The Structure of The Group of Polynomial Matrices
Unitary In The Indefinite Metric of Index 1

Boris D. Lubachevsky

bdl@bell-labs.com

Bell Laboratories
600 Mountain Avenue
Murray Hill, New Jersey

Abstract

We consider the group $M$ of all polynomial $\nu \times \nu$ matrices $U(\omega) = \sum_{\kappa=0}^{\nu} U_{i} \omega^{i}$, with diagonal $\nu \times \nu$ matrix $D = \text{diag}\{-1, 1, 1, ..., 1\}$. Here $\nu \geq 2$, $U(\omega)^{*} = \sum_{\kappa=0}^{\nu} U_{i}^{*} \omega^{i}$ and $U_{i}^{*}$ is the Hermitean conjugate of $U_{i}$. We show that the subgroup $M_{0}$ of those $U(\omega) \in M$, that are normalized by the condition $U(0) = I$, is the free product of certain groups $M_{z}$ where $z$ is a $\nu$-vector drawn from the set $\Xi \overset{\text{def}}{=} \{z = \text{column}\{\zeta_{1}, \zeta_{2}, ..., \zeta_{\nu}\} | \zeta_{1} = 1, z^{*}Dz = 0\}$. Matrices in each $M_{z}$ are explicitly and uniquely parametrized in the paper. Thus, every $\nu \times \nu$ matrix $U = U(\omega) \in M_{0}$ can be represented in the form $U = G_{z_{1}} \cdot G_{z_{2}} \cdot ... \cdot G_{z_{\eta}}$ with $\nu \times \nu$ polynomial matrix multiples $G_{z_{i}} \in M_{z_{i}}$, $z_{i} \in \Xi$, for $i = 1, ..., \eta$, so that $z_{i} \neq z_{i+1}$ for $i = 1, ..., \eta-1$ and this representation is unique. The uniqueness includes the number of multiples $\eta$ where $\eta = 0, 1, 2, ...$, their particular sequence $G_{z_{1}}, G_{z_{2}}, ..., G_{z_{\eta}}$, and the multiples themselves with their respective parametrizations; all these items can be defined in only one way once the $U$ is given.

Key words: polynomial matrix, unitary, indefinite metric, free product, group, factorization

AMS subject classification: primary ??, secondary ??
1. Let

\[ U(\omega) = U_0 + U_1 \omega + U_2 \omega^2 + \cdots + U_\kappa \omega^\kappa \]  

(1)

be a polynomial matrix (p.m.) of size \( \nu_1 \times \nu_2 \). Here \( U_i \) for \( i = 0, 1, \ldots, \kappa \), are complex \( \nu_1 \times \nu_2 \) matrices, \( \omega \) is a scalar variable, and \( \kappa = 0, 1, \ldots \). Under assumption that variable \( \omega \) takes on only real values, \( \omega^* = \omega \), we extend the Hermitian conjugation operation \( * \) to a p.m. \( U(\omega) \) as follows:

\[ U(\omega)^* \overset{\text{def}}{=} U_0^* + U_1^* \omega + U_2^* \omega^2 + \cdots + U_\kappa^* \omega^\kappa \].

For \( \nu \geq 2 \), let \( D \) be the diagonal \( \nu \times \nu \) matrix of the form \( \omega = \text{diag}\{-1, 1, \ldots\} \). In this paper we shall describe the group \( \mathcal{M} \) of all p.m.’s \( U(\omega) \) of size \( \nu \times \nu \) which satisfy the equation

\[ U(\omega) \cdot B \cdot U(\omega)^* = D, \]  

(2)

i.e., the group of p.m.’s unitary in the indefinite metric of index 1. The description is effective in that every matrix in \( \mathcal{M} \) is parametrized in a unique canonical form. The main results are stated in §2, 3. This problem relates to the problem \( \mathbb{1} \) of finding a factorization of a p.m. \( A(\omega) = A(\omega)^* \) in the form

\[ A(\omega) = U(\omega) \cdot C \cdot U(\omega)^* \]  

(3)

with a p.m. \( U(\omega) \) and a constant matrix \( C = C^* \), both of size \( \nu \times \nu \). Note that matrix \( C \) in (3) is not necessarily positive definite. As a special case, one might seek to factorize a constant matrix \( A(\omega) = C = C^* \), \( \det C \neq 0 \). In this case, without loss of generality, we can assume \( C \) to be a diagonal matrix

\[ C = \text{diag}\{\pm 1, \pm 1, \ldots, \pm 1\}. \]  

(4)

The structure of the set of the solutions \( U(\omega) \) of the equation (3), where \( A(\omega) \equiv C \), depends on how many \( +1 \)'s and \( -1 \)'s are on the diagonal of \( C \). Indeed, if the elements of the diagonal of \( C \) are all of the same sign, then equation (3), where \( A(\omega) \equiv C \), coupled with the normalizing condition \( \omega(0) = I \), has only the trivial solution \( U(\omega) \equiv I \). The set of the solutions becomes non-trivial when \( C \) has both \( +1 \)'s and \( -1 \)'s on the diagonal. In this paper we consider the case of exactly one \( -1 \) with the rest of diagonal elements being \( +1 \). The same set of the solutions \( U \) will result if \( D \) is substituted with \( -D \), a diagonal matrix that has exactly one \( +1 \) with the rest of diagonal elements being \( -1 \). These two cases exhaust the set of all indefinite matrices \( C \) of the form (4) in dimensions \( \nu \leq 3 \). Without loss of generality, we assume that the diagonal matrix \( C \) has the single \( -1 \) being at the top of the diagonal.

\footnote{For a constant \( \nu_1 \times \nu_2 \) matrix \( B \), its Hermitian conjugate \( B^* \) is defined as the \( \nu_2 \times \nu_1 \) transpose of the matrix whose elements are complex conjugates of the elements of \( B \). If \( B \neq 0 \) in (1) then \( \kappa \) is the degree of \( U(\omega) \), denoted \( \kappa = \text{deg} U(\omega) \). The following notations and conventions are assumed. Greek lower case denotes scalars, Latin lower case denotes column vectors of height \( \nu \), e.g., \( z = \text{column}\{\zeta_1, \zeta_2, \ldots, \zeta_\nu\} \). Latin capitals denote matrices, \( i = \sqrt{-1} \), \( I \) is the identity \( \nu \times \nu \) matrix. The symbol \( \ni \) is read “such that”. The notation \( \{\Gamma_1 \mid \Gamma_2\} \) is used for the set with elements \( \Gamma_1 \) satisfying the defining property \( \Gamma_2 \).}
and the rest of the diagonal consists of +1's, i.e., $C = D$ for the matrix $D$ defined above. The task of finding a symmetric factorization (3) recurs in many contexts, for example in the synthesis of linear optimal control in differential games [2].

2. Introduce the sets:

of vectors

$$\Xi \overset{\text{def}}{=} \{ z = \text{column}\{\zeta_1, \ldots, \zeta_\nu\} \mid \zeta_1 = 1, z^*Dz = 0 \} ,$$

$$\Delta_0^z \overset{\text{def}}{=} \{ d \mid d^*z = d^*Dz = 0 \} , \text{for any } z \in \Xi ,$$

of vector polynomials

$$\Delta_z \overset{\text{def}}{=} \{ g = g_1\omega + g_2\omega^2 + \ldots \mid g_i \in \Delta_0^z \} , \text{for any } z \in \Xi ,$$

and of scalar polynomials

$$\Phi \overset{\text{def}}{=} \{ \phi = \phi(\omega) \mid \phi(0) = 0, \phi(\omega)^* = -\phi(\omega) \} .$$

Then:

1) the set of generatrices of the cone

$$\Lambda \overset{\text{def}}{=} \{ x \mid x^*Dx = 0 \}$$

is parametrized in a one-to-one way by $\Xi$;

2) $\Delta_z \neq 0$ only if $\nu > 2$ and if $g \in \Delta_z$ then $g = Dg$;

3) an element $\phi \in \Phi$ is of the form $\phi(\omega) = i \sum_{1 \leq i \leq \kappa} \rho_i \omega^i$, where $i = \sqrt{-1}$, the $\rho_i$ are real numbers, and $\kappa = 1, 2, \ldots$.

For $z \in \Xi, \phi \in \Phi, g \in \Delta_z$ define the following $\nu \times \nu$ polynomial matrix

$$G_z(\phi, g) \overset{\text{def}}{=} D[z(\phi - (1/2) \cdot g^*g)z^* + (zg^* - gz^*)] + I . \quad (5)$$

The significance of the matrix in (5) is that, as will be shown, any matrix $U \in \mathcal{M}$ can be decomposed into p.m.’s of this form and a constant matrix.

Define the sets of matrices:

$$\mathcal{M}_z \overset{\text{def}}{=} \{ G_z(\phi, g) \mid \phi \in \Phi, g \in \Delta_z \} \text{ for any } z \in \Xi ,$$
\[ M_0 \overset{\text{def}}{=} \{ U(\omega) \in M \mid U(0) = I \}, \]
\[ N \overset{\text{def}}{=} \{ V \in M \mid \deg V = 0 \}, \]
\[ \Upsilon \overset{\text{def}}{=} \{ W \in N \mid W = \text{diag}\{1, L\}, L \in \{ L \mid L^*L = I_{(\nu-1) \times (\nu-1)} \} \}. \]

Clearly, every p.m. \( U(\omega) \in M \) may be represented uniquely in the form \( U(\omega) = U_0(\omega) \cdot V \) where \( U_0(\omega) \in M_0, V \in N \).

The main result on the structure of \( M \) is the following

**Theorem 1.** 1). For every \( z \in \Xi \), the set \( M_z \) is a group under matrix multiplication. The group \( M_0 \) is the free product of groups \( M_z \) for all \( z \in \Xi \). This means that any \( U = U(\omega) \in M_0 \) can be decomposed in the form

\[ U = G_{z_1}(\phi_1, g_1) \cdot \ldots \cdot G_{z_\eta}(\phi_\eta, g_\eta), \] (6)

where \( G_{z_i} \in M_{z_i} \) for \( i = 1, \ldots \eta \). Moreover, if by aggregating the consecutive multiples in the sequence in (6) that belong to the same group we make sure that \( z_i \neq z_{i+1}, i = 1, \ldots \eta - 1 \), then the obtained decomposition (6) becomes unique for a given \( U \). The uniqueness includes the number of multiples \( \eta \), the particular sequence \( z_1, z_2, \ldots z_\eta \) in (6) with the sequence \( M_{z_1}, M_{z_2}, \ldots M_{z_\eta} \) of the corresponding groups, and the multiples \( G_{z_i} \in M_{z_i} \) themselves, each of which is a unique p.m. for a given \( U(\omega) \in M_0 \).

2). Multiplication in each \( M_z, z \in \Xi \), satisfies the condition

\[ G_z(\phi, g) \cdot G_z(\psi, h) = G_z(\phi + \psi + (1/2) \cdot (h^*g - g^*h), g + h), \] (7)

where \( \phi, \psi \in \Phi, g, h \in \Delta_z \).

3). Different groups \( M_z \) are isomorphic; the connecting isomorphism is as follows

\[ W \cdot G_z(\phi, g) \cdot W^{-1} = G_{Wz}(\phi, Wg), \] (8)

where \( W \in \Upsilon, z \in \Xi, \phi \in \Phi, g \in \Delta_z \), and necessarily \( W^{-1} \in \Upsilon, Wz \in \Xi \) and \( Wg \in \Delta_{Wz} \).

4). The mapping \( (z, \phi, g) \mapsto G_z(\phi, g) \) is one-to-one on the set

\[ \{(z, \phi, g) \mid z \in \Xi, \phi \in \Phi, g \in \Delta_z, \phi \neq 0 \text{ or } g \neq 0\} \]

If \( U(\omega) \) is decomposed in the form (6) with \( z_i \neq z_{i+1}, i = 1, \ldots \eta - 1 \), then

\[ \deg U(\omega) = \deg G_{z_1}(\phi_1, g_1) + \ldots + \deg G_{z_\eta}(\phi_\eta, g_\eta) = \deg(\phi_1 - g_1^*g_1) + \ldots + \deg(\phi_\eta - g_\eta^*g_\eta). \]

Here \( z_i \in \Xi, \phi_i \in \Phi, g_i \in \Delta_{z_i}, i = 1, \ldots \eta \).
5). The center of \( M_z \), \( z \in \Xi \), is the set \( \Psi_z \) defined as \( \{ G_z(\phi, 0) \mid \phi \in \Phi \} \). If \( \nu > 2 \), then the commutant of \( M_z \) coincides with the center of \( M_z \). If \( \nu = 2 \), then the group \( M_z \) is commutative and coincides with \( \Psi_z \).

3. Real cases. Let \( M' \) be the subgroup of \( \mathcal{M} \) which consists of p.m.'s with real coefficients. Let \( \Xi', \Phi', \Delta_z', z \in \Xi' \) be the real analogs of the sets \( \Xi, \Phi, \Delta_z \) introduced in §2. Clearly, \( \Phi' = \{0\} \). As in §2, let us define \( M'_z \) and \( M_0 \). The structure of the group \( M_0 \) is described in the following

Theorem 2. If \( \nu = 2 \), then the group \( M_0 \) is trivial. If \( \nu > 2 \), then the group \( M_0 \) is not trivial and is the free product of groups \( M_z \) for all \( z \in \Xi' \). Groups \( M'_z \), \( z \in \Xi' \), are commutative.

It is also interesting to describe the subgroup of p.m.'s in \( \mathcal{M} \) with real coefficients not with respect to variable \( \omega \), but with respect to variable \( \lambda = i\omega \). Note, that whereas \( \omega^* = \omega \), we have \( \lambda^* = -\lambda \). Thus, a p.m. \( \mathfrak{I} \) has real coefficients with respect to variable \( \omega \) if all \( U_i \) are real matrices; a p.m. \( \mathfrak{I} \) has real coefficients with respect to variable \( \lambda \) if \( U_i \) are real for even \( i \) and \( iU_i \) are real for odd \( i \). As above, we introduce \( \Phi^\nu = \{ \phi = \phi(\lambda) \mid \phi(0) = 0, \phi^* = -\phi \} \). The general form of such a \( \phi \) is \( \phi(\lambda) = \rho_1\lambda + \rho_2\lambda^3 + \rho_3\lambda^5 + \ldots \), where \( \rho_i \) are real numbers, and the sum is finite. In the \( \omega \) representation the general form of a \( \phi \in \Phi^\nu \) is \( i(\rho_1\omega + \rho_2\omega^3 + \rho_3\omega^5 + \ldots) \), with real \( \rho_i \). We define \( \Delta'_z \), \( M'_z \) for \( z \in \Xi'' \) and also \( M'_0 \).

Theorem 3. 1). The group \( M'_0 \) is a free product of groups \( M'_z \) for all \( z \in \Xi'' \).

2). If \( \nu = 2 \), then \( \Xi'' \) consists of two elements \( \Xi'' = \{z_1, z_2\} \), \( z_1 = \text{column} \{1, 1\} \) and \( z_2 = \text{column} \{1, -1\} \), so that the free product just mentioned contains two groups \( M'_z \) only, and each of these two groups is commutative;

3). If \( \nu > 2 \), then each group \( M'_z \) is not commutative. The commutant of \( M'_z \) coincides with the center and is equal to \( \{ G_z(\phi, 0) \mid \phi \in \Phi^\nu \} \).

As an example of application of Theorem 3, let us present a general form of a \( 2 \times 2 \) p.m. \( U(\lambda) \in M'_0 \) of degree at most 2. We have: either \( U(\lambda) = G_{z_1}(\alpha\lambda) \cdot G_{z_2}(\beta\lambda) \) or \( U(\lambda) = G_{z_2}(\alpha\lambda) \cdot G_{z_1}(\beta\lambda) \). Here \( z_1 \) and \( z_2 \) are the two elements in \( \Xi' \) when \( \nu = 2 \); \( \alpha\lambda \) and \( \beta\lambda \) are \( \lambda \)-monomials in \( \Phi^\nu \) with real coefficients \( \alpha \) and \( \beta \). Each monomial is of degree 1, if the coefficient is non-zero. Otherwise the monomial is zero. We can go further and represent these two types of \( U(\lambda) \) in the per-component form. In the first case, we calculate that

\[
U(\lambda) = \begin{bmatrix}
-2\alpha\beta\lambda^2 - (\alpha + \beta)\lambda + 1 & 2\alpha\beta\lambda^2 - (\alpha - \beta)\lambda \\
2\alpha\beta\lambda^2 + (\alpha - \beta)\lambda & -2\alpha\beta\lambda^2 + (\alpha + \beta)\lambda + 1
\end{bmatrix},
\]

and in the second case,

\[
U(\lambda) = \begin{bmatrix}
-2\alpha\beta\lambda^2 - (\alpha + \beta)\lambda + 1 & -2\alpha\beta\lambda^2 + (\alpha - \beta)\lambda \\
-2\alpha\beta\lambda^2 - (\alpha - \beta)\lambda & -2\alpha\beta\lambda^2 + (\alpha + \beta)\lambda + 1
\end{bmatrix}.
\]
Proofs. Let $a \parallel b$ denote the existence of a linear dependence between vectors $a$ and $b$; we shall write $a \parallel b$ to mean that $a \parallel b$ and, moreover, that the coefficients of the nontrivial linear combination of $a$ and $b$ can be chosen real. The proofs of Lemmas 1 to 2 below present no difficulties.

**Lemma 1.** $(a, b \in \Lambda) \iff ((a^*Db = 0) \iff (a \parallel b))$.

**Lemma 2.** $(ab^* = ba^*) \iff (a \parallel b)$.

**Lemma 3.** $(XD^*X = X^*DX = 0) \implies (\exists y, z \in \Xi, \alpha \ni X = \alpha Dyx^*)$.

The lemma is proved by applying Lemma 1 to the rows and, separately, columns of $X$.

**Lemma 4.** Let $y, z \in \Xi$ and $a$) $X^*D = 0$; $b) \forall k \in G$ $(XDk \parallel Dy)$. Then $\exists s \in G$, $r \in G$, $\alpha \ni X = D(r^* - y^*s^* + \alpha yz^*)$.

**Lemma 5.** Let $z_0 \in \Xi$, $i = 1, \ldots, \eta$. Then

$$(Dz_0^* \cdot Dz_1^* \cdots \cdot Dz_\eta^* = 0) \iff (\exists \iota_0(1 \leq \iota_0 < \eta) \ni z_{\iota_0} = z_{\iota_0+1}).$$

The product of diadic matrices $Dz_0 \cdots Dz_\eta$, $i = 1, \ldots, \eta$, in the lemma is equal to $(Dz_0z_\eta^*)^\xi$, where $\xi = (z_0^*Dz_0) \cdot \cdots \cdot (z_\eta^*Dz_\eta)$. But $Dz_0z_\eta^* \neq 0$ and the condition $\xi = 0$ is equivalent to the existence of an $i$, $1 \leq i \leq \eta$, such that $z_i^*Dz_i = 0$. Since $z_i, z_{i+1} \in \Lambda$, it follows from Lemma 1 that $z_i^*z_{i+1}$; hence, in view of the normalization, $z_i = z_{i+1}$, and the implication is true. Obviously the $\iff$ implication is also true.

**Lemma 6.** Let $\phi \in \Theta$, $g \in G$, $z \in \Xi$ and suppose $\phi$ and $g$ are not both zeros. Then 1) the leading coefficient of a p.m. $G_z(\phi, g)$ is proportional to $Dzz^*$; 2) $\deg G_z(\phi, g) = \deg (\phi - g^*g) > 0$.

The most laborous is the proof of

**Lemma 7.** If $\deg U(\omega) > 0$, then the degree of the p.m. $U(\omega) \in M$ may be decreased by a right or left multiplication by a p.m. of the form $1$ (and real if $U(\omega)$ is real).

**Proof of Lemma 7.** Let $U(\omega) = \sum_{\iota=0}^\kappa X_{\kappa-\iota} \omega^\iota$, where $X_{\iota}, \iota = 0, 1, \ldots, \kappa$, are complex (or real, or $\lambda$-real, depending on the case) constant $\nu \times \nu$ matrices, and $X_0 \neq 0$, so that $\deg U(\omega) = \kappa$. It follows from (2) that $U(\omega)^* \cdot D \cdot U(\omega) = D$. Extending definition of $X_{\iota}$ for $\iota > \kappa$ to be null matrices, we obtain a family of equalities

$$X_0 \cdot D \cdot X_{\gamma} + X_1 \cdot D \cdot X_{\gamma-1} + \ldots + X_\gamma \cdot D \cdot X_0^* = 0 \quad (9_{\gamma})$$

$$X_0^* \cdot D \cdot X_{\gamma} + X_1^* \cdot D \cdot X_{\gamma-1} + \ldots + X_\gamma^* \cdot D \cdot X_0 = 0 \quad (10_{\gamma})$$

Here $\gamma = 0, 1, \ldots, 2\kappa - 1$. Having (90) and (100) we can apply Lemma 3 to $X = X_0$. For the leading coefficient $X_0$, this yields its diadic representation $X_0 = a_0Dyz^*$ for some vectors
Using (12) it is easy to verify that 
\[ \text{deg } U(\omega) = \kappa. \]

The \( y \) and \( z \) will be our candidates for the subscript of the p.m. of the form (8) which should decrease the degree of \( U(\omega) \) after multiplying \( U(\omega) \) on the left or on the right, respectively. Specifically, we will prove the lemma if we show that at least one of the two possibilities holds: 
I) \( \exists \phi \in \Phi, \exists g \in \Delta_2 \ni \text{deg } [U(\omega) \cdot G_2(\phi, g)] < \kappa, \)
II) \( \exists \psi \in \Phi, \exists h \in \Delta_y \ni \text{deg } [G_y(\psi, h) \cdot U(\omega)] < \kappa. \)
(In these statements, sets \( \Xi, \Phi, \Delta_x, \Delta_y \) should be appropriately modified in the cases of reals.)

For integer positive numbers \( \tau, \mu, \) and \( \xi, \) such that \( \tau \leq \mu \) and \( \tau \leq \xi, \) consider the following conditions:
A) \( \exists \alpha_i, 0 \leq i \leq \tau - 1, \ni X_i = \alpha_i Dyz^*, \) i.e., the first \( \tau \) leading coefficients of p.m. \( U(\omega) \) are diadic matrices proportional to \( X_0; \)
B) \( X_\mu Dz = 0 \) for all \( i, \tau \leq i \leq \mu - 1, \) and \( X_\mu Dz \neq 0, \) i.e., the \( \mu - \tau \) coefficients of p.m. \( U(\omega), \) that follow the last coefficient \( X_{\tau - 1} \) mentioned in A), turn into 0, when multiplied on the right by the vector-column \( Dz, \) but this does not happen for the \( \mu - \tau + 1 \)st coefficient;
C) \( y^*X_i = 0, \) for all \( i, \tau \leq i \leq \xi - 1, \) and \( y^*X_\xi \neq 0, \) i.e., the \( \xi - \tau \) coefficients of p.m. \( U(\omega), \) that follow the last coefficient \( X_{\xi - 1} \) mentioned in A), turn into 0, when multiplied on the left by the vector-row \( y^*, \) but this does not happen for the \( \xi - \tau + 1 \)st coefficient;
D) \( \exists s \in \Delta_0^0, r \in \Delta_0^r, \alpha_r \ni X_r = D(rz^* - ys^* + \alpha_r z^*), \) i.e., the coefficient \( X_r \) is the sum of three diadic matrices as stated.

Consider the largest \( \tau \ni A). \) Since \( X_0 = \alpha_0 Dyz^*, \) the \( \tau \) is at least 1. It can not be larger than \( \kappa, \) though. Hence, \( 1 \leq \tau \leq \kappa. \) Let \( \mu \ni B), \xi \ni C). \) Clearly, \( \tau \leq \mu, \xi \leq \kappa. \) The way the proof proceeds further depends of whether the two inequalities
\[ \mu \geq 2\tau \text{ and } \xi \geq 2\tau \]
both hold or not. First we consider the easier

**Case 1: at least one inequality in (11) fails.** Suppose, for example, that \( \mu < 2\tau. \) We will then verify I). Equality (9\( \mu \)) and Lemma 2 imply that there exists a real number \( \rho \) such that
\[ \alpha_0 Dy + i\rho X_\mu Dz = 0. \] (12)

Using (12) it is easy to verify that \( \text{deg } [U(\omega) \cdot (Dz(i\rho \omega^\mu)z^* + I)] < \kappa, \) i.e., as stipulated in I), the degree decreases when \( U(\omega) \) is multiplied on the right by \( G_z(\phi, g) \) with \( \phi(\omega) \) and \( g(\omega) \) taken here as \( \phi(\omega) = i\rho \omega^\mu, g(\omega) = 0. \) (When \( z, y \) and \( X_\mu \) are real, since \( \rho \) is real, (12) implies that \( i\alpha_0 \) is real and hence all elements of \( iX_0 \) are real. It follows, that if \( U(\omega) \) has only real coefficients, inequality \( \mu < 2\tau \) can not occur. In case of reals with variable \( \lambda = i\omega, \) inequality \( \mu < 2\tau \) can only occur for odd \( \mu. \) Analogously we verify II) if the second inequality in (11) fails.

**Case 2: both inequalities in (11) hold.** This case will be more laborous. First, we will verify D) by using Lemma 4 where \( X = X_r. \) The conditions a) and b) in Lemma 4 are obviously satisfied. Let us verify the condition c). Pick a vector \( k \in \Delta_0^0 \) and denote \( v \ni X_r Dk. \)
Condition c) is then rewritten as $v \| Dy$. To verify the latter, we are going to use Lemma 1 with $a = Dy$, $b = v$. Obviously $a \in \Lambda$, so we have to check that also $\Gamma') b \in \Lambda$, i.e., $v^* D v = 0$, and that $\Pi') a^* D b = 0$, i.e., $y^* v = 0$.

Multiply the equality (9.2) by $Dk$ on the right and by $k^* D$ on the left. It follows from (11) that on the left-hand side all the summands but one will turn to zero. In other words, $\Gamma'$ holds. We also have $y^* v = (y^* X_\tau D k = 0$, that is, $\Pi'$ holds.

Now, as the pre-conditions of Lemma 4 are satisfied, we obtain representation

$$X_\tau = D (r z^* - y s^* + \alpha y z^*) .$$

(13) Observe, that vectors $s$ and $r$ can not be both zero in (13), because it would have contradicted to the definition of $\tau$ which could have been possible to increase in such a case. Let us assume that $s \neq 0$ in (13) and show $\Gamma$). If $r \neq 0$ in (13) we can similarly show $\Pi$).

Define $w \overset{\text{def}}{=} X_{2\tau} D z$, $p \overset{\text{def}}{=} \alpha_0 D y$. Using Lemma 1, we will show that $w \| p$. Obviously $p^* D p = 0$ and we will also show that $\Gamma'' w^* D w = 0$ $\Pi'' p^* D w = 0$ which are the preconditions in Lemma 1 applied to $a = w$, $b = p$.

Using (10.4) immediately yields $\Gamma''$. To obtain $\Pi''$, we begin with (10.2) which we multiply on the right at $D z$. This yields $\alpha_0^* z y^* w = 0$ which implies $\Pi''$). Now we can apply Lemma 1 and obtain

$$w = (\sigma_0 + i \rho_0) \cdot p ,$$

(14) where $\sigma_0$ and $\rho_0$ some real numbers. (In the case of reals with respect to variable $\omega$ the resulting $\rho_0$ will be zero, in the case of reals with respect to variable $\lambda$ the resulting $\rho_0$ will be zero for $????$).

We can now use the obtained expressions for substituting in (9.2)

$$X_0 = p z^* , \quad X_{2\tau} = (\sigma_0 + i \rho_0) \cdot p \quad X_\tau = -p s_0 + n z^*$$

where we denoted $n \overset{\text{def}}{=} D (r + \alpha_\tau y)$ and $s_0 \overset{\text{def}}{=} s / \alpha_0^*$. We then obtain an equation $p p^* (2 \sigma_0 + s_0^* D s_0) = 0$ from which, taking into account equality $s_0^* D s_0 = |s_0|^2$, we obtain $\sigma_0 = -(1/2) |s_0|^2$.

We are now ready to verify $\Gamma$). Set $\phi(\omega) \overset{\text{def}}{=} i \rho \omega^{2r}$ and $g(\omega) \overset{\text{def}}{=} d \omega^r$, where $d \in \Delta_0^z$ and real $\rho$ are constants to be defined. The definition should satisfy condition

$$X_0 + X_{2\tau} D z z^* (i \rho - (1/2)|d|^2) + X_\tau D (z d^* - d z^*) \overset{\text{def}}{=} p z^* \epsilon = 0 .$$

(15) Parameter $\epsilon$ which is defined in (17) is equal to

$$\epsilon = 1 + s_0^* d + (i \rho_0 - (1/2) |s_0|^2) \cdot (i \rho - (1/2)|d|^2).$$
Our goal is to set parameters $\rho$ and $d$ so that $\epsilon$ will become zero. We are looking for $d$ of the form $d = \theta s_0$, where $\theta_0$ is a constant to be defined. It follows, that $\epsilon = \epsilon_1 - i \cdot (1/2)|s_0|^2 \cdot \epsilon_2$, where $\epsilon_1 = |1 + (1/2)\theta \cdot |s_0|^2|^2 - \rho \cdot \rho_0$ and $\epsilon_2 = \rho_0 \cdot |\theta|^2 - 2 \cdot \text{Im}\theta + \rho$. If $\rho_0 = 0$, we set $\rho = 0$ and $\theta = -2|s_0|^{-2}$ to obtain $\epsilon = 0$. If $\rho_0 \neq 0$, then by setting $\rho = |1 + (1/2)\theta \cdot |s_0|^2|^2/\rho_0$ we obtain $\epsilon_1 = 0$. Such setting of $\rho$ results in

$$\epsilon_2 = \rho_0^{-1}\{1 + |\theta|^2 \cdot [\rho_0^2 + ((1/2)|s_0|^2)^2] - 2[\rho_0 \cdot \text{Im}\theta - (1/2)|s|^2) \cdot \text{Re}\theta]\}$$

(16)

Now setting

$$\theta = (i\rho_0 - (1/2)|s_0|^2) \cdot (\rho_0^2 + ((1/2)|s_0|^2)^2)^{-1}$$

results in $\epsilon_2 = 0$ as required. Lemma 5 is proved.

**Proof of Theorem 1.** Let us first establish (16). We have

$$G_z(\phi, g) \cdot G_z(\psi, h) = (DA + I)(DB + I) = DADB + D(A + B) + I,$$

where $A = z(\phi - (1/2)\cdot g^*g)z^* + (zg^* - gz^*)$ and $B = z(\psi - (1/2)\cdot h^*h)z^* + (zh^* - hz^*)$ according to (16). Using identities $z^*Dz = 0$, $z^*Dh = 0$, and $g^*Dz = 0$, that follow from the definitions, and multiplying each summand in $A$ by each summand in $B$ on the right, we establish that the only non-zero term in the resulting sum for $DADB$ is $-Dzg^*Dhz^* = -Dz(g^*h)z^*$ where $Dh$ is substituted with $h$ since $h \in \Delta_z$. Using the identity,

$$(g + h)^*(g + h) - (h^*g - g^*h) = g^*g + h^*h + 2g^*h,$$

the sought product then becomes

$$G_z(\phi, g) \cdot G_z(\psi, h) = -Dzg^*hz^* + D(A + B) + I =$$

$$= D[z(\phi + \psi - (1/2) \cdot (g^*g + h^*h - 2g^*h))z^* + (z(g + h)^* - (g + h)z^*)$$

$$= G_z(\phi + \psi + (1/2) \cdot (h^*g - g^*h), g + h)$$

as required in (16). Similar and even simpler arguments show that for $U = G_z(\phi, g)$ (2) holds. Hence sets $\mathcal{M}_z$ for $z \in \Xi$ are indeed groups under the matrix multiplication and $\mathcal{M}_z \subset \mathcal{M}_0$ for any $z \in \Xi$.

It is easy to see that $\text{deg} G_z(\phi, g) = \text{deg}(\phi - g^*g)$ unless both $\phi$ and $g$ are identical zeros in which case $G_z(\phi, g)$ is identical to the matrix $I$. Hence if $G_z(\phi, g)$ is not identical to $I$, then the leading coefficient of $G_z(\phi, g)$ is proportional to the diadic matrix $Dzz^*$. If a product of matrices $G_z(\phi, g)$ is formed, like that in the right-hand side of (3) with adjacent multiples belonging to different groups $\mathcal{M}_z$, then the leading term of this product will be proportional to the product of the diadic $Dzz^*$ as in the left-hand side of the statement in Lemma 5. According to that lemma, the leading term can not degenerate to zero. Because
if it did, then a pair of adjacent multiples would have belonged to the same group \( M_z \). Therefore, the products like those in the right-hand side of (6) can never degenerate to the identity matrix \( I \), given that the number of multiples is not zero and that adjacent multiples belong to different groups \( M_z \). Hence the product of group \( M_z \) where \( z \) runs over set \( \Xi \), we temporarily denote this product as \( \tilde{M} \), is free (i.e., has no relations, see, e.g., [3] for the definition). Lemma 7 tells that \( M_0 \subset \tilde{M} \) and obviously we have \( \tilde{M} \subset M_0 \), thus \( \tilde{M} = M_0 \) and statement 1) of theorem 1 is proved.

Acknowledgment. The author is indebted to V. A. Yakubovitch and D.K. Faddeev for their interest in this paper.

References

[1] V. A. Yakubovitch, Dokl. Akad. Nauk SSSR 194 (1970), 532 = Soviet Math. Dokl. 11 (1970), 1261. MR 42 #6012.

[2] V. A. Yakubovitch, Dokl. Akad. Nauk SSSR 195 (1970), 296 = Soviet Math. Dokl. 11 (1970), 1478. MR 42 #7300.

[3] A.G. Kuros, Theory of Groups, 3d ed., “Nauka”, Moscow, 1967; English transl. of 2nd ed., Chelsea, New York, 1960. MR 22 #727; 40 #2740.