x - y| - |y - z| \\
\geq \delta_D(x) - t \delta_D(x) - \frac{1 - t}{2} \delta_D(x) \\
= \frac{1 - t}{2} \delta_D(x),$$

whence

$$k_D(x, y) \leq \int_\gamma \frac{|dz|}{\delta_D(z)} \leq \frac{\ell(\gamma)}{\frac{1 - t}{2} \delta_D(x)} \leq \frac{8\nu}{1 - t}. \tag*{(5.18)}$$

Now, we assume that $u_0 \in (0, \ell(\gamma))$. Under this assumption, obviously, we see that for all $u \in [0, u_0]$,

$$\delta_D(\gamma(u)) \geq \delta_D(y) - |\gamma(u) - y| \geq \delta_D(x) - |x - y| - \frac{1 - t}{2} \delta_D(x) = \frac{1 - t}{2} \delta_D(x).$$

To proceed with this proof, we set

$$u_1 = \inf \{ u \in [0, \ell(\gamma)] : \gamma_{\cdot|[u,\ell(\gamma)]} \subseteq \overline{B}_G(x, t \delta_D(x)) \}.$$ 

Then for any $u \in [u_1, \ell(\gamma)]$, we have

$$\delta_D(\gamma(u)) \geq \delta_D(x) - |x - \gamma(u)| \geq (1 - t) \delta_D(x).$$

If $u_1 \leq u_0$, then

$$k_D(x, y) \leq \int_\gamma \frac{|dz|}{\delta_D(z)} = \int_0^{u_0} \frac{du}{\delta_D(\gamma(u))} + \int_{u_0}^{\ell(\gamma)} \frac{du}{\delta_D(\gamma(u))} \\
\leq \frac{u_0}{\frac{1 - t}{2} \delta_D(x)} + \frac{\ell(\gamma) - u_0}{(1 - t) \delta_D(x)} \leq \frac{2 \ell(\gamma)}{(1 - t) \delta_D(x)} \\
\leq \frac{8\nu}{1 - t}. \tag*{(5.19)}$$

If $u_1 > u_0$, then for any $u \in [u_0, u_1]$,

$$\delta_D(\gamma(u)) \geq \frac{u}{24\mu\nu} \geq \frac{u_0}{24\mu\nu} \geq \frac{\delta_D(x)}{48\mu\nu}(1 - t),$$
whence

\begin{equation}
(5.20) \quad k_D(x, y) \leq \int_\gamma \frac{|dz|}{\delta_D(z)}
\end{equation}

\begin{align*}
= & \int_0^{u_0} \frac{du}{\delta_D(\gamma_s(u))} + \int_{u_0}^{u_1} \frac{du}{\delta_D(\gamma_s(u))} + \int_{u_1}^{\ell(\gamma)} \frac{du}{\delta_D(\gamma_s(u))} \\
\leq & \frac{u_0}{\frac{1-\delta}{2}} + \frac{48\mu\nu(u_1 - u_0)}{(1-t)\delta_D(x)} + \frac{\ell(\gamma) - u_1}{(1-t)\delta_D(x)} \\
\leq & \frac{2u_0 + 48\mu\nu(u_1 - u_0) + \ell(\gamma) - u_1}{(1-t)\delta_D(x)} < \frac{48\mu\nu\ell(\gamma)}{(1-t)\delta_D(x)} \\
\leq & \frac{192}{1-t}\mu\nu^2.
\end{align*}

It follows from (5.18), (5.19) and (5.20) that Claim 5.1.5.1 holds.

Now, we are ready to construct the needed homeomorphism. Let

\begin{equation}
\theta'(t) = \begin{cases} 
\frac{9c^2}{3c - 1}\delta t, & \text{if } 0 < t \leq \frac{1}{3c}, \\
\frac{1}{1-t}, & \text{if } \frac{1}{3c} < t < 1,
\end{cases}
\end{equation}

where \( \delta = 192\mu\nu^2 \). Obviously, \( \theta' = \theta'_{c,\kappa,\lambda} \). Then it follows from (5.17) and Claim 5.1.5.1 that for \( t \in (0, 1) \),

\[ k_D(x, y) \leq \theta'(t). \]

Hence the proof of Step 5.1.5 is complete. \( \square \)

Now, we have enough preparation for the proof of the implication \( P_1 \Rightarrow P_5 \). Since \( f \) is fully \( \varphi \)-semisolid, as we have indicated that, to prove the implication \( P_1 \Rightarrow P_5 \), it suffices to show that \( f \) is \( \theta \)-relative in domain \( D \), where \( \theta = \theta_{c,\kappa,\lambda,\varphi} \) is a homeomorphism from \( [0, 1) \) to \( [0, \infty) \). The existence of such a homeomorphism follows from the following step.

**Step 5.1.6.** There exists a homeomorphism \( \theta : [0, 1) \to [0, \infty) \) such that \( f \) is \( \theta \)-relative in \( D \), where \( \theta = \theta_{c,\kappa,\lambda,\varphi} \).

For \( x \) and \( y \) in \( D \) with \( |x - y| = t\delta_D(x) \), set \( \psi(t) = e^t - 1 \). Then (2.4) and Step 5.1.5 lead to

\[ \frac{|f(x) - f(y)|}{\delta_D(f(x))} \leq \psi(k_D'(f(x), f(y))) \leq \psi \circ \varphi(k_D(x, y)) \leq \psi \circ \varphi \circ \theta'(t). \]

Obviously, \( \theta = \psi \circ \varphi \circ \theta' \) is a homeomorphism from \( [0, 1) \) to \( [0, \infty) \) and \( \theta = \theta_{c,\kappa,\lambda,\varphi} \). Hence the proof of Theorem 3 is complete. \( \square \)

6. **An application**

In this section, as an application of the characterizations of full semisolidity obtained in Theorem 3, we prove that the composition of two locally weakly quasisymmetric mappings is locally \( \eta \)-quasisymmetric, and also it is quasiconformal.
We begin with the following result showing invariance of full semisolidity under the composition operator of mappings.

**Lemma 6.1.** Suppose $X_i$ are $c_i$-quasiconvex metric spaces, and $G_i \subset X_i$ are domains, where $i = 1, 2, 3$. If $f : G_1 \rightarrow G_2$ is fully $\varphi_1$-semisolid and $g : G_2 \rightarrow G_3$ is fully $\varphi_2$-semisolid. Then the composition $g \circ f : G_1 \rightarrow G_3$ is fully $\varphi_2 \circ \varphi_1$-semisolid.

**Proof.** For any domain $D \subset G_1$, by the hypotheses, we see that $f : D \rightarrow f(D) \subset G_2$ is $\varphi_1$-semisolid and $g : f(D) \rightarrow g \circ f(D) \subset G_3$ is $\varphi_2$-semisolid, which shows that

$$k_{f(D)}(f(x), f(y)) \leq \varphi_1(k_D(x, y))$$

and

$$k_{g \circ f(D)}(g \circ f(x), g \circ f(y)) \leq \varphi_2(k_{f(D)}(f(x), f(y)))$$

for all $x, y \in D$. Hence

$$k_{g \circ f(D)}(g \circ f(x), g \circ f(y)) \leq \varphi_2 \circ \varphi_1(k_D(x, y)),$$

and so the proof of Lemma 6.1 is complete since the domain $D$ is arbitrarily taken from $G_1$. □

**The proof of Theorem 4** Under the assumptions, by Theorem 2, $f$ is fully $\varphi_1$-semisolid and $g$ is fully $\varphi_2$-semisolid with some homeomorphisms $\varphi_i$, quantitatively, for $i = 1, 2$. Lemma 6.1 gives that the composition $g \circ f$ is fully $\varphi_2 \circ \varphi_1$-semisolid. It follows from Theorem 8 that there are $0 < q < 1$ and a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$ such that $g \circ f$ is $q$-locally $\eta$-quasisymmetric. Obviously, $g \circ f$ is $q$-locally weakly $H$-quasisymmetric, where $H = \eta(1)$. It follows from Remark 4.1(2) that $g \circ f$ is also $H$-quasiconformal. □

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