Dynamic Model and Equilibrium Stability of an Inverted Double Pendulum System

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Abstract. Most inverted pendulum types, such as single or double pendulums, translational or rotational, are underactuated, nonlinear, and unstable. To handle this inverted pendulum, much effort has to be done, especially in stabilization aspects. An appropriate control strategy based on suitable modeling is the key to stabilization’s handling. Modeling and stabilizing an inverted double pendulum need more complex efforts than those of a single type. The obtained model is indispensable for control design in future works, and the success of the method depends on the accuracy of the developed model. The result showed that the dynamic model is fit enough, and the eigenvalue of the system is unstable, as predicted earlier.

1. Introduction

Most inverted pendulum types like single or double pendulums, translational or rotational pendulums are underactuated, nonlinear, and open-loop unstable. Therefore, research topics on dynamics inverted pendulum are very challenging, and it is proven that they have been founded in many reports on scheme control theory and implementations. Notably, a mathematical model of an inverted pendulum can be likened to many other unrelated nonlinear physical systems like a synchronous generator model, phase-locked loop model, and so on [1]. The crucial issue in inverted pendulum applications is system stability. To handle these problems, much effort has to be done, especially in stabilization aspects. An appropriate control strategy that must be chosen based on suitable modeling is the key to stabilization’s handling. Before deciding on a control strategy, an accurate model should be obtained to avoid loss of control implementation both in simulation and realization. Modeling and stabilizing an inverted double pendulum need more complex efforts than those of a single type. The obtained model is indispensable for control design in future works, and the success of the method depends on the accuracy of the developed model.

This paper describes an inverted double pendulum’s modeling and stability, which certainly has more complexity than a single pendulum. To mention research reports on the control method and the realization of a single inverted pendulum moved by a cart, we can see the most straightforward inverted pendulum [2, 3]. According to potential and kinetic energies, the dynamic model’s derivation is often based on the Lagrange equation. In this paper, a mathematical model’s realization via SimMechanics is presented to analyze stability and its control design in future works. Stability analysis for nonlinear is ensured with Jacobian matrix in the sense of Lyapunov [1]. References and tutorial videos on physical models with SimMechanics Second Generation are available freely. One of them is in [4].
2. Dynamic Model

To derive a mathematical model of an inverted double pendulum, one can start by deriving a dual pendulum model as its inner and outer rod are downward. Figure 1 is presented to illustrate the dynamics of the system. Physical parameter data and their visualization are shown in Fig 2 and listed in Table 1. An inverted pendulum is in an upright position. It needs a control method to move clockwise or counterclockwise until reaching around the equilibrium vertical position. The objective of the control solution for swinging up is to bring a rod or link in part called a homoclinic orbit ($\theta_i = 2\pi n$, and $\dot{\theta}_i = 0$) [5,6]. The method applied in this research is not only dynamic but also needs to be considered. The following researchers [17–19] have discussed the stability test in their study. According to Figure 1 with parameter data in Table 1, one can derive a double pendulum’s dynamics in the following equations [7]. At inner pendulum rod $L_1$, the dynamic in the trigonometric sense is

$$x_1 = \frac{L_1}{2}\sin\theta_1, \quad y_1 = -\frac{L_1}{2}\cos\theta_1$$

Trigonometric equation for outer pendulum rod $L_2$ is

$$x_2 = x_1 + L_2\sin\theta_2, \quad y_2 = y_1 - L_2\cos\theta_2$$

$$x_2 = \frac{L_1}{2}\sin\theta_1 + L_2\sin\theta_2, \quad y_2 = -\frac{L_1}{2}\cos\theta_1 - L_2\cos\theta_2$$

Total kinetic energy $T$ for inner and outer pendulum rods is

$$T = \frac{1}{2}m_1(v_1^2 + v_y^2) + \frac{1}{2}m_2(v_2^2 + v_y^2)$$

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2}m_1[(\frac{L_1}{2}\sin\dot{\theta}_1)^2 + (\frac{L_1}{2}\cos\dot{\theta}_1)^2] + \frac{1}{2}m_2[(\frac{L_1}{2}\sin\dot{\theta}_1 + L_2\sin\dot{\theta}_2)^2 + (\frac{L_1}{2}\cos\dot{\theta}_1 + L_2\cos\dot{\theta}_2)^2]$$

Total potential energy $V$ for inner and outer pendulum rods is as follows

$$V = m_1gy_1 + m_2gy_2 = -g[m_1\frac{L_1}{2}\cos\theta_1 + m_2\frac{L_1}{2}\cos\theta_1 + m_2L_2\cos\theta_2]$$

The Lagrangian $L$ is the total energy of kinetic and potential for both inner and outer pendulum rod

$$L = T - V$$

Then the dynamics of an inverted double pendulum is written based on the Euler Lagrange equation, and for an inner pendulum with external input force $\tau$ applies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = \tau$$

The external force $\tau$ serves to rotate the inner pendulum to lead both inner and outer rods to an upright vertical position. The Euler Lagrange of an outer pendulum, since there is no external force, is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0$$
By substituting (1)-(4) to (5) and with some mathematic manipulations, we have (6) and (7) in these detailed equations

\[
\left( m_1 + m_2 \right) \frac{L_1}{2} \ddot{\theta}_1 + m_2 L_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin \theta_1 = \tau
\]  

(8)

\[
m_2 L_2 \ddot{\theta}_2 + m_2 \frac{L_1}{2} \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 \frac{L_1}{2} \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin \theta_2 = 0
\]  

(9)

Equations (8) and (9) can be written in a preferred matrix type as follows:

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + h(q) = F
\]

where

\[
M(q) = \begin{bmatrix}
(m_1 + m_2) \frac{L_1}{2} & m_2 L_2 \cos(\theta_1 - \theta_2) \\
m_2 \frac{L_1}{2} \cos(\theta_1 - \theta_2) & m_2 L_2
\end{bmatrix}
\]

\[
C(q, \dot{q}) = \begin{bmatrix}
0 & m_2 L_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\
-m_2 \frac{L_1}{2} \dot{\theta}_1 \sin(\theta_1 - \theta_2) & 0
\end{bmatrix}
\]

\[
h(q) = \begin{bmatrix}
(m_1 + m_2) g \sin \theta_1 \\
-m_2 g \sin \theta_2
\end{bmatrix}
\]

\[
F = \begin{bmatrix}
\tau \\
0
\end{bmatrix}, \quad q = \begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
\]

(10)

The trigonometric function of the inner pendulum angular \( \theta_1 \) and outer pendulum angular \( \theta_2 \) leads equations (8)-(9) to a nonlinear model.

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**Figure 1.** An inverted double pendulum trigonometric construction  
**Figure 2.** Visualization of an inverted double pendulum in SimMechanics

State trajectories of a nonlinear system can be computed and simulated scientifically with Matlab [8]. Based on Table 1, with the unit step input \( \tau \), the responses are shown in Figure 3. It can be seen that the system response angular position \( \theta_1, \theta_2 \) and angular velocity \( \dot{\theta}_1, \dot{\theta}_2 \) are unstable.

To simulate two pendulum rods’ motion, we can build a mathematical model using SimMechanics Second Generation, displayed in Figure 4. Furthermore, the trajectory of
pendulum angle $\theta_1$ and pendulum angle $\theta_2$ for the system without input, where the double pendulum's motion only depends on the force of gravity has been seen in Figure 6.

Furthermore, we can verify the physical model realized with SimMechanics whether it fits the mathematical model enough or not, we can compare it with the conservative model via Simulink. Researchers widely use Simulink to get a more real algorithmic result in graphs and numbers [9–15]. In the current New Era, Simulink can be applied to obtain the best algorithm for the introduction of Covid-19. The rotational force of the pendulum is related to the kinetic energy:

$$T = J\ddot{\theta}_1$$ and $$J = \frac{1}{2}m_1L_1^2$$ (11)

### Table 1: Parameters in an inverted double pendulum

| Parameters       | Value          |
|------------------|----------------|
| inner pendulum   |                |
| Length $L_1$     | 24 cm          |
| Cylinder radius $r_1$ | 0.707 cm  |
| Density $\rho_1$ | 2700 kg/m$^3$ |
| Angular $\theta_1$ |            |
| outer pendulum   |                |
| Length $L_2$     | 48 cm          |
| Cylinder radius $r_2$ | 0.25 cm  |
| Density $\rho_2$ | 10 kg/m$^3$   |
| Angular $\theta_2$ |            |
| sphere           |                |
| Cylinder radius $r_3$ | 3 cm        |
| Density $\rho_3$ | 5 kg/m$^3$    |

The translational force is related to the potential energy in the inner pendulum:

$$V = -m_1gx_1 = -m_1gL_1 \sin \theta_1$$ (12)

The conservation concept of kinetic and potential energies is conservatively formulated as $T = E - V = 0$, yields

$$\dot{\theta}_1 = \frac{3}{2}g\frac{L_1}{L_1} \sin \theta_1$$ (13)

The diagram block of the conservative model (13) is shown in Figure 5. The results of this concept can be depicted in Figure 6, and it can be seen that the angular trajectory has similar to the response from SimMechanics.

3. Stability

Stability problems often arise in many studies of dynamics systems. The stability types are equilibrium point, input-output strength, and stability of periodic orbits [1]. In the case of an inverted pendulum, stability of equilibrium point is mainly concerned in the sense of Lyapunov that gives a sufficient condition. To ensure the next state of the system $\dot{x} = f(x,t)$ with initial condition $x(t_0) = x_0$ has a unique solution, this inequality must be fulfilled

$$\|f(t, y) - f(t, x)\| \leq L\|y - x\|$$ (14)

which are $(t, x)$ and $(t, y)$ is nearby of $(t_0, x_0)$ And $L$ is Lipschitz constant. This Lipschitz can be obtained from Jacobian matrix $\partial f/\partial x$ of dynamics system based on the following lemma.

**Lemma 1**
Let $f : [a, b] \times D \to \mathbb{R}^m$ be continuous for some dimensions $D \to \mathbb{R}^n$. If $\partial f/\partial x$ exists and is continuous in $[a, b] \times D$, and a subset convex $W \subset D$ there is $L \geq 0$ such that $\|\partial f/\partial x(t, x)\| \leq L$ on $[a, b] \times W$ then equation (1) is fulfilled $\forall t \in [a, b]; x, y \in W$.

Jacobian matrix $\partial f/\partial x$ can be used to evaluate the equilibrium point whether the system is in stable, unstable, or saddle point nodes. The nonlinear system $\dot{x} = f(x, t)$ in the system of an
inverted double pendulum (8), (9), (10) is:

\[
\dot{x} = M^{-1}(F - C(q, \dot{q})\dot{q} - h(q))
\]

(15)

As an inverted double pendulum rod position is around an upright equilibrium, i.e., \(t=0.905\) second, we obtained measured data \(\theta_1 = 128^0, \dot{\theta}_1 = 0.899\text{ rad/sec}, \) and \(\theta_2 = 87.2^0, \dot{\theta}_2 = -19.2\text{ rad/sec}.\) The eigenvalue of the Jacobian matrix is 0.1652 and -0.1652, and the system is stated unstable because one of the eigenvalue systems is nonnegative \(\text{Re}\lambda_i > 0\) although pendulum is around the upright vertical position. However, the role of the outer pendulum is stated stable with eigenvalue negative \((-0.1652, \text{Re}\lambda > 0)\) and around an equilibrium, see Figure 7.
4. Conclusion
An inverted double pendulum is a kind of nonlinear and unstable system. This can be found by evaluating the system stabilization via the eigenvalue of the Jacobian matrix. The nonlinear model was derived mathematically, and its physical model is realized in SimMechanics. Equilibrium stability was analyzed and verified using these models as an inverted double pendulum is around an upright vertical equilibrium position. Obtaining the model and its equilibrium stability enables us to develop a control strategy to stabilize well. The result showed that the dynamic model is fit enough, and the eigenvalue of the system is unstable, as predicted earlier.

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