HOLOMORPHIC FORMS ON THREEFOLDS

TIE LUO AND QI ZHANG

Abstract. Two conjectures relating the Kodaira dimension of a smooth projective
variety and existence of number of nowhere vanishing 1-forms on the variety are
proposed and verified in dimension 3.

§1 Introduction

Let $X$ be a smooth projective variety over the complex number field $\mathbb{C}$. Let
$\omega \in H^0(X, \Omega^i_X)$ be a nontrivial global holomorphic differential $i$-form. We would
like to understand the nature of zero locus of $\omega$ in terms of the birational geometry
of $X$. On one hand algebraic varieties are birationally classified according to their
Kodaira dimensions using high multiples of holomorphic form of top degree-the
canonical bundle and Mori’s theory of extremal contractions and flips are birational
operations done to part of the base locus of (multiples of) canonical bundle, on the
other hand very little is known about the impact of zero locus of holomorphic forms
of lower degree has on the birational geometry of the underlying variety. The first
question one may ask is: What makes $\omega$ to have zero locus? It is proved in [Z] that
for any global holomorphic 1-form $0 \neq \omega \in H^0(X, \Omega^1_X)$, the zero locus $Z(\omega)$ is not
empty provided that the canonical bundle $K_X$ is ample. It is natural to suspect
that the same conclusion should hold for varieties of general type.

Indeed we propose two conjectures:

Conjecture 1. Assume $X$ is of general type and $0 \neq \omega \in H^0(X, \Omega^1_X)$. Then $Z(\omega) \neq \phi$.

and more precisely

2000 Mathematics Subject Classification. 14E30 14F10.
Conjecture 2. Assume there is a subspace $V$ of dimension $k$ in $H^0(X, \Omega_X)$ for some $1 \leq k \leq \dim(X)$ and $Z(\omega) = \emptyset$ for any $0 \neq \omega \in V$. Then the Kodaira dimension of $X$ is bounded from above by $\dim(X) - k$.

These conjectures are trivially true for curves. As an exercise we ask readers to check them for surfaces. The purpose of the present paper is to verify that they also hold in dimension 3 by proving:

Theorem 1. Let $X$ be a threefold of general type. For any global holomorphic 1-form $0 \neq \omega \in H^0(X, \Omega_X)$, its zero locus $Z(\omega)$ is not empty.

and

Theorem 2. Let $X$ be a threefold admitting a $k$-dimensional subspace $V \subset H^0(X, \Omega^1_X)$ such that $z(\omega) = \emptyset$ for any $0 \neq \omega \in V$. Then the Kodaira dimension of $X$ is at most $3 - k$.

Some of the techniques used in proving these results can not be generalized to higher dimensions. We plan to discuss similar results in higher dimensions using different tools in a future paper.

In contrast to the result on 1-forms, there are examples (see section 4 below) of threefolds with ample canonical bundle and carrying nowhere vanishing 2-forms. Using a completely different approach and viewing a 2-form as a meromorphic vector field, we have

Theorem 3. Let $X$ be a threefold of general type. $\omega \in H^0(X, K_X \otimes \Omega^2_X)$ is a canonically twisted holomorphic 2-form. Then the zero locus $Z(\omega)$ of $\omega$ is not empty.

Acknowledgments. We thank S. Keel, J. McKernan, and M. Reid for their interest in and comments on our results.

§2 Proof of Theorem 1

The following result uses the classification of extremal contractions in dimension 3. It is related to the ideas used in [L] and reduces the proof of Theorem 1 to those $X$ admitting smooth minimal models.
Lemma 2.1. Assume $X$ is smooth and of general type. If $X$ does not admit a smooth minimal model, then every holomorphic 1-form on $X$ has nonempty zero locus.

Proof. Since $K_X$ is not nef, there is an extremal contraction $\pi : X \to Y$. Let $E$ be the exceptional locus. Mori [M] says that $f$ is divisorial with the exceptional divisor $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$ or $\mathcal{O}(-2)$, $\mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \otimes \mathcal{O}(-1)$, a cone $E$ over rational normal curve of degree 2 with normal bundle $\mathcal{O}(-1)$, a $\mathbb{P}^1$ bundle over a smooth curve.

One has the following exact sequence

$$0 \to N^*_E/X \to \Omega_X \to \Omega_E \to 0.$$ 

Note that unless $E$ is a $\mathbb{P}^1$ bundle over a smooth curve, $H^0(E, \Omega_E) = 0$. This implies that restrictions of a nontrivial global 1-form on $E$ provides a nontrivial global section of positive line bundle $N^*_E/X$, which has nontrivial zero locus.

When $E$ is a $\mathbb{P}^1$ bundle over a smooth curve, the resulting $Y$ is smooth. From our assumption, $K_Y$ is not nef. One can repeat the argument done for $X$ and the process has to terminate after finitely many steps. So every holomorphic 1-form on $X$ has nonempty zero locus.

$\square$

Remark. If $X$ admits a 1-form with isolated zeros, $X$ must be minimal. In [L] we showed that it is possible to have a nonminimal threefold carrying a 2-form with isolated zeros.

Thus we may assume that the canonical bundle of $X$ is nef and big. Let $f : X \to Y$ be the birational morphism to the canonical model of $X$. $Y$ is known to have Gorenstein canonical singularities. Reid [R1] constructed explicitly a crepant resolution for such singularities. Let $g : X' \to Y$ be the resolution such that $g$ is crepant and $X'$ has at most isolated cDV points. Since both $K_X$ and $K_{X'}$ are nef, it is known that $X$ and $X'$ are different by flops. One way to prove Theorem 1 is to show that any 1-form on $X'$ has to have nontrivial zero locus and flops do not change the existence of the zero locus.

Here we take a different approach using Kodaira-Akizuki-Nakano type vanishing theorem for semismall morphisms proved by Migliorini as in [Mi]. A birational mor-
phism \( \phi \) is called semismall if for every irreducible component \( E \) in the exceptional locus,

\[
\operatorname{codim} E \geq \dim \phi^{-1}(p)
\]

for any \( p \in \phi(E) \). For example in dimension 3 a semismall morphism contracts curves to points or divisor to curves.

**Proposition 2.2 (Theorem 4.6 in [Mi]).** Let \( X \) be a smooth projective variety over \( \mathbb{C} \) with a semiample divisor \( L \). Assume the morphism \( \phi|_{mL} \) is semismall for some \( m >> 0 \). Then

\[
H^p(X, L^{-1} \otimes \Omega^q_X) = 0
\]

for \( p + q < \dim X \).

The following general result deals with maps which contract divisors to points.

**Lemma 2.3.** Let \( \pi : X \to Y \) be a projective birational morphism with \( X \) smooth and \( Y \) normal. Assume an irreducible divisor \( E \subset X \) is contracted to a point. Let \( \omega \in H^0(X, \Omega_X) \). Then \( Z(\omega) \) is not trivial along \( E \).

**Proof.** Assume \( \pi(E) = p \). Let us embed a neighborhood of \( p \) in some \( \mathbb{C}^N \) so that \( p \) is mapped to the origin \( O \) and still call the composition \( \pi \). For any \( q \in E \), one has \( \pi^*: m_{O,\mathbb{C}^N}/m_{O,\mathbb{C}^N}^2 \to m_{q,X}/m_{q,X}^2 \) between the differentials.

If \( q \) is a singularity of \( E \) and assume \( E \) is locally defined by some \( f \), then \( f \in m_{q,X}^2 \). This implies \( \pi^* = 0 \) and \( q \in z(\omega) \). If \( E \) is smooth at \( q \), then the rank of \( \pi^* \) is either 0 or 1. When the rank is 0, again \( q \in z(\omega) \). So we may reduce the proof of the desired result to the case in which the rank of \( \pi^* \) is 1 along \( E \) (in particular \( E \) is smooth and \( \pi^* \) of any element of \( m_{O,\mathbb{C}^N}/m_{O,\mathbb{C}^N}^2 \) is a scalar multiple of the class \( x_1 + m_{q,X}^2 \) where \( x_1 \) locally defines \( E \). The map \( m_{q,X}/m_{q,X}^2 \to m_{q,E}/m_{q,E}^2 \) is simply dropping \( x_1 \) component, hence sending a scalar multiple of the class \( x_1 + m_{q,X}^2 \) to 0).

In this case \( q \notin Z(\omega) \) for any \( q \in E \) implies that \( \omega \) defines a nowhere vanishing section of \( N^{*}_{E/X} \). This is against the adjuction lemma of Kawamata in [K].

\( \square \)

Now we can finish the proof of Theorem 1.
**Theorem 2.4**=**Theorem 1.** Let $X$ be a threefold of general type. For any global holomorphic 1-form $0 \neq \omega \in H^0(X, \Omega_X)$, its zero locus $Z(\omega)$ is not empty.

**Proof.** If $K_X$ is not nef, Lemma 2.1 says $Z(\omega)$ is not empty. Assume $K_X$ is nef, then $K_X$ is semiample. Let $\phi_m = \phi_{mK_X}$ for $m \gg 0$. By Lemma 2.3 we may assume $\phi_m$ is semismall. Assuming $Z(\omega)$ is empty, one has exact sequence

$$0 \to \mathcal{O} \xrightarrow{\omega} \Omega^1_X \xrightarrow{\wedge \omega} \Omega^2_X \xrightarrow{\wedge \omega} \Omega^3_X \to 0.$$ 

Arguing as in [Z], using Proposition 2.2 when $L = K_X$ one gets an embedding of $\Omega^3_X$ into $\Omega^2_X$. It follows that $H^0(X, K_X^{-1} \otimes \Omega^2_X) \neq 0$ which is impossible again by 2.2.

\[\square\]

It is easy to construct a threefold of general type with ample canonical bundle admitting a 1-form whose zero locus is of dimension 0, 1, or 2.

**§3 Proof of Theorem 2**

On a threefold $X$, the maximal dimension of a subspace $V \subset H^0(X, \Omega^1_X)$ with the property that any $0 \neq \omega \in V$ has no zero locus is 3 because the rank of $\Omega^1_X$ is 3. By taking a product of a variety of general type and an Abelian variety, one gets examples of threefold of Kodaira dimension $3 - k$ and having a $k$-dimensional subspace of 1-forms with the property that any nonzero element has no zero.

**Theorem 3.1**=**Theorem 2.** Let $X$ be a threefold admitting a $k$-dimensional subspace $V \subset H^0(X, \Omega^1_X)$ such that $z(\omega) = \emptyset$ for any $0 \neq \omega \in V$. Then the Kodaira dimension of $X$ is at most $3 - k$.

**Proof.** When $k = 1$, the claim is implied by Theorem 1. When $k = 3$, let $\omega_1, \omega_2, \omega_3$ be a basis of $V$, $\omega_1 \wedge \omega_2 \wedge \omega_3$ provides a nowhere zero section of $K_X$. So $K_X$ is trivial and $\kappa(X) = 0$.

Let us assume $k = 2$. By Lemma 2.1 we see that $X$ is minimal because of existence of $V$. We also know $\kappa(X) \neq 3$ by Theorem 1. Thus it is enough to show $\kappa(X) \neq 2$. Assuming the contrary holds. Let $\phi : X \to S$ be the Iitaka fibration defined by high multiples of $K_X$. $\phi^{-1}(s)$ is an elliptic curve for a general $s \in S$. $S$ is surface of general type with at worst canonical singularities. Let $\alpha : X \to \text{Alb}(X)$
be the Albanese map. The existence of \( V \) implies that \( \alpha(X) \) is at least of dimension 2.

We consider two cases:

Case I. \( \alpha(\phi^{-1}(s)) \) is a point for a general \( s \in S \). In this case \( \alpha(X) \) has dimension 2. Let \( X \to Y \to \alpha(X) \) be the Stein factorization of \( \alpha \). It is easy to see that \( S \) and \( Y \) are birationally equivalent. In particular 1-forms on \( X \) are pullbacks of 1-forms (on the smooth locus) of \( S \). Assume the subspace \( V \) has a basis \( \omega_1, \omega_2 \), which correspond to \( \omega_1', \omega_2' \) on \( S \). \( \omega_1', \omega_2' \) span a 2-dimensional subspace \( V' \) of \( H^0(S, \Omega^1_S) \) having the property that for any \( 0 \neq \omega' \in V' \) the zero locus on the smooth locus of \( S \) is empty. Then \( \omega_1' \wedge \omega_2' \) generates a nowhere zero section of \( K_S \) on the smooth locus. Hence \( \kappa(S) = 0 \), a contradiction.

Case II. \( \alpha(\phi^{-1}(s)) \) is a curve for a general \( s \in S \). In this case \( 0 = \kappa(\phi^{-1}(s)) \geq \kappa(\alpha(\phi^{-1}(s))) \geq 0 \). So \( \alpha(\phi^{-1}(s)) = E_s \) is also an elliptic curve in \( A = \text{Alb}(X) \). Since \( A \) contains at most countably many sub Abelian varieties of dimension 1, \( E_s \) are translations of a particular sub Abelian variety \( A_1 \) of dimension 1. Let \( \pi : A \to A/A_1 \) be the quotient map. One has the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & A \\
\phi \downarrow & & \downarrow \pi \\
S & \xrightarrow{\pi} & A/A_1 \\
\end{array}
\]

If \( \alpha(X) \) is of dimension 3, \( \pi(\alpha(X)) \) is of dimension 2. Again let \( X \to Y \to \pi(\alpha(X)) \) be the Stein factorization of \( \pi \circ \alpha \). \( S \) is birationally equivalent to \( Y \) and we are done by case I. If \( \alpha(X) \) is of dimension 2, \( C = \pi(\alpha(X)) \) is of dimension 1 in the Abelian variety \( A/A_1 \). Let \( \omega_1', \omega_2' \) be the 1-forms on \( A \) which pullback to a basis of \( V \). There is a nontrivial linear combination \( \omega' = c_1 \omega_1' + c_2 \omega_2' \) which is identically zero on the fibers of \( \pi \). \( \omega' \) is a pullback of some \( \omega'' \) on \( A/A_1 \). Assume \( C \) is of general type and we may assume \( C \) is smooth, \( \omega''|C \) vanishes at some point \( p \in C \), which in turn implies \( c_1 \omega_1 + c_2 \omega_2 \) vanishes on the fiber of \( \pi \circ \alpha \) in \( X \) over \( p \). If \( C \) is smooth elliptic curve, the pullback of \( \omega' = c_1 \omega_1' + c_2 \omega_2' \) (which is the same as the pullback of \( \omega'' \)) would have a zero unless the morphism \( \pi \circ \alpha \) is smooth and \( \omega'' \) does not vanish on \( C \). So let us assume \( \pi \circ \alpha \) is smooth. Note that each fiber the morphism has to be a surface of general type in order to keeping \( \kappa(X) = 2 \). Furthermore each fiber has to be minimal since \( K_X \) is nef. Thus \( \pi \circ \alpha : X \to C \) represents a smooth family of
minimal surfaces of general type over an elliptic curve. It has to be isotrivial thanks to Theorem 1.1 in [Mi]. So up to an étale covering, \( X \) is isomorphic to the product of \( C \) and a fiber of \( \pi \circ \alpha \). It implies that \( V \) contains a nonzero member \( \omega \) coming from a 1-form on the fiber which is a surface of general type. As any 1-form on a surface of general type has to have a zero, \( Z(\omega) \) contains a curve. A contradiction.

\[ \Box \]

\S 4 2-FORMS ON THREEFOLDS OF GENERAL TYPE

After the discussion regarding zeros of 1-forms, it is quite natural to wonder whether similar results on 1-forms still hold true for higher orders forms, i.e;

**Question:** Let \( X \) be a threefold of general type. Is it true that any global differential 2-form on \( X \) must have nonempty zero locus?

Campana and Peternell [CP] constructed the following example which shows that a global section of \( \Omega^2_X \) may not have zero locus even when \( K_X \) is ample:

**Example.** Let \( Y \) be projective holomorphic symplectic fourfolds. Let \( X \) be a hypersurface of sufficiently high degree. Then the nondegenerate 2-form on \( Y \) provides a nowhere vanishing 2-form on \( X \).

However by virtue of [L] it is known that when \( K_X \) is not nef, any 2-form has nontrivial zero locus. It would be interesting to see whether there are examples with nef canonical bundle and a 2-form without any zero.

Here we will show that any canonically twisted 2-form \( \omega \in H^0(X, K_X \otimes \Omega^2_X) \) on a threefold of general type has nontrivial zero locus.

Let \( X \) be a smooth projective variety of dimension \( n \). \( \theta \in H^0(X, T_X \otimes L) \) with isolated zeros. Then locally around each zero \( p \), under coordinates \( z_1, z_2, \ldots, z_n \), one can express \( \theta \) as

\[
\theta = \sum_{i=1}^n f_i s \otimes \frac{\partial}{\partial z_i}
\]

where \( f_i \) are holomorphic functions and \( s \) is a local section of \( L \). The Jacobian at the zero is defined as

\[
A_p = \left( \frac{\partial f_i}{\partial z_j} \right)|_p
\]

which is an \( n \times n \) matrix. \( A_p \) is invariant under coordinate change up to a nowhere zero holomorphic function. \( p \) is nondegenerate if \( \det(A_p) \neq 0 \).
The following lemma is a corollary of the residue theorem due to Bott (see [C]).

**Lemma 4.1.** Notations as above and assume \( \theta \) has only nondegenerate zeros. Then

\[
\int_X \bar{c}_1(T_X \otimes L)^n = \sum_{p \text{ a zero}} \frac{(Tr(A_p))^n}{\det(A_p)}
\]

and

\[
\int_X \bar{c}_n(T_X \otimes L) = \sum_{p \text{ a zero}} \frac{\det(A_p)}{\det(A_p)}
\]

where \( \bar{c}_r(T_X \otimes L) = c_r(T_X) + c_{r-1}c_1(L) + ... + c_1(L)^r \).

**Remark.** Indeed Chern’s proof [C] shows that away from the zero locus, \( \bar{c}_1(T_X \otimes L)^n \) and \( \bar{c}_n(T_X \otimes L) \) can be written as derived differential forms. This fact is used in the proof of the following

**Theorem 4.2 = Theorem 3.** Let \( X \) be a threefold of general type. \( \omega \in H^0(X, K_X \otimes \Omega^2_X) \) is a canonically twisted holomorphic 2-form. Then the zero locus \( Z(\omega) \) of \( \omega \) is not empty.

**Proof.** We may assume \( K_X \) is nef since otherwise there is a divisorial extremal contraction and an analysis similar to that in section 2 of [L] shows that for all possible divisorial extremal contraction any canonically twisted 2-form has nontrivial zero locus along the exceptional divisor.

Since \( \Omega^2_X \) is isomorphic to \( K_X \otimes T_X \), \( \omega \) corresponds to a global section \( \theta \) in \( H^0(X, 2K_X \otimes T_X) \), which can be viewed as a meromorphic vector field with coefficient in the line bundle \( 2K_X \). Assume \( \theta \) (or equivalently \( \omega \)) does not have zero, the remark about the above lemma says

\[
\bar{c}_1(2K_X \otimes T_X)^3 = 0.
\]

However the left hand side is

\[
\bar{c}_1(2K_X \otimes T_X)^3 = (c_1(T_X) + c_1(2K_X))^3 = K_X^3 > 0,
\]

a contradiction.

\( \square \)

Finally the geometric meaning of \( c_3(\Omega^2_X) \) (which is \( c_3 - c_1c_2 \) of \( X \)) in terms of singularities appearing on a minimal model is explained in:
Theorem 4.3. Let $X$ be a threefold of general type. Assume $\Omega^2_X$ admits a section $\omega$ with isolated nondegenerate zeros. Then $c_3(\Omega^2_X)$ is greater or equal to the number of singularities whose indices are bigger than 1 on a minimal model.

Proof. Let $X_{\text{min}}$ be a minimal model of $X$. Assuming $\omega$ is a 2-form with only isolated nondegenerate zeros on $X$. By [L], $\omega$ corresponds to a (global locally invariant) 2-form on $X_{\text{min}}$ not vanishing at the singularities whose indices are bigger than 1. Moreover every such singularity produces at least one isolated zero on $X$. Bott’s residue formula, when taking the top Chern class, says

$$\bar{c}_3(K_X \otimes T_X) = \sum_{p \text{ a zero}} \frac{\text{det}(A_p)}{\text{det}(A_p)} = \text{number of zeros of } \omega.$$ 

This implies

$$\bar{c}_3(K_X \otimes T_X) = c_3(T_X) + c_2(T_X)c_1(K_X) + c_1(T_X)c_1(K)^2 + c_1(K_X)^3 = c_3 - c_1c_2$$

$$\geq \text{number of singularities whose indices are bigger than 1}.$$ 

□

References

[C] Chern, S.S, Meromorphic vector fields and characteristic numbers, Scripta Mathematica 29 (1973), 243-251.
[CP] Campana, F and Peternell, T, Holomorphic 2-forms on complex threefolds, Journal of Algebraic Geometry 9 (2000), 223-264.
[K] Kawamata, Y, On the length of an extremal rational curve, Invent. Math. 105 (1991), 609-611.
[L] Luo, T, Global 2-forms and pluricanonical systems on threefolds, Math. Ann. (2000), to appear.
[M] Migliorini, L, A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial, J. Algebraic Geom. 4 (1995), 356-363.
[Mo] Mori, S, Threefolds whose canonical bundles are not numerically effective, Ann. of Math. 116 (1982), 133-176.
[R1] Reid, M, Canonical 3-folds, Journée de Géométrie algébrique d’Angers, 1980, pp. 273-310.
[R2] Reid, M, Minimal models of canonical 3-folds, Algebraic Varieties and Analytic Varieties, Adv. Stud. Pure Math., vol. 1, 1980, pp. 131-180.
[R3] Reid, M, Young person’s guide to canonical singularities, Algebraic Geometry-Bowdoin 1985, vol. 46, 1987, pp. 333-414.
[U] Ueno, K, Classification theory of algebraic varieties and compact complex spaces, Lecture Notes in Mathematics 439.
[Z] Zhang, Q, Global holomorphic one-forms on projective manifolds with ample canonical bundles, Journal of Algebraic Geometry 6 (1997), 777-787.