ON CD SPACES WITH NONNEGATIVE CURVATURE OUTSIDE A COMPACT SET

MAURICIO CHE AND JESÚS NÚÑEZ-ZIMBRÓN

Abstract. In this paper we adapt work of Z.-D. Liu to prove a ball covering property for non-branching CD spaces with nonnegative curvature outside a compact set. As a consequence we obtain uniform bounds on the number of ends of such spaces.

1. Introduction

In [9][10], Z.-D. Liu proved that Riemannian manifolds with nonnegative Ricci curvature outside a compact set satisfy the following ball covering property. In the following we denote the metric ball of radius $r$ centered at $p \in M$ by $B_r(p)$ and the closed metric ball with the same radius and center by $\overline{B}_r(p)$.

**Theorem 1.1.** Let $M^n$ be a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set $B$. Assume that $\text{Ric}_M \geq (n-1)H$ and that $B \subset B_{D_0}(p_0)$ for some $p_0 \in M$ and $D_0 > 0$. Then for any $\mu > 0$ there exists $C = C(n, HD_0^2, \mu) > 0$ such that for any $r > 0$, the following property is satisfied: If $S \subset B_r(p_0)$, there exist $p_1, \ldots, p_k \in S$ with $k \leq C$ such that

$$S \subset \bigcup_{j=1}^k B_{\mu r}(p_j).$$

We state and prove this result in the more general context of non-branching metric measure spaces satisfying the curvature-dimension condition introduced by Lott-Sturm-Villani [11][13][14] (see the section on preliminaries below for the definitions). This class of spaces contains the class of RCD spaces, as it was recently shown that these are non-branching (see [5] Theorem 1.3), so, a fortiori, it also includes Alexandrov spaces [12][16] and weighted Riemannian manifolds. More precisely, we prove the following theorem.

**Theorem A.** Let $(X,d,m)$ be a non-branching metric measure space satisfying the CD$(K,N)$ condition for some $N > 1$ and $K \in \mathbb{R}$. Assume that $B$ is a compact subset of $X$ with $B \subset B_{D_0}(p_0)$ for some $p_0 \in M$ and $D_0 > 0$ and such that the CD$_{loc}(0,N)$ condition is satisfied on $X \setminus B$ (see Definition 2.5). Then for any $\mu > 0$, there exists $C = C(N, K D_0^2, \mu) > 0$ such that for any $r > 0$, the following property is satisfied: If $S \subset \overline{B}_r(p_0)$, there exist $p_1, \ldots, p_k \in S$ with $k \leq C$ and such that

$$S \subset \bigcup_{j=1}^k B_{\mu r}(p_j).$$

The proof follows the arguments of [9][10] almost verbatim, albeit with some needed adaptations to account for the more general hypotheses. The main tools we need are a version of the local-to-global theorem for the CD condition (see Lemma 2.2) and a Bishop-Gromov inequality for certain
star-shaped sets (see Theorem 2.4). In general one can prove that $\text{CD}(K,N)$ spaces support a Bishop-Gromov inequality for star-shaped sets following the proof of [14] Theorem 2.3, just as is done in [4] Proposition 3.5 to get a timelike Bishop-Gromov inequality in the context of Lorentzian synthetic spaces. However, it is important to notice that Lemma 2.4 is not a direct consequence of this fact. Namely, since the $\text{CD}(K,N)$ condition implies a Bishop-Gromov inequality with parameters $K,N$ and we are interested in the corresponding inequality with parameters $0,N$, we need to follow the original proof of the Bishop-Gromov inequality in [14] and make sure that all optimal transports involved remain in the region where $\text{CD}_{\text{loc}}(0,N)$ holds.

Finally, a direct consequence is that spaces satisfying the hypotheses of Theorem A have a uniformly bounded number of ends (see Definition 3.1).

**Corollary B.** Let $(X,d,m)$ be a metric measure space satisfying the $\text{CD}(K,N)$ with $N \geq 1$ and $K \in \mathbb{R}$. Assume that $B$ is a compact subset of $X$ with $B \subset B_{D_0}(p_0)$ for some $p_0 \in M$ and $D_0 > 0$ and such that the $\text{CD}_{\text{loc}}(0,N)$ condition is satisfied on $X \setminus B$. Then there exists $C = C(N,KD_0^2) > 0$ such that $(X,d)$ has at most $C$ ends.

In [3] a result bounding the number of ends of manifolds with nonnegative Ricci curvature outside a compact set was obtained using different techniques. This argument was extended in [15] to the case of smooth metric measure spaces with nonnegative Brakry-Émery Ricci curvature outside of a compact set. A related result bounding the number of ends of Alexandrov spaces with nonnegative sectional curvature outside a compact set was obtained in [8]. More recently, a result bounding the number of ends of RCD$(0,N)$ spaces was obtained in [7].

**Acknowledgements.** The authors wish to thank Fabio Cavalletti, Fernando Galaz-García, Nicola Gigli and Guofang Wei for very useful communications.

## 2. Preliminaries

In this section we provide a brief overview of the definitions and results we will need to prove Theorem A. Throughout the article we consider complete and geodesic metric measure spaces $(X,d,m)$ such that $m$ is finite on bounded sets and $\text{supp}(m) = X$. We begin by recalling the definition of the so-called Wasserstein space.

**Definition 2.1.** Let $\mathcal{P}(X,d)$ be the set of Borel probability measures on $X$ and $\mathcal{P}_2(X,d,m) \subset \mathcal{P}(X,d)$ the space of those probability measures that are absolutely continuous with respect to $m$ and have finite second moment, i.e. for some (and therefore for any) $x_0 \in X$ the following holds:

$$\int d^2(x,x_0) \, dm(x) < \infty.$$  

This set $\mathcal{P}_2(X,d,m)$ is endowed with the 2-Wasserstein metric

$$W_2(\mu_0,\mu_1) = \inf \left( \int_{X \times X} d^2(x_0,x_1) \, d\pi(x_0,x_1) \right)^{1/2}$$

where the infimum is taken over all couplings $\pi \in \mathcal{P}(X \times X)$ from $\mu_0$ to $\mu_1$, i.e. probability measures on $X \times X$ having first and second marginals equal to $\mu_0$ and $\mu_1$ respectively.

**Remark 2.1.** It turns out that $(\mathcal{P}_2(X),W_2)$ is also a complete separable geodesic space. Moreover, in this case, the distance $W_2(\mu_0,\mu_1)$ can be characterized as

$$W_2^2(\mu_0,\mu_1) = \min_{\pi} \int_0^1 \int_{\Gamma_\pi} |\gamma_s|^2 \, dt \, d\pi(\gamma),$$

where the minimum is taken among all $\pi \in \mathcal{P}(C([0,1],X))$ such that $(\epsilon_t)_# \pi = \mu_i$, $i = 0,1$. Here $\epsilon_t$ denotes the usual evaluation map at time $t$. The set of minimizers is denoted by $\text{OptGeo}(\mu_0,\mu_1)$, and minimizers, which are always supported in $\text{Geo}(X)$ (the set of geodesics of $(X,d)$), are called
optimal plans. It is known that \((\mu_t)_{t\in[0,1]}\) is a geodesic connecting \(\mu_0\) to \(\mu_1\) if and only if there exists \(\pi \in \text{OptGeo}(\mu_0, \mu_1)\) such that \(\mu_t = (e_t)_# \pi\) (see [1]).

In order to recall the definition of the CD condition, we now recall the volume distortion coefficients:

\[
\sigma_{K,N}(\theta) = \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} \theta) & \text{if } \kappa > 0, \\
\frac{1}{\sqrt{\kappa}} \sinh(\sqrt{-\kappa} \theta) & \text{if } \kappa < 0,
\end{cases}
\]

\[
\tau_{K,N}^{(t)}(\theta) = \begin{cases} 
+\infty & \text{if } K\theta^2 > N\pi^2, \\
t & \text{if } K\theta^2 = 0 \text{ or } K\theta^2 < 0 \text{ and } N = 0, \\
\frac{\#_{K,N}(\theta)}{\#_{K,N}(0)} & \text{if } K\theta^2 < N\pi^2 \text{ and } K\theta^2 \neq 0,
\end{cases}
\]

\[
\gamma_{K,N}^{(t)}(\theta) = \frac{1}{N^2} \sigma_{K,N}^{(t)}(\theta)^{1-1/N}.
\]

**Definition 2.2.** Given parameters \(K \in \mathbb{R}\) and \(N \geq 1\), we say that \((X, d, m)\) satisfies the CD\((K, N)\) condition if for any \(\mu_0, \mu_1 \in P_2(X, d, m)\) there exists an optimal plan \(\pi \in \text{OptGeo}(\mu_0, \mu_1)\) such that for any \(t \in [0,1]\) and any \(N' \geq N\)

\[
(1) \quad \int \rho_t^{1-1/N'}(x) \, d\mathfrak{m}(x) \geq \int \tau_{K,N}^{(t)}(d(\gamma_0, \gamma_1)) \rho_t^{1-1/N'}(\gamma_0) + \tau_{K,N}^{(t)}(d(\gamma_0, \gamma_1)) \rho_t^{1-1/N'}(\gamma_1) \, d\pi(\gamma)
\]

where \(\rho_t\) is the density of the absolutely continuous part of \((e_t)_# \pi\) with respect to \(m\).

Let us also recall the definition of the reduced curvature-dimension condition CD\(^*\) due to Bacher-Sturm [2].

**Definition 2.3.** Given parameters \(K \in \mathbb{R}\) and \(N \geq 1\), we say that \((X, d, m)\) satisfies the CD\(^*\)(\(K, N\)) if for any \(\mu_0, \mu_1 \in P_2(X, d, m)\) there exists an optimal plan \(\pi \in \text{OptGeo}(\mu_0, \mu_1)\) such that for any \(t \in [0,1]\) and any \(N' \geq N\)

\[
(2) \quad \int \rho_t^{1-1/N'}(x) \, d\mathfrak{m}(x) \geq \int \sigma_{K,N'}^{(1-t)}(d(\gamma_0, \gamma_1)) \rho_t^{1-1/N'}(\gamma_0) + \sigma_{K,N'}^{(t)}(d(\gamma_0, \gamma_1)) \rho_t^{1-1/N'}(\gamma_1) \, d\pi(\gamma)
\]

where \(\rho_t\) is the density of the absolute continuous part of \((e_t)_# \pi\) with respect to \(m\).

**Definition 2.4.** Given parameters \(K \in \mathbb{R}\) and \(N > 1\), we say that \((X, d, m)\) satisfies the CD\(_{\text{loc}}\)(\(K, N\)) condition if each point \(x \in X\) has a neighbourhood \(M(x) \subset \Omega\) such that for each \(\mu_0, \mu_1 \in P_2(X, d, m)\) supported in \(M(x)\) there exists an optimal plan \(\pi \in \text{OptGeo}(\mu_0, \mu_1)\) satisfying (1) for all \(t \in [0,1]\) and \(N' \geq N\).

We will assume that Definition 2.4 holds outside a compact set \(B \subset X\) in the following sense.

**Definition 2.5.** We say that CD\(_{\text{loc}}\)(\(K, N\)) condition holds in an open set \(\Omega \subset X\) if each point \(x \in \Omega\) has a neighbourhood \(M(x) \subset \Omega\) such that for each \(\mu_0, \mu_1 \in P_2(X, d, m)\) supported in \(M(x)\) there exists an optimal plan \(\pi \in \text{OptGeo}(\mu_0, \mu_1)\) satisfying (1) for all \(t \in [0,1]\) and \(N' \geq N\).

From this point on, we will also assume that \((X, d, m)\) is non-branching in the following sense.

**Definition 2.6.** We say that a metric space \((X, d)\) is non-branching if whenever we have a tuple \((z, x_0, x_1, x_2)\) such that \(z\) is a midpoint of \(x_0, x_1\) and of \(x_0, x_2\), this implies that \(x_1 = x_2\).

In [3] it was proved that non-branching CD spaces have unique optimal plans in the following sense.

**Theorem 2.1.** Let \((X, d, m)\) be a complete, separable and non-branching CD\((K, N)\)-space for some \(K \in \mathbb{R}\) and \(N \geq 1\). Then for any \(\mu_0, \mu_1 \in P_2(X, d, m)\) there is a unique optimal plan \(\pi \in \text{OptGeo}(\mu_0, \mu_1)\) and this \(\pi\) is induced by a map, i.e. there exists a \(\mu_0\)-measurable map \(F : X \to \text{Geo}(X)\) such that \(\pi = F_# \mu_0\).
In [2] it was proved that the $CD_{\text{loc}}(K, N)$ condition implies the reduced curvature-dimension condition $CD^*(K, N)$. In a similar fashion, and emulating the arguments in [2], we can prove the following key result that will allow us to generalize Theorem 2.4. Below, we let $\mathcal{P}_\infty(X, d, m)$ denote the space of probability measures which are absolutely continuous with respect to $m$ and have bounded support. For the next lemma, note that we cannot directly apply the local-to-global property since we are using the restricted metric on $X \setminus B$ and this might not be a geodesic space unless $X \setminus B$ is geodesically convex, for example. However, the proof of [2, Theorem 5.1] applies verbatim as we are assuming that all the measures involved are connected by a geodesic in $\mathcal{P}_\infty(X, d, m)$. \footnote{In fact the proof of [2, Claim 5.2] simplifies in our case as by Theorem 2.1 the geodesic joining $\mu_0$ and $\mu_1$ is unique, so there is no need to construct the sequence $\Gamma^{(i)}$.}

**Lemma 2.2.** Assuming the hypotheses from Theorem A, if $\mu_0, \mu_1 \in \mathcal{P}_\infty(X, d, m)$ are supported on $X \setminus B$ and the optimal plan $\pi \in \text{OptGeo}(\mu_0, \mu_1)$ given by Theorem 2.1 is supported in geodesics contained in $X \setminus B$, then $\pi$ satisfies the condition (2) for $K = 0$ and for all $t \in [0, 1]$ and $N' \geq N$.

Now we can follow the arguments in [14] to prove a generalized Bishop-Gromov result for star-shaped sets outside a compact set. To this end, we need the following modified version of the Brunn-Minkowski inequality.

**Theorem 2.3** (Brunn–Minkowski inequality). Let $(X, d, m)$ be a metric measure space and $\Omega \subset X$ an open set such that $CD_{\text{loc}}(K, N)$ condition holds in $\Omega$. Then, for all measurable sets $A_0, A_1 \subset \Omega$ such that $m(A_0)m(A_1) > 0$ and $A_1 \subset \Omega$ for all $t \in [0, 1]$,

\begin{equation}
(3) \quad m(A_t)^{1/N'} \geq \sigma_{K,N'}^{(1-t)}(\theta)m(A_0)^{1/N'} + \sigma_{K,N'}^{(t)}m(A_1)^{1/N'},
\end{equation}

holds for all $t \in [0, 1]$ and all $N' \geq N$, where $A_t$ denotes the set of points which divide geodesics starting in $A_0$ and ending in $A_1$ with ratio $t : (1 - t)$ and where $\theta$ denotes the minimal/maximal length of such geodesics, that is,

$$A_t := \{ y \in X : \exists (x_0, x_1) \in A_0 \times A_1 \text{ s.t. } d(y, x_0) = td(x_0, x_1), \ d(y, x_1) = (1 - t)d(x_0, x_1) \}$$

and

$$\theta := \begin{cases} 
\inf_{x_0 \in A_0, x_1 \in A_1} d(x_0, x_1), & \text{if } K \geq 0, \\
\sup_{x_0 \in A_0, x_1 \in A_1} d(x_0, x_1), & \text{if } K < 0.
\end{cases}$$

In particular, if $K \leq 0$ then

$$m(A_t)^{1/N'} \geq (1 - t)m(A_0)^{1/N'} + tm(A_1)^{1/N'}$$

**Proof.** Assuming that $0 < m(A_0)m(A_1) < \infty$, and thanks to Theorem 2.2, we can apply $CD^*(K, N)$ to $\mu_t := (1/m(A_t))1_{A_t}m$ for $i = 0, 1$ and proceed as in [14, Proposition 2.1], just replacing the coefficients $\tau_{K,N'}^{(i)}(\cdot)$ by $\sigma_{K,N'}^{(i)}(\cdot)$. The general case follows by approximation of $A_t$ by sets of finite volume.

Recall that, given a metric space $(X, d)$ and $x \in X$, it is said that $W_x \subset X$ is star-shaped at $x$ if $x \in W_x$ and for any $y \in W_x$ not in the cut-locus of $x$, the minimal geodesic joining $x$ and $y$ is contained in $W_x$. In that case, we set

$$v(W_x, r) = m(\overline{B}_r(x) \cap W_x) \quad \text{and} \quad s(W_x, r) = \limsup_{\eta \to 0} m((\overline{B}_{r+\eta}(x) \setminus B_r(x)) \cap W_x)$$

**Theorem 2.4** (Bishop-Gromov theorem for star-shaped sets). Let $(X, d, m)$ be a metric measure space and $\Omega \subset X$ an open set such that $CD_{\text{loc}}(K, N)$ condition holds in $\Omega$. Let $x \in \Omega$ and $W_x \subset \Omega$
be a star-shaped set at \( x \) such that for some \( \epsilon_0 > 0 \) every geodesic connecting points in \( B_{\epsilon_0}(x) \) with points in \( W_x \) is contained in \( \Omega \). Then
\[
\frac{v(W_x, r)}{v(W_x, R)} > \left( \frac{\delta \mu_{N}(r)}{\delta \mu_{N}(R)} \right)^N \quad \text{and} \quad \frac{s(W_x, r)}{s(W_x, R)} \geq \frac{\int_{0}^{R} \delta \mu_{N}(t)^N \, dt}{\int_{0}^{R} \delta \mu_{N}(t)^N \, dt}
\]
for all \( 0 < r \leq R \leq \text{rad}_x(W_x) := \sup \{d(x, y) : y \in W_x\} \)

**Proof.** Let \( 0 < r < R \leq \text{rad}_x(W_x) \), \( 0 < \epsilon \leq \epsilon_0 \) and \( \eta > 0 \) and let
\[
A_0 := B_r(x) \cap W_x \quad \text{and} \quad A_1 := (\overline{B}_{(1+\eta)}R(x) \setminus B_R(x)) \cap W_x.
\]
In particular, the \( (r/R) \)-intermediate set \( A_{r/R} \) between \( A_0 \) and \( A_1 \) is contained in \( \Omega \), so we can apply Theorem 2.3 with \( K = 0 \) to get
\[
m(A_{r/R})^{1/N} \geq (1 - r/R)m(A_0)^{1/N} + (r/R)m(A_1)^{1/N}.
\]
We can take \( \epsilon \to 0 \), which yields
\[
m((\overline{B}_{(1+\eta)}R(x) \setminus B_r(x)) \cap W_x) \geq (r/R)^N m((\overline{B}_{(1+\eta)}R(x) \setminus B_R(x)) \cap W_x).
\]
We thus can conclude just as in the proof of [11] Theorem 2.3. \( \square \)

### 3. Proofs

In this section we proceed to prove Theorem A. The proof follows almost verbatim the arguments in [10]. For the convenience of the reader, we elaborate on this argument and stress the needed changes due to the more general hypotheses. We proceed with the following technical Lemma assuming the hypothesis of Theorem A.

**Lemma 3.1.** Assume that \( p \in X \) and \( R > 0 \) are such that \( B_{2R}(p) \subset X \setminus B_{2D}(p_0) \). Then for every \( m > 0 \) there exists \( \delta = \delta(m) \in (0, 1) \) such that whenever a subset \( W \subset B_R(p) \) satisfies
\[
(4) \quad m(W) \geq \frac{1}{m} m(B_R(p))
\]
then there exists \( q \in W \) such that \( d(q, p) \leq \delta R \), and in particular \( B_{(1-\delta)R}(q) \subset B_R(p) \).

**Proof.** Let \( \delta \in (0, 1) \) and \( W \subset B_R(p) \setminus \overline{B}_{\delta R}(p) \) be such that [4] holds. Since \( B_{2R}(p) \cap B_{2D}(p_0) = \emptyset \) then the optimal plan between any two probability measures supported on \( B_{(1+\eta)R}(p) \) for sufficiently small \( \eta > 0 \) is concentrated in geodesics outside \( B_D(p_0) \). Therefore, applying Theorem 2.4 yields
\[
\frac{1}{m} \leq \frac{m(W)}{m(B_R(p))} \leq \frac{m(B_R(p) \setminus \overline{B}_{\delta R}(p))}{m(B_R(p))} \leq \frac{\int_{0}^{R} \delta t^{N-1} \, dt}{\int_{0}^{R} \delta t^{N-1} \, dt} = 1 - \delta^N,
\]
that is, \( \delta \leq (1 - 1/m)^{1/N} \). Thus, if we take \( \delta = (1 - 1/(2m))^{1/N} \) and any \( W \) satisfying [4], then \( W \cap \overline{B}_{\delta R}(p) \neq \emptyset \), i.e there is some \( q \in W \) such that \( d(q, p) \leq \delta R \). In particular, for such \( q \in W \) and any \( x \in B_{(1-\delta)R}(q) \) we have
\[
d(x, p) \leq d(x, q) + d(q, p) < (1 - \delta)R + \delta R = R,
\]
so \( B_{(1-\delta)R}(q) \subset B_R(p) \). \( \square \)

**Proof of Theorem A.** Clearly we can assume that \( K < 0 \). Moreover, by rescaling the metric in \( X \) by \( \sqrt{-K} \), we can further assume that \( (X, d, m) \) satisfies CD\((-1, N)\) condition. In particular, we get that \( B \subset B_D(p_0) \) where \( D = \sqrt{-K}D_0 \).

For \( \mu > 2 \) the result follows from the fact that \( B_{r}(p_0) \subset B_{\mu \mu}(p) \) for any \( p \in B_{r}(p_0) \), so we can set \( C(N, K D_0^2, \mu) = 1 \). Therefore, we will assume from now on that \( 0 < \mu \leq 2 \).

We now divide \( S \) into the union \( S_1 \) and \( S_2 \) where
\[
S_1 = S \cap B_{\mu/2}(p_0), \quad S_2 = S \setminus S_1.
\]
If \( S_1 \neq \emptyset \) then it can be covered by just one \( B_{\mu r}(p) \) with \( p \in S_1 \). In any case, we only need to estimate the covering number of \( S_2 \). We will actually estimate the number of \( (\mu r/4) \)-balls needed to cover \( S_2 \), so for simplicity let us denote \( t = \mu r/4 \).

Now, fix some \( \lambda > 2 \). The case when \( tr \leq \lambda D \) follows exactly as in [10, Page 11] (where instead of \( \lambda \) it suffices to consider 2). Therefore we assume that \( tr > \lambda D \). In particular, for \( q \in S_2 \),

\[
B_{tr}(q) \cap B_{\lambda D}(p_0) = \emptyset.
\]

Write \( \partial B_{\lambda D}(p_0) \) as the union of subsets \( \{ U_1, \ldots, U_m \} \) such that \( d(x, y) < 2D \) for any \( x, y \in U_a \). This can be done as follows. Take a maximal set of points \( \{ q_1, \ldots, q_m \} \subset \partial B_{\lambda D}(p_0) \) such that \( d(q_a, q_b) \geq D, a \neq b \). Then

\[
\partial B_{\lambda D}(p_0) \subset \bigcup_{j=1}^m B_D(p_0),
\]

\[
B_{D/2}(q_a) \cap B_{D/2}(q_b) = \emptyset, \quad a \neq b.
\]

Suppose \( B_{D/2}(q_a) \) has the smallest volume among all \( B_{D/2}(q_i) \). Since \( \bigcup_{i=1}^m B_{D/2}(q_i) \subset B_{(1/2+2\lambda)D}(q_a) \), Bishop-Gromov comparison corresponding to the condition \( \text{CD}(-1, N) \) yields

\[
m < \frac{V^{-1}((1/2 + 2\lambda)D)}{V^{-1}(D/2)}.
\]

Note that the right-hand side of (5) depends only on \( n \) and \( D \). We define

\[
U_a = B_D(q_a) \cap \partial B_{\lambda D}(p_0), \quad a = 1, \ldots, m.
\]

Let \( M_r \) be the subset of \( M \) consisting of all points on any minimal geodesic emanating from \( p_0 \) that is no shorter than \( r \). Note that \( M - B_r(p_0) \subset M_r \), and \( M_r \) is star-shaped at \( p_0 \).

We now divide \( M_{AD} \) into \( m \) cones \( K_a \) by defining \( K_a \) to be the subset consisting of all points on any minimal geodesic emanating from \( p_0 \) that intersects \( U_a \). Observe that, by the triangle inequality, if \( d(x_1, p_0) > \lambda D \), \( x_1 \in K_a \), \( i = 1, 2 \), then any minimal geodesic connecting \( x_1 \) and \( x_2 \) will not pass through \( B_{\lambda D/2}(p_0) \). Indeed, let \( \gamma_i \) be a minimal geodesic from \( p_0 \) to \( x_i \) with \( \gamma_i(\lambda D) \in U_a \), \( i = 1, 2 \). Then the broken geodesic from \( x_1 \) to \( \gamma_1(\lambda D) \) to \( \gamma_2(\lambda D) \) to \( x_2 \) has length no greater than

\[
d(x_1, p_0) + d(x_2, p_0) - 2\lambda D + 2D.
\]

On the other hand, if a minimal geodesic connecting \( x_1 \) and \( x_2 \) intersects \( B_{\lambda D/2}(p_0) \), then it would have a length greater than

\[
d(x_1, p_0) + d(x_2, p_0) - \lambda D,
\]

which is a contradiction.

Now we can estimate the covering number just as in [9, 10]. For the convenience of the reader, we will repeat some of the constructions.

Take a maximal set of points \( \{ p_1, \ldots, p_k \} \) in \( S_2 \) such that \( d(p_i, p_j) > tr, i \neq j \). Then

\[
S_2 \subset \bigcup_j B_{tr}(p_j),
\]

\[
B_{tr/2}(p_i) \cap B_{tr/2}(p_j) = \emptyset, \quad i \neq j.
\]

We then divide the points \( p_j \) into \( m \) families as follows: for each ball \( B_{tr/2}(p_j) \), look at \( m(B_{tr/2}(p_j) \cap K_a) \), \( a = 1, \ldots, m \). Fix an \( a_j \) such that \( m(B_{tr/2}(p_j) \cap K_{a_j}) \) is maximal. Then

\[
m(B_{tr/2}(p_j) \cap K_{a_j}) \geq \frac{1}{m}m(B_{tr/2}(p_j)).
\]

We denote

\[
B_{p_j}^{L,a_j} := B_{tr/2}(p_j) \cap K_{a_j},
\]
Adding up the contributions from the \( p_j \) in the \( a_j \)-th family, call it \( \mathcal{F}_a \). Fix a \( K_a \). Suppose \( B^L_{p_j} \) has the smallest volume among all \( B^L_{p_j} \) in this cone. By Lemma 3.1, we can find a \( q \in B^L_{p_j} \) such that

\[
B(1-\delta tr/2)(q) \subset B_{tr/2}(p).
\]

Let \( W_q \) be the star-shaped set such that \( y \in W_q \) if and only if there is a point \( x \) belonging to either \( B(1-\delta tr/2)(q) \) or \( B^L_{p_j} \) for some \( p_j \in \mathcal{F}_a \) and there is a minimal geodesic \( \gamma \) connecting \( q \) and \( x \) which passes \( y \).

Observe that for \( \epsilon_0 = \min\{1 - \delta tr, (\lambda - 2)D/2\} \geq 0 \), \( z \in B_{\epsilon_0}(q) \), \( z \in B_{\epsilon_0}(y) \), any geodesic joining \( z \) with \( y \) is outside \( B_{\lambda D/2}(p_0) \). Indeed, if \( x \in B(1-\delta tr/2) \cup \bigcup B_{p_j} \) is such that \( y \) is in a geodesic \( \gamma \) joining \( q \) with \( x \) and \( \gamma_1 \) is a geodesic joining \( z \) with \( y \) and passing through \( B_{\lambda D/2}(p_0) \) then the broken geodesic from \( q \) to \( z \) to \( y \) to \( x \) will have length greater than

\[
d(q, p_0) + d(x, p_0) - \lambda D.
\]

However, it also has length no greater than

\[
e_0 + \epsilon_0 + d(q, y) + d(y, x) = 2\epsilon_0 + L(\gamma) \leq 2\epsilon_0 + d(q, p_0) + d(x, p_0) - 2\lambda D + 2D
\]

which is a contradiction.

By a simple triangle inequality we get that \( d(q, y) \leq (2 + t)r \) for all \( y \in W_q \). Therefore, applying Theorem 2.2 with \( K = 0 \), we get

\[
\frac{m(W_q)}{m(B(1-\delta tr/2)(q))} \leq 2^N(2 + t)^N(1 - \delta)^{-N}t^{-N}
\]

However,

\[
\frac{m(W_q)}{m(B(1-\delta tr/2)(q))} \geq \sum_{p_j \in \mathcal{F}_a} \frac{m(B^L_{p_j})}{m(B_{tr/2}(p))} = \#\mathcal{F}_a \frac{m}{m}
\]

thus we get

\[
\#\mathcal{F}_a \leq 2^N(2 + t)^N(1 - \delta)^{-N}t^{-N}m.
\]

Adding up the contributions from the \( m \) families \( \mathcal{F}_a \), we get that

\[
k \leq 2^N(2 + t)^N(1 - \delta)^{-N}t^{-N}m^2.
\]

The right hand side of (7) depends on \( N, \mu \) and \( m \). However \( m \) is bounded above by the right hand side of (5), which is a function depending on \( KD_0^2 \) and \( \lambda \), increasing with respect to \( \lambda \). Taking \( \lambda \lesssim 2 \), we get the required constant \( C = C(N, KD_0^2, \mu) > 0 \). \( \square \)

**Definition 3.1.** Let \( (X, d) \) be a metric space and \( k \in \mathbb{N} \). We say that \( X \) has \( k \) ends if both the following are true:

1. for any \( K \) compact, \( X \setminus K \) has at most \( k \) unbounded connected components,
2. there exists \( K' \) compact such that \( X \setminus K' \) has exactly \( k \) unbounded connected components.

**Proof of Corollary 3.2** If the result does not hold true, we take \( r \) large enough so that \( X \setminus \overline{B}_r(p_0) \) has \( n > C(N, KD_0^2, 1/2) \) unbounded connected components. It is clear that each such unbounded connected component \( E \) requires at least one ball of radius \( r \) to cover \( E \cap \partial B_{2r}(p_0) \). This contradicts Theorem 4. \( \square \)

**References**

[1] L. Ambrosio and N. Gigli. A user’s guide to optimal transport. In *Modelling and Optimisation of Flows on Networks*, volume 2062 of Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, 2013.

[2] K. Bacher and K. T. Sturm. Localization and tensorization properties of the curvature-dimension condition for metric measure spaces. *Journal of Functional Analysis*, 259:28–56, 2010.

[3] M. Cai. Ends of riemannian manifolds with nonnegative ricci curvature outside a compact set. *Bull. Amer. Math. Soc. (N.S.)*, 24(2):371–377, 1991.

[4] F. Cavalletti and A. Mondino. Optimal transport in lorentzian synthetic spaces, synthetic timelike ricci curvature lower bounds and applications, 2020.
[5] Q. Deng. Hölder continuity of tangent cones in $\text{RCD}(K,N)$ spaces and applications to non-branching. ArXiv:2009.07956.

[6] N. Gigli. Optimal maps in non branching spaces with ricci curvature bounded from below. Geometric and Functional Analysis, 22:990–999, 2012.

[7] N. Gigli and I. Y. Violo. Monotonicity formulas for harmonic functions in $\text{RCD}(0,N)$ spaces. ArXiv:2101.03331.

[8] L.-K. Koh. Alexandrov spaces with nonnegative curvature outside a compact set. Manuscr. Math., 94:401–407, 1997.

[9] Z.-D. Liu. Ball covering on manifolds with nonnegative Ricci curvature near infinity. Proc. Amer. Math. Soc., 115(1):211–219, 1992.

[10] Z.-D. Liu. Ball covering property and nonnegative Ricci curvature outside a compact set. In Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), volume 54 of Proc. Sympos. Pure Math., pages 459–464. Amer. Math. Soc., Providence, RI, 1993.

[11] J. Lott and C. Villani. Ricci curvature for metric measure spaces via optimal transport. Ann. of Math., 169:903–991, 2009.

[12] A. Petrunin. Alexandrov meets Lott-Villani-Sturm. Münster J. Math., 4:53–64, 2011.

[13] K.-T. Sturm. On the geometry of metric measure spaces. I. Acta Math., 196:65–131, 2006.

[14] K.-T. Sturm. On the geometry of metric measure spaces. II. Acta Math., 196:133–177, 2006.

[15] J.-Y. Wu. Counting ends on complete smooth metric measure spaces. Proc. Amer. Math. Soc., 144(5):2231–2239, 2016.

[16] H.-C. Zhang and X.-P. Zhu. Ricci curvature on Alexandrov spaces and rigidity theorems. Comm. Anal. Geom., 18(3):503–553, 2010.

(M. Che) Durham University, Durham, United Kingdom
Email address: mauricio.a.che-moguel@durham.ac.uk

(J. Núñez-Zimbrón) Centro de Investigación en Matemáticas, Guanajuato, Mexico
Email address: jesus.nunez@cimat.mx