Quantum Complexity of Weighted Diameter and Radius in CONGEST Networks

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Abstract

This paper studies the round complexity of computing the weighted diameter and radius of a graph in the quantum CONGEST model. We present a quantum algorithm that $(1 + o(1))$-approximates the diameter and radius with round complexity $\tilde{O} \left( \min \{ n^{9/10} D^{3/10}, n \} \right)$, where $D$ denotes the unweighted diameter. This exhibits the advantages of quantum communication over classical communication since computing a $(3/2 - \varepsilon)$-approximation of the diameter and radius in a classical CONGEST network takes $\tilde{\Omega}(n)$ rounds, even if $D$ is constant [2]. We also prove a lower bound of $\tilde{\Omega}(n^{2/3})$ for $(3/2 - \varepsilon)$-approximating the weighted diameter/radius in quantum CONGEST networks, even if $D = \Theta(\log n)$. Thus, in quantum CONGEST networks, computing weighted diameter and weighted radius of graphs with small $D$ is strictly harder than unweighted ones due to Le Gall and Magniez’s $\tilde{O} \left( \sqrt{\pi D} \right)$-round algorithm for unweighted diameter/radius [12].

1 Introduction

Quantum distributed computing has received great attention in the past decade [4, 24, 10, 12, 13, 18, 19, 7, 20, 14]. A large body of work has been devoted to investigating the quantum advantages in distributed computing. In this paper, we are concerned with the CONGEST networks, which are one of the most fundamental models in distributed computing. In a classical CONGEST network, the nodes synchronously exchange classical messages, and each channel has $O(\log n)$-bit bandwidth, where $n$ is the number of nodes in the network. Quantum CONGEST networks were first introduced by Elkin, Klauck, Nanongkai, and Pandurangan [10], where the only difference is that the nodes exchange quantum messages and the bandwidth of each channel is $O(\log n)$ qubits. The round complexity of diameter and radius of unweighted graphs in classical CONGEST networks has been extensively studied [2, 3, 17, 22, 11, 15]. Le Gall and Magniez [12] proved that quantum communication may save the round complexity in CONGEST networks if the graph has a low diameter.

In this paper, we further investigate the round complexity of computing the diameters and radius of weighted graphs in quantum CONGEST networks. We prove that quantum communication may also save the round complexity for both problems.

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1.1 Our Results

The following is one of our main results which asserts that quantum communication may save the round complexity for computing the weighted diameter and radius of a graph given that the graph has a low unweighted diameter.

**Theorem 1.1.** There exists a $\tilde{O} \left( \min \{ n^{9/10} D^{3/10}, n \} \right)$-round distributed algorithm computing a $(1 + o(1))$-approximation of the weighted diameter/radius with probability at least $1 - 1/poly(n)$, in the quantum CONGEST model, where $D$ denotes the unweighted diameter.

Holzer and Pinsker in [16] and Abboud, Censor-Hillel and Khoury in [2] proved that $(2 - o(1))$-approximating the diameter and $(3/2 - \varepsilon)$-approximating the (even unweighted) radius in the classical CONGEST network require $\Omega(n)$ rounds, even when $D$ is constant. Therefore, Theorem 1.1 exhibits the advantages of quantum communication over classical communication in approximating the weighted diameter/radius when $D = o(n^{1/3})$.

We prove Theorem 1.1 by applying the framework of distributed quantum optimization introduced by Le Gall and Magniez in [12]. Note that the diameter and radius are the maximum and minimum of eccentricities respectively. It will not give a sublinear-time algorithm if we simply apply a quantum search algorithm, because evaluating the eccentricity of one node takes $\Theta(\sqrt{n})$ rounds (the lower bound is due to [10]), and the searching process should require another $\Theta(\sqrt{n})$ times of evaluation (the number of nodes with maximum/minimum eccentricity maybe $O(1)$). Thus the number of rounds in total will be $\Theta(n)^1$. Our algorithm is inspired by Nanongkai’s algorithm [21] for approximating the weighted shortest paths in a classical network. The algorithm constructs several small vertex sets and searches the node achieving the maximum/minimum eccentricity within those sets, which turns out to be a good approximation of diameter/radius. Our algorithm quantizes Nanongkai’s algorithm using the standard technique [5] and further combines with the framework of distributed quantum optimization in [12].

We also prove lower bounds for approximating weighted diameter and radius.

**Theorem 1.2.** Any algorithm computing a $(3/2 - o(1))$-approximation of the weighted diameter/radius requires $\tilde{\Omega}(n^{2/3})$ rounds, in the quantum CONGEST model, even when $D = \Theta(\log n)$.

The hardness of both problems is proved via the communication complexity of quantum Server models. The Server model is a variant of two-party communication complexity models introduced in [10]. Combining with the graph gadget in [2], we get a reduction from the communication complexity of certain read-once functions to the round complexity of approximating the weighted diameter and radius. We further apply a lifting theorem of quantum communication complexity [10] to obtain the desired lower bounds.

Compared with Le Gall and Magniez’s algorithm [12] for unweighted diameter/radius with $O(\sqrt{nD})$ rounds, Theorem 1.2 says that computing weighted diameter/radius is strictly harder than unweighted diameter/radius, when $D$ is small. While in the classical setting, computing weighted and unweighted diameter/radius have the same round complexity $\Theta(n)$ [2, 6].

1.2 Related Works

A series of works started with the distance computation in the classical CONGEST network. Earlier, Frischknecht, Holzer, and Wattenhofer [11] showed that computing the diameter of an un-
Table 1: Complexity of computing diameter and radius in the CONGEST model.

| Problem | Variant | Approx. | Upper bound $\tilde{O}(-)$ | Lower bound $\tilde{\Omega}(-)$ |
|---------|---------|---------|-----------------------------|-------------------------------|
|          |         |         | Classical | Quantum | Classical | Quantum |
| diameter | exact   | $n$ [17, 22] | $\sqrt{nD}$ [12] | $n$ [11] | $\sqrt{nD^2} + \sqrt{n}$ [20] |
|          | $3/2 - \epsilon$ | $n$ | $\sqrt{nD}$ | $n$ | $\sqrt{nD} + D$ [12] |
|          | $3/2$ | $\sqrt{nD}$ | open | open |
|          | weighted | exact | $n$ | $n$ | $n^{3/3}$ (This work) |
|          | weighted | $1, 3/2$ | $\min \left\{ n^{9/10}D^{3/10}, n \right\}$ (This work) | $n$ | $n^{2/3}$ (This work) |
|          | weighted | $2 - \epsilon$ | $\min \left\{ n^{9/10}D^{3/10}, n \right\}$ | $n$ [16] | $\sqrt{nD}$ |
|          | weighted | $2$ | $\sqrt{\piD} + D$ | $\sqrt{\piD} + D$ | open | open |
| radius   | exact   | $n$ [17, 22] | $\sqrt{nD}$ | $n$ | $\sqrt{nD^2} + \sqrt{n}$ |
|          | $3/2 - \epsilon$ | $n$ | $\sqrt{nD}$ | $n$ | $\sqrt{nD} + D$ |
|          | $3/2$ | $\sqrt{nD}$ | open | open |
|          | weighted | exact | $n$ | $n$ | $n^{3/3}$ |
|          | weighted | $1, 3/2$ | $\min \left\{ n^{9/10}D^{3/10}, n \right\}$ (This work) | $n$ | $n^{2/3}$ (This work) |
|          | weighted | $2$ | $\sqrt{\piD} + D$ | $\sqrt{\piD} + D$ | open | open |

weighted graph with constant diameter requires $\tilde{\Omega}(n)$ rounds, which is tight up to logarithmic factors since even computing All-Pairs Shortest Paths (ASAP) on an unweighted graph can be resolved in $O(n)$ rounds [17, 22]. Abboud, Censor-Hillelet, and Khoury [2] later gave the same lower bound of $\tilde{\Omega}(n)$ for $(3/2 - \epsilon)$-approximating the diameter/radius in sparse networks. Bernstein and Nanongkai [6] provided a $\tilde{O}(n)$-round algorithm computing the exact APSP on any weighted graph. As a result, computing unweighted diameter/radius and weighted diameter/radius (exactly or with a small approximation ratio) have an almost tight complexity of $\tilde{\Omega}(n)$ in the classical CONGEST network. If a larger approximation ratio is allowed, there are $\tilde{O}(\sqrt{nD} + D)$-round algorithms for $3/2$-approximating the diameter/radius on any unweighted graph [15, 3]. Besides, Chechik and Mukhtar [8] showed a $\tilde{O}(\sqrt{\piD} + D)$-round algorithm computing Single-Source Shortest Paths (SSSP) exactly on any weighted graph, which also gives a 2-approximation of the diameter/radius.

As for the quantum setting, while quantum computation offers advantages over classical computation in various settings such as query complexity and two-party communication complexity, the power of quantum computation in distributed computing has not been fully explored. In the quantum CONGEST network, Elkin et al. [10] gave negative results for several problems such as minimum spanning tree, minimum cut, and SSSP, i.e., quantum communication does not speed up distributed algorithms for these problems. Le Gall and Magniez [12] presented a $\tilde{O}\left(\sqrt{\piD}\right)$-round algorithm computing the diameter/radius on any unweighted graph, along with a $\tilde{O}\left(\sqrt{\piD} + D\right)$-round algorithm $3/2$-approximating the diameter. They also proved a $\tilde{\Omega}(\sqrt{\pi} + D)$-lower bound for computing the unweighted diameter, which was later improved to $\tilde{\Omega}\left(\sqrt{nD^2} + \sqrt{n}\right)$ by Magniez and Nayak [20]. The above results are listed on Table 1.
2 Preliminaries

2.1 Graph Notations

Given a weighted graph \( (G, w) \) where \( G = (V, E) \) and \( w : E \rightarrow \mathbb{N}^+ \). The length of a path is defined to be the sum of weights of edges on it, and the distance between nodes \( u \) and \( v \), denoted by \( d_{G,w}(u,v) \), is the least length over all paths between them. The eccentricity of a node \( u \) is denoted by \( e_{G,w}(u) = \max_{v \in V} d_{G,w}(u,v) \). The radius of weighted graph \( (G,w) \), denoted by \( R_{G,w} \), is the minimum eccentricity over all nodes, i.e., \( R_{G,w} = \min_{u \in V} e_{G,w}(u) \), while the diameter of \( (G,w) \), denoted by \( D_{G,w} \), is the maximum eccentricity of nodes, or equally, the maximum distance between any two nodes, i.e., \( D_{G,w} = \max_{u,v \in V} d_{G,w}(u,v) \). The unweighted diameter of graph \( G \) is denoted by \( D_G = D_{G,w^*} \) where \( w^*(e) = 1 \) for all \( e \in E \), which is an essential parameter when \( G \) represents the underlying graph of a distributed network.

2.2 CONGEST Model

In the classical CONGEST model, the communication network is a graph \( G = (V, E) \) with \( n \) nodes, and every node is assigned with a unique identifier. Each node represents a processor with unlimited computational power, i.e., the consumption of any local computation in a single processor is ignored. Each edge connecting two nodes represents a communication channel with \( B = O(\log n) \) bits of bandwidth. In this article, we further consider the weighted graph \( (G,w) \) as underlying network, where the weight of each edge is initially known to both of its endpoints. For quantum version of the CONGEST model defined in [10], adjacent nodes are allowed to exchange qubits (quantum bits), i.e., the classical channels are now quantum channels with the same bandwidth \( B = O(\log n) \). Each node can locally do some quantum computation, and distinct nodes may own qubits with entanglement. In this paper, we assume that initially all nodes do not share any entanglement, but the nodes can, for example, locally create a pair of entangled qubits, and send one to others.

For both classical and quantum CONGEST models, the algorithm is implemented round by round in a synchronous manner. In each round, each node sends/receives a message of \( O(\log n) \) (qu)bits to/from each neighbor, and then does local computation according to local knowledge. The algorithm halts when all nodes halt, and at the end of the algorithm, each node has its own output. We say an algorithm computes the diameter/radius if all nodes output the correct answer. The round complexity of an algorithm in this model is defined to be the number of communication rounds needed. And the round complexity of a distributed problem is the least round complexity of any algorithm solving it. Our focus here is the distance problems, mainly the computation of diameter and radius mentioned in Section 2.1.

2.3 Server Model

The Sever model is a variant of the two-party communication model, which was introduced by Elkin et al. [10] to prove lower bounds in the CONGEST model. There are three players in the Server model: Alice, Bob, and the server. Alice and Bob receive the inputs \( x \) and \( y \) respectively, and want to compute \( F(x,y) \) for some function \( F \). The server receives no input. Alice and Bob can exchange messages with the server. The catch here is that the server can send messages for free. Thus, the communication complexity counts only messages sent by Alice and Bob. Note that Alice and Bob can talk to each other by considering the server as a communication channel, so any protocol in the traditional two-party communication model can be implemented in the Server model with the same complexity.
For a two-argument function $F$ and $0 \leq \epsilon < 1$, we let $Q^\epsilon_F$ denote the communication complexity (in the quantum setting) of computing $F$ where for any inputs $x, y$, the algorithm must output $F(x, y)$ with probability at least $1 - \epsilon$. For Boolean function $f : \{0, 1\}^n \to \mathbb{R}$ and $0 \leq \epsilon < 1$, the $\epsilon$-approximate degree of $f$, denoted by $\deg_\epsilon(f)$, is the smallest degree of any polynomial $p$ that $\epsilon$-represents $f$, i.e., $|p(x) - f(x)| \leq \epsilon$ for any input $x \in \{0, 1\}^n$.

3 Algorithm

We first introduce the framework of distributed quantum optimization in [12]. Given function $f : X \to \mathbb{Z}$, where $X$ is a finite set, let $G = (V, E)$ be a network with a pre-defined node leader $v \in V$. We write $|\phi\rangle_v$ to denote a state in the memory space of node $v$. A specific register $|\cdot\rangle_1$ called internal and the control of the algorithm are centralized by the node leader. Assume that the following three quantum procedures are given as black boxes.

- **Initialization**: Prepare an initial state $|0\rangle_1 |\text{init}\rangle$ with some pre-computed information $|\text{init}\rangle$.

- **Setup**: Produce a superposition from the initial state:

$$|0\rangle_1 |\text{init}\rangle \mapsto \sum_{x \in X} \alpha_x |x\rangle_1 |\text{data}(x)\rangle |\text{init}\rangle,$$

where the $\alpha_x$’s are arbitrary amplitudes and data$(x)$ are information depending on $x$.

- **Evaluation**: Perform the transformation

$$|x, 0\rangle_1 |\text{data}(x)\rangle |\text{init}\rangle \mapsto |x, f(x)\rangle_1 |\text{data}(x)\rangle |\text{init}\rangle.$$

The following lemma provides an algorithm to search $x \in X$ with high value $f(x)$ given the three procedures above.

**Lemma 3.1** (Theorem 2.4 in [12]). Assume that Initialization can be implemented within $T_0$ rounds in the quantum CONGEST model, and that unitary operators Setup and Evaluation and their inverses can be implemented within $T$ rounds. Let $\rho > 0$ be such that $\sum_{x \in X} |f(x) - M| |\alpha_x|^2 \geq \rho$ where $M$ is unknown to all nodes. Then, for any $\delta > 0$, the node leader can find, with probability at least $1 - \delta$, some element $x$ such that $f(x) \geq M$, in $T_0 + O(\sqrt{\log(1/\delta)}/\rho) \times T$ rounds.

The three procedures will be described as deterministic or randomized procedures that combine the subroutines provided by Nanongkai [21] (also presented in Appendix A). They can be quantized using the standard technique [5], with potentially additional garbage whose size is of the same order as the initial memory space.

Given a weighted graph $(G, w)$ where $G = (V, E)$ is a network and $w : E \to \mathbb{N}$, we show a quantum algorithm approximating $D_{G,w}$ and $R_{G,w}$ by proving Theorem 1.1. We only show the algorithm approximating the diameter. The proof for radius is basically the same except that it finds the minimum (approximate) eccentricity instead of the maximum one.

We choose the parameters throughout this section.

$$\epsilon = 1/\log n, \quad r = n^{2/5}D_G^{1/5}, \quad \ell = n \log n/r, \quad k = \sqrt{D_G}.$$  

\footnote{Although Le Gall and Magniez write a slightly weaker statement, the lemma we claim here can be proven by the same argument in [12].}
As mentioned in Section 1.1, finding a node with maximum eccentricity among all nodes by directly applying a quantum search algorithm can hardly be done in $o(n)$ rounds. We instead try to find a vertex set containing a node with maximum approximate eccentricity among $n$ vertex sets $S_1, \ldots, S_n$, and then search such a node in this vertex set. Each set $S_i$ for $i \in [1, n]$ is sampled by having each node $v \in V$ join it independently with probability $r/n$. For such a random set and a node $s$ in it, Nanongkai showed in [21] an efficient classical procedure to approximate its eccentricity (actually every node $v \in V$ can know an approximation of the distance from $s$ to $v$).

### 3.1 Computation of Approximate Eccentricity

For convenience, we need to introduce several graph notations. Given a weighted graph $(G, w)$, the hop distance between nodes $u$ and $v$, denoted by $h_{G,w}(u,v)$, is the minimum number of edges over all shortest paths between them. The hop diameter of the weighted graph, denoted by $H_{G,w}$, is the maximum hop distance between any two nodes. For $\ell > 0$, the $\ell$-hop distance between $u$ and $v$, denoted by $d_{G,w}^\ell(u,v)$, is the least length over all paths between them containing at most $\ell$ edges. Note that $d_{G,w}^\ell(u,v) = d_{G,w}(u,v)$ when $h_{G,w}(u,v) \leq \ell$.

In general, Nanongkai [21] would approximate the bounded-hop distance, and sample a random set of key nodes as skeleton. Then it could approximate the distance from any key node $s$ to any node $v$ since, with high probability, any shortest path from $s$ to $v$ can be partitioned into bounded-hop shortest paths between key nodes, along with a tail path from some key node to $v$, as long as the number of key nodes is sufficiently large.

Here we only list the necessary definitions of approximate bounded-hop distance, approximate distance, and approximate eccentricity. We claim that these are good approximations. The algorithms evaluating these quantities are presented in Appendix A, and the detailed proof should be found in [21, arXiv version]. Note that we are given a weighted graph $(G,w)$ where $G = (V,E)$ and $w : E \to \mathbb{N}^+$.

#### Lemma 3.2 (Theorem 3.3 in [21])

Given an integer $\ell > 0$. For integer $i \geq 0$, define $w_i : E \to \mathbb{N}^+$ where $w_i(e) = \left\lceil \frac{2h_{G,w}(e)}{\ell + 2} \right\rceil$ for $e \in E$. For any $u,v \in V$, the approximate bounded-hop distance is defined as

$$\overline{d}_{G,w}^\ell(u,v) = \min_i \left\{ d_{G,w_i}(u,v) : \frac{\varepsilon}{2\ell} : d_{G,w_i}(u,v) \leq \left(1 + \frac{\varepsilon}{2}\right)\ell \right\}.$$  

Then $d_{G,w}(u,v) \leq \overline{d}_{G,w}^\ell(u,v) \leq (1 + \varepsilon)d_{G,w}^\ell(u,v)$.

#### Lemma 3.3 (Theorem 4.2 in [21])

Given a vertex set $S \subseteq V$. Let the weighted complete graph $(G'_S, w'_S)$ be such that

$G'_S = (S, (\set{S})_{\ell}), w'_S : (\set{S})_{\ell} \to \mathbb{N}^+$,

$w'_S([u,v]) = \overline{d}_{G,w}^\ell(u,v), \forall [u,v] \in (\set{S})_{\ell}$.

For node $v \in S$, let $N^K_S(v)$ be the set of the $k$ nodes with the least distance from $v$ on $(G'_S, w'_S)$. And let the weighted complete graph $(G''_S, w''_S)$ be such that

$G''_S = (S, (\set{S})_{\ell}), w''_S : (\set{S})_{\ell} \to \mathbb{N}^+$,

$w''_S([u,v]) = \begin{cases} d_{G'_S,w'_S}(u,v), & u \in N^K_S(v) \text{ or } v \in N^K_S(u) \\ w'_S([u,v]), & \text{otherwise} \end{cases}, \forall [u,v] \in (\set{S})_{\ell}$.
For any \( s \in S \) and \( v \in V \), the approximate distance is defined as

\[
\tilde{d}_{G,w,S}(s,v) = \min_{u \in S} \left\{ \tilde{d}^{\lfloor |S|/k \rfloor}_{G',w_S'}(s,u) + \tilde{d}^\ell_{G,w}(u,v) \right\}.
\]

If \( \ell = n \log n/r \) and \( S \) is sampled by having each node \( v \in V \) join it independently with probability \( r/n \), then \( d_{G,w}(s,v) \leq \tilde{d}_{G,w,S}(s,v) \leq (1 + \varepsilon)^2 d_{G,w}(s,v) \) for all \( s \in S \) and \( v \in V \), with probability at least \( 1 - 2^{-cn^r} \), for some constant \( c > 0 \) and sufficiently large \( n \).

**Remark.** We briefly explain why \( \tilde{d}_{G,w,S}(\cdot) \) is a good approximation. By the choice of \( \ell \) and \( S \), Lemma 4.3 in [21] says that, with high probability, any \( s \in S \), \( v \in V \) and shortest path \( (s \sim v) \) on \( (G,w) \) is of the form \( (s = s_1 \sim \cdots \sim s_m = u \sim v) \) such that \( s_i \in S \) for \( i \in [1,m] \), \( h_{G,w}(s_{i-1},s_i) \leq \ell \) for \( i \in [2,m] \), and \( h_{G,w}(u,v) \leq \ell \). Apparently \( \tilde{d}_{G,w,S} \geq d_{G,w}(s,v) \). On the other side,

\[
\tilde{d}_{G,w,S}(s,v) = \tilde{d}^{\lfloor |S|/k \rfloor}_{G',w_S'}(s,u) + \tilde{d}^\ell_{G,w}(u,v)
\]

\[
\leq (1 + \varepsilon)^2 d_{G',w_S'}(s,u) + \tilde{d}^\ell_{G,w}(u,v)
\]

\[
= (1 + \varepsilon)^2 d_{G',w_S'}(s,u) + \tilde{d}^\ell_{G,w}(u,v)
\]

\[
\leq (1 + \varepsilon)^2 \sum_{i=2}^m w_S'(s_{i-1},s_i) + \tilde{d}^\ell_{G,w}(u,v)
\]

\[
\leq (1 + \varepsilon)^2 \sum_{i=2}^m \tilde{d}^\ell_{G,w}(s_{i-1},s_i) + \tilde{d}^\ell_{G,w}(u,v)
\]

\[
\leq (1 + \varepsilon)^2 \left( \sum_{i=2}^m d_{G,w}(s_{i-1},s_i) + d_{G,w}(u,v) \right)
\]

\[
= (1 + \varepsilon)^2 \left( \sum_{i=2}^m d_{G,w}(s_{i-1},s_i) + d_{G,w}(u,v) \right)
\]

\[
= (1 + \varepsilon)^2 d_{G,w}(s,v).
\]

The second and sixth lines are due to Lemma 3.2. The third line is due to Theorem 3.10 in [21], which says that \( H_{G',w_S'} < 4|S|/k \) since \((G',w_S')\) is the \( k\)-shortcut graph of \((G,w_S')\).

For \( i \in [1,n] \), we rewrite \( G'_{S_i}, w'_{S_i}, \tilde{d}_{G,w,S_i}(\cdot) \) as \( G_{S_i}', w_{S_i}', \tilde{e}_{G,w,i}(\cdot) \) for short. For any \( s \in S_i \), the approximate eccentricity is defined as \( e_{G,w,i}(s) = \max_{v \in V} \tilde{d}_{G,w,i}(s,v) \). Define two good events:

- **Good-Scale:** For all \( i \in [1,n] \), \( |S_i| = \Theta(r) \). Besides, let \( v^* \in V \) be a node with maximum eccentricity, i.e., \( e_{G,w}(v^*) = D_{G,w} \) then \( v^* \) joins \( \beta = \Theta(r) \) sets \( S_{i_1}, \cdots, S_{i_k} \).

- **Good-Approximation:** For all \( i \in [1,n] \) and \( s \in S_i, v \in V \), \( d_{G,w}(s,v) \leq \tilde{d}_{G,w,i}(s,v) \leq (1 + \varepsilon)^2 d_{G,w}(s,v) \), thus \( e_{G,w}(s) \leq \tilde{e}_{G,w,i}(s) \leq (1 + \varepsilon)^2 e_{G,w}(s) \).

By Chernoff inequality and a union bound, the event Good-Scale occurs with probability at least \( 1 - 1/poly(n) \). By Lemma 3.3 and a union bound, the event Good-Approximation occurs with probability at least \( 1 - 1/poly(n) \). Therefore, we can assume that the two events all happen in the following context.
3.2 Quantization

For each \( i \in [1, n] \), we define \( f_i : S_i \to \mathbb{Z} \) where \( f_i(s) = \bar{e}_{G,w,i}(s) \) for \( s \in S_i \), and \( f : [1, n] \to \mathbb{Z} \) where \( f(i) = \max_{s \in S_i} f_i(s) \) for \( i \in [1, n] \).

**Lemma 3.4.** The number of \( i \in [1, n] \) satisfying \( f(i) \geq D_{G,w} \) is \( \Theta(r) \). Moreover, \( f(i) \leq (1 + \varepsilon)^2 D_{G,w} \) for all \( i \in [1, n] \).

**Proof.**

\[
\begin{align*}
  f(i) & = \max_{s \in S_i} \bar{e}_{G,w,i}(s) \geq \max_{s \in S_i} e_{G,w}(s) \geq e_{G,w}(v^*) = D_{G,w}, \quad \forall j \in [1, \beta]; \\
  f(i) & = \max_{s \in S_i} \bar{e}_{G,w,i}(s) \leq \max_{s \in S_i} (1 + \varepsilon)^2 e_{G,w}(s) \leq (1 + \varepsilon)^2 D_{G,w}, \quad \forall i \in [1, n].
\end{align*}
\]

\( \square \)

**Lemma 3.5.** Given \( i \in [1, n] \), there exists a quantum procedure performing the transformation

\[
\bigotimes_{v \in V} |i\rangle_v |0\rangle_{\text{leader}} \mapsto \bigotimes_{v \in V} |i\rangle_v |f(i)\rangle_{\text{leader}}
\]

in the quantum CONGEST model, and taking \( \tilde{O}(D_G + \frac{n}{\varepsilon^r} + rk + \sqrt{r} (\varepsilon^r \cdot D_G + r)) \) rounds, with probability at least \( 1 - \frac{1}{\text{poly}(n)} \).

**Proof.** We give the quantum procedure maximizing \( f_i \) (thus evaluating \( f(i) \)) by following the framework of distributed quantum optimization:

- **Initialization**: Perform the transformation

  \[
  \bigotimes_{v \in V} |i\rangle_v \mapsto \bigotimes_{v \in V} |i\rangle_v |0\rangle_{\text{init}_i},
  \]

  where

  \[
  |\text{init}_i\rangle = \bigotimes_{v \in V, u \in S_i} |d_{G,w}(u, v)\rangle_v |G_i''', w_i''\rangle,
  \]

  and \( d_{G,w}(u, v) \) is given in Lemma 3.2, \( G_i''' \) and \( w_i''' \) are given in lemma 3.3.

- **Setup**: Perform the transformation

  \[
  \bigotimes_{v \in V} |i\rangle_v |0\rangle_{\text{init}_i} \mapsto \bigotimes_{v \in V} |i\rangle_v \left( \sum_{s \in S_i} \frac{1}{|S_i|} |s\rangle_I |\text{data}_i(s)\rangle \right) |\text{init}_i\rangle,
  \]

  where \( |\text{data}_i(s)\rangle = \bigotimes_{v \in V} |s\rangle_v \bigotimes_{v \in V, u \in S_i} |d_{G,w}''(s, u)\rangle_v \).

- **Evaluation**: Perform the transformation

  \[
  \bigotimes_{v \in V} |i\rangle_v (|s, 0\rangle_I |\text{data}_i(s)\rangle) |\text{init}_i\rangle \mapsto \bigotimes_{v \in V} |i\rangle_v (|s, f_i(s)\rangle_I |\text{data}_i(s)\rangle) |\text{init}_i\rangle.
  \]

We now analyze the round complexity:
• In $\widetilde{O}(D_G + \frac{n}{\epsilon_T} + r)$ rounds, each $v \in V$ can know $\tilde{d}_{G,w}^f(u,v)$ for each $u \in S_i$, with high probability, due to Lemma A.2. After that, the overlay network $(G''', w''')$ can be embedded in $O(D_G + r k)$ rounds due to Lemma A.3 (we say that the network $G = (V, E)$ embeds an overlay network $G' = (V', E')$ with a weight function $w': E' \to \mathbb{N}^+$ if $V' \subseteq V$ and for each $v \in V'$, it stores each $e \in E'$ incident to $v$ along with $w'(e)$ in the local memory). Therefore, the procedure Initialization, can be implemented in $T_0 = \widetilde{O}(D_G + \frac{n}{\epsilon_T} + r k)$ rounds.

• The node leader can collect $S_i$ in $O(D_G + r)$ rounds. It then prepares the quantum state $\sum_{s \in S_i} \frac{1}{|S_i|} |s\rangle_1$ and broadcasts to all nodes using CNOT copies, in $O(D_G)$ rounds. Thus, the transformation

$$
\bigotimes_{v \in V} |i\rangle_v |0\rangle_{1|\text{init}_i} \mapsto \bigotimes_{v \in V} |i\rangle_v \left(\sum_{s \in S_i} \frac{1}{|S_i|} |s\rangle_1 \bigotimes_{v \in V} |s\rangle_v\right)|\text{init}_i\rangle
$$

can be implemented in $O(D_G + r)$ rounds. Besides, the transformation

$$
\bigotimes_{v \in V} |i\rangle_v \bigotimes_{s \in V} |s\rangle_v |\text{init}_i\rangle \mapsto \bigotimes_{v \in V} |i\rangle_v \bigotimes_{s \in V} |s\rangle_v \bigotimes_{s \in V, u \in S_i} |d_{G''', w'''} |s\rangle_v\rangle_v |\text{init}_i\rangle
$$

can be implemented in $T_1 = \widetilde{O}\left(\frac{r}{\epsilon_T} \cdot D_G + r\right)$ rounds since Lemma A.4 implies that, after the overlay network $(G''', w''')$ is embedded, each $v \in V$ can know $\tilde{d}_{G''', w'''}(s, u)$ for each $u \in S_i$ within $T_1$ rounds. Therefore, the procedure Setup, can be implemented in $T_1 = \widetilde{O}\left(\frac{r}{\epsilon_T} \cdot D_G + r\right)$ rounds.

• For the procedure Evaluation, recall that $f_i(s) = \max_{v \in V} \tilde{d}_{G, w, i}(s, v)$ where

$$
\tilde{d}_{G, w, i}(s, v) = \min_{u \in S_i} \left\{ \frac{|d_{G''', w'''}(s, u)|}{k} + \tilde{d}_{G, w}^f(u, v) \right\}.
$$

By definition, for any $v \in V$ and $u \in S_i$, $d_{G''', w'''}(s, u)$ and $\tilde{d}_{G, w}^f(u, v)$ have been stored in the local memory of $v$, i.e., $|\rangle_v$. Thus, each $v \in V$ can locally compute $d_{G, w, i}(s, v)$, and the node leader can compute the maximum by converge-casting in $O(D_G)$ rounds. So the procedure Evaluation, can be implemented in $T_2 = O(D_G)$ rounds.

By Lemma 3.1, there exists a quantum procedure maximizing $f_i$ in $\widetilde{O}(T_0 + \sqrt{T_1 + T_2})$ rounds with high probability.

**Proof of Theorem 1.1.** We give a quantum procedure maximizing $f$ also by following the framework of distributed quantum optimization:

• **Initialization** is a classical procedure which samples vertex sets $S_1, \ldots, S_n$, and $|\text{init}\rangle$ represents the corresponding classical information.

• **Setup:** Perform the transformation

$$
|0\rangle_{1|\text{init}} \mapsto \sum_{i=1}^{n} \frac{1}{n} |i\rangle_{1} |\text{data}(i)\rangle |\text{init}\rangle,
$$

where $|\text{data}(i)\rangle = \bigotimes_{v \in V} |i\rangle_v$. 

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• **Evaluation**: Perform the transformation

\[ |i, 0\rangle_1 |\text{data}(i)\rangle |\text{init}\rangle \rightarrow |i, f(i)\rangle_1 |\text{data}(i)\rangle |\text{init}\rangle. \]

We now analyze the round complexity:

• \(S_1, \cdots, S_n\) are sampled locally in parallel, and the procedure Initialization is free, i.e., \(T_0 = 0\).

• The node leader prepares the quantum state \(\sum_{i=1}^{n} \frac{1}{\sqrt{2^n}} |i\rangle_1\) and broadcast using CNOT copies to all nodes. Therefore, the procedure Setup can be implemented in \(T_1 = O(D_G)\) rounds.

• The procedure Evaluation can be of \(T_2 = \bar{O} (D_G + \frac{n}{\varepsilon \cdot r} + rk + \sqrt{r} (\frac{r}{\varepsilon \cdot k} \cdot D_G + r))\) rounds by Lemma 3.5.

By Lemma 3.1 and Lemma 3.4, there exists a quantum procedure that find, with high probability, some \(i \in [1, n]\) such that \(D_{G,w} \leq f(i) \leq (1 + \varepsilon)^2 D_{G,w}\), in

\[\bar{O}(T_0 + \sqrt{n/r(T_1 + T_2)}) = \bar{O} (\sqrt{n/r} (D_G + \frac{n}{\varepsilon \cdot r} + rk + \sqrt{r} (\frac{r}{\varepsilon \cdot k} \cdot D_G + r)))\]
rounds. By the choice of the parameters in Eq. (1), Theorem 1.1 follows.

4 **Lower Bound**

To prove the lower bound on the round complexity of approximating (weighted) diameter in the quantum CONGEST model, we combine the reduction in [9, 23, 10] and the graph gadget in [2].

4.1 **Reduction from Server Model**

We briefly outline the reduction introduced by Elkin et al. [9, 23, 10] from the Server model to prove the hardness of certain graph problems such as diameter and radius. We will introduce a distributed network \(G = (V, E)\) and embed a certain two-argument function \(F : \{0, 1\}^k \times \{0, 1\}^k \rightarrow \{0, 1\}\) into the network by showing that if the instance on the network \(G\) has a low round-complexity protocol in the quantum CONGEST model, then there exists a low communication complexity protocol for \(F\) in the quantum Server model. Thus, the hardness of diameters and radius in the quantum CONGEST model is reduced to proving the lower bounds the communication complexity in the quantum Server model.

The network \(G = (V, E)\) is depicted by Figure 1 where \(V = V_S \cup V_A \cup V_B\) and \(E = E_S \cup E_A \cup E_B \cup E'\). We use \(G[U]\) to denote the subgraph induced by vertex set \(U \subseteq V\), then \(E_S, E_A, E_B\) are the edges in \(G[V_S], G[V_A], G[V_B]\) respectively. And \(E'\) denotes the edges between \(V_S\) and \(V_A \cup V_B\).

\(G[V_S]\) includes a full binary tree of height \(h\) and \(m\) disjoint paths of length \(2^h - 1\). Each of the \(2^h\) leaves of the binary tree is connected to the nodes on the paths as depicted in Figure 1. Suppose nodes of depth \(i\) on the tree are \(t_{i,1}, \cdots, t_{i,2^i}\) and nodes on the \(i\)-th path are \(p_{i,1}, \ldots, p_{i,2^i}\) from left to right. Then \(t_{h,1}, \ldots, t_{h,2^h}\) are the leaves of the binary tree in \(G[V_S]\). For each \(i \in [1, m]\) and \(j \in [1, 2^h]\), there is an edge between \(t_{h,j}\) and \(p_{i,j}\) Thus,

\[
V_S = \{t_{ij} : i \in [0, h], j \in [1, 2^i]\} \\
\cup \{p_{ij} : i \in [1, m], j \in [1, 2^i]\},
\]

\[
E_S = \{(t_{ij}, t_{i-1,j2^i}) : i \in [1, h], j \in [1, 2^i]\} \\
\cup \{p_{ij}, p_{i,j-1} : i \in [1, m], j \in [2, 2^m]\} \\
\cup \{p_{t_{h,j}, t_{p_{ij}}} : i \in [1, m], j \in [1, 2^m]\}.
\]
Alice simulates \( V \) in has \( T \leq 2 \). The protocol we will construct simulates the distributed algorithm round by round. Thus, it also simulates the right side. More formally, in the end of the \( k \)-th round, the server simulates \( p_{i,r} \) for \( 1 \leq i \leq m \). Those 2m edges are contained in \( E' \). The subgraphs \( G[V_A] \) and \( G[V_B] \) are decided by Alice’s input and Bob’s input, respectively.

The following lemma gives an efficient simulation of algorithms on network \( G \) by the protocols in the quantum Server model.

**Lemma 4.1 (Quantum Simulation Lemma).** Suppose Alice and Bob are given \( (V_A, E_A) \) and \( (V_B, E_B) \), respectively. For any \( T \)-round (\( T < 2^h/2 \)) distributed algorithm on network \( G \) described above, there exists a communication protocol for Alice and Bob in the quantum Server model to simulate the algorithm with communication complexity \( O(T \cdot h \cdot B) \), where \( B \) denotes the bandwidth in the CONGEST model.

**Proof.** The proof of Lemma 4.1 follows closely with the proof in [10, Proof of Theorem 3.5]. The protocol we will construct simulates the distributed algorithm round by round. Thus, it also has \( T < 2^h/2 \) rounds of communication. In the beginning, the server simulates all the nodes in \( V_S \) which are independent of Alice and Bob’s inputs. And in the end of the \( r \)-th round, the server simulates \( p_{i,1}, p_{i,2}, \ldots, p_{i,2^{h-r}} \) on the i-th path and nodes \( t_{h,1}, t_{h,2}, \ldots, t_{h,2^{2h-r}} \) along with their ancestors on the binary tree, while Alice simulates the nodes on the left side and Bob simulates on the right side. More formally, in the end of the \( r \)-th round, the server simulates

\[
\{ p_{i,j} : i \in [1, m], j \in [1 + r, 2^h - r] \} \cup \{ t_{i,j} : i \in [0, h], j \in \left[ 
\left\lceil (1 + r)/2^{h-i} \right\rceil, \left\lceil (2^h - r)/2^{h-i} \right\rceil \right] \}
\]

Alice simulates

\[
V_A \cup \{ p_{i,j} : i \in [1, m], j \in [1, 1 + r] \} \cup \{ t_{i,j} : i \in [0, h], j \in \left[ (1 + r)/2^{h-i} \right] \};
\]

Bob simulates

\[
V_B \cup \{ p_{i,j} : i \in [1, m], j \in (2^h - r, 2^h] \} \cup \{ t_{i,j} : i \in [0, h], j \in \left( (2^h - r)/2^{h-i}, 2^h \right) \}.
\]

We describe the simulation of the computation and communication of a processor \( v \) in the \( r \)-th round, and count the total communication complexity.
Hence, a total of $O(n \log n)$ messages, each of size $O(1)$, are sent from Alice or Bob to the server. 

### 4.2 Hardness of Approximating Diameter

We will use $G$ constructed above as a gadget to prove a lower bound on round complexity of approximating weighted diameter in the quantum CONGEST model. The specific graph depicted in Figure 2 will contain $n = (2^{h+1} - 1) + (2s + \ell)(2^h + 2) + 2 \cdot 2^s$ nodes, where parameters $h, s, \ell$ are chosen as follows throughout this section.

\[
\text{h is some even number, } s = 3h/2, \ell = 2^s - h. \tag{2}
\]

This choice makes $2^h = \tilde{\Theta}(n^{2/3}), 2^s = \tilde{\Theta}(n)$ and $\ell = \tilde{\Theta}(n^{1/3})$.

**Theorem 4.2 (Restated).** For any constant $\epsilon \in [0, \frac{1}{2}]$, any algorithm, with probability at least $\frac{11}{12}$, computing a $(\frac{3}{2} - \epsilon)$-approximation of the weighted diameter in the quantum CONGEST model requires $\Omega\left(\frac{n^{2/3}}{\log n}\right)$ rounds, even when the unweighted diameter is $\Theta(\log n)$, where $n$ denotes the number of nodes.

On network $G = (V, E)$ described in Section 4.1, we specify $G[V_A]$ and $G[V_B]$. Let

\[
\begin{align*}
V_A &= \{a_1, \cdots, a_{2^s}\} \cup \{a_1^0, a_1^1, \cdots, a_1^0, a_1^1\} \cup \{a_1^2, \cdots, a_2^0\}, \\
V_B &= \{b_1, \cdots, b_{2^s}\} \cup \{b_1^0, b_1^1, \cdots, b_2^0, b_2^1\} \cup \{b_3^0, \cdots, b_4^0\}.
\end{align*}
\]
Figure 2: Graph G for approximating diameter. The black edges are of weight 1; the blue edges are of weight $\alpha$; and weights of red edges are determined by inputs $x, y$, i.e., $w((a_i, a^*_j)) = \alpha$ if $x_{i,j} = 1$ and $w((a_i, a^*_j)) = \beta$ if $x_{i,j} = 0$, and $w((b_i, b^*_j)) = \alpha$ if $y_{i,j} = 1$ and $w((b_i, b^*_j)) = \beta$ if $y_{i,j} = 0$, for $i \in [1, 2^s]$ and $j \in [1, \ell]$.

The edges $E_A$, $E_B$ and $E'$ are specified as follows.

$$E_A = \{ (a_i, a^*_j) : i \in [1, 2^s], j \in [1, s] \}$$

$$\cup \{ (a_i, a^*_j), (a_i, a_j) : i, j \in [1, 2^s], i \neq j \},$$

$$E_B = \{ (b_i, b^*_j) : i \in [1, 2^s], j \in [1, s] \}$$

$$\cup \{ (b_i, b^*_j), (b_i, b_j) : i, j \in [1, 2^s], i \neq j \},$$

$$E' = \{ (a^*_i, p_{2i-1, 1}), (b^*_i, p_{2i-1, 2^h}) : i \in [1, s] \}$$

$$\cup \{ (a^*_i, p_{2i, 1}), (b^*_i, p_{2i, 2^h}) : i \in [1, s] \}$$

$$\cup \{ (a^*_i, p_{2s+i, 1}), (b^*_i, p_{2s+i, 2^h}) : i \in [1, \ell] \},$$

where $\text{bin}(i, j)$ denote the $j$-th bit in binary expression of integer $i - 1$.

The node pairs $(a^*_i, p_{2i-1, 1}), (a^*_i, p_{2i, 1}), (b^*_i, p_{2i, 2^h}), (b^*_i, p_{2i-1, 2^h})$ for $1 \leq i \leq s$, and $(a^*_i, p_{2s+i, 1}), (b^*_i, p_{2s+i, 2^h})$ for $1 \leq i \leq \ell$. 

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Lemma 4.3. the following lemma. An edge is contracted if the two endpoints are merged to one node, and the as there are at most $|V|$ contracting all edges of weight 1 due to the proof.

\[
\|G\|_{\alpha} = \sum_{i \in |V|} \text{weight}_i
\]

Proof. Recall that Alice and Bob receive inputs $x, y \in \{0, 1\}^{2s+\ell}$ respectively. $x$ and $y$ are indexed by $x_{i,j}$ and $y_{i,j}$ for $i \in [1, 2^s], j \in [1, \ell]$ where $s$ and $\ell$ are given in Eq. (2). For each $i \in [1, 2^s], j \in [1, \ell], w((a_i, a_j^\dagger)) = \alpha$ if $x_{i,j} = 1$ and $w((a_i, a_j^\dagger)) = \beta$ if $x_{i,j} = 0 (\alpha < \beta)$; weights of edges between $\{b_1, \ldots, b_2^s\}$ and $\{\lambda^*, \ldots, b_2^s\}$ are assigned according to $y$ in the same way (the red edges in Figure 2).

- The edges between the binary tree and the 2s + $\ell$ paths, those between $\{a_1, \ldots, a_2^s\}$ and $\{a_1^0, a_1^1, \ldots, a_s^0, a_s^1\}$, and those between $\{b_1, \ldots, b_2^s\}$ and $\{b_1^0, b_1^1, \ldots, b_s^0, b_s^1\}$ are of weight $\alpha$; weights of edges inside $G[[a_1, \ldots, a_2^s]]$ and $G[[b_1, \ldots, b_2^s]]$ are also $\alpha$ (the blue edges in Figure 2).

It is sufficient to analyze the diameter of graph after contracting all edges of weight 1 due to the following lemma. An edge is contracted if the two endpoints are merged to one node, and the adjacent edges of the two endpoints are incident to it. If there are parallel edges after contraction, we only keep the one with the lowest weight.

Lemma 4.3. Given a weighted graph $(G, w)$ where $G = (V, E)$ and $w : E \rightarrow \mathbb{N}^+$. Let $G'$ be the graph after contracting all edges of weight 1. We have $D_{G', w} \leq D_{G, w} \leq D_{G', w} + n$ and $R_{G', w} \leq R_{G, w} \leq R_{G', w} + n$, where $n = |V|$.

Proof. For any path $P$ in $G$, let $P'$ be the path in $G'$ obtained from $P$ after contraction. Then

\[
\text{length}(P') \leq \text{length}(P) \leq \text{length}(P') + n
\]
as there are at most $n - 1$ 1-weight edges. Thus we conclude the result.

For inputs $x, y \in \{0, 1\}^{2s+\ell}$ received by Alice and Bob, define

\[
F(x, y) = \bigwedge_{i \in [1, 2^s]} \left( \bigvee_{j \in [1, \ell]} (x_{i,j} \wedge y_{i,j}) \right),
\]
i.e., $F = \text{AND}_{2^n} \circ (\text{OR}_\ell \circ \text{AND}_{2^s})^{2s}$. We have the following lemma.

Lemma 4.4. $D_{G, w} \leq \max(2\alpha, \beta) + n$ if $F(x, y) = 1$, and $D_{G, w} \geq \min(\alpha + \beta, 3\alpha)$ otherwise.

Proof. The graph $G'$ after contraction is given in Figure 3. The binary tree is contracted to node $t$. The 2s + $\ell$ paths are contracted to nodes $a_1^0, a_1^1, \ldots, a_s^0, a_s^1$ and $a_1^*, \ldots, a_\ell^*$ respectively. Note that $b_i$ is connected to $a_j^{\bin(i,j)}$ for $i \in [1, 2^s], j \in [1, s]$. we list upper bounds of the distances between any two nodes $u$ and $v$ in $G'$ on Table 2 with the corresponding paths, except for the distance between $a_i$ and $b_i$ with $i \in [1, 2^s]$.

Regarding the distance between $a_i$ and $b_i$ for $i \in [1, 2^s]$, if there exists $j \in [1, \ell]$ such that $x_{i,j} = y_{i,j} = 1$, then $w((a_i, a_j^\dagger)) = w((b_i, a_j^\dagger)) = \alpha$ and $d_{G', w}(a_i, b_i) \leq 2\alpha$ because of the path $(a_i \rightarrow a_j^\dagger \rightarrow b_i)$ in $G'$. If there is no $j \in [1, \ell]$ such that $x_{i,j} = y_{i,j} = 1$, we claim that $d_{G', w}(a_i, b_i) \geq \max(2\alpha, \beta) + n$.
The diameter is at most $\max\{\alpha + \beta, 3\alpha\}$. Therefore, the diameter is at most $\max\{2\alpha, \beta\}$ if, for any $i \in [1, 2^\ell]$, there exists $j \in [1, \ell]$ such that $x_{i,j} = y_{i,j} = 1$, otherwise it is at least $\min\{\alpha + \beta, 3\alpha\}$. Therefore, the diameter is at most $\max\{2\alpha, \beta\}$ if, for any $i \in [1, 2^\ell]$, there exists $j \in [1, \ell]$ such that $x_{i,j} = y_{i,j} = 1$, otherwise it is at least $\min\{\alpha + \beta, 3\alpha\}$.

For any path between $a_i$ and $b_i$, if it contains exactly two edges, it is of the form $(a_i \rightarrow a_j^* \rightarrow b_i)$ for some $j \in [1, \ell]$ by the construction of $G'$, and it is of length at least $\alpha + \beta$ by the assumption. If it contains at least three edges, it is of length at least $3\alpha$.

If $F(x, y) = 0$, then for any $i \in [1, 2^\ell]$, there exists $j \in [1, \ell]$ such that $x_{i,j} = y_{i,j} = 0$. Hence,

$$d_{G',w}(a_i, b_i) = \min_{P \text{ path from } a_i \text{ to } b_i} \text{length}(P) \geq \min\{\alpha + \beta, 3\alpha\},$$

$$D_{G',w} = \max_{u,v} d_{G',w}(u, v) \geq d_{G',w}(a_i, b_i) \geq \min\{\alpha + \beta, 3\alpha\}.$$ Therefore, $D_{G',w} \geq D_{G',w} \geq \min\{\alpha + \beta, 3\alpha\}$ by Lemma 4.3.

Combining Lemma 4.1 and Lemma 4.4, we have a reduction from computing $F$ in the Server model to approximating diameter in the quantum CONGEST model. To prove the communication complexity of $F$ in the Server model, we adopt the following lemma.

**Lemma 4.5** (Lemma B.4 in [10], arXiv version). Function $\text{VER} : \{0, 1, 2, 3\} \times \{0, 1, 2, 3\} \rightarrow \{0, 1\}$ is defined by $\text{VER}(x, y) = 1$ if and only if $x + y$ is equivalent to 0 or 1 modulo 4, where $x, y \in \{0, 1, 2, 3\}$. Let
Lemma 4.6 (Theorem 6 in [1]). For any read-once formula $\ell : \{0, 1\}^k \to \{0, 1\}$, $\deg_{1/3}(\ell) = \Theta\left(\sqrt{k}\right)$.

Lemma 4.7. Given $s, \ell$ defined in Eq. (2) where $\ell$ is a multiple of 4, $F = \text{AND}_{2^s} \circ (\text{OR}_{\ell} \circ \text{AND}_2)_{2^s}$ with inputs $x, y \in \{0, 1\}^{2^s \cdot \ell}$, set

$$F(x, y) = \bigwedge_{i \in [1, 2^s]} \left( \bigvee_{j \in [1, \ell]} (x_{i,j} \land y_{i,j}) \right).$$

It holds that

$$Q_{1/12}^{sv}(F) = \Omega\left(\sqrt{2^s \cdot \ell}\right).$$

Table 2: Distance between nodes in $G'$. Let router be any node in $\{a_i^0, a_i^1, \cdots, a_i^0, a_i^1, \cdots, a_i^r\}$. $\text{adj}(i, j)$ denotes the integer after changing the $j$-th bit in binary expression of integer $i - 1$, and $\text{ind}(i, j)$ is the smallest $z \in [1, s]$ satisfying $\text{bin}(i, z) \neq \text{bin}(j, z)$.

| $u$ | $v$ | $d_{G', w}(u, v)$ | Path |
|-----|-----|-------------------|------|
| $t$ | router | $\leq \alpha$ | $(t \to v)$ |
| $a_i$ (i $\in [1, 2^s]$) | router | $\leq 2\alpha$ | $(t \to a_0^{\text{bin}(i,0)} \to a_i)$ |
| $b_i$ (i $\in [1, 2^s]$) | router | $\leq 2\alpha$ | $(t \to a_0^{\text{bin}(i,0)\oplus 1} \to b_i)$ |
| $a_j$ (j $\neq i, j \in [1, 2^s]$) | $a_j^{\text{bin}(i,j)\oplus 1}$ (j $\in [1, s]$) | $\leq \alpha$ | $(a_i \to a_j)$ |
| $b_j$ (j $\neq i, j \in [1, 2^s]$) | $b_j^{\text{bin}(i,j)}$ (j $\in [1, s]$) | $\leq 2\alpha$ | $(a_i \to b_j^{\text{bin}(i,j)})$ |
| $a_i^*$ (j $\in [1, \ell]$) | $a_j^{\text{bin}(i,j)}$ (j $\in [1, s]$) | $\leq \beta$ | $(a_i \to a_j^*)$ |
| $b_i$ (i $\in [1, 2^s]$) | $b_j^{\text{bin}(i,j)}$ (j $\in [1, s]$) | $\leq \alpha$ | $(b_i \to b_j)$ |
| $a_j^*$ (j $\in [1, \ell]$) | $a_j^{\text{bin}(i,j)}$ (j $\in [1, s]$) | $\leq \beta$ | $(b_i \to a_j^*)$ |
| router | router | $\leq 2\alpha$ | $(u \to t \to v)$ |

$f : \{0, 1\}^k \to \{0, 1\}$ be an arbitrary function. Then

$$Q_{\varepsilon}^{sv}(f \circ \text{VER}^k) \geq \frac{1}{2} \deg_{4\varepsilon}(f) - O(1)$$

for any $0 < \varepsilon < 1/4$.

A read-once formula, which consists of AND gates, OR gates, and NOT gates, is a formula in which each variable appears exactly once. We will need the following conclusion for approximate degree of read-once formulas.
Proof. The function $F$ can be rewritten as $F = f \circ \text{GDT}^{2^s \cdot \ell/4}$, where $f = \text{AND}_{2^s} \circ \text{OR}_{7/4}^s$ and $\text{GDT} = \text{OR}_4 \circ \text{AND}_{5}^4$. Obviously the function $f$ is a read-once formula. It can be seen that the function $\text{VER}$ is actually a promise version of the function $\text{GDT}$ where inputs $x, y \in \{0, 1\}^4$ satisfy

$$x \in \{0011, 1001, 1100, 0110\}, y \in \{0001, 0010, 0100, 1000\}.$$  

Thus, the lower bound for $f \circ \text{VER}^{2^s \cdot \ell/4}$ clearly implies the lower bound for $f \circ \text{GDT}^{2^s \cdot \ell/4}$. Therefore,

$$Q_{1/12}^{sv}(f \circ \text{GDT}^{2^s \cdot \ell/4}) \geq Q_{1/12}^{sv}(f \circ \text{VER}^{2^s \cdot \ell/4}) \geq \frac{1}{2} \deg_{1/3}(f) - O(1) = \Omega \left( \sqrt{2^s \cdot \ell} \right).$$  

The second inequality is due to Lemma 4.5 and the last inequality is due to Lemma 4.6.

Proof of Theorem 4.2. Let $A$ be a $T$-round algorithm ($T < 2^h/2$) in the quantum CONGEST model which, for any weighted graph $(G, w)$, computes a $(\frac{3}{2} - \varepsilon)$-approximation of $D_{G,w}$ (constant $\varepsilon \in (0, 1/2]$) with probability at least $11/12$. Alice and Bob, who receive $x, y \in \{0, 1\}^{2^s \cdot \ell}$, respectively, construct the network $G$ as described above with parameters $h, s, \ell$ given in Eq. (2). The number of nodes is

$$n = (2^h + 1 - 1) + (2s + \ell) (2^h + 2) + 2 \cdot 2^s = \Theta \left( 2^{3h/2} \right).$$  

And the unweighted diameter is $D_G = \Theta(h) = \Theta(\log n)$. Let $w$ be the weight function. Due to Lemma 4.1, they can simulate $A$ on $(G, w)$ in the quantum Server model with communication complexity $O(T \cdot h \cdot B)$ where $B$ denotes the bandwidth. With probability at least $\frac{11}{12}$, Alice and Bob output an approximation $\tilde{D}_{G,w}$ satisfying $D_{G,w} \leq \tilde{D}_{G,w} \leq (\frac{3}{2} - \varepsilon)D_{G,w}$. We set $\alpha = n^2$ and $\beta = 2n^2$. By Lemma 4.4,

$$\text{if } F(x, y) = 1, \tilde{D}_{G,w} \leq \left( \frac{3}{2} - \varepsilon \right)D_{G,w} \leq \left( \frac{3}{2} - \varepsilon \right) \left( \max(2\alpha, \beta) + n \right);$$
$$\text{if } F(x, y) = 0, \tilde{D}_{G,w} \geq D_{G,w} \geq \min(\alpha + \beta, 3\alpha) = 3n^2.$$  

For large enough $n$, Alice and Bob can distinguish whether $F(x, y) = 1$ or not with probability at least $\frac{11}{12}$ in the Server model, and thus $Q_{1/12}^{sv}(F) = O(T \cdot h \cdot B)$. Due to Lemma 4.7,

$$T = \Omega \left( \frac{\sqrt{2^s} \cdot \ell}{h \cdot B} \right) = \Omega \left( \frac{2^h}{h \cdot B} \right) = \Omega \left( \frac{n^{2/3}}{\log^2 n} \right),$$  

where the last equality is by the choice of $h$ and the the bandwidth $B = \Theta(\log n)$. Therefore, the round complexity of approximating diameter is $\Omega \left( \min \left\{ 2^h/2, \frac{n^{2/3}}{\log^2 n} \right\} \right) = \Omega \left( \frac{n^{2/3}}{\log^2 n} \right)$.  

4.3 Hardness of Approximating Radius

We choose the same set of parameters $h, s, \ell$ given in Eq. (2). The argument is very close to the one for diameter.

Theorem 4.8 (Restated). For any constant $\varepsilon \in (0, \frac{1}{2}]$, any algorithm, with probability at least $\frac{11}{12}$, computing a $(\frac{3}{2} - \varepsilon)$-approximation of radius in the quantum CONGEST model requires $\Omega \left( \frac{n^{2/3}}{\log^2 n} \right)$ rounds, even when the unweighted diameter is $\Theta(\log n)$, where $n$ denotes the number of nodes.
The weighted graph \((G, w)\) that we construct for showing hardness of approximating radius is almost the same except that we add a node \(a_0\) in \(V_A\) along with edges \((a_0, a_1), \ldots, (a_0, a_{2^s})\) of weight \(2\alpha\). Here we only show in Figure 4 the graph \(G'\) after contracting all edges of weight 1 (the green edges are the new-added edges).

Figure 4: Graph \(G'\) (after contraction) for approximating radius. The additional green edges are of weight \(2\alpha\). The eccentricity of any node, except \(a_i\) for \(i \in [1, 2^s]\), is at least \(3\alpha\); and the eccentricity of \(a_i\) is at most \(\max(2\alpha, \beta)\) if there exists \(j \in [1, \ell]\) such that \(x_{i,j} = y_{i,j} = 1\), otherwise it is at least \(\min(\alpha + \beta, 3\alpha)\). Therefore, the radius is at most \(\max(2\alpha, \beta)\) if there exist \(i \in [1, 2^s], j \in [1, \ell]\) such that \(x_{i,j} = y_{i,j} = 1\), otherwise it is at least \(\min(\alpha + \beta, 3\alpha)\).

For inputs \(x, y \in \{0, 1\}^{2^s-1}\) define

\[ F'(x, y) = \bigvee_{i \in [1,2^s], j \in [1,\ell]} (x_{i,j} \land y_{i,j}). \]

We have the following lemma.

**Lemma 4.9.** \(R_{G,w} \leq \max(2\alpha, \beta) + n\) if \(F'(x, y) = 1\), and \(R_{G,w} \geq \min(\alpha + \beta, 3\alpha)\) otherwise.

**Proof.** It suffices to estimate the radius of \((G', w)\) by Lemma 4.3. For any node \(v \notin \{a_0, a_1, \ldots, a_{2^s}\}\), \(d_{G',w}(a_0, v) \geq 3\alpha\). This is because that any path from \(a_0\) to \(v\) is of the form \((a_0 \rightarrow a_i \sim v)\) for some \(i \in [1, 2^s]\), where \(w((a_0, a_i)) = 2\alpha\), and the remaining edges on the path have total weight at least \(\alpha\). Therefore, \(c_{G',w}(v) \geq 3\alpha\) for any \(v \notin \{a_1, \ldots, a_{2^s}\}\). To estimate the eccentricity of \(a_i\) for \(i \in [1, 2^s]\), we have \(d(a_i, v) \leq \max(2\alpha, \beta)\) for any \(v \neq b_i\) as shown on Table 2, and \(d_{G',w}(a_i, b_i) \leq 2\alpha\) if there exists \(j \in [1, \ell]\) such that \(x_{i,j} = y_{i,j} = 1\), and \(d_{G',w}(a_i, b_i) \geq \min(\alpha + \beta, 3\alpha)\) otherwise.
Therefore, the complexity of approximating radius is \( \Omega(n) \) for large enough \( n \).

Proof of Theorem 4.8. Given quantified Server model. The function \( F \) can be rewritten as \( \log \alpha \) with probability at least \( \epsilon_n \). Due to Lemma 4.9, Alice and Bob can simulate \( F \) with communication complexity \( \Theta(2^h) \). The unweighted diameter \( D_G = \Theta(\log n) \). Due to Lemma 4.1, Alice and Bob can simulate \( A \) in the quantum Server model with communication complexity \( O(T \cdot h \cdot B) \). Then with probability at least \( \Omega(n) \), Alice and Bob compute \( G_{G,G} \), satisfying \( G_{G,G} \leq G_{G,G} \leq (2^{-\epsilon}) G_{G,G} \). We set \( \alpha = n^2 \) and \( \beta = 2n^2 \). Due to Lemma 4.9,

\[
\begin{align*}
&\text{if } F(x,y) = 1, G_{G,G} \leq 3n^2 - \left(2\epsilon n^2 - \left(\frac{3}{2} - \epsilon\right) n\right); \\
&\text{if } F(x,y) = 0, G_{G,G} \geq 3n^2.
\end{align*}
\]

For large enough \( n \), Alice and Bob can compute \( F \) with probability at least \( \Omega(n) \) in the Server model, and thus \( Q_{1/2}(F) = O(T \cdot h \cdot B) \). Due to Lemma 4.10, \( T = \Omega\left(\frac{n^{2/3}}{\log^2 n}\right) \). Therefore, the round complexity of approximating radius is \( \Omega\left(\min\left\{\frac{2^h}{\log n}, \frac{n^{2/3}}{\log n}\right\}\right) = \Omega\left(\frac{n^{2/3}}{\log^3 n}\right) \).
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A Toolkits in Nanongkai’s Algorithm

Let $G = (V, E)$ be a distributed network with a weight function $w : E \to \mathbb{N}^+$ and a pre-defined node leader $s \in V$. We assume that each node initially knows $n = |V|$ and $W = \max_{e \in E} w(e)$. The parameters $\epsilon, r, \ell, k$ are chosen the same as in Eq. (1). We follow the background of Section 3.1. Given a vertex set $S \subseteq V$, let $d_{G,w}^\ell(\cdot, (G_S, w_S'), N_S^k(\cdot), (G_{S}', w_{S}', \bar{d}_{G,w,S}(\cdot))$ be as defined in Lemma 3.2 and Lemma 3.3. The following lemmas and algorithms are summarized from [21, arXiv version].

Lemma A.1 (Theorem 3.2 in [21]). For $s \in V$ known to all nodes, there exists an algorithm (Algorithm 1) such that in $O(\ell/\epsilon)$ rounds, each $v \in V$ knows $\bar{d}_{G,w}^\ell(s, v)$, and during the whole computation, each node broadcasts $O(\log n)$ messages of size $O(\log n)$ to its neighbors.

Algorithm 1 Bounded-Hop SSSP $(G, w, s, \ell, \epsilon)$

Input: Network $(G, w)$, source node $s$ and parameters $\ell, \epsilon > 0$.

Output: Each node $v$ knows $\bar{d}_{G,w}^\ell(s, v)$.

1: Initially, $\bar{d}_{G,w}^\ell(s, v) \leftarrow \infty$ for each $v \in V$.
2: for $i = 0$ to $\log \frac{2nW}{\epsilon}$ do
3: \hspace{1em} Run bounded-distance SSSP with parameters $(G, w_i, s, (1 + 2/\epsilon)\ell)$ using Algorithm 2.
4: \hspace{1em} for each $v \in V$ do in parallel
5: \hspace{2em} if $d_{G,w_i}(s, v) \leq (1 + 2/\epsilon)\ell$ then
6: \hspace{3em} $\bar{d}_{G,w}^\ell(s, v) \leftarrow \min\{\bar{d}_{G,w}^\ell(s, v), d_{G,w_i}(s, v)\}$.
7: \hspace{2em} end if
8: \hspace{1em} end for
9: end for
Algorithm 2 Bounded-Distance SSSP \((G, w, s, L)\)

**Input:** Network \((G, w)\), source node \(s\) and parameter \(L > 0\).

**Output:** Each node \(v\) knows whether \(d_{G,w}(s, v) \leq L\), and if so, it further knows \(d_{G,w}(s, v)\).

1: Initially, \(d_{G,w}(s, s) \leftarrow 0\) and \(d_{G,w}(s, v) \leftarrow \infty\) for each \(v \neq s\).
2: Let \(t\) be the time this algorithm starts.
3: for round \(r = t\) to \(t + L\) do
4:    for each \(v \in V\) do in parallel
5:        for each message \((u, d_{G,w}(s, u))\) received in the previous round do
6:            if \(d_{G,w}(s, u) + w((u, v)) \leq L\) then
7:                \(d_{G,w}(s, v) \leftarrow \min\{d_{G,w}(s, v), d_{G,w}(s, u) + w((u, v))\}\).
8:        end if
9:    end for
10:    if \(d_{G,w}(s, v) = r - t\) then
11:        \(v\) broadcasts message \((v, d_{G,w}(s, v))\) to all neighbors.
12:    end if
13: end for
14: end for

**Lemma A.2** (Theorem 3.6 and Lemma 3.7 in [21]). There exist an algorithm (Algorithm 3) such that in \(\tilde{O}(D_G + \ell/\varepsilon + |S|)\) rounds, each node \(v \in V\) knows \(\tilde{d}^\ell_{G,w}(s, v)\) for each \(s \in S\), with probability of failure at most \(n^{-c}\), for any constant \(c > 0\) and sufficiently large \(n\).

Algorithm 3 Bounded-Hop Multi-Source Shortest Paths \((G, w, S, \ell, \varepsilon)\)

**Input:** Network \((G, w)\), set of source nodes \(S\) and parameters \(\ell, \varepsilon > 0\).

**Output:** With high probability, each node \(v\) knows \(\tilde{d}^\ell_{G,w}(s, v)\) for each \(s \in S\).

1: Assume that \(S = \{s_1, \ldots, s_b\}\). Let \(A_i\) be the Algorithm 1 with parameters \((G, w, s_i, \ell, \varepsilon)\) for each \(i \in [1, k]\) (each \(A_i\) is of \(T = \tilde{O}(\ell/\varepsilon)\) rounds, and during the whole computation of \(A_i\), each node broadcasts \(O(\log n)\) messages to its neighbors due to Lemma A.1).
2: The node leader samples \(\Delta_1, \ldots, \Delta_b \in [0, b \log n]\) independently and uniformly at random for delaying algorithms \(A_1, \ldots, A_k\), and broadcasts them by pipelining in \(O(D_G + b)\) rounds.
3: for \(r = 1\) to \(T + b \log n\) do
4:    for each \(v \in V\) do in parallel
5:        Let \(a = \lfloor i \in [1, b] : v\) broadcasts a message in the \((r - \Delta_i)\)-th round of \(A_i\) \rfloor\).
6:        if \(a < \lfloor \log n \rfloor\) then
7:            \(v\) broadcasts these \(a\) messages in the next \(\lfloor \log n \rfloor\) rounds.
8:        else
9:            The algorithm fails.
10:    end if
11: end for
12: end for

Lemma A.3 (Theorem 4.5 in [21]). After the overlay network \((G'_S, w'_S)\) is embedded, there exists an algorithm (Algorithm 4) which further embeds the overlay network \((G''_S, w''_S)\) in \(\tilde{O}(D_G + |S|k)\) rounds.

Algorithm 4 Embedding Overlay Network \((G, w, S, G'_S, w'_S, k)\)

**Input:** Network \((G, w)\), set of source nodes \(S\), overlay network \((G'_S, w'_S)\) and parameter \(k > 0\).

**Output:** It embeds the overlay network \((G''_S, w''_S)\).

1. Each node \(s \in S\) broadcasts the \(k\) shortest edges incident to it on \((G'_S, w'_S)\) (this can be done in \(O(D_G + |S|k)\) rounds).
2. **for** each \(s \in S\) do locally
3. \(\quad s\) computes \(N^k_S(s)\), along with the weight \(w''_S((s,v)) = d_{G'_S,w'_S}(s,v)\) for each \(v \in N^k_S(s)\) (this can be done due to Observation 3.12 in [21]).
4. **end for**

Lemma A.4 (Lemma 4.6 in [21]). For node \(s \in S\) known to all nodes, after the overlay network \((G''_S, w''_S)\) is embedded, there exists an algorithm (Algorithm 5) such that in \(\tilde{O}\left(\frac{|S|}{\varepsilon k} \cdot D_G + |S|\right)\) rounds, each node \(v \in V\) knows for each \(u \in S\) the value of \(\tilde{d}_{G''_S,w''_S}(s,u)\).

Algorithm 5 SSSP on Overlay Network \((G, w, S, \varepsilon, k, G''_S, w''_S, s)\)

**Input:** Network \((G, w)\), set of source nodes \(S\), parameters \(\varepsilon, k > 0\), overlay network \((G''_S, w''_S)\) and source node \(s \in S\).

**Output:** Each node \(v\) knows \(\tilde{d}_{G''_S,w''_S}(s,u)\) for each \(u \in S\).

1. Let \(A\) be the Algorithm 1 with parameters \((G''_S, w''_S, s, 4|S|/k, \varepsilon)\) (\(A\) is of \(T = \tilde{O}\left(\frac{|S|}{\varepsilon k}\right)\) rounds, and during the whole computation of \(A\), each node broadcasts \(O(\log n)\) messages to its neighbors due to Lemma A.1).
2. **for** \(r = 1\) to \(T\) **do**
3. \(\quad\) Let \(a\) be the number of nodes in \(G''_S\) that want to broadcast a message to its neighbors in \(G''_S\) in the \(r\)-th round of \(A\). Count \(a\) and make every nodes in \(G\) knows \(a\) in \(O(D_G)\) rounds.
4. \(\quad\) Each node in \(G'_S\), which wants to send a message to each of its neighbors in \(G''_S\), broadcasts such message to all nodes in \(G\) (this takes \(O(D_G + a)\) rounds).
5. **end for**