We review the determinations of the pseudoscalar glueball and eventual radial excitation of the \( \eta \) masses and decay constants from QCD spectral sum rules (QSSR). The glueball mass is \( (2.05 \pm 0.19) \) which one can compare with the eventual experimental candidate \( X(1835) \), while the \( \eta(1400) \) is likely a radial excitation of the \( \eta^\prime \)-meson. Their effects on the estimates of \( U(1)_A \) topological susceptibility and its slope as well as the impact of the latter in the estimate of the spin of the proton is discussed. We predict the singlet polarized parton distributions to be \( a^0 (Q^2 = 4 \text{ GeV}^2) = 0.32 \pm 0.02 \), which is about a factor two smaller than the OZI value, but comparable with the COMPASS measurement of \( 0.24 \pm 0.02 \).

Prologue

It is a great honour and pleasure for me to write this review for Professor Adriano Di Giacomo for celebrating his 70th birthday. My scientific connection with Adriano started when I studied at CERN in 82, the SVZ-expansion of QSSR\(^1\),\(^2\), and where the Pisa group\(^3\) has presented the first quenched lattice results values of the gluon condensates \( \langle \alpha_s G^2 \rangle \) and \( g^3 f_{abc} \langle G^a G^b G^c \rangle \) at the 1st Montpellier Conference on Non-perturbative methods in 85. About the same time, my previous interest\(^4\) on the topological susceptibility \( \chi(0) \) of the \( U(1)_A \) anomaly\(^5\) and its slope \( \chi'(0) \) continued after several discussions with Gabriele Veneziano during my stay at CERN. A program on the estimate of these quantities using QSSR has started\(^7\). In parallel, the Pisa group has runned their lattice simulations, and, for his 2nd participation at the Montpellier conference (QCD 90), Adriano has presented the lattice Pisa group\(^8\) results for \( \chi(0) \) and \( \chi'(0) \) which confirmed our previous results in\(^7\). Since then, there was a common interest between the Pisa group and Montpellier on these non-perturbative aspects of QCD. More recently, these common interests concern the proton spin problem\(^9\),\(^10\),\(^11\), the estimate of the gluon condensate using QSSR\(^12\) and lattice\(^13\), and the estimate of the quark-gluon mixed condensates using QSSR\(^14\) and field correlators\(^15\).

In 94, Adriano joined the international organizing committee of the QCD-Montpellier Series of Conferences and, in 2001, the one of the Madagascar High-Energy Physics (HEP-MAD) Series of Conferences. He is among the few committee members who send regularly speakers and participate continuously to these conferences. The organization team of these meetings has always appreciated his human kindness and friendship.

1 Introduction

The \( U(1)_A \) anomaly is one the most fundamental and fascinating problem of QCD, which has been solved in the large \( N_c \) limit by ’t Hooft-Witten-Veneziano\(^5\),\(^6\), where Veneziano...
found a solution without the uses of instantons. The $U(1)_A$ topological susceptibility is defined as:

$$\chi(0) \equiv \psi_{gg}(0) \equiv i \int d^4x \langle T Q(x) Q^T(0) \rangle ,$$

(1)

where:

$$Q(x) = \frac{1}{8\pi} \alpha_s G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} ,$$

(2)

is the $U(1)_A$ anomaly of the singlet axial-current:

$$\partial^\mu A_\mu(x) = \sum_{q=u,d,s} \left[ J_q \equiv 2m_q \bar{\psi}_q(i\gamma_5)\psi_q \right] + 2n_f Q(x) ,$$

(3)

where $n_f$ is the number of light quark flavours, $\psi_q$ the quark fields, $m_q$ the current quark mass and: $\tilde{G}^a_{\mu\nu} \equiv (1/2)\epsilon_{\mu\nu\rho\sigma} G^a_{\rho\sigma}$ . The topological charge is defined as:

$$Q = \int d^4x \, Q(x) ,$$

(4)

which is an integer for classical field configurations (2nd Chern number). Arguments based on large $N_c$ for $SU(N_c) \times SU(n_f)$ lead to:

$$\chi(0) = \frac{f_\pi^2}{2n_f} \left( M^2_{\eta'} + M^2_{\eta} - 2M^2_K \right) .$$

(5)

where, $f_\pi = (92.42 \pm 0.26)$ MeV. For $n_f = 3$:

$$\chi(0) \simeq (180 \text{ MeV})^4 .$$

(6)

Gabriele Veneziano and, later on, Giancarlo Rossi have challenged us to analyze the validity of this approximation at finite $N_c = 3$, by computing $\chi(0)$ and $\chi'(0)$ using QSSR, because the previous result is obtained at $q^2 = M^2_{\eta'}$ but not at $q^2 = 0$, where, they have used the expansion:

$$\chi(q^2) \simeq \chi(0) + q^2 \chi'(0) + ...,$$

(7)

2 The pseudoscalar gluonium/glueball mass from QSSR

QSSR has been used to calculate the topological charge in pure Yang-Mills QCD. In so doing, one has first extracted the mass and decay constant of the pseudoscalar glueball/gluonium using either a numerical fitting procedure or sum rule optimization criteria from the unsubtracted sum rules (USR):

$$\mathcal{L}(\tau) = \int^\tau_0 dt e^{-\tau t} \text{Im} \chi(t) \quad \text{and} \quad \mathcal{R}(\tau) = -\frac{d}{d\tau} \log \mathcal{L}(\tau) ,$$

(8)

\text{A generalization of the result including } SU(3) \text{ breakings for the decay constants can be found in.}
where the QCD model coming from the discontinuity of the QCD graphs has been used for the continuum. The analysis gives:

\[ M_G = (2.05 \pm 0.19) \text{ GeV} \quad \text{and} \quad f_G = (8 - 17) \text{ MeV}, \tag{9} \]
corresponding to \( t_c = (6 - 7) \text{ GeV}^2 \), and where \( f_G \) is normalized as:

\[ \langle 0 | Q(x) | G \rangle = \sqrt{2} f_G M_G^2, \tag{10} \]
i.e. like \( f_\pi / (2n_f) \) with \( f_\pi = (92.42 \pm 0.26)(92.42 \pm 0.26) \text{ MeV} \). The positivity of the spectral function gives:

\[ M_G \leq (2.34 \pm 0.42) \text{ GeV}. \tag{11} \]

One can compare the previous values with the quenched lattice results:

\[ M_G = (2.1 - 2.5) \text{ GeV}. \tag{12} \]

Recent sum rule analysis using instanton liquid model leads to similar results: \( M_G \simeq 2.2 \text{ GeV} \) and \( f_G \simeq 17 \text{ MeV} \) in our normalization, though the same approach leads to some inconsistencies in the scalar channel. One should note that in the previous analysis and, the value of the gluonium decay constant is smaller than the corresponding value of the \( \eta' \) one which is about 24 MeV in the chiral limit (see section 5.2). In a recent paper, the mass of the eventual experimental pseudoscalar gluonium candidate \( X(1835) \) has been used as input and the Laplace sum rule \( L \) has been exploited for fixing the corresponding decay constant. Though (a priori) self-consistent, this analysis is less constrained than the previous sum rules used in, where the two sum rules \( L \) and \( R \) have been simultaneously used for constraining the gluonium mass and decay constant. The resulting value of the gluonium decay constant is about (8-12) MeV and agrees with the previous values. On the other, a \( G-\eta_c \) mixing due to direct instanton has been considered for pushing the unmixed gluonium mass of about 2.1 GeV down to 1.8 GeV with a mixing angle of about 17°. This value is comparable with the early value of about \( \theta_P = 12° \) of the meson-gluonium mixing angle using the OPE of the off-diagonal light quark-gluon correlator:

\[ \psi_{gq}(q^2) \equiv \frac{i}{2n_f} \int d^4xe^{iqx} \langle TQ(x)J^g_+(0) \rangle, \tag{13} \]

which is proportional to \( m_s \). Indeed, it is also plausible that the \( X(1845) \) comes from a complicated mixing of the glueball with the \( \eta(1440) \) and \( \eta_c \). A confirmation of the gluonium nature of the \( X(1835) \) requires some more independent tests.

### 3 On the natures of the \( \eta(1295) \) and \( \eta(1400) \) from QSSR

There are two other experimental candidates which are the \( \eta(1295) \) and \( \eta(1400) \), which one can intuitively interpret as the first radial excitations of the \( \eta(547) \) and \( \eta'(958) \). Here, we shall test if the \( \eta(1400) \) satisfies this interpretation. Using the same approach, the nature of the \( \eta(1400) \) has been tested by measuring its coupling to the \( U(1)_A \) singlet current \( Q(x) \) in the chiral limit. In this case, we can work with the SSR:

\[ \int_0^{t_c} dt \frac{e^{-tr}}{t} \frac{1}{\pi} \text{Im } \chi(t), \tag{14} \]
as one expects from large $N_c$ arguments that the topological charge $\chi(0)$ vanishes due to the $\theta$-independence of the QCD Lagrangian, thanks to the presence of the singlet $\eta_1$. Including in the sum rule, the contributions of the $\eta'$, $\eta(1400)$, $G$ and the QCD continuum with $t_c$ fixed previously, one cannot find any room to put the $\eta(1400)$, i.e. $f_{\eta} \approx 0$. Relaxing the constraint by replacing the QCD continuum with the $\eta(1400)$, one can deduce:

$$f_{\eta(1400)} \leq 16 \text{ MeV} \ll f_{\eta'} \simeq (24.1 \pm 3.5) \text{ MeV},$$  \hspace{1cm} (15)

which should be a weak upper bound within this assumption. This feature indicates that the $\eta(1400)$ is likely the first radial excitation of the $\eta'$, as intuitively expected, while the glueball has a higher mass.

4 The $U(1)_A$ topological susceptibility from QSSR

4.1 The pure Yang-Mills result

Applying the Borel/Laplace operator to the subtracted quantity:

$$\chi(q^2) - \chi(0)|_{\text{no quarks}}$$  \hspace{1cm} (16)

one can derive a combination of the unsubtracted sum rule (USR) and subtracted sum rule (SSR) for the topological susceptibility in pure Yang-Mills without quarks$^{4,7}$:

$$\chi(0)|_{\text{no quarks}} = \int_0^{t_c} \frac{dt}{t} e^{-t \tau} \left(1 - \frac{t \tau}{2}\right) \frac{1}{\pi} \text{Im} \chi(t) - \left(\frac{\alpha_s}{8\pi}\right)^2 \frac{2}{\tau^2} \left\{ \frac{1}{\log \tau \Lambda^2} + 2\pi^2 \langle G^2 \rangle + 6\pi^2 g \langle G^3 \rangle \tau^3 \right\}. \hspace{1cm} (17)$$

The value of the gluon $\langle \alpha_s G^2 \rangle = (0.07 \pm 0.01) \text{ GeV}^2$ from $e^+e^-$ data and heavy-quark mass-splitting$^{12,2}$ has been confirmed by lattice calculations with dynamical fermions$^{13}$, while the value of the triple gluon condensate $g^3 f_{abc} \langle G^a G^b G^c \rangle$ is about $1.5 \text{ GeV}^2 \langle \alpha_s G^2 \rangle$ from instanton model$^{26}$ and lattice calculations$^{3}$. The previous combination of sum rules is more interesting than the alone SSR, as the effect of the QCD continuum is minimized here. At the sum rule optimization scale of $\tau$ about 0.5 GeV$^{-2}$, it leads to the value$^{4,7}$:

$$\chi(0)|_{\text{no quarks}} \simeq -[(106 - 122) \text{ MeV}]^4 \approx -4 \left(\frac{\alpha_s}{8\pi}\right)^2 \langle G^2 \rangle, \hspace{1cm} (18)$$

indicating the role of the gluon condensate in the determination of $\chi(0)$. The sign and the size of this result, though inaccurate are in agreement with the large $N_c$ results obtained previously. In pure gauge, lattice gives the value$^{8,27,28}$:

$$\chi(0)|_{SU(2)} \simeq -[(167 \pm 25) \text{ MeV}]^4, \hspace{1cm} \chi(0)|_{SU(3)} \simeq -[(191 \pm 5) \text{ MeV}]^4, \hspace{1cm} (19)$$

which one can compare with the two former ones from large $N_c$ and from the sum rules.
4.2 Result in the presence of quarks

In this case the analysis is more involved as one also has to consider the diagonal quark-quark correlator:

\[ \psi_{qq}(q^2) \equiv \frac{i}{(2n_f)^2} \int d^4x \ e^{iqx} \langle T J_q(x) J_{q}^\dagger(0) \rangle , \] (20)

and off-diagonal quark-gluon correlator in Eq. (13). Then, the full correlator reads:

\[ \psi_{5}(q^2) = \psi_{gg}(q^2) + 2\psi_{gq}(q^2) + \psi_{qq}(q^2) . \] (21)

In this case, \( \psi_{5}(0) \) is not vanishing and can be deduced from chiral Ward identities to be\(^6\,10\):

\[ \psi_{5}(0) = -\frac{4}{(2n_f)^2} \sum_{q=u,d,s} m_q \langle \bar{q}q \rangle , \] (22)

Therefore, the topological charge \( \chi_{(0)} \) vanishes in the chiral limit due to the \( \theta \) independence of the QCD Lagrangian. Lattice calculations in full QCD for two degenerate dynamical fermions find\(^29\):

\[ \psi_{5}(0) = \left[ (163 \pm 6) \text{ MeV} \right]^4 , \] (23)

in agreement with the large \( N_c \) result\(^5\,6\) but does not have enough accuracy for checking the linear \( m_q \)-dependence expected from current algebra.

In this case of massive quarks, we include \( SU(3) \) breakings to the sum rule in order to see the effect of the physical \( \eta' \). Including both the gluonium and QCD continuum contribution to the appropriate sum rule, one can see from\(^4\,7\) that their contributions tend to compensate, and then give the approximate numerical formula to leading order:

\[ 2M_{\eta'}^2 f_{\eta'}^2 e^{-M_{\eta'}^2 \tau} \left( 1 - \frac{M_{\eta'}^2 \tau}{2} \right) \simeq \psi_{5}(0) + \frac{3}{4\pi^2} \left( \frac{m_s}{2n_f} \right)^2 \tau^{-1} . \] (24)

Using the quark model prediction:

\[ f_{\eta'} \simeq \frac{1}{2n_f} \sqrt{3} f_\pi \simeq 27 \text{ MeV} , \] (25)

the physical \( \eta' \) mass and the value \( \overline{m}_s(\tau) \simeq 100 \text{ MeV} \)\(^30\), we can deduce at the sum rule optimization scale \( \tau \simeq 0.5 \text{ GeV}^{-2} \):

\[ \psi_{5}(0) \simeq \left[ 157 \text{ MeV} \right]^4 , \] (26)

in good agreement with the large \( N_c \) and lattice results, then showing the consistency of the sum rule approach. Armed with these consistency checks, we shall now evaluate with confidence the slope of the topological susceptibility \( \chi'(0) \).

\(^{\text{c}}\)These correlators include in their definitions the quark mass through \( J_q \), while in\(^10\), this quark mass is factorized out.
5 The slope $\chi'(0)$ of the topological susceptibility from QSSR

Using the $q^2$ expansion in Eq. (7), one can derive a sum rule for $\chi'(0)$ by applying, the Laplace sum rule operator, to the twice subtracted quantity:

$$\frac{\chi(q^2) - \chi(0) - q^2 \chi'(0)}{(q^2)^2} = \int \frac{dt}{t} \frac{1}{t - q^2 - i\epsilon} \frac{1}{\pi} \text{Im} \chi(t),$$  \hspace{1cm} (27)

5.1 The pure Yang-Mills case

One obtains in this case:

$$\int_0^\infty \frac{dt}{t^2} e^{-t\tau} \frac{1}{\pi} \text{Im} \chi(t) = F_1(\tau) - \chi(0)\tau + \chi'(0),$$  \hspace{1cm} (28)

where $F_1(\tau)$ comes from the OPE expression of the correlator $\chi(q^2)$. Eliminating $\chi(0)$ by using Eq. (17), one can deduce the sum rule:

$$\int_0^\infty \frac{dt}{t^2} e^{-t\tau} \left[1 - t\tau \left(1 - \frac{t\tau}{2}\right)\frac{1}{\pi} \text{Im} \chi(t)\right] \simeq \chi'(0) + \left(\frac{\alpha_s}{8\pi}\right)^2 \frac{2}{\pi^2} \tau^{-1} \times \left\{1 + \left(\frac{\alpha_s}{\pi}\right) \left[10 - 2\gamma_E\beta_1 - 2\frac{\beta_2}{\beta_1} \log \left(-\log \tau\Lambda^2\right)\right] + 4\pi^2 g(G^3)\tau^3\right\},$$  \hspace{1cm} (29)

where $\beta_1 = -11/2$ and $\beta_2 = -51/4$ for pure gauge $SU(3)$. Using $\tau$- and $t_c$-stability criteria, and saturating the spectral function by the pseudoscalar glueball and QCD continuum, one can deduce at $\tau \simeq 0.5$ GeV:

$$\chi'(0)_{\text{no quarks}} \simeq -[7 \pm 3] \text{ MeV}^2,$$  \hspace{1cm} (30)

indicating that: $\chi'(0)M_{\eta'}^2 \ll \chi(0)$, which justifies the accuracy of the large $N_c$ result. This value and the sign has been confirmed by lattice calculations:

$$\chi'(0)_{\text{quenched}} \simeq -[(9.8 \pm 0.9) \text{ MeV}]^2.$$

5.2 Result in the presence of massless quarks

In this case, quark loops enter into the value of the $\beta$ function and of the QCD scale $\Lambda$. The $\eta'$ enters now into the spectral function. Its coupling to the gluonic current can be estimated from the SSR:

$$\int_0^{t_c} \frac{dt}{t} e^{-t\tau} \left(1 - \frac{t\tau}{2}\right)\frac{1}{\pi} \text{Im} \chi(t),$$  \hspace{1cm} (32)

taking into account the fact that in the chiral limit $\chi(0) = 0$. Therefore, one obtains:

$$f_{\eta'} \left(\tau \simeq 0.5 \text{ GeV}^{-2}\right) \simeq (24.1 \pm 3.5) \text{ MeV},$$  \hspace{1cm} (33)

where the decay constant has a weak $\tau$-dependence because of renormalization:

$$f_{\eta'}(\tau) \simeq \hat{f}_{\eta'} \exp \left(\frac{8}{-\beta_1^2 \log \tau\Lambda^2}\right),$$  \hspace{1cm} (34)
where \( \hat{f}_{\eta'} \) is RG invariant. This result can be compared with the quark model prediction in Eq. (25). Using the Laplace and FESR versions of the twice subtracted sum rule, one finds:

\[
\sqrt{\chi'(0)_{\text{massless}}(\tau)} \simeq [(26.5 \pm 3.1) \text{ MeV}]^2. \tag{35}
\]

The result has changed sign compared to the one from pure Yang-Mills, while its absolute value is about 12 times higher. The main effect is due to the \( \eta' \) which gives the dominant contribution to the spectral function compared to the gluonium and QCD continuum. That can be understood because of its low mass and of the fact that its decay constant \( f_{\eta'} \) is larger than the gluonium decay constant \( f_G \).

Similar estimate of \( \psi_\eta(0) \) in the chiral limit for the singlet channel has been done in [31], where the value of \( \chi(0)' \) is significantly larger by about a factor 2.5 than ours. However, we found some inconsistencies in the two sides of the SR: in the experimental side the contributions of the singlet and octet mesons have been considered and the phenomenological value of the decays constants, masses and mixing angles have been used; while in the QCD side, only the correlator associated to the singlet current \( Q(x) \), or to massless quark has been accounted for.

On the other, some criticisms raised in a series of paper [32], due to direct instanton breaking of the OPE, do not apply in our case: our different sum rules optimize at a large scale \( \tau \lesssim 0.5 \text{ GeV}^{-2} \), where this contribution become irrelevant like other high-dimension condensates.

In [20], it has been argued that screening corrections cancel the direct instanton contributions, indicating that these effects are not yet well understood. However, in your approach, the introduction of the new \( 1/q^2 \)-term in the OPE, which is also negligible, is an alternative to the direct instanton contributions [33,34]. Some answers to these criticisms have been already explicitly given in [10].

5.3 Result in the presence of massive quarks

As can be seen from the expression of the two-point correlator \( \psi_\eta(q^2) \) in Eq. (21), the analysis is more involved. In this case, its slope has been estimated from the substracted Laplace sum rules. At the stability point \( \tau \simeq (0.2 - 0.4) \text{ GeV}^{-2} \), one finds:

\[
\sqrt{\psi_\eta'(0)} = (33.5 \pm 3.9) \text{ MeV}, \tag{36}
\]

while the \( \eta' \)-decay constant becomes:

\[
f_{\eta'}(\tau) \simeq (27.4 \pm 3.7) \text{ MeV}, \tag{37}
\]

compared with the results for massless quarks in Eqs. (35) and (33) and agrees quite well with the quark model prediction in Eq. (25). One can notice that the \( SU(3) \) breaking effect is about 10\% and 20\% respectively showing a smooth dependence on the strange quark mass as expected. Similar analysis can be done in the flavour non-singlet case by working with the correlator associated to the \( \eta \)-meson current:

\[
(0|\partial^\mu f_{\mu 3}^8|\eta) = f_\eta M_\eta^2, \tag{38}
\]

where in the \( SU(3) \) limit and in this normalization without \( 1/2n_f \), \( f_\eta = f_\pi = (92.42 \pm 0.26) \text{ MeV}. \) Therefore, one obtains [10]:

\[
\sqrt{\psi_\eta^{(8)}(0)} = (43.8 \pm 5.0) \text{ MeV}, \quad \text{and} \quad f_\eta/f_\pi = 1.37 \pm 0.16. \tag{39}
\]
Instead of the value of \( f^{\exp}_{\pi} = (92.42 \pm 0.26) \) MeV, we have used the sum rule prediction\(^{10}\),

\[
 f_{\pi} = (107 \pm 12) \text{ MeV},
\]  
(40)

for a self-consistency of the whole results. The value of the \( SU(3) \) breaking ratio \( f_{\eta}/f_{\pi} \) is in line with \( f_{K}/f_{\pi} = 1.2 \), where we expect bigger effects for the \( \eta \) than for the \( K \).

### 6 Applications of the QSSR results to the proton spin

The previous results have been applied to the proton spin problem, where one expects that the gluon content of the proton is due to the \( U(1)_{A} \) anomaly. This property can be made explicit in the approach of DIS where the matrix elements from the OPE are factorised into composite operators and proper vertex functions\(^{35}\). In the case of polarised \( \mu p \) scattering, the composite operator can be identified with the slope \( \chi'(0) \) of the topological susceptibility, which is an universal quantity and then target independent, while the corresponding proper vertex is renormalisation group invariant. The first moment of the polarised structure function reads, in terms of the axial charges of the proton:

\[
\Gamma_{p}^{p}(Q^{2}) \equiv \int_{0}^{1} dx \, g_{1}^{p}(x, Q^{2}) = \frac{1}{12} C_{1}^{NS}(\alpha_{s}(Q^{2})) \left( a^{3} + \frac{1}{3} a^{8} \right) + \frac{1}{9} C_{1}^{S}(\alpha_{s}(Q^{2})) a^{0}(Q^{2}),
\]  
(41)

where the Wilson coefficients arise from the OPE of the two electromagnetic currents:

\[
C_{1}^{NS} = 1 - a_{s} - 3.583 a_{s}^{2} - 20.215 a_{s}^{3}, \quad C_{1}^{S} = 1 - \frac{1}{3} a_{s} - 0.550 a_{s}^{2}, \quad (42)
\]

where \( a_{s} \equiv \alpha_{s}/\pi \). The axial charge are defined from the forward matrix elements as:

\[
\langle p, s | J_{\mu 5}^{3} | p, s \rangle = \frac{1}{2} a^{3} s_{\mu}, \quad \langle p, s | J_{\mu 5}^{8} | p, s \rangle = \frac{1}{2 \sqrt{3}} a^{8} s_{\mu}, \quad \langle p, s | J_{\mu 5}^{0} | p, s \rangle = a^{0}(Q^{2}) s_{\mu},
\]  
(43)

where \( J_{\mu 5}^{a} \) are the axial currents and \( s_{\mu} \) the proton polarisation vector. Using QCD parton model, the axial charges read, in terms of moments of parton distributions\(^{36}\):

\[
a^{3} = \Delta u - \Delta d, \quad a^{8} = \Delta u + \Delta d - 2\Delta s, \quad a^{0} = [\Delta \Sigma \equiv \Delta u + \Delta d + \Delta s] - \frac{n_{f}}{2} a_{s} \Delta g(Q^{2}).
\]  
(44)

\( a^{3} \) and \( a^{8} \) are known in terms of the \( F \) and \( D \) coefficients from beta and hyperon decays:

\[
a^{3} = F + D \quad \text{and} \quad a^{8} = 3F - D,
\]  
(45)

where\(^{37}\):

\[
F + D = 1.257 \pm 0.008 \quad \text{and} \quad F/D = 0.575 \pm 0.016,
\]  
(46)

so that an experimental determination of the first moment of \( g_{1}^{p} \) in polarised deep inelastic scattering (DIS) allows a determination of the singlet axial charge \( a^{0}(Q^{2}) \). In the naïve quark or valence quark, parton model, one expects that \( \Delta s = \Delta g = 0 \), and then \( a^{0} = a^{8} \), which is the OZI prediction. The proton spin problem is that the experimental value of
$a_0(Q^2)$ is much smaller than $a^8$, which would be its expected value if the OZI rule were exact in this channel. The first SMC data found $^{39, 40}\colon$

$$a_0(Q^2 = 10 \text{ GeV}^2) = 0.19 \pm 0.17 \quad \text{and} \quad a_0(Q^2 = 5 \text{ GeV}^2) = 0.19 \pm 0.06 \ , \quad (47)$$

which are confirmed and improved by the recent COMPASS data at $Q^2 = 4 \text{ GeV}^2$ $^{41}$:

$$a_0(Q^2 = 4 \text{ GeV}^2) = 0.24 \pm 0.02 \ . \quad (48)$$

These results are much smaller than the OZI prediction ($\Delta g = 0$):

$$\Delta \Sigma |_{\text{OZI}} = 3F - D = 0.579 \pm 0.021 \ . \quad (49)$$

Inserted into Eq. (41), the OZI results lead to the Ellis-Jaffe sum rule $^{38}$. It has been conjectured in $^{36}$ that the suppression of $a_0$ with respect to $a^8$ is due to the gluon distribution. However, the SMC measurement of $\Delta \Sigma$ gives a value $^{40}$:

$$\Delta \Sigma |_{\text{SMC}} \simeq 0.38 \pm 0.04 \ , \quad (50)$$

indicating that independently of $\Delta g$, there is also a large difference between the OZI prediction and the data. In the following, we will not try to solve this discrepancy but will show our prediction for $a_0$, where the sum of the quark and gluon components are concerned.

6.1 Prediction in the chiral limit

Using the composite operator $\oplus$ proper vertex functions approach $^9$, one can write the decomposition, in the chiral limit:

$$\Gamma_{1 \text{ singlet}}^p = \frac{1}{9} \frac{1}{2M_N} 2n_f C^S_T(\alpha_s(Q^2)) \sqrt{\chi'(0)|_{Q^2} \hat{\Gamma}_{\eta^p NN}} \ . \quad (51)$$

The key assumption is that the vertex is well approximated by its OZI value:

$$\hat{\Gamma}_{\eta^p NN} = \sqrt{2} \hat{\Gamma}_{\eta^s NN} \ , \quad (52)$$

while all OZI violation in $\Gamma_{1 \text{ singlet}}^p$ is contained inside $\sqrt{\chi'(0)}$. Comparing the result with the OZI prediction of $a^8$, one can deduce:

$$\frac{a_0(Q^2)}{a^8} = \frac{\sqrt{6}}{f_\pi} \sqrt{\chi'(0)|_{Q^2}} = 0.60 \pm 0.12 \ , \quad (53)$$

which gives our original proton spin sum rule $^9$:

$$a_0(Q^2 = 10 \text{ GeV}^2) = 0.35 \pm 0.05 \quad \Rightarrow \quad \Gamma_{1}^p(Q^2 = 10 \text{ GeV}^2) = 0.143 \pm 0.005 \ , \quad (54)$$

where we have used:

$$\sqrt{\chi'(0)|_{Q^2=10 \text{ GeV}^2}} = (23.2 \pm 2.4) \text{ MeV} \ , \quad (55)$$

after running the result in Eq. (35) from 2 to 10 GeV$^2$. These results can be compared with the OZI value in Eq. (49) and agree with the last SMC data at $Q^2 = 10 \text{ GeV}^2$ $^{40}$:

$$\Gamma_{1}^p(Q^2) = 0.145 \pm 0.008 \pm 0.011 \quad \Rightarrow \quad a_0(Q^2) = 0.37 \pm 0.07 \pm 0.10 \ , \quad (56)$$

$^a$In $^{42}$, an estimate of the scalar sea quark content of the nucleon indicates that it is suppressed compared to the valence one. Similar results may apply here.
6.2 The case of massive quarks

In this case, the relation in Eq. (53) is replaced by:

\[
\frac{a^0(Q^2)}{a^8} = \frac{1}{\sqrt{2}} \frac{\sqrt{\psi_5^0(0)}}{\sqrt{\psi_{88}^0(0)}} = 0.55 \pm 0.02 ,
\]

(57)

where \( \psi_5(q^2) \) has been defined in Eq. (21) and \( \psi_{88}^0(q^2) \) is the two-point correlator for the octet current. The running of the subtraction constant \( \psi_5^0(0) \) is very smooth from \( \tau^{-1}=3 \) to \( 10 \) GeV\(^2\). Using \( \psi_{88}^0(0) \equiv a^8 = 0.58 \), the previous relation leads to:

\[
a^0(Q^2 = 4 \text{ GeV}^2) \simeq a^0(Q^2 = 10 \text{ GeV}^2) \simeq 0.32 \pm 0.02 ,
\]

(58)

which is almost equal to its value in the chiral limit. This prediction is comparable with the COMPASS result \((0.24 \pm 0.02)^{11}\) given in Eq. (48) but is about a factor 2 smaller than its OZI value in Eq. (49). Using this result, we predict:

\[
\Gamma_p^0(Q^2 = 4 \text{ GeV}^2) = 0.145 \pm 0.002 ,
\]

(59)

where we have used the value of \( \alpha_s(M_\tau) \simeq 0.347 \pm 0.03^{43,44,23} \), and the previous values of F and D from \(^{37}\). Improvements of these results at COMPASS energy require a good control of the higher twist corrections in the PT expressions of \( \Gamma_p^0 \) which are expected to be bigger here than at SMC. Using the sum rule estimate in \(^{45}\) of about \(-0.03\) of the coefficient of the 1st power \( 1/Q^2 \) corrections, one obtains a correction of about \((-0.008 \pm 0.008)\) to \( \Gamma_p^0 \) at \( 4 \) GeV\(^2\), where the error is an educated guess taking into account larger absolute value obtained from the fit of the Bjorken sum rule in \(^9\). A control of the corrections in the assumption of the validity of OZI for the singlet and non-singlet vertex functions is not yet testable\(^6\). Including the power correction, we consider as a final estimate:

\[
\Gamma_p^0(Q^2 = 4 \text{ GeV}^2) = 0.137 \pm 0.008 .
\]

(60)

Several tests of the approach have been also proposed in the literature\(^{47}\). Another crucial test of our result will be a lattice measurement of \( \psi_5^0(0) \) with massive dynamical fermions, which we wish that the Pisa group puts in its agenda. Unfortunately, an attempt to measure \( a^0(Q^2) \) on the lattice\(^{11}\) has not been conclusive, as it gives a value:

\[
a^0(Q^2) = 0.04 \pm 0.04 \pm 0.20 ,
\]

(61)

where the last error is an educated guess of the effective error expected by the authors.

7 Conclusions

We have used QSSR for predicting the gluonium decay constant and mass, and for arguing that the \( \eta(1440) \) is likely the radial excitation of the \( \eta'(958) \). These results have been used for predicting the topological susceptibility and its slope, which are useful inputs in the resolution of the “proton spin problem”.

\( ^{6} \)A QSSR evaluation of the \( \eta N N \) coupling has not been conclusive\(^{46}\).
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