Spherical harmonic analysis of stochastic gravitational wave background anisotropies in interferometry experiments

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We study the interferometric observation of intensity and polarization anisotropies of a stochastic gravitational wave background (SGWB). We show that the observed correlated data is defined in the group manifold of the three-dimensional rotation. Explicit correlation between two detectors in the interferometry experiments such as LIGO-VIRGO and KAGRA is constructed in terms of the Wigner D-functions. Our results reveal the topology of the data structure and provide a systematic way for constructing data pipelines to estimate the power spectra of the SGWB anisotropies.

I. INTRODUCTION

The LIGO detectors have firstly observed gravitational waves (GWs) emitted by a binary black hole merger as predicted in general relativity. Since then, a handful of GW events from compact binary coalescences has been observed in Advanced LIGO and Advanced Virgo O2 and O3 observing runs. This achievement has opened up a new era of GW astronomy and cosmology. Future experimental plans such as Einstein Telescope, Cosmic Explorer, LISA, DECIGO, Taiji, Tianqin, pulsar-timing arrays, and SKA will bring us a precision science in GW observation.

Stochastic gravitational wave background (SGWB) is a key target in GW experiments. There have been many studies on possible astrophysical and cosmological sources for SGWB such as distant compact binary coalescences, early-time phase transitions, cosmic string or defect networks, second-order primordial scalar perturbations, and inflationary GWs. GWs have very weak gravitational interaction, so they decouple from matter at the time of production and then travel to us almost without being perturbed. At the present, they remain as a SGWB that carries original information of the process of production in the very early universe.

In general, the SGWB can be anisotropic and polarized. The method adopted in current GW experiments for detecting SGWB is to correlate the responses of a pair of detectors to the GW strain amplitude. This allows us to filter out detector noises and obtain a large signal-to-noise ratio. The correlation between the GW strain data from a pair of detectors is a convolution of the sky map of the SGWB with the overlap reduction function. By correlating outputs from two different GW detectors, it is possible to detect these intensity and polarization anisotropies or the Stokes parameters of the SGWB. In this article, we provide an unified frame-work to calculate the overlap reduction functions for the Stokes parameters in the spherical harmonic basis.

II. FORMALISM

In the Minkowskian vacuum, the metric perturbation $h_{ij}$ in the transverse traceless gauge depicts GWs propagating at the speed of light $c = \omega/k$. At a given spacetime point $(t, \vec{x})$, it can be expanded in terms of its Fourier modes:

$$h_{ij}(t, \vec{x}) = \sum_A \int_{-\infty}^{\infty} df \int_{S^2} d\hat{k} \, h_A(f, \hat{k}) e^{i\omega t - i\hat{k} \cdot \vec{x}/c},$$

where $A$ stands for the polarization or the helicity of GWs described by the corresponding basis tensors $e^{ij}_A(\hat{k})$, which are transverse to the direction of the wave propagation denoted by $\hat{k}$. Since $h_{ij}$ is real, its Fourier components are not fully independent with each other. For our application, we require those Fourier components with negative frequencies to be $h_A(-f, \hat{k}) = h^*_A(f, \hat{k})$ for all $f \geq 0$. The GWs are considered as stochastic as long as $h_{ij}$ are random fields thus characterized by their ensemble averages. Besides, assuming the probability distribution of the random amplitude $h_{ij}$ be Gaussian, then only the two-point correlation function $\langle h_{ij}(t, \vec{x}_1) h_{ij}(t, \vec{x}_2) \rangle$ is needed to describe its statistical behavior. Furthermore, if the waves are homogeneous, i.e., having translational symmetry, the ensemble average can be evaluated by doing spatial averages. As a result, the two-point correlation function of the Fourier modes should have the following form

$$\langle h_A(f, \hat{k}) h^*_A(f', \hat{k'}) \rangle = \delta(f - f') \delta(\hat{k} - \hat{k'}) P_{AA'}(f, \hat{k}),$$

where the $\delta(f - f')$ arises from the delta function of the magnitude of their 3-momenta $\delta(\hat{k} - \hat{k'})$ and the assump-
tion made in Eq. (1) that these waves satisfy the equation of motion in vacuum. Also, the presence of \( \delta(f - f') \) implies that the signal is stationary. This is a fairly good approximation during an observing period for a typical experiment. For example, a period of about 9 months of the LIGO second observing run (O2) contains 99 days of clean data [13].

For GWs coming from the sky direction \(-\hat{k}\) with wave vector \(\hat{k}\), it is customary to write the polarization basis tensors in terms of the basis vectors in the spherical coordinates:

\[
e^+(\hat{k}) = \hat{e}_\theta \otimes \hat{e}_\phi - \hat{e}_\phi \otimes \hat{e}_\theta, \tag{3}
\]
\[
e^x(\hat{k}) = \hat{e}_\theta \otimes \hat{e}_\phi + \hat{e}_\phi \otimes \hat{e}_\theta, \tag{4}
\]
in which \(\hat{e}_\theta, \hat{e}_\phi,\) and \(\hat{k}\) form a right-handed orthonormal basis. Also, we can define the complex circular polarization basis tensors as

\[
e_R = \frac{(e_+ + ie_{\times})}{\sqrt{2}}, \quad e_L = \frac{(e_+ - ie_{\times})}{\sqrt{2}}, \tag{5}
\]

where \(e_R\) stands for the right-handed GW with a positive helicity while \(e_L\) stands for the left-handed GW with a negative helicity. The corresponding amplitudes in Eq. (1) in the two different bases are related to each other via:

\[
h_R = \frac{(h_+ - ih_{\times})}{\sqrt{2}}, \quad h_L = \frac{(h_+ + ih_{\times})}{\sqrt{2}}. \tag{6}
\]

Analogous to the case in electromagnetic waves [14], the coherency matrix \(P_{AA'}\) in Eq. (2) is related to the Stokes parameters, \(I, Q, U,\) and \(V\) as

\[
I = \langle [h_R h_R^*] + [h_L h_L^*] \rangle / 2, \tag{7}
\]
\[
Q + iU = \langle h_L h_R^* \rangle, \tag{8}
\]
\[
Q - iU = \langle h_R h_L^* \rangle, \tag{9}
\]
\[
V = \langle [h_R h_R^*] - [h_L h_L^*] \rangle / 2. \tag{10}
\]

They are functions of the frequency, \(f\), and the propagation direction, \(k\). To get some flavor of the meaning of these Stokes parameters, we may take a look at an example for an unpolarized quasi monochromatic GW signal with a constant intensity. It should have a constant \(I\) which represents the total intensity regardless of its polarization. We have \(V = 0\) since the power in the right-handed and the left-handed modes should be identical. Also, because the relative phase between the left-handed and the right-handed modes \(\arg(h_L) - \arg(h_R)\) is random for an unpolarized source, the ensemble average \(\langle h_L h_R^* \rangle \sim \langle e^{i(\arg(h_L) - \arg(h_R))} \rangle\) becomes zero, thereby making \(Q = U = 0\).

Presumably, if one can point a GW telescope with a finite resolution and a polarization capability to a certain direction on the sky, it would be possible to measure the Stokes parameters of the incoming GWs from different patches of the sky. Unfortunately, neither a physical GW polarizer nor a directional GW detector is feasible with current technology. Alternatively, we may extract these anisotropies by combining or correlating the outputs from different existing GW detectors.

GW detectors that use laser interferometers to measure the differential length change along two different directions, such as LIGO, VIRGO, and KAGRA, provide the so-called strain data \(s(t) = h(t) + n(t)\) representing the fractional change of the differential arm length, where \(h(t)\) is the signal due to the GW and \(n(t)\) considered as noise is anything else than the signal. The signal \(h_a(t_a, \vec{x}_a)\) in a GW detector \(a\) located at \(\vec{x}_a\) can be expressed as the contraction of the metric perturbation \(h_{ij}(t, \vec{x})\) and the detector tensor \(d^a_{ij}\) of the detector:

\[
h_a(t_a, \vec{x}_a) = d^a_{ij} h_{ij}(t_a, \vec{x}_a)
= d^a_{ij} \sum_A \int_{-\infty}^{\infty} df \int_{S^2} d\hat{k} \ h_A(f, \hat{k}) e^{i\hat{k}(f, \hat{k})} e^{-2\pi i f(t_a - \vec{k} \cdot \vec{x}_a / c)}, \tag{11}
\]

where the detector tensor is

\[
d^a_{ij} = \frac{1}{2} (X^i_a \otimes X^j_a - Y^i_a \otimes Y^j_a), \tag{12}
\]

with \(X^i_a\) being the \(i\)-th component of the unit vector along the X-arm of the detector, while \(Y^i_a\) representing the Y-arm.

The correlation of signals in a pair of detectors \(a\) and \(b\) can be expressed in terms of the baseline vector \(\vec{r} \equiv \vec{x}_a - \vec{x}_b\) and the time delay \(\tau \equiv t_a - t_b\). In frequency domain, one have
where the Fourier integral is taken over an interval $T$ within which the orientation and the condition of the detectors are approximately fixed. In addition, the interval $T$ has to be large enough when compared with the period of GW signals in the detectors. In Eq. (13), the polarization tensors $\mathbb{E}$ associated with the corresponding Stokes parameters are defined as

$$
\mathbb{E}^I_{ijkl}(\hat{k}) = e^E_{ij}(\hat{k})e^R_{kl}(\hat{k}) + e^E_{ik}(\hat{k})e^L_{jl}(\hat{k}),
$$

$$
\mathbb{E}^V_{ijkl}(\hat{k}) = e^E_{ij}(\hat{k})e^R_{kl}(\hat{k}) - e^E_{ik}(\hat{k})e^L_{jl}(\hat{k}),
$$

$$
\mathbb{E}^{+U}_{ijkl}(\hat{k}) = e^E_{ij}(\hat{k})e^R_{kl}(\hat{k}),
$$

$$
\mathbb{E}^{-U}_{ijkl}(\hat{k}) = e^E_{ij}(\hat{k})e^L_{kl}(\hat{k}),
$$

while the detector tensor $\mathbb{D}$ is the direct product of two detector tensors

$$
\mathbb{D}(\mathbb{R}_a; \mathbb{R}_b) = a_{ij}^a d_{kl}^b, \tag{18}
$$

which is a function of the orientations of the two detectors relative to the sky, determined by two three-dimensional rotations $\mathbb{R}_a$ and $\mathbb{R}_b$. To be more specific, for a pair of interferometry detectors, the fundamental degrees of freedom regarding to its geometry include the opening angle between the two arms of each detector, the orientation of each detector relative to the sky; and the baseline vector connecting the two detectors. In practice, the opening angles are usually fixed. It is 90° for LIGO, Virgo, and KAGRA, and 60° for LISA, Cosmic Explorer, and Einstein Telescope. As the relative orientation between the two detectors and the baseline vector are fixed, it is convenient to factor out an overall SO(3) rotation of the whole pair, which can be realized by three Euler angles. In the literature, we use $\mathbb{R}_B$, which is an element of SO(3), to represent such an overall rotation. Furthermore, for a pair of ground-based detectors, we can choose the polar coordinates of the first detector $(\theta_a, \phi_a)$ and the angle $\alpha$, which gives the direction pointing to the second detector, as the rotational angles of $\mathbb{R}_B(\theta_a, \phi_a, \alpha)$. The rest degrees of freedom can be described by the other three angles, $\sigma_a$, $\sigma_b$, and $\beta$, as illustrated in Fig. 1. The numerical values of the six angles for detector pairs among LIGO, Virgo, and KAGRA are listed in Table I. In this way, the baseline vector $\vec{r}$ can be determined by $\mathbb{R}_B$ and $\beta$. To simplify the expression, we use $\mathbb{R}_{ab}$ to denote those internal angles, i.e. $\sigma_a$, $\sigma_b$, and $\beta$. With the help of these angular parameters, the detector tensor can be expressed as

$$
\mathbb{D}_{ijkl}^{ab} = d_{ij}^a d_{kl}^b, \tag{19}
$$

where

$$
\mathbb{D}_0(\mathbb{R}_{ab}) = [\mathbb{R}_Z(\sigma_a)d_0] \otimes [\mathbb{R}_Y(\beta)\mathbb{R}_Z(\sigma_b)d_0] \tag{20}
$$

is the detector tensor of pair a-b when we rotate the pair of detectors such that the detector-a is located at the north pole of the Earth while the detector-b is stayed on the $\phi = 0$ meridian. In this configuration, the corresponding baseline direction $\vec{r}_0$ is

$$
\vec{r}_0 = (\theta_{r_0}, \phi_{r_0}) = \left(\frac{\beta - \pi}{2}, 0\right) = \left(\frac{\pi - \beta}{2}, \pi\right). \tag{21}
$$

In the expression, the $d_0 = \frac{1}{2}(\hat{X} \otimes \hat{X} - \hat{Y} \otimes \hat{Y})$ is the detector tensor for a detector located at the north pole with its X-arm pointing to the X-axis of the celestial coordinate system, while the $\mathbb{R}(\alpha, \beta, \gamma) \equiv [\hat{Z}(\alpha) \hat{Y}(\beta) \hat{Z}(\gamma)]$ is the Euler rotation matrix, and $\mathbb{R}_X$, $\mathbb{R}_Y$, $\mathbb{R}_Z$ are three-dimensional rotation matrices that actively rotate tensors around fixed celestial $X$, $Y$, and $Z$-axes correspondingly. A brief review of the Euler rotation is given in the Appendix B.

To investigate, for given $I(f, \hat{k})$, $Q(f, \hat{k})$, $U(f, \hat{k})$, and $V(f, \hat{k})$, how $\xi_{ab}$ vary with the geometrical configuration or the orientation of the detector pair a-b, it is convenient to rewrite the convolutional integral Eq. (13) into the following form:

\begin{table}[h]
\centering
\begin{tabular}{|c|cccccc|}
\hline
Detectors & $\theta_a$ & $\phi_a(t=0)$ & $\alpha$ & $\beta$ & $\sigma_a$ & $\sigma_b$ \\
\hline
K-H & 53.6 & 137.3 & 135.3 & 72.4 & -15.7 & 160.7 \\
K-L & 53.6 & 137.3 & 139.5 & 99.3 & -19.9 & 250.4 \\
V-K & 46.4 & 10.5 & 139.8 & 86.5 & 20.8 & 84.1 \\
L-V & 59.4 & -90.8 & 145.6 & 79.6 & 70.4 & 128.1 \\
H-L & 43.5 & -119.4 & 154.5 & 151.6 & 241.5 \\
H-V & 43.5 & -119.4 & 145.6 & 79.6 & 70.4 & 128.1 \\
\hline
\end{tabular}
\caption{Angular parameters for different pairs of detectors formed by KAGRA(K), Virgo(V), LIGO-Hanford(H), and LIGO-Livingston(L). The data is converted from LAL-Suite [15] assuming that the Earth is a perfect sphere.}
\end{table}
FIG. 1. Convention of Angles. \( \vec{x}_a, \vec{x}_b \) represent the positions of detector-a and detector-b, respectively. \( \vec{r} \) is the baseline. \( \sigma_a \) and \( \sigma_b \) are the angles between the great circle connecting the pair a-b and the X-arms of detector-a and detector-b, respectively.

\[
\xi_{ab}(f, R_D; R_{ab}) = \sum_{I=\{I,V,Q\}} \int_{S^2} d\hat{k} I(\hat{k}, \hat{k}) \gamma_{I}^{(f, R_D; R_{ab})},
\]

where

\[
\gamma_{I}^{(f, R_D; R_{ab})} = \int_{S^2} d\hat{k} Y_{\ell m}(\hat{k}) \gamma_{I}^{(f, R_D; R_{ab})},
\]

of ordinary and spin-weighted spherical harmonics as

\[
I(\hat{k}, \hat{k}) = \sum_{\ell m} I_{\ell m}(f) Y_{\ell m}(\hat{k}),
\]

\[
V(\hat{k}, \hat{k}) = \sum_{\ell m} V_{\ell m}(f) Y_{\ell m}(\hat{k}),
\]

\[
(Q + iU)(\hat{k}, \hat{k}) = \sum_{\ell m} (Q + iU)_{\ell m}(f) + \frac{i}{4} Y_{\ell m}(\hat{k}),
\]

\[
(Q - iU)(\hat{k}, \hat{k}) = \sum_{\ell m} (Q - iU)_{\ell m}(f) - \frac{i}{4} Y_{\ell m}(\hat{k}),
\]

so as the overlap reduction functions:

\[
\gamma_{I}^{(f, R_D; R_{ab})} = \int_{S^2} d\hat{k} Y_{\ell m}(\hat{k}) \gamma_{I}^{(f, R_D; R_{ab})},
\]

\[
\gamma_{Q}^{(f, R_D; R_{ab})} = \int_{S^2} d\hat{k} Y_{\ell m}(\hat{k}) \gamma_{Q}^{(f, R_D; R_{ab})},
\]

The specific combinations, \( Q \pm iU \), make them become spin \( \pm 4 \) objects so that we can expand them nicely by the corresponding spin-weighted spherical harmonics.

By plugging Eq. (23) into Eqs. (29) and (30), and ex-
respectively. In Eqs. (37) and (38), the Wigner-D matrices \( D_{m' m}^{\ell} (\mathbf{R}^{-1}_D) \) account for the coupling coefficients between different spherical harmonics [16]. It is worth noting that the properties of the Wigner-3j symbols in Eq. (36) imply that \( L, \ell_c, \) and \( \ell \) have to satisfy the triangular condition, i.e. \( \ell + \ell_c \geq L \geq \ell - \ell_c \), while \( M + m_c + m = 0 \).

Nevertheless, a rotation of the pair of detectors on the Earth is equivalent to rotating the sky in the reverse sense. This fact enables us to choose a convenient coordinate system to evaluate Eq. (35). A convenient choice is to place the pair of detectors in the position described by Eq. (20). Equivalently, we can perform the rotation \( \mathbf{R}^{-1}_D \) on the detector pair, the baseline, and the SGWB sky simultaneously by using Eqs. (B12) and (B13), turning \( \hat{k} \) and \( \hat{r} \) into \( \mathbf{R}^{-1}_D \hat{k} \) and \( \mathbf{R}^{-1}_D \hat{r} = \hat{r}_0 \), respectively. In this coordinate system, the explicit forms of \( \gamma_{\ell m}^{I}, \gamma_{\ell m}^{Q+iU}, \gamma_{\ell m}^{Q-iU}, \) and \( \gamma_{\ell m}^{V} \) are given by

\[
\gamma_{\ell m}^{I} (f; \mathbf{R}_D; \mathbf{R}_{ab}) = (4\pi) \sum_{m'} D_{m' m}^{\ell} (\mathbf{R}_D^{-1}) \sum_{\ell_c m_c} \mathbb{D}_0 (\mathbf{R}_{ab}) \cdot \mathbb{D}_{\ell_c m_c} \sum_{L M} i^L j^L \left( \frac{2\pi f r}{c} \right) Y_{LM}(\hat{r}_0) D_{L M, s m_c}^{\ell e} \left( \begin{array}{ccc} L & \ell & 0 \\ M & 0 & -s m_i \\ 0 & -s_1 & -s_2 \end{array} \right),
\]

\[
\gamma_{\ell m}^{Q+iU} (f; \mathbf{R}_D; \mathbf{R}_{ab}) = (4\pi) \sum_{m'} D_{m' m}^{\ell} (\mathbf{R}_D^{-1}) \sum_{\ell_c m_c} \mathbb{D}_0 (\mathbf{R}_{ab}) \cdot \mathbb{D}_{\ell_c m_c} \sum_{L M} i^L j^L \left( \frac{2\pi f r}{c} \right) Y_{LM}(\hat{r}_0) D_{L M, s m_c}^{\ell e} \left( \begin{array}{ccc} L & \ell & 0 \\ M & 0 & m_i \\ 0 & +4 m_c & \pm 4 m_i' \end{array} \right),
\]

\[
\gamma_{\ell m}^{Q-iU} (f; \mathbf{R}_D; \mathbf{R}_{ab}) = (4\pi) \sum_{m'} D_{m' m}^{\ell} (\mathbf{R}_D^{-1}) \sum_{\ell_c m_c} \mathbb{D}_0 (\mathbf{R}_{ab}) \cdot \mathbb{D}_{\ell_c m_c} \sum_{L M} i^L j^L \left( \frac{2\pi f r}{c} \right) Y_{LM}(\hat{r}_0) D_{L M, s m_c}^{\ell e} \left( \begin{array}{ccc} L & \ell & 0 \\ M & 0 & -s m_i \\ 0 & -s_1 & -s_2 \end{array} \right),
\]

\[
\gamma_{\ell m}^{V} (f; \mathbf{R}_D; \mathbf{R}_{ab}) = (4\pi) \sum_{m'} D_{m' m}^{\ell} (\mathbf{R}_D^{-1}) \sum_{\ell_c m_c} \mathbb{D}_0 (\mathbf{R}_{ab}) \cdot \mathbb{D}_{\ell_c m_c} \sum_{L M} i^L j^L \left( \frac{2\pi f r}{c} \right) Y_{LM}(\hat{r}_0) D_{L M, s m_c}^{\ell e} \left( \begin{array}{ccc} L & \ell & 0 \\ M & 0 & m_i \\ 0 & +4 m_c & \pm 4 m_i' \end{array} \right),
\]
the degrees of freedom reflecting the free rotation of the whole pair of detectors. In the case of ground-based GW detectors, the two Euler angles, \( \theta_a \) and \( \alpha \), are fixed with respect to the geographical locations of the detectors, while \( \phi_a \) changes azimuthally as the Earth rotates. Besides, the frequency dependency of these \( \gamma \)'s, caused by the time delay of GW signal arriving at each detector, is taken cared of by the projection into the spherical Bessel functions \( j_L(2\pi f r/c) \).

### III. SGWB SKY MAP

A SGWB sky map can be constructed from a time series of the correlation output from a pair of GW detectors through a convolutional integral (22) over the whole sky. Each pointing on the SGWB sky map has a time-accumulated data output in an element of three Euler angles \((\alpha, \theta_a, \phi_a)\) in the group manifold of the three-dimensional rotation. However, for ground-based detectors the sky coverage is limited to a ring along the azimuthal angle \( \phi_a \) about the celestial pole. Regarding to different polarizations or Stokes parameters, the corresponding kernels, i.e., the overlap reduction functions, can be expanded in terms of the spherical harmonic basis to convert the convolutional integral into a summation over multipole moments in Eq. (24). The multipole moments of the overlap reduction functions are given in Eqs. (37) and (38). They depend on an overall SO(3) rotation \( \mathbf{R}_B \) characterizing the relative orientation between the whole pair of detectors and the SGWB sky, the antenna pattern function \( \mathbf{D} \cdot \mathbf{E} \) in the unrotated frame defined by Eqs. (20) and (21), and the spherical Bessel function \( j_L(2\pi f r/c) \) term that gives the frequency dependency of the overlap reduction function for a given baseline vector. All these three terms are coupled through the coefficients (36) involving Wigner-3j symbols.

### IV. EXAMPLE - UNPOLARIZED ISOTROPIC CASE

In the case of isotropic and unpolarized SGWB, the only relevant overlap reduction function is the \( \gamma_{00}^L \), which can be calculated from Eq. (37) as

\[
\gamma_{00}^L(f, \mathbf{R}_B; \mathbf{R}_{ab}) = (4\pi) \sum_{m'} D^0_{m'0}(\mathbf{R}_B^{-1}) \sum_{\ell, m_e} \mathbb{D}_0(\mathbf{R}_{ab}) \cdot \mathbb{E}_\ell^{\ell} m_e \sum_{LM} i^{\ell} j_L(2\pi f r/c) Y_{LM}(\hat{r}_0) \left\langle \frac{L}{M} \frac{\ell_e}{0} \frac{m_e}{0} \frac{m'}{0} \right\rangle
\]

where

\[
\mathbb{D}_0(\mathbf{R}_{ab}) \cdot \mathbb{E}_\ell^{\ell} m_e = (4\pi) \sum_{\ell, m_e} \mathbb{D}_0(\mathbf{R}_{ab}) \cdot \mathbb{E}_\ell^{\ell} m_e \sum_{LM} i^{\ell} j_L(2\pi f r/c) Y_{LM}(\hat{r}_0) \delta_{LL} \delta_{M m_e}
\]

The result does not depend on the orientation of the whole pair, i.e., the overall SO(3) rotation \( \mathbf{R}_B \). This is due to the fact that there is no preferred direction for an isotropic unpolarized SGWB. In Eq. (39), the expression of \( \mathbb{D}_0(\mathbf{R}_{ab}) \cdot \mathbb{E}_\ell^{\ell} m_e \) can be found in Appendix D 1.

### V. GW SPACE INTERFEROMETRY

By summing all 15 terms whose \( \ell_e = 0, 2, 4 \) and \( m_e \) are integers from \(-\ell_e\) to \( \ell_e \), one can get the standard overlap reduction function of unpolarized isotropic SGWB as same as, e.g., the result given in Ref. [11]:

\[
\gamma_{00}^L(f, \mathbf{R}_B; \mathbf{R}_{ab}) = \cos(2(\sigma_1 - \sigma_2)) \\
\times \frac{8\pi}{5} \left( j_0 + \frac{5j_2}{7} + \frac{3j_4}{112} \right) \cos^4 \left\langle \frac{\beta}{2} \right\rangle \\
+ \cos(2(\sigma_1 + \sigma_2) + \pi) \\
\times \frac{8\pi}{5} \left( -\frac{3j_0}{8} + \frac{45j_2}{56} - \frac{169j_4}{896} \right) + \left( j_0 - \frac{5j_2}{7} - \frac{27j_4}{224} \right) \cos(\beta) + \left( -\frac{j_0}{8} - \frac{5j_2}{56} - \frac{3j_4}{896} \right) \cos(2\beta) \\
\times \frac{8\pi}{5} \left( j_0 + \frac{5j_2}{7} + \frac{3j_4}{112} \right) \cos^4 \left\langle \frac{\beta}{2} \right\rangle.
\]

Basiclly, our formalism can be equally applied to space-based GW detectors. For example, in the LISA space mission, the three spacecrafts form an equilateral
triangle ABC, as illustrated in Fig. 2, where the center of mass is located at the origin of the rectangular coordinates XYZ. There are three independent baseline vectors: AB, BC, and CA, which are also the detector arms. The correlation output between spacecrafts A and B are then labelled by the pointing angles \((\alpha, \theta, \phi)\) of the spacecraft A and so on for other pairs of detectors. The final product of the space mission would be a SGWB sky map constructed from all correlation signals, with a sky coverage determined by the design of the spacecrafts orbit.

VI. CONCLUSION

We have studied the interferometric observation of stochastic gravitational wave background anisotropies. Different from previous works, we have expanded the polarization tensors of gravitational waves in terms of the spin-weighted spherical harmonics. This allows us to avoid tackling complicated tensor calculus and hence provide a systematic way for calculating the antenna pattern functions for all Stokes parameters. Our formalism has explicitly revealed the topology of the data structure that observed correlated signals are defined in the group manifold of the three-dimensional rotation. The correlations between two detectors in the interferometry experiments such as LIGO-VIRGO and KAGRA are explicitly constructed in terms of the Wigner D-functions and the Wigner-3j symbols. Our results are useful for constructing data pipelines to estimate the power spectra of stochastic gravitational wave background anisotropies.

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Appendix A: Spin-Weighted Spherical Harmonics

The explicit form of the spin-weighted spherical harmonics that we use is

\[
\mathbf{s}Y_{\ell m}(\theta, \phi) = (-1)^{s+m} e^{im\phi} \sqrt{\frac{(2\ell + 1)(\ell - m)!}{(4\pi)(\ell + m)!}} \sin^2\left(\frac{\ell}{2}\right) \sum_r \left(\begin{array}{c} \ell - s \\ r \\ \ell + s - m \end{array}\right) (-1)^{\ell - r - s} \cot^{2r + s - m} \left(\frac{\theta}{2}\right) .
\]

(A1)

When \(s = 0\), it reduces to the ordinary spherical harmonics,

\[
Y_{\ell m} = \sqrt{\frac{(2\ell + 1)(\ell - m)!}{(4\pi)(\ell + m)!}} P_{\ell m}(\cos \theta) e^{im\phi} .
\]

(A2)

Spin-weighted spherical harmonics satisfy the orthogonal relation,

\[
\int_{S^2} d\hat{n} \mathbf{s}Y^*_{\ell m} (\hat{n}) \mathbf{s}Y_{\ell' m'} (\hat{n}) = \delta_{\ell \ell'} \delta_{m m'} ,
\]

(A3)

and the completeness relation,

\[
\sum_{\ell m} \mathbf{s}Y^*_{\ell m} (\hat{n}) \mathbf{s}Y_{\ell m} (\hat{n'}) = \delta (\hat{n} - \hat{n'}) .
\]

(A4)

Its complex conjugate is

\[
\mathbf{s}Y^*_{\ell m} (\hat{n}) = (-1)^{s+m} \mathbf{s}Y_{\ell - m} (\hat{n}) .
\]

(A5)

Also, we have the spherical wave expansion:

\[
e^{i\hat{k} \cdot \hat{r}} = 4\pi \sum_{\ell = 0}^{\infty} \sum_{m = -\ell}^{\ell} i^{\ell} j_{\ell}(kr) Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r}) ,
\]

(A6)

where \(j_{\ell}(x)\) is the spherical Bessel function.
Appendix B: Three-Dimensional Rotation

A three-dimensional rotation can be parameterized by three Euler angles, \( \alpha, \beta, \) and \( \gamma. \) To rotate a vector in the real space, we may use the Euler matrix \( R(\alpha, \beta, \gamma) \), which can be decomposed into three consecutive rotations around fixed global axes as

\[
R(\alpha, \beta, \gamma) = R_Z(\alpha)R_Y(\beta)R_Z(\gamma). \tag{B1}
\]

For example,

\[
\hat{Z} = R(0, \frac{\pi}{2}, \frac{\pi}{2}) \hat{Y}. \tag{B2}
\]

For a tensor constructed by the direct product of a number of vectors, we adopt the convention self-explained by the following example,

\[
\left[ \hat{Z} \otimes \hat{Y} \right]^{ij} = R_{kl}^{ij}(0, \frac{\pi}{2}, \frac{\pi}{2}) \left[ \hat{Y} \otimes \hat{X} \right]^{kl}. \tag{B3}
\]

Furthermore, we can perform the similar rotation to a function defined on a sphere \( S^2, \)

\[
f^R(\hat{n}) = R(\alpha, \beta, \gamma)f(\hat{n}) = \langle \hat{n} | R(\alpha, \beta, \gamma) | f \rangle = f(R^{-1}(\alpha, \beta, \gamma)\hat{n}) = f(R(-\beta, -\gamma, -\alpha)\hat{n}). \tag{B4}
\]

For instance,

\[
f^R(\theta, \phi) = R(0, 0, \delta)f(\theta, \phi) = f(\theta, \phi - \delta), \tag{B5}
\]

in which the function \( f \) have been actively rotated about the fixed Z-axis by an angle \( \delta. \)

If we expand the function in terms of spherical harmonics:

\[
f(\theta, \phi) = \langle \theta, \phi | f \rangle = \sum_{\ell m} \langle \theta, \phi | \ell, m \rangle \langle \ell, m | f \rangle = \sum_{\ell m} Y_{\ell m}(\theta, \phi) f_{\ell m}, \tag{B6}
\]

the active Euler rotation \( R(\alpha, \beta, \gamma) \) can be acted on either the expansion coefficients or the spherical harmonics:

\[
\langle \theta, \phi | R | f \rangle = \sum_{\ell m} \sum_{\ell m'} \langle \theta, \phi | \ell, m \rangle \langle \ell, m | R | \ell, m' \rangle \langle \ell, m' | f \rangle
\]

\[
= \sum_{\ell m} \sum_{\ell m'} Y_{\ell m}(\theta, \phi) D_{\ell m'}^{\ell m}(R)f_{\ell m'}
\]

\[
= \sum_{\ell m'} Y_{\ell m'}(\theta, \phi) R_{\ell m'},
\]

the active Euler rotation \( R(\alpha, \beta, \gamma) \) can be acted on either the expansion coefficients or the spherical harmonics:

\[
\langle \theta, \phi | R | f \rangle = \sum_{\ell m} \sum_{\ell m'} Y_{\ell m}(\theta, \phi) D_{\ell m'}^{\ell m}(R)f_{\ell m'}.
\]

where the Wigner-D matrix is defined by

\[
\langle \ell, m | R | \ell', m' \rangle = D_{m m'}^{\ell \ell'}(R) = D_{m m'}^{\ell \ell'}(\alpha, \beta, \gamma), \tag{B8}
\]

which is a \( \ell + 1 \)-dimensional irreducible unitary representation of the rotation operator \( R(\alpha, \beta, \gamma). \) The Wigner-D matrix is closely related to the spin-weighted spherical harmonics:

\[
D_{s m}^{\ell}(\alpha, \beta, \gamma) = \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell m}(\hat{\beta}, -\gamma, -\alpha) e^{-is\alpha}. \tag{B9}
\]

Since the multiplication of two successive rotations is still a rotation, the matrix representation should reflect the closure property,

\[
D_{s m'}^{\ell}(R^{-1}) = \sum_{m''} D_{s m'}^{\ell}(R_{k}^{-1}R_{k})D_{m'' m}(R_{k}^{-1}). \tag{B10}
\]

Combining Eq. (B9), one can relate the spin-weighted spherical harmonics in different coordinates as

\[
y_{\ell m}(\theta, \phi) e^{-is\gamma_k} = \sum_{m'} y_{\ell m'}(\theta', \phi') e^{-is\gamma_{k'}} D_{m' m}(R_{k}^{-1}), \tag{B11}
\]

in which \( R_{k}^{-1}(\phi_{k'}, \theta_{k'}, \gamma_{k'}) \equiv (R_{k}^{-1}R_{k}). \) By choosing \( \gamma_{k} = 0, \) i.e., making \( R_{k} = R_{k}(\phi_{k}, \theta_{k}, 0), \) the LHS of Eq. (B11) becomes the spin-s spherical harmonics in the unprimed coordinates. However, on the RHS, an extra phase factor \( e^{-is\gamma_{k'}} \) appears in addition to the Wigner-D matrix of the coordinate transformation. The angle \( \gamma_{k'}(\theta_{k'}, \phi_{k'}; R_{k}) \) is the angle between the two great circles that connect the point \((\theta_{k'}, \phi_{k'}) \) to the unprimed and the primed north poles as shown in Fig. 3. Also see a similar result, the Eq. (5.4) in Ref. [17], derived in a different context.

Here, we summarize some results used in the literature:

\[
y_{\ell m}(R\hat{n}) = \sum_{m'} y_{\ell m'}(\hat{n}) D_{m m'}^{\ell}(R^{-1}), \tag{B12}
\]

\[
s_{\ell} y_{\ell m}(\hat{n}) = \sum_{m'} s_{\ell} y_{\ell m'}(R^{-1}\hat{n}) e^{-is\gamma_{m'}} D_{m' m}(R^{-1}), \tag{B13}
\]

\[
f_{\ell m}^{R^{-1}} = \sum_{m'} D_{m m'}^{\ell}(R^{-1}) f_{\ell m'}, \tag{B14}
\]
FIG. 3. The angle relates the spin-weighted spherical harmonics in two different coordinates. A point \( P \) can be described by both the primed and unprimed coordinates where the \( Z' \) and \( Z \) indicate their north poles correspondingly. The angle \( \gamma \) is the angle between the two great circles that connect the point \( P \) to the two north poles.

Appendix C: Multipole Moments of Polarization Tensor

1. \( ijkl E_{lm}^{ij} \)

The only non-zero coefficients are \( \ell = 0, 2, 4 \) cases for \( ijkl E_{lm}^{ij} \), which are symmetric under exchanging between \( i \leftrightarrow j, k \leftrightarrow l, \) and \( ij \leftrightarrow kl \).

\[
\begin{align*}
\frac{16}{15} \sqrt{\frac{\pi}{5}} &= xxxx E_{00} = yyy y E_{00} = zzz z E_{00} \\
- \frac{8}{15} \sqrt{\frac{\pi}{5}} &= xxyy E_{00} = xxx z E_{00} = yyzz E_{00}
\end{align*}
\]

\[
\begin{align*}
\frac{16}{21} \sqrt{\frac{\pi}{5}} &= xxxx E_{20} = yyy y E_{20} = zzz z E_{20} = yyzz E_{20} \\
- \frac{32}{21} \sqrt{\frac{\pi}{5}} &= xxyy E_{20} = xxyz E_{20}
\end{align*}
\]

\[
\begin{align*}
\frac{4}{7} \sqrt{\frac{2\pi}{15}} &= xxxz E_{21} = xzzz E_{21} \\
\frac{4i}{7} \sqrt{\frac{2\pi}{15}} &= yyyz E_{21} = yzzz E_{21} \\
\frac{16}{7} \sqrt{\frac{6\pi}{5}} &= xxyz E_{21} = xyyz E_{21}
\end{align*}
\]

\[
\begin{align*}
\frac{8}{7} \sqrt{\frac{2\pi}{15}} &= xxxz E_{40} = yyy y E_{40} \\
- \frac{8}{7} \sqrt{\frac{2\pi}{15}} &= xxxz E_{22} = yyy z E_{22} \\
- \frac{8i}{7} \sqrt{\frac{2\pi}{15}} &= xxxy E_{22} = yzzz E_{22} \\
\frac{4i}{7} \sqrt{\frac{2\pi}{15}} &= xxxx E_{22} = xyy E_{22} \\
\frac{8}{105} \sqrt{\frac{\pi}{5}} &= xxxz E_{40} = yyy z E_{40} \\
- \frac{16}{105} \sqrt{\frac{\pi}{5}} &= xxxz E_{40} = zzz z E_{40}
\end{align*}
\]
\[
\begin{align*}
\frac{1}{7} \sqrt{\frac{\pi}{5}} &= xxxzE_{41} \\
-i \frac{1}{7} \sqrt{\frac{\pi}{5}} &= yyyzE_{41} \\
\frac{1}{21} \sqrt{\frac{\pi}{5}} &= xyyzE_{41} \\
-i \frac{1}{21} \sqrt{\frac{\pi}{5}} &= xxyzE_{41} \\
-\frac{4}{21} \sqrt{\frac{\pi}{5}} &= xzzzE_{41} \\
\frac{4i}{21} \sqrt{\frac{\pi}{5}} &= yzzzE_{41} \\
2i \frac{2\pi}{21} \sqrt{\frac{\pi}{5}} &= yyyzE_{42} = xzzzE_{42} \\
-i \frac{2\pi}{21} \sqrt{\frac{\pi}{5}} &= xxyzE_{42} \\
i \frac{2\pi}{21} \sqrt{\frac{\pi}{5}} &= xyyzE_{42} = xyyE_{42} \\
1 \frac{3}{35} \sqrt{\pi} &= yyyzE_{43} \\
-1 \frac{3}{35} \sqrt{\pi} &= xxxzE_{43} \\
-i \frac{3}{35} \sqrt{\pi} &= yyyzE_{43} \\
i \frac{3}{35} \sqrt{\pi} &= xyyzE_{43} \\
1 \frac{3}{35} \sqrt{\pi} &= xyyzE_{43} \\
-1 \frac{3}{35} \sqrt{\pi} &= xxxzE_{43} \\
-i \frac{3}{35} \sqrt{\pi} &= yyyzE_{43} \\
i \frac{3}{35} \sqrt{\pi} &= xyyzE_{43} \\
1 \frac{2\pi}{35} \sqrt{\frac{\pi}{5}} &= xxxzE_{44} = yyyzE_{44} \\
-i \frac{2\pi}{35} \sqrt{\frac{\pi}{5}} &= xyyzE_{44} \\
-\frac{1}{3} \frac{2\pi}{35} \sqrt{\frac{\pi}{5}} &= xxyyE_{44} \\
i \frac{2\pi}{35} \sqrt{\frac{\pi}{5}} &= xyyzE_{44} \\
2. \ ijkl \ E_{\ell m}
\end{align*}
\]

The only non-zero coefficients are \( \ell = 1, 3 \) cases for \( ijkl E_{\ell m} \), which are symmetric under exchanging between 
\( i \leftrightarrow j, k \leftrightarrow l, \) and \( ij \leftrightarrow kl. \)

\[
\begin{align*}
-\frac{8i}{5} \sqrt{\frac{\pi}{3}} &= xxxyE_{10} = xyyzE_{10} \\
-\frac{4}{5} \sqrt{\frac{2\pi}{3}} &= xxxzE_{11} \quad xzzzE_{11} \\
4i \sqrt{\frac{2\pi}{3}} &= yyyzE_{11} = yzzzE_{11} \\
-\frac{4}{5} \sqrt{\frac{2\pi}{3}} &= xyyzE_{11} \\
-\frac{2i}{5} \sqrt{\frac{\pi}{7}} &= xxxyE_{30} = xyyzE_{30} \\
-\frac{4}{5} \sqrt{\frac{2\pi}{3}} &= xxxzE_{31} \quad xzzzE_{31} \\
-\frac{i}{5} \sqrt{\frac{3\pi}{7}} &= xyyzE_{31} \\
-\frac{4}{5} \sqrt{\frac{2\pi}{3}} &= yyyzE_{31} \\
i \sqrt{\frac{2\pi}{105}} &= xyyxE_{31} \\
2 \sqrt{\frac{2\pi}{105}} &= xxzyE_{31} = yzzzE_{31} \\
-2 \sqrt{\frac{2\pi}{105}} &= xxxzE_{31} \\
-\frac{i}{2} \sqrt{\frac{3\pi}{105}} &= xyyzE_{31} \\
-\frac{2i}{5} \sqrt{\frac{\pi}{7}} &= xyyzE_{31} \\
i \sqrt{\frac{2\pi}{105}} &= xyyxE_{31} \\
2 \sqrt{\frac{2\pi}{105}} &= xxzyE_{31} = yzzzE_{31}
\end{align*}
\]
i√\frac{2\pi}{105} = xxxyE_{32}
2√\frac{2\pi}{105} = xxyyE_{32} = yyzzE_{42}
−2√\frac{2\pi}{105} = xzzzE_{32}
−i√\frac{2\pi}{105} = xyyyE_{32}
2i√\frac{2\pi}{105} = xyyzE_{32}

\sqrt{\frac{\pi}{35}} = xxxzE_{33}
−i\sqrt{\frac{\pi}{35}} = xxyzE_{33}
−\sqrt{\frac{\pi}{35}} = xyyzE_{33}
i\sqrt{\frac{\pi}{35}} = ygyzE_{33}

3. \ ijk\ell E_{m\ell n}^{Q\pm_i}\mu_j

The only non-zero coefficients are \ell = 4 cases for \ ijk\ell E_{m\ell n}^{Q\pm_i}\mu_j, which are symmetric under exchanging between \ i \leftrightarrow j, k \leftrightarrow l, and \ ij \leftrightarrow kl.

\sqrt{\frac{2\pi}{35}} = xxxzE_{40} = yyyzE_{40}
\frac{1}{3}\sqrt{\frac{2\pi}{35}} = xxyyE_{40}
−\frac{4}{3}\sqrt{\frac{2\pi}{35}} = xzzzE_{40} = yyzzE_{40}
8\frac{2\pi}{35} = zzzzE_{40}

\sqrt{\frac{\pi}{14}} = xxxzE_{41}
−\frac{i}{3}\sqrt{\frac{\pi}{14}} = xxyyE_{41}
\frac{1}{3}\sqrt{\frac{\pi}{14}} = xyyzE_{41}
−\frac{2}{3}\sqrt{\frac{2\pi}{7}} = xzzzE_{41}
−i\sqrt{\frac{\pi}{14}} = yyyzE_{41}
\frac{2}{3}\sqrt{\frac{\pi}{7}} = yyyzE_{41}

\frac{2}{3}\sqrt{\pi} = xyyzE_{43} = xyyE_{43}
\frac{i}{3}\sqrt{\frac{\pi}{2}} = xyyE_{43} = yyyE_{43}
\frac{1}{3}\sqrt{\frac{\pi}{2}} = xyyzE_{43} = xyyE_{43}

\frac{1}{3}\sqrt{\pi} = xxxzE_{44} = ygyzE_{44}
−\frac{1}{3}\sqrt{\pi} = xyyzE_{44}
i\frac{1}{3}\sqrt{\pi} = yyyzE_{44}
−\frac{i}{3}\sqrt{\pi} = xyyzE_{44}

Appendix D: Antenna Pattern Functions

In most literature, the inner product between the detector tensor and the polarization basis tensor is referred as the antenna pattern function.
1. $\mathbf{D}E^I$

For the Stokes-I parts, the only non-vanishing $\mathbf{D}_0(\sigma_1, \sigma_2, \beta) \cdot \mathbf{E}^I_{\ell m}$ are 15 components with $\ell = 0, 2, 4$, which satisfy $\mathbf{D}E^I_{\ell - m} = (-1)^m \mathbf{D}E^I_{\ell m}$.

\[
\begin{align*}
\mathbf{D}E^I_{00} &= \frac{4}{5} \sqrt{\pi} \left( \cos^4 \left( \frac{\beta}{2} \right) \cos(2\sigma_1 - 2\sigma_2) + \sin^4 \left( \frac{\beta}{2} \right) \cos(2\sigma_1 + 2\sigma_2) \right) \\
\mathbf{D}E^I_{20} &= \frac{8}{7} \sqrt{\frac{\pi}{5}} \left( \cos^4 \left( \frac{\beta}{2} \right) \cos(2\sigma_1 - 2\sigma_2) + \sin^4 \left( \frac{\beta}{2} \right) \cos(2\sigma_1 + 2\sigma_2) \right) \\
\mathbf{D}E^I_{21} &= -\frac{2}{7} \sqrt{6\pi} e^{-2i\sigma_1} \sin(\beta)(\cos(\beta) \cos(2\sigma_2) + i \sin(2\sigma_2)) \\
\mathbf{D}E^I_{22} &= \frac{2}{7} \sqrt{6\pi} e^{-2i\sigma_1} \sin^2(\beta) \cos(2\sigma_2) \\
\mathbf{D}E^I_{40} &= \frac{2}{105} \sqrt{\pi} \left( \cos^4 \left( \frac{\beta}{2} \right) \cos(2\sigma_1 - 2\sigma_2) + \sin^4 \left( \frac{\beta}{2} \right) \cos(2\sigma_1 + 2\sigma_2) \right) \\
\mathbf{D}E^I_{41} &= -\frac{1}{21} \sqrt{\frac{\pi}{5}} e^{-2i\sigma_1} \sin(\beta)(\cos(\beta) \cos(2\sigma_2) + i \sin(2\sigma_2)) \\
\mathbf{D}E^I_{42} &= \frac{1}{7} \sqrt{\frac{\pi}{10}} e^{-2i\sigma_1} \sin^2(\beta) \cos(2\sigma_2) \\
\mathbf{D}E^I_{43} &= \frac{1}{3} \sqrt{\frac{\pi}{35}} e^{-2i\sigma_1} \sin(\beta)(\cos(\beta) \cos(2\sigma_2) - i \sin(2\sigma_2)) \\
\mathbf{D}E^I_{44} &= \frac{1}{6} \sqrt{\frac{\pi}{70}} e^{-2i\sigma_1} ((\cos(2\beta) + 3) \cos(2\sigma_2) - 4i \cos(\beta) \sin(2\sigma_2))
\end{align*}
\]

2. $\mathbf{D}E^V$

For the Stokes-V parts that correspond to the circular polarized signal, the only non-vanishing $\mathbf{D}_0(\sigma_1, \sigma_2, \beta) \cdot \mathbf{E}^V_{\ell m}$ are 10 components with $\ell = 1, 3$, which satisfy $\mathbf{D}E^V_{\ell - m} = (-1)^{m+1} \mathbf{D}E^V_{\ell m}$.

\[
\begin{align*}
\mathbf{D}E^V_{10} &= \frac{8}{5} i \sqrt{\frac{\pi}{3}} \left( \cos^4 \left( \frac{\beta}{2} \right) \sin(2\sigma_1 - 2\sigma_2) + \sin^4 \left( \frac{\beta}{2} \right) \sin(2\sigma_1 + 2\sigma_2) \right) \\
\mathbf{D}E^V_{11} &= \frac{2}{5} \sqrt{\frac{2\pi}{3}} e^{2i\sigma_1} \sin(\beta)(\cos(\beta) \cos(2\sigma_2) - i \sin(2\sigma_2)) \\
\mathbf{D}E^V_{30} &= \frac{2}{5} i \sqrt{\frac{\pi}{7}} \left( \cos^4 \left( \frac{\beta}{2} \right) \sin(2\sigma_1 - 2\sigma_2) + \sin^4 \left( \frac{\beta}{2} \right) \sin(2\sigma_1 - 2\sigma_2) \right) \\
\mathbf{D}E^V_{31} &= \frac{1}{5} \sqrt{\frac{3\pi}{7}} e^{2i\sigma_1} \sin(\beta)(\cos(\beta) \cos(2\sigma_2) - i \sin(2\sigma_2)) \\
\mathbf{D}E^V_{32} &= \sqrt{\frac{3\pi}{70}} e^{2i\sigma_1} \sin^2(\beta) \cos(2\sigma_2) \\
\mathbf{D}E^V_{33} &= -\sqrt{\frac{\pi}{35}} e^{2i\sigma_1} \sin(\beta)(\cos(\beta) \cos(2\sigma_2) + i \sin(2\sigma_2))
\end{align*}
\]
3. $\mathbb{D} E^{\pm iU}$

For the linear polarized signal, the only non-vanishing $D_0(\sigma_1, \sigma_2, \beta) \cdot E_{\ell m}^{Q \pm iU}$ are 9 components with $\ell = 4$, which satisfy $\mathbb{D} E_{-m} = (-1)^m \mathbb{D} E_{\ell m}$.

\begin{align*}
\mathbb{D} E_{40}^{Q \pm iU} &= \frac{2\pi}{35} \left( \cos^4 \left( \frac{\beta}{2} \right) \cos(2\sigma_1 - 2\sigma_2) + \sin^4 \left( \frac{\beta}{2} \right) \cos(2\sigma_1 + 2\sigma_2) \right) \\
\mathbb{D} E_{41}^{Q \pm iU} &= -\frac{1}{3} \sqrt{\frac{\pi}{14}} e^{-2i\sigma_1} \sin(\beta) (\cos(\beta) \cos(2\sigma_2) + i \sin(2\sigma_2)) \\
\mathbb{D} E_{42}^{Q \pm iU} &= \frac{1}{2} \sqrt{\frac{\pi}{7}} e^{-2i\sigma_1} \sin^2(\beta) \cos(2\sigma_2) \\
\mathbb{D} E_{43}^{Q \pm iU} &= \frac{1}{3} \sqrt{\frac{\pi}{2}} e^{-2i\sigma_1} \sin(\beta) (\cos(\beta) \cos(2\sigma_2) - i \sin(2\sigma_2)) \\
\mathbb{D} E_{44}^{Q \pm iU} &= \frac{1}{12} \sqrt{\frac{\pi}{e^{-2i\sigma_1}}} ((\cos(2\beta) + 3) \cos(2\sigma_2) - 4i \cos(\beta) \sin(2\sigma_2))
\end{align*}

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