THE $\varepsilon$ - REVISED SYSTEM OF THE RIGID BODY WITH THREE LINEAR CONTROLS

Dan COMĂNESCU, Mihai IVAN and Gheorghe IVAN

Abstract. In this paper we introduce the $\varepsilon$ - revised system associated to a Hamilton- Poisson system. The $\varepsilon$ - revised system of the rigid body with three linear controls is defined and some of its geometrical and dynamical properties are investigated.

1 Introduction

It is well known that many dynamical systems can be formulated using a Poisson structure (see for instance, R. Abraham and J. E. Marsden [1] and M. Puta [11]).

The metriplectic systems was introduced by P. J. Morrison in the paper [8]. These systems combine both the conservative and dissipative systems.

A metriplectic system is a differential system of the form $\dot{x} = PdH + gdC$, where $P$ is a Poisson tensor on a manifold $M$, $g$ is a symmetric tensor of type $(2, 0)$ on $M$, and $H$ and $C$ are two smooth functions on $M$ with the additional requirements:

(a) $PdC = 0$; (b) $gdH = 0$ and (c) $dC\cdot gdC \leq 0$.

The differential systems of the form $\dot{x} = PdH + gdC$ which satisfies only the conditions (a) and (b) are called almost metriplectic systems (see Fish, [2]; Marsden, [7]; Ortega and Planas - Bielsa, [9]). An interesting class of almost metriplectic systems are so-called the revised dynamical systems associated to Hamilton-Poisson systems (see Gh. Ivan and D. Oprea, [5]).

The control of the rotation rigid body is one of the problems with a large practical applicability. For this reason, in this paper we study the $\varepsilon$ - revised dynamical system associated to the rigid body with three linear controls.

2 Almost metriplectic systems

We start this section with the presentation of the concept of almost metriplectic manifold (see Ortega and Planas- Bielsa, [9]).

Let $M$ be a smooth manifold of dimension $n$ and let $C^\infty(M)$ be the ring of smooth real-valued functions on $M$.

A Leibniz manifold is a pair $(M, [\cdot, \cdot])$, where $[\cdot, \cdot]$ is a Leibniz bracket on $M$, that is $[\cdot, \cdot]: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a $\mathbb{R}$ - bilinear operation satisfying the following two conditions:

(i) the left Leibniz rule:

$$[f_1 \cdot f_2, f_3] = [f_1, f_3] \cdot f_2 + f_1 \cdot [f_2, f_3] \quad \text{for all } f_1, f_2, f_3 \in C^\infty(M);$$

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We define the map 
\[ [\cdot, (\cdot, \cdot)]_\varepsilon : C^\infty(M) \times (C^\infty(M) \times C^\infty(M)) \to C^\infty(M) \]
by:
\[ [f, (h_1, h_2)]_\varepsilon = P(df, dh_1) + \varepsilon g(df, dh_2), \quad \text{for all } f, h_1, h_2 \in C^\infty(M). \quad (1) \]

**Proposition 2.1.** The map 
\[ [\cdot, (\cdot, \cdot)]_\varepsilon \]
given by (1) satisfy the following relations:

(i) \[ [af_1 + bf_2, (h_1, h_2)]_\varepsilon = a[f_1, (h_1, h_2)]_\varepsilon + b[f_2, (h_1, h_2)]_\varepsilon; \]

(ii) \[ [f, a(h_1, h_2) + b(h_1', h_2')]_\varepsilon = a[f, (h_1, h_2)]_\varepsilon + b[f, (h_1', h_2')]_\varepsilon; \]

(iii) \[ [ff_1, (h_1, h_2)]_\varepsilon = f[f_1, (h_1, h_2)]_\varepsilon + f_1[f, (h_1, h_2)]_\varepsilon; \]

(iv) \[ [f, h(h_1, h_2)]_\varepsilon = h[f, (h_1, h_2)]_\varepsilon + h_1 P(df, dh) + \varepsilon h_2 g(df, dh), \]
for all \( f, f_1, f_2, h_1, h_2, h_1', h_2' \in C^\infty(M) \) and \( a, b \in \mathbb{R} \).

**Proof.** Applying the properties of the differential of functions and using that \( P \) and \( g \) are \( \mathbb{R} \)-bilinear maps, it is easy to establish the relations (i) – (iv). \( \square \)

We consider the map 
\[ [[\cdot, \cdot]]_\varepsilon : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \]
defined by:
\[ [[f, h]]_\varepsilon = [f, (h, h)]_\varepsilon, \quad \text{for all } f, h \in C^\infty(M). \quad (2) \]

Therefore, the map 
\[ [[\cdot, \cdot]]_\varepsilon : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \]
is given by:
\[ [[f, h]]_\varepsilon = P(df, dh) + \varepsilon g(df, dh), \quad \text{for all } f, h \in C^\infty(M). \quad (3) \]

**Proposition 2.2.** The bracket 
\[ [[\cdot, \cdot]]_\varepsilon \]
on \( M \) given by (3) verify the right Leibniz rule:
\[ [[f, hh']]_\varepsilon = h[[f, h']]_\varepsilon + h'[[f, h]]_\varepsilon, \quad \text{for all } f, h, h' \in C^\infty(M). \]

**Proof.** Indeed, \( [[f, hh']]_\varepsilon = [f, (hh', hh')]_\varepsilon = [f, h(h', h')]_\varepsilon \). Putting \( h_1 = h_2 = h' \) in the relation (iv) from Proposition 2.1, we have succesive:
\[ [f, h(h', h')]_\varepsilon = h[f, (h', h')]_\varepsilon + h'P(df, dh) + \varepsilon h'g(df, dh) = h[f, (h', h')]_\varepsilon + h'P(df, dh) + \varepsilon g(df, dh) = h[f, (h', h')]_\varepsilon + h'[f, (h, h)]_\varepsilon = h[[f, h']]_\varepsilon + h'[[f, h]]_\varepsilon. \] \( \square \)

By Proposition 2.1, (i), (ii) and (iii) and Proposition 2.2, we have that the map 
\[ [[\cdot, \cdot]]_\varepsilon \]
given by (3) is a Leibniz bracket on \( M \). Hence, \( [[\cdot, \cdot]]_\varepsilon \) defines a Leibniz structure on the manifold \( M \) and \( (M, P, g, [[\cdot, \cdot]]_\varepsilon) \) is a Leibniz manifold for each \( \varepsilon \in \mathbb{R} \).
A Leibniz manifold \((M, P, g, [[\cdot, \cdot]]_\epsilon)\) such that \(P\) is a skewsymmetric tensor field and \(g\) is a symmetric tensor field is called \textit{almost metriplectic manifold}. In other words, given a skewsymmetric tensor field \(P\) of type \((2, 0)\) and a symmetric tensor field \(g\) of type \((2, 0)\) on a manifold \(M\), we can define an almost metriplectic structure on \(M\).

If the tensor field \(P\) is Poisson and the tensor field \(g\) is nondegenerate, then \((M, P, g, [[\cdot, \cdot]]_\epsilon)\) is a \textit{metriplectic manifold}, see Ortega & Planas - Bielsa \([9]\).

\textbf{Proposition 2.3.} Let \((M, P, g, [[\cdot, \cdot]]_\epsilon)\) be an almost metriplectic manifold. If there exist \(h_1, h_2 \in C^\infty(M)\) such that \(P(df, dh_2) = 0\) and \(g(df, dh_1) = 0\) for all \(f \in C^\infty(M)\), then the bracket \([[\cdot, \cdot]]\) given by \((3)\) satisfies the relation:

\[
[[f, h_1 + h_2]]_\epsilon = [f, (h_1, h_2)]_\epsilon, \quad \text{for all } f \in C^\infty(M).
\]

\textbf{Proof.} Indeed, \([[f, h_1 + h_2]]_\epsilon = P(df, dh_1 + dh_2) + \epsilon g(df, dh_1 + dh_2) = P(df, dh_1 + dh_2) + \epsilon g(df, dh_1) + P(df, dh_2) + \epsilon g(df, dh_2) = P(df, dh_1) + \epsilon g(df, dh_2) = f, (h_1, h_2)]_\epsilon.
\]

Let \((M, P, g, [[\cdot, \cdot]]_\epsilon)\) be an almost metriplectic manifold and let \(h_1, h_2 \in C^\infty(M)\) two functions such that \(P(df, dh_2) = 0\) and \(g(df, dh_1) = 0\) for all \(f \in C^\infty(M)\). The vector field \(X_{h_1, h_2}\) given by:

\[X_{h_1, h_2}(f) = [[f, h_1 + h_2]]_\epsilon \quad \text{for any } f \in C^\infty(M)\]

is called the \textit{Leibniz vector field} associated to the triple \((h_1, h_2, \epsilon)\) on \(M\).

Taking account into Proposition 2.3 and \((1)\), \(X_{h_1, h_2}\) is given by:

\[X_{h_1, h_2}(f) = [f, (h_1, h_2)]_\epsilon = P(df, dh_1) + \epsilon g(df, dh_2), \quad \text{for all } f \in C^\infty(M).
\]

In local coordinates on \(M\), the differential system given by:

\[\dot{x}^i = [x^i, h_1 + h_2]]_\epsilon = [x^i, (h_1, h_2)]_\epsilon
\]

where

\[x^i, (h_1, h_2)]_\epsilon = X_{h_1, h_2}(x^i) = P^{ij}\frac{\partial h_1}{\partial x^j} + \epsilon g^{ij}\frac{\partial h_2}{\partial x^j}, \quad i, j = 1, n
\]

with \(P^{ij} = P(dx^i, dx^j)\) and \(g^{ij} = g(dx^i, dx^j)\), is called the \textit{almost metriplectic system} on \(M\) associated to the Leibniz vector field \(X_{h_1, h_2}\) with the bracket \([[\cdot, \cdot]]\).

We denote the matrix of the tensor fields \(P\) and \(g\) respectively by \(P = (P^{ij})\) and \(g = (g^{ij})\). We have that \(P\) is a skewsymmetric matrix and \(g\) is a symmetric matrix.

We give now a way for to produce almost metriplectic manifolds.

\textbf{Proposition 2.4.} For a skewsymmetric tensor \(P\) of type \((2, 0)\) on a manifold \(M\) and two functions \(h_1, h_2 \in C^\infty(M)\) such that \(P(df, dh_2) = 0\) for all \(f \in C^\infty(M)\), there exists a symmetric tensor \(g\) of type \((2, 0)\) on \(M\) such that \(g(df, dh_1) = 0\) for all \(f \in C^\infty(M)\) and \((M, P, g, [[\cdot, \cdot]]_\epsilon)\) is an almost metriplectic manifold.
Proof. In a system of local coordinates on $M$, let $g = (g^{ij})$ the matrix of the symmetric tensor $g$ which must to be determined. Then, the components $g^{ij}$, $i, j = 1, n$ verify the system of differential equations $g^{ij} \frac{\partial h_1}{\partial x^j} = 0$, $i, j = 1, n$.

In a chart $U$ such that $\frac{\partial h_1}{\partial x^j}(x) \neq 0$, the components $g^{ij}$ are given by:

$$
\begin{align*}
\left\{
\begin{array}{l}
 g^{ii}(x) = -\sum_{k=1, k \neq i}^{n} \left( \frac{\partial h_1}{\partial x^k} \right)^2 \\
 g^{ij}(x) = \frac{\partial h_1}{\partial x^i} \frac{\partial h_1}{\partial x^j}, \text{ for } i \neq j
\end{array}
\right.
\end{align*}
$$

Applying now Proposition 2.3 we obtain the result. \qed

Proposition 2.4 is useful when we consider the $\varepsilon$- revised system of a Hamilton-Poisson system.

For this, let be a Hamilton-Poisson system on $M$ described by the Poisson tensor $P$ having the matrix $P = (P^{ij})$ and by the Hamiltonian function $h_1 \in C^\infty(M)$ with the Casimir function $h_2 \in C^\infty(M)$ (i.e. $P^{ij} \frac{\partial h_2}{\partial x^j} = 0$ for $i, j = 1, n$). The differential equations of the Hamilton-Poisson system are the following:

$$
\dot{x}^i = P^{ij} \frac{\partial h_1}{\partial x^j}, \ i, j = 1, n.
$$

Using (8), we determine the matrix $g = (g^{ij})$ and we have:

$$
\frac{\partial h_1}{\partial x^j} = 0, \ i, j = 1, n.
$$

Applying now Proposition 2.4, for each $\varepsilon \in \mathbb{R}$, we obtain an almost metriplectic structure on $M$ associated to system (9). The differential system associated to this structure is called the $\varepsilon$ - revised system of the Hamilton - Poisson system.

Hence, the $\varepsilon$ - revised system of the Hamilton - Poisson system defined by (9) is:

$$
\dot{x}^i = P^{ij} \frac{\partial h_1}{\partial x^j} + \varepsilon g^{ij} \frac{\partial h_2}{\partial x^j}, \ i, j = 1, n.
$$

The terms $g^{ij} \frac{\partial h_2}{\partial x^j}$, $i, j = 1, n$ from the $\varepsilon$ - revised system (11) describe a cube perturbation of the Hamilton - Poisson system.

Remark 2.1. We observe that the 0- revised system (11) coincide with the Hamilton - Poisson system (9). \qed
3 The \( \varepsilon \)-revised system associated to the rigid body with three linear controls

The rigid body equations with three linear controls (see, M. Puta and D. Comănescu [12]) are given by:

\[
\begin{align*}
\dot{x}^1 &= (a_3 - a_2)x^2x^3 + cx^2 - bx^3 \\
\dot{x}^2 &= (a_1 - a_3)x^1x^3 - cx^1 + ax^3 \\
\dot{x}^3 &= (a_2 - a_1)x^1x^2 + bx^1 - ax^2
\end{align*}
\] (12)

where \( x(t) = (x^1(t), x^2(t), x^3(t)) \in \mathbb{R}^3 \) and \( a_1 = \frac{1}{I_1}, \ a_2 = \frac{1}{I_2}, \ a_3 = \frac{1}{I_3} \) with \( I_1 > I_2 > I_3 > 0 \) (\( I_1, I_2, I_3 \) being the principal moments of inertia of the body) and \( a, b, c \in \mathbb{R} \) are feedback parameters. We have \( 0 < a_1 < a_2 < a_3 \).

The dynamics (12) is described by the Poisson tensor \( \Pi \) and by the Hamiltonian \( H \) on \( \mathbb{R}^3 \) given by:

\[
\Pi(x) = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix},
\]

(13)

\[
H(x) = \frac{1}{2}[a_1(x^1)^2 + a_2(x^2)^2 + a_3(x^3)^2] + ax^1 + bx^2 + cx^3.
\]

(14)

Using (13) and (14), the dynamics (12) can be written in the matrix form:

\[
\dot{x}(t) = \Pi(x(t)) \cdot \nabla H(x(t)),
\]

(15)

where \( \dot{x}(t) = (\dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t))^T \) and \( \nabla H(x(t)) \) is the gradient of the Hamiltonian function \( H \) with respect to the canonical metric on \( \mathbb{R}^3 \).

Therefore, the dynamics (12) has the Hamilton-Poisson formulation \( (\mathbb{R}^3, \Pi, H) \), where \( \Pi \) and \( H \) are given by (13) and (14).

The function \( C \in C^\infty(\mathbb{R}^3) \) given by:

\[
C(x) = \frac{1}{2}[(x^1)^2 + (x^2)^2 + (x^3)^2]
\]

(16)

is a Casimir of the configuration \( (\mathbb{R}^3, \Pi) \), i.e.

\[
C(x) \cdot \nabla H(x) = O.
\]

(17)

Applying the relations (8) for \( P = \Pi, h_1(x) = H(x) \) and \( h_2(x) = C(x) \), the symmetric tensor \( g \) is given by the matrix:

\[
g = \begin{pmatrix}
-(a_2x^2 + b)^2 & -(a_3x^3 + c)^2 & (a_1x^1 + a)(a_2x^2 + b) & (a_1x^1 + a)(a_3x^3 + c) \\
(a_1x^1 + a)(a_2x^2 + b) & -(a_1x^1 + a)^2 & -(a_3x^3 + c)^2 & (a_2x^2 + b)(a_3x^3 + c) \\
(a_1x^1 + a)(a_3x^3 + c) & (a_2x^2 + b)(a_3x^3 + c) & -(a_1x^1 + a)^2 & -(a_2x^2 + b)^2
\end{pmatrix}
\]
the vector form: where \( \times \)

\[
\text{system of the free rigid body, see [5].}
\]

\[
\text{three linear controls. Taking}
\]

\[
\text{For all}
\]

\[
\text{We have}
\]

\[
\text{Using the relations (22) with}
\]

\[
\text{The} \varepsilon \text{- revised system associated to dynamics (12) is:}
\]

\[
\begin{align*}
\dot{x}^1 &= [(a_3 - a_2)x_2^2x_3 + cx_2 - bx_3] + \varepsilon v_1(x) \\
\dot{x}^2 &= [(a_1 - a_3)x_1^2x_3 - cx_1 + ax_2] + \varepsilon v_2(x) \\
\dot{x}^3 &= [(a_2 - a_1)x_1^2x_2 + bx_1 - ax_2] + \varepsilon v_3(x)
\end{align*}
\]

\[
\text{The differential system (20) is called the} \varepsilon \text{- revised system of the rigid body with}
\]

\[
\text{three linear controls. Taking} a = b = c = 0 \text{ and } \varepsilon = 1 \text{ in (20), we obtain the revised}
\]

\[
\text{system of the free rigid body, see [5].}
\]

**Vector writing of the \( \varepsilon \) - revised system (20).** We introduce the following notations:

\[
x = (x^1, x^2, x^3), \quad v = (v_1, v_2, v_3), \quad a = (a, b, c), \quad m(x) = (a_1x^1 + a_2x^2 + b, a_3x^3 + c).
\]

For all \( u = (u_1, u_2, u_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3 \), the following relation holds:

\[
u \cdot [w \times (u \times w)] = (u \times w)^2, \quad \text{with} \quad (u \times w)^2 = (u \times w) \cdot (u \times w)\
\]

where " \( \times \) " and " \( \cdot \) " denote the cross product resp. inner product in \( \mathbb{R}^3 \); that is:

\[
u \times w = (u_2w_3 - u_3w_2, u_3w_1 - u_1w_3, u_1w_2 - u_2w_1), \quad u \cdot w = u_1w_1 + u_2w_2 + u_3w_3.
\]

With the above notations, the dynamics (12) has the vector form:

\[
\dot{x} = x \times m(x).
\]

It is not hard to verify the following equality:

\[
v = (x \times m(x)) \times m(x).
\]

Using the relations (22), (23) and (20), we can written the \( \varepsilon \) - revised system (20) in the vector form:

\[
\dot{x} = x \times m(x) + \varepsilon [ (x \times m(x)) \times m(x) ].
\]
4 The equilibrium points of the $\varepsilon$ - revised system

The equilibrium points of the Hamilton - Poisson system (12) ( or (22) ) are solutions of the vector equation:

$$x \times m(x) = 0.$$  \hspace{1cm} (25)

The equilibrium points of the $\varepsilon$ - revised system (20) ( or (24) ) are solutions of the vector equation:

$$x \times m(x) + \varepsilon[(x \times m(x)) \times m(x)] = 0.$$  \hspace{1cm} (26)

**Theorem 4.1.** The Hamilton - Poisson system (12) and its revised system (20) have the same equilibrium points.

**Proof.** Let $x_0$ be an equilibrium point of the system (12). According with (25) follows $x_0 \times m(x_0) = 0$. We have that $x_0$ is a solution of the vector equation (26), since $x_0 \times m(x_0) + (x_0 \times m(x_0)) \times m(x_0) = 0 + 0 \times m(x_0) = 0$. Hence $x_0$ is an equilibrium point of the $\varepsilon$ - revised system (20).

Conversely, let $x_0$ be an equilibrium point for (20). Using (25) it follows

(a) $x_0 \times m(x_0) + \varepsilon[(x_0 \times m(x_0)) \times m(x_0)] = 0$

The relation (a) can be written in the form:

(b) $x_0 \times m(x_0) - \varepsilon[m(x_0) \times (x_0) \times m(x_0)] = 0$

Multiplying the relation (b) with the vector $x_0$, we obtain:

(c) $x_0 \cdot (x_0 \times m(x_0)) - \varepsilon x_0 \cdot [m(x_0) \times (x_0) \times m(x_0)] = 0$.

Using the equality (21), the relation (c) is equivalent with:

(d) $-\varepsilon (x_0 \times m(x_0))^2 = 0$.

From (d) (if $\varepsilon \neq 0$), follows $x_0 \times m(x_0) = 0$, that is $x_0$ is an equilibrium point for (20).

□

The equilibrium points of the dynamics (12) are well-known (see M. Puta and D. Comănescu, [12]) and these are presented in the following proposition.

**Proposition 4.1.** ([12]) The equilibrium points of the Hamilton - Poisson system (12) are the following:

(i) $e_1 = (0, 0, 0)$;

(ii) $e_2 = \left(\frac{a}{\lambda - a_1}, \frac{b}{\lambda - a_2}, \frac{c}{\lambda - a_3}\right)$ for $\lambda \in \mathbb{R} \setminus \{a_1, a_2, a_3\}$;

(iii) $e_3 = (\alpha, -\frac{b}{a_2 - a_1}, \frac{c}{a_1 - a_3})$ for $\alpha \in \mathbb{R}$, if $a = 0$;

(iv) $e_4 = \left(\frac{a}{a_2 - a_1}, \alpha, -\frac{c}{a_3 - a_2}\right)$ for $\alpha \in \mathbb{R}$, if $b = 0$;

(v) $e_5 = \left(-\frac{a}{a_1 - a_2}, \frac{b}{a_3 - a_2}, \alpha\right)$ for $\alpha \in \mathbb{R}$, if $c = 0$. 
By Theorem 4.1, the equilibrium points of the \( \varepsilon \)-revised system (20) are \( e_1, ..., e_5 \) indicated in the Proposition 4.1.

It is well-known that the dynamics (12) have the first integrals \( H \) and \( C \) given by (14) and (16). These first integrals may be written thus:

\[
H(x^1, x^2, x^3) = \frac{1}{2} x \cdot \mathbf{I}^{-1} x + a \cdot x \quad \text{and} \quad C(x^1, x^2, x^3) = \frac{1}{2} x^2
\]

where \( \mathbf{I} \) is inertia tensor and \( \mathbf{I}^{-1} \) is its inverse. We have:

\[
\frac{dH}{dt}(x) = \mathbf{m}(x) \cdot \dot{x} \quad \text{and} \quad \frac{dC}{dt}(x) = x \cdot \dot{x}.
\]

Indeed, \( \frac{dH}{dt} = (a_1 x^1 + a)\dot{x}^1 + (a_2 x^2 + b)\dot{x}^2 + (a_3 x^3 + c)\dot{x}^3 = \mathbf{m}(x) \cdot \dot{x} \) and

\[
\frac{dC}{dt} = x^1\dot{x}^1 + x^2\dot{x}^2 + x^3\dot{x}^3 = x \cdot \dot{x}.
\]

**Theorem 4.2.** (i) For each \( \varepsilon \in \mathbb{R} \), the function \( H \) given by (14) is a first integral for the \( \varepsilon \)-revised system (20).

(ii) If \( x : \mathbb{R} \to \mathbb{R}^3 \) is a solution of the \( \varepsilon \)-revised system, then:

\[
\frac{d}{dt} \left( \frac{1}{2} x^2 \right) = -\varepsilon (x \times \mathbf{m}(x))^2.
\]

(iii) For \( \varepsilon \in \mathbb{R}^* \), the function \( C \) is not a first integral for the \( \varepsilon \)-revised system.

**Proof.** (i) Multiplying the relation (24) with the vector \( \mathbf{m}(x) \), we have:

\[
\mathbf{m}(x) \cdot \dot{x} = \mathbf{m}(x) \cdot (x \times \mathbf{m}(x)) + \varepsilon \mathbf{m}(x) \cdot [(x \times \mathbf{m}(x)) \times \mathbf{m}(x)] = 0.
\]

Applying now (28), we obtain \( \frac{dH}{dt} = \mathbf{m}(x) \cdot \dot{x} = 0 \). Hence \( H \) is a first integral for the system (20).

(ii) Multiplying the relation (24) with the vector \( x \), we have

\[
x \cdot \dot{x} = x \cdot (x \times \mathbf{m}(x)) + \varepsilon x \cdot [(x \times \mathbf{m}(x)) \times \mathbf{m}(x)] = -\varepsilon x \cdot [(\mathbf{m}(x) \times (x \times \mathbf{m}(x))].
\]

Using now the equality (21), we obtain \( x \cdot \dot{x} = -\varepsilon (x \times \mathbf{m}(x))^2 \). Then, we have

\[
\frac{d}{dt} \left( \frac{1}{2} x^2 \right) = x \cdot \dot{x} = -(x \times \mathbf{m}(x))^2.
\]

(iii) This assertion follows from the second relation of (28) and (ii).

**Remark 4.1.** The function \( H \) given by (14) can be put in the equivalent form:

\[
H(x^1, x^2, x^3) = \frac{1}{2} \left[ a_1(x^1 + \frac{a}{a_1})^2 + a_2(x^2 + \frac{b}{a_2})^2 + a_3(x^3 + \frac{c}{a_3})^2 \right] - \frac{1}{2} \left( \frac{a^2}{a_1} + \frac{b^2}{a_2} + \frac{c^2}{a_3} \right).
\]

For a given constant \( k \in \mathbb{R} \), the geometrical image of the surface:
is an ellipsoid, since \(a_1 > 0, a_2 > 0, a_3 > 0\).

**Proposition 4.2.** The set of equilibrium points which belong to the ellipsoid \(H(x_1, x_2, x_3) = k\) is finite.

**Proof.** Following the description of the equilibrium points given in Proposition 4.1, we remark that:

(i) the equilibrium points of the form \(e_3\) (similarly, for \(e_4\) and \(e_5\)) make a straight line; the intersection between a straight line and an ellipsoid have at most two points; we deduce that on the chosen ellipsoid there exist at most two points of the form \(e_3\).

(ii) the equilibrium points of the form \(e_2\) can be obtained by solving with respect \(\lambda\) the following equation:

\[
\frac{1}{2}[a_1(\frac{a}{\lambda - a_1})^2 + a_2(\frac{b}{\lambda - a_2})^2 + a_3(\frac{c}{\lambda - a_3})^2] + \frac{a_2}{\lambda - a_1} + \frac{b_2}{\lambda - a_2} + \frac{c_2}{\lambda - a_3} = k.
\]

The above equation is equivalent with the determination of roots of a polynomial of degree at most 6; therefore on the chosen ellipsoid there exist at most 6 equilibrium points of the form \(e_2\). □

5 The behaviour of the solutions of the \(\varepsilon\) - revised system

**Theorem 5.1**

(i) The solutions of the \(\varepsilon\)-revised system are bounded.

(ii) The maximal solutions of the \(\varepsilon\)-revised system are globally solutions (i.e. these are defined on \(\mathbb{R}\)).

**Proof** (i) Given a solution of (20), there exists a constant \(k\) such that its trajectory lie on the ellipsoid \(H(x^1, x^2, x^3) = k\). From this we deduce that all solutions are bounded.

(ii) Let \(x : (m, M) \subseteq \mathbb{R} \rightarrow \mathbb{R}^3\) be a maximal solution. We assume that \(x\) is not globally. It follows \(m > -\infty\) or \(M < \infty\). In these situations, we know that there exists \(k \in \mathbb{R}\) such that \(H(x^1, x^2, x^3) = k\) for all \(t \in \mathbb{R}\) and the graph of the solution is contained in a compact domain. According with [6] (theorem 3.2.5, p.141) we obtain a contradiction with the fact that \(x\) admit a prolongation on the right or the left (also, can be applied the theorem of Chilingworth (1976), see theorem 1.0.3, p.7 in [3]). □

In the sequel we study the asymptotic behaviour of the globally solutions of the \(\varepsilon\)-revised system.
Denote by $E$ the set of equilibrium points of the $\varepsilon$-revised system (20) and by $\Gamma$ the trajectory of a solution $x: \mathbb{R} \to \mathbb{R}^3$ of (20). By theory of differential equations (see [10] p. 174-176), the $\omega$-limit set and $\alpha$-limit set of $\Gamma$ are:

$$
\omega(\Gamma) = \{y \in \mathbb{R}^3 / \exists t_n \to \infty \text{ such that } x(t_n) \to y\},
$$

$$
\alpha(\Gamma) = \{z \in \mathbb{R}^3 / \exists t_n \to -\infty \text{ such that } x(t_n) \to z\}.
$$

**Theorem 5.2** Let $x : \mathbb{R} \to \mathbb{R}^3$ be a solution of the $\varepsilon$-revised system with $\varepsilon \neq 0$. There exist the equilibrium points $x_m, x_M \in \mathbb{R}^3$ of the system (20) such that $\lim_{t \to -\infty} x(t) = x_M$ and $\lim_{t \to \infty} x(t) = x_m$.

**Proof** The theorem is proved in the following steps:

(i) $\alpha(\Gamma) \neq \emptyset$ and $\omega(\Gamma) \neq \emptyset$.

(ii) $\alpha(\Gamma) \cap \omega(\Gamma) \subset E$.

(iii) The sets $\alpha(\Gamma)$ and $\omega(\Gamma)$ contains exactly one element.

Taking account into that each solution is bounded (hence it is contained in a compact domain) and applying theorem 1, p. 175 in [10], we obtain immediately the assertions (i).

(ii) For demonstration consider the case when $\varepsilon > 0$. Using the relation (29), we deduce that the function $t \to x_2(t)$ is a strictly decreasing function. Being bounded it follows that there exists $\lim_{t \to \infty} x_2(t) = L$ and $L$ is finite.

For each $y \in \omega(\Gamma)$ there exists the sequence $t_n \to \infty$ such that $x(t_n) \to y$. Then $x_2(t_n) \to y_2$ and hence $y_2 = L$.

By theorem 2, p.176 in [10], we have that the trajectory $\Gamma_y$ of the solution $x_y$ which verifies the initial condition $x_y(0) = y$, satisfies the relation $\Gamma_y \subset \omega(\Gamma)$.

If we assume that $y$ is not an equilibrium point, then we deduce (using the relation (29)) that for $t > 0$ we have $x_2^y(t) < L$ and this is in contradiction with the above result. Therefore, we have $\omega(\Gamma) \subset E$.

Similarly, we prove that $\alpha(\Gamma) \subset E$. Hence the assertion (ii) holds.

The case $\varepsilon < 0$ is similar.

(iii) There exists a constant $k$ such that the sets $\alpha(\Gamma)$ and $\omega(\Gamma)$ are included in the ellipsoid $H(x_1, x_2, x_3) = k$. By (ii), we deduce that $\alpha(\Gamma)$ and $\omega(\Gamma)$ are included in the set of equilibrium points which lies of the above ellipsoid. On the other hand, applying Proposition 4.2 and using the fact that $\alpha(\Gamma)$ and $\omega(\Gamma)$ are connected (see theorem 1, p.175 in [10]), we obtain that $\alpha(\Gamma)$ and $\omega(\Gamma)$ are formed by only one element.

**Remark 5.1** Using the relation (29) it is easy to observe that the following assertions hold:

(i) if $\varepsilon > 0$ then $x_2_M > x_2_m$;

(ii) if $\varepsilon < 0$ then $x_2_M < x_2_m$. 

□
As an immediate consequence we obtain the following theorem.

**Theorem 5.3** If $\varepsilon \neq 0$, then for each solution $x : \mathbb{R} \rightarrow \mathbb{R}^3$ of the $\varepsilon$-revised system we have:

\[
\begin{cases}
\text{if } t \rightarrow \infty \Rightarrow d(x(t), E) \rightarrow 0 \\
\text{if } t \rightarrow -\infty \Rightarrow d(x(t), E) \rightarrow 0.
\end{cases}
\tag{31}
\]

□

**Remark 5.2** From Theorem 5.3 follows that the set $E$ of the equilibrium points is an attracting set (see definition 2, p.178 in [10]) and also is a repelling set (see [3], p.34). Thus, the space $\mathbb{R}^3$ is simultaneously a domain of attraction and a domain of repulsion of $E$.

6 The Lyapunov stability of equilibrium points of the $\varepsilon$-revised system in the case $\varepsilon > 0$

**The stability of the point** $e_1 = (0,0,0)$. We have the following results.

**Theorem 6.1** The equilibrium point $e_1$ is Lyapunov stable.

**Proof** Let $\gamma > 0$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$ such that $|x_0| < \gamma$, where $| \cdot |$ denotes the euclidian norm in $\mathbb{R}^3$. Denote by $t \rightarrow x(t,t_0,x_0)$ the solution of the $\varepsilon$-revised system which verifies the initial condition $x(0,t_0,x_0) = x_0$.

Using the relation $|x(t,t_0,x_0)| = \sqrt{x^2(t,t_0,x_0)}$ and according with the relation (29), we observe that the function $t \rightarrow x(t,t_0,x_0)$ is a decreasing function and hence we have:

$|x(t,t_0,x_0)| \leq |x_0| < \gamma$ for $t > t_0$.

Then (see [4], p.22) we have that $e_1$ is a Lyapunov stable equilibrium point. □

**Remark 6.1** The equilibrium point $e_1$ is not asymptotical stable.

Indeed, if $a = b = c = 0$ then the coordinates axis are formed from equilibrium points. If at least of one of the numbers $a, b, c$ is non null, then:

\[
\text{if } |\lambda| \rightarrow \infty \Rightarrow \left( \frac{a}{\lambda - a_1}, \frac{b}{\lambda - a_2}, \frac{c}{\lambda - a_3} \right) \rightarrow (0,0,0).
\]

Hence, in all neighbourhood of $e_1$ there exist an infinity of equilibrium points. □

**The stability of the point** $x_0 = (-\frac{a}{a_1}, -\frac{b}{a_2}, -\frac{c}{a_3})$. The equilibrium point $x_0$ is an equilibrium point of the form $e_2$ and it is obtained for $\lambda = 0$.

**Theorem 6.2** The equilibrium point $x_0$ is Lyapunov stable.
Proof Using the relation (30) and the inequality \(0 < a_1 < a_2 < a_3\), we deduce:

\[
\frac{a_1}{2} |x - x_0| \leq H(x) - H(x_0) \leq \frac{a_3}{2} |x - x_0|
\]

For \(t_0 \in \mathbb{R}\) and \(x_0 \in \mathbb{R}^3\) denote with \(x(t, t_0, x_0)\) a solution of \(\varepsilon\)-revised system which verifies the initial condition \(x(0, t_0, x_0) = x_0\).

Let \(\gamma > 0\) and \(\delta(\gamma) = \frac{2 \gamma}{a_1}\). Let \(x_0 \in \mathbb{R}^3\) such that:

\[
H(x_0) - H(x) \leq \delta(\gamma).
\]

From the fact that \(H\) is a first integral we deduce that:

\[
H(x(t, t_0, x_0) - H(x_0)) = H(x_0) - H(x_0).
\]

Hence for all \(t \in \mathbb{R}\) the following inequality holds:

\[
\frac{a_1}{2} |x - x_0| \leq \delta(\gamma)
\]

and we obtain that \(x_0\) is Lyapunov stable. \(\square\)

Remark 6.2 The stable equilibrium point \(x_0\) realizes the absolute minimum of the function \(H\). \(\square\)

The unstability of equilibrium points of the form \(e_2\) with \(\lambda \in (0, a_1)\). For the demonstration of this results we use the Theorem 6.3 and Lemma 6.1.

Theorem 6.3 If \(x_0 \in E\) such that there exists \(y \in E\) with the properties:

(i) \(H(y) = H(x_0)\) and (ii) \(|y| < |x_0|\)

then \(x_0\) is an unstable equilibrium point.

Proof For \(k \in \mathbb{R}\) denote by \(E_k = \{x \in E \mid H(x) = k\}\). The set \(E_{H(x_0)}\) is finite (by Proposition 4.2). We denote:

\[
\gamma_0 = \min\{|x - x_0| / x \in E_{H(x_0)} - \{x_0\} \}
\]

Let \(z \in \mathbb{R}^3\) such that \(H(z) = H(x_0)\) and \(|z| < |x_0|\). Then:

\[
\lim_{t \to \infty} x(t, 0, z) \in E
\]

and

\[
if \ t > 0 \Rightarrow |x(t, 0, z)| < |z|
\]

and we deduce that there exists \(t_z > 0\) such that:

\[
|x(t, 0, z) - x_0| > \frac{\gamma_0}{2} if \ t > t_z
\]
It follows that $x_0$ is unstable. □

We assume that $(a, b, c) \neq (0, 0, 0)$ and we introduce the notation:

$$e_{2\lambda} = \left(\frac{a}{\lambda - a_1}, \frac{b}{\lambda - a_2}, \frac{c}{\lambda - a_3}\right) \text{ for all } \lambda \in \mathbb{R} - \{a_1, a_2, a_3\}$$

**Lemma 6.1**

(i) If $\sigma < \mu < a_1$, then $|e_{2\sigma}| < |e_{2\mu}|$.

(ii) If $\sigma, \mu > 0$, $(\frac{\sigma}{\mu})^2 > a_3/a_1$ and $H(e_{2\sigma}) = H(e_{2\mu})$, then $|e_{2\sigma}| > |e_{2\mu}|$.

(iii) If $0 < \sigma < a_1 < a_3 < \mu$ and $H(e_{2\sigma}) = H(e_{2\mu})$, then $|e_{2\sigma}| > |e_{2\mu}|$.

**Proof**

(i) Consider the function $g : (-\infty, a_1) \to \mathbb{R}$ given by:

$$g(\lambda) = \left(\frac{a}{\lambda - a_1}\right)^2 + \left(\frac{b}{\lambda - a_2}\right)^2 + \left(\frac{c}{\lambda - a_3}\right)^2.$$

The derivative of the function $g$ is:

$$g'(\lambda) = -\frac{2a^2}{(\lambda - a_1)^3} - \frac{2b^2}{(\lambda - a_2)^3} - \frac{2c^2}{(\lambda - a_3)^3}.$$

We observe that $g'(\lambda) > 0$ and we obtain that $g$ is a strictly increasing function. We have:

$$g(\sigma) = |e_{2\sigma}|^2, \quad g(\mu) = |e_{2\mu}|^2$$

and we obtain the desired result.

(ii) From hypothesis $H(e_{2\sigma}) = H(e_{2\mu})$ follows that there exists a constant $q > 0$ with the following properties:

$$\frac{1}{a_1} \frac{a^2}{(\sigma - a_1)^2} + \frac{1}{a_2} \frac{b^2}{(\sigma - a_2)^2} + \frac{1}{a_3} \frac{c^2}{(\sigma - a_3)^2} = \frac{q}{\sigma^2}$$

$$\frac{1}{a_1} \frac{a^2}{(\mu - a_1)^2} + \frac{1}{a_2} \frac{b^2}{(\mu - a_2)^2} + \frac{1}{a_3} \frac{c^2}{(\mu - a_3)^2} = \frac{q}{\mu^2}$$

Using $a_1 < a_2 < a_3$, we obtain the inequalities:

$$|e_{2\sigma}|^2 > \frac{a_1q}{\sigma}, \quad |e_{2\mu}|^2 < \frac{a_3q}{\mu}$$

and we observe that the assertion (ii) holds.

(iii) This assertion follows immediately from (ii). □

**Theorem 6.4** The equilibrium point $e_{2\lambda}$ with $0 < \lambda < a_1$ is unstable.
Proof Consider the function \( h : (-\infty, a_1) \cup (a_3, \infty) \rightarrow \mathbb{R} \) given by:

\[
h(\sigma) = H(e^{2\sigma}).
\]

Using the relation (30) for \( H \), we find:

\[
h(\sigma) = \sigma^2 \left[ \frac{a^2}{a_1(\sigma - a_1)^2} + \frac{b^2}{a_2(\sigma - a_2)^2} + \frac{c^2}{a_3(\sigma - a_3)^2} - \frac{1}{2} \left( \frac{a^2}{a_1} + \frac{b^2}{a_2} + \frac{c^2}{a_3} \right) \right]
\]

The function \( h \) have the following properties:

- 0 is an absolute minimum point.
- \( \lim_{\sigma \to -\infty} h(\sigma) = \lim_{\sigma \to \infty} h(\sigma) = 0. \)
- \( \lim_{\sigma \to a_1} h(\sigma) = \lim_{\sigma \to a_3} h(\sigma) = \infty. \)
- \( h \) is strictly decreasing on \((-\infty, 0)\), strictly increasing on \((0, a_1)\) and strictly decreasing on \((a_3, \infty)\).

The demonstrations divided on three cases.

(I) Assume that \( h(\lambda) < 0 \). In this situation there exists \( \sigma < 0 < \lambda < a_1 \) such that \( h(\lambda) = h(\sigma) \) and imply \( H(e^{2\lambda}) = H(e^{2\sigma}) \). Hence the equilibrium points \( e^{2\lambda} \) and \( e^{2\sigma} \) belong to same ellipsoid.

On the other hand, by Lemma 6.1 (ii), follows \( |e^{2\sigma}| < |e^{2\lambda}| \). Applying now Theorem 6.3, deduce that \( e^{2\lambda} \) is an unstable equilibrium point.

(II) Assume that \( h(\lambda) = 0 \) we have \( H(e^{2\lambda}) = H(0, 0, 0) \) and it is clearly that \( |(0, 0, 0)| < |e^{2\lambda}| \). By Theorem 6.3 we find the desired result.

(III) Assume that \( h(\lambda) > 0 \). Then there exists \( \sigma > a_3 \) such that \( h(\lambda) = h(\sigma) \) and hence \( H(e^{2\lambda}) = H(e^{2\sigma}) \). Applying Lemma 6.1 (iii) follows \( |e^{2\lambda}| > |e^{2\sigma}| \) and by Theorem 6.3 we deduce that \( e^{2\lambda} \) is unstable.

\[\Box\]

The stability of equilibrium points of the form \( e^2 \) with \( \lambda < 0 \).

Theorem 6.5 The equilibrium points of the form \( e^2 \) with \( \lambda < 0 \) are Lyapunov stables.

Proof Let \( \lambda < 0 \) and the equilibrium point \( x_0 = (\frac{a}{\lambda - a_1}, \frac{b}{\lambda - a_2}, \frac{c}{\lambda - a_3}) \) of the form \( e^2 \).

It is well-known that the study of stability of \( x_0 \) in the Lyapunov sense is equivalent with the study of stability of the null solution \((0, 0, 0)\) for the differential system obtained from the \( \varepsilon \)-revised system by transformation of variables:

\[ z = x - x_0 \]

The system obtained in this manner is called the perturbed \( \varepsilon \)-revised system.
Consider the function $K : \mathbb{R}^3 \to \mathbb{R}$ given by
\[
K(z) = \frac{1}{2} z \cdot I^{-1} z - \frac{\lambda}{2} z^2
\]
Since the tensor $I^{-1}$ is strictly positive definite and $\lambda < 0$ we obtain that $K$ is a quadratic form strictly positive definite.

Next we prove that if $z : \mathbb{R} \to \mathbb{R}^3$ is a solution for the perturbed $\varepsilon$-revised system, then:
\[
\frac{d}{dt} K(z(t)) < 0
\]

By a direct computation and taking account into the relations (27) we have:
\[
K(z) = H(x) - \lambda C(x) - \frac{1}{2} x_0 \cdot I^{-1} x_0 - a \cdot x_0 + \frac{\lambda}{2} x_0^2
\]
Applying now Theorem 4.2, we obtain:
\[
\frac{d}{dt} K(z(t)) = \varepsilon \lambda (x \times m(x))^2
\]
and follows that $\frac{d}{dt} K(z(t)) < 0$, since $\varepsilon > 0$, $\lambda < 0$.

It is easy to see that $K^*(t) = K(z(t))$ is a strictly decreasing function. By theorem 1.1, p.21 in the paper [4] we deduce that $x_0$ is Lyapunov stable. □

Conclusion - the stability problem for the Hamilton Poisson system (12) versus the $\varepsilon$- revised system (20) with $\varepsilon > 0$

Concerning to the equilibrium points of the system (12) are established the following results (see, theorem 1.1, [12]):
(1) $e_1$ is Lyapunov stable;
(2) $e_2$ are Lyapunov stables for $\lambda \in (-\infty, a_1) \cup (a_3, \infty)$;
(3) $e_3$ are Lyapunov stables;
(4) $e_4$ are unstables;
(5) $e_5$ are Lyapunov stables.

By Remark 4.1 (see, [12]), there exist cases for which the problem to decide the nonlinear stability or unstability are not discussed.

For the stability of equilibrium points of the $\varepsilon$- revised system (20) with $\varepsilon > 0$ have proved the following assertions:
(1) $e_1$ is Lyapunov stable;
(2) the equilibrium points of the form $e_2$ with $\lambda \leq 0$ are Lyapunov stables;
(3) the equilibrium points of the form $e_2$ with $0 < \lambda < a_1$ are unstables.

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Seminarul de Geometrie și Topologie
West University of Timișoara
Bd-ul V. Pârvan no.4, 300223, Timișoara
Romania
E-mail: comanescu@math.uvt.ro, ivan@math.uvt.ro and mihai31ro@yahoo.com