Certification of three black boxes with unsharp measurements using $3 \to 1$ sequential quantum random access codes

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Abstract  Unsharp measurements play an increasingly important role in quantum information theory. In this paper, we study a three-party prepare-transform-measure experiment with unsharp measurements based on $3 \to 1$ sequential random access codes (RACs). We derive optimal trade-off between the two correlation witnesses in $3 \to 1$ sequential quantum random access codes (QRACs), and use the result to complete the self-testing of quantum preparations, instruments and measurements for three sequential parties. We also give the upper and lower bounds of the sharpness parameter to complete the robustness analysis of the self-testing scheme. In addition, we find that classical correlation witness violation based on $3 \to 1$ sequential RACs cannot be obtained by both correlation witnesses simultaneously. This means that if the second party uses strong unsharp measurements to overcome the classical upper bound, the third party cannot do so even with sharp measurements. Finally, we give the analysis and comparison of the random number generation efficiency under different sharpness parameters based on the determinant value, $2 \to 1$ and $3 \to 1$ QRACs separately. This letter sheds new light on generating random numbers among multi-party in semi-device independent framework.

1 Introduction

It is an important task of quantum mechanics to obtain relevant information of the system by quantum measurement [1]. Sharp measurements, obtaining the maximum amount of information about the system, collapse the quantum states randomly into one of the eigenstates of the measured observable. On the other hand, unsharp measurements cause little disturbance to the system and allow us to obtain partial information of the system. Unsharp measurements that aim at bringing minimal disturbance to the system are often called “weak measurements” [2]. Unsharp, and especially weak measurements play an important role in many quantum information processing tasks, for instance, quantum random number generation [3–6], state tomography [7,8], sequential quantum correlations [9–12] and others.

Recently, Mohan et al. [13] discussed unsharp measurements in a three-party prepare-transform-measure experiment with $2 \to 1$ random access codes (RACs) [14,15], which has been experimentally demonstrated [16,17]. In Ref. [13], they characterized optimal trade-off between the two $2 \to 1$ quantum random access codes (QRACs), and applied the result to realise semi-device independent (SDI) self-testing of quantum measurement instruments. Self-testing is the task of characterizing unknown quantum states and measurements solely from the measurement statistics.

The original idea of testing states and measurements was proposed by Popescu et al. [18] based on Clauser-Horne-Shimony-Holt (CHSH) inequality [19]. If we obtain the maximal violation of the CHSH inequality, we can determine that the state measured is a two-qubit maximally entangled one and the measurements are two anticommuting Pauli observables. Since then, a growing number of quantum states have also been proved to
be self-testable \[20\]–\[25\]. All of the above self-testing schemes rely on quantum nonlocality within the device independent framework. Beyond those based on nonlocality, Tavakoli et al. presented a self-testing method for quantum prepare-and-measure experiment in 2018 \[20\]. After that, various self-testing schemes for different quantum states have been proposed under such framework \[13,27–33\]. These schemes consider quantum systems in fixed dimensions and belong to SDI framework, which opens interesting possibilities for quantum information processing.

In this paper, we study a three-party prepare-transform-measure experiment with \(3 \rightarrow 1\) sequential RACs, in which three black boxes in the safe area have been considered. For convenience, we call these black boxes Alice, Bob and Charlie sequentially. Alice randomly prepares one of the quantum states and sends it to Bob. Bob applies quantum instrument on it, and gets both a classical and quantum output, then Bob sends his post-measurement state to Charlie who will make further measurement. In our scenario, Alice encodes a three-bit long random sequence into an one-bit message while both Bob and Charlie aim to decode any of the three bits held by Alice, i.e. they individually implement a \(3 \rightarrow 1\) RAC with Alice. To analyze the optimization problem on the third party’s correlation witness, we derive optimal trade-off between the two correlation witnesses in \(3 \rightarrow 1\) sequential QRACs. This result allows us to self-test Alice’s preparations, Bob’s instruments and Charlie’s measurements within the SDI framework. Moreover, if the pair of these two correlation witnesses is suboptimal (here, we can regard it as the deviation caused by noise which is characterized as the sharpness parameter of Bob’s quantum measurement instruments), we give the upper and lower bounds of the sharpness parameter, and complete the robustness analysis of the self-testing scheme.

Whether the three-party protocol can be extended to any number of parties has always been a hot research topic in recent years if all transformers use weak measurements \[34\]–\[35\]. We find that the double classical correlation witness violation cannot be obtained based on \(3 \rightarrow 1\) sequential QRACs. If Bob uses unsharp measurements strong enough to overcome the classical upper bound, Charlie cannot do so even with maximal strength. Finally, the analysis of all our results sheds new light on the interaction between the three-party quantum dimension witness and the unsharp measurement technology. We apply it to the generation of SDI quantum random numbers, and give the local randomness on Bob’s side and Charlie’s side respectively. We give the analysis and comparison of the random number generation efficiency under different sharpness parameters based on the determinant value, \(2 \rightarrow 1\) and \(3 \rightarrow 1\) QRACs separately.

## 2 The \(3 \rightarrow 1\) sequential RACs

To better explain our theory, we first introduce a three-party prepare-transform-measure experiment based on \(3 \rightarrow 1\) sequential RACs in detail.

In our experiment, Alice has the freedom to choose one of eight preparations \(\{\rho_x\}\) where \(x = x_0x_1x_2 (x_0, x_1, x_2 \in \{0, 1\})\), but knows nothing about these quantum states apart from their dimensionality 2. For a given input \(x\), Alice prepares a quantum state \(\rho_x\) and sends it to Bob. Bob performs one of three quantum measurement instruments on \(\rho_x\) based on his input \(y\) \((y \in \{0, 1, 2\})\), and gets both a classical binary outcome \(b\) \((b \in \{0, 1\})\) and a qubit output \(\rho_z^{y,b}\) \[11\]. Since these quantum measurement instruments are completely positive trace-preserving maps, we characterise the quantum measurement instrument by Kraus operators \(\{K_{b|y}\}\)

Therefore, we obtain the Bob’s qubit post-measurement state

\[
\rho_z^{y,b} = \frac{K_{b|y}\rho_xK_{b|y}^\dagger}{\text{tr}(K_{b|y}^\dagger K_{b|y}\rho_x)}.
\]

Then Bob sends the qubit post-measurement state \(\rho_z^{y,b}\) to Charlie, Charlie performs one of three sharp measurements \(\{C_z\}\) on \(\rho_z^{y,b}\) depending on his input \(z\) \((z \in \{0, 1, 2\})\) and gets a measurement result denoted as \(c\) \((c \in \{0, 1\})\). All the random bits \(x_0, x_1, x_2, y, z\) and \(z\) are independent and uniformly distributed. This scenario is schematically depicted in Fig.\[4\]

After repeating this procedure many times, Alice, Bob and Charlie can estimate the conditional probability distribution \(p(b,c|x,y,z)\)\(=\text{tr}[K_{b|y}\rho_xK_{b|y}^\dagger C_{c|z}]\), which denotes the probability of Bob and Charlie obtaining the outcome \(b, c\) when the Kraus operators \(\{K_{b|y}\}\) and measurements \(\{C_{c|z}\}\) are performed on Alice’s prepared state \(\rho_x\) sequentially.

Alice encodes three bits of information into a single bit of information while both Bob and Charlie aim to recover some randomly chosen subset of the data held by Alice, i.e. they individually implement a \(3 \rightarrow 1\) RAC.
To achieve the maximum value instrument on the received qubit depending on his input (according to her three classical bits) and gets the measurement result denoted as \(b\) (\(b \in \{0, 1\}\)) and a qubit post-measurement state \(\rho_x^b\). Charlie performs his measurement on \(\rho_x^b\) depending on his input \(z\) (\(z \in \{0, 1, 2\}\)) and gets the measurement result denoted as \(c\) (\(c \in \{0, 1\}\)).

with Alice. We are interested in two separate correlation witnesses in \(3 \rightarrow 1\) sequential RACs. The correlation witness is the average guessing probability. The two respective average success rates read

\[
A_{AB} = \frac{1}{24} \sum_{x_0, x_1, x_2, y} p(b = x_y | x_0 x_1 x_2, y).
\]

\[
A_{AC} = \frac{1}{24} \sum_{x_0, x_1, x_2, z} p(c = x_z | x_0 x_1 x_2, z).
\]

This means that, upon receiving input \(y\) (\(z\)), Bob’s (Charlie’s) measurement device should get the output \(b = x_y\) (\(c = x_z\)), i.e., the \(y\)-th (\(z\)-th) of the input bit-string \(x\) received by the Alice’s preparation device.

Let’s firstly consider the \(3 \rightarrow 1\) RAC between Alice and Bob. Since Bob’s quantum instrument (a completely positive trace-preserving map) realises a measurement, the Kraus operators must satisfy the completeness relation \(\forall y : B_{b|y} + B_{1|y} = I\), where \(B_{b|y} = K_{b|y}^\dagger K_{b|y}\) are the corresponding elements of the positive operator-valued measures (POVMs). Moreover, the observable \(B_y\) is defined as \(B_y = B_{0|y} - B_{1|y}\). Therefore, we have

\[
A_{AB} = \frac{1}{24} \sum_{x, y} \text{tr}[K_{x|y} \rho_x K_{x|y}^\dagger] = \frac{1}{24} \sum_{x, y} \text{tr}[\rho_x B_{x|y}].
\]

In a two-dimensional system, the upper bound of \(A_{AB}\) corresponding to the quantum system is \(A_{AB}^Q = \frac{1}{2} + \frac{1}{2\sqrt{3}} \approx 0.79\) [36], and the maximum value of \(A_{AB}\) corresponding to the classical system is \(A_{AB}^C = \frac{3}{4}\). The quantum bound \(A_{AB}^Q\) can be obtained via the following “ideal” strategy. Alice’s eight preparations are chosen as

\[
\rho_{x_0 x_1 x_2}^{\text{ideal}} = \frac{I + \vec{n}_{x_0 x_1 x_2} \cdot \vec{\sigma}}{2},
\]

where \(\vec{n}_{x_0 x_1 x_2} = \frac{1}{\sqrt{3}}((-1)^{x_0}, (-1)^{x_1}, (-1)^{x_2})\) is Bloch vector, \(\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\) denotes the Pauli matrix vector. To achieve the maximum value \(\frac{1}{2} + \frac{1}{2\sqrt{3}}\), the corresponding measurements in Bob’s side are:

\[
B_0^{\text{ideal}} = \sigma_x, B_1^{\text{ideal}} = \sigma_y, B_2^{\text{ideal}} = \sigma_z.
\]

This means that Alice’s preparations are pure states, and correspond to the eigenvector of \(L_x = \sum x (-1)^x B_x^{\text{ideal}}\) associated to its largest eigenvalue. Such a set of preparations correspond to Bloch vectors forming a cube on the Bloch sphere and the measurements correspond to three mutually unbiased bases (i.e., three pairwise anti-commuting Pauli observables). It should be emphasized that this set of quantum states and measurements is uniquely determined in the sense of a unitary and a complex conjugation.

Correspondingly, the \(3 \rightarrow 1\) RAC between Alice and Charlie is considered. Charlie can also guess the average probability of success of a bit of Alice. In a classical model, the state at all times is diagonal in the same basis. Bob can interact with Alice’s preparations without disturbing her states. In this case, \(\mathcal{A}_{AB}, \mathcal{A}_{AC} \in \left\{\frac{1}{2}, \frac{3}{4}\right\}\).
However, in a quantum model, Bob’s instrument disturbs the physical state of Alice’s qubit, and therefore he cannot relay Alice’s original quantum message to Charlie. The effective state \( \tilde{\rho}_x \) received by Charlie is the post-measurement state of Bob averaged over \( y \) and \( b \), we write

\[
\tilde{\rho}_x = \frac{1}{3} \sum_{y,b} \rho(b|y) \rho_x^{b,y} = \frac{1}{3} \sum_{y,b} K_{b|y} \rho_x K_{b|y}^\dagger.
\]

Therefore, we have

\[
A_{AC} = \frac{1}{24} \sum_{x,z} \text{tr}[\tilde{\rho}_x C_{x,z}] = \frac{1}{72} \sum_{x,y,b,z} \text{tr}[K_{b|y} \rho_x K_{b|y}^\dagger C_{x,z}].
\]

In this case, \( A_{AB}, A_{AC} \in [1/2, (1 + 1/\sqrt{3})]/2 \). Evidently, \( A_{AB} \) is independent of Charlie. However, \( A_{AC} \) is not independent of Bob because he operates on Alice’s original preparation that reaches Charlie. In other words, Charlie’s ability to access the desired information depends on Bob’s preceding interaction. Furthermore, we are interested in the relation between \( A_{AB} \) and \( A_{AC} \). We intuitively know that in order to make the value of \( A_{AB} \) larger, the quantum states prepared by Alice should be close to the quantum states in equation (5), and Bob’s quantum measurement instruments should be close to the measurements in equation (6) in the sense of local isometry. This means that Bob’s measurements should be reasonably sharp. Once \( A_{AB} \) achieves the maximum value \( 1/2 + \frac{1}{\sqrt{3}} \), we can know that the disturbance caused by Bob to the original quantum state prepared by Alice is maximal, and accordingly, the value of \( A_{AC} \) that we can obtain should be minimal. The converse is also true. It is therefore natural to ask what is the optimal trade-off between the two correlation witnesses in a three-party prepare-transform-measure experiment. We derive optimal trade-off relation between the two correlation witnesses based on \( 3 \rightarrow 1 \) sequential QRACs in the next section.

### 3 Optimal trade-off between the two correlation witnesses based on \( 3 \rightarrow 1 \) sequential QRACs

In this section, we analyze what values are attainable for the pair of the two correlation witnesses \( (A_{AB}, A_{AC}) \) based on \( 3 \rightarrow 1 \) sequential QRACs in detail. From the above discussion, we can rephrase the problem as follows: for a given value of \( A_{AB} \in [1/2, 1/2(1 + 1/\sqrt{3})] \), what is the optimal value of \( A_{AC} \) in quantum theory? We will solve this problem by considering the related optimization problem

\[
A_{AC}^{A_{AB}} = \max_{\rho, U, M, C} A_{AC}
\]

subject to:

\[
\forall x : \rho_x \in C^2, \quad \rho_x \geq 0, \quad \text{tr}\rho_x = 1,
\]

\[
\forall y, b : U_{yb} \in SU(2), \quad B_{0|y} \geq 0, \quad B_{0|y} + B_{1|y} = I,
\]

\[
\forall z, c : C_{cz} \geq 0, \quad C_{0|z} + C_{1|z} = I,
\]

\[
\frac{1}{2} \leq A_{AB} \leq \frac{1}{2}(1 + \frac{1}{\sqrt{3}}).
\]

The optimization of the process takes over all Alice’s preparations, Bob’s instruments and Charlie’s measurements. We solve this optimization problem by first giving a lower bound on \( A_{AC}^{A_{AB}} \) and then matching it with an upper bound. We can obtain the optimal value \( A_{AC}^{A_{AB}} \) and give the following proposition.

**Proposition 1.** The optimal trade-off between the pair of the two correlation witnesses \( (A_{AB}, A_{AC}) \) based on \( 3 \rightarrow 1 \) sequential QRACs corresponds to

\[
A_{AC}^{A_{AB}} = \frac{1}{2} + \frac{\sqrt{3}}{18}(1 + 2 \sqrt{12A_{AB} - 12A_{AB}^2 - 2}),
\]

where \( A_{AB} \in [1/2, (1 + 1/\sqrt{3})]/2 \).

The proof of Proposition 1 is too lengthy to be included in the main text and is given in Appendix A. The respective trade-offs between the two correlation witnesses based on \( 3 \rightarrow 1 \) RACs in a classical and quantum model are given in Fig[2]. We should point out that all the presented bounds are tight in the sense that there
Fig 2. The trade-off relations between the two $3 \rightarrow 1$ RACs in a classical and quantum model respectively.

exist states and measurements reaching these average success probabilities. In particular, when $A_{AB}$ reaches the maximum value $\frac{1}{2} + \frac{1}{2\sqrt{3}}$, the value of $A_{AC}$ is $\frac{1}{2}(1 + \frac{\sqrt{3}}{3}) > \frac{1}{2}$, that is, Charlie’s average guessing probability is still better than random guessing. In this case, the effective ensemble relayed by Bob (the first decoder) to Charlie (the second decoder) corresponds to that originally prepared states (given in equation (5)) by Alice, but with Bloch vectors of $\frac{1}{3}$ the original length. Moreover, if Charlie performs the same Kraus operators (given in equation (6)) as Bob, we find $A_3 = \frac{1}{2}(1 + \frac{\sqrt{3}}{3})$. Similarly, the effective ensemble to David (the third decoder) relayed by Charlie will be identical to that relayed by Alice, except that the Bloch vectors will be shrunk to $\frac{1}{9}$ of the original length. Continuing the sequence in this manner, we find that the worst average guessing probability obtained by the $k$-th decoder is

$$A_k = \frac{1}{2}(1 + \frac{\sqrt{3}}{3^k}). \quad (11)$$

In addition, we find that both $3 \rightarrow 1$ QRACs cannot always outperform the $3 \rightarrow 1$ classical RACs. In order to understand these two pairs of correlation witnesses ($A_{AB}$, $A_{AC}$) under $3 \rightarrow 1$ classical RACs and QRACs more intuitively, the average success probability as a function of the sharpness parameter $\eta$ is studied and shown in Fig.3. Note that the green area in Fig.3 corresponds to the gray area above the red line in Fig.2 which represents that neither $A_{AB}$ nor $A_{AC}$ can realize the classical correlation witness violation. Furthermore, $A_{AB}$ and $A_{AC}$ cannot achieve the double classical correlation witness violation simultaneously. This implies that if Bob uses unsharp measurements strong enough to achieve the maximal classical violation of the correlation witness, Charlie cannot do so even with maximal strength. This is entirely different from the situation of $2 \rightarrow 1$ sequential QRACs [13].

To understand why this is so, we need to consider the constructions of the optimal $2 \rightarrow 1$ and $3 \rightarrow 1$ QRACs in Ref. [15]. In a two-dimensional system, both the states and the measurements can be represented by the unit vectors on Bloch sphere. From Ref. [15], we notice that compared with the optimal $3 \rightarrow 1$ QRAC, the states and the measurements for the optimal $2 \rightarrow 1$ QRAC lie in one plane and do not use the full size of the space. Our result seems to be match with the results in Ref. [37]. Li et al. calculated the dependence of the effectiveness of the randomness generation on $n$ ($n \rightarrow 1$ QRAC) and found it optimal for $n = 3$, and provided a similar explanation for this fact. Here, we consider a case where the most information can be obtained with the least interference with the original quantum state of the system. In order to understand our result intuitively, we can consider the following scenario: the maximum amount of randomness generated (the maximum amount of information) is obtained when $n = 3$, Bob causes the most disturbance to the original quantum state, resulting in the minimum amount of information obtained by Charlie’s measurement (i.e., $A_{AC}$ is the minimum). Besides, the upper bound of $A_{AB}^{3 \rightarrow 1}(A_{AB}^{3 \rightarrow 1})$ corresponding to the quantum system based on $2 \rightarrow 1(3 \rightarrow 1)$ QRAC is approximately 0.85(0.79). The classical upper bound of both is the same, which is 0.75. Obviously, compared with $2 \rightarrow 1$ RAC, the interval between quantum upper bound and classical upper bound is narrower under $3 \rightarrow 1$ RAC. Therefore, compared with the situation of $2 \rightarrow 1$ sequential RACs, it seems reasonable that classical correlation witness violation based on $3 \rightarrow 1$ sequential RACs cannot be obtained by both correlation
witnesses simultaneously.

This is expected to spark widespread interest in more general scenarios in future studies. For example, whether this property exists for \( n \rightarrow 1 \) QRACs and more general scenarios (higher-dimensional and many-input QRACs, as well as longer sequences of observers). It is potentially useful in recycling quantum resources in the context of various information processing tasks. Besides, this work also has played an important role in promoting the study of sequential QRACs and has far-reaching influence in related fields.

4 Self-testing and robustness analysis of three black boxes with \( 3 \rightarrow 1 \) sequential QRACs

According to the derivation of the optimal trade-off relation between the two correlation witnesses based on \( 3 \rightarrow 1 \) sequential QRACs in the previous sections, we find that this process implies a set of self-testing schemes for Alice’s preparations, Bob’s instruments and Charlie’s measurements under the SDI framework. In order to obtain a self-testing scheme, we must additionally prove that the optimal correlation witness pair \((A_{AB}, A_{AC}^\alpha)\) only allows one implementation with a specific set of Alice’s preparations, Bob’s instruments and Charlie’s measurements (up to the collective unitary transformation). We will discuss it in detail below.

Firstly, we have already shown that Alice’s preparations should be pure and pairwise antipodal. In Lemma 1, we obtain \( \mu = \arccos \frac{1}{3}, \varphi = \frac{\pi}{4} \). Thus, Alice’s preparations correspond to a cube on the surface of the Bloch sphere. The above arguments fully characterise Alice’s preparations up to a reference frame. Next, Bob’s instrument realises a measurement \( B_y = \alpha_y I + \vec{t}_y \cdot \vec{\sigma} \) where \( \vec{t}_y = (t_{y0}, t_{y1}, t_{y2}) \). From Lemma 1, we get \( \alpha_y = 0 \) and \( t_{yj} = \begin{cases} \eta & y = j, y, j \in \{0, 1, 2\} \\ 0 & y \neq j \end{cases} \). Therefore, \( B_0 = \eta \sigma_x, B_1 = \eta \sigma_y, B_2 = \eta \sigma_z \). we also get \( \gamma_0 = \sigma_x, \gamma_1 = \sigma_y, \gamma_2 = \sigma_z \). Moreover, we have made the optimal choice of letting \( V_{yzb} = U_{yb}^\dagger C_{0|z} U_{yb} \) project onto the eigenvector of \( \sqrt{B_{yb}^{\dagger} B_b} \) with the largest eigenvalue \( \lambda_{\text{max}} \). Thus, \( V_{yzb} = U_{yb}^\dagger C_{0|0} U_{yb} = \frac{1}{2}(|+\rangle \langle +| + i |\rangle \langle -|) \). Hence, For Charlie’s measurements \( C_z = C_{0|z} - C_{1|z} \), we get \( \forall y,b : U_{yb} = U \) and \( C_0 = U \sigma_x U^\dagger, C_1 = U \sigma_y U^\dagger, C_2 = U \sigma_z U^\dagger \). Finally, We are ready to present the result, which is given by the following proposition.

**Proposition 2.** Once the optimal trade-off between the pair of the two correlation witnesses \((A_{AB}, A_{AC})\) based on \( 3 \rightarrow 1 \) sequential QRACs is obtained, we can implement the self-test of the following unique Alice’s preparations, Bob’s instruments and Charlie’s measurements (up to collective unitary transformations)

(a) Alice’s states are pure, which correspond to a set of the states forming a cube on the surface of the Bloch sphere. These eight states are given in equation (5).
(b) Bob’s instruments are Kraus operators \( K_{by} = U_{yb} \sqrt{B_{by}} \), where \( B_{by} = K_{by}^\dagger K_{by} \) are the corresponding elements of the positive operator-valued measures (POVMs), \( B_y = B_{0y} - B_{1y} = K_{0y}^\dagger K_{0y} - K_{1y}^\dagger K_{1y} \).

Speciﬁcally, \( \forall y, b : U_{yb} = U, B_0 = \eta \sigma_x, B_1 = \eta \sigma_y, B_2 = \eta \sigma_z \) where \( \eta = \sqrt{3}(2A_{AB} - 1) \).

(c) Charlie’s measurements are rank-one sharp measurements, where \( C_0 = U_\sigma x U^\dagger, C_1 = U_\sigma y U^\dagger, C_2 = U_\sigma z U^\dagger \). This set of strategies corresponds to the red line in Fig[2]. The unitarity of these operators in Proposition 2 holds only if the statistics are ideal, however, we can never have perfect statistics in the real case. An interesting question is how to make this result have noise-tolerance. To solve this problem, we can bound the sharpness parameter of Bob’s instruments from noisy correlations. We rewrite the equation \( (A12) \) as

\[
A_{AB} = \frac{1}{2} + \frac{1}{24} (|\vec{s}_0||\vec{t}_0 + |\vec{s}_1||\vec{t}_1 + |\vec{s}_2||\vec{t}_2),
\]

(12)

where \( \vec{s}, \vec{t} \) are the normalized form of \( \vec{s}, \vec{t} \) respectively. From the above, it is not diﬃcult to get \( \forall y : \eta = |\vec{t}_y| \), thus,

\[
\eta = \frac{24A_{AB} - 12}{|\vec{s}_0\vec{t}_0 + |\vec{s}_1\vec{t}_1 + |\vec{s}_2\vec{t}_2|},
\]

(13)

when \( |\vec{s}_0| = |\vec{s}_1| = |\vec{s}_2| = \frac{4}{\sqrt{3}} \) and \( |\vec{t}_0 = |\vec{t}_1 = |\vec{t}_2 = 1 \), we can maximize the denominator to get the lower bound of \( \eta \). Therefore, the lower bound of \( \eta \) is

\[
\eta \geq \sqrt{3}(2A_{AB} - 1).
\]

(14)

This lower bound is nontrivial whenever \( A_{AB} > \frac{1}{2} \). Next, we consider the witness \( A_{AC} \). Rewriting inequality \( (A23) \) we can get the upper bound of \( \eta \), we show that this upper bound reads

\[
\eta \leq \frac{1}{2} \sqrt{3(6\sqrt{3}A_{AC} - 3\sqrt{3} + 1)(-2\sqrt{3}A_{AC} + \sqrt{3} + 1)}.
\]

(15)

This lower bound is nontrivial whenever \( \frac{1}{2}(1 + \frac{2\sqrt{3}}{3}) \leq A_{AC} \leq \frac{1}{2}(1 + \frac{2\sqrt{3}}{3}) \). Notice that the upper bound \( (15) \) coincides with the lower bound \( (14) \) for optimal trade-oﬀ between the pair of the two correlation witnesses \( (A_{AB}, A_{AC}) \) as given in equation \( (A11) \).

In order to have a more intuitive feeling of the noise-tolerance of this scheme in real experiments, we take a simple example to illustrate. When \( \eta = \frac{1}{\sqrt{3}} \), we attempt to implement the quantum strategies \( (A22) \) and \( (A23) \) for the optimal correlation witness pair \( (A_{AB}, A_{AC}) \). In an ideal case, we can easily know \( (A_{AB}, A_{AC}) \approx (0.6667, 0.7534) \). However, noise is unavoidable in real experiments, and here we can take a 95% visibility in Alice’s preparations, 90% visibility in Bob’s instruments, and 95% visibility in Charlie’s measurements. In this case, we obtain \( (A_{AB}, A_{AC}) \approx (0.6425, 0.7156) \). Therefore, we ﬁnd \( \eta \in [0.4936, 0.7844] \). This interval is fairly wide. Note that the certiﬁcation is more precise (the interval is smaller) as the sharpness parameter increases. It is worth mentioning that the experimental demonstrations for a three-party prepare-transform-measure protocol with 2 \( \rightarrow \) 1 sequential QRACs has been completed in Ref. [16][17]. Therefore, we can set the corresponding experimental scheme of our works by referring to the relevant experimental parameters in the three-party prepare-transform-measure experiment with 2 \( \rightarrow \) 1 sequential QRACs. This work will be studied in the future.

5 Random number generation eﬃciency in the three-party prepare-transform-measure experiment

Before analyzing the randomness of a three-party prepare-transform-measure experiment, it should be emphasized ﬁrstly, that an optimal weak measurement (that is, the most information can be obtained with the least disturbance to the original quantum state of the system. See Ref. [5][9][38] for more details) is mathematically
equivalent to POVMs formalism \[34\] and this is the basis of the realistic experiments. In our scenario, the measurement quality factor \( F = \sqrt{1 - \eta^2} \) and the precision of the measurement \( G = \eta \) correspond to the definitions in Ref. \[34\]. Furthermore, we have \( F^2 + G^2 = 1 \). Thus, unsharp measurement yields the maximum information about the system while disturbing the original state minimally. Although Li et al. \[4\] have analyzed the randomness of the classic dimension witnesses (based on the 2 \( \rightarrow \) 1 QRAC and the nonlinear determinant value respectively) violation in the three-observer protocol by using non-optimal weak measurements, our analysis is different from theirs because we use unsharp measurements, which are mathematically equivalent to the optimal weak measurement.

According to the observed probabilities, dimension witness inequality \( W \) based on the nonlinear determinant value test in the two-observer system is given by \[4\][5][39]

\[
W = \begin{vmatrix}
    p(1|00,0) - p(1|01,0) & p(1|10,0) - p(1|11,0) \\
    p(1|00,1) - p(1|01,1) & p(1|10,1) - p(1|11,1)
\end{vmatrix}.
\] \ (16)

Specifically, in the two-dimensional Hilbert space, the upper bound of the quantum dimension witness value is 1, while the classical dimension witness value is 0. This dimension witness can be used to estimate the genuine randomness generated if the quantum random number generator system satisfied two assumptions \[5][39][40\].

(1) The state preparation device and measurement device are assumed to be independent (they have no shared randomness), and their hidden variables are independent of any other devices. For three-party protocol, Alice, Bob, and Charlie must be independent with each other. (2) The dimension of the quantum system is restricted to two.

In our scheme, we can obtain the values of the dimension witnesses (based on the determinant value) between Alice and Bob as follows

\[ W_{AB} = \eta^2, \] \ (17)

while the dimension witness value (based on the determinant value) between Alice and Charlie is given by

\[ W_{AC} = \left( \frac{1 + \sqrt{1 - \eta^2}}{2} \right)^2. \] \ (18)

Meanwhile, we find that the sharpness parameters \( \eta \) in this paper are mathematically equivalent to the optimal weak measurement parameters \( \theta \) in Ref. \[5\], and they satisfy

\[ \eta = \cos 2\theta. \] \ (19)

The detailed quantum dimension witnesses values \( W_{AB} \) and \( W_{AC} \) with different sharpness parameters \( \eta \) are shown in Fig.[4]

To get more randomness than in Ref. \[5\], we use the tighter bound of quantum randomness certification given in Ref. \[11\], the relation between the dimension witness \( W \) (based on the determinant value) and the randomness generation efficiency \( H_{min} \) is given by

\[ H_{min}(W) = -\log_2 \left( \frac{1}{2} + \frac{1}{2} \sqrt{\frac{2 - W}{2}} \right), \] \ (20)

where \( 0 \leq W \leq 1 \). Therefore, we can take the dimensional witnesses values of \( W_{AB} \) and \( W_{AC} \) into equation \[20\], and then we can get the local randomness between on Bob’side and Charlie’side respectively.

As is well-known, another quantum dimension witness inequality based on the QRAC in the two-observer system is given in Ref. \[37][42\]. And we also know that the relation between dimensional witnesses and QRACs (see Ref. \[43][44\]). Therefore, according to the value of \( A_{AB} \) and \( A_{AC} \), we obtain the two-dimensional quantum dimensional witnesses between Alice-Bob and between Alice-Charlie based on 2 \( \rightarrow \) 1 QRAC and 3 \( \rightarrow \) 1 QRAC respectively, as follows

\[ T_{AB}^{2 \rightarrow 1} = 2\sqrt{2}\eta, \]
\[ T_{AC}^{2 \rightarrow 1} = \sqrt{2}(1 + \sqrt{1 - \eta^2}), \] \ (21)
Fig 4. Dimension witness $W$ (based on the determinant value) as a function of the sharpness parameter $\eta$. The blue line is the dimension witness between Alice and Bob. The red line is the dimension witness between Alice and Charlie.

Fig 5. Local random number generation efficiency with different sharpness parameters $\eta$.

and

$$
T_{AB}^{3 \rightarrow 1} = 4\sqrt{3}\eta, \\
T_{AC}^{3 \rightarrow 1} = \frac{4\sqrt{3}}{3} (1 + 2\sqrt{1 - \eta^2}),
$$

(22)

where $2 \leq T_{2 \rightarrow 1}^{2 \rightarrow 1} \leq 2\sqrt{2}$ and $6 \leq T_{3 \rightarrow 1}^{3 \rightarrow 1} \leq 4\sqrt{3}$.

In Ref. [45], the general analytical relationship between dimension witnesses $T_{2 \rightarrow 1}^{2 \rightarrow 1}$ and the randomness generation efficiency $H'_{\min}$ is given by

$$
H'_{\min}(T_{2 \rightarrow 1}^{2 \rightarrow 1}) = -\log_2\left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1 + \sqrt{1 - \left(\frac{(T_{2 \rightarrow 1}^{2 \rightarrow 1})^2 - 4}{4}\right)^2}}{2}}\right).
$$

(23)

Hence, we can take the dimensional witnesses values of $T_{AB}^{2 \rightarrow 1}$ and $T_{AC}^{2 \rightarrow 1}$ into equation (23), and then we can get the local randomness on Bob’s side and Charlie’s side respectively.

However, in the SDI scenario based on 3 $\rightarrow$ 1 QRACs, although the relation between the randomness generation and the dimension witnesses is also given by using analytic [46] and numerical analysis [36, 42, 47] respectively, we find that the existing analytic relation is not better than the numerical relation. Here, we use the numerical method given in Ref. [42]. By solving the minimization problem with the Levenberg-Marquardt
algorithm [48], we get the min-entropy bound of the measurement outcome for the given $3 \rightarrow 1$ QRAC. Local random number generation efficiency with different sharpness parameters $\eta$ on Bob’s side and Charlie’s side based on dimensional witnesses values, $2 \rightarrow 1$ QRAC and $3 \rightarrow 1$ QRAC are shown in Fig.5 respectively. Obviously, if the sharpness parameter is equal to 0, the random generation efficiency is 0.3425, which is same as the result in the SDI randomness expansion with sharp measurements proposed in Ref. [37]. It should be noted that in Fig.5, the random generation efficiency of $3 \rightarrow 1$ QRAC is better than the results of $2 \rightarrow 1$ QRAC on Bob’s side (Charlie’s side) only when the sharpness parameters $0.9956 < \eta \leq 1$ ($0 \leq \eta < 0.1105$). In other words, the random generation efficiency of $3 \rightarrow 1$ QRAC is not always better than the results of $2 \rightarrow 1$ QRAC, probably because the tighter analytic relation (similar to equation (23)) between random generation efficiency of $3 \rightarrow 1$ QRAC and the dimension witness is missing.

The research on the random generation efficiency here is of fundamental significance, and it sheds new light on generating random number among multi-user in the network environment. This also supplies much space to imagine its application in the area of quantum cryptography and quantum randomness generation research.

6 Conclusion

In conclusion, we derive the optimal trade-off between the pair of two correlation witnesses $(A_{AB}, A_{AC})$ based on $3 \rightarrow 1$ sequential QRACs in a three-party prepare-transform-measure experiment. Based on the trade-off, we have completed the self-testing of preparations, instruments and measurements for three sequential parties. We give the upper and lower bounds of the sharpness parameter, and complete the robustness analysis of the self-testing scheme. We find that classical correlation witness violation based on $3 \rightarrow 1$ sequential RACs cannot be obtained by both correlation witnesses $A_{AB}$ and $A_{AC}$ simultaneously. This implies that if Bob uses unsharp measurements strong enough to achieve the maximal classical violation of the correlation witness, Charlie cannot do so even with maximal strength. Besides, we give the analysis and comparison of the random number generation efficiency under different sharpness parameters based on the determinant value, $2 \rightarrow 1$ and $3 \rightarrow 1$ QRACs respectively, and the analysis method can also be applied to future multi-party quantum network studies.

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Appendix A. Proof of Proposition 1

In this section we provide the proof of Proposition 1, which for completeness we also state here.

**Proposition 1.** The optimal trade-off between the pair of the two correlation witnesses $(A_{AB}, A_{AC})$ based on $3 \rightarrow 1$ sequential QRACs corresponds to

$$A_{AC}^{AB} = \frac{1}{2} + \frac{\sqrt{3}}{18} (1 + 2 \sqrt{12A_{AB} - 12A_{AB}^2 - 2}),$$

(A1)

where $A_{AB} \in [1/2, (1 + 1/\sqrt{3})/2]$.

**Proof:** We use the polar decomposition to write the Kraus operators as $K_{b|y} = U_{yb} \sqrt{B_{b|y}}$ for the unitary operator $U_{yb}$ and the element of POVM $B_{b|y}$ in the above. Kraus operators of this form correspond to extremal quantum instruments in the considered scenario [18]. We can then use the cyclicity of the trace along with the substitution $C_{1|z} = I - C_{0|z}$ to write equation (8) as

$$A_{AC} = \frac{1}{2} + \frac{1}{72} \sum_{x,y,b,z} (-1)^{xz} \text{tr}[\sqrt{B_{b|y} \rho_x \sqrt{B_{b|y}} U_{yb} U_{yb}^\dagger C_{0|z} U_{yb}}]$$

$$= \frac{1}{2} + \frac{1}{72} \sum_{x,y,b,z} \text{tr}[\sqrt{B_{b|y} \gamma_z \sqrt{B_{b|y}} V_{y\delta b}}],$$

(A2)
where $\gamma_z = \sum_x (-1)^x \rho_x$, $V_{yzb} = U_1 C_{0z} U_{yrb}$. We can now consider the optimisation over $U_{yrb}$ and $C_{0z}$ as a single optimisation over $V_{yzb}$. To this end, we note that the set of measurements $\{C_{0z}\}$ is convex. Therefore, every nonextremal (interior point) measurement can be written as a convex combination of extremal measurements (on the boundary). Due to linearity, no nonextremal POVM can lead to a larger value of $A_{AC}$ than some extremal POVM. The extremal binary-outcome qubit measurements are rank-one projectors. Therefore, we can consider the optimisation over $V_{yzb}$ as an optimisation over general rank-one projectors. This gives

$$
\max A_{AC} = \frac{1}{2} + \max_{\rho, y, b, z} \frac{1}{2} \sum_{x, y, b, z} \text{tr}[\sqrt{B_{b|x}} \gamma_z |V_{yzb}|] = \frac{1}{2} + \max_{\rho, y, b} \frac{1}{2} \sum_{x, y, b, z} \lambda_{\text{max}}[\sqrt{B_{b|x}} \gamma_z |B_{b|y}|],
$$

(A3)

where the optimal choice of $V_{yzb}$ is aligned with the eigenvector of $\sqrt{B_{b|x}} \gamma_z \sqrt{B_{b|y}}$ corresponding to the largest eigenvalue $\lambda_{\text{max}}$.

The general representation of a qubit can be illustrated by using the density matrix formalism $\rho_x = \frac{I + \vec{n}_x}{2}$, Bloch vector $\vec{n}_x \in R^3$, $|\vec{n}_x| \leq 1$. Thus, we have

$$
\begin{align*}
\gamma_0 &= \frac{1}{2}[\vec{n}_{000} + \vec{n}_{001} + \vec{n}_{010} + \vec{n}_{011} - (\vec{n}_{111} + \vec{n}_{110} + \vec{n}_{101} + \vec{n}_{100})] \cdot \vec{\sigma} = \vec{s}_0 \cdot \vec{\sigma}, \\
\gamma_1 &= \frac{1}{2}[\vec{n}_{000} + \vec{n}_{001} + \vec{n}_{010} + \vec{n}_{011} - (\vec{n}_{110} + \vec{n}_{111} + \vec{n}_{101} + \vec{n}_{100})] \cdot \vec{\sigma} = \vec{s}_1 \cdot \vec{\sigma}, \\
\gamma_2 &= \frac{1}{2}[\vec{n}_{000} + \vec{n}_{010} + \vec{n}_{110} + \vec{n}_{111} - (\vec{n}_{011} + \vec{n}_{011} + \vec{n}_{101} + \vec{n}_{100})] \cdot \vec{\sigma} = \vec{s}_2 \cdot \vec{\sigma},
\end{align*}
$$

(A4)

where

$$
\begin{align*}
\vec{s}_0 &= \frac{1}{2} [\vec{n}_{000} + \vec{n}_{001} + \vec{n}_{010} + \vec{n}_{011} - (\vec{n}_{111} + \vec{n}_{110} + \vec{n}_{101} + \vec{n}_{100})], \\
\vec{s}_1 &= \frac{1}{2} [\vec{n}_{000} + \vec{n}_{001} + \vec{n}_{010} + \vec{n}_{011} - (\vec{n}_{110} + \vec{n}_{111} + \vec{n}_{101} + \vec{n}_{100})], \\
\vec{s}_2 &= \frac{1}{2} [\vec{n}_{000} + \vec{n}_{010} + \vec{n}_{110} + \vec{n}_{111} - (\vec{n}_{011} + \vec{n}_{011} + \vec{n}_{101} + \vec{n}_{100})].
\end{align*}
$$

(A5)

As we know, given any set of preparations $\{\vec{n}_x\}$, we can consider other preparations $\{\vec{n}_x\}$ choosen such that $\vec{n}_{000} = -\vec{n}_{111}, \vec{n}_{001} = -\vec{n}_{110}, \vec{n}_{010} = -\vec{n}_{101}, \vec{n}_{011} = -\vec{n}_{100}$. Therefore, we can reduce the number of operator equalities (A5) by exploiting the apparent symmetries in the expressions for $\vec{s}_z$. Moreover, it is evident that if not all preparations are pure, one cannot obtain optimal correlations. Thus, without loss of generality, we define

$$
\begin{align*}
\vec{n}_{000} &= (\sin \mu \cos \varphi, \sin \mu \sin \varphi, \cos \mu), \\
\vec{n}_{001} &= (\sin \mu \cos \varphi, \sin \mu \sin \varphi, -\cos \mu), \\
\vec{n}_{010} &= (\sin \mu \cos \varphi, -\sin \mu \sin \varphi, \cos \mu), \\
\vec{n}_{011} &= (\sin \mu \cos \varphi, -\sin \mu \sin \varphi, -\cos \mu),
\end{align*}
$$

(A6)

where $\mu, \varphi \in [0, \frac{\pi}{2}]$. This lead to

$$
|\vec{s}_0| = 4 \sin \mu \cos \varphi, |\vec{s}_1| = 4 \sin \mu \sin \varphi, |\vec{s}_2| = 4 \cos \mu.
$$

(A7)

To further derive the upper bound of equation (A3), we can use the following relation

$$
\forall B, \forall \vec{a} \in R^3 : \sum_{b=0,1} \lambda_{\text{max}}[\sqrt{B_{b|a}} \vec{\sigma}] \leq |\vec{a}|,
$$

(A8)

with equality if and only if $\vec{a}$ is aligned with the Bloch vector of the POVM. Identifying $\vec{a}$ with $\vec{s}_z$, we apply it twice to equation (12) corresponding to the terms in which $z = y$. This gives

$$
A_{AC} \leq \frac{1}{2} + \frac{1}{72}(|\vec{s}_0| + |\vec{s}_1| + |\vec{s}_2| + \sum_{y, b} \lambda_{\text{max}}[\sqrt{B_{b|y}} |\vec{s}_0 \cdot \vec{\sigma}]) \sqrt{B_{b|y}}),
$$

(A9)
where \( y, \tilde{y} \in \{0, 1, 2\} \) and \( y \neq \tilde{y} \).

For the convenience of the following analysis, we define \( B_y = \alpha_y I + \vec{t}_y \cdot \vec{\sigma} \) where \( \vec{t}_y = (t_{y0}, t_{y1}, t_{y2}) \), \( |\vec{t}_y| \leq 1 \), \( |\vec{t}_y| - 1 \leq \alpha_y \leq 1 - |\vec{t}_y| \). The sharpness parameter of Bob’s measurements is defined as \( \eta_y = |\vec{t}_y| \). Notice that for \( \eta_y \in \{0, 1\} \) the measurements are noninteractive and sharp measurement respectively, whereas \( \eta_y \in (0, 1) \) corresponds to intermediate cases. Furthermore, we have

\[
B_{ib} = f_{ib}|\vec{t}_y\rangle\langle\vec{t}_y| + h_{ib} - |\vec{t}_y\rangle\langle -\vec{t}_y|,
\]

where \( |\vec{t}_y\rangle \) is the pure state corresponding to the Bloch sphere direction \( \vec{t}_y \), and

\[
f_{ib} = \frac{1}{2}(1 + (-1)^b \alpha_y + (-1)^b |\vec{t}_y|),
\]

\[
h_{ib} = \frac{1}{2}(1 + (-1)^b \alpha_y - (-1)^b |\vec{t}_y|).
\]

From the above analysis, the equation (4) can be written as

\[
\eta_y = (\frac{1}{2} + \frac{1}{24} (|\vec{s}_0\rangle\langle \vec{t}_0| + |\vec{s}_1\rangle\langle \vec{t}_1| + |\vec{s}_2\rangle\langle \vec{t}_2|))
\]

\[
= \frac{1}{2} + \frac{1}{24} (|\vec{s}_0|t_{00} + |\vec{s}_1|t_{11} + |\vec{s}_2|t_{22}).
\]

Then, in order to derive the upper bound of \( A_{AC} \), we need to further analyze inequality (A9), consider the characteristic equation \( \text{det}(\sqrt{B_{ib}}(\vec{s}_y \cdot \vec{\sigma})\sqrt{B_{ib} - \mu I}) = 0 \) and after a complicated derivation and simplification, we can obtain

\[
T = \sum_{y,b} \lambda_{\text{max}}(\sqrt{B_{ib}}(\vec{s}_y \cdot \vec{\sigma})\sqrt{B_{ib}})
\]

\[
= \sum_{y,b} \frac{|\vec{s}_y|}{2} \sqrt{(1 + (-1)^b \alpha_y)^2 - |\vec{t}_y|^2(1 - \langle \vec{t}_y | \vec{s}_y \rangle \langle \vec{s}_y | \vec{t}_y \rangle)^2},
\]

where \( \vec{s} \) is the normalized form of \( \vec{t} \). We can now consider the optimisation over \( \alpha_y \) by separately considering the three terms corresponding to \( y = 0, y = 1 \) and \( y = 2 \) respectively. This amounts to maximising expressions of the form \( \sqrt{(1 + x)^2 - R} + \sqrt{(1 - x)^2 - R} \), for some positive constant \( R \). It is easily shown that the value of function is maximal if and only if \( x = 0 \). Thus, we require \( \alpha_0 = \alpha_1 = \alpha_2 = 0 \). Moreover, since \( \vec{s}_0 \propto (1, 0, 0), \vec{s}_1 \propto (0, 1, 0) \) and \( \vec{s}_2 \propto (0, 0, 1) \), and we also separately maximise search square root expression above by standard differentiation, it is seen from (A12) and (A13) that one optimally chooses \( t_{00} = t_{02} = 0, t_{10} = t_{12} = 0, t_{20} = t_{21} = 0 \). Finally, we can get

\[
\max T = |\vec{s}_0|((1 - t_{11}^2 + \sqrt{1 - t_{22}^2}) + |\vec{s}_1|((1 - t_{00}^2 + \sqrt{1 - t_{22}^2}) + |\vec{s}_2|((1 - t_{00}^2 + \sqrt{1 - t_{11}^2})),
\]

Inequality (A9) can be written as

\[
A_{AC} \leq \frac{1}{2} + \frac{1}{72} (|\vec{s}_0| + |\vec{s}_1| + |\vec{s}_2| + |\vec{s}_0|((1 - t_{11}^2 + \sqrt{1 - t_{22}^2}) + |\vec{s}_1|((1 - t_{00}^2 + \sqrt{1 - t_{22}^2}) + |\vec{s}_2|((1 - t_{00}^2 + \sqrt{1 - t_{11}^2})),
\]

\[
\equiv A_C.
\]

Without loss of generality, we define \( t_{00} = \cos \phi_y, t_{11} = \cos \phi_y, t_{22} = \cos \phi_2 \) where \( \phi_y \in [0, \frac{\pi}{2}] \). By plugging this in inequality (A15), \( A_C \) can be reexpressed

\[
A_C = \frac{1}{2} + \frac{1}{18} |\sin \mu \cos \varphi + \sin \mu \sin \varphi + \cos \mu + \sin \mu \cos \varphi| \sin \phi_1 + \sin \phi_2) + |\sin \mu \sin \varphi (\sin \phi_0 + \sin \phi_2) + \cos \mu (\sin \phi_0 + \sin \phi_1)|.
\]

Accordingly, equation (A12) can be rewritten as

\[
A_{AB} = \frac{1}{2} + \frac{1}{6} (\sin \mu \cos \varphi \cos \phi_0 + \sin \mu \sin \varphi \cos \phi_1 + \cos \mu \cos \phi_2).
\]
Next, the optimization problem is transformed into taking the values of all parameters \( \mu, \varphi, \phi_0, \phi_1, \phi_2 \), and finding the maximum value of \( \mathcal{A}_C \). To solve this problem, we use the following theorem

**Lemma 1.** For every tuple \((\mu, \varphi, \phi_0, \phi_1, \phi_2)\) corresponding to \((\mathcal{A}_{AB}, \mathcal{A}_C)\), there exists another tuple \((\mu, \varphi, \phi_0, \phi_1, \phi_2) = (\arccos \frac{1}{\sqrt{3}}, \frac{\pi}{4}, \phi, \phi, \phi)\) that always produces \((\mathcal{A}_{AB}, \mathcal{A}_C)\) with \(\mathcal{A}_C \geq \mathcal{A}_A\).

To prove this statement, we must show that for all \(\mu, \varphi, \phi_0, \phi_1, \phi_2 \in [0, \frac{\pi}{2}]\) there exists a \(\phi \in [0, \frac{\pi}{2}]\) such that

\[
\sin \mu \cos \varphi \cos \phi_0 + \sin \mu \sin \varphi \cos \phi_1 + \cos \mu \cos \phi_2 = \sqrt{3} \cos \phi,
\]

\[
\sin \mu \cos \varphi (\sin \phi_1 + \sin \phi_2) + \sin \mu \sin \varphi (\sin \phi_0 + \sin \phi_2)
+ \cos \mu (\sin \phi_0 + \sin \phi_1) \leq \sqrt{3} + 2 \sqrt{3} \sin \phi.
\] (A18)

As far as we know, it trivially holds that \(\sin \mu \cos \varphi + \sin \mu \sin \varphi + \cos \mu \leq \sqrt{3}\) with equality if and only if \(\mu = \arccos \frac{1}{\sqrt{3}}, \varphi = \frac{\pi}{4}\). Furthermore, we obtain

\[
\sin \mu \cos \varphi \sin \phi_0 + \sin \mu \sin \varphi \sin \phi_2 + \cos \mu \sin \phi_0 \leq \sqrt{3} \sin \phi,
\]

\[
\sin \mu \cos \varphi \sin \phi_2 + \sin \mu \sin \varphi \sin \phi_0 + \cos \mu \sin \phi_1 \leq \sqrt{3} \sin \phi,
\]

\[
\sin \mu \cos \varphi \cos \phi_0 + \sin \mu \sin \varphi \cos \phi_1 + \cos \mu \cos \phi_2 \leq \sqrt{3} \cos \phi.
\] (A19)

Then, by squaring inequations [A18] and inequations [A19], we have

\[
\frac{1}{3} (\sin^2 \phi_1 + \sin^2 \phi_2 + \sin^2 \phi_0) + \frac{2}{3} (\sin \phi_1 \sin \phi_2 + \sin \phi_0 \sin \phi_1 + \sin \phi_0 \sin \phi_2) \leq 3 \sin^2 \phi,
\]

\[
\frac{1}{3} (\sin^2 \phi_0 + \sin^2 \phi_1 + \sin^2 \phi_2) + \frac{2}{3} (\sin \phi_0 \sin \phi_2 + \sin \phi_1 \sin \phi_2 + \sin \phi_0 \sin \phi_1) \leq 3 \sin^2 \phi,
\]

\[
\frac{1}{3} (\cos^2 \phi_0 + \cos^2 \phi_1 + \cos^2 \phi_2) + \frac{2}{3} (\cos \phi_1 \cos \phi_2 + \cos \phi_0 \cos \phi_1 + \cos \phi_0 \cos \phi_2) \leq 3 \cos^2 \phi.
\] (A20)

we can combine inequations [A20] into a single equation in which \(\phi\) is eliminated. The statement reduces to the inequality

\[
\sin \phi_1 \sin \phi_2 + \sin \phi_0 \sin \phi_1 + \sin \phi_0 \sin \phi_2 + \cos \phi_1 \cos \phi_2 + \cos \phi_0 \cos \phi_1 + \cos \phi_0 \cos \phi_2 \leq 3.
\] (A21)

After a detailed derivation, one finds that the optimum of the left hand side is attained for \(\phi_0 = \phi_1 = \phi_2\). Then, the theorem 1 can be derived.

Putting these together we can reduce our consideration of [A16] and [A17] to \(\mu = \arccos \frac{1}{\sqrt{3}}, \varphi = \frac{\pi}{4}\), \(t_{00} = \cos \phi_0 = \eta_0, t_{11} = \cos \phi_1 = \eta_1, t_{22} = \cos \phi_2 = \eta_2\) and \(\eta_0 = \eta_1 = \eta_2 \equiv \eta\). Therefore, equation [A17] reduces to

\[
\mathcal{A}_{AB} = \frac{1}{2} + \frac{\sqrt{3}}{6} \eta,
\] (A22)

and equation [A16] reduces to

\[
\mathcal{A}_{AC} = \frac{1}{2} + \frac{\sqrt{3}}{18} (1 + 2 \sqrt{1 - \eta^2}).
\] (A23)

According to equation [A22], we have \(\eta = \sqrt{3}(2\mathcal{A}_{AB} - 1)\). By plugging this in equation [A23], we finally get

\[
\mathcal{A}_{AC}^{\mathcal{A}_{AB}} = \frac{1}{2} + \frac{\sqrt{3}}{18} (1 + 2 \sqrt{12\mathcal{A}_{AB} - 12\mathcal{A}_{AB}^2 - 2}).
\] (A24)

This finishes the proof of Proposition 1.

\[\square\]

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