Fuzzy project planning and scheduling using critical path technique

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Abstract. In this article, we intend to determine the total duration of completion of the project for fuzzy networking problem with the activity duration represented as trapezoidal fuzzy number using Critical Path technique. A new fuzzy arithmetic operation, ranking method and a precise step by step plan is designed to obtain the total fuzzy duration, utilizing the available resources. Finally the proposed study is discussed with a numerical example.

1. Introduction
Network Analysis or Network planning and scheduling techniques used for planning, scheduling and controlling large and complex projects. These techniques are based on the representation of the projects as a network of activities. A project involves a large number of interrelated activities that must be completed on or before a specified time limit, in a specified sequence, with specified quantity and minimum cost of using resources such as money, material, facilities and space. Network representations are widely used for problems in areas such as production, distribution, project planning, resource management and financial planning.

A Network is a graphical representation of arrows and nodes provides a powerful visual and conceptual aid for portraying the relationships between the components of systems and shows the logical sequence of various activities to be performed to achieve project objectives. Planning is a phase which involves listing of tasks or jobs that must be performed to complete a project under consideration. Scheduling is a phase which involves the laying out of the actual activities of the project in a logical sequence of time in which they have to be performed. Control is a phase consists of reviewing the progress of the project whether the actual performance is according to the planned schedule, and finding the reason for difference, if any, between the schedule and performance. Total project time is the time taken to complete a project and found from the sequence of critical activities. The path connecting the first initial node to the very last terminal node of the longest duration in a project network is called critical path.

Critical path method provides the solution for complicated project network when the decision parameters are crisp. But in reality, due to uncertainty of information as well as the variation of management scenario, it is difficult for the decision maker to obtain the exact time duration for the project activities. Hence we cannot use the traditional classical method for solving the project networking problems. Zadeh [10] introduced the concept of fuzzy set in 1965, which plays a vital
role in handling such situations effectively. Buckley [1] has applied trapezoidal fuzzy number in linear programming. The basic concepts of fuzzy sets and its application are discussed by Dubois et.al [2]. Kikuchi [3] deals the method of defuzzification of fuzzy numbers. Stefan Chanas et.al [8] found critical path in the network where the duration of each activity are fuzzy numbers. Revathi et.al [5] used octagonal fuzzy number to find fuzzy critical path in fuzzy project networking problem. Rajendran et.al [4] used hexagonal fuzzy number to find fuzzy critical path in fuzzy project networking problem. Shin-Pin-Chen et.al, [7] proposed fuzzy LPP formulation to find the fuzzy critical path. Stephan Dinagar et.al,[9] obtained fuzzy critical path for the problem with duration of each activities represented by fuzzy number. Selvakumari et.al, [6] solved the fuzzy networking problem with hexagonal fuzzy number using job sequencing technique.

In this paper, we investigate a more realistic problem, namely project scheduling problem with fuzzy duration represented as trapezoidal fuzzy number. The paper is arranged as follows: In section 2, we review the basic definitions, ranking of trapezoidal fuzzy number and arithmetic operation of trapezoidal fuzzy number. In section 3, we proposed an algorithm to find the critical path and maximum duration to complete the project, where the duration of activities are trapezoidal fuzzy number. In section 4, a numerical problem is provided to find the efficiency of the proposed approach.

2. Preliminaries

In this section, we represent the idea about the fuzzy numbers and definitions which involved in the research work. Fuzzy set theory have been defined by Zadeh [10] and the fundamental concepts are given by Dubois and Prade [2] are applied to fuzzy environment.

**Definition 2.1**

A fuzzy set \( \tilde{A} \) in \( X \) is a set of ordered pair defined by, 

\[
\tilde{A}(x) = (x, \mu_{\tilde{A}}(x)) : x \in X, \mu_{\tilde{A}}(x) \in [0,1]
\]

where \( \mu_{\tilde{A}}(x) \) is a membership function.

**Definition 2.2**

A fuzzy set \( \tilde{A} \) defined on a set of real number \( R \) is said to be a fuzzy number, if its membership function, \( \tilde{A} : R \rightarrow [0,1] \) has the following characteristic:

1) \( \tilde{A} \) is convex (i.e.,)

\[
\lambda x_1 + (1-\lambda)x_2 \geq \min \{\tilde{A}(x_1), \tilde{A}(x_2)\}
\]

for all \( x_1, x_2 \in R \) and \( \lambda \in [0,1] \)

2) \( \tilde{A} \) is normal, i.e., there exist an \( x \in R \) such that \( \tilde{A}(x) = 1 \)

3) \( \tilde{A} \) is piecewise continuous.

**Definition 2.3**

A fuzzy number \( \tilde{A} \) is a trapezoidal fuzzy number denoted by \( \tilde{A} = (a_1, a_2, a_3, a_4) \) where \( a_1, a_2, a_3, a_4 \) are real numbers and its membership function \( \tilde{A}(x) \) is given below,

\[
\tilde{A}(x) =
\begin{cases}
\frac{x-a_1}{a_2-a_1} , & \text{if } a_1 \leq x \leq a_2 \\
1 , & \text{if } a_2 \leq x \leq a_3 \\
\frac{a_4-x}{a_4-a_3} , & \text{if } a_3 \leq x \leq a_4 \\
0 , & \text{otherwise}
\end{cases}
\]
In this paper, the trapezoidal fuzzy number is represented as, \( \tilde{A} = (a_1, a_2, a_3, a_4) = (m, w, a^*, a*) \)
where \( m = \frac{(a_1 + a_3)}{2} \) and \( w = \frac{(a_3 - a_2)}{2} \) are the midpoint and width of the core \([a_2, a_3]\) respectively. Also \( a^* = (a_2 - a_1) \) and \( a^* = (a_4 - a_3) \) represents the left and right spreads respectively of the trapezoidal fuzzy number \( \tilde{A} = (a_1, a_2, a_3, a_4) \).

### 2.1 Ranking of trapezoidal fuzzy number

The ranking function \( \Re: F(R) \rightarrow R \) is defined by its graded mean as \( R(\tilde{A}) = \left[ \frac{(a_2 + a_3)}{2} + \frac{(a_4 - a_1)}{4} \right] \), where \( T(R) \) is the set of all trapezoidal fuzzy numbers defined on \( R \). A trapezoidal fuzzy number \( \tilde{A} = (a_1, a_2, a_3, a_4) \) in \( T(R) \) is said to be positive, if and only if \( \Re(\tilde{A}) > 0 \) and is denoted by \( \tilde{A} > \emptyset \). For any two trapezoidal fuzzy number \( \tilde{A} = (a_1, a_2, a_3, a_4) \) and \( \tilde{B} = (b_1, b_2, b_3, b_4) \) in \( T(R) \), we have the following comparison:

(i) \( \tilde{A} \geq \tilde{B} \) if and only if \( \Re(\tilde{A}) \geq \Re(\tilde{B}) \)

(ii) \( \tilde{A} \leq \tilde{B} \) if and only if \( \Re(\tilde{A}) \leq \Re(\tilde{B}) \)

(iii) \( \tilde{A} \approx \tilde{B} \) if and only if \( \Re(\tilde{A}) = \Re(\tilde{B}) \)

### 2.2 Arithmetic operations on trapezoidal fuzzy number

For arbitrary trapezoidal fuzzy numbers \( \tilde{A} = (m(\tilde{a}), w(\tilde{a})), a^*, a^* \) and \( \tilde{B} = (m(\tilde{b}), w(\tilde{b})), a^*, a^* \) and \(* = \{+,-,\cdot, \div\}\), the arithmetic operations on the trapezoidal fuzzy numbers are defined by,

\[
\tilde{A} \circ \tilde{B} = \left( m(\tilde{a}) \circ m(\tilde{b}), \text{max} \{w(\tilde{a}), w(\tilde{b})\}, \text{max} \{a^*, b^*\}, \text{max} \{a^*, b^*\} \right)
\]

In particular, for any two trapezoidal fuzzy numbers, \( \tilde{A} = (m(\tilde{a}), w(\tilde{a})), a^*, a^* \) and \( \tilde{B} = (m(\tilde{b}), w(\tilde{b})), b^*, b^* \) we define,
Addition: 
\[ \tilde{A} + \tilde{B} = \left( m(\tilde{a}) + m(\tilde{b}), \max \{ w(\tilde{a}), w(\tilde{b}) \}, \max \{ a_*, b_* \}, \max \{ a^*, b^* \} \right) \]

Subtraction: 
\[ \tilde{A} - \tilde{B} = \left( m(\tilde{a}) - m(\tilde{b}), \max \{ w(\tilde{a}), w(\tilde{b}) \}, \max \{ a_*, b_* \}, \max \{ a^*, b^* \} \right) \]

Multiplication: 
\[ \tilde{A} \times \tilde{B} = \left( m(\tilde{a}) \times m(\tilde{b}), \max \{ w(\tilde{a}), w(\tilde{b}) \}, \max \{ a_*, b_* \}, \max \{ a^*, b^* \} \right) \]

Division: 
\[ \tilde{A} \div \tilde{B} = \left( m(\tilde{a}) \div m(\tilde{b}), \max \{ w(\tilde{a}), w(\tilde{b}) \}, \max \{ a_*, b_* \}, \max \{ a^*, b^* \} \right) \]

3. Network Scheduling and Planning

Any individual's operation, which utilizes resources and has an end and a beginning, is called activity. An event represents the start and the end point of an activity. It is shown by a circle called nodes.

3.1 Fuzzy Critical Path Analysis

The objective of fuzzy critical path analysis is to estimate the total project duration and to assign starting and finishing time of all activities involved in the project. This helps to check the actual progress against the scheduled duration of the project.

The following factors in fuzzy should be known in order to prepare the project scheduling:

1. Total completion time of the project.
2. Earlier and latest start time of each activity.
3. Critical activities and critical path.
4. Float for each activity (i.e), the amount of time by which the completion of a non-critical activity can be delayed, without delaying the total project completion time.

3.2. Notations to represent fuzzy factors

\[ FES_{ij} = \text{Fuzzy Earliest start time of an activity (i, j)} \]
\[ FLS_{ij} = \text{Fuzzy Latest start time of an activity (i, j)} \]
\[ FEF_{ij} = \text{Fuzzy Earliest finish time of an activity (i, j)} \]
\[ FLF_{ij} = \text{Fuzzy Latest finish time of an activity (i, j)} \]
\[ Ft_{ij} = \text{Fuzzy Duration of an activity (i, j)} \]

3.3. Fuzzy Critical Path Algorithm

Step 1: Construct the fuzzy project network with predecessor and successor events.
Step 2: Calculate fuzzy earliest start time, \( FES_{ij} = \max \left[ FES_i + t_{ij} \right], \text{ i = number of preceding nodes} \).
Step 3: Calculate fuzzy earliest finish time, \( FEF_{ij} = \left[ FES_i + t_{ij} \right] \).
Step 4: Calculate fuzzy latest finish time, \( FLF_{ij} = \min \left[ FLF_j - t_{ij} \right], \text{ j = number of succeeding nodes} \).
Step 5: Calculate fuzzy latest start time, \( FLS_{ij} = \left[ FLF_j - t_{ij} \right] \).
Step 6: Calculate fuzzy total floating time (i.e), 
\[ F_T F_{ij} = F_{L} F_{ij} - F_{E} F_{ij} \] or 
\[ F_T F_{ij} = F_{L} S_{ij} - F_{E} S_{ij} \]
Step 7: If \( F_T F_{ij} = 0 \) then those activities are called as fuzzy critical activities.
Step 8: Connect all the fuzzy critical activities from source node to destination node. The path connecting all the fuzzy critical activities is called as fuzzy critical path.
Step 9: The maximum duration to complete the fuzzy network project is obtained by adding all the fuzzy activity duration of the critical activities in the critical path.

4. Numerical example
The proposed technique is illustrated by the following example. Consider fuzzy networking problem were each activity duration is trapezoidal fuzzy number.

**Table 1:** Fuzzy network problem with fuzzy activity duration as trapezoidal fuzzy number

| Activity | 1-2 | 1-3 | 2-4 | 2-5 | 3-4 |
|----------|-----|-----|-----|-----|-----|
| Fuzzy duration | (25,28,32,35) | (40,55,65,70) | (32,37,43,48) | (35,38,42,45) | (20,25,35,40) |

| Activity | 3-6 | 4-7 | 5-7 | 6-7 |
|----------|-----|-----|-----|-----|
| Fuzzy duration | (42,45,49,60) | (60,65,75,85) | (65,75,85,90) | (15,18,22,26) |

**Table 2:** Trapezoidal fuzzy activity duration of each activity is represented as \( \left( \left[ m(a), w(a) \right], a^*, a^* \right) \)

| Activity | 1-2 | 1-3 | 2-4 | 2-5 | 3-4 |
|----------|-----|-----|-----|-----|-----|
| Fuzzy duration | <30,2,3,3> | <60,5,15,5> | <40,3,5,5> | <40,2,3,3> | <30,5,5,5> |
| Activity | 3-6 | 4-7 | 5-7 | 6-7 |
| Fuzzy duration | <47,2,3,11> | <70,5,5,10> | <80,5,10,5> | <20,2,3,4> |
Table 3: Total float for each activity in a Trapezoidal Fuzzy Project Network and Critical Path

| Activity | Fuzzy duration | Fuzzy Earliest Start | Fuzzy Earliest finish | Fuzzy Latest Start | Fuzzy Latest Finish | Fuzzy Total Float |
|----------|----------------|----------------------|-----------------------|--------------------|---------------------|------------------|
| (1,2)    | <30,2,3,3>     | <0,0,0,0>            | <30,2,3,3>            | <10,5,15,10>      | <40,5,15,10>        | <10,5,15,10>     |
| (1,3)    | <60,5,15,5>    | <0,0,0,0>            | <60,5,15,5>           | <0,5,15,10>       | <60,5,15,10>        | <0,5,15,10>      |
| (2,4)    | <40,3,5,5>     | <30,2,3,3>            | <70,3,5,5>           | <50,5,15,10>      | <90,5,15,10>        | <20,5,15,10>     |
| (2,5)    | <40,2,3,3>     | <30,2,3,3>            | <70,2,3,3>           | <40,5,15,10>      | <80,5,15,10>        | <10,5,15,10>     |
| (3,4)    | <30,5,5,5>     | <60,5,15,5>           | <90,5,15,5>          | <60,5,5,10>       | <90,5,15,10>        | <0,5,15,10>      |
| (3,6)    | <47,2,3,11>    | <60,5,15,5>           | <107,5,15,11>        | <93,5,15,11>      | <140,5,15,10>       | <33,5,5,11>      |
| (4,7)    | <70,5,5,10>    | <90,5,15,5>           | <160,5,15,10>        | <90,5,15,10>      | <160,5,15,10>       | <0,5,15,10>      |
| (5,7)    | <80,5,10,5>    | <70,2,3,3>            | <150,5,10,5>         | <80,5,15,10>      | <160,5,15,10>       | <10,5,15,10>     |
| (6,7)    | <20,2,3,4>     | <107,5,15,1>          | <127,5,15,11>        | <140,5,15,10>     | <160,5,15,10>       | <33,5,15,11>     |
4.1. Result

According to the fuzzy total float, the fuzzy critical activities are 1-3, 3-4 and 4-7.

Therefore, connecting all the fuzzy critical activities we get the critical path for fuzzy project network problem as 1→3→4→7.

Thus, the total duration of completion of the fuzzy networking problem is (160,5, 15,10).

Hence the optimal fuzzy project duration in terms of trapezoidal fuzzy number $(a_1, a_2, a_3, a_4) = (140,155,165,175)$.

Our results are better comparing with the project duration obtained by Revathi and Saravanan [6], where they have converted the fuzzy duration of trapezoidal fuzzy number into crisp number.

5. Conclusion

In this paper, we proposed a new critical path algorithm, to obtain an optimal duration of completion of a fuzzy networking problem with trapezoidal fuzzy number, without converting the problem as a classical networking problem. The numerical illustration clearly shows that, the proposed methodology is more efficient than the existing method and provides a better solution to the fuzzy networking problem.

6. References

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