On the bounds of the sum of eigenvalues for a Dirichlet problem involving mixed fractional Laplacians

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Abstract
Our purpose in this paper is to study of the eigenvalues \( \{ \lambda_i(\mu) \} \) of the Dirichlet problem

\[
(-\Delta)^{s_1} u = \lambda ((-\Delta)^{s_2} u + \mu u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Omega,
\]

where \( 0 < s_2 < s_1 < 1 \), \( N > 2s_1 \) and \((-\Delta)^s\) is the fractional Laplacian operator defined in the principle value sense.

We first show the existence of a sequence of eigenvalues, which approaches infinity. Secondly we provide a Berezin–Li–Yau type lower bound for the sum of the eigenvalues of the above problem. Furthermore, using a self-contained and novel method, we establish an upper bound for the sum of eigenvalues of the problem under study.

Keywords: Dirichlet eigenvalues; Fractional Laplacian, Berezin-Li-Yau method, mixed nonlocal operator, mixed fractional Laplacian.

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1 Introduction and main results
Let \( 0 < s_2 < s_1 < 1 \), \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N \) with the integer \( N \geq 1 \), \( N > 2s_1 \). The main goal of this paper is to study the lower bounds of the eigenvalues for the Dirichlet problem

\[
\begin{cases}
(-\Delta)^{s_1} u = \lambda ((-\Delta)^{s_2} u + \mu u) & \text{in} \quad \Omega, \\
u = 0 & \text{in} \quad \mathbb{R}^N \backslash \Omega,
\end{cases}
\]  

(1.1)

where \((-\Delta)^s\) is the fractional laplacian defined in the following sense (principle value)

\[
(-\Delta)^s u(x) = c_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N + 2s}} dy
\]

(1.2)

with \( s \in (0, 1) \), \( c_{N,s} = 2^{2s} \pi^{-N/2} s \Gamma(\frac{N + 2s}{2}) \Gamma(1 - s) \) and \( \Gamma \) being the Gamma function, see e.g. [40]. Recall that, for \( s \in (0, 1) \), the fractional Laplacian of a function \( u \in C^\infty_c(\mathbb{R}^N) \) can also be defined by:

\[
(-\Delta)^s u(\xi) = \mathcal{F}^{-1}\left( |\xi|^{2s} \mathcal{F}(u)(\xi) \right) \quad \text{for all} \quad \xi \in \mathbb{R}^N,
\]

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where $F(f)$ denotes the Fourier transform of $f$.

Problem (1.1) involves two fractional Laplacians with two different powers. The terminology “mixed operators” refers to the differential or pseudo-differential order of the operator, and to the type of the operator, which can combine classical and fractional features. When $\lambda < 0$, (1.1) involves a sum of two fractional Laplacians of different orders. Indeed, such sum of operators arises naturally from superposition of two stochastic processes with a different random walk and a Lévy flight, this is the case when a particle can follow either of these two processes according to a certain probability, the associated limit diffusion equation is described by a sum of fractional Laplacians, see for e.g [4]. While if $\lambda > 0$, (1.1) models a difference of two fractional Laplacians. The sum and the difference of two fractional Laplacians appear in many circumstances. To mention few, problems of blood circulation in the heart, responsible for causing heart problems and in many circumstances coronary bypass surgeries, can be modelled by two to five mixed fractional Laplacians, for e.g. see [16, 33–35]. It also appears in many circumstances because of the anomalous blood circulation, but it is often not the same anomaly in all the arteries, and the blood can follow either of the five arteries. Other applications of mixed fractional operators with different orders include plasma physics and population dynamics, ways to reduce pandemics and so on. In view of these important applications, we strongly believe that equation (1.1) and some of its variants described above will get an increasing interest in the near future.

The most primitive model of (1.1) for the eigenvalues is

\[
\begin{align*}
-\Delta u &= \lambda u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

which has attracted the attention of mathematicians since 1912. Indeed, Weyl in [43] showed that the $k$-th eigenvalue, $\lambda_{1,k}(\Omega)$ (1 stands for $s = 1$) of (1.3), has the asymptotic behavior

\[
\lambda_{1,k}(\Omega) \sim C_N(k!|\Omega|)^{\frac{2}{N}} \quad \text{as } k \to +\infty,
\]

where

\[
C_N = (2\pi)^2 |B_1|^{-\frac{2}{N}}
\]

and $|B_1|$ is the volume of unit ball in $\mathbb{R}^N$. Later, Pólya [38] (in 1960) provided a lower bound for the $k$-th eigenvalue,

\[
\lambda_{1,k}(\Omega) \geq C(k!/|\Omega|)^{\frac{2}{N}}
\]

for any plane covering domain $D$ in $\mathbb{R}^2$ with $C = C_N$, (his proof also works in dimension $N \geq 3$). $D$ is called a plane covering domain in $\mathbb{R}^2$ if an infinity of domains congruent to $D$ cover the whole plane without gaps and without overlapping except a set of measure zero. In [38], Pólya also made a conjecture that (1.4) holds for any bounded domain in $\mathbb{R}^N$ with $C = C_N$. To answer this conjecture, Lieb [32] proved (1.4) with a positive constant $C$ for general bounded domain and Li-Yau [31] improved the constant $C$ to $\frac{N}{N+2}C_N$. Until now, this constant for lower bound is the best and (1.4) with $C = \frac{N}{N+2}C_N$ is called Berezin-Lieb-Yau inequality. More related estimates on lower bounds for the eigenvalues under various setting can be found in [14,18,32,36]. On the other hand, the upper bounds of Dirichlet eigenvalues were derived by Kröger in [29] by calculating the Rayleigh quotient by using a sequence of functions approaching the characterized function of $\Omega$. We also refer to Yang’s upper bounds of the Dirichlet’s eigenvalues in [8,13] in the following way:

\[
\lambda_{1,k}(\Omega) \leq c(N,k)\lambda_{1,1}(\Omega)k^{\frac{2}{N}} \quad \text{for some } c(N,k) > 0.
\]

The unstopped interest in finding bounds for eigenvalues of the Dirichlet problem is in part due to the following fact: The Hilbert-Pólya conjecture is to associate the zero of the Riemann Zeta function with the eigenvalue of a Hermitian operator. This quest initiated the mathematical interest for estimating the sum of Dirichlet eigenvalues of the Laplacian while in physics the question is
related to count the number of bound states of a one body Schrödinger operator and to study their asymptotic distribution. The latter constitutes in itself a branch in nonlinear analysis. During the last decade, there has been a renewed and increasing interest in the study of linear and nonlinear integral operators. The prototype is the fractional Laplacian. This has been motivated by numerous applications in different fields motivated by connections to real world life applications and by important advances in the theory of linear and nonlinear partial differential equations, see basic properties \cite{24,25,37}, regularities \cite{5,39}, Liouville property \cite{6}, general nonlocal operator \cite{11}, fractional Pohozaev identity \cite{10}, singularities \cite{17,9}, uniqueness \cite{20}, fractional variational setting \cite{3,15,19,27,41} and the references therein.

When $\mu = 0$, $s_2 = 0$ and $s = s_1 \in (0,1)$, (1.1) reduces to the fractional Laplacian problem

\[
\begin{cases}
(-\Delta)^s u = \lambda u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(1.5)

for which the asymptotic behavior of eigenvalues $\lambda_{s,i}$ has been studied, for Klein-Gordon operator i.e. $s = \frac{1}{2}$ in \cite{23} Proposition 3.1] or for general order $s \in (0,1)$ in \cite{21} Theorem 1,

\[
\lim_{k \to +\infty} k^{-\frac{2s}{N}} \lambda_{s,k} = a(N, s) |\Omega|^{-\frac{2s}{N}}
\]

(1.6)

where

\[
a(N, s) = \left(\frac{2\pi}{2s}\right)^{2s} |B_1|^{-\frac{2s}{N}}.
\]

(1.7)

Moreover, a refinement of Berezin–Li–Yau-type lower bound for the sum of eigenvalues was built by Yolcu and Yolcu in \cite{45} Theorem 1.4 as follows

\[
\sum_{j=1}^{k} \lambda_{s,j} \geq \frac{N}{N + 2s} a(N, s) |\Omega|^{-\frac{2s}{N}} k^{1+\frac{2s}{N}} + ck^{1-\frac{2s}{N}},
\]

(1.8)

for some $c > 0$ depending on $|\Omega|$. In a recent work \cite{44}, Hajaiej and Wang provided the asymptotic behavior of the sum of the eigenvalues of (1.5)

\[
\lim_{k \to +\infty} k^{-1-\frac{2s}{N}} \sum_{j=1}^{k} \lambda_{s,j}(\Omega) = \frac{N}{N + 2s} a(N, s) |\Omega|^{-\frac{2s}{N}}.
\]

(1.9)

For more estimates on eigenvalues of the fractional Dirichlet problem, we refer the readers to \cite{12,21,23,45,46}, a review \cite{20} and the references therein.

To analyze the fractional Dirichlet eigenvalues of (1.1), we denote $\mathbb{H}_0^s(\Omega)$ the space of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ with $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and

\[
\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} dx dy < +\infty.
\]

It is well known that $\mathbb{H}_0^s(\Omega)$ is a Hilbert space equipped with the inner product

\[
\mathcal{E}_s(u, w) = \frac{c_{N,s}}{2} \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N + 2s}} dx dy
\]

and the induced norm

\[
\|u\|_s := \sqrt{\mathcal{E}_s(u, u)}.
\]

We say a function $u \in \mathbb{H}_0^s(\Omega)$ be an eigenfunction of (1.1) corresponding to the eigenvalue $\lambda$ if

\[
\mathcal{E}_{s_1}(u, w) = \lambda \left( \mathcal{E}_{s_2}(u, w) + \mu \int_{\Omega} uw \, dx \right) \quad \text{for all } w \in \mathbb{H}_0^s(\Omega).
\]
For $\mu \geq -\lambda_{s_2,1}$, we denote by $\mathbb{H}^{s_2}_{\mu,0}(\Omega)$ the space of all measurable functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ with $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and

$$\|u\|_{s_2,\mu} := \left(\mathcal{E}_{s_2}(u, u) + \mu \int_{\Omega} u^2 \, dx\right)^{\frac{1}{2}} < \infty.$$ 

The corresponding inner product in $\mathbb{H}^{s_2}_{\mu,0}(\Omega)$ is given by

$$\langle u, w \rangle_{s_2,\mu} := \mathcal{E}_{s_2}(u, w) + \mu \int_{\Omega} uw \, dx, \quad \forall \ u, w \in \mathbb{H}^{s_2}_{0,0}(\Omega).$$

Let $\lambda_{s_2,1}$ be the first eigenvalue of (1.5) corresponding to $s = s_2$. We note that $\|\|_{s_2,\mu}$ is equivalent to $\|\|_{s_2}$ for $\mu > -\lambda_{s_2,1}$.

Our first aim is to show the existence of a sequence of discrete eigenvalues of (1.1) as follows.

**Theorem 1.1.** Let $\mu > -\lambda_{s_2,1}$, where $\lambda_{s_2,1} > 0$ be the first eigenvalue of (1.5) with $s = s_2$. Then problem (1.1) admits a sequence of real eigenvalues

$$0 < \lambda_1(\mu) \leq \lambda_2(\mu) \leq \cdots \leq \lambda_j(\mu) \leq \lambda_{j+1}(\mu) \leq \cdots$$

and the corresponding eigenfunction $\phi_i$, $i \in \mathbb{N}$. Moreover, we have the following properties:

(i) $\lambda_j(\mu) = \min\{\mathcal{E}_{s_1}(u, u) : u \in \mathbb{H}^{s_1}_{0,j}(\Omega), \|u\|_{H^{s_2}_{\mu,0}(\Omega)} = 1\}$, where

$$\mathbb{H}^{0,1}_{0}(\Omega) = \mathbb{H}^{0,1}_{0,0}(\Omega), \quad \mathbb{H}^{0,j}_{0}(\Omega) := \{u \in \mathbb{H}^{0,j}_{0}(\Omega) : \langle u, \phi_m \rangle_{s_2,0} = 0 \text{ for } m = 1, \ldots, j - 1\} \text{ for } j > 1;$$

(ii) $\{\phi_j : j \in \mathbb{N}\}$ is an orthonormal basis of $\mathbb{H}^{s_2}_{\mu,0}(\Omega)$;

(iii) $\lim_{j \to \infty} \lambda_j(\mu) = +\infty$;

(iv) For $\mu \in (-\lambda_{s_2,1}, +\infty)$, the map $\mu \mapsto \lambda_1(\mu)$ is decreasing and $\lim_{\mu \to -\lambda_{s_2,1}^+} \lambda_1(\mu) < +\infty$.

We remark that

(a) from the appendix in [11], problem (1.5) has the property that the first eigenvalue is simple and the corresponding eigenfunction $\phi_{s,1}$ is positive; these properties are derived by the following: $\mathcal{E}_{s}(\phi_{s,1}) < \mathcal{E}_{s}(\phi_{s,1})$, if $\phi_{s,1}$ is sign-changing. However, this argument fails for problem (1.1) due to the presence of multiple fractional Laplacians, and it is very interesting but challenging to determine the one-fold of $\lambda_1(\mu)$ and the positivity of the eigenfunction $\phi_1$ corresponding to the first eigenvalue $\lambda_1$ for problem (1.1);

(b) thanks to the monotonicity and boundedness of $\lambda_1(\mu)$, assertion (iv) indicates that it is possible to obtain the existence of $\{\lambda_j(\mu)\}_{j \in \mathbb{N}}$ for $\mu \leq -\lambda_{s_2,1};$

(c) it is known that eigenfunctions of (1.5) are $C^\infty(\Omega)$. To see this, one uses bootstraps method to prove solutions of (1.5) are in $L^\infty(\Omega)$ and then uses regularity results of [39]. While the regularity for (1.1) seems to be much more complicated, because bootstraps iteration has to work between different order fractional Laplacians.

We now provide a lower bound for the sum of eigenvalues of (1.1).

**Theorem 1.2.** Let $\mu \geq 0$ and $\{\lambda_j(\mu)\}_{j \in \mathbb{N}}$ be the increasing sequence of eigenvalues of problem (1.1) and $\omega_{N-1}$ denote the surface area of the unit sphere in $\mathbb{R}^N$. Then for $k \in \mathbb{N}$

$$\sum_{j=1}^{k} \lambda_j(\mu) \geq b_1|\Omega| \frac{2s_1 - 2s_2}{8} k^{\frac{2s_1 - 2s_2}{2s_2 + N}} - b_2|\Omega| \frac{2s_1}{8} k^{\frac{2s_1 + 4s_2}{2s_2 + N}}, \quad (1.10)$$

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where
\[ b_1 = \frac{(2s_2 + N)^{2s_2 + N}}{2s_1 + N} \left( 2^{-\left(N+1+2s_2\right)} \pi^{-\frac{3N}{2}} \omega_{N-1}^{-\frac{2-2s_2}{2s_2 + N}} \frac{\Gamma\left(\frac{N-2s_2}{2}\right)}{\Gamma\left(s_2 + 1\right)} \right)^{\frac{2s_2}{2s_2 + N}} \]

and
\[ b_2 = b_1 \cdot \frac{2s_1 + N}{N(2s_2 + N)^{2s_2 + N}} \left( 2^{-\left(N+1+2s_2\right)} \pi^{-\frac{3N}{2}} \omega_{N-1}^{-\frac{2-2s_2}{2s_2 + N}} \frac{\Gamma\left(\frac{N-2s_2}{2}\right)}{\Gamma\left(s_2 + 1\right)} \right)^{\frac{2s_2}{2s_2 + N}}. \]

We remark that when \( s_2 = 0 = \mu \), the constant \( b_1(s_1, 0) \) in Theorem 1.2 reduces to \( \frac{N}{N+2s_1} a(N, s_1) \), where \( a(N, s_1) \) is defined as in (1.7). Moreover, (1.10) coincides with Berezin-Li-Yau estimate for (1.9) (with \( s_1 = 1 \)).

The proof of Theorem 1.2 is inspired by the Berezin–Li–Yau method (see [11]) in which the authors treated mainly the function \( \Psi_k(x, y) = \sum_j \psi_j(x)\psi_j(y) \), where \( \psi_j \)'s are the eigenfunctions corresponding to the eigenvalues \( \lambda_j \) of (1.3). Denote \( F = \int_{\mathbb{R}^N} |(\mathcal{F}_x \Psi_k)(z, y)|^2 dy \), the key estimate is the following
\[ \int_{\mathbb{R}^N} Fdz \leq \left( \frac{N + 2}{N} \right)^{\frac{N}{N+2}} \left( |B_1| \| F \|_{L^\infty} \right)^{\frac{N}{N+2}}, \tag{1.11} \]
where \( |B_1| \) denotes the volume of unit ball in \( \mathbb{R}^N \).

For our problem (1.1), the situation is much more complicated. More precisely, to apply Berezin-Li-Yau method, we need to consider the function \( f = \int_{\mathbb{R}^N} |(\mathcal{F}_x \Psi_k)(z, y)|^2 dy \), where
\[ \Phi_k(x, y) = \sum_{j=1}^k \phi_j(x)\tilde{\phi}_j(y), \quad \tilde{\phi}_j := \left( (\Delta)^{s_2} + \mu \right)^{\frac{1}{2}} \phi_j \]
and \( \phi_j \)'s are eigenfunctions corresponding to the eigenvalues \( \lambda_j \) of (1.1). The most important estimate is \( \int_{\mathbb{R}^N} |z|^{2s_2 + \mu} f(z) dz \), which is controlled by \( \int_{\mathbb{R}^N} |z|^{2s_1} f dz \) and \( \| f \|_{L^\infty} \). These difficulties arise from the non homogeneous property of lower order term \( |z|^{2s_2 + \mu} f \) and the estimate \( \| f \|_{L^\infty} \).

For \( \| f \|_{L^\infty} \), the original tool is the Bessel's inequality, which requires orthonormal property for \( \{ \phi_j \} \) in \( L^2(\Omega) \), while \( \{ \phi_j \} \) is not an orthonormal sequence in \( L^2(\Omega) \) (it is orthonormal in \( H^s_{\mu, 0}(\Omega) \)). To overcome this difficulty, we transform \( \phi_j \) to \( \tilde{\phi}_j \) which is orthonormal in \( L^2(\mathbb{R}^N) \) but not supported in \( \Omega \). Further using certain delicate estimate, we have overcome the difficulty (see Section 3.2).

The following is a direct corollary from Theorem 1.2 using the monotonicity of map \( j \to \lambda_j(\mu) \).

**Corollary 1.3.** Let \( \mu \geq 0 \) and \( \{ \lambda_j(\mu) \}_{j \in \mathbb{N}} \) be the increasing sequence of eigenvalues of problem (1.1). Then for \( k \in \mathbb{N} \)
\[ \lambda_k(\mu) \geq b_1|\Omega|^{-\frac{2(s_1-s_2)}{N}} k^{-\frac{2(s_2-s_1)}{2s_2+N}} - \mu b_2|\Omega|^{-\frac{2s_1}{2s_2+N}} - \mu b_2|\Omega|^{-\frac{2s_1-4s_2}{2s_2+N}}, \tag{1.12} \]
where \( b_1 \) and \( b_2 \) are same as in Theorem 1.2.

For the upper bounds, due to the numerous challenges mentioned earlier, we only address the case that \( \mu = 0 \) and a restriction on the upper range of \( s_1 \). More specifically, we have the following estimates on upper bounds.

**Theorem 1.4.** Let \( 0 < s_2 < s_1 < \frac{1+s_1}{2} \), \( \mu = 0 \) and \( \{ \lambda_i(\mu) \}_{i \in \mathbb{N}} \) be the increasing sequence of eigenvalues of problem (1.1) and \( \Omega \) be a bounded \( C^2 \) domain. Then there exists \( c_0 = c_0(N, s_1, s_2, \Omega) > 0 \) and \( c_3 \in (0, 1 + \frac{2s_1-4s_2}{2s_2+N}) \) such that for \( k \in \mathbb{N} \)
\[ \sum_{j=1}^k \lambda_j(0) \leq b_3|\Omega|^{-\frac{2(s_1-s_2)}{N}} k^{1-\frac{2(s_1-s_2)}{2s_2+N}} + c_0 b_3, \tag{1.13} \]
where
\[ b_3 = (2\pi)^{2(s_1-s_2)} \omega_{N-1}^{-\frac{2(1+s_2)}{N}} \frac{N^{1-\frac{2(s_1-s_2)}{N}}}{N + 2(s_1-s_2)}. \]
It is worth noting that

(d) the constant $b_3$ in the upper bound \([1.13]\) coincides with \([1.9]\) with $s = s_1 - s_2$;

(e) the upper bound and the lower bound for our problem \([1.1]\) obtained in Theorem \([1.2]\) and Theorem \([1.4]\) are not enough to determine $\sum_{j=1}^{k} \lambda_j(\mu)$, even for $\mu = 0$.

(f) from our proofs, it isn’t too difficult to see that Theorem \([1.1]\) and Theorem \([1.2]\) can be extended to the case $s_1 = 1$. However, it fails for the upper bound in Theorem \([1.4]\) because of the restriction of $s_1 < \frac{1+s_2}{2} < 1$, which is not essential since it appears due to the technique difficulty.

The rest of the paper is organized as follows. In Section 2, we study qualitative properties of the eigenvalues and the corresponding eigenfunctions of problem \([1.1]\), namely Theorem \([1.1]\). Section 3 is devoted to show the lower bound of the sum of eigenvalues, namely Theorem \([1.2]\). Finally, in section 4, we discuss the upper bounds aspects, namely Theorem \([1.4]\).

**Notations:** Throughout this paper, $\omega_{N-1}$ denotes the surface area of unit sphere in $\mathbb{R}^N$, $B_r(x) \subset \mathbb{R}^N$ is an open ball of radius $r$ centered at $x \in \mathbb{R}^N$, and we set $B_r := B_r(0)$ for $r > 0$. For any set $A$ of $\mathbb{R}^N$, $|A|$ denotes the Lebesgue measure of $A$ and $\mathcal{F}(f)$ denotes the Fourier transform of a function $f$. By $C_c^\infty(\mathbb{R}^N)$, we denote $C^\infty$ functions in $\mathbb{R}^N$ with compact support.

## 2 Existence

We set $Q := \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$, where $\Omega^c = \mathbb{R}^N \setminus \Omega$ and for $1 \leq p < \infty$ define

$$W^{s,p}_0(\Omega) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable } \left| u = 0 \text{ a.e. in } \Omega^c \text{ and } \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy < \infty \right\}.$$  

Note that from the fractional Poincaré inequality, see \([15]\), the space $W^{s,p}_0(\Omega)$ is endowed with the norm defined as

$$\|u\|_{W^{s,p}_0(\Omega)} := \left( \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{1/p}.$$  

**Theorem 2.1.** Let $0 < s_2 < s_1 < 1$, $p > 1$ and $\Omega$ be a bounded domain in $\mathbb{R}^N$, then $W^{s_1,p}_0(\Omega) \hookrightarrow W^{s_2,p}_0(\Omega)$ is continuous and compact.

Before starting to prove the above theorem, we need to introduce the Besov Space over $\Omega$. We follow the notations of \([12]\), Section 2.3.1.

**Definition 2.2.** Let $\mathcal{S}$ denote the Schwartz class functions on $\mathbb{R}^N$ and $\mathcal{A}$ be collection of all sequence $\eta = \{\eta_i\}_{i=0}^{\infty} \in \mathcal{S}(\mathbb{R}^N)$ such that

$$\supp(\eta_0) \subset \{x : |x| \leq 2\}, \supp(\eta_j) \subset \{x : 2^{j-1} \leq |x| \leq 2^{j+1}\} \text{ for } j = 1, 2, \ldots,$$

$$\sum_{i=0}^{\infty} \eta_j(x) = 1,$$

and for every multi-index $\alpha$ there exists a positive number $c_\alpha$ such that

$$2^{j|\alpha|} |D^\alpha \eta_j(x)| \leq c_\alpha \quad \forall \, j = 0, 1, 2, \ldots \quad \text{and} \quad \forall \, x \in \mathbb{R}^N.$$

**Definition 2.3.** Let $s \in (-\infty, \infty)$ and $p, q \in (0, \infty]$ and $\{\eta_i\}_{i=0}^{\infty} \in \mathcal{A}$. Then

$$B^s_{p,q}(\mathbb{R}^N) := \left\{ f \in \mathcal{S}'(\mathbb{R}^N) : \|f\|_{B^s_{p,q}(\mathbb{R}^N)} := \|2^{sj} \mathcal{F}^{-1}(\eta_j \mathcal{F}(f))\|_{L^p(\mathbb{R}^N)} < \infty \right\}.$$  

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It can be shown that the quasi-norm $\|f\|_{L^p_{\eta}(\mathbb{R}^N)}$ does not depend on the choice of $\eta \in A$ (see [42, Proposition 1 in 2.3.2]).

**Definition 2.4.** Let $\mathcal{D}'(\Omega)$ denote the set of all distributions over $\Omega$. For $1 \leq p, q \leq \infty$, and $0 < s < 1$, we set

$$B^s_{p,q}(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) : \exists g \in B^s_{p,q}(\mathbb{R}^N) \text{ with } g|_{\Omega} = u \right\}$$

and

$$\|u\|_{B^s_{p,q}(\Omega)} = \inf_{g \in B^s_{p,q}(\mathbb{R}^N), g|_{\Omega} = u} \|g\|_{B^s_{p,q}(\mathbb{R}^N)}.$$

Here $B^s_{p,q}(\Omega)$ is called the Besov Space over $\Omega$.

**Lemma 2.5.**[42, Theorem 3.1.1(i)] Let $p_0, q_0, p_1, q_1 \in (0, \infty)$, $s_0, s_1 \in (-\infty, \infty)$ and $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$. Then the following embedding

$$B^{s_0}_{p_0, q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1, q_1}(\Omega)$$

is continuous if $s_0 - \frac{N}{p_0} > s_1 - \frac{N}{p_1}$.

**Lemma 2.6.**[42, pg.233] Let $0 < s_2 < s_1$ and $p, q \in (0, \infty)$ and $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$. Then the embedding $B^{s_1}_{p,q}(\Omega) \hookrightarrow B^{s_2}_{p,q}(\Omega)$ is compact.

**Proof of Theorem 2.1.** From [2, Lemma 2.2], we know that the embedding $W^{s_1,p}_0(\Omega) \subset W^{s_2,p}_0(\Omega)$ is continuous. Moreover, from [42, pg. 209]), it follows that

$$\|u\|_{W^{s,p}_0(\Omega)} := \|u\|_{L^p(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \right)^{\frac{1}{p}}$$

is an equivalent norm for $\|u\|_{B^s_{p,q}(\Omega)}$ for $p \in (1, \infty), s \in (0, 1)$. Therefore, by Lemma 2.6, we have that

$$W^{s_1,p}_0(\Omega) \subset W^{s_2,p}_0(\Omega)$$

is compact. (2.1)

Now let $\{u_n\}$ be a bounded sequence in $W^{s_1,p}_0(\Omega)$, to prove the theorem we need to extract a convergent subsequence in $W^{s_2,p}_0(\Omega)$. By Rellich compactness, up to a subsequence $u_n \to u$ in $L^p(\Omega)$ for some $u \in W^{s_1,p}_0(\Omega)$. For that subsequence we define $v_n := u_n - u$. Therefore, we need to show $v_n \to 0$ in $W^{s_2,p}_0(\Omega)$. As $v_n = 0$ in $\Omega^c$,

$$\|v_n\|_{W^{s_2,p}_0(\Omega)}^p = \int_{\Omega \times \Omega} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + 2 \int_{\Omega \times \Omega} \int_{\Omega^c} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy$$

$$\leq \|v_n\|_{W^{s_2,p}(\Omega)}^p + 2 \int_{\Omega \times \Omega} \int_{\Omega^c} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy. \quad (2.2)$$

By Poincaré inequality in [45, Theorem 6.7] there exists a positive constant $c > 0$ such that

$$\|v\|_{L^p(\Omega)} \leq c \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \quad \text{for any } v \in W^{s,p}_0(\Omega).$$

Thus, from the definition of $W^{s_1,p}_0(\Omega)$ and $W^{s_1,p}(\Omega)$, there exists $c > 0$ such that

$$\|v_n\|_{W^{s_1,p}_0(\Omega)} \leq c \|v_n\|_{W^{s_1,p}(\Omega)}.$$

Therefore, $\{v_n\}$ is a bounded sequence in $W^{s_1,p}_0(\Omega)$. Consequently, by [231], up to a subsequence $v_n \to 0$ in $W^{s_2,p}(\Omega)$. Therefore, to complete the proof, we only have to show that

$$\int_{\Omega \times \Omega} \int_{\Omega^c} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \to 0.$$
To that end, let $\epsilon > 0$ be arbitrary and we define $M := \sup_n \| v_n \|_{W^{s_1,p}_0(\Omega)}$. Then
\[
\int_{x \in \Omega} \int_{y \in \Omega^c} \frac{|v_n(x) - v_n(y)|^p}{|x - y|^{N + s_2p}} dxdy = \int_{x \in \Omega} \int_{y \in \Omega^c \cap |x - y| < \epsilon} \frac{|v_n(x) - v_n(y)|^p}{|x - y|^{N + s_2p}} dxdy
+ \int_{x \in \Omega} \int_{y \in \Omega^c \cap |x - y| \geq \epsilon} \frac{|v_n(x) - v_n(y)|^p}{|x - y|^{N + s_2p}} dxdy
= \int_{x \in \Omega} \int_{y \in \Omega^c \cap |x - y| < \epsilon} \frac{|v_n(x) - v_n(y)|^p}{|x - y|^{N + s_1p}} |x - y|^{(s_1 - s_2)p} dxdy
+ \int_{x \in \Omega} \left( \int_{y \in \Omega^c \cap |x - y| \geq \epsilon} \frac{|v_n(x) - v_n(y)|^p}{|x - y|^{N + s_2p}} \right) |v_n(x)|^p dx
\leq \epsilon^{(s_1 - s_2)p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x - y|^{N + s_2p}} dxdy + C(\epsilon) \| v_n \|_{L^p(\Omega)}^{p}\| v_n \|_{L^p(\Omega)}^{p-1}
\leq \epsilon^{(s_1 - s_2)p} M + o(1),
\]
where $o(1) \to 0$ as $n \to \infty$. Therefore, the above integral can be made arbitrary small. Hence from (2.2), we conclude that $\| v_n \|_{W^{s_2,p}_0(\Omega)} \to 0$. This completes the proof. \hfill \Box

In the particular case that $p = 2$, for $s \in (0, 1)$ we set
\[
\mathbb{H}^s_0(\Omega) = W^{s_1,2}_0(\Omega)
\]
which is a Hilbert space with the inner product $E_s(u, v)$ for $u, v \in \mathbb{H}^s_0(\Omega)$.

**Proof of Theorem 1.1.** The functional
\[
\mathcal{M}_1 := \{ u \in \mathbb{H}^s_0(\Omega), \| u \|_{s_2} + \mu \| u \|_{L^2(\Omega)} = 1 \},
\]
where $\| \cdot \|_{s_2} + \mu \| \cdot \|_{L^2(\Omega)}$ is equivalent to $\| \cdot \|_{s_2}$ in the space $\mathbb{H}^{s_2}_0(\Omega)$ for $\mu > -\lambda_{s_2,1}$. Then we have that
\[
\Phi(u) < +\infty \quad \text{for } u \in \mathcal{M}_1.
\]
Put $\lambda_1(\mu) := \inf_{\mathcal{M}_1} \Phi$. By Theorem 2.1 the embedding $W^{s_1,p}_0(\Omega) \hookrightarrow W^{s_2,p}_0(\Omega)$ is compact and therefore, it follows that $\lambda_1(\mu)$ is attained by a function $\phi_1 \in \mathcal{M}_1$. Consequently, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that
\[
E_{s_1}(\phi_1, w) = \frac{1}{2} \Phi'(\phi_1)w = \lambda \left( E_{s_2}(\phi_1, w) + \mu \int_{\Omega} \phi_1 w dx \right) \quad \text{for all } w \in \mathbb{H}^{s_1}_0(\Omega).
\]
Choosing $w = \phi_1$ yields $\lambda = \lambda_1(\mu)$. Hence $\phi_1$ is an eigenfunction of (1.1) corresponding to the eigenvalue $\lambda_1(\mu)$. Moreover $\lambda_1(\mu) > 0$. Next we proceed inductively and assume that $\phi_2, \ldots, \phi_k$ are already given for some $k \in \mathbb{N}$ with the properties that for $j = 2, \ldots, k$, the function $\phi_j$ is a minimizer of $\Phi$ within the set
\[
\mathcal{M}_j := \{ u \in \mathbb{H}^{s_1}_0(\Omega) : \| u \|_{s_2} + \| u \|_{L^2(\Omega)} = 1, \ E_{s_2}(u, \phi_m) + \mu \int_{\Omega} u \phi_m dx = 0 \text{ for } m = 1, \ldots, j - 1 \},
\]
and
\[
E_{s_1}(\phi_j, \varphi) = \lambda_j(\mu) \left( E_{s_2}(\phi_j, \varphi) + \mu \int_{\Omega} \phi_j \varphi dx \right) \quad \text{for all } \varphi \in \mathbb{H}^{s_1}_0(\Omega).
\] (2.3)
Again by the compact embedding in Theorem 2.1, the value $\lambda_{k+1}$ is attained by a function $\phi_{k+1} \in \mathcal{M}_{k+1}$. Consequently, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\mathcal{E}_1(\phi_{k+1}, \varphi) = \lambda \left( \mathcal{E}_2(\phi_{k+1}, \varphi) + \mu \int_{\Omega} \phi_{k+1} \varphi \, dx \right) \quad \text{for all } \varphi \in \mathcal{M}_{k+1}(\Omega). \tag{2.4}$$

Choosing $\varphi = \phi_{k+1}$, we have that $\lambda = \lambda_{k+1}(\mu)$. Moreover, for $j = 1, \ldots, k$, we have, by (2.3) and the definition of $\mathcal{M}_{k+1}(\Omega)$, that

$$\mathcal{E}_1(\phi_{k+1}, \phi_j) = \mathcal{E}_1(\phi_j, \phi_{k+1}) = \lambda_j(\mu) \left( \mathcal{E}_2(\phi_j, \phi_{k+1}) + \mu \int_{\Omega} \phi_j \phi_{k+1} \, dx \right)$$

$$= 0$$

$$= \lambda_{k+1}(\mu) \left( \mathcal{E}_2(\phi_{k+1}, \phi_j) + \mu \int_{\Omega} \phi_j \phi_{k+1} \, dx \right).$$

Hence (2.4) holds with $\lambda = \lambda_{k+1}(\mu)$ for all $\varphi \in \mathbb{H}^s_0(\Omega)$. Inductively, we have now constructed a normalized sequence $(\phi_k)_k$ in $\mathbb{H}^s_0(\Omega)$ and a nondecreasing sequence $\{\lambda_k\}_k$ in $\mathbb{R}$ such that property (i) holds and such that $\lambda_k$ is an eigenfunction of (1.1) corresponding to $\lambda = \lambda_k(\mu)$ for every $k \in \mathbb{N}$. Moreover, by construction, the sequence $(\phi_k)_k$ forms an orthonormal system in $\mathbb{H}^s_\mu(\Omega)$.

Next we show property (iii), i.e., $\lim_{k \to +\infty} \lambda_k(\mu) = +\infty$. Supposing by contradiction that $c := \lim_{k \to +\infty} \lambda_k(\mu) < +\infty$, we deduce that $\mathcal{E}_1(\phi_k, \phi_k) \leq c$ for every $k \in \mathbb{N}$. Hence the sequence $(\phi_k)$ is bounded in $\mathbb{H}^s_0(\Omega)$, and therefore by Rellich compactness theorem, $(\phi_k)$ contains a convergent subsequence $(\phi_{k_l})_l$ in $L^2(\Omega)$. This however is impossible since the functions $(\phi_{k_l})_l$ are orthonormal in $H^s_\mu(\Omega)$. Hence (iii) is proved.

Next, to prove that $\{\phi_k : k \in \mathbb{N}\}$ is an orthonormal basis of $H^s_\mu(\Omega)$, we first suppose by contradiction that there exists $v \in \mathbb{H}^s_\mu(\Omega)$ with $\|v\|_2 + \mu \|v\|_L^2(\Omega) = 1$ and $\mathcal{E}_2(v, \phi_k) + \mu \int_{\Omega} v \phi_k \, dx = 0$ for any $k \in \mathbb{N}$. Since $\lim_{k \to +\infty} \lambda_k(\mu) = +\infty$, there exists an integer $k_0 > 0$ such that

$$\Phi(v) < \lambda_{k_0}(\mu) = \inf_{\mathcal{M}_{k_0}} \Phi(u),$$

which by definition of $\mathcal{M}_{k_0}$ implies that $\mathcal{E}_2(v, \phi_k) + \mu \int_{\Omega} v \phi_k \, dx \neq 0$ for some $k \in \{1, \ldots, k_0 - 1\}$. This is a contradiction to $\mathcal{E}_2(v, \phi_k) + \mu \int_{\Omega} v \phi_k \, dx = 0$ for any $k \in \mathbb{N}$. Thus, we conclude that $\mathbb{H}^s_\mu(\Omega)$ is contained in the $H^s_\mu$-closure of the span of $\{\phi_k : k \in \mathbb{N}\}$. Since $\mathbb{H}^s_\mu(\Omega)$ is dense in $H^s_\mu(\Omega)$, we conclude that the span of $\{\phi_k : k \in \mathbb{N}\}$ is dense in $H^s_\mu(\Omega)$, and hence $\{\phi_k : k \in \mathbb{N}\}$ is an orthonormal basis of $H^s_\mu(\Omega)$. This proves (ii).

Finally, we show (iv). It follows by the definition of the first eigenvalue $\lambda_1(\mu)$ that for $\mu \in (-\lambda_{s_2,1}, +\infty)$, the map $\mu \mapsto \lambda_1(\mu)$ is decreasing.

Let $(\lambda_{s_2,1}, \varphi_{s_2,1})$ be the first eigenvalue and the corresponding eigenfunction of (1.3) with $s = s_2$. We may assume that $\varphi_{s_{2,1}} > 0$ in $\Omega$, $\int_{\Omega} \varphi_{s_{2,1}}^2 \, dx = 1$. Let $\eta_0$ be a smooth function with compact support in $B_{r_0}(x_0)$ such that $B_{2r_0}(x_0) \subset \Omega$ with $\int_{\Omega} \eta_0^2 \, dx = 1$. Note that $\eta_0 \not\equiv \varphi_{s_{2,1}}$ in $\Omega$. Then $\mathcal{E}_2(\eta_0, \eta_0) > \lambda_{s_2,1}$, otherwise $\eta_0 \equiv \varphi_{s_{2,1}}$ since the first eigenfunction of (1.3) with $s = s_2$ is simple.

Thus, we have that

$$\lambda_1(\mu) \leq \frac{\mathcal{E}_1(\eta_0, \eta_0)}{\mathcal{E}_2(\eta_0, \eta_0) + \mu \int_{\Omega} \eta_0^2 \, dx}$$

$$\leq \frac{\mathcal{E}_1(\eta_0, \eta_0)}{\mathcal{E}_2(\eta_0, \eta_0) - \lambda_{s_2,1}} < +\infty \quad \text{as } \mu \to -\lambda_{s_{2,1}}^+. $$

This completes the proof. \qed
3 Lower bounds

3.1 Important estimate

To prove the Li-Yau’s type lower bound for (1.1), we need the following results.

**Proposition 3.1.** Let \( \mu \geq 0, 0 < s_2 < s_1 < 1 \) and \( f \) be a real-valued function defined on \( \mathbb{R}^N \) with \( 0 \leq f \leq M_1 \),

\[
\int_{\mathbb{R}^N} f(z)|z|^{2s_1}dz \leq M_2,
\]
then

\[
\int_{\mathbb{R}^N} (|z|^{2s_2} + \mu) f(z)dz \leq \frac{\omega_{N-1}M_1}{2s_2 + N} \left( \frac{(2s_1 + N)M_2}{M_1\omega_{N-1}} \right)^{\frac{2s_2 + N}{2s_1 + N}} \left[ 1 + \mu \frac{2s_2 + N}{N} \left( \frac{(2s_1 + N)M_2}{M_1\omega_{N-1}} \right)^{\frac{2s_2}{2s_1 + N}} \right].
\]

**Proof of Proposition 3.1.** Let

\[
h(z) = \begin{cases} 
(\frac{|z|^{2s_2} + \mu}{|z|^{2s_1}})M_1 & \text{for } |z| < R, \\
0 & \text{for } |z| \geq R,
\end{cases}
\]

where \( R > 0 \) such that

\[
\int_{B_R} M_1 |z|^{2s_1}dz = M_2.
\]

Direct computation shows that

\[
R^{2s_1 + N} = \frac{(2s_1 + N)M_2}{M_1\omega_{N-1}}.
\]

Since

\[
|z|^{2s_2} \left( |z|^{2(s_1 - s_2)} - R^{2(s_1 - s_2)} \right) \left( f(z) - (|z|^{2s_2} + \mu)^{-1}h(z) \right) \geq 0
\]
and

\[
\mu \left( |z|^{2s_1} - R^{2s_1} \right) \left( f(z) - (|z|^{2s_2} + \mu)^{-1}h(z) \right) \geq 0,
\]

it follows that

\[
\int_{\mathbb{R}^N} |z|^{2s_2} \left( f(z) - (|z|^{2s_2} + \mu)^{-1}h(z) \right)dz 
\]

\[
\leq \frac{1}{R^{2(s_1 - s_2)}} \int_{\mathbb{R}^N} |z|^{2s_2} |z|^{2s_1} \left( f(z) - (|z|^{2s_2} + \mu)^{-1}h(z) \right)dz 
\]

\[
\leq \frac{1}{R^{2(s_1 - s_2)}} \int_{\mathbb{R}^N} |z|^{2s_1} (f(z) - M_1\chi_{B_R})dz \leq 0, \quad (3.1)
\]

and

\[
\mu \int_{\mathbb{R}^N} \left( f(z) - (|z|^{2s_2} + \mu)^{-1}h(z) \right)dz 
\]

\[
\leq \frac{\mu}{R^{2s_1}} \int_{\mathbb{R}^N} |z|^{2s_1} \left( f(z) - (|z|^{2s_2} + \mu)^{-1}h(z) \right)dz 
\]

\[
\leq \frac{\mu}{R^{2s_1}} \int_{\mathbb{R}^N} |z|^{2s_1} (f(z) - M_1\chi_{B_R})dz \leq 0. \quad (3.2)
\]
Combining (3.1) and (3.2), we have
\[
\int_{\mathbb{R}^N} |z|^{2s_2} + \mu f(z) dz \leq \int_{\mathbb{R}^N} h(z) dz = \frac{\omega_{N-1} M_1}{2s_2 + N} R^{2s_2 + N} \left( 1 + \frac{2s_2 + N}{N} R^{-2s_2} \right)
\]
\[
= \frac{\omega_{N-1} M_1}{2s_2 + N} \left( \frac{(2s_1 + N) M_2}{M_1 \omega_{N-1}} \right)^{2s_2 + N} \left[ 1 + \frac{2s_2 + N}{N} \left( \frac{(2s_1 + N) M_2}{M_1 \omega_{N-1}} \right)^{-2s_2 + N} \right].
\]
We complete the proof. □

We also need the following Lemma.

**Lemma 3.2.** Assume that \(1 > \tau_1 > \tau_2 > 0\) and \(d_1 > 0\).
Let \(r_1\) be the solution of
\[
r_{\tau_1} \left( 1 + r^{-\tau_2} \right) = d_1,
\]
then
\[
\frac{d_1^{\tau_1}}{\tau_1} \left( 1 - \frac{1}{\tau_1} \frac{2}{\tau_1} \right) < r_1 < \frac{d_1^{\tau_1}}{\tau_1}.
\]

**Proof.** Let
\[
f(r) = r_{\tau_1} \left( 1 + r^{-\tau_2} \right),
\]
then \(f(0) = 0\), \(\lim_{r \to +\infty} f(r) = +\infty\) and \(f'(r) = \tau_1 r_{\tau_1-1} + (\tau_1 - \tau_2) r_{\tau_1 - \tau_2 - 1} > 0\), \(f''(r) = \tau_1 (\tau_1 - 1) r_{\tau_1 - 2} + (\tau_1 - \tau_2) (\tau_1 - \tau_2 - 1) r_{\tau_1 - \tau_2 - 2} < 0\). As a consequence, \(f\) is strictly increasing, concave in \((0, +\infty)\) and for any \(d_1 > 0\), there exists a unique solution \(r_1\) such that \(f(r_1) = d_1\). Moreover, since \(f(r) > 0\) for \(r > 0\), we conclude \(r_1 > 0\).

Let
\[
R_1 = \frac{d_1^{\tau_1}}{\tau_1} \left( 1 - \frac{1}{\tau_1} \frac{2}{\tau_1} \right) + \quad \text{and} \quad R_2 = \frac{d_1^{\tau_1}}{\tau_1},
\]
where \(a_+ = \max\{0, a\}\). Note that
\[
\frac{f(R_1)}{d_1} = 1 + d_1^{\frac{\tau_2}{\tau_1}} > 1
\]
and if \(\frac{d_1^{\tau_2}}{\tau_1} < 1\), then
\[
\frac{f(R_1)}{d_1} = \left( 1 - \frac{1}{\tau_1} \frac{2}{\tau_1} \right)^{\tau_1} + d_1^{\frac{\tau_2}{\tau_1}} \left( 1 - \frac{1}{\tau_1} \frac{2}{\tau_1} \right)^{\tau_1 - \tau_2}
\]
\[
\leq 1 - d_1^{\frac{\tau_2}{\tau_1}} + d_1^{\frac{\tau_2}{\tau_1}} - \frac{\tau_1 - \tau_2}{\tau_1} d_1^{\frac{2}{\tau_1}}
\]
\[
< 1,
\]
where we used the fact that for \(\tau \in (0, 1)\),
\[
(1 - t)^{\tau} \leq 1 - \tau t \quad \text{for any } t \in (0, 1).
\]
On the other hand, if \(\frac{1}{\tau_1} \frac{2}{\tau_1} \geq 1\), then \(R_1 = 0\). Therefore, \(\frac{f(R_1)}{d_1} = 0 < 1\). Thus, in both the cases \(f(R_1) < d_1 < f(R_2)\). Now using the fact that \(f\) is strictly increasing, continuous and \(f(r_1) = d_1\), we conclude \(R_1 < r_1 < R_2\). This completes the proof. □

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3.2 Lower bound

The Bessel inequality plays an important role in the Li-Yau method for the lower bound of eigenvalues: Let $\mathbb{H}$ be an inner product space or a Hermitian product space together with its product function $\langle \cdot, \cdot \rangle$. Let $e_1, e_2, \cdots$ be any (finite or infinite) orthonormal sequence. Then for any $x \in \mathbb{H}$,
\[
\sum_{j \geq 1} |\langle x, e_j \rangle|^2 \leq \langle x, x \rangle = \|x\|_H^2.
\]

Proof of Theorem 1.2. Let
\[
\Phi_k(x, y) = \sum_{j=1}^k \phi_j(x)\hat{\phi}_j(y),
\]
where
\[
\hat{\phi}_j := \left( (-\Delta)^{s_2} + \mu \right)^{\frac{1}{2}} \phi_j = \mathcal{F}^{-1}\left[ \left( |z|^{2s_2} + \mu \right)^{\frac{1}{2}} \mathcal{F}(\phi_j) \right].
\]

Using the Fourier transform, we have that
\[
\mathcal{F}_x(\Phi_k)(z, y) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \Phi_k(x, y)e^{ix \cdot z} dx
\]
and
\[
\mathcal{F}\left( \left( (-\Delta)^{s_2} + \mu \right)^{\frac{1}{2}} \phi_j \right)(z) = \left( |z|^{2s_2} + \mu \right)^{\frac{1}{2}} \mathcal{F}(\phi_j(z)) \quad \text{for} \quad \mu \geq 0.
\]

Therefore, doing a straightforward computation we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( |z|^{s_2} + \mu \right)^{\frac{1}{2}} \mathcal{F}_x(\Phi_k)(z, y) dydz = \sum_{j=1}^k \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \hat{\phi}_j(x)\hat{\phi}_j(y) dydx = k, \quad (3.3)
\]
by the orthonormality of $\{\hat{\phi}_j\}_{j \in \mathbb{N}}$ in $L^2(\mathbb{R}^N)$. Next, we estimate $\left\| \int_{\mathbb{R}^N} \mathcal{F}_x(\Phi_k)(z, y) \right\|_{L^\infty(\mathbb{R}^N)}^2$.

Here the main difficulty comes from the fact that $\{\phi_j\}_{j}$ is not an orthonormal sequence in $L^2(\Omega)$.

Let $G_\mu$ be the fundamental solution of $(-\Delta)^{s_2} + \mu$ in $\mathbb{R}^N$. Then
\[
0 < G_\mu(x) \leq G_0(x) = a_{N, s_2}|x|^{2s_2-N} \quad \text{for} \quad \mu \geq 0,
\]
\[
a_{N, s_2} = 2^{-2s_2}\pi^{-\frac{N}{2}} \frac{\Gamma(N-2s_2)}{\Gamma(s_2)} = 2^{-2s_2}\pi^{-\frac{N}{2}} \frac{\Gamma(N-2s_2)}{\Gamma(s_2 + 1)} s_2 := \tilde{a}_{N, s_2}.
\]

Note that
\[
((-\Delta)^{s_2} + \mu)^{-1} g = G_\mu * g \quad \text{in} \quad \mathbb{R}^N
\]
for $g \in L^2(\mathbb{R}^N)$. Using Fourier transform, we can see that
\[
\mathcal{F}\left( (\cdot) (-\Delta)^{s_2} + \mu \right)^{-1} g \right)(z) = \left( |z|^{2s_2} + \mu \right)^{-1} \mathcal{F}(g(z)).
\]

Consequently, $((-\Delta)^{s_2} + \mu)^{-1/2} g$ can be defined as follows
\[
((-\Delta)^{s_2} + \mu)^{-1/2} g = \mathcal{F}^{-1}\left[ \left( |z|^{2s_2} + \mu \right)^{-1/2} \right] \mathcal{F}^{-1}\left[ (\cdot)^{-1/2} \mathcal{F}(g) \right].
\]
∫_{\mathbb{R}^N} |\mathcal{F}_x(\Phi_k)(z,y)|^2 dy = (2\pi)^{-N} \sum_{j=1}^{k} \left| \int_{\mathbb{R}^N} \phi_j(x) (e^{ixz} \chi_\Omega(x)) dx \right|^2 \int_{\mathbb{R}^N} |\hat{\phi}_j(y)|^2 dy

= (2\pi)^{-N} \sum_{j=1}^{k} \left| \int_{\mathbb{R}^N} \tilde{\phi}_j(x) ((-\Delta)^{s_2} + \mu)^{\frac{1}{2}} \phi_j(x) ((-\Delta)^{s_2} + \mu)^{-\frac{1}{2}} (e^{ixz} \chi_\Omega(x)) dx \right|^2

\leq (2\pi)^{-N} \int_{\mathbb{R}^N} \left| ((-\Delta)^{s_2} + \mu)^{\frac{1}{2}} (e^{ixz} \chi_\Omega(x)) \right|^2 dx

= (2\pi)^{-N} \int_{\Omega} \left| G_\mu (e^{ixz} \chi_\Omega(x)) \right| dx

\leq (2\pi)^{-N} \int_{\Omega} G_0 \chi_\Omega dx

= (2\pi)^{-N} a_{N,s_2} \int_{\Omega} \int |x - y|^{2s_2-N} dy dx

\leq (2\pi)^{-N} a_{N,s_2} |\Omega| \sup_{x \in \Omega} \int_{\Omega} |x - y|^{2s_2-N} dy.

Now we choose \( r > 0 \) such that \( |\Omega| = \frac{\omega_{N-1} r^N}{N} \) and using rearrangement inequality, we have

\[
\sup_{x \in \Omega} \int_{\Omega} |x - y|^{2s_2-N} dy \leq \int_{B_r(x)} |x - y|^{2s_2-N} dy
\]

\[
= \frac{\omega_{N-1} r^{2s_2}}{2s_2} = \frac{\omega_{N-1} N|\Omega|}{2s_2} \left( \frac{2s_2}{\omega_{N-1}} \right)^{\frac{2s_2}{N}} = a_0 |\Omega|^{\frac{2s_2}{N}},
\]

where

\[
a_0 = \frac{1}{2s_2} N^{\frac{2s_2}{N}} \frac{1}{\omega_{N-1}^{\frac{2s_2}{N}}}.
\] (3.5)

As a consequence, we obtain that

\[
\int_{\mathbb{R}^N} |\mathcal{F}_x(\Phi_k)(z,y)|^2 dy \leq (2\pi)^{-N} a_{N,s_2} a_0 |\Omega|^{1+\frac{2s_2}{N}} = \tilde{a}_{N,s_2} \frac{N^{\frac{2s_2}{N}} 1 - \frac{2s_2}{\omega_{N-1}^{\frac{2s_2}{N}}} |\Omega|^{1+\frac{2s_2}{N}}.}
\] (3.6)

Meanwhile, using the definition of \( \Phi_k \), it also follows that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |z|^{2s_1} |\mathcal{F}_x(\Phi_k)(z,y)|^2 dy dz = \int_{\mathbb{R}^N} \int_{\Omega} \Phi_k(x,y) (-\Delta)^{s_1} \Phi_k(x,y) dy dx
\]

\[
= \sum_{j=1}^{k} \int_{\Omega} \phi_j(-\Delta) \phi_j dx \int_{\Omega} |\hat{\phi}_j(y)|^2 dy
\]

\[
= \sum_{j=1}^{k} \chi_j(\mu).
\]

Now we apply Proposition 3.1 to the function

\[
f(z) = \int_{\mathbb{R}^N} |(\mathcal{F}_z \Phi_k)(z,y)|^2 dy
\]

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with
\[ M_1 = (2\pi)^{-N} a_{N,s_2} a_0 |\Omega|^{\frac{N+2s_2}{N}} \quad \text{and} \quad M_2 = \sum_{j=1}^{k} \lambda_j(\mu), \]
then we conclude that
\[ k \leq \frac{\omega_{N,-1} M_1}{2s_2 + N} \left( \frac{(2s_2 + N)M_2}{M_1 \omega_{N,-1}} \right)^{\frac{2s_2+N}{2s_2+1}} \left( 1 + \mu \frac{2s_2 + N}{N} \left( \frac{(2s_2 + N)M_2}{M_1 \omega_{N,-1}} \right)^{-\frac{2s_2}{2s_2+1}} \right). \quad (3.7) \]

**Case: \( \mu = 0 \).** In this case, (3.7) reduces to
\[ k \leq \frac{\omega_{N,-1} M_1}{2s_2 + N} \left( \frac{(2s_2 + N)M_2}{M_1 \omega_{N,-1}} \right)^{\frac{2s_2+N}{2s_2+1}}, \]
that is
\[ M_2 \geq \frac{(2s_2 + N)^{\frac{2s_2+N}{2s_2+1}}}{2s_1 + N} (M_1 \omega_{N,-1})^{-\frac{2(s_1-s_2)}{2s_2+1} k^{1+\frac{2(s_1-s_2)}{2s_2+1}}}, \]
which implies that
\[ \sum_{j=1}^{k} \lambda_j(0) \geq \frac{(2s_2 + N)^{\frac{2s_2+N}{2s_2+1}}}{2s_1 + N} ((2\pi)^{-N} a_{N,s_2} a_0 \omega_{N,-1})^{-\frac{2(s_1-s_2)}{2s_2+1} k^{1+\frac{2(s_1-s_2)}{2s_2+1}}} \]
\[ = b_1 |\Omega|^{\frac{2(s_1-s_2)}{2s_2+1} k^{1+\frac{2(s_1-s_2)}{2s_2+1}}}. \]

**Case: \( \mu > 0 \).** By Lemma 3.2 with
\[ \tau_1 = \frac{2s_2 + N}{2s_1 + N}, \quad \tau_2 = \frac{2s_2}{2s_1 + N} \]
and
\[ r = \left( \mu \frac{2s_2 + N}{N} \right)^{\frac{2s_2+N}{2s_2}} \left( \frac{2s_1 + N)M_2}{M_1 \omega_{N,-1}} \right)^{\frac{2s_2+N}{2s_2}}, \quad d_1 = \frac{2s_2 + N}{\omega_{N,-1} M_1} \left( \frac{2s_2 + N}{N} \right)^{-\frac{2s_2+N}{2s_2} k} > 0. \]

Then
\[ \left( \mu \frac{2s_2 + N}{N} \right)^{\frac{2s+1+N}{2s_2}} \left( \frac{(2s_1 + N)M_2}{M_1 \omega_{N,-1}} \right)^{\frac{2s_2+N}{2s_2}} \geq \left( \frac{2s_2 + N}{\omega_{N,-1} M_1} \left( \mu \frac{2s_2 + N}{N} \right)^{\frac{2s_2+N}{2s_2}} \right)^{\frac{2s_2+N}{2s_2}} \left( \frac{2s_1 + N)M_2}{M_1 \omega_{N,-1}} \right)^{\frac{2s_2+N}{2s_2}} \left[ 1 - \frac{N-2s_2 + N}{\omega_{N,-1} M_1} \left( \mu \frac{2s_2 + N}{N} \right)^{-\frac{2s_2+N}{2s_2} k} \right]^{\frac{2s_2+N}{2s_2}}, \]
which is equivalent to
\[ M_2 \geq \frac{(2s_2 + N)^{\frac{2s_2+N}{2s_2+1}}}{2s_1 + N} (M_1 \omega_{N,-1})^{-\frac{2(s_1-s_2)}{2s_2+1} k^{1+\frac{2(s_1-s_2)}{2s_2+1}}} \]
that is,
\[ \sum_{j=1}^{k} \lambda_j(\mu) \geq b_1 |\Omega|^{-\frac{2(s_1-s_2)}{2s_2+1} k^{1+\frac{2(s_1-s_2)}{2s_2+1}}} \]
\[ - \mu b_1 \frac{2s_2 + N}{N(2s_2 + N)^{\frac{2s_2+N}{2s_2+1}}} \left( (2\pi)^{-N} \omega_{N,-1} c_{s_2} a_0 \right)^{\frac{2s_2+N}{2s_2+1} k^{1+\frac{2(s_1-s_2)}{2s_2+1}}} |\Omega|^{-\frac{2s_1}{N} k^{1+\frac{2(s_1-s_2)}{2s_2+1}}}]. \]

Substituting the value of \( a_0 \) and \( a_{N,s} \) from (3.5) and (3.4) respectively, we complete the proof. \( \Box \)

**Proof of Corollary 1.3** It follows by the nondecreasing monotonicity of \( k \rightarrow \lambda_k(\mu) \) that
\[ \lambda_k(\mu) \geq \frac{1}{k} \sum_{j=1}^{k} \lambda_j(\mu). \]
Applying the above inequality in (1.10) yields (1.12). \( \Box \)
4 Upper bounds

In order to prove the upper bounds, we need following lemmas.

**Lemma 4.1.** Let \( s \in (0, 1) \) and for fixed \( z \in \mathbb{R}^N \setminus \{0\} \)

\[
v_z(x) = e^{ixz}, \quad \forall x \in \mathbb{R}^N,
\]

then

\[
(-\Delta)^s v_z(x) = |z|^{2s} v_z(x), \quad \forall x \in \mathbb{R}^N.
\]

**Proof.** Without loss of generality, it is enough to prove (4.2) with \( z = te_1 \), where \( t > 0 \) and \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N \). For this, we write

\[
v_t(x) = \mu_z(x_1) = e^{ix_1}, \quad x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}.
\]

Using [10] Lemma 3.1 we obtain that

\[
(-\Delta)^s v_t(x) = (-\Delta)^s v_t(x_1).
\]

Now we claim that

\[
(-\Delta)^s v_t(x_1) = t^{2s} v_t(x_1), \quad \forall x_1 \in \mathbb{R}.
\]

Indeed, observe that \(-\Delta^R v_t := -(v_t)_{x_1 x_1} = t^2 v_t \) in \( R \) and then

\[
(|\xi_1|^2 - t^2) \hat{v}_t(\xi_1) = F \left( -\Delta^R v_t - t^2 v_t \right)(\xi_1) = 0,
\]

which implies that

\[
\text{supp}(\hat{v}_t) \subset \{ \pm t \},
\]

which in turn implies

\[
(|\xi_1|^{2s} - t^{2s}) \hat{v}_t(\xi_1) = 0 = F \left( (-\Delta)^s v_t - t^{2s} v_t \right)(\xi_1).
\]

and finally

\[
\left( (-\Delta)^s v_t - t^{2s} v_t \right)(\xi_1) = 0 \quad \text{in} \quad \mathbb{R},
\]

which yields

\[
(-\Delta)^s v_t(x) = (-\Delta)^s v_t = t^{2s} v_t(x), \quad \forall x \in \mathbb{R}^N,
\]

which completes the proof. \( \square \)

Let \( \eta_0 \) be a \( C^2 \) increasing function such that \( \|\eta_0\|_{C^1}, \|\eta_0\|_{C^2} \leq 2 \),

\[
\eta_0(t) = 1 \quad \text{if} \quad t \geq 1, \quad \eta_0(t) = 0 \quad \text{if} \quad t \leq 0.
\]

For \( \sigma > 0 \), denote

\[
w_\sigma(x) = \eta_0(\sigma^{-1} \rho(x)), \quad \forall x \in \mathbb{R}^N,
\]

where

\[
\rho(x) = \text{dist}(x, \mathbb{R}^N \setminus \Omega) \quad \text{for} \quad x \in \mathbb{R}^N.
\]

Since \( \Omega \) is a \( C^2 \) domain, then \( \rho \) is \( C^2 \) in \( \{ x \in \mathbb{R}^N : \rho(x) < \delta_0 \} \) for some \( \delta_0 > 0 \). Therefore, there is \( \sigma_0 \in (0, 1] \) such that for \( \sigma \in (0, \sigma_0] \)

\[
w_\sigma \in C^2(\mathbb{R}^N).
\]

Notice that

\[
w_\sigma \to 1 \quad \text{in} \quad \Omega \quad \text{as} \quad \sigma \to 0^+.
\]

Moreover, we have that

\[
|\Omega| \geq \int_\Omega w_\sigma dx \geq \int_\Omega w_\sigma^2 dx \geq |\Omega_\sigma|,
\]

thanks to \( w_\sigma = 1 \) in \( \Omega_\sigma \).
Lemma 4.2. Let $s \in (0,1)$ and $\Omega$ be a $C^2$ domain, then for $\sigma \in (0, \sigma_0)$
\[ |(-\Delta)^s w_\sigma(x) | \leq 2 c_{N,s} \omega_{N-1} \sigma^{-2s} \quad \text{for } x \in \Omega. \]

Proof. For $x \in \Omega$, we have that
\[ |2w_\sigma(x) - w_\sigma(x + \zeta) - w_\sigma(x - \zeta)| \leq \min \{4, \|w_\sigma\|_{C^2} |\zeta|^2\} \leq \min \{4, \sigma^{-2}\|\eta_0\|_{C^2} |\zeta|^2\}. \]

We use an equivalent definition
\[
\frac{2}{c_{N,s}} |(-\Delta)^s w_\sigma(x)| = \left| \int_{\mathbb{R}^N} \frac{2w_\sigma(x) - w_\sigma(x + \zeta) - w_\sigma(x - \zeta)}{|\zeta|^{N+2s}} d\zeta \right|
\leq \int_{\mathbb{R}^N} \frac{\min \{4, \sigma^{-2}\|\eta_0\|_{C^2} |\zeta|^2\}}{|\zeta|^{N+2s}} d\zeta
\leq 2\sigma^{-2} \int_{B_\sigma} \frac{|\zeta|^2}{|\zeta|^{N+2s}} d\zeta + \int_{\mathbb{R}^N \setminus B_\sigma} \frac{4}{|\zeta|^{N+2s}} d\zeta
\leq 4\omega_{N-1} \sigma^{-2s},
\]
where $\|\eta_0\|_{C^2} \leq 2$. This completes the proof. $\square$

Note that if $w_\sigma$ and $v_z$ are defined by (4.1) and (4.4) respectively then
\[ (-\Delta)^s (w_\sigma v_z)(z) = v_z(x)(-\Delta)^s w_\sigma(x) + w_\sigma(x)(-\Delta)^s v_z(x) + \mathcal{L}_z^s w_\sigma(x), \]
where
\[ \mathcal{L}_z^s w_\sigma(x) = c_{N,s} \int_{\mathbb{R}^N} \frac{(w_\sigma(x) - w_\sigma(\tilde{x}))(e^{i\tilde{x} \cdot z} - e^{ix \cdot z})}{|x - \tilde{x}|^{N+2s}} d\tilde{x}. \quad (4.6) \]

Lemma 4.3. Let $s \in (0,1)$, $\Omega$ be a $C^2$ domain and $R \geq 1$ be such that $\Omega \subset B_R(0)$, then $\sigma \in (0, \sigma_0)$,
\[ x \in \Omega \text{ and } |z| > 1. \]

(i) for $s \in \left(\frac{1}{2}, 1\right)$,
\[ \frac{1}{c_{N,s}} |\mathcal{L}_z^s w_\sigma(x)| \leq \frac{\omega_{N-1} \sigma^{-1}}{1 - s} |z|^{2s-1} + \frac{4\sigma^{-1}}{2s-1} |z|^{2s-1} + \frac{\omega_{N-1} R^{-2s}}{2s}; \]

(ii) for $s = \frac{1}{2}$,
\[ \frac{1}{c_{N,s}} |\mathcal{L}_z^s w_\sigma(x)| \leq \frac{\omega_{N-1} \sigma^{-1}}{1 - s} |z|^{2s-1} + 4\sigma^{-1} \omega_{N-1} (\log |z| + \log(4R)) + \frac{\omega_{N-1}}{2s} R^{-1}; \]

(iii) for $s \in (0, \frac{1}{2})$,
\[ \frac{1}{c_{N,s}} |\mathcal{L}_z^s w_\sigma(x)| \leq \frac{\omega_{N-1} \sigma^{-1}}{1 - s} |z|^{2s-1} + 4\sigma^{-1} \frac{\omega_{N-1}}{1 - 2s} (4R)^{1-2s} + \frac{\omega_{N-1}}{2s} R^{-2s}. \]

Proof. Note that
\[ |e^{i\tilde{x} \cdot z} - e^{ix \cdot z}| \leq \min \{2, |z||\tilde{x} - x|\} \]
and
\[ |w_\sigma(x) - w_\sigma(\tilde{x})| \leq \frac{2}{\sigma} |x - \tilde{x}|, \quad |\tilde{x}| < 3R. \]
For $x \in \Omega$ and $|z| > 1$, we have that

$$
\frac{1}{CN^s}|L_\sigma^s w_\sigma(x)| \leq \int_{\mathbb{R}^N} \left| \frac{w_\sigma(x) - w_\sigma(\tilde{x})}{|x - \tilde{x}|^{N+2s}} \right| e^{i\sigma z} - e^{i\sigma z} \, d\tilde{x}
$$

$$
\leq \int_{B_4} 2\sigma^{-1} |x - \tilde{x}| \min\{2, |z||\tilde{x} - x|\} \, d\tilde{x} + \int_{\mathbb{R}^N \setminus B_4} \frac{2}{|x - \tilde{x}|^{N+2s}} \, d\tilde{x}
$$

$$
\leq 2\sigma^{-1} |z| \int_{B_{3/4} \setminus \{x\}} |x - \tilde{x}|^{2-N-2s} \, d\tilde{x} + 4\sigma^{-1} \int_{B_4 \setminus B_{3/4} \setminus \{x\}} |x - \tilde{x}|^{1-N-2s} \, d\tilde{x}
$$

$$
+ \int_{\mathbb{R}^N \setminus B_4} \frac{2}{|x|^{N+2s}} \, d\tilde{x},
$$

where

$$
2\sigma^{-1} |z| \int_{B_{3/4} \setminus \{x\}} |x - \tilde{x}|^{2-N-2s} \, d\tilde{x} \leq \frac{\sigma^{-1} \omega_{N-1}}{1-s} |z|^{2s-1},
$$

$$
\int_{\mathbb{R}^N \setminus B_4} \frac{2}{|x|^{N+2s}} \, d\tilde{x} \leq \frac{\omega_{N-1}}{2s} R^{-2s}
$$

and

$$
4\sigma^{-1} \int_{B_4 \setminus B_{3/4} \setminus \{x\}} |x - \tilde{x}|^{1-N-2s} \, d\tilde{x} \leq \begin{cases} 
4 \frac{\sigma^{-1} \omega_{N-1}}{2s-1} |z|^{2s-1} & \text{if } s \in \left(\frac{1}{2}, 1\right), \\
4 \sigma^{-1} \omega_{N-1} (\log |z| + \log(4R)) & \text{if } s = \frac{1}{2}, \\
4 \frac{\sigma^{-1} \omega_{N-1}}{1-2s} (4R)^{1-2s} & \text{if } s \in \left(0, \frac{1}{2}\right).
\end{cases}
$$

This completes the proof. \qed

**Proof of Theorem 1.4.** Denote

$$
\Psi_k(x, y) = \sum_{j=1}^{k} \phi_j(x) \phi_j(y)
$$

and

$$
\mathcal{F}_\sigma(\Psi_k)(z, y) = (2\pi)^{-N} \mathcal{F} \int_{\mathbb{R}^N} \Psi_k(x, y) e^{i\sigma x, z} \, dx.
$$

Let $v_\sigma(x, y)$ be the solution of

$$
\begin{cases}
(-\Delta)^{\frac{d}{2}} u = w_\sigma e^{i\sigma z} & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
$$

(4.7)

We claim that $v_\sigma \in H^s_\sigma(\Omega)$ for any $s \in (0, \frac{1+s}{2})$.

In fact, from [39], Corollary 1.6], for $\beta \in \left[\frac{s}{2}, \frac{1+s}{2}\right]$ there exists $C_\beta > 0$ such that

$$
[v_\sigma]_{C^s(\Omega_t)} \leq C_\beta t^{s/2 - \beta}, \quad \forall t \in (0, r_0)
$$

for some $r_0 > 0$, where

$$
\Omega_t = \{ x \in \Omega : \rho(x) > t \}.
$$

From [39] Proposition 1.1], we also have $v_\sigma \in C^{s/2}(\mathbb{R}^N)$. First we note that for $\beta \in \left[\frac{s}{2}, \frac{1+s}{2}\right]$, it holds

$$
|v_\sigma(x) - v_\sigma(y)| \leq C_\beta \max \left\{ \rho^{s/2 - \beta}(x), \rho^{s/2 - \beta}(y) \right\} |x - y|^{\beta} \text{ for any } x, y \in \Omega. \quad (4.8)
$$

Indeed, to see the above estimate, note that if $|x - y| \leq 2\rho(x)$ then $\rho(y) \leq 3\rho(x)$, i.e., $x, y \in \Omega_{\frac{3\rho(x)}{2}}$, and thus

$$
|v_\sigma(x) - v_\sigma(y)| \leq C \rho^{s/2 - \beta}(y) |x - y|^{\beta}.
$$
Similarly if $|x - y| \leq 2\rho(y)$ then it follows
\[ |v_\sigma(x) - v_\sigma(y)| \leq C|\rho^{s_2/2 - \beta}(x)|x - y|^{\beta}. \]

Finally when $|x - y| > 2 \max\{\rho(x), \rho(y)\}$, then from [39] we have
\[ |v_\sigma(x) - v_\sigma(y)| \leq |v_\sigma(x)| + |v_\sigma(y)| \leq C(\rho^{s_2/2 - \beta}(x) + \rho^{s_2/2 - \beta}(y)) \leq C \max\{\rho^{s_2/2 - \beta}(x), \rho^{s_2/2 - \beta}(y)\}|x - y|^{\beta}. \]

Hence [4,8] follows.

Next, for any given $s \in (\frac{s_2}{2}, \frac{1+s_2}{2})$. We fix some $\beta > 0$ such that $\beta \in (s, \frac{1+s_2}{2})$. Then using (4.8), we have that
\[
\int_{\Omega} \int_{\Omega} \frac{|v_\sigma(x) - v_\sigma(y)|^2}{|x - y|^{N + 2s}} \, dy \, dx \leq C \int_{\Omega} \int_{\Omega} \left( \rho^{s_2/2 - \beta}(x) + \rho^{s_2/2 - \beta}(y) \right) |x - y|^{2\beta - N - 2s} \, dy \, dx \\
= 2 \int_{\Omega} \int_{\Omega} \rho^{s_2/2 - \beta}(y)|x - y|^{2\beta - N - 2s} \, dy \, dx \\
\leq 2 \int_{\Omega} \left( \int_{|x - y| \leq \rho(y)} |x - y|^{2\beta - N - 2s} \, dx \right) \rho^{s_2/2 - \beta}(y) \, dy \\
+ 2 \int_{\Omega} \left( \int_{|x - y| > \rho(y)} |x - y|^{2\beta - N - 2s} \, dx \right) \rho^{s_2/2 - \beta}(y) \, dy \\
\leq C \int_{\Omega} \rho^{s_2/2 - \beta}(y) \, dy + C \int_{\Omega} \rho^{2\beta/2 + s_2/2 - \beta}(y) \, dy \\
\leq C \int_{t=0}^{T_0} \int_{\rho(y) = t} t^{s_2/2 - \beta} \, dS \, dt + C \int_{t=0}^{T_0} \int_{\rho(y) = t} t^{s_2/2} \, dS \, dt \\
< \infty,
\]
for our choice that $\frac{s_2}{2} < \beta < \frac{1+s_2}{2}$. Similarly, we also show that
\[
\int_{\Omega^c} \int_{\Omega^c} \frac{|v_\sigma(x) - v_\sigma(y)|^2}{|x - y|^{N + 2s}} \, dy \, dx = \int_{\Omega^c} \int_{\Omega^c} \frac{|v_\sigma(y)|^2}{|x - y|^{N + 2s}} \, dy \, dx \leq \int_{\Omega^c} \int_{\Omega^c} \rho(y)^{2s_2|x - y|^{-2s_2}} \, dy \, dx < \infty,
\]
as $s < \frac{1+s_2}{2}$ and for any $y \in \Omega$, it follows $\{ x \in \Omega^c : |x - y| < \rho(y) \} = \emptyset$. Hence, $v_\sigma \in H_0^s(\Omega)$ for $s < \frac{1+s_2}{2}$.

Denote
\[ \tilde{v}_\sigma(x, z) = (-\Delta)^{s_2/2} v_\sigma(x, z) \]
and
\[ v_{\sigma,k}(z, y) := v_\sigma(y, z) - \sum_{j=1}^k \langle \tilde{v}_\sigma(\cdot, z), \hat{\phi}_j \rangle_{L^2(\mathbb{R}^N)} \hat{\phi}_j(y). \]

Note that
\[ (-\Delta)^{s_2} \tilde{v}_{\sigma,k}(z, y) = \tilde{v}_\sigma(y, z) - \sum_{j=1}^k \langle \tilde{v}_\sigma(\cdot, z), \hat{\phi}_j \rangle_{L^2(\mathbb{R}^N)} \hat{\phi}_j(y), \]
thus we get $v_{\sigma,k}(z, \cdot) \in H_{0,k+1}(\Omega)$ and the Rayleigh-Ritz formula shows that
\[ \lambda_{k+1} \left| \int_{\mathbb{R}^N} (-\Delta)^{s_2} v_{\sigma,k}(z, y) \, dy \right|^2 \leq \left| \int_{\mathbb{R}^N} (-\Delta)^{s_2} v_{\sigma,k}(z, y) \, dy \right|^2 \]
for any $z \in \mathbb{R}^N$ and $\sigma > 0$. Thus we can conclude that
\[ \lambda_{k+1}(\mu) \leq \inf_{\sigma > 0} \frac{\int_{B_r} \int_{\mathbb{R}^N} (-\Delta)^{s_1} v_{\sigma,k}(z, y) v_{\sigma,k}(z, y) \, dy \, dz}{\int_{B_r} \int_{\mathbb{R}^N} (-\Delta)^{s_2} v_{\sigma,k}(z, y) v_{\sigma,k}(z, y) \, dy \, dz}. \]
An elementary calculation yields that

\[
\int_{B_r} \int_{\mathbb{R}^N} (-\Delta)_{y} \sigma_{r,k}(z,y) v_{\sigma,k}(z,y) dydz = \int_{B_r} \int_{\mathbb{R}^N} \left| (-\Delta)_{y} \sigma_{r,k}(z,y) \right|^2 dydz
\]

\[
= \int_{B_r} \int_{\mathbb{R}^N} \left| (-\Delta)_{y} \sigma_{r,k}(z,y) \right|^2 dydz - \int_{B_r} \sum_{j=1}^{k} \left| \langle \tilde{v}_\sigma(.,z), \tilde{\phi}_j \rangle \right|_{L^2(\mathbb{R}^N)}^2 \int_{\Omega} \tilde{\phi}_j^2 dx dz
\]

\[
= \int_{B_r} \int_{\mathbb{R}^N} w_\sigma^2 dx dz - \sum_{j=1}^{k} \int_{B_r} \left| \langle \tilde{v}_\sigma(.,z), \tilde{\phi}_j \rangle \right|_{L^2(\mathbb{R}^N)}^2 dz
\]

and

\[
\int_{B_r} \left( \int_{\mathbb{R}^N} (-\Delta)_{y} \sigma_{r,k}(z,y) v_{\sigma,k}(z,y) dydz \right)
\]

\[
= \int_{B_r} \int_{\mathbb{R}^N} \left( (-\Delta)_{y} \sigma_{r,k}(y,z) v_{\sigma,k}(y,z) \right) dydz - \int_{B_r} \sum_{j=1}^{k} \left| \langle \tilde{v}_\sigma(.,z), \tilde{\phi}_j \rangle \right|_{L^2(\mathbb{R}^N)}^2 \int_{\Omega} \tilde{\phi}_j^2 dx dz
\]

\[
= \int_{B_r} \int_{\mathbb{R}^N} \left( (-\Delta)_{y} \sigma_{r,k}(y,z) v_{\sigma,k}(y,z) \right) dydz - \sum_{j=1}^{k} \lambda_j(\mu) \int_{B_r} \left| \langle \tilde{v}_\sigma(.,z), \tilde{\phi}_j \rangle \right|_{L^2(\mathbb{R}^N)}^2 dz
\]

\[
= \int_{B_r} \int_{\mathbb{R}^N} \left( \mathcal{L}_{z}^{s_1-s_2} w_\sigma \right) \left( \mathcal{L}_{z}^{s_1-s_2} w_\sigma \right) + w_\sigma (-\Delta)_{y}^{s_1-s_2} w_\sigma + z^{2(s_1-s_2)} w_\sigma^2(x) dx dz
\]

\[
- \sum_{j=1}^{k} \lambda_j(\mu) \int_{B_r} \left| \langle \tilde{v}_\sigma(.,z), \tilde{\phi}_j \rangle \right|_{L^2(\mathbb{R}^N)}^2 dz.
\]

Since \( \Omega \) is \( C^2 \), there exists \( t_0 \in (0,1) \) such that

\[
|\partial \Omega_t| \leq 2|\partial \Omega| \quad \text{for } t \in (0,t_0).
\]

From (4.3),

\[
|\Omega| \geq \int_{\Omega} w_\sigma^2(y) dy > |\Omega_\sigma| \geq |\Omega| - 2\sigma|\partial \Omega| > \frac{|\Omega|}{2},
\]

when \( \sigma \) is small enough. We choose \( r > r_0 \) for some \( r_0 > 1 \) such that \( r^{-\frac{s_1-s_2}{2}} = \sigma \) satisfies the above relation and \( \sigma < \frac{1}{4|\partial \Omega|} |\Omega| \). Therefore,

\[
|\Omega| \geq \int_{\Omega} w_\sigma^2(y) dy \geq |\Omega| - 2r^{-\frac{s_1-s_2}{2}}|\partial \Omega| > \frac{|\Omega|}{2}. \quad (4.9)
\]

Note that for \( s = s_1 - s_2 \) and \( r > r_0 \)

\[
\frac{1}{c_{N,s}} \int_{B_r} \int_{\mathbb{R}^N} \mathcal{L}_{z}^{s} w_\sigma \mathcal{L}_{z}^{s} w_\sigma e^{iz} dydz
\]

\[
\leq \frac{\omega_{N-1}r^{N+2s-1}}{\sigma(N+2s-1)(1-s)} |\Omega| + \frac{\omega_{N-1}r^{N}}{2sN} R^{-2s} |\Omega|
\]

\[
\leq \frac{\omega_{N-1} |\Omega|}{(N+2s-1)(1-s)} e^{N+2s+\frac{\alpha}{2}-1} + r^{\frac{\alpha}{2} \varphi_\alpha(r,R)|\Omega|} + \frac{\omega_{N-1}r^{N}}{2sN} R^{-2s} |\Omega|
\]

\[
\leq c_1(N, s_1, s_2) |\Omega| e^{N+\max\{2s+\frac{\alpha}{2}-1, \frac{\alpha}{2}\}},
\]

where
Moreover, we have that

\[
\int_{B_r} \int_{\mathbb{R}^N} |w_\sigma(-\Delta)^s w_\sigma| dy dz \leq 2c_{N,s} \frac{\omega_{N-1}}{N} r^N \sigma^{-2s} |\Omega| = c_2(N, s_1, s_2) |\Omega| r^{N+2s}.
\]

Furthermore,

\[
\int_{B_r} \int_{\mathbb{R}^N} |z|^{2s} w_\sigma^2(x) dy dz = \frac{\omega_{N-1} r^{N+2s}}{N + 2s} \int_{\Omega} w_\sigma^2(y) dy.
\]

Let

\[
\delta_0 = \max \left\{ 2s + \frac{s}{2} - 1, \frac{s}{2}, s^2 \right\} = \frac{s}{2},
\]

where we have used the hypothesis that \( s_1 < \frac{1+s_2}{2} \).

Let

\[
P_j := \int_{B_r} \left| \left\langle \tilde{v}_\sigma(., z), \tilde{\phi}_j \right\rangle \right|_{L^2(\mathbb{R}^N)}^2 dz
\]

\[
\leq \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} w_\sigma e^{iz \cdot z} \tilde{\phi}_j \right|^2 dz \leq (2\pi)^N \int_{\mathbb{R}^N} \left| \mathcal{F}(w_\sigma \tilde{\phi}_j) \right|^2 dz
\]

\[
= (2\pi)^N \int_{\mathbb{R}^N} (w_\sigma \tilde{\phi}_j)^2 dx \leq (2\pi)^N,
\]

using the Parseval’s inequality. Note that

\[
\lambda_{k+1}(0) \leq \frac{\omega_{N-1}}{N + 2(s_1 - s_2)} \int_{\Omega} w_\sigma^2(y) dy + O(1) |\Omega| r^{N+\frac{s}{2}} - \sum_{j=1}^k \lambda_j(0) P_j
\]

\[
- \frac{\omega_{N-1}}{N} \int_{\Omega} w_\sigma^2(y) dy - \sum_{j=1}^k P_j,
\]

and now taking

\[
Q_1 = \frac{\omega_{N-1}}{N + 2(s_1 - s_2)} r^{N+2(s_1 - s_2)} \int_{\Omega} w_\sigma^2(y) dy + O(1) |\Omega| r^{N+\frac{s}{2}} \quad \text{and} \quad Q_2 = \frac{\omega_{N-1}}{N} r^N \int_{\Omega} w_\sigma^2(y) dy,
\]

we have that

\[
0 \leq \frac{Q_1 - \sum_{j=1}^k \lambda_j(0) P_j}{Q_2 - \sum_{j=1}^k P_j} - \lambda_{k+1}(0)
\]

\[
= \frac{(Q_1 - Q_2 \lambda_{k+1}(0)) + \sum_{j=1}^k \left( \lambda_{k+1}(0) - \lambda_j(0) \right) P_j}{Q_2 - \sum_{j=1}^k P_j}
\]

\[
\leq \frac{(Q_1 - Q_2 \lambda_{k+1}(0)) + (2\pi)^N \sum_{j=1}^k \left( \lambda_{k+1}(0) - \lambda_j(0) \right)}{Q_2 - (2\pi)^N k},
\]

20
since $\lambda_{k+1}(0) \geq \lambda_j(\mu)$ for $j < k + 1$ and $P_j \in (0,(2\pi)^N]$. Therefore

$$0 < \lambda_{k+1}(0) \leq \frac{\frac{\omega_{N-1}}{\sqrt{2(s_1-s_2)}} r^{N+2(s_1-s_2)} \int_{\Omega} w^2\sigma(y)dy + O(1)|\Omega| r^{N+\frac{4(s_1-s_2)}{2}} - (2\pi)^N \sum_{j=1}^k \lambda_j(0)}{\omega_{N-1}^N \int_{\Omega} w^2\sigma(y)dy - (2\pi)^N k}.$$ 

There exists $k_0 \geq 1$ such that for $k \geq k_0$, we can choose $r > r_0$ satisfying

$$\frac{\omega_{N-1}}{N} r^{N} \int_{\Omega} w^2\sigma(y)dy = (2\pi)^N (k + 1).$$

(4.11)

As a consequence, using [449] and the above relation, we obtain for $k \geq k_0$

$$\sum_{j=1}^k \lambda_j(0) \leq (2\pi)^{-N} \frac{\omega_{N-1}}{N + 2(s_1 - s_2)} r^{N+2(s_1-s_2)} \int_{\Omega} w^2\sigma(y)dy + c|\Omega| r^{N+\frac{4(s_1-s_2)}{2}}$$

$$= (2\pi)^{2(s_1-s_2)} \frac{N}{N + 2(s_1 - s_2)} \left(\int_{\Omega} w^2\sigma(y)dy\right)^{-\frac{2(s_1-s_2)}{N}} (k + 1)^{1+\frac{2(s_1-s_2)}{N}}$$

$$+ c_3 k^{1+\frac{2(s_1-s_2)}{N}} + c_4 k^{1+\frac{2(s_1-s_2)}{N}} + c_5 k^{1+\frac{2(s_1-s_2)}{N}},$$

where $c_3, c_4 > 0$ depends on $N, s_1, s_2, \Omega$,

$$(k + 1)^{1+\frac{2(s_1-s_2)}{N}} \leq k^{1+\frac{2(s_1-s_2)}{N}} + c_5 k^{2(s_1-s_2)}$$

and we use the fact that for $c_7, c_8 > 0$

$$\left(\int_{\Omega} w^2\sigma(y)dy\right)^{-\frac{2(s_1-s_2)}{N}} \leq \left(|\Omega| - 2r^{s_1-s_2} (\partial\Omega)\right)^{-\frac{2(s_1-s_2)}{N}}$$

$$\leq |\Omega|^{-\frac{2(s_1-s_2)}{N}} + c_6 r^{-\frac{s_1-s_2}{2}} + c_7 k^{-\frac{s_1-s_2}{2N}}.$$ 

Here for $k \leq k_0$, we only have to adjust the constant $c_3$ or $c_4$. 

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