Abstract

In this paper we consider an algorithmic technique more general than that proposed by Zharkov and Blinkov for the involutive analysis of polynomial ideals. It is based on a new concept of involutive monomial division which is defined for a monomial set. Such a division provides for each monomial the self-consistent separation of the whole set of variables into two disjoint subsets. They are called multiplicative and non-multiplicative. Given an admissible ordering, this separation is applied to polynomials in terms of their leading monomials. As special cases of the separation we consider those introduced by Janet, Thomas and Pommaret for the purpose of algebraic analysis of partial differential equations. Given involutive division, we define an involutive reduction and an involutive normal form. Then we introduce, in terms of the latter, the concept of involutivity for polynomial systems. We prove that an involutive system is a special, generally redundant, form of a Gröbner basis. An algorithm for construction of involutive bases is proposed. It is shown that involutive divisions satisfying certain conditions, for example, those of Janet and Thomas, provide an algorithmic construction of an involutive basis for any polynomial ideal. Some optimization in computation of involutive bases is also analyzed. In particular, we incorporate Buchberger’s chain criterion to avoid unnecessary reductions. The implementation for Pommaret division has been done in Reduce.
1 Introduction

In modern times the Gröbner bases method invented by Buchberger [1] has become one of the most universal algorithmic tools for analyzing and solving polynomial equations [2, 3]. Even in the general case, when the roots cannot be exactly computed, the method is still able to obtain valuable information about the solutions. In particular, it allows one to verify compatibility of the initial equations and compute the dimension of the solution space. For the last few years notable progress has been achieved in extension of the Gröbner bases method to non-commutative [4, 5] and differential algebras [6, 7].

On the other hand, already by the early 20s the foundation of a constructive approach to algebraic analysis of partial differential equations was laid by Riquier [8] and Janet [9] giving, among other things, answers to the same general questions of compatibility and dimension. Later on, this approach, in the context of partial differential equations, was developed by Thomas [10] and more recently by Pommaret [11]. The main idea of the approach, as with the computation of a Gröbner basis, is rewriting the initial differential system into another, so-called, involutive form [12].

In the involutive approach, unlike the Gröbner basis method, independent variables for each equation are separated into two distinct groups called multiplicative and non-multiplicative. Such a separation is determined by the structure of the leading derivative terms. A differential system is called involutive if its non-multiplicative derivatives are algebraic consequences of multiplicative ones. In doing so, Janet [9], Thomas [10] and Pommaret [11] used different separations of variables.

Zharkov and Blinkov [13, 14] argued that the involutive technique along with the Gröbner bases one can be used in commutative algebra. Based on Pommaret definition of multiplicative and non-multiplicative variables [11], they proved, among other things, that an involutive basis is a Gröbner one. Moreover, their computational experience demonstrated a reasonably high efficiency of the new algorithm when it terminates. The termination, however, does not hold, generally, for positive dimensional ideals, while for zero-dimensional ones it does for any degree-compatible monomial orderings [14]. Apart from that, the Pommaret involutive form of Gröbner bases for zero-dimensional polynomial ideals reveals a number of rather attractive features [15].

In the present paper we consider an algorithmic technique more general than that proposed in [13, 14] for the involutive analysis of polynomial ideals. First of all, we introduce a new concept of involutive monomial division (Sect.3) which leads to the self-consistent separation of the whole set of variables into multiplicative and non-multiplicative subsets. Given an admissible ordering, the separation is applied to polynomials in terms of their leading monomials. That concept generalizes the particular choice used by Janet [9], Thomas [10] and Pommaret [11] for analysis of partial differential equations. We characterize also important properties of noetherity, continuity and constructivity for involutive divisions (Sect.4). Noetherity provides for the existence of a finite involutive basis for any polynomial ideal. The other two properties allows one to construct that basis algorithmically. It is shown that all the above three divisions are
continuous and constructive. Thomas and Janet divisions are also noetherian whereas Pommaret division is not.

Given an involutive division, we define an involutive reduction and an involutive normal form (Sect.5). As this takes place, we show that much like the Pommaret normal form, investigated in [13], the general involutive normal form is also unique and linear. Then we define involutive systems by analogy with differential equations (Sect.6). To be involutive, systems are required to satisfy the involutivity conditions, which form the basis for their algorithmic construction.

We prove (Sect.7) that any involutive basis, if it exists, is a special, generally extended, form of the reduced Gröbner basis. Though it is unique for Pommaret division [14], generally, it may not be the case, as it is shown by an explicit example. We propose an algorithm for construction of involutive polynomial bases (Sect.8). Its correctness is proved for any continuous involutive division and for arbitrary admissible monomial ordering, while its termination holds, generally, for noetherian divisions. The algorithm is an improved and generalized version of one proposed in [14, 15], and has been implemented in Reduce for Pommaret division. The main improvement is the incorporation of Buchberger’s chain criterion [16].

2 Preliminaries

Let $\mathbb{R} = K[x_1, \ldots, x_n]$ be a polynomial ring over the field $K$ of characteristic zero. In this paper we use the notations:

- $f, g, h, p, q$ are polynomials in $\mathbb{R}$.
- $a, b, c$ are elements in $K$.
- $F, G, H$ are finite subsets of $\mathbb{R}$.
- $\mathbb{N}$ is the set of non-negative integers.
- $\mathbb{M} = \{ x_1^{d_1} \cdots x_n^{d_n} \mid d_i \in \mathbb{N}, i = 1, \ldots, n \}$ is the set of monomials in $\mathbb{R}$.
- $\mathbb{T} = \{ au \mid u \in \mathbb{M}, a \in K \}$ is the set of terms in $\mathbb{R}$.
- $u, v, w, s, t$ are monomials or terms with nonzero coefficients.
- $U, V, W$ are finite subsets of $\mathbb{M}$.
- $\deg_i(u)$ is the degree of $x_i$ in $u$.
- $\deg(u)$ is the total degree of $u$.
- $cf(f, u) \in K$ is the coefficient of the term $u$ of the polynomial $f$.
- $Id(F)$ is the ideal in $\mathbb{R}$ generated by the polynomial set $F$.
- $\triangleright$ is an admissible monomial ordering with $x_1 \triangleright x_2 \triangleright \cdots \triangleright x_n$.
- $lt(f)$ is the leading term of $f$ w.r.t. the ordering $\triangleright$.
- $lc(f) = cf(f, lt(f))$ is the leading coefficient of $f$.
- $lm(f) = lt(f)/lc(f)$ is the leading monomial of $f$.
- $lm(F) = \{ lm(f) \mid f \in F \}$ is the set of the leading monomials of $F$.
- $lcm(F)$ is the least common multiple of the set $\{ lm(f) \mid f \in F \}$. 

If the monomial $u$ divides the monomial $v$ we shall write $u|v$. 

3


3 Involutive Monomial Division

Definition 3.1 We shall say that an involutive division \( L \) or \( L-division \) is given on \( \mathbb{M} \) if for any finite set \( U \subset \mathbb{M} \) a relation \( \mid_L \) is defined on \( U \times \mathbb{M} \) such that for any \( u, u_1 \in U \) and \( v, w \in \mathbb{M} \) the following holds:

(i). \( u \mid_L w \) implies \( u \mid w \).

(ii). \( u \mid_L u \) for any \( u \in U \).

(iii). \( u \mid_L (uv) \) and \( u \mid_L (uw) \) if and only if \( u \mid_L (uvw) \).

(iv). If \( u \mid_L w \) and \( u_1 \mid_L w \), then \( u \mid_L u_1 \) or \( u_1 \mid_L u \).

(v). If \( u \mid_L u_1 \) and \( u_1 \mid_L w \), then \( u \mid_L w \).

(vi). If \( V \subseteq U \) and \( u \in V \), then \( u \mid_L w \) w.r.t. \( U \) implies \( u \mid_L w \) w.r.t. \( V \).

If \( u \mid_L (w = uv) \), we say \( u \) is an involutive divisor of \( w \), \( w \) is an involutive multiple of \( u \), and \( v \) is multiplicative for \( u \). In such an event we shall write \( w = u \times v \). If \( u \) is a conventional divisor of \( w \) but not an involutive one we shall write, as usual, \( w = u \cdot v \).

Then \( v \) is said to be non-multiplicative for \( u \).

The conventional monomial division, obviously, satisfies condition (iv) only in the univariate case. The simplest bivariate example: \( x \mid (xy) \) and \( y \mid (xy) \) but \( -x \mid y \) and \( -y \mid x \).

Definition 3.1 for each \( u \in U \) provides separation of the set of variables

\[ \{ x_1, \ldots, x_n \} = M_L(u, U) \cup NM_L(u, U) \]

into two disjoined subsets \((M_L(u, U) \cap NM_L(u, U) = \emptyset)\) of multiplicative \( M_L(u, U) \) and non-multiplicative \( NM_L(u, U) \) variables. It is convenient to define an involutive division for a monomial set just by specifying the subsets of multiplicative and non-multiplicative variables to satisfy the conditions (iv)-(vi). The other conditions will be fulfilled by the construction.

Given an involutive division \( L \) and a finite set \( U \), for each \( u \in U \) let \( L(u, U) \subseteq \mathbb{M} \) be the set of multiplicative monomials for \( u \), that is,

\[ u \mid_L v \iff v \in uL(u, U) \quad (1) \]

Then it is easy to see that Definition 3.1 admits another form:

Definition 3.2 An involutive division \( L \) on \( \mathbb{M} \) is given, if for any finite \( U \subset \mathbb{M} \) and for any \( u \in U \) there is given a submonoid \( L(u, U) \) of \( \mathbb{M} \) satisfying the conditions:

(a). If \( w \in L(u, U) \) and \( v \mid w \), then \( v \in L(u, U) \).
(b). If \( u, v \in U \) and \( uL(u, U) \cap vL(v, U) \neq \emptyset \), then \( u \in vL(v, U) \) or \( v \in uL(u, U) \).

(c). If \( v \in U \) and \( v \in uL(u, U) \), then \( L(v, U) \subseteq L(u, U) \).

(d). If \( V \subseteq U \), then \( L(u, U) \subseteq L(u, V) \) for all \( u \in V \).

We consider three different examples of involutive division introduced by Janet \cite{9}, Thomas \cite{10} and Pommaret \cite{11} for analysis of algebraic differential equations. In doing so, we give, firstly, the definition of multiplicative and non-multiplicative variables for each of the divisions, and, secondly, prove the fulfillment of the three extra conditions (iv)-(vi) in Definition 3.1 equivalent to (b)-(d) in Definition 3.2.

**Definition 3.3** Thomas division \cite{10}. Given a finite set \( U \), let

\[
h_i(U) = \max \{ \deg_i(u) \mid u \in U \}.
\]

A variable \( x_i \) is considered as multiplicative for \( u \in U \) if \( \deg_i(u) = h_i(U) \) and non-multiplicative, otherwise.

**Definition 3.4** Janet division \cite{9}. Let \( U \) be a finite set. For each \( 1 \leq i \leq n \) divide \( U \) into groups labeled by non-negative integers \( d_1, \ldots, d_i \):

\[
[d_1, \ldots, d_i] = \{ u \in U \mid \deg_j(u) = d_j, \ 1 \leq j \leq i \}.
\]

A variable \( x_i \) is multiplicative for \( u \in U \) if \( i = 1 \) and \( \deg_1(u) = \max \{ \deg_1(v) \mid v \in U \} \), or if \( i > 1 \), \( u \in [d_1, \ldots, d_{i-1}] \) and

\[
\deg_i(u) = \max \{ \deg_i(v) \mid v \in [d_1, \ldots, d_{i-1}] \}.
\]

**Definition 3.5** Pommaret division \cite{11}. For a monomial \( x_1^{d_1} \cdots x_k^{d_k} \) with \( d_k > 0 \) the variables \( x_j \) with \( j \geq k \) are considered as multiplicative and \( x_j \) with \( j < k \) as non-multiplicative. For \( u = 1 \) all the variables are multiplicative.

We note that

- Thomas division does not depend on the ordering on the variables \( x_i \). Janet and Pommaret divisions, as defined, are based on the ordering of the variables assumed in Sect.2.

- The separation of variables into multiplicative and non-multiplicative ones for Thomas and Janet divisions are defined in terms of the whole set \( U \). Contrastingly, Pommaret division is determined in terms of the monomial itself, regardless of the others, and, by this reason, admits extension to infinite monomial sets, unlike Thomas and Janet divisions.
To distinguish the above divisions the related subscripts \( T, J, P \) will be used.

**Proposition 3.6** Thomas, Janet and Pommaret monomial divisions are involutive.

**Proof** According to the above remark we must prove that the conditions (iv)-(vi) in Definition 3.1 are satisfied.

Let \( u \in U \) be a Thomas divisor of \( w \in \mathbb{M} \), that is, \( w = u \times v \). Then \( \deg_i(v) = \deg_i(w) - h_i(U) \) if \( \deg_i(w) \geq h_i(U) \) and \( \deg_i(v) = 0 \) if \( \deg_i(w) < h_i(U) \). Thus, if \( w \) has an involutive divisor \( u \), then \( w/u \) is uniquely defined, and, hence, \( u \) is unique in \( U \). It implies also the property (v) for Thomas division, since \( u|_T v \) for \( u, v \in U \) if and only if \( u = v \). The property (vi) also holds since any \( h_i \) for \( V \) is less than or equal to the corresponding \( h_i \) for \( U \).

Let now \( u, v \in U \) be two different Janet divisors of \( w \), such that \( \deg_i(u) = \deg_i(v) = d_i \) for \( 1 \leq i < k \leq n \) and assume, for definiteness, that \( \deg_k(u) > \deg_k(v) \). Then, since both \( u, v \) are members of the same group \([d_1, \ldots, d_{k-1}]\), the variable \( x_k \) is non-multiplicative for \( v \). Hence, if \( u \) is a Janet divisor of \( w \) such that \( \deg_k(w) \geq \deg_k(u) > \deg_k(v) \), then \( v \) is not Janet divisor of \( w \). In other words, similar to Thomas division, any monomial \( w \in \mathbb{M} \) cannot have different Janet divisors in any set \( U \). A monomial group may only be decreased by diminishing the set \( U \), which implies the relation (vi).

Lastly, consider a Pommaret divisor \( u \) of the monomial \( w = x_1^{d_1} \cdots x_m^{d_m} \) with \( m \leq n \). By definition, \( u \) constitutes a left subset of the string representation for \( w \) as it is shown.

\[
\begin{align*}
  w = \overbrace{x_1 \cdots x_{d_1}}^{u} \cdots \overbrace{x_m \cdots x_{d_m}}^{n}.
\end{align*}
\]

(2)

It makes evident the fulfillment of the conditions (iv) and (v) for Pommaret division while the condition (vi) trivially holds since the division does not depend on the set \( U \) at all.

\[ \Box \]

**Proposition 3.7** For any finite set \( U \) and for any \( u \in U \), the inclusion \( M_T(u, U) \subseteq M_J(u, U) \) and, respectively, \( N M_J(u, U) \subseteq N M_T(u, U) \) holds.

**Proof** If \( x_i \in M_J(u, U) \), \( u \in [d_1, \ldots, d_{i-1}] \), then, by definition,

\[
\deg_i(u) = \max \{ \deg_i(v) \mid v \in [d_i, \ldots, d_{i-1}] \} \leq \max \{ \deg_i(v) \mid v \in U \}.
\]

Hence, \( x_i \in M_T(u, U) \) implies \( x_i \in M_J(u, U) \).

\[ \Box \]

**Definition 3.8** A set \( U \) is called **involutively autoreduced** with respect to division \( L \) or \( L \)–**autoreduced** if it does not contain elements \( L \)–divisible by other elements in \( U \).

**Proposition 3.9** If \( U \) is \( L \)–**autoreduced**, then any monomial \( w \in \mathbb{M} \) has at most one \( L \)–**involutive divisor in \( U \).
Proof This follows immediately from the property (iv) of involutive division. In terms of Definition 3.2 it means that \( uL(u, U) \cap vL(v, U) = \emptyset \) for all distinct \( u, v \in U \), if \( U \) is involutively autoreduced.

Proposition 3.10 \[17\]. If a set \( U \) is autoreduced with respect to Pommaret division, then for any \( u \in U \) \( M_P(u, U) \subseteq M_J(u, U) \) and \( NM_J(u, U) \subseteq NM_P(u, U) \), respectively.

Proof Let \( u = x_1^{d_1} \cdots x_k^{d_k} \in M \) be a monomial with \( d_k > 0 \) and \( v \in U \) be its Pommaret divisor. Then, as follows from the representation (4), \( v = x_1^{d_1} \cdots x_{m-1}^{d_{m-1}} x_m^r \) with \( 1 \leq m \leq k \) and \( 1 \leq r \leq d_m \). It means that \( v \in [d_1, \ldots, d_{m-1}] \). Since \( U \) is autoreduced by Pommaret division, there are no other members of the same group with degree in \( x_m \) higher than \( r \). Therefore, \( v \) is also a Janet divisor of \( u \), and \( u/v \), being Pommaret multiplicative for \( v \), is also Janet multiplicative.

Example 3.11 \( U = \{xy, y^2, z\} \ (x > y > z) \).

| monomial | Thomas \( M_T \) | \( NM_T \) | Janet \( M_J \) | \( NM_J \) | Pommaret \( M_P \) | \( NM_P \) |
|----------|-----------------|-------------|-----------------|-------------|-----------------|-------------|
| \( xy \)  | \( x \)         | \( y, z \)  | \( x, y, z \)   | \( - \)      | \( y, z \)      | \( x \)      |
| \( y^2 \) | \( y \)         | \( x, z \)  | \( y, z \)      | \( x \)      | \( y, z \)      | \( x \)      |
| \( z \)   | \( z \)         | \( x, y \)  | \( z \)         | \( x, y \)   | \( z \)         | \( x, y \)   |

4 Involutive Monomial Sets

**Definition 4.1** Given an involutive division \( L \), a set \( U \) is called **involutive** with respect to \( L \) or **\( L-\)involutive**, if any multiple of some element \( u \in U \), is also \( (L-)\)involutively multiple of an element \( v \in U \), generally, different from \( u \). It means that

\[
(\forall u \in U) \ (\forall w \in M) \ (\exists v \in U) \ [ \ v|L(uw) \ ] \tag{3}
\]
or, in accordance with (4) and Definition 3.2,

\[
\bigcup_{u \in U} uM = \bigcup_{u \in U} uL(u, U).
\]

**Definition 4.2** We shall call the set \( \bigcup_{u \in U} uM \) the **cone** generated by \( U \) and denote it by \( C(U) \). The set \( \bigcup_{u \in U} uL(u, U) \) will be called the **involutive cone** of \( U \) with respect to \( L \) and denoted by \( C_L(U) \).

Thus, the set \( U \) is \( (L-)\)involutive if and only if its cone \( C(U) \) coincides with its involutive cone \( C_L(U) \).
Definition 4.3 A finite $L$–involutive set $\tilde{U} \subset \mathbb{M}$ will be called $L$–completion of a set $U \subseteq \tilde{U}$ if $C(\tilde{U}) = C(U)$. If there exists an $L$–completion $\tilde{U}$ of the set $U$, then the latter is said to be finitely generated with respect to $L$. An involutive division $L$ is called noetherian if every finite set $U$ is finitely generated.

Proposition 4.4 Given a noetherian involutive division $L$, every monomial ideal $U$ has a finite involutive basis.

Proof This is an immediately consequence of Definition 4.3 and Dickson’s lemma [3].

Proposition 4.5 Thomas and Janet divisions are noetherian.

Proof Given a finite set $U$, consider the monomial $h = x_1^{h_1} \cdots x_n^{h_n}$ where, as given in the definition of Thomas division, $h_i = \max \{ \deg_i(u) \mid u \in U \}$, and form the finite set $V \subset \mathbb{M}$ of all the different monomials $v$ such that $v|h$ and $u|v$ for some $u \in U$. The set $V$, which contains, in particular, the monomial $h$ and the initial set $U$, is involutive for Thomas division. Indeed, let $w = x_1^{d_1} \cdots x_n^{d_n}$ be a multiple of some $u \in V$. If $w \in V$, then, obviously, $w \in C_T(V)$. Otherwise, let $\{d_1, \ldots, d_k\}$ $(k \leq n)$ be the nonempty set which contains all the exponents $d_i$ $(1 \leq i \leq n)$ in $w$ such that $d_i > h_i$. Then there exists $v \in V$ satisfying

$$w = v x_1^{d_1 - h_1} \cdots x_k^{d_k - h_k}.$$ 

Since $\deg_i(v) = h_i$, $\deg_i(v) = h_i$, $v$ is a Thomas involutive divisor of $w$, and, hence, $w \in C_T(V)$.

Furthermore, from Proposition 4.5 it follows that there is a set of $V_1 \subseteq V$ which is a Janet completion of $U$. 

Definition 4.6 Multiplication of a monomial $u \in U$ by a variable $x$ is called a prolongation of $u$. Given an involutive division specified by the set $U$, the prolongation is called multiplicative if $x$ is multiplicative for $u$ and non-multiplicative, otherwise.

In the construction of involutive sets the following concept of local involutivity plays the crucial role and admits the direct extension to polynomial sets (see Sect.6).

Definition 4.7 A set $U$ is called locally involutive with respect to the involutive division $L$ if any non-multiplicative prolongation of any element in $U$ has an involutive divisor in $U$, that is,

$$\forall u \in U \ (\forall x_i \in NM_L(u, U)) \ (\exists v \in U) \ [ v|L(u \cdot x_i) ] \tag{4}$$
In accordance with Definition 4.1, the conditions (4), apparently, are necessary for involutivity of $U$. Generally, however, they are not sufficient, as the next simple example shows.

**Example 4.8** Let $L$ be an involutive division on $M \subset K[x, y, z]$ defined by the table

| monomial | $M$ | $NM$ |
|----------|-----|------|
| 1        | $x, y, z$ | $-$ |
| $x$      | $x, z$  | $y$  |
| $y$      | $x, y$  | $z$  |
| $z$      | $y, z$  | $x$  |
| $u \in M$ | $\deg(u) \geq 2$ | $-$ | $x, y, z$ |

It is easy to see that all properties listed in Definition 3.1 (3.2) are satisfied, and the set $U = \{x, y, z\}$ is locally involutive. For instance, $x \cdot y = y \times x$. However, $U$ is not involutive since none $u \in M$ with $\deg_x(u) > 0, \deg_y(u) > 0, \deg_z(u) > 0$, e.g. $xyz$, has involutive divisors in $U$.

The following definition and theorem enable one to reveal involutive divisions providing involutivity of every locally involutive set.

**Definition 4.9** An involutive division $L$ will be called *continuous* if for any finite set $U$ and for any finite sequence $\{u_i\}_{1 \leq i \leq k}$ of elements in $U$ such that

$$\forall i < k \ (\exists x_j \in NM_L(u_i, U)) \ [ u_{i+1} \mid L u_i \cdot x_j ]$$

the inequality $u_i \neq u_j$ for $i \neq j$ holds.

**Theorem 4.10** If an involutive division $L$ is continuous then local involutivity of any set $U$ implies its involutivity.

**Proof** Let set $U$ be locally involutive, and such that any sequence in $U$ satisfying (3) has no coinciding elements. We must prove that $U$ satisfies (3). Take any $u \in U$ and any $w \in M$ and show that there is $v \in U$ such that $v \mid_L (uw)$. If $u \mid_L (uw)$ we are done. Otherwise, there is $x_{k_1} \in NM_L(u, U)$ such that $w$ contains $x_{k_1}$. Then $u \cdot x_{k_1}$ has an involutive divisor $v_1 \in U$. If $v_1 \mid_L (uw)$ we are done. Otherwise, there are $x_{k_2} \in NM_L(v_1, U)$ and $v_2 \in U$ such that $uw/v_1$ contains $x_{k_2}$ and $v_2 \mid_L (v_1 \cdot x_{k_2})$. Going on, we obtain the sequence $u, v_1, v_2, \ldots$ of elements in $U$ satisfying (3). By construction, each element of the sequence divides $uw$. Since all the elements are distinct and $uw$ has a finite number of distinct divisors, it follows that the above sequence in $U$ is finite, and, hence, it ends up with an involutive divisor of $uw$.

**Corollary 4.11** Thomas, Janet and Pommaret divisions are continuous.
Proof Let $U$ be a finite set, and $\{u_i\}_{1 \leq i \leq k}$ be a sequence of elements in $U$ satisfying the conditions (3). We shall show that there cannot be coinciding elements in the sequence for three divisions.

It is easy to see that $u_{i+1} | T_i (u_i \cdot x_{k_i}) \implies u_{i+1} = u_i \cdot x_{k_i}$. Indeed, suppose that $u_i \cdot x_{k_i} = u_{i+1} \times v_{i+1}$ which means $-x_{k_i} | u_{i+1}$. If $u_{i+1}$ would contain any other variable $x_{j_i}$, then it would mean that $\deg_{x_{j_i}} (u_i) > \deg_{x_{j_i}} (u_{i+1})$, and, hence, $x_{j_i}$ could not be multiplicative for $u_{i+1}$. Therefore, any Thomas sequence satisfying (3) consists of distinct elements.

If $u_{i+1} | T_i (u_i \cdot x_{k_i})$, then from definition of Janet division it follows that $u_{i+1} \succ_{\text{Lex}} u_i$, where $\succ_{\text{Lex}}$ is the lexicographical ordering corresponding to the choice of variable order $x_1 \succ x_2 \succ \cdots \succ x_n$ as assumed in Sect.2. It is now obvious that $u_i \neq u_j$ for $i \neq j$ for Janet division.

Let now $u_{i+1} | p_i (u_i \cdot x_{k_i})$. Then the representation (2) shows clearly that $u_{i+1} \succ_{\text{RevLex}} u_i$ where $\succ_{\text{RevLex}}$ is the reverse lexicographical ordering on $M$ induced by the assumed variable order.

With an eye to the below described algorithms based on examination of non-multiplicative prolongations only, we impose, in addition to continuity, one more requirement on an involutive division.

**Definition 4.12** We shall say that a continuous involutive division $L$ is **constructive** if for any $U \subset M$, $u \in U$, $x_i \in NML(u, U)$ such that $u \cdot x_i \not\in C_L(U)$ and

$$\forall v \in U \ (\forall x_j \in NML(v, U)) \ (v \cdot x_j | u \cdot x_i, \ v \cdot x_j \neq u \cdot x_i) \ [\ v \cdot x_j \in C_L(U)]$$

the following condition holds:

$$\forall w \in C_L(U) \ [u \cdot x_i \not\in wL(w, U \cup \{w\})].$$

**Proposition 4.13** Thomas, Janet and Pommaret divisions are constructive.

Proof Let $T$ be Thomas division. Suppose there is $u_1 \in U$, and $v \in T(u_1, U)$ such that $u \cdot x_i = u_1 v \times w$, $w \in T(u_1 v, U \cup \{u_1 v\})$. From Definition 3.3 it follows that if there exists $x_j | w$ and $-x_j | v$ for some $1 \leq j \leq n$, then $x_j \in M_T(u_1, U)$. This implies $w \in T(u_1, U)$ and $u \cdot x_i \in u_1 T(u_1, U)$.

Consider now Janet division $J$, and let $u \cdot x_i$ be a non-multiplicative prolongation which has no Janet divisors in $U$, and for which the condition (3) holds. Assume for a contradiction that there is $u_1 \in U$ and $v \in J(u_1, U)$ satisfying

$$u \cdot x_i = u_1 v \times w_1, \ w_1 \in J(u_1 v, U \cup \{u_1 v\}).$$

Because $v \neq 1$ and $w_1 \neq 1$, select minimal $j, m$ such that $x_j | v$ and $x_m | w_1$. It is easy to see that $i < \min \{j, m\}$. Otherwise, by Definition 3.4, we would have either
$x_j \not\in J(u_1, U)$ if $i \geq j$ or $x_m \not\in J(u_1v, U \cup \{u_1v\})$ if $i > m$. Note that the equality $i = m$ impossible since $U \cup \{u_1v\}$ as well as any other monomial set is Janet autoreduced. Thus, $u_1 \succ_{\text{Lex}} u$ where $\succ_{\text{Lex}}$ is the lexicographical ordering induced by the variable order $x_1 \succ \cdots \succ x_n$. If monomial $ux_i$ is obtained by non-multiplicative prolongations of several elements in $U$, then we suppose that $u$ is lexicographically maximal from all of them. Since $w_1$ is non-multiplicative for $u_1$, there is $x_{k_1}|w_1$ such that $x_{k_1} \in \text{NM}_I(u_1, U \cup \{u_1v\})$. Then, by condition (3), we can rewrite

$$u \cdot x_i = (u_1 \cdot x_{k_1}) \frac{vw_1}{x_{k_1}} = (u_2 \cdot x_{k_2}) \frac{vw_1}{x_{k_1}} = (u_2 \cdot x_{k_2}) \frac{vw_1w_2}{x_{k_1}x_{k_2}} = \cdots,$$

where $u \prec_{\text{Lex}} u_1 \prec_{\text{Lex}} u_2 \prec_{\text{Lex}} \cdots$. Continuity of Janet division implies termination of this chain with some $u_t \in U$ such that $u \cdot x_i \in u_tJ(u_t, U)$ what contradicts our assumption $u \cdot x_i \not\in C_J(U)$.

For Pommaret division condition (7) follows directly from the property (v) in Definition 3.1.

\[\Box\]

**Theorem 4.14** Let $U$ be a non-involutive finitely generated set with respect to a constructive division $L$. Then there is a procedure of completing $U$ to an $L$–involutive set $\tilde{U} \supset U$ based on enlargement of $U$ by non-multiplicative prolongations of its elements.

**Proof** Given $U$, by Definition 4.13, there exists a finite $L$–completion $\tilde{U}$ of $U$. We claim that $\tilde{U}$ contains some non-multiplicative prolongations of elements in $U$. Assume for a contradiction that there are no such elements in $\tilde{U}$. Since set $U$ is not involutive, there exist non-multiplicative prolongations of elements in $U$ which have no $L$–divisors in $U$.

Take any admissible ordering $\prec$ and select $u \in U$ with a non-multiplicative prolongation $u \cdot x_i$ which is not $L$–multiple of any element in $U$, and which is the lowest with respect to $\prec$. Because $\tilde{U}$ is involutive, and, by the above assumption, $u \cdot x_i \not\in \tilde{U}$, there is $v \in \tilde{U} \setminus U$ and $1 \prec w \in \mathbb{M}$ such that $u \cdot x_i = v \cdot w$, $w \in L(v, \tilde{U})$. From the condition $C(U) = C_L(\tilde{U})$ it follows that $v$ is multiple of some $u_1 \in U$ with $\text{deg}(u_1) < \text{deg}(v)$.

Show that $v \in C_L(U)$. If $u_1 L$–divides $v$, then we are done. Otherwise, there exists $x_{k_1}|(v/u_1)$, $x_{k_1} \in \text{NM}_L(u_1, U)$, and we can rewrite

$$v = u_1 \cdot \frac{v}{u_1} = (u_1 \cdot x_{k_1}) \frac{v}{u_1x_{k_1}} = (u_2 \cdot x_{k_2}) \frac{v}{u_1x_{k_1}} = (u_2 \cdot x_{k_2}) \frac{vw_2}{u_1x_{k_1}x_{k_2}} = \cdots$$

until, by continuity of $L$, we come to an involutive divisor $u_m \in U$ of $v$ at some step of this rewriting procedure. This contradicts the constructivity condition (7), and, hence $u \cdot x_i \in \tilde{U}$.

Now instead of $U$ take $U_1 = U \cup \{u \cdot x_i\}$ where $u_1 \in U$ and $u_1 \cdot x_i \in \tilde{U} \setminus U$. If set $U_1$ is not involutive, then it can be further completed by the corresponding lowest non-multiplicative prolongation in $U_1$. Since the set $\tilde{U}$ is finite, by repeating this
completion procedure, in a finite number of steps we construct the set \( \bar{U} \subseteq \tilde{U} \) which is an \( L \)-completion of \( U \).

As an immediate consequence of the above described procedure of completing a set \( U \) by non-multiplicative prolongations of its elements we have the following corollary.

**Corollary 4.15** If \( U \) is a finitely generated set with respect to a constructive involutive division, then there is the unique minimal involutive completion \( \bar{U} \) of \( U \) such that for any other completion \( \tilde{U} \) the inclusion \( U \subseteq \tilde{U} \) holds.

The following algorithm, given a constructive division \( L \), computes the minimal involutive completion \( \tilde{U} \) for any finitely generated set \( U \) and any fixed admissible ordering \( \prec \). Its correctness and termination are provided by Theorem 4.14.

**Algorithm InvolutiveCompletion:**

**Input:** \( U \), a finite monomial set  
**Output:** \( \bar{U} \), an involutive completion of \( U \)  
**begin**  
\( \bar{U} := U \)  
while exist \( u \in \bar{U} \) and \( x \in NM_L(u, \bar{U}) \) such that \( u \cdot x \) has no involutive divisors in \( \bar{U} \)  
do  
choose such \( u \) and \( x \) with the lowest \( u \cdot x \) w.r.t. \( \prec \)  
\( \bar{U} := \bar{U} \cup \{u \cdot x\} \)  
**end**

**Example 4.16** (Continuation of Example 3.11). The minimal involutive bases of the set \( U = (xy, y^2, z) \) \((x \succ y \succ z)\) for Thomas, Janet and Pommaret divisions are  
\( \bar{U}_T = \{xy, y^2, z, xz, yz, xy^2, xyz, y^2z, xy^2z\} \),  
\( \bar{U}_J = \{xy, y^2, z, xz, yz\} \),  
\( \bar{U}_P = \{xy, y^2, z, xz, yz, x^2y, x^2z, \ldots, x^ky, \ldots, x^mz, \ldots\} \),

where \( k, m \in \mathbb{N} \). These bases can be easily derived from \( U \) using algorithm InvolutiveCompletion. Note that \( \bar{U}_J \subset \bar{U}_T \) and \( \bar{U}_J \subset \bar{U}_P \) in agreement with Propositions 3.7 and 3.10. This example explicitly shows that Pommaret division is not noetherian. However, for another ordering \( z \succ y \succ x \) the set \( U \) is finitely generated, and then \( \bar{U}_P = U \).

## 5 Polynomial Reduction

In this section we generalize the results obtained in \([13, 14]\) for Pommaret division to arbitrary involutive division.
**Definition 5.1** Given a finite polynomial set \( F \subset \mathbb{R} \) and an admissible ordering \( \succ \), the concept of multiplicative and non-multiplicative variables for \( f \in F \) is to be defined in terms of \( \text{lm}(f) \) and the leading monomial set \( \text{lm}(F) \).

Therefore, as soon as we have polynomials rather than monomials, any involutive division is to be determined on the basis of some admissible ordering, even when it does not depend on the latter for the pure monomial case, as with Thomas division.

The concepts of involutive polynomial reduction and involutive normal form are introduced similar to their conventional analogues (Buchberger, 1985) with the use of involutive division instead of the conventional one.

**Definition 5.2** Let \( L \) be an involutive division \( L \) on \( M \), and let \( F \) be a finite set of polynomials. Then we shall say:

(i). \( p \) is \( L \)-reducible modulo \( f \in F \) if \( p \) has a term \( t = au \in T \) \( (a \neq 0) \) such that \( u = \text{lm}(f) \times v \), \( v \in L(\text{lm}(f), \text{lm}(F)) \). It yields the \( L \)-reduction \( p \rightarrow g = p - (a/\text{lc}(f))f \times v \).

(ii). \( p \) is \( L \)-reducible modulo \( F \) if there exists \( f \in F \) such that \( p \) is \( L \)-reducible modulo \( f \).

(iii). \( p \) is in \( L \)-normal form modulo \( F \) if \( p \) is not \( L \)-reducible modulo \( F \).

We denote an \( L \)-normal form of \( p \) modulo \( F \) by \( NF_L(p, F) \). In contrast, a conventional normal form will be denoted by \( NF(p, F) \). As an involutive normal form algorithm one can use, for example, the following:

**Algorithm** \texttt{InvolutiveNormalForm}:

**Input:** \( p \), \( F \)  
**Output:** \( h = NF_L(p, F) \)  
**begin**  
\( h := p \)  
**while** exist \( f \in F \) and a term \( u \) of \( h \) such that \( \text{lm}(f)|_L(u/cf(h, u)) \) do  
**choose** the first such \( f \)  
\( h := h - (u/\text{lt}(f))f \)  
**end**  
**end**

**Correctness** and **termination** of this algorithm can be proved, apparently, as they do for the conventional normal form algorithm [1-3]. Since involutive reductions form a fixed subset of the conventional ones, generally, \( NF_L(p, F) \neq NF(p, F) \).

**Definition 5.3** A set \( F \) is called \textit{involutively autoreduced} with respect to the given involutive division \( L \), or \( L \)-autoreduced, if the set \( \text{lm}(F) \) is \( L \)-autoreduced and every \( f \in F \) has no terms \( t = cf(f, t)u \neq \text{lt}(f) \) with \( cf(f, t) \neq 0 \) and \( u \in CL(\text{lm}(F)) \).
Given an involutive division $L$ and a finite set $F$, the following algorithm returns an $L$-autoreduced set $H$, denoted by $H = \text{Autoreduce}_L(F)$, and such that $\text{Id}(F) = \text{Id}(H)$.

Correctness of the algorithm is obvious from the while-loop structure. Since the underlying set of involutive interreductions is a subset of the conventional interreductions, its termination follows from that for the conventional autoreduction \cite{2,3}.

Algorithm InvolutiveAutoreduction:

**Input:** $F$

**Output:** $H = \text{Autoreduce}_L(F)$

begin

$H := F$

while exist $h \in H$ and $g \in H \setminus \{h\}$ such that $h$ is reducible modulo $g$ do

choose the first such $h$

$H' := H \setminus \{h\}$

$h' := NF_L(h, H)$

if $h' = 0$ then $H := H'$

else $H := H' \cup \{h'\}$

end

end

Theorem 5.4 If set $F$ is $L$-autoreduced, then $NF_L(p, F) = 0$ if and only if $p$ is presented in terms of a finite sum of the form

$$p \in S_F \subset \mathbb{R}, \quad S_F = \{ \sum_{ij} f_i \times u_{ij} \mid f_i \in F, u_{ij} \in \mathbb{T} \} \quad (8)$$

with $\text{lm}(u_{ij}) \neq \text{lm}(u_{ik})$ for $j \neq k$.

Proof $\implies$: If $NF_L(p, F) = 0$, then, by Definition 5.2 of involutive reductions, at each intermediate reduction step the current value $p'$ of $p$ is rewritten as $p' \rightarrow p'' = p' - f_i \times u_{ij}$. Since the reduction chain is finite by admissibility of an ordering $\succ$, the representation (8) holds.

$\impliedby$: Let $p$ is given by expression (8). Firstly, we show that $\text{lm}(p)$ has an involutive divisor in the set $\text{lm}(F)$. For this purpose select the leading term in the right hand side of (8). It has the form $s = \text{lt}(f_i \times u_{ij}) = \text{lt}(f_i) \times u_{ij}$ with some $i, j$ and cannot appear in any other term $\text{lt}(f_k) \times u_{kl}$. Otherwise, the underlying monomial $s/\text{lc}(s)$ would have two involutive divisors $\text{lm}(f_i)$ and $\text{lm}(f_k)$ what, by Proposition 3.3, would contradict the involutive autoreduction of $F$. Secondly, since $p$ is involutely reducible, after each reduction step the representation (8), obviously, still holds providing the further reductions until the chain stops when we obtain zero at a certain step. It just means that $NF_L(p, F) = 0$. \hfill \square
Corollary 5.5 If set $F$ is $L$–autoreduced, then the $L$–normal form, for an arbitrary algorithm of its computation and for any polynomials $p_1, p_2$ and $p$, has the properties:

(i). **Uniqueness:** if $h_1 = NF_L(p, F)$ and $h_2 = NF_L(p, F)$ then $h_1 = h_2$.

(ii). **Linearity:** $NF_L(p_1 + p_2, F) = NF_L(p_1, F) + NF_L(p_2, F)$.

**Proof** (i) By an involutive normal form algorithm, $h_1 = p - \sum_{ij} f_i \times u_{ij}$ and $h_2 = p - \sum_{ij} f_i \times v_{ij}$. Therefore, $h_1 - h_2$ has the representation (8), and $NF_L(h_1 - h_2, F) = 0$ by Theorem 5.4. On the other hand, since $h_1$ and $h_2$ are normal forms, they have no involutive divisors and so does $h_1 - h_2$. Hence, we have $h_1 = h_2$.

(ii) Denote $p_1 + p_2$ by $p_3$ and let

$$h_1 = NF_L(p_1, F), \quad h_2 = NF_L(p_2, F), \quad h_3 = NF_L(p_3, F).$$

Then $NF_L(h_3 - h_1 - h_2, F) = h_3 - h_1 - h_2$, since none of $h_1, h_2, h_3$ has involutive divisors in $lm(F)$. In addition, because $h_k = p_k - \sum_{ij} f_i \times v_{kij}$ ($k = 1, 2, 3$), we have $h_3 - h_1 - h_2 \in S_F$. Thus, by Theorem 5.4, $NF_L(h_3 - h_1 - h_2, F) = 0$, and, hence, $h_3 = h_1 + h_2$. \hfill \Box

### 6 Involutivity Conditions

**Definition 6.1** Multiplication of a polynomial $f \in F$ by a variable $x$ is called the **prolongation** of $f$. Given an involutive division specified by the set $lm(F)$, the prolongation is called **multiplicative** if $x$ is multiplicative for $lm(f)$, and **non-multiplicative**, otherwise.

**Definition 6.2** An $L$–autoreduced set $F$ is called $(L–$)involutive basis of $Id(F)$ if

$$\forall f \in F \ \forall u \in M \ [ NF_L(fu, F) = 0 ]. \quad (9)$$

**Proposition 6.3** Let $F$ be an involutive polynomial basis. Then the monomial set $lm(F)$ is also involutive.

**Proof** It follows immediately from Definitions 4.1, and 6.2 \hfill \Box

It is clear from Definition 6.2 and the linearity of the involutive normal form, by Corollary 5.5, that an involutive basis provides decision of the ideal membership problem. Hence, we have the following corollary.
Corollary 6.4 If set $F$ is $L$–involutive, then $p \in Id(F)$ if and only if $NF_L(p, F) = 0$. In this case, obviously, the equality $S_F = Id(F)$ holds.

The definition of involutive polynomial sets is the direct extension of that for involutive monomial sets in Sect.4. The theorem below imparts the constructive characterization of involutivity, which is the heart of the involutive algorithms.

Theorem 6.5 An $L$–autoreduced set $F$ is involutive with respect to a continuous involutive division $L$ if and only if the following conditions of local involutivity hold

$$\forall f \in F \ (\forall x_i \in NM_L(lm(f), lm(F))) \ [ NF_L(f \cdot x_i, F) = 0 ]. \ (10)$$

Proof $\implies$: Since $x_i \in \mathbb{M}$ we are done.

$\iff$: An immediate consequence of (10) is local involutivity of the set $lm(F)$ in accordance with Definition 4.7. Then, by continuity of division $L$, this set is involutive. Thus, for any $f \in F$ and any $u \in \mathbb{M}$ the monomial $lm(f) \cdot u$ has the involutive divisor $lm(g), g \in F$.

We claim that the polynomial $f \cdot u$ can be presented as follows

$$f \cdot u = g \times v + \sum_{ij} f_i v_{ij}, \ (11)$$

where $v, v_{ij} \in T, f_i \in F$ and relation $lm(f \cdot u) = lm(g \times v) \succ lm(f_i v_{ij})$ holds for any term of the sum. Indeed, if $u$ is multiplicative for $f$ we are trivially done. Otherwise $u$ contains $x_k \in NM_L(f, lm(F))$. Then, the local involutivity of $F$, by Theorem 5.4, yields the representation

$$f \cdot x_k = g_1 \times u_1 + \sum_{ij} f_i \times u_{ij} \ (12)$$

with $g_1 \in F$ and $lm(f \cdot x_k) = lm(g_1 u_1) \succ f_i u_{ij}$ for any term under the summation sign. If monomial $u/x_k$ is multiplicative for $g_1$, then (11) immediately follows from (12) with $g = g_1$ and $v = u_1 u/x_k$. Otherwise, multiply both sides of (12) by $u/x_k$, take a variable $x_m \in NM_L(g_1, lm(F))$, which is contained in $u/x_k$, and apply the local involutivity conditions for $g_1 \cdot x_m$. It gives the relation

$$f \cdot u = (g_2 \times u_2) u_1 u/(x_k x_m) + \sum_{ij} f_i \tilde{u}_{ij} \ (13)$$

where inequality $lm(g_2) uu_1 u_2/(x_k x_m) \succ lm(f_i \tilde{u}_{ij})$ holds for all $i, j$. If $uu_1/(x_k x_m)$ is still non-multiplicative for $g_2$ the relation (13) can be further rewritten by using the local involutivity conditions until we obtain relation (11). This is guaranteed by continuity of involutive division $L$, because all the polynomials $g_1, g_2, \ldots \in F$ are distinct, since their leading monomials, by construction, form the sequence satisfying (5).
Next, similar rewriting the every term $f_i v_{ij}$ in (11) gives $f_i v_{ij} = f_k \times w_k + \sum_{lm} f_l w_{lm}$ with $lm(f_i v_{ij}) = lm(f_k \times w_k) \succ lm(f_l w_{lm})$. Proceeding with this way, by admissibility of ordering $\prec$, we find, in a finite number of steps, that $f \cdot u \in S_F$. □

The next definition of partial involutivity is useful for the algorithmic construction of involutive bases as we show below.

**Definition 6.6** Given $v \in \mathbb{M}$ and an $L$–autoreduced set $F$, if there exist $f \in F$ such that $lm(f) \prec v$ and

$$(\forall f \in F) \ (\forall u \in \mathbb{M}) \ (lm(f) \cdot u \prec v) \ [ \ NF_L(fu, F) = 0 ] ,$$

(14)

then $F$ is called *partially involutive up to the monomial $v$* with respect to the admissible ordering $\prec$. $F$ is still said to be partially involutive up to $v$ if $v \prec lm(f)$ for all $f \in F$.

Looking at the proofs of Theorems 4.10 and 6.5 it is easy to see that they prove also the following *conditions of partial involutivity*.

**Corollary 6.7** Given a continuous involutive division $L$, an $L$–autoreduced set $F$ is partially involutive up to the monomial $v$ if and only if

$$(\forall f \in F) \ (\forall x_i \in NML(lm(f), lm(F))) \ (lm(f) \cdot x_i \prec v) \ [ \ NF_L(f \cdot x_i, F) = 0 ] .$$

(15)

### 7 Gröbner Bases and Involutive Bases

In [13, 14] it was shown that a *Pommaret basis*, that is, involutive basis for Pommaret division, is also a Gröbner basis, though, generally, not the reduced one. A similar property of a Janet basis was noticed in [17]. The following theorem shows that such a relation holds for any involutive division.

**Theorem 7.1** If set $F$ is $L$–involutive, then the equality of the conventional and $L$–normal forms

$$(\forall p \in \mathbb{R}) \ [ \ NF(p, F) = NF_L(p, F) ]$$

holds for any normal form algorithm.

**Proof** To prove the theorem it is sufficient to show that any polynomial $p$ is reducible modulo $F$ if and only if it is involutively reducible. But the latter statement is an easy consequence of Definitions 3.1 or 3.2 and 6.2. Indeed, if $p$ is involutively reducible, then it is conventionally reducible. Conversely, let the term $u$ have a divisor among the leading monomials of $F$, that is, $u = lc(u) \cdot lm(f) \cdot v$ for some $f \in F$ and $v \in \mathbb{M}$. By the condition (9) and Theorem 5.4, it implies $f \cdot v = \sum_{ij} f_i \times u_{ij}$. Hence, $u$ has also the involutive divisor in $lm(F)$. It is just that $f_i$ which satisfies the condition $lm(f_i) \times u_{ij} = lm(f) \cdot v$ and is unique. □
Corollary 7.2 An involutive basis is a Gröbner basis.

Proof According to the algorithmic characterization of Gröbner bases [1, 2, 3] consider the S-polynomial of \( f_i, f_j \in F \)

\[
S(f_i, f_j) = \frac{lcm(f_i, f_j)}{lt(f_i)} f_i - \frac{lcm(f_i, f_j)}{lt(f_j)} f_j.
\]  

(17)

From \( S(f_i, f_j) \in Id(F) \), Corollary 6.4 and Theorem 7.1, we have \( NF(S(f_i, f_j), F) = 0 \).

Corollary 7.3 If set \( F \) is partially involutive up to the monomial \( v \), then

\[
(\forall p \in \mathbb{R}) \ (lm(p) \prec v) \ [ \ NF(p, F) = NF_L(p, F) ].
\]  

(18)

Proof It follows by perfect analogy to the proof of Theorem 7.1. 

Note that while a Pommaret basis, if it exists for the given ideal, is unique [14], this may not hold for other involutive divisions. We demonstrate it by the following explicit example.

Example 7.4 Two lexicographical \((x \succ y)\) Janet bases \( F_1 \) and \( F_2 \)

\[
F_1 = \{ xy^3 - y, xy^2 - 1, xy - y^2, x - y, y^3 - 1 \},
\]

\[
F_2 = \{ x^2 y^3 - y^2, x^2 y^2 - y, x^2 y - 1, x^2 - y^2, xy^3 - y, xy^2 - 1, xy - y^2, x - y, y^3 - 1 \},
\]

with indicated non-multiplicative variables, are involutive. It can easily be verified. Both of them generate, obviously, the same ideal with the Gröbner basis \((x - y, y^3 - 1)\), which is also a Janet basis and, in this particular case, coincides with the Pommaret basis.

As it was shown in Sect. 4, given a polynomial set \( F \) and an arbitrary involutive division, the ideal \( Id(F) \) may not have a finite involutive basis. For example, while a finite Pommaret basis exists for any zero-dimensional ideal [11, 14, 18], it may not exist for a positive dimensional one. Generally, for positive dimensional ideals, the existence of finite Pommaret basis can be achieved by means of an appropriate linear transformation of variables [11, 18].

On the other hand, a noetherian involutive division, for example, a Thomas or Janet one, implies the existence of finite involutive bases for any polynomial ideals as the following proposition shows.
Proposition 7.5 If involutive division \( L \) is noetherian, then any polynomial ideal \( \text{Id}(F) \) has a finite \( L \)-involutive basis.

Proof Let \( G \) be the reduced Gröbner basis of \( \text{Id}(F) \) which is finite for any polynomial ideal \([3, 3]\). If set \( G \) is not involutive, then complete it by non-multiplicative prolongations of its elements just as it done in algorithm InvolutiveCompletion. This means that at every step of the completion we select a non-multiplicative prolongation with the lowest leading term which is \( L \)-irreducible modulo the current leading monomial set. By noetherity of \( L \), in a finite number of steps, a polynomial set \( \tilde{G} \) will be produced such that \( \text{lm}(\tilde{G}) \) be an \( L \)-autoreduced involutive completion of \( \text{lm}(G) \). Finally, \( L \)-autoreduction of the tales in \( \tilde{G} \) will give an \( L \)-involutive basis of \( \text{Id}(F) \).

8 Basic Algorithm

In this section we describe an algorithm for the construction of an involutive basis. The algorithm is an improved version of one presented in \([14]\) for Pommaret division and generalized to any continuous noetherian division \( L \) and any admissible ordering \( \succ \). The main optimization is based on the use of Buchberger’s chain criterion for avoiding unnecessary reductions introduced in \([16]\) (see also \([2, 3]\)).

Corollary 7.3 shows that for any S-polynomial \( S(f_i, f_j) \), given by formula \([17]\), both its conventional and \( L \)-normal forms are vanishing as soon as the conditions \([15]\) are satisfied up to the monomial \( \text{lcm}(f_i, f_j) \). According to Theorem 5.4 and Corollary 5.5 the conditions \([15]\) can be presented as \( \text{NF}_L(S_L(f_i, f_j), F) = 0 \), where \( S_L(f_i, f_j) \) are just (\( L \)-involutive) S-polynomials of the special form

\[
S_L(f_i, f_j) = f_i \cdot x - f_j \times u_{jk}.
\]  

The following theorem gives the involutive form of Buchberger’s chain criterion.

Theorem 8.1 Let \( F \) be a finite \( L \)-autoreduced polynomial set, and let \( g \cdot x \) be a non-multiplicative prolongation of \( g \in F \). Then \( \text{NF}_L(g \cdot x, F) = 0 \) if the following holds

\[
(\forall h \in F) \ (\forall u \in M) \ (\text{lm}(h) \cdot u \prec \text{lm}(g \cdot x)) \ [ \ \text{NF}_L(h \cdot u, F) = 0 \],
\]  

\[
(\exists f, f_0, g_0 \in F) \left[ \begin{array}{c}
\text{lm}(f_0) | \text{lm}(f) \\
\text{lm}(g_0) | \text{lm}(g)
\end{array} \right] \left[ \begin{array}{c}
\text{lm}(f)_L | \text{lm}(g \cdot x) \\
\text{lm}(f)_L | \text{lm}(g \cdot x)
\end{array} \right] \left[ \begin{array}{c}
\text{lm}(f_0) \cdot \text{lt}(f_0, F) \\
\text{lm}(g_0) \cdot \text{lt}(g_0, F)
\end{array} \right] = 0.
\]  

Proof Condition \([21]\) yields that at least one of polynomials \( f, g \) can be considered as derived from \( f_0, g_0 \) by prolongations with at least one non-multiplicative among them. If, for example, \( \text{lm}(f_0) \neq \text{lm}(f) \), it leads to the equality \( f = f_0 \cdot (\text{lm}(f) / \text{lm}(f_0)) \) modulo \( F \).
Thus, if the condition \((21)\) holds, there is a chain of polynomials in \(F\) of the form
\[
f \equiv f_k, f_{k-1}, \ldots, f_0, g_0, \ldots, g_{m-1}, g_m \equiv g,
\]
where \(k + m > 0\). Here \(f\) or \(g\) or both of them are produced by prolongations, including non-multiplicative ones, of the polynomials \(f_i\) or \(g_j\) in the chain whose indices are less than \(k\) or \(m\), respectively.

The chain \((22)\) has the property
\[
NF(S_L(f, f_{k-1}), F) = \cdots = NF(S(f_0, g_0), F) = \cdots = NF(S_L(g_{m-1}, g), F) = 0.
\]
This property is resulted from the observations as follow. Consider relation
\[
lm(g) \cdot x = \lm(f) \times w,
\]
which means that \(w\) does not contain \(x\). Otherwise, \(g\) would be reducible by \(f\), and, hence, \(F\) could not be \(L\)-autoreduced. Thus, \(\text{lcm}(f, g) = \lm(g) \cdot x\). By admissibility of the monomial ordering \(\prec\), the least common multiple of the leading monomials for pair of the neighboring polynomials in the chain \((22)\) is less than or equal to \(g \cdot x\). Then the above property of the chain follows immediately from partial involutivity \((20)\) of \(F\) and Corollary \((7.3)\). Furthermore, conditions \((20-21)\) imply \(NF_L(S(f_0, g_0), F) = NF(S(f_0, g_0), F) = 0\), and \(NF_L(S(g_{i-1}, g_i), F) = NF(S(g_{i-1}, g_i), F) = 0\) \((1 \leq i \leq k)\) as well as \(NF_L(S_L(g_{i-1}, g_i), F) = NF(S(g_{i-1}, g_i), F) = 0\) \((1 \leq i \leq m)\).

By construction, \(\text{lcm}(f_1, f_0, g_0, g_1, \ldots, g) = \text{lcm}(f, g)\) what leads \((9)\) to the representation \(S(f, g) = \sum_{ij} f_i u_{ij}\) where \(f_i \in F\) and \(\lm(f_i u_{ij}) \prec \text{lcm}(f, g) = \lm(g) \cdot x\). Then, condition \((20)\), by Corollaries \((5.3)\) and \((7.3)\), yields
\[
NF_L(S_L(f, g), F) = NF(S(f, g), F) = 0
\]
in accordance with \((4), (9)\).

Before analysis of correctness and termination of the below algorithm, we give some necessary clarifications.

First of all, the conventional autoreduction of the initial polynomial set is done. It removes, in particular, all the predecessors of every polynomial from the initial set.

Set \(T\) collects all the triples \((g, u, P)\); \(g\) is an element in the current basis \(G\); \(u = \lm(f)\) where \(f \in G\) is the predecessor of \(g\), by a non-multiplicative prolongation of which \(g\) was derived, or \(u = \lm(g)\) if \(g\) has no such predecessor in \(G\); \(P\) is a set containing the non-multiplicative variables of \(g\) have been used for its prolongations.

The current non-multiplicative prolongation \(g \cdot x\) is selected to be the lowest with respect to the ordering \(\prec\). If there are several different non-multiplicative prolongations with the same leading term, then any of them may be selected. This selection strategy will be called normal.

If the leading monomial of the current prolongation \(g \cdot x\) is involutively reducible by the basis element \(f \in G\), then the other conditions in \((21)\) are verified. The verification
is done in the form of comparison of \( \text{lcm}(u, v) \) with \( \text{lcm}(f, g) \), where \( u \) and \( v \) are the second elements of the triples containing \( g \) and \( f \), respectively. By Theorem 8.1, the criterion (21) is false if and only if \( \text{lcm}(u, v) = \text{lcm}(f, g) = g \cdot x \). One should be also noted that Buchberger’s second criterion [2] can be applied in the involutive approach only in exceptional cases. Relation (23) shows that \( \text{lcm}(f, g) = \text{lm}(f)\text{lm}(g) \) if and only if \( \text{lm}(f) = x \) and \( \text{lm}(g) = w \).

If the current prolongation is not reducible to zero, that is, \( h = NF_L(g \cdot x, G) \neq 0 \), then \( h \) is added to \( G \).

After involutive autoreduction of the enlarged set \( G \) an adjustment of the set \( T \) is done. For an element \( g \in G \) whose leading monomials was not mutually reduced, the second element \( u \) in the triple is kept, if the leading term of the corresponding predecessor of \( g \) was also not reduced. Otherwise, \( u \) is replaced by its involutive divisor in \( \text{lm}(G) \). Essentially new leading monomials, that is, those not multiple of any others occurring in \( T \) before the autoreduction, are included in the refreshed \( T \) with their actual leading monomials as the second elements of the triples.

Algorithm \textbf{InvolutiveBasis}:

\textbf{Input}: \( F \), a finite polynomial set  
\textbf{Output}: \( G \), an involutive basis of the ideal \( \text{Id}(F) \)

\begin{verbatim}
begin  
  G := Autoreduce(F)  
  T := {}  
  for each \( g \in G \) do  
    T := T \cup \{(g, \text{lm}(g), \emptyset)\}  
    while exist \((g, u, P)\) \in T and \( x \in \text{NM}_L(\text{lm}(g), \text{lm}(G))\) do  
      choose such \((g, u, P)\) and \( x \) with the lowest \( \text{lm}(g) \cdot x \)  
      T := T \{\{(g, u, P)\} \cup \{(g, u, P \cup \{x\}\}\}  
      if exist \((f, v, D)\) \in T such that \( \text{lm}(f) \mid L \text{lm}(g \cdot x) \) then  
        if \( \text{lcm}(u, v) = \text{lm}(g) \cdot x \) then \( h := NF_L(g \cdot x, G) \)  
          if \( h \neq 0 \) then T := T \cup \{(h, \text{lm}(h), \emptyset)\}  
        else \( h := NF_L(g \cdot x, G) \)  
      T := T \cup \{(h, u, \emptyset)\}  
      G := Autoreduce_L(G \cup \{h\})  
    end  
    Q := T  
    T := {}  
    for each \( g \in G \) do  
      if exist \((f, u, P)\) \in Q such that \( \text{lm}(f) = \text{lm}(g) \) then  
        choose \( g_1 \in G \) such that \( \text{lm}(g_1) \mid L u \)  
        T := T \cup \{(g, \text{lm}(g_1), P)\}  
      else T := T \cup \{(g, \text{lm}(g), \emptyset)\}  
    end  
end  
\end{verbatim}

\textit{Correctness.} As we have shown, criterion (21) is used in algorithm \textbf{InvolutiveBasis}
in accordance with Theorem 8.1. It is easy to show that there is the unique polynomial \( g_1 \in G \) which is chosen in the inner for each-loop such that \( \text{lm}(g_1) \) involutively divides \( u \). Indeed, if the leading term of the predecessor \( h \) of \( g \) with \( u = \text{lm}(h) \) has not been reduced, then \( g_1 = h \). Otherwise, there is \( g_1 \in G \) such that \( g_1 \neq h \) and \( \text{lm}(g_1) \mid _{L} u \). The uniqueness of \( g_1 \) for the autoreduced set \( G \) is an immediate consequence of the property (v) in Definition 3.1. Besides, the replacement of \( u \) by \( g_1 \) does not violate, obviously, the conditions for applicability of the criterion. Furthermore, from Corollary 7.3 it follows:

(i) a leading monomial, being involutively reducible at some step of the algorithm, will never appear again among the leading monomials; (ii) there is no need in recomputing zero reductions after enlargement of an intermediate polynomial set. This enables one to assign the set \( \tilde{P} \) of the used non-multiplicative variables for polynomial \( f \) to the corresponding polynomial \( g \) with \( \text{lm}(g) = \text{lm}(f) \) as it is done in the inner for each-loop. Such an optimization allows one to avoid the repeated prolongations.

Therefore, if division \( L \) is continuous, and the algorithm terminates, then it produces, by Theorem 3.3, the involutive basis. The termination holds if and only if the set \( \tilde{P} \) in each triple \( (g, u, P) \in T \) contains all non-multiplicative variables for basis element \( g \). It just means that any non-multiplicative prolongation of every element in \( G \) is reduced to zero, and, hence, \( G \) is involutive.

**Termination.** Note that the initial value of the leading monomial set

\[
U_0 = \text{lm} \left( \text{Autoreduce}(F) \right)
\]

is determined by the input set \( F \) subjected to the conventional autoreduction. Since only those monomials occur in the leading monomial set which have not been reducible at some step of the algorithm, the change in set \( U = \text{lm}(G) \) after running the while-loop may take place only in two cases:

(i). \( \text{lm}(g) \cdot x \) has no involutive divisors in \( U \). In this case \( U \) is enlarged to include \( \text{lm}(g) \cdot x \).

(ii). \( g \cdot x \) is reducible by elements of \( U \). Then \( U \) is enlarged to include \( \text{lm}(h) \), where \( h = NF_L(g \cdot x, G) \neq 0 \) and \( \text{lm}(h) \) is not multiple, in the conventional sense, of any elements in \( U_0 \).

The number of different \( \text{lm}(h) \) occurring in case (ii) is finite by Dickson’s lemma (Becker, Weispfenning and Kredel, 1993). Recall also that algorithms InvolutiveAutoreduction and InvolutiveNormalForm always terminate (Sect.5).

Thus, the algorithm termination is determined by that of algorithm InvolutiveCompletion considered in Sect.4. It follows that algorithm InvolutiveBasis terminates for any noetherian division and arbitrary input polynomial set \( F \). If \( L \) is not noetherian, then termination may not hold if an intermediate set \( U = \text{lm}(G) \) is not finitely generated with respect to \( L \) as the below Example 8.2 shows. In the case of Pommaret division the algorithm terminates, however, for any degree compatible ordering and any zero-dimensional ideal \([14]\). Because the involutive division \( L \) is continuous, once algorithm InvolutiveCompletion terminates, an \( L \)-completion \( \tilde{U} \) of
$U$ will be constructed such that autoreduction of the corresponding set $G$ does not produce new leading monomials. $G$ is, obviously, the output involutive basis.

Proposition [13] implies, in particular, the algorithm termination for Thomas and Janet divisions. However, for Pommaret division, which is not noetherian, the algorithm may not terminate even in the case when there is a finite Pommaret basis but the ordering is not degree compatible as the following simple example shows.

**Example 8.2** The set $F = \{x^2 - 1, xy - 1, z\}$ generates a zero-dimensional ideal with the lexicographical Pommaret basis $(x \succ y \succ z)$ given by $G = \{x - y, y^2 - 1, yz, z\}$. However, following the above algorithm we have to choose $z \cdot y$ as the first prolongation which is lexicographically lowest. Since polynomial $h = yz$ has no Pommaret divisors among $\text{lm}(F)$, we find $F \cup \{yz\}$ as an intermediate basis. The next lowest prolongation is $yz \cdot y$ again has no Pommaret divisors among the leading monomials of the enlarged set. Exploring this procedure further produces the infinite involutively irreducible set

$$\{x^2 - 1, xy - 1, z, yz, y^2z, \ldots, y^kz, \ldots\} \quad k \in \mathbb{N}.$$

It is well-known [11, 13, 14, 18] that positive dimensional ideals may not have finite Pommaret bases. Example [4,16] illustrates this fact at the monomial level. The following more non-trivial example shows the output of algorithm `InvolutiveBasis` for Pommaret and Janet divisions in the case of polynomial ideal.

**Example 8.3** Cyclic 4-th roots.

| $NM_J$ | $NM_P$ | Initial Polynomial Set |
|--------|--------|------------------------|
| $x_2$  | $x_1$  | $x_1 + x_2 + x_3 + x_4$ |
| $x_3$  | $x_1$  | $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$ |
| $x_4$  | $x_1, x_2$ | $x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2$ |
| $x_1, x_2, x_3$ | $x_1x_2x_3x_4 - 1$ |

Here we choose the degree-reverse-lexicographical-ordering with the order of variables as in Sect.2. Note that, since the initial set is not autoreduced, the inclusion $NM_J \subseteq NM_P$ (see Proposition [3.10]) does not hold.

Application of algorithm `InvolutiveBasis` gives the following form of Janet and Pommaret bases
The Janet basis consists of the upper seven polynomials and coincides with the Gröbner basis, while the Pommaret basis is infinite and contains also prolongations of the seventh polynomial with respect to its non-multiplicative variable \( x_3 \). Note that the ideal is one-dimensional, that is why it does not have a finite Pommaret basis.

The algorithm \textbf{InvolutiveBasis} has been implemented in Reduce 3.5 for the degree-reverse-lexicographical-ordering and Pommaret division refined in a certain way to provide the algorithm termination for any polynomial ideal. This refinement is equivalent to the dynamical incorporation of some noetherian involutive division in the computational process. Its detailed description will be given elsewhere. In addition, the current package called INVBASE is considerably faster than previous version \([14]\), in particular, since it uses the criterion \([21]\).

Experimentally, we observed much smoother behavior of the algorithm \textbf{Involutive Basis} with respect to Buchberger algorithm\footnote{More precisely, with respect to its implementation in Reduce 3.5.} as the ordering changes. Consider, for instance, the following example.

\textbf{Example 8.4} Cyclic 6-th roots.

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6, \\
x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_1, \\
x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_1 + x_6x_1x_2, \\
x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + x_4x_5x_6x_1 + x_5x_6x_1x_2 + x_6x_1x_2x_3, \\
x_1x_2x_3x_4x_5 + x_2x_3x_4x_5x_6 + x_3x_4x_5x_6x_1 + x_4x_5x_6x_1x_2 + x_5x_6x_1x_2x_3 + x_6x_1x_2x_3x_4, \\
x_1x_2x_3x_4x_5x_6 - 1. 
\end{align*}
\]

The next table gives the timings of INVBASE on an 66 Mhz MS-DOS based AT/486 computer for different degree-reverse-lexicographical-orderings.
Comparison with the package GROEBNER implementing Buchberger algorithm on
the same Reduce 3.5 platform shows that its corresponding timings are not only much
larger than those presented in the table, but also vary dramatically with the order of
the variables. This fact was already observed in [14] where some comparative data for
GROEBNER and the previous version of the INVBASE package are presented.

9 Conclusion

Buchberger algorithm and the involutive one are based on different rewriting tech-
niques, namely, on the use of S-polynomials and prolongations, respectively, as well as
on distinct reduction processes. Nevertheless, as we demonstrate in this paper, they
are in fact very interconnected. If, as we propose in the algorithm InvolutiveBasis,
we choose the current prolongation in increasing order with respect to given monomial
ordering, then the conventional and involutive normal form will coincide. What is
more, the involutive reduction of the prolongation is equivalent to the consideration of
a certain S-polynomial. Just this fact makes it possible to use Buchberger’s criteria.

Recently another interesting facet of interrelation of both methods was discovered
by Apel [15], namely, that Pommaret bases can be associated with Gröbner ones in
appropriate graded structures. Earlier such Gröbner bases were intensively investi-
gated in more general context by Mora [1]. That observation gives an opportunity to
algorithmically construct Pommaret bases whenever they exist [18]. Though such an
analogy also enables one to take advantage of Buchberger’s criteria, it is restricted to
Pommaret division.

Thus, all the above, as well as computer experiments with both techniques, offers
a clearer view of the most optimal computational procedures.

There is no question that any algorithmic improvement of the Gröbner basis and
involutive techniques at the algebraic level has an analogous optimization at the dif-
erential level, at least for linear partial differential equations [2].
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