New Developments in Interval Arithmetic and Their Implications for Floating-Point Standardization*

M.H. van Emden

Department of Computer Science,
University of Victoria, Victoria, Canada
vanemden@cs.uvic.ca
http://www.cs.uvic.ca/~vanemden/

Abstract. We consider the prospect of a processor that can perform interval arithmetic at the same speed as conventional floating-point arithmetic. This makes it possible for all arithmetic to be performed with the superior security of interval methods without any penalty in speed. In such a situation the IEEE floating-point standard needs to be compared with a version of floating-point arithmetic that is ideal for the purpose of interval arithmetic. Such a comparison requires a succinct and complete exposition of interval arithmetic according to its recent developments. We present such an exposition in this paper. We conclude that the directed roundings toward the infinities and the definition of division by the signed zeros are valuable features of the standard. Because the operations of interval arithmetic are always defined, exceptions do not arise. As a result neither Nans nor exceptions are needed. Of the status flags, only the inexact flag may be useful. Denormalized numbers seem to have no use for interval arithmetic; in the use of interval constraints, they are a handicap.

Keywords: interval arithmetic, IEEE floating-point standard, extended interval arithmetic, exceptions

1 Introduction

Continuing advances in process technology have caused a tremendous increase in the number of transistors available to the designer of a processor chip. As a result, multiple parallel floating-point units become feasible. The time will soon come when interval arithmetic can be done as fast as conventional arithmetic.

However, to properly utilize the newly available number of transistors, chip designers need to spend ever more time iterating through cycles of synthesis, place-and-route, and physical verification than current design methodology allows. This design bottleneck makes it desirable to simplify floating-point units.

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Recent developments in the theory of interval arithmetic suggest possibilities for simplification. As far as interval arithmetic is concerned, certain parts the 1985 standard are essential, whereas other parts are superfluous or even a liability.

In this paper we present a consolidated, self-contained account of the new developments in interval arithmetic not available elsewhere. In the conclusions we give a sketch of what would be an ideal standard from the point of view of interval arithmetic when arranged in this way.

2 Why interval arithmetic?

Conventional, non-interval, numerical analysis is marvelously cheap and it works most of the time. This was exactly what was needed in the 1950s when computers needed a demonstration of feasibility. A lot has changed since that time. Numerical computation no longer needs to be cheap. It has become more important that it always works. As a result interval arithmetic is becoming an increasingly compelling alternative. For example, civil, mechanical, and chemical engineers are liable for damage due to unsound design. So far, they have been able to get away with the use conventional numerical analysis, appealing to what appears to be best practice. As interval methods mature, it is becoming harder to ignore them when defining “best practice”.

Recent developments, which we call “modern interval arithmetic” provide a practical and mathematically compelling basis. In conjunction with this it has become clear that some aspects of IEEE standard 754 are not needed or are detrimental, whereas other aspects are marvelously suited to interval arithmetic. If these latter are preserved in future development of the standard, then interval arithmetic can help bridge the design gap and lead to the situation where all arithmetic can be faster due to interval methods.

3 A theory of approximation

Conventional numerical analysis approximates each real by a single floating-point number. It approximates each of the elementary operations on reals by the floating-point operation that has the same name, but not the same effect. Let us call this approach “point approximation”. It has been amply documented that this approach, though often satisfactory, can lead to catastrophic errors. This can happen because it is not known to what degree a floating-point variable approximates its real-valued counterpart.

Interval arithmetic is based on a theory of approximation, set approximation, that ensures that for every real-valued variable $x$ in the mathematical model, there is a machine-representable set $X$ of reals that contains $x$. Such arithmetic is exact in the sense that $x \in X$ is and remains a true statement in the sense of mathematics. It is of course not exact in the sense that $X$ typically contains many reals.

Conversely, in case of numerical difficulties, it will turn out that continued iteration does not reduce the size of $X$, in which case we have a notification
that there is a problem with the algorithm or with the model. Because of this property, we call this manifest approximation: there is always a known lower bound to the quality of the approximation.

In this way the operations can be interpreted as inference rules of logic; for example, \( x \in X \) and \( y \in Y \) imply that \( x + y \in Z \), where \( Z \) is computed from \( X \) and \( Y \).

### 4 Approximation structures

We address the situation where we need to solve a mathematical model in which a variable takes on values that are not representable in a computer, but where it is possible to so represent sets of values. We then approximate a variable by a representable set that contains all the values that are possible according to the model. Models with real-valued variables are but one example of such a situation.

As the theory of set approximation applies to sets in general, we first present it this way.

**Definition 1.** Let \( T \) be the type of a variable \( x \), that is, the set of the possible values for \( x \). A finite set \( A \) of subsets of \( T \) that contains \( T \) and that is closed under intersection is called an approximation structure in \( T \).

In a typical practical application of this theory, \( A \) is a set of computer-representable subsets of \( T \).

**Theorem 1.** If \( A \) is an approximation structure for \( T \), then for every \( S \subset T \) there exists a unique least (in the sense of the set-inclusion partial order) element \( S' \) of \( A \) such that \( S \subset S' \).

**Definition 2.** For every \( S \subset T \), \( \phi(S) \) is the unique least element of \( A \) that exists according to theorem [4].

We regard \( \phi(S) \) as the approximation of \( S \). As \( S \) can be a singleton set, this theory provides approximations both of elements and subsets of \( T \).

### 5 An approximation structure for the reals

We seek a set of subsets of the reals that can serve as approximation structure. A first step, not yet computer-representable, is that of the closed, connected sets of reals.

**Theorem 2.** The closed, connected subsets of \( R \) are an approximation structure for \( R \).

**Proof** According to a well-known result in topology, all closed connected subsets of \( R \) have one of the following four forms: \( \{ x \in \mathbb{R} \mid a \leq x \leq b \} \), \( \{ x \in \mathbb{R} \mid x \leq b \} \), \( \{ x \in \mathbb{R} \mid a \leq x \} \), and \( \mathbb{R} \) where \( a \) and \( b \) are reals. Here we do not exclude \( a > b \), because the empty set is also included among the
closed connected sets. Clearly the conditions for an approximation structure are satisfied: $\mathcal{R}$ is included and the intersection of any two sets of this form is again closed and connected. This completes the proof.

The significance of the closed connected sets of reals as approximation structure is that they can be represented as a pair of extended reals.

**Definition 3.** The extended reals are the set obtained by adding to the reals the two infinities. As with the reals, the extended reals are totally ordered. When two extended reals are finite, then they are ordered within the extended reals as they are in the reals. Furthermore, $-\infty$ is less than any real and $+\infty$ is greater than any real.

Theorem 2 together with definition 3 suggest the following notation for the four forms of the closed connected sets of reals. In this notation we do not include the empty interval. The reason is that if it is found in an interval computation that an interval is empty, any further operations involving the corresponding variable yield the same result, so that the computation can be halted.

**Definition 4.** Let $a$ and $b$ be reals such that $a \leq b$.

\[
\langle a, b \rangle \overset{\text{def}}{=} \{ x \in \mathbb{R} \mid a \leq x \leq b \}
\]

\[
(-\infty, b) \overset{\text{def}}{=} \{ x \in \mathbb{R} \mid x \leq b \}
\]

\[
(a, +\infty) \overset{\text{def}}{=} \{ x \in \mathbb{R} \mid a \leq x \}
\]

\[
(-\infty, +\infty) \overset{\text{def}}{=} \mathbb{R}
\]

Note that each of these pairs denote sets of reals, even though in their notation the infinities are used. These are not reals.

### 6 Floating-point intervals: a finite approximation structure for the reals

Let $F$ be a finite set of reals.

**Theorem 3.** The sets of the form $\emptyset$, $(-\infty, b)$, $\langle a, b \rangle$, $\langle a, +\infty \rangle$, and $(-\infty, +\infty)$ are an approximation structure when $a$ and $b$ are restricted to elements of $F$ such that $a \leq b$.

**Definition 5.** The real intervals are sets of the form described in theorem 4. The floating-point intervals are sets of the form described in theorem 4, where $a$ and $b$ are finite IEEE754 floating point numbers (according to a choice of format: single-length, double-length, extended) such that $a \neq -0$, $b \neq +0$, and $a \leq b$.

From definitions 4 and 5 and theorem 4 we conclude:

- The restriction on the sign of zero bounds in definition 4 is there to make the notation unambiguous. We will see that disambiguating the notation in this way has an advantage for interval division.
– \{0\} is written as \((+0, -0)\).
– When \((a, b)\) is a floating-point interval, then \(a \neq +\infty\) and \(b \neq -\infty\).

Let us take care to distinguish “real intervals” from “floating-point intervals”. Both are sets of reals. The latter are a subset of the former.

From now on we assume the floating-point intervals as approximation structure when we rely on the fact that for any set \(S\) of reals there is a unique least floating-point interval \(\phi(S)\) containing it.

**Definition 6.** For any real \(x\), \(x^- (x^+)\) is the left (right) bound of \(\phi(\{x\})\).

This operation is implemented by performing a floating-point operation that yields \(x\) in rounding mode toward \(-\infty (+\infty)\).

### 7 Interval Arithmetic

Much of the standard is concerned with defining, signaling, and trapping exceptions caused by overflow, underflow, and undefined operations. What distinguishes modern interval arithmetic from the old is that *no exceptions occur*. As we will see, no operation can result in Nan. Every operation is defined on all operands. Moreover, it is defined in such a way that the floating-point endpoints bound the set of the real numbers that are the possible values of the associated variable in the mathematical model.

This property is based on the use of *set extensions* of the arithmetical operations. It is helped by the use of *relational definition* rather than functional ones of these operations. We discuss these in turn.

*Set extensions of functions* Whenever a function \(f\) is defined on a set \(S\) and has values in a set \(T\), there exists the canonical set extension \(\hat{f}\), which is a function defined on the subsets of \(S\) and has as values subsets of \(T\) according to \(\hat{f}(X) = \{f(x) \mid x \in X\}\) for any \(X \subset S\). This definition is of interest because it also carries over to partial functions and to multivalued functions.

Though \(X\) may be an approximation of \(x\), \(\hat{f}(X)\) may not be an element of an approximation structure of \(T\), so is not necessarily an approximation of \(f(x)\). But \(\phi(\hat{f}(X))\) does approximate \(f(x)\). Thus \(\phi\) induces a transformation among functions. It changes \(f\) to the function that maps \(x\) to \(\phi(\hat{f}(\{x\}))\).

The *inverse canonical set extension* of \(f\) is defined as \(f^{-1}(Y) = \{x \mid f(x) \in Y\}\). This definition is of interest because such an inverse is defined even when \(f\) itself has no inverse.

By using the canonical set extensions of a function, one ensures that undefined cases never arise. By considering instead of the arithmetical operations on the reals their canonical set extensions to suitably selected sets of reals (namely, floating-point intervals), undefined cases are eliminated.

An example of a set extension for arithmetical operations is \(X + Y = \{x + y \mid x \in X \land y \in Y\}\). Though \(X\) and \(Y\) may be floating-point intervals, that is typically not the case for \(\{x + y \mid x \in X \land y \in Y\}\). So to ensure that addition is
closed in the set of floating-point intervals, we need to apply $\phi$, as shown below in the formulas for interval operations that go back to R.E. Moore [7].

$$X + Y = \phi(\{x + y \mid x \in X \land y \in Y\})$$
$$X - Y = \phi(\{x - y \mid x \in X \land y \in Y\})$$
$$X \ast Y = \phi(\{x \ast y \mid x \in X \land y \in Y\})$$
$$X/Y = \phi(\{x/y \mid x \in X \land y \in Y\})$$

Regarded as a set extension, the above definition of $X/Y$ is correct and unambiguous: set extensions are defined just as well for partial functions, functions that are not everywhere defined. Yet many authors have subjected it to the condition $0 \notin Y$, making it useless in practice. Others have taken a less restrictive stance by changing the definition to:

$$X/Y = \phi(\{x/y \mid x \in X \land y \in Y \land y \neq 0\}).$$

Relational definitions Ratz [9] has avoided such difficulties by using a relational form of the above definitions. Although not necessary, this relational form also makes it possible to define both addition and subtraction with the same ternary relation $x + y = z$. This leads to an attractive uniformity in the definition of the interval arithmetic operations.

**Definition 7.** Let $X$ and $Y$ be non-empty floating-point intervals. Then interval addition, subtraction, multiplication, and division are defined as follows.

$$X + Y \overset{\text{def}}{=} \phi(\{z \mid \exists x \in X \land \exists y \in Y. x + y = z\})$$
$$X - Y \overset{\text{def}}{=} \phi(\{z \mid \exists x \in X \land \exists y \in Y. z + y = x\})$$
$$X \ast Y \overset{\text{def}}{=} \phi(\{z \mid \exists x \in X \land \exists y \in Y. x \ast y = z\})$$
$$X \odot Y \overset{\text{def}}{=} \phi(\{z \mid \exists x \in X \land \exists y \in Y. z \ast y = x\})$$

We use the symbol $\odot$ in $X \odot Y$ here for interval division rather than the $X/Y$ defined earlier. There is only a difference between the two definitions when $(0,0)$ occurs as an operand. For details, see [8]. The difference is immaterial, as intuition fails in these cases, anyway.

The operations thus defined form an interval arithmetic that is sound in the sense that the resulting sets contain all the real values they should contain according the set extension definition. They are closed in the sense that they are defined for all interval arguments and yield only interval results. Such an interval arithmetic never yields an exception.

It remains to show that these definitions can be efficiently computed by IEEE standard floating-point arithmetic while avoiding the undefined floating-point operations $\infty - \infty$, $\pm \infty / \pm \infty$, $0 \ast \pm \infty$, and $0/0$. This we do in the next sections.
7.1 The algorithm for interval addition and subtraction

**Theorem 4.** If $X = \langle a, b \rangle$ and $Y = \langle c, d \rangle$ are non-empty floating-point intervals, then $X + Y$ and $X - Y$ according to definition 7 are equal to $\langle (a + c)^-, (b + d)^+ \rangle$ and $\langle (a - d)^-, (b - c)^+ \rangle$, respectively.

See [4]. The interesting part of the proof takes into account that adding $a$ and $c$ is undefined if they are infinities with opposite signs. As, according to definition 4, $a$ and $c$ are not $+\infty$, this cannot happen. Similar reasoning shows that $b + d$ is always defined and that the formula for subtraction cannot give an undefined result. Thus, in interval addition and subtraction we achieve the ideal: *Never a Nan*, and this without the need to test.

7.2 The algorithm for interval multiplication

If $\langle a, b \rangle$ and $\langle c, d \rangle$ are bounded, real intervals, then

$$\langle a, b \rangle \ast \langle c, d \rangle = \langle \min(S), \max(S) \rangle,$$

where $S = \{a \ast c, a \ast d, b \ast c, b \ast d\}$.

This formula holds for real rather than floating-point intervals. It is several steps away from interval arithmetic. When we allow the bounds to be any floating-point number, we introduce the possibility that they are infinite. In that case we need to be assured that all four products in $S$ are defined. Moreover, we want, as much as possible, to perform only two multiplications, one for each bound. The above formula always requires four.

To attain these goals, we classify the intervals $\langle a, b \rangle$ and $\langle c, d \rangle$ according to the signs of their elements, as shown in the table in Figure 1. This classification creates many cases in which intervals can be multiplied with only one multiplication for each bound.

| Class of $\langle u, v \rangle$ | at least one negative | at least one positive | Signs of endpoints |
|--------------------------------|-----------------------|----------------------|-------------------|
| $M$                            | yes                   | yes                  | $u < 0 \land v > 0$ |
| $Z$                            | no                    | no                   | $u = 0 \land v = 0$ |
| $P$                            | no                    | yes                  | $u \geq 0 \land v > 0$ |
| $P_0$                          | no                    | yes                  | $u = 0 \land v > 0$ |
| $P_1$                          | no                    | yes                  | $u > 0 \land v > 0$ |
| $N$                            | yes                   | no                   | $u < 0 \land v \leq 0$ |
| $N_0$                          | yes                   | no                   | $u < 0 \land v = 0$ |
| $N_1$                          | yes                   | no                   | $u < 0 \land v < 0$ |

**Fig. 1.** Classification of nonempty intervals according to whether they contain at least one real of the sign indicated at the top of the second and third columns. Classes $P$ and $N$ are further decomposed according to whether they have a zero bound. As only non-empty intervals are classified, we have $u \leq v$. 
The classification yields four cases (for multiplication the subdivision of $P$ and $N$ do not matter) for each of the operands, giving at first sight 16 cases. However, when at least one of the operands classifies as $Z$, several cases collapse. As a result, we are left with 11 cases.

**Theorem 5.** If $⟨a, b⟩$ and $⟨c, d⟩$ are real intervals, then $⟨a, b⟩ ∗ ⟨c, d⟩$ is a real interval whose endpoints are given by the expressions, to be evaluated as extended reals, in Figure 2.

| Class of $⟨a, b⟩$ | Class of $⟨c, d⟩$ | Left Endpoint of $⟨a, b⟩ ∗ ⟨c, d⟩$ | Right Endpoint of $⟨a, b⟩ ∗ ⟨c, d⟩$ | Symmetry |
|-------------------|-------------------|-------------------------------|-------------------------------|----------|
| $P$               | $P$               | $a ∗ c$                        | $b ∗ d$                        | proved directly |
| $P$               | $M$               | $b ∗ c$                        | $b ∗ d$                        | proved directly |
| $P$               | $N$               | $b ∗ c$                        | $a ∗ d$                        | $x ∗ y = -(x ∗ -y)$ |
| $M$               | $P$               | $a ∗ d$                        | $b ∗ d$                        | $x ∗ y = y ∗ x$ |
| $M$               | $M$               | $\min(a ∗ d, b ∗ c)$           | $\max(a ∗ c, b ∗ d)$           | proved directly |
| $M$               | $N$               | $b ∗ c$                        | $a ∗ c$                        | $x ∗ y = -(x ∗ -y)$ |
| $N$               | $P$               | $a ∗ d$                        | $b ∗ c$                        | $x ∗ y = -(x ∗ -y)$ |
| $N$               | $M$               | $a ∗ d$                        | $a ∗ c$                        | $x ∗ y = -(x ∗ -y)$ |
| $N$               | $N$               | $b ∗ d$                        | $a ∗ c$                        | $x ∗ y = -(x ∗ -y)$ |
| $Z$               | $P, M, N, Z$      | $0$                            | $0$                            | proved directly |
| $P, M, N$         | $Z$               | $0$                            | $0$                            | proved directly |

**Fig. 2.** Case analysis for multiplication of real intervals, $⟨a, b⟩ ∗ ⟨c, d⟩$. Results for floating-point intervals are obtained by performing the lower-bound (upper-bound) computations rounded toward $-∞$ ($+∞$).

In [4] the cases indicated as such in the table in Figure 2 are proved directly. The other cases can be proved by symmetry from the case proved already. The symmetries applied are based on the identities $x ∗ y = -(x ∗ -y)$ or similar ones shown in the last column in the table.

The proofs first show the correctness of the scalar products for bounded real intervals. To allow for floating-point intervals, which can be unbounded, we have to consider whether the products are defined. Let us consider as example the top line according to which $⟨a, b⟩ ∗ ⟨c, d⟩ = ⟨a ∗ c, b ∗ d⟩$. The undefined cases occur when one operand is 0 and the other $\infty$. It is possible for $a$ or $c$ to equal 0, but neither can be infinite: because of the classification $P$, they cannot be $-\infty$; because of their being lower bounds, they cannot be $+\infty$.

Let us now consider $b ∗ d$. It is possible for $b$ or $d$ to equal $+\infty$, but neither can be 0 because of the classification $P$. One may verify that in every case of the table in Figure 2 undefined values are avoided by a combination of definitions [4] and [5] and the classification of the case concerned.

We need tests to identify the right case in the table anyway to minimize the number of multiplications. We obtain as a bonus the saving of tests to avoid
undefined values. Thus, in interval multiplication we achieve the ideal: Never a Nan, and this without the need to test.

### 7.3 Division

For interval multiplication the classification of the interval operands in the classes $P$, $M$, $N$, and $Z$ is sufficient. For interval division it turns out that the further subdivision of $P$ into $P_0$ and $P_1$ and of $N$ into $N_0$ and $N_1$ (see the table in figure [1]) is relevant for the dividend.

**Theorem 6.** If $\langle a, b \rangle$ and $\langle c, d \rangle$ are real intervals, then $\langle a, b \rangle \odot \langle c, d \rangle$ is the least floating-point interval containing the real interval whose endpoints are given as the “general formula” column in Figure 3 unless the specified condition in the next column holds, in which case the result is given by the exception case in column 5.

| Class of $\langle a, b \rangle$ | Class of $\langle c, d \rangle$ | $\langle a, b \rangle \odot \langle c, d \rangle$ general formula | unless | $\langle a, b \rangle / \langle c, d \rangle$ exception case |
|-----------------------------|-----------------------------|---------------------------------|--------|--------------------------|
| $P_1$           | $P$            | $\langle a/d, b/c \rangle \setminus \{0\}$ | $c = 0$ | $\langle a/d, \infty \rangle \setminus \{0\}$ | $D$ |
| $P_0$           | $P$            | $\langle 0, b/c \rangle$ | $c = 0$ | $\langle -\infty, \infty \rangle$ | $D$ |
| $M$             | $P$            | $\langle a/c, b/c \rangle$ | $c = 0$ | $\langle -\infty, \infty \rangle$ | $D$ |
| $N_0$           | $P$            | $\langle a/c, 0 \rangle$ | $c = 0$ | $\langle -\infty, \infty \rangle$ | $S_2$ |
| $N_1$           | $P$            | $\langle a/c, b/d \rangle \setminus \{0\}$ | $c = 0$ | $\langle -\infty, b/d \rangle \setminus \{0\}$ | $S_2$ |
| $P_0$           | $M$            | $\langle -\infty, a/c \rangle \cup \langle a/d, \infty \rangle \setminus \{0\}$ | | | $D$ |
| $P_0$           | $M$            | $\langle -\infty, +\infty \rangle$ | | | $D$ |
| $M$             | $M$            | $\langle -\infty, +\infty \rangle$ | | | $D$ |
| $N_0$           | $M$            | $\langle -\infty, +\infty \rangle$ | | | $S_2$ |
| $N_1$           | $M$            | $\langle -\infty, b/d \rangle \cup \langle b/c, \infty \rangle \setminus \{0\}$ | | | $S_2$ |
| $P_0$           | $N$            | $\langle b/d, a/c \rangle \setminus \{0\}$ | $d = 0$ | $\langle -\infty, a/c \rangle \setminus \{0\}$ | $S_1$ |
| $P_0$           | $N$            | $\langle b/d, 0 \rangle$ | $d = 0$ | $\langle -\infty, \infty \rangle$ | $S_1$ |
| $M$             | $N$            | $\langle b/d, a/d \rangle$ | $d = 0$ | $\langle -\infty, \infty \rangle$ | $S_1$ |
| $N_0$           | $N$            | $\langle 0, a/d \rangle$ | $d = 0$ | $\langle -\infty, \infty \rangle$ | $S_2$ |
| $N_1$           | $N$            | $\langle b/c, a/d \rangle \setminus \{0\}$ | $d = 0$ | $\langle b/c, \infty \rangle \setminus \{0\}$ | $S_2$ |
| $Z$             | $P_1, N_1$     | $\langle 0, 0 \rangle$ | | | |
| $Z$             | $P_0, M, N_0, Z$ | $\langle -\infty, +\infty \rangle$ | | | |
| $P_1, N_1$     | $Z$            | $\langle 0 \rangle$ | | | |
| $P_0, M, N_0, Z$ | $Z$            | $\langle -\infty, +\infty \rangle$ | | | |

**Fig. 3.** Case analysis for relational division of real intervals, $\langle a, b \rangle / \langle c, d \rangle$ when $a \leq b$, $c \leq d$. The last column refers to how the formula has been proved (“$D$” for a direct proof, “$S_1$” and “$S_2$” refer to a symmetry used to reduce it to an earlier case.) The “class” labels, $N, N_1, N_0, M, P_0, P_1, P$ are as in Figure 1.

In [4] the cases indicated as such in the table in Figure 3 are proved directly. The other cases can be proved by symmetry from the case proved already. The
symmetries used are based on the identities \(x/y = -(x/-y)\) (indicated as \(S_1\)) and \(x/y = -(-x/y)\) (indicated by \(S_2\)).

The proofs first show the correctness of the scalar products for bounded real intervals. To allow for floating-point intervals, which can be unbounded, we have to consider whether the products are defined. In the column labelled “unless” we find the values for which an undefined value occurs. In the “exception case” column we find the correct value for the exception case. In every case, evaluating the formula in the third column in IEEE standard floating-point arithmetic in the exception case is defined and gives the infinity of the right sign, as shown in column 5. This property depends on a zero lower bound being +0 and a zero upper bound being –0, as required by definition \(\text{Prom}\).

Let us now consider potentially undefined cases. In case of division these are \(\infty/\infty\) and \(0/0\). Consider for example the top line according to which \(\langle a, b \rangle \odot \langle c, d \rangle = \langle a/d, b/c \rangle \setminus \{0\}\). Because of the classification \(P_1\), \(a\) can be neither infinite nor zero. This ensures that \(a/d\) is defined. Because of the \(P_1\) classification, \(b\) cannot be zero. It is possible for \(b\) to be infinite, but not for \(c\) because of the \(P\) classification. This ensures that \(b/c\) is defined.

One may verify that in every case of the table in Figure 3 undefined values are avoided by a combination of definition \(\text{Prom}\) and the classification of the case concerned. Thus, in relational interval division we achieve the ideal: Never a \(\text{Nan}\), and this without the need to test.

8 Related work

For most of the time since the beginning of interval arithmetic, two systems have coexisted. One was the official one, where intervals were bounded, and division by an interval containing zero was undefined. Recognizing the unpracticality of this approach, there was also a definition of “extended” interval arithmetic \(\mathcal{E}\) where these limitations were lifted. Representative of this state of affairs are the monographs by Hansen \(\mathcal{H}\) and Kearfott \(\mathcal{K}\). However, here the specification of interval division is quite far from an efficient implementation that takes advantage of the IEEE floating-point standard. The specification is indirect via multiplication by the interval inverse. There is no consideration of the possibility of undefined operations: presumably one is to perform a test before each operation.

Steps beyond this were taken by Older \(\mathcal{O}\) in connection with the development of BNR Prolog. A different approach has been taken by Walster \(\mathcal{W}\), who pioneered the idea that intervals are sets of values rather than abstract elements of an interval algebra. In Walster shares our objective to obtain a closed system of arithmetic without exceptions. He attains this objective in a different way: by including the infinities among the possible values of the variables. In our approach, the variables can only take reals as values; the infinities are only used for the representation of unbounded sets of reals. In this way, the conventional framework of calculus, where variables are restricted to the reals, needs no modification.
9 Conclusions

We have presented the result of some recent developments in interval arithmetic that lead to a system with the following properties.

- **Correctness** The interval operations are such that their result includes all real numbers that are possible as values of the variables according to the mathematical model.

- **Freedom of exceptions** No floating-point operation needs raise an exception. All divisions by zero are defined and give the correct result: an infinity of the correct sign. This is achieved by a zero lower (upper) bound being +0 (−0). Mathematically speaking, the system is a closed interval algebra. We do not emphasize the algebra aspect, because it is not important whether it has any interesting properties. Other approaches have limited the applicability of interval arithmetic in their pursuit of a presentable algebra.

- **Efficiency** The system is efficient in that tests are only needed to determine the right case in the tables in Figures 2 and 3. Tests are not necessary to avoid exceptions.

These properties lead to several observations about the floating-point standard from the point of view of interval arithmetic:

- **Exceptions** Freedom from exceptions has interesting implications for the standard. A considerable part of the definition effort, and presumably also of the implementation effort, is concerned with defining, signaling, and trapping exceptions caused by overflow, underflow and undefined operations. A processor where the floating-point arithmetic is interval arithmetic can omit this as unnecessary ballast.

  Let us review the five exceptions. *Invalid Operation* is prevented by the design of the algorithms. *Division by Zero* does occur in our interval arithmetic and is designed to yield the correct result. So it should not be an exception. *Overflow* occurs in the sense that a real \(x\) can result in real arithmetic such that \(\phi(x)\) is the interval between the greatest finite floating-point number and +\(\infty\). This result is mathematically correct and therefore the desired one. There is no reason to terminate computation: it should not be an exception. *Underflow* means that a lower bound zero is substituted for a nonzero bound with very small absolute value. This is correct and no reason to terminate computation. *Inexact* result: this might be of some use, but is certainly not essential for interval arithmetic.

- **Signed zeros** Often signed zeros are regarded as an unavoidable, but regrettable artifact of the sign-magnitude format of floating-point numbers. It is fortunate that the drafters of the standard have nonetheless taken them seriously and defined sensible conventions for operations involving zeros. Especially having the right sign of a zero bound turns out to be useful in interval division.
Denormalized numbers of view of interval arithmetic, denormalized numbers seem to be neither useful nor harmful. It is different from the point of view of interval constraints. This a method for using interval arithmetic to solve systems of constraints with real-valued variables. Interval arithmetic is used for the basic operations in constraint propagation. This is an iteration that can be slowed down by denormalized numbers when the limit is zero, even when operations on denormalized numbers are performed at normal speed. Thus the presence of denormalized numbers only plays a role as a performance bug that occurs gratuitously, and fortunately rarely, in this special case.

An argument that is advanced in favour of denormalized numbers is that it justifies compiler optimizations that rely on certain mathematical equivalences that hold only in the presence of denormalized numbers. This is of no interest from the point of view of interval constraints. Any mathematically correct transformation can be performed on the set of constraints without changing the set of solutions obtained by a correctly implemented interval constraint system. This correctness is not dependent on the presence of denormalized numbers. In fact, it only depends on the finite floating-point numbers being some subset $F$ of the reals, as described in this paper.

Because of this independence, elaborate symbolic processing far beyond currently contemplated compiler optimizations is taken for granted in interval constraints.

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