ON THE EKEDAHL-OORT STRATIFICATION OF SHIMURA CURVES

BENJAMIN HOWARD

ABSTRACT. We study the Hodge-Tate period domain associated to a quaternionic Shimura curve at a prime of bad reduction, and give an explicit description of its Ekedahl-Oort stratification.

1. Introduction

Fix a prime $p$, and let $C$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Denote by $\mathcal{O} \subset C$ its ring of integers, and by $k = \mathcal{O}/\mathfrak{m}$ its residue field.

1.1. Stratifications of $p$-adic periods domains. Let $G$ be a $p$-divisible group over $\mathcal{O}$. It has a $p$-adic Tate module

$$T_p(G) = \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G)$$

and a module of invariant differential forms $\Omega(G)$. These are free of finite rank over $\mathbb{Z}_p$ and $\mathcal{O}$, respectively. Using the canonical trivialization $\Omega(\mu_{p^\infty}) \cong \mathcal{O}$, we define the Hodge-Tate morphism

$$(1.1.1) \quad T_p(G) \cong \text{Hom}(G^\vee, \mu_{p^\infty}) \xrightarrow{\text{HT}} \text{Hom}(\Omega(\mu_{p^\infty}), \Omega(G^\vee)) \cong \Omega(G^\vee),$$

where $G^\vee$ is the Cartier dual of $G$.

Theorem A (Scholze-Weinstein [SW13]). There is an equivalence between the category of $p$-divisible groups over $\mathcal{O}$ and the category of pairs $(T, W)$ in which

- $T$ is a free $\mathbb{Z}_p$-module of finite rank,
- $W \subset T \otimes_{\mathbb{Z}_p} C$ is a $C$-subspace.

The equivalence sends $G$ to its $p$-adic Tate module $T = T_p(G)$, endowed with its Hodge-Tate filtration

$$W = \ker(T_p(G) \otimes_{\mathbb{Z}_p} C \xrightarrow{\text{HT}} \Omega(G^\vee) \otimes_{\mathcal{O}} C).$$

Fix a free $\mathbb{Z}_p$-module $T$ of finite rank, and consider the $\mathbb{Q}_p$-scheme

$$X = \text{Gr}_d(T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

parametrizing subspaces of $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of some fixed dimension $d \leq \text{rank}(T)$. By the theorem of Scholze-Weinstein, every point $W \in X(C)$ determines a $p$-divisible group $G$ over $\mathcal{O}$, whose reduction to the residue field we denote by $G_k$. Let $G_k[p]$ be the group scheme of $p$-torsion points in $G_k$. 

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If we declare two points \( W, W' \in X(C) \) to be equivalent when the corresponding reductions \( G_k \) and \( G'_k \) are isogenous, the resulting partition is the \textit{Newton stratification} of \( X(C) \). Alternatively, if we declare \( W, W' \in X(C) \) to be equivalent when the \( p \)-torsion group schemes \( G_k[p] \) and \( G'_k[p] \) are isomorphic, the resulting partition is the \textit{Ekedahl-Oort stratification} of \( X(C) \).

There are similar partitions when \( X \) is replaced by a more sophisticated flag variety, called the \textit{Hodge-Tate period domain}, associated to a Shimura datum of Hodge type and a prime \( p \). This period domain and its Newton stratification were studied by Caraiani-Scholze [CS17], who proved that each Newton stratum in \( X(C) \) can be realized as the \( C \)-points of a locally closed subset of the associated adic space. For the Ekedahl-Oort stratification of \( X(C) \) there is nothing in the existing literature, and it is not known if it has any structure other than set-theoretic partition.

In the case of modular curves, the Hodge-Tate period domain is the projective line \( \mathbb{P}^1 \) over \( \mathbb{Q}_p \). In this case the Newton stratification and the Ekedahl-Oort stratification agree, and there are two strata: the \textit{ordinary locus} \( \mathbb{P}^1(\mathbb{Q}_p) \), and the \textit{supersingular locus} \( \mathbb{P}^1(C) \setminus \mathbb{P}^1(\mathbb{Q}_p) \).

For the compact Shimura curve determined by an indefinite quaternion algebra over \( \mathbb{Q} \), and a prime \( p \) at which the quaternion algebra is ramified, the Hodge-Tate period domain \( X \) is a twisted form of \( \mathbb{P}^1 \). All points of \( X(C) \) give rise to supersingular \( p \)-divisible groups over \( k \), and so the Newton stratification contains a single stratum, \( X(C) \) itself. In contrast, the Ekedahl-Oort stratification is nontrivial, and the goal of this paper is to make it explicit.

Although the methods used here are fairly direct and elementary, it is not clear to the author how far they can be extended. For example, the case of Hilbert modular surfaces may already require new ideas.

For background on the classical Ekedahl-Oort stratification of reductions of Shimura varieties (as opposed to their Hodge-Tate period domains), we refer the to work of Oort [Oor01], Moonen [Moo04], Viehmann-Wedhorn [VW13], Zhang [Zha18], and the references found therein.

1.2. The Shimura curve period domain. Let \( \mathbb{Q}_{p^2} \subset C \) be the unique unramified quadratic extension of \( \mathbb{Q}_p \), and let \( \mathbb{Z}_{p^2} \subset O \) be its ring of integers. Denote by \( x \mapsto \pi \) the nontrivial automorphism of \( \mathbb{Q}_{p^2} \). Define a non-commutative \( \mathbb{Z}_p \)-algebra of rank 4 by

\[
\Delta = \mathbb{Z}_{p^2}[\Pi],
\]

where \( \Pi \) is subject to the relations \( \Pi^2 = p \) and \( \Pi \cdot x = \pi \cdot \Pi \) for all \( x \in \mathbb{Z}_{p^2} \). In other words, \( \Delta \) is the unique maximal order in the unique quaternion division algebra over \( \mathbb{Q}_p \).
Let $T$ be a free $\Delta$-module of rank one, and let $X$ be the smooth projective variety over $\mathbb{Q}_p$ with functor of points

$$X(R) = \left\{ \begin{array}{l} \text{R-module local direct summands} \\ W \subset T \otimes_{\mathbb{Z}_p} R \\ \text{of rank 2 that are stable under } \Delta \end{array} \right\}$$

for any $\mathbb{Q}_p$-algebra $R$. This is the Hodge-Tate period domain associated to a quaternionic Shimura curve.

As we explain in §4.1, our period domain becomes isomorphic to the projective line after base change to $\mathbb{Q}_p^2$, and any choice of $\Delta$-module generator $\lambda \in T$ determines a bijection

$$X(C) \cong C \cup \{ \infty \}.$$ 

After fixing such a choice, we normalize the valuation $\text{ord} : C \to \mathbb{R} \cup \{ \infty \}$ by $\text{ord}(p) = 1$, extend it to $C \cup \{ \infty \}$ by $\text{ord}(\infty) = -\infty$, and use (1.2.2) to view $\text{ord}$ as a function

$$\text{ord} : X(C) \to \mathbb{R} \cup \{-\infty, \infty \}.$$ 

The theorem of Scholze-Weinstein gives us a canonical bijection

$$X(C) \cong \text{isomorphism classes of } p\text{-divisible groups } G \text{ over } \mathcal{O} \text{ of height 4 and dimension 2, endowed with an action of } \Delta \text{ and a } \Delta\text{-linear isomorphism } T_p(G) \cong T.$$ 

By forgetting the level structure $T_p(G) \cong T$, reducing to the residue field, and then taking $p$-torsion subgroups, we obtain a function

$$X(C) \to \text{isomorphism classes of finite group schemes over } k, \text{ endowed with an action of } \Delta/p\Delta$$

sending $G \mapsto G_k[p]$, whose fibers are the Ekedahl-Oort strata of $X(C)$.

**Hypothesis.** For the rest of this introduction, we assume $p > 2$. Theorems B and C below are presumably true without this hypothesis, but we are unable to provide a proof. See the remarks following Theorem 2.3.6.

It is convenient to organize the strata into two types: those on which the $p$-torsion group scheme $G_k[p]$ is superspecial (in the sense of §3.2), and those on which it is not. The two theorems that follow show that there are three superspecial strata, and two infinite families of non-superspecial strata. These results are proved in §4.2 where the reader will also find an explicit recipe for computing the Dieudonné module of the $p$-torsion group scheme $G_k[p]$ attached to a point of $X(C)$.

**Theorem B.** The conditions

$$\frac{1}{p+1} < \text{ord}(\tau) < \frac{p}{p+1}$$

on $\tau \in X(C)$ define an Ekedahl-Oort stratum, as do each one of

$$\text{ord}(\tau) < \frac{1}{p+1}, \quad \frac{p}{p+1} < \text{ord}(\tau).$$
The union of these three strata is the locus of points with superspecial reduction. In particular, the isomorphism class of the finite group scheme $G_k[p]$ is the same all on three strata, but the isomorphism class of $G_k[p]$ with its $\Delta$-action is not.

Now consider the locus of points
\[
\left\{ \tau \in C : \text{ord}(\tau) = \frac{1}{p + 1} \right\} \cup \left\{ \tau \in C : \text{ord}(\tau) = \frac{p}{p + 1} \right\} \subset X(C)
\]
at which the corresponding $p$-divisible group does not have superspecial reduction. The isomorphism class of the $p$-torsion group scheme $G_k[p]$ is constant on (1.2.3), but the isomorphism class of $G_k[p]$ with its $\Delta$-action varies. In fact, the $\Delta$-action varies so much that (1.2.3) decomposes as an infinite disjoint union of Ekedahl-Oort strata.

**Theorem C.** The fibers of the composition
\[
\left\{ \tau \in C : \text{ord}(\tau) = \frac{1}{p + 1} \right\} \xrightarrow{\tau \mapsto \tau^{p+1}} \mathcal{O}^\times \rightarrow k^\times
\]
are Ekedahl-Oort strata, as are the fibers of the composition
\[
\left\{ \tau \in C : \text{ord}(\tau) = \frac{p}{p + 1} \right\} \xrightarrow{\tau \mapsto \tau^{p+1}/p^p} \mathcal{O}^\times \rightarrow k^\times.
\]
Both unlabeled arrows are reduction to the residue field.

**Remark 1.2.1.** The infinitude of Ekedahl-Oort strata is a pathology arising from the non-smooth reduction of compact Shimura curves. Similar pathologies for the reductions of Hilbert modular varieties at ramified primes are described in the appendix to [AG03].

1.3. **Notation and conventions.** Throughout the paper $p$ is a fixed prime. We allow $p = 2$ unless otherwise stated. Let $k = \mathcal{O}/\mathfrak{m}$ as above, and denote by $\sigma : k \rightarrow k$ the absolute Frobenius $\sigma(x) = x^p$.

The rings $\mathbb{Z}_{p^2} \subset \mathcal{O}$ and $\Delta = \mathbb{Z}_{p^2}[\Pi]$ have the same meaning as above. We label the embeddings
\[
(1.3.1) \quad j_0, j_1 : \mathbb{Z}_{p^2} \rightarrow \mathcal{O}
\]
in such a way that $j_0$ is the inclusion and $j_1(x) = j_0(\overline{x})$ is its conjugate.

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2. **Integral $p$-adic Hodge theory**

In this section we study the integral $p$-adic Hodge theory of an arbitrary $p$-divisible group $G$ over $\mathcal{O}$. The quaternion order $\Delta$ plays no role whatsoever. Following [SW17], [Par15], and [Lau17], we will attach to $G$ a Breuil-Kisin-Fargues module, and explain how to extract from it invariants of $G$ such as its Hodge-Tate morphism (1.1.1), and the Dieudonné module of its reduction to $k$. 
2.1. A ring of periods. Let $C^h$ be the tilt of $C$, with ring of integers $\mathcal{O}^h$. Thus

$$\mathcal{O}^h = \lim_{x \to x^p} \mathcal{O}/(p)$$

is a local domain of characteristic $p$, fraction field $C^h$, and residue field $k = \mathcal{O}^h/m^h$. An element $x \in \mathcal{O}^h$ is given by a sequence $(x_0, x_1, x_2, \ldots)$ of elements $x_k \in \mathcal{O}/(p)$ satisfying $x_{k+1}^p = x_k$. After choosing arbitrary lifts $x_k \in \mathcal{O}$, set

$$x^h = \lim_{k \to \infty} x_k^h.$$ 

The construction $x \mapsto x^h$ defines a multiplicative function $\mathcal{O}^h \to \mathcal{O}$, and we define $\text{ord} : \mathcal{O}^h \to \mathbb{R} \cup \{\infty\}$ by $\text{ord}(x) = \text{ord}(x^h)$.

Denote by $\sigma : \mathcal{O}^h \to \mathcal{O}^h$ the absolute Frobenius $x \mapsto x^p$, and in the same way the induced automorphism of the local domain $A_{\text{inf}} = W(\mathcal{O}^h)$.

There is a canonical homomorphism of $\mathbb{Z}_p$-algebras

$$\Theta : A_{\text{inf}} \to \mathcal{O}$$

satisfying $\Theta([x]) = x^h$, where $[\cdot] : \mathcal{O}^h \to A_{\text{inf}}$ is the Teichmuller lift.

The kernel of $\Theta$ is a principal ideal. To construct a generator, first fix a $\mathbb{Z}_p$-module generator

$$\zeta = (\zeta_p, \zeta_p^2, \zeta_p^3, \ldots) \in T_p(\mu_p^\infty)$$

and define $\epsilon = (1, \zeta_p, \zeta_p^2, \ldots) \in \mathcal{O}^h$. The element

$$\xi = [1] + [\epsilon^{1/p}] + [\epsilon^{2/p}] + \cdots + [\epsilon^{(p-1)/p}] \in A_{\text{inf}}$$

generates $\text{ker}(\Theta)$. If we denote by

$$\varpi = 1 + \epsilon^{1/p} + \epsilon^{2/p} + \cdots + \epsilon^{(p-1)/p} \in \mathcal{O}^h$$

its image under the reduction map $A_{\text{inf}} \to A_{\text{inf}}/(p) = \mathcal{O}^h$, then $\text{ord}(\varpi) = 1$, and there are canonical isomorphisms

$$\mathcal{O}/(p) \cong A_{\text{inf}}/(\xi, p) \cong \mathcal{O}^h/(\varpi).$$

2.2. Breuil-Kisin-Fargues modules. There is an equivalence of categories between $p$-divisible groups over $\mathcal{O}$ and Breuil-Kisin-Fargues modules, whose definition we now recall.

**Definition 2.2.1.** A Breuil-Kisin-Fargues module is a triple $(M, \phi, \psi)$, in which $M$ is a free module of finite rank over $A_{\text{inf}}$, and

$$\phi, \psi : M \to M$$

are homomorphisms of additive groups satisfying

$$\phi(am) = \sigma(a)\phi(m), \quad \psi(\sigma(a)m) = a\psi(m),$$

for all $a \in A_{\text{inf}}$ and $m \in M$, as well as $\phi \circ \psi = \xi$. 
Suppose $(M, \phi, \psi)$ is a Breuil-Kisin-Fargues module. Let

$$\sigma^* M = A_{\text{inf}} \otimes_{\sigma, A_{\text{inf}}} M$$

be the Frobenius twist, as usual. Denote by $N$ the image of the $A_{\text{inf}}$-linear map

$$M \xrightarrow{x \mapsto 1 \otimes \psi(x)} \sigma^* M.$$

It is easy to see that

$$\xi \sigma^* M \subset N \subset \sigma^* M.$$

Define the de Rham, étale, and crystalline realizations of $(M, \phi, \psi)$ by

$$M_{\text{dR}} = \sigma^* M / \xi \sigma^* M$$

$$M_{\text{et}} = M_{\psi=1}$$

$$M_{\text{crys}} = W(k) \otimes_{\sigma, A_{\text{inf}}} M.$$

Abbreviating $h = \text{rank}_{A_{\text{inf}}}(M)$, these satisfy the following properties.

- The de Rham realization is a free $O$-module of rank $h$, and sits in the short exact sequence

$$0 \to N / \xi \sigma^* M \to M_{\text{dR}} \to \sigma^* M / N \to 0$$

of free $O$-modules. Indeed, the freeness of $M_{\text{dR}}$ is clear, the freeness of $\sigma^* M / N$ follows from the proof of [Lau17, Lemma 9.5], and the freeness of $N / \xi \sigma^* M$ is a consequence of this.

- The étale realization $M_{\text{et}}$ is a free $\mathbb{Z}_p$-module of rank $h$ (see the remarks below Theorem 2.2.2). Its Hodge-Tate filtration $F_{\text{HT}}(M) \subset M_{\text{et}} \otimes_{\mathbb{Z}_p} C$ is the kernel of the $C$-linear extension of

$$M_{\text{et}} \xrightarrow{x \mapsto 1 \otimes \psi(x)} N / \xi \sigma^* M.$$

- The crystalline realization is a free $W(k)$-module of rank $h$, endowed with operators

$$F(a \otimes m) = a^p \otimes \phi(m), \quad V(a \otimes m) = \sigma^{-1}(a) \otimes \psi(m),$$

giving it the structure of a Dieudonné module.

The following theorem is due to Fargues [Far15] and Scholze-Weinstein [SW13, SW17]. See also the work of Lau [Lau17].

**Theorem 2.2.2.** There is an equivalence of categories from Breuil-Kisin-Fargues modules to $p$-divisible groups over $O$. It is constructed by composing the functor

$$\begin{align*}
(2.2.1) \quad (M, \phi, \psi) & \mapsto (M_{\text{et}}, F_{\text{HT}}(M))
\end{align*}$$

from Breuil-Kisin-Fargues modules to the category of pairs $(T, W)$ appearing in Theorem A with the equivalence from that category to the category of $p$-divisible groups.
The equivalence of categories of Theorem 2.2.2 is established in [SW17], but with a different notion of étale realization, and hence a different functor in place of (2.2.1). We explain the connection between our functor (2.2.1) and the one appearing in [SW17].

The inclusion $O^0 \subset C^0$ induces a ring homomorphism $A_{\text{inf}} \to W(C^0)$. Given a Breuil-Kisin-Fargues module $(M, \phi, \psi)$ one can define an alternate étale realization

$$M'_\text{et} = (M \otimes_{A_{\text{inf}}} W(C^0))^{\phi=1}. $$

Set $\mathcal{M} = M'_\text{et} \otimes A_{\text{inf}}$. It is proven in [SW17] that $M'_\text{et}$ is a free $\mathbb{Z}_p$-module of rank $h = \text{rank}_{A_{\text{inf}}}(M)$, and that the inclusion of $M'_\text{et}$ into $M \otimes_{A_{\text{inf}}} W(C^0)$ induces an isomorphism

$$\mathcal{M} \otimes_{A_{\text{inf}}} W(C^0) \cong M \otimes_{A_{\text{inf}}} W(C^0).$$

Define $\mu = [\varepsilon] - 1 \in A_{\text{inf}}$. This element satisfies $\mu = \xi \sigma^{-1}(\mu)$. In fact

$$\mathbb{Z}\sigma^{-1}(\mu) = \{ a \in A_{\text{inf}} : \sigma(a) = \xi a \},$$

and inside the $W(C^0)$-module (2.2.2) we have inclusions of $A_{\text{inf}}$-submodules

$$\sigma^{-1}(\mu) \mathcal{M} \subset M \subset \mathcal{M}.$$ 

As $\sigma^{-1}(\mu)$ has nonzero image under $\theta : A_{\text{inf}} \to O$, we obtain canonical isomorphisms

$$M \otimes_{A_{\text{inf}}, \theta} C \cong \mathcal{M} \otimes_{A_{\text{inf}}, \theta} C \cong M'_\text{et} \otimes_{\mathbb{Z}_p} C.$$ 

This allows us to define an alternate Hodge-Tate filtration

$$F'_{\text{HT}}(M) \subset M'_\text{et} \otimes_{\mathbb{Z}_p} C$$

as the $C$-span of the image of

$$\sigma^* M \xrightarrow{x \mapsto 1 \otimes \sigma(x)} M \to M/\xi M \subset M \otimes_{A_{\text{inf}}, \theta} C = M'_\text{et} \otimes_{\mathbb{Z}_p} C.$$ 

It is proven in [SW17] that the functor

$$\mathcal{F} \to (M'_\text{et}, F'_{\text{HT}}(M))$$

establishes an equivalence of categories from Breuil-Kisin-Fargues modules to pairs $(T, W)$ as in Theorem A. To relate this to the functor (2.2.1), note that (2.2.3) implies the equality

$$M'_\text{et} = \sigma^{-1}(\mu) M'_{\text{et}}$$

of $\mathbb{Z}_p$-submodules of $M \otimes_{A_{\text{inf}}} W(C^0)$. In fact, it is easy to verify that multiplication by $\sigma^{-1}(\mu)$ establishes an isomorphism of functors from (2.2.4) to (2.2.1), and hence this latter functor is also an equivalence of categories.
2.3. More on Breuil-Kisin-Fargues modules. To better understand the properties of the equivalence of categories of Theorem 2.2.2, we introduce the category of windows, following Zink [Zin01] and Lau [Lau17].

Let $A_{\text{crys}}$ be the $p$-adic completion of the subring

$$A_{\text{crys}}^0 = A_{\text{inf}}[\xi^n/n! : n = 1, 2, 3, \ldots] \subset A_{\text{inf}}[1/p].$$

It is an integral domain endowed with a ring homomorphism

$$(2.3.1) \quad \theta_{\text{crys}} : A_{\text{crys}} \rightarrow \mathcal{O}$$

extending $\theta : A_{\text{inf}} \rightarrow \mathcal{O}$, and divided powers on the kernel $I = \ker(\theta_{\text{crys}})$.

The subring $A_{\text{crys}}^0 \subset A_{\text{inf}}[1/p]$ is stable under $\sigma$, and there is unique continuous extension to an injective ring homomorphism $\sigma : A_{\text{crys}} \rightarrow A_{\text{crys}}$ reducing to the usual $p$-power Frobenius on $A_{\text{crys}}/pA_{\text{crys}}$. Moreover,

$$(2.3.2) \quad \sigma(I) \subset pA_{\text{crys}} \quad \text{and} \quad \frac{\sigma(\xi)}{p} \in A_{\text{crys}}^\times.$$  

**Definition 2.3.1.** A window is a quadruple $(P, Q, \Phi, \Phi_1)$ consisting of a projective $A_{\text{crys}}$-module $P$ of finite rank, a submodule $Q \subset P$, and $\sigma$-semi-linear maps

$$\Phi : P \rightarrow P, \quad \Phi_1 : Q \rightarrow P$$

satisfying

- $IP \subset Q$, and both $Q/IP$ and $P/Q$ are free over $A_{\text{crys}}/I \cong \mathcal{O}$,
- $a \otimes x \mapsto a\Phi_1(x)$ defines an isomorphism $\sigma^*Q \cong P$ of $A_{\text{crys}}$-modules,
- $\Phi(ax) = p\Phi_1(ax)$ for all $a \in I$ and $x \in P$.

**Remark 2.3.2.** Taking $a = \xi$ in the final condition, and using (2.3.2), yields

$$\Phi(x) = \frac{p}{\sigma(\xi)} \cdot \Phi_1(\xi x),$$

for all $x \in P$. This implies $\Phi(x) = p\Phi_1(x)$ for all $x \in Q$, and also shows that each one of $\Phi$ and $\Phi_1$ determines the other.

**Remark 2.3.3.** In [Lau10] and [Lau17], what we have called a window would be called a window over the frame

$$A_{\text{crys}} = (A_{\text{crys}}, I, \mathcal{O} = A_{\text{crys}}/I, \sigma, \sigma_1),$$

where $\sigma_1 : I \rightarrow A_{\text{crys}}$ is $\sigma_1(x) = \sigma(x)/p$.

Suppose $G$ is a $p$-divisible group over $\mathcal{O}$. Let $P$ be its covariant Grothendieck-Messing crystal evaluated at the divided power thickening (2.3.1). This is a projective $A_{\text{crys}}$-module of rank equal to the height of $G$, equipped with a $\sigma$-semi-linear operator $\Phi : P \rightarrow P$ and a short exact sequence

$$0 \rightarrow \Omega(G^\vee) \rightarrow P/IP \rightarrow \text{Lie}(G) \rightarrow 0$$

of free $\mathcal{O}$-modules. Define $Q \subset P$ as the kernel of $P \rightarrow \text{Lie}(A)$. One can show that $\Phi(Q) \subset pP$, allowing us to define $\Phi_1 : Q \rightarrow P$ by

$$\Phi_1(x) = \frac{1}{p} \cdot \Phi(x).$$
The following is a special case of the main results of [Lau17].

**Theorem 2.3.4** (Lau). The construction $G \mapsto (P, Q, \Phi, \Phi_1)$ just given defines a functor from the category of $p$-divisible groups over $\mathcal{O}$ to the category of windows. It is an equivalence of categories if $p > 2$.

Now suppose we start with a Breuil-Kisin-Fargues module $(M, \phi, \psi)$. Set $P = A_{\text{crys}} \otimes_{\sigma, \text{inf}} M$, and define $Q \subset P$ as the kernel of the composition

$$
A_{\text{crys}} \otimes_{\sigma, \text{inf}} M \longrightarrow A_{\text{crys}}/IA_{\text{crys}} \otimes_{\sigma, \text{inf}} M \\
\cong A_{\text{inf}}/\xi A_{\text{inf}} \otimes_{\sigma, \text{inf}} M \\
\cong \sigma^* M/\xi \sigma^* M \longrightarrow \sigma^* M/N.
$$

Equivalently, $Q \subset P$ is the $A_{\text{crys}}$-submodule generated by all elements of the form $1 \otimes \psi(m)$ and $a \otimes m$ with $m \in M$ and $a \in I$. There is a unique $\sigma$-semi-linear map $\Phi_1 : Q \to P$ whose effect on these generators is

$$
\Phi_1(1 \otimes \psi(m)) = 1 \otimes m, \quad \Phi_1(a \otimes m) = \frac{\sigma(a)}{\sigma(\xi)} \otimes \phi(m).
$$

Note that $\sigma(a)/\sigma(\xi) \in A_{\text{crys}}$ by (2.3.2). Finally, define $\Phi : P \to P$ by

$$
\Phi(a \otimes m) = \frac{p \sigma(a)}{\sigma(\xi)} \otimes \phi(m)
$$

for all $a \in A_{\text{crys}}$ and $m \in M$. The following is a special case of the main results of [Lau17].

**Theorem 2.3.5** (Lau). The construction $(M, \phi, \psi) \mapsto (P, Q, \Phi, \Phi_1)$ just given defines a functor from the category of Breuil-Kisin-Fargues modules to the category of windows. It is an equivalence of categories if $p > 2$.

Given a window $(P, Q, \Phi, \Phi_1)$, define its \textit{étale realization}

$$
P_{\text{et}} = \{ x \in Q : \Phi_1(x) = x \}.
$$

This is a torsion-free $\mathbb{Z}_p$-module equipped with a \textit{Hodge-Tate filtration}

$$
F_{\text{HT}}(P_{\text{et}}) \subset P_{\text{et}} \otimes_{\mathbb{Z}_p} \mathbb{C},
$$

defined as the kernel of the $\mathbb{C}$-linear extension of $P_{\text{et}} \to Q/IP$.

Denote by HTpair the category of pairs $(T, W)$ in which $T$ is a torsion-free $\mathbb{Z}_p$-module, and $W \subset T \otimes_{\mathbb{Z}_p} \mathbb{C}$ is a subspace. Using the obvious notation
for the categories of Breuil-Kisin-Fargues modules, $p$-divisible groups over $\mathcal{O}$, and windows, we now have functors

\[
\begin{array}{c}
\text{BKF-Modules} \\
\downarrow a \\
p-\text{DivGrp} \quad \downarrow c \\
\text{Win} \\
\downarrow d \\
\text{HTpair.} \\
\downarrow e \\
\end{array}
\]

Here $a$ is the equivalence of Theorem 2.2.2, $b$ is given by Theorem 2.3.5, $c$ is given by 2.3.4, $d$ sends a Breuil-Kisin-Fargues module to its étale realization, $e$ does the same for windows, and $f$ sends a $p$-divisible group over $\mathcal{O}$ to its $p$-adic Tate module endowed with its Hodge filtration.

The outer triangle $adf$ commutes, just by definition of the functor $a$, but it is not obvious from the definitions that any of the inner triangles commute. The following theorem is no doubt known to the experts, but for lack of a reference we give the proof.

**Theorem 2.3.6.** Assume that $p > 2$. The diagram (2.3.3) commutes, and the Breuil-Kisin-Fargues module $(M, \phi, \psi)$ associated to $G$ by Theorem 2.2.2 enjoys the following properties.

1. There are isomorphisms of $\mathcal{O}$-modules

   \[
   \Omega(G^\circ) \cong N/\xi \sigma^* M, \quad \text{Lie}(G) \cong \sigma^* M / N.
   \]

2. If $G_k$ denotes the reduction of $G$ to the residue field $k = \mathcal{O} / \mathfrak{m}$, the covariant Dieudonné module of $G_k$ is isomorphic to $M_{\text{crys}}$.

3. There is an isomorphism $T_p (G) \cong M_{\text{et}}$ making the diagram

   \[
   \begin{array}{ccc}
   T_p (G) & \xrightarrow{\text{HT}} & M_{\text{et}} \\
   \downarrow \Omega (G^\circ) & & \downarrow N / \xi \sigma^* M \\
   \end{array}
   \]

   commute, where the vertical arrow on the right is the restriction to $M_{\text{et}} \subset M$ of the $\mathcal{O}$-linear map

   \[
   M \xrightarrow{x \mapsto 1 \otimes \psi (x)} N \rightarrow N / \xi \sigma^* M.
   \]

**Proof.** The functors labeled $b$ and $c$ are equivalences of categories by Theorems 2.3.4 and 2.3.5.

Consider the constant $p$-divisible group $\mathbb{Q}_p / \mathbb{Z}_p$ over $\mathcal{O}$. The window corresponding to $\mathbb{Q}_p / \mathbb{Z}_p$ consists of $P^0 = A_{\text{crys}}$ and $Q^0 = A_{\text{crys}}$ endowed with the operators $\Phi : P^0 \rightarrow P^0$ and $\Phi_1 : Q^0 \rightarrow P^0$ defined by

\[
\Phi (x) = p \sigma (x) \quad \text{and} \quad \Phi_1 (x) = \sigma (x).
\]
If $G$ is a $p$-divisible group over $\mathcal{O}$ with associated window $(P, Q, \Phi, \Phi_1)$, the full faithfulness of the arrow $c$ allows us to identify

$$\tag{2.3.4} T_p(G) \xrightarrow{\text{HT}} \Omega(G^\vee)$$

$$\text{Hom}_{p-\text{DivGrp}}(\mathbb{Q}_p/\mathbb{Z}_p, G) \xrightarrow{} \text{Hom}(\Omega(\mu_p), \Omega(G^\vee))$$

$$\text{Hom}_{\text{Win}}(P^0, P) \xrightarrow{} \text{Hom}(Q^0/IP^0, Q/IP)$$

From this it follows that the triangle $cef$ is commutative.

Similarly, the Breuil-Kisin-Fargues module of $\mathbb{Q}_p/\mathbb{Z}_p$ consists of $M^0 = A_{\text{inf}}$ endowed with its operators

$$\phi(x) = \xi \sigma(x) \quad \text{and} \quad \psi(x) = \sigma^{-1}(x).$$

If $(M, \phi, \psi)$ is a Breuil-Kisin-Fargues module with associated window $(P, Q, \Phi, \Phi_1)$, the full faithfulness of $b$ allows us to identify

$$\tag{2.3.5} M_{\text{et}} \xrightarrow{} N/\xi \sigma^* M$$

$$\text{Hom}_{\text{BKF}}(M^0, M) \xrightarrow{} \text{Hom}(N^0/\xi \sigma^* M^0, N/\xi \sigma^* M)$$

$$\text{Hom}_{\text{Win}}(P^0, P) \xrightarrow{} \text{Hom}(Q^0/IP^0, Q/IP)$$

From this it follows that the triangle $bde$ is commutative.

As $c$ is an equivalence of categories and $f$ is fully faithful, the commutativity of the triangle $cef$ implies that $e$ is fully faithful. It follows from this, along with commutativity of the triangles $bde$ and $cef$, and commutativity of the outermost triangle $adf$ (which is just a restatement of the definition of $a$) that triangle $abc$ commutes. Hence the whole diagram (2.3.3) commutes.

Suppose $G$ is a $p$-divisible group over $\mathcal{O}$, with Breuil-Kisin-Fargues module $(M, \phi, \psi)$ and window $(P, Q, \Phi, \Phi_1)$. Using the commutativity of (2.3.3), we deduce that

$$\Omega(G^\vee) \cong \ker(P/IP \to \text{Lie}(G)) = Q/IP \cong N/\xi \sigma^* M$$

and

$$\text{Lie}(G) \cong P/Q \cong \sigma^* M/N.$$
The quotient map $\mathcal{O} \to k$ induces a ring homomorphism $A_{\text{inf}} \to W(k)$ sending $\xi \mapsto p$. It follows that there is a unique continuous extension to $A_{\text{crys}} \to W(k)$, and standard results from Grothendieck-Messing theory identify the Dieudonné module of $G_k$ with

$$W(k) \otimes_{A_{\text{crys}}} P \cong W(k) \otimes_{\sigma,A_{\text{inf}}} M = M_{\text{crys}}.$$ 

Finally, combining (2.3.4) and (2.3.5) identifies

$$T_p(G) \xrightarrow{\text{HT}} \Omega(G^\vee) \xrightarrow{\Pi} \Omega(G^\vee),$$

This proves all parts of Theorem 2.3.6.

The invocation of Theorem 2.3.6 in the calculations below is the only reason why the assumption $p > 2$ is imposed in the introduction. Our approach in the sequel will be to allow arbitrary $p$, but to take the conclusions of Theorem 2.3.6 as hypotheses.

3. Bounding the Hodge-Tate periods

Let $G$ be a $p$-divisible group of height four and dimension two over $\mathcal{O}$, endowed with an action $\Delta \to \text{End}(G)$. Its $p$-adic Tate module is free of rank one over $\Delta$, and we fix a generator $\lambda \in T_p(G)$.

Throughout §3 we assume that $G$ and its Breuil-Kisin-Fargues module enjoy the properties listed in Theorem 2.3.6. For example, it is enough to assume that $p > 2$.

3.1. Hodge-Tate periods. The embeddings (1.3.1) determine a decomposition

(3.1.1) $\Omega(G^\vee) = \Omega_0(G^\vee) \oplus \Omega_1(G^\vee),$

in which each summand is free of rank one over $\mathcal{O}$, and $\mathbb{Z}_p^2 \subset \Delta$ acts on them through $j_0$ and $j_1$, respectively. The operator $\Pi$ maps each summand injectively into the other. Applying $\otimes \mathcal{O} k$ to (3.1.1) yields a decomposition

$\Omega(G^\vee_k) = \Omega_0(G^\vee_k) \oplus \Omega_1(G^\vee_k)$

into one dimensional $k$-vector spaces.

Composing the Hodge-Tate morphism (1.1.1) with the two projections yields two partial Hodge-Tate morphisms

$$T_p(G) \xrightarrow{\text{HT}_0} \Omega_0(G^\vee), \quad T_p(G) \xrightarrow{\text{HT}_1} \Omega_1(G^\vee).$$

By fixing isomorphisms

(3.1.2) $\Omega_0(G^\vee) \cong \mathcal{O}, \quad \Omega_1(G^\vee) \cong \mathcal{O},$

we view these as $\mathcal{O}$-valued linear functionals on $T_p(G)$. 
Using the chosen $\Delta$-module generator $\lambda \in T_p(G)$, define
\[
\tau_0 = \frac{\text{HT}_0(\Pi \lambda)}{\text{HT}_0(\lambda)}, \quad \tau_1 = \frac{\text{HT}_1(\Pi \lambda)}{\text{HT}_1(\lambda)}.
\]
These are the Hodge-Tate periods of $G$. In each fraction the numerator or denominator may vanish, but not simultaneously. Thus the Hodge-Tate periods lie in $\mathbb{P}^1(C) = C \cup \{x\}$. They do not depend on the choice of (3.1.2), but do depend on the choice of generator $\lambda$.

**Proposition 3.1.1.** The Hodge-Tate periods satisfy $\tau_0 \cdot \tau_1 = p$.

**Proof.** The action of $\Pi$ on $\Omega_0(G^\vee) \oplus \Omega_1(G^\vee)$ is given by
\[
(\omega_0, \omega_1) \mapsto (s_0 \omega_1, s_1 \omega_0)
\]
for some $s_0, s_1 \in \mathcal{O}$ satisfying $s_0 s_1 = p$. From the $\Delta$-linearity of the Hodge-Tate morphism we deduce first
\[
\text{HT}_0(\Pi \lambda) = s_0 \cdot \text{HT}_1(\lambda), \quad \text{HT}_1(\Pi \lambda) = s_1 \cdot \text{HT}_0(\lambda),
\]
and then
\[
\tau_0 \cdot \tau_1 = \frac{\text{HT}_0(\Pi \lambda)}{\text{HT}_0(\lambda)} \cdot \frac{\text{HT}_1(\Pi \lambda)}{\text{HT}_1(\lambda)} = s_0 \cdot s_1 = p,
\]
as desired. \qed

3.2. Reduction to the residue field. Let $G_k$ be the reduction of $G$ to the residue field $k = \mathcal{O}/\mathfrak{m}$.

**Definition 3.2.1.** Let $H$ be the $p$-divisible group of a supersingular elliptic curve over $k$. In other words, $H$ is the unique connected $p$-divisible group of height two and dimension one. The reduction $G_k$ is said to be

1. supersingular if it is isogenous to $H \times H$,
2. superspecial if it is isomorphic to $H \times H$.

**Remark 3.2.2.** Our notions of supersingular and superspecial depend only on the $p$-divisible group $G_k$, and not on its $\Delta$-action. This differs from the meaning of superspecial in some literature on Shimura curves.

We have defined superspecial using the entire $p$-divisible group $G_k$, but it is really a property only of the $p$-torsion subgroup $G_k[p]$. This is a consequence of the following proposition.

**Proposition 3.2.3.** Let $(D, F, V)$ be the Dieudonné module of $G_k$. The reduction $G_k$ is superspecial, and the following are equivalent:

1. $G_k$ is superspecial,
2. $FD = VD$,
3. the operator $V^2$ kills $D/pD$.

**Proof.** All parts of this are well-known. The first claim follows from the Dieudonné-Manin classification of isocrystals: one can list all isogeny classes of $p$-divisible groups over $k$ of height four and dimension two, and the supersingular isogeny class is the only one whose endomorphism algebra contains
a quaternion division algebra. The second claim follows from the arguments of \[Oor75\].

Let \((M, \phi, \psi)\) be the Breuil-Kisin-Fargues module of \(G\). The quotient

\[ M^b = M/pM \]

is a free module over \(O^b \cong A_{\text{inf}}/(p)\), endowed with operators \(\phi, \psi : M^b \to M^b\) satisfying \(\phi \circ \psi = \varpi\). Denote by \(N^b = N/pN\) the image of

\[ M^b \xrightarrow{m \mapsto 1 \otimes \psi(m)} \sigma^* M^b. \]

Each of our embeddings \(j_0, j_1 : \mathbb{Z}_p^2 \to O\) determines a map

\[ \mathbb{Z}_p^2 \to O/pO \cong O^b/\varpi O^b, \]

and these two maps lift uniquely to \(j_0, j_1 : \mathbb{Z}_p^2 \to O^b\). The action of \(\Delta\) on \(G\) determines an action on \(M^b\), which induces a decomposition

\[ M^b = M_0^b \oplus M_1^b \]

analogous to (3.1.1). It follows from the next proposition that each factor free of rank two over \(O^b\).

**Proposition 3.2.4.**

1. \(D\) is free of rank one over \(\Delta \otimes_{\mathbb{Z}_p} W(k)\).
2. \(M\) is free of rank one over \(\Delta \otimes_{\mathbb{Z}_p} A_{\text{inf}}\).

**Proof.** The \(p\)-divisible group \(G_k\) appears as a \(k\)-valued point in the Rapoport-Zink space associated to a Shimura curve with bad reduction at \(p\). The first claim therefore follows from the Drinfeld’s explicit parametrization of such points and their Dieudonné modules. See [KR00] for a brief summary, and [BC91] for the full details.

Theorem 2.3.6 gives us an isomorphism

\[ D/pD \cong \sigma^*(M/mM) \]

of \(\Delta \otimes_{\mathbb{Z}_p} k\)-modules, and from what was said above we deduce that \(M/mM\) is free of rank one over \(\Delta \otimes_{\mathbb{Z}_p} k\). The second claim of the proposition follows easily from this and Nakayama’s lemma. □

### 3.3. The case \(\Pi\Omega(G^\vee_k) = 0\)

We assume throughout 3.3 that \(\Pi\Omega(G^\vee_k) = 0\).

We will analyze the structure of \(M^\phi\), with its operators \(\phi\) and \(\psi\), and use this to bound the Hodge-Tate periods of \(G\). The first step is to choose a convenient basis.

**Lemma 3.3.1.** There are \(O^b\)-bases \(e_0, f_0 \in M_0^b\) and \(e_1, f_1 \in M_1^b\) such that the operator \(\Pi \in \Delta\) satisfies

\[ \Pi e_0 = 0, \quad \Pi e_1 = 0, \quad \Pi f_0 = e_1, \quad \Pi f_1 = e_0, \]

(3.3.1)
and such that $\psi$ satisfies

$$
\psi(e_0) = t_0e_1, \quad \psi(e_1) = t_1e_0, \quad \psi(f_0) = e_1 + t_1f_1, \quad \psi(f_1) = e_0 + t_0f_0
$$

for scalars $t_0, t_1 \in \mathcal{O}^\circ$ satisfying $\text{ord}(t_0) > 0$, $\text{ord}(t_1) > 0$, and

$$
\text{ord}(t_0) + \text{ord}(t_1) = 1/p.
$$

**Proof.** As $M^\circ$ is free of rank one over $\Delta \otimes_{\mathbb{Z}_p} \mathcal{O}^\circ$, we may choose a basis such that \((3.3.1)\) holds, and the relation $\psi \circ \Pi = \Pi \circ \psi$ then implies

$$
\psi(e_0) = t_0e_1, \quad \psi(e_1) = t_1e_0, \quad \psi(f_0) = u_1e_1 + t_1f_1, \quad \psi(f_1) = u_0e_0 + t_0f_0
$$

for some $u_0, u_1, t_0, t_1 \in \mathcal{O}^\circ$. The submodule $N^\circ \subset \sigma^*M^\circ$ is generated by

$$
\begin{align*}
1 \otimes \psi(e_0) &= t_0^p \otimes e_1 \\
1 \otimes \psi(e_1) &= t_1^p \otimes e_0 \\
1 \otimes \psi(f_0) &= u_1^p \otimes e_1 + t_1^p \otimes f_1 \\
1 \otimes \psi(f_1) &= u_0^p \otimes e_0 + t_0^p \otimes f_0.
\end{align*}
$$

It is easy to see that

$$
\sigma^*M^\circ/N^\circ \cong (\sigma^*M/N) \otimes \mathcal{O}/(p)
$$

is free of rank two as a module over $\mathcal{O}/(p) \cong \mathcal{O}^\circ/(\varpi)$, and using this an easy calculation shows that $\text{ord}(t_0t_1) = 1/p$.

Recall that $m^\circ \subset \mathcal{O}^\circ$ is the maximal ideal. Theorem \([2.3.6]\) identifies $\Omega(G^\vee_k)$ with the image of $N^\circ$ in $(\sigma^*M^\circ)/m^\circ(\sigma^*M^\circ)$, and by hypothesis this $k$-vector space is annihilated by $\Pi$. It is easy to see from this that $\text{ord}(t_0)$ and $\text{ord}(t_1)$ are positive.

Again using the fact that $\sigma^*M^\circ/N^\circ$ is free of rank two over $\mathcal{O}^\circ/(\varpi)$, and that $t_0, t_1 \in \mathcal{O}^\circ$ are nonunits, an easy calculation shows that $u_0$ and $u_1$ are units. Using this, another easy calculation shows that our basis elements may be rescaled in order to make $u_0$ and $u_1$ equal to 1. \(\square\)

Fix a basis as in Lemma \([3.3.1]\). Theorem \([2.3.6]\) identifies

$$
T_p(G)/pT(G) = M^{\psi=1}/pM^{\psi=1} \subset (M^\circ)^{\psi=1},
$$

and the image of our fixed generator $\lambda \in T_p(G)$ has the form

$$
a_0e_0 + a_1e_1 + b_0f_0 + b_1f_1 \in M^\circ
$$

for some coefficients $a_0, a_1, b_0, b_1 \in \mathcal{O}^\circ$ satisfying

$$
\begin{align*}
a_0^p &= a_1t_1^p + b_1 \\
a_1^p &= a_0t_0^p + b_0 \\
b_0^p &= b_1t_0^p \\
b_1^p &= b_0t_1^p.
\end{align*}
$$

Theorem \([2.3.6]\) also identifies

$$
\Omega(G^\vee)/p\Omega(G^\vee) = N/(pN + \xi\sigma^*M) = N^\circ/\varpi\sigma^*M^\circ
$$
with the direct summand of $\sigma^* M^\flat / \omega \sigma^* M^\flat$ generated by the reductions of
\[ 1 \otimes \psi(f_0) = 1 \otimes e_1 + t_1^p \otimes f_1 \in \sigma^* M^\flat \]
\[ 1 \otimes \psi(f_1) = 1 \otimes e_0 + t_0^p \otimes f_0 \in \sigma^* M^\flat. \]

If we use this basis to identify
\[ \Omega(G^\vee)/p\Omega(G^\vee) = N^\flat / \omega \sigma^* M^\flat \cong \mathcal{O}^\flat / (\omega) \]
then, again using Theorem 2.3.6, the partial Hodge-Tate morphisms
\[ T_p(G)/pT_p(G) \xrightarrow{\text{HT}_0} \Omega_0(G^\vee)/p\Omega_0(G^\vee) \cong \mathcal{O}^\flat / (\omega) \]
\[ T_p(G)/pT_p(G) \xrightarrow{\text{HT}_1} \Omega_1(G^\vee)/p\Omega_1(G^\vee) \cong \mathcal{O}^\flat / (\omega) \]
are given by
\[ \text{HT}_0(\lambda) = a_1^p \quad \text{HT}_0(\Pi \lambda) = b_0^p \]
\[ \text{HT}_1(\lambda) = a_0^p \quad \text{HT}_1(\Pi \lambda) = b_1^p. \]

**Lemma 3.3.2.** For $i \in \{0, 1\}$, we have
\[ \text{ord}(b_i) = \frac{1}{p^2 - 1} + \frac{p \cdot \text{ord}(t_i)}{p + 1}. \]

**Proof.** As $\Pi \lambda \in T_p(G)$ has nonzero image in
\[ T_p(G)/pT_p(G) \subset M^\flat, \]
we must have $b_0 e_1 + b_1 e_0 \neq 0$. Therefore one of $b_0$ and $b_1$ is nonzero. The relations 3.3.2 then imply first that both $b_0$ and $b_1$ are nonzero, and then that
\[ b_i^{p^2 - 1} = (t_0 t_1)^p \cdot t_i^{p-1}. \]
The claim follows by applying ord to both sides of this equality. \qed

**Lemma 3.3.3.** If we assume that
\[ \frac{1}{p^2 (p - 1)} < \text{ord}(t_1), \]
then
\[ \text{ord}(a_0) = \frac{1}{p(p^2 - 1)} + \frac{\text{ord}(t_1)}{p + 1}, \quad \text{ord}(a_1) = \frac{1}{p^2 - 1} - \frac{\text{ord}(t_1)}{p + 1}. \]

Of course there is a similar result if $t_1$ is replaced by $t_0$.

**Proof.** Recall the equality $a_0^p = a_1 t_1^p + b_1$ from 3.3.2. The only way this can hold is if (at least) one of the three relations
\[ p \cdot \text{ord}(a_0) = \text{ord}(b_1) \leq \text{ord}(t_1^p a_1) \]
\[ p \cdot \text{ord}(a_0) = \text{ord}(t_1^p a_1) \leq \text{ord}(b_1) \]
\[ \text{ord}(b_1) = \text{ord}(t_1^p a_1) \leq p \cdot \text{ord}(a_0) \]
is satisfied. The second and third relations cannot be satisfied, as each implies

\[ 0 \leq \text{ord}(a_1) \leq \text{ord}(b_1) - p \cdot \text{ord}(t_1) = \frac{1}{p^2 - 1} - \frac{p^2 \cdot \text{ord}(t_1)}{p + 1} < 0. \]

Hence the first relation holds, and Lemma 3.3.2 shows that

\[ p \cdot \text{ord}(a_0) = \text{ord}(b_1) = \frac{1}{p^2 - 1} + \frac{p \cdot \text{ord}(t_1)}{p + 1}. \]

This proves the first equality.

For the second equality, the relations (3.3.2) imply

\[
\begin{align*}
a_0^2 &= a_1^p \cdot (t_1^p + t_1) - (t_0 t_1)^p a_0 \\
a_1^2 &= b_0^p \cdot (t_0^p + t_0) - (t_0 t_1)^p a_1.
\end{align*}
\]

Using the second of these, along with

\[ \text{ord}(a_0^p \cdot (t_0^p + t_0)^p) = \text{ord}(b_1) + p \cdot \text{ord}(t_0) = \frac{p^2}{p^2 - 1} - \frac{p^2 \cdot \text{ord}(t_1)}{p + 1} < 1 \leq \text{ord}((t_0 t_1)^p a_1), \]

we find that

\[ \text{ord}(a_1) = \frac{\text{ord}(a_0^p \cdot (t_0^p + t_0)^p)}{p^2} = \frac{1}{p^2 - 1} - \frac{\text{ord}(t_1)}{p + 1}. \]

Now we can prove the main result of this subsection.

**Proposition 3.3.4.** If we assume, as above, that \( \Pi \Omega(G^\vee_k) = 0 \) then

\[ \frac{1}{p + 1} < \text{ord}(\tau_0) < \frac{p}{p + 1} \quad \text{and} \quad \frac{1}{p + 1} < \text{ord}(\tau_1) < \frac{p}{p + 1}. \]

**Proof.** First assume that

\[ (3.3.4) \quad \frac{1}{p^2(p - 1)} < \text{ord}(t_1). \]

The discussion leading to (3.3.3) provides us with an \( \mathcal{O} \)-module isomorphism

\[ \Omega_0(G^\vee)/\mathcal{O}_0(G^\vee) \cong \mathcal{O}^b/(\varpi) \cong \mathcal{O}/(p), \]

and we fix any lift to an isomorphism \( \Omega_0(G^\vee) \cong \mathcal{O} \).

It is easy to see from Lemmas 3.3.2 and 3.3.3 that \( \text{ord}(a_1) \) and \( \text{ord}(b_0) \) lie in the open interval \((0, 1/p, \), and so \( a_0^p \) and \( b_0^p \) have nonzero images in \( \mathcal{O}^b/(\varpi) \). By (3.3.3) these images agree with the images of \( \text{HT}_0(\lambda) \) and \( \text{HT}_0(\Pi \lambda) \) under

\[ \mathcal{O} \to \mathcal{O}/(p) \cong \mathcal{O}^b/(\varpi). \]

Thus

\[ \text{ord}(\text{HT}_0(\lambda)) = \text{ord}(a_1^p) = \frac{p}{p^2 - 1} - \frac{p \cdot \text{ord}(t_1)}{p + 1}. \]
and
\[ \text{ord}(\HT_0(\Pi \lambda)) = \text{ord}(b_0^p) = \frac{p}{p^2 - 1} + \frac{p^2 \cdot \text{ord}(t_0)}{p + 1}. \]

It follows that
\[ \text{ord}(\tau_0) = \text{ord}(\HT_0(\Pi \lambda)) - \text{ord}(\HT_0(\lambda)) = \frac{p}{p + 1} - \frac{(p - 1)}{p + 1} \cdot \text{ord}(p_1^p), \]

and so
\[ \frac{1}{p + 1} < \text{ord}(\tau_0) < \frac{p}{p + 1}. \]

The analogous inequalities for \( \text{ord}(\tau_1) \) follow from \( \tau_0 \tau_1 = p \). This proves Proposition 3.3.4 under the assumption (3.3.4), and the proof when
\[ (3.3.5) \quad \frac{1}{p^2(p - 1)} < \text{ord}(t_0) \]
is entirely similar.

Thus we are left to prove the claim under the assumption that both (3.3.4) and (3.3.5) fail. This assumption implies that
\[ \frac{1}{p} = \text{ord}(t_0) + \text{ord}(t_1) \leq \frac{2}{p^2(p - 1)}, \]

which implies that we are in the highly degenerate case of \( p = 2 \) and
\[ \text{ord}(t_0) = \frac{1}{4} = \text{ord}(t_1), \]

and so Lemma 3.3.2 simplifies to
\[ \text{ord}(b_0) = \frac{1}{2} = \text{ord}(b_1). \]

Consider the equality \( a_0^2 = a_1 t_0^2 + b_1 \) of (3.3.2). As in the proof of Lemma 3.3.3 the only way this can hold is if (at least) one of the relations
\[ \begin{align*}
& \text{ord}(a_0) = 1/4 \\
& \text{ord}(a_0) = 0 \text{ and ord}(a_1) \geq 1/4
\end{align*} \]
holds. Similarly, the equality \( a_1^2 = a_0 t_0^2 + b_0 \) implies that (at least) one of the relations
\[ \begin{align*}
& \text{ord}(a_1) = 1/4 \\
& \text{ord}(a_1) = 0 \text{ and ord}(a_0) \geq 1/4.
\end{align*} \]
holds. Combining these shows that \( \text{ord}(a_0) \geq 1/4 \) and \( \text{ord}(a_1) \geq 1/4 \).

In particular, \( a_1^p \) has nonzero image in \( \mathcal{O}^p/(\pi \overline{\pi}) \), and
\[ \text{ord}(\HT_0(\lambda)) = \text{ord}(a_1^p) \leq \frac{1}{2}. \]

On the other hand, \( b_0^p \) has trivial image in \( \mathcal{O}^p/(\wp) \), and it follows that
\[ \text{ord}(\HT_0(\Pi \lambda)) \geq 1. \]

Therefore
\[ \text{ord}(\tau_0) = \text{ord}(\HT_0(\Pi \lambda)) - \text{ord}(\HT_0(\lambda)) \geq \frac{1}{2}. \]
The same reasoning shows that \( \text{ord}(\tau_1) \geq 1/2 \), and the relation \( \text{ord}(\tau_0) + \text{ord}(\tau_1) = 1 \) then yields

\[
\text{ord}(\tau_0) = \frac{1}{2} = \text{ord}(\tau_1).
\]

This completes the proof of Proposition 3.3.3. \( \square \)

3.4. The case \( \Pi\Omega_1(G^\vee_k) \neq 0 \). We assume throughout [3.3] that

\[
\Pi\Omega_1(G^\vee_k) \neq 0.
\]

Once again, we will analyze the structure of \( M^b = M/pM \), and use this to bound the Hodge-Tate periods of \( G \). As in [3.3] the first step is to choose a convenient basis for \( M^b \).

**Lemma 3.4.1.** There are \( \mathcal{O}^b \)-bases \( e_0, f_0 \in M^b_0 \) and \( e_1, f_1 \in M^b_1 \) such that the operator \( \Pi \in \Delta \) satisfies

\[
\begin{align*}
\Pi e_0 &= 0, & \Pi e_1 &= 0, & \Pi f_0 &= e_1, & \Pi f_1 &= e_0,
\end{align*}
\]

and such that \( \psi \) satisfies

\[
\begin{align*}
\psi(e_0) &= e_1, & \psi(e_1) &= te_0, & \psi(f_0) &= tf_1, & \psi(f_1) &= se_0 + f_0
\end{align*}
\]

for some scalars \( s, t \in \mathcal{O}^b \) with \( \text{ord}(t) = 1/p \). Moreover:

1. For any such basis, \( G_k \) is superspecial if and only if \( \text{ord}(s) > 0 \).
2. If \( G_k \) is not superspecial such a basis can be found with \( s = 1 \).

**Proof.** Exactly as in the proof of Lemma 3.4.1, we may choose a basis such that (3.4.1) holds, and such that

\[
\begin{align*}
\psi(e_0) &= t_0 e_1, & \psi(e_1) &= t_1 e_0, & \psi(f_0) &= u_1 e_1 + t_1 f_1, & \psi(f_1) &= u_0 e_0 + t_0 f_0
\end{align*}
\]

for some \( u_0, u_1, t_0, t_1 \in \mathcal{O} \) with \( \text{ord}(t_0) + \text{ord}(t_1) = 1/p \).

The \( \Delta \)-module \( \Omega(G^\vee_k) \) is identified with the image of

\[
N^b \to (\sigma^* M^b)/\mathfrak{m}^b(\sigma^* M^b),
\]

and this identifies \( \Omega_1(G^\vee_k) \) with the (one-dimensional) \( k \)-span of the vectors

\[
1 \otimes \psi(e_1) = t_1^0 \otimes e_0, & \quad & 1 \otimes \psi(f_1) = u_0^b \otimes e_0 + t_0^b \otimes f_0
\]

in \( (\sigma^* M^b_0)/\mathfrak{m}^b(\sigma^* M^b) \). The assumption that \( \Pi \) does not annihilate \( \Omega_1(G^\vee_k) \) implies that \( \text{ord}(t_0) = 0 \), which allows us to rescale our basis vectors to make \( t_0 = 1 \), and then add a multiple of \( e_0 \) to \( f_0 \) to make \( u_1 = 0 \). Setting \( t = t_1 \) and \( s = u_0 \), the relations (3.4.2) now hold.

It follows from Proposition 3.2.3 and Theorem 2.3.6 that

\[
G_k \text{ is superspecial } \iff V^2(D/pD) = 0
\]

\[
\iff \psi^2(M^b/\mathfrak{m}^b M^b) = 0
\]

\[
\iff \text{ord}(s) > 0.
\]

Finally, if \( \text{ord}(s) = 0 \) it is an easy exercise in linear algebra to see that the given basis elements can be rescaled to make \( s = 1 \). \( \square \)
As in §3.3, our fixed generator $\lambda \in T_p(G)$ determines an element

$$a_0e_0 + a_1e_1 + b_0f_0 + b_1f_1 \in M^b,$$

where the coefficients $a_0, a_1, b_0, b_1 \in \mathcal{O}^b$ satisfy

$$a_0^p = a_1t^p + b_1s^p$$

(3.4.3)

$$a_1^p = a_0$$

$$b_0^p = b_1$$

$$b_1^p = b_0t^p.$$

As in §3.3 we may identify

$$\Omega(G^\vee)/p\Omega(G^\vee) = N/(pN + \xi\sigma^*M) = N^b/\varpi\sigma^*M^b$$

with the direct summand of $\sigma^*M^p/\varpi\sigma^*M^b$ generated by the reductions of

$$1 \otimes \psi(e_0) = 1 \otimes e_1 \in \sigma^*M^b$$

$$1 \otimes \psi(f_1) = s^p \otimes e_0 + 1 \otimes f_0 \in \sigma^*M^b.$$

If we use this basis to identify

$$\Omega(G^\vee)/p\Omega(G^\vee) = N^b/\varpi\sigma^*M^b \cong \mathcal{O}^b/(\varpi) \oplus \mathcal{O}^b/(\varpi)$$

then, using Theorem 2.3.6, the partial Hodge-Tate morphisms

$$T_p(G)/pT_p(G) \overset{\text{HT}_0}{\longrightarrow} \Omega_0(G^\vee)/p\Omega_0(G^\vee) \cong \mathcal{O}^b/(\varpi)$$

$$T_p(G)/pT_p(G) \overset{\text{HT}_1}{\longrightarrow} \Omega_1(G^\vee)/p\Omega_1(G^\vee) \cong \mathcal{O}^b/(\varpi)$$

satisfy

(3.4.4)

$$\text{HT}_0(\lambda) = a_0 \quad \text{HT}_0(\Pi\lambda) = b_1$$

$$\text{HT}_1(\lambda) = b_1 \quad \text{HT}_1(\Pi\lambda) = 0.$$  

**Lemma 3.4.2.** We have

$$\text{ord}(b_0) = \frac{1}{p^2 - 1}, \quad \text{ord}(b_1) = \frac{p}{p^2 - 1}.$$  

Moreover,

$$\text{ord}(a_0) \geq \frac{1}{p^2 - 1}, \quad \text{ord}(a_1) \geq \frac{1}{p(p^2 - 1)},$$

and $G_k$ is superspecial if and only if one (equivalently, both) of these inequalities is strict.

**Proof.** Exactly as in the proof of Lemma 3.3.2, both $b_0$ and $b_1$ are nonzero. The relations (3.4.3) therefore imply that

$$b_0^{p^2 - 1} = t^p,$$

from which the stated formulas for $\text{ord}(b_0)$ and $\text{ord}(b_1) = \text{ord}(b_0^p)$ are clear.
The relations (3.4.3) imply that \( a_0 \) is a root of \( x^{p^2} - xt^{p^2} - b_1^p s^{p^2} \), and by examination of the Newton polygon we see that

\[
\text{ord}(a_0) \geq \frac{1}{p^2 - 1}
\]

with strict inequality if and only if \( \text{ord}(s) > 0 \). Combining this with \( a_1^p = a_0 \) completes the proof.

□

Lemma 3.4.3. If \( G_k \) is not superspecial then

\[
\varpi (a_0/b_1)^{p+1} \in (\mathcal{O}^\times)^{\varpi} \quad \text{and} \quad \varpi s^{p+1}/t^p \in (\mathcal{O}^\times)^{\varpi},
\]

and these units have the same reduction to \( k^\times \).

Proof. We have already noted that (3.4.3) implies \( t^p = b_1^{p^2 - 1} \), from which one easily deduces the equality

\[
\left( \frac{a_1}{b_0} \right)^{p^2} = \frac{a_1}{b_0} + \frac{s^p}{b_0^{p-1}}
\]

in the fraction field of \( \mathcal{O}^b \). It follows from this and Lemma 3.4.2 that

\[
\varpi^{p+1} \left( \frac{a_1}{b_0} \right)^{p^2} \quad \text{and} \quad \left( \frac{\varpi s^{p+1}/t^p}{b_0^{p-1}} \right)^p
\]

are units in \( \mathcal{O}^b \) with the same reduction to \( k^\times \), hence the same is true after raising both to the power \( (p+1)/p \). The lemma follows easily from this and the relations (3.4.3).

□

Proposition 3.4.4. If we assume, as above, that \( \Pi \Omega_1(G_k^\times) \neq 0 \) then

(3.4.5)

\[
\frac{p}{p+1} \leq \text{ord}(\tau_1)
\]

with strict inequality if and only if \( G_k \) is superspecial. Moreover, if equality holds then

\[
\frac{p}{\tau_0^{p+1}} \in \mathcal{O}^\times \quad \text{and} \quad \frac{\varpi s^{p+1}}{t^p} \in (\mathcal{O}^b)^{\varpi}
\]

have the same reduction to \( k^\times/\mathbb{F}_p^\times \).

Proof. Using (3.4.4) and Lemma 3.4.2 we find that

\[
\text{ord}(\text{HT}_0(\Pi \lambda)) = \frac{p}{p^2 - 1},
\]

and that

\[
\text{ord}(\text{HT}_0(\lambda)) \geq \frac{1}{p^2 - 1}
\]

with strict inequality if and only if \( G_k \) is superspecial. This implies that

\[
\text{ord}(\tau_0) = \text{ord}(\text{HT}_0(\Pi \lambda)) - \text{ord}(\text{HT}_0(\lambda)) \leq \frac{1}{p+1}
\]
with strict inequality if and only if \(G_k\) is superspecial. The inequality \((3.4.5)\) follows from this and the relation \(\tau_0 \tau_1 = p\) of Proposition \(3.1.1\) with strict inequality if and only if \(G_k\) is superspecial.

Suppose that equality holds in \((3.4.5)\), so that \(G_k\) is not superspecial. Choose an \(\alpha \in \mathcal{O}\) satisfying \(\alpha^{p^2-1} = \varpi\). The construction of \(2.1\) determines an element \(\alpha^\sharp \in \mathcal{O}\) whose image in \(\mathcal{O}/(p) \cong \mathcal{O}/(\varpi)\) agrees with \(\alpha\).

Combining the relations \((3.4.4)\) with Lemma \(3.4.2\) shows that

\[
\frac{\text{HT}_0(\Pi \lambda)}{(\alpha^\sharp)^p} \in \mathcal{O}^\times \quad \text{and} \quad \frac{b_1}{\alpha^p} \in (\mathcal{O}^\mathfrak{s})^\times
\]

have the same reduction to \(k^\times\), as do

\[
\frac{\text{HT}_0(\lambda)}{\alpha^\sharp} \in \mathcal{O}^\times \quad \text{and} \quad \frac{a_0}{\alpha} \in (\mathcal{O}^\mathfrak{s})^\times.
\]

It then follows that

\[
\frac{\tau_0}{(\alpha^\sharp)^{p-1}} \in \mathcal{O}^\times \quad \text{and} \quad \frac{b_1}{a_0 \alpha^{p-1}} \in (\mathcal{O}^\mathfrak{s})^\times
\]

have the same reduction to \(k^\times\). Raising both to the power \(p+1\) and applying Lemma \(3.4.3\) proves that

\[
\frac{\varpi^{\sharp}}{\tau_0^{p+1}} \in \mathcal{O}^\times \quad \text{and} \quad \frac{\varpi^{\mathfrak{s}p+1}}{\mathfrak{t}^p} \in (\mathcal{O}^\mathfrak{s})^\times
\]

have the same reduction to \(k^\times\).

The only thing left to check is that the reduction of \(\varpi^\sharp/p \in \mathcal{O}^\times\) lies in the prime subfield \(\mathbb{F}_p \subset k\). This is clear, as \(\varpi^\sharp/p\) lies in the closure of the subring \(\mathbb{Z}_p[\mu_{p\mathfrak{s}}] \subset \mathcal{O}\) by construction. \(\square\)

4. The main results

We now formulate and prove our main results on the Ekedahl-Oort stratification of the Hodge-Tate period domain \((1.2.1)\). Throughout \(\S 4\) we assume that the conclusions of Theorem \(2.3.6\) hold. For example, it is enough to assume that \(p > 2\).

4.1. The setup. Let \(T\) be a free \(\Delta\)-module of rank one, and fix a generator \(\lambda \in T\). Use the embeddings \((1.3.1)\) to decompose

\[
T \otimes_{\mathbb{Z}_p} \mathbb{C} = T_{C,0} \oplus T_{C,1}
\]

as a direct sum of 2-dimensional \(\mathbb{C}\)-subspaces, in such a way that the action of \(\mathbb{Z}_p^2 \subset \Delta\) on the summands is through \(j_0\) and \(j_1\), respectively. Using the projection maps to the two factors, we obtain injective \(\mathbb{Z}_p\)-linear maps

\[
q_0 : T \to T_{C,0}, \quad q_1 : T \to T_{C,1}.
\]

To each \(\tau \in C \cup \{\infty\}\) we associate the \(\Delta\)-stable plane

\[
W_{\tau} \subset T \otimes_{\mathbb{Z}_p} \mathbb{C}
\]
spanned by the two vectors
\[ \tau q_0(\lambda) - q_0(\Pi \lambda) \in T_{C,0}, \quad p q_1(\lambda) - \tau q_1(\Pi \lambda) \in T_{C,1}. \]
The construction \( \tau \mapsto W_\tau \) establishes a bijection
\[ C \cup \{ \infty \} \cong X(C). \]

Remark 4.1.1. It is not hard to see that the above bijection \( \mathbb{P}^1(C) \cong X(C) \) arises from an isomorphism of schemes over \( \mathbb{Q}_p \). The isomorphism cannot descend to \( \mathbb{Q}_p \), for the simple reason that \( X(\mathbb{Q}_p) = \emptyset \).

For the rest of \( \text{§} 4.1 \) and \( \text{§} 4.2 \) we hold \( \tau \in C \cup \{ \infty \} \) fixed, and let \( G \) be the \( p \)-divisible group over \( \mathcal{O} \) determined by the pair \( (T, W_\tau) \). Thus \( G \) comes equipped with an action of \( \Delta \), and \( \Delta \)-linear identifications
\[
\begin{align*}
T_p(G) & \to \text{HT} \to \Omega(G^\vee) \otimes \mathcal{O} C \\
T & \to (T \otimes \mathbb{Z}_p C)/W_\tau.
\end{align*}
\]
In the notation of \( \text{§} 3.1 \) the Hodge-Tate periods of \( G \) are
\[
(4.1.1) \quad \tau_0 = \tau \quad \text{and} \quad \tau_1 = p/\tau.
\]

4.2. Computing the reduction. Let \( G_k \) be the reduction of \( G \) to the residue field \( k = \mathcal{O}/\mathfrak{m} \), and let \( (D, F, V) \) be its covariant Dieudonné module. We will show how to compute the isomorphism class of \( G_k[p] \) from the Hodge-Tate periods \( (4.1.1) \).

Let \( \mathbb{D} = \Delta \otimes_{\mathbb{Z}_p} k \) with its natural action of \( \Delta \) by left multiplication. The embeddings \( (1.3.1) \) induce a decomposition
\[
\mathbb{D} = \mathbb{D}_0 \oplus \mathbb{D}_1
\]
in which \( \mathbb{Z}_p^2 \subset \Delta \) acts on \( \mathbb{D}_1 \) through the composition of \( \mathfrak{j}_i : \mathbb{Z}_p^2 \to \mathcal{O} \) with the reduction map \( \mathcal{O} \to k \). Choose \( k \)-bases
\[
e_0, f_0 \in \mathbb{D}_0, \quad e_1, f_1 \in \mathbb{D}_1
\]
in such a way that \( \Pi \in \Delta \) acts as
\[
(4.2.1) \quad \Pi e_0 = 0, \quad \Pi e_1 = 0, \quad \Pi f_0 = e_1, \quad \Pi f_1 = e_0.
\]

Theorem 4.2.1. The inequalities
\[
(4.2.2) \quad \frac{1}{p + 1} < \text{ord}(\tau) < \frac{p}{p + 1}
\]
hold if and only if \( \Pi \Omega(G^\vee_k) = 0 \). When these conditions hold, there is a \( \Delta \)-linear isomorphism \( D/pD \cong \mathbb{D} \) under which
\[
F e_0 = 0, \quad F f_0 = e_1, \quad F e_1 = 0, \quad F f_1 = e_0,
\]
\[
V e_0 = 0, \quad V f_0 = e_1, \quad V e_1 = 0, \quad V f_1 = e_0.
\]
Proof. If \( \Pi \Omega(G_k^*) \neq 0 \) then either \( \Pi \Omega_1(G_k^*) \neq 0 \) or \( \Pi \Omega_0(G_k^*) \neq 0 \). In the first case Proposition 3.3.4 implies

\[
\frac{p}{p+1} \leq \text{ord}(\tau_1).
\]

In the second case the same proof, with indices 0 and 1 interchanged throughout, shows that

\[
\frac{p}{p+1} \leq \text{ord}(\tau_0).
\]

In either case, these bounds imply that (4.2.2) fails.

Now assume that \( \Pi \Omega(G_k^*) = 0 \). We have already proved in Proposition 3.3.4 that (4.2.2) holds, and so it only remains to prove that \( D/pD \) admits an isomorphism to \( D \) with the prescribed properties.

Let \( e_0, f_0 \in M_0^p \) and \( e_1, f_1 \in M_1^p \) be the bases of Lemma 3.3.1. Using the formula for \( \psi : M^p \to \mathcal{D} \) prescribed in that lemma, and the relation \( \phi \circ \psi = \varnothing \), one can write down an explicit formula for \( \phi \), and then see that the induced operators on the reduction \( M^p / \mathfrak{m}^p M^p \) are given by

\[
\phi(e_0) = 0, \quad \phi(f_0) = u e_1, \quad \phi(e_1) = 0, \quad \phi(f_1) = u e_0,
\]

\[
\psi(e_0) = 0, \quad \psi(f_0) = e_1, \quad \psi(e_1) = 0, \quad \psi(f_1) = e_0,
\]

where \( u^{-1} \in k^\times \) is the reduction of \( -\ell_0^p \ell_1^p / \varnothing \in (\mathcal{O}^p)^\times \).

The images of \( e_0, f_0, e_1, f_1 \) under the injection

\[
M^p / \mathfrak{m}^p M^p \xrightarrow{\pi^{-1} \otimes x} \sigma^*(M^p / \mathfrak{m}^p M^p) \cong D/pD
\]

provided by Theorem 2.3.6 form a \( k \)-basis of \( D/pD \), denoted the same way, satisfying the relations (4.2.1) and

\[
F e_0 = 0, \quad F f_0 = w^p e_1, \quad F e_1 = 0, \quad F f_1 = w^p e_0,
\]

\[
V e_0 = 0, \quad V f_0 = e_1, \quad V e_1 = 0, \quad V f_1 = e_0.
\]

We remain to prove that \( u = 1 \). The two embeddings (4.3.1) reduce to morphisms \( j_0, j_1 : \mathbb{Z}_{p^2} \to k \), which then admit unique lifts to \( j_0, j_1 : \mathbb{W}(k) \). This allows us to decompose \( D = D_0 \oplus D_1 \) as \( \mathcal{W} \)-modules, where \( \mathbb{Z}_{p^2} \subset \Delta \) acts on the two summands via \( j_0 \) and \( j_1 \), respectively. Choose arbitrary lifts

\[
\tilde{f}_0 \in D_0, \quad \tilde{f}_1 \in D_1
\]

of \( f_0 \) and \( f_1 \), and then define

\[
\tilde{e}_0 = \Pi \tilde{f}_1 \in D_0, \quad \tilde{e}_1 = \Pi \tilde{f}_0 \in D_1.
\]

Using the fact that \( \Pi \) and \( V \) commute, we see that

\[
V \tilde{e}_0 = p b_1 \tilde{e}_1 + p a_1 \tilde{f}_1 \quad V \tilde{f}_0 = a_1 \tilde{e}_1 + p b_1 \tilde{f}_1
\]

\[
V \tilde{e}_1 = p b_0 \tilde{e}_0 + p a_0 \tilde{f}_0 \quad V \tilde{f}_1 = a_0 \tilde{e}_0 + p b_0 \tilde{f}_0
\]

for scalars

\[
a_0, a_1 \in 1 + p \mathcal{W}(k), \quad b_0, b_1 \in \mathcal{W}(k).
\]
Denote again by $\sigma : W(k) \to W(k)$ the lift of the Frobenius on $k$. From the above equalities and $FV = p$, one can see that
\[ p\tilde{f}_1 = \sigma(a_0)F\tilde{e}_0 + \sigma(pb_0)F\tilde{f}_0, \quad p\tilde{f}_1 = \sigma(pb_0)F\tilde{e}_0 + \sigma(pa_0)F\tilde{f}_0. \]
Solving for $F\tilde{f}_0$ results in
\[(\sigma(a_0)^2 - p\sigma(b_0)^2) \cdot F\tilde{f}_0 = \sigma(a_0)\tilde{e}_1 - p\sigma(b_0)\tilde{f}_1,\]
and reducing this modulo $p$ proves that $Ff_0 = e_1$. Hence $u = 1$. □

**Theorem 4.2.2.** The inequality

\[(4.2.3) \quad \text{ord}(\tau) \leq \frac{1}{p+1} \]

holds if and only if $\Pi\Omega_1(G_k^\vee) \neq 0$. Moreover:

1. If strict inequality holds in \[(4.2.3) \], there is a $\Delta$-linear isomorphism $D/pD \cong \mathbb{D}$ under which

\[ Fe_0 = e_1, \quad Ff_0 = 0, \quad Fe_1 = 0, \quad Ff_1 = f_0, \]
\[ V\tilde{e}_0 = e_1, \quad Vf_0 = 0, \quad V\tilde{e}_1 = 0, \quad Vf_1 = f_0. \]

2. If equality holds in \[(4.2.3) \], there is a $\Delta$-linear isomorphism $D/pD \cong \mathbb{D}$ under which

\[(4.2.4) \quad Fe_0 = u^pe_1, \quad Ff_0 = -u^pf_0, \quad Fe_1 = 0, \quad Ff_1 = u^pf_0, \]
\[ V\tilde{e}_0 = e_1, \quad Vf_0 = 0, \quad V\tilde{e}_1 = 0, \quad Vf_1 = e_0 + f_0, \]

where $u$ is the image of $p/\tau_0^{p+1} = p/\tau^{p+1}$ under $O^\times \to k^\times$.

**Proof.** If \[(4.2.3) \] holds then Theorem 4.2.1 implies that $\Pi\Omega(G_k^\vee) \neq 0$, and so either

\[ \Pi\Omega_0(G_k^\vee) \neq 0 \quad \text{or} \quad \Pi\Omega_1(G_k^\vee) \neq 0. \]

The first possibility cannot occur, as then the proof of Proposition 3.4.4, with the indices 0 and 1 reversed everywhere, would give the bound

\[ \frac{p}{p+1} \leq \text{ord}(\tau_0), \]

contradicting \[(4.2.3) \]. Conversely, if $\Pi\Omega_1(G_k^\vee) \neq 0$ then \[(4.2.3) \] holds by Proposition 3.4.4.

Assume now that \[(4.2.3) \] holds, and that $\Pi\Omega_1(G_k^\vee) \neq 0$. Let $e_0, f_0 \in M_0^p$ and $e_1, f_1 \in M_1^p$ be the bases of Lemma 3.4.1. As in the proof of Theorem 4.2.1, the operator $\phi$ on $M^p$ can be computed from the formula for $\psi$ given in the lemma. The induced operators on the reduction $M^p/m^pM^p$ are found to be

\[ \phi(e_0) = ue_1, \quad \phi(f_0) = -u^pe_1, \quad \phi(e_1) = 0, \quad \phi(f_1) = uf_0, \]
\[ \psi(e_0) = e_1, \quad \psi(f_0) = 0, \quad \psi(e_1) = 0, \quad \psi(f_1) = ve_0 + f_0, \]
where \( u \in k^\times \) is the reduction of \( \varpi/p \in (O^\times)^k \), and \( v \in k \) is the reduction of \( s \in O^p \). By the final claim of Lemma \[3.4.1\], we may further assume that

\[
v = \begin{cases} 
0 & \text{if } G_k \text{ is superspecial} \\
1 & \text{otherwise}.
\end{cases}
\]

Suppose that strict inequality holds in \[4.2.3\]. Proposition \[3.4.4\] tells us that \( G_k \) is superspecial, and so \( v = 0 \). The images of \( e_0, f_0, e_1, f_1 \) under the injection

\[
\left. \begin{array}{c}
M^p/M^p \mapsto \sigma^\ast(M^p/M^p) = D/pD \\
\end{array} \right\}
\]

provided by Theorem \[2.3.6\] form a \( k \)-basis of \( D/pD \), denoted the same way, satisfying the relations \[4.2.1\] and

\[
\begin{align*}
F_{e_0} &= u^p e_1, & F_{f_0} &= 0, & F_{e_1} &= 0, & F_{f_1} &= u^p f_0, \\
V_{e_0} &= e_1, & V_{f_0} &= 0, & V_{e_1} &= 0, & V_{f_1} &= f_0.
\end{align*}
\]

One can prove that \( u = 1 \) by lifting the basis elements to \( D \) and arguing exactly as in Theorem \[4.2.1\].

Suppose now that equality holds in \[4.2.3\]. Proposition \[3.4.4\] implies that \( G_k \) is not superspecial, and so \( v = 1 \). As above, we obtain a \( k \)-basis \( e_0, f_0, e_1, f_1 \) of \( D/pD \) satisfying \[4.2.1\] and \[4.2.3\]. Moreover, Proposition \[3.4.4\] implies that the reduction map sends \( p/\tau^{p+1}_0 \mapsto u \), up to scaling by \( \mathbb{F}_p^\times \). One can easily check that the isomorphism class of \( D \), with its \( \Delta \) action and operators \[4.2.4\], depend only on the \( \mathbb{F}_p^\times \)-orbit of \( u \), and so this extra scaling factor can be ignored. \( \square \)

**Theorem 4.2.3.** The inequality

\[
\frac{p}{p+1} \leq \text{ord}(\tau)
\]

holds if and only if \( \Pi O_0(G_k^\times) \neq 0 \). Moreover:

1. If strict inequality holds in \[4.2.5\], there is a \( \Delta \)-linear isomorphism \( D/pD \cong D \) under which

\[
\begin{align*}
F_{e_0} &= 0, & F_{f_0} &= f_1, & F_{e_1} &= e_0, & F_{f_1} &= 0, \\
V_{e_0} &= 0, & V_{f_0} &= f_1, & V_{e_1} &= e_0, & V_{f_1} &= 0.
\end{align*}
\]

2. If equality holds in \[4.2.5\], there is a \( \Delta \)-linear isomorphism \( D/pD \cong D \) under which

\[
\begin{align*}
F_{e_0} &= 0, & F_{f_0} &= u^p f_1, & F_{e_1} &= w^p e_0, & F_{f_1} &= -w^p e_0, \\
V_{e_0} &= 0, & V_{f_0} &= e_1 + f_1 & V_{e_1} &= e_0, & V_{f_1} &= 0.
\end{align*}
\]

where \( u \) is the image of \( p/\tau^{p+1}_1 = \tau^{p+1}/p^p \) under \( O^\times \rightarrow k^\times \).

**Proof.** Recalling \[4.1.1\], the inequality \[4.2.5\] is equivalent to

\[
\text{ord}(\tau_1) \leq \frac{1}{p+1}.
\]

Using this observation, the proof is identical to that of Theorem \[4.2.2\] but with the indices 0 and 1 reversed everywhere. \( \square \)
Corollary 4.2.4. The subset of $X(C)$ defined by
\[
\frac{1}{p+1} < \text{ord}(\tau) < \frac{p}{p+1}
\]
is an Ekedahl-Oort stratum. The same is true of the subset defined by
\[
\text{ord}(\tau) < \frac{1}{p+1},
\]
and the subset defined by
\[
\text{ord}(\tau) > \frac{p}{p+1}.
\]
The union of these three strata is the locus of points in $X(C)$ for which the corresponding $p$-divisible group has superspecial reduction.

Proof. This is immediate from Theorems 4.2.1, 4.2.2, and 4.2.3. For the claim about superspecial reduction use Proposition 3.2.3. □

Now consider the locus of points
\[
\left\{ \tau \in C : \text{ord}(\tau) = \frac{1}{p+1} \right\} \cup \left\{ \tau \in C : \text{ord}(\tau) = \frac{p}{p+1} \right\} \subset X(C)
\]
at which the corresponding $p$-divisible group does not have superspecial reduction. This set is a union of infinitely many Ekedahl-Oort strata.

Corollary 4.2.5. The fibers of the composition
\[
\left\{ \tau \in C : \text{ord}(\tau) = \frac{1}{p+1} \right\} \xrightarrow{\tau \mapsto \tau^{p+1}/p} \mathcal{O}^\times \to k^\times
\]
are Ekedahl-Oort strata, as are the fibers of the composition
\[
\left\{ \tau \in C : \text{ord}(\tau) = \frac{p}{p+1} \right\} \xrightarrow{\tau \mapsto \tau^{p+1}/p^p} \mathcal{O}^\times \to k^\times.
\]

Proof. For each $u \in k^\times$ let $F_u$ and $V_u$ be the operators on $\mathbb{D}$ defined by (4.2.4). Note that $V_u$ is actually independent of $u$. We claim that the existence of a $\Delta$-linear isomorphism
\[
(\mathbb{D}, F_u, V_u) \xrightarrow{\phi} (\mathbb{D}, F_{u'}, V_{u'})
\]
implies $u = u'$. To see this one checks that the first relation in
\[
\phi \circ V_u = V_{u'} \circ \phi, \quad \phi \circ F_u = F_{u'} \circ \phi
\]
implies that $\phi$ has the form
\[
\phi(e_0) = ae_0, \quad \phi(e_1) = ae_1, \quad \phi(f_0) = af_0, \quad \phi(f_1) = af_1 + be_1
\]
for some $a \in \mathbb{F}_p$ and $b \in k$. Using this, one checks that $\phi$ commutes with both $F_u$ and $F_{u'}$. The second relation in (4.2.8) then implies that $F_u = F_{u'}$, and hence $u = u'$.

The same is true if we replace the operators of (4.2.4) with those of (4.2.6), and the corollary then follows from Theorem 4.2.2 and Theorem 4.2.3 □
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Department of Mathematics, Boston College, 140 Commonwealth Ave., Chestnut Hill, MA 02467, USA
E-mail address: howardbe@bc.edu