Linear Depth Deduction with Subformula Property for Intuitionistic Epistemic Logic

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Abstract
In their seminal paper Artemov and Protopopescu provide Hilbert formal systems, Brower–Heyting–Kolmogorov and Kripke semantics for the logics of intuitionistic belief and knowledge. Subsequently Krupski has proved that the logic of intuitionistic knowledge is PSPACE-complete and Šu and Sano have provided calculi enjoying the subformula property. This paper continues the investigations around sequent calculi for Intuitionistic Epistemic Logics by providing sequent calculi that have the subformula property and that are terminating in linear depth. Our calculi allow us to design a procedure that for invalid formulas returns a Kripke model of minimal depth. Finally we also discuss refutational sequent calculi, that is sequent calculi to prove that formulas are invalid.

Keywords
Intuitionistic epistemic logic · Sequent calculi · Subformula property · Counter-models generation

1 Introduction
In [1] the epistemic logics $IEL$ and $IEL^-$ are introduced with the aim to study knowledge from the intuitionistic point of view. The authors remark that Brower–Heyting–Kolmogorov (BHK) semantics is the intended semantics of intuitionistic logic, where a proposition is true if it is proved and thus, in the intuitionistic perspective, knowledge and belief are the product of verification. From the idea that intuitionistic proof is a form of verification that implies intuitionistic knowledge (represented by the modality $K$) follows that the co-reflexion principle (or constructivity of the proof) $A \rightarrow K.A$ is assumed both for belief and knowledge.

On the other hand, in the intuitionistic perspective, the verification of a statement does not guarantee to have a proof. A possibility is that a verified statement cannot be false and this is the intuitionistic reflexion principle $KA \rightarrow \neg \neg A$, formally stating that if $A$ has a verification, then $A$ has a proof which is not necessarily obtainable from the verification process. If the intuitionistic reflexion principle is assumed, then known formulas cannot be false. If co-reflexion and intuitionistic reflexion are assumed, then intuitionistic truth...
implies intuitionistic knowledge and intuitionistic knowledge implies classical truth. The intuitionistic systems of epistemic logic IEL and IEL differ because IEL does not assume the intuitionistic reflection principle thus we can have false beliefs. The investigations in [1] also include characterisations of IEL and IEL by means of both Hilbert axiom systems and Kripke semantics, where a binary relation E between worlds is added to the usual elements of the intuitionistic Kripke frame.

Logics IEL and IEL have attracted people engaged in type theory: paper [2] provides a formal analysis of the computational content of intuitionistic belief by introducing a natural deduction calculus for IEL; paper [10] constructs a type system which is Curry-Howard isomorphic to IEL. The investigations of our paper are more in line with those in [9], where it is proved that IEL is PSPACE-complete. Moreover, paper [9] extends Gentzen calculus LJ for propositional intuitionistic logic [12, 13] with two rules for the connective K that do not fulfil the subformula property. We also quote [11], that presents a calculus for First Order IEL that extends LJ with new rules for the connective K and the result is a logical apparatus that fulfils the subformula property. Apart from the aspects related to the subformula property and computational complexity, in the quoted papers there is no investigation about efficiency in proof search.

In this paper we propose sequent calculi that have the subformula property and whose deductions are all depth-bounded in the number of connectives occurring in the formula to be decided. To this aim, we use the ideas from [4] where a sequent calculus for propositional intuitionistic logic is provided. Paper [4] uses an extension of the ordinary sequent that has similarity with nested sequents [7]. Because of connective K and its Kripkean meaning, to handle IEL and IEL we have to extend the object language used in [4] by adding a new type of sequent that roughly speaking stores semantical information related to the relation E. The sequent calculi we present can be explained by Kripke semantics of Intuitionistic Epistemic Logics and our proofs follow model theoretic techniques, that is we provide correctness and completeness theorems using the Kripke semantics for the logics at hand.

The paper is organised as follows: in Sect. 2 we recall the definitions of IEL and IEL; in Sect. 3 we analyse the case of logic IEL and we provide a sequent calculus, Liel, and then, in Sect. 4, a procedure, Piel, that systematically builds trees of sequents and returns a proof of linear depth, if the given formula is valid in IEL, or a counter-model of minimal depth witnessing the invalidity of the given formula. In Sect. 5, we discuss Riel, a calculus to prove that a formula is invalid in IEL. Calculus Riel is tied to Liel: it has the subformula property and the deductions are depth-bounded in the number of connectives occurring in the formula to be proved. In Sect. 6, the sequent calculi Liel and Riel to prove validity and invalidity of IEL are provided. They are related to Liel and Riel respectively and they are obtained by following the Kripke semantics of the logics. We conclude with Sect. 7 by discussing some possible future works.

2 Definitions and Notations

Let V be a denumerable set of propositional variables. We consider the propositional language L built by using the set of atoms At = V ∪ {⊥} and the set of connectives {∨, ∧, →, K}. When convenient we write ¬A in place of A → ⊥.

We assume that the reader is familiar with Hilbert systems and related definitions [12, 13]. Logic IEL is proof-theoretically defined in [1] as an Hilbert system where the rule is Modus Ponens and the axioms are the following:
(Ax 1) axioms for propositional intuitionistic logic Int;
(Ax 2) \( \text{K}(A \rightarrow B) \rightarrow (\text{KA} \rightarrow KB) \);
(Ax 3) \( A \rightarrow \text{KA} \);
(Ax 4) \( \text{KA} \rightarrow \neg\neg A \).

We call (Kripke) models for IEL the structures \( K = \langle S, \leq, \models, E \rangle \) defined as follows:

- \( \langle S, \leq, \models \rangle \) is the usual Kripke model for Int. We recall that \( (S, \leq) \) is a non-empty partial order where the elements of \( S \) are called worlds or states of \( K \), and \( \models \subseteq S \times V \) is the forcing relation satisfying the monotonicity property: for every \( p \in V \) and for every \( \alpha, \beta \in S \), if \( \alpha \models p \) and \( \alpha \leq \beta \), then \( \beta \models p \);
- \( E \subseteq S \times S \) fulfils the following properties:

\[(1) \text{ for every } \alpha, \beta \in S, \alpha E \beta \text{ implies } \alpha \leq \beta; \]
\[(2) \text{ for every } \alpha, \beta, \gamma \in S, \text{ if } \alpha \leq \beta \text{ and } \beta E \gamma, \text{ then } \alpha E \gamma; \]
\[(3) \text{ for every } \alpha \in S, \text{ there exists } \beta \in S \text{ such that } \alpha E \beta. \]

The meaning of \( \bot \) and the connectives is explained by extending \( \models \) to \( \mathcal{L} \) as:

- for every \( \alpha \in S \), \( \alpha \not\models \bot \);
- for every \( \alpha \in S \), \( \alpha \models A \land B \) iff \( \alpha \models A \) and \( \alpha \models B \);
- for every \( \alpha \in S \), \( \alpha \models A \lor B \) iff \( \alpha \models A \) or \( \alpha \models B \);
- for every \( \alpha \in S \), \( \alpha \models A \rightarrow B \) iff for every \( \beta \in S \), if \( \alpha \leq \beta \), then \( \beta \not\models A \) or \( \beta \models B \);
- for every \( \alpha \in S \), \( \alpha \models \text{KA} \) iff for every \( \beta \in S \), if \( \alpha E \beta \) then \( \beta \models A \).

Following [12], it is easy to show that the monotonicity property extends to all the elements in \( \mathcal{L} \). The case of \( \text{KA} \) is also easy and exploits properties (Im 1)–(Im 3).

Let \( K \) be a model for IEL and \( A \in \mathcal{L} \). We say that \( K \) satisfies \( A \) (or \( A \) is satisfied by \( K \)) and we write \( K \models A \), if \( \alpha \models A \) holds for every world \( \alpha \) of \( K \).

In [1] it is proved that IEL can be semantically characterised by models for IEL, that is

\[ \text{IEL} = \{ A \in \mathcal{L} \mid \text{ for every model } K \text{ for IEL, } K \models A \} \]

As regards the notation, given \( \alpha, \beta \in S \), for \( E \) we use interchangeably \( \alpha E \beta \), \( (\alpha, \beta) \in E \) and \( \beta \in E(\alpha) \), where \( E(\alpha) = \{ \beta \in S \mid \alpha E \beta \} \); we write \( \alpha < \beta \) to mean \( \alpha \leq \beta \) and \( \alpha \neq \beta \) hold. If \( \alpha < \beta \) and for every \( \gamma \in S \), \( \alpha \leq \gamma \) and \( \gamma \leq \beta \) imply \( \alpha = \gamma \) or \( \beta = \gamma \), then we say that \( \beta \) is an immediate successor of \( \alpha \).

In a partial order \( \langle S, \leq \rangle \) of a model \( K \) we distinguish the final states and the inner states: \( \alpha \in S \) is a final state iff for every \( \beta \in S \), if \( \alpha \leq \beta \), then \( \alpha = \beta \). The inner states are the elements of \( S \) that are not final states. It is well-known [12] that Int has the finite model property, that is Int can be characterised by Kripke models with a finite number of worlds.

This implies that all the ascending (w.r.t. \( \leq \) ) chains of different worlds are finite and bounded (w.r.t. \( \leq \) ) by a state. The finiteness of the structure allows us to define the depth of a world \( \alpha \in S \), and we write depth(\( \alpha \)), as the number of different states in the longest path (through \( \leq \) ) from \( \alpha \) to a final state. The depth of a finite Kripke model \( K \), denoted by depth(\( K \)), is the number of different states in the longest path that can be build in \( K \).

We use the notion of depth only in relation to finite structures because results we present in this paper rely on the finite model property for IEL. In the semantical characterisation of IEL provided by [1] there is no reference to the finite model property.

In Sect. 3 our first step is to prove that IEL fulfils the finite model property (Theorem 1) by using the model theoretic technique of [3].

As regards the calculi, our ideas exploit the finite model property of IEL and the following consequence of Property (Im 3): for every final state \( \gamma \) of a model \( K \) we have that \( \gamma E \gamma \) holds.
The logical apparatus for \textbf{Int} in Figure 2 and

Axioms

\[
\begin{align*}
\Theta; \bot, \Gamma & \Rightarrow E \Delta \quad (\text{eIrr}) & \Theta; A, \Gamma & \Rightarrow E A, \Delta \quad (\text{eId}) \\
\end{align*}
\]

Rules

\[
\begin{align*}
\bot; A, \Gamma & \Rightarrow E \Delta & \bot; A, \Theta, \Gamma & \Rightarrow E \bot & (K \ L) \\
\Theta; A_1, \ldots, A_n, \Gamma & \Rightarrow E B, \Delta & \emptyset; A_1, \ldots, A_n, \Theta, \Gamma & \Rightarrow E B & (K \ R)
\end{align*}
\]

where \( n \geq 0 \)

\[
\begin{align*}
\Theta; A, \Gamma & \Rightarrow E \Delta & \Theta; \Gamma & \Rightarrow E B, \Delta & (eK \ L) & (eK \ R) \\
\Theta; \ K A, \Gamma & \Rightarrow E \Delta & \Theta; \Gamma & \Rightarrow E K B, \Delta & \Theta; A, \Gamma & \Rightarrow E A \land B \Rightarrow \Delta & (e \land L) & (e \land R) \\
\Theta; A, \Gamma & \Rightarrow E \Delta & \Theta; B, \Gamma & \Rightarrow E \Delta & (e \lor L) & (e \lor R) \\
\Theta; A \lor B, \Gamma & \Rightarrow E \Delta & \Theta; A \rightarrow B, \Gamma & \Rightarrow E \Delta & (e \rightarrow L) & (e \rightarrow R)
\end{align*}
\]

\textbf{Fig. 1} The calculus Liel for \textbf{IEL}

Thus, if \( \alpha \Vdash K A \), then necessarily all the final states reachable from \( \alpha \) force \( A \) and if \( \alpha \nVdash K A \), then there exists at least a world \( \beta \) such that \( \alpha E \beta \) and \( \beta \nVdash A \) hold. By Property (Im 1), it follows that \( \alpha \leq \beta \) holds.

Property (Im 3) is crucial to the validity of (Ax4). The logic \textbf{IEL}− is the logic proof-theoretically characterised by the axioms (Ax 1)–(Ax 3). The corresponding semantical characterisation is by Kripke models where relation \( E \) satisfies (Im 1) and (Im 2).

In Fig. 1 we present the rules of Liel, a calculus to prove validity of formulas in \textbf{IEL}, and in Fig. 3 we present the rules for Riel, a calculus to prove invalidity of formulas in \textbf{IEL}.

The object language of Liel and Riel is based on sequents of the kind \( \Theta; \Gamma \Rightarrow \Delta \) with three compartments. We refer to the compartments respectively as first, second and third compartment. For our purposes, the compartments are always finite subsets of \( \mathcal{L} \). We call \( E \)-sequents the sequents of the kind \( \Theta; \Gamma \Rightarrow \Delta \).

The presence of the \( E \)-sequents in the logical apparatus is related to the presence of relation \( E \) in the definition of Kripke semantics for \textbf{IEL}. The proof of the completeness theorem in the part related to rule \( K R \) uses the \( E \)-sequents. In Fig. 2 the calculus LSJ of [4] is provided. Calculus LSJ handles sequents of the kind \( \Theta; \Gamma \Rightarrow \Delta \) that are an extension
Axioms

\[ \Theta; \bot, \Gamma \Rightarrow \Delta \quad (\text{Irr}) \quad \Theta; A, \Gamma \Rightarrow A, \Delta \quad (\text{Id}) \]

Rules

\[
\begin{align*}
\Theta; A, B, \Gamma \Rightarrow \Delta & \quad (\land L) \\
\Theta; A \land B, \Gamma \Rightarrow \Delta & \quad (\land R) \\
\Theta; A, \Gamma \Rightarrow \Delta \quad \Theta; B, \Gamma \Rightarrow \Delta & \quad (\lor L) \\
\Theta; B \lor, \Gamma \Rightarrow \Delta & \quad (\lor R) \\
\Theta; B, \Gamma \Rightarrow \Delta \quad B, \Theta; \Gamma \Rightarrow A, \Delta & \quad (\rightarrow L) \\
\Theta; A \rightarrow B, \Gamma \Rightarrow \Delta & \quad (\rightarrow R)
\end{align*}
\]

Fig. 2 The calculus LSJ for \textit{Int}

of standard sequent \( \Gamma \Rightarrow \Delta \). The meaning of the sequents can be defined by means of Kripke models. Let \( \mathcal{K} = (S, \leq, \models, E) \) be a Kripke model and \( \alpha \in S \). We say that \( \alpha \) realizes \( \Gamma \Rightarrow \Delta \), and we write \( \alpha \models \Gamma \Rightarrow \Delta \), iff for every \( A \in \Gamma \), \( \alpha \models A \) and for every \( B \in \Delta \), \( \alpha \not\models B \). We also say that \( \mathcal{K} \) realizes \( \Gamma \Rightarrow \Delta \) and that \( \Gamma \Rightarrow \Delta \) is realizable (by \( \mathcal{K} \)). For sake of completeness, we recall that this corresponds to the following definition: \( \alpha \models \Gamma \Rightarrow \Delta \) iff \( \alpha \not\models \land \Gamma \lor \Delta \). where \( \land \Gamma \lor \Delta \) is the formula obtained by the conjunction of all the formulas in \( \Gamma \) and \( \lor \Delta \) is the formula obtained by the disjunction of all the formulas in \( \Delta \) (with the proviso that if \( \Gamma = \emptyset \) then \( \land \Gamma \) is any tautological formula, such as \( \bot \rightarrow \bot \), and if \( \Delta = \emptyset \) then \( \lor \Delta \) is any contradictory formula, such as \( \top \rightarrow \top \)). We say that \( \mathcal{K} \) realizes \( \Theta; \Gamma \Rightarrow \Delta \), and we write \( \alpha \models \Theta; \Gamma \Rightarrow \Delta \), iff \( \alpha \models \Theta \Rightarrow \Delta \) and for every \( C \in \Theta \) and for every \( \beta \in S \), if \( \alpha < \beta \), then \( \beta \models C \) (equivalently \( \alpha \models \Theta; \Gamma \Rightarrow \Delta \) iff \( \alpha \models \Theta \Rightarrow \Delta \) and for every \( \beta \in S \), if \( \alpha < \beta \), then \( \beta \models \ land \Theta \). The consequence of the semantical meaning of the sequents of the kind \( \Theta; \Gamma \Rightarrow \Delta \) is that the rules handling the implication have one more premise than the rules handling the standard sequent \( \Gamma \Rightarrow \Delta \), because the semantics of implication is defined considering worlds \( \beta \) that are equal or greater than \( \alpha \) and thus the rules take into account the cases \( \alpha = \beta \) and \( \alpha < \beta \). In the case of logic IEL, rules handling K have to take into account the relation \( E \), which is a subset of \( \leq \). If \( KA \) is satisfied by \( \alpha \), to draw a correct deduction we need to know if \( E(\alpha, \alpha) \) holds. Thus our calculi use the sequent \( \Theta; \Gamma \Rightarrow \Delta \). We say that \( \alpha \) realizes \( \Theta; \Gamma \Rightarrow \Delta \), and we write \( \alpha \models \Theta; \Gamma \Rightarrow \Delta \), iff \( E(\alpha, \alpha) \) and \( \alpha \models \Theta; \Gamma \Rightarrow \Delta \) hold. Thus \( \Theta; \Gamma \Rightarrow \Delta \) stores the information that it is realized in a world that \( E \)-reaches itself. If \( \alpha \models \Theta; \Gamma \Rightarrow \Delta \) holds, then we conclude that \( E(\alpha, \alpha) \) holds; if \( \alpha \models \Theta; \Gamma \Rightarrow \Delta \) holds, then we cannot draw any conclusion about \( E(\alpha, \alpha) \).

To define the deduction in our calculi, we need to identify a particular type of sequents that we call \textit{terminal}. We divide terminal sequents in two disjoint categories: \textit{axioms} and \textit{flat}. In the case of calculus Liel, the axioms are of the kind \( \text{(Irr)}, \text{(Id)}, \text{(elrr)} \) or \( \text{(eld)} \) and the flat sequents fulfil the following conditions: \( \Gamma \subseteq V, \Delta \subseteq At \) and \( \Gamma \cap \Delta = \emptyset \), where \( \Gamma \) is the second compartment and \( \Delta \) is the third. In the case of calculus Riel, the axioms are \( \text{(Sat)} \).
and (esSat) and the flat sequents contain ⊥ in the second compartment or their second and third compartments share formulas.

As regards proof construction, we use the rules bottom-up, thus by instantiating the rule R with a sequent σ we mean that σ is not a terminal sequent and we use σ to instantiate the conclusion of R. We call the result of the instantiation of R with σ the sequents σ₁, ..., σᵣ occurring in the premise of R. We consider the sequents occurring in the premise of the rules enumerated from left to right, thus σ₁ is the leftmost premise of R and σᵣ is the rightmost. We assume that in the instantiation the formulas in evidence in a compartment are all different among them and do not occur in the set in evidence in the same compartment. In practice, the instantiation does not use the formal notion of set to perform surreptitiously a contraction.

Let C be a sequent calculus presented in this paper. Given a sequent σ, a (C-)tree T of sequents for σ fulfils the following properties: (i) the root of T is σ; (ii) for every sequent σ₀ occurring in T as non-leaf node, if σ₁, ..., σᵣ are the children of σ₀, then there exists a rule R of C such that if σ₀ instantiates R, then σ₁, ..., σᵣ is the result of the instantiation of R with σ₀ and the sequents σ₁, ..., σᵣ are enumerated considering from left to right the sequents in the premise of R. T is a completed tree of sequents if the leaves of T are terminal sequents. T is a C-proof of σ if all the leaves are axiom sequents. In this case we say that σ is provable (in C) or C proves σ.

3 A Calculus to Prove the Validity of IEL

In this section we discuss the problem of proving the validity in IEL by means of the calculus Liel provided in Fig. 1 (we recall that the rules in Fig. 2 are the calculus for Int proposed in [4]). To decide the validity of A ∈ L we look for a (Liel-)proof of ∅; ∅ ⊩ A. If such a proof exists we say that A is provable in Liel or Liel proves A.

To prove the correctness of the calculus, in particular the correctness of the rules in Fig. 1, we have to prove the Finite Model Property for IEL.

Theorem 1 (Finite Model Property for IEL)

IEL = {A ∈ L | for every finite model K for IEL, K ⊨ A}

Proof the key point is to prove that if a model K = ⟨S, ≤, ⊨, E⟩ for IEL does not satisfy a formula A, then there exists a finite model K′ = ⟨S′, ≤′, ⊨′, E′⟩ for IEL such that K′ does not satisfy A. We apply the filtration method along the lines of [3]. Let α ∈ S such that α ∤ A. Let Subf(A) be the set of subformulas of A. We define the relation ≡⊆ S × S as follows: α ≡ β iff for every B ∈ Subf(A), α ⊨ B iff β ⊨ B. It is easy to prove that ≡ is an equivalence relation on S. Every equivalence class of ≡ characterises exactly a subset of formulas in Subf(A). As usual, the equivalence class of α ∈ S is denoted by [α].

We define K′ = ⟨S′, ≤′, ⊨′, E′⟩ as follows:

- S′ = {[α] | α ∈ S};
- ≤′ is the transitive closure of {(x, y) ∈ S′ × S′ | there exist α ∈ x and β ∈ y such that α ≤ β};
- ⊨′ = {(x, p) ∈ S′ × V | p ∈ Subf(A) and α ⊨ p, where α ∈ x};
- We define E′ iteratively starting from the final worlds of S′:
  - for every final state x ∈ S′, we put E′(x) = {x};
let us suppose that we have defined \( E'(y) \) for every \( y \in S' \) such that \( x <' y \). We put 
\[
E'_0(x) = \{ y \in S' \mid \text{there exist } \alpha \in x \text{ and } \beta \in y \text{ s.t. } \alpha E \beta \text{ holds} \}
\]
\[
E'(x) = E'_0(x) \cup \left( \bigcup_{y \in \{z \in S' \mid x <' z\}} E'(y) \right)
\]
As regards the definition of \( \models' \), we remark that the elements of a given equivalence class of \( \equiv \) force exactly the same elements of \( \text{Subf}(A) \). Thus the definition of \( \models' \) does not depend on the representative of the class on which the relation \( \models' \) is defined.

First we have to prove that the structure \( K' \) built from \( K \) via \( \equiv \) is a finite model for \( \text{IEL} \).

Since \( \text{Subf}(A) \) is finite, the equivalence classes of \( \equiv \) are finite, thus \( S' \) is finite.

To prove that \( \leq' \) is a reflexive and transitive partial order on \( S' \) one uses the fact that \( \leq \) is reflexive and transitive.

As regards the monotonicity of \( \models' \), we sketch the proof. We have to prove that if \( x \models' p \) and \( x \leq' y \), then \( y \models' p \). Since \( x \models' p \) holds, then for every \( \alpha \in x \), \( \alpha \models p \) holds; since \( x \leq' y \), then there exist \( \alpha \in x \) and \( \beta \in y \) such that \( \alpha \leq \beta \) holds. The monotonicity of \( \models' \) implies \( \beta \models p \). The definition of \( \models' \) implies \( y \models' p \).

To prove that \( E' \) fulfills Properties (Im 1)–(Im 3) we proceed by induction:

- Let \( x \in S' \) be a final world. Properties (Im 1) and (Im 3) easily follow from the definitions of \( E'(x) \) (in particular \( E'_0(x) \)), \( \leq' \) and the fact that \( E \) fulfills (Im 1)–(Im 3);
- Let \( x \in S' \) be a world that is not final. Let us suppose that Properties (Im 1)–(Im 3) are fulfilled by every world \( y \in S' \) such that \( x <' y \). If \( x \in E'z \) holds, then by definition of \( E' \) we have two cases:
  (i) \( z \in E'_0(x) \). Then there exist \( \alpha \in x \) and \( \beta \in z \) such that \( \alpha E \beta \). Since \( E \) fulfills Property (Im 1) we have \( \alpha \leq \beta \); by definition of \( \leq' \) we have \( x \leq' z \);
  (ii) there exists \( y \in S' \) such that \( x <' y \) and \( z \in E'(y) \). By induction hypothesis Property (Im 1) is satisfied on \( y \), thus \( y \leq' z \) and by transitivity of \( \leq' \) we get \( x \leq' z \).

Thus we have proved that \( E' \) fulfills Property (Im 1). As regards Property (Im 2) the non-trivial case is to prove that if \( x <' y \) and \( y \in E'z \) hold, then \( x \in E'z \) holds. The definition of \( E' \) implies that \( E'(x) \) includes \( E'(y) \). This proves that \( E'(x) \) fulfills Property (Im 2) and similarly follows Property (Im 3) for \( E'(x) \) (that is, the non-emptiness of \( E'(x) \)).

Thus we have proved that \( K' \) is a finite model for \( \text{IEL} \). Now to prove that \( K' \) does not satisfy \( A \), we show that on the subformulas of \( A \), the forcing relation is preserved, that is: for every \( B \in \text{Subf}(A) \), \( \alpha \models B \) iff \( [\alpha] \models' B \).

We go by induction on the the structure of \( B \):

- \( B \in V \). Then \( \alpha \models p \) iff \([\alpha] \models' p \) by definition of \( \models' \);
- Let \( B = \text{KC} \).
  - If \( \alpha \not\models \text{KC} \) then to prove that \([\alpha] \not\models' \text{KC} \) we show that there exists an element \( \beta' \in E'(\{\alpha\}) \) such that \( \beta' \not\models' C \). From \( \alpha \not\models \text{KC} \) follows that there exists \( \beta \in E(\alpha) \) such that \( \beta \not\models C \). By induction hypothesis \([\beta] \not\models' C \). Since \( E \) fulfills (Im 1), \( \alpha \leq \beta \) holds. By definition of \( \leq' \) it follows that \([\alpha] \leq' [\beta] \). By definition of \( E' \) we have that \([\beta] \in E'(\{\alpha\}) \) (in particular \([\beta] \in E'_0(\{\alpha\}) \) holds), thus, by the semantics of \( K \), we get \([\alpha] \not\models' \text{KC} \).
  - If \( \alpha \models \text{KC} \), then to prove that \([\alpha] \models' \text{KC} \) we have to show that for every \( \beta' \in E'(\{\alpha\}) \), \( \beta' \models' C \) holds. We recall that by (Im 1), for every element \( \beta' \in E'(\{\alpha\}) \), \( [\alpha] \leq' \beta' \) holds.

By construction of \( E' \), for every element \( \beta' \in E'(\{\alpha\}) \) we have two cases:
(i) $\beta' \in E_0'(\{\alpha\})$. Thus there exists $\gamma \in \{\alpha\}$ and $\beta \in \beta'$ such that $\beta \in E (\gamma)$. This implies that $\beta \models C$ holds. By induction hypothesis we get $\beta' \models' C$;

(ii) $\beta' \notin E_0'(\{\alpha\})$. By the finiteness of $K'$ and the definition of $E'$, there exists $\gamma' \in S'$ such that $\{\alpha\} \prec' \gamma'$ and $\beta' \in E_0'(\gamma')$. By definition of $E_0'(\gamma')$ there exist $\gamma \in \gamma'$ and $\beta \in \beta'$ such that $\beta \in E (\gamma)$. Since $\{\alpha\} \leq' \gamma$ holds and by definition of $\leq'$, we have that there exists a finite sequence $z_0, \ldots, z_n$ of elements of $S'$ fulfilling the following: $[\alpha] = z_0 \prec' z_1 \prec' \cdots \prec' z_n = \gamma'$ and, for $i = 0, \ldots, n - 1$, there exist $x \in z_i$ and $y \in z_{i+1}$ such that $x < y$. Since $\alpha$ and all the elements in $[\alpha]$ force $KC$, by monotonicity of $\models$ and the definition of $\leq'$ follows that all the elements in $\gamma'$ force $KC$. Thus $\gamma \models' KC$ holds. The semantics of $K$ and $\beta \in E (\gamma)$ imply that $\beta \models C$ holds. By induction hypothesis we conclude that $\beta' \models' C$ holds.

By Points (i) and (ii) we conclude that $[\alpha] \models' KC$ holds.

The other cases of $B$ are easy.

This concludes the proof of the preservation of forcing relation between $K$ and $K'$. Now it is immediate that if we have a model for $IEL$ that does not satisfy $A \in L$ then we can build a finite model for $IEL$ that does not satisfy $A$. \qed

By using the techniques from [12] one can prove that $IEL$ is semantically characterized by rooted models. Although the correctness of the calculi we present is not tied to models with a minimum world, since we use model theoretic constructions that glue together models it is handy to work with rooted models, in particular in the description of Procedure Piel in Sect. 4. Thus, hereafter we consider rooted models for $IEL$ and we write $\langle S, \leq, \rho, \models, E \rangle$ to denote that $\rho \in S$ is a distinguished element, the root, such that for every $\alpha \in S$, $\rho \leq \alpha$. By the monotonicity property, follows that $\rho \models A$ means $K \models A$.

Now we come to the properties of Liel. We start by discussing the depth of the trees built with Liel. We define the depth of a tree $T$ of sequents for $\sigma$ as the longest path from the $\sigma$ to a leaf. By inspection of the rules of Liel, it is easy to prove that the depth of every tree of sequents $T$ for $\sigma$ is bounded by the number of connectives occurring in $\sigma$. As a matter of fact for every rule $R$, the number of connectives of the sequents occurring in the premise of $R$ is lower than the number of connectives occurring in the sequent of the conclusion. From this it follows that the length of every path in $T$ (the number of sequents in every branch of $T$) is bounded by the number of connectives of $\sigma$.

The rules of Liel in Fig. 2 characterise propositional intuitionistic logic and are discussed in [4]. Thus in the following only the correctness of the rules in Fig. 1 is studied and to this aim we exploit the Kripke semantics of the connective $K$.

**Theorem 2** (Correctness) Let $K = \langle S, \leq, \rho, \models, E \rangle$ be a finite Kripke model for $IEL$ and $\alpha \in S$. For every rule $R$ of Liel, if $\alpha$ realizes the sequent in the conclusion of $R$, then there exists $\gamma \in S$ such that $\alpha \leq \gamma$ and $\gamma$ realizes at least a sequent in the premise.

**Proof** We only provide the cases related to $IEL$. The correctness of the rules for $K$ exploits the fact that from properties (Im 1)–(Im 3), follows that for every $\alpha \in S$, there exists $\beta \in S$ such that $\alpha \leq \beta$ and $E (\alpha, \beta)$.

Rule ($eKL$): let us suppose that $\alpha \in S$ realizes the sequent in the bottom of rule $eKL$. Thus, by definition of $E$-sequent, $E (\alpha, \alpha)$ holds and by semantical definition of $K$, $\alpha \models A$ holds. Thus we have proved that $\alpha$ realizes the $E$-sequent in the premise of ($eKL$).

Rule ($KL$): let us suppose that $\alpha \in S$ realizes the sequent in the bottom of rule ($KL$). We recall that by definition of realizability of a sequent, $\alpha$ forces the formulas in the second
compartments and for every $\beta \in S$, if $\alpha < \beta$, then $\beta \not\models \bigwedge \Theta$. Moreover, by semantical
definition of $K$, for every $KB$ in the second compartment and for every $\beta \in S$, if $E(\alpha, \beta)$
holds, then $\beta \not\models B$. Let us consider a final world $\gamma$ of $K$ such that $\alpha \leq \gamma$. Since $\gamma$
is a final
world, by $IEL$ semantics it follows that $E(\gamma, \gamma)$ holds, because on the final worlds relation
$E$ is reflexive. We have two cases: (i) $\alpha = \gamma$. Thus $\alpha$ has no immediate successors and $\alpha$
realizes the leftmost premise of $(KL)$; (ii) $\alpha < \gamma$. Thus $\gamma$ realizes the rightmost premise of
$(KL)$. Rule $(KR)$: let us suppose that $\alpha \in S$ realizes the sequent in the bottom of rule $(KR)$.
This implies that $\alpha$ forces the formulas in the second compartment and for every $\beta \in S$, if $\alpha < \beta$, then $\beta \not\models \bigwedge \Theta$. Moreover, $\alpha$ does not force any formula in the third compartment.
Since by hypothesis $\alpha \not\models KB$, there exists a world $\beta \in S$ such that $\alpha \leq \beta$, $E(\alpha, \beta)$, and $\beta \not\models B$. By the semantics of connective $K$, we have that if $\alpha \not\models KA$, then $\beta \not\models A$. Now, for $\beta$ we have two cases: (i) $\alpha = \beta$. Then $\alpha$ realizes sequent $\Theta; \Gamma, A_1, \ldots, A_n \implies B, \Delta$ and $E(\alpha, \alpha)$ holds. Thus the leftmost premise is realized; (ii) $\alpha < \beta$. Then $\beta$ realizes sequent
$\emptyset; \Theta, \Gamma, A_1, \ldots, A_n \implies B$. Thus the rightmost premise is realized.

Rule $(eKR)$: let us suppose that $\alpha \in S$ realizes the sequent in the bottom of rule $(eKR)$.
This implies that $E(\alpha, \alpha)$ holds and $\alpha$ does not force any formula in the third compartment.
Thus there exists a world $\beta \in S$ such that $\alpha \leq \beta$, $E(\alpha, \beta)$ and $\beta \not\models B$. By the property
of forcing relation, $\alpha \not\models B$ follows. Since $E(\alpha, \alpha)$ holds, also $\alpha \not\models \Theta; \Gamma \implies B, \Delta$ holds.

Rule $(e \rightarrow L)$: let us suppose that $\alpha \in S$ realizes the sequent in the bottom of the rule
$(e \rightarrow L)$. Thus $E(\alpha, \alpha)$ holds. By definition of intuitionistic implication at least one of the
following points holds: (i) $\alpha \not\models B$. Thus $\alpha$ realizes the leftmost premise of $e \rightarrow L$; (ii)
$\alpha \not\models A$ and for every $\beta \in S$, if $\alpha < \beta$, then $\beta \not\models A$. Thus $\beta \not\models B$. Moreover by the hypothesis
and $\alpha < \beta$ follows $\beta \not\models \bigwedge \Theta$. Hence we have proved that $\alpha$ realizes the second premise
of $e \rightarrow L$; (iii) there exists $\beta \in S$ such that $\alpha < \beta$, $\beta \not\models B$ and for every $\gamma \in S$, if $\beta < \gamma$, then $\gamma \not\models B$. Moreover, from $\alpha < \beta$ and the hypothesis, follows that $\beta \not\models \bigwedge \Theta$.
Thus we have proved that $\alpha$ realizes the rightmost premise of $(e \rightarrow L)$. □

We remark that in the proof of the correctness of rule $(KR)$, we do not claim that $\beta$ is
a final world. This explains the occurrence $\emptyset$ in the second premise of $(KR)$. The presence
of $\bot$ in the first compartment of the premises of $(KL)$ expresses the fact that the sequents,
if realizable, must be realized by Kripke models containing exactly one world. The effect
in rule application is that when this kind of sequent instantiates the rules $(\rightarrow R)$, $(e \rightarrow R)$,
$(\rightarrow L)$, $(e \rightarrow L)$, $(KR)$ or $(KL)$, the resulting $\sigma_r$ (the rightmost premise) is the axiom $(Irr)$
or $(eIrr)$.

**Example 1** Calculus Liel proves $(Ax 2) K(A \rightarrow B) \rightarrow (KA \rightarrow KB)$.

We remark that when $\Theta$ and $\Delta$ are equal to $\emptyset$, the two premises of rule $(\rightarrow R)$ coincide,
thus we only show one branch.

\[
\begin{align*}
[\text{Ded}_1] & \quad \emptyset; B, A \implies B \ (\text{Id}) \quad B; A \implies A, B \ (\text{Id}) \quad B; A \implies A \ (\text{Id}) \\
& \quad \emptyset; A \rightarrow B, A \implies B \quad (e \rightarrow L) \\
[\text{Ded}_2] & \quad \emptyset; B, A \implies B \ (\text{Id}) \quad B; A \implies A, B \ (\text{Id}) \quad B; A \implies A \ (\text{Id}) \\
& \quad \emptyset; A \rightarrow B, A \implies B \quad (\rightarrow L) \\
& \quad [\text{Ded}_1] \quad [\text{Ded}_2] \\
& \quad \emptyset; K(A \rightarrow B), KA \implies KB \quad (KR) \\
& \quad \emptyset; \emptyset \implies K(A \rightarrow B) \implies KA \rightarrow KB \quad (\rightarrow R)
\end{align*}
\]
Example 2 Calculus Liel proves (Ax 4) $K A \rightarrow \neg \neg A$.

Rules $(KL)$ and $(\rightarrow R)$ are instantiated with $\Delta = \{ \bot \}$ and $\Theta = \emptyset$. Thus the premises are equal and to save space we only show one branch.

\[
\begin{array}{c}
\emptyset; A, \bot \quad \begin{array}{c}
\text{E} \\
\text{(eIrr)}
\end{array} \quad \bot; A \quad \begin{array}{c}
\text{E} \\
\text{(eId)}
\end{array} \quad \bot; A \quad \begin{array}{c}
\text{E} \\
\text{(eId)}
\end{array} \\
\hline
\emptyset; A, \neg A \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array} \\
\emptyset; K A, \neg A \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array} \\
\emptyset; \emptyset \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array}
\end{array}
\]

\[
\emptyset; A \quad \begin{array}{c}
\text{eIrr}
\end{array} \]

\[
\emptyset; \bot \quad \begin{array}{c}
\text{eId}
\end{array} \]

\[
\emptyset; \bot \quad \begin{array}{c}
\text{eId}
\end{array}
\]

Example 3 Calculus Liel proves the double negation of the classical reflection: $\neg \neg (K A \rightarrow A)$. The instantiation of $(\rightarrow R)$ gives two equal premises, thus we do not show the branch. Moreover, since $\bot$ in the third compartment is irrelevant, we disregard it in the second premise of $(\rightarrow L)$ application.

\[
\begin{array}{c}
\bot; A \quad \begin{array}{c}
\text{E} \\
\text{(eId)}
\end{array} \quad \bot; A \quad \begin{array}{c}
\text{E} \\
\text{(eId)}
\end{array} \quad \bot; A \quad \begin{array}{c}
\text{E} \\
\text{(eId)}
\end{array} \\
\hline
\emptyset; K A \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array} \quad \bot; \bot \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array} \\
\emptyset; \bot \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array} \\
\emptyset; \emptyset \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array}
\end{array}
\]

\[
\emptyset; \bot \quad \begin{array}{c}
\text{eIrr}
\end{array} \]

\[
\emptyset; K A \quad \begin{array}{c}
\text{eIrr}
\end{array}
\]

\[
\emptyset; \emptyset \quad \begin{array}{c}
\text{eIrr}
\end{array}
\]

Example 4 Liel does not prove $K p \rightarrow p$ (where $p \in V$), an instance of the classical reflection $K A \rightarrow A$. As a matter of fact, the following is the unique sequent tree for $K p \rightarrow p$ that we can build with the rules of Liel:

\[
\begin{array}{c}
\bot; p \quad \begin{array}{c}
\text{E} \\
\text{(eId)}
\end{array} \quad \bot; p \quad \begin{array}{c}
\text{E} \\
\text{(eId)}
\end{array} \quad \bot; p \quad \begin{array}{c}
\text{E} \\
\text{(eId)}
\end{array} \\
\hline
\emptyset; K p \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array} \quad \bot; \bot \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array} \\
\emptyset; \emptyset \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array}
\end{array}
\]

\[
\emptyset; K p \quad \begin{array}{c}
\text{eIrr}
\end{array} \]

We remark that in the construction of the sequent tree for $K p \rightarrow p$ there are no choices in rule application. This implies that there is exactly one sequent tree for $K p \rightarrow p$. We conclude that Liel does not prove $\emptyset; \emptyset \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array}$ and hence Liel fails to prove the classical reflection principle $K A \rightarrow A$.

As regards last example, we wonder if Liel does not prove $K p \rightarrow p$ because of the invalidity of the formula or because of Liel being incomplete. We address the question of the completeness of Liel in the following section.

4 Completeness

In this section we prove that given a formula $A$, if there is no proof-tree for $\emptyset; \emptyset \quad \begin{array}{c}
\text{E} \\
\text{(KL)}
\end{array}$, then $A$ is not a valid formula. In particular, we design the procedure Piel that given a sequent $\sigma$ uses Liel to build a proof of $\sigma$, if any, otherwise it returns a model whose root realizes $\sigma$. In general, a given sequent $\sigma$ can instantiate more than one rule, moreover $\sigma$ can instantiate a given rule in different ways. Thus every step in proof search is a backtracking point. It is well-known that backtracking can be avoided without losing completeness when the existence of a model realizing a sequent in the premise of a rule $R$ implies the existence of a model realizing the sequent in the conclusion of $R$. If all the premises of $R$ have this feature,
then R is called invertible [13]. It is easy to check that the rules $(\rightarrow R)$, $(\rightarrow L)$, $(KR)$, $(e \rightarrow R)$, $(e \rightarrow L)$ and $(KL)$ are not invertible, because the existence of a model realizing their rightmost premise does not imply the existence of a model realizing the conclusion. Provide Piel bounds backtracking by applying rule $(KL)$ in Step (6) when no other rule is applicable and by applying the rules $(\rightarrow R)$, $(\rightarrow L)$, $(KR)$, $(e \rightarrow R)$ and $(e \rightarrow L)$ in Step 5 when no other rule of the calculus apart $(KL)$ is applicable. This order in rule application bounds the backtracking and preserves completeness (further comments related to avoiding backtracking and Step 6 are provided after the description of Piel).

A consequence of the rules of Liel for the the connectives $\rightarrow$ and $KA$ is that the failure in proof search allows getting a model of minimum depth. This is possible if all the branches generated by rule instantiation are developed in a systematic way. Procedure Piel is designed to return models of minimum depth. To this aim, when rules with more than one premise are instantiated, some machinery related to model construction is required in order to compare the depth of the models in hand.

Procedure Piel($\sigma$)

1. if $\sigma$ can instantiate an axiom, then return the proof $\sigma(R)$;

where $R \in \{\text{Irr, Id, elrr, eId}\}$;

2. if $\sigma$ is a flat sequent $\Theta; \Gamma \models \Delta$ or $\Theta; \Gamma \overset{E}{\models} \Delta$, then return the structure $\mathcal{K} = \langle S, \rho, \leq, \models, E \rangle$

where $S = \{\rho\}$, $\leq = \{(\rho, \rho)\}$, $E = (\rho, \rho)$ and $\models = \{\rho\} \times \Gamma$;

3. if $\sigma$ can instantiate R $\in \{\land L, \lor R, e \land L, e \lor R, eKL, eKR\}$, then let $\sigma_1$ be the result of an instantiation of R with $\sigma$. Let $U = \text{Piel}(\sigma_1)$. If $U$ is a structure, then return $U$, otherwise return $\bigcup\limits_{\sigma} U(\sigma)$;

4. if $\sigma$ can instantiate R $\in \{\lor L, \land R, e \lor L, e \land R\}$, then let $\sigma_1$ and $\sigma_2$ be the result of an instantiation of R with $\sigma$. For $i = 1, 2$, let $U_i = \text{Piel}(\sigma_i)$;

(a) if $U_1$ and $U_2$ are proofs, then return $U_1$;

(b) if exactly one between $U_1$ and $U_2$ is a structure, then return the structure $U_i$, where $i \in \{1, 2\}$;

(c) if depth($U_1$) < depth($U_2$), then return $U_1$, else return $U_2$;

5. if $\sigma$ can instantiate one of the rules in $\{(\rightarrow R), (\rightarrow L), (KR), (e \rightarrow R), (e \rightarrow L)\}$ then:

(a) let nonInv = $\emptyset$ and Inv = $\emptyset$;

(b) for every $R \in \{\rightarrow R, KR, e \rightarrow R\}$, for every possible instantiation of R with $\sigma$:

(i) let $\sigma_1$ and $\sigma_2$ be the result of the instantiation of R with $\sigma$, let $U_1 = \text{Piel}(\sigma_1)$ and $U_2 = \text{Piel}(\sigma_2)$;

(ii) if $U_1$ and $U_2$ are proofs, then return $U_1$;

(iii) if $U_1$ is a structure, then let Inv = Inv $\cup$ $\{U_1\}$;

(iv) if $U_2$ is a structure, then let nonInv = nonInv $\cup$ $\{U_2\}$

(c) for every $R \in \{\rightarrow L, e \rightarrow L\}$, for every possible instantiation of R with $\sigma$:

(i) let $\sigma_1, \sigma_2$ and $\sigma_3$ be the result of the instantiation and let $U_1 = \text{Piel}(\sigma_1)$, $U_2 = \text{Piel}(\sigma_2)$ and $U_3 = \text{Piel}(\sigma_3)$;
(ii) if $U_1, U_2$ and $U_3$ are proofs, then return $\frac{U_1 \quad U_2 \quad U_3}{\sigma} (\rightarrow L)$;

(iii) if $U_1$ is a structure, then let $\text{Inv} = \text{Inv} \cup \{U_1\}$;

(iv) if $U_2$ is a structure, then let $\text{Inv} = \text{Inv} \cup \{U_2\}$;

(v) if $U_3$ is a structure, then let $\text{nonInv} = \text{nonInv} \cup \{U_3\}$;

(d) if the cardinality of nonInv coincides with the number of possible instantiations of the rules ($\rightarrow L$), ($\rightarrow R$), ($\mathbf{K} R$), ($e \rightarrow R$) and ($e \rightarrow L$) with $\sigma$, then:

- let $M_1, \ldots, M_n$ be an enumeration of the structures in nonInv, where for $i = 1, \ldots, n$, $M_i = (S_i, \rho_i, \leq_i, \models_i, E_i)$. We suppose that for $i, j = 1, \ldots, n$, if $i \neq j$, then $S_i \cap S_j = \emptyset$. Let $\mathcal{M} = \langle S, \rho, \leq, \models, E \rangle$ be defined as follows:

$$S = \{\rho\} \cup \bigcup_{i=1}^n S_i;$$

$$\leq = \left(\{\rho\} \times S\right) \cup \bigcup_{i=1}^n \leq_i;$$

$$E' = \bigcup_{i=1}^n E_i \cup \{(\rho, \alpha) \in \{\rho\} \times S \mid \text{there exists } \beta \in S \text{ such that } (\beta, \alpha) \in \bigcup_{i=1}^n E_i\}$$

if $\sigma = \emptyset$; $\Gamma \overset{E}{\longrightarrow} \Delta$, then $E = E' \cup \{(\rho, \rho)\}$ else $E = E'$;

$$\models = \left\{(\rho, p) \in \{\rho\} \times V \mid p \in \Gamma\right\} \cup \bigcup_{i=1}^{n} \models_i,$$ where $\Gamma$ is the second compartment of $\sigma$;

otherwise let $\mathcal{M}$ be undefined;

(e) if $\text{Inv} = \emptyset$, then return $\mathcal{M}$;

(f) let $\text{mindepth} = \min\{\text{depth}(U) \mid U \in \text{Inv}\}$ and $U \in \{U' \in \text{Inv} \mid \text{depth}(U') = \text{mindepth}\}$;

(g) if $\mathcal{M}$ is undefined, then return $U$;

(h) if $\text{depth}(\mathcal{M}) < \text{mindepth}$, then return $\mathcal{M}$, else return $U$;

(6) let $\sigma_1$ and $\sigma_2$ be the result of an instantiation of rule ($\mathbf{K} L$) with $\sigma$. Let $U_i = \text{Piel}(\sigma_i)$, for $i = 1, 2$;

(a) if $U_1$ and $U_2$ are proofs, then return $\frac{U_1 \quad U_2}{\sigma} (\mathbf{K} L)$;

(b) if $U_1$ is a structure and $U_2$ a proof, then return $U_1$;

(c) let $U_2$ be the structure $\langle S', \rho', \leq', \models', E' \rangle$. Let $\mathcal{M}$ be the structure $\langle S, \rho, \leq', \models', E \rangle$ defined as follows:

$$S = \{\rho\} \cup S';$$

$$\leq = \left(\{\rho\} \times S\right) \cup \leq';$$

$$E = \left\{(\rho, \alpha) \in \{\rho\} \times S \mid \text{there exists } (\rho', \alpha) \in E' \right\} \cup E';$$

$$\models = \left\{(\rho, p) \in \{\rho\} \times V \mid p \in \Gamma\right\} \cup \models',$$ where $\Gamma$ is the second compartment of $\sigma$.

(d) if $U_1$ is a proof, then return $\mathcal{M}$;

(e) if $\text{depth}(U_1) < \text{depth}(\mathcal{M})$, then return $U_1$, else $\mathcal{M}$.

End Procedure Piel

To summarize the steps of Piel: Step (1) is non-recursive and applies the axioms; Step (2) is non-recursive and it is performed when $\sigma$ cannot instantiate any rule. All the subsequent steps are recursive: Step (3) applies the invertible rules with one premise; Step (4) applies the
invertible rules with more than one premise. To return the model with minimal depth some checks are performed; Step (5) applies the non-invertible rules of Liel but rule ($K_L$). The step is lengthy because backtracking is required. Backtracking is implemented in the iterations at Steps (5.b) and (5.c) and it is broken if the condition at Step (5.b.ii) or (5.c.ii) is fulfilled. If no proof is found, then $\text{Inv} \cup \text{nonInv} \neq \emptyset$ (note that the cases $\text{Inv} = \emptyset$ or $\text{nonInv} = \emptyset$ are possible) and a structure is returned (in Theorems 3 and 5 we prove that the structure is a Kripke model of minimal depth). If there exists an iteration in Step (5.b) such that $U_2$ is a proof or there exists an iteration in Step (5.c) such that $U_3$ is a proof, then in Step (5.d) the structure $M$ is not built because it would not fulfill the statement of Proposition 1 (Proposition 1 is the main result to get the completeness of Liel). If $M$ is not built, then Step (5.g) returns an element of Inv. This is analogous to model theoretic constructions for $\text{Int}$ [4–6]. Step (6) instantiate rule ($K_L$) and no backtracking is performed. It is easy to check that when Step (6) is reached, the sequent $\sigma$ can instantiate rule ($K_L$) only because second compartment contains propositional variables and $K$-formulas only and third compartment contains atoms only. In Proposition 1 we prove that under the stated conditions, if rule ($K_L$) is instantiated, then the backtracking can be avoided because from the realizability of the rightmost premise we deduce the realizability of the conclusion. In the proof of Theorem 5 we discuss that Piel returns a minimal model despite rule ($K_L$) is not part of Step (5).

We start by proving that if Piel returns a structure $K$, then $K$ fulfils the definition of Kripke model for IEL. This is the first point to prove the completeness theorem.

Theorem 3 (Kripke Model) Let $\sigma$ be a sequent . If Piel($\sigma$) returns a structure $K$, then $K$ is a Kripke model for IEL.

Proof We proceed by induction on the depth of the recursive calls.

Base: Piel($\sigma$) returns $K$ without performing any recursive call. Then $K$ is the result of the construction in Step (2) and $\sigma$ is flat. It is immediate to check that the definition of $K$ at Step (2) fulfills the definition of Kripke model for IEL.

Induction: Piel returns $K$ by performing recursive calls that by induction hypothesis return a Kripke model. We proceed by considering every step of Piel where a structure is returned.

$K$ is the result of performing Step (3), Step (4.b) or Step (4.c). Immediate, since the elements returned by Piel are built by the recursive calls and hence, by induction hypothesis, they are Kripke models.

$K$ is the result of performing Step (5.e). We have to show that the returned structure $M$ is a Kripke model. Here we have two main cases: (i) $\sigma$ is an E-sequent. The rules that can be instantiated with $\sigma$ are $(e \rightarrow L)$ and $(e \rightarrow R)$; (ii) $\sigma$ is not an E-sequent. The rules that can be instantiated with $\sigma$ are $(\rightarrow L), (\rightarrow R), (K_L)$ and $(K_R)$. In the following we handle the two cases in a row, without any further distinction. Since $M$ at Step (5.e) is returned, $\text{Inv} = \emptyset$ holds. This means that the sequents $\sigma_1$ and $\sigma_2$ resulting from every possible instantiation of the rules $\{(\rightarrow L), (e \rightarrow L)\}$ with $\sigma$ and the sequent $\sigma_1$ resulting from every possible instantiation of the rules $\{(\rightarrow R), (e \rightarrow R), (K_L), (K_R)\}$ with $\sigma$ are provable, that is there is no Kripke model realizing them. By construction of Piel, it follows that if Step (5.e) is performed, then $\sigma_r$ resulting from every possible instantiation of the rules $\{(\rightarrow L), (e \rightarrow L), (\rightarrow R), (e \rightarrow R), (K_L), (K_R)\}$ with $\sigma$ has a Kripke model (otherwise a proof would have been returned). Procedure Piel collects all these models in nonInv. We remark that for every rule $R$ involved in Step (5), the propositional variables occurring in the second compartment of the conclusion of $R$ also occur in the second compartment of the premises. Thus, for every $U \in \text{nonInv}$, the root $\rho'$ of $U$ forces the propositional variables in $\Gamma$. Since $\rho'$ is an immediate successor of $\rho$ and by definition of forcing in $\rho$, it follows that $\models$ fulfills the monotonicity property and the returned structure $M$ obeys the definition
of Kripke model for propositional intuitionistic logic. As regards the definition of $E$, for every $U \in \text{nonInv}$, if $E'(\rho', \alpha)$ holds, where $\rho'$ is the root of $U$, $\alpha$ a world of $U$ and $E'$ the $E$-relation of $U$, then, $E(\rho, \alpha)$ holds by construction of $\mathcal{M}$. Thus $\mathcal{M}$ obeys the definition of Kripke model for IEL and we have proved the statement of the theorem.

$K$ is the result of performing Step (5.g), Step (5.h) or Step (6.b)

Immediate by using induction hypothesis.

$K$ is the result of performing Step (6.d). Analogous to the proof for Step (e) applied to only one model (that is $n = 1$).

$K$ is the result of performing Step (6.e). Immediate since $U_1$ and $\mathcal{M}$ are structures that Piel returns in Step (b) or Step (d) and we have already proved that they satisfy the statement of the theorem.

By Theorem 3 we have that given a sequent $\sigma$, if $\text{Piel}(\sigma)$ returns a structure $K$, then $K$ is a Kripke model for IEL. In the following Proposition 1 we prove that the structure $K$ realizes $\sigma$. This is the main step to prove Theorem 4, the completeness of calculus IEL.

**Proposition 1** (Realizability) Let $\sigma$ be a sequent. If $\text{Piel}(\sigma)$ returns a structure $K$, then the root of $K$ realizes $\sigma$.

**Proof** Let $K = \langle S, \preceq, \rho, \Vdash, E \rangle$. The sequent $\sigma$ is of the kind $\Theta; \Gamma \implies \Delta$ or $\Theta; \Gamma \implies E \Delta$. We recall from Sect. 2 that if $\Theta; \Gamma \implies E \Delta$ is realized, then $\Theta; \Gamma \implies \Delta$ is realized. We proceed by induction on the depth of the recursive calls.

**Base**: $\text{Piel}(\sigma)$ returns $K$ without performing any recursive call. Then $K$ is the result of the construction in Step (2) and $\sigma$ is flat. Since $\sigma$ is flat, from $\Gamma \subseteq V$ and by definition of $\Vdash$, immediately follows that $\rho \Vdash \left\langle \right. \Gamma$ holds; from $\Delta \subseteq \text{At}$ and $\Gamma \cap \Delta = \emptyset$ follows that for every $B \in \Delta$, $\rho \nVdash B$. By the fact that $K$ only contains the world $\rho$, it holds that for every $\alpha \in S$, if $\rho < \alpha$, then $\alpha \nVdash \left\langle \right. \Theta$. Moreover $E(\rho, \rho)$ holds. Thus we have proved that $\rho \Vdash \sigma$.

**Induction**: Piel returns $K$ by performing recursive calls that by induction hypothesis return Kripke models fulfilling the statement of the proposition. We proceed by considering every step of Piel where a Kripke model is returned.

$K$ is the result of performing Step (3). By induction hypothesis $K$ realizes $\sigma_1$, where $\sigma_1$ is the result of the instantiation of $R \in \{ (\land), (\lor), (\vee) \}, (e \land L), (e \lor R), (eK), (eKR) \}$ with $\sigma$. We proceed by cases: if $R$ is $(eK)$, then $\sigma_1$ is of the kind $\Theta; \Gamma \implies B. \Delta$. Since by induction hypothesis $K$ realizes $\sigma_1$, we have that $\rho \nVdash B$ and $E(\rho, \rho)$ hold. By the semantics of K follows that $\rho \Vdash KB$ and thus $\rho$ realizes $\sigma$; if $R$ is $(eK)$, then $\sigma_1$ is of the kind $\Theta; \Gamma, A \implies \Delta$. Since $\rho \Vdash A$, by monotonicity, $A$ holds in every world of $K$ and we immediately get $\rho \Vdash KA$ and $\rho$ realizes $\sigma$. The other cases are similar.

$K$ is the result of performing Step (4.b). By induction hypothesis, $K$ realizes $\sigma_i$. To prove that $K$ realizes $\sigma$ we proceed by cases according to the possible values of $R$. If $R$ is $(e \lor L)$, then, since by induction hypothesis $K$ realizes $\sigma_i$, it follows that $E(\rho, \rho)$ and $\rho \Vdash A \lor B$ hold. Thus we have proved that $K$ realizes $\sigma$. The other cases of $R$ are similar.

$K$ is the result of performing Step (4.c). By induction hypothesis $U_1$ and $U_2$ fulfil the statement of the proposition on respectively $\sigma_1$ and $\sigma_2$. At this point we get the statement of the proposition on $\sigma$ by applying to the returned model the proof provided for Step (4.b).

$K$ is the result of performing Step (5.e). At this stage of the procedure, the formulas in $\Gamma \cup \Delta$ are atoms, implications and $K$-formulas. We remark that if $\sigma = \Theta; \Gamma \implies \Delta$, then no $K$-formula belongs to $\Gamma$. We prove that $\rho$ realizes $\sigma$ by analyzing the forcing relation between $\rho$ and the formulas occurring in every compartment of $\sigma$:

- for every $A \in \Delta \cap \text{At}$, $\rho \nVdash A$ follows from the definition of $\Vdash$ and $\Gamma \cap \Delta = \emptyset$;
– For every rule $R$ involved in Step (5), the second compartment of the rightmost premise of $R$ includes the first compartment of the conclusion of $R$. Thus, by induction hypothesis, every root of every model in nonInv forces the formulas in the first compartment of $\sigma$;
– for every $A \rightarrow B \in \Gamma$, let $\sigma_2$ be the result of instantiating $\sigma$ with $(\rightarrow L)$ or $(e \rightarrow L)$ according to the type of $\sigma$. The recursive call $\text{Piel}(\sigma_2)$ returns a model $U$ that is collected in nonInv. Let $\rho'$ be the root of $U$. By induction hypothesis $\rho' \not\models \sigma_2$, hence $\rho' \not\models A$ and for every world $\alpha$ of $U$ different from $\rho'$, $\alpha \models B$. By the meaning of implication, we have that $\rho' \not\models A \rightarrow B$. We remark that all the $K$-formulas in $\Gamma$ are in the second compartment of $\sigma_2$, thus, by induction hypothesis, for every $KB \in \Gamma$, $\rho' \not\models KB$. By construction of $\mathcal{M}$ and Theorem 3, $\rho \not\models A$. For every model $U' \in \text{nonInv}$ different from $A$, we have that $U' = \text{Piel}(\sigma')$ and $A \rightarrow B$ is in the second compartment of $\sigma'$. By induction hypothesis $U'$ realizes $\sigma'$ and thus the root of $U'$ forces $A \rightarrow B$. Summarising, we have proved that for every $A \rightarrow B \in \Gamma$, $\rho \not\models A \rightarrow B$;
– for every $A \rightarrow B \in \Delta$, the recursive call $\text{Piel}(\sigma_2)$ returns a model $U$ collected in nonInv, with $\sigma_2 = \emptyset; \Gamma, \Theta, A \Rightarrow B$. Let $\rho'$ be the root of $U$. By induction hypothesis, $\rho' \not\models \sigma_2$, hence $\rho' \not\models A$ and $\rho' \not\models B$. Thus $\rho' \not\models A \rightarrow B$. By construction of $\mathcal{M}$ and Theorem 3 we have that $\rho \not\models A \rightarrow B$. Thus we have proved that for every $A \rightarrow B \in \Delta$, $\rho \not\models A \rightarrow B$;
– or every $KB \in \Delta$, the recursive call $\text{Piel}(\sigma_2)$ returns a model $U$ collected in nonInv, with $\sigma_2 = \emptyset; \Gamma, \Theta, A_1, \ldots, A_n \Rightarrow B$. By induction hypothesis, $\rho' \not\models A_1, \ldots, \rho' \not\models A_n$, thus $\rho' \not\models KA_1, \ldots, \rho' \not\models KA_n$, and $\rho' \not\models B$, with $\rho'$ root of $U$. By construction of $\mathcal{M}$, $E(\rho, \rho')$ holds. We get $\rho \not\models KB$. Thus we have proved that for every $KB \in \Delta$, $\rho \not\models KB$.

In the points above we have proved that for every $KA \in \Gamma$ and for every model $U \in \text{nonInv}$, the root of $U$ forces $KA$. Moreover by the last point above, if $E(\rho, \rho')$ holds, then $\rho' \not\models A$. Thus we have proved that for every $KA \in \Gamma$, $\rho \not\models KA$. Summarising, the points above prove that $\rho$ realizes $\sigma = \Theta$; $\Gamma \Rightarrow \Delta$. Finally, by construction of $\mathcal{M}$, if $\sigma = \Theta$; $\Gamma \Rightarrow \Delta$, $E(\rho, \rho')$ holds and thus we have proved that $\rho \not\models \sigma$ for any kind of $\sigma$.

$K$ is the result of performing Step (5.g). The point is proved by cases. We provide two of them:
– $U = \text{Piel}(\sigma_1)$, where $\sigma_1$ is the result of instantiating $KR$ with $\sigma$. Since $U = \mathcal{K}$, by induction hypothesis the root $\rho$ of $K$ realizes $\sigma_1$. This implies that $\rho \not\models B$, $\rho \models A_1, \ldots, \rho \models A_n$ and $E(\rho, \rho')$ hold. We immediately get that $\rho \models KA_1, \ldots, \rho \models KA_n$ and $\rho \not\models KB$ hold. Thus we have proved that $\rho$ realizes $\sigma$;
– $U = \text{Piel}(\sigma_2)$, where $\sigma_2$ is one of the sequent in the result of the instantiation of $(e \rightarrow L)$ with $\sigma$. Since $U = \mathcal{K}$, by induction hypothesis the following hold: $\rho \models \bigwedge \Theta$; $E(\rho, \rho)$; $\rho \not\models A$; for every $\alpha \in S$, if $\rho < \alpha$, then $\alpha \models B$. By the meaning of implication we have proved that $\rho \models A \rightarrow B$ and hence $\rho \not\models \sigma$.

The other cases are similar.

$K$ is the result of performing Step (5.h). If $\text{Piel}(\sigma)$ returns $\mathcal{M}$, then the proof goes as in the case of Step (5.e), otherwise the proof goes as in the case of (5.g).

$K$ is the result of performing Step (6.b). By induction hypothesis, $K$ realizes $\sigma_1$. Therefore, $\rho \models A$ and $E(\rho, \rho)$ and by the fact that the first compartment of $\sigma_1$ contains $\bot$, follows that $S = \{\rho\}$. Thus $\rho \models KA$. Since $\rho$ is the only world of $K$, the following holds: for every $\alpha \in S$, if $\rho < \alpha$, then $\alpha \models \bot$. Thus we have proved that $\rho$ realizes $\sigma$.

$K$ is the result of performing Step (6.d). By induction hypothesis $U_2$ is a Kripke model that realizes $\sigma_2$. This implies that $S' = \{\rho'\}$ and $E^*(\rho', \rho')$ hold. $E^*(\rho', \rho')$ implies that for every $KB \in \Gamma$, $\rho' \models B$. By construction of $\mathcal{M}$, $E(\rho, \rho')$ and $\rho \leq \rho'$ hold. Moreover the following holds: for every $KB \in \Gamma$, $\rho' \models B$. Thus by the semantical meaning of $K$ we have:
for every $KA$ occurring in the second compartment of $\sigma$, $\rho \models KA$. We recall that if we reach Step (6.d) the second compartment of $\sigma$ is included in $\forall \cup \{KA \in L\}$ and $\Delta \subseteq At$. By definition of $\models$, $\rho$ forces all the propositional variables in the second compartment of $\sigma$ and $\rho$ does not force any element in $\Delta$. Summarising we have proved that $\rho$ realizes $\sigma$.

$K$ is the result of performing (6.e). Immediate since $U_1$ and $M$ are structures that Piel returns in Step (6.b) or Step (6.d) and we have already proved that they satisfy the statement of the proposition.

From the proposition above we have the completeness theorem:

**Theorem 4** (Completeness) Let $A \in L$. If Piel($\emptyset$; $\emptyset \rightarrow A$) = $K$, then $A$ is not valid in $IEL$.

**Proof** By Proposition 1, the root $\rho$ of $K$ realizes $\emptyset$; $\emptyset \rightarrow A$. By definition of realizability, we have $\rho \not\models A$ and hence $A$ is not valid in $IEL$. □

**Remark 1** Let $h$ be the number of $K$-formulas in the second compartment of a given sequent $\sigma$. Rule $(KR)$ can be instantiated with $\sigma$ by choosing the value of $n$ between zero and $h$. Every possible choice of $n$ is correct. From the proof of Proposition 1 easily follows that completeness is preserved if for every $K$-formula in the third compartment of $\sigma$, rule $(KR)$ is instantiated once by setting the parameter $n$ to $h$, that is by considering in the rule application all the $K$-formulas occurring in the second compartment of $\sigma$.

**Remark 2** The following remark is useful to better understand the completeness of Piel and model construction in particular in connection to the rules $(\rightarrow L)$, $(\rightarrow R)$, $(KR)$ and $(KL)$.

It also introduces the proof of minimality, in particular as regards Steps (5.e), (5.g) and (5.h) of Piel.

In the discussion we assume that the formal parameter $\sigma$ of Piel is realizable and that $\sigma$ can instantiate rule $(KL)$ and at least one of the rules $(\rightarrow R)$, $(\rightarrow L)$ and $(KR)$. In practice we assume that $\sigma$ fulfills the conditions to perform Steps (5) and (6).

We highlight that despite rule $(KL)$ is a non-invertible rule and it is not involved in the backtracking phase, Proposition 1 shows that Piel is complete. Moreover, when Step (6) is performed and a model is returned, then the depth of the model is either one or two. Because of the occurrence of $\bot$ in the first compartment, the premises of $(KL)$ are realizable only by models of depth one.

In view of Theorem 5 some further remarks on the depth of the models realizing the premises of $(KL)$ are in order.

The leftmost premise $\sigma_1$ of rule $(KL)$ is invertible. Let us suppose that a model $M$ realizes $\sigma_1$. Then, by the invertibility of $\sigma_1$, $M$ realizes $\sigma$. By the fact that $\bot$ occurs in the first compartment of $\sigma_1$ we have that $M$ has one world. Let $\rho$ be the root of $M$.

If $A \rightarrow B \in \Delta$ holds, it follows that $\rho \models A$ and $\rho \not\models B$ hold and thus $M$ realizes the leftmost premise of $(\rightarrow R)$. If $A \rightarrow B \in \Gamma$ holds, we have two cases: (i) $\rho \models B$. Then it follows that $M$ realizes the leftmost premise of $(\rightarrow L)$; (ii) $\rho \not\models A$. Then it follows that $M$ realizes the second premise of $(\rightarrow L)$; if $KA \in \Delta$ holds, it follows that $\rho \not\models A$ and for every $KB \in \Gamma$, $\rho \models B$ and thus $M$ realizes the leftmost premise of $(KR)$.

Thus if there exists a model of depth one for the leftmost premise of $(KL)$, then there exists a model of depth one for at least one of the invertible premises of the rules in $\{(\rightarrow R), (\rightarrow L), (KR)\}$, it is collected in Inv and Piel can return it via Step (5).

**Remark 3** This remark is related to the proof of minimality for Step (5.h) of Piel. If the root $\rho$ of a model $K = (S, \rho, \leq, \models, E)$ realizes the conclusion $\sigma = \emptyset$; $\Gamma \rightarrow \Delta$ of rule $(KR)$ and

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for every $KA \in \Gamma$, $\rho \models A$ holds, then the root of the model $\mathcal{K}' = \langle S', \rho', \leq', \models', E' \cup \{(\rho, \rho')\} \rangle$ realizes both $\sigma$ and the premise $\sigma_1$ of $(KR)$. Thus, if the root of $\mathcal{K}$ realizes both $\sigma$ and the premise $\sigma_r$ of $(KR)$, then the root of the model $\mathcal{K}'$ realizes $\sigma$ and both premises of $(KR)$. This follows from the fact that if $\beta$ is a world such that $\alpha \leq \beta$ and for every $KA \in \mathcal{L}$, if $\alpha \models KA$ holds, then $\beta \models A$ holds, then the forcing on $\alpha$ does not change if $\beta$ is added to $E(\alpha)$. We also remark that for every $R \in \{(\rightarrow L), (\rightarrow R), (e \rightarrow L), (e \rightarrow R), (KL)\}$, if the root of $\mathcal{K}$ realizes both conclusion and the premise $\sigma_r$ of $R$, then the root of $\mathcal{K}$ also realizes one invertible premise of $R$. For the sake of clarity, we highlight that if we look at the premises $\sigma_{r-1}$ and $\sigma_r$ state the same property, but $\sigma$ and $\sigma_{r-1}$ refer to the same world, whereas $\sigma_r$ refers to a world which is subsequent to that $\sigma$ refers to.

Suppose $\mathcal{K} = \langle S, \rho, \leq, \models, E \rangle$ is a Kripke model and $\alpha \in S$. Following [12], we call truncated of $\mathcal{K}$ at $\alpha$ the structure $\mathcal{K}' = \langle S', \alpha, \leq', \models', E' \rangle$, where $S' = \{\beta \in S : \alpha \leq \beta\}$, $\leq' = \leq \cap (S' \times S')$, $\models' = \{(\beta, p) \in \models | \beta \in S'\}$, and $E' = E \cap (S' \times S')$. It is easy to show that $\mathcal{K}'$ is a model for IEL.

Now we show that Piel returns a Kripke model of minimum depth.

**Theorem 5** (Minimality) Let $\sigma$ be a sequent. If Piel($\sigma$) returns a structure $\mathcal{K}$, then

$$\text{depth}(\mathcal{K}) = \min\{\text{depth}(U) \mid U \text{ is a Kripke model with root } \rho \text{ and } \rho \models \sigma\}.$$  

**Proof** We proceed by induction on the depth of the recursive calls.

**Base:** Piel($\sigma$) returns $\mathcal{K}$ without performing any recursive call. Then $\mathcal{K}$ is the result of the construction in Step (2) and $\sigma$ is flat. The statement of the theorem immediately follows by the fact that $\mathcal{K}$ has one world.

**Induction:** Piel returns $\mathcal{K}$ by performing recursive calls that by induction hypothesis return Kripke models fulfilling the statement of the theorem.

$\mathcal{K}$ is the result of performing Step (3). Let us assume that there exists a model $\mathcal{K}' = \langle S', \rho', \leq', \models', E' \rangle$ that realizes $\sigma$ and $\text{depth}(\mathcal{K}') < \text{depth}(\mathcal{K})$. We use the induction hypothesis that $\mathcal{K}$ realizes the statement of the theorem on $\sigma_1$ to get a contradiction. We proceed by cases according to $R$: if $R$ is $(eKR)$, then $\sigma$ is of the kind $\Theta : \Gamma \Rightarrow E KB, \Delta$ and thus $\rho' \models E' KB$. This implies that there exists $\alpha \in S'$ such that $\rho' \leq' \alpha$, $E'(\rho', \alpha)$ and $\alpha \not\models B$. Thus $\rho' \not\models B$ and $\mathcal{K}'$ realizes $\sigma_1$. $\mathcal{K}'$ is a model such that $\text{depth}(\mathcal{K}') < \text{depth}(\mathcal{K})$ and this contradicts the induction hypothesis that there is no model realizing $\sigma_1$ with depth less than the depth of $\mathcal{K}$; if $R$ is $(eKL)$, then $\sigma$ is of the kind $\Theta : \Gamma, KA \Rightarrow E \Delta$ and thus $\rho' \not\models A$. Since $E'(\rho', \rho')$ holds, then $\rho' \not\models A$ holds and thus $\mathcal{K}'$ realizes $\sigma_1$. This is absurd because by induction hypothesis there is no model realizing $\sigma_1$ with depth less than the depth of $\mathcal{K}$. The other cases are similar.

$\mathcal{K}$ is the result of performing Step (4.b). Let us assume that there exists $\mathcal{K}'$ such that $\text{depth}(\mathcal{K}') < \text{depth}(\mathcal{K})$ and $\mathcal{K}'$ realizes $\sigma$. We go by cases on $R$: if $R$ is $(e \lor L)$, then $\mathcal{K}' \models A \lor B$. Moreover $E'(\rho', \rho')$ holds. Thus $\mathcal{K}'$ realizes $\sigma_1$ (since $\sigma_{3-i}$ has a proof). This contradicts the induction hypothesis on $\mathcal{K}$ for $\sigma_1$. The other cases of $R$ are similar.

$\mathcal{K}$ is the result of performing Step (4.c). By induction hypothesis $U_1$ and $U_2$ fulfill the statement of the theorem respectively for $\sigma = \sigma_1$ and $\sigma = \sigma_2$. Procedure Piel returns the model of minimum depth between $U_1$ and $U_2$. At this point we get the statement of the theorem by applying to the returned model the proof provided for Step (4.b).

$\mathcal{K}$ is the result of performing Steps (5.e)–(5.h). We remark that by induction hypothesis, the models collected in Inv and nonInv are of minimal depth, that is, for every instantiation
of the rules of the calculus with $\sigma$ and for every sequent $\sigma'$ in the result of the instantiation, if $\text{Piel}(\sigma')$ is a model, then $\text{Piel}(\sigma')$ is a model of minimal depth among the models that realize $\sigma'$. We recall that when Step (5) is reached, if $\sigma = \emptyset$ ; $\Gamma \models \Delta$, then $\sigma$ instantiates at least one among ($\rightarrow L$), ($\rightarrow R$), and $\text{(KL)}$, and possibly $\text{(KL)}$. Otherwise $\sigma$ instantiate at least one between ($e \rightarrow L$) and ($e \rightarrow R$). The full proof requires to consider Steps (5.e)-(5.h) and, for every step, the cases $\sigma = \emptyset$ ; $\Gamma \models \Delta$ and $\sigma = \emptyset$ ; $\Gamma \models L$. We start to prove the minimality of $\mathcal{K}$ when $\mathcal{K}$ is the result of performing Step (5.h). First we consider the case $\sigma = \emptyset$ ; $\Gamma \models \Delta$.

Since Step (5.h) is performed, sequent $\sigma$ can instantiate the rules ($\rightarrow L$), ($\rightarrow R$), $\text{(KL)}$ and $\text{(KR)}$ and $\sigma$ instantiates at least one among ($\rightarrow L$), ($\rightarrow R$) and $\text{(KR)}$. We proceed by giving a proof by contradiction. Let us suppose that there exists a model $\mathcal{K}' = \{S', \rho', \leq', \vdash', E'\}$ such that $\text{depth}(\mathcal{K}') < \text{depth}(\mathcal{K})$ and $\rho' \triangleright \sigma$. We show that the existence of $\mathcal{K}'$ is against the induction hypothesis that Inv and nonInv collect models of minimal depth. We recall that $\mathcal{K}$ is either $\mathcal{M}$ or $\mathcal{U}$ according to the result of the comparison performed in Step (5.h).

For every rule $R \in \{((\rightarrow L), (\rightarrow R), (\text{KR})\}$ and for every instantiation of $R$ with $\sigma$, the correctness of the calculus and $\rho' \triangleright \sigma$ imply that there exists at least a sequent in the result of the instantiation of $R$ that is realized by a world of $\mathcal{K}'$. The following holds: (i) For every rule $R \in \{((\rightarrow L), (\rightarrow R), (\text{KR})\}$ and for every possible instantiation of $R$ with $\sigma$, $\rho'$ does not realize any (instantiation of) the invertible premises $\sigma'$ of $R$, otherwise $\text{Piel}(\sigma')$ (we recall $\text{Piel}(\sigma') \in \text{Inv}$) is not model of minimal depth that realizes $\sigma'$, against the induction hypothesis. Summarizing, if we assume the existence of a model $\mathcal{K}'$ of depth lower than $\mathcal{K}$, then, by induction hypothesis, we conclude that for every $R \in \{((\rightarrow L), (\rightarrow R), (\text{KR})\}$ and for every possible instantiation of $R$ with $\sigma$, $\mathcal{K}'$ cannot realize the invertible premises of $R$.

Thus we remain with the case: (ii) for every rule $R \in \{((\rightarrow L), (\rightarrow R), (\text{KR})\}$ and for every possible instantiation of $R$ with $\sigma$, $\mathcal{K}' \triangleright \sigma_r$ holds (we recall that $\sigma_r$ denotes the rightmost premise of $R$ that, in the case of the rules involved in Step (5), are non-invertible). Then, there exists a world $\alpha$ of $\mathcal{K}'$, such that $\rho' \leq \alpha$ and $\alpha \triangleright \sigma_r$. Since by previous point a model with depth less than $\mathcal{K}$ cannot realize the invertible premises of $R$, we have that $\alpha \neq \rho'$ holds (see Remark 3).

Thus the depth of $\rho'$, and hence the depth of $\mathcal{K}'$, is at least the depth of $\alpha$ plus one. Now, to get the contradiction, we only need to consider a non-invertible premise $\sigma'_r$ such that $\text{depth}(\text{Piel}(\sigma'_r)) = \max \{\text{depth}(V) | V \in \text{nonInv}\}$, that is, it is sufficient to consider a $\sigma'_r$ such that the model $\text{Piel}(\sigma'_r)$ collected in nonInv is both a model of minimum depth (induction hypothesis) among the models that realize $\sigma'_r$ and a model of maximum depth among the models collected in nonInv. By construction, the depth of the model $\mathcal{M}$ defined by $\text{Piel}(\sigma)$ in Step (5.d) is $\text{depth}(\text{Piel}(\sigma'_r)) + 1$. Since we have proved that $\rho'$ does not realize any invertible premise, it follows that if the depth of $\mathcal{K}'$ is less than the depth of $\mathcal{M}$, then the depth of the world $\beta$ of $\mathcal{K}'$ that realizes $\sigma'_r$ is less than $\text{depth}(\text{Piel}(\sigma_r))$. Thus the truncated of $\mathcal{K}'$ at $\beta$ is a model that realizes $\sigma'_r$, hence $\text{Piel}(\sigma'_r)$ is not model of minimum depth among the models that realize $\sigma'_r$, against the induction hypothesis. Thus the depth of $\mathcal{K}'$ cannot be lower than $\mathcal{M}$. Since in Step (5.h) $\text{Piel}$ performs a comparison between the depth of $\mathcal{M}$ and $\mathcal{U}$, we have proved that the model returned in Step (5.h), with $\sigma = \emptyset$ ; $\Gamma \models \Delta$, has minimal depth. The case $\sigma$ $E$-sequent goes similarly to the proof given above and do not need to consider the case of the instantiation of $\text{(KL)}$ with $\sigma$.

$\mathcal{K}$ is the result of performing Step (5.e). First we consider the case $\sigma = \emptyset$ ; $\Gamma \models \Delta$.

$\text{Inv} = \emptyset$ immediately implies that the invertible premises of ($\rightarrow L$), ($\rightarrow R$), $\text{(KR)}$ and $\text{(KL)}$ are not realizable. The case of non-invertible premise is analogous to Point (ii) above. The proof for case $\sigma$ $E$-sequent goes as for Step (5.h).
\( \mathcal{K} \) is the result of performing Step (5.g). We handle the two cases of \( \sigma \) in a row. By construction of Piel, model \( \mathcal{K} \) is the model of minimal depth among those collected in Inv and there exists an instantiation of \( R \in \{ \langle \rightarrow L \rangle, \langle \rightarrow R \rangle, \langle K \rangle R \} \) with \( \sigma \) such that Piel(\( \sigma_r \)) is a proof. Thus \( \sigma_r \) is not realizable. By the correctness of the rules, there exists an invertible premise \( \sigma' \) of \( R \) such that \( \mathcal{K}' \triangleright \sigma' \). If \( \text{depth}(\mathcal{K}') < \text{depth}(\text{Piel}(\sigma')) \), then \( \text{Piel}(\sigma') \) does not return a model of minimal depth for \( \sigma' \), against the induction hypothesis. Thus \( \mathcal{K}' \) does not exist and \( \mathcal{K} \) is a model of minimal depth. Note that we have provided the proof for both kinds of sequents.

\( \mathcal{K} \) is the result of performing Step (6.b). Proved by contradiction using the facts: (a) there is no Kripke model realizing \( \sigma_2 \); (b) by induction hypothesis, \( \mathcal{K} \) is model of minimum depth realizing \( \sigma_1 \).

\( \mathcal{K} \) is the result of performing Step (6.d). By induction hypothesis \( U_2 \) is a model of minimum depth among the models realizing \( \sigma_2 \) and if we are at this step of Piel, then there is no model realizing \( \sigma_1 \). Let us suppose that there exists a model \( \mathcal{K} \) of depth one that realizes \( \sigma \). Let \( \rho \) be the root of \( \mathcal{K} \). Since we are at Step (d), \( \sigma = \Theta : \Gamma \implies \Delta \). Since \( \mathcal{K} \) has depth one, we have that \( \rho \) forces all the formulas in \( \Gamma \) and for every \( KA \in \Gamma, \rho \vdash A \) holds. Thus \( \mathcal{K} \) realizes \( \sigma_1 \), but this is impossible since by hypothesis no model realizes \( \sigma_1 \).

\( \mathcal{K} \) is the result of performing (6.e). In the previous cases we have already proved that \( U_1 \) and \( \mathcal{M} \) satisfy the statement of the theorem. \( \square \)

Example 5 In previous section we have argued that Liel does not prove the instance \( KA \to p \) of the classical reflection principle \( KA \to A \). There the argument was combinatorial. Now, by using Piel and the completeness theorem, we formalise that \( KA \to p \) is not a valid formula of the logic by providing a (counter)model that does not force it. We describe the steps performed by Piel(\( \varnothing ; \varnothing \implies KA \to p \)).

The sequent \( \varnothing ; \varnothing \implies KA \to p \), is used as actual parameter of Piel. Step (5.b.i) is reached and the recursive calls \( U_1 = \text{Piel}(\varnothing ; KA \to p \implies p) \) and \( U_2 = \text{Piel}(\varnothing ; KA \to p \implies p) \) are performed. With \( \varnothing ; KA \to p \to p \) as actual parameter, Step (6) is reached and the recursive calls \( U_1 = \text{Piel}(\varnothing ; p \implies p) \) and \( U_2 = \text{Piel}(\varnothing ; p \implies p) \) are performed:

- with actual parameter \( \bot ; p \implies p \), procedure Piel reaches Step (1), thus the proof \( \bot ; p \implies p ) \) is returned.

- with actual parameter \( \bot ; p \implies p \), procedure Piel reaches Step (2) because \( \bot ; p \implies p ) \) is a flat sequent and a Kripke model is returned.

After the two recursive calls at Step (6) terminate, since \( U_1 \) is a proof, Step (6.c) is performed and from the Kripke model

\[ U_2 = \{ \langle \rho' \rangle, \rho, \{ (\rho, \rho) \}, \{ (\rho', p) \}, \{ (\rho', \rho') \} \} \]

the Kripke model \( \mathcal{M} = \{ S, \rho, \leq, \models, E \} \) is defined as follows:

\[ S = \{ \rho, \rho' \}, \leq = \{ (\rho, \rho), (\rho, \rho'), (\rho', \rho'), (\rho', \rho') \}, \models = \{ (\rho', p) \}, E = \{ (\rho, \rho'), (\rho', \rho') \} \]

Since \( U_1 \) is a proof, \( \mathcal{M} \) is returned in Step (6.d). At this point we have that the two calls \( U_1 = \text{Piel}(\varnothing ; KA \to p) \) and \( U_2 = \text{Piel}(\varnothing ; KA \to p) \) in Step (5.b.i) return in both cases \( \mathcal{M} \). The sets Inv and nonInv are respectively updated at Steps (5.b.iii) and (5.b.iv). Since nonInv is not empty, Step (5.d) is performed and a structure of depth three is built.
Since Inv = \{U_1\} and U_1 has depth two, we get that Piel(∅; ∅ → KP → p) is the structure U_1 = M of depth two in Inv and it is returned performing Step (5.h).

Example 6 KP is invalid in IEL. We perform the call Piel(∅; ∅ → Kp). Since the actual parameter is Piel(∅; ∅ → Kp), Step (5.b.i) is reached and the recursive calls U_1 = Piel(∅; ∅ → p) and U_2 = Piel(∅; ∅ → p) are performed. Since ∅; ∅ → p and ∅; ∅ → p are flat sequents, in both calls Step (2) is reached and a Kripke model of depth one is returned. The sets Inv and nonInv are respectively updated at Steps (5.b.iii) and (5.b.iv). Since nonInv is not empty, Step (5.d) is performed and M of depth two is built. Finally Step (5.h) is reached and U_1 of depth one is returned as result of Piel(∅; ∅ → Kp). Note that the model proving the invalidity in IEL of Kp has a single world that E-reaches himself.

Example 7 K(p ∨ q) → (Kp ∨ Kq) is invalid. There are two possible completed trees of sequents. A part of them is provided in the following (the instantiation of R → gives two identical premises, thus we show only one).

\[
\begin{align*}
&\frac{\vdots}{\emptyset; p \lor q \rightarrow E p, Kq} \quad (KR) \quad \frac{\emptyset; p \rightarrow p (Id) \emptyset; q \rightarrow p}{\emptyset; p \lor q \rightarrow p} \quad (\lor L)\\
&\frac{\emptyset; K(p \lor q) \rightarrow Kp, Kq}{\emptyset; (Kp \lor Kq) \rightarrow (Kp \lor Kq)} \quad (\lor R)\\
&\frac{\emptyset; \emptyset \rightarrow K(p \lor q) \rightarrow (Kp \lor Kq) \rightarrow (Kp \lor Kq)}{\emptyset; p \lor q \rightarrow (Kp \lor Kq)} \quad (\lor L)\\
&\frac{\emptyset; K(p \lor q) \rightarrow Kp, Kq}{\emptyset; (Kp \lor Kq) \rightarrow (Kp \lor Kq)} \quad (\lor R)\\
&\frac{\emptyset; \emptyset \rightarrow K(p \lor q) \rightarrow (Kp \lor Kq) \rightarrow (Kp \lor Kq)}{\emptyset; \emptyset \rightarrow K(p \lor q)}
\end{align*}
\]

Note that there are two possible instantiations of KR, this explains the two trees of sequents. In both cases, the tree of sequents ends with a flat sequent. Thus we get two models. The collection of the trees of sequents is a way to represent the recursive calls of Piel. The rightmost branch of the first tree of sequents ends with the flat sequent ∅; q → p. From the construction of Piel there is a model with one world that realizes ∅; q → p. The same model realizes ∅; p ∨ q → p. Similarly for the second tree of sequents starting from the flat sequent ∅; p → q. Sequent ∅; K(p ∨ q) → Kp, Kq instantiates KR in two different ways, thus this is a backtracking point. The two models we have at hand are “glued” together using a new world in Step (5.d) and a model with three worlds is built. This model realizes the remaining sequents in the two branches we have considered in the two trees of sequents.

5 Refutational Calculus for IEL

Usually we are interested in designing calculi that prove the validity of a given formula A. As a result, in a model theoretic approach, the validity of A is witnessed by a proof and the invalidity is witnessed by a model that does not satisfy A.
This asymmetry between the validity, where a tree of sequents is returned and the invalidity, where a relational structure is returned, can be adjusted by designing a logical calculus to prove invalidity.

To keep the two aspects apart, calculi aimed to prove invalidity are called refutational calculi. The proofs built with refutational calculi are called refutations. A formula provable in a refutational calculus is called refutable. Since a refutable calculus aims to prove invalidity, if a formula is refutable, then there exists a world of a Kripke model that does not force it. This is the correctness of the refutational calculus. We also want that a formula which is invalid is refutable. This is the completeness of the refutational calculus.

The aim of this section is to present a refutational calculus for IEL. The work developed to prove the correctness and the completeness of Liel is useful also to prove correctness and completeness of the refutational calculus Riel provided in Fig. 3.

We remark that Liel and Riel share the same object language. The names of the rules of Riel are derived from those of Liel. The idea behind the calculus Riel is that if a given formula A is provable in Riel, then from the Riel-proof of an A A we can construct a model in which a world does not force A. In practice we are exploiting Proposition 1. We remark that Liel and Riel have the single-premise rules (L) and (R). Thus disjunctions on the left are backtracking points in Riel proof search. The motivation for this is clear in light of Proposition 1: roughly speaking, to prove the invalidity of a disjunction on the left it is sufficient to prove that one of the disjuncts is invalid. The same applies to conjunctions on the right.

Clearly, the duality between Liel and Riel is related to the fact that Liel is designed to prove validity, Riel to prove invalidity.

To prove that Riel is correct, we have to show that if a formula A is Riel-provable, then A is not valid in IEL. First of all we recall from Sect. 2 that in the cases of calculus Riel, the axioms are and flat sequents contain ⊥ in the second compartment or their second and third compartment share formulas. Note that in Riel the notions of axiom and flat are switched w.r.t. Liel.

Now we are ready to prove the correctness of Riel. This requires to prove that from a Riel-proof π of σ we can define a Kripke model for IEL. After that, by Proposition 2 we prove that the root of the Kripke model extracted from π realizes σ. Both these tasks have been already done:

**Proposition 2** (Realizability) Let σ be a sequent in the object language of Riel. Let π be a Riel proof of σ. Then there exists a Kripke model K = ⟨S, ρ, ≤, ⊩, E⟩, such that ρ ∨ σ.

**Proof** we proceed by induction on the depth of π. Note that this proposition corresponds to prove Theorem 3 and Proposition 1 for Liel, where π has the role of Piel. Thus we exploit the work already done and we provide only a proof sketch.
Axioms

\[ \Theta; \Gamma \rightarrow \Delta \quad (\text{Sat}) \quad \Theta; \Delta \rightarrow \Delta \quad (\text{eSat}) \]

provided \( \Gamma \subseteq V, \Delta \subseteq \text{At} \) and \( \Gamma \cap \Delta = \emptyset \)

Rules

(Proviso: rules apply iff \( \bot \) does not occur in the second compartment and
the second and third compartments do not share formulas)

\[
\begin{align*}
\Theta; A, B, \Gamma \rightarrow \Delta & \quad (\land L) \\
\Theta; A \land B, \Gamma \rightarrow \Delta & \quad (\land R_1) \\
\Theta; A_1, \Gamma \rightarrow \Delta & \quad (\lor L_1) \\
\Theta; A, \Gamma \rightarrow \Delta & \quad (\lor R_1) \\
\Theta; B, \Gamma \rightarrow \Delta & \quad (\rightarrow L_1) \\
\Theta; A \rightarrow B, \Gamma \rightarrow \Delta & \quad (\rightarrow R_1)
\end{align*}
\]

\[ \bot; A, \Gamma \rightarrow \Delta \quad (K L_1) \]

\[ \bot; \Theta; A, \Gamma \rightarrow \Delta \quad (K L_2), \text{ provided } \Delta \subseteq \text{At} \]

\[ \Theta; A, \Gamma \rightarrow B, \Delta \quad (\rightarrow R_1) \]

\[ \Theta; \Gamma \rightarrow A \rightarrow B, \Delta \quad (\rightarrow R_1) \]

\[ \Theta; A, \Gamma \rightarrow \Delta \quad (eK L) \]

\[ \Theta; A, \Gamma \rightarrow \Delta \quad (eK R) \]

\[ \Theta; A_1 \land A_2, \Gamma \rightarrow \Delta \quad (e \land L) \]

\[ \Theta; A \land B, \Gamma \rightarrow \Delta \quad (e \land R_i) \]

\[ \Theta; A_1 \lor A_2, \Gamma \rightarrow \Delta \quad (e \lor L_1) \]

\[ \Theta; A_1 \lor A_2, \Gamma \rightarrow \Delta \quad (e \lor R) \]

\[ \Theta; B, \Gamma \rightarrow \Delta \quad (e \rightarrow L_1) \]

\[ \Theta; A \rightarrow B, \Gamma \rightarrow \Delta \quad (e \rightarrow R_1) \]

\[ \Theta; A \rightarrow B, \Gamma \rightarrow \Delta \quad (e \rightarrow L_2) \]

\[ \Theta; A_1 \rightarrow A_2, \Gamma \rightarrow \Delta \quad (e \rightarrow R_1) \]

\[ \Theta; A \rightarrow B, \Gamma \rightarrow \Delta \quad (e \rightarrow L_2) \]

\[ \Theta; A \rightarrow B, \Gamma \rightarrow \Delta \quad (e \rightarrow R_1) \]

and the rules in Figure 4

Fig. 3 The refutational calculus Riel for IEL

**Base:** no rule is applied. Thus \( \pi \) coincides with \( \sigma \). Sequent \( \sigma \) is an axiom of Riel. This means that \( \sigma \) is a flat sequent of Liel. Let \( \mathcal{K} \) be defined as in Step (2) of Piel. By Theorem 3, \( \mathcal{K} \) is a Kripke model. Now the statement of the theorem is proved as in the base case of Proposition 1.

**Induction:** we proceed by cases according to the rule R that \( \sigma \) instantiates in the construction of \( \pi \). We assume that for every sequent \( \sigma' \) in the set resulting from the instantiation of R with \( \sigma \), there exists a Kripke model \( \mathcal{K}' \) with root \( \rho' \) such that \( \rho' \triangleright \sigma' \). We notice that for every possible value of R we have already proved the result in Theorem 3 and Proposition 1.
\[
B \cup A \implies \{0 \mid \Theta, \Gamma \implies B \} A \cup B \in \Delta \\
\Theta, \Gamma \implies \Delta \\
\text{where}
\]
\[
\Gamma \subseteq V \cup \{A \implies B \in \mathcal{L}\} \cup \{KA \in \mathcal{L}\} \\
\Delta \subseteq \mathcal{A} \cup \{A \implies B \in \mathcal{L}\} \cup \{KA \in \mathcal{L}\} \\
\Gamma_1 = \Gamma \setminus \{A \implies B\} \\
\Gamma_2 = (\Gamma \setminus \{KA \in \Gamma\}) \cup \{A \mid KA \in \Gamma\}
\]
\[
\{B ; \Theta, \Gamma_1 \implies A\} A \cup B \in \Delta \\
\{0 ; \Theta, \Gamma \implies B \} A \cup B \in \Delta \\
\Theta; \Gamma \implies \Delta \\
\text{where}
\]
\[
\Gamma \subseteq V \cup \{A \implies B \in \mathcal{L}\} \\
\Delta \subseteq \mathcal{A} \cup \{A \implies B \in \mathcal{L}\} \\
\Gamma_1 = \Gamma \setminus \{A \implies B\}
\]

Fig. 4 Glue rules for the refutational calculus Riel

As regards the rules in the sets \{(\land L), (\lor R), (e \land L), (e \lor R), (e K L), (e K R), (e L \lor I), (e \land I), \}, \{(\rightarrow L_1), (\rightarrow L_2), (\rightarrow R), (e \rightarrow L_1), (e \rightarrow L_2), (K R)\} and \{K L_1\} we extract a structure \(\mathcal{K}\) that coincides with \(\mathcal{K}'\) respectively following the Steps (3), (4.b), (5.g) and (6.b) of Piel. Theorem 3 proves that \(\mathcal{K}\) is a model immediately. As regards the realizability, in Proposition 1 see the proof respectively for \(\mathcal{K}\) is the result of performing Step (3), \(\mathcal{K}\) is the result of performing Step (4.b), \(\mathcal{K}\) is the result of performing Step (5.g) and \(\mathcal{K}\) is the result of performing Step (6.b).

As regards the remaining rules:

Rule \((K L_2)\): from \(\pi\) the structure \(\mathcal{K}\) is defined as in Step (d) of Piel using \(\mathcal{K}'\). Structure \(\mathcal{K}\) is proved to fulfill the definition of Kripke model as in Theorem 3. As regards the realizability, we proceed as in the proof of Proposition 1 for the case \(\mathcal{K}\) is the result of performing Step (d).

Rules \((Glue)\) and \((eGlue)\): the case \(\sigma = \Theta\); \(\Gamma \implies \Delta\) and \(\sigma = \Theta\); \(\Gamma \overset{E}{\implies} \Delta\) are analogous. The models obtained by induction from the premises of the rules are glued together as in Step (5.e) of Piel to define \(\mathcal{K}\). Now, we proceed as in Theorem 3 to prove that \(\mathcal{K}\) is a Kripke model and as in \(\mathcal{K}\) is the result of performing Step (5.e) of Proposition 1 to prove the realizability.

**Theorem 6** (Correctness of Riel) Let \(A \in \mathcal{L}\). If Riel proves \(\emptyset ; \emptyset \implies A\), then \(A\) is not valid in IEL.

**Proof** By Proposition 2, from a proof of Riel we can extract a Kripke model \(\mathcal{K} = (\mathcal{S}, \rho, \preceq, \models_E)\) such that \(\rho \models \emptyset ; \emptyset \implies A\). By the meaning of realizability of a sequent follows that \(\rho \not\models A\) and hence \(A\) is not valid in IEL.

As regards the completeness of Riel, we must show that every unprovable (irrefutable) formula in Riel is valid in IEL. To this aim we exploit the work already done for Liel providing the procedure PR that given \(\emptyset ; \emptyset \implies A\), returns a proof of Liel if \(A\) is valid and a proof of Riel if \(A\) is not valid. Procedure PR is a rewriting of procedure Piel, where for terseness we disregard the part related to minimality. We remark that the order in rule application is relevant and backtracking is required. Our comments for Piel also applies to PR.
Procedure PR(σ)

1. if σ can instantiate a Liel axiom, then return the Liel proof
   \[ \sigma(\text{R}) \]
   where R ∈ \{Irr, Id, eIrr, eId\};

2. if σ can instantiate a Riel axiom, then return the Riel proof
   \[ \sigma(\text{R}) \]
   where R ∈ \{Sat, eSat\};

3. if σ can instantiate R ∈ \{∧L, ∨R, e ∧ L, e ∨ R, eKL, eKR\}, then let \( σ_1 \) be the result of an instantiation of R with σ. Let \( U = PR(σ_1) \). Return \( \frac{U}{σ}(\text{R}) \), where \( \frac{U}{σ}(\text{R}) \) is a Liel or Riel proof according to the kind of U;

4. if σ can instantiate R ∈ \{∥L, ∧R, e ∥ L, e ∧ R\}, then let \( σ_1 \) and \( σ_2 \) be the result of an instantiation of R with σ. For \( i = 1, 2 \), let \( U_i = PR(σ_i) \);
   (a) If \( U_1 \) and \( U_2 \) are Liel proofs, then return \( U_1 U_2(\text{R}) \);
   (b) if \( U_1 \) is a Riel proof, then return \( U_1(\text{R}_1) \);
   (c) return \( U_2(\text{R}_2) \);

5. if σ can instantiate one of the rules in \{(→ R), (→ L), (KR), (e R), (e L)\} then:
   (a) let nonInv = ∅;
   (b) for every R ∈ \{→ R, KR, e R\}, for every possible instantiation of R with σ:
      (i) let \( σ_1 \) and \( σ_2 \) be the result of the instantiation of R with σ, let \( U_1 = PR(σ_1) \) and \( U_2 = PR(σ_2) \);
      (ii) if \( U_1 \) and \( U_2 \) are Liel proofs, then return \( \frac{U_1 U_2}{σ}(\text{R}) \);
      (iii) if \( U_1 \) is a Riel proof, then return \( \frac{U_1}{σ}(\text{R}_1) \);
      (iv) collect \( U_2 \): nonInv = nonInv ∪ \{U_2\}
   (c) for every R ∈ \{→ L, e → L\}, for every possible instantiation of R with σ:
      (i) let \( σ_1 \), \( σ_2 \) and \( σ_3 \) be the result of the instantiation and let \( U_1 = PR(σ_1) \), \( U_2 = PR(σ_2) \) and \( U_3 = PR(σ_3) \);
      (ii) if \( U_1 \), \( U_2 \) and \( U_3 \) are Liel proofs, then return \( \frac{U_1 U_2 U_3}{σ}(\text{L}) \);
      (iii) if \( U_1 \) is a Riel proof, then return \( \frac{U_1}{σ}(\text{L}_1) \);
      (iv) if \( U_2 \) is a Riel proof, then return \( \frac{U_2}{σ}(\text{L}_2) \);
      (v) collect \( U_3 \): nonInv = nonInv ∪ \{U_3\};
   (d) Let \( σ_1, \ldots, σ_n \) be an enumeration of nonInv. Return \( \frac{σ_1 \ldots σ_n}{σ}(\text{R}) \), where R ∈ \{(Glue), (eGlue)\};

6. let \( σ_1 \) and \( σ_2 \) be the result of an instantiation of KL with σ. Let \( U_i = PR(σ_i) \), for \( i = 1, 2 \);
   (a) If \( U_1 \) and \( U_2 \) are Liel proofs, then return \( \frac{U_1 U_2}{σ}(\text{KL}) \);
   (b) if \( U_1 \) is a Riel proof, then return \( \frac{U_1}{σ}(\text{KL}_1) \);
(c) return \( \frac{U_2}{\sigma} (KL_2) \);

End Procedure PR

By induction on the number of recursive calls, it is easy to show that given a sequent \( \sigma \), PR returns a Liel proof of \( \sigma \) or a Riel proof of \( \sigma \). By the correctness of Liel and Riel we can get in a row the completeness of Riel, which is the result that we have to prove, but also, in another form the completeness of Liel:

**Theorem 7** (Completeness of Riel (and Liel)) *Let \( A \) be a formula. If \( PR(\emptyset; \emptyset \implies A) \) returns a Riel proof, then \( A \) is not valid, otherwise \( A \) is valid*

**Proof** we have already remarked that given a sequent, PR returns a Liel proof or a Riel proof. Let \( A \) be an invalid formula. By the correctness of Liel, PR cannot return a proof of Liel for \( \emptyset; \emptyset \implies A \), thus PR returns a proof of Riel. Thus we have proved that all the invalid formulas have a proof in the refutational calculus Riel.

Note that with a dual argument we can prove the completeness of Liel, by using PR and the correctness of Riel.

\( \Box \)

As regards procedure PR, we remark that for every possible instantiation of \( \{ (\rightarrow R), (\rightarrow L), (KR), (e \rightarrow R) (e \rightarrow L) \} \) with \( \sigma \), we consider the Riel proof \( PR(\sigma_r) \). Note that the collection of the sequents \( \sigma_r \) coincides with the result of instantiating \( Glue \) or \( eGlue \) with \( \sigma \). Riel proofs collected in nonInv are the result of \( PR(\sigma_r) \).

**Example 8** Riel proves the invalidity of \( Kp \rightarrow p \) (where \( p \in V \)), an instance of the classical reflection \( KA \rightarrow A \).

\[
\begin{align*}
\emptyset; p &\quad \vdash (\text{eSat}) \\
\emptyset; Kp &\quad \rightarrow p \\
\emptyset; \emptyset &\quad \rightarrow Kp \rightarrow p \quad (\rightarrow R_1)
\end{align*}
\]

Note that in this proof we have two backtracking points. As regards the first backtracking point, sequent \( \emptyset; \emptyset \implies Kp \rightarrow p \) can also instantiate rule \( (Glue) \). In this case our choice is irrelevant because the result of the instantiation of \( (Glue) \) with \( \emptyset; \emptyset \implies Kp \rightarrow p \) is \( \emptyset; Kp \implies p \).

As regards the second backtracking point, sequent \( \emptyset; Kp \implies p \) can also instantiate \( (KL_1) \). In this case our choice is relevant, because the result of the instantiation of \( (KL_1) \) with \( \emptyset; Kp \implies p \) is the sequent \( \bot; p \implies p \), which is irrefutable.

**Example 9** Riel proves the invalidity of \( K(p \lor q) \rightarrow (Kp \lor Kq) \). There are two possible completed trees of sequents that are provided in the following (where in the \( (R \rightarrow) \) application we have two equal sequents, thus we show one branch only).

\[
\begin{align*}
\emptyset; p &\quad \rightarrow q \quad (\text{Sat}) \\
\emptyset; p \lor q &\quad \rightarrow q \quad (\lor L_1) \\
\emptyset; q &\quad \rightarrow p \quad (\text{Sat}) \\
\emptyset; p \lor q &\quad \rightarrow p \quad (\lor L_2) \\
\emptyset; K(p \lor q) &\quad \rightarrow Kp, Kq \\
\emptyset; K(p \lor q) &\quad \rightarrow Kp \lor Kq \quad (\lor R) \\
\emptyset; \emptyset &\quad \rightarrow K(p \lor q) \rightarrow (Kp \lor Kq) \quad (\rightarrow R_1)
\end{align*}
\]

We remark that in proof construction there are some backtracking points:

1. \( \emptyset; \emptyset \implies K(p \lor q) \rightarrow (Kp \lor Kq) \) can also instantiate rule \( (Glue) \). If one is not interested in minimality, this choice makes no difference;
(2) $\emptyset; K(p \lor q) \implies Kp \lor Kq$ can also instantiate $(KL_1)$;
(3) $\emptyset; K(p \lor q) \implies Kp, Kq$ can also instantiate $(KL_1)$, and, in two ways, $(KR_1)$;
(4) $\emptyset; p \lor q \implies p$ can also instantiate $(\lor L_1)$;
(5) $\emptyset; p \lor q \implies q$ can also instantiate $(\lor L_2)$.

By the work developed in Sect. 4 and in Proposition 2 it should be straightforward how-to extract a Kripke model whose root does not realize a sequent provable in Riel.

We conclude by remarking that by inspection of the rules, the depth of all Riel-trees $T$ is bounded by the number of connectives occurring in the sequent in the root of $T$.

6 Calculus for $\text{IEL}^-$

Logic $\text{IEL}^-$ lacks axiom (Ax 4). Semantically this means that property $(\text{Im} \; 3)$ on Kripke models does not hold.

We show that we get the complete calculus $\text{Liel}^-$ for $\text{IEL}^-$ by removing rule $KL$ from the logical apparatus for Liel. The intuition is related to Piel, that uses $KL$ in Step (6) when no other rule is applicable. When Piel performs Step (6), the following statement $(S)$ on $\Theta; \Gamma \implies \Delta$ holds:

$$(S) \; \Delta \subseteq \text{At}, \text{the second compartment is included in } \{KA \in \mathcal{L}\} \cup V \text{ and the intersection between second and third compartment is empty.}$$

The following formally justifies our choice and is part of the completeness theorem:

**Proposition 3** Any sequent $\sigma = \Theta; KA, \Gamma \implies \Delta$ that satisfies statement $(S)$ is realizable by the root of the Kripke model $\mathcal{K} = \langle S, \rho, \leq, \models, E \rangle$, where $S = \{\rho\}, \leq = \{(\rho, \rho)\}, \models = \{(\rho, p) | p \in \Gamma \cap V\}$ and $E = \emptyset$.

**Proof** Since $\Gamma$ and $\Delta$ do not share any element and by definition of forcing, $\rho \not\models \bigvee \Delta$. By definition of forcing, for every $p \in V$, if $p \in \Gamma$, then $\alpha \models p$. By the fact that $E = \emptyset$ and the semantics of $K$, for every $KA \in \Gamma, \alpha \models KA$. Since $K$ has a single world it is immediate that for every $\alpha \in S$, if $\rho < \alpha$, then $\alpha \models \bigwedge \Theta$. Hence we have proved that $\rho$ realizes $\sigma$. $\square$

Since there is no rule to handle directly $K$-formulas on the left, we have a new definition of *flat* sequent for calculus $\text{Liel}^-$ that extends the definition of flat sequent for Liel. Flat sequents for $\text{Liel}^-$ are $E$-sequent $\Theta; \Gamma \implies \Delta$ fulfilling $\Gamma \subseteq V, \Delta \subseteq \text{At}$ and $\Gamma \cap \Delta = \emptyset$ or sequents $\emptyset; \Gamma \implies \Delta$ fulfilling $\Gamma \subseteq \{KA \in \mathcal{L}\} \cup V, \Delta \subseteq \text{At}$ and $\Gamma \cap \Delta = \emptyset$. As usual, the definition of flat sequent is tailored to characterise a non-axiom sequent that does not instantiate any rule of the calculus.

We use the new definition of flat sequent to provide procedure $\text{Piel}^-$ that returns a proof of $\text{Liel}^-$ or a Kripke model for $\text{IEL}^-$. Because of the close relationships between Liel and $\text{Liel}^-$, we have that Procedure $\text{Piel}^-$ is a slight modification of Piel:

– erase Step (6) from procedure Piel;
– replace Step (2) of Piel with the following two steps:

(2a) if $\sigma$ is a flat sequent $\Theta; \Gamma \implies \Delta$, then return the structure

$$\mathcal{K} = \langle S, \rho, \leq, \models, E \rangle$$

where $S = \{\rho\}, \leq = \{(\rho, \rho)\}, \models = \{\rho\} \times (\Gamma \cap V)$ and $E = \{(\rho, \rho)\}$.
(2b) if $\sigma$ is a flat sequent $\Theta; \Gamma \Longrightarrow \Delta$, then return the structure

$$K = \langle S, \rho, \leq, \models, E \rangle$$

where $S = \{\rho\}$, $\leq = \{(\rho, \rho)\}$, $\models = \{\rho\} \times (\Gamma \cap V)$ and $E = \emptyset$.

Now, the proof that for every invalid formula $\text{Piel}^-$ returns a Kripke model for $\text{IEL}^-$ of minimal depth follows from Proposition 3 and is analogous to the proofs given for Theorems 3, 4 and 5.

**Example 10** Calculus $\text{Liel}^-$ does not prove $Kp \rightarrow \neg \neg p$, an instance of $(\text{Ax 4}) K A \rightarrow \neg \neg A$.

Rules $(\text{KL})$ and $(\rightarrow R)$ are instantiated with $\Delta = \{\bot\}$ and $\Theta = \emptyset$. Thus the premises are equal and to save space we only show one branch.

$$\emptyset; Kp, \bot \Longrightarrow \bot \; (\text{Irr}) \quad \emptyset; Kp \Longrightarrow p, \bot \; (e \rightarrow L)$$

$$\emptyset; Kp, \neg p \Longrightarrow \bot \; (\rightarrow R)$$

$$\emptyset; \emptyset \Longrightarrow Kp \rightarrow \neg \neg p \; (\rightarrow R)$$

From the flat sequent on the middle we extract the model of minimum depth that does not forces the given formula: it is a model with a single world that does not force any propositional variable and $E = \emptyset$.

To conclude the section, we remark that to get a refutational calculus $\text{Riel}^-$ for $\text{IEL}^-$ we modify the refutational calculus $\text{Riel}$ for Liel as follows:

- the sequents fulfilling the property stated in $(S)$ are axioms. Thus a sequent of the kind

$$\Theta; \Gamma \Longrightarrow \Delta \; (\text{kSat})$$

provided $\Gamma \subseteq \{KA \in L\} \cup V \cup A$ and $\Delta \subseteq A$ and $\Gamma \cap \Delta = \emptyset$.

is an axiom of the refutational calculus $\text{Riel}^-$;

- the rules $\text{KL}_1$ and $\text{KL}_2$ of Riel are not rules of $\text{Riel}^-$.

Now, by proceeding as for the case of Riel, we can prove that $\text{Riel}^-$ is a calculus that proves invalidity of formulas with respect to $\text{IEL}^-$.

### 7 Conclusions and Future Works

In this paper we have presented sequent calculi to prove validity and invalidity for the intuitionistic propositional logics of belief and knowledge $\text{IEL}$ and $\text{IEL}^-$. As for the case of propositional intuitionistic logic [4], we have shown that $\text{IEL}$ and $\text{IEL}^-$ have terminating calculi whose trees all have depth bounded by the number of connectives in the formula to be proved and obey the subformula property. For invalid formulas, our calculi allow us to get Kripke models of minimal depth. Compared with [4], the particular Kripke semantics characterising $\text{IEL}$ and $\text{IEL}^-$ requires an extension of the object language employed by the logical apparatus. As in the case of [4], the sequents are not standard and they have some features related to nested sequents [7]. Roughly speaking, a single sequent to be fulfilled (realized, in the terminology used in the paper) requires that the formulas in the first compartment are forced in all the successors of the world the second and third compartment refer to.

A possible future investigation is to get terminating calculi for intuitionistic epistemic logic following the ideas in [8, 14]. These calculi do not have the subformula property but
they use standard sequents. A feature of the calculi in [8] is that to get proofs in linear depth it can be required to introduce new propositional variables. Finally, to continue the investigation in terminating calculi for \( \text{IEL} \) fulfilling the subformula property the results of [5] for \( \text{Int} \) can be considered. Paper [5] presents a terminating strategy for the sequent calculus \( \text{G3i} \). The strategy builds finite trees by using the information in the sequent at hand only, no history mechanisms are required. The depth of the returned proofs is quadratic in the number of connectives occurring in the formula to be proved. In the case of \( \text{IEL} \), we could add to \( \text{G3i} \) a rule for connective \( \text{K} \) analogous to \( \text{K}_{\text{IEL}} \) of [11] and then modify accordingly the machinery of [5] to get the termination.

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