ALGEBRIZATION OF SOME COMPLETE MODULES

MOHSEN ASGHARZADEH

Abstract. We give classes of finitely generated \( \hat{R} \)-modules that are extended from a finitely generated \( R \)-module. Applications are given.

1. Introduction

In this note \( (R, m) \) is a commutative noetherian local ring. By completion we mean completion with respect to \( m \)-adic topology. By a complete module we mean a finitely generated \( \hat{R} \)-module. Also, a complete module \( \mathcal{M} \) is called algebraic if there is a finitely generated \( R \)-module \( M \) such that \( \hat{M} \cong \mathcal{M} \) (some times \( \mathcal{M} \) is called extended from a finitely generated \( R \)-module). A complete module such that each of its direct summand is algebraic is called strongly algebraic.

Observation 1.1. Let \( R \) be a local domain whose integral closure \( \overline{R} \) is a finite \( R \)-module but \( R \) is not local. If \( R \) is of isolated singularity, then \( \overline{R} \) is algebraic but not strongly algebraic.

Let \( R \) be a 1-dimensional local ring with algebraically closed residue field such that \( R_{\text{red}} \) is analytically irreducible. Algebrization helps to show \( G_0(R) \cong \oplus_{\text{min}(R)} \mathbb{Z} \). One may call this as a Herzog’s conjecture. In the case \( R \) is a 1-dimensional local domain this is in \cite{25} Ex. II.6.9]. As another application of algebrization, see Corollary 2.6.

A module \( \mathcal{M} \) is called weakly algebraic, if it is a direct summand of an algebraic module. The following extends the recent result \cite{5} Proposition 4.3] by dropping torsion-free assumption via a new argument:

Observation 1.2. Let \( R \) be analytically unramified and of dimension one. Then any complete module is weakly algebraic. Conversely, assume \( R \) is essentially of finite type over a prime field and any complete \( \hat{R} \)-module is weakly algebraic, then \( \text{dim } R \leq 1 \).

This has a connection to derived category (see Proposition 3.5 and Corollary 3.6). Also, we show:

Observation 1.3. Let \( R \) be \( d \)-dimensional and \( \mathcal{M} \) a complete module of finite projective dimension. Suppose one of the following holds: i) \( d = 1 \), ii) \( d = 2 \) and \( \mathcal{M} \) is torsion-less, iii) If \( d = 3 \) and \( \mathcal{M} \) is reflexive. Then \( \mathcal{M} \) is strongly algebraic.

Despite of a restriction on projective dimension, part i) may compare with a result of Levy and Odenthal \cite{11} where they worked over 1-dimensional analytically unramified rings. Also, part ii) may compare with a result of Weston \cite{26} who worked over 2-dimensional analytically normal domains. The final section collects some remarks on the algebrization of formal regular functions. Let \( \mathcal{X} = \text{Spec}(R) \setminus \{m\} \) and \( (\hat{\mathcal{X}}, \mathcal{O}_{\hat{\mathcal{X}}}) \) denote the formal completion of \( \mathcal{X} \) along with \( \mathcal{Y} := \mathcal{V}(a) \setminus \{m\} \). Following Hironaka and

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2010 Mathematics Subject Classification. Primary 13B35, 13J10.
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Key words and phrases. algebraic modules; completion; descent-method; Ext-modules; formal regular functions.
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Matsumura \cite{17}, \( \mathcal{Y} \) is G1 in \( \mathcal{X} \) if \( \text{H}^0(\mathcal{X}, \mathcal{O}_\mathcal{X}) \xrightarrow{\sim} \text{H}^0(\hat{\mathcal{X}}, \mathcal{O}_{\hat{\mathcal{X}}}) \). We present situations that \( \mathcal{Y} \) is (not) G1 in \( \mathcal{X} \). For example, over complete Cohen-Macaulay local ring of dimension \( d > 1 \) we show \( \mathcal{Y} \) is G1 in \( \mathcal{X} \) if and only if \( \text{cd}(a) \leq d - 2 \). Recall from \cite{3}, Corollary 4.2 that the Cohen-Macaulay property of a nonzero module \( M \) is equivalent with the following property

\[ \text{fgrade}(b, M) + \text{cd}(b, M) = \dim(M) \quad \forall b \triangleleft R \quad (\ast) \]

Here, \( \text{fgrade}(b, -) \) stands for the notion of formal grade. It was conjectured in \cite{3} page 1098] that the Cohen-Macaulay condition in \( (\ast) \) can be replaced by a weaker condition if we put some restrictions on the family of ideals:

**Observation 1.4.** Let \( R \) be a local domain of dimension \( d \geq 3 \) and \( M \) a torsion-free satisfies Serre’s \( S_r \) condition. Suppose \( R \) is complete with respect to \( a \)-adic topology and \( R \) is homomorphic image of a Gorenstein ring. If \( \text{cd}(a, M) \leq d - r \), then \( \text{fgrade}(a, M) \geq r \).

The case \( r = 2 \) is due to Faltings \cite{8} Corollary 2 where he worked with the assumptions \( \mu(a) \leq d - 2 \) and homomorphic image of a regular ring instead of \( \text{cd}(a) \leq d - 2 \) and homomorph image of a Gorenstein ring: So, \( \text{H}^0(\hat{\mathcal{X}}, \hat{\mathcal{F}}) \simeq \text{H}^0(\mathcal{X}, \mathcal{F}) \), where \( \mathcal{F} \) is a sheaf associated to \( M \) over \( \mathcal{X} \) and \( \hat{\mathcal{F}} \) is its formal completion along with \( \mathcal{Y} \). We should remark that Observation 1.4 uses Faltings’ argument on the algebrization of formal vector bundles. In this regards, we present some examples.

### 2. Algebraic Modules

The notation \( \text{mod}(-) \) stands for the category of finitely generated modules. Free modules are algebraic modules. Also, the class of finite length modules is algebraic. More generally:

**Discussion 2.1.** i) Let \( \mathcal{M} \) be a complete module. Suppose \( \mathcal{M} \) is finitely generated as an \( R \)-module. Then \( \mathcal{M} \) is strongly algebraic. Indeed, let \( \mathcal{N} \) a direct summand of \( \mathcal{M} \). Note that \( \mathcal{N} \) is finitely generated as an \( R \)-module. In view of \cite{10} Theorem 1.8, \( \mathcal{N} \simeq \mathcal{N} \otimes_R \widehat{R} \). So, \( \mathcal{N} \) is algebraic. In particular, any finite length \( \widehat{R} \)-module is algebraic.

ii) Let \( A \) an artinian \( R \)-module. This is well-known that \( A \) equipped with the structure of an \( \widehat{R} \)-module such that we recover the origin \( R \)-module structure from \( R \to \widehat{R} \). In this regard \( A \simeq A \otimes_R \widehat{R} \).

iii) Let \( (\hat{R}, m_{\hat{R}}) \) be any complete ring and \( a \) primary to the maximal ideal. Then \( a \) is algebraic. Indeed, we look at \( 0 \to a \to \hat{R} \to \hat{R}/a \to 0 \). Since \( \hat{R}/a \) is of finite length it extended from an \( R \)-module (see Discussion 2.1). Also, \( \text{Hom}_R(\hat{R}, \hat{R}/a) \simeq \hat{R}/a \) is of finite length as an \( \hat{R} \)-module and as an \( R \)-module. In particular, \( \text{Hom}_R(\hat{R}, \hat{R}/a) \) is finitely generated as an \( R \)-module. In view of \cite{10} Proposition 3.2(ii)] we see \( a \) is algebraic.

**Remark 2.2.** Suppose any ideal of a complete ring is algebraic. Is any complete module algebraic?

i) The question has negative answer even over 2-dimensional regular rings.

ii) Let \( R \) be a two-dimensional complete normal local domain. Suppose any ideal is algebraic. Then any torsion-free module is algebraic.

**Proof.** i) To see a counter-example, we look at \( R := \mathbb{Q}[X, Y]_{(X, Y)} \). It is shown in \cite{10} Example 3.6] that there are non algebraic modules over \( \widehat{R} \). Here, we check that any ideal of \( \widehat{R} \) is algebraic. Indeed, flatness behaves nicely with respect to intersection. This implies that intersection of any algebraic collection
of ideals is algebraic. We apply these along with the primary decomposition to reduce things to the primary ideals. By Discussion 2.1(iii) we may assume in addition that the height of the primary ideal is one. Recall that height-one primary ideal in a UFD is principal. It remains to recall that principal ideals over an integral domain are free and that free modules are algebraic (for a modern argument, see Fact A in Example 2.3 below).

ii) Any torsion-free module $M$ has a free submodule as $F$ such that $M/F$ is an ideal $I$ of $A$. First, we look at the sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ to see $\text{Ext}^1_R(I, R) \simeq \text{Ext}^2_R(R/I, R)$. Note that $\text{gl. dim}(R_p) = 1$ for all $p \in \text{Spec}(R) \setminus \{m_R\}$. From this $\text{Supp}(\text{Ext}^2_R(R/I, R)) \subset \{m_R\}$. We conclude by this that $\text{Ext}^1_R(I, R) \simeq \text{Ext}^2_R(R/I, R)$ is of finite length. Let $R$ be such that $R = \hat{R}$. There is a short exact sequence $D := 0 \rightarrow F \rightarrow M \rightarrow I \rightarrow 0$. Then $D \in \text{Ext}^1_R(I, F)$. Free modules are algebraic: there is a free $R$-module $F$ such that $\hat{F} \simeq F$. By assumption $I \simeq \hat{I}$. Since $\text{Ext}^1_R(I, F) \simeq \text{Ext}^1_F(I, \hat{F}) \otimes_R \hat{R}$ we have $\text{Ext}^1_R(I, F)$ is finite length as an $R$-module. In view of Discussion 2.4 $D \in \text{Ext}^1_R(I, F) \simeq \text{Ext}^1_F(I, \hat{F}) \otimes_R \hat{R} \simeq \text{Ext}^1_F(I, F)$. By Yoneda’s definition of $\text{Ext}^1$, there is $D := 0 \rightarrow F \rightarrow M \rightarrow I \rightarrow 0$ such that $D \simeq D \otimes_R \hat{R}$. By 5-lemma, $M \simeq \hat{M}$. □

**Example 2.3.** Let $R$ be a two-dimensional complete normal local domain such that $\text{Cl}(R) \neq 0$. There is a torsion-free algebraic module $M$ with a direct summand $N$ such that $N$ is not algebraic.

**Proof.** Any complete local domain of depth at least two can be realize as a completion of a unique factorization domain. This is in [15]. Let $R$ be a UFD such that its completion is $\hat{R}$. Let $I$ be such that $[I] \in \text{Cl}(\hat{R})$ is nonzero and put $J$ be such that $[J] = [I]^{-1}$. We look at the $R$-modules $M := I \oplus J$ and $N := I$. Note that $0 = [M] \in \text{im}(\text{Cl}(R) \rightarrow \text{Cl}(\hat{R}))$ and $0 \neq [N] \notin \text{im}(\text{Cl}(R) \rightarrow \text{Cl}(\hat{R}))$. We bring the following result of Weston (see [12] Proposition 2.15):

Fact A): Let $L$ be a torsion-free and complete $\hat{R}$-module. Then $L$ is algebraic if and only if $[L] \in \text{im}(\text{Cl}(R) \rightarrow \text{Cl}(\hat{R}))$.

Since $R$ is UFD, its classical group is zero. Thus, $\text{im}(\text{Cl}(R) \rightarrow \text{Cl}(\hat{R})) = 0$. We are going to apply Fact A) to deduce that $M$ is algebraic and that its direct summand $N$ is not algebraic. □

**Discussion 2.4.** There is a useful principle (see [10] Proposition 3.1): Let $M$ and $N$ be complete modules. If two of $\{M, N, M \oplus N\}$ are algebraic, then so the third. We call this the principle of 2 of 3.

We follow Auslander’s approach to show:

**Proposition 2.5.** Let $R$ be a local domain whose integral closure $\overline{R}$ is a finite $R$-module but $\overline{R}$ is not local. If $R$ is of isolated singularity, then $(\overline{R})$ is algebraic but not strongly algebraic.

**Proof.** There are only finitely many prime ideals $\{p_1, \ldots, p_n\} \subseteq \text{Spec}(\overline{R})$ lying over $m$, because $\overline{R}$ is finitely generated over $R$ as a module. Since $\overline{R}$ is not local, $n > 1$. Recall that $\overline{R}$ is indecomposable as an $R$-module but $(\overline{R}) \simeq (\overline{R}_{p_1}) \oplus \cdots \oplus (\overline{R}_{p_n})$ is decomposable as an $\hat{R}$-module. Suppose on the contrary that $(\overline{R}_{p_i})$ is algebraic for some $i_0$. By the principle of 2 of 3 (see Discussion 2.4), $\bigoplus_{i \neq i_0} (\overline{R}_{p_i})$ is algebraic. Let $P$ and $Q$ be finitely generated $R$-modules such that $\hat{P} \simeq (\overline{R}_{p_{i_0}})$ and $\hat{Q} \simeq (\bigoplus_{i \neq i_0} (\overline{R}_{p_i})$. Thus, $(Q \oplus P) \simeq (\overline{R})$. Let $q \in \text{Spec}(R) \setminus \{m\}$. Recall that $(\overline{R}_q) = (\overline{R})_q$. Regular rings are normal. Since $R$ is of isolated singularity we deduce that $\overline{R}_q = (\overline{R}_q) = (\overline{R})_q$. Conclude by this that $\overline{R}$ is locally free on
the punctured spectrum. Since \( \overline{R} \) is locally free, \( \text{Ext}^1_R(\overline{R}, -) : \text{mod} \ (R) \rightarrow \text{mod} \ (R) \) is of finite length. Let \( L \in \text{mod} \ (R) \). Due to Discussion 2.1, \( \text{Ext}^1_R(\overline{R}, L) \cong \text{Ext}^1_R(\overline{R}, L) \otimes_R \hat{R} \). We have

\[
\text{Ext}^1_R(\overline{R}, L) \cong \text{Ext}^1_R(\overline{R}, L) \otimes_R \hat{R} \\
\cong \text{Ext}^1_R((\overline{R}), L \otimes_R \hat{R}) \\
\cong \text{Ext}^1_R((\overline{Q} \oplus P), L \otimes_R \hat{R}) \\
\cong \text{Ext}^1_R((\overline{Q} \otimes_R \hat{R}, L \otimes_R \hat{R}) \oplus \text{Ext}^1_R(P \otimes_R \hat{R}, L \otimes_R \hat{R}).
\]

From this we see \( \text{Ext}^1_R(\overline{Q}, \hat{L}) \cong \text{Ext}^1_R(Q, L) \otimes_R \hat{R} \) is finite length as an \( \hat{R} \)-module. Conclude by this that \( \text{Ext}^1_R(Q, L) \) is finite length as an \( R \)-module. In view of Discussion 2.1,

\[
\text{Ext}^1_R(\overline{Q}, \hat{L}) \cong \text{Ext}^1_R(Q, L) \otimes_R \hat{R} \cong \text{Ext}^1_R(Q, L)
\]

Similarly, \( \text{Ext}^1_R(\overline{P}, \hat{L}) \cong \text{Ext}^1_R(P, L) \otimes_R \hat{R} \cong \text{Ext}^1_R(P, L) \). Combine these together to see

\[
\text{Ext}^1_R(\overline{R}, L) \cong \text{Ext}^1_R(Q, L) \oplus \text{Ext}^1_R(P, L) \ (\ast)
\]

We borrow some lines from Hartshorne's coherent functors [13]; \( \ast \) implies that \( \overline{R} \) and \( P \oplus Q \) are stably equivalent modules. Projective modules are free over local rings. By splitting off free modules, every stable equivalence class has a unique smallest element containing no free direct summands. Since \( \overline{R} \) is minimal, we assume that \( P \) and \( Q \) are also minimal. Thus \( \overline{R} \cong P \oplus Q \). To see a contradiction recall that \( \overline{R} \) is of rank one. Due to this contradiction, we observe that \( \overline{R} \) is not strongly algebraic.

Here, we present two applications of algebraization. Recall that Ferrand and Raynaud constructed 1-dimensional Cohen-Macaulay local rings that are not homomorphic image of Gorenstein local rings.

**Corollary 2.6.** Let \( \hat{R} \) be any 1-dimensional Cohen-Macaulay local ring. Suppose \( \hat{R} \) is generically Gorenstein. Then \( R \) is a homomorphic image of a Gorenstein local ring.

**Proof.** Complete rings admit a canonical module. Let \( \omega_{\hat{R}} \) be the canonical module of \( \hat{R} \). Since \( \hat{R} \) is generically Gorenstein, \( \omega_{\hat{R}} \) has a rank. From this, either \( \omega_{\hat{R}} \) is an ideal of height one or \( \omega_{\hat{R}} = \hat{R} \). The second possibility implies that \( R \) is Gorenstein. Without loss of the generality we may assume that \( \omega_{\hat{R}} \) is an ideal of height one. Since the ring is 1-dimensional, \( \omega_{\hat{R}} \) is primary to the maximal ideal. In view of Discussion 2.1 1iii) \( \omega_{\hat{R}} \) is algebraic. Let \( M \) be such that \( \overline{M} = \omega_R \). It is easy to see that \( M \) is Minimal Cohen-Macaulay. In view of \( (\overline{R}, \overline{M}) \cong \text{Ext}^1_R((\overline{R}), \overline{M}) \cong \text{Ext}^1_R(R/m, M) \), we see \( \text{Ext}^1_R(R/m, M) \cong \overline{R} \), i.e., \( M \) is of type 1. Also, \( \text{Ext}^1_R(R/m, M) \cong \text{Ext}^1_R(\overline{R}, \overline{M}) = 0 \). So, \( \text{Ext}^1_R(R/m, M) = 0 \), i.e., \( M \) is of finite injective dimension. From this, \( R \) admits a canonical module. It turns out that \( R \) is a homomorphic image of a Gorenstein local ring.

Recall that \( G_0(R) := \left( \sum_{\text{max}(R)} Z[M] \right) \).

**Corollary 2.7.** Let \( R \) be a 1-dimensional local ring with algebraically closed residue field such that \( R_{\text{red}} \) is analytically reducible. Then \( G_0(R) \cong \sum_{\min(R)} Z \).

The following argument may not be the best one:

**Proof.** In the case \( R \) is complete and equi-characteristic this is in [16 Proposition 2.2] (the proof works in the complete case). Clearly, the residue field of \( R \) coincides with the residue field of \( R_{\text{red}} \). Since \( G_0(R) = G_0(R_{\text{red}}) \) and \( \min(R) = \min(R_{\text{red}}) \) we may assume that \( R \) is reduced. Clearly, the residue
field of \( R \) coincides with the residue field of \( \hat{R} \). The assignment \([M] \mapsto [M \otimes_R \hat{R}]\) induces a morphism \( \varphi : G_0(R) \to G_0(\hat{R}) \). For simplicity of the reader we bring the following fact:

Fact A): Let \( A \) be a local ring such that both of \( A \) and \( \hat{A} \) are of isolated singularity. Then \( \varphi : G_0(A) \to G_0(\hat{A}) \) is injective, see [20, Claim 5.1].

Both of \( R \) and \( \hat{R} \) are of isolated singularity. We apply Fact A) along with [16, Proposition 2.2] to see \( \varphi : G_0(R) \leftarrow G_0(\hat{R}) \simeq \bigoplus_{\text{min}(\hat{R})} \mathbb{Z} \). In fact [16, Proposition 2.2] shows that \( \hat{\mathbb{A}}_R = 0 \). Since the map is injective, we have \( \hat{\mathbb{A}}_R = 0 \). Any module has a filtration with prime factors. By prime filtration,

\[
G_0(R) = \langle [R/p] : p \in \text{spec}(R) \rangle = \langle [R/p] : p \in \text{min}(R) \rangle.
\]

To finish the proof, we claim that \( \{[R/p] : p \in \text{min}(R)\} \) are \(*\)-linearly independent. To this end, and in view of Claim A), we need to show \( \{\varphi([R/p]) : p \in \text{min}(R)\} \) are \(*\)-linearly independent. Note that \( \hat{R}/\hat{p} \) has a filtration with prime factors. This prime factors belong to support of \( \hat{R}/\hat{p} \). On the other hand \( \text{Supp}(\hat{R}/\hat{p}) = \text{Ass}_{\hat{R}}(R/p \otimes_R \hat{R}) \cup \{\hat{m}\} \). Since \( \hat{\mathbb{A}}_R = 0 \) we see that \( \{\varphi([R/p]) : p \in \text{Ass}(\hat{R}/\hat{p})\} \). In view of [21, Theorem 23.2], \( \text{Ass}_{\hat{R}}(R/p \otimes_R \hat{R}) = \{P \in \text{Spec}(\hat{R}) : P \cap R = p\} \). These belongs to \( \text{min}(\hat{R}) \). Note that if \( p \) are \( q \) are distinct minimal primes, then \( \text{Ass}_{\hat{R}}(R/p \otimes_R \hat{R}) \cap \text{Ass}_{\hat{R}}(R/q \otimes_R \hat{R}) = \emptyset \).

We conclude by this that \( \{\varphi([R/p]) : p \in \text{min}(R)\} \) are \(*\)-linearly independent. The proof is now complete.

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3. \text{ WEAKLY AND STRONGLY ALGEBRAIC MODULES}
\]

Evans asked which local rings satisfy Krull-Remak-Schmidt property.

Observation 3.1. Let \( R \) be a local ring that fails Krull-Remak-Schmidt property. Then there is a weakly algebraic module which is not algebraic.

Proof. By \( X/Y \) we mean \( X \) is a direct summand of \( Y \). Due to the assumption there are finite family of indecomposable modules \( \{M_i\}_i \) and \( \{N_j\}_j \) such that \( \bigoplus_i M_i \simeq \bigoplus_j N_j \) but \( M_i \) is not isomorphic to each of \( N_j \). Since completion commutes with finite direct sum, \( \bigoplus_i \hat{M}_i \simeq \bigoplus_j \hat{N}_j \). Suppose on the contrary that \( \hat{M}_i \) is indecomposable as an \( \hat{R} \)-module. As complete local rings satisfy Krull-Remak-Schmidt (see [12, Corollary 1.10]), we should have \( \hat{M}_1 \simeq \hat{N} \) where \( \hat{N} \) is for some \( j \). Thus, \( \hat{M}_1 \simeq \hat{N} \).

Fact A): (see the proof of [10, Proposition 3.1]) Let \( A \to B \) be a flat ring homomorphism, \( L_1 \) and \( L_2 \) be two finitely generated \( A \)-modules. If \( L_1 \otimes_B L_2 \otimes_B \), then \( L_1 \otimes L_2 \).

In view of Fact A), \( M_1 | N_j \). This contradiction shows that \( \hat{M}_1 \) is decomposable: \( \hat{M}_1 \simeq A \oplus \hat{B} \) for some \( \hat{R} \)-modules \( A \) and \( B \). It is now clear that both of \( A \) and \( B \) are weakly algebraic. Suppose on the contradiction both of \( A \) and \( B \) are algebraic. There are finitely generated \( R \)-modules \( A \) and \( B \) such that \( A \simeq \hat{A} \) and \( B \simeq \hat{B} \). This shows that \( \hat{A} \simeq \hat{M}_1 \). Another use of Fact A) shows that \( A \simeq \hat{M}_1 \). This contradicts the fact that \( M_1 \) is indecomposable. By principle 2 of 3, \( A \) (resp. \( B \)) is a weakly algebraic module and it is not algebraic.

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\Box
\]

Corollary 3.2. (Levy-Odenthal) Let \( R \) be a 1-dimensional analytically irreducible local ring. Then \( R \) satisfies Krull-Remak-Schmidt property.

Proof. By the above observation its enough to show any module is algebraic. This is in [11] (see [10, Proposition 3.3] for a simple proof).

\[
\Box
\]
We need the following implicit result from [10]:

**Fact 3.3.** Let $\mathcal{M}$ be a complete module locally free over punctured spectrum. Then $\mathcal{M}$ is weakly algebraic. Indeed, the map $R \to \hat{R}$ factors through the henselization $R^h$. By a result of Elkik (see [12, Theorem 10.10]) there is a finitely generated $R^h$-module $M_h$ such that $M_h \otimes_R \hat{R} \simeq \mathcal{M}$. By [10, Corollary 3.5] there is a finitely generated $R$-module $M$ such that $M_h \oplus L_h \simeq M \otimes_R R^h$ for some $R^h$-module $L_h$. So,

$$\mathcal{M} \oplus (L_h \otimes_{R^h} \hat{R}) \simeq (M_h \otimes_{R^h} \hat{R}) \oplus (L_h \otimes_{R^h} \hat{R}) \simeq (M \otimes_R R^h) \otimes_{R^h} \hat{R} \simeq M \otimes_R \hat{R}.$$ 

**Proposition 3.4.** Let $(R, \mathfrak{m})$ be a local ring. The following assertions are true:

i) Suppose $R$ is analytically unramified and of dimension one. Then any complete module is weakly algebraic.

ii) Suppose $R$ is essentially of finite type over a prime field and any complete $\hat{R}$-module is weakly algebraic. Then $\dim(\hat{R}) \leq 1$.

**Proof.** i) Let $\mathcal{M}$ be any complete module. Since $\hat{R}$ is reduced, $\hat{R}$ satisfies Serre’s condition $R_0$. This means that $\hat{R}$ is regular over $\mathcal{X} := \text{Spec}(\hat{R}) \setminus \{\hat{m}\}$. Zero-dimensional regular local rings are field. In particular, any module over a such ring is a vector space. We conclude by this that $M_p$ is free over $\hat{R}_p$ for all $p \in \mathcal{X}$. In view of Fact 3.3, $\mathcal{M}$ is weakly algebraic.

ii) Suppose $R$ is essentially of finite type over a prime field and any complete $\hat{R}$-module is weakly algebraic. Then the prime field in characteristic zero is $\mathbb{Q}$ and in the characteristic $p$ is $\mathbb{F}_p$. Since $R$ is essentially of finite type over a prime field, we conclude that $R$ is countable. We are going to use a trick that we learnt from [10, Example 3.6]. As $R$ is countable, the class of finitely generated modules is countable up to the isomorphism class. This implies that the class of algebraic modules is countable, again up to the isomorphism class. The same thing holds for weakly algebraic modules. Suppose on the contradiction that $\dim R > 1$. It follows by countable prime avoidance for complete rings that there are uncountable family of height one prime ideals $\{\mathcal{P} : \mathcal{P} \in \Gamma\}$. This implies that $\{\hat{R}_p : \mathcal{P} \in \Gamma\}$ is an uncountable family of complete modules up to isomorphism. So, there is a complete module that is not weakly algebraic, a contradiction.

If $x \in \hat{R}$ is not zero-divisor then $(x) \simeq \hat{R}$. So, it is algebraic. What can say when $x$ is zero-divisor? Let us revisit $R_0 := C[x, y]|_{(x, y)}/(y^2 - x^3 - x^2)$ and recall that $\hat{R}_0 \simeq C[[u, v]]/(uv)$. It is shown in [19, Example A.5] that $(v)$ is not algebraic. They showed that $(u) \oplus (v)$ is algebraic via three different arguments: 1) Thomason’s localization theorem, 2) Levy-Odenthal’s criterion, and 3) a direct calculation over a triangulated category.

**Proposition 3.5.** Let $\hat{R}$ be any 1-dimensional reduced local ring. Let $0 \neq x \in \hat{R}$ be a zero-divisor. Let $p \in \text{Ass}(\hat{R})$ be such that $x \in p$. Then $\hat{R}/(x) \oplus \hat{R}/(y)$ is algebraic for any $y \in \bigcap_q \text{Ass}(\hat{R}) \setminus p$.

**Proof.** Since $\hat{R}$ has a zero divisor $0 \notin \text{Ass}(\hat{R})$. Reduced rings satisfy Serre’s condition $S_1$. Hence $\text{Ass}(\hat{R}) = \min(\hat{R})$. If $\text{Ass}(\hat{R})$ were be singleton e.g. $\text{Ass}(\hat{R}) = \{p\}$ for some $P$, then we should had $P \neq 0$ (because there are non-trivial zero-divisors) and so there are nilpotent elements. But the ring is reduced. This shows that $|\text{Ass}(\hat{R})| \geq 2$. Let $\text{Ass}(\hat{R}) = \min(\hat{R}) =: \{p, q_1, \ldots, q_n\}$. Due to the primary decomposition, $(x) \cap (y) \subseteq p \cap (\bigcap q_i) = \text{nil}(\hat{R}) = 0$. Look at the exact sequence

$$0 \longrightarrow \hat{R} \longrightarrow \hat{R}/(x) \oplus \hat{R}/(y) \longrightarrow \hat{R}/(x, y) \longrightarrow 0 \quad (*)$$

Recall that $p \cap (\bigcap q_i) = 0$. As $x \notin q_i$ and $y \notin p$, we have $\text{rad}(x, y) = m$. Thus, $\hat{R}/(x, y)$ is of finite length. Due to Discussion 2.1 $\hat{R}/(x, y)$ is algebraic. Suppose it extended from an $R$-module $M$. 

Since $\text{Ext}^1_\mathcal{R}(\mathcal{R}/(x,y), \mathcal{R})$ is of finite length as an $\hat{\mathcal{R}}$-module, we have $\text{Ext}^1_\mathcal{R}(M, R) \simeq \text{Ext}^1_\mathcal{R}(M, \mathcal{R}) \otimes \mathcal{R} \hat{\mathcal{R}} \simeq \text{Ext}^1_\mathcal{R}(\mathcal{R}/(x,y), \mathcal{R})$. We deduce from Yoneda’s definition of $\text{Ext}^1$ that there is an exact sequence $0 \to R \to N \to M \to 0$ such that its completion is ($\ast$). By 5-lemma $\hat{N} \simeq \mathcal{R}/(x) \oplus \mathcal{R}/(y)$. □

The singular category of a ring $A$ is the Verdier quotient $D_{\text{SG}}(A) := \text{projdim}(A)/\text{projdim}(A)$. By $\Sigma : D_{\text{SG}}(A) \to D_{\text{SG}}(A)$ we mean the suspension functor. The following was proved by using derived-categorical methods in [19 Example A.5]:

**Corollary 3.6.** Let $\mathcal{R}_0 \simeq k[[u,v]]/(uv)$. Then $(u) \oplus \Sigma(u) = k \in (D_{\text{SG}}(\mathcal{R}_0), \Sigma)$.

**Proof.** Recall that $\mathcal{R}_u \simeq (v)$ and $\mathcal{R}_v \simeq (u)$. From this $\Sigma(u) = (v)$. In $D_{\text{SG}}(\mathcal{R}_0)$ the class of free modules realized as a zero object. Apply this along with the exact sequence $0 \to \mathcal{R} \to \mathcal{R}_u \oplus \mathcal{R}_v \to k \to 0$ to see $(u) \oplus \Sigma(u) = k \in D_{\text{SG}}(\mathcal{R}_0)$. □

Let $(\mathcal{R}, m)$ be a regular local ring. Horrocks proved that any complete module free over $\text{Spec}(\mathcal{R}) \setminus \{m_{\mathcal{R}}\}$ is algebraic, see [18], [8]. This extends in the following sense:

**Theorem 3.7.** (Auslander-Brüder, Peskine-Szpiro) Let $\mathcal{M}$ be a complete module of finite projective dimension such that $\text{Ext}^1_\mathcal{R}(\mathcal{M}, \mathcal{R})$ is of finite length. Then $\mathcal{M}$ is strongly algebraic.

**Proof.** It is shown in [6] Proposition 5.44 that there is an $n \in \mathbb{N}_0$ and a finitely generated $\mathcal{R}$-module $\mathcal{N}$ such that $\mathcal{M} \oplus \mathcal{R}^n \simeq \tilde{N}$. Recall that complete free modules are algebraic. By principle 2 of 3 (see Discussion 2.4) $\mathcal{M}$ is algebraic. We apply this for any direct summand of $\mathcal{M}$ to conclude that $\mathcal{M}$ is strongly algebraic. □

Let us revisit again $\mathcal{R}_0 := C[[x,y]]/(x^2 + y^2 - z^3)$. Then $\mathcal{R}_0 \simeq C[[u,v]]/(uv)$. Since $\mathcal{M} := \mathcal{R}_0/(u)$ is locally free on the punctured spectrum, we have $\ell(\text{Ext}^1_{\mathcal{R}_0}(\mathcal{M}, \mathcal{R}_0)) < \infty$. But, $\mathcal{M}$ is not algebraic (see [10] Page 335). Thus, finiteness of projective dimension in Theorem 3.7 is important.

**Discussion 3.8.** Let $\mathcal{M}$ and $\mathcal{N}$ be finitely generated modules such that either $\text{p. dim}(\mathcal{M}) < \infty$ or $\text{id}(\mathcal{N}) < \infty$ over any commutative noetherian ring. Let $i \leq d := \dim \mathcal{R}$. Then $\text{dim}(\text{Ext}^1_\mathcal{R}(\mathcal{M}, \mathcal{N})) \leq d - i$. The origin source of this is [14] by Grothendieck (claim in the case $\text{id}(\mathcal{N}) < \infty$ follows easily, see e.g., [2]).

**Corollary 3.9.** Let $\mathcal{R}$ be a $d$-dimensional local ring and $\mathcal{M}$ a complete module of finite projective dimension which is $(d - 1)$-syzygy. Then $\mathcal{M}$ is strongly algebraic.

**Proof.** We assume that $\mathcal{M} \neq 0$ and that $d > 0$. First, suppose that $d > 1$. By definition of syzygy modules, there is a complete module $\mathcal{N}$ and the exact sequence $0 \to \mathcal{M} \to \mathcal{R}^{d-2} \to \ldots \to \mathcal{R}^0 \to \mathcal{N} \to 0$. We have $\text{p. dim}(\mathcal{M}) \leq 1$. In the case $d = 1$, we apply Auslander-Buchsbaum formula to see $\text{p. dim}(\mathcal{M}) \leq \text{depth}(\mathcal{R}) \leq \dim(\mathcal{R}) = 1$. In both cases $\text{p. dim}(\mathcal{M}) \leq 1$. Projective modules are free over local rings, and free modules are strongly algebraic. Without loss of generality we assume that $\text{p. dim}(\mathcal{M}) = 1$. We have $\text{Ext}^1_{\mathcal{R}}(\mathcal{M}, \mathcal{R}) \simeq \text{Ext}^1_{\mathcal{R}}(\mathcal{N}, \mathcal{R})$. By Discussion 3.8

$$\dim(\text{Ext}^1_{\mathcal{R}}(\mathcal{M}, \mathcal{R})) = \dim(\text{Ext}^1_{\mathcal{R}}(\mathcal{N}, \mathcal{R})) \leq \dim \mathcal{R} - d = 0.$$ Consequently, $\text{Ext}^1_{\mathcal{R}}(\mathcal{M}, \mathcal{R})$ is of finite length. By Theorem 3.7 $\mathcal{M}$ is strongly algebraic. □
Corollary 3.10. Let $R$ be $d$-dimensional and $M$ a complete module of finite projective dimension. Suppose one of the following holds:

i) $d = 1$,

ii) $d = 2$ and $M$ is torsion-less, or

iii) If $d = 3$ and $M$ is reflexive.

Then $M$ is strongly algebraic.

One may replace ii) with the following statement: If $d = 2$, $M$ is torsion-free and $\hat{R}$ satisfies Serre’s condition $R_1$ (resp. $\hat{R}$ is an integral domain), then $M$ is strongly algebraic.

Proof. i) We may assume that $\dim(M) = 1$. In the light of Discussion 3.8 we observe that $\dim(\text{Ext}^1_R(M, \hat{R})) = 0$. Therefore, $\text{Ext}^1_R(M, \hat{R})$ is of finite length. Theorem 3.7 yields the claim.

ii) Let $\hat{R}^n \to M^* \to 0$ be a presentation. Applying $\text{Hom}_{\hat{R}}(-, \hat{R})$ to the presentation implies that $0 \to M^{**} \to \hat{R}^n$ is exact. By torsion-less, $M \subset M^{**}$. Combine these to see $M$ is 1-syzygy. It remains to use Corollary 3.9.

iii) Let $\hat{R}^m \to \hat{R}^m \to M^* \to 0$ be a finite presentation. Then $0 \to M^{**} \to \hat{R}^m \to \hat{R}^n$ is exact. Since $M$ is reflexive it follows that it is second syzygy. Due to Corollary 3.9 $M$ is strongly algebraic.

Let $R$ be a ring, an ideal with a generating set $\underline{a} := a_1, \ldots, a_r$. By $H^i_a(M)$, we mean the $i$-th cohomology of the Čech complex of a module $M$ with respect to $\underline{a}$. This is independent of the choice of the generating set. For simplicity, we denote it by $H^i_a(M)$.

Corollary 3.11. Let $\hat{R}$ be an analytically irreducible local ring which is Gorenstein over its punctured spectrum. Let $M$ be a generalized Cohen-Macaulay module and of finite projective dimension. If $\dim(M) = \dim(\hat{R})$, then $M$ is strongly algebraic. In fact, $M$ extended from a generalized Cohen-Macaulay $R$-module.

Proof. We use a trick taken from 23. Set $f_\underline{a}(N) := \inf\{i : H^i_a(N) \text{ is not finitely generated}\}$. Recall that $\hat{R}$ is a quotient of a regular ring. By Grothendieck’s finiteness theorem

$$f_\underline{a}(M) = \inf\{\text{depth}(M_q) + \text{ht}(\frac{a+q}{q}) : q \in \text{Supp}(M) \setminus V(\underline{a})\}.$$ 

Then, $\dim(\hat{R}) = f_{\underline{m}}(M)$. Since $\dim(M) = \dim(\hat{R})$ and $\hat{R}$ is domain, we see that $\text{Spec}(\hat{R}) = \text{Supp}(M)$. Let $q \in \text{Spec}(\hat{R}) \setminus V(\underline{m}).$ Then $q \in \text{Supp}(M) \setminus V(\underline{m}).$ and so

$$\dim(\hat{R}) \leq \text{depth}(M_q) + \text{dim}(\hat{R}_q) \leq \text{dim}(M_q) + \text{ht}(\frac{\hat{R}_q}{q}) \leq \text{ht}(q) + \text{ht}(\frac{\hat{R}_q}{q}) \leq \dim(\hat{R}).$$

We conclude that $\text{depth}(M_q) = \dim(\hat{R}_q)$. We combine this and the Gorenstein property of $\hat{R}_q$ along with the local duality to see $\text{Ext}^i_{\hat{R}}(M, \hat{R})_q \simeq \text{Ext}^i_{\hat{R}_q}(M_q, \hat{R}_q) \simeq H^{\dim\hat{R}_q-i}_q(M_q) = 0$ for all $i > 0$. From this we conclude that $\text{Ext}^i_{\hat{R}}(M, \hat{R})$ is of finite length. We deduce from Theorem 3.7 that $M$ is strongly algebraic. Let $M$ be such that $\hat{M} = M$. To show $M$ is generalized Cohen-Macaulay, its enough to note that $H^n_a(M) \simeq H^n_a(M) \otimes_R \hat{R} \simeq H^n_{\underline{m}}(M)$ for all $i < \dim M$.

Remark 3.12. In Corollary 3.11 the assumption $\dim(M) = \dim(\hat{R})$ is important even over regular rings: We look at $R := \mathbb{Q}[X, Y]_{(X, Y)}$. By Proposition 3.4 there is a height-one prime ideal $p$ such that $M := \hat{R}/p$ is not algebraic. Since $p$ is prime, $H^0_a((X, Y))(M) = 0$ and so $M$ is (generalized) Cohen-Macaulay. Note that $p. \dim(M) = \dim(M) = 1 < 2 = \dim(\hat{R})$. 
Recall that an $R$-module $M$ of projective dimension $r$ is called $r$-spherical if $\text{Ext}^i_R(M, R) = 0$ for all $i \neq 0, r$. Now, let $M$ be an $r$-spherical complete module such that $\text{Ext}^i_R(M, \hat{R})$ is algebraic. In view of [22 Proposition I. 5.8] $M$ is strongly algebraic. Now, let $M$ be a complete module of finite projective dimension such that $\text{Ext}^i_R(M, \hat{R})$ is algebraic for all $i > 0$. When is $M$ algebraic?

Assume that $R_0 = k$ is a field of characteristic zero. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded $k$-algebra. Here, by completion we mean completion with respect to the graded maximal ideal. Let $M \in \text{mod} (\hat{R})$ be such that $\text{Ext}^1_R(M, M) = 0$. In view of [19 Proposition 6.1] $M$ is completion of a finitely generated graded $R$-module $M$. The graded structure on $R$ is important. To find an example, again we revisit $R_0 := \mathbb{C}[x, y]/(y^2 - x^3 - x^2)$. Recall that $\hat{R}_0 := \mathbb{C}[u, v]/(uv)$ and $\text{Ext}^1_{\hat{R}_0}(\hat{R}_0/(u), \hat{R}_0/(u)) = 0$. We ask: When algebrization depends on the characteristic?

4. ALGEBRIZATION OF FORMAL REGULAR FUNCTIONS

In this section, $(R, m)$ is a complete local ring, and $a \triangleleft R$ (in fact we only need $R$ is complete with respect to $a$-adic topology). Let $X := \text{Spec}(R) \setminus \{m\}$ and $(\hat{X}, \mathcal{O}_{\hat{X}})$ denote the formal completion of $X$ along with $Y := V(a) \setminus \{m\}$. Recall that $Y$ is G1 in $X$ if $H^0(\hat{X}, \mathcal{O}_{\hat{X}}) \xrightarrow{\sim} H^0(\hat{X}, \mathcal{O}_{\hat{X}})$. This means that formal regular functions are the ordinary regular functions.

**Lemma 4.1.** Adopt the above notation. The following holds:

i) If $\text{ht}(a) = \dim R - 1$, then $H^0(\hat{X}, \mathcal{O}_{\hat{X}}) \simeq (R_x)^{\wedge a}$ for some $x \in m$.

ii) Assume in addition to i) that $R$ is normal. Then $H^0(\hat{X}, \mathcal{O}_{\hat{X}})$ is not finitely generated over $R$.

iii) The module $H^0(X, O_X)$ is algebraic over $R$ provided it is finitely generated over $R$.

**Proof.** i) Under some extra assumptions this can be find in [3]. By $D_{\triangleleft}(\cdot)$ we mean the ideal transformation, and recall that if $I = (a)$ is principal, it is just the localization. Now, let $x + p$ be a parameter element for the 1-dimensional ring $R_p$, so that $\text{rad}(a + xR) = m$. By the theorem on formal functions,

$$H^0(\hat{X}, \mathcal{O}_{\hat{X}}) \simeq \lim_{\longleftarrow n} H^0(\hat{X}, \mathcal{O}_{\hat{X}}/a^n) = \lim_{\longleftarrow n} \bigoplus_{\rho \in \mathbb{R}} D_{\triangleleft |_{\rho a^n}} \bigoplus_{\rho / a^n} (\mathcal{R}_x/(\mathcal{R}_x/a^n)) = (R_x)^{\wedge a}.$$  

ii) Suppose on the contrary that $H^0(\hat{X}, \mathcal{O}_{\hat{X}})$ is finitely generated. Then $(R_x)^{\wedge a}$ is finitely generated. Hence, its submodule $R_x$ is as well. Thus, $1/x$ is integral over $R$. Since $R$ is normal, $1/x \in R$. This contradicts the fact that $x \in m$.

iii) Let $R$ be such that $\hat{R} \simeq \hat{R}$. There is an exact sequence

$$0 \rightarrow \Gamma_m(R) \rightarrow R \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^1_m(R) \rightarrow 0 \quad (\ast)$$

Recall that $\Gamma_m(R)$ is algebraic. Clearly, $\mathcal{R}$ is algebraic. Also, $\text{Hom}_R(\Gamma_m(R), \mathcal{R})$ is of finite length. In particular, $\text{Hom}_R(\Gamma_m(R), \mathcal{R})$ is finitely generated as an $R$-module. Set $\overline{\mathcal{R}} := \frac{R}{\mathcal{R}}$, and look at $0 \rightarrow \Gamma_m(R) \rightarrow R \rightarrow \overline{\mathcal{R}} \rightarrow 0$. In view of [11 Proposition 3.2(ii)] we see $\overline{\mathcal{R}}$ is algebraic. Suppose it extends from $M$. Since $H^0(X, \mathcal{O}_X)$ is finitely generated, we see $H^1_m(R)$. From this we deduce that $H^1_m(R)$ is of finite length. Due to Discussion [23], $H^1_m(R)$ is algebraic. Suppose it extended from $H$. Now, we look at $D : 0 \rightarrow \mathcal{R} \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^1_m(R) \rightarrow 0$. Then $D \in \text{Ext}^1_D(D, \hat{M}) \simeq \text{Ext}^1_D(H, M) \otimes_R \hat{R} \simeq \text{Ext}^1_D(H, M)$ corresponds to completion of $0 \rightarrow M \rightarrow D \rightarrow H \rightarrow 0$ where $D$ is a finitely generated $R$-module. By 5-lemma, $D \simeq H^0(X, \mathcal{O}_X)$. So, $H^0(X, \mathcal{O}_X)$ is algebraic. \qed
Corollary 4.2. Let \((\mathcal{R}, m)\) be a complete normal local domain of dimension \(d \geq 2\) and let \(a\) be of height \(d - 1\). Then \(\mathcal{Y}\) is not \(G_1\) in \(\mathcal{X}\).

Proof. Normal rings are \(S_2\). Apply this along with \(d \geq 2\) to see \(\Gamma_m(\mathcal{R}) = H^1_m(\mathcal{R}) = 0\). In view of (\ast) in Lemma 3.1(iii) we see \(H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \mathcal{R}\). Due to Lemma 3.1(ii) \(H^0(\hat{\mathcal{X}}, \mathcal{O}_{\hat{\mathcal{X}}})\) is not finitely generated over \(\mathcal{R}\). This implies that \(H^0(\hat{\mathcal{X}}, \mathcal{O}_{\hat{\mathcal{X}}}) \not\cong H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})\). By definition, \(\mathcal{Y}\) is not \(G_1\) in \(\mathcal{X}\). \(\square\)

Discussion 4.3. Let \(\mathcal{N}\) be a module.

i) The cohomological dimension of \(\mathcal{N}\) with respect to a \(cd(a, \mathcal{N})\) is the supremum of \(i\)'s such that \(H^i_m(\mathcal{N}) \neq 0\). We use \(cd(a)\) instead of \(cd(a, \mathcal{R})\).

ii) Following Peskine and Szpiro, formal grade of \(\mathcal{N}\) with respect to \(a\) is the least integer \(i\) such that 
\[ \lim_{\leftarrow n} H^i_m(\mathcal{N}/a^n\mathcal{N}) \neq 0 \] and denoted by \(fgrade(a, \mathcal{N})\).

Observation 4.4. Let \((\mathcal{R}, m)\) be a Cohen-Macaulay local ring of dimension \(d > 1\) and complete with respect to \(a\)-adic topology. Then \(\mathcal{Y}\) is \(G_1\) in \(\mathcal{X}\) if and only if \(cd(a) \leq d - 2\).

Proof. Recall that \(\text{depth}(\mathcal{R}) > 1\). The depth condition implies that \(H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong \mathcal{R}\). We look at the following exact sequence (see \([\underline{22}]\) \(\S\)III.2)
\[ 0 \to \lim_\leftarrow n \Gamma_m(\mathcal{R}) \to \lim_\leftarrow n \frac{\mathcal{R}}{a^n} \to \lim_\leftarrow n H^0(\mathcal{X}, \mathcal{R}) \to \lim_\leftarrow n H^1_m(\mathcal{R}) \to 0. \]
Recall that \(\lim_\leftarrow n (\mathcal{R}) = \mathcal{R}\). Recall from \([\underline{3}]\) Corollary 4.2 that the Cohen-Macaulay property of a nonzero module \(\mathcal{N}\) is equivalent with the following property \(fgrade(b, \mathcal{N}) + cd(b, \mathcal{N}) = \dim(\mathcal{N}) \forall b \in \mathcal{R} \) (\ast). Recall that \(G_1\) means that the natural map \(\varphi : H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to H^0(\hat{\mathcal{X}}, \mathcal{O}_{\hat{\mathcal{X}}})\) is an isomorphism. Note that \(\ker(\varphi) = \lim_\leftarrow n \Gamma_m(\mathcal{R})\) and \(\operatorname{coker}(\varphi) = \lim_\leftarrow n H^1_m(\mathcal{R})\). Thus \(\mathcal{Y}\) is \(G_1\) in \(\mathcal{X}\) if and only if \(fgrade(a, \mathcal{R}) \geq 2\). In view of (\ast) this holds if and only if \(cd(a) \leq d - 2\), as claimed. \(\square\)

Let us extend the previous observation:

Proposition 4.5. Let \(\mathcal{R}\) be a local domain of dimension \(d \geq 3\) and \(\mathcal{M}\) is torsion-free satisfies Serre’s condition \(S_d\). Suppose \(\mathcal{R}\) is complete with respect to \(a\)-adic topology and \(\mathcal{R}\) is homomorphic image of a Gorenstein ring. If \(\text{cd}(a, \mathcal{M}) \leq d - r\), then \(fgrade(a, \mathcal{M}) \geq r\).

Proof. Let \(A\) be a ring \(I, b \triangleleft A\) and \(s, \ell \in \mathbb{N}\). We say \((A, I, T := V(b), M, s, \ell)\) satisfies the property \(\mathcal{P}\) if we are in the situation of \([\underline{23}]\) Tag 0EFU]. Here, we collect the basic properties of \(\mathcal{P}\):

Fact A): Let \(I \subset b\) be ideals of a ring \(A\). Let \(M\) be a finite \(A\)-module. Let \(s\) and \(\ell\) be integers. Also, we assume: i) \(A\) has a dualizing complex, ii) \(\text{cd}(A, I) \leq \ell\), iii) \(p \not\in V(I)\) and \(q \in V(p) \cap V(b)\) then \(\text{depth}_{A_p}(M_p) > s\) or \(\text{depth}_{A_p}(M_p) + \dim((A/p)_q) > \ell + s\). Then \((A, I, V(b), M, s, \ell)\) has property \(\mathcal{P}\) (see \([\underline{24}]\) Tag 0EFW]).

We are going to check \(A := \mathcal{R}, I := a, V(b) = m, M := \mathcal{R}, s := r - 1, \ell := d - 2\) has property \(\mathcal{P}\). Since \(\mathcal{R}\) is homomorphic image of a Gorenstein ring i) holds. Since \(\mathcal{M}\) is torsion-free, \(0 \in \text{Supp}(\mathcal{M})\). Hence \(\text{Supp}(\mathcal{M}) = \text{Spec}(\mathcal{R})\). Also, cohomological dimension is support sensitive. Part ii) follows by these. To show iii), let \(p \not\in V(a)\) and \(q \in V(p) \cap V(m) = \{m\}\). Suppose \(\text{depth}(\mathcal{M}_p) > r - 1\) is not the case. That is \(\text{depth}(\mathcal{M}_p) \leq r - 1\). In the light of \(\text{depth}(\mathcal{M}_p) \geq \min\{r, \dim(\mathcal{M}_p)\}\) we see \(\dim(\mathcal{M}_p) = \text{depth}(\mathcal{M}_p)\).
We may assume that \( p \neq 0 \). Clearly, \( \mathcal{M}_p \) is torsion-free over \( R_p \). Thus, \( \text{Supp}_{R_p}(\mathcal{M}_p) = \text{Spec}(R_p) \). Therefore, \( \text{ht}(p) = \text{Supp}(\mathcal{M}_p) \) and consequently

\[
\text{depth}(\mathcal{M}_p) + \dim((R/p)_{q}) = \text{ht}(p) + \dim(R/p) \overset{(1)}{=} d > d - 1 = (d - 2) + 1 =: \ell + s.
\]

Regarding (\( \ell \)): we use the fact that any homomorphic image of a Gorenstein ring is catenary. This clarifies iii).

Fact B): Assume \( P \) holds over a local ring \((A, m)\) and \( T = \{ m \} \). Then \( H^i_m(M) \to \lim_{\leftarrow n} H^i_m(M/I^nM) \) is an isomorphism for \( i \leq s \) (see [24] Tag 0EFR).

We apply Fact B) to see that \( \lim_{\leftarrow n} H^r_{m}(\frac{M}{n^n M}) \cong H^r_{m}(\lim_{\leftarrow n} \frac{M}{n^n M}) \cong H^r_{m}(M) \overset{(\ast)}{=} 0 \) where (\( \ast \)) follows from \( S_r \). By definition, \( \text{fgrade}(a, \mathcal{M}) \geq r \).

Let \( F \) be a sheaf associated to \( \mathcal{M} \) over \( \mathcal{X} \) and \( \hat{F} \) be its formal completion along with \( \mathcal{Y} \).

**Corollary 4.6.** (Faltings) Let \( R \) be a local domain of dimension \( d \geq 3 \), \( \mathcal{M} \) be torsion-free and \( S_2 \). Suppose the following holds:

i) \( \mathcal{M} \) is complete with respect to \( a \)-adic topology,

ii) \( R \) is homomorphic image of a Gorenstein ring, and

iii) \( \text{cd}(a, \mathcal{M}) \leq d - 2 \).

Then \( H^0(\hat{\mathcal{X}}, \hat{F}) \cong H^0(\mathcal{X}, F) \).

**Proof.** By the proposition \( \text{fgrade}(a, \mathcal{M}) \geq 2 \). We conclude from

\[
\begin{array}{cccccccc}
0 = \lim_{\leftarrow n} \Gamma_m(\frac{M}{n^n M}) & \to & \lim_{\leftarrow n} \frac{M}{n^n M} & \to & \lim_{\leftarrow n} H^0(\mathcal{X}, \frac{F}{n^n F}) & \to & \lim_{\leftarrow n} H^1_m(\frac{M}{n^n M}) = 0 \\
\cong & & \cong & & \cong & & \\
0 = \Gamma_m(\mathcal{M}) & \to & \mathcal{M} & \to & H^0(\mathcal{X}, F) & \to & H^1_m(\mathcal{M}) = 0
\end{array}
\]

and the theorem of formal functions to deduce that \( H^0(\hat{\mathcal{X}}, \hat{F}) \cong \lim_{\leftarrow n} H^0(\mathcal{X}, \frac{F}{n^n F}) \cong H^0(\mathcal{X}, F) \).

We found that the following is in [24] Tag 0EII]. However, the following proof is quite elementary:

**Proposition 4.7.** Let \((R, \mathfrak{m})\) be a local ring, \( a := (x) \) and \( \mathcal{M} \) a finitely generated module with the following properties:

i) \( x \) is \( \mathcal{M} \)-regular,

ii) \( \mathcal{M} \) is complete with respect to \( a \)-adic topology, and

iii) \( H^1_{\mathfrak{m}}(\mathcal{M}) \) and \( H^2_{\mathfrak{m}}(\mathcal{M}) \) are annihilated by some power of \( x \).

Then \( H^0(\hat{\mathcal{X}}, \hat{F}) \cong H^0(\mathcal{X}, F) \).

**Proof.** After changing \( x \) with its high powers we may assume that \( xH^1_m(M) = xH^2_m(M) = 0 \). Let \( \theta_n: \frac{M}{x^n+1aM} \to \frac{M}{x^nM} \) be the natural surjection. The following diagram:

\[
\begin{array}{cccccccc}
0 & \to & \mathcal{M} & \xrightarrow{x} & \mathcal{M} & \to & \frac{M}{x^n M} & \to & 0 \\
& & x & & \leftarrow & & \theta_n & & \\
0 & \to & \mathcal{M} & \xrightarrow{x^{n+1}} & \mathcal{M} & \to & \frac{M}{x^{n+1}aM} & \to & 0
\end{array}
\]

\]
We apply 5-lemma to deduce that \( H^0_0 \to H^1_0 \to \cdots \) is an exact sequence, where \( \pi_n := H^1_0(\theta_n) \). Since \( x H^1_0(\mathcal{M}) = x H^2_0(\mathcal{M}) = 0 \) we have

\[
0 \to H^1_0(\mathcal{M}) \to H^1_0(\mathcal{M}) \to H^2_0(\mathcal{M}) \to 0
\]

Note that \( H^0_0(\mathcal{M}) \to H^2_0(\mathcal{M}) \) is a zero map. By Mittag-Leffler, we have

\[
0 \to \lim_{\leftarrow n}\left( H^1_0(\mathcal{M}), \text{id} \right) \to \lim_{\leftarrow n}\left( H^1_0(\mathcal{M}), \pi_n \right) \to \lim_{\leftarrow n}\left( H^2_0(\mathcal{M}), \text{zero} \right) \to 0.
\]

Since \( \lim_{\leftarrow n}\left( H^1_0(\mathcal{M}), \text{zero} \right) = 0 \) we get that

\[
H^1_0(\mathcal{M}) \simeq \lim_{\leftarrow n}\left( H^1_0(\mathcal{M}), \text{id} \right) \simeq \lim_{\leftarrow n}\left( H^1_0(\mathcal{M}), \pi_n \right).
\]

Similarly, \( H^0_0(\mathcal{M}) \simeq \lim_{\leftarrow n} H^0_0(\mathcal{M}) \). This is zero, since \( \text{depth}(\mathcal{M}) > 0 \). Now, we put these in the following diagram:

\[
0 = \lim_{\leftarrow n}\Gamma(\mathcal{M}, \mathcal{M}) \to \lim_{\leftarrow n} H^0(\mathcal{X}, \mathcal{F}) \to 0
\]

This implies that:

**Corollary 4.8.** Let \((\mathcal{R}, \mathfrak{a})\) be a generalized Cohen-Macaulay complete local ring of dimension \(> 2\) and \(a = (x)\) for some regular element \(x\). Then \(Y\) is G1 in \(X\).

Corollary 4.8 is not true in 2-dimensional case, even the ring is regular (see Corollary 4.2).

**Corollary 4.9.** Let \((\mathcal{R}, \mathfrak{a})\) be a complete local ring. If \(\text{depth}(\mathcal{R}/\mathfrak{a}^n) > 1\) for all \(n \gg 0\), then \(Y\) is G1 in \(X\). In particular, suppose \(\text{depth}(\mathcal{R}) > 2\) and \(a\) is generated by a regular element, then \(Y\) is G1 in \(X\).

**Proof.** Recall that \(\Gamma_0(\mathcal{R}/\mathfrak{a}^n) = H^1_0(\mathcal{R}/\mathfrak{a}^n) = 0\) for \(n \gg 0\). In view of the following exact sequence

\[
0 \to \Gamma_0(\mathcal{R}/\mathfrak{a}^n) \to \mathcal{R}/\mathfrak{a}^n \to H^0(\mathcal{X}, \mathcal{R}/\mathfrak{a}^n) \to H^1_0(\mathcal{R}/\mathfrak{a}^n) \to 0,
\]

we see \(H^0(\mathcal{X}, \mathcal{R}/\mathfrak{a}^n) \simeq \mathcal{R}/\mathfrak{a}^n\) for \(n \gg 0\). Therefore,

\[
H^0(\mathcal{X}, \mathcal{O}_x) \simeq \lim_{\leftarrow n} H^0(\mathcal{X}, \mathcal{R}/\mathfrak{a}^n) \simeq \lim_{\leftarrow n} (\mathcal{R}/\mathfrak{a}^n) = \mathcal{R}^{(\ast)} \simeq \mathcal{R},
\]

where \(\ast\) follows by: if a module is complete with respect to \(m\)-adic topology, then it is complete with respect to \(a\)-adic topology too.
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E-mail address: mohsenasgharzadeh@gmail.com