Bounded-Regret MPC via Perturbation Analysis: Prediction Error, Constraints, and Nonlinearity

Yiheng Lin
California Institute of Technology
Pasadena, CA, USA
yihengl@caltech.edu

Yang Hu
Harvard University
Cambridge, MA, USA
yanghu@g.harvard.edu

Guannan Qu
Carnegie Mellon University
Pittsburgh, PA, USA
gqu@andrew.cmu.edu

Tongxin Li
The Chinese University of Hong Kong (Shenzhen)
Shenzhen, Guangdong, China
litongxin@cuhk.edu.cn

Adam Wierman
California Institute of Technology
Pasadena, CA, USA
adamw@caltech.edu

Abstract

We study Model Predictive Control (MPC) and propose a general analysis pipeline to bound its dynamic regret. The pipeline first requires deriving a perturbation bound for a finite-time optimal control problem. Then, the perturbation bound is used to bound the per-step error of MPC, which leads to a bound on the dynamic regret. Thus, our pipeline reduces the study of MPC to the well-studied problem of perturbation analysis, enabling the derivation of regret bounds of MPC under a variety of settings. To demonstrate the power of our pipeline, we use it to generalize existing regret bounds on MPC in linear time-varying (LTV) systems to incorporate prediction errors on costs, dynamics, and disturbances. Further, our pipeline leads to regret bounds on MPC in systems with nonlinear dynamics and constraints.

1 Introduction

Model Predictive Control (MPC) is an optimal control approach that solves a Finite-Time Optimal Control Problem (FTOCP) using future predictions in a receding horizon manner [1]. It is a flexible approach that is able to accommodate nonlinear and time-varying dynamics, state and actuation constraints, and general cost functions [2–5]. As a result, it is broadly applied in a wide spectrum of control problems, including robotics [6–10], autonomous vehicles [11–17], power systems [18–24], process control [25–27], etc.

Despite the popularity of MPC, its theoretic analysis has been quite challenging. Early works along this line focused on the stability and recursive feasibility of MPC [28,31]. More recently, there has been tremendous interest in providing finite-time learning-theoretic performance guarantees for MPC, such as regret and/or competitive ratio bounds [32,33]. For example, progress has recently been

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made toward (i) regret analysis of MPC in linear time-invariant (LTI) systems with prediction errors on the trajectory to track [34], (ii) the dynamic regret and competitive ratio bounds of MPC under linear time-varying (LTV) dynamics with exact predictions [35], and (iii) exponentially decaying perturbation bounds of the finite-time optimal control problem in time-varying, constrained, and non-linear systems [36–37]. Beyond MPC, providing regret and/or competitive ratio guarantees for a variety of (predictive) control policies has been a focus in recent years. Examples include RHGC [38–39] and AFHC [20–40] for online control/optimization with prediction horizons, OCO-based controllers [41–42] for no-regret online control, and variations of ROBD for competitive online control without predictions [43–44] or with delayed observations [45]. In addition, regret lower bounds have been studied in known LTI systems [46] and unknown LTV systems [47].

A promising analysis approach that has emerged from the literature studying MPC and, more generally, predictive control, is the use of perturbation analysis techniques, or more particularly, the use of so-called exponential decaying perturbation bounds. Such techniques underlie the results in [34,37]. This research direction is particularly promising since perturbation bounds exist for FTOCP in many dynamical systems, e.g., [48–52], and thus it potentially allows the derivation of regret and/or competitive ratio bounds in a variety of settings. However, to this point the approach has only yielded results in unconstrained linear systems with no prediction errors (e.g., [35]), and often requires adjusting MPC to include a counter-intuitively large re-planning window due to technical challenges in the analysis (e.g., [48,49]).

Thus, though perturbation analysis techniques might seem promising, many important questions about applying them for the study of predictive control remain open. Firstly, one of the major reasons for the extensive application of MPC is its flexibility in incorporating constraints and nonlinear dynamics [53]. However, none of the existing results and approaches can analyze the performance of MPC under constraints and/or nonlinear dynamics. In fact, the analysis of MPC under constraints or nonlinearity has long been known to be challenging because of the intractable form of cost-to-go functions and optimal solutions. Secondly, prediction error is inevitable for real-world implementations of MPC due to unpredictable noise and model mismatch, yet the analysis of MPC subject to prediction errors is limited. Thirdly, existing approaches analyze MPC in a case-by-case manner and, in most cases, the analysis framework is specific to the assumptions of the particular case (e.g., quadratic costs, perfect predictions, etc) in a way that does not generalize to other settings [33,35,48,49].

Contributions. In this paper, we propose a general analysis pipeline (Section 3) that converts perturbation bounds for an FTOCP into dynamic regret bounds for MPC across a variety of settings. More specifically, the pipeline consists of three steps (see Figure 1). In Step 1, we obtain the required perturbation bounds for the specific setting. In Step 2, as shown in Lemma 3.1, the perturbation bounds are used to bound the per-step error, which is defined to be the error of the MPC action against the clairvoyant optimal action (see Definition 3.1). In Step 3, the per-step error bound is converted to a dynamic regret bound for MPC, as shown in Lemma 3.2. The full pipeline is summarized into a Pipeline Theorem (Theorem 3.3), which directly converts perturbation bounds into bounds on the dynamic regret of MPC in general settings, including those with time-variation, prediction error, constraints, and nonlinearities. The key technical insight that enables the pipeline is the following recursive relationship between Step 2 and Step 3 (Lemma 3.1 and Lemma 3.2): Step 2 guarantees a “small” per-step error once the current state $x_t$ of MPC is “near” the offline optimal trajectory (OPT), while Step 3 guarantees the next state $x_{t+1}$ of MPC will be near OPT if all previous per-step errors $\{e_{\tau}\}_{\tau \leq t}$ are small. Thus Step 2 and Step 3 work together to guarantee MPC states are always near OPT and thus MPC per-step errors are always small (Theorem 3.3).

To demonstrate the power of the proposed pipeline, we apply it to a range of settings, as summarized in Table 1. Our first applications are to two settings with linear time-varying (LTV) dynamics and prediction errors on (i) disturbances, Section 4.1, and (ii) the dynamical matrices and cost functions, Section 4.2. The state-of-the-art results in the LTV setting are [35], which requires exact knowledge of the disturbances and of the dynamics. To the best of our knowledge, our work provides the first regret result for MPC with prediction error on the dynamics (see Theorem 4.2), a result that enables the bounds in settings where MPC is applied to learned dynamics [53,54].

Our second application is to a setting with nonlinear dynamics and constraints (Section 5). We show the first dynamic regret bound for MPC under state and actuation constraints in nonlinear systems with general costs (Theorem 5.1). Very few prior results exist for MPC in this setting, even without nonlinear dynamics or constraints individually. The most related works are [48], which studies constrained
2 Preliminaries

In this section, we first introduce the general predictive online control problem including the settings, the objective, available information, and the predictive controller class. Then, we introduce the MPC algorithm, which is a widely-used predictive controller that we focus on in this work. Specifically, we consider a general, finite-horizon, discrete-time optimal control problem with time-varying costs, dynamics and constraints, namely

\[
\min_{x_0:T, u_{0:T-1}} \sum_{t=0}^{T-1} f_t(x_t, u_t; \xi^*_t) + F_T(x_T; \xi^*_T)
\]

\[
\text{s.t. } x_{t+1} = g_t(x_t, u_t; \xi^*_t), \quad \forall 0 \leq t < T,
\]

\[
s_t(x_t, u_t; \xi^*_t) \leq 0, \quad \forall 0 \leq t < T,
\]

\[
x_0 = x(0).
\]

Here, \(x_t \in \mathbb{R}^n\) is the state, \(u_t \in \mathbb{R}^m\) is the control input or action; \(f_t\) is a time-varying stage cost function, \(g_t\) is a time-varying dynamical function, and \(s_t\) is a time-varying constraint function, all parameterized by a ground-truth parameter \(\xi^*_t\) (unknown to an online controller); and \(F_T\) is a terminal cost function parameterized by \(\xi^*_T\) that regularizes the terminal state.

The offline optimal trajectory \(\text{OPT}\) is obtained by solving (I) with the full knowledge of the true parameters \(\xi^*_{0:T}\). In contrast, an online controller can only observe noisy estimations of the parameters in a fixed prediction horizon to decide its current action \(u_t\) at each time step \(t\). For example, MPC picks \(u_t\) by calculating the optimal sub-trajectory confined to the prediction horizon. The objective is to design an online controller that can compete against the offline optimal trajectory \(\text{OPT}\). We use dynamic regret as the performance metric, which is widely used to evaluate the performance of online controllers/algorithms in the literature of online control [32, 34, 35] and online optimization [38, 43, 55]. Specifically, for a concrete problem instance \((x(0), \xi^*_{0:T})\), let \(\text{cost}(\text{OPT})\) denote the total cost incurred by \(\text{OPT}\), and \(\text{cost}(\text{ALG})\) denote the total cost incurred by an online controller \(\text{ALG}\). The dynamic regret is defined as the worst-case additional cost incurred by \(\text{ALG}\) against \(\text{OPT}\), i.e., \(\sup_{x(0), \xi^*_{0:T}} (\text{cost}(\text{ALG}) - \text{cost}(\text{OPT})).\)

The formulation in (I) is general enough to include a variety of challenging settings. In this paper, we consider three important settings to illustrate how to apply our analysis pipeline. The settings differ in (a) the form of costs, dynamics, and constraints, and (b) the quantities in the system to be predicted (i.e., parameterized by \(\xi^*_t\)), and the prediction error allowed. An overview of the settings is presented in Table 1 below.

| Section | Costs | Dynamics | Constraints | Prediction \(\xi^*_t\) | Prediction error |
|---------|-------|----------|-------------|----------------------|-----------------|
| 4.1     | decomposable | LTV | none | disturbance: \(w_t\) | arbitrary |
| 4.2     | quadratic             | LTV | none | cost: \(Q_t, R_t, x_t\) | sufficiently small |
| 5       | general | non-linear | non-linear stage constraint | cost: \(f_t\) | sufficiently small |
|         |        | time-varying | dynamics: \(A_t, B_t\) | constraints: \(s_t\) | |

In each setting, we impose different assumptions on cost functions, dynamical systems, constraints, and properties of the predicted quantities as functions of parameter \(\xi_t\). In general, we require well-defined costs, Lipschitz and uniformly controllable dynamics, and Lipschitzness of the predicted quantities with regard to \(\xi_t\). For constraints, additional assumptions characterizing the active constraints along and near the optimal trajectory are imposed. Detailed definitions and statements are deferred to Appendix B and Sections 3, 4, and 5. To facilitate the statement of the pipeline, we assume the following universal properties hold throughout the paper:

- **Stability of \(\text{OPT}\):** there exists a constant \(D_x\) such that \(\|x^*\| \leq D_x\) for every state \(x^*_t\) on the offline optimal trajectory \(\text{OPT}\).
We formally define the quality of predictions by introducing the following notion of prediction error.

We represent the uncertainties in cost functions, dynamics, constraints, and terminal costs as function families parameterized by \( \xi \): \( F_\xi \) := \( \{ f(x_t, u_t; \xi_t) \mid \xi_t \in \Xi_t \} \), \( G_\xi \) := \( \{ g_t(x_t, u_t; \xi_t) \mid \xi_t \in \Xi_t \} \), \( S_\xi := \{ s_t(x_t, u_t; \xi_t) \mid \xi_t \in \Xi_t \} \), and \( F_{T, \xi} := \{ F_T(x_T; \xi_T) \mid \xi_T \in \Xi_T \} \). The online controller knows the function families \( F_{0:T}, G_{0:T-1} \), and \( S_{0:T-1} \) as prior knowledge, but it does not know the true parameters \( \xi^\ast_{0: T} \in \prod_{t=0}^T \Xi_t \). Instead, at time step \( t \), the online controller has access to noisy predictions of these parameters for the future \( k \) time steps (where \( k \) is called the prediction horizon), represented by \( \xi_{t:t+k|t} \in \prod_{t=0}^T \Xi_t \). The parameter space \( \Xi_t \) at each time step \( t \) may have different dimensions.

We formally define the quality of predictions by introducing the following notion of prediction error.

**Definition 2.1.** The prediction error is defined as \( \rho_{t, \tau} := \| \xi_{t+\tau|t} - \xi^\ast_{t+\tau} \| \) for an integer \( \tau \geq 0 \). The power of \( \tau \)-step-away predictions (for parameter \( \xi \)) is defined as \( P(\tau) := \sum_{\tau=0}^\infty \rho_{t, \tau}^2 \).

Under this noisy prediction model, a general predictive online controller \( \text{ALG} \) decides the control action based on the current state and the latest available predictions of future parameters. We formally define the class of predictive online controllers considered in this paper in Definition 2.2, which includes MPC as a special case.

**Definition 2.2.** A predictive online controller \( \text{ALG} \) is a function that takes the current state \( x_t \) and the available predictions \( \xi_{t:t+k|t} \) as inputs at time \( t \) and outputs the current control action \( u_t \), i.e., \( u_t = \text{ALG}(x_t, \xi_{t:t+k|t}) \). We use \( x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} \cdots \xrightarrow{u_{T-1}} x_T \) to denote the trajectory achieved by \( \text{ALG} \), and use \( x_0 \xrightarrow{u^\ast_0} x_1 \xrightarrow{u^\ast_1} \cdots \xrightarrow{u^\ast_{T-1}} x^\ast_T \) to denote the offline optimal trajectory \( \text{OPT} \).

A core component of both the design of online controllers and our analysis is the following finite-time optimal control problem (FTOCP). Given a time interval \([t_1, t_2]\), the FTOCP solves the optimal sub-trajectory subject to the given initial state \( z \), terminal cost \( F \), and a sequence of (potentially noisy) parameters \( \xi_{t_1:t_2-1}, \xi_{t_2} \), as formalized in the following definition.

**Definition 2.3.** The finite-time optimal control problem (FTOCP) over the horizon \([t_1, t_2]\), with initial state \( z \), parameters \( \xi_{t_1:t_2-1} \), and terminal cost \( F(\cdot; \cdot) \), is defined as

\[
\ell^F_{t_1}(z; \xi_{t_1:t_2-1}, \xi_{t_2}: F) := \min_{y_{t_1:t_2-1}, u_{t_1:t_2-1}} \sum_{t=t_1}^{t_2-1} f_t(y_t, u_t; \xi_t) + F(y_{t_2}; \xi_{t_2})
\]

subject to

\[
\begin{align*}
    y_{t+1} &= g_t(y_t, u_t; \xi_t), & \quad & \forall t_1 \leq t < t_2, \\
    s_t(y_t, u_t; \xi_t) &\leq 0, & \quad & \forall t_1 \leq t < t_2, \\
    y_{t_1} &= z,
\end{align*}
\]

and the corresponding optimal solution as \( \psi^F_{t_1}(z; \xi_{t_1:t_2-1}, \xi_{t_2}; F) \). We shall use the shorthand notation \( \psi^F_{t_1}(z; \xi_{t_1:t_2-1}, \xi_{t_2}; F) := \psi^F_{t_1}(z; \xi_{t_1:t_2-1}, \xi_{t_2}; F) \) when the context is clear.

Note that the formulation of the FTOCP in Definition 2.3 does not include a terminal constraint set. To compensate for this, we allow the terminal cost \( F(\cdot; \xi_{t_2}) \) to take value \( +\infty \) in some subset of \( \mathbb{R}^n \), and \( \xi_{t_2} \) is not necessarily an element in \( \Xi_{t_2} \). For example, a terminal cost function that we frequently use later is the indicator function of the terminal parameter \( \xi_{t_2} \), where \( \xi_{t_2} \in \mathbb{R}^n \). We use \( I \) to denote such indicator terminal cost (i.e., \( I(y_{t_2}; \xi_{t_2}) = 0 \) if \( y_{t_2} = \xi_{t_2} \) and \( I(y_{t_2}; \xi_{t_2}) = +\infty \) otherwise).
Finally, given the definition of the FTOCP, we are ready to formally introduce MPC. The pseudocode of this online controller is given in Algorithm 1. Basically, at time step \( t \), \( \text{MPC}_k \) solves a \( k \)-step predictive FTOCP using the latest available parameter predictions, and commits the first control action in the solution. When there are only fewer than \( k \) steps left, \( \text{MPC}_k \) directly solves a \((T - t)\)-step FTOCP at time \( t \) until the end of the horizon, using the predicted real terminal cost \( F_T(\cdot; \xi_{T|t}) \). This MPC controller (and its variants) has a wide range of real-world applications.

**Algorithm 1 Model Predictive Control (\( \text{MPC}_k \))**

**Require:** Specify the terminal costs \( F_t \) for \( k \leq t < T \).

1: for \( t = 0, 1, \ldots, T - 1 \) do
2: \( t' \leftarrow \min\{t + k, T\} \)
3: Observe current state \( x_t \) and obtain predictions \( \xi_{t:t'|t} \).
4: Solve and commit control action \( u_t := \psi_t'(x_t, \xi_{t:t'|t}; F_T)_{v_1} \).

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### 3 The Pipeline: Bounded Regret via Perturbation Analysis

The goal of this section is to give an overview of a novel analysis pipeline that converts a perturbation bound into a bound on the dynamic regret. We begin by highlighting the form of perturbation bounds required in the pipeline, and then describe the 3-step process of applying the pipeline. In subsequent sections, we apply this pipeline to obtain new regret bounds for MPC in different settings.

#### 3.1 Per-Step Error and Perturbation Bounds

A key challenge when comparing the performance of an online controller against the offline optimal trajectory is that the online controller’s state \( x_t \) is different from the offline optimal state \( x_t^* \) at time step \( t \). Due to such discrepancy in states, we cannot simply evaluate the online controller’s action \( u_t \) via comparison against the offline optimal action \( u_t^* \). To address this challenge, our pipeline uses the notion of per-step error (Definition 3.1) inspired by the performance difference lemma and its proofs in reinforcement learning (RL) \cite{35}. Specifically, we compare \( u_t \) to the clairvoyant optimal action one may adopt at the same state \( x_t \) if all true future parameters \( \xi_{t:T} \) are known, which leads to the definition of per-step error as follows.

**Definition 3.1.** The per-step error \( e_t \) incurred by a predictive online controller \( \text{ALG} \) at time step \( t \) is defined as the distance between its actual action \( u_t \) and the clairvoyant optimal action, i.e.,

\[
e_t := \| u_t - \psi_t'(x_t, \xi_{t:T}; F_T)_{v_1} \|, \quad \text{where } u_t = \text{ALG}(x_t, \xi_{t:t+k|t}).
\]

The clairvoyant optimal trajectory starting from \( x_t \) is defined as \( x_{t:T|t}^* := \psi_t^T(x_t, \xi_{t:T}; F_T)_{v_1} \).

Note that the clairvoyant optimal trajectory can be viewed as being generated by an MPC controller with long enough prediction horizon and exact predictions. This notion highlights the reason why MPC can compete against the clairvoyant optimal trajectory, since the per-step error in a system controlled by \( \text{MPC}_k \) becomes

\[
e_t = \| \psi_t^t+k(x_t, \xi_{t:t+k|t}; F_{t+k})_{v_1} - \psi_t^t(x_t, \xi_{t:T}; F_T)_{v_1} \| \text{.}
\]

Intuitively, the per-step error converges to zero as the prediction horizon \( k \) increases and the quality of predictions improves (i.e. \( \|\xi_{t:t+k|t} - \xi_{t:T}^*\| \to 0 \)).

This intuition highlights the important role of perturbation bounds in comparing online controllers against (offline) clairvoyant optimal trajectories. As we have discussed in Section 1, many previous works \cite{36,37,43,49} have established (local) decaying sensitivity/perturbation bounds for different instances of the FTOCP \cite{4}. These bounds may take different forms, but for the application of our pipeline we require two types of perturbation bounds that are both common in the literature:

(a) **Perturbations of the parameters** \( \xi_{t_1:t_2} \) **given a fixed initial state** \( z \):

\[
\| \psi_t^{t_2}(z, \xi_{t_1:t_2}; F)_{v_1} - \psi_{t_1}^{t_2}(z, \xi_{t_1:t_2}; F)_{v_1} \| \leq \sum_{t=t_1}^{t_2} q_1(t - t_1)\delta_t, \quad \sum_{t=1}^{\infty} q_1(t) \leq C_t, \quad i = 1, 2.
\]

where \( \delta_t := \| \xi_t - \xi_t^* \| \) for \( t \in [t_1, t_2] \), and scalar functions \( q_1 \) and \( q_2 \) satisfy \( \lim_{t \to \infty} q_1(t) = 0 \), \( \sum_{t=0}^{\infty} q_i(t) \leq C_i \) for constants \( C_i \geq 1 \). This perturbation bound is useful in bounding the per-step error \( e_t \), as we will discuss in Lemma 3.1.
(b) Perturbation of the initial state \(z\) given fixed parameters \(\xi_{t_1:t_2}\):

\[
\left\| \psi_{t_2}^{t_1}(z, \xi_{t_1:t_2}; F)_{y_{t_1}/v_{t_1}} - \psi_{t_1}^{t_1}(z', \xi_{t_1:t_2}; F)_{y_{t_1}/v_{t_1}} \right\| \leq q_3(t - t_1) \|z - z'\|, \quad \text{for } t \in [t_1, t_2],
\]

where the scalar function \(q_3\) satisfies \(\sum_{t=0}^{\infty} q_3(t) \leq C_3\) for some constant \(C_3 \geq 1\). This bound is useful in preventing the accumulation of per-step errors \(e_t\) throughout the horizon (see Lemma 3.2).

Compared with (3), the right hand side of (4) has a simpler form.

Existing perturbation bounds usually combine the above two types ((3) and (4)) into a single equation that characterizes perturbations on \(z\) and \(\xi_{t_1:t_2}\) simultaneously, e.g., [35, 37]. Here, we decompose them into two separate types because they are used in different parts of our pipeline.

### 3.2 A 3-Step Pipeline from Perturbation Bounds to Regret

An overview of the pipeline is given in Figure 1, which illustrates the high-level ideas of the pipeline that starts by obtaining perturbation bounds, proceeds to bound the per-step error using perturbation bounds, and finally combines the per-step error and perturbation bounds to bound the dynamic regret. In the following we describe each step in detail.

**Step 1: Obtain the perturbation bounds given in (3) and (4).**

The form of the perturbation bounds depends heavily on the specific form of the FTOCP, and thus the derivation requires case-by-case study (e.g., see Section 4 and Section 5). However, off-the-shelf bounds are available in most cases, as there has been a rich literature on perturbation analysis of control systems (e.g., [35, 37, 48, 49] and the references therein). The following property summarizes precisely what is expected to be derived for bounds (3) and (4) in Steps 2 and 3.

**Property 3.1.** Suppose there exists a positive constant \(R\) such that the perturbation bound (3) holds for the following specifications: with \(t_1 = t\) and \(t_2 = t + k\) for \(t < T - k\), (3) holds for \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) be the identity function \(I_2\), and

\[
z \in B(x_t^*, R); \quad \xi_{t:t+k-1} \in \Xi_{t:t+k-1}; \quad \xi_{t:t+k-1} = \xi^*_{t:t+k-1}; \quad \xi_{t:t+k} \in B(x_{t+k}^*, R) \subseteq \mathbb{R}^n;
\]

with \(t_1 = t\) and \(t_2 = T\) for \(t \geq T - k\), (3) holds for \(z \in B(x_t^*, R); \quad \xi_{t:T} \in \Xi_{t:T}; \quad \xi_{t:T} = \xi^*_{t:T}; \quad F = F_{t+T}\). Further, perturbation bound (4) holds for any \(z, z' \in B(x_t^*, R)\) and \(\xi_{t:t+k}\) is set to be the indicator function of some state \(\xi_{t:k}\) that satisfies\(\xi_{t:k}\) live in the space \(\mathbb{R}^n\) rather than \(\Xi_{t:k}\) because they represent the target terminal state of the FTOCP solved by \(\text{MPC}_k\). Intuitively, Property 3.1 states that perturbation bounds (3) and (4) hold in a small neighborhood (specifically, a ball with radius \(R\)) around the offline optimal trajectory \(\text{OPT}\), which is much weaker than the global exponentially decaying perturbation bounds required by previous work (e.g., [35]) in the following sense: (i) in the general settings where the dynamical function \(g_t\) is non-linear, or where there are constraints on states and actions, one cannot hope the perturbation bound to hold globally for all possible parameters (37, 49, 50); (ii) the decay functions \(\{q_i\}_{i=1,2,3}\) are only required to converge to zero and satisfy \(\sum_{t=0}^{\infty} q_i(t) \leq C_i\), which means the exponential decay rate as in (35) is not necessary — in fact, polynomial decay rates can also satisfy these properties, which greatly broadens the applicability of our pipeline.

**Step 2: Bound the per-step error \(e_t\).** The core of the analysis is to apply the perturbation bounds to bound the per-step error. For \(\text{MPC}_k\), under Property 3.1 this step can be done in a universal way, as summarized in Lemma 3.1 below. A complete proof of Lemma 3.1 can be found in Appendix C.

**Lemma 3.1.** Let Property 3.1 hold. Suppose the current state \(x_t\) satisfies \(x_t \in B(x_t^*, R/C_3)\) and the terminal cost \(F_{t+k}\) of \(\text{MPC}_k\) is set to be the indicator function of some state \(\bar{y}(\xi_{t+k})\) that satisfies \(\bar{y}(\xi_{t+k}) \in B(x_{t+k}^*, R)\) for \(t < T - k\). Then, the per-step error of \(\text{MPC}_k\) is bounded by

\[
e_t \leq \sum_{\tau=0}^{k} \left( \left( \frac{R}{C_3} + D_{x^*} \right) \cdot q_1(\tau) + q_2(\tau) \right) \rho_{t,\tau} + 2R \left( \left( \frac{R}{C_3} + D_{x^*} \right) \cdot q_1(k) + q_2(k) \right). \quad (5)
\]
Lemma 3.1 is a straight-forward implication of perturbation bound (3) specified in Property 3.1. To see this, for \( t < T - k \), note that the per-step error \( e_t \) can be bounded by

\[
e_t = \left\| \psi_t^{t+k}(x_t, \xi_{t:t+k-1}; \bar{y}(\xi_{t+k|t}; \bar{B}_T)e_t) - \psi_t^T(x_t, \xi_{t:T}; F_T)e_t \right\|
\]

(6a)

\[
= \left\| \psi_t^{t+k}(x_t, \xi_{t:t+k-1}; \bar{y}(\xi_{t+k|t}; \bar{B}_T)e_t) - \psi_t^{t+k}(x_t, \xi_{t:t+k-1}; x^*_{t+k|t}; \bar{B}_T)e_t \right\|
\]

(6b)

\[
\leq \sum_{\tau = 0}^{k-1} \left( \| x_t \cdot q_1(\tau) + q_2(\tau) \| \rho_{t,\tau} + (\| x_t \| \cdot q_1(k) + q_2(k)) \right) \left\| \bar{y}(\xi_{t+k|t}) - x^*_{t+k|t} \right\|.
\]

(6c)

Here, we apply the principle of optimality to conclude that the optimal trajectory from \( x_t \) to \( x^*_{t+k|t} \) (i.e., \( \psi_t^{t+k}(x_t, \xi_{t:t+k-1}; x^*_{t+k|t}; \bar{B}_T) \) in (6b) is a sub-trajectory of the clairvoyant optimal trajectory from \( x_t \) (i.e., \( \psi_t^T(x_t, \xi_{t:T}; F_T) \) in (6a)), and (6c) is obtained by directly applying perturbation bound (3). Note that \( \| x_t \| \leq R/C_3 + D_x \), and that both \( \bar{y}(\xi_{t+k|t}) \) and \( x^*_{t+k|t} \) are in \( B(x^*_t; R) \) by assumption and by perturbation bound (4) specified in Property 3.1, we conclude that (5) hold for \( t < T - k \). The case \( t \geq T - k \) can be shown similarly. We defer the detailed proof to Appendix C.

**Step 3: Bound the dynamic regret by \( \sum_{t=0}^{T-1} e_t^2 \).** This final step builds upon perturbation bound (4), and aims at deriving dynamic regret bounds in a universal way, as stated in Lemma 3.2 below. Specifically, under the assumption that a local decaying perturbation bound in the form of (4) holds around the offline optimal trajectory OPT, and the property that per-step errors \( e_t \) are sufficiently small, we can show that the online controller will not leave the “safe region” near the offline optimal trajectory as specified in Property 3.1 and thus the dynamic regret of ALG is bounded as in (7) (note that ALG is not confined to MPC, but is allowed to be any algorithm with bounded per-step errors). A complete proof of Lemma 3.2 can be found in Appendix D.

**Lemma 3.2.** Let Property 3.1 hold. If the per-step errors of ALG satisfy \( e_t \leq L / (C_3^2 L_g) \) for all time steps \( \tau < t \), the trajectory of ALG will remain close to OPT at time \( t \), i.e. \( x_t \in B(x^*_t; R/C_3) \). Further, if \( e_t \leq L / (C_3^2 L_g) \) for all \( t < T \), the dynamic regret of ALG is upper bounded by

\[
\text{cost(ALG)} - \text{cost(OPT)} = O \left( \sqrt{\text{cost(OPT)} + \sum_{t=0}^{T-1} e_t^2} \right).
\]

(7)

**Summary.** Combining Steps 2 and 3 of the pipeline yields the following Pipeline Theorem for MPC_k (see Theorem 3.3). Basically it states that, when the prediction horizon \( k \) is sufficiently large and the prediction errors \( \rho_{t,\tau} \) are sufficiently small, Lemma 3.1 and Lemma 3.2 can work together to make sure that MPC_k never leaves a \( (R/C_3) \)-ball around the offline optimal trajectory OPT; thus we obtain a dynamic regret bound.

**Theorem 3.3 (The Pipeline Theorem).** Let Property 3.1 hold. Suppose the terminal cost \( F_{t+k} \) of MPC_k is set to be the indicator function of some state \( y(\xi_{t+k|t}) \) that satisfies \( y(\xi_{t+k|t}) \in B(x^*_t; R) \) for all time steps \( t < T - k \). Further, suppose the prediction errors \( \rho_{t,\tau} \) are sufficiently small and the prediction horizon \( k \) is sufficiently large, such that

\[
\sum_{\tau = 0}^{k-1} \left( \left( \frac{R}{C_3} + D_x \right) \cdot q_1(\tau) + q_2(\tau) \right) + 2 \rho_{t,\tau} + 2 \left( \left( \frac{R}{C_3} + D_x \right) \cdot q_1(k) + q_2(k) \right) \leq \frac{R}{C_3^2 L_g}.
\]

Then, the trajectory of MPC_k will remain close to OPT, i.e. \( x_t \in B(x^*_t; R/C_3) \) for all time steps \( t \), and the dynamic regret of MPC_k is upper bounded by

\[
\text{cost(MPC_k)} - \text{cost(OPT)} = O \left( \sqrt{\text{cost(OPT)} \cdot E + E} \right),
\]

(8)

where \( E := \sum_{t=0}^{T-1} (q_1(\tau) + q_2(\tau)) P(\tau) + (q_1(k)^2 + q_2(k)^2) T \).

The proof of Theorem 3.3 can be found in Appendix E. To interpret the dynamic regret bound in (8), note that we have \( \text{cost(OPT)} = O(T) \) as a result of our model assumptions. Thus, the dynamic regret of ALG is in the order of \( \sqrt{TE} + E \). When there is no prediction error, the bound \( O((q_1(k) + q_2(k)) \cdot T) \) reproduces the result in [35], and the bound will degrade as the prediction error increases. It is also worth noticing that, when the prediction power improves over time as the online controller learns the system better and \( k = \Omega(\ln T) \), the dynamic regret can be \( o(T) \).
4 Unconstrained LTV Systems

We now illustrate the use of the Pipeline Theorem by applying it in the context of (unconstrained) LTV systems with prediction errors, either on disturbances or the dynamical matrices.

4.1 Prediction Errors on Disturbances

In this section, we consider the following special case of problem (1), where the dynamics is LTV and the prediction error can only occur on the disturbances $w_t$:

$$\min_{x_0:T, u_0:T-1} \sum_{t=0}^{T-1} \left( f_t^L(x_t) + f_t^u(u_t) \right) + F_T(x_T)$$

s.t. $x_{t+1} = A_t x_t + B_t u_t + w_t(\xi_t^*), \quad \forall 0 \leq t < T,$

$$x_0 = x(0).$$

All necessary assumptions on the system are summarized below in Assumption 4.1.

**Assumption 4.1.** Assume the following holds for the online control problem instance (9):

- Cost functions: $\{f_t^L\}_{t=0}^{T-1}, \{f_t^u\}_{t=0}^{T-1}, F_T$ are nonnegative $\mu$-strongly convex and $l$-smooth. And we assume $f_t^L(0) = f_t^u(0) = 0$ without the loss of generality.
- Dynamical systems: the LTV system $\{A_t, B_t\}$ is $\sigma$-uniform controllable with controllability index $d$, and $|A_t| \leq a$, $|B_t| \leq b$, and $|B_t^r| \leq b'$ hold for all $t$, where $B_t^r$ denotes the Moore–Penrose inverse of matrix $B_t$. The detailed definitions can be found in Assumption F.1 in Appendix F.
- Predicted quantities: $\|w_t(\xi_t)\| \leq D_w$ holds for all $\xi_t \in \Xi_t$ and all $t$. For every time step $t$, $w_t(\xi_t)$ is a $L_w$-Lipschitz function in $\xi_t$, i.e., $\|w_t(\xi_t) - w_t(\xi_t')\| \leq L_w \|\xi_t - \xi_t'\|, \forall \xi_t, \xi_t' \in \Xi_t$.

Under Assumption 4.1, we can again apply the perturbation bounds shown in [35] to show Property 3.1. In particular, we already know that for some constants $H_1 \geq 1$ and $\lambda_1 \in (0, 1)$, perturbation bounds (3) and (4) hold globally, for $q_1(t) = 0$, $q_2(t) = H_1 \lambda_1^t$, and $q_3(t) = H_1 \lambda_1^t$. Since both of these perturbation bounds hold globally, radius $R$ in Property 3.1 can be set arbitrarily, and we shall take $R := \max \left\{ D_x \cdot \frac{2L_w H_1^3}{(1-\lambda_1)^2} \right\}$ so that Theorem 3.3 can be applied to MPC_k with terminal cost $F_{t+k}(\cdot; \xi_{t+k}) \equiv \|\cdot\|_0$. This leads to the following dynamic regret bound:

**Theorem 4.1.** In the unconstrained LTV setting (9), under Assumption 4.1, when the prediction horizon $k$ is sufficiently large such that $k \geq \ln \left( \frac{4H_1^3 L_w}{(1-\lambda_1)^2} \right) / \ln(1/\lambda_1)$, the dynamic regret of MPC_k (Algorithm I) with terminal cost $F_{t+k}(\cdot; \xi_{t+k}) \equiv \|\cdot\|_0$ is bounded by $\text{cost}(\text{MPC}_k) - \text{cost}(\text{OPT}) \leq O \left( \sqrt{T \cdot \sum_{t=0}^{k-1} \lambda_1^t P(\tau)} + \lambda_1^k T^2 + \sum_{t=0}^{k-1} \lambda_1^t P(\tau) \right)$.

A complete proof of Theorem 4.1 can be found in Appendix F. When there are no prediction errors, the bound in Theorem 4.1 reduces to $O(\lambda_1^k T)$, which reproduces the result of [35]. Further, it is also worth noticing that due to the form of discounted sum $\sum_{t=0}^{k-1} \lambda_1^t P(\tau)$, prediction errors for the near future matter more than those for the far future.

4.2 Prediction Error on Costs and Dynamical Matrices

We now consider prediction errors on cost functions and dynamics, rather than disturbances. Specifically, we consider the following instance of problem (1):

$$\min_{x_0:T, u_0:T-1} \sum_{t=0}^{T-1} \left( \langle x_t - \bar{x}_t(\xi_t^*) \rangle^T Q_t(\xi_t^*) (x_t - \bar{x}_t(\xi_t^*)) + u_t^T R_t(\xi_t^*) u_t \right) + F_T(x_T; \xi_t^*)$$

s.t. $x_{t+1} = A_t(\xi_t^*) \cdot x_t + B_t(\xi_t^*) \cdot u_t + w_t(\xi_t^*), \quad \forall 0 \leq t < T,$

$$x_0 = x(0),$$

where the terminal cost is given by $F_T(x_T; \xi_t^*) := \langle x_T - \bar{x}_T(\xi_t^*) \rangle^T P_T(\xi_t^*) (x_T - \bar{x}_T(\xi_t^*)).$

All necessary assumptions on the system are summarized below in Assumption 4.2.
Assumption 4.2. Assume the following holds for the online control problem instance (10):

- Cost: $\mu I \preceq Q_t(\xi_t) \preceq I, \mu I \preceq R_t(\xi_t) \preceq I, \text{ and } \mu I \preceq P_T(\lambda T) \preceq I, \forall \xi_t \in \Xi_t, \forall t$.
- Dynamical systems: both the ground-truth LTV system $\{A_t(\xi_t), B_t(\xi_t)\}_{t=0}^{T-1}$ and any predicted LTV system $\{A_t(\tilde{\xi}_{t:t+1}), B_t(\tilde{\xi}_{t:t+1})\}_{t=0}^{T-1}$ (for all $\xi_t \in \Xi_t$ and all $t$) satisfy the controllability assumptions in Assumption G.7 in Appendix G.
- Assumption H.1 in Appendix H. Perhaps surprisingly, decaying perturbation bounds can hold even in this case. In particular, using Theorem 4.5 in [50], we can show that there exists a small constant $w = O(1)$ such that, for some constants $L_A, L_B$, $L_{R_T}$, $L_{x}$, $L_{w}$ are defined similarly.

Under Assumption 4.2, we can show that for some constants $H_2 \geq 1$ and $\lambda_2 \in (0, 1)$, perturbation bounds (3) and (4) hold globally for $q_1(t) \equiv H_2 \lambda_2^2, q_2(t) \equiv H_2 \lambda_2^3$, and $q_3(t) \equiv H_2 \lambda_2^4$ under the specifications of Property 3.1. Thus, Property 3.1 holds for arbitrary $R$, and we can set $R = D_2^* + D_\xi$, so that Theorem 3.3 can be applied to MPC$_k$ with terminal cost $F_{t+k}(\cdot; \tilde{\xi}_{t:t+k}) = I(\cdot; \tilde{x}(\tilde{\xi}_{t:t+k}))$, which leads to the following dynamic regret bound:

**Theorem 4.2.** In the unconstrained LTV setting (10), under Assumption 4.2, the prediction horizon $k \geq O(1)$ and the prediction errors satisfy $\sum_{\tau=0}^{k} \lambda_2^2 \rho_{t,\tau} \leq O(1)$ for all $t$, the dynamic regret of MPC$_k$ (Algorithm 7) with terminal cost $F_{t+k}(\cdot; \tilde{\xi}_{t:t+k}) = I(\cdot; \tilde{x}(\tilde{\xi}_{t:t+k}))$ is bounded by

$$
\text{cost}(\text{MPC}_k) - \text{cost}(\text{OPT}) \leq O \left( \sqrt{T \cdot \sum_{\tau=0}^{k-1} \lambda_2^2 P(\tau) + \lambda_2^2 T^2 + \sum_{\tau=0}^{k-1} \lambda_2^2 P(\tau)} \right).
$$

The exact constants and a complete proof of Theorem 4.2 can be found in Appendix C. Compared with Theorem 4.1 in Theorem 4.2, additionally requires the total prediction errors $\sum_{\tau=0}^{k} \lambda_2^2 \rho_{t,\tau}$ to be less than or equal to some constant. This is actually expected, and emphasizes the critical difference between the prediction errors on dynamical matrices ($A_t, B_t$) and the prediction errors on $w_t$, since an online controller cannot even stabilize the system when the predictions on ($A_t, B_t$) can be arbitrarily bad. It is worth noting that Assumption 4.2 requires the uniform controllability to hold for the unknown ground-truth LTV dynamics and any predicted dynamics. The goal is to ensure the perturbation bounds for KKT matrix inverse hold in Lemma G.2. Intuitively, this assumption is necessary because otherwise the solution of MPC (by solving FTOPC induced by the predicted dynamics) can be unbounded. We provided two examples (Example G.4 and G.5) that satisfy Assumption 4.2 while the true dynamics are unknown.

5 General Dynamical Systems

We now move beyond unconstrained linear systems to constrained nonlinear systems given by the general online control problem (1) in Section 2. All necessary assumptions are summarized in Assumption 3.1 in Appendix H. Perhaps surprisingly, decaying perturbation bounds can hold even in this case. In particular, using Theorem 4.5 in [50], we can show that there exists a small constant $R$ such that, for some constants $H_3 \geq 1$ and $\lambda_3 \in (0, 1)$, perturbation bounds (3) and (4) hold for $q_1(t) = 0, q_2(t) = H_3 \lambda_3^2$, and $q_3(t) = H_3 \lambda_3^3$. Thus, Property 3.1 holds (see Appendix H for formal statements) and we can apply Theorem 3.3 to obtain the following dynamic regret bound:

**Theorem 5.1.** In the general system (1), under Assumption 3.1 in Appendix H. Property 3.1 holds for some positive constant $R$ and that satisfies $\sum_{\tau=0}^{k-1} \lambda_3^2 \rho_{t,\tau} \leq \sqrt{T \cdot \sum_{\tau=0}^{k-1} \lambda_3^2 P(\tau) + \lambda_3^2 T^2 + \sum_{\tau=0}^{k-1} \lambda_3^2 P(\tau)}$. Then, the dynamic regret of MPC$_k$ is upper bounded by

$$
\text{cost}(\text{MPC}_k) - \text{cost}(\text{OPT}) \leq O \left( \sqrt{T \cdot \sum_{\tau=0}^{k-1} \lambda_3^2 P(\tau) + \lambda_3^2 T^2} \right).
$$

A complete proof of Theorem 5.1 can be found in Appendix H. An assumption in Theorem 5.1 that is difficult to satisfy in general is that the reference terminal states $\tilde{y}(\tilde{\xi}_{t:t+k})$ of MPC$_k$ must be close enough to the offline optimal state $x^*_{t+k}$, i.e., $\tilde{y}(\tilde{\xi}_{t:t+k}) \in B(x^*_{t+k}, R)$, while the offline optimal state $x^*_{t+k}$ is generally unknown. This can be achieved in some special cases, for example, when we

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1When we say $z \geq O(1)$, we mean there exists $c = O(1)$ such that $z \geq c$ holds.
know $\|\psi_t^*\|$ is sufficiently small. In this case, one can first solve FTOCP $\psi_0^T(x_0, 0; F_T)$ and use it as a reference to set the terminal states of MPC$_k$. This intuition is formally shown in Appendix H. Another limitation is that Theorem 5.1 is only a bound on the cost of MPC, not its feasibility. There are many ways to guarantee recursive feasibility of MPC [53], which we leave as future work. We also discuss how to verify Assumption [H.1] in two simple examples that arise from a simple inventory dynamics in Appendix I. The first positive example shows that Assumption [H.1] is not vacuous, and the second negative example shows exponentially decaying perturbation bounds may not hold when Assumption [H.1] is not satisfied.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper’s contributions and scope? [Yes] See Contributions in Section 1 and the references within.
   (b) Did you describe the limitations of your work? [Yes] See the last paragraph of Section 5.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A] The goal of our work is to advance the theoretical understanding for predictive online controllers. We did not see any potential negative societal impacts of our work.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes] We did not find any aspect of our work that may lead to ethics violations.

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See Assumption 4.1 and 4.2 in the main body, Appendix B and Assumption F.1, G.1, and H.1 in the appendix.
   (b) Did you include complete proofs of all theoretical results? [Yes] See Appendices for complete proofs of our results. We also add references to the relevant sections of the Appendix when we introduced each theorem or lemma.

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [N/A]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you’re using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]

5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]
A Notation Summary

In this paper, we use \( \alpha_{t_1:t_2} \) \((t_2 \geq t_1)\) to denote a sequence of vectors \((\alpha_{t_1}, \alpha_{t_1+1}, \ldots, \alpha_{t_2})\). For ease of reference, we summarize in the following table all the notations used in the paper.

| Notation | Meaning |
|----------|---------|
| \( \xi_t \) | The uncertainty parameter of the system, used to parameterize costs, dynamics, and constraints. |
| \( \xi^*_t \) | The ground-truth parameter of the system, unknown to the controller. |
| \( \xi_{\tau|t} \) | The prediction of \( \xi^*_t \) revealed to the controller at time step \( \tau \geq t \). |
| \( \Xi_t \) | The space of uncertainty parameters, \( \xi^*_t \) and \( \xi_{\tau|t}, \tau \leq t \) are in \( \Xi_t \). We assume the diameter of \( \Xi_t \) is less than or equal to 1 without the loss of generality, i.e., \( ||\xi^*_t - \xi_{\tau|t}|| \leq 1 \) for all \( \xi^*_t, \xi_{\tau|t} \in \Xi_t \). |
| \( k \) | The prediction horizon. At time \( t \), the controller observes predictions \( \xi_{t:t'|t}, \) where \( t' := \min\{t + k, T\} \). |
| \( \rho_{t,\tau} \) | The error of predicting the system parameter after \( \tau \) steps at time \( t \), i.e., \( \rho_{t,\tau} = ||\xi^*_t - \xi_{t+\tau|t}|| \). We adopt the convention that \( \rho_{t,\tau} := 0 \) if \( t + \tau > T \). |
| \( P(\tau) \) | The total error of predicting the system parameter after \( \tau \) steps (the power of \( \tau \)-step-away predictions), i.e., \( P(\tau) := \sum_{\tau=0}^{T-\tau} \rho_{t,\tau}^2 \). |
| \( f_t(x_t, u_t; \xi_t) \) | The stage cost of FTOCP at time step \( t \), parameterized by \( \xi_t \in \Xi_t \). The true stage cost is \( f_t(x_t, u_t; \xi^*_t) \). |
| \( g_t(x_t, u_t; \xi_t) \) | The dynamical function at time step \( t \), parameterized by \( \xi_t \in \Xi_t \). The true dynamics is \( x_{t+1} = g_t(x_t, u_t; \xi^*_t) \). |
| \( s_t(x_t, u_t; \xi_t) \) | The constraint function at time step \( t \), parameterized by \( \xi_t \in \Xi_t \). The true constraint is \( s_t(x_t, u_t; \xi^*_t) \leq 0 \). |
| \( F_T \) and \( \{F_{t+k}\}_{t=0}^{T-k-1} \) | \( F_T \) is the true terminal cost function defined by the original online control problem \( \mathbb{P} \), while \( F_{t+k} \) for \( t < T - k \) is the terminal cost function used by MPC\(k\) at time \( t \). |
| \( F_{t+k} \) \((t \geq 1, \xi_t, \xi_{t+1:k-1}, \xi_{t+k}; F) \) | The FTOCP defined on the time interval \([t_1, t_2]\), where \( z \) is the initial state at time \( t_1 \), and \( F \) is some terminal cost function at time \( t_2 \). \( \xi_{t_1:t_2-1} \) are the parameters for the cost, dynamics, and constraints at time \([t_1, t_2-1]\), while \( \xi_t \) is the parameter for the terminal cost \( F \). |
| \( \psi_{t_1}^{t_2}(z, \xi_{t_1:t_2-1}; \xi_{t_2}; F) \) | An optimal solution to the FTOCP \( \psi_{t_1}^{t_2}(z, \xi_{t_1:t_2-1}; \xi_{t_2}; F) \). The entries are indexed by \( y_{t_1:t_2} \) (for states) and \( u_{t_1:t_2-1} \) (for actions). |
| \( \psi_{t_1}^{t_2}(z, \xi_{t_1:t_2}; F) \) | The shorthand notation of \( \psi_{t_1}^{t_2}(z, \xi_{t_1:t_2-1}; \xi_{t_2}; F) \). |

B Assumptions Overview

In this section, we give a more detailed overview of the assumptions that the online control problem \( \mathbb{P} \) should satisfy in general so that our pipeline in Section 5.2 works. Specific assumptions in each specific setting will be presented separately in Assumptions \( \mathbf{F.1} \), \( \mathbf{G.1} \), and \( \mathbf{H.1} \).

Cost functions. In general, we require the stage cost functions \( f_t \) and the terminal cost \( F_T \) to be well-conditioned, which includes non-negativity, strong convexity, smoothness (Lipschitz continuous gradient), and twice continuous differentiability. Note that these assumptions are equivalent to bounded Hessian (\( \mu I \preceq \nabla^2 f_t \preceq \Lambda I \)) and non-negative minimizer of the cost functions. Specifically, for quadratic costs \( \nabla^2 f_t \) are constant, and the assumptions are further equivalent to bounded spectra of the cost matrices.

Dynamical systems. A basic requirement of the dynamical function \( g_t \) is Lipschitzness in \( u_t \), i.e.,

\[
\|g_t(x_t, u_t; \xi^*_t) - g_t(x_t, u'_t; \xi^*_t)\| \leq L_g \|u_t - u'_t\|.
\]

We point out that only Lipschitzness in control action \( u_t \) is needed for the Pipeline Theorem to hold, which guarantees that an error on a control action \( u_t \) has a bounded impact on the next state \( x_{t+1} \).

A more non-trivial assumption on dynamics is that the dynamical system should be (uniformly) controllable. Intuitively, this means the online controller should be able to steer the system to some target state in a finite number of time steps with some bounded control actions.
**Definition B.1** (uniform controllability). Consider a general dynamics \( x_{t+1} = g_t(x_t, u_t; \xi_t) \). For any time steps \( t_2 \geq t_1 \) and fixed \((x_t, u_t)\), define \( A_t := \nabla_{x_t}^T g_t(x_t, u_t; \xi_t) \) and \( B_t := \nabla_{u_t}^T g_t(x_t, u_t; \xi_t) \), and we further define transition matrix \( \Phi(t_2, t_1) \in \mathbb{R}^{n \times n} \) at \((x_t, u_t)\) as

\[
\Phi(t_2, t_1) := \begin{cases} A_{t_2-1}A_{t_2-2} \cdots A_{t_1} & \text{if } t_2 > t_1, \\
I & \text{otherwise}. \end{cases}
\]

For any time \( t \) and time interval \( p \geq 0 \), define controllability matrix \( M(t, p; x_{t:t+p}, u_{t:t+p}) \in \mathbb{R}^{n \times (mp)} \) as

\[
M(t, p; x_{t:t+p}, u_{t:t+p}) := [\Phi(t + p, t + 1)B_t, \Phi(t + p, t + 2)B_{t+1}, \ldots, \Phi(t + p, t + p)B_{t+p}].
\]

We say the system is **controllable** if there exists a positive integer \( d \), such that the controllability matrix \( M(t, d; x_t, u_t) \) is of full row rank for any \( t \) and any \((x_t, u_t)\). The smallest such constant \( d \) is called the **controllability index** of the system. Further, we say the system is \( \sigma \)-uniformly controllable if exists a positive constant \( \sigma \) such that \( \sigma_{\min}(M(t, d)) \geq \sigma \) holds for all \( t = 0, \ldots, T - d \).

The definition has a clear control-theoretic interpretation for linear dynamics (where \( A_t \) and \( B_t \) are independent of \((x_t, u_t)\)), but might seem trickier for non-linear dynamics (where \( A_t \) and \( B_t \) are functions of \((x_t, u_t)\)). For the latter case, uniform controllability may be assumed for the offline optimal trajectory only, or for state-action pairs in a small neighborhood around it.

**Constraints.** Recursive feasibility is a well-known challenge for the design of online controllers in constrained systems [53]: at some time \( t \), the controller may encounter an absence of feasible trajectories to continue from the current state \( x_t \). Many solutions have been proposed for different controllers in a variety of systems. Since the purpose of this work is to establish dynamic regret guarantees for an online controller, and for the purpose of this paper, we would expect that there is a solution, potentially via a combination of proper controller design (e.g., setting the terminal cost/constraint of MPC) and some additional assumptions on the system (e.g., the SSOSC, strong second-order sufficient conditions, and LICQ, linear independent constraint qualification, which will be introduced in Section 5), so that we could focus on the sub-optimality of the online controller against the offline optimal trajectory.

We also need to point out that, although the additional assumptions on system that involve constraints might seem tricky, sometimes they are exactly the implications of previous assumptions on costs and dynamics that is actually needed in the proof. For example, Lemma 12 in [37] shows that Lipschitzness of dynamics and uniform controllability together imply uniform LICQ property of the system. For the clarity of exposition, these implications might be directly assumed in place of the low-level ones.

**Parameter \( \xi_t \).** In general, we require that all predicted quantities, which might include cost functions, dynamical functions, and constraints, should be **Lipschitz** in \( \xi_t \), so that these quantities get closer to their ground truth value in a linearly-bounded way as the prediction error on the parameter \( \xi_t \) decreases. For a specific example of parameterized linear dynamics \( x_{t+1} = A_t(\xi_t)x_t + B_t(\xi_t)u_t + w_t(\xi_t) \), the requirement is realized by assuming Lipschitzness of \( A_t(\cdot), B_t(\cdot), w_t(\cdot) \) in \( \xi_t \).

**Offline optimal trajectory.** We require the offline optimal trajectory \( \text{OPT} \) to be **stable**; i.e., there exists a constant \( D_x \) such that \( \|x_t^*\| \leq D_x \) for any state \( x_t^* \) visited by \( \text{OPT} \). While this can be shown under some assumptions in unconstrained LTV systems (see [55]), we introduce this assumption to simplify and unify the presentation for more complex systems.

### C Proof of Lemma 3.1

We have already shown [5] holds for all time step \( t < T - k \) in the main body. For \( t \geq T - k \), we see that

\[
e_t = \|\psi^T_t(x_t, \xi_{t:T}; F_T) - \psi^T_t(x_t, x^*_{t:T}; F_T)\| \leq \sum_{\tau=0}^{k} (\|x_t\| \cdot q_1(\tau) + q_2(\tau))p_{t,\tau} \leq \sum_{\tau=0}^{k} \left( \left( \frac{R}{C^3} + D_x^* \right) \cdot q_1(\tau) + q_2(\tau) \right)p_{t,\tau},
\]

where \( C, R \) are the cost functions of the MPC. For the latter case, uniform controllability may be assumed for the offline optimal trajectory only, or for state-action pairs in a small neighborhood around it.
where we used the definition of per-step error $e_t$ in (11a); we used the perturbation bound (4) specified by Property 3.1 in [11B]; we used the assumption $x_t \in B \left( x_t^*, \frac{R}{C_5} \right)$, $\|x_t^*\| \leq D_{x*}$, and the convention $\rho_{t,\tau} := 0$ if $t + \tau > T$ in (11c). Thus $e_t$ also satisfies (5) for $t \geq T - k$.

D Proof of Lemma 3.2

To simplify the notation, we will use $\psi^T_t(z)$ as a shorthand notation of $\psi^T_t(z, e_{t:T}; F_T)$ in the proof of Lemma 3.2, since the proof only relies on the perturbation bound (4).

Note that for any time step $t + 1$, by Lipschitzness of the dynamics we have

$$
\left\| x_{t+1} - \psi^T_t(x_t) \right\|_{y_{t+1}} = \left\| g_t(x_t, u_t, w_t) - g_t(x_t, \psi^T_t(x_t) v_t, w_t) \right\| \\
\leq L_g \left\| u_t - \psi^T_t(x_t) v_t \right\| \\
= L_g e_t.
$$

(12)

Therefore, we can show the statement that $x_t \in B \left( x_t^*, \frac{R}{C_5} \right)$ holds if $e_t \leq R/(C_3^2 L_g)$, $\forall t < T$ by induction. Note that this statement clearly holds for $t = 0$ since $x_0^* = x_0$. Suppose it holds for $0, 1, \ldots, t - 1$. Then, we see that

$$
\left\| x_t - x_t^* \right\| = \left\| x_t - \psi^T_0(x_0) \right\| \\
\leq \left\| x_t - \psi^T_{t-1}(x_{t-1}) y_{t-1} \right\| + \sum_{i=1}^{t-1} \left\| \psi^T_{i-1}(x_{t-i}) y_{t-i} - \psi^T_{i-1}(x_{t-i}) y_{t-i} \right\| \\
\leq \left\| x_t - \psi^T_{t-1}(x_{t-1}) y_{t-1} \right\| + \sum_{i=1}^{t-1} q_3(i) \left\| x_{t-i} - \psi^T_{i-1}(x_{t-i}) y_{t-i} \right\| \\
\leq \sum_{i=0}^{t-1} q_3(i) \left\| x_{t-i} - \psi^T_{i-1}(x_{t-i}) y_{t-i} \right\| \\
= L_g \sum_{i=0}^{t-1} q_3(i) e_{t-i-1},
$$

(13a)

where in (13a), we apply the perturbation bound (4) specified by Property 3.1. To see why it can be applied, note that for $i \in [1, t - 1]$, $x_{t-i} \in C_3$ by the induction assumption, thus we have $\psi^T_{i-1}(x_{t-i}) y_{t-i} \in B \left( x_{t-i}^*, R \right)$ because $q_3(1) \leq \sum_{\tau=0}^{\infty} q_3(\tau) \leq C_3$. Therefore, we can apply the perturbation bound (4) specified by Property 3.1 to compare the optimization solution vectors $\psi^T_{t-i}(x_{t-i})$ and $\psi^T_{i-1}(\psi^T_{i-1}(x_{t-i}) y_{t-i})$, and by the principle of optimality, we see that

$$
\psi^T_{t-i}(\psi^T_{i-1}(x_{t-i}) y_{t-i}) = \psi^T_{t-i}(x_{t-i}) y_{t-i}.
$$

We also used $q_3(0) \geq 1$ in (13b) and (12) in (13c). Recall that we assume $e_{t-i} \leq \frac{R}{C_3^2 L_g}$. Substituting this into (13) gives that

$$
\left\| x_t - x_t^* \right\| \leq L_g \cdot \frac{R}{C_3^2 L_g} \sum_{i=0}^{t-1} q_3(i) \leq \frac{R}{C_3}.
$$

Hence we have shown $x_t \in B \left( x_t^*, \frac{R}{C_5} \right)$ holds if $e_t \leq R/(C_3^2 L_g)$, $\forall t < T$ by induction. An implication of this result is that $x_t \in B \left( x_t^*, \frac{R}{C_5} \right)$ holds for all $t \leq T$ if $e_t \leq R/(C_3^2 L_g)$ holds for all $t < T$.

Similar with (13), we see the following inequality holds for all $t \leq T$ if $e_t \leq R/(C_3^2 L_g), \forall t < T$:

$$
\left\| u_t - u_t^* \right\| = \left\| u_t - \psi^T_0(x_0) v_t \right\|
$$
where the second inequality holds for the same reason as (13a).

By (13), we see that
\[ \eta > 0 \]
inputs, by Lemma F.2 in \[35\], we see that the following inequality holds for arbitrary \(f_t\) and \(g_t\):
\[ \frac{\eta}{C} ≤ \frac{\sum_{i=0}^{t-1} q_3(i) e_{i-1}}{e_{t-1}} \]

Similarly, by (14), we see that
\[ \frac{\eta}{C} ≤ \frac{\sum_{i=0}^{t-1} q_3(i) e_{i-1}}{e_{t-1}} \]

where we use the Cauchy-Schwarz inequality in (15a), and \(\sum_{i=0}^{t-1} q_3(i) ≤ C_3\) in (15b).

Summing (15) and (16) over time steps \(t\) gives that
\[ \sum_{t=0}^{T} \| x_t - x_t^* \|^2 + \sum_{t=0}^{T-1} \| u_t - u_t^* \|^2 \]
\[ ≤ C_3 L_g^2 \sum_{t=0}^{T} \left( \sum_{i=0}^{t-1} q_3(i) e_{i-1}^2 \right) + (1 + C_3 L_g^2) \cdot \sum_{t=0}^{T-1} \left( e_t^2 + \sum_{i=0}^{t-1} q_3(i) e_{i-1}^2 \right) \]
\[ ≤ (1 + 2C_3 L_g^2) \cdot (1 + C_3) \cdot \sum_{t=0}^{T-1} e_t^2, \]

where we rearrange the terms and use \(\sum_{j=0}^{\infty} q_3(j) ≤ C_3\) in the last inequality.

Since the cost function \(f_t(:, \ldots, x_t^*)\) and \(F_T(:, x_T^*)\) are nonnegative, convex, and \(\ell\)-smooth in their inputs, by Lemma F.2 in \[35\], we see that the following inequality holds for arbitrary \(\eta > 0\):

\[ \text{cost}(\text{ALG}) - \text{cost}(\text{OPT}) ≤ \left( \sum_{t=0}^{T-1} f_t(x_t, u_t; x_t^*) + F_T(x_T; x_T^*) \right) - \left( \sum_{t=0}^{T-1} f_t(x_t^*, u_t^*; x_t^*) + F_T(x_T^*; x_T^*) \right) \]
\[ ≤ \eta \left( \sum_{t=0}^{T-1} f_t(x_t^*, u_t^*; \xi_t^*) + F_T(x_T^*; \xi_T^*) \right) \]
We first use induction to show that the following two conditions holds for all time steps:

\[\sum_{t=1}^{T} \|x_t - x_t^*\|^2 + \sum_{t=0}^{T-1} \|u_t - u_t^*\|^2 \leq \frac{\ell}{2} \left( 1 + \frac{1}{\eta} \right) \left( \sum_{t=1}^{T} \|x_t - x_t^*\|^2 + \sum_{t=0}^{T-1} \|u_t - u_t^*\|^2 \right)\]  

\[\leq \eta \cdot \text{cost}(\text{OPT}) + \left( 1 + \frac{1}{\eta} \right) \cdot \frac{\ell}{2} \cdot (1 + 2C_3L_g^2) \cdot (1 + C_3) \cdot \sum_{t=0}^{T-1} e_t^2\]  

\[= \eta \cdot \text{cost}(\text{OPT}) + \frac{1}{\eta} \cdot \frac{\ell}{2} \cdot (1 + 2C_3L_g^2) \cdot (1 + C_3) \cdot \sum_{t=0}^{T-1} e_t^2\]  

\[+ \frac{\ell}{2} \cdot (1 + 2C_3L_g^2) \cdot (1 + C_3) \cdot \sum_{t=0}^{T-1} e_t^2,\]  

where we apply Lemma F.2 in [35] in (18a), and we use (17) in (18b). Setting the tunable weight \(\eta\) in (18c) to be

\[\eta = \left( \frac{\frac{\ell}{2} \cdot (1 + 2C_3L_g^2) \cdot (1 + C_3) \cdot \sum_{t=0}^{T-1} e_t^2}{\text{cost}(\text{OPT})} \right)^{\frac{1}{2}}\]

gives that

\[\text{cost}(\text{ALG}) - \text{cost}(\text{OPT}) \leq \sqrt{\left( \frac{\frac{\ell}{2} \cdot (1 + 2C_3L_g^2) \cdot (1 + C_3) \cdot \sum_{t=0}^{T-1} e_t^2}{\text{cost}(\text{OPT})} \right) + \frac{\ell}{2} \cdot (1 + 2C_3L_g^2) \cdot (1 + C_3) \cdot \sum_{t=0}^{T-1} e_t^2}.\]  

This finishes the proof of Lemma 3.2.

**E Proof of Theorem 3.3**

We first use induction to show that the following two conditions holds for all time steps \(t < T\):

\[x_t \in B\left( x_t^*, \frac{R}{C_3} \right),\]  

\[e_t \leq \sum_{\tau=0}^{k} \left( \frac{R}{C_3} + D_{x^*} \right) \cdot q_1(\tau) + q_2(\tau)\rho_{t,\tau} + 2R \left( \frac{R}{C_3} + D_{x^*} \right) \cdot q_1(k) + q_2(k).\]  

At time step 0, (20a) holds because \(x_0 = x_0^*\), and (20b) holds by Lemma 3.1 and the assumption on the terminal cost \(F_k\) of MPC\(_k\).

Suppose (20a) and (20b) hold for all time steps \(\tau < t\). For time step \(t\), by the assumption on the prediction errors \(\rho_{t,\tau}\) and prediction horizon \(k\) in Theorem 3.3, we know that \(e_{2\tau} \leq \frac{R}{C_3}e_{2\tau}\) holds for all \(\tau < t\) because (20b) holds for all \(\tau < t\). Thus, we know that (20a) holds for time step \(t\) by Lemma 3.2. Then, since (20a) holds for time step \(t\), and the terminal cost \(F_{t+k}\) of MPC\(_k\) is set to be the indicator function of some state \(\bar{y}(\xi_{t+k})\) that satisfies \(\bar{y}(\xi_{t+k}) \in B(x_t^*, R)\) if \(t < T - k\), we know (20b) also holds for time step \(t\) by Lemma 3.1. This finishes the induction proof of (20).

To simplify the notation, let \(R_0 := \frac{R}{C_3} + D_{x^*}\). Note that (20b) implies that

\[e_t^2 \leq \left( \sum_{\tau=0}^{k} (R_0 \cdot q_1(\tau) + q_2(\tau)) + 2R(R_0 + 1) \right) \cdot \left( \sum_{\tau=0}^{k} (R_0 \cdot q_1(\tau) + q_2(\tau)) \rho_{t,\tau}^2 + 2R (R_0 \cdot q_1(k)^2 + q_2(k)^2) \right)\]  

\[\leq (R_0C_1 + C_2 + 2R(R_0 + 1))\]
We assume the LTV system 

\[ (R_0 \cdot q_1(\tau) + q_2(\tau)) \rho_{t,\tau} + (2R + 1) \left( R_0 \cdot q_1(k)^2 + q_2(k)^2 \right) \]  

(21b)

where we use the Cauchy-Schwarz inequality in (21a); we use the bounds \( \sum_{\tau=0}^{k} q_1(\tau) \leq C_1 \), \( \sum_{\tau=0}^{k} q_2(\tau) \leq C_2 \), and \( \rho_{t,\tau} \leq 1 \) in (21b).

Since (20) and (21) holds for all time steps \( t < T \), we can apply Lemma 3.2 to obtain that 

\[ \text{cost}(\text{MPC}_k) - \text{cost}(\text{OPT}) \leq \sqrt{\text{cost}(\text{OPT})} \cdot E_0 + E_0, \]

where 

\[ E_0 := (R_0C_1 + C_2 + 2R(R_0 + 1)) \]

\[ \cdot \left( \sum_{\tau=0}^{k} (R_0 \cdot q_1(\tau) + q_2(\tau)) P(\tau) + (2R + 1) \left( R_0 \cdot q_1(k)^2 + q_2(k)^2 \right) T \right). \]

This finishes the proof of Theorem 3.3

F Assumptions and Proofs of Section 4.1

The formal definition of the controllability index \( d \) and \( \sigma \)-uniform controllable are given in [35]. For completeness, we restate them for LTV dynamics in Assumption F.1 below.

**Assumption F.1.** For time steps \( t_2 \geq t_1 \), we define the transition matrix \( \Phi(t_2, t_1) \in \mathbb{R}^{n \times n} \) as

\[ \Phi(t_2, t_1) := \begin{cases} A_{t_2-1}A_{t_2-2} \cdots A_{t_1} & \text{if } t_2 > t_1 \\ I & \text{otherwise}. \end{cases} \]

For any positive integer \( p \), we define the controllability matrix \( M(t, p) \in \mathbb{R}^{n \times (mp)} \) as

\[ M(t, p) := [\Phi(t + p, t + t + 1)B_t, \Phi(t + p, t + t + 2)B_{t+1}, \ldots, \Phi(t + p, t + p)B_{t+p}]. \]

We assume the LTV system \( \{A_t, B_t\} \) is \( \sigma \)-uniform controllable with controllability index \( d \), i.e., \( d \) is the smallest positive integer such that \( \sigma_{\min}(M(t, d)) > 0 \) holds for all \( t \in [0, T - d] \), and \( \sigma_{\min}(M(t, d)) \geq \sigma \) holds for all \( t \in [0, T - d] \).

As a remark, the Assumption F.1 is a special case of Definition B.1 in unconstrained LTV systems. [35] has established a perturbation bound for the LTV system in (9) which implies the our requirements in Property 3.1. Thus we can use Theorem 3.3 to show Theorem 4.1

**Proof of Theorem 4.1** By Theorem 3.3 in [35], we know Property 3.1 holds under Assumption 4.1 for arbitrary \( R \) and \( q_1(t) = 0, q_2(t) = H_1 \lambda_1^d \), where \( H_1 = \max \{ D_x^{\ast}, 2L_w \} \), \( \lambda_1 > 0 \), and \( \lambda_1 = \lambda_1(\mu, \ell, d, \sigma, a, b, b', L_w) \) is the decay rate. Here, \( H_1 \) corresponds to \( C \) and \( \lambda_1 \) corresponds to \( \lambda \) in Theorem 3.3 in [35].

By setting \( R := \max \{ D_x^{\ast}, 2L_w \} \), we guarantee that the terminal state 0 of MPC\(_k\) is always in the closed ball \( B(x^\ast_{t+k}, R) \), and the condition

\[ \sum_{\tau=0}^{k} \left( \frac{R}{C_3 + D_x^{\ast}} \cdot q_1(\tau) + q_2(\tau) \right) \rho_{t,\tau} + 2R \left( \frac{R}{C_3 + D_x^{\ast}} \cdot q_1(k) + q_2(k) \right) \leq \frac{R}{C_3 + D_x^{\ast}} \]

holds once \( k \geq \ln \left( \frac{4H_1^2L_w}{(1-\lambda_1)^2} \right) / \ln(1/\lambda_1) \) because \( \rho_{t,\tau} \leq 1 \). Therefore, we can apply Theorem 3.3 to finish the proof of Theorem 4.1

G Assumptions and Proofs of Section 4.2

In this section, we give the detailed assumptions and proofs of the results in Section 4.2. Before we present the assumption on the uncertain LTV systems in (10), we first define several quantities that we will use heavily in the rest of this section:

\[ \text{cost}(\text{MPC}_k) - \text{cost}(\text{OPT}) \leq \sqrt{\text{cost}(\text{OPT})} \cdot E_0 + E_0, \]
For time steps $t_1 \leq t_2$ and $\xi_{t_1:t_2} \in \Xi_{t_1:t_2}$, define
\[
N_{t_1}^{t_2}(\xi_{t_1:t_2}) := \begin{bmatrix}
I & -A_{t_1}(\xi_{t_1}) & -B_{t_1}(\xi_{t_1}) & I \\
& \ddots & \ddots & \ddots \\
& & -A_{t_2}(\xi_{t_2}) & -B_{t_2}(\xi_{t_2}) & I \\
\end{bmatrix}.
\] (22)

This matrix is closely related to the stability of the LTV system in the time interval $[t_1, t_2 + 1]$. To see this, note that $N_{t_1}^{t_2}(\xi_{t_1:t_2})$ always has full row rank, i.e., given any disturbance vector $w = (x_{t_1}, w_{t_1}, w_{t_1+1}, \ldots, w_{t_2})^T$, one can always find a feasible sub-trajectory $z = (x_{t_1}, u_{t_1}, x_{t_1+1}, \ldots, u_{t_2}, x_{t_2})^T$ that satisfies $N_{t_1}^{t_2}(\xi_{t_1:t_2})z = w$. If for any vector $w$, there exists a feasible sub-trajectory $z$ such that $\|z\| \leq (1/\sigma) \cdot \|w\|$ for some positive constant $\sigma$, then the smallest singular value of $N_{t_1}^{t_2}(\xi_{t_1:t_2})$ is lower bounded by $\sigma$.

Similar with (22), we define matrix
\[
\hat{N}_{t_1}^{t_2}(\xi_{t_1:t_2}) := \begin{bmatrix}
I & -A_{t_1}(\xi_{t_1}) & -B_{t_1}(\xi_{t_1}) & I \\
& \ddots & \ddots & \ddots \\
& & -A_{t_2}(\xi_{t_2}) & -B_{t_2}(\xi_{t_2}) & I \\
\end{bmatrix}
\] (23)

for any time steps $t_1 \leq t_2$ and $\xi_{t_1:t_2} \in \Xi_{t_1:t_2}$, which removes the last column of (22). The matrix $\hat{N}_{t_1}^{t_2}(\xi_{t_1:t_2})$ is closely related to the controllability of the LTV system in the time interval $[t_1, t_2 + 1]$. To see this, given any disturbance vector $\tilde{w} = (x_{t_1}, u_{t_1}, w_{t_1}, \ldots, w_{t_2})^T$ whose first/last entry depends on the initial/terminal state, a feasible sub-trajectory $\tilde{z} = (x_{t_1}, u_{t_1}, x_{t_1+1}, \ldots, u_{t_2})^T$ must satisfy that $\hat{N}_{t_1}^{t_2}(\xi_{t_1:t_2})\tilde{z} = \tilde{w}$. Different from $N_{t_1}^{t_2}(\xi_{t_1:t_2})$, $\hat{N}_{t_1}^{t_2}(\xi_{t_1:t_2})$ is not guaranteed to have full row rank. If for any vector $\tilde{w}$, there exists a feasible sub-trajectory $\tilde{z}$ such that $\|\tilde{z}\| \leq (1/\sigma) \cdot \|\tilde{w}\|$ for some positive constant $\sigma$, then the smallest singular value of $\hat{N}_{t_1}^{t_2}(\xi_{t_1:t_2})$ is lower bounded by $\sigma$.

We make the following assumption on the smallest singular values of matrices $N_{t_1}^{t_2}(\xi_{t_1:t_2})$ and $\hat{N}_{t_1}^{t_2}(\xi_{t_1:t_2})$ so that the LTV system possesses uniform stability and controllability properties under any uncertainty parameters:

**Assumption G.1.** There exists some universal constant $\sigma > 0$ such that $\sigma_{\text{min}} \left( N_{t}^{T-1}(\xi_{t:T-1}) \right) \geq \sigma$ for any $t < T$, and $\sigma_{\text{min}} \left( \hat{N}_{t}^{T+k}(\xi_{t:t+k}) \right) \geq \sigma$ for any $t < T - k$.

While Assumption G.1 may seem more restricted than the uniform controllability defined in Definition B.1, it can actually be derived from Definition B.1 by Lemma 12 in [37].

In order to formulate (10) as a quadratic programming problem with equality constraints, we also need to define the matrix for the cost functions:
\[
M^T_T(\xi_{t:T}) := \text{diag} \left( Q(t, \xi_{t}), R(t, \xi_{t}), Q_{t+1}(\xi_{t+1}), \ldots, R_{T-1}(\xi_{T-1}), P_T(\xi_{T}) \right), \forall t < T, \\
\hat{M}^{t+k}_{t}(\xi_{t:t+k}) := \text{diag} \left( Q(t, \xi_{t}), R(t, \xi_{t}), Q_{t+1}(\xi_{t+1}), \ldots, R_{T+k-1}(\xi_{T+k-1}) \right), \forall t < T - k. \tag{24}
\]

To write down the KKT conditions for the equality constrained quadratic programming problem, we also need to define
\[
H^T_t(\xi_{t:T}) := \begin{bmatrix}
M^T_T(\xi_{t:T}) & N_{t}^{T-1}(\xi_{t:T-1}) \\
N_{t}^{T-1}(\xi_{t:T-1}) & 0 \\
\end{bmatrix}, \\
\hat{H}^{t+k}_t(\xi_{t:t+k}) := \begin{bmatrix}
\hat{M}^{t+k}_{t}(\xi_{t:t+k}) & \hat{N}_{t}^{t+k-1}(\xi_{t:t+k-1}) \\
\hat{N}_{t}^{t+k-1}(\xi_{t:t+k-1}) & 0 \\
\end{bmatrix}, \\
\hat{b}^T_t(z, \xi_{t:T}) := (Q_t(\xi_T)\rho_1(\xi_T), 0, \ldots, P(\xi_T)\rho_T(\xi_T), z, w_t(\xi_T), \ldots, w_{T-1}(\xi_{T-1}))^T, \\
\hat{b}^{t+k}_t(z, \xi_{t:t+k}) := (Q_t(\xi_T)\rho_1(\xi_T), 0, \ldots, 0, z, w_t(\xi_T), \ldots, w_{T+k-1}(\xi_{T+k-1}) - \xi_{t+k})^T, \\
\chi^T_t := (y_t, v_t, y_{t+1}, \ldots, v_{T-1}, y_T, \eta_t, \eta_{t+1}, \ldots, \eta_T)^T, \\
\chi^{t+k}_t := (y_t, v_t, y_{t+1}, \ldots, v_{T+k-1}, \eta_t, \eta_{t+1}, \ldots, \eta_{T+k})^T.
\]
According to the KKT condition, the optimal primal-dual solution to $\iota_t^T(z, \xi_{t:T}; F_T)$ ($t < T$) is the unique solution $\chi_t^T$ to the linear equation $H_t^T(\xi_{t:T})\chi_t^T = b_t^T(z, \xi_{t:T})$. Similarly, the optimal primal-dual solution to $\iota_t^{t+k}(z, \xi_{t:t+k}; \Xi)$ ($t < T-k$) is the unique solution $\chi^{t+k}_t$ to the linear equation $H_t^{t+k}(\xi_{t:t+k})\chi^{t+k}_t = b_t^{t+k}(z, \xi_{t:t+k})$. We provide an illustrative example for $\chi^T_t$ with $(t, T) = (0, 3)$ below:

$$
\begin{bmatrix}
Q_0 & R_0 & I & -A_0^T & -B_0^T \\
I & -A_0 & I & -A_1^T & -B_1 \\
-A_0 - B_0 & -A_1 & -B_1 - A_2 - B_2 & I \\
\end{bmatrix}
= 
\begin{bmatrix}
y_0 \\
v_0 \\
y_1 \\
v_1 \\
v_2 \\
\eta_0 \\
\eta_1 \\
\eta_2 \\
\eta_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
Q_0 \bar{x}_0 \\
0 \\
z \\
\bar{x}_1 \\
0 \\
\bar{x}_2 \\
0 \\
\bar{x}_3 \\
\end{bmatrix}
$$

where we omit the parameters $\xi_{0:3}$ to simplify the notations. Rearranging the rows and columns of the matrix on the left hand side gives the equation:

$$
\begin{bmatrix}
Q_0 & R_0 & I & -A_0^T & -B_0^T \\
I & -A_0 & I & -A_1^T & -B_1 \\
-A_0 - B_0 & -A_1 & -B_1 - A_2 - B_2 & I \\
\end{bmatrix}
= 
\begin{bmatrix}
y_0 \\
v_0 \\
y_1 \\
v_1 \\
v_2 \\
\eta_0 \\
\eta_1 \\
\eta_2 \\
\eta_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
Q_0 \bar{x}_0 \\
0 \\
z \\
\bar{x}_1 \\
0 \\
\bar{x}_2 \\
0 \\
\bar{x}_3 \\
\end{bmatrix}
$$

Let $\Phi^T_t$ denote the permutation matrix that permute $(y_t, v_t, y_{t+1}, \ldots, v_{T-1}, y_T; \eta_t, \ldots, \eta_T)^T$ to $(y_t, v_t, y_{t+1}, v_{t+1}, y_{t+1}, \ldots, y_T, \eta_T)^T$. We use $\Upsilon^T_t(\xi_{t:T}) := (\Phi^T_t)^T H_t^T(\xi_{t:T}) (\Phi^T_t)^T$ to denote the rearrangement of $H_t^T(\xi_{t:T})$ as illustrated in the above equation, and use $\beta^T_t(z, \xi_{t:T}) := (\Phi^T_t)^T b_t^T(z, \xi_{t:T})$ to denote the corresponding rearrangement of $b_t^T(z, \xi_{t:T})$.

We also provide an illustrative example for $\chi^{t+k}_t$ with $(t, k) = (0, 3)$ below:

$$
\begin{bmatrix}
Q_0 & R_0 & I & -A_0^T & -B_0^T \\
I & -A_0 & I & -A_1^T & -B_1 \\
-A_0 - B_0 & -A_1 & -B_1 - A_2 - B_2 & I \\
\end{bmatrix}
= 
\begin{bmatrix}
y_0 \\
v_0 \\
v_1 \\
v_1 \\
v_2 \\
\eta_0 \\
\eta_1 \\
\eta_2 \\
\eta_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
Q_0 \bar{x}_0 \\
0 \\
z \\
\bar{x}_1 \\
0 \\
\bar{x}_2 \\
0 \\
\bar{x}_3 \\
\end{bmatrix}
$$

where we omit the parameters $\xi_{0:3}$ to simplify the notations. Rearranging the rows and columns of the matrix on the left hand side gives the equation:
Let \( \hat{\Phi}_t^{+k} \) denote the permutation matrix that permute \((y_t, v_t, y_{t+1}, \ldots, y_{t+k-1}; \eta_t, \ldots, \eta_{t+k})^T\) to \((y_t, v_t, \eta_t, y_{t+1}, \ldots, y_{t+k-1}, v_{t+k-1}, \eta_{t+k-1}, \eta_{t+k})^T.\) We use \( \hat{\Upsilon}_t^{+k}(\xi_{t:t+k}) := (\hat{\Phi}_t^{+k})H_t^{+k}(\xi_{t:t+k})(\hat{\Phi}_t^{+k})^T \) to denote the rearrangement of \( H_t^{+k}(\xi_{t:t+k}) \) as illustrated in the above equation, and use \( \beta_t^{+k}(z, \xi_{t:t+k}) := (\hat{\Phi}_t^{+k})b_t^{+k}(z, \xi_{t:t+k}) \) to denote the corresponding rearrangement of \( b_t^{+k}(z, \xi_{t:t+k}).\)

Before showing the main result about the per-step error, we first show a technical lemma about the singular values of a block matrix in Lemma G.1.

**Lemma G.1.** Consider a block matrix

\[
H = \begin{bmatrix} M & N^T \\ N & 0 \end{bmatrix}.
\]

Here \( M \in \mathbb{R}^{n_0 \times n_0} \) is a symmetric positive definite matrix that satisfies \( \underline{\sigma}_M I \preceq M \preceq \overline{\sigma}_M I \) with \( \underline{\sigma}_M > 0, \) and \( N \in \mathbb{R}^{n_1 \times n_0} \) with with \( n_1 \leq n_0 \) satisfies that \( \underline{\sigma}_N \leq \sigma(N) \leq \overline{\sigma}_N \) with \( \underline{\sigma}_N > 0. \) Then \( H \) satisfies that

\[
\min(\underline{\sigma}_M, 1) \cdot \sigma_N \cdot \sqrt{\frac{\overline{\sigma}_M}{2\underline{\sigma}_M \overline{\sigma}_M + \sigma_M (\sigma_N)^2}} \leq \sigma(H) \leq \sqrt{2}(\overline{\sigma}_M + \sigma_N).
\]

**Proof of Lemma G.1.** We first establish the lower bound on \( \sigma(H): \) Suppose the singular value decomposition of \( NM^{-\frac{1}{2}} \) is given by

\[
NM^{-\frac{1}{2}} = U \Sigma V^T,
\]

where \( U \in \mathbb{R}^{n_1 \times n_1} \) and \( V \in \mathbb{R}^{n_0 \times n_0} \) are unitary matrices and \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}^{n_1 \times n_0} \) with \( \frac{\overline{\sigma}_M}{\sqrt{\Sigma}} \geq \sigma_1 \geq \cdots \geq \sigma_m \geq \frac{\underline{\sigma}_N}{\sqrt{\Sigma}}. \) We can decompose \( H \) as

\[
H = \begin{bmatrix} M^\frac{1}{2} & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & M^{-\frac{1}{2}}F^T \\ FM^{-\frac{1}{2}} & 0 \end{bmatrix} \cdot \begin{bmatrix} M^\frac{1}{2} & 0 \\ 0 & I \end{bmatrix}.
\]

Note that for any \( \alpha \in \mathbb{R}^{n_0} \) and \( \beta \in \mathbb{R}^{n_1}, \) we have

\[
\left\| \begin{bmatrix} I & \Sigma^T \\ \Sigma & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|^2 = \sum_{i=1}^{n_1} (\alpha_i^2 + \sigma_i^2 \beta_i^2) + \sum_{i=n_1+1}^{n_0} \beta_i^2 = \sum_{i=1}^{n_1} \left(1 + \frac{\sigma_i^2}{2} \right) \left( \alpha_i^2 + \frac{2\sigma_i^2}{2 + \sigma_i^2} \beta_i^2 \right) + \sum_{i=n_1+1}^{n_0} \alpha_i^2 \geq \left( \min_i \frac{\sigma_i^2}{2 + \sigma_i^2} \right) \left\| \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|^2 \geq \frac{\underline{\sigma}_M (\sigma_N)^2}{2\underline{\sigma}_M \overline{\sigma}_M + \sigma_M (\sigma_N)^2} \left\| \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|^2.
\]

Therefore, by (25), we see that

\[
\left\| H \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\| \geq \min(\underline{\sigma}_M, 1) \cdot \sigma_N \cdot \sqrt{\frac{\overline{\sigma}_M}{2\underline{\sigma}_M \overline{\sigma}_M + \sigma_M (\sigma_N)^2}} \left\| \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|.
\]

This finishes the proof of the lower bound.

For the upper bound, note that

\[
\left\| H \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\| \leq \left\| M\alpha + N^T \beta \right\| + \left\| N\alpha \right\|
\]

\[
\leq \left\| M\alpha \right\| + \left\| N^T \beta \right\| + \left\| N\alpha \right\|
\]

\]

23
Since the primal-dual optimal solution to $\iota_t^T (z, \xi_{t:T}; F_T)$ and $\iota_{t+k}^T (z, \xi_{t:t+k}; \bar{U})$ are given by $(\Upsilon_t^T (\xi_{t:T}))^{-1} \beta_t^T (z, \xi_{t:T})$ and $(\hat{\Upsilon}_{t+k}^T (\xi_{t:t+k}))^{-1} \beta_{t+k}^T (z, \xi_{t:t+k})$ respectively, it is critical to establish the exponentially decaying bounds for the matrices $(\Upsilon_t^T (\xi_{t:T}))^{-1}$ and $(\hat{\Upsilon}_{t+k}^T (\xi_{t:t+k}))^{-1}$.

Note that after the rearrangement, $\Upsilon_t^T (\xi_{t:T})$ is a block matrix with $(T - t + 1) \times (T - t + 1)$ blocks, indexed by $(i, j) \in [t, T]^2$; $\hat{\Upsilon}_{t+k}^T (\xi_{t:t+k})$ is a block matrix with $(k + 1) \times (k + 1)$ blocks, indexed by $(i, j) \in [t, t + k]^2$.

**Lemma G.2.** Under Assumption [5.2] the following inequalities hold for the norm of the block entries of $(\Upsilon_t^T (\xi_{t:T}))^{-1}$ and $(\hat{\Upsilon}_{t+k}^T (\xi_{t:t+k}))^{-1}$:

$$\left\| (\Upsilon_t^T (\xi_{t:T}))^{-1}_{ij} \right\| \leq C_2 \lambda_2^{-1-j}, \forall (i, j) \in [t, T]^2, \forall \xi_{t:T} \in \Xi_{t:T},$$

$$\left\| (\hat{\Upsilon}_{t+k}^T (\xi_{t:t+k}))^{-1}_{ij} \right\| \leq C_2 \lambda_2^{-1-j}, \forall (i, j) \in [t, t+k]^2, \forall \xi_{t:t+k} \in \Xi_{t:t+k}. \quad (26)$$

Further, the following inequalities hold for the norm of differences between the block entries of $(\Upsilon_t^T (\xi_{t:T}))^{-1}$ and $(\hat{\Upsilon}_{t+k}^T (\xi_{t:t+k}))^{-1}$: For all $(i, j) \in [t, T]^2$ and $\xi_{t:T} \in \Xi_{t:T}$, we have

$$\left\| (\Upsilon_t^T (\xi_{t:T}))^{-1} - (\hat{\Upsilon}_{t+k}^T (\xi_{t:t+k}))^{-1}_{ij} \right\| \leq C_2' \sum_{\tau = t}^{T} \lambda_2^{\tau-i+j} \cdot \| \xi_{\tau} - \xi_{\tau}' \|, \quad (27)$$

For all $(i, j) \in [t, t+k]^2$ and $\xi_{t:t+k} \in \Xi_{t:t+k}$ with $t < T - k$, we have

$$\left\| (\hat{\Upsilon}_{t+k}^T (\xi_{t:t+k}))^{-1} - (\hat{\Upsilon}_{t+k}^T (\xi_{t:t+k}'))^{-1}_{ij} \right\| \leq C_2' \sum_{\tau = t}^{t+k} \lambda_2^{\tau-i+j} \cdot \| \xi_{\tau} - \xi_{\tau}' \|, \quad (28)$$

where the constants $C_2, C_2'$, and $\lambda_2$ are given by

$$\lambda_2 = \left( \frac{\sigma_H - \sigma_H}{\sigma_H + \sigma_H} \right)^{\frac{1}{2}}, C_2 = \frac{4(\ell + 1 + a + b)}{\sigma_H^2 \cdot \lambda_2},$$

$$C_2' = C_2^2 \left( \max \{L_Q + L_R + L_P\} + \frac{2}{\lambda_2} (L_A + L_B) \right).$$

where $\sigma_H$ and $\sigma_H$ are defined as

$$\sigma_H := \min(\mu, 1) \cdot (a + b + 1) \cdot \sqrt{\frac{\ell}{2\mu \ell + \mu \sigma^2}}, \text{ and } \sigma_H := \sqrt{2}(\ell + a + b + 1).$$

**Proof of Lemma G.2.** In the proof, we only show the results for $\Upsilon_t^T$. The results for $\hat{\Upsilon}_{t+k}^T$ can be shown using the same method.

We first show (26) holds. By Lemma G.1, we know that $\Upsilon_t^T (\xi_{t:T})^2$ is a positive definite matrix that has band width 4 and satisfies

$$\sigma_H^2 I \preceq \Upsilon_t^T (\xi_{t:T})^2 \preceq \sigma_H^2 I.$$

Using the same method as the proof of Lemma B.1 in [35], one can show that for any $(i, j) \in [t, T]^2$,

$$\left( (\Upsilon_t^T (\xi_{t:T})^2)^{-1} \right)_{ij} \leq \frac{2}{\sigma_H^2} \lambda_2^{i-j}. \quad (29)$$
Note that $\Upsilon_\ell^T(\xi; T)^{-1} := (\Upsilon_\ell^T(\xi; T))^2)^{-1}$. Thus we see that

\[
(Y_i^T(\xi; T)^{-1})_{ij} = \sum_{k=1}^T Y_i^T(\xi; T)_{ik} \cdot (Y_i^T(\xi; T)^{-1})_{kj}.
\]

Therefore, by (29), we see that

\[
\| (Y_i^T(\xi; T)^{-1})_{ij} \| \leq \frac{4(\ell + 1 + a + b)}{g^2_T \cdot \lambda_2} \cdot \lambda^{i-j} \cdot \forall (i, j) \in [t, T]^2. \tag{30}
\]

Note that we have

\[
\Upsilon_i^T(\xi'_{t'; T})^{-1} - (\Upsilon_i^T(\xi'; T) - (\Upsilon_i^T(\xi'; T))^{-1} \cdot (\Upsilon_i^T(\xi'; T) - (\Upsilon_i^T(\xi'; T))^{-1} = \Upsilon_i^T(\xi'_{t'; T})^{-1}.
\]

To simplify the notation, we define

\[
E_\tau := (Y_i^T(\xi; T) - (Y_i^T(\xi'; T))^{-1}, \forall \tau \in [t, T],
\]

and

\[
E_\tau' := (Y_i^T(\xi; T) - (Y_i^T(\xi'; T))^{-1}, \forall \tau \in [t, T - 1].
\]

The right hand side of (31) is a linear equation. Thus we can study the following 3 equations separately and sum them up:

\[
\Phi_\tau := (Y_i^T(\xi; T))^{-1} \begin{bmatrix} 0 & \cdots & 0 & E_\tau & 0 & \cdots & 0 \\ 0 & \cdots & 0 & E_\tau & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (E_\tau')^T & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} (Y_i^T(\xi; T))^{-1}, \forall \tau \in [t, T], \tag{32a} \\ (Y_i^T(\xi; T))^{-1}, \forall \tau \in [t, T - 1], \tag{32b} \\ (Y_i^T(\xi; T))^{-1}, \forall \tau \in [t, T - 1]. \tag{32c} \end{bmatrix}
\]

By (32a), (32b), and (32c), we see that

\[
\| (\Phi_\tau)_{ij} \| \leq \| (Y_i^T(\xi'_{t'; T})^{-1})_{ij} \| \cdot \| E_\tau \| \cdot \| (Y_i^T(\xi; T)^{-1})_{ij} \|
\]

\[
\| (\Phi_\tau^L)_{ij} \| \leq \| (Y_i^T(\xi'_{t'; T})^{-1})_{ij} \| \cdot \| E_\tau \| \cdot \| (Y_i^T(\xi; T)^{-1})_{ij} \|
\]

\[
\| (\Phi_\tau^U)_{ij} \| \leq \| (Y_i^T(\xi'_{t'; T})^{-1})_{ij} \| \cdot \| E_\tau \| \cdot \| (Y_i^T(\xi; T)^{-1})_{ij} \|.
\]

\[
\leq C_2^2 \cdot \lambda_2^{i-j} \cdot \| E_\tau \|,
\]

\[
\leq C_2^2 \cdot \lambda_2^{i-j} \cdot \| E_\tau \|,
\]

\[
\leq C_2^2 \cdot \lambda_2^{i-j} \cdot \| E_\tau \|,
\]

25
\[
\| (\Phi^L)_{ij} \| \leq \| (\Upsilon^T_i (\xi_{t+T})^{-1})_{i(t+1)} \| \cdot \| E^*_t \| \cdot \| (\Upsilon^T_i (\xi_{t+T})^{-1})_{ij} \| \\
\leq C^2 \cdot \lambda^2 \cdot \| \| E^*_t \| \|,
\]
where we use the bound (30) on the norm of individual block entries in (33a), (33b), and (33c). Summing these inequalities up over \( \tau \), we see that
\[
\| (\Upsilon^T_i (\xi_{t+T})^{-1} - \Upsilon^T_i (\xi'_{t+T})^{-1})_{ij} \|
= \left\| \sum_{\tau=t}^{T} (\Phi_{ij})_{ij} + \sum_{\tau=t}^{T-1} (\Phi^L_{ij})_{ij} \right\|
\leq \sum_{\tau=t}^{T} \| (\Phi_{ij})_{ij} \| + \sum_{\tau=t}^{T-1} \| (\Phi^L_{ij})_{ij} \|
\leq C^2 \left( \sum_{\tau=t}^{T} \lambda^2_{ij} \cdot \| E^*_t \| \cdot \| (\Phi^L_{ij})_{ij} \| \right)
\leq C^2 \left( \max\{L_{Q} + L_{R}, L_{P}\} + \frac{2}{\lambda^2} \sum_{\tau=t}^{T-1} \lambda^2_{ij} \cdot \| \xi_{\tau} - \xi'_{\tau} \| \right),
\]
where we use (31) in (34a); we use the triangle inequality in (34b); we use (33) in (34c); we use the Lipschitzness of dynamical and cost matrices in \( \xi \) (Assumption 4.2) in (34d). \( \square \)

With Lemma G.2 we can derive the perturbation bounds specified by Property 3.1

**Theorem G.3.** Under Assumption 4.2 Property 3.1 holds for arbitrary positive constant \( R \) and \( q_1(t) = H_2 \lambda^2_{ij}, q_2(t) = H_2 \lambda^2_{ij}, \) and \( q_3(t) = H_2 \lambda^2_{ij}, \) where \( \lambda^2_{ij} \) is defined in Lemma G.2 and \( H_2 \) is given by
\[
H_2 = C^2 \left( \frac{2(\ell D_x + D_w)}{1 - \lambda^2} + R + D_x + 1 \right) + C^2 \left( L_{w} + \ell L_{x} + D_{w} L_{Q} + 1 \right).
\]

**Proof of Theorem G.3** For \( t < T - k \), under the specification of Property 3.1 we see that
\[
\| \psi^t+k(z, \xi_{t:t+k}; 1)_{vt} - \psi^t+k(z, \xi'_{t:t+k}; 1)_{vt} \|
\leq \left\| (\Upsilon^T_i (\xi_{t:t+k})^{-1} - \Upsilon^T_i (\xi'_{t:t+k})^{-1}) \right\| \cdot \| E^*_t \| \cdot \| (\Upsilon^T_i (\xi_{t:t+k})^{-1})_{ij} \|
\leq C^2 \left( \max\{L_{Q} + L_{R}, L_{P}\} + \frac{2}{\lambda^2} \sum_{\tau=t}^{T-1} \lambda^2_{ij} \cdot \| \xi_{\tau} - \xi'_{\tau} \| \right),
\]
where we used the KKT condition in (35a) and the triangle inequality in (35b).

For the first term in (35b), we see that
\[
\left\| (\Upsilon^T_i (\xi_{t:t+k})^{-1} - \Upsilon^T_i (\xi'_{t:t+k})^{-1}) \cdot \beta^t+k(z, \xi_{t:t+k}) \right\|
\leq \sum_{\tau=t}^{T} \left\| (\Upsilon^T_i (\xi_{t:t+k})^{-1} - \Upsilon^T_i (\xi'_{t:t+k})^{-1}) \right\| \cdot \| \beta^t+k(z, \xi_{t:t+k}) \|
\leq C^2 \left( \sum_{\tau=t}^{T} \lambda^2_{ij} \cdot \| \xi_{\tau} - \xi'_{\tau} \| \right) \cdot \| E^*_t \| + \sum_{\tau=t}^{T} C^2 \left( \sum_{i=t}^{T} \lambda^2_{ij} \cdot \| \xi_{\tau} - \xi'_{\tau} \| \right) \cdot (\ell D_x + D_w)
\leq C^2 \lambda^2 \cdot \left( \sum_{\tau=t}^{T} \| \xi_{\tau} - \xi'_{\tau} \| \right) \cdot \| E^*_t \|
\]
\[
+ C^2 \lambda^2 \cdot \left( \sum_{\tau=t}^{T} \| \xi_{\tau} - \xi'_{\tau} \| \right) \cdot \| E^*_t \|
\]
\[
\leq C_2^{k} \sum_{\tau=0}^{\infty} \lambda_2^{2\tau} \delta_{t+\tau} \cdot \|z\| + C_2^{k} \left( \frac{2(\ell D_x + D_{uw})}{1 - \lambda_2} + R + D_x^2 \right) \sum_{\tau=0}^{k} \lambda_2^{2\tau} \delta_{t+\tau}, \tag{36c}
\]

where we use the triangle inequality in (36a); we use Lemma G.2 and the bounds on each entry of \(\hat{\beta}_t^{t+k}(z, \xi_{t:t+k})\) in (36b); we rearrange the terms and use \(\xi_{t+k} \in B(x^*_t+k, R)\) in (36c). For the second error term (36b), we see that

\[
\left\| \left( \hat{\gamma}_t^{t+k}(\xi_t^{t+k})^{-1} \left( \hat{\beta}_t^{t+k}(z, \xi_{t:t+k}) - \hat{\beta}_t^{t+k}(z, \xi_t^{t+k}) \right) \right) \right\|_{v_t} \leq C_2^{k} \sum_{\tau=0}^{k} \lambda_2^{2\tau} \left( L_w + \ell L_x + D_x L_Q \right) \delta_{t+\tau} + C_2^{k} \lambda_2^{2\tau} \delta_{t+\tau}, \tag{37}
\]

where we use the following inequality to bound the difference between \(\hat{\beta}_t^{t+k}(z, \xi_{t:t+k})\) and \(\hat{\beta}_t^{t+k}(z, \xi_t^{t+k})\):

\[
\|Q_T(\xi_t^T)\bar{x}_t(\xi_t^T) - \|Q_T(\xi_t^T)\bar{x}_t(\xi_t^T)\| \leq \|Q_T(\xi_t^T)\bar{x}_t(\xi_t^T) - \|Q_T(\xi_t^T)\bar{x}_t(\xi_t^T)\| \leq (L_Q D_x + \ell L_x) \delta_{t+\tau}.
\]

Substituting (36) and (37) into (35) gives that for any \(t < T - k\),

\[
\left\| \psi_t^{T+k}(z, \xi_{t:t+k}; \|) - \psi_t^{T+k}(z, \xi_t^{t+k}; \|) \right\|_{v_t} \leq \left( \sum_{\tau=0}^{k} q_1(\tau) \delta_{t+\tau} \right) \|z\| + \sum_{\tau=0}^{k} q_2(\tau) \delta_{t+\tau}
\]

under the specification that \(\xi_{t:t+k-1} \in \Xi_{t:t+k-1}, \xi_t^{t+k-1} = \xi_t^{t+k-1}; \xi_{t+k}, \xi_t^{t+k} \in B(x^*_t+k, R)\).

We can use a similar method to show that for any \(t \geq T - k\),

\[
\left\| \psi_t^{T}(z, \xi_{t:T}; F_T) - \psi_t^{T}(z, \xi_{t:T}; F_T) \right\| \leq \left( \sum_{\tau=0}^{T-t} q_1(\tau) \delta_{t+\tau} \right) \|z\| + \sum_{\tau=0}^{T-t} q_2(\tau) \delta_{t+\tau}
\]

under the specification that \(\xi_{t:T} \in \Xi_{t:T}, \xi_{t:T}^T = \xi_{t:T}^T\).

For any \(t < T\), we see that

\[
\left\| \psi_t^{T}(z, \xi_{t:T}; F_T)_{y_r/v_r} - \psi_t^{T}(z', \xi_{t:T}; F_T)_{y_r/v_r} \right\| \leq \left( \sum_{\tau=0}^{T-t} q_1(\tau) \delta_{t+\tau} \right) \|z - z'\| + \sum_{\tau=0}^{T-t} q_2(\tau) \delta_{t+\tau}, \tag{38a}
\]

where we use the KKT condition in (38a); we use Lemma G.2 in (38b).

Now we come back to the proof of Theorem 4.2.

**Proof of Theorem 4.2** By Theorem G.3, Property 5.1 holds for arbitrary positive constant \(R\) and \(q_1(t) = H_2\lambda_2^{2\tau}, q_2(t) = H_2\lambda_2^{2\tau}, q_3(t) = H_2\lambda_2^{2\tau}\), where the decay rate \(\lambda_2 \in (0, 1)\) and constant \(H_2\) depends on \(R\). We set \(R = \frac{D_x^* + D_x}{2} \) so that MPC\(k\) with terminal state \(\bar{x}_{t+k}(\xi_{t+k})\) satisfies the assumption of Theorem 3.3. The constant \(H_2\) is given by

\[
H_2 = C_2^{k} \left( \frac{2(\ell D_x + D_{uw})}{1 - \lambda_2} + 2D_x^2 + D_x^2 + 1 \right) + C_2 (L_w + \ell L_x + D_x L_Q + 1).
\]

By Theorem 3.3, in order to achieve the claimed dynamic regret bound in Theorem 4.2, a sufficient condition is that the prediction errors \(\rho_{t,\tau}\) satisfy

\[
\sum_{\tau=0}^{k} \lambda_2^{2\tau} \rho_{t,\tau} \leq \frac{(1 - \lambda_2)^2(D_x^* + D_x)}{2H_2^2L_Q ((1 - \lambda_2)(D_x^* + D_x) + H_2(D_x^* + 1))}.
\]

27
and the prediction horizon $k$ satisfies that
\[
\lambda_2^k \leq \frac{(1 - \lambda_2)^2}{4H^2_2 L_g ((1 - \lambda_2)(D_{x^*} + D_x) + H_2(D_{x^*} + 1))}.
\]

Therefore, we can use Theorem 1 in [37] to show

We provide two example systems where the dynamics is unknown but the uniform controllability assumption (Assumption G.1) is satisfied for all possible uncertainty parameters.

**Example G.4** (Inverted pendulum with unknown mass). We consider the linearized inverted pendulum dynamics in discrete state space. The dynamics of the system is given by

\[
\begin{bmatrix}
  x_{t+1} \\
  \dot{x}_{t+1} \\
  \phi_{t+1}
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & -m_l b & I(M+m) + Mm_t^2
\end{bmatrix}
\begin{bmatrix}
  x_t \\
  \dot{x}_t \\
  \phi_t
\end{bmatrix} +
\begin{bmatrix}
  0 \\
  (I+m_m^2)\delta \\
  m_l g (M+m_m)\delta
\end{bmatrix}
\begin{bmatrix}
  u_t \\
  1 \\
  0
\end{bmatrix},
\]

where $M$ is the mass of the cart, $m$ is the mass of the pendulum, $b$ is the coefficient of friction for cart, $l$ is the length to the center of the mass of the pendulum, $I$ is the mass moment of inertia of the pendulum, $x_t$ is the position of the cart, $\phi_t$ is the angle of the pendulum, and $\delta$ is step size of discretization. For simplicity, we assume all system parameters are known except the mass of the cart $M$, which is an unknown value in the interval $[M, \bar{M}]$ for some constants $\bar{M} > M > 0$. The controllability matrix of the dynamical system satisfies that

\[
\det \begin{bmatrix}
  B(M) \\
  A(M)B(M) \\
  A(M)^2B(M) \\
  A^3(M)B(M)
\end{bmatrix} = \frac{\delta^{10} g^{214} m^4}{(IM + m(I + l^2 M))^4} \geq \frac{\delta^{10} g^{214} m^4}{(IM + m(I + l^2 M))^4} > 0.
\]

Therefore, we can use Theorem 1 in [37] to show

\[
\sigma_{\min} \begin{bmatrix} B(M), A(M)B(M), A(M)^2B(M), A^3(M)B(M) \end{bmatrix} \geq \sigma_0
\]

for some positive constant $\sigma_0$, which implies Assumption G.1 by Lemma 12 in [37].

**Example G.5** (Frequency regulation with unknown inertia). We consider the power grid dynamics with $n$ nodes studied in [18]:

\[
\begin{bmatrix}
  \dot{\omega} \\
  \dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
  -M^{-1}_{q(t)} L & -M^{-1}_{q(t)} D \\
  I & 0
\end{bmatrix}
\begin{bmatrix}
  \omega \\
  \theta
\end{bmatrix} +
\begin{bmatrix}
  0 \\
  M^{-1}_{q(t)}
\end{bmatrix}
\begin{bmatrix}
  \rho_{\omega} \\
  u_{\omega}
\end{bmatrix}.
\]

Here, $\theta, \omega \in \mathbb{R}^n$ are the vectors of voltage phase angles and frequencies. The state $x(t)$ is a stacked vector of $\theta$ and $\omega$, and $u(t) = \rho_{\omega}$ is the power input. $D$ is a diagonal matrix whose entries represent droop control coefficients, and $L$ is the Laplacian matrix of the network of nodes. $M_q(t)$ is the inertia matrix in mode $q(t) \in \{1, \ldots, m\}$ and is time-varying. It is further assumed to be in the form $M_q(t) = m_q(t) I$, where $m_q(t)$ is a scalar that represents the inertia coefficient at time $t$. We assume all system parameters are known except the inertia coefficient, and we define $\xi_t := m_q(t)$.

For simplicity, we use a different discretization technique with [18] so that it is easier to verify uniform controllability. Specifically, we write the discrete-time system as

\[
x_{t+1} = (I + \delta A(t)) x_t + \delta B(t) u_t,
\]

where $\delta$ is the step size of discretization. We see that when $0 < m < m_q(t) \leq m < +\infty$ holds for some positive constants $m$, $m_q(t)$ for all possible inertia coefficients, we have

\[
|\det [B(\xi_{t+1}), A(\xi_{t+1})] B(\xi_{t})] \geq \frac{\delta^3 n}{m^{2n}} > 0.
\]

Thus, we can use Theorem 1 in [37] to show that

\[
\sigma_{\min} [B(\xi_{t+1}), A(\xi_{t+1})] \geq \sigma_0
\]

holds for all possible inertia coefficient at all time steps for some positive constant $\sigma_0$, which implies Assumption G.1 by Lemma 12 in [37].
H Assumptions and Proofs of Section 5

To introduce the SSOSC assumption, we first define the reduced Hessian of the Lagrangian.

Definition H.1 (reduced Hessian). For a constrained optimization problem with primal variable $z$ and dual variable $\eta$, let $H = \nabla^2_{zz}L$ denote the Hessian of the Lagrangian $L(z, \eta; \xi)$. Let $G$ denote the active constraints Jacobian, i.e. Jacobian of all equality constraints and active inequality constraints, and let $Z$ be the null-space matrix of $G$ (i.e., the column vectors of $Z$ form an orthonormal basis of the null space of $G$). Then the reduced Hessian is defined as $H_{re}(z, \eta; \xi) := Z^\top H Z$.

We define the concept of singular spectrum bounds for a specific instance of FTOCP:

Definition H.2 (singular spectrum bounds). Consider the FTOCP $\nu^2(z, \xi_{t_1:t_2}; F)$. The positive real numbers $\sigma_H, \sigma_R, \sigma_Z$ are called singular spectrum bounds for this specific instance of FTOCP if they satisfy that

$$\sigma_H \geq \| H(z, \xi_{t_1:t_2}) \|, \quad \sigma_R \geq \| R(z, \xi_{t_1:t_2}) \|, \quad 0 < \sigma_Z \leq \sigma_H(z, \xi_{t_1:t_2}),$$

where $\sigma_H, \sigma_R,$ and $\sigma_Z$ are defined in (4.16a-c) in [50].

Assumption H.1. We make the following assumptions on the costs, dynamics, and constraints of an FTOCP $\nu^2(z, \xi_{t_1:t_2}; F)$:

1. All cost functions, dynamical functions, and constraint functions are twice continuously differentiable in $(x_t, u_t)$ and $\xi_t$.

2. (SSOSC) The reduced Hessian at the optimal primal-dual solution is positive-definite.

3. (LICQ) The active constraints Jacobian $G$ at the optimal primal-dual solution has full row rank, i.e. $\sigma_{\min}(G) > 0$.

4. (Uniform singular spectrum bounds) There exist positive singular spectrum bounds $\sigma_H, \sigma_R, \sigma_Z$ for all FTOCP specifications below:

   (a) $t_1 = t, t_2 = t + k$ for $t < T - k$:

   $$z \in B(x^*_t, R), \xi_{t:t+k-1} \in \Xi_{t:t+k-1}, \xi_{t+k} \in B(x^*_t, R), F = I.$$

   (b) $t_1 = t, t_2 = T$ for $t < T$:

   $$z \in B(x^*_t, R), \xi_{t:T} \in \Xi_{t:T}, F = F_T.$$

We remind the readers that Lemma 12 in [37] shows that Lipschitzness of dynamics and uniform controllability together imply uniform LICQ property of the system.

Under Assumption H.1 we know Property 3.1 holds for $q_1(t) = 0$, $q_2(t) = H_3 \lambda_3^1$, and $q_3(t) = H_3 \lambda_3^1$ for some $H_3 > 0$ and $\lambda_3^1 \in (0, 1)$ by Theorem 4.5 in [50].

Theorem H.1. Under Assumption H.1 that holds for some $R > 0$, Property 3.1 holds for $q_1(t) = 0$, $q_2(t) = H_3 \lambda_3^1$, and $q_3(t) = H_3 \lambda_3^1$ with the same $R$. The coefficient $H_3$ and decay factor $\lambda_3$ are given by

$$H_3 := \left( \frac{\sigma_H \sigma_R}{\sigma_Z^2} \right)^{\frac{1}{2}}, \quad \lambda_3 := \left( \frac{\sigma_H^2 - \sigma_Z^2}{\sigma_H^2 + \sigma_Z^2} \right)^{\frac{1}{2}}.$$

Combining Theorem H.1 with the Pipeline Theorem (Theorem 3.3) finishes the proof of Theorem 5.1

Note that when $\xi_t^* \leq (1 - \lambda_3^1) R_{H_3}$, we know that $\left\| \psi_T^*(x_0, 0; F_T)_{g_{t+k}} - x^*_t \right\| \leq R$. Thus using $\psi_T^*(x_0, 0; F_T)_{g_{t+k}}$ as the terminal state of MPC at time step $t$ can satisfy the requirement of Theorem 5.1.

---

1. We remind the reader that the functions $\sigma_H, \sigma_R$, and $\sigma_Z$ depend on the form of FTOCP, i.e., different horizon $[t_1, t_2]$ and different terminal cost function $F$.

2. If the terminal function $F$ is an indicator function of some state, we view it as a constraint instead of cost.
We can set
\[ x_{t+1} = x_t + u_t, \quad \text{s.t. } x_t \in [-1, 1], u_t \geq -\frac{4}{5} \]
for all \( t \). The stage cost is given by \( f_t(x_t, u_t; \xi_t) \), where \( f_t \) is convex and \( \ell \)-smooth. We also assume that \( f_t \) is \( \mu \)-strongly convex in its first variable \( x_t \). Then, for any positive integer \( p \geq 3 \), we have
\[
\left| \psi^p_0 \left( x_0, \xi_0; \zeta_p; \mathbb{I} \right)_{x_0} - \psi^p_0 \left( x_0', \xi'_0; \zeta'_p; \mathbb{I} \right)_{x_0} \right| \\
\leq C \left( \lambda^h |x_0 - x_0'| + \sum_{\tau=0}^{p-1} \lambda^{h-\tau} |\xi_\tau - \xi'_\tau| + \lambda^{p-h} |\zeta_p - \zeta'_p| \right),
\]
for all \( x_0, \zeta_p \in [-1, 1] \), where \( h \in \{1, \ldots, p\} \) and \( C > 0, \lambda \in (0, 1) \) are some constants.

\textbf{Proof of Theorem I.1} We can rewrite the optimization problem to remove the equality constraints as following:
\[
\min_{x_{1:p-1}} \sum_{t=0}^{p-1} f_t(x_t, x_{t+1} - x_t; \xi_t) \\
\text{s.t. } -1 \leq x_t \leq 1, \forall t \in \{1, 2, \ldots, p-1\},
\]
\[
x_t - x_{t-1} \geq -\frac{4}{5}, \forall t \in \{1, 2, \ldots, p\},
\]
where \( x_p = \zeta_p \).

Note that for any time index \( t \in \{1, 2, \ldots, p-1\} \), at most 2 constraints that involves \( x_t \) can be active. They can be chosen from the 4 possible constraints that involves \( x_t \):
\[
x_t \geq -1, x_t \leq 1, x_t - x_{t-1} \geq -\frac{4}{5}, x_{t+1} - x_t \geq -\frac{4}{5}.
\]

And for any time index \( t \in \{1, 2, \ldots, p-2\} \), the 3 consecutive “coupling” constraints
\[
x_t - x_{t-1} \geq -\frac{4}{5}, x_{t+1} - x_t \geq -\frac{4}{5}, x_{t+2} - x_{t+1} \geq -\frac{4}{5},
\]
cannot activate simultaneously. Let \( \sigma_0 \) denote the smallest singular value of matrix
\[
\begin{bmatrix}
1 & 1 \\
-1 & -1
\end{bmatrix}.
\]

Therefore, in the context of Theorem 4.5 in [50], we see that \( \sigma \left( \nabla_{x,y} \mathcal{L} \left( z^z(\xi); \mathcal{B} \right) \right) \) is lower bounded by \( \sigma_0 \) and upper bounded by 2. Since we also have that
\[
\mu I \preceq \nabla_{x,x} \mathcal{L} \left( z^z(\xi); \mathcal{B} \right) \preceq 5\mu I.
\]

By Lemma G.1 we further see that we can set \( \sigma_H \) and \( \sigma_R \) as
\[
\sigma_H := 2 \min(\mu, 1) \sqrt{\frac{5\mu}{10\mu + \mu \sigma_0^2}}, \sigma_R := \sqrt{2(5\ell + 2)}.
\]

We can set \( \sigma_R := \ell \). Applying Theorem 4.5 in [50] finishes the proof.
As a remark, we have already certified Assumption $H.1$ in the proof of Theorem $I.1$ and the perturbation bound provided by Theorem $I.1$ is more general than the statement of Theorem $H.1$.

While Theorem $I.1$ shows that Assumption $H.1$ is not vacuous, and we present a negative result in Theorem $I.2$ which shows exponentially decaying perturbation bounds may not hold when Assumption $H.1$ is not satisfied.

**Theorem I.2.** Consider the optimal control problem where the state $x_t$ and the control input $u_t$ are both in $\mathbb{R}$. The dynamics and constraints are given by

$$x_{t+1} = x_t + u_t, \text{ s.t. } x_t \in [-1, 1], u_t \in \left[-\frac{4}{5}, \frac{4}{5}\right]$$

for all $t$. The stage cost is given by $f_t(x_t, u_t; \xi_t) = (x_t - \xi_t)^2$, where $\xi_t = \frac{1}{2}$ if $t$ is odd, and $\xi_t = -\frac{1}{2}$ if $t$ is even. For any $p$ is even, we have

$$\left| \psi_0^p \left(0, \xi_0; p-1, -\frac{2}{5} \right)_{x_h} - \psi_0^p \left(0, \xi_0; p-1, -\frac{2}{5} + \epsilon \right)_{x_h} \right| = \epsilon$$

(42)

holds for any $\epsilon \in \left[0, \frac{2}{5(p-1)}\right]$ and $h \in \{1, \ldots, p\}$. For any $p$ is odd, we have

$$\left| \psi_0^p \left(0, \xi_0; p-1, \frac{2}{5} \right)_{x_h} - \psi_0^p \left(0, \xi_0; p-1, \frac{2}{5} + \epsilon \right)_{x_h} \right| = \epsilon$$

(43)

holds for any $\epsilon \in \left[0, \frac{2}{5p}\right]$ and $h \in \{1, \ldots, p\}$.

Before presenting the proof of Theorem $I.2$, we want to add a remark about why a similar proof as Theorem $I.1$ cannot work here. Note that a key property we leveraged in the proof of Theorem $I.1$ is that any three consecutive “coupling” constraints (41) cannot activate simultaneously. This is no longer the case when $u_t$ has two sides of constraints, i.e., $u_t \in \left[-\frac{4}{5}, \frac{4}{5}\right]$. As a result, the smallest singular value of matrix $\nabla_x L \{z^*| z; \xi\} |B, \overline{B}|$ can be arbitrarily small (i.e., decaying w.r.t. the horizon length $p$). Thus, Assumption $H.1$ is not satisfied because $\sigma_{\xi H}$ cannot be set as a positive constant, and the same proof as Theorem $I.1$ can no longer work. We will leverage this intuition to construct a counterexample to show Theorem $I.2$. We construct a sequence of cost functions so that $u_t$ reaches either its lower bound $-\frac{2}{5}$ or its upper bound $\frac{2}{5}$ at every time step.

**Proof of Theorem I.2.** We first show that (42) holds by induction on $p$. Specifically, we will show that the following holds for any $q \in \mathbb{Z}_+$

$$t_0^{2q} \left(0, \xi_0; 2q-1, -\frac{2}{5} + \epsilon; \mathbb{I}\right) \begin{cases} = \frac{8(q+2)}{25} + 2q(\epsilon) & \text{if } \epsilon \in \left[0, \frac{2}{5(2q-1)}\right], \\ \geq \frac{8(q+2)}{25} + \frac{8q}{25(2q-1)^2} & \text{if } \epsilon \in \left(\frac{2}{5(2q-1)}, \frac{2}{5}\right], \\ \geq \frac{8(q+2)}{25} & \text{if } \epsilon \in \left(-\frac{2}{5}, 0\right], \\ \end{cases}$$

(44)

by induction on $q$.

It is straightforward to check that (44) holds for $q = 1$. Suppose it holds for $q$. For $q + 1$, we consider the following three cases separately.

**Case 1:** $0 \leq \epsilon \leq \frac{2}{5(2q+1)}$.

Suppose $x_{2q} = -\frac{2}{5} + \epsilon^*$. When $\epsilon^* \in \left[0, \epsilon\right]$, we should choose $x_{2q+1} = \frac{2}{5} + \epsilon^*$ to minimize the total cost. The total cost is given by

$$\frac{8(q+2)}{25} + 2q(\epsilon) + \left(\frac{2}{5} - \epsilon^*\right)^2 + \left(\frac{2}{5} + \epsilon^*\right)^2,$$

and it is minimized at $\epsilon^* = \epsilon$. Thus, we achieve the total cost of $\frac{8(q+2)}{25} + 2(q+1)\epsilon^*$. When $\epsilon^* > \epsilon$, note that the optimal choice of $x_{2q+1}$ is $\frac{2}{5} + \epsilon$, which is the same as when $\epsilon^* = \epsilon$. By the induction assumption on $t_0^{2q}$, we see that the total cost incurred is lower bounded by $\frac{8(q+2)}{25} + 2(q+1)\epsilon^*$. When
When we see that $\epsilon$, the total cost is lower bounded by

$$\frac{8(q+2)}{25} + \frac{4}{25} + \left(\frac{2}{5} + \epsilon\right)^2 = \frac{8(q+3)}{25} + \epsilon^2 + \frac{4}{5} \epsilon \geq \frac{8(q+1)}{25} + 2(q+1)\epsilon^2.$$  

Thus, we have shown that

$$\mathcal{I}_0^q \left(0, v_{0:2q-1}, -\frac{2}{5} + \epsilon; \|I\| \right) = \frac{8(q+3)}{25} + 2(q+1)\epsilon^2, \forall \epsilon \in \left[0, \frac{2}{5(2q+1)}\right].$$

**Case 2:** $\frac{2}{5(2q+1)} < \epsilon \leq \frac{2}{5}$. 

Suppose $x_{2q} = -\frac{2}{5} + \epsilon$. When $\epsilon \leq \frac{2}{5}$, we know that $x_{2q+1} \leq \frac{2}{5} + \epsilon \leq \frac{4}{5}$. The total cost lower bounded by

$$\frac{8(q+2)}{25} + 2q(\epsilon')^2 + \left(\frac{2}{5} - \epsilon'\right)^2 \geq \frac{8(q+2)}{25} + 2q\left(\frac{2}{5(2q+1)}\right)^2 + \left(\frac{2}{5} - \frac{2}{5(2q+1)}\right)^2.$$  

Thus the total cost is lower bounded by

$$\frac{8(q+3)}{25} + \frac{8(q+1)}{25(2q+1)^2}.$$  

When $\epsilon > \frac{2}{5}$, we see that the total cost is lower bounded by

$$\frac{8(q+2)}{25} + 2q(\epsilon')^2 + \left(\frac{2}{5} + \epsilon\right)^2 \geq \frac{8(q+2)}{25} + 2q\left(\frac{2}{5(2q+1)}\right)^2 + \left(\frac{2}{5} - \frac{2}{5(2q+1)}\right)^2 + \left(\frac{2}{5} + \frac{2}{5(2q+1)}\right)^2.$$  

Thus the total cost is lower bounded by

$$\frac{8(q+3)}{25} + \frac{8(q+1)}{25(2q+1)^2}.$$  

**Case 3:** $-\frac{3}{5} \leq \epsilon < 0$. 

Note that the total cost of steps 0 to 2q is uniformly lower bounded by $\frac{8(q+2)}{25}$ regardless of the choice of $x_{2q}$, and the total cost of steps $(2q+1)$ and $(2q+2)$ is uniformly lower bounded by $\frac{8}{25}$. Therefore, we see that

$$\mathcal{I}_0^{2(q+1)} \left(0, \xi_{0:2q+1}, -\frac{2}{5} + \epsilon; \|I\| \right) \geq \frac{8(q+3)}{25}.$$  

Therefore, by combining the three cases, we have shown that (44) holds for all $q$ by induction.

By rolling out the optimal states that minimize the total cost, one can show the unique optimal solution is given by

$$\psi_0^p \left(0, \xi_{0:p-1}, -\frac{2}{5} + \epsilon; \|I\| \right)_{x_h} = \begin{cases} \frac{2}{5} + \epsilon & \text{if } h \text{ is odd}, \\ -\frac{2}{5} + \epsilon & \text{if } h \text{ is even}, \end{cases}$$

when $\epsilon \in \left[0, \frac{2}{5(2q+1)}\right]$. This finishes the proof of (42).

For (43), suppose $p = 2q + 1$. It is straightforward to verify that (43) holds for $q = 0$. When $q \geq 1$, by (44), we know that in the optimal solution, we have $x_{2q} = -\frac{2}{5} + \epsilon$. We can further derive that the unique optimal solution is given by

$$\psi_0^p \left(0, \xi_{0:p-1}, -\frac{2}{5} + \epsilon; \|I\| \right)_{x_h} = \begin{cases} \frac{2}{5} + \epsilon & \text{if } h \text{ is odd}, \\ -\frac{2}{5} + \epsilon & \text{if } h \text{ is even}, \end{cases}$$

when $\epsilon \in \left[0, \frac{2}{5(2q+1)}\right]$. This finishes the proof of (43).