VECTOR FIELDS LIFTABLE OVER FINITELY DETERMINED
MULTIGERMS OF CORANK AT MOST ONE

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Abstract. In this paper, we propose one index $i_1(f) - i_2(f)$ which measures how well-behaved a given finitely determined multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ ($n \leq p$) of corank at most one is from the viewpoint of liftable vector fields; and we answer the following problems when the index indicates that the given multigerm $f$ is best-behaved.

(1) When is the module of vector fields liftable over $f$ finitely generated?

(2) How can we characterize the minimal number of generators when the module of vector fields liftable over $f$ is finitely generated?

(3) How can we calculate the minimal number of generators when the module of vector fields liftable over $f$ is finitely generated?

(4) How can we construct generators when the module of vector fields liftable over $f$ is finitely generated?

Mathematics Subject Classification (2010): 58K40 (primary), 57R45, 58K20 (secondary).

Key words: liftable vector field, higher version of the reduced Kodaira-Spencer-Mather map, module, generator, finitely determined multigerm, corank at most one.

1. Introduction

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, $S$ be a finite subset of $\mathbb{K}^n$. Let $C_S$ (resp., $C_0$) be the set of smooth (that is, $C^\infty$ if $\mathbb{K} = \mathbb{R}$ or holomorphic if $\mathbb{K} = \mathbb{C}$) function-germs $(\mathbb{K}^n, S) \rightarrow \mathbb{K}$ (resp., $(\mathbb{K}^p, 0) \rightarrow \mathbb{K}$) and let $m_S$ (resp., $m_0$) be the subset of $C_S$ (resp., $C_0$) consisting of smooth function-germs $(\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ (resp., $(\mathbb{K}^p, 0) \rightarrow (\mathbb{K}, 0)$). The sets $C_S$ and $C_0$ have natural $\mathbb{K}$-algebra structures induced by the $\mathbb{K}$-algebra structure of $\mathbb{K}$.

For a smooth multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$, let $f^* : C_0 \rightarrow C_S$ be the $\mathbb{K}$-algebra homomorphism defined by $f^*(u) = u \circ f$. Put $Q(f) = C_S/f^*m_0C_S$.

For a smooth multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, T)$ such that $f(S) \subset T$, where $S$ (resp., $T$) is a finite subset of $\mathbb{K}^n$ (resp., $\mathbb{K}^p$), let $\theta_S(f)$ be the $C_S$-module consisting of germs of smooth vector fields along $f$. We may identify $\theta_S(f)$ with $C_S \oplus \cdots \oplus C_S$.

We put $\theta_S(n) = \theta_S(id_{(\mathbb{K}^n, S)})$ and $\theta_0(p) = \theta_{(0)}(id_{(\mathbb{K}^p, 0)})$, where $id_{(\mathbb{K}^n, S)}$ (resp., $id_{(\mathbb{K}^p, 0)}$) is the germ of the identity mapping of $(\mathbb{K}^n, S)$ (resp., $(\mathbb{K}^p, 0)$).

For a given smooth multigerm $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$, following Mather ([10]), we define $tf$ and $\omega f$ as follows:

$$tf : \theta_S(n) \rightarrow \theta_S(f), \quad tf(\eta) = df \circ \eta,$$

$$\omega f : \theta_0(p) \rightarrow \theta_S(f), \quad \omega f(\xi) = \xi \circ f.$$
where $df$ is the differential of $f$. For the $f$, following Wall ([19]), we put

$$
TR(f) = tf(m_S \theta_S(n)), \quad TR_e(f) = tf(\theta_S(n)),
$$
$$
TL(f) = \omega f(m_0 \theta_0(p)), \quad TL_e(f) = \omega f(\theta_0(p)),
$$
$$
TA(f) = TR(f) + TL(f), \quad TA_e(f) = TR_e(f) + TL_e(f),
$$
$$
TK(f) = TR(f) + f^*m_0 \theta_S(f), \quad TK_e(f) = TR_e(f) + f^*m_0 \theta_S(f).
$$

For a given smooth multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ a vector field $\xi \in \theta_0(p)$ is said to be liftable over $f$ if $\xi \circ f$ belongs to $TR_e(f)$. The set of vector fields liftable over $f$ has naturally a $C_0$-module structure. In this paper, we consider the following problems on the module of liftable vector fields.

**Problem 1.** Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a smooth multigerm.

1. When is the module of vector fields liftable over $f$ finitely generated?
2. How can we characterize the minimal number of generators when the module of vector fields liftable over $f$ is finitely generated?
3. How can we calculate the minimal number of generators when the module of vector fields liftable over $f$ is finitely generated?
4. How can we construct generators when the module of vector fields liftable over $f$ is finitely generated?

In order to study Problem 1 we generalize Mather’s homomorphism ([11])

$$
\overline{\varphi} f : \frac{\theta_0(p)}{m_0 \theta_0(p)} \to \frac{\theta_S(f)}{TK_e(f)}
$$

defined by $\overline{\varphi} f([\xi]) = [\omega f(\xi)]$. Note that

$$
\frac{\theta_S(f)}{TK_e(f)} \cong \frac{\theta_S(f)}{TR_e(f)} \frac{TR_e(f)}{f^*m_0} \frac{f^*m_0}{TH_e(f)}
$$

as finite dimensional vector spaces over $\mathbb{K}$ for any smooth multigerm $f$ satisfying $\dim_\mathbb{K} \theta_S(f)/TK_e(f) < \infty$. Thus, by the preparation theorem (for instance, see [2]), we have that $\theta_S(f) = TA_e(f)$ if and only if $\overline{\varphi} f$ is surjective for any smooth multigerm $f$ satisfying $\dim_\mathbb{K} \theta_S(f)/TK_e(f) < \infty$. In the case that $\mathbb{K} = \mathbb{C}$, $n \geq p$ and $S = \{\text{one point}\}$, the map $\hat{\omega} f : \theta_0(p) \to \frac{\theta_S(f)}{TK_e(f)}$ given by $\hat{\omega} f(\xi) = [\omega f(\xi)]$ is called the Kodaira-Spencer map of $f$ and Mather’s homomorphism $\overline{\varphi} f$ is called the reduced Kodaira-Spencer map of $f$ ([9]). Thus, $\overline{\varphi} f$ is a generalization of the reduced Kodaira-Spencer map of $f$ and the module of vector fields liftable over $f$ is the kernel of $\hat{\omega} f$. We would like to consider higher versions of $\overline{\varphi} f$. For a non-negative integer $i$, an element of $m_S^i$ or $m_0^i$ is a germ of smooth function such that the terms of the Taylor series of it up to $(i - 1)$ are zero. Thus, $m_S^i = C_S$ and $m_0^i = C_0$. For any non-negative integer $i$ and a given smooth multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, we let

$$
\overline{i} \varphi f : \frac{m_0^i \theta_0(p)}{m_0^{i+1} \theta_0(p)} \to \frac{f^*m_0^i \theta_S(f)}{TR_e(f) \cap f^*m_0^i \theta_S(f) + f^*m_0^{i+1} \theta_S(f)}
$$

be a homomorphism of $C_0$-modules via $f$ defined by $\overline{i} \varphi f([\xi]) = [\omega f(\xi)]$. Then, it is clearly seen that $\overline{i} \varphi f$ is well-defined. Note that $\omega \overline{i} \varphi f = \overline{\varphi} f$. Similarly as the target module of $\overline{\varphi} f$, for any non-negative integer $i$ and any smooth multigerm $f$
satisfying \( \dim_K \theta_S(f)/TK_c(f) < \infty \), the target module of \( \omega f \) is isomorphic to the following:

\[
\frac{f^*m_0^i \theta_S(f)}{TK_c(f) \cap f^*m_0^i \theta_S(f)}.
\]

Thus, again by the preparation theorem, we have that \( f^*m_0^i \theta_S(f) \subset TA_c(f) \) if and only if \( \omega f \) is surjective. The following holds clearly:

**Lemma 1.1.** Let \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \) be a smooth multigerm satisfying \( \dim_K \theta_S(f)/TK_c(f) < \infty \). Then, the following hold:

1. Suppose that there exists a non-negative integer \( i \) such that \( \omega f \) is surjective. Then, \( \omega f \) is surjective for any integer \( j \) such that \( i < j \).
2. Suppose that there exists a non-negative integer \( i \) such that \( \omega f \) is injective. Then, \( \omega f \) is injective for any non-negative integer \( j \) such that \( i > j \).

**Definition 1.1.** Let \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \) be a smooth multigerm satisfying the condition \( \dim_K \theta_S(f)/TK_c(f) < \infty \).

1. Put \( I_1(f) = \{ i \in \{0\} \cup \mathbb{N} \mid \omega f \) is surjective\}. Define \( i_1(f) \) as

\[
i_1(f) = \begin{cases} \infty & \text{(if } I_1(f) = \emptyset) \\ \min I_1(f) & \text{(if } I_1(f) \neq \emptyset) \end{cases}
\]

2. Put \( I_2(f) = \{ i \in \{0\} \cup \mathbb{N} \mid \omega f \) is injective\}. Define \( i_2(f) \) as

\[
i_2(f) = \begin{cases} -\infty & \text{(if } I_2(f) = \emptyset) \\ \max I_2(f) & \text{(if } 0 \neq I_2(f) \neq \{0\} \cup \mathbb{N}) \\ \infty & \text{(if } I_2(f) = \{0\} \cup \mathbb{N}) \end{cases}
\]

A smooth multigerm \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \) is said to be *finitely determined* if there exists a positive integer \( k \) such that the inclusion \( m^k_S \theta_S(f) \subset TA_c(f) \) holds. The proof of the assertion (ii) of proposition 4.5.2 in [19] works well to show the following:

**Proposition 1.** Let \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \) be a finitely determined multigerm satisfying \( \theta_S(f) \neq TA_c(f) \). Then, \( i_2(f) \geq 0 \).

In the rest of this paper we concentrate on the case \( n \leq p \). It is reasonable to do so since there is already an extensive theory in the case \( n > p \) and, in the case \( n < p \) there seems to have been no answers to 1–4 of Problem 1 for a given multigerms which is well-behaved from the viewpoint of liftable vector fields (Note that in the case \( n < p \) the minimal number of generators is greater than \( p \) in general. For details, see §3. Thus, the target dimension \( p \) is not an answer to 2 of Problem 1). We want to answer all of 1–4 of Problem 1 for well-behaved multigerms in the case \( n \leq p \).

Suppose that \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \) is finitely determined. Then, it is clear that \( f^*m_0C_S \subset m_S \), there exists a positive integer \( k \) such that the inclusion \( f^*m_0^k \theta_S(f) \subset TA_c(f) \) holds. Thus, \( k\omega f \) is surjective. Conversely, suppose that there exists a positive integer \( k \) such that \( k\omega f \) is surjective for a smooth multigerm \( f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0) \) satisfying \( \dim_K \theta_S(f)/TK_c(f) < \infty \). Then, as we have confirmed, the inclusion \( f^*m_0^k \theta_S(f) \subset TA_c(f) \) holds by the preparation theorem. In the case \( n \leq p \), by Wall’s estimate (theorem 4.6.2 in [19]), the condition \( \dim_K \theta_S(f)/TK_c(f) < \infty \) implies that there exists an integer \( \ell \) such that \( m_0^\ell \subset f^*m_0C_S \). Hence, we have the following:
Proposition 2. Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a smooth multigerm satisfying the condition $\dim \ker(\omega_{f})/\ker(f) < \infty$. Suppose that $n \leq p$. Then, $i_1(f) < \infty$ if and only if $f$ is finitely determined.

A smooth multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ ($n \leq p$) is said to be of corank at most one if $\max\{n - \text{rank} Jf(s_j) \mid 1 \leq j \leq |S|\} \leq 1$ holds, where $Jf(s_j)$ is the Jacobian matrix of $f$ at $s_j \in S$ and $|S|$ stands for the number of distinct points of $S$.

Proposition 3. Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ ($n \leq p$) be a finitely determined multigerm of corank at most one. Then, $i_1(f) \geq i_2(f)$.

Proposition 3 is proved in §2. Proposition 3 yields the following corollary.

Corollary 1. Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ ($n \leq p$) be a finitely determined multigerm of corank at most one. Suppose that there exists a non-negative integer $i$ such that $i\omega f$ is bijective. Then, the following hold:

1. For any non-negative integer $j$ such that $j < i$, $j\omega f$ is injective but not surjective.
2. For any non-negative integer $j$ such that $i < j$, $j\omega f$ is surjective but not injective.

Example 1.1. Let $e : \mathbb{K} \to \mathbb{K}^2$ be the embedding defined by $e(x) = (x, 0)$ and for any real number $\theta$ let $R_{\theta} : \mathbb{K}^2 \to \mathbb{K}^2$ be the linear map which gives the rotation of $\mathbb{K}^2$ around the origin with respect to the angle $\theta$.

$$R_{\theta} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$ 

For any non-negative integer $\ell$ put $S = \{s_0, \ldots, s_{\ell+1}\}$ ($s_j \neq s_k$ if $j \neq k$). For the $\ell$ define $\theta_j = \frac{j\pi}{\ell + 1}$ and $e_j : (\mathbb{K}, s_j) \to (\mathbb{K}^2, 0)$ as $e_j(x_j) = R_{\theta_j} \circ e(x_j)$ for any $j$ ($0 \leq j \leq \ell + 1$), where $x_j = x - s_j$. Then, $E_\ell = \{e_0, \ldots, e_{\ell+1}\} : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ is a finitely determined multigerm of corank at most one. The image of $E_\ell$ is a line arrangement and hence the Euler vector field of the defining equation of the image of $E_\ell$ is a liftable vector field over $E_\ell$. It follows that $\omega E_\ell$ is not injective. Furthermore, it is easily seen that $0\omega E_\ell$ is injective even in the case $\ell = 0$ (in the case $\ell \geq 1$ this is a direct corollary of Proposition 1). Thus, $i_2(E_\ell) = 0$. On the other hand, it is seen that $i_1(E_\ell) = \ell$. Therefore, $i_1(E_\ell) - i_2(E_\ell) = \ell$.

This example shows that there are no upper bound of $i_1(f) - i_2(f)$ for a finitely determined multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ ($n \leq p$) of corank at most one in general. By this example one may guess that the integer $i_1(f) - i_2(f)$ can be a candidate of index to measure how well-behaved a given finitely determined multigerm of corank at most one is from the viewpoint of liftable vector fields.

In the rest of this paper we concentrate on the case that the index $i_1(f) - i_2(f)$ is the smallest.

Theorem 1. Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ ($n \leq p$) be a finitely determined multigerm of corank at most one. Suppose that there exists a non-negative integer $i$ such that $i\omega f$ is bijective (namely, $i_1(f) = i_2(f) = i$). Then, the minimal number of generators for the module of vector fields liftable over $f$ is exactly $\dim \ker(i_{i+1}\omega f)$. 
Note that the embedding $e$ in Example 1.1 is not an example of Theorem 1. Actually, since $c_e$ is surjective but not injective, $i_1(e) = 0$ and $i_2(e) = -\infty$. On the other hand, the multigerm $E_0$ in Example 1.1 is an example of Theorem 1 although $E_0$ does not satisfy the assumption of Proposition 1. Furthermore, a lot of examples of Theorem 1 are given in §3. Example 1.1 and examples in §3 suggest that Theorem 1 may be regarded as an answer to 1, 2 of Problem 1 for a smooth multigerm $f$ which is best-behaved from the viewpoint of liftable vector fields.

In order to answer 3 of Problem 1 for a given finitely determined multigerm $f$ of corank at most one satisfying the condition $i_1(f) = i_2(f)$, we generalize Wall’s homomorphism ([19])

$$\overline{t}f : Q(f)^n \to Q(f)^p, \quad \overline{t}f([\eta]) = [tf(\eta)]$$

as follows. For a given smooth multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ satisfying the condition $\dim_{\mathbb{K}} Q(f) < \infty$, let $\delta(f)$ (resp., $\gamma(f)$) be the dimension of the vector space $Q(f)$ (resp., the dimension of the kernel of $\overline{t}f$). For the $f$ and a non-negative integer $i$, we put $iQ(f) = f^*m_0^iC_S/f^*m_0^{i+1}C_S$ and $i\delta(f) = \dim_{\mathbb{K}} iQ(f)$. Thus, we have that $\delta(f) = Q(f)$ and $\delta(f) = \delta(f) = \dim_{\mathbb{K}} Q(f)$. The $Q(f)$-modules $iQ(f)^n$ and $iQ(f)^p$ may be identified with the following respectively.

$$\frac{f^*m_0^i\theta_S(n)}{f^*m_0^{i+1}\theta_S(n)} \quad \text{and} \quad \frac{f^*m_0^i\theta_S(f)}{f^*m_0^{i+1}\theta_S(f)}.$$  

We let $i\gamma(f)$ be the dimension of the kernel of the following well-defined homomorphism of $Q(f)$-modules.

$$i\overline{t}f : iQ(f)^n \to iQ(f)^p, \quad i\overline{t}f([\eta]) = [tf(\eta)].$$

Then, we have that $i\delta(f) < \infty$ if $\delta(f) < \infty$ and $i\gamma(f) < \infty$ if $\gamma(f) < \infty$. For details on $iQ(f), i\delta(f), i\overline{t}f$ and $i\gamma(f)$, see [16].

**Proposition 4.** Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a smooth multigerm of corank at most one satisfying the condition $\dim_{\mathbb{K}} Q(f) < \infty$. Suppose that there exists a non-negative integer $i$ such that $i+1 \overline{t}f$ is surjective. Then, the following holds:

$$\dim_{\mathbb{K}} \ker(i+1 \overline{t}f) = p \cdot \left(\binom{p + i}{i + 1} - (p - n) \cdot i+1 \delta(f) + i \gamma(f) - \gamma(f)\right),$$

where the dot in the center stands for the multiplication.

**Proposition 5.** Let $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a smooth multigerm of corank at most one satisfying the condition $\dim_{\mathbb{K}} Q(f) < \infty$. Then, the following hold:

1. $\alpha \gamma(f) = \gamma(f) = \delta(f) - |S|.$
2. $i\delta(f) = \left(\binom{n + i - 1}{i}\right) \cdot \delta(f), \quad i\gamma(f) = \left(\binom{n + i - 1}{i}\right) \cdot \gamma(f) \quad (i \in \mathbb{N} \cup \{0\}).$

By combining Propositions 4 and 5, for a smooth multigerm $f$ of corank at most one such that $\dim_{\mathbb{K}} Q(f) < \infty$, the $A$-invariant “$\dim_{\mathbb{K}} \ker(i+1 \overline{t}f)”$ can be calculated easily by using $K$-invariants “$\delta(f), \gamma(f)”$ when there exists a non-negative integer $i$ such that $i+1 \overline{t}f$ is surjective.

In Section 2, proofs of Propositions 3, 4 and 5 are given. In Section 3, examples for which actual calculations of minimal numbers of generators are carried out are given. Theorem 1 is proved in Sections 4. In Section 5, by constructing concrete
generators for several examples, it is shown that the proof of Theorem 1 may be regarded as an answer to 4 of Problem 1 in principle.

2. Proofs of Propositions 3, 4 and 5

Proof of Proposition 5.

Put \( S = \{ s_1, \ldots, s_{|S|} \} \) (\( s_j \neq s_k \) if \( j \neq k \)) and for any \( j \) (\( 1 \leq j \leq |S| \)) let \( f_j \) be the restriction \( f\mid_{(\mathbb{K}^n, s_j)} \). Then, we have the following:

\[
\delta(f) = \dim_{\mathbb{K}} Q(f) = \sum_{j=1}^{|S|} \dim_{\mathbb{K}} Q(f_j) = \sum_{j=1}^{|S|} \delta(f_j).
\]

\[
\gamma(f) = \dim_{\mathbb{K}} \ker(Tf) = \sum_{j=1}^{|S|} \dim_{\mathbb{K}} \ker(Tf_j) = \sum_{j=1}^{|S|} \gamma(f_j)
= \sum_{j=1}^{|S|} (\delta(f_j) - 1) = \delta(f) - |S|.
\]

This completes the proof of the assertion 1 of Proposition 5.

Next we prove the assertion 2 of Proposition 5. Since \( f \) is of corank at most one, for any \( j \) (\( 1 \leq j \leq |S| \)) there exist germs of diffeomorphism \( h_j : (\mathbb{K}^n, s_j) \rightarrow (\mathbb{K}^n, s_j) \) and \( H_j : (\mathbb{K}^p, 0) \rightarrow (\mathbb{K}^p, 0) \) such that \( H_j \circ f_j \circ h_j^{-1} \) has the following form:

\[
H_j \circ f_j \circ h_j^{-1}(x, \ldots, x_{n-1}, y) = (x_1, \ldots, x_{n-1}, y^{\delta(f_j)} + f_{j,n}(x_1, \ldots, x_{n-1}, y), f_{j,n+1}(x_1, \ldots, x_{n-1}, y), \ldots, f_{j,p}(x_1, \ldots, x_{n-1}, y)).
\]

Here, \( x_1, \ldots, x_{n-1}, y \) are local coordinates of the coordinate neighborhood \((U_j, h_j)\) at \( s_j \) and \( f_{j,q} \) satisfies \( f_{j,q}(0, \ldots, 0, y) = o(y^{\delta(f_j)}) \) for any \( q \) (\( n \leq q \leq p \)). By the preparation theorem, \( C_{s_j} \) is generated by \( 1, y, \ldots, y^{\delta(f_j)-1} \) as \( C_0 \)-module via \( f_j \). Thus, \( f_j^* m_0 C_{s_j} \) is generated by elements of the following set as \( C_0 \)-module via \( f_j \).

\[
\begin{cases}
  x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} y^{k_n \delta(f_j) + \ell} & \text{if } k_m \geq 0, \sum_{m=1}^n k_m = i, 0 \leq \ell \leq \delta(f_j) - 1
\end{cases}
\]

Thus, the following set is a basis of \( iQ(f_j) \).

\[
\begin{cases}
  [x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} y^{k_n \delta(f_j) + \ell}] & \text{if } k_m \geq 0, \sum_{m=1}^n k_m = i, 0 \leq \ell \leq \delta(f_j) - 1
\end{cases}
\]
Therefore, we have the following:

\[ i \delta(f) = \dim K_i Q(f) = \sum_{j=1}^{[S]} \dim K_i Q(f_j) = \sum_{j=1}^{[S]} \dim K_i f_j^* m_0 C_s j \]

\[ = \sum_{j=1}^{[S]} \binom{n+i-1}{i} \cdot \delta(f_j) \]

\[ = \left( \frac{n+i-1}{i} \right) \cdot \sum_{j=1}^{[S]} \delta(f_j) \]

\[ = \left( \frac{n+i-1}{i} \right) \cdot \delta(f). \]

Next we prove the formula for \( i \gamma(f) \). Since it is clear that \( i \gamma(f_j) \) does not depend on the particular choice of coordinate systems of \((K^n, s_j)\) and of \((K^p, 0)\), we may assume that \( f_j \) has the above form from the first. Then, it is easily seen that the following set is a basis of \( \ker i tf_j \).

\[
\left\{ \begin{array}{c}
0 \oplus \cdots \oplus 0 \oplus x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} y^k \delta(f_j) + \ell \\
(n-1) \text{ tuples}
\end{array} \right\} \quad \text{where} \quad k_m \geq 0, \sum_{m=1}^{n} k_m = i, 1 \leq \ell \leq \delta(f_j) - 1.
\]

Therefore, we have the following:

\[ i \gamma(f) = \dim K \ker(i \overline{f}) = \sum_{j=1}^{[S]} \dim K \ker(i \overline{f}_j) = \sum_{j=1}^{[S]} \binom{n+i-1}{i} \cdot (\delta(f_j) - 1) \]

\[ = \left( \frac{n+i-1}{i} \right) \cdot \sum_{j=1}^{[S]} \gamma(f_j) \]

\[ = \left( \frac{n+i-1}{i} \right) \cdot \gamma(f). \]

Q.E.D.

Secondly, we prove Proposition 4.

**Proof of Proposition 4**

Consider the linear map \( i+1 \overline{f} \). Then, we have the following:

\[ \dim K i+1 Q(f)^n = i+1 \gamma(f) + \dim K \Image(i+1 \overline{f}). \]

Since \( \dim K Q(f) < \infty \) and \( f \) is of corank at most one, it is easily seen that \( tf \) is injective. Hence we see that

\[ \dim K \frac{\Image(f) \cap f^* m_0^{i+1} \theta_s(f)}{\Image(f) \cap f^* m_0^{i+2} \theta_s(f)} = \dim K \Image(i+1 \overline{f}) + \gamma(f). \]

Therefore, we have the following:

\[ \frac{f^* m_0^{i+1} \theta_s(f)}{\Image(f) \cap f^* m_0^{i+1} \theta_s(f) + f^* m_0^{i+2} \theta_s(f)} = (p-n) \cdot i+1 \delta(f) + i+1 \gamma(f) - i \gamma(f). \]
Hence and since $i+1\varpi f$ is surjective, we have the following:

\[
\dim_k \ker (i+1\varpi f) = \dim_k \frac{m_{0+1}^i \theta_0(p)}{m_{0+2}^i \theta_0(p)} - \dim_k \frac{f^*m_{0+1}^i \theta_S(f)}{TR^e_0(f) \cap f^*m_{0+2}^i \theta_S(f) + f^*m_{0+2}^i \theta_S(f)}
\]

\[
= p \cdot \left( \frac{p+i}{i+1} \right) - (p-n) \cdot i+1 \delta(f) + i+1 \gamma(f) - i \gamma(f).
\]

Q.E.D.

Finally, we prove Proposition 3.

**Proof of Proposition 3**

By Lemma 1.1, it suffices to show that for any $i$ and any finitely determined multigerm $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ of corank at most one satisfying that $i\varpi f$ is surjective, $i+1\varpi f$ is not injective if $i\varpi f$ is not injective. Thus, we may assume that $i\varpi f$ is bijective.

We first prove Proposition 3 in the case $i = 0$. Since we have assumed that $0\varpi f$ is bijective, the following holds (see [13] or [19]):

\[
p \cdot \left( \frac{p-1}{0} \right) = \dim_k \frac{\theta_0(p)}{m_0 \theta_0(p)} = (p-n) \cdot 0 \delta(f) + o \gamma(f).
\]

Note that the above equality can not be obtained by Proposition 4. Note further that at least one of $p-n > 0$ or $0 \gamma(f) > 0$ holds by this equality. We have the following:

\[
p \cdot \left( \frac{p}{1} \right) = p^2 \cdot \left( \frac{p-1}{0} \right)
= p \cdot ((p-n) \cdot 0 \delta(f) + o \gamma(f))
= \frac{p}{n} \cdot ((p-n) \cdot i \delta(f) + 1 \gamma(f)) \quad \text{(by 2 of Proposition 5)}
\geq (p-n) \cdot i \delta(f) + 1 \gamma(f) \quad \text{(by } n \leq p)
\geq (p-n) \cdot i \delta(f) + 1 \gamma(f) - o \gamma(f) \quad \text{(by } o \gamma(f) \geq 0).
\]

Since we have confirmed that at least one of $p-n > 0$ or $0 \gamma(f) > 0$ holds, we have the following sharp inequality:

\[
p \cdot \left( \frac{p}{1} \right) > (p-n) \cdot i \delta(f) + 1 \gamma(f) - o \gamma(f).
\]

Hence $i\varpi f$ is not injective by Lemma 1.1 and Proposition 4.

Next we prove Proposition 3 in the case $i \geq 1$. Since we have assumed that $i\varpi f$ is bijective, we have the following equality by Proposition 4:

\[
p \cdot \left( \frac{p+i-1}{i} \right) = (p-n) \cdot i \delta(f) + i \gamma(f) - i-1 \gamma(f)
= (p-n) \cdot i \delta(f) + \left( 1 - \frac{i}{n+i-1} \right) \cdot i \gamma(f) \quad \text{(by 2 of Proposition 5)}
= (p-n) \cdot i \delta(f) + \frac{n-1}{n+i-1} \cdot i \gamma(f).
\]
Note that at least one of $p - n > 0$ or $(n - 1) \cdot \gamma(f) > 0$ holds by this equality. We have the following:

$$p \cdot \left( \frac{p + i}{i + 1} \right) = \frac{p + i}{i + 1} \cdot p \cdot \left( \frac{p + i - 1}{i} \right)$$

$$= \frac{p + i}{i + 1} \cdot \left( (p - n) \cdot \delta(f) + \frac{n - 1}{n + i - 1} \cdot \gamma(f) \right)$$

$$= \frac{p + i}{i + 1} \cdot \left( \frac{i + 1}{n + i} \cdot (p - n) \cdot \delta(f) + \frac{n - 1}{n + i - 1} \cdot \frac{i + 1}{n + i} \cdot i + 1 \cdot \gamma(f) \right)$$

(by 2 of Proposition 5)

$$= \frac{p + i}{n + i} \cdot (p - n) \cdot i + 1 \cdot \delta(f) + \frac{p + i}{n + i - 1} \cdot \frac{n - 1}{n + i} \cdot i + 1 \cdot \gamma(f)$$

(by $n \leq p$)

$$\geq (p - n) \cdot i + 1 \cdot \delta(f) + \frac{n - 1}{n + i} \cdot i + 1 \cdot \gamma(f)$$

(by $n \leq p$ and $(n - 1) \cdot i + 1 \cdot \gamma(f) \geq 0$)

$$= (p - n) \cdot i + 1 \cdot \delta(f) + i + 1 \cdot \gamma(f) - i \cdot \gamma(f)$$

(by 2 of Proposition 5).

Since we have confirmed that at least one of $p - n > 0$ or $(n - 1) \cdot \gamma(f) > 0$ holds and $i + 1 \cdot \gamma(f) = \frac{n + i}{n + i - 1} \cdot \gamma(f)$ by the assertion 2 of Proposition 5, we have the following sharp inequality:

$$p \cdot \left( \frac{p + i}{i + 1} \right) \geq (p - n) \cdot i + 1 \cdot \delta(f) + i + 1 \cdot \gamma(f) - i \cdot \gamma(f).$$

Hence, $\varphi f$ is not injective by Lemma 1.1 and Proposition 4. Q.E.D.

3. Examples

Example 3.1. Let $\varphi : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ be the map-germ given by $\varphi(x_1, \ldots, x_{n-1}, y) = (x_1, \ldots, x_{n-1}, y^{n+1} + \sum_{i=1}^{n-1} x_i y^i)$. Then $0 \varphi \varphi$ is known to be bijective by [1] or [2]. By Corollary 1, Propositions 4, 5 and Lemma 1.1, the minimal number of generators for the module of vector fields liftable over $\varphi$ can be calculated as follows:

$$n \cdot \left( \frac{n}{1} \right) - ((n - n) \cdot 1 \cdot \delta(\varphi) + 1 \gamma(\varphi) - 0 \gamma(\varphi))$$

$$= n^2 - ((n - n) \cdot n \cdot (n + 1) + n \cdot (n + 1 - 1) - (n + 1 - 1))$$

$$= n.$$

It has been verified in [1] that the minimal number of generators for the module of vector fields liftable over $\varphi$ is exactly $n$ in the complex case.

Example 3.2. Let $\varphi_k : (\mathbb{K}^{2k-2}, 0) \to (\mathbb{K}^{2k-1}, 0)$ be given by

$$\varphi_k(u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-1}, y) = (u_1, \ldots, u_{k-2}, v_1, \ldots, v_{k-1}, y^k + \sum_{i=1}^{k-2} u_i y^i, \sum_{i=1}^{k-1} v_i y^i).$$

Then $0 \varphi \varphi_k$ is known to be bijective by [1] or [2]. By Corollary 1, Propositions 4, 5 and Lemma 1.1, the minimal number of generators for the module of vector
fields liftable over \( \varphi_k \) can be calculated as follows:

\[
(2k - 1) \cdot \left( \begin{array}{c} 2k - 1 \\ 1 \end{array} \right) - (((2k - 1) - (2k - 2)) \cdot 1\delta(\varphi_k) + 1\gamma(\varphi_k) - \alpha\gamma(\varphi_k))
\]

\[
= (2k - 1)^2 - (((2k - 1) - (2k - 2)) \cdot (k - 1) \cdot (k - 1))
\]

\[
= 3k - 2.
\]

It has been verified in [11] that the minimal number of generators for the module of vector fields liftable over \( \varphi_k \) is exactly \( 3k - 2 \) in the complex case and in the case a set of generators has been obtained in [17] (see also [4]).

**Example 3.3.** Let \( \psi_n : (K^n, 0) \to (K^{2n-1}, 0) \) be given by

\[
\psi_n(v_1, \ldots, v_{n-1}, y) = (v_1, \ldots, v_{n-1}, y^2, v_1 y, \ldots, v_{n-1} y).
\]

Then \( \varphi \circ \psi_n \) is known to be bijective by [20] or [21] or [12]. By Corollary 1, Propositions 4, 5 and Lemma 1.1, the minimal number of generators for the module of vector fields liftable over \( \psi_n \) can be calculated as follows:

\[
(2n - 1) \cdot \left( \begin{array}{c} 2n - 1 \\ 1 \end{array} \right) - (((2n - 1) - n) \cdot 1\delta(\varphi) + 1\gamma(\varphi) - \alpha\gamma(\varphi))
\]

\[
= (2n - 1)^2 - ((n - 1) \cdot n \cdot 2 + n \cdot (2 - 1) - (2 - 1))
\]

\[
= 2n^2 - 3n + 2.
\]

In the case that \( n = 2 \), \( \psi_2 \) equals \( \varphi_2 \) of Example 3.2. Thus, in this case, It has been verified in [4] and [6] that the minimal number of generators for the module of vector fields liftable over \( \psi_2 \) is exactly 4 in the complex case and a set of generators has been obtained in [4] and [7].

**Example 3.4.** Examples 3.1, 3.2 and 3.3 can be generalized as follows. Let \( f : (K, 0) \to (K^p, 0) \) be a smooth map-germ such that \( 2 \leq \delta(f) < \infty \) and let \( F : (K \times K, 0) \to (K^p \times K, 0) \) be a \( K \)-miniversal unfolding of \( f \), where \( K \)-\textit{miniversal unfolding of} \( f \) is a map-germ given by (5.8) of [11] with \( c = r \). Then, by [11] or [12] \( \varphi \circ F \) is bijective. Note that \( c = p\delta(f) - 1 - p \) by theorem 4.5.1 of [19]. By Corollary 1, Propositions 4, 5 and Lemma 1.1, the minimal number of generators for the module of vector fields liftable over \( F \) can be calculated as follows:

\[
(p + c) \cdot \left( \begin{array}{c} p + c \\ 1 \end{array} \right) - (((p + c) - (1 + c)) \cdot 1\delta(F) + 1\gamma(F) - \alpha\gamma(F))
\]

\[
= (p + c)^2 - ((p - 1) \cdot (1 + c) \cdot \delta(f) + c \cdot \delta(f) - 1))
\]

\[
= p^2 \cdot \delta(f) - p \cdot \delta(f) + \delta(f) - p.
\]

A smooth multigerm \( g : (K^n, S) \to (K^p, 0) \) is said to be \textit{stable} if \( \theta_S(g) = T_A(g) \) is satisfied. By Mather's classification theorem (theorem A of [11]), proposition (1.6) of [11], Mather's normal form theorem for a stable map-germ (theorem (5.10) of [11]), the fact that the sharp inequality \( p^2 \delta(f) - p \delta(f) + \delta(f) - p > p + c \) holds (since \( p, \delta(f) \geq 2 \)) and the fact that the module of liftable vector fields over an immersive stable multigerm is a free module if and only if \( p = n + 1 \), we have the following:

**Proposition 6.** Let \( f : (K^n, S) \to (K^p, 0) \) \( (n < p) \) be a stable multigerm of corank at most one. Then, the module of liftable vector fields over \( f \) is a free module if and only if the properties \( p = n + 1 \) and \( \delta(f) = |S| \) are satisfied.
Example 3.5. Let $f : (\mathbb{K}, S) \to (\mathbb{K}^2, 0)$ be any one of the following three.

1. $x \mapsto (x^4, x^5 + x^7)$ (taken from [3]).
2. $x \mapsto (x^2, x^3), x \mapsto (x^3, x^2)$ (taken from [8]).
3. $x \mapsto (x, 0), x \mapsto (0, x), x \mapsto (x^2, x^3 + x^4)$ (taken from [8]).

It has been shown in [3] or [8] that $TK(f) = TA(f)$ is satisfied. Thus, $1\omega f$ is surjective. We can confirm easily that the following equality holds.

$$2 \cdot \left( \begin{array}{c} 2 \\ 1 \end{array} \right) = (2 - 1) \cdot 1 \delta(f) + 1 \gamma(f) - 0 \gamma(f).$$

Thus, $1\omega f$ is injective by Proposition 4. By Corollary 1, Propositions 4, 5 and Lemma 1.1, the minimal number of generators for the module of vector fields liftable over $f$ can be calculated as follows:

$$2 \cdot \left( \begin{array}{c} 3 \\ 2 \end{array} \right) - ((2 - 1) \cdot 2 \delta(f) + 2 \gamma(f) - 1 \gamma(f))$$

$$= 2 \cdot 3 - ((2 - 1) \cdot 1 \cdot 4 + (4 - |S|)) - (4 - |S|))$$

$$= 2.$$

In the case $\mathbb{K} = \mathbb{C}$, it has been known that any plane curve is a free divisor by [18]. Thus, by combining [3] and [18], it has been known that the minimal number of generators for the module of vector fields liftable over $f$ is 2 in the complex case.

Example 3.6. Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ be given by $f(x, y) = (x, xy + y^5 \pm y^7)$ (taken from [16]). It has been shown in [16] that $TK(f) = TA(f)$ is satisfied. Thus, $1\omega f$ is surjective. It is easily seen that the following equality holds.

$$2 \cdot \left( \begin{array}{c} 2 \\ 1 \end{array} \right) = (2 - 2) \cdot 1 \delta(f) + 1 \gamma(f) - 0 \gamma(f).$$

Thus, $1\omega f$ is injective by Proposition 4. By Corollary 1, Propositions 4, 5 and Lemma 1.1, the minimal number of generators for the module of vector fields liftable over $f$ can be calculated as follows:

$$2 \cdot \left( \begin{array}{c} 3 \\ 2 \end{array} \right) - ((2 - 2) \cdot 2 \delta(f) + 2 \gamma(f) - 1 \gamma(f))$$

$$= 2 \cdot 3 - ((2 - 2) \cdot 3 \cdot 5 + 3 \cdot (5 - 1) - 2 \cdot (5 - 1))$$

$$= 2.$$

As same as Example 3.5, it has been known that the minimal number of generators for the module of vector fields liftable over $f$ is 2 in the complex case.

Example 3.7. Let $f : (\mathbb{K}^4, 0) \to (\mathbb{K}^5, 0)$ be given by the following:

$$f(x_1, x_2, x_3, y) = (x_1, x_2, x_3, y^4 + x_1y, y^5 + y^7 + x_2y + x_3y^2).$$

This example is taken from [17] where the property $TK(f) = TA(f)$ has been shown. Thus, $1\omega f$ is surjective. It is easily seen that the following equality holds.

$$5 \cdot \left( \begin{array}{c} 5 \\ 1 \end{array} \right) = (5 - 4) \cdot 1 \delta(f) + 1 \gamma(f) - 0 \gamma(f).$$

Thus, $1\omega f$ is injective by Proposition 4. By Corollary 1, Propositions 4, 5 and Lemma 1.1, the minimal number of generators for the module of vector fields liftable over $f$ can be calculated as follows:

$$5 \cdot \left( \begin{array}{c} 3 \\ 2 \end{array} \right) - ((5 - 4) \cdot 2 \delta(f) + 2 \gamma(f) - 1 \gamma(f))$$

$$= 5 \cdot 3 - ((5 - 4) \cdot 3 \cdot 5 + 3 \cdot (5 - 2) - 2 \cdot (5 - 2))$$

$$= 2.$$
over \( f \) can be calculated as follows:
\[
5 \cdot \binom{6}{2} - (5 - 4) \cdot 2\delta(f) + 2\gamma(f) - 1\gamma(f)
\]
\[
= 5 \cdot 15 - ((5 - 4) \cdot 10 + 4 \cdot (4 - 1) - 4 \cdot (4 - 1))
\]
\[
= 17.
\]

4. Proof of Theorem 1

Since \( j f \) is surjective, by Lemma 1.1 we have that \( j f \) is surjective for any \( j > i \). Since \( j f \) is injective, any \( \xi \in \theta_0(p) \) such that \( \omega f(\xi) \in TR_e(f) \) is contained in \( m_0^{i+1} \theta_0(p) \). Put \( \rho(f) = \dim K \ker(i+1 f) \). Then, since \( i f \) is surjective, \( \rho(f) \) must be positive by Corollary 1. Let \( \{\xi_1 + m_0^{i+2} \theta_0(p), \ldots, \xi_{\rho(f)} + m_0^{i+2} \theta_0(p)\} \) be a basis of \( \ker(i+1 f) \). Then, we have that
\[
\xi_j \circ f \in TR_e(f) \cap f^{*}m_0^{i+1} \theta S(f) + f^{*}m_0^{i+2} \theta S(f) \quad (1 \leq j \leq \rho(f)).
\]

Since \( i+2 f \) is surjective, we have the following:
\[
TR_e(f) \cap f^{*}m_0^{i+1} \theta S(f) + f^{*}m_0^{i+2} \theta S(f)
\]
\[
= TR_e(f) \cap f^{*}m_0^{i+1} \theta S(f) + TR_e(f) \cap f^{*}m_0^{i+2} \theta S(f) + \omega f(m_0^{i+2} \theta_0(p)).
\]

Thus, for any \( j (1 \leq j \leq \rho(f)) \) there exists \( \xi_j \in m_0^{i+2} \theta_0(p) \) such that \( (\xi_j + \xi_j) \circ f \in TR_e(f) \cap T_L(f). \) Let \( A \) be the \( C_0 \)-module generated by \( \xi_j + \xi_j \) (\( 1 \leq j \leq \rho(f) \)).

Let \( \hat{\omega} f : \theta_0(p) \rightarrow \frac{\theta_0(f)}{TR_e(f)} \) be given by \( \hat{\omega} f(\xi) = \omega f(\xi) + TR_e(f). \) Then, \( \ker(\hat{\omega} f) \) is the set of vector fields liftable over \( f \). In order to show that \( \ker(\hat{\omega} f) = A \), we consider the following commutative diagram.

\[
\begin{array}{cccccc}
0 & \longrightarrow & 0 & \longrightarrow & \ker(b_3) \\
\downarrow & & \downarrow & & \downarrow \\
m_0 \ker(\hat{\omega} f) & \xrightarrow{a_2} & m_0^{i+2} \theta_0(p) & \xrightarrow{c_2} & f^{*}m_0^{i+2} \theta S(f) & \longrightarrow 0 \\
\downarrow{b_1} & & \downarrow{b_2} & & \downarrow{b_3} \\
0 & \longrightarrow & \ker(\hat{\omega} f) & \xrightarrow{a_1} & m_0^{i+1} \theta_0(p) & \xrightarrow{c_1} & f^{*}m_0^{i+1} \theta S(f) & \longrightarrow 0 \\
\downarrow{d_1} & & \downarrow{d_2} & & \downarrow{d_3} \\
coker(b_1) & \longrightarrow & coker(b_2) & \longrightarrow & coker(b_3)
\end{array}
\]

Here, \( a_j (j = 1, 2) \), \( b_j (j = 1, 2) \) are inclusions, \( b_3 \) is defined by \( b_3(\xi) = [\xi]_{i+j} \) and \( c_j (j = 1, 2) \) are defined by \( c_j(\xi) = [\omega f(\xi)]_{i+j} \), where \( [\xi]_{i+j} = \xi + TR_e(f) \cap f^{*}m_0^{i+1} \theta S(f) \) and \( [\xi]_{i+2} = \xi + f^{*}m_0 \left( TR_e(f) \cap f^{*}m_0^{i+1} \theta S(f) \right) \).

Lemma 4.1.

\[
f^{*}m_0 \left( TR_e(f) \cap f^{*}m_0^{i+1} \theta S(f) \right) = TR_e(f) \cap f^{*}m_0^{i+2} \theta S(f).
\]

Proof of Lemma 4.1. It is clear that \( f^{*}m_0 \left( TR_e(f) \cap f^{*}m_0^{i+1} \theta S(f) \right) \subset TR_e(f) \cap f^{*}m_0^{i+2} \theta S(f) \). Thus, in the following we concentrate on showing its converse. Let \( \xi \) be an element of \( TR_e(f) \cap f^{*}m_0^{i+2} \theta S(f) \). Let \( f_j \) be a branch of \( f \), namely, \( f_j = f_{j(S)} \) (\( 1 \leq j \leq |S| \)). Then, we have that \( \xi \in TR_e(f_j) \cap f_j^{*}m_0^{i+2} \theta_s (f_j) \) for any \( j (1 \leq j \leq |S|) \). Thus, there exists \( \eta_j \in \theta_s (n) \) such that \( \eta_j(\eta_j) = \xi \).
Since $f$ is of corank at most one, for any $j$ ($1 \leq j \leq |S|$) there exist germs of diffeomorphism $h_j : (\mathbb{K}^n, s_j) \to (\mathbb{K}^n, s_j)$ and $H_j : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ such that $H_j \circ f_j \circ h_j^{-1}$ has the following form:

$$H_j \circ f_j \circ h_j^{-1}(x, \ldots, x_{n-1}, y) = (x_1, \ldots, x_{n-1}, y^\delta(f_j) + f_j,n(x_1, \ldots, x_{n-1}, y), f_j,n+1(x_1, \ldots, x_{n-1}, y), \ldots, f_j,p(x_1, \ldots, x_{n-1}, y)).$$

Here, $x_1, \ldots, x_{n-1}, y$ are local coordinates of the coordinate system $(U_j, h_j)$ at $s_j$ and $f_j,q$ satisfies $f_j,q(0, \ldots, 0, y) = o(y^\delta(f_j))$ for any $q$ ($n \leq q \leq p$). Put

$$\eta_j = \sum_{m=1}^{n-1} \eta_{j,m} \frac{\partial}{\partial x_m} + \eta_{j,n} \frac{\partial}{\partial y}$$

and $\xi = \sum_{q=1}^p \xi_q \frac{\partial}{\partial X_q}$. Then, by the above form of $H_j \circ f_j \circ h_j^{-1}$ and the equality $tf_j(\eta_j) = \xi$, the following hold:

$$\eta_{j,m}(x_1, \ldots, x_{n-1}, y) = \xi_{m}(x_1, \ldots, x_{n-1}, y) \quad (1 \leq m \leq n-1) \quad (4.1)$$

$$\lambda(x_1, \ldots, x_{n-1}, y)\eta_{j,n}(x_1, \ldots, x_{n-1}, y) = \mu(x_1, \ldots, x_{n-1}, y), \quad (4.2)$$

where $\lambda = \delta(f_j)y^\delta(f_j)+1 + \frac{\partial f_{j,n}}{\partial x_n}$ and $\mu = \xi_n - \sum_{m=1}^{n-1} \eta_{j,m} \frac{\partial f_{j,n}}{\partial x_m}$. Since $\xi_q \in f^*m_0^{q+2}C_{s_j}$ for any $q$ ($1 \leq q \leq p$), by (4.1) we have that $\eta_{j,m} \in f^*m_0^{q+2}C_{s_j}$ for any $m$ ($1 \leq m \leq n-1$).

Since $f_{j,q}(0, \ldots, 0, y) = o(y^\delta(f_j))$ for any $q$ ($n \leq q \leq p$), we have the following properties:

1. $Q(f_j) = Q(x_1, \ldots, x_{n-1}, y^\delta(f_j)) = Q(x_1, \ldots, x_n, y\lambda)$.
2. $[1], [y], \ldots, [y^\delta(f_j)+2], [\lambda]$ constitute a basis of $Q(f_j)$.

Thus, by the preparation theorem, $C_{s_j}$ is generated by $1, y, \ldots, y^\delta(f_j)-2, \lambda$ as $C_{0}$-module via $f_j$. Therefore, for any positive integer $r$, $f_j^*m_0^rC_{s_j}$ is generated by elements of the union of the following three sets $U_r, V_r, W_r$ as $C_{0}$-module via $f_j$.

$$U_r = \left\{ x_1^{k_1} \ldots x_{n-1}^{k_{n-1}} \lambda^{k_n}y^\ell \bigg| k_m \geq 0, \sum_{m=1}^{n-1} k_m = r - k_n < r, 0 \leq \ell \leq \delta(f_j) - 2 \right\},$$

$$V_r = \left\{ x_1^{k_1} \ldots x_{n-1}^{k_{n-1}} y^{k_n} \bigg| k_m \geq 0, \sum_{m=1}^{n-1} k_m = r - k_n, \right\},$$

$$W_r = \left\{ x_1^{k_1} \ldots x_{n-1}^{k_{n-1}} \ell \bigg| k_m \geq 0, \sum_{m=1}^{n-2} k_m = r, 0 \leq \ell \leq \delta(f_j) - 2 \right\}.$$

Then, by using these notations, for any $m$ ($1 \leq m \leq n-1$) $\eta_{j,m}$ can be expressed as follows:

$$\eta_{j,m} = \sum_{u \in U_{r+2}} \phi_{u,j,m} u + \sum_{v \in V_{r+2}} \phi_{v,j,m} v + \sum_{w \in W_{r+2}} \phi_{w,j,m} w,$$

where $\phi_{u,j,m}, \phi_{v,j,m}, \phi_{w,j,m}$ are some elements of $C_{s_j}$.

Next, we investigate $\eta_{j,n}$. Since $\mu$ has the form $\mu = \xi_n - \sum_{m=1}^{n-1} \eta_{j,m} \frac{\partial f_{j,n}}{\partial x_m}$ and $\xi_n, \eta_{j,m}$ are contained in $f^*m_0^{q+2}C_{s_j}$, $\mu$ is contained in $f^*m_0^{q+2}C_{s_j}$. On the other hand, $\lambda$ must divide $\mu$ by (4.2). Thus, $\mu$ is generated by elements of $U_{r+2} \cup V_{r+2}$. Hence, $\eta_{j,n} = \frac{\mu}{\lambda}$ can be expressed as follows:

$$\eta_{j,n} = \sum_{u \in U_{r+2}} \phi_{u,j,n} u \lambda + \sum_{v \in V_{r+2}} \phi_{v,j,n} v \lambda.$$
Lemma 4.1 implies that $\phi$ where $\psi$ Thus and since $\delta$ is a finite set, we have the following:

$$A \ni d$$

such that $\tilde{S} \leq |S|$ we have the following:

$$\sum_{u,j,n} u/j,m$$

$$\sum_{w,j,m} w/j,m$$

where $\tilde{u},j,m, \tilde{v},j,m, \tilde{w},j,m$ are some elements of $C_s$. Furthermore, for any $j (1 \leq j \leq |S|)$ we have the following:

$$t_j \left( \eta_{j,m} \frac{\partial}{\partial x_m} \right)$$

$$t_j \left( \psi_{u,j} \frac{\partial}{\partial y} \right)$$

$$t_j \left( \psi_{v,j,n} \frac{\partial}{\partial y} \right)$$

where $\psi_{u,j}, \psi_{v,j,n}, \psi_{w,j,n}$ are elements of $C_s$. Since the union $U_{i+1} \cup V_{i+1} \cup W_{i+1}$ is a finite set, we have the following:

$$\xi = t_j \left( \eta_{j,m} \frac{\partial}{\partial x_m} + \eta_{j,n} \frac{\partial}{\partial y} \right) \in f_j^* m_0^{i+1} (T \mathcal{R}_c (f_j) \cap f_j^* m_0^i \theta_S (f_j)).$$

Thus and since $i + 1 \geq 1$ and $f_j$ is any branch of $f$, we have that

$$\xi \in f^* m_0 (T \mathcal{R}_c (f) \cap f^* m_0^{i+1} \theta_S (f)).$$

Lemma 4.1 implies that $c_2$ is surjective, thus even the second row sequence is exact. Lemma 4.1 implies also that $b_3$ is injective and thus $\ker (b_3) = 0$. Hence, by the snake lemma, we see that $d_1$ is injective. On the other hand, since there exists an isomorphism

$$\varphi: \frac{f^* m_0^{i+1} \theta_S (f)}{T \mathcal{R}_c (f) \cap f^* m_0^{i+1} \theta_S (f) + f^* m_0^{i+2} \theta_S (f)} \to \coker (b_3)$$

such that $d_2 = \varphi \circ i+1 \varphi f$ we have that $\ker (d_2) = \ker (\varphi \circ i+1 \varphi f) = \ker (i+1 \varphi f)$. Therefore, we have the following:

$$\dim_k \ker (\varphi f) = \dim_k \ker (i+1 \varphi f) = \rho (f) = \dim_k \frac{A}{m_0 A}.$$
5. How to Construct Generators

In principle, the proof of Theorem 1 provides how to construct generators for the module of liftable vector fields over a given finitely determined multigerm \( f \) satisfying the assumption of Theorem 1. In this section, we examine it by some examples.

5.1. Generators for the module of vector fields liftable over \( \psi_n \) of Example 3.3. We let \( (V_1, \ldots, V_{n-1}, W, X_1, \ldots, X_{n-1}) \) be the standard coordinates of \( \mathbb{K}^{2n-1} \). Since \( \omega_\psi \) is bijective we first look for a basis of \( \ker(1 \omega_\psi) \). We can find out easily a basis of \( \ker(1 \omega_\psi) \) which is (for instance) the following:

\[
V_i \frac{\partial}{\partial V_j} + X_i \frac{\partial}{\partial X_j} + m_0^2 \theta_0(2n-1) \quad (1 \leq i, j \leq n-1),
\]
\[
X_i \frac{\partial}{\partial V_j} + m_0^2 \theta_0(2n-1) \quad (1 \leq i, j \leq n-1),
\]
\[
2X_i \frac{\partial}{\partial W} + m_0^2 \theta_0(2n-1) \quad (1 \leq i \leq n-1),
\]
\[
2W \frac{\partial}{\partial W} + \sum_{j=1}^{n-1} X_j \frac{\partial}{\partial X_j} + m_0^2 \theta_0(2n-1).
\]

Since any component function of \( \psi_n \) is a monomial, we can determine easily the desired higher terms of liftable vector fields and thus we see that the following constitute a set of generators for the module of vector fields liftable over \( \psi_n \).

\[
V_i \frac{\partial}{\partial V_j} + X_i \frac{\partial}{\partial X_j}, \quad (1 \leq i, j \leq n-1)
\]
\[
X_i \frac{\partial}{\partial V_j} + V_i W \frac{\partial}{\partial X_j} \quad (1 \leq i, j \leq n-1),
\]
\[
2X_i \frac{\partial}{\partial W} + \sum_{j=1}^{n-1} V_j \frac{\partial}{\partial X_j} \quad (1 \leq i \leq n-1),
\]
\[
2W \frac{\partial}{\partial W} + \sum_{j=1}^{n-1} X_j \frac{\partial}{\partial X_j}.
\]

5.2. Generators for the module of vector fields liftable over \( f \) of Example 3.5.2. Recall that the multigerm \( f \) of Example 2.6.2 is \( f_1(x) = (x^2, x^3), f_2(x) = (x^3, x^2) \). Let \( (X, Y) \) be the standard coordinates of \( \mathbb{K}^2 \). Since \( \omega_\psi \) is bijective we first look for a basis of \( \ker(2 \omega_\psi) \). We can find out easily a basis of \( \ker(2 \omega_\psi) \) which is (for instance) the following:

\[
6XY \frac{\partial}{\partial X} + 4Y^2 \frac{\partial}{\partial Y} + m_0^3 \theta_0(2), \quad 4X^2 \frac{\partial}{\partial X} + 6XY \frac{\partial}{\partial Y} + m_0^3 \theta_0(2).
\]

Put \( \eta_{1,1,1} = 3x^4 \frac{\partial}{\partial x}, \eta_{1,2,1} = 2x^3 \frac{\partial}{\partial x} \) and \( \xi_{1,1} = 6XY \frac{\partial}{\partial X} + 4Y^2 \frac{\partial}{\partial Y} \). Then, we have the following:

\[
\xi_{1,1} \circ f_1 - df_1 \circ \eta_{1,1,1} = -5x^6 \frac{\partial}{\partial X},
\]
\[
\xi_{1,1} \circ f_2 - df_2 \circ \eta_{1,2,1} = 0.
\]
Put $\xi_{1,2} = 5X^3 \frac{\partial}{\partial Y}$. Then we have the following:

$$(\xi_{1,1} + \xi_{1,2}) \circ f_1 - df_1 \circ \eta_{1,1,1} = 0,$$  \hspace{1cm} (5.1)

$$(\xi_{1,1} + \xi_{1,2}) \circ f_2 - df_2 \circ \eta_{1,2,1} = 5x^9 \frac{\partial}{\partial Y}.$$  \hspace{1cm} (5.2)

Put $\xi_{1,3} = -5X^3 Y \frac{\partial}{\partial X}$ and $\eta_{1,1,2} = -\frac{5}{9} x^9 \frac{\partial}{\partial x}$. Then we have the following:

$$(\xi_{1,1} + \xi_{1,2} + \xi_{1,3}) \circ f_1 - df_1 \circ (\eta_{1,1,1} + \eta_{1,1,2}) = \frac{10}{3} x^{10} \frac{\partial}{\partial X},$$

$$(\xi_{1,1} + \xi_{1,2} + \xi_{1,3}) \circ f_2 - df_2 \circ \eta_{1,2,1} = 0.$$  \hspace{1cm} (5.3)

Put $\xi_{1,4} = -\frac{10}{3}X^2 Y^2 \frac{\partial}{\partial X}$ and $\eta_{1,2,2} = -\frac{10}{9} x^8 \frac{\partial}{\partial x}$. Then we have the following:

$$(\xi_{1,1} + \xi_{1,2} + \xi_{1,3} + \xi_{1,4}) \circ f_1 - df_1 \circ (\eta_{1,1,1} + \eta_{1,1,2}) = 0,$$  \hspace{1cm} (5.4)

Note that the right hand side of (5.3) (resp., the right hand side of (5.4)) is the right hand side of (5.1) (resp., the right hand side of (5.2)) multiplied by $\left(\frac{2}{3}\right)^2$. Thus, the following vector field $\xi_{1}$ must be liftable over $f$.

$$\xi_1 = \xi_{1,1} + \xi_{1,2} + \left(1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right) + \cdots\right)(\xi_{1,3} + \xi_{1,4})$$

$$= (6XY - 6X^2 Y^2) \frac{\partial}{\partial X} + (4Y^2 + 5X^3 - 9XY^3) \frac{\partial}{\partial Y}.$$  \hspace{1cm} (5.5)

Next, put $\eta_{2,1,1} = 2x^3 \frac{\partial}{\partial x}$ and $\eta_{2,1,1} = 3x^4 \frac{\partial}{\partial x}$ and $\xi_{2,1} = 4X^2 \frac{\partial}{\partial X} + 6XY \frac{\partial}{\partial Y}$. Then, we have the following:

$$\xi_{2,1} \circ f_1 - df_1 \circ \eta_{2,1,1} = 0,$$

$$\xi_{2,1} \circ f_2 - df_2 \circ \eta_{2,2,1} = 5x^6 \frac{\partial}{\partial X}.$$  \hspace{1cm} (5.6)

Put $\xi_{2,2} = 5Y^3 \frac{\partial}{\partial X}$. Then we have the following:

$$(\xi_{2,1} + \xi_{2,2}) \circ f_1 - df_1 \circ \eta_{2,1,1} = 5x^9 \frac{\partial}{\partial X},$$  \hspace{1cm} (5.5)

$$(\xi_{2,1} + \xi_{2,2}) \circ f_2 - df_2 \circ \eta_{2,2,1} = 0.$$  \hspace{1cm} (5.6)

Put $\xi_{2,3} = -5X^3 Y \frac{\partial}{\partial X}$ and $\eta_{2,2,2} = -\frac{5}{3} x^9 \frac{\partial}{\partial x}$. Then we have the following:

$$(\xi_{2,1} + \xi_{2,2} + \xi_{2,3}) \circ f_1 - df_1 \circ (\eta_{2,1,1}) = 0,$$

$$(\xi_{2,1} + \xi_{2,2} + \xi_{2,3}) \circ f_2 - df_2 \circ (\eta_{2,2,1} + \eta_{2,2,2}) = \frac{10}{3} x^{10} \frac{\partial}{\partial Y}.$$  \hspace{1cm} (5.7)

Put $\xi_{2,4} = -\frac{10}{3}X^2 Y^2 \frac{\partial}{\partial Y}$ and $\eta_{2,1,2} = -\frac{10}{9} x^8 \frac{\partial}{\partial x}$. Then we have the following:

$$(\xi_{2,1} + \xi_{2,2} + \xi_{2,3} + \xi_{2,4}) \circ f_1 - df_1 \circ (\eta_{2,1,1} + \eta_{2,1,2}) = \frac{20}{9} x^9 \frac{\partial}{\partial X},$$

$$(\xi_{2,1} + \xi_{2,2} + \xi_{2,3} + \xi_{2,4}) \circ f_2 - df_2 \circ (\eta_{2,2,1} + \eta_{2,2,2}) = 0.$$  \hspace{1cm} (5.8)
Note that the right hand side of (5.7) (resp., the right hand side of (5.8)) is the right hand side of (5.5) (resp., the right hand side of (5.6)) multiplied by \((\frac{2}{3})^2\). Thus, the following vector field \(\xi_2\) must be liftable over \(f\).

\[
\xi_2 = \xi_{2,1} + \xi_{2,2} + \left(1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 + \cdots\right) \left(\xi_{2,3} + \xi_{2,4}\right)
\]

\[
= (4X^2 + 5Y^3 - 9X^3Y) \frac{\partial}{\partial X} + (6XY - 6X^2Y^2) \frac{\partial}{\partial Y}.
\]

Therefore, the following constitute a set of generators for the module of vector fields liftable over \(f\).

\[
\xi_1 = (6XY - 6X^2Y^2) \frac{\partial}{\partial X} + (4Y^2 + 5X^3 - 9XY^3) \frac{\partial}{\partial Y},
\]

\[
\xi_2 = (4X^2 + 5Y^3 - 9X^3Y) \frac{\partial}{\partial X} + (6XY - 6X^2Y^2) \frac{\partial}{\partial Y}.
\]

REFERENCES

[1] V. I. Arnol’d, ‘Wave front evolution and equivariant Morse lemma’, Commun Pure Appl. Math. 29 (1976) 557–582.
[2] V. I. Arnol’d, S. M. Gusein-Zade and A. N. Varchenko, Singularities of Differentiable Maps I (Monographs in Mathematics 82) (Birkhäuser, Boston, 1985).
[3] J. W. Bruce and T. J. Gaffney, ‘Simple singularities of mappings \(C, 0 \to C^2, 0\)’, J. London Math. Soc. 26 (1982) 465–474.
[4] J. W. Bruce and J. M. West, ‘Functions on a crosscap’, Math. Proc. Cambridge Philos. Soc. 125 (1998) 19–39.
[5] J. Damon, ‘A-equivalence and the equivalence of sections of images and discriminants’, Singularity theory and its applications, Part I (Coventry, 1988/1989) 93–121 Lecture Notes in Math. 1462 Springer, Berlin, 1991.
[6] M. P. Holland and D. Mond, ‘Stable mappings and logarithmic relative symplectic forms’, Math. Z. 231 (1999) 605–623.
[7] K. Houston and D. Littlestone, ‘Vector fields liftable over corank 1 stable maps’, arXiv.org math (2009) no. 0905.0556.
[8] P. A. Kolgushkin and R. R. Sadykov, ‘Simple singularities of multigerms of curves’, Rev. Mat. Complut. 14 (2001) 311–344.
[9] E. J. N. Looijenga, Isolated Singular Points on Complete Intersections, London Mathematical Society Lecture Note Series, 77, Cambridge University Press, Cambridge, 1984.
[10] J. Mather, ‘Stability of \(C^\infty\) mappings, III. Finitely determined map-germs’, Publ. Math. Inst. Hautes Études Sci. 35 (1969) 127–156.
[11] J. Mather, ‘Stability of \(C^\infty\) mappings, IV. Classification of stable map-germs by \(\mathbb{R}\)-algebras’, Publ. Math. Inst. Hautes Études Sci. 37 (1970) 223–248.
[12] J. Mather, ‘Stability of \(C^\infty\)-mappings V. Transversality’, Adv. in Math. 4 (1970) 301–336.
[13] J. Mather, ‘Stability of \(C^\infty\)-mappings VI. The nice dimensions’, Lecture Notes in Math. 192 (1971) 207–253.
[14] B. Morin, ‘Formes canoniques des singularites d’une application differentiable’, Comptes Rendus 260(1965) 5662–5665 and 6503–6506.
[15] T. Nishimura, ‘\(A\)-simple multigerms and \(L\)-simple multigerms’, Yokohama Math. J. 55 (2010) 93–104.
[16] J. H. Rieger, ‘Families of maps from the plane to the plane’, J. London Math. Soc. 36 (1987) 351–369.
[17] J. H. Rieger, M. A. S. Ruas and R. Wik Atique, ‘\(M\)-deformations of \(A\)-simple germs from \(\mathbb{R}^n\) to \(\mathbb{R}^{n+1}\)’, Math. Proc. Camb. Phil. Soc. 144 (2008) 181–196.
[18] K. Saito, ‘Theory of logarithmic differential forms and logarithmic vector fields’, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980) 265–291.
[19] C. T. C. Wall, ‘Finite determinacy of smooth map-germs’, Bull. London Math. Soc. 13 (1981) 481–539.
[20] H. Whitney, ‘The general type of singularities of a set of $2n - 1$ smooth functions of $n$ variables’, Duke Math. J. 10 (1943) 161–172.
[21] H. Whitney, ‘The singularities of a smooth $n$-manifold in $(2n - 1)$-space’, Ann. of Math. 45 (1944) 247–293.
[22] H. Whitney, ‘On singularities of mappings of Euclidean spaces. Mappings of the plane to the plane’, Ann. of Math. 62 (1955) 374–410.

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