Hamilton–Jacobi theory for contact systems: autonomous and non-autonomous

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Abstract

In this paper we obtain two Hamilton–Jacobi equations for time dependent contact Hamiltonian systems. In these systems there is a dissipation parameter and the fact of obtaining two equations reflects whether we are looking for solutions that depend on this parameter or not. We also study the existence of complete solutions and the integrability problem. The theory is illustrated with two examples.

Keywords: Hamilton–Jacobi equation, contact Hamiltonian systems, integrability

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1 Introduction

Contact Hamiltonian systems have been widely used in the study of thermodynamics. Recently there has been a renewed interest in these systems, since, unlike symplectic Hamiltonian systems, they lead to dissipated rather than conserved quantities [25, 37, 38]. These systems are also relevant to describe mechanical systems with certain types of damping [23, 25, 37], quantum mechanics [9], circuit theory, control theory [22, 56, 60] and black holes [39], among many others. The underlying variational principle is the so-called Herglotz principle [24, 43], a generalization of the well-known Hamilton principle, which gives rise to action-dependent Lagrangian systems, which are becoming popular in theoretical physics [47–49].

Hamilton–Jacobi theory provides a remarkably powerful method to integrate the dynamics of many Hamiltonian systems. In particular, for a completely integrable system, if one knows a complete solution of the Hamilton–Jacobi problem, the dynamics of the system can be easily reduced to quadratures [40–42]. Geometrically, the Hamilton–Jacobi problem consists on finding a section γ of πQ: T∗Q → Q which transforms integral curves of a projected vector field $X_H^*$ on Q into integral curves of the dynamical vector field $X_H$ on $T^*Q$ [1, 6]. This idea can be naturally extended to other vector bundles. As a matter of fact, it has been applied in many other different contexts, such as nonholonomic systems [7, 13, 44, 57], singular Lagrangian systems [27, 28, 50], higher-order systems [12], field theories [4, 26, 31–33, 61, 62] or systems with external forces [19, 20]. A unifying Hamilton–Jacobi theory for almost-Poisson manifolds was developed in reference [29]. Hamilton–Jacobi theory has also been extended to Hamiltonian systems with non-canonical symplectic structures [55], non-Hamiltonian systems [58], locally conformally symplectic manifolds [35], Nambu–Poisson [14] and Nambu–Jacobi [15] manifolds, Lie algebroids [51] and implicit differential systems [34]. The applications of Hamilton–Jacobi theory include the relation between classical and quantum mechanics [3, 8, 54], information geometry [10, 11], control theory [59] and the study of phase transitions [46].

We have recently initiated the study of time-dependent contact Hamiltonian systems [16], and the underlying geometric structures, which we call co-contact, since they are a combination of cosymplectic (the setting for studying time-dependent Hamiltonian systems) and contact structures. Such structures thus consist of two 1-forms, $\tau$ and $\eta$, where $\tau$ is closed and $\tau \wedge \eta \wedge d\eta^n$ is a volume form, in a 2n + 2-dimensional manifold. Obviously, the geometric model is $\mathbb{R} \times T^*Q \times \mathbb{R}$ with canonical structures. In fact, in [16] we have been able to identify that such structures in turn define a Jacobi structure such that the characteristic foliation is formed by contact leaves.

The aim of the present paper is to develop a Hamilton–Jacobi theory for this type of Hamiltonian systems. We follow the line undertaken in previous papers, considering sections of the canonical fibrations $\mathbb{R} \times T^*Q \times \mathbb{R} \to Q$ and $\mathbb{R} \times T^*Q \times \mathbb{R} \to Q \times \mathbb{R}$ which allow us to project the Hamiltonian vector field to the base and by comparing the values on the section, we obtain the corresponding Hamilton–Jacobi equations. This study has led us to the symmetries and first integrals of the system as well as to the existence of complete solutions and the corresponding integrability problems.

The paper is structured as follows. Section 2 is devoted to review time-dependent contact Hamiltonian systems introducing the basic elements necessary to follow the rest of the paper. In Section 3 we study symmetries and dissipated quantities in cocontact systems. In Sections 4 and 5 we deal with the two main results of the paper: the action-independent approach and the action-dependent approach to the Hamilton–Jacobi equation, respectively. Finally, in Section 6 we discuss two examples that illustrate the theory previously developed.

From now on, all the manifolds and mappings are assumed to be smooth and second-countable. Sum over crossed repeated indices is understood.

2 Review on time-dependent contact systems

In this section we are going to review some fundamentals on cocontact geometry and time-dependent contact Hamiltonian systems introduced in [16].
2.1 Cocontact manifolds

Definition 2.1. A cocontact structure on a \((2n + 2)\)-dimensional manifold \(M\) is a couple \((\tau, \eta)\), where \(\tau, \eta \in \Omega^1(M)\) and \(d\tau = 0\), such that \(\tau \wedge \eta \wedge (d\eta)^n\) is a volume form on \(M\). In this case, \((M, \tau, \eta)\) is called a cocontact manifold.

Given a cocontact manifold \((M, \tau, \eta)\), the distribution \(\mathcal{H} = \ker \eta\) is called the horizontal or contact distribution. Notice that this distribution is non-integrable.

Example 2.2. Let \((P, \eta_0)\) be a contact manifold and consider the product manifold \(M = \mathbb{R} \times P\). Denoting by \(d\tau\) the pullback to \(M\) of the volume form in \(\mathbb{R}\) and by \(\eta\) the pullback of \(\eta_0\) to \(M\), we have that \((d\tau, \eta)\) is a cocontact structure on \(M\).

Example 2.3. Let \((P, \tau, -d\theta)\) be an exact cosymplectic manifold and consider the product manifold \(M = P \times \mathbb{R}\). Denoting by \(z\) the coordinate in \(\mathbb{R}\) we define the 1-form \(\eta = dz - \theta\). Then, \((\tau, \eta)\) is a cocontact structure on \(M = P \times \mathbb{R}\).

Example 2.4 (Canonical cocontact manifold). Let \(Q\) be an \(n\)-dimensional smooth manifold with local coordinates \((q^i)\) and its cotangent bundle \(T^*Q\) with induced natural coordinates \((q^i, p_i)\). Consider the product manifolds \(\mathbb{R} \times T^*Q\) with coordinates \((t, q^i, p_i)\), \(T^*Q \times \mathbb{R}\) with coordinates \((q^i, p_i, z)\) and \(\mathbb{R} \times T^*Q \times \mathbb{R}\) with coordinates \((t, q^i, p_i, z)\) and the canonical projections

\[
\begin{array}{ccc}
\mathbb{R} \times T^*Q \times \mathbb{R} & \xrightarrow{\rho_1} & \mathbb{R} \times T^*Q \\
\pi & \xleftarrow{\pi_1} & T^*Q \\
\pi_2 & \xrightarrow{\rho_2} & T^*Q \times \mathbb{R}
\end{array}
\]

Let \(\theta_0 \in \Omega^1(T^*Q)\) be the canonical 1-form of the cotangent bundle, which has local expression \(\theta_0 = p_i dq^i\). Denoting by \(\theta_2 = \pi_2^* \theta\), we have that \((dt, \eta)\) is a cosymplectic structure in \(\mathbb{R} \times T^*Q\). On the other hand, denoting by \(\theta_1 = \pi_1^* \theta_0\), we have that \(\eta_1 = dz - \theta_1\) is a contact form in \(T^*Q \times \mathbb{R}\).

Finally, consider the 1-form \(\theta = \rho_1^* \theta_2 = \rho_2^* \theta_1 = \pi^* \theta_0 \in \Omega^1(\mathbb{R} \times T^*Q \times \mathbb{R})\) and let \(\eta = dz - \theta\). Then, \((dt, \eta)\) is a cocontact structure in \(\mathbb{R} \times T^*Q \times \mathbb{R}\). The local expression of the 1-form \(\eta\) is

\[
\eta = dz - p_i dq^i.
\]

Given a cocontact manifold \((M, \tau, \eta)\), we have the flat isomorphism

\[
b : v \in TM \mapsto (\iota_v \tau) \tau + \iota_v \eta + (\iota_v \eta) \eta \in \Omega^1(M).
\]

This isomorphism can be trivially extended to an isomorphism of \(\mathcal{C}^\infty(M)\)-modules \(b : \mathfrak{X}(M) \to \Omega^1(M)\). The inverse of the flat morphism is denoted by \(b^{-1} : \Omega^1(M) \to \mathfrak{X}(M)\) and called the sharp isomorphism.

Moreover, we have the following results, whose proofs can be found in [16].

Proposition 2.5. On every cocontact manifold \((M, \tau, \eta)\) there exist two vector fields \(R_t, R_z\) on \(M\) such that

\[
\begin{align*}
\iota_{R_t} \tau &= 1, & \iota_{R_t} \eta &= 0, & \iota_{R_t} d\eta &= 0, \\
\iota_{R_z} \tau &= 0, & \iota_{R_z} \eta &= 1, & \iota_{R_z} d\eta &= 0.
\end{align*}
\]

or, equivalently, \(R_t = b^{-1}(\tau)\) and \(R_z = b^{-1}(\eta)\). These vector fields \(R_t\) and \(R_z\) are called time and contact Reeb vector fields respectively.
Theorem 2.6 (Cocontact Darboux theorem). Given a cocontact manifold \((M, \tau, \eta)\), around every point \(p \in M\) there exist local coordinates \((t, q^i, p_i, z)\) such that
\[
\tau = dt, \quad \eta = dz - p_i dq^i.
\]
These coordinates are called canonical or Darboux coordinates. In addition, in Darboux coordinates, the Reeb vector fields read
\[
R_t = \frac{\partial}{\partial t}, \quad R_z = \frac{\partial}{\partial z}.
\]

Proposition 2.7. Let \((M, \tau, \eta)\) be a cocontact manifold. Then, \((M, \Lambda, E)\) is a Jacobi manifold, where
\[
\Lambda(\alpha, \beta) = -d\eta(\sharp\alpha, \sharp\beta), \quad E = -R_z.
\]
The bivector \(\Lambda\) induces a \(C^\infty(M)\)-module morphism \(\hat{\Lambda} : \Omega^1(M) \to \mathfrak{X}(M)\) given by
\[
\hat{\Lambda}(\alpha) = \Lambda(\alpha, \cdot) = \sharp\alpha - \alpha(R_z)R_z - \alpha(R_t)R_t.
\]
(2)
It can be seen that \(\ker \hat{\Lambda} = \langle \tau, \eta \rangle\). The morphism \(\hat{\Lambda}\) is also denoted by \(\sharp\Lambda\) in the literature [18, 23].

Taking Darboux coordinates \((t, q^i, p, z)\), the bivector \(\Lambda\) has local expression
\[
\Lambda = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial z},
\]
and the Jacobi bracket reads
\[
\{f, g\} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} - \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial z} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial z} \right) - f \frac{\partial g}{\partial z} + g \frac{\partial f}{\partial z}.
\]
In particular, one has
\[
\{q^i, q^j\} = \{p_i, p_j\} = 0, \quad \{q^i, p_j\} = \delta^i_j, \quad \{q^i, z\} = -q^i, \quad \{p_i, z\} = -2p_i.
\]

2.2 Cocontact Hamiltonian systems

Definition 2.8. A cocontact Hamiltonian system is family \((M, \tau, \eta, H)\) where \((\tau, \eta)\) is a cocontact structure on \(M\) and \(H : M \to \mathbb{R}\) is a Hamiltonian function. The cocontact Hamilton equations for a curve \(\psi : I \subset \mathbb{R} \to M\) are
\[
\begin{align*}
\iota_{\psi'} d\eta &= dH - (\mathcal{L}_{R_z} H) \eta - (\mathcal{L}_{R_t} H) \tau, \\
\iota_{\psi'} \eta &= -H, \\
\iota_{\psi'} \tau &= 1,
\end{align*}
\]
(3)
where \(\psi' : I \subset \mathbb{R} \to TM\) is the canonical lift of \(\psi\) to the tangent bundle \(TM\). The cocontact Hamiltonian equations for a vector field \(X_H \in \mathfrak{X}(M)\) are:
\[
\begin{align*}
\iota_{X_H} d\eta &= dH - (\mathcal{L}_{R_z} H) \eta - (\mathcal{L}_{R_t} H) \tau, \\
\iota_{X_H} \eta &= -H, \\
\iota_{X_H} \tau &= 1,
\end{align*}
\]
(4)
or equivalently, \(\flat(X_H) = dH - (\mathcal{L}_{R_z} H + H) \eta + (1 - \mathcal{L}_{R_t} H) \tau\). The unique solution to these equations is called the cocontact Hamiltonian vector field.
Given a curve $\psi$ with local expression $\psi(r) = (f(r), q^i(r), p_i(r), z(r))$, the third equation in (3) imposes that $f(r) = r + cnt$, thus we will denote $r \equiv t$, while the other equations read:

\[
\begin{align*}
\dot{q}^i &= \frac{\partial H}{\partial p_i}, \\
\dot{p}_i &= -\left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right), \\
\dot{z} &= p_i \frac{\partial H}{\partial p_i} - H.
\end{align*}
\]

(5)

On the other hand, the local expression of the cocontact Hamiltonian vector field is

\[X_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right) \frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right) \frac{\partial}{\partial z}.
\]

The integral curves of the cocontact Hamiltonian vector field satisfy the following variational principle [53], which is a Hamiltonian version of the Herglotz principle [43].

**Theorem 2.9** (Hamiltonian formulation of the Herglotz principle). Given a Hamiltonian $H : \mathbb{R} \times T^*Q \times \mathbb{R} \to \mathbb{R}$, a curve $c = (\text{Id}_{\mathbb{R}}, q, p, z) : [0, T] \to T^*Q \times \mathbb{R}$ is an integral curve of $X_H$ if and only if it is a critical point of the action map:

\[\mathcal{A}(c) = \left(\int_0^T p(t)\dot{q}(t) - H(t, q(t), p(t), z(t))\right) dt\]

among all curves satisfying $c(0) = c_0$, $c(T) = c_T$ and $\dot{z} = p(t)\dot{q}(t) - H(t, q(t), p(t), z(t))$.

3 Symmetries and dissipated quantities in cocontact systems

There are several notions of symmetries in contact mechanics depending on the structures they preserve [17, 36]. However, in the present paper we will restrict ourselves to what we call generalized dynamical symmetries (see [38] for other symmetries). In some cases we will restrict ourselves to the case of cocontact manifolds of the form $M = \mathbb{R} \times N$ where $N$ is a contact manifold (see Example 2.2). In this case, the natural projection $\mathbb{R} \times N \to \mathbb{R}$ defines a global canonical coordinate $t$.

**Definition 3.1.** Let $(M, \tau, \eta, H)$ be a cocontact Hamiltonian system and let $X_H$ be its cocontact Hamiltonian vector field.

- If $M = \mathbb{R} \times N$ with $N$ a contact manifold, a **generalized dynamical symmetry** is a diffeomorphism $\Phi : M \to M$ such that $\eta(\Phi_*X_H) = \eta(X_H)$ and $\Phi^*t = t$.

- An **infinitesimal generalized dynamical symmetry** is a vector field $Y \in \mathfrak{X}(M)$ such that $\eta([Y, X_H]) = 0$ and $i_Y\tau = 0$. In particular, if $M = \mathbb{R} \times N$ with $N$ a contact manifold, the flow of $Y$ is made of generalized dynamical symmetries.

**Definition 3.2.** Let $(M, \tau, \eta, H)$ be a cocontact Hamiltonian system. A **dissipated quantity** is a function $f \in C^\infty(M)$ such that

\[\mathcal{L}_{X_H} f = -R_z(H)f.
\]

It is worth pointing out that, unlike in the contact case, the Hamiltonian function is not, in general, a dissipated quantity. Indeed, using that

\[\mathcal{L}_{X_H} H = -R_z(H)H + R_t(H),
\]

it is clear that $H$ is a dissipated quantity if and only if it is time-independent, i.e. $R_t(H) = 0$. This resembles the cosymplectic case, where the Hamiltonian function is conserved if, and only if, it is time-independent (see [5]).
**Proposition 3.3.** A function $f \in \mathcal{C}^\infty(M)$ is a dissipated quantity if and only if $\{H, f\} = R_t(f)$, where $\{\cdot, \cdot\}$ is the Jacobi bracket associated to the cocontact structure $(\tau, \eta)$.

**Proof.** The Jacobi bracket of $f$ and $H$ is given by

$$\{f, H\} = \Lambda(df, dH) + fE(H) - HE(f) = -d\eta(\sharp df, \sharp dH) - fR_z(H) + HR_z(f),$$

but

$$\sharp df = X_f + (R_z(f) + f)R_z - (1 - R_t(f))R_t,$$

so

$$\iota_{\sharp df} \eta = \iota_{X_f} \eta = df - R_z(f)\eta - R_t(f)\tau,$$

and thus

$$d\eta(\sharp df, \sharp dH) = X_H(f) + R_z(f)H - R_t(f).$$

Hence,

$$\{H, f\} + R_t(f) = X_H(f) + R_z(H)f.$$

In particular, the right-hand side vanishes if and only if $f$ is a dissipated quantity. \(\square\)

**Theorem 3.4** (Noether’s theorem). Let $Y$ be an infinitesimal generalized dynamical symmetry of the cocontact Hamiltonian system $(M, \tau, \eta, H)$. Then, $f = -\iota_Y \eta$ is a dissipated quantity of the system. Conversely, given a dissipated quantity $f \in \mathcal{C}^\infty(M)$, the vector field $Y = X_f - R_t$, where $X_f$ is the Hamiltonian vector field associated to $f$, is an infinitesimal generalized dynamical symmetry and $f = -\iota_Y \eta$.

**Proof.** Let $f = -\iota_Y \eta$, where $Y$ is an infinitesimal generalized dynamical symmetry. Then,

$$\mathcal{L}_{X_H} f = -\mathcal{L}_{X_H} \iota_Y \eta = -\iota_Y \mathcal{L}_{X_H} \eta - \iota_{[X_H, Y]} \eta = \iota_Y (R_z(H)\eta + R_t(H)\tau) = R_z(H)\iota_Y \eta = R_z(H)f,$$

and thus $f$ is a dissipated quantity.

On the other hand, given a dissipated quantity $f$, let $Y = X_f - R_t$. Then, it is clear that $f = -\iota_Y \eta$.

In addition, $\iota_Y \tau = 0$, and

$$\iota_{[X_H, Y]} \eta = \mathcal{L}_{X_H} \iota_Y \eta - \iota_Y \mathcal{L}_{X_H} \eta = -\mathcal{L}_{X_H} f + \iota_Y (R_z(H)\eta + R_t(H)\tau) = R_z(H)f - R_z(H)\iota_Y \eta = 0.$$  \(\square\)

The symmetries presented yield dissipated quantities. However, we are also interested in finding conserved quantities. The latter are important due to their elation with complete solutions of the Hamilton–Jacobi problem (see Section 5).

**Definition 3.5.** A **conserved quantity** of a cocontact Hamiltonian system $(M, \tau, \eta, H)$ is a function $g \in \mathcal{C}^\infty(M)$ such that

$$\mathcal{L}_{X_H} g = 0.$$

Taking into account that every dissipated quantity changes with the same rate $R_z(H)$, we have the following result, whose proof is straightforward.

**Proposition 3.6.** Consider a cocontact Hamiltonian system $(M, \tau, \eta, H)$.

- If $f_1, f_2$ are two dissipated quantities and $f_2 \neq 0$, then $f_1/f_2$ is a conserved quantity.

- If $f$ is a dissipated quantity and $g$ is a conserved quantity, then $fg$ is a dissipated quantity.

Note that if $H \neq 0$, given an infinitesimal generalized dynamical symmetry $Y$, we have a conserved quantity $g = -\frac{1}{H} \iota_Y \eta$. 

6
4 Hamilton–Jacobi theory: the action-independent approach

Let \((\mathbb{R} \times T^*Q \times \mathbb{R}, \tau, \eta, H)\) be a cocontact Hamiltonian system. Consider a section \(\gamma\) of the bundle \(\pi^* : \mathbb{R} \times T^*Q \times \mathbb{R} \to \mathbb{R} \times Q\), locally given by

\[
\gamma : \mathbb{R} \times Q \to \mathbb{R} \times T^*Q \times \mathbb{R} \quad (t, q^i) \mapsto (t, q^i, \gamma_i(t, q), S(t, q)).
\]

Let us introduce the vector field \(X^\gamma_H\) on \(\mathbb{R} \times Q\) given by

\[
X^\gamma_H = T\pi_Q \circ X_H \circ \gamma,
\]

where \(X_H\) is the Hamiltonian vector field of \((\mathbb{R} \times T^*Q \times \mathbb{R}, \tau, \eta, H)\). Suppose that \(X^\gamma_H\) and \(X_H\) are \(\gamma\)-related, i.e.,

\[
X_H \circ \gamma = T\gamma \circ X^\gamma_H,
\]

so that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R} \times T^*Q \times \mathbb{R} & \xrightarrow{X_H} & T(\mathbb{R} \times T^*Q \times \mathbb{R}) \\
\gamma \circ \pi^* & \xrightarrow{T\pi_Q \circ \gamma} & T(\mathbb{R} \times \mathbb{R}) \\
\mathbb{R} \times Q & \xrightarrow{X^\gamma_H} & T(\mathbb{R} \times Q)
\end{array}
\]

Locally,

\[
X_H \circ \gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left( \frac{\partial H}{\partial q^i} + \gamma_i \frac{\partial H}{\partial z} \right) \frac{\partial}{\partial p_i} + \left( \gamma_i \frac{\partial H}{\partial p_i} - H \right) \frac{\partial}{\partial z},
\]

and

\[
T\gamma \circ X^\gamma_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} + \left( \gamma_i \frac{\partial }{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial \gamma_i}{\partial q^i} \right) \frac{\partial}{\partial p_i} + \left( \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i} \frac{\partial H}{\partial q^i} \right) \frac{\partial}{\partial z},
\]

so Eq. (14) holds if and only if

\[
- \left( \frac{\partial H}{\partial q^i} + \gamma_i \frac{\partial H}{\partial z} \right) = \gamma_i \frac{\partial H}{\partial p_i} + \frac{\partial \gamma_i}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial \gamma_i}{\partial q^j},
\]

\[
\gamma_i \frac{\partial H}{\partial p_i} - H = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i} \frac{\partial H}{\partial q^i}.
\]

**Definition 4.1.** Given a section \(\alpha : \mathbb{R} \times Q \times \mathbb{R} \to \mathbb{R} \times \bigwedge^k(T^*Q) \times \mathbb{R}\) and \(t, z \in \mathbb{R}\), let

\[
\alpha_{(t,z)} : Q \to \bigwedge^k(T^*Q) \quad q \mapsto \text{pr}_{\Lambda^k(T^*Q)}(\alpha(t, q, z)),
\]

where \(\text{pr}_{\Lambda^k(T^*Q)} : \mathbb{R} \times \bigwedge^k(T^*Q) \times \mathbb{R} \to \bigwedge^k(T^*Q)\) is the canonical projection. The **exterior derivative at fixed** \(z\) is the section of \(\mathbb{R} \times \bigwedge^{k+1}(T^*Q) \times \mathbb{R} \to Q \times \mathbb{R}\) given by

\[
d_Q \alpha_{(t, z)} = (t, d\alpha_{(t, z)}(q), z).
\]

In local coordinates, for \(f : \mathbb{R} \times Q \times \mathbb{R} \to \mathbb{R}\) and \(\alpha : \mathbb{R} \times Q \times \mathbb{R} \to \mathbb{R} \times T^*Q \times \mathbb{R}\), the expressions are

\[
d_Q f = \frac{\partial f}{\partial q^i} dq^i,
\]

\[
d_Q (\alpha dq^i) = \frac{\partial \alpha_i}{\partial q^j} dq^i \wedge dq^j.
\]
Definition 4.2. Let \( f \in \mathcal{C}^\infty(\mathbb{R} \times T^*Q \times \mathbb{R}) \) The fixed-time 1-jet of \( f \) is given by
\[
j^1_t f(t, q) = (t, q, dQf, f).
\]

Proposition 4.3. Let \( \gamma \) be a section of \( \pi^*_Q : \mathbb{R} \times T^*Q \times \mathbb{R} \times \mathbb{R} \to Q \). Then, \( \text{Im} \gamma(t, \cdot) \) for every \( t \in \mathbb{R} \) is a Legendrian submanifold of \( (T^*Q \times \mathbb{R}, \eta) \) if and only if it is the image of the fixed-time 1-jet of a function, namely, \( N = \text{Im} \gamma \), where
\[
\gamma(t, q) = j^1_t f(t, q) = (t, q, dQf, f).
\]

Proof. Let \( t \in \mathbb{R} \) and let \( \gamma : Q \times \mathbb{R} \to \mathbb{R} \times T^*Q \times \mathbb{R} \) such that \( \gamma(g, z) = \gamma(t, q, z) = (t, q, \alpha(t, q, z), f(t, q, z)) \). Clearly, \( \gamma^* \tau = 0 \), hence \( \text{Im} \gamma \) is Legendrian if and only if \( \gamma^* \eta = 0 \). Thus,
\[
\gamma^* \eta = f^* dz - \alpha^* \theta_Q = dQf - \alpha,
\]
so \( \gamma^* \eta_Q \) vanishes precisely when \( \alpha = dQf \).

Now, suppose that \( \text{Im} \gamma \) is a Legendrian submanifold. By Proposition 4.3, we have that
\[
\gamma_i = \frac{\partial S}{\partial q^i},
\]
so Eqs. (17) can be written as
\[
- \left( \frac{\partial H}{\partial q^i} + \frac{\partial S}{\partial q^i} \frac{\partial H}{\partial z} \right) = \frac{\partial^2 S}{\partial t \partial q^i} + \frac{\partial H}{\partial p_j} \frac{\partial S}{\partial q^i} \frac{\partial q^j}{\partial q^i},
\]
\[
\left( \frac{\partial S}{\partial q^i} \frac{\partial H}{\partial p_i} - H \right) = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i} \frac{\partial H}{\partial q^i} \frac{\partial p_i}{\partial q^i}.
\]

Eq. (23a) implies that
\[
d_Q(H \circ j^1 S) + d_Q(\mathcal{L}_H S) = 0,
\]
while Eq. (23b) yields
\[
H = -\frac{\partial S}{\partial t},
\]
that is,
\[
H \circ j^1 S + \mathcal{L}_H S = 0.
\]
Clearly, Eq. (24) is implied by Eq. (26). We have thus proven the following.

Theorem 4.4 (Action-independent Hamilton–Jacobi Theorem). Let \( \gamma \) be a section of \( \pi^*_Q : \mathbb{R} \times T^*Q \times \mathbb{R} \to \mathbb{R} \times Q \) such that, for every \( t \in \mathbb{R} \), \( \text{Im} \gamma(t, \cdot) \) is a Legendrian submanifold of \( (T^*Q \times \mathbb{R}, \eta) \). Then, \( X^\gamma_H \) and \( X_H \) are \( \gamma \)-related if and only if Eq. (26) holds. This equation will be called the action-independent Hamilton–Jacobi equation for \( (\mathbb{R} \times T^*Q \times \mathbb{R}, \tau, \eta, H) \). If \( \gamma = j^1 S \), the function \( S \) is called a generating function for \( H \).

4.1 The variational interpretation of the solution to Hamilton–Jacobi equation

Suppose that \( q : [0, T] \to Q \) is a trajectory given by the cocontact Hamilton equations (5) for the Hamiltonian function \( H : \mathbb{R} \times T^*Q \times \mathbb{R} \to \mathbb{R} \). If \( \gamma = j^1 S \) is a solution to the Hamilton–Jacobi problem for \( H \), the generating function \( S \) can be interpreted as the action of the lifted curve \( j^1 S \circ q \) up to a constant.

Theorem 4.5. Suppose that \( S \in \mathcal{C}^\infty(\mathbb{R} \times T^*Q) \) is a generating function for \( H \). Let \( q : [0, T] \to Q \) be a curve such that \( c : t \mapsto (t, q(t)) \) is an integral curve of \( X^\gamma_H \). Then,
\[
S(t, q(t)) = A(j^1 S \circ q)(t) + S_0
\]
for some \( S_0 \in \mathbb{R} \), where \( A \) denotes the action map (6).
Proof. Assume that \( S \in \mathcal{C}^\infty(\mathbb{R} \times T^*Q) \) is a generating function for \( H \). Then,

\[
\frac{d}{dt} S(t, q(t)) = \frac{\partial S(t, q(t))}{\partial t} + \frac{\partial S(t, q(t))}{\partial q^i} \dot{q}^i = \frac{\partial S(t, q(t))}{\partial q^i} \dot{q}^i - H \circ j^1_i S = \frac{d}{dt} A(j^1_i S \circ q)(t),
\]

where we have used the Hamilton–Jacobi equation (26) on the second step, and the definition of the action map (6) on the last step. Hence,

\[
S(t, q(t)) = A(j^1_i S \circ q)(t) + S_0
\]

for some constant \( S_0 \).

4.2 A new approach for the Hamilton–Jacobi problem in time-independent contact Hamiltonian systems

The analogous of Theorem 4.4 for autonomous contact Hamiltonian systems was developed in [30] (see also [21]):

**Theorem 4.6** (Hamilton–Jacobi Theorem for autonomous systems). Let \((T^*Q \times \mathbb{R}, \eta, H)\) be a contact Hamiltonian system with contact Hamiltonian vector field \( X_H \). Consider a section \( \gamma \) of \( \pi_Q : T^*Q \times \mathbb{R} \to Q \) such that \( \text{Im} \gamma \) is a Legendrian submanifold of \((T^*Q \times \mathbb{R}, \eta)\). Then, \( X_H \gamma \) and \( X_H \) are \( \gamma \)-related if and only if

\[
H \circ \gamma = 0.
\]

(29)

The problem with this approach is that it cannot be used to completely integrate the system. Indeed, Eq. (29) implies that every integral curve of \( X_H \circ \gamma \) is contained in \( H^{-1}(0) \). This can be solved by regarding the contact Hamiltonian system \((T^*Q \times \mathbb{R}, \eta, H)\) as the cocontact Hamiltonian system \((\mathbb{R} \times T^*Q \times \mathbb{R}, dt, \eta, H)\) such that \( R_t(H) = 0 \) and making use of Theorem 4.4. Suppose that \( S \) is of the form \( S(t, q) = \alpha(q) + \beta(t) \). Then, Eq. (26) yields

\[
H \circ j^1 \alpha + \mathcal{L}_\alpha \beta = 0,
\]

that is,

\[
H \left(q^i, \frac{\partial \alpha}{\partial q^i}, z \right) + \dot{\beta}(t) = 0.
\]

(31)

With a suitable choice of \( \alpha \) and \( \beta \), one can cover energy levels distinct from \( H = 0 \).

**Example 4.7** (The free particle with linear dissipation). Consider the contact Hamiltonian system \((T^*\mathbb{R} \times \mathbb{R}, \eta, H)\) with Hamiltonian function

\[
H(q, p, z) = \frac{p^2}{2} + \delta z.
\]

(32)

Then, Eq. (31) becomes

\[
\frac{\alpha'(q)^2}{2} + \dot{\beta}(t) + \delta(\alpha(q) + \beta(t)) = 0.
\]

(33)

Suppose that \( \dot{\beta}(t) + \delta \beta(t) = 0 \). Then,

\[
\alpha(q) = -\frac{\delta}{2} q^2, \quad \beta(t) = \lambda e^{-\delta t},
\]

(34)

for arbitrary \( \lambda \in \mathbb{R} \). Thus, \( S(t, q) = \lambda e^{-\delta t} - \frac{\delta}{2} q^2 \).
Now,

$$X^\gamma_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p} \frac{\partial}{\partial q} = \frac{\partial}{\partial t} + p \frac{\partial}{\partial q} = \frac{\partial}{\partial t} - \delta q \frac{\partial}{\partial q},$$

whose integral curves are of the form $\sigma(t) = (t, q_0 e^{-\delta t})$. Thus, the integral curves of $X_H|_{\text{Im}(\gamma)}$ are given by $\gamma \circ \sigma(t) = (t, q_0 e^{-\delta t}, -\delta q_0 e^{-\delta t}, -\frac{\delta^2}{2} q_0 e^{-2\delta t} + \lambda e^{-\delta t})$.

5 Hamilton–Jacobi theory: the action-dependent approach

In Section 4 we have introduced a Hamilton–Jacobi theory for time-dependent contact Hamiltonian systems. This approach was shown to be especially useful for dealing with time-independent contact Hamiltonian systems, where time is used as a free parameter. Nevertheless, this approach has a couple of drawbacks. First, one cannot define a notion of complete solution in the same fashion as it is done for symplectic Hamiltonian systems [6]. Additionally, time-independent solutions only cover the zero-energy level.

In order to solve these problems, in this section we propose an alternative approach, considering solutions of the Hamilton–Jacobi problem depending on the action variable $z$. Let us consider a section $\gamma$ of the bundle $\pi_{t,s}^T : \mathbb{R} \times T^*Q \times \mathbb{R} \rightarrow \mathbb{R} \times Q \times \mathbb{R}$, locally given by

$$\gamma : \mathbb{R} \times Q \times \mathbb{R} \rightarrow \mathbb{R} \times T^*Q \times \mathbb{R}$$

$$(t, q, z) \mapsto (t, q, \gamma_i(t, q, z), z).$$

As in the previous approach, assume that $X^\gamma_H$ and $X_H$ are $\gamma$-related, so that the following diagram commutes:

$$\begin{array}{cccc}
\mathbb{R} \times T^*Q \times \mathbb{R} & \xrightarrow{X_H} & T(\mathbb{R} \times T^*Q \times \mathbb{R}) \\
\gamma \mapsto \pi_{t,s}^T & & T_{\gamma} \pi_{t,s}^T \\
\mathbb{R} \times Q \times \mathbb{R} & \xrightarrow{X^\gamma_H} & T(\mathbb{R} \times Q \times \mathbb{R})
\end{array}$$

Locally,

$$X_H \circ \gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} + \left( \frac{\partial H}{\partial q} - \gamma_i \frac{\partial}{\partial z} \right) \frac{\partial}{\partial p_i} + \left( \gamma_i \frac{\partial H}{\partial p_i} - H \right) \frac{\partial}{\partial z},$$

and

$$T_{\gamma} \circ X^\gamma_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} + \left( \frac{\partial \gamma_i}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial \gamma_j}{\partial q^i} + \left( \frac{\partial H}{\partial p_j} - H \right) \frac{\partial \gamma_i}{\partial s} \right) \frac{\partial}{\partial p_i} + \left( \gamma_i \frac{\partial H}{\partial p_i} - H \right) \frac{\partial}{\partial z},$$

so $X^\gamma_H$ and $X_H$ are $\gamma$-related if and only if

$$- \left( \frac{\partial H}{\partial q^i} + \gamma_i \frac{\partial H}{\partial z} \right) = \frac{\partial \gamma_i}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial \gamma_j}{\partial q^i} + \frac{\partial \gamma_i}{\partial s} \left( \frac{\partial H}{\partial p_j} - H \right).$$

Note that $\text{Im} \gamma$ is $(n + 2)$-dimensional, so it no longer makes sense to require it to be Legendrian [16]. Demanding it to be coisotropic is not enough to obtain a satisfactory Hamilton–Jacobi equation. This motivates the following.

Let us recall that, given a Jacobi manifold $(M, \Lambda, E)$ and a distribution $D \subseteq TM$, the orthogonal complement $D^\perp$ of $D$ is given by [18, 52]

$$D^\perp = \Lambda(D^o),$$

where $D^o = \{ \alpha \in T_x^*M \mid \alpha(v) = 0, \forall v \in D_x \}$ denotes the annihilator. In particular, a cocotact manifold is a Jacobi manifold (see Proposition 2.7) and its morphism $\Lambda$ is given by Eq. (2).
Definition 5.1. Let \((M, \tau, \eta)\) be a cocontact manifold and \(i: N \hookrightarrow M\) a submanifold. The submanifold \(N\) is said to be coisotropic if \(TN^\perp \subseteq TN\).

The coisotropic condition can be written in local coordinates as follows.

**Lemma 5.2.** Assume that an \((n+2)\)-dimensional submanifold \(N\) of a \((2n+2)\)-dimensional cocontact submanifold \((M, \tau, \eta)\) is locally the zero set of the constraint functions \(\{\phi_a\}_{a=1}^n\). Then, \(N\) is coisotropic if and only if the following equation holds in Darboux coordinates:

\[
\left( \frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial \tau} \right) \frac{\partial \phi_b}{\partial p_i} - \frac{\partial \phi_a}{\partial \tau} \left( \frac{\partial \phi_b}{\partial q^i} + p_i \frac{\partial \phi_b}{\partial \tau} \right) = 0. \tag{40}
\]

**Proof.** Assume that \((M, \tau, \eta)\) is a \((2n+2)\)-dimensional cocontact manifold. Let \(N \subseteq M\) be a \(k\)-dimensional submanifold given locally by the zero set of functions \(\phi_a: U \to \mathbb{R}\), with \(a \in \{1, \ldots, 2n+1-k\}\). We have that

\[
TN^\perp = \left\{ \{Z_a\}_{a=1}^{2n+1-k} \right\},
\]

where

\[
Z_a = \hat{\Lambda}(d\phi_a) = \left( \frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial \tau} \right) \frac{\partial \phi_b}{\partial p_i} - \frac{\partial \phi_a}{\partial \tau} \left( \frac{\partial \phi_b}{\partial q^i} + p_i \frac{\partial \phi_b}{\partial \tau} \right).
\]

Therefore, \(N\) is coisotropic if and only if \(Z_a(\phi_b) = 0\) for all \(a, b\), which in Darboux coordinates yields Eq. (40).

\[\square\]

**Proposition 5.3.** Let \(\gamma\) be a section of \(\mathbb{R} \times T^*Q \times \mathbb{R}\) over \(\mathbb{R} \times Q \times \mathbb{R}\). Then \(\text{Im} \gamma\) is a coisotropic submanifold if and only if

\[
\frac{\partial \gamma_i}{\partial q^j} + \gamma_j \frac{\partial \gamma_i}{\partial \tau} = \frac{\partial \gamma_j}{\partial q^i} + \gamma_i \frac{\partial \gamma_j}{\partial \tau}. \tag{41}
\]

**Proof.** Eq. (41) is obtained by applying the previous result to the submanifold \(N = \text{Im} \gamma\), which is locally defined by the constraints \(\phi_i = p_i - \gamma_i\).

\[\square\]

Now suppose that \(\text{Im} \gamma\) is coisotropic in Eq. (39). Then, by means of Eq. (41) we obtain

\[
\frac{\partial H}{\partial q^i} + \frac{\partial \gamma_i}{\partial \tau} + \frac{\partial H}{\partial p_j} \frac{\partial \gamma_i}{\partial q^j} + \gamma_i \left( \frac{\partial \gamma_j}{\partial q^i} \frac{\partial H}{\partial \tau} \frac{\partial \gamma_j}{\partial p_j} + \frac{\partial H}{\partial \tau} \frac{\partial \gamma_j}{\partial q^j} \right) - H \frac{\partial \gamma_i}{\partial \tau} = 0, \tag{42}
\]

or, globally,

\[
d_Q(H \circ \gamma) + \mathcal{L}_\gamma (H \circ \gamma) + \mathcal{L}_H \gamma = (H \circ \gamma) \mathcal{L}_\gamma \gamma. \tag{43}
\]

**Theorem 5.4** (Action-dependent Hamilton–Jacobi Theorem). Let \(\gamma\) be a section of \(\pi_Q^*: \mathbb{R} \times T^*Q \times \mathbb{R} \to \mathbb{R} \times Q \times \mathbb{R}\) such that \(\text{Im} \gamma\) is a coisotropic submanifold of \((\mathbb{R} \times T^*Q \times \mathbb{R}, \tau, \eta)\). Then, \(X_H^\gamma\) and \(X_H\) are \(\gamma\)-related if and only if Eq. (43) holds. This equation will be called the action-dependent Hamilton–Jacobi equation for \((\mathbb{R} \times T^*Q \times \mathbb{R}, \tau, \eta, H)\).

**Definition 5.5.** Let \((\mathbb{R} \times T^*Q \times \mathbb{R}, \tau, \eta, H)\) be a cocontact Hamiltonian system. A complete solution of the action-dependent Hamilton–Jacobi problem for \((\mathbb{R} \times T^*Q \times \mathbb{R}, \tau, \eta, H)\) is a local diffeomorphism \(\Phi: \mathbb{R} \times Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times T^*Q \times \mathbb{R}\) such that, for each \(\lambda \in \mathbb{R}\),

\[
\Phi_\lambda: \mathbb{R} \times Q \times \mathbb{R} \longrightarrow \mathbb{R} \times T^*Q \times \mathbb{R}
\]

\[
(t, q^i, z) \longmapsto \Phi(t, q^i, \lambda, z) \tag{44}
\]

is a solution of the action-dependent Hamilton–Jacobi problem for \((\mathbb{R} \times T^*Q \times \mathbb{R}, \tau, \eta, H)\).
Let $\alpha: \mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and $\pi_i: \mathbb{R}^n \to \mathbb{R}$ denote the canonical projections. Let us define the functions $f_i = \pi_i \circ \alpha \circ \Phi^{-1}$ on $\mathbb{R} \times T^*Q \times \mathbb{R}$, so that the following diagram commutes:

\[ \begin{array}{ccc}
\mathbb{R} \times Q \times \mathbb{R}^n & \xrightarrow{\Phi} & \mathbb{R} \times T^*Q \times \mathbb{R} \\
\downarrow{\alpha} & & \downarrow{f_i} \\
\mathbb{R}^n & \xrightarrow{\pi_i} & \mathbb{R}
\end{array} \]

**Theorem 5.6.** Let $\Phi: \mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times T^*Q \times \mathbb{R}$ be a complete solution of the Hamilton–Jacobi problem for $(\mathbb{R} \times T^*Q \times \mathbb{R}, \tau, \eta, H)$. Then,

(i) For each $i \in \{1, \ldots, n\}$, the function $f_i = \pi_i \circ \alpha \circ \Phi^{-1}$ is a constant of the motion. However, these functions are not necessarily in involution, i.e., $\{f_i, f_j\} \neq 0$.

(ii) For each $i \in \{1, \ldots, n\}$, the function $\hat{f}_i = g f_i$, where $g$ is a dissipated quantity, is also a dissipated quantity. Moreover, if $R_k(H) = 0$ and taking $g = H$, these functions are in involution, i.e., $\{\hat{f}_i, \hat{f}_j\} = 0$.

**Proof.** Observe that

\[ \text{Im} \Phi_{\lambda} = \bigcap_{i=1}^n f_i^{-1}(\lambda_i), \]

where $\lambda = (\lambda, \ldots, \lambda_n) \in \mathbb{R}^n$. In other words,

\[ \text{Im} \Phi_{\lambda} = \{x \in \mathbb{R} \times T^*Q \times \mathbb{R} \mid f_i(x) = \lambda_i, i = 1, \ldots, n\}. \]

Therefore, since $X_H$ is tangent to any of the submanifolds $\text{Im} \Phi_{\lambda}$, we deduce that

\[ X_H(f_i) = 0. \]

Moreover, we can compute

\[ \{f_i, f_j\} = \Lambda(df_i, df_j) - f_i R_z(f_j) + f_j R_z(f_i), \]

but

\[ \Lambda(df_i, df_j) = \hat{\Lambda}(df_i) f_j = 0, \]

since $(T \text{Im} \Phi_{\lambda})^\perp = \hat{\Lambda}(T \text{Im} \Phi_{\lambda})^\perp \subset T \text{Im} \Phi_{\lambda}$, so

\[ \{f_i, f_j\} = -f_i R_z(f_j) + f_j R_z(f_i). \quad (45) \]

We already know that the product of a conserved quantity and a dissipated quantity is a dissipated quantity 3.6. Let $f_i$ and $f_j$ be conserved quantities and take $g = H$. Then,

\[ \{\hat{f}_i, \hat{f}_j\} = \{H f_i, H f_j\} = f_j \{H f_i, H\} + H \{H f_i, f_j\} - f_j H R_z(H f_i) 
\]

\[ = -f_j H \{f_i, f_i\} + f_i f_j H R_z(H) - f_i H \{f_j, f_i\} - H^2 \{f_j, f_i\} + f_i H^2 R_z(f_j) - f_j H R_z(H f_i) \]

\[ = 0. \]

\[ \square \]

From a complete solution of the Hamilton–Jacobi problem one can reconstruct the dynamics of the system. Given an integral curve $\sigma$ of the vector field $X_{H}^\perp$, $\Phi_{\lambda} \circ \sigma$ is an integral curve of $X_H$, thus recovering the dynamics of the original system.
5.1 Integrable contact Hamiltonian systems

Let \((T^*Q \times \mathbb{R}, \eta, H)\) be a contact Hamiltonian system. Let \(\Phi: Q \times \mathbb{R}^n \times \mathbb{R} \to T^*Q \times \mathbb{R}\) be a complete solution of the Hamilton–Jacobi problem for \((T^*Q \times \mathbb{R}, \eta, H)\). Then,

\[
\mathcal{F} = \{ \mathcal{F}_\lambda = \text{Im} \Phi_\lambda \mid \lambda \in \mathbb{R}^n \} \subseteq T^*Q \times \mathbb{R}
\]

is a foliation in coisotropic submanifolds.

In the symplectic case the images of the complete solution for each choice of parameters \(\lambda\) form a Lagrangian foliation invariant under the Hamiltonian flow. This structure is called an integrable system. In analogy, we introduce the following definition:

**Definition 5.7.** Let \((M, \eta, H)\) be a contact Hamiltonian system and let \(\mathcal{F}\) be a foliation consisting of \((n + 1)\)-dimensional coisotropic leaves invariant under the flow of the Hamiltonian vector field \(X_H\). Then we call \((M, \eta, H, \mathcal{F})\) an **integrable system**.

This definition can be compared to the ones given in [2, 45]:

- In [2], Boyer proposes a concept of completely integrable system in which he insists that the Hamiltonian vector field and the constants of motion are in involution. This is a particular case of our definition, in which both the Hamiltonian and the constants of the motion do not depend on \(z\).

- In [45], Khesin and Tabachnikov call a foliation \(F^n\) **co-Legendrian** when it is transverse to \(\mathcal{H}\) and \(TF \cap \mathcal{H}\) is integrable. Then they define an integrable system as a particular of a co-Legendrian foliation with some extra regularity conditions. In the case that the dimension of the leaves is \(n + 1\), the following proposition shows that co-Legendrian foliations are particular cases of coisotropic foliations.

**Proposition 5.8.** Let \(i: N \hookrightarrow M\) be a submanifold of a \((2n + 1)\)-dimensional contact manifold \((M, \eta)\). If \(N\) is an \((n + 1)\)-dimensional co-Legendrian submanifold, then it is also a coisotropic submanifold.

**Proof.** Let us write \(TN = D\mathcal{H} + \mathcal{E}\), where \(D\mathcal{H} = TN \cap \mathcal{H}\). Then, \(TN^\perp = D\mathcal{H}^\perp \cap \mathcal{E}^\perp\). Obviously, \(\eta\) vanishes in \(TN \cap \mathcal{H}\). Moreover, since \(D\mathcal{H}\) is integrable,

\[
0 = \eta([v, w]) = \iota_{[v, w]}\eta = \mathcal{L}_v\iota_w\eta - \iota_w\mathcal{L}_v\eta = -\iota_w\iota_v\eta - \iota_v\iota_w\eta = -\iota_v\iota_w\eta,
\]

for any \(v, w \in D\mathcal{H}\), so \(d\eta|_{D\mathcal{H}} = 0\). Observe that \(\hat{\mathcal{L}}|_{D\mathcal{H}} = \hat{\iota}_{D\mathcal{H}}\mathcal{H}\), and \(\iota_{D\mathcal{H}}: \mathcal{H} \to \mathcal{R}\), \(\iota_{D\mathcal{H}}(v) = \iota_v\eta\) is an isomorphism. Since \(d\eta|_{D\mathcal{H}} = 0\), \(\iota_{D\mathcal{H}}^{-1}(D\mathcal{H}) \subseteq D\mathcal{H}^\perp\). Thus, \(D\mathcal{H} \subseteq \hat{\mathcal{L}}(D\mathcal{H}) = D\mathcal{H}^\perp\). By a dimension counting argument, we can see that both spaces are equal and, thus, \(D\mathcal{H} = D\mathcal{H}^\perp\).

We also note that a foliation \(\tilde{\mathcal{F}}\) by Legendrian submanifolds can never be invariant by the Hamiltonian flow. Indeed, let \(\tilde{F} \in \tilde{\mathcal{F}}\). The leaves of \(\tilde{\mathcal{F}}\) are Lagrangian, thus \(TF_0 \subseteq \ker \eta\). Since \(\eta(X_H) = -H\), \(X_H\) can only be tangent to the leaves at the zero set of \(H\), hence its flow cannot leave invariant the whole foliation.

6 Examples

6.1 Freely falling particle with linear dissipation

Consider a particle of time-dependent mass \(m(t)\) which is freely falling and subject to a dissipation linear in the velocity with proportionality constant \(\gamma\). The Hamiltonian function \(H : \mathbb{R} \times T^*\mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is given by

\[
H(t, q, p, z) = \frac{p^2}{2m(t)} + m(t)gq + \frac{\gamma}{m(t)}z,
\]
where \( g \) is the gravity. The Hamiltonian vector field corresponding to this Hamiltonian function is

\[
X = \frac{\partial}{\partial t} + \frac{p}{m(t)} \frac{\partial}{\partial q} - \left( m(t)g + \frac{\gamma}{m(t)}p \right) \frac{\partial}{\partial p} + \left( \frac{p^2}{2m(t)} - m(t)gq - \frac{\gamma}{m(t)}z \right) \frac{\partial}{\partial s}.
\]

Its integral curves \((t(r), q(r), p(r), z(r))\) satisfy the system of differential equations

\[
i = 1, \quad \dot{q} = \frac{p}{m(t)}, \quad \dot{p} = -m(t)g - \frac{\gamma}{m(t)}p, \quad \dot{z} = \frac{p^2}{2m(t)} - m(t)gq - \frac{\gamma}{m(t)}z.
\]

Combining the second and third equations, we get

\[
\frac{d}{dt}(m(t)q) = -m(t)g - \gamma q.
\]

In order to solve the Hamilton–Jacobi problem, we look for a conserved quantity linearly independent from the Hamiltonian, i.e., a function \( f \) on \( \mathbb{T} \times \mathbb{R} \) such that \( X_H(f) = 0 \). For the sake of simplicity, one can assume that \( f \) does not depend on \( q \) or \( z \). Indeed, one can verify that

\[
f(t, q, p, z) = e^{\int_1^t \frac{\gamma}{m(s)} ds} \left( p + ge^{\int_1^t \frac{\gamma}{m(s)} ds} \int_1^t e^{-\int_s^t \frac{\gamma}{m(s)} ds} m(u) du \right)
\]

is a conserved quantity. We can thus express the momentum \( p \) as a function of \( t \) and the real parameter \( \lambda \), namely,

\[
P(t, \lambda) = e^{-\int_1^t \frac{\gamma}{m(s)} ds} \left( \lambda - ge^{\int_1^t \frac{\gamma}{m(s)} ds} \int_1^t e^{\int_s^t \frac{\gamma}{m(s)} ds} m(u) du \right),
\]

and obtain a complete solution of the Hamilton–Jacobi problem for \( H \):

\[
\phi_\lambda : (t, q, z) \mapsto \left( t, q, p \equiv e^{-\int_1^t \frac{\gamma}{m(s)} ds} \left( \lambda - ge^{\int_1^t \frac{\gamma}{m(s)} ds} \int_1^t e^{\int_s^t \frac{\gamma}{m(s)} ds} m(u) du \right), z \right).
\]

In this case, Eq. (41) holds trivially, so \( \Phi_\lambda \) is coisotropic.

In addition, one can verify that

\[
k(t, q, p, z) = p + ge^{\int_1^t \frac{\gamma}{m(s)} ds} \int_1^t e^{\int_s^t \frac{\gamma}{m(s)} ds} m(u) du = e^{\int_1^t \frac{\gamma}{m(s)} ds} f(t, q, p, z)
\]

is a dissipated quantity, that is, \( \{k, H\} = \mathcal{L}_{R_t}k = 0 \).

### 6.2 Damped forced harmonic oscillator

Consider the product manifold \( \mathbb{R} \times \mathbb{T}Q \times \mathbb{R} \) with natural coordinates \((t, q, p, z)\). The Hamiltonian function

\[
H(t, q, p, s) = \frac{p^2}{2m} + \frac{k}{2}q^2 - qF(t) + \frac{\gamma}{m}s
\]

describes a harmonic oscillator with elastic constant \( k \), friction coefficient \( \gamma \) and subjected to an external time-dependent force \( F(t) \) [16].

The Hamiltonian vector field is

\[
X = \frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial q} + \left(-kq + F(t) - \frac{p}{m} \gamma \right) \frac{\partial}{\partial p} + \left( \frac{p^2}{2m} - \frac{k}{2}q^2 + qF(t) - \frac{\gamma}{m}s \right) \frac{\partial}{\partial s},
\]

and its integral curves \((t(r), q(r), p(r), s(r))\) satisfy

\[
i = 1, \quad \dot{q} = \frac{p}{m}, \quad \dot{p} = -kq + F(t) - \frac{p}{m} \gamma, \quad \dot{s} = \frac{p^2}{2m} - \frac{k}{2}q^2 + qF(t) - \frac{\gamma}{m}s.
\]
Combining the second and the third equations above, we obtain the second-order differential equation
\[ m\ddot{q} + \gamma \dot{q} + kq = F(t), \]
which corresponds to a damped forced harmonic oscillator.

One can check that the function
\[ g(t,q,p,z) = e^{\frac{\gamma t}{2m}} \left( \sinh \left( \frac{\kappa t}{2m} \right) \left( 2kmq + \gamma p \right) + p \cosh \left( \frac{\kappa t}{2m} \right) \right) - \int_1^t \left( \cosh \left( \frac{\kappa s}{2m} \right) + \frac{\gamma \sinh \left( \frac{\kappa s}{2m} \right)}{\kappa} \right) ds, \]
(50)
where \( \kappa = \sqrt{\gamma^2 - 4km} \), is a conserved quantity. It is worth noting that, since \( \sinh x = x + O(x^3) \) and \( \cosh x = 1 + O(x^2) \) near \( x = 0 \), \( \sinh(ix) = i\sin x \) and \( \cosh(ix) = \cos x \), the equation above is well-defined for any value of \( \kappa \in \mathbb{C} \). Thus, we can write \( p \) in terms of \( t, q, z \) and a real parameter \( \lambda \) as
\[ P(t,q,\lambda,z) = e^{\frac{\gamma t}{2m}} \left( \kappa \int_1^t e^{\frac{\gamma s}{2m}} F(s) \left( \cosh \left( \frac{\kappa s}{2m} \right) + \frac{\gamma \sinh \left( \frac{\kappa s}{2m} \right)}{\kappa} \right) ds - 2kmqe^{\frac{\gamma t}{2m}} \sinh \left( \frac{\kappa t}{2m} \right) + \kappa \lambda \right) \gamma \sinh \left( \frac{\kappa t}{2m} \right) + \kappa \cosh \left( \frac{\kappa t}{2m} \right), \]
and obtain a complete solution of the Hamilton–Jacobi problem:
\[ \Phi_\lambda: (t,q,\lambda,z) \mapsto (t,q,p \equiv P(t,q,\lambda,z),z). \]
(51)

Obviously Eq. (41) is satisfied, hence \( \Phi_\lambda \) is coisotropic.

In addition,
\[ f(t,q,p,z) = e^{-\frac{\gamma t}{2m}} \left[ e^{\frac{\gamma t}{2m}} \left( \sinh \left( \frac{\kappa t}{2m} \right) \left( 2kmq + \gamma p \right) + p \cosh \left( \frac{\kappa t}{2m} \right) \right) - \int_1^t e^{\frac{\gamma s}{2m}} F(s) \left( \cosh \left( \frac{\kappa s}{2m} \right) + \frac{\gamma \sinh \left( \frac{\kappa s}{2m} \right)}{\kappa} \right) ds \right], \]
is a dissipated quantity.

### 7 Conclusions and outlook

The main contributions of the present paper are the following ones:

- We have obtained two different Hamilton–Jacobi equations for time-dependent contact Hamiltonian systems: the so-called action independent and action-dependent approaches. In particular, the action-independent approach is useful for time-independent contact Hamiltonian systems, where the use of time as a free parameter allows to integrate the system at non-zero energy levels.

- The action-dependent approach permits to introduce a natural notion of complete solutions to the Hamilton–Jacobi problem. Each of these complete solutions is associated with a family of \( n \) independent dissipated quantities in involution (where \( n \) is the number of degrees of freedom of the system). Moreover, the image of a complete solution is a coisotropic submanifold.

- We introduce a new notion of integrable system in a contact manifold, taking into account the dynamics given by the Hamiltonian vector field, and extending the concept of complete solution. This allows us to study the dynamics outside the zero-energy level.

In future works, we will deepen in the study of infinitesimal symmetries [38], and in the reduction problem, in the Hamilton–Jacobi equations for the evolution vector field and its possible applications to thermodynamics as well as the extension to higher order systems. The study of the discrete Hamilton–Jacobi equations and applications to the construction of geometric integrators is also on the agenda.
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References

[1] R. Abraham and J. Marsden. *Foundations of Mechanics*. AMS Chelsea Publishing. AMS Chelsea Pub./American Mathematical Society, 2008. ISBN: 978-0-8218-4438-0.

[2] C. P. Boyer. “Completely Integrable Contact Hamiltonian Systems and Toric Contact Structures on $S^2 \times S^3$”. *SIGMA Symmetry Integrability Geom. Methods Appl.* (June 15, 2011). ISSN: 18150659. DOI: 10.3842/SIGMA.2011.058.

[3] A. Budiyono. “Quantization from Hamilton–Jacobi Theory with a Random Constraint”. *Phys. A*, 391(20), pp. 4583–4589 (Oct. 15, 2012). ISSN: 0378-4371. DOI: 10.1016/j.physa.2012.05.046.

[4] C. M. Campos, M. de León, D. Martín de Diego, and M. Vaquero. “Hamilton-Jacobi Theory in Cauchy Data Space”. *Rep. Math. Phys.*, 76(3), pp. 359–387 (Dec. 1, 2015). ISSN: 0034-4877. DOI: 10.1016/S0034-4877(15)30038-0.

[5] F. Cantrijn, M. de León, and E. A. Lacomba. “Gradient Vector Fields on Cosymplectic Manifolds”. *J. Phys. A: Math. Gen.*, 25(1), pp. 175–188 (Jan. 7, 1992). ISSN: 0305-4470, 1361-6447. DOI: 10.1088/0305-4470/25/1/022.

[6] J. F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. C. Muñoz-Lecanda, and N. Román-Roy. “Geometric Hamilton–Jacobi Theory”. *Int. J. Geom. Methods Mod. Phys.*, 03(07), pp. 1417–1458 (Nov. 2006). ISSN: 0219-8878, 1793-6977. DOI: 10.1142/S0219887806001764.

[7] J. F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. C. Muñoz-Lecanda, and N. Román-Roy. “Geometric Hamilton–Jacobi Theory for Nonholonomic Dynamical Systems”. *Int. J. Geom. Methods Mod. Phys.*, 07(03), pp. 431–454 (May 2010). ISSN: 0219-8878, 1793-6977. DOI: 10.1142/S021988781000434X.

[8] J. F. Cariñena, F. X. Gràcia Sabaté, E. Martínez, G. Marmo, M. C. Muñoz Lecanda, and N. Román Roy. *Hamilton–Jacobi Theory and the Evolution Operator*. Prensas Univ. Zaragoza, Apr. 2009, pp. 177–186. ISBN: 978-84-92774-04-3.

[9] F. M. Ciaglia, H. Cruz, and G. Marmo. “Contact Manifolds and Dissipation, Classical and Quantum”. *Ann. Physics*, 398, pp. 159–179 (Nov. 1, 2018). ISSN: 0003-4916. DOI: 10.1016/j.aop.2018.09.012.

[10] F. M. Ciaglia, F. Di Cosmo, D. Felice, S. Mancini, G. Marmo, and J. M. Pérez-Pardo. “Hamilton–Jacobi Approach to Potential Functions in Information Geometry”. *J. Math. Phys.*, 58(6), p. 063506 (June 1, 2017). ISSN: 0022-2488. DOI: 10.1063/1.4984941.

[11] F. M. Ciaglia, F. Di Cosmo, and G. Marmo. “Hamilton-Jacobi Theory and Information Geometry”. In: *Geometric Science of Information*. Ed. by F. Nielsen and F. Barbaresco. Lecture Notes in Computer Science. Cham: Springer International Publishing, 2017, pp. 495–502. ISBN: 978-3-319-68445-1. DOI: 10.1007/978-3-319-68445-1_58.

[12] L. Colombo, M. de León, P. D. Prieto-Martínez, and N. Román-Roy. “Geometric Hamilton–Jacobi Theory for Higher-Order Autonomous Systems”. *J. Phys. A: Math. Theor.*, 47(23), p. 235203 (May 2014). ISSN: 1751-8121. DOI: 10.1088/1751-8113/47/23/235203.
M. de León, J. C. Marrero, and D. Martín de Diego. “Linear Almost Poisson Structures and Hamilton–Jacobi Equation. Applications to Nonholonomic Mechanics”. *J. Geom. Mech.*, 2(2), p. 159 (2010). DOI: 10.3934/jgm.2010.2.159.

M. de León and C. Sardón. “Geometric Hamilton–Jacobi Theory on Nambu–Poisson Manifolds”. *J. Math. Phys.*, 58(3), p. 033508 (Mar. 1, 2017). ISSN: 0022-2488. DOI: 10.1063/1.4978853.

M. de León and C. Sardón. “A Geometric Hamilton–Jacobi Theory on a Nambu–Jacobi Manifold”. *Int. J. Geom. Methods Mod. Phys.*, 16, p. 1940007 supp01 (Feb. 1, 2019). ISSN: 0219-8878. DOI: 10.1142/S0219887819400073.

M. de León, J. Gaset, X. Gràcia, M. C. Muñoz-Lecanda, and X. Rivas. *Time-dependent contact mechanics*. May 19, 2022. arXiv: 2205.09454 [math-ph].

M. de León, J. Gaset, M. Lainz, X. Rivas, and N. Román-Roy. “Unified Lagrangian-Hamiltonian Formalism for Contact Systems”. *Fortschritte der Phys.*, 68(8), p. 2000045 (2020). ISSN: 1521-3978. DOI: 10.1002/prop.202000045.

M. de León and M. Lainz. “A Review on Contact Hamiltonian and Lagrangian Systems”. Feb. 22, 2021. arXiv: 2011.05579 [math-ph].

M. de León, M. Lainz, and A. López-Gordón. “Discrete Hamilton–Jacobi Theory for Systems with External Forces”. *J. Phys. A: Math. Theor.* (2022). ISSN: 1751-8121. DOI: 10.1088/1751-8121/ac6240.

M. de León, M. Lainz, and A. López-Gordón. “Geometric Hamilton–Jacobi Theory for Systems with External Forces”. *J. Math. Phys.*, 63(2), p. 022901 (Feb. 1, 2022). ISSN: 0022-2488. DOI: 10.1063/5.0073214.

M. de León, M. Lainz, and Á. Muñiz-Brea. “The Hamilton–Jacobi Theory for Contact Hamiltonian Systems”. *Mathematics*, 9(16), p. 1993 16 (Jan. 2021). ISSN: 2227-7390. DOI: 10.3390/math9161993.

M. de León, M. Lainz, and M. C. Muñoz-Lecanda. *Optimal Control, Contact Dynamics and Her-glotz Variational Problem*. June 25, 2020. DOI: 10.48550/arXiv.2006.14326. arXiv: 2006.14326 [math-ph].

M. de León and M. Lainz Valcázar. “Contact Hamiltonian Systems”. *J. Math. Phys.*, 60(10), p. 102902 (Oct. 2019). ISSN: 0022-2488. DOI: 10.1063/1.5096475.

M. de León and M. Lainz Valcázar. “Singular Lagrangians and Precontact Hamiltonian Systems”. *Int. J. Geom. Methods Mod. Phys.*, 16(10), p. 1950158 (Oct. 2019). ISSN: 0219-8878. DOI: 10.1142/S0219887819501585.

M. de León and M. Lainz Valcázar. “Infinitesimal Symmetries in Contact Hamiltonian Systems”. *J. Geom. Phys.*, 153, p. 103651 (July 2020). ISSN: 03930440. DOI: 10.1016/j.geomphys.2020.103651.

M. de León, J. C. Marrero, and D. Martín de Diego. “A Geometric Hamilton–Jacobi Theory for Classical Field Theories”. In: *Variations, Geometry and Physics*. New York: Nova Sci. Publ., 2009, pp. 129–140. ISBN: 978-1-60456-920-9. arXiv: 0801.1181 [math-ph].

M. de León, J. C. Marrero, D. Martín de Diego, and M. Vaquero. “On the Hamilton–Jacobi Theory for Singular Lagrangian Systems”. *J. Math. Phys.*, 54(3), p. 032902 (Mar. 1, 2013). ISSN: 0022-2488. DOI: 10.1063/1.4796088.

M. de León, D. Martín de Diego, and M. Vaquero. “A Hamilton–Jacobi Theory for Singular Lagrangian Systems in the Skinner and Rusk Setting”. *Int. J. Geom. Methods Mod. Phys.*, 09(08), p. 1250074 (Dec. 1, 2012). ISSN: 0219-8878. DOI: 10.1142/S0219887812500740.

M. de León, D. Martín de Diego, and M. Vaquero. “A Hamilton–Jacobi Theory on Poisson Manifolds”. *J. Geom. Mech.*, 6(1), pp. 121–140 (2014). ISSN: 1941-4897. DOI: 10.3934/jgm.2014.6.121.

M. de León, D. Martín de Diego, and M. Vaquero. “Hamilton–Jacobi Theory, Symmetries and Coisotropic Reduction”. *J. Math. Pures Appl.*, 107(5), pp. 591–614 (May 1, 2017). ISSN: 0021-7824. DOI: 10.1016/j.matpur.2016.07.009.
[31] M. de León, P. D. Prieto-Martínez, N. Román-Roy, and S. Vilarriño. “Hamilton–Jacobi Theory in Multisymplectic Classical Field Theories”. J. Math. Phys., 58(9), p. 092901 (Sept. 1, 2017). ISSN: 0022-2488. DOI: 10.1063/1.5004260.

[32] M. de León and S. Vilarriño. “Hamilton–Jacobi Theory in k-Cosymplectic Field Theories”. Int. J. Geom. Methods Mod. Phys., 11(01), p. 1450007 (Jan. 1, 2014). ISSN: 0219-8878. DOI: 10.1142/S0219887814500078.

[33] M. de León and M. Zajac. “Hamilton–Jacobi Theory for Gauge Field Theories”. J. Geom. Phys., 152, p. 103636 (June 2020). ISSN: 03930440. DOI: 10.1016/j.geomphys.2020.103636.

[34] O. Esen, M. de León, and C. Sardón. “A Hamilton–Jacobi Theory for Implicit Differential Systems”. J. Math. Phys., 59(2), p. 022902 (Feb. 1, 2018). ISSN: 0022-2488. DOI: 10.1063/1.4999669.

[35] O. Esen, M. de León, C. Sardón, and M. Zajsc. “Hamilton–Jacobi Formalism on Locally Conformally Symplectic Manifolds”. J. Math. Phys., 62(3), p. 033506 (Mar. 1, 2021). ISSN: 0022-2488. DOI: 10.1063/5.0021790.

[36] J. Gaset, X. Gràcia, M. C. Muñoz-Lecanda, X. Rivas, and N. Román-Roy. “A Contact Geometry Framework for Field Theories with Dissipation”. Ann. Phys., 414, p. 168092 (Mar. 1, 2020). ISSN: 0003-4916. DOI: 10.1016/j.aop.2020.168092.

[37] J. Gaset, X. Gràcia, M. C. Muñoz-Lecanda, X. Rivas, and N. Román-Roy. “New Contributions to the Hamiltonian and Lagrangian Contact Formalisms for Dissipative Mechanical Systems and Their Symmetries”. Int. J. Geom. Methods Mod. Phys., 17(06), p. 2050090 (May 2020). ISSN: 0219-8878. DOI: 10.1142/S0219887820500905.

[38] J. Gaset, A. López-Gordón, and X. Rivas. “Symmetries, conservation and dissipation in time-dependent contact systems”. Under preparation.

[39] A. Ghosh and C. Bhamidipati. “Contact Geometry and Thermodynamics of Black Holes in AdS Spacetimes”. Phys. Rev. D, 100(12), p. 126020 (Dec. 19, 2019). DOI: 10.1103/PhysRevD.100.126020.

[40] S. Grillo. “Non-Commutative Integrability, Exact Solvability and the Hamilton–Jacobi Theory”. Regul. Chaotic Dyn., 15(2), p. 2050090 (May 2020). ISSN: 1560-3547, 1468-4845. DOI: 10.1007/s11071-019-05036-z.

[41] S. Grillo. “Non-Commutative Integrability, Exact Solvability and the Hamilton–Jacobi Theory”. Regul. Chaotic Dyn., 15(2), p. 2050090 (May 2020). ISSN: 1560-3547, 1468-4845. DOI: 10.1007/s11071-019-05036-z.

[42] S. Grillo and E. Padrón. “A Hamilton–Jacobi Theory for General Dynamical Systems and Integrability by Quadratures”. Mathematics, 9(12), p. 1357 12 (Jan. 2021). DOI: 10.3390/math9121357.

[43] S. Grillo and E. Padrón. “A Hamilton–Jacobi Theory for General Dynamical Systems and Integrability by Quadratures in Symplectic and Poisson Manifolds”. J. Geom. Phys., 110, pp. 101–129 (Dec. 1, 2016). ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2016.07.010.

[44] G. Herglotz. “Berührungstransformationen”. Lecture notes. University of Göttingen, 1930.

[45] D. Iglesias-Ponte, M. de León, and D. M. de Diego. “Towards a Hamilton–Jacobi Theory for Nonholonomic Mechanical Systems”. J. Phys. A: Math. Theor., 41(1), p. 015205 (Jan. 11, 2008). ISSN: 1751-8113, 1751-8121. DOI: 10.1088/1751-8113/41/1/015205.

[46] B. Khesin and S. Tabachnikov. “Contact Complete Integrability”. Regul. Chaotic Dyn., 15(4-5), pp. 504–520 (Oct. 2010). ISSN: 1560-3547, 1468-4845. DOI: 10.1134/S1560354710040076.

[47] R. Kraaij, F. Redig, and W. van Zuijlen. “A Hamilton–Jacobi Point of View on Mean-Field Gibbs-non-Gibbs Transitions”. Trans. Amer. Math. Soc., 374(08), pp. 5287–5329 (Aug. 2021). ISSN: 0002-9947, 1088-6850. DOI: 10.1090/tran/8408.

[48] M. J. Lazo, J. Paiva, and G. S. F. Frederico. “Noether Theorem for Action-Dependent Lagrangian Functions: Conservation Laws for Non-Conservative Systems”. Nonlinear Dyn., 97(2), pp. 1125–1136 (July 1, 2019). ISSN: 1573-269X. DOI: 10.1007/s11071-019-05036-z.

[49] M. J. Lazo, J. Paiva, J. T. S. Amaral, and G. S. F. Frederico. “From an Action Principle for Action-dependent Lagrangians toward Non-Conservative Gravity: Accelerating Universe without Dark Energy”. Phys. Rev. D, 95(10), p. 101501 (May 31, 2017). ISSN: 2470-0010, 2470-0029. DOI: 10.1103/PhysRevD.95.101501. arXiv: 1705.04604 [gr-qc, physics:math-ph].
[49] M. J. Lazo, J. Paiva, J. T. S. Amaral, and G. S. F. Frederico. “An Action Principle for Action-dependent Lagrangians: Toward an Action Principle to Non-Conservative Systems”. J. Math. Phys., 59(3), p. 032902 (Mar. 2018). ISSN: 0022-2488, 1089-7658. DOI: 10.1063/1.5019936. arXiv: 1803.08308 [math-ph, physics:physics].

[50] M. Leok, T. Ohsawa, and D. Sosa. “Hamilton–Jacobi Theory for Degenerate Lagrangian Systems with Holonomic and Nonholonomic Constraints”. J. Math. Phys., 53(7), p. 072905 (July 1, 2012). ISSN: 0022-2488. DOI: 10.1063/1.4736733. arXiv: 1803.08308 [math-ph, physics:physics].

[51] M. Leok and D. Sosa. “Dirac Structures and Hamilton–Jacobi Theory for Lagrangian Mechanics on Lie Algebroids”. J. Geom. Mech., 4(4), p. 421 (2012). DOI: 10.3934/jgm.2012.4.421.

[52] P. Libermann and C.-M. Marle. Symplectic Geometry and Analytical Mechanics. Dordrecht: Springer Netherlands, 1987. ISBN: 978-90-277-2439-7. DOI: 10.1007/978-94-009-3807-6.

[53] Q. Liu, P. J. Torres, and C. Wang. “Contact Hamiltonian Dynamics: Variational Principles, Invariants, Completeness and Periodic Behavior”. Ann. Phys., 395, pp. 26–44 (Aug. 1, 2018). ISSN: 0003-4916. DOI: 10.1016/j.aop.2018.04.035.

[54] G. Marmo, G. Morandi, and N. Mukunda. “The Hamilton–Jacobi Theory and the Analogy between Classical and Quantum Mechanics”. J. Geom. Mech., 1(3), p. 317 (2009). DOI: 10.3934/jgm.2009.1.317.

[55] A. A. Martínez-Merino and M. Montesinos. “Hamilton–Jacobi Theory for Hamiltonian Systems with Non-Canonical Symplectic Structures”. Ann. Phys., 321(2), pp. 318–330 (Feb. 1, 2006). ISSN: 0393-0491. DOI: 10.1016/j.aop.2005.08.008.

[56] B. Maschke and A. van der Schaft. “Homogeneous Hamiltonian Control Systems Part II: Application to Thermodynamic Systems”. IFAC-PapersOnLine. 6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control LHMNC 2018, 51(3), pp. 7–12 (Jan. 1, 2018). ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2018.06.002.

[57] T. Ohsawa, O. E. Fernandez, A. M. Bloch, and D. V. Zenkov. “Nonholonomic Hamilton–Jacobi Theory via Chaplygin Hamiltonization”. J. Geom. Phys., 61(8), pp. 1263–1291 (Aug. 1, 2011). ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2011.02.015.

[58] S. Rashkovskiy. “Hamilton–Jacobi Theory for Hamiltonian and Non-Hamiltonian Systems”. J. Geom. Mech., 12(4), p. 563 (2020). DOI: 10.3934/jgm.2020024.

[59] N. Sakamoto. “Analysis of the Hamilton–Jacobi Equation in Nonlinear Control Theory by Symplectic Geometry”. SIAM J. Control Optim., 40(6), pp. 1924–1937 (Jan. 1, 2002). ISSN: 0363-0129. DOI: 10.1137/S0363012999362803.

[60] A. van der Schaft and B. Maschke. “Homogeneous Hamiltonian Control Systems Part I: Geometric Formulation”. IFAC-PapersOnLine. 6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control LHMNC 2018, 51(3), pp. 1–6 (Jan. 1, 2018). ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2018.06.001.

[61] L. Vitagliano. “Geometric Hamilton–Jacobi Field Theory”. Int. J. Geom. Methods Mod. Phys., 09(02), p. 1260008 (Mar. 1, 2012). ISSN: 0219-8878. DOI: 10.1142/S0219887812600080.

[62] V. Zatloukal. “Classical Field Theories from Hamiltonian Constraint: Canonical Equations of Motion and Local Hamilton–Jacobi Theory”. Int. J. Geom. Methods Mod. Phys., 13(06), p. 1650072 (July 1, 2016). ISSN: 0219-8878. DOI: 10.1142/S0219887816500729.