Exact form factors for the scaling \(Z_N\)-Ising and the affine \(A_{N-1}\)-Toda quantum field theories

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Abstract

Previous results on form factors for the scaling Ising and the sinh-Gordon models are extended to general \(Z_N\)-Ising and affine \(A_{N-1}\)-Toda quantum field theories. In particular result for order, disorder parameters and para-fermi fields \(\sigma_Q(x), \mu_Q(x)\) and \(\psi_Q(x)\) are presented for the \(Z_N\)-model. For the \(A_{N-1}\)-Toda model form factors for exponentials of the Toda fields are proposed. The quantum field equation of motion is proved and the mass and wave function renormalization are calculated exactly.

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In the framework of the bootstrap program the quantum field theories are defined by the S-matrices. However, we may motivate the models as follows: The \(Z(N)\)-Ising quantum field theory in 1+1 dimensions is considered as the scaling limit of a classical statistical lattice model in 2-dimensions given by the partition function

\[
Z = \sum_{\{\sigma\}} \exp \left( -\frac{1}{kT} \sum_{\langle ij \rangle} E(\sigma_i, \sigma_j) \right) ; \quad \sigma_i \in \{1, \omega, \ldots, \omega^{N-1}\}, \quad \omega = e^{2\pi i/N}
\]

as a generalization of the Ising model. It was conjectured by Köberle and Swieca # that there exists a \(Z(N)\)-invariant interaction \(E(\sigma_i, \sigma_j)\) such that the resulting quantum field theory is integrable. This model has also been

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discussed as a deformation \[2\] of a conformal \(Z_N\) para-fermi field theory \[3\]. The classical \(A_{N-1}\)-Toda model is defined by the Lagrangian

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \tilde{\varphi} \right)^2 + \frac{\alpha}{\beta} \sum_{j=0}^{N-1} \exp \left( \beta \tilde{\alpha}_j \tilde{\varphi} \right)
\]

where \(\tilde{\varphi} = (\varphi^1, \ldots, \varphi^{N-1})\) are real fields, \(\tilde{\alpha}_j (|\tilde{\alpha}_j| = \sqrt{2})\) are the simple positive \(A_{N-1}\)-roots and \(\tilde{\alpha}_0 = -\sum_{j=1}^{N-1} \tilde{\alpha}_j\). The field equations are

\[
\square \tilde{\alpha}_j \tilde{\varphi} + \frac{\alpha}{\beta} \left( 2 e^{\beta \tilde{\alpha}_j \tilde{\varphi}} - e^{\beta \tilde{\alpha}_{j+1} \tilde{\varphi}} - e^{\beta \tilde{\alpha}_{j-1} \tilde{\varphi}} \right) = 0.
\]

The \(Z_N\)-Ising model in the scaling limit and the affine \(A_{N-1}\)-Toda model possess the same particle content. There are \(N-1\) types of particles: \(a = 1, \ldots, N-1\) of charge \(a\), mass \(m_a = M \sin \pi a / N\) and \(\bar{a} = N - a\) is the antiparticle of \(a\). In particular for \(N = 2\) the scaling \(Z_2\)-Ising model is the well investigated model \[5, 6\] which is equivalent to a massive free Dirac field theory. The affine \(A_1\)-Toda model is the sinh-Gordon model which is equivalent to the lowest breather sector of the sine-Gordon model for imaginary couplings. The \(n\)-particle S-matrices factorize in terms of two-particle ones since the models are integrable. The two-particle S-matrix for the \(Z_N\)-Ising model has been proposed by Köberle and Swieca \[1\]. The scattering of two particles of type 1 is

\[
S^Z(\theta) = \frac{\sinh \left( \frac{\theta + 2\pi i}{N} \right)}{\sinh \left( \frac{\theta - 2\pi i}{N} \right)}.
\]

The two-particle S-matrix for the \(A_{N-1}\)-Toda model has been proposed by Arin-shtein, Fateev and Zamolodchikov \[7\] (see also \[8\]). The scattering of two particles of type 1 is

\[
S^T(\theta) = S^Z(\theta) \frac{\sinh \left( \frac{\theta - \pi i B}{2} \right) \sinh \left( \frac{\theta - \pi i (2 - B)}{2} \right)}{\sinh \left( \frac{\theta + \pi i B}{2} \right) \sinh \left( \frac{\theta + \pi i (2 - B)}{2} \right)}, \quad B = \frac{2\beta^2}{4\pi + \beta^2}.
\]

Both S-matrices are consistent with the picture that the bound state of \(N-1\) particles of type 1 is the anti-particle of 1. This will be essential also for the construction of form factors below.

The form factor bootstrap program has been applied in \[6\] to the \(Z_2\)-model. Form factors for the \(Z_3\)-model were investigated by one of the present authors in \[9\]. There the form factors of the order parameter \(\sigma_1\) were proposed up to four particles. Kirilov and Smirnov \[10\] proposed all form factors of the \(Z_3\)-model in terms of determinants. For general \(N\) form factors for chargeless states (\(n\) particles of type 1 and \(n\) particles of type \(N-1\)) were calculated in \[11\]. Low particle number form factors of \(A_{N-1}\)-Toda models\(^1\) where investigated by Destri and de Vega \[12\], Oota \[13\] and Lukyanov \[14\]. In the present letter we

\(^1\)For other Toda models see \[26, 27\] and for sinh-Gordon also \[28, 29\].
present integral representations for all matrix elements of field operators for the $Z_N$-Ising and the $A_{N-1}$-Toda models.

For the $Z_N$-model we consider the fields $\psi_{QQ}(x)$, $Q \bar{Q} = 0, \ldots, N-1$ with charge $Q$, spin $Q\bar{Q}/N$ and statistics factor $2$ (with respect to the particle $a = 1$) $e^{2\pi i \bar{Q}/N}$. There are in particular the order parameters $\sigma_Q(x) = \psi_{QQ}(x)$, the disorder parameters $\mu_{\bar{Q}}(x) = \psi_{\bar{Q}Q}(x)$ and the para-fermi fields $\psi_Q(x) = \psi_{QQ}(x)$ (for $Q = 1, \ldots, N-1$). They satisfy the space like commutation rules:

\[
\begin{align*}
\sigma_Q(x)\sigma_Q(y) &= \sigma_Q(y)\sigma_Q(x) \\
\mu_{\bar{Q}}(x)\mu_{\bar{Q}}(y) &= \mu_{\bar{Q}}(y)\mu_{\bar{Q}}(x) \\
\sigma_Q(x)\mu_{\bar{Q}}(y) &= \mu_{\bar{Q}}(y)\sigma_Q(x) e^{\theta(x^1-y^1)2\pi i Q\bar{Q}/N} \\
\psi_Q(x)\psi_Q(y) &= \psi_Q(y)\psi_Q(x) e^{\theta(x^1-y^1)2\pi i Q^2/N}.
\end{align*}
\]

For the $A_{N-1}$-Toda models we present integral representations for all matrix elements of normal ordered exponentials of the fields

\[
\exp \left( \gamma_1 \varphi^1 + \cdots + \gamma_{N-1} \varphi^{N-1} \right)
\]

where the $\varphi^a$ are the fundamental Toda fields.

The generalized form factors $O_n(\theta_1, \ldots, \theta_n)$ are defined by the vacuum $n$-particle matrix elements

\[
\langle 0 | O(x) | p_1, \ldots, p_n \rangle^{in}_{a_1 \ldots a_n} = e^{-i\pi(p_1 + \cdots + p_n)} O_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_n)
\]

where the $a_i$ denote the type (charge) and the $\theta_i$ are the rapidities of the particles ($p_i = m(cosh \theta_i, sinh \theta_i)$). This definition is meant for $\theta_1 > \cdots > \theta_n$, in the other sectors of the variables the functions $O_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_n)$ are given by analytic continuation with respect to the $\theta_i$. General matrix elements are obtained from $O_{a_2 \bar{Q}}$ by crossing which means in particular the analytic continuation $\theta_i \rightarrow \theta_i \pm i\pi$. The form factor equations which have to be solved are:

\begin{enumerate}
\item The form factor function $O_a(\theta)$ is meromorphic with respect to all variables $\theta_1, \ldots, \theta_n$.
\item It satisfies Watson’s equations

\[
O_{a_1 a_2 \ldots a_n}(\theta_1, \theta_2, \ldots, \theta_n) = O_{a_2 a_3 \ldots a_n}(\ldots, \theta_j, \theta_1, \ldots) S_{a_1 a_2}(\theta_{ij}).
\]
\item The crossing relation means for the connected part (see e.g. [19]) of the matrix element

\[
a_i \langle p_1 | O(0) | p_2, \ldots, p_n \rangle^{\text{in,conn.}}_{a_2 \ldots a_n} = \sigma_{O1} O_{a_1 a_2 \ldots a_n}(\theta_1 + i\pi, \theta_2, \ldots, \theta_n)
\]

where $\sigma_{O1}$ is the statistics factor of the operator $O$ with respect to the particle 1.
\end{enumerate}

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2 Another model with nontrivial statistics is the Federbusch model (see e.g. [15]).
3 These formulae have been proposed in [17] as a generalization of formulae in [16] and they has been proven in [18] using LSZ assumptions.
(iii) The function $O_n(\theta)$ has poles determined by one-particle states in each sub-channel. In particular, if 1 is the antiparticle of 2, it has the so-called annihilation pole at $\theta_{12} = i\pi$ which implies the recursion formula

$$\text{Res}_{\theta_{12}=i\pi} O_n(\theta_1, \ldots, \theta_n) = 2i O_{n-2}(\theta_3, \ldots, \theta_n) (1 - \sigma \sigma_1 S(\theta_{2n}) \ldots S(\theta_{23})) .$$

(iv) Bound state form factors yield another recursion formula

$$\text{Res}_{\theta_{12}=iu} O_{ab \ldots}(\theta) = \sqrt{2} O_{c \ldots}(\theta_c, \theta') \Gamma^c_{ab}$$

where $\Gamma^c_{ab}$ is the bound state intertwiner (see e.g. [19]) defined by

$$i \text{Res}_{\theta_{12}=iu} S_{ab}(\theta) = \Gamma^c_{c \ldots} \Gamma^c_{ab}$$

if $iu$ is the position of the bound state pole.

(v) Since we are dealing with relativistic quantum field theories Lorentz covariance in the form

$$O_n(\theta_1, \ldots, \theta_n) = e^{s \mu} O_n(\theta_1 + \mu, \ldots, \theta_n + \mu)$$

holds if the local operator transforms as $O \rightarrow e^{s \mu} O$ where $s$ is the "spin" of $O$.

We investigate generalized form factors of an operator $O(x)$ and $n$ particles of type 1 and for simplicity we write $O_n(\theta) = O_{1 \ldots 1}(\theta)$. Note that all further matrix elements with different particle states of the field operator $O(x)$ are obtained by the crossing formula (ii) and the bound state formula (iv). The form factors $O_n(\theta)$ are of the form [16]

$$O_n(\theta) = K^O_n(\theta) \prod_{1 \leq i < j \leq n} F(\theta_{ij}) , \quad (\theta_{ij} = \theta_i - \theta_j) \quad (5)$$

where $F(\theta)$ is the 'minimal' form factor function. It is the solution of Watsons equation [20] and the crossing relation for $n = 2$

$$F(\theta) = F(-\theta) S(\theta)$$

$$F(i \pi - \theta) = F(i \pi + \theta) \quad (6)$$

with no poles and zeros in the physical strip $0 < \text{Im} \theta \leq \pi$ (and a simple zero at $\theta = 0$). We obtain the solutions (see [16] for the procedure to solve (6))

$$F^Z(\theta) = c_Z \exp \int_0^\infty \frac{dt}{t} 2 \sinh t \left( - \frac{1}{N} \cosh \frac{1}{N} \left( 1 - \cosh t \left( 1 - \frac{\theta}{i \pi} \right) \right) \right) \quad (7)$$

$$F^T(\theta) = \exp \int_0^\infty \frac{dt}{t} 4 \sinh t \left( - \frac{1}{N} \sinh \frac{1}{2N} \cosh \frac{1}{N} \left( 1 - \cosh t \left( 1 - \frac{\theta}{i \pi} \right) \right) \right) \quad (8)$$
for the $Z_N$ and $A_{N-1}$ S-matrices given by (2) and (3), respectively. The normalization constant $c_Z$ will be fixed below and $F_T$ is normalized by $F_T(\infty) = 1$. In eq. (5) $K_n^O(\theta)$ is a rational function of $e^{\theta_i}$ and has the 'physical poles'\(^4\) in $0 < \text{Im} \theta_{ij} \leq \pi$ corresponding to the form factor properties (iii) and (iv). The form factor equations (i) and (ii) hold because of the equations (6). We propose $K_n^O(\theta)$ as linear combinations of the integrals

\[
I_{nm}(\theta \mid p_n^O) = \frac{1}{m_1! \cdots m_{N-1}!} \left( \prod_{k=1}^{N-1} \prod_{j=1}^{m_k} \frac{dz_{ki}}{R} \right) \times \prod_{k=1}^{N-1} \left( \prod_{j=1}^{m_k} \phi(z_{kj} - \theta_i) \prod_{1 \leq i < j \leq m_k} \tau(z_{ki} - z_{kj}) \right) \times \prod_{1 \leq k < l \leq N-1} \prod_{i=1}^{m_k} \prod_{j=1}^{m_l} \kappa(z_{ki} - z_{lj}) p_{nm}^O(\theta, z)
\]

with

\[
\kappa(z) = \frac{1}{\phi(-z)}, \quad \tau(z) = \frac{1}{\phi(z)\phi(-z)}, \quad R = 2\pi i \text{Re} \phi(z).
\]

The integration contour $C_\theta$ encloses the points $z = \theta_i$. The $p$-functions $p_{nm}^O(\theta, z)$ are symmetric with respect to the $\theta_i$ and the $z_{kj}$ for fixed $j$. They depend on the operator $O(x)$ whereas the other functions are determined by the S-matrix only. For both models the bound state of $N-1$ particles of type 1 is the anti-particle of 1. Together with the form factor recursion relations (iii) and (iv) this property implies the following equation for the function $\phi(z)$

\[
\prod_{k=1}^{N-1} \phi(\theta + 2\pi i k/N) \prod_{k=0}^{N-1} F(\theta + 2\pi i k/N) = 1
\]

where $F(\theta)$ is the 'minimal' form factor function, here given by eqs. (7,8) for the $Z_N$- and the $A_{N-1}$-model, respectively. One easily verifies that the functions

\[
\phi^Z(z) = \frac{1}{\sinh \frac{1}{2} z \sinh \frac{1}{2} (z - 2\pi i/N)}
\]

\[
\phi^T(z) = \phi^Z(z) \sinh \frac{1}{2} (z - i\pi B/N) \sinh \frac{1}{2} (z - i\pi (2 - B)/N)
\]

solve the equation (10) for the $Z_N$- and the $A_{N-1}$-model, respectively, and moreover they satisfy the 'Jost-function property' \[
\frac{\phi(\theta)}{\phi(-\theta)} = S(\theta)
\]

Equation (10) also determines the normalization constant $c_Z$. The following lemma will be used to construct solutions of the form factor equations (i) – (v).

\(^4\)For bound state form factors there are also higher order 'physical poles' (see e.g. [21, 8, 26, 27].
Lemma 1 The functions $I_{nm}(\vartheta, p_{nm}^O)$ satisfy the recursion relation

$$\text{Res}_{\theta_{N-1N} = iu} \cdots \text{Res}_{\theta_{12} = iu} I_{nm}(\vartheta, p_{nm}^O) = c \left( \prod_{j=N+1}^{n} \prod_{i=1}^{N} F_{11}^{\min}(\theta_{ij}) \right)^{-1}$$

$$\times I_{n-Nm-1}(\theta', p_{n-Nm-1}^O) \left( 1 - \sigma_{\mathcal{O}} \prod_{i=N+1}^{n} S(\theta_{Ni}) \right)$$

(11)

if eq. (10) holds, the $p$-functions are analytic and satisfy

$$p_{nm}^O(\theta, z') = p_{n-Nm-1}^O(\theta', z'')$$

(12)

$$p_{nm}^O(\theta, z'') = \sigma_{\mathcal{O}} p_{nm}^O(\theta, z')$$

(13)

where $c$ and $c'$ are a normalization constants, $\sigma_{\mathcal{O}}$ is the statistics factor of the operator $\mathcal{O}$ with respect to the particle of type 1 and

$$z' = z'' = z; \text{ except } z'_{i1} = \theta_i, \ z''_{i1} = \theta_N, \ z'_{i+1} = \theta_i$$

$$z'' = (z_{12}, \ldots, z_{1m_1}; \ldots; z_{N-12}, \ldots, z_{N-1m_{N-1}})$$

$$p_{n-Nm-1}^O(\theta', z'') = \sigma_{\mathcal{O}} p_{nm}^O(\theta, z')$$

(11)

The proof of this lemma will be published elsewhere [22]. We propose the following solutions of (i) – (v) and identify the operators by means of the quantum numbers as charge, spin and statistics factor. For the Toda model we also consider the asymptotic behavior and field equations.

The scaling $Z_N$-Ising model: The scaling $Z_2$-Ising model is well known [5, 6], it is equivalent to a free fermi field theory. As a generalization we propose for the general $Z_N$-model the form factors for $n$ particles of type 1 (up to normalizations) as

$$\mathcal{O}_n(\vartheta) = I_{nm}^Z(\vartheta) \prod_{1 \leq i < j \leq n} F^Z(\theta_{ij})$$

where $I_{nm}^Z(\vartheta)$ is defined by eq. (9) in terms of $\phi^Z$. The p-functions and $\mathfrak{m}$ are given by the following correspondences to operators. For $n = Nm + Q$ particles of type 1 and $Q, \tilde{Q} = 0, \ldots, N - 1$ the p-functions

$$p_{nm}^{Q\tilde{Q}}(\vartheta, z) = \exp \left( \frac{\tilde{Q}}{N} \sum_{i=1}^{n} \theta_i - \frac{\tilde{Q}Q}{N} \sum_{k=1}^{m} \sum_{j=1}^{k} z_{kj} \right)$$

(14)

(with $m_k = m + 1$ for $1 \leq k < Q$, $m_k = m$ for $Q \leq k$) belong to operators

$$\psi_{Q, \tilde{Q}}(x) \text{ with } \begin{cases} \text{charge} & Q \\ \text{spin mod 1} & \tilde{Q}Q/N \\ \text{statistics factor} & e^{2\pi i \tilde{Q}/N} \end{cases}$$
This follows from (v) and (13). Also (12) holds which means that the form factor equations (i) – (v) are satisfied. In particular we have for
\[
\hat{Q} = 0 \quad \text{the order parameters} \quad \sigma_Q(x)
\]
\[
Q = 0 \quad \text{the disorder parameter} \quad \mu_Q(x)
\]
\[
\hat{Q} = \tilde{Q} \quad \text{the para-fermi fields} \quad \psi_Q(x)
\]
with the commutation rules (4). One can show [22] that for \(N = 2\) the formulae (9) and (14) reproduce the well known results for the scaling Ising model as mentioned above. Note also that the properties of the fields for the \(Z_N\) Ising model are consistent with the results for the conformal \(Z_N\) para-fermi field theory [4]. For the higher conserved currents \(J^\pm_L(x)\) see equations (18) and (19) below.

**Examples:** Up to normalization constants we calculate for general \(N\) for the order parameters
\[
\langle 0 | \sigma_1(0) | p \rangle = 1
\]
\[
\langle 0 | \sigma_2(0) | p_1, p_2 \rangle^{in} = \frac{F^Z(\theta_{12})}{\sinh \frac{1}{2}(\theta_{12} - 2\pi i/N) \sinh \frac{1}{2}(\theta_{12} + 2\pi i/N)}
\]
and for \(N = 3\)
\[
\langle 0 | \sigma_1(0) | p_1, p_2, p_3, p_4 \rangle^{in} = \left( \sum e^{\theta_1} \sum e^{-\theta_i} - 1 \right)
\]
\[
\times \prod_{1 \leq i < j \leq 4} \frac{F^Z(\theta_{ij})}{\sinh \frac{1}{2}(\theta_{ij} - 2\pi i/3) \sinh \frac{1}{2}(\theta_{ij} + 2\pi i/3)}
\]
which agrees with the results of [9, 10]. For disorder parameters we have
\[
\langle 0 | \mu_\tilde{Q}(0) | 0 \rangle = 1
\]
and for \(N = 3\) we obtain
\[
\langle 0 | \mu_1,2(0) | p_1, p_2, p_3 \rangle^{in} = \left( e^{\pm \theta_1} + e^{\pm \theta_2} + e^{\pm \theta_3} \right) e^{\pm \frac{1}{2} \sum_{i=1}^3 \theta_i}
\]
\[
\times \prod_{1 \leq i < j \leq 3} \frac{F^Z(\theta_{ij})}{\sinh \frac{1}{2}(\theta_{ij} - 2\pi i/3) \sinh \frac{1}{2}(\theta_{ij} + 2\pi i/3)}
\]
For the para-fermi fields we have for example
\[
\langle 0 | \psi_1(0) | p \rangle = e^{\hat{Q}_\psi}
\]
\[
\langle 0 | \psi_2(0) | p_1, p_2 \rangle^{in} = \frac{\left( e^{-\theta_1} + e^{-\theta_2} \right) e^{\frac{Z}{2}(\theta_{12} + \theta_2)} e^{\frac{1}{2} \sum_{i=1}^2 \theta_i}}{\sinh \frac{1}{2}(\theta_{12} - 2\pi i/N) \sinh \frac{1}{2}(\theta_{12} + 2\pi i/N)}
\]
and for \(N = 3\)
\[
\langle 0 | \psi_1(0) | p_1, p_2, p_3, p_4 \rangle^{in} = e^{\hat{Q}_\psi \sum_{i=1}^3 \theta_i} \sum_{i<j} e^{-\theta_i - \theta_j}
\]
\[
\times \prod_{1 \leq i < j \leq 4} \frac{F^Z(\theta_{ij})}{\sinh \frac{1}{2}(\theta_{ij} - 2\pi i/3) \sinh \frac{1}{2}(\theta_{ij} + 2\pi i/3)}
\]
The affine $A_{N-1}$-Toda model: This model possesses only bosonic fields. Therefore we consider constant $p$-functions and generalize results of our investigations \[23\] on the sinh-Gordon model which is the $A_1$-Toda model. Again we consider first form factors for $n$ particles of type 1. We write the $K$-function in \[4\] as a linear combination of the integrals $I_{nm}^T(\theta, 1)$ given by eq. \[9\] in terms of $\phi^T$

$$K_n^C(\theta) = N_n \sum_{m_1=0}^n \cdots \sum_{m_{N-1}=0}^n \prod_{k=1}^{N-1} q_k^{m_k} I_{nm}^T(\theta, 1)$$

(15)

for $N-1$ parameters $q_k$. In particular $K_1(\theta) = N_1 \left(1 + \sum_{k=1}^{N-1} q_k\right)$. We propose that this $K$-function yields the $n$-particle (charge 1) form factors of the general exponential of the fields $O(x) = :\exp \vec{\gamma} \vec{\varphi}(x):$.

This is motivated as follows: In \[23\] (see also \[16\]) we argued that the form factors of exponentials of bose fields satisfy the momentum space ‘clustering’ property for $\text{Re} \theta_1 \rightarrow \infty$

$$[e^{\gamma \varphi}]_n(\theta_1, \theta_2, \ldots, \theta_n) = [e^{\gamma \varphi}]_1(\theta_1) [e^{\gamma \varphi}]_{n-1}(\theta_2, \ldots, \theta_n) + O(e^{-\theta_1}).$$

The form factors given by \[5\] and \[15\] show just this asymptotic behavior because of

$$\lim_{\theta \rightarrow \infty} F^T(\theta) = 1$$

$$\lim_{\theta_1 \rightarrow \infty} I_{nm}^T(\theta) = I_{n-1,m}^T(\theta') + \sum_{k=1}^{N-1} I_{n-1,m,k}^T(\theta')$$

where $\theta' = (\theta_2, \ldots, \theta_n)$ and $m_k = m_1, \ldots, m_{k-1}, \ldots, m_{N-1}$. The normalization constants are obtained from this equation and \[11\] as $N_n = N_1^n$ and

$$N_1 = \left(\frac{2i}{r}\right)^{1/2} \left(\frac{1}{F^T(iu)} \prod_{k=2}^{N-1} \frac{\phi^T(kiu)}{\phi^T(kiu)} \prod_{k=1}^{N-1} q_k\right)^{1/N}$$

(16)

where $u = \frac{2\pi}{N}$, $r = \text{Res} \phi^T(z)$ and the intertwiner $\Gamma^{k+1}_{k1} = i \left(\frac{r}{z \phi^T(-kiu)}\right)^{1/2}$ have been used.

The exponentials of the special linear combination of fields $\hat{\gamma} \vec{\varphi}(x) = \gamma \vec{\alpha}(x)$ are obtained for the choice

$$q_k = \omega^{-k} \omega^{N-\gamma}(\delta_{kj} - \delta_{kj+1})$$

(17)

(For $j = 0$ and $N - 1$ there are extra factors $\omega^{\pm \gamma}$). As in \[23\] we obtain the fields $\vec{\alpha}(x)$ by expansion in terms of $\gamma$. The quantum version of the field equation \[11\] is satisfied for all matrix elements if we take $\hat{\gamma} = \frac{B}{2\sin \frac{\pi}{N}}$ and relate the renormalized and the bare mass by

$$m^2 = 4 \alpha \sin^2 \frac{\pi}{N} \frac{2N \sin \frac{\pi}{N} B \sin \frac{\pi}{N}(2 - B)}{\pi B \sin \frac{\pi}{N}}$$
which agrees with the results of [14, 25]. The proof is analogous to the one in [23]. The proposal [17] is in addition motivated by the fact that the vacuum 1-particle (of charge $b$) matrix element for the field with charge $a$ turns out to be proportional to $\delta_{ab}$.

\[
\langle 0 | \varphi^a(0) | p \rangle_b = \frac{-1}{\sqrt{N} 2 \sin \frac{a\pi}{N}} \sum_{j=0}^{N-1} \omega^{ja} \langle 0 | \bar{a}_j \varphi(0) | p \rangle_b = \delta_{ab} \sqrt{Z^a}.
\]

The form factors of the bound states $| p \rangle_b$ are obtained from (15) with (17) by applying iteratively (iv) as

\[
\langle 0 | :e^{i \gamma \bar{a}_j \varphi(0)} : | p \rangle_b = \frac{\beta}{\pi B} \sqrt{N Z^b} \chi_b(\gamma)
\]

\[
\chi_b(\gamma \bar{a}_j) = \prod_{k=1}^{N-1} q_k^{-b/N} \sum_{0 \leq k_1 < \cdots < k_b < N} q_{k_1} \cdots q_{k_b}
\]

\[-4 \omega^{-b} \sin \left( \frac{\pi}{N} \frac{B \gamma}{2 \beta} \right) \sin \left( \frac{\pi}{N} \frac{1 - B \gamma}{2 \beta} \right).
\]

The function $\chi_b$ may be written as a character $\chi_b(\gamma \bar{a}_j) = \text{tr}_{\chi_b} \omega(\bar{a} \gamma \bar{a}_j)$ [14]. By expansion with respect to $\gamma$ we obtain

\[
\langle 0 | \bar{a}_j \varphi(0) | p \rangle_b = -\sqrt{Z^b} \omega^{-b} 2 \sin \frac{\pi b}{N}.
\]

The wave function renormalization constants $Z^a$ are calculated from (vi) and (16) as

\[
Z^a = \frac{\pi B(2 - B)}{4 \sin \frac{\pi B}{2}} \exp \int_0^\infty dt \frac{2 \sinh \frac{t B}{2 N} \sinh \frac{t B - a}{2 N} \sinh \frac{t B + a}{2 N}}{\sinh^2 t \frac{B}{N} \sinh \frac{a}{N}} (\sinh^2 t \frac{a}{N} + \sinh^2 t \frac{a}{N})
\]

which satisfies the charge conjugation symmetry $a \leftrightarrow \bar{a} = N - a$ and agree with the results of [12, 14].

The higher conserved currents $J^\pm_L(x)$ which are characteristic for integrable models are obtained by (5) and (9) and the K- and p-function

\[
K_n^{J^\pm_L}(\theta) = \sum_{m_1=0}^{n} \cdots \sum_{m_{N-1}=0}^{n} \prod_{k=1}^{N-1} \omega^{km_b} I_{nm_b}^T(\theta, p_n^c)\tag{18}
\]

\[
p_n^{J^\pm_L}(\theta, z) = \sum_{i=1}^{n} e^{\pm \theta_i} \sum_{k=1}^{m_b} e^{Lz_k j} \tag{19}
\]

5This corresponds to the complex representation of the roots $\alpha^a_j = \frac{1}{\sqrt{N}} 2 \sin \frac{a\pi}{N} \omega^{-ja}$ and fields with $(\varphi^a)^\dagger = \varphi^{N-a}$. 

9
The higher charges $Q_L = \int dx J^0_L(x)$ satisfy the eigenvalue equation

$$\left( Q_L - \sum_{i=1}^{n} e^{L \theta_i} \right) |p_1, \ldots, p_n \rangle^m = 0.$$  

Since the currents are $Z_N$-chargeless the number $n$ of particles of charge 1 is $0 \mod N$. Obviously we get the energy momentum tensor from $J^\pm_{\pm 1}(x)$. The higher conserved currents $J^\pm_L(x)$ for the $Z_N$-Ising model are obtained by same $p$-function with the additional factor $\prod_{k=1}^{N-1} \delta_{m_k m}$ for $n = Nm$. In a more detailed version of this letter [22], we will present the proofs and further results.

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