AN ALGEBRAIC GEOMETRIC CLASSIFICATION OF SUPERINTEGRABLE SYSTEMS IN THE EUCLIDEAN PLANE

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Abstract. We prove that the set of non-degenerate second order maximally superintegrable systems in the complex Euclidean plane carries a natural structure of a projective variety, equipped with a linear isometry group action. This is done by deriving the corresponding system of homogeneous algebraic equations. We then solve these equations explicitly and give a detailed analysis of the algebraic geometric structure of the corresponding projective variety. This naturally associates a unique planar line triple arrangement to every superintegrable system, providing a geometric realisation of this variety and an intrinsic labelling scheme. In particular, our results confirm the known classification by independent, purely algebraic means.

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1. Introduction

1.1. Superintegrable systems. In classical and quantum mechanics exact solutions to the equations of motion play a central role, providing models with which to explore the properties of such systems and as a basis for perturbations. Two systems in particular stand out, the harmonic oscillator, which represents the lowest order term in the Taylor expansion of any non-singular potential, and the Kepler-Coulomb potential of the classical celestial motion and the quantum Hydrogen atom.

Finding exact solutions to PDEs generally requires the use of symmetry methods. In classical mechanics, continuous symmetries describe flows in the phase space that provide conserved quantities. In quantum mechanics, differential operators commuting with the Hamiltonian preserve its energy eigenspaces. Superintegrable systems are those systems with the greatest number of independent symmetries and the harmonic oscillator and Kepler-Coulomb system are the best known examples.
For a natural Hamiltonian system on an $n$-dimensional manifold, this maximum number is $2n - 1$.

In the classical case, superintegrability confines the trajectories of the system to the level sets of $2n - 1$ conserved quantities in the $2n$-dimensional phase space and hence to one-dimensional orbits which necessarily close if they are confined. In the quantum case, maximal superintegrability is associated with quasi-exact solvability and in many cases, the energy spectrum has been determined from the symmetry algebra alone.

Not only are the harmonic oscillator and Kepler-Coulomb systems superintegrable, but their superintegrability is due to symmetries that are quadratic in the momenta, or second order differential operators in the quantum case. Second order symmetries allow the equations of motion to be solved by separation of variables and (non-degenerate) second order superintegrable systems in two and three dimensions allow separation in more than one distinct system of coordinates [KKPM01, KKM06].

The separated solutions of the harmonic oscillator and Kepler-Coulomb system are given in terms of an orthonormal basis of special functions. When separation occurs in more than one system, the interbasis expansion coefficients are also given in terms of special functions in the form of orthogonal polynomials. These connections with orthogonal polynomials have been exploited by Post et al to “explain” the Askey-Wilson scheme of hypergeometric orthogonal polynomials with Wigner-İnönü like contractions between second order superintegrable systems providing the degenerations between the classes of orthogonal polynomials [KMP13]. It is this observation that motivates the current work that seeks to describe second order superintegrable systems as a projective variety.

Recently, there has been rapid growth in the number of known families of superintegrable systems, in particular those with higher order symmetries. For an overview see the topical review of Miller, Post and Winternitz [MPW13]. However, many questions remain open and even the complete classification for non-degenerate second order superintegrability in constant curvature and conformally flat spaces is only known for two and three dimensions [KKM05, CK14]. This paper aims to develop new techniques that will provide a classification in any dimension.

1.2. Motivation. Our approach is strongly motivated by recent results on separable systems. While the classification of separable systems on constant scalar curvature manifolds has been known for over 30 years [KM86, Kal86], a recent algebraic geometric approach developed by the second author [Sch12, Sch14, Sch15, Sch16] revealed a deep algebraic and geometric structure underlying this classification. Together with A. P. Veselov he proved that the space of all separable systems on an $n$-dimensional sphere (in normal form) carries a natural structure of a projective variety, isomorphic to the real part $\mathcal{M}_{0,n+2}(\mathbb{R})$ of the Deligne-Mumford moduli space $\mathcal{M}_{0,n+2}$ of stable algebraic curves of genus zero with $n + 2$ marked points [SV15]. Exploiting the structure of these moduli spaces led to a topological classification of separable systems on spheres by Stasheff polytopes and to a simple explicit construction based on a natural operad structure on the sequence of moduli spaces $\mathcal{M}_{0,n}(\mathbb{R})$.

We regard the present work as a proof of concept that the same ideas apply to the classification of superintegrable systems, initiating an “algebro-geometrisation”
of the classification of superintegrable systems and its applications. Indeed, leading experts in superintegrability consider algebraic geometric methods the most promising route to mayor advances in this field [MPW13]:

“The possibility of using methods of algebraic geometry to classify superintegrable systems is very promising and suggests a method to extend the analysis in arbitrary dimension as well as a way to understand the geometry underpinning superintegrable systems.”

Even though, a concrete and detailed concept how to achieve this goal has never been proposed. This gap is filled with our proof of concept, which can be better extended to higher dimensions and more general context than the classical approach.

In the case of separable systems, a thorough analysis of the least non-trivial example, the 3-sphere, provided enough information for a generalisation to arbitrary dimensions. Since separable systems are closely related to superintegrable systems, this suggests that an algebraic geometric description of superintegrable systems in low dimensions will reveal sufficient additional structure to push the classification further to higher dimensions, a task which currently seems intractable by standard methods. The present article is a first step in this direction.

1.3. Results. Traditionally, classifying superintegrable systems of a certain type always meant to give lists of normal forms for the equivalence classes of such systems under isometries. This sense of the word “classification” ignores the fact that the set of superintegrable systems has a topological structure and possibly an even more fine-grained geometric structure. The first main result we prove in this article is that the set of non-degenerate second order maximally superintegrable systems in the Euclidean plane has the structure of (a linear bundle over) a projective variety, equipped with a linear isometry group action. We call this variety the variety of superintegrable systems.

In other words, the natural category in which to consider the classification problem for superintegrable systems is the category of projective varieties equipped with linear group actions. This indicates that the classification problem can be better treated by algebraic geometric means, studying the underlying algebraic equations instead of partial differential equations. The remaining part of this article is dedicated to substantiate this claim by showing that a consequent algebraic geometric treatment not only simplifies the classification problem considerably, but also reveals a deep and previously hidden geometric structure. We show, for instance, that non-degenerate superintegrable systems on the plane are (essentially) parametrised by a completely reducible ternary cubic. This associates a planar arrangement of projective line triples to each superintegrable system in the plane, which supports the singular locus of the corresponding superintegrable potential.

In this sense our work not only confirms the known classification by independent means, but also enriches it with additional algebraic and geometric structure.

1.4. Comparison to prior work. It should be noted that the idea to use varieties in order to classify superintegrable systems is not new and appeared in earlier works of the first author, along with Kalnins et al [KKM07a, KKM07b] and Capel [CK14], where inhomogeneous polynomial integrability conditions were used to classify superintegrable systems in two and three dimensions. Those varieties depend on a non-canonical choice of some generic base point in the manifold and carry an intricate non-linear isometry group action. This makes the use of computer algebra
inevitable and is the principal obstruction for a generalisation to higher dimensions. We remark that the algebraic geometric structure of these varieties has never been exploited for the classification.

While we make use of the same polynomial integrability conditions, our approach is fundamentally different in that here we use algebraic geometry right from the beginning to define the setting and not merely as a tool to analyse these conditions. This has a number of advantages. First, our variety does not depend on any additional choices and it carries a linear isometry group action. Moreover, although the polynomial integrability conditions are inhomogeneous, it turns out to be a projective variety. Second, our variety is simple enough to be analysed without the help of computer algebra. Third, the line triple arrangements and the multiplicities of the ternary cubic mentioned above provide intrinsic geometric and algebraic labelling schemes for superintegrable systems respectively their isometry classes, in contrast to the known classification scheme where isometry classes were labelled arbitrarily. Note that the more structure on the set of superintegrable systems that is revealed, the more likely it is to find patterns that generalise to higher dimensions.

1.5. Prospects. It has been observed recently that the families of hypergeometric orthogonal polynomials in the Askey scheme can be associated to second order superintegrable systems on spaces of constant curvature, such that limiting cases of these polynomials can be derived from contractions of superintegrable systems [KMP13]. On the other hand, an extension of the methods and results in the present article will eventually reveal a natural structure of a projective variety on the set of second order superintegrable systems on spaces of constant curvature. Combined with the above mentioned observation, we arrive at the following conjecture.

**Conjecture.** The set of hypergeometric orthogonal polynomials in the Askey scheme carries a natural structure of a projective variety, equipped with a linear action of the general linear group GL(3). Moreover, the graph of orbits under this action and their degenerations reproduces the Askey scheme as well as the classification of second order superintegrable systems in dimension two.

This conjecture is supported by the fact that the Askey scheme carries the structure of three manifolds with corners glued together [Koo09]. As of yet, there is no generalisation of the Askey scheme to higher dimensions or more general special functions known, but superintegrable systems seem to provide the right context for that.

A parametrisation via a projective variety would allow to study hypergeometric orthogonal polynomials using a broad range of algebraic geometric methods and on a global level. What we propose here is a paradigm shift. While hypergeometric polynomials or, more generally, special functions have always been regarded as many different families, each of them of the form $P_\alpha(x)$ with the parameter $\alpha$ varying in some open subset of $\mathbb{R}^n$, our results suggest to consider them as a single family $P_\alpha(x)$, where $\alpha$ now varies in an algebraic variety.

It is expected that a two-variable generalisation of the Askey scheme will be revealed in the structure of second order superintegrable systems on constant curvature spaces in three dimensions [Pos]. Current work in progress extending the results reported here would then suggest a natural generalisation of the above conjecture to a multivariable Askey scheme.
1.6. Method. As will be explained in Section 2, a second order maximally superintegrable system on an $n$-dimensional Riemannian manifold is given by a (non-trivial) solution to the $2n - 1$ equations

$$dV^{(\alpha)} = K^{(\alpha)}dV, \quad \alpha = 0, 1, \ldots, 2n - 2,$$

for linearly independent second order Killing tensors $K^{(\alpha)}$ and arbitrary potential functions $V^{(\alpha)}$, where $V = V^{(0)}$ is the potential defining the Hamiltonian of the system. The point of departure for our approach to a classification of superintegrable systems is the observation that this system, which at first glance is a non-linear system of partial differential equations, is actually equivalent to a system of algebraic equations on a finite dimensional vector space. This can be seen as follows.

First note that by definition the $K^{(\alpha)}$ belong to the finite dimensional vector space of Killing tensors and are therefore, essentially, algebraic objects – as opposed to the arbitrary functions $V^{(\alpha)}$. We will therefore eliminate first the $V^{(\alpha)}$ for $\alpha \neq 0$ and then $V = V^{(0)}$ from the above equations, leaving equations on the $K^{(\alpha)}$ alone.

If $K^{(\alpha)}$ and $V$ are given, then the Equation (1.1) can be used to obtain $V^{(\alpha)}$, provided the integrability condition

$$d(K^{(\alpha)}dV) = 0$$

is met. This is the so-called Bertrand-Darboux condition and already eliminates the unknown functions $V^{(\alpha)}$ for $\alpha \neq 0$.

If only the $K^{(\alpha)}$ are given, the Equations (1.2) are second order linear differential equations for $V$, the coefficients being linear in the $K^{(\alpha)}$ and their first derivatives. Following [KKM07a] we can use them to express all but one of $V$’s second derivatives as linear combinations of $V$’s first derivatives, where the coefficients are rational in the $K^{(\alpha)}$ and their first derivatives. In the case of the Euclidean plane considered here, for example, this reads (in complex coordinates)

$$\begin{bmatrix} V_{zz} \\ V_{ww} \end{bmatrix} = \frac{3}{2} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} V_z \\ V_w \end{bmatrix}. \quad (1.3)$$

If the values of $V$, its first derivatives as well as the “missing” second derivative (here $V_{zz} = \Delta V$) are prescribed at some generic point, the above equations determine all higher derivatives of $V$ and hence $V$ itself. The integrability conditions for the System (1.3) are algebraic equations in the coefficients $C_{ij}$ and their first derivatives, c.f. Equations (3.9). Now remember that the $C_{ij}$ were rational in the $K^{(\alpha)}$ and their first derivatives. Consequently, the integrability conditions for the System (1.3) are algebraic equations in the $K^{(\alpha)}$ and their derivatives. This also allows us to eliminate the unknown function $V$.

Now observe that the (covariant) derivative is a linear operation. The integrability conditions of (1.3) are therefore algebraic in the $K^{(\alpha)}$ alone. This defines a system of algebraic equations for $2n - 1$ Killing tensors. These equations determine whether $2n - 1$ linearly independent Killing tensors $K^{(\alpha)}$ can be completed to a solution of the Equations (1.1), necessary to define a superintegrable system.

Our next observation is that a superintegrable system is a linear space, expressed by the linearity of (1.1) in $(K^{(\alpha)}, V^{(\alpha)})$. This translates to the fact that the above algebraic equations can be written in terms of the Plücker coordinates for the vector space spanned by the $K^{(\alpha)}$. Hence they define a subvariety in the Grassmannian of

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1The factor $3/2$ chosen here for convenience differs from the convention used in [KKM07a].
(2n − 1)-dimensional subspaces in the space of Killing tensors. It is this subvariety which we will call the \emph{variety of superintegrable systems}.

In Section 3 we make the above explicit for superintegrable systems in the Euclidean plane, in particular the algebraic equations defining the variety of superintegrable systems. For spaces where the classification of superintegrable systems is known, it is a priori clear that these algebraic equations can be solved explicitly. However, the present example of the Euclidean plane shows that it is much simpler to solve these equations from scratch than to rewrite the known normal forms. This is the content of Section 4.

In Section 5 we give a detailed description of the algebraic geometric structure of the variety of superintegrable systems, such as the irreducible components, their intersections, birational structure and singularities.

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2. Preliminaries

2.1. Superintegrable systems. A Hamiltonian system is a dynamical system characterised by a Hamiltonian function $H(p, q)$ on the phase space of positions $q = (q_1, \ldots, q_n)$ and momenta $p = (p_1, \ldots, p_n)$. Its temporal evolution is governed by the equations of motion

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = +\frac{\partial H}{\partial p}.$$ 

A function $F(p, q)$ on the phase space is called \emph{a constant of motion} or \emph{first integral}, if it is constant under this evolution, i.e. if

$$\dot{F} = \frac{\partial F}{\partial q} \dot{q} + \frac{\partial F}{\partial p} \dot{p} = \frac{\partial F}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial H}{\partial q} = 0$$

or

$$\{F, H\} = 0,$$

where

$$\{F, G\} = \sum_{i=1}^{n} \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \right)$$

is the canonical Poisson bracket. Such a constant of motion restricts the trajectory of the system to a hypersurface in phase space. If the system possesses the maximal number of $2n - 1$ functionally independent constants of motion $F^{(0)}, \ldots, F^{(2n-2)}$, then its trajectory in phase space is the (unparametrised) curve given as the intersection of the hypersurfaces $F^{(\alpha)}(p, q) = c^{(\alpha)}$, where the constants $c^{(\alpha)}$ are determined by the initial conditions. For such systems we can solve the equations of motion exactly and in a purely algebraic way, without having to solve explicitly any differential equation.

Definition 2.1. A \emph{maximally superintegrable system} is a Hamiltonian system admitting $2n - 1$ functionally independent constants of motion $F^{(\alpha)}$,

$$\{F^{(\alpha)}, H\} = 0 \quad \alpha = 0, 1, \ldots, 2n - 2,$$

one of which we can take to be the Hamiltonian itself:

$$F^{(0)} = H.$$
A superintegrable system is *second order* if the constants of motion $F^{(\alpha)}$ are of the form

$$F^{(\alpha)} = K^{(\alpha)} + V^{(\alpha)},$$  \hspace{1cm} (2.2)

where

$$K^{(\alpha)}(p, q) = \sum_{i=1}^{n} K_{ij}^{(\alpha)}(q)p^{i}p^{j}$$

is quadratic in momenta and

$$V^{(\alpha)}(p, q) = V^{(\alpha)}(q)$$

a potential function depending only on the positions. In particular,

$$H = g + V,$$  \hspace{1cm} (2.3)

where

$$g(p, q) = \sum_{i=1}^{n} g_{ij}(q)p^{i}p^{j}$$

is given by the Riemannian metric $g_{ij}(q)$ on the underlying manifold. We call $V$ a superintegrable potential if the Hamiltonian (2.3) defines a superintegrable system.

In this article we will be concerned exclusively with second order maximally superintegrable systems and thus omit the terms “second order” and “maximally” without further mentioning.

The condition (2.1) for (2.2) and (2.3) splits into two parts, which are cubic respectively linear in $p$:

$$\{K^{(\alpha)}, g\} = 0$$  \hspace{1cm} (2.4a)

$$\{K^{(\alpha)}, V\} + \{V^{(\alpha)}, g\} = 0$$  \hspace{1cm} (2.4b)

### 2.2. Killing tensors.

The condition $\{K, g\} = 0$ for $K(p, q) = K_{ij}(q)p^{i}p^{j}$ is equivalent to $K_{ij}$ being a Killing tensor in the following sense.

**Definition 2.2.** A (second order) Killing tensor is a symmetric tensor field on a Riemannian manifold satisfying the Killing equation

$$K_{ij,k} + K_{jk,i} + K_{ki,j} = 0,$$

where the comma denotes covariant derivatives.

Note that the metric $g$ is trivially a Killing tensor, since it is covariantly constant.

### 2.3. Bertrand-Darboux condition.

The metric $g$ also allows us to identify symmetric forms and endomorphisms. Interpreting a Killing tensor in this way as an endomorphism on 1-forms, equation (2.4b) can be written in the form

$$dV^{(\alpha)} = K^{(\alpha)}dV,$$

and shows that, once the Killing tensors $K^{(\alpha)}$ are known, the potentials $V^{(\alpha)}$ can be recovered from $V = V^{(0)}$ (up to an irrelevant constant), provided the integrability condition

$$d(K^{(\alpha)}dV) = 0$$  \hspace{1cm} (2.5)

is satisfied. This eliminates the potentials $V^{(\alpha)}$ for $\alpha \neq 0$ from our equations. In fact, as we will see below, the remaining potential $V = V^{(0)}$ can be eliminated as well, leaving equations on the Killing tensors $K^{(\alpha)}$ alone.
2.4. **Non-degeneracy.** For simplicity of notation, let us write $V_i$ and $V_{ij}$ for the first and second derivatives of the scalar function $V$. The Equations (2.5) can be used to express the second derivatives $V_{ij}$, for $i \neq j$, and $V_{ii} - V_{jj}$ as linear combinations of the first derivatives $V_i$, the coefficients being rational expressions in $K^{(a)}_{ij}$ and $K^{(a)}_{ij,k}$. This determines all higher derivatives of $V$ at a non-singular point if $V$, $\nabla V$ and $\Delta V$ are known at this point. Therefore an analytic potential $V$ is uniquely determined by the Killing tensors $K^{(a)}$ if $V$, $\nabla V$ and $\Delta V$ are prescribed at a non-singular point of $V$. This motivates the following definition, following [KPM00].

**Definition 2.3.** A superintegrable system is called non-degenerate, if $\Delta V$ and the components of $\nabla V$ are linearly independent functions.

This article is concerned with the classification of superintegrable systems which are non-degenerate.

Note that, by the above, the Killing tensors $K^{(a)}$ of a non-degenerate superintegrable system uniquely define an $(n + 2)$-dimensional linear space of corresponding superintegrable potentials $V$, parametrised by the values of $V$, $\nabla V$ and $\Delta V$ at a fixed non-singular point.

**Definition 2.4.** We call the $(2n - 1)$-dimensional subspace in the space of Killing tensors that is spanned by the Killing tensors $K^{(a)}$ of a (non-degenerate) superintegrable system the associated (non-degenerate) free superintegrable system. The $(n + 2)$-dimensional space of superintegrable systems with the same associated free superintegrable system will be called the fibre over this free superintegrable system.

**Remark 2.5.** Setting $V$ and $V^{(a)}$ identically zero gives a solution of the Conditions (2.4) for any choice of Killing tensors $K^{(a)}$. Thus a $(2n - 1)$-dimensional subspace in the space of Killing tensors is a (not necessarily non-degenerate) free superintegrable system in the above sense if it contains the metric $g$.

2.5. **Special conformal Killing tensors.** In dimension two Killing tensors can be described equivalently via special conformal Killing tensors. This will considerably simplify our characterisation of superintegrable systems.

**Definition 2.6.** A special conformal Killing tensor is a symmetric tensor field $L_{ij}$ satisfying

$$L_{ij,k} = \frac{1}{2}(\lambda_i g_{jk} + \lambda_j g_{ik}), \quad (2.6a)$$

where $\lambda_i$ denotes the covariant derivative of

$$\lambda := \text{tr} L, \quad (2.6b)$$

as can be seen from contracting $i$ and $j$ in (2.6a).

Note that the metric $g$ is trivially a special conformal Killing tensor as well, since it is covariantly constant.

**Lemma 2.7.** A special conformal Killing tensor $L$ determines a Killing tensor $K$ via

$$K := L - (\text{tr} L)g. \quad (2.7)$$

On 2-dimensional constant curvature manifolds this defines an isomorphism between the space of Killing tensors and the space of special conformal Killing tensors, mapping $g$ to $-g$. 
Proof. The Killing equation for $K$ is a direct consequence of (2.6). For the second part observe that the map defined by (2.7) is injective and that the dimension of both spaces is known to be six. □

We can rewrite the Bertrand-Darboux condition (2.5) for (2.7) in terms of $L$ as

$$d(LdV) = d\lambda \wedge dV$$

or, in local coordinates, as

$$\sum_{i=1}^{n} \left( L_{[j}^{i} V_{k]}^{i} + L_{[j,k]}^{i} V_{i}^{i} \right) = \lambda[k V_{j}],$$

where the square brackets denote antisymmetrisation in the enclosed indices. Using (2.6) this becomes

$$\sum_{i=1}^{n} L_{[j}^{i} V_{k]}^{i} = \frac{3}{2} \lambda[k V_{j}].$$

(2.8)

2.6. Complex Euclidean plane. We will consider the Euclidean plane, i.e. a complex 2-dimensional vector space equipped with a complex scalar product $g$. The scalar product defines a complex Riemannian metric which, by abuse of notation, will be denoted by $g$ as well. In the Euclidean plane every special conformal Killing tensor is of the form

$$L = A + bx^T + cx^T x$$

(2.9a)

with trace

$$\lambda = tr A + 2b^T x + cx^T x$$

(2.9b)

where $A$, $b$ and $c$ are, respectively, a symmetric $2 \times 2$ matrix, a vector and a scalar. We will combine these parameters into a symmetric $3 \times 3$ matrix

$$\hat{L} := \begin{bmatrix} c & b^T \\
 b & A \end{bmatrix}, \quad A^T = A$$

(2.10)

parametrising the space of special conformal Killing tensors in the plane. In particular, as a special conformal Killing tensor the metric $g$ is given by the symmetric matrix

$$\hat{g} = \begin{bmatrix} 0 & 0 \\
 0 & g \end{bmatrix}.$$  

(2.11)

For convenience we will choose a null basis and corresponding coordinates $z$ and $w$:

$$g_{zz} = g_{ww} = 0 \quad g_{zw} = g_{wz} = 1.$$  

(2.12)

The Bertrand-Darboux condition in the form (2.8) then reads

$$L_{ww} V_{zz} - L_{zz} V_{ww} = \frac{3}{2} (\lambda_z V_{w} - \lambda_w V_{z}).$$

(2.13)

2.7. Functional independence. So far we have ignored the distinction between linear and functional independence of the constants of motion. Recall that functional independence means that their differentials are linearly independent almost everywhere. This condition can be formulated in a purely algebraic fashion and thus be incorporated into our algebraic geometric description. Being more pragmatic, we would say we can check functional independence a posteriori. However, it turns out (or is known) that in our case functional and linear independence are equivalent. That is why we will ignore this distinction right from the beginning.
2.8. **The variety of free superintegrable systems.** Let us denote by $S^2\mathbb{C}^3$ the space of complex symmetric $3 \times 3$ matrices and by $S_0^2\mathbb{C}^3$ the subspace of matrices of the form (2.10) with $\text{tr} \ A = 0$. In a null basis we have $\text{tr} \ A = 2 \ A_{zw}$, so that we can represent elements in the five-dimensional space $S_0^2\mathbb{C}^3$ as vectors

$$\hat{L} = (A_{zz}, 2b_z, c, 2b_w, A_{ww}).$$

We have seen that the space of Killing tensors on the Euclidean plane is naturally isomorphic to $S^2\mathbb{C}^3$. Hence a free superintegrable system on the Euclidean plane in the sense of Remark 2.5 is given by a three-dimensional subspace in $S^2\mathbb{C}^3$ containing (2.11) or, equivalently, by a two-dimensional subspace in $S_0^2\mathbb{C}^3$. Therefore the free superintegrable systems on the Euclidean plane constitute a projective variety isomorphic to the Grassmannian $G_2(S_0^2\mathbb{C}^3)$ of 2-planes in $S_0^2\mathbb{C}^3$, which can be embedded into $\mathbb{P}(\Lambda^2 S_0^2\mathbb{C}^3)$ under the Plücker embedding

$$G_2(S_0^2\mathbb{C}^3) \hookrightarrow \mathbb{P}(\Lambda^2 S_0^2\mathbb{C}^3)$$

span{\(\hat{L}^{(1)}, \hat{L}^{(2)}\)} $\mapsto \hat{L}^{(1)} \wedge \hat{L}^{(2)}$.

For simplicity of notation let us use the simple superscripts “1” and “2” instead of “(1)” and “(2)”. Then the image of the above map is the projective variety of rank two skew symmetric 5 × 5 matrices:

$$\hat{L}^1 \wedge \hat{L}^2 = \begin{bmatrix}
0 & 2(A_{zz} b_z^2 - b_z^4 A_{zz}^2) & A_{zz}^2 c^2 - c^4 A_{zz}^4 & A_{zz}^2 A_{ww}^2 - A_{ww}^2 A_{zz}^2 \\
0 & 2(b_z^2 c^2 - c^4 b_z^2) & 4(b_z^2 b_w^2 - b_z^4 b_w^2) & 2(b_z^2 A_{ww}^2 - A_{ww}^2 b_z^2) \\
\text{(skew)} & 0 & 2(c^2 b_w^2 - b_z^4 c^2) & 2(b_w^2 A_{ww}^2 - A_{ww}^2 b_w^2) \\
\end{bmatrix}$$

(2.14)

This variety is defined by the Plücker relations, given by the vanishing of the Pfaffians of the five principal minors of the above matrix:

$$a_{03}a_{21} - a_{02}a_{11} + a_{01}a_{12} = 0$$
$$a_{03}a_{20} - a_{02}a_{10} + a_{00}a_{12} = 0$$
$$a_{03}a_{30} - a_{01}a_{10} + a_{00}a_{11} = 0$$
$$a_{02}a_{30} - a_{01}a_{20} + a_{00}a_{21} = 0$$
$$a_{12}a_{30} - a_{11}a_{20} + a_{10}a_{21} = 0.$$ 

(2.15)

These are the algebraic equations assuring that the skew symmetric matrix (2.14) is of rank two (or zero). Let us summarise the above in the following Proposition.

**Proposition 2.8.** The free superintegrable systems in the Euclidean plane constitute a projective variety isomorphic to the Grassmannian $G_2(S_0^2\mathbb{C}^3)$ of 2-planes in $S_0^2\mathbb{C}^3$, embedded into $\mathbb{P}(\Lambda^2 S_0^2\mathbb{C}^3)$ as the 6-dimensional variety of rank two skew symmetric matrices (2.14) given by the Equations (2.15).
In the next section we will show that free superintegrable systems which are non-degenerate not only form a subset, but a subvariety in the above variety. The non-degenerate superintegrable systems then form a fibre bundle with 4-dimensional linear fibres over this subvariety, explaining the denomination in Definition 2.4.

3. The algebraic superintegrability conditions

3.1. Derivation. We consider the Bertrand-Darboux conditions (2.5) for two Killing tensors \(K^{(1)}\) and \(K^{(2)},\) rewritten in the form (2.13) for the corresponding special conformal Killing tensors \(L^{(1)}\) and \(L^{(2)}:\)

\[
\begin{align*}
L_{zz,w}^1 V_{zz} - L_{zz}^1 V_{ww} &= \frac{3}{2} (\lambda_z^1 V_w - \lambda_w^1 V_z) \\
L_{ww}^2 V_{zz} - L_{zz}^2 V_{ww} &= \frac{3}{2} (\lambda_z^2 V_w - \lambda_w^2 V_z). 
\end{align*}
\] (3.1)

As before, we write simple superscripts for simplicity of notation, as there is no risk of confusion with exponents. From (2.9b) we deduce that the second derivative of \(\lambda\) is \(\lambda_{ij} = c_{ij}.\) Together with (2.6) and (2.12) we obtain for the derivatives of the coefficient determinant in (3.1):

\[
\begin{align*}
L_{zz,z}^i &= 0 & L_{zz,w}^i &= \lambda_z^i \\
L_{ww,w}^i &= 0 & L_{ww,z}^i &= \lambda_w^i \\
\lambda_{iij} &= c^i \\
\lambda_{iij} &= c^i 
\end{align*}
\] (3.2)

for \(i = 1, 2.\) Hence the derivatives of the coefficient determinant

\[
D := \det \begin{bmatrix} L_{zz}^1 & L_{zz}^2 \\
L_{ww}^1 & L_{ww}^2 \end{bmatrix} = L_{zz}^1 L_{ww}^2 - L_{ww}^1 L_{zz}^2 
\] (3.3a)

are

\[
\begin{align*}
D_z &= L_{zz}^1 \lambda_z^2 - \lambda_w^1 L_{zz}^2 \\
D_\lambda^1 &= L_{zz}^1 \lambda_z^2 - \lambda_w^1 L_{zz}^2 \\
D_\lambda^2 &= L_{zz}^1 \lambda_z^2 - \lambda_w^1 L_{zz}^2 \end{align*}
\] (3.3b)

and

\[
\begin{align*}
D_w &= \lambda_w^1 L_{ww}^2 - L_{ww}^1 \lambda_z^2 \\
D_{zzw} &= \lambda_z^1 \lambda_z^2 - \lambda_w^1 \lambda_z^2 \\
D_{wwz} &= \lambda_z^1 \lambda_z^2 - \lambda_w^1 \lambda_z^2
\end{align*}
\] (3.3c)

This shows that \(D = D(z, w)\) is a cubic polynomial, but only quadratic in \(z\) respectively in \(w.\) In terms of the entries of the skew symmetric matrix (2.14), this cubic is given by

\[
D(z, w) = \sum a_{ij} z^i w^j,
\] (3.4)

where the sum runs over all pairs \((i, j) \neq (2, 2)\) with \(0 \leq i, j \leq 2.\)

Solving the linear system (3.1) for \(V_{zz}\) and \(V_{ww},\) we arrive at the system

\[
\begin{bmatrix} V_{zz} \\ V_{ww} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} V_z \\ V_w \end{bmatrix}
\] (3.5)

with

\[
\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \frac{1}{L_{zz}^1 L_{ww}^2 - L_{ww}^1 L_{zz}^2} \begin{bmatrix} \lambda_w^1 L_{zz}^2 - \lambda_z^1 L_{ww}^2 & L_{zz}^1 \lambda_z^2 - \lambda_w^1 L_{ww}^2 \\ \lambda_w^1 L_{ww}^2 - L_{ww}^1 \lambda_z^2 & L_{zz}^1 \lambda_z^2 - \lambda_w^1 L_{ww}^2 \end{bmatrix}.
\]

Comparing with (3.3), we see that the coefficient matrix can be written as

\[
\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} -D_z & A_z \\ B_w & -D_w \end{bmatrix}
\] (3.6)
where
\[ A := L_z^1 x^2 - L_z^1 z^2 \quad \quad B := \lambda^1 L_w^1 w^2 - L_w^1 w^2 \lambda^2. \]
By (3.2) the derivatives of \( A_z \) and \( B_w \) are
\[ A_{zz} = 0 \quad A_{zw} = D_{zz} \quad B_{ww} = 0 \quad B_{wz} = D_{ww} \quad (3.7) \]
and hence \( A_z = A_z(w) \) and \( B_w = B_w(z) \) are quadratic polynomials. In terms of the entries of the skew symmetric matrix (2.14) they are given by
\[ A_z(w) = a_{21} w^2 + 2a_{20} w + a_{30} \quad B_w(z) = a_{12} z^2 + 2a_{02} z + a_{03}. \quad (3.8) \]
That is, we can find the coefficients of \( A_z, D \) and \( B_w \) in, respectively, the (overlapping) upper left, upper right and lower right 3 x 3 submatrices of the skew symmetric 5 x 5 matrix (2.14).

From the expression (3.6) together with (3.7) and the Plücker relations (2.15) in the form (3.11a) we easily derive the following relations, where we have introduced two new symbols \( C_{122} = C_{12,w} \) and \( C_{211} = C_{21,z} \):
\[ C_{11,w} = C_{12} C_{21} \quad C_{12,z} = C_{12} C_{11} \quad C_{11,z} = C_{12} C_{22} + (C_{11})^2 - C_{122} \]
\[ C_{22,z} = C_{12} C_{21} \quad C_{21,w} = C_{21} C_{22} \quad C_{22,w} = C_{21} C_{11} + (C_{22})^2 - C_{211}. \]
The integrability conditions for these derivatives allow the derivatives of \( C_{122} \) and \( C_{211} \) to be expressed as
\[ C_{122,w} = C_{21} C_{122} + C_{22} C_{12} C_{21} \]
\[ C_{122,z} = 2 C_{11} C_{12} C_{21} + C_{12} (C_{22})^2 + C_{22} C_{122} - C_{12} C_{211} \]
\[ C_{211,w} = 2 C_{22} C_{12} C_{21} + C_{21} (C_{11})^2 + C_{11} C_{211} - C_{21} C_{112} \]
\[ C_{211,z} = C_{12} C_{211} + C_{11} C_{12} C_{21}. \]
We now have all derivatives of \( C_{11}, C_{12}, C_{21}, C_{22}, C_{122} \) and \( C_{211} \) expressed in terms of these symbols. The remaining integrability conditions are generated by
\[ 3 C_{21} C_{122} - C_{11} C_{211} - C_{22} C_{12} C_{21} - C_{21} C_{11} C_{11} = 0 \quad (3.9a) \]
\[ 3 C_{12} C_{211} - C_{22} C_{122} - C_{11} C_{21} C_{12} - C_{12} C_{22} C_{22} = 0 \]
and their differential consequence
\[ 2 C_{122} C_{211} - C_{11} C_{21} C_{122} - C_{22} C_{12} C_{211} + C_{12} C_{12} C_{21} C_{21} - C_{11} C_{22} C_{12} C_{21} = 0. \quad (3.9b) \]
Substituting (3.6) into (3.9) and taking (3.7) into account, we obtain
\[ 3 A_z D D_{ww} - 2 A_z B_w D_z - 2 A_z D_w^2 + D D_w D_{zz} = 0 \]
\[ 3 B_w D D_{zz} - 2 A_z B_w D_w - 2 B_w D_w^2 + D D_z D_{ww} = 0 \quad (3.10a) \]
as well as
\[ 2 D_z D_{zz} D_{ww} + B_w D D_z D_{zz} - A_z D D_w D_{ww} - A_z B_w D_z D_w + A_z B_w^2 = 0. \quad (3.10b) \]
Recall that \( D, A_z \) and \( B_w \) are polynomials in \( z \) and \( w \) with coefficients \( a_{ij} \). The partial differential equations (3.10) therefore define homogeneous algebraic equations in the \( a_{ij} \). In combination with Proposition 2.8 we now get our first main result.
Theorem 3.1. The set of non-degenerate (second order maximally) free superintegrable systems on the Euclidean plane has a natural structure of a projective variety, isomorphic to the subvariety in the variety of rank two skew symmetric $5 \times 5$ matrices (2.14) defined by the algebraic equations (3.10) for the polynomials (3.4) and (3.8).

Definition 3.2. For brevity, we will call the variety defined in the above proposition the variety of superintegrable systems.

Some remarks concerning this definition are in order.

Remark 3.3. We can regard the variety of superintegrable systems as a subvariety in the projective space of skew symmetric $5 \times 5$ matrices given by the homogeneous equations (3.10) together with the Plücker relations (2.15), the latter assuring that the matrix rank is two.

Remark 3.4. Degenerate superintegrable systems in dimension two turn out to be particular instances of non-degenerate systems [KKM05, KKMP09]. This is why we omit “non-degenerate” from the name of the variety.

Remark 3.5. By construction, every non-degenerate free superintegrable system defines a point on the above variety. However, there are two valid solutions of the superintegrability conditions (3.10) with $D$ vanishing identically. Indeed, by (3.7) the polynomials $A_z$ and $B_w$ are constant in this case and by (2.15) or (3.10b) one of them must vanish. Regarding (3.5) and (3.6), the variety of superintegrable systems therefore contains two points which do not correspond to a non-degenerate free superintegrable system. We call them the two degenerate points and denote their union by $V_\emptyset$ for reasons to become clear later.

Remark 3.6. Working over the complex numbers allows us to treat both real cases at once: In the Euclidean case we impose that $z$ and $w$ as well as $a_{ij}$ and $a_{ji}$ be complex conjugates and in the Minkowski case we impose them to be real. The corresponding involutions $a_{ij} \mapsto \overline{a_{ji}}$ and $a_{ij} \mapsto a_{ij}$ define two real forms of the variety of superintegrable systems, which classify superintegrable systems on the real Euclidean plane respectively the Minkowski plane.

The involution $a_{ij} \mapsto \overline{a_{ij}}$ is equivalent to exchanging $z$ and $w$ as well as $A$ and $B$. We will refer to this operation as conjugation.

The variety of superintegrable systems captures the essential (difficult) part of the classification problem, since all non-degenerate (free and non-free) superintegrable systems form a fibre bundle with 4-dimensional linear fibres over this variety (excluding the two degenerate points). Obtaining the fibre over a point in the base amounts to a (simple) integration of the System (3.5) for given $C_{ij}$, see Section 4.5.

3.2. Simplification. Writing

\[ l^i = (L^i_{zz}, \lambda^i_z, c, \lambda^i_w, L^i_{ww}), \quad i = 1, 2, \]

we can arrange the derivatives of $D$ together with $A_z$ and $B_w$ in the rank two matrix

\[ l^1 \wedge l^2 = \begin{bmatrix}
0 & A_z & D_{zz} & D_z & D \\
0 & D_{zzw} & D_{zw} & D_w & D \\
0 & D_{wwz} & D_{ww} & D_w & D \\
\text{(skew)} & 0 & 0 & B_w & 0
\end{bmatrix}, \]
Having rank two implies that the Pfaffians of its five principal minors vanish. For the \((3, 3)\) minor this yields the identity
\[
A_z B_w = D_z D_w - D D_{zw}.
\] (3.11a)

The remaining four principal minors are differential consequences of this identity, namely the two derivatives of this identity with respect to \(z\) respectively \(w\):
\[
A_z D_{ww} = D_w D_{zz} - D D_{zzw} \quad A_z D_{wwz} = D_{zz} D_{zw} - D_z D_{zzw} \quad B_w D_{zz} = D_z D_{ww} - D D_{wwz} \quad B_w D_{zzw} = D_{zw} D_{ww} - D_w D_{wwz}.
\] (3.11b)

All other derivatives are identically satisfied. Note that this is nothing but a local (differential) version of the Plücker relations (2.15), which are equivalent to (3.11a).

With the above identities we can transform the cubic superintegrability condition (3.10a) into
\[
4 D D_{ww} D_{zz} - 3 D^2 D_{zzw} - 2 D_w D_z^2 + 2 D D_z D_{zw} - 2 A_z D_w^2 = 0
\]
\[
4 D D_{zz} D_{ww} - 3 D^2 D_{wwz} - 2 D_z D_w^2 + 2 D D_w D_{zw} - 2 B_w D_z^2 = 0.
\] (3.12)

Differentiating the first condition with respect to \(w\) and replacing the term containing \(A_z\) using the Plücker relations (3.11b) yields
\[
D(2 D_{zz} D_{ww} + D_z D_{wwz} + D_w D_{zzw} + D_z^2) - (D_z^2 D_{ww} + D_w^2 D_{zz} + D_z D_w D_{zw}) = 0.
\] (3.13)

Doing similarly with the second condition gives the same result. In the same way we can replace all terms containing \(A_z\) or \(B_w\) in the quartic superintegrability condition (3.10b) using the Plücker relations (3.11) to obtain Equation (3.13) multiplied by \(D\). Consequently, given the Plücker relations (3.11), we can confirm that the quartic superintegrability condition (3.10b) is a differential consequence of the cubic superintegrability conditions (3.10a). Rewriting Equations (3.12) and (3.13), we can summarise the above as follows.

**Lemma 3.7.** The Plücker relations (2.15) are equivalent to (3.11a). If they are satisfied, then the conditions (3.10) are equivalent to
\[
A_z D_w^2 = 2 D D_w D_{zz} - \frac{3}{2} D^2 D_{zzw} - D_w D_z^2 + D D_z D_{zw} \quad B_w D_z^2 = 2 D D_z D_{ww} - \frac{3}{2} D^2 D_{wwz} - D_z D_w^2 + D D_w D_{zw}
\] (3.14a)
\[
\quad \text{and imply}
\]
\[
D_z^2 D_{ww} + D_w^2 D_{zz} + D_z D_w D_{zw} = D(2 D_{zz} D_{ww} + D_z D_{wwz} + D_w D_{zzw} + D_z^2).
\] (3.15)

Note that one can solve Equation (3.15) for \(D\) and then substitute the solution into the Equations (3.14) or into the Plücker relations (3.11) in order to determine \(A_z\) and \(B_w\).

**Corollary 3.8.** The variety of superintegrable systems is isomorphic to the subvariety in the projective space of skew symmetric \(5 \times 5\) matrices (2.14) given by the Plücker relations (3.11) and the Equations (3.14) for the polynomials (3.4) and (3.8).
Recall that $D$, $A_z$ and $B_w$ are polynomials in $z$ and $w$ with coefficients $a_{ij}$. The Equations (3.11) and (3.14) therefore define homogeneous algebraic equations in the $a_{ij}$. All together, this gives a set of 5 quadratic and 32 cubic equations. There will be no need here to write them down explicitly.

**Definition 3.9.** We call the defining equations of the variety of superintegrable systems the *algebraic superintegrability conditions*.

4. Solution, normal forms and classification

4.1. Splitting of the ternary cubic. The following lemma is the key observation for most of what follows.

**Lemma 4.1.** The algebraic superintegrability conditions imply that the ternary cubic $D(z, w)$ can be decomposed into linear factors.

**Proof.** Differentiating Condition (3.15) with respect to $z$ respectively $w$ results in

\begin{align}
D_z D_{zw} D_{zw} &= D(D_{zz} D_{wuw} + D_{zw} D_{zzw}) \tag{4.1a} \\
D_z D_{uw} D_{zw} &= D(D_{uw} D_{zzw} + D_{zw} D_{wuw}). \tag{4.1b}
\end{align}

Differentiating (4.1a) with respect to $w$ or (4.1b) with respect to $z$ we obtain

\begin{equation}
D_z D_{zw} D_{uw} = 2D D_{zzw} D_{wuw}. \tag{4.2}
\end{equation}

Recall that $D(z, w)$ is a cubic polynomial in $z$ and $w$. Hence the second derivatives $D_{zz}$, $D_{zw}$ and $D_{uw}$ are linear while the third derivatives $D_{zzw}$ and $D_{wuw}$ are constants. We distinguish four cases, depending on whether these constants are zero or not.

1. If $D_{wuw} \neq 0 \neq D_{zzw}$, Equation (4.2) shows that $D$ decomposes into linear factors.

2. If $D_{wuw} = 0 \neq D_{zzw}$ we have $D_{zw} \neq 0$ and from (4.1a) we deduce that $D$ is a constant multiple of $D_w D_{zz}$. In particular we can assume $D_{zz} \neq 0$ so that $D_{wuw} = 0$ by (4.2). Consequently, $D_w$ only depends on $z$ and hence decomposes (over $\mathbb{C}$) into linear factors. But then $D_w D_{zz}$ and therefore $D$ also decompose into linear factors.

3. The case $D_{zzw} = 0 \neq D_{wuw}$ becomes case (2) after interchanging $z$ and $w$.

4. In the remaining case $D_{zzw} = D_{wuw} = 0$ the polynomial $D(z, w)$ is quadratic. Hence the first derivatives $D_z$ and $D_w$ are linear and the second derivatives $D_{zz}$, $D_{zw}$ and $D_{uw}$ are constants. Equation (3.15) then shows that $D$ is a constant multiple of $D_z D_{zw} + D_w D_{zzw}$. If $D_{zw} \neq 0$ then the Equations (4.1) show that $D_z D_{uw} = D_w D_{zz} = 0$. Therefore $D$ is a constant multiple of $D_z D_{zw}$, which is a product of linear factors. If $D_{zw} = 0$ then $D$ is a constant multiple of $D_z^2 D_{uw} + D_w^2 D_{zz}$, which can also be decomposed (over $\mathbb{C}$) into linear factors. □

The above lemma allows us to write the cubic (3.3) as a product of three linear forms,

\begin{equation}
D(z, w) = (a_1 z + b_1 w + c_1)(a_2 z + b_2 w + c_2)(a_3 z + b_3 w + c_3). \tag{4.3}
\end{equation}

The isometry group acts on each linear form as the dual of the standard representation, c.f. Section 4.3. This action has three orbits: One orbit consisting of linear factors depending on $z$ alone (i.e. $b_i = 0$), one consisting of linear factors depending on both $z$ and $w$ and one consisting of factors depending on $w$ alone ($a_i = 0$). Accordingly we denote the multiplicities of the linear factors in (4.3) by a triple
with one label for each orbit (in this order). The label “1” stands for a single factor, the label “2” for two proportional factors, “11” for two non-proportional factors and “0” for a constant. Higher multiplicities cannot appear due to the fact that \( D(z, w) \) contains neither of the cubic monomials \( z^3 \) and \( w^3 \). See Table 3 for some examples.

By (3.14), the cubic \( D(z, w) \) completely determines the superintegrable system except for the degenerate cases where it does not depend on \( z \) or \( w \). That is why we will use these multiplicities to label isometry classes of superintegrable systems. Moreover, the factorisation (4.3) provides a geometric way to classify superintegrable systems in the plane: Since each linear factor determines a projective line in the plane, superintegrable systems can be labelled by planar arrangements of three (possibly coinciding) projective lines, c.f. Figure 1. From (3.5) and (3.6) we see that these arrangements actually have an interpretation in terms of the superintegrable potential:

**Proposition 4.2.** The singular set of a superintegrable potential \( V(z, w) \) in the fibre over a non-degenerate free superintegrable system is contained in a planar arrangement of (up to) three projective lines, given by the equation \( D(z, w) = 0 \).

In summary, the splitting of the ternary cubic \( D(z, w) \) provides an intrinsic (algebraic as well as geometric) labelling scheme for superintegrable systems. In the next section we will see that it also allows for a relatively simple solution of the algebraic superintegrability conditions.

### 4.2. Solution of the algebraic superintegrability conditions.

Owing to the fact that the cubic polynomial \( D(z, w) \) is completely reducible and only quadratic in \( z \) and \( w \), it must be of one of the following two forms:

\[
D(z, w) = (a_1 z + c_1)(a_2 z + b_2 w + c_2)(b_3 w + c_3) \tag{4.4a}
\]

\[
D(z, w) = (a_1 z + b_1 w + c_1)(a_3 z + b_3 w + c_3). \tag{4.4b}
\]

In this parametrisation Equation (3.15) for \( D \) is easily solved.

**Proposition 4.3.** As a set, the variety \( D \) of solutions to the Equation (3.15) is the union

\[
D = D_{(1,1,1)} \cup D_{(11,0,1)} \cup D_{(1,0,11)} \cup D_{(0,11,0)} \tag{4.5}
\]

of four classes, consisting of completely reducible ternary cubics of the following form.

- \( D_{(1,1,1)} \): \( D(z, w) = (a_1 z + c_1)(a_2 z + b_2 w + c_2)(b_3 w + c_3) \) subject to
  \[
  \det \begin{bmatrix} a_1 & 0 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{bmatrix} = 0. \tag{4.6a}
  \]

- \( D_{(11,0,1)} \): \( D(z, w) = (a_1 z + c_1)(a_2 z + c_2)(b_3 w + c_3) \)

- \( D_{(1,0,11)} \): \( D(z, w) = (a_1 z + c_1)(b_2 w + c_2)(b_3 w + c_3) \)

- \( D_{(0,11,0)} \): \( D(z, w) = (a_1 z + b_1 w + c_1)(a_3 z + b_3 w + c_3) \) subject to
  \[
  \det \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 1 \\ a_3 & -b_3 & c_3 \end{bmatrix} = a_1 b_3 + b_1 a_3 = 0. \tag{4.6b}
  \]

\(^2\)The minus sign in front of \( b_3 \) is correct.
Figure 1. Inclusion graph for classes of solutions of Equation (3.15). Inclusions are from bottom to top between classes joined by an edge. Two contiguous lines symbolise a double line.

Remark 4.4. The above classes are not disjoint. Their intersections are given by Figure 1 with the following subclasses (and their conjugates):

- \( D_{(11,0,0)}: \quad D(z, w) = (a_1 z + c_1)(a_2 z + c_2) \)
- \( D_{(2,0,1)}: \quad D(z, w) = (a_1 z + c_1)^2(b_3 w + c_3) \)
- \( D_{(1,0,1)}: \quad D(z, w) = (a_1 z + c_1)(b_3 w + c_3) \)

These subclasses are not disjoint either. Their intersections are given by the following subsubclasses (and their conjugates):

- \( D_{(2,0,0)}: \quad D(z, w) = (a_1 z + c_1)^2 \)
- \( D_{(1,0,0)}: \quad D(z, w) = (a_1 z + c_1) \)
- \( D_{(0,1,0)}: \quad D(z, w) = a_2 z + b_2 w + c_2 \)

Finally, these subsubclasses intersect in the class \( D_{(0,0,0)} \) consisting of constant polynomials \( D(z, w) \).

Corollary 3.8 now yields a complete solution of the algebraic superintegrability conditions and hence a parametrisation of the variety of superintegrable systems.
Theorem 4.5. As a set, the projective variety of superintegrable systems is the union

\[ V = \mathcal{V}_{(1,1,1)} \cup \mathcal{V}_{(11,0,1)} \cup \mathcal{V}_{(11,0,0)} \cup \mathcal{V}_{(0,11,0)} \]

\[ \cup \mathcal{V}_{(1,0,11)} \cup \mathcal{V}_{(0,0,11)} \]

of the six classes given in Table 2 (up to conjugates).

Proof. Recall that \( D(z, w) \) determines the superintegrable system up to the two constants \( a_{30} \) and \( a_{03} \). These can be determined from the superintegrability conditions (3.14a) and (3.14b) or from the the Plücker relations (3.11). Solving the algebraic superintegrability conditions is therefore straightforward, so we will only justify the completeness of the list.

Suppose \( D \) lies in class \((1,1,1)\) with \( a_1 = 0 \) or \( b_3 = 0 \). Without loss of generality we may suppose the latter. In this case \( c_3 \neq 0 \), since otherwise \( D \) would be identically zero, i.e. in class \((11,0,0)\). Condition (4.6a) then implies \( a_1 = 0 \) or \( b_2 = 0 \), i.e. that \( D \) lies in class \((0,1,0)\) or \((11,0,0)\).

Note that if \( D \) lies in class \((11,0,1)\) with \( b_3 = 0 \) then it lies in class \((11,0,0)\) and similarly for class \((1,0,11)\).

Suppose now that \( D \) is of class \((0,11,0)\) with \( a_1 a_3 b_1 b_3 = 0 \). Due to Condition (4.6b) we may assume without loss of generality that \( b_1 = a_1 = 0 \) or \( b_1 = b_3 = 0 \). In the first case \( D \) lies in class \((0,1,0)\), in the second case in class \((11,0,0)\).

Suppose finally that \( D \) lies in class \((0,1,0)\) with \( a_2 = 0 \) or \( b_2 = 0 \). In the first case \( D \) also lies in class \((0,0,11)\) and in the second in class \((11,0,0)\).

4.3. Normal forms. The set of superintegrable systems is invariant under isometries. The variety of superintegrable systems on the plane is therefore equipped with a natural action of the Euclidean group. On the polynomials \( D, A \) and \( B \) this action is induced by the standard action of the Euclidean group on the plane. In the null basis (2.12) translations and rotations are given by shifts and shears, respectively, i.e. by

\[ (z, w) \mapsto (z + c, w + d) \quad c, d \in \mathbb{C}, \]

\[ (z, w) \mapsto (\lambda z, w/\lambda) \quad \lambda \in \mathbb{C} \setminus \{0\}. \]

The different orbits and normal forms of this action can easily be derived from Table 2 and are listed in Table 3.

4.4. Relative invariants. In [KKM07a] a complete set of relative invariants for isometry classes of superintegrable systems was constructed. We point out that in our algebraic description this task is trivial: The variables \( a_{ij} \) already constitute a complete set of relative invariants, as shown in Table 1.

4.5. Classification of superintegrable potentials. By a separation of variables in \( x = z + w \) and \( y = z - w \) the system (3.5) can be integrated for each normal form to yield the corresponding superintegrable potentials as listed in Table 4. This perfectly matches the list in [KKPM01] and thereby confirms the present approach (or the known classification).

\[ \text{The ring accents indicate that the corresponding sets are not Zariski closed.} \]
class | vanishing relative invariants | relations
--- | --- | ---
(1, 1, 1) | ∅ | 
(11, 0, 1) | $a_{12}, a_{02}, a_{03}$ | $a_{21}a_{01} = 4a_{11}^2, a_{20}a_{00} = 4a_{10}^2$
(2, 0, 1) | $a_{12}, a_{02}, a_{03}$ | $a_{21}a_{01} = 4a_{11}^2, a_{20}a_{00} = 4a_{10}^2$
(0, 11, 0) | $a_{12}, a_{21}, a_{11}$ | 
(11, 0, 0) | $a_{ij}$ unless $j = 0$ | 
(2, 0, 0) | $a_{ij}$ unless $j = 0$ | $4a_{20}a_{00} = a_{10}^2$
(1, 0, 1) | $a_{ij}$ unless $0 \leq i, j \leq 1$ | 
(0, 1, 0) | $a_{20}, a_{21}, a_{11}, a_{12}, a_{02}$ | 
(1, 0, 0) | $a_{ij}$ unless $0 \leq i \leq 1$ and $j = 0$ | 
(0, 0, 0) | $a_{ij}$ unless $i = j = 0$ | 

Table 1. Relative invariants for isometry classes of superintegrable systems in the plane (up to conjugation).

5. The variety of superintegrable systems

After having solved the algebraic superintegrability conditions we now study the geometric structure of the corresponding variety, i.e. the variety $\mathcal{V}$ of superintegrable systems. Recall that $\mathcal{V}$ is a subvariety in the Grassmannian $G_2(S^2_0 \mathbb{C}^3)$ of 2-planes in $S^2_0 \mathbb{C}^3$, embedded into $\mathbb{P}(\Lambda^2 S^2_0 \mathbb{C}^3)$ via the Plücker embedding, and that a point on $\mathcal{V}$ is given by three polynomials $D$, $A_z$ and $B_w$. Mapping $(D, A_z, B_w) \mapsto D$ defines a projection

$$
\pi : \mathbb{P}(\Lambda^2 S^2_0 \mathbb{C}^3) \rightarrow \mathbb{P}(S^3 \mathbb{C}^3)
$$

to the space of ternary cubics. This map is defined on the complement of the subspace $D = 0$, which intersects $\mathcal{V}$ in the union $\mathcal{V}_0$ of the two degenerate points.

By Lemma 4.1 the image $D = \pi(\mathcal{V})$ under this projection is contained in the subvariety of ternary cubics that are decomposable into linear factors. Denoting the symmetric product of three projective planes by

$$
\Sigma_3 \mathbb{P}^2 := (\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)/S_3,
$$

this subvariety is the image of the embedding of $\Sigma_3 \mathbb{P}^2$ into the space of ternary cubics, given by mapping the three linear factors to their product.

We now study the irreducible components $D_c \subset D$ and their preimages under $\pi$. The fact that generically the polynomials $A_z$ and $B_w$ are uniquely defined by $D$ will provide rational right inverses $\sigma_c$ to the projection $\pi$ over each irreducible component $D_c$ and thereby a description of the irreducible components in $\mathcal{V}$.

**Proposition 5.1.** Let $\hat{D}_{(1,1,1)}$ be the variety of points

$$
((a_1 : c_1), (a_2 : b_2 : c_2), (b_3 : c_3)) \in \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^1
$$

(5.1a)
for which
\[ \det \begin{bmatrix} a_1 & 0 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{bmatrix} = 0. \] (5.1b)

Then the map given by sending (5.1a) to the cubic (4.4a) defines a regular birational map
\[ \hat{D}_{(1,1,1)} \to D_{(1,1,1)}. \]

In particular, \( D_{(1,1,1)} \) is birational to \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) and hence irreducible.

**Proof.** The above regular map is given explicitly by expanding (4.4a) and comparing it to (3.4):
\[ \begin{align*}
  a_{21} &= a_1 a_2 b_3 \\
  a_{10} &= a_1 c_2 c_3 + c_1 a_2 c_3 \\
  a_{11} &= a_1 b_2 c_3 + a_1 c_2 b_3 + c_1 a_2 b_3 \\
  a_{12} &= a_1 b_2 b_3 \\
  a_{01} &= c_1 b_2 c_3 + c_1 c_2 b_3 \\
  a_{02} &= c_1 b_2 b_3 \\
  a_{00} &= c_1 c_2 c_3.
\end{align*} \]

By (5.1b) we have \( a_{11} = 2a_1 c_2 b_3 \), which gives the rational inverse
\[ (a_{ij}) \mapsto \left( (a_{12} : a_{02}), (a_{21} : a_{12} : \frac{1}{2} a_{11}), (a_{21} : a_{20}) \right). \]

Finally, a birational isomorphism \( \hat{D}_{(1,1,1)} \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) is given by the projection \( (a_{2} : b_{2} : c_{2}) \mapsto (a_{2} : b_{2}) \) in the middle factor. \( \square \)

Under the duality of points and lines in \( \mathbb{P}^2 \), \( \hat{D}_{(1,1,1)} \) is the variety of triples of collinear points, the first of them confined to the line \( w = 0 \), the third to \( z = 0 \). The second point is then confined to the line between the other two unless they both coincide with the origin.

In view of Hironaka’s Theorem, the following proposition shows that the map \( \hat{D}_{(1,1,1)} \to D_{(1,1,1)} \) above is “almost” a resolution of the singularities in \( D_{(1,1,1)} \).

**Proposition 5.2.** The variety \( \hat{D}_{(1,1,1)} \) is smooth on the complement of the point \( ((0 : 1), (0 : 0 : 1), (0 : 1)) \), which is the preimage of the point \( D_{(0,0,0)} \).

In the above dual picture, this singularity corresponds to the configuration when all three points coincide with the origin.

**Proof.** \( \hat{D}_{(1,1,1)} \) is the zero locus of the determinant map \( \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{C} \), given by sending (5.1a) to the left hand side of (5.1b), and its singularities are those points where the tangent map vanishes. The tangent of the determinant map \( A \mapsto \det A \) is given by Jacobi’s formula as \( X \mapsto \text{tr} \, XC^T \), where \( C \) is the cofactor matrix of \( A \). Hence \( \hat{D}_{(1,1,1)} \) is singular at points (5.1a) for which the cofactor matrix of the matrix in (5.1b) is orthogonal to the the space of matrices \( X \) of the form
\[ \begin{bmatrix}
  * & 0 & * \\
  * & * & * \\
  0 & * & *
\end{bmatrix} \]

with respect to the usual Hermitian inner product on matrices. That is, the singular locus is given by the minors of the non-zero entries of the matrix in (5.1b). In particular, we have \( a_1 b_3 = a_1 c_3 = 0 \). Now \( a_1 \neq 0 \) would imply \( b_3 = c_3 = 0 \), which is impossible. So \( (a_1 : c_1) = (0 : 1) \) and similarly \( (b_3 : c_3) = (0 : 1) \). We also have
$c_1a_2 = a_1c_2 = 0$ and $b_2c_3 = c_2b_3 = 0$, from which we conclude $a_2 = b_2 = 0$ since $c_1, c_3 \neq 0$.

**Proposition 5.3.** $\mathcal{D}_{(1,0,1)}$ is a variety biregular to $\mathbb{P}^2 \times \mathbb{P}^1$. So obviously, $\mathcal{D}_{(1,0,1)}$ is irreducible and smooth. The same holds for $\mathcal{D}_{(1,1,0)}$, since it is conjugated to $\mathcal{D}_{(1,0,1)}$.

**Proof.** This follows from the fact that $\mathcal{D}_{(1,0,1)}$ is the set of cubics of the form $(a_1z + c_1)(a_2z + c_2)(b_3w + c_3)$ and therefore biregular to the variety $(\Sigma_2 \mathbb{P}^1) \times \mathbb{P}^1$, which is biregular to $\mathbb{P}^2 \times \mathbb{P}^1$.

Under the duality of points and lines in $\mathbb{P}^2$ we can regard $\mathcal{D}_{(1,0,1)}$ as the variety of triples of unordered points with two of them confined to the line $w = 0$ and the third one to the line $z = 0$.

**Proposition 5.4.** $\mathcal{D}_{(0,11,0)}$ is a variety biregular to the cubic threefold of points

$$(a_{20} : a_{02} : a_{10} : a_{01} : a_{00}) \in \mathbb{P}^4$$

for which

$$\det \begin{bmatrix} a_{02} & a_{01} & 0 \\ a_{01} & 4a_{00} & a_{10} \\ 0 & a_{10} & a_{20} \end{bmatrix} = 0.$$  (5.2b)

In particular, $\mathcal{D}_{(0,11,0)}$ is birational to $\mathbb{P}^3$ and hence irreducible. Moreover, it is singular in the pair of intersecting lines $\mathcal{D}_{(2,0,0)} \cup \mathcal{D}_{(0,0,2)}$.

**Proof.** $\mathcal{D}_{(0,11,0)}$ is the variety of ternary quadrics of the form (4.4b) subject to (4.6b) and hence a subvariety in the variety $\Sigma_2 \mathbb{P}^2$ of completely reducible ternary quadrics. Writing a ternary quadric as

$$D(z, w) = a_{20}z^2 + a_{02}w^2 + a_{11}zw + a_{10}z + a_{01}w + a_{00},$$

$\Sigma_2 \mathbb{P}^2 \subset \mathbb{P}(S^2 \mathbb{C}^3) \cong \mathbb{P}^5$ is the hypersurface given by

$$\det \begin{bmatrix} a_{02} & a_{01} & \frac{1}{2}a_{11} \\ a_{01} & 4a_{00} & a_{10} \\ \frac{1}{2}a_{11} & a_{10} & a_{20} \end{bmatrix} = 0.$$  (5.2b)

Expanding (4.4b), comparing it to (3.4) and taking (4.6b) into account, the subvariety $\mathcal{D}_{(0,11,0)} \subset \Sigma_2 \mathbb{P}^2$ is seen to be the linear section $a_{11} = 0$.

A birational isomorphism $\mathcal{D}_{(0,11,0)} \subset \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ is given, for example, by projecting (5.2a) onto the first four homogeneous coordinates.

The singular locus of $\mathcal{D}_{(0,11,0)}$ can be computed similarly to that of $\mathcal{D}_{(1,1,1)}$ above. It is given by the minors of the non-zero entries of the matrix in (5.2b), which define the subvariety $\mathcal{D}_{(2,0,0)} \cup \mathcal{D}_{(0,0,2)}$.

Since all four components are irreducible and cover $\mathcal{D}$, we have the following.

**Corollary 5.5.** The decomposition (4.5) of $\mathcal{D} = \pi(\mathcal{V})$ is a decomposition into irreducible components.

We finally state our last main result, the structure theorem for the variety of superintegrable systems in the Euclidean plane. It shows that the non-trivial components of $\mathcal{V}$ are blowups of the components of $\mathcal{D}$ in certain pairs of intersecting lines.
Theorem 5.6. The variety of superintegrable systems has a decomposition into six irreducible components,

\[ V = V_{(1,1,1)} \cup V_{(11,0,1)} \cup V_{(11,0,0)} \cup V_{(0,11,0)} \]
\[ \cup V_{(1,0,11)} \cup V_{(0,0,11)}, \]

which are the Zariski closures of the corresponding classes given in Table 2. The projection \( \pi : V \to D \) restricts to regular maps

(i) \( V_{(1,1,1)} \to D_{(1,1,1)} \)
(ii) \( V_{(11,0,1)} \to D_{(11,0,1)} \)
(iv) \( V_{(0,11,0)} \to D_{(0,11,0)} \)
(iii) \( V_{(1,0,11)} \to D_{(1,0,11)} \),

each of which is an isomorphism over the complement of a pair of intersecting lines, namely:

(i - iii) \( D_{(1,0,0)} \cup D_{(0,0,1)} \)
(iv) \( D_{(2,0,0)} \cup D_{(0,0,2)} \)

On the remaining two components, the projection \( \pi : V \to D \) restricts to central projections

(v) \( V_{(11,0,0)} \to D_{(11,0,0)} \)
(vi) \( V_{(0,0,11)} \to D_{(0,0,11)} \),

each from one of the two degenerate points.

Proof. The projection \( \pi \) is regular on the complement of the line where \( D \) is identically zero. This line intersects \( V \) exactly in the two degenerate points, which are contained in \( V_{(11,0,0)} \) respectively \( V_{(0,0,11)} \), but not in the other components. To define the required birational inverses, recall that the projection map \( \pi \) “forgets” the two coefficients \( a_{30} \) and \( a_{03} \), so that we have to recover them from the remaining \( a_{ij} \). From the Plücker relations (2.15) we get

\[ a_{30} = \frac{a_{20}a_{11} - a_{21}a_{10}}{a_{12}} = \frac{a_{20}a_{01} - a_{21}a_{00}}{a_{02}} \]

and from evaluating (3.14a) at \( z = w = 0 \) we obtain

\[ a_{30} = \frac{a_{00}(4a_{10}a_{02} - 3a_{00}a_{12}) + a_{01}(a_{00}a_{11} - a_{10}a_{01})}{a_{01}^2}. \]

On the other hand, from the explicit solution in Table 2 we see that

on \( V_{(1,1,1)} \): \( a_{30} = \frac{a_{20}}{a_{21}}a_{20} \)

on \( V_{(11,0,1)} \): \( a_{30} = \frac{a_{20}}{a_{21}}a_{20} = \frac{a_{10}}{a_{11}}a_{20} = \frac{a_{00}}{a_{01}}a_{20} \)

on \( V_{(0,11,0)} \): \( a_{30} = \frac{a_{20}}{a_{02}}a_{01} = \frac{4a_{00}a_{20} - a_{10}^2}{a_{01}}. \)

The coefficient \( a_{30} \) is therefore well defined on the complement of the common zero locus of all nominators and denominators in the above quotients. On \( D_{(11,0,1)} \) and \( D_{(0,11,0)} \) this is readily seen to be \( D_{(1,0,0)} \) respectively \( D_{(2,0,0)} \). On \( D_{(1,1,1)} \) the square of the left hand side of (5.1b) can be expressed as

\[ a_{11}^2 + 8a_{20}a_{02} - 4(a_{01}a_{21} + a_{10}a_{12}) = 0 \]

and implies that also \( a_{11} = 0 \). Therefore \( a_{30} \) is well defined on the complement of \( D_{(1,0,0)} \) in \( D_{(1,1,1)} \). On \( V_{(1,0,11)} \) we have \( a_{30} = 0 \), which is well defined anyway. Similar statements hold for \( a_{03} \) by interchanging \( a_{ji} \) and \( a_{ij} \). This gives a rational
Table 2. Complete solution of the algebraic superintegrability conditions. The ring accents indicate that the corresponding sets are not Zariski closed. We define the class \( \mathcal{V}_{(1,1,1)} \) to comprise the class \( \mathcal{V}_{(0,1,0)} \), since it is a limiting case.

| class          | polynomials                                                                 | conditions |
|----------------|------------------------------------------------------------------------------|------------|
| \( \mathcal{V}_{(1,1,1)} \) | \( D(z, w) = (a_1 z + c_1)(a_2 z + b_2 w + c_2)(b_3 w + c_3) \) \hspace{1cm} (4.6a)  
|                | \( A_z(w) = a_1 a_2 (b_3 w + c_3)^2 / b_3 \) & \( b_3 \neq 0 \)  
|                | \( \cup B_w(z) = b_2 b_3 (a_1 z + c_1)^2 / a_1 \) & \( a_1 \neq 0 \) |
| \( \mathcal{V}_{(0,1,0)} \)  | \( D(z, w) = a_2 z + b_2 w + c_2 \) & \( b_2 \neq 0 \)  
|                | \( A_z(w) = -a_2^2 / b_2 \) & \( b_2 \neq 0 \)  
|                | \( B_w(z) = -b_2^2 / a_2 \) & \( a_2 \neq 0 \) |
| \( \mathcal{V}_{(11,0,1)} \) | \( D(z, w) = (a_1 z + c_1)(a_2 z + c_2)(b_3 w + c_3) \)  
|                | \( A_z(w) = a_1 a_2 (b_3 w + c_3)^2 / b_3 \) & \( b_3 \neq 0 \)  
|                | \( B_w(z) = 0 \) |
| \( \mathcal{V}_{(0,11,0)} \)  | \( D(z, w) = (a_1 z + b_1 w + c_1)(a_3 z + b_3 w + c_3) \) \hspace{1cm} (4.6b)  
|                | \( A_z(w) = a_1 a_3 (2 w + c_3 / b_3 + c_1 / b_1) \) & \( b_1 b_3 \neq 0 \)  
|                | \( B_w(z) = b_1 b_3 (2 z + c_3 / a_3 + c_1 / a_1) \) & \( a_1 a_3 \neq 0 \) |
| \( \mathcal{V}_{(11,0,0)} \)  | \( D(z, w) = (a_1 z + c_1)(a_2 z + c_2) \) & \( a_{1j} = 0 \) for \( j \neq 0 \)  
|                | \( A_z(w) = 2 a_1 a_2 w + a_{30} \) & none  
|                | \( B_w(z) = 0 \) |
Table 3. Normal forms for solutions of the algebraic superintegrability conditions with corresponding labels from [KKPM01].

| class     | $D(z, w)$           | $A_z(w)$ | $B_w(z)$ | label |
|-----------|---------------------|----------|----------|-------|
| (1, 1, 1) | $z(z + w)w$         | $w^2$    | $z^2$    | E16   |
| (11, 0, 1)| $z(z + 1)w$         | $w^2$    | 0        | E19   |
| (1, 0, 11)| $z(w + 1)w$         | 0        | $z^2$    | E19   |
| (2, 0, 1) | $z^2w$              | $w^2$    | 0        | E17   |
| (1, 0, 2) | $zw^2$              | 0        | $z^2$    | E17   |
| (0, 11, 0)| $(z + w)(z - w)$    | $2w$     | $-2z$    | E1    |
| (11, 0, 0)| $z(z + 1)$          | $2w$     | 0        | E7    |
| (0, 0, 11)| $w(w + 1)$          | 0        | $2z$     | E7    |
| (2, 0, 0) | $z^2$               | $2w$     | 0        | E8    |
| (0, 0, 2) | $w^2$               | 0        | $2z$     | E8    |
| (1, 0, 1) | $zw$                | 0        | 0        | E20   |
| (0, 1, 0) | $z + w$             | 1        | 1        | E2    |
| (1, 0, 0) | $z$                 | 1        | 0        | E9    |
| (1, 0, 0) | $z$                 | 0        | 0        | E11   |
| (0, 0, 1) | $w$                 | 0        | 1        | E9    |
| (0, 0, 1) | $w$                 | 0        | 0        | E11   |
| (0, 0, 0) | 1                   | 1        | 0        | E10   |
| (0, 0, 0) | 1                   | 0        | 1        | E10   |
| (0, 0, 0) | 1                   | 0        | 0        | E3    |
Table 4. Superintegrable potentials in the plane (up to conjugation). For each class a basis of non-constant superintegrable potentials is given.
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SUPERINTEGRABLE SYSTEMS IN THE EUCLIDEAN PLANE

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