SUBCRITICAL POLARISATIONS OF SYMPLECTIC MANIFOLDS HAVE DEGREE ONE

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ABSTRACT. We show that if the complement of a Donaldson hypersurface in a closed, integral symplectic manifold has the homology of a subcritical Stein manifold, then the hypersurface is of degree one. In particular, this demonstrates a conjecture by Biran and Cieliebak on subcritical polarisations of symplectic manifolds. Our proof is based on a simple homological argument using ideas of Kulkarni–Wood.

1. DONALDSON HYPERSONSURFACES AND SYMPLECTIC POLARISATIONS

Let $(M, \omega)$ be a closed, connected, integral symplectic manifold, that is, the de Rham cohomology class $[\omega]_{\text{dR}}$ lies in the image of the homomorphism $H^2(M) \to H^2_{\text{dR}}(M) = H^2(M; \mathbb{R})$ induced by the inclusion $\mathbb{Z} \to \mathbb{R}$. The cohomology classes in $H^2(M)$ mapping to $[\omega]_{\text{dR}}$ are called integral lifts, and by abuse of notation we shall write $[\omega]$ for any such lift. Following McDuff and Salamon [10, Section 14.5], we call a codimension 2 symplectic submanifold $\Sigma \subset M$ a Donaldson hypersurface if it is Poincaré dual to $d[\omega] \in H^2(M)$ for some integral lift $[\omega]$ and some (necessarily positive) integer $d$. Donaldson [4] has established the existence of such hypersurfaces for any sufficiently large $d$.

The pair $(M, \Sigma)$ is called a polarisation of $(M, \omega)$, and the number $d \in \mathbb{N}$, the degree of the polarisation. Biran and Cieliebak [2] studied these polarisations in the Kähler case, where $\omega$ admits a compatible integrable almost complex structure $J$. In that setting, the complement $(M \setminus \Sigma, J)$ admits in a natural way the structure of a Stein manifold.

As shown recently by Giroux [7], building on work of Cieliebak–Eliashberg, even in the non-Kähler case the complement of a symplectic hypersurface $\Sigma \subset M$ found by Donaldson’s construction admits the structure of a Stein manifold. Here, of course, the complex structure on $M \setminus \Sigma$ does not, in general, extend over $\Sigma$. Complements of Donaldson hypersurfaces are also studied in [3].

2. SUBCRITICAL POLARISATIONS

The focus of Biran and Cieliebak [2] lay on subcritical polarisations of Kähler manifolds, which means that $(M \setminus \Sigma, J)$ admits a plurisubharmonic Morse function $\varphi$ all of whose critical points have, for $\dim M = 2n$, index less than $n$. (They also assumed that $\varphi$ coincides with the plurisubharmonic function defining the natural Stein structure outside a compact set containing all critical points of $\varphi$.)

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More generally, McDuff and Salamon [10, p. 504] propose the study of polarisations \((M, \Sigma)\) where the complement \(M \setminus \Sigma\) is homotopy equivalent to a subcritical Stein manifold (of finite type). We relax this condition a little further and call \((M, \Sigma)\) **homologically subcritical** if \(M \setminus \Sigma\) has the homology of a subcritical Stein manifold, that is, of a CW-complex containing finitely many cells up to dimension at most \(n - 1\). This means that there is some \(\ell \leq n - 1\) such that \(H_k(M \setminus \Sigma)\) vanishes for \(k \geq \ell + 1\) and \(H_{\ell}(M \setminus \Sigma)\) is torsion-free.

Motivated by the many examples they could construct, Biran and Cieliebak [2, p. 751] conjectured that subcritical polarisations necessarily have degree 1. They suggested an approach to this conjecture using either symplectic or contact homology. A rough sketch of a proof along these lines, in the language of symplectic field theory, was given by Eliashberg–Givental–Hofer [5, p. 661]. A missing assumption \(c_1(M \setminus \Sigma) = 0\) of that argument and a few more details — still short of a complete proof — were added by J. He [8, Proposition 4.2], who appeals to Gromov–Witten theory and polyfolds.

Here is our main result, which entails the conjecture of Biran–Cieliebak.

**Theorem 1.** Let \((M, \omega)\) be a closed, integral symplectic manifold, and \(\Sigma \subset M\) a compact symplectic submanifold of codimension 2, Poincaré dual to the integral cohomology class \(d[\omega]\) for some (positive) integer \(d\). If \((M, \Sigma)\) is homologically subcritical, then \(d[\omega]/\text{torsion}\) is indivisible in \(H^2(M)/\text{torsion}\). In particular, \(d = 1\).

Our proof is devoid of any sophisticated machinery. The assumption on \((M, \Sigma)\) to be homologically subcritical guarantees the surjectivity of a certain homomorphism in homology described by Kulkarni and Wood [9]; this implies the claimed indivisibility.

3. **The Kulkarni–Wood homomorphism**

We consider a pair \((M, \Sigma)\) consisting of a closed, connected, oriented manifold \(M\) of dimension \(2n\), and a compact, oriented hypersurface \(\Sigma \subset M\) of codimension 2. No symplectic assumptions are required in this section.

Write \(i: \Sigma \to M\) for the inclusion map. The Poincaré duality isomorphisms on \(M\) and \(\Sigma\) from cohomology to homology, given by capping with the fundamental class, are denoted by \(\text{PD}_M\) and \(\text{PD}_\Sigma\), respectively.

In their study of the topology of complex hypersurfaces, Kulkarni and Wood [9] used the following composition, which we call the **Kulkarni–Wood homomorphism**:

\[
\Phi_{KW}: H^k(M) \xrightarrow{i^*} H^k(\Sigma) \xrightarrow{\text{PD}_\Sigma} H_{2n-2-k}(\Sigma) \xrightarrow{i_*} H_{2n-2-k}(M) \xrightarrow{\text{PD}_M^{-1}} H^{k+2}(M).
\]

**Lemma 2.** The Kulkarni–Wood homomorphism equals the cup product with the cohomology class \(\sigma := \text{PD}_M^{-1}(i_*[\Sigma]) \in H^2(M)\).

**Proof.** For \(\alpha \in H^k(M)\) we compute

\[
\Phi_{KW}(\alpha) = \text{PD}_M^{-1}i_* \text{PD}_\Sigma i^* \alpha = \text{PD}_M^{-1}i_* (i^* \alpha \cap [\Sigma]) = \text{PD}_M^{-1}(\alpha \cap \text{PD}_M(\sigma)) = \text{PD}_M^{-1}(\alpha \cap (\sigma \cap [M])) = \text{PD}_M^{-1}(\alpha \cup \sigma) \cap [M].
\]

\(\Box\)
Lemma 3. If the complement $M \setminus \Sigma$ has the homology type of a CW-complex of dimension $\ell$ for some $\ell \leq n - 1$, then $\Phi_{KW}: H^k(M) \to H^{k+2}(M)$ is surjective in the range $\ell - 1 \leq k \leq 2n - \ell - 2$.

Proof. Write $\nu \Sigma$ for an open tubular neighbourhood of $\Sigma$ in $M$. By homotopy, excision, duality, and the universal coefficient theorem we have
\[
H_k(M, \Sigma) \cong H_k(M, \nu \Sigma) \cong H_k(M \setminus \nu \Sigma, \partial(\nu \Sigma)) \cong H^{2n-k}(M \setminus \Sigma) \oplus TH_{2n-k-1}(M \setminus \Sigma),
\]
where $F$ and $T$ denotes the free and the torsion part, respectively. This vanishes for $2n - k - 1 \geq \ell$, that is, for $k \leq 2n - \ell - 1$. It follows that the homomorphism $i_*: H_{2n-2-k}(\Sigma) \to H_{2n-2-k}(M)$ is surjective for $2n - 2 - k \leq 2n - \ell - 1$, or $k \geq \ell - 1$.

Similarly (or directly by Poincaré–Lefschetz duality) we have
\[
H^k(M, \Sigma) \cong H_{2n-k}(M \setminus \Sigma),
\]
which vanishes for $2n - k - 1 \geq \ell$, that is, for $k \leq 2n - \ell - 1$. Hence, the homomorphism $i^*: H^k(M) \to H^k(\Sigma)$ is surjective for $k + 1 \leq 2n - \ell - 1$, that is, for $k \leq 2n - \ell - 2$. \qed

4. Proof of Theorem [1]

Under the assumptions of Theorem [1] the homomorphism $\Phi_{KW}: H^k(M) \to H^{k+2}(M)$ is surjective at least in the range $n - 2 \leq k \leq n - 1$; simply set $\ell = n - 1$ in Lemma [3]. Thus, we can pick an even number $k = 2m$ in this range. The free part of $H^{2m+2}(M)$ is non-trivial, since this cohomology group contains the element $[\omega]^{m+1}$ of infinite order.

On the other hand, $\Phi_{KW}$ is given by the cup product with $d[\omega]$, as shown in Lemma [2]. If $d[\omega]/\text{torsion}$ were divisible, so would be all elements in the image of $\Phi_{KW}$ in $H^{2m+2}(M)/\text{torsion}$, and $\Phi_{KW}$ would not be surjective.

Remark 4. The real Euler class of the circle bundle $\partial(\nu \Sigma)$ equals $d[\omega]_{\text{dR}}$, and the natural Boothby–Wang contact structure on this bundle has an exact convex filling by the complement $M \setminus \nu \Sigma$, see [6] Lemma 3], [7] Proposition 5] and [8] Lemma 2.2]. With [1] Theorem 1.2] the condition ‘homologically subcritical’ of Theorem [1] may be replaced by assuming the existence of some subcritical Stein filling of this Boothby–Wang contact structure.

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