STOCHASTIC PROCESSES UNDER PARAMETER UNCERTAINTY

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ABSTRACT. In this paper we study a family of nonlinear (conditional) expectations that can be understood as a stochastic process with uncertain parameters. We develop a general framework which can be seen as a version of the martingale problem method of Stroock and Varadhan with parameter uncertainty. To illustrate our methodology, we explain how it can be used to model nonlinear Lévy processes in the sense of Neufeld and Nutz, and we introduce the new class of stochastic partial differential equations under parameter uncertainty. Moreover, we study properties of the nonlinear expectations. We prove the dynamic programming principle, i.e., the tower property, and we establish conditions for the (strong) USC\_b–Feller property and a strong Markov selection principle.

1. Introduction

We study conditional nonlinear expectations on the path space $\Omega$ of continuous or càdlàg functions from $\mathbb{R}_+$ into a Polish space $F$. More specifically, we are interested in nonlinear expectations of the form

\[ \mathcal{E}_t(\psi)(\omega) = \sup_{P \in \mathcal{C}(t, \omega)} E^P[\psi], \]

where $\psi$ is a suitable real-valued function on $\Omega$ and $\mathcal{C}(t, \omega)$ is a set of laws of stochastic processes whose paths coincide with $\omega \in \Omega$ till time $t \in \mathbb{R}_+$. We think of the nonlinear conditional expectation $\mathcal{E}$ as a nonlinear stochastic process or a stochastic process under parameter uncertainty.

The systematic study of nonlinear stochastic processes started with the seminal work of Peng [45, 46] on the $G$-Brownian motion. More recently, larger classes of nonlinear semimartingales have been investigated in [8, 9, 10, 20, 24, 25, 32, 39]. At this point we also highlight the articles [17, 38, 42] where abstract tools for the study were developed. Next to the measure theoretic approach based on the nonlinear expectation (1.1), nonlinear Markov processes have also been constructed by analytic methods via nonlinear semigroups, see [15, 36, 37]. This construction is based on fundamental ideas of Nisio [41]. In the recent paper [33], it was shown that the two approaches are equivalent for the class of nonlinear Lévy processes from [15, 39]. A key property of the construction via (1.1) is its relation to a class of stochastic processes given through the set $\mathcal{C}$. This connection opens the door to analyze properties of $\mathcal{E}$ with powerful tools from probability theory such as stochastic calculus.

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In this paper, we propose a new framework for the set $C$ which extends the nonlinear expectation approach to nonlinear stochastic processes beyond semimartingales. The framework is inspired by the martingale problem method of Stroock and Varadhan [53]. Consider the family

$$f(X) - \int_0^t g(X_s)ds, \quad (f, g) \in A \subset C_b(F; \mathbb{R}) \times C_b(F; \mathbb{R}),$$

of test processes, where $X$ is the coordinate process on $\Omega$. The set $A$ of test functions is often called a pregenerator. A probability measure $P$ is said to be a solution to the martingale problem associated to $A$ if all test processes in (1.2) are $P$-martingales. Our idea is to incorporate uncertainty to the pregenerator $A$.

Let $U$ be a countable index set and let $\{Y^u; u \in U\}$ be a family of càdlàg processes on $\Omega$, which serve as test processes for the martingale problem under uncertainty. If $P$ is a probability measure such that $Y^u$ is a special $P$-semimartingale, then there exists a $P$-unique predictable càdlàg process $A^P(Y^u)$ of finite variation, starting in zero, such that $Y^u - Y^u_0 - A^P(Y^u)$ is a $P$-local martingale. In case the process in (1.2) is a (local) $P$-martingale, the process $f(X)$ is a special semimartingale and $A^P(f(X)) = \int_0^t g(X_s)ds$.

Motivated by this observation, we define $C$ to be the set of all probability measures $P$ under that each test process $Y^u$ is a special $P$-semimartingale with absolutely continuous compensator $A^P(Y^u)$ such that $(\lambda \otimes P)$-a.e.

$$\left(\frac{dA^P(Y^u)}{d\lambda}\right)_{u \in U} \in \Theta.$$

Here, $\Theta$ is a time and path-dependent set-valued mapping on $\mathbb{R}_+ \times \Omega$, which captures the uncertainty.

Our framework is tailor made for adding uncertainty to any class of stochastic processes which can be characterized by a martingale problem. For instance, this is the case for semimartingales (with absolutely continuous characteristics), solutions to stochastic partial differential equations and many Markov processes. In particular, it is possible to recover nonlinear semimartingale settings which are built from suitably parameterized absolutely continuous semimartingale characteristics. To illustrate this, we explain how nonlinear Lévy processes, in the sense of [39], can be modeled with our framework. Further, we introduce a novel class of nonlinear Markov processes which are not necessarily semimartingales: the class of nonlinear (in the sense of uncertainty) stochastic partial differential equations (NSPDE). We think this class is of specific interest and deserve further investigation.

The first main result in this paper is the dynamic programming principle (DPP) for the nonlinear expectation (1.1), i.e., the tower property

$$\mathcal{E}_s(\mathcal{E}_t(\psi))(\omega) = \mathcal{E}_s(\psi)(\omega), \quad \omega \in \Omega, \quad s \leq t.$$

In case the uncertainty set $\Theta$ has a Markovian structure in the sense that $\Theta(t, \omega)$ depends on $(t, \omega)$ only through the value $\omega(t-)$, the DPP induces a nonlinear Markov property given by

$$\mathcal{E}^x(\mathcal{E}^{X_t}(\psi(X_s))) = \mathcal{E}^x(\psi(X_{t+s})), \quad \mathcal{E}^x(\psi) \triangleq \mathcal{E}_0(\psi)(\omega)\big|_{\omega(0)=x}.$$

As in the theory of (linear) Markov processes, there is a strong link to semigroups. Indeed, in this Markovian case, the nonlinear Markov property ensures the semigroup property $T_tT_s = T_{t+s}$ of the family $(T_t)_{t \geq 0}$ defined by

$$T_t(\psi)(x) \triangleq \mathcal{E}^x(\psi(X_t)).$$
where \( \psi \) runs through the class of bounded upper semianalytic functions. It is an important question when the semigroup \( (T_t)_{t \geq 0} \) preserves some regularity. For a continuous path setting, we provide general conditions for the \( \text{USC}_b \)-Feller property of the semigroup \( (T_t)_{t \geq 0} \), which means that it is a sublinear Markovian semigroup on the space of bounded upper semicontinuous functions. Further, we establish a strong Markov selection principle, i.e., we prove that, for every bounded upper semicontinuous function \( \psi : F \to \mathbb{R} \) and any time \( t > 0 \), there exists a (time inhomogeneous) strong Markov family \( \{P_{(s,x)} : (s,x) \in \mathbb{R}_+ \times F\} \) such that \( P_{0,x} \in C(0,x) \) and

\[
T_t(\psi)(x) = E^{P_{(0,x)}}[\psi(X_t)].
\]

We stress that the nonlinear structure of our setting is reflected by the fact that the strong Markov selection \( \{P_{(s,x)} : (s,x) \in \mathbb{R}_+ \times F\} \) depends on the input elements \( \psi \) and \( t \).

Thereafter, we investigate the Markovian class of NSPDEs in more detail. We establish continuity and linear growth conditions for the \( \text{USC}_b \)-Feller property and the strong Markov selection principle. Moreover, for the subclass of infinite-dimensional nonlinear Cauchy problems with drift uncertainty, we derive conditions for the strong \( \text{USC}_b \)-Feller property, which means that \( T_t(\text{USC}_b) \subset C_b \) for all \( t > 0 \). The strong \( \text{USC}_b \)-Feller property can be seen as a smoothing property of the sublinear semigroup \( (T_t)_{t \geq 0} \). To the best of our knowledge, such an effect was first discovered in [8] for nonlinear one-dimensional diffusions. We also emphasize that the strong \( \text{USC}_b \)-Feller property entails the \( C_b \)-Feller property, i.e., \( T_t(C_b) \subset C_b \) for all \( t \in \mathbb{R}_+ \). This property is considered to be of fundamental importance for the study of sublinear Markovian semigroups via pointwise generators, see [9, Section 2.4] for some comments.

Let us now comment on some technical aspects of our proofs. To establish the DPP, we invoke a general theorem from [17] which requires that we check a measurable graph condition and stability under conditioning and concatenation. Hereby, we benefit from technical ideas developed in [8, 39]. In the context of classical optimal control, the relation of a value function and a nonlinear semigroup was discovered in [40]. For a nonlinear Markov process framework, the Markov property (1.3) appeared, to the best of our knowledge, for the first time in the thesis [24] for nonlinear Markovian semimartingales. We adapt the proof from [24] to our setting. For the \( \text{USC}_b \)-Feller property, we rely on the theory of correspondences and, for the strong Markov selection principle, we invoke a technique from [16, 22] for controlled diffusions, which is based on ideas of Krylov [31] and Stroock and Varadhan [53] about Markovian selection. Again, we also benefit from technical ideas developed in [9] for one-dimensional nonlinear diffusions. The strong \( \text{USC}_b \)-Feller property for nonlinear Cauchy problems is proved by means of a strong Feller selection principle, which adapts ideas from [9, 10, 53] for finite-dimensional (nonlinear) diffusions to an infinite-dimensional non-semimartingale setting.

We end this section with a summary of the structure of this paper. In Section 2, we introduce our framework and formulate the DPP. Thereafter, in Section 3, we comment on nonlinear Lévy processes and we introduce the new class of NSPDEs. We discuss nonlinear Markov processes in Section 4. In particular, in Section 4.2, we establish the \( \text{USC}_b \)-Feller property and, in Section 4.3, we provide the strong Markov selection principle. In Section 5, these results are tailored to the special case of NSPDEs and the strong \( \text{USC}_b \)-Feller property for nonlinear Cauchy problems with drift uncertainty is established. The remaining sections of this paper are devoted to the proofs of our results. The location of each proof is indicated before the statement of the respective result.
2. Nonlinear Processes and the Dynamic Programming Principle

2.1. The Ingredients. Let \( F \) be a Polish space and define \( \Omega \) to be either the space of all càdlàg functions \( \mathbb{R}_+ \to F \) or the space of all continuous functions \( \mathbb{R}_+ \to F \) endowed with the Skorokhod \( J_1 \) topology.\(^1\) The canonical process on \( \Omega \) is denoted by \( X_t(\omega) = \omega(t) \) for \( \omega \in \Omega \) and \( t \in \mathbb{R}_+ \). It is well-known that \( \mathcal{F} \triangleq \mathcal{B}(\Omega) = \sigma(X_t, t \geq 0) \). We define \( \mathcal{F} \triangleq (\mathcal{F}_t)_{t \geq 0} \) to be the canonical filtration generated by \( X \), i.e., \( \mathcal{F}_t \triangleq \sigma(X_s, s \leq t) \) for \( t \in \mathbb{R}_+ \). To lighten our notation, for two stopping times \( S \) and \( T \), we also define the stochastic interval

\[
[S, T] := \{ (t, \omega) \in \mathbb{R}_+ \times \Omega : S(\omega) \leq t < T(\omega) \}.
\]

The stochastic intervals \( [S, T], [S, T], [S, T] \) are defined accordingly. In particular, \( [0, \infty[ = \mathbb{R}_+ \times \Omega \). The set of all probability measures on \( (\Omega, \mathcal{F}) \) is denoted by \( \mathfrak{P}(\Omega) \) and endowed with the usual topology of convergence in distribution, i.e., the weak topology. The shift operator on \( \Omega \) is denoted by \( \vartheta = (\vartheta_t)_{t \geq 0} \), i.e., \( \vartheta_t(\omega) = \omega(\cdot + t) \) for all \( \omega \in \Omega \) and \( t \in \mathbb{R}_+ \).

Let \( K \) be either the real line \( \mathbb{R} \) or the complex plane \( \mathbb{C} \). Further, let \( U \) be a countable index set and, for every \( u \in U \), let \( Y_u \) be a \( K \)-valued càdlàg \( \mathcal{F} \)-adapted process on \( \Omega \) such that \( Y_{t+} = Y_{\cdot +} \circ \vartheta_t \) for all \( t \in \mathbb{R}_+ \). We endow \( K^U \) with the product Euclidean topology.

Finally, let \( \Theta : \mathbb{R}_+ \times \Omega \to K^U \) be a correspondence, i.e., a set-valued mapping.

**Standing Assumption 2.1.** The correspondence \( \Theta \) has a measurable graph, i.e., the graph

\[
\text{gr } \Theta = \{ (t, \omega, g) \in \mathbb{R}_+ \times \Omega \times K^U : g \in \Theta(t, \omega) \}
\]

is Borel.

Recall that a \( K \)-valued càdlàg adapted process \( Y = (Y_t)_{t \geq 0} \) is said to be a *special semimartingale* if \( Y = Y_0 + M + A \), where \( M \) is a local martingale and \( A \) is a predictable process of (locally) finite variation, both starting in zero. The decomposition is unique up to indistinguishability. A \( K \)-valued càdlàg process \( Y \) is called a *special semimartingale after a time* \( t^* \in \mathbb{R}_+ \) if the process \( Y_{t+} = (Y_{t+})_{t \geq 0} \) is a special semimartingale for the right-continuous natural filtration of \( X_{t+} = (X_{t+})_{t \geq 0} \). For \( P \in \mathfrak{P}(\Omega) \), the set of special semimartingales after time \( t^* \) on the probability space \( (\Omega, \mathcal{F}, P) \) is denoted by \( \mathcal{S}_{np}(t^*, P) \). We also write \( \mathcal{S}_{np}(P) \triangleq \mathcal{S}_{np}(0, P) \). For a given measure \( P \) and a given process \( Y \in \mathcal{S}_{np}(t^*, P) \), denote by \( \mathcal{A}_{sp}(Y_{t+}) \) the predictable part of (locally) finite variation in the semimartingale decomposition of \( Y_{t+} \), and denote by \( \mathcal{S}_{np}^{ac}(t^*, P) \) the set of all \( Y \in \mathcal{S}_{np}(t^*, P) \) such that \( \mathcal{A}_{sp}(Y_{t+}) \) is absolutely continuous w.r.t. to the Lebesgue measure \( \lambda \).

Again, we write \( \mathcal{S}_{nc}^{ac}(P) \triangleq \mathcal{S}_{nc}(0, P) \). For \( \omega, \omega' \in \Omega \) and \( t \in \mathbb{R}_+ \), we define the concatenation

\[
\omega \otimes_t \omega' \triangleq \omega I_{[0,t]} + (\omega(t) + \omega' - \omega'(t)) I_{[t,\infty)},
\]

and, finally, for \( (t, \omega) \in [0, \infty[ \), we define \( \mathcal{C}(t, \omega) \in \mathfrak{P}(\Omega) \) by

\[
\mathcal{C}(t, \omega) \triangleq \{ P \in \mathfrak{P}(\Omega) : P(\lambda(\omega)) = 1, \forall u \in U \ Y_u \in \mathcal{S}_{nc}^{sp}(t, P), (\lambda \otimes P) - \text{a.e. } (d\mathcal{A}_{sp}(Y_{t+})/d\lambda)_{u \in U} \in \Theta(\cdot + t, \omega \otimes_t X) \}.
\]

**Standing Assumption 2.2.** \( \mathcal{C}(t, \omega) \neq \emptyset \) for all \( (t, \omega) \in [0, \infty[ \).

**Remark 2.3.**

(i) For every \( (t, \omega) \in [0, \infty[ \), the set \( \mathcal{C}(t, \omega) \) depends only on the path \( \omega \) up to time \( t \).

\(^1\) When restricted to the space of continuous functions \( \mathbb{R}_+ \to F \), the Skorokhod \( J_1 \) topology coincides with the local uniform topology.
holds for and assume that there exists a controlled coefficient, this interpretation is made explicit by the following proposition.

## Proposition 2.4.

Let $G$ be a metrizable Souslin space and assume that there exists a predictable Carathéodory function $\mathfrak{g} = (\mathfrak{g}^u)_{u \in U} : G \times [0, \infty] \rightarrow \mathbb{R}$, i.e., $\mathfrak{g}$ is continuous in the first and predictable in the second variable, such that

$$
\Theta(t, \omega) = \{ \mathfrak{g}(g, t, \omega) : g \in G \}, \quad (t, \omega) \in [0, \infty].
$$

Here, we still assume that Standing Assumption 2.1 holds for $\Theta$, i.e., that $\text{gr} \Theta$ is measurable. For $t \in \mathbb{R}_+$, let $\mathcal{P}_t$ be the predictable $\sigma$-field on $[0, \infty]$ corresponding to the filtration generated by $X_{+t}$. Then, for every $(t, \omega) \in [0, \infty]$ and $P \in \mathcal{C}(t, \omega)$, there exists a $\mathcal{P}_t$-measurable function $g : [0, \infty] \rightarrow G$ such that, for all $u \in U$, the process

$$
Y^u_{t+} - \mathfrak{g}(g(s, X), s + t, \omega \otimes t, X) ds
d$$

is a $P$-local martingale for the right-continuous natural filtration of $X_{+t}$.

**Proof.** For $(s, \alpha) \in [0, \infty]$, we set $\alpha^s = \alpha I_{[0, s)} + \alpha(s-)I_{s, \infty}$ and

$$
H = \{(s, \alpha) \in [0, \infty] : (s + t, \omega \otimes t, \alpha^{(s+t)-}, (dA^P_s(Y^u_{t+})/d\lambda)_{u \in U}) \not\subseteq \text{gr} \Theta \}.
$$

By Standing Assumption 2.1, the graph $\text{gr} \Theta$ is measurable and therefore, $H \in \mathcal{B}([0, \infty])$. Further, we get from [14, Theorem IV.97] that $H \in \mathcal{P}_t$, that the function $(g, s, \alpha) \rightarrow$ 

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2That is a metrizable space that is the continuous image of a Polish space. Any analytic subspace of a Polish space is a Souslin space.
\[ \mathcal{g}(g, s + t, \omega \otimes_t \alpha) \text{ is continuous in the } G\text{-variable and is } \mathcal{P}_t\text{-measurable in the } [0, \infty[\text{-variable and that} \]

\[ H = \left\{ (s, \alpha) \in [0, \infty[ : (s + t, \omega \otimes_t \alpha, (dA^P_s(Y^u_{s+t})(\alpha)/d\lambda)_{u \in U}) \notin \text{ gr } \Theta \right\}. \]

For some arbitrary, but fixed, \( g_0 \in G \), we set

\[ \pi(s, \alpha) \triangleq \begin{cases} \mathcal{g}(g_0, s + t, \omega \otimes_t \alpha), & \text{if } (s, \alpha) \in H, \\ (dA^P_s(Y^u_{s+t})(\alpha)/d\lambda)_{u \in U}, & \text{if } (s, \alpha) \notin H, \end{cases} \]

which is a \( \mathcal{P}_t\text{-measurable map such that } \pi(s, \alpha) \in \Theta(s + t, \omega \otimes_t \alpha) \text{ for all } (s, \alpha) \in [0, \infty[. \]

As \( \lambda \otimes P \) is \( \sigma \)-finite on \( ([0, \infty[ , \mathcal{P}_t) \) and \( G \) is a metrizable Souslin space, the measurable implicit function theorem [23, Theorem 7.2] yields the existence of a \( \mathcal{P}_t\text{-measurable function } \mathcal{g} : [0, \infty[ \to G \) such that \( \pi(s, \alpha) = \mathcal{g}(g, s, \alpha), s + t, \omega \otimes_t \alpha) \) for \( (\lambda \otimes P)\text{-a.a. } (s, \alpha) \in [0, \infty[. \]

As \( P \in \mathcal{C}(t, \omega) \), by virtue of (2.2), we have \( (\lambda \otimes P)\text{-a.e. } \pi = (dA^P_s(Y^u_{s+t})/d\lambda)_{u \in U} \) and the claim follows. \( \square \)

For general martingale problems, the pregenerator \( A \) is not necessarily a countable set.

In the diffusion-type setting of Stroock and Varadhan [53], for example, the pregenerator \( A \) is given by

\[ A = \left\{ (f, g) : f \in C^2_\beta(\mathbb{R}^d; \mathbb{R}), \ g = \langle b, \nabla f \rangle + \frac{1}{2} \text{tr } [\sigma \sigma^* \nabla^2 f] \right\}, \]

where \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times r} \) are Borel measurable coefficients. In the definition of the set \( \mathcal{C} \) we take only countably many test functions into account. In typical situations this is no restriction, as it is often possible to pass to a countable subset of the pregenerator that determines the same martingale problem. For example, in the diffusion-type setting explained above, it is sufficient to consider the set

\[ A = \left\{ (f, g) : f(x) = x_i^k x_j^k, \ k = 0, 1, i, j = 1, \ldots, d, \ g = \langle b, \nabla f \rangle + \frac{1}{2} \text{tr } [\sigma \sigma^* \nabla^2 f] \right\}, \]

cf. [30, Proposition 5.4.6]. We also refer to [19, Proposition 4.3.1] for an abstract result in this direction. In Section 3.1 below, we will also discuss a way to characterize the class of nonlinear Lévy processes as introduced in [39] with a countable pregenerator under parameter uncertainty.

Let us also comment on the question to what extend the pregenerator has to determine the martingale problem uniquely. In the classical theory on Feller–Dynkin processes, the (pre)generators characterize the law of the processes in a unique manner, cf., for example, [34, Theorem 3.33]. In the classical theory on martingale problems (as in [19, 53], for instance), such a uniqueness property is an important question rather than intrinsically given. In particular, there is interest in martingale problems that are not well-posed (i.e., do not have unique solutions for each deterministic initial value). Indeed, by the strong Markov selection principle (see Chapter 12 in [53]), many non-well-posed martingale problems give rise to strong Markov families which can be selected from the set of solutions. Our approach is fully independent of a uniqueness property, i.e., we can also introduce uncertainty to martingale problems that have infinitely many solutions (by convexity, a martingale problem has none, one or infinitely many solutions). In fact, considering martingale problems with infinitely many solutions can lead to interesting effects. We explain an example of such an effect in Remark 4.7 below.
2.2. Dynamic Programming Principle. We now define a nonlinear conditional expec-
tation and we provide the corresponding dynamic programming principle (DPP), i.e., the
tower property of the nonlinear conditional expectation. Suppose that $\psi: \Omega \to [-\infty, \infty]$ is
an upper semianalytic function, i.e., the set $\{\omega \in \Omega: \psi(\omega) > c\}$ is analytic for every $c \in \mathbb{R}$, and define the value function $v$ by

$$v(t, \omega) \triangleq \sup_{P \in \mathcal{C}(t, \omega)} E^P[\psi], \quad (t, \omega) \in [0, \infty].$$

The following theorem is proved in Section 6.

**Theorem 2.5 (Dynamic Programming Principle).** The value function $v$ is upper semian-
alytic. Moreover, for every $(t, \omega) \in [0, \infty]$ and every stopping time $\tau$ with $t \leq \tau < \infty$, we have

$$v(t, \omega) = \sup_{P \in \mathcal{C}(t, \omega)} E^P[v(\tau, X)]. \quad (2.3)$$

The value function $v$ induces a nonlinear expectation $\mathcal{E}$ via the formula

$$\mathcal{E}_t(\psi)(\omega) \triangleq v(t, \omega), \quad (t, \omega) \in [0, \infty],$$

and the DPP provides its tower property, i.e., (2.3) means that

$$\mathcal{E}_t(\psi) = \mathcal{E}_t(\mathcal{E}_\tau(\psi)), \quad t \leq \tau \leq \infty.$$ 

By the pathwise structure of this property, it follows also immediately that

$$\mathcal{E}_\rho(\psi) = \mathcal{E}_\rho(\mathcal{E}_\tau(\psi))$$

for all finite stopping times $0 \leq \rho \leq \tau < \infty$.

For $x \in F$, we define

$$\mathcal{R}(x) \triangleq \{P \in \mathcal{P}(\Omega): P(X_0 = x) = 1, \forall u \in U \ Y^u \in S^p_u(P), \ (\lambda \otimes P)\text{-a.e.} \ (dA^P(Y^u)/d\lambda)_{u \in U} \in \Theta\},$$

and

$$\mathcal{E}^x(\psi) \triangleq \sup_{P \in \mathcal{R}(x)} E^P[\psi].$$

Thinking of classical martingale problems, we interpret $\{\mathcal{E}^x: x \in F\}$ as a nonlinear sto-
chastic process. In Section 4, we specify our setting to a Markovian situation and we
establish more properties of $\{\mathcal{E}^x: x \in F\}$ to justify our interpretation. Before we start this
program, let us discuss some important examples.

3. Examples

In this section we explain how our framework can be used to model nonlinear Lévy
processes in the sense of [39] and we introduce some new classes of nonlinear processes,
namely the class of nonlinear Markov chains and the class of nonlinear (in the sense of
uncertainty) stochastic partial differential equations (NSPDEs).
3.1. Nonlinear Lévy processes. Nonlinear Lévy processes have been introduced in [39] via an uncertain set of Lévy–Khinchine triplets. We now explain how such nonlinear processes can be constructed in our framework. Let us emphasis that the discussion can be extended to more general classes of semimartingales under parameter uncertainty.

To fix the basic setting, we consider \( F \triangleq \mathbb{R}^d, K \triangleq \mathbb{C}, U \triangleq \mathbb{Q}^d \) and we take \( \Omega \) to be the space of all càdlàg functions \( \mathbb{R}_+ \to F = \mathbb{R}^d \). Let \( S^d \) be the space of symmetric non-negative definite real-valued \( d \times d \) matrices. Further, let \( L \) be the space of all Lévy measure on \( \mathbb{R}^d \), i.e., the space of all measures \( K \) on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) such that

\[
K(\{0\}) = 0 \quad \text{and} \quad \int (1 \land \|x\|^2)K(dx) < \infty.
\]

As in [38, 39], we endow \( L \) with the weakest topology under which all maps

\[
K \mapsto \int f(x)(1 \land \|x\|^2)K(dx), \quad f \in C_b(\mathbb{R}^d; \mathbb{R}),
\]

are continuous. By Lemma A.1 below, the space \( L \) is Polish. Let \( \tilde{\Theta} \subset \mathbb{R}^d \times S^d \times L \) be an analytic subset, where the product space is endowed with the product topology (and \( S^d \) and \( \mathbb{R}^d \) are endowed with their usual topologies), such that

\[
sup_{(b, c, K) \in \tilde{\Theta}} \left( \|b\| + \|c\| + \int (1 \land \|x\|^2)K(dx) \right) < \infty.
\]

For every \( u \in \mathbb{Q}^d \), we set \( Y^u \triangleq e^{i(u, x)} \) and we define

\[
A^u(b, c, K) \triangleq -i(u, b) + \frac{1}{2}(u, cu) - \int (e^{i(u, x)} - 1 - i(u, h(x)))K(dx),
\]

and

\[
\Theta(t, \omega) \triangleq \{(e^{i(u, \omega(t-))}A^u(b, c, K))_{u \in \mathbb{Q}^d} : (b, c, K) \in \tilde{\Theta}\}, \quad (t, \omega) \in [0, \infty[,
\]

where \( h : \mathbb{R}^d \to \mathbb{R}^d \) is a fixed continuous truncation function and \( i \) denotes the imaginary number. We also assume that Standing Assumption 2.1 holds, which is for instance the case when \( \tilde{\Theta} \) is compact (by [8, Lemma 2.12]).

We now relate this setting to those from [39]. First, for a given \( P \in \mathcal{R}(x_0) \), we show that the coordinate process \( X \) is an \( \mathbb{R}^d \)-valued \( P \)-semimartingale whose characteristics \((B^P, C^P, \nu^P)\) are absolutely continuous with densities \((b^P, c^P, K^P)\) such that \((\lambda \otimes P)\)-a.e. \((b^P, c^P, K^P) \in \tilde{\Theta}\). Take \( P \in \mathcal{R}(x_0) \). For every \( k \in \{1, \ldots, d\} \) and \( n \in \mathbb{N} \), the process \( \sin(X^{(k)}/n) \) is a \( P \)-semimartingale. Here, \( X^{(k)} \) denotes the \( k \)-th coordinate of \( X \). There exists a function \( f \in C^2(\mathbb{R}; \mathbb{R}) \) such that \( f(\sin(x)) = x \) for all \( |x| \leq 1/2 \). Hence, for

\[
\tau_n \triangleq \inf\{t \geq 0 : |X_t^{(k)}| > n/2\},
\]

the process \( X^{(k)} = n f(\sin(X^{(k)}/n)) \) is a \( P \)-semimartingale on \([0, \tau_n]\). Consequently, by part c) of [28, Proposition I.4.25], it follows that \( X^{(k)} \) is a \( P \)-semimartingale. Of course, this means that \( X \) is an \( \mathbb{R}^d \)-valued \( P \)-semimartingale. For every \( u \in \mathbb{R}^d \), by Lemma A.2 below, the map

\[
\mathbb{R}^d \times S^d \times L \ni (b, c, K) \mapsto A^u(b, c, K) \in \mathbb{C}
\]
is continuous. Using this observation, we deduce from Proposition 2.4 that there exists a predictable function \( g = g(P) : [0, \infty) \to \Theta \) such that
\[
e^{i(u,x)} - e^{i(0,x_0)} - \int_0^t e^{i(u,x_s)} A^u(g(s,X))ds, \quad u \in \mathbb{Q}^d,
\]
are \( P \)-local martingales. For every \( u \in \mathbb{Q}^d \), we define
\[
\mathfrak{A}^P(u) \triangleq -i(u,B^P) + \frac{1}{2} \langle u, C^P u \rangle - \int \left( e^{i(u,x)} - 1 - i(u, h(x)) \right) \nu^P([0,\cdot] \times dx).
\]
By virtue of [28, Theorem II.2.42], we get that \( P \)-a.s.
\[
\mathfrak{A}^P(u) = \int_0^t A^u(g(s,X))ds, \quad u \in \mathbb{Q}^d.
\]
It is clear that \( P \)-a.s. the map \( u \mapsto \mathfrak{A}^P(u) \) is continuous for every \( t \in \mathbb{R}_+ \). Further, thanks to (3.1), the dominated convergence theorem implies that \( u \mapsto \int_0^t A^u(g(s,X))ds \) is continuous for every \( t \in \mathbb{R}_+ \). Hence, the equality in (3.2) even holds for all \( u \in \mathbb{R}^d \) and we may use the uniqueness lemma of Gnedenko and Kolmogorov ([28, Lemma II.2.44]) to conclude that \( P \)-a.s. \( (B^P, C^P, \nu^P) \ll \lambda \) with \( (\lambda \otimes P) \)-a.e. \( (b^P, c^P, F^P) = g(\cdot, X) \).

This completes the proof of the first direction.

Conversely, if \( P \) is the law of a semimartingale (for its canonical filtration), starting at \( x_0 \in \mathbb{R}^d \), with absolutely continuous characteristics whose densities \( (b^P, c^P, F^P) \) are such that \( (\lambda \otimes P) \)-a.e. \( (b^P, c^P, F^P) \in \Theta \), then [28, Theorem II.2.42] yields that \( P \in \mathcal{R}(x_0) \).

3.2. Nonlinear Markov Chains. A (continuous-time) Markov chain is a strong Markov process on some countable discrete state space \( F \). Typically, Markov chains are modeled via \( Q \)-matrices, which encode their infinitesimal dynamics. In particular, provided the Markov chain is a Feller–Dynkin process, the \( Q \)-matrix corresponds to the generator of the chain (see [50]). In the following we adapt the underlying martingale problem (see, e.g., [51, Theorem IV.20.6] or [54, Lemma 2.4]) to a nonlinear setting. For simplicity, we will restrict our attention to nonlinear Markov chains with a finite state space \( F \).

Let \( U \triangleq F \triangleq \{1, \ldots, N\} \) with \( N \in \mathbb{N} \), let \( \Omega \) be the space of all càdlàg functions \( \mathbb{R}_+ \to F \), and set \( Y^u = 1_{\{X = u\}} \) for \( u \in U \). Denote by \( M_q \) the set of all \( Q \)-matrices on \( F \), i.e., the set of all matrices \( Q = (q(k,n))_{k,n=1}^N \subseteq \mathbb{R}^{N \times N} \) such that \( q(k,n) \geq 0 \) for all \( k \neq n \) and \( Q \mathds{1} = 0 \).

Let \( G \) be a set, let \( Q = (q(k,n))_{k,n=1}^N : G \to M_q \) be a map and define, for \( (t,\omega) \in [0,\infty[, \Theta(t,\omega) \triangleq \{ (q(\omega(t^{-}),u)(g))_{u \in U} : g \in G \} \).

In case \( G \) is the singleton \( \{g_0\} \), then \( \mathcal{R}(x_0) \) is also a singleton whose only element is the law of the continuous-time Markov chain with starting value \( x_0 \) and \( Q \)-matrix \( Q(g_0) \), see, e.g., [54, Lemma 2.4]. By virtue of Proposition 2.4, this observation confirms our interpretation as a nonlinear Markov chain, i.e., a Markov chain with uncertain \( Q \)-matrix.

3.3. Nonlinear SPDEs. Let \((H_1, \| \cdot \|_{H_1}, \langle \cdot, \cdot \rangle_{H_1})\) and \((H_2, \| \cdot \|_{H_2}, \langle \cdot, \cdot \rangle_{H_2})\) be two separable Hilbert spaces over the real numbers, and let \((A, D(A))\) be the generator of a \( C_0 \)-semigroup on \( H_2 \). In the following we introduce drift and volatility uncertainty to the class of (semilinear) stochastic partial differential equations (SPDEs) of the type
\[
dY_t = AY_t dt + \mu(Y_t) dt + \sigma(Y_t) dW_t,
\]
where $W$ is a cylindrical Brownian motion over $H_1$ and $\mu$ and $\sigma$ are suitable measurable coefficients. To define our setting, we rely on the cylindrical martingale problem associated to semilinear SPDEs, see, e.g., [6, 11] for more details.

We denote the set of linear bounded operators from $H_1$ into $H_2$ by $L(H_1, H_2)$. Let $F \triangleq H_2$, let $\Omega$ be the space of all continuous functions $\mathbb{R}_+ \to F$, take a set $G$ and let $\mu: G \times H_2 \to H_2$ and $\sigma: G \times H_2 \to L(H_1, H_2)$ be functions. The adjoint of $(A, D(A))$ is denoted by $(A^*, D(A^*))$. By virtue of [7, Lemma 7.3], there exists a countable set $D \subset D(A^*) \subset H_2$ such that, for every $y \in D(A^*)$, there exists a sequence $(y_n)_{n=1}^\infty \subset D$ such that $y_n \to y$ and $A^*y_n \to A^*y$. In other words, $D$ is a countable subset of $D(A^*)$ which is dense in $D(A^*)$ for the graph norm. We set $f_i(x) \triangleq x^i$ for $x \in \mathbb{R}$ and $i = 1, 2$,

$$U \triangleq \{(H_2 \ni x : f_i((y, x)_{H_2}) : y \in D, i = 1, 2, x^i \triangleq u(X), u \in U,$$

and

$$\mathcal{L}^{\mu, \sigma}(x) \triangleq \langle (A^* y, x)_{H_2} + \langle y, \mu(g(x), x) \rangle_{H_2} \rangle_{H_2} + \frac{1}{2} \|\sigma^*(g(x), y\|_{H_2}^2, \langle y, x \rangle_{H_2},$$

for $(x, g) \in H_2 \times G$ and $u = f_i((y, \cdot)_{H_2}) \in U$. Finally, for $(t, \omega) \in [0, \infty]$, we define

$$\Theta(t, \omega) \triangleq \{(\mathcal{L}^{\mu, \sigma}(\omega(t))))_{u \in U} : g \in G\}.$$

In Lemma 11.2 below, we show that, under suitable assumptions on the coefficients and the parameter space $G$, any $P \in \mathcal{P}(x_0)$ is the law of a mild solution processes (see [13, Section 6.1]) to an SPDE of the form

$$dY_t = AY_t dt + \mu(g(t, Y), Y_t) dt + \sigma(g(t, Y), Y_t) dW_t, Y_0 = x_0,$$

where $g = g(P): [0, \infty] \to G$ is a predictable function. This observation confirms our interpretation as an SPDE with uncertain coefficients.

4. Nonlinear Markov Processes

In this section we investigate a class of nonlinear Markov processes. That is, we study the family $\{\mathcal{E}^x : x \in F\}$ under the following standing assumption.

**Standing Assumption 4.1.** For every $(t, \omega) \in [0, \infty]$, the set $\Theta(t, \omega)$ depends on $(t, \omega)$ only through $\omega(t-)$, i.e., $\Theta(t, \omega) = \Theta(\omega(t-))$, where $\Theta : F \to \mathbb{R}^U$.

The program of this section is the following. First, we explain the Markov property of $\{\mathcal{E}^x : x \in F\}$, which further leads to a sublinear Markovian semigroup $(T_t)_{t \geq 0}$ on the cone of bounded upper semianalytic functions. Afterwards, we establish conditions for $(T_t)_{t \geq 0}$ to be a sublinear Markovian semigroup on the cone of bounded upper semicontinuous functions and we derive a strong Markov selection principle.

4.1. The nonlinear Markov property. The following proposition provides the Markov property of $\{\mathcal{E}^x : x \in F\}$. The proof is given in Section 8.

**Proposition 4.2.** For every upper semianalytic function $\psi : \Omega \to [-\infty, \infty]$, the equality $\mathcal{E}^x(\psi \circ \theta_t) = \mathcal{E}^{\mathcal{E}^x(\psi)}(\psi)$ holds for every $(t, x) \in \mathbb{R}_+ \times F$.

To the best of our knowledge, in the context of nonlinear stochastic processes, the nonlinear Markov property was first observed in [24, Lemma 4.32] for nonlinear Markovian semimartingales, see also [9, Proposition 2.8] for a setting with continuous paths. Similar to the fact that linear Markov processes have a canonical relation to linear semigroups, nonlinear Markov processes can be related to nonlinear semigroups.
Definition 4.3. Let $\mathcal{H}$ be a convex cone of functions $f : F \to \mathbb{R}$ containing all constant functions. A family of sublinear operators $T_t : \mathcal{H} \to \mathcal{H}$, $t \in \mathbb{R}_+$, is called a sublinear Markovian semigroup on $\mathcal{H}$ if it satisfies the following properties:

(i) $(T_t)_{t \geq 0}$ has the semigroup property, i.e., $T_s T_t = T_{s+t}$ for all $s, t \in \mathbb{R}_+$ and $T_0 = \text{id}$;
(ii) $T_t$ is monotone for each $t \in \mathbb{R}_+$, i.e., $f, g \in \mathcal{H}$ with $f \leq g$ implies $T_t f \leq T_t g$;
(iii) $T_t$ preserves constants for each $t \in \mathbb{R}_+$, i.e., $T_t(c) = c$ for each $c \in \mathbb{R}$.

For a bounded upper semianalytic function $\psi : F \to \mathbb{R}$, $x \in F$ and $t \in \mathbb{R}_+$, we set
\begin{equation}
T_t(\psi)(x) \triangleq \mathcal{E}_t^{\psi}(\psi(X_t)) = \sup_{P \in \mathcal{P}(x)} E^P[\psi(X_t)].
\end{equation}

The following proposition should be compared to [24, Remark 4.33] and [9, Proposition 2.9], where it has been established for nonlinear Markovian semimartingale frameworks. Its proof can be found in Section 9.

Proposition 4.4. The family $(T_t)_{t \geq 0}$ defines a sublinear Markovian semigroup on the set of bounded upper semianalytic functions from $F$ into $\mathbb{R}$.

Linear semigroups with suitable regularity properties are well-known to be uniquely characterized by their (infinitesimal) generators. For nonlinear Lévy processes, a unique characterization of their nonlinear semigroups via (pointwise) generators has been established in [33, Proposition 6.5]. It is very interesting to prove such a result also for other classes of nonlinear processes such as NSPDEs. We leave this question for future investigations.

In the following section we establish a regularity preservation property of $(T_t)_{t \geq 0}$ in a continuous path setting. More precisely, we extend Proposition 4.4 via conditions for $(T_t)_{t \geq 0}$ to be a sublinear Markovian semigroup on the space of bounded upper semicontinuous functions from $F$ into $\mathbb{R}$.

4.2. The $\text{USC}_b$–Feller Property. It is natural to ask whether the sublinear Markovian semigroup $(T_t)_{t \geq 0}$ preserves some regularity. In case $F$ is endowed with the discrete topology, as it is the case for the class of nonlinear Markov chains from Section 3.2, it is trivial that $T_t$ is a selfmap on the space of bounded continuous functions for every $t \in \mathbb{R}_+$. In general, however, such a preservation property is non-trivial to establish. In the following we provide general conditions for a type of regularity preservation property of nonlinear stochastic processes with continuous paths, namely the so-called $\text{USC}_b$–Feller property.

Before we formulate our conditions, let us introduce a last bit of notation. We fix an $\mathbb{R}_+$-valued continuous adapted process $L = (L_t)_{t \geq 0}$ on $(\Omega, F, \mathbb{F})$ and, for $M > 0$, we set
\[\rho_M(\omega) \triangleq \inf\{t \geq 0 : L_t \geq M\} \wedge M, \quad \omega \in \Omega.\]

When $\Omega$ is the Wiener space of continuous paths, which will be assumed below, a typical choice for $L$ could be $d_F(X, y_0)$, where $d_F$ is a metric on $F$ and $y_0 \in F$ is an arbitrary reference point.

Condition 4.5.

(i) The underlying path space $\Omega$ is the space of all continuous functions $\mathbb{R}_+ \to F$, for every $u \in U$, the process $Y^u$ has continuous paths and $\Omega \ni \omega \mapsto Y^u(\omega) \in C(\mathbb{R}_+; \mathbb{R})$ and $\Omega \ni \omega \mapsto L(\omega) \in C(\mathbb{R}_+; \mathbb{R})$ are continuous (where the image and the inverse image spaces are endowed with the local uniform topology). Furthermore, for every $u \in U$, $M > 0$ and any compact set $K \subset F$, the process $Y_{\lambda M}^u 1_{(X_0 \in K)}$ is bounded.

(ii) The correspondence $\Theta : F \to \mathbb{K}^U$ is convex-valued.
For every $M > 0$, there exists a family $\{K^{u,M}: u \in U\} \subset \mathbb{R}$ of bounded sets such that

$(iii)$ \[ \lim_{M \to \infty} \sup_{x \in K} \sup_{P \in \mathcal{R}(x)} P\left(\sup_{s \in [0,T]} L_s \geq M\right) = 0. \]

$(iv)$ For every $\omega \in \Omega$, the correspondence $\mathbb{R}^+ \ni t \mapsto \Theta(\omega(t))$ is upper hemicontinuous and compact-valued, and, for every $t \in \mathbb{R}^+$ and $m \in \mathbb{N}$, the correspondence

$\omega \mapsto \overline{\Theta([t, t + 1/m], \omega)} \triangleq \overline{\bigcup_{s \in [t, t + 1/m]} \Theta(\omega(s))}$

is upper hemicontinuous and compact-valued. Here, $\overline{\Theta}$ denotes the closure of the convex hull.

$(v)$ For every compact set $K \subset F$, the set $\bigcup_{x \in K} \mathcal{R}(x) \subset \Psi(\Omega)$ is relatively compact.

For nonlinear one-dimensional diffusions, a version of the following theorem was established in [9]. Moreover, for nonlinear semimartingales with jumps some conditions were proved in [24, Theorem 4.41, Lemma 4.42]. In the context of controlled diffusions, upper semicontinuity of the value function has been established in [16]. We provide a result which reaches beyond semimartingale settings. Its proof can be found in Section 10.

**Theorem 4.6.** Suppose that Condition 4.5 holds. Then, $(T_t)_{t \geq 0}$ is a sublinear Markovian semigroup on the space $\text{USC}_b(F; \mathbb{R})$ of bounded upper semicontinuous functions $F \to \mathbb{R}$.

**Remark 4.7.** It is interesting to note that Condition 4.5 is not sufficient for the $C_b$–Feller property of $(T_t)_{t \geq 0}$, i.e., it does not imply $T_t(\text{USC}_b(F; \mathbb{R})) \subset C_b(F; \mathbb{R})$ for all $t > 0$. A counterexample is given in [53, Exercise 12.4.2]. In fact, the counterexample reveals an interesting effect when passing from linear to nonlinear settings. To explain this, we recall the example. Let $b: \mathbb{R} \to \mathbb{R}$ be a bounded continuous function such that $b(x) = \text{sgn}(x) \sqrt{|x|}$ for $|x| \leq 1$ and which is continuously differentiable off $(-1, 1)$. For $x \in \mathbb{R}$, define

$\mathcal{R}(x) \triangleq \left\{ \delta_y : g \in C^1(\mathbb{R}^+; \mathbb{R}), \text{dg}(t) = b(g(t))dt, g(0) = x \right\} \subset \Psi(C(\mathbb{R}^+; \mathbb{R})).$

In other words, the set $\mathcal{R}(x)$ contains all laws of solutions to the (deterministic) ordinary differential equation

$dY_t = b(Y_t)dt, \quad Y_0 = x.$

Notice that this is a special case of the NSPDE framework discussed in Section 3.3. We say that a family $(P_x)_{x \in \mathbb{R}} \subset \Psi(C(\mathbb{R}^+; \mathbb{R}))$ is $C_b$–Feller if the map $x \mapsto E^{P_x}[f(X_t)]$ is continuous for every $f \in C_b(\mathbb{R}; \mathbb{R})$ and $t > 0$. According to [53, Exercise 12.4.2], it is not possible to select a $C_b$–Feller family $(P_x)_{x \in \mathbb{R}}$ such that $P_x \in \mathcal{R}(x)$. By the linearity of the expectation map $P \mapsto E^P[\cdot]$, it is easy to see that a family $(P_x)_{x \in \mathbb{R}}$ of probability measures is a $C_b$–Feller family if and only if it is a $\text{USC}_b$–Feller family in the sense that $x \mapsto E^{P_x}[f(X_t)]$ is upper semicontinuous for every $f \in \text{USC}_b(\mathbb{R}; \mathbb{R})$ and $t > 0$. Thus, [53, Exercise 12.4.2] also tells us that it is not possible to select a $\text{USC}_b$–Feller family $(P_x)_{x \in \mathbb{R}}$ such that $P_x \in \mathcal{R}(x)$. However, Theorem 4.6 (or Corollary 5.4 below) shows that the nonlinear semigroup

$T_t(\psi)(x) = \sup_{P \in \mathcal{R}(x)} E^P[\psi(X_t)]$

It seems that there is a gap in the proof of [24, Lemma 4.42]. Indeed, it is claimed that the map $\omega \mapsto \omega(t)$ is upper semicontinuous on the Skorokhod space. However, this is not the case. Indeed, by linearity, upper semicontinuity would already imply continuity, which is false.
has the $USC_b$–Feller property (in the sense that $(T_t)_{t \geq 0}$ is a nonlinear semigroup on the cone $USC_b(|\mathbb{R}|)$). Broadly speaking, passing from the linear to the nonlinear setting provides a smoothing effect in the sense that the nonlinear semigroup has the $USC_b$–Feller property although it is not possible to select a linear semigroup with this property from the uncertainty set.

In Section 5 below, we return to the class of NSPDEs from Section 3.3 and we provide some general parametric conditions which imply Condition 4.5. From a practical point of view, (i) – (iv) from Condition 4.5 are mainly structural assumptions, while checking part (v) requires an argument.

4.3. The strong Markov selection principle. For a probability measure $P$ on $(\Omega, \mathcal{F})$, a kernel $\Omega \ni \omega \mapsto Q_{\omega} \in \mathcal{P}(\Omega)$, and a finite stopping time $\tau$, we define the pasting measure

$$
(P \otimes Q)(\omega) = \int \mathbb{I}_A(\omega \otimes \tau(\omega)) Q_{\omega'}(d\omega') P(d\omega)
$$

for all $A \in \mathcal{F}$.

**Definition 4.8** (Time inhomogeneous Markov Family). A family $\{P_{(s,x)} : (s,x) \in \mathbb{R}_+ \times F\} \subset \mathcal{P}(\Omega)$ is said to be a strong Markov family if $(t,x) \mapsto P_{(t,x)}$ is Borel and the strong Markov property holds, i.e., for every $(s,x) \in \mathbb{R}_+ \times F$ and every finite stopping time $\tau \geq s$,

$$
P_{(s,x)}(\cdot | \mathcal{F}_\tau)(\omega) = \omega \otimes \tau(\omega) P_{(\tau(\omega), \omega(\tau(\omega)))}
$$

for $P_{(s,x)}$-a.a. $\omega \in \Omega$.

We introduce a correspondence $\mathcal{K} : \mathbb{R}_+ \times F \to \mathcal{P}(\Omega)$ by

$$
\mathcal{K}(t,x) = \left\{ P \in \mathcal{P}(\Omega) : P(X = x \text{ on } [0,t]) = 1, \forall u \in U, Y^u \in \mathcal{S}_ac(t,P), \right.
$$

$$(\lambda \otimes P)-\text{a.e. } ((d\lambda)^{\mathcal{K}}(Y^u_{t+})) / d\lambda) \in \Theta(\cdot + t, X) \bigg\},
$$

where $(t,x) \in \mathbb{R}_+ \times F$. The following theorem is proved in Section 7.

**Theorem 4.9** (Strong Markov Selection Principle). Suppose that Condition 4.5 holds. For every $\psi \in USC_b(\mathbb{R}; F)$ and every $t > 0$, there exists a strong Markov family $\{P_{(s,x)} : (s,x) \in \mathbb{R}_+ \times F\}$ such that, for all $(s,x) \in \mathbb{R}_+ \times F$, $P_{(s,x)} \in \mathcal{K}(s,x)$ and

$$
E^{P_{(s,x)}}[\phi(X_t)] = \sup_{P \in \mathcal{K}(s,x)} E^P[\phi(X_t)].
$$

In particular, for every $x \in F$,

$$
T_t(\psi)(x) = E^{P_{(0,x)}}[\psi(X_t)].
$$

In a relaxed framework for finite-dimensional controlled diffusions, a strong Markov selection principle has been established in [16]. A strong Markov selection principle for nonlinear one-dimensional diffusions was proved in [9]. Theorem 4.9 covers the result from [9] as a special case (see Section 5 below).

In the next section we tailor the Theorems 4.6 and 4.9 to the NSPDE setting from Section 3.3. Furthermore, we are going to discuss the strong $USC_b$–Feller property, which provides a smoothing effect.
5. NSPDEs: USC\textsubscript{b}–Feller Properties and Strong Markov Selections

We return to the class of nonlinear SPDEs from Section 3.3 and provide explicit conditions (in terms of the drift and diffusion coefficients) for the USC\textsubscript{b}–Feller property and the strong Markov selection principle.

We recall the precise setting. Let \((H_1, \| \cdot \|_{H_1}, \langle \cdot , \cdot \rangle_{H_1})\) and \((H_2, \| \cdot \|_{H_2}, \langle \cdot , \cdot \rangle_{H_2})\) be two separable Hilbert spaces over the real numbers, and let \((A, D(A))\) be the generator of a \(C_0\)-semigroup \((S_t)_{t \geq 0}\) on the Hilbert space \(H_2\). Moreover, let \(\Omega\) be the space of all continuous functions \(\mathbb{R}_+ \to F\), let \(G\) be a topological space and let \(\mu: G \times H_2 \to H_2\) and \(\sigma: G \times H_2 \to L(H_1, H_2)\) be Borel functions, where, for the latter, we mean that \(\sigma h_1\) is a Borel function for every \(h_1 \in H_1\). Further, let \(D \subset D(A^*)\) be a countable set such that for every \(y \in D(A^*)\) there exists a sequence \((y_n)_{n=1}^\infty \subset D\) such that \(y_n \to y\) and \(A^*y_n \to A^*y\).

Recall that such a set exists by \cite[Lemma 7.3]{7}. We set \(f_i(x) \equiv x^i\) for \(x \in \mathbb{R}\) and \(i = 1, 2\),

\[
U \equiv \{(H_2 \ni x \mapsto f_i((y,x)_{H_2})): y \in D, i = 1, 2\}, \quad Y^u \equiv u(X), \; u \in U,
\]

and

\[
\mathcal{L}^{u,g}(x) \equiv \langle (A^*y, x)_{H_2} + \langle y, \mu(g,x)_{H_2} \rangle, f_i((y,x)_{H_2}) + \frac{1}{2}\|\sigma^*(g,x)\|_2 H_1, f''_i((y,x)_{H_2}) \rangle,
\]

for \((x,g) \in H_2 \times G\) and \(u = f_i((y,x)_{H_2}) \in U\). Finally, for \(x \in H_2\) and \((t, \omega) \in [0, \infty]\), we define

\[
\Theta(x) \equiv \{(\mathcal{L}^{u,g}(x))_{u \in U}: g \in G\}, \quad \Theta(t, \omega) \equiv \Theta(\omega(t)).
\]

We denote the operator norm on \(L(H_1, H_2)\) by \(\| \cdot \|_{L(H_1, H_2)}\) and the Hilbert–Schmidt norm by \(\| \cdot \|_{L_2(H_1, H_2)}\).

**Condition 5.1.**

(i) \(G\) is a compact metrizable space.

(ii) For every \(x \in H_2\), the set \(\{(\mu(g,x), (\sigma^*)(g,x)): g \in G\}\) is convex.

(iii) For every \(y \in D\), the functions \(\langle y, \mu \rangle_{H_2}\) and \(\|\sigma^* y\|_{H_1}\) are continuous.

(iv) There exists a constant \(C > 0\) such that

\[
\|\mu(g,x)\|_{H_2} + \|\sigma(g,x)\|_{L(H_1,H_2)} \leq C(1 + \|x\|_{H_2})
\]

for all \((g,x) \in G \times H_2\).

(v) The semigroup \(S_t\) is compact, i.e., \(S_t\) is compact for every \(t > 0\), and there exists an \(\alpha \in (0, 1/2)\) and a Borel function \(f: (0, \infty) \to [0, \infty]\) such that

\[
\int_0^T \left[ \frac{f(s)}{s^\alpha} \right]^2 ds < \infty, \quad \forall T > 0,
\]

and, for all \(t > 0, x \in H_2\) and \(g \in G\),

\[
\|S_t\sigma(g,x)\|_{L_2(H_1, H_2)} \leq f(t)(1 + \|x\|_{H_2}).
\]

**Lemma 5.2.** If Condition 5.1 holds, then both Standing Assumptions 2.1 and 2.2 are satisfied.

**Proof.** Thanks to (i) and (iii) of Condition 5.1, Standing Assumption 2.1 is implied by \cite[Lemma 2.12]{8}. Moreover, thanks to (iii) – (v) from Condition 5.1, Standing Assumption 2.2 follows from \cite[Theorem 2.5]{7} and Lemmata 7.3 and 7.9 below. \(\square\)

The following proposition is proved in Section 11.

**Proposition 5.3.** Condition 5.1 implies Condition 4.5 with \(L \equiv \|X\|_{H_2}\).
Thanks to Proposition 5.3, the Theorems 4.6 and 4.9 yield the following result.

**Corollary 5.4.** Suppose that Condition 5.1 holds. Then, \((T_t)_{t \geq 0}\) is a sublinear Markovian semigroup on \(\text{USC}_b(H_2; \mathbb{R})\) and, for every \(\psi \in \text{USC}_b(H_2; \mathbb{R})\) and every \(t > 0\), there exists a strong Markov family \(\{P_{(s,x)}: (s,x) \in \mathbb{R}_+ \times H_2\}\) such that, for every \((s,x) \in \mathbb{R}_+ \times H_2\), \(P_{(s,x)} \in \mathcal{K}(s,x)\) and

\[
E^{P_{(s,x)}}[\phi(X_t)] = \sup_{P \in \mathcal{K}(s,x)} E^P[\phi(X_t)].
\]

In particular, \(T_t(\psi)(x) = E^{P_{(0,x)}}[\psi(X_t)]\) for every \(x \in H_2\).

Finally, we establish a smoothing effect for stochastic nonlinear Cauchy problems with drift uncertainty, which are infinite-dimensional processes of the form

\[
dY_t = AY_t dt + \mu(Y_t) dt + dW_t,
\]

with drift uncertainty.

**Definition 5.5.** We say that \((T_t)_{t \geq 0}\) has the strong \(\text{USC}_b\)–Feller property if

\[
\mathcal{T}_t(\text{USC}_b(F; \mathbb{R})) \subset C_b(F; \mathbb{R})
\]

for every \(t > 0\).

To the best of our knowledge, the strong \(\text{USC}_b\)–Feller property of sublinear Markovian semigroups was discovered in [9] for nonlinear one-dimensional elliptic diffusions. In [10] it was established for nonlinear multidimensional diffusions with fixed (strongly) elliptic diffusion coefficients. Adapting some ideas from [9, 10, 53], we establish the seemingly first result for infinite-dimensional nonlinear stochastic processes.

**Condition 5.6.**

(i) \(G\) is a compact metrizable space and \(H_1 \equiv H_2 \equiv H\).

(ii) For every \(x \in H\), the set \(\{\mu(g, x): g \in G\}\) is convex.

(iii) For every \(y \in D, \langle y, \mu \rangle_H\) is continuous and \(\sigma\) equals constantly the identity operator on \(H\), i.e., \(\sigma(g, x) \equiv \text{id}\) for all \((g, x) \in G \times H\).

(iv) There exists a constant \(C > 0\) such that

\[
\|\mu(g, x)\|_H \leq C(1 + \|x\|_H)
\]

for all \((g, x) \in G \times H\).

(v) There exists an \(\alpha \in (0, 1/2)\) such that

\[
\int_0^T \frac{\|S_s\|_{L^2(H, H)}^2 ds}{s^{2\alpha}} < \infty, \quad \forall T > 0.
\]

**Remark 5.7 ([6]).** Part (v) of Condition 5.6 implies that \(S\) is a compact semigroup. Moreover, the condition

\[
\int_0^T \frac{\|S_s\|_{L^2(H, H)}^2 ds}{s^{2\alpha}} < \infty
\]

holds for all \(T > 0\) once it holds for some \(T > 0\).

The proof for the next theorem is given in Section 12.

**Theorem 5.8.** If Condition 5.6 holds, then \((T_t)_{t \geq 0}\) has the strong \(\text{USC}_b\)–Feller property.

Under suitable additional conditions (see [13, Hypothesis 9.1]), Theorem 5.8 can be extended to NSPDEs with certain diffusion coefficient, i.e., to the case \(\sigma(g, x) \equiv \sigma(x)\).
Example 5.9 ([21]). Let $O \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Part (v) of Condition 5.6 holds when $H = L^2(O)$ and $A$ is a strongly elliptic operator of order $2m > d$, see [21, Remark 3, Example 3] for some details. In particular, this is the case when $d = 1$ and $A$ is the Laplacian, i.e., Condition 5.6 holds for one-dimensional stochastic heat equations with drift uncertainty.

6. **Proof of the Dynamic Programming Principle: Theorem 2.5**

Recall that the Standing Assumptions 2.1 and 2.2 are in force. The proof of Theorem 2.5 is based on an application of the general [17, Theorem 2.1], which provides three abstract conditions on the set $C$ implying the DPP. In the following we verify these conditions. For reader’s convenience, let us restate them.

(i) **Measurable graph condition:** The set $\{(t, \omega, P) \in [0, \infty] \times \mathcal{P}(\Omega) : P \in C(t, \omega)\}$ is analytic.

(ii) **Stability under conditioning:** For any $t \in \mathbb{R}_+$, any stopping time $\tau$ with $t \leq \tau < \infty$, and any $P \in C(t, \alpha)$ there exists a family $\{P(\cdot | \mathcal{F}_\tau)(\omega) : \omega \in \Omega\}$ of regular $P$-conditional probabilities given $\mathcal{F}_\tau$ such that $P$-a.s. $P(\cdot | \mathcal{F}_\tau) \in C(\tau, X)$.

(iii) **Stability under pasting:** For any $t \in \mathbb{R}_+$, any stopping time $\tau$ with $t \leq \tau < \infty$, any $P \in C(t, \alpha)$ and any $\mathcal{F}_\tau$-measurable map $\Omega \ni \omega \mapsto Q_\omega \in \mathcal{P}(\Omega)$ the following implication holds:

$$P\text{-a.s. } Q \in C(\tau, X) \implies P \otimes \tau Q \in C(t, \alpha).$$

Here, the pasting measure $P \otimes \tau Q$ was defined in (4.3).

In the following three sections we check these properties. In the fourth (and last) section, we finalize the proof of Theorem 2.5. Hereby, although technically different, we follow a strategy from [39] and we also use some technical ideas from [8] about the measurability of the semimartingale property in time.

6.1. **Measurable graph condition.** The proof of the measurable graph condition is split into several parts.

**Lemma 6.1.** For every $\mathbb{K}$-valued càdlàg $\mathbf{F}$-adapted process $Y^*$, the set

$$\mathcal{P}_{sp\text{ sem}}(Y^*) \triangleq \{P \in \mathcal{P}(\Omega) : Y^* \in \mathcal{S}_{ac}^P(\mathcal{P})\}$$

is Borel. Moreover, there exists a Borel map

$$[0, \infty] \times \mathcal{P}_{sp\text{ sem}}(Y^*) \ni (t, \omega, P) \mapsto a^P_t(Y^*)\omega \in \mathbb{K}$$

such that, for every $P \in \mathcal{P}_{sp\text{ sem}}(Y^*)$, $a^P$ is predictable and $(\lambda \otimes P)$-a.e. $a^P(Y^*) = dA^P(Y^*)/d\lambda$.

**Proof.** For $P \in \mathcal{P}(\Omega)$, denote the set of $\mathbb{K}$-valued $P$-$\mathbf{F}_+$-semimartingales by $\mathcal{S}(P)$. The set $\mathcal{P}_{\text{sem}}(Y^*) \triangleq \{P \in \mathcal{P}(\Omega) : Y^* \in \mathcal{S}(P)\}$ is Borel by [38, Theorem 2.5]. Furthermore, by the very same theorem, there are Borel maps (with values in a suitable Polish space; see Lemma A.1 below)

$$\mathbb{R}_+ \times \Omega \times \mathcal{P}_{\text{sem}}(Y^*) \ni (t, \omega, P) \mapsto (B^P_t(\omega), C^P_t(\omega), \nu^P_t(\omega))$$

such that $(B^P, C, \nu^P)$ are the $P$-$\mathbf{F}_+$-semimartingale characteristics of $Y^*$ (corresponding to a given truncation function $h$). By virtue of [28, Proposition II.2.29], we have

$$\mathcal{P}_{sp\text{ sem}}(Y^*) \triangleq \{P \in \mathcal{P}(\Omega) : Y^* \in \mathcal{S}_{ac}(\mathcal{P})\}$$

$$= \{P \in \mathcal{P}_{\text{sem}}(Y^*) : \forall t \in \mathbb{N} P\text{-a.s. } ((|x|^2 \wedge |x|) \ast \nu^P_t < \infty\},$$
where, as usual,
\[ (|x|^2 \wedge |x|) \ast \nu^P_t = \int (|x|^2 \wedge |x|) \nu^P_t ([0, t] \times dx). \]
Consequently, \( \mathfrak{P}_{sp\,sem}(Y^*) \) is Borel by [4, Theorem 8.10.61]. Thanks again to [28, Proposition II.2.29], in case \( Y^* \in \mathcal{S}^{sp}(P) \), the predictable part of (locally) finite variation in the semimartingale decomposition is given by
\[ A^P(Y^*) = B^P + (x - h(x)) \ast \nu^P. \]
Hence,
\[ \mathfrak{P}_{sp\,sem}(Y^*) = \mathcal{P} \in \mathfrak{P}_{sp\,sem}(Y^*) : P \text{-a.s. } B^P + (x - h(x)) \ast \nu^P \ll \lambda. \]
We deduce from [2, Theorem 8.4.4] that \( P \)-a.s. there exists a decomposition
\[ B^P + (x - h(x)) \ast \nu^P = \int_0^t \phi^P_s ds + \psi^P, \]
where \( t \mapsto \psi^P_t \) is singular w.r.t. the Lebesgue measure, \( (t, \omega, P) \mapsto \phi^P_t (\omega) \in K \) is Borel and \( \phi^P \) is predictable. Now,
\[ \mathfrak{P}_{sp\,sem}(Y^*) = \left\{ P \in \mathfrak{P}_{sp\,sem}(Y^*) : \forall t \in \mathbb{Q}_+ \text{ P-a.s. } B^P_t + (x - h(x)) \ast \nu^P_t = \int_0^t \phi^P_s ds \right\}, \]
and the latter set is Borel by [4, Theorem 8.10.61]. Finally, we can take \( \alpha^P(Y^*) = \phi^P \) and the proof is complete. \( \square \)

**Lemma 6.2.** The map \( (t, P) \mapsto P \circ \theta_t^{-1} \triangleq P_t \) is Borel.

**Proof.** As the map \( (t, \omega) \mapsto \theta_t(\omega) \) is Borel, the claim follows from [4, Theorem 8.10.61]. \( \square \)

The following lemma is a restatement of [27, Lemma 2.9 a)].

**Lemma 6.3.** Let \( Y = (Y_t)_{t \geq 0} \) be an \( F \)-valued càdlàg process and set \( F^Y_t = \sigma(Y_s, s \leq t) \) for \( t \in \mathbb{R}_+ \). Then, \( Y^{-1}(F^Y_\cdot) = F^Y_{\cdot+} \) for all \( s \in \mathbb{R}_+ \).

The following lemma is a restatement of [27, Lemma 2.9 a)].

**Lemma 6.4.** Take two filtered probability spaces \( \mathbb{B}^* = (\Omega^*, \mathcal{F}^*, \mathbb{F}^* = (\mathcal{F}_t^*)_{t \geq 0}, \mathbb{P}^*) \) and \( \mathbb{B}' = (\Omega', \mathcal{F}', \mathbb{F}' = (\mathcal{F}'_t)_{t \geq 0}, \mathbb{P}') \) with right-continuous filtrations and the property that there is a map \( \phi : \Omega' \rightarrow \Omega^* \) such that \( \phi^{-1}(\mathcal{F}^*) \subset \mathcal{F}' \), \( \mathbb{P}^* = \mathbb{P}' \circ \phi^{-1} \) and \( \phi^{-1}(\mathcal{F}_t^*) = \mathcal{F}'_t \) for all \( t \in \mathbb{R}_+ \). Then, \( X^* \) is a \( d \)-dimensional semimartingale on \( \mathbb{B}^* \) if and only if \( X' = X^* \circ \phi \) is a \( d \)-dimensional semimartingale on \( \mathbb{B}' \). Moreover, \( (B^*, C^*, \nu^*) \) are the characteristics of \( X^* \) if and only if \( (B^* \circ \phi, C^* \circ \phi, \nu^* \circ \phi) \) are the characteristics of \( X' = X^* \circ \phi \).

**Lemma 6.5.** The set
\[ \left\{ (t, \omega, P) \in [0, \infty] \times \mathfrak{P}(\Omega) : \forall u \in U \ Y^u \in \mathcal{S}^{sp}_a(t, P), \right\}
\[ (\lambda \otimes \mathbb{P}) \text{-a.e. } (dA^u_t(Y^u_t))/d\lambda \in \Theta(\cdot + t, \omega \otimes_t X) \}
\]
is Borel.

**Proof.** For \( \omega, \omega' \in \Omega \) and \( t \in \mathbb{R}_+ \), we define the concatenation
\[ \omega \otimes_t \omega' \triangleq \omega 1_{[0, t]} + (\omega(t) + \omega' (\cdot - t) - \omega'(0)) 1_{[t, \infty]}, \]
and we notice that \((\omega \otimes t) X) \circ \theta_t = \omega \otimes t X\) for all \((t,\omega) \in [0,\infty[.\) Further, recall that, for every \(u \in U\), we assume that \(Y^u = Y^u \circ \theta_t\) for all \(t \in \mathbb{R}_+\). Let \(a^P_t(Y)\) be the map from Lemma 6.1. Thanks to Lemmata 6.3 and 6.4, we get that
\[
\{(t,\omega,P): \forall u \in U \ Y^u \in \mathcal{S}^\text{sp}_\text{ac}(t,P), (\lambda \otimes P)\text{-a.e. } (dP_t(Y^u))/d\lambda) \in \Theta(\cdot + t, \omega \otimes t X)\}
\]
\[
= \{(t,\omega,P): \forall u \in U \ P_t \in \mathcal{Q}^\text{sp}_{\text{ac}}(Y^u), (\lambda \otimes P_t)\text{-a.e. } (a^P_t(Y^u)) \in \Theta(\cdot + t, \omega \otimes t X)\}.
\]
The latter set is Borel by Lemmata 6.1 and 6.2, and [4, Theorem 8.10.61].

**Corollary 6.6.** *The measurable graph condition holds.*

### 6.2. Stability under Conditioning

Next, we check stability under conditioning. In this section, we fix \((t^*,\omega^*) \in [0,\infty[,\) a stopping time \(\tau \) with \(t^* \leq \tau < \infty\), and a probability measure \(P\) on \((\Omega,F)\) such that \(P(X = \omega^* \mid [0,t^*]) = 1\). We denote by \(P(\cdot |F_\tau)\) a version of the regular conditional \(P\)-probability given \(F_\tau\). We recall some simple observations.

**Lemma 6.7.**

(i) There exists a \(P\)-null set \(N \subseteq F_\tau\) such that \(P(A | F_\tau) (\omega) = 1_A(\omega)\) for all \(A \subseteq F_\tau\) and \(\omega \notin N\).

(ii) If \(\omega \mapsto Q_\omega\) is a kernel from \(F\) into \(\mathbb{F}\) such that \(Q_\omega(\omega = X \mid [0,\tau(\omega)]) = 1\) for \(P\)-a.a. \(\omega \in \Omega\), then \(Q_\omega(\omega \otimes \tau(\omega) X = X) = 1\) for \(P\)-a.a. \(\omega \in \Omega\). In particular, there exists a \(P\)-null set \(N \subseteq F_\tau\) such that \(P(\omega \otimes \tau(\omega) X = X | F_\tau)(\omega) = 1\) for all \(\omega \notin N\).

(iii) A measurable process \(H = (H_t)_{t \geq 0}\) on \((\Omega,F)\) is optional (resp. predictable) if and only if, for all \(t \in \mathbb{R}_+, H_t(\omega)\) depends on \(\omega\) only through the values \((\omega(s))_{s \leq t}\) (resp. \((\omega(s))_{s < t}\)).

**Proof.** Part (i) is classical ([53, Theorem 1.1.8]), part (ii) is obvious and part (iii) follows from [14, Theorem IV.97].

In the following we use the standard notation \(M^s = M_{\Lambda s}\) for a process \(M\) and a time \(s\). The next lemma is implied by [53, Theorem 1.2.10].

**Lemma 6.8.** Let \(G = (G_t)_{t \geq 0}\) be the filtration on \((\Omega,F)\) that is generated by a càdlàg process with values in a Polish space. Furthermore, let \(\rho\) be a \(G\)-stopping time such that \(\rho \geq t^*\). If \(M = M^{\rho}\) is a \(P,G\)-martingale, then there exists a \(P\)-null set \(N\) such that \(M - M^{\rho} = P(\cdot | G_{\rho}) (\omega)\)-\(G\)-martingale for all \(\omega \notin N\).

**Lemma 6.9.** Suppose that \(Y \in \mathcal{S}^\text{sp}(t^*,P)\). Then, there exists a \(P\)-null set \(N\) such that, for every \(\omega \notin N\), \(Y \in \mathcal{S}^\text{sp}(\tau(\omega),P(\cdot | F_\tau)(\omega))\) and \(P(\cdot | F_\tau)(\omega)\text{-a.s.}\)
\[
A_t \tau(\omega)(Y_{\tau(\omega)}) = (A_{t^*} Y_{\tau(\omega)} - t^* A_{t} (Y_{\tau(\omega)} - t^* A_{t^*} (Y_{\tau(\omega)})))(\omega \otimes \tau(\omega) X).
\]
Moreover, when \(\nu^P\) denotes the third \(P\)-characteristic of \(Y_{\tau(\omega)}\), then, for every \(\omega \notin N\), the third \(P(\cdot | F_\tau)(\omega)\)-characteristic of \(Y_{\tau(\omega)}\) (for the right-continuous natural filtration generated by \(X_{\tau(\omega)}\)) is given by
\[
\nu^P(\omega \otimes \tau(\omega) X; \tau(\omega) - t^* + dt, dx).
\]

**Proof.** Let \(G^{t^*} = (G^t_t)_{t \geq 0}\) be the filtration generated by \(X_{\tau(\omega)}\) and let \((\tau_n)_{n=1}^{\infty}\) be a localizing sequence for the \(P,G^{t^*}\)-local martingale
\[
M \triangleq Y_{\tau(\omega)} - A^P_{t^*}(Y_{\tau(\omega)}).
\]
By Lemma 6.8, there exists a \(P\)-null set \(N\) such that \(M^{\tau_n} = M^{\tau_n \wedge (\tau(\omega) - t^*)}\) is a \(P(\cdot | G^n_{\tau(\omega)})(\omega)\)-\(G^{t^*}\)-martingale for every \(\omega \notin N\). As \(P(X = \omega^* \mid [0,t^*)] = 1\), we have \(P\text{-a.s. } P(\cdot | G^n_{\tau(\omega)})(\omega)\)-
Lemma 6.10. \[ \Box \]

The process \( \mathbb{P} (\cdot | \mathcal{F}_\tau) \). Hence, possibly making \( N \) a bit larger, \( M^{\tau_n} - M^{\tau_{n+1}} \) is a \( \mathbb{P} (\cdot | \mathcal{F}_\tau) \) martingale for every \( \omega \notin N \). Inserting \( s \leftrightarrow s + \tau(\omega) - t^* \) with \( \omega \notin N \) shows that the process

\[
Y_{\land \rho_n + \tau(\omega)} - Y_\tau(\omega) - (A^{\rho_n \land \tau(\omega) - t^*}_P(Y_{t^*}) - A^{\rho_n \land \tau(\omega) - t^*}_P(Y_{t^*}))
\]

is a \( \mathbb{P} (\cdot | \mathcal{F}_\tau) \) martingale, where

\[
\rho_n \equiv \lceil \tau_n - \tau(\omega) + t^* \rceil \forall 0, \quad n = 1, 2, \ldots.
\]

By Lemma 6.7, possibly making \( N \) again larger, we have \( \mathbb{P} (\omega \land \tau(\omega) X = X | \mathcal{F}_\tau) = 1 \) for all \( \omega \notin N \). Hence, for all \( \omega \notin N \), using that \( \mathcal{G}^\tau(\omega) \subset \mathcal{G}^\tau_{t^*} \), andLemma 6.7 (iii), the tower rule shows that the process

\[
(Y_{\land \rho_n + \tau(\omega)} - Y_\tau(\omega) - (A^{\rho_n \land \tau(\omega) - t^*}_P(Y_{t^*}) - A^{\rho_n \land \tau(\omega) - t^*}_P(Y_{t^*}))) \land \tau(\omega) X
\]

is a \( \mathbb{P} (\cdot | \mathcal{F}_\tau) \) martingale. Notice that \( \rho_n (\omega \land \tau(\omega) X) \) is a \( \mathcal{G}^\tau(\omega) \)-stopping time by Galmarino’s test ([14, Theorem IV.100]). Finally, since

\[
\mathbb{E} \left[ \mathbb{P} \left( \lim \inf_{n \to \infty} \tau_n < \infty | \mathcal{F}_\tau \right) \right] = \mathbb{P} \left( \lim \inf_{n \to \infty} \tau_n < \infty \right) = 0,
\]

we can enlarge \( N \) a last time such that also \( \mathbb{P} (\tau_n \to \infty | \mathcal{F}_\tau) = 1 \) for all \( \omega \notin N \). Of course, \( \tau_n \to \infty \) implies that \( \rho_n \to \infty \). In summary, by virtue of Lemma 6.7 (iii), for all \( \omega \notin N \), the process \( Y_{t^*} \) is a special \( \mathbb{P} (\cdot | \mathcal{F}_\tau) \) martingale with predictable compensator

\[
(A^{\rho_n \land \tau(\omega) - t^*}_P(Y_{t^*}) - A^{\rho_n \land \tau(\omega) - t^*}_P(Y_{t^*}))) \land \tau(\omega) X.
\]

This finishes the proof of the first claim.

To prove the second claim, let \( g : \mathbb{R} \to \mathbb{R} \) be a bounded Borel function which vanishes around the origin. Then, arguing as above, possibly making \( N \) a bit larger, for every \( \omega \notin N \), the process

\[
\sum_{s \leq t^*} g(Y_{t^*}) \Delta s - \int_0^{t^*} \int g(x) \mathbb{P} (\omega \land \tau(\omega) X; \tau(\omega) - t^* + dt, dx)
\]

is a local \( \mathbb{P} (\cdot | \mathcal{F}_\tau) \) martingale. The third characteristic can be characterized by these local martingale properties for countably many test functions \( g \), see [28, Theorem II.2.21]. Thus, the formula for the third \( \mathbb{P} (\cdot | \mathcal{F}_\tau) \)-characteristic follows. The proof is complete. \[ \Box \]

We are in the position to deduce stability under conditioning.

Lemma 6.10. If \( P \in \mathcal{C}(t^*, \omega^*) \), then \( \mathbb{P} (\cdot | \mathcal{F}_\tau) \in \mathcal{C}(\tau, X) \).

Proof. By Lemma 6.7 (i), \( P(\tau = \tau(\omega) | \mathcal{F}_\tau) = 1 \) and \( P(X = \omega) = 1 \) for \( P \)-a.a. \( \omega \in \Omega \). Thanks to Lemma 6.9, as the set \( U \) is assumed to be countable, for \( P \)-a.a. \( \omega \in \Omega \) and all \( u \in U \), we have \( Y^u \in \mathcal{S}^\mathbb{P}_{\mathbb{R}^\infty}(\tau(\omega), P(\cdot | \mathcal{F}_\tau) \omega) \) and \( P(\cdot | \mathcal{F}_\tau) \)-a.s.

\[
d\mathbb{P} (\cdot | \mathcal{F}_\tau) (Y^u_{t^*}) / d\lambda = (dA^P(Y^u_{t^*}) / d\lambda)(\cdot + \tau(\omega) - t^*, \omega \land \tau(\omega) X).
\]

We get from Fubini’s theorem, the tower rule and \( P \in \mathcal{C}(t^*, \omega^*) \) that

\[
\int_0^\infty P((dA^P(Y^u_{t^*}) / d\lambda)_{u \in U} (t, X) \notin \Theta(t + \tau(\omega), X) | \mathcal{F}_\tau) \omega) dt P(d\omega)
\]

is equal to

\[
\int_0^\infty P((dA^P(Y^u_{t^*}) / d\lambda)_{u \in U} (t + \tau(\omega) - t^*, X) \notin \Theta(t + \tau(\omega), X) | \mathcal{F}_\tau) \omega) dt P(d\omega)
\]
6.3. Stability under Pasting. In this section we check stability under pasting. The proof is split into several steps. Throughout this section, we fix \((t^*, \omega^* \in [0, \infty[\), a probability measure \(P\) on \((\Omega, \mathcal{F})\) such that \(P(X = \omega^* \text{ on } [0, t^*) = 1\), a stopping time \(\tau\) with \(t^* \leq \tau < \infty\), and an \(\mathcal{F}_\tau\)-measurable map \(\Theta : \omega^* \mapsto \Theta(\omega) \in \mathcal{B}(\Omega)\) such that \(\Theta_\omega(X = \omega^* \text{ on } [0, \tau(\omega)]) = 1\) for \(P\)-a.a. \(\omega \in \Omega\). To simplify our notation, we set \(\bar{P} := P \otimes_\tau Q\). The following corollary is an immediate consequence of part (ii) of Lemma 6.7.

Corollary 6.11. \(\bar{P} = E^P[Q(\cdot)].\)

The next lemma follows identically to [8, Lemma 3.14]. We omit the details.

Lemma 6.12. \(P = \bar{P}\) on \(\mathcal{F}_\tau\) and \(\bar{P}\)-a.s. \(P(\cdot | \mathcal{F}_\tau) = Q\).

Clearly, Lemma 6.12 yields that \(\bar{P}(X = \omega^* \text{ on } [0, t^*) = P(X = \omega^* \text{ on } [0, t^*)) = 1\).

Lemma 6.13. Suppose that \(Y \in \mathcal{S}^{sp}(t^*, P)\). If \(Y \in \mathcal{S}^{sp}(\tau(\omega), Q_\omega)\) for \(P\)-a.a. \(\omega \in \Omega\), then \(Y \in \mathcal{S}^{sp}(t^*, \bar{P})\).

Proof. Step 1: Semimartingale Property. Let \(T > t^*\), and take a sequence \((H^n)^{\infty}_{n=0}\) of simple predictable processes on \([t^*, T]\) such that \(H^n \rightarrow H^0\) uniformly in time and \(\omega\). Then, by Lemma 6.12, we get

\[
E^\bar{P}\left[\int_{t^*}^{T} (H^n_s - H_s) dY_s \wedge 1\right] \leq E^P\left[\int_{t^*}^{T} (H^n_s - H_s) dY_s \wedge 1\right] + \int E^{Q_\omega}\left[\int_{\tau(\omega) \wedge T}^{T} (H^n_s - H_s)(\omega \otimes_{\tau(\omega)} X) dY_s \wedge 1\right] dP(d\omega).
\]

The first term converges to zero as \(Y \in \mathcal{S}^{sp}(t^*, P)\) and thanks to the Bichteler–Dellacherie (BD) Theorem ([49, Theorem III.43]). The second term converges to zero by dominated convergence, the assumption that \(P\)-a.s. \(Y \in \mathcal{S}^{sp}(\tau, Q)\) and, by virtue of part (iii) of Lemma 6.7, again the BD Theorem. Consequently, invoking the BD Theorem a third time, but this time the converse direction, yields that \(Y \in \mathcal{S}(t^*, \bar{P})\).

Step 2: Special Semimartingale Property. Denote by \(\nu^\bar{P}\) (resp. \(\nu^P\)) the third characteristic of \(Y_{t^*} \tau\) under \(\bar{P}\) (resp. under \(P\)). By Lemma 6.12, we obtain

\[
(6.1) \quad \bar{P}((|x|^2 \wedge |x|) \ast \nu^\bar{P}_{t^*} \tau < \infty) = P((|x|^2 \wedge |x|) \ast \nu^P_{t^*} \tau < \infty) = 1.
\]

Furthermore, as \(Y \in \mathcal{S}(t^*, \bar{P})\) and \(\bar{P}(\cdot | \mathcal{F}_\tau) = Q\) by Lemma 6.12, we deduce from Lemma 6.9 that, for \(\bar{P}\)-a.a. \(\omega \in \Omega\), the third \(Q_\omega\)-characteristic of \(Y_{t^*} \tau\) is given by

\[
\nu^\bar{P}(\omega \otimes_{\tau(\omega)} X; \tau(\omega) - t^* + dt, dx).
\]
As $P$-a.s. $Y \in S^{\text{sp}}(\tau, Q)$, we have, for $\overline{P}$-a.a. $\omega \in \Omega$ and all $T > \tau(\omega) - t^*$, that

$$Q_\omega\left( \int_\tau^{T} \int (|x|^2 \wedge |x|) \nu^{\overline{P}}(\omega \otimes_{\tau(\omega)} X; dt, dx) < \infty \right) = 1.$$  

By Lemmata 6.7 [(i), (ii)] and 6.12, this yields, for all $T > 0$, that

$$\overline{P}\left( \int_{\tau - t^*}^{T} \int (|x|^2 \wedge |x|) \nu^{\overline{P}}(\omega \otimes_{\tau(\omega)} X; dt, dx) < \infty, T > \tau - t^* \right)$$

(6.2)

$$= \int_{\{T > \tau - t^*\}} \overline{P}\left( \int_{\tau(\omega) - t^*}^{T} \int (|x|^2 \wedge |x|) \nu^{\overline{P}}(\omega \otimes_{\tau(\omega)} X; dt, dx) < \infty | \mathcal{F}_\tau \right) (\omega) \overline{P}(d\omega)$$

$$= \int_{\{T > \tau - t^*\}} Q_\omega\left( \int_{\tau(\omega) - t^*}^{T} \int (|x|^2 \wedge |x|) \nu^{\overline{P}}(\omega \otimes_{\tau(\omega)} X; dt, dx) < \infty \right) \overline{P}(d\omega)$$

$$= \overline{P}(T > \tau - t^*).$$

Finally, (6.1) and (6.2) yield that

$$\overline{P}(\int (|x|^2 \wedge |x|) * \nu^{\overline{P}} < \infty) = 1, \quad T > 0.$$  

By virtue of [28, Proposition II.2.29], this implies that $Y \in S^{\text{sp}}(t^*, \overline{P})$.

**Step 3: Absolutely Continuous Compensator.** Let

$$A^{\overline{P}}(Y_{t^*}) = \int_0^{t^*} \phi_s ds + \psi$$

be the Lebesgue decomposition of (the paths) of $A^{\overline{P}}(Y_{t^*})$, cf. [2, Theorem 8.4.4]. Since $Y \in S^{\text{sp}}_{\text{ac}}(t^*, P)$ and $\overline{P} = P$ on $\mathcal{F}_\tau$ by Lemma 6.12, we get that $A^{\overline{P}}(Y_{t^*}) \ll \lambda$ on $[0, \tau - t^*]$. Hence, by Corollary 6.11, it suffices to show that

$$D \triangleq \left\{ A^{\overline{P}}_{-\tau-t^*}(Y_{-t^*}) - A^{\overline{P}}_{-t^*}(Y_{-t^*}) \neq \int_{-\tau-t^*}^{-t^*} \phi_s ds \right\}$$

is a $Q_\omega$-null set for $\overline{P}$-a.a. $\omega \in \Omega$. Due to Lemmata 6.7 (ii), 6.9 and 6.12, for $\overline{P}$-a.a. $\omega \in \Omega$, we have $Q_\omega$-a.s.

$$A^{\overline{P}}_{-\tau(\omega)-t^*}(Y_{-t^*}) - A^{\overline{P}}_{-t^*}(Y_{-t^*}) = A^{Q_\omega}(Y_{-\tau(\omega)}).$$

Since $P$-a.s. $Y \in S^{\text{sp}}_{\text{ac}}(\tau, Q)$, the uniqueness of the Lebesgue decomposition yields that, for $\overline{P}$-a.a. $\omega \in \Omega$,

$$Q_\omega\left( A^{\overline{P}}_{-\tau(\omega)-t^*}(Y_{-t^*}) - A^{\overline{P}}_{-t^*}(Y_{-t^*}) \neq \int_{-\tau(\omega)-t^*}^{-\tau(\omega)-t^*} \phi_s ds \right)$$

$$= Q_\omega\left( A^{Q_\omega}(Y_{-\tau(\omega)}) \neq \int_{\tau(\omega)-t^*}^{\tau(\omega)-t^*} \phi_s ds \right) = 0.$$  

Since $Q_\omega(\tau = \tau(\omega)) = 1$ for $\overline{P}$-a.a. $\omega \in \Omega$, we have $\overline{P}$-a.s. $Q(D) = 0$. This shows that $A^{\overline{P}}(Y_{t^*}) \ll \lambda$ and hence, completes the proof of $Y \in S^{\text{sp}}_{\text{ac}}(t^*, \overline{P})$. \qed

**Lemma 6.14.** Let $P \in \mathcal{C}(t^*, \omega^*)$. If $Q_\omega \in \mathcal{C}(\tau(\omega), \omega)$ for $P$-a.e. $\omega \in \Omega$, then $\overline{P} \in \mathcal{C}(t^*, \omega^*)$.

**Proof.** Lemma 6.13 implies that $Y^u \in S^{\text{sp}}_{\text{ac}}(t^*, \overline{P})$ for all $u \in U$. Recall that

$$\overline{P}(X = \omega^* \text{ on } [0, t^*]) = P(X = \omega^* \text{ on } [0, t^*]) = 1.$$
Consequently, as $\overline{P} = P$ on $\mathcal{F}_\tau$, it suffices to show that
\[ R \triangleq \{(t, \omega) \in [\tau - t^*, \infty[, (d\overline{\mathcal{P}}(Y_{u+t}^\omega))/d\lambda)_{u \in U} (t, \omega) \notin \Theta(t + t^*, \omega)\} \]
is a ($\lambda \otimes \overline{P}$)-null set. By virtue of Lemmata 6.9 and 6.12, the assumption that $P$-a.s.
$\mathcal{Q} \in \mathcal{C}(\tau, X)$, which is always in force).
by virtue of Lemmata 6.9 and 6.12, the assumption that $P$-a.s. $\mathcal{Q} \in \mathcal{C}(\tau, X)$ yields, for $P$-a.a. $\omega \in \Omega$, that
\[ (\lambda \otimes \mathcal{Q}_\omega)((t, \omega')) \in [\tau(\omega) - t^*, \infty[, (d\overline{\mathcal{P}}(Y_{u+t}^\omega))/d\lambda)_{u \in U} (t, \omega') \notin \Theta(t + t^*, \omega') \]
\[ = (\lambda \otimes \mathcal{Q}_\omega)((t, \omega')) \in [0, \infty[, (d\overline{\mathcal{P}}_{t+\tau(\omega) - t^*}(Y_{u+t}^\omega))/d\lambda)_{u \in U} (t, \omega') \notin \Theta(t + \tau(\omega), \omega') \]
\[ = (\lambda \otimes \mathcal{Q}_\omega)((t, \omega')) \in [0, \infty[, (d\overline{\mathcal{Q}}_{t+\tau(\omega)}(Y_{u+t}^\omega))/d\lambda)_{u \in U} (t, \omega') \notin \Theta(t + \tau(\omega), \omega') \] = 0.
Finally, as $Q_\omega(\tau = \tau(\omega)) = 1$ for $P$-a.a. $\omega \in \Omega$, Corollary 6.11 and Fubini’s theorem yield that
\[ (\lambda \otimes \overline{P})(R) = E^P \left[ \int_{\tau - t^*}^\infty 1 \{(d\overline{\mathcal{P}}(Y_{u+t}^\omega))/d\lambda)_{u \in U}(t, X) \notin \Theta(t + t^*, X)\} dt \right] \]
\[ = E^P \left[ E^Q \left[ \int_{\tau - t^*}^\infty 1 \{(d\overline{\mathcal{P}}(Y_{u+t}^\omega))/d\lambda)_{u \in U}(t, X) \notin \Theta(t + t^*, X)\} dt \right] \right] \]
\[ = E^P \left[ \int_{\tau - t^*}^\infty Q((d\overline{\mathcal{P}}(Y_{u+t}^\omega))/d\lambda)_{u \in U}(t, X) \notin \Theta(t + t^*, X)\} dt \right] = 0. \]
This completes the proof. □

6.4. **Proof of Theorem 2.5.** Corollary 6.6 and Lemmata 6.10 and 6.14 yield that the prerequisites of [17, Theorem 2.1] are fulfilled and this theorem implies the DPP. □

7. **Proof of the Strong Markov Selection Principle: Theorem 4.9**

The proof of the strong Markov selection principle is based on some fundamental ideas of Krylov [31] on Markovian selections as worked out in the monograph [53] by Stroock and Varadhan. The main technical steps in the argument are to establish stability under conditioning and pasting for a suitable sequence of correspondences. Hereby, we adapt the proof of [9, Theorem 2.19] to our more general framework.

This section is split into two parts. In the first, we study some properties of the correspondence $K$ and thereafter, we finalize the proof.

### 7.1 Preparations.

Let us stress that in this section we do not assume Condition 4.5. In each of the following results, we indicate precisely which prerequisites are used (except for our Standing Assumptions 2.1, 2.2 and 4.1, which are always in force).

For $t \in \mathbb{R}_+$, we define $\gamma_t: \Omega \to \Omega$ by $\gamma_t(\omega) \triangleq \omega((\cdot - t)^+)$ for $\omega \in \Omega$. Moreover, for $P \in \mathcal{P}(\Omega)$ and $t \in \mathbb{R}_+$, we set
\[ P^t \triangleq P \circ \gamma_t^{-1}. \]

Recall also the notation $P_t = P \circ \theta_t^{-1}$. In case the path space $\Omega$ is the space of continuous functions from $\mathbb{R}_+$ into $F$, we have the following extension of Lemma 6.2.

**Lemma 7.1.** Suppose that $\Omega = C(\mathbb{R}_+; F)$. The maps $(t, P) \mapsto P_t$ and $(t, P) \mapsto P^t$ are continuous.

**Proof.** Notice that $(t, \omega) \mapsto \theta_t(\omega)$ and $(t, \omega) \mapsto \gamma_t(\omega)$ are continuous by the Arzelà–Ascoli theorem. Now, the claim follows from [4, Theorem 8.10.61]. □

**Lemma 7.2.** For every $(t, \omega) \in [0, \infty[$, $P \in C(\tau, \omega)$ implies $P_t \in K(0, \omega(t))$. □
Proof. Let \((t, \omega) \in [0, \infty[,\) take \(P \in \mathcal{C}(t, \omega)\) and fix \(u \in U\). It is clear that \(P_t \circ X_0^{-1} = \delta_{\omega(t)}\).

Since, by hypothesis, \(Y^u_{\cdot+t} = Y^u \circ \theta_t\), we deduce from Lemma 6.4 and [28, Proposition II.2.29] that \(Y^u \in S^2_M(P_t)\) and \(\mathbb{P}\)-a.s. \(A^P(Y^u_{\cdot+t}) = A^P(Y^u) \circ \theta_t\). Hence, using also Standing Assumption 4.1, it follows that

\[
(\lambda \otimes P_t)((A^P(Y^u))_{u \in U} \not\in \Theta) = (\lambda \otimes P)((A^P(Y^u) \circ \theta_t)_{u \in U} \not\in \Theta(\cdot, X \circ \theta_t))
= (\lambda \otimes P)((A^P(Y^u_{\cdot+t}))_{u \in U} \not\in \Theta(\cdot + t, X)) = 0.
\]

We conclude that \(P_t \in \mathcal{K}(0, \omega(t))\). □

Lemma 7.3. For every \((t, x) \in \mathbb{R}_+ \times F\), we have \(P \in \mathcal{K}(0, x)\) if and only if \(P^t \in \mathcal{K}(t, x)\).

Proof. Take \(P \in \mathcal{K}(0, x)\). As \(\gamma^{-1}_t(\{X = x\} \cap [0, t]) = \{X_0 = x\}\), we have \(P^t(X = x) = 1\). Take \(u \in U\) and notice that \(Y^u = Y^u \circ \theta_t \circ \gamma_t = Y^u_{\cdot+t} \circ \gamma_t\), as \(\theta_t \circ \gamma_t = \mathbb{I}\). Hence, by Lemma 6.4 and [28, Proposition II.2.29], \(Y^u \in S^2_M(t, P^t)\) and \(\mathbb{P}\)-a.s. \(A^P(Y^u) = A^{P^t}(Y^u_{\cdot+t}) \circ \gamma_t\). From the last equality, and Standing Assumption 4.1, we deduce that

\[
(\lambda \otimes P)((A^P(Y^u_{\cdot+t}))_{u \in U} \not\in \Theta(\cdot + t, X)) = (\lambda \otimes P)((A^{P^t}(Y^u_{\cdot+t}))_{u \in U} \not\in \Theta((X \circ \gamma_t)(\cdot + t)))
= (\lambda \otimes P)((A^P(Y^u))_{u \in U} \not\in \Theta)
= 0.
\]

This proves that \(P^t \in \mathcal{K}(t, x)\). Conversely, take \(P^t \in \mathcal{K}(t, x)\). Due to the identity \(\theta_t \circ \gamma_t = \mathbb{I}\), we have \(P = (P^t)_t\) and Lemma 7.2 implies that \(P \in \mathcal{K}(0, x)\). The proof is complete. □

Lemma 7.4. For every \((t, x) \in \mathbb{R}_+ \times F\), we have \(\mathcal{K}(t, x) = \{P^t : P \in \mathcal{K}(0, x)\}\).

Proof. Lemma 7.3 yields the inclusion \(\{P^t : P \in \mathcal{K}(0, x)\} \subseteq \mathcal{K}(t, x)\). Conversely, take \(P \in \mathcal{K}(t, x)\). As \(P(X = x) = 1\), we have \(P = (P^t)_t\). Now, since \(P_t \in \mathcal{K}(0, x)\) by Lemma 7.2, we get \(\mathcal{K}(t, x) \subseteq \{P^t : P \in \mathcal{K}(0, x)\}\). □

Lemma 7.5. Suppose that \(\Omega = C(\mathbb{R}_+; F)\) and that \(\Omega \ni \omega \mapsto L(\omega) \in C(\mathbb{R}_+; \mathbb{R}_+)\) is continuous. Let \(P\) be a Borel probability measure on \(\Omega \times C(\mathbb{R}_+; \mathbb{R})\) and set

\[
\zeta_M(\omega, \alpha) \triangleq \sup \left\{ \frac{\alpha(t \wedge \rho_M(\omega)) - \alpha(s \wedge \rho_M(\omega))}{t - s} : 0 \leq s < t \right\},
\]

for \((\omega, \alpha) \in \Omega \times C(\mathbb{R}_+; \mathbb{R})\), where

\[
\rho_M = \inf\{t \geq 0 : L_t \geq M\}, \quad M > 0.
\]

Then, there exists a dense set \(D \subseteq \mathbb{R}_+\) such that, for every \(M \in D\), there exists a \(P\)-null set \(N = N(M)\) such that \(\zeta_M\) is lower semicontinuous at every \(\omega \not\in N\).

Proof. In the first part of this proof, we adapt an argument from [53, Lemma 11.1.2]. The map \(\omega \mapsto \rho_M(\omega)\) is lower semicontinuous. To see this, recall that

\[
\{\rho_M \leq t\} = \left\{ \inf_{s \in \mathbb{Q} \cap [0, t]} \inf_{z \geq M} |L_s - z| = 0 \right\}, \quad t \in \mathbb{R}_+,
\]

and notice that the set on the right hand side is closed, as \(\omega \mapsto \inf_{s \in \mathbb{Q} \cap [0, t]} \inf_{z \geq M} |L_s(\omega) - z|\) is continuous. Next, set

\[
\rho_M^+ \triangleq \inf\{t \geq 0 : L_t > M\}.
\]
Since, for every $t > 0$,
\[
\{ \rho^+_M < t \} = \bigcup_{s < t} \{ L_s > y \},
\]
where the set on the right hand side is open by the continuity of $\omega \mapsto L(\omega)$, we conclude that $\rho^+_M$ is upper semicontinuous. Notice that $\rho^+_M = \rho_{M^+} \triangleq \lim_{K \to M} \rho_K$. Define $\Psi: \Omega \times C(\mathbb{R}_+; \mathbb{R}) \to \Omega$ by $\Psi(\omega, \alpha) = \omega$, and $\phi: \mathbb{R}_+ \to [0, 1]$ by $\phi(M) \triangleq E^P[e^{-\rho_M \circ \Psi}]$, and set
\[
D \triangleq \left\{ M \in \mathbb{R}_+ : E^P[e^{-\rho_M \circ \Psi}] = E^P[e^{-\rho_{M^+} \circ \Psi}] \right\}.
\]
The dominated convergence theorem yields that
\[
D = \{ M \in \mathbb{R}_+ : \phi(M) = \phi(M+) \}, \quad \phi(M+) \triangleq \lim_{K \to M} \phi(K).
\]
The set $\mathbb{R}_+ \setminus D$ is countable, as monotone functions, such as $\phi$, have at most countably many discontinuities. Since $\rho_M \leq \rho^+_M$, for every $M \in D$ we have $P$-a.s. $\rho_M \circ \Psi = \rho^+_M \circ \Psi$.

To see this, notice that
\[
E^P\left[\left(e^{-\rho^+_M \circ \Psi} - e^{-\rho_M \circ \Psi}\right) 1_{\{\rho_M \circ \Psi < \rho^+_M \circ \Psi\}}\right] = E^P\left[e^{-\rho^+_M \circ \Psi} - e^{-\rho_M \circ \Psi}\right] = 0,
\]
which implies that $P(\rho_M \circ \Psi < \rho^+_M \circ \Psi) = 0$.

Now, take $\omega \in \{ \rho_M = \rho^+_M \}$. Since $\rho_M \leq \rho^+_M$, for every sequence $(\omega^n)_{n=1}^\infty \subset \Omega$ with $\omega^n \to \omega$, we obtain
\[
\rho_M(\omega) \leq \liminf_{n \to \infty} \rho_M(\omega^n) \leq \limsup_{n \to \infty} \rho_M(\omega^n) \leq \limsup_{n \to \infty} \rho^+_M(\omega^n) \leq \rho^+_M(\omega) = \rho_M(\omega),
\]
which implies that $\rho_M$ is continuous at $\omega$.

Finally, let $\Omega \times C(\mathbb{R}_+; \mathbb{R}) \ni (\omega^n, \alpha^n) \to (\omega, \alpha) \in \{ \rho_M \circ \Psi = \rho^+_M \circ \Psi \}$. Then,
\[
\zeta_M(\omega, \alpha) = \sup \left\{ \liminf_{n \to \infty} \frac{|\alpha^n(t \wedge \rho_M(\omega^n)) - \alpha^n(s \wedge \rho_M(\omega^n))|}{t-s} : 0 \leq s < t \right\} \leq \liminf_{n \to \infty} \zeta_M(\omega^n, \alpha^n).
\]

The proof is complete. \(\square\)

**Proposition 7.6.** Assume that (i) – (iv) from Condition 4.5 hold. Then, for every closed set $K \subset F$, the set $\mathcal{R}(K) \triangleq \bigcup_{x \in K} \mathcal{R}(x)$ is closed (in $\mathcal{F}(\Omega)$ endowed with the weak topology).

**Proof.** We adapt the proof strategy from [9, Proposition 3.8]. Let $(P^n)_{n=1}^\infty \subset \mathcal{R}(K)$ be such that $P^n \to P$ weakly. By definition of $\mathcal{R}(K)$, for every $n \in \mathbb{N}$, there exists a point $x^n \in K$ such that $P^n \in \mathcal{R}(x^n)$. Since $P^n \to P$ and $\{ \delta_{x^n} : x^n \in K \}$ is closed ([1, Theorem 15.8]), there exists a point $x^0 \in K$ such that $P \circ X_0^{-1} = \delta_{x^0}$. In particular, $x^n \to x^0$. We now prove that $P \in \mathcal{R}(x^0)$. Before we start our program, let us fix some auxiliary notation. We set $\Omega^* \triangleq \Omega \times C(\mathbb{R}_+; \mathbb{R})^U$ and endow this space with the product local uniform topology. In this case, the Borel $\sigma$-field $B(\Omega^*) \triangleq \mathcal{F}^*$ coincides with $\sigma(\mathcal{Z}_t, t \geq 0)$, where $\mathcal{Z} = (Z^{(1)}, (Z^{(2, u)})_{u \in U})$ denotes the coordinate process on $\Omega^*$. Furthermore, we define $\mathcal{F}^* \triangleq (\mathcal{F}_t^*)_{t \geq 0}$ to be the right-continuous filtration generated by the coordinate process $\mathcal{Z}$.

**Step 1.** In this first step, we show that the family $\{ P^n \circ (X, (A^{P^n}(Y^n))_{n \in U})^{-1} : n \in \mathbb{N} \}$ is tight when seen as a family of probability measures on $(\Omega^*, \mathcal{F}^*)$. As $\Omega^*$ is endowed with a product topology and since $P^n \to P$, it suffices to prove that, for every fixed $u \in U$, the family $\{ P^n \circ A^{P^n}(Y^n)^{-1} : n \in \mathbb{N} \}$ is tight when seen as a family of Borel probability measures on $C(\mathbb{R}_+; \mathbb{R})$, endowed with the local uniform topology. First, for every $M > 0$,
we deduce tightness of \( \{P^n \circ A^n_{\lambda \rho_M} (Y^u)^{-1} : n \in \mathbb{N}\} \) from Kolmogorov’s tightness criterion ([29, Theorem 23.7]). By the first part from Condition 4.5 (iii), there exists a constant \( C > 0 \) such that, for all \( s < t, \)
\[
\sup_{n \in \mathbb{N}} E^{P^n} [ |A^n_{\lambda \rho_M} (Y^u) - A^n_{\lambda \rho_M} (Y^u)|^2 ] \leq \sup_{n \in \mathbb{N}} E^{P^n} \left[ \left( \int_{s \wedge \rho_M}^{t \wedge \rho_M} |dA^n_{\lambda \rho_M} (Y^u)| d\lambda \right)^2 \right] \leq C(t - s)^2.
\]
We conclude the tightness of \( \{P^n \circ A^n_{\lambda \rho_M} (Y^u)^{-1} : n \in \mathbb{N}\} \), for every \( M > 0 \). Using the Arzelà–Ascoli tightness criterion given by [29, Theorem 23.4], we transfer this observation to the global family \( \{P^n \circ A^n_{\lambda \rho_M} (Y^u)^{-1} : n \in \mathbb{N}\} \). For a moment, fix \( \varepsilon, N > 0 \). By virtue of Condition 4.5 (iii), more precisely (4.2), there exists an \( M_0 > N \) such that
\[
\sup_{n \in \mathbb{N}} P^n (\rho_{M_0} \leq N) = \sup_{n \in \mathbb{N}} P^n \left( \sup_{s \in [0,N]} L_s \geq M_0 \right) \leq \varepsilon/2.
\]

Thanks to the tightness of the family \( \{P^n \circ A^n_{\lambda \rho_M} (Y^u)^{-1} : n \in \mathbb{N}\} \), there exists a compact set \( K_N \subset \mathbb{R} \) such that
\[
\sup_{n \in \mathbb{N}} P^n (A^n_{\lambda \rho_M} (Y^u) \notin K_N) \leq \varepsilon/2.
\]
Consequently, we obtain that
\[
\sup_{n \in \mathbb{N}} P^n (A^n_{\lambda \rho_M} (Y^u) \notin K_N) \leq \sup_{n \in \mathbb{N}} P^n (A^n_{\lambda \rho_M} (Y^u) \notin K_N) + \sup_{n \in \mathbb{N}} P^n (\rho_{M_0} \leq N) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

This observation shows that the family \( \{P^n \circ A^n_{\lambda \rho_M} (Y^u)^{-1} : n \in \mathbb{N}\} \) is tight. For \( r > 0, \omega \in C(\mathbb{R}_+; \mathbb{R}) \) and \( h > 0 \), we define
\[
w_{[0,r]}(\omega, h) \triangleq \sup \{ |\omega(s) - \omega(t)| : 0 \leq s, t \leq r, |s - t| \leq h \}.
\]
As \( \{P^n \circ A^n_{\lambda \rho_M} (Y^u)^{-1} : n \in \mathbb{N}\} \) is tight, [29, Theorem 23.4] yields that
\[
\sup_{n \in \mathbb{N}} E^{P^n} \left[ w_{[0,N]} (A^n_{\lambda \rho_M} (Y^u), h) \wedge 1 \right] \leq \sup_{n \in \mathbb{N}} E^{P^n} \left[ w_{[0,N]} (A^n_{\lambda \rho_M} (Y^u), h) \wedge 1 \right] + \sup_{n \in \mathbb{N}} P^n (\rho_{M_0} \leq N) \leq \sup_{n \in \mathbb{N}} E^{P^n} \left[ w_{[0,N]} (A^n_{\lambda \rho_M} (Y^u), h) \wedge 1 \right] + \varepsilon/2 \rightarrow \varepsilon/2
\]
as \( h \to 0 \). Using [29, Theorem 23.4] once again, but this time the converse direction, we conclude tightness of \( \{P^n \circ A^n_{\lambda \rho_M} (Y^u)^{-1} : n \in \mathbb{N}\} \), which implies those of \( \{P^n \circ (X, (A^n_{\lambda \rho_M} (Y^u))_{u \in U})^{-1} : n \in \mathbb{N}\} \) (on the respective probability space). Up to passing to a subsequence, we can assume that \( (P^n \circ (X, (A^n_{\lambda \rho_M} (Y^u))_{u \in U})^{-1})_{n=1}^\infty \) converges weakly (on the space \( (\Omega^*, F^*) \)) to a probability measure \( Q \).

**Step 2.** Recall that \( Z = (Z^{(1)}, (Z^{(2,u)})_{u \in U}) \) denotes the coordinate process on \( \Omega^* \) and fix some \( u \in U \). Next, we show that \( Z^{(2,u)} \) is \( Q \)-a.s. locally Lipschitz continuous. Thanks to Lemma 7.5, there exists a dense set \( D \subset \mathbb{R}_+ \) such that, for every \( M \in D \), the map \( \zeta_M \) is \( Q \circ (Z^{(1)}, Z^{(2,u)})^{-1} \)-a.s. lower semicontinuous. By part (iii) of Condition 4.5, for every \( M > 0 \), there exists a constant \( C = C(M) > 0 \) such that
\[
P^n(\zeta_M (X, A^n_{\lambda \rho_M} (Y^u)) \leq C) = 1
\]
all \( n \in \mathbb{N} \). Hence, for every \( M \in D \), using the \( Q \circ (Z^{(1)}, Z^{(2,u)})^{-1} \)-a.s. lower semicontinuity of \( \zeta_M \), we deduce from [47, Example 17, p. 73] that
\[
0 = \liminf_{n \to \infty} P^n(\zeta_M(X, A^n(Y^u)) > C) \geq Q(\zeta_M(Z^{(1)}, Z^{(2,u)}) > C).
\]
Further, since \( D \) is dense in \( \mathbb{R}_+ \), we can conclude that \( Z^{(2,u)} \) is \( Q \)-a.s. locally Lipschitz continuous and hence, in particular, locally absolutely continuous and of finite variation.

**Step 3.** Define the map \( \Phi : \Omega^* \to \Omega \) by \( \Phi(\omega^{(1)}, \omega^{(2)}) := \omega^{(1)} \) for \( \omega = (\omega^{(1)}, \omega^{(2)}) \in \Omega^* \).

In this step, we prove that \( (\lambda \otimes Q)\)-a.e. \((dZ^{(2,u)}/d\lambda)_{u \in U} \in \Theta \circ \Phi \). By virtue of [52, Theorems II.4.3 and II.6.2], \( P^n \)-a.s. for all \( t \in \mathbb{R}_+ \), we have
\[
(m(A^n_{t+1/m}(Y^u) - A^n_t(Y^u)))_{u \in U} \in \overline{\text{co}} \left( (dA^n_{t+1/m}(Y^u)/d\lambda)_{u \in U}[|t, t + 1/m]| \right) \subset \overline{\text{co}} \Theta([t, t + 1/m], X^0).
\]
(7.1)

Recall that \( \overline{\text{co}} \) denotes the closure of the convex hull. By Skorokhod’s coupling theorem ([29, Theorem 5.31]), there exist random variables
\[
(X^0, (B^{0,u})_{u \in U}), (X^1, (B^{1,u})_{u \in U}), (X^2, (B^{2,u})_{u \in U}), \ldots
\]
defined on some probability space \((\Sigma, \mathcal{G}, P)\) such that, for every \( n \in \mathbb{N} \), \((X^n, (B^{n,u})_{u \in U})\) has distribution \( P^n \circ (X, (A^n(Y^u))_{u \in U})^{-1} \), \((X^0, (B^{0,u})_{u \in U})\) has distribution \( Q \), and \( P^n \)-a.s. \((X^n, (B^{n,u})_{u \in U}) \to (X^0, (B^{0,u})_{u \in U})\). Thanks to the assumption (see (iv) of Condition 4.5) that the correspondence
\[
\omega \mapsto \overline{\text{co}} \Theta([t, t + 1/m], \omega)
\]
is upper hemicontinuous with compact values, we deduce from (7.1) and [1, Theorem 17.20] that, for every \( m \in \mathbb{N} \), \( P^n \)-a.s. for all \( t \in \mathbb{R}_+ \)
\[
(m(B^{0,u}_{t+1/m} - B^{0,u}_t))_{u \in U} = \lim_{n \to \infty} (m(B^{n,u}_{t+1/m} - B^{n,u}_t))_{u \in U} \in \overline{\text{co}} \Theta([t, t + 1/m], X^0).
\]
(7.2)

Notice that \( (\lambda \otimes P)\)-a.e.
\[
(dB^{0,u}/d\lambda)_{u \in U}(t) = \lim_{m \to \infty} (m(B^{0,u}_{t+1/m} - B^{0,u}_t))_{u \in U}.
\]
(7.3)

Using (7.2) and (7.3), we deduce from (ii) and (iv) of Condition 4.5, and [9, Lemma 3.4], that \( P^n \)-a.s. for \( \lambda \)-a.a. \( t \in \mathbb{R}_+ \)
\[
(dB^{0,u}/d\lambda)_{u \in U}(t) \in \bigcap_{m \in \mathbb{N}} \overline{\text{co}} \Theta([t, t + 1/m], X^0) \subset \Theta(t, X^0).
\]

This proves that \( (\lambda \otimes Q)\)-a.e. \((dZ^{(2,u)}/d\lambda)_{u \in U} \in \Theta \circ \Phi \).

**Step 4.** In the final step of the proof, we show that \( Y^u \in S_{ac}^\infty(P) \) and we relate \( B^{0,u} \) to \( A^P(Y^u) \). Notice that \( Q \circ \Phi^{-1} = P \). For a moment, we fix \( u \in U \). As in the proof of Lemma 7.5, we obtain the existence of a dense set \( D = D^u \subset \mathbb{R}_+ \) such that \( \rho_M \circ \Phi \) is \( Q \)-a.s. continuous for all \( M \in D \). Take some \( M \in D \). Since \( Y^u \in S_{ac}^\infty(P^n) \), the process \( Y^u_{\lambda;\rho_M} - A^n_{\lambda;\rho_M}(Y^u) \) is a \( P^n\)-local martingale. Furthermore, by (i) and (iii) from Condition 4.5, we see that \( Y^u_{\lambda;\rho_M} - A^n_{\lambda;\rho_M}(Y^u) \) is \( P^n\)-a.s. bounded by a constant independent of \( n \), which in particular, implies that it is a true \( P^n\)-martingale. Recall from part (i) of Condition 4.5, that \( \omega \mapsto Y^u(\omega) \) is continuous. Now, it follows\footnote{[28, Proposition IX.1.4] is stated for \( F = \mathbb{R}^d \) but the argument needs no change to work for more general state spaces.} from [28, Proposition IX.1.4] that
\[
Y^u_{\lambda;\rho_M} \circ \Phi - Z^{(2,u)}_{\lambda;\rho_M} \circ \Phi
\]
is a $Q$-$\mathbf{F}^\ast$-martingale. Since $Z^{(2,u)}$ is $Q$-a.s. locally absolutely continuous by Step 2, this means that $Y^u \circ \Phi$ is a $Q$-$\mathbf{F}^\ast$-semimartingale whose first characteristic is given by $Z^{(2,u)}$.

Next, we relate this observation to the probability measure $P$ and the filtration $\mathbf{F}_+$. For a process $A$ on $(\Omega^*, \mathcal{F}^*)$, we denote by $A^e \Phi^{-1}(\mathbf{F}_+)$ its dual predictable projection to the filtration $\Phi^{-1}(\mathbf{F}_+)$. Recall from [26, Lemma 10.42] that, for every $t \in \mathbb{R}_+$, a random variable $V$ on $(\Omega^*, \mathcal{F}^*)$ is $\Phi^{-1}(\mathcal{F}_t)$-measurable if and only if it is $\mathcal{F}_t$-measurable and $V(\omega^{(1)}, \omega^{(2)})$ does not depend on $\omega^{(2)}$. Thanks to Stricker’s theorem ([27, Lemma 2.7]), the process $Y^u \circ \Phi$ is a $Q$-$\Phi^{-1}(\mathbf{F}_+)$-semimartingale. Recall from Step 3 that $(\mathcal{X} \otimes \mathcal{Q})$-a.e. $(dZ^{(2,u)}/d\lambda)_{u \in U} \in \Theta \circ \Phi$. By virtue of (iii) from Condition 4.5, for every $M \in \mathcal{D}$, we have

$$E^Q[\text{Var}(Z^{(2,u)}_{\rho_M \circ \Phi})] = E^Q\left[\int_0^{\rho_M \circ \Phi} \left| \frac{dZ^{(2,u)}}{d\lambda} \right| d\lambda \right] < \infty,$$

where $\text{Var}(\cdot)$ denotes the total variation process. Hence, we get from [26, Proposition 9.24] that the first $Q$-$\Phi^{-1}(\mathbf{F}_+)$-characteristic of $Y^u \circ \Phi$ is given by $(Z^{(2,u)}_{\rho_M \circ \Phi})$. Lemma 6.4 yields that $Y^u$ is a $P$-$\mathbf{F}_+$-semimartingale whose first characteristic $A^P(Y^u)$ satisfies

$$A^P(Y^u) \circ \Phi = (Z^{(2,u)}_{\rho_M \circ \Phi}).$$

Consequently, we deduce from the Steps 2 and 3 that $P$-a.s. $A^P(Y^u) \ll \lambda$ and

$$(\mathcal{X} \otimes P)((A^P(Y^u))_{u \in U} \not\in \Theta) = (\mathcal{X} \otimes \Phi^{-1})((dA^P(Y^u)/d\lambda)_{u \in U} \not\in \Theta)
= (\mathcal{X} \otimes \Phi)[E^Q[(dZ^{(2,u)}/d\lambda)_{u \in U}|\Phi^{-1}(\mathbf{F}_+)] \not\in \Theta \circ \Phi] = 0,$$

where we use (ii) and (iii) from Condition 4.5 and [52, Theorems II.4.3 and II.6.2] for the final equality. This observation completes the proof. \hfill \Box

Given the previous observations, the following result can be proved similar to [9, Proposition 5.7]. For reader’s convenience, we provide the details.

**Proposition 7.7.** Suppose that Condition 4.5 holds. The correspondence $(t, x) \mapsto \mathcal{K}(t, x)$ is upper hemicontinuous with nonempty and compact values.

**Proof.** The correspondence $x \mapsto \mathcal{K}(0, x)$ has nonempty values by Standing Assumption 2.2. Further, it has compact values by Proposition 7.6 and part (v) of Condition 4.5. Thus, Lemmata 7.1 and 7.4 yield that the same is true for $(t, x) \mapsto \mathcal{K}(t, x)$.

It remains to show that $\mathcal{K}$ is upper hemicontinuous. Let $G \subset \Psi(\Omega)$ be closed. We need to show that $\mathcal{K}(G) = \{(t, x) \in \mathbb{R}_+ \times \mathcal{F}: \mathcal{K}(t, x) \cap G \neq \emptyset\}$ is closed. Suppose that the sequence $(t^n, x^n)_{n=1}^\infty \subset \mathcal{K}(G)$ converges to $(t^0, x^0) \in \mathbb{R}_+ \times \mathcal{F}$. For each $n \in \mathbb{N}$, there exists a probability measure $P^n \in \mathcal{K}(t^n, x^n) \cap G$. By part (v) of Condition 4.5 and Proposition 7.6, the set $\mathcal{R}^\circ \triangleq \bigcup_{n=0}^\infty \mathcal{R}(x^n)$ is compact. Hence, by Lemma 7.1, so is the set

$$\mathcal{K}^\circ \triangleq \{ P^n: (t, P) \in \{ t^n: n \in \mathbb{Z}_+ \} \times \mathcal{R}^\circ \}.$$

By virtue of Lemma 7.4, we conclude that $\{ P^n: n \in \mathbb{N} \} \subset \mathcal{K}^\circ$ is relatively compact. Hence, passing to a subsequence if necessary, we can assume that $P^n \rightarrow P$ weakly for some $P \in \mathcal{K} \cap \mathcal{G}$. Let $d_F$ be a metric on $\mathcal{F}$ which induces its topology. For every $\varepsilon \in (0, t^0)$, the set $\{ d_F(X_s, x^0) \leq \varepsilon \text{ for all } s \in [0, t^0]-\varepsilon\} \subset \Omega$ is closed. Consequently, by the Portmanteau theorem, for every $\varepsilon \in (0, t^0)$, we get

$$1 = \limsup_{n \rightarrow \infty} P^n(d_F(X_s, x^0) \leq \varepsilon \text{ for all } s \in [0, t^0]-\varepsilon)\notag
\leq P(d_F(X_s, x^0) \leq \varepsilon \text{ for all } s \in [0, t^0]-\varepsilon).$$
It follows that $P(X = x^0$ on $[0,t^0]) = 1$, which implies that $P = (P_t^\omega)^{t^0}$. By Lemmata 7.1 and 7.2, we have $(P^\omega)_t \in \mathcal{K}(0,x^0)$ and $(P^\omega)_t \to P^\omega$ weakly. Further, since $P^\omega \circ X_0^{-1} = \delta_{x^0}$, Proposition 7.6 yields that $P^\omega \in \mathcal{K}(0,x^0)$. Thus, by virtue of Lemma 7.3, $P \in \mathcal{K} \cap G \cap \mathcal{K}(t^0,x^0) = \mathcal{K}(t^0,x^0) \cap G$, which implies $(t^0,x^0) \in \mathcal{K}^2(G)$. We conclude that $\mathcal{K}$ is upper hemicontinuous.

\[\Box\]

**Lemma 7.8.** Suppose that part (ii) of Condition 4.5 holds. The correspondence $(t,x) \mapsto \mathcal{K}(t,x)$ has convex values.

**Proof.** By virtue of Lemma 7.4, it suffices to prove that $\mathcal{K}(0,x)$ is convex for every $x \in F$. Indeed, for every $P,Q \in \mathcal{K}(t,x)$ and $\alpha \in (0,1)$, there are probability measures $P,Q \in \mathcal{K}(0,x)$ such that $P^t = P$ and $Q^t = Q$. Then, $\alpha P + (1 - \alpha)Q = (\alpha P + (1 - \alpha)Q)^t$ and consequently, from Lemma 7.3, we get $\alpha P + (1 - \alpha)Q \in \mathcal{K}(t,x)$ once $\alpha P + (1 - \alpha)Q \in \mathcal{K}(0,x)$. As $\mathcal{O}$ is assumed to be convex-valued by Condition 4.5 (ii), the convexity of $\mathcal{K}(0,x)$ follows from [28, Lemma III.3.38, Theorem III.3.40]. To make the argument precise, take $P,Q \in \mathcal{K}(0,x)$ and $\alpha \in (0,1)$, and set $R \triangleq \alpha P + (1 - \alpha)Q$. Clearly, $R \circ X_0^{-1} = \delta_x$. Moreover, we have $P \ll R$ and $Q \ll R$ and, by [28, Lemma III.3.38], there are versions $Z^P$ and $Z^Q$ of the Radon–Nikodym densities $dP/dR$ and $dQ/dR$, respectively, such that

$$\alpha Z^P + (1 - \alpha)Z^Q = 1, \quad 0 \leq Z^P \leq 1/\alpha, \quad 0 \leq Z^Q \leq 1/(1 - \alpha).$$

For every $u \in U$, [28, Proposition II.2.29, Theorem III.3.40] yields that $Y^u \in \mathcal{S}_{ac}^F(R)$ and that $(\mathcal{O} \otimes R)$-a.e.

$$dA^R(Y^u)/d\lambda = \alpha Z^P dA^P(Y^u)/d\lambda + (1 - \alpha)Z^Q dA^Q(Y^u)/d\lambda.$$ 

Notice that

$$\int \int Z^P \mathbb{1}\{(dA^P(Y^u)/d\lambda)_{u \in U} \notin \Theta, Z^P > 0\}d(\mathcal{O} \otimes R) = (\mathcal{O} \otimes P)(\{(dA^P(Y^u)/d\lambda)_{u \in U} \notin \Theta, Z^P > 0\} = 0.\text{ }}$$

Consequently, $(\mathcal{O} \otimes R)$-a.e. $\mathbb{1}\{(dA^Q(Y^u)/d\lambda)_{u \in U} \notin \Theta, Z^Q > 0\} = 0$. In the same manner, we obtain that $(\mathcal{O} \otimes R)$-a.e. $\mathbb{1}\{(dA^Q(Y^u)/d\lambda)_{u \in U} \notin \Theta, Z^Q > 0\} = 0$. Finally, as $\alpha Z^P + (1 - \alpha)Z^Q = 1$, using the convexity of $\Theta$, we get that

$$(\mathcal{O} \otimes (dA^R(Y^u)/d\lambda) u \in U \notin \Theta) - (\mathcal{O} \otimes (dA^R(Y^u)/d\lambda) u \in U \notin \Theta, Z^P > 0, Z^Q = 0)$$

$$+ (\mathcal{O} \otimes (dA^Q(Y^u)/d\lambda) u \in U \notin \Theta, Z^P = 0, Z^Q > 0)$$

$$+ (\mathcal{O} \otimes (dA^R(Y^u)/d\lambda) u \in U \notin \Theta, Z^P > 0, Z^Q > 0)$$

$$= (\mathcal{O} \otimes (dA^R(Y^u)/d\lambda) u \in U + (1 - \alpha)Z^Q dA^Q(Y^u)/d\lambda) u \in U \notin \Theta)$$

$$= (dA^R(Y^u)/d\lambda) u \in U \notin \Theta, (dA^Q(Y^u)/d\lambda) u \in U \in \Theta, Z^P > 0, Z^Q > 0)$$

$$= 0.$$

We conclude that $R \in \mathcal{K}(0,x)$. The proof is complete. \[\Box\]

**Lemma 7.9.** Let $Q \in \mathcal{P}(\Omega)$ and take $t \in \mathbb{R}_+, \omega, \alpha \in \Omega$ such that $\omega(t) = \alpha(t)$. Then,

$$\delta_\omega \otimes t Q \in \mathcal{C}(t,\omega) \iff \delta_\alpha \otimes t Q \in \mathcal{C}(t,\omega).$$

**Proof.** Set $Q \triangleq \delta_\omega \otimes t Q$ and $P \triangleq \delta_\alpha \otimes t Q$. Suppose that $Q \in \mathcal{C}(t,\alpha)$. Thanks to Lemma 7.2, we have $Q_t \in \mathcal{C}(0,\alpha(t)) = \mathcal{C}(0,\omega(t))$. Since $Q_t = P_t$, we also have $P_t \in \mathcal{C}(0,\omega(t))$. We
deduce from Lemma 6.4 that $Y^u \in \mathcal{S}^u_{\text{sp}}(t, P)$ and that $P$-a.s. $A^P(Y^u_{\tau + t}) = A^P(Y^u) \circ \theta_t$. Hence, using Standing Assumption 4.1, we get that

$$(\lambda \otimes P)((A^P(Y^u_{\tau + t}))_{u \in U} \notin \Theta(\cdot, \tau), X)) = (\lambda \otimes P)((A^P(Y^u) \circ \theta_t)_{u \in U} \notin \Theta(\cdot, X \circ \theta_t)) = (\lambda \otimes P)((A^P(Y^u))_{u \in U} \notin \Theta) = 0.$$}

In summary, $P \in \mathcal{C}(t, \omega)$. The converse implication follows by symmetry. □

**Definition 7.10.** A correspondence $U: \mathbb{R}_+ \times F \to \mathfrak{F}(\Omega)$ is said to be

(i) stable under conditioning if for any $(t, x) \in \mathbb{R}_+ \times F$, any stopping time $\tau$ with $t \leq \tau < \infty$, and any $P \in \mathcal{U}(t, x)$, there exists a $P$-null set $N \in \mathcal{F}_\tau$ such that $\delta_{\omega(\tau(\omega))} \otimes_{\omega(\tau(\omega))} P(\cdot | \mathcal{F}_\tau)(\omega) \in U(\tau(\omega), \omega(\tau(\omega)))$ for all $\omega \notin N$;

(ii) stable under pasting if for any $(t, x) \in \mathbb{R}_+ \times F$, any stopping time $\tau$ with $t \leq \tau < \infty$, any $P \in \mathcal{U}(t, x)$ and any $\mathcal{F}_\tau$-measurable map $\Omega \ni \omega \mapsto Q_\omega \in \mathfrak{F}(\Omega)$ the following implication holds:

$$P$-$\text{a.a.} \; \omega \in \Omega \; \delta_{\omega(\tau(\omega))} \otimes_{\omega(\tau(\omega))} Q_\omega \in U(\tau(\omega), \omega(\tau(\omega))) \implies P \otimes_{\tau} Q \in \mathcal{U}(t, x).$$

**Lemma 7.11.** The correspondence $\mathcal{K}$ is stable under conditioning and pasting.

**Proof.** Stability under conditioning follows from Lemmata 6.10 and 7.9, and stability under pasting follows from Lemmata 6.14 and 7.9. □

Recall from [1, Definition 18.1] that a correspondence $U: \mathbb{R}_+ \times F \to \mathfrak{F}(\Omega)$ is called measurable if the lower inverse $\{(t, x) \in \mathbb{R}_+ \times F: U(t, x) \cap G \neq \emptyset\}$ is Borel measurable for every closed set $G \subset \mathfrak{F}(\Omega)$. The proof of the following result is similar to those of [9, Lemma 5.12]. We added it for reader’s convenience and because it explains why both stability properties, i.e., stability under conditioning and pasting, are needed, see also Remark 7.13 below.

**Lemma 7.12.** Suppose that $U: \mathbb{R}_+ \times F \to \mathfrak{F}(\Omega)$ is a measurable correspondence with nonempty and compact values such that, for all $(t, x) \in \mathbb{R}_+ \times F$ and $P \in \mathcal{U}(t, x)$, $P(X = x \; \text{on} \; [0, t]) = 1$. Suppose that $U$ is stable under conditioning and pasting. Then, for every $\phi \in \text{USC}_b(F; \mathbb{R})$, the correspondence

$$U^*(t, x) \triangleq \left\{ P \in \mathcal{U}(t, x): E^P[\phi(X_T)] = \sup_{Q \in \mathcal{U}(t, x)} E^Q[\phi(X_T)] \right\}, \; (t, x) \in \mathbb{R}_+ \times F,$$

is also measurable with nonempty and compact values and it is stable under conditioning and pasting. Further, if $U$ has convex values, then so does $U^*$.

**Proof.** We adapt the proof of [53, Lemma 12.2.2], see also the proofs of [22, Lemma 3.4 (c), (d)]. As $\psi$ is assumed to be upper semicontinuous, [1, Theorem 2.43] implies that $U^*$ has nonempty and compact values. Further, [1, Theorem 18.10] and [53, Lemma 12.1.7] imply that $U^*$ is measurable. The final claim about convexity is obvious. It is left to show that $U^*$ is stable under conditioning and pasting. Take $(t, x) \in \mathbb{R}_+ \times F, P \in U^*(t, x)$ and let $\tau$ be a stopping time such that $t \leq \tau < \infty$. We define

$$N \triangleq \left\{ \omega \in \Omega: P_\omega \triangleq \delta_{\omega(\tau(\omega))} \otimes_{\omega(\tau(\omega))} P(\cdot | \mathcal{F}_\tau)(\omega) \notin U(\tau(\omega), \omega(\tau(\omega))) \right\},$$

$$A \triangleq \left\{ \omega \in \Omega \setminus N: P_\omega \notin U^*(\tau(\omega), \omega(\tau(\omega))) \right\}.$$ 

As $U$ is stable under conditioning, we have $P(N) = 0$. By [53, Lemma 12.1.9], $N, A \in \mathcal{F}_\tau$. As we already know that $U^*$ is measurable, by virtue of [53, Theorem 12.1.10], there
exists a measurable map \((s, y) \mapsto R(s, y)\) such that \(R(s, y) \in U^*(s, y)\). We set \(R_\omega \triangleq R(\tau(\omega), \omega(\tau(\omega)))\), for \(\omega \in \Omega\), and note that \(\omega \mapsto R_\omega\) is \(\mathcal{F}_\tau\)-measurable. Further, we set

\[
Q_\omega \triangleq \begin{cases} 
R_\omega, & \omega \in N \cup A, \\
\varnothing, & \omega \notin N \cup A.
\end{cases}
\]

By definition of \(R\) and \(N\), \(Q_\omega \in U(\tau(\omega), \omega(\tau(\omega)))\) for all \(\omega \in \Omega\). As \(U\) is stable under pasting, \(P \otimes \tau Q \in U(t, x)\). We obtain

\[
\sup_{Q^* \in U(t, x)} E^{Q^*}[\phi(X_T)] \\
\geq E^{P \otimes \tau Q}[\phi(X_T)] \\
= \int_{N \cup A} E^{\delta_{\omega \tau(\omega)} R_\omega}[\phi(X_T)] P(d\omega) + E^P \left[1_{N \cap A} E^P[\phi(X_T) | \mathcal{F}_\tau]\right] \\
= \int_A E^{\delta_{\omega \tau(\omega)} R_\omega}[\phi(X_T)] P(d\omega) + E^P \left[1_A E^P[\phi(X_T) | \mathcal{F}_\tau]\right] \\
= \int_A \left[E^{\delta_{\omega \tau(\omega)} R_\omega}[\phi(X_T)] - E^{\delta_{\omega \tau(\omega)} P_\omega}[\phi(X_T)]\right] P(d\omega) + \sup_{Q^* \in U(t, x)} E^{Q^*}[\phi(X_T)] \\
= \int_A \left[E^{R_\omega}[\phi(X_T)] - E^{P_\omega}[\phi(X_T)]\right] P(d\omega) + \sup_{Q^* \in U(t, x)} E^{Q^*}[\phi(X_T)].
\]

As \(E^{R_\omega}[\phi(X_T)] > E^{P_\omega}[\phi(X_T)]\) for all \(\omega \in A\), we conclude that \(P(A) = 0\). This proves that \(U^*\) is stable under conditioning.

Next, we prove stability under pasting. Let \((t, x) \in \mathbb{R}_+ \times F\), take a stopping time \(\tau\) with \(t \leq \tau < \infty\), a probability measure \(P \in U^*(t, x)\) and an \(\mathcal{F}_\tau\)-measurable map \(\Omega \ni \omega \mapsto Q_\omega \in \mathfrak{P}(\Omega)\) such that, for \(P\)-a.a. \(\omega \in \Omega\), \(\delta_{\omega(\tau(\omega))} \otimes_{\tau(\omega)} Q_\omega \in U^*(\tau(\omega), \omega(\tau(\omega)))\). As \(U\) is stable under pasting, we have \(P \otimes \tau Q \in U(t, x)\). Further, recall that \(\delta_{\omega(\tau(\omega))} \otimes_{\tau(\omega)} P(\cdot | \mathcal{F}_\tau)(\omega) \in U(\tau(\omega), \omega(\tau(\omega)))\) for \(P\)-a.a. \(\omega \in \Omega\), as \(U\) is stable under conditioning. Thus, we get

\[
\sup_{Q^* \in U(t, x)} E^{Q^*}[\phi(X_T)] \\
\geq E^{P \otimes \tau Q}[\phi(X_T)] \\
= \int E^{\delta_{\omega(\tau(\omega))} Q_\omega}[\phi(X_T)] P(d\omega) \\
= \int E^{\delta_{\omega(\tau(\omega))} \otimes_{\tau(\omega)} Q_\omega}[\phi(X_T)] P(d\omega) + E^P \left[\phi(X_T) 1_{\tau(\omega) < T} \right] \\
= \sup_{Q^* \in U(\tau(\omega), \omega(\tau(\omega)))} \int E^{Q^*}[\phi(X_T)] P(d\omega) + E^P \left[\phi(X_T) 1_{\tau(\omega) \leq T} \right] \\
\geq \int \left[E^{\delta_{\omega(\tau(\omega))} \otimes_{\tau(\omega)} P(\cdot | \mathcal{F}_\tau)(\omega)}[\phi(X_T)] 1_{\tau(\omega) < T} \right] P(d\omega) + E^P \left[\phi(X_T) 1_{\tau(\omega) \leq T} \right] \\
= E^P \left[\phi(X_T) 1_{\tau(\omega) \leq T} \right] \\
= \sup_{Q^* \in U(t, x)} E^{Q^*}[\phi(X_T)].
\]

This implies that \(P \otimes \tau Q \in U^*(t, x)\). The proof is complete.
7.2. Proof of Theorem 4.9. We follow the proof of [9, Theorem 2.19], which adapts the proofs of [53, Theorems 6.2.3 and 12.2.3], cf. also the proofs of [16, Proposition 6.6] and [22, Proposition 3.2]. In the following, Condition 4.5 is in force.

We fix a finite time horizon \( T > 0 \) and a function \( \psi \in USC_b(F; \mathbb{R}) \). As \( F \) is a Polish space, there exists an equivalent metric \( d_F \) with respect to which \( F \) is totally bounded (see [44, p. 43] or [53, p. 9]). Let \( U_{d_F}(F) \) be the space of all bounded uniformly continuous functions on \((F, d_F)\) endowed with the uniform topology. Then, by [44, Lemma 6.3], \( U_{d_F}(F) \) is separable. Furthermore, by [44, Theorem 5.9], \( U_{d_F}(F) \) is measure determining.

Let \( \{\sigma_n : n \in \mathbb{N}\} \) be a dense subset of \((0, \infty)\), let \( \{\phi_n : n \in \mathbb{N}\} \) be a dense subset of \( U_{d_F}(F) \) and let \( (\lambda_N, f_N)_{N=1}^\infty \) be an enumeration of \( \{(\sigma_m, \phi_n) : n, m \in \mathbb{N}\} \). For \((t, x) \in \mathbb{R}_+ \times F\), define inductively

\[
U_0(t, x) \triangleq \left\{ P \in \mathcal{K}(t, x) : E^P[\psi(X_T)] = \sup_{Q \in \mathcal{K}(t, x)} E^Q[\psi(X_T)] \right\}
\]

and

\[
U_{N+1}(t, x) \triangleq \left\{ P \in U_N(t, x) : E^P[f_{N+1}(X_{\lambda_{N+1}})] = \sup_{Q \in U_N(t, x)} E^Q[f_{N+1}(X_{\lambda_{N+1}})] \right\}
\]

for \( N \in \mathbb{Z}_+ \). Furthermore, we set

\[
U_\infty(t, x) \triangleq \bigcap_{N=0}^\infty U_N(t, x).
\]

Thanks to Proposition 7.7 and Lemmata 7.8 and 7.11, the correspondence \( \mathcal{K} \) is measurable with nonempty convex and compact values and it is further stable under conditioning and pasting. Thus, by Lemma 7.12, the same is true for \( U_0 \) and, by induction, also for every \( U_N, N \in \mathbb{N} \). As (arbitrary) intersections of convex and compact sets are itself convex and compact, \( U_\infty \) has convex and compact values. By Cantor’s intersection theorem, \( U_\infty \) has nonempty values, and, by [1, Lemma 18.4], \( U_\infty \) is measurable. Moreover, it is clear that \( U_\infty \) is stable under conditioning, as this is the case for every \( U_N, N \in \mathbb{Z}_+ \).

We now show that \( U_\infty \) is singleton-valued. Take \( P, Q \in U_\infty(t, x) \) for some \((t, x) \in \mathbb{R}_+ \times F\). By definition of \( U_\infty \), we have

\[
E^P[f_N(X_{\lambda_N})] = E^Q[f_N(X_{\lambda_N})], \quad N \in \mathbb{N}.
\]

This implies that \( P \circ X_s^{-1} = Q \circ X_s^{-1} \) for all \( s \in \mathbb{R}_+ \). Next, we prove that

\[
E^P\left[ \prod_{k=1}^n g_k(X_{t_k}) \right] = E^Q\left[ \prod_{k=1}^n g_k(X_{t_k}) \right]
\]

for all \( g_1, \ldots, g_n \in C_b(F; \mathbb{R}) \), \( t_1 < t_2 < \cdots < t_n < \infty \) and \( n \in \mathbb{N} \). We use induction over \( n \). For \( n = 1 \) the claim is implied by the equality \( P \circ X_s^{-1} = Q \circ X_s^{-1} \) for all \( s \in \mathbb{R}_+ \). Suppose that the claim holds for \( n \in \mathbb{N} \) and take test functions \( g_1, \ldots, g_{n+1} \in C_b(F; \mathbb{R}) \) and times \( t \leq t_1 < \cdots < t_{n+1} < \infty \). We define

\[
\mathcal{G}_n \triangleq \sigma(X_{t_k}, k = 1, \ldots, n).
\]

Since

\[
E^P\left[ \prod_{k=1}^{n+1} g_k(X_{t_k}) \right] = E^P\left[ E^P[g_{n+1}(X_{t_{n+1}}) \mid \mathcal{G}_n] \prod_{k=1}^n g_k(X_{t_k}) \right],
\]

it suffices to show that \( P \)-a.s.

\[
E^P[g_{n+1}(X_{t_{n+1}}) \mid \mathcal{G}_n] = E^Q[g_{n+1}(X_{t_{n+1}}) \mid \mathcal{G}_n].
\]
As $\mathcal{U}_\infty$ is stable under conditioning, there exists a null set $N_1 \in \mathcal{F}_{t_n}$ such that $\delta_{\omega(t_n)} \otimes t_n P(\cdot | \mathcal{F}_{t_n})(\omega) \in \mathcal{U}_\infty(t_n, \omega(t_n))$ for all $\omega \notin N_1$. Notice that, by the tower rule, there exists a $P$-null set $N_2 \in \mathcal{G}_n$ such that, for all $\omega \notin N_2$ and all $A \in \mathcal{F}$,

$$
\int \delta_{\omega(t_n)} \otimes t_n P(A | \mathcal{F}_{t_n})(\omega') P(d\omega' | \mathcal{G}_n)(\omega) \\
= \int \int 1_A(\omega' \otimes t_n) P(d\alpha | \mathcal{F}_{t_n})(\omega') P(d\omega' | \mathcal{G}_n)(\omega) \\
= \int \int 1_A(\omega(t_n) \otimes t_n) P(d\alpha | \mathcal{F}_{t_n})(\omega') P(d\omega' | \mathcal{G}_n)(\omega) \\
= \int \int 1_A(\omega(t_n) \otimes t_n) P(d\omega' | \mathcal{G}_n)(\omega) \\
= (\delta_{\omega(t_n)} \otimes t_n) P(\cdot | \mathcal{G}_n)(\omega')(A).
$$

Let $N_3 \triangleq \{ P(N_1 | \mathcal{G}_n) > 0 \} \in \mathcal{G}_n$. Clearly, $E^P[P(N_1 | \mathcal{G}_n)] = P(N_1) = 0$, which implies that $P(N_3) = 0$. For a moment, take $\omega \notin N_2 \cup N_3$. Using that $\mathcal{U}_\infty$ has convex and compact values and that $\delta_{\omega(t_n)} \otimes t_n P(\cdot | \mathcal{F}_{t_n})(\omega') \in \mathcal{U}_\infty(t_n, \omega'(t_n))$ for all $\omega' \notin N_1$, we obtain that

$$
\int \delta_{\omega'(t_n)} \otimes t_n P(A | \mathcal{F}_{t_n})(\omega') P(d\omega' | \mathcal{G}_n)(\omega) \in \mathcal{U}_\infty(t_n, \omega(t_n)).
$$

Consequently, by virtue of (7.4), we also have $\delta_{\omega(t_n)} \otimes t_n P(\cdot | \mathcal{G}_n)(\omega) \in \mathcal{U}_\infty(t_n, \omega(t_n))$. Similarly, there exists a $Q$-null set $N_4 \in \mathcal{G}_n$ such that $\delta_{\omega(t_n)} \otimes t_n Q(\cdot | \mathcal{G}_n)(\omega) \in \mathcal{U}_\infty(t_n, \omega(t_n))$ for all $\omega \notin N_4$. Set $N \triangleq N_2 \cup N_3 \cup N_4$. As $P = Q$ on $\mathcal{G}_n$, we get that $P(N) = 0$. For all $\omega \notin N$, the induction base implies that

$$
E^P[g_{n+1}(X_{t_{n+1}}) | \mathcal{G}_n](\omega) = E^{\delta_{\omega(t_n)} \otimes t_n} P(\cdot | \mathcal{G}_n)(\omega)[g_{n+1}(X_{t_{n+1}})] \\
= E^{\delta_{\omega(t_n)} \otimes t_n} Q(\cdot | \mathcal{G}_n)(\omega)[g_{n+1}(X_{t_{n+1}})] \\
= E^Q[g_{n+1}(X_{t_{n+1}}) | \mathcal{G}_n](\omega).
$$

The induction step is complete and hence, $P = Q$.

We proved that $\mathcal{U}_\infty$ is singleton-valued and we write $\mathcal{U}_\infty(s, y) = \{ P_{(s, y)} \}$. By the measurability of $\mathcal{U}_\infty$, the map $(s, y) \mapsto P_{(s, y)}$ is measurable. It remains to show the strong Markov property of the family $\{ P_{(s, y)} : (s, y) \in \mathbb{R}_+ \times F \}$. Take $(s, y) \in \mathbb{R}_+ \times F$. As $\mathcal{U}_\infty$ is stable under conditioning, for every finite stopping time $\tau \geq s$, there exists a $P_{(s, x)}$-null set $N$ such that, for all $\omega \notin N$,

$$
\delta_{\omega(\tau(\omega))} \otimes \tau(\omega) P_{(s, y)}(\cdot | \mathcal{F}_{\tau})(\omega) \in \mathcal{U}_\infty(\tau(\omega), \omega(\tau(\omega))) = \{ P_{(\tau(\omega), \omega(\tau(\omega)))} \}.
$$

This yields, for all $\omega \notin N$, that

$$
P_{(s, y)}(\cdot | \mathcal{F}_{\tau})(\omega) = \delta_{\omega} \otimes \tau(\omega) \left[ \delta_{\omega(\tau(\omega))} \otimes \tau(\omega) P_{(s, y)}(\cdot | \mathcal{F}_{\tau})(\omega) \right] = \delta_{\omega} \otimes \tau(\omega) P_{(\tau(\omega), \omega(\tau(\omega)))}.
$$

This is the strong Markov property and consequently, the proof is complete. \hfill \Box

**Remark 7.13.** Notice that the strong Markov property of the selection $\{ P_{(s, y)} : (s, y) \in \mathbb{R}_+ \times F \}$ follows solely from the stability under conditioning property of $\mathcal{U}_\infty$. We emphasize that the stability under pasting property of each $\mathcal{U}_N, N \in \mathbb{Z}_+$, is crucial for its proof. Indeed, in Lemma 7.12, the fact that $\mathcal{U}$ is stable under pasting has been used to establish that $\mathcal{U}^*$ is stable under conditioning.
8. Proof of the Nonlinear Markov Property: Proposition 4.2

In this section, the Standing Assumptions 2.1, 2.2 and 4.1 are in force. We need a refinement of Lemma 7.2.

**Lemma 8.1.** For every $(t, \omega) \in [0, \infty[, \mathcal{K}(0, \omega(t)) = \{ P_t : P \in \mathcal{C}(t, \omega) \}$.

**Proof.** The inclusion $\mathcal{K}(0, \omega(t)) \supset \{ P_t : P \in \mathcal{C}(t, \omega) \}$ follows from Lemma 7.2. For the converse inclusion, take $P \in \mathcal{K}(0, \omega(t))$. Then, by Lemmata 7.3 and 7.9, we have $Q \triangleq \delta_\omega \otimes_t P^t \in \mathcal{C}(t, \omega)$. Since, for every $A \in \mathcal{F}$, it holds that

$$Q_t(A) = \int 1_{\theta^{-1}_t A}(\omega \otimes_t \omega') P^t(d\omega') = (P^t)_t(A) = P(A),$$

the proof is complete. \hfill \Box

We can now follow the proof of [9, Proposition 2.8] to deduce Proposition 4.2 from the DPP (Theorem 2.5).

**Proof of Proposition 4.2.** For every upper semianalytic function $\psi : \Omega \rightarrow [-\infty, \infty]$ and any $(t, \omega) \in [0, \infty[$, we deduce from Lemma 8.1 that

$$\mathcal{E}_t(\psi \circ \theta_t)(\omega) = \sup_{P \in \mathcal{C}(t, \omega)} E^{P_t}[\psi] = \sup_{P \in \mathcal{K}(0, \omega(t))} E^{P}[\psi],$$

which means nothing else than

$$\mathcal{E}_t(\psi \circ \theta_t)(\omega) = \mathcal{E}^{\omega(t)}(\psi).$$

Now, the DPP (Theorem 2.5) yields that

$$\mathcal{E}^x(\psi \circ \theta_t) = \mathcal{E}^x(\mathcal{E}_t(\psi \circ \theta_t)) = \mathcal{E}^x(\mathcal{E}^{X^t}(\psi)).$$

The proof is complete. \hfill \Box

9. Proof of the Sublinear Semigroup Property: Proposition 4.4

First, let us show that $T_t$, for every $t \in \mathbb{R}_+$, is a selfmap on the space of bounded upper semianalytic functions. Take a bounded upper semianalytic function $\psi : F \rightarrow \mathbb{R}$ and $t \in \mathbb{R}_+$. Clearly, $T_t(\psi)$ is bounded. It remains to show that $T_t(\psi)$ is also upper semianalytic. Define a function $\phi : F \times \Omega \rightarrow [-\infty, \infty]$ by

$$\phi(x, \omega) \triangleq \psi(\omega(t))1_{\{\omega(0) = x\}} + (-\infty)1_{\{\omega(0) \neq x\}}.$$

Evidently, $\Omega \ni \omega \mapsto \omega(t)$ and $F \times \Omega \ni (x, \omega) \mapsto 1_{\{\omega(0) = x\}}$ are Borel. Hence, by [3, Lemma 7.30], the function $\phi$ is upper semianalytic. We set

$$\mathcal{K} \triangleq \{ P \in \mathfrak{P}(\Omega) : P \in \mathcal{C}(0, \omega) \text{ for some } \omega \in \Omega \}.$$

By Lemma 6.5, the correspondence $\mathcal{C}$ has a Borel measurable graph. Thus, the set

$$K \triangleq \{ (\omega, P) \in \Omega \times \mathfrak{P}(\Omega) : P \in \mathcal{C}(0, \omega) \}$$

is Borel measurable, and consequently, the set $\mathcal{K}$ is analytic as it is the image of $K$ under the projection to the second coordinate ([3, Proposition 7.43]). Notice that

$$T_t(\psi)(x) = \sup_{P \in \mathcal{K}} E^P[\phi(x, X)], \quad x \in F.$$

We conclude from [3, Propositions 7.47 and 7.48] that $x \mapsto T_t(\psi)(x)$ is upper semianalytic.
Finally, we discuss the properties (i) – (iii) from Definition 4.3. The properties (ii) and (iii) are trivially satisfied and the first property (i) is implied by Proposition 4.2. The proof is complete.

10. Proof of the USC\(_b\)-Feller Property: Theorem 4.6

Fix \( \psi \in \text{USC}_b(F; \mathbb{R}) \) and \( t \in \mathbb{R}_+ \), and notice that \( \omega \mapsto \psi(\omega(t)) \) is upper semicontinuous and bounded. Thus, by \([4, \text{Theorem 8.10.61}]\), the map \( \mathcal{P}(\Omega) \ni P \mapsto E_P[\psi(X_t)] \) is upper semicontinuous, too. Thanks to Proposition 7.7, the correspondence \( x \mapsto \mathcal{R}(x) \) is upper hemicontinuous and compact-valued. Thus, upper semicontinuity of \( x \mapsto T_t(\psi)(x) \) follows from \([1, \text{Lemma 17.30}]\). The proof is complete.

11. Proof of Proposition 5.3

The following section is devoted to the proof of Proposition 5.3, which is split into several parts. In this section, we presume that Condition 5.1 holds.

11.1. Proof of parts (i) and (ii) from Condition 4.5. Take \( u = (y, \cdot)^\alpha_{H_2} \in U \). By virtue of \([19, \text{Problem 13, p. 151}]\), the maps \( \omega \mapsto Y^u(\omega) = (y, \omega)^\alpha_{H_2} \) and \( \omega \mapsto L(\omega) = \||\omega\||_{H_2} \) are continuous. Furthermore, for every \( R, M > 0 \), we have
\[
|Y^u_{\lambda \wedge \tau_M} I_{\{\|X_s\|_{H_2} \leq R\}}| \leq \||y\||_{H_2}(R + M)^i.
\]
Thus, part (i) of Condition 4.5 holds.

The correspondence \( \Theta \) is convex-valued by part (ii) of Condition 5.1. Hence, also part (ii) of Condition 4.5 holds.

11.2. Proof of part (iii) from Condition 4.5. The linear growth assumption (iv) from Condition 5.1 implies the first part of Condition 4.5 (iii). The second part, i.e., (4.2) for \( L = \||X\||_{H_2} \), follows directly from the following lemma and Chebyshev’s inequality.

**Lemma 11.1.** For every bounded set \( K \subset H_2 \), every \( T > 0 \) and every \( p > 1/\alpha \), where \( \alpha \in (0, 1/2) \) is as in part (v) of Condition 5.1,
\[
\sup_{x \in K} \sup_{P \in \mathcal{R}(x)} E_P \left[ \sup_{s \in [0, T]} \|X_s\|_{H_2}^p \right] < \infty.
\]

Before we prove this lemma, we record another useful observation which is used in the proof.

**Lemma 11.2.** For every \( x \in H_2 \) and \( P \in \mathcal{R}(x) \), there exists a predictable map \( g : [0, \infty] \to G \) such that, possibly on a standard extension of the filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, P)\), the coordinate process \( X \) admits the dynamics
\[
X_t = S_t x + \int_0^t S_{t-s} \mu(g(s, X), X_s) ds + \int_0^t S_{t-s} \sigma(g(s, X), X_s) dW_s, \quad t \in \mathbb{R}_+,
\]
where \( W \) is a cylindrical standard Brownian motion.

**Proof.** Take \( x \in H_2 \) and \( P \in \mathcal{R}(x) \). Thanks to Proposition 2.4, there exists a predictable function \( g : [0, \infty] \to G \) such that the processes
\[
u(X) - \int_0^t \mathcal{L}^{nu}(g(s, X))(X_s) ds, \quad u = (y, \cdot)^\alpha_{H_2}, y \in D, i = 1, 2,
\]
are \( P \)-local martingales. Recall that \( D \subset D(A^*) \) has the property that for every \( y \in D(A^*) \) there exists a sequence \((y_n)_{n=1}^\infty \subset D \) such that \( y_n \to y \) and \( A^*y_n \to A^*y \). Hence, as ucp
(uniformly on compacts in probability) limits of continuous local martingales are again continuous local martingales (see [5, Lemma B.11]), we conclude that all processes of the form
\begin{equation}
(11.1) \quad u(X) - \int_0^T \mathcal{L}^{u,g(s,X)}(X_s)\,ds, \quad u = \langle y, \cdot \rangle_{H_2}, \quad y \in D(A^*), \quad i = 1, 2,
\end{equation}
are $P$-local martingales. Thanks to [43, Theorem 13], in our setting the concepts of analytically weak and mild solutions are equivalent for the SPDE
\[ dY_t = AY_t\,dt + \mu(g(t,Y), Y_t)\,dt + \sigma(g(t,Y), Y_t)\,dW_t, \]
where $W$ is a cylindrical standard Brownian motion. Using the $P$-local martingale properties of the processes from (11.1), the claim of the lemma now follows as in the proof of [11, Lemma 3.6]. We omit the details. \hfill $\square$

**Proof of Lemma 11.1. Step 1: Recap on the Factorization Method.** In this step, let $P$ be a probability measure on some filtered probability space which supports a cylindrical standard $P$-Brownian motion $W$ over the Hilbert space $H_2$. Take $1/p < \lambda \leq 1$ and set, for $h \in L^p([0,T]; H_2)$,
\begin{equation}
(11.2) \quad R_\lambda(h)(t) \triangleq \int_0^t (t-s)^{\lambda-1}S_{t-s}h(s)\,ds, \quad t \in [0,T].
\end{equation}

Notice that $R_\lambda$ is well-defined, as, by Hölder’s inequality,
\begin{equation}
(11.3) \quad \int_0^t (t-s)^{\lambda-1}\|S_{t-s}h(s)\|_{H_2}\,ds
\leq \left( \int_0^T s^{p(\lambda-1)/(p-1)}\|S_s\|_{L(H_2,H_2)}^{p/(p-1)}\,ds \right)^{(p-1)/p} \left( \int_0^t \|h(s)\|_{H_2}^p\,ds \right)^{1/p}.
\end{equation}

This inequality shows that
\[ \|R_\lambda h(t)\|_{H_2} \leq \left( \int_0^T s^{p(\lambda-1)/(p-1)}\|S_s\|_{L(H_2,H_2)}^{p/(p-1)}\,ds \right)^{(p-1)/p} \left( \int_0^t \|h(s)\|_{H_2}^p\,ds \right)^{1/p}, \]
which means that $R_\lambda$ is a bounded linear operator on $L^p([0,T]; H_2)$. The following lemma provides more properties of $R_\lambda$, which turn out to be useful in Section 11.4 below.

**Lemma 11.3 ([21, Proposition 1]).** The operator $R_\lambda$ maps $L^p([0,T]; H_2)$ into $C([0,T]; H_2)$. Moreover, if the semigroup $S$ is compact, then $R_\lambda$ is compact.

We now recall the factorization formula from [12]. Take $\alpha \in (0,1/2)$ and let $p > 2$ be large enough such that $1/p < \alpha$. Moreover, let $f: (0,T] \to [0,\infty)$ be a Borel function such that
\[ \int_0^T \left[ \frac{f(s)}{s^{\alpha}} \right]^2\,ds < \infty, \]
let $\phi$ be a predictable $L_1(H_1,H_2)$-valued process and let $\psi$ be a real-valued predictable process such that
\[ \|S_t\phi_s\|_{L_2(H_1,H_2)} \leq f(t)|\psi_s|, \]
for all $t,s \in (0,T]$, and
\[ E^P \left[ \int_0^T |\psi_s|^p\,ds \right] < \infty. \]
The factorization formula given by [13, Theorem 5.10] shows that
\begin{equation}
\int_0^t S_{t-s} \phi_s dW_s = \frac{\sin(\pi \alpha)}{\pi} R_\alpha(Y)(t), \quad t \in [0,T],
\end{equation}
with
\[ Y_t \triangleq \int_0^t (t-s)^{-\alpha} S_{t-s} \phi_s dW_s, \]
see also [7, Step 0 in Section 4] for more details. By Eq. (4.4) in [7], it also holds that
\begin{equation}
E^P\left[ \int_0^T \|Y_t\|_{\mathcal{E}}^p dt \right] \leq c_p \left( \int_0^T \left[ \frac{f(s)}{s^{\alpha}} \right]^2 ds \right)^{p/2} E^P \left[ \int_0^T |\psi_s|^p ds \right],
\end{equation}
where the constant \( c_p \) only depends on \( p \). In particular, (11.3) and (11.5) yield, for all \( r \in [0,T] \), that
\begin{equation}
E^P \left[ \sup_{t \in [0,r]} \left\| \int_0^t S_{t-s} \phi_s dW_s \right\|_{H^2}^p \right] \leq CE^P \left[ \int_0^r |\psi_s|^p ds \right],
\end{equation}
where \( C > 0 \) is a constant which only depends on \( T, S, f, \alpha \) and \( p \).

Step 2: A Gronwall Argument. We are in the position to prove Lemma 11.1. Take \( x \in K \) and \( P \in \mathcal{R}(x) \) and let \( g \) be as in Lemma 11.2. Then, using the inequalities (11.3) and (11.6), for every \( M > 0 \) and all \( r \in [0,T] \), we obtain
\begin{align*}
E^P \left[ \sup_{t \in [0,r \wedge \rho_M]} \|X_t\|_{H^2}^p \right] & \leq C \left( \|x\|_{H^2}^p + E^P \left[ \sup_{t \in [0,r \wedge \rho_M]} \left\| \int_0^t S_{t-s} \mu(g(s,X),X_s) ds \right\|_{H^2}^p \right] 
+ E^P \left[ \sup_{t \in [0,r \wedge \rho_M]} \left\| \int_0^t S_{t-s} \sigma(g(s,X),X_s) dW_s \right\|_{H^2}^p \right] \right) \\
& \leq C \left( \|x\|_{H^2}^p + E^P \left[ \int_0^T \|\mu(g(s,X),X_s \wedge \rho_M)\|_{H^2}^p ds \right] 
+ E^P \left[ \sup_{t \in [0,r]} \left\| \int_0^t S_{t-s} \sigma(g(s,X),X_s \wedge \rho_M) dW_s \right\|_{H^2}^p \right] \right) \\
& \leq C \left( \|x\|_{H^2}^p + 1 + \int_0^r E^P \left[ \sup_{s \leq t \wedge \rho_M} \|X_s\|_{H^2}^p \right] dt \right),
\end{align*}
where the constant \( C > 0 \) depends only on the constant from Condition 5.1 (iv), \( T, S, f, \alpha \) and \( p \). Finally, we conclude the claim of the lemma from Gronwall’s and Fatou’s lemma. The proof is complete. \( \square \)

11.3. Proof of part (iv) from Condition 4.5. By part (iii) of Condition 5.1, the map \((g,x) \mapsto (\mathcal{L}^{nu}g(x))_{u \in U}\) is continuous. Thus, thanks to [9, Lemma 3.2], for every \( (t,\omega) \in [0,\infty] \) and \( m \in \mathbb{N} \), the correspondences \( s \mapsto \Theta(\omega(s)) \) and \( \alpha \mapsto \Theta(\alpha([t,t+1/m])) \) are continuous with compact values, and, since \( \mathbb{R}^U \) is a Fréchet space ([1, p. 206]), it follows also that \( \overline{\Theta}(\omega([t,t+1/m])) \) is compact ([1, Theorem 5.35]). Finally, we deduce from [1, Theorem 17.35] that \( \omega \mapsto \overline{\Theta}(\omega([t,t+1/m]), \omega) \) is upper hemicontinuous for every \( t \in \mathbb{R}_+ \) and \( m \in \mathbb{N} \). This completes the proof. \( \square \)
11.4. Proof of part (v) from Condition 4.5. We adapt the compactness method from [21]. Fix a finite time horizon \( T > 0 \) and take \( p > 1/\alpha \), where \( \alpha \) is as in part (v) of Condition 5.1. Let \( K \subset H_2 \) be a compact set and, for \( R > 0 \), define
\[
G_R \triangleq \left\{ \omega \in C([0,T];H_2) : \omega = Sx_0 + R_1(\psi) + \frac{\sin(\pi \alpha)}{\pi}R_\alpha(\phi), \ x_0 \in K, \right. \\
\left. \phi, \psi \in L^p([0,T];H_2) \text{ with } \int_0^T \|\psi(s)\|^p_{H_2}ds \leq R, \int_0^T \|\phi(s)\|^p_{H_2}ds \leq R \right\},
\]
where the operators \( R_1 \) and \( R_\alpha \) are defined as in (11.2). For every \( t \in [0,T] \), the set \( \{S_t x_0 : x_0 \in K\} \) is compact, because \( S \) is a compact semigroup by Condition 5.1 (v). By [18, Lemma I.5.2], the map \([0,T] \times K \ni (t,x) \mapsto S_t x \in E\) is uniformly continuous. Thus, the Arzelà–Ascoli theorem ([29, Theorem A.5.2]) yields that the set \( \{(0,T) \ni t \mapsto S_t x_0 : x_0 \in K\} \) is relatively compact in \( C([0,T];H_2) \). Recall from Lemma 11.3 that \( R_1 \) and \( R_\alpha \) are compact operators from \( L^p([0,T];H_2) \) into \( C([0,T];H_2) \). Hence, for every \( R > 0 \), the set \( G_R \) is relatively compact in \( C([0,T];H_2) \).

For any \( P \in \mathcal{R}(x_0) \), by Lemma 11.2 and the factorization formula (11.4), possibly on an enlargement of the filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, P)\), we have a.s.
\[
X_t = S_t x_0 + R_1(\mu(g,X))(t) + \frac{\sin(\pi \alpha)}{\pi}R_\alpha(Y)(t), \ t \in [0,T],
\]
where
\[
Y_t \triangleq \int_0^t (t-s)^{-\alpha}S_{t-s}\sigma(g(s,X),X_s)dB_s, \ t \in [0,T].
\]
Using this observation and Chebyshev’s inequality, we deduce from (11.3) and (11.5), the linear growth assumptions on \( \mu \) and \( \sigma \), and Lemma 11.1, that, for every \( \varepsilon > 0 \), there exists an \( R > 0 \) such that
\[
\sup_{x \in K} \sup_{P \in \mathcal{R}(x)} P(\{(0,T) \ni t \mapsto X_t \in G_R\}) \geq 1 - \varepsilon.
\]
By virtue of [29, Theorem 23.4], since \( T > 0 \) was arbitrary, we conclude that the set \( \bigcup_{x \in K} \mathcal{R}(x) \) is tight and hence, relatively compact by Prohorov’s theorem. The proof is complete. \( \square \)

12. Proof of Strong USC\(_h\)-Feller property: Theorem 5.8

The following proof adapts the idea from [10] that the strong USC\(_h\)-Feller property can be deduced from the the strong Markov selection principle and a change of measure. Throughout this proof, we fix a function \( \psi \in USC_h(F;\mathbb{R}) \) and a time \( T > 0 \). W.l.o.g., assume that \( |\psi| \leq 1 \). We start with some auxiliary observations. Notice that Condition 5.6 implies Condition 5.1. Hence, by Corollary 5.4, there exists a strong Markov family \( \{P_{(s,x)} : (s,x) \in \mathbb{R}_+ \times H\} \) such that \( P_{(s,x)} \in \mathcal{K}(s,x) \) and \( T_T(\psi)(x) = E^{P_{(0,x)}}[\psi(X_T)] \). To simplify our notation, we set \( P_x \triangleq P_{(0,x)} \). By Lemma 11.2, with some abuse of notation, for every \( x \in H \), there exists a predictable map \( g^x : [0,\infty[ \rightarrow G \) and a cylindrical standard \( P_x\)-Brownian motion \( W \) such that \( P_x\)-a.s.
\[
X_t = S_t x + \int_0^t S_{t-s} \mu(g^x(s,X),X_s)ds + \int_0^t S_{t-s} dW_s, \ t \in \mathbb{R}_+.
\]
We define
\[
Z^{P_x} \triangleq \exp\left(-\int_0^t \langle \mu(g^x(s,X),X_s),dW_s \rangle_H - \frac{1}{2} \int_0^t \|\mu(g^x(s,X),X_s)\|^2_H ds \right),
\]
and, for $M > 0$, we set

$$\rho_M \triangleq \inf\{t \geq 0: \|X_t\|_H \geq M\} \land M.$$ 

For every $M > 0$, [26, Theorem 8.25] (or Novikov’s condition [29, Theorem 19.24]) yields that $Z_{\rho_M}^x$ is a uniformly integrable $P_x$-martingale. We define a probability measure $Q^n_M$ on $(\Omega, F)$ via the Radon–Nikodym derivative $dQ^n_M / dP_x = Z^n_x$. Thanks to Girsanov’s theorem (cf. [35, Proposition I.0.6]), the process

$$B^M \triangleq W + \int_0^{\wedge \rho_M} \mu(g^+(s, X), X_s) ds$$

is a cylindrical standard $Q^n_M$-Brownian motion. Thus, under $Q^n_M$, we have

$$X_t = S_t x + \int_0^t S_{t-s} dB^M_s, \quad t \leq \rho_M.$$ 

Now, for every $t \in [0, M)$, we obtain

$$Q^n_M(\rho_M \leq t) \leq \frac{1}{M} E^{Q^n_M} \left[ \sup_{s \in [0,t] \wedge \rho_M} \|X_s\|_H \right]$$

$$= \frac{1}{M} E^{Q^n_M} \left[ \sup_{s \in [0,t] \wedge \rho_M} \left\|S_s x + \int_0^s S_{s-r} B^M_r \right\|_H \right]$$

$$\leq \frac{1}{M} E^{Q^n_M} \left[ \sup_{s \in [0,t]} \left\|S_s x + \int_0^s S_{s-r} B^M_r \right\|_H \right]$$

$$= \frac{1}{M} E^{P_x} \left[ \sup_{s \in [0,t]} \left\|S_s x + \int_0^s S_{s-r} W_r \right\|_H \right] \to 0 \text{ as } M \to \infty.$$ 

Hence,

$$E^{P_x}[Z^n_T] = \lim_{M \to \infty} E^{P_x}[Z^n_T \mathbb{1}_{\{\rho_M > t\}}] = \lim_{M \to \infty} Q^n_M(\rho_M > t) = 1,$$

where we use the monotone convergence theorem for the first equality. This shows that $Z^n_T$ is a true $P_x$-martingale. Consequently, by a standard extension theorem ([29, Lemma 19.19]), there exists a probability measure $Q^n_x$ on $(\Omega, F)$ such that $Q^n_x(G) = E^{P_x}[Z^n_T \mathbb{1}_G]$ for all $G \in F_T$ and $T > 0$. Using again Girsanov’s theorem (cf. [35, Proposition I.0.6]), we obtain that the process

$$B \triangleq W + \int_0^t \mu(g^+(s, X), X_s) ds$$

is a cylindrical standard $Q^n_x$-Brownian motion and, under $Q^n_x$,

$$X_t = S_t x + \int_0^t S_{t-s} dB_s, \quad t \in \mathbb{R}_+.$$ 

Now, we are in the position to prove Theorem 5.8. Let $(x^n)_{n=0}^{\infty} \subset H$ be a sequence such that $x^n \to x^0$ and fix $\varepsilon > 0$. By virtue of (12.1), there exists an $M' > 0$ such that

$$\sup_{n \in \mathbb{Z}_+} Q^n_{x^n}(\rho_{M'} \leq T) \leq \varepsilon.$$ 

Furthermore, by Lemma 11.1, there exists an $M^o > 0$ such that

$$\sup_{n \in \mathbb{Z}_+} P_{x^n}(\rho_{M^o} \leq T) \leq \varepsilon.$$
We set $M \triangleq M' \lor M^c$. By the linear growth part of Condition 5.6, there exists a constant $C = C_M > 0$ such that
\[
\|\mu(g, x)\|_H \leq C
\]
for all $g \in G$ and $x \in H$: $\|x\|_H \leq M$. Take $\beta \in (0, T)$ and notice that, for all $n \in \mathbb{Z}_+$,
\[
\left( E^{P_{T_n}} [1 - Z^{P_{T_n}}_{\beta \land \rho_M}] \right)^2 \leq E^{P_{T_n}} [1 - Z^{P_{T_n}}_{\beta \land \rho_M}] = E^{P_{T_n}} [(Z^{P_{T_n}}_{\beta \land \rho_M})^2] - 1 \leq e^\beta c^2 - 1.
\]

Now, let $\beta \in (0, T)$ small enough such that, for all $n \in \mathbb{Z}_+$,
\begin{equation}
E^{P_{T_n}} [1 - Z^{P_{T_n}}_{\beta \land \rho_M}] \leq \varepsilon.
\end{equation}

Define
\[
\Psi_\beta(x) \triangleq E^{P_{T_n}} [\psi(X_T)], \quad x \in H.
\]

Then, for every $n \in \mathbb{N}$, using the (strong) Markov property of $\{P_{(s, x)} : (s, x) \in \mathbb{R}_+ \times H\}$ and (12.2), (12.3) and (12.4), we obtain
\[
\begin{align*}
&\left| E^{P_{T_n}} [\psi(X_T)] - E^{P_{T_0}} [\psi(X_T)] \right| \\
&= \left| E^{P_{T_n}} [\Psi_\beta(X_{\beta})] - E^{P_{T_0}} [\Psi_\beta(X_{\beta})] \right| \\
&\leq \left| E^{P_{T_n}} [\Psi_\beta(X_{\beta}) \mathbf{1}_{\{\rho_M > \beta\}}] - E^{P_{T_0}} [\Psi_\beta(X_{\beta}) \mathbf{1}_{\{\rho_M > \beta\}}] \right| + 2\varepsilon \\
&\leq \left| E^{P_{T_n}} [Z^{P_{T_n}}_{\beta} \Psi_\beta(X_{\beta}) \mathbf{1}_{\{\rho_M > \beta\}}] - E^{P_{T_0}} [Z^{P_{T_0}}_{\beta} \Psi_\beta(X_{\beta}) \mathbf{1}_{\{\rho_M > \beta\}}] \right| \\
&\quad + E^{P_{T_n}} [1 - Z^{P_{T_n}}_{\beta \land \rho_M}] + E^{P_{T_0}} [1 - Z^{P_{T_0}}_{\beta \land \rho_M}] + 2\varepsilon \\
&\leq \left| E^{Q_{T_n}} [\Psi_\beta(X_{\beta}) \mathbf{1}_{\{\rho_M > \beta\}}] - E^{Q_{T_0}} [\Psi_\beta(X_{\beta}) \mathbf{1}_{\{\rho_M > \beta\}}] \right| + 4\varepsilon \\
&\leq \left| E^{Q_{T_n}} [\Psi_\beta(X_{\beta})] - E^{Q_{T_0}} [\Psi_\beta(X_{\beta})] \right| + 6\varepsilon.
\end{align*}
\]

Thanks to [13, Theorem 9.32], we have
\[
\left| E^{Q_{T_n}} [\Psi_\beta(X_{\beta})] - E^{Q_{T_0}} [\Psi_\beta(X_{\beta})] \right| \rightarrow 0
\]
as $n \rightarrow \infty$. Hence, there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$,
\[
\left| E^{P_{T_n}} [\psi(X_T)] - E^{P_{T_0}} [\psi(X_T)] \right| \leq 7\varepsilon.
\]

We conclude that the map $x \mapsto E^{P_{T_n}} [\psi(X_T)]$ is continuous. This finishes the proof of Theorem 5.8. \hfill \Box

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**Appendix A. Some facts on the space of Lévy measures**

As in Section 3.1, let $\mathcal{L}$ be the space of all Lévy measure on $\mathbb{R}^d$, i.e., the space of all measures $K$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that
\[
K(\{0\}) = 0 \quad \text{and} \quad \int (1 \land \|x\|^2) K(dx) < \infty.
\]

We endow $\mathcal{L}$ with the weakest topology under which all maps
\[
K \mapsto \int f(x)(1 \land \|x\|^2) K(dx), \quad f \in C_b(\mathbb{R}^d; \mathbb{R}),
\]
are continuous. The following lemma shows that this topology has a convenient structure. It extends [38, Lemma 2.3], where $\mathcal{L}$ is shown to be a separable metrizable space.
**Lemma A.1.** The space $\mathcal{L}$ is a Polish space.

**Proof.** The proof is split into two steps.

**Step 1.** Let $\mathcal{M}^f(\mathbb{R}^d)$ be the set of all finite measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ endowed with the weak topology. This space is Polish by [48, Theorem 1.11]. We now prove that

$$\mathcal{M}_0^f(\mathbb{R}^d) \triangleq \{ K \in \mathcal{M}^f(\mathbb{R}^d) : K(\{0\}) = 0 \}$$

is a $G_\delta$ subset of $\mathcal{M}^f(\mathbb{R}^d)$ and consequently, a Polish subspace. Let $(g_n)_{n=1}^\infty$ be a sequence of continuous functions on $\mathbb{R}^d$ into $[0, 1]$ such that $g_n \searrow 1_{\{0\}}$. It is well-known that such a sequence exists. Now, set

$$H \triangleq \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty \left\{ K \in \mathcal{M}^f(\mathbb{R}^d) : \int g_m dK < 1/n \right\}.$$

The map $K \mapsto \int g_m dK$ is continuous by the definition of the weak topology. Thus, $\{ K \in \mathcal{M}^f(\mathbb{R}^d) : \int g_m dK < 1/m \}$ is open and consequently, $H$ is a $G_\delta$ set. We now explain that $\mathcal{M}_0^f(\mathbb{R}^d) = H$. If $K \in \mathcal{M}_0^f(\mathbb{R}^d)$, then

$$\lim_{m \to \infty} \int g_m dK = K(\{0\}) = 0,$$

by the dominated convergence theorem. Hence, $K \in H$. Conversely, take $K \in H$. For every $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $\int g_m dK < 1/n$. As $k \mapsto g_k$ decreases, we get $\int g_k dK < 1/n$ for all $k \geq m$. This proves that $\lim_{m \to \infty} \int g_m dK = 0$ and it follows from the dominated convergence theorem that $K(\{0\}) = 0$. In summary, $K \in \mathcal{M}_0^f(\mathbb{R}^d)$, which completes the proof of $\mathcal{M}_0^f(\mathbb{R}^d) = H$.

**Step 2.** Let $d$ be a metric that induces the weak topology on $\mathcal{M}_0^f(\mathbb{R}^d)$. Then,

$$(K_1, K_2) \mapsto d_{\mathcal{L}}(K_1, K_2) \triangleq d(gdK_1, gdK_2), \quad g(x) \triangleq 1 \wedge \|x\|^2,$$

induces the topology of $\mathcal{L}$. In other words, the function $K \mapsto gdK$ is an isometry between $\mathcal{L}$ and $\mathcal{M}_0^f(\mathbb{R}^d)$. Further, it is a bijection whose inverse is given by $K \mapsto 1_{\mathbb{R}^d \setminus \{0\}}dK/g$. Hence, $\mathcal{L}$ and $\mathcal{M}_0^f(\mathbb{R}^d)$ are isometric, which implies that $\mathcal{L}$ is a Polish space.

We also identify a class of continuous functions on $\mathcal{L}$.

**Lemma A.2.** Let $g : \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that $|g(x)| \leq C(1 \wedge \|x\|^2)$ for some constant $C > 0$. Then, the map

$$K \mapsto \int g(x)K(dx)$$

is continuous from $\mathcal{L}$ into $\mathbb{R}$.

**Proof.** Suppose that $K^n \to K^0$ in $\mathcal{L}$. Then, by definition of the topology on $\mathcal{L}$, $G^n \triangleq (1 \wedge \|x\|^2)K^n(dx) \to (1 \wedge \|x\|^2)K^0(dx) \triangleq G^0$ in $\mathcal{M}^f(\mathbb{R}^d)$. As $G^0(\{0\}) = 0$, the function

$$f(x) \triangleq \begin{cases} g(x)/(1 \wedge \|x\|^2), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is bounded and $G^0$-a.e. continuous. Hence, the continuous mapping theorem for $\mathcal{M}^f(\mathbb{R}^d)$ (see [48, Theorem 1.8]) yields that $\int f dG^n \to \int f dG^0$. By definition of $f$, this yields the claim. \qed
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