QUANTUM SL(3, C)'S: THE MISSING CASE

CHRISTIAN OHN

Abstract. We study the only missing case in the classification of quantum SL(3, C)'s undertaken in our paper [J. of Algebra 213 (1999), 721–756], thereby completing this classification.

Introduction

The aim of this paper is to complete the classification of quantum SL(3)'s undertaken in [8].

Roughly speaking, we call a quantum SL(3) any Hopf algebra (over C) whose (finite-dimensional) comodules are “similar” to the modules of the (ordinary) group SL(3, C) (see Definition 1.1). Given such a Hopf algebra A, it will have, among other things, two simple (nonisomorphic) comodules V, W of dimension 3, and the usual decomposition rules for tensor products will imply the existence of A-comodule morphisms (1.1), satisfying certain compatibility conditions (1.2). The tuple LA, consisting of those two comodules and eight morphisms, will be called the basic quantum SL(3) datum (BQD for short) associated to A.

Conversely, starting from an “abstract” BQD L (i.e. where V, W are just vector spaces and (1.1) just linear maps satisfying the aforementioned conditions), we may reconstruct a Hopf algebra AL via the usual Tannakian procedure.

We may now state our main result.

Theorem A.

(a) If A is a quantum SL(3), then LA is a BQD.
(b) If L is a BQD, then AL is a quantum SL(3).
(c) The correspondences A ↦ LA and L ↦ AL are inverse of each other between quantum SL(3)'s (up to Hopf algebra isomorphism) and BQDs (up to equivalence; see Definition 1.2).
(d) BQDs can be explicitly classified up to equivalence, yielding a classification of quantum SL(3)'s up to Hopf algebra isomorphism.

An almost complete version of this theorem was stated and proved in [8]. More precisely, we found a classification of all BQDs, except for one class related to elliptic curves, and we proved the theorem for all BQDs outside this class. In the present paper, we settle the study of this last class, thereby proving Theorem A in full.

Moreover, a geometric analysis of this class of BQDs yields the following contribution to a question raised in the Introduction of [2].
Theorem B. Let \( B \) be the quantum three-space associated to an arbitrary quantum \( SL(3) \). Then \( B \) cannot be a Sklyanin algebra. In other words, the scheme of point modules \([3,4]\) of \( B \) (which is a cubic divisor in \( \mathbb{P}^2 \)) cannot be an elliptic curve.

The paper is organized as follows. After some recollections from [8] in Section 1 (definition of a quantum \( SL(3) \) and of a BQD, and the correspondences between them), Section 2 recalls and studies the form of the only class of BQDs not covered by the results of [8]; this will finish the classification of BQDs. In Section 3, we introduce the shape algebra [8, Section 5] for this class of BQDs and we determine the associated flag variety (in the sense of [9]). These geometric data suggest to view the shape algebra as a twist (in the sense of [10]) of another algebra, which we show in Section 4 to be isomorphic to the shape algebra of another BQD, already covered by the results of [8]. In Section 5, we finish the proof of Theorem A by carrying over the necessary properties from the untwisted to the twisted shape algebra. Theorem B is proved in Section 6 by picking up some leftovers from Section 2.

In the Appendix, we show a result on twists of Koszul algebras that is needed in Section 5, but may also be of independent interest.

Conventions. We denote by \( \mathbb{Z} \) (resp. \( \mathbb{N}, \mathbb{C} \)) the set of integers (resp. nonnegative integers, complex numbers). All vector spaces, algebras and tensor products are over \( \mathbb{C} \).

1. Recollections from [8]

Recall that the group \( SL(3) \) is linearly reductive and that its simple modules are parametrized by their highest weights, which are pairs \((k,l)\in\mathbb{N}^2\). Recall further that the dimension of the simple module of highest weight \((k,l)\) is given by \( d_{(k,l)} := (k+1)(l+1)(k+l+2)/2 \). For any \( \lambda, \mu, \nu \in \mathbb{N}^2 \), denote by \( m^\nu_{\lambda\mu} \) the multiplicity of the simple module of highest weight \( \nu \) inside the tensor product of those of (respective) highest weights \( \lambda \) and \( \mu \). (Recall also that \( m^\nu_{\lambda\mu} \) can, in principle, be determined in a purely combinatorial way.)

Our main objects of interest may now be defined as follows (see [8]).

Definition 1.1. We call a quantum \( SL(3) \) any (not necessarily commutative) Hopf algebra \( A \) (over \( \mathbb{C} \)) such that

(a) there is a family \( \{ V_{\lambda} \mid \lambda \in \mathbb{N}^2 \} \) of simple and pairwise nonisomorphic \( A \)-comodules, with \( \dim V_{\lambda} = d_{\lambda} \),

(b) every \( A \)-comodule is isomorphic to a direct sum of these,

(c) for every \( \lambda, \mu \in \mathbb{N}^2 \), \( V_{\lambda} \otimes V_{\mu} \) is isomorphic to \( \bigoplus_{\nu} m^\nu_{\lambda\mu} V_{\nu} \).

In particular, if we write \( V := V_{(1,0)} \) and \( W := V_{(0,1)} \), then Condition (c) implies the existence of \( A \)-comodule morphisms

\[
A : V \otimes V \to W, \quad a : W \to V \otimes V \\
B : W \otimes W \to V, \quad b : V \to W \otimes W \\
C : W \otimes V \to C, \quad c : C \to V \otimes W \\
D : V \otimes W \to C, \quad d : C \to W \otimes V,
\]

(1.1)
each being unique up to a scalar. We showed in [8, Propositions 3.1 and 3.2] that these maps must, for an appropriate choice of these scalars, satisfy the following.
compatibility conditions:

\[(1.2a) \quad (1_V \otimes C)(c \otimes 1_V) = 1_V, \quad (D \otimes 1_V)(1_V \otimes d) = 1_V\]

\[(1.2b) \quad Aa = 1_V,\]

\[(1.2c) \quad C(A \otimes 1_V) = \omega D(1_V \otimes A), \quad (1_V \otimes a)c = \omega (a \otimes 1_V)d\]

\[(1.2d) \quad \omega(C \otimes 1_V)(1_W \otimes a) = B, \quad (A \otimes 1_W)(1_V \otimes c) = b\]

\[(1.2e) \quad De = \kappa 1_C, \quad Cd = \kappa 1_C\]

\[(1.2f) \quad (1_V \otimes A)(a \otimes 1_V)(A \otimes 1_V)(1_V \otimes a) = \rho(1_{V \otimes W} + cD)\]

\[(1.2g) \quad (A \otimes 1_V)(1_V \otimes a)(1_V \otimes A)(a_V \otimes 1) = \rho(1_{W \otimes V} + dC),\]

where

- \(\omega\) is a 3rd root of unity,
- \(\kappa = q^{-2} + 1 + q^2\) and \(\rho = (q + q^{-1})^{-2}\) for some \(q \in \mathbb{C}\), \(q \neq 0\), with \(q^2\) either 1 or not a root of unity.

**Definition 1.2.** A basic quantum SL(3) datum (BQD for short) is a tuple \(\mathcal{L} = (V, W; A, a, B, b, C, c, D, d)\) consisting of two vector spaces \(V\) and \(W\) of dimension 3, and of eight linear maps (1.1), satisfying Conditions (1.2) (with \(\omega, \kappa, q\) as indicated). Two BQDs are called equivalent if one can be obtained from the other through (any combination of) base change, rescaling of the maps (1.1), and interchanging \(V \leftrightarrow W\), \(A \leftrightarrow B\), \(a \leftrightarrow b\), \(C \leftrightarrow D\), \(c \leftrightarrow d\).

Thus, each quantum SL(3), \(A\), gives rise to a BQD \(\mathcal{L}_A\) (which is really defined only up to equivalence).

Conversely, start with a BQD \(\mathcal{L}\), and define an algebra \(\mathcal{A}_L\) with \((9+9)\) generators \(t_j^i (i, j = 1, 2, 3)\) and \(u_{ij}^\alpha (\alpha, \beta = 1, 2, 3)\), and relations

\[
A_{ij}^{\alpha} t_k^i t_l^j = u_{ij}^\alpha A_{kl}^{\beta}, \quad t_k^i t_l^j a_{\beta}^{ij} = a_{ij}^{ij} u_{ij}^\alpha
\]

\[
B_{\alpha \beta}^{ij} u_{ij}^\alpha u_{ij}^\beta = t_j^i B_{\gamma \delta}^{ij}, \quad u_{ij}^\alpha u_{ij}^\beta b_{ij}^{\gamma \delta} = b_{ij}^{\gamma \delta} t_j^i
\]

\[
C_{\alpha i} u_{ij}^\alpha t_j^i = C_{\beta j}, \quad t_j^i u_{ij}^\alpha C_{\beta j} = c_{\alpha i}
\]

\[
D_{\alpha i} t_j^i u_{ij}^\alpha = D_{\beta j}, \quad u_{ij}^\alpha t_j^i d_{ij}^{\beta j} = d_{ij}^{\beta j}
\]

(Here, we have chosen bases \(x_1, x_2, x_3\) of \(V\) and \(y_1, y_2, y_3\) of \(W\), and set \(A(x_i \otimes x_j) = A_{ij}^{\alpha} y_\alpha\), etc.) Then \(\mathcal{A}_L\) possesses a Hopf algebra structure given by

\[
\Delta(t_j^i) = t_k^i \otimes t_j^k, \quad \Delta(u_{ij}^\alpha) = u_j^\alpha \otimes u_j^\alpha
\]

\[
\varepsilon(t_j^i) = \delta_j^i, \quad \varepsilon(u_{ij}^\alpha) = \delta_j^\alpha
\]

\[
S(t_j^i) = c_{ij}^{\beta} u_j^\beta C_{\alpha j}, \quad S(u_{ij}^\alpha) = d_{ij}^{\beta j} t_j^i D_{\beta j}
\]

(1.3)

The main difficulty is to prove that the Hopf algebra \(\mathcal{A}_L\) is indeed a quantum SL(3) in the sense of Definition 1.1. We were able to do this in [8] for all BQDs, except for one class, which we will describe in the next section.

### 2. Classification of Case I.H

If we set \(Q_j^i := c_{ij}^{\alpha} D_{\alpha j}\), then (1.2a) implies \((Q^{-1})_j^i = d_{ij}^{\alpha} C_{\alpha j}\). Now it follows from (1.3) that

\[
S^2(t_j^i) = Q_j^i t_j^i (Q^{-1})_j^i
\]

so \(S^2\) is “encoded” by the linear map \(Q : V \rightarrow V\).
In [8, Section 10], we used the possible Jordan normal forms of $Q$ and the value of $\omega$ as first criteria for the classification of BQDs. One possibility, called Type I in [8], consists in taking $Q = 1_V$ and $\omega = 1$. The condition $Q = 1_V$ amounts to setting $W = V^*$, with $C, c, D, d$ the obvious canonical maps. (In particular, we now have $S^2 = 1_A$, reflecting the fact that $W$ is both the left dual and the right dual of $V$.) In this case, (1.2a) and (1.2e) are automatically satisfied (for $k = 3$, so $q^2 = 1$ and $\rho = \frac{1}{4}$).

Now consider the “quantum determinants,” appearing in (1.2c):

$$ e := (1_V \otimes a)c = \omega(a \otimes 1_V)d : C \to V \otimes V \otimes V $$

$$ E := C(A \otimes 1_V) = \omega D(1_V \otimes A) : V \otimes V \otimes V \to C. $$

If $L$ is of Type I, then Conditions (1.2c) imply that

$$ e = \lambda + s, \quad E = \Lambda + S, $$

with $\lambda, \Lambda$ totally antisymmetric and $s, S$ totally symmetric. In particular, choosing dual bases $x_i$ in $V$ and $y_a$ in $W (= V^*)$, we may view $s$ and $S$ as homogeneous polynomials of degree 3, i.e. as cubic curves in the projective plane $\mathbb{P}^2$ and in its dual plane $\mathbb{P}^2^*$, respectively (unless $s = 0$ or $S = 0$).

Using the standard classification of cubic curves in $\mathbb{P}^2$, we were then able in [8] to classify all possible forms of $e$ and $E$ that satisfy Conditions (1.2bf)g, except in the case where $s$ is an elliptic curve.

In this particular case, called Case I. in [8], the bases of $V$ and $W$ may be chosen in such a way that $a$ reads

$$ a : y_1 \mapsto \alpha x_2 \otimes x_3 + \beta x_3 \otimes x_2 + \gamma x_1 \otimes x_1 $$

$$ y_2 \mapsto \alpha x_3 \otimes x_1 + \beta x_1 \otimes x_3 + \gamma x_2 \otimes x_2 $$

$$ y_3 \mapsto \alpha x_1 \otimes x_2 + \beta x_2 \otimes x_1 + \gamma x_3 \otimes x_3, $$

with $\gamma \neq 0$ and $(\alpha + \beta)^3 + \gamma^3 \neq 0$ (so that $s$ is indeed elliptic). It then follows from (1.2b) that $A$ must read

$$ A : \begin{align*}
  x_1 \otimes x_1 &\mapsto \gamma' y_1, \\
  x_2 \otimes x_1 &\mapsto \beta' y_3, \\
  x_3 \otimes x_1 &\mapsto \alpha' y_2,
\end{align*} $$

$$ \begin{align*}
  x_1 \otimes x_2 &\mapsto \alpha' y_3, \\
  x_2 \otimes x_2 &\mapsto \gamma' y_2, \\
  x_3 \otimes x_2 &\mapsto \beta' y_1, \\
  x_3 \otimes x_3 &\mapsto \gamma' y_3,
\end{align*} $$

with

$$ (2.1) \quad \alpha \alpha' + \beta \beta' + \gamma \gamma' = 1. $$

Moreover, Conditions (1.2fg) now read

$$ P_0 := \alpha^2 \alpha'^2 + \beta^2 \beta'^2 + \gamma^2 \gamma'^2 - 2 \alpha \alpha' \beta \beta' - 2 \alpha \alpha' \gamma \gamma' - 2 \beta \beta' \gamma \gamma' = 0 $$

(2.2)

$$ P_1 := \alpha^2 \beta' \gamma' + \beta^2 \alpha' \gamma' + \gamma^2 \alpha' \beta' = 0 $$

$$ P_2 := \alpha^2 \beta \gamma + \beta^2 \alpha \gamma + \gamma^2 \alpha \beta = 0. $$

**Proposition 2.1.** Up to base change, we may assume that $\alpha = \alpha' = 0$.

**Proof.** Case 1 : $\alpha = 0$ or $\alpha' = 0$. Substituting into $P_2$ or $P_1$ shows that $\alpha = \alpha' = 0$.

Case 2 : $\beta = 0$ or $\beta' = 0$. Permuting two of the three basis vectors takes us back to Case 1.
Case 3: \( \gamma' = 0 \) (recall that \( \gamma = 0 \) has been ruled out by ellipticity of \( s \)). Substituting into \( P_1 \) shows that \( \alpha' = 0 \) or \( \beta' = 0 \), which takes us back to Case 1 or Case 2.

Case 4: none of \( \alpha, \alpha', \beta, \beta', \gamma, \gamma' \) equals zero. Viewing \( P_0, P_1, P_2 \) as polynomials in \( \alpha' \), their resultants must vanish:

\[
0 = \text{Res}_{\alpha'}(P_0, P_1) = \begin{vmatrix}
\alpha^2 & -2\alpha(\beta' + \gamma') & (\beta' - \gamma')^2 \\
\beta^2 \gamma' + \gamma^2 \beta' & \alpha^2 \beta' \gamma' & 0 \\
0 & \beta^2 \gamma' + \gamma^2 \beta' & \alpha^2 \beta' \gamma'
\end{vmatrix}
\]

\[
= \beta^2 \gamma^4 \beta^4 + 2(\alpha^3 + \beta^3 - \gamma^3)\beta^2 \beta^3 \gamma' + (\alpha^6 + \beta^6 + \gamma^6 + 2\alpha^3 \beta^3 + 2\alpha^3 \gamma^3 - 4\beta^3 \gamma^3)\beta^2 \gamma'^2 + 2(\alpha^3 + \gamma^3 - \beta^3)\beta^2 \beta' \gamma'^2 + \beta^4 \gamma' \gamma'^4
\]

\[=:Q_1\]

\[
0 = \text{Res}_{\alpha'}(P_1, P_2) = \begin{vmatrix}
\beta^2 \gamma' + \gamma^2 \beta' & \alpha^2 \beta' \gamma' & 0 \\
0 & \beta^2 \gamma' + \gamma^2 \beta' & \alpha^2 \beta' \gamma' \\
0 & \alpha(\beta^2 \gamma + \gamma^2 \beta)
\end{vmatrix}
\]

\[
= \alpha \gamma^5 \beta^4 + 2\alpha \beta^2 \gamma^3 \beta^3 \gamma' + (\alpha^3 + \beta^3 + \gamma^3)\alpha \beta \gamma \beta^2 \gamma'^2 + 2\alpha \beta^3 \gamma^2 \beta' \gamma'^2 + \alpha \beta^5 \gamma' \gamma'^4
\]

\[=:Q_2.\]

(These resultants make sense, because the leading coefficients of \( P_0, P_1, P_2 \) are nonzero: in particular, if we had \( \beta^2 \gamma' + \gamma^2 \beta' = 0 \), then substituting into \( P_1 \) would imply \( \alpha = 0, \beta' = 0, \) or \( \gamma' = 0 \).)

Viewing \( Q_1, Q_2 \) as polynomials in \( \beta' \), their resultant must again vanish:

\[
0 = \text{Res}_{\beta'}(Q_1, Q_2) = (a 6 \times 6 \text{ determinant})
\]

\[(2.3) = \gamma^{10}(\alpha \beta \gamma)^{10} \left[ (\alpha^3 + \beta^3 + \gamma^3)^3 - (3\alpha \beta \gamma)^3 \right].\]

(The author confesses not to have computed this \( 6 \times 6 \) determinant by hand!)

Therefore, \( \alpha^3 + \beta^3 + \gamma^3 = 3\zeta \alpha \beta \gamma \) for some 3rd root of unity \( \zeta \), i.e.

\[
(\zeta \gamma + \alpha + \beta)(\zeta \gamma + j \alpha + j^2 \beta)(\zeta \gamma + j^2 \alpha + j \beta) = 0,
\]

where \( j \) is a primitive 3rd root of unity. But \( \zeta \gamma + \alpha + \beta = 0 \) is ruled out by ellipticity of the curve \( s \), so we may assume that \( \zeta \gamma + j \alpha + j^2 \beta = 0 \) (exchanging \( j \leftrightarrow j^2 \) if necessary).

Now consider the following change of basis in \( V \):

\[
\begin{cases}
x'_1 = \zeta x_1 + x_2 + x_3 \\
x'_2 = \zeta x_1 + j x_2 + j^2 x_3 \\
x'_3 = \zeta x_1 + j^2 x_2 + j x_3
\end{cases}
\]

(together with the dual basis \( y'_1, y'_2, y'_3 \) in \( W \)). Then the map \( a \) reads

\[
(3\zeta) a : y'_1 \mapsto (\zeta \gamma + j^2 \alpha + j \beta) x'_3 \otimes x'_2 + (\zeta \gamma + \alpha + \beta) x'_1 \otimes x'_1
\]

\[
y'_2 \mapsto (\zeta \gamma + j^2 \alpha + j \beta) x'_1 \otimes x'_3 + (\zeta \gamma + \alpha + \beta) x'_2 \otimes x'_2
\]

\[
y'_3 \mapsto (\zeta \gamma + j^2 \alpha + j \beta) x'_2 \otimes x'_1 + (\zeta \gamma + \alpha + \beta) x'_3 \otimes x'_3,
\]

so we are taken back to Case 1. \( \square \)
Now we still have one degree of freedom to rescale the maps $a$ and $A$, so we are left with one essential parameter, say $\gamma$. Now let us determine the flag variety of $L$ as defined in [9]. To do this, modify Relations (3.1) as follows: in each relation, adorn the right factor of each term with $'$ if the left factor is an $x_i$ and with $''$ if the left factor is a $y_\alpha$ (e.g. the first relation on the fourth line now reads $y_1 x_1 = x_2 y_2$, $y_2 x_1 = t x_3 y_3$, $t y_3 x_1 = x_2 y_1$).

View $(x_1 : x_2 : x_3)$ and $(y_1 : y_2 : y_3)$ as homogeneous coordinates in $\mathbb{P}^2$ and in $\mathbb{P}^2*$, respectively. Since Relations (3.1) are $\mathbb{N}^2$-homogeneous, their modified version may now be seen as defining equations for a subscheme

$$\Gamma \subset (\mathbb{P}^2 \times \mathbb{P}^2*) \times (\mathbb{P}^2 \times \mathbb{P}^2*) \times (\mathbb{P}^2 \times \mathbb{P}^2*).$$

One easily checks that $\Gamma$ is of the form

$$\Gamma = \{(p, \sigma_1(p), \sigma_2(p)) \mid p \in X\}$$

for some subscheme $X \subset \mathbb{P}^2 \times \mathbb{P}^2*$ and some automorphisms $\sigma_1, \sigma_2$ of $X$: indeed, the scheme $X$ is the variety with nine irreducible components pictured in Figure 1, and the automorphisms $\sigma_1, \sigma_2$ naturally decompose into $\sigma_1 = \tau_1 \sigma_1^*, \sigma_2 = \tau_2 \sigma_2^*$, where $\tau_1, \tau_2$ are the automorphisms of $\mathbb{P}^2 \times \mathbb{P}^2*$ given by

$$\tau_1 : \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \mapsto \begin{pmatrix} x_3 & x_1 & x_2 \\ y_3 & y_1 & y_2 \end{pmatrix}, \quad \tau_2 : \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \mapsto \begin{pmatrix} x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \end{pmatrix}.$$
and where $\sigma_1^0, \sigma_2^0$ are also described in Figure 1, e.g. $\sigma_1^0 : \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\sigma_2^0 : \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

**Informal remark 3.1.** Since $\mathcal{M}_\mathcal{L}$ is a quantum analogue of the multihomogeneous coordinate ring of the (ordinary) flag variety of $\text{SL}(3)$, nontrivial characters of $\mathcal{M}_\mathcal{L}$ should, in some sense, correspond to “quantum Borel subgroups” of $\mathcal{A}_\mathcal{L}$. On the other hand, such characters obviously correspond to simultaneous fixed points of $\sigma_1, \sigma_2$. But in the case considered here, there is no such fixed point: so once Theorem A will be proved, we will have a quantum group with the same representations as $\text{SL}(3)$, although in some sense, it has no quantum Borel subgroup to induce them from!

4. **Twisting the shape algebra**

The natural decompositions $\sigma_1 = \tau_1 \sigma_1 \tau_1^{-1}$ suggest to consider $\mathcal{M}_{\mathcal{L}}$ as the twist $R^\tau$ (in the sense of [10]) of another $\mathbb{N}^2$-graded algebra $R$, whose associated flag variety will be $(X, \sigma_1^0, \sigma_2^0)$ instead of $(X, \sigma_1, \sigma_2)$ (this works by an obvious multigraded version.
of [4, Proposition 8.9], noting that the $\tau_1, \tau_2$ commute with each other and with $\sigma_1^\#, \sigma_2^\#$. The algebra $R$ will also be generated by $x_i$ ($i = 1, 2, 3$) and $y_\alpha$ ($\alpha = 1, 2, 3$), with defining relations obtained by twisting (3.1) backwards:

\[
\begin{align*}
  x_3x_1 &= tx_1x_3, \\
  x_1x_2 &= tx_2x_1, \\
  x_2x_3 &= tx_3x_2, \\
  y_1x_2 &= x_2y_1, \\
  y_2x_2 &= tx_2y_2, \\
  y_3x_2 &= x_2y_3 \\
  y_1x_1 &= tx_1y_1, \\
  y_2x_1 &= x_1y_2, \\
  y_3x_1 &= x_1y_3 \\
  x_1y_3 + x_2y_1 + x_3y_2 &= 0.
\end{align*}
\]

(4.1)

A straightforward computation shows that relations (4.1), with $y_1, y_2, y_3$ renamed to $y_2, y_3, y_1$, turn out to define the shape algebra $M_{L^\circ}$, where $L^\circ$ is the BQD of Type I defined by the maps

\[
\begin{align*}
  a : y_1 &\mapsto \lambda x_2 \otimes x_3 + \mu x_3 \otimes x_2 \\
  y_2 &\mapsto \lambda x_3 \otimes x_1 + \mu x_1 \otimes x_3 \\
  y_3 &\mapsto \lambda x_1 \otimes x_2 + \mu x_2 \otimes x_1 \\
  A : x_1 \otimes x_1 &\mapsto 0, \\
  x_1 \otimes x_2 &\mapsto \lambda' y_3, \\
  x_1 \otimes x_3 &\mapsto \mu' y_2 \\
  x_2 \otimes x_1 &\mapsto \mu' y_3, \\
  x_2 \otimes x_2 &\mapsto 0, \\
  x_2 \otimes x_3 &\mapsto \lambda' y_1 \\
  x_3 \otimes x_1 &\mapsto \lambda' y_2, \\
  x_3 \otimes x_2 &\mapsto \mu' y_1, \\
  x_3 \otimes x_3 &\mapsto 0
\end{align*}
\]

(Case I.e in [8, Section 10]), with $\lambda\lambda' = \mu\mu' = \frac{1}{2}$ and $t = -\frac{4}{\lambda} = -\frac{\lambda'}{\mu'}$.

5. Proof of Theorem A

In [8], all intermediate results in the (almost complete) proof of Theorem A are valid for an arbitrary BQD $L$, except for [8, Proposition 5.3] (and the resulting [8, Corollary 5.4]), which relies on a case by case analysis that is not valid in Case I.h. More precisely, the proof of Theorem A will be complete if we show the following facts for this particular case:

(a) the shape algebra $M_L$ is a Koszul algebra (when viewed as an $\mathbb{N}$-graded algebra via the total grading),

(b) $\dim V_{(k,l)} = d_{(k,l)}$ for all $(k,l) \in \mathbb{N}^2$.

Since the BQD $L^\circ$ introduced in Section 4 is already covered by [8], Properties (a) and (b) are true for $M_{L^\circ}$.

Therefore, $M_L$, being a twist of $M_{L^\circ}$, also satisfies those two properties: for Property (b), this is automatic, and for Property (a), it follows from Proposition A.1 in the Appendix. Theorem A follows.

6. Proof of Theorem B

To each BQD $L$ are associated two “quantum three-spaces,” namely the quadratic algebras $B_L := T(V)/(\text{Im}a)$ and $C_L := T(W)/(\text{Im}b)$. For Case I.h, $B_L$ is
defined by the following relations:

\[ \alpha x_2 x_3 + \beta x_3 x_2 + \gamma x_1^2 = 0 \]
\[ \alpha x_3 x_1 + \beta x_1 x_3 + \gamma x_2^2 = 0 \]
\[ \alpha x_1 x_2 + \beta x_2 x_1 + \gamma x_3^2 = 0, \]

(The defining relations of \( C_L \) are similar.) This algebra is one of the regular algebras of dimension 3 studied in [1], where the matrix \( Q \) (see Section 2) also played a role in the classification of such algebras. Algebras for which \( Q = 1 \) (called of Type A in [1]) give rise to the cubic curve \( s \) in \( \mathbb{P}^2 \) as in Section 2, so let us call this curve the AS-curve. The AS-curve associated to \( B_L \) is given by

\[ \gamma (x_3^3 + x_2^3 + x_1^3) = 3(\alpha + \beta)x_1 x_2 x_3. \]

Another cubic curve in \( \mathbb{P}^2 \) has been associated to \( B_L \) in [3], namely the scheme of its point modules, given by

\[ (\alpha \beta \gamma)(x_3^3 + x_2^3 + x_1^3) = (\alpha^3 + \beta^3 + \gamma^3)x_1 x_2 x_3. \]

Call this curve the ATV-curve associated to \( B_L \).

**Remark 6.1.** The ATV-curve is defined for an arbitrary regular algebra of dimension 3, whereas the AS-curve is defined only for those of Type A. Note however, as observed in [3], that even when both curves are defined, they do not coincide in general!

Recall [3, 7] that \( B_L \) is called a Sklyanin algebra if its ATV-curve is elliptic. But by Proposition 2.1, this is impossible: the ATV-curve of \( B_L \) must degenerate to a triangle (which may also be viewed as the image of the flag variety \( X \) under the projection \( \mathbb{P}^2 \times \mathbb{P}^2^* \to \mathbb{P}^2 \)).

On the other hand, a case by case analysis shows that none of the other BQDs, classified in [8, Section 10], can give rise to an elliptic ATV-curve. This proves Theorem B.

**Remark 6.2.** The fact that, for the BQD \( L \) of Case I.h, the ATV-curve of \( B_L \) cannot be elliptic was already visible in (2.3), which was obtained via an elimination procedure from Conditions (1.2fg). In turn, the latter were shown in [8, Section 3] to be related to the existence of an endomorphism of \( V \otimes V \) satisfying the braid relation.

One would of course like to see a more direct and natural link, for an arbitrary BQD \( L \), between the braid relation and the fact that the ATV-curve of \( B_L \) cannot be elliptic.

**Appendix: Multitwists preserve the Koszul property**

Let \( \Gamma \) be a monoid and \( A = \bigoplus_{\gamma \in \Gamma} A_{\gamma} \), a \( \Gamma \)-graded algebra.

Assume that to each \( \gamma \in \Gamma \), we associate a graded automorphism \( \tau_\gamma \) of \( A \), in such a way that \( \tau_{\gamma' \gamma} = \tau_{\gamma'} \tau_\gamma \), for all \( \gamma, \gamma' \in \Gamma \). (This is really only a special case of the notion introduced in [10], but it will be sufficient for our purposes: in Section 4, we take \( \tau_{(k,l)} := \tau_k \tau_l \) for each \( (k,l) \in \mathbb{N}^2 \).)

Recall [10] that the twisted algebra \( A^\tau \) is defined to be the vector space \( \bigoplus_{\gamma \in \Gamma} A_{\gamma} \), endowed with the following new multiplication:

\[ x \ast y := x \tau_\gamma(y) \quad \text{for all } x \in A_{\gamma}, y \in A. \]
Proposition A.1. Consider $A$ and $A^\tau$ as $\mathbb{N}$-graded algebras via some morphism $h : \Gamma \to \mathbb{N}$, and assume that they are generated by $A_1(= A^\tau_1) = \bigoplus_{h(\gamma) = 1} A_\gamma$. Then $A^\tau$ is Koszul if and only if $A$ is Koszul.

Remark A.2. If $\Gamma = \mathbb{N}$, then the categories of $\mathbb{N}$-graded modules of $A$ and of $A^\tau$ are equivalent thanks to [10, Theorem 3.1], and the Koszul property is a homological property in this category, so the result is immediate.

However, for arbitrary $\Gamma$, [10, Theorem 3.1] shows that the categories of $\Gamma$-graded modules are equivalent, but this does not imply that those of $\mathbb{N}$-graded modules are. Therefore, a proof is still needed.

Proof of Proposition A.1. Of course, since $A = (A^\tau)^{\tau^{-1}}$, we only need to show one way, so assume that $A$ is Koszul. Then $A$ is quadratic, say $A = T(A_1)/(R)$ with $R \subset A_1 \otimes A_1$. Moreover, for each $k \geq 2$, the sublattice (w.r.t. $\cap$ and $+$) of $A_1 \otimes (i-1) \otimes R \otimes A_1 \otimes (k-i-1)$ is distributive thanks to Backelin’s criterion [5] (see also [6, Lemma 4.5.1]).

Now define $v : A_1 \otimes A_1 \to A_1 \otimes A_1$ as follows: view $A_1 \otimes A_1$ as the direct sum of all subspaces $A_{\gamma_1} \otimes \cdots \otimes A_{\gamma_k}$ with $h(\gamma_1) = \cdots = h(\gamma_k) = 1$, and set

$$v|_{A_{\gamma_1} \otimes \cdots \otimes A_{\gamma_k}} := 1_{A_{\gamma_1}} \otimes \tau_{\gamma_1} \otimes (\tau_{\gamma_2} \tau_{\gamma_1}) \otimes \cdots \otimes (\tau_{\gamma_k-1} \cdots \tau_{\gamma_1}).$$

By construction, $A^\tau = T(A_1)/(v^{-1}(R))$, so $A^\tau$ is again quadratic. Moreover, the sublattice of $A_1 \otimes A_1$ generated by $v^{-1}(R)$ is the image under $v^{-1}$ of that generated by $R$, so it is still distributive. Applying Backelin’s criterion in the reverse direction, we conclude that $A^\tau$ is Koszul. □

References

[1] M. Artin and W. F. Schelter, Graded algebras of global dimension 3, Adv. Math. 66 (1987), 171–216
[2] M. Artin, W. F. Schelter, and J. Tate, Quantum deformations of $GL_n$, Commun. Pure Appl. Math. 44 (1991), 879–895
[3] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, in “The Grothendieck Festschrift,” vol. I, Birkhäuser, Basel, 1990
[4] M. Artin, J. Tate, and M. Van den Bergh, Modules over regular algebras of dimension 3, Invent. Math. 106 (1991), 335–388
[5] J. Backelin, A distributiveness property of augmented algebras and some related homological results, Ph. D. Thesis, Stockholm, 1982
[6] A. A. Beilinson, V. A. Ginsburg, and V. V. Schechtman, Koszul duality, J. Geom. Phys. 5 (1988), 317–350
[7] A. V. Odesskii and B. L. Feigin, Sklyanin’s elliptic algebras, Funct. Anal. Phys. 23 (1999), 207–214
[8] Ch. Ohn, Quantum $SL(3, \mathbb{C})$’s with classical representation theory, J. of Algebra 213 (1999), 721–756
[9] Ch. Ohn, “Classical” flag varieties for quantum groups: the standard quantum $SL(n, \mathbb{C})$, Adv. in Math. (to appear)
[10] J. J. Zhang, Twisted graded algebras and equivalences of graded categories, Proc. London Math. Soc. 72 (1996), 281–311