The electromagnetic aspect for Yang-Mills fields

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Abstract

Let $\mathcal{A}$ be the space of irreducible connections (vector potentials) over the principal bundle $M \times SU(n)$ on a compact three-dimensional manifold $M$. The Yang-Mills field $\mathcal{F}$ is defined as a subspace of the Whitney’s direct sum $\mathbb{T} = T\mathcal{A} \times \mathcal{A} \times T^*\mathcal{A}$ of the tangent and cotangent bundles of $\mathcal{A}$. We shall prove the Maxwell equations

\begin{align}
  d^*A B + \dot{E} &= 0, \\  d_A E - \dot{B} &= 0, \\  d_A B &= 0, \\  d^*A E &= 0,
\end{align}

(-1.1)

\begin{align}
  \frac{d}{dt} A B &= 0, \\  \frac{d}{dt} A E &= 0, 
\end{align}

(-1.2)

on $\mathcal{F}$. Where the point of $\mathbb{T}$ is enoted by $(A, E, B)$ with $E \in T\mathcal{A} \simeq \Omega^1(M, su(n))$ and $B \in T^*\mathcal{A} \simeq \Omega^2(M, su(n))$. The first two equations are the Hamilton equations of motion derived from a symplectic structure on $\mathbb{T}$, and the second equations which are the defining equations of $\mathcal{F}$ come from the action of the group of gauge transformations $G$. 

2010 Mathematics Subject Classification. Primary 70S15; Secondary 53D20, 70S05.

Key Words and Phrases. Symplectic structures, Yang-Mills fields, Maxwell’s equations.
on \( \mathcal{A} \). The symplectic structure on \( \mathbb{T} \) is given by the 2-form:

\[
\Omega_{(A,E,B)} \left( \begin{array}{c} a_1 \\ e_1 \\ \beta_1 \\ \\ a_2 \\ e_2 \\ \beta_2 \\ \end{array} \right) = (e_2 \wedge a_1, B) - (e_1 \wedge a_2, B) + (e_2, d_A^* \beta_1) - (e_1, d_A^* \beta_2),
\]

for \((A, E, B) \in \mathbb{T} \) and \( a_i, e_i, \beta_i \in T_A \mathcal{A} \) and \( \beta_i \in T_A^* \mathcal{A} \). The corresponding Poisson bracket on \( \mathcal{F} \) is

\[
\{ \Phi, \Psi \}_{(E,B)} \mathbb{T} = \left( \frac{\delta \Phi}{\delta B}, d_A^* \frac{\delta \Psi}{\delta E} \right)_1 - \left( \frac{\delta \Psi}{\delta B}, d_A^* \frac{\delta \Phi}{\delta E} \right)_1.
\]

This is a parallel formula of Marsden-Weinstein in case of the electric-magnetic field. We shall investigate the Clebsch parametrization of the Yang-Mills field \((\mathcal{F}, \Omega)\). We show that the action of \( \mathcal{G} \) on \((\mathcal{F}, \Omega)\) is Hamiltonian with the moment map \( \mathbb{J}(E, B) = [d_A^* B, E] \). This gives a conserved quantity \( \int_M [d_A^* B, E] \) which is due to the non-commutativity of the gauge group.

MSC: 70S15, 53D42, 70S05.

Subj. Class: Global analysis, Quantum field theory.

Keywords Symplectic structures, Yang-Mills fields, Maxwell equation.

0 Introduction

We shall investigate the electric-magnetic paradigm on the Yang-Mills field \( \mathcal{F} \) over a compact three-dimensional manifold \( M \). We shall give the Maxwell equations:

\[
d_A^* B + \dot{E} = 0, \quad d_A E - \dot{B} = 0, \quad (0.1)
\]

\[
d_A B = 0, \quad \partial_A^* E = 0, \quad (0.2)
\]

at \((A, E, B) \in \mathcal{F} \). The first equations \((0.1)\) are the Hamilton equations of motion derived from a symplectic structure on \( \mathcal{F} \), and the second equations \((0.2)\) that represent the conservation of electric and magnetic charges come from the action of the group of gauge transformations.
Let $\mathcal{A}$ be the space of irreducible connections (vector potentials) on $M \times SU(n)$ and let $T\mathcal{A} \simeq \Omega^1(M, su(n))$ be the tangent space and $T^*\mathcal{A} \simeq \Omega^2(M, su(n))$ be the cotangent space over $\mathcal{A}$. Our Yang-Mills field $\mathcal{F}$ is realized as a subspace of the Whitney’s direct sum

$$\mathbb{T} = T\mathcal{A} \times \mathcal{A} T^*\mathcal{A} \rightarrow \mathcal{A}.$$ 

A point of $\mathbb{T}$ is denoted by $(A, E, B)$ with $E \in T_A\mathcal{A}$ and $B \in T^*_A\mathcal{A}$. Then the subspace $\mathcal{F}$ of $\mathbb{T}$ is defined by the equations (0.2).

$\mathbb{T}$ becomes a symplectic manifold endowed with the following symplectic form:

$$\Omega_{(A,E,B)} \left( \begin{array}{c} a_1 \\ e_1 \\ \beta_1 \\ a_2 \\ e_2 \\ \beta_2 \end{array} \right) = (e_2 \wedge a_1 , B)_2 - (e_1 \wedge a_2 , B)_2 + (e_2 , d_A^*\beta_1)_1 - (e_1 , d_A^*\beta_2)_1,$$

for $\begin{pmatrix} a_i \\ e_i \\ \beta_i \end{pmatrix} \in T_{(A,E,B)}\mathbb{T}$, $i = 1, 2$, where the bracket $( , )_k$ is the inner product on the Sobolev space of differential $k$-forms; $\Omega^k_{s}(M, su(n))$. $\mathcal{F}$ is a symplectic subspace. The corresponding Poisson bracket on $\mathbb{T}$, so on $\mathcal{F}$, becomes

$$\{ \Phi, \Psi \}^\mathbb{T}_{(E,B)} = \left( \frac{\delta \Phi}{\delta B} , d_A^*\frac{\delta \Psi}{\delta E} \right)_1 - \left( \frac{\delta \Psi}{\delta B} , d_A^*\frac{\delta \Phi}{\delta E} \right)_1. \quad (0.3)$$

This is the Yang-Mills counterpart to the Poisson bracket on the electromagnetic field $\mathcal{F}_{Max}$ discussed by Marsden-Weinstein, [2,12]:

$$\{ \Phi, \Psi \}^\mathbb{T}_{(E,B)} = \left( \frac{\delta \Phi}{\delta B} , curl \frac{\delta \Psi}{\delta E} \right)_1 - \left( \frac{\delta \Psi}{\delta B} , curl \frac{\delta \Phi}{\delta E} \right)_1. \quad (0.4)$$

As is well known the Maxwell’s equation is given by

$$curl B + \dot{E} = 0, \quad curl E - \dot{B} = 0, \quad (0.5)$$

$$\text{div} B = 0, \quad \text{div} E = 0. \quad (0.6)$$

The electric-magnetic field $\mathcal{F}_{Max}$ is the subspace characterized by the second line equations (0.6).
We shall give a detailed explanation of our investigation. Let \( M \) be a compact \( m \)-dimensional manifold. Let \( \mathcal{A} \) be the space of irreducible connections (vector potentials) over the trivial principal bundle \( M \times G \) and let \( T\mathcal{A} \) be the tangent space and \( T^*\mathcal{A} \) be the cotangent space over \( \mathcal{A} \). \( T\mathcal{A} \) is a vector space isomorphic to \( \Omega^1_s(M, \text{Lie} G) \); the Sobolev space of differential 1-forms on \( M \), and \( T^*\mathcal{A} \) is a vector space isomorphic to \( \Omega^{m-1}_s(M, \text{Lie} G) \).

The dual coupling is given by

\[
\int_M tr a \wedge \beta, \quad a \in T\mathcal{A}, \beta \in T^*\mathcal{A}.
\]

In the following we shall abbreviate to write \( R = T\mathcal{A} \) and \( S = T^*\mathcal{A} \). But we retain the freedom to use the original notations. A point of \( S \) is denoted by \((A, \lambda)\) with \( A \in \mathcal{A} \) and \( \lambda \in T^*\mathcal{A} \). A tangent vector to \( S \) at \((A, \lambda)\) is \( (a, \alpha) \) with \( a \in T\mathcal{A} \) and \( \alpha \in T^*\mathcal{A} \). There exists a canonical 1-form \( \theta \) on \( S \) characterized by \( \varphi^*\theta = \varphi \) for any section \( \varphi \) of \( T^*\mathcal{A} \rightarrow \mathcal{A} \). The exterior derivative (on \( \mathcal{A} \)) of \( \theta \) gives a canonical symplectic form \( \omega = d\theta \) on \( S \). Then the Hamiltonian vector vector field of a function \( \Phi = \Phi(A, \lambda) \) is given by

\[
X_\Phi = \left( \begin{array}{c} \frac{\delta \Phi}{\delta \lambda} \\ -\frac{\delta \Phi}{\delta A} \end{array} \right),
\]

Where \( \frac{\delta}{\delta A} \) indicates the partial derivative (in the sense of Frechet-Gateau) to the direction of \( \mathcal{A} \), and \( \frac{\delta}{\delta \lambda} \) is the partial derivative along the fiber \( T^*_A \mathcal{A} \).

For example, if \( \text{dim } M = 3 \) and

\[
H(A, B) = \frac{1}{2} \int_M tr F_A \wedge \ast F_A + \frac{1}{2} \int_M tr B \wedge \ast B
\]

for \( A \in \mathcal{A} \), \( B \in T^*_A \mathcal{A} \), then the Hamiltonian vector field of \( H \) becomes

\[
X_H = \left( \begin{array}{c} \ast B \\ -d_A \ast F_A \end{array} \right),
\]

and the Hamilton’s equation of motion is

\[
\dot{A} = \ast B, \quad \dot{B} = -d_A \ast F_A, \quad \text{for } A \in \mathcal{A}, B \in T\mathcal{A}.
\]

(0.7)

The group of (pointed) gauge transformations \( \mathcal{G} = \mathcal{G}(M) = \Omega^0_s(M, \text{Ad} G) \) acts on \( \mathcal{A} \) (from the right) by \( g \cdot A = g^{-1}Ag + g^{-1}dg \). \( \mathcal{G} \) acts on \( T\mathcal{A} \) by

4
the adjoint representation; \( a \rightarrow \text{Ad}_{g^{-1}} a = g^{-1}ag \), and on \( T^*_A \mathcal{A} \) by its dual \( \alpha \rightarrow \alpha \). The dual space of \( \text{Lie} \ \mathcal{G} \) is given by \( (\text{Lie} \ \mathcal{G})^* = \Omega^m(M, \text{Lie} \mathcal{G}) \) with the dual pairing:

\[
\langle \mu, \xi \rangle = \int_M \text{tr} (\mu \xi), \quad \forall \xi \in \mathcal{G}, \mu \in \Omega^m(M, \text{Lie} \mathcal{G}).
\] (0.8)

We see that the action of the group of gauge transformations \( \mathcal{G} \) on the symplectic space \( (S = T^* \mathcal{A}, \omega) \) is an hamiltonian action with the moment map \( J : S \rightarrow (\text{Lie} \ \mathcal{G})^* = \Omega^m(M, \text{Lie} \mathcal{G}) \) given by

\[
J^*(A, \lambda) = -d_A \lambda.
\] (0.9)

From Marsden-Weinstein reduction theorem the reduced space \( (J^*)^{-1}(\rho)/\mathcal{G} \) for a \( \rho \in (\text{Lie} \ \mathcal{G})^* \) becomes a symplectic manifold endowed with the induced symplectic form \( \omega \), and coincides with the space \( \{(A, \lambda) \in S; -d_A \lambda = \rho \} \).

A parallel argument is valid on the tangent space \( R = T \mathcal{A} \) as we shall show in the following, but this is not a canonical one. The point of \( R \) is denoted by \((A, p)\) with \( A \in \mathcal{A} \) and \( p \in T_A \mathcal{A} \). The tangent space at \((A, p)\) is \( T_{(A, p)} R = T_A \mathcal{A} \oplus T_A \mathcal{A} \), so any tangent vector \( a \in T_{(A, p)} R \) is of the form \( a = \begin{pmatrix} a \\ x \end{pmatrix} \) with \( a, x \in T_A \mathcal{A} \). The symplectic structure on \( R \) is defined by the formula

\[
\sigma_{(A,p)} \left( \begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix} \right) = (b, x)_1 - (a, y)_1 \quad (0.10)
\]

for all \( \begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix} \in T_{(A, p)} R \).

If \( \dim M = 3 \), the hamiltonian function

\[
H(A, p) = \frac{1}{2} (F_A, F_A)_2 + \frac{1}{2} (p, p)_1.
\] (0.11)

gives the Hamilton’s equation of motion

\[
\dot{A} = -p, \quad \dot{p} = d_A^* F_A.
\] (0.12)

Actually we find that the symplectic manifolds \( (S, \omega) \) and \( (R, \sigma) \) are isomorphic via the Hodge operator

\[
*: T^*_A \mathcal{A} \simeq \Omega^1(M, su(n)) \leftrightarrow T^*_A \mathcal{A} \simeq \Omega^{m-1}(M, su(n)).
\] (0.13)
When \( \dim M = 3 \) the symplectic isomorphism given by Hodge * changes the Hamilton equations of motion (0.7) and (0.12) each other:

\[
\begin{align*}
\dot{A} &= *B \\
\dot{B} &= -*d^*_AF_A \\
p = B \\
\dot{p} &= -d^*_AF_A.
\end{align*}
\] (0.14)

One of our purpose in this paper is to write down the equations of motion in the form that do not contain the potential variable \( A \), but contain only the field variables \( E \) and \( B \). So both (0.7) and (0.12) are insufficient for us.

The action of \( \mathcal{G} \) on the symplectic space \( (R = TA, \sigma) \) is an hamiltonian action with the moment map given by

\[
J(A, p) = d^*_Ap. \quad (0.15)
\]

This time the image of the moment map is \( (\text{Lie} \mathcal{G})^* \simeq \text{Lie} \mathcal{G} = \Omega^0(M, \text{Lie} G) \) by virtue of the inner product \( (,)_0 \). The symplectic reduction \( (J^{-1}(0)/\mathcal{G}, \sigma) \) becomes a symplectic manifold and coincides with the subbundle

\[
R^0 = \{(A, p) \in R; \ d^*_Ap = 0 \}. \quad (0.16)
\]

Now we suppose that \( M \) is a compact manifold with \( \dim M = 3 \). We shall introduce the space where the electric magnetic future of Yang-Mills theory is relevant. We consider the direct sum

\[
\begin{align*}
\mathbb{T} &= R \times_A S \\
&\quad \longrightarrow (S, \omega) \\
\pi_* \downarrow &\quad \quad \quad \pi \downarrow \\
(R, \sigma) &\quad \longrightarrow A.
\end{align*}
\] (0.17)

A point of \( \mathbb{T} \) will be denoted by \( (A, E, B) \) with \( E \in R \) and \( B \in S \) that are over \( A \in \mathcal{A} \). The tangent space of \( \mathbb{T} \) at the point \( (E, B) \in \mathbb{T} \) (over \( A \in \mathcal{A} \)) is \( T_{(A,E,B)}\mathbb{T} = T_{A}\mathcal{A} \oplus T_{A}^*\mathcal{A} \). So a vector in \( T_{(A,E,B)}\mathbb{T} \) is denoted by \( a = \begin{pmatrix} a \\ e \\ \beta \end{pmatrix} \),

with \( a, e \in T_{A}\mathcal{A} \simeq \Omega^1(M, \text{Lie} G) \) and \( \beta \in T_{A}^*\mathcal{A} \simeq \Omega^2(M, \text{Lie} G) \). The inner product on the fiber \( T_{(E,B)}\mathbb{T} \) over \( A \in \mathcal{A} \) is given by

\[
\left( \begin{pmatrix} e_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ \beta_2 \end{pmatrix} \right)_T = (e_2, d^*_A\beta_1)_1 + (e_1, d^*_A\beta_2)_1. \quad (0.18)
\]
The partial derivative of a function $\Phi = \Phi(E, B)$ over $T$ to the direction $e \in T_{AA}$ is defined as the vector $\frac{\delta \Phi}{\delta E} \in T_{AA}$. Respectively that to the direction $\beta \in T_{A}^*$ is defined as the vector $\frac{\delta \Phi}{\delta B} \in T_{AA}$. They satisfy the defining equation:

\[
(\tilde{d}\Phi)_{(E, B)} \left( \begin{array}{c}
e \\
\beta \end{array} \right) = \left( \begin{array}{c}
\frac{\delta \Phi}{\delta B} \\
\frac{\delta \Phi}{\delta E} \end{array} \right) \left( \begin{array}{c}
e \\
\beta \end{array} \right) \right)_{T} \tag{0.19}
\]

Now we endow the space $T$ with a symplectic structure given by the following 2-form $\Omega$:

\[
\Omega_{(A, E, B)} \left( \begin{array}{c}
a_1 \\
e_1 \\
\beta_1 \end{array} , \begin{array}{c}
a_2 \\
e_2 \\
\beta_2 \end{array} \right) = (e_2 \wedge a_1, B) - (e_1 \wedge a_2, B)_{2} + (e_2, d^*_{A} \beta_1)_{1} - (e_1, d^*_{A} \beta_2)_{1},
\]

for $\left( \begin{array}{c}
e_i \\
\beta_i \end{array} \right) \in T_{(A, E, B)} T$, $i = 1, 2$, where the bracket $(, )_k$ is the inner product on the Sobolev space of differential $k$-forms; $\Omega^k(M, su(n))$. $\Omega$ is a non-degenerate skew-symmetric 2-form and we have a symplectic structure on $(T, \Omega)$. For $\Phi, \Psi \in C^\infty(T)$ the Poisson bracket is defined by the formula:

\[
\{ \Phi, \Psi \}^T_{(E, B)} = \Omega_{(E, B)} (X_\Phi, X_\Psi). \tag{0.20}
\]

We have the following representation of Poisson bracket:

\[
\{ \Phi, \Psi \}^T_{(E, B)} = \left( \frac{\delta \Phi}{\delta B}, d^*_{A} \frac{\delta \Psi}{\delta E} \right)_1 - \left( \frac{\delta \Psi}{\delta B}, d^*_{A} \frac{\delta \Phi}{\delta E} \right)_1. \tag{0.21}
\]

If we take the Hamiltonian function on $T$ written in the vortex representation;

\[
H(E, B) = \frac{1}{2} \left( \begin{array}{c}
E \\
B \end{array} \right), \left( \begin{array}{c}
d^*_{A} B \\
d_{A} B \end{array} \right) \right)_T = \frac{1}{2} \{ (d_{A} E, d_{A} E)_1 + (d^*_{A} B, d^*_{A} B)_1 \}, \tag{0.22}
\]

We obtain the following equation of motion on the strength field $T$:

\[
\dot{E} = \frac{\delta H}{\delta B} = d^*_{A} B, \quad \dot{B} = \frac{\delta H}{\delta E} = d_{A} E. \tag{0.23}
\]

Which is nothing but the Maxwell equation (0.1) over $T$: 7
The Yang-Mills field is the symplectic subspace of $\mathbb{T}$ defined by

$$\mathcal{F} = \{(E, B) \in \mathbb{T} :\ d_AB = 0,\ d_A^*E = 0 \text{ with } \pi_*(E, B) = A\} \quad (0.24)$$

It is a $G$-invariant subspace. We shall give a symplectomorphism from the reduced space $(\mathbb{R}^0, \sigma)$, \((0.15)\), to $(\mathcal{F}, \Omega)$, that is, $(\mathbb{R}^0, \sigma)$ is a symplectic variable (Clebsch parametrization) of the Yang-Mills field $\mathcal{F}$.

We find that the action of $G$ on $\mathcal{F}$ is Hamiltonian with the moment map $J(E, B) = [d_A^*B, E]$. We have an invariant

$$\int_M [d_A^*B, E] \quad (0.26)$$

which reflects the non-commutativity of the gauge group $G = SU(n)$.

1 Calculation on the space of connections \([5, 8, 9]\)

Let $M$ be a compact, connected and oriented $m$-dimensional riemannian manifold possibly with boundary $\partial M$. Let $P \xrightarrow{\pi} M$ be a principal $G$-bundle, $G = SU(N)$, $N \geq 2$.

We write $\mathcal{A} = \mathcal{A}(M)$ the space of irreducible $L^2_{s-1}$ connections over $P$, which differ from a smooth connection by a $L^2_{s-1}$ section of $T_M \otimes \text{Lie} G$, hence the tangent space of $\mathcal{A}$ at $A \in \mathcal{A}$ is

$$T_A \mathcal{A} = \Omega^1_{s-1}(M, \text{Lie} G). \quad (1.1)$$

The cotangent space of $\mathcal{A}$ at $A$ is

$$T_A^* \mathcal{A} = \Omega^{m-1}_{s-1}(M, \text{Lie} G), \quad (1.2)$$

where the pairing $\langle a, \alpha \rangle_A$ of $\alpha \in T_A^* \mathcal{A}$ and $a \in T_A \mathcal{A}$ is given by the symmetric bilinear form $(X, Y) \rightarrow tr(XY)$ of $\text{Lie} G$ and the Sobolev norm $(\cdot, \cdot)_{s-1}$ on the Hilbert space $L^2_{s-1}(M)$:

$$\langle \phi \otimes X, \psi \otimes Y \rangle = (\phi, \psi)_{s-1} tr(XY),$$
for $\psi \in \Omega^{n-1}(M)$, $\phi \in \Omega^1(M)$, and $X, Y \in \text{Lie} G$. We shall write it by $\langle a, \alpha \rangle_A = \int_M \text{tr}(a \wedge \alpha)$, or simply by $\int_M \text{tr}(a \alpha)$.

A vector field $a$ on $A$ is a section of the tangent bundle; $a(A) \in T_A A$, and a 1-form $\phi$ on $A$ is a section of the cotangent bundle; $\phi(A) \in T^*_A A$.

For a smooth map $F = F(A)$ on $A$ valued in a vector space $V$ the derivation $\partial_A F$ is defined by the functional variation of $A \in A$:

$$\partial_A F : T_A A \rightarrow V,$$

$$(\partial_A F) a = \lim_{t \to 0} \frac{1}{t} (F(A + ta) - F(A)), \quad \text{for } a \in T_A A. \quad (1.3)$$

For example,

$$(\partial_A A) a = a,$$

since the derivation of an affine function is defined by its linear part. The curvature of $A \in A$ is given by

$$F_A = dA + \frac{1}{2} [A \wedge A] \in \Omega^2_s(M, \text{Lie} G).$$

So it holds that

$$F_{A+a} = F_A + d_A a + a \wedge a,$$

and we have

$$(\partial_A F_A)a = d_A a.$$

The derivation of a vector field $v$ on $A$ and that of a 1-form $\varphi$ are defined similarly:

$$(\partial_A v)a \in T_A A, \quad (\partial_A \varphi)a \in T^*_A A, \quad \forall a \in T_A A.$$

It follows that the derivation of a function $F = F(A)$ by a vector field $v$ is given by

$$(vF)_A = (\partial_A F)(v_A).$$

We have the following formulas, [4, 8].

$$[v, w]_A = (\partial_A v)w_A - (\partial_A w)v_A, \quad (1.5)$$

$$(v\langle \varphi, u \rangle)_A = \langle \varphi, (\partial_A u)v_A \rangle + \langle (\partial_A \varphi) v_A, u_A \rangle. \quad (1.6)$$

Let $\tilde{d}$ be the exterior derivative on $A(M)$. For a function $F$ on $A(M)$,

$$(\tilde{d}F)_A a = (\partial_A F) a.$$
For a 1-form $\Phi$ on $\mathcal{A}(M)$,

$$(\tilde{d}\Phi)_A(a, b) = (\partial_A < \Phi, b >)a - (\partial_A < \Phi, a >)b - < \Phi, [a, b] >$$

$$= < (\partial_A\Phi)a, b > - < (\partial_A\Phi)b, a > .$$  \hspace{1cm} (1.7)$$

This follows from (1.5) and (1.6). Likewise, if $\varphi$ is a 2-form on $\mathcal{A}(M)$ then it holds that

$$(\tilde{d}\varphi)_A(a, b, c) = (\partial_A\varphi(b, c))a + (\partial_A\varphi(c, a))b + (\partial_A\varphi(a, b))c .$$  \hspace{1cm} (1.8)$$

We write the group of $L^2$-gauge transformations by $\mathcal{G}'(M)$:

$$\mathcal{G}'(M) = \Omega^0_s(M, Ad P).$$  \hspace{1cm} (1.9)$$

Where $Ad P = P \times_G G$ is the adjoint bundle associated to the principal bundle $P$. In this paper we shall mainly deal with the trivial principal bundle. In this case $\mathcal{G}'(M) = \Omega^0_s(M, G)$. $\mathcal{G}'(M)$ acts on $\mathcal{A}(M)$ by

$$g \cdot A = g^{-1}dg + g^{-1}Ag = A + g^{-1}dAg .$$  \hspace{1cm} (1.10)$$

By Sobolev lemma one sees that $\mathcal{G}'(M)$ is a Banach Lie Group and its action is a smooth map of Banach manifolds.

In the following we choose a fixed point $p_0 \in M$ and deal with the group of gauge transformations that are identity at $p_0$:

$$\mathcal{G} = \mathcal{G}(M) = \{ g \in \mathcal{G}'(M); g(p_0) = 1 \} .$$

$\mathcal{G}$ act freely on $\mathcal{A}$. Let $\mathcal{C}(M) = \mathcal{A}(M)/\mathcal{G}(M)$ be the quotient space of this action. It is a smooth infinite dimensional manifold.

Let $\mathcal{G}_0(M)$ be the group of gauge transformations that are identity on the boundary of $M$. When $M$ has no boundary $\mathcal{G}(M) = \mathcal{G}_0(M)$.

We have

$$Lie(\mathcal{G}) = \Omega^0_s(M, ad P) .$$

Where $ad P = P \times_G Lie G$ is the derived bundle of $Ad P$. When $P$ is trivial $Lie(\mathcal{G}) = \Omega^0_s(M, Lie G)$. The Lie algebra of $\mathcal{G}_0$ is

$$Lie \mathcal{G}_0 = \{ \xi \in Lie \mathcal{G}; \xi|\partial M = 0 \} = \{ \xi \in \Omega^0_s(M, ad P); \xi|\partial M = 0 \} .$$

The infinitesimal action of $\mathcal{G}$ on $\mathcal{A}$ is described by

$$\xi \cdot A = d_A\xi = d\xi + [A \wedge \xi], \hspace{1cm} \forall \xi \in Lie \mathcal{G}, \forall A \in \mathcal{A} .$$  \hspace{1cm} (1.11)$$
The fundamental vector field on $\mathcal{A}$ corresponding to $\xi \in \text{Lie}(\mathcal{G})$ is given by
\[
d_\mathcal{A}\xi = \frac{d}{dt}|_{t=0}(\exp t\xi) \cdot \mathcal{A},
\]
and the tangent space to the orbit at $A \in \mathcal{A}$ is
\[
T_A(\mathcal{G} \cdot \mathcal{A}) = \{d_\mathcal{A}\xi ; \xi \in \Omega^0_\mathcal{G}(M, \text{ad} P)\}. \tag{1.12}
\]

2 Canonical structure on $T^*\mathcal{A}$

On the cotangent bundle of any manifold we have the notion of canonical symplectic form, and the standard theory of Hamiltonian mechanics and its symmetry follows from it. We apply these standard notions to our infinite dimensional manifold $\mathcal{A}(M)$ and write up their explicit formulas.

2.1 Canonical 1-form and 2-form on $T^*\mathcal{A}$

Let $M$ be a manifold of dim $M = m$ possibly with the non-empty boundary $\partial M$. Let $T^*\mathcal{A} \xrightarrow{\pi} \mathcal{A}$ be the cotangent bundle. We denote the pairing of $T_\mathcal{A}\mathcal{A}$ and $T_{\mathcal{A}}^*\mathcal{A}$ by
\[
\langle a, \alpha \rangle_\mathcal{A} = \int_M \text{tr} a \wedge \alpha, \quad \forall a \in T_\mathcal{A}\mathcal{A}, \alpha \in T_{\mathcal{A}}^*\mathcal{A}. \tag{2.1}
\]

In the following we shall denote the cotangent space $T^*\mathcal{A}$ by $S$. The point of $S$ will be denoted by $(A, \lambda)$ with $A \in \mathcal{A}$ and $\lambda \in T_{\mathcal{A}}^*\mathcal{A}$. The tangent space to the cotangent space $S$ at the point $(A, \lambda) \in S$ becomes
\[
T_{(A,\lambda)}S = T_\mathcal{A}\mathcal{A} \oplus T_{\mathcal{A}}^*\mathcal{A} = \Omega^1(M, \text{Lie} \mathcal{G}) \oplus \Omega^{m-1}(M, \text{Lie} \mathcal{G}). \tag{2.2}
\]

Any tangent vector $a \in T_{(A,\lambda)}S$ has the form $a = \begin{pmatrix} a \\ \alpha \end{pmatrix}$ with $a \in T_\mathcal{A}\mathcal{A}$ and $\alpha \in T_{\mathcal{A}}^*\mathcal{A}$.

The canonical 1-form on the cotangent space $S$ is defined as follows:
\[
\theta_{(A,\lambda)}\left( \begin{pmatrix} a \\ \alpha \end{pmatrix} \right) = \langle \lambda, \pi_* \left( \begin{pmatrix} a \\ \alpha \end{pmatrix} \right) \rangle_A = \int_M \text{tr} a \wedge \lambda, \tag{2.3}
\]
for any tangent vector $\begin{pmatrix} a \\ \alpha \end{pmatrix} \in T_{(A,\lambda)}S$. 

Let $\phi$ be a 1-form on $\mathcal{A}$. By definition, $\phi$ is a section of the cotangent bundle $T^*\mathcal{A}$, so the pullback by $\phi$ of $\theta$ is a 1-form on $\mathcal{A}$. We have the following characteristic property:

$$\phi^* \theta = \phi. \quad (2.4)$$

**Lemma 2.1.** The derivation of the 1-form $\theta$ is given by

$$(\partial_{(A,\lambda)} \theta) \left( \begin{array}{c} a \\ \alpha \end{array} \right) = \int_M tr a \wedge \alpha, \quad \text{for } \forall \left( \begin{array}{c} a \\ \alpha \end{array} \right) \in T_{(A,\lambda)}S. \quad (2.5)$$

In fact,

$$(\partial_{(A,\lambda)} \theta) \left( \begin{array}{c} a \\ \alpha \end{array} \right) = \lim_{t \to 0} \frac{1}{t} \int_M (tr a \wedge (\lambda + t\alpha) - tr a \wedge \lambda) = \int_M tr a \wedge \alpha. \quad (2.6)$$

The canonical 2-form is defined by

$$\omega = \tilde{d}\theta. \quad (2.7)$$

Lemma 2.1 and (1.7) yields the following

**Proposition 2.2.**

$$\omega_{(A,\lambda)} \left( \begin{array}{c} a \\ \alpha \end{array} \right), \left( \begin{array}{c} b \\ \beta \end{array} \right) = \int_M tr [b \wedge \alpha - a \wedge \beta] \quad (2.8)$$

$\omega$ is a non-degenerate closed 2-form on the cotangent space $S$. We see the non-degeneracy as follows. Let $\left( \begin{array}{c} a \\ \alpha \end{array} \right) \in T_{(A,\lambda)}T^*\mathcal{A}$, then $a \in \Omega^1(M,LieG)$ and $\alpha \in \Omega^{m-1}(M,LieG)$. Hence $*\alpha \in \Omega^1(M,LieG)$ and $*a \in \Omega^{m-1}(M,LieG)$ and we have

$$\omega_{(A,\lambda)} \left( \begin{array}{c} a \\ \alpha \end{array} \right), \left( \begin{array}{c} *\alpha \\ *a \end{array} \right) = ||\alpha||^2_{m-1} - ||a||^2_1,$$

where $|| \cdot ||_k$ is the $L^2$-metric on $\Omega^k(M,LieG)$. This formula implies the non-degeneracy of $\omega$.

Let $\Phi = \Phi(A,\lambda)$ be a function on the cotangent space $S$. The Hamiltonian vector field $X_\Phi$ of $\Phi$ is defined by the formula:

$$(\tilde{d}\Phi)_{(A,\lambda)} = \omega(\cdot, X_\Phi(A,\lambda)). \quad (2.9)$$
Here the directional derivative of $\Phi$ at the point $(A, \lambda)$ to the direction \(\begin{pmatrix} a \\ \alpha \end{pmatrix} \in T_{(A,\lambda)}S\) is defined by the formula

\[
(\partial\Phi)_{(A,\lambda)} \begin{pmatrix} a \\ \alpha \end{pmatrix} = \lim_{t \to 0} \frac{1}{t} (\Phi(A + ta, \lambda + t\alpha) - \Phi(A, \lambda)).
\] (2.9)

Hence the partial derivatives \(\frac{\partial \Phi}{\partial \alpha} \in T_A^*A\) and \(\frac{\partial \Phi}{\partial \lambda} \in TA^*A\) are given respectively by the formulas

\[
\langle \frac{\partial \Phi}{\partial \lambda}, \alpha \rangle_A = \lim_{t \to 0} \frac{1}{t} (\Phi(A, \lambda + t\alpha) - \Phi(A, \lambda)),
\]

\[
\langle a, \frac{\partial \Phi}{\partial \alpha} \rangle_A = \lim_{t \to 0} \frac{1}{t} (\Phi(A + ta, \lambda) - \Phi(A, \lambda)).
\]

(2.10) (2.11)

It holds that

\[
(\tilde{d}\Phi)_{(A,\lambda)} \begin{pmatrix} a \\ \alpha \end{pmatrix} = \langle a, \frac{\partial \Phi}{\partial \alpha} \rangle_A + \langle \frac{\partial \Phi}{\partial \lambda}, \alpha \rangle_A.
\] (2.12)

So the Hamiltonian vector field of $\Phi$ is given by

\[
X_\Phi = \begin{pmatrix} \frac{\partial \Phi}{\partial \lambda} \\ -\frac{\partial \Phi}{\partial \alpha} \end{pmatrix}.
\] (2.13)

Example

Let $M$ be a compact three dimensional manifold. We look at the following Hamiltonian function

\[
H(A, B) = \frac{1}{2} \int_M \text{tr} [F_A \wedge *F_A] + \frac{1}{2} \int_M \text{tr} [B \wedge B],
\] (2.14)

for $A \in \mathcal{A}$, $B \in T^*_A\mathcal{A}$. Then, since $\frac{\delta H}{\delta A} = d_A * F_A = *(d_A^* F_A)$ and $\frac{\delta H}{\delta B} = *B$, the Hamiltonian vector field of $H$ becomes

\[
X_H = \begin{pmatrix} *B \\ - *d_A^* F_A \end{pmatrix}.
\] (2.15)

The Hamilton’s equation of motion is

\[
\dot{A} = *B, \quad \dot{B} = - *d_A^* F_A, \quad \text{for } A \in \mathcal{A}, B \in T_A\mathcal{A}.
\] (2.16)
It follows that the critical points of the Hamiltonian function \( H(A,0) = \frac{1}{2} \int_M \text{tr}[F_A \wedge *F_A] \) are given by the Yang-Mills equation on the 3-dimensional manifold:

\[
d_A F_A = d_A^* F_A = 0.
\] (2.17)

The group of (pointed) gauge transformations \( G(M) = \Omega^0(M, \text{Ad} G) \) acts on \( T_A A \) by the adjoint representation; \( a \rightarrow \text{Ad}_g^{-1} a = g^{-1} a g \), and on \( T^*_A A \) by its dual \( \alpha \rightarrow g^{-1} \alpha g \). Hence the canonical 1-form and 2-form are \( G \)-invariant. The infinitesimal action of \( \xi \in \text{Lie} G = \Omega^0(M, \text{Lie} G) \) on the cotangent space \( S = T^* A \) gives a vector field \( \xi_S \) (called fundamental vector field) on \( S \) that is defined at the point \((A, \lambda)\) by the equation:

\[
\xi_S(A, \lambda) = \frac{d}{dt} \exp t \xi \cdot \begin{pmatrix} A \\ \lambda \end{pmatrix} = \begin{pmatrix} d_A \xi \\ [\lambda, \xi] \end{pmatrix}.
\] (2.18)

Remember that \( G_0(M) \) is the group of gauge transformations that are identity on the boundary of \( M \). When \( M \) has no boundary \( G(M) = G_0(M) \). If \( \xi \in \text{Lie} G_0 \), the vector field \( \xi_{T^*_A A} \) is null on the boundary.

The dual space of \( \text{Lie} G_0 \) is given by \( (\text{Lie} G_0)^* = \Omega^m(M, \text{Lie} G) \) with the dual pairing:

\[
\langle \mu, \xi \rangle = \int_M \text{tr} (\mu \xi), \quad \forall \xi \in G_0, \mu \in \Omega^m(M, \text{Lie} G).
\] (2.19)

The moment map for the action of \( G_0 \) on the symplectic space \((S, \omega)\) is the map \( J : S \rightarrow (\text{Lie} G_0)^* \approx \Omega^m(M, \text{Lie} G) \), such that, if we denote \( J^\xi(A, \lambda) = \langle J(A, \lambda), \xi \rangle \) for \( \xi \in \text{Lie} G_0 \).

1. \( J^\xi \) is \( \text{Ad}^* G \)-equivariant:

\[
J^{A g \xi}(g \cdot A, g \cdot \lambda) = J^\xi(A, \lambda),
\] (2.20)

2. \( J^\xi \) satisfies the relation

\[
\tilde{d} J^\xi = \omega (\cdot, \xi_S).
\] (2.21)

**Proposition 2.3.**

The action of the group of gauge transformations \( G_0(M) \) on the symplectic space \((S, \omega)\) is an hamiltonian action with the moment map given by

\[
J(A, \lambda) = - d_A \lambda.
\] (2.22)
Proof

The equivariance of $J^\xi$ follows easily. We shall verify the condition (2.21). Stokes’ theorem yields

$$J^\xi(A, \lambda) = \langle J(A, \lambda), \xi \rangle = -\int_M \tr (d_A \lambda) \xi = \int_M \tr (d_A \xi \wedge \lambda).$$

Since

$$\lim_{t \to 0} \frac{1}{t} \int_M \tr (d_A + ta \xi \wedge (\lambda + t \alpha) - d_A \xi \wedge \lambda) = \int_M \tr (a \wedge [\xi, \lambda] + d_A \xi \wedge \alpha),$$

we have

$$\left( \tilde{d}J^\xi \right)_{(A, \lambda)} \left( \begin{array}{c} a \\ \alpha \end{array} \right) = \omega_{(A, \lambda)} \left( \begin{array}{c} a \\ \alpha \end{array} \right), \left( \begin{array}{c} d_A \xi \\ [\lambda, \xi] \end{array} \right).$$

(2.23)

Define 2.1.

The above moment map will be denoted by $J^* : S \longrightarrow (\text{Lie } G)^*.$

$$J^*(A, \lambda) = -d_A \lambda, \quad \forall (A, \lambda) \in S.$$ (2.24)

Remark 2.1.

$(J^*)^\xi (A, \lambda)$ is given by

$$(J^*)^\xi (A, \lambda) = -\theta_{(A, \lambda)} (\xi T_A).$$ (2.25)

Remark 2.2. From Marsden-Weinstein reduction theorem the reduced space $(J^*)^{-1}(\rho)/G_0$ for a $\rho \in (\text{Lie } G_0)^*$ becomes a symplectic manifold endowed with the induced symplectic form $\omega$, and coincides with the space $\{(A, \lambda) \in S ; -d_A \lambda = \rho \}$.

2.2

Let $M$ be a compact $m$-dimensional riemannian manifold and $G = SU(n)$, $n \geq 2$, the special unitary group. The Lie algebra of $G$ is $n \times n$-matrices with vanishing trace. Let $P \xrightarrow{\pi} M$ be the principal $G$-bundle over $M$, and let $\mathcal{A}$ be the space of irreducible connections over $M$. We assume that $P$ is a trivial bundle $P = M \times G$ though the same argument applies to the non-trivial case by a small change. $\mathcal{A}$ is an affine space modelled by the vector
space $\Omega^1(M, ad P = \Omega^1(M, su(n))$. The tangent space at the point $A \in \mathcal{A}$ is

$$T_A \mathcal{A} = \Omega^1(M, su(n)).$$

(2.26)

The inner product on $T_A \mathcal{A}$ is given by

$$(a, b)_1 = \int_M \text{Tr} a \wedge * b \quad \forall a, b \in T_A \mathcal{A}.$$  

(2.27)

This is expressed by the product of differential forms and the multiplication of matrices.

We put $R = T \mathcal{A}$. The point of $R$ is denoted by $(A, p)$ with $A \in \mathcal{A}$ and $p \in T_A \mathcal{A}$. The tangent space at $(A, p) \in R$ is $T_{(A,p)}R = T_A \mathcal{A} \oplus T_A \mathcal{A}$, so any tangent vector $a \in T_{(A,p)}R$ is of the form $a = \begin{pmatrix} a \\ x \end{pmatrix}$ with $a, x \in T_A \mathcal{A}$. The symplectic structure on $R$ is defined by the formula

$$\sigma_{(A,p)} \left( \begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix} \right) = (b, x)_1 - (a, y)_1,$$

(2.28)

for all $\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix} \in T_{(A,p)}R$.

The directional derivative to the direction $\begin{pmatrix} a \\ x \end{pmatrix}$ of a function $\varphi$ over $R$ is defined by

$$(\partial \varphi)_{(A,p)} \left( \begin{pmatrix} a \\ x \end{pmatrix} \right) = \lim_{t \to 0} \frac{1}{t} \left( \varphi(A + ta, p + ta) - \varphi(A, p) \right).$$

(2.29)

Hence the partial derivatives $\frac{\delta \varphi}{\delta A}, \frac{\delta \varphi}{\delta p} \in T_A \mathcal{A} \simeq \Omega^1(M, su(n))$ is defined by the equations:

$$\partial \varphi_{(A,p)} \left( \begin{pmatrix} a \\ 0 \end{pmatrix} \right) = \left( \frac{\delta \varphi}{\delta A}, a \right)_1, \quad \partial \varphi_{(A,p)} \left( \begin{pmatrix} 0 \\ x \end{pmatrix} \right) = \left( \frac{\delta \varphi}{\delta p}, x \right)_1.$$  

(2.30)

The hamiltonian vector field $X_\varphi$ corresponding to $\varphi$ is given by

$$X_\varphi = \begin{pmatrix} \frac{\delta \varphi}{\delta p} \\ -\frac{\delta \varphi}{\delta A} \end{pmatrix}.$$  

(2.31)

Example
If we take the hamiltonian function

$$H(A, p) = \frac{1}{2} (F_A, F_A)_2 + \frac{1}{2} (p, p)_1.$$  \hfill (2.32)

Then

$$\partial H_{(A, p)} \left( \begin{array}{c} a \\ x \end{array} \right) = (d_A a, F_A)_2 + (p, x)_1 = (a, d_A^* F_A)_1 + (p, x)_1,$$  \hfill (2.33)

and the corresponding hamiltonian vector field is

$$\left( X_H \right)_{(A, p)} = \left( \begin{array}{c} p \\ -d_A^* F_A \end{array} \right).$$  \hfill (2.34)

Therefore the Hamilton’s equation of motion becomes

$$\dot{A} = p, \quad \dot{p} = -d_A^* F_A.$$  \hfill (2.35)

The group of gauge transformation $G = Aut_0(P) = \Omega^0(M, AdP)$ acts on the symplectic manifold $(R, \omega)$:

$$g \cdot (A, p) = (A + g^{-1} d_A g, g^{-1} p g), \quad g \in G.$$  \hfill (2.36)

Under the action of $G$ the hamiltonian $H$ is invariant.

**Proposition 2.4.** The action of $G$ on the symplectic space $(R = TA, \sigma)$ is an hamiltonian action with the moment map $J : R \to (Lie G)^* \simeq Lie G$ given by

$$J(A, p) = d_A^* p.$$  \hfill (2.37)

**Proof**

We note that we consider the dual of $Lie G$ as $Lie G = \Omega^0(M, Lie G)$ itself given by the inner product $(\ , \ )_0$. We shall prove that $J^\xi(A, p) = (d_A^* p, \xi)_0$ gives the the moment map $J(A, p) : Lie G \ni \xi \to J^\xi(A, p) \in R$ of the action of $G$ on $R$. The fundamental vector field $\xi_R$ corresponding to $\xi \inLie G$ is

$$\xi_R(A, p) = \frac{d}{dt} |_{t=0}(\exp t \xi \cdot A, \exp t \xi \cdot p) = \left( \begin{array}{c} d_A \xi \\ -[\xi, p] \end{array} \right) \in T_{(A, p)} R.$$  \hfill (2.38)

Then, for any $\left( \begin{array}{c} a \\ x \end{array} \right) \in T_{(A, p)} R$,

$$\sigma(\left( \begin{array}{c} a \\ x \end{array} \right), \xi_R(A, p)) = (d_A \xi, x)_1 - (a, -[\xi, p])_1.$$  \hfill (2.39)
On the other hand, since
\[
(\tilde{d}J^\xi)_{(A,p)} \begin{pmatrix} a \\ 0 \end{pmatrix} = \lim_{t \to 0} \frac{1}{t} \left( (d^*_A + ta)p, \xi \right)_0 - (d^*_A p, \xi)_0 = \lim_{t \to 0} \frac{1}{t}(p, d_A + ta \xi - d_A \xi)_1 \\
= (p, [a, \xi])_1 = (a, [\xi, p])_1,
\]
and
\[
(\tilde{d}J^\xi)_{(A,p)} \begin{pmatrix} 0 \\ x \end{pmatrix} = \lim_{t \to 0} \frac{1}{t} \left( (d^*_A (p + tx), \xi)_0 - (d^*_A p, \xi)_0 \right) = (d^*_A x, \xi)_0 = (x, d_A \xi)_1,
\]
we have
\[
(\tilde{d}J^\xi)_{(A,p)} \begin{pmatrix} a \\ x \end{pmatrix} = (d_A \xi, x)_1 + (a, [\xi, p])_1.
\]
Therefore
\[
(\tilde{d}J^\xi)_{(A,p)} \begin{pmatrix} a \\ x \end{pmatrix} = \sigma \left( \begin{pmatrix} a \\ x \end{pmatrix}, \xi_R(A, p) \right), \quad \forall \begin{pmatrix} a \\ x \end{pmatrix} \in T(A, p) R.
\]
\[\]
\[\]
\[J(A, p) = d^*_A p \text{ is the moment map.} \]

If \( \rho \in (\text{Lie} \ G)^* \simeq \Omega^0(M, \text{Lie} \ G) \) then \( J^{-1}(\rho) = \{(A, p) \in R; d^*_A p = \rho\} \).

So \((\text{Lie} \ G)^* \) is the space of charge densities and the equation \( d^*_A E = \rho \) for \( E = -p \) is the counterpart of Gauss’s law.

### 2.3 Duality

Let \( M \) be a compact \( m \)-dimensional riemannian manifold. Let \( \mathcal{A} \) be the space of irreducible connections over the trivial \( G = SU(n) \)-bundle \( P = M \times G \), and let \( \mathcal{G} \) be the group of gauge transformations over \( P \). In previous sections we have investigated the symplectic manifolds \((S = T^* \mathcal{A}, \omega)\) and \((R = T \mathcal{A}, \sigma)\).

There correspond Hamiltonian actions of the group of gauge transformations \( \mathcal{G} \) on these space with the moment maps \( J, (2.37) \), and \( J^*, (2.22) \), respectively. We must note that the dual space of \( \text{Lie} \ G = \Omega^1(M, \text{Lie} \ G) \) viewed as the image of moment map \( J^* \) is \( \Omega^m(M, \text{Lie} \ G) \), while that as the image of \( J \) is \( \Omega^1(M, \text{Lie} \ G) \).

**Proposition 2.5.** The symplectic manifolds \((S, \omega)\) and \((R, \sigma)\) are isomorphic via the Hodge operator:
\[
* : T_A \mathcal{A} \simeq \Omega^1(M, su(n)) \longrightarrow T_A^* \mathcal{A} \simeq \Omega^{m-1}(M, su(n)), \quad A \in \mathcal{A}. \quad (2.38)
\]
In fact the correspondence $T_A A \ni a \longrightarrow \ast a \in T^*_A A$ gives the symplectic isomorphism:

$$\sigma_{(A,p)} \left( \left( \begin{array}{c} a \\ x \end{array} \right), \left( \begin{array}{c} b \\ y \end{array} \right) \right) = (b, x) - (a, y) = \omega_{(A,\ast p)} \left( \left( \begin{array}{c} a \\ \ast x \end{array} \right), \left( \begin{array}{c} b \\ \ast y \end{array} \right) \right).$$

Proposition 2.6.

1. (a) We have the following orthogonal decomposition of $T_A A$:

$$T_A A = \{d_A \xi; \xi \in \text{Lie } G \} \oplus H_A,$$

with $H_A = \{ x \in \Omega^1(M, \text{Lie } G); d_A^* x = 0 \}$.

(b) Let

$$R^0 = \bigcup_{A \in A} H_A.$$  

Then $R^0$ coincides with the symplectic reduction of $R$ by the moment map $J$, (2.37):

$$J^{-1}(0)/G \simeq R^0.$$  

(2.41)

2. (a) We have the following orthogonal decomposition of $T^*_A A$:

$$T^*_A A = \{d^*_A \lambda; \lambda \in (\text{Lie } G)^* = \Omega^m(M, \text{Lie } G) \} \oplus H^*_A,$$

with $H^*_A = \{ w \in \Omega^{m-1}(M, \text{Lie } G); d_A w = 0 \}$.

(b) Let

$$S^0 = \bigcup_{A \in A} H^*_A.$$  

Then $S^0$ coincides with the symplectic reduction of $S$ by the moment map $J^*$, (2.22):

$$(J^*)^{-1}(0)/G \simeq S^0.$$  

(2.44)

3. The symplectic isomorphism given by Hodge $\ast$, (2.38), changes the Hamilton equations of motion (2.16) and (2.35) together:

$$\begin{align*}
\dot{A} &= \ast B \\
\dot{B} &= - \ast d_A^* F_A
\end{align*}$$

$$\Longleftrightarrow$$

$$\begin{align*}
\dot{p} &= -d_A^* F_A \\
p &= \ast B
\end{align*}.$$  

(2.45)
Proof

We shall prove only the orthogonal decomposition of $T_A\mathcal{A}$ and that of $T^*A$. The rest is easy to see. Let $H_A$ be the orthogonal complement of the subspace $d_A\text{Lie}\mathcal{G}$ in $T_A\mathcal{A} = \Omega^1(M, \text{Lie}\mathcal{G})$. So $x \in \Omega^1(M, \text{Lie}\mathcal{G})$ is decomposed to $x = d_A\xi + y$ with $\xi \in \Omega^0(M, \text{Lie}\mathcal{G})$ and $y \in H_A$. It implies $H_A = \{ y \in T_A\mathcal{A} : d_A^*y = 0 \}$. It is easy to verify $x = G_A d_A^*x$. Where $G_A$ is the Green operator that will be precisely explained at the beginning of section 3.3. Next let $u \in T^*\mathcal{A} = \Omega^{m-1}(M, \text{Lie}\mathcal{G})$ and put $x = *u$. Then $x \in \Omega^1(M, \text{Lie}\mathcal{G})$ is decomposed to

$$x = d_A\xi + y, \quad \text{with} \quad \xi = G_A d_A^*x \in \Omega^0(M, \text{Lie}\mathcal{G}), \quad y \in H_A.$$ 

Let $\lambda = *\xi$, $w = *y$. Since $d_Aw = *d_A^*y = 0$ and $d_A^*\lambda = *d_A\xi$, we have $u = d_A^*\lambda + w$ with $w \in H_A$. The decomposition is orthogonal with respect to $(, )_{m-1}$. We have also $\lambda = d_A G_A u$.

3 Electronic Magnetic paradigm of Yang-Mills fields

3.1 Symplectic structure over $T_A \times_A T^*A$

We introduce the Yang-Mills field over the 3-dimensional manifold $M$, that becomes the configuration space of the Yang-Mills equation. This is a counterpart of the configuration space for the Hamiltonian description of Maxwell’s equation, [6]. For that we consider the fiber product of the cotangent space $T^*\mathcal{A}$ and the tangent space $T\mathcal{A}$ over $\mathcal{A}$. In the following we shall endow it with a symplectic structure.

We introduce the Whittney’s direct sum of tangent and cotangent bundles: $T = T\mathcal{A} \times_A T^*\mathcal{A} \longrightarrow \mathcal{A}$:

$$\mathbb{T} = T\mathcal{A} \times_A T^*\mathcal{A} \longrightarrow (T^*\mathcal{A}, \omega) = S$$

$$\pi_s \downarrow \quad \pi \downarrow$$

$$R = (T\mathcal{A}, \sigma) \longrightarrow \mathcal{A}.$$ 

A point of $\mathbb{T}$ will be denoted by $(A, E, B)$ with $A \in \mathcal{A}$, $E \in T_A\mathcal{A}$ and $B \in T^*_A\mathcal{A}$. The tangent space of $\mathbb{T}$ at the point $(A, E, B) \in \mathbb{T}$ is $T_{(A,E,B)}\mathbb{T} = \ldots$
$T_A \mathbb{A} \oplus (T_A \mathbb{A} \oplus T_A \mathbb{A})$. So a vector in $T_{(A,E,B)} \mathbb{T}$ is denoted by $a = \begin{pmatrix} a \\ e \\ \beta \end{pmatrix}$, with $a, e \in T_A \mathbb{A} \simeq \Omega^1(M, \text{Lie } G)$ and $\beta \in T_A^* \mathbb{A} \simeq \Omega^2(M, \text{Lie } G)$. Remember that each fibers $T_A \mathbb{A} \simeq \Omega^1(M, \text{Lie } G)$ and $T_A^* \mathbb{A} \simeq \Omega^2(M, \text{Lie } G)$ are endowed with the structure of Sobolev space $L^2_{s-1}$ with $s > 2$.

Now we shall endow the space $\mathbb{T}$ with a symplectic structure that is written only by the field variables $(E,B)$, so the potential variable $A$ is implicit.

Let $\Theta$ be the following 1-form on $\mathbb{T}$ defined by

$$\Theta_{(A,E,B)} \begin{pmatrix} a \\ e \\ \beta \end{pmatrix} = (e, d^*_A B)_1, \quad \forall \begin{pmatrix} a \\ e \\ \beta \end{pmatrix} \in T_{(E,B)} \mathbb{T}. \quad (3.2)$$

Since $(\partial_A d^*_A)_{(A,E,B)} a = *(a \ast \cdot)$, the partial derivative with respect to $A$ of the 1-form $\Theta$ to the direction $a \in T_A \mathbb{A}$ is

$$(\partial_A \Theta)_{(A,E,B)} \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = (\cdot, *(a \ast B))_1 = (\cdot \land a, B)_2, \quad (3.3)$$

The partial derivative with respect to $E$ of the 1-form $\Theta$ to any direction vanishes;

$$(\partial_E \Theta)_{(E,B)} = 0, \quad (3.4)$$

while the partial derivative with respect to $B$ of $\Theta$ is given by

$$(\partial_B \Theta)_{(E,B)} \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} = (\cdot, d^*_A \beta)_1. \quad (3.5)$$

Hence the exterior derivative of the 1-form $\Theta$ becomes

$$\tilde{(d\Theta)}_{(E,B)} \begin{pmatrix} a_1 \\ e_1 \\ \beta_1 \\ a_2 \\ e_2 \\ \beta_2 \end{pmatrix} = (e_2 a_1, B)_2 - (e_1 a_2, B)_2 + (e_2, d^*_A \beta_1)_1 - (e_1, d^*_A \beta_2)_1, \quad (3.7)$$

see the formula (1.7) in about the exterior differentiation of a 1-form.
Definition 3.1. We define the 2-form Ω on \( T \) by the following formula:

\[
\Omega = \tilde{d} \Theta. \tag{3.8}
\]

Then

\[
\Omega = \Omega^1 + \Omega^2 \tag{3.9}
\]

with

\[
\Omega^1_{(A,E,B)} \left( \begin{pmatrix} a_1 \\ e_1 \\ \beta_1 \\ e_2 \\ e_3 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} a_2 \\ e_2 \\ \beta_2 \end{pmatrix} \right) = (e_2 a_1, B) - (e_1 a_2, B),
\]

\[
\Omega^2_{(A,E,B)} \left( \begin{pmatrix} a_1 \\ e_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ e_2 \\ \beta_2 \end{pmatrix} \right) = (e_2 a_1, B) - (e_1 a_2, B)
\]

Since

\[
(\partial_B \Omega^1)_{(A,E,B)/\beta_1} \left( \begin{pmatrix} a_2 \\ e_2 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} a_3 \\ e_3 \\ \beta_3 \end{pmatrix} \right) = (e_3 a_2, \beta_1) - (e_2 a_3, \beta_1),
\]

and

\[
(\partial_A \Omega^2)_{(A,E,B)/a_1} \left( \begin{pmatrix} a_2 \\ e_2 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} a_3 \\ e_3 \\ \beta_3 \end{pmatrix} \right) = (e_3 a_1, \beta_2) - (e_2 a_1, \beta_3),
\]

and other directional derivations are 0, we have \( \tilde{d} \Omega = \tilde{d} \Omega^1 + \tilde{d} \Omega^2 = 0 \) from [1,8]. It is easy to see that \( \Omega \) is a non-degenerate skew linear form. Thus we have the following

Theorem 3.1.

(\( T = T_A \times_A T^*A, \Omega \)) is a symplectic manifold.

We define the inner product on each fiber of \( T \rightarrow A \).

\[
\left( \begin{pmatrix} e_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ \beta_2 \end{pmatrix} \right)_{T} = (e_2, d^{*}_{\beta_1})_1 + (e_1, d^{*}_{\beta_2})_1, \tag{3.10}
\]

for \( E \in T_A, B \in T^*_A \) and \( \forall \begin{pmatrix} e_i \\ \beta_i \end{pmatrix} \in T_{(A,E,B)} T, i = 1, 2, \) and . Here a tangent vector to the fiber \( T_A \) is denoted by \( \begin{pmatrix} e_1 \\ \beta_1 \end{pmatrix} \).

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The partial derivative of a function $\Phi = \Phi(E, B)$ over $T$ to the direction $e \in T_A$ is defined as the vector $\frac{\delta \Phi}{\delta E} \in T^*_A$ that satisfies the equation

$$\left( \partial \Phi \right)(E, B) \left( \begin{array}{c} e \\ 0 \end{array} \right) = (e, d_A^* \frac{\delta \Phi}{\delta E})_1, \quad \forall e \in T_A,$$

where $\pi_*(E, B) = A$. Respectively the directional derivative to $\beta \in T^*_A$ is defined as the vector $\frac{\delta \Phi}{\delta B} \in T_A$ that satisfies the equation

$$\left( \partial \Phi \right)(E, B) \left( \begin{array}{c} 0 \\ \beta \end{array} \right) = (\frac{\delta \Phi}{\delta E}, d_A^* \beta)_1, \quad \forall \beta \in T^*_A.$$  

We have then

$$\left( \theta \Phi \right)(E, B) \left( \begin{array}{c} e \\ \beta \end{array} \right) = \left( \begin{array}{c} \frac{\delta \Phi}{\delta E} \\ \frac{\delta \Phi}{\delta B} \end{array} \right) \left( \begin{array}{c} e \\ \beta \end{array} \right)_T.$$  

**Example**

Let $H = H(E, B)$ be the Hamiltonian function on $T$ written in the vortex representation;

$$H(E, B) = \frac{1}{2} \left( \begin{array}{c} E \\ B \end{array} \right), \left( \begin{array}{c} d_A^* B \\ d_A E \end{array} \right)_T = \frac{1}{2} \{(d_A E, d_A E)_1 + (d_A^* B, d_A^* B)_1\},$$

we have

$$\frac{\delta H}{\delta B} = d_A^* B, \quad \frac{\delta H}{\delta E} = d_A E.$$  

**Proposition 3.2.**

Let $\Phi = \Phi(E, B)$ be a function on the fields $T$. Then the Hamiltonian vector field $X_\Phi$ of $\Phi$ is given by

$$X_\Phi(E, B) = \left( \begin{array}{c} -\frac{\delta \Phi}{\delta B} \\ \frac{\delta \Phi}{\delta E} \end{array} \right).$$

The formulae (3.13) and (3.10) imply (3.16).

**Definition 3.2.**

The Poisson bracket on $T$ is defined by the formula

$$\{ \Phi, \Psi \}_{(E, B)} = \Omega_{(E, B)}(X_\Phi, X_\Psi),$$

for $\Phi, \Psi \in C^\infty(T)$.  

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The following formula is our counterpart to the Marsden-Weinsrein’s vortex formula for the Poisson bracket of Maxwell’s fields.\cite{12}

**Proposition 3.3.**

\[
\{ \Phi, \Psi \}_{(E,B)}^{T} = \left( \frac{\delta \Phi}{\delta B}, d_{\Lambda}^{*} \frac{\delta \Psi}{\delta E} \right)_{1} - \left( \frac{\delta \Psi}{\delta B}, d_{\Lambda}^{*} \frac{\delta \Phi}{\delta E} \right)_{1}
\]
\[
= \left( d_{\Lambda} \frac{\delta \Phi}{\delta B}, \frac{\delta \Psi}{\delta E} \right)_{2} - \left( d_{\Lambda} \frac{\delta \Psi}{\delta B}, \frac{\delta \Phi}{\delta E} \right)_{2}
\]

(3.18)

Proposition follows from (3.16).

**Example**

Let \( H = H(E, B) \) be the Hamiltonian function of (3.14). The equation of motion on the strength field \( T \) is written in the form

\[
\dot{\Phi} = \{ \Phi, H \}_{(E,B)}^{T}.
\]

Which implies by the formula (3.15) the Maxwell equation over \( T \):

\[
\dot{E} = -d_{\Lambda}^{*} B, \quad \dot{B} = d_{\Lambda} E.
\]

(3.19)

The group of gauge transformations \( G \) acts on \( T \) by

\[
g \cdot (E, B) = (Ad_{g} E, Ad_{g}^{*} B) = (g^{-1} E g, g^{-1} B g).
\]

(3.20)

It is a symplectic action because of

\[
(g \cdot e, d_{\Lambda}^{*} (g \cdot \beta))_{1} = (g \cdot e, d_{\Lambda}^{*} \beta)_{1} = (e, d_{\Lambda} \beta),
\]

for any \((e, \beta) \in T_{(E,B)} T\).

The Lie algebra of infinitesimal gauge transformations \( Lie \, G = \Omega^{0}(M, Lie \, G) \) acts on \( T \) by

\[
\xi \cdot \begin{pmatrix} A \\ E \\ B \end{pmatrix} = \begin{pmatrix} d_{\Lambda} \xi \\ [E, \xi] \\ [B, \xi] \end{pmatrix},
\]

(3.21)

that is, the fundamental vector field on \( T \) corresponding to \( \xi \in Lie \, G \) becomes

\[
\xi_{T} (A, E, B) = \begin{pmatrix} d_{\Lambda} \xi \\ [E, \xi] \\ [B, \xi] \end{pmatrix}.
\]

(3.22)
3.2 Yang-Mills fields

We shall introduce the Yang-Mills field as a subspace of $T$. We have investigated the moment map over the space of vector potentials $T\mathcal{A}$ and that over the space of vortex potentials $T^*\mathcal{A}$:

$$J : R = T\mathcal{A} \rightarrow (\text{Lie } \mathcal{G})^* \simeq \Omega^0(M, \text{Lie } G),$$

and

$$J^* : T^*\mathcal{A} \rightarrow (\text{Lie } \mathcal{G})^* \simeq \Omega^3(M, \text{Lie } G).$$

For a Hamiltonian system with symmetries the range of a moment map represents conserved quantities of the system. In our case, we take $(\text{Lie } \mathcal{G})^* \simeq \Omega^3(M, \text{Lie } G)$ as the space of charges for the action $\mathcal{G}$ over $T$, that has two component corresponding to the electric charge (times the volume form $\ast 1$) and the magnetic charge. So given $\rho \ast 1, \rho' \in (\text{Lie } \mathcal{G})^*$, we have the configuration space $\{(E, B) \in T; \ J(E) = \rho, \ J^*(B) = \rho'\}$. On the component which contains the field strength $F_A$ we must have $\rho' = 0$.

**Definition 3.3.**

The Yang-Mills field is the subspace of $T$ defined by

$$\mathcal{F} = \{(E, B) \in T : \ d_A B = 0, \ d_A^* E = 0 \ \text{with } \pi_*(E, B) = \mathcal{A}\} \ (3.23)$$

The Yang-Mills field $\mathcal{F}$ is a symplectic subspace of $(T, \Omega)$ that is $\mathcal{G}$-invariant because of the relation:

$$d_{g \cdot A}(g \cdot B) = g \cdot (d_A B), \quad d_{g \cdot A}(g \cdot E) = g \cdot (d_A E).$$

On Yang-Mills field $\mathcal{F}$ we have the analogous formula of Maxwell’s equations:

$$d^*_A B + \dot{E} = 0, \quad d^*_A E = 0, \quad (3.24)$$

$$d_A E - \dot{B} = 0, \quad d_A B = 0. \quad (3.25)$$

Now we shall investigate the Hamiltonian action of $\mathcal{G}$ on the Yang-Mills field $\mathcal{F}$.

A map

$$\mathcal{J} : \mathcal{F} \rightarrow (\text{Lie } \mathcal{G})^* \simeq \Omega^3(M, \text{Lie } G) \quad (3.26)$$

is by definition a moment map for the symplectic action of $\mathcal{G}$ on $\mathcal{F}$ provided
1. If we put $\mathcal{J}(E, B) = \langle \mathcal{J}(E, B), \xi \rangle$, the Hamiltonian vector fields of $\mathcal{J}$ coincides with the fundamental vector field $\xi_T$, (3.1).

2. $\mathcal{J}$ is $\text{Ad}^*$-equivariant:

$$
\mathcal{J}(g^{-1}Eg, g^{-1}Bg) = \mathcal{J}(E, B).
$$

In this case we say that the action of $G$ is Hamiltonian.

**Proposition 3.4.** The action of $G$ on $\mathcal{F}$ is Hamiltonian with the moment map

$$
\mathcal{J}(E, B) = [d_A * B, E].
$$

(3.27)

**Proof**

It holds the relation

$$
\mathcal{J}(E, B) = \Theta_{(A,E,B)}(\xi_T (A, E, B)).
$$

(3.28)

We have

$$
(d \mathcal{J})_{(A,E,B)} \begin{pmatrix}
a \\
e \\
\beta 
\end{pmatrix} = ([E, \xi] \land a, B)_2 + ([e, \xi], d_A^* B)_1 + ([E, \xi], d_A^* \beta)_1,
$$

$$
(i_{\xi_T} \Omega) \begin{pmatrix}
a \\
e \\
\beta 
\end{pmatrix} = ([e, d_A \xi], B)_2 - ([E, \xi] \land a, B)_2
$$

$$
+ (e, d_A^*[B, \xi])_1 - ([E, \xi], d_A^* \beta)_1.
$$

Hence $d \mathcal{J} = -i_{\xi_T} \Omega$. The equivariance of $\mathcal{J}$ is easy to verify.

**Corollary 3.5.** On the $G$-orbit passing through a solution of equation (3.24) we have $\mathcal{J}(E, B) = [\dot{B}, * B]$.

### 3.3 Symplectic variable $\gamma : R \to \mathcal{F}$

Since any $A \in \mathcal{A}$ is an irreducible connection we have the Green operator $G_A$ defined on $\Omega^k((M, \text{Lie} G), k = 1, 2,$

$$(d_A d_A^*) + d_A^* d_A^*) G_A \alpha = \alpha, \ \forall \alpha \in \Omega^k((M, \text{Lie} G)).$$

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$G_A$ is a self-adjoint operator; $(G_A u, v)_k = (u, G_A v)_k$ for any $u$, $v \in \Omega^k(M, \text{Lie } G)$, $k = 1, 2$. We note also the fact that $G_A$ commutes with $d_A$ and $d_A^*$:

$$d_A G_A = G_A d_A, \quad d_A^* G_A = G_A d_A^*.$$  

Restricted to the space $\mathcal{F}$ we have

$$d_A d_A^* G_A \beta = \beta, \quad d_A^* d_A G_A e = e, \quad (3.29)$$

for $e \in T_A \mathcal{A}$ and $\beta \in T_A^* \mathcal{A}$.

**Definition 3.4.**

1. $\phi : R \rightarrow \mathcal{F} \subset \mathbb{T}$ is the map defined by

$$\phi (A, p) = (E = -p, B = F_A). \quad (3.30)$$

2. Let $\phi_* : TR \rightarrow T\mathbb{T}$ be the tangent map of $\phi$:

$$(\phi_*)_p (\begin{array}{c} a \\ x \end{array}) = \left( \begin{array}{c} -x \\ d_A a \end{array} \right),$$

and let $G_A : T_A \mathcal{A} \rightarrow T_A \mathcal{A}$ be the Green operator.

We define the modified tangent map $\gamma : TR \rightarrow T\mathbb{T}$ of $\phi$ as follows

$$\gamma = \phi_* \circ \left( \begin{array}{cc} 1 & 0 \\ 0 & G_A \end{array} \right) = \left( \begin{array}{cc} 0 & -G_A \\ d_A & 0 \end{array} \right), \quad (3.31)$$

that is,

$$T_{(a, x)} \supset \left( \begin{array}{c} a \\ x \end{array} \right) \rightarrow \gamma_{(a, x)} = \left( \begin{array}{c} -G_A x \\ d_A a \end{array} \right) \in T_{\phi(a, x)} \mathbb{T}.$$

**Lemma 3.6.**

$$\gamma^* \Omega = \sigma. \quad (3.32)$$

In fact, we have, for any

$$\left( \begin{array}{c} a_i \\ x_i \end{array} \right) \in T_{(a, x)} \supset \left( \begin{array}{c} a_i \\ x_i \end{array} \right) \rightarrow \gamma_{(a, x)} = \left( \begin{array}{c} -G_A x_i \\ d_A a_i \end{array} \right) = \left( \begin{array}{c} x_i, a_1 \\ x_i, a_2 \end{array} \right) - \left( \begin{array}{c} x_i, a_1 \\ x_i, a_2 \end{array} \right) = \sigma_{(a, x)} \left( \begin{array}{c} a_i \\ x_i \end{array} \right), \left( \begin{array}{c} a_i \\ x_i \end{array} \right).$$

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Let $R^0 = \bigcup_{A \in A} H^0_A$ be the reduction of $R$, (2.41). Remember that the symplectic reduction of $R$ by the moment map $J$ is isomorphic to $R^0$, Proposition [2.6].

**Theorem 3.7.** $(R^0, \sigma)$ is symplectomorph to $(\mathcal{F}, \Omega)$.

**Proof**

Since $d^*_A(-p) = 0$ and $d_A F_A = 0$ for $(A,p) \in R^0$, $\phi$ maps the subspace $R^0$ into $\mathcal{F}$. The tangent space of $H^0_A$ consist of those vectors $\begin{pmatrix} a \\ x \end{pmatrix} \in T_{(A,p)} R$ such that $d^*_A x = 0$, and the tangent space $T\mathcal{F}$ consists of those vectors $\begin{pmatrix} e \\ \beta \end{pmatrix} \in T\mathcal{T}$ such that $d^*_A e = 0$ and $d_A \beta = 0$. If $\begin{pmatrix} a \\ x \end{pmatrix}$ is tangent to $H^0_A$ then $d^*_A G_A x = 0$ and $d_A (d_A a) = 0$, (the latter follows from the derivation of $d_A F_A = 0$). So $\gamma$ maps $TR^0$ into $T\mathcal{F}$. Moreover $\gamma$ is a bijective map of $TR^0$ onto $T\mathcal{F}$. In fact we have the inverse map given by

$$T\mathcal{F} \ni \begin{pmatrix} e \\ \beta \end{pmatrix} \rightarrow (-d^*_A \circ \gamma) \begin{pmatrix} e \\ \beta \end{pmatrix} = \begin{pmatrix} d^*_A G_A \beta \\ -d^*_A d_A e \end{pmatrix} \in TR^0.$$

By virtue of the implicit function theorem in Banach space the vector spaces $R$ and $\mathcal{F}$ are diffeomorphic. Let $\tilde{\gamma}: R^0 \rightarrow \mathcal{F}$ be the diffeomorphism. Lemma [3.6] implies that $\tilde{\gamma}$ is a symplectomorphism.

**Lemma 3.8.** Let $\Phi \in C^\infty(\mathcal{F})$ and let $\varphi \in C^\infty(R)$ be the pullback of $\Phi$ by $\phi: R \rightarrow \mathcal{F}$:

$$\varphi(A,p) = \phi^* \Phi = \Phi(-p, F_A).$$

Then the Hamiltonian vector field $X^R_\varphi$ of $\varphi$ has the formula

$$X^R_\varphi(A, p) = \begin{pmatrix} d^*_A \Phi \frac{\partial \Phi}{\partial E} \\ d^*_A d_A \Phi \frac{\partial \Phi}{\partial B} \end{pmatrix}_{(E,B) = \phi(A,p)}.$$ (3.33)

**Proof**
\[(\partial \varphi)_{(A,p)} \begin{pmatrix} a \\ 0 \end{pmatrix} = \lim_{t \to 0} \frac{1}{t} (\Phi(-p, F_{A+ta}) - \Phi(-p, F_A)) \]
\[
= \lim_{t \to 0} \frac{1}{t} (\Phi(E, B + t\beta) - \Phi(E, B))|_{E=-p, B=F_A, \beta=d_Aa} 
\]
\[
= (\partial \Phi)(-p,F_A) \begin{pmatrix} 0 \\ d_Aa \end{pmatrix} = \begin{pmatrix} \frac{\delta \Phi}{\delta B} \\ d_A^*d_Aa \end{pmatrix} \bigg|_{(-p,F_A)} 
\]
\[
= \left( a, (d_A^*d_A \frac{\delta \Phi}{\delta B}) \right)_{(-p,F_A)} .
\]

Hence \[\frac{\delta \varphi}{\delta A} = (d_A^*d_A \frac{\delta \Phi}{\delta B})|_{(-p,F_A)}\]. And
\[(\partial \varphi)_{(A,p)} \begin{pmatrix} 0 \\ x \end{pmatrix} = \lim_{t \to 0} \frac{1}{t} (\Phi(E + te), B) - \Phi(E, B))|_{E=-p, e=-x, B=F_A} 
\]
\[
= \left( -x, (d_A^* \frac{\delta \Phi}{\delta E}) \bigg|_{(-p,F_A)} \right)_{1} = \left( -(d_A^* \frac{\delta \Phi}{\delta E}) \bigg|_{(-p,F_A)} , x \right)_{1} .
\]

Therefore
\[\frac{\delta \varphi}{\delta A} = (d_A^*d_A \frac{\delta \Phi}{\delta E})|_{(-p,F_A)}, \quad \frac{\delta \varphi}{\delta p} = -(d_A^* \frac{\delta \Phi}{\delta E})|_{(-p,F_A)} .\]

(3.33) follows from (2.31).

Corollary 3.9.
\[
\gamma^{-1}X^\gamma_{\Phi} = \begin{pmatrix} d_A^*G_A \frac{\delta \Phi}{\delta E} \\ d_A^*d_A \frac{\delta \Phi}{\delta B} \end{pmatrix}_{(E,B) = \phi(A,p)} = \begin{pmatrix} G_A & 0 \\ 0 & 1 \end{pmatrix} X^R_{\varphi} . \tag{3.34}
\]

Proposition 3.10.
\[\{ \Phi, \Psi \}^T \circ \tilde{\gamma} = \sigma(\gamma^{-1}X^\gamma_{\Phi}, \gamma^{-1}X^\gamma_{\Psi}) = \{ \Phi \circ \tilde{\gamma} , \Psi \circ \tilde{\gamma} \} . \tag{3.35}\]

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