POLAR AND COISOTROPIC ACTIONS ON KÄHLER MANIFOLDS

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Abstract. The main result of the paper is that a polar action on a compact irreducible homogeneous Kähler manifold is coisotropic. This is then used to give new examples of polar actions and to classify coisotropic and polar actions on quadrics.

§1. Introduction

The aim of the present paper is to investigate the relationship between polar and coisotropic actions on compact Kähler manifolds.

The action of a compact Lie group $G$ of isometries on a Riemannian manifold $(M, g)$ is called polar if there exists a properly embedded submanifold $\Sigma$ which meets every $G$-orbit and is orthogonal to the $G$-orbits in all common points. Such a submanifold $\Sigma$ is called a section (see [PT1,2]) and if it is flat, the action is called hyperpolar. It is of course meaningful to relax the definition of polar actions and not require that the section is properly embedded. Our results are correct for weakly polar actions in this sense, making Theorems 1.1 and 1.3 stronger, but Theorem 1.2 weaker.

If $(M, g)$ is a compact Kähler manifold with Kähler form $\omega$ and $G$ is a compact subgroup of its full isometry group, then the $G$-action is called multiplicity-free ([GS2]) or coisotropic ([HW]) if the principal $G$-orbits are coisotropic with respect to $\omega$. Notice that the existence of one coisotropic principal $G$-orbit implies the same property for all principal $G$-orbits, see [HW].

In this paper we shall consider the special case of $M$ being a simply connected, compact homogeneous Kähler manifold; such manifolds are diffeomorphic to quotient spaces $L/K$, where $L$ is a compact semisimple Lie group and $K$ is the centralizer in $L$ of some torus. We also recall that a compact homogeneous Kähler manifold cannot be written as a nontrivial Riemannian product of two Riemannian manifolds if and only if its full isometry group is a compact simple Lie group; in this last case the manifold is simply connected (see e.g. [On], p. 238 ff).

In section two we will prove the following

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Theorem 1.1. A polar action on an irreducible compact homogeneous Kähler manifold is coisotropic.

It is obvious that the theorem is not true without the assumption that $M$ is irreducible. The converse of Theorem 1.1 is not true. In fact, we will give an example of a coisotropic action on $\mathbb{P}_n(\mathbb{C})$ which is not polar in section three and there are further such examples on quadrics in Theorem 1.3. In certain situations, we can prove that coisotropic actions are polar and we will use this to give many new examples of polar actions. We will for example show that every nonoriented compact surface $N$ is the section of a polar action on some four dimensional compact Kähler manifold. In section three we will prove the following

Theorem 1.2. An effective and isometric action of the (real) torus $T^n$ on a compact Kähler manifold $M$ with positive Euler characteristic and complex dimension $n$ is polar.

The condition on the Euler characteristic is to ensure that the action is Poisson; see the proof in section three. We also remark that this condition is not restrictive, since a compact, symplectic $2n$-dimensional manifold which is acted on effectively by an $n$-dimensional torus in a Poisson fashion has positive Euler characteristic (see [De]).

As an application of Theorem 1.1, we will classify polar actions on quadrics in section four. We do this by first classifying the coisotropic actions and then deciding which of them are polar. An interesting consequence of this classification is that polar actions on quadrics are hyperpolar. This is in contrast with complex projective spaces or more generally to rank one symmetric spaces that admit many polar actions that are not hyperpolar (see [PTh]). Another interesting consequence of the classification is that all coisotropic actions on quadrics are related to transitive actions on spheres. The classification of coisotropic and polar actions on quadrics is given in the following

Theorem 1.3. Let $G$ be a connected compact Lie subgroup of $\text{SO}(n)$ ($n \geq 5$) and let $Q_{n-2}$ be the quadric $\text{SO}(n)/\text{SO}(2) \times \text{SO}(n-2)$. Then the action of $G$ on $Q_{n-2}$ is nontransitive and coisotropic if and only if one of the following occurs.

(i) The action of $G$ on $\mathbb{R}^n$ is irreducible and $G$ is conjugate to

$$\text{SU}(\frac{n}{2}), \text{U}(\frac{n}{2}), \text{Sp}(1) \cdot \text{Sp}(\frac{n}{4}), T^1 \cdot \text{Sp}(n/4), \text{Spin}(9).$$

Only the action of the last one of these groups is not polar on $Q_{n-2}$.

(ii) The action of $G$ on $\mathbb{R}^n$ has a one dimensional fixed point set and the group $G$ as a subgroup of $\text{SO}(n-1)$ is conjugate to one of the following

$$\text{SO}(n-1), \text{G}_2, \text{Spin}(7),$$
$$\text{SU}(\frac{n-1}{2}), \text{U}(\frac{n-1}{2}), \text{Sp}(1) \cdot \text{Sp}(\frac{n-1}{4}),$$
$$T^1 \cdot \text{Sp}(\frac{n-1}{4}), \text{Spin}(9).$$
Only the actions of the first three of these groups are polar on $Q_{n-2}$.

(iii) The action of $G$ on $\mathbb{R}^n$ is reducible and $\mathbb{R}^n$ is the sum of two irreducible nontrivial $G$-submodules $V_1, V_2$; the group $G$ splits as the product $G_1 \times G_2$ with $G_i \subset \text{SO}(V_i)$, $i = 1, 2$ and each $G_i$ is conjugate in $\text{SO}(V_i)$ to one of the following

$\text{SO}(p), G_2, \text{Spin}(7), \text{Sp}(1) \cdot \text{Sp}(p), \text{Spin}(9)$

for suitable $p \in \mathbb{N}$. Only the actions obtained choosing each $G_i$, $i = 1, 2$, among the first three of these groups are polar on $Q_{n-2}$.

Moreover if the $G$-action on $Q_{n-2}$ is polar, then it is hyperpolar.

We now recall some basic definition and results from the paper [HW]. A submanifold $W$ of a symplectic manifold $(M, \omega)$ is called coisotropic if

$$T_pW^{\perp_\omega} \subseteq T_pW$$

for all $p$ in $W$. Here $T_pW^{\perp_\omega}$ denotes the subspace of $T_pM$ that is $\omega$-orthogonal to $T_pW$. If $M$ is a Kähler manifold and $\omega$ its Kähler form, then it is easy to see that a submanifold $W$ of $M$ is coisotropic if and only if

$$J(N_pW) \subseteq T_pW$$

for all $p$ in $W$, where $J$ denotes the complex structure of $M$, and $N_pW$ the normal space of $W$ in $p$. One calls a real subspace $V$ in a complex vector space with a Hermitian scalar product totally real if $V$ is perpendicular to $J(V)$. It follows from this discussion that a submanifold $W$ in a Kähler manifold is coisotropic if and only if all its normal spaces are totally real.

Let $(M, \omega)$ be a symplectic manifold. Then one can associate to every function $f$ in $C^\infty(M)$ its symplectic gradient $X_f$ defined as the unique vector field on $M$ such that

$$df(Y) = \omega(X_f, Y)$$

for every vector field $Y$ on $M$. If $f$ and $g$ are two functions in $C^\infty$, then one defines their Poisson bracket as $\{f, g\} = \omega(X_f, X_g)$. It follows that $C^\infty(M)$ with the Poisson bracket is a Lie algebra.

Now let $G$ act on $M$ preserving the symplectic structure $\omega$. Then the action is called Poisson if there is a Lie algebra homomorphism $\lambda : \mathfrak{g} \to C^\infty(M)$ such that $X_{\lambda(\xi)}$ agrees with the infinitesimal action of $\xi$ on $M$. The moment map of a Poisson action is defined as

$$\Phi : M \to \mathfrak{g}^*; \quad \Phi(p)(\xi) = \lambda(\xi)(p).$$

We have $\Phi(gp) = \text{Ad}^*(g)\Phi(p)$, i.e., the moment map is $G$-equivariant.

Let $G$ be a compact group acting isometrically on a compact Kähler manifold. This action is automatically holomorphic by a theorem of Kostant (see [KN], vol. I, p. 247) and it induces by compactness of $M$ an action of the complexified group $G^C$ on $M$. We say that $M$ is $G^C$-almost homogeneous if $G^C$ has an open orbit in $M$. If all Borel subgroups of $G^C$ act with an open orbit on $M$, then the open orbit $\Omega$ in
$M$ under the action of $G^C$ is called a **spherical homogeneous space** and $M$ is called a **spherical embedding** of $\Omega$. In general, given a complex Lie group $U$ and a closed subgroup $H \subset U$, the pair $(U, H)$ is called a **spherical pair** if every Borel subgroup of $U$ has an open orbit in $U/H$; all such pairs $(U, H)$, where $U$ is semisimple and $H$ is reductive, have been classified by Brion ([Br2]).

One of the basic results on spherical embeddings is that if the orbit $\Omega$ of $G^C$ is a spherical homogeneous space, then $G^C$ and all Borel subgroups of $G^C$ have finitely many orbits in $M$, see [Br1].

The following theorem is proved in [HW], p. 275.

**Theorem 1.4 (Equivalence Theorem of [HW]).** Let $M$ be a connected compact Kähler manifold with an isometric action of a connected compact group $G$ that is also Poisson. Then the following conditions are equivalent:

(i) The space $C^\infty(M)^G$ of $G$-invariant functions on $M$ is abelian with respect to the Poisson bracket.

(ii) The $G$-action is coisotropic.

(iii) The cohomogeneity of the $G$-action is equal to the difference between the rank of $G$ and the rank of a regular isotropy subgroup of $G$.

(iv) The moment map $\Phi : M \rightarrow g^*$ separates orbits.

(v) The Kähler manifold $M$ is projective algebraic, $G^C$-almost homogeneous and a spherical embedding of the open $G^C$-orbit.

We remark here that the conditions (i) to (iii) are equivalent even without the hypothesis of compactness for $M$ (see [HW]).

Under the hypothesis of Theorem 1.4, the following convexity theorem holds (see [GS1] and [Kir]): if $T$ denotes a maximal torus of $G$, $t^*$ is the fixed point set of $T$ in $g^*$ and $t^*_+$ denotes a fundamental domain for the action of the Weyl group on $t^*$, then the image $\Phi(M)$ under the moment mapping $\Phi$ intersects $t^*_+$ in a convex polytope.

On the other hand, if $G$ is a compact group of isometries of a Riemannian manifold and if the $G$-action is polar, then we can associate to any section $\Sigma$ a finite group $W$, also called Weyl group, which is defined as $W := N_G(\Sigma)/Z_G(\Sigma)$; here $N_G(\Sigma)$ (resp. $Z_G(\Sigma)$) denotes the subgroup of $G$ given by those elements which map $\Sigma$ onto itself (resp. act trivially on $\Sigma$). It is well known that the intersection of a $G$-orbit with $\Sigma$ coincides with a $W$-orbit and we can identify the two orbit spaces $M/G$ and $\Sigma/W$; for general properties of the Weyl group we refer to [PT1], [PTh], [Co].

¿From Theorem 1.1 and Theorem 1.4 we obtain the following

**Corollary 1.5.** Let $M$ be an irreducible homogeneous Kähler manifold which is acted on polarly by a compact Lie group of isometries $G$. Then

1. the cohomogeneity of the $G$-action does not exceed the rank of $G$; moreover the complexified group $G^C$ and any Borel subgroup of $G^C$ act on $M$ with finitely many orbits;
2. if $\Phi : M \rightarrow g^*$ denotes the moment map, $\Sigma$ is any section for the $G$-action and $W$ its Weyl group, then the moment map induces a homeomorphism between $\Sigma/W$ and the convex polytope $\Phi(M) \cap t^*_+$. 
To prove the corollary one only has to observe that the action of $G$ is Poisson since the full group of isometries of $M$ acts in a Poisson fashion: indeed any flag manifold is a coadjoint orbit whose moment map is simply the inclusion.

We end this section pointing out one common property of polar and coisotropic linear actions of a compact Lie group $G$ which will be used in section four. It is known that a polar linear action of $G$ on a (real) vector space does not contain two distinct irreducible equivalent $G$-submodules (see [HPTT], Proposition 2.10); if $G$ is realized as a subgroup of $U(N)$ for some $N \geq 1$ and if the corresponding action on $\mathbb{C}^N$ is coisotropic with respect to the standard symplectic form of $\mathbb{C}^N$, then every irreducible complex $G$-module appears in $\mathbb{C}^N$ with multiplicity at most one, i.e. every linear coisotropic action of $G$ is \textit{multiplicity free} (see for instance [HW], Theorem 1, p. 275). We also note here that the restriction of a linear coisotropic action to a complex $G$-submodule is still coisotropic (see the Equivariant Mapping Lemma in [HW], p. 272); as a consequence, a linear coisotropic action has no nontrivial fixed vectors, in contrast with the polar actions.

The following result states that a coisotropic linear action is multiplicity free also from the ‘real’ point of view.

\textbf{Proposition 1.6.} Let $G$ be a compact subgroup of $U(N)$ for some $N \geq 1$. If the corresponding action of $G$ on $\mathbb{C}^N$ is coisotropic with respect to the standard symplectic form on $\mathbb{C}^N$, then the real $G$-module $\mathbb{C}^N \simeq \mathbb{R}^{2N}$ does not contain two distinct equivalent $G$-submodules.

The proof of this proposition will be given in the next section.

\section*{\S 2. Proofs of Theorem 1.1 and Proposition 1.6}

Let $M$ be a compact, irreducible homogeneous Kähler manifold with Kähler metric $g$ and complex structure $J$; if we denote by $L$ the identity component of the full isometry group of $(M, g)$, then $L$ preserves the complex structure $J$ (see the introduction) and we can represent $M$ as $M = L/K$, where $K$ is the centralizer in $L$ of a torus. Note again that the irreducibility of $(M, g)$ implies that $L$ is simple.

We consider a compact connected Lie subgroup $G \subset L$ of holomorphic isometries and we shall suppose that the $G$-action on $M$ is polar with a section $\Sigma$.

We saw in the introduction that a submanifold of a Kähler manifold is coisotropic if and only if all of its normal spaces are totally real. The proof of Theorem 1.1 is therefore a direct consequence of the following proposition.

\textbf{Proposition 2.1.} The section $\Sigma$ is totally real.

The proof of this proposition requires some lemmas.

We assume that the cohomogeneity of the action is at least two since the claim in the proposition is trivially true for cohomogeneity one actions. First of all, if $\Sigma$ is a section, we may define the complex subspace

$$\mathcal{H}_p = T_p \Sigma \cap JT_p \Sigma \quad (2.1)$$

for $p \in \Sigma$. We next prove that $\mathcal{H}$ is a complex subbundle over $\Sigma$. It suffices to show that if $p$ and $q$ are two points in $\Sigma$ and $\sigma$ a path in $\Sigma$ joining them, then
parallel transport along $\sigma$ sends $\mathcal{H}_p$ to $\mathcal{H}_q$. This is clear since $\Sigma$ is totally geodesic and parallel transport in $M$ commutes with $J$. An important consequence of this argument that we will use later is that $\mathcal{H}$ is a parallel subbundle of $T(\Sigma)$.

We can now define a complex subbundle $\mathcal{H}$ of $T(M)_{|M_{\text{reg}}}$ by setting $\mathcal{H}_q = T_q\Sigma(q) \cap JT_q\Sigma(q)$ for $q \in M_{\text{reg}}$, where $\Sigma(q)$ denotes the unique section through $q$ and $M_{\text{reg}}$ denotes the set of regular points of the $G$-action.

Our aim is to prove that $\mathcal{H}_p = \{0\}$ for all $p$. In order to do this, we will first show that $\mathcal{H}$ extends to a smooth integrable subbundle of the whole $T(M)$, whose orthogonal complement is also integrable, and then we will use a result on foliations to prove our claim.

**Lemma 2.2.** The subbundle $\mathcal{H}$ can be extended to a differentiable subbundle of the tangent bundle $T(M)$.

**Proof.** We consider a nonregular point $q \in M$ and fix two sections $\Sigma, \Sigma'$ passing through $q$. If we denote by $\mathcal{H}_q(\Sigma)$ and $\mathcal{H}_q(\Sigma')$ the complex subspaces of $T_q\Sigma$ and $T_q\Sigma'$, respectively, defined as in (2.1), we first must prove that they coincide and then that the extension of the bundle to the whole manifold $M$ is differentiable.

We will use the fact that the isotropy representation of $G_q$ is polar (see [PT1,2]).

We start proving the following

**Sublemma 2.3.** The subspace $\mathcal{H}_q(\Sigma)$ is contained in the intersection of all singular hyperplanes of $T_q\Sigma$ with respect to the isotropy representation of $G_p$.

**Proof.** Indeed, if $Y \subset T_q\Sigma$ is a singular hyperplane, then there exists an element $g \in G_q$ which leaves $\Sigma$ invariant and such that $dg_q$ is a reflection in $Y$. Since $g$ is holomorphic, it leaves the maximal holomorphic subspace of $T_q\Sigma$ invariant; moreover, since the fixed point set $Y$ has real codimension one, it must contain $\mathcal{H}_q(\Sigma)$. □

**Sublemma 2.4.** Let $G$ be a compact Lie group acting linearly and polarly on a vector space $V$. If $\Sigma$ and $\Sigma'$ are two different sections with nontrivial intersection, then the intersection $\Sigma \cap \Sigma'$ contains the intersection of all singular hyperplanes of $\Sigma$ (and of $\Sigma'$).

**Proof.** We use induction on the dimension of $V$, the case when $\dim V = 2$ being trivially true. First of all, we may suppose that the fixed point set $V^G$ is equal to $\{0\}$. Indeed, if $V^G \neq \{0\}$, we decompose $V = V^G \oplus W$, with $W = (V^G)^{\perp}$ and we note that any section splits as the sum of $V^G$ plus a section for the $G$-action in $W$; using the induction hypothesis on $W$, we get our claim. So, we suppose that $V^G = \{0\}$. If we fix a nonzero vector $v \in \Sigma \cap \Sigma'$, we clearly have that $v$ is singular for the $G$-action and moreover the orbit $Gv$ has positive dimension. Therefore the isotropy $G_v$ acts polarly on the normal space $N_v$ to the orbit $Gv$ and, since $\dim N_v < \dim V$, by the induction hypothesis, we have that $\Sigma \cap \Sigma'$ contains the intersection of all singular hyperplanes in $\Sigma$ passing through $v$. A fortiori, it then follows that $\Sigma \cap \Sigma'$ contains the intersection of all singular hyperplanes in $\Sigma$. □

We now extend the bundle $\mathcal{H}$ to the singular points. We fix a singular point $q$ and consider two sections $\Sigma, \Sigma'$ through $q$. They correspond to two linear sections,
also denoted by $\Sigma$, $\Sigma'$, in the normal spaces $N_q$ of the orbit $Gq$ for the polar action of the isotropy subgroup $G_q$. The $K$-cycles of Bott and Samelson ([BS]) can be used to prove that there is a sequence of linear sections $\Sigma_0 = \Sigma, \ldots, \Sigma_l = \Sigma'$ such that $\Sigma_l \cap \Sigma_{l+1}$ is nontrivial since we are assuming cohomogeneity greater than one (see the proof of Lemma 1B.3 in [PTh]). From Sublemma 2.4, we see that $\Sigma_i \cap \Sigma_{i+1}$ contains the intersection of all singular hyperplanes of $\Sigma_i$ and therefore, by Sublemma 2.3, it contains $\mathcal{H}_q(\Sigma_i)$; similarly, it contains $\mathcal{H}_q(\Sigma_{i+1})$. Since $\mathcal{H}_q(\Sigma_i)$ is the maximal complex subspace of $\Sigma_i$, it follows that $\mathcal{H}_q(\Sigma_{i+1}) \subseteq \mathcal{H}_q(\Sigma_i)$ and we also have the opposite inclusion, so that $\mathcal{H}_q(\Sigma_i) = \mathcal{H}_q(\Sigma_{i+1})$. Therefore $\mathcal{H}_q(\Sigma) = \mathcal{H}_q(\Sigma')$.

We are left with proving that the extension is differentiable.

We fix a singular point $q \in M$ and a basis $\{v_1, \ldots, v_l\}$ of $\mathcal{H}_q$; we will show that there exist smooth vector fields $v_1^\ast, \ldots, v_l^\ast$ on some suitable neighborhood $U$ of $q$, which extend $v_1, \ldots, v_l$ respectively and span $\mathcal{H}_y$ for all $y \in U$.

We denote by $N_q$ the normal space $T_q(G \cdot q) \perp$ and choose a real positive number $r > 0$ so that $\exp_q|_{B_r} : B_r \to M$ is a diffeomorphism onto its image, where $B_r := \{v \in N_q; ||v|| < r\}$. We set $N_r := \exp_q(B_r)$ and note that $N_r = \bigcup \mathcal{N}_r \cap \Sigma$, where the union is taken over all sections $\Sigma$ passing through $q$. If $y \in N_r$, then $y = \exp_q(Y)$ for some $Y \in B_r$ and for $i = 1, \ldots, l$ we define $\hat{v}_i(y)$ to be the parallel transport of $v_i$ along the geodesic $\exp_q(tY)$, $t \in [0, 1]$; note that if $\Sigma$ is any section through $q$ with $Y \in T_y\Sigma$, then $y \in \Sigma$ and $\hat{v}_i(y) \in \mathcal{H}_y \subseteq T_y\Sigma \subset T_yN_r$, since $\mathcal{H}$ is parallel along the sections.

Let now $g_q$ be the Lie algebra of the isotropy subgroup $G_q$ and fix a complement $m_q$ of $g_q$ in $g$ so that $g = g_q + m_q$; it is clear that, if $r$ is sufficiently small, there exists a neighborhood $V$ of $O \in m_q$ so that the map

$$V \times N_r \to M$$

$$(X, y) \mapsto \exp^G(X) \cdot y$$

is a diffeomorphism onto an open neighborhood $U$ of $q$ in $M$. If $x \in U$, say $x = \exp^G(X) \cdot y$ with $X \in V$ and $y \in N_r$, we define $v_i^\ast(x) = \exp^G(X)_*(\hat{v}_i(y)) \in \mathcal{H}_x$ for all $i = 1, \ldots, l$ and this proves our claim.

We now prove the following

**Sublemma 2.5.** The distribution given by $\mathcal{H}$ is integrable and totally geodesic; the distribution given by $\mathcal{H}^\perp$ is integrable.

*Proof.* It is enough to prove our claims on the regular set $M_{\text{reg}}$ which is dense in $M$.

The claims about $\mathcal{H}$ follow from the fact that $\mathcal{H}$ is a parallel subbundle of $T(\Sigma)$ for all sections $\Sigma$.

We now put $k = \dim \Sigma - \dim \mathcal{H}$ and $m$ the dimension of a regular $G$-orbit. Then $\dim \mathcal{H}^\perp = k + m$. We pick a regular point $q$ and consider the section $\Sigma$ through $q$; we then fix a set $\{Z_1, \ldots, Z_k\}$ of smooth independent sections of $\mathcal{H}^\perp$ which are tangent to $\Sigma$ over a suitable neighborhood $V \cap \Sigma$, where $V$ is an open neighborhood of $q$ in $M_{\text{reg}}$. We may extend the sections $\{Z_1, \ldots, Z_k\}$ to vector fields on a neighborhood of $q$ by using the $G$-action, since the slice representation of $G_p$ is trivial for every $p \in V$. 


On some suitable open neighborhood $V'$ of $q$ in $V$, we can choose Killing vector fields $\{X_1, \ldots, X_m\}$, induced by the $G$-action, which span the tangent space to every orbit $Gp$ for $p \in V'$; the vector fields $\{Z_1, \ldots, Z_k, X_1, \ldots, X_m\}$ are therefore linearly independent and span $\mathcal{H}_p^\perp$ for all $p \in V'$. We now prove that $\mathcal{H}_p^\perp$ is integrable. First of all we note that, by the definition of $J$, the Lie derivative $\mathcal{L}_{X_i} Z_i = 0$ and therefore $[Z_i, X_j] = 0$ for all $i = 1, \ldots, k, j = 1, \ldots, m$; moreover, $[X_i, X_j]$ is tangent to the $G$-orbits and is therefore a section of $\mathcal{H}_p^\perp$ for all $i, j = 1, \ldots, m$. It is left to prove that $[Z_i, Z_j]$ are sections of $\mathcal{H}_p^\perp$ for all $i, j = 1, \ldots, k$.

Given any section $X \in \Gamma(\mathcal{H})$ on $V'$, we have

$$g([Z_i, Z_j], X) = g(Z_i, \nabla_{Z_j} X) - g(Z_j, \nabla_{Z_i} X),$$

where $\nabla$ denotes the Levi-Civita connection of the metric $g$. If $x \in V'$ and $\Sigma(x)$ denotes the (unique) section through $x$, then $Z_i, Z_j$ and $X$ are tangent to $\Sigma(x)$ and $\nabla_{Z_i} X, \nabla_{Z_j} X$ are sections of $\mathcal{H}$, since $\mathcal{H}$ is a parallel subbundle of $T(\Sigma(x))$; therefore $g([Z_i, Z_j], X) = 0$ and our claim follows. \hfill \Box

**Lemma 2.6.** We have $\mathcal{H}_p = \{0\}$ for all $p \in M$.

*Proof.* In [BH] the following result is proved: given a totally geodesic foliation $\mathcal{F}$ on a compact Riemannian manifold $(M, g)$ such that the distribution given by $T\mathcal{F}^\perp$ is integrable, then the universal covering manifold $\tilde{M}$ splits topologically as the product of two manifolds that are the universal coverings of leaves of the two distributions $\mathcal{F}$ and $\mathcal{F}^\perp$. In our situation, the manifold $M$ is simply connected; moreover, it is known (see [On], p. 279 and [Sh]) that the cohomology ring $H^*(M, \mathbb{R})$ is indecomposable, i.e. can not be written as a tensor product $A \otimes B$, where $A, B$ are differential graded algebras of dimension bigger than two. This implies that the leaves of the distribution $\mathcal{H}$ are zero dimensional. \hfill \Box

*Proof of Proposition 2.1.* Since the manifold $M$ is compact, the complexified Lie group $G^\mathbb{C}$ acts on $M$. Moreover, we can consider an Iwasawa decomposition $g^\mathbb{C} = g + s$, where $s$ is a solvable Lie subalgebra. We recall that the manifold $M = L/K$ can be $L^\mathbb{C}$-equivariantly embedded into some complex projective space. If $S$ denotes the Lie subgroup of $G^\mathbb{C}$ with Lie algebra $s$, then by the Borel Fixed Point Theorem, $S$ has a fixed point in $M$ and therefore there exists a point $q \in M$ such that the $G^\mathbb{C}$-orbit $G^\mathbb{C}q$ coincides with the $G$-orbit $Gq$, which is therefore complex. We denote by $N_q$ the normal space of the orbit $Gq$ at $q$; then $N_q$ is a complex subspace of $T_q M$ and the isotropy $G_q$ acts on it polarly with a section $\Sigma$.

First of all, we note that the fixed point set $N_q^{G_q}$ is equal to $\{0\}$, since otherwise it would be a complex subspace contained in a section, contradicting Lemma 2.6. Therefore the $G_q$-module $N_q$ splits as a sum of irreducible, nontrivial complex submodules; these are also irreducible over the reals since otherwise $N_q$ would contain two equivalent real submodules, which is not possible by [Da], see also [HPTT], p. 168. Our claim will now follow from the following lemma.

**Lemma 2.7.** Let $K$ be a compact Lie subgroup of $O(V)$, where $V$ is a real vector space. If the action of $K$ on $V$ is irreducible, polar and leaves some complex structure $J$ on $V$ invariant, then the sections are totally real subspaces with respect to $J$. 
Proof. We know from Dadok's Theorem that the $K$-action is orbit equivalent to the isotropy representation of a symmetric space $U/H$ and that $K$ can be considered to be a subgroup of $H$. Furthermore the representation of $K$ on $V$ is the restriction of the isotropy representation of $U/H$. If $K$ itself is the isotropy subgroup of a symmetric space $U/K$, then $U/K$ is Hermitian symmetric and the sections are totally real: indeed if $u = \mathfrak{t} + \mathfrak{m}$ is the corresponding Cartan decomposition, we identify $V$ with $\mathfrak{m}$ and the complex structure $J$ can be represented as the adjoint action of some element $z \in \mathfrak{t}$; therefore, if $\Sigma$ is any section and $v, w \in \Sigma$, then $< Jv, w > = < [z, v], w > = 0$, since $w$ belongs to $\Sigma$ which is orthogonal to the tangent space of the orbit $\text{Ad}(K)$-orbit through $v$.

If, on the other hand, there exists a symmetric pair $(U, H)$ such that $K$ is a proper subgroup of $H$ and $H, K$ have the same orbits in $V$, where $u = \mathfrak{h} + \mathfrak{v}$ is the corresponding Cartan decomposition, then the pairs $(K, H)$ have been completely classified in [EH] (see also [GT]). We note here that our claim is obviously true when the cohomogeneity is one. The list of such pairs $(K, H)$ with cohomogeneity of the $K$-action at least two is given in the following table, where we also indicate the group $U$, such that $(U, H)$ is a symmetric pair.

| $K$          | $H$       | $U$      |
|--------------|-----------|----------|
| $SO(2) \times G_2$ | $SO(2) \times SO(7)$ | $SO(9)$  |
| $SO(2) \times \text{Spin}(7)$ | $SO(2) \times SO(8)$ | $SO(10)$  |
| $SO(3) \times \text{Spin}(7)$ | $SO(3) \times SO(8)$ | $SO(11)$  |
| $SU(p) \times SU(q), p \neq q$ | $SU(U(p) \times U(q))$ | $SU(p + q)$  |
| $SU(n), n \text{ odd}$ | $U(n)$ | $SO(2n)$ |
| $SO(10)$ | $T^1 \cdot \text{SO}(10)$ | $E_6$ |

Since the corresponding symmetric spaces $U/H$ are Hermitian symmetric with the only exception of $H = SO(3) \times SO(8)$ and $K = SO(3) \times \text{Spin}(7)$, it will be enough to prove that the representation space $V = \mathbb{R}^{24}$ of the group $K$ (here the representation is the tensor product of the standard representation of $SO(3)$ on $\mathbb{R}^3$ and the spin representation of Spin$(7)$ on $\mathbb{R}^8$) does not carry any invariant complex structure. If $V$ carries such a complex structure, then $K$ would have an irreducible representation in $\mathbb{C}^{12}$, which would be a tensor product of two irreducible representations of $SO(3)$ and Spin$(7)$ respectively; this contradicts the fact that the least dimensional irreducible complex representation of Spin$(7)$ is seven dimensional. □

Remark. We point out here that the homogeneity of the Kähler manifold $M$ was used only in the proof of Lemma 2.6 and at the beginning of the proof of Proposition 2.1 where we use the Borel Fixed Point Theorem. Actually, looking at the proof of Lemma 2.6, we see that Theorem 1.1 holds true for every compact, projective algebraic Kähler manifold whose universal covering space is not diffeomorphic to a product of two Kähler manifolds of lower dimension.

Proof of Proposition 1.6. We denote by $J$ the complex structure and by $\langle, \rangle$ the scalar product on the real $G$-module $\mathbb{R}^{2N}$. We split $\mathbb{C}^N = \sum_{i=1}^k W_i$ as the sum of mutually inequivalent complex irreducible $G$-submodules; we recall that each $(W_i)_{\mathbb{R}}$, i.e. the real $G$-module obtained from $W_i$ by restricting the scalars, is either
irreducible or splits as the sum of two equivalent submodules which are interchanged by $J$. Suppose that the $G$-module $\mathbb{R}^{2N}$ contains two equivalent irreducible $G$-submodules; then we can suppose that there exist two equivalent $G$-submodules $E_1, E_2$ such that either $E_1, E_2$ are both complex or $E_1 + E_2$ is complex and $J$ interchanges $E_1, E_2$. Since the restriction of the $G$-action to a complex submodule is still coisotropic we can restrict ourselves to the complex submodule $W := E_1 + E_2$. We identify both $E_1, E_2$ with the irreducible $G$-module $V$ endowed with the invariant scalar product $g$ and $W$ with the orthogonal sum $V \oplus V$ so that $< , >$ restricts to $g$ on the first factor and with $c^2 g$ on the second one for $c \in \mathbb{R} \setminus \{0\}$. If $E_1$ and $E_2$ are not complex submodules, then $Hom(V, V)^G = \mathbb{R}$ (see e.g. [BtD], p. 99) and it is easy to see that $J$ is given up to the sign by $J(v, w) = (-cw, \frac{1}{c}v)$ for $v, w \in V$; if both $E_1, E_2$ are complex submodules, then $V$ has a complex structure $\bar{J}$ and $\bar{J}(v, w) = (\bar{J}v, -\bar{J}w)$, since $E_1 \cong E_2$ as complex modules.

We now construct two functions $f_1, f_2 \in C^\infty(W)^G$ which do not commute with respect to the Poisson bracket, contradicting Theorem 1.4. We define $f_1(v, w) = g(v, v)$ and $f_2(v, w) = g(v, w)$ for $v, w \in V$. Then $X_{f_1}|_{(v,w)}$ is given by $(2\bar{J}v, 0)$ if $V$ is complex and by $(0, \frac{2}{c}v)$ if $JE_1 = E_2$. It is then easy to check that $\{f_1, f_2\} = -df_2(X_{f_1})$ does not vanish identically, proving our claim. □

§3. New examples of polar actions

In this section we first prove Theorem 1.2 and then give new examples of polar actions. We also show that there is a coisotropic action which is not polar.

Proof of Theorem 1.2. We will denote by $J$ and $\omega$ the complex structure and the Kähler form of $M$ respectively and denote the torus $T^n$ by $G$. The commutativity of $G$ implies that the action has only one principal isotropy group which is contained in all other isotropy groups. It follows that the principal isotropy group is trivial since the action is effective. Hence the principal orbits are $n$-dimensional. Since the Euler characteristic of $M$ is positive, the $G$-action has a fixed point and it is therefore Poisson by a result of Frankel [Fr]; this together with the commutativity of $G$ implies that $\omega(X, Y) = 0$ for all $X, Y \in T_p(Gp)$ and therefore $JT_p(Gp) = N_p(Gp)$ for every regular point $p$, where $N_p(Gp)$ denotes the normal space of $Gp$ at $p$. We denote by $\Phi$ the corresponding moment map, which separates $G$-orbits by Theorem 1.4.

Let $P = \Phi(M)$ be the moment polytope of the $T^n$-space $M$. Delzant constructs in [De] a ‘canonical’ smooth projective variety $M_P$ that is $G$-equivariantly symplectomorphomorphic to $M$. The symplectic manifold $M_P$ is naturally equipped with a Kähler metric $g_P$, a complex structure $J_P$ and a symplectic form $\omega_P$; moreover there exists an antiholomorphic involution $\tau_P$ on $M_P$ such that $\tau_P g = g^{-1}\tau_P$ for all $g \in G$ and $\Phi \tau_P = \Phi$. We denote by $Q_P$ the fixed point set of $\tau_P$. The compact, properly embedded submanifold $Q_P$ is not empty: indeed, if $z \in M_P^G$, the fixed point set of $G$ in $M_P$, then $\tau_P z \in M_P^G$ and $\Phi \tau_P z = \Phi z$. Since $\Phi$ separates orbits, we have $\Phi^{-1}\Phi w = w$ for all $w \in M_P^G$. Hence $\tau_P z = z$.

It follows from [Du] that $\Phi(Q_P) = \Phi(M_P)$ since $Q_P$ is not empty. Hence $Q_P$ contains regular points; if $q \in Q_P$ is regular, then we have the orthogonal splitting $T_q M_P = T_q Gq \oplus T_q Q_P$ (with respect to $g_P$) and $J_P(T_q Q_P) = T_q Gq$. 


Abreu shows in [Ab] that the complex manifolds \((M, J)\) and \((M_P, J_P)\) are \(G\)-equivariantly biholomorphic. If \(\phi\) is such a biholomorphism, then \(\tau := \phi^{-1} T_P\phi\) is an antiholomorphic involution of \(M\) (which is not necessarily isometric). If \(Q\) denotes the fixed point set of \(\tau\), namely \(Q := \phi^{-1}(Q_P)\), then for every regular point \(q \in Q\) we have \(T_q Q = J T_q(Gq) = N_q(Gq)\). This shows that a connected component \(\Sigma\) of \(Q\) is an integral submanifold for the normal distribution to the regular orbits. It is a well known fact that \(\Sigma\) meets every \(G\)-orbit; moreover \(\Sigma\) intersects the regular and hence all \(G\)-orbits orthogonally. □

We will now give examples of torus actions, in which we can explicitly describe the sections.

**Example 1.** We consider the complex projective space \(\mathbb{P}_n(\mathbb{C})\) together with the polar action of the \(n\)-dimensional torus \(T^n\), which is simply the maximal torus in the group \(SU(n + 1)\); the regular orbits are \(n\)-dimensional tori and any section is a totally geodesic real projective space \(\mathbb{P}_n(\mathbb{R})\).

Our aim is to produce new examples of polar actions on nonhomogeneous Kähler manifolds, which are not hyperpolar. In order to do this, we choose a \(T^n\)-fixed point \(p \in \mathbb{P}_n(\mathbb{C})\) and we denote by \(M_1\) the complex manifold which is obtained blowing up \(\mathbb{P}_n(\mathbb{C})\) in \(p\); note that \(M_1\) is a projective variety and that \(M_1\) is not homogeneous with respect to the full group of holomorphic automorphisms (hence with respect to the full group of isometries of any Kähler metric compatible with the given complex structure), see [Bl]. The \(T^n\)-action lifts to a holomorphic action on \(M_1\); moreover, since \(M_1\) is Kähler and \(T^n\) is compact, we can find a \(T^n\)-invariant Kähler metric \(g\) on \(M_1\) by averaging; we remark that the metric \(g\) has *a priori* no relation with the Fubini-Study metric on \(\mathbb{P}_n(\mathbb{C})\). It follows from Theorem 1.2 that the action of \(T^n\) on \(M_1\) is polar. In the following we will determine the section of the action.

We denote by \(\pi : M_1 \to \mathbb{P}_n(\mathbb{C})\) the holomorphic projection. We fix a section \(\Sigma\) in \(\mathbb{P}_n(\mathbb{C})\) and denote by \(\Sigma_1\) the closed submanifold of \(M_1\) which coincides with \(\Sigma\) outside \(\{p\}\) and has the property that \(\Sigma_1 \cap \pi^{-1}(p) = \mathbb{P}_C(T_p \Sigma)\), where we have identified \(\pi^{-1}(p)\) with \(\mathbb{P}_C(T_p \mathbb{P}_n(\mathbb{C}))\). It is clear that \(\Sigma_1\) meets every \(T^n\)-orbit. We are now going to show that \(\Sigma_1\) is a section for the \(T^n\)-action with respect to the Kähler metric \(g\) and we will prove this in the dense open set of regular points.

First of all we remark that the \(T^n\)-action on \(M_1\) is a Poisson action, since the manifold \(M_1\) is simply connected (see e.g. [GS3]); moreover the \(T^n\)-action is coisotropic by Theorem 1.4, (v). We denote by \(J\) the complex structure on \(M_1\) and fix a regular point \(y \in \pi^{-1}(p)\). If we denote by \(N_y\) the normal space to the orbit \(T^n \cdot y\), then \(J N_y = T_y(T^n \cdot y)\) since the action is coisotropic. On the other hand, \(J(T_y \Sigma_1)\) also coincides with \(T_y(T^n \cdot y)\), so that \(N_y = T_y(\Sigma_1)\) and therefore \(\Sigma_1\) is a section. We remark that the section \(\Sigma_1\), as a differentiable manifold, is the connected sum \(\mathbb{P}_n(\mathbb{R}) \# \mathbb{P}_n(\mathbb{R})\).

Since the \(T^n\)-action on \(M_1\) has fixed points in \(\pi^{-1}(p)\), we can iterate this construction blowing up \(M_1\) in one of these fixed points. In this way, we get a sequence of compact Kähler manifolds \(M_k\) on which the group \(T^n\) acts polarly with the sections being the connected sum of \(k + 1\) copies of \(\mathbb{P}_n(\mathbb{R})\). If \(n \geq 3\), then \(\pi_1(\Sigma_1)\) is a free group \(\mathbb{Z} \ast \mathbb{Z} \ast \ldots \ast \mathbb{Z}\) \((k + 1\) times). Therefore it can not carry a flat metric by the Bieberbach Theorem (see [Wo]).

**Remark.** We point out that the lifted action of \(T^n\) on \(M_k\) is polar with respect to *any*
invariant Kähler metric. We also remark that this construction yields polar actions on four dimensional Kähler manifolds with any nonorientable compact surface as a section.

We would now like to give examples of polar actions of nonabelian Lie groups using the method of blow-ups.

**Example 2.** We first describe the general principle which will allow us to obtain new examples. We consider a compact Lie group \( G \) acting polarly on a homogeneous Kähler manifold \( M \) with at least one fixed point \( p \); we denote by \( K \) the isotropy subgroup of a regular point \( q \in M \). We recall the following well-known facts on moment maps (see [GS3] and also [HW]). Denote by \( \Phi : M \to g^* \) the moment map with respect to the given Poisson-Kähler action of \( G \) and recall that \( \Phi \) maps \( G \)-orbits onto coadjoint orbits in \( g^* \); moreover, for any regular point \( y \in M \), the isotropy subgroup \( H_y := G_{\Phi(y)} \) contains \( G_y \) and, at the level of Lie algebras, we may decompose \( h_y \) (with respect to a fixed biinvariant scalar product on \( g \)) as

\[
h_y = g_y + a_y,
\]

where \( a_y = g_y^\perp \cap h_y \) is abelian and centralizes \( g_y \). If we denote by \( N_y \) the normal space to the regular orbit \( G \cdot y \) and by \( J \) the complex structure on \( M \), then

\[
J(N_y) = T_y(H_y \cdot y) \subset T_y(G \cdot y),
\]

since the \( G \)-action on \( M \) is coisotropic by Theorem 1.1. It follows that the dimension of \( a_y \) is equal to the cohomogeneity of the action of \( G \). The tangent space to the orbit \( G \cdot y \) can be identified with the complement \( g_y^\perp \) and we have the decomposition

\[
g = g_y + a_y + m_y,
\]

where \( m_y \) corresponds to the maximal complex subspace of the tangent space \( T_y(G \cdot y) \). In the spirit of S. Lie and F. Engel ([LE], p. 501 ff), we call the \( G \)-action \( \mathbb{C} \)-asystatic if for a regular orbit \( G/K \) there is no nontrivial complex subspace of the tangent space \( T_{[e,K]}(G/K) \) which is left fixed by the isotropy representation of \( K \). (For a modern treatment of asystatic actions, see [AA] and [PT1]). It follows that if a polar actions as above is also \( \mathbb{C} \)-asystatic, then the space of all tangent vectors to a regular orbit which are fixed under the isotropy representation has dimension equal to the cohomogeneity of the \( G \)-action.

We now assume that the \( G \)-action is \( \mathbb{C} \)-asystatic and we consider the blow-up \( M_1 \) of \( M \) in the fixed point \( q \). If \( \Sigma \) denotes a section for the \( G \)-action on \( M \), then the submanifold \( \Sigma_1 \) of \( M_1 \) constructed exactly as in Example 1 is a closed submanifold which meets every \( G \)-orbit in \( M_1 \). If we now endow \( M_1 \) with any \( G \)-invariant Kähler metric \( g \), we show that \( \Sigma_1 \) is a section, namely that it meets every \( G \)-orbit orthogonally. Indeed, the lifted action of \( G \) on \( M_1 \) is Poisson, since \( M_1 \) is simply connected; moreover the \( G \)-action is coisotropic because \( M_1 \) is a projective algebraic variety, \( G^\mathbb{C} \) acts on \( M \), hence on \( M_1 \), with an open orbit and the same is true for a Borel subgroup of \( G^\mathbb{C} \) (see Theorem 1.4). We fix a regular point \( y \in M_1 \) and we denote by \( N_y \) the normal space to \( T_y(G \cdot y) \) with respect to \( g \), so that
\[ J \tilde{N}_y \subseteq T_y(G \cdot y); \] since the action of \( G \) on \( M \) is also coisotropic, we have that \( J(T_y \Sigma_1) \subseteq T_y(G \cdot y) \) and we claim that \( J(\tilde{N}_y) = J(T_y \Sigma_1) \). Both subspaces \( J(\tilde{N}_y) \) and \( J(T_y \Sigma_1) \) are fixed under the isotropy representation of \( G_y \) and have the same dimension, namely the cohomogeneity of \( G \) and therefore, by the assumption that the action is \( \mathbb{C} \)-asystatic, they coincide.

As an application, we consider the quadric \( M := Q_n = SO(n+2)/SO(2) \times SO(n) \). The isotropy subgroup \( G = SO(2) \times SO(n) \) acts polarly on \( Q_n \) with sections being two dimensional flat tori. It is easy to check that a regular isotropy subgroup \( K \) is given by \( K = \mathbb{Z}_2 \times SO(n-2) \) and therefore the isotropy representation splits as \( 2\mathbb{R} + 2\mathbb{R}^{n-2} \), where \( \mathbb{R} \) is a trivial submodule, while \( K \) acts on \( \mathbb{R}^{n-2} \) in the standard way; therefore the \( G \)-action is \( \mathbb{C} \)-asystatic. The blown up manifold \( M_1 \) is acted on polarly by \( G \) with sections being the connected sum \( T^2 \# \mathbb{P}_2(\mathbb{R}) \). Note that \( M_1 \) is not homogeneous, as already pointed out in Example 1.

**Example 3.** We will now give examples of coisotropic actions on complex projective spaces that are not polar. Let \( G \) denote the compact Lie group \( U(2) \times Sp(n), n \geq 2 \), acting on \( \mathbb{C}^{4n} = \mathbb{C}^2 \otimes \mathbb{C}^{2n} \) as a tensor product of the standard representations of \( U(2) \) and \( Sp(n) \) respectively. This representation is not polar. (It has cohomogeneity three and is shown to have taut orbits in [GT], but these properties will not be important here.) We consider the complexification \( G^\mathbb{C} = GL(2, \mathbb{C}) \times Sp(n, \mathbb{C}) \) of \( G \) and the corresponding representation on \( \mathbb{C}^{4n} \) which is multiplicity free by the classification of such irreducible representations of reductive noncompact complex Lie groups in [Ka]. This means that \( G^\mathbb{C} \) acts on \( \mathbb{C}^{4n} \) with an open orbit as well as every Borel subgroup of \( G^\mathbb{C} \). The induced action of \( G^\mathbb{C} \) on \( \mathbb{P}_{4n-1}(\mathbb{C}) \) clearly also has the property that every Borel subgroup of \( G^\mathbb{C} \) acts on \( \mathbb{P}_{4n-1}(\mathbb{C}) \) with an open orbit. This implies by Theorem 1.4 that the action of \( G \) on \( \mathbb{P}_{4n-1}(\mathbb{C}) \) is coisotropic.

This action is not polar since otherwise the action of \( T^1 \times G \) on \( \mathbb{C}^{4n} \), where \( T^1 \) acts as multiplication by scalars, would have to be polar by [PTh]; since the center of \( G \) already acts on \( \mathbb{C}^{4n} \) as multiplication by scalars by Schur Lemma, \( T^1 \times G \) and \( G \) have the same orbits and therefore the \( G \)-action on \( \mathbb{P}_{4n-1}(\mathbb{C}) \) is not polar.

It is not difficult to classify coisotropic actions on complex projective spaces. If \( G \) is a compact subgroup of \( U(n+1) \), then the action of \( G \) on \( \mathbb{P}_n(\mathbb{C}) \) is coisotropic if and only if the Borel subgroups of \( \mathbb{C}^* \times G^\mathbb{C} \) have dense orbits in \( \mathbb{C}^{n+1} \) which in turn is equivalent to the action of \( \mathbb{C}^* \times G^\mathbb{C} \) being multiplicity free on \( \mathbb{C}^{n+1} \) (see e.g. [Kr], p. 199). One can now use the classification of multiplicity free representations of reductive groups that can be found in [Ka] for the irreducible case and in [BR] and [Le] for the reducible case.

§4. Polar and coisotropic actions on quadrics

The aim of this section is to prove Theorem 1.3, or more precisely to classify coisotropic and polar actions on the Hermitian symmetric space \( Q_{n-2} = SO(n)/SO(2) \times SO(n-2) \) \((n \geq 5)\), which can be represented as the \((n-2)\)-dimensional complex quadric in \( \mathbb{P}_{n-1}(\mathbb{C}) \), i.e., the set of all totally isotropic lines in \( \mathbb{C}^n \) with respect to the invariant symmetric bilinear form \( \langle , \rangle \) given by \( \langle z, w \rangle = \sum_{i=1}^n z_i w_i \), or equivalently as the Grassmannian of oriented two-planes in \( \mathbb{R}^n \).
Hyperpolar actions of compact Lie groups on irreducible compact symmetric spaces were classified by Kollross ([Ko]). We state his result in the special case of complex quadrics in the following theorem.

**Theorem 4.1 (Kollross).** Let $G \subset SO(n)$ be a compact connected Lie subgroup acting on the symmetric space $Q_{n-2} = SO(n)/SO(2) \times SO(n-2)$ in a hyperpolar fashion. Then there exists a compact connected Lie subgroup $G'$ of $SO(n)$ containing $G$ such that the $G'$-action on $Q_{n-2}$ has the same orbits as the $G$-action and $G'$ is conjugate to one of the following subgroups

1. $G' = SO(p) \times SO(q) \subset SO(p+q)$ with $p, q \geq 1$;
2. $G' = U(n) \subset SO(2n)$;
3. $G' = Sp(1) \cdot Sp(n) \subset SO(4n)$;
4. $G' = Spin(9) \subset SO(16)$.

**Remark.** The actions listed in Theorem 4.1 all have cohomogeneity one except case (1) when $p, q > 1$ which has cohomogeneity two. Recall that a compact Lie subgroup of $SO(n)$ acts transitively on $Q_{n-2}$ if and only if $G = SO(n)$ with the two exceptions $G = G_2$ ($n = 7$) and $G = Spin(7)$ ($n = 8$) as can be seen from [On], Theorem 2, p. 226.

We now consider a compact connected Lie subgroup $G$ of $SO(n)$ acting coisotropically on $Q_{n-2}$. The complexification $G^C$ acts on $Q_{n-2}$ with an open orbit $\Omega$ and there exists a Borel subgroup of $G^C$ which has an open orbit in $\Omega$.

In order to prove Theorem 1.3, we will make an extensive use of the results by Kimelfeld ([Ki]), who classified all reductive subgroups $G^C$ of $SO(n, \mathbb{C})$ having an open orbit on the quadric $Q_{n-2}$.

We will distinguish between the two cases when $G \subset SO(n)$ acts irreducibly on $\mathbb{R}^n$ and when $G$ acts reducibly on $\mathbb{R}^n$.

**First case: the $G$-action on $\mathbb{R}^n$ is irreducible**

In this case the complexified action of $g^C$ on $\mathbb{C}^n$ can be irreducible or reducible (as the sum of two submodules $W \oplus W^*$, where $W \simeq \mathbb{C}^{n/2}$). These two subcases appear as separate cases in Kimelfeld’s paper (see Tables I and II in [Ki], p. 535-536) and we too will reproduce them separately in Tables I and II below.

As for notation, we will use the standard symbols $A_s, B_s, C_s, D_s$ ($s \geq 1$) and $G_2, F_4$ to denote complex simple Lie algebras; by $R(\phi)$ we will denote the irreducible representation of a simple Lie algebra with highest weight $\phi$ and $\phi_i$ will indicate the highest weight of the $i$th fundamental representation of a simple complex Lie algebra (the standard representation of $so(n)$ on $\mathbb{C}^n$ will be call simplest); by $t_n$ we will indicate the Lie algebra of the $n$-dimensional complex torus and $\epsilon$ will denote the faithful one dimensional representation of $t_1$.

In the following Table I we list all complex reductive Lie algebras $g^C$ with the corresponding complex irreducible representation such that the induced action on the quadric $Q$ has an open orbit $\Omega$; moreover we will also indicate a generic stability subalgebra $h$, with $p$-dimensional nilradical $U_p$, so that $\Omega = G^C/H$, where $H$ is an appropriate subgroup of $G^C$ having $h$ as Lie algebra.
We start noting that the cases 1, 3 and 4 correspond to transitive actions of the compact real form $G$, while the cases 2 and 5 are known as cohomogeneity one actions (see Kollross’ theorem).

In the cases 6 to 8, the group $G$ acts on $\mathfrak{g}$ via adjoint representation; the cohomogeneity of the $G$-action on the corresponding quadric $Q$ is easily seen to be at least four, hence bigger than the rank of $G$. This means that the $G$-action on $Q$ is not coisotropic by Theorem 1.4.

We are now going to exclude all other cases from the above table; in order to do this, we use the fact that a Borel subgroup of $G^C$ has an open orbit in $G^C/B$, see Theorem 1.4. This immediately excludes cases 9 and 13.

The following result (see [Ki], p. 572 ff) is well known:

**Lemma 4.2.** Given a semisimple complex algebraic group $U$ and a Borel subgroup $B$, a reductive subgroup $H$ of $U$ has an open orbit in $U/B$ if and only if the space of $H$-fixed vectors in every irreducible $U$-module is at most one-dimensional.

Note that, if a Borel subgroup $B$ of $G^C$ has an open orbit in $G^C/H$, then $H$ has an open orbit in $G^C/B$.

In the remaining cases 10 to 12, we note that these are complexifications of isotropy representations of rank two symmetric spaces and that $\mathfrak{h}$ is exactly the complexification of a regular stability subalgebra for these representations. Since $H$ stabilizes a two dimensional subspace of the corresponding irreducible representation of $G^C$, the $G$-action on the quadric is not coisotropic by Theorem 1.4 (v) and Lemma 4.2.

We now come to the subcase that the complexified action of $\mathfrak{g}^C$ on $\mathbb{C}^n$ is reducible (as the sum of two submodules $W \oplus W^*$, where $W \simeq \mathbb{C}^{n/2}$). In the following Table II, we reproduce all such cases having an open orbit on the corresponding quadric, only indicating the irreducible representation $W$ and the generic stability subalgebra $\mathfrak{h}$ for the $G^C$-action on the corresponding quadric; we will indicate by $U_p$ the $p$-dimensional nilradical of the generic stability subalgebra $\mathfrak{h}$.

| $n.$ | $\mathfrak{g}^C$ | Represen. | $\mathfrak{h}$ |
|------|-----------------|------------|----------------|
| 1    | $\mathfrak{so}(s)$, $s \geq 3$ | Simplest | $(\mathfrak{so}(s-1) \oplus \mathfrak{t}_1) + U_{s-2}$ |
| 2    | $\mathfrak{g}_2 \oplus \mathfrak{a}_1$, $s \geq 2$ | $R(\phi_1) \times R(\phi_1)$ | $(\mathfrak{g}_2 \oplus \mathfrak{a}_1) \oplus \mathfrak{t}_1) + U_{4s-5}$ |
| 3    | $\mathfrak{g}_2$ | $R(\phi_1)$ | $\mathfrak{a}_2$ |
| 4    | $\mathfrak{b}_3$ | $R(\phi_4)$ | $\mathfrak{g}_2$ |
| 5    | $\mathfrak{b}_3$ | $R(\phi_4)$ | $\mathfrak{b}_3$ |
| 6    | $\mathfrak{a}_2$ | $R(\phi_1 + \phi_2)$ | $\mathfrak{t}_2$ |
| 7    | $\mathfrak{c}_2$ | $R(2\phi_1)$ | $\mathfrak{t}_2$ |
| 8    | $\mathfrak{g}_2$ | $R(\phi_4)$ | $\mathfrak{t}_2$ |
| 9    | $\mathfrak{a}_1$ | $R(4\phi_1)$ | 0 |
| 10   | $\mathfrak{c}_3$ | $R(\phi_2)$ | $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1$ |
| 11   | $\mathfrak{f}_4$ | $R(\phi_1)$ | $\mathfrak{d}_4$ |
| 12   | $\mathfrak{c}_2 \oplus \mathfrak{g}_2$, $s \geq 2$ | $R(\phi_1) \times R(\phi_1)$ | $(\mathfrak{c}_2 \oplus \mathfrak{g}_2) \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1$ |
| 13   | $\mathfrak{a}_1 \oplus \mathfrak{a}_1$ | $R(\phi_1) \times R(3\phi_1)$ | 0 |

Table I
We start noting that case 2 is of cohomogeneity one (see Theorem 4.1) and the same holds for case 1: indeed, the action of $U(s)$ on $SO(2s)/SO(2) \times SO(2s - 2)$ has a complex singular orbit which is $\mathbb{P}_{s-1}(\mathbb{C}) = U(s)/U(1) \times U(s - 1)$; this orbit also is an orbit of the semisimple part $SU(s)$ and the stabilizer $S(U(1) \times U(s - 1))$ still acts with cohomogeneity one on the normal space.

Cases 6 to 8 and 11 to 14 can be ruled out simply because the dimension of a Borel subgroup of $G^C$ is strictly less than the dimension of the open orbit $G^C/H$.

In cases 3 and 4 we consider the action of the compact group $G$ acting on the quadric $Q := Q_{4s+2}$ and we will prove that in case 4 this $G$-action is not coisotropic; by Theorem 1.4 this implies that the $G$-action in case 3 is also not coisotropic. We may take $G$ as $S(U(2) \times U(s + 1))$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^{s+1} \simeq \mathbb{R}^{4s+4}$: we then fix two unit vectors $e_1 \in \mathbb{C}^2$, $v \in \mathbb{C}^{s+1}$ and we take the oriented two plane $\pi$ in $\mathbb{R}^{4s+4}$ given by the complex line $\mathbb{C} \cdot e_1 \perp v$. It is easy to see that $G_\pi$ is isomorphic to $S(U(1) \times U(1) \times U(1) \times U(s))$, so that the orbit $G/G_\pi$ is complex and biholomorphic to $S^2 \times \mathbb{P}_s(\mathbb{C})$. Now, if $V$ denotes the subspace $(\mathbb{C} \cdot v)^\perp \subset \mathbb{C}^{s+1}$ and $e_2$ is a unit vector generating $(\mathbb{C} \cdot e_1)^\perp \subset \mathbb{C}^2$, then we have

$$T_\pi Q = Hom(\pi, \pi^\perp) = \pi^* \otimes_{\mathbb{R}} [(e_1 \otimes V)_{\mathbb{R}} + (e_2 \otimes V)_{\mathbb{R}} + (\mathbb{C} \cdot e_2 \otimes v)_{\mathbb{R}}],$$

which contains two pairs of equivalent 2s-dimensional $G_\pi$-submodules. Since the isotropy representation of $G_\pi$ on $T_{eG_\pi}G/G_\pi$ contains only one 2s-dimensional submodule, it then follows that the slice representation of $G_\pi$ at $\pi$ contains at least two distinct equivalent submodules. If the $G$-action on $Q$ is coisotropic, then the slice representation of $G_\pi$ is also coisotropic (see [HW], p. 274), contradicting Proposition 1.6.

Case 5 can be ruled out because the pair $(\mathfrak{so}(s) + t_1, \mathfrak{so}(s - 2))$ is not a spherical pair; indeed, if it were spherical, the pair $(\mathfrak{so}(s), \mathfrak{so}(s - 2))$ would be spherical too, contradicting Lemma 4.2.

In case 9, we show that the action of $G = Sp(s)$ is not coisotropic. We consider the $G$-action on $\mathbb{H}^s \simeq \mathbb{C}^{2s} \simeq \mathbb{R}^{4s}$ and the corresponding action on the quadric $Q := Q_{4s-2}$. We fix an oriented two plane $\pi$ in $\mathbb{R}^{4s}$, which is invariant under

| $n.$ | $\mathfrak{g}^C$ | Represen. | $\mathfrak{h}$ |
|------|-----------------|------------|---------------|
| 1    | $A_s, s \geq 2$ | $R(\phi_1)$ | $(A_{s-2} + t_1) + U_{2s-1}$ |
| 2    | $A_s + t_1, s \geq 2$ | $R(\phi_1) \times \epsilon$ | $(A_{s-2} + t_2) + U_{2s-1}$ |
| 3    | $A_s + A_1, s \geq 2$ | $R(\phi_1) \times R(\phi_1)$ | $A_{s-2} + t_1$ |
| 4    | $A_s + A_1 + t_1, s \geq 2$ | $R(\phi_1) \times R(\phi_1) \times \epsilon$ | $A_{s-2} + t_2$ |
| 5    | $\mathfrak{so}(s) \otimes t_1$ | Simplest $\times \epsilon$ | $\mathfrak{so}(s-2)$ |
| 6    | $A_1 \otimes t_1$ | $R(2\phi_1) \times \epsilon$ | 0 |
| 7    | $B_3 \otimes t_1$ | $R(\phi_3) \times \epsilon$ | $A_2$ |
| 8    | $G_2 \otimes t_1$ | $R(\phi_1) \times \epsilon$ | $A_1$ |
| 9    | $C_s, s \geq 3$ | $R(\phi_1)$ | $(C_{s-2} + t_1) + U_{4s-5}$ |
| 10   | $C_s + t_1, s \geq 3$ | $R(\phi_1) \times \epsilon$ | $(C_{s-2} + t_2) + U_{4s-5}$ |
| 11   | $A_4$ | $R(\phi_2)$ | $A_1 + A_1$ |
| 12   | $A_4 \otimes t_1$ | $R(\phi_2) \times \epsilon$ | $A_1 + A_1 + t_1$ |
| 13   | $D_5$ | $R(\phi_5)$ | $A_3$ |
| 14   | $D_5 \otimes t_1$ | $R(\phi_5) \times \epsilon$ | $A_3 + t_1$ |

Table II
the complex structure of \( \mathbb{C}^{2s} \). Then \( G_\pi = T^1 \times \text{Sp}(s-1) \) and the orbit \( G \cdot \pi \) is diffeomorphic to a complex projective space \( \mathbb{P}_{2s-1}(\mathbb{C}) \). The tangent space \( T_\pi Q \) can be identified with \( \text{Hom}(\pi, \pi^\perp) \) and we have the orthogonal splitting \( \pi^\perp = \hat{\pi} + W \), where \( \pi + \hat{\pi} \) is the quaternionic span of \( \pi \), while \( W \simeq \mathbb{R}^{4(s-1)} \) is the standard representation space of \( \text{Sp}(s-1) \).

If we denote by \( \rho \) the representation of the center \( T^1 \) of \( G_\pi \) on \( \pi \), then \( T^1 \) acts on \( \hat{\pi} \) by means of \( \rho^{-1} \); the tangent space \( T_\pi Q \) splits as a \( G_\pi \) module as \( 2\mathbb{R} + \mathbb{R}^2 + 2W \), where \( \mathbb{R} \) denotes the trivial module, \( \mathbb{R}^2 \) is acted on by \( T^1 \) by means of \( \rho^2 \), while \( W \) is the standard representation space of \( G_\pi \). Since the isotropy representation of \( G_\pi \) on the orbit \( G \cdot \pi \) is \( \mathbb{R}^2 + W \), we see that the normal space \( N_\pi \) is isomorphic to \( 2\mathbb{R} + W \) as a \( G_\pi \)-module; it follows that a regular isotropy subgroup \( K \) of the \( G \)-action on \( Q \) is isomorphic to \( T^1 \times \text{Sp}(s-2) \) of corank one in \( G \), while the \( G \)-cohomogeneity on \( Q \) is three. This proves that the \( G \)-action on \( Q \) is not coisotropic.

In case 10 the group \( G = T^1 \times \text{Sp}(s) \) acts on \( \mathbb{C}^{2s} \simeq \mathbb{R}^{4s} \) in a standard way; for notational reasons we will denote by \( Z \) the one dimensional center of \( G \). We will show that the \( G \)-action on \( Q := Q_{4s-2} \) is coisotropic but not polar. As in the previous case, we fix an oriented two plane \( \pi \subset \mathbb{R}^{4s} \) which is \( Z \)-invariant; the isotropy subgroup \( G_\pi \) is \( Z \times T^1 \times \text{Sp}(s-1) \) and the orbit \( O := G \cdot \pi \) is a complex projective space \( \mathbb{P}_{2s-1}(\mathbb{C}) \). If we compute the slice representation along the same lines as in case 9, we see that the normal space \( N_\pi \) splits as a \( G_\pi \)-module as \( \mathbb{R}^2 + \mathbb{R}^{4(s-1)} \), where \( G_\pi \) acts nontrivially on each factor (this action can be shown to be actually polar). A regular isotropy subgroup \( K \) is isomorphic to \( T^1 \times \text{Sp}(s-2) \) of corank two in \( G \) and the cohomogeneity is two, hence the \( G \)-action on \( Q \) is coisotropic.

We now show that this action is not polar. It is not difficult to see that there exists a totally geodesic submanifold \( F \subset Q \) with \( T_\pi F = N_\pi \) such that \( F \) is isometric to the complex projective space \( \mathbb{P}_{2s-1}(\mathbb{C}) \) after an appropriate rescaling of the metrics. Moreover we observe that the group \( G \) lies in \( \text{U}(2s) \), which acts on \( Q \) with cohomogeneity one; the two singular \( \text{U}(2s) \)-orbits are \( O \) and \( O^* := \text{U}(2s) \cdot \pi^* = G \cdot \pi^* \), where \( \pi^* \) denotes the two-plane \( \pi \) with reversed orientation. The orbit \( O^* \) consists actually of all points of maximal distance \( d \) from \( O \) and \( d \) is the diameter of \( F \). Now suppose that the \( G \)-action is polar. If \( \Sigma \) is a section passing through \( \pi \), then \( \Sigma \subset F \) and we know from \cite{[PT]} that \( \Sigma \) is a real projective plane. The submanifold \( \Sigma \) contains a circle \( S \) of points at distance \( d \) from \( \pi \), hence \( S \) is contained in \( O^* \); this means that \( \Sigma \) intersects the \( G \)-orbit in a circle, hence not orthogonally; a contradiction.

So far we have proved the following

**Proposition 4.3.** Let \( G \) be a compact connected Lie subgroup of \( \text{SO}(n) \), where \( n \geq 5 \). Suppose the action of \( G \) on \( \mathbb{R}^n \) is irreducible. Then the action of \( G \) on \( Q := Q_{n-2} \) is nontransitive and polar if and only if it is hyperpolar and \( G = \text{U}(n/2), \text{SU}(n/2), G = \text{Sp}(1) \cdot \text{Sp}(n/4) \) or \( G = \text{Spin}(9) \subset \text{SO}(16) \).

The action of \( G \) on \( Q \) is nontransitive and coisotropic if and only if the action is polar or \( G = T^1 \cdot \text{Sp}(n/4) \)

**Second case:** the \( G \)-action on \( \mathbb{R}^n \) is reducible
We will now suppose that the linear $G$-action on $\mathbb{R}^n$ is reducible. Here the proof will be more direct than in the first case. We will first deal with the particular case when $G$ has a fixed nonzero vector $e$ in $\mathbb{R}^n$, so that we can write $\mathbb{R}^n = \mathbb{R} \cdot e + \mathbb{R}^{n-1}$ (note that we are not supposing that $\mathbb{R}^{n-1}$ is $G$-irreducible). We prove the following

**Proposition 4.4.** Let $G$ be a compact connected Lie group acting on $V := \mathbb{R}^n$ with nonzero fixed point set. Then

(i) the action of $G$ on $Q := Q_{n-2}$ is coisotropic if and only if the fixed point set $V^G$ is one dimensional and $G$ acts transitively on the unit sphere of $(V^G)^\perp \simeq \mathbb{R}^{n-1}$ with the exception of $G = \text{Sp}(k)$ acting in a standard way on $\mathbb{R}^{4k}$, $n = 4k + 1$.

(ii) the action of $G$ on $Q := Q_{n-2}$ is polar if and only if the fixed point set $V^G$ is one dimensional and $G = \text{SO}(n-1), G = G_2$ ($n = 8$) or $G = \text{Spin}(7)$ ($n = 9$); in particular the action of $G$ is polar if and only if it is hyperpolar.

**Proof.** We suppose that the $G$-action on $Q_{n-2}$ is coisotropic. We first show that $G$ must act transitively on the unit sphere of $\mathbb{R}^{n-1}$. We fix a nonzero vector $v \in \mathbb{R}^{n-1}$ which is regular for the $G$-action on $\mathbb{R}^{n-1}$ and consider the oriented two plane $\pi = e \wedge v$; we note that $G_v = G_\pi$. We may identify the tangent space $T_\pi Q$ with $e \wedge W + v \wedge W$, where $W \subset \mathbb{R}^{n-1}$ is the set of all vectors orthogonal to $v$ in $\mathbb{R}^{n-1}$; moreover $T_\pi G \cdot \pi$ is given by

$$T_\pi G \cdot \pi = \{ X e \wedge v + e \wedge X v; \ X \in \mathfrak{g} \} = \{ e \wedge X v; \ X \in \mathfrak{g} \}.$$ 

We note that $\mathfrak{g} \cdot v \subset W$ and we split orthogonally $W = \mathfrak{g} \cdot v + V$, where $\dim V + 1$ is equal to the cohomogeneity of $G$ in $\mathbb{R}^{n-1}$. Then

$$T_\pi Q = T_\pi G \cdot \pi + e \wedge V + v \wedge V + v \wedge \mathfrak{g} \cdot v.$$ 

We recall then that the complex structure $J$ on $T_\pi Q$ is given (up to sign) by

$$J(e \wedge w_1 + v \wedge w_2) = e \wedge w_2 - v \wedge w_1,$$

for $w_1, w_2 \in W$. Therefore the subspace $e \wedge V + v \wedge V$ is a complex subspace of $(T_\pi G \cdot \pi)^\perp$ on which $G_\pi$ acts trivially. The slice representation of $G_\pi$ is coisotropic since the action of $G$ on $Q_{n-2}$ is coisotropic (see [HW], p. 274) and can therefore not leave a nontrivial subspace invariant. This implies that $e \wedge V + v \wedge V = \{0\}$, hence $V = \{0\}$ and our first claim follows.

We now use the fact that $G$ induces a coisotropic, possibly transitive action on the quadric $Q(\mathbb{R}^{n-1}) = Q_{n-3}$. An inspection of the lists of compact groups acting transitively on spheres (see e.g. [Bo]) and quadrics (see the remark after Theorem 4.1) together with Proposition 4.3 shows that $G$ can be either

(a) $G = \text{SO}(n-1), G_2$, or $\text{Spin}(7)$ which all give rise to a cohomogeneity one action on $Q$ (those of $G_2$ and Spin(7) being orbit equivalent to the one of $\text{SO}(n-1)$ for an appropriate $n$),

(b) $G = \text{SU}(\frac{n-1}{2}), \text{U}(\frac{n-1}{2}), \text{Sp}(1) \cdot \text{Sp}(\frac{n-1}{4}), T^1 \cdot \text{Sp}(\frac{n-1}{4})$ or Spin(9).

It is easy to check that all groups listed in (b) yield coisotropic actions; their cohomogeneity on the quadric is given by the cohomogeneity of $G_v$ acting on $\mathfrak{g} \cdot v$. 

We will now show that the $G$-action on the quadric is not polar in all the cases enumerated in $(b)$. Let now $G$ be one of the groups listed in $(b)$; we already know that the induced action of $T^1 \cdot \text{Sp}(\frac{n-1}{4})$ on the subquadric $Q_{n-3}$ is not polar if $n > 5$, while, for $n = 5$, $G$ coincides with $U(2)$, so that this case can be ruled out. Using the same notation as above, it is easy to see that $G_v$ acts on $g \cdot v$ polarly and with cohomogeneity two. We choose two orthogonal vectors $w_1, w_2$ of unit length in $g \cdot v$ which span a $G_v$-section in $g \cdot v$; if the $G$-action on $Q$ is polar, then there exists a section $\Sigma$ passing through $\pi$ with $T_\pi \Sigma = \text{Span}(v \wedge w_1, v \wedge w_2)$. The unique totally geodesic submanifold of $Q$ with such properties is given by $\Sigma = \{ v \wedge (ae + bw_1 + cw_2); \ a^2 + b^2 + c^2 = 1 \}$. If we now consider the two plane $\pi_1 = v \wedge w_1$, then $T_{\pi_1} \Sigma$ is generated by $v \wedge e, v \wedge w_2$, while $T_{\pi_1} G \cdot \pi_1$ is given by $g v \wedge w_1 + v \wedge gw_1$. Since the scalar product $< v \wedge gw_1, v \wedge w_2 > = < gw_1, w_2 >$ is not zero, we see that $\Sigma$ cannot be a section and this contradiction proves our last claim.

We now suppose that $G$ acts on $\mathbb{R}^n$ reducibly without fixed vectors and that it acts on the quadric $Q_{n-2}$ coisotropically. We split $\mathbb{R}^n = V_1 \oplus V_2$ into two nontrivial invariant subspaces with $\dim V_i \geq 2$ and consider first the case when both $V_i$ are $G$-irreducible. The restriction of the $g$-action onto $V_i$ defines homomorphisms $\rho_i : g \to \mathfrak{so}(V_i)$ for $i = 1, 2$. We put $g_i := \rho_i(g)$, $i = 1, 2$, so that $g \subset g_1 + g_2$. If $\dim V_i \geq 5$, Proposition 4.3 and the fact that the action of $G$ restricted to $V_i$ induces a coisotropic action on the corresponding quadric imply that $g_i$ is one of the following Lie algebras

$$\mathfrak{so}(p), \mathfrak{g}_2, \mathfrak{spin}(7), \mathfrak{spin}(9), \mathfrak{su}(p), \mathfrak{u}(p), \mathbb{R} + \mathfrak{sp}(p), \mathfrak{sp}(1) + \mathfrak{sp}(p)$$

for some integer $p$. On the other hand, if $2 \leq \dim V_i \leq 4$, the list in $(*)$ contains all possible subalgebras of $\mathfrak{so}(V_i)$ which act on $V_i$ irreducibly, so that $(*)$ gives the list of all possible candidates for $g_i$, $i = 1, 2$.

**Lemma 4.5.** We have $g_i \neq \mathfrak{su}(p), \mathfrak{u}(p), \mathbb{R} + \mathfrak{sp}(p)$.

**Proof.** We suppose that $g_1 \in \{ \mathfrak{su}(p), \mathfrak{u}(p), \mathbb{R} + \mathfrak{sp}(p) \}$. We fix an oriented two plane $\pi$ in $V_2$ and let $g_\pi$ denote the stability subalgebra. If we identify $T_\pi Q$ with $\text{Hom}(\pi, \pi^\perp)$, we see that the slice representation at $\pi$ contains a submodule isomorphic to $\text{Hom}(\pi, V_1)$. Let $J_0$ and $J_1$ denote $g_\pi$-invariant complex structures in $\pi$ and $V_1$ respectively and denote by $W_k$, $k = 1, 2$, the $g_\pi$-submodules of $\text{Hom}(\pi, V_1)$ consisting of elements $f$ such that $f(J_0 v) = (-1)^k J_1 f(v)$ for all $v \in V_1$. It is easy to see that $W_1$ and $W_2$ are equivalent $g_\pi$-modules. This implies that the $G$-action cannot be coisotropic by Proposition 1.6.

We now prove the following

**Lemma 4.6.** If $g$ is not isomorphic to $g_1$, then $g_2 \subset g$.

**Proof.** Since $g$ is not isomorphic to $g_1$, the kernel of $\rho_1$, given by $g \cap g_2$, is not trivial. Therefore we can write $g = \hat{g}_1 + (g \cap g_2)$, sum of two ideals of $g$, where $\hat{g}_1$ is isomorphic to $g_1$. Now $\rho_2(g \cap g_2)$ is an ideal of $g_2$; if $g_2$ is simple, then
\( \rho_2(\mathfrak{g} \cap \mathfrak{g}_2) = \mathfrak{g} \cap \mathfrak{g}_2 = \mathfrak{g}_2 \), hence \( \mathfrak{g}_2 \subseteq \mathfrak{g} \). If \( \mathfrak{g}_2 \) is not simple, namely \( \mathfrak{g}_2 \cong \mathfrak{sp}(1) + \mathfrak{sp}(p) \) and \( \mathfrak{g}_2 \not\subseteq \mathfrak{g} \), then \( \rho_2(\mathfrak{g}_1) \) is a nontrivial ideal of \( \mathfrak{g}_2 \). We note that \( \rho_2(\mathfrak{g}_1) \) cannot coincide with \( \mathfrak{g}_2 \); indeed, if so, we have \( \mathfrak{g}_1 \cong \mathfrak{sp}(1) + \mathfrak{sp}(p) \cong \mathfrak{g}_2 \) and \( \rho_2|_{\mathfrak{g}_1} \) is an isomorphism; since \( \ker \rho_2 = \mathfrak{g} \cap \mathfrak{g}_1 \) is an ideal of \( \mathfrak{g} \) and \( \mathfrak{g} \cap \mathfrak{g}_1 \cap \mathfrak{g}_1 = \{0\} \), we have \( \ker \rho_2 = \{0\} \) and \( \dim \mathfrak{g} = \dim \mathfrak{g}_2 = \dim \mathfrak{g}_1 \), forcing \( \mathfrak{g} \cong \mathfrak{g}_1 \), a contradiction.

Therefore we have the following possibilities:

- (a) \( \rho_2(\mathfrak{g}_1) = \mathfrak{sp}(1) \). Then \( \mathfrak{g} \cap \mathfrak{g}_2 = \mathfrak{sp}(p) \). Now \( \mathfrak{g}_1 \cong \mathfrak{g}_1 \) contains an ideal isomorphic to \( \mathfrak{sp}(1) \), hence we have the following subcases:
  - (a1) \( \mathfrak{g} = \mathfrak{sp}(1) + \mathfrak{sp}(p) \), where \( \rho_1(\mathfrak{g}) = \mathfrak{sp}(1) \) acting on \( \mathbb{R}^3 \) by Lemma 4.5, \( \rho_2(\mathfrak{g}) = \mathfrak{sp}(1) + \mathfrak{sp}(p) \);
  - (a2) \( \mathfrak{g} = \mathfrak{sp}(1) + \mathfrak{sp}(p) + \mathfrak{sp}(q) \), where \( \mathfrak{g}_1 = \mathfrak{sp}(1) + \mathfrak{sp}(q) \) and \( \mathfrak{g}_2 = \mathfrak{sp}(1) + \mathfrak{sp}(p) \).

- (b) \( \rho_2(\mathfrak{g}_1) = \mathfrak{sp}(p) \). Then we have the following subcases:
  - (b1) \( \mathfrak{g} = \mathfrak{sp}(1) + \mathfrak{sp}(p) \), where \( \mathfrak{g}_1 = \mathfrak{sp}(p) \), \( p = 1, 2 \);
  - (b2) \( \mathfrak{g} = \mathfrak{sp}(1) + \mathfrak{sp}(p) + \mathfrak{sp}(1) \) and \( \mathfrak{g}_1 \cong \mathfrak{g}_2 \).

We now show that the above cases do not yield coisotropic actions. Indeed, for (a1), if we fix a two plane \( \pi \subset \mathbb{R}^3 = V_1 \), then \( \rho_2(\mathfrak{g}_\pi) \subset \mathbb{R} + \mathfrak{sp}(p) \) leaves a complex structure invariant and the same arguments as in Lemma 4.5 can be applied. Cases (a2) and (b2) can be dealt with similarly. As for (b1) a simple computation shows that the cohomogeneity \( c \) of \( G \) is at least 4 if \( p = 1 \) and 9 if \( p = 2 \), contradicting the fact that \( c \) does not exceed the rank of \( G \). □

Therefore, if \( \mathfrak{g} \) is not isomorphic to \( \mathfrak{g}_i \) for \( i = 1, 2 \), then \( \mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 \).

**Lemma 4.7.** If \( \mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 \), then

- (i) the action of \( G \) on \( Q_{n-2} \) is coisotropic if and only if
  \[ \mathfrak{g}_i \in \{ \mathfrak{so}(V_i), \mathfrak{G}_2, \mathfrak{spin}(7), \mathfrak{spin}(9), \mathfrak{sp}(1) + \mathfrak{sp}(p) \} \]
  for \( i = 1, 2 \);

- (ii) the action of \( G \) on \( Q_{n-2} \) is polar if and only if
  \[ \mathfrak{g}_i \in \{ \mathfrak{so}(V_i), \mathfrak{G}_2, \mathfrak{spin}(7) \} \]
  for \( i = 1, 2 \).

**Proof.** (i) We use the list (*) together with Lemma 4.5.

Now if \( \mathfrak{g}_i \in \{ \mathfrak{so}(p), \mathfrak{G}_2, \mathfrak{spin}(7) \} \) for \( i = 1, 2 \), then it can be easily checked that \( G \) has the same orbits in the quadric \( Q = Q_{n-2} \) as \( \text{SO}(V_1) \times \text{SO}(V_2) \), which acts on \( Q \) hyperpolarly, hence coisotropically.

Now we suppose that, say, \( \mathfrak{g}_1 = \mathfrak{spin}(9) \). We choose an oriented two plane \( \pi \subset V_2 \) such that the orbit \( G \cdot \pi \) is complex; then the stability subalgebra \( \mathfrak{g}_\pi = \mathfrak{spin}(9) + (\mathfrak{g}_2)_\pi \) and \( \text{rank}(\mathfrak{g}_\pi) = \text{rank}(\mathfrak{g}_2) \) since \( G \cdot \pi \) is a flag manifold.

We first claim that the \( \mathfrak{g}_\pi \)-action on \( \pi \) is not trivial.

Indeed, when \( \mathfrak{g}_2 = \mathfrak{so}(V_2), \mathfrak{G}_2 \) or \( \mathfrak{spin}(7) \), then our claim follows from the fact the groups \( \text{SO}(V_2), \mathfrak{G}_2 \) and \( \mathfrak{spin}(7) \) act transitively on the Stiefel manifold of two frames in \( V_2 \) (see e.g. [On], p. 90, 93). When \( \mathfrak{g}_2 = \mathfrak{spin}(9) \), then the action of \( \mathfrak{spin}(9) \)
on $Q_{14}$ has cohomogeneity one and we claim that there exists only one complex singular orbit in $Q_{14}$ given by $G \cdot \pi = \text{Spin}(9)/U(4)$, where $U(4)$ acts nontrivially on $\pi$. Indeed, we see the spin representation of Spin(9) as the isotropy representation of the Cayley plane $\mathbb{P}_2(\mathbb{O}) = F_4/\text{Spin}(9)$ at the origin $o$ and, if we choose a two plane $\pi$ inside the tangent space $W \cong \mathbb{R}^8$ at $o$ of a projective line in $\mathbb{P}_2(\mathbb{O})$ through $o$, then the stabilizer Spin(9)$_\pi$ leaves $W$ invariant, since two different projective lines cannot be tangent along a two plane; this means that Spin(9)$_\pi$ is contained in Spin(8) and therefore it covers $T^1 \times SO(6) \subset SO(W)$. Hence Spin(9)$_\pi = U(4)$ and it acts nontrivially on $\pi$. Moreover, since $\chi(Q_{14}) = 16$ is the sum of the Euler characteristics of the singular orbits in $Q_{14}$, and since $\chi(\text{Spin}(9)/U(4)) = 16$, we see that there is exactly one complex singular orbit $G \cdot \pi = \text{Spin}(9)/U(4)$. When $g_2 = \mathfrak{sp}(1) + \mathfrak{sp}(p)$, then the action of $\text{Sp}(1) \cdot \text{Sp}(p)$ on $Q_{4p-2}$ has cohomogeneity one; if $\pi$ is a complex line, viewed as a real two plane, inside a quaternionic line in $\mathbb{H}^p$, then it is easy to see that $(g_2)_\pi = \mathbb{R}^2 + \mathfrak{sp}(p-1)$ acts nontrivially on $\pi$. The same argument as in the previous case shows that $\text{Sp}(1) \cdot \text{Sp}(p)$ has only one complex singular orbit in $Q_{4p-2}$.

So we may write $(g_2)_\pi = \mathbb{R} + h_2$, where $h_2$ denotes the kernel of the $(g_2)_\pi$-action on $\pi$. The slice representation of $(g_2)_\pi$ on $N_{\pi}$ splits as $(\mathbb{R}^2)^* \otimes V_1 + N'$, where $N'$ is acted on by $\mathbb{R} + h_2$; note that $N' = \{0\}$ if and only if $g_2 \in \{\mathfrak{so}(p), \mathfrak{O}_2, \mathfrak{spin}(7)\}$. Now $(\mathbb{R}^2)^* \otimes V_1$ is acted on by $\mathbb{R} + \mathfrak{spin}(9)$, while $h_2$ acts trivially on it; the representation of $\mathbb{R} + \mathfrak{spin}(9)$ on $(\mathbb{R}^2)^* \otimes V_1$ is coisotropic, see [Ka] or notice that a regular subalgebra is given by $\mathfrak{su}(3)$ of corank three and the representation has cohomogeneity three (see [Ya], p. 324). If $g_2 \in \{\mathfrak{so}(p), \mathfrak{O}_2, \mathfrak{spin}(7)\}$, then the $G$-action on the quadric $Q$ is coisotropic; indeed, a regular isotropy subalgebra of $\mathfrak{g}$ is $\mathfrak{su}(3) + h_2$ of corank three in $\mathfrak{g}_\pi$, hence of corank three in $\mathfrak{g}$, while $N' = \{0\}$ implies that the cohomogeneity of the $G$-action on $Q$ is the cohomogeneity of $\mathbb{R} + \mathfrak{spin}(9)$ acting on $(\mathbb{R}^2)^* \otimes V_1$.

If $g_2 = \mathfrak{spin}(9)$, then $(g_2)_\pi = \mathbb{R} + \mathfrak{su}(4)$, $N' = \mathbb{R}^8$ and $\mathfrak{su}(4)$ acts on it by cohomogeneity one; hence a regular isotropy subalgebra of $\mathfrak{g}$ is $\mathfrak{su}(3) + \mathfrak{su}(3)$ of corank four, while the cohomogeneity of the $G$-action is also four.

If $g_2 = \mathfrak{sp}(1) + \mathfrak{sp}(p)$ ($p \geq 2$), then $(g_2)_\pi = \mathbb{R}^2 + \mathfrak{sp}(p-1)$ and $N' = \mathbb{R}^{4p-4}$. So $(g_\pi) = \mathfrak{su}(3) + \mathbb{R} + \mathfrak{sp}(p-2)$ of corank four in $\mathfrak{g}$ and the cohomogeneity of the $G$-action on $Q$ is also four.

Next we suppose that $g_1 = \mathfrak{sp}(1) + \mathfrak{sp}(p)$. If we argue in the same way as in the case $g_2 = \mathfrak{spin}(9)$, we see that we only need to prove that the representation of $\mathbb{R} + \mathfrak{sp}(1) + \mathfrak{sp}(p)$ on $(\mathbb{R}^2)^* \otimes \mathbb{R}^{4p}$ is coisotropic (see also [Ka]). This representation has cohomogeneity three (see [Ya], p. 318); a regular isotropy subalgebra $\mathfrak{k}$ contains a copy of $\mathfrak{sp}(p-2)$ as an ideal, so that we can write $\mathfrak{k} = \mathfrak{k}' + \mathfrak{sp}(p-2)$ and a simple dimension count shows that $\mathfrak{k}' = \mathbb{R}$. So the corank of $\mathfrak{k}$ in $\mathbb{R} + \mathfrak{sp}(1) + \mathfrak{sp}(p)$ is also three and the representation is coisotropic.

(ii) Using the same arguments as in case (i), we only need to show that the representation of $\mathbb{R} + \mathfrak{spin}(9)$ on $\mathbb{R}^2 \otimes \mathbb{R}^{16}$ and of $\mathbb{R} + \mathfrak{sp}(1) + \mathfrak{sp}(p)$ on $\mathbb{R}^2 \otimes \mathbb{R}^{4p}$ are not polar. Actually these representations are neither isotropy representations of irreducible symmetric spaces nor appear in the list of irreducible polar representations which are not isotropy representations of symmetric spaces (see the table from [EH] in the proof of Lemma 2.7). □

Next we suppose that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_1$. 
If \( g_1 \) is simple, then \( \rho_1, \rho_2 \) are isomorphisms and \( g_1 \simeq g_2 \). In particular, \( g_1 \in \{ \mathfrak{so}(V_1), \mathfrak{so}_2, \mathfrak{spin}(7), \mathfrak{spin}(9) \} \). Now, if \( \rho_1 \) is equivalent to \( \rho_2 \), we select a two plane \( \pi \subset V_1 \) and observe that the slice representation of \( \mathfrak{g}_\pi \) contains \( \text{Hom}(\pi, V_2) \) as a complex submodule; if we denote by \( \pi \) the same oriented two plane in \( V_2 \), then \( \text{Hom}(\pi, \pi) \) contains a trivial \( \mathfrak{g}_\pi \)-submodule. This means that the slice representation contains a trivial complex submodule, preventing the \( G \)-action from being coisotropic. If on the other hand \( \rho_1 \) is not equivalent to \( \rho_2 \), then \( g = \mathfrak{spin}(7) \) or \( g = \mathfrak{spin}(9) \); in this case a simple computation shows that the dimension of a Borel subalgebra of \( \mathfrak{g}^C \) is strictly less than the complex dimension of the quadric and the action is therefore not coisotropic.

If \( g_1 \) is not simple, i.e. \( g_1 = \mathfrak{sp}(1) + \mathfrak{sp}(p) \), then \( g_2 \) can be \( \mathfrak{sp}(1) \simeq \mathfrak{so}(3), \mathfrak{sp}(2) \approx \mathfrak{so}(5) \) or \( \mathfrak{sp}(1) + \mathfrak{sp}(p) \). The first two cases already appeared in the proof of Lemma 4.6 and the last case can be ruled out with the same arguments.

In order to complete the proof of Theorem 1.3, we need to prove the following lemma.

**Lemma 4.8.** Let \( G \) be a compact Lie group acting reducibly on \( \mathbb{R}^n \) \((n \geq 5)\) and coisotropically on the quadric \( Q_{n-2} \). Then the number of irreducible submodules of \( \mathbb{R}^n \) is at most two.

**Proof.** We assume that \( \mathbb{R}^n \) contains at least three irreducible submodules \( V_1, V_2, V_3 \), none of which is trivial. We then restrict ourselves to the coisotropic action of \( G \) on the quadric corresponding to the \( G \)-module \( V = \sum_{i=1}^3 V_i \). We can apply the results obtained so far to the quadrics corresponding to the sums \( V_i + V_j, i, j \in \{1, 2, 3\} \), showing that \( g = g_1 + g_2 + g_3 \), where each \( g_i \in \{ \mathfrak{so}(V_i), \mathfrak{so}_2, \mathfrak{spin}(7), \mathfrak{spin}(9), \mathfrak{sp}(1) + \mathfrak{sp}(p) \} \). We consider a two plane \( \pi \subset V_1 \) and its isotropy subalgebra \( \mathfrak{g}_\pi = (g_1)_{\pi} + g_2 + g_3 \). The slice representation of \( \mathfrak{g}_\pi \) contains \( \text{Hom}(\pi, V_2 + V_3) \) as a complex submodule which is coisotropic; this means that \( \mathfrak{g}_\pi \) acts nontrivially on \( \pi \), since otherwise \( \text{Hom}(\pi, V_2 + V_3) \) would contain equivalent \( \mathfrak{g}_\pi \)-submodules, contradicting Proposition 1.6. The action of \( \mathfrak{g}_\pi \) on \( \text{Hom}(\pi, V_2 + V_3) \) is the action of \( \mathfrak{h} := \mathbb{R} + g_2 + g_3 \) on \( (\mathbb{R}^2)^* \otimes (V_2 + V_3) \). We set \( W_i := (\mathbb{R}^2)^* \otimes V_i \) and \( \mathfrak{t}_i \) the regular isotropy subalgebras of \( \mathbb{R} + g_i \) acting on \( W_i \) for \( i = 2, 3 \); it is known that the regular isotropy subalgebra of \( \mathbb{R} + \mathfrak{so}(n) \) acting on \( \mathbb{R}^2 \otimes \mathbb{R}^n \) is \( \mathfrak{so}(n-2) \), hence \( \mathfrak{t}_i \subset g_i \) for \( i = 2, 3 \).

Now a regular isotropy subalgebra \( \mathfrak{t} \) of \( \mathfrak{h} \) acting on \( \text{Hom}(\pi, V_2 + V_3) \) is given by \( \mathfrak{t}_1 + \mathfrak{t}_2 \). Since the \( \mathbb{R} + g_i \)-actions on \( W_i \) are coisotropic, we have that the cohomogeneity \( c \) of the \( \mathfrak{h} \)-action on \( \text{Hom}(\pi, V_2 + V_3) \) is given by

\[
c = \dim(W_1 + W_2) - \dim(g_1 + g_2) - 1 + \dim(\mathfrak{t}_1 + \mathfrak{t}_2)
= \dim W_1 - \dim(\mathbb{R} + g_1) + \dim \mathfrak{t}_1
+ \dim W_2 - \dim(\mathbb{R} + g_2) + \dim \mathfrak{t}_2 + 1
= \text{rk}(\mathbb{R} + g_1) - \text{rk}(\mathfrak{t}_1) + \text{rk}(\mathbb{R} + g_2) - \text{rk}(\mathfrak{t}_2) + 1
= \text{rk}(\mathbb{R} + g_1 + g_2) - \text{rk}(\mathfrak{t}_1 + \mathfrak{t}_2) + 2.
\]

This means that the \( G_\pi \) action on \( \text{Hom}(\pi, V_2 + V_3) \) is not coisotropic. Hence the \( G \)-action on \( Q_{n-2} \) is not coisotropic. \( \square \)
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