A parametrization of the baryon octet and decuplet masses

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Abstract.

We construct a general parametrization of the baryon octet and decuplet mass operators including the three-body terms using the unit operator and the symmetry-breaking factors $M^d = \text{diag}(0,1,0)$ and $M^s = \text{diag}(0,0,1)$ in conjunction with the spin operators. Our parametrization has the minimal number of operators needed to describe all the octet and decuplet masses. Investigating the likely size of the three-body terms, we find that contributions of the three-body hypercharge splittings are comparable to those from the one- and two-body isospin splittings and that contributions of the three-body isospin splitting operators are very small. We prove that, in dynamical calculations, one must go to three loops to get the three-body terms. We also find that the suggested hierarchy of sizes for terms in the most general expression for baryon masses that involve multiple factors of $M^d$ and/or $M^s$ does not hold strictly for dynamical calculations in heavy baryon chiral perturbation theory: terms of a given order in a meson loop expansion may appear both with the expected factors of $M^d$ and $M^s$, and with one factor more.

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1. Introduction

The study of baryon masses is one of the fundamentally important problems in nuclear and particle physics and has been of interest for many years. Progress on the modern theory began with the introduction of flavor SU(3) symmetry and of the electromagnetic mass relation derived by Coleman and Glashow [1]. It was later found in the nonrelativistic quark model [2, 3, 4, 5] that, by including only two-body interactions among the quarks in the baryons, there exist nine mass formulas (a.k.a. sum rules) connecting the eighteen charge states of the baryon octet and decuplet. One of them is the remarkably accurate Coleman-Glashow relation. The baryon octet and decuplet masses and the sum rules have been also investigated in various versions of chiral perturbation theory [6, 7, 8, 9, 10, 11, 12]. As remarked elsewhere, the chiral expansion gives a complete parametrization of the static properties of the baryons when carried to high enough order, but no dynamical information unless the new couplings can be calculated in the underlying theory (see, for example, the analysis of the magnetic moment problem in [13]).

It was Morpurgo who first constructed a general parametrization of baryon masses for the case of hypercharge breaking. His general parametrization method [14], derived exactly from the QCD Lagrangian employing only few general properties‡, expresses the possible mass operators in terms of flavor-dependent terms proportional to powers of the strange-quark projection operator $P_s$ (denoted by $M_s$ in our work), and nonrelativistic-appearing products of Pauli spin operators $\sigma$. The results are completely general and relativistic. Morpurgo also considered the purely electromagnetic contributions to the baryon masses and showed that the known sum rules for mass splittings within isospin multiplets hold in the absence of three-body terms [15]. Dillon and Morpurgo later included the light quark mass terms in some of their work. For example, they briefly discussed the effects of $m_d - m_u$ on the Coleman-Glashow and other baryon mass formula in [16] where the first order of the light quark mass difference $m_d - m_u$ is considered.

As already mentioned, baryon masses have been studied extensively in chiral perturbation theory (ChPT). See, for example, [6, 7, 8, 9, 10, 11, 12] and the references in those papers. The possible mass operators are customarily expressed in terms of traces of products of baryon effective fields and their derivatives using a matrix representation of the fields, and mesonic corrections to the initial operators are calculated in perturbation theory. This effective field theory approach is based on a well-defined low-momentum approximation to QCD with a structure determined by the general properties of QCD in the limit of small quark masses and momenta [17], and gives a completely general description of the low-energy properties of the baryons.

In [18, 19, 20], we reconsidered the mass problem using the heavy-baryon version of chiral effective field theory (HBChPT) [21]. We used a “quark” representation in which the baryon effective fields are labeled using three flavor indices $i, j, k$ and spin

‡ The general properties are listed in the first footnote in [28].
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indices rather than the usual matrix representation §, and calculated the perturbative corrections in a meson-loop expansion. We found that this approach leads to a (relativistic) description of the possible mass operators in HBChPT in terms of simple flavor and spin operators which is identical to that developed by Morpurgo [14]. This equivalence is expected since HBChPT retains the low-momentum structure of QCD, and the effective-field results must be consistent with the general structure of operators and matrix elements in QCD, and give a complete description of those quantities at low energies where the effective field methods apply. HBChPT also provides a formalism in which one can make practical dynamical calculations perturbatively, an important consideration in our earlier work.

In the present paper, we use Morpurgo’s method [14] to construct a general parametrization of the baryon masses including three-body terms using the unit operator and the symmetry-breaking factors, \( M^d = \text{diag}(0, 1, 0) \) and \( M^s = \text{diag}(0, 0, 1) \), in conjunction with the spin operators. Our general expression includes all orders of the light quark mass difference \( m_d - m_u \) which are not explicitly shown in the general parametrization for baryon masses constructed by Dillon and Morpurgo. Using a Gram matrix analysis, we establish a minimal set of 18 independent operators to describe 18 baryon octet and decuplet mass states. The present work gives the overall setting without relying on HBChPT.

Our objective in our earlier studies of mass splittings (and moments) was to see the extent to which the observed splittings (other than explicit quark mass terms) could be explained in terms of dynamical interactions in HBChPT and the added electromagnetic interactions [20, 22], rather than just parametrized. Our approach was therefore mainly calculational, restricted to one- or two-loop corrections to the initial mass operators, and we did not include an explicit analysis of the properties behind the parametrization, or of the most general forms allowed. We found in that work that all the one- and two-body operators were generated in a meson loop expansion in HBChPT carried to two loops, but no three-body operators were generated. We will show here that one must go to three loops in HBChPT to get three-body terms. Our general demonstration of this and the examples of nontrivial three-body terms are also new.

We will also investigate the likely sizes of the coefficients of the various operators, particularly the three-body terms, in the most general expression for baryon masses. We find that powers of \( M^d \) or \( M^s \) do not correlate strictly with powers of the quark masses \( m_d - m_u \) or \( m_s - m_u \) and, as a result, the suggested hierarchy of sizes for terms in the most general expression for baryon masses that involve multiple factors of \( M^d \) and/or \( M^s \) does not hold strictly for dynamical calculations in HBChPT. Numerical calculations of the sum rules that give constraints on the three-body terms indicate

§ The connection of this representation to the usual effective-field methods was discussed in detail in [18, 19]. The results in each case can be summarized in terms of a set of effective interactions that have the appearance of interactions between quarks in the familiar semirelativistic or nonrelativistic quark models for the baryons [29, 30, 31], but the corresponding matrix elements can be translated back to expressions in terms of the relativistic effective baryon fields.
that contributions of the three-body hypercharge splittings are comparable to those from the one- and two-body isospin splittings. In addition, the calculations show that contributions of the three-body isospin splitting operators to baryon masses are very small. Ignoring terms associated with the three-body isospin splitting operators, we do phenomenological fits to the experimental data using our general expression. Our fits provide the values of the parameters that must be explained in the loop expansion in ChPT or any other dynamical model. The results of the fits also confirm that three-loop contributions will necessarily play a role in explaining masses at the scale of about 1 MeV.

Our parametrization has the minimal number of operators needed to describe all the octet and decuplet masses and can be translated back into HBChPT \cite{18,19}. Our general expression for baryon masses is particularly useful to an analysis of the baryon mass splittings due to simultaneous hypercharge-breaking and isospin-breaking effects.

The paper is organized as follows. In Sec. 2 we discuss how the general parametrization of the baryon octet and decuplet masses is constructed. A reduction of the general expression to the independent one- and two body operators at two-loop level and a general proof on the cancellation of the three-body terms are shown in Sec. 3. Then, we briefly discuss the hierarchy of sizes of the coefficients of the various operators in the general expression for baryon masses, investigate the likely size of the three-body terms, and do the phenomenological fits to the experimental data in Sec. 4. Concluding remarks are presented in Sec. 5.

2. General parametrization of the baryon octet and decuplet masses

2.1. Construction of the mass operators

First of all, we want to remind the reader that the operators that appear, while nonrelativistic in appearance, are all that are needed to describe the structure of matrix elements in exact QCD.

Before introducing our general parametrization of the baryon masses, we need to show how to construct the possible mass operators using the unit operator, $1$, and the symmetry-breaking factors, $M^d = \text{diag}(0,1,0)$ and $M^s = \text{diag}(0,0,1)$, in conjunction with the quark spin operators $\sigma$. Note that $M^u = \text{diag}(1,0,0) = 1 - M^d - M^s$, so symmetry breaking terms in $M^u$ can be transformed to $M^d$ and $M^s$. We are only interested in the mass operators that can have nonzero matrix elements in or between the octet and decuplet states.

If isospin breaking is ignored, employing the Morpurgo’s general parametrization method as discussed in \cite{14}, one can easily find that the following mass operators intervene in the expression of the baryon masses:

$$1, \sum_{i \neq j} \sigma_i \cdot \sigma_j,$$
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\[ \sum_i M_i^s \sum_{i \neq j} M_i^s \sigma_i \cdot \sigma_j , \sum_i M_i^s M_j^s , \sum_i M_i^s M_j^s M_k^s , (1) \]

\[ \sum_{i \neq j} M_i^s \sigma_j \cdot \sigma_k , \sum_{i \neq j} M_i^s M_j^s \sigma_j \cdot \sigma_k , \sum_{i \neq j} M_i^s M_j^s M_k^s , \sum_{i \neq j} M_i^s M_j^s M_k^s \sigma_j \cdot \sigma_k , \]

where \( i, j, k \in u, d, s \) are flavor indices, \( \mathbf{1} \) and \( \sum_{i \neq j} \sigma_i \cdot \sigma_j \) are the flavor-symmetric operators, the first line is the result for no flavor symmetry breaking, and the second and third lines introduce the hypercharge-breaking operators. Note that the operators carry the “quark” labels of the effective fields that indicate the flavor index on which they are to act. There are implied unit flavor matrices for the quarks whose labels do not appear explicitly. For example, \( \sum_i M_i^s \) has the complete flavor structure \( (M_i^s, 1)_{j,k} 1_{k,k} + 1_{i,j} M_j^s 1_{k,k} + 1_{i,j} M_j^s M_k^s) \). In Eq. (1), terms in the first and second lines are the one- and two-body operators and terms in the third line are the three-body operators in the sense that they act nontrivially on one, two, or three indices.

It is straightforward to generalize the result shown in Eq. (1) to find the possible isospin splitting operators - terms with one, two, or three factors of \( M^d \). That can be done by replacing in turn the matrix \( M^s \) by \( M^d \) at every place it appears in a flavor splitting operator. For example, the flavor splitting operators \( \sum_i M_i^s \), \( \sum_{i \neq j} M_i^s M_j^s \), and \( \sum_{i \neq j \neq k} M_i^s M_j^s \sigma_j \cdot \sigma_k \) generate the followings

\[ \sum_i M_i^d \to \sum_i M_i^d , \]

\[ \sum_{i \neq j} M_i^s M_j^s \to \sum_{i \neq j} M_i^d M_j^d , \sum_{i \neq j} M_i^d M_j^d , \sum_{i \neq j} M_i^d M_j^d , \sum_{i \neq j} M_i^d M_j^d , (2) \]

\[ \sum_{i \neq j \neq k} M_i^s M_j^s \sigma_j \cdot \sigma_k \to \sum_{i \neq j \neq k} M_i^d M_j^d \sigma_j \cdot \sigma_k , \sum_{i \neq j \neq k} M_i^d M_j^d \sigma_j \cdot \sigma_k , \sum_{i \neq j \neq k} M_i^d M_j^d \sigma_j \cdot \sigma_k . \]

Since the indices \( i, j, \) and \( k \) of the sums are dummy, one can easily find that some of the generated operators are identical. For example,

\[ \sum_{i \neq j} M_i^d M_j^d = \sum_{i \neq j} M_i^s M_j^s , \sum_{i \neq j} M_i^d M_j^d \sigma_i \cdot \sigma_j = \sum_{i \neq j} M_i^s M_j^s \sigma_i \cdot \sigma_j , (3) \]

and so on. Keeping in mind Eq. (3), we find the possible one- and two-body isospin splitting operators

\[ \sum_i M_i^d , \sum_{i \neq j} M_i^d \sigma_i \cdot \sigma_j , \sum_{i \neq j} M_i^d M_j^d , \sum_{i \neq j} M_i^d M_j^d , \sum_{i \neq j \neq k} M_i^d M_j^d \sigma_j \cdot \sigma_k , \sum_{i \neq j \neq k} M_i^d M_j^d M_k^d , (4) \]

and the possible three-body isospin splitting operators

\[ \sum_{i \neq j \neq k} M_i^d \sigma_j \cdot \sigma_k , \sum_{i \neq j \neq k} M_i^d M_j^d \sigma_i \cdot \sigma_k , \sum_{i \neq j \neq k} M_i^d M_j^d \sigma_i \cdot \sigma_k , \sum_{i \neq j \neq k} M_i^d M_j^d M_k^d \sigma_j \cdot \sigma_k , \sum_{i \neq j \neq k} M_i^d M_j^d M_k^d \sigma_j \cdot \sigma_k , \sum_{i \neq j \neq k} M_i^d M_j^d M_k^d \sigma_j \cdot \sigma_k , \]
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\[ \sum_{i \neq j \neq k} M_i^d M_j^d \sigma_j \cdot \sigma_k, \quad \sum_{i \neq j \neq k} M_i^d M_j^d M_k^s, \quad \sum_{i \neq j \neq k} M_i^d M_j^d M_k^s \sigma_i \cdot \sigma_j, \quad \sum_{i \neq j \neq k} M_i^d M_j^d M_k^s \sigma_i \cdot \sigma_k, \]
\[ \sum_{i \neq j \neq k} M_i^d M_j^d M_k^d, \quad \sum_{i \neq j \neq k} M_i^d M_j^d M_k^d \sigma_i \cdot \sigma_j. \] (5)

We note that some isospin splitting operators listed in Eqs. (4) and (5) (e.g., terms involving multiple factors of \( M^d \)) are not explicitly shown in the parametrization constructed by Dillon and Morpurgo. However, it is worth pointing out that, using the identities

\[ Q = \frac{2}{3} - M^d - M^s, \quad Q^2 = \frac{4}{9} - \frac{1}{3} M^d - \frac{1}{3} M^s \] (6)

for the charge operator \( Q = \text{diag}(2/3, -1/3, -1/3) \), and also the matrix relations \( M_i^d M_i^d = M_i^d M_i^s = 0, \ M_i^d M_i^d = M_i^d, \) and \( M_i^d M_i^s = M_i^s \), the electromagnetic operators introduced by Morpurgo in [15] can be reduced to sums of the flavor-symmetric operators, the hypercharge splitting operators shown in Eq. (1), and the isospin splitting operators (except those with three factors of \( M^d \)) listed in Eqs. (4) and (5), respectively.

2.2. A general expression of the baryon masses

Not all the splitting operators shown in Eqs. (4), (4), and (5) are independent. Using the identity \( M_{i,j}^p M_{i,j}^p = M_{i,j}^p M_{i,j}^p \), where \( p = u, d, s \), and the exchange operator \( P_{ij} = (1 + \sigma_i \cdot \sigma_j)/2 \) from [19] to rearrange indices, we can easily show that

\[ M_i^p M_j^p = M_i^p M_j^p \sigma_i \cdot \sigma_j. \] (7)

As a result, we obtain the following identities

\[ \sum_{i \neq j} M_i^p M_j^p = \sum_{i \neq j} M_i^p M_j^p \sigma_i \cdot \sigma_j, \]
\[ \sum_{i \neq j \neq k} M_i^p M_j^p M_k^d = \sum_{i \neq j \neq k} M_i^p M_j^p M_k^d \sigma_i \cdot \sigma_j, \] (8)

where \( p, q = u, d, s \). Taking into account these identities, we find that the numbers of operators in Eqs. (4), (4) and (4) are reduced to 8, 5, and 9, respectively. Hereafter, for the operators related by Eq. (8), we choose to work with those without spin operators.

We now have 22 operators (2 symmetric, 6 hypercharge splitting, 5 one- and two-body isospin splitting, and 9 three-body isospin splitting) to describe a total of 18 baryon octet and decuplet masses. Therefore, four operators must be redundant. To determine what operators are independent, we make the tables of their contributions to the baryon masses and then consider the corresponding Gram matrix. For the eight flavor-symmetric operators and the hypercharge splitting operators, their contributions to the baryon masses are given in Table I where the three-body hypercharge splitting operators are denoted as

\[ s_1 = \sum_{i \neq j \neq k} M_i^s \sigma_j \cdot \sigma_k, \quad s_2 = \sum_{i \neq j \neq k} M_i^s M_j^s \sigma_j \cdot \sigma_k, \quad s_3 = \sum_{i \neq j \neq k} M_i^s M_j^s M_k^s. \] (9)
We present in Table 2 the contributions of the thirteen isospin splitting operators where, for simplicity, the three-body isospin splitting operators are labelled as

\[ t_1 = \sum_{i \neq j \neq k} M_i^d M_j^s M_k^s, \quad t_2 = \sum_{i \neq j \neq k} M_i^d M_j^s M_k^s \sigma_i \cdot \sigma_j, \quad t_3 = \sum_{i \neq j \neq k} M_i^d M_j^s \sigma_j \cdot \sigma_k, \]

\[ t_4 = \sum_{i \neq j \neq k} M_i^d M_j^d M_k^s, \quad t_5 = \sum_{i \neq j \neq k} M_i^d M_j^d M_k^s \sigma_i \cdot \sigma_k, \quad t_6 = \sum_{i \neq j \neq k} M_i^d M_j^d M_k^d, \]

\[ t_7 = \sum_{i \neq j \neq k} M_i^d \sigma_j \cdot \sigma_k, \quad t_8 = \sum_{i \neq j \neq k} M_i^d M_j^s \sigma_i \cdot \sigma_k, \quad t_9 = \sum_{i \neq j \neq k} M_i^d M_j^s \sigma_j \cdot \sigma_k. \]

We then consider the 22 \times 22 Gram matrix \( \mathcal{M}_\Gamma = \mathcal{M}_T \mathcal{M} \) associated with the 18 \times 22 matrix \( \mathcal{M} \) defined by joining the 18 \times 8 matrix in Table 1 (extended to all states in each hypercharge multiplet) and the 18 \times 14 matrix in Table 2. \( \mathcal{M}_\Gamma \) has a vanishing determinant and four zero eigenvalues. This indicates that there are four relations among the 22 operators. Applying the conditions for linear dependence of the mass operators, \( \mathcal{M}_\Gamma \chi = 0 \) for the eigenvectors \( \chi \) with zero eigenvalues, we find the four following relations:

\[ 0 = -6 \mathbf{1} + \sum_{i \neq j} \sigma_i \cdot \sigma_j + 4 \sum_i M_i^s \sigma_i - 2 \sum_i M_i^s \sigma_i \cdot \sigma_j \]

\[ + 4 \sum_i M_i^d - 2 \sum_i M_i^d \sigma_i \cdot \sigma_j - 2 \sum_i M_i^d M_j^s + 2 \sum_i M_i^d M_j^s \sigma_i \cdot \sigma_j, \]  

\[ 0 = -6 \mathbf{1} + \sum_{i \neq j} \sigma_i \cdot \sigma_j + 6 \sum_i M_i^s \sigma_i - 2 \sum_i M_i^s \sigma_i \cdot \sigma_j - 2 \sum_i M_i^s M_j^s - s_1 + 2s_2 \]

\[ + 4 \sum_i M_i^d - 2 \sum_i M_i^d \sigma_i \cdot \sigma_j - 4 \sum_i M_i^d M_j^s + 2t_8 + 2t_9, \]  

\[ 0 = -6 \mathbf{1} + \sum_{i \neq j} \sigma_i \cdot \sigma_j + 6 \sum_i M_i^s \sigma_i - 2 \sum_i M_i^s \sigma_i \cdot \sigma_j - 2 \sum_i M_i^s M_j^s - s_1 + 2s_2 \]

\[ + 4 \sum_i M_i^d - 2 \sum_i M_i^d \sigma_i \cdot \sigma_j - 4 \sum_i M_i^d M_j^s + 2t_8 + 2t_9. \]
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Table 2. Contributions of the isospin splitting operators listed in Eqs. (1) and (8) to the baryon masses. Here, for simplicity, we denote $c_1 = \sum_i M_i^d$, $c_2 = \sum_{i \neq j} M_i^d \sigma_i \cdot \sigma_j$, $c_3 = \sum_{i \neq j} M_i^d M_j^s$, $c_4 = \sum_{i \neq j} M_i^d M_j^s \sigma_i \cdot \sigma_j$, and $c_5 = \sum_{i \neq j} M_i^d M_j^d$. Contributions of the dependent operators identified by Eq. (8) are not shown.

| Baryon | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ | $t_7$ | $t_8$ | $t_9$ |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $p$    | 1     | -4    | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 2     | 0     |
| $n$    | 2     | -2    | 0     | 0     | 2     | 0     | 0     | -4    | 0     | 0     | 0     | -8    | 0     | 0     |
| $\Lambda$ | 1   | -3    | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | -3    | 0     |
| $\Sigma^+$ | 0 | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $\Sigma^0$ | 1  | -1    | 1     | -2    | 0     | 0     | 0     | 0     | 0     | 0     | 0     | -4    | 1     | -2    |
| $\Sigma^-$ | 2  | -2    | 2     | -4    | 2     | 0     | 0     | -4    | 2     | -4    | 0     | -8    | 2     | -4    |
| $\Xi^0$ | 0 | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $\Xi^-$ | 1  | -4    | 2     | -4    | 0     | 2     | -4    | 0     | 0     | 0     | 2     | -4    | 2     | 0     |
| $\Delta^{++}$ | 0 | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $\Delta^+$ | 1 | 2     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 2     | 0     | 0     |
| $\Delta^0$ | 2 | 4     | 0     | 0     | 2     | 0     | 0     | 2     | 0     | 0     | 0     | 4     | 0     | 0     |
| $\Delta^-$ | 3 | 6     | 0     | 0     | 6     | 0     | 0     | 6     | 0     | 0     | 6     | 6     | 0     | 0     |
| $\Sigma^{++}$ | 0 | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $\Sigma^{*0}$ | 1 | 2     | 1     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 2     | 1     | 1     |
| $\Sigma^{*-}$ | 2 | 4     | 2     | 2     | 2     | 0     | 2     | 2     | 2     | 0     | 4     | 2     | 2     | 0     |
| $\Xi^{*0}$ | 0 | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $\Xi^{*-}$ | 1 | 2     | 2     | 2     | 2     | 0     | 2     | 2     | 2     | 0     | 0     | 0     | 2     | 2     |
| $\Omega^-$ | 0 | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |

\[
+2 \sum_i M_i^d - 2 \sum_{i \neq j} M_i^d \sigma_i \cdot \sigma_j - 2 \sum_{i \neq j} M_i^d M_j^s + 2 \sum_{i \neq j} M_i^d M_j^d + t_7 + 2t_8 - 2t_3 - 2t_4 + 2t_5,
\]

\[
0 = 2 \sum_i M_i^s + 2 \sum_{i \neq j} M_i^s M_j^s + s_1 - 2s_2 + 2 \sum_{i \neq j} M_i^d M_j^s - 2t_8 - 2t_1 + 2t_2.
\]

Using these relations, we can select 4 operators to be dependent. The chosen operators are $\sum_{i \neq j} M_i^d M_j^s \sigma_i \cdot \sigma_j$, $t_7$, $t_8$, and $t_9$. Note that the chosen two-body operator is related in Eq. (11) only to other one- and two-body operators and does not change the values of any three-body operators. On the other hand, it follows from the remaining equations that the matrix elements of $t_7$, $t_8$, and $t_9$ can be absorbed in the matrix elements of one- and two-body operators and the remaining three-body operators $s_1 - 2s_2$, $t_1$ to $t_9$. No one- or two-body operators are mixed into the three-body operators, so the expected smallness of the remaining three-body terms should be maintained.

We now have a following set of 18 independent mass operators

**Flavor-symmetric**

\[
1, \sum_{i \neq j} \sigma_i \cdot \sigma_j,
\]

**Hypercharge splitting**

**One and two-body:**

\[
\sum_i M_i^s, \sum_{i \neq j} M_i^s \sigma_i \cdot \sigma_j, \sum_{i \neq j} M_i^s M_j^s.
\]
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Three-body: $\sum_{i \neq j \neq k} M_i^s \sigma_j \cdot \sigma_k + \sum_{i \neq j \neq k} M_i^s M_j^s \sigma_j \cdot \sigma_k + \sum_{i \neq j \neq k} M_i^s M_j^s M_k^s$. \hfill (16)

Isospin splitting

One and two-body: $\sum_i M_i^d$, $\sum_{i \neq j} M_i^d \sigma_i \cdot \sigma_j$, $\sum_i M_i^d M_j^d$.

Three-body: $\sum_{i \neq j \neq k} M_i^d M_j^d \sigma_j \cdot \sigma_k$, $\sum_{i \neq j \neq k} M_i^d M_j^d M_k^d$. \hfill (17)

Mixed splitting

Two-body: $\sum_{i \neq j} M_i^d M_j^s$.

Three-body: $\sum_{i \neq j \neq k} M_i^d M_j^s M_k^s$, $\sum_{i \neq j \neq k} M_i^d M_j^s M_k^s \sigma_i \cdot \sigma_j$, $\sum_{i \neq j \neq k} M_i^d M_j^d M_k^s$, $\sum_{i \neq j \neq k} M_i^d M_j^d M_k^s \sigma_i \cdot \sigma_k$. \hfill (18)

Note that the mixed splitting operators affect both hypercharge and isospin splittings and involve at least one factor each of $M^s$ and $M^d$.

We can now construct a general expression for baryon masses. Using the flavor-symmetric operators and other independent mass operators given in Eqs. (15) - (18), we write the most general expression for baryon masses as

$$\mathcal{H}_B = m_0 \mathbf{1} + A \sum_{i \neq j} \sigma_i \cdot \sigma_j + \mathcal{H}_{\text{Inter}} + \mathcal{H}_{\text{Intra}},$$

where $\mathcal{H}_{\text{Inter}}$ and $\mathcal{H}_{\text{Intra}}$ are the general expressions for the intermultiplet splittings and intramultiplet splittings, respectively. Namely,

$$\mathcal{H}_{\text{Inter}} = B \sum_i M_i^s + C \sum_{i \neq j} M_i^s \sigma_i \cdot \sigma_j + D \sum_{i \neq j} M_i^s M_j^s + a s_1 + b s_2 + c s_3,$$ \hfill (20)

and

$$\mathcal{H}_{\text{Intra}} = A' \sum_i M_i^d + B' \sum_{i \neq j} M_i^d \sigma_i \cdot \sigma_j + C' \sum_{i \neq j} M_i^d M_j^s + D' \sum_{i \neq j} M_i^d M_j^d + \sum_{i=1}^6 d_i t_i. \hfill (21)$$

Here $m_0, A, B, C, D, A', B', C', D', a, b, c,$ and $d_i (i = 1, \ldots, 6)$ are the parameters.

3. A parametrization of baryon masses to two loops

3.1. Generation of the one- and two-body operators

In [19], we analyzed the structure of meson loop corrections to the $O(m_s)$ expressions for the baryon masses. Our approach was based on the standard chiral Lagrangian of the heavy-baryon chiral effective field theory [21]. However, we used a spin- and flavor-index or “quark” representation of the effective octet and decuplet baryon fields rather than the usual matrix expressions for the fields. There, to one loop for equal mass light quarks in an expansion in $m_s$, we found the appearance of all one- and two-body mass operators listed in Eq. (1), but no three-body operators.
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For the case of nonzero $m_d - m_u$, it is straightforward to show that one-loop corrections involving $M^d$ insertions and the effects of meson mass differences in loop integrals introduce all possible one- and two-body isospin splitting operators shown in Eq. (4). We find no appearance of the three-body isospin splitting operators at this level.

Recall that, in [20], our calculations of the electromagnetic contributions to baryon masses were carried out including the one-loop mesonic corrections to the basic electromagnetic interactions, so to two loops overall. To this order, the electromagnetic contributions also produce a complete set of the one- and two-body mass splitting operators, but again no three-body operators are generated. A general proof on the cancellation of the three-body terms up to two loops will be given in the next subsection.

As a result, a general parametrization of baryon masses to two loops is

$$H_B = m_0 1 + A \sum_{i \neq j} \sigma_i \cdot \sigma_j + B \sum_i M^s_i + C \sum_{i \neq j} M^s_i \sigma_i \cdot \sigma_j + D \sum_i M^s_i M^s_j + H'_{IB},$$

(22)

where $H'_{IB}$ is the isospin splitting mass Hamiltonian consisting of the one- and two-body operators only,

$$H'_{IB} = A' \sum_i M^d_i + B' \sum_{i \neq j} M^d_i \sigma_i \cdot \sigma_j + C' \sum_i M^d_i M^s_j + D' \sum_{i \neq j} M^d_i M^d_j.$$  

(23)

Again, $m_0$, $A$, $B$, $C$, $D$, $A'$, $B'$, $C'$, and $D'$ are the parameters.

Before moving on to the next subsection, we want to make clear that, in this subsection, we are not deriving the parametrization from the calculations, but are summarizing what happens in actual dynamical calculations. We emphasize that the form of the parametrization above follows rigorously from properties of QCD as shown by Morpurgo [14], and is encompassed also in the most general chiral Lagrangian in effective field theory. While we obtained the same one- and two-body operators in our earlier calculations in HBChPT [18, 19, 20] as summarized here, the form of the general parametrization, whether approached from QCD or effective field theory, is independent of the results of particular dynamical calculations. The situation is different in the next subsection, which deals specifically with the origin of three-body effects in HBChPT.

3.2. Cancellation of the three-body terms

We present here a general proof on the cancellation of the three-body terms through two loops in HBChPT. Our analysis is done using an expansion in meson (or photon) loops in time-ordered perturbation theory. The four possible types of time-ordered diagram that involve all three indices $i$, $j$, $k$ (“three-body operators”) and one- or two meson (or photon) loops are shown in Fig. 1. We are going to show that these diagrams cancel exactly with the renormalization diagrams with the same topology.

In Fig. 1, a solid vertical line represents a “quark” (a flavor index) moving upward in time from an initial state with flavor index $i$ to a final state with index $i'$ rather than a propagating QCD quark. These flavor indices $i, j, k$ and $i', j', k'$ are connected in the effective field theory to the corresponding indices on initial and final effective baryon
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11 fields $\psi^\lambda_{i,j,k}$, $\psi'^\lambda_{i',j',k'}$ as described in detail in [18, 19], and the spin matrix elements are calculated with respect to these fields. The baryon fields are suppressed in the present graphical notation, and only the connections of initial and final flavor indices and the structure of the spin-dependent operators are shown explicitly.

A bold solid line connecting flavor lines represents the exchange of a particle in the chiral effective theory such as a meson or photon between flavor (or “quark”) lines. This exchange corresponds in the baryon picture to a diagram with a meson or photon loop connecting the baryon to itself, with the vertex operators acting on different flavor and spin indices at different times in time-dependent perturbation theory. The counting of loops is done in this baryonic context. Note that the horizontal dot-dashed lines pick out the intermediate states. The vertex operators typically involve flavor operators such as Gell-Mann $\lambda$ matrices or charge or mass matrices, and spin operators expressed in terms of Pauli matrices.

Similarly, a squared dot represents an operator or mass insertion that acts only on one index, while a zigzag line represents a “two-particle” or two-index insertion with no propagator or time ordering.

Note that in Figs. (a), (c), and (d), only one of the possible time orderings is shown and in Fig. (b) both the possible orderings are included. There are two, sixteen, and again sixteen possible orderings for each type of diagram in (a), (c), and (d), respectively.

Firstly, let us consider Fig. (a) depicting a one-loop exchange diagram with a one-particle “mass” insertion. As will be illustrated below, the value of the diagram can be expressed as a product of energy denominators and a matrix element of vertex operators dependent on the flavor and spin indices. In the figure, the particle exchange connects the quark lines $j$ and $k$ and the “mass” insertion is on the quark line $i$ at a time between two vertices of the exchange line. The “mass” insertion could include a mass matrix, a spin, a charge matrix, a moment operator, and so on; the latter two appeared in our analysis of baryon moments. The matrix element involves all three indices $i, j, k$ so the diagram is three-body in Morpurgo’s classification. However, the three vertices are on separate lines so they are not constrained topologically. Moving the $j$ and $k$ vertices along those quark lines to above or below the $i$ vertex, which can be done freely, gives diagrams topologically equivalent to renormalization diagrams, with the same vertex factors. In addition, the energy denominator for the diagram corresponds to the one in the renormalization constant associated with the exchange. Multiplying the single particle mass insertion amplitude, which corresponds diagrammatically to Fig. (a) with the particle exchange deleted, by the renormalization constants for the initial and final states associated with the particle exchange on the other lines gives a term with the same energy denominator. This cancels the three-body contribution from the diagram under consideration.

As a demonstration, we now show the time-ordered perturbation theory calculations in HBChPT for the diagram in Fig. (a) when the exchange is a meson. This corresponds in the baryon picture to a one-loop diagram with the meson loop connecting
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Figure 1. Time-ordered one- and two-loop three-body diagrams which cancel exactly with renormalization diagrams. A solid vertical line represents a quark moving upwards toward later times. The horizontal dot-dashed lines pick out the intermediate states. A squared dot indicates a one-particle or one-index insertion. The zigzag line is a two-particle or two-index insertion with no propagator or time ordering. A bold solid line represents the exchange of a particle in the HBChPT Lagrangian between the baryon and itself (a loop diagram in the baryon picture). (a): One-loop exchange diagram with an one-particle “mass” insertion. (b): The exchange diagrams with a two-particle “mass” insertion where both time orderings of the particle exchange vertices are shown. (c): A double exchange diagram with the exchanges connecting the quark lines \( i \) and \( j \), and \( j \) and \( k \). (d): A double exchange diagram similar to (c), but with an one-particle insertion on one of the quark lines. Note that in (a), (c), and (d), only one of the possible time orderings is shown and in (b) both the possible orderings are included. There are two, sixteen, and sixteen possible orderings for each type of diagram in (a), (c), and (d), respectively.
the baryon to itself. An energy denominator $1/(E_0 - E_n)$ appears in the expression for the perturbed energy for each intermediate state $|n\rangle$ of the baryon system between successive vertices. In the heavy-baryon approximation, any internal baryon momenta $k$ resulting from baryon interactions are small on the scale of the baryon mass $M$ and can be neglected [21]. The baryon energy is then always $M$ for the initial baryon at rest, and cancels out in the difference $E_0 - E_n$. Hence, the energy factor reduces simply to $-1/ \sum_i E_i$, where the sum is over the energies of the lines cut in the intermediate state, but with no contribution from the quark lines. In this example, $E_i = E_i(k) = \sqrt{k^2 + M_i^2}$, where $M_i$ is the mass of the meson in the loop and $k$ is its momentum in the baryon rest frame. We find that a contribution from the diagram shown in Fig. 1(a) is

$$\frac{\beta^2}{4f^2} \sum_l \int \frac{d^3k}{(2\pi)^3} \frac{F^2(k^2)}{\sqrt{k^2 + M_l^2}} \delta_{\nu i} \left( \lambda^l_{i,j} \sigma_j \cdot k \sigma_k \cdot k' \right),$$

where $O_{\nu i}$ is the “mass” insertion vertex, $\lambda$ is a Gell-Mann matrix in flavor space, $F(k^2)$ is a form factor used to regularize the integral, and $I'_l$ is a modified integral defined as

$$I'_l = \frac{\beta^2}{16\pi^2 f^2} \int_0^\infty dk \frac{k^4}{(k^2 + M_l^2)^{3/2}} F^2(k^2).$$

A contribution from the renormalization diagram can be found by multiplying the “mass”-insertion amplitude $O_{\nu i}$ by the wave function renormalization constants $Z = 1 - \delta Z$ for the initial and final states associated with the meson exchange on line $j$ and $k$. The result is

$$-\frac{1}{2} \left( O_{\nu i} \delta Z_{j^l k' ; jk} + \delta Z_{j^l k' ; jk} O_{\nu i} \right) = -O_{\nu i} \delta Z_{j^l k' ; jk},$$

where

$$\delta Z_{j^l k' ; jk} = \frac{1}{3} \sum_l I'_l \lambda^l_{i,j} \sigma_k \cdot k' \left( \sigma_j \cdot \sigma_k \right).$$

It is obvious that contributions from the diagram in Fig. 1(a) and from its renormalization diagram cancel exactly as suggested by its topology.

Secondly, we turn to the case of the two-particle “mass” insertion. Again, we consider the exchange diagrams with a particle exchange connecting the quark lines $j$ and $k$, but with a two-particle “mass” insertion “connecting” quark lines $i$ and $j$ as shown in Fig. 1(b). Note that the two-particle insertion, denoted by $O_{\nu i,j^l,j'}$, is an instantaneous short-distance term (e.g., $O_{\nu i,j^l,j'} = (M^p \sigma \cdot \sigma)_{\nu i,j^l,j'}$, where $p = d, s$) but with one-loop structure, so we count it as equivalent to one loop. For the first (second) ordering in Fig. 1(b), since $O_{\nu i,j^l,j'}$ has no effect on the vertex on line $k$, and its vertex on line $j$ is above (below) the exchange vertex on that line, the exchange vertices can be moved freely down (up) on lines $j$ and $k$ to obtain a figure with the topology of a renormalization diagram. On the other hand, because the two-particle insertion has no
energy denominator and does not contribute to $\delta Z$, the energy denominators for both the orderings in Fig. 1(b) are identical to those for the renormalization diagrams with the same topology. As a result, when all contributions are added together, the three-body terms from the exchange diagrams and their renormalization diagrams cancel each other as expected. For example, if the exchange particles are the mesons, the diagrams in Fig. 1(b) give the following contribution

$$\frac{1}{3} \sum_l I_l' \left( O_{\ell,i,j',j''} l_{j',j''} \sigma_{j''} \cdot \sigma_{k} l_{k',k} + l_{j,j''} \cdot \sigma_{j''} \cdot (O_{\ell,i,j''} l_{j''} \sigma_{k} l_{k,k'}) \right), \quad (28)$$

that cancels exactly with the contribution from the renormalization diagrams

$$- \left( O_{\ell,i,j',j''} \delta Z_{j',j'';k,k} + \delta Z_{j',j'';k,k} O_{\ell,i,j''} \right), \quad (29)$$

where $\delta Z$ is given by Eq. (27).

Thirdly, in Fig. 1(c), we consider a double exchange diagram with the exchanges connecting, say, $ij$ and $jk$. The vertices on lines $i$ and $k$ can be freely slid along those lines, while the two vertices on $j$ can be slid in opposite directions. Moving the $ij$ exchange up leads to a figure topologically equivalent to that encountered for an $ij$ exchange multiplied by the renormalization constant for a $jk$ exchange, or conversely. Recall that there are sixteen possible time orderings for this type of diagram. When we combine the energy denominators for various orderings, we in fact get those for products of single exchange diagrams and renormalization factors. We also need to include explicitly the contributions of single $ij$ and $jk$ exchanges multiplied by renormalization factors. The cancellation occurs only when all contributions are added together.

As an example, we take the exchanges to be the mesons and find the following contribution from all the time orderings of the double exchange diagram

$$- \frac{2}{9} \sum_{l,l'} (I_l I_{l'} + I_{l'} I_l) V_{\ell,i,j',i;j;k,i} \quad (30)$$

where $I_l$ is the common integral for the exchange of meson $l$,

$$I_l = \frac{\beta^2}{16\pi^2 f^2} \int_0^{\infty} dk \frac{k^4}{k^2 + M^2_l} F^2(k^2), \quad (31)$$

the modified integral $I_{l'}$ is given by Eq. (25), and the group factor $V_{\ell,j',j'';i;j;k,i}$ is

$$V_{\ell,j',j'';i;j;k,i} = \left( \sigma_i \cdot \sigma_j \right) \left( \sigma_{j'} \cdot \sigma_{k} \right) \left( \lambda_{j',j}^{i} \sigma_{j'}^{i} \lambda_{k,k'}^{j} + \lambda_{j',j}^{i} \lambda_{k,k'}^{j} \sigma_{j'}^{i} \right). \quad (32)$$

To determine a contribution from the renormalization diagrams, the amplitudes for the two separate $ij$ and $jk$ exchanges are multiplied with the parts of the renormalization constant $Z$ associated with other exchanges. Sum of the two products gives terms that cancel the original ”three-body” amplitude shown in Eq. (30).

Fourthly, it is straightforward to generalize the above arguments for a double exchange diagram with the exchanges connecting two pairs of lines and an extra one-particle “mass” insertion as depicted in Fig. 1(d). Since we can always slide one exchange line topologically outside the rest of the diagram, whatever the initial time
orderings of the vertices, the cancellation occurs when all time orders are summed and all the $\delta Z$’s associated with that line in the various time-orders are included.

Finally, we present in Fig. 2 the three-exchange diagrams that cannot be cancelled by renormalizations. The same three cases also appear when one of the exchanged lines is replaced by an instantaneous two-particle “mass” insertion. One can see that there are three distinct topologies involved: i) two vertices on each line with the loops entangled; ii) one vertex on each of two lines and four vertices on the remaining line with a loop from that line to itself enclosing the other two vertices; and iii) two vertices on each of two lines with crossed exchanges, and a third trapped vertex on one of the lines with the exchange connecting to the third line. In the remaining three-loop diagrams, one line can always be slid outside the remainder of the diagram, leading topologically to a renormalization-type diagram, and the final result is actually cancelled by renormalization terms.

We conclude this subsection by noting that since the three-body contributions only appear at three loops, one can expect their contributions to the baryon masses to be very small.

4. Hierarchy and Fits

4.1. Hierarchy of sizes

In the case of hypercharge breaking, Morpurgo [23,24] argued on the basis of ideas on the interactions of gluons with structure (rather than elementary) quarks and fits to some of the coefficients that the sizes of the coefficients of the various operators shown in the most general expression for baryon masses (see Eqs. (19) - (21)) should satisfy a set of hierarchical relations, with suppressions by a factor of $\sim 1/3$ for each factor of $M^4$ and by a factor in the range from $0.22 - 0.37$ for each extra gluon exchange needed.
to get the number of indices and spin factors in the operator $[16, 28]$. 

Somewhat different arguments apply in the HBChPT approach when the higher-order operators are generated by meson loop corrections to a basic set, as in [21] and [19]. Factors of $M^s$ then appear multiplying differences of loop integrals involving kaons or eta mesons and pions. The coefficients are small because these differences would vanish for equal-mass mesons. $M^s$ also appears in explicit non-calculable mass insertions with fairly small coefficients, smaller for spin-dependent than spin-independent insertions. These effects were studied in detail in our previous work [19].

Similar results hold in the case of isospin breaking, with factors of $M^d$ now multiplying either differences of pion (or kaon) loop integrals for mesons within the isospin multiplet, or explicit $d$-quark mass insertions. Since $m_d << m_u$, and the mass differences within the pion and kaon multiplets are much smaller than those between different multiplets, the coefficients of $M^d$ are expected to be much smaller than the corresponding coefficients of $M^s$. The coefficients are also expected to be smaller for spin-dependent than for spin-independent forms. The results are consistent with the observed accuracy of various sum rules for the masses.

It is important to note that the sizes of terms proportional to $M^d M^d$, $M^d M^s$, or $M^s M^s$ do not go strictly as $(m_d - m_u)^2$, $(m_d - m_u)(m_s - m_u)$, or $(m_s - m_u)^2$ as mass insertion arguments would seem to suggest. The first two can both be proportional instead in the loop expansion of HBChPT to $m_d - m_u$ because of the structure of the loop corrections, while the last can vary as $m_s - m_u$. This can be seen explicitly in the $M^s \sigma \cdot \sigma$ and $M^s M^s \sigma \cdot \sigma$ terms obtained from the one-loop mesonic corrections to the baryon masses in [19]. The leading $M^s$-dependent corrections, given in Eq. (3.14) in that paper, are all proportional to differences of meson loop integrals which vanish as $m_s - m_u$ as the quark and meson masses become equal. A single factor of $M^s$ would therefore be expected. However, because of the particular tensor structures involved in the products of Gell-Mann $\lambda$ matrices in the two-body couplings, the loop corrections that involve meson exchanges between quarks contribute to both the $M^s \sigma \cdot \sigma$ and $M^s M^s \sigma \cdot \sigma$ terms, with similar sized corrections. The first is similar in size to other contributions to the $M^s \sigma \cdot \sigma$ coefficient. The second is much larger than would be expected from the simple QCD argument.

Similar results hold for the meson loop corrections that involve isospin breaking. Since terms in $M^d M^d$ and $M^d M^s$ can be of the same order as terms with a single factor of $M^d$, it is not justified in applications to drop them without further analysis. Thus, while contributions that explicitly involve higher powers of the quark or meson mass differences, or higher loop integrals, are clearly (or presumably) suppressed, we do not expect to find Morpurgo’s strict hierarchy for terms that involve multiple factors of $M^d$ and $M^s$.

$\parallel$ Dillon and Morpurgo have adopted the value of 0.3 for the suppression factor due to each extra gluon exchange [28].
4.2. Fits to the data

Before doing the fits, we want to study the constraints on the three-body terms. Note that without the three-body terms, we have nine parameters \( m_0, A, B, C, D, A', B', C', \) and \( D' \) to describe eighteen mass states of the baryon octet and decuplet. Hence, we expect there to be nine sum rules among the baryon masses. To find these sum rules, we construct a \( 18 \times 18 \) matrix \( \mathcal{M} \) by replacing all columns associated with the three-body operators in the \( 18 \times 18 \) matrix defined earlier in Sect. 2.2 with zero entries. The obtained \( \mathcal{M} \) matrix has nine zero eigenvalues. The sum rules are just the inner products of the left null eigenvectors \( \tilde{x}_0 \) with \( \mu = \mathcal{M} \nu, \tilde{x}_0 \mu = \tilde{x}_0 \mathcal{M} \nu = 0 \), where \( \nu \) is a eighteen-component column vector of coefficients \( m_0, A, B, C, D, a, b, c, A', B', C', D', \) and \( d_i \) \((i = 1, \ldots, 6)\). We find

\[
\begin{align*}
\Delta^0 - \Delta^+ &= n - p \\
\Delta^- - \Delta^{++} &= 3(n - p) \\
\Delta^0 - \Delta^{++} &= 2(n - p) + (\Sigma^0 - \Sigma^+) - (\Sigma^- - \Sigma^0) \\
\Xi^- - \Xi^0 &= (\Sigma^- - \Sigma^+) - (n - p) \\
\Xi^{+-} - \Xi^{*0} &= (\Sigma^{*-} - \Sigma^{*+}) - (n - p) \\
2\Sigma^{*0} - \Sigma^{*+} - \Sigma^{*-} &= 2\Sigma^0 - \Sigma^+ - \Sigma^- \\
\Xi^{*0} - \Xi^0 &= \Sigma^{*+} - \Sigma^+ \\
\Omega^- - \Delta^{++} &= 3(\Xi^{*0} - \Sigma^{*+}) \\
(3p - n)/2 + \Xi^0 - (3\Lambda + \Sigma^+ + \Sigma^0 - \Sigma^-)/2 &= (3\Xi^{*0} - \Xi^{*+})/2 - 2\Sigma^{*+} + \Delta^{++}.
\end{align*}
\]

The first six relations are the well-known sum rules for isospin splittings. The fourth is the Coleman-Glashow relation, suggested originally on the basis of an unbroken SU(3) flavor symmetry \[1\] (another original SU(3) sum rule is the Gell-Mann - Okubo (GMO) mass formula \[25, 26\]). All the sum rules were later established for nonrelativistic quark models with only one- and two-body interactions independently of the flavor symmetry breaking \[2, 3, 4, 5\]. Note that the ninth relation is the modified GMO sum rule that first appeared in the quark model context \[3\]. If there are no intramultiplet splittings, it reduces to the usual GMO rule.

The above sum rules are violated by three-body terms. If we transfer all the terms to the left hand sides of the equations and denote the differences by \( \delta_i \) \((i = 1, \ldots, 9)\), the results for these sum rules in terms of the three-body coefficients are

\[
\begin{align*}
\delta_1 &= 6d_3, \\
\delta_2 &= 18d_3 + 6d_6, \\
\delta_3 &= 6d_3 + 2d_4 - 4d_5, \\
\delta_4 &= 2d_1 - 4d_2 - 2d_4 - 4d_5, \\
\delta_5 &= 2d_1 + 2d_2 - 6d_3 - 2d_4 - 2d_5, \\
\delta_6 &= -6d_3 - 6d_5, \\
\delta_7 &= -12a - 6b, \\
\delta_8 &= 6c, \\
\delta_9 &= -6b + d_1 + d_2 + d_4 - 2d_5.
\end{align*}
\]

**Isospin splittings**: We have mentioned at the beginning of this section that the coefficients of the mass operators in Eq. (19) are expected to satisfy an approximate hierarchy of sizes. As discussed below, the sizes of some three-body coefficients can be estimated by evaluating the \( \delta \)'s using the experimental values of the accurately known baryon masses.
The $\Delta$ baryon masses are not determined with sufficient accuracy for the first three sum rules to give a real test of this expectation. The results for the next three as written are, in order, $\delta_4 = -0.31 \pm 0.25$ MeV, $\delta_5 = 0.09 \pm 0.93$ MeV, and $\delta_6 = -1.06 \pm 1.18$ MeV, all consistent with zero within the experimental uncertainties. No significant violations of the sum rules are evident.

Note that the coefficients $d_i$ ($i = 1, ..., 6$) associated with the three-body isospin splitting operators are expected to be very small. In particular, the coefficient $d_6$ of the three-body operator $t_6 = \sum_{i \neq j \neq k} M_i^d M_j^d M_k^d$ that contributes only to the $\Delta^-$ mass, is expected to be very small since $t_6$ carries three factors of $M^d$ (its size is proportional at least to $(m_d - m_u)^2$). We also expect that the coefficients $d_5$ associated with the three-body operators $t_5 = \sum_{i \neq j \neq k} M_i^d M_j^d M_k^d \sigma_i \cdot \sigma_j \cdot \sigma_k$ should be smaller than the other $d_i$ ($i = 1, ..., 4$) since $t_5$ involves two factors of $M^d$, one factor of $M^s$, and a spin interaction with the third particle. Therefore, we will ignore $d_5$ and $d_6$ when trying to estimate the other coefficients $d_i$. Using the values of $\delta_4$, $\delta_5$, and $\delta_6$, we find $d_4 - d_4 = 0.33 \pm 0.50$ MeV, $d_2 = 0.24 \pm 0.25$ MeV, and $d_3 = 0.18 \pm 0.20$ MeV. These results show that contributions of the three-body isospin splitting operators to baryon masses are indeed very small and can be ignored.

**Hypercharge splittings:** If we set $d_i = 0$ ($i = 1, ..., 6$), as the above results suggest, we are left with the relations

$$\delta_7 = -12a - 6b, \ \delta_8 = 6c, \ \delta_9 = -6b.$$  

To estimate the coefficients of the three-body hypercharge splitting terms $a$, $b$, and $c$, we use the average mass of the $\Delta$'s and other accurately known baryon masses to evaluate $\delta_7$, $\delta_8 - \delta_9$ (note that $\Delta^{++}$ does not appear in this combination), and $\delta_9$. The result are $\delta_7 = -23.54 \pm 0.55$ MeV, $\delta_8 - \delta_9 = 3.40 \pm 1.08$ MeV, and $\delta_9 = -9.95 \pm 2.24$ MeV. Then, Eq. (35) gives $a = 1.13 \pm 0.06$ MeV, $b = 1.66 \pm 0.37$ MeV, and $c = -1.09 \pm 0.41$ MeV. These contributions of the three-body hypercharge splitting terms are statistically significant and as large as observed isospin splittings, so they should be kept in a complete analysis of the baryon octet and decuplet masses.

**General constrained fit:** We now consider a fit to the baryon masses using the general expression in Eq. (19) neglecting the coefficients $d_i$ ($i = 1, ..., 6$). Using 12 parameters $m_0$, $A$, $B$, $C$, $D$, $A'$, $B'$, $C''$, $D'$, $a$, $b$, and $c$, we can do a weighted least-squares fit to the 15 measured quantities (14 accurately known baryon masses plus the average mass of the $\Delta$'s).

A best fit is obtained at values (in MeV)

$$m_0 = 1082.9 \pm 1.0,$$
$$A = 24.66 \pm 0.17, \ B = 183.66 \pm 1.05, \ C' = -17.10 \pm 0.34, \ D' = -2.94 \pm 0.71,$$
$$a = 1.22 \pm 0.17, \ b = 1.52 \pm 0.35, \ c = -0.93 \pm 0.40,$$
$$A' = 0.95 \pm 0.18, \ B' = -0.60 \pm 0.08, \ C'' = 1.69 \pm 0.24, \ D' = 0.78 \pm 0.04.$$  

(36)

The fitted masses have an average deviation from experiment of only 0.07 MeV and a $\chi^2 = 1.73$ (with 3 degrees of freedom). The calculated values of the baryon masses are
Table 3. Baryon masses in units of MeV. Excluding the $\Delta$ resonances, the average deviation $|\Delta M_B|$ = 0.07 MeV, where $\Delta M_B = M_B^{\text{theory}} - M_B^{\text{expt.}}$. The experimental data are from the listings given by the Particle Data Group [27]. No definite masses or uncertainties are given for the $\Delta$ resonances as different experimental analyses are in conflict. Only ranges of mass are given here.

| Baryon | Theory | Expt. | $|\Delta M_B|$ |
|--------|--------|-------|--------------|
| $p$    | 938.27 $\pm$ 1.50 | 938.27 $\pm$ 0.00 | 0.00 |
| $n$    | 939.57 $\pm$ 1.51 | 939.57 $\pm$ 0.00 | 0.00 |
| $\Lambda$ | 1115.68 $\pm$ 2.10 | 1115.68 $\pm$ 0.01 | 0.00 |
| $\Sigma^+$ | 1189.39 $\pm$ 2.28 | 1189.37 $\pm$ 0.06 | 0.02 |
| $\Sigma^0$ | 1192.64 $\pm$ 2.30 | 1192.64 $\pm$ 0.02 | 0.00 |
| $\Sigma^-$ | 1197.45 $\pm$ 2.36 | 1197.45 $\pm$ 0.03 | 0.00 |
| $\Xi^0$ | 1314.64 $\pm$ 3.58 | 1314.83 $\pm$ 0.20 | 0.19 |
| $\Xi^-$ | 1321.39 $\pm$ 3.62 | 1321.31 $\pm$ 0.13 | 0.08 |
| $\Delta$ | 1232.00 $\pm$ 1.48 | 1232.00 $\pm$ 2.00 | 0.00 |
| $\Delta^{++}$ | 1230.82 $\pm$ 1.46 | 1230.5 $\pm$ 2.9 | 0.29 |
| $\Delta^+$ | 1230.57 $\pm$ 1.47 | 1231.6 $\pm$ 1.4 | 0.00 |
| $\Delta^0$ | 1231.87 $\pm$ 1.53 | 1233.1 $\pm$ 0.75 | 0.00 |
| $\Delta^-$ | 1234.73 $\pm$ 1.64 | 1233.1 $\pm$ 2.9 | 0.00 |
| $\Sigma^{++}$ | 1382.74 $\pm$ 1.95 | 1382.80 $\pm$ 0.40 | 0.06 |
| $\Sigma^{+0}$ | 1384.18 $\pm$ 2.02 | 1383.70 $\pm$ 1.00 | 0.48 |
| $\Sigma^{-+}$ | 1387.18 $\pm$ 2.06 | 1387.20 $\pm$ 0.50 | 0.02 |
| $\Xi^{+0}$ | 1531.81 $\pm$ 3.37 | 1531.80 $\pm$ 0.32 | 0.01 |
| $\Xi^{-+}$ | 1534.95 $\pm$ 3.41 | 1535.00 $\pm$ 0.60 | 0.05 |
| $\Omega^+$ | 1672.45 $\pm$ 6.78 | 1672.45 $\pm$ 0.29 | 0.00 |

Given in Table 3, where the experimental data are from the listings given by the Particle Data Group [27]. No definite masses or uncertainties are given for the $\Delta$ resonances as different experimental analyses are in conflict. Instead, we give here their mass ranges except for $\Delta^-$. Our predictions for the $\Delta$ resonances agree with the experimental values within the margin of error.

Note that the best-fit values of the parameters $A'$, $B'$, $C'$, and $D'$ show that terms in $M^d M^d$ and $M^d M^s$ are of the same order as terms with a single factor of $M^d$, as expected in the meson loop expansion, rather than much smaller as suggested by the QCD hierarchy arguments.

Baryon mass differences: We remark that one can use the general expression for baryon masses in Eq. (19) to describe the baryon mass differences. Then the parameters associated with two flavor-symmetric operators and with the intermultiplet splitting operators drop out, so there remain 10 parameters ($A'$, $B'$, $C'$, $D'$, and $d_i$ ($i = 1, \ldots, 6$)). Using contributions of the isospin splitting operators to the mass differences given in Table 2, we easily find the followings

$$
n - p = A' + 2B' + 2D' - 4d_3, \\
\Sigma^- - \Sigma^+ = 2(A' - B' + C' + D' - 2d_3 + d_4 - 2d_5), \\
\Sigma^- - \Sigma^0 = A' - B' + C' + 2D' - 4d_3 + 2d_4 - 4d_5,
$$
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\[
\Xi^- - \Xi^0 = A' - 4B' + 2C' + 2d_1 - 4d_2, \\
\Sigma^- - \Sigma^{++} = 2(A' + 2B' + C' + D' + d_3 + d_4 + d_5), \\
\Sigma^* - \Sigma^{*0} = A' + 2B' + C' + 2D' + 2d_3 + 2d_4 + 2d_5, \\
\Xi^- - \Xi^{*0} = A' + 2B' + 2C' + 2d_1 + 2d_2, \\
\Delta^0 - \Delta^{++} = 2(A' + 2B' + D' + d_3), \\
\Delta^- - \Delta^{++} = 3(A' + 2B' + 2D' + 2d_3 + 2d_6), \\
\Delta^0 - \Delta^+ = A' + 2B' + 2D' + 2d_3.
\]

(37)

If the three-body isospin splitting terms \(d_i\) are neglected, we have 4 parameters \(A', B', C', D'\) to describe the baryon mass differences. We studied this case earlier in [20, 22] using the one- and two-body electromagnetic operators. A best weighted fit to the seven known mass splittings other than those for the \(\Delta\) baryons is obtained at values (in MeV) of \(A' = 0.91 \pm 0.19, B' = -0.59 \pm 0.08, C' = 1.73 \pm 0.23,\) and \(D' = 0.78 \pm 0.05\) with an average deviation from experiment of 0.13 MeV and a \(\chi^2 = 1.67\) (with 3 degrees of freedom). The calculated values of the baryon mass splittings are the same as those shown in Table I in [22].

Our above fits provide the parameters that must be explained in the loop expansion in chiral effective field theory or any other dynamical model, and show that three-body contributions will necessarily play a role in explaining masses at the scale of about 1 MeV. The scale for three-body isospin effects is at least ten times smaller.

5. Conclusions

We have constructed the general parametrization of the baryon octet and decuplet mass operators including three-body terms using the unit operator and the symmetry-breaking factors, \(M^d\) and \(M^s\), in conjunction with the spin operators. Our general expression for baryon masses includes all order of the light quark mass difference \(m_d - m_u\). Using the Gram matrix analysis, we established a minimal set of 18 independent operators to describe 18 baryon octet and decuplet mass states.

We have demonstrated the cancellation of the three-body terms through two loops in HBChPT and have identified three distinct topologies of the three-loop diagrams that generate the three-body terms. We also investigated the likely size of the three-body terms using the sum rules that give constraints on their coefficients. Numerical calculations of the sum rules indicate that contributions of the three-body hypercharge splittings, while small, are comparable to those from the one- and two-body isospin splittings. The contributions of three-body isospin splitting operators are statistically consistent with zero, and these operators can be neglected at the present level of experimental accuracy. We find that the suggested hierarchy of sizes for terms that involve multiple factors of \(M^d\) and/or \(M^s\) may not be evident because powers of \(M^d\) and \(M^s\) do not correlate strictly with powers of the quark masses \(m_d - m_u\) and \(m_s - m_u\), respectively.
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We have done the phenomenological fits to the experimental data using our general expression, but ignoring terms associated with the three-body isospin splitting operators. Our fits provide the values of the parameters that must be explained in the loop expansion in ChPT or any other dynamical model. The results of the fits also confirm that three-loop contributions will necessarily play a role in explaining masses at the scale of about 1 MeV.

Finally, our parametrization has the minimal number of operators needed to describe all the octet and decuplet masses. It can be translated back into the language of the heavy-baryon effective field theory and is particularly useful to an analysis of the baryon mass splittings due to both hypercharge-breaking and isospin-breaking effects.

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