Parameter Identification in MHD Duct Flow Cauchy Problem using the DRBEM

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Abstract. In this study, the Hartmann number (Ha) is identified through the inverse formulation of Cauchy problem for magnetohydrodynamic (MHD) flow in a duct with insulated but no-slip or variably slipping walls. The solutions for direct and inverse problems are obtained using the dual reciprocity boundary element method (DRBEM), and Tikhonov regularization with L-curve method is used for solving ill-conditioned linear system of equations resulted from the inverse problem. The velocity and the induced magnetic field profiles are simulated from the direct solution depicting the well-known MHD characteristics. The Hartmann number is reconstructed from the inverse problem with an accuracy of $10^{-4}$ to $10^{-6}$. The DRBEM provides overdetermined information needed in the inverse MHD flow problem since it gives both the solution and its normal derivatives on the boundary points.

1. Introduction
MHD channel flow has some industrial and biological applications such as MHD generator, flowmeters, cooling of nuclear fusion apparatus and blood flow measurements [1]. There are quite a number of DRBEM and BEM solutions of MHD duct flow with no-slip velocity and different combinations of wall conductivities [2, 3]. MHD duct flow equations with under or overspecified wall conditions, especially for slipping velocity and variably conducting induced magnetic field boundary conditions, form the boundary inverse MHD flow problems and can be solved with DRBEM [4]. There are also other classes of inverse problems such as parameter identification in the governing equations [5].

In this paper, the problem parameter, Hartmann number, is regained from the inverse MHD duct flow equations by employing the DRBEM with insulated and variable slipping walls. The velocity, induced magnetic field and their normal derivatives on the boundary, and also their interior values are obtained from the direct solution for a specified Hartmann number using DRBEM. The DRBEM is the only numerical method providing these overspecified boundary and interior information in one stroke. The regained Ha values from the Cauchy MHD problem have a maximum error of order $10^{-4}$ to $10^{-6}$ for $5 \leq Ha \leq 50$. The direct solutions exhibit the MHD flow characteristics as the flattening tendency of the flow when Ha increases.

2. The MHD Duct Flow Problem
An electrically conducting fluid is considered in a long channel of square cross-section (duct) under the effect of external uniform magnetic field applied perpendicular to the channel. The
non-dimensional governing equations for the velocity \( V(x, y) \) and induced magnetic field \( B(x, y) \) are \[6\]

\[
\nabla^2 V + Ha \frac{\partial B}{\partial y} = -1 \\
\n\nabla^2 B + Ha \frac{\partial V}{\partial y} = 0
\]

in \(-1 < x, y < 1\) \(1\)

where \( Ha \) is the Hartmann number containing the intensity of the applied magnetic field. Two types of wall conditions are considered as Problem 1 and Problem 2.

2.1. Problem 1

The MHD direct problem is solved with the equations (1) and wall conditions \( V = B = 0 \) on \( x = \mp 1, y = \pm 1 \). The DRBEM solution gives \( \frac{\partial V}{\partial n}, \frac{\partial B}{\partial n} \) on the boundary and \( V, B \) interior values.

2.2. Problem 2

The MHD direct problem is solved with the equations (1) and wall conditions \( V + \alpha \frac{\partial V}{\partial n} = 0, B = 0 \) on \( x = \mp 1, y = \pm 1 \) where \( \alpha \) is the slip length. The DRBEM solution provides both \( V, B \) and \( \frac{\partial V}{\partial n}, \frac{\partial B}{\partial n} \) on the boundary and \( V, B \) interior values.

The inverse MHD problems are constructed with these overspecified information for the velocity and induced magnetic field obtained from the direct solutions.

3. The DRBEM Application

The DRBEM is applied to the MHD flow equations (1) by using the fundamental solution of the Laplace equation which is \( u^* = ln(\frac{1}{r})/2\pi \). All the terms other than Laplacian are treated as inhomogeneity and approximated by using radial basis functions which are related to some particular solutions through Laplace operator. Thus, the BEM idea \[7\] is applied to both sides (transforming domain integrals to boundary integrals using Divergence theorem) of equations (1). The boundary is discretized by constant elements and some arbitrarily selected interior points are used. The resulting matrix-vector system of equations are

\[
\begin{bmatrix}
    H & K \\
    K & H 
\end{bmatrix}
\begin{bmatrix}
    V \\
    B 
\end{bmatrix} =
\begin{bmatrix}
    G & 0 \\
    0 & G 
\end{bmatrix}
\begin{bmatrix}
    \frac{\partial V}{\partial n} \\
    \frac{\partial B}{\partial n} 
\end{bmatrix} +
\begin{bmatrix}
    b \\
    0 
\end{bmatrix}
\]

(2)

where \( G, H \) are BEM matrices with entries composed of boundary integrals of fundamental solution and its normal derivative, respectively. \( K = (H\hat{U} - G\hat{Q})F^{-1}Ha\frac{\partial F}{\partial y}F^{-1}, b = -(H\hat{U} - G\hat{Q})F^{-1}(1, 1, ..., 1)^T \) where \( \hat{U} \) and \( \hat{Q} \) are constructed from the particular solutions and their normal derivatives columnwise, respectively. \( F \) is the coordinate matrix defined by linear radial basis functions \( f_{ij} = 1 + r_{ij} \) columnwise, \( r_{ij} \) being the distance from node \( i \) to \( j \).
The insertion of boundary conditions for Problem 1 or Problem 2 to the system (2) gives direct solutions for these problems in terms of $V, B, \frac{\partial V}{\partial n}, \frac{\partial B}{\partial n}$ boundary values and interior $V, B$ values. For reconstructing the Hartmann number $Ha$, the system (2) is rewritten as

$$
\begin{bmatrix}
H & 0 \\
0 & H \\
\end{bmatrix}
\begin{bmatrix}
B \\
V \\
\end{bmatrix} =
\begin{bmatrix}
G & 0 \\
0 & G \\
\end{bmatrix}
\begin{bmatrix}
\frac{\partial B}{\partial n} \\
\frac{\partial V}{\partial n} \\
\end{bmatrix} +
\begin{bmatrix}
K' & 0 \\
0 & K'' \\
\end{bmatrix}
\begin{bmatrix}
Ha_1 \\
Ha_2 \\
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
b \\
\end{bmatrix}
$$

(3)

where $K' = (H\hat{U} - G\hat{Q})F^{-1}\frac{\partial F}{\partial y} F^{-1} \text{diag}(V)$, $K'' = (H\hat{U} - G\hat{Q})F^{-1}\frac{\partial F}{\partial y} F^{-1} \text{diag}(B)$. The system (3) is solved for $(Ha_1, Ha_2)^T$ as an inverse problem with the overdetermined known solution obtained from the direct problem. Since the system (3) is ill-posed, the Tikhonov regularization with L-curve method is used in solving the system (3) for regaining the vector $(Ha_1, Ha_2)^T$.

4. Numerical Results

The velocity and the induced magnetic field from the direct solution are simulated for Problem 1 and Problem 2 in Figures 1 and 2, respectively. In Figure 1, we see the effects of increasing $Ha$ which are the formation of boundary layers, the enlargement of core stagnant region for the flow and the flattening of both the flow and induced current. These behaviors are in well-agreement with the ones in [1] for $V = B = 0$ wall conditions. Figure 2 shows the effect of increasing slip length for a fixed $Ha$ value. It is observed that when the slip length increases, the magnitude of the velocity increases, and the boundary layer formation is weakened.

Then, the inverse MHD problem is solved from the DRBEM discretized system (3) both in coupled and decoupled forms for $(Ha_1, Ha_2)^T$. The regularization parameter $\lambda$ is found as $10^{-7}$ or $10^{-8}$ for Problem 1 and $\lambda = 0$ is taken for Problem 2. The maximum errors are presented in Table 1 and Table 2 for Problem 1 and Problem 2, respectively. Maximum errors are defined as $e_1 = \max_i |Ha_i - Ha_{1i}|$, $e_2 = \max_i |Ha_i - Ha_{2i}|$ where $i$ is the number of interior points. It is seen that, $Ha$ is regained with an accuracy of order at least $10^{-4}$ and $10^{-6}$ for Problem 1 and Problem 2, respectively, from the second half of the vector $(Ha_1, Ha_2)^T$.

![Figure 1: Velocity and induced magnetic field profiles, $\alpha = 0$, Prob. 1](image-url)
Figure 2: Velocity and induced magnetic field profiles for $Ha = 10$, $c = 0$, Prob. 2

Table 1: The maximum errors in $Ha$, $\alpha = 0$, Prob. 1

| Coupled | Decoupled |
|---------|-----------|
| $e_1$   | $e_2$     | $e_1$   | $e_2$     |
| 5       | $6.2 \times 10^{-5}$ | $2.3 \times 10^{-7}$ | $4.1 \times 10^{-5}$ | $3.4 \times 10^{-5}$ |
| 10      | $1.3 \times 10^{-2}$ | $3.5 \times 10^{-6}$ | $2.1 \times 10^{-4}$ | $0.7 \times 10^{-4}$ |
| 30      | $2.2 \times 10^{-2}$ | $2.6 \times 10^{-6}$ | $7.8 \times 10^{-4}$ | $1.5 \times 10^{-4}$ |
| 50      | $2.4 \times 10^{-2}$ | $5.4 \times 10^{-6}$ | $0.1 \times 10^{-2}$ | $1.1 \times 10^{-4}$ |

Table 2: The maximum errors in $Ha$, $\alpha = 0.2$, Prob. 2

| Coupled | Decoupled |
|---------|-----------|
| $e_1$   | $e_2$     | $e_1$   | $e_2$     |
| 5       | $1 \times 10^{-7}$ | $3.3 \times 10^{-7}$ | $1.2 \times 10^{-7}$ | $3.3 \times 10^{-7}$ |
| 10      | $4.4 \times 10^{-8}$ | $2 \times 10^{-7}$ | $4.4 \times 10^{-8}$ | $2 \times 10^{-7}$ |
| 30      | $9.1 \times 10^{-7}$ | $1.5 \times 10^{-6}$ | $7.9 \times 10^{-7}$ | $1.5 \times 10^{-6}$ |
| 50      | $1.2 \times 10^{-7}$ | $5.4 \times 10^{-7}$ | $1.2 \times 10^{-7}$ | $5.4 \times 10^{-7}$ |

5. References

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