NONLOCAL FULLY NONLINEAR DOUBLE OBSTACLE PROBLEMS

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ABSTRACT. We prove the existence and $C^{1,\alpha}$ regularity of solutions to nonlocal fully nonlinear elliptic double obstacle problems. We also obtain boundary regularity for these problems. The obstacles are assumed to be Lipschitz semi-concave/semi-convex functions, and we do not require them to be $C^1$. Our approach is to adapt a penalization method to be applicable to the setting of nonlocal equations and their viscosity solutions.

MATHEMATICS SUBJECT CLASSIFICATION. 35R35, 47G20, 35B65.

1. Introduction

In this paper we consider the existence and regularity of solutions to the double obstacle problem

$$
\begin{cases}
\max\{\min\{-Iu-f, u-\psi^-\}, u-\psi^+\} = 0 & \text{in } U, \\
u = \varphi & \text{in } \mathbb{R}^n - U.
\end{cases}
$$

(1.1)

Here $I$ is a nonlocal elliptic operator, of which a prototypical example is the fractional Laplacian

$$
-(\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy.
$$

Nonlocal operators appear naturally in the study of discontinuous stochastic processes as the jump part of their infinitesimal generator. These operators have also been studied extensively in recent years from the analytic viewpoint of integro-differential equations. The foundational works of Caffarelli and Silvestre [4, 5, 6] paved the way and set the framework for such studies. They provided an appropriate notion of ellipticity for nonlinear nonlocal equations, and obtained their $C^{1,\alpha}$ regularity. They also obtained Evans-Krylov-type $C^{2s+\alpha}$ regularity for convex equations. An interesting property of their estimates is their uniformity as $s \uparrow 1$, which provides a new proof for the corresponding classical estimates for local equations.

Free boundary problems involving nonlocal operators have also seen many advancements. Silvestre [25] obtained $C^{1,\alpha}$ regularity of the obstacle problem for fractional Laplacian. Caffarelli et al. [7] proved the optimal $C^{1,s}$ regularity for this problem when the obstacle is smooth enough. Björland, Caffarelli, and Figalli [3] studied a double obstacle problem for the infinity

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fractional Laplacian which appear in the study of a nonlocal version of the tug-of-war game. Korvenpää et al. \cite{13} studied the obstacle problem for operators of fractional $p$-Laplacian type. Petrosyan and Pop \cite{17} considered the obstacle problem for the fractional Laplacian with drift in the subcritical regime $s \in (\frac{1}{2}, 1)$, and Fernández-Real and Ros-Oton \cite{11} studied the critical case $s = \frac{1}{2}$. There has also been some works on other types of nonlocal free boundary problems, like the work of Rodrigues and Santos \cite{19} on nonlocal linear variational inequalities with constraint on the fractional gradient.

A major breakthrough in the study of nonlocal free boundary problems came with the work of Caffarelli et al. \cite{8}, in which they obtained the regularity of the solution and of the free boundary of the obstacle problem for a large class of nonlocal elliptic operators. These problems appear naturally when considering optimal stopping problems for Lévy processes with jumps, which arise for example as option pricing models in mathematical finance. We should mention that in their work, the boundary regularity results of Ros-Oton and Serra \cite{20, 21} for nonlocal elliptic equations were also essential.

In this paper we prove $C^{1,\alpha}$ regularity of the double obstacle problem \cite{18} for a large class of nonlocal fully nonlinear operators $I$. In contrast to \cite{8}, we do not require the operator to be convex. We also allow less smooth obstacles, and do not require them to be $C^1$. And in addition to the interior regularity, we obtain the boundary regularity too. Our estimates in this work are uniform as $s \uparrow 1$; hence they provide a new proof for the corresponding regularity results for local double obstacle problems. We use these regularity results in an upcoming work to study nonlocal equations with constraint on the (classical) gradient. A particular consequence of those results is that the regularity for nonlocal double obstacle problems can be proved for even less smooth obstacles.

Let us also briefly mention some of the works on the (double) obstacle problem for local equations, and local equations with gradient constraints. The study of elliptic equations with gradient constraints was initiated by Evans \cite{10} when he considered the problem

$$\max\{Lu - f, |Du| - g\} = 0,$$

where $L$ is a (local) linear elliptic operator. Equations of this type stem from dynamic programming in a wide class of singular stochastic control problems. Recently, there has been new interest in these types of problems. Hynd and Mawi \cite{12} studied (local) fully nonlinear elliptic equations with strictly convex gradient constraints of the form

$$\max\{F(x, D^2u) - f, H(Du)\} = 0.$$

Closely related to these problems are variational problems with gradient constraints. An important example among them is the well-known elastic-plastic torsion problem, which is the problem of minimizing the functional $\int_U \frac{1}{2} |Dv|^2 - v \, dx$ over the set

$$W_{B_1} := \{v \in W^{1,2}_0(U) : |Dv| \leq 1 \text{ a.e.}\}.$$
An interesting property of variational problems with gradient constraints is that under mild conditions they are equivalent to double obstacle problems. For example the minimizer of \( \int_{U} G(Dv) \, dx \) over \( W_{B_{1}} \), also satisfies \(-d \leq v \leq d\) and

\[
\begin{cases}
-D_{i}(D_{i}G(Dv)) = 0 & \text{in } \{-d < v < d\}, \\
-D_{i}(D_{i}G(Dv)) \leq 0 & \text{a.e. on } \{v = d\}, \\
-D_{i}(D_{i}G(Dv)) \geq 0 & \text{a.e. on } \{v = -d\},
\end{cases}
\]

where \( d \) is the distance to \( \partial U \); see for example [22]. This problem can be more compactly written as

\[
\max\{\min\{F(x, D^{2}v), v + d\}, v - d\} = 0,
\]

where \( F(x, D^{2}v) = -D_{i}(D_{i}G(Dv)) = -D_{ij}^{2}G(x)D_{ij}^{2}v.\)

Variational problems with gradient constraints have also seen new developments in recent years. De Silva and Savin [9] investigated the minimizers of some functionals subject to gradient constraints, arising in the study of random surfaces. In their work, the functional is allowed to have certain kinds of singularities. Also, the constraints are given by convex polygons; so they are not strictly convex. In [24] we have studied variational problems with non-strictly convex gradient constraints, and we obtained their optimal \( C^{1,1} \) regularity. This has been partly motivated by the above-mentioned problem about random surfaces. There has also been similar interests in elliptic equations with gradient constraints which are not strictly convex. These problems emerge in the study of some singular stochastic control problems appearing in financial models with transaction costs; see for example [2, 18]. In [23] we extended the results of [12] and proved the optimal regularity for (local) fully nonlinear elliptic equations with non-strictly convex gradient constraints. Our approach was to obtain a link between double obstacle problems and elliptic equations with gradient constraints. This link has been well known in the case where the double obstacle problem reduces to an obstacle problem. However, we have shown that there is still a connection between the two problems in the general case. In this approach, we also studied (local) fully nonlinear double obstacle problems with singular obstacles. Finally, let us also mention the recent works [1, 15] on (local) double obstacle problems.

Now let us introduce the problem in more detail. First we recall some of the definitions and conventions about nonlocal operators introduced in [4]. Let

\[
\delta u(x, y) := u(x + y) + u(x - y) - 2u(x).
\]

A linear nonlocal operator is an operator of the form

\[
Lu(x) = \int_{\mathbb{R}^{n}} \delta u(x, y)K(y) \, dy,
\]

where the kernel \( K \) is a positive function which satisfies \( K(-y) = K(y) \), and

\[
\int_{\mathbb{R}^{n}} \min\{1, |y|^{2}\}K(y) \, dy < \infty.
\]
We say a function $u$ belongs to $C^{1,1}(x)$ if there are quadratic polynomials $P, Q$ such that $P(x) = u(x) = Q(x)$, and $P \leq u \leq Q$ on a neighborhood of $x$. A nonlocal operator $I$ is an operator for which $Iu(x)$ is well-defined for bounded functions $u \in C^{1,1}(x)$, and $Iu(\cdot)$ is a continuous function on an open set if $u$ is $C^2$ over that open set. The operator $I$ is uniformly elliptic with respect to a family of linear operators $\mathcal{L}$ if for any bounded functions $u, v \in C^{1,1}(x)$ we have

$$M^-_\mathcal{E}(u - v)(x) \leq Iu(x) - Iv(x) \leq M^+_\mathcal{E}(u - v)(x),$$

where the extremal Pucci-type operators $M^\pm_\mathcal{E}$ are defined as

$$M^-_\mathcal{E}u(x) = \inf_{L \in \mathcal{L}} Lu(x), \quad M^+_\mathcal{E}u(x) = \sup_{L \in \mathcal{L}} Lu(x).$$

Let us also note that $\pm M^\pm_\mathcal{E}$ are subadditive.

An important family of linear operators is the class $\mathcal{L}_0$ of linear operators whose kernels are comparable with the kernel of fractional Laplacian $-(\Delta)^s$, i.e.

$$(1 - s)\frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq (1 - s)\frac{\Lambda}{|y|^{n+2s}},$$

where $0 < s < 1$ and $0 < \lambda \leq \Lambda$. It can be shown that in this case the extremal operators $M^\pm_{\mathcal{L}_0}$, which we will simply denote by $M^\pm$, are given by

$$M^+u = (1 - s)\int_{\mathbb{R}^n} \frac{\Lambda\delta u(x, y)^+ - \lambda\delta u(x, y)^-}{|y|^{n+2s}} dy,$$

$$M^-u = (1 - s)\int_{\mathbb{R}^n} \frac{\lambda\delta u(x, y)^+ - \Lambda\delta u(x, y)^-}{|y|^{n+2s}} dy,$$

where $r^\pm = \max\{\pm r, 0\}$ for a real number $r$. Another important class is $\mathcal{L}_* \subset \mathcal{L}_0$ which consists of operators with homogeneous kernel, i.e.

$$K(y) = a(y/|y|)|y|^{n+2s} \quad \text{with} \quad (1 - s)\lambda \leq a(\cdot) \leq (1 - s)\Lambda.$$

The class $\mathcal{L}_*$ consists of all infinitesimal generators of stable Lévy processes belonging to $\mathcal{L}_0$, and appears in the study of boundary regularity of nonlocal equations (see [20, 21] for more details). We will denote the extremal operators $M^\pm_{\mathcal{L}_*}$ simply by $M^\pm_*$. Note that we have

$$M^-u \leq M^-_*u \leq M^+_*u \leq M^+u.$$

We will also only consider “constant coefficient” nonlocal operators, i.e. we assume that $I$ is translation invariant:

$$I(\tau_z u) = \tau_z(Iu)$$

for every $z$, where $\tau_z u(x) := u(x - z)$ is the translation operator. In addition, without loss of generality we can assume that $I(0) = 0$, i.e. the action of $I$ on the constant function 0 is 0. Because by translation invariance $I(0)$ is constant, and we can consider $I - I(0)$ instead of $I$. 

Now let us state our main results. We denote the distance to $\partial U$ by $d(\cdot) = d(\cdot, \partial U)$.

**Theorem 1.** Suppose $I$ is a translation invariant operator which is uniformly elliptic with respect to $L_0$, and $0 < s_0 < s < 1$. Also, suppose $U, \varphi, \psi^\pm$ satisfy Assumption 1 and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Then the double obstacle problem (1.1) has a viscosity solution $u$, and

$$u \in C^{1,\alpha}_{\text{loc}}(U)$$

for some $\alpha > 0$ depending only on $n, \lambda, \Lambda, s_0$. And for an open subset $V \subset U$ we have

$$\|u\|_{C^{1,\alpha}(V)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(U)} + C_V),$$

where $C$ depends only on $n, \lambda, \Lambda, s_0$, and $d(V, \partial U)$; and $C_V$ depends only on these constants together with $\|\psi^\pm\|_{L^\infty(\mathbb{R}^n)}$ and the semi-concavity norms of $\pm \psi^\pm$ on $V$.

**Remark.** Note that our estimate is uniform for $s > s_0$; so we can retrieve the interior estimate for local double obstacle problems as $s \to 1$.

Next we state our result about boundary regularity. As it is well known (see [20, 21]), the boundary regularity for nonlocal equations should be stated in terms of $u/d^s$, instead of $u$.

**Theorem 2.** Suppose in addition to the assumptions of Theorem 1, $\psi^\pm$ are $C^2$ on a neighborhood of $\partial U$ in $\mathbb{R}^n$, $\varphi$ is $C^2$ on $\mathbb{R}^n$, and $I$ is elliptic with respect to $L_*$. Then for $x_0 \in \partial U$ and $r$ small enough we have

$$\frac{(u - \varphi)}{d^s} \in C^{\tilde{\alpha}}_{\text{loc}}(B_r(x_0)),$$

where $d$ is the distance to $\partial U$, and $\tilde{\alpha} > 0$ depends only on $n, \lambda, \Lambda, s_0$. In addition, we have

$$\|(u - \varphi)/d^s\|_{C^{\tilde{\alpha}}(B_r(x_0))} \leq \tilde{C}(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(U)} + C_0),$$

where $\tilde{C}$ depends only on $n, \lambda, \Lambda, s, r, U$; and $C_0$ depends only on $n, \lambda, \Lambda, s_0$, and $\varphi, \psi^\pm$.

**Remark.** The constant $\tilde{C}$ in the above estimate can be chosen uniformly for $s > s_0$ if for example $\partial U \cap B_r(x_0)$ is flat, or $I$ is of the form $\inf_{\alpha} \sup_{\beta} L_{\alpha\beta}$ for a family of linear operators (see [20] for the exact statements and assumptions). However, to the best of author’s knowledge, currently there is no uniform in $s$ boundary estimate for arbitrary elliptic operators with respect to $L_*$ over arbitrary domains with $C^2$ boundary.

**Remark.** Note that unlike the local case (see [23]), here we need to require $\psi^\pm$ to be $C^2$ on a neighborhood of $\partial U$ in $\mathbb{R}^n$, not merely on a neighborhood of $\partial U$ in $\mathcal{U}$. This extra restriction, which is not satisfied by the obstacles arising in problems with gradient constraints, is imposed on us by the nonlocal nature of the problem.

Finally let us provide a brief sketch of the proof. We first approximate the obstacles with smoother obstacles, and use them to solve some approximate double obstacle problems. To solve the approximate problems, we use a penalization method, which is adapted to be applicable to the setting of nonlocal equations and their viscosity solutions. Next we obtain uniform estimates for the solutions to the approximate problems, and show that they converge to a solution of (1.1).
2. Proof of the Main Results

Let us start with defining the notion of viscosity solutions of nonlocal equations. We want to study the nonlocal double obstacle problem

$$\max\{\min\{-Iu - f, u - \psi\}, u - \psi^+\} = 0.$$  

We are also going to consider penalized versions of a nonlocal equation. So we state the definition of viscosity solution in the more general case of a nonlocal operator $\tilde{I}(x, u(x), u(\cdot))$ whose value also depends on the pointwise values of $x, u(x)$ (see [16] for more details). For example, in the case of the double obstacle problem we have

$$-\tilde{I}(x, r, u(\cdot)) = \max\{\min\{-Iu(x) - f(x), r - \psi^-(x)\}, r - \psi^+(x)\},$$

where we replaced $u(x)$ with $r$ to clarify the dependence of $\tilde{I}$ on its arguments.

**Definition 1.** An upper semi-continuous (USC) function $u$ is a viscosity subsolution of $-\tilde{I} \leq 0$ in $U$ if whenever $\phi$ is a bounded $C^2$ function and $u - \phi$ has a maximum over $\mathbb{R}^n$ at $x_0 \in U$ we have

$$-\tilde{I}(x_0, u(x_0), \phi(\cdot)) \leq 0.$$  

And a lower semi-continuous (LSC) function $u$ is a viscosity supersolution of $-\tilde{I} \geq 0$ in $U$ if whenever $\phi$ is a bounded $C^2$ function and $u - \phi$ has a minimum over $\mathbb{R}^n$ at $x_0 \in U$ we have

$$-\tilde{I}(x_0, u(x_0), \phi(\cdot)) \geq 0.$$  

A continuous function $u$ is a viscosity solution of $-\tilde{I} = 0$ in $U$ if it is a subsolution of $-\tilde{I} \leq 0$ and a supersolution of $-\tilde{I} \geq 0$ in $U$.

Now let us state our assumptions about $U$, the obstacles $\psi^\pm$, and $\phi$.

**Assumption 1.** We assume that $U \subset \mathbb{R}^n$ is a bounded open set with $C^2$ boundary, and $\psi^\pm, \varphi : \mathbb{R}^n \to \mathbb{R}$ are bounded Lipschitz functions which satisfy

(a) For every $x, y \in \mathbb{R}^n$ we have

$$|\psi^\pm(x) - \psi^\pm(y)| \leq C_1|x - y|.$$  

And similarly $|\varphi(x) - \varphi(y)| \leq C_1|x - y|.$

(b) $\psi^\pm = \varphi$ on $\mathbb{R}^n - U$, and for all $x \in U$ we have

$$0 < \psi^+(x) - \psi^-(x) \leq 2C_1d(x),$$

where $d$ is the distance to $\partial U$.

(c) For every $x \in U$ and $|y| \leq d(x) - \epsilon$ we have

$$\pm \delta \psi^\pm(x, y) \leq C|y|^2,$$

where the constant $C$ depends only on $\epsilon$. In other words, $\psi^+, \psi^-$ are respectively semi-concave and semi-convex on compact subsets of $U$.  

Remark. A prototypical example of obstacles satisfying this assumption is given by \( \varphi = 0 \) and \( \psi^\pm = \pm \rho \), where \( \rho \) is the distance to \( \partial U \) with respect to some suitable norm. These kinds of obstacles appear for example in the study of equations with gradient constraints (see [23, 24]).

Let \( \eta_\varepsilon \) be a standard mollifier whose support is \( B_\varepsilon(0) \). Then we define

\[
\begin{align*}
\psi^+_\varepsilon(x) &= (\eta_\varepsilon * \psi^+)(x) := \int_{|y| \leq \varepsilon} \eta_\varepsilon(y) \psi^+(x - y) \, dy, \\
\psi^-_\varepsilon(x) &= (\eta_\varepsilon * \psi^-)(x), \quad \varphi_\varepsilon(x) := (\eta_\varepsilon * \varphi)(x).
\end{align*}
\]

(2.3)

Note that \( \psi^\pm_\varepsilon, \varphi_\varepsilon \) are uniformly bounded smooth functions on \( \mathbb{R}^n \) which uniformly converge to \( \psi^\pm, \varphi \). More explicitly, for every \( x \) we have

\[
|\psi^\pm_\varepsilon(x) - \psi^\pm(x)| \leq \int_{|y| \leq \varepsilon} \eta_\varepsilon(y)|\psi^\pm(x - y) - \psi^\pm(x)| \, dy
\]

\[
\leq \int_{|y| \leq \varepsilon} C_1 |y| \eta_\varepsilon(y) \, dy \leq C_1 \varepsilon.
\]

(2.4)

And similarly \( |\varphi_\varepsilon(x) - \varphi(x)| \leq C_1 \varepsilon \). In addition note that since \( \psi^\pm \) are Lipschitz functions we have \( |D\psi^\pm| \leq C_1 \) a.e. Thus

\[
|D\psi^\pm_\varepsilon(x)| \leq \int_{|y| \leq \varepsilon} |\eta_\varepsilon(y)D\psi^\pm(x - y)| \, dy
\]

\[
= \int_{|y| \leq \varepsilon} \eta_\varepsilon(y)|D\psi^\pm(x - y)| \, dy \leq C_1 \int_{|y| \leq \varepsilon} \eta_\varepsilon(y) \, dy = C_1.
\]

Similarly \( |D\varphi_\varepsilon| \leq C_1 \). Now let

\[
U_\varepsilon := \{ x \in \mathbb{R}^n : d(x, \overline{U}) < \varepsilon \}.
\]

Then for \( x \in U_\varepsilon \) we have \( \psi^-_\varepsilon(x) < \psi^+_\varepsilon(x) \), and for \( x \in \mathbb{R}^n - U_\varepsilon \) we have \( \psi^\pm_\varepsilon(x) = \varphi_\varepsilon(x) \). In addition, since the distance function is \( C^2 \) on a tubular neighborhood of \( \partial U \) and its derivative is the normal to the boundary, the \( \partial U_\varepsilon \) is also \( C^2 \).

Lemma 1. Suppose that Assumption 1 holds. Also suppose \( 0 < s_0 < s < 1 \). Then for every bounded open set \( V \subset \subset U \) and \( \varepsilon < \frac{1}{3}d(V, \partial U) \) there is a constant \( C_V \) such that

\[
\pm I \psi^\pm_\varepsilon \leq C_V
\]

on \( V \), where the constant \( C_V \) depends only on \( n, \lambda, \Lambda, s_0, \) and \( d(V, \partial U) \).

Proof. Let \( \varepsilon_0 := \frac{1}{3}d(V, \partial U) \). Let \( x \in V \). Then for \( |y|, |z| \leq \varepsilon_0 \) we have \( d(x - z, \partial U) \geq 2 \varepsilon_0 \geq |y| + \varepsilon_0 \). Hence by (2.2) we get

\[
\pm \delta \psi^\pm_\varepsilon(x, y) = \pm \int_{|z| \leq \varepsilon} \eta_\varepsilon(z) \delta \psi^\pm(x - z, y) \, dz \leq \int_{|z| \leq \varepsilon} \eta_\varepsilon(z) C|y|^2 \, dz = C|y|^2,
\]

where \( C \) depends only on \( \varepsilon_0 \). Hence we have

\[
\delta \psi^\pm_\varepsilon(x, y)^+ - \delta \psi^\pm_\varepsilon(x, y)^- = \delta \psi^\pm_\varepsilon(x, y) \leq C|y|^2.
\]
for $|y| \leq \varepsilon_0$. We can make $C$ larger if necessary so that for $|y| > \varepsilon_0$ we have $\delta \psi^+_\varepsilon(x,y) \leq C\varepsilon^2_0$, since $\psi^+_\varepsilon$ is bounded independently of $\varepsilon$. (It suffices to take $C \geq 4\|\psi^+\|_{L^\infty}/\varepsilon_0^2$.) Thus by the ellipticity of $I$ we get

$$I \psi^+_\varepsilon(x) \leq I0(x) + M^+ \psi^+_\varepsilon(x)$$

$$= 0 + (1 - s) \int_{\mathbb{R}^n} \frac{\Lambda \delta \psi^+_\varepsilon(x,y)^+ - \lambda \delta \psi^+_\varepsilon(x,y)^-}{|y|^{n+2s}} dy$$

$$\leq (1 - s) \int_{\mathbb{R}^n} \frac{(\Lambda + \lambda)C \min\{\varepsilon^2_0, |y|^2\}}{|y|^{n+2s}} dy$$

$$= (1 - s)(\Lambda + \lambda)C \int_{\mathbb{R}^n} \int_0^\infty \frac{\min\{\varepsilon^2_0, r^2\}}{r^{n+2s}} r^{n-1} dr ds$$

$$= (1 - s) \hat{C} \int_0^\infty \min\{\varepsilon^2_0, r^2\} r^{-1-2s} dr = \frac{\hat{C}}{2s} \varepsilon^2_0 \varepsilon^2 = \frac{\hat{C}}{2s} \varepsilon^2 \varepsilon^2 \leq \frac{\hat{C} \varepsilon^{-2s}}{2s} = : C_V < \infty,$$

as desired. We can similarly obtain a uniform bound for $-I \psi^-_\varepsilon$. □

**Remark.** Note that if in the inequality $\delta \psi^+_\varepsilon(x,y) \leq C|y|^2$ we divide by $|y|^2$ and let $|y| \to 0$ (without changing the direction of $y$) we get $D^2_{yy} \psi^+_\varepsilon(x) \leq C$, where $\hat{y} = y/|y|$ is the unit vector in the direction of $y$. Conversely, a bound for the second derivative of a $C^2$ function like $\psi^+_\varepsilon$ implies a bound for $\delta \psi^+_\varepsilon$, since we can easily see that

$$\delta \psi^+_\varepsilon(x,y) = |y|^2 \int_0^1 \int_{-1}^1 t D^2_{yy} \psi^+_\varepsilon(x + sty) ds dt.$$

**Lemma 2.** Suppose $I$ is a translation invariant operator which is uniformly elliptic with respect to $\mathcal{L}_0$. Also, suppose $U, \varphi, \psi^\pm$ satisfy Assumption [2], and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Then the double obstacle problem

$$(2.7)\quad \begin{cases} \max\{|-Iu_\varepsilon - f, u_\varepsilon - \psi^-_\varepsilon\}, u_\varepsilon - \psi^+_\varepsilon = 0 & \text{in } U_\varepsilon, \\ u_\varepsilon = \varphi_\varepsilon & \text{in } \mathbb{R}^n - U_\varepsilon. \end{cases}$$

has a viscosity solution $u_\varepsilon$, and

$$u_\varepsilon \in C^{1,0}_{loc}(U_\varepsilon).$$

**Proof.** Fix $\varepsilon > 0$. For $\delta > 0$ let $\beta_\delta$ be a smooth increasing function that vanishes on $(-\infty, 0]$ and equals $\frac{1}{\delta}t$ for $t \geq \delta$. Then the equation

$$(2.8)\quad \begin{cases} -I\bar{u}_\delta - f - \beta_\delta(\psi^-_\varepsilon - \bar{u}_\delta) + \beta_\delta(\bar{u}_\delta - \psi^+_\varepsilon) = 0 & \text{in } U_\varepsilon, \\ \bar{u}_\delta = \varphi_\varepsilon & \text{in } \mathbb{R}^n - U_\varepsilon, \end{cases}$$

has a viscosity solution (see Theorem 5.6 of [16]). (Here we are also using the fact that $\partial U_\varepsilon$ is $C^2$.) To simplify the notation we set

$$\bar{u} = \bar{u}_\delta, \quad \beta = \beta_\delta.$$
Now let us show that
\[ \|\beta(\pm(\tilde{u} - \psi^\pm_\varepsilon))\|_{L^\infty(U_\varepsilon)} \leq \tilde{C}_0, \]
where \( \tilde{C}_0 \) is independent of \( \delta \). Note that \( \beta(\pm(\tilde{u} - \psi^\pm_\varepsilon)) \) is zero on \( \mathbb{R}^n - U_\varepsilon \). So assume that \( \beta(\pm(\tilde{u} - \psi^\pm_\varepsilon)) \) attains its positive maximum at \( x_0 \in U_\varepsilon \). Let us consider \( \beta(\tilde{u} - \psi^+_\varepsilon) \); the other case is similar. Since \( \beta \) is increasing, \( \tilde{u} - \psi^+_\varepsilon \) has a positive maximum at \( x_0 \) too. Therefore by the definition of viscosity solution we have
\[ -I\psi^+_\varepsilon(x_0) - f(x_0) - \beta(\psi^-_\varepsilon(x_0) - \tilde{u}(x_0)) + \beta(\tilde{u}(x_0) - \psi^+_\varepsilon(x_0)) \leq 0. \]
So at \( x_0 \) we have
\[ -I\psi^+_\varepsilon(x_0) - f(x_0) \leq \beta(\psi^-_\varepsilon - \tilde{u}) - \beta(\tilde{u} - \psi^+_\varepsilon) = -\beta(\tilde{u} - \psi^+_\varepsilon), \]
since by our assumption \( \psi^-_\varepsilon(x_0) < \psi^+_\varepsilon(x_0) < \tilde{u}(x_0) \). Thus \( \beta(\tilde{u} - \psi^+_\varepsilon) \leq I\psi^+_\varepsilon + f \) at \( x_0 \). Therefore \( \beta(\tilde{u} - \psi^+_\varepsilon) \) is bounded independently of \( \delta \), as desired, because \( I\psi^+_\varepsilon, f \) are continuous functions.

The bound \( \beta(\pm(\tilde{u} - \psi^\pm_\varepsilon)) \rangle \leq \tilde{C}_0 \) and the definition of \( \beta \) imply that
\[ (2.9) \quad \tilde{u} - \psi^+_\varepsilon \leq \delta(\tilde{C}_0 + 1), \quad \psi^-_\varepsilon - \tilde{u} \leq \delta(\tilde{C}_0 + 1). \]
This also shows that \( \tilde{u} \) is uniformly bounded independently of \( \delta \). In addition, we can choose \( \tilde{C}_0 \) large enough so that \( |f| \leq \tilde{C}_0 \). Then from the equation (2.8) we conclude that
\[ -3\tilde{C}_0 \leq I\tilde{u} \leq 3\tilde{C}_0 \]
in the viscosity sense.

Thus by Theorem 4.1 of [14] if \( B_r(x_0) \subset U_\varepsilon \) we have
\[ \|\tilde{u}\|_{C^{1,\alpha}(B_{r/2}(x_0))} \leq \frac{C}{r^{1+\alpha}}(\|\tilde{u}\|_{L^\infty(\mathbb{R}^n)} + 3\tilde{C}_0 r^{2s}), \]
where \( C, \alpha \) depend only on \( n, s, \lambda, \Lambda \) (actually, \( C, \alpha \) can be chosen uniformly for \( s > s_0 \)). For simplicity we are assuming that \( r \leq 1 \). (Note that by considering the scaled operator \( I_r v(\cdot) = r^{2s} I v(\tilde{\cdot}) \), which has the same ellipticity constants \( \lambda, \Lambda \) as \( I \), and using the translation invariance of \( I \), we have obtained the estimate on the domain \( B_{r/2}(x_0) \) instead of \( B_{l/2}(0) \).) Then we can cover an open subset \( V \subset \subset U_\varepsilon \) with finitely many open balls contained in \( U_\varepsilon \) and obtain
\[ \|\tilde{u}\|_{C^{1,\alpha}(\overline{V})} \leq C(\|\tilde{u}\|_{L^\infty(\mathbb{R}^n)} + 3\tilde{C}_0), \]
where \( C \) depends only on \( n, s, \lambda, \Lambda, \) and \( d(V, \partial U_\varepsilon) \). Therefore \( \tilde{u} = \tilde{u}_\delta \) is bounded in \( C^{1,\alpha}(\overline{V}) \) independently of \( \delta \), due to the uniform boundedness of \( \tilde{u} \) in \( L^\infty \).

Now, we choose a decreasing sequence \( \delta_j \to 0 \), and let
\[ V_j := \{x \in U_\varepsilon: d(x, \partial U_\varepsilon) > \delta_j\}. \]
For convenience we also set \( \tilde{u}_j := \tilde{u}_{\delta_j} \). Consider the sequence \( \tilde{u}_j|_{V_j} \). Then \( \|\tilde{u}_j\|_{C^{1,\alpha}(\overline{V}_j)} \) is bounded independently of \( j \). Hence there is a subsequence of \( \tilde{u}_j \)'s, which we denote by \( \tilde{u}_{j_1} \), that is convergent in \( C^1 \) norm to a function \( u_{\varepsilon,1} \) in \( C^{1,\alpha}(\overline{V}_1) \). Now we can repeat this process.
with $\tilde{u}_j|_{V_2}$ and get a function $u_{\varepsilon, 2}$ in $C^{1, \alpha}(\bar{V}_2)$, which agrees with $u_{\varepsilon, 1}$ on $V_1$. Continuing this way with subsequences $\tilde{u}_j$, for each positive integer $l$, we can finally construct a function $u_{\varepsilon}$ in $C^{1, \alpha}_{\text{loc}}(U_\varepsilon)$. Note that the diagonal sequence $\tilde{u}_l$, which we denote by $\tilde{u}_l$, converges pointwise to $u_{\varepsilon}$ on $U_\varepsilon$, and converges uniformly to $u_{\varepsilon}$ on compact subsets of $U_\varepsilon$. Also note that if we let $\delta_l \to 0$ in (2.39) we get $\psi^-_{\varepsilon} \leq u_{\varepsilon} \leq \psi^+_{\varepsilon}$. Consequently, as we approach $\partial U_\varepsilon$, $u_{\varepsilon}$ converges to $\varphi_\varepsilon$. We extend $u_{\varepsilon}$ to all of $\mathbb{R}^n$ by setting it equal to $\varphi_\varepsilon$ on $\mathbb{R}^n - U_\varepsilon$. Note that $u_{\varepsilon}$ is a continuous function.

Let us show that $\tilde{u}_l$ converges uniformly to $u_{\varepsilon}$ on $\mathbb{R}^n$. Suppose $\varepsilon$ is given and we want to show that $\sup_{\mathbb{R}^n} |\tilde{u}_l - u_{\varepsilon}| < \varepsilon$ for large enough $l$. We know that $\tilde{u}_l = \varphi_\varepsilon = u_{\varepsilon}$ on $\mathbb{R}^n - U_\varepsilon$. And by (2.9), the fact that $\psi^-_{\varepsilon} \leq u_{\varepsilon} \leq \psi^+_{\varepsilon}$, and (2.1) we have

$$|\tilde{u}_l - u_{\varepsilon}| \leq \psi^+_{\varepsilon} - \psi^-_{\varepsilon} + \delta_l(\tilde{C}_0 + 1) \leq 2C_1d(\cdot) + \delta_l(\tilde{C}_0 + 1).$$

Now let $V \subset \subset U$ be such that for $x \in U - V$ we have $2C_1d(x) < \varepsilon/2$. Then if $l$ is large enough so that $\sup_V |\tilde{u}_l - u_{\varepsilon}| < \varepsilon$ and $\delta_l(\tilde{C}_0 + 1) < \varepsilon/2$, we get the desired.

Finally, let us show that $u_{\varepsilon}$ satisfies the double obstacle problem (2.7). Suppose $\phi$ is a bounded $C^2$ function and $u_{\varepsilon} - \phi$ has a maximum over $\mathbb{R}^n$ at $x_0 \in U$. Let us first consider the case where $u_{\varepsilon} - \phi$ has a strict maximum at $x_0$. We must show that at $x_0$ we have

$$\max\{\min\{-I\phi(x_0) - f, u_{\varepsilon} - \psi^-_{\varepsilon}\}, u_{\varepsilon} - \psi^+_{\varepsilon}\} \leq 0. \quad (2.10)$$

Now we know that $\tilde{u}_l - \phi$ takes its global maximum at a point $x_l$ where $x_l \to x_0$; because $\tilde{u}_l$ uniformly converges to $u_{\varepsilon}$ on $\mathbb{R}^n$.

We also know that $\psi^-_{\varepsilon} \leq u_{\varepsilon} \leq \psi^+_{\varepsilon}$. If $\psi^-_{\varepsilon}(x_0) = u_{\varepsilon}(x_0)$ then (2.10) holds trivially. So suppose $\psi^-_{\varepsilon}(x_0) < u_{\varepsilon}(x_0)$. Then for large $l$ we have $\psi^-_{\varepsilon}(x_l) < \tilde{u}_l(x_l)$. On the other hand, since $\tilde{u}_l$ is a viscosity solution of the equation (2.8), at $x_l$ we have

$$-I\phi(x_l) - f - \beta_{\delta_l}(\psi^-_{\varepsilon} - \tilde{u}_l) + \beta_{\delta_l}(\tilde{u}_l - \psi^+_{\varepsilon}) \leq 0.$$ 

But $\beta_{\delta_l}(\psi^-_{\varepsilon} - \tilde{u}_l) = 0$ at $x_l$, so

$$-I\phi(x_l) - f \leq -\beta_{\delta_l}(\tilde{u}_l - \psi^+_{\varepsilon}) \leq 0.$$ 

Thus by letting $l \to \infty$ and using the continuity of $I\phi$ and $f$ we see that (2.10) holds in this case too.

Now if the maximum of $u_{\varepsilon} - \phi$ at $x_0$ is not strict, we can approximate $\phi$ with $\phi_{\varepsilon} = \phi + \varepsilon \tilde{\phi}$, where $\tilde{\phi}$ is a bounded $C^2$ functions which vanishes at $x_0$ and is positive elsewhere. Then, as we have shown, when $\psi^-_{\varepsilon}(x_0) < u_{\varepsilon}(x_0)$ we have $-I\phi_{\varepsilon}(x_0) \leq f(x_0)$. Hence by the ellipticity of $I$ we get

$$-I\phi(x_0) \leq M^+(\varepsilon \tilde{\phi})(x_0) - I\phi_{\varepsilon}(x_0) \leq \varepsilon M^+\tilde{\phi}(x_0) + f(x_0) \to f(x_0),$$

as desired. Similarly, we can show that when $u_{\varepsilon} - \phi$ has a global minimum at $x_0 \in U$ we have

$$\max\{\min\{-I\phi(x_0) - f, u_{\varepsilon} - \psi^-_{\varepsilon}\}, u_{\varepsilon} - \psi^+_{\varepsilon}\} \geq 0.$$ 

Therefore $u_{\varepsilon}$ is a viscosity solution of equation (2.7) as desired.
Finally we have

**Proof of Theorem 1**: Let $u_\epsilon$ be as in Lemma 2. Let us first show that for every bounded open set $V \subset U$ and every small enough $\epsilon$ we have

$$(2.11) \quad - C_V - \|f\|_{L^\infty(U)} \leq Iu_\epsilon \leq \|f\|_{L^\infty(U)} + C_V$$

in the viscosity sense on $V$, with the constant $C_V$ given in Lemma 1.

Suppose $\phi$ is a bounded $C^2$ function and $u_\epsilon - \phi$ has a maximum over $\mathbb{R}^n$ at $x_0 \in V$. We must show that

$$-I\phi(x_0) \leq \|f\|_{L^\infty(U)} + C_V.$$  

We can assume that $u_\epsilon(x_0) - \phi(x_0) = 0$ without loss of generality, since we can consider $\phi + c$ instead of $\phi$ without changing $I$ (because $M^+(c) = 0$). So we can assume that $u_\epsilon - \phi \leq 0$, or $u_\epsilon \leq \phi$. We also know that at $x_0$ we have

$$\max\{\min\{-I\phi(x_0) - f, u_\epsilon - \psi^-_\epsilon, u_\epsilon - \psi^+_\epsilon\} \leq 0,$$

since $u_\epsilon$ is a viscosity solution of (2.7). In addition remember that $\psi^-_\epsilon \leq u_\epsilon \leq \psi^+_\epsilon$. Now if $\psi^-_\epsilon(x_0) < u_\epsilon(x_0) \leq \psi^+_\epsilon(x_0)$ then we must have $-I\phi(x_0) \leq f(x_0) \leq \|f\|_{L^\infty(U)}$. And if $u_\epsilon(x_0) = \psi^-_\epsilon(x_0)$ then $\phi$ is also touching $\psi^-_\epsilon$ from above at $x_0$, since $\psi^-_\epsilon \leq u_\epsilon \leq \phi$. But by Lemma 1, we know that $-I\psi^-_\epsilon \leq C_V$ on $V$. So we must have

$$-I\phi(x_0) \leq -I\psi^-_\epsilon(x_0) + M^+(\psi^-_\epsilon - \phi)(x_0) \leq C_V,$$

since it is easy to see that $M^+(\psi^-_\epsilon - \phi)(x_0) \leq 0$ as $L(\psi^-_\epsilon - \phi)(x_0) \leq 0$ for any linear operator $L$. Thus in either case we must have $-I\phi(x_0) \leq \|f\|_{L^\infty(U)} + C_V$, and therefore $-Iu_\epsilon \leq \|f\|_{L^\infty(U)} + C_V$ in the viscosity sense. (Heuristically, note that on the contact set $\{u_\epsilon = \psi^+_\epsilon\}$ although a priori we do not have a lower bound for the second derivative of $\psi^+_\epsilon$, and hence an upper bound for $-I\psi^+_\epsilon$, we can obtain the desired bound for $-Iu_\epsilon$ from the equation.)

Thus by Theorem 4.1 of \[14\], and similarly to the proof of Lemma 2, we can show that there is $\alpha$ depending only on $n, \lambda, \Lambda, s_0$, such that for an open subset $V \subset U$ we have

$$\|u_\epsilon\|_{C^{1,\alpha}(V)} \leq C(\|u_\epsilon\|_{L^\infty(V)} + \|f\|_{L^\infty(U)} + C_V),$$

where $C$ depends only on $n, \lambda, \Lambda, s_0$, and $d(V, \partial U)$. In particular $C$ does not depend on $\epsilon$. Therefore $u_\epsilon$ is bounded in $C^{1,\alpha}(V)$ independently of $\epsilon$, because $\|u_\epsilon\|_{L^\infty}$ is bounded by $\|\psi^\pm\|_{L^\infty}$, and $\|\psi^\pm\|_{L^\infty}$ are uniformly bounded by $\|\psi^\pm\|_{L^\infty}$. Now we choose a decreasing sequence $\epsilon_k \rightarrow 0$, and a sequence $V_k \subset V_{k+1}$ such that $U = \bigcup_{k \geq 0} V_k$. We also assume that $\epsilon_k < \frac{1}{3}d(V_k, \partial U)$. For convenience we denote $u_{\epsilon_k}, \psi_{\epsilon_k}$ by $u_k, \psi_{\pm_k}$. Consider the sequence $u_k|_{V_1}$. Then $\|u_k\|_{C^{1,\alpha}(V_1)}$ is bounded independently of $k$. Hence there is a subsequence of $u_k$'s, which we denote by $u_{k_1}$, that is convergent in $C^1$ norm to a function in $C^{1,\alpha}(V_1)$. Now we can repeat this process with $u_{k_1}|_{V_2}$ and get a function in $C^{1,\alpha}(V_2)$, which agrees with the previous limit on $V_1$. Continuing this way with subsequences $u_{k_l}$ for each positive integer $l$, we can finally construct a function $u$ in $C^{1,\alpha}_{\text{loc}}(U)$. 11
Note that the diagonal sequence $u_l$, which we denote by $u_l$, converges pointwise to $u$ on $U$, and converges uniformly to $u$ on compact subsets of $U$. It is also obvious that $\psi^- \leq u \leq \psi^+$, since $\psi^-_l \leq u_l \leq \psi^+_l$ for every $l$. Consequently, as we approach $\partial U$, $u$ converges to $\varphi$. We extend $u$ to all of $\mathbb{R}^n$ by setting it equal to $\varphi$ on $\mathbb{R}^n - U$. Note that $u$ is a continuous function. Let us also show that $u_l$ converges uniformly to $u$ on $\mathbb{R}^n$. Suppose $\varepsilon$ is given and we want to show that $\sup_{\mathbb{R}^n} |u_l - u| < \varepsilon$ for large enough $l$. Since $u, u_l$ are between their corresponding obstacles, by using (2.1), (2.4) we get

$$|u_l - u| \leq \max \{|\psi^+_l - \psi^-_l|, |\psi^-_l - \psi^+_l|\}$$

$$\leq |\psi^+ - \psi^-| + \max \{|\psi^+_l - \psi^+_l|, |\psi^-_l - \psi^-_l|\}$$

$$\leq \begin{cases} 2C_1d(\cdot) + C_1\varepsilon_l & \text{in } U, \\ C_1\varepsilon_l & \text{in } \mathbb{R}^n - U, \end{cases}$$

because $|\psi^+ - \psi^-| = 0$ outside of $U$. Now let $V \subset \subset U$ be such that for $x \in U - V$ we have $2C_1d(x) < \varepsilon/2$. Then if $l$ is large enough so that $\sup_V |u_l - u| < \varepsilon$ and $C_1\varepsilon_l < \varepsilon/2$, we get the desired.

Finally, due to the stability of viscosity solutions, $u$ must satisfy the double obstacle problem (1.1). (We can also show this similarly to the proof of Lemma 2.)

**Proof of Theorem 2** We are assuming that $\psi^\pm$ are $C^2$ on a neighborhood $W$ of $\partial U$. Let $W_1 \subset \subset W$ be a smaller neighborhood of $\partial U$. Let $0 \leq \zeta \leq 1$ be a $C^\infty$ function with compact support in $W$ which equals 1 on $W_1$. Set

$$\hat{\psi}_\varepsilon^\pm := \zeta \psi^\pm + (1 - \zeta) \psi^\pm_\varepsilon$$

for $\varepsilon$ small enough. Note that $\hat{\psi}_\varepsilon^\pm$ are $C^2$ on a neighborhood of $\overline{U}$, and agree on $\partial U$. Also, $\psi^\pm_\varepsilon$ uniformly converge to $\psi^\pm$ as $\varepsilon \to 0$. In fact, by (2.4) we can easily see that $|\hat{\psi}_\varepsilon^\pm(x) - \psi^\pm(x)| \leq C_1\varepsilon$. It is obvious that $\hat{\psi}_\varepsilon^- = \psi^- < \psi^+ = \hat{\psi}_\varepsilon^+$ on $W_1$. Let $W_2 \subset \subset W_1$ be a yet smaller neighborhood of $\partial U$. Then due to compactness, on $U - W_2$ we have $\psi^+ - \psi^- \geq c > 0$ for some $c$. Hence for $x \in U - W_1$ and small enough $\varepsilon$ we have

$$\psi^+_\varepsilon(x) - \psi^-_\varepsilon(x) = \int_{|y| \leq \varepsilon} \eta_\varepsilon(y) [\psi^+(x - y) - \psi^-(x - y)] dy \geq c \int_{|y| \leq \varepsilon} \eta_\varepsilon(y) dy = c,$$

and thus we get

$$\hat{\psi}_\varepsilon^+ - \hat{\psi}_\varepsilon^- = \zeta (\psi^+ - \psi^-) + (1 - \zeta) (\psi^+_\varepsilon - \psi^-_\varepsilon) \geq c(\zeta + 1 - \zeta) = c > 0.$$

Therefore we have $\hat{\psi}_\varepsilon^- < \hat{\psi}_\varepsilon^+$ on $U$. 

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Furthermore, note that by the proof of Lemma 1 and the remark after it, \( \pm D^2 \psi^\pm_\varepsilon \) is uniformly bounded from above on \( U - W_1 = U \cap \{1 - \zeta > 0\} \). Hence
\[
\pm D^2 \psi^\pm_\varepsilon = \pm \zeta D^2 \psi^\pm + 2D \zeta D \psi^\pm + \psi^\pm D^2 \zeta \\
\pm (1 - \zeta) D^2 \psi^\pm + 2D \zeta D \psi^\pm + \psi^\pm D^2 \zeta
\]
is uniformly bounded from above on \( U \cup W_1 \supseteq \overline{U} \). Because \( D^2 \psi^\pm \) is bounded on the support of \( \zeta \), and \( D \psi^\pm_\varepsilon \) is uniformly bounded due to (2.5). So similarly to the proof of Lemma 1 and employing the remark after it, we can easily show that \( \pm I \psi^\pm_\varepsilon \leq C_U \) on \( U \), for some constant \( C_U \) independent of \( \varepsilon \).

Now we can repeat the construction of \( u_\varepsilon \) with \( \hat{\psi}^\pm_\varepsilon \) instead of \( \psi^\pm_\varepsilon \). Note that in this case we have \( U_\varepsilon = U \) for every \( \varepsilon \). So we can find viscosity solutions \( u_\varepsilon \in C^{1,\alpha}_{\text{loc}}(U) \) of
\[
\begin{cases}
\max\{\min\{-Iu_\varepsilon - f, u_\varepsilon - \hat{\psi}^+_\varepsilon\}, u_\varepsilon - \hat{\psi}^-_\varepsilon\} = 0 & \text{in } U, \\
u_\varepsilon = \varphi & \text{in } \mathbb{R}^n - U.
\end{cases}
\]
(2.12)
Then similarly to the proof of Theorem 1 we can show that for every small enough \( \varepsilon \) we have
\[
-C_U - \|f\|_{L^\infty(U)} \leq Iu_\varepsilon \leq \|f\|_{L^\infty(U)} + C_U
\]
in the viscosity sense on \( U \). Then, again similarly to the proof of Theorem 1, we can construct a function \( u \) in \( C^{1,\alpha}_{\text{loc}}(U) \) satisfying \( \psi^- \leq u \leq \psi^+ \), which we extend to all of \( \mathbb{R}^n \) by setting it equal to \( \varphi \) on \( \mathbb{R}^n - U \). In addition, due to the stability of viscosity solutions, \( u \) satisfies the double obstacle problem (I.I).

Now let \( x_0 \in \partial U \) and suppose \( B_r(x_0) \subset W_1 \). Also set \( v_\varepsilon := u_\varepsilon - \varphi \). Let us show that
\[
\begin{cases}
M^+_\varepsilon v_\varepsilon \geq -\|f\|_{L^\infty(U)} - C_0 & \text{in } B_r(x_0) \cap U, \\
M^-_\varepsilon v_\varepsilon \leq \|f\|_{L^\infty(U)} + C_0 & \text{in } B_r(x_0) \cap U
\end{cases}
\]
in the viscosity sense, for some constant \( C_0 \) independent of \( \varepsilon \). Suppose \( \phi \) is a bounded \( C^2 \) function and \( v_\varepsilon - \phi \) has a maximum over \( \mathbb{R}^n \) at \( x \in B_r(x_0) \cap U \). We must show that
\[
-M^+_\varepsilon \phi(x) \leq \|f\|_{L^\infty(U)} + C_0.
\]
Now note that \( u_\varepsilon - (\varphi + \phi) \) has a maximum over \( \mathbb{R}^n \) at \( x \in B_r(x_0) \cap U \), and \( \phi + \varphi \) is a bounded \( C^2 \) function. Hence by (2.13) we must have
\[
-I(\phi + \varphi)(x) \leq \|f\|_{L^\infty(U)} + C_U.
\]
Then by the subadditivity of \( M^+_\varepsilon \) and ellipticity of \( I \) with respect to \( L_* \), at \( x \) we have
\[
-M^+_\varepsilon \phi - M^+_\varepsilon \varphi \leq -M^+_\varepsilon (\phi + \varphi) \leq -I(\phi + \varphi) \leq \|f\|_{L^\infty(U)} + C_U.
\]
However, since \( \varphi \) is \( C^2 \), similarly to the proof of Lemma 1 we can show that \( M^+_\varepsilon \varphi \leq M^+ \varphi \leq C \) on \( B_r(x_0) \). Hence we get the desired bound for \( -M^+_\varepsilon \phi(x) \). We can similarly prove the other inequality in (2.12).
Next note that in addition to (2.14), $v_\varepsilon = 0$ in $B_r(x_0) - U$. Thus by Theorem 1.5 of [21] (also see [20]) there is $\bar{\alpha} > 0$ depending only on $n, \lambda, \Lambda, s_0$ so that
\[
\|v_\varepsilon/d^s\|_{C^{\bar{\alpha}}(B_{r/2}(x_0) \cap \overline{U})} \leq \tilde{C} (\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(U)} + C_0),
\]
where $\tilde{C}$ depends only on $n, \lambda, \Lambda, s, r, U$. (Note that we have rescaled the functions in order to obtain the estimate on the domain $B_{r/2}(x_0)$ instead of $B_{1/2}(0)$.) Now remember that $u$ is the pointwise limit of a sequence $u_{\varepsilon_i}$ on $U$ (the diagonal sequence constructed in the proof of Theorem 1). Considering this sequence in the above estimate, it follows that there is a subsequence corresponding to $\varepsilon_j \to 0$ such that $v_{\varepsilon_j}/d^s$ uniformly converges to a function in $C^{\bar{\alpha}}(B_{r/2}(x_0) \cap \overline{U})$. But the limit must be equal to $(u - \varphi)/d^s$. So $(u - \varphi)/d^s$ is $C^{\bar{\alpha}}$ up to $\partial U$, and satisfies the desired estimate. □

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