Insulating charge density wave for a half-filled SU(N) Hubbard model with an attractive on-site interaction in one dimension

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(Dated: October 16, 2018)

We study a one-dimensional SU(N) Hubbard model with an attractive on-site interaction and \( N > 2 \) at half-filling on the bipartite lattice using density-matrix renormalization-group method and a perturbation theory. We find that the ground state of the SU(N) Hubbard model is a charge density wave state with two-fold degeneracy. All the excitations are found to be gapful, resulting in an insulating ground state, on contrary to that in the SU(2) case. Moreover, the charge gap is equal to the Cooperon gap, which behaves as \(-2Nt^2/(N-1)U\) in the strong coupling regime. However, the spin gap \( \Delta_s \) and the quasiparticle gap \( \Delta_1 \) as well open exponentially in the weak coupling region, while in the strong coupling region, they linearly depend on \( U \) such that \( \Delta_s \sim -U(N-1) \) and \( \Delta_1 \sim -U(N-1)/2 \).

I. INTRODUCTION

Correlation effects in electronic systems have been of long-term interest in the condensed matter physics. In recent years, important progress has been made experimentally in ultra-cold atomic systems where strong correlations lead to some novel physical phenomena. In particular, interacting fermionic atoms can be trapped in an optical lattice. More interestingly, the interaction in the ultra-cold atomic systems is tunable through the Feshbach resonance, which allows for a full exploration of various fundamental properties of strongly correlated models. Moreover, the nuclear spin of the atoms can be larger than the electronic spin. One could expect richer physics induced by the degree of freedom of higher spin. Therefore, one would like naturally to generalize the Hubbard model with two spin components to the one with \( N \)-components.

In this paper, we investigate low energy properties of a one-dimensional half-filled SU(N) Hubbard model with an attractive on-site interaction and \( N > 2 \). This is a generalization of our previous work in which we have focused on the SU(2) case. The Hamiltonian of the one-dimensional SU(N) Hubbard model is represented by

\[
\mathcal{H} = -t \sum_{i=1}^{L} \sum_{\sigma} (\hat{c}^\dagger_{i\sigma} \hat{c}_{i+1\sigma} + h.c.) + \frac{U}{2} \sum_{i=1}^{L} \sum_{\sigma \neq \sigma'} \hat{n}_{i\sigma} \hat{n}_{i\sigma'},
\]

where \( t > 0 \) is the hopping matrix set as the energy unit, \( U < 0 \) the coupling constant of the attractive on-site interaction, \( L \) the number of lattice sites, \( \sigma \) and \( \sigma' \) the spin indices, which take the values \((N-1)/2, (N-1)/2 - 1, \ldots, 1 - (N-1)/2, -(N-1)/2, \) respectively. \( \hat{c}^\dagger_{i\sigma} \) and \( \hat{c}_{i\sigma} \) denote the creation and annihilation operators, respectively, for a particle with spin \( \sigma \) at the site \( i \) and \( \hat{n}_{i\sigma} = \hat{c}^\dagger_{i\sigma} \hat{c}_{i\sigma} \) indicates the operator of the particle number. The Hamiltonian (1) has \( U(1) \otimes SU(N) \) symmetry. The generator for the \( U(1) \) symmetry is \( \mathcal{N} = \sum_{i\sigma} \hat{n}_{i\sigma} \) and the \( U(1) \) symmetry implies that the particle number \( \mathcal{N} \) is conserved. The generator for the SU(N) symmetry is \( S^A = \sum_{i\sigma\sigma'} \hat{c}^\dagger_{i\sigma} T^A_{\sigma\sigma'} \hat{c}_{i\sigma'} \), where \( T^A \) are the generators of the SU(N) group in its fundamental representation. These symmetries are useful for simplifying numerical calculations and classifying excitations for the present system.

Although this model is exactly solvable for \( N = 2 \), it seems that physical features obtained for \( N = 2 \) are not immediately applicable to more general cases with \( N > 2 \). Recently, this model with \( N > 2 \) and \( U > 0 \) has been studied by using several analytic as well as numerical approaches. At half-filling, a renormalization-group analysis shows that both the charge and spin coupling constants are renormalized to a large value, resulting in gaps in both sectors. An analytic perturbative renormalization group treatment in the fermionic representation alternatively gives rise to a gapful spectrum for all \( N > 2 \) and \( U > 0 \). Moreover, by employing bosonization method and quantum Monte Carlo simulations, Assaraf et al found for a 1/N filling that the spin excitation is gapless for any positive \( U \), whereas the charge excitation is gapful only for \( U > U_c \) where \( U_c = 0 \). However, Buchta et al obtained gapless spin as well as gapful charge excitations from accurate density-matrix renormalization group(DMRG) calculations with \( N = 3, 4 \) and 5 for any \( U > 0 \).

For the attractive interaction, on the other hand, it is known that the one-dimensional attractive half-filled SU(2) Hubbard model is described by a Luther-Emery liquid model, in which the charge excitation is gapless, whereas the spin excitation is gapful. By the hidden SU(2) transformation, the SU(2) Hubbard model with \( U \) can be mapped to the one with \(-U\). However, for the SU(N) Hubbard model with \( N > 2 \) such a mapping does not exist so that one cannot obtain any insight into the low-energy properties for the attractive case through the mapping from the repulsive case. In this paper, we
will show that the SU(N) Hubbard model at half filling with an attractive interaction belongs to a different universality class from the SU(2) one and all the excitations are gapful. We expect that our findings are not only of fundamental interest, but also useful for experimentalists, since the attractive SU(N) Hubbard model may be possibly realized by experiments in ultra-cold atomic systems.

The rest of this paper is organized as follows. In Sec. II, we analyze low-energy properties of the Hamiltonian \( \mathcal{H} \) in the strong coupling limit \((N-1)|U| \gg t \) first and then in the weak coupling limit \((N-1)|U| \ll t \) by the perturbation treatments. Particularly we discuss the dependence of charge gap, Cooperon gap, spin gap and quasiparticle gap on \( U \) in both regimes. In Sec. III, we present our numerical results which are obtained from the DMRG calculations and compare them with analytic behavior in both weak and strong coupling regime. A summary is finally given in Sec. IV.

II. PERTURBATION CALCULATIONS

In this section, we study the low-energy properties of the Hamiltonian \( \mathcal{H} \) by a perturbation theory. For this purpose, we rewrite it as

\[
\mathcal{H} = \mathcal{H}_t + \mathcal{H}_u, \tag{2}
\]

where \( \mathcal{H}_t = -t \sum_{\sigma \alpha} (\hat{c}_{i \sigma} ^{\dagger} \hat{c}_{i+1 \sigma} + h.c.) \) is the hopping term and \( \mathcal{H}_u = (U/2) \sum_{\sigma \neq \sigma'} \hat{n}_{i \sigma} \hat{n}_{i \sigma'} \) the on-site interaction. We start with the on-site interaction part \( \mathcal{H}_u \). Since the on-site interaction is attractive, \( N \) particles with different \( \sigma \) tend to stay on one site and form a SU(N) singlet. The energy of the SU(N) singlet is \( UN(N-1)/2 \). In the half-filling case where \( N/2 \) particles per site, the ground states of \( \mathcal{H}_u \) are highly degenerate, involving half of the lattice sites occupied by the SU(N) singlets and the other half being empty. In the strong coupling region, i.e. \( |U|/(N-1) \gg t \), \( \mathcal{H}_u \) is taken as the zeroth order Hamiltonian, while \( \mathcal{H}_t \), being the order of \( Nt \), is regarded as a perturbation. Up to the second order, we obtain the first order effective Hamiltonian

\[
\mathcal{H}^{(1)}_{\text{eff}} = P \mathcal{H}_t P = 0,
\]

where \( P \) is a projection operator which restricts the effective Hamiltonian in the subspace spanned by the ground states of \( \mathcal{H}_u \). The second order effective Hamiltonian reads

\[
\mathcal{H}^{(2)}_{\text{eff}} = P \mathcal{H}_t \left( \frac{1}{E_0 - \mathcal{H}_u} (1 - P) \mathcal{H}_t P \right) = \frac{2t^2}{(N-1)U} \sum_i \hat{n}_i P
\]

where \( \hat{n}_i = \sum_{\sigma} \hat{n}_{i \sigma} \) the number operator at the site \( i \). The degeneracy of the ground state of \( \mathcal{H}_u \) is thus lifted by \( \mathcal{H}_t \). At half filling, one can easily obtain the ground state energy correction per site which is given as \( Nt^2/(N-1)U \) by the first term of Eq. (3). Since the second term introduces an effective repulsion interaction between the particles at nearest neighbor (NN) sites, it results in a charge density wave (CDW) state in which the SU(N) singlet and the empty site occur alternatively to exhibit a long range order resulting from translational symmetry breaking. It turns out that the ground state is two-fold degenerate. The configuration of the ground state is schematically shown in Fig. II(a).

![Fig. 1: Schematic illustrations of the configurations, and the empty circle here represents one SU(N) singlet formed by \( N \) particles with different \( \sigma \). (a) For the CDW ground state. Shifting all the circles by one site, one can obtain the other one of two-fold degenerate ground states. (b) For the charge excitation: the SU(N) singlet at site \( i \) shifts to the site \( i+1 \). (c) For the Cooperon excitation: one SU(N) singlet is added to the site \( i+1 \).](image)

It is well-known that both the charge and Cooperon excitations for the attractive SU(2) Hubbard model are gapless. However for the SU(4) case, it has been shown that the charge and Cooperon excitations are gapful and equal to each other at half-filling. In the following, one can see that this conclusion is also valid for other attractive SU(N) Hubbard models with \( N > 2 \). In particular, the charge gap and Cooperon gap can be easily derived from the effective Hamiltonian (3). The charge gap \( \Delta_c \) is defined as the lowest excitation in the SU(N) singlet subspace as follows

\[
\Delta_c = E_1(L, NL/2, 0) - E_0(L, NL/2, 0),
\]

where \( E_0(L, M, S) \) is the ground state energy in the spin-S channel with \( L \) sites and \( M \) particles, and \( E_n(L, M, S) \) the \( n \)-th excitation energy. The gap for Cooperon excitations for the SU(N) Hubbard model is given from the energy difference between states by adding \( N \) particles or \( N \) holes to the system, which is defined as

\[
\Delta_N = \frac{1}{2} \left[ E_0(L, NL/2 + N, 0) + E_0(L, NL/2 - N, 0) \right] - E_0(L, NL/2, 0).
\]

(4)

(5)
The charge gap $\Delta_c$ is given by shifting one of SU(N) singlets in the ground state configuration to its nearest neighbor site as shown in Fig. 4(b). Then one has
\begin{equation}
\Delta_c = -\frac{2Nt^2}{(N-1)U}.
\end{equation}
Similarly, one obtains the SU(N) Cooperon gap
\begin{equation}
\Delta_N = -\frac{2Nt^2}{(N-1)U}.
\end{equation}
which is shown in Fig. 4(c). One can see that $\Delta_c = \Delta_N$ for all $N > 2$, which was previous shown for the $N = 4$ case. In the large-$N$ limit,
\begin{equation}
\Delta_c = \Delta_N = \lim_{N \to \infty} \frac{2Nt^2}{(N-1)U} = -\frac{2t^2}{U}.
\end{equation}
We note that the difference for the low-lying excitations between $N = 2$ and $N > 2$ results from the effective interaction between the singlets at the NN sites as involved in effective Hamiltonian (14). For the $N = 2$ case, one has the effective Hamiltonian
\begin{equation}
\mathcal{H}^{(2)}_{s^{-}} = \frac{2t^2}{U} \sum_{i\sigma} \hat{n}_{i\sigma} P \sum_{j\sigma} \hat{n}_{j\sigma} P - \frac{t^2}{U} \sum_{(ij)\sigma} (\hat{n}_{i\sigma} \hat{n}_{j\sigma} - \hat{c}^{\dagger}_{i\sigma} \hat{c}^{\dagger}_{j\sigma} \hat{c}_{j\sigma} \hat{c}_{i\sigma}) P.
\end{equation}
Compared to the effective Hamiltonian (3), one has additionally a pair hopping term, which involves the same amplitude as the NN repulsion term and eventually destroys the CDW long range order for $N = 2$. On the other hand, for $N > 2$ cases, although a similar hopping term emerges at the $N$-th order perturbation, it has a smaller amplitude than the NN repulsion term so that one can has a stable CDW ground state.

Now we turn to study the spin and quasiparticle excitations. The spin gap is defined in correspondence to the lowest excitation with different spin quantum number from the ground state. The quasiparticle gap is defined as an energy change by adding one particle or hole to the system. Since the ground state is CDW, in order to obtain these two gaps, we resort to the following Hartree-Fock(HF) approximation.
\begin{equation}
\langle \hat{n}_{i\sigma} \hat{n}_{i'\sigma'} \rangle \zeta \sim \sum_{\sigma} \langle \hat{n}_{\sigma} \hat{n}_{\sigma'} \rangle + \langle \hat{n}_{\sigma} \hat{n}_{\sigma'} \rangle - \langle \hat{n}_{\sigma} \rangle \langle \hat{n}_{\sigma'} \rangle, \tag{10}
\end{equation}
where $\langle \hat{n}_{\sigma} \rangle = n_0 + (-1)^{\delta n}$, and $\delta n$ is the order parameter, $\langle \cdots \rangle$ is the average over the ground state. The HF Hamiltonian then reads
\begin{equation}
\mathcal{H}^{HF} = -t \sum_{i\sigma} \langle \hat{c}^{\dagger}_{i\sigma} \hat{c}_{i+1\sigma} + h.c. \rangle + U \sum_{i\sigma \neq i'\sigma} \langle \hat{n}_{i\sigma} \hat{n}_{i'\sigma} \rangle + \langle \hat{n}_{i\sigma} \hat{n}_{i'\sigma} \rangle - \langle \hat{n}_{\sigma} \rangle \langle \hat{n}_{\sigma'} \rangle. \tag{11}
\end{equation}
At half-filling, one has $n_0 = \frac{1}{2}$ and $\sum_{i\sigma} \hat{n}_{i\sigma} = \frac{L}{2}$. Eq. (11) can be further simplified as
\begin{equation}
\mathcal{H}^{HF} = -t \sum_{i\sigma} \langle \hat{c}^{\dagger}_{i\sigma} \hat{c}_{i+1\sigma} + h.c. \rangle + U(N-1)\delta n \sum_{i\sigma} (-1)^i \hat{n}_{i\sigma} + \text{const.} \tag{12}
\end{equation}
Introducing $\hat{a}_{i\sigma} = \hat{c}_{2i\sigma}, \hat{b}_{i\sigma} = \hat{c}_{2i+1\sigma}$ and taking a Fourier transformation $\hat{a}_{i\sigma} = \frac{1}{\sqrt{L}} \sum_k \hat{a}_{k\sigma} e^{ik}, \hat{b}_{i\sigma} = \frac{1}{\sqrt{L}} \sum_k \hat{b}_{k\sigma} e^{ik}$, we obtain
\begin{equation}
\mathcal{H}^{HF} = -t \sum_{k\sigma} ((1 + e^{-ik})\hat{a}^{\dagger}_{k\sigma} \hat{b}_{k\sigma} + h.c.) + U(N-1)\delta n \sum_{k\sigma} (\hat{a}^{\dagger}_{k\sigma} \hat{a}_{k\sigma} - \hat{b}^{\dagger}_{k\sigma} \hat{b}_{k\sigma}) + \text{const.} \tag{13}
\end{equation}
Diagonalizing this HF Hamiltonian, we can obtain two bands for each spin species with the following dispersion
\begin{equation}
\omega_{k\sigma} = \pm \sqrt{\Delta_1^2 + 4t^2 \cos^2 \frac{k}{2}}, \tag{14}
\end{equation}
where $\Delta_1 = -(N-1)U\delta n$ is the quasiparticle gap. Under the HF approximation, one has that $\Delta_c = \Delta_1 = 2\Delta_1$. By solving the self-consistent equation for $\delta n = \langle \hat{a}^{\dagger}_{i\sigma} \hat{a}_{i\sigma} - \hat{b}^{\dagger}_{i\sigma} \hat{b}_{i\sigma} \rangle$ one can determine the order parameter $\delta n$. In the weak coupling limit $t \ll |U|(N-1)$, we can solve it approximately and obtain
\begin{equation}
\Delta_1 \simeq 2\pi t e^{-\frac{2\pi}{(N-1)}}, \tag{15}
\end{equation}
In the strong coupling limit, one can put $\delta n \simeq 0.5$, and then obtain the quasiparticle gap
\begin{equation}
\Delta_1 \simeq -U(N-1)/2, \tag{16}
\end{equation}
and the spin gap
\begin{equation}
\Delta_s \simeq -U(N-1). \tag{17}
\end{equation}
In the HF approximation, since $\Delta_c = 2\Delta_1$, one obtains $\Delta_c \simeq -U(N-1)$ which is inconsistent with the results Eq. (16) obtained from the strong coupling perturbation theory. This is because in the presence of the strong attractive on-site interaction between particles, the single particle picture is not valid for the charge excitations, in which $N$ particles forming a SU(N) singlet shift as a whole to the NN site, as illustrated in Fig. 4(b). However, we would expect that the HF results are still qualitatively correct for weak coupling limit.

### III. NUMERICAL CALCULATIONS

In this section, we present our numerical results for $N = 3, 4, 5$ and 6 cases and compare them with those analytic predictions from the perturbation theories and the
HF approximation. Our numerical results are obtained from a large scale DMRG computation. It is well known that DMRG method is the most powerful numerical tool for accurate exploration on low-energy properties of one dimensional systems at zero temperature. However, it is nontrivial to apply this method to the SU(N) Hubbard model. On one hand, there is a large number of degrees of freedom per site. For instance, the degree of freedom per site is 32 when \( N = 5 \). On the other hand, although a open boundary condition (OBC) in DMRG calculations may lead to more accurate results than a periodic boundary condition (PBC), while the emergence of edge states under the OBC makes the calculation of various excitations practically more difficult.

For these reasons, we have made some additional considerations to the standard DMRG algorithm as follows. When \( N = 3 \), i.e. SU(3) case, one has 8 degrees of freedom per site. In this case, we use the PBC with keeping about 2000 \( \sim 3500 \) optimal states in the DMRG procedure and two sites are added in order to enlarge the chain length at each DMRG step. The maximal truncation error is of the order \( 10^{-5} \). Similarly, the charge gap for \( N = 4 \) is calculated with at most 1600 states kept. The other excitations are calculated with the OBC. At each step we add one lattice site to the chain, breaking the lattice site into two pseudo sites. To obtain accurate results, we preselect some specific chain lengths, and perform sweeping at these preselected lengths, the final results are got by extrapolating the results at the preselected lengths to the thermodynamic limit. To avoid cumbersome edge excitations when the OBC is employed, the preselected lengths are odd instead of even. Correspondingly the definitions of the excitations are changed. For example, the Cooperon gap for \( N = 4 \) is redefined as 
\[
\Delta_N = E_0(L,2L+6,0) - E_0(L,2L+2,0) - 6U,
\]
where the particle-hole symmetry is taken into account and odd \( L \) are used.

### A. Energy correction and degeneracy of the ground states

The analysis based on the strong coupling perturbation theory in the above section shows that the hopping term, lifting the degeneracy of \( H_u \), results in an energy correction per site \( N t^2/(N-1) U \). To verify this prediction, we calculate the energy corrections for \( N = 3, 4 \) and 5 using DMRG method. Both numerical and analytic results are shown in Fig. 2. One can see that the energy correction is negative and decreases monotonically with respect to increasing of \( U \). In the strong coupling region, the results given by the perturbation theory agree well with the DMRG data. For \( N = 3 \), the deviation between analytic and numerical results is within the numerical accuracy for \( U < -2 \). With increasing of \( N \), the deviation becomes smaller and smaller. This implies that the perturbation theory gives rise to better results for larger \( N \) for the strong coupling regime. In the weak coupling region, the analytic results from the effective Hamiltonian \( \mathcal{H}_i \) severely deviates from the DMRG results. This is reasonable since in the weak coupling region, \( \mathcal{H}_i \) is no longer a small perturbation. Nevertheless, when \( N \) increases, the valid region of the strong coupling perturbation theory increases simultaneously.

![Figure 2: Ground state energy correction per site obtained from both the DMRG calculation and the strong coupling perturbation theory \( \mathcal{H}_i^{(2)} \) for \( N=3, N=4 \) and \( N=5 \) are shown as a function of \( U \).](image)

![Figure 3: Two lowest excitation gaps in the SU(N) singlet subspace for \( N=3 \) case with two different values of \( U \).](image)
effective Hamiltonian \( H \). Moreover, for a given chain length \( L \), we find that the energy difference between two states for the Hamiltonian \( H \) decrease rapidly with increasing \( |U| \), since the effective Hamiltonian Eq. \( H \) becomes more accurate for larger \( |U| \). In Fig. 3 and 4, we demonstrate this feature with two different values of \( U \) under OBC for \( N = 3 \) and 4, respectively. Note that the energy difference for the second excitation, which is finite in the large-\( L \) limit, corresponds to edge excitations rather than bulk excitations.

![FIG. 4: Two lowest excitation gaps in the SU(N) singlet subspace for \( N = 4 \) case with two different values of \( U \).](image1)

B. Charge gaps and Cooperon gaps

The prediction that the charge gap \( \Delta_c \) is equal to the Cooperon gap \( \Delta_N \) is confirmed numerically for \( N=4 \). However, the strong coupling perturbation theory gives rise to \( \Delta_c = \Delta_N \) for general \( N \), it is still interesting to examine numerically whether this equation is valid for both even and odd \( N \) and all regimes of the coupling constant \( U \). Fig. 5 demonstrates the charge and Cooperon gaps for \( N = 3 \) as comparison to \( N = 4 \) as a function of \( U \). One can see that both charge and Cooperon gaps for both \( N = 3 \) and \( N = 4 \) are finite for all \( U < 0 \). When \( |U| \) is increased, these gaps increase first in the weak coupling region. After they reach their maxima, they decrease following the asymptotic behavior \( -\frac{2N^2}{(N-1)t} \) resulting from the strong coupling perturbation theory. We would mention that even beyond the valid range of the perturbation theory, our numerical data still show that the charge gaps are equal to the corresponding Cooperon gaps for both odd \( N(=3) \) and even \( N(=4) \). Therefore, one may conclude that \( \Delta_c = \Delta_N \) for all \( N \) and \( U < 0 \).

Moreover we calculate the Cooperon gaps for \( N = 5 \) and 6 in order to explore the large-\( N \) behavior of the charge as well as Cooperon gaps. The results of the Cooperon gap for \( N = 5 \) and 6 are shown in Fig. 6. Comparing Fig. 5 and 6 one can see that the larger \( N \) is, the closer Cooperon gaps to that derived from the effective Hamiltonian \( H \). For \( N = 3 \) as shown in Fig. 5, the deviation of the asymptotic behavior from the DMRG data is visible even at \( U = -10 \), while for \( N = 4 \) case, the deviation is within the numerical accuracy up to \( U \approx -5 \). For \( N = 5 \) and 6, the good agreement can be seen up to \( U \approx -3 \) and \( U \approx -2 \), respectively. Furthermore, as \( N \) increases, the height of the peak of the Cooperon gaps increases and the position of the peak shifts toward \( U = 0 \). These features can be easily understood since the criteria for the strong coupling perturbation \( t \ll |U|(N-1) \) depends on \( |U|(N-1) \) rather than \( U \) only. The DMRG results here also verify that the charge and Cooperon gap
TABLE I: Decomposition of some irreducible representations of SU(3) to SO(3).

| SU(3)  | SO(3) | ν |
|--------|-------|---|
| [1^1]  | 0     | 1 |
| [2^11^1] | 2 ⊕ 1 | 8 |
| [3^1]  | 3 ⊕ 1 | 10|
| [1^2]  | 1     | 3 |
| [2^2]  | 2 ⊕ 0 | 6 |

behaves asymptotically as $-\frac{2\Delta_U^2}{U}$ in the large-$N$ limit.

C. Quasiparticle gap and spin gap

In the following, we discuss the quasiparticle and spin gaps. In order to improve numerical efficiency, we need to carry out DMRG calculations in a subspace which can be obtained by decomposing the irreducible representations of SU(N) into the irreducible representation of SO(3)\textsuperscript{18,19}. Numerically, one can determine the irreducible representations for short chains by varying $z$-component. Table II shows the decomposition of some irreducible representations of SU(3) to SO(3).

TABLE II: Decomposition of some irreducible representations of SU(4) to SO(3).

| SU(4)  | SO(3) | ν |
|--------|-------|---|
| [1^4]  | 0     | 1 |
| [2^2]  | 4 ⊕ 2 ⊕ 2 ⊕ 0 | 20|
| [2^11^2] | 3 ⊕ 2 ⊕ 1 | 15|
| [4^1]  | 0 ⊕ 2 ⊕ 3 ⊕ 4 ⊕ 6 | 35|
| [3^11^1] | 1 ⊕ 1 ⊕ 2 ⊕ 3 ⊕ 3 ⊕ 4 ⊕ 5 | 45|
| [1^2]  | 1 ⊕ 2 | 4 |
| [2^11^1] | 1 ⊕ 2 ⊕ 2 ⊕ 2 | 20|
| [3^11^2] | 1 ⊕ 1 ⊕ 2 ⊕ 3 ⊕ 3 ⊕ 4 ⊕ 5 | 36|

For $N = 3$ that the spin excitation belongs to the representation $[2^11^1]$ and it is 8-fold degenerate, while the quasiparticle excitation belongs to the representation $[1^1]$ and it is 3-fold degenerate. Table II shows the decomposition of some irreducible representations of SU(4) to SO(3). In this case, the spin excitation belongs to the representation $[2^11^2]$ and its degeneracy is 15-fold, whereas the quasiparticle excitation belongs to the representation $[1^1]$ and its degeneracy 4-fold. Fig. 7 shows the quasiparticle and spin gaps for $N = 3$ and 4 as well. In weak coupling region, one can see that both the quasiparticle and spin gaps open exponentially, as predicted by the HF approximation, although we cannot determine precise critical behavior for the opening of the gaps from our numerical data due to limited numerical accuracy. In the strong coupling region, both the quasiparticle gap and spin gap depend linearly on $U$ as seen in the inset. This is also consistent with the HF results. Moreover, one also see that the ratio $\Delta_s/|U|$ approaches 1 for $N = 3$ asymptotically, while it approaches 1.5 for $N = 4$. These asymptotic behaviors qualitatively coincide with Eq. (16) and (17) and the relation $\Delta_s = 2\Delta_1$ derived from the HF approximation. For finite $U$, however, the relation is just approximately valid since there remains residual interaction between quasiparticles.

IV. CONCLUSIONS

In this paper, we have investigated low-energy properties of the SU(N)(N$\geq$2) Hubbard model with attractive on-site interaction at half-filling. By using the perturbation theory and DMRG method, we show that the ground state is a CDW state with two-fold degeneracy. The CDW long range order reflects the broken translational symmetry. On one hand, the strong coupling perturbation theory predicts that both the charge excitations and Cooperon excitations are gapful and equal to each other with the asymptotic behavior $-2Nt^2/(N - 1)U$. Combining this with numerical results for $N = 3, 4, 5$ and 6, we conclude that the charge gap is equal to the Cooperon gap for all $U < 0$ and $N > 2$. In the large $N$ limit, both the charge and Cooperon gaps behaves as $-2t^2/U$. Considering the CDW ground state, we obtain qualitatively correct behavior for the spin and quasiparticle gaps from HF approximation, which are confirmed by our numerical data. In the weak coupling region, they open exponentially, whereas in the strong coupling region, they depend linearly on $U$ with the spin gap $\Delta_s \sim -(N-1)U$ and quasiparticle gap $\Delta_1 \sim -(N-1)U/2$.

Our findings indicate that at half filling, the
SU(N)(N>2) Hubbard model belongs to a different universality class from the SU(2) case. This can be easily understood by observing the difference between the effective Hamiltonians for the SU(N)(N>2) and the SU(2) cases. In particular, the effective Hamiltonian which for the SU(2) case involves an effective hopping term has the same order as the effective repulsion interaction for particles between nearest-neighbor sites. The hopping term destroys the CDW order. For N > 2, the repulsive interaction dominates over the effective hopping term.

J. Zhao would thank H. Tsunetsugu for helpful discussion. X.Q. Wang is supported in part by NFC2005CB32170X, and NSFC10425417 & C10674142.

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