GEOMETRIC ASPECTS OF THE PERIODIC $\mu$DP EQUATION

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Abstract. We consider the periodic $\mu$DP equation (a modified version of the Degasperis-Procesi equation) as the geodesic flow of a right-invariant affine connection $\nabla$ on the Fréchet Lie group $\text{Diff}^\infty(S^1)$ of all smooth and orientation-preserving diffeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. On the Lie algebra $C^\infty(S^1)$ of $\text{Diff}^\infty(S^1)$, this connection is canonically given by the sum of the Lie bracket and a bilinear operator. For smooth initial data, we show the short time existence of a smooth solution of $\mu$DP which depends smoothly on time and on the initial data. Furthermore, we prove that the exponential map defined by $\nabla$ is a smooth local diffeomorphism of a neighbourhood of zero in $C^\infty(S^1)$ onto a neighbourhood of the unit element in $\text{Diff}^\infty(S^1)$. Our results follow from a general approach on non-metric Euler equations on Lie groups, a Banach space approximation of the Fréchet space $C^\infty(S^1)$, and a sharp spatial regularity result for the geodesic flow.

Dedicated to Herbert Amann on the occasion of his 70th birthday

1. Introduction

In recent years, several nonlinear equations arising as approximations to the governing model equations for water waves attracted a considerable amount of attention in the fluid dynamics research community (cf. [22]). The Korteweg-de Vries (KdV) equation is a well-known model for wave-motion on shallow water with small amplitudes and a flat bottom. This equation is completely integrable, allows for a Lax pair formulation and the corresponding Cauchy problem was the subject of many studies. However, it was observed in [3] that solutions of the KdV equation do not break as physical water waves do: The flow is globally well posed for square integrable initial data (see also [23, 24] for further results). The Camassa-Holm (CH) equation

$$u_t + 3uu_x = 2u_xu_{xx} + uu_{xxx} + u_{txx}$$

was introduced to model the shallow-water medium-amplitude regime (see [4]). Closely related to the CH equation is the Degasperis-Procesi (DP) equation

$$u_t + 4uu_x = 3u_xu_{xx} + uu_{xxx} + u_{txx}$$

was discovered in a search for integrable equations similar to the CH equation (see [12]). Both equations are higher order approximations in a small amplitude.
expansion of the incompressible Euler equations for the unidirectional motion of waves at a free surface under the influence of gravity (cf. [10]). They have a bi-Hamiltonian structure, are completely integrable and allow for wave breaking and peaked solitons, [3, 13, 16, 21]. The Cauchy problem for the periodic CH equation in spaces of classical solutions has been studied extensively (see, e.g., [7, 33]); in [6] and [11] the authors explain that this equation is also well posed in spaces which include peakons, showing in this way that peakons are indeed meaningful solutions of CH. Well-posedness for the periodic DP equation and various features of solutions of the DP on the circle are discussed in [20]. Both, the CH equation and the DP equation, are embedded into the family of $b$-equations

$$m_t = -(m_x u + b m u_x), \quad m := u - u_{xx},$$

where $u(t, x)$ is a function of a spatial variable $x \in S^1$ and a temporal variable $t \in \mathbb{R}$. Note that the family (1) can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any $b \neq -1$ by an appropriate Kodama transformation, [14]. For $b = 2$, we recover the CH equation and for $b = 3$, we get the DP equation. Note that the $b$-equation is integrable only if $b = 2$ or $b = 3$. For further results and references we refer to [19].

Since the pioneering works [1, 15], geometric interpretations of evolution equations led to several interesting results in the applied analysis literature. A detailed discussion of the CH equation in this framework was given by [26]. The geometrical aspects of some metric Euler equations are explained in [8, 9, 25, 32]. Studying the $b$-equations as a geodesic flow on the diffeomorphism group $\text{Diff}^\infty(S^1)$, it was shown recently in [17] that for smooth initial data $u_0(x) = u(0, x)$, there is a unique short-time solution $u(t, x)$ of (1), depending smoothly on $(t, u_0)$. The crucial idea is to define an affine (not necessarily Riemannian) connection $\nabla$ on $	ext{Diff}^\infty(S^1)$ given at the identity by the sum of the Lie bracket and a bilinear symmetric operator $B$ so that $B(u, u) = -u_t$. Most importantly, this approach also works for $b$-equations of non-metric type and it motivates the study of geometric quantities like curvature or an exponential map for the family (1). In particular, the authors of [17] proved that the exponential map for $\nabla$ is a smooth local diffeomorphism near zero in $C^\infty(S^1)$. Recently it has been shown in [18] that the $b$-equation can be realized as a metric Euler equation only if $b = 2$. In all other cases $b \neq 2$ there is no Riemannian metric on $\text{Diff}^\infty(S^1)$ such that the corresponding geodesic flow is re-expressed by the $b$-equation. Geometric aspects of some novel nonlinear PDEs related to CH and DP are discussed in [31].

In this paper, we study the $\mu$DP equation

$$(2) \quad \mu(u_t) - u_{txx} + 3\mu(u)u_x - 3u_x u_{xx} - uu_{xxx} = 0,$$

where $\mu$ denotes the projection $\mu(u) = \int_0^1 u \, dx$ and $u(t, x)$ is a spatially periodic real-valued function of a time variable $t \in \mathbb{R}$ and a space variable $x \in S^1$. The $\mu$DP equation belongs to the family of $\mu$-$b$-equations which follows from (1) by replacing $m = \mu(u) - u_{xx}$. The study of $\mu$-variants of (1) is motivated by the following key
observation: Letting $m = -\partial_x^2 u$, equation (1) for $b = 2$ becomes the Hunter-Saxton (HS) equation

$$2u_x u_{xx} + uu_{xxx} + u_{txx} = 0,$$

which possesses various interesting geometric properties, see, e.g., [29, 30], whereas the choice $m = (1 - \partial_x^2)u$ leads to the CH equation as explained above. In the search for integrable equations that are given by a perturbation of $-\partial_x^2$, the $\mu$-$b$-equation has been introduced and it could be shown that it behaves quite similarly to the $b$-equation; see [31] where the authors discuss local and global well-posedness as well as finite time blow-up and peakons. Our study of the $\mu$DP is inspired by the results in [17]. In fact using the approach of [17] we shall conceptualise a geometric picture of the $\mu$DP equation.

Our study is mostly performed in the $C^\infty$-category. Elements of $C^\infty(S^1)$ are sometimes also called smooth for brevity.

We will reformulate the $\mu$DP equation in terms of a geodesic flow on $\text{Diff}^\infty(S^1)$ to obtain the following main result: Given a smooth initial data $u_0(x)$, for which $\|u_0\|_{C^3(S^1)}$ is small, there is a unique smooth solution $u(t, x)$ of (2) which depends smoothly on $(t, u_0)$. More precisely, we have

**Theorem 1.1.** There exists an open interval $J$ centered at zero and $\delta > 0$ such that for each $u_0 \in C^\infty(S^1)$ with $\|u_0\|_{C^3(S^1)} < \delta$, there exists a unique solution $u \in C^\infty(J, C^\infty(S^1))$ of the $\mu$DP equation such that $u(0) = u_0$. Moreover, the solution $u$ depends smoothly on $(t, u_0) \in J \times C^\infty(S^1)$.

It is known that the Riemannian exponential mapping on general Fréchet manifolds fails to be a smooth local diffeomorphism from the tangent space back to the manifold, cf. [8]. Therefore the following result is quite remarkable.

**Theorem 1.2.** The exponential map $\exp$ at the unity element for the $\mu$DP equation on $\text{Diff}^\infty(S^1)$ is a smooth local diffeomorphism from a neighbourhood of zero in $C^\infty(S^1)$ onto a neighbourhood of $\text{id}$ in $\text{Diff}^\infty(S^1)$.

Our paper is organized as follows: In Section 2, we rewrite (2) in terms of a local flow $\varphi \in \text{Diff}^n(S^1)$, $n \geq 3$, and explain the geometric setting. The resulting equation is an ordinary differential equation and in Section 3, we apply the Theorem of Picard and Lindelöf to obtain a solution of class $C^n(S^1)$ with smooth dependence on $t$ and $u_0(x)$. In addition, we show that this solution in $\text{Diff}^n(S^1) \times C^n(S^1)$ does neither lose nor gain spatial regularity as $t$ varies through the associated interval of existence. We then approximate the Fréchet Lie group $\text{Diff}^\infty(S^1)$ by the topological groups $\text{Diff}^n(S^1)$ and the Fréchet space $C^\infty(S^1)$ by the Banach spaces $C^n(S^1)$ to obtain an analogous existence result for the geodesic equation on $\text{Diff}^\infty(S^1)$. Finally, in Section 4, we make again use of a Banach space approximation to prove that the exponential map for the $\mu$DP is a smooth local diffeomorphism from zero as a map $C^\infty(S^1) \to \text{Diff}^\infty(S^1)$.

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2. Geometric reformulation of the $\mu$DP equation

We write $\operatorname{Diff}^\infty(S^1)$ for the smooth orientation-preserving diffeomorphisms of the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ and $\operatorname{Vect}^\infty(S^1)$ for the space of smooth vector fields on $S^1$. Clearly, $\operatorname{Diff}^\infty(S^1)$ is a Lie group and it is easy to see that its Lie algebra is $\operatorname{Vect}^\infty(S^1)$: If $t \mapsto \varphi(t)$ is a smooth path in $\operatorname{Diff}^\infty(S^1)$ with $\varphi(0) = \text{id}$, then $\varphi_t(0,x) \in T_xS^1$ for all $x \in S^1$ and thus the Lie algebra element $\varphi_t(0,\cdot)$ is a smooth vector field on $S^1$. Furthermore, since $T_S^1 \simeq S^1 \times \mathbb{R}$ is trivial, we can identify the Lie algebra $\operatorname{Vect}^\infty(S^1)$ with $C^\infty(S^1)$. Note that $[u,v] = u_x v - v_x u$ is the corresponding Lie bracket. In the following, we will also use that $\operatorname{Diff}^\infty(S^1)$ has a smooth manifold structure modeled over the Fréchet space $C^\infty(S^1)$. In particular, $\operatorname{Diff}^\infty(S^1)$ is a Fréchet Lie group and thus it is parallelizable, i.e., $\mathcal{T}\operatorname{Diff}^\infty(S^1) \simeq \operatorname{Diff}^\infty(S^1) \times C^\infty(S^1)$. Let $\operatorname{Diff}^n(S^1)$ denote the group of orientation-preserving diffeomorphisms of $S^1$ which are of class $C^n(S^1)$. Similarly, $\operatorname{Diff}^n(S^1)$ has a smooth manifold structure modeled over the Banach space $C^n(S^1)$. Note that $\operatorname{Diff}^n(S^1)$ is only a topological group but not a Banach Lie group, since the composition and inversion maps are continuous but not smooth. Furthermore, the trivialization $\mathcal{T}\operatorname{Diff}^n(S^1) \simeq \operatorname{Diff}^n(S^1) \times C^n(S^1)$ is only topological and not smooth.

In this section, we write (2) as an ordinary differential equation on the tangent bundle $\operatorname{Diff}^n(S^1) \times C^n(S^1)$, where $n \geq 3$. In a first step, we rewrite (2) using the operator $A := \mu - \partial_x^2$. Here $\mu$ denotes the linear map given by $f \mapsto \int_0^1 f(t,x) \, dx$ for any function $f(t,x)$ depending on time $t$ and space $x \in S^1$. Observe that $\mu(\partial_x^k f) = 0$, $k \geq 1$, if $f$ and its derivatives are continuous functions on $S^1$. Furthermore, $\mu(f)$ is still depending on the time variable $t$. The following lemma establishes the invertibility of $A$ as an operator acting on $C^n(S^1)$ for $n \geq 2$.

Lemma 2.1. Given $n \geq 2$, the operator $A = \mu - \partial_x^2$ maps $C^n(S^1)$ isomorphically onto $C^{n-2}(S^1)$. The inverse is given by

$$(A^{-1} f)(x) = \left( \frac{1}{2} x^2 - \frac{1}{2} x + \frac{13}{12} \right) \int_0^1 f(a) \, da + \left( x - \frac{1}{2} \right) \int_0^1 \int_0^a f(b) \, db \, da$$

$$- \int_0^x \int_0^a f(b) \, db \, da + \int_0^1 \int_0^a \int_0^b f(c) \, dc \, db \, da.$$  

Proof. Clearly, $\mu(A^{-1} f) = \mu(f)$ and $(A^{-1} f)_{xx} = \mu(f) - f$ so that $A(A^{-1} f) = f$. To verify that $A$ is surjective, we observe that $\partial_x^k (A^{-1} f)(0) = \partial_x^k (A^{-1} f)(1)$ for all $k \in \{0, \ldots, n\}$. To see that $A$ is injective, assume that $A u = 0$ for $u \in C^n(S^1)$ and $n \geq 2$. Then there are constants $c, d \in \mathbb{R}$ such that $u = \frac{1}{2} \mu(u) x^2 + c x + d$. By periodicity we first conclude that $c = 0$ and $\mu(u) = 0$. Hence $d$ has to vanish as well. \hfill \Box

Lemma 2.2. Assume that $u \in C((-T,T), C^n(S^1)) \cap C^1((-T,T), C^{n-1}(S^1))$ is a solution of (2) for some $n \geq 3$ with $T > 0$. Then the $\mu$DP equation can be written
as
\begin{equation}
\tag{3}
u_t = -A^{-1}(u(Au)_x + 3(Au)u_x).
\end{equation}

Proof. Writing (2) in the form
\[\mu(u_t) - u_{txx} = uu_{xxx} - 3u_x(\mu - u_{xx}),\]
we see that it is equivalent to
\[Au_t = -u(Au)_x - 3(Au)u_x.\]
Thus \(u\) is a solution of (2) if and only if (3) holds true. \(\square\)

As explained in [27, 28], the vector field \(u(t, x)\) admits a unique local flow \(\varphi\) of class \(C^n(S^1)\), i.e.,
\[\varphi_t(t, x) = u(t, \varphi(t, x)), \quad \varphi(0, x) = x\]
for all \(x \in S^1\) and all \(t\) in some open interval \(J \subset \mathbb{R}\). We will use the short-hand notation \(\varphi_t = u \circ \varphi\) for \(\varphi_t(t, x) = u(t, \varphi(t, x))\); i.e., \(\circ\) denotes the composition with respect to the spatial variable. Particularly, we have that \(u = \varphi_t \circ \varphi^{-1}\). Moreover, given \((\varphi, \xi) \in C^1(J, \text{Diff}^\infty(S^1) \times C^n(S^1))\), then \(\varphi^{-1} \circ \varphi\) is a \(C^n(S^1)\)-diffeomorphism for all \(t \in J\) and \(\xi \circ \varphi^{-1} \in C^1(J, C^n(S^1))\).

In this paper, we are mainly interested in smooth diffeomorphisms on \(S^1\). For the reader’s convenience we briefly recall the basic geometric setting. Let us consider the Fréchet manifold \(\text{Diff}^\infty(S^1)\) and a continuous non-degenerate inner product \(\langle \cdot, \cdot \rangle\) on \(C^\infty(S^1)\), i.e., \(u \mapsto \langle u, u \rangle\) is continuous (and hence smooth) and \(\langle u, v \rangle = 0\) for all \(v \in C^\infty(S^1)\) forces \(u = 0\). To define a weak right-invariant Riemannian metric on \(\text{Diff}^\infty(S^1)\), we extend the inner product \(\langle \cdot, \cdot \rangle\) to any tangent space by right-translations, i.e., for all \(g \in \text{Diff}^\infty(S^1)\) and all \(u, v \in T_g \text{Diff}^\infty(S^1)\), we set
\[\langle u, v \rangle_g = \langle (R_{g^{-1}})_* u, (R_{g^{-1}})_* v \rangle_e,\]
where \(e\) denotes the identity. Observe that any open set in the topology induced by this inner product is open in the Fréchet space topology of \(C^\infty(S^1)\) but the converse is not true. We therefore call \(\langle \cdot, \cdot \rangle\) a weak Riemannian metric on \(\text{Diff}^\infty(S^1)\), cf. [8].

We next define a bilinear operator \(B : \text{Vect}^\infty(S^1) \times \text{Vect}^\infty(S^1) \to \text{Vect}^\infty(S^1)\) by
\[B(u, v) = \frac{1}{2}((\text{ad}_u)^*(v) + (\text{ad}_v)^*(u)),\]
where \((\text{ad}_u)^*\) is the adjoint (with respect to \(\langle \cdot, \cdot \rangle\)) of the natural action of the Lie algebra on itself given by \(\text{ad}_u : v \mapsto [u, v]\). Observe that \(B\) defines a right-invariant affine connection \(\nabla\) on \(\text{Diff}^\infty(S^1)\) by
\begin{equation}
\tag{4}
\nabla_{\xi_u} \xi_v = \frac{1}{2} [\xi_u, \xi_v] + B(\xi_u, \xi_v),
\end{equation}
where \(\xi_u\) and \(\xi_v\) are the right-invariant vector fields on \(\text{Diff}^\infty(S^1)\) with values \(u, v\) at the identity. It can be shown that a smooth curve \(t \mapsto g(t)\) in \(\text{Diff}^\infty(S^1)\) is a geodesic if and only if \(u = (R_{g^{-1}})_* \dot{g}\) solves the Euler equation
\begin{equation}
\tag{5}
u_t = -B(u, u);
\end{equation}
here, \( u \) is the \textit{Eulerian velocity} (cf. \cite{2}). Hence the Euler equation \((\mathbf{3})\) corresponds to the geodesic flow of the affine connection \( \nabla \) on the diffeomorphism group \( \text{Diff}^\infty(\mathbb{S}^1) \). Paradigmatic examples are the following: In \cite{3}, the authors show that the Euler equation for the right-invariant \( L^2 \)-metric on \( \text{Diff}^\infty(\mathbb{S}^1) \) is given by the inviscid Burgers equation. Equipping on the other hand \( C^\infty(\mathbb{S}^1) \) with the \( H^1 \)-metric, one obtains the Camassa-Holm equation. Similar correspondences for the general \( H^k \)-metrics are explained in \cite{4}.

Conversely, starting with an equation of type \( u_t = -B(u, u) \) with a bilinear operator \( B \), one associates an affine connection \( \nabla \) on \( \text{Diff}^\infty(\mathbb{S}^1) \) by formula \((\mathbf{4})\). It is however by no means clear that this connection corresponds to a Riemannian structure on \( \text{Diff}^\infty(\mathbb{S}^1) \). It is worthwhile to mention that the connection \( \nabla \) corresponding to the family of \( b \)-equations is compatible with some metric only for \( b = 2 \): In \cite{18} the authors explain that for any \( b \neq 2 \), the \( b \)-equation \((\mathbf{1})\) cannot be realized as an Euler equation on \( \text{Diff}^\infty(\mathbb{S}^1) \) for any regular inertia operator. This motivates the notion of \textit{non-metric Euler equations}. An analogous result holds true for the \( \mu \)-\( b \)-equations from which we conclude that the \( \mu \text{DP} \) equation belongs to the class of non-metric Euler equations. Although we have no metric for the \( \mu \text{DP} \) equation, we will obtain some geometric information by using the connection \( \nabla \), defined in the following way.

Let \( X(t) = (\varphi(t), \xi(t)) \) be a vector field along the curve \( \varphi(t) \in \text{Diff}^\infty(\mathbb{S}^1) \). Furthermore let

\[
B(v, w) := \frac{1}{2} A^{-1}(v(Aw)_x + w(Av)_x + 3(Av)w_x + 3(Aw)v_x).
\]

Lemma \cite{2.2} shows that

\[
B(u, u) = A^{-1}(u(Au)_x + 3(Au)u_x) = -u_t,
\]

if \( u \) is a solution to the \( \mu \text{DP} \) equation. Next, the covariant derivative of \( X(t) \) in the present case is defined as

\[
\frac{DX}{Dt}(t) = \left( \varphi(t), \xi_t + \frac{1}{2} [u(t), \xi(t)] + B(u(t), \xi(t)) \right),
\]

where \( u = \varphi_t \circ \varphi^{-1} \). We see that \( u \) is a solution of the \( \mu \text{DP} \) if and only if its local flow \( \varphi \) is a geodesic for the connection \( \nabla \) defined by \( B \) via \((\mathbf{1})\).

Although we are mainly interested in the smooth category, we will first discuss flows \( \varphi(t) \) on \( \text{Diff}^n(\mathbb{S}^1) \) for technical purposes. Regarding \( \text{Diff}^n(\mathbb{S}^1) \) as a smooth Banach manifold modelled over \( C^n(\mathbb{S}^1) \), the following result has to be understood locally, i.e., in any local chart of \( \text{Diff}^n(\mathbb{S}^1) \).

\textbf{Proposition 2.3.} Given \( n \geq 3 \), the function \( u \in C(J, C^n(\mathbb{S}^1)) \cap C^1(J, C^{n-1}(\mathbb{S}^1)) \) is a solution of \((\mathbf{2})\) if and only if \((\varphi, \xi) \in C^1(J, \text{Diff}^n(\mathbb{S}^1) \times C^n(\mathbb{S}^1)) \) is a solution of

\[
(\mathbf{6}) \quad \left\{ \begin{array}{l}
\varphi_t = \xi, \\
\xi_t = -P_\varphi(\xi),
\end{array} \right.
\]

where \( P_\varphi := R_\varphi \circ P \circ R_{\varphi^{-1}} \) and \( P(f) := 3A^{-1}(f_xf_{xx} + (Af)f_x) \).
Proof. The function \( u \) and the corresponding flow \( \varphi \in \text{Diff}^n(S^1) \) satisfy the relation \( \varphi_t = u \circ \varphi \). Setting \( \varphi_t = \xi \), the chain rule implies that

\[
\xi_t = (u_t + uu_x) \circ \varphi.
\]

Applying Lemma 2.2, we see that \( u \) is a solution of the \( \mu \)-DP equation (2) if and only if

\[
u_t + uu_x = -A^{-1}(u(Au)_x - A(uu_x) + 3(Au)_x)
\]

\[
= -3A^{-1}(u_x + (Au)u_x)
\]

\[
= -3A^{-1}(u_x + (Au)_x).
\]

Recall that

\[
\mu(uu_x) = \int_0^1 uu_x \, dx = \frac{1}{2} \int_0^1 \partial_x(u^2) \, dx = \frac{1}{2}(u^2(1) - u^2(0)) = 0,
\]

since \( u \) is continuous on \( S^1 \). With \( u = \xi \circ \varphi^{-1} \) the desired result follows. \( \square \)

3. Short time existence of geodesics

We now define the vector field

\[
F(\varphi, \xi) := (\xi, -P(\mu))
\]

such that \( (\varphi_t, \xi_t) = F(\varphi, \xi) \). We know that

\[
F : \text{Diff}^n(S^1) \times C^n(S^1) \to C^n(S^1) \times C^n(S^1),
\]

since \( P \) is of order zero. We aim to prove smoothness of the map \( F \). It is worth to mention that this will not follow from the smoothness of \( P \) since neither the composition nor the inversion are smooth maps on \( \text{Diff}^n(S^1) \). The following lemma will be crucial for our purposes.

Lemma 3.1. Assume that \( p \) is a polynomial differential operator of order \( r \) with coefficients depending only on \( \mu \), i.e.,

\[
p(u) = \sum_{\mu(u) \in \mathbb{N} \cup \{0\}, \| \mu \| \leq K} a_I(\mu(u)) u^{\alpha_0}(u')^{\alpha_1} \cdots (u(r))^{\alpha_r}.
\]

Then the action of \( p_\varphi := R_\varphi \circ p \circ R_{\varphi^{-1}} \) is

\[
p_\varphi(u) = \sum_I a_I \left( \int_0^1 u(y) \varphi_x(y) \, dy \right) q_I(u; \varphi_x, \ldots, \varphi^{(r)}),
\]

where \( q_I \) are polynomial differential operators of order \( r \) with coefficients being rational functions of the derivatives of \( \varphi \) up to the order \( r \). Moreover, the denominator terms only depend on \( \varphi_x \).

Proof. It is sufficient to consider a monomial

\[
m(u) = a(\mu(u)) u^{\alpha_0}(u')^{\alpha_1} \cdots (u(r))^{\alpha_r}.
\]

We have

\[
m_\varphi(u) = a(\mu(u \circ \varphi^{-1})) u^{\alpha_0}[(u \circ \varphi^{-1})' \circ \varphi]^{\alpha_1} \cdots [(u \circ \varphi^{-1})^{(r)} \circ \varphi]^{\alpha_r},
\]
where \( \circ \) denotes again the composition with respect to the spatial variable. First, we observe that
\[
\mu(u \circ \varphi^{-1}) = \int_{\mathbb{S}^1} u(\varphi^{-1}(x)) \, dx = \int_{0}^{1} u(y) \varphi_x(y) \, dy,
\]
where we have omitted the time dependence of \( u \) and \( \varphi \). Recall that \( \varphi(\mathbb{S}^1) = \mathbb{S}^1 \), \( \varphi_x > 0 \) and that \( \mu(u \circ \varphi^{-1}) \) is a constant with respect to the spatial variable \( x \in \mathbb{S}^1 \).

Let us introduce the notation
\[
a_k = (u \circ \varphi^{-1})(k) \circ \varphi, \quad k = 1, 2, \ldots, r.
\]
Then, by the chain rule,
\[
a_1 = (\partial_x (u \circ \varphi^{-1})) \circ \varphi = \frac{u_x \circ \varphi^{-1}}{\varphi_x \circ \varphi^{-1}} \circ \varphi = \frac{u_x}{\varphi_x},
\]
and
\[
a_{k+1} = (\partial_x (u \circ \varphi^{-1})^k) \circ \varphi = (\partial_x (a_k \circ \varphi^{-1})) \circ \varphi = \frac{\partial_x a_k}{\varphi_x},
\]
so that our theorem follows by induction.

Recall that in the Banach algebras \( C^n(\mathbb{S}^1) \), \( n \geq 1 \), addition and multiplication as well as the mean value operation \( \mu \) and the derivative \( \frac{d}{dx} \) are smooth maps. We therefore conclude that if the coefficients \( a_I \) are smooth functions for any multi-index \( I \) and \( u \) and \( \varphi \) are at least \( r \) times continuously differentiable, then \( p_{\varphi}(u) \) depends smoothly on \( (\varphi, u) \).

**Proposition 3.2.** The vector field
\[
F : \text{Diff}^n(\mathbb{S}^1) \times C^n(\mathbb{S}^1) \to C^n(\mathbb{S}^1) \times C^n(\mathbb{S}^1)
\]
is smooth for any \( n \geq 3 \).

**Proof.** We write \( F = (F_1, F_2) \). Since \( F_1 : (\varphi, \xi) \mapsto \xi \) is smooth, it remains to check that \( F_2 : (\varphi, \xi) \mapsto -P_{\varphi}(\xi) \) is smooth. For this purpose, we consider the map
\[
\tilde{P} : \text{Diff}^n(\mathbb{S}^1) \times C^n(\mathbb{S}^1) \to \text{Diff}^n(\mathbb{S}^1) \times C^n(\mathbb{S}^1)
\]
defined by
\[
\tilde{P}(\varphi, \xi) = (\varphi, (R_{\varphi} \circ P \circ R_{\varphi^{-1}})(\xi)).
\]
Observe that we have the decomposition \( \tilde{P} = \tilde{A}^{-1} \circ \tilde{Q} \) with
\[
\tilde{A}(\varphi, \xi) = (\varphi, (R_{\varphi} \circ A \circ R_{\varphi^{-1}})(\xi))
\]
and
\[
\tilde{Q}(\varphi, \xi) = (\varphi, (R_{\varphi} \circ Q \circ R_{\varphi^{-1}})(\xi)),
\]
where \( Q(f) := 3(f_x f_{xx} + (A f)_x) \). We now apply Lemma 3.1 to deduce that \( \tilde{A}, \tilde{Q} : \text{Diff}^n(\mathbb{S}^1) \times C^n(\mathbb{S}^1) \to \text{Diff}^n(\mathbb{S}^1) \times C^{n-2}(\mathbb{S}^1) \)
are smooth. To show that \( \tilde{A}^{-1} : \operatorname{Diff}^n(S^1) \times C^{n-2}(S^1) \rightarrow \operatorname{Diff}^n(S^1) \times C^n(S^1) \) is smooth, we compute the derivative \( D \tilde{A} \) at an arbitrary point \((\varphi, \xi)\). We have the following directional derivatives of the components \( \tilde{A}_1 \) and \( \tilde{A}_2 \):

\[
D_{\varphi} \tilde{A}_1 = \text{id}, \quad D_{\xi} \tilde{A}_1 = 0, \quad D_{\xi} \tilde{A}_2 = R_{\varphi} \circ A \circ R_{\varphi^{-1}}.
\]

It remains to compute \( (D_{\varphi} \tilde{A}_2(\varphi, \xi))(\psi) = \frac{d}{d\varepsilon} \tilde{A}_2(\varphi + \varepsilon \psi, \xi) |_{\varepsilon=0} \). In a first step, we calculate

\[
\partial_x^2 (\xi \circ (\varphi + \varepsilon \psi)^{-1}) = \partial_x \left[ \left( \frac{\xi}{\varphi_x + \varepsilon \psi_x} \right) \circ (\varphi + \varepsilon \psi)^{-1} \right]
\]

\[
= \left( \frac{\xi_{xx} + \varepsilon \psi_{xx} \varphi_x}{(\varphi_x + \varepsilon \psi_x)^2} - \frac{\xi_x}{(\varphi_x + \varepsilon \psi_x)^3} \right) \circ (\varphi + \varepsilon \psi)^{-1},
\]

from which we get

\[
\frac{d}{d\varepsilon} \left[ \partial_x^2 (\xi \circ (\varphi + \varepsilon \psi)^{-1}) \circ (\varphi + \varepsilon \psi) \right] = \frac{d}{d\varepsilon} \left( \frac{\xi_{xx} \psi_x}{(\varphi_x + \varepsilon \psi_x)^3} - \frac{\xi_x \psi_{xx}}{(\varphi_x + \varepsilon \psi_x)^3} \right)
\]

\[
= -2 \frac{\xi_{xx} \psi_x}{(\varphi_x + \varepsilon \psi_x)^3} - 3 \frac{\xi_x \psi_{xx}}{(\varphi_x + \varepsilon \psi_x)^3} + 3 \frac{\xi_x \psi_x}{(\varphi_x + \varepsilon \psi_x)^2}
\]

and finally

\[
\frac{d}{d\varepsilon} \left[ \partial_x^2 (\xi \circ (\varphi + \varepsilon \psi)^{-1}) \circ (\varphi + \varepsilon \psi) \right] |_{\varepsilon=0} = -2 \frac{\xi_{xx} \psi_x}{\varphi_x^3} - \frac{\xi_x \psi_{xx}}{\varphi_x^3} + 3 \frac{\xi_x \psi_x}{\varphi_x^2}.
\]

Secondly, we observe that

\[
\frac{d}{d\varepsilon} \mu(\xi \circ (\varphi + \varepsilon \psi)^{-1}) |_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_{S^1} \xi(y)(\varphi_x + \varepsilon \psi_x)(y) \, dy |_{\varepsilon=0}
\]

\[
= \int_{S^1} \xi(y) \psi_x(y) \, dy,
\]

since \( \varphi + \varepsilon \psi \in \operatorname{Diff}^n(S^1) \) for small \( \varepsilon > 0 \). Hence

\[
(D_{\varphi} \tilde{A}_2(\varphi, \xi))(\psi) = \int_{S^1} \xi(y) \psi_x(y) \, dy + 2 \frac{\xi_{xx} \psi_x}{\varphi_x^3} + \frac{\xi_x \psi_{xx}}{\varphi_x^3} - 3 \frac{\xi_x \psi_x}{\varphi_x^2}
\]

and

\[
D \tilde{A}(\varphi, \xi) = \begin{pmatrix}
\text{id} & 0 \\
D_{\varphi} \tilde{A}_2(\varphi, \xi) & R_{\varphi} \circ A \circ R_{\varphi^{-1}} \end{pmatrix}.
\]

It is easy to check that \( D \tilde{A}(\varphi, \xi) \) is an invertible bounded linear operator \( C^n(S^1) \times C^n(S^1) \rightarrow C^n(S^1) \times C^{n-2}(S^1) \). By the open mapping theorem, \( D \tilde{A} \) is a topological isomorphism and, by the inverse mapping theorem, \( \tilde{A}^{-1} \) is smooth.

Since \( F \) is smooth, we can apply the Banach space version of the Picard-Lindelöf Theorem (also known as Cauchy-Lipschitz Theorem) as explained in [27], Chapter XIV-3. This yields the following theorem about the existence and uniqueness of integral curves for the vector field \( F \).
Theorem 3.3. Given \( n \geq 3 \), there is an open interval \( J_n \) centered at zero and an open ball \( B(0, \delta_n) \subset C^n(S^1) \) such that for any \( u_0 \in B(0, \delta_n) \) there exists a unique solution \( (\varphi, \xi) \in C^n(J_n, \text{Diff}^n(S^1) \times C^n(S^1)) \) of (3) with initial conditions \( \varphi(0) = \text{id} \) and \( \xi(0) = u_0 \). Moreover, the flow \( (\varphi, \xi) \) depends smoothly on \( (t, u_0) \).

From Theorem 3.3 we get a unique short-time solution \( u = \xi \circ \varphi^{-1} \) of \( \mu \text{DP} \) in \( C^n(S^1) \) with continuous dependence on \( (t, u_0) \). We now aim to obtain an analogous result for smooth initial data \( u_0 \). But since \( C^n(S^1) \) is a Fréchet space, classical results like the Picard-Lindelöf Theorem or the local inverse theorem for Banach spaces are no longer valid in \( C^n(S^1) \). In the proof of our main theorem, we will make use of a Banach space approximation of the Fréchet space \( C^n(S^1) \). First we shall establish that any solution \((\varphi, \xi)\) of the \( \mu \text{DP} \) equation (3) does not lose nor gain spatial regularity as \( t \) increases or decreases from zero. For this purpose, the following conservation law is quite useful. In its formulation we use the notation \( m_0(x) := (Au)(0, x) = \mu(u_0) - (u_0)_{xx} \).

Lemma 3.4. Let \( u \) be a \( C^3(S^1) \)-solution of the \( \mu \text{DP} \) equation on \( (-T, T) \) and let \( \varphi \) be the corresponding flow. Then

\[
(Au)(t, \varphi(t, x)) \varphi_x^3(t, x) = m_0,
\]

for all \( t \in (-T, T) \).

Proof. We compute

\[
\begin{align*}
\frac{d}{dt} \left[ (\mu(u) - u_{xx} \circ \varphi) \varphi_x^3 \right] &= [\mu(u_t) - u_{xxx} \circ \varphi] \varphi_x^3 + 3\varphi_x^2 \varphi_{xx}(\mu(u) - u_{xx} \circ \varphi) \\
&= [\mu(u) - u_{xx} \circ \varphi - (u_{xxx} \circ \varphi) \varphi_t] \varphi_x^3 + 3\varphi_x^2 \varphi_{xx}(\mu(u) - u_{xx} \circ \varphi) \\
&= [(\mu(u_t) - u_{xxx} - u_{xxx} u) \circ \varphi] \varphi_x^3 + 3\varphi_x^2 (u_{xx} \circ \varphi) \varphi_x (\mu(u) - u_{xx} \circ \varphi) \\
&= [(\mu(u_t) - u_{xxx} - u_{xxx} u) \circ \varphi] \varphi_x^3 + 3\varphi_x^2 (u_{xx} \circ \varphi) \varphi_x (\mu(u) - u_{xx} \circ \varphi) \\
&= [(3u_{xx} - 3\mu(u) u_{xx}) \circ \varphi] \varphi_x^3 - 3\varphi_x^3 (u_{xx} u_{xx} - \mu(u) u_{xx}) \circ \varphi \\
&= 0.
\end{align*}
\]

Since \( \varphi(0) = \text{id} \) and \( \varphi_x(0) = 1 \), the proof is completed. \( \square \)

Lemma 3.5. Let \( (\varphi, \xi) \in C^\infty(J_3, \text{Diff}^3(S^1) \times C^3(S^1)) \) be a solution of (3) with initial data \( (\text{id}, u_0) \), according to Theorem 3.3. Then, for all \( t \in J_3 \),

\[
\varphi_{xx}(t) = \varphi_x(t) \left( \int_0^t \mu(u) \varphi_x(s) \, ds - m_0 \int_0^t \varphi_x(s)^{-2} \, ds \right)
\]

and

\[
\xi_{xx}(t) = \xi_x(t) \frac{\varphi_{xx}(t)}{\varphi_x(t)} + \varphi_x(t) \left[ \mu(u) \varphi_x(t) - m_0 \varphi_x(t)^{-2} \right].
\]

Proof. We have

\[
\frac{d}{dt} \left( \frac{\varphi_{xx}}{\varphi_x} \right) = \frac{\varphi_{xxx} \varphi_x - \varphi_{xx} \varphi_{xx}}{\varphi_x^3}.
\]
Since \( \varphi_t = u \circ \varphi \),
\[
\varphi_{xt} = \varphi_{tx} = \partial_x (u \circ \varphi) = (u_x \circ \varphi) \varphi_x
\]
and
\[
\varphi_{xxt} = \varphi_{txx} = \partial^2_x (u \circ \varphi) = \partial_x [(u_x \circ \varphi) \varphi_x] = (u_{xx} \circ \varphi) \varphi_x^2 + (u_x \circ \varphi) \varphi_{xx}.
\]
Hence
\[
\frac{d}{dt} \left( \frac{\varphi_{xx}}{\varphi_x} \right) = (u_{xx} \circ \varphi) \varphi_x.
\]
According to the previous lemma, we know that
\[
u_{xx} = \mu(u) - m_0 \varphi_x^{-3}.
\]
Integrating
\[
\frac{d}{dt} \left( \frac{\varphi_{xx}}{\varphi_x} \right) = \mu(u) \varphi_x - m_0 \varphi_x^{-2}
\]
over \([0, t]\) leads to equation (7) and taking the time derivative of (7) yields (8).

Remark 3.6. Since the \( \mu \)DP equation is equivalent to the quasi-linear evolution equation
\[
u_t + uu_x + 3 \mu(u) \partial_x A^{-1} u = 0,
\]
we see that \( \mu(u) = 0 \) and hence \( \mu(u) = \mu(u_0) \) so that \( \mu(u) \) can in fact be written in front of the first integral sign in equation (7).

Corollary 3.7. Let \((\varphi, \xi)\) be as in Lemma 3.5. If \(u_0 \in C^n(S^1)\) then we have
\((\varphi(t), \xi(t)) \in \text{Diff}^n(S^1) \times C^n(S^1)\) for all \(t \in J_3\).

Proof. We proceed by induction on \(n\). For \(n = 3\) the result is immediate from our assumption on \((\varphi(t), \xi(t))\). Let us assume that \((\varphi(t), \xi(t)) \in \text{Diff}^n(S^1) \times C^n(S^1)\) for some \(n \geq 3\). Then Lemma 3.5 shows that, if \(u_0 \in C^{n+1}(S^1)\), then \((\varphi(t), \xi(t)) \in \text{Diff}^{n+1}(S^1) \times C^{n+1}(S^1)\), finishing the proof.

Corollary 3.8. Let \((\varphi, \xi)\) be as in Lemma 3.5. If there exists a nonzero \(t \in J_3\) such that \(\varphi(t) \in \text{Diff}^n(S^1)\) or \(\xi(t) \in C^n(S^1)\) then \(\xi(0) = u_0 \in C^n(S^1)\).

Proof. Again, we use a recursive argument. For \(n = 3\), there is nothing to do. For some \(n \geq 3\), suppose that \(u_0 \in C^n(S^1)\). By the previous corollary, \((\varphi(t), \xi(t)) \in \text{Diff}^n(S^1) \times C^n(S^1)\) for all \(t \in J_3\). Assume that there is \(0 \neq t_0 \in J_3\) such that \(\varphi(t_0) \in \text{Diff}^{n+1}(S^1)\) or \(\xi(t_0) \in C^{n+1}(S^1)\). Since \(\varphi_x > 0\), Lemma 3.5 immediately implies that also \(u_0 \in C^{n+1}(S^1)\). □

Now we discuss Banach space approximations of Fréchet spaces.
**Theorem 3.11.** By Corollary 3.7, we have for any \( n \) zero and an open ball \( U \)

**Proof.** Let \( C \) the flow \( (\Phi, \xi) \) unique solution \( (1) \Phi \) the restriction \( (2) \) the restriction \( \Phi := \Phi |_{U_n} : U_n \to V_n \) is a smooth map. Then \( \Phi_0(U) \subset V \) and the map \( \Phi := \Phi_0 |_{U} : U \to V \) is smooth.

Now we come to our main theorem which we first formulate in the geometric picture.

**Theorem 3.11.** There exists an open interval \( J \) centered at zero and \( \delta > 0 \) such that for all \( u_0 \in C^\infty(S^1) \) with \( \|u_0\|_{C^3(S^1)} < \delta \), there exists a unique solution \( (\varphi, \xi) \in C^\infty(J, \text{Diff}^3(S^1) \times C^3(S^1)) \) of (6) such that \( \varphi(0) = \text{id} \) and \( \xi(0) = u_0 \). Moreover, the flow \( (\varphi, \xi) \) depends smoothly on \( (t, u_0) \in J \times C^\infty(S^1) \).

**Proof.** Theorem 3.3 for \( n = 3 \) shows that there is an open interval \( J \) centered at zero and an open ball \( U_3 = B(0, \delta) \subset C^3(S^1) \) so that for any \( u_0 \in U_3 \) there exists a unique solution \( (\varphi, \xi) \in C^\infty(J, \text{Diff}^3(S^1) \times C^3(S^1)) \) of (6) with initial data \( (\text{id}, u_0) \) and a smooth flow

\[
\Phi_3 : J \times U_3 \to \text{Diff}^3(S^1) \times C^3(S^1).
\]

Let

\[
U_n := U_3 \cap C^n(S^1) \quad \text{and} \quad U_\infty := U_3 \cap C^\infty(S^1).
\]

By Corollary 3.7 we have

\[
\Phi_3(J \times U_n) \subset \text{Diff}^n(S^1) \times C^n(S^1)
\]

for any \( n \geq 3 \) and the map

\[
\Phi_n := \Phi_3 |_{J \times U_n} : J \times U_n \to \text{Diff}^n(S^1) \times C^n(S^1)
\]
is smooth. Lemma 3.10 implies that
\[ \Phi_3(J \times U) \subset \text{Diff}^\infty(S^1) \times C^\infty(S^1), \]
completing the proof of the short-time existence for smooth initial data \( u_0 \). Moreover, the mapping
\[ \Phi_\infty := \Phi_3|_{J \times U} : J \times U \to \text{Diff}^\infty(S^1) \times C^\infty(S^1), \]
is smooth, proving the smooth dependence on time and on the initial condition. □

Under the assumptions of Theorem 3.11, the map
\[ \text{Diff}^\infty(S^1) \times C^\infty(S^1) \to C^\infty(S^1), \quad (\varphi, \xi) \mapsto \xi \circ \varphi^{-1} = u \]
is smooth. Thus we obtain the result stated in Theorem 1.1.

4. The exponential map

For a Banach manifold \( M \) equipped with a symmetric linear connection, the exponential map is defined as the time one of the geodesic flow, i.e., if \( t \mapsto \gamma(t) \) is the (unique) geodesic in \( M \) starting at \( p = \gamma(0) \) with velocity \( \gamma_t(0) = u \in T_pM \) then \( \exp_p(u) = \gamma(1) \). Roughly speaking, the map \( \exp_p(\cdot) \) is a projection from \( T_pM \) to the manifold \( M \). Since the derivative of \( \exp_p \) at zero is the identity, the exponential map is a smooth diffeomorphism from a neighbourhood of zero of \( T_pM \) to a neighbourhood of \( p \in M \). However, this fails for Fréchet manifolds like \( \text{Diff}^\infty(S^1) \) in general. We know that the Riemannian exponential map for the \( L^2 \)-metric on \( \text{Diff}^\infty(S^1) \) is not a local \( C^1 \)-diffeomorphism near the origin, cf. [9]. For the Camassa-Holm equation and more general for the \( H^k \)-metrics, \( k \geq 1 \), the Riemannian exponential map in fact is a smooth local diffeomorphism. This result was generalized to the family of \( b \)-equations, see [17], and in this section we obtain a similar result for the \( \mu \)DP equation.

The basic idea of the proof of Theorem 1.2 is to consider a perturbed problem: Let \( (\varphi^\varepsilon, \xi^\varepsilon) \) denote the local expression of an integral curve of (6) in \( T\text{Diff}^n(S^1) \) with initial data \( (\text{id}, u + \varepsilon w) \), where \( u, w \in C^n(S^1) \). Let
\[ \psi(t) := \frac{\partial \varphi^\varepsilon(t)}{\partial \varepsilon} \big|_{\varepsilon=0}. \]
By the homogeneity of the geodesics,
\[ \varphi^\varepsilon(t) = \exp(t(u + \varepsilon w)), \]
so that
\[ \psi(t) = D(\exp(tu))tw =: L_n(t, u)w, \]
where \( L_n(t, u) \) is a bounded linear operator on \( C^n(S^1) \).

Lemma 4.1. Suppose that \( u \in C^{n+1}(S^1) \). Then, for \( t \neq 0 \),
\[ L_n(t, u)(C^n(S^1) \setminus C^{n+1}(S^1)) \subset C^n(S^1) \setminus C^{n+1}(S^1). \]
Proof. First, we write down equation (7) for \( \varphi^\varepsilon(t) \),

\[
\varphi_{xx}^\varepsilon(t) = \varphi_x^\varepsilon(t) \left[ \mu(u + \varepsilon w) \int_0^t \varphi_x^\varepsilon(s) \, ds - m_0^\varepsilon \int_0^t \varphi_x^\varepsilon(s)^{-2} \, ds \right],
\]

and take the derivative with respect to \( \varepsilon \),

\[
\frac{\partial \varphi_{xx}^\varepsilon}{\partial \varepsilon}(t) = \frac{\partial \varphi_x^\varepsilon}{\partial \varepsilon}(t) \left[ \mu(u + \varepsilon w) \int_0^t \varphi_x^\varepsilon(s) \, ds - m_0^\varepsilon \int_0^t \varphi_x^\varepsilon(s)^{-2} \, ds \right]
+ \varphi_x^\varepsilon(t) \left[ \mu(u) \int_0^t \varphi_x^\varepsilon(s) \, ds + \mu(u + \varepsilon w) \int_0^t \frac{\partial \varphi_x^\varepsilon}{\partial \varepsilon}(s) \, ds \right]
- \varphi_x^\varepsilon(t) \left[ \frac{\partial m_0^\varepsilon}{\partial \varepsilon} \int_0^t \varphi_x^\varepsilon(s)^{-2} \, ds + m_0^\varepsilon \int_0^t \frac{\partial \varphi_x^\varepsilon}{\partial \varepsilon}(s)^{-2} \, ds \right].
\]

Notice that

\[
\frac{\partial m_0^\varepsilon}{\partial \varepsilon} = \mu(w) - w_{xx} = Aw
\]

and that \( m_0^\varepsilon \to m_0 = Au \) as \( \varepsilon \to 0 \). Hence

\[
\psi_{xx}(t) = \psi_x(t) \left[ \mu(u) \int_0^t \varphi_x(s) \, ds - m_0 \int_0^t \varphi_x(s)^{-2} \, ds \right]
+ \varphi_x(t) \left[ \mu(u) \int_0^t \varphi_x(s) \, ds + \mu(u + \varepsilon w) \int_0^t \psi_x(s) \, ds \right]
- \varphi_x(t) \left[ (\mu(w) - w_{xx}) \int_0^t \varphi_x(s)^{-2} \, ds - 2m_0 \int_0^t \psi_x(s)\varphi_x(s)^{-3} \, ds \right]
= a(t)\psi_x(t) + b(t) \int_0^t c(s)\psi_x(s) \, ds + d(t) + e(t)w_{xx}
\]

with \( a(t), b(t), c(t), d(t), e(t) \in C^{n-1}(S^1) \) and \( e(t) \neq 0 \) for \( t \neq 0 \). Finally, if

\[
w \in C^n(S^1) \backslash C^{n+1}(S^1),
\]

then

\[
\psi(t) = L_n(t, w) \in C^n(S^1) \backslash C^{n+1}(S^1).
\]

\( \square \)

Let us now turn to the proof of Theorem 1.2. Since \( C^3(S^1) \) is a Banach space and \( \text{Diff}^3(S^1) \) is a Banach manifold modelled over \( C^3(S^1) \), we know that the exponential map is a smooth diffeomorphism near zero, i.e., there are neighbourhoods \( U_3 \) of zero in \( C^3(S^1) \) and \( V_3 \) of \( \text{id} \) in \( \text{Diff}^3(S^1) \) such that

\[
\exp_3 := \exp|_{U_3} : U_3 \to V_3
\]

is a smooth diffeomorphism. For \( n \geq 3 \), we now define

\[
U_n := U_3 \cap C^n(S^1) \quad \text{and} \quad V_n = V_3 \cap \text{Diff}^n(S^1).
\]

Let \( \exp_n := \exp_{|U_n} \). Since \( \exp_n \) is a restriction of \( \exp_3 \), it is clearly injective. We now use Corollary 3.7 and Corollary 3.8 to deduce that \( \exp_n \) is also surjective, more precisely, \( \exp_n(U_n) = V_n \). If the geodesic \( \varphi \) with \( \varphi(1) = \exp(u) \) starts at \( \text{id} \in \text{Diff}^n(S^1) \) with velocity vector \( u \) belonging to \( C^n(S^1) \), then \( \varphi(t) \in \text{Diff}^n(S^1) \) for any \( t \) and hence \( \exp_n(U_n) \subset V_n \). Conversely, if \( v \in V_n \) is given, then there
is \( u \in U_3 \) with \( \exp_3(u) = v \). Corollary 3.8 immediately implies that \( u \in C^n(S^1) \); hence \( u \in U_n \) and \( \exp_n(u) = v \). Note that \( \exp_n \) is a bijection from \( U_n \) to \( V_n \). Furthermore, \( \exp_n \) is a smooth map and diffeomorphic \( U_n \to V_n \). We now show that \( \exp_n \) is a smooth diffeomorphism; precisely we show that \( \exp_n^{-1}: V_n \to U_n \) is smooth by virtue of the inverse mapping theorem. For each \( u \in C^n(S^1) \), \( D\exp_n(u) \) is a bounded linear operator \( C^n(S^1) \to C^n(S^1) \). Notice that

\[
D\exp_n(u) = D\exp_3(u)|_{C^n(S^1)},
\]

from which we conclude that \( D\exp_n(u) \) is injective. Let us prove the surjectivity of \( D\exp_n(u) \), \( n \geq 3 \), by induction. For \( n = 3 \), this follows from the fact that \( \exp_3 : U_3 \to V_3 \) is diffeomorphic and hence a submersion. Assume that \( D\exp_n(u) \) is surjective for some \( n \geq 3 \) and that \( u \in C^{n+1}(S^1) \). We have to show that this implies the surjectivity of \( D\exp_{n+1}(u) \). But this is a direct consequence of \( D\exp_n(u) = L_n(1,u) \) and the previous lemma: Let \( f \in C^{n+1}(S^1) \). We have to find \( g \in C^n(S^1) \) such that \( D\exp_n(u)g = f \). By our assumption, there is \( g \in C^n(S^1) \) such that \( D\exp_n(u)g = f \). Suppose that \( g \notin C^{n+1}(S^1) \). But then \( f = L_n(1,u)g \notin C^{n+1}(S^1) \) in contradiction to the choice of \( f \). Thus \( g \in C^{n+1}(S^1) \) and \( D\exp_{n+1}(u)g = f \). Now we can apply the open mapping theorem to deduce that for any \( n \geq 3 \) and any \( u \in C^n(S^1) \) the map

\[
D\exp_n(u) : C^n(S^1) \to C^n(S^1)
\]

is a topological isomorphism. By the inverse function theorem, \( \exp_n : U_n \to V_n \) is a smooth diffeomorphism. If we define

\[
U_\infty := U_3 \cap C^\infty(S^1) \quad \text{and} \quad V_\infty := V_3 \cap \text{Diff}^{\infty}(S^1),
\]

Lemma 3.10 yields that

\[
\exp_\infty := \exp_3|_{U_\infty} : U_\infty \to V_\infty
\]

as well as

\[
\exp_\infty^{-1} : V_\infty \to U_\infty
\]

are smooth maps. Thus \( \exp_\infty \) is a smooth diffeomorphism between \( U_\infty \) and \( V_\infty \).
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