Remarks on the Blowup Criteria for Oldroyd Models

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Abstract

We provide a new method to prove and improve the Chemin-Masmoudi criterion for viscoelastic systems of Oldroyd type in [5] in two space dimensions. Our method is much easier than the one based on the well-known losing a priori estimate and is expected to be easily adopted to other problems involving the losing a priori estimate.

Keyword: Losing a priori estimate, blowup criteria, Oldroyd model.

1 Introduction

In this paper, we are going to study the non-blowup criteria of solutions of a type of incompressible non-Newtonian fluid flows described by the Oldroyd-B model in the whole 2D space:

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= \nu \Delta v + \mu_1 \nabla \cdot \tau, \\
\partial_t \tau + v \cdot \nabla \tau + a \tau &= Q(\tau, \nabla v) + \mu_2 D(v), \\
\nabla \cdot v &= 0,
\end{aligned}
\]

(1.1)

where \(v\) is the velocity field, \(\tau\) is the non-Newtonian part of the stress tensor and \(p\) is the pressure. The constants \(\nu\) (the viscosity of the fluid), \(a\) (the reciprocal of the relaxation time), \(\mu_1\) and \(\mu_2\) (determined by the dynamical viscosity of the fluid, the retardation time and \(a\)) are assumed to be non-negative. The bilinear term \(Q\) has the following form:

\[
Q(\tau, \nabla v) = W(v)\tau - \tau W(v) + b(D(v)\tau + \tau D(v)).
\]

(1.2)

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Here $b \in [-1, 1]$ is a constant, $D(v) = \frac{\nabla v + (\nabla v)^T}{2}$ is the deformation tensor and $W(v) = \frac{\nabla v - (\nabla v)^T}{2}$ is the vorticity tensor. Fluids of this type have both elastic properties and viscous properties. More discussions and the derivation of Oldroyd-B model (1.1) can be found in Oldroyd [22] or Chemin and Masmoudi [5].

There has been a lot of work on the existence theory of Oldroyd model [8, 9, 10, 17, 5, 14]. In particular, the following Theorem is established by Chemin and Masmoudi in [5]:

Theorem (Chemin and Masmoudi): In two space dimensions, the solutions to the Oldroyd model (1.1) with smooth initial data do not develop singularities for $t \leq T$ provided that

$$\int_0^T \|\tau(t, \cdot)\|_{L^\infty} + |b| \|\tau(t, \cdot)\|_{L^2}^2 dt < \infty.$$ (1.3)

To establish the blowup criterion (1.3), the authors in [5] use a losing a priori estimate for solutions of transport equations which was developed by Bahouri and Chemin [1] and used later on by a lot of authors (for example, see [6 5 7 16 21 18 19] and the references therein). Our purpose of this paper is to provide a simple method which avoids using the complicated losing a priori estimate and to improve the blowup criterion (1.3) for Oldroyd model (1.1) established by Chemin and Masmoudi [5]. To best illustrate our ideas and for simplicity, we will take $a = 0$ and $\nu = \mu_1 = \mu_2 = b = 1$ throughout this paper. More precisely, we study the following system

$$\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= \Delta v + \nabla \cdot \tau, \\
\partial_t \tau + v \cdot \nabla \tau &= \nabla v \tau + \tau(\nabla v)^t + D(v), \\
\nabla \cdot v &= 0.
\end{aligned}$$ (1.4)

We point out here that the results in this paper are obviously true for general constants $a, \mu_1, \mu_2 \geq 0, \nu, b > 0$ from our proofs.

Our main result concerning system (1.4) is:

**Theorem 1.1.** Assume that $(v, \tau)$ is a local smooth solution to the Oldroyd model (1.4) on $[0, T)$ and $\|v(0, \cdot)\|_{L^{2,\infty}(\mathbb{R}^2)} + \|\tau(0, \cdot)\|_{L^1(\mathbb{R}^2)} < \infty$ for some $\alpha \in (0, 1)$. Then one has

$$\|v(t, \cdot)\|_{\dot{C}^{1+\alpha}} + \|\tau(t, \cdot)\|_{\dot{C}^\alpha} < \infty$$

for all $0 \leq t \leq T$ provided that

$$\|\tau(t, \cdot)\|_{L^1(BMO)} < \infty \quad \text{and} \quad \|\tau\|_{L^\infty(L^1)} < \infty.$$ (1.5)
Remark 1.2. This result is in the spirit of the Beale-Kato-Majda [2] non blowup criterion for 3D Euler equations. There were many subsequent results improving the criterion, see for instance [13, 12, 23]. In particular our result still holds if we replace $BMO$ with the Besov space $B_{\infty, \infty}^0$ used in H. Kozono, T. Ogawa, and Y. Taniuchi. [12] or if we replace the condition by the one introduced in Planchon [23]. In other words, the first condition in (1.5) in the above theorem can be weakened to

$$\int_0^T \|\tau\|_{B_{\infty, \infty}^0}^q \leq \int_0^T \sup_q \|\Delta_q \tau(t, \cdot)\|_{L^\infty} dt < \infty,$$

or to

$$\lim_{\delta \to 0} \sup_q \int_{T-\delta}^T \|\Delta_q \tau(t, \cdot)\|_{L^\infty} dt < \epsilon,$$

for some sufficiently small $\epsilon > 0$. The second condition in (1.5) can be replaced by

$$\|\tau\|_{L^2_T(L^2)} < \infty,$$

which was used in [5].

Remark 1.3. It is easy to check that smooth solutions to (1.4) enjoy the following energy law:

$$\int_{\mathbb{R}^2} |v(t, \cdot)|^2 + tr\tau(t, \cdot) dx + \int_0^t \int_{\mathbb{R}^2} |\nabla v|^2 dx ds = \int_{\mathbb{R}^2} |v(0, \cdot)|^2 + tr\tau(0, \cdot) dx,$$  \hspace{1cm} (1.6)

which means that

$$v \in L^\infty_T(L^2) \cap L^2_T(\dot{H}^1)$$  \hspace{1cm} (1.7)

for all $T > 0$ under the second condition of (1.5). The a priori estimate (1.7) will be important to apply lemma 3.1.

Finally, it is well-known that if $A = 2\tau + I$ is a positive definite symmetric matrix at $t = 0$ (which is actually the physical case), then this property is conserved for later times. Indeed, $A$ satisfies the equation

$$\partial_t A + v \cdot \nabla A = \nabla v A + A(\nabla v)^t.$$  

Also, if at $t = 0$, we have $\det(A(0)) > 1$ and $A$ is positive definite, then this will also hold for later times (see [11]). In particular this implies that $\text{tr}(\tau) > 0$ (or one has $-1 < \text{tr}(\tau) \leq 0$, which contradicts with $\det(A) > 1$). Hence, we have the following corollary where we also use the improved criterion of Planchon.
Corollary 1.4. There exists an $\epsilon > 0$, such that if $(v, \tau)$ is a local smooth solution to the Oldroyd model (1.4) on $[0, T)$, $\|v(0, \cdot)\|_{L^2_t\dot{C}^{1+\alpha}} + \|\tau(0, \cdot)\|_{L^1_t\dot{C}^{\alpha}} < \infty$ for some $\alpha \in (0, 1)$ and that $\det(I + 2\tau(0)) > 1$, $A = I + 2\tau(0)$ is positive definite symmetric. Then one has

$$\|v(t, \cdot)\|_{\dot{C}^{1+\alpha}} + \|\tau(t, \cdot)\|_{\dot{C}^{\alpha}} < \infty$$

for all $0 \leq t \leq T$ provided that

$$\limsup_{\delta \to 0} \sup_q \int_{T-\delta}^T \Delta_q \tau(t, \cdot) \|L^\infty dt < \epsilon. \quad (1.8)$$

Our proof is based on careful Hölder estimates of heat and transport equations and the standard Littlewood-Paley theory, which is much easier than the extensively used losing a priori estimates (for example, see [1, 6, 7, 16]). In fact, the main innovation of this paper is that our analysis may be viewed as a replacement of the losing a priori estimate. Our method is expected to be easily adopted to other problems via the losing a priori estimate. Moreover, our criterion slightly improves the one established by Chemin and Masmoudi (see [5]).

Finally, let us make a remark on MHD:

$$\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla p = \nu \Delta v + \nabla \cdot (H \times H), \\
\partial_t H + v \cdot \nabla H = H \cdot \nabla v, \\
\nabla \cdot v = \nabla \cdot H = 0,
\end{cases} \quad (1.9)$$

where $H$ denotes the magnetic field. A direct corollary of Theorem 1.1 for MHD (1.9) is the following:

Corollary 1.5. Assume that $(v, H)$ is a local smooth solution to MHD (1.9) on $[0, T)$ and $\|v(0, \cdot)\|_{L^2_t\dot{C}^{1+\alpha}} + \|H(0, \cdot)\|_{L^2_t\dot{C}^{\alpha}} < \infty$ for some $\alpha \in (0, 1)$. Then one has

$$\|v(t, \cdot)\|_{\dot{C}^{1+\alpha}} + \|H(t, \cdot)\|_{\dot{C}^{\alpha}} < \infty$$

all $0 \leq t \leq T$ provided that

$$\int_0^T \|(H \times H)(t, \cdot)\|_{BMO} dt < \infty. \quad (1.10)$$

The proof of this corollary is given in section 4.Unfortunately, at present we are not able to improve (1.10) as

$$\int_0^T \|H(t, \cdot)\|_{BMO} dt < \infty,$$

and this is still an open problem.

The paper is organized as follows: Section 2 is devoted to recalling some basic properties of Littlewood-Paley theory and proving two interpolation inequalities. The proof of Theorem 1.1 is given in section 3. In the last section we sketch the proof of Corollary 1.5.
2 Preliminaries

Let $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^2)$, its Fourier transform $\mathcal{F} f = \hat{f}$ (inverse Fourier transform $\mathcal{F}^{-1} g = \check{g}$, respectively) is defined by $\hat{f}(\xi) = \int e^{-ix \cdot \xi} f(x) dx$ ($\check{g}(x) = \int e^{ix \cdot \xi} g(\xi) d\xi$, respectively). Now let us recall the Littlewood-Paley decomposition (see [3, 4]). Choose two nonnegative radial functions $\psi, \phi \in \mathcal{S}(\mathbb{R}^2)$, supported respectively in $B = \{ \xi \in \mathbb{R}^2 : |\xi| \leq 4^3 \}$ and $C = \{ \xi \in \mathbb{R}^2 : 3^4 \leq |\xi| \leq 8^3 \}$ such that

$$\psi(\xi) + \sum_{j \geq 0} \phi(\xi/2^j) = 1 \text{ for } \xi \in \mathbb{R}^2, \quad \sum_{-\infty \leq j \leq \infty} \phi(\xi/2^j) = 1 \text{ for } \xi \in \mathbb{R}^2/\{0\}.$$

The frequency localization operator is defined by

$$\Delta_q f = \int_{\mathbb{R}^2} \check{\phi}(y) f(x - 2^{-q} y) dy, \quad S_q f = \int_{\mathbb{R}^2} \check{\psi}(y) f(x - 2^{-q} y) dy. \quad (2.1)$$

The following Lemma is well-known (for example, see [4]).

Lemma 2.1. For $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and integer $q$, one has

$$\begin{align*}
&c 2^q \|\Delta_q f\|_{L^p} \leq \|\nabla^s \Delta_q f\|_{L^p} \leq C 2^q \|\Delta_q f\|_{L^p}, \\
&\|\nabla^s S_q f\|_{L^p} \leq C 2^q \|f\|_{L^p}, \\
&c e^{-C 2^q t} \|\Delta_q f\|_{L^\infty} \leq \|e^{t\Delta} \Delta_q f\|_{L^\infty} \leq Ce^{-c 2^q t} \|\Delta_q f\|_{L^\infty}.
\end{align*} \quad (2.2)$$

Here $C$ and $c$ are positive constants independent of $s$, $p$ and $q$.

We also need the following Lemma (see also [21, 20] where similar estimates were used.)

Lemma 2.2. Assume that $\beta > 0$. Then there exists a positive constant $C > 0$ such that

$$\begin{align*}
&\|f\|_{L^\infty} \leq C \left( 1 + \|f\|_{L^2} + \|f\|_{\text{BMO}} \ln(e + \|f\|_{C^\beta}) \right), \\
&\int_0^T \|\nabla g(s, \cdot)\|_{L^\infty} ds \leq C \left( 1 + \int_0^T \|g(s, \cdot)\|_{L^2} ds \\
&\quad + \sup_q \int_0^T \|\Delta_q \nabla g(s, \cdot)\|_{L^\infty} ds \ln(e + \int_0^T \|\nabla g(s, \cdot)\|_{C^\beta} ds) \right). \quad (2.3)
\end{align*}$$

Proof. The first inequality is well-known. For example, see [21, 13, 15]. To prove the
second inequality, we use the Littlewood-Paley theory to compute that

\[
\int_0^T \| \nabla g(s, \cdot) \|_{L^\infty} ds \leq C \int_0^T \| \sum_{q \leq 0} \nabla \Delta_q g(s, \cdot) \|_{L^\infty} ds
\]

\[
+ CN \max_{1 \leq q \leq N} \int_0^T \| \Delta_q \nabla g(s, \cdot) \|_{L^\infty} ds + \int_0^T \sum_{q \geq N+1} 2^{-\beta q} 2^\beta q \| \Delta_q \nabla g(s, \cdot) \|_{L^\infty} ds
\]

\[
\leq C \left( \int_0^T \| g(s, \cdot) \|_{L^2} ds + \sup_{1 \leq q \leq N} \int_0^T N \| \Delta_q \nabla g(s, \cdot) \|_{L^\infty} ds
\]

\[
+ 2^{-\beta N} \int_0^T \| \nabla g(s, \cdot) \|_{\dot{C}^{1/3}} ds \right).
\]

Then the second inequality in Lemma 2.2 follows by choosing

\[
N = \frac{1}{\beta} \log_2 (e + \int_0^T \| \nabla g(s, \cdot) \|_{\dot{C}^{1/3}} ds) \leq C \ln (e + \int_0^T \| \nabla g(s, \cdot) \|_{\dot{C}^{1/3}} ds).
\]

\[
\square
\]

3 Blowup Criteria for Oldroyd-B Model

This section is devoted to establishing the blowup criterion for the Oldroyd-B Model (1.1) and proving Theorem 1.1. Our analysis is based on careful H"older estimates of heat and transport equations and the standard Littlewood-Paley theory, which is much easier than the extensively used losing a priori estimates (for example, see [1, 6, 5, 7, 16]). Moreover, our criterion slightly improves the one established by Chemin and Masmoudi (see [5]). We divide our proof into two steps. The first step is focused on establishing some a priori estimates for 2-D Navier-Stokes equations. Then we establish Hölder estimates for the velocity field \(v\) and the stress tensor \(\tau\) in the second step.

Step 1. The a priori estimates for 2-D Navier-Stokes equations.

We need the following Lemma which is basically established by Chemin and Masmoudi in [3]. For completeness, the proof will be also sketched here.

Lemma 3.1. (Chemin-Masmoudi) Let \(v\) be a a solution of the Navier-Stokes equations with initial data in \(L^2\) and an external force \(f \in \dot{L}^1_T(C^{-1}) \cap \dot{L}^2_T(H^{-1})\):

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= \Delta v + f, \\
\nabla \cdot v &= 0,
\end{aligned}
\]

\[
v(0, x) = v_0(x).
\]
Then we have the following a priori estimate:

\[
\|v\|_{L^2_t(C^1)} \leq C \left( \sup_q \|\Delta_q v_0\|_{L^2} (1 - \exp\{-c 2^{2q} T\}) \right) \\
+ \left( \|v_0\|_{L^2} + \|f\|_{L^2_t(H^{-1})} \|\nabla v\|_{L^2_t(L^2)}^2 + \sup_q \int_0^T \|2^{-q} \Delta_q f(s)\|_{L^\infty} ds \right).
\]

**Proof.** First of all, applying the operator $\Delta_q$ to the 2-D Navier-Stokes equations (3.1) and then using Lemma 2.1 and the standard energy estimate, we deduce that

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q v\|_{L^2}^2 + c 2^{2q} \|\Delta_q v\|_{L^2}^2 \\
\leq \|2^q \Delta_q f\|_{L^2} \left( \|2^{-q} \Delta_q f\|_{L^2} + \|\Delta_q (v \otimes v)\|_{L^2} \right) \\
\leq c 2^{2q} \|\Delta_q v\|_{L^2}^2 + C \left( \|2^{-q} \Delta_q f\|_{L^2}^2 + \|\Delta_q (v \otimes v)\|_{L^2}^2 \right).
\]

Integrating with respect to time and summing over $q$, we get

\[
\sum_q \|\Delta_q v\|_{L^2_t(L^2)}^2 \leq \|v_0\|_{L^2}^2 + C \left( \|f\|_{L^2_t(H^{-1})}^2 + \|v \otimes v\|_{L^2_t(L^2)}^2 \right) \\
\leq \|v_0\|_{L^2}^2 + C \left( \|f\|_{L^2_t(H^{-1})}^2 + \|v\|_{L^2_t(L^2)}^2 \|\nabla v\|_{L^2_t(L^2)}^2 \right),
\]

where we used the standard interpolation inequality $\|v\|_{L^4}^2 \leq C \|v\|_{L^2} \|\nabla v\|_{L^2}$. Recalling the basic energy estimate

\[
\|v\|_{L^2_t(L^2)}^2 + \|\nabla v\|_{L^2_t(L^2)}^2 \leq \|v_0\|_{L^2}^2 + \|f\|_{L^2_t(H^{-1})}^2,
\]

one has

\[
\sum_q \|\Delta_q v\|_{L^2_t(L^2)}^2 \leq C \left( \|v_0\|_{L^2}^2 + \|f\|_{L^2_t(H^{-1})}^2 \right) \\
\times \left( 1 + \|v_0\|_{L^2}^2 + \|f\|_{L^2_t(H^{-1})}^2 \right).
\]

Next, let us apply $\Delta_q$ to (3.1) and use Lemma 2.1 to estimate

\[
\|\Delta_q v(t)\|_{L^\infty} \leq C \|\Delta_q v_0\|_{L^\infty} e^{-c 2^{2q} t} \\
+ \int_0^t \left( \|\Delta_q f(s)\|_{L^\infty} + \|\Delta_q \nabla \cdot (v \otimes v)(s)\|_{L^\infty} \right) e^{-c 2^{2q} (t-s)} ds,
\]

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which yields

\[ \| v \|_{L^1_T(C)} \leq C \sup_q \int_0^T \| \Delta_q v_0 \|_{L^\infty} 2^q e^{-c2^q t} dt \]

\[ + C \sup_q \int_0^T \int_0^t \| \Delta_q f(s) \|_{L^\infty} 2^q e^{-c2^q (t-s)} ds dt \]

\[ + C \sup_q \int_0^T \int_0^t \| \Delta_q \nabla \cdot (v \otimes v)(s) \|_{L^\infty} 2^q e^{-c2^q (t-s)} ds dt \]

\[ \leq C \sup_q \| \Delta_q v_0 \|_{L^2} (1 - e^{-c2^q T}) \]

\[ + C \sup_q \int_0^T \| \Delta_q (v \otimes v)(s) \|_{L^\infty} ds + \| f \|_{L^1_T(C^{-1})}. \]

Using the Bony’s decomposition, one can write

\[ \| \Delta_q (v \otimes v)(s) \|_{L^\infty} = \sum_{|p-r| \leq 1} \| \Delta_q (\Delta_p v \otimes \Delta_r v)(s) \|_{L^\infty} \]

\[ + \sum_{p-r \geq 2} \| \Delta_q (\Delta_p v \otimes \Delta_r v)(s) \|_{L^\infty} + \sum_{r-p \geq 2} \| \Delta_q (\Delta_p v \otimes \Delta_r v)(s) \|_{L^\infty}. \]

A straightforward computation gives

\[ \int_0^T \sum_{|p-r| \leq 1} \| \Delta_q (\Delta_p v \otimes \Delta_r v)(s) \|_{L^\infty} ds \]

\[ \leq C \int_0^T \sum_{|p-r| \leq 1} 2^q \| \Delta_q (\Delta_p v \otimes \Delta_r v)(s) \|_{L^2} ds \]

\[ \leq C \int_0^T \sum_{|p-r| \leq 1, \ p \geq q-3} 2^{q-\frac{p+r}{2}} \| 2^p \Delta_p v \|_{L^\infty} \| \Delta_r v \|_{L^2} \| \Delta_p v \|_{L^2} \| 2^r \Delta_r v \|_{L^2} ds \]

\[ \leq C \int_0^T \sum_{|p-r| \leq 1, \ p \geq q-3} 2^{q-\frac{p+r}{2}} \| 2^p \Delta_p v \|_{L^\infty} \| \Delta_r v \|_{L^2} \| 2^p \Delta_p v \|_{L^2} \| 2^r \Delta_r v \|_{L^2} ds \]

\[ \leq C \| v \|_{L^p_T(L^2)} \| \nabla v \|_{L^2_T(L^2)} \| v \|_{L^1_T(C^{-1})}. \]
Similarly, one has
\[
\int_0^T \left( \sum_{p-r \geq 2} \| \Delta_q(\Delta_p v \otimes \Delta_r v)(s) \|_{L^\infty} + \sum_{r-p \geq 2} \| \Delta_q(\Delta_p v \otimes \Delta_r v)(s) \|_{L^\infty} \right) ds
\]
\[
\leq C \int_0^T \sum_{p-r \geq 2, |p-q| \leq 2} \| \Delta_p v \|_{L^\infty} \| \Delta_r v \|_{L^\infty} ds
\]
\[
\leq C \int_0^T \sum_{p-r \geq 2, |p-q| \leq 2} \| 2^p \Delta_p v \|_{L^\infty}^{\frac{1}{2}} \| 2^p \Delta_p v \|_{L^2}^{\frac{1}{2}} 2^{p-q} \| \Delta_r v \|_{L^2} ds
\]
\[
\leq C \| v \|_{L^2_T(L^2)} \| \nabla v \|_{L^2_T(L^2)} \| v \|_{L^4_T(C^1)}^{\frac{3}{2}}.
\]

Using the above two estimates, one can improve (3.5) as
\[
\| v \|_{L^4_T(C^1)} \leq C \left( \sup_q \| \Delta_q v_0 \|_{L^2} \left( 1 - e^{-c2^{2q}T} \right) + \| v \|_{L^\infty_T(L^2)} \| \nabla v \|_{L^2_T(L^2)}^2 + \| f \|_{L^1_T(C^{1-1})} \right).
\]

Consequently, one can deduce (3.2) from and the basic energy estimate (3.3) and the above inequality.

Now let us assume that \( f \in L^1_T(\dot{C}^{-1}) \cap L^2_T(H^{-1}) \). By Lemma 3.1, it is easy to see that
\[
\| v \|_{L^1_{[t_0,T]}(C^1)} \leq C \left( \sup_q \| \Delta_q v(t_0) \|_{L^2} \left( 1 - \exp \{-c2^{2q}(T - t_0)\} \right) \right.
\]
\[
+ \int_{t_0}^T \sup_q \| 2^{-q} \Delta_q f(s) \|_{L^\infty} ds + \left( \| v(t_0) \|_{L^2} + \| f \|_{L^1_{[t_0,T]}(H^{-1})} \right) \| \nabla v \|_{L^2_{[t_0,T]}(L^2)}^2
\]
holds for any \( t_0 \in [0, T) \). By (3.4), one can choose some \( q_0 \) such that
\[
\sup_{q > q_0} \| \Delta_q v \|_{L^2_{[t_0,T]}(L^2)}^2 \leq \frac{\epsilon}{4C}.
\]

Furthermore, by the basic energy estimate (3.3), one can choose some \( t_1 \in [t_0, T) \) such that
\[
\sup_{t_1 \leq t \leq T} \sup_{q \leq q_0} \| \Delta_q v(t) \|_{L^2} \left( 1 - \exp \{-c2^{2q}(T - t)\} \right) \leq \frac{\epsilon}{4C}.
\]

Consequently, one has
\[
\sup_{t_1 \leq t \leq T} \sup_{q \leq q_0} \| \Delta_q v(t) \|_{L^2} \left( 1 - \exp \{-c2^{2q}(T - t)\} \right) \leq \frac{\epsilon}{2C}.
\]
On the other hand, it is obvious that one can choose some $t_2 \in [t_1, T)$ such that

$$\left( \sup_{t_2 \leq t \leq T} \|v(t)\|_{L^2} + \|f\|_{L^2_{[t_2, T]}(H^{-1})} \right) \|\nabla v\|_{L^2_{[t_2, T]}(L^2)}^2 \quad (3.8)$$

$$+ \int_{t_2}^T \sup_{q} \|2^{-q} \Delta_q f(s)\|_{L^\infty} ds \leq \frac{\epsilon}{2C}.$$  

Combining (3.7) and (3.8) with (3.6), one arrives at

$$\|v\|_{L^1_{[t_2, T]}(C^1)} \leq \epsilon. \quad (3.9)$$

**Step 2. H"older estimate for $v$ and $\tau$.**

First of all, by (3.9) and the assumption (1.5), one can choose $t_* \in [t_2, T)$ such that

$$\|v\|_{L^1_{[t_2, T]}(C^1)} \leq \epsilon, \quad \|\tau\|_{L^1_{[t_2, T]}(BMO)} \leq \epsilon. \quad (3.10)$$

For $0 \leq t < T$, define

$$A(t) = \sup_{0 \leq s < t} \|v(t, \cdot)\|_{C^{1+\alpha}}, \quad B(t) = \sup_{0 \leq s < t} \|\tau(t, \cdot)\|_{C^{\alpha}}.$$ 

We are about to estimate $A(t)$ and $B(t)$ for $0 \leq t < T$. For this purpose, let us apply $\Delta_q$ to the Oldroyd-B system (1.4) to get

$$\begin{cases}
\partial_t \Delta_q v - \Delta \Delta_q v + \nabla \Delta_q p = \nabla \cdot \Delta_q (\tau - v \otimes v), \\
\partial_t \Delta_q \tau + v \cdot \nabla \Delta_q \tau = \Delta_q \left( \nabla v \tau + \tau (\nabla v)^{t} + D(v) \right) + [v \cdot \nabla, \Delta_q] \tau.
\end{cases} \quad (3.11)$$

Let us first estimate $\|v(t, \cdot)\|_{C^{1+\alpha}}$. By the first equation in (3.11) and Lemma 2.1, one has

$$\|\Delta_q v\|_{L^\infty} \leq C e^{-c2^q t} \|\Delta_q v(0)\|_{L^\infty}$$

$$+ \int_0^t e^{-c2^q (t-s)} \|\nabla \cdot \Delta_q (\tau - v \otimes v)\|_{L^\infty}(s) ds. \quad (3.12)$$

Multiplying $2^{q(1+\alpha)}$ to both sides of (3.12), we have

$$\|\Delta_q v(t, \cdot)\|_{C^{1+\alpha}} \leq C \|v(0, \cdot)\|_{C^{1+\alpha}}$$

$$+ C \int_0^t 2^{2q} e^{-c2^q (t-s)} \|\Delta_q \tau\|_{C^{\alpha}} ds$$

$$+ C \int_0^t 2^{2q} e^{-c2^q (t-s)} \|v \otimes v\|_{C^{1+\alpha}} ds$$

$$\leq C \left( \|v(0, \cdot)\|_{C^{1+\alpha}} + B(t) \right) + C \left( \int_0^t \|v\|_{L^4}^4 \|v\|_{C^{1+\alpha}}^4 ds \right)^{\frac{1}{2}},$$
Consequently, there holds
\[ \|v(t, \cdot)\|_{C_t^{1+\alpha}}^4 \leq C\left(\|v(0, \cdot)\|_{C_t^{1+\alpha}} + B(t)\right)^4 + C \int_0^t \|v\|_{L^4}^4 \|v\|_{C_t^{1+\alpha}}^4 ds \]
\[ \leq C\left(\|v(0, \cdot)\|_{C_t^{1+\alpha}} + B(\tilde{t})\right)^4 + C \int_0^t \|v(s, \cdot)\|_{L^2}^2 \|\nabla v(s, \cdot)\|_{L^2}^2 \|v(s, \cdot)\|_{C_t^{1+\alpha}}^4 ds \]
for any fixed \( \tilde{t} : 0 \leq \tilde{t} < T \) and \( t \leq \tilde{t} < T \). Here we used the fact that \( B(t) \) is nondecreasing. Consequently, Gronwall’s inequality gives that
\[ A(\tilde{t})^4 = \sup_{0 \leq t \leq \tilde{t}} \|v(t, \cdot)\|_{C_t^{1+\alpha}}^4 \leq C\left(\|v(0, \cdot)\|_{C_t^{1+\alpha}} + B(\tilde{t})\right)^4 \exp \left\{C \int_0^t \|v(s, \cdot)\|_{L^2}^2 \|\nabla v(s, \cdot)\|_{L^2}^2 ds \right\}. \]
Since \( \tilde{t} \in [0, T) \) is arbitrary, using the basic energy inequality (1.7), we in fact have
\[ A(t) \leq C\left(\|v(0, \cdot)\|_{C_t^{1+\alpha}} + B(t)\right), \quad 0 \leq t < T. \] (3.13)

Next, by the second equation in (3.11), we have
\[ \|\Delta_q \tau(t, \cdot)\|_{L^\infty} \leq \|\Delta_q \tau(0, \cdot)\|_{L^\infty} + \int_0^t \left(2^n \|\Delta_q v(s, \cdot)\|_{L^\infty} + \|\Delta_q (\nabla v \tau + \tau (\nabla v)^t)(s, \cdot)\|_{L^\infty} + \|[v \cdot \nabla, \Delta_q] \tau(s, \cdot)\|_{L^\infty}\right) ds, \]
which implies that
\[ \|\Delta_q \tau(t, \cdot)\|_{C_t^\alpha} \leq C\|\tau(0, \cdot)\|_{C_t^\alpha} + \int_0^t \left(\|v\|_{C_t^{1+\alpha}} + \|\nabla v\|_{L^\infty}\|\tau\|_{C_t^\alpha} + \|\tau\|_{L^\infty}\|v\|_{C_t^{1+\alpha}} + 2^n \|[v \cdot \nabla, \Delta_q] \tau(s, \cdot)\|_{L^\infty}\right) ds. \] (3.14)

By Bony’s decomposition, one has
\[ [v \cdot \nabla, \Delta_q] \tau = \sum_{|\rho - q| \leq 1} [\Delta_p v \cdot \nabla, \Delta_q] \Delta_{q'} \tau + \sum_{p \leq q'-2} [\Delta_p v \cdot \nabla, \Delta_q] \Delta_{q'} \tau + \sum_{p \leq q'-2} [\Delta_{q'} v \cdot \nabla, \Delta_q] \Delta_p \tau \]
\[ = \sum_{|q'-q| \leq 2} \left([S_{q'-1} v \cdot \nabla, \Delta_q] \Delta_{q'} \tau + [\Delta_{q'} v \cdot \nabla, \Delta_q] S_{q'-1} \tau\right) + \sum_{|p-q'| \leq 1} [\Delta_p v \cdot \nabla, \Delta_q] \Delta_{q'} \tau. \]
Noting that

\[ [S_{q'-1}v, \Delta_q]f = \int h(y) \left( (S_{q'-1}v)(x) - (S_{q'-1}v)(x - 2^{-q}y) \right) f(x - 2^{-q}y) dy, \]

one has

\[ \|[S_{q'-1}v, \Delta_q]f\|_{L^\infty} \leq C 2^{-q} \|\nabla S_{q'-1}v\|_{L^\infty} \|f\|_{L^\infty}. \]

Consequently, we have

\[
\sum_{|q'| \leq 2} \int_0^t \left( 2^{aq} \|[S_{q'-1}v \cdot \nabla, \Delta_q]\Delta_q \tau(s, \cdot)\|_{L^\infty} \right) ds
\]

\[
\leq \sum_{|q'| \leq 2} \int_0^t 2^{aq} 2^{-q} \|\nabla S_{q'-1}v\|_{L^\infty} \|\Delta_q \nabla \tau\|_{L^\infty} (s, \cdot) ds
\]

\[
\leq C \sum_{|q'| \leq 2} \int_0^t \|\nabla S_{q'-1}v\|_{L^\infty} \|2^{aq} \|\Delta_q \tau\|_{L^\infty} (s, \cdot) ds
\]

\[
\leq C \int_0^t \|\nabla v\|_{L^\infty} \|\tau\|_{\dot{C}^{\alpha}} ds.
\]

Similarly, we have

\[
\sum_{|q'| \leq 2} \int_0^t \left( 2^{aq} \|[\Delta_q v \cdot \nabla, \Delta_q]S_{q'-1} \tau(s, \cdot)\|_{L^\infty} \right) ds
\]

\[
\leq \sum_{|q'| \leq 2} \int_0^t 2^{aq} 2^{-q} \|\Delta_q \nabla v\|_{L^\infty} \|S_{q'-1} \nabla \tau\|_{L^\infty} (s, \cdot) ds
\]

\[
\leq C \int_0^t \|\tau\|_{L^\infty} \|\nabla v\|_{\dot{C}^{1+\alpha}} ds.
\]

At last, one computes that

\[
\sum_{|p-q| \leq 1} \int_0^t \left( 2^{aq} \|[\Delta_p v \cdot \nabla, \Delta_q]D_q \tau(s, \cdot)\|_{L^\infty} \right) ds
\]

\[
\leq \sum_{p,q} \int_0^t \left( 2^{(1+\alpha)q} \|[\Delta_p v, \Delta_q]D_q \tau(s, \cdot)\|_{L^\infty} \right) ds
\]

\[
+ \sum_{p,q} \int_0^t \left( 2^{(1+\alpha)q} \|[\Delta_p v, \Delta_q]D_q \tau(s, \cdot)\|_{L^\infty} \right) ds
\]

\[
\leq C \int_0^t \|\tau\|_{L^\infty} \|\nabla v\|_{\dot{C}^{1+\alpha}} ds.
\]
The above inequalities yield an improvement of (3.14):

\[
\|\tau(t, \cdot)\|_{\mathcal{C}_\alpha} \leq C\|\tau(0, \cdot)\|_{\mathcal{C}_\alpha} + \int_0^t (\|\nabla v\|_{L^\infty} + \|\tau\|_{L^\infty}) (\|\tau(s, \cdot)\|_{\mathcal{C}_\alpha} + \|v(s, \cdot)\|_{\mathcal{C}_1}) ds.
\] (3.15)

Now let us insert (3.13) into (3.15) to get

\[
B(t) = \sup_{0 \leq s < t} \|\tau(t, \cdot)\|_{\mathcal{C}_\alpha} 
+ C \int_0^t (\|\nabla v\|_{L^\infty} + \|\tau\|_{L^\infty}) (\|v(0, \cdot)\|_{\mathcal{C}_1} + B(s)) ds.
\]

Noting that by the inequalities in Lemma 2.2, we can estimate

\[
\int_0^t (\|\nabla v\|_{L^\infty} + \|\tau\|_{L^\infty}) ds 
\leq \int_0^{t_*} (\|\nabla v\|_{L^\infty} + \|\tau\|_{L^\infty}) ds + C \int_{t_*}^t (1 + \|v\|_{L^2} + \|\tau\|_{L^2}) ds 
+ C \sup_q \int_{t_*}^t \|\nabla \Delta_q v\|_{L^\infty} ds \ln (e + \int_0^t \|v\|_{\mathcal{C}_1} ds) 
+ C \int_{t_*}^t \|\tau\|_{BMO} \ln (e + \|\tau\|_{\mathcal{C}_\alpha}) ds 
\leq C_* + C\epsilon \ln [e + Ct(\|v(0, \cdot)\|_{\mathcal{C}_1} + B(t))] + C\epsilon \ln (e + B(t)) 
\leq C_* + C\epsilon \ln (e + \|v(0, \cdot)\|_{\mathcal{C}_1} + B(t)).
\]

Here \(C_*\) is a positive constant depending on the solution \((v, \tau)\) on \([0, t_*]\). Consequently, we have

\[
B(t) \leq C_* \left(1 + \|\tau(0, \cdot)\|_{\mathcal{C}_\alpha} + \|v(0, \cdot)\|_{\mathcal{C}_1}\right) + C \int_0^t (\|\nabla v\|_{L^\infty} + \|\tau\|_{L^\infty}) B(s) ds.
\]

Then Gronwall’s inequality yields that

\[
e + \|v(0, \cdot)\|_{\mathcal{C}_1} + B(t) 
\leq C_* \left(e + \|\tau(0, \cdot)\|_{\mathcal{C}_\alpha} + \|v(0, \cdot)\|_{\mathcal{C}_1}\right) \exp \left\{C \int_{t_*}^t (\|\nabla v\|_{L^\infty} + \|\tau\|_{L^\infty}) ds\right\} 
\leq C_* \left(e + \|\tau(0, \cdot)\|_{\mathcal{C}_\alpha} + \|v(0, \cdot)\|_{\mathcal{C}_1}\right) e^{C\epsilon \ln \left(e + \|v(0, \cdot)\|_{\mathcal{C}_1} + B(t)\)} 
\leq C_* \left(e + \|\tau(0, \cdot)\|_{\mathcal{C}_\alpha} + \|v(0, \cdot)\|_{\mathcal{C}_1}\right) \left(e + \|v(0, \cdot)\|_{\mathcal{C}_1} + B(t)\right)^{C\epsilon}.
\]

From the above inequalities and (3.13), we have

\[
A(t) + B(t) \leq C_* \left(1 + \|\tau(0, \cdot)\|_{\mathcal{C}_\alpha} + \|v(0, \cdot)\|_{\mathcal{C}_1}\right)^2
\] (3.16)

by choosing \(\epsilon = \frac{1}{2C_*}\).
4 Proof of Corollary 1.5

In fact, it is easy to see that the tensor $H \otimes H$ satisfies the following transport equation:

$$\partial_t (H \otimes H) + v \cdot \nabla (H \otimes H) = \nabla v (H \otimes H) + (H \otimes H)(\nabla v)^t.$$  (4.1)

Hence, the tensor $H \otimes H - \frac{1}{2}I$ plays the role of $\tau$ (However, it seems not being able to directly apply Theorem 1.1 to get Corollary 1.5). The rest part of the proof of Corollary 1.5 is similar as that of Theorem 1.1.

In fact, by the assumption $\|v(0, \cdot)\|_{L^2 \cap \dot{C}^{1+\alpha}} + \|H(0, \cdot)\|_{L^2 \cap \dot{C}^{\alpha}} < \infty$, one can easily derive that

$$\|v(0, \cdot)\|_{L^2 \cap \dot{C}^{1+\alpha}(\mathbb{R}^2)} + \|H \otimes H(0, \cdot)\|_{L^1 \cap \dot{C}^{\alpha}(\mathbb{R}^2)} < \infty.$$  (4.2)

Moreover, one has the following energy law

$$\|(v, H)\|_{L^\infty_T(L^2)} + 2\|\nabla v\|_{L^2_T(L^2)} = \|(v, H)(0, \cdot)\|_{L^2},$$  (4.3)

which gives that

$$\|H \otimes H\|_{L^\infty_T(L^1)} < \infty.$$  (4.4)

Having (4.2), (4.3) and (4.4) in hand, and noticing the assumption (1.10) and the transport equation (4.1) for $H \otimes H$, one has

$$\|v(t, \cdot)\|_{\dot{C}^{1+\alpha}} + \|H \otimes H(t, \cdot)\|_{\dot{C}^{\alpha}} < \infty$$

by an exactly same manner as in section 3. Coming back to the transport equation for $H$ in (1.9), we have

$$\|H\|_{\dot{C}^{\alpha}} < \infty$$

in a standard manner. This completes the proof of Corollary 1.5.

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