THE TWO-DIMENSIONAL ANALOGUE OF GENERAL RELATIVITY

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Abstract

General Relativity in three or more dimensions can be obtained by taking the limit $\omega \to \infty$ in the Brans-Dicke theory. In two dimensions General Relativity is an unacceptable theory. We show that the two-dimensional closest analogue of General Relativity is a theory that also arises in the limit $\omega \to \infty$ of the two-dimensional Brans-Dicke theory.
1. Introduction

It is known that in two-dimensional (2D) spacetimes the Einstein-Hilbert action, \( S = \int d^2x \sqrt{-g} R \), is a topological invariant; variation with respect to the metric \( g_{ab} \) yields the Einstein tensor, \( G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R \), which is identically zero for any 2D metric. The energy-momentum tensor is defined as

\[
T_{ab} \equiv \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_{\text{matter}})}{\delta g_{ab}},
\]

where \( \mathcal{L}_{\text{matter}} \) is the Lagrangean for the matter fields. Then, assuming Einstein’s equations, \( G_{ab} - \Lambda g_{ab} = T_{ab} \), the vanishing of \( G_{ab} \) implies that \( T_{ab} \) should be proportional to \( \Lambda g_{ab} \), giving an unacceptable theory. Thus, one has to consider alternative 2D theories of gravitation.

Teitelboim and Jackiw [1] have proposed the simplest model for a 2D theory of gravitation, given by the constant curvature equation \( R - \Lambda = 0 \). Introducing an auxiliary scalar field, \( \Phi(x) \), one can derive this equation through an action given by

\[
S = \int d^2x \sqrt{-g} \Phi(x) (R - \Lambda).
\]

To include matter one can add the trace \( T \) of an energy-momentum tensor \( T_{ab} \) to yield the equation \( R - \Lambda = T \) [2, 3]. However, to derive this equation from a variational principle it is now difficult to justify the inclusion of the trace \( T \) inside the action of eq. (2). To remedy this problem a different action with a different auxiliary field \( \psi \) has been proposed [4, 5]

\[
S = \int d^2x \sqrt{-g} \left( \psi R + \frac{1}{2} (\partial \psi)^2 + \Lambda + 2 \mathcal{L}_{\text{matter}} \right).
\]

This action gives the desired equation \( R - \Lambda = T \), where \( T \) is the trace of the energy-momentum tensor defined in eq. (2).

Now, a general (not the most general) 2D action can be written as

\[
S = \int d^2x \sqrt{-g} e^{-2\phi} \left( R - 4\omega (\partial \phi)^2 + 4\lambda^2 \right),
\]

where \( \omega \) is a parameter, \( \lambda \) is a constant usually related to \( \Lambda \) as \( \Lambda = -4\lambda^2 \), and \( \Phi \) and \( \phi \) are related through \( \Phi = e^{-2\phi} \). This action includes several important cases: for \( \omega = 0 \) one recovers the Teitelboim-Jackiw theory given
in eq. (2), $\omega = -\frac{1}{2}$ gives planar General Relativity \[6\] and for $\omega = -1$ one obtains the first order string theory \[7, 8\]. This 2D Brans-Dicke theory has been analysed by the authors \[9\] for all values of $\omega$ and of the cosmological constant $\lambda$. It admits various types of black holes with different types of singularities.

One question that is still open, is which of the 2D theories of gravity is the analogue of General Relativity. In $n$-dimensions ($n \geq 3$) the Einstein’s theory of gravity can be obtained from the Brans-Dicke theory by taking in the latter the limit $\omega \to \infty$. In particular, this holds for our four-dimensional world. So, it is natural to consider that the analogue of General Relativity in two-dimensions is also the theory of gravity obtained from 2D Brans-Dicke theory in the abovementioned limit. In this letter we show that the theory given by action (3) is, in that sense, the 2D analogue of Einstein’s General Relativity. Indeed, in the limit $\omega \to \infty$ the 2D Brans-Dicke theory (4) (generalized to include a matter Lagrangean) gives rise to the equation $\mathbf{R} - \Lambda = \mathbf{T}$. This theory has been called the $\mathbf{R} = \mathbf{T}$ theory \[10\].

In section 2 we perform a heuristic derivation of the 2D Brans-Dicke action, and in section 3 we show how in the limit $\omega \to \infty$ of Brans-Dicke theory one obtains the 2D analogue of General Relativity. Finally we summarize our results in a concluding section.

2. An Heuristic Derivation of the 2D Brans-Dicke Action

Let us start with Newtonian dynamics in order to arrive at a relativistic theory. In one spatial dimension, Laplace’s equation implies that the force $\mathbf{F}$ between particles on a line is constant. This in turn implies that there are no tidal forces on a line, $d\mathbf{F} = 0$. Now, inferring that the 2D world is also special relativistic, one has to generalize the 2D Newtonian theory. If one associates tidal forces with the curvature, then Newton’s law implies that the scalar curvature vanishes, $\mathbf{R} = 0$. A theory with this equation cannot be derived from the Hilbert action, $S = \int d^2x \sqrt{-g} \mathbf{R}$, since it gives $\mathbf{G}_{ab} = \mathbf{R}_{ab} - \frac{1}{2}g_{ab}\mathbf{R} \equiv 0$, which is always true in 2D. However, it can be derived from $S = \int d^2x \sqrt{-ge^{-2\phi}} \mathbf{R}$, where $\phi$ is a scalar field. Variation with respect to $\phi$, $\frac{\delta S}{\delta \phi}$, yields $\mathbf{R} = 0$. One can generalize it immediately to give a spacetime with constant curvature by adding a cosmological constant, $\lambda^2$, to the action. If we want to add dynamical content to the scalar field then a
term proportional to \((\partial \phi)^2\) have to be introduced in the action. If the action is to be homogeneous in \(\phi\) such that \(S[\text{constant} + \phi] = \text{constant} \times S[\phi]\), then the proportionality term in \((\partial \phi)^2\) is a constant which we call \(\omega\). The action can then be written as

\[
S = \int d^2 x \sqrt{-g} \left( e^{-2\phi} \left( R - 4\omega(\partial \phi)^2 + 4\lambda^2 \right) \right).
\]

This is the Brans-Dicke action which of course can be generalized even more to contain a potential term of the scalar field, \(4\lambda^2 \rightarrow U(\phi)\), and to have a \(\phi\)-dependent Brans-Dicke parameter, \(\omega \rightarrow \omega(\phi)\).

### 3. The Analogue of General Relativity

General Relativity is the particular case of Brans-Dicke theory when \(\omega \rightarrow \infty\). We now show that the action given in eq. (5) can be derived in two-dimensions from the Brans-Dicke action in the same limit. Let us generalize action (4) to include matter,

\[
S = \int d^2 x \sqrt{-g} \left\{ e^{-2\phi} \left[ R - 4\omega(\partial \phi)^2 + 4\lambda^2 \right] + 4\mathcal{L}_{\text{matter}} \right\},
\]

where \(g\) is the determinant of the 2D metric, \(R\) is the curvature scalar, \(\phi\) is a scalar field, \(\lambda\) and \(\omega\) are constants. Variation of this action with respect to \(g_{ab}\) and \(\phi\) gives, respectively,

\[
e^{2\phi} T_{ab} = -2(\omega + 1) D_a \phi D_b \phi + D_a D_b \phi - g_{ab} D_c D^c \phi + (\omega + 2) g_{ab} D_c \phi D^c \phi - g_{ab} \lambda^2,
\]

\[
R - 4\omega D_c D^c \phi + 4\omega D_c \phi D^c \phi + 4\lambda^2 = 0,
\]

where \(D\) represents the covariant derivative. The limit \(\omega \rightarrow \infty\) in eq. (8) makes sense if one makes

\[
\phi = \phi_0 + \frac{\phi}{4\omega} + O\left(\frac{1}{\omega^2}\right),
\]

where the numerical factor \(\frac{1}{4}\) was chosen for later convenience. In this case we have

\[
R + 4\lambda^2 = \Box \phi.
\]
The trace of eq. (7) together with (9) gives
\[ 4\omega e^{2\phi_0} \frac{\omega}{2} T = -\Box \varphi + \frac{1}{2\omega} (\nabla \varphi)^2 - 8\omega \lambda^2. \] (11)

Then, comparing (10) and (11) we obtain
\[ R + 4\omega e^{2\phi_0} \frac{\omega}{2} T + 8\omega \lambda^2 = -4\lambda^2 + \frac{1}{2\omega} (\nabla \varphi)^2. \] (12)

Finally, redefining
\[ \Lambda = -8\omega \lambda^2, \] (13)
\[ \bar{T} = -4\omega e^{2\phi_0} T, \] (14)
and taking the limit \( \omega \to \infty \) one has
\[ R = \Box \varphi, \] (15)
\[ R - \Lambda = T, \] (16)
where we have dropped the bar over \( T \). Note that the limit \( \omega \to -\infty \) gives the same theory.

Substituting (9) into action (6) one obtains
\[ S = -\frac{1}{2\omega} e^{-2\phi_0} \int d^2 x \sqrt{-g} \left\{ -2\omega R + \varphi R + \frac{(\partial \varphi)^2}{2} - 8\omega \lambda^2 - 8\omega e^{2\phi_0} \mathcal{L}_{\text{matter}} \right\}. \] (17)

In order to simplify eq. (17) we perform the following three steps: (i) drop out the term \(-2\omega R\) inside curly brackets, since variation with respect to \( \varphi \) and \( g_{ab} \) yields zero; (ii) renormalize the cosmological constant as in eq. (13) and the matter Lagrangean using eq. (14); and (iii) absorb into the action the constant spurious coefficients in front of the integral. The new effective action is then
\[ S = \int d^2 x \sqrt{-g} \left( \varphi R + \frac{1}{2} (\partial \varphi)^2 + \Lambda + 2 \mathcal{L}_{\text{matter}} \right), \] (18)
which is precisely equal to eq. (3). Since in four dimensions the Einstein’s Theory of Relativity can be obtained from the Brans-Dicke theory taking the limit \( \omega \to \infty \), it is natural to consider the theory given by action (6) in the
limit \( \omega \to \infty \) (the R=T theory) as the 2D analogue of General Relativity. Thus, equation (16) is the 2D equivalent of Einstein equations; equation (15) determines the auxiliary field \( \varphi \).

5. Conclusions

The theory we have shown to be 2D natural analogue of General Relativity has been explored by Mann and collaborators. It admits gravitational collapse \([12, 13]\), gravitational waves \([12]\), particle solutions with horizons and their causal structure \([8, 14]\), cosmological solutions \([12, 15]\), and semiclassical approximations \([5]\).

We have found in another work \([6]\) that this theory also admits a vacuum black hole with two rather surprising properties: (i) it is a black hole of constant curvature, and (ii) it has no spacetime singularities. Black holes of constant curvature have also been found in three dimensions by identifying certain points of the anti-de Sitter spacetime \([17]\). The two dimensional constant curvature black hole version arises through a similar procedure and is analogous to the \( \omega = 0 \) (see eq. (4)) black hole \([16]\). On the other hand non-singular black holes have also appeared in the framework of an exact solution of 2D string theory \([18]\). Thus, it seems that the non-singular character of these black holes is a property of the 2D world.

As a final remark we would like to point out, that whereas we have discussed here the 2D analogue of General Relativity, the theory given by \( \omega = -\frac{1}{2} \) (see eq. (4)) is equal to vacuum planar General Relativity \([8]\).

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References

[1] C. Teitelboim, in: Quantum Theory of Gravity, essays in honour of the 60th birthday of B. DeWitt, ed. S. Christensen (Adam Hilger-Bristol,
1984) p. 327; R. Jackiw, in: Quantum Theory of Gravity, essays in honour of the 60th birthday of B. DeWitt, ed. S. Christensen (Adam Hilger-Bristol, 1984) p. 403.

[2] J. D. Brown, M. Henneaux and C. Teitelboim, *Phys. Rev. D* **33** (1986) 319.

[3] R. B. Mann, A. Shiekh and L. Tarasov, *Nucl. Phys. B* **341** (1990) 134.

[4] C. G. Torre, *Phys. Rev. D* **40** (1989) 2588.

[5] R. B. Mann, S. M. Morsink, A. S. Sikkema and T. G. Steele, *Phys. Rev. D* **43** (1991) 3948.

[6] J. P. S. Lemos, “Two-Dimensional Black Holes and General Relativity”, preprint DF/IST-13.93, 1993.

[7] G. Mandal, A. M. Sengupta and S. R. Wadia, *Mod. Phys. Lett. A* **6** (1991) 1685.

[8] E. Witten, *Phys. Rev. D* **44** (1991) 314.

[9] J. P. S. Lemos and P. M. Sá, “The Black Holes of a General Two-Dimensional Dilaton Gravity Theory”, preprint DF/IST-9.93, 1993.

[10] R. B. Mann and S. F. Ross, *Phys. Rev. D* **47** (1993) 3312.

[11] T. Banks and M. O’Loughlin, *Nucl. Phys. B* **362** (1991) 649.

[12] A. E. Sikkema and R. B. Mann, *Class. Quant. Grav.* **8** (1991) 219.

[13] R. B. Mann and S. F. Ross, *Class. Quant. Grav.* **9** (1992) 2335.

[14] D. Christensen and R. B. Mann, *Class. Quant. Grav.* **9** (1992) 1769.

[15] K. C. K. Chan and R. B. Mann, *Class. Quant. Grav.* **10** (1993) 913.

[16] J. P. S. Lemos and P. M. Sá, “Non-Singular Constant Curvature Two-Dimensional Black Hole”, preprint DF/IST-8.93 (revised version), 1993.
[17] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanneli, *Phys. Rev. D* **48** (1993) 1506.

[18] M. J. Perry and E. Teo, *Phys. Rev. Lett.* **70** (1993) 2669.