DYNAMICAL NETWORKS, ISOSPECTRAL GRAPH REDUCTIONS, AND IMPROVED ESTIMATES OF MATRICES’ SPECTRA

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Abstract. Dynamical networks are characterized by large complex graphs of interactions. We suggest a procedure of simplifying the structure of such graphs while preserving the spectrum of their weighted adjacency matrix. As the process of isospectral graph reductions maintains the spectrum of the matrix up to some known set it is possible to estimate the spectrum of the original matrix by considering Gershgorin-type estimates associated with the reduced matrix. The main result of this paper is that eigenvalue estimates improve for all known methods as the matrix size is reduced. Moreover, our procedure of isospectral graph reductions is very flexible and in particular can be used to obtain better eigenvalue estimates of a matrix with complex valued entries to whatever degree is desired.

1. Introduction

A simple and geometrically intuitive method for estimating the eigenvalues of a matrix with complex valued entries was introduced by Gershgorin in [10]. This method gives a result equivalent to a nonsingularity result for diagonally dominant matrices (see theorem 1.4 [17]) which can be traced back to earlier work done by Lévy, Desplanques, Minkowski, and Hadamard [13, 6, 4, 11]. Gershgorin’s estimate was later improved on by Brauer, Brualdi, and Varga [3, 4, 17]. Their results were similar in spirit to Gershgorin’s in that each assigned to every matrix $A \in \mathbb{C}^{n \times n}$ a region of the complex plane containing the spectrum of this matrix.

For Gershgorin and Brauer we will denote these regions by $\Gamma(A)$ and $\mathcal{K}(A)$ respectively. For the extension of Brualdi’s result given by Varga we denote the corresponding set by $\mathcal{B}(A)$. We note that the Gershgorin, Brauer, and Brualdi regions have the property that $\mathcal{B}(A) \subseteq \mathcal{K}(A) \subseteq \Gamma(A)$ for any complex valued matrix $A$ (see [17] for details).

In this paper, we first extend these classical results to a larger class of square matrices with entries in the set $\mathbb{W}$ consisting of complex rational functions. We then use this extension to improve these original estimates. The motivation for considering the class of matrices with entries in $\mathbb{W}$ arises from the following.

Dynamical networks are typically described by large and often complex graphs of interactions. In the studies of such systems it has been found that an important characteristic of a network’s structure is the spectrum of the network’s adjacency matrix [2, 16, 1, 15]. With this in mind we note that to each matrix $M$ with entries in $\mathbb{W}$ that there is an associated graph $G \in \mathcal{G}$. That is, the matrix $M = M(G)$.
is the adjacency matrix of $G$. We denote by $\mathbb{G}$ the class of graphs with adjacency matrices in $\mathbb{W}^{n \times n}$.

Using the theory developed in [6] it is possible to reduce the graph $G \in \mathbb{G}$ to another smaller graph $R \in \mathbb{G}$. The main result of [6] is that the spectra of $G$ and $R$ differ at most by some known finite set. We refer to this reduction process as an **isospectral graph reduction** or simply a graph reduction of $G$.

In the present paper we show that by using graph reductions one can improve Gershgorin, Brauer, and Brualdi-type estimates of the spectra of matrices in $\mathbb{W}^{n \times n}$. Specifically, for $M(G) \in \mathbb{W}^{n \times n}$ the regions in the complex plane for both Gershgorin and Brauer estimates of the eigenvalues of $M(G)$ shrink as the graph $G$ is reduced (see theorems 5.1 and 5.3 for exact statements). Moreover, for Brualdi-type estimates we give a sufficient condition under which such estimates also improve as the underlying graph is reduced (see theorem 5.4). Importantly, we note that as square matrices with complex entries belong to $\mathbb{W}^{n \times n}$ then this method allows for improved estimates of these matrices spectra as a special case.

We also note that, for a given graph, many graph reductions are typically possible. Hence this process is quite flexible and can be used to systematically improve eigenvalue estimates. Moreover, graph reductions on a typical $G \in \mathbb{G}$ can be used to estimate the spectrum of $M(G)$ with increasing accuracy depending on the extent to which $G$ is reduced. In particular, if $G$ is reduced as much as possible the corresponding Gershgorin region is a finite set of points in the complex plane that differs from the actual spectrum of $M(G)$ by some other finite but known set of points.

Moreover, the computational procedure for improving these Gershgorin, Brauer, and Brualdi estimates via graph reduction does not require much effort especially if some special structural features of the graph are known (see section 6). In particular, we note that an isospectral reduction of a graph has less sets from with its Gershgorin and Brauer regions are built than does the unreduced graph.

This paper is organized as follows. Section 2 defines the sets $\mathbb{W}$ and $\mathbb{G}$ as well as the spectrum for the matrices in $\mathbb{W}^{n \times n}$ or equivalently the graphs $G \in \mathbb{G}$. Section 3 extends the results of Gershgorin, Brauer, Brualdi, and Varga to the class of matrices with entries in $\mathbb{W}$. Section 4 summarizes the theory of isospectral graph reductions developed in [6] and uses this to improve the eigenvalue estimates of section 3. Section 5 contains the main results of this paper demonstrating that our procedure of isospectral graph reduction gives better estimates of the spectra of matrices than these other methods. Section 6 gives some natural applications of the theorems of section 5. These include estimating the spectrum of a Laplacian matrix of graph, estimating the spectral radius of a matrix, and determining which specific reductions to use for a given matrix.

2. Preliminaries

In this paper we consider two equivalent mathematical objects. The first is the set of graphs (i.e. structures or “topologies” of networks) consisting of all finite weighted digraphs with or without loops having no parallel edges and edge weights in the set $\mathbb{W}$ of complex rational functions described below. We denote this class of graphs by $\mathbb{G}$ where $\mathbb{G}^n$ is the set of graphs in $\mathbb{G}$ having $n \in \mathbb{N}$ vertices. The second set of objects we consider are the weighted adjacency matrices associated with the graphs in $\mathbb{G}$.
By way of notation we let the weighted digraph $G \in \mathbb{G}$ be the triple $(V, E, \omega)$ where $V$ and $E$ are the finite sets denoting the vertices and edges of $G$ respectively, the edges corresponding to ordered pairs $(v, w)$ for $v, w \in V$. Furthermore, $\omega : E \to \mathbb{W}$ where $\omega(e)$ is the weight of the edge $e$ for $e \in E$. We will use the convention that $\omega(e) = 0$ if and only if $e \notin E$.

For convenience, any graph that is denoted by some triple, e.g. $G = (V, E, \omega)$, will be assumed to be in $\mathbb{G}$. Moreover, if the vertex set of the graph $G = (V, E, \omega)$ is labeled $V = \{v_1, \ldots, v_n\}$ then we denote the edge $(v_i, v_j)$ by $e_{ij}$. For the remainder of this paper if $G = (V, E, \omega)$ is a graph in $\mathbb{G}^n$ then we will assume that its vertex set has some labeling $V = \{v_1, \ldots, v_n\}$.

In order to describe the set of weights $\mathbb{W}$ let $\mathbb{C}[\lambda]$ denote the set of polynomials in the single complex variable $\lambda$ with complex coefficients. We define the set $\mathbb{W}$ to be the set of rational functions of the form $p/q$ where $p, q \in \mathbb{C}[\lambda]$ such that $p$ and $q$ have no common factors and $q$ is nonzero.

The set $\mathbb{W}$ is then a field under addition and multiplication with the convention that common factors are removed when two elements are combined. That is, if $p/q, r/s \in \mathbb{W}$ then $p/q + r/s = (ps + rq)/(qs)$ where the common factors of $(ps + rq)$ and $(qs)$ are removed. Similarly, in the product $pr/qs$ of $p/q$ and $r/s$ the common factors of $(pr)$ and $(qs)$ are removed. We let $\mathbb{C}[\lambda]^{n \times n}$ and $\mathbb{W}^{n \times n}$ denote the set of $n \times n$ matrices with entries in $\mathbb{C}[\lambda]$ and $\mathbb{W}$ respectively.

In order to stress the generality of considering the set $\mathbb{G}$ we note that graphs, which are either undirected, unweighted or have parallel edges, can be considered to be graphs in $\mathbb{G}$. This is done by making an undirected graph $G$ into a directed graph by orienting each of its edges in both directions. Similarly, if $G$ is unweighted then it can be made weighted by giving each edge unit weight. Also multiple edges between two vertices of $G$ may be considered a single edge by adding the weights of the multiple edges and setting this to be the weight of this single equivalent edge.

To introduce the spectrum associated to a graph $G \in \mathbb{G}$ we will use the following notation. If $G = (V, E, \omega)$ then the matrix $M(G) = M(G, \lambda)$ defined entrywise by

$$M(G)_{ij} = \omega(e_{ij})$$

is the weighted adjacency matrix of $G$.

We let the spectrum of a matrix $A = A(\lambda) \in \mathbb{W}^{n \times n}$ be the solutions including multiplicities of the equation

$$(1) \quad \det(A(\lambda) - \lambda I) = 0$$

and for the graph $G$ we let $\sigma(G)$ denote the spectrum of $M(G)$. The spectrum of a matrix with entries in $\mathbb{W}$ is therefore a generalization of the spectrum of a matrix with complex entries.

Moreover, the spectrum is a list of numbers. That is,

$$\sigma(G) = \{ (\sigma_i, n_i) : 1 \leq i \leq p, \sigma_i \in \mathbb{C}, n_i \in \mathbb{N} \}$$

where $n_i$ is the multiplicity of the solutions $\sigma_i$ to equation (1), $p$ the number of distinct solutions, and $(\sigma_i, n_i)$ the elements in the list. In what follows we may write a list as a set with multiplicities if this is more convenient.

As we are mainly concerned with the properties of the adjacency matrix of graphs in $\mathbb{G}$ we note, as we have previously suggested, that there is a one-to-one correspondence between the graphs in $\mathbb{G}^n$ and the matrices $\mathbb{W}^{n \times n}$. Therefore, we
may talk of a graph \( G \in \mathbb{G}^n \) associated with a square matrix \( M = M(G) \) in \( \mathbb{W}^{n \times n} \) and vice-versa without ambiguity.

3. Spectra Estimation of Graphs in \( \mathbb{G} \).

In this section we extend the classical results of Gershgorin, Brauer, and Brualdi (see for instance [17]) on the spectra of matrices with complex entries to matrices in \( \mathbb{W}^{n \times n} \) as well as some related results. To do so we will first define the notion of a polynomial extension of a graph \( G \in \mathbb{G} \).

**Definition 3.1.** If \( G \in \mathbb{G}^n \) and \( M(G)_{ij} = p_{ij}/q_{ij} \) where \( p_{ij}, q_{ij} \in \mathbb{C}[\lambda], q_{ij} \neq 0 \) let \( L_i(G, \lambda) = \prod_{j=1}^{n} q_{ij} \) for \( 1 \leq i \leq n \). We call the graph \( \tilde{G} \) with adjacency matrix

\[
M(\tilde{G}, \lambda)_{ij} = \begin{cases} L_i(G, \lambda)M(G, \lambda)_{ij} & i \neq j \\ L_i(G, \lambda)M(G, \lambda)_{ij} - L_i(G, \lambda)\lambda & i = j \end{cases}, 
1 \leq i, j \leq n
\]

the polynomial extension of \( G \).

To justify this name note that each \( M(\tilde{G}, \lambda)_{ij} \) is an element of \( \mathbb{C}[\lambda] \) or \( M(\tilde{G}, \lambda) \) has complex polynomial entries. Moreover, we have the following result.

**Lemma 3.2.** If \( G \in \mathbb{G} \) then \( \sigma(G) \subseteq \sigma(\tilde{G}) \).

**Proof.** For \( G \in \mathbb{G}^n \) note that the matrix \( M(\tilde{G}, \lambda) - \lambda I \) is given by

\[
(M(\tilde{G}, \lambda) - \lambda I)_{ij} = \begin{cases} L_i(G, \lambda)M(G, \lambda)_{ij} & i \neq j \\ L_i(G, \lambda)M(G, \lambda)_{ij} - L_i(G, \lambda)\lambda & i = j \end{cases}, 
1 \leq i, j \leq n.
\]

The matrix \( M(\tilde{G}, \lambda) - \lambda I \) is then the matrix \( M(G, \lambda) - \lambda I \) whose \( i \)th row has been multiplied by \( L_i(\lambda) \). It follows that

\[
\det(M(\tilde{G}, \lambda) - \lambda I) = \left( \prod_{i=1}^{n} L_i(G, \lambda) \right) \det(M(G, \lambda) - \lambda I)
\]

implying \( \sigma(G) \subseteq \sigma(\tilde{G}) \). \( \square \)

3.1. Gershgorin-Type Regions. As previously mentioned, a well known result of Gershgorin’s originating from [10] gives a simple method whereby the eigenvalues of a matrix with complex valued entries can be estimated. This result is the following theorem which we formulate after introducing some standard notation.

If \( A \in \mathbb{C}^{n \times n} \) let

\[
r_i(A) = \sum_{j=1, j \neq i}^{n} |A_{ij}|, \quad 1 \leq i \leq n
\]

be the \( i \)th row sum of \( A \).

**Theorem 3.3. (Gershgorin)** Let \( A \in \mathbb{C}^{n \times n} \). Then all eigenvalues of \( A \) are contained in the set

\[
\Gamma(A) = \bigcup_{i=1}^{n} \{ z \in \mathbb{C} : |z - A_{ii}| \leq r_i(A) \}.
\]
Geometrically, Gershgorin’s theorem states that all eigenvalues of a matrix $A \in \mathbb{C}^{n \times n}$ are contained in the (Gershgorin) discs of the complex plane centered at $A_{ii}$ of radius $r_i(A)$ for $1 \leq i \leq n$.

In order to extend theorem 3.3 to the class of matrices $W^{n \times n}$ we use the following adaptation of the notation given in (2). For $G \in \mathbb{G}^n$ let

$$r_i(G, \lambda) = \sum_{j=1, j \neq i}^n |M(G, \lambda)_{ij}| \text{ for } 1 \leq i \leq n$$

be the $i$th row sum of $M(G)$.

**Theorem 3.4.** Let $G \in \mathbb{G}^n$. Then $\sigma(G)$ is contained in the set

$$BW_G(G) = \bigcup_{i=1}^n \{\lambda \in \mathbb{C} : |\lambda - M(G, \lambda)_{ii}| \leq r_i(G, \lambda)\}.$$  

**Proof.** Since $M(\bar{G}, \lambda) \in \mathbb{C}[\lambda]^{n \times n}$ then for $\alpha \in \sigma(G)$ it follows that $M(\bar{G}, \alpha)$ is a complex valued matrix. As Lemma 3.2 implies that $\alpha$ is an eigenvalue of the matrix $M(G, \alpha)$ then by an application of Gershgorin’s theorem the inequality $|\alpha - M(\bar{G}, \alpha)_{ii}| \leq r_i(\bar{G}, \alpha)$ holds for some $1 \leq i \leq n$. Hence, $\alpha \in BW_G(G)$. \(\square\)

Because it will be useful later in comparing different regions in the complex plane for $G \in \mathbb{G}^n$ we denote

$$BW_G(G)_i = \{\lambda \in \mathbb{C} : |\lambda - M(\bar{G}, \lambda)_{ii}| \leq r_i(\bar{G}, \lambda)\} \text{ where } 1 \leq i \leq n$$

and call this the $i$th Gershgorin-type region of $G$. Similarly, we call the union of these $n$ sets, given in theorem 3.4 as $BW_G(G)$, to be the Gershgorin-type region of the graph $G$.

As an illustration of theorem 3.4 consider the following example. Let $G \in \mathbb{G}$ be the graph with adjacency matrix

$$M(G) = \begin{bmatrix} \lambda + 1 & 1 & \lambda + 1 \\ 2\lambda & 1 \lambda & \lambda \\ 0 & 1 & 0 \end{bmatrix}.$$  

As $\text{det}(M(G, \lambda) - \lambda I) = (-\lambda^5 + 2\lambda^3 + 2\lambda^2 + 3\lambda + 2)/(\lambda^2)$ then one can compute $\sigma(G) = \{-1, -1, i, -i, 2\}$. The corresponding Gershgorin-type region $BW_G(G)$ is shown in figure 1.

We note here that $BW_G(G)$ is the union of the three regions $BW_G(G)_1, BW_G(G)_2$, and $BW_G(G)_1$ whose boundaries are shown in black (on line in color). Additionally, the interior shading of these regions reflect their intersections and the eigenvalues of $M(G)$ are indicated as points. In the figures that follow we will use the same technique to display similar regions.

**Remark 1.** From the point of view of estimating the spectrum of a graph $G \in \mathbb{G}$ only the union $BW_G(G)$ of the regions $BW_G(G)_i$ matter. This is also true of the other methods presented in this paper (e.g. Brauer and Bruaudi-type regions).

### 3.2. Brauer Cassini-Type Regions

Following Gershgorin, Brauer was able to give the following eigenvalue inclusion result for matrices with complex valued entries (see [17]).
Proof. As in the proof of theorem 3.4, if

Also, \( \sigma(BW_\Gamma) \) consists of one or two distinct components. Moreover, there are \( \binom{\sigma(G)}{2} \) such regions for any \( n \times n \) matrix with complex entries. As with Gershgorin’s theorem we prove an extension to Brauer’s theorem for matrices in \( \mathbb{W}^{n \times n} \).

Theorem 3.5. (Brauer) Let \( A \in \mathbb{C}^{n \times n} \) where \( n \geq 2 \). Then all eigenvalues of \( A \) are located in the set

\[
K(A) = \bigcup_{1 \leq i,j \leq n} \{ z \in \mathbb{C} : |z - A_{ii}| + |z - A_{jj}| \leq r_i(A)r_j(A) \}.
\]

The individual regions given by \( \{ z \in \mathbb{C} : |z - A_{ii}| + |z - A_{jj}| \leq r_i(A)r_j(A) \} \) in equation (4) are known as Cassini ovals and may consist of one or two distinct components. Moreover, there are \( \binom{\sigma(G)}{2} \) such regions for any \( n \times n \) matrix with complex entries. As with Gershgorin’s theorem we prove an extension to Brauer’s theorem for matrices in \( \mathbb{W}^{n \times n} \).

Theorem 3.6. Let \( G \in \mathbb{G}^n \) where \( n \geq 2 \). Then \( \sigma(G) \) is contained in the set

\[
BW_K(G) = \bigcup_{1 \leq i,j \leq n, i \neq j} \{ z \in \mathbb{C} : |\lambda - M(\bar{G}, \lambda)_{ii}| + |\lambda - M(\bar{G}, \lambda)_{jj}| \leq r_i(\bar{G}, \lambda)r_j(\bar{G}, \lambda) \}.
\]

Also, \( BW_K(G) \subseteq BW_\Gamma(G) \).

Proof. As in the proof of theorem 3.4 if \( \alpha \in \sigma(G) \) then \( \alpha \in \sigma(\bar{G}) \) and the matrix \( M(\bar{G}, \alpha) \in \mathbb{C}^{n \times n} \). Brauer’s theorem therefore implies that

\[
|\alpha - M(\bar{G}, \alpha)_{ii}| + |\alpha - M(\bar{G}, \alpha)_{jj}| \leq r_i(\bar{G}, \alpha)r_j(\bar{G}, \alpha)
\]

for some pair of distinct integers \( i \) and \( j \). It then follows that, \( \alpha \in BW_K(G) \) or \( \sigma(G) \subseteq BW_K(G) \).

To prove the assertion that \( BW_K(G) \subseteq BW_\Gamma(G) \) let

\[
BW_K(G)_{ij} = \{ \lambda \in \mathbb{C} : |\lambda - M(\bar{G}, \lambda)_{ii}| + |\lambda - M(\bar{G}, \lambda)_{jj}| \leq r_i(\bar{G}, \lambda)r_j(\bar{G}, \lambda) \}
\]

for distinct \( i \) and \( j \). The claim then is that \( BW_K(G) \subseteq BW_\Gamma(G) \).

To see this assume \( \lambda \in BW_K(G)_{ij} \) or

\[
|\lambda - M(\bar{G}, \lambda)_{ii}| + |\lambda - M(\bar{G}, \lambda)_{jj}| \leq r_i(\bar{G}, \lambda)r_j(\bar{G}, \lambda)
\]

If \( r_i(\bar{G}, \lambda)r_j(\bar{G}, \lambda) = 0 \) then either \( \lambda - M(\bar{G}, \lambda)_{ii} = 0 \) or \( \lambda - M(\bar{G}, \lambda)_{jj} = 0 \). As \( \lambda = M(\bar{G}, \lambda)_{ii} \) implies \( \lambda \in BW_\Gamma(G) \) and \( \lambda = M(\bar{G}, \lambda)_{jj} \) implies \( \lambda \in BW_\Gamma(G) \) then \( \lambda \in BW_\Gamma(G) \).
If \( r_i(\bar{G}, \lambda) r_j(\bar{G}, \lambda) > 0 \) then it follows that
\[
\left| \frac{|\lambda - M(\bar{G}, \lambda)|_{ii}}{r_i(\bar{G}, \lambda)} \right| \left( \frac{|\lambda - M(\bar{G}, \lambda)|_{jj}}{r_j(\bar{G}, \lambda)} \right) \leq 1.
\]
Since at least one of the two quotients on the left must be less than or equal to 1 then \( \lambda \in BW_{K}(G) \cup BW_{\Gamma}(G) \) which verifies the claim and the result follows.

We call the region \( BW_{K}(G) \) given in theorem 3.6 the \textit{Brauer-type region} of the graph \( G \) and the region \( BW_{K}(G)_{ij} \) given in (5) the \textit{ij}th Brauer-type region of \( G \).

Using theorem 3.6 on the graph \( G \) given in figure 1 we have the Brauer-type region shown in the left hand side of figure 2. On the right is a comparison between \( BW_{K}(G) \) and \( BW_{\Gamma}(G) \) where the inclusion \( BW_{K}(G) \subseteq BW_{\Gamma}(G) \) is demonstrated.

3.3. 

**Brualdi-Type Regions.** In this section we extend a result of Varga [17] which is itself an extension of a result of Brualdi [4] relating the spectrum of a graph with complex weights to its cycle structure. In order to state this result we need the following.

A \textit{path} \( P \) in the graph \( G = (V, E, \omega) \) is a sequence of distinct vertices \( v_1, \ldots, v_m \in V \) such that \( e_{i,i+1} \in E \) for \( 1 \leq i \leq m - 1 \). In the case that the vertices \( v_1, \ldots, v_m \) are distinct, except that \( v_1 = v_m \), \( P \) is a \textit{cycle}. If \( \gamma \) is a cycle of \( G \) we denote it by its ordered set of vertices. That is, if \( e_{i,i+1} \in E \) for \( 1 \leq i \leq m - 1 \) and \( e_{m1} \in E \) then we write this cycle as the ordered set of vertices \( \{v_1, \ldots, v_m\} \) up to cyclic permutation. Moreover, a cycle consisting of a single vertex is a \textit{loop}.

A \textit{strong cycle} of \( G \) is a cycle \( \{v_1, \ldots, v_m\} \) such that \( m \geq 2 \). Furthermore, if \( v_i \in V \) has no strong cycle passing through it then we define its associated \textit{weak cycle} as \( \{v_i\} \) irregardless of whether \( e_{ii} \in E \). For \( G \in \mathbb{G} \) we let \( C_s(G) \) and \( C_w(G) \) denote the set of strong and weak cycles of \( G \) respectively and let \( C(G) = C_s(G) \cup C_w(G) \).

A directed graph is \textit{strongly connected} if there is a path (possibly of length zero) from each vertex of the graph to every other vertex. The \textit{strongly connected components} of \( G = (V, E) \) are its maximal strongly connected subgraphs. Moreover, its vertex set \( V = \{v_1, \ldots, v_n\} \) can always be labeled in such a way that \( M(G) \) has the following triangular block structure.

\[
\begin{pmatrix}
A & B \\
B & C
\end{pmatrix}
\]
where $S_i(G)$ is a strongly connected component of $G$ and $*$ are block matrices with possibly nonzero entries (see [12] or [17] for more details).

As the strongly connected components of a graph are unique then for $G \in \mathbb{G}^n$ we define

$$\tilde{r}_i(G, \lambda) = \sum_{j \in N_i, j \neq i} |M(S_i(G), \lambda)_{ij}|$$

for $1 \leq i \leq n$, where $i \in N_t$ and $N_t$ is the set of indices indexing the vertices in $S_t(G)$. That is, $\tilde{r}_i(G, \lambda)$ is $r_i(G, \lambda)$ restricted to the strongly connected component containing $v_i$.

If $A \in \mathbb{C}^{n \times n}$ then we write $\tilde{r}_i(A) = \tilde{r}_i(A)$ where $A = M(G)$. This allows us to state the following theorem by Varga [17] which is an extension of Brualdi’s original theorem [11].

**Theorem 3.7. (Varga)** Let $A \in \mathbb{C}^{n \times n}$. Then the eigenvalues of $A$ are contained in the set

$$B(A) = \bigcup_{\gamma \in C(A)} \left\{ z \in \mathbb{C} : \prod_{v_i \in \gamma} |z - A_{ii}| \leq \prod_{v_i \in \gamma} \tilde{r}_i(A) \right\}.$$  

As with the theorems of Gershgorin and Brauer this result can also be extended to matrices in $\mathbb{W}^{n \times n}$ as follows.

**Theorem 3.8.** Let $G \in \mathbb{G}^n$. Then $\sigma(G)$ is contained in the set

$$\mathcal{BW}_B(G) = \bigcup_{\gamma \in C(G)} \left\{ \lambda \in \mathbb{C} : \prod_{v_i \in \gamma} |\lambda - M(\tilde{G}, \lambda)_{ii}| \leq \prod_{v_i \in \gamma} \tilde{r}_i(\tilde{G}, \lambda) \right\}.$$  

Also, $\mathcal{BW}_B(G) \subseteq \mathcal{BW}_K(G)$.

For $G \in \mathbb{G}^n$ we call $\mathcal{BW}_B(G)$ the Brualdi-type region of the graph $G$ and

$$\mathcal{BW}_B(G)_\gamma = \left\{ \lambda \in \mathbb{C} : \prod_{v_i \in \gamma} |\lambda - M(\tilde{G}, \lambda)_{ii}| \leq \prod_{v_i \in \gamma} \tilde{r}_i(\tilde{G}, \lambda) \right\}$$

the Brualdi-type region associated with the cycle $\gamma$.

**Proof.** For $G \in \mathbb{G}^n$, note that its polynomial extension $\tilde{G}$ can be considered to be a graph with polynomial weights in the variable $\lambda$. With this in mind let $\tilde{G} = \tilde{G}(\lambda)$.

Then for fixed $\alpha \in \mathbb{C}$, $G(\alpha)$ is the graph with adjacency matrix $M(G, \alpha) \in \mathbb{C}^{n \times n}$. Furthermore, since $C(\tilde{G}(\lambda))$ is the cycle set of $\tilde{G}(\lambda)$ then for any $\gamma = \{v_1, \ldots, v_m\}$ in $C(\tilde{G}(\lambda))$ and fixed $\alpha \in \mathbb{C}$ let $\gamma(\alpha)$ be the set of vertices $\{v_1, \ldots, v_m\}$ in the graph $\tilde{G}(\alpha)$.

Using this notation, if $\alpha \in \sigma(G)$ then by lemma 3.2 and theorem 3.7 there exists a $\gamma' \in C(\tilde{G}(\alpha))$ such that

$$\prod_{v_i \in \gamma'} |\alpha - M(\tilde{G}, \alpha)_{ii}| \leq \prod_{v_i \in \gamma'} \tilde{r}_i(\tilde{G}, \alpha).$$

(7)

There are then two possibilities. Either $\gamma' \in C(\tilde{G})$ i.e. the set of vertices $\gamma'(\alpha)$ is also a cycle in $\tilde{G}$ in which case $\alpha \in \mathcal{BW}_B(G)$ by equation (6) or $\gamma' \notin C(\tilde{G})$. Suppose that the latter is the case or $\gamma' \notin C(\tilde{G})$.
Note that if \( \gamma' \in C_s(\bar{G}(\alpha)) \) then as \( M(\bar{G}, \alpha)_{ij} \neq 0 \) implies \( M(\bar{G}, \lambda)_{ij} \neq 0 \) for \( i \neq j \) then \( \gamma' \in C_s(\bar{G}) \) which is not possible. Hence, \( \gamma' \in C_w(\bar{G}(\alpha)) \) or \( \gamma' \) must be a loop of some vertex \( v_j \) where the graph induced by \( \{v_j\} \) in \( \bar{G}(\alpha) \) is a strongly connected component of \( \bar{G}(\alpha) \). Therefore, equation (7) is equivalent to \( |\alpha - M(\bar{G}, \alpha)_{jj}| \leq 0 \).

Hence, \( \alpha = M(\bar{G}, \alpha)_{jj} \). As some cycle \( \gamma \in C(\bar{G}) \) contains the vertex \( v_j \) then \( \alpha \) is contained in the set

\[
\{ \lambda \in \mathbb{C} : \prod_{v_i \in \gamma} |\lambda - M(\bar{G}, \lambda)_{ii}| \leq \prod_{v_i \in \gamma} \tilde{r}_i(\bar{G}, \lambda) \}
\]

implying that \( \alpha \in BW_B(\bar{G}) \).

To show that \( BW_B(\bar{G}) \subseteq BW_K(\bar{G}) \) let \( \gamma \in C(\bar{G}) \). Supposing that \( \gamma \in C_w(\bar{G}) \) then \( \lambda = \{v_i\} \) for some vertex \( v_i \) of \( G \) and

\[
BW_B(\bar{G}) = \{ \lambda \in \mathbb{C} : |\lambda - M(\bar{G}, \lambda)_{ii}| = 0 \}
\]

as \( v_i \) is the vertex set of some strongly connected component of \( \bar{G} \). It follows from equation (5) that \( BW_B(\bar{G}) \subseteq BW_K(\bar{G})_{ij} \) for any \( 1 \leq j \leq n \) where \( i \neq j \). In particular, note that if \( \tilde{r}_i(\bar{G}, \lambda) = 0 \) then \( \lambda \in BW_K(\bar{G})_{ij} \) for any \( 1 \leq j \leq n \) where \( i \neq j \).

If on the other hand, \( \gamma \in C_s(\bar{G}) \) then for convenience let \( \gamma = \{v_1, \ldots, v_p\} \) where \( p > 1 \) and note that

\[
BW_B(\bar{G}) = \{ \lambda \in \mathbb{C} : \prod_{i=1}^{p} |\lambda - M(\bar{G}, \lambda)_{ii}| \leq \prod_{i=1}^{p} \tilde{r}_i(\bar{G}, \lambda) \}.
\]

Assuming \( 0 < \tilde{r}_i(\bar{G}, \lambda) \) for all \( 1 \leq i \leq p \) then for fixed \( \lambda \in BW_B(\bar{G}) \) it follows by raising both sides of the inequality in (8) to the \((p - 1)st\) power that

\[
\prod_{1 \leq i,j \leq p \atop i \neq j} \left( \frac{|\lambda - M(\bar{G}, \lambda)_{ii}| |\lambda - M(\bar{G}, \lambda)_{jj}|}{\tilde{r}_i(\bar{G}, \lambda) \tilde{r}_j(\bar{G}, \lambda)} \right) \leq 1
\]

As not all the terms of the product in (9) can exceed unity then for some pair of indices \( \ell \) and \( k \) where \( 1 \leq \ell, k \leq p \) and \( \ell \neq k \) it follows that

\[
|\lambda - M(\bar{G}, \lambda)_{kk}| |\lambda - M(\bar{G}, \lambda)_{\ell\ell}| \leq \tilde{r}_k(\bar{G}, \lambda) \tilde{r}_\ell(\bar{G}, \lambda).
\]

Using the fact that \( \tilde{r}_i(\bar{G}, \lambda) \leq r_i(\bar{G}, \lambda) \) for all \( 1 \leq i \leq n \) we conclude that \( \lambda \in BW_K(\bar{G})_{kk} \) completing the proof.

The Brualdi-type region for the graph \( \bar{G} \) with adjacency matrix \( \bar{G} \) is shown in figure 3 (left) where we note that \( BW_B(\bar{G}) \subseteq BW_K(\bar{G}) \) or in this particular case that \( BW_B(\bar{G}) = BW_K(\bar{G}) \) (right).

4. ISOSPECTRAL GRAPH REDUCTIONS

Here we present a method developed in [8] which allows for the reduction of a graph \( G \in \mathcal{G} \) while maintaining the graph’s spectrum up to some known set. To do this we first introduce some definitions as well as some terminology that allow us to be precise in the formulation of an isospectral graph reduction. We note that all results in this section, with the exception of proposition 1, can be found in [8] as well as their proofs.

In the following if \( S \subseteq V \) where \( V \) is the vertex set of a graph we let \( \bar{S} \) denote the complement of \( S \) in \( V \). Also if \( \{v_1, \ldots, v_m\} \) is a path in \( G \in \mathcal{G} \) let the vertices \( v_2, \ldots, v_{m-1} \) of \( P \) be its interior vertices. If \( P = \{v_1, \ldots, v_m\} \) is a cycle where we
Figure 3. Left: The Brualdi-type region $BW_B(G)$ for $G$ in figure 1. Right: $BW_B(G) = BW_K(G)$.

fix some $v_i \in P$ then we say $P$ is a cycle from $v_i$ to $v_i$ where $P \setminus \{v_i\}$ are its interior vertices.

Recall from section 2 that if we write the graph $G$ as some triple $(V, E, \omega)$ then we are assuming $G \in \mathbb{G}$. With this in mind we give the following definitions.

**Definition 4.1.** For $G = (V, E, \omega)$ let $\ell(G)$ be the digraph $G$ with all loops removed. We say the nonempty vertex set $S \subseteq V$ is a structural set of $G$ if $\bar{S}$ induces no cycles in $\ell(G)$ and for each $v_i \in \bar{S}$, $\omega(e_{ii}) \neq \lambda$. We denote by $st(G)$ the set of all structural sets of $G$.

**Definition 4.2.** For $G = (V, E, \omega)$ with $S = \{v_1, \ldots, v_m\} \in st(G)$ let $B_{ij}(G; S)$ be the set of paths or cycles from $v_i$ to $v_j$ in $G$ having no interior vertices in $S$. Furthermore, let

$$B_S(G) = \bigcup_{1 \leq i,j \leq m} B_{ij}(G; S).$$

We call the set $B_S(G)$ the set of all branches of $G$ with respect to $S$.

**Definition 4.3.** Let $G = (V, E, \omega)$ and $\beta \in B_S(G)$ for some $S \in st(G)$. If $\beta = v_1, \ldots, v_m$, $m > 2$ we define

$$P_\omega(\beta) = \omega(e_{12}) \prod_{i=2}^{m-1} \frac{\omega(e_{i,i+1})}{\lambda - \omega(e_{ii})}$$

as the branch product of $\beta$. If $m = 2$ we define $P_\omega(\beta) = \omega(e_{12})$.

The reason we require $w(e_{ii}) \neq \lambda$ in definition 4.1 is that we need $P_\omega(\beta)$ to be defined in the following.

**Definition 4.4.** Let $G = (V, E, \omega)$ with structural set $S = \{v_1, \ldots, v_m\}$. Define the graph $R_S(G) = (S, E, \mu)$ to be the graph such that $e_{ij} \in E$ if $B_{ij}(G; S) \neq \emptyset$ and

$$\mu(e_{ij}) = \sum_{\beta \in B_{ij}(G; S)} P_\omega(\beta), \quad 1 \leq i, j \leq m.$$  

We call $R_S(G)$ the isospectral reduction of $G$ over $S$. 
By proposition 1 it then follows that

\[ \text{Proposition 1.} \]

In order to understand the extent to which the spectrum of a graph is maintained under different reductions, that is reductions of the same graph over different structural sets, we introduce the following. If \( G \) is a structural set of the graph \( G = (V,E,\omega) \) where \( V = \{v_1, \ldots, v_n\} \) let

\[ \mathcal{N}(G;S) = \bigcup_{v_i \in S} \{ \lambda \in \mathbb{C} : \lambda = \omega(e_{i,i}) \text{ or } \omega(e_{i,i}) \text{ is undefined} \}. \]

That is, \( \mathcal{N}(G;S) \) is the set of \( \lambda \in \mathbb{C} \) for which there is some vertex \( v_i \in \bar{S} \) where \( \omega(e_{i,i}) = \lambda \) or, as \( \omega(e_{i,i}) = p_i(\lambda)/q_i(\lambda) \in \mathbb{W} \), the values of \( \lambda \) at which \( q_i(\lambda) = 0 \).

Note that, \( \omega(e_{i,i}) \) may be zero for a given graph i.e. \( e_{i,i} \) could be a loop with weight zero and therefore not appear on the graph. However, it is important to note that if such is the case and \( v_i \in \bar{S} \), then we consider the following example.

Let \( \mathcal{R}_G(S) \) be in \( \mathbb{G} \) where \( S \in \text{st}(G) \) and \( S \neq V \). Then

\[ \text{Proposition 1.} \]

To better understand the extent to which the spectrum of a graph \( G \) and the spectrum of its reduction \( \mathcal{R}_G(S) \) differ we give the following.

**Theorem 4.5.** Let \( G \in \mathbb{G} \) with \( S \in \text{st}(G) \). Then \( \sigma(G) \) and \( \sigma(\mathcal{R}_S(G)) \) differ at most by \( \mathcal{N}(G;S) \).

To better understand the extent to which the spectrum of a graph \( G \) and the spectrum of its reduction \( \mathcal{R}_S(G) \) differ we give the following.

**Proposition 1.** Let \( G = (V,E,\omega) \) be in \( \mathbb{G} \) where \( S \in \text{st}(G) \) and \( S \neq V \). Then

\[ \frac{\det(M(G) - \lambda I)}{\prod_{v_i \in S} (\omega(e_{i,i}) - \lambda)} = \det(M(\mathcal{R}_S(G)) - \lambda I). \]

That is, the characteristic polynomial associated with the reduced graph \( \mathcal{R}_S(G) \) is the characteristic polynomial of \( G \) divided by the product \( \prod_{v_i \in S} (\omega(e_{i,i}) - \lambda) \). As a consequence, for a given graph \( G \) and structural set \( S \in \text{st}(G) \) it is possible that either \( \sigma(G) = \sigma(\mathcal{R}_S(G)) \), \( \sigma(G) \subset \sigma(\mathcal{R}_S(G)) \), or \( \sigma(\mathcal{R}_S(G)) \subset \sigma(G) \). To illustrate this we consider the following example.

Let \( A \in \mathbb{G} \) be given above as in figure 4. One can compute that

\[ \det(M(A) - \lambda I) = 2\lambda - \lambda^3 \quad \text{implying} \quad \sigma(A) = \{0, \pm \sqrt{2}\}. \]

By proposition 1 it then follows that

\[ \det(M(\mathcal{R}_{\{v_1,v_2\}}(A)) - \lambda I) = (2\lambda - \lambda^3)/\lambda \]

as \( \omega(e_{33}) = 0 \), since \( e_{33} \) is a loop of weight zero. It follows that \( \sigma(\mathcal{R}_{\{v_1,v_2\}}(A)) = \{\pm \sqrt{2}\} \subset \sigma(A) \).
If the graph $\mathcal{R}_{\{v_1,v_2\}}(A)$ is reduced over the vertex $\{v_1\}$ then proposition 1 implies

$$\det\left( M(\mathcal{R}_{\{v_1\}}(\mathcal{R}_{\{v_1,v_2\}}(A))) - \lambda I \right) = \frac{\lambda^2 - 2}{1/\lambda - \lambda} = \frac{2\lambda - \lambda^3}{\lambda^2 - 1}. $$

Therefore, $\sigma(\mathcal{R}_{\{v_1,v_2\}}(A)) \subset \sigma(\mathcal{R}_{\{v_1\}}(\mathcal{R}_{\{v_1,v_2\}}(A))) = \{0, \pm \sqrt{2}\}$.

Note that for both of these reductions theorem 4.5 indicates the sets by which the eigenvalues of these graphs can differ. However, proposition 1 can be used to find not only these numbers but also the multiplicity of these numbers by which these spectrums can differ.

For example, by proposition 1 the graph $\mathcal{R}_{\{v_2\}}(A)$ has two less zeros in its spectrum than $A$ provided that $\sigma(A)$ contains at least two zeros, otherwise $\sigma(\mathcal{R}_{\{v_2\}}(A))$ contains no zeros. As $\sigma(A)$ contains only one zero this is in fact the case.

It is also possible for the spectrum of a graph to remain unchanged under reduction. This happens for instance in the reduction of the graph $G$ to the graph $G_1$ where $M(G_0)$ and $M(G_1)$ are given on page 14 of this paper.

Now, as any reduction $\mathcal{R}_S(G)$ of a graph $G \in \mathbb{G}$ is again a graph in $\mathbb{G}$ it is natural, as in the example above, to consider sequences of reductions on a graph as well as to what degree a graph can be reduced.

**Definition 4.6.** For $G = (V,E,\omega)$ suppose the sequence of sets $S_1, \ldots, S_m \subseteq V$ are such that $S_1 \in st(G)$, $R_1(G) = \mathcal{R}_{S_1}(G)$ and

$$S_{i+1} \in st(R_i(G)) \text{ where } \mathcal{R}_{S_{i+1}}(R_i(G)) = R_{i+1}(G), \ 1 \leq i \leq m - 1. $$

If this is the case then we say $S_1, \ldots, S_m$ induces a sequence of reductions on $G$ and we write $R_i(G) = \mathcal{R}(G;S_1,\ldots,S_i)$ for $1 \leq i \leq m - 1$. Moreover, we let $N(G;S_1,\ldots,S_i) = N(G;S_1,\ldots,S_{i-1}) \cup N(R_{i-1}(G);S_i), \ 2 \leq i \leq m.$

**Definition 4.7.** For $p \in \mathbb{C}[\lambda]$ let $\deg(p)$ be the degree of $p$ and for $\omega = p/q \in \mathbb{W}$ let $\pi(\omega) = \deg(p) - \deg(q)$. Let $\mathbb{G}_n \subseteq \mathbb{G}^n$ be the set of graphs where for any $G \in \mathbb{G}_n$, $\pi(M(G)_{ij}) \leq 0$ for all $1 \leq i,j \leq n$. Furthermore, let $\mathbb{G}_\pi = \bigcup_{n \geq 1} \mathbb{G}_n^\pi$.

**Remark 2.** It is important to note that any graph $G$ where $M(G) \in \mathbb{C}^{n \times n}$ is a graph in the set $\mathbb{G}_\pi$.

**Remark 3.** Note that if the sets $S_1, \ldots, S_m$ induce a sequence of reductions on a graph $G \in \mathbb{G}$ then by repeated use of theorem 4.5 it follows that $\sigma(G)$ and $\sigma(\mathcal{R}(G;S_1,\ldots,S_m))$ differ at most by $N(G;S_1,\ldots,S_m)$. Moreover, if the graph $G \in \mathbb{G}_\pi$ then the set $N(G;S_1,\ldots,S_m)$ is a finite set of points in the complex plane.

The following theorem shows that sequential reductions are in a certain sense commutative.

**Theorem 4.8.** Let $G = (V,E,\omega)$ and $V$ be any nonempty subset of $V$. If $G \in \mathbb{G}_\pi$ then there exists a sequence of sets $S_1, \ldots, S_{m-1},V \subseteq V$ inducing a sequence of reductions on $G$. Moreover, for any such sequence $T_1,\ldots,T_{n-1},V$ there is a unique graph $\mathcal{R}_V[G] = \mathcal{R}(G;T_1,\ldots,T_{n-1},V)$ independent of the particular sets $T_1,\ldots,T_{m-1}$.

That is, the final vertex set in a sequence of reductions completely specifies the reduced graph irrespective of the specific sequence of reductions.
5. Main Results

In this section we give the main results of this paper. Specifically, we show that a reduced graph has a smaller Gershgorin and Brauer-type regions respectively than the associated unreduced graph. That is, the eigenvalue estimates given in section 3.1 and 3.2 can be improved by use of graph reductions.

Moreover, if $H$ is a nontrivial reduction of the graph $G$ then there are less regions of the form $BW_{r}(H)_{ij}$ and $BW_{e}(H)_{ij}$ then of the form $BW_{r}(G)_{ij}$ and $BW_{e}(G)_{ij}$ respectively. So although there is some effort involved in reducing a graph this is offset by the fact that there are less Gershgorin and Brauer-type regions to compute.

For Brualdi-type regions the situation is more complicated. For certain reductions the Brualdi-type region of a graph may decrease in size similar to Gershgorin and Brauer-type regions. In other cases the Brualdi-type region of a graph may do the opposite and increase in size when the graph is reduced. We give an example of both of these possibilities in section 5.3.

5.1. Improving Gershgorin-Type Estimates. We first consider the effect of reducing a graph on its associated Gershgorin region. Our main result in this direction is the following theorem.

**Theorem 5.1. (Improved Gershgorin Regions)** Let $G = (V, E, \omega)$ where $V$ is any nonempty subset of $V$. If $G \in \mathbb{G}_\pi$ then $BW_{r}(\mathcal{R}_{\mathcal{V}[G]}) \subseteq BW_{r}(G)$.

Theorem 5.1 has the following corollary.

**Corollary 1.** Let $G = (V, E, \omega)$ where $V$ is a nonempty subset of $V$. If the graph $\mathcal{R}_{\mathcal{V}[G]} = \mathcal{R}(G; S_1, \ldots, S_m, V)$ for some sequence $S_1, \ldots, S_m, V$ then the spectrum $\sigma(G) \subseteq BW_{r}(\mathcal{R}_{\mathcal{V}[G]}) \cup N(G; S_1, \ldots, S_m, V)$.

In order to understand in which situations $BW_{r}(\mathcal{R}_{\mathcal{V}[G]})$ is strictly contained in $BW_{r}(G)$ we consider the following. For $G \in \mathbb{G}_\pi$ let

$$\partial BW_{r}(G)_{i} = \{ \lambda \in \mathbb{C} : |\lambda - M(H)_{ii}| = r_i(G, \lambda) \} \text{ for } 1 \leq i \leq n.$$  

We note that the (topological) boundary of the region $BW_{r}(G)_{i}$ in the complex plane is contained in the set $\partial BW_{r}(G)_{i}$.

**Theorem 5.2.** Let $G = (V, E, \omega)$ be in $\mathbb{G}_\pi$ and $V$ be a nonempty proper subset of $V$. If $V_i = V \setminus \{v_i\}$ then at most finitely many points of the set

$$\bigcup_{v_i \in V} \left( \partial BW_{r}(G)_{i} \setminus \bigcup_{v_j \in V_i} BW_{r}(G)_{j} \right)$$

are contained in $BW_{r}(\mathcal{R}_{\mathcal{V}[G]})$.

For a nontrivial $G = (V, E, \omega)$ we note that there is typically some region $BW_{r}(G)_{i}$ whose boundary is not contained in the union of the other $j$th Gershgorin regions. Moreover, as this piece of the boundary usually contains infinitely many points then assuming this is the case theorems 5.1 and 5.2 imply

$$BW_{r}(\mathcal{R}_{\mathcal{V}[G] \setminus \{v_i\}}(G)) \subsetneq BW_{r}(G).$$

By another application of theorem 5.1 it follows that

$$BW_{r}(\mathcal{R}_{\mathcal{V}[G]}) \subsetneq BW_{r}(G)$$

for any nonempty subset $V \subset V$ where $V$ does not contain $v_i$. That is, reducing over such sets strictly improve the estimates given by Gershgorin-type regions.
As an example consider the graph $G_0 \in \mathcal{G}_\pi$ with adjacency matrix
\[ M(G_0) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}. \]

Denoting the graphs $G_1 = \mathcal{R}_{\{v_1,v_2,v_3\}}[G_0]$ and $G_2 = \mathcal{R}_{\{v_1,v_2\}}[G_1]$ it follows that
\[ M(G_1) = \begin{bmatrix} \frac{\lambda+1}{\lambda} & \frac{1}{\lambda} & \frac{\lambda+1}{\lambda} \\ \frac{2\lambda+1}{\lambda} & \frac{1}{\lambda} & \frac{2\lambda+1}{\lambda} \\ 0 & \frac{1}{\lambda} & \frac{1}{\lambda} \end{bmatrix} \quad \text{and} \quad M(G_2) = \begin{bmatrix} \frac{\lambda+1}{\lambda} & \frac{1}{\lambda} & \frac{\lambda+1}{\lambda} \\ \frac{2\lambda+1}{\lambda} & \frac{1}{\lambda} & \frac{2\lambda+1}{\lambda} \\ 0 & \frac{1}{\lambda} & \frac{1}{\lambda} \end{bmatrix}. \]

The Gershgorin regions of $G_0, G_1,$ and $G_2$ are shown in figure 5 where by theorem 5.1 $BW_T(G_2) \subseteq BW_T(G_1) \subseteq BW_T(G_0)$.

Moreover, as
\[ \partial BW_T(G_0) = \bigcup_{j=1}^4 BW_T(G_0) \quad \text{and} \quad \partial BW_T(G_1) = \bigcup_{j=1}^2 BW_T(G_1) \]
are both infinite sets then this implies in fact that $BW_T(G_2) \subseteq BW_T(G_1) \subseteq BW_T(G_0)$.

Lastly, note that $\mathcal{N}(G_0;\{v_1,v_2,v_3\}) = \mathcal{N}(G_1;\{v_1,v_2\}) = \{0\}$. As the point \{0\} $\subseteq BW_T(G_1), BW_T(G_2)$ then both $BW_T(G_1)$ and $BW_T(G_2)$ contain $\sigma(G_0)$ by corollary 1. (Note that $M(G_1) = M(G)$ where $M(G)$ is given by [3].)

In addition an important implication of theorem 5.1 is that graph reductions on some $G \in \mathcal{G}_\pi$ can be used to obtain estimates of $\sigma(G)$ with increasing precision depending on how much one is willing to reduce the graph $G$. Bearing this in mind suppose $v_i$ is a vertex of $G \in \mathcal{G}_\pi$. Then the graph $\mathcal{R}_{\{v_i\}}[G] = (\{v_i\}, \mathcal{E}, \mu)$ consists of a single vertex $v_i$ and possibly a loop. We note that this is the furthest extent to which $G$ may be reduced. Moreover, as $\pi(\mu(e_{ii})) < 0$, under the assumption $G \in \mathcal{G}_\pi$, it follows that $BW_T(\mathcal{R}_{\{v_i\}}[G])$ is a finite set of points in the complex plane. Furthermore, if $\mathcal{R}_{\{v_i\}}[G] = \mathcal{R}(G; S_1, \ldots, S_{m-1}, \{v_i\})$ then corollary 1 implies $\sigma(G) \subseteq BW_T(\mathcal{R}_{\{v_i\}}[G]) \cup \mathcal{N}(G; S_1, \ldots, S_{m-1}, \{v_i\})$. As $\mathcal{N}(G; S_1, \ldots, S_{m-1}, \{v_i\})$ is also a finite set of points in the complex plane we summarize this in the following remark.
Remark 4. Theorem 5.1 implies that graph reductions on some $G \in \mathbb{G}_\pi$ can be used to estimate $\sigma(G)$ with increasing accuracy depending on the extent to which $G$ is reduced. Moreover, if $G$ is reduced as much as possible, i.e., to a graph on a single vertex, the corresponding Gershgorin-type region is a finite set of points in the complex plane. By corollary 1 this set differs from $\sigma(G)$ by some other finite but known set of points.

As an example, if $G_3 = R_{\{e_1\}}[G_2]$ then one can compute $\sigma(G_3) = \{-1, -1, -i, i, 2\}$ and $N(G_2, \{v_1\}) = \{0, 1.3247, -6623 \pm 0.5622i\}$. Corollary 1 then implies that $\sigma(G_0) \subseteq \{-1, -i, i, 2, 0, 1.3247, -6623 \pm 0.5622i\}$.

Remark 5. The computation involved in reducing a graph $G$ from $n$ to $m$ vertices is offset by the fact that there are $n - m$ less $i$th Gershgorin-type regions to calculate.

5.2. Improving Brauer-Type Estimates. We now consider Brauer-type regions for which we give results similar to those given in section 5.1.

Theorem 5.3. (Improved Brauer Regions) Let $G = (V, E, \omega)$. If $G \in \mathbb{G}_\pi$ where $\mathcal{V} \subseteq V$ contains at least two vertices, then $BW_K(R_{\mathcal{V}}[G]) \subseteq BW_K(G)$.

Theorem 5.3 has the following corollary.

Corollary 2. Let $G = (V, E, \omega)$ where $\mathcal{V} \subseteq V$ contains at least two vertices. If the graph $R_{\mathcal{V}}[G] = R(G; S_1, \ldots, S_{m-1}, \mathcal{V})$ for some sequence $S_1, \ldots, S_{m-1}, \mathcal{V}$ then $\sigma(G) \subseteq BW_K(R_{\mathcal{V}}[G]) \cup N(G; S_1, \ldots, S_{m-1}, \mathcal{V})$.

Continuing our example, the Brauer-type regions of $G_0, G_1, G_2$ are shown in figure 6 where by theorem 5.3 $BW_K(G_2) \subseteq BW_K(G_1) \subseteq BW_K(G_0)$. Moreover, theorem 5.3 implies $BW_K(G_2) \subseteq BW_T(G_2), BW_K(G_1) \subseteq BW_T(G_1)$, and $BW_K(G_0) \subseteq BW_T(G_0)$.

We note that a graph $G \in \mathbb{G}$ must have at least two vertices for $BW_K(G)$ to exist. However, a graph in $\mathbb{G}$ with two vertices may have a Brauer-type region consisting of an infinite set of points (e.g., figure 6 right hand side). That is, if $G$ is reduced to a graph on two vertices, i.e., as much as is possible such that the Brauer-type region still exists, then this Brauer-type region will not be a finite set of points. This is in contrast to the situation mentioned in remark 4 where Gershgorin-type regions of fully reduced graphs have this property.

Furthermore, we note that if a graph is reduced from $n$ to $m$ vertices then there are $\binom{n}{2} - \binom{m}{2}$ $i$th Brauer-type regions to calculate. Hence, the number of regions
In this example we have the strict inclusions of the associated Brualdi-type regions given by $\ BW_B(\mathcal{R}_S(\mathcal{H})) \subsetneq BW_B(\mathcal{R}_T(\mathcal{H}))$ (see figure 8). In particular, as $BW_B(\mathcal{R}_S(\mathcal{H})) \subsetneq BW_B(\mathcal{R}_T(\mathcal{H}))$ then reducing the graph $\mathcal{H}$ over $S$ increases the size of its Brualdi-type region. So graph reductions do not always improve Brualdi-type estimates of a graph’s spectrum.

However, Brualdi-type estimates still do a better job than Gershgorin and Brauer-type regions in estimating spectra. For instance, despite the fact that $BW_B(\mathcal{H}) \not\subseteq BW_B(\mathcal{R}_S(\mathcal{H}))$ in the example above, theorems $\text{3.6}$ and $\text{3.8}$ still imply that the region $BW_B(\mathcal{R}_S(\mathcal{H})) \subsetneq BW_B(\mathcal{R}_T(\mathcal{H}))$. In order to give a sufficient condition under which a graph’s Brualdi-type region shrinks as the graph is reduced we consider the following. Let $G = (V, E, \omega)$ where $V = \{v_1, \ldots, v_n\}$ for some $n \geq 1$ and $G$ has strongly connected components $S_1(G), \ldots, S_m(G)$. Define

$$E^{\text{sc}} = \{e \in E : e \in S_i(G), 1 \leq i \leq m\}.$$ 

The cycle $\gamma \in C(G)$ is said to be adjacent to $v_i \in V$ if $v_i \notin \gamma$ and there is some vertex $v_j \in \gamma$ such that $e_{ji} \in E^{\text{sc}}$. For any $v_i \in V$ we denote

$$\mathcal{A}(v_i, G) = \{\gamma \in C(G) : \gamma \text{ is adjacent to } v_i\}.$$ 

Moreover, if $C(v_i, G) = \{\gamma \in C(G) : v_i \in \gamma\}$ then let $S(v_i, G) \subseteq C(v_i, G)$ be the set of cycles containing the following elements.

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**Figure 7.** Left: $BW_B(\mathcal{G}_0)$. Middle: $BW_B(\mathcal{G}_1)$. Right: $BW_B(\mathcal{G}_2)$, where in each the spectrum $\sigma(\mathcal{G}_0) = \{-1, -1, -i, 2\}$ is indicated.
For fixed $i$, let $\gamma = \{v_{\alpha_1}, \ldots, v_{\alpha_m}\}$ be a cycle in $C(v_i, G)$ where $v_i = v_{\alpha_1}$. If $m = 1$, that is $\gamma = \{v_i\}$, then $\gamma \in S(v_i, G)$ only if $\gamma \in C(v_i, G)$. Otherwise, supposing $1 < m \leq n$ then relabel the vertices of $G$ such that $v_{\alpha_j}$ is $v_j$ for $1 \leq j \leq m$ and denote this relabelled graph by $G_r = (V_r, E_r, \omega_r)$. Then $\gamma \in S(v_i, G)$ if and only if $e_{j1} \notin E_r$ for $1 < j < m$ and $e_{mk} \notin E_r^{scc}$ for $m < k \leq n$. With this in place we give the following theorem.

**Theorem 5.4. (Improved Brualdi Regions)** Let $G = (V, E, \omega)$ where $G \in \mathcal{G}_{\pi}$ and $V$ contains at least two vertices. If $v \in V$ such that both $A(v, G) = \emptyset$ and $C(v, G) = S(v, G)$ then $BW_B(\mathcal{R}_{\mathcal{V}\setminus v}(G)) \subseteq BW_B(G)$.

That is, if the vertex $v$ is adjacent to no cycle in $C(G)$ and each cycle passing through $v$ is in $S(v, G)$ then removing this vertex improves the Brualdi-type region of $G$. We note that for graph $\mathcal{H}$ in figure 8 the set $A(v_1, \mathcal{H}) = \{v_2, v_3\} \neq \emptyset$. Hence, theorem 5.4 does not apply to the reduction of $\mathcal{H}$ over $S$.

However, the vertex $v_4$ has the property that $A(v_4, \mathcal{H}) = \emptyset$ as well as $S(v_4, \mathcal{H}) = C(v_4, \mathcal{H})$. Therefore, reducing $\mathcal{H}$ over the vertex set $\mathcal{T} = \{v_1, v_2, v_3\}$ improves the Brualdi-type region of this graph which can be seen on the upper right hand side of figure 8.

As an example for why the condition $C(v, G) = S(v, G)$ is necessary in theorem 5.4 consider the following. Let $\mathcal{J}, \mathcal{R}_S(\mathcal{J}) \in \mathcal{G}$ be the matrices given by

$$M(\mathcal{J}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \text{then} \quad M(\mathcal{R}_S(\mathcal{J})) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

where $S = \{v_2, v_3, v_4\}$. In this case one can compute that $BW_B(\mathcal{R}_S(\mathcal{J})) \nsubseteq BW_B(\mathcal{J})$. We note that $A(v_1, \mathcal{J}) = \emptyset$ but $S(v_1, \mathcal{J})$ consists of the cycle $\{v_1, v_2, v_3\}$ whereas the cycle set $C(v_1, \mathcal{J}) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_1\}\}$.

Aside from this, we observe that graph reductions can increase, decrease or maintain the number of cycles a graph has in its cycle set. For instance the graph
\( G_0 \) in our previous example has 12 cycles in its cycle set whereas \( G_1 \) has 3 and \( G_2 \) has 2 (see figure 7).

On the other hand the graphs \( P, RU(P) \in G \) with adjacency matrices given by

\[
M(P) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}, \quad \text{and} \quad M(RU(P)) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1/\lambda & 0 & 1/\lambda & 0 \\
0 & 0 & 0 & 1 \\
1/\lambda & 0 & 1/\lambda & 0
\end{bmatrix}
\]

where \( U = \{v_1, v_2, v_3, v_4\} \) have the following cycle structure. The set \( C(P) = \{\{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\} \) whereas \( C(RU(P)) = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\} \). That is, reducing \( P \) over \( U \) increases the number of cycles needed to compute the associated Brualdi-type region from 2 to 3.

5.4. **Proofs.** Here we give the proofs of the theorems in section 5.1, 5.2, and 5.3. The proof of theorem 5.1 is the following.

**Proof.** For \( G = (V, E, \omega) \) where \( G \in G_0^n \) and \( n \geq 2 \) note that the vertex set \( V \setminus \{v_1\} \in st(G) \). In order to simplify notation let \( RU(V \setminus \{v_1\}) = R_1, L_i(G, \lambda) = L_i, L_i(R_1, \lambda) = L_i^1 \) and \( M(G, \lambda) = \omega_{ij} \). Then it follows that

\[
BW_\Gamma(R_1) = \{\lambda \in \mathbb{C} : |(\lambda - \omega_{ii} - \frac{\omega_{i1}\omega_{1j}}{\lambda - \omega_{11}})L_i^1| \leq \sum_{j=2}^{n} |(\omega_{ij} + \frac{\omega_{i1}\omega_{1j}}{\lambda - \omega_{11}})L_i^1| \}
\]

for \( 2 \leq i \leq n \). Supposing \( \lambda \in BW_\Gamma(R_1) \), for fixed \( \lambda \) and \( 1 \leq i \leq n \) then from the inequality above

\[
|(\lambda - \omega_{ii})L_i^1 - \frac{\omega_{i1}\omega_{1j}}{\lambda - \omega_{11}}L_i^1| \leq \sum_{j=2, j \neq i}^{n} |\omega_{ij}L_i^1| + \sum_{j=2, j \neq i}^{n} |\frac{\omega_{i1}\omega_{1j}}{\lambda - \omega_{11}}L_i^1|.
\]

By the triangle inequality

\[
|(\lambda - \omega_{ii})L_i^1| - |\frac{\omega_{i1}\omega_{1j}}{\lambda - \omega_{11}}L_i^1| \leq \sum_{j=1, j \neq i}^{n} |\omega_{ij}L_i^1| - |\omega_{1i}L_i^1| + \sum_{j=2}^{n} |\frac{\omega_{i1}\omega_{1j}}{\lambda - \omega_{11}}L_i^1| - |\frac{\omega_{i1}\omega_{1j}}{\lambda - \omega_{11}}L_i^1|.
\]

Therefore

\[
(10) \quad |(\lambda - \omega_{ii})L_i^1| \leq \sum_{j=1, j \neq i}^{n} |\omega_{ij}L_i^1| - \left( |\omega_{1i}L_i^1| - \sum_{j=2}^{n} |\frac{\omega_{i1}\omega_{1j}}{\lambda - \omega_{11}}L_i^1| \right).
\]

In order to use the inequality given in (10) let \( w_{ij} = \frac{p_{ij}}{q_{ij}} \) where \( p_{ij}, q_{ij} \in \mathbb{C}[\lambda] \) and where \( L_{ij} = \prod_{\ell=1, \ell \neq j}^{n} q_{ij} \) for \( 1 \leq i, j \leq n \). Then

\[
(11) \quad BW_\Gamma(G) = \{\lambda \in \mathbb{C} : |L_{1i}(q_{ii} \lambda - p_{ii})| \leq \sum_{j=1, j \neq i}^{n} |p_{ij}L_{ij}| \}.
\]

Moreover,

\[
M(R_1, \lambda)_{ij} = \frac{p_{1i}p_{1j}q_{ij}q_{11} + q_{i1}q_{1j}p_{ij}(q_{11} \lambda - p_{11})}{q_{i1}q_{1j}q_{ij}(q_{11} \lambda - p_{11})}, \quad 2 \leq i, j \leq n
\]
from which \( L^i_1 = \prod_{j=2}^{n} q_{ij} q_{j}(q_{11} \lambda - p_{11}) \).

Suppose then that
\[
|\omega_{i1} L^i_1| - \sum_{j=2}^{n} \frac{|\omega_{i1} \omega_{ij}|}{\lambda - \omega_{11}} L^i_1 \geq 0.
\]

By (10) this then implies that
\[
|(\lambda - \omega_{ii}) L^i_1| \leq \sum_{j=1, j \neq i}^{n} |\omega_{ij} L^i_1|.
\]

As this can be written
\[
|q_{i1}^{-2}(q_{11} \lambda - p_{11}) L_{i1}| |L_{i1}(q_{ii} \lambda - p_{ii})| \leq |q_{i1}^{-2}(q_{11} \lambda - p_{11}) L_{i1}| \sum_{j=1, j \neq i}^{n} |p_{ij} L_{ij}|
\]

we consider the following.

First, if \( q_{i1} = 0 \) then \( L_{i1} = 0 \) implying \( \lambda \in B \Gamma_{1}(G) \). Similarly, if \( L_{i1} = 0 \) then \( \lambda \in B \Gamma_{1}(G) \) by (11). This is also the case if \( q_{11} \lambda - p_{11} = 0 \). Conversely, if \( |q_{i1}^{-2}(q_{11} \lambda - p_{11}) L_{i1}| \neq 0 \) then inequalities (11) and (12) imply that \( \lambda \in B \Gamma_{1}(G) \).

Hence, \( \lambda \in B \Gamma_{1}(G) \cup B \Gamma_{1}(G) \) in this case.

On the other hand if
\[
|\omega_{i1} L^i_1| - \sum_{j=2}^{n} \frac{|\omega_{i1} \omega_{ij}|}{\lambda - \omega_{11}} L^i_1 < 0
\]

we note the following. If \( |q_{i1}^{-2}(q_{11} \lambda - p_{11}) L_{i1}| = 0 \) then by the above it follows that \( \lambda \in B \Gamma_{1}(G) \cup \Gamma(G) \). If \( |q_{i1}^{-2}(q_{11} \lambda - p_{11}) L_{i1}| \neq 0 \) then \( \omega_{11} \), and \( \omega_{j1} \) are defined at \( \lambda \) for \( 2 \leq j \leq n \) and (13) may be rewritten as
\[
|L^i_1| < \frac{L^i_1}{\lambda - \omega_{11}} \sum_{j=2}^{n} |\omega_{ij}|
\]
since \( \omega_{11} \neq 0 \) in this case. Moreover, as \( L^i_1/(\lambda - \omega_{11}) \) cannot be zero then this inequality can in turn be written as \( |\lambda - \omega_{11}| < \sum_{j=2}^{n} |\omega_{ij}| \) implying \( \lambda \in B \Gamma_{1}(G) \).

Hence, \( B \Gamma_{1}(R_1) \subseteq B \Gamma_{1}(G) \cup B \Gamma_{1}(G) \) and therefore \( B \Gamma_{1}(R_1) \subseteq B \Gamma_{1}(G) \).

By sequentially removing single vertices of \( V \) from the graph \( G \) an application of theorem 4.8 completes the proof.

We now give a proof of theorem 5.2.

Proof. Let \( G \in \mathbb{G}_{n}^{>0} \) for \( n > 1 \). As in the proof of theorem 5.1 let \( R_{V \setminus \{v_1\}}(G) = R_1 \), \( L_i(G, \lambda) = L_i, L_i(R_1, \lambda) = L^i_1 \) and \( M(G, \lambda)_{ij} = \omega_{ij} \).

Let \( \lambda \in \mathbb{C} \) be fixed such that
\[
\lambda \in \partial B \Gamma_{1} \setminus \bigcup_{j=2}^{n} B \Gamma_{1}(G)_{j}.
\]
It then follows that
\begin{equation}
|(\lambda - \omega_{11}) L_1| = \sum_{j=2}^{n} |\omega_{1j} L_1|, \tag{14}
\end{equation}
\begin{equation}
|(\lambda - \omega_{ii}) L_i| > \sum_{j=1, j \neq i}^{n} |\omega_{ij} L_i|, \quad 1 < i \leq n. \tag{15}
\end{equation}

Supposing that \( \lambda \in BW_T(R_1)_j \) for some \( 1 < i \leq n \) then the inequality given by (10) holds. Multiplying this inequality on both sides by \( |L_1||L_i| \) yields
\begin{equation}
|L_1^i L_1| \left| (\lambda - \omega_{ii}) L_i - \sum_{j=1, j \neq i}^{n} \omega_{ij} L_i \right| \leq |\omega_{11} L_1| \left( -|L_1^i L_1| + \sum_{j=2}^{n} \left| \frac{\omega_{1j} L_1}{\lambda - \omega_{11}} \right| \right). \tag{16}
\end{equation}
Moreover, if both sides (14) are multiplied by \( |L_1^i/(\lambda - \omega_{11})| \) then we have
\begin{equation}
|L_1^i L_1| = \sum_{j=2}^{n} \left| \frac{\omega_{1j} L_1}{\lambda - \omega_{11}} \right|. \tag{17}
\end{equation}

Therefore, if \( |L_1^i L_1| \neq 0 \) then (15), (16), and (17) yield a contradiction implying \( \lambda \notin BW_T(R_1)_i \).

If \( |L_1^i L_1| = 0 \) then as both \( L_1^i \) and \( L_1 \) are polynomials in the variable \( \lambda \) there are only finitely many points in the complex plane where this holds. Therefore, for any fixed \( 1 < i \leq n \) the set
\[
\left( \partial BW_T(G)_1 \setminus \bigcup_{j=2}^{n} BW_T(G)_j \right) \bigcap BW_T(R_1)_i
\]
consists of at most finitely many points. If \( \mathcal{V} = \{ v_1, \ldots, v_m \} \) is some nonempty, proper subset of the vertices of \( G \) the result follows via theorem 4.8 and 5.1 by sequentially removing single vertices from \( G \) until \( \mathcal{V} \) has been removed.

Next we give the proof of theorem 5.3.

**Proof.** For \( G = (V, E, \omega) \) where \( G \in G_+^n \) and \( n \geq 3 \) note that the vertex set \( V \setminus \{ v_1 \} \in st(G) \). For the sake of simplicity let \( R_{V \setminus \{ v_1 \}}(G) = R_1, L_k(G, \lambda) = L_k, K_k(R_1, \lambda) = L_1^k, M(G, \lambda)_{k \ell} = \omega_{k \ell} \) as well as \( R_k(G) = \sum_{\ell=1, \ell \neq k}^{n} |\omega_{k \ell} L_1^k| \) and \( \lambda - \omega_{kk} = \lambda_{kk} \).

The claim is that \( K_{ij}(R_1) \subseteq K_{ii}(G) \cup K_{jj}(G) \cup K_{ij}(G) \) for \( 2 \leq i, j \leq n \) where \( i \neq j \). To see this let \( \lambda \in K_{ij}(R_1) \) from which it follows that
\begin{equation}
\left| (\lambda_{ii} - \frac{\omega_{1i} \omega_{1j}}{\lambda_{11}}) L_1^i \right| \left| (\lambda_{jj} - \frac{\omega_{1j} \omega_{1j}}{\lambda_{11}}) L_1^j \right| \leq \sum_{\ell=2}^{n} \left| (\omega_{1\ell} + \frac{\omega_{1i} \omega_{1j}}{\lambda_{11}}) L_1^i \right| \left| (\omega_{1\ell} + \frac{\omega_{1i} \omega_{1j}}{\lambda_{11}}) L_1^j \right|. \tag{18}
\end{equation}

Multiplying both sides of (18) by \( |\lambda_{11} L_1 L_i| \) and \( |\lambda_{11} L_1 L_j| \) it follows that
\[
|\lambda_{ii} \lambda_{11} L_1 L_i L_1^i - \omega_{1i} \omega_{1j} L_1^i L_1 L_i | \left| \lambda_{jj} \lambda_{11} L_1 L_j L_1^j - \omega_{1i} \omega_{1j} L_1^j L_1 L_j \right| \leq \left( \sum_{\ell=2}^{n} \left| (\omega_{1\ell} + \omega_{1i} \omega_{1j}) L_1 L_1^i \right| \right) \left( \sum_{\ell=2}^{n} \left| (\omega_{1\ell} + \omega_{1i} \omega_{1j}) L_1 L_1^j \right| \right).
\]
From the triangle inequality we have
\[ \prod_{k \in \{i, j\}} \left( |\lambda_{kk}\lambda_{11}L_1L_k| - |\omega_{k1}\omega_{1k}L_1L_kL_k^1| \right) \leq \prod_{k \in \{i, j\}} \left( \sum_{\ell=2}^n |\omega_{k\ell}\lambda_{11}L_1L_k^1| + \sum_{\ell=2}^n |\omega_{k1}\omega_{1\ell}L_1L_kL_k^1| - |\omega_{k1}\omega_{1k}L_kL_k^1| \right). \]

Let \( \lambda \notin \mathcal{K}_{1i}(G) \), \( \mathcal{K}_{1j}(G) \) and note then that \( |\lambda_{11}| |\lambda_{kk}| > R_1(G)R_k(G) \) for \( k = i, j \). If \( |\lambda_{11}| \leq R_1(G) \) then by the previous inequality
\[ \prod_{k \in \{i, j\}} \left( R_1(G)R_k(G)|L_k^1| - |\omega_{k1}\omega_{1k}L_k^1L_kL_1| \right) < \prod_{k \in \{i, j\}} \left( R_1(G) \sum_{\ell=2}^n |\omega_{k\ell}L_k^1L_k| + \sum_{\ell=2}^n |\omega_{k1}\omega_{1\ell}L_1L_kL_k^1| - |\omega_{k1}\omega_{1k}L_kL_k^1| \right) \]

From the fact that
\[ R_1(G)R_k(G)|L_k^1| - |\omega_{k1}\omega_{1k}L_k^1L_kL_1| = \]

(19)
\[ R_1(G) \sum_{\ell=2}^n |\omega_{k\ell}L_k^1L_k| + \sum_{\ell=2}^n |\omega_{k1}\omega_{1\ell}L_1L_kL_k^1| \]

it follows that the previous inequality does not hold. Therefore, if \( \lambda \notin \mathcal{K}_{1i}(G) \), \( \mathcal{K}_{1j}(G) \) then \( |\lambda_{11}| > R_1(G) \).

Proceeding as before, where we assume again that \( \lambda \in \mathcal{K}_{1i}(R_1) \), then note (18) via the triangle inequality implies
\[ \prod_{k \in \{i, j\}} \left( |\lambda_{kk}L_k^1| - |\frac{\omega_{k1}\omega_{1k}}{\lambda_{11}}L_k^1| \right) \leq \prod_{k \in \{i, j\}} \left( \sum_{\ell=1}^n |\omega_{k\ell}L_k^1| - \frac{\omega_{k1}\omega_{1k}}{\lambda_{11}}L_k^1| + \sum_{\ell=2}^n |\omega_{k1}\omega_{1\ell}L_1L_kL_k^1| - |\omega_{k1}\omega_{1k}L_kL_k^1| \right). \]

Multiplying both sides by \( |\lambda_{11}L_1L_i| \) and \( |\lambda_{ii}L_iL_j| \) furthermore implies
\[ \left( |\lambda_{11}L_1||\lambda_{ii}L_i||L_i^1| - |\omega_{11}\omega_{1i}L_1L_iL_i^1| \right) \left( |\lambda_{ii}L_i||\lambda_{jj}L_j||L_j^1| - |\omega_{jj}\omega_{1j}L_iL_jL_j^1| \right) \leq \left( |\lambda_{11}L_1|R_1(G)|L_i^1| - |\omega_{11}\omega_{1i}L_1L_iL_i^1| + |\omega_{i1}L_iL_i^1|(R_1(G) - |\lambda_{11}|) \right). \]

\[ \left( |\lambda_{ii}L_i|R_j(G)|L_j^1| - |\omega_{jj}\omega_{1j}L_iL_jL_j^1| + n |\omega_{j1}\omega_{1\ell}L_iL_jL_j^1| - |\omega_{j1}L_j\lambda_{ii}L_i| \right) \]

Suppose that \( \lambda \notin \mathcal{K}_{1i}(G) \), \( \mathcal{K}_{1j}(G) \). Then from the above \( |\lambda_{11}| > R_1(G) \) and moreover if it is also the case that \( \lambda \notin \mathcal{K}_{ij}(G) \) then from the previous inequality it follows that
\[ R_i(G)R_j(G)|L_j^1| < |\lambda_{ii}L_i|R_j(G)|L_j^1| + \sum_{\ell=2}^n |\omega_{j1}\omega_{1\ell}L_iL_jL_j^1| \frac{\lambda_{ii}}{\lambda_{11}} - |\omega_{j1}L_j\lambda_{ii}L_i|. \]
If $|\lambda_i L_i| \leq R_i(G)$ then this inequality implies

$$\sum_{\ell=2}^{n} |\omega_{i\ell} L_{\lambda_{1\ell}}^1| < |\lambda_{11} L_1| \leq R_1(G).$$

Note that if $|\lambda_{11} L_1| = 0$ then $\lambda \in K_{11}(G)$ for $k = i, j$ which is not possible. Hence, $|\lambda_{11} L_1| \neq 0$ and multiplying both sides of (21) by $|\lambda_{11} L_1|$ yields

$$|\lambda_{11} L_1||L_j^1| < \sum_{\ell=2}^{n} |\omega_{i\ell} L_1||L_j^1|$$

implying $|\lambda_{11} L_1| < R_1(G)$, a contradiction.

Therefore, if $\lambda \notin K_{11}(G), K_{1j}(G),$ and $K_{ij}(G)$ then $|\lambda_i L_i| > R_i(G)$. Moreover, by switching the indices $i$ and $j$ in this argument it similarly follows that if $\lambda \notin K_{11}(G), K_{1j}(G),$ and $K_{ij}(G)$ then $|\lambda_{jj} L_j| > R_j(G)$.

Let $\lambda \in K_{ij}(R_1)$ and suppose that $\lambda \notin K_{1i}(G), K_{1j}(G), K_{ij}(G)$. From the above it follows that $|\lambda_{kk} L_k| > R_k(G)$ for $k = 1, i, j$. Hence, by multiplying (20) on both sides by $|L_i||L_j|$ this implies

$$\prod_{k \in \{i,j\}} (R_k(G)|L_k^1| - |\omega_{kk} L_k^1|) <$$

$$\prod_{k \in \{i,j\}} (R_k(G)|L_k^1| - |\omega_{kk} L_k^1| + \sum_{\ell=2}^{n} |\omega_{k\ell} L_k^1|).$$

Note that if

$$\sum_{\ell=2}^{n} |\omega_{k\ell} L_k^1| - |\omega_{kk} L_k^1| > 0$$

and assuming as before that $|\lambda_{11} L_1| \neq 0$ then by multiplying both sides of this inequality by $|\lambda_{11} L_1|$ we have

$$|\omega_{kk} L_k^1| \sum_{\ell=2}^{n} |\omega_{i\ell} L_1| > |\omega_{kk} L_k^1||\lambda_{11} L_1|$$

for $k = i, j$.

As this implies that $R_1(G) > |\lambda_{11} L_1|$, which is not possible, then by (22) it follows that $\lambda \in K_{11}(G) \cup K_{1j}(G) \cup K_{ij}(G)$.

This verifies the claim which completes the proof. \hfill \square

In order to prove theorem 5.4 we first give the following lemma.

**Lemma 5.5.** Let $G = (V, E, \omega)$ where $G \in \mathbb{G}_n$ and $V$ contains at least two vertices. Moreover, suppose $v_1 \in V$ such that both $A(v_1, G) = \emptyset$ and $C(v_1, G) = S(v_1, G)$. If $\gamma_2 = \{v_2, \ldots, v_m\} \subset C(R_{V\setminus \{v\}}(G)), m \geq 2,$ and there exists a $\gamma_1 = \{v_1, v_2, \ldots, v_m\} \subset C(G)$ then $\mathcal{BW}_B(R_{V\setminus \{v\}}(G))_{\gamma_2} \subset \mathcal{BW}_B(G)$

**Proof.** Suppose first that the hypotheses of the lemma hold for some $G \in \mathbb{G}_n$. We then make the observation that the edges belonging to no strongly connected component of $G$ are not used to calculate to $\mathcal{BW}_B(G)$. Furthermore, any cycle of $G$ is contained in exactly one strongly connected component of this graph. This implies that the Brualdi-type region of the graph is the union of the Brualdi-type regions of its strongly connected components. Therefore, we may without loss in generality assume that $G$ consists of a single strongly connected component.
As in the previous proofs let $R_{V \backslash \{v_1\}}(G) = R_1$, $L_k(G, \lambda) = L_k$, $L_k(R_1, \lambda) = L_k^1$, $M(G, \lambda)_{\ell \ell} = \omega_{\ell \ell}$ as well as $R_k(G) = \sum_{\ell = 1, \ell \neq k}^n |\omega_{\ell \ell}L_k|$ and $\lambda - \omega_{kk} = \lambda_{kk}$. In addition, suppose that both $\gamma_1 = \{v_1, \ldots, v_m\}$ and $\delta = \{v_1, v_m\}$ are cycles in $C(v_1, G)$ for some $1 < m \leq n$. The fact that $\gamma_1 \in C(v_1, G)$ implies in particular that $\gamma_2 = \{v_2, \ldots, v_m\}$ is a cycle in $C(R_1)$.

From the assumption that $v_1$ has no adjacent cycles it follows that $\omega_{mi} = 0$ for $1 < i < m$ since otherwise $\{v_i, v_{i+1}, \ldots, v_m\} \in A(v_1, G)$. Also, as $\gamma_1 \in C(v_1, G) = S(v_1, G)$ then $\omega_{mi} = 0$ for $m < i < n$ and $\omega_{jj} = 0$ for $1 < j < m$. Therefore,

\[
BWB(G)_{\gamma_1} = \{ \lambda \in \mathbb{C} : \prod_{i=1}^m |\lambda_i L_i| \leq |\omega_{m1} L_m| \prod_{i=1}^{m-1} R_i(G) \},
\]

(23) \[BWB(G)_{\delta} = \{ \lambda \in \mathbb{C} : |\lambda_{11} L_1| |\lambda_{mm} L_m| \leq |\omega_{m1} L_m| R_1(G) \}.
\]

In addition, the region $BWB(R_1)_{\gamma_2}$ is given by the set

\[
\{ \lambda \in \mathbb{C} : (|\lambda_{mm} - \frac{\omega_{m1} \omega_{11} L_m}{\lambda_{11}}| L_m) \prod_{i=2}^m |\lambda_i L_i| \leq \sum_{i=2}^m (|\omega_{m1} \omega_{11} L_m|) \prod_{i=2}^m R_i(G) \}.
\]

(25) In this case $L_1^1 = L_1$ for $1 < i < m$ since for each such $i$ the edge $e_{i1} \notin E$.

Letting $\omega_{ij} = p_{ij} / q_{ij}$ for all $1 \leq i, j \leq n$ then

\[
L_m^1 = (q_{m1} (q_{11} \lambda - p_{11}))^{n-1} \prod_{i=2}^n q_{i1}.
\]

If $q_{11} \lambda - p_{11} = 0$ or $q_{m1} = 0$ then $|\lambda_{11}| = 0$. Similarly $q_{ii} = 0$ implies that $|\lambda_{ii}| = 0$ for all $2 \leq i \leq n$. Therefore, if it is the case that $L_m^1 = 0$ then $\lambda \in BWB(G)_{\gamma_1}$.

Suppose then that both $\lambda \in BWB(R_1)_{\gamma_2}$ and $L_m^1 \neq 0$. By multiplying both sides of (25) by $|\lambda_{11} L_1|$ and dividing out $|L_m^1|$ then by the triangle inequality

\[
(|\lambda_{11} L_1 \lambda_{mm} L_m| - |\omega_{m1} L_1 \omega_{m1} L_m|) \prod_{i=2}^m |\lambda_i L_i| \leq \prod_{i=2}^m |\omega_{m1} L_m| (R_1(G) - |\omega_{11} L_1|) \prod_{i=2}^m R_i(G).
\]

(26)

Note that if $\prod_{i=2}^m R_i(G) = 0$ then $\lambda \in BWB(G)_{\gamma_1}$ and if $|\lambda_{11} L_1 \lambda_{mm} L_m| - |\omega_{m1} L_1 \omega_{m1} L_m| = 0$ or $|\lambda_{11} L_1 \lambda_{mm} L_m| = 0$ then $\lambda \in BWB(G)_{\delta}$. Otherwise, (26) implies

\[
\prod_{i=2}^m |\lambda_i L_i| \leq \frac{|\omega_{m1} L_m| (R_1(G) - |\omega_{11} L_1|)}{|\lambda_{11} L_1 | \lambda_{mm} L_m| - |\omega_{m1} L_1 \omega_{m1} L_m|} \prod_{i=2}^m R_i(G).
\]

(27) With this in mind, we note that if

\[
\frac{|\omega_{m1} L_m| (R_1(G) - |\omega_{11} L_1|)}{|\lambda_{11} L_1 | \lambda_{mm} L_m| - |\omega_{m1} L_1 \omega_{m1} L_m|} \leq \frac{|\omega_{m1} L_m| R_1(G)}{|\lambda_{11} L_1 | \lambda_{mm} L_m|}
\]

then it follows from (27) that $\lambda \in BWB(G)_{\gamma_1}$. On the other hand if this inequality does not hold then $|\lambda_{11} L_1 | \lambda_{mm} L_m| \leq |\omega_{m1} L_m| R_1(G)$ or $\lambda \in BWB(G)_{\delta}$. Hence, if $\delta \in C(G)$ then $BWB(R_{V \backslash \{v_1\}}(G))_{\gamma_1} \subseteq BWB(G)_{\gamma_1} \cup BWB(G)_{\delta}$.

Conversely, if $\delta \notin C(G)$ then $|\omega_{11} L_1 L_m| = 0$. From (26) it follows that $BWB(G)_{\gamma_2} = BWB(G)_{\gamma_1}$. This completes the proof. \qed
We now give a proof of theorem 5.4.

**Proof.** First suppose that the graph \( G = (V, E, \omega) \) in \( \mathbb{G}_n^d \) consists of a single strongly connected component that is nontrivial. Moreover, for the vertex \( v_1 \in V \) suppose both \( \mathcal{A}(v_1, G) = \emptyset \) and \( C(v_1, G) = \mathcal{S}(v_1, G) \).

Using the notation \( \mathcal{R}_1 = \mathcal{R}_{V \setminus \{v_1\}}(G) \) let \( \gamma = \{v_2, \ldots, v_m\} \) be a cycle in \( C(\mathcal{R}_1) \) for some \( 1 < m \leq n \). If \( \gamma \in C(G) \) then as \( \mathcal{A}(v_1, G) = \emptyset \) it follows that \( M(G, \lambda)_{ij} = M(\mathcal{R}_1, \lambda)_{ij} \) for all \( 2 \leq i \leq m \) and \( 1 \leq j \leq n \). Hence, \( \mathcal{B}W_B(\mathcal{R}_1)_{\gamma} = \mathcal{B}W_B(G)_{\gamma} \) or \( \mathcal{B}W_B(\mathcal{R}_1)_{\gamma} \subseteq \mathcal{B}W_B(G) \).

On the other hand if \( \gamma \notin C(G) \) then for at least one \( 1 \leq i \leq m \) both of the edges \( e_{i-1,1}, e_{1,i} \in E \). Suppose then that this is the case for only one such \( 0 \leq i \leq m \) and without loss in generality that this happens at \( i = 1 \). Therefore, the cycle \( \delta = \{v_1, \ldots, v_m\} \in C(G) \). Moreover, as \( C(v_1, G) = \mathcal{S}(v_1, G) \) and \( \mathcal{A}(v_1, G) = \emptyset \) by assumption then lemma 5.5 implies that \( \mathcal{B}W_B(\mathcal{R}_1)_{\gamma} \subseteq \mathcal{B}W_B(G) \).

If there are multiple \( 0 \leq i \leq m \) such that \( e_{i,1}, e_{1,i+1} \in E \) then we define the following. Denoting \( v_0 = v_m \) then let \( I = \{i : e_{i-1,1}, e_{1,i} \in E, 1 \leq i \leq m\} \). We give the set \( I \) the ordering \( I = \{i_1, \ldots, i_{\ell}\} \) such that \( i_j < i_k \) if and only if \( j < k \). Then the sets

\[
\gamma_j = \{v_k \in \gamma : i_j < k < i_{j+1}\} \quad \text{for} \quad 1 \leq j < \ell
\]

along with \( \gamma_{\ell} = \{v_k \in \gamma : v_k < i_1 \text{ or } v_k \geq i_{\ell}\} \) partition the vertices of the cycle \( \gamma \).

Moreover, let \( \gamma_j^1 = \gamma_j \cup \{v_i\} \) be given the ordering \( \{v_1, v_i, v_{i+1}, \ldots, v_{i+1-1}\} \) for \( 1 \leq j < \ell \) and \( \gamma_j^2 = \gamma_j \cup \{v_i\} \) the ordering \( \{v_1, v_i, v_{i+1}, \ldots, v_k, v_2, \ldots, v_{i-1}\} \). Then each \( \gamma_j^1 \in C(v_1, G) \) for \( 1 \leq j \leq \ell \). Furthermore, let \( \gamma_j \) be given this same ordering as \( \gamma_j^1 \) for \( 1 \leq j \leq \ell \) with the vertex \( v_1 \) removed. If this is the case then each \( \gamma_j \in C(\mathcal{R}_1) \). We note that by another application of lemma 5.5 it follows that \( \mathcal{B}W_B(\mathcal{R}_1)_{\gamma_j} \subseteq \mathcal{B}W_B(G) \) for each \( 1 \leq j \leq \ell \).

The claim then is that the region

\[
\mathcal{B}W_B(\mathcal{R}_1)_{\gamma} \subseteq \bigcup_{j=1}^{\ell} \mathcal{B}W_B(\mathcal{R}_1)_{\gamma_j}.
\]

To see this denote \( \lambda^1_{ij} = (\lambda - \omega_{ii} - \omega_{ii}/\lambda_{ii})L^1_{ij} \) and \( R^1_i = \sum_{j=2, j \neq i}^n |M(\mathcal{R}_1, \lambda)_{ij}|. \) Then using this notation we have

\[
\mathcal{B}W_B(\mathcal{R}_1)_{\gamma} = \{\lambda \in \mathbb{C} : \prod_{i=2}^m |\lambda^1_{ii}| \leq \prod_{i=2}^m R^1_i\} \quad \text{and} \quad \mathcal{B}W_B(\mathcal{R}_1)_{\gamma_j} = \{\lambda \in \mathbb{C} : \prod_{i \in \gamma_j^1} |\lambda^1_{ii}| \leq \prod_{i \in \gamma_j^1} R^1_i\} \quad \text{for} \quad 1 \leq j \leq \ell.
\]

Therefore, assuming that \( \lambda \notin \mathcal{B}W_B(\mathcal{R}_1)_{\gamma_j} \) for all \( 1 \leq j \leq \ell \) implies \( \lambda \notin \mathcal{B}W_B(\mathcal{R}_1)_{\gamma} \). This follows from the fact that the vertices of the cycles \( \gamma_j \) for \( 1 \leq j \leq \ell \) partition the vertices of \( \gamma \). Hence, (28) holds or \( \mathcal{B}W_B(\mathcal{R}_1)_{\gamma} \) is contained in the union of the regions \( \mathcal{B}W_B(\mathcal{R}_1)_{\gamma_j} \) which are themselves contained in \( \mathcal{B}W_B(G) \). Therefore, \( \mathcal{B}W_B(\mathcal{R}_1)_{\gamma} \subseteq \mathcal{B}W_B(G) \) for any \( \gamma \in C(\mathcal{R}_1) \).

To end this section we give a proof of proposition 1.
Proof. Assuming the hypotheses of the proposition suppose that $\bar{S} = \{v_1\}$ and note that this in particular implies $\omega(e_{11}) \neq \lambda$. Let $d_{in}(v_1)$ and $d_{out}(v_1)$ be the in and out degree of $v_1$ respectively. Using the notion of a branch expansion from [6] let $\mathcal{X}$ be a branch expansion of $G$ with respect to $S$. Then there are $n = d_{in}(v_1) \cdot d_{out}(v_1)$ branches in $\mathcal{B}_S(\mathcal{X})$ of length two each with a single intermediate vertex having a loop of weight $\omega(e_{11})$. Moreover, by repeated use of the proof of lemma 4.4 in [6] we have

$$(\omega(e_{11}) - \lambda)^{n-1} \det (M(G) - \lambda I) = \det (M(\mathcal{X}) - \lambda I)$$

as there are $n-1$ more vertices in $\mathcal{X}$ than in $G$ each adding the solutions of $\lambda = \omega(e_{11})$ to $\sigma(G)$.

Contracting the $n$ branches of length two in $\mathcal{B}_S(G)$ to single edges yields

$$\det (M(\mathcal{X}) - \lambda I) = (\omega(e_{11}) - \lambda)^n \det (M(R_S(G)) - \lambda I)$$

by use of the proof of lemma 4.5 in [6]. Therefore, the result follows in this case.

For general structural sets $S \in st(G)$ the result follows by sequentially reducing $G$ over single vertices of $S$ by use of theorem 4.8.

6. Some Applications

In this section we discuss some natural applications of using graph reductions to improve estimates of the spectra of certain graphs or equivalently matrices in $\mathbb{R}^{n \times n}$. Our first application deals with estimating the spectra of the Laplacian matrix of a given graph. Following this we give an algorithm for estimating the spectral radius of a matrix using graph reductions. Last, we use the results of theorem 5.2 as well as some structural knowledge of a graph to identify particularly useful structural sets. The motivation for finding such sets in general is that reducing over them allows for better eigenvalue estimates with minimal effort. This is especially useful for establishing eigenvalue estimates for large graphs with known structural properties.

6.1. Laplacian Matrices. An important application of theorem 4.5 is that one may reduce not only the graph $G$ but also the graphs associated with both the combinatorial Laplacian matrix and the normalized Laplacian matrix of $G$. Such matrices are typically defined for undirected graphs without loops or weights but this definition can be extended to graphs in $G$ (see remark 6 below). However, here we give the standard definitions as these are of interest in their own right (see [7, 8]).

Let $G = (V, E)$ be an unweighted undirected graph without loops, i.e. a simple graph. If $G$ has vertex set $V = \{v_1, \ldots, v_n\}$ and $d(v_i)$ is the degree of vertex $v_i$ then its combinatorial Laplacian matrix $M_L(G)$ is given by

$$M_L(G)_{ij} = \begin{cases} 
  d(v_i) & \text{if } i = j \\
  -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\
  0 & \text{otherwise}
\end{cases}$$

On the other hand the normalized Laplacian matrix $M_L(G)$ of $G$ is defined as

$$M_L(G)_{ij} = \begin{cases} 
  1 & \text{if } i = j \text{ and } d(v_j) \neq 0 \\
  \frac{-1}{\sqrt{d(v_i)d(v_j)}} & \text{if } v_i \text{ is adjacent to } v_j \\
  0 & \text{otherwise}
\end{cases}$$
Figure 9. Left: $BW_1(L(H))$. Right: $BW_1(R_S(L(H)))$, where in each the spectrum $\sigma(L(H)) = \{0, 1, 2, 4, 5\}$ is indicated.

The interest in the eigenvalues of $M_L(G)$ is that $\sigma(M_L(G))$ gives structural information about $G$ (see [7]). On the other hand knowing $\sigma(M_L(G))$ is useful in determining the behavior of algorithms on the graph $G$ among other things (see [8]).

Let $L(G)$ be the graph with adjacency matrix $M_L(G)$ and similarly let $L(G)$ be the graph with adjacency matrix $M_G(G)$. Since both $L(G), L(G) \in G_\pi$ either may be reduced over any subset of their respective vertex sets.

For example if $H \in G_\pi$ is the simple graph with adjacency matrix

$$M(H) = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & \ \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}$$

then the graph $L(H)$, has the structural set $S = \{v_1, v_2, v_3, v_4\}$. Reducing over this set yields $R_S(L(H))$ where

$$M(R_S(L(H))) = \begin{bmatrix}
\lambda - 3 & \frac{1}{\lambda - 1} & \frac{1}{\lambda - 4} \\
\frac{1}{\lambda - 1} & \lambda - 7 & \frac{1}{\lambda - 4} \\
\frac{1}{\lambda - 4} & \frac{1}{\lambda - 4} & \lambda - 11 \\
\frac{1}{\lambda - 1} & \frac{1}{\lambda - 4} & \lambda - 11 \\
\frac{1}{\lambda - 4} & \frac{1}{\lambda - 4} & \lambda - 11
\end{bmatrix}.$$  

Figure 9 shows the Gershgorin regions for $L(H)$ as well as $R_S(L(H))$.

Note that the adjacency matrix of $H$ is symmetric so its eigenvalues must be real numbers. With this in mind the Gershgorin-type region associated with simple graphs and their reductions can be reduced to intervals of the real number line.

**Remark 6.** It is possible to generalize $M_L(G)$ to any $G \in G$ if $G$ has no loops and $n$ vertices by setting $M_L(G)_{ij} = -M(G)_{ij}$ for $i \neq j$ and $M_L(G)_{ii} = \sum_{j=1, j \neq i}^n M(G)_{ij}$. This generalizes and is consistent with what is done for weighted digraphs in [18] for example.
6.2. Estimating the Spectral Radius of a Matrix. For $G \in \mathbb{G}_p$, the spectral radius of $G$ denoted $\rho(G)$ is the maximum among the absolute values of the elements in $\sigma(G)$ i.e.

$$\rho(G) = \max_{\lambda \in \sigma(G)} |\lambda|.$$ 

For many graphs $G \in \mathbb{G}_p$, it is possible to find some structural set $S \in \text{st}(G)$ such that each vertex of $S$ has no loop. If $S$ is such a set then $\sigma(G)$ and $\sigma(\mathcal{R}_S(G))$ differ at most by $N(G; S) = \{0\}$. Moreover, as $\mathcal{N}(G; S)$ is in some sense the error in estimating $\sigma(G)$ by $\sigma(\mathcal{R}_S(G))$ then it follows that $\rho(G) = \rho(\mathcal{R}_S(G))$.

For example consider the graph $K$ shown in figure 10 and note that the vertices $v_1, v_2, v_3$ are the vertices of $K$ without loops. As $\{v_1, v_2, v_3\} \in \text{st}(K)$ then $\mathcal{N}(G; \{v_1, v_2, v_3\}) = \{0\}$ where $\mathcal{R}_{\{v_1, v_2, v_3\}}(G)$ is the graph shown on the lower right of the figure.

By employing the region $\mathcal{BW}_T(K)$ it is possible to estimate $\rho(K) \leq 3$. However, via $\mathcal{BW}_T(\mathcal{R}_{\{v_1, v_2, v_3\}}(K))$ it follows that $\rho(K) \leq 2$ (see the top left and right of figure 10). The important idea here is that each vertex without loop may be removed from a graph without effecting its spectral radius.

It should be noted that for a given graph there is often no unique set of vertices without loops which is simultaneously a structural set. Therefore, there may be many ways to reduce a graph without effecting its spectral radius.

Moreover, it is possible to continue reducing the graph even if, as in figure 10, the graph has been reduced to a graph in which each vertex has a loop. For example, the graph $\tilde{K} = \mathcal{R}_{\{v_1, v_2, v_3\}}(K)$ in this figure can be further reduced over the set $\{v_1, v_3\}$ where one can compute $\mathcal{N}(\tilde{K}; \{v_1, v_3\}) = \{(1 + \sqrt{5})/2\}$.

Since $(1 + \sqrt{5})/2 \in \mathcal{BW}_T(\tilde{K})$ then it follows from theorem 4.5 that

$$\sigma(G) \subseteq \mathcal{BW}_T(\mathcal{R}_{\{v_1, v_3\}}(\tilde{K})) \cup \{0, (1 + \sqrt{5})/2\}.$$
Figure 11. Left: The graph $N$. Right: $BW_{Γ}(N)$.

Therefore, $ρ(\bar{K})$ can be estimated using $BW_{Γ}(\mathcal{R}_{\{\bar{v}_{1}, \bar{v}_{3}\}}(\bar{K})) \cup \{(1 \pm \sqrt{5})/2\}$.

Had it been the case that some $k \in N(\bar{K}; \{\bar{v}_{1}, \bar{v}_{3}\})$ was not contained in $BW_{Γ}(\bar{K})$ then by our previous calculations $k \in σ(\bar{K})$ and we could then ignore it. This process can then be continued in the same way over some further structural set of $\mathcal{R}_{\{\bar{v}_{1}, \bar{v}_{3}\}}(\bar{K})$ if desired.

To summarize this method, the first step in estimating the spectral radius of a graph $G \in Ω$ via graph reductions is to remove those vertices of $G$ having no loop. This can be accomplished either by a single reduction or sequence of reductions and results in a graph $\mathcal{G}$ where $ρ(G) = ρ(\mathcal{G})$. That is, $ρ(G)$ can be estimated via $BW_{Γ}(\mathcal{G})$ rather than $BW_{Γ}(G)$.

If greater accuracy in estimating $ρ(G)$ is desired it is possible to improve this estimate by further reducing $\mathcal{G}$ over some $S \in st(\mathcal{G})$. This is done by using the set $BW_{Γ}(\mathcal{R}(\mathcal{G})) \cup (N(\mathcal{G}; S) \cap BW_{Γ}(\mathcal{G}))$ to estimate $ρ(G)$. This is possible since $σ(\mathcal{G}) \subseteq BW_{Γ}(\mathcal{R}(\mathcal{G})) \cup (N(\mathcal{G}; S) \cap BW_{Γ}(\mathcal{G})) \subseteq BW_{Γ}(\mathcal{G})$.

This second step can then be repeated by reducing $\mathcal{R}(\mathcal{G})$ over some structural set and so on if improved estimates are desired.

6.3. Targeting Specific Structural Sets of Graphs (and Networks). In this section we consider reducing graphs over specific structural sets in order to improve eigenvalue estimates when some structural feature of the graph is known. To do so consider $G = (V, E, ω)$ where $V = \{v_{1}, \ldots, v_{n}\}$.

If the sets $BW_{Γ}(G)$, for $1 \leq i \leq n$ are known or can be estimated by some structural knowledge of $G$ then it is possible to make decisions on which structural sets to reduce over based on these sets. That is, it may be possible to identify structural sets $V \subset V$ such that

$$\bigcup_{v_{i} \in V} \left(∂(BW_{Γ}(G))_{i} \setminus \bigcup_{v_{j} \in V_{i}} BW_{Γ}(G)_{j}\right)$$

is some infinite set. If this can be done, theorem 5.2 then implies that a strictly better estimate of $σ(G)$ can be achieved by reducing over $V$.

For example consider the graph $N = (V, E, ω)$ in the left hand side of figure 11 where $V = \{v_{1}, \ldots, v_{n}\}$ for some $n > 5$. If it is known that $N$ is a simple graph
such that \( d(v_1) = 4 \), \( d(v_2) = d(v_3) = d(v_4) = d(v_5) = 3 \) and \( d(v_i) \in \{0, 1, 2, 3\} \) for all \( 6 \leq i \leq n \) then the sets \( BW_{\Gamma}(G)_i \) are each discs of radius either 0,1,2,3 or 4 (see right hand side of figure 11). Moreover, as

\[
\partial BW_{\Gamma}(G)_1 \setminus \bigcup_{i=2}^{n} BW_{\Gamma}(G)_i = \{ \lambda \in \mathbb{C} : |\lambda| = 4 \}
\]

then theorem 5.2 implies that \( \mathcal{R}_{V \setminus \{v_1\}}(N) \) has a strictly smaller Gershgorin-type region than does \( N \) which can be seen in figure 12. Considering the fact that \( n \) may be quite large this example is intended to illustrate that eigenvalues estimates can be improved with little effort if some simple structural feature(s) of the graph are known.

7. Concluding Remarks

A considerable amount of work has gone into the study of matrices with complex valued entries (e.g. [12, 17, 5]). It is quite possible that there are many more results in this area, besides those contained in section 3, which can be extended to the class of matrices \( W_{n \times n} \). However, such results are not the focus of this paper.

The main results of this paper demonstrate that isospectral graph reductions can be used to improve each of the classical eigenvalue estimates of Gershgorin, Brauer, and Brualdi. Moreover, these graph reductions are general enough that this process can be applied to almost any graph \( G \in \mathbb{G}^n \) and in particular to any graph with complex valued weights. Hence, the eigenvalue estimates for most matrices in \( W_{n \times n} \) and all matrices in \( \mathbb{C}^{n \times n} \) can be improved via our isospectral reduction process. Additionally, this process is sufficiently flexible to improve such eigenvalue estimates to whatever degree is desired.

Furthermore, graph reductions computationally do not require much effort especially if some particular structural feature of the graph is known. In fact, the number of calculations required by such estimates may even be reduced by our procedure since nontrivial reductions typically produce fewer regions used to estimate the graph’s spectrum. These properties indicate the potential usefulness of graph reductions with respect to applications and particularly to dynamical networks.
Moreover, this paper also raises new questions related to graph reductions and eigenvalue estimates. For instance, what algorithms related to choosing structural sets and sequences of structural sets can be developed to improve the speed or accuracy of such estimates.

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