Exotic Bialgebra $S_{03}$: Representations, Baxterisation and Applications

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Abstract

The exotic bialgebra $S_{03}$, defined by a solution of the Yang-Baxter equation, which is not a deformation of the trivial, is considered. Its FRT dual algebra $s_{03_F}$ is studied. The Baxterisation of the dual algebra is given in two different parametrisations. The finite-dimensional representations of $s_{03_F}$ are considered. Diagonalisations of the braid matrices are used to yield remarkable insights concerning representations of the $L$-algebra and to formulate the fusion of finite-dimensional representations. Possible applications are considered, in particular, an exotic eight-vertex model and an integrable spin-chain model.

Dedicated to our friend Daniel Arnaudon

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1 Introduction

For several years [1–3] our collaboration studied the algebraic structures coming from 4x4 $R$-matrices (solutions of the Yang–Baxter equation) that are not deformations of classical ones (i.e., the identity up to signs). According to the classification of Hietarinta [4] there are five such 4x4 $R$-matrices that are invertible. These matrices were obtained first by Hlavatý [5] without classification claims. In the present paper we consider in more detail one of these cases which seems most interesting, namely, the matrix bialgebra $S_{03}$ and its FRT [6] dual $s_{03_F}$.

The paper is organised as follows. In Section 2 we introduce the matrix bialgebra $S_{03}$, its FRT dual $s_{03_F}$, and an affinisation for the latter. We give also a basis of $s_{03_F}$ suitable to define the class of representations when $s_{03_F}$ acts on itself. In Section 3 we give alternative parametrisation of the baxterised $R$ and $L$ matrices. This allows to introduce a diagonalisation of the braid matrix which gives remarkable insights concerning representations of the L-algebra. In Section 4 we study the finite-dimensional representations of $s_{03_F}$. In Section 5 we use the diagonalisation of the permuted $R$-matrix in order to formulate the fusion of certain representations. In Section 6 we consider some of the possible applications: an exotic eight-vertex model and an integrable spin-chain model are discussed.

2 FRT Duality

2.1 Preliminaries

Our starting point is the following $4 \times 4$ $R$-matrix:

$$R = R_{S_{03}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \quad (2.1)$$

This $R$-matrix appears in the classification of [4] which gives all (up to equivalence) $4 \times 4$ matrix solutions of the Yang-Baxter equation. Obviously, (2.1) is not a deformation of the identity.\(^5\)

In this subsection we introduce various quantities that we need later. First two standard matrices $R^\pm$ defined by:

$$R^+ \equiv PRP = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad R^- \equiv R^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad (2.2)$$

\(^5\)Higher dimensional ($N^2 \times N^2$ matrices for all $N > 2$) exotic braid matrices (which are not deformations of some “classical limits”) have been presented and studied in [7,8].
where \( P \) is the permutation matrix:

\[
P ≡ \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (2.3)

Then we introduce the Baxterised \( R \)-matrix:

\[
R(x) = R_{S03}(x) = x^{-1/2}R + x^{1/2}R_{21}^{-1} =
\]

\[
= \frac{1}{\sqrt{2x}} \begin{pmatrix}
x + 1 & 0 & 0 & 1 - x \\
0 & 1 - x & x + 1 & 0 \\
0 & x + 1 & x - 1 & 0 \\
x - 1 & 0 & 0 & x + 1
\end{pmatrix}.
\] (2.4)

It satisfies the spectral parameter dependent Yang–Baxter equation

\[
R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x)
\] (2.5)

Finally, we define the braid matrix \( \hat{R} \equiv PR \), and its Baxterisation \( \hat{R}(x) \):

\[
\hat{R} \equiv P_{12}R \\
\hat{R}(x) \equiv P_{12}R(x) = x^{-1/2}\hat{R} + x^{1/2}\hat{R}^{-1} =
\]

\[
= \frac{1}{\sqrt{2x}} \begin{pmatrix}
x + 1 & 0 & 0 & 1 - x \\
0 & x + 1 & x - 1 & 0 \\
0 & 1 - x & x + 1 & 0 \\
x - 1 & 0 & 0 & x + 1
\end{pmatrix}.
\] (2.7)

Note that \( \hat{R}^{-1} = R^{-1}P \).

We also record two identities involving \( \hat{R} \):

\[
\hat{R} + \hat{R}^{-1} = \sqrt{2}I \\
\hat{R}^2 + \hat{R}^{-2} = 0
\] (2.8)

### 2.2 The bialgebra S03

Here we recall the matrix bialgebra \( S03 \) which we obtained in [2] by applying the RTT relations of [6]:

\[
RT_1 T_2 = T_2 T_1 R, \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\] (2.9)
where $T_1 = T \otimes 1_2$, $T_2 = 1_2 \otimes T$, for the case when $R = R_{S03}$. The relations which follow from (2.9) and (2.11) are:

$$b^2 + c^2 = 0, \quad a^2 - d^2 = 0,$$
$$cd = ba, \quad dc = -ab,$$
$$bd = ca, \quad db = -ac,$$
$$da = ad, \quad cb = -bc.$$ (2.10)

### 2.3 The FRT dual s03_F

The FRT dual $s03_F$ of $S03$ is given in terms of $L^\pm$ which are matrices of operators $L_{ij}^\pm$ $(i, j = 1, 2)$ satisfying the so-called RLL relations [6]:

$$R^+ L_1^+ L_2^+ = L_2^+ L_1^+ R^+$$
$$R^+ L_1^- L_2^- = L_2^- L_1^- R^+$$
$$R^+ L_1^+ L_2^- = L_2^- L_1^+ R^+$$ (2.11)

with $L_1 \equiv L \otimes 1$, $L_2 \equiv 1 \otimes L$.

Encoding $L^+$ and $L^-$ in $L(x) = x^{-1/2}L^+ + x^{1/2}L^-$, the equations (2.11) are equivalent to

$$R_{12}(x/y)L_1(x)L_2(y) = L_2(y)L_1(x)R_{12}(x/y)$$ (2.12)

Explicitly, these RLL relations read

$$(L_{11}^\pm)^2 = (L_{22}^\pm)^2 \quad [L_{11}^\pm, L_{22}^\pm] = 0$$
$$(L_{12}^\pm)^2 = -(L_{21}^\pm)^2 \quad [L_{12}^\pm, L_{21}^\pm]^+ = 0$$
$$L_{11}^\pm L_{12}^\pm = L_{22}^\pm L_{21}^\pm \quad L_{11}^\pm L_{21}^\pm = L_{22}^\pm L_{12}^\pm$$
$$L_{12}^\pm L_{11}^\pm = -L_{22}^\pm L_{22}^\pm \quad L_{12}^\pm L_{22}^\pm = -L_{21}^\pm L_{11}^\pm$$ (2.13)

and for the $RL^+L^-$ ones

$$L_{ij}^\pm L_{ki}^- - L_{ij}^- L_{ki}^+ + \theta_1 L_{ij}^+ L_{ki}^- + \theta_2 L_{ij}^- L_{ki}^+ = 0$$ (2.14)

with $\bar{n} \equiv 3 - n$, $\theta_1 = 1$, $\theta_2 = -1$. The $RL^+L^+$ relations are to be compared with (2.10).

Introducing

$$\tilde{L}_{11} = L_{11}^\pm + L_{22}^\pm \quad \tilde{L}_{22} = L_{22}^\pm - L_{11}^\pm$$
$$\tilde{L}_{12}^\pm = L_{12}^\pm + L_{21}^\pm \quad \tilde{L}_{21}^\pm = L_{21}^\pm - L_{12}^\pm$$ (2.15)

the relations (2.13) become

$$\tilde{L}_{11}^\pm \tilde{L}_{22}^\pm = 0 \quad \tilde{L}_{22}^\pm \tilde{L}_{11}^\pm = 0$$
$$(\tilde{L}_{12}^\pm)^2 = 0 \quad (\tilde{L}_{21}^\pm)^2 = 0$$
$$\tilde{L}_{11}^\pm \tilde{L}_{21}^\pm = 0 \quad \tilde{L}_{12}^\pm \tilde{L}_{11}^\pm = 0$$
$$\tilde{L}_{21}^\pm \tilde{L}_{22}^\pm = 0 \quad \tilde{L}_{22}^\pm \tilde{L}_{12}^\pm = 0$$ (2.16)
the relations (2.14) become

\begin{align*}
[L_{11}^-, L_{11}^+] &= 0, & L_{21}^+ L_{11}^- &= L_{21}^+ L_{11}^-, \\
\tilde{L}_{11}^- L_{12}^+ &= \tilde{L}_{11}^+ L_{12}^-, & \tilde{L}_{21}^- L_{12}^+ &= \tilde{L}_{21}^+ L_{12}^-, \\
L_{11}^- L_{21}^+ &= \tilde{L}_{21}^+ L_{21}^-, & L_{21}^- L_{21}^+ &= -L_{11}^- L_{22}^-, \\
\tilde{L}_{11}^- L_{22}^+ &= \tilde{L}_{21}^+ L_{22}^- & L_{21}^- L_{22}^+ &= -L_{11}^- L_{22}^- \quad (2.17)
\end{align*}

We would like to introduce a basis for the FRT dual algebra. We need the following notation:

\begin{align*}
F_n(k_i; l_i) &\equiv \prod_{i=1}^{n} L_{11}^{+k_i} L_{12}^{+l_i} L_{22}^{+k_i} L_{21}^+, \quad n \geq 1, \\
G_n(l_i; k_i) &\equiv \prod_{i=1}^{n} \tilde{L}_{22}^{+l_i} \tilde{L}_{21}^{+k_i} \tilde{L}_{11}^+, \quad n \geq 1, \\
F_0(k_i; l_i) &\equiv 1; \quad G_0(l_i; k_i) \equiv 1. \quad (2.18)
\end{align*}

The basis elements of the algebra generated by the \( \tilde{L}^- \)'s are:

\begin{align*}
F_n(k_i; l_i) \tilde{L}_{11}^{+k_n} &\equiv F_{n-1}(k_i; l_i) \tilde{L}_{11}^{+k_n} \tilde{L}_{12}^+ \tilde{L}_{22}^+ l_n, \\
G_n(l_i; k_i) \tilde{L}_{22}^{+l_n} &\equiv G_{n-1}(l_i; k_i) \tilde{L}_{22}^{+l_n} \tilde{L}_{21}^+ \tilde{L}_{11}^+. \quad (2.19)
\end{align*}

Defining also \( K_n = \sum_{i=1}^{n} k_i, \quad L_n = \sum_{i=1}^{n} l_i \) the actions of generators \( \tilde{L}^- \) on the basis elements are, e.g.,

\begin{align*}
\tilde{L}_{11}^- F_n(k_i; l_i) &= F_{n-1}(k_1 + 1, k_i; l_i) \tilde{L}_{11}^{+k_n} \tilde{L}_{12}^+ \tilde{L}_{22}^+ l_n, \\
\tilde{L}_{12}^- F_n(k_i; l_i) &= (-1)^{K_n + L_n + 1} G_{n-1}(k_1 + 1, k_i; l_i) \tilde{L}_{22}^{+l_n} \tilde{L}_{21}^+ \tilde{L}_{11}^+, \\
\tilde{L}_{21}^- F_n(k_i; l_i) &= G_n(0, \ldots, l_{n-1}, k_i) \tilde{L}_{22}^{+l_n} \tilde{L}_{21}^+, \\
\tilde{L}_{22}^- F_n(k_i; l_i) &= (-1)^{K_n + L_n} F_n(0, \ldots, l_{n-1}, k_i) \tilde{L}_{11}^{+l_n} \tilde{L}_{22}^+, \quad (2.20)
\end{align*}

\begin{align*}
\tilde{L}_{11}^- G_n(l_i; k_i) &= (-1)^{K_n + L_n} G_n(0, \ldots, l_{n-1}, k_i) \tilde{L}_{11}^{+k_n}, \\
\tilde{L}_{12}^- G_n(l_i; k_i) &= F_n(0, \ldots, k_{n-1}, l_i) \tilde{L}_{11}^{+k_n} \tilde{l}_{12}, \\
\tilde{L}_{21}^- G_n(l_i; k_i) &= (-1)^{K_n + L_n + 1} F_n(1 + 1, l_i; k_i) \tilde{L}_{11}^{+l_n} \tilde{L}_{12}^+ \tilde{L}_{22}^+ \tilde{L}_{11}^-, \\
\tilde{L}_{22}^- G_n(l_i; k_i) &= G_{n-1}(l_1 + 1, l_i; k_i) \tilde{L}_{22}^{+l_n} \tilde{L}_{21}^+ \tilde{L}_{11}^{+k_n} \tilde{L}_{12}^+. \quad (2.21)
\end{align*}

These equations allow one to order the \( \tilde{L}^- \) with respect to the \( \tilde{L}^+ \). For the \( \tilde{L}^- \) among themselves, there exists a basis similar to (2.14). Thus, this basis gives the class of representations when \( s03_F \) acts on itself.
2.4 Affine $s03_F$

$L^\pm(x)$ are now matrices of operators $L^\pm_{ij}(x)$ $(i, j = 1, 2)$ satisfying the relations

\begin{align*}
R^+(x_1/x_2) L^+_i(x_1) L^+_j(x_2) &= L^+_i(x_2) L^+_j(x_1) R^+(x_1/x_2) \\
R^-(x_1/x_2) L^-_i(x_1) L^-_j(x_2) &= L^-_i(x_2) L^-_j(x_1) R^+(x_1/x_2) \\
R^+(x_1/x_2) L^+_i(x_1) L^-_j(x_2) &= L^-_i(x_2) L^+_j(x_1) R^+(x_1/x_2)
\end{align*}

i.e.

\begin{align*}
(x_1 + x_2) \left( L^+_a(x_1) L^+_b(x_2) - L^+_a(x_2) L^+_b(x_1) \right) + \\
\theta_a(x_2 - x_1) L^+_a(x_1) L^+_b(x_2) + \theta_b(x_2 - x_1) L^+_a(x_1) L^+_b(x_2) = 0
\end{align*}

(2.22)

In particular,

\begin{align*}
[L^+_1(x_1), L^-_2(x_2)] - [L^+_2(x_1), L^-_1(x_2)] &= 0 \\
[L^+_1(x_1), L^-_1(x_2)] + [L^+_2(x_1), L^-_2(x_2)] &= 0
\end{align*}

(2.24)

3 Alternative parametrisation

Another parametrisation of the braid matrix is:

\[ \hat{R}(z) = \frac{(1 + z)\hat{R} + (1 - z)\hat{R}^{-1}}{(2(1 + z^2))^{1/2}} = \frac{1}{\sqrt{1 + z^2}} \begin{pmatrix} 1 & 0 & 0 & z \\ 0 & 1 & -z & 0 \\ 0 & z & 1 & 0 \\ -z & 0 & 0 & 1 \end{pmatrix} \]

(3.1)

Some advantages of this parametrisation are:

\[ \hat{R}(\pm 1) = \hat{R}^{\pm 1} \]

(3.2)

and

\[ \hat{R}_{12}(z')\hat{R}_{23}(z)\hat{R}_{12}(z') = \hat{R}_{23}(z')\hat{R}_{12}(z)\hat{R}_{23}(z'') \]

(3.3)

where

\[ z'' = \frac{z - z'}{1 - zz'} \]

(3.4)

A Baxterisation for $L^\pm$ is:

\[ L(z) = \frac{(1 + z)L^+ + (1 - z)L^-}{(2(1 + z^2))^{1/2}} \]

(3.5)
here

\[ L(\pm 1) = L^\pm \quad (3.6) \]

and

\[ \hat{R}(z'')L_2(z)L_1(z') = L_2(z')L_1(z)\hat{R}(z'') \quad (3.7) \]

If we accept the convention \( z'' = 1 \) for \( z = \pm 1 \) and \( z' = \pm 1 \) we can reproduce the formulae for \( L^\pm \) \( (2.11,2.12,2.13,2.14) \).

For \( \hat{R} \) matrices satisfying a minimal quadratic equation (the first equation of \( (2.8) \) being an example) there exist two possibilities of defining co-products of \( L \). Here they correspond to A) and B) below:

A)

\[ \delta L_{ij}(z) = \sum_k L_{ik}(z) \otimes L_{kj}(z) \quad (3.8) \]

B)

\[ \tilde{\delta} L_{ij}(z) = \frac{1}{(2(1 + z^2)^{1/2})}((1 - z)\delta L^+_{ij} + (1 + z)\delta L^-_{ij}) \quad (3.9) \]

where

\[ L(z) = \begin{pmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{pmatrix} \quad (3.10) \]

Both types of coproducts satisfy the equations

\[ \hat{R}(z'')(\delta L(z))_2(\delta L(z'))_1 = (\delta L(z'))_2(\delta L(z))_1 \hat{R}(z'') \quad (3.11) \]

and exactly the same for \( \tilde{\delta} L \).

A two-dimensional representation for the algebra generated by the \( L \)-operators is provided by the \( R \)-matrix itself, setting \( \pi(L^+) = R_{21}, \pi(L^-) = R^{-1} \) (see \([6,7]\)). Thus, if for \( L \) is used the fundamental representation

\[ L(z) = \begin{pmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{pmatrix} = \hat{R}(z)P = \frac{1}{\sqrt{1 + z^2}} \begin{pmatrix} 1 & 0 & 0 & z \\ 0 & -z & 1 & 0 \\ 0 & 1 & z & 0 \\ -z & 0 & 0 & 1 \end{pmatrix} \quad (3.12) \]
we have the following explicit forms

\[
\begin{pmatrix}
1 & 0 & 0 & z \\
0 & -z & -z^2 & 0 \\
0 & 1 & -z & 0 \\
-z & 0 & 0 & z^2
\end{pmatrix}, \quad \tilde{\delta}L_{11}(z) =
\begin{pmatrix}
1 & 0 & 0 & z \\
0 & -z & -1 & 0 \\
0 & 1 & -z & 0 \\
-z & 0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & z & z^2 & 0 \\
1 & 0 & 0 & z \\
z & 0 & 0 & -z^2 \\
0 & 1 & -z & 0
\end{pmatrix}, \quad \tilde{\delta}L_{12}(z) =
\begin{pmatrix}
0 & z & 1 & 0 \\
-1 & 0 & 0 & -z \\
z & 0 & 0 & 1 \\
0 & 1 & -z & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
-2 & 0 & 0 & 1 \\
z & 0 & 0 & 1 \\
0 & -z^2 & z & 0 \\
-z & 0 & 0 & 1
\end{pmatrix}, \quad \tilde{\delta}L_{22}(z) =
\begin{pmatrix}
1 & 0 & 0 & z \\
0 & z & 1 & 0 \\
0 & -1 & z & 0 \\
-z & 0 & 0 & 1
\end{pmatrix}
\]

These two sets coincide for \(z = \pm 1\) but except for these limits they can be shown to be inequivalent.

We use the diagonaliser \(M\) (cf. [8]):

\[
M = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & i \\
0 & 1 & -i & 0 \\
0 & -i & 1 & 0 \\
i & 0 & 0 & 1
\end{pmatrix}.
\]

(3.14)

to define the matrices \(X\)

\[
ML_2(z)L_1(z')M^{-1} = X(z, z'), \quad ML_2(z')L_1(z)M^{-1} = X'(z', z)
\]

(3.15)

so that

\[
X_{ij}(z, z') = X'_{ij}(z', z) \quad i, j = 1, 2, 3, 4.
\]

(3.16)
Denote \( L(z) = L \) and \( L(z') = L' \). Then we have for the components of \( X \):

\[
\begin{pmatrix}
X_{11} \\
X_{44}
\end{pmatrix} = (L_{11} L'_{11} + L_{22} L'_{22}) \pm i(L_{21} L'_{21} - L_{12} L'_{12})
\]

\[
\begin{pmatrix}
X_{12} \\
X_{43}
\end{pmatrix} = \pm (L_{12} L'_{11} - L_{21} L'_{22}) + i(L_{22} L'_{21} + L_{11} L'_{12})
\]

\[
\begin{pmatrix}
X_{21} \\
X_{34}
\end{pmatrix} = \pm (L_{21} L'_{11} - L_{12} L'_{22}) - i(L_{11} L'_{21} + L_{22} L'_{12})
\]

\[
\begin{pmatrix}
X_{22} \\
X_{33}
\end{pmatrix} = (L_{22} L'_{11} + L_{11} L'_{22}) \mp i(L_{12} L'_{21} - L_{21} L'_{12})
\]

\[
\begin{pmatrix}
X_{13} \\
X_{42}
\end{pmatrix} = \pm (L_{11} L'_{12} - L_{22} L'_{21}) + i(L_{21} L'_{11} + L_{12} L'_{22})
\]

\[
\begin{pmatrix}
X_{23} \\
X_{32}
\end{pmatrix} = (L_{12} L'_{21} + L_{21} L'_{12}) \pm i(L_{22} L'_{11} - L_{11} L'_{22})
\]

\[
\begin{pmatrix}
X_{14} \\
X_{41}
\end{pmatrix} = (L_{21} L'_{21} + L_{12} L'_{12}) \mp i(L_{11} L'_{11} - L_{22} L'_{22})
\]

\[
\begin{pmatrix}
X_{24} \\
X_{31}
\end{pmatrix} = \mp (L_{11} L'_{21} - L_{22} L'_{12}) - i(L_{21} L'_{11} + L_{12} L'_{22}). \tag{3.17}
\]

Then the Yang-Baxter equation (3.4) reads:

\[
(\hat{R}(z''))_{\alpha}X = X'(\hat{R}(z''))_{\alpha}
\]

which gives for the \( \hat{R}L_{2}L_{1} \) relations the following explicit formulae

\[
X_{ij}(z, z') = X'_{ij}(z', z), \quad (ij) = (11, 12, 21, 22; 33, 34, 43, 44), \tag{3.19}
\]

\[
((1 - z z') - i(z - z'))X_{ij} = ((1 - z z') + i(z - z'))X'_{ij}, \quad (ij) = (13, 14, 23, 24),
\]

\[
((1 - z z') + i(z - z'))X_{ij} = ((1 - z z') - i(z - z'))X'_{ij}, \quad (ij) = (31, 41, 32, 42)
\]

Having in mind (3.17) the general structure of the \( X_{ij} \) (up to normalisation factors) is:

\[
X_{ij} = (1 + zz') A_{ij} + (1 + zz') B_{ij} + (z + z') C_{ij}, \quad (ij) = (11, 12, 21, 22; 33, 34, 43, 44)
\]

\[
X_{ij} = ((1 - z z' + i(z - z')) Q_{ij}, \quad (ij) = (13, 14, 23, 24)
\]

\[
X_{ij} = ((1 - z z' - i(z - z')) N_{ij}, \quad (ij) = (31, 41, 32, 42) \tag{3.20}
\]

where the matrices \( \{A_{ij}, B_{ij}, C_{ij}, Q_{ij}, N_{ij}\} \) do not depend on \( z \). However they can not be arbitrary, but should be compatible with the definitions (3.17) for \( X_{ij} \).
An example: If we take the fundamental representation for $L(z)$ (3.12) we obtain

$$
\begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix} = ((1 + zz') - i(z + z'))
\begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
0 & -i & i & 0 \\
-i & 0 & 0 & -i
\end{pmatrix}
\begin{pmatrix}
X_{33} & X_{34} \\
X_{43} & X_{44}
\end{pmatrix} = ((1 + zz') + i(z + z'))
\begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & -1 \\
-i & 0 & 0 & i \\
0 & -i & -i & 0
\end{pmatrix}
\begin{pmatrix}
X_{13} & X_{14} \\
X_{23} & X_{24}
\end{pmatrix} = i((1 - zz') + i(z - z'))
\begin{pmatrix}
0 & i & i & 0 \\
-i & 0 & 0 & i \\
1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
X_{31} & X_{32} \\
X_{41} & X_{42}
\end{pmatrix} = -i((1 - zz') + i(z - z'))
\begin{pmatrix}
0 & i & i & 0 \\
-i & 0 & 0 & i \\
1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0
\end{pmatrix}
$$

(3.21)

So in that case:

$$
B_{ij} = 0, \quad C_{ij} = \mp iA_{ij} .
$$

(3.22)

A class of representations for arbitrary dimensions can be considered. Based on formulae (3.13) and subsequent use of the diagonaliser $M$ we have:

$$
M(\delta L_{ij})M^{-1} = \begin{pmatrix}
U_{ij} & 0 \\
0 & D_{ij}
\end{pmatrix}
$$

(3.23)

where (neglecting normalisation factors):

$$
\begin{pmatrix}
U_{11}, U_{22}, U_{12}, U_{21}
\end{pmatrix} = (1 - iz)
\begin{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -i
\end{pmatrix}, 
\begin{pmatrix}
1 & 0 \\
0 & i
\end{pmatrix}, 
\begin{pmatrix}
0 & i \\
1 & 0
\end{pmatrix}, 
\begin{pmatrix}
0 & i \\
-1 & 0
\end{pmatrix}
\end{pmatrix},
$$

$$
\begin{pmatrix}
D_{11}, D_{22}, D_{12}, D_{21}
\end{pmatrix} = i(1 + iz)
\begin{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -i
\end{pmatrix}, 
\begin{pmatrix}
-1 & 0 \\
0 & -i
\end{pmatrix}, 
\begin{pmatrix}
0 & i \\
1 & 0
\end{pmatrix}, 
\begin{pmatrix}
0 & -i \\
1 & 0
\end{pmatrix}
\end{pmatrix}.
$$

(3.24)

For $z = \pm 1$ $U_{ij}^{\pm}$, $D_{ij}^{\pm}$ give a particular class of complex $2 \times 2$ representations.

In view of (3.24), where there are factors $(1 \pm iz)$ appearing, we try the Ansatz:

$$
L_{ij} = (1 + kz)\hat{L}_{ij}
$$

(3.25)

where $\hat{L}_{ij}$ is $z$-independent. Then each $X_{ij}$ would be proportional to $(1 + kz)(1 + kz')$ and relations (3.20) are satisfied with:

$$
(A_{ij} - B_{ij}) = k^2(A_{ij} + B_{ij}) = kC_{ij} , \quad Q_{ij} = 0, \quad N_{ij} = 0 .
$$

(3.26)
Now consider as an example the $3 \times 3$ case:

$$
L_{11} = (1 + kz) \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad L_{12} = (1 + kz) \begin{pmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ e & 0 & f \end{pmatrix},
$$

$$
L_{22} = \pm (1 + kz) \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad L_{21} = \pm (1 + kz) \begin{pmatrix} 0 & d & 0 \\ e & 0 & f \\ 0 & g & 0 \end{pmatrix}, \quad L_{21} = \pm (1 + kz) \begin{pmatrix} 0 & d & 0 \\ e & 0 & f \\ 0 & g & 0 \end{pmatrix}, \quad (3.27)
$$

It can be seen that equations (3.26) are satisfied.

The analogue of (3.27) for the general $n \times n$ case is:

$$
(L_{11})_{mn} = (1 + kz) \delta_{m,n} a_m,
$$

$$
(L_{22})_{mn} = \epsilon (1 + kz) \delta_{m,n} (-1)^{m-1} a_m,
$$

$$
(L_{12})_{mn} = (1 + kz) (\delta_{m,n+1} u_m + \delta_{n,m-1} v_m),
$$

$$
(L_{21})_{mn} = \epsilon (1 + kz) (\delta_{m,n+1} (-1)^{m-1} u_m + \delta_{n,m-1} (-1)^{m-1} v_m) \quad (3.28)
$$

### 4 Finite dimensional representations

#### 4.1 Representations on $S_{03}$

Here we shall study the representations of $s_{03_F}$ obtained by the use of its right regular action (RRA) on the dual bialgebra $S_{03}$. The RRA is defined as follows:

$$
\pi_R(L^\pm_{ij}) f = f(1) < L^\pm_{ij}, f(2) >
$$

where we use Sweedler’s notation for the co-product: $\delta(f) = f(1) \otimes f(2)$. More explicitly, for the generators of $s_{03_F}$ we have:

$$
\pi_R(L^\pm_{11}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \mp b \\ c & \mp d \end{pmatrix}, \quad \pi_R(L^\pm_{12}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & \pm a \\ d & \pm c \end{pmatrix},
$$

$$
\pi_R(L^\pm_{21}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mp b & a \\ \mp d & c \end{pmatrix}, \quad \pi_R(L^\pm_{22}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm a & b \\ \pm c & d \end{pmatrix} \quad (4.2)
$$

Obviously, the above representation is the direct sum of two equivalent two-dimensional irreps (with vector spaces spanned by $\{a, b\}$ and $\{c, d\}$, respectively) such that the representation matrices (acting on $(a, b)$ or $(c, d)$ from the right) are given by:

$$
\pi_R(L^\pm_{11}) = \begin{pmatrix} 1 & 0 \\ 0 & \mp 1 \end{pmatrix}, \quad \pi_R(L^\pm_{12}) = \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix},
$$

$$
\pi_R(L^\pm_{21}) = \begin{pmatrix} 0 & 1 \\ \mp 1 & 0 \end{pmatrix}, \quad \pi_R(L^\pm_{22}) = \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.3)
$$
Further, we would like to consider polynomials in the elements $a, b, c, d$ of degree $N > 1$. Superficially, for fixed $N$ such polynomials would span a vector space of dimension $4^N$, however, due to the relations (4.10) such polynomials actually span a vector space of dimension $2^{N+1}$. Explicitly, these vector spaces are spanned by:

$$\{a, b\}^{\otimes N}, \{c, d\} \otimes \{a, b\}^{\otimes N-1}. \tag{4.4}$$

These vector spaces split into irreducible representations for which the most suitable bases are complex linear combinations of the above. We give explicitly some examples of small $N$ and then formulate a general statement.

For $N = 2$ from the vector space of dimension 8 one can extract four two-dimensional irreducible representations (two by two equivalent). The representation spaces are spanned over the elements $V_1^\epsilon = a^2 + i\epsilon b^2$, $V_2^\epsilon = ab - i\epsilon ba$, where $\epsilon = \pm 1$ labels the two non-equivalent irreducible representations. Another two irreducible representations are spanned over the elements $\tilde{V}_1^\epsilon = ca + i\epsilon db$, $\tilde{V}_2^\epsilon = cb - i\epsilon da$, however, they are equivalent to the first two for coinciding values of $\epsilon$. In a matrix form the representations are as follows:

$$\pi_R(L_{11}^\pm) = (1 \pm i\epsilon) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pi_R(L_{12}^\pm) = (1 \pm i\epsilon) \begin{pmatrix} 0 & -i\epsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{4.5}$$

$$\pi_R(L_{21}^\pm) = (1 \pm i\epsilon) \begin{pmatrix} 0 & -i\epsilon & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pi_R(L_{22}^\pm) = (1 \pm i\epsilon) \begin{pmatrix} 1 & 0 & 0 \\ 0 & i\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.5}$$

(In these formulae the $\epsilon$ just indicates the the representation concerned.)

Having these elements one can proceed further to construct all representations for any fixed $N$. For $N = 3$ the overall vector space is 16-dimensional. We first consider the 8-dimensional vector space $\{a, b\}^{\otimes 3}$ for which the convenient basis is:

$$U_1^\epsilon = aV_1^\epsilon, \quad U_2^\epsilon = bV_2^\epsilon, \quad U_3^\epsilon = aV_2^\epsilon, \quad U_4^\epsilon = bV_1^\epsilon. \tag{4.6}$$

These elements form two four-dimensional irreducible representations, labelled again by the index $\epsilon$. In matrix form these representations can be written as:

$$\pi_R(L_{11}^\pm) = (1 \pm i\epsilon) \begin{pmatrix} 1 & \mp i\epsilon & 0 & 0 \\ -1 & \mp i\epsilon & 0 & 0 \\ 0 & 0 & i\epsilon & \mp 1 \\ 0 & 0 & -i\epsilon & \mp 1 \end{pmatrix}, \quad \pi_R(L_{12}^\pm) = (1 \pm i\epsilon) \begin{pmatrix} 0 & 0 & -i\epsilon & \pm 1 \\ 0 & 0 & i\epsilon & \mp 1 \\ 1 & \mp i\epsilon & 0 & 0 \\ 1 & \mp i\epsilon & 0 & 0 \end{pmatrix} \tag{4.7}$$

$$\pi_R(L_{21}^\pm) = (1 \pm i\epsilon) \begin{pmatrix} 0 & 0 & \mp i\epsilon & 1 \\ 0 & 0 & \mp i\epsilon & -1 \\ \pm 1 & i\epsilon & 0 & 0 \\ \mp 1 & -i\epsilon & 0 & 0 \end{pmatrix}, \quad \pi_R(L_{22}^\pm) = (1 \pm i\epsilon) \begin{pmatrix} \pm 1 & -i\epsilon & 0 & 0 \\ \mp 1 & -i\epsilon & 0 & 0 \\ 0 & 0 & \mp i\epsilon & 1 \\ 0 & 0 & \pm i\epsilon & 1 \end{pmatrix}. \tag{4.7}$$

Clearly, these four-dimensional representations are irreducible. For the remaining 8-dimensional vector space coming from $\{c, d\} \otimes \{a, b\}^{\otimes 2}$ the convenient basis is:

$$\tilde{U}_1^\epsilon = cV_1^\epsilon, \quad \tilde{U}_2^\epsilon = dV_2^\epsilon, \quad \tilde{U}_3^\epsilon = cV_2^\epsilon, \quad \tilde{U}_4^\epsilon = dV_1^\epsilon. \tag{4.8}$$
The transformation rules for the elements $\hat{U}^{\epsilon}$ are the same as those for $U^{\epsilon}$ given in (4.17) for the same values of $\epsilon$. Thus, again we have four irreducible representations, which are two by two equivalent.

For $N = 4$ the overall vector space is 32-dimensional. Using the elements $V_{i}^{\epsilon}$ and $\hat{V}_{i}^{\epsilon}$ it can be split into the following 8 four-dimensional representations:

$$\omega_{ij}^{\epsilon, \epsilon} = V_{i}^{\epsilon} V_{j}^{\epsilon}, \quad \hat{\omega}_{ij}^{\epsilon, \epsilon} = \hat{V}_{i}^{\epsilon} V_{j}^{\epsilon}, \quad \tilde{\omega}_{ij}^{\epsilon, -\epsilon} = \hat{V}_{i}^{\epsilon} V_{j}^{\epsilon -\epsilon}, \quad \epsilon = \pm 1,$$

(4.9)

(four sets doubled by $\epsilon$), where the indices $ij$ enumerate the four elements of a representation. In a matrix form the representation formulae for $\omega$ are:

$$\pi_{R}(L_{11}^{\pm}) = \pm 2 \begin{pmatrix} i \epsilon & 0 & 0 & -i \epsilon \\ 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -i \epsilon & 0 & 0 & -i \epsilon \end{pmatrix}, \quad \pi_{R}(L_{12}^{\pm}) = \pm 2 \begin{pmatrix} 0 & 1 & -1 & 0 \\ i \epsilon & 0 & 0 & i \epsilon \\ i \epsilon & 0 & 0 & i \epsilon \\ 0 & 1 & -1 & 0 \end{pmatrix} \quad (4.10)$$

$$\pi_{R}(L_{21}^{\pm}) = \pm 2 \begin{pmatrix} 0 & 1 & 1 & 0 \\ -i \epsilon & 0 & 0 & i \epsilon \\ -i \epsilon & 0 & 0 & -i \epsilon \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad \pi_{R}(L_{22}^{\pm}) = \pm 2 \begin{pmatrix} i \epsilon & 0 & 0 & -i \epsilon \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -i \epsilon & 0 & 0 & -i \epsilon \end{pmatrix}.$$

To write this matrix formula we used the conventional ordering of the elements $\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}$. The analogous formulae for the $\hat{\omega}$ read:

$$\pi_{R}(L_{11}^{\pm}) = 2 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -i \epsilon & i \epsilon & 0 \\ 0 & i \epsilon & i \epsilon & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad \pi_{R}(L_{12}^{\pm}) = 2 \begin{pmatrix} 0 & i \epsilon & -i \epsilon & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & i \epsilon & i \epsilon & 0 \end{pmatrix} \quad (4.11)$$

$$\pi_{R}(L_{21}^{\pm}) = 2 \begin{pmatrix} 0 & i \epsilon & i \epsilon & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & i \epsilon & i \epsilon & 0 \end{pmatrix}, \quad \pi_{R}(L_{22}^{\pm}) = \pm 2 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i \epsilon & i \epsilon & 0 \\ 0 & -i \epsilon & -i \epsilon & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

One can check that the four-dimensional representations given by (4.10) or (4.11) are irreducible. The representations $\tilde{\omega}_{i}^{\epsilon, \epsilon}, (\tilde{\omega}_{i}^{\epsilon, -\epsilon})$, are equivalent to $\omega_{i}^{\epsilon, \epsilon}, (\hat{\omega}_{i}^{\epsilon, -\epsilon})$, for the respective value of $\epsilon$.

We have carried out explicitly also the cases $N = 5, 6$ and all these results lead us to the following general statements.

In general the bases of degree $N = 2n$ can be written in the form:

$$\Omega_{i_{1}i_{2},...,i_{n}}^{\epsilon} = V_{i_{1}}^{\epsilon_{1}} V_{i_{2}}^{\epsilon_{2}} ... V_{i_{n}}^{\epsilon_{n}}, \quad \tilde{\Omega}_{i_{1}i_{2},...,i_{n}}^{\epsilon} = \hat{V}_{i_{1}}^{\epsilon_{1}} V_{i_{2}}^{\epsilon_{2}} ... V_{i_{n}}^{\epsilon_{n}} \quad (4.12)$$

where the set of indices $\{\epsilon_{1}, \epsilon_{2}, ..., \epsilon_{n}\}$ labels the $2^n$ representations, while the indices $\{i_{1}, i_{2}, ..., i_{n}\}$ enumerate the $2^n$ elements within a given representation.
The bases of odd order $N = 2n + 1$ are constructed multiplying the above even elements $\Omega$ from the left by $a$ and $b$, then by $c$ and $d$, the second batch of representations being equivalent to the first.

Thus, we can formulate the following general statement:

**Proposition 4.1**

- Tensor products of $2^{2n}$ $2$-dimensional representations of the type described in (4.3) (constructed using the coproduct structure) decompose into sums of $2^{n+1}$ $2^n$-dimensional irreducible representations. These are $2$-by-$2$ equivalent, i.e., the number of non-equivalent irreducible representations is $2^n$.

- Tensor products of $2^{2n+1}$ $2$-dimensional representations of the type described in (4.3) decompose into sums of $2^{n+1}$ $2^{n+1}$-dimensional irreducible representations. These are $2$-by-$2$ equivalent, i.e., the number of non-equivalent irreducible representations is $2^n$.

### 4.2 Finite dimensional irreducible representations (other constructions)

#### 4.2.1 Generalities

$L^+_1$ and $L^-_1$ commute, so they have a common eigenvector $v_0$.

\[
L^+_1 v_0 = \lambda^+ v_0 \quad L^-_1 v_0 = \lambda^- v_0
\]  

(4.13)

**A.** Let us first suppose $\lambda^+ \neq 0$ and $\lambda^- \neq 0$. Then

\[
\tilde{L}^+_2 v_0 = \tilde{L}^-_2 v_0 = 0
\]

(4.14)

On the whole representation,

\[
\tilde{L}^-_{ij} = \frac{\lambda^-}{\lambda^+} \tilde{L}^+_{ij}
\]  

(4.15)

Indeed,

\[
\tilde{L}^-_{i1} v_0 = \frac{1}{\lambda^+} \tilde{L}^-_{i1} \tilde{L}^+_1 v_0 = \frac{1}{\lambda^+} \tilde{L}^+_1 \tilde{L}^-_{i1} v_0 = \frac{\lambda^-}{\lambda^+} \tilde{L}^+_1 v_0
\]  

(4.16)

and, by recursion,

\[
\tilde{L}^-_{ij1} (\tilde{L}^+_1 \tilde{L}^+_2 \tilde{L}^+_j \tilde{L}^+_j \cdots \tilde{L}^+_j) v_0 = \tilde{L}^+_{ij1} \tilde{L}^-_{j1j2} (\tilde{L}^+_j \tilde{L}^+_j \cdots \tilde{L}^+_j) v_0
\]

\[
= \frac{\lambda^-}{\lambda^+} \tilde{L}^-_{ij1} \tilde{L}^+_1 \tilde{L}^+_2 \tilde{L}^+_j \tilde{L}^+_j \cdots \tilde{L}^+_j v_0
\]  

(4.17)

when consecutive indices coincide, whereas in the other case

\[
\tilde{L}^-_{ij1} (\tilde{L}^+_1 \tilde{L}^+_2 \tilde{L}^+_j \tilde{L}^+_j \cdots \tilde{L}^+_j) v_0 = \pm \tilde{L}^+_1 \tilde{L}^-_{ij1} \tilde{L}^+_j \tilde{L}^+_j \cdots \tilde{L}^+_j v_0 = 0
\]  

(4.18)

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B. Let us consider the case $\lambda^+ \neq 0$ and $\lambda^- = 0$. This case is not yet completely understood. Let us mention that the regular representation on linear terms in $a$, $b$, $c$ and $d$ described by (4.3) falls in this case.

A particular class of such representations corresponds to the choice

$$L_{ii}^- = L_{ii}^+ \quad L_{ii}^- = xL_{ii}^+$$

(4.19)

i.e.

$$\tilde{L}_{i1}^- = \tilde{L}_{i2}^+ , \quad \tilde{L}_{i2}^- = x\tilde{L}_{i2}^+ , \quad \tilde{L}_{21}^- = -x\tilde{L}_{12}^+ , \quad \tilde{L}_{22}^- = -\tilde{L}_{11}^-$$

(4.20)

leading to supplementary relations for $\tilde{L}^+$ (with respect to (2.16))

$$\tilde{L}_{12}^+\tilde{L}_{21}^- = -x^{-1}(\tilde{L}_{11}^+)^2 \quad \tilde{L}_{21}^+\tilde{L}_{12}^- = -x^{-1}(\tilde{L}_{22}^+)^2$$

(4.21)

### 4.2.2 2-dim irreps

Two dimensional representations fall again into three cases.

- Those on which $\tilde{L}^\pm v_0 = \lambda^\pm v_0$ with both $\lambda^+$, $\lambda^-$ non-zero. They are described by

  $$\pi(\tilde{L}_{11}^+) = \begin{pmatrix} \lambda^+ & 0 \\ 0 & 0 \end{pmatrix} \quad \pi(\tilde{L}_{12}^+) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

  $$\pi(\tilde{L}_{21}^+) = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \quad \pi(\tilde{L}_{22}^+) = \begin{pmatrix} 0 & 0 \\ 0 & \mu^+ \end{pmatrix}$$

  (4.22)

  and $\pi(\tilde{L}_{ij}^-) = \frac{\lambda^-}{\lambda^+} \pi(\tilde{L}_{ij}^+)$.  

- Those on which there exists $v_0$ such that $\tilde{L}^\pm v_0 = \lambda^\pm v_0$ with $\lambda^+ \neq 0$, $\lambda^- = 0$.

  $$\pi(\tilde{L}_{11}^+) = \begin{pmatrix} \lambda^+ & 0 \\ 0 & 0 \end{pmatrix} \quad \pi(\tilde{L}_{12}^+) = \begin{pmatrix} 0 & x\lambda^+ \\ 0 & 0 \end{pmatrix}$$

  $$\pi(\tilde{L}_{21}^+) = \begin{pmatrix} 0 & 0 \\ x\lambda^+ & 0 \end{pmatrix} \quad \pi(\tilde{L}_{22}^+) = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda^+ \end{pmatrix}$$

  (4.23)

  $$\pi(\tilde{L}_{11}^-) = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} \quad \pi(\tilde{L}_{12}^-) = \begin{pmatrix} 0 & x^{-1}\mu \\ x^{-1}\mu & 0 \end{pmatrix}$$

  $$\pi(\tilde{L}_{21}^-) = \begin{pmatrix} 0 & -x^{-1}\mu \\ 0 & 0 \end{pmatrix} \quad \pi(\tilde{L}_{22}^-) = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}$$

  (4.24)
Those representations with only 0 eigenvalues for $\tilde{L}^{\pm}_{11}$. In the case of Jordan form for $\tilde{L}^{\pm}_{11}$, the corresponding representation can be proved not to be irreducible. Hence

$$\pi(\tilde{L}^+_{11}) = 0, \quad \pi(\tilde{L}^+_{22}) = 0$$

(4.25)

Then $\tilde{L}^+_{12}$ and $\tilde{L}^+_{21}$ should be of the form

$$\pi(\tilde{L}^+_{12}) = \begin{pmatrix} 0 & \ell^+_{12} \\ 0 & 0 \end{pmatrix}, \quad \pi(\tilde{L}^+_{21}) = a^\pm \begin{pmatrix} 1 & b \\ -b & -1 \end{pmatrix}$$

(4.26)

with $a^+\ell^-_{12} = a^-\ell^+_{12}$.

### 4.2.3 Other irreps

**Example 4.2** Here is an example (in the $\tilde{L}$ basis) of a finite dimensional irreducible representation of arbitrary dimension $N_1 + N_2$, where $N_1$ and $N_2$ are two non-negative integers:

$$\pi(\tilde{L}_{11}) = \text{diag}(\rho_1, \ldots, \rho_{N_1}, 0, \ldots, 0)$$

$\rho_i \neq \rho_j$ for $i \neq j$

$$\pi(\tilde{L}_{22}) = \text{diag}(0, \ldots, 0, \lambda_1, \ldots, \lambda_{N_2})$$

$\lambda_i \neq \lambda_j$ for $i \neq j$

$$\left(\pi(\tilde{L}_{12})\right)_{ij} \neq 0 \quad \text{iff} \quad i \in \{1, \ldots, N_1\}, \quad j \in \{N_1 + 1, \ldots, N_1 + N_2\}$$

$$\left(\pi(\tilde{L}_{21})\right)_{ij} \neq 0 \quad \text{iff} \quad i \in \{N_1 + 1, \ldots, N_1 + N_2\}, \quad j \in \{1, \ldots, N_1\}.$$  

(4.27)

### 5 Diagonalisation of $\hat{R}$, fusion and evaluation representations

Due to (2.7), if $\hat{R}$ is diagonalisable with the matrix $M$, then $\hat{R}(x)$ will be diagonalisable with the same matrix $M$ independent of $x$. Actually

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}. \quad (5.1)$$

is such that $M\hat{R}(x)M^{-1} = \frac{1 + i}{2\sqrt{2}} \text{diag}(x - i, x - i, 1 - ix, 1 - ix)$.

Let $\mu_1(x), \mu_2(x)$ be the eigenvalues of $\hat{R}(x)$, then

$$\Pi^{(1)} = \frac{\hat{R}(x) - \mu_2(x)}{\mu_1(x) - \mu_2(x)} \quad \Pi^{(2)} = \frac{\hat{R}(x) - \mu_1(x)}{\mu_2(x) - \mu_1(x)} \quad (5.2)$$

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are projectors ($\Pi^{(i)}_i^2 = \Pi^{(i)}_i$ and $\Pi^{(1)} + \Pi^{(2)} = 1$) on the eigenspaces of $\hat{R}(x)$. They are independent of $x$.

Taking the representations $\pi(L)$ given by (4.3),

\[
(\pi L^+_{ij})^p_m = (R_{21})^{jp}_{im} \quad \quad \quad (\pi L^-_{ij})^p_m = (R^{-1})^{jp}_{im} \quad (5.3)
\]

\[
(\pi \otimes \pi) (\delta(L^+_{ij}))^pq_{mn} = \pi (L^+_{ik})^p_m \pi (L^+_{kj})^q_n = R_{mi}^{pk} R_{nj}^{ql} = R_{21} R_{31} \quad \text{(formally)} \quad (5.4)
\]

Using the Yang–Baxter equation $\hat{R}_{23} R_{21} R_{31} = R_{21} R_{31} \hat{R}_{23}$, one has

\[
\hat{R}_{23} (\pi \otimes \pi) \delta(L^+) = (\pi \otimes \pi) \Delta(L^+) \hat{R}_{23} \quad (5.5)
\]

Similarly

\[
(\pi \otimes \pi) (\delta(L^-_{ij}))^pq_{mn} = \pi (L^-_{ik})^p_m \pi (L^-_{kj})^q_n = (R^{-1})^{kp}_{im} (R^{-1})^{jq}_{kn} = (R^{-1})_{21} (R^{-1})_{31} \quad \text{(formally)} \quad (5.6)
\]

Using the Yang–Baxter equation $\hat{R}_{23} (R^{-1})_{12} (R^{-1})_{13} = (R^{-1})_{12} (R^{-1})_{13} \hat{R}_{23}$, one has also

\[
\hat{R}_{23} (\pi \otimes \pi) \delta(L^-) = (\pi \otimes \pi) \delta(L^-) \hat{R}_{23} \quad (5.7)
\]

Hence

\[
[\Pi^{(i)}, (\pi \otimes \pi) \delta(L^\pm)] = 0 \quad (5.8)
\]

so that the eigenstates of $\hat{R}$ are left invariant by the tensor product of the fundamental representation.

We turn now to evaluation representations. Noting that the characteristic polynomial of $\hat{R}$ is of degree two, we can define an evaluation representation by

\[
L^+(x) = x^{-1}L^+ + L^- \quad , \quad L^-(x) = L^+ + xL^- \quad . \quad (5.9)
\]

If $L^\pm$ are representations of the relations (2.11) then $L^\pm(x)$ are representations of (2.23).

Using the fact that $L^+$ and $L^-$ become identical on the tensor product of the 2-dimensional fundamental representation given by (4.3), it is straightforward to see that the $x$ dependence completely factorises out for the corresponding evaluation representation.

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6 Possible applications

We repeat the Baxterised $R$-matrix of S03 \(2.4\) in order to introduce necessary notation:

\[
R(u) = \begin{pmatrix}
    a_1(u) & 0 & 0 & d_1(u) \\
    0 & b_1(u) & c_1(u) & 0 \\
    0 & c_2(u) & b_2(u) & 0 \\
    d_2(u) & 0 & 0 & a_2(u)
\end{pmatrix} = \frac{1}{\sqrt{2u}} \begin{pmatrix}
    u + 1 & 0 & 0 & 1 - u \\
    0 & u + 1 & u - 1 & 0 \\
    0 & 1 - u & u + 1 & 0 \\
    u - 1 & 0 & 0 & u + 1
\end{pmatrix}. \tag{6.1}
\]

It satisfies the Yang–Baxter equation with spectral parameter

\[
R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z) \tag{6.2}
\]

In this section we use some ingredients of the quantum inverse scattering method [9] (for a book exposition, see [10]), however, we would not be able to follow it throughout, due to the peculiarities of our exotic algebra, and, on the other hand, we are able to use some simple procedures, which work just in our situation.

6.1 An exotic eight-vertex model

An integrable vertex model can be constructed in the following way:

The entries of $R$ are interpreted as the Boltzmann weight of a statistical model. The Yang–Baxter equation for $R$ leads to a kind of star-triangle equation (in Baxter’s terms) for the weights of the model.

We define the row-to-row transfer matrix on a closed chain as $t(u) = Tr_0 \mathcal{T}(u)$, where $\mathcal{T}(u)$ is the monodromy matrix given by

\[
\mathcal{T}(u) = R_{01}(u)R_{02}(u) \cdots R_{0L}(u). \tag{6.3}
\]

The Yang–Baxter algebra satisfied by $\mathcal{R}$ ensures that transfer matrices with different spectral parameters commute, i.e.

\[
[Tr_0 \mathcal{T}(u), Tr_0 \mathcal{T}(v)] = 0, \quad \forall u, v. \tag{6.4}
\]
This commutativity relies on the so-called “rail-way” proof.

An integrable vertex model with open boundary conditions can also be defined using a double-row monodromy matrix, cf. [11, 12], (see also [13–16])

\[ T(u) = \mathcal{R}_{01}(u) \mathcal{R}_{02}(u) \cdots \mathcal{R}_{0L}(u) K(u) \mathcal{R}_{L0}(u) \mathcal{R}_{L-1,0}(u) \cdots \mathcal{R}_{10}(u). \quad (6.5) \]

and transfer matrix \( t(u) = T r_0 K'(u) T(u) \). where \( K \) and \( K' \) are boundary reflection matrices. One should be able to prove that the double-row transfer matrices commute among themselves for any values of the spectral parameters, at least in the case \( K(u) = 1 \) and \( K'(u) = 1 \). This would use the reflection equation [17]

\[ \mathcal{R}_{12}(u-v) K_1^{-}(u) \mathcal{R}_{21}(u+v) K_2^{-}(v) = K_2^{-}(v) \mathcal{R}_{12}(u+v) K_1^{-}(u) \mathcal{R}_{21}(u-v) \quad (6.6) \]

and a so-called crossing-unitarity relation for \( R(u) \).

The commutativity for open chain has been checked with the computer for some values of \( L \) (up to now \( L = 2, \cdots, 6. \))

In this exotic model the weights cannot be all non-negative except for the trivial limit \( u=1 \). So negative weights have to be suitably interpreted, e.g., as in [18]. We leave this for future investigations.

6.2 An integrable spin chain

6.2.1 The model

A Hamiltonian of spin chain can be defined as the first term in the expansion of \( T(u) \) around \( u = 1 \)

\[ \mathcal{H}_{\text{per}} = \left. \frac{d}{du} \right|_{u=1} T(u).T(1)^{-1} = \sum_{i=1}^{L-1} \mathcal{H}_{i \ i+1} + \mathcal{H}_{L \ 1}, \quad (6.7) \]

(for closed boundary conditions).

This Hamiltonian by construction commutes with the transfer matrices \( T(u) \) for any \( u \). It has a high degeneracy of spectrum (experimental-computer fact).

Similarly, an open chain Hamiltonian can be defined using the derivative of \( T(u) \)

\[ \mathcal{H}_{\text{open}} = \left. \frac{d}{du} \right|_{u=1} T(u).T(1)^{-1} = \sum_{i=1}^{L-1} \mathcal{H}_{i \ i+1}, \quad (6.8) \]

(up to normalisation).

Let us recall the elements \( B, C, D \) of the standard dual s03 of S03 from [2]. They are duals to the elements \( b, c, d \), resp., (cf. [24]), and their two-dimensional representation is related to the standard sigma matrices:

\[ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -i \sigma_2, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3. \quad (6.9) \]
Then the Hamiltonian on two sites may be written as: $H = B \otimes C$. The operators $B$, $C$, $D$ satisfy

$$B^2 = -C^2 = D^2 = 1, \quad DB = -BD = C, \quad DC = -CD = B, \quad CB = -BC = D$$

(6.10)

### 6.2.2 Eigenstates and eigenvalues

The eigenvalues $\lambda$ and eigenstates of the Hamiltonian on two sites are

$$\lambda = -i \quad |\uparrow\uparrow\rangle + i |\downarrow\downarrow\rangle$$

$$\lambda = i \quad |\uparrow\uparrow\rangle + i |\downarrow\downarrow\rangle$$

(6.11)

The eigenvalues $\lambda$ and eigenstates of the Hamiltonian on three sites are

$$\lambda = \pm i \sqrt{2} \quad w_1^\pm = |\downarrow\downarrow\uparrow\rangle \pm i \sqrt{2} |\uparrow\uparrow\uparrow\rangle$$

$$w_2^\pm = |\downarrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle \mp i \sqrt{2} |\downarrow\downarrow\downarrow\rangle$$

$$w_3^\pm = |\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle \mp i \sqrt{2} |\downarrow\downarrow\downarrow\rangle$$

$$w_4^\pm = |\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle \pm i \sqrt{2} |\uparrow\downarrow\uparrow\rangle$$

(6.12)

The characteristic polynomial of the open chain Hamiltonian is

2 sites : $(x^2 + 1)^2$

3 sites : $(x^2 + 2)^4$

4 sites : $(x^2 + 5)^4(x^2 + 1)^4$

5 sites : $(x^4 + 8x^2 + 4)^8$

6 sites : $(x^2 + 1)^8(x^6 + 19x^4 + 83x^2 + 1)^8$

(6.13)

The characteristic polynomial of the periodic chain Hamiltonian is

2 sites : $x^2(x^2 + 4)$

3 sites : $(x^2 + 3)^4$

4 sites : $x^4(x^2 + 8)^2(x^2 + 4)^4$

5 sites : $(x^4 + 10x^2 + 5)^8$

6 sites : $x^{24}(x^2 + 16)^4(x^2 + 4)^8(x^2 + 12)^8$

(6.14)
6.2.3 Symmetries

It can be checked that the Hamiltonian on two sites commutes with

\[ B \otimes 1, \quad D \otimes B, \quad D \otimes D \]  

(6.15)

and the algebra generated by those, including \( 1 \otimes C, C \otimes D, B \otimes C, C \otimes B \).

The Hamiltonian on \( L \) sites (with open boundary conditions) hence commutes with

\[ B_i \equiv D \otimes D \otimes \cdots \otimes D \otimes B \otimes 1 \otimes \cdots \otimes 1 \]  

(6.16)

with \( B \) on \( i \)-th position and with

\[ B_{L+1} \equiv D \otimes D \otimes \cdots \otimes D \otimes D \otimes \cdots \otimes D \]  

(6.17)

These \( L+1 \) operators generate a Clifford algebra \( \mathcal{C}_{L+1} \), i.e.

\[ \{ B_i, B_j \} = 2 \delta_{ij} \quad i, j \in \{1, \ldots, L+1\}. \]  

(6.18)

For even \( L \), we have a Casimir given by

\[ C = \prod_{j=1}^{L+1} B_j. \]  

(6.19)

The left regular representation of \( \mathcal{C}_{L+1} \), of dimension \( 2^{L+1} \) and with basis elements \( B_1^{n_1} B_2^{n_2} \cdots B_{L+1}^{n_{L+1}} (n_j = 0, 1) \) decomposes into \( 2^{\lfloor \frac{L+2}{2} \rfloor} \) irreducible representations, each of dimension \( 2^{\lfloor \frac{L+1}{2} \rfloor} \).

For even \( L \), an irreducible representation can be described with the following set of basis vectors

\[ B_1^{n_1} B_2^{n_2} \cdots B_{L+1}^{n_{L+1}} (1 + \alpha B_{L+1}) (1 + i \epsilon_1 B_1 B_L) (1 + i \epsilon_2 B_2 B_{L-1}) \cdots (1 + i \epsilon_{\frac{L}{2}} B_L B_{L+1}) \]  

(6.20)

on which the \( B_j \) act by left multiplication. The parameters \( \epsilon_j \) satisfy \( \epsilon_j^2 = 1 \). The exponents \( n_j \) take the values 0 and 1. On this representation, the Casimir operator \( C \) acts as \( \prod_{j=1}^{\lfloor \frac{L+1}{2} \rfloor} (-i \epsilon_j) \).

We can use properties like

\[ (1 + i \epsilon_1 B_1 B_L) (1 + i \epsilon_2 B_2 B_{L-1}) = (1 + i \epsilon_1 B_1 B_L) (1 - \epsilon_1 \epsilon_2 B_1 B_L B_2 B_{L-1}) \]  

(6.21)

to change the expressions, in particular to get an explicit appearance of

\[ 1 - \prod_{j=1}^{\lfloor \frac{L}{2} \rfloor} (-i \epsilon_j) B_1 B_2 \cdots B_{L+1} = 1 - \prod_{j=1}^{\lfloor \frac{L}{2} \rfloor} (-i \epsilon_j) C \]
For odd $L$, an irreducible representation can be described with the following set of basis vectors

$$(1 + (-1)^{nL+1}B_{L+1}) B_1^{n_1} B_2^{n_2} \cdots B_{\lfloor \frac{L-1}{2} \rfloor}^{n_{\lfloor \frac{L-1}{2} \rfloor}} (1 + i\epsilon_1 B_1 B_L) (1 + i\epsilon_2 B_2 B_{L-1}) \cdots \left( 1 + i\epsilon_{\frac{L-1}{2}} B_{\frac{L-1}{2}} B_{\frac{L+1}{2}} \right) (1 + \alpha B_{L+1})$$

For even $L$ denote the basis vector:

$$A^{n_1, \ldots, n_L} = \left\{ \prod_{k=1}^{\frac{L}{2}} B_k^{n_k} \right\} (1 + \alpha B_{L+1}) (1 + i\epsilon_j B_j B_{L+1-j})$$

where $\alpha^2 = 1$.

Then for $j = 1, \ldots, \frac{L}{2}$ we have:

$$B_j A^{n_1, \ldots, n_L} = (-1)^{n_1 + \cdots + n_j} A^{n_1, \ldots, n_j} A^{n_{j+1}, \ldots, n_L}$$

$$B_{L+1-j} A^{n_1, \ldots, n_L} = -i\epsilon_j (-1)^{n_1 + \cdots + n_j} A^{n_1, \ldots, n_j} A^{n_{j+1}, \ldots, n_L}$$

The action of the Casimir is

$$\mathcal{C} A^{n_1, \ldots, n_L} = \alpha \Pi_{j=1}^{\frac{L}{2}} (-i\epsilon_j) A^{n_1, \ldots, n_L}$$

For odd $L$ we denote the basis vector:

$$A^{n_{L+1}; n_1, \ldots, n_L} = (1 + (-1)^{nL+1} B_{L+1}) \prod_{k=1}^{\frac{L}{2}} B_k^{n_k} \times \left\{ \prod_{j=1}^{\frac{L}{2}} (1 + i\epsilon_j B_j B_{L+1-j}) \right\} (1 + i\epsilon_{\frac{L+1}{2}} B_{\frac{L+1}{2}} B_{L+1})$$

Then for $j = 1, \ldots, \frac{L}{2}$ we have:

$$B_j A^{n_{L+1}; n_1, \ldots, n_L} = (-1)^{n_1 + \cdots + n_j} A^{n_{L+1}; n_1, \ldots, n_j} A^{n_{j+1}, \ldots, n_{L}}$$

$$B_{L+1-j} A^{n_{L+1}; n_1, \ldots, n_L} = -i\epsilon_j (-1)^{n_1 + \cdots + n_j} A^{n_{L+1}; n_1, \ldots, n_j} A^{n_{j+1}, \ldots, n_{L}}$$

In the derivation of the above relations were used also the following formulae:

For even $L$:

$$B_{L+1-j} (1 + i\epsilon_j B_j B_{L+1-j}) = -i\epsilon_j B_j (1 + i\epsilon_j B_j B_{L+1-j})$$

$$B_{L+1} (1 + \alpha B_{L+1}) = \alpha (1 + \alpha B_{L+1})$$

For odd $L$:

$$B_{L+1-j} (1 + i\epsilon_j B_j B_{L+1-j}) = -i\epsilon_j B_j (1 + i\epsilon_j B_j B_{L+1-j})$$

$$B_{L+1} (1 + i\epsilon_{\frac{L+1}{2}} B_{\frac{L+1}{2}} B_{L+1}) = i\epsilon_{\frac{L+1}{2}} B_{L+1} (1 + i\epsilon_{\frac{L+1}{2}} B_{\frac{L+1}{2}} B_{L+1})$$

$$B_{L+1} (1 + (-1)^{nL+1} B_{L+1}) = (-1)^{nL+1} (1 + (-1)^{nL+1} B_{L+1})$$

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