RATIONALLY SMOOTH SCHUBERT VARIETIES AND INVERSION HYPERPLANE ARRANGEMENTS

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ABSTRACT. We show that an element \( w \) of a finite Weyl group \( W \) is rationally smooth if and only if the hyperplane arrangement \( I \) associated to the inversion set of \( w \) is inductively free, and the product \((d_1+1) \cdots (d_l+1)\) of the coexponents \( d_1, \ldots, d_l \) is equal to the size of the Bruhat interval \([e, w]\) where \( e \) is the identity in \( W \). As part of the proof, we describe exactly when a rationally smooth element in a finite Weyl group has a chain Billey-Postnikov decomposition. For finite Coxeter groups, we show that chain Billey-Postnikov decompositions are connected with certain modular coatoms of \( I \).

1. INTRODUCTION

Let \( W \) be a finite Weyl group with length function \( \ell \). The Poincare polynomial of an element \( w \in W \) is \( P_w(q) = \sum_{x \in [e, w]} q^{\ell(x)} \), where \([e, w]\) is the interval between the identity \( e \) and \( w \) in Bruhat order. If \( X(w) \) is a Schubert variety indexed by \( w \), then \( P_w(q^2) = \sum q^i \dim H^i(X(w)) \). A theorem of Carrell-Peterson states that \( X(w) \) is rationally smooth if and only if \( P_w(q) \) is palindromic, meaning that \( q^{\ell(w)} P_w(q^{-1}) = P_w(q) \) (note that \( \ell(w) \) is the degree of \( P_w(q) \)). We say that \( w \in W \) is rationally smooth if this condition is satisfied.

Let \( w_0 \) be the maximal element of \( W \). It is well-known that the Poincare polynomial \( P_{w_0}(q) = \prod_i [m_i + 1]_q \), where \([m_i]_q \) is the \( q \)-integer \((1 + q + q^2 + \ldots + q^{m_i-1}) \) and \( m_1, \ldots, m_l \) are the exponents of \( W \). In finite type, it follows from results primarily of Gasharov [Gas98], Billey [Bil98], Billey-Postnikov [BP05], and Akyildiz-Carrell [AC12] that \( P_w(q) \) factors in this way for every rationally smooth element \( w \). The integers \( m_1, \ldots, m_l \) arising in this factorization can be uniquely determined from \( P_w(q) \). We call these integers the exponents of the rationally smooth element \( w \) (see Theorem 2.1 and Definition 2.2).

A central hyperplane arrangement \( A \) in a Euclidean space \( V \) is said to be free if the module of derivations of the complexified arrangement \( A_C \) is free as a module over the polynomial ring \( \mathbb{C}[V_C] \). If \( A \) is free, then the module of derivations has a homogeneous basis, and the degrees \( d_1, \ldots, d_l \) of the basis elements are called the coexponents of \( A \). The Poincare polynomial \( Q_A(t) \) of \( A \) is \( \sum_i t^i \dim H^i(M(A)) \), where \( M(A) \) is the complement of \( A_C \). The polynomial \( Q_A(t) \) is related to the chromatic polynomial \( \chi_A(t) \) by the identity \( t^i Q_A(-t^{-1}) = \chi_A(t) \). If \( A \) is free with coexponents \( d_1, \ldots, d_l \), then the Poincare polynomial \( Q_A(t) = \prod_i (1 + d_i t) \) by a result of Terao [Ter81]. Note that \( Q_A(q) \) is the number of chambers of the arrangement \( A \).

Let \( R \) be a root system associated to \( W \), with ambient Euclidean space \( V \). For convenience we identify \( V \) and \( V^* \) using the Euclidean form. Let \( R^+ \) and \( R^- \) denote
the positive and negative root sets, respectively. The inversion set of \( w \in W \) is defined to be

\[
I(w) = \{ \alpha \in R^+ : w^{-1}\alpha \in R^- \}.
\]

Let \( \mathcal{I}(w) \) denote the inversion hyperplane arrangement

\[
\mathcal{I}(w) = \bigcup_{\alpha \in I(w)} \ker \alpha
\]

in \( V^\perp \). For the maximal element \( w_0 \) the inversion set is \( I(w_0) = R^+ \), so \( \mathcal{I}(w_0) \) is the Coxeter arrangement corresponding to the root system \( R \). In this case, it is well-known that \( \mathcal{I}(w_0) \) is free, and \( Q_{\mathcal{I}(w_0)}(t) = \prod_i (1 + m_i t) \), where the coexponents \( m_1, \ldots, m_l \) are equal to the exponents of \( W \). The Poincare polynomial \( Q_{\mathcal{I}(w_0)}(t) \) can be written strictly in terms of the Weyl group as

\[
\sum_{w \in W} t^{\ell(w)} = \prod_{i} (1 + m_i t),
\]

where \( m_1, \ldots, m_l \) are the exponents of \( w \) \cite{Car72} \cite{ST54}.

Hultman, Linusson, Shareshian, and Sj"{o}strand (type A, \cite{HLSS09}) and Hultman (all Coxeter groups, \cite{Hul11}) have shown that if \( Q_{\mathcal{I}(w)}(1) \) is equal to the size of the Bruhat interval \([e, w]\), then

\[
Q_{\mathcal{I}(w)}(t) = \sum_{u \in [e, w]} t^{al(u, w)} = \sum_{u \in [e, w]} t^{\ell(uw^{-1})},
\]

where \( al(u, w) \) is the distance from \( u \) to \( w \) in the Bruhat graph of \([e, w]\). In particular Equation (1) holds if \( w \) is rationally smooth. We say that \( w \) satisfies the HLSS condition if \( Q_{\mathcal{I}(w)}(1) \) is equal to the size of \([e, w]\). In type A, the HLSS condition holds for \( w \) if and only if the Schubert variety \( X(w) \) is defined by inclusions \cite{HLSS09}, a condition introduced by Gasharov and Reiner which now has a number of equivalent formulations \cite{GR02} \cite{Sj07}.

In type A, Oh, Postnikov, and Yoo show that if \( w \) is rationally smooth, then \( Q_{\mathcal{I}(w)}(t) = \prod_i (1 + m_i t) \), where \( m_1, \ldots, m_l \) are the exponents of \( w \) \cite{OPY08}. Their proof implicitly implies that \( \mathcal{I}(w) \) is free. Yoo has conjectured that \( Q_{\mathcal{I}(w)}(t) \) factors in this way for rationally smooth elements in all finite Weyl groups \cite[Conjecture 1.7.3]{Yoo11}. Oh, Postnikov, and Yoo also show that, in type A, if \( w \) is rationally smooth then the Poincare polynomial \( P_w(q) \) is equal to the wall-crossing polynomial of \( I(w) \)\footnote{Since \( V \) and \( V^* \) are identified, \( \ker \alpha = \{ \beta \in V : (\beta, \alpha) = 0 \} \).}

This result has been extended to all finite-type Weyl groups by Oh and Yoo \cite{OY10}, using what we will call chain Billey-Postnikov (BP) decompositions (this is a modest variation on the terminology in \cite{OY10}).

Inspired by \cite{OY10}, we list the rationally smooth elements in finite type which do not have a chain BP decomposition. We also show that an element \( w \) of an arbitrary finite Coxeter group has a chain BP decomposition if and only if \( \mathcal{I}(w) \) has a modular coatom of a certain form. From these two results, we prove Yoo’s conjecture by showing that \( \mathcal{I}(w) \) is inductively free when \( w \) is rationally smooth, with coexponents equal to the exponents of \( w \). Using the root system pattern avoidance avoidance

1The wall-crossing polynomial is always palindromic, so in fact \( w \) is rationally smooth if and only if \( P_w(q) \) is equal to the wall-crossing polynomial of \( I(w) \).
criterion for rational smoothness due to Billey and Postnikov [BP05], we prove a converse: \( w \) is rationally smooth if and only if \( \mathcal{I}(w) \) is free and \( w \) satisfies the HLSS condition. Note that when \( \mathcal{I}(w) \) is free, \( w \) satisfies the HLSS condition if and only if \( \prod_i (1 + d_i) = ||e, w|| \). Finally, we show that \( w \) has a complete chain BP decomposition if and only if \( w \) is rationally smooth and \( \mathcal{I}(w) \) is supersolvable.

1.1. Organization. We start in Section 1.3 by giving some additional background and terminology. The main results are stated in Section 2. Section 3 contains results on chain BP decompositions. In Section 4 we give some more background on the HLSS condition. Section 5 develops the connection between chain BP decompositions and modular elements, and proves one direction of the main theorem. Section 6 discusses root-system pattern avoidance and the HLSS condition. The proofs of the main theorems are finished in Subsection 6.1. Examples and further directions are discussed in Section 7.

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1.3. Background and notation. The notation and concepts used in the introduction will be used throughout the paper. Before proceeding, we give some background on the concepts introduced above, and introduce some additional notation.

1.3.1. Root systems and Weyl groups. \( W \) will be a finite Coxeter group with generating set \( S \) and root system \( R \). We use \( l = |S| \) to refer to the rank of \( W \).

We let \( \leq \) denote Bruhat order on \( W \). The Bruhat graph of \( W \) is the directed graph with vertex set \( W \), and an edge from \( u \) to \( tw \) whenever \( t \) is a reflection with \( \ell(tw) > \ell(w) \). By definition, the distance \( d(u, w) \) from \( u \) to \( w \) in the Bruhat graph is infinite unless \( u \leq w \) in Bruhat order.

Given a root \( \alpha \in R \), we let \( t_\alpha \) denote the corresponding reflection. If \( s \in S \) is a simple reflection, we let \( \alpha_s \) denote the corresponding simple positive root. We let \( D_L(w) \) and \( D_R(w) \) denote the left and right descent sets of \( w \in W \), and \( S(w) \) denote the support of \( w \) (i.e. the set of all simple reflections which appear in some reduced expression for \( w \)).

Given a subset \( J \subset S \), we let \( W_J \) denote the parabolic subgroup generated by \( J \), \( W^J \) denote the set of minimal length left coset representatives, and \( ^JW \) denote the set of minimal length right coset representatives. Every element \( w \in W \) can be written uniquely as \( w = vu \) where \( v \in W^J \) and \( u \in W_J \). This factorization is called the right parabolic decomposition of \( w \). Left parabolic decompositions are defined similarly.

We refer to [BL00] for more background on the combinatorics of rationally smooth Schubert varieties.
1.3.2. **Hyperplane arrangements.** A central hyperplane arrangement $\mathcal{A}$ is a union of linear hyperplanes in some vector space $V$. For convenience, we will also use $\mathcal{A}$ to refer to the set of hyperplanes which it contains. Given a central hyperplane arrangement $\mathcal{A}$ in $V$, we can choose a defining linear form $\alpha_H$ for each $H \in \mathcal{A}$. Then $\mathcal{A}$ is cut out by the polynomial $Q = \prod \alpha_H$, where the product is over all $H \in \mathcal{A}$. The complexification $\mathcal{A}_C$ is the arrangement cut out by $Q$ in $V_C = V \otimes \mathbb{C}$, and $\text{Der}(\mathcal{A}_C)$ is the $\mathbb{C}[V_C]$-modules of derivations of $\mathbb{C}[V_C]$ which preserve the ideal $\mathbb{C}[V_C]Q$. The module $\text{Der}(\mathcal{A})$ is graded by polynomial degree.

Let $L(\mathcal{A})$ denote the intersection lattice of $\mathcal{A}$. By convention, the maximal element of $L(\mathcal{A})$ is the center $\cap_{H \in \mathcal{A}} H$ of $\mathcal{A}$. If $X \in L(\mathcal{A})$, we let $\mathcal{A}^X$ denote the restriction of $\mathcal{A}$ to $X$, and $\mathcal{A}_X$ the localization of $\mathcal{A}$ at $X$ (the localization $\mathcal{A}_X$ consists of the hyperplanes $H \in \mathcal{A}$ such that $X \subset H$). Given $H \in \mathcal{A}$, we let $\mathcal{A} \setminus H$ denote the deletion by $H$, which is the arrangement containing all hyperplanes of $\mathcal{A}$ except $H$. Finally, if $Y \subset V$ is a subspace of the center of $\mathcal{A}$, we let $\mathcal{A}/Y$ be the quotient of $\mathcal{A}$ by $Y$, an arrangement in $V/Y$. The arrangement $\mathcal{A}$ is free if and only if $\mathcal{A}/Y$ is free. If $\mathcal{A}$ is free, then 0 is a coexponent of $\mathcal{A}$ of multiplicity at least $\dim Y$, and the remaining coexponents of $\mathcal{A}$ are equal to the coexponents of $\mathcal{A}/Y$.

The addition theorem states that if $\mathcal{A} \setminus H$ is free with coexponents $d_1, \ldots, d_{l-1}, d_l - 1$, and $\mathcal{A}^H$ is free with coexponents $d_1, \ldots, d_{l-1}$, then $\mathcal{A}$ is free with coexponents $d_1, \ldots, d_l$ [OT92]. An arrangement is said to be inductively free if either (a) $\mathcal{A}$ contains no hyperplanes (in which case all exponents are zero), or (b) there is some hyperplane $H \subset \mathcal{A}$ such that $\mathcal{A} \setminus H$ is inductively free with coexponents $d_1, \ldots, d_l - 1$, and $\mathcal{A}^H$ is inductively free with coexponents $d_1, \ldots, d_{l-1}$. The addition theorem implies that inductively free arrangements are free.

Given an order on the set $\{\alpha_H : H \in \mathcal{A}\}$, a broken circuit is defined to be an ordered subset $\{\alpha_{H_1} < \ldots < \alpha_{H_k}\}$ such that there is $\alpha_{H_{k+1}} > \alpha_{H_k}$ for which $\{\alpha_{H_1}, \ldots, \alpha_{H_{k+1}}\}$ is a minimal linearly dependent set in $\{\alpha_H\}$[3]. An nbh-set is an ordered subset $\{\alpha_{H_1} < \ldots < \alpha_{H_k}\}$ which does not contain a broken circuit. The Orlik-Solomon algebra is the free exterior algebra generated by the forms $\alpha_H$, modulo the relations

$$\partial(\alpha_{H_1} \wedge \ldots \wedge \alpha_{H_k}) = 0 \text{ if } \text{codim } H_1 \cap \ldots \cap H_k < k.$$ 

Here $\partial$ is the unique derivation which sends $\alpha_H \mapsto 1$. It is well-known that the elements

$$\alpha_{H_1} \wedge \ldots \wedge \alpha_{H_k} : \{\alpha_{H_1}, \ldots, \alpha_{H_k}\} \text{ is an nbh-set}$$

form a basis of this algebra, and that the Orlik-Solomon algebra is isomorphic to the cohomology ring $H^*(\mathcal{M}(\mathcal{A}))$ of the complement $\mathcal{M}(\mathcal{A}) = V_C \setminus \mathcal{A}_C$. In particular, the number of nbh-sets does not depend on the order on $\{\alpha_H\}$.

2. **Main results**

The starting point for this paper is the following theorem:

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3The order is often reversed in this definition. We are using the order convention from [Hul11].
Theorem 2.1 ([AC12, Gas98, Bil98, BP05]). Let $W$ be a finite Weyl group. An element $w \in W$ is rationally smooth if and only if

$$P_w(q) = \prod_{i=1}^{l} [m_i + 1]_q$$

for some collection of non-negative integers $m_1, \ldots, m_l$.

The proof of Theorem 2.1 splits into cases, and is due to a variety of authors. As far as the author is aware, the complete theorem has not been previously stated before, so we give a full account of the proof in Section 3. The integers $m_1, \ldots, m_l$ appearing in Theorem 2.1 are uniquely determined by Equation (2), leading to the following definition.

Definition 2.2. Let $W$ be a finite Weyl group. If $w \in W$ is rationally smooth, then the exponents of $w$ are the integers $m_1, \ldots, m_l$ appearing in Theorem 2.1.

The main result of this paper is then:

Theorem 2.3. Let $W$ be a finite Weyl group. An element $w \in W$ is rationally smooth if and only if the inversion hyperplane arrangement $I(w)$ is free, and the product $\prod_i (1 + d_i)$ of the coexponents is equal to the size of the Bruhat interval $[e, w]$. Furthermore, if $w$ is rationally smooth then the coexponents $d_1, \ldots, d_l$ are equal to the exponents of $w$.

As mentioned in the introduction, one direction of Theorem 2.3 is implicitly proved for type $A$ in [OPY08]. In the proof of Theorem 2.3 we will show that if $w$ is rationally smooth, then $I(w)$ is in fact inductively free. As an immediate consequence of Theorem 2.3 we have that:

Corollary 2.4. Let $W$ be a finite Weyl group. If $w \in W$ is rationally smooth, then

$$Q_{I(w)}(t) = \prod_{i=1}^{l} (1 + m_i t),$$

where $m_1, \ldots, m_l$ are the exponents of $w$.

We can also characterize when $w$ is rationally smooth and $I(w)$ is not just free, but supersolvable. If $J \subset S$, and $v \in W^J$, define

$$P_v^J(q) = \sum_{x \in [e, v] \cap W^J} q^{\ell(x)}.$$

We define $^J P_v(q)$ for $v \in ^J W$ similarly.

Definition 2.5. Let $w = vu$ be a right parabolic decomposition of $w$ with respect to $J \subset S$, so $u \in W_J$ and $v \in W^J$. Then $w = vu$ is a (right) Billey-Postnikov (BP) decomposition if $P_w(q) = P_v^J(q) P_u(q)$. If in addition $[e, v] \cap W^J$ is a chain, we say that $w = vu$ is a chain BP decomposition. Left chain BP decompositions are defined similarly.

Note that a left BP decomposition of $w$ is a right BP decomposition of $w^{-1}$. 
Definition 2.6. We say that $w$ has a complete chain BP decomposition if either $w$ is the identity, or $w$ has a right or left chain BP decomposition $w = vu$ or $w = uv$, where $u \in W_J$ has a complete chain BP decomposition.

Let $W$ be an arbitrary finite Coxeter group. The interval $[e, v] \cap W^J$ is a chain if and only if $P^J_v(q) = \ell(v) + 1$, so if $w$ has a complete chain BP decomposition, then $P_w(q)$ is a product of $q$-integers. Hence if $w$ has a complete chain BP decomposition, then $w$ is rationally smooth, and it is also possible to define exponents of $w$ as in Definition 2.2. In types $A$, $B$, $C$, and $G_2$, every rationally smooth has a complete chain BP decomposition, but this is not true in types $D$, $E$, or $F_4$ (see Theorem 3.3).

Recall that an arrangement $\mathcal{A}$ is supersolvable if and only if $L(\mathcal{A})$ has a complete chain of modular elements.

Theorem 2.7. Let $W$ be a finite Weyl group. An element $w \in W$ has a complete chain BP decomposition if and only if $w \in W$ is rationally smooth and $I(w)$ is supersolvable.

If $w = uv$ is a parabolic decomposition, then the inversion set $I(w)$ is the disjoint union of $I(u)$ and $uI(v)$, and

$$X = \bigcap I(u) = \bigcap_{\alpha \in I(u)} \ker \alpha$$

is a flat of $L(I(w))$. The proofs of Theorems 2.3 and 2.7 are based on a connection between chain BP decompositions and certain modular coatoms of the inversion arrangement:

Theorem 2.8. Suppose that $w = uv$ is a left parabolic decomposition with respect to $J$, so $u \in W_J$ and $v \in J^W$. Let $X = \bigcap I(u)$. Then $w = uv$ is a chain BP decomposition if and only if $X$ is a modular coatom of $L(I(w))$.

Theorem 2.8 holds for arbitrary finite Coxeter groups. As will be explained in Section 5, the key point of Theorem 2.8 is that $[e, v] \cap J^W$ is a chain if and only if a certain linear condition on $I(v)$ holds. Also note that $I(w^{-1}) = -w^{-1}I(w)$, so $I(w)$ and $I(w^{-1})$ are linearly isomorphic, and Theorem 2.8 could equivalently be stated in terms of right parabolic decompositions. Although we do not develop this further, Theorem 2.8 implies that an element $w$ of an arbitrary finite Coxeter group has a complete chain BP decomposition if and only if $L(I(w))$ has a complete chain of modular elements of a certain form.

3. Chain Billey-Postnikov decompositions

If $w = vu$ is the parabolic decomposition of $w$ with respect to $J \subset S$, then multiplication gives an injective map

$$(3) \quad ([e, v] \cap W^J) \times [e, u] \to [e, w].$$

If $x = v_1u_1$ is the parabolic decomposition of an element of $[e, w]$, then $v_1 \leq v$. However, it is not true that $u_1 \leq u$, even though $u_1 \leq w$ and $u_1 \in W_J$. 
Lemma 3.1 ([BP05], [OY10], [RS13b]). Let \( w = vu \) be the parabolic decomposition of \( w \) with respect to \( J \). The following are equivalent:

(a) The map in equation (3) is surjective (hence bijective).
(b) \( u \) is the unique maximal element of \([e, w] \cap W_J\).
(c) \( S(v) \cap J \subset D_L(u) \).
(d) \( w = vu \) is a BP decomposition, so \( P_w(q) = P_j^v(q)P_u(q) \).

We refer to [OY10] or [RS13a] for more information about BP decompositions.

The fundamental theorem for Billey-Postnikov decompositions is:

Theorem 3.2 ([Gas98] [Bil98] [BP05]). If \( w \in W \) is rationally smooth, \( \ell(w) \geq 2 \), then \( w \) has either a left or right BP decomposition with respect to \( J = S \setminus \{s\} \), where \( s \in S(w) \) is a leaf of the Dynkin diagram for \( W_{S(w)} \).

Theorem 3.2 was proved by Gasharov [Gas98] for type A, by Billey [Bil98] in classical types, and by Billey-Postnikov [BP05] in the exceptional types. Note that if \( w = vu \) is a BP decomposition with respect to \( J \), then \( u \) is rationally smooth, and \( v \) is rationally smooth relative to \( J \) (meaning that the relative Schubert variety is rationally smooth, or equivalently that \( P_j^v(q) \) is palindromic) [BP05].

We note three particular details of Theorem 3.2:

- If \( W \) is of type A, \( w \in W \) is smooth, and \( s \) is a fixed leaf of \( W_{S(w)} \), then \( w \) has either a left or right BP decomposition with respect to \( J = S \setminus \{s\} \).
  (In other words, it is possible to choose the leaf \( s \) used in the decomposition, which is not necessarily possible in other types.) [Gas98] [Bil98]

- If \( W \) is simply-laced and \( w \in W \) is rationally smooth with a BP decomposition \( w = vu \) with respect to \( J \setminus \{s\} \), \( s \) a leaf of \( S(w) \), then \( v \) is the maximal element of \( W_{J \setminus S(v)} \) ([BP99] [OY10], see also [RS13a]).

- If \( W \) is of type A, B, or \( G_2 \), \( s \) is a leaf of \( W \), \( w \) belongs to \( W' \), where \( J = S \setminus \{s\} \), and \( P_j^v(q) \) is palindromic, then \([e, w] \cap W'\) is a chain in Bruhat order (or equivalently, \( P_j^v(q) = \ell(w) + 1 \)) ([Bil98] [OY10], see also [RS13a]). This is also true in type \( F_4 \), except when \( w \) is the maximal element of \( W' \) ([OY10], see also [RS13a]).

By the second fact above, if \( w \in W \) is rationally smooth and does not have a chain BP decomposition, then \( W \) must have type \( D \), \( E \), or \( F_4 \). The main result of this section is a list of all the elements in these types which do not have a chain BP decomposition. To list these elements, we use the following labelling for the Dynkin diagram of \( E_8 \):

\[
\begin{array}{cccccccc}
1 & & & & & & & \\
& 8 & 7 & 6 & 5 & 4 & 3 & 2 \\
\end{array}
\]

We let \( S_k = \{s_1, \ldots, s_k\} \), where \( s_i \) is the simple reflection corresponding to the node labelled by \( i \), and take the convention that \( E_6 \) and \( E_7 \) are embedded inside \( E_8 \) as \( W_{S_6} \) and \( W_{S_7} \). Finally, we let \( J_k = S_k \setminus \{s_2\} \), and let \( \tilde{u}_k \) be the maximal element of \( W_{J_k} \), and \( \tilde{v}_k \) be the maximal element of \( W_{S_k}^J \). Note that \( \tilde{u}_k \) belongs to a parabolic decomposition.
subgroup of type $D$, while $\tilde{v}_k$ belongs to a parabolic subgroup of type $D$ when $k = 5$, and a parabolic subgroup of type $E$ when $k \geq 6$.

**Theorem 3.3.** Suppose that $w \in W$ is a rationally smooth element of a finite Weyl group $W$, that $\ell(w) \geq 2$, and that $w$ has no chain BP decomposition. Then $w$ is one of the following elements:

- The maximal element of $D_n$, $n \geq 4$.
- The maximal element of $E_n$, $n = 6, 7, 8$.
- The element $w_{kl} = \tilde{v}_l \tilde{v}_k$ in $E_8$, or its inverse $w_{kl}^{-1}$, $5 \leq l < k \leq 8$.
- The maximal element of $F_4$.

For the proof, we use the following terminology and lemma from [RS13a]: We say that a subset $T \subset S$ is connected if the Dynkin diagram restricted to $T$ is connected. Note that if $J = S \setminus \{s\}$ and $w \in W^J$ then $S(w)$ is connected. We say that two elements $s, t \in S$ are adjacent if they are connected by an edge in the Dynkin diagram.

**Lemma 3.4 (RS13a, Lemma 5.3).** Let $w = vu$ be a parabolic decomposition with respect to some $J$, and suppose $s \in S$ is adjacent to an element of $S(v)$, but is not contained in $S(v)$. Then $s$ does not belong to $D_L(w)$.

**Proof of Theorem 3.3.** By [Bil98], it is sufficient to check type $E$. While this can be checked on the computer, we give a complete proof using Lemma 3.1. Suppose $w$ is a rationally smooth element of $E_8$, with $S(w) = S_k$ for some $k \geq 6$. Take a BP decomposition of $w$ with respect to $J = S_k \setminus \{s\}$, where $s$ is a leaf of $S(w)$. We can assume without loss of generality that $w$ has a right BP decomposition of this form, so $w = vu$, where $v \in W^J_{S_k}$ and $u \in W_J$. We consider the different possibilities for $s$ separately.

**Case 1:** $s = s_k$. As noted above, if $[e, v] \cap W^J$ is not a chain, then $W_{S(v)}$ must have type $D$ or $E$. If in addition $w$ is not maximal of type $E$, then we must have $S(v) = J_k$, so $W_{S(v)}$ is of type $D$. Now $J \cap S(v) = J_{k-1} \subset D_L(u)$ by Lemma 3.1 so we can write $u = u_0 u_1$, where $u_0$ is the maximal element of $W_{J_{k-1}}$. Then $v$ is the maximal element of $W_{J_k}^{-1}$, so $vu_0 = \tilde{u}_k$ is the maximal element of $W_{J_k}$. Because $u_0$ is maximal, $w = (vu_0)u_1$ is a left BP decomposition with respect to $S(w) \setminus \{s_2\}$. If $[e, u_1] \cap W^J$ is not a chain, then $S(u_1) \supset S_5$, and $u_1 = \tilde{v}_k^{-1}$ for some $l < k$. So $w = \tilde{u}_k \tilde{v}_k^{-1} = w_{kl}^{-1}$.

**Case 2:** $s = s_1$. If $[e, v] \cap W^J$ is not a chain, and $w$ is not maximal, then we must have $S(v) \supset J_5$, and $s_2$ and $s_5$ cannot both belong to $S(v)$. Suppose $S(v)$ contains one of $\{s_2, s_5\}$, and let $t$ be the element which is not contained in $S(v)$. Since $W_{S(v)}$ has type $A$, the element $u$ has a BP decomposition with respect to $J' = J \setminus \{t\}$. If this is a left BP decomposition $u = u_0 u_1$, $u_0 \in W_{J'}, u_1 \in J' W_J$, then $w = (vu_0)u_1$ is a left chain BP decomposition. Otherwise, we have a right BP decomposition $u = u_1 u_0$, $u_0 \in W_{J'}, u_1 \in J' W_J$. Then $W_{S(u_1)}$ has type $A$ and $[e, u_1] \cap W^J$ is a chain. If $S(u_1)$ and $S(v)$ pairwise commute, then $w = u_1 (vu_0)$ is a chain BP decomposition. Otherwise, an element of $S(v)$ must be adjacent to or contained in $S(u_1)$. But since $S(v) \setminus \{s_1\}$ is connected and is contained in $D_L(u)$ by Lemma 3.1, we can conclude...
that $S(v) \setminus \{s_1\} \subset S(u_1)$ by Lemma 3.4. Consequently $S(u_1)$ contains $\{s_2, s_k\}$, so $u_0$ is the maximal element of $W_{J'}$, and $u$ is the maximal element of $W_J$. Therefore $u$ has a left BP decomposition with respect to $J'$.

This leaves the possibility that neither $s_2$ or $s_k$ lies in $S(v)$. Take a BP decomposition of $u$ with respect to $J' = J \setminus \{s_k\}$. If we get a left decomposition, or a right decomposition $u = u_1u_0$ where $S(u_1)$ pairwise commutes with $S(v)$, then we get a chain decomposition of $w$ as above. Suppose we get a right BP decomposition $u = u_1u_0$, $u_1 \in W_{J'}$, $u_0 \in W_J$, where $S(v)$ and $S(u_1)$ do not pairwise commute. By Lemma 3.4, $S(u_1) \supset S(v) \setminus \{s_1\}$. If $S(u_1) = J$, then as above $u$ is the maximal element of $W_J$, so we can take a left BP decomposition of $u$ to get a chain decomposition of $w$. Otherwise $S(u_1) = J \setminus \{s_2\}$ since $J_5 \subset S(v) \setminus \{s_1\}$ and $S(u_1)$ is connected. Write $u_0 = u'_0u''_0$, where $u'_0$ is the maximal element of $W_{J' \setminus \{s_2\}}$ and $u''_0 \in J' \setminus \{s_2\}W_{J'}$. Then $w = (vu'_0)u''_0$ is a chain BP decomposition.

Case 3: $s = s_2$. In this case, $v \in W_{S_{s}}$. If $[e, v] \cap W_{J_k}$ is not a chain and $w$ is not maximal, then $S(v) = S_t$ for some $k > l \geq 5$, and $v = \tilde{v}_t$. If $u$ has a left BP decomposition $u = u_0u_1$ with respect to $J_{k-1} = S(u) \setminus \{s_k\}$, then either $W_{S(u_1)}$ is of type $A$, in which case $w = (vu_0)u_1$ is a chain BP decomposition, or $S(u_1) = J_{k}$, in which case $u = \tilde{u}_k$, so that $w = w_{k}$.

If $u$ has a right BP decomposition $u = u_1u_0$ with respect to $J_{k-1}$ and $S(u_1)$ pairwise commutes with $S(v)$, then $w = u_1(vu_0)$ is a chain BP decomposition (since $S(u_1)$ cannot meet $S(v)$, $W_{S(u_1)}$ is of type $A$). If $S(u_1)$ does not pairwise commute with $S(v)$ then $S(u_1) \supset J_t$, so $S(u_1) = J_k$ by Lemma 3.4 and again $u = \tilde{u}_k$.

If $u$ does not have a BP decomposition with respect to $J_{k-1}$, then $u$ has a BP decomposition with respect to $J' = J_k \setminus \{t\}$, where $t \in \{s_1, s_3\}$. Suppose $u$ has a right BP decomposition with respect to $J'$, so $u = u_1u_0$, $u_1 \in W_{J'_k}$, $u_0 \in W_{J'}$. Then $S(u_1)$ and $S(v)$ both contain $t$, so by Lemma 3.4, $S(u_1) \supset S(v) \setminus \{s_2\} = J_t$. If $s_k \in S(u_1)$ then $S(u) = J_k$ and $u = \tilde{u}_k$. Otherwise $W_{S(u_0)}$ is of type $A$, so we can take a decomposition of $u_0$ with respect to $K = J' \setminus \{s_k\}$. If we get a left BP decomposition of $u_0$, or a right BP decomposition $u_0 = u''_0u'_0$, where $u'_0 \in KW_{J'}$, $u''_0 \in W_K$, then $w = u'_0v$ is a chain BP decomposition. Otherwise by Lemma 3.4 we must have a right BP decomposition $u_0 = u''_0u'_0$ where $S(u''_0) \supset S(u_1) \setminus \{t\}$. Since $s_k \in S(u''_0)$, this means that $u_0$ is the maximal element of $W_J$, and hence has a left BP decomposition with respect to $K$.

Finally, suppose that $u$ has a left BP decomposition $u = u_0u_1$ where $u_1 \in J'W_{J_k}$, $u_0 \in W_{J'}$. If $W_{S(u_1)}$ is of type $A$, then $w$ has a chain BP decomposition $w = (vu_0)u_1$. If $W_{S(u_1)}$ is not of type $A$ then $S(u_1)$ must contain $J_5$. If $s_k \in S(u_1)$ then $u = \tilde{u}_k$. Suppose $s_k \notin S(u_1)$. Since $S(u_0)$ is of type $A$, we can take a BP decomposition of $u_0$ with respect to $K$. If $u_0 = u''_0u'_0$, where $u''_0 \in W_{J'_k}$ and $u'_0 \in W_K$, and $S(u''_0)$ pairwise commutes with $S(u_1)$ then $w = (vu''_0u'_0)u''_0$ is a chain BP decomposition. If $S(u''_0)$ does not pairwise commute with $S(u_1)$ then by Lemma 3.4, $S(u''_0) \supset S(u_1) \setminus \{t\}$, so $u_0$ is the maximal element of $W_{J'}$ and $J' \subset D_L(u)$. Since $t \in S(v)$, we must have $t \in D_L(u)$ and hence $u = \tilde{u}_k$. Similarly if $u_0 = u''_0u'_0$ where $u''_0 \in W_{J'_k}$ and $u'_0 \in W_K$, then $t \notin S(u''_0)$. Since $s_k \notin S(u_1)$, $u = u''_0u'_0v$ is a parabolic decomposition with
respect to $J_{k-1}$, and it follows from Lemma 3.4 that $S(u'_0)$ must pairwise commute with $S(v)$, so $w = u'_0(vu'u_1)$ is a chain BP decomposition. \hfill \Box

3.1. Proof of Theorem 2.1. We can now give a short account of the proof of Theorem 2.1. If $w$ has a chain BP decomposition $w = vu$ or $w = uv$, then $P_w(q) = [l(v) + 1]qP_u(q)$, and since $u$ is also rationally smooth, we can proceed by induction. Thus we only need to check Theorem 2.1 for elements which do not have a chain BP decomposition. If $w$ is the maximal element in $W_{S(w)}$, then as noted in the introduction the conclusion of Theorem 2.1 is well-known. This leaves the elements $w_{kl}$ and $w_{kl}^{-1}$ from Theorem 3.3. The inverse map gives an order-isomorphism $[e, w] \cong [e, w^{-1}]$, so $P_w(q) = P_{w^{-1}}(q)$. Thus we only need to check the theorem for the elements $w_{kl}$, $5 \leq l < k \leq 8$. Let $\tilde{w}_l$ be the maximal element of $S_l$. Then

$$P_{w_{kl}}(q) = P_{\tilde{w}_l}(q) P_{\tilde{w}_l}(q) = P_{\tilde{w}_l}(q) P_{\tilde{w}_l}(q) P_{\tilde{w}_l}^{-1},$$

where the last equality uses the fact that $\tilde{w}_l = \tilde{w}_l \tilde{w}_l$. The elements $\tilde{w}_l$ are maximal of type $D$, and the elements $\tilde{w}_l$ are maximal of types $D$ ($l = 5$) and $E$ ($l > 5$). We can check that the exponents of $\tilde{w}_l$ are, counting with multiplicity, contained in the union of the exponents of $\tilde{w}_l$ and $\tilde{w}_l$, so that $P_{w_{kl}}(q)$ is a product of $q$-integers. The exponents of the elements $w_{kl}$ are given in Table 1.

| $w$ | Exponents of $w$ (zeroes omitted) |
|-----|----------------------------------|
| $w_{65}$ | 1, 4, 4, 5, 7, 7 |
| $w_{75}$ | 1, 4, 5, 5, 7, 7, 9 |
| $w_{85}$ | 1, 4, 5, 6, 7, 7, 9, 11 |
| $w_{76}$ | 1, 5, 5, 7, 8, 9, 11 |
| $w_{86}$ | 1, 5, 6, 7, 8, 9, 11, 11 |
| $w_{87}$ | 1, 6, 7, 9, 11, 11, 13, 17 |

Table 1. Table of exponents for elements $w_{kl}$.

It is not necessary to use Theorem 3.3 to prove Theorem 2.1. When the Schubert variety $X(w)$ is smooth, Theorem 2.1 follows from a result of Akyildiz-Carrell [AC12], which states that the exponents can be calculated from the torus weights $\Omega_w$ of the tangent space $T_e X(w)$ to $X(w)$ at the identity, analogously to how the exponents of $W$ can be determined from the heights of the positive roots of $R$. In the simply-laced types, all rationally smooth Schubert varieties are smooth by a theorem of Peterson [CK03], so the Akyildiz-Carrell theorem covers Theorem 2.1. The non-simply-laced types are covered by [Bi98] (classical types) and [OY10] (type $F_4$).

4. The HLSS condition and nbc-sets

In this section we give some background on the HLSS condition which is necessary in the subsequent sections. Given a reduced expression $s_1 \cdots s_k$ for an element $w \in W$, we can order the inversion set $I(w)$ by $\beta_1 < \cdots < \beta_{l(w)}$, where $\beta_l = s_{l-1} \alpha_s l_i$. A total order on $I(w)$ constructed in this way is called a convex order. Let $2^{I(w)}$ denote the power set of $I(w)$. Given a convex order, we can define a surjective map

$$\phi : 2^{I(w)} \rightarrow [e, w] : \{\beta_1, \ldots, \beta_k\} \mapsto t_{\beta_1} \cdots t_{\beta_k} w$$

when $\beta_1 < \cdots < \beta_k$. 


Theorem 4.1 (Hultman-Linusson-Shareshian-Sjöstrand [HLSS09]). Choose a convex order for $I(w)$, and let $\text{nb}(I(w))$ denote the set of nb-sets of $I(w)$ with respect to the chosen order. Then the restriction of $\phi$ to $\text{nb}(I(w))$ is injective.

In particular the number of nb-sets of $I(w)$ is less than the size of the Bruhat interval $[e, w]$. The restriction of $\phi$ to $\text{nb}(I(w))$ will be surjective if and only if the number of nb-sets is equal to the size of the Bruhat interval. Since the size of $\text{nb}(I(w))$ is independent of the choice of convex order, if there is some convex order for which the restriction of $\phi$ to $\text{nb}(I(w))$ is surjective, then this happens for all convex orders.

A theorem of Hultman-Linusson-Shareshian-Sjöstrand (type A) and Hultman (all finite Coxeter groups) characterizes when the restriction is surjective. Recall that $\ell'(w)$ is the absolute length of $w$, and $\text{al}(u, w)$ is the distance from $u$ to $w$ in the directed Bruhat graph. The distance from $u$ to $w$ in the undirected Bruhat graph is simply $\ell'(uw^{-1})$, so $\text{al}(u, w) \geq \ell'(uw^{-1})$.

Theorem 4.2 ([HLSS09] [Hul11]). The restriction of $\phi$ to $\text{nb}(I(w))$ is surjective (and hence bijective) if and only if $\text{al}(u, w) = \ell'(uw^{-1})$ for all $u \leq w$.

Definition 4.3. We say that $w$ satisfies the HLSS condition if $\text{al}(u, w) = \ell'(uw^{-1})$ for all $u \leq w$, or equivalently if the restriction of $\phi$ to $\text{nb}(I(w))$ is surjective.

The HLSS condition is weaker than being rationally smooth:

Theorem 4.4 ([Hul11]). The HLSS condition is satisfied if $w$ is rationally smooth.

The following theorem of Carter, used by Hultman in the proof of Theorem 4.2 implies that if $\{\beta_{j_1}, \ldots, \beta_{j_k}\}$ is linearly independent, then $\ell'(\phi(\beta_{j_1}, \ldots, \beta_{j_k})w^{-1}) = k$.

Theorem 4.5 ([Car72]). Suppose $w = r_{\beta_1} \cdots r_{\beta_m}$, where $r_{\beta_i}$ refers to reflection through the root $\beta_i$. Then

(a) $\ell'(w) = m$ if and only if $\{\beta_1, \ldots, \beta_m\}$ is linearly independent.

(b) The fixed point space of $w$ on $V$ contains the orthogonal complement of $\text{span}\{\beta_1, \ldots, \beta_m\}$.

(c) The codimension of the fixed point space of $w$ is equal to $\ell'(w)$, so if $\ell'(w) = m$ then the fixed point space of $w$ is equal to the orthogonal complement of $\text{span}\{\beta_1, \ldots, \beta_m\}$.

We will use Theorem 4.5 later in Section 6. To finish the section, we note Equation (1) from the introduction follows immediately from the results outlined in this section.

Corollary 4.6. [HLSS09] If $w$ satisfies the HLSS condition then

$$Q_{I(w)}(q) = \sum_{u \in [e, w]} q^{\ell(uw^{-1})} = \sum_{u \in [e, w]} q^{\text{al}(u, w)}.$$
Lemma 5.1. The linear span of the inversion set \( I(w) \) in \( V \) is \( V_{S(w)} \), and the center of the inversion hyperplane arrangement \( \mathcal{I}(w) \) is the orthogonal complement of \( V_{S(w)} \).

Proof. The proof is by induction on \( \ell(w) \). The lemma is clearly true for the identity. Given \( w \neq e \), choose \( s \in D_L(w) \). Then \( I(w) = \{ \alpha \} \cup sI(sw) \), so the span of \( I(w) \) is
\[
\mathbb{R}\alpha + \text{span } I(sw) = \mathbb{R}\alpha + V_{S(sw)} = V_{S(w)}.
\]
The corresponding statement for \( \mathcal{I}(w) \) follows immediately.

Lemma 5.1 implies that the rank of \( \mathcal{I}(w) \) is the size of the support set \( S(w) \). Recall that a coatom of \( L(\mathcal{I}(w)) \) is an element of \( L(\mathcal{I}(w)) \) of rank \( |S(w)| - 1 \), or in other words is an \((l - |S(w)| + 1)\)-dimensional subspace of \( V \) which can be written as an intersection of hyperplanes in \( \mathcal{I}(w) \). The flat \( X = \cap \mathcal{I}(u) \) has rank \( |S(u)| \), and hence \( X \) will be a coatom if and only if \( |S(u)| = |S(w)| - 1 \).

To prove Theorem 2.3 we also need the following characterization of modular coatoms from [CDF+09]:

Lemma 5.2 ([CDF+09]). Let \( \mathcal{A} \) be an arrangement, and let \( \{ \alpha_H \} \) be a set of defining forms for \( \mathcal{A} \). Let \( X \in L(\mathcal{A}) \) be a coatom. Then \( X \) is modular if and only if for every distinct pair \( H_1, H_2 \notin \mathcal{A}_X \), there is \( H_3 \in \mathcal{A}_X \) such that \( \alpha_{H_1}, \alpha_{H_2}, \alpha_{H_3} \) are linearly dependent.

Proof. Lemma 3.20 of [CDF+09] states that \( X \) is modular if and only if the vector space sum \( X + Y \) belongs to \( L(\mathcal{A}) \) for every \( Y \in L(\mathcal{A}) \) of rank 2, or in other words for all \( Y = H_1 \cap H_2 \), where \( H_1, H_2 \in \mathcal{A} \). If, say, \( H_1 \in \mathcal{A}_X \) then \( H_1 \cap H_2 + X \subset H_1 \), and thus \( H_1 \cap H_2 + X = \text{either } H_1 \text{ or } H_1 \cap H_2 \). Thus the condition only needs to be checked for \( H_1, H_2 \notin \mathcal{A}_X \). But if \( X \) is not contained in \( H_1 \) or \( H_2 \) then \( H_1 \cap H_2 + X \) is a hyperplane, so it is in \( L(\mathcal{A}) \) if and only if there is \( H_3 \in \mathcal{A} \) such that \( H_1 \cap H_2 + X = H_3 \). This means that \( H_3 \) must contain both \( X \) and \( H_1 \cap H_2 \), and the latter condition is equivalent to the condition that \( \alpha_{H_3} \) be in the span of \( \alpha_{H_1} \) and \( \alpha_{H_2} \).

Proof of Theorem 2.3 If \( w = uv \) is a chain BP decomposition, then \( |S(w) \cap J| = |S(w)| - 1 \), and \( S(v) \cap J \subset S(u) \) so \( |S(u)| = |S(w)| - 1 \). Hence we can assume throughout that \( X \) is a coatom.

Recall that \( I(w) \) is the union of \( I(u) \) and \( uI(v) \). The set \( I(u) \) is equal to \( I(w) \cap V_J \), and \( w \in W \) if and only if \( I(w) \cap V_J \) is empty. The hyperplanes of \( \mathcal{I}(w) \) which do not contain \( X \) correspond precisely to the roots in \( uI(v) \). By Lemma 5.2 \( X \) is modular if and only if for all \( \alpha, \beta \in uI(v) \), there is \( \gamma \in I(u) \) such that \( \alpha, \beta, \gamma \) are linearly dependent. The inversion set \( I(u^{-1}) = -u^{-1}I(u) \), so applying \( -u^{-1} \) we see that \( X \) is modular if and only if for all \( \alpha, \beta \in I(v) \) there is \( \gamma \in I(u^{-1}) \) such that \( \alpha, \beta, \gamma \) are linearly dependent.

Now suppose that \( w = uv \) is a chain BP decomposition. We first assume that \( u \) satisfies the HLSS condition. Order \( I(w) \) so that all the elements of \( I(u) \) come after
the elements of $uI(v)$. Every element $\alpha \in uI(v)$ is independent from the span of $I(u)$. Thus if $\{\gamma_1, \ldots, \gamma_k\}$ is an nbc-set for $I(u)$, then $\{\alpha, \gamma_1, \ldots, \gamma_k\}$ is an nbc-set for $I(w)$. So

$$|\text{nbc}(I(w))| \geq (1 + \ell(v)) \cdot |\text{nbc}(I(u))|.$$ 

Since $u$ satisfies the HLSS condition, $|\text{nbc}(I(u))| = ||e, u||$, while $|\text{nbc}(I(w))| \leq ||e, w||$. But $w = uv$ is a chain BP decomposition, so

$$||e, w|| = P_v(1) = P_u(1) \cdot J P_v(1) = (1 + \ell(v)) \cdot ||e, u|| = (1 + \ell(v)) \cdot |\text{nbc}(I(u))|.$$ 

We conclude that all nbc-sets of $I(w)$ are either nbc-sets of $I(u)$, or are of the form $\{\alpha, \gamma_1, \ldots, \gamma_k\}$ for $\alpha \in uI(v)$, and $\{\gamma_1, \ldots, \gamma_k\}$ an nbc-set of $I(u)$.

In particular, if $\alpha, \beta \in uI(v)$, $\alpha < \beta$, then $\{\alpha, \beta\}$ is not an nbc-set, and hence there must be some $\gamma \in I(u)$, $\gamma > \alpha, \beta$ such that $\alpha, \beta, \gamma$ is linearly dependent. If $\gamma \notin I(u)$ then $\gamma \in uI(v)$, and we can repeat this process by replacing $\alpha, \beta$ with $\beta, \gamma$, until we find $\gamma' \in I(u)$ such that $\beta, \gamma, \gamma'$ is linearly dependent. Since no two positive roots are colinear, $\gamma'$ must be in the span of $\beta, \gamma$, and $\gamma$ is in the span of $\alpha, \beta$. Hence $\alpha, \beta, \gamma'$ are linearly dependent. By Lemma 5.2, $X$ is modular.

If $u$ does not satisfy the HLSS condition, we can replace $w$ with $w' = u_0v$, where $u_0$ is the maximal element of $W_{J \cap S(v)}$. Since $u_0$ is rationally smooth, it satisfies the HLSS condition, and the above argument, the coatom $X' = \bigcap I(u_0)$ is modular. This means that for all $\alpha, \beta \in I(v)$, there is $\gamma \in I(u_0^{-1})$ such that $\alpha, \beta, \gamma$ are linearly dependent. But since $w = uv$ is a BP decomposition, $u$ has a reduced decomposition $u = u_1u_0$, and hence $I(u_0^{-1}) \subset I(u^{-1})$. Thus $X$ is modular.

Now suppose that $X$ is a modular coatom. We want to show that $w = uv$ is a chain BP decomposition. We start with the following claim: if $v$ has a reduced decomposition $v = v_0sv_1$, where $s, t \in S$, and $t \notin D_R(v_0)$, then $v_0t \notin J W$, and in fact $v_0t = t'v_0$, where $t' \in D_R(u)$. Indeed, let $\alpha = v_0\alpha_s$ and $\beta = v_0\alpha_t$. Since $\alpha, \beta \in I(v)$ and $X$ is modular, there is $\gamma \in I(u^{-1})$ such that $\alpha, \beta, \gamma$ are linearly dependent. Now

$$\beta = v_0\left(\alpha_t - \frac{2(\alpha_s, \alpha_t)}{(\alpha_s, \alpha_s)}\alpha_s\right) = v_0\alpha_t - \frac{2(\alpha_s, \alpha_t)}{(\alpha_s, \alpha_s)}\alpha.$$ 

Hence the span of $\alpha$ and $\beta$ is the same as the span of $v_0\alpha_s$ and $v_0\alpha_t$, and we can find $a, b \in \mathbb{R}$ such that

$$av_0\alpha_s + bv_0\alpha_t = \gamma. \tag{4}$$ 

Since $\gamma \in V_J$ and $v_0 \in J W$, the root $\gamma$ cannot be in $I(v_0)$, and hence $a\alpha_s + b\alpha_t = v_0^{-1}\gamma \in R^+$. It follows that $a, b \geq 0$. Let $S(v) \setminus S(u) = \{r\}$. Now $v_0\alpha_s$ is in $I(v)$, and in particular is positive and does not belong to $V_J$. It follows that if we write $\alpha$ as a linear combination of simple roots, the coefficient of $\alpha_r$ will be positive. Since $t \notin D_R(v_0)$, $v_0t$ is reduced and $v_0\alpha_t \in I(v)$ is also positive. Hence the coefficient of $\alpha_r$ in $v_0\alpha_t$ is non-negative. But $av_0\alpha_s + bv_0\alpha_t = \gamma$ belongs to $V_J$, so we must have $a = 0$, $b = 1$, and $v_0\alpha_t = \gamma \in I(u^{-1})$. Since $v_0\alpha_t \in I(v_0t) \cap V_J$, the element $v_0t$ does not belong to $J W$, and $v_0t$ has a non-trivial parabolic decomposition $v_0t = v_0'v_0''$, $v_0' \in W_J$, $v_0'' \in J W$. Since $v_0 \leq v_0t$, we have $v_0 \leq v_0'$, and by comparing lengths we see that $v_0 = v_0'$, while $v_0' = t'$ for some $t' \in J$. The simple reflection $\alpha_{t'}$ is the unique element
of $I(v_0 t) \cap V_J$, so $\alpha' = v_0 \alpha \in I(u^{-1})$, and consequently $t' \in D_L(u^{-1}) = D_R(u)$. This finishes the proof of the claim.

Next we show that $w = uv$ is a BP decomposition. Indeed, if $t \in S(v) \cap J$, we can find a reduced decomposition $v = v_0 s t u_1$, $s \in S$, where $t \notin S(v_0 s)$. This latter fact implies that $v_0 t$ is reduced, so by the above claim $v_0 t = t' v_0$ where $t' \in D_R(u)$. But $t \in S(t' v_0)$ and $t \notin S(v_0)$, so $t = t'$. Hence $w = uv$ is a BP decomposition.

It remains to show that $[e, v] \cap JW$ is a chain. We use induction on $\ell(v)$. If $\ell(v) = 1$ the claim is obvious, so suppose $\ell(v) > 1$. Let $v = s_1 \cdots s_k$ be a reduced factorization of $v$ into simple reflections, and let $\hat{v}_i = s_1 \cdots s_i$. We want to show that $[e, v] \cap JW = \{ \hat{v}_i : i = 1, \ldots, k\}$. Suppose that $v' = s_1 \cdots s_{i-1} s_i \cdots s_k$ is an element of $[e, v] \cap JW$ of length $\ell(v) - 1$. Since $|S(v_0) \setminus J| = 1$, we must have $i > 1$. If $i < k$ then we get a reduced decomposition $v = \hat{v}_{i-1} s_i s_{i+1} v_1$, where $v_1 = s_{i+1} \cdots s_k$. Since $v_0 s_{i+1} v_1$ is reduced, $s_{i+1} \notin D_R(v_0)$. Hence by the above claim, $v_0 s_{i+1} \notin JW$. But this contradicts the fact that $v' \in JW$. Hence the only element of $[e, v] \cap JW$ of length $\ell(v) - 1$ is $\hat{v}_{k-1}$. The flat $X$ remains a modular coatom of $I(w \hat{v}_{k-1})$, so by induction $[e, \hat{v}_{k-1}] \cap JW = \{ \hat{v}_i : i = 1, \ldots, k-1\}$. If $v'$ is a general element of $[e, v] \cap JW$, $v' \neq v$, then by Corollary 3.8 of [Deo77] there is a chain $v' = x_0 < x_1 < \cdots < x_m = v$, where $x_i \in JW$ and $\ell(x_i) = \ell(x_{i-1}) + 1$ for $i = 1, \ldots, m$. As just shown, we must have $x_{m-1} = \hat{v}_{k-1}$, and hence $v' \in [e, \hat{v}_{k-1}] \cap JW$. We conclude that $v' = \hat{v}_i$ for some $i$.  

**Remark 5.3.** The proof of Theorem 2.8 implies that $[e, v] \cap JW$ is a chain for $v \in JW$ if and only if the span of every pair of elements $\alpha, \beta \in I(v)$ contains an element of $R_J$. Thus, checking whether or not $[e, v] \cap JW$ is a chain reduces to a linear condition on $I(v)$.

**Remark 4.4.** Recall that an arrangement $\mathcal{A}$ is supersolvable if $L(\mathcal{A})$ has a complete chain of modular elements. Equivalently, $\mathcal{A}$ is supersolvable if and only if $L(\mathcal{A})$ has a modular coatom $X$, and the localization $\mathcal{A}_X$ is supersolvable. Hence Theorem 2.8 implies immediately that $I(w)$ is supersolvable when $w$ has a complete chain BP decomposition.

The proof of Theorem 2.8 given above is quite lengthy. For the purposes of proving Theorems 2.3 and 2.1, we can assume that $u$ and $w$ satisfy the HLSS condition. With these assumptions, there is a much shorter proof of Theorem 2.8 as follows: Suppose $X$ is modular. Because $u$ and $w$ satisfy the HLSS condition,

$$|[e, w]| = |\text{nbc}(I(w))| = |\text{nbc}(I(u))| \cdot (|\mathcal{A} \setminus \mathcal{A}_X| + 1) = |[e, u]| \cdot (\ell(v) + 1).$$

On the other hand, $|[e, w]| \geq |[e, u]| \cdot |[e, v] \cap JW|$, and $|[e, v] \cap JW| \leq \ell(v) + 1$. So $|[e, v] \cap JW| = \ell(v) + 1$, and consequently $JP_v(q) = [\ell(v) + 1]_q$. Furthermore, the multiplication map $[e, u] \times ([e, v] \cap JW) \to [e, w]$ will be surjective, so $w = uv$ is a BP decomposition.

We now use Theorem 2.8 to prove one direction of Theorem 2.3.

**Theorem 5.3.** If $w \in W$ is rationally smooth then $I(w)$ is inductively free, and the coexponents of $I(w)$ are equal to the exponents of $w$.

Our motivation for studying modular coatoms is the following lemma. Although this lemma is likely well-known, we give the proof for completeness.
Lemma 5.6. If $X$ is a modular coatom for $\mathcal{A}$, and $A_X$ is inductively free with coexponents $0, m_1, \ldots, m_{l-1}$, then $\mathcal{A}$ is inductively free with coexponents $m_1, \ldots, m_{l-1}$, $m_l = |\mathcal{A}| - |A_X|$.

Note that 0 is a coexponent of $A_X$ of multiplicity at least one, since the center of $A_X$ is non-trivial.

Proof of Lemma 5.6. The proof is by induction on the size of $A - A_X$. If $H \in A \setminus A_X$, then $\alpha_H$ is independent from $\{\alpha_{H'} : H' \subset A_X\}$. By Lemma 5.2 if $X$ is modular then $A^H \cong A_X / X$, since the restriction of any hyperplane in $A \setminus A_X$ to $H$ agrees with the restriction of a hyperplane in $A_X$. Using Lemma 5.2 again, we see that $X$ is modular in $A \setminus H$, so the lemma follows from the addition theorem.

Proof of Theorem 5.5. The proof is by induction on $|S(w)|$. Clearly the proposition is true if $|S(w)| \leq 1$. Suppose $w$ has a chain BP decomposition. The element $w$ is rationally smooth if and only if $w^{-1}$ is rationally smooth. Since $I(w^{-1}) = -w^{-1}I(w)$, the arrangements $I(w)$ and $I(w^{-1})$ are linearly equivalent. Thus the proposition holds for $w$ if and only if it holds for $w^{-1}$, and we can assume without loss of generality that $w$ has a right chain BP decomposition $w = ws$. Then $u$ is also rationally smooth, and $P_w(q) = \ell(v) + 1]qP_u(q)$, so if the exponents of $u$ are $0, m_1, \ldots, m_{l-1}$, then the exponents of $w$ are $m_1, \ldots, m_{l-1}, \ell(v)$. The coatom $X = \bigcap I(u)$ is modular by Theorem 2.8. The arrangement $I(w)_X$ is simply $I(u)$, which by induction is free with coexponents equal to $0, m_1, \ldots, m_{l-1}$. By Lemma 5.6 the arrangement $I(w)$ is free with coexponents $m_1, \ldots, m_{l-1}, \ell(v)$, since $|I(w) \setminus I(w)_X| = \ell(w) - \ell(u) = \ell(v)$.

This leaves the possibility that $w$ is one of the elements listed in Theorem 3.3. If $w$ is the maximal element of $D_n$, $F_4$, or $F_4$, then as mentioned in the introduction the proposition is a well-known theorem due to Terao [Ter81]. As in the previous paragraph, the proposition holds for $w_{kl}$ if and only if it holds for $w_{kl}^{-1}$. The proof is finished by checking the elements $w_{kl}$ on a computer for $5 \leq l < k \leq 8$ (see the remarks in Subsection 5.1).

Although it’s not necessary to our argument, the result of applying Theorem 2.8 and Lemma 5.6 in the above proof can be explained in terms of the group $W$. Suppose $w$ has a chain BP decomposition $w = ws$ with respect to $J$, where $w$ satisfies the HLSS condition. Since $[e, v] \cap JW$ is a chain, $v$ has a unique right descent $s$, and $[e, vs] \cap JW$ is still a chain. Let $\alpha = -w\alpha_s \in I(w)$ be the inversion corresponding to the appearance of $s$ in the last position of a reduced expression for $w$. If $\beta \in uI(v)$, then as in the proof of Theorem 2.8 there is $\gamma \in I(u)$ such that $\alpha, \beta, \gamma$ are linearly dependent. Hence if $H = \ker \alpha$, then $I(w)^H = I(u)$, while $I(w) \setminus H = I(ws)$. The element $ws$ has length one less than $w$, and chain BP decomposition $w = u(vs)$, so applying the addition theorem repeatedly we find that if $I(u)$ is free, then $I(w)$ is free, and the coexponents of $I(w)$ are the coexponents of $I(u)$ along with $\ell(v)$.

5.1. Inductive freeness of the elements $w_{kl}$. To check if an arrangement $\mathcal{A}$ is inductively free, we go through the hyperplanes $H \subset A$ and see if $A \setminus H$ and $A^H$ are inductively free. We refer to this procedure as the naive algorithm. We refer to the hyperplane $H$ used at each step of this procedure as the pivot. We say that $H \subset A$
is a good pivot if $A^H$ is inductively free with exponents $m_1, \ldots, m_{l-1}$, and $A \setminus H$ is inductively free with exponents $m_1, \ldots, m_l - 1$. Using this terminology, the naive algorithm can be described as follows:

(1) If $A$ is empty, then $A$ is inductively free with all exponents zero.
(2) Otherwise, for each $H \in A$:
   (i) Determine if $A^H$ and $A \setminus H$ are inductively free.
   (ii) If $H$ is a good pivot, stop: $A$ is inductively free, and the exponents can be determined from the exponents of $A^H$ and $A \setminus H$.
(3) If we do not find a good pivot in step 2, then $A$ is not inductively free.

Un fortunately, the number of hyperplanes in the arrangements $I(w_{kl})$, shown in Table 2, make it impossible to apply the above naive algorithm to the arrangements $I(w_{kl})$. To show that the arrangements $I(w_{kl})$ are inductively free, we make three improvements on the naive algorithm:

- Memoization: Whenever we determine the inductive freeness of an arrangement, we record the result so we do not have to traverse through the induction tree a second time.
- Heuristic for pivots: If $A$ is inductively free, then the Poincare polynomial $Q_A(t)$ splits, and the exponents can be recovered from the roots. The Poincare polynomial of an arrangement can be computed relatively quickly, so at each stage of the recursion we keep track of the roots of $Q_A(t)$. Before checking the inductive freeness of $A^H$ and $A \setminus H$, we first make sure that their Poincare polynomials split, and that the roots have the correct form.
- Terminate early: Since all arrangements of rank $\leq 2$ are inductively free, we stop the recursion at rank 2.

Using a C++ program incorporating these features, we can show that $I(w_{87})$ is inductively free in under 30 minutes. However, memoization requires around 26 gigabytes of random access memory (RAM). It is likely that this computation time can be improved. Barakat and Cuntz report times of under 5 minutes for showing that $E_8$ is inductively free, using a good heuristic initial ordering of the roots [BC12].

The complexity of the above approach makes it difficult to guarantee that our program does not have a bug. As in [BC12], we address this problem by using the complicated program outlined above to generate a certificate of inductive freeness for each arrangement. Specifically, we say that a string $C$ is a certificate of inductive freeness for $A$ if $C$ is empty when the effective rank of $A$ is $\leq 2$, and otherwise is of the

| $w$ | $\ell(w) = |I(w)|$ |
|-----|-------------------|
| $w_{65}$ | 28 |
| $w_{75}$ | 38 |
| $w_{85}$ | 50 |
| $w_{76}$ | 46 |
| $w_{86}$ | 58 |
| $w_{87}$ | 75 |

Table 2. Length of the elements $w_{kl}$. 
form $[H, C_1, C_2]$, where $H$ is a good pivot for $A$, $C_1$ is a certificate for $A \setminus H$, and $C_2$ is a certificate for $A^H$. The validity of a certificate can be checked with a much simpler program (although the computation time can be longer, since no memoization is used). Certificates for the elements $w_{kl}$ are available on the author’s webpage. The largest certificate file is 50 megabytes (MB) ($< 5$ MB when compressed).

6. The flattening map

In this section, we recall the definition of the flattening map, and show that the HLSS condition and freeness are preserved by flattening. Let $U$ be a subspace of $V$. The intersection $R_U = R \cap U$ is also a root system, with positive and negative roots $R_U^+ = R^+ \cap U$ and $R_U^- = R^- \cap U$ respectively. Let $W_U$ be the Weyl group of $R_U$. The Weyl group $W_U$ is isomorphic to the parabolic subgroup $\langle t_\beta : \beta \in R_U \rangle$ of $W$, or equivalently can be regarded as the subgroup of $W$ which acts identically on the orthogonal complement of $U$ inside of $V$ [Ste64]. A subset $I$ of $R^+$ is convex if $\alpha, \beta \in I$, $\alpha + \beta \in R^+$ implies that $\alpha + \beta \in I$. A subset $I$ of $R^+$ is coconvex if the complement $R^+ \setminus I$ is convex. Finally, a subset $I$ is biconvex if it is both convex and coconvex. A subset of $R^+$ is biconvex if and only if it is an inversion set $I(w)$ for some $w \in W$. Since biconvexity is a linear condition, the intersection $I(w) \cap U$ is biconvex in $R_U$, and thus there is an element $w' \in W_U$ such that $I(w') = I(w) \cap U$. The element $w'$ is called the flattening of $w$, and will be denoted by $fl_U(w)$ [BP05]. If $U = V_J$, then $fl_U(w) = u$, where $w = uw$ is the left parabolic decomposition of $w$ with respect to $J$.

**Lemma 6.1.** If $u \in W_U$, $w \in W$, then $fl_U(uw) = u fl_U(w)$.

**Proof.** The inversion set of $I(uw)$ is the symmetric difference $I(u) \oplus uI(w)$. When $u \in W_U$, the intersection $uI(w) \cap U = u(I(w) \cap U)$, so that

$$(I(u) \oplus uI(w)) \cap U = (I(u) \cap U) \oplus u(I(w) \cap U) = I(u) \oplus u(I(fl_U(w)))$$

which is the inversion set of $u fl_U(w)$ in $W_U$. \hfill \Box

Recall from the definition of the HLSS condition that a convex order on an inversion set $I(w)$ is an order coming from a reduced expression. An arbitrary total order $\prec$ on $I(w)$ is convex if and only if it satisfies two conditions [Pap94]:

- if $\alpha \prec \beta \in I(w)$, and $\alpha + \beta \in R^+$, then $\alpha \prec \alpha + \beta \prec \beta$.
- if $\alpha \in I(w)$, $\beta \notin I(w)$, and $\alpha - \beta \in R^+$, then $\alpha - \beta \prec \alpha$.

Because these conditions are linear, we immediately get the following lemma.

**Lemma 6.2.** If $\prec$ is a convex order on $I(w)$, then the induced order on $I(fl_U(w)) = I(w) \cap U$ is also convex.

This leads to the main result of this section:

**Proposition 6.3.** Let $U \subset V$ be any subspace. If $w$ satisfies the HLSS condition, then so does $fl_U(w)$. 
Proof. Choose a convex order $\prec$ on $I(w)$, and take the induced convex order on $I(\text{fl}_U(w))$. If $x \in [e, \text{fl}_U(w)]$, we can always find $u = t_{\beta_1} \cdots t_{\beta_m}$, where $\beta_1 \prec \cdots \prec \beta_m$ in $I(\text{fl}_U(w))$, such that $x = u \text{fl}_U(w)$. We want to show that we can take $\{\beta_1, \ldots, \beta_m\}$ to be an nbc-set with respect to the given convex order. Now $x' = uw$ is less than $w$ in Bruhat order, so since the HLSS condition is satisfied we can find an nbc-set $\{\gamma_1, \ldots, \gamma_k\}$ such that $x' = yw$, where $y = t_{\gamma_1} \cdots t_{\gamma_k}$, $\gamma_1 \prec \cdots \prec \gamma_k$. Let $X$ denote the fixed point subspace of $y$. By Theorem 4.5. $X$ is equal to $\text{span}\{\gamma_1, \ldots, \gamma_k\}$. But $y = u$, so using Theorem 4.5 again, we get that $X$ contains $\text{span}\{\beta_1, \ldots, \beta_m\}$. It follows that $\text{span}\{\gamma_1, \ldots, \gamma_k\} \subseteq \text{span}\{\beta_1, \ldots, \beta_m\} \subseteq U$, and hence $y \in W_U$.

Now $\{\gamma_1, \ldots, \gamma_k\}$ is an nbc-set in $I(\text{fl}_U(w))$, and $x = u \text{fl}_U(w) = \text{fl}_U(uw) = \text{fl}_U(yw) = y \text{fl}_U(w)$ by Lemma 6.1. We conclude that the restriction of the map $\phi : 2^I(\text{fl}_U(w)) \to [e, \text{fl}_U(w)]$ to $\text{nbc}(I(\text{fl}_U(w)))$ is surjective. $\square$

Now we turn to freeness. For the inversion hyperplane arrangement, flattening is the same as localization up to a quotient.

Lemma 6.4. Suppose $U \subset V$, and let $U^\perp$ be the orthogonal complement to $U$. Given $w \in W$, let

$$X = \bigcap_{\alpha \in I(w) \cap R_U} \ker \alpha,$$

so that $U^\perp \subset X$. Then the inversion hyperplane arrangement $I(\text{fl}_U(w))$ is equal to the localization $I(w)_X/U^\perp$.

If $U$ is spanned by $I(w) \cap U$, then $U^\perp$ is equal to $X$, and $I(\text{fl}_U(w))$ is simply the quotient of $I(w)_X$ by its center. Conversely, given $X \in L(I(w))$, if we let $U = \text{span}\{\alpha \in I(w) : X \subset \ker \alpha\}$ then $I(\text{fl}_U(w)) \cong I(w)_X/X$.

It is well-known that localization preserves freeness:

Theorem 6.5 (OT92, Theorem 4.37). If $\mathcal{A}$ is a free arrangement, and $X \in L(\mathcal{A})$, then $\mathcal{A}_X$ is also free.

As an immediate consequence of Theorem 6.5 and Lemma 6.4 we have:

Corollary 6.6. If $I(w)$ is free, then so is $I(\text{fl}_U(w))$.

Localization also preserves supersolvability:

Lemma 6.7. If $I(w)$ is supersolvable and $U \subset V$ then $I(\text{fl}_U(w))$ is supersolvable.

Proof. By a theorem of Björner and Ziegler [BZ91], an arrangement $\mathcal{A}$ is supersolvable if and only if there is an order on the set $\{\alpha_H : H \in \mathcal{A}\}$ of defining forms such that every minimal broken circuit with respect to inclusion has size 2.

Given an order with this property on $I(w)$, take the induced order on $I(w) \cap U$. Since $I(w) \cap U$ is linearly closed, a subset of $I(w) \cap U$ is a broken circuit if and only if it is a broken circuit in $I(w)$. Hence all minimal broken circuits in $I(w) \cap U$ have size 2, and $I(\text{fl}_U(w))$ is supersolvable. $\square$
6.1. Proof of main results. Let $R$ and $R'$ be two root systems with Weyl groups $W(R)$ and $W(R')$ respectively, and let $w' \in W(R')$. An element $w \in W(R')$ is said to contain the pattern $(w', R')$ if there is a subspace $U$ of the ambient space of $R$ such that $R_U$ is isomorphic to $R'$, and $f_U(w) = w'$ when $R_U$ is identified with $R'$. If this does not happen for any subspace $U$, then $w$ is said to avoid $(w', R')$. This notion of root system pattern avoidance due to Billey and Postnikov [BP05] generalizes the usual notion of pattern avoidance for permutations.

The main result of [BP05] is that an element $w \in W$ is rationally smooth if and only if it avoids a finite list of patterns in the classical types $A_3$, $B_3$, $C_3$, and $D_4$. In Table 3 below, we list each bad pattern $w'$, along with the Poincare polynomial $Q_{I(w')}(t)$ (shown in factored form) of the inversion arrangement, the number $Q_{I(w')}(1)$ of nbc-sets of the inversion arrangement, and the size of the Bruhat interval $|[e, w']|$. Since rational smoothness depends only on the Coxeter system, not on the choice of root system, the patterns for types $B_3$ and $C_3$ come in pairs $(w', B_3)$ and $(w', C_3)$. If we take $R = B_n$, then $C_n$ is the dual root system $R^\vee = \{\alpha^\vee : \alpha \in R\}$, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. From this fact, it is not hard to see that the inversion arrangement of an element $w \in W(B_n)$ is isomorphic to the inversion arrangement of the same element in $W(C_n)$. Thus we include only one line for each pair of patterns in $B_3$ and $C_3$. The elements $w'$ are written using the following Dynkin diagram labelling, taken from [BP05]:

\[
\begin{align*}
A_3 &: \quad \begin{array}{c} \alpha_1 \end{array} \quad 2 \quad 3 \\
B_3/C_3 &: \quad \begin{array}{c} \alpha_1 \end{array} \quad 2 \quad 3
\end{align*}
\]

| $R'$ | $w'$ | $Q_{I(w')}(t)$ | $|\text{nbc}(I(w'))|$ | $|[e, w']|$ |
|-----|-----|-----------------|-----------------|-----------------|
| $A_3$ | $s_2s_1s_3s_2$ | $(1 + t)(1 + 3t + 3t^2)$ | 14 | 14 |
| $A_3$ | $s_1s_2s_3s_1s_2$ | $(1 + t)(1 + 2t)^2$ | 18 | 20 |
| $D_4$ | $s_2s_1s_3s_4s_2$ | $(1 + t)(1 + 2t)(1 + 2t + 2t^2)$ | 30 | 30 |
| $B_3/C_3$ | $s_2s_1s_3s_2$ | $(1 + t)(1 + 3t + 3t^2)$ | 14 | 14 |
| $B_3/C_3$ | $s_3s_2s_1s_3s_2s_2$ | $(1 + t)(1 + 4t + 5t^2)$ | 20 | 20 |
| $B_3/C_3$ | $s_2s_1s_3s_2s_3$ | $(1 + t)(1 + 4t + 5t^2)$ | 20 | 20 |
| $B_3/C_3$ | $s_3s_2s_1s_3s_2s_3$ | $(1 + t)(1 + 5t + 7t^2)$ | 26 | 26 |
| $B_3/C_3$ | $s_3s_2s_1s_3s_2s_3$ | $(1 + t)(1 + 5t + 7t^2)$ | 26 | 28 |
| $B_3/C_3$ | $s_2s_1s_3s_2s_1s_3s_2$ | $(1 + t)(1 + 5t + 7t^2)$ | 26 | 28 |
| $B_3/C_3$ | $s_3s_2s_1s_3s_2s_3s_2$ | $(1 + t)(1 + 6t + 10t^2)$ | 34 | 36 |
| $B_3/C_3$ | $s_1s_2s_3s_2s_1$ | $(1 + t)(1 + 2t)^2$ | 18 | 20 |
| $B_3/C_3$ | $s_1s_2s_3s_2s_1s_3$ | $(1 + t)(1 + 2t)(1 + 3t)$ | 24 | 28 |
| $B_3/C_3$ | $s_1s_2s_3s_2s_1s_3s_2$ | $(1 + t)(1 + 3t)^2$ | 32 | 36 |
| $B_3/C_3$ | $s_1s_2s_3s_2s_1s_3s_2s_3$ | $(1 + t)(1 + 3t)^2$ | 32 | 36 |
| $B_3/C_3$ | $s_1s_2s_3s_2s_1s_3s_2s_3s_2$ | $(1 + t)(1 + 3t)(1 + 4t)$ | 40 | 42 |
| $B_3/C_3$ | $s_1s_2s_3s_2s_1s_3s_2s_3$ | $(1 + t)(1 + 3t)(1 + 4t)$ | 40 | 44 |

Table 3. Patterns characterizing rational smoothness.
Suppose \( \text{Proof of Theorem 2.3.} \) Suppose \( w \) is rationally smooth. Then by Theorem 5.5, \( \mathcal{I}(w) \) is free with coexponents equal to the exponents of \( \mathcal{I}(w) \). It follows that \( Q_{\mathcal{I}(w)}(t) = \prod_i (1 + m_i t) \), so the number of nbc-sets is \( Q_{\mathcal{I}(w)}(1) = \prod_i (1 + m_i) = P_w(1) = |e, w| \).

Conversely, suppose that \( \mathcal{I}(w) \) is free with coexponents \( d_1, \ldots, d_l \), and \( \prod_i (1 + d_i) = |e, w| \). The product \( \prod_i (1 + d_i) \) is the number of nbc-sets, so \( w \) satisfies the HLSS condition. By Proposition 6.3 and Corollary 6.6 if \( w \) contains \( (w', R') \) then \( \mathcal{I}(w') \) is free and \( w' \) satisfies the HLSS condition. For every element \( w' \) listed in Table 3, either the Poincare polynomial \( Q_{\mathcal{I}(w')}(t) \) does not factor, implying that \( \mathcal{I}(w') \) is not free, or the number of nbc-sets is less than \( |e, w'| \), implying that \( w' \) does not satisfy the HLSS condition. Hence if \( \mathcal{I}(w) \) is free and \( w \) satisfies the HLSS condition, then \( w \) cannot contain any of the patterns listed in Table 3, so \( w \) is rationally smooth. \( \square \)

\text{Proof of Theorem 2.7.} As mentioned in Remark 5.4, if \( w \) has a complete chain decomposition, then \( \mathcal{I}(w) \) is supersolvable. Conversely, suppose that \( w \) is rationally smooth and \( \mathcal{I}(w) \) is supersolvable. The inversion arrangements of the maximal elements of type \( D_4 \) and \( F_4 \) are not supersolvable [HR]. If \( w \) is the maximal elements of type \( D_n \) or \( E_n \), or one of the elements \( w^{-1}_{kl} \), then there is a subset \( J \subset S \) such that \( h_{V_J}^\dagger(w) \) is the maximal element in \( D_4 \). By Lemma 6.7, this cannot happen if \( \mathcal{I}(w) \) is supersolvable. Since \( \mathcal{I}(w_{kl}) \) is linearly equivalent to \( \mathcal{I}(w_{kl}^{-1}) \), \( w \) cannot be one of the elements \( w_{kl} \). By Theorem 3.3, we conclude that \( w \) has a right or left chain BP decomposition \( w = vu \) or \( w = uv \) with respect to some \( J \). Now \( u \) is rationally smooth and \( \mathcal{I}(u) \) is supersolvable by Lemma 6.7 so we can repeat this process with \( u \) to show that \( w \) has a complete chain BP decomposition. \( \square \)

7. Further directions

The positive roots \( R^+ \) of \( \mathcal{R} = A_n \) can be realized as \( \{e_i - e_j : 1 \leq i < j \leq n + 1\} \), where \( e_1, \ldots, e_{n+1} \) is a standard basis of \( \mathbb{R}^n \). Subsets of \( R^+ \) can be identified with graphs on vertex set \( 1, \ldots, n + 1 \) by identifying \( e_i - e_j \) with the edge \( ij \). For this reason, a union of hyperplanes \( H = \ker \alpha \), \( \alpha \) a positive root of \( A_n \), is called a graphic arrangement. A theorem of Stanley states that a graphic arrangement is free if and only if the corresponding graph is chordal, meaning that every cycle of length \( \geq 4 \) has a chord [CDF+09].

\text{Lemma 7.1.} If \( I \) is a biconvex set in \( A_n \), then every chordless cycle in the corresponding graph has length \( \leq 4 \).

\text{Proof.} Suppose \( C \) is a cycle of length \( \geq 5 \) in the graph \( G \) corresponding to \( I \). Let \( i \) be the smallest vertex in \( C \), and let \( j < k \) be the two vertices adjacent to \( i \) in \( C \). Finally, let \( m \neq k \) be the other vertex adjacent to \( j \) in \( C \). Since \( C \) has length \( \geq 5 \), \( m, j, i, k \) are all distinct, and any edge between these vertices which is not already in \( C \) will be a chord.
If \( m > j \), then the root corresponding to edge \( jm \) is \( e_j - e_m \), so \( e_j - e_m \in I \). Similarly \( e_i - e_j \in I \). Since \( i < m \), \( e_i - e_m = (e_j - e_i) + (e_j - e_m) \) is a positive root, and since \( I \) is biconvex, \( e_i - e_m \in I \). Hence \( C \) has a chord in \( G \).

If \( m < j \), then \( e_m - e_k \) is a positive root. Since \( I \) is coconvex and \( e_i - e_k = (e_m - e_k) + (e_i - e_m) \) belongs to \( I \), one of \( e_m - e_k \) or \( e_i - e_m \) must belong to \( I \), and again \( C \) will have a chord in \( G \). \( \square \)

If \( R \) is of type \( A \), then any subsystem \( R_U \) also has type \( A \) (or is a product of subsystems of type \( A \)). Hence pattern avoidance conditions in type \( A \) are usually written in terms of permutations. For example, Billey and Postnikov’s root system pattern avoidance criterion simplifies in type \( A \) to the earlier pattern avoidance criterion of Lakshmibai and Sandhya:

**Theorem 7.2 (\cite{LS90}).** If \( w \) is a permutation in \( A_n \), then \( w \) is rationally smooth if and only if \( w \) avoids the permutations \( 3412 \) and \( 4231 \).

Using Lemma 7.1 it is easy to give a pattern characterization of freeness in type \( A \):

**Proposition 7.3.** If \( w \) is a permutation in \( A_n \), then \( I(w) \) is free if and only if \( w \) avoids the permutation \( 3412 \).

**Proof.** Let \( G \) be the graph corresponding to \( I(w) \). By Stanley’s theorem and Lemma 7.1 \( I(w) \) is free if and only if every 4-cycle in \( G \) has a chord. If \( U \) is a subspace of the ambient space, let \( G_U \) denote the subgraph of \( G \) with edge set \( \{ij : e_i - e_j \in I(w) \cap U \} \). If connected vertices, \( G_U \) is the corresponding graph to \( I(\text{fl}_U(w)) \). If \( C \) is a 4-cycle, then the space \( U = \text{span}(e_i - e_j : ij \in C) \) has dimension 3, so \( I(w) \) is free if and only if \( I(\text{fl}_U(w)) \) is free for all subspaces \( U \) of rank 3.

The inversion set \( I(3412) = \{e_2 - e_3, e_1 - e_3, e_2 - e_4, e_1 - e_4\} \), and the corresponding graph is the 4-cycle \( 2 - 3 - 1 - 4 \). It is not hard to check that this is the only inversion set in \( A_3 \) with a chordless 4-cycle. \( \square \)

Note that the reduced word for \( 3412 \) is \( s_2s_1s_3s_2 \). Proposition 7.3 makes it clear that the freeness of \( I(w) \) is not equivalent to rational smoothness. For example, in \( A_3 \) the permutation \( 4231 \) has reduced word \( s_1s_2s_3s_2s_1 \) and inversion set \( \{e_1 - e_2, e_3 - 3_4, e_1 - e_3, e_2 - e_4, e_1 - e_4\} \), which corresponds to the complete graph minus an edge. Hence \( I(4231) \) is free with exponents 1, 2, 2 (see Table 3).

In type \( A \), The HLSS condition can also be can also be characterized by pattern avoidance:

**Theorem 7.4 (\cite{HLSS09}).** If \( w \) is a permutation in type \( A_n \), then \( w \) satisfies the HLSS condition if and only if \( w \) avoids \( 4231, 35142, 42513, \) and \( 351624 \).

The patterns 35142, 42513, and 351624 all contain the pattern 3412. By combining Theorem 7.2, Proposition 7.3, and Theorem 7.4 we get another proof that \( w \) is rationally smooth if and only if \( I(w) \) is free and \( w \) satisfies the HLSS condition. However, this approach does not show that the coexponents and exponents agree.

Proposition 6.3 and Corollary 6.6 suggest that it should be possible to extend Proposition 7.3 and Theorem 7.4 to arbitrary finite type. The problem of finding a pattern avoidance criterion for freeness will be addressed in \cite{Slo13}. This leaves:
**Problem 7.5.** Find a root system pattern avoidance criterion for the HLSS condition in arbitrary finite type.

Finally, we have the following question:

**Question 7.6.** Do Theorems 2.1, 2.3 and 2.7 hold in arbitrary (finite) Coxeter groups? In particular, does the Poincare polynomial $Q_{\pi(w)}(t)$ have a linear factorization when $P_{\pi(w)}(q)$ is palindromic?

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