Analytic Deformation and Stresses Solutions of Functionally Graded Rotating Disk under Mechanical and Thermal Load

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Abstract. This article presents analytical solutions for radial displacement and stresses of thin-walled functionally graded rotating disk subjected to pressure and temperature difference on its boundary surfaces. The thickness, elasticity modulus, material density, thermal expansion and thermal conduction coefficients of disk are represented by exponential forms of radial variable. Numerical results of radial displacement and stresses are shown in figures. Separate and combined effects of both pressure and temperature difference on radial displacement and stresses will be discussed graphically.

Introduction

Functionally graded material (FGM) disks have high strength and thermal resistance, their varying material properties make they can simultaneously work in high pressure and temperature environments. Mechanical-related analyses of disks made of an isotropic and homogeneous materials have gained great attention from many investigators. Zenkour [1] presented the stress and displacement analytical solutions for the rotating exponentially graded annular disks with various boundary conditions and varying material properties. Bayat et al. [2] analyze functionally graded rotating disks which material properties and disk thickness profile are assumed to be represented by two power-law distributions. Thawait [3] investigated deformation and stresses behavior of functionally graded shells which properties vary in radial direction according to exponential distribution law. Jalali and Shahriari [4] presented finite difference method to investigate elastic stress analysis of rotating variable thickness annular disk made of functionally graded materials. Thawait et al. [5] considered three types of distribution laws on a concave thickness profile FGM rotating disk, and the resulting deformation and stresses are evaluated for clamped-free boundary condition.

Thermal-related analyses in the basic structural components of FGM disk also attract researcher’s interests. Jabbari et al. [6] solved the Navier equation of functionally graded rotating cylinder which material property was represented in power form of radius and was subjected to axisymmetric thermal and mechanical loads. Bayat et al. [7] considered a rotating functionally graded disk with variable thickness under steady temperature field. Thermo elastic stresses for the solid disk and the hollow disks with different boundary conditions are obtained. Leu and Chien [8] presented elastic behavior of functionally graded rotating disks subjected to non-uniform thermal load, the thickness is a power function of radius. Akbari and Ghanbari [9] investigate functionally graded rotating disks under thermal, pressure and non-symmetric inertial load, the load on the outer edge is a harmonic sinusoidal load. Yildirim [10] investigated a power-law graded disk under centrifugal and steady state thermal loads. However, if the indexes of the material parameters equal to zero, all properties vanish. Power-law graded disks are not applicable for material which discussed domain includes disk center.
Theoretical Formulation and Solution

In this article, we consider a thin-walled hollow functionally graded material annular disk, the disk rotates at constant angular velocity $\omega$. Pressure and temperature difference at inner surface are $P_i$ and $T_a$, at outer surface are $P_o$ and $T_b$, respectively.

Material properties are assumed to be functions of radius. The thickness $h(r)$, elasticity modulus $E(r)$, material density $\rho(r)$, coefficient of thermal expansion $\alpha(r)$ and coefficient of thermal conduction $k(r)$ of the rotating disks are assumed to vary radially in the following exponentially-varying functions.

$$h(r) = h_m e^{m_1 (r-a)/b}$$ (1)

$$E(r) = E_m e^{m_2 (r-a)/b}$$ (2)

$$\rho(r) = \rho_m e^{m_3 (r-a)/b}$$ (3)

$$\alpha(r) = \alpha_m e^{m_4 (r-a)/b}$$ (4)

$$k(r) = k_m e^{m_5 (r-a)/b}$$ (5)

where $r$ is disk radius, $h_m$, $E_m$, $\rho_m$, $\alpha_m$ and $k_m$ are thickness, elasticity modulus, material density, coefficient of thermal expansion and coefficient of thermal conduction at inner surface of hollow disk. The index $m_1$ is geometric parameter, indexes $m_2$, $m_3$, $m_4$ and $m_5$ are the material parameters.

Poisson’s ratio $\nu$ is assumed to be constant in this article.

According to axis symmetry and plane stress assumption, the strain–displacement and constitutive equations are

$$\varepsilon_r = \frac{du}{dr}, \varepsilon_\theta = \frac{u}{r}$$ (6)

$$\sigma_r = \frac{E(r)}{1-\nu^2} \left[ \varepsilon_r + \nu \varepsilon_\theta - (1 + \nu) \alpha(r) T(r) \right]$$ (7)

$$\sigma_\theta = \frac{E(r)}{1-\nu^2} \left[ \varepsilon_\theta + \nu \varepsilon_r - (1 + \nu) \alpha(r) T(r) \right]$$ (8)

where $u$ is radial displacement, $(\varepsilon_r, \varepsilon_\theta)$ are radial and circumferential strains, $(\sigma_r, \sigma_\theta)$ are radial and circumferential stresses, respectively, $T(r)$ is the change in temperature. Assuming there is no heat generation in the structure, for axisymmetric problem and steady-state condition, the governing differential equation of $T(r)$ for disk has variable thickness is

$$\frac{d}{dr} \left[ rh(r)k(r) \frac{dT(r)}{dr} \right] = 0$$ (9)

Let $\gamma = (m_1 + m_5)/b$, then

$$T(r) = \frac{T_a-T_b}{\Delta(a)-\Delta(b)} \Delta(r) + \frac{T_b\Delta(a) - T_a\Delta(b)}{\Delta(a)-\Delta(b)}$$ (10)

$$\Delta(r) = \ln r + \sum_{n=1}^{\infty} \frac{(-\gamma r)^n}{n!n!}$$ (11)

In the absence of body forces, the equation of equilibrium for disk with variable thickness in radial direction can be described as in the following

$$\frac{d}{dr} \left[ h(r)r \sigma_r \right] - h(r)\sigma_\theta + h(r)\rho(r)\omega^2 r^2 = 0$$ (12)
Using Eqs. (1)-(11), let \( \beta = (m_1 + m_2)/b, \delta = (m_1 + m_2 + m_4)/b, \Omega = b\sqrt{\rho_m(1 - \nu^2)/E_m}, \zeta = (1 + \nu)\alpha m b, r = bR, u = bU \) and \( \bar{a} = a/b \), the governing equation of radial displacement becomes

\[
R^2 \frac{d^2 U}{dR^2} + R(1 + \beta bR) \frac{dU}{dR} + (\nu \beta bR - 1)U = -\Omega^2 \omega^2 R^3 e^{(m_3 - m_2)(R-a)} + \zeta R^2 e^{m_4(R-a)} \left\{ \frac{\delta (\Delta(bR) - \Delta(b)) + e^{-\gamma bR/bR}}{\Delta(a) - \Delta(b)} \right\} T_a = \frac{\delta (\Delta(bR) - \Delta(a)) + e^{-\gamma bR/bR}}{\Delta(a) - \Delta(b)} T_b \]

(13)

The above equation is a generalized confluent hypergeometric differential equation and the solution is

\[
U(R) = A \tilde{W}(R) + B \tilde{M}(R) + \omega^2 \tilde{\Phi}(R) + T_a \tilde{\Phi}(R) + T_b \tilde{\Psi}(R)
\]

(14)

\[
\tilde{W}(R) = e^{-\frac{x}{2}x^2}W_{i,j}(x)
\]

(15)

\[
\tilde{M}(R) = e^{-\frac{x}{2}x^2}M_{i,j}(x)
\]

(16)

\[
\tilde{F}(R) = e^{-\frac{x}{2}x^2} \left[ W_{i,j}(x) \int F(R) M_{i,j}(x) dR - M_{i,j}(x) \int F(R) W_{i,j}(x) dR \right]
\]

(17)

\[
F(R) = \frac{\Omega^2 e^{(m_3 - m_2)(R-a)} R^2 x^2 e^x}{(1 + \nu) M_{i+1,j}(x) W_{i,j}(x) + M_{i,j}(x) W_{i+1,j}(x)}
\]

(18)

\[
\tilde{\Phi}(R) = e^{-\frac{x}{2}x^2} \left[ W_{i,j}(x) \int \Phi(R) M_{i,j}(x) dR - M_{i,j}(x) \int \Phi(R) W_{i,j}(x) dR \right]
\]

(19)

\[
\Phi(R) = \frac{\zeta}{\Delta(a) - \Delta(b)} e^{m_4(x-a)} \left( \frac{\delta (\Delta(bR) - \Delta(b)) + e^{-\gamma bR/bR}}{\Delta(a) - \Delta(b)} \right) \Re e^{\frac{x}{2}x^2}
\]

(20)

\[
\tilde{\Psi}(R) = e^{-\frac{x}{2}x^2} \left[ W_{i,j}(x) \int \Psi(R) M_{i,j}(x) dR - M_{i,j}(x) \int \Psi(R) W_{i,j}(x) dR \right]
\]

(21)

\[
\Psi(R) = \frac{\zeta}{\Delta(a) - \Delta(b)} e^{m_4(x-a)} \left( \frac{\delta (\Delta(bR) - \Delta(a)) + e^{-\gamma bR/bR}}{\Delta(a) - \Delta(b)} \right) \Re e^{\frac{x}{2}x^2}
\]

(22)

\[
M_{i,j}(x) \text{ and } W_{i,j}(x) \text{ are Whittaker’s functions presented by Abramowitz and Stegun [11]}
\]

(23)

\[
W_{i,j}(x) = e^{-\frac{x}{2}x^2} U(j - i + 1/2, 1 + 2j, x)
\]

(24)

in which \( M(j - i + 1/2, 1 + 2j, x) \) and \( U(j - i + 1/2, 1 + 2j, x) \) are the hypergeometric and Kummer function with \( i = (2\nu - 1)/2, j = 1 \) and \( x = \beta bR \). The Whittaker’s functions \( M_{i,j}(x) \) and \( W_{i,j}(x) \) converge for \( |x| < 1 \). Coefficients \( A \) and \( B \) are arbitrary constants can to be determined from mechanical loads on boundary surface. Substituting Eq. (14) into Eqs. (6)-(8) and let \( (\Sigma_r, \Sigma_\theta) = (1 - \nu^2)/E_m, (\sigma_r, \sigma_\theta) \), the radial and circumferential stresses in dimensionless forms are

\[
\Sigma_r = l_1 P_i + l_2 P_0 + l_3 \omega^2 + l_4 T_a + l_5 T_b
\]

(25)

\[
\Sigma_\theta = \lambda_1 P_i + \lambda_2 P_0 + \lambda_3 \omega^2 + \lambda_4 T_a + \lambda_5 T_b
\]

(26)

\[
l_1 = A_{P_i} \Sigma_{r W} + B_{P_i} \Sigma_{r \bar{W}}
\]

(27)

\[
l_2 = A_{P_0} \Sigma_{r W} + B_{P_0} \Sigma_{r \bar{W}}
\]

(28)
\[ l_3 = A_\omega \Sigma_{r \bar{w}} + B_\omega \Sigma_{r \bar{M}} + \Sigma_{r \bar{F}} \]  
(29)

\[ l_4 = A_T a \Sigma_{r \bar{w}} + B_T a \Sigma_{r \bar{M}} - \Sigma_{r T a} \]  
(30)

\[ l_5 = A_T a \Sigma_{r \bar{w}} + B_T a \Sigma_{r \bar{M}} + \Sigma_{r T b} \]  
(31)

\[ \Sigma_{r \bar{w}} = -e^{m_2(R - \bar{a})} R^{-1} e^{-\frac{x}{2} x^{-\frac{1}{2}} W_{i+1,j}(x)} \]  
(32)

\[ \Sigma_{r \bar{M}} = (1 + \nu) e^{m_2(R - \bar{a})} R^{-1} e^{-\frac{x}{2} x^{-\frac{1}{2}} M_{i+1,j}(x)} \]  
(33)

\[ \Sigma_{r \bar{F}} = -e^{m_2(R - \bar{a})} R^{-1} e^{-\frac{x}{2} x^{-\frac{1}{2}} [W_{i+1,j}(x) \int F(R) M_{i,j}(x) dR + (1 + \nu) M_{i+1,j}(x) \int F(R) W_{i,j}(x) dR]} \]  
(34)

\[ \Sigma_{r T a} = -e^{m_2(R - \bar{a})} R^{-1} e^{-\frac{x}{2} x^{-\frac{1}{2}} [W_{i+1,j}(x) \int \Phi(R) M_{i,j}(x) dR + (1 + \nu) M_{i+1,j}(x) \int \Phi(R) W_{i,j}(x) dR] + (1 + \nu) \alpha_m e^{(m_4 + m_2)(R - \bar{a})} \frac{\Delta(b R) - \Delta(b)}{\Delta(a) - \Delta(b)}} \]  
(35)

\[ \Sigma_{r T b} = -e^{m_2(R - \bar{a})} R^{-1} e^{-\frac{x}{2} x^{-\frac{1}{2}} [W_{i+1,j}(x) \int \Psi(R) M_{i,j}(x) dR + (1 + \nu) M_{i+1,j}(x) \int \Psi(R) W_{i,j}(x) dR] + (1 + \nu) \alpha_m e^{(m_4 + m_2)(R - \bar{a})} \frac{\Delta(b R) - \Delta(b)}{\Delta(a) - \Delta(b)}} \]  
(36)

\[ A_{p i} = -\frac{(1 - \nu^2) D_{22}}{E_m(D_{11} D_{22} - D_{12} D_{21})}, B_{p i} = \frac{(1 - \nu^2) D_{21}}{E_m(D_{11} D_{22} - D_{12} D_{21})} \]  
(37)

\[ A_{p o} = \frac{(1 - \nu^2) D_{12}}{E_m(D_{11} D_{22} - D_{12} D_{21})}, B_{p o} = -\frac{(1 - \nu^2) D_{11}}{E_m(D_{11} D_{22} - D_{12} D_{21})} \]  
(38)

\[ A_\omega = \frac{-D_{13} D_{22} + D_{12} D_{23}}{D_{11} D_{22} - D_{12} D_{21}}, B_\omega = \frac{-D_{11} D_{22} + D_{13} D_{21}}{D_{11} D_{22} - D_{12} D_{21}} \]  
(39)

\[ A_T a = \frac{D_{14} D_{22} - D_{12} D_{24}}{D_{11} D_{22} - D_{12} D_{21}}, B_T a = \frac{D_{12} D_{23} - D_{13} D_{22}}{D_{11} D_{22} - D_{12} D_{21}} \]  
(40)

\[ A_T b = \frac{D_{12} D_{23} - D_{15} D_{22}}{D_{11} D_{22} - D_{12} D_{21}}, B_T b = \frac{D_{12} D_{23} - D_{15} D_{22}}{D_{11} D_{22} - D_{12} D_{21}} \]  
(41)

\[ D_{11} = \Sigma_{r \bar{w}} \bigg|_{R = \bar{a}}, D_{12} = \Sigma_{r \bar{M}} \bigg|_{R = \bar{a}}, D_{13} = \Sigma_{r \bar{F}} \bigg|_{R = \bar{a}}, D_{14} = \Sigma_{r T a} \bigg|_{R = \bar{a}}, D_{15} = \Sigma_{r T b} \bigg|_{R = \bar{a}} \]  
(42)

\[ D_{21} = \Sigma_{r \bar{w}} \bigg|_{R = 1}, D_{22} = \Sigma_{r \bar{M}} \bigg|_{R = 1}, D_{23} = \Sigma_{r \bar{F}} \bigg|_{R = 1}, D_{24} = \Sigma_{r T a} \bigg|_{R = 1}, D_{25} = \Sigma_{r T b} \bigg|_{R = 1} \]  
(43)

\[ \lambda_1 = A_{p i} \Sigma_{\theta \bar{w}} + B_{p i} \Sigma_{\theta \bar{M}} \]  
(44)

\[ \lambda_2 = A_{p o} \Sigma_{\theta \bar{w}} + B_{p o} \Sigma_{\theta \bar{M}} \]  
(45)

\[ \lambda_3 = A_\omega \Sigma_{\theta \bar{w}} + B_\omega \Sigma_{\theta \bar{M}} + \Sigma_{\theta \bar{F}} \]  
(46)

\[ \lambda_4 = A_T a \Sigma_{\theta \bar{w}} + B_T a \Sigma_{\theta \bar{M}} - \Sigma_{\theta T a} \]  
(47)
\[ \lambda_5 = A_{TB} \Sigma \theta_\Omega + B_{TB} \Sigma \theta_\Omega + \Sigma \theta_{TB} \]  

(48)

\[ \Sigma \theta_\Omega = e^{m_2(R-a)} R^{-1} e^{\frac{x}{2} \frac{1}{x_2}} [(1-v^2) W_{i,j}(x) - \nu W_{i+1,j}(x)] \]  

(49)

\[ \Sigma \theta_M = e^{m_2(R-a)} R^{-1} e^{\frac{x}{2} \frac{1}{x_2}} [(1-v^2) M_{i,j}(x) + (\nu + v^2) M_{i+1,j}(x)] \]  

(50)

\[ \Sigma \theta_F = e^{m_2(R-a)} R^{-1} \left\{ (1-v^2) \Phi(R) - e^{\frac{x}{2} \frac{1}{x_2}} W_{i+1,j}(x) \int F(R) M_{i,j}(x) dR + (1 + \nu) M_{i+1,j}(x) \right\} \]  

(51)

\[ \Sigma \theta_T_a = e^{m_2(R-a)} R^{-1} \left\{ (1-v^2) \Phi(R) - e^{\frac{x}{2} \frac{1}{x_2}} W_{i+1,j}(x) \int \Phi(R) M_{i,j}(x) dR + (1 + \nu) M_{i+1,j}(x) \right\} + (1 + v) \alpha_m e^{(m_4+m_2)(R-a)} \frac{\Delta(bR)-\Delta(b)}{\Delta(a)-\Delta(b)} \]  

(52)

\[ \Sigma \theta_T_b = e^{m_2(R-a)} R^{-1} \left\{ (1-v^2) \Phi(R) - e^{\frac{x}{2} \frac{1}{x_2}} W_{i+1,j}(x) \int \Phi(R) M_{i,j}(x) dR + (1 + \nu) M_{i+1,j}(x) \right\} + (1 + v) \alpha_m e^{(m_4+m_2)(R-a)} \frac{\Delta(bR)-\Delta(a)}{\Delta(a)-\Delta(b)} \]  

(53)

Illustrative Example

In the computations, the inner and outer radius of the considered functionally graded material hollow disk are \( a = 0.2 \) m and \( b = 1 \) m, respectively. The material thickness and properties at inner surface of FGM are \( h_m = 0.01 \) m, \( m_1 = 0.3 \), \( E_m = 70 \) GPa, \( m_2 = 0.2 \), \( \rho_m = 2,700 \) kg/m\(^3\), \( m_3 = 0.4 \), \( \alpha_m = 23 \times 10^{-6} \) /K, \( m_4 = 0.2 \), \( k_m = 209 \) W/m°C, \( m_5 = 0.1 \). Poisson’s ratio is \( \nu = 0.3 \). Angular velocity of rotating disk is \( \omega = 80 \) rad/sec, inner and outer surface of disk are subjected to \( P_i = 50 \) MPa, \( T_a = 100 \) K, \( P_o = 10 \) MPa and \( T_b = 5 \) K respectively.

Fig.1-4 show that radial displacement and stresses of disk subjected to \( P_i \), \( P_o \), \( T_a \) and \( T_b \), respectively. In Fig.1, disk subjected to internal pressure \( P_i \). The radial stress is compressive and the circumferential stress is tensile. Their maximums occurring at inner surface. In Fig.2, disk subjected to external pressure \( P_o \), the radial and circumferential stresses are all compressive. The maximal radial stress occurs at outer surface and maximal circumferential stress occurs at inner surface. In Fig.3, disk subjected to internal temperature \( T_a \), the compressive radial stress with its maximum occurring at \( R \approx 0.38 \). The circumferential stress has maximal compressive stress at inner surface and maximal tensile stress at outer surface. In Fig.4, disk subjected to external temperature \( T_b \), the tensile radial stress with its maximum occurring at \( R \approx 0.37 \). The circumferential stress has maximal tensile stress at inner surface and maximal compressive stress at outer surface. Fig.5 compares the rotating FGM and homogeneous isotropic annular disks subjected to \( P_i \), \( P_o \), \( T_a \) and \( T_b \) altogether. We can observe that FGM disk has lower radial displacement and stresses than homogeneous isotropic disk.

![Figure 1. FGM subjected to \( P_i \) only.](image1.png)

![Figure 2. FGM subjected to \( P_o \) only.](image2.png)
Conclusion

This study is bound up with the analysis of inhomogeneous rotating FGM hollow disk which is subjected to pressure and temperature difference at inner and outer surface, there is no heat generation in the structure. Material properties and thickness of the rotating disks are assumed to vary radially in exponentially-varying functions. Closed form solutions for radial displacement and stresses of the disk have been obtained. Separate and combined effects of each loading are shown in diagrams. As shown in illustrative examples, the maximal displacement occurs at inner surface. The radial stress is not a monotonic increasing function of radius, its maximum occurs inside the hollow disk. The circumferential stress is monotonic decreasing with its maximum occurring at inner surface.

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