Remarks on mapping properties for the Bargmann transform on modulation spaces

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We investigate the mapping properties for the Bargmann transform and prove that this transform is isometric and bijective from modulation spaces to convenient Banach spaces of analytic functions.

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1. Introduction

Bargmann [1] introduced a transform $\mathcal{V}$ which is bijective and isometric from $L^2(\mathbb{R}^d)$ into the Hilbert space $A^2(\mathbb{C}^d)$ of all entire analytic functions $F$ on $\mathbb{C}^d$ such that $F \cdot e^{-\left| \cdot \right|^2/2} \in L^2(\mathbb{C}^d)$. (We use the usual notation for the usual function and distribution spaces [12], and refer to Section 2 for other notation). Furthermore, several important properties for $\mathcal{V}$ were established. For example:

- the Hermite functions were mapped into the normalized analytical monomials. Furthermore, the latter set forms an orthonormal basis for $A^2(\mathbb{C}^d)$;
- the creation and annihilation operators, and harmonic oscillator on appropriate elements in $L^2$, were transformed by $\mathcal{V}$ into simple operators;
- there is a convenient reproducing formula for elements in $A^2$.

Bargmann [2] continued his work and discussed mapping properties for $\mathcal{V}$ on more general spaces. For example, he proved that $\mathcal{V}(S')$, the image of $S'$ under the Bargmann transform, is given by the formula

$$\mathcal{V}(S') = \bigcup_{\omega \in \mathcal{P}} A^\infty_{(\omega)} \cdot$$

(1)

Here $A^p_q(\mathbb{C}^d)$ is the set of all entire functions $F$ on $\mathbb{C}^d$ such that $F \cdot e^{-\left| \cdot \right|^2/2} \cdot \omega_0$ belongs to the mixed Lebesgue space $L^1_p(\mathbb{C}^d)$, for some appropriate modification $\omega_0$ of the weight function $\omega$. We refer to [12] and Section 2 for specific definitions.

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The Bargmann transform can easily be reformulated in terms of the short-time Fourier transform, with a particular Gauss function as window function. In this context, we remark that the modulation space $M_{(ω)}^{p,q}(R^d)$ is obtained by imposing the $L_{(ω)}^{p,q}$ norm on the short-time Fourier transform. From these arguments, it follows easily that the Bargmann transform is continuous and injective from $M_{(ω)}^{p,q}(R^d)$ to $A_{(ω)}^{p,q}(C^d)$. Furthermore, by choosing the window function as a particular Gaussian function in the $M_{(ω)}^{p,q}$ norm, it follows that $V : M_{(ω)}^{p,q} \rightarrow A_{(ω)}^{p,q}$ is isometric.

These facts and several other mapping properties for the Bargmann transform on modulation spaces were established and proved by Feichtinger and Gröchenig [5], Feichtinger et al. [7], Gröchenig [8] and Gröchenig and Walnut [11]. In fact, here they state and motivate that the Bargmann transform from $M_{(ω)}^{p,q}(R^d)$ to $A_{(ω)}^{p,q}(C^d)$ is not only injective, but in fact bijective. In their proof of the surjectivity, they recall from [1] that the Bargmann transform is bijective from $S_0(R^d)$ to $P(C^d)$, where $S_0(R^d)$ is the set of finite linear combinations of the Hermite functions and $P(C^d)$ is the set of analytic polynomials on $C^d$. Then they use duality in combination with the argument that $P(C^d)$ is dense in $A_{(ω)}^{p,q}$ when $p, q < \infty$. Since $S_0$ is dense in $M_{(ω)}^{p,q}$, the asserted surjectivity easily follows from these arguments.

We are convinced that, somewhere in the literature, it is proved that $P(C^d)$ is dense in $A_{(ω)}^{p,q}$ (e.g. a proof might occur in [5,8,11,13]). On the other hand, so far we are unable to find any explicit proof of this fact. In particular, we could not find any explicit references in the papers [5,8,11].

Feichtinger et al. [7], Gröchenig [8] and Gröchenig and Walnut [11] also give an other motivation for the surjectivity. More precisely, they use the arguments that the Bargmann–Fock representation of the Heisenberg group is unitarily equivalent to the Schrödinger representation with $V$ as the intertwining operator. Then they explain that the general intertwining theorem [5, Theorem 4.8] applied to the Schrödinger representation and the Bargmann–Fock representation implies that $V$ extends to a Banach space isomorphism from $M_{(ω)}^{p,p}(R^d)$ to $A_{(ω)}^{p,p}(C^d)$ and the asserted surjectivity follows.

It is obvious that these arguments are sufficient to conclude that $V$ is a Banach space isomorphism from $M_{(ω)}^{p,q}(R^d)$ to $\mathcal{V}(M_{(ω)}^{p,q})$. On the other hand, so far we are unable to understand that these arguments are sufficient to conclude that indeed $\mathcal{V}(M_{(ω)}^{p,q})$ is the same as $A_{(ω)}^{p,q}$.

In this paper, we present some of the arguments in [15] for the proof of this bijectivity. The main part in [15] is to prove that (1') can be improved into

$$\mathcal{V}(S') = \bigcup_{ω \in \mathbb{R}} A_{(ω)}^{p,q}, \quad (1')$$

when $p, q \in [1, \infty]$. Admitting this, it follows that each element in $A_{(ω)}^{p,q}$ is a Bargmann transform of a tempered distribution. The fact that the Bargmann transform is continuous and injective from $M_{(ω)}^{p,q}$ to $A_{(ω)}^{p,q}$ then shows that this tempered distribution must belong to $M_{(ω)}^{p,q}$, and the result follows.

The paper is organized as follows. In Section 2, we recall some facts for modulation spaces and the Bargmann transform. In Section 3, we present some links to the proof of the main result, that is, that the Bargmann transform is bijective from $M_{(ω)}^{p,q}$ to $A_{(ω)}^{p,q}$.

2. Preliminaries

In this section, we give some definitions and recall some basic results. The proofs are in general omitted.
2.1. The short-time Fourier transform and Toeplitz operators

The Fourier transform \( \mathcal{F} \) is the linear and continuous mapping on \( S'(\mathbb{R}^d) \), which takes the form

\[
(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i(x,\xi)} \, dx,
\]
when \( f \in L^1(\mathbb{R}^d) \). Here \( (x, \xi) \) denotes the usual scalar product on \( \mathbb{R}^d \) between \( x \in \mathbb{R}^d \) and \( \xi \in \mathbb{R}^d \). The map \( \mathcal{F} \) is a homeomorphism on \( S'(\mathbb{R}^d) \), which restricts to a homeomorphism on \( S(\mathbb{R}^d) \) and to a unitary operator on \( L^2(\mathbb{R}^d) \).

Let \( \phi \in S(\mathbb{R}^d) \setminus 0 \) be fixed. For every \( f \in S'(\mathbb{R}^d) \), the short-time Fourier transform \( V_{\phi}f \) is the distribution on \( \mathbb{R}^{2d} \) defined by the formula

\[
(V_{\phi}f)(x, \xi) = \mathcal{F}(f \phi(\cdot - x))(\xi).
\]

We note that the right-hand side defines an element in \( S'(\mathbb{R}^{2d}) \cap C_\infty(\mathbb{R}^{2d}) \). If in addition \( f \in L^1(\mathbb{R}^d) \), then \( V_{\phi}f \) takes the form

\[
V_{\phi}f(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) \phi(y - x) e^{-i(y,\xi)} \, dy.
\]

If \( a \in S(\mathbb{R}^{2d}) \) and \( \phi \in S(\mathbb{R}^d) \setminus 0 \), then the Toeplitz operator \( T_p(a) = T_p(\phi) \) is the continuous and linear operator from \( S(\mathbb{R}^d) \) to \( S(\mathbb{R}^d) \), defined by the formula

\[
(T_p(\phi)(a)f, g)_{L^2(\mathbb{R}^d)} = (a V_{\phi}f, V_{\phi}g)_{L^2(\mathbb{R}^d)}.
\]

There are several ways to extend the definition of Toeplitz operators (see, e.g. [10] and the references therein). For example, the definition of \( T_p(\phi) \) is uniquely extendable to every \( a \in S'(\mathbb{R}^{2d}) \), and then \( T_p(\phi)(a) \) is continuous from \( S(\mathbb{R}^d) \) to \( S'(\mathbb{R}^d) \).

2.2. Modulation spaces

Next we present appropriate conditions of the involved weight functions. Assume that \( \omega, v \in L^\infty_{\text{loc}}(\mathbb{R}^d) \) are positive functions. Then \( \omega \) is called \( v \)-moderate if

\[
\omega(x + y) \leq C \omega(x)v(y)
\]
for some constant \( C \) which is independent of \( x, y \in \mathbb{R}^d \). If \( v \) in (3) can be chosen as a polynomial, then \( \omega \) is called polynomially moderate. We let \( \mathcal{P}(\mathbb{R}^d) \) be the set of all polynomially moderated functions on \( \mathbb{R}^d \). We say that \( v \) is submultiplicative when (3) holds with \( \omega = v \). Throughout, we assume that the submultiplicative weights are even. Furthermore, \( v \) and \( v_0 \) always stand for submultiplicative weights, if nothing else is stated.

For each \( \omega \in \mathcal{P}(\mathbb{R}^d) \) and \( p \in [1, \infty] \), we let \( L^p(\omega)(\mathbb{R}^d) \) be the Banach space which consists of all \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) such that \( \|f\|_{L^p(\omega)} := \|f\|_{L^p} \) is finite.

The definition of modulation spaces is the following. Here \( L^{p,q}(\mathbb{R}^{2d}) \) is the mixed-norm space which consists of all \( F \in L^1_{\text{loc}}(\mathbb{R}^{2d}) \) such that

\[
\|F\|_{L^{p,q}} \equiv \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \xi)|^p \, d\xi \right)^{q/p} \, dx \right)^{1/q} < \infty.
\]
(with obvious modifications when \( p = \infty \) or \( q = \infty \).
Let \( p, q \in [1, \infty] \) and \( \omega \in \mathcal{P}(\mathbb{R}^{2d}) \), and let \( \phi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \). Then the modulation space \( M^{p,q}_{(\omega)}(\mathbb{R}^d) \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^d) \) such that

\[
\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_{\phi}f \omega\|_{L^{p,q}} < \infty. \tag{4}
\]

We set \( M^{p}_{(\omega)} = M^{p,p}_{(\omega)} \). We also set \( M^{p,q} = M^{p,q}_{(\omega)} \) and \( M^p = M^{p,p}_{(\omega)} \), when \( \omega = 1 \). An important type of weight functions are

\[
\sigma_s(x, \xi) \equiv \langle x, \xi \rangle^s = (1 + |x|^2 + |\xi|^2)^{s/2}, \tag{5}
\]

when \( s \in \mathbb{R} \), and for convenience we set \( M^{p,q}_s = M^{p,q}_{(\sigma_s)} \) and \( M^p_s = M^{p,p}_{(\sigma_s)} \).

In the following proposition, we recall some facts about modulation spaces. We omit the proof, since the result can be found in [4–6,9,16].

**Proposition 2.2** Let \( p, q \in [1, \infty] \) and \( \omega, v \in \mathcal{P}(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate. If \( \phi \in M^{1}_{(\omega)}(\mathbb{R}^d) \setminus \{0\} \), then \( f \in M^{p,q}_{(\omega)} \) if and only if (4) holds, i.e. \( M^{p,q}_{(\omega)} \) is independent of the choice of \( \phi \). Moreover, \( M^{p,q}_{(\omega)} \) is a Banach space under the norm in (4), and different choices of \( \phi \) give rise to equivalent norms.

**Remark 1** Let \( p, q \in [1, \infty] \) and let \( \omega_j \) for \( j \in J \) be a family of elements in \( \mathcal{P}(\mathbb{R}^{2d}) \) such that for each \( s \geq 0 \), there is a constant \( C > 0 \), and \( j_1, j_2 \in J \) such that \( \omega_{j_1}(x, \xi) \leq C \langle x, \xi \rangle^{-s} \) and \( C^{-1} \langle x, \xi \rangle^s \leq \omega_{j_2}(x, \xi) \). Then \( \bigcup_{j \in J} M^{p,q}_{(\omega_j)} = \mathcal{S}'(\mathbb{R}^d) \) and \( \cap_{j \in J} M^{p,q}_{(\omega_j)} = \mathcal{S}(\mathbb{R}^d) \) (cf. [18, Remark 1.3 (5)])

### 2.3. The Bargmann transform

We shall now consider the Bargmann transform, which is defined by the formula

\[
(\mathfrak{M} f)(z) = \pi^{-d/4} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2} \langle z, z \rangle + |y|^2 \right) f(y) \, dy, \tag{6}
\]

when \( f \in L^2(\mathbb{R}^d) \). We note that if \( f \in L^2(\mathbb{R}^d) \), then the Bargmann transform \( \mathfrak{M} f \) of \( f \) is the entire function on \( \mathbb{C}^d \), given by \( (\mathfrak{M} f)(z) = \int \mathfrak{A}_d(z, y) f(y) \, dy \), or

\[
(\mathfrak{M} f)(z) = \langle f, \mathfrak{A}_d(z, \cdot) \rangle, \tag{7}
\]

where the Bargmann kernel \( \mathfrak{A}_d \) is given by \( \mathfrak{A}_d(z, y) = \pi^{-d/4} \exp\left(-\frac{1}{2} \langle z, z \rangle + |y|^2 \right) + 2^{1/2} \langle z, y \rangle \). Here \( \langle z, w \rangle = \sum_{j=1}^d z_j w_j \), when \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \) and \( w = (w_1, \ldots, w_d) \in \mathbb{C}^d \), and \( \langle \cdot, \cdot \rangle \) denotes the duality between elements in \( \mathcal{S}'(\mathbb{R}^d) \) and elements in \( \mathcal{S}'(\mathbb{R}^d) \). We note that the right-hand side in (7) makes sense when \( f \in \mathcal{S}'(\mathbb{R}^d) \) and defines an element in \( A(\mathbb{C}^d) \), since \( y \mapsto \mathfrak{A}_d(z, y) \) can be interpreted as an element in \( \mathcal{S}(\mathbb{R}^d) \) with values in \( A(\mathbb{C}^d) \). Here and in what follows, \( A(\mathbb{C}^d) \) denotes the set of all entire functions on \( \mathbb{C}^d \).

From now on, we assume that \( \phi \) in (2'), (2) and (4) is given by

\[
\phi(x) = \pi^{-d/4} e^{-|x|^2/2}, \tag{8}
\]

if nothing else is stated. Then it follows that the Bargmann transform can be expressed in terms of the short-time Fourier transform \( f \mapsto V_{\phi} f \). More precisely, for such choice of \( \phi \), it follows
by straightforward computations that
\[(\mathfrak{M} f)(z) = (\mathfrak{M} f)(x + i\xi) = (2\pi)^{d/2}e^{(|x|^2 + |\xi|^2)/2}e^{-i(x,\xi)}V_\phi f(2^{1/2}x, -2^{1/2}\xi)\]
\[= (2\pi)^{d/2}e^{(|x|^2 + |\xi|^2)/2}e^{-i(x,\xi)}(S^{-1}(V_\phi f))(x, \xi),\]
(9)
or equivalently,
\[V_\phi f(x, \xi) = (2\pi)^{-d/2}e^{-i(x,\xi)/2}(\mathfrak{M} f)(2^{-1/2}x, -2^{-1/2}\xi).\]
(10)
Here \(S\) is the dilation operator given by
\[(SF)(x, \xi) = F(2^{-1/2}x, -2^{-1/2}\xi).\]
(11)
We are now prepared to make the following definition.

**Definition 2.3** Let \(\omega \in \mathcal{P}(\mathbb{R}^d)\) and let \(p, q \in [1, \infty]\).

1. The space \(A_{(\omega)}^{p,q}(\mathbb{C}^d)\) consists of all \(F \in A(\mathbb{C}^d)\) such that
\[
\|F\|_{L_{(\omega),\infty}^{p,q}} = (2\pi)^{-d/2}\|\mathcal{S}(F e^{-|\cdot|^2/2})\omega\|_{L_{(\omega),\infty}^{p,q}} < \infty;
\]
(2) The space \(A_{(\omega),0}^{p,q}(\mathbb{C}^d)\) is given by
\[A_{(\omega),0}^{p,q}(\mathbb{C}^d) \equiv \{F = \mathfrak{M} f; f \in M_{(\omega)}^{p,q}(\mathbb{R}^d)\},\]
and is equipped by the norm \(\|F\|_{A_{(\omega),0}^{p,q}} \equiv \|f\|_{M_{(\omega)}^{p,q}}\), when \(F = \mathfrak{M} f\).

The following result shows that the norm in \(A_{(\omega),0}^{p,q}(\mathbb{C}^d)\) is well defined.

**Proposition 2.4** Let \(\omega \in \mathcal{P}(\mathbb{R}^d)\), let \(p, q \in [1, \infty]\) and let \(\phi\) be as in (8). Then \(A_{(\omega),0}^{p,q}(\mathbb{C}^d) \subseteq A_{(\omega)}^{p,q}(\mathbb{C}^d)\), and the map \(\mathfrak{M}\) is an isometric injection from \(M_{(\omega)}^{p,q}(\mathbb{R}^d)\) to \(A_{(\omega)}^{p,q}(\mathbb{C}^d)\).

**Proof** The result is an immediate consequence of (9), (10) and Definition 2.3. \(\blacksquare\)

In the case \(\omega = 1\) and \(p = q = 2\), then it follows from [1] that Proposition 2.4 holds, after the inclusion is replaced by the equality. That is, we have \(A_{(\omega),0}^{2} = A_{(\omega)}^{2}\) which is called the Bargmann–Foch space, or just the Foch space. In the next section, we improve the latter property and show that for every \(p, q \in [1, \infty]\) and every \(\omega \in \mathcal{P}\), we have \(A_{(\omega),0}^{p,q}(\mathbb{C}^d) = A_{(\omega)}^{p,q}(\mathbb{C}^d)\).

### 3. Mapping results for the Bargmann transform on modulation spaces

In this section, we present some links to the proof in [15] that \(A_{(\omega),0}^{p,q}\) agrees with \(A_{(\omega)}^{p,q}\) for every choice of \(\omega\) and \(p, q\). That is, we have the following.

**Theorem 3.1** Let \(p, q \in [1, \infty]\) and let \(\omega \in \mathcal{P}(\mathbb{R}^d)\). Then \(A_{(\omega),0}^{p,q}(\mathbb{C}^d) = A_{(\omega)}^{p,q}(\mathbb{C}^d)\), and the map \(f \mapsto \mathfrak{M} f\) from \(M_{(\omega)}^{p,q}(\mathbb{R}^d)\) to \(A_{(\omega)}^{p,q}(\mathbb{C}^d)\) is isometric and bijective.
We need some preparations for the proof and start with giving some remarks on the images of $S(R^d)$ and $S'(R^d)$ under the Bargmann transform. We denote these images by $A_S(C^d)$ and $A'_S(C^d)$, respectively, i.e. $A_S(C^d) = \{ \mathcal{U} f; f \in S(R^d) \}$ and $A'_S(C^d) = \{ \mathcal{U} f; f \in S'(R^d) \}$. As a consequence of Remark 1 and (9), we have

$$A'_S(C^d) \subseteq \left\{ F \in A(C^d); \| F e^{-\frac{1}{2} |x|^2} \|_{L^p} < \infty \text{ for some } N \geq 0 \right\} \tag{12}$$

(cf. (5)). We recall that in [2] it is proved that (12) holds with equality when $p = \infty$. An essential part of our investigations concerns to prove that equality is attained in (12) for each $p \in [1, \infty]$.

### 3.1. The image of harmonic oscillator on $M_{2N}^2$

Next we discuss mapping properties for a modified harmonic oscillator on modulation spaces of the form $M_{2N}^2(R^d)$, when $N$ is an integer. The operator which we shall consider is $H = |x|^2 - \Delta + 4d + 1$. We recall that if $\phi$ is given by (8) and $a(x, \xi) = \sigma_2(x, \xi) = |x|^2 + |\xi|^2 + 1$, then $H = Tp_\phi(a)$ (cf. [17, Section 3]).

Let $B$ be a translation invariant BF-space and $\omega \in \mathcal{P}(R^{2d})$. By Theorem 3.1 in [10], it follows that $H = Tp(a) = Tp(\sigma_2)$ is a continuous isomorphism from $M_{\sigma_2, \omega}(R^d)$ to $M(\omega, B)$. Since this is true for any weight $\omega$, it follows by induction and Banach’s theorem that the following is true.

**Proposition 3.2** Let $N$ be an integer, $\omega \in \mathcal{P}(R^{2d})$ and let $B$ be an invariant BF-space. Then $H^N$ on $S'(R^d)$ restricts to a continuous isomorphism from $M(\sigma_2, \omega, B)$ to $M(\omega, B)$. In particular, the set $\{ f \in S'(R^d); H^N f \in L^2(R^d) \}$ is equal to $M_{2N}^2(R^d)$, and the norm $f \mapsto \| H^N f \|_{L^2}$ is equivalent to $\| f \|_{M_{2N}^2}$.

We remark that the second part of Proposition 3.2 is proved in [3].

From now on, we assume that the norm and scalar product of $M_{2N}^2(R^d)$ are given by $\| f \|_{M_{2N}^2} = \| H^N f \|_{L^2}$ and $(f, g)_{M_{2N}^2} = (H^N f, H^N g)_{L^2}$, respectively. Then it follows from Proposition 3.2 that $(e_J^j)_{j \in J}$ is an orthonormal basis for $M_{2N}^2$, if and only if $(H^N e_J^j)_{j \in J}$ is an orthonormal basis for $L^2$. In the following, we use this fact to find appropriate orthonormal basis to $M_{2N}^2(R^d)$ in terms of the Hermite functions.

More precisely, we recall that the Hermite function $h_\alpha$ with respect to the multi-index $\alpha \in N^d$ is defined by $h_\alpha(x) = \pi^{-d/4} (-1)^{|\alpha|} (2|\alpha|!_d)^{1/2} e^{x^2/2} (\partial^\alpha e^{-|x|^2})$. Then $(h_\alpha)_{\alpha \in N^d}$ is an orthonormal basis for $L^2$, and by the definitions, it follows that $H h_\alpha = 2(|\alpha| + 2d + 1) h_\alpha$ [14]. The following result is now an immediate consequence of these observations.

**Lemma 3.3** Let $N$ be an integer. Then $\{ 2^{-N} (|\alpha| + 2d + 1)^{-N} h_\alpha \}_{\alpha \in N^d}$ is an orthonormal basis for $M_{2N}^2(R^d)$.

### 3.2. Mapping properties of \(C\) on $M_{2N}^2$

We shall now prove $A_{2N}^2 = A_{N}^2$ when $N$ is a non-zero even integer. Important parts of these investigations are based on serie expansions for analytic functions. We recall that each $F \in A(C^d)$ is equal to its Taylor series, i.e.

$$F(z) = \sum_{\alpha \in N^d} a_\alpha \frac{z^\alpha}{(\alpha!)^{1/2}}, \quad a_\alpha = \frac{(\partial^\alpha F)(0)}{(\alpha!)^{1/2}}. \tag{13}$$
We recall that \( F \in A^2_0 = A^2 \) if and only if the coefficients in (13) satisfy \( \| (a_\alpha)_{\alpha \in \mathbb{N}^d} \|_2^2 = \sum_{\alpha \in \mathbb{N}^d} |a_\alpha|^2 < \infty \). Furthermore, for \( F \in A^2 \), we have that \( F = \mathcal{V} f \) for some \( f \in L^2 \) and

\[
f(x) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha h_\alpha, \tag{14}\]

is fulfilled if and only if (13) holds, and then

\[
\| F \|_{A^2} = \| f \|_{L^2} = \| (a_\alpha)_{\alpha \in \mathbb{N}^d} \|_2^2. \tag{15}\]

We have now the following result. We refer to [15] for the proof.

**Proposition 3.4** Let \( N \) be an integer. Then the following is true:

1. \( A^2_{(2\sigma N), 0} \) consists of all \( F \in A(\mathbb{C}^d) \) with expansion given by (13), where

\[
\| F \| \| (a_\alpha^{(\sigma N)})_{\alpha \in \mathbb{N}^d} \|_2 < \infty. \tag{16}\]

Furthermore, \( \| \cdot \| \) and \( \| \cdot \|_{A^2_{(2\sigma N), 0}} \) are equivalent norms;

2. \( A^2_{(2\sigma N)} = A^2_{(\sigma N) \ast} \).

**Corollary 3.5** The set in the right-hand side of (12) agrees with \( \{ \mathcal{B} f : f \in S'(\mathbb{R}^d) \} \).

**Proof** The result follows from Proposition 3.4 and the fact that \( \cup_{N \in \mathbb{Z}} M^2_{2N}(\mathbb{R}^d) = S'(\mathbb{R}^d) \).

### 3.3. Mapping properties of \( \mathcal{B} \) on \( S' \) and the proof of the main theorem

We shall now consider relation (12) and motivate that we indeed have equality when \( 1 \leq p \leq 2 \). In order to do this, we need the following lemma. Here we let \( B_r(z) \) denote the open ball in \( \mathbb{C}^d \) with radius \( r \) and centre at \( z \in \mathbb{C}^d \). We refer to [15] for its proof.

**Lemma 3.6** There is a family \( (B_j)_{j \in J} \) of open balls \( B_j \) such that the following conditions are fulfilled:

1. \( \mathcal{C} B_j(0) \subseteq \cup B_j \);
2. \( B_j = B_{r_j}(z_j) \) for some \( r_j \) and \( z_j \) such that \( |z_j| \geq 4, r_j \leq 1/|z_j| \);
3. there is a bound of overlapping \( B_{4r_j}(z_j) \).

We have now the following result.

**Proposition 3.7** Let \( p \in [1, 2] \) be fixed. Then \( A'_S(\mathbb{C}^d) \) agrees with the set in the right-hand side of (12).

**Proof** Let \( \Omega_p \) be the set on the right-hand side of (12). In view of Corollary 3.5, it suffices to prove that \( \Omega_p \) is independent of \( p \). First assume that \( p_1 \leq p_2 \), and let \( \rho = [1, \infty] \) be such that \( 1/p_2 + 1/\rho = 1/p_1 \). Then it follows from Hölder’s inequality that

\[
\| F e^{-|\cdot|^{1/2}} \cdot)^{-N-d-1} \|_{L^{p_2}} = \| (F e^{-|\cdot|^{1/2}} \cdot)^{-N} \|_{L^{p_1}} \cdot \| \cdot \|_{L^{p_2}} \cdot \leq C \| F e^{-|\cdot|^{1/2}} \cdot)^{-N} \|_{L^{p_2}} ,
\]

where \( C = \| \cdot \|_{L^{p_2}} < \infty \). This proves that \( \Omega_{p_2} \subseteq \Omega_{p_1} \).
The opposite inclusion follows by a combination of Lemma 3.6 and mean-value properties for analytic functions. We refer to [15, Proposition 2.7] for the verifications.

Proof of Theorem 3.1 By Proposition 2.4, it follows that the map $f \mapsto \mathcal{V} f$ is an isometric injection from $M_{p,q}^{(ω)}(\mathbb{R}^d)$ to $A_{p,q}^{(ω)}(\mathbb{C}^d)$. We have to show that this mapping is surjective.

Therefore, assume that $F \in A_{p,q}^{(ω)}(\mathbb{C}^d)$. By Remark 1 and Propositions 3.4 and 3.7, there is an element $f \in S'(\mathbb{R}^d)$ such that $F = \mathcal{V} f$. We have $\|f\|_{M_{p,q}^{(ω)}(\mathbb{R}^d)} = \|\mathcal{V} f\|_{A_{p,q}^{(ω)}(\mathbb{C}^d)} = \|F\|_{A_{p,q}^{(ω)}(\mathbb{C}^d)} < \infty$. Hence, $f \in M_{p,q}^{(ω)}(\mathbb{R}^d)$, and the result follows. The proof is complete.

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