Hydroelastic response of a circular sandwich plate interacting with a liquid layer

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Abstract. We considered the formulation and solution of the forced oscillations hydroelasticity problem for a three-layered circular plate contacting with a viscous incompressible fluid layer, the pressure in which varies according to the harmonic law. The plate is the bottom wall of a narrow channel completely filled with a viscous fluid. The axisymmetric coupled hydroelasticity problem consisting of the plate dynamics equation, the viscous fluid layer dynamics equation, and their corresponding boundary conditions was investigated. We obtained the plate dynamics equations taking into account inertia forces in the radial and normal directions in the framework of zigzag kinematic theory. In these equations, the load was expressed by the stresses of the viscous fluid contacting with the three-layered circular plate. The fluid dynamics equations were represented by the Navier-Stokes equations and continuity equation written for the case of creeping fluid flow in a channel. We obtained the forced radial and bending hydroelastic oscillations equations of the circular three-layered plate using the perturbation method. The solution of these equations was represented by a series of eigenfunctions of the corresponding Sturm-Liouville problem. We have also presented the numerical study results of the radial and bending vibrations amplitude dependence on the frequency for the main steady oscillations mode of the plate.

Keywords Hydroelasticity, Vibrations, Three-layered circular plate, Viscous fluid, Narrow channel.

1. Introduction

Beams and plates are the main design elements in the schemes of actual engineering structures. Nowadays, composite structures, and in particular, sandwich plates, are widely used in order to reduce weight and size parameters, as well as to protect against various aggressive factors. The historical review of developing the model for the multilayered structural elements deformation based on the zigzag kinematic theories is given in [1]. The problems of studying the statics and dynamics of three-layered structural elements were considered in the monograph [2]. In reference [3], the problem of bending a three-layered beam under concentrated and distributed load action, placed in a temperature field, was solved. In reference [4], the deformation of an elastic-plastic three-layered circular plate in a temperature field and under the distributed load influence was studied. In reference [5], deformations of an annular sandwich plate under local loads were studied. The axisymmetric vibrations of a three-layered circular plate resting on the Winkler foundation were investigated in [6]. Considering the interaction of elastic structure with dissimilar bodies, we can specify hydroelasticity problems. Historically, one of the first problems of the kind was considered in [7]: it reviewed free oscillations of a circular plate contacting with unlimited volume of ideal liquid. In reference [8], the above-mentioned problem was solved on the basis of the coupled hydroelasticity problem, and in addition, in [9], the fluid viscosity was estimated. The circular plate oscillations immersed in ideal fluid located in a rigid
cylinder and possessing a free surface were studied in [10]. The stability problems of rectangular plates interacting with ideal and viscous fluid were studied in [11, 12]. In reference [13], the hydroelastic vibrations of a disk and a circular plate with viscous liquid in between them, caused by the foundation vibration were considered. The oscillations of a circular plate resting on a Winkler foundation and interacting with a viscous fluid layer were studied in [14]. The hydroelastic vibrations of the sensor elastic element in the form of a circular plate were investigated in [15]. However, there were fewer studies of the hydroelasticity problems for composite plates. For example, in [16-18], free oscillations and stability of the multilayer composite cantilevered beams and plates in air and water were studied both analytically and numerically. In references [19, 20], the hydroelastic vibrations problems of three-layered beams and plates forming the wall of a narrow channel filled with viscous liquid were considered, taking into account the opposite wall vibration and the fluid pressure pulsation. However, the shear stresses of the liquid, as well as the inertia forces of the sandwich plate in the longitudinal direction, were excluded from consideration in above-mention publications.

2. Statement of the Problem

Let us consider a narrow channel formed by a three-layered circular plate and a rigid disk parallel and coaxial to it, as shown in Fig. 1. The radii of the plate and disk are \( R \). We assume that the channel walls are clamped along the contour and consider the axisymmetric problem taking into account the channel axial symmetry. The sandwich circular plate is formed by the upper and lower sheets with a thickness of \( h_1, h_2 \), and the core with a thickness of \( 2c \). We assume the plate core is lightweight and incompressible, and take into account the core work in tangential direction. Let us associate the cylindrical coordinate system center with the middle surface centre of the plate core. The distance between the channel walls in the unperturbed state is \( h_0 \).

We obtained the circular three-layered plate dynamics equations from the known equations of its equilibrium [2] using d'Alembert principle, i.e. taking into account the inertia forces of the upper and lower plate sheets in the radial and normal directions. These equations are:

\[
\begin{align*}
\mathcal{L}_2 \left( a _2 u + a _3 \varphi - a _3 \frac{\partial w}{\partial r} \right) - M_0 \frac{\partial ^2 u}{\partial r^2} &= - q_z , \\
\mathcal{L}_2 \left( a _2 u + a _3 \varphi - a _3 \frac{\partial w}{\partial r} \right) - G_3 c \varphi &= 0 , \\
\mathcal{L}_3 \left( a _3 u + a _4 \varphi - a _4 \frac{\partial w}{\partial r} \right) - M_0 \frac{\partial ^2 w}{\partial r^2} &= - q_z , \\
L_3(g) &= \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial L_3(g)}{\partial r} \right] .
\end{align*}
\]

where \( q_z \) are shear and normal fluid stresses, respectively, acting on the plate surface; \( G_k \) is the shear modulus of the k-th layer; \( K_k \) is the bulk modulus of the k-th layer; \( \rho_k \) is the density of the k-th layer material; \( u \) is the radial three-layered circular plate displacement; \( w \) is the three-layered circular plate deflection; \( \varphi \) is the rotation angle of the deformed normal in the three-layered circular plate core. The accepted notation for \( a_1-a_6 \):

\[
\begin{align*}
L_3 (g) &= \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial L_3 (g)}{\partial r} \right] .
\end{align*}
\]
\[ a_i = h_i K_i + h_i K_i^2 + 2c K_i, \quad a_z = c(h_i K_i - h_i K_i) \] 
\[ a_z = h_i \left( c + \frac{1}{2} h_i \right) K_i - h_i \left( c + \frac{1}{2} h_i \right) K_i^2, \]
\[ a_4 = c^2 \left( h_i K_i + h_i K_i^2 + \frac{2}{3} c K_i \right), \quad a_z = c \left( h_i \left( c + \frac{1}{2} h_i \right) K_i + h_i \left( c + \frac{1}{2} h_i \right) K_i^2 + \frac{2}{3} c^2 K_i \right), \]
\[ a_6 = h_i \left( c^2 + c h_i + \frac{1}{3} h_i^2 \right) K_i + h_i \left( c^2 + c h_i + \frac{1}{3} h_i^2 \right) K_i^2 + \frac{2}{3} c^3 K_i, \quad K_i^2 = K_i + \frac{4}{3} G_i, \quad M_0 = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3. \]

The stresses \( q_{rr}, q_{zr} \) on the plate surface according to \([21]\) are written as
\[ q_{rr} = \rho V \left( \frac{\partial V}{\partial r} + \frac{\partial V}{\partial z} \right) \text{ at } z = w + c + h, \quad q_{zr} = -p + 2 \rho V \frac{\partial V}{\partial z} \text{ at } z = w + c + h. \]

The boundary conditions of Eq. (1) are:
\[ w = u = \phi = 0 \text{ at } r = R, \quad r \frac{\partial w}{\partial r} = 0 \text{ at } r = 0. \]

We assume the creeping fluid motion in a narrow channel, i.e. viscous fluid dynamics equations have the form \([21]\):
\[ \frac{1}{\rho} \frac{\partial p}{\partial r} = V \left( \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} - \frac{V}{r^2} \right), \quad \frac{1}{\rho} \frac{\partial p}{\partial \zeta} = V \left( \frac{\partial^2 V}{\partial \zeta^2} + \frac{1}{r} \frac{\partial V}{\partial \zeta} + \frac{\partial^2 V}{\partial \zeta^2} \right), \quad \frac{\partial V}{\partial r} + \frac{1}{r} V + \frac{\partial V}{\partial \zeta} = 0, \]
where \( V_r, V_z \) are the fluid vector velocity projections on the axes of the cylindrical coordinate system, \( \nu \) is the kinematic viscosity coefficient of the fluid, \( \rho \) is the fluid density.

The boundary conditions of Eq. (4) are non-slip ones:
\[ V_r = 0, \quad V_z = 0 \text{ at } z = h_0 + c + h, \quad V_r = \frac{\partial u}{\partial r}, \quad V_z = \frac{\partial w}{\partial r} \text{ at } z = w + c + h. \]

In addition, we formulate the conditions for pressure at the channel contour and the symmetry axis:
\[ p = p_0 + p_i (\omega \zeta) \text{ at } r = R, \quad r \frac{\partial p}{\partial r} = 0 \text{ at } r = 0. \]

3. Determining Circular Sandwich Plate Response
Let us introduce dimensionless variables and small parameters of the problem
\[ W = \frac{h_0}{R}, \quad u = \frac{w}{h_0}, \quad \phi = \frac{w}{h_0}, \quad V_r = w_c \omega U_r, \quad V_z = \frac{w_c \omega}{h_0} U_z, \]
\[ w = w_c W, \quad u = u_c U, \quad \phi = \omega \Phi, \quad p = p_0 + p_i (\tau) + \frac{\rho \nu w_c \omega}{h_0 \psi \rho} P, \]
where \( W, \psi, \lambda \) are small parameters characterizing the problem.

By substituting variables (7) into Eqs. (1)-(6), and neglecting the small terms \([22]\), we obtain:
– dynamics equations of viscous incompressible fluid layer
\[ \frac{\partial P}{\partial \zeta} = \frac{\partial^2 U_r}{\partial \zeta^2}, \quad \frac{\partial P}{\partial \zeta} = 0, \quad \frac{\partial U_r}{\partial \zeta} + \frac{1}{\zeta} U_r + \frac{\partial U_z}{\partial \zeta} = 0. \]

– dynamics equations of a three-layered circular plate
\[ L_\zeta \left( a_1 u_m U + a_2 \Phi - \frac{a_3 w_c}{R} \frac{\partial W}{\partial \zeta} \right) - M_0 \dot{\omega}^2 u_m \frac{\partial^2 U}{\partial \zeta^2} = -q, \]
\[
L_2 \left( a_1 u_{xx} + a_4 \varphi_x \Phi - \frac{a_1 w_m}{R} \frac{\partial W}{\partial \xi} \right) - G_3 \frac{2c \varphi_x \Phi}{\xi} = 0, \quad L_3 \left( a_1 u_{xx} + a_4 \varphi_x \Phi - \frac{a_1 w_m}{R} \frac{\partial W}{\partial \xi} \right) - M_\omega^2 w_m \frac{\partial^2 W}{\partial \xi^2} = -q_z, \quad L_2(g) = \frac{\partial}{R^\xi \partial \xi^2} \left[ \frac{1}{2 \xi} \frac{\partial}{\partial \xi} \left( \xi \frac{\partial}{\partial \xi} \right) \right], \quad L_3(g) = \frac{\partial}{\partial \xi} \left[ \frac{\partial}{\partial \xi} g \left( \xi \right) \right], \quad q_{\nu} = \frac{\partial v w_m}{h_0 \psi} \frac{\partial U_j}{\partial \xi} \bigg|_{\xi = 0}, \quad q_{\nu} = -p_0 - p_1(\tau) \frac{\partial v w_m}{h_0 \psi^2} P, \quad r = \text{boundary conditions of Eqs. (8), (9)}.
\]

\[
U_j = 0, \quad U_j = 0, \quad U_j = 0 \quad \text{at } \xi = 1, \quad U_j = 0, \quad P = 0 \quad \text{at } \xi = 0, \quad \xi \frac{\partial P}{\partial \xi} = 0 \quad \text{at } \xi = 0, \quad (10)
\]

Solving the fluid dynamics equations (8) with boundary conditions (10), we obtain:

\[
U_j = \frac{\partial P}{\partial \xi} \left( \xi - 1 \right), \quad \xi = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \xi \frac{\partial}{\partial \xi} \left( \frac{3 \varphi^2 - 3 \varphi^2}{12} \right) \right), \quad P = 12 \sum \left[ \frac{1}{\xi} \frac{\partial^2 W}{\partial \xi^2} \right] \frac{\partial U_j}{\partial \xi} \bigg|_{\xi = 0}, \quad \frac{\partial^2 W}{\partial \xi^2} \bigg|_{\xi = 0} = \frac{\xi}{\xi} \frac{\partial^2 W}{\partial \xi^2} \bigg|_{\xi = 0}. \quad (12)
\]

The plate elastic displacements and the angle rotation of the plate core normal are determined from the solution of Eq. (9). Taking into account the boundary conditions (11), elastic displacements can be represented as a series of eigenfunctions of the Sturm-Liouville problem:

\[
w = w_m \sum_{k=1}^{\infty} \left( R_k^0 + Q_k(\tau) \right) \left[ J_0(\beta_k \xi) \right] \left[ I_0(\beta_k \xi) \right], \quad u = -u_m \sum_{k=1}^{\infty} \beta_k \left( Q_k^0 + Q_k(\tau) \right) \left[ \frac{J_0(\beta_k \xi)}{I_0(\beta_k \xi)} + I_0(\beta_k \xi) \right], \quad \varphi = -\varphi_m \sum_{k=1}^{\infty} \beta_k \left( \beta_k \xi \right) = \beta_k \left( \beta_k \xi \right) + T_k(\tau) \left[ \frac{J_0(\beta_k \xi)}{I_0(\beta_k \xi)} + I_0(\beta_k \xi) \right]
\]

Here \( J_0 \) is the zero-order Bessel function of the first kind; \( I_0 \) is a modified zero-order Bessel function; \( \beta_k \) is the root of the transcendental equation \( J_1(\beta_k \xi)/I_0(\beta_k \xi) = J_1(\beta_k \xi)/I_0(\beta_k \xi), k = 1, 2, \ldots \), where \( J_1(\beta_k), I_1(\beta_k) \) are the first-order Bessel functions [2]. The coefficients \( R_k^0, Q_k^0, T_k \) correspond to static plate movements at a constant pressure level \( p_0 \), while harmonic time functions \( R_k, Q_k, T_k \) correspond to plate movements due to pressure pulsation.

Substituting (13) in (12) and (9) and re-performing representation of the remaining constants and functions as the series (13), we obtain the following equations for the main mode of oscillations:

\[
(a_1 u_{xx} (Q_k^0 + Q_k) + a_2 \varphi_m (T_k^0 + T_k) - a_1 w_m \frac{R_k^0 + R_k}{R} \beta_k \xi)^{d_{11}^0} \left[ \frac{J_0(\beta_k \xi)}{I_0(\beta_k \xi)} + I_0(\beta_k \xi) \right] + \]

\[+ R^2 w_m M_\omega^2 \beta_k \xi \left[ J_1(\beta_k \xi) / J_0(\beta_k \xi) + I_0(\beta_k \xi) / I_0(\beta_k \xi) \right] = \frac{R^2 \partial^2 \omega}{h_0 \psi^2} w_m \left[ J_0(\beta_k \xi) / J_0(\beta_k \xi) + I_0(\beta_k \xi) / I_0(\beta_k \xi) \right], \quad \left( a_1 u_{xx} (Q_k^0 + Q_k) + a_2 \varphi_m (T_k^0 + T_k) - a_1 w_m \frac{R_k^0 + R_k}{R} \beta_k \xi^{d_{11}^0} + G_3 c \varphi_m (T_k^0 + T_k) \right) \left[ J_0(\beta_k \xi) / J_0(\beta_k \xi) + I_0(\beta_k \xi) / I_0(\beta_k \xi) \right] = 0, \]

\[= R^2 (p_0 + p_1(\tau)) \left[ J_0(\beta_k \xi) / J_0(\beta_k \xi) - I_0(\beta_k \xi) / I_0(\beta_k \xi) \right] - 12 R^2 \frac{\partial \omega}{h_0 \psi^2} w_m \left[ J_0(\beta_k \xi) / J_0(\beta_k \xi) - I_0(\beta_k \xi) / I_0(\beta_k \xi) \right]. \]
Here accepted notation, \( d_{11}^{(1)} = \left( J_1^2(\beta_1) - 4 J_1(\beta_1) \right) / \beta_1^2 \), and \( d_{11}^{(2)} = \left( J_1^2(\beta_1) - 4 J_1(\beta_1) \right) / \beta_1^2 \).

Using the second equation of the system (14), we find that
\[
T_0 = \frac{a_3 w_0 R_0^3 - a_5 w_0 Q_0 R}{\varphi_0 R (a_4 + G_2 c / (\beta_1^2 d_{11}^{(1)}))},
\]
and considering the steady-state harmonic oscillations, i.e. given that
\[
\frac{\partial^2 Q}{\partial t^2} = -Q_1, \quad \frac{\partial^2 R}{\partial t^2} = -R_1,
\]
we define \( u_0 Q_0, w_0 Q_1, w_0 R_0, w_0 R_1 \), which allows us presenting the radial displacement and deflection of the three-layered circular plate in the form:
\[
u = -\frac{p_0 R_0^3}{b_{21}} A_1(0, \xi) - \frac{p_0 R_0^3}{b_{21}} A_4(\omega, \xi) \sin(\alpha t + \phi_1(\omega)),
\]
\[
w = -\frac{p_0 R_0^3}{b_{21}} A_3(0, \xi) - \frac{p_0 R_0^3}{b_{21}} A_4(\omega, \xi) \sin(\alpha t + \phi_1(\omega)),
\]
\[
\begin{align*}
A_1(\omega, \xi) & = \frac{b_{21}^2 ((b_1^2)^2 + (K_1,\omega)^2) - 2 J_1(\beta_1) J_1(\beta_1, \xi) + I_1(\beta_1, \xi)}{(b_1^2 b_{22} - b_2^2 b_{21})^2 + (b_1^2 b_{22} - b_2^2 b_{21})^2}, \\
A_4(\omega, \xi) & = \frac{(b_1^2 b_{21})^2 + (b_1^2 b_{22} - b_2^2 b_{21})^2 + (b_1^2 b_{22} - b_2^2 b_{21})^2 + (b_1^2 b_{22} - b_2^2 b_{21})^2}{(b_1^2 b_{21})^2 + (b_1^2 b_{22} - b_2^2 b_{21})^2 + (b_1^2 b_{22} - b_2^2 b_{21})^2 + (b_1^2 b_{22} - b_2^2 b_{21})^2}.
\end{align*}
\]
\[
\tan \phi_1(\omega) = \frac{K_1,\omega (b_1 b_{22} - b_2 b_{21})}{b_1 (b_1 b_{22} - b_2 b_{21}) + K_1,\omega (b_1 b_{22} - b_2 b_{21})}, \quad \tan \phi_3(\omega) = \frac{-b_1 K_1,\omega + b_2 K_1,\omega}{b_1 b_{22} - b_2 b_{21}},
\]
where the following symbols are entered
\[
b_{11} = (a_1 - a_2^2 / (a_4 + G_2 c / (\beta_1^2 d_{11}^{(1)}))) - \frac{M_0 a_1^3 R^3}{b_{11}^3},
b_{22} = (a_4 - a_2^2 / (a_4 + G_2 c / (\beta_1^2 d_{11}^{(1)}))),
b_{21} = (a_4 / (a_4 + G_2 c / (\beta_1^2 d_{11}^{(1)}))),
b_{21} = (a_4 / (a_4 + G_2 c / (\beta_1^2 d_{11}^{(1)}))) - \frac{M_0 a_1^3 R^3}{b_{11}^3},
\]
\[
K_{11} = 6 \rho \nu R^2 / h_y \rho \phi_1 R^3, \quad K_{21} = \rho \nu R^2 / h_y \phi_1 R^3.
\]

4. Calculation results
To illustrate the obtained results, we perform a numerical study of \( A_1(\omega, \xi), A_2(\omega, \xi), \) considering the channel with the following parameters: \( R = 0.1 \) m, \( h_y/R = 0.08, \) \( h_1/R = 0.01, \) \( h_2/R = 0.015, \) \( \rho_0 = 10^3 \) kg/m\(^3\), \( \rho_1 = \rho_2 = 2.7 \cdot 10^3 \) kg/m\(^3\), \( \rho_3 = 10^3 \) kg/m\(^3\), \( K_1 = K_2 = 8 \cdot 10^4 \) Pa, \( K_3 = 4.7 \cdot 10^4 \) Pa, \( G_1 = G_2 = 2.67 \cdot 10^5 \) Pa, \( G_3 = 9 \cdot 10^7 \) Pa, \( v = 10^3 \) m/s. In the calculations, we introduce the dimensionless amplitude frequency responses of radial movement and deflection of the sandwich plate as the ratio of \( A_1(\omega, \xi), A_2(\omega, \xi) \) to their values at a static pressure, i.e. \( \alpha_1(\omega) = A_1(\omega, \xi) / A_1(0, \xi), \) \( \alpha_2(\omega) = A_2(\omega, \xi) / A_1(0, \xi). \) These amplitude frequency responses allow us determining the resonant frequencies of the main mode of the radial and bending vibrations, and to estimate the vibrations amplitude at those frequencies. The results of the calculations are shown in Fig.2 and Fig.3.
5. Summary and Conclusion

Thus, we constructed the equations for the radial and bending vibrations of the sandwich plate and amplitude-frequency responses for the main mode of the plate’s oscillations. Computations have shown that two resonant frequencies were observed on the main oscillation mode of the plate, which could be explained by the cross-influence of inertia and stiffness forces in the radial and normal directions. Therefore we conclude that, in contrast to the hydroelasticity problems of homogeneous plates, for three-layered plates, it is important to take into account the inertia forces and the fluid stresses in the radial direction.

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