A GEOMETRIC INTERPRETATION OF STANLEY’S MONOTonicity theorem

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Abstract. We present a new geometric proof of Stanley’s monotonicity theorem for lattice polytopes, using an interpretation of δ-polynomials of lattice polytopes in terms of orbifold Chow rings.

1. Introduction

Let $P$ be a $d$-dimensional lattice polytope in a lattice $N$ of rank $n$. That is, $P$ is the convex hull of finitely many points in $N \cong \mathbb{Z}^n$. If $m$ is a positive integer, then let $f_P(m) := \#(mP \cap N)$ denote the number of lattice points in the $m$’th dilate of $P$. A famous theorem of Ehrhart [6] asserts that $f_P(m)$ is a polynomial in $m$ of degree $d$, called the Ehrhart polynomial of $P$. The generating series of $f_P(m)$ can be written in the form

$$\frac{\delta_P(t)}{(1-t)^{d+1}} = \sum_{m \geq 0} f_P(m) t^m,$$

where $\delta_P(t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$ is a polynomial of degree at most $d$ with integer coefficients, called the $\delta$-polynomial of $P$. Using techniques from commutative algebra, Stanley proved that the coefficients $\delta_i$ are non-negative [12] and proved that $\delta$-polynomials of lattice polytopes satisfy the following monotonicity property [13, Theorem 3.3]. An alternative combinatorial proof of these results was given by Beck and Sottile in [2]. If $f(t) = \sum_i f_i t^i$ and $g(t) = \sum_i g_i t^i$ are polynomials with integer coefficients, then we write $f(t) \leq g(t)$ if $f_i \leq g_i$ for all $i \geq 0$.

Theorem 1.1 (Stanley’s Monotonicity Theorem). If $Q \subseteq P$ are lattice polytopes, then $\delta_Q(t) \leq \delta_P(t)$.

We now present a new geometric proof of Stanley’s theorem. We first recall the following geometric interpretation of $\delta$-polynomials of lattice polytopes. After replacing $N$ with its intersection with the affine span of $P$, we may assume that $N$ has rank $d$. Let $T$ be a regular, lattice triangulation of $P$ and let $\sigma$ denote the cone over $P \times \{1\}$ in $N_{\mathbb{R}} \times \mathbb{R}$, where $N_{\mathbb{R}} = N \otimes \mathbb{R}$. The triangulation $T$ induces a simplicial fan refinement $\Delta$ of $\sigma$, with cones given by the cones over the faces of $T$, and we may consider the $(d+1)$-dimensional, simplicial toric variety $Y = Y(\Delta)$ associated to $\Delta$. The toric variety $Y$ is semi-projective in the sense that it contains a torus-fixed point and is projective over its affinisation $Y(\sigma)$ [8]. The cohomology ring $H^*(X, \mathbb{Q})$ of a semi-projective, simplicial toric variety $X$ was computed by Hausel and Sturmfels in [8], and it was observed by Jiang and Tseng [9].

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Lemma 2.7] that Hausel and Sturmfel’s proof, along with the results in [7, Section 5.1], imply that $H^\ast(X, \mathbb{Q})$ is isomorphic to the Chow ring $A^\ast(X, \mathbb{Q})$.

The orbifold Chow ring of a Deligne-Mumford stack was introduced by Abramovich, Graber and Vistoli [1] as the algebraic analogue of Chen and Ruan’s orbifold cohomology ring [5]. Borisov, Chen and Smith introduced the notion of a toric stack in [4] and showed that any simplicial, semi-projective toric variety $X$ has the canonical structure of a Deligne-Mumford stack. The orbifold Chow ring $A^\ast_{\text{orb}}(X, \mathbb{Q})$ of $X$ is a $\mathbb{Q}$-graded $\mathbb{Q}$-algebra and was computed by Jiang and Tseng in [9], generalising results in [4] (Remark 2.2). The following combinatorial observation follows from [14, Theorem 4.6] (c.f. [10, Corollary 1.2]).

(1) $\delta_P(t) = \sum_{i \in \mathbb{Q}} \dim_{\mathbb{Q}} A^i_{\text{orb}}(Y, \mathbb{Q}) t^i.$

If $Q$ is a lattice polytope contained in $P$, then let $N'$ denote the intersection of $N$ with the affine span of $Q$ and let $\sigma'$ denote the cone over $Q \times \{1\}$ in $(N')_\mathbb{R} \times \mathbb{R}$. One verifies that we may choose a regular, lattice triangulation $T$ of $P$ which restricts to a regular, lattice triangulation of $Q$. In this case, the fan $\Delta$ refining $\sigma$ restricts to a fan $\Sigma$ refining $\sigma'$ and we may consider the semi-projective toric variety $Y' = Y'((\Sigma))$. The inclusion of $N'$ in $N$ induces a locally closed toric immersion $Y' \hookrightarrow Y$ and a restriction map between the corresponding orbifold Chow rings. We will prove the following lemma in Section 2.

**Lemma 1.2.** The morphism $Y' \hookrightarrow Y$ induces a surjective graded ring homomorphism $A^\ast_{\text{orb}}(Y, \mathbb{Q}) \to A^\ast_{\text{orb}}(Y', \mathbb{Q})$.

By (1), $\delta_P(t) = \sum_{i \in \mathbb{Q}} \dim_{\mathbb{Q}} A^i_{\text{orb}}(Y, \mathbb{Q}) t^i$ and $\delta_Q(t) = \sum_{i \in \mathbb{Q}} \dim_{\mathbb{Q}} A^i_{\text{orb}}(Y', \mathbb{Q}) t^i$, and we conclude that $\delta_Q(t) \leq \delta_P(t)$, as desired.

**Remark 1.3.** If we regard the empty face as a face of the triangulation $T$ of dimension $-1$, then the $h$-vector of $T$ is defined by

$$h_T(t) = \sum_F t^{\dim F + 1} (1 - t)^{d - \dim F},$$

where the sum ranges over all faces $F$ in $T$. It is a well known fact that $0 \leq h_T(t) \leq \delta_P(t)$ and $h_T(t) = \delta_P(t)$ if and only if $T$ is a unimodular triangulation [3,11]. We have the following geometric interpretation of this result.

It follows from the definition of the orbifold Chow ring (see Section 2) that $A^\ast(Y, \mathbb{Q})$ is a direct summand of $A^\ast_{\text{orb}}(Y, \mathbb{Q})$ and $A^\ast(Y, \mathbb{Q}) = A^\ast_{\text{orb}}(Y, \mathbb{Q})$ if and only if $Y$ is smooth. The result now follows from the fact that $h_T(t) = \sum_{i \geq 0} \dim_{\mathbb{Q}} A^i(Y, \mathbb{Q}) t^i$ [5, Corollary 2.12] and the fact that $Y$ is smooth if and only if $T$ is a unimodular triangulation.

All varieties and stacks will be over the complex numbers. In Section 2, we will review orbifold Chow rings and prove Lemma [2].
2. Orbifold Chow Rings

The orbifold Chow ring $A_{\text{orb}}^*(\mathcal{X}, \mathbb{Q})$ of a Deligne-Mumford stack $\mathcal{X}$ was introduced by Abramovich, Graber and Vistoli as the degree 0 piece of the small quantum cohomology ring of $\mathcal{X}$ [1]. We will review the structure of $A_{\text{orb}}^*(\mathcal{X}, \mathbb{Q})$ as a $\mathbb{Q}$-graded vector space and refer the reader to [1] for the relevant details and the description of the ring structure of $A_{\text{orb}}^*(\mathcal{X}, \mathbb{Q})$. The inertia stack $\mathcal{I}\mathcal{X}$ of $\mathcal{X}$ is a Deligne-Mumford stack whose objects consist of pairs $(x, \alpha)$, where $x$ is an object of $\mathcal{X}$ and $\alpha$ is an automorphism of $x$. If $\mathcal{X}_1, \ldots, \mathcal{X}_r$ denote the connected components of $\mathcal{I}\mathcal{X}$, then

$$A_{\text{orb}}^*(\mathcal{X}, \mathbb{Q}) = \bigoplus_{j=1}^r A^*(|\mathcal{X}_j|, \mathbb{Q})[s_j],$$

where $|\mathcal{X}_j|$ is the coarse moduli space of $\mathcal{X}_j$, $s_j \in \mathbb{Q}$ is the age of $\mathcal{X}_j$ and $[s_j]$ denotes a grading shift by $s_j$. If we identify $\mathcal{X}$ as the connected component of $\mathcal{I}\mathcal{X}$ whose objects consist of pairs $(x, \text{id})$, where $x$ is an object of $\mathcal{X}$ and $\text{id}$ is the identity automorphism of $x$, then the age of $\mathcal{X}$ is 0 and $A^*(|\mathcal{X}|, \mathbb{Q})$ is a direct summand of $A_{\text{orb}}^*(\mathcal{X}, \mathbb{Q})$.

Continuing with the notation of the introduction, recall that $P$ is a $d$-dimensional lattice polytope in a lattice $N$ of rank $d$ and $T$ is a regular lattice triangulation of $P$. Recall that $T$ induces a fan refinement $\Delta$ of the cone $\sigma$ over $P \times \{1\}$ in $N_{\mathbb{R}} \times \mathbb{R}$, and that $Y = Y(\Delta)$ is the associated $(d+1)$-dimensional, semi-projective, simplicial toric variety. There is a canonical Deligne-Mumford stack $\mathcal{Y}$ with coarse moduli space $Y$ [4]. If $F$ is a non-empty face of $T$ with vertices $v_1, \ldots, v_s$, then set

$$\text{Box}(F) = \{ w \in N_{\mathbb{R}} \times \mathbb{R} \mid w = \sum_{i=1}^s q_i(v_i, 1) \text{ for some } 0 < q_i < 1 \},$$

and let $\text{Box}(\emptyset) = \{ 0 \in N_{\mathbb{R}} \times \mathbb{R} \}$. Borisov, Chen and Smith [4] showed that the inertia stack of $\mathcal{Y}$ decomposes into connected components as $\mathcal{I}\mathcal{Y} = \coprod_{F \in T} \prod_{w \in \text{Box}(F) \cap (N \times \mathbb{Z})} Y_w$, where $Y_w = Y$ if $w = 0$ and, if $w \neq 0$, then $|Y_w|$ is isomorphic to the torus-invariant subvariety $V(F)$ of $Y$ corresponding to the cone over $F \times \{1\}$ in $\Delta$. Moreover, if $\psi : N_{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$ denotes projection onto the second co-ordinate, then the age of $Y_w$ is $\psi(w) \in \mathbb{Z}$.

Recall that if $Q$ is a lattice polytope contained in $P$, then $N'$ is the intersection of $N$ with the affine span of $Q$ and the fan $\Delta$ restricts to a fan $\Sigma$ refining the cone $\sigma'$ over $Q \times \{1\}$ in $(N')_{\mathbb{R}} \times \mathbb{R}$. If $\mathcal{Y}'$ denotes the canonical Deligne-Mumford stack with coarse moduli space $Y' = Y'(\Sigma)$, then the inclusion of $N'$ in $N$ induces an inclusion of $\mathcal{Y}'$ as a closed substack of $\mathcal{Y} \times (\mathbb{C}^*)^{\dim P - \dim Q}$ and an inclusion of $\mathcal{Y} \times (\mathbb{C}^*)^{\dim P - \dim Q}$ as an open substack of $\mathcal{Y}$. These inclusions induce a corresponding restriction map $\nu : A_{\text{orb}}^*(\mathcal{Y}, \mathbb{Q}) \to A_{\text{orb}}^*(\mathcal{Y}', \mathbb{Q})$, which we describe below (c.f. Remark 2.2).

If $T|_Q$ denotes the restriction of $T$ to $Q$, then the inertia stack of $\mathcal{Y}'$ decomposes into connected components as $\mathcal{I}\mathcal{Y}' = \coprod_{F \in T|_Q} \prod_{w \in \text{Box}(F) \cap (N \times \mathbb{Z})} Y'_w$, where $Y'_w = Y'$ if $w = 0$ and, if $w \neq 0$, then the age of $Y'_w$ is $\psi(w)$ and $|Y'_w|$ is isomorphic to the torus-invariant subvariety $V(F)'$ of $Y'$ corresponding to the cone over $F \times \{1\}$ in $\Sigma$. For each face $F \in T|_Q$, the inclusion of $N'$ in $N$ induces a closed immersion $V(F)' \hookrightarrow V(F) \times (\mathbb{C}^*)^{\dim P - \dim Q}$ and an open immersion $V(F)' \times (\mathbb{C}^*)^{\dim P - \dim Q} \hookrightarrow V(F)$. The corresponding restriction map $\nu_F : A^*(V(F), \mathbb{Q}) \to A^*(V(F)', \mathbb{Q})$ is surjective since if $W'$ is an irreducible closed subvariety of $V(F)'$ and $W$ denotes the closure
of $W' \times (\mathbb{C}^*)^{\dim P - \dim Q}$ in $V(F)$, then $\nu_F([W]) = [W']$. The restriction map $\iota : A^*_\text{orb}(Y, Q) \to A^*_\text{orb}(Y', Q)$ has the form

$$\iota : \prod_{F \in \mathcal{T}} \prod_{w \in \text{BOX}(F) \cap (N \times Z)} A^*((\mathcal{Y}_w), Q)[\psi(w)] \to \prod_{F \in \mathcal{T}|_Q} \prod_{w \in \text{BOX}(F) \cap (N \times Z)} A^*((\mathcal{Y}'_w), Q)[\psi(w)],$$

where for each $F \in \mathcal{T}$ and $w \in \text{BOX}(F) \cap (N \times Z)$, $\iota$ restricts to $\nu_F$ (with a grading shift) on $A^*((\mathcal{Y}_w), Q)[\psi(w)]$ if $F \subseteq Q$ and restricts to zero otherwise. One can verify from the description of the ring structure of an orbifold Chow ring in [1] that $\iota$ is a ring homomorphism. We conclude that $\iota$ is a surjective ring homomorphism, thus establishing Lemma 2.2.

**Remark 2.1.** The dimensions of the graded pieces of $A^*(V(F), Q)$ are equal to the coefficients of an $h$-vector of a fan [3 Corollary 2.12]. The analogous combinatorial proof of Stanley’s theorem goes as follows: one can express $\delta_P(t)$ and $\delta_Q(t)$ as sums of shifted $h$-vectors [3 11], and then apply Stanley’s monotonicity theorem for $h$-vectors [13] to conclude the result.

**Remark 2.2.** Consider the deformed group ring $Q[N \times Z]^\Delta := \bigoplus_{v \in \sigma \cap (N \times Z)} Q \cdot y^v$, with ring structure defined by

$$y^v \cdot y^w = \begin{cases} y^{v+w} & \text{if there exists a cone } \tau \in \Delta \text{ containing } v \text{ and } w \\ 0 & \text{otherwise.} \end{cases}$$

If $v_1, \ldots, v_s$ denote the vertices of $T$ and $M = \text{Hom}_Z(N, Z)$, then Jiang and Tseng [9 Theorem 1.1] showed that there is an isomorphism of rings

$$A^*_\text{orb}(Y, Q) \cong \frac{Q[N \times Z]^\Delta}{\left\{ \sum_{i=1}^s ((v_i, 1), u)y^{(v_i, 1)} \mid u \in M \times Z \right\}}.$$  

Similarly, if $v_1, \ldots, v_s$ are the vertices of $T|_Q$ and $M' = \text{Hom}_Z(N', Z)$, then

$$A^*_\text{orb}(Y', Q) \cong \frac{Q[N' \times Z]^\Sigma}{\left\{ \sum_{i=1}^s ((v_i, 1), u)y^{(v_i, 1)} \mid u \in M' \times Z \right\}}.$$  

Consider the surjective ring homomorphism $j : Q[N \times Z]^\Delta \to Q[N' \times Z]^\Sigma$ satisfying $j(y^v) = y^w$ if $v \in \Sigma$ and $j(y^v) = 0$ if $v \notin \Sigma$. The induced ring homomorphism

$$\frac{Q[N \times Z]^\Delta}{\left\{ \sum_{i=1}^s ((v_i, 1), u)y^{(v_i, 1)} \mid u \in M \times Z \right\}} \to \frac{Q[N' \times Z]^\Sigma}{\left\{ \sum_{i=1}^s ((v_i, 1), u)y^{(v_i, 1)} \mid u \in M' \times Z \right\}}$$

corresponds to the restriction map $\iota : A^*_\text{orb}(Y, Q) \to A^*_\text{orb}(Y', Q)$ under the above isomorphisms. The existence of such a ring homomorphism was used by Stanley in his original commutative algebra proof of Theorem 2.1 [13].

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