PRIMITIVE IDEALS FOR W-ALGEBRAS IN TYPE A

IVAN LOSEV

Abstract. In this note we classify the primitive ideals in finite W-algebras of type A.

1. Introduction

Let $g$ be a semisimple Lie algebra over an algebraically closed field $K$ of characteristic 0 and $e \in g$ be a nilpotent element. Then to the pair $(g, e)$ one can assign an associative algebra $\mathcal{W}$ called the W-algebra. This algebra was defined in full generality by Premet in [P1]. Two equivalent definitions of W-algebras are provided in Section 2. For other details on W-algebras the reader is referred to review [L5].

One of the reasons to be interested in $\mathcal{W}$ are numerous connections between this algebra and the universal enveloping algebra $U$ of $g$. For example, the sets of primitive ideals $Pr(\mathcal{W})$ and $Pr(U)$ of $\mathcal{W}$ and $U$ are closely related. Recall that an ideal in an associative algebra is called primitive if it is the annihilator of some irreducible module. The structure of $Pr(U)$ was studied extensively in 70’s and 80’s.

One of manifestations of a relationship between $Pr(\mathcal{W})$ and $Pr(U)$ is a map $I \mapsto I^\dagger : Pr(\mathcal{W}) \rightarrow Pr(U)$ constructed in [L1]. One can describe the image of this map. Namely, to each primitive ideal $\mathcal{J} \in Pr(U)$ one assigns its associated variety $V(U/\mathcal{J})$. According to a theorem of Joseph, the associated variety is the closure of a single nilpotent orbit in $g^* \cong g$. Thanks to [L1], Theorem 1.2.2(vii), an element $\mathcal{J} \in Pr(U)$ is of the form $I^\dagger$ for some $I \in Pr(\mathcal{W})$ if and only if $O \subset V(U/\mathcal{J})$, where $O$ stands for the adjoint orbit of $e$.

In general, the map $\bullet^\dagger$ is not injective. However, the following result holds.

Theorem 1.1. The map $\bullet^\dagger : Pr(\mathcal{W}) \rightarrow Pr(U)$ is an injection provided $g \cong sl_n$.

This theorem provides a classification of primitive ideals in $\mathcal{W}$ because the set of primitive ideals $\mathcal{J} \subset U$ with $V(U/\mathcal{J}) = \emptyset$ is known thanks to the work of Joseph, [J].

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2. W-algebras and the map between ideals

2.1. Quantum slice. Let $Y$ be an affine Poisson scheme equipped with a $K^\times$-action such that the Poisson bracket has degree $-2$. Let $A_h$ be an associative flat graded $K[h]$-algebra (where $h$ has degree 1) such that $[A_h, A_h] \subset h^2 A_h$ and $K[Y] = A_h/(h)$ as a graded Poisson algebra. Pick a point $\chi \in Y$. Let $I_\chi$ be the maximal ideal of $\chi$ in $K[Y]$ and let $\tilde{I}_\chi$ be the inverse image of $I_\chi$ in $A_h$. Consider the completion $A_h^{\wedge} := \lim_{\leftarrow n \rightarrow \infty} A_h/\tilde{I}_\chi^n$. This is a space.

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MSC 2010: Primary 16S99, 17B35. 
Address: Northeastern University, Department of Mathematics, 360 Huntington Avenue, Boston, MA 02115. 
E-mail: i.losev@neu.edu.]
complete topological $\mathbb{K}[[\hbar]]$-algebra with $\mathcal{A}_h^\wedge \times (\hbar) = \mathbb{K}[Y]^{\wedge, \times}$, where on the right hand side we have the usual commutative completion. Moreover, as we have seen in [L4], Lemma 2.11, the algebra $\mathcal{A}_h^\wedge$ is flat over $\mathbb{K}[[\hbar]]$.

The cotangent space $T^*_\chi Y = I_\chi/I_\chi^2$ comes equipped with a natural skew-symmetric form, say $\omega$. Fix a maximal symplectic subspace $V \subset T^*_\chi Y$. One can choose an embedding $V \hookrightarrow \tilde{I}_\chi^\wedge$ such that $[\iota(u), \iota(v)] = \hbar^2 \omega(u, v)$ and whose composition with the projection $\tilde{I}_\chi^\wedge \to T^*_\chi Y$ is the identity. This is proved similarly to Proposition 3.3 in [Ka] (or can be deduced from that proposition, compare with the argument of Subsection 7.2 in [L3]). Consider the homogenized Weyl algebra $\mathcal{A}_h(V) = T(V)[\hbar]/(u \otimes v - v \otimes u - \hbar^2 \omega(u, v))$ and its completion $\Lambda_{\hbar, h}^\wedge(V)$ at zero. It is easy to show that

$$\Lambda_{\hbar, h}^\wedge := \Lambda_{\hbar, h}^\wedge(V) \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathcal{A}_h^\prime,$$

where $\mathcal{A}_h^\prime$ is the centralizer of $V$ in $\Lambda_{\hbar, h}^\wedge$. We remark that the algebra $\mathcal{A}_h^\prime$ is complete with respect to the topology induced by its maximal ideal. The symbol $\widehat{\otimes}$ stands for the completed tensor product of topological vector spaces.

The argument in the proof of [L6], Proposition 6.6.1, Step 2, shows that any two embeddings $\iota^1, \iota^2 : V \hookrightarrow \tilde{I}_\chi^\wedge$ satisfying the conditions in the previous paragraph differ by an automorphism of $\Lambda_{\hbar, h}^\wedge$ of the form $\exp(\frac{1}{\hbar^2} \text{ad}(z))$ with $z \in (\tilde{I}_\chi^\wedge)^3$. In particular, the algebra $\mathcal{A}_h^\prime$ is defined uniquely up to a $\mathbb{K}[[\hbar]]$-linear isomorphism.

Now let us consider a compatibility of our construction with certain derivations. Suppose $\mathcal{A}_h$ is equipped with a derivation $D$ such that $D\hbar = \hbar$. The derivation extends to $\Lambda_{\hbar, h}^\wedge$. According to [L4], there is a derivation $D'$ of $\mathcal{A}_h^\prime$, with the following properties. First, $D'\hbar = \hbar$. Second, we have $D - D' = -\frac{1}{\hbar^2} \text{ad}(a)$ for some element $a \in \Lambda_{\hbar, h}^\wedge$, where $D'$ means the derivation that equals $D'$ on $\mathcal{A}_h^\prime$ and acts by 1 on the symplectic space $V$ generating $\Lambda_{\hbar, h}^\wedge(V)$.

An easy special case of the previous construction is when the derivation $D$ comes from a $\mathbb{K}^\times$-action on $\mathcal{A}_h$ preserving $\chi$. Here we can choose a $\mathbb{K}^\times$-stable $V$ and a $\mathbb{K}^\times$-equivariant embedding $\iota : V \to \tilde{I}_\chi^\wedge$ and so we get a $\mathbb{K}^\times$-action on $\mathcal{A}_h^\prime$. Moreover, the algebra $\mathcal{A}_h^\prime$ is now defined uniquely up to a $\mathbb{K}^\times$-equivariant $\mathbb{K}[[\hbar]]$-linear isomorphism.

Now let $Z(\mathcal{A}_h)$ denote the center of $\mathcal{A}_h$. Consider a $\mathbb{K}^\times$-equivariant $\mathbb{K}[[\hbar]]$-linear homomorphism $\lambda : Z(\mathcal{A}_h) \to \mathbb{K}[[\hbar]]$ and the corresponding central reduction $\mathcal{A}_{\lambda, h} : = \mathcal{A}_h/\ker \lambda$. Until the end of the subsection we assume that $\chi$ lies in the spectrum of $\mathcal{A}_{\lambda, h}^\wedge(\hbar)$.

Consider the induced homomorphism $Z(\mathcal{A}_h) \to \mathcal{A}_{\hbar, h}^\wedge = \mathcal{A}_{\hbar, h}^\wedge(V) \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathcal{A}_h^\prime$. The image is central. The center of $\mathcal{A}_{\hbar, h}^\wedge(V)$ coincides with $\mathbb{K}[[\hbar]]$, so the image of $Z(\mathcal{A}_h)$ is contained in $\mathcal{A}_h^\prime$. Set $\mathcal{A}^\wedge_{\lambda, h} = \mathcal{A}_{\hbar, h}^\wedge/\ker \lambda$.

Then we have the completion $\mathcal{A}^\wedge_{\lambda, h}$ of $\mathcal{A}_{\lambda, h}$, and $\mathcal{A}^\wedge_{\lambda, h} = \mathcal{A}^\wedge_{\hbar, h}/\ker \lambda$. Furthermore, we have the following commutative diagram, where the horizontal arrows are isomorphisms and the vertical arrows are the natural quotients.
Now suppose that we have a reductive group $Q$ that acts on $A_h$ rationally by $\mathbb{K}[h]$-algebra automorphism fixing $\chi$. Further, suppose that there is quantum moment map $\Phi^A : q \to A_h$, i.e., a $Q$-equivariant linear map with the property that $[\Phi^A(\xi), a] = \hbar^2 \xi.a$, where on the right hand side $\xi$ is the derivation of $A_h$ coming from the $Q$-action. Composing $\Phi^A$ with a natural homomorphism $A_h \to A_h^\times$, we get a quantum moment map $q \to A_h^\times$ again denoted by $\Phi^A$.

We remark that we can choose $V$ to be $Q$-stable. This gives rise to a $Q$-action on $V$ by linear symplectomorphisms and hence to an action of $Q$ on $A_h(V)$ by $\mathbb{K}[h]$-linear algebra automorphisms. There is a quantum moment map $\Phi^A : q \to A_h(V)$ that is the composition $q \to \text{sp}(V) = S^2V \hookrightarrow A_h(V)$.

Further, since $Q$ is reductive, we can assume that the embedding $\iota : V \hookrightarrow A_h^\times$ is $Q$-equivariant. So we get a $Q$-action on $A_h'$. Let us produce a quantum moment map for this action. For $\xi \in q$ set $\Phi(\xi) = \Phi^A(\xi) - \Phi^A(\xi)$. The quantum moment map conditions for $\Phi^A$, $\Phi^A$ imply that $\Phi(\xi)$ commutes with $A_h(V)^{\hbar^0}$ and hence the image of $\Phi$ is in $A_h'$. Now it is clear that $\Phi : q \to A_h'$ is a quantum moment map for the action of $Q$ on $A_h'$.

We remark that the $Q$-action and the map $\Phi$ are defined by the previous construction uniquely up to an isomorphism of the form $\exp(\frac{1}{\hbar} \text{ad}(z))$, where, in addition to conditions mentioned above, $z$ is $Q$-invariant. Also we remark that if we have a $\mathbb{K}^\times$-action on $A_h$ as above but additionally commuting with $Q$ and such that $\Phi^A(\xi)$ has degree 2 for each $\xi$, then the $Q$-action on $A_h'$ also may be assumed to commute with $\mathbb{K}^\times$ and $\Phi(\xi)$ may be assumed to have degree 2.

### 2.2. W-algebras via quantum slices.

We are going to consider a special case of the construction explained in the previous subsection.

Let $G$ be a simply connected semisimple algebraic group and let $\mathfrak{g}$ be the Lie algebra of $G$. Set $Y := \mathfrak{g}^\ast$. We remark that we can identify $\mathfrak{g}$ with $\mathfrak{g}^\ast$ by means of the Killing form. Pick a nilpotent orbit $O \subset \mathfrak{g}$ and an element $e \in O$. We take $e$ for $\chi$. Let us equip $Y$ with a Kazhdan $\mathbb{K}^\times$-action defined as follows. Pick an $\mathfrak{sl}_2$-triple $(e, h, f)$, where $h$ is semisimple. Let $\gamma : \mathbb{K}^\times \to G$ be the one-parameter corresponding to $h$. We define a $\mathbb{K}^\times$-action on $\mathfrak{g}^\ast$ by $t.\alpha = t^{-2}\gamma(t)\alpha, \alpha \in \mathfrak{g}^\ast, t \in \mathbb{K}^\times$. We remark that $t.\chi = \chi$.

For $A_h$ we take the homogenized version $U_h$ of the universal enveloping algebra defined by $U_h := T(\mathfrak{g})[h]/(\chi \otimes y - y \otimes \chi - h^2[\chi, y])$. We can extend the Kazhdan action to $U_h$. Explicitly, for $\xi \in \mathfrak{g}$ with $[h, \xi] = i\xi$ we have $t.\xi = t^{i+2}\xi$ and we set $t.h := th$.

Apply the construction of the previous subsection to $A_h, \chi$. We get the $\mathbb{K}[[h]]$-algebra $A_h'$ acted on by $\mathbb{K}^\times$ together with a $\mathbb{K}^\times$-equivariant isomorphism $A_h^\times = A_h^{\hbar^0}(V) \otimes_{\mathbb{K}[[h]]} A_h'$. Here $V$ has the same meaning as before but we can describe it explicitly: namely, for $V$ we take the subspace $[\mathfrak{g}, f] \subset \mathfrak{g} = T_\chi^\ast Y$.

Let $A_h'$ denote the subalgebra of all $\mathbb{K}^\times$-finite vectors in $A_h'$. It turns out that the algebra $A_h'/hA_h'$, or, more precisely, the corresponding variety is well-known in the theory of nilpotent orbits – this is a so called Slodowy slice. In more detail, set $S := e + 3g(f)$ and view $S$ as a subvariety in $\mathfrak{g}^\ast$ via the identification $\mathfrak{g} \cong \mathfrak{g}^\ast$. Then $S$ is $\mathbb{K}^\times$-stable and is transverse to $Ge$ in $e$: $\mathfrak{g} = T_eS \oplus T_eGe$. Moreover, the Kazhdan action contracts $S$ to $e$, meaning that $\lim_{t\to\infty} t.s = e$ for all $s \in S$. This implies that $S$ is transversal to any $G$-orbit it intersects – $\mathfrak{g} = T_eS + T_eGe$ for all $s \in S$. Also the contraction property means that the grading on $\mathbb{K}[S]$ induced by the $\mathbb{K}^\times$-action is positive: there are no negative degrees, and the only elements in degree $0$ are constants.
The contraction property for the $\mathbb{K}^x$-action on $S$ implies that the subalgebra of $\mathbb{K}[S]^{\wedge x} = A_h'/hA_h'$ consisting of the $\mathbb{K}^x$-finite elements coincides with $\mathbb{K}[S]$. Moreover, since the degree of $h$ is positive, this implies that $A_h'/hA_h = \mathbb{K}[S]$.

We set $A := A_h/(h-1)A_h$. This is a filtered associative algebra whose associated graded is $\mathbb{K}[S]$. The algebra $A_A$ can be recovered as the Rees algebra of $A$, while $A_h'$ is the completion $\mathcal{A}_h^{\wedge x}$.

Let us remark that the algebra $A$ comes equipped with a homomorphism $Z \to A$, where $Z$ is the center of the universal enveloping algebra $U(= U_h/(h-1)U_h)$ of $\mathfrak{g}$. Indeed, set $Z_h := U_h^G$. This is the center of $U_h$. Consider $Z_h$ as a subalgebra of $A_h^{\wedge x}$. According to the previous subsection, we have $Z_h \subset A_h$. It is easy to see that $Z_h$ consists of $\mathbb{K}^x$-finite vectors so $Z_h \subset A_h$. This gives rise to an embedding $Z \to A$. We remark that this embedding does not depend on the choice (that of the embedding $V \to \tilde{V}^{\wedge x}$) we have made. This is because that the two choices are conjugate by an automorphism that commutes with all elements of the center.

Consider the subgroup $Q := Z_G(e, h, f) \subset G$. This group acts on $U_h$ and stabilizes $\chi$ (and $S$ as well). The $Q$-action commutes with $\mathbb{K}^x$ and there is a quantum moment map $q \to A_h$ whose image consists of functions of degree 2 with respect to the $\mathbb{K}^x$-action. So we get a $Q$-action on $A_h$ as well as a quantum comoment map $q \to A_h$. Both the action and the quantum moment map descend to $A$.

2.3. Equivalence with a previous definition. Recall that the cotangent bundle $T^*G$ carries a natural symplectic form $\omega$. This form is invariant with respect to the natural $G \times G$-action. Moreover, for the $\mathbb{K}^x$-action by fiberwise dilations we have $t.\omega = t^{-1}\omega$, $t \in \mathbb{K}^x$.

We remark that we can trivialize $T^*G$ by using left-invariant 1-forms hence $T^*G = G \times \mathfrak{g}^*$. Consider $S$ as a subvariety in $\mathfrak{g}^*$ and set $X := G \times S$. It turns out that the subvariety $X \subset T^*G$ is symplectic. It is easy to see that $X$ is stable with respect to the left $G$-action, as well as to the Kazhdan $\mathbb{K}^x$-action on $T^*G$ given by $t.(g, \alpha) = (g\gamma(t)^{-1}, t^{-2}\gamma(t)\alpha)$. Clearly, $\omega$ has degree 2 with respect to the Kazhdan action. Also $Q$ acts on $T^*G$ by $q.(g, \alpha) = (gq^{-1}, q\alpha)$ and $X$ is $Q$-stable.

We remark that $\mu_G : T^*G \to \mathfrak{g}^*, \mu_G(g, \alpha) = g\alpha$ is a moment map, i.e., a $G$-equivariant map such that for any $\xi \in \mathfrak{g}$ the derivation $\mu_G^k(\xi)$ of $\mathbb{K}[T^*G]$ coincides with the derivation produced by $\xi$ via the $G$-action. Similarly, $\mu_Q : T^*G \to \mathfrak{q}^*, \mu_Q(x, \alpha) = \alpha|_q$.

In [L1] the author proved that there is an associative product $*$ on $\mathbb{K}[X][h]$, where $h$ is an independent variable, satisfying the following properties:

1. $* \text{ is } G \times \mathbb{K}^x$-equivariant, where $G \times \mathbb{K}^x$ acts on $\mathbb{K}[X]$ as usual, $g.h = h, t.h = th$.
2. For $f, g \in \mathbb{K}[X]$ we have $f \ast g = \sum_{i=0}^{\infty} D_i(f, g) h^i$, where $D_i$ is a bi-differential operator of order at most $i$.
3. $f \ast g \equiv fg \mod h^2$.
4. $f \ast g - g \ast f \equiv h^2 \{f, g\} \mod h^4$.
5. The map $\mu^*_G : \mathfrak{g} \to \mathbb{K}[X][h], \mu^*_Q : \mathfrak{q} \to \mathbb{K}[X][h]$ are quantum moment maps.

The last property was established in [L2]. By definition, the W-algebra $\mathcal{W}$ is the quotient of the invariant subalgebra $\mathbb{K}[X][h]^G$ by $h - 1$. The quantum moment map $\mu^* : \mathfrak{g} \to \mathbb{K}[X][h]$ gives rise to the homomorphism $Z = U(\mathfrak{g})^G \to \mathcal{W}$.

Proposition 2.1. We have a filtration preserving $Q$-equivariant isomorphism $\mathcal{W} \to A$ intertwining the embeddings of $Z$ and the quantum moment maps from $q$. 
Proof. Similarly to the above we have a star-product on $T^*G$ having the properties analogous to (1)-(5). Set $x := (1, \chi) \in X \subset T^*G$. Consider the completions $\mathbb{K}[T^*G][[\hbar]]^\wedge_{Gx}, \mathbb{K}[X][[\hbar]]^\wedge_{Gx}$ of the corresponding algebras (w.r.t. star-products) at the ideals of $Gx$. According to Theorem 2.3.1 from [L2], we have a $G \times \mathbb{K}^\times$-equivariant (where we consider the Kazhdan $\mathbb{K}^\times$-actions) topological $\mathbb{K}[[\hbar]]$-algebra isomorphism

$$\mathbb{K}[T^*G][[\hbar]]^\wedge_{Gx} \sim \mathcal{A}_h^\wedge(V) \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathbb{K}[X][[\hbar]]^\wedge_{Gx},$$

and this isomorphism intertwines the quantum moment maps for the $G$-action and $Q$-action (the Weyl algebra component of the quantum moment map for $G$ on the right hand side is 0). The algebra of $G$-invariants of the left hand side is $\mathcal{U}_h^\wedge$, while on the right hand side we get $\mathcal{A}_h^\wedge(V) \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathcal{W}_h^\wedge$, see loc. cit. So we can take $\mathcal{W}_h^\wedge$ for $\mathcal{A}_h^\wedge$. It follows that we have $Q$-equivariant isomorphisms $\mathcal{W}_h \cong \mathcal{A}_h$ of graded $\mathbb{K}[[\hbar]]$-algebras and $\mathcal{W} \cong \mathcal{A}$ of filtered algebras. Both isomorphisms intertwine the quantum moment maps from $q$. Moreover, the embedding $\mathcal{U}_h \hookrightarrow \mathcal{U}_0 = \mathbb{K}[T^*G][[\hbar]]^G$ induced by the moment map is just the inclusion. This completes the proof of the proposition. \qed

2.4. Map between the set of ideals. Let us construct the map $\bullet^\dagger$ mentioned in Theorem 1.4. We will start with the general setting explained in Subsection 2.1.

Consider the set $\mathfrak{J}_h(\mathcal{A}_h)$ of all $\mathbb{K}^\times$-stable $h$-saturated ideals $\mathcal{J}_h \subset \mathcal{A}_h$, where “$h$-saturated” means that $\mathcal{A}_h / \mathcal{J}_h$ is flat over $\mathbb{K}[[\hbar]]$. Similarly, consider the set $\mathfrak{I}_h(\mathcal{A}_h')$ of all $D'$-stable $h$-saturated ideals in $\mathcal{A}_h'$. The discussion of $D'$ in Subsection 2.1 implies that an $h$-saturated $\mathfrak{I}_h \subset \mathcal{A}_h'$ is $D'$-stable if and only if $\mathcal{A}_h^\wedge \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathfrak{I}_h \subset \mathcal{A}_h^\wedge$ is $D'$-stable. In particular, the set $\mathfrak{I}_h(\mathcal{A}_h')$ does not depend on the choice of $D'$.

We have maps between $\mathfrak{J}_h(\mathcal{A}_h)$ and $\mathfrak{I}_h(\mathcal{A}_h')$ constructed as follows. Take an ideal $\mathcal{J}_h \subset \mathcal{A}_h$ and form its closure $\mathcal{J}_h^\wedge \subset \mathcal{A}_h^\wedge$. This ideal is $D'$-stable but also one can check that it is actually $h$-saturated. As such, the ideal $\mathcal{J}_h^\wedge$ has the form $\mathcal{A}_h^\wedge \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathfrak{I}_h$ for a unique two-sided ideal $\mathfrak{I}_h$ in $\mathcal{A}_h$. The ideal $\mathfrak{I}_h$ is automatically $D'$-stable and $h$-saturated. We consider the map $\bullet^\dagger : \mathfrak{J}_h(\mathcal{A}_h) \to \mathfrak{I}_h(\mathcal{A}_h')$ sending $\mathcal{J}_h$ to $\mathfrak{I}_h$.

Let us produce a map in the opposite direction. Take $\mathfrak{I}_h' \in \mathfrak{I}_h(\mathcal{A}_h')$. Then $\mathfrak{J}_h := \mathcal{A}_h \cap \mathcal{A}_h^\wedge \widehat{\otimes}_{\mathbb{K}[[\hbar]]} \mathfrak{I}_h'$ is a $\mathbb{K}^\times$-stable $h$-saturated ideal in $\mathcal{A}_h$. Consider the map $\bullet^\dagger : \mathfrak{I}_h(\mathcal{A}_h') \to \mathfrak{J}_h(\mathcal{A}_h)$ sending $\mathfrak{I}_h'$ to $\mathfrak{J}_h$.

Now suppose that the grading on $\mathbb{K}[Y]$ induced by the $\mathbb{K}^\times$-action is positive. Then $\mathfrak{J}_h(\mathcal{A}_h)$ is in bijection with the set $\mathfrak{J}(\mathcal{A})$ of two-sided ideals in $\mathcal{A} := \mathcal{A}_h / (h - 1)$. Under this bijection, the ideal in $\mathcal{A}$ corresponding to $\mathfrak{J}_h \in \mathfrak{J}_h(\mathcal{A}_h)$ is $\mathfrak{J}_h / (h - 1)\mathfrak{J}_h$.

Similarly, suppose that $D'$ is also induced from some $\mathbb{K}^\times$-action such that $\mathcal{A}_h'$ is the projective limit of some positively graded algebras (this is the case in the situation considered in Subsection 2.2). Then we have natural identifications $\mathfrak{J}_h(\mathcal{A}_h') \cong \mathfrak{J}_h(\mathcal{A}_h) \cong \mathfrak{J}(\mathcal{A})$. So we have maps between $\mathfrak{J}(\mathcal{A})$, $\mathfrak{I}(\mathcal{A})$ that still will be denoted by $\bullet^\dagger$, $\bullet^\dagger$.

In a special case we have some additional information about the maps $\bullet^\dagger$, $\bullet^\dagger$. Suppose that $Y$ is still equipped with a contracting $\mathbb{K}^\times$-action and, moreover, has only finitely many symplectic leaves. For a symplectic leaf $\mathcal{L}$ let $\mathfrak{J}_\mathcal{L}(\mathcal{A}_h)$ denote the subset of $\mathfrak{J}(\mathcal{A})$ consisting of all ideals $\mathcal{J}$ such that $gr(\mathcal{A}/\mathcal{J})$ is supported on the closure of $\mathcal{L}$. The maximal elements in $\mathfrak{J}_\mathcal{L}(\mathcal{A})$ are precisely prime (=primitive by [L4]) ideals.

Now let $\mathcal{L}$ be the leaf containing $\chi$. Then $\bullet^\dagger$ defines a map $\mathfrak{J}_\mathcal{L}(\mathcal{A}) \to \mathfrak{J}_{h,fin}(\mathcal{A}_h')$, where, by definition, the target set consists of all ideals $\mathfrak{I}_h'$ such that $\mathcal{A}_h'/\mathfrak{I}_h'$ is free of finite rank over $\mathbb{K}[[\hbar]]$. For a prime ideal $\mathcal{J} \in \mathfrak{J}_\mathcal{L}(\mathcal{A})$ and any minimal prime ideal $\mathfrak{I}_h'$ of $\mathcal{J}_h$ we have
\( \mathcal{J} = (\mathcal{T}_h)^\dagger \), see [L4], Lemma A4. Corollary 3.17 from [ES] implies that there are finitely many prime ideals in \( \mathfrak{A}_{h,fin}(\mathcal{A}') \) and so \( \mathfrak{A}_{\mathcal{C}}(\mathcal{A}) \) also contains finitely many prime ideals.

We are interested in the special case when \( \mathcal{A} \) is a central reduction of \( \mathcal{U} \) at some central character \( \lambda \). Consider a unique \( \mathbb{K}^x \)-equivariant \( \mathbb{K}[h] \)-linear homomorphism \( Z(\mathcal{U}_\theta) \to \mathbb{K}[h] \) specializing to \( \lambda \) at \( h = 1 \). This homomorphism will also be denoted by \( \lambda \). So \( \mathcal{A}_h = \mathcal{U}_{\lambda,h} \) is the Rees algebra of \( \mathcal{U}_\lambda \). The underlying variety \( Y \) is the nilpotent cone \( \mathcal{N} \) of \( \mathfrak{g} \) and so contains finitely many symplectic leaves (=nilpotent orbits). Using the usual identification \( \mathfrak{g} = \mathfrak{sl}_n \), see [L4], Lemma A4. Corollary 3.17 from [ES] implies that there are finitely many prime ideals in \( \mathfrak{A}_{h,fin}(\mathcal{A}') \) and so \( \mathfrak{A}_{\mathcal{C}}(\mathcal{A}) \) also contains finitely many prime ideals.

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Proof. We will check that the intersection \( S \cap \overline{O}_1 \) is normal. Since \( S \cap \overline{O}_1 \) is \( \mathbb{K}^\times \)-stable and hence connected, the normality implies that \( S \cap \overline{O}_1 \) is irreducible. Being an open subvariety in \( S \cap \overline{O}_1 \), the variety \( S \cap O_1 \) is also irreducible.

Again, thanks to the contracting \( \mathbb{K}^\times \)-action it is enough to show that the completion \((S \cap \overline{O}_1)^\wedge_e \) at the point \( e \) is normal. Recall that the intersection \( S \cap O \) is transversal at \( e \). This implies that \( \overline{O}_1^\wedge_e \) decomposes into the direct product \( O^\wedge_e \times (S \cap \overline{O}_1)^\wedge_e \). According to Kraft and Procesi, [KP], the variety \( \overline{O}_1 \) is normal. Hence the formal scheme \( \overline{O}_1^\wedge_e \) is normal as well. The direct product decomposition now implies that \((S \cap \overline{O}_1)^\wedge_e \) is normal. \( \square \)

Remark 3.2. In fact, the techniques used below to prove Theorem 1.1 allow one to show that \( S \cap O_1 \) is irreducible for any (not necessarily nilpotent) orbit \( O_1 \).

3.2. Quantum level. Pick some nilpotent orbit \( \widetilde{O} \subset g \) whose closure contains \( O \), let \( \tilde{\chi} \in \widetilde{O} \cap S \), and let \( \widetilde{W} \) be the corresponding \( W \)-algebra. We claim that the inequalities

\[
(3.1) \quad |\Pr_{\tilde{\mathcal{O}} \cap S}(W_\lambda)| \leq |\Pr_{\tilde{\chi}}(\widetilde{W}_\lambda)|
\]

(for all possible \( \tilde{\mathcal{O}} \)) imply that \( \Pr_{\tilde{\mathcal{O}} \cap S}(W_\lambda) \) to \( \Pr_{\tilde{\chi}}(\widetilde{W}_\lambda) \) and is a bijection between the two sets.

Indeed, thanks to Proposition 2.2(3), the map \( \bullet^\dagger: \Pr_{\tilde{\chi}}(\widetilde{W}_\lambda) \to \Pr_{\tilde{\mathcal{O}}}(U_\lambda) \) is a bijection. On the other hand, the preimage of \( J \in \Pr_{\tilde{\mathcal{O}}}(U_\lambda) \) in \( \Pr(W_\lambda) \) contains at least one element from \( \Pr_{\tilde{\mathcal{O}} \cap S}(W_\lambda) \), assertion (2). Finally, by assertion (1), both sets \( \Pr(W_\lambda), \Pr(U_\lambda) \) are finite. So inequalities (3.1) imply the claim of the previous paragraph.

To prove the inequality \( |\Pr_{\tilde{\mathcal{O}} \cap S}(W_\lambda)| \leq |\Pr_{\tilde{\chi}}(\widetilde{W}_\lambda)| \) we will apply the general construction of Subsection 2.1 to \( Y = S \cap \mathcal{N}, \mathcal{A} = W_\lambda \) and the point \( \tilde{\chi} \in S \cap \widetilde{O} \).

Lemma 3.3. \( \mathcal{A}_h' \cong \widetilde{W}_h^\wedge, \) where \( \widetilde{W}_h \) is the homogenized version (=the Rees algebra) of the central reduction \( \mathcal{W}_h \) of \( \mathcal{W} \).

Proof. Set \( V_1 := T_{\tilde{\chi}}(S \cap \widetilde{O}) \) and let \( V_2 \) be the skew-orthogonal complement of \( V_1 \) in \( T_{\tilde{\chi}}\widetilde{O} \). We can form the corresponding completed homogenized Weyl algebras \( A_h(V_1)^\wedge, A_h(V_2)^\wedge \). Then we have

\[
(3.2) \quad U_h^\wedge = A_h(V_1 \oplus V_2)^\wedge \otimes_{\mathbb{K}[\hbar]} \widetilde{W}_h^\wedge,
\]

\[
(3.3) \quad W_h^\wedge = A_h(V_1)^\wedge \otimes_{\mathbb{K}[\hbar]} \mathcal{A}_h'.
\]

Applying the construction used in the proof of Proposition 2.1 to the point \( (1, \tilde{\chi}) \in G \times S \subset T^*G \) we see that \( U_h^\wedge \cong A_h(V_2)^\wedge \otimes_{\mathbb{K}[\hbar]} \mathcal{W}_h^\wedge. \) From here and the description of the embedding \( Z \to \mathcal{W} \) provided in Subsection 2.3 one can see that

\[
(3.4) \quad U_h^\wedge \cong A_h(V_2)^\wedge \otimes_{\mathbb{K}[\hbar]} \widetilde{W}_h^\wedge.
\]

Combining (3.3) and (3.4), we get

\[
(3.5) \quad U_h^\wedge \cong A_h(V_1 \oplus V_2)^\wedge \otimes_{\mathbb{K}[\hbar]} \mathcal{A}_h'.
\]

From Subsection 2.1 we see that there is a \( \mathbb{K}[[\hbar]] \)-linear isomorphism \( \widetilde{W}_h^\wedge \cong \mathcal{A}_h' \). \( \square \)

Now we have two derivations of \( \widetilde{W}_h \), the derivation \( \tilde{D} \) induced by the Kazhdan action defined for the nilpotent element \( \tilde{\chi} \), and the derivation \( D' \) coming from an isomorphism \( \widetilde{W}_h^\wedge \cong \mathcal{A}_h' \). Both satisfy \( \tilde{D} \hbar = D' \hbar = \hbar \). Consider the sets \( \Pr_{\text{fin},h}(\widetilde{W}_h^\wedge), \Pr_{\text{fin},h}(\widetilde{W}_h^\wedge) \) that
consist of all prime (=maximal) $h$-saturated ideals $\mathcal{T}'_h \subset \widetilde{W}^\Lambda_h$ such that $\widetilde{W}^\Lambda_h / \mathcal{T}'_h$ is of finite rank over $\mathbb{K}[[h]]$ and such that $\mathcal{T}'_h$ is, respectively, $\widetilde{D}$- and $D'$-stable. The set $\operatorname{Pr}_{f_{\text{fin}},h}(\mathcal{W}_h^\Lambda)$ is in natural bijection with $\operatorname{Pr}_{f_{\text{fin}}}(\mathcal{W}_h)$. On the other hand, by the results recalled in Subsection 2.4 the cardinality of $\operatorname{Pr}_{f_{\text{fin}},h}(\mathcal{W}_h^\Lambda)$ is bigger than or equal to that of $\operatorname{Pr}_{S\cap \mathfrak{G}}(\mathcal{W})$. So it remains to show that the two sets coincide.

It is enough to check that any derivation $d$ of $\mathcal{A}'_h = \widetilde{W}^\Lambda_h$ with $d(h) = h$ fixes any maximal $h$-saturated ideal of finite corank. Consider the quotient $(\mathcal{A}'_h)^{(n)}$ of $\mathcal{A}'_h$ by the ideal generated by the elements $s_{2n}(x_1, \ldots, x_{2n}) = \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma)x_{\sigma(1)} \ldots x_{\sigma(2n)}, x_1, \ldots, x_{2n} \in \mathcal{A}'_h$. This ideal is clearly $d$-stable. Also consider the analogous quotient $\mathcal{A}_h^{(n)}$ of $\mathcal{A}_h := \mathcal{W}_h$. It follows from Section 7.2 of [L3] that $\mathcal{A}_h^{(n)}$ has finite rank over $\mathbb{K}[[h]]$. But $(\mathcal{A}_h^{(n)})$ is the completion of $\mathcal{A}_h^{(n)}$ at $\tilde{\chi}$. So $(\mathcal{A}_h^{(n)})$ has finite rank over $\mathbb{K}[[h]]$. Therefore the localization $(\mathcal{A}_h^{(n)})[h^{-1}]$ is a finite dimensional $\mathbb{K}[h^{-1}, h]$-algebra.

Maximal $h$-saturated ideals of finite corank in $\mathcal{A}_h'$ are in a natural one-to-one correspondence with maximal ideals of finite codimension in the $\mathbb{K}[h^{-1}, h]$-algebra $\mathcal{A}_h'[h^{-1}]$. Clearly $\mathcal{A}_h'[h^{-1}]^{(n)} = (\mathcal{A}_h'[h^{-1}])[h^{-1}]$. Thanks to the Amitsur-Levitzki theorem, every maximal ideal of finite codimension in $\mathcal{A}_h'[h^{-1}]$ is the preimage of an ideal in $\mathcal{A}_h'[h^{-1}]^{(n)}$ for some $n$. Of course, $d$ induces a $\mathbb{K}[h^{-1}, h]$-linear derivation of $\mathcal{A}_h'[h^{-1}]^{(n)}$. Now it remains to use a fact that a maximal ideal in a finite dimensional algebra is stable under any derivation of this algebra. For reader’s convenience we will provide a proof here.

Let $A$ be a finite dimensional algebra over some field $K$ and let $\mathfrak{m}$ be its maximal ideal. Replacing $A$ with $A/\bigcap_{i=1}^{\infty} \mathfrak{m}^i$, we may assume that $\mathfrak{m}$ is a nilpotent ideal and hence the radical of $A$. To complete the proof apply Lemma 3.3.3 from [D].

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