An improved bound for the Gaussian concentration inequality

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Abstract

In this note, we provide an improved bound for the Gaussian concentration inequality. Our proof is short and bases on the integration-by-parts formula for the gaussian measure.

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1 Introduction

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a Lipschitz function with Lipschitz constant \( K \) and \( Y = (Y_1, ..., Y_n) \) be an \( n \)-dimensional standard Gaussian random vector. The Gaussian concentration inequality states that, for any \( x \geq 0 \),

\[
P ( f(Y) - E[f(Y)] \geq x ) \leq e^{-\frac{x^2}{2K^2}}. \tag{1.1}
\]

This is one of the most fundamental inequalities in the theory of Gaussian processes and its proof can be found in many textbooks, see e.g. Appendix A in [2]. We also refer the reader to [4] and the references therein for recent results. Let us observe that if \( f(Y) \) is a linear combination of \( Y_1, ..., Y_n \), then \( \text{Var}(f(Y)) = K^2 \). In this case, we can rewrite (1.1) as follows

\[
P ( f(Y) - E[f(Y)] \geq x ) \leq e^{-\frac{x^2}{2\text{Var}(f(Y))}}, \quad x \geq 0. \tag{1.2}
\]

In general case, by the Poincaré inequality, we have \( \text{Var}(f(Y)) \leq K^2 \). This relation implies

\[
e^{-\frac{x^2}{4K^2}} \leq e^{-\frac{x^2}{2K^2}}.
\]

So an interesting question arising here is that under which conditions on \( f \), the bound (1.1) can be improved to (1.2). Of course, from practical point of view, these condition should be simple and easy to check. Motivated by this question, the aim of the present paper is to prove the following theorem.

**Theorem 1.1.** Let \( Y = (Y_1, ..., Y_n) \) be an \( n \)-dimensional standard Gaussian random vector. Suppose that the function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is twice differentiable and its derivatives have subexponential growth at infinity. In addition, we assume that

(i) \( E[e^{\lambda f(Y)}] < \infty \) for all \( \lambda \geq 0 \),

(ii) \( \frac{\partial f}{\partial y_i}(x) \frac{\partial f}{\partial y_j}(y) \frac{\partial^2 f}{\partial y_i \partial y_j}(z) \leq 0 \) for all \( x, y, z \in \mathbb{R}^n \) and \( 1 \leq i, j \leq n \).

Then, the concentration inequality (1.2) holds true.

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The proof of Theorem above will be given in the next Section. Let us end up this Section by giving some remarks and examples.

Remark 1.1. Given a standard Gaussian random variable $Z$, we always have $E[e^{\lambda |Z|^\beta}] < \infty$ for all $\lambda \geq 0$ and $\beta \in [0, 2)$. Hence, by the independence, the condition (i) is fulfilled if

$$|f(Y)| \leq M(1 + \sum_{j=1}^m \frac{\partial f}{\partial y_j} Y_j^{\beta_j})^n,$$

where $\beta_j \in [0, 2), 1 \leq i \leq n, 1 \leq j \leq m$. In particular, if $f$ is Lipschitz then the condition (i) is itself satisfied.

Remark 1.2. The condition (ii) is easy to check. For example, it is satisfied if $\frac{\partial f}{\partial y_i} \geq 0$ (or $\frac{\partial f}{\partial y_i} \leq 0$) and $\frac{\partial^2 f}{\partial y_i \partial y_j} \leq 0$ for all $1 \leq i, j \leq n$. In one-dimensional, the condition (ii) means that $f$ is a monotonic function with $f'' \leq 0$.

Example 1.1. Let $\sigma : \mathbb{R} \to \mathbb{R}_+$ be a continuously differentiable function with $\|\sigma\|_\infty := \sup_{x \in \mathbb{R}} |\sigma(x)| \in (0, +\infty)$, $\sigma'$ has subexponential growth at infinity and $\sigma'(x) \leq 0$ for all $x \in \mathbb{R}$. Define the function

$$f(x) := \int_{0}^{x} \sigma(z)dz, \ x \in \mathbb{R}.$$

Let $Z$ be a standard Gaussian random variable. Since $f$ has the Lipschitz constant $\|\sigma\|_\infty$, the inequality (1.1) gives us

$$P(f(Z) - E[f(Z)] \geq x) \leq \exp(-\frac{x^2}{2\|\sigma\|_\infty^2}), \ x \geq 0. \tag{1.3}$$

On the other hand, we have

$$\text{Var}(f(Z)) = E[f(Z)^2] - (E[f(Z)])^2 \leq \|\sigma\|_\infty^2 E[Z^2] - E[f(Z)]^2 = \|\sigma\|_\infty^2 - E[f(Z)]^2.$$

Since all conditions of Theorem 1.1 are fulfilled, we obtain from (1.2) that

$$P(f(Z) - E[f(Z)] \geq x) \leq \exp\left(-\frac{x^2}{2\|\sigma\|_\infty^2 - 2E[f(Z)]^2}\right), \ x \geq 0. \tag{1.4}$$

Clearly, our bound (1.4) significantly improves (1.3) whenever $E[f(Z)] \neq 0$. In addition, the bound (1.4) points out that the mean of the random variable of interest plays an important role in Gaussian concentration inequalities for Lipschitz functions.

## 2 Proofs

The proof of Theorem 1.1 is pretty simple and bases on the following technical lemma.

**Lemma 2.1.** Let $Y = (Y_1, \ldots, Y_n)$ be an $n$-dimensional standard Gaussian random vector. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be absolutely continuous functions such that $E[e^{\lambda f(Y)}] < \infty$ for all $\lambda \geq 0$, and $E[g(Y)] = 0$. Assume that $g$ and the partial derivatives of $f$ and $g$ have subexponential growth at infinity. Define the function

$$T(y) := \int_{0}^{1} \frac{1}{2\sqrt{t}} E\left[\frac{\partial f}{\partial y_i}(y) \frac{\partial g}{\partial y_i}(\sqrt{t}y + \sqrt{1-t}Y')\right] dt, \ y \in \mathbb{R}^n,$$

where $Y'$ is an independent copy of $Y$. Then, it holds that

$$E[e^{\lambda f(Y)}g(Y)] = \lambda E[e^{\lambda f(Y)}T(Y)].$$
Proof. This lemma is not new, its proof is similar to that of Lemma 5.3 in [1]. Note that the sub-exponential growth condition ensures the existence of all expectations below. We have

\[
E[e^{\lambda f(Y)} g(Y)] = E[e^{\lambda f(Y)} g(Y) - e^{\lambda f(Y)} g(Y')]
\]
\[
= E \left[ \int_0^1 e^{\lambda f(Y)} \frac{d}{dt} g(\sqrt{t} Y + \sqrt{1 - t} Y') dt \right]
\]
\[
= E \left[ \int_0^1 e^{\lambda f(Y)} \sum_{i=1}^n \left( \frac{Y_i}{2\sqrt{t}} - \frac{Y_i'}{2\sqrt{1 - t}} \right) \frac{\partial g}{\partial y_i}(\sqrt{t} Y + \sqrt{1 - t} Y') dt \right].
\]

Let \( U_t = \sqrt{t} Y + \sqrt{1 - t} Y' \) and \( V_t = \sqrt{1 - t} Y - \sqrt{t} Y' \). Then \( U_t \) and \( V_t \) are independent standard gaussian random vectors. We obtain

\[
E[e^{\lambda f(Y)} g(Y)] = E \left[ \int_0^1 \frac{1}{2\sqrt{t}(1 - t)} e^{\lambda f(\sqrt{t} U_t + \sqrt{1 - t} V_t)} \sum_{i=1}^n V_t, \frac{\partial g}{\partial y_i}(U_t) dt \right].
\]

For each \( 1 \leq i \leq n \), by using Stein’s identity (see, e.g. Appendix A.6 in [3]), we obtain

\[
E \left[ e^{\lambda f(\sqrt{t} U_t + \sqrt{1 - t} V_t)} V_t, \frac{\partial g}{\partial y_i}(U_t) \right] = E \left[ E \left[ e^{\lambda f(\sqrt{t} u_t + \sqrt{1 - t} v_t)} V_t, \frac{\partial g}{\partial y_i}(u_t) \right]_{u_t = U_t} \right]
\]
\[
= E \left[ \partial e^{\lambda f(\sqrt{t} u_t + \sqrt{1 - t} v_t)} \frac{\partial g}{\partial y_i}(U_t) \right]_{u_t = U_t}
\]
\[
= \lambda \sqrt{1 - t} E \left[ e^{\lambda f(\sqrt{t} U_t + \sqrt{1 - t} V_t)} \frac{\partial f}{\partial y_i}(\sqrt{t} U_t + \sqrt{1 - t} V_t) \frac{\partial g}{\partial y_i}(U_t) \right]
\]
\[
= \lambda \sqrt{1 - t} E \left[ e^{\lambda f(Y)} \frac{\partial f}{\partial y_i}(Y) \frac{\partial g}{\partial y_i}(\sqrt{t} Y + \sqrt{1 - t} Y') \right].
\]

So we can conclude that

\[
E[e^{\lambda f(Y)} g(Y)] = \lambda E \left[ \int_0^1 \frac{1}{2\sqrt{t}} e^{\lambda f(Y)} \sum_{i=1}^n \frac{\partial f}{\partial y_i}(Y) \frac{\partial g}{\partial y_i}(\sqrt{t} Y + \sqrt{1 - t} Y') dt \right].
\]

This completes the proof.

Proof of Theorem 1.1. The condition (i) allows us to define the function

\[
\varphi(\lambda) = E[e^{\lambda f(Y)} - E[f(Y)]], \quad \lambda \geq 0.
\]

Moreover, we have

\[
\varphi'(\lambda) = E[e^{\lambda f(Y)} - E[f(Y)]](f(Y) - E[f(Y)]), \quad \lambda \geq 0.
\]

Applying Lemma 2.4 to \( g(Y) = f(Y) - E[f(Y)] \) yields

\[
\varphi'(\lambda) = \lambda E[e^{\lambda f(Y)} - E[f(Y)]] T(Y), \quad \lambda \geq 0,
\]

where

\[
T(y) := \int_0^1 \frac{1}{2\sqrt{t}} E \left[ \sum_{i=1}^n \frac{\partial f}{\partial y_i}(y) \frac{\partial f}{\partial y_i}(\sqrt{t} y + \sqrt{1 - t} Y') \right] dt.
\]

It is known from the proof of Lemma 5.3 in [1] that \( E[T(Y)] = Var(f(Y)) \). Hence, we can apply Lemma 2.4 to \( g(Y) = T(Y) - Var(f(Y)) \) and we obtain

\[
\varphi'(\lambda) = \lambda Var(f(Y)) \varphi(\lambda) + \lambda E[e^{\lambda f(Y)} - E[f(Y)]](T(Y) - Var(f(Y))))
\]
\[
= \lambda Var(f(Y)) \varphi(\lambda) + \lambda^2 E[e^{\lambda f(Y)} - E[f(Y)]] T(Y'], \quad \lambda \geq 0, \tag{2.1}
\]
where, for some independent copy $Y''$ of $Y$,

$$
\tilde{T}(y) := \int_0^1 \frac{1}{2\sqrt{s}} \mathbb{E} \left[ \sum_{j=1}^n \frac{\partial f}{\partial y_j}(y) \frac{\partial T}{\partial y_j}(\sqrt{sy} + \sqrt{1-sY''}) \right] \, ds.
$$

For each $1 \leq j \leq n$, we have

$$
\frac{\partial T}{\partial y_j}(y) = \int_0^1 \frac{1}{2\sqrt{t}} \mathbb{E} \left[ \sum_{i=1}^n \left( \frac{\partial^2 f}{\partial y_j \partial y_i}(y) \frac{\partial f}{\partial y_i}(\sqrt{ty} + \sqrt{1-tY''}) + \sqrt{t} \frac{\partial f}{\partial y_i}(y) \frac{\partial^2 f}{\partial y_j \partial y_i}(\sqrt{ty} + \sqrt{1-tY''}) \right) \right] \, dt.
$$

So the condition (ii) implies $\tilde{T}(Y) \leq 0$ a.s. This, combined with (2.1), gives us

$$
\varphi'(\lambda) \leq \lambda \text{Var}(f(Y)) \varphi(\lambda), \quad \lambda \geq 0.
$$

Since $\varphi(0) = 1$, we obtain

$$
\mathbb{E}[e^{\lambda(f(Y) - \mathbb{E}[f(Y)])}] = \varphi(\lambda) \leq e^{\lambda \text{Var}(f(Y)) \frac{\lambda^2}{2}}, \quad \lambda \geq 0.
$$

This, together with Markov’s inequality, gives us

$$
P \left( f(Y) - \mathbb{E}[f(Y)] \geq x \right) = P \left( e^{\lambda(f(Y) - \mathbb{E}[f(Y)])} \geq e^{\lambda x} \right) \leq e^{\lambda \text{Var}(f(Y)) \frac{\lambda^2}{2} - \lambda x}, \quad \lambda \geq 0, x \in \mathbb{R}.
$$

When $x \geq 0$, the function $\lambda \to e^{\lambda \text{Var}(f(Y)) \frac{\lambda^2}{2} - \lambda x}$ attains its minimum value at $\lambda_0 := x/\text{Var}(f(Y))$.

Choosing $\lambda = \lambda_0$, we get

$$
P \left( f(Y) - \mathbb{E}[f(Y)] \geq x \right) \leq e^{-\frac{x^2}{2 \text{Var}(f(Y))}}, \quad x \geq 0.
$$

Thus the proof of Theorem 1.1 is complete.

References

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