STABILITY OF SUPersonic CONTACT DISCONTINuity FOR TWO-DIMENSIONAL STEADY COMPRESSIBLE EULER FLOWS IN A FINITE NOZZLE

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Abstract. In this paper, we study the stability of supersonic contact discontinuity for the two-dimensional steady compressible Euler flows in a finitely long nozzle of varying cross-sections. We formulate the problem as an initial-boundary value problem with the contact discontinuity as a free boundary. To deal with the free boundary value problem, we employ the Lagrangian transformation to straighten the contact discontinuity and then the free boundary value problem becomes a fixed boundary value problem. We develop an iteration scheme and establish some novel estimates of solutions for the first order of hyperbolic equations on a cornered domain. Finally, by using the inverse Lagrangian transformation and under the assumption that the incoming flows and the nozzle walls are smooth perturbations of the background state, we prove that the original free boundary problem admits a unique weak solution which is a small perturbation of the background state and the solution consists of two smooth supersonic flows separated by a smooth contact discontinuity.

1. Introduction

The two-dimensional steady full Euler system of compressible flows is of the following form:

\[ \begin{align*}
\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) &= 0, \\
\frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho uv) &= 0, \\
\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + p) &= 0, \\
\frac{\partial}{\partial x}((\rho E + p)u) + \frac{\partial}{\partial y}((\rho E + p)v) &= 0,
\end{align*} \tag{1.1} \]

where \((u, v)\), \(p\), \(\rho\) stand for the velocity, pressure, and density, respectively, and

\[ E = \frac{1}{2}(u^2 + v^2) + \frac{p}{(\gamma - 1)\rho} \]

is the energy with adiabatic exponent \(\gamma > 1\). Here, the pressure \(p\) and density \(\rho\) satisfy the constitutive relation: \(p = A(S)\rho^{\gamma}\), with \(S\) the entropy. If the solution of (1.1) is classical, i.e., \(U = (u, v, p, \rho) \in C^1\), we have the following transport equations:

\[ (u, v) \cdot \nabla B = 0, \quad (u, v) \cdot \nabla \left( \frac{p}{\rho^{\gamma}} \right) = 0, \]

where

\[ B = \frac{1}{2}(u^2 + v^2) + \frac{\gamma p}{(\gamma - 1)\rho}. \]
This means that the entropy is preserved and the Bernoulli law holds along each streamline. We denote the sonic speed of the flow by \( c = \sqrt{\frac{\gamma p}{\rho}} \). The flow is called supersonic if \( u^2 + v^2 > c^2 \), subsonic if \( u^2 + v^2 < c^2 \), and sonic if \( u^2 + v^2 = c^2 \). For the supersonic flow the system (1.1) is hyperbolic, while for the subsonic flow the system (1.1) is elliptic, and in general the system (1.1) is of hyperbolic-elliptic mixed type.

In this paper, we are concerned with the stability of supersonic contact discontinuity governed by the two-dimensional steady full Euler equations in a finite nozzle (see Fig. 1.1). The domain in the nozzle can be described by

\[
\Omega := \{(x, y) \in \mathbb{R}^2 : g_-(x) < y < g_+(x), \ 0 < x < L\},
\]

where \( g_- \), \( g_+ \) are functions of \( x \). The lower and upper boundaries of the nozzle are denoted by \( \Gamma_- \) and \( \Gamma_+ \), i.e.,

\[
\Gamma_{\pm} := \{(x, y) : y = g_{\pm}(x), \ 0 < x < L\}.
\]

**Figure 1.1. Supersonic contact discontinuity in a finite nozzle**

There have been many studies in literature on the subsonic flows in nozzles. For the subsonic flow in infinitely or finitely long nozzles, L. Bers in [3] first studied the existence and uniqueness of subsonic irrotational flow in a two-dimensional infinitely long nozzle for a given appropriate incoming mass flux. This argument was verified rigorously in [37] for two-dimensional nozzles, in [38] for axially symmetric nozzles, and in [25] for multi-dimensional nozzles. The existence and uniqueness of isentropic flow was obtained in a two-dimensional infinitely long nozzle with small non-zero vorticity in [45], and in the axi-symmetric case with zero swirl in [21]. For the full Euler flow, similar results were obtained in [7] and then [22, 24]. M. Bae in [1] showed the stability of straight contact discontinuities in a two-dimensional almost flat and infinity long nozzles. Recently, the sign condition and the small assumption were removed in [12] by applying the compensated compactness argument. The sonic-subsonic limit for subsonic flow in infinitely long nozzles was also studied in [6,11,27,34,35], while the incompressible limit was studied in [13]. For the subsonic flow in a finitely long nozzle, there are only a few results. It is well-known that it is ill-posed when the pressure is assigned at both the inlet and the outlet. So how to find an appropriate boundary condition is one of the major issues; see [23] for more details. Another important problem for the steady Euler flows is the stability of the transonic shock in nozzles. G.-Q. Chen and M. Feldman [39] studied the multi-dimensional transonic shocks in a finite or infinitely long straight nozzles for the potential flows, and
some further related results were obtained in [10,43]. The transonic shock governed by the Euler equations was studied by S.-X. Chen in [16], where he studied the two-dimensional Euler flows with the uniform Bernoulli constant for the incoming flow with some symmetric structure and proved the stability of the transonic shock in a finitely long duct. Then the result was generalized in [16] to finitely long nozzles as small perturbations of a straight one. The uniqueness for the transonic shocks in two-dimensional steady Euler flows in a finitely long duct was considered in [26] for a class of piecewise $C^1$ smooth functions. For the three-dimensional case we refer to [17,20,31,42,44,45]. The transonic shock in finitely and infinitely long nozzles was studied in [4,5] for the full Euler system. Moreover, there are also some works on the transonic shock in a divergent or a De laval nozzle in [2,18,29,30] and references therein.

For the supersonic flow in nozzles there are only a few results. The reason is that it is very difficult to control the solutions due to the reflection of the characteristics. The solutions are expected to blow up if the nozzles are sufficiently long. Actually, we show that the non-constant irrotational flow will generally blow up in the semi-ininitely long flat nozzle in the Appendix (see Theorem A.1). The problem of the supersonic flow in nozzles is different from the problem of supersonic flow past a solid wall (see in [14,33]), or the local stability of straight contact discontinuity without boundary (see in [40,41]), for which the decay estimates are expected or there is no reflection on the characteristics. So far almost all the results on the supersonic flow in nozzles are about the nozzles with special structure, for example, the expanding nozzles. Chen and Qu in [19] studied steady supersonic potential flow with rarefaction waves in an expanding and infinitely long nozzles and found a sufficient condition to determine whether the vacuum state appears or not, which is devoted to one of the questions proposed by Courant and Friedrichs in [15]. The smooth potential flow without rarefaction waves was considered in [36] for the two-dimensional case and in [39] for the multi-dimensional case.

To our best knowledge this paper is the first on the supersonic steady contact discontinuity in nozzles. The problem of two-dimensional supersonic contact discontinuity in nozzles governed by the steady Euler equations can be formulated as a nonlinear initial-boundary value problem for the first order quasilinear hyperbolic system, with the contact discontinuity as a free boundary as well as the two nozzle walls as fixed boundaries. For the study of the nonlinear stability of supersonic contact discontinuity in a finitely long nozzles, the main difficulties are caused by the facts that the states on the both sides of the contact discontinuity are unknown and it is not clear how to locate the position of the contact discontinuity due to loss of the normal velocity on the contact discontinuity. However, noticing that the tangent of the contact discontinuity is parallel to the velocity of the flow from the both sides, we can apply the Euler-Lagrange coordinates transformation to fix the free boundary and reformulate the original free boundary value problem as a fixed boundary value problem in the Lagrangian coordinates. To solve the fixed boundary value problem, we shall develop new ideas and arguments inspired by [32]. Since the Bernoulli law and entropy are invariant along the stream lines for the $C^1$-smooth flows, the Bernoulli’s function and the entropy can be determined completely by the incoming flows. With this observation, we can introduce the generalized Riemann invariants $z_\pm$ (see Section 3.2 below) which are invariant along the corresponding characteristics to reduce the Euler system in the Lagrangian coordinates to a diagonal form. To solve the equivalent boundary value problem for the Riemann invariants $z_\pm$ we shall introduce an iteration scheme to construct a sequence of approximate solutions. Then, the remaining
tasks are to show that the iteration scheme is well-defined, and the sequence of the approximate solutions is convergent. For this purpose we shall establish various $C^2$ estimates for the approximate solutions of the initial boundary value problem, using the characteristics method carefully for all cases depending on the reflection of the characteristics on the nozzle walls or on the contact discontinuity. The reason we need the $C^2$-estimates is that the inhomogeneous terms in the difference of equations of two approximate solutions depend on the first order derivatives of the approximates solutions. With these estimates, one can show that the iteration map is a contraction map in $C^1$-norm, which leads to the convergence of the approximate solutions. The limit is unique, and is actually the unique solution of the nonlinear initial-boundary value problem.

The rest of the paper is organized as follows. In section 2, we introduce the mathematical problem and state the main results of this paper. In section 3, we further reformulate this problem by introducing the Euler-Lagrangian coordinates transformation and the Riemann invariants to reduce the system to a fixed boundary value problem governed by equations of diagonalized form, and finally introduce the iteration scheme by linearizing the nonlinear boundary value problem near the background solution. Section 4 is devoted to the study of the linearized boundary value problem introduced in Section 3 by deriving the a priori estimates for the approximate solutions case by case. In section 5, we show the convergence of the approximate solutions obtained in Section 4 and then complete the proof of the main theorem by applying the Banach contraction mapping theorem. Finally, in the Appendix, we give an example to show that the solution for the infinity long nozzle will blow-up in general.

2. Problems and Main Result

In this section, we shall formulate the supersonic contact discontinuity problem in a finitely long nozzle in the Eulerian coordinates and introduce the main results. First, we consider a special case of a supersonic contact discontinuity in a finitely long flat nozzle (see Fig. 2.1). The special solution is the background solution in this paper.

![Supersonic contact discontinuity in a finitely long straight nozzle](image)

**Figure 2.1.** Supersonic contact discontinuity in a finitely long straight nozzle

Suppose that the nozzle with flat boundaries is described as:

$$\Omega := \{(x, y) \in \mathbb{R}^2 : 0 < x < L, -1 < y < 1\}.$$
The two layers of uniform flow in $\Omega$ separated by a contact discontinuity are:

$$U^{(i)} := (u^{(i)}, 0, p^{(i)}, \rho^{(i)})^T, \quad i = a, b.$$ 

They are two constant states. $U^{(a)}$ is the top layer and $U^{(b)}$ is the bottom layer. The horizontal velocity and density of both the top and bottom layers $u^{(i)}$, $\rho^{(i)}$, $i = a, b$ are positive. The pressure $p^{(i)}$ of both the top and bottom layers are given by the same positive constant, i.e., $p^{(a)} = p^{(b)} = p$. Finally, both the two layers are supersonic, i.e., there is a constant $\delta_0 > 0$, such that

$$u^{(i)} - c^{(i)} > \delta_0,$$

where $c^{(i)} = \sqrt{\gamma p^{(i)}}$, for $(i = a, b)$. Let

$$U(x, y) = \begin{cases} U^{(a)}(x, y), & (x, y) \in (0, L) \times (0, 1), \\ U^{(b)}(x, y), & (x, y) \in (0, L) \times (-1, 0). \end{cases} \quad (2.1)$$

Then, $U$ is a weak solution of the Euler system (1.1) in $\Omega$ in the distribution sense, with the discontinuity line $y = 0$ as the supersonic contact discontinuity. The line $y = 0$ divides the domain $\Omega$ into two parts, $\Omega^{(i)}(i = a, b),

$$\Omega^{(a)} := \Omega \cap \{0 < y < 1\}, \quad \Omega^{(b)} := \Omega \cap \{-1 < y < 0\},$$

and the flows are supersonic in $\Omega^{(a)}$ and in $\Omega^{(b)}$, respectively. We call the solution $U(x, y)$ defined by (2.1) is the background solution.

2.1. Mathematical problems and the main results. The incoming flow $U_0$ at the inlet $x = 0$, which is divided by $y = 0$, is given by

$$U_0(y) = \begin{cases} U_0^{(a)}(y), & 0 < y < g_+(0), \\ U_0^{(b)}(y), & g_-(0) < y < 0, \end{cases} \quad (2.2)$$

where $U_0^{(i)}(y) = (u^{(i)}, v^{(i)}, p^{(i)}, \rho^{(i)})$, $i = a, b$, satisfies

$$\left(\frac{v_0^{(a)}}{u_0^{(a)}}\right)(0) = \left(\frac{v_0^{(b)}}{u_0^{(b)}}\right)(0), \quad p_0^{(a)}(0) = p_0^{(b)}(0). \quad (2.3)$$

So the point $(0, 0)$ is the starting point of the contact discontinuity.

Let the location of the contact discontinuity be $\Gamma_{cd} = \{y = g_{cd}(x)\}$, which divides the domain $\Omega$ into two subdomains:

$$\Omega^{(a)} := \Omega \cap \{g_{cd}(x) < y < g_+(x)\}, \quad \Omega^{(b)} := \Omega \cap \{g_-(x) < y < g_{cd}(x)\}. \quad (2.4)$$

Let $U^{(i)} = (u^{(i)}, v^{(i)}, p^{(i)}, \rho^{(i)})$ be smooth and supersonic in $\Omega^{(i)}$, $i = a, b$, respectively.

On the boundaries $\Gamma_-$, $\Gamma_+$, the flow satisfies the impermeable slip boundary conditions:

$$(u, v) \cdot n_-=0, \quad (u, v) \cdot n_+=0, \quad (2.5)$$

where $n_- = (g'_-, -1)$ and $n_+ = (-g'_+, 1)$ represent the outer normal vectors of the low and upper boundaries $\Gamma_-$, $\Gamma_+$, respectively. In addition, along the contact discontinuity $\Gamma_{cd}$, the following conditions which are derived from Rankine-Hugoniot conditions hold:

$$(u, v) \cdot n_{cd}|_{\Gamma_{cd}} = 0, \quad [p]|_{\Gamma_{cd}} = 0, \quad (2.6)$$

where $n_{cd} = (g'_{cd}, -1)$ is the normal vector on $\Gamma_{cd}$ and $[\cdot]$ denotes the jump of the quantity between the two states across the contact discontinuity.
According to the above setting, we will study the following nonlinear free boundary problem with the supersonic contact discontinuity.

**Problem A.** Given a supersonic incoming flow $U_0(y)$ at the entrance $\{x = 0\}$ by (2.2), which satisfies (2.3) and (2.5), find a piecewise smooth solutions $(U(x, y), g_{cd}(x))$ separated by a contact discontinuity $\Gamma_{cd}$ such that, the flows are smooth and satisfy the Euler equations (1.1) in $\Omega$, the Rankine-Hugoniot condition (2.6) on $\Gamma_{\pm}$, and the Rankine-Hugoniot condition (2.6) on $\Gamma_{cd}$; moreover, the flows are supersonic, i.e., $\sqrt{u^2 + v^2} > c$ in $\Omega(a) \cup \Omega(b)$, where $c$ is the sonic speed.

We remark that a function $U(x, y) = (u, v, p, \rho)^T$ of Problem A is a weak solution of the Euler equations (1.1) in the weak sense:

$$\int_{\Omega} W(U) \partial_x \zeta + H(U) \partial_y \zeta dxdy = 0,$$

for any $\zeta \in C^\infty_0(\Omega)$, where

$$W(U) = \left( \rho u, \rho u^2 + p, \rho u v, (\rho E + p) u \right)^T, \quad H(U) = \left( \rho v, \rho u v, \rho v^2 + p, (\rho E + p) v \right)^T.$$

Our main result in this paper is the following theorem.

**Theorem 2.1.** There exist constants $\varepsilon_0 > 0$ and $C_0 > 0$ depending only on $L$, $L$ and $\gamma$, such that, for any $\varepsilon \in (0, \varepsilon_0)$, if the incoming flow and the boundaries of the nozzle satisfy

$$\| U_0^{(a)} - U_t^{(a)} \|_{C^2([0, g_+ + \varepsilon)(0))} + \| U_0^{(b)} - U_t^{(b)} \|_{C^2([0, g_+ + \varepsilon)(0))}$$

$$+ \| g_+ - 1 \|_{C^3([0, L])} + \| g_- + 1 \|_{C^3([0, L])} \leq \varepsilon,$$

then Problem A admits a unique solution $U(x, y)$ with contact discontinuity $y = g_{cd}(x)$ satisfying:

(i) Solution $U$ consists of two smooth supersonic flows $U^{(a)} \in C^1(\Omega^{(a)})$ and $U^{(b)} \in C^1(\Omega^{(b)})$ with $y = g_{cd}(x)$ as the contact discontinuity, and the following estimate holds:

$$\| U^{(a)} - U_t^{(a)} \|_{C^1(\Omega^{(a)})} + \| U^{(b)} - U_t^{(b)} \|_{C^1(\Omega^{(b)})} \leq C_0 \varepsilon,$$

(ii) The contact discontinuity $y = g_{cd}(x)$ is a stream line and satisfies

$$\| g_{cd} \|_{C^2([0, L])} \leq C_0 \varepsilon,$$

where $g_{cd}(0) = 0$.

### 3. Mathematical Formulation and the Iteration Scheme

**3.1. Mathematical problems in the Lagrangian coordinates.** Since the tangent of the contact discontinuity $\Gamma_{cd}$ is parallel to the velocity of the flow on the both sides of $\Gamma_{cd}$ by (2.4), it is convenient to apply the Euler-Lagrange coordinates transformation to fix the free boundary $\Gamma_{cd}$ and hence reformulate Problem A into a fixed boundary value problem in the Lagrangian coordinates.

Assume that $(U(x, y), g_{cd}(x))$ is a solution of Problem A. By the conservation of mass, i.e., (1.1), for any $0 < x < L$, it holds that

$$\int_{g_{cd}(x)}^{g_+(x)} \rho u(x, \tau) d\tau = m^{(a)}, \quad \int_{g_{cd}(x)}^{g_-(x)} \rho u(x, \tau) d\tau = m^{(b)},$$

where $m^{(a)}$ and $m^{(b)}$ are the mass influx on the both sides of the contact discontinuity, respectively.
and that $$\int_{g^-(x)}^{g^+(x)} \rho u(x, \tau) d\tau = m^{(a)} + m^{(b)},$$ where $$m^{(a)} = \int_0^{g^+(0)} \rho^{(a)} u^{(a)}(0) d\tau, \quad m^{(b)} = \int_{g^-(0)}^0 \rho^{(b)} u^{(b)}(0) d\tau,$$

are the mass fluxes at the inlet above and below the contact discontinuity respectively.

Let

$$\eta(x, y) = \int_{g^-(x)}^y \rho u(x, \tau) d\tau - m^{(b)}.$$ 

By (1.1), it is easy to see that

$$\frac{\partial \eta(x, y)}{\partial x} = -\rho v, \quad \frac{\partial \eta(x, y)}{\partial y} = \rho u.$$

Now, we can introduce the Lagrangian coordinates transformation $\mathcal{L}$ as

$$\mathcal{L} : \begin{cases} \xi = x, \\ \eta = \eta(x, y). \end{cases}$$

Notice that

$$\frac{\partial (\xi, \eta)}{\partial (x, y)} = \begin{pmatrix} 1 & 0 \\ -\rho v & \rho u \end{pmatrix},$$

so the Lagrangian coordinates transformation is invertible if and only if $\rho u \neq 0$, which is guaranteed in Theorem 3.1, i.e., Remark 3.1.

![Figure 3.1. Supersonic contact discontinuity nozzle flows in Lagrangian coordinates](image.png)

Under this coordinates transformation, domain $\Omega$ becomes

$$\tilde{\Omega} = \{(\xi, \eta) \in \mathbb{R}^2 : 0 < \xi < L, \ -m^{(b)} < \eta < m^{(a)}\},$$

and the lower and upper boundaries are

$$\tilde{\Gamma}_- := \{(\xi, \eta) : \eta = -m^{(b)}, \ 0 < \xi < L\}, \quad \tilde{\Gamma}_+ := \{(\xi, \eta) : \eta = m^{(b)}, \ 0 < \xi < L\}.$$ 

On $\Gamma_{cd}$, we have

$$\eta(x, g_{cd}(x)) = \int_{g^-(x)}^{g^+(x)} \rho u(x, \tau) d\tau - m^{(b)} = 0.$$
Thus, the free boundary $\Gamma_{cd}$ is transformed into the following fixed straight line:

$$\Gamma_{cd} := \{ (\xi, \eta) : \eta = 0, \ 0 < \xi < L \}. \quad (3.2)$$

Define

$$\tilde{\Omega}^{(a)} := \Omega \cap \{ 0 < \eta < m^{(a)} \}, \quad \tilde{\Omega}^{(b)} := \Omega \cap \{ -m^{(b)} < \eta < 0 \},$$

and let

$$\tilde{U}^{(i)}(\xi, \eta) = (\tilde{u}^{(i)}, \tilde{v}^{(i)}, \tilde{p}^{(i)})^{\top}(\xi, \eta), \quad (\xi, \eta) \in \tilde{\Omega}^{(i)}, \ i = a, b,$$

be the corresponding solutions in $\tilde{\Omega}^{(a)}$ and $\tilde{\Omega}^{(b)}$, respectively. Then the corresponding background state in the new coordinates corresponding to (2.1) is

$$\tilde{U}(\xi, \eta) = \begin{cases} U^{(a)}(\xi, \eta), \quad (\xi, \eta) \in (0, L) \times (0, m^{(a)}), \\ U^{(b)}(\xi, \eta), \quad (\xi, \eta) \in (0, L) \times (-m^{(b)}, 0), \end{cases} \quad (3.3)$$

where $m^{(i)} = \rho^{(i)} \tilde{u}^{(i)}, \ i = a, b$. The flow at the inlet $\xi = 0$ is given by

$$\tilde{U}_0(\eta) = \begin{cases} \tilde{U}_0^{(a)}(\eta), \quad 0 < \eta < m^{(a)}, \\ \tilde{U}_0^{(b)}(\eta), \quad -m^{(b)} < \eta < 0, \end{cases} \quad (3.4)$$

where $\tilde{U}_0^{(i)}(\eta) = (\tilde{u}_0^{(i)}, \tilde{v}_0^{(i)}, \tilde{p}_0^{(i)})(\eta), \ i = a, b$, satisfies

$$\left( \begin{array}{c} \tilde{v}_0^{(a)} \\ \tilde{u}_0^{(a)} \end{array} \right)(0) = \left( \begin{array}{c} \tilde{v}_0^{(b)} \\ \tilde{u}_0^{(b)} \end{array} \right)(0), \quad \tilde{p}_0^{(a)}(0) = \tilde{p}_0^{(b)}(0). \quad (3.5)$$

Notice that

$$\partial_x = \partial_{\xi} - \rho \nu \partial_{\eta}, \quad \partial_y = \rho \nu \partial_{\eta},$$

then system (1.1) in the Lagrangian coordinates becomes

$$\begin{cases} \partial_{\xi} \left( \frac{1}{\rho} \tilde{u}^{(a)} \right) - \partial_{\eta} \left( \frac{\tilde{v}^{(a)}}{\tilde{u}^{(a)}} \right) = 0, \\ \partial_{\xi} \left( \tilde{u} + \frac{\tilde{p}}{\rho} \right) - \partial_{\eta} \left( \frac{\tilde{v} \tilde{u}}{\rho} \right) = 0, \\ \partial_{\xi} \tilde{v} + \partial_{\eta} \tilde{p} = 0, \\ \tilde{p} = A(S)\tilde{p}^{-1}, \end{cases} \quad (3.6)$$

together with the Bernoulli law:

$$\frac{1}{2}(\tilde{u}^2 + \tilde{v}^2) + \frac{\gamma \tilde{p}}{(\gamma - 1)\rho} = \left\{ \begin{array}{ll} \tilde{B}_0^{(a)}(\eta), & (\xi, \eta) \in \tilde{\Omega}^{(a)}, \\ \tilde{B}_0^{(b)}(\eta), & (\xi, \eta) \in \tilde{\Omega}^{(b)} \end{array} \right. \quad (3.7)$$

Here we use the fact that $\tilde{B}_0^{(i)}(\eta)$ for $i = a, b$ are conserved along the streamlines by (1.1), thus they depend only on $\gamma$ and the incoming flow $\tilde{U}_0(\eta)$ at the entrance $\xi = 0$.

The boundary conditions in (2.4) become

$$\frac{\tilde{v}^{(a)}}{\tilde{u}^{(a)}} \bigg|_{\Gamma_+} = g_+(\xi), \quad \frac{\tilde{v}^{(b)}}{\tilde{u}^{(b)}} \bigg|_{\Gamma_-} = g_-(\xi), \quad (3.8)$$

and the Rankine-Hugoniot conditions on $\Gamma_{cd}$ read as

$$\frac{\tilde{v}^{(a)}}{\tilde{u}^{(a)}} \bigg|_{\Gamma_{cd}} = \frac{\tilde{v}^{(b)}}{\tilde{u}^{(b)}} \bigg|_{\Gamma_{cd}} = g_{cd}(\xi), \quad \tilde{p}^{(a)} \bigg|_{\Gamma_{cd}} = \tilde{p}^{(b)} \bigg|_{\Gamma_{cd}}, \quad (3.9)$$

which indicates that $\tilde{u}$ and $\tilde{p}$ are continuous across $\Gamma_{cd}$. 
Then, the free boundary value problem, Problem A, in the Eulerian coordinates can be reformulated as the following nonlinear fixed boundary value problem in the Lagrangian coordinates.

**Problem B.** Given a supersonic incoming flow $\tilde{U}_0(\eta)$ at the entrance $\{\xi = 0\}$ by (3.4) satisfying (3.5) and (3.8), find a piecewise smooth solution $\tilde{U}(\xi, \eta)$ with a contact discontinuity along the straight line $\tilde{\Gamma}_{cd}$ such that, the flows are smooth and supersonic, and satisfy the Euler equation (3.6) and (3.7) in $\tilde{\Omega}^{(a)} \cup \tilde{\Omega}^{(b)}$, the slip boundary condition (3.8) on $\tilde{\Gamma}_\pm$, and the Rankine-Hugoniot condition (3.9) on $\tilde{\Gamma}_{cd}$.

Thus Theorem 2.1 becomes the following theorem:

**Theorem 3.1.** There exist two constants $\tilde{\varepsilon}_0 > 0$ and $\tilde{C}_0 > 0$ depending only on $\tilde{U}$, $L$ and $\gamma$, such that for any $\tilde{\varepsilon} \in (0, \tilde{\varepsilon}_0)$ if

$$
\|\tilde{U}^{(a)}_0 - L^{(a)}\|_{C^2([0, m^{(a)})]} + \|\tilde{U}^{(b)}_0 - L^{(b)}\|_{C^2([-m^{(b)}, 0)]} + \|g_- + 1\|_{C^3([0, L])} + \|g_+ - 1\|_{C^3([0, L])} \leq \tilde{\varepsilon},
$$

(3.10)

then Problem B admits a unique piecewise smooth solution $\tilde{U}(\xi, \eta)$ consisting of two smooth supersonic flow $\tilde{U}^{(a)} \in C^1(\tilde{\Omega}^{(a)})$ and $\tilde{U}^{(b)} \in C^1(\tilde{\Omega}^{(b)})$ with $\eta = 0$ as the discontinuity. Moreover,

$$
\|\tilde{U}^{(a)} - L^{(a)}\|_{C^1(\tilde{\Omega}^{(a)})} + \|\tilde{U}^{(b)} - L^{(b)}\|_{C^1(\tilde{\Omega}^{(b)})} \leq \tilde{C}_0 \tilde{\varepsilon}.
$$

(3.11)

**Remark 3.1.** Theorem 3.1 and Theorem 2.1 are equivalent under the Lagrangian coordinates transformation $\mathcal{L}$. The reason is that for small $\tilde{\varepsilon}$, we have

$$
\det \left( \frac{\partial (\xi, \eta)}{\partial (x, y)} \right) = pu > 0.
$$

So the inverse Lagrangian coordinates transformation $\mathcal{L}^{-1}$ exists and can be given explicitly by

$$
\mathcal{L}^{-1} : \begin{cases}
    x = \xi, \\
y = \int_{-m^{(b)}}^{\gamma} \left( \frac{1}{pu} \right)(\xi, \tau) d\tau + g_-(\xi).
\end{cases}
$$

Therefore, if Theorem 3.1 holds, then one can use $\mathcal{L}^{-1}$ to define $U(x, y) = \tilde{U}(\xi(x, y), \eta(x, y))$ and define

$$
g_{cd}(x) = \int_{-m^{(b)}}^{0} \left( \frac{1}{pu} \right)(x, \tau) d\tau + g_-(x), \quad x \in [0, L].
$$

Obviously,

$$
g'_{cd}(x) = \frac{u}{u}(x, g_{cd}(x)), \quad x \in [0, L],
$$

and $g'_{cd}(x) \in C^2([0, L])$. It concludes the proof of Theorem 2.1. Therefore, in the rest of the paper, we only need to consider Problem B and prove Theorem 3.1.

### 3.2. Riemann invariants

In this subsection, we will use the Riemann invariants to diagonalize the system (3.6). First, the system (3.6) can be rewritten as the following first order non-divergence symmetric system.

$$
A(\tilde{U})\partial_x \tilde{U} + B(\tilde{U})\partial_\eta \tilde{U} = 0,
$$

(3.12)
where \( \tilde{U}(\xi, \eta) = (\tilde{u}, \tilde{v}, \tilde{p})^\top(\xi, \eta) \) and
\[
A(\tilde{U}) = \begin{pmatrix}
\tilde{u} & 0 & \frac{1}{\rho} \\
0 & \tilde{u} & 0 \\
\frac{1}{\rho} & 0 & \frac{\tilde{u}}{c^2 \rho^2}
\end{pmatrix}, \quad B(\tilde{U}) = \begin{pmatrix}
0 & 0 & -\tilde{v} \\
0 & 0 & \tilde{u} \\
-\tilde{v} & \tilde{u} & 0
\end{pmatrix}.
\]
The eigenvalues of (3.12) are
\[
\lambda_- = \frac{\rho u c^2}{u^2 - c^2} \left( \frac{\tilde{v}}{\tilde{u}} - \frac{\sqrt{u^2 + \tilde{v}^2 - c^2}}{c} \right), \quad \lambda_0 = 0, \quad \lambda_+ = \frac{\rho u c^2}{u^2 - c^2} \left( \frac{\tilde{v}}{\tilde{u}} + \frac{\sqrt{u^2 + \tilde{v}^2 - c^2}}{c} \right),
\]
and the associated right and left eigenvectors are
\[
\begin{align*}
\begin{pmatrix} r_- \end{pmatrix} &= \left( \frac{\lambda_-}{\rho} + \tilde{v}, -\tilde{v}, -\lambda_- \tilde{u} \right)^\top, \\
r_0 &= \left( \tilde{u}, \tilde{v}, 0 \right)^\top, \\
r_+ &= \left( \frac{\lambda_+}{\rho} + \tilde{v}, -\tilde{v}, -\lambda_+ \tilde{u} \right)^\top,
\end{align*}
\]
and
\[
l_\pm = (r_\pm)^\top, \quad l_0 = (r_0)^\top.
\]
Multiply system (3.12) by \( l_\pm \) and \( l_0 \) to get
\[
\tilde{v}(\partial_\xi \tilde{u} + \lambda_\pm \partial_\eta \tilde{u}) - \tilde{u}(\partial_\xi \tilde{v} + \lambda_\pm \partial_\eta \tilde{v}) \pm \frac{\sqrt{u^2 + \tilde{v}^2 - c^2}}{\rho c} (\partial_\xi \tilde{p} + \lambda_\pm \partial_\eta \tilde{p}) = 0, \quad (3.13)
\]
and
\[
\tilde{u} \partial_\xi \tilde{u} + \tilde{v} \partial_\xi \tilde{v} + \frac{\partial_\xi \tilde{p}}{\rho} = 0. \quad (3.14)
\]
By the Bernoulli law (3.7), (3.14) can be reduced to
\[
\partial_\xi \tilde{S} = 0,
\]
which implies that \( \tilde{S} \) is constant in \( \tilde{\Omega}^{(a)} \) or \( \tilde{\Omega}^{(b)} \), i.e.,
\[
\tilde{S} = \tilde{S}_0(\eta). \quad (3.15)
\]
Denote
\[
\tilde{w} := \frac{\tilde{v}}{\tilde{u}}, \quad \Lambda := \frac{\sqrt{u^2 + \tilde{v}^2 - c^2}}{\rho c u^2},
\]
and define the operator,
\[
\mathcal{D}_- = \partial_\xi + \lambda_- \partial_\eta, \quad \mathcal{D}_+ = \partial_\xi + \lambda_+ \partial_\eta.
\]
Then, we can further rewrite the equations (3.13) as
\[
\mathcal{D}_- \tilde{w} - \Lambda \mathcal{D}_- \tilde{p} = 0, \quad \mathcal{D}_+ \tilde{w} + \Lambda \mathcal{D}_+ \tilde{p} = 0. \quad (3.16)
\]
**Remark 3.2.** Once \( \tilde{w} \) and \( \tilde{p} \) are solved, then by the Bernoulli law (3.7), we can obtain \( \tilde{u} \) and \( \tilde{v} \) as the following:
\[
\tilde{u} = \sqrt{\frac{2 \left( (\gamma - 1) \tilde{B}_0 - \gamma \tilde{A}_0 \tilde{v}^{\frac{\gamma - 1}{\gamma}} \right)}{(\gamma - 1)(1 + \tilde{w}^2)}}, \quad \tilde{v} = \tilde{w} \sqrt{\frac{2 \left( (\gamma - 1) \tilde{B}_0 - \gamma \tilde{A}_0 \tilde{p}^{\frac{\gamma - 1}{\gamma}} \right)}{(\gamma - 1)(1 + \tilde{w}^2)}},
\]
where \( \tilde{A}_0(\eta) = A(\tilde{S}_0(\eta)) \) and \( \tilde{S}_0(\eta) \) is given by (3.15).
Therefore, we only need to solve \( \bar{w} \) and \( \bar{p} \). Set \( \mathcal{W} = (\bar{w}, \bar{p})^\top \), then the system \( (3.16) \) can be rewritten in the following:

\[
\partial_\zeta \mathcal{W} + \mathcal{A} \partial_\eta \mathcal{W} = 0,
\]

where

\[
\mathcal{A} = \begin{pmatrix}
\frac{\tilde{\rho}\tilde{u}^2}{\tilde{u}^2 - \tilde{c}^2} & \frac{\tilde{\rho}\tilde{v}^2}{\tilde{u}^2 - \tilde{c}^2} \\
\frac{\tilde{\rho}\tilde{c}^2\tilde{u}^2}{\tilde{u}^2 - \tilde{c}^2} & \frac{\tilde{\rho}\tilde{u}^2}{\tilde{u}^2 - \tilde{c}^2}
\end{pmatrix}.
\]

Direct computation shows that the eigenvalues of \( (3.18) \) are \( \lambda_\pm \) and the corresponding right eigenvectors \( \tilde{r}_\pm \) are

\[
\tilde{r}_\pm = (\sqrt{\tilde{u}^2 + \tilde{v}^2 - \tilde{c}^2}, \pm \tilde{\rho}\tilde{c}\tilde{u}^2)^\top.
\]

Then we can define the Riemann invariants \( z_\pm \) for the system \( (3.17) \) as the following form:

\[
z_- = \arctan \bar{w} + \Theta(\bar{p}; \tilde{S}_0, \tilde{B}_0), \quad z_+ = \arctan \bar{w} - \Theta(\bar{p}; \tilde{S}_0, \tilde{B}_0),
\]

where

\[
\Theta(\bar{p}; \tilde{S}_0, \tilde{B}_0) = \int_{\tilde{p}} \frac{\sqrt{2\tilde{B}_0 - \frac{\gamma(\gamma+1)}{\gamma-1} A_0^\frac{1}{\gamma} \tilde{p}^{\frac{2-\gamma}{\gamma}}}}{2\gamma^\frac{1}{2} A_0^\frac{1}{2\gamma} \left( \tilde{B}_0 - \frac{\gamma-1}{\gamma-1} A_0^\frac{1}{2} \tilde{p}^{-1} \tilde{p}^{\frac{1}{2\gamma}} \right)^\frac{2+\gamma}{2\gamma}} d\tau.
\]

By \( (3.19) \) and \( (3.20) \), we have

\[
\bar{w} = \tan \left( \frac{z_- + z_+}{2} \right), \quad \Theta(\bar{p}; \tilde{S}_0, \tilde{B}_0) = \frac{1}{2}(z_- - z_+).
\]

Set \( z = (z_-, z_+)^\top \). By the implicit function theorem, we have the following lemma.

**Lemma 3.1.** For any given \( z \), if the flow is supersonic, then equation \( (3.21) \) admits a unique solution \( \bar{p} = \bar{p}(z; \tilde{S}_0, \tilde{B}_0) \) and \( \bar{w} = \bar{w}(z) \).

**Proof.** By the straightforward computation, we have

\[
\frac{\partial \bar{p}}{\partial z_-} = \frac{1}{2\partial_\bar{p}\Theta(\bar{p}; \tilde{S}_0, \tilde{B}_0)}, \quad \frac{\partial \bar{p}}{\partial z_+} = -\frac{1}{2\partial_\bar{p}\Theta(\bar{p}; \tilde{S}_0, \tilde{B}_0)},
\]

where

\[
\partial_\bar{p}\Theta(\bar{p}; \tilde{S}_0, \tilde{B}_0) = \frac{\sqrt{2\tilde{B}_0 - \frac{\gamma(\gamma+1)}{\gamma-1} A_0^\frac{1}{\gamma} \tilde{p}^{\frac{2-\gamma}{\gamma}}}}{2\gamma^\frac{1}{2} A_0^\frac{1}{2\gamma} \left( \tilde{B}_0 - \frac{\gamma-1}{\gamma-1} A_0^\frac{1}{2} \tilde{p}^{-1} \tilde{p}^{\frac{1}{2\gamma}} \right)^\frac{2+\gamma}{2\gamma}} > 0.
\]

Then the lemma follows from the implicit function theorem. \( \square \)
By Remark 3.2 and Lemma 3.1 we only need to consider the following nonlinear boundary value problem:

\[
\begin{cases}
    \partial_\xi z^a + \text{diag}(\lambda^+ a, \lambda^a) \partial_\eta z^a = 0, & \text{in } \tilde{\Omega}^{(a)}, \\
    \partial_\xi z^b + \text{diag}(\lambda^+ b, \lambda^b) \partial_\eta z^b = 0, & \text{in } \tilde{\Omega}^{(b)}, \\
    z^a = \tilde{z}^a_0(\eta), & \text{on } \xi = 0, \\
    z^b = \tilde{z}^b_0(\eta), & \text{on } \xi = 0, \\
    z^a + z^b_+ = 2 \arctan g^+(\xi), & \text{on } \tilde{\Gamma}^+, \\
    z^b + z^a_+ = 2 \arctan g^-(\xi), & \text{on } \tilde{\Gamma}^-, \\
    \tilde{p}(z^a; \tilde{S}^a_0, \tilde{B}^a_0) = \tilde{p}(z^b; \tilde{S}^b_0, \tilde{B}^b_0), & \text{on } \tilde{\Gamma}_{cd}.
\end{cases}
\] (3.24)

Here, \(z^{(i)}, \lambda^{(i)}_\pm, \tilde{S}^{(i)}_0, \tilde{B}^{(i)}_0\), \(i = a, b\), represent the states taking values in \(\tilde{\Omega}^{(a)}\) and \(\tilde{\Omega}^{(b)}\), respectively.

Then Problem B is reformulated into the following problem.

**Problem C.** Given a supersonic incoming flow \((z^a_0, z^b_0)(\eta)\) at the entrance \(\{\xi = 0\}\) satisfying (3.15) and (3.9), find a piecewise smooth supersonic flow \((z^a, z^b)(\xi, \eta)\) of the nonlinear boundary value problem \((\tilde{P})\).

Then, we have the following theorem for Problem C.

**Theorem 3.2.** There exist two constants \(\tilde{\varepsilon}_1 > 0\) and \(\tilde{C}_1 > 0\) depending only on \(\tilde{U}, L\) and \(\gamma\) such that, for each \(\varepsilon_1 \in (0, \tilde{\varepsilon}_0]\), if

\[
\sum_{i=a,b} \left( \|z^i_0 - \tilde{z}^i\|_{C^2(\Sigma_i)} + \|\tilde{S}^i_0 - \tilde{S}^i\|_{C^2(\Sigma_i)} + \|\tilde{B}^i_0 - \tilde{B}^i\|_{C^2(\Sigma_i)} \right)
\]

\[
+ \|g_+ - 1\|_{C^3([0,L])} + \|g_- + 1\|_{C^3([0,L])} \leq \tilde{\varepsilon}_1,
\] (3.25)

where \(\Sigma_a = (0, m^{(a)}), \Sigma_b = (-m^{(b)}, 0)\), then Problem C admits a unique solution \((z^a, z^b) \in C^1(\tilde{\Omega}^{(a)}) \times C^1(\tilde{\Omega}^{(b)})\) satisfying

\[
\|z^a - \tilde{z}^a\|_{C^1(\tilde{\Omega}^{(a)})} + \|z^b - \tilde{z}^b\|_{C^1(\tilde{\Omega}^{(b)})} \leq \tilde{C}_1 \tilde{\varepsilon}_1.
\] (3.26)

**Remark 3.3.** By Remark 3.2 and Lemma 3.1, Theorem 3.2 directly follows from Theorem 3.2. Thus, to solve Problem B, it suffices to show the existence and uniqueness of solutions of problem \((\tilde{P})\). The next several sections are devoted to the study of the non-linear boundary value problem \((\tilde{P})\).

### 3.3. Iteration scheme

In order to solve the nonlinear boundary value problem: Problem C, we first linearize the problem \((\tilde{P})\) to construct a sequence of approximate solutions, and then show that the approximate solutions are convergent and their limit is actually the solution of Problem C. The linearized hyperbolic boundary value problem for \(z\) is solved in \(\tilde{\Omega}^{(a)}\) and \(\tilde{\Omega}^{(b)}\) at the same time (see Fig 3.2).
First, the linearized boundary value problem of (P) is

\[
\begin{array}{l}
\partial_\xi \delta z^{a,(n)} + \text{diag}(\tilde{\lambda}_+^{a,(n-1)}, \tilde{\lambda}_-^{a,(n-1)}) \partial_\eta \delta z^{a,(n)} = 0, \quad \text{in } \tilde{\Omega}^{(a)}, \\
\partial_\xi \delta z^{b,(n)} + \text{diag}(\tilde{\lambda}_+^{b,(n-1)}, \tilde{\lambda}_-^{b,(n-1)}) \partial_\eta \delta z^{b,(n)} = 0, \quad \text{in } \tilde{\Omega}^{(b)}, \\
\delta z^{a,(n)} = \delta z_0^{a}(\eta), \\
\delta z^{b,(n)} = \delta z_0^{b}(\eta), \\
\delta z_-^{a,(n)} + \delta z_-^{a,(n)} = 2 \arctan g'_+(\xi), \quad \text{on } \tilde{\Gamma}_+, \\
\delta z_-^{b,(n)} + \delta z_-^{b,(n)} = 2 \arctan g'_-(\xi), \quad \text{on } \tilde{\Gamma}_-, \\
\delta z_+^{a,(n)}-\delta z_+^{b,(n)} = \delta z_+^{a,(n)}-\delta z_+^{b,(n)}, \quad \text{on } \tilde{\Gamma}_{cd}, \\
\alpha^{(n-1)} \delta z_-^{a,(n)} + \beta^{(n-1)} \delta z_-^{b,(n)} = \alpha^{(n-1)} \delta z_+^{a,(n)} + \beta^{(n-1)} \delta z_+^{b,(n)} + c(\xi), \quad \text{on } \tilde{\Gamma}_{cd},
\end{array}
\]

(3.27)

where

\[
\delta z^{i,(n)} := z^{(n)} - \bar{z}^i, \quad \delta z_0^{i}(\eta) := z_0^i(\eta) - \bar{z}^i, \quad \tilde{\lambda}_\pm^{i,(n-1)} := \lambda_\pm^i(z_-^{i,(n-1)}, z_+^{i,(n-1)}),
\]

for \(i = a, b\),

\[
\alpha^{(n-1)} := \alpha(z^{a,(n-1)}; \tilde{S}_0^a, B_0^a) = \frac{1}{2 \int_0^1 \partial_\tau \Theta(p_\tau + \tau (p^{a,(n-1)} - \bar{p}); \tilde{S}_0^a, B_0^a) d\tau},
\]

(3.28)

\[
\beta^{(n-1)} := \beta(z^{b,(n-1)}; \tilde{S}_0^b, B_0^b) = \frac{1}{2 \int_0^1 \partial_\tau \Theta(p_\tau + \tau (p^{b,(n-1)} - \bar{p}); \tilde{S}_0^b, B_0^b) d\tau},
\]

and

\[
c(\xi) = p(z^{a}; \tilde{S}_0^a, B_0^a) - p(z^{a}; \tilde{S}_0^a, B_0^b) + p(z^{b}; \tilde{S}_0^a, B_0^b) - p(z^{b}; \tilde{S}_0^b, B_0^b).
\]

(3.29)

(3.30)

Define the iteration set as

\[
\mathcal{M}_\varepsilon = \{ (z^{a,(n)}, z^{b,(n)}) : \|z^{a,(n)} - \bar{z}^a\|_{C^2(\tilde{\Omega})} + \|z^{b,(n)} - \bar{z}^b\|_{C^2(\tilde{\Omega})} \leq \varepsilon \},
\]

for some \(0 < \varepsilon < 1\).

Next, let us introduce the map

\[
\mathcal{T} : \mathcal{M}_{2\sigma} \rightarrow \mathcal{M}_{2\sigma}.
\]

(3.31)

For given functions \((z^{a,(n-1)}, z^{b,(n-1)}) \in \mathcal{M}_{2\sigma}\), we solve the linearized boundary value problem \((\mathcal{P}_n)\) to obtain functions \((z^{a,(n)}, z^{b,(n)}) \in \mathcal{M}_{2\sigma}\). Then the map \(\mathcal{T}\) is defined such
that
\[(z^{a,(n)}, z^{b,(n)}) := T(z^{a,(n-1)}, z^{b,(n-1)}). \tag{3.32}\]

For the map \(T\), we shall show the following two facts:
(1) \(T\) is well-defined, i.e., \(T\) exists and maps from \(M_{2\sigma}\) to itself;
(2) \(T\) is a contraction map.

Once the above facts are proved, the convergence follows from the Banach fixed point theorem and one can show that the limit is the solution of the boundary value problem (\(\tilde{P}\)). The first fact will be proved in Theorem 4.1 and the second fact will be proved in Proposition 5.1.

4. Estimates for the Solutions to the Problem \((\tilde{P}_n)\) in \(\tilde{\Omega}^{(a)} \cup \tilde{\Omega}^{(b)}\)

In this section, we shall consider the solutions of Problem \((\tilde{P}_n)\) near the background state \(\tilde{z}\) and prove the following theorem.

**Theorem 4.1.** There exist positive constants \(C^*\) and \(\sigma^*_0\) depending only on \(\tilde{U}\), \(L\) and \(\gamma\) such that, for any \(\sigma \in (0, \sigma^*_0)\), if \((\delta z^{a,(n-1)}, \delta z^{b,(n-1)}) \in M_{2\sigma}\), then the solutions \((\delta z^{a,(n)}, \delta z^{b,(n)})\) to the problem \((\tilde{P}_n)\) satisfy
\[
\|\delta z^{a,(n)}\|_{C^2(\Omega^{(a)})} + \|\delta z^{b,(n)}\|_{C^2(\Omega^{(b)})} \\
\leq C^* \left( \sum_{i=a,b} \left( \|\tilde{z}^i_0 - \tilde{z}^i\|_{C^2(\Sigma_i)} + \|\tilde{S}^i_0 - \tilde{S}^i\|_{C^2(\Sigma_i)} \right) \\
+ \sum_{i=a,b} \|\tilde{B}^i_0 - \tilde{B}^i\|_{C^2(\Sigma_i)} + \|g_+ - 1\|_{C^3([0,L])} + \|g_- + 1\|_{C^3([0,L])} \right). \tag{4.1}\]
\]

where \(\Sigma_a = (0, m^{(a)})\), \(\Sigma_b = (-m^{(b)}, 0)\).

We shall divide the proof of the Theorem 4.1 into several parts for different subdomains.

As shown in Fig. 4.1, the subdomains are determined as follows. Let \(\Omega_I\) be the union of two triangles which are bounded by the inlet \(\xi = 0\), the two characteristics of (3.27)
corresponding to \( \lambda_\pm \) starting from point \((0, -m^{(b)})\) (denoted by \( l_1^+ \)) and from point \((0, 0)\) (denoted by \( l_0^+ \)), and the other two characteristics of \( \lambda_- \) starting from point \((0, m^{(a)})\) (denoted by \( l_1^- \)) and from point \((0, 0)\) (denoted by \( l_0^- \)). Let \( \Omega_{II} \) be the union of two triangles bounded by the nozzle walls \( \eta = m^{(a)} \) (or \( \eta = -m^{(b)} \)), \( l_0^+ \) (or \( l_0^- \)), and \( l_1^\pm \) (or \( l_1^- \)). Let \( \Omega_{III} \) be the diamond bounded by \( l_0^\pm \) and \( l_1^\pm \). Let \( \Omega_{IV} \) be the union of two diamonds bounded by \( l_0^\pm, l_1^\pm \), the characteristic corresponding to \( \lambda_+ \) starting from the intersection point of \( l_0^- \) and \( \eta = -m^{(b)} \), and the characteristic corresponding to \( \lambda_- \) starting from the intersection point of \( l_0^+ \) and \( \eta = m^{(a)} \).

4.1. Estimates of solutions of the boundary value problem \((\tilde{P}_n)\) in \( \Omega_I \). In this subsection, we study the problem \((\tilde{P}_n)\) in \( \Omega_I \) by the characteristic method. Notice that \( \Omega_I \) is bounded by the inlet and the characteristics issuing from the inlet, so if we regard \( \xi \) as the time, then the problem \((\tilde{P}_n)\) in \( \Omega_I \) can be regarded as the following initial value problem \((\tilde{P}_n)_1\):

\[
\begin{align*}
(\tilde{P}_n)_1: & \quad \partial_\xi \delta z^{a,(n)} + \text{diag}(\tilde{\lambda}^{a,(n-1)}_+, \tilde{\lambda}^{a,(n-1)}_-) \partial_\eta \delta z^{a,(n)} = 0, \quad \text{in } \tilde{\Omega}^{(a)}, \\
& \quad \partial_\xi \delta z^{b,(n)} + \text{diag}(\tilde{\lambda}^{b,(n-1)}_+, \tilde{\lambda}^{b,(n-1)}_-) \partial_\eta \delta z^{b,(n)} = 0, \quad \text{in } \tilde{\Omega}^{(b)}, \\
& \quad \delta z^{a,(n)} = \delta z^{a}_0, \quad \delta z^{b,(n)} = \delta z^{b}_0, \quad \text{on } \xi = 0,
\end{align*}
\]

(4.2)

For \( \eta_0 \in [-m^{(b)}, m^{(a)}] \), let \( \eta = \chi^{i,(n)}_\pm(\xi, \eta_0) \) be the characteristic curves corresponding to \( \tilde{\lambda}^{i,(n-1)}_\pm \), issuing from the point \((0, \eta_0)\), i.e., for \( i = a \) or \( b \),

\[
\begin{align*}
\frac{d\chi^{i,(n)}_\pm}{d\xi} &= \tilde{\lambda}^{i,(n-1)}_\pm(\xi, \chi^{i,(n)}_\pm), \\
\chi^{i,(n)}_\pm(0, \eta_0) &= \eta_0.
\end{align*}
\]

Then, along the characteristics one has

\[
\eta = \eta_0 + \int_0^\xi \tilde{\lambda}^{i,(n-1)}_\pm(\tau, \chi^{i,(n)}_\pm(\tau, \eta_0)) d\tau.
\]

(4.3)

By \( \| \), for any given point \((\xi, \eta) \) in \( \Omega_I \), there exist unique \( \eta^{\pm}_0 \) such that the characteristics corresponding to \( \lambda_\pm \) issuing from \((0, \eta^{\pm}_0) \) respectively pass through \((\xi, \eta) \). Thus we can regard \( \eta^{\pm}_0 \) as a function of \((\xi, \eta) \) in \( \Omega_I \). For simplicity of notation, we write \( \eta^{\pm}_0 \) as \( \eta_0 \).

We have the following estimates for \( \eta_0 \).

**Lemma 4.1.** For any \( \delta z^{i,(n-1)} \in M_{20} \) \((i = a, b)\), there exist constants \( C_{0,1} \) and \( C_{0,2} \) depending only on \( \tilde{L} \), \( L \) and \( \sigma \) such that

\[
\| D^k \eta_0 \|_{C^0(\bar{\Omega}_0 \cup \Omega_{10})} \leq C_{0,k}, \quad k = 1, 2.
\]

(4.4)

**Proof.** Without loss of the generality, we only consider the estimates of \( \| \) in \( \Omega_I \), since otherwise we should consider the reflection of the characteristics by the nozzle walls which can be uniformly bounded by a constant depending only on the background solution. The discussion of the reflection of the characteristics by the nozzle walls is postponed to Lemma 4.2.

Taking derivatives on (4.3) with respect to \( \xi, \eta \) and by the direct computation we can obtain the formulas of the first and second order partial derivatives of \( \eta_0 \) with respect to
\( \xi, \eta \), for example,
\[
\frac{\partial \eta_0}{\partial \eta} = \frac{1}{1 + \int_0^\xi \partial \eta_i \partial \lambda^{i,(n-1)} \overline{c} ds d\tau}, \quad i = a, b,
\]
and
\[
\frac{\partial^2 \eta_0}{\partial \eta^2} = \frac{-\int_0^\xi \partial \eta_i \partial \lambda^{i,(n-1)} \overline{c} ds d\tau}{\left(1 + \int_0^\xi \partial \eta_i \partial \lambda^{i,(n-1)} \overline{c} ds d\tau \right)^2} d\eta_n.
\]

Other derivatives have the similar formulas. From these formulas we can deduce the estimates for \( \|D^2 \eta_0\|_{C^0(\Omega)} \). This complete the proof. \( \square \)

Let \( \xi^{a,+}_1 \) be the intersection point of the characteristics corresponding to \( \lambda_+ \) starting from \((0, 0)\) and the upper nozzle wall \( \eta = m^{(a)} \). Let \( \xi^{b,-}_1 \) be the intersection point of the characteristics corresponding to \( \lambda_- \) starting from \((0, 0)\) and the lower nozzle wall \( \eta = -m^{(b)} \), i.e., it holds that
\[
\int_0^{\xi^{a,+}_1} \lambda^{a,(n-1)}_+(\tau, \lambda^{a,(n)}_+(\tau, 0)) d\tau = m^{(a)},
\]
\[
\int_0^{\xi^{b,-}_1} \lambda^{b,(n-1)}_-(\tau, \lambda^{b,(n)}_-(\tau, 0)) d\tau = -m^{(b)}. \tag{4.5}
\]

Then we have the following proposition.

**Proposition 4.1.** There exist positive constants \( \tilde{C}_1 \) and \( \sigma_0 \) depending only on \( \tilde{U} \) and \( L, \gamma \) such that, for any \( \sigma \in (0, \sigma_0) \), if \( \delta z^{i,(n-1)} \in M_{2\sigma} \), the solution \( \delta z^{i,(n)} \) to the initial value problem \((\tilde{P}_n)_1\) satisfies
\[
\|\delta z^{a,(n)}\|_{C^2(\tilde{\Omega}^{(a)} \cap (\Omega_I \cup \Omega_{II}))} + \|\delta z^{b,(n)}\|_{C^2(\tilde{\Omega}^{(b)} \cap (\Omega_I \cup \Omega_{II}))} \leq \tilde{C}_1 \left(\|\delta z^{a,(0)}\|_{C^2([0,m^{(a)})]} + \|\delta z^{b,(0)}\|_{C^2([-m^{(b)},0])}\right), \tag{4.6}
\]
and
\[
\|\delta z^{a,(n)}\|_{C^2(\tilde{\Omega}^{(a)} \cap (\Omega_I \cup \Omega_{II}))} + \|\delta z^{b,(n)}\|_{C^2(\tilde{\Omega}^{(b)} \cap (\Omega_I \cup \Omega_{II}))} \leq \tilde{C}_1 \left(\|\delta z^{a,(0)}\|_{C^2([0,m^{(a)})]} + \|\delta z^{b,(0)}\|_{C^2([-m^{(b)},0])}\right). \tag{4.7}
\]

**Proof.** We only consider the estimates of the solution for \( z^{a,(n)}_\gamma \) in \( \tilde{\Omega}^{(a)} \cap (\Omega_I \cup \Omega_{II}) \) since the proof for the others are the same as \( z^{a,(n)} \). The proof is divided into three steps.

**Step 1.** For any point \((0, \eta_0)\) with \( \eta_0 \in [0, m^{(a)}] \), along the characteristic \( \eta = \lambda^{a,(n)}_+(\xi, \eta_0) \), we have \( \delta z^{a,(n)}_\gamma (\xi, \eta) = \delta z^{a,(n)}_0(\eta_0) \). Therefore, we have
\[
\|\delta z^{a,(n)}_\gamma \|_{C^0(\tilde{\Omega}^{(a)} \cap (\Omega_I \cup \Omega_{II}))} = \|\delta z^{a,(n)}_0\|_{C^0([0,m^{(a)}])}. \tag{4.8}
\]

**Step 2.** We now work on the estimate of \( \|D\delta z^{a,(n)}\|_{C^0(\tilde{\Omega}^{(a)} \cap (\Omega_I \cup \Omega_{II}))} \). To do this, we differentiate the equation \((4.2)\) for \( \delta z^{a,(n)} \) with respect to \( \xi, \eta \) to obtain
\[
\partial_\xi (\partial_\xi \delta z^{a,(n)}_\gamma) + \lambda^{a,(n-1)}_+ \partial_\eta (\partial_\xi \delta z^{a,(n)}_\gamma) = -\partial_\xi \tilde{\lambda}^{a,(n-1)}_+ \partial_\eta \delta z^{a,(n)}_\gamma, \tag{4.9}
\]
and

$$\partial_\xi (\partial_\eta \delta z_{-}^{a,n}) + \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n}) = -\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n}).$$

(4.10)

Then, along the $\eta = \chi_+^{a,n}(\xi, \eta_0)$ characteristic, it holds that

$$\partial_\xi \delta z_{-}^{a,n}(\xi, \eta) = \partial_\xi \delta z_{-}^{a,n}(0, \eta_0) - \int_0^\xi (\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n})(\tau, \chi_+^{a,n}(\tau, \eta_0)))d\tau$$

$$= \frac{\partial \eta_0}{\partial \xi} \partial_\eta \delta z_{-0}^{a} (\eta) - \int_0^\xi (\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n}) (\tau, \eta_0))d\tau,$$  

(4.11)

and

$$\partial_\eta \delta z_{-}^{a,n}(\xi, \eta) = \partial_\eta \delta z_{-}^{a,n}(0, \eta_0) - \int_0^\xi (\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n})(\tau, \chi_+^{a,n}(\tau, \eta_0)))d\tau$$

$$= \frac{\partial \eta_0}{\partial \eta} \partial_\eta \delta z_{-0}^{a} (\eta) - \int_0^\xi (\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n}) (\tau, \eta_0))d\tau.$$  

(4.12)

Now, we need to estimate each term in (4.11) and (4.12). First, let us consider the estimates for $\int_0^\xi (\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n}))(\tau, \eta_0))d\tau$ and $\int_0^\xi (\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n})(\tau, \eta_0))d\tau.$ Note that

$$\int_0^\xi (|\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n})(\tau, \eta_0)|)d\tau + \int_0^\xi (|\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n})(\tau, \eta_0)|)d\tau$$

$$\leq C |D\delta z_{-}^{a,n}(\xi, \eta)|_{C^0([0, \eta_0])} \int_0^\xi |D\delta z_{-}^{a,n}(\tau, \eta_0)|d\tau.$$  

(4.13)

where $C$ is a positive constant that depends only on the background solution $\tilde{U}$ and $\sigma.$ Hence $C$ essentially depends only on $\tilde{U}$ if $\sigma$ is small. Then by (4.13) and Lemma 4.1, we have

$$|D\delta z_{-}^{a,n}(\xi, \chi_+^{a,n}(\xi, \eta_0))| \leq \|D\eta_0\|_{C^0([\tilde{U}])} \|\partial_\eta \delta z_{-0}^{a} \|_{C^0([0, \eta_0])}$$

$$+ \int_0^\xi (|\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n})(\tau, \eta_0)|)d\tau$$

$$+ \int_0^\xi (|\partial_\eta \tilde{\lambda}_+^{a,n-1}(\partial_\eta \delta z_{-}^{a,n})(\tau, \eta_0)|)d\tau$$

$$\leq C |\partial_\eta \delta z_{-0}^{a} \|_{C^0([0, \eta_0])} + 2C\sigma \int_0^\xi |D\delta z_{-}^{a,n}(\tau, \eta_0)|d\tau,$$

where $C$ is a positive constant that depends only on the background solution $\tilde{U}$ and $L, \gamma$ for $\sigma$ sufficiently small. Applying the Gronwall inequality yields

$$\|D\delta z_{-}^{a,n}(\xi, \chi_+^{a,n}(\Omega_1, \Omega_2))\|_{C^0([\tilde{U}])} \leq C \|\partial_\eta \delta z_{-0}^{a} \|_{C^0([0, \eta_0])} e^{2CL\sigma}.$$  

(4.14)
Step 3. Finally, let us consider the estimate for \( \| D^2 \delta z_{-}^{a(n)} \|_{C^0([\bar{\Omega}(\omega) \cap (\Omega(t) \cup \Omega(t'))])} \). To do this, we differentiate equations (4.9) and (4.11) with respect to \((\xi, \eta)\) again to obtain

\[
\begin{align*}
\partial_\xi (\partial_{\xi \xi}^2 \delta z_{-}^{a(n)}) + \tilde{\lambda}_+^{a(n-1)} \partial_\eta (\partial_{\xi \eta}^2 \delta z_{-}^{a(n)}) &= -2 \partial_\xi \tilde{\lambda}_+^{a(n-1)} \partial_{\xi \eta}^2 \delta z_{-}^{a(n)} - \partial_{\xi \xi} \tilde{\lambda}_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)}, \\
\partial_\xi (\partial_{\eta \eta}^2 \delta z_{-}^{a(n)}) + \lambda_+^{a(n-1)} \partial_\eta (\partial_{\eta \eta}^2 \delta z_{-}^{a(n)}) &= -\partial_\eta \lambda_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)} - \delta_\eta \tilde{\lambda}_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)},
\end{align*}
\]

and

\[
\begin{align*}
\partial_\xi (\partial_{\eta \eta} \delta z_{-}^{a(n)}) + \tilde{\lambda}_+^{a(n-1)} \partial_\eta (\partial_\eta \delta z_{-}^{a(n)}) &= -2 \partial_\eta \lambda_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)} - \partial_\eta \tilde{\lambda}_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)}.
\end{align*}
\]

Then, integrating equations (4.15) and (4.16) along the characteristic \( \eta = \lambda_+^{a(n)}(\xi, \eta_0) \) from 0 to \( \xi \), we have

\[
\begin{align*}
\partial_{\xi \xi}^2 \delta z_{-}^{a(n)} &= \partial_{\xi \xi} \delta z_{-}^{a(n)}(0, \eta_0) - 2 \int_0^\xi \partial_\tau \tilde{\lambda}_+^{a(n-1)} \partial_{\tau \eta} \delta z_{-}^{a(n)} d\tau \\
&\quad - \int_0^\xi \partial_{\tau \tau} \tilde{\lambda}_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)} d\tau, \\
&= \left( \frac{\partial \eta_0}{\partial \xi} \right)^2 \partial_{\eta \eta} \delta z_{-}^{a,0} + \frac{\partial^2 \eta_0}{\partial \xi \partial \eta} \partial_\eta \delta z_{-}^{a,0} - 2 \int_0^\xi \partial_\tau \tilde{\lambda}_+^{a(n-1)} \partial_{\tau \eta} \delta z_{-}^{a(n)} d\tau \tag{4.17} \\
&\quad - \int_0^\xi \partial_{\tau \tau} \tilde{\lambda}_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)} d\tau := J_{11} + E_{11},
\end{align*}
\]

\[
\begin{align*}
\partial_{\xi \eta} \delta z_{-}^{a(n)} &= \frac{\partial \eta_0}{\partial \xi} \frac{\partial \eta_0}{\partial \eta} \partial_{\eta \eta} \delta z_{-}^{a,0} + \frac{\partial^2 \eta_0}{\partial \xi \partial \eta} \partial_\eta \delta z_{-}^{a,0} - \int_0^\xi \partial_\eta \tilde{\lambda}_+^{a(n-1)} \partial_{\tau \eta} \delta z_{-}^{a(n)} d\tau \\
&\quad - \int_0^\xi \partial_{\tau \tau} \tilde{\lambda}_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)} d\tau - \int_0^\xi \partial_{\eta \eta} \tilde{\lambda}_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)} d\tau \\
&:= J_{12} + E_{12},
\end{align*}
\]

and

\[
\begin{align*}
\partial_{\eta \eta} \delta z_{-}^{a(n)} &= \left( \frac{\partial \eta_0}{\partial \eta} \right)^2 \partial_{\eta \eta} \delta z_{-}^{a,0} + \frac{\partial^2 \eta_0}{\partial \xi \partial \eta} \partial_\eta \delta z_{-}^{a,0} - 2 \int_0^\xi \partial_\eta \tilde{\lambda}_+^{a(n-1)} \partial_{\tau \eta} \delta z_{-}^{a(n)} d\tau \\
&\quad - \int_0^\xi \partial_{\tau \tau} \tilde{\lambda}_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)} d\tau - \int_0^\xi \partial_{\eta \eta} \tilde{\lambda}_+^{a(n-1)} \partial_\eta \delta z_{-}^{a(n)} d\tau \tag{4.19} \\
&:= J_{22} + E_{22}.
\end{align*}
\]

Here \( J_{11}, J_{12}, J_{22} \) denote the first two terms on the right hands of (4.17)-(4.19) and \( E_{11}, E_{12}, E_{22} \) represent the rest of them.

For \( J_{11}, J_{12}, J_{22} \), we have, using Lemma 4.1

\[
|J_{11} + J_{12} + J_{22}| \leq C \left( \| D \eta_0 \|_{C^0([\bar{\Omega}_a] \cap \Omega(t))} + \| D^2 \eta_0 \|_{C^0([\bar{\Omega}_a])} \right) \| \delta z_{-}^{a,0} \|_{C^2([0, m(a))}) \]

\[
\leq C \| \delta z_{-}^{a,0} \|_{C^2([0, m(a))}),
\]

where \( C \) depends only on \( \widetilde{\Omega} \) and \( L, \gamma \).
Next, for $E_{11}$, $E_{12}$, $E_{22}$, we have

$$|E_{11} + E_{12} + E_{22}|$$

$$\leq C\|\delta z_{a,(n-1)}\|_{C^2(\tilde{\Omega}^a)} \int_0^\xi |D^2\delta z_{a,(n)}(\tau; \eta_0)|d\tau + C\|\delta z_{a,(n-1)}\|_{C^2(\tilde{\Omega}^a)} \int_0^\xi |D\delta z_{a,(n)}(\tau; \eta_0)|d\tau$$

$$\leq C\|\delta z_{a,0}\|_{C^2([0, m(a)])} + 2C\|\delta z_{a,(n)}(\tau; \eta_0)|d\tau,$$

where the positive constant $C$ depends only on $\tilde{U}$ and $L$, $\gamma$, using the estimate (4.14) and taking $0 < \sigma < 1$ sufficiently small.

Combining (4.20) and (4.21) together, we have

$$|D^2\delta z_{a,(n)}(\xi, \chi_{a,(n)}(\xi, \eta_0))| \leq \sum_{i,j=1,2} J_{ij} |E_{ij}|$$

$$\leq C\|\delta z_{a,0}\|_{C^2([0, m(a)])} + 2C\|\delta z_{a,(n)}(\tau; \eta_0)|d\tau.$$ 

Then by the Gronwall inequality, we have

$$|D^2\delta z_{a,(n)}(\xi, \chi_{a,(n)}(\xi, \eta_0))| \leq C\|\delta z_{a,0}\|_{C^2([0, m(a)])} e^{2C\sigma \xi}.$$ 

Hence, we can choose $\sigma''_0 > 0$ depending only on $\tilde{U}$ and $L$, $\gamma$ small enough such that for $\sigma \in (0, \sigma''_0)$, it holds that

$$\|D^2\delta z_{a,(n)}(\xi, \chi_{a,(n)}(\xi, \eta_0))\|_{C^0(\tilde{\Omega}^a)} \leq 2C\|\delta z_{a,0}\|_{C^2([0, m(a)])}.$$ 

Finally, from the estimates (4.9), (4.14) and (4.22) together, there exist constants $\tilde{C}_{a_1,1} > 0$ and $\sigma_0 = \min\{\sigma_0', \sigma''_0\}$ depending only on $\tilde{U}$ and $L$ such that for $\sigma \in (0, \sigma_0)$, it holds that

$$\|\delta z_{a,(n)}(\xi, \chi_{a,(n)}(\xi, \eta_0))\|_{C^2(\tilde{\Omega}^a)} \leq \tilde{C}_{a_1,1} \|\delta z_{a,0}\|_{C^2([0, m(a)])}.$$ 

This completes the proof of the proposition. \qed

4.2. Estimates for the boundary value problem $(\tilde{P}_n)$ in $\Omega_{II}$. In this subsection, we shall analyze the boundary value problem $(\tilde{P}_n)$ with the boundary conditions on $\tilde{\Gamma}_-$ and $\tilde{\Gamma}_+$ respectively. Note that the mathematical problem $(\tilde{P}_n)$ in $\Omega_{II}$ can be reformulated as the following problem:

$$\begin{cases}
\tilde{\Omega}^a, \\
\tilde{\Omega}^b, \\
\tilde{\Gamma}_+, \\
\tilde{\Gamma}_-, \\
\xi = 0, \\
\xi = 0.
\end{cases}$$

Similarly to §4.1, we shall study the problem $(\tilde{P}_n)_2$ also via the characteristics method but different from the one in §4.1 due to the reflection of the characteristics by the nozzle walls.
Let \( \xi = \psi^{i,n}_-(\eta, \xi^a) \) (or \( \xi = \psi^{i,n}_+(\eta, \xi^b) \)) be the characteristic curves corresponding to \( \lambda_- \) (or \( \lambda_+ \)) and passing through the points \((\xi^a, m^{(a)})\) (or \((\xi^b, -m^{(b)})\)), i.e., defined by

\[
\begin{aligned}
\frac{d\psi^{a,n}_-(\eta)}{d\eta} &= \frac{1}{\lambda_-^{n-1}(\eta, \psi^{a,n}_-)}; \\
\psi^{a,n}_-(m^{(a)}, \xi^a) &= \xi^a,
\end{aligned}
\tag{4.24}
\]

and

\[
\begin{aligned}
\frac{d\psi^{b,n}_+(\eta)}{d\eta} &= \frac{1}{\lambda_+^{n-1}(\eta, \psi^{b,n}_+)}; \\
\psi^{b,n}_+(\eta, \xi^b) &= \xi^b,
\end{aligned}
\tag{4.25}
\]

where \( \xi^a, \xi^b \in [0, L] \). Then,

\[
\xi = \xi^a + \int_{m^{(a)}}^\eta \frac{1}{\lambda_-^{n-1}(\tau, \xi^a)} \, d\tau, \quad \xi = \xi^b + \int_{m^{(b)}}^\eta \frac{1}{\lambda_+^{n-1}(\tau, \xi^b)} \, d\tau.
\]

For any point \((\xi, \eta)\) in \( \bar{\Omega}^{(a)} \cap \Omega_{II} \) (or in \( \bar{\Omega}^{(b)} \cap \Omega_{II} \)), there exists a unique \( \xi^a \) (or \( \xi^b \)) such that the characteristic corresponding to \( \lambda_- \) (or \( \lambda_+ \)) passes through \((\xi, \eta)\). Thus we can consider \( \xi^a \) (or \( \xi^b \)) as a function of \((\xi, \eta)\) in \( \bar{\Omega}^{(a)} \cap \Omega_{II} \) (or in \( \bar{\Omega}^{(b)} \cap \Omega_{II} \)), and we have the following lemma for \( \xi^a \) and \( \xi^b \).

**Lemma 4.2.** For any \( \delta z^{i,n-1} \in \mathcal{M}_{2\sigma} \) (i.e., \( \delta z^{i,n-1} \in \mathcal{M}_{2\sigma} \)), there exist constants \( C_{1,1}^i \) and \( C_{1,2}^i \) depending only on \( U, L \) and \( \sigma \) such that

\[
\|D^k \xi^i\|_{C^0(\bar{\Omega}^{(i)} \cap \Omega_{II})} \leq C_{1,k}^i, \quad k = 1, 2,
\]

where \( i = a, \) or \( b. \)

**Proof.** By the equation (4.24), the straightforward computation gives us the formulas of the first order and the second order derivatives of \( \xi^a \) with respect to \( \xi, \eta \), for example,

\[
\frac{\partial \xi^a}{\partial \xi} = \frac{1}{1 + \int_{m^{(a)}}^\eta \partial_\xi \left( \frac{1}{\lambda_-^{n-1}(\tau)} \right) \, d\tau},
\]

and

\[
\frac{\partial^2 \xi^a}{\partial \xi^2} = -\frac{\int_{m^{(a)}}^\eta \partial_\xi \left( \frac{1}{\lambda_-^{n-1}(\tau)} \right) \, d\tau \partial_\xi \frac{1}{\lambda_-^{n-1}(\tau)}}{(1 + \int_{m^{(a)}}^\eta \partial_\xi \left( \frac{1}{\lambda_-^{n-1}(\tau)} \right) \, d\tau)^2},
\]

and the other derivatives are similar. Then the estimates for \( D\xi^a \) and \( D^2\xi^a \) follow from the above formulas. In the same way, we can also get the estimates for \( D\xi^b \) and \( D^2\xi^b \). \( \Box \)

Our main result in this subsection is as follows.

**Proposition 4.2.** There exist constants \( \tilde{C}_{2,-} > 0, \tilde{C}_{2,+} > 0 \) and \( \sigma_1 > 0 \) depending only on \( U \) and \( L, \gamma \) such that for \( \sigma \in (0, \sigma_1) \), if \( \delta z^{i,n-1} \in \mathcal{M}_{2\sigma} \) and \((\delta z^{a,n}_+, \delta z^{b,n}_-)\) are the solutions to the boundary value problem \((F_n)_{2}\),

\[
\|\delta z^{a,n}_+\|_{C^2(\bar{\Omega}^{(a)} \cap \Omega_{II})} \leq \tilde{C}_{2,-} \left( \|\delta z^{a,n}_-\|_{C^2([0, \xi^{a,+}_1] \times \{m^{(a)}\})} + \|g_{+} - 1\|_{C^3([0, \xi^{a,+}_1])} \right), \quad (4.26)
\]
and
\[
\|\delta z_{+}^{b(n)}\|_{C^{2}(\tilde{\Omega}) \cap \Omega_{II}} \leq \bar{C}_{2}^{b} \left( \|\delta z_{+}^{b(n)}\|_{C^{2}([0,\xi_{b}^{+}] \times \{-m^{(a)}\})} + \|g_{-} + 1\|_{C^{3}([0,\xi_{b}^{+}] \times \{-m^{(a)}\})} \right),
\]
(4.27)
where \(\xi_{1}^{a,+}\) and \(\xi_{b}^{+}\) are given in (4.15).

**Remark 4.1.** Together with Proposition 4.1, we actually have the estimates of \(\delta z_{+}^{i(n)}\) in \(\Omega_{I} \cup \Omega_{II}\) for \(i = a\) or \(b\).

**Proof.** Similarly to the proof of Proposition 4.1, without loss of the generality, we only consider the estimate of \(\delta z_{+}^{a(n)}\). By (4.23),
\[
\partial_{\eta} \delta z_{+}^{a(n)} + \frac{1}{\lambda_{-}^{a(n-1)}} \partial_{\xi} \delta z_{+}^{a(n)} = 0.
\]
(4.28)
Therefore along the characteristic \(\xi = \psi_{1}(\eta, \xi^{a})\), where \(\xi^{a} \in [0, \xi_{1}^{a,+}]\), we have
\[
\|\delta z_{+}^{a(n)}\|_{C^{0}(\tilde{\Omega}) \cap \Omega_{II}} \leq \|\delta z_{-}^{a(n)}\|_{C^{0}([0,\xi_{1}^{a,+}] \times \{m^{(a)}\})} + 2\|\arctan g_{+}'\|_{C^{0}([0,\xi_{1}^{a,+}])},
\]
(4.29)
where we have used the boundary condition (4.23) on \(\tilde{\Gamma}_{+}\).

Next, we are going to estimate \(\|D\delta z_{+}^{a(n)}\|_{C^{0}(\tilde{\Omega}) \cap \Omega_{II}}\). To do this, we need to take the derivatives on \(\tilde{\Gamma}_{+}\) with respect to \((\xi, \eta)\) and then integrate it along the characteristic \(\xi = \psi_{1}(\eta, \xi^{a})\) from \(m^{(a)}\) to \(\eta\) to obtain
\[
\partial_{\xi} \delta z_{+}^{a(n)}(\psi_{1}(\eta, \xi^{a}), \eta) = \partial_{\xi} \delta z_{+}^{a(n)}(\xi^{a}, m^{(a)}) - \int_{m^{(a)}}^{\eta} \partial_{\xi} \left( \frac{1}{\lambda_{-}^{a(n-1)}} \right) \partial_{\xi} \delta z_{+}^{a(n)}(\tau; \xi^{a}) d\tau,
\]
(4.30)
and
\[
\partial_{\eta} \delta z_{+}^{a(n)}(\psi_{1}(\eta, \xi^{a}), \eta) = \partial_{\eta} \delta z_{+}^{a(n)}(\xi^{a}, m^{(a)}) - \int_{m^{(a)}}^{\eta} \partial_{\eta} \left( \frac{1}{\lambda_{-}^{a(n-1)}} \right) \partial_{\eta} \delta z_{+}^{a(n)}(\tau; \xi^{a}) d\tau,
\]
(4.31)
where \(\partial_{\tau} \delta z_{+}^{a(n)}(\tau; \xi^{a}) = \partial_{\tau} \delta z_{+}^{a(n)}(\psi_{1}(\eta, \xi^{a}), \tau)\). Then by the argument in Step 2 of the proof of Proposition 4.1 with Lemma 4.2, we have in \(\tilde{\Omega}_{a} \cap \Omega_{II}\),
\[
|D\delta z_{+}^{a(n)}(\psi_{1}(\eta, \xi^{a}), \eta)| \leq C|D\delta z_{+}^{a(n)}|_{C^{0}([0,\xi_{1}^{a,+}] \times \{m^{(a)}\})},
\]
(4.32)
where the constant \(C\) is positive and only depends on \(\tilde{U}\) and \(L, \gamma\) provided that \(\sigma\) is small depending only \(\tilde{U}\) and \(L, \gamma\).

Notice that
\[
\partial_{\xi} \delta z_{+}^{a(n)}(\xi^{a}, m^{(a)}) = \frac{\partial c_{a}}{\partial \xi} \partial_{\xi} \delta z_{+}^{a(n)}(\xi^{a}, m^{(a)})
\]
\[
= \frac{\partial c_{a}}{\partial \xi} \left( \frac{g_{+}'}{1 + (g_{+}')^{2}} - \partial_{\xi} \delta z_{+}^{a(n)}(\xi^{a}, m^{(a)}) \right),
\]
\[
\partial_{\eta} \delta z_{+}^{a(n)}(\xi^{a}, m^{(a)}) = \frac{\partial c_{a}}{\partial \eta} \partial_{\eta} \delta z_{+}^{a(n)}(\xi^{a}, m^{(a)})
\]
\[
= \frac{\partial c_{a}}{\partial \eta} \left( \frac{g_{+}'}{1 + (g_{+}')^{2}} - \partial_{\xi} \delta z_{+}^{a(n)}(\xi^{a}, m^{(a)}) \right).
\]
Then by Lemma 4.2 and the boundary condition (4.23) on $\Gamma_+$, we have
\[
\|D\delta_z^{a,\Omega}(\cdot, m)\|_{C^0([0,\xi_1])} \\
\leq \|D\xi\|_{C^0([0,\xi_1])} \|\partial_\xi \delta z^{a,\Omega}_{\pm} \|_{C^0([0,\xi_1])} \\
\leq C \left( \|g_+\|_{C^0([0,\xi_1])} + \|\partial_\xi \delta z^{a,\Omega}_{\pm} \|_{C^0([0,\xi_1])} \right)
\]
(4.33)

Combining estimates (4.32) and (4.33) together yields
\[
\|D\delta_z^{a,\Omega}(\cdot, m)\|_{C^0(\Omega_+ \cap \Omega_{II})} \leq C \left( \|D\delta z^{a,\Omega}(\cdot, m)\|_{C^0([0,\xi_1])} + \|g_+\|_{C^0([0,\xi_1])} \right),
\]
(4.34)

where the constant $C$ depends only on $U$, $L$ and $\gamma$.

Finally, we turn to the estimate of $\|D^2\delta z^{a,\Omega}(\cdot, m)\|_{C^0(\Omega_+ \cap \Omega_{II})}$. We first take the derivatives with respect to $(\xi, \eta)$ on (4.30) and (4.31) twice and then integrate them along the characteristic $\xi = \psi^{a,\Omega}_{\pm}(\eta, \xi)$ from $m$ to $\eta$ to get
\[
\partial_{\xi\xi}^2 \delta z^{a,\Omega}(\eta, \xi) = \partial_{\xi\xi}^2 \delta z^{a,\Omega}(\xi, m) - 2 \int_{m}^{\eta} \partial_\xi \left( \frac{1}{\lambda_{\pm}(\eta, \xi)} \right) \partial_{\xi\xi}^2 \delta z^{a,\Omega}(\xi, \zeta) d\zeta \\
- \int_{m}^{\eta} \partial_{\xi\xi}^2 \left( \frac{1}{\lambda_{\pm}(\eta, \xi)} \right) \partial_\xi \delta z^{a,\Omega}(\xi, \zeta) d\zeta,
\]
(4.35)

\[
\partial_{\xi\eta}^2 \delta z^{a,\Omega}(\eta, \xi) = \partial_{\xi\eta}^2 \delta z^{a,\Omega}(\xi, m) - \int_{m}^{\eta} \left( \partial_\eta \left( \frac{1}{\lambda_{\pm}(\eta, \xi)} \right) + \partial_\xi \left( \frac{1}{\lambda_{\pm}(\eta, \xi)} \right) \right) \partial_{\xi\eta}^2 \delta z^{a,\Omega} d\eta \\
- \int_{m}^{\eta} \partial_{\xi\eta}^2 \left( \frac{1}{\lambda_{\pm}(\eta, \xi)} \right) \partial_\xi \delta z^{a,\Omega} d\eta,
\]
(4.36)

and
\[
\partial_{\eta\eta}^2 \delta z^{a,\Omega}(\eta, \xi) = \partial_{\eta\eta}^2 \delta z^{a,\Omega}(\xi, m) - 2 \int_{m}^{\eta} \partial_\eta \left( \frac{1}{\lambda_{\pm}(\eta, \xi)} \right) \partial_{\eta\eta}^2 \delta z^{a,\Omega} d\eta \\
- \int_{m}^{\eta} \partial_{\eta\eta}^2 \frac{1}{\lambda_{\pm}(\eta, \xi)} \partial_\xi \delta z^{a,\Omega} d\eta.
\]
(4.37)

Then by the argument in Step 3 of the proof of Proposition 4.1 with Lemma 4.2, we have in $\Omega_+ \cap \Omega_{II}$,
\[
\|D^2\delta z^{a,\Omega}(\cdot, m)\|_{C^0([0,\xi_1]) \times \{m\}} \leq C \|D^2\delta z^{a,\Omega}(\cdot, m)\|_{C^0([0,\xi_1]) \times \{m\}},
\]
(4.38)
where the positive constant $C$ only depends on $\overline{U}$, $L$ and $\gamma$ provided that $\sigma$ is small depending only $\overline{U}$, $L$ and $\gamma$. By the boundary condition \ref{4.23}, one has

\[
\partial_{\xi}^2 \delta z_+^{a(n)}(\xi^a, m^{(a)}) = \partial_{\xi}^2 \left( \frac{g''}{1 + (g'_+)^2} - \partial_{\xi} \delta z_-^{a(n)}(\cdot, m^{(a)}) \right)
\]
\[
+ \left( \frac{\partial \xi}{\partial \eta} \right)^2 \left( \frac{(1 + (g'_+)^2)g'' - 2g'_+(g''')^2}{(1 + (g'_+)^2)^2} - \partial_{\xi} \delta z_-^{a(n)}(\cdot, m^{(a)}) \right),
\]
\[
\partial_{\eta}^2 \delta z_+^{a(n)}(\xi^a, m^{(a)}) = \partial_{\xi} \partial_{\eta} \left( \frac{g''}{1 + (g'_+)^2} - \partial_{\xi} \delta z_-^{a(n)}(\cdot, m^{(a)}) \right)
\]
\[
+ \partial \xi \partial \eta \left( \frac{(1 + (g'_+)^2)g'' - 2g'_+(g''')^2}{(1 + (g'_+)^2)^2} - \partial_{\xi} \delta z_-^{a(n)}(\cdot, m^{(a)}) \right),
\]
and

\[
\partial_{\eta}^2 \delta z_+^{a(n)}(\xi^a, m^{(a)}) = \partial_{\eta}^2 \left( \frac{g''}{1 + (g'_+)^2} - \partial_{\xi} \delta z_-^{a(n)}(\cdot, m^{(a)}) \right)
\]
\[
+ \left( \frac{\partial \xi}{\partial \eta} \right)^2 \left( \frac{(1 + (g'_+)^2)g'' - 2g'_+(g''')^2}{(1 + (g'_+)^2)^2} - \partial_{\xi} \delta z_-^{a(n)}(\cdot, m^{(a)}) \right).
\]

From Lemma \ref{4.2} we have

\[
\|D^2 \delta z_+^{a(n)}(\cdot, m^{(a)})\|_{C^0([0, \xi^a_1]+)}
\]
\[
\leq \|D^2 \xi\|_{C^0(\Omega^a)}(\|g'_+\|_{C^0([0, \xi^a_1]+)} + \|D \delta z_-^{a(n)}(\cdot, m^{(a)})\|_{C^0([0, \xi^a_1]+)})
\]
\[
+ \|(D \xi)^2\|_{C^0(\Omega^a)}(\|g''_+\|_{C^0([0, \xi^a_1]+)} + \|D^2 \delta z_-^{a(n)}(\cdot, m^{(a)})\|_{C^0([0, \xi^a_1]+)})
\]
\[
\leq C \left( \|g'_+\|_{C^1([0, \xi^a_1]+)} + \|D \delta z_-^{a(n)}(\cdot, m^{(a)})\|_{C^1([0, \xi^a_1]+)} \right),
\]

where the constant $C$ depends only on $\overline{U}$, $L$ and $\gamma$. The estimates \ref{1.39} and \ref{4.39} lead to the following estimate:

\[
\|D^2 \delta z_+^{a(n)}\|_{C^0(\Omega^a \cap \Omega_{III})} \leq C \left( \|D \delta z_-^{a(n)}\|_{C^1([0, \xi^a_1]+)} + \|g''_+\|_{C^1([0, \xi^a_1]+)} \right).
\]

Therefore, by \ref{1.29}, \ref{4.34} and \ref{4.31}, there exist positive constants $\tilde{C}_{2,-}^a$ and $\sigma_1$ depending only on $\overline{U}$ and $L$ such that for $\sigma \in (0, \sigma_1)$, the estimate \ref{4.26} holds. This completes the proof of the proposition. \hfill \Box

**Remark 4.2.** Following the same argument in the proof above, taking the integration along the characteristics up to the contact discontinuity $\{\eta = 0\} \cap \Omega_{III}$ for the equations \ref{1.28}, \ref{4.30}–\ref{4.31} and \ref{4.35}–\ref{4.37}, we can actually obtain

\[
\|\delta z_+^{a(n)}\|_{C^2(\Omega_{III} \cap \{\eta = 0\})} + \|\delta z_-^{b(n)}\|_{C^2(\Omega_{III} \cap \{\eta = 0\})}
\]
\[
\leq \tilde{C}_{2,-}^a \left( \|\delta z_-^{a(n)}\|_{C^2([0, \xi^a_1]+ \times (m^{(a)}))} + \|g_+ - 1\|_{C^3([0, \xi^a_1]+)} \right)
\]
\[
+ \tilde{C}_{2,+}^a \left( \|\delta z_-^{a(n)}\|_{C^2([0, \xi^a_1]+ \times (-m^{(a)}))} + \|g_- + 1\|_{C^3([0, \xi^a_1]+)} \right),
\]

4.3. **Estimates for the boundary value problem** ($\tilde{P}_n$) **in** $\Omega_{III}$. Based on Propositions \ref{4.1} and \ref{4.2} in this subsection we will analyze the boundary value problem ($\tilde{P}_n$) in $\Omega_{III}$
which involves the contact discontinuity on $\eta = 0$. The boundary value problem \((\tilde{P}_n)\) in $\Omega_{III}$ can be described as the following:

\[
\begin{cases}
\partial_z \delta z_{a,n} + \text{diag}(\tilde{\lambda}_+^{a,n}, \tilde{\lambda}_-^{a,n}) \partial_t \delta z_{a,n} = 0, & \text{in } \tilde{\Omega}^{(a)}, \\
\partial_z \delta z_{b,n} + \text{diag}(\tilde{\lambda}_+^{b,n}, \tilde{\lambda}_-^{b,n}) \partial_t \delta z_{b,n} = 0, & \text{in } \tilde{\Omega}^{(b)}, \\
\delta z_{a,n} - \delta z_{b,n} = (\alpha^{a,n}) \delta z_{a,n} + (\beta^{b,n}) \delta z_{b,n} + c(\xi), & \text{on } \tilde{\Gamma}_{cd},
\end{cases}
\]

(4.41)

where $\alpha^{a,n}, \beta^{b,n}$ and $c(\xi)$ are given in (3.29) and (3.30).

The way of studying the problem \((\tilde{P}_n)\) in $\Omega_{III}$ is similar to the way in §4.2 except the complicated boundary conditions \((4.41)_3\) and \((4.41)_4\) on $\tilde{\Gamma}_{cd}$. Moreover, following the argument in §4.1 and §4.2, we can obtain the estimate of $\delta z_{a,n}^{i}$ in $\Omega_a \cap \Omega_{III}$ and the estimate of $\delta z_{b,n}^{i}$ in $\tilde{\Omega}_b \cap \Omega_{III}$. Then we only need to prove the following proposition.

**Proposition 4.3.** There exist constants $\tilde{C}_{2,a} > 0$, $\tilde{C}_{2,b} > 0$ and $\sigma > 0$ depending only on $\tilde{U}$, $L$ and $\gamma$ such that for $\sigma \in (0, \sigma_2)$, if $\delta z_{i}^{(n-1)} \in M_{2\sigma}$ and $(\delta z_{a,n}^{i}, \delta z_{b,n}^{i})$ is the solution to the free boundary value problem \((\tilde{P}_n)\) in $\Omega_{III}$, then

\[
\|\delta z_{a,n}^{i}\|_{C^2(\tilde{\Omega}(\tilde{\Omega}) \cap \Omega_{III})} \leq \tilde{C}_{2,a} \left( \|\delta z_{a,n}^{i}\|_{C^2(\Omega_{III} \cap (\eta=0))} + \|\delta z_{b,n}^{i}\|_{C^2(\Omega_{III} \cap (\eta=0))} \right) + \sum_{i=a,b} \left( \|\tilde{S}_i^{a} - S_i^{a}\|_{C^2(\Sigma_i)} + \|\tilde{B}_i^{a} - B_i^{a}\|_{C^2(\Sigma_i)} \right),
\]

(4.42)

and

\[
\|\delta z_{b,n}^{i}\|_{C^2(\tilde{\Omega}(\tilde{\Omega}) \cap \Omega_{III})} \leq \tilde{C}_{2,b} \left( \|\delta z_{a,n}^{i}\|_{C^2(\Omega_{III} \cap (\eta=0))} + \|\delta z_{b,n}^{i}\|_{C^2(\Omega_{III} \cap (\eta=0))} \right) + \sum_{i=a,b} \left( \|\tilde{S}_i^{b} - S_i^{b}\|_{C^2(\Sigma_i)} + \|\tilde{B}_i^{b} - B_i^{b}\|_{C^2(\Sigma_i)} \right),
\]

(4.43)

where $\Sigma_a = (0, m^{(a)})$ and $\Sigma_b = (-m^{(b)}, 0)$.

Before giving the proof of Proposition 4.3, let us introduce the characters arising from $\eta = 0$, that is, $\xi = \psi_{+}^{a,n}(\eta, \xi_0^{a})$ and $\xi = \psi_{-}^{b,n}(\eta, \xi_0^{b})$ are the characteristic curves passing through the points $(0, \xi_0)$ defined by

\[
\left\{ \begin{array}{l}
\frac{d\psi_{+}^{a,n}(\eta)}{d\eta} = \frac{1}{\tilde{\lambda}_{+}^{a,n-1}(\eta, \psi_{+}^{a,n})}, \\
\psi_{+}^{a,n}(0, \xi_0^{a}) = \xi_0^{a},
\end{array} \right.
\]

(4.44)

and

\[
\left\{ \begin{array}{l}
\frac{d\psi_{-}^{b,n}(\eta)}{d\eta} = \frac{1}{\tilde{\lambda}_{-}^{b,n-1}(\eta, \psi_{-}^{b,n})}, \\
\psi_{-}^{b,n}(0, \xi_0^{b}) = \xi_0^{b},
\end{array} \right.
\]

(4.45)

where $\xi_0^{a}, \xi_0^{b} \in [0, L]$. Similarly, we can regard $\xi_0^{a}$ and $\xi_0^{b}$ as the functions of $(\xi, \eta)$. Then similarly to Lemma 4.2, we also have following estimates for $\xi_0^{a}$ and $\xi_0^{b}$.

**Lemma 4.3.** For any $\delta z_{i}^{(n-1)} \in M_{2\sigma}$ $(i = a, b)$, there exist constants $C_{2,1}^i$ and $C_{2,2}^i$ depending only on $\tilde{U}$, $L$ and $\sigma$ such that

\[
\|D^k \xi_0^i\|_{C^0(\tilde{\Omega}(\tilde{\Omega}) \cap \tilde{\Omega}(\tilde{\Omega}))} \leq C_{2,k}^i, \quad k = 1, 2,
\]

(4.46)

for $i = a, b$. 
where \( \delta z \) since the estimate of \( \xi, \eta \)\( \eta \) \( \eta \)

Notice that on the contact discontinuity

\[
\frac{\partial a, n}{\partial t} + \frac{\partial b, n}{\partial x} = \frac{\partial c, n}{\partial t} + \frac{\partial c, n}{\partial x} = 0,
\]

(4.47)

First of all, from the boundary conditions (4.41) and (4.44), we obtain

\[
\delta z_{a, n}(0) = \frac{\alpha(n-1) \xi}{\alpha(n-1) + \beta(n-1)}, \quad \gamma(n-1) = \frac{2\alpha(n-1)}{\alpha(n-1) + \beta(n-1)},
\]

(4.48)

\[
\gamma(n-1) = \frac{2\beta(n-1)}{\alpha(n-1) + \beta(n-1)}, \quad \gamma(n-1) = \frac{1}{\alpha(n-1) + \beta(n-1)}.
\]

As in the proof of Proposition 4.2, we also rewrite the equation for \( \delta z_{a, n} \) as

\[
\partial_\eta \delta z_{a, n} + \frac{1}{\chi_{a, n}(n-1)} \partial_\xi \delta z_{a, n} = 0.
\]

(4.49)

Then along the characteristic \( \xi = \psi_{+}^{a, n}(\eta, \xi^0) \), we have

\[
\|\delta z_{a, n}\|_{C^0(\tilde{\Omega}(\xi) \cap \Omega_{I/II})} = \|\delta z_{a, n}\|_{C^0(\Omega_{I/II} \cap \{\eta = 0\})}
\]

\[
= \|\gamma(n-1) \delta z_{a, n} + \chi(n-1) \delta z_{b, n} + \delta z_{c, n}\|_{C^0(\tilde{\Omega}(\xi) \cap \Omega_{I/II})}
\]

\[
\leq C \|\delta z_{a, n}\|_{C^0(\tilde{\Omega}(\xi) \cap \Omega_{I/II})} + \|\delta z_{b, n}\|_{C^0(\tilde{\Omega}(\xi) \cap \Omega_{I/II})}
\]

\[
+ \sum_{i=a,b} (\|S^i_0 - S^i\|_{C^0(\Sigma_i)} + \|D^i_0 - D^i\|_{C^0(\Sigma_i)}),
\]

(4.50)

where we have used the boundary condition (4.43) on \( \tilde{\Omega}_{cd} \) and the constant \( C \) depends only on \( \tilde{\Omega}, L, \) and \( \gamma \).

Next, let us consider the estimate of \( \|D\delta z_{a, n}\|_{C^0(\tilde{\Omega}(\xi) \cap \Omega_{I/II})} \). Taking the derivatives on equation (4.49) with respect to \( (\xi, \eta) \) and then integrating the resulting equations along the characteristic \( \xi = \psi_{+}^{a, n}(\eta, \xi^0) \) determined by (4.44), we obtain

\[
\partial_\xi \delta z_{a, n} = \partial_\xi \delta z_{a, n}(\xi^0, 0) - \int_0^\eta \partial_\xi \left( \frac{1}{\chi_{a, n}(n-1)} \partial_\tau \delta z_{a, n}(\tau, \cdot) \right) d\tau,
\]

\[
\partial_\eta \delta z_{a, n} = \partial_\eta \delta z_{a, n}(\xi^0, 0) - \int_0^\eta \partial_\tau \left( \frac{1}{\chi_{a, n}(n-1)} \partial_\tau \delta z_{a, n}(\tau, \cdot) \right) d\tau,
\]

where

\[
\partial_\xi \delta z_{a, n}(\xi^0, 0) = \frac{\partial c_{a, n}}{\partial \xi} \delta z_{a, n}(\xi^0, 0), \quad \partial_\eta \delta z_{a, n}(\xi^0, 0) = \frac{\partial c_{a, n}}{\partial \eta} \delta z_{a, n}(\xi^0, 0).
\]

Notice that on the contact discontinuity \( \eta = 0 \), by (4.47),

\[
\partial_\xi \delta z_{a, n}(\xi^0, 0) = \partial_\xi \gamma_1(n-1) \partial z_{a, n} + \gamma_1(n-1) \partial_\xi \delta z_{a, n} + \partial_\xi \gamma_2(n-1) \delta z_{b, n}
\]

\[
+ \gamma_3(n-1) \partial_\xi \delta z_{b, n} + \partial_\xi \gamma_4(n-1) \delta z_{c, n} + \partial_\xi \gamma_4(n-1) \partial_\xi \partial_\xi \delta z_{c, n}.
\]
Then

\[
|D\delta z^{a,n}(\xi^0, 0)| \leq \|D\xi^0\|_{C^0(\tilde{\Omega}_a \cap \Omega_{III})}\|\partial_{\xi^0} \delta z^{a,n}(\xi^0, 0)|
\leq C \left( \|\delta z_{+}^{a,n}\|_{C^1((\Omega_{III} \cap \{\eta=0\})} + \|\delta z_{-}^{b,n}\|_{C^1((\Omega_{III} \cap \{\eta=0\})}
+ \sum_{i=a,b} (\|S^i_0 - S^i\|_{C^1(\Sigma_1)} + \|B^i_0 - B^i\|_{C^1(\Sigma_1)}) \right),
\]

where constant \( C \) only depends on \( \overline{\Omega} \), \( L \) and \( \gamma \). Following Step 2 in the proof of Proposition 4.1 and applying the Gronwall type inequality, we can choose \( \sigma_1' \) such that for any \( \sigma \in (0, \sigma_1') \), we have

\[
\|D\delta z^{a,n}\|_{C^0(\tilde{\Omega}(\sigma) \cap \Omega_{III})} \leq C \|D\delta z^{a,n}\|_{C^0(\Omega_{III} \cap \{\eta=0\})}
\leq C \left( \|\delta z_{+}^{a,n}\|_{C^1((\Omega_{III} \cap \{\eta=0\})} + \|\delta z_{-}^{b,n}\|_{C^1((\Omega_{III} \cap \{\eta=0\})}
+ \sum_{i=a,b} (\|S^i_0 - S^i\|_{C^1(\Sigma_1)} + \|B^i_0 - B^i\|_{C^1(\Sigma_1)}) \right),
\]

where constant \( C \) only depends on \( \overline{\Omega} \), \( L \) and \( \gamma \).

Finally, let us consider the estimate of \( \|D^2\delta z^{a,n}\|_{C^0(\tilde{\Omega}(\sigma) \cap \Omega_{III})} \). Taking the derivatives on \( \xi^0 \) with respect to \((\xi, \eta)\) twice and then integrating the resulting equations along the characteristic \( \xi = \psi^{a,n}(\eta, \xi^0) \) from 0 to \( \eta \), we obtain

\[
\partial^2_{\xi \xi} \delta z^{a,n}(\xi^0, 0) = \partial^2_{\xi \xi} \delta z^{a,n}(\xi^0, 0) - 2 \int_0^\eta \partial_{\xi} \left( \frac{1}{\lambda_{a(n-1)}} \right) \partial^2_{\xi \xi} \delta z^{a,n}(\xi^0, 0) d\tau
- \int_0^\eta \partial^2_{\xi \xi} \left( \frac{1}{\lambda_{a(n-1)}} \right) \partial_{\xi} \delta z^{a,n}(\xi^0, 0) d\tau,
\]

\[
\partial^2_{\xi \eta} \delta z^{a,n}(\xi^0, 0) = \partial^2_{\xi \eta} \delta z^{a,n}(\xi^0, 0) - \int_0^\eta \left( \partial_{\eta} \left( \frac{1}{\lambda_{a(n-1)}} \right) + \partial_{\xi} \left( \frac{1}{\lambda_{a(n-1)}} \right) \right) \partial^2_{\xi \eta} \delta z^{a,n}(\xi^0, 0) d\tau
- \int_0^\eta \partial^2_{\xi \eta} \left( \frac{1}{\lambda_{a(n-1)}} \right) \partial_{\xi} \delta z^{a,n}(\xi^0, 0) d\tau,
\]

and

\[
\partial^2_{\eta \eta} \delta z^{a,n}(\xi^0, 0) = \partial^2_{\eta \eta} \delta z^{a,n}(\xi^0, 0) - 2 \int_0^\eta \partial_{\eta} \left( \frac{1}{\lambda_{a(n-1)}} \right) \partial^2_{\xi \eta} \delta z^{a,n}(\xi^0, 0) d\tau
- \int_0^\eta \partial^2_{\eta \eta} \left( \frac{1}{\lambda_{a(n-1)}} \right) \partial_{\xi} \delta z^{a,n}(\xi^0, 0) d\tau.
\]

For the terms \( D^2 \delta z^{a,n}(\xi^0, 0) \) in above equations, we first have

\[
\partial^2_{\xi \xi} \delta z^{a,n}(\xi^0, 0) = \frac{\partial^2 \xi^0}{\partial \xi^2} \partial_{\xi^0} \delta z^{a,n}(\xi^0, 0) + \left( \frac{\partial \xi^0}{\partial \xi} \right)^2 \partial_{\xi \xi} \delta z^{a,n}(\xi^0, 0),
\]

\[
\partial^2_{\xi \eta} \delta z^{a,n}(\xi^0, 0) = \frac{\partial^2 \xi^0}{\partial \xi \partial \eta} \partial_{\xi^0 \eta} \delta z^{a,n}(\xi^0, 0) + \frac{\partial \xi^0}{\partial \xi} \frac{\partial \xi^0}{\partial \eta} \partial_{\xi \eta} \delta z^{a,n}(\xi^0, 0),
\]

\[
\partial^2_{\eta \eta} \delta z^{a,n}(\xi^0, 0) = \frac{\partial^2 \xi^0}{\partial \eta^2} \partial_{\xi^0 \eta} \delta z^{a,n}(\xi^0, 0) + \left( \frac{\partial \xi^0}{\partial \eta} \right)^2 \partial_{\eta \eta} \delta z^{a,n}(\xi^0, 0).
\]
For the term $\partial_{\xi_0}^2 a \delta z_{-}^{a(n)}(\xi_0,0)$, taking the derivatives twice on (4.47) gives

$$
\partial_{\xi_0}^2 a \delta z_{-}^{a(n)}(\xi_0,0) = \partial_{\xi_0}^2 \left( \partial_{\xi_0} a \delta z_{+}^{a(n)} + \gamma_1^{(n-1)} \partial_{\xi_0} \delta z_{+}^{a(n)} + \partial_{\xi_0} \delta z_{-}^{a(n)} \right) \\
+ \gamma_3^{(n-1)} \partial_{\xi_0} \delta z_{-}^{b(n)} + \partial_{\xi_0} \gamma_4^{(n-1)} c + \gamma_4^{(n-1)} \partial_{\xi_0} c \\
= \partial_{\xi_0}^2 \left( \gamma_1^{(n-1)} \delta z_{+}^{a(n)} + 2\partial_{\xi_0} \gamma_1^{(n-1)} \partial_{\xi_0} \delta z_{+}^{a(n)} + \gamma_1^{(n-1)} \partial_{\xi_0}^2 \delta z_{+}^{a(n)} \\
+ \partial_{\xi_0}^2 \gamma_3^{(n-1)} \delta z_{-}^{b(n)} + 2\partial_{\xi_0} \gamma_3^{(n-1)} \partial_{\xi_0} \delta z_{-}^{b(n)} + \gamma_3^{(n-1)} \partial_{\xi_0}^2 \delta z_{-}^{b(n)} \\
+ \partial_{\xi_0}^2 \gamma_4^{(n-1)} c + 2\partial_{\xi_0} \gamma_4^{(n-1)} \partial_{\xi_0} c + \gamma_4^{(n-1)} \partial_{\xi_0}^2 c. \right)
$$

Thus,

$$
|\partial_{\xi_0}^2 a \delta z_{-}^{a(n)}(\xi_0,0)| \leq C \left( \|\delta z_{+}^{a(n)}\|_{C^2(\Omega_{III}\cap\{\gamma=0\})} + \|\delta z_{-}^{b(n)}\|_{C^2(\Omega_{III}\cap\{\gamma=0\})} \right) \\
+ \sum_{i=a,b} \left( \|\tilde{S}_i - S_i\|_{C^2(\Sigma_i)} + \|\tilde{B}_i - B_i\|_{C^2(\Sigma_i)} \right).
$$

Then, by Lemma 4.3 on $\Omega_{III} \cap \{\eta = 0\}$, one has

$$
|D^2 \delta z_{-}^{a(n)}(\xi_0,0)| \leq \|D^2 c_{\xi_0}\|_{C^0(\tilde{\Omega}(\xi_0))} \|\partial_{\xi_0} a \delta z_{-}^{a(n)}(\xi_0,0)\|_{C^0(\Omega_{III}\cap\{\gamma=0\})} \\
+ \|(D^2 c_{\xi_0})^2\|_{C^0(\tilde{\Omega}(\xi_0))} \|\partial_{\xi_0}^2 \delta z_{-}^{a(n)}(\xi_0,0)\|_{C^0(\Omega_{III}\cap\{\gamma=0\})} \\
\leq C \left( \|\delta z_{+}^{a(n)}\|_{C^2(\Omega_{III}\cap\{\gamma=0\})} + \|\delta z_{-}^{b(n)}\|_{C^2(\Omega_{III}\cap\{\gamma=0\})} \right) + \sum_{i=a,b} \left( \|\tilde{S}_i - S_i\|_{C^2(\Sigma_i)} + \|\tilde{B}_i - B_i\|_{C^2(\Sigma_i)} \right),
$$

where the constant $C$ depends only on $\tilde{U}$, $L$ and $\gamma$.

With (4.52) and following Step 3 in the proof of Proposition 4.1 and further applying the Gronwall inequality, we can choose the constant $\sigma_i''$ sufficiently small such that for any $\sigma \in (0,\sigma_i'')$, we have

$$
\|D^2 \delta z_{+}^{a(n)}\|_{C^0(\tilde{\Omega}(\xi_0)\cap\Omega_{III})} \leq 2C \left( \|\delta z_{+}^{a(n)}\|_{C^2(\Omega_{III}\cap\{\gamma=0\})} + \|\delta z_{-}^{b(n)}\|_{C^2(\Omega_{III}\cap\{\gamma=0\})} \right) + \sum_{i=a,b} \left( \|\tilde{S}_i - S_i\|_{C^2(\Sigma_i)} + \|\tilde{B}_i - B_i\|_{C^2(\Sigma_i)} \right).
$$

From the estimates (4.50), (4.51) and (4.53) together, there exist constants $\tilde{C}_{2,+} > 0$ and $\sigma_2 = \min\{\sigma_1',\sigma_1'',1\} > 0$ depending only on $\tilde{U}, L$ and $\gamma$, such that for $\sigma \in (0,\sigma_2)$, the estimate (4.12) holds. This completes the proof of the proposition.

With Propositions 4.1, 4.3, we can prove Theorem 4.1.

Proof of Theorem 4.1. We will prove inequality (4.11) by the induction procedure together with Propositions 4.1, 4.3.

Note that by Remark 4.2, $\|\delta z_{+}^{a(n)}\|_{C^2(\Omega_{III}\cap\{\gamma=0\})} + \|\delta z_{-}^{b(n)}\|_{C^2(\Omega_{III}\cap\{\gamma=0\})}$ in (4.12)–(4.13) can be bounded by the terms on the right hand side of the inequality (4.26)–(4.27),
5. Convergence of the Approximate Solution and Existence of Problem (P)

In this section, we shall prove that the mapping \( T \) defined by (3.31) is a contraction map, so that the sequence of solutions obtained in \( \S 3.3 \) converges to a limit function which is actually the solution to the problem \( (P) \).

First we will show the following proposition.

Let \( \xi^*_1 := \min \{ \xi^a_1, \xi^b_1 \} \) and \( \Omega^i_1 := \bar{\Omega}^i \cap \{ \xi \leq \xi^*_1 \} \) for \( i = a \) or \( b \). In \( \Omega_{IV} \cap \Omega^i_1 \), we can repeat the proof of Proposition 4.1 to show that inequality (4.54) also holds. Then, there exist constants \( C^*_1 \) and \( \sigma^*_1 \) which only depend on \( \bar{U}, L \) and \( \gamma \), such that for \( \sigma \in (0, \sigma^*_1) \), we have

\[
\| \delta z^{a,(n)} \|_{C^2(\Omega^a_1)} + \| \delta z^{b,(n)} \|_{C^2(\Omega^b_1)} \leq C^*_1 \left( \sum_{i=a,b} \left( \| z_0^i - z^i \|_{C^2(\Sigma_i)} + \| S_0^i - S^i \|_{C^2(\Sigma_i)} \right) + \sum_{i=a,b} \| B_0^i - B^i \|_{C^2(\Sigma_i)} + \| g_+ - 1 \|_{C^3([0, \xi^*_1])} + \| g_- + 1 \|_{C^3([0, \xi^*_1])} \right) .
\]

(4.55)

Let \( \xi^a_2 \) be the intersection point of the characteristic corresponding to \( \lambda_+ \) starting from \( (\xi^*_1, 0) \) and the upper nozzle wall \( \eta = m(a) \), \( \xi^b_2 \) be the intersection point of the characteristic corresponding to \( \lambda_- \) starting from \( (\xi^*_1, 0) \) and the lower nozzle wall \( \eta = -m(b) \), and \( \Omega^i_2 := \bar{\Omega}^i \cap \{ \xi_2 \leq \xi \leq \xi^*_2 \} \) for \( i = a \) or \( b \). Regarding the line \( \xi = \xi^*_1 \) as the initial line \( \xi = 0 \) and repeating the argument of the proof of Proposition 4.1 again, one has

\[
\| \delta z^{a,(n)} \|_{C^2(\Omega^a_2)} + \| \delta z^{b,(n)} \|_{C^2(\Omega^b_2)} \leq C^*_2 \left( \sum_{i=a,b} \left( \| z_0^i - z^i \|_{C^2(\Omega^i_2 \cap \{ \xi = \xi^*_1 \})} + \| S_0^i - S^i \|_{C^2(\Omega^i_2 \cap \{ \xi = \xi^*_1 \})} \right) + \sum_{i=a,b} \| B_0^i - B^i \|_{C^2(\Omega^i_2 \cap \{ \xi = \xi^*_1 \})} + \| g_+ - 1 \|_{C^3([\xi^*_1, \xi^*_2])} + \| g_- + 1 \|_{C^3([\xi^*_1, \xi^*_2])} \right) .
\]

(4.55)

Then, we can repeat this procedure for \( \ell \) times with \( \ell = \left\lceil \frac{\xi^*_1}{\xi^*_2} \right\rceil + 1 \). Define \( \xi^*_\ell \) and \( \Omega^i_\ell \) for \( i = a \), or \( b \). Obviously \( \bar{\Omega}^i = \bigcup_{1 \leq k \leq \ell} \Omega^i_k \). Summing all the estimates (4.55) together for \( k = 1, ..., \ell \), we finally obtain (4.1). This completes the proof of the theorem. \( \square \)
Proposition 5.1. Under the assumptions of Theorem 3.2, there exists a constant \( \overline{\sigma}_0 > 0 \) depending only on \( \overline{U}_0 \), \( L \) and \( \gamma \) such that for any \( \sigma \in (0, \overline{\sigma}_0) \), the sequence \( \{(z^{a,n}, z^{b,n})\}_{n=1}^{\infty} \) constructed by the iteration scheme \( (\mathcal{P}_n) \) is well defined in \( \mathcal{M}_{2\sigma} \). Moreover, the sequence \( \{(z^{a,n}, z^{b,n})\}_{n=1}^{\infty} \) is convergent in \( C^1(\overline{\Omega}^a) \times C^1(\overline{\Omega}^b) \) and its limit function is the solution of the problem \( (\mathcal{P}) \).

Proof. Firstly, by Theorem 4.1, we have

\[
\|\delta z^{a,n}\|_{C^2(\overline{\Omega}(a))} + \|\delta z^{b,n}\|_{C^2(\overline{\Omega}(b))} \leq C^* \left( \sum_{i=a,b} (\|\dot{z}^i - \ddot{z}^i\|_{C^2(\Sigma_i)} + \|\dot{S}^i_0 - \ddot{S}^i\|_{C^2(\Sigma_i)}) + \sum_{i=a,b} \|\ddot{B}^i_0 - \ddot{B}^i\|_{C^2(\Sigma_i)} + \|g_+ - 1\|_{C^3([0,L])} + \|g_- + 1\|_{C^3([0,L])} \right)
\]

where the constant \( C^* \) only depends on \( \overline{U}_0 \), \( L \) and \( \gamma \). Thus, we can choose \( \overline{\sigma} \) sufficiently small such that \( C^* \overline{\sigma} \leq 2\sigma \), then this implies that the iteration scheme \( (\mathcal{P}_n) \) is well defined in \( \mathcal{M}_{2\sigma} \), that is, the map \( \mathcal{T} : \mathcal{M}_{2\sigma} \mapsto \mathcal{M}_{2\sigma} \) given in [334] is well defined. We set \( (z^{a,(n+1)}, z^{b,(n+1)}) := \mathcal{T}(z^{a,(n)}, z^{b,(n)}) \).

Secondly, we will show that \( \mathcal{T} \) is a contraction mapping which implies that the sequence \( \{(z^{a,n}, z^{b,n})\}_{n=0}^{\infty} \) is convergent in \( C^1(\overline{\Omega}^a) \times C^1(\overline{\Omega}^b) \). To this end, let

\[
\Delta z_i^{i,(n)} := \delta z_i^{i,(n)} - \delta z_i^{i,(n-1)}, \quad i = a, b.
\]

Then, \( (\Delta z^{a,(n)}, \Delta z^{b,(n)}) \) satisfies following boundary value problem,

\[
(\Delta P_n) \begin{cases}
\partial_\xi \Delta z^{a,(n+1)} + \text{diag}(\dot{\lambda}^{a,(n)}, \ddot{\lambda}^{a,(n)}) \partial_\eta \Delta z^{a,(n+1)} \\
= \text{diag}(\dot{\lambda}^{a,(n-1)}, \ddot{\lambda}^{a,(n-1)}) \partial_\eta \Delta z^{a,(n)}, & \text{in } \overline{\Omega}^a,
\partial_\xi \Delta z^{b,(n+1)} + \text{diag}(\dot{\lambda}^{b,(n)}, \ddot{\lambda}^{b,(n)}) \partial_\eta \Delta z^{b,(n+1)} \\
= \text{diag}(\dot{\lambda}^{b,(n-1)}, \ddot{\lambda}^{b,(n-1)}) \partial_\eta \Delta z^{b,(n)}, & \text{in } \overline{\Omega}^b,
\Delta z^{a,(n+1)}_{\pm} = 0, & \text{on } \xi = 0,
\Delta z^{b,(n+1)}_{\pm} = 0, & \text{on } \xi = 0,
\Delta z^{a,(n+1)}_{\pm} + \Delta z^{a,(n+1)}_{\pm} = 0, & \text{on } \Gamma_+, \\
\Delta z^{b,(n+1)}_{\pm} + \Delta z^{b,(n+1)}_{\pm} = 0, & \text{on } \Gamma_-, \\
\alpha^{(n)} \Delta z^{a,(n+1)}_{\pm} + \beta^{(n)} \Delta z^{b,(n+1)}_{\pm} \\
= \alpha^{(n)} \Delta z^{a,(n+1)}_{\pm} + \beta^{(n)} \Delta z^{b,(n+1)}_{\pm}, & \text{on } \Gamma_{cd},
\end{cases}
\]
Similarly to the proof of Theorem 4.1, there exists a constant $C > 0$ depending only on $\mathcal{O}$ and $L$, such that
\[
\|\Delta z^{a,(n+1)}\|_{C^1(\tilde{\Omega}^{(0)})} + \|\Delta z^{b,(n+1)}\|_{C^1(\tilde{\Omega}^{(0)})} \\
\leq C\left(\sum_{i=a, b} \|\delta z^{i,(n)}\|_{C^2(\tilde{\Omega}^{(0)})}\right) \left(\|\Delta z^{a,(n+1)}\|_{C^1(\tilde{\Omega}^{(0)})} + \|\Delta z^{b,(n+1)}\|_{C^1(\tilde{\Omega}^{(0)})}\right) \\
\leq 2C\sigma\left(\|\Delta z^{a,(n)}\|_{C^1(\tilde{\Omega}^{(0)})} + \|\Delta z^{b,(n)}\|_{C^1(\tilde{\Omega}^{(0)})}\right).
\]

Take $\sigma_0 = \frac{1}{4\alpha}$. Then for any $0 < \sigma \leq \sigma_0$, we get
\[
\|\Delta z^{a,(n+1)}\|_{C^1(\tilde{\Omega}^{(0)})} + \|\Delta z^{b,(n+1)}\|_{C^1(\tilde{\Omega}^{(0)})} \leq \frac{1}{2} \left(\|\Delta z^{a,(n)}\|_{C^1(\tilde{\Omega}^{(0)})} + \|\Delta z^{b,(n)}\|_{C^1(\tilde{\Omega}^{(0)})}\right). 
\]
i.e.,
\[
\|\mathcal{T}(\Delta z^{a,(n)}, \Delta z^{b,(n)})\|_{C^1(\tilde{\Omega}^{(a)}) + \tilde{\Omega}^{(b)}} \leq \frac{1}{2} \|\Delta z^{a,(n)}, \Delta z^{b,(n)}\|_{C^1(\tilde{\Omega}^{(a)}) + \tilde{\Omega}^{(b)}}.
\]
This implies that $\{(\delta z^{a,(n)}, \delta z^{b,(n)})\}_{n=1}^{\infty}$ is a Cauchy sequence and has a limit which is denoted by $(\delta z^a, \delta z^b) \in C^1(\tilde{\Omega}^{(a)}) \times C^1(\tilde{\Omega}^{(b)})$.

Finally, define
\[
z^a := \delta z^a + \bar{z}^a, \quad z^b := \delta z^b + \bar{z}^b.
\]
Obviously, $(z^a, z^b) \in \tilde{\Omega}^{(a)} \times \tilde{\Omega}^{(b)}$ is a weak solution of problem $(\tilde{P})$ in $\tilde{\Omega}$. This completes the proof of the proposition. \qed

Now we are ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** The existence is a direct consequence of Proposition 5.1. For the uniqueness, we consider two solutions $z_1 = (z^a_1, z^b_1)$ and $z_2 = (z^a_2, z^b_2)$ which both satisfy the estimate (3.26). Define
\[
\Delta z^{(i)} := \delta z_1^{(i)} - \delta z_2^{(i)}, \quad i = a, b.
\]
Obviously, following the argument in the proof of Proposition 5.1, we know that $\Delta z^{(i)}$ satisfies estimates (5.1) for $C^0$-estimates, where $\Delta z^{(i),(n+1)}$ and $\Delta z^{(i),(n)}$ are replaced by $\Delta z^{(i)}$ for $i = a$ or $b$. It means that $\Delta z^{(i)} = 0$, then we conclude that $z_1 = z_2$. This completes the proof of Theorem 3.2. \qed

**APPENDIX A. BLOW-UP OF SOLUTION IN A SEMI-INFINITY LONG NOZZLE**

In this appendix, we give an example to show that the solution for the semi-infinity long nozzle problem will blow-up in general. More precisely, the irrotational flow in a semi-infiniately long flat nozzle (see Fig. A.1) with compatibility conditions will blow up if it is not a constant flow.

Consider the supersonic incoming flow past a semi-infinity long flat nozzle $\Omega := \{(x, y) \in \mathbb{R}^2 : 0 < y < 1, \ x > 0\}$, whose lower and upper boundaries are denoted by $\Gamma_a$ and $\Gamma_b$, with $\Gamma_a := \{(x, y) : y = 1, \ x > 0\}$ and $\Gamma_b := \{(x, y) : y = 0, \ x > 0\}$. Assume that the flow is irrotational satisfying
\[
\begin{align*}
\frac{\partial z}{\partial y} + \frac{\partial y}{\partial x} \rho v &= 0, \\
\frac{\partial z}{\partial x} - \frac{\partial x}{\partial y} \rho u &= 0,
\end{align*}
\]
(A.1)
STABILITY OF SUPERSONIC CONTACT DISCONTINUITY IN A NOZZLE

Figure A.1. Supersonic flow past a semi-infinitely long straight nozzle
together with the Bernoulli law
\[
\frac{u^2 + v^2}{2} + \frac{c^2}{\gamma - 1} = \frac{1}{2} \hat{q}^2,
\] (A.2)
where \((u, v), \rho\) stand for the velocity and density respectively, and \(c\) represents the sonic speed which is given by \(c = \rho \frac{\hat{q}}{\gamma} \), while \(\hat{q} = \sqrt{\gamma + 1} \hat{c}\) and \(\hat{c}\) is the critical sonic speed.

The incoming flow at the inlet \(x = 0\) is given by
\[
(u, v, \rho)(0, y) := (u_0, v_0, \rho_0)(y) \in C^2([0, 1]),
\] (A.3)
and along the upper and lower walls of the straight semi-infinitely long nozzle the flow speeds satisfy the following boundary condition
\[
v(x, 1) = v(x, 0) = 0,
\] (A.4)
for \(x \geq 0\). Then by (A.2) and (A.4), we impose the compatibility condition on the first order derivatives of the incoming flows:
\[
(\partial_y u, \partial_y \rho)(x, 0+) = (\partial_y u, \partial_y \rho)(x, 1-) = (0, 0).
\] (A.5)
In addition, we impose the compatibility condition on the second order derivatives of the incoming flows such that
\[
\partial_y^2 v_0(0+) = \partial_y^2 v_0(1-) = 0.
\] (A.6)

Now, we can define the following periodic extension of the incoming flow.
\[
(\tilde{u}_0, \tilde{v}_0, \tilde{\rho}_0)(y) = \begin{cases} (u_0, v_0, \rho_0)(y), & y \in [0, 1], \\ (u_0, -v_0, \rho_0)(-y), & y \in [-1, 0], \\ (u_0, v_0, \rho_0)(y - 2j), & y \in [2j - 1, 2j + 1], \end{cases}
\] (A.7)
for \(j = \pm 1, \pm 2, \cdots\). Then \((\tilde{u}_0, \tilde{v}_0, \tilde{\rho}_0)(y) \in C^2(\mathbb{R})\).

We will show the blow-up of solutions in general in a semi-infinitely long nozzle by a contradiction argument, if they satisfy
\[
\| (u, v, \rho) - (\underline{u}, \underline{0}, \underline{\rho}) \|_{C^1([0, +\infty) \times [0, 1])} \leq \varepsilon,
\] (A.8)
where \(\underline{u}, \underline{\rho}\) are the constant background state. More precisely, we suppose that the solutions to the initial-boundary value problem (A.1)-(A.6) exist globally. Then for any \((x, y) \in \Omega\)
For a given $\delta Z_{+},\delta Z_{-}$ are two solutions to

\[
\begin{aligned}
\partial_{+}\delta Z_{+} &= (\lambda_{-}(Z_{+1},Z_{-1}) - \lambda_{-}(Z_{+2},Z_{-2}))\partial_{y}Z_{+2}, \\
\partial_{+}\delta Z_{-} &= (\lambda_{+}(Z_{+1},Z_{-1}) - \lambda_{+}(Z_{+2},Z_{-2}))\partial_{y}Z_{-2}, \\
\delta Z_{+}(0,y) &= 0, \\
\delta Z_{-}(0,y) &= 0,
\end{aligned}
\]

So, we can employ the characteristic methods to get that

\[
\|\langle\delta Z_{+},\delta Z_{-}\rangle\|_{C^{0}(\mathbb{R}^{+}\times\mathbb{R})} \leq CT\|\langle\partial_{y}Z_{+2},\partial_{y}Z_{-2}\rangle\|_{C^{0}(\mathbb{R}^{+}\times\mathbb{R})}\|\langle\delta Z_{+},\delta Z_{-}\rangle\|_{C^{0}(\mathbb{R}^{+}\times\mathbb{R})} \leq C\varepsilon\|\langle\delta Z_{+},\delta Z_{-}\rangle\|_{C^{0}(\mathbb{R}^{+}\times\mathbb{R})},
\]

where $C$ depends on $T$ and $u,\rho$. This implies that $Z_{\pm,0}(-1) = Z_{\pm,0}(1)$. Thus, if $\tilde{\alpha}_{0}(y),\tilde{\nu}_{0}(y),\tilde{\rho}_{0}(y)$ are not constants, then there is a point in $[-1,1]$ such that $\partial_{y}\tilde{Z}_{+,0} < 0$ or $\partial_{y}\tilde{Z}_{-,0} < 0$. Then, following the result of P. Lax in [28], we obtain the contradiction since the $C^{1}$
solution of the Cauchy problem \([A.14]\) will blow up. Therefore, we have the following theorem:

**Theorem A.1.** For \(\bar{q}_0 > \bar{c}_0\) and \(\gamma > 1\), if the incoming flow is not a constant and satisfies \([A.6]\), then \(\partial_\gamma Z_+\) or \(\partial_\gamma Z_-\) will become infinity in a finite time, i.e., the solutions to the problem \((P)\) will blow-up in a finite time.

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