Abstract

For a weighted directed multigraph, let \( f_{ij} \) be the total weight of spanning converging forests that have vertex \( i \) in a tree converging to \( j \). We prove that \( f_{ij} f_{jk} = f_{ik} f_{jj} \) if and only if every directed path from \( i \) to \( k \) contains \( j \) (a graph bottleneck equality). Otherwise, \( f_{ij} f_{jk} < f_{ik} f_{jj} \) (a graph bottleneck inequality). In a companion paper [1] (P. Chebotarev, A new family of graph distances, arXiv preprint math.CO/0810.2717, 2008. [http://arXiv.org/abs/0810.2717](http://arXiv.org/abs/0810.2717). Submitted), this inequality underlies, by ensuring the triangle inequality, the construction of a new family of graph distances. This stems from the fact that the graph bottleneck inequality is a multiplicative counterpart of the triangle inequality for proximities.

Keywords: Spanning converging forest; Matrix forest theorem; Laplacian matrix

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1 Introduction

Let \( \Gamma \) be a weighted directed multigraph with vertex set \( V(\Gamma) = \{1, \ldots, n\}, n > 1 \). We assume that \( \Gamma \) has no loops. For \( i, j \in V(\Gamma) \), let \( n_{ij} \in \{0, 1, \ldots\} \) be the number of arcs emanating from \( i \) to \( j \) in \( \Gamma \); for every \( p \in \{1, \ldots, n_{ij}\} \), let \( w_{ij}^p > 0 \) be the weight of the \( p \)-th arc directed from \( i \) to \( j \) in \( \Gamma \); let \( w_{ij} = \sum_{p=1}^{n_{ij}} w_{ij}^p \) (if \( n_{ij} = 0 \), we set \( w_{ij} = 0 \)) and \( W = (w_{ij})_{n \times n} \). \( W \) is the matrix of total arc weights. The outdegree and indegree of vertex \( i \) are \( \text{od}(i) = \sum_{j=1}^{n} n_{ij} \) and \( \text{id}(i) = \sum_{j=1}^{n} n_{ji} \), respectively.

A converging tree is a weakly connected weighted digraph in which one vertex, called the root, has outdegree zero and the remaining vertices have outdegree one. A converging forest is a weighted digraph all of whose weakly connected components are converging trees. The roots of these trees are referred to as the roots of the converging forest. A spanning converging forest of \( \Gamma \) is called an in-forest of \( \Gamma \).

By the weight of a weighted digraph \( H, w(H) \), we mean the product of the weights of all its arcs. If \( H \) has no arcs, then \( w(H) = 1 \). The weight of a set \( S \) of digraphs, \( w(S) \), is
the sum of the weights of the digraphs belonging to \( S \); the weight of the empty set is zero. If the weights of all arcs are unity, i.e., the graphs in \( S \) are actually unweighted, then \( w(S) \) reduces to the cardinality of \( S \).

For a fixed \( \Gamma \), by \( \mathcal{F}^{-\bullet} \) and \( \mathcal{F}^{i-j} \) we denote the set of all in-forests of \( \Gamma \) and the set of all in-forests of \( \Gamma \) that have vertex \( i \) belonging to a tree rooted at \( j \), respectively. Let \( f = w(\mathcal{F}^{-\bullet}) \) and

\[
    f_{ij} = w(\mathcal{F}^{i-j}), \quad i, j \in V(\Gamma);
\]

by \( F \) we denote the matrix with entries \( f_{ij} \): \( F = (f_{ij})_{n \times n} \). \( F \) is called the matrix of in-forests of \( \Gamma \).

Let \( L = (\ell_{ij}) \) be the Laplacian matrix of \( \Gamma \), i.e.,

\[
    \ell_{ij} = \begin{cases} 
    -w_{ij}, & j \neq i, \\
    \sum_{k \neq i} w_{ik}, & j = i. 
    \end{cases}
\]

Consider the matrix

\[
    Q = (q_{ij}) = (I + L)^{-1}. \tag{2}
\]

By the matrix forest theorem\[^2\] \[^3\], for any weighted digraph \( \Gamma \), \( Q \) does exist and

\[
    q_{ij} = \frac{f_{ij}}{f}, \quad i, j = 1, \ldots, n. \tag{3}
\]

Therefore \( F = fQ = f \cdot (I + L)^{-1} \). The matrix \( Q \) can be considered as a proximity (similarity) matrix of \( \Gamma \) \[^2\] \[^6\].

In Section 2, we present the graph bottleneck inequality involving the \( f_{ij} \)'s and a necessary and sufficient condition of its reduction to equality.

### 2 A graph bottleneck inequality and a graph bottleneck equality

**Theorem 1** Let \( \Gamma \) be a weighted directed multigraph and let the values \( f_{ij} \) be defined by \(^1\). Then for every \( i, j, k \in V(\Gamma) \),

\[
    f_{ij} f_{jk} \leq f_{ik} f_{jj}. \tag{4}
\]

Moreover,

\[
    f_{ij} f_{jk} = f_{ik} f_{jj} \tag{5}
\]

if and only if every directed path from \( i \) to \( k \) contains \( j \).

\[^1\] Versions of this theorem for undirected (multi)graphs can be found in \[^4\] \[^5\].
Since (4) reduces to (5) when \( j \) is a kind of bottleneck in \( \Gamma \), (5) is called a \textit{graph bottleneck equality}; by the same reason, (4) is referred to as a \textit{graph bottleneck inequality}. It is readily seen that the graph bottleneck inequality is a multiplicative counterpart of the triangle inequality for proximities (see, e.g., [2]).

It turns out that it is not easy to construct a direct bijective proof to Theorem 1. We present a different proof; it requires some additional notation and two propositions given below.

For a fixed multidigraph \( \Gamma \), let us choose an arbitrary \( \varepsilon > 0 \) such that \( 0 \leq \varepsilon \cdot \max_{1 \leq i \leq n} \ell_{ii} < 1 \).

It is easy to verify that the matrix \( P = (p_{ij}) = I - \varepsilon L \) (6) is row stochastic: \( 0 \leq p_{ij} \leq 1 \) and \( \sum_{k=1}^{n} p_{ik} = 1, \ i, j = 1, \ldots, n \).

Denote by \( \Gamma^\circ \) the weighted multidigraph with loops whose matrix \( W(\Gamma^\circ) \) of total arc weights is \( (1+\varepsilon)^{-1}P \). More specifically, every vertex \( i \) of \( \Gamma^\circ \) has a loop with weight \( (1+\varepsilon)^{-1}p_{ii} \); the remaining arcs of \( \Gamma^\circ \) are the same as in \( \Gamma \), their weights being the corresponding weights in \( \Gamma \) multiplied by \( (1+\varepsilon)^{-1} \varepsilon \).

Recall that a \( v_0 \to v_k \) \textit{route} in a multidigraph with loops is an alternating sequences of vertices and arcs \( v_0, x_1, v_1, \ldots, x_k, v_k \) where each arc \( x_i \) is \( (v_{i-1}, v_i) \). The length of a route is the number \( k \) of its arcs (including loops). The \textit{weight} of a route is the product of the weights of all its arcs. We assume that for every vertex \( i \), there is a unique route of length 0 from \( i \) to \( i \), the weight of this route being 1. The \textit{weight of a set of routes} is the total weight of the routes the set contains.

Let \( r_{ij} \) be the weight of the set \( \mathcal{R}^{ij} \) of all \( i \to j \) routes in \( \Gamma^\circ \), provided that this weight is finite (this reservation is essential because the set of \( i \to j \) routes is infinite whenever \( j \) is reachable from \( i \)). \( R = (r_{ij})_{n \times n} \) will denote the \textit{matrix of the total weights of routes}.

**Proposition 1** For every weighted multidigraph \( \Gamma \) and every \( \varepsilon > 0 \) such that \( 0 \leq \varepsilon \max_{1 \leq i \leq n} \ell_{ii} < 1 \), the matrix \( R \) of the total weights of routes in \( \Gamma^\circ \) exists and it is proportional to the matrix \( F \) of \textit{in-forests} of \( \Gamma \).

**Proof.** Observe that for every \( k = 0, 1, 2, \ldots \), the matrix of total weights of \( k \)-length routes in \( \Gamma^\circ \) is \( ((1+\varepsilon)^{-1}P)^k \). Therefore the matrix \( R \), whenever it exists, can be expressed as follows:

\[
R = \sum_{k=0}^{\infty} ((1+\varepsilon)^{-1}P)^k. \tag{7}
\]

Since the spectral radius of \( P \) is 1 and \( 0 < (1+\varepsilon)^{-1} < 1 \), the sum in (7) does exist\(^2\); therefore

\(^2\)On counting routes see [7]. Related finite topological representations that involved paths were obtained in [8]. For a connection with matroid theory see, e.g., [9].
\( R = (I - (1 + \varepsilon)^{-1})^{-1} = (I - (1 + \varepsilon)^{-1}(I - \varepsilon L))^{-1} \)
\[
= \left( \frac{\varepsilon}{1 + \varepsilon} (I + L) \right)^{-1} = (1 + \varepsilon^{-1}) Q = (1 + \varepsilon^{-1}) f^{-1} F,
\]
which completes the proof.

Proposition 2 For any weighted multidigraph with loops and any vertices \( i, j, \) and \( k, \) if the total weights of routes \( r_{ij}, r_{jj}, r_{jk}, \) and \( r_{ik} \) are finite, then
\[
r_{ij} r_{jk} \leq r_{ik} r_{jj}. \tag{8}
\]
Moreover,
\[
r_{ij} r_{jk} = r_{ik} r_{jj} \tag{9}
\]
if and only if every directed path from \( i \) to \( k \) contains \( j.\)

Proof. Suppose that the total weights of routes \( r_{ij}, r_{jj}, r_{jk}, \) and \( r_{ik} \) are finite. Let \( R_{ij}^{(1)} \) be the set of all \( i \to j \) routes that contain only one appearance of \( j. \) Let \( r_{ij}^{(1)} = w(R_{ij}^{(1)}). \) Then every \( i \to j \) route \( r_{ij} \in R_{ij}^{ij} \) can be uniquely decomposed into a route \( r_{ij}^{(1)} \in R_{ij}^{ij}(1) \) and a route (possibly, of length 0) \( r_{jj} \in R_{jj}^{jj}. \) And vice versa, linking an arbitrary route \( r_{ij}^{(1)} \in R_{ij}^{ij}(1) \) with an arbitrary \( r_{jj} \in R_{jj}^{jj} \) results in a certain route \( r_{ij} \in R_{ij}^{ij}. \) This determines a natural bijection between \( R_{ij}^{ij} \) and \( R_{ij}^{ij(1)} \times R_{jj}^{jj}. \) Therefore
\[
r_{ij} = r_{ij}^{(1)} r_{jj}. \tag{10}
\]

Let \( R_{ijk} \) and \( R_{ijk}^{jk} \) be the sets of all \( i \to k \) routes that contain and do not contain \( j, \) respectively. Then \( R_{ik} = R_{ijk} \cup R_{ijk}^{jk} \) and \( R_{ijk} \cap R_{ijk}^{jk} = \emptyset, \) consequently,
\[
r_{ik} = r_{ijk} + r_{ijk}^{jk}, \tag{11}
\]
where \( r_{ijk} = w(R_{ijk}) \) and \( r_{ijk}^{jk} = w(R_{ijk}^{jk}). \)

Furthermore, by the argument similar to that justifying (10) one obtains
\[
r_{ijk} = r_{ij}^{(1)} r_{jk}. \tag{12}
\]

Combining (11), (12), and (10) yields
\[
r_{ik} r_{jj} = (r_{ijk} + r_{ijk}^{jk}) r_{jj} = r_{ij}^{(1)} r_{jk} r_{jj} + r_{ijk} r_{jj} = r_{ij} r_{jk} + r_{ijk} r_{jj} \geq r_{ij} r_{jk},
\]
with the equality if and only if \( R_{ijk}^{jk} = \emptyset. \)

Proof of Theorem 1. Theorem 1 follows immediately by combining Propositions 1 and 2.

Finally, consider the graph bottleneck inequality and the graph bottleneck equality for undirected graphs.
Corollary 1 (to Theorem 1) Let $G$ be a weighted undirected multigraph and let $f_{ij}$, $i, j \in V(G)$, be the total weight of all spanning rooted forests of $G$ that have vertex $i$ belonging to a tree rooted at $j$. Then for every $i, j, k \in V(G)$,

$$f_{ij} f_{jk} \leq f_{ik} f_{jj}.$$  \hspace{1cm} (13)

Moreover,

$$f_{ij} f_{jk} = f_{ik} f_{jj} \hspace{1cm} (14)$$

if and only if every path from $i$ to $k$ contains $j$.

Proof. Consider the weighted multidigraph $\Gamma$ obtained from $G$ by replacing every edge by two opposite arcs carrying the weight of that edge. Then, by the matrix forest theorems for weighted and unweighted graphs, $f_{ij}(G) = f_{ij}(\Gamma)$, $i, j \in V(G)$. Observe that for every $i, j, k \in V(G)$, every path from $i$ to $k$ contains $j$ if and only if every directed path in $\Gamma$ from $i$ to $k$ contains $j$. Therefore, by virtue of Theorem 1, inequality (13) follows for $G$; moreover, equality (14) holds true if and only if every path in $G$ from $i$ to $k$ contains $j$. \hfill \square

In a companion paper [1], the graph bottleneck inequality for undirected graphs is used to ensure the triangle inequality for a new parametric family $\{d_\alpha(\cdot, \cdot)\}$ of graph distances. In turn, the bottleneck equality provides a necessary and sufficient condition under which the triangle inequality $d_\alpha(i, j) + d_\alpha(j, k) \geq d_\alpha(i, k)$ for a triple $i, j, k$ of graph vertices reduces to equality.

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