On Super-Liouville Operator Constructions

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Abstract

We review the construction of field operators of the N=1 supersymmetric Liouville theory in terms of the components of a quantized free superfield.
1 Classical N=1 super-Liouville theory

In the following we give a short review on the main features of the N = 1 supersymmetric extension of the previously discussed bosonic Liouville operator constructions [1]. The derivation of the Liouville action from the Super-Weyl anomaly of a superstring in d dimensions [13] in "superconformal gauge" of the super-zweibein fields goes quite parallel to the bosonic model. Similar to that case the prefactor $\gamma^{-2}$ of the Liouville action is shifted from its generic value $(10-d)/8\pi$ to $(9-d)/8\pi$ by taking into account effects from the nontrivial functional measure of the super-Weyl (or Liouville) field itself [14, 9, 6]. Here $d$ denotes the embedding dimension of the coupled $N = 1$ superstring theory or $d = \frac{2}{3}c$ where $c$ is the Virasoro central charge of a coupled superconformal matter theory. The effective action for the Liouville superfield $L$ induced this way is (up to a numerical factor)

$$S_{L,N=1} = \gamma^{-2} \int d^2 Z_+ d^2 Z_- (D_+ L D_- L - i\mu e^L) \quad (\gamma^2 = \frac{8\pi}{9-d})$$  \hfill (1)

The appearance of a "super"-cosmological term $\propto \mu$ is not as uncontroversially motivated as in the bosonic case, as there is no divergence of that type in the anomaly and therefore no unescapable need to consider $\mu \neq 0$ [10]. We are going to insist on $\mu \neq 0$ as there is no real argument for not having it, either, and to stay as close as possible at the bosonic case.

We use the following conventions ($\theta_a$ are odd Grassmann variables)

$$\xi_a = \frac{i}{2}(\tau + a\sigma) \quad (a = +, -)$$

$$Z_a = (\xi_a, \theta_a)$$

$$d^2 Z_a = d\xi_a d\theta_a$$

$$D_a = \frac{\partial}{\partial \theta_a} + \theta_a \frac{\partial}{\partial \xi_a} \equiv \partial_{\theta_a} + \theta_a \partial_{\xi_a} \quad (D_a^2 \equiv \partial_a)$$  \hfill (2)

From the action (1) the super-Liouville equation emerges:

$$D_+ D_- L(Z_+, Z_-) + i\mu e^{L(Z_+, Z_-)} = 0$$  \hfill (3)

Using the component representation for the superfield $L$

$$L(Z_+, Z_-) = \phi(\xi_+, \xi_-) + \theta_+ \lambda_+(\xi_+, \xi_-) + \theta_- \lambda_-(\xi_+, \xi_-) + \theta_+ \theta_- a(\xi_+, \xi_-)$$  \hfill (4)

and eliminating the auxiliary field $a(\xi_+, \xi_-)$ the super-Liouville equation (3) is equivalent to the following coupled system for the bosonic ($\phi$) and fermionic ($\lambda_\pm$) components.

$$\partial^2 \phi + \mu^2 e^{2\phi} = -i\mu \lambda_+ \lambda_- e^\phi \quad (\partial^2 \equiv -\partial_+ \partial_- \equiv \partial^2_+ - \partial^2_-)$$

$$\partial_\pm \lambda_\mp = \pm i\mu \lambda_\pm e^\phi$$  \hfill (5)

As we need the most general solution for quantization, we follow the strategy that proved to be viable in the purely bosonic case ($\lambda_\pm \equiv 0$) already: Choose a simple solution
\( \phi_0(\xi_+, \xi_-) \), transform the variables analytically (i.e. \( \xi_+ \to A_+(\xi_+) \) and take into account the (inhomogeneous) transformation behaviour of the Liouville field under this map. Here we have to generalize to superanalytic maps both in the "+" and the "-" sectors [11]:

\[
Z = (\xi, \theta) \to \tilde{Z} = (\tilde{\xi}(\xi, \theta), \tilde{\theta}(\xi, \theta))
\]  

(6)

with the constraints (sometimes called super Cauchy-Riemann eqs.)

\[
D_{\pm} \tilde{\xi}_{\pm} = \tilde{\theta}_{\pm} D_\mp
\]  

(7)

corresponding to a homogeneous behaviour of the supercovariant derivatives and differentials (see [2]).

\[
D_{\pm} = (D_\pm \tilde{\theta}_{\pm}) D_\mp
\]

\[
d^2 Z_{\pm} = (D\tilde{\theta}_{\pm})^{-1} d^2 \tilde{Z}_{\pm}
\]  

(8)

The transformation law for \( L \) ensuring the invariance of (1,3) under (6) is given by

\[
e^{L(Z_+, Z_-)} = (D_+ \tilde{\theta}_+)(D_- \tilde{\theta}_-) e^{L(\tilde{Z}_+, \tilde{Z}_-)}.
\]  

(9)

This characterizes (\( \exp L \)) as a superconformal primary field with weights \((\Delta_+, \Delta_-) = (1/2, 1/2)\). A simple solution of the super-Liouville equation easily is written down in terms of a "distance" in superspace defined generally as

\[
Z - Z' = \xi - \xi' - \theta \theta'.
\]  

(10)

One easily verifies, that one solution of (3) \( L_0(Z_+, Z_-) \) is given by

\[
e^{-L_0(Z_+, Z_-)} = i\mu (Z_+ - Z_-).
\]  

(11)

Two independent superanalytic maps

\[
Z_{\pm} = (\xi_{\pm}, \theta_{\pm}) \to \tilde{Z}_{\pm} = (A_{\pm}(Z_{\pm}), \alpha_{\pm}(Z_{\pm}))
\]  

(12)

provide a parametrization of the general solution of (3), where the superfields called \( \alpha \) are Grassmann odd. Taking eq.(9) into account, one finds [3, 2] for \( L(Z_+, Z_-) = \tilde{L}_0(\tilde{Z}_+, \tilde{Z}_-) \)

\[
e^{-L(Z_+, Z_-)} = i\mu \frac{A_+(Z_+) - A_-(Z_-) - \alpha_+(Z_+) \alpha_-(Z_-)}{D_+ \alpha_+(Z_+) D_- \alpha_-(Z_-)}
\]  

(13)

with the superanalyticity constraints [7], i. e.

\[
D_{\pm} A_{\pm} = \alpha_{\pm} D_{\pm} \alpha_{\pm}
\]  

(14)

We find it technically more appropriate to bring (13) by some special "super- Möbius" transformation in the "-"-sector (essentially \( A_- \to -1/A_- \)) together and some minor rescalings to the form

\[
e^{-L(Z_+, Z_-)} = \frac{1 - \mu^2 A_+(Z_+) A_-(Z_-) - i\mu \alpha_+(Z_+) \alpha_-(Z_-)}{D_+ \alpha_+(Z_+) D_- \alpha_-(Z_-)}
\]  

(15)
This is the complete analog of the "scalar" parametrization used for the bosonic model \[15, 5, 7\]. Again following this analogy, the unavoidable choice for a free superfield in terms of which the Liouville field has to be expressed is

\[
e^{F_{\pm}(Z_{\pm})} = D_{\pm}\alpha_{\pm} \\
D_{\pm}D_{-}(F_{+} + F_{-}) = 0
\] (16)

For later use we introduce components

\[
F_{\pm}(Z_{\pm}) = \psi_{\pm}(\xi_{\pm}) + \theta_{\pm}\chi(\xi_{\pm})
\] (17)
2 Super Liouville Fieldoperators

Up to now we have written down only classical relations. Going over to operators defined in the Fock space of the modes of the free superfield \( F^+ + F^- = F \) one tries to find well defined operators of the type

\[
(e^{-L})^{\text{op}} = : e^{-F} : [1 - \mu^2 A_+ A_- - i\mu \alpha_+ \alpha_-]^{\text{op}}
\]

and generally

\[
(e^{\nu L})^{\text{op}} = : e^{\nu F} : [1 - \mu^2 A_+ A_- - i\mu \alpha_+ \alpha_-]^{-\nu}^{\text{op}}
\]

The normal product refers to the free theory ground state.

Similiar to the purely bosonic case we now establish as a construction principle that the vertex operators (19) are superconformal primaries. That means, that the corresponding weights

\[
\Delta_+(\nu) = \Delta_-(\nu) = \frac{\nu}{2} \left( 1 - \nu \frac{\gamma^2}{4\pi} \right)
\]

derive from the free theory vertex operator : \( e^{\nu F} : \). The free field theory for \( F \) couples to a background charge, leading to an ”improvement” term in the energy-momentum supercurrent and resulting in the term linear in \( \nu \) in eq. (20) [16]. All the operators in brackets [... in (18,19)] must be \((\Delta_+ = \Delta_- = 0)\)-operators. It is the construction of these ”screening operators” in terms of the free field \( F \) we are focusing on in the rest of this talk.

Let us first express the classical superfields \( A_+(Z_+), \alpha_+(Z_+) \) in terms of \( F_+(Z_+) \). (The corresponding operators in the ”-” -sector can be treated in completely the same way. We are using the well known trick [4] of using independent sets of zero modes \((Q_+, P_+)\) and \((Q_-, P_-)\) for the ”+” and the ”-” sector after replacing \( Q_{\pm} \) by \( 2Q_{\pm} \).

Then at the level of classical fields eq. (16) leads to

\[
\alpha_+ = (D_+)^{-1}(e^{F_+}) = (\partial_+)^{-1}[D_+ e^{F_+}]
\]

We rewrite this in terms of components of \( F_+ \) (17) and for \( A_+, \alpha_+ \):

\[
A_+ = A_+^b(\xi_+) + \theta_+ A_+^f(\xi_+),
\]

\[
\alpha_+ = \alpha_+^b(\xi_+) + \theta_+ \alpha_+^b(\xi_+).
\]

(Here upper indices \( b, f \) stand for Grassman even and odd fields, respectively). We obtain

\[
\alpha_+^b(\xi_+) = e^{\psi_+}(\xi_+)
\]

\[
\alpha_+^f(\xi_+) = (\partial_+)^{-1}[e^{\psi_+} \chi_+](\xi_+)
\]

with

\[
(\partial_+)^{-1} = \text{const} + \int_0^{\xi_+} d\xi_+
\]

and the integration constant has to match the periodicity requirements both of the integrand and of the integral.
To obtain the bosonic superfield $A_+(Z_+)$ we have to integrate the superanalyticity constraints (14). This results in

$$A^f_+(\xi_+) = \alpha^f_+(\xi_+)\alpha^b_+(\xi_+)$$

In order to find the operator $(\alpha_+(\xi_+))^\text{op}$ transforming as a $(\Delta_+ = 0)$-superfield the components $\alpha^f_+, \alpha^b_+$ have to be conformal fields with weights $\Delta_+ = 0$ and $\Delta_+ = 1/2$, respectively.

We introduce therefore a dressing factor $\eta$ so that (see eqs.(1,20)

$$\alpha^b_+(\xi_+)^\text{op} = e^{i\psi_+(\xi_+)} :$$

The resulting solutions for $\eta$ are the conventional ones [2, 9] for $N = 1$ worldsheet supersymmetry :

$$\eta_{\pm} = \frac{1}{4}(9 - d \pm \sqrt{(9 - d)(1 - d)})$$

The operator expressions for

$$\alpha^f_+(\xi_+)^\text{op} = (\partial_+)^{-1}[(\alpha^b_+(\xi_+))^\text{op}(\alpha^b_+)\text{op}]$$

automatically come up with the correct conformal weights. Short distance singularities can be rendered integrable (under $(\partial_+)^{-1}$) by suitable analytic continuation in $g = \hbar \eta^2 \equiv \frac{\gamma^2 \eta^2}{4\pi} = \eta - 1$.

The only more serious problem is posed by the construction of $A^b_+(\xi_+)$ (27) as conformal primary field with $\Delta_+ = 0$ due to the appearance of $(\alpha^b_+(\xi_+))^2$ in the integrand. At operator level this should be translated into an expression being both well defined and of weight $\Delta_+ = 1$. Both [$(\alpha^b_+(\xi_+))^\text{op}$]² (see (29)) and $e^{2i\psi_+(\xi_+)} :$ (see (29)) are not acceptable. The brute force way of introducing another ”dressing parameter” $\eta'$ with weight $\Delta(2\eta') = 1$ for the tentative operator : $e^{2i\psi_+(\xi_+)} :$ would destroy supersymmetry and shift the region of direct applicability from $d \leq 1$ ($\eta$ real) to $d \leq -7$ ($\eta'$ real).

A way out is opened up by the fact, that the leading short distance singularities in the point-split version of the complete integrand of eq.(27) cancel, the remaining short-distance singularities can be made integrable by analytic continuation in $g \propto \eta^2$ (i.e. in $d$) as usual. As the argument in the component language is too lengthy here, we give a shorter but more implicit version in the conformal superfield formalism [4].

Leaving aside the $\pm$ -indices, the problem is to construct a well defined $\Delta = 1/2$ fermionic superfield operator (i.e. $\Delta = 1/2$ for the lowest (fermionic) and $\Delta = 1$ for the bosonic component) for the r.h.s. of

$$(DA(Z))^\text{op} = (\alpha(Z))\text{op}(D\alpha(Z))^\text{op}$$

($Z \equiv (z, \theta), \ D \equiv \partial_0 + \theta \partial_z$) (34)
on the basis of the bosonic superfield \((\Delta = 1/2)\) - operator

\[(D\alpha(Z))^{op} = e^{\eta F(Z)} : \]

and the fermionic \((\Delta = 0)\) - "screening"-operator

\[(\alpha(Z))^{op} = D^{-1} : e^{\eta F} := (\partial_z)^{-1}[D : e^{\eta F} :] (Z) \]

For the time being we use coordinates on the complex plane \((z = e^{\xi}, \tau \rightarrow -i\tau)\). Formally, the product of (33) and (36) fits the requirements for a \((\Delta = 1/2)\) - superfield but a closer look immediately reveals the appearance of the square of \((\alpha^{\mu}_{\nu})^{op} (29)\) being an undefined operator. Let us write the inverse of the supercovariant derivative as an integral operator (The integration constant is irrelevant here)

\[D^{-1} = \int z \, dz' \, \int d\theta (1 + \theta' \theta \partial_{z'}) \equiv \int^{(Z)} dZ' \]

Then we can write formally for the product of (33) and (36)

\[\int^{(Z)} dZ' : e^{\eta F(Z')} :: e^{\eta F(Z)} : \]

With the propagator (adapted to the field normalizations used here, continuation from the euclidean case with radial time ordering is understood)

\[G(Z, Z') = -\frac{\gamma^2}{4\pi} \ln(Z - Z') = -\frac{\gamma^2}{4\pi} [\ln(z - z') - \frac{\theta \theta'}{z - z'}] \]

we obtain

\[: e^{\eta F(Z')} :: e^{\eta F(Z)} := (Z - Z')^{-g} : e^{\eta(F(Z)+F(Z'))} : \quad (g \equiv \frac{(\gamma \eta)^2}{4\pi}) \]

The bilocal normal product at the r. h. s. can be super-Taylor- expanded around \(Z = Z'\) using

\[H(Z) = \exp \{ (Z' - Z) \partial_y + (\theta' - \theta) D_y \} H(Y)|_{Y = Z} \]

Plugging all this into (38), we see that all the terms with powers \((n \geq 0, \text{ integer})\) of the superdistance \((Z - Z')^{n-g}\) integrate to zero, while the surviving terms are of the form

\[\int z \, dz' (z - z')^{n-g} \times \text{(regular operator)} \]

and can be treated by the usual continuation in \(g\). The short distance factor corresponds to the well known power of some sinus- function of the difference in the "cylindrical" \(\sigma\)-variables \((\parallel \parallel \parallel \parallel)\) due to the map \(z = e^{\xi} \propto exp(\frac{\sigma}{2})\).

The explicit operator construction of the bosonic and fermionic Liouville operators in terms of their free field counterparts can now go ahead parallel to the bosonic construction. What makes things quantitatively even more complex, is the consideration of different periodicity conditions for the Neveu-Schwarz and Ramond sectors in both chiral sectors of the free theory, respectively. At the present (low) stage of applicability even of the much simpler bosonic Liouville operator formalism a motivation for working out all the details in the supersymmetric case, too, is not obvious.
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