Higher regularity of solutions
to the singular $p$-Laplacean parabolic system

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Abstract - We study existence and regularity properties of solutions to the singular $p$-Laplacean parabolic system in a bounded domain $\Omega$. The main purpose is to prove global $L^r(\varepsilon, T; L^q(\Omega))$, $\varepsilon \geq 0$, integrability properties of the second spatial derivatives and of the time derivative of the solutions. Hence, for suitable $p$ and exponents $r, q$, by Sobolev embedding theorems, we deduce global regularity of $u$ and $\nabla u$ in Hölder spaces. Finally we prove a global pointwise bound for the solution under the assumption $p > \frac{2n}{n+2}$.

Keywords: parabolic system, singular $p$-Laplacean, higher integrability, global regularity.

M.R.:

1. Introduction

This note deals with the existence and regularity of solutions to a singular non-linear, second order, parabolic system, under Dirichlet boundary conditions, of the type

$$u_t - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0, \quad \text{in } (0, T) \times \Omega,$$

$$u(t, x) = 0, \quad \text{on } (0, T) \times \partial \Omega,$$

$$u(0, x) = u_0(x), \quad \text{on } \{0\} \times \Omega,$$

where the $p$-growth exponent belongs to the interval $(1, 2)$. Here we assume that $\Omega$ is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, whose boundary is $C^2$-smooth, and $u : (0, T) \times \Omega \to \mathbb{R}^N$, $N \geq 1$, is a scalar or a vector field. The data $u_0$ belongs to $L^2(\Omega)$.

Our main purpose is to prove “global”, that is on the cylinder $(\varepsilon, T) \times \Omega$, $\varepsilon \geq 0$, $L^r(\varepsilon, T; L^q(\Omega))$ integrability properties of the second spatial derivatives and of the time derivative of solutions to problem (1.1). Hence, for suitable $p \in (1, 2)$ and exponents $r, q$, by Sobolev embedding theorems, we deduce “global” regularity of $u$ and $\nabla u$ in Hölder spaces. Our results are developed under two main assumptions. The former is that we consider a $p$-parabolic system with $p$-Laplacean operator and not more general elliptic operators, whose structural properties have the $p$-Laplacean as prototype. The latter concerns the bounded domain. These assumptions are made just to develop in a simpler way a new technique that leads to the high integrability of second derivatives and of the first time derivative, which are the chief results of this paper, and, as far as we know, they are new in the literature. The proof is performed under the assumption of

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an homogeneous right-hand side. This is not a limit of the technique employed, which works for a nonnull right-hand side as well, but a choice to develop the proofs in a more readable way. As a consequence of the integrability properties, by embedding theorems, we deduce the Hölder regularity of the solutions to problem (1.1). This important topic has been developed in a wide literature. We refer to the monograph [8] and to the more recent essay [12] for a general review, and, for regularity results, more specifically to the papers [9, 10, 11]. For solutions to more general singular or degenerate parabolic systems, as far as the local integrability properties are concerned, we quote the papers [1, 14, 18, 22, 23], and, as far as the Hölder regularity is concerned, we quote the paper [2] and the very complete and recent memoirs [13].

Before giving the statement of our results, we would like to say a few words about the technique. Firstly we point out that we do not prove that a weak solution has more regularity properties, but we prove the existence of regular solutions, and, as a consequence of the uniqueness of weak solutions, the same regularity is enjoyed by the weak solution too. The existence theorem is proved by using the Galerkin method in the way suggested by Prodi in [21], where the special basis of eigenfunctions is proposed, in our case of the Laplacean operator. In this way, provided that the initial data \( u_o \in L^2(\Omega) \), we are able to furnish a solution to problem (1.1) which has more regularity of the usual weak solutions. Indeed, we obtain \( D^2u(t,x) \in L^2(\varepsilon,T;L^p(\Omega)) \), \( \varepsilon \geq 0 \). This gives an advantage in establishing further regularity properties of the solutions. As far as the integrability properties are concerned, we employ a duality technique which is a suitable modification of the one employed in [20] to estimate the second derivatives in (space-time) anisotropic Sobolev spaces. For this task we define suitable adjoint problems of (1.1). This approach firstly gives estimates for the time derivatives \( u_t \), subsequently, as in [20], viewing the first equation of (1.1) as an equation of elliptic type with data \( f = u_t \), allows us to establish estimates of the second derivatives.

To better explain our results, we shortly introduce the space \( V = W^{1,p}_0 \cap L^2(\Omega) \). \( V \) is a reflexive Banach space endowed with the norm \( \| \cdot \|_V = \| \cdot \|_1,p + \| \cdot \|_2 \). Moreover we denote by \( V' = W^{-1,p'}(\Omega) + L^2(\Omega) \) its strong dual. For further notations and positions see the next section.

**Definition 1.1** Let \( u_o \in L^2(\Omega) \). A field \( u: (0,T) \times \Omega \to \mathbb{R}^N \) is said a solution of system (1.1) if

\[
\begin{align*}
    u & \in L^p(0,T;V) \cap C(0,T;L^2(\Omega)), \quad t^{1/p} \nabla u \in L^\infty(0,T;L^p(\Omega)), \\
    u_t & \in L^p(0,T;V'), \quad t u_{tt} \in L^\infty(0,T;L^2(\Omega)), \quad t^{p/2} \nabla u_t \in L^2(0,T;L^p(\Omega)), \\
    \int_0^t [(u,\psi) - (|\nabla u|^{p-2}\nabla u,\nabla \psi)] \, d\tau = (u(t),\psi(t)) - (u_o,\psi(0)), \\
    \quad \forall \psi \in W^{1,2}(0,T;L^2(\Omega)) \cap L^p(0,T;V),
\end{align*}
\]

and

\[
\lim_{t \to 0^+} \|u(t) - u_o\|_2 = 0.
\]

The above definition is different from the usual formulation of a weak solution to problem (1.1), actually the properties indicated for a solution are wider than the ones given in [19] and considered by other authors, for instance in [8, 13]. This is in connection
with the fact that we are able to prove that, for all $p \in (1, 2)$, the set of solutions is not empty, provided that $u_\circ \in L^2(\Omega)$. Of course, a solution in the sense of the Definition 1.1 is a weak solution in the sense given in [19]. Hence the uniqueness of the weak solution makes unique the functional class of existence and the related properties of the solutions.

We set
\[
p_\circ := \max \left\{ \frac{3}{2}, \frac{2n}{n+2} \right\},
\]
and
\[
\mathcal{P} := 2 - \frac{1}{H},
\]
with the constant $H$ introduced in (6.2).

**Theorem 1.1** Assume that $u_\circ$ belongs to $L^2(\Omega)$. Then there exists a unique solution $u$ to problem (1.1) in the sense of Definition 1.1. Moreover, if $p > p_\circ$, then
\[
t^{\alpha_1} \nabla u \in C(0, T; L^2(\Omega)),
\]
and
\[
t^{\alpha_2} u \in L^2(0, T; W^{2,p}(\Omega)),
\]
where $\alpha_1 = \frac{2}{2p-n(2-p)}$ and $\alpha_2 = \alpha_1 + \frac{2-p}{2p}$.

**Corollary 1.1** Assume that $u_\circ$ belongs to $W_0^{1,2}(\Omega)$. Then there exists a unique solution $u$ to problem (1.1) in the sense of Definition 1.1. Moreover, for $p > \frac{3}{2}$,
\[
\nabla u \in C(0, T; L^2(\Omega)),
\]
and
\[
u \in L^2(0, T; W^{2,p}(\Omega)).
\]

Theorem 1.1 says in particular that if the initial data is just in $L^2(\Omega)$, then $\nabla u$, $D^2 u$, $u_t$ and $\nabla u_t$ have a singularity in the origin $t = 0$, that we explicitly compute. If the data is more regular, as in Corollary 1.1, we remove the singularity in $t = 0$. This result completely agrees with the known results for the linear case, with an obvious rescaling due to the exponent $p$. In the case of a more regular initial data, we limit ourselves to the claims in Corollary 1.1 for the sake of brevity. However under the assumption $u_\circ \in W_0^{1,2}(\Omega)$, we could give further regularity properties, that we consider unessential for the development of the paper. We point out that these cannot be considered like results on the asymptotic behavior of the solution, since, as it is well known, for all $p \in (1, 2)$, if $\Omega$ is bounded there is the extinction of the solution in a finite time (cf. [8]).

We also observe that the introduction of a force term $f \in L^p(0, T; V')$ on the right-hand side would be easy to handle and would lead to the same $L^2(\varepsilon, T; W^{2,p}(\Omega))$, $\varepsilon > 0$, integrability for second derivatives. Obviously, under this weak assumption on $f$, the solution as in Definition 1.1 would lost the regularity properties of $u_t$ given in (1.3)$_{1,2}$.

The next theorem and its corollary are our chief results and concern the “global” high regularity of the solutions furnished by Theorem 1.1. For the definition of the Hölder seminorm in (1.11) we refer to the next section. We set
\[
p_1 := \frac{7(n-2)+1-\sqrt{4(n-1)^2-3}}{3(n-2)}.
\]
Theorem 1.2 Let \( n \geq 3 \). Let \( p > \max\{p_0, p_1\} \), and let \( q \in \left[ \frac{7 - 3p}{4 - 2p}, \frac{7 - 3p}{4 - 2p} \right] \). Assume that \( \Omega \) is a convex domain. If \( u(t, x) \) is the solution of Theorem 1.1, then, for all \( \varepsilon > 0 \),
\[
    u \in L^\infty(\varepsilon, T; W^2,q(\Omega)) \quad \text{with} \quad u_t \in L^\infty(\varepsilon, T; L^q(\Omega)),
\]
with \( \hat{q} = \frac{nq(p - 1)}{n - q(2 - p)} \) if \( q < n \), \( \hat{q} < n \) if \( q = n \), and \( \hat{q} = q \) if \( q > n \).

Under the further assumption \( p > \overline{p} \), the same result holds for \( \Omega \) non-convex domain.

Theorem 1.3 Let \( n = 2 \). Let \( p > \frac{3}{2}, \) and let \( q \in (2, \frac{7 - 3p}{4 - 2p}] \). Assume that \( \Omega \) is a convex domain. If \( u(t, x) \) is the solution of Theorem 1.1, then, for all \( \varepsilon > 0 \),
\[
    u \in L^\infty(\varepsilon, T; W^2,q(\Omega)) \quad \text{with} \quad u_t \in L^\infty(\varepsilon, T; L^q(\Omega)).
\]

Under the further assumption \( p > \overline{p} \), the same result holds for \( \Omega \) non-convex domain.

We set
\[
p_2 := \frac{2n + 7 - \sqrt{(2n - 7)^2 + 8n}}{6}.
\]

Corollary 1.2 Assume that \( \Omega \) is a convex domain and \( u(t, x) \) is the solution of Theorem 1.1. Let \( p > \frac{3}{2} \) for \( n = 3 \) and \( p > p_2 \) for \( n > 3 \), and \( q_0 \in (\frac{n}{p}, n) \). Then, for each \( t > t_0 > 0 \) we get
\[
    \left[ u \right]_{\lambda_0, t, x} \leq c \left( t_0^{-1 - \gamma_0} \| u_0 \|_2^{(2-p)\gamma_0 + 1} + t_0^{-\frac{\gamma_0 + 1}{p}} \| u_0 \|_2^{(2-p)\gamma_0 + 1} \right),
\]
where \( \lambda_0 = \frac{2 - \frac{n}{q_0}}{\hat{q}_0} \), with \( \hat{q}_0 = \frac{nq(p - 1)}{n - q(2 - p)} \) if \( q_0 < n \), \( \hat{q}_0 \in (\frac{5}{2}, n) \) if \( q_0 = n \), and \( \gamma_0 = \frac{n(q_0 - 2)}{q_0(2p - 2n + np)} \). Moreover, let \( p > \max\{p_0, \frac{4n - 7}{2n - 3}\} \) and \( q_1 \in (n, \frac{7 - 3p}{4 - 2p}] \). Then, we get
\[
    \left[ \nabla u \right]_{\lambda_1, t, x} \leq c \left( t_0^{-1 - \gamma_1} \| u_0 \|_2^{(2-p)\gamma_1 + 1} + t_0^{-\frac{\gamma_1 + 1}{p}} \| u_0 \|_2^{(2-p)\gamma_1 + 1} \right)\)
\]
where \( \lambda_1 = 1 - \frac{n}{q_1} \) and \( \gamma_1 = \frac{n(q_1 - 2)}{q_1(2p - 2n + np)} \). The constant \( c \) in (1.10)-(1.11) is independent of \( t_0 \) and \( u_0 \).

Under the further assumption \( p > \overline{p} \), the same results hold for \( \Omega \) non-convex domain.

Corollary 1.3 Let \( n = 2 \). Let \( p > \frac{3}{2} \), and \( q \in (2, \frac{7 - 3p}{4 - 2p}] \). Assume that \( \Omega \) is a convex domain and \( u(t, x) \) is the solution of Theorem 1.1. Then, for each \( t > t_0 > 0 \) we get
\[
    \left[ \nabla u \right]_{\lambda, t, x} \leq c \left( t_0^{-1 - \gamma_1} \| u_0 \|_2^{(2-p)\gamma_1 + 1} + t_0^{-\frac{\gamma_1 + 1}{p}} \| u_0 \|_2^{(2-p)\gamma_1 + 1} \right),
\]
where \( \lambda = 1 - \frac{2}{q} \), and \( \gamma_1 = \frac{(q-2)}{2q(p-1)} \). The constant \( c \) in (1.12) is independent of \( t_0 \) and \( u_0 \).

Under the further assumption \( p > \overline{p} \), the same results hold for \( \Omega \) non-convex domain.

Our result of “high regularity” is expressed by means of the existence of the second derivatives, that is estimates (1.8) and (1.9). These estimates for suitable \( p \) and \( q \) imply the H"older regularity of \( u \) and \( \nabla u \). We point out that our H"older exponent \( \lambda \) depends on \( p, n \). This is in accordance with the result given in [4]. However we are not able to compare the two exponents since, as far as we know, in [4] a functional dependence for \( \lambda \) on \( p \) is not given.
We have recently seen paper [3], where, for \( n \geq 3 \), under the assumption of \( f \in L^p(0,T; W^{-1,p}(\Omega)) \cap L^2(0,T; L^2(\Omega)) \), it is proved that a weak solution of (1.1) belongs to \( L^{2(p-1)}(0,T; W^{2,r}(\Omega)) \) with \( r = \frac{2n(p-1)}{n-2(p-2)} \) under suitable constraints on \( p, r \). Hence, for \( n = 3 \), \( r \) belongs to \( (p,0) \), and, for \( n > 3 \), \( r < p \).

Finally, we prove a global pointwise bound for the solution.

**Theorem 1.4** Let \( u \) be the solution of (1.1) corresponding to \( u_\circ \in L^\infty(\Omega) \). Then

\[
\|u(t)\|_\infty \leq \|u_\circ\|_\infty.
\]

Moreover, if \( p > \frac{2n}{n+2} \) then, corresponding to an initial data \( u_\circ \in L^q(\Omega) \), for some \( q \in [2, +\infty] \) one has

\[
\|u(t)\|_\infty \leq c \|u_\circ\|_q \|u_\circ\|_2^{2(2-p)\beta} t^{-\frac{2p}{\beta}q}, \quad \forall t > 0,
\]

with \( \beta := \frac{n}{p(n+2)-2n} \).

The precise aim of Theorem 1.4 is to prove a \( L^\infty(\Omega) \)-bound for a weak solution with no investigations of high regularity properties of solutions. Of course, estimate (1.14) holds for \( t \in (0,T) \), where \( T \) is the instant of extinction. Analogous results are proved in [10, 12, 8] for equations, and locally in [5] for systems.

Finally, we shortly describe the plan of the paper and the strategy of the proofs. In Sec. 2, we introduce some notations and auxiliary lemmas. In particular we prove Lemma 2.1, which is an important tool to estimate the second derivatives, and semigroup properties for a suitable linear parabolic system with regular coefficients. In Sec. 3 we introduce two approximating systems, (3.1) and (3.2). They are both non-singular quasi-linear systems, for which the Galerkin approximation method, and in particular a suitable choice of the basis for this approximation, together with weighted estimates in \( W^{m,r}(\Omega) \) and Lemma 2.1 are the essential tools to get existence and regularity. Since the proof of the basic properties of these systems is standard for people acquainted with the Galerkin method, we confine it in the appendix, at the end of the paper. The proof of our existence result is then given in Sec. 4. The crucial step in the proof of the higher integrability of second derivatives is the derivation of an \( L^\infty(0,T; L^q(\Omega)) \)-estimate on the time derivative \( u_t \), which is done in Sec. 5, by using the semigroup properties of Sec. 2. Once this regularity has been derived, the corresponding \( L^\infty(0,T; W^{2,\tilde q}(\Omega)) \)-integrability of the second derivatives is obtained in Sec 6, and it relies on the regularity results on the \( p \)-Laplacean elliptic system studied in [7]. For \( \tilde q > n \) this result gives us the H"older continuity of the solution. Finally, the maximum modulus theorem is proved in Sec. 7, employing a duality arguments.

**2. Notations and some auxiliary results**

Throughout the paper we denote by \( p \) the growth exponent, with \( p \in (1,2) \). We denote by \( \Omega \subset \mathbb{R}^n \) a bounded domain whose boundary \( \partial \Omega \) is \( C^2 \)-smooth. For a function \( v(t,x) \), by \( \partial_k v \) and \( \partial_t v \) we mean \( \frac{\partial}{\partial x_k} v(t,x) \) and \( \frac{\partial}{\partial t} v(t,x) \), respectively. We set \( v \cdot \nabla v = v_k \partial_k v \), and \( \nabla v \cdot v = v \cdot \partial_t v \). For \( m \in \mathbb{N} \cup \{0\} \), \( C^m(\Omega) \) (\( C(\Omega) \) for \( m = 0 \)) is the usual space of functions which are bounded and uniformly continuous on \( \Omega \) together with their derivatives up
to the order $m$. The norm in $C^m(\Omega)$ is denoted by $| \cdot |_m := \sum_{|\alpha|=0}^{m} \sup_{\Omega} |D^\alpha u(x)|$. For $\lambda \in (0, 1)$, by $C^{m,\lambda}(\Omega)$ we mean the set of functions of $C^m(\Omega)$ such that, for $|\alpha| = m$, $D^\alpha u \in C^{0,\lambda}(\Omega)$, that is $|D^\alpha u|_\lambda + |D^\alpha u|_\lambda < \infty$, where $[\cdot]_\lambda$ is the Hölder seminorm. The norm of an element of $C^{m,\lambda}(\Omega)$ is denoted by $|u|_{m,\lambda,\Omega} := |u|_m + |D^\alpha u|_\lambda, |\alpha| = m$. We denote by $C^m(a, b; X)$ the Banach space (endowed with the natural norm) of all functions bounded and continuous on $(a, b) \subseteq \mathbb{R}$ with value in a Banach space $X$, together with all derivatives $D^k$, $k \leq m$. For $\lambda \in (0, 1)$ we set

$$
[g]_{\lambda,t,x} = \sup_{t \in (0,T)} \frac{|g(t, \tau) - g(t, \bar{\tau})|}{|x - \bar{x}|^\lambda} + \sup_{\tau \in (0,T)} \frac{|g(\tau, x) - g(\tau, \bar{x})|}{|\tau - \bar{\tau}|^\lambda}, \quad (2.1)
$$

provided that the right-hand side is finite. The $L^p$-norm is denoted by $\| \cdot \|_p$ and, if $m \geq 0$, the $W^{m,p}$ and $W^{m,p}_0$-norms are denoted by $\| \cdot \|_{m,p}$. We introduce the space $V = W^{1,p}_0 \cap L^2(\Omega)$. $V$ is a reflexive Banach space endowed with the norm $\| \cdot \|_V = \| \cdot \|_{1,p} + \| \cdot \|_2$, where $\| \cdot \|_{1,p}$ represents a semi-norm on $V$. Moreover we denote by $V' = W^{-1,p'}(\Omega) + L^2(\Omega)$ its strong dual. Note that $W^{0,p}_0 \subseteq L^2(\Omega)$ only if $p \geq \frac{2n}{n+2}$. On the other hand $V$ is dense and continuously embedded in $L^2$ and $V \subset L^2 \subset V'$. Let $q \in [1, \infty)$, let $X$ be a Banach space with norm $\| \cdot \|_X$. We denote by $L^q(a, b; X)$ the set of all function $f : (a, b) \to X$ which are measurable and such that the Lebesgue integral $\int_a^b \| f(\tau) \|_X^q d\tau = \| f \|_{L^q(a, b; X)} < \infty$.

As well as, if $q = \infty$ we denote by $L^\infty(a, b; X)$ the set of all function $f : (a, b) \to X$ which are measurable and such that $\ess \sup_{t \in (a, b)} \| f(t) \|_X^q = \| f \|_{L^\infty(a, b; X)} < \infty$.

In the remaining part of this section we give some preliminary results, which represent fundamental tools in our proofs. The first is the following lemma, which, for $p = 2$, gives a well known estimate (see [15] and [16]).

**Lemma 2.1** Let $\mu > 0$. Assume that $v \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$. Then, for any $\eta > 0$,

$$
\left\| (\mu + |\nabla v|^2)^{\frac{p-2}{2}} D^2 v \right\|_2 \leq C_1 \left\| (\mu + |\nabla v|^2)^{\frac{p-2}{2}} \Delta v \right\|_2 + \frac{C_2}{\eta} \left( \|\nabla v\|_p^p + \mu^\frac{p}{2} |\Omega| \right)^{\frac{1}{2}}.
$$

where

$$
C_1 := \left( \frac{p}{p(p-1)^{2-\eta}} \right)^{\frac{1}{2}}.
$$

If $\Omega$ is a convex domain the inequality holds with $C_2 = 0, \eta = 0$.

**Proof.** We prove the result for sufficiently smooth functions. It can be extended to functions in $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ by density arguments. So, let $v$ be a function which is
continuously differentiable three times and vanishes on \( \partial \Omega \). Integration by parts gives

\[
\int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |\Delta v|^2 \, dx = - \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} \frac{\partial \Delta v}{\partial x_k} \cdot \frac{\partial v}{\partial x_k} \, dx \\
-(p-2) \int (\mu + |\nabla v|^2)^{\frac{(p-4)}{2}} \Delta v \cdot \frac{\partial v}{\partial x_k} \nabla v \cdot \frac{\partial \nabla v}{\partial x_k} \, dx + \int_{\partial \Omega} (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} \Delta v \cdot \frac{\partial v}{\partial n} \, d\sigma
\]

By applying Hölder’s and Cauchy’s inequalities to the last integral one readily has

\[
\int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |\Delta v|^2 \, dx = \int (\mu + |\nabla v|^2)^{\frac{(p-4)}{2}} \Delta v \cdot \frac{\partial v}{\partial x_k} \nabla v \cdot \frac{\partial \nabla v}{\partial x_k} \, dx + (p-2) \int (\mu + |\nabla v|^2)^{\frac{(p-4)}{2}} \left( \frac{\partial \nabla v \cdot \nabla v}{\partial x_j} \right)^2 \, dx
\]

Denote the boundary integral in the previous estimate by \( I_{\partial \Omega} \). Since \( p > 1 \), one can estimate the right-hand side as follows

\[
(p-1) \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |D^2 v|^2 \, dx \\
\leq \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |\Delta v|^2 \, dx + (2-p) \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |D^2 v| |\Delta v| \, dx - I_{\partial \Omega}.
\]

\( \Omega \) convex - By using the arguments in \([15]\), based on a localization technique, one can show that the boundary integral \( I_{\partial \Omega} \) is non-negative if \( \Omega \) is convex. Therefore from (2.2) one gets

\[
(p-1) \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |D^2 v|^2 \, dx \\
\leq \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |\Delta v|^2 \, dx + (2-p) \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |D^2 v| |\Delta v| \, dx.
\]

By applying Hölder’s and Cauchy’s inequalities to the last integral one readily has

\[
\left[ p - 1 - \frac{\varepsilon}{2} (2-p)^2 \right] \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |D^2 v|^2 \, dx \\
\leq \left( 1 + \frac{1}{2\varepsilon} \right) \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |\Delta v|^2 \, dx,
\]

hence

\[
\int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |D^2 v|^2 \, dx \leq C(\varepsilon) \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |\Delta v|^2 \, dx,
\]

with

\[
C(\varepsilon) := \frac{1+2\varepsilon}{\varepsilon[2(p-1)-\varepsilon(2-p)^2]}.
\]

By an easy computation, one can verify that the minimum of \( C(\varepsilon) \) equals \( 1/(p-1)^2 \) and it is attained for \( \varepsilon = (p-1)/(2-p) \). Therefore we get

\[
\int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |D^2 v|^2 \, dx \leq \frac{1}{(p-1)^2} \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |\Delta v|^2 \, dx.
\]

\( \Omega \) non-convex - If \( \Omega \) is not convex, starting from (2.2) and using the above arguments (see (2.3)–(2.5)) we have

\[
\int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |D^2 v|^2 \, dx \leq \frac{1}{(p-1)^2} \int (\mu + |\nabla v|^2)^{\frac{(p-2)}{4}} |\Delta v|^2 \, dx - \frac{2}{p(p-1)} I_{\partial \Omega}.
\]
Again following [15], the integral \( I_{\partial \Omega} \) can be estimated as follows

\[
I_{\partial \Omega} \leq C \int_{\partial \Omega} (\mu + |\nabla v|^2)^{\frac{p-2}{2}} \left( \frac{\partial v}{\partial n} \right)^2 \, d\sigma \leq C \| \nabla v \|_{L^p(\partial \Omega)}^p
\]

\[
\leq C \int_{\Omega} |\nabla v|^p \, dx + C \int_{\Omega} |\nabla v|^{p-1} |D^2 v| \, dx. \tag{2.7}
\]

Multiplying and dividing by \((\mu + |\nabla v|^2)^{\frac{2-p}{2}}\), using Hölder’s and then Cauchy’s inequalities we have, for any \( \eta > 0 \),

\[
\int_{\Omega} |\nabla v|^{p-1} |D^2 v| \, dx \leq \left( \int_{\Omega} (\mu + |\nabla v|^2)^{\frac{2-p}{2}} |\nabla v|^{2(p-1)} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (\mu + |\nabla v|^2)^{\frac{2-p}{2}} |D^2 v|^{2} \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2\eta} \int_{\Omega} (\mu + |\nabla v|^2)^{\frac{2-p}{2}} \, dx + \frac{\eta}{2} \int_{\Omega} (\mu + |\nabla v|^2)^{\frac{2-p}{2}} |D^2 v|^{2} \, dx,
\]

for any \( \eta > 0 \). Therefore,

\[
I_{\partial \Omega} \leq C \left( \| \nabla v \|_{L^p}^p + \frac{1}{2\eta} \| (\mu + |\nabla v|^2)^{\frac{2-p}{2}} \|_{L^p}^p + \frac{\eta}{2} \| (\mu + |\nabla v|^2)^{\frac{2-p}{2}} D^2 v \|_{L^p}^2 \right).
\]

By replacing the above estimate in (2.6) we get

\[
\int_{\Omega} (\mu + |\nabla v|^2)^{\frac{2-p}{2}} |D^2 v|^{2} \, dx \leq \frac{1}{(p-1)^2} \int_{\Omega} (\mu + |\nabla v|^2)^{\frac{2-p}{2}} |\Delta v|^2 \, dx
\]

\[
+ \frac{\eta}{p(p-1)} \int_{\Omega} (\mu + |\nabla v|^2)^{\frac{2-p}{2}} |D^2 v|^{2} \, dx + C \left( \| \nabla v \|_{L^p}^p + \frac{1}{2\eta} \| (\mu + |\nabla v|^2)^{\frac{2-p}{2}} \|_{L^p}^p \right),
\]

which easily gives the result.

Our second kind of results is concerned with the analysis of semigroup properties for the following parabolic system with regular coefficients

\[
\begin{align*}
\varphi_t - \nu \Delta \varphi - \nabla \cdot (B_\eta(s,x) \nabla \varphi) &= 0, & \text{in } (0,t) \times \Omega, \\
\varphi(s,x) &= 0, & \text{on } (0,t) \times \partial \Omega, \\
\varphi(0,x) &= \varphi_0(x), & \text{on } \{0\} \times \Omega,
\end{align*}
\tag{2.8}
\]

with \( \nu \geq 0 \) and \( B_\eta(s,x) = (B_\eta)_{i\alpha,j\beta}(s,x) \) satisfying the following conditions

\[
B_\eta \text{ is continuous in } [0,t] \times \overline{\Omega},
\]

\[
\| B_\eta \|_\infty = \max_{i\alpha,j\beta} \| (B_\eta)_{i\alpha,j\beta} \|_\infty < +\infty,
\tag{2.9}
\]

\[
B_\eta(s,x) \text{ is uniformly elliptic}.
\]

Lemma 2.2 Assume that \( \nu > 0 \) and let \( \varphi_0(x) \in C_0^\infty(\Omega) \). Then, there exists a unique solution \( \varphi \) of (2.8), such that \( \varphi \in L^2(0,t;W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)), \varphi_x \in L^2(0,t;L^2(\Omega)) \). The existence and regularity follow from well known regularity results for linear parabolic systems with uniformly continuous and bounded coefficients. We refer, for instance, to [17], Theorem IV.9.1.
For \( p \in (1, 2) \) and \( \mu > 0 \), set
\[
a(\mu, v) := (\mu + |(\nabla v)|^2)^{\frac{\mu - 1}{2}}, \tag{2.10}
\]
and
\[
a_{\eta}(\mu, v) := (\mu + |J_\eta(\nabla v)|^2)^{\frac{\mu - 1}{2}}, \tag{2.11}
\]
with \( J_\eta \) space-time Friederich’s mollifier, and assume that
\[
s \|(\mu + |\nabla v|^2)^{\frac{1}{2}}\|_p^2 \leq M, \quad \forall s \in [0, t], \tag{2.12}
\]
with a positive constant \( M \).

Let \( p \in (1, 2), \mu > 0 \), and define
\[
(B_\eta(s, x))_{\alpha\beta} := \frac{\delta_{ij} \delta_{\alpha\beta}}{(\mu + |J_\eta(\nabla v)(t, x)|^2)^{\frac{2-\mu}{2}}} - b(2 - p)\frac{(J_\eta(\nabla v)(t, x))}{(\mu + |J_\eta(\nabla v)(t, x)|^2)^{\frac{2-\mu}{2}}}
\]
with \( J_\eta \) space-time Friederich’s mollifier and \( b = 0, 1 \). Note that \( B_\eta \) defined in (2.13), by means of (2.11), satisfies condition (2.9), with
\[
\|B_\eta\|_\infty < (3 - p)\mu^{\frac{\mu - 2}{2}} < +\infty, \quad \forall \mu > 0.
\]

**Lemma 2.3** Assume that \( \varphi_0(x) \in C_0^\infty(\Omega) \) and let \( \varphi \) be the unique solution \( \varphi \) of (2.8), corresponding to \( B_\eta \) as in (2.13). Then, for all \( r \in [1, 2] \) if \( b = 0 \), and for all \( r \in [\frac{7-3p}{3p}, 2] \) if \( b = 1 \),
\[
\|\varphi(s)\|_r \leq ||\varphi_0\|_r, \quad \forall s \in [0, t], \text{ uniformly in } \nu > 0 \text{ and } \eta > 0. \tag{2.14}
\]

**Proof.** Let us multiply (2.8) by \( \varphi(\delta + |\varphi|^2)^{\frac{\mu - 2}{2}} \), for some \( \delta > 0 \). Then
\[
\frac{1}{r} \frac{d}{ds} \|(\delta + |\varphi|^2)^{\frac{1}{2}}\|_r + \nu \int_\Omega (\delta + |\varphi|^2)^{\frac{\mu - 2}{2}} |\nabla \varphi|^2 dx
\]
\[
+ \nu(\nu - 2) \int_\Omega (\delta + |\varphi|^2)^{\frac{\mu - 4}{2}} (\nabla \varphi \cdot \varphi)^2 dx + \int_\Omega a_{\eta}(\mu, v)(\delta + |\varphi|^2)^{\frac{\mu - 2}{2}} |\nabla \varphi|^2 dx
\]
\[
+ (r - 2) \int_\Omega a_{\eta}(\mu, v)(\delta + |\varphi|^2)^{\frac{\mu - 2}{2}} (\nabla \varphi \cdot \varphi)^2 dx
\]
\[
+ b(2 - p) \left[ \int_\Omega (\mu + |J_\eta(\nabla v)|^2)^{\frac{\mu - 4}{2}} (\delta + |\varphi|^2)^{\frac{\mu - 2}{2}} (J_\eta(\nabla v) \cdot \nabla \varphi)^2 dx
\]
\[
+ (r - 2) \int_\Omega \eta \frac{(\delta + |\varphi|^2)^{\frac{\mu - 4}{2}}}{(\mu + |J_\eta(\nabla v)|^2)^{2} \mu} (J_\eta(\nabla v) \cdot \nabla \varphi)(J_\eta(\nabla v) \cdot \varphi)(\nabla \varphi \cdot \varphi) dx \right] = 0. \tag{2.15}
\]
Taking into account that \( r \geq 1 \), the differential equation (2.15) gives
\[
\frac{1}{r} \frac{d}{ds} \|(\delta + |\varphi|^2)^{\frac{1}{2}}\|_r + (r - 1) \int_\Omega a_{\eta}(\mu, v)(\delta + |\varphi|^2)^{\frac{\mu - 2}{2}} |\nabla \varphi|^2 dx
\]
\[
\leq b(2 - p)(3 - r) \int_\Omega a_{\eta}(\mu, v)(\delta + |\varphi|^2)^{\frac{\mu - 2}{2}} |\nabla \varphi|^2 dx.
\]
Therefore, for all \( r \in [1, 2] \) if \( b = 0 \), and for all \( r \in \left[ \frac{7-3p}{3-p}, 2 \right] \) if \( b = 1 \), a straightforward computation gives the existence of a constant \( C \geq 0 \) such that
\[
\frac{1}{r} \int_{\Omega} |(\delta + |\varphi|)^{\frac{r}{p}} + C \int_{\Omega} a_\eta(\mu, v)(\delta + |\varphi|^{2}) |\nabla \varphi|^{2} dx \leq 0 ,
\]
from which one obtains
\[
\| (\delta + |\varphi(s)|^{2})^{\frac{r}{p}} \|_{r} \leq \| (\delta + |\varphi_{0}|^{2})^{\frac{r}{p}} \|_{r} , \quad \forall s \in [0, t] , \forall \delta > 0 ,
\]
which gives (2.14). \( \square \)

**Lemma 2.4** Let \( p > \frac{2n}{n+2} \). Under the assumptions of Lemma 2.3
\[
\| \varphi(s) \|_{2} \leq c M^{\frac{2-n}{2}} \| \varphi_{0} \|_{1} [t^{\frac{1}{p}} - (t - s)^{\frac{1}{p}}]^{-\gamma} , \quad \forall s \in (0, t] , \quad (2.16)
\]
\[
\gamma = \gamma(r) := \frac{n(2-r)}{2p - 2nr + mp} . \quad (2.17)
\]
Moreover, if \( b = 0 \), then
\[
\| \varphi(s) \|_{r} \leq c M^{\frac{2-n}{2}} \| \varphi_{0} \|_{1} [t^{\frac{1}{p}} - (t - s)^{\frac{1}{p}}]^{-2p\beta(1 - \frac{1}{p})} , \quad \forall s \in (0, t] , \quad (2.18)
\]
with
\[
\beta := \gamma(1) = \frac{n}{p(n+2) - 2n} . \quad (2.19)
\]
and \( r' \) conjugate exponent of \( r \).

**Proof.** Let us first observe that, multiplying equation (2.8) by \( \varphi \), taking into account (2.13) and integrating over \( \Omega \), we obtain
\[
\frac{1}{2} \frac{d}{ds} \| \varphi \|^{2} + \nu \int_{\Omega} |\nabla \varphi|^{2} dx + C \int_{\Omega} a_\eta(\mu, v) |\nabla \varphi|^{2} dx \leq 0 , \quad (2.20)
\]
hence
\[
\| \varphi(s) \|_{2}^{2} + 2\nu \int_{0}^{s} \| \nabla \varphi(\tau) \|_{2}^{2} d\tau + 2C \int_{0}^{s} \| a_\eta(\mu, v(t - \tau))^{\frac{1}{2}} \nabla \varphi(\tau) \|_{2}^{2} d\tau \leq \| \varphi_{0} \|_{2}^{2} . \quad (2.21)
\]
By H"{o}lder's inequality
\[
\int_{\Omega} |\nabla \varphi|^{p} dx = \int_{\Omega} (\mu + J_\eta|\nabla \varphi|^{2})^{\frac{p-2}{2}} |\nabla \varphi|^{p} (\mu + |J_\eta(\nabla \varphi)|^{2})^{\frac{2}{4-p}} dx \\
\leq \left( \int_{\Omega} (\mu + |J_\eta(\nabla \varphi)|^{2})^{\frac{p-2}{2}} |\nabla \varphi|^{2} \right)^{\frac{p}{2}} \left( \int_{\Omega} (\mu + |J_\eta(\nabla \varphi)|^{2})^{\frac{2}{4-p}} dx \right)^{\frac{2-p}{p}} \quad (2.22)
\]
\[
\leq \| (\mu + |J_\eta(\nabla \varphi)|^{2}) \|^{\frac{p}{2}} \| \nabla \varphi \|_{2}^{p} \| (\mu + |J_\eta(\nabla \varphi)|^{2}) \|^{\frac{p}{4-p}} .
\]
By using Minkowski's inequality, the last term can be treated as follows
\[
\| (\mu + |J_\eta(\nabla \varphi)(t - s)|^{2}) \|^{\frac{1}{2}} \leq \| J_\eta(\sqrt{\mu + |\nabla \varphi|})(t - s) \|_{p} \leq \int_{\mathbb{R}} J_{\eta}(t - s - \tau) \| \sqrt{\mu + |\nabla \varphi(\tau)|} \|_{p} d\tau .
\]
Therefore, using (2.12), it can be further estimated as

\[ \|(\mu + |J_n(\nabla v)(t-s)|^2)|\frac{t}{2} \leq 2 \int_0^t |J_n(t-s-\tau)|\|(\mu + |\nabla v(\tau)|^2)\|^p_p d\tau \leq c \left( \frac{M}{t-s} \right)^{\frac{b}{p}}, \]

which, raised to the power \((2-p)\) gives, a.e. in \(s > 0\),

\[ \|(\mu + |J_n(\nabla v)(t-s)|^2)|\frac{t}{2} \leq c \left( \frac{M}{t-s} \right)^{\frac{2-p}{p}}. \]

Using this estimate in (2.22) we end-up with

\[ c \left( \frac{t-s}{M} \right)^{\frac{2-p}{p}} \|\nabla \varphi\|^2_p \leq \|(\mu + J_n)|\nabla p|^2\frac{2-p}{p} \|\nabla \varphi\|^2_p. \tag{2.23} \]

From the differential inequality (2.20) and (2.23) we obtain

\[ \frac{1}{2} \frac{d}{ds} \|\varphi\|^2_s + c \left( \frac{t-s}{M} \right)^{\frac{2-p}{p}} \|\nabla \varphi\|^2_p \leq 0. \tag{2.24} \]

By the well known Gagliardo-Nirenberg inequality and estimate (2.14) we have

\[ \|\varphi\|^2_p \leq C \left( \frac{t-s}{M} \right)^{\frac{2-p}{p}} \|\nabla \varphi\|^2_p \leq C \|\nabla \varphi\|^2_p \|\varphi\|^\frac{2-p}{p}. \tag{2.25} \]

with \(a = \frac{np(2-p)}{2(np+rp-np)}\). From (2.24) and (2.25) we arrive at the differential inequality

\[ \frac{1}{2} \frac{d}{ds} \|\varphi\|^2_s + C \left( \frac{t-s}{M} \right)^{\frac{2-p}{p}} \|\nabla \varphi\|^2_p \|\varphi\|^\frac{2-p}{p} \leq 0 \forall s \in [0,t]. \]

Integrating from \(0\) to \(s\) and performing straightforward calculations, we get

\[ \|\varphi(s)\|_p \leq c M^{\frac{2-p}{2-p}} \|\varphi_0\|_r \left[ (t^\frac{p}{2} - (t-s)\frac{b}{2-p}) \right]^{-\frac{1}{2-p}}, \]

whence

\[ \|\varphi(s)\|_p \leq c M^{\frac{2-p}{2-p}} \|\varphi_0\|_r \left[ (t^\frac{p}{2} - (t-s)\frac{b}{2-p}) \right]^{-\frac{1}{2-p}}, \tag{2.26} \]

which, by the expression of \(M\), gives (2.16).

Assume that \(b = 0\). Then, from Lemma 2.3, estimate (2.14) holds for all \(r \in [1,2]\). The proof of (2.18) follows step by step the above proof. One has just to use (2.25) with \(r = 1\):

\[ \|\varphi\|_2 \leq c \|\nabla \varphi\|_p \|\varphi\|_1^{1-a} \leq c \|\nabla \varphi\|_p \|\varphi\|_1^{1-a}, \]

where now

\[ a = \frac{np}{2(np+np-np)}, \quad 1-a = \frac{np+2p-2n}{2(np+np-np)}. \]

Note that \(a < 1\) if and only if \(p > \frac{2n}{n+2}\). With the previous calculations we arrive at (2.26) with \(r = 1\), which, by setting \(\gamma = \gamma(1) = \beta\) gives

\[ \|\varphi(s)\|_2 \leq c M^{\frac{2-p}{2-p}} \|\varphi_0\|_1 \left[ (t^\frac{p}{2} - (t-s)\frac{b}{2-p}) \right]^{-\beta}, \tag{2.27} \]
Lemma 2.6 Therefore, from the Lebesgue dominated convergence theorem we obtain the result.

Lemma 2.5 Let $\nabla \psi \in L^2((0, t) \times \Omega)$, $\nabla v \in L^r((0, t) \times \Omega)$, for some $r > 1$, and let $h^m$ be a sequence with $h^m \to \nabla v$ in $L^2((0, t) \times \Omega)$, uniformly in $m \in \mathbb{N}$. Then, there exists a subsequence $h^{m_k}$ such that

$$\lim_{k \to \infty} \int_0^t \int_\Omega \left( (\mu + |J_{\frac{1}{m_k}}(\nabla v)|^2)^{\frac{p}{2} - \frac{2}{r}} - (\mu + |\nabla v|^2)^{\frac{p}{2} - \frac{2}{r}} \right) \nabla h^{m_k} \cdot \nabla \psi \, dx \, d\tau = 0.$$  

Proof. By assumption $\nabla \psi \in L^p((0, t) \times \Omega)$. Therefore one has $J_{\frac{1}{m_k}}(\nabla v) \to \nabla v$ in $L^p((0, t) \times \Omega)$ as $m$ goes to $\infty$. This ensures the existence of a subsequence $J_{\frac{1}{m_k}}(\nabla v)$ converging to $\nabla v$ a.e. in $(t, x) \in (0, T) \times \Omega$. Therefore, along this subsequence,

$$(\mu + |J_{\frac{1}{m_k}}(\nabla v)|^2)^{\frac{p}{2} - \frac{2}{r}} \to (\mu + |\nabla v|^2)^{\frac{p}{2} - \frac{2}{r}} , \text{ a.e. in } (s, x) \in (0, t) \times \Omega.$$  

By Hölder’s inequality one has

$$\int_0^t \int_\Omega \left( (\mu + |J_{\frac{1}{m_k}}(\nabla v)|^2)^{\frac{p}{2} - \frac{2}{r}} - (\mu + |\nabla v|^2)^{\frac{p}{2} - \frac{2}{r}} \right) \nabla h^{m_k} \cdot \nabla \psi \, dx \, d\tau$$

$$\leq \left( \int_0^t \int_\Omega |\nabla \psi(\tau)|^2 \left( (\mu + |J_{\frac{1}{m_k}}(\nabla v)|^2)^{\frac{p}{2} - \frac{2}{r}} - (\mu + |\nabla v|^2)^{\frac{p}{2} - \frac{2}{r}} \right)^2 \, dx \, d\tau \right)^{\frac{1}{2}} \|\nabla h^{m_k}\|_{L^2((0, t) \times \Omega)}.$$  

Further, since $\nabla \psi \in L^2((0, t) \times (\Omega))$, we have

$$|\nabla \psi(\tau)|^2 \left( (\mu + |J_{\frac{1}{m_k}}(\nabla v)|^2)^{\frac{p}{2} - \frac{2}{r}} - (\mu + |\nabla v|^2)^{\frac{p}{2} - \frac{2}{r}} \right)^2$$

$$\leq \left( 2 \mu^{p-2} \right)^2 |\nabla \psi(\tau)|^2 \in L^1(0, t; L^1(\Omega)).$$  

Therefore, from the Lebesgue dominated convergence theorem we obtain the result.  

Lemma 2.6 Let $\nu > 0$, $\mu > 0$ and $p > \frac{2m}{m+2}$. Let $v(x)$ satisfy

$$\|\mu + |\nabla v|^2\|^\frac{1}{p} \leq M,$$  

with a positive constant $M$. Then, for any $\varphi_0(x) \in C_0^\infty(\Omega)$, there exists a unique solution $\varphi \in C(0, t; L^2(\Omega)) \cap L^2(0, t; W_0^{1, 2}(\Omega))$ of the following integral equation

$$\int_0^s (\varphi, \varphi_\tau) d\tau - \nu \int_0^s (\nabla \varphi, \nabla \psi) d\tau - \int_0^s (\mu + |\nabla v|^2)^{\frac{p}{2} - \frac{2}{r}} \nabla \varphi, \nabla \psi) d\tau$$

$$= (\varphi(s), \psi(s)) - (\varphi_0, \psi(0)), \quad \forall \psi \in C_0^\infty([0, t] \times \Omega).$$  

Moreover, for any $r \in [1, 2]$, one has

$$\|\varphi(s)\|_r \leq \|\varphi_0\|_r, \quad \forall s \in [0, t].$$  

(2.30)
Using this estimate in (2.33) we get
\[ \| \varphi(s) \|_r \leq c M^{(2-p)\beta} \| \varphi_0 \|_1 s^{-p\beta(1 - \frac{1}{r})}, \quad \forall s \in (0,t), \] 
(2.31)
and
\[ \| \nabla \varphi(s) \|_p \leq c M^{\frac{2-p}{2} \beta + \frac{1}{2}} \| \varphi_0 \|_1 s^{-\frac{p+1}{2}}, \quad \forall s \in (0,t), \] 
(2.32)
with \( \beta \) given in (2.19) and \( r' \) conjugate exponent of \( r \).

**Proof.** Let us consider the unique solution of system (2.8) given in Lemma 2.2. The solution \( \varphi = \varphi(\eta) \) satisfies estimates (2.30) and (2.31), uniformly in \( \eta > 0 \). The proof is the same of Lemma 2.4 in the case where \( b = 0 \), replacing the assumption (2.12) by (2.28). Let us multiply (2.8) by \( \varphi_s \) and integrate over \( \Omega \). We get
\[ \frac{\nu}{2} \frac{d}{ds} \| \nabla \varphi \|_2^2 + \frac{1}{2} \frac{d}{ds} \| (\mu + |J_\eta(\nabla v)|^2)^{\frac{p-2}{2}} \nabla \varphi \|_2^2 + \| \varphi_s \|_2^2 = 0. \] 
(2.33)
Let us multiply (2.8) by \( \varphi \) and integrate over \( \Omega \). By using estimate (2.31) and recalling (2.11) we get
\[ \nu \| \nabla \varphi \|_2^2 + \| a_\eta(\mu, v)^{\frac{p}{2}} \nabla \varphi \|_2^2 \leq \| \varphi_s \|_2 \| \varphi \|_2 \leq c \| \varphi_s \|_2 \| \varphi_0 \|_1 M^{\frac{2-p}{2} \beta} s^{-\frac{p}{2}}, \quad \forall s \in (0,t), \]
hence
\[ c \left( \frac{\nu \| \nabla \varphi \|_2^2 + \| a_\eta(\mu, v)^{\frac{p}{2}} \nabla \varphi \|_2^2 }{\| \varphi_0 \|_1 M^{(2-p)\beta}} \right)^{\beta p} \leq \| \varphi_s \|_2^2, \quad \forall s \in (0,t). \]
Using this estimate in (2.33) we get
\[ \frac{1}{2} \frac{d}{ds} \left( \nu \| \nabla \varphi \|_2^2 + \| a_\eta(\mu, v)^{\frac{p}{2}} \nabla \varphi \|_2^2 \right) + c \left( \frac{\nu \| \nabla \varphi \|_2^2 + \| a_\eta(\mu, v)^{\frac{p}{2}} \nabla \varphi \|_2^2 }{\| \varphi_0 \|_1 M^{(2-p)\beta}} \right)^{\beta p} \leq 0, \]
which, integrated from 0 to \( s \) gives
\[ \nu \| \nabla \varphi \|_2^2 + \| a_\eta(\mu, v)^{\frac{p}{2}} \nabla \varphi \|_2^2 \leq c \left( M^{(2-p)\beta} \right) \| \varphi_0 \|_1^2 s^{-\left(\beta p + 1\right)}. \]
By using the arguments in (2.22), the above estimate and assumption (2.12) on \( v \), we have
\[ \| \nabla \varphi \|_p \leq \| (\mu + |J_\eta(\nabla v)|^2)^{\frac{p-2}{2}} \nabla \varphi \|_2 \| (\mu + |J_\eta(\nabla v)|^2)^{\frac{2-p}{2}} \| \leq c \left( M^{(2-p)\beta} \right) \| \varphi_0 \|_1^2 s^{-\frac{p+1}{2}}, \quad \forall s \in (0,t), \]
which gives (2.32). In order to obtain the result, the next step is to prove that, denoting for any \( \eta > 0 \) by \( \varphi^\eta \) the unique solution of (2.8), which satisfies estimates (2.30), (2.31) and (2.32), the sequence \( \{ \varphi^\eta \} \) converges in some sense to the solution of the integral equation (2.29) as \( \eta \) goes to zero. The proof is straightforward. We avoid the details and just show the following convergence, along a suitable subsequence,
\[ \int_0^s \left( (\mu + |J_\eta(\nabla v)|^2)^{\frac{p-2}{2}} \nabla \varphi^\eta, \nabla \psi \right) d\tau \to \int_0^s \left( (\mu + |\nabla v|^2)^{\frac{p-2}{2}} \nabla \varphi, \nabla \psi \right) d\tau. \]
Indeed, writing
\[ \int_0^s \left( (\mu + |J_\eta(\nabla v)|^2)^{\frac{p-2}{2}} \nabla \varphi^\eta, \nabla \psi \right) d\tau - \int_0^s \left( (\mu + |\nabla v|^2)^{\frac{p-2}{2}} \nabla \varphi, \nabla \psi \right) d\tau = \int_0^s \left( (\mu + |J_\eta(\nabla v)|^2)^{\frac{p-2}{2}} - (\mu + |\nabla v|^2)^{\frac{p-2}{2}} \right) \nabla \varphi^\eta \cdot \nabla \psi \ dx \ d\tau - \int_0^s \left( (\mu + |\nabla v|^2)^{\frac{p-2}{2}} (\nabla \varphi^\eta - \nabla \varphi) \cdot \nabla \psi \ dx \ d\tau, \right)
the first integral goes to zero, thanks to Lemma 2.5 and the second integral goes to zero thanks to the weak convergence of $\nabla \varphi^\eta$ to $\nabla \varphi$ in $L^2((0, t) \times \Omega)$ (using (2.21)).

We also give a useful inequality, referring, for instance, to [6].

**Lemma 2.7** Let $\mu > 0$. For any given real numbers $\xi, \eta \geq 0$, the following inequality holds:

$$\left| \frac{1}{(\mu + \xi)^{2-p}} - \frac{1}{(\mu + \eta)^{2-p}} \right| \leq \frac{2 - p}{3 - p} \frac{\mu^3}{|\xi - \eta|}.$$

Below we recall some well known results for Bochner spaces.

**Lemma 2.8** Let $u$ belong to $L^\infty(\varepsilon, T; W^{2,\overline{q}}(\Omega) \cap W_0^{1,\overline{q}}(\Omega))$ with $u_t \in L^\infty(\varepsilon, T; L^q(\Omega))$. For $m = 0, 1$, if $\lambda = 2 - m - \frac{3}{q} \in (0, 1)$, then

$$[\nabla^m u]_{\lambda, \varepsilon, t, x} \leq C \left[ \sup_{(\varepsilon, T)} (\|u(t)\|_q + \|D^2 u(t)\|_{\overline{q}}) + \sup_{(\varepsilon, T)} \|u(t)\|_q \right],$$

with $C$ independent of $u$.

**Proof.** The proof is a trivial generalization of Theorem 2.1 proved in [25].

For the following embedding results we refer, for instance, to [24], Ch. 3.

**Lemma 2.9** Let $X$ be a Banach space and let $X'$ be its dual. Assume that $X$ is dense and continuously embedded in an Hilbert space $H$. We identify $H$ with $H'$, which is continuously embedded in $X'$. If $u \in L^q(0, T; X)$ and $u' \in L^q(0, T; X')$, with $q, q' \in (1, +\infty)$, $\frac{1}{q} + \frac{1}{q'} = 1$, then $u$ is almost everywhere equal to a continuous function from $[0, T]$ into $H$.

**Lemma 2.10** [Aubin-Lions]- Let $X, X_1, X_2$ be Banach spaces. Assume that $X_1$ is compactly embedded in $X$ and $X$ is continuously embedded in $X_2$, and that $X_1$ and $X_2$ are reflexive. For $1 < q, s < \infty$, set

$$W = \{ \psi \in L^s(0, T; X_1) : \psi_t \in L^q(0, T; X_2) \}.$$

Then the inclusion $W \subset L^s(0, T; X)$ is compact.

### 3. Approximating systems

Let us study the approximating systems

$$u_t - \nabla \cdot \left( (\mu + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right) = 0, \quad \text{in } (0, T) \times \Omega,$$

$$u(t, x) = 0, \quad \text{on } (0, T) \times \partial \Omega,$$

$$u(0, x) = u_0(x), \quad \text{on } \{0\} \times \Omega,$$

(3.1)

with $\mu > 0$, and
\[ v_t - \nu \Delta v - \nabla \cdot \left( \left( \mu + |\nabla v|^2 \right)^{\frac{p-2}{2}} \nabla v \right) = 0, \quad \text{in } (0, T) \times \Omega, \]

\[ v(t, x) = 0, \quad \text{on } (0, T) \times \partial \Omega, \quad (3.2) \]

\[ v(0, x) = v_0(x), \quad \text{on } \{0\} \times \Omega, \]

with \( \mu > 0 \) and \( \nu > 0 \).

Let us introduce the operators from \( V \) to \( V' \) defined as

\[ A(v) := -\nabla \cdot \left( |\nabla v|^{p-2} \nabla v \right), \quad (3.3) \]

\[ A_{\mu}(v) := -\nabla \cdot \left( \left( \mu + |\nabla v|^2 \right)^{\frac{p-2}{2}} \nabla v \right). \quad (3.4) \]

They are both monotonous and emicontinuous operators\(^1\). Set

\[ B(\mu, w) := c \|w\|_2^2 + c(\Omega, T)\mu \tilde{x}. \quad (3.5) \]

Recall that, from (2.10),

\[ a(\mu, v) = \left( \mu + |\nabla v|^2 \right)^{\frac{p-2}{2}}. \]

**Definition 3.1** Let \( \mu > 0 \). Let \( u_0 \in L^2(\Omega) \). A field \( u: (0, T) \times \Omega \to \mathbb{R}^N \) is said a solution of system (3.1) if

\[ u \in L^p(0, T; V) \cap C(0, T; L^2(\Omega)), \quad t^\frac{1}{p} \nabla u \in L^\infty(0, T; L^p(\Omega)), \]

\[ t u_t \in L^\infty(0, T; L^2(\Omega)), \quad t^{\frac{p+2}{p}} \nabla u_t \in L^2(0, T; L^p(\Omega)), \]

\[ \int_0^t [(u, \psi_t) - (a(\mu, u) \nabla u, \nabla \psi)] \, d\tau = (u(t), \psi(t)) - (u_0, \psi(0)), \]

\[ \forall \psi \in W^{1,2}(0, T; L^2(\Omega)) \cap L^p(0, T; V), \]

and

\[ \lim_{t \to 0^+} \|u(t) - u_0\|_2 = 0. \]

**Definition 3.2** Let \( \mu > 0, \nu > 0 \). Let \( v_0 \in L^2(\Omega) \). A field \( v: (0, T) \times \Omega \to \mathbb{R}^N \) is said a solution of system (3.2) if

\[ v \in C(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}_0(\Omega)), \]

\[ v_t \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)), \]

\[ \int_0^t [(v, \psi_t) - \nu(\nabla v, \nabla \psi) - (a(\mu, v) \nabla v, \nabla \psi)] \, d\tau = (v(t), \psi(t)) - (v_0, \psi(0)), \]

\[ \forall \psi \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}_0(\Omega)), \]

and

\[ \lim_{t \to 0^+} \|v(t) - v_0\|_2 = 0. \]

\(^1\)For the sake of brevity for the corresponding definitions we refer, for instance, to [19] Ch. II, Sec.1.2.
We have introduced the approximating systems (3.1) and (3.2) in order to prove Theorem 1.1. In particular, the introduction of this second kind of approximation is necessary, for our technique, to obtain the crucial estimate \( u_t \in L^\infty(\varepsilon, T; L^p(\Omega)) \), which is one of the key tools to get higher order integrability for the second derivatives (Theorem 1.2 and Theorem 1.3) and, further, space-time Hölder regularity (see Corollary 1.2). If we limit ourselves just to Theorem 1.1, where the \( L^2(\varepsilon, T; W^{2,p}(\Omega)) \) integrability of the second derivatives is shown, then we could avoid the study of system (3.2), making the proof easier.

In order to study the existence and regularity of a solution of (3.1), firstly we study the same issues for the parabolic approximating system (3.2). The existence and regularities for the solution of this latter system are obtained in the following Proposition 3.1, Corollary 3.1 and Corollary 3.2. These results are a fundamental step for the proof of our main results. On the other hand their proofs rely on the Galerkin approximation method, with a suitable choice of the basis functions, and related weighted estimates in \( W^{m,r} \)-spaces, \( m \in \mathbb{N} \cup \{0\} \). Since these arguments are standard, we confine the proofs in the appendix. We observe that in the propositions below we will assume that the initial data of problems (3.1) and (3.2) are in \( C^\infty_0(\Omega) \). This assumption could be weakened, for the validity of the same results. This will be done in the next sections, where we deal with the solutions of problem (1.1) and of problem (3.1), and consider the completion of \( C^\infty_0(\Omega) \) in \( L^2(\Omega) \) and \( W^{1,2}_0(\Omega) \), so realizing a suitable generalization of the following results.

**Proposition 3.1** Let be \( \nu > 0 \) and \( \mu > 0 \). Assume that \( v_0 \) belongs to \( C^\infty_0(\Omega) \). Then there exists a unique solution \( v \) of system (3.2) in the sense of Definition 3.2. In particular:

i) \( v \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; V) \), uniformly in \( \nu \) and \( \mu \);

ii) \( v_t \in L^\infty(0, T; L^2(\Omega)), \nabla v_t \in L^2(0, T; L^p(\Omega)) \), uniformly in \( \nu \), non-uniformly in \( \mu \);

iii) For all \( \nu > 0, v \in L^2(0, T; W^{1,2}_0(\Omega)), v_t \in L^2(0, T; W^{-1,2}(\Omega)), \nabla v_t \in L^2(0, T; L^2(\Omega)) \).

**Corollary 3.1** Under the assumptions of Proposition 3.1, we have

i) \( t^{p/2} \nabla v \in L^\infty(0, T; L^p(\Omega)), t v_t \in L^\infty(0, T; L^2(\Omega)), t^{p/2} v_t \in L^2(0, T; L^p(\Omega)) \), \( t^{p+2/2} \nabla v_t \in L^2(0, T; L^p(\Omega)) \), uniformly in \( \nu > 0 \) and \( \mu > 0 \);

ii) \( t \nabla v_t \in L^2(0, T; L^2(\Omega)) \), non-uniformly in \( \nu > 0 \).

Set

\[
\alpha := \frac{4}{2p-n(2-p)}, \quad (3.6)
\]

\[
\beta_1(p) := \frac{1}{2} \left[ \alpha - \frac{n(2-p)^2}{p(2p-n(2-p))} \right], \quad \beta_2(p) := \frac{1}{2} \left[ \alpha + \frac{2-p}{p} \right]. \quad (3.7)
\]

**Corollary 3.2** Let \( \mu > 0 \). Assume that \( v_0 \) belongs to \( C^\infty_0(\Omega) \). If \( p > \frac{3}{2} \), then \( \nabla v \in C(0, T; L^2(\Omega)) \), \( v \in L^2(0, T; W^{2,p}(\Omega)) \), with

\[
\| \nabla v \|_{C(0, T; L^2(\Omega))} + \| v \|_{L^2(0, T; W^{2,p}(\Omega))} \leq M_3(\| v_0 \|_{1,2}, B(\mu, v_0)).
\]
If \( p > p_o \), then \( t^{\beta_1(p)} \nabla v \in C(0,T;L^2(\Omega)) \), \( t^{\beta_2(p)} v \in L^2(0,T;W^{2,p}(\Omega)) \), with

\[
\|v(t)\|_2^2 \leq C \mu^{\frac{n(2-p)}{2p-n(2-p)} - \beta_1} \|B(\mu, v_0)\|_p + C \|B(\mu, v_0)\|_p + cB(\mu, v_0) t^\alpha (1 + t),
\]

where \( \alpha, \beta_1 \) and \( \beta_2 \) are given by (3.6)–(3.7).

In the next proposition, starting from the existence and regularities of the solution of system (3.2), given in Proposition 3.1, and passing to the limit as \( \nu \) goes to zero, we deduce analogous existence and regularity properties for the solution of system (3.1).

**Proposition 3.2** Let be \( \mu > 0 \). Assume that \( u_o \) belongs to \( C^\infty_0(\Omega) \). Then there exists a unique solution \( u \) of system (3.1) in the sense of Definition 3.1, such that

i) \( u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V) \), uniformly in \( \mu \);

ii) for all \( \mu > 0 \), \( u_t \in L^\infty(0,T;L^2(\Omega)) \), \( \nabla u_t \in L^2(0,T;L^p(\Omega)) \).

**Proof.** From Proposition 3.1 for all \( \mu > 0 \), the function \( v \), solution of (3.2) corresponding to the initial data \( u_o \in C^\infty_0(\Omega) \), satisfies the bounds collected below, uniformly in \( \nu > 0 \),

\[
\|v\|_{L^\infty(0,T;L^2(\Omega))} + \|v\|_{L^p(0,T;V)} \leq M(T,\Omega,\|u_o\|_2),
\]

\[
\|v\|_{L^\infty(0,T;L^2(\Omega))} + \|v\|_{L^2(0,T;W^{1,p}(\Omega))} + \|v\|_{L^p(0,T;V')} \leq M_1(\mu, T, \Omega, \|u_o\|_2, 2),
\]

where the constant \( M_1 \) blows up as \( \mu \to 0 \). Hence we can extract a subsequence, still denoted by \( \{v^{\nu}\} \), weakly or weakly-* converging, in the above norms, to a function \( u \), as \( \nu \) tends to zero. Further, from \( v^{\nu} \in L^p(0,T;V) \), it follows that \( a(\mu, v^{\nu}) \nabla v^{\nu} \in L^p(0,T;L^p(\Omega)) \), hence \( A_\mu(v^{\nu}) \in L^p(0,T;V') \) and (see (A.12))

\[
A_\mu(v^{\nu}) \rightharpoonup \bar{\chi} \quad \text{in} \quad L^p(0,T;V') \quad \text{weakly}. \tag{3.8}
\]

Using the monotonicity trick as in [19], we show that the non-linear part \( A_\mu(v^{\nu}) \) actually (weakly) converges to \( A_\mu(u) \) and that the limit \( u \) is a solution of system (3.1). In particular the regularities stated for \( u \) follow from the analogous regularities of \( v^{\nu} \) and the lower semi-continuity of the norm for the weak convergence.

Hence, set

\[
X_\nu = \int_0^t (A_\mu(v^{\nu}) - A_\mu(\varphi), v^{\nu} - \varphi) d\tau \geq 0, \quad \forall \varphi \in L^p(0,T;V).
\]

By using that \( v^{\nu} \) is a solution of (3.2), we can write \( X_\nu \) as follows

\[
X_\nu = \int_0^t (A_\mu(v^{\nu}), v^{\nu}) d\tau - \int_0^t (A_\mu(v^{\nu}), \varphi) d\tau - \int_0^t (A_\mu(\varphi), v^{\nu} - \varphi) d\tau
\]

\[
= \frac{1}{2} \|v^{\nu}(0)\|_2^2 - \frac{1}{2} \|v^{\nu}(t)\|_2^2 - \nu \int_0^t \|\nabla v^{\nu}(\tau)\|_2^2 d\tau
\]

\[
+ \int_0^t (A_\mu(v^{\nu}), \varphi) d\tau - \int_0^t (A_\mu(\varphi), v^{\nu} - \varphi) d\tau.
\]
Passing to the lim sup and observing that
\[ \limsup(-\nu \int_0^T \| \nabla v'(\tau) \|_2^2 d\tau) = -\liminf(\nu \int_0^T \| \nabla v'(\tau) \|_2^2 d\tau) \leq 0, \]
we get
\[ 0 \leq \limsup X_\nu \leq \frac{1}{2} \| u_0 \|_2^2 - \frac{1}{2} \| u(t) \|_2^2 + \limsup(-\nu \int_0^t \| \nabla v'(\tau) \|_2^2 d\tau) \]
\[ - \int_0^t (\bar{\chi}, \varphi) d\tau - \int_0^t (A_\mu(\varphi), u - \varphi) d\tau \]
\[ \leq \frac{1}{2} \| u_0 \|_2^2 - \frac{1}{2} \| u(t) \|_2^2 - \int_0^t (\bar{\chi}, \varphi) d\tau - \int_0^t (A_\mu(\varphi), u - \varphi) d\tau. \]

On the other hand, from the weak convergence it is easy to see that the limit \( u \) satisfies
\[ \int_0^t (\bar{\chi}, u) d\tau = \frac{1}{2} \| u_0 \|_2^2 - \frac{1}{2} \| u(t) \|_2^2. \]  
(3.9)

Therefore
\[ \int_0^t (\bar{\chi} - A_\mu(\varphi), u - \varphi) d\tau \geq 0. \]
Taking \( \varphi = u + \lambda w \), for \( \lambda > 0 \) and for some \( w \in L^p(0, T, V) \), and then letting \( \lambda \) tend to zero the thesis follows.

Next, we improve property \( ii) \) of Proposition 3.2, by using Corollary 3.1 and Corollary 3.2. Indeed we are going to show that time weighted estimates for \( u_t \) and \( \nabla u_t \) hold for any \( \mu \) > 0.

**Proposition 3.3** Let be \( \mu > 0 \). Assume that \( u_0 \) belongs to \( C_0^\infty(\Omega) \). Then the solution \( u \) of system (3.1) of Proposition 3.2 satisfies, uniformly in \( \mu \), \( t^\frac{p}{2} \nabla u \in L^\infty(0, T; L^p(\Omega)) \)
\( t u_t \in L^\infty(0, T; L^2(\Omega)), t^{\frac{p+2}{p}} \nabla u_t \in L^2(0, T; L^p(\Omega)). \)
Furthermore, for \( p > \frac{3}{2} \), \( \nabla u \in C(0, T; L^2(\Omega)), u \in L^2(0, T; W^{2,p}(\Omega)), \) with
\[ \| \nabla u \|_{C(0, T; L^2(\Omega))} + \| u \|_{L^2(0, T; W^{2,p}(\Omega))} \leq M_3(\| u_0 \|_{1, 2}, B(\mu, u_0)). \]
For \( p > p_0 \), \( t^{\beta_1(p)} \nabla u \in C(0, T; L^2(\Omega)), t^{\beta_2(p)} u \in L^2(0, T; W^{2,p}(\Omega)), \) with
\[ t^\alpha \| \nabla u(t) \|_2^2 \leq C \mu \frac{\alpha + \alpha(p-2, p)}{p-2} B^\frac{p}{2}(\mu, u_0) t^{\frac{\alpha + \alpha(p-2, p)}{p-2}} \]
\[ + C(\beta(\mu, u_0))^\frac{\alpha + \alpha(p-2, p)}{p-2} + cB(\mu, u_0) t^\alpha (1 + t), \]
\[ \int_0^t t^{\alpha + \frac{2-p}{p}} \| D^2 u(\tau) \|_2^2 d\tau \leq C(B(\mu, u_0), T), \]
where \( \alpha, \beta_1 \) and \( \beta_2 \) are given by (3.6)–(3.7).

**Proof.** Let us consider system (3.2), with initial data \( v_0 = u_0 \). From Corollary 3.1,
\[ \| t^\frac{p}{2} \nabla v' \|_{L^\infty(0, T; L^p(\Omega))} + \| t v' \|_{L^\infty(0, T; L^2(\Omega))} + \| t^{\frac{p+2}{2p}} \nabla v' \|_{L^2(0, T; L^p(\Omega))} \leq M_2(T, \Omega, \| u_0 \|_2) \]
and, from Corollary 3.2, for \( p > \frac{3}{2} \)
\[
\| \nabla v' \|_{C(0,T;L^2(\Omega))} + \| v' \|_{L^2(0,T;W^{2,p}(\Omega))} \leq M_3(\| u_0 \|_{1,2}; B(\mu, u_0)),
\]
while for \( p > p_0 \)
\[
t^\alpha \| \nabla v(t) \|^2 \leq C \mu^\frac{\alpha^2 n (2 - p)^2}{2n(2 - p) - p} B(\mu, u_0) t^\frac{\alpha}{\mu^\frac{\alpha}{2n(2 - p) - p}}
+ C (B(\mu, u_0))^{\frac{4 - n (2 - p)}{4 - n (2 - p)}} + cB(\mu, u_0) t^\alpha (1 + t),
\]
\[
\int_0^t \tau^{\alpha + \frac{2 - p}{2}} \| D^2 u'(\tau) \|^2_p d\tau \leq C(B(\mu, u_0), T),
\]
uniformly in \( \nu > 0 \). Therefore, passing to the limit as \( \nu \) tends to zero, and then reasoning as in the proof of Proposition 3.2 we get that the limit \( u \), solution of system (3.1), satisfies the same bounds. The proof is then completed.

\( \square \)

4. Proof of Theorem 1.1 and Corollary 1.1

In the next proposition, under the assumption of an initial data \( u_0 \in L^2(\Omega) \) or \( u_0 \in W^{1,2}_0(\Omega) \), we study the existence and regularity of the solution of system (3.1), for any \( \mu > 0 \), already obtained in Proposition 3.2 and Proposition 3.3 under the stronger assumption of \( u_0 \in C^\infty_0(\Omega) \).

**Proposition 4.1** Let \( \mu > 0 \). Assume that \( u_0 \) belongs to \( L^2(\Omega) \). Then there exists a unique solution \( u \) of system (3.1) in the sense of Definition 3.1. Moreover, for \( p > p_0 \), \( \mu^{\beta_1(p)} \nabla u \in C(0,T;L^2(\Omega)) \), \( \mu^{\beta_2(p)} u \in L^2(0,T;W^{2,p}(\Omega)) \), with
\[
t^\alpha \| \nabla u(t) \|^2 \leq C \mu^\frac{\alpha^2 n (2 - p)^2}{2n(2 - p) - p} B(\mu, u_0) t^\frac{\alpha}{\mu^\frac{\alpha}{2n(2 - p) - p}}
+ C (B(\mu, u_0))^{\frac{4 - n (2 - p)}{4 - n (2 - p)}} + cB(\mu, u_0) t^\alpha (1 + t),
\]
\[
\int_0^t \tau^{\alpha + \frac{2 - p}{2}} \| D^2 u(\tau) \|^2_p d\tau \leq C(B(\mu, u_0), T),
\]
where \( \alpha, \beta_1 \) and \( \beta_2 \) are given by (3.6)–(3.7). Finally, assume that \( u_0 \in W^{1,2}_0(\Omega) \). Then, for \( p > \frac{3}{2} \), \( \nabla u \in C(0,T;L^2(\Omega)) \), \( u \in L^2(0,T;W^{2,p}(\Omega)) \), with
\[
\| \nabla u \|_{C(0,T;L^2(\Omega))} + \| u \|_{L^2(0,T;W^{2,p}(\Omega))} \leq M_3(\| u_0 \|_{1,2}; B(\mu, u_0)).
\]

**Proof.** Let \( \{ u_0^m(x) \} \) be a sequence in \( C^\infty_0(\Omega) \) strongly converging to \( u_0 \) in \( L^2(\Omega) \) and let \( \{ u^m \} \) be a sequence of solutions of system (3.1) corresponding to the initial data \( \{ u_0^m(x) \} \). The existence and regularity of such solutions has been gained in Proposition 3.2. In particular we have \(^2\)
\[
\| u^m \|_{L^\infty(0,T;L^2(\Omega))} + \| u^m \|_{L^2(0,T;V')} \leq M(T,\Omega,\| u_0 \|_2), \quad \forall m \in \mathbb{N}.
\]
\(^2\)Note that we also have, as in the proof of Proposition 3.2,
\[
\| u_0^m \|_{L^\infty(0,T;L^2(\Omega))} + \| u^m \|_{L^2(0,T;W^{1,p}(\Omega))} + \| u^m \|_{L^p(0,T;V')} \leq M_3(\mu,T,\Omega,\| u_0^m \|_{2,2}),
\]
but this estimate is not uniform in \( m \) and \( \mu \).
Moreover, from Proposition 3.3,
\[
\|t^{\alpha} \nabla u^m\|_{L^\infty(0,T;L^p(\Omega))} + \|u^m_t\|_{L^\infty(0,T;L^2(\Omega))} + \|t^{\frac{\alpha}{2}} \nabla u^m_t\|_{L^2(0,T;L^p(\Omega))} \leq M_2(T,\Omega,\|u_0\|_2), \quad \forall m \in \mathbb{N},
\]
and, for \( p > p_0 \),
\[
t^{\alpha}\|\nabla u^m(t)\|_2^2 \leq C \mu^{n(2-p)^2} B^\Phi(\mu,u_0) t^{n(2-p)^2} + C (B(\mu,u_0))^{\frac{4-n(2-p)}{2}} + cB(\mu,u_0)t^\alpha(1 + t),
\]
\[
\int_0^t r^{\alpha-1} \frac{2}{r} \|D^2 u^m(\tau)\|_p^2 d\tau \leq C(B(\mu,u_0),T). \quad (4.4)
\]

As the estimates in the norms (4.1)–(4.4) are uniform in \( m \), we can extract a subsequence, still denoted by \( \{u^m\} \), weakly or weakly-* converging in the same norms. Further, \( u^m \in L^p(0,T;V) \), implies \( \alpha(\mu,u^m) \nabla u^m \in L^p_0(0,T;L^p(\Omega)) \), hence \( A(\mu(u^m)) \in L^p(0,T;V') \) and \( u^m_t \in L^p(0,T;V') \) and, as \( m \to \infty \),
\[
A(\mu(u^m)) \to \chi \quad \text{in} \quad L^p(0,T;V') \quad \text{weakly}.
\]

Moreover, we have
\[
\|u^m(t) - u^k(t)\|_2^2 + \int_0^t (A(\mu(u^m)) - A(\mu(u^k)),u^m - u^k) d\tau \leq \|u_0^m - u_0^k\|_2^2, \quad \forall t \geq 0, \forall m,k \in \mathbb{N}. \quad (4.5)
\]

From this inequality and taking into account the \( L^2 \)-strong convergence of the sequence \( \{u^m\} \) to \( u_0 \) and the monotonicity of the operator, it follows the strong convergence of the sequence \( \{u^m\} \) to \( u \) in \( L^2(\Omega) \), uniformly in \( t \geq 0 \). Now, exactly as in [19] one proves that the limit \( u \) is the unique solution of (3.1), corresponding to the initial data \( u_0 \in L^2(\Omega) \).

If \( u_0 \in W^{1,2}_0(\Omega) \), we can reason as before, choosing a sequence \( \{u^m_0\} \) strongly converging to \( u_0 \) in \( W^{1,2}_0(\Omega) \), and replacing estimates (4.3), (4.4) by the following one, which, from Proposition 3.3, holds for any \( p > \frac{2}{\alpha} \)
\[
\|\nabla u^m\|_{C(0,T;L^2(\Omega))} + \|u^m\|_{L^2(0,T;W^{1,p}(\Omega))} \leq M_3(T,\Omega,\|u_0\|_{1,2}). \quad (4.6)
\]

We omit further details.

**Proof of Theorem 1.1** - From Proposition 4.1, for any fixed \( \mu > 0 \), there exists a unique solution of (3.1). Let us denote by \( \{u^\mu\} \) the sequence of solutions of (3.1) for the different values of \( \mu > 0 \). This sequence satisfies the bounds in (4.1)–(4.4), uniformly in \( \mu \). Hence, we can extract a subsequence, still denoted by \( \{u^\mu\} \), weakly or weakly-* converging in the same norms, as \( \mu \) goes to zero. In particular, in the limit as \( \mu \) goes to zero, estimate (4.3) gives
\[
t^\alpha\|\nabla u(t)\|_2^2 \leq C \|u_0\|_2^{2(4-n(2-p))} + c\|u_0\|_2^2 t^\alpha(1 + t),
\]
hence \( t^{\frac{\alpha}{2}} \nabla u \in C(0,T;L^2(\Omega)) \). Let us show that the limit \( u \) is the unique solution of system (1.1). Recall that, from (3.3),
\[
A(\psi) := -\nabla \cdot (|\nabla \psi|^{\frac{2}{\alpha}} \nabla \psi).
\]
Since $u^\mu$ belongs to $L^p(0,T;V)$ and it is $\mu$-uniformly bounded, then $a(\mu,u^\mu)\nabla u^\mu \in L^{p'}(0,T;L^{p'}(\Omega))$, and therefore $A_\mu(u^\mu) \in L^{p'}(0,T;V')$ and, along a subsequence,

$$A_\mu(u^\mu) \rightharpoonup \chi \text{ in } L^{p'}(0,T;V') \text{ weakly.}$$

Let us show that $\chi = A(u)$. Firstly we observe that

$$|(\mu + |\nabla \psi|^{p-2}) \nabla \psi - |\nabla \psi|^{p-2} \nabla \psi'| \to 0, \text{ a.e. in } (0,T) \times \Omega,$$

and

$$|(\mu + |\nabla \psi|^{p-2}) \nabla \psi - |\nabla \psi|^{p-2} \nabla \psi'| \leq 2p|\nabla \psi|^p.$$

The Lebesgue dominated convergence theorem ensures that

$$(\mu + |\nabla \psi|^{p-2}) \nabla \psi \to |\nabla \psi|^{p-2} \nabla \psi \text{ in } L^{p'}((0,T) \times \Omega) \text{ strongly,}$$

hence

$$A_\mu(\psi) \to A(\psi) \text{ in } L^{p'}(0,T;V') \text{ strongly.} \quad (4.7)$$

Set

$$X_\mu := \int_0^t (A_\mu(u^\mu) - A_\mu(\psi), u^\mu - \psi) \, d\tau \geq 0, \forall \psi \in L^p(0,T,V).$$

By using that $u^\mu$ is a solution of (3.1), we write $X_\mu$ as follows

$$X_\mu = \int_0^t (A_\mu(u^\mu), u^\mu) - (A_\mu(u^\mu), \psi) - (A_\mu(\psi), u^\mu - \psi) \, d\tau$$

$$= \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|u^\mu(t)\|^2 - \int_0^t [(A_\mu(u^\mu), \psi) + (A_\mu(\psi), u^\mu) - (A_\mu(\psi), \psi)] \, d\tau.$$

Let us pass to the lim inf. Observe that, since

$$(A_\mu(\psi), u^\mu) - (A(\psi), u) = (A_\mu(\psi) - A(\psi), u^\mu) + (A(\psi), u^\mu - u),$$

from the strong convergence (4.7), the uniform bound of $u^\mu$ in $L^p(0,T;V)$ and the weak convergence of $u^\mu$ to $u$ in $L^p(0,T;V)$, we get

$$\lim_{\mu \to 0} \int_0^t (A_\mu(\psi), u^\mu) \, d\tau = \int_0^t (A(\psi), u) \, d\tau.$$

Hence

$$0 \leq \liminf X_\mu \leq \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|u(t)\|^2 - \int_0^t (\chi, \psi) + (A(\psi), u - \psi) \, d\tau.$$

On the other hand, it is easy to see that the limit $u$ satisfies

$$\int_0^t (\chi, u) \, d\tau = \frac{1}{2}\|u_0\|^2 - \frac{1}{2}\|u(t)\|^2. \quad (4.8)$$

Therefore we obtain

$$\int_0^t (\chi - A(\psi), u - \psi) \, d\tau \geq 0, \forall \psi \in L^p(0,T;V).$$
Proposition 5.1

Let \( \psi = u + \lambda w \), for \( \lambda > 0 \) and for some \( w \in L^p(0, T, V) \), and then letting \( \lambda \) tend to zero the thesis follows.

\[ \square \]

Proof of Corollary 1.1 - From Proposition 4.1, for any fixed \( \mu > 0 \), there exists a solution of (3.1). Let us denote by \( \{u^n\} \) the sequence of solutions of (3.1) for the different values of \( \mu > 0 \). This sequence satisfies the bounds in (4.1), (4.2) and (4.6), uniformly in \( \mu \). Hence, we can extract a subsequence, still denoted by \( \{u^n\} \), weakly or weakly-* converging in the same norms, as \( \mu \) goes to zero. That the limit \( u \) is the unique solution of system (1.1) can be proved as in the proof of Theorem 1.1.

5. A crucial estimate: \( u_t \in L^\infty(0, T; L^q(\Omega)) \)

Proposition 5.1 Let \( p > p_0 \), with \( p_0 \) given in (1.4), and \( q \in [2, \frac{T-3p}{2p}) \). Let \( u \) be the unique solution of (1.1) corresponding to \( u_o \in L^2(\Omega) \). Then \( t^{1+\gamma} u_t \in L^\infty(0, T; L^q(\Omega)) \), with \( \gamma = \gamma(q') \) given by (2.17). Moreover the following estimate holds

\[ \|u_t(t)\|_q \leq C t^{1+\gamma} \|u_0\|_1^{(2-p)\gamma+1}, \forall t \in (0, T). \]  

(5.1)

Proof. Let \( \{u_n^m(x)\} \) be a sequence in \( C_0^\infty(\Omega) \) strongly converging to \( u_0 \) in \( L^2(\Omega) \) and let \( \{v^m\} \) be the sequence of solutions of system (3.2) corresponding to the initial data \( \{u_n^m(x)\} \). Under our assumptions on \( p \), from Proposition 3.1 and Corollary 3.2 we know that, for any data \( u_n^m \), there exists a unique solution \( v^m \) of (3.2) such that \( t^{\beta(p)} v_n^m \in L^2(0, T; W^{2,p}(\Omega)) \) and \( t v_n^m \in L^\infty(0, T; L^2(\Omega)) \). Therefore \( v^m \) satisfies system (3.2) a.e. in \( (0, T) \times \Omega \). In the sequel, for simplicity we suppress the superscript \( m \).

Let us mollify equation (3.2) with respect to \( t \), and denote the mollifier by \( J_\delta \). We can derive the regularized system (3.2) with respect to \( t \) and \( \eta \) and obtain, a.e. in \( \Omega \times (0, T) \),

\[ \partial_t J_\delta(v_t) - \nu \partial_t \Delta(J_\delta v) - \nabla \cdot \partial_t J_\delta \left( (\mu + |\nabla v|^2)^{\frac{p-2}{2}} \nabla v \right) = 0. \]  

(5.2)

For any fixed \( \eta \), let \( \varphi^\eta(s, x) \) be the unique solution of system (2.8) with \( \nu > 0 \), corresponding to an initial data \( \varphi_o \in C_0^\infty(\Omega) \), and \( B_{\eta} \), given by (2.13), with \( v \) as above. For simplicity we also drop the superscript \( \eta \). From Lemma 2.2, \( \varphi \in L^2(0, T; W^{1,2}_0(\Omega)) \) and \( \varphi_{\eta} \in L^2((0, t) \times \Omega) \). Therefore, setting \( \tau = t - s \), \( s \in (0, \frac{t}{2}) \), we can multiply equation (5.2) by \( \varphi(t-\tau) \), for \( \tau \in (\frac{t}{2}, t) \), and integrate in \( (\frac{t}{2}, t) \times \Omega \). Integrations by parts give the following identity

\[ (J_\delta(v_t)(t), \varphi(0)) = (J_\delta(v_t)(\frac{t}{2}), \varphi(\frac{t}{2})) + \int_0^t (J_\delta(v_\tau)(\tau), \varphi(t-\tau)) d\tau \]

\[ + \nu \int_0^t (J_\delta(\nabla v_\tau)(\tau), \nabla \varphi(t-\tau)) d\tau + \int_0^t \left( J_\delta \left( \frac{\nabla v_\tau}{(\mu + |\nabla v|^2)^{\frac{p-2}{2}}} \right)(\tau), \nabla \varphi(t-\tau) \right) d\tau \]

\[ + (p-2) \int_0^t \left( J_\delta \left( \frac{\nabla v \otimes \nabla v}{(\mu + |\nabla v|^2)^{\frac{p-2}{2}}} \right)(\tau), \nabla \varphi(t-\tau) \right) d\tau. \]  

(5.3)

Let us pass to the limit as \( \delta \) goes to zero. Note that from Corollary 3.1 one gets \( t v_t \in C([0, T); L^2(\Omega)) \). Hence, since, from Lemma 2.3, we also have \( \varphi \in C([0, t); L^2(\Omega)) \), we get

\[ \lim_{\delta \to 0} (J_\delta(v_t)(\frac{t}{2}), \varphi(\frac{t}{2})) = (v_t(\frac{t}{2}), \varphi(\frac{t}{2})). \]
as well as
\[ \lim_{\delta \to 0} (J_\delta(v_\delta)(t), \varphi_\delta) = (v(t), \varphi). \]

Further, from Corollary 3.1, one has \( \tau^{\frac{1}{r}} v_\tau \in L^2((0, T) \times \Omega) \), hence \( J_0(t^{\frac{1}{r}} v_t) \) strongly converges to \( t^{\frac{1}{r}} v_t \) in \( L^2((0, T) \times \Omega) \), as \( \delta \to 0 \). Since, from Lemma 2.2, one has \( \tau^{-\frac{1}{r}} \varphi_\tau(t- \tau) \in L^2((\frac{1}{2}, t) \times \Omega) \), we get

\[ \lim_{\delta \to 0} \int_{\frac{1}{2}}^t (J_\delta(v_\tau)(\tau), \varphi_\tau(t- \tau)) d\tau = \int_{\frac{1}{2}}^t (v_\tau(\tau), \varphi_\tau(t- \tau)) d\tau. \]

Recall that, from Corollary 3.1,

\[ \|J_\delta(\nabla v_\tau)\|_{L^2(\frac{1}{2}, T; L^2(\Omega))} \leq \|\nabla v_\tau\|_{L^2(\frac{1}{2}, T; L^2(\Omega))} \leq \frac{c}{t} B^\frac{2}{2}(\mu, u_0^m). \]

Therefore

\[ \|J_\delta(\nabla v_\tau)\|_{L^2((\frac{1}{2}, T) \times \Omega)} \leq \frac{c}{t} \mu^{p-2} B^\frac{2}{2}(\mu, u_0^m), \]

and, similarly,

\[ \|J_\delta(\nabla v \otimes \nabla v) \cdot (\nabla v_\tau)\|_{L^2((\frac{1}{2}, T) \times \Omega)} \leq \frac{c}{t} \mu^{p-2} B^\frac{2}{2}(\mu, u_0^m). \]

Moreover, observing that \( \tau^{-1} \nabla \varphi_\tau \in L^2(\frac{1}{2}, T; L^2(\Omega)) \), we can pass to the limit as \( \delta \) tends to zero in the last two integrals on the right-hand side of (5.3). Summarizing, in the limit as \( \delta \) goes to zero, (5.3) gives

\[
(v_\tau(t), \varphi(0)) = (v_\tau(t), \varphi(t)) + \int_{\frac{1}{2}}^t (v_\tau(\tau), \varphi_\tau(t- \tau)) d\tau \\
+ \nu \int_{\frac{1}{2}}^t (\nabla v_\tau(\tau), \nabla \varphi(t- \tau)) d\tau + \int_{\frac{1}{2}}^t \left( \frac{\nabla v_\tau(\tau)}{(\mu + |\nabla v(\tau)|^2)^{\frac{2-\rho}{2}}} \cdot \nabla \varphi(t- \tau) \right) d\tau \\
+ (p-2) \int_{\frac{1}{2}}^t \left( (\nabla v(\tau) \otimes \nabla v(\tau)) : \nabla \varphi_\tau(\tau), \nabla \varphi(t- \tau) \right) d\tau. \leq \frac{c}{t} \mu^{p-2} B^\frac{2}{2}(\mu, u_0^m). \]

(5.4)

Let us write the last two integrals on the right-hand side of (5.4) as follows

\[
\int_{\frac{1}{2}}^t \left( \frac{\nabla v_\tau(\tau)}{(\mu + |\nabla v|^2)^{\frac{2-\rho}{2}}} \cdot \nabla \varphi(t- \tau) \right) d\tau + (p-2) \int_{\frac{1}{2}}^t \left( (\nabla v(\tau) \otimes \nabla v) \cdot \nabla \varphi_\tau(\tau), \nabla \varphi(t- \tau) \right) d\tau \\
= \int_{\frac{1}{2}}^t (\nabla v_\tau(\tau), B_\eta(\tau, x) \varphi(t- \tau)) d\tau \\
+ \int_{\frac{1}{2}}^t \left( \nabla v_\tau(\tau), \frac{1}{(\mu + |\nabla v|^2)^{\frac{2-\rho}{2}}} - \frac{1}{(\mu + |J_\eta(\nabla v)|^2)^{\frac{2-\rho}{2}}} \right) \nabla \varphi(t- \tau) d\tau \\
+ (p-2) \int_{\frac{1}{2}}^t \left( \nabla v_\tau, \frac{(\nabla v) \otimes (\nabla v)}{(\mu + |\nabla v|^2)^{\frac{1}{2}}} - \frac{J_\eta(\nabla v) \otimes J_\eta(\nabla v)}{(\mu + |J_\eta(\nabla v)|^2)^{\frac{1}{2}}} \right) \cdot \nabla \varphi(t- \tau) d\tau. \leq \frac{c}{t} \mu^{p-2} B^\frac{2}{2}(\mu, u_0^m). \]
Then, since \( \varphi \) is solution of (2.8) with \( B_\eta \) as in (2.13), identity (5.4) becomes

\[
(v_t(t), \varphi_o) = (v_t(\frac{t}{2}), \varphi(\frac{t}{2})) + \int_0^t \left( \nabla v_\tau(\tau), \left[ \frac{1}{(\mu + |\nabla v|^2)^{\frac{p}{2}}} - \frac{1}{(\mu + |J_\eta(\nabla v)|^2)^{\frac{p}{2}}} \right] \nabla \varphi(t - \tau) \right) d\tau \\
+ (p - 2) \int_0^t \left( \nabla v_\tau, \left[ \frac{(\nabla v) \otimes (\nabla v)}{(\mu + |\nabla v|^2)^{\frac{p}{2}}} - \frac{J_\eta(\nabla v) \otimes J_\eta(\nabla v)}{(\mu + |J_\eta(\nabla v)|^2)^{\frac{p}{2}}} \right] \cdot \nabla \varphi(t - \tau) \right) d\tau \\
= (v_t(\frac{t}{2}), \varphi(\frac{t}{2})) + I_{\eta}^1 + I_{\eta}^2
\]

We claim that the integrals \( I_{\eta}^1 \) and \( I_{\eta}^2 \) go to zero along a subsequence. This follows from Lemma 2.5, with \( h^\eta = \tau^{-1} \nabla \varphi^\eta \), which is bounded in \( L^2((\frac{t}{2}, T) \times \Omega) \), uniformly in \( \eta > 0 \), and \( \psi = \tau \nabla v_r \), which is also in \( L^2((\frac{t}{2}, T) \times \Omega) \), due to Corollary 3.1.

Therefore, by using estimate (2.16) in Lemma 2.4 with \( r = q' \), \( s = \frac{t}{2} \) and \( M = B \), and using that, from i) in Corollary 3.1, \( tv_t \in L^\infty(0, T; L^2(\Omega)) \) (see estimate (A.23)), then passing to the limit as \( \eta \) goes to zero in (5.5) we get

\[
(v_t(t), \varphi_o) = (v_t(\frac{t}{2}), \varphi(\frac{t}{2})) \leq \|v_t(\frac{t}{2})\| \|\varphi(\frac{t}{2})\| \leq \frac{c}{t^{1+\gamma(q')}} \|\varphi_o\|q' \left( \|u_\eta^m\|_2^2 + c(\Omega, T)\mu^{\frac{2-\gamma(q')}2} \right)^{\frac{(2-\gamma(q'))+1}{2}},
\]

with \( \gamma(q') \) given in (2.17). Let us give back the superscript \( m \) to the sequence \( \{v^m\} \). The last estimate ensures that \( t^{1+\gamma} v_t^m \in L^\infty(0, T; L^4(\Omega)) \) and that the following estimate holds

\[
\|v_t^m(t)\|_q \leq \frac{c}{t^{1+\gamma(q')}} \left( \|u_\eta\|_2^2 + c(\Omega, T)\mu^{\frac{2-\gamma(q')}2} \right)^{\frac{(2-\gamma(q'))+1}{2}}, \text{ uniformly in } \nu > 0, \mu > 0, m \in \mathbb{N}.
\]

Now we pass to the limit, firstly as \( \nu \) goes to zero, thus obtaining the sequence of solutions \( \{u^m\} \) of Proposition 3.2, whose time derivative also satisfies estimate (5.6). Then we pass to the limit as \( m \) tends to infinity, finding the solution \( u = u^\mu \) of system (3.1) and satisfying (5.6). Finally, we pass to the limit as \( \mu \) goes to zero, thus obtaining that the solution \( u \), found in Theorem 1.1, satisfies (5.1).

\[\Box\]

**Remark 5.1** Note that the main reason that leads us to resort to system (3.2) as approximation of system (3.1) is due to the possibility of handling the integrals \( I_{\eta}^1 \) and \( I_{\eta}^2 \).

### 6. Proof of Theorem 1.2, Theorem 1.3 and their corollaries

In order to prove Theorem 1.2 and its Corollary 1.2, firstly we recall a regularity result obtained in [7], related to the singular elliptic \( p \)-Laplace system

\[
-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f, \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial \Omega,
\]

(6.1)
with \( p \in (1, 2) \). We recall that, if \( v \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \), then the following estimate holds
\[
\|D^2v\|_2 \leq H\|\Delta v\|_2,
\]
where the constant \( H \) depends on the size of \( \Omega \). If \( \Omega \) is a convex domain the inequality holds with \( H = 1 \). For details we refer to to [15] or [16]. We recall that
\[
p = 2 - \frac{1}{H},
\]
where \( H \) is the above constant. The results in [7] can be stated as follows.

**Theorem 6.1** Let \( \Omega \subset \mathbb{R}^n, n \geq 3 \), be convex. Assume that \( f \in L^q(\Omega) \), with \( q \in \left[\frac{2n}{n(p-1)+2(2-p)}\infty\right) \). There exists a unique solution \( u \in W^{1,2}_0(\Omega) \cap W^{2,\hat{q}}(\Omega) \) of system (6.1), with
\[
\|u\|_{2,\hat{q}} \leq C\|f\|_{\hat{q}^{-1}},
\]
where \( C \) is a constant independent of \( u \) and \( \hat{q} = \hat{q}(q) = \frac{nq(p-1)}{n-q(2-p)} \) if \( q < n \), \( \hat{q} = q \) if \( q = n \), and \( \hat{q} = q \) if \( q > n \).

The same result holds for non-convex domains \( \Omega \), provided that \( p > p_0 \).

**Proof of Theorem 1.2** - Since \( p > p_0 \), we can apply Theorem 1.1, and find that the unique solution of system (1.1) satisfies the following system, a.e. in \( t \in (0,T) \),
\[-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = u_t, \quad \text{in } \Omega,\]
\[u = 0, \quad \text{on } \partial \Omega.\]

We set
\[\eta := \frac{2n}{n(p-1)+2(2-p)},\]
and observe that \( \eta \geq 2 \) and, since \( p > p_1 \), then \( \eta \leq \frac{7-3p}{p-2} \). From Proposition 5.1, for \( p > p_0 \) and \( q \in [\eta, \frac{7-3p}{p-2}] \), we have \( t^{1+\gamma}u_t \in L^\infty(0,T; L^q(\Omega)) \), where \( \gamma = \frac{n(q-2)}{q(2p-2n+np)} \), and \( u_t \) satisfies estimate (5.1). By applying the above Theorem 6.1 we find that: if \( \Omega \) is convex, then \( u \in W^{2,\hat{q}}(\Omega) \), a.e. in \( t \in (0,T) \); if \( \Omega \) is not convex, the same result holds if \( p > p_0 \) too. In both cases, using estimate (5.1) in (6.4) we find
\[
\|u\|_{2,\hat{q}} \leq C\|u_t\|_{\hat{q}^{-1}} \leq \frac{C}{t^{1+\gamma}} \|u_0\|_2 \frac{(2-p)_+}{p+1}, \quad \text{a.e. in } t \in (0,T).\]
\[
\square
\]

For the proof of Theorem 1.3 we argue exactly in the same way, employing Theorem 6.2 in place of Theorem 6.1. For the sake of completeness we give the proof.
Proposition 3.1 and Corollary 3.1 hold. In the sequel, for simplicity we suppress the solution \( \phi \) of system (3.2) corresponding to the initial data \( \{ \theta \} \). Then \( u \in W^{2, \tilde{q}}(\Omega) \), with \( \tilde{q} = q \), a.e. in \( t \in (0, T) \), ; if \( \Omega \) is not convex, the same result holds for \( p > \overline{q} \). In both cases, using estimate (5.1) in (6.5), we find \( u \in W^{2, \tilde{q}}(\Omega) \), with \( \tilde{q} = q \) if \( q_0 = n \), \( q_0 = \left( \frac{2}{p} - \frac{1}{n} \right) \) if \( q_0 = \left( \frac{2}{p} - \frac{1}{2} \right) \). From Proposition 5.1, for \( p > p_0 \) and \( q > 2(\frac{p-1}{p}) \), \( u \in L^\infty(\Omega) \). Hence, by applying Lemma 2.8 with \( m, \nu \), we obtain

\[
\| u \|_{L^q(\Omega)} \leq C \| u \|_{L^q(\Omega)} + \| D^2 u \|_{L^q(\Omega)} + \| u \|_{L^q(\Omega)}
\]

\( \lambda_0 = 2 - \frac{q_0}{2} \), from which estimate (1.10) easily follows by increasing the right-hand side via estimate (6.7). Assume now that \( p > \max \{ p_0, \frac{4n-q_0}{2n-q_0} \} \) and \( q_1 = (\frac{2}{p} - \frac{1}{n}) \). Under the assumption \( p > \frac{4n-q_0}{2n-q_0} \), \( \frac{4n-q_0}{2n-q_0} > n \) is ensured. Therefore we can apply Theorem 1.2 with \( q_1 = q > n \), and obtain \( u \in L^\infty(0, T; L^q(\Omega)), \ u \in L^\infty(0, T; W^{2, q_1}(\Omega)) \). By applying Lemma 2.8 with \( m = 1 \) and \( \overline{q} = q_1 \) we get estimate (1.11).

Proof of Corollary 1.3 - Theorem 1.3 gives, for \( q > 2 \), \( u \in L^\infty(0, T; W^{2, q}(\Omega)) \) and \( u \in L^\infty(0, T; L^q(\Omega)) \). Hence, by applying Lemma 2.8 with \( m = 1 \) and \( \overline{q} = q \) we get estimate (1.12).

7. The Maximum Modulus Theorem: Proof of Theorem 1.4

Proposition 7.1 Let \( \mu > 0 \). Let \( u \) be the solution of (3.1) corresponding to \( u_0 \in L^\infty(\Omega) \). Then

\[
\| u(t) \|_\infty \leq \| u_0 \|_\infty.
\]

Proof. We argue as in Proposition 5.1. Hence we consider a sequence \( \{ u_n(t) \} \in C^\infty_0(\Omega) \) strongly convergent to \( u_0 \) in \( L^2(\Omega) \), and such that \( \| u_n \|_\infty \leq \| u_0 \|_\infty \), the sequence \( \{ v_\mu \} \) of solutions of system (3.2) corresponding to the initial data \( \{ u_n \} \), for which Proposition 3.1 and Corollary 3.1 hold. In the sequel, for simplicity we suppress the superscripts \( m \) and \( \nu \) for the sequence \( \{ v_\mu \} \). Let us consider, for a fixed \( \eta > 0 \), the solution \( \phi^\eta(s, x) \) of system (2.8) corresponding to a data \( \phi_0 \in C^\infty_0(\Omega) \), where \( B_\eta \) is given...
by (2.13) with $b = 0$, and set $\dot{\varphi}^0(\tau) = \varphi^0(t - \tau)$. Note that, from estimate (A.21), $v$ satisfies (2.12). Then, using $\dot{\varphi}^0(\tau)$ as test function in the weak formulation of (3.2), we have

\[
(v(t), \varphi_0) - (v(0), \varphi^0(t)) - \int_0^t (v(\tau), \dot{\varphi}^0(\tau)) d\tau = -\nu \int_0^t (\nabla v(\tau), \nabla \dot{\varphi}^0(\tau)) d\tau - \int_0^t (a(\mu, v(\tau)) \nabla v(\tau), \nabla \dot{\varphi}^0(\tau)) d\tau.
\]

Writing the second term on the right-hand side of (7.1) as

\[
\int_0^t (a(\mu, v) \nabla v(\tau), \nabla \dot{\varphi}^0(\tau)) d\tau = \int_0^t (\nabla v(\tau), a_\varphi(\mu, v(\tau)) \nabla \dot{\varphi}^0(\tau)) d\tau + \int_0^t (\nabla v(\tau), \nabla \dot{\varphi}^0(\tau)) [a(\mu, v) - a_\varphi(\mu, v)] d\tau,
\]

and recalling that $\varphi^0$ is solution of (2.8), identity (7.1) becomes

\[
(v(t), \varphi(0)) = (v(0), \varphi^0(t))
- \int_0^t (\nabla v(\tau), \nabla \dot{\varphi}^0(\tau)) [a(\mu, v(\tau)) - a_\varphi(\mu, v(\tau))] d\tau = (v(0), \varphi^0(t)) + I_\eta.
\]

The integral $I_\eta$ goes to zero, as $\eta$ goes to zero, along a subsequence, thanks to Lemma 2.5 with $h^\eta = \varphi^0 \in L^2(0, T; L^2(\Omega))$, due to Lemma 2.2 and with $\psi = \nabla v \in L^2(0, T; L^2(\Omega))$, due to Proposition 3.1. Finally, using (2.14) and then passing to the limit as $\eta$ tends to zero in (7.2), along a subsequence, we get

\[
|(v(t), \varphi_0)| \leq \|u^m_\omega\| \|\varphi_0\|_{r'}, \forall \varphi_0 \in C^\infty_0(\Omega).
\]

Giving back the superscripts to the sequence $\{v^{m, \nu}\}$, this last estimate together with the bound $\|u^m_\omega\| \leq \|u_\omega\|_{\infty}$ imply

\[
\|v^{m, \nu}(t)\|_{r'} = \sup_{\varphi_0 \in C^\infty_0(\Omega)} \|v^{m, \nu}(t, \varphi_0)| \leq \|u^m_\omega\| \|\varphi_0\|_{r'} \leq \|\Omega\|^{\frac{1}{p'}}\|u^m_\omega\|_{\infty} \leq \|\Omega\|^{\frac{1}{p'}}\|u_\omega\|_{\infty}.
\]

Since the the right-hand side is uniform with respect to $r'$, letting $r'$ go to $\infty$, we obtain

\[
\|v^{m, \nu}(t)\|_{\infty} \leq \|u_\omega\|_{\infty}.
\]

Now we pass to the limit, firstly as $\nu$ goes to zero. We obtain the sequence of solutions $\{u^m\}$ as in Proposition 3.2. From Proposition 3.1, we have, uniformly in $\nu > 0$, that $v^{m, \nu} \in L^\infty(0, T; V)$, $v^{m, \nu}_t \in L^2(0, T; L^p(\Omega))$. Since $V$ is compactly embedded in $L^s(\Omega)$, for any $p < s < \frac{n(1-p)}{n-p}$, and $L^s(\Omega)$ is continuously embedded in $L^p(\Omega)$, from Lemma 2.10, the sequence $\{v^{m, \nu}\}$ converges to $u^m$, as $\nu$ goes to zero, strongly in $L^s((0, T) \times \Omega)$, hence almost everywhere in $t, x$, along to a subsequence. Therefore, along such a subsequence, we find

\[
|u^m(t, x)| \leq |u^m(t, x) - v^{m, \nu}(t, x)| + |v^{m, \nu}(t, x)|
\leq |u^m(t, x) - v^{m, \nu}(t, x)| + \|v^{m, \nu}(t)\|_{\infty}.
\]

\[
|(v(t), \varphi_0) - (v(0), \varphi^0(t)) - \int_0^t (v(\tau), \dot{\varphi}^0(\tau)) d\tau = -\nu \int_0^t (\nabla v(\tau), \nabla \dot{\varphi}^0(\tau)) d\tau - \int_0^t (a(\mu, v(\tau)) \nabla v(\tau), \nabla \dot{\varphi}^0(\tau)) d\tau.
\]
a.e. in \((t, x) \in (0, T) \times \Omega\). Passing to the limit on \(\nu\), we easily get
\[
\|u^m(t)\|_\infty \leq \|u_0\|_\infty .
\] (7.4)

We now pass to the limit as \(m\) tends to infinity. Then \(\{u^m\}\) strongly converges to \(u_0\) and, from (4.5), the corresponding sequence of solution converges strongly in \(L^2(\Omega)\), uniformly in \(t \geq 0\), hence, along a subsequence, a.e. in \((t, x) \in (0, T) \times \Omega\). Therefore, reasoning as in (7.3), we find that the solution \(u = u^\mu\) of system (3.1) satisfies (7.4).

\[\Box\]

**Proposition 7.2** Let \(p > \frac{2n}{n+2}\) and \(\mu > 0\). Let \(u\) be the solution of (3.1) corresponding to \(u_0 \in L^q(\Omega)\), for some \(q \in [2, +\infty]\). Then
\[
\|u(t)\|_\infty \leq c \left(\|u_0\|_q^2 + c(\Omega, T)|\mu|^\frac{2}{\beta}\right)^{\frac{(2-p)\beta}{q}} \|u_0\|_q t^{-\frac{\beta}{q}} , \quad \forall t \in (\varepsilon, T),
\]
with \(\beta\) given in (2.19).

**Proof.** We consider a sequence \(\{u^m_0(x)\} \subset C^\infty_0(\Omega)\) strongly converging to \(u_0\) in \(L^q(\Omega)\), the sequence \(\{v^m,\nu\}\) of solutions of system (3.2), corresponding to the initial data \(\{u^m_0\}\), satisfying Proposition 3.1 and Corollary 3.1. Arguing as in Proposition 7.1, we arrive at the following estimate
\[
(v^m,\nu)(t, \varphi_0) = (v^m,\nu(0), \varphi(t)) + I_\eta.
\] (7.5)

From Lemma 2.4 with \(r = q'\) and \(M = B(\mu, u^m_0)\), and introducing two arbitrary conjugate exponents \(\overline{\sigma} > 2\) and \(\sigma'\), we can estimate the first term on the right-hand side of (7.5) as
\[
\left|(v^m,\nu(0), \varphi(t))\right| \leq \|u^m_0\|_q \|\varphi(t)\|_{q'} \leq c \left(B(\mu, u^m_0)\right)^{\frac{(2-p)\beta}{q}} \|u^m_0\|_q \|\varphi_0\|_1 t^{-\frac{\beta}{q}},
\]
\[
\leq c |\Omega|^\frac{1}{\overline{\sigma}} \left(B(\mu, u^m_0)\right)^{\frac{(2-p)\beta}{q}} \|u^m_0\|_q \|\varphi_0\|_\sigma' t^{-\frac{\beta}{\overline{\sigma}}},
\]
while the term \(I_\eta\) goes to zero along a subsequence, as \(\eta\) tends to zero. Therefore, we get
\[
\frac{\left|(v^m,\nu(t), \varphi_0)\right|}{\|\varphi_0\|_\sigma'} \leq c |\Omega|^\frac{1}{\overline{\sigma}} \left(B(\mu, u^m_0)\right)^{\frac{(2-p)\beta}{q}} \|u^m_0\|_q t^{-\frac{\beta}{\overline{\sigma}}} \text{ for all } \varphi_0 \in C^\infty_0(\Omega),
\]
which implies
\[
\|v^m,\nu(t)\|_\sigma \leq c |\Omega|^\frac{1}{\overline{\sigma}} \left(B(\mu, u^m_0)\right)^{\frac{(2-p)\beta}{q}} \|u^m_0\|_q t^{-\frac{\beta}{\overline{\sigma}}}.\]

Since the right-hand side is uniform with respect to \(\overline{\sigma}\), letting \(\overline{\sigma} \to \infty\), we obtain
\[
\|v^m,\nu(t)\|_\infty \leq c \left(B(\mu, u^m_0)\right)^{\frac{(2-p)\beta}{q}} \|u^m_0\|_q t^{-\frac{\beta}{q}} .
\]

As in the previous proof, taking into account suitable strong convergences, we may pass to the limit firstly as \(\nu \to 0\), then as \(m \to \infty\), and get the result.

\[\Box\]

**Proof of Theorem 1.4** - Let \(\mu > 0\). From Proposition 4.1, \(u^\mu \in L^\infty(0, T; V)\) and \(u^\mu_\nu \in L^2(\varepsilon, T; L^p(\Omega))\), uniformly in \(\mu > 0\). Moreover, the sequence \(\{u^\mu\}\) converges to the solution \(u\) of system (1.1), in suitable norms, as \(\mu\) tends to zero (see the proof of Theorem 1.1). From the compact embedding of \(V\) in \(L^s(\Omega)\), for any \(p < s < \frac{np}{n-p}\),
and the continuous embedding of $L^s(\Omega)$ in $L^p(\Omega)$, using Lemma 2.10, the sequence $\{u^\mu\}$ converges to $u$, as $\mu$ goes to zero, strongly in $L^s((\varepsilon, T) \times \Omega)$. Hence, $\{u^\mu\}$ converges to $u$ almost everywhere in $t$, $x$, along a subsequence. Therefore, along such a subsequence, we find
\[
|u(t, x)| \leq |u^\mu(t, x) - u(t, x)| + |u^\mu(t, x)| \leq |u^\mu(t, x) - u(t, x)| + \|u^\mu(t)\|_\infty, \quad (7.6)
\]
a.e. in $(t, x) \in (\varepsilon, T) \times \Omega$. Passing to the limit as $\mu$ goes to zero, and using the estimate in Proposition 7.1, we easily get
\[
\|u(t)\|_\infty \leq \|u_0\|_\infty.
\]
Therefore we find that $u$ satisfies (1.13).

In order to obtain estimate (1.14), we can repeat verbatim the previous arguments, employing Proposition 7.2 in place of Proposition 7.1 to estimate the $L^\infty$-norm of the sequence $\{u^\mu\}$ in (7.6).

\[\Box\]

\section*{Appendix}

This appendix is designed for the proof of Proposition 3.1 and its corollaries. Firstly we introduce an easy lemma, useful in the sequel.

\begin{lemma}
Let $g(t, x)$ and $F(t, x)$ be such that
\[
\|t^{\delta_1}(\mu + |F|^2)^{\frac{p}{2}}\|_{L^\infty(0,T;L^p(\Omega))} = K_1, \quad \|t^{\delta_2}(\mu + |F|^2)^{\frac{p-2}{4}}g\|_{L^2((0,T) \times \Omega)}^2 = K_2, \quad (A.1)
\]
and $t^\delta g(t, x) \in L^2(0,T;L^p(\Omega))$, with nonnegative constants $\mu, \delta_1, \delta_2, \delta$ such that $\delta := \frac{4-p}{2\delta_2} \delta_1 + \delta_2$. Then, one has
\[
\|t^\delta g(t, x)\|_{L^2((0,T) \times \Omega)}^2 \leq K_1^{2-p} K_2. \quad (A.2)
\]
\end{lemma}

\begin{proof}
Let us write the $L^2(0,T;L^p(\Omega))$ norm of $t^\delta g(t, x)$ as follows
\[
\int_0^t \tau^{2\delta} \|g(\tau)\|_p^2 d\tau = \int_0^t \tau^{2\delta} \left[ \int_\Omega (\mu + |F(\tau, x)|^2)^{\frac{p(\mu-2)}{4}} |g(\tau, x)|^p (\mu + |F(\tau, x)|^2)^{\frac{p(\mu-2)}{4}} dx \right]^{\frac{2}{p}} d\tau.
\]
By applying Hölder’s inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, we get
\[
\int_0^t \tau^{2\delta} \|g(\tau)\|_p^2 d\tau \leq \int_0^t \tau^{\delta_1(2-p)} \left[ \int_\Omega \tau^{2\delta_2} (\mu + |F(\tau, x)|^2)^{\frac{p-2}{2}} |g(\tau, x)|^2 dx \right] \left[ \int_\Omega (\mu + |F(\tau, x)|^2)^{\frac{p-2}{2}} dx \right]^{\frac{2}{p}} d\tau.
\]
Hence, from (A.1) we get
\[
\int_0^t \tau^{2\delta} \|g(\tau)\|_p^2 d\tau \leq \|t^{\delta_1}(\mu + |F|^2)^{\frac{p}{2}}\|_{L^\infty(0,T;L^p(\Omega))}^{2-p} \int_0^t \tau^{2\delta_2} \|g(\tau)\|_2^2 d\tau \leq K_1^{2-p} K_2.
\]
\[\Box\]
Proof of Proposition 3.1 - In the sequel we adopt the idea introduced by Prodi [21] in the context of Navier-Stokes equations, where the existence of a solution was proved by the Galerkin method with eigenfunctions of the Stokes operator as basis functions. Obviously, we replace this basis with the one given by eigenfunctions of the Laplace operator. So let \( \{a_j\} \) be the eigenfunctions of \( \Delta \), and denote by \( \lambda_j \) the corresponding eigenvalues:

\[-\Delta a_j = \lambda_j a_j, \quad \text{in } \Omega,\]

\[a_j = 0, \quad \text{on } \partial \Omega.\]

Recall that \( a_j \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \). We consider the Galerkin approximations related to system (3.2) of the form

\[v^k(t, x) = \sum_{j=1}^{k} c_{jk}(t) a_j(x) \quad k \in \mathbb{N}, \quad (A.3)\]

where the coefficients \( c_{jk}(t) \) satisfy the following system of ordinary differential equations

\[\begin{align*}
\dot{c}_{jk}(t) &= -\nu \sum_{i=1}^{k} b_{ji} c_{ik}(t) - \sum_{i=1}^{k} d_{ji} c_{ik}(t) = 0, \quad j = 1, \ldots, k, \\
c_{jk}(0) &= (v_v, a_j),
\end{align*}\]

(A.4)

with \( b_{ji} = (\nabla a_i, \nabla a_j) \), \( d_{ji} = ((\mu + |\nabla (c_{\nu n}(t)a_r)|^2)^{\frac{p-2}{2}} \nabla a_i, \nabla a_j) \). With this choice of \( c_{jk}(t) \) we impose that \( v^k(t, x) \) are solutions of the following system of \( k \)-differential equations

\[(v^k_t, a_j) - \nu (\Delta v^k, a_j) + (A_\mu(v^k), a_j) = 0, j = 1, \ldots, n, \quad (A.5)\]

with initial conditions \( c_{jk}(0) = (v_v, a_j) \), \( j = 1, \ldots, k \), where \( (\cdot, \cdot) \) denotes the standard \( L^2 \)-inner product. As the right-hand side of (A.4) is a Lipschitz function, due to the assumption \( \mu > 0 \) and using Lemma 2.7, the existence of a solution in a time interval \([0, t_k] \), \( t_k \in (0, T] \), follows by standard results on ordinary differential equations. The following a priori estimates (see (A.9)) will ensure that \( t_k = T \), for all \( k \in \mathbb{N} \).

\textbf{A priori estimates} - Let us multiply (A.5) by \( c_{jk} \), by \( dc_{jk}/dt \) and sum over \( j \). We get, respectively:

\[\frac{1}{2} \frac{d}{dt} \|v_t^k\|^2_2 + \nu \|\nabla v^k_t\|^2_2 + ((\mu + |\nabla v^k|^2)^{\frac{p-2}{2}} \nabla v^k)^2 = 0, \quad (A.6)\]

\[\|v_t^k\|^2_2 + \nu \frac{d}{dt} \|\nabla v^k_t\|^2_2 + \frac{1}{p} \frac{d}{dt} ((\mu + |\nabla v^k|^2)^{\frac{p}{2}})^{\frac{p}{2}} = 0. \quad (A.7)\]

Differentiating (A.5) with respect to \( t \), multiplying by \( dc_{jk}/dt \), then summing over \( j \), and observing that

\[\partial_t (a(\mu, v^k) \nabla v^k) = a(\mu, v^k) \nabla v^k_t + (p-2)((\mu + |\nabla v^k|^2)^{\frac{p-2}{2}} (\nabla v^k \cdot \nabla v^k) \cdot \nabla v^k, \]

we also have

\[\frac{1}{2} \frac{d}{dt} \|v^k_t\|^2_2 + \nu \|\nabla v^k_t\|^2_2 + (p-1)(\mu + |\nabla v^k|^2)^{\frac{p-2}{2}} \nabla v^k_t = 0. \quad (A.8)\]
The energy identity (A.6) implies

$$\|v^k(t)\|^2 + 2\nu \int_0^t \|\nabla v^k(\tau)\|^2 d\tau + 2 \int_0^t \|a(\mu, v^k(\tau)) \frac{\partial}{\partial \tau} v^k(\tau)\|^2 d\tau = \|v^k(0)\|^2,$$  

(A.9)

for any $t \in [0, t_k]$. Since $\|v^k(0)\|_2 \leq \|v_0\|_2$, (A.9) gives

$$\|v^k(t)\|^2 = |c_k(t)|^2 \leq \|v_0\|^2, \forall \ t \in [0, T].$$

Moreover

$$\|\nabla v^k\|_{L^2(0, T; L^2(\Omega))} \leq c(\nu) \|v_0\|_2,$$  

(A.10)

and, using standard arguments, we have

$$\|\nabla v^k\|^p_{L^p(0, T; L^p(\Omega))} \leq c \int_0^T \|a(\mu, v^k(\tau)) \frac{\partial}{\partial \tau} v^k(\tau)\|^2 d\tau + c(\Omega, T) \mu^\frac{2}{p} \leq B(\mu, v_0),$$

(A.11)

uniformly in $\nu > 0$, with $B$ given by (3.5). From this estimate it also follows that $v^k \in L^p(0, T; V)$ uniformly with respect to $k$, and

$$\|(\mu + |\nabla v^k|^2)^{\frac{p-2}{2}} \nabla v^k\|_{L^p'(0, T; L^p'(\Omega))} \leq \|\nabla v^k\|_{L^p(0, T; L^p(\Omega))} \leq B(\mu, v_0).$$

Hence

$$\|A_\mu(v^k)\|_{L^p'(0, T; V')},$$

(A.12)

which implies $A_\mu(v^k) \in L^p'(0, T; V')$ uniformly with respect to $k$. Moreover, since $p \in (1, 2)$, using (A.10) we get

$$\|(\mu + |\nabla v^k|^2)^{\frac{p-2}{2}} \nabla v^k\|_{L^2(0, T; L^2(\Omega))] \leq \|\nabla v^k\|_{L^2(0, T; L^2(\Omega))} \leq c(\nu) \|v_0\|^2.$$

Hence we also get

$$\|A_\mu(v^k)\|_{L^2(0, T; W^{-1, 2}(\Omega))} \leq c(\nu) \|v_0\|^2.$$

Further $\Delta v^k \in L^2(0, T; W^{-1, 2}(\Omega))$ since, for any $\varphi \in L^2(0, T; W^{1, 2}_0(\Omega))$

$$(\Delta v^k, \varphi) = -(\nabla v^k, \nabla \varphi),$$

and $\nabla v^k \in L^2(0, T; L^2(\Omega))$ from (A.10). This ensures that $v^k \in L^2(0, T; W^{-1, 2}(\Omega)).$

Integrating (A.7) from 0 to $t$, we get

$$\int_0^t \|v^k(\tau)\|^2 d\tau + \frac{\nu}{2} \|\nabla v^k(t)\|^2 + \frac{1}{p} \| (\mu + |\nabla v^k(\tau)|^2)\|_{L^p}^p \leq \frac{\nu}{2} \|\nabla v^k(0)\|^2 + \frac{1}{p} \| (\mu + |\nabla v^k(\tau)|^2)\|_{L^p}^p \leq c \|v_0\|^2 + c(\Omega, T) \mu^\frac{2}{p},$$

(A.13)

We are using the inequalities

$$\int_\Omega |\nabla u|^p dx = \int_{|\nabla u|^2 \geq \mu} |\nabla u|^p dx + \int_{|\nabla u|^2 \leq \mu} |\nabla u|^p dx \leq 2^{\frac{2-p}{p}} \int_\Omega a(\mu, u) |\nabla u|^2 dx + \int_{|\nabla u|^2 \leq \mu} \mu^\frac{2}{p} dx \leq c \int_\Omega a(\mu, u) |\nabla u|^2 dx + \mu^\frac{2}{p} |\Omega|.$$
where, observing that \( c_{ik}(0) = (v_0, a_l) = \frac{1}{N_l}\langle \nabla v_0, \nabla a_l \rangle = \frac{\langle \nabla v_0, \nabla a_l \rangle}{\| \nabla a_l \|_2^2} \), we have used
\[
\| \nabla v^k(0) \|_2 \leq \| \nabla v_0 \|_2. 
\]

Integrating inequality (A.8) from 0 to t, we get
\[
\| v^k(t) \|_2^2 + 2\nu \int_0^t \| \nabla v^k(\tau) \|_2^2 d\tau + 2(p-1) \int_0^t \| a(\mu, v^k(\tau)) \cdot \nabla v^k(\tau) \|_2^2 d\tau \leq \| v^k(0) \|_2^2. 
\] (A.14)

Let us estimate the right-hand side. Multiplying (A.5) by \( dc_{jk}/dt \) and summing over \( j \) we have
\[
\| v^k(t) \|_2^2 \leq \nu \| \Delta v^k \|_2 \| v^k \|_2 + \| A_{ij}(v^k) \|_2 \| v^k \|_2 \leq (\nu + c(\mu)) \| \Delta v^k \|_2 \| v^k \|_2, 
\]

since
\[
\| A_{ij}(v^k) \|_2^2 = \left\| \frac{\Delta v^k}{(\mu + |\nabla v^k|)^2} \right\|_2^2 + (p-2) \frac{\nabla v^k \cdot \nabla \nabla v^k \cdot \nabla v^k}{(\mu + |\nabla v^k|)^2} \left\| \frac{\Delta v^k}{(\mu + |\nabla v^k|)^2} \right\|_2^2 
\leq 2 \mu^{p-2} \| \Delta v^k \|_2 + 2(2-p) \mu^{p-2} \| D^2 v^k \|_2 \leq c(\mu) \| \Delta v^k \|_2^2. 
\]

Observing that, as \( c_{ik}(0) = (v_0, a_l) = \frac{\langle \Delta v_0, \Delta a_l \rangle}{\| \nabla a_l \|_2^2} \), then \( \| \Delta v^k(0) \|_2 \leq \| \Delta v_0 \|_2 \), we get
\[
\| v^k(t) \|_2^2 \leq (\nu + c(\mu)) \| \Delta v_0 \|_2^2. 
\] (A.15)

Finally, by using estimates (A.13), (A.14) and (A.15), we can apply Lemma A.1, with \( g = \nabla v^k, F = \nabla v^k, \delta_1 = \delta_2 = \delta = 0 \), and obtain
\[
\int_0^t \| \nabla v^k(\tau) \|_2^2 d\tau \leq (\nu + c(\mu)) \| v_0 \|_2^2 \left( \nu v \| \nabla v_0 \|_2^2 + c(\Omega, T) \mu \right)^{\frac{2-p}{p}}. 
\]

Passage to the limit - Using the above estimates we can extract a subsequence, still denoted by \( \{v^k\} \), such that, in the limit as \( k \) tends to \( \infty \), uniformly in \( \nu > 0 \),
\[
v^k \rightharpoonup v \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weakly} - *; \\
v^k \rightharpoonup v \text{ in } L^p(0, T; V) \text{ weakly}; \\
v^k(t) \rightharpoonup v_t \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weakly} - *; \\
v^k(t) \rightharpoonup \xi \text{ in } L^2(\Omega) \text{ weakly}; \\
\nabla v^k \rightharpoonup \nabla v_t \text{ in } L^2(0, T; L^p(\Omega)) \text{ weakly}; \\
A_{ij}(v^k) \rightharpoonup \chi \text{ in } L^p(0, T; V') \text{ weakly}. 
\] (A.16) (A.17)

The convergence (A.17) implies convergence in \( L^p(0, T; L^2(\Omega)) \), hence
\[
v^k(t) \rightharpoonup v_t \text{ in } L^p(0, T; V') \text{ weakly}, 
\] (A.18)
and \( \xi = v(t) \). Further, from (A.16) and (A.18), \( v \in C(0, T; L^2(\Omega)) \). Moreover, non-uniformly in \( \nu \), we also have
\[
v^k \rightharpoonup v \text{ in } L^2(0, T; W^{1,2}_0(\Omega)) \text{ weakly}, 
\]

\[ A_\mu(v^k) \rightharpoonup \chi \text{ in } L^2(0,T;W^{-1,2}(\Omega)) \text{ weakly,} \]
\[ v^k_t \rightharpoonup v_t \text{ in } L^2(0,T;W^{-1,2}(\Omega)) \text{ weakly,} \]

Following usual arguments from the theory of monotone operators (see [19], Chapter 2, Sec. 1), one shows that the non-linear part \( A_\mu(v^k) \) actually (weakly) converges to \( A_\mu(v) \), as \( k \) tends to \( \infty \) and that the limit \( v \) is a weak solution of system (3.2). The regularities stated for \( v \) follows from the analogous regularities of \( \{v^k\} \) and the lower semi-continuity of the norm for the weak convergence. Finally, the uniqueness follows from the monotonicity of \( A_\mu \).

\[ \square \]

**Proof of Corollary 3.1** - We argue as in the proof of Proposition 3.1, without exploiting the \( C^2_0 \)- regularity of the initial data. So, let us consider (A.7). Multiplying by \( t \), a simple computation gives
\[ t \|v^k_t\|_2^2 + \frac{\nu}{2} \frac{d}{dt}(t \|\nabla v^k\|_2^2) + \frac{1}{p} \frac{d}{dt}(t \|\mu + |\nabla v^k|^2\|_p^p) \]
\[ = \frac{\nu}{2} \|\nabla v^k\|_2^2 + \frac{1}{p} \|\mu + |\nabla v^k|^2\|_p^p. \]  
(A.19)

Integrating inequality (A.19) from 0 to \( t \) and estimating the right-hand side of (A.19) via inequalities (A.9) and (A.11), we get
\[ \int_0^t \tau \|v^k_\tau\|_2^2 d\tau + \frac{\nu}{2} t \|\nabla v^k(t)\|_2^2 + \frac{L}{p} \|\mu + |\nabla v^k(t)|^2\|_p^p \]
\[ = \frac{\nu}{2} \int_0^t \|\nabla v^k(\tau)\|_2^2 d\tau + \frac{1}{p} \int_0^t \|\mu + |\nabla v^k(\tau)|^2\|_p^p d\tau \leq c B(\mu, v_0), \]  
(A.20)

with \( B \) given in (3.5). Hence, in particular, we obtain
\[ \|t^{\frac{\nu}{2}} v^k\|_{L^2(0,T;L^2(\Omega))} + t \|\nabla v^k(t)\|_p^p \leq c B(\mu, v_0). \]  
(A.21)

Let us consider (A.8). Multiplication by \( t^2 \) gives
\[ \frac{1}{2} \frac{d}{dt}(t^2 \|v^k_t\|_2^2) + \nu t^2 \|\nabla v^k\|_2^2 + (p - 1)t^2 \|\mu + |\nabla v^k|^2\|_p^p \leq \frac{\nu}{2} t \|\nabla v^k\|_2^2. \]  
(A.22)

Integrating inequality (A.22) from 0 to \( t \), and then estimating the right-hand side via (A.20), we get
\[ t^2 \|v^k_t(t)\|_2^2 + 2 \nu \int_0^t \tau^2 \|\nabla v^k_\tau(\tau)\|_2^2 d\tau \]
\[ + (p - 1) \int_0^t \tau^2 \|\mu + |\nabla v^k(\tau)|^2\|_p^p \leq 2 \int_0^t \tau \| v^k_\tau(\tau)\|_2^2 d\tau \leq c B, \]  
(A.23)

which ensures that \( t \nabla v^k_t \in L^2(0,T;L^2(\Omega)) \), non-uniformly in \( \nu > 0 \). Finally, by using estimates (A.20) and (A.23), we can apply Lemma A.1, with \( g = \nabla v^k_t, F = \nabla v^k, \delta_1 = \frac{1}{p}, \delta_2 = 1, \) hence \( \delta = \frac{\nu + 2}{2p} \), and obtain
\[ \int_0^t \tau^{\frac{\nu + 2}{2p}} \|\nabla v^k_\tau(\tau)\|_p^p d\tau \leq c B^{\frac{2}{p}}(\mu, v_0). \]  
(A.24)
The previous bounds (A.21), (A.23) and (A.24) ensure that, up to subsequences, in the limit of $k \to \infty$:

$$
t^\frac{1}{r} \nabla v^k \rightharpoonup t^\frac{1}{r} \nabla v \quad \text{in} \quad L^\infty(0, T; L^p(\Omega)) \quad \text{weakly-*},$$

$$
t v_t^k \rightharpoonup t v_t \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)) \quad \text{weakly-*},$$

$$
t A_\mu(v_t^k) \rightharpoonup t \chi_t \quad \text{in} \quad L^2(0, T; W^{-1,2}(\Omega)) \quad \text{weakly},$$

$$
t^\frac{\nu + 2}{2} \nabla v_t^k \rightharpoonup t^\frac{\nu + 2}{2} \nabla v_t \quad \text{in} \quad L^2(0, T; L^p(\Omega)) \quad \text{weakly},$$

and completes the proof.

Proof of Corollary 3.2 - Let us consider the Galerkin approximations (A.3). Let us multiply (A.5) by $\lambda_j c_{jk}$ and sum over $j$. By observing that $(v_t^k, \sum_{j=1}^k \lambda_j c_{jk}a_j) = (v_t^k, -\Delta v^k) = \frac{d}{dt} \|\nabla v^k\|_2^2$, recalling Lemma 2.1 and using Cauchy’s inequality, we get

$$
\frac{1}{2} \frac{d}{dt} \|\nabla v^k\|_2^2 + \nu \|\Delta v^k\|_2^2 + \|\mu + |\nabla v^k|^2\|_2^2 \Delta v^k \|_2^2
\leq \left( (2 - p) C_1 + \frac{\delta}{2} \right) \|\mu + |\nabla v^k|^2\|_2^2 \Delta v^k \|_2^2 + C \|\nabla v^k\|_p^p + C \mu^{\frac{2}{p}} |\Omega|, \tag{A.25}
$$

for any $\delta > 0$. Hence, since $p > p_0 \geq \frac{3}{2}$, choosing $\delta = 1 - (2 - p)C_1 \equiv \overline{C}(p)$ we get the inequality

$$
\frac{d}{dt} \|\nabla v^k\|_2^2 + \overline{C}(p) \|\mu + |\nabla v^k|^2\|_2^2 \Delta v^k \|_2^2 \leq C \|\nabla v^k\|_p^p + C \mu^{\frac{2}{p}} |\Omega|. \tag{A.26}
$$

By using that $v_0 \in W^{1,2}_0(\Omega)$, integrating (A.26) from 0 to $t$, we have

$$
\|\nabla v^k(t)\|_2^2 + \overline{C}(p) \int_0^t \|\mu + |\nabla v^k(\tau)|^2\|_2^2 \Delta v^k(\tau) \|_2^2 d\tau
\leq \|\nabla v_0\|_2^2 + C \int_0^t (\|\nabla v^k(\tau)\|_p^p + C \mu^{\frac{2}{p}} |\Omega|)d\tau,
$$

which ensures that $\nabla v^k \in L^\infty(0, T; L^2(\Omega))$ uniformly with respect to $k$, and, applying ones again Lemma 2.1, $(\mu + |\nabla v^k|^2)^{\frac{2}{p+2}} D^2 v^k \in L^2(\Omega)$ uniformly with respect to $k$.

Moreover, by using estimates (A.13), (A.14) and (A.15), and employing Lemma A.1, with $g = D^2 v^k$, $F = \nabla v^k$, $\delta_1 = \delta_2 = \delta = 0$, we obtain

$$
\int_0^t \|D^2 v^k(\tau)\|_p^2 d\tau
\leq \|\mu + |\nabla v^k|^2\|_L^\infty(0,T;L^p(\Omega)) \int_0^t \|\nabla v^k(\tau)\|_p^p + C \mu^{\frac{2}{p}} |\Omega|)d\tau, \tag{A.26}
$$

If we do not exploit the $W^{1,2}_0$-regularity of the initial data, but just its $L^2$-integrability, we can argue as follows. Throughout this proof, we denote $\tilde{B}(\mu, v_0)$ just by $B$. Multiplication of (A.26) by $t^{\alpha}$, for some $\alpha \geq 1$ that will be specified later, gives

$$
\frac{d}{dt}(t^{\alpha} \|\nabla v^k\|_2^2) + \overline{C}(p) t^{\alpha} \|\mu + |\nabla v^k|^2\|_2^2 \Delta v^k \|_2^2
\leq \alpha t^{\alpha - 1} \|\nabla v^k\|_2^2 + C t^{\alpha} \|\nabla v^k\|_p^p + C t^{\alpha} \mu^{\frac{2}{p}} |\Omega|.
$$
An integration from 0 to \( t \) gives

\[
t^\alpha \| \nabla v^k(t) \|_2^2 + \int_0^t \tau^\alpha \| (\mu + |\nabla v^k(\tau)|^2)^{\frac{\alpha-2}{2}} \Delta v^k(\tau) \|_2^2 \, d\tau \\
\leq \alpha \int_0^t \tau^{\alpha-1} \| \nabla v^k(\tau) \|_2^2 \, d\tau + C \int_0^t \left( \tau^\alpha \| \nabla v^k(\tau) \|_p^p + \tau^\alpha \mu \mathbf{f} \|\Omega\| \right) \, d\tau .
\]  

(A.27)

For the first integral on the right-hand side we argue as follows. Observing that \( p > \frac{2n}{n+2} \), by using a Gagliardo-Nirenberg inequality we get

\[
\| \nabla v^k \|_2 \leq C \| D^2 v^k \|_p^{2a} \| \nabla v^k \|_p^{2(1-a)} , \quad a = n \left( \frac{1}{p} - \frac{1}{2} \right) \in (0, 1) .
\]

Observing that

\[
\| D^2 v^k \|_p^2 \leq \| (\mu + |\nabla v^k|^2)^{\frac{\alpha-2}{2}} D^2 v^k \|_2^2 \| (\mu + |\nabla v^k|^2)^{\frac{1}{2}} \|_p^{2-p} ,
\]

we have

\[
\| \nabla v^k(\tau) \|_2 \leq C \| (\mu + |\nabla v^k|^2)^{\frac{\alpha-2}{2}} D^2 v^k \|_2^2 \| (\mu + |\nabla v^k|^2)^{\frac{1}{2}} \|_p^{2-p} \| \nabla v^k \|_p^{2(1-a)} ,
\]

and, by Cauchy’s inequality, for any \( \delta > 0 \), we get

\[
\tau^{\alpha-1} \| \nabla v^k(\tau) \|_2^2 \\
\leq \frac{\delta}{2} \| \nabla v^k(\tau) \|_2^2 + \frac{C}{2\delta} \tau^\alpha \left( \| (\mu + |\nabla v^k|^2)^{\frac{1}{2}} \|_p^{2-p} \| \nabla v^k \|_p \right) \| (\mu + |\nabla v^k|^2)^{\frac{\alpha-2}{2}} D^2 v^k \|_2^2 \| (\mu + |\nabla v^k|^2)^{\frac{1}{2}} \|_p^{2-p} \| \nabla v^k \|_p^{2(1-a)}
\]

(A.28)

The first term on the right-hand side can be estimated using Lemma 2.1. Let us integrate the last two terms on the right-hand side of (A.28) from 0 and \( t \). Since

\[
\int_0^t \tau^\alpha \| \nabla v^k(\tau) \|_2^2 \, d\tau = \int_0^t \tau^\alpha \| \nabla v^k(\tau) \|_p^{2-a} \| \nabla v^k(\tau) \|_p^a \, d\tau ,
\]

by choosing \( \alpha \) in such a way that

\[
\alpha = \frac{1}{1-a} + \frac{2-p}{2p(1-a)},
\]

hence \( \alpha \) as in (3.6), and then using (A.21) and (A.11), we get

\[
\int_0^t \tau^\alpha \| \nabla v^k(\tau) \|_p^{2-a} \, d\tau \leq B \frac{2-p}{2p(1-a)} \int_0^t \| \nabla v^k(\tau) \|_p^a \, d\tau \leq B \frac{2-p}{2p(1-a)}. 
\]

Fixed \( \alpha \), in a similar way we also get

\[
\int_0^t \tau^\alpha \| \nabla v^k(\tau) \|_p^2 \, d\tau = \int_0^t \tau^\frac{2-2p}{2p(1-a)} | \nabla v^k(\tau) |^{2-p} \| \nabla v^k(\tau) \|_p^2 \, d\tau \\
\leq B \frac{2-p}{2p(1-a)} \int_0^t \| \nabla v^k(\tau) \|_p^2 \, d\tau \leq B \frac{2-p}{2p(1-a)}. 
\]
Therefore, integrating (A.28) from 0 to $t$ and using the previous estimates for the terms on the right-hand side, we obtain
\[
\int_0^t \tau^\alpha \|\nabla v^k(\tau)\|_p^2 \, d\tau \leq C \left( \frac{\delta}{2} \int_0^t \tau^\alpha \|a(\mu, v^k(\tau))^{\frac{2}{p}} \Delta v^k(\tau)\|_p^2 \, d\tau \right) + C \delta \int_0^t \tau^\alpha (\|\nabla v(\tau)\|_p^p + \mu^{\frac{2}{p}} |\Omega|) \, d\tau + \frac{C}{2\delta} \mu \left( \frac{2(p-\alpha)}{p} \right) B_{\tau}^{\frac{2-p}{p}} \cdot t^{\frac{2-p}{p}} \cdot \frac{\alpha}{2} + \frac{C}{2\delta} B_{\tau}^{\frac{2-p}{p}} \cdot t^{\frac{2-p}{p}} \cdot \frac{\alpha}{2}.
\]

Inserting this estimate in (A.27), then choosing $\delta = \frac{C(p)}{\alpha C}$ and using (A.20), we arrive at
\[
t^\alpha \|\nabla v^k(t)\|_p^2 \leq C \int_0^t \tau^\alpha \|a(\mu, v^k)^{\frac{2}{p}} \Delta v^k(\tau)\|_p^2 \, d\tau \leq C \mu \frac{(2(p-\alpha))}{(p)} B_{\tau}^{\frac{2-p}{p}} \cdot t^{\frac{2-p}{p}} \cdot \frac{\alpha}{2} + C B t^\alpha + c B t^{\alpha+1}.
\]

Observing that $\frac{2-p}{p} < \alpha$, and defining $\beta_1(p) = \frac{1}{\alpha} (\alpha - \frac{2-p}{p} - \frac{\alpha}{2}) = \frac{1}{\alpha} (\alpha - \frac{n(2-p)^2}{p(2p-n)(2-p)})$, as in (3.7) the above estimate shows in particular that $t^{\beta_1(p)} \nabla v^k \in L^\infty(0, T; L^2(\Omega))$.

This estimate, together with $\nabla v^k \in C(\varepsilon, T; L^p(\Omega))$, which follows from Corollary 3.1, gives $t^{\beta_1(p)} \nabla v^k \in C_w(0, T; L^2(\Omega))$. The strong continuity can be obtained as follows.

From (A.26) we have, for any $t > s > 0$,
\[
\|\nabla v^k(t)\|_p^2 \leq C \int_s^t \left( \|\nabla v^k(\tau)\|_p^p + C \mu^{\frac{2}{p}} |\Omega| \right) \, d\tau + \|\nabla v^k(s)\|_p^2.
\]  

(A.29)

By using the identity
\[
\|\nabla v^k(t) - \nabla v^k(s)\|_p^2 = \|\nabla v^k(t)\|_p^2 + \|\nabla v^k(s)\|_p^2 - 2(\nabla v^k(t), \nabla v^k(s)),
\]
then estimate (A.29) and the weak continuity of $\nabla v^k(t)$ in $L^2(\Omega)$ we get the result.

Finally, by applying Lemma A.1, with $\mu = D^2 v^k$, $F = \nabla v^k$, $\delta_1 = \frac{1}{p}$, $\delta_2 = \frac{2}{2}$, hence $\delta = \frac{p+2}{2p}$, and obtain
\[
\int_0^t \tau^\alpha \left( \frac{2\alpha}{p} \right) \|D^2 v^k(\tau)\|_p^p \, d\tau \leq t^{\frac{2-p}{p}} \left( \mu + \|\nabla v^k\|_p^2 \right) \frac{2-p}{p} \int_0^t \tau^\alpha \|a(\mu, v^k)^{\frac{2}{p}} D^2 v^k(\tau)\|_p^2 \, d\tau \leq C(B, T).
\]

The previous bounds, all uniform with respect to $k$, ensure the weak-* convergence of a subsequence of $\{t^{\beta_1(p)} \nabla v^k\}$ in the space $L^\infty(0, T; L^2(\Omega))$ and, recalling the expression of $\beta_2(p)$ given in (3.7), the weak convergence of a subsequence of $\{t^{\beta_2(p)} D^2 v^k\}$ in the space $L^2(0, T; L^p(\Omega))$, as $k \to \infty$, uniformly in $\nu$, $\mu$. From Proposition 3.1, we get that the limit solution $v$ of (3.2) satisfies $t^{\beta_1(p)} \nabla v \in L^\infty(0, T; L^2(\Omega))$ and $t^{\beta_2(p)} v \in L^2(0, T; W^{2,p}(\Omega))$.

\[\square\]

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