Heavy Baryons in SU(2) × SU(6)

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Abstract

The spectrum of baryons containing heavy quarks of one flavor is described in terms of representations of the group SU(2) × SU(6), where the two factor groups refer to spin rotations of the heavy quarks and spin-flavor rotations of the light quarks, respectively. This symmetry has a natural interpretation in the heavy quark limit. We exhibit the decomposition of baryon mass operators under this symmetry and compare to experimental results. We discuss the relation of this analysis to that of large-$N_c$ QCD as well as four-flavor SU(8), and indicate the generalization of this work to other properties of heavy baryons.
I. INTRODUCTION

Recent experiments continue rapidly to expand our knowledge of the properties of heavy-quark hadrons. For example, the past few years have seen evidence for the first observations of numerous ground-state charmed and bottom baryons, including both charmed and bottom cascades and bottom Σ's. One important task of current theoretical efforts is to perform a critical analysis of whether we understand the information this new data is providing. A natural starting point is to develop an understanding of the mass spectra of heavy-quark hadrons. In this paper we propose a symmetry group for the heavy baryons and explore its mathematical and phenomenological consequences.

The symmetry paradigm we adopt is the group SU(2) × SU(6), where the first factor refers to the spin of the heavy quark \( Q = c \) or \( b \), and the second factor is the spin-flavor symmetry of the three light quarks \( u, d, \) and \( s \). By organizing the representations (hereafter \( \text{reps} \)) of the symmetry group in this way, we recognize the fundamental phenomenological difference between heavy and light quarks. In particular, we appear to inhabit a world in which one may accurately calculate physical quantities by performing expansions about the massless quark limit for light quarks (chiral perturbation theory) and about the infinite mass limit for heavy quarks (heavy quark effective theory). The symmetry SU(6) for baryons has a long and illustrious history, and appears to accurately model reality in describing features like the closeness of the octet and decuplet of light baryons, the magnetic moment ratio \( \mu_p/\mu_n \approx -3/2 \), and the axial current coefficient ratio \( F/D \approx 2/3 \). The decomposition of light baryon bilinear operators in SU(6), analogous to the analysis performed here for heavy-quark baryons, appears in Ref. [1]. The use of SU(6) × O(3) to describe just the light quarks (including orbital angular momentum) in heavy baryons has recently been advocated by Körner [2] to increase predictive power beyond heavy quark effective theory.

The ground-state charmed baryons (and by inference, the bottom baryons) appear to fall into multiplets determined by the SU(4) flavor symmetry of the \( u, d, \) and \( c \) quarks [3]. How can this be when the charm quark is so much heavier than the other quarks? In fact,
the same spectrum arises from the much milder assumption of approximate SU(3) symmetry for light quarks and the assertion that the color wavefunction for each baryon is completely antisymmetric in all of the quark indices. Then the (spin × flavor × space) wavefunction must be completely symmetric under exchange of quark indices; since the ground-state baryons are assumed to have no internal orbital angular momentum, the space wavefunction is symmetric. The two possible spins from three quarks are 1/2 (mixed symmetry) and 3/2 (completely symmetric), and working out the corresponding SU(3) reps for 3, 2, 1, and 0 light quarks that leave the product of spin and flavor wavefunctions symmetric gives the multiplet structure indicated in Ref. [3]. Unlike for SU(4) or its corresponding spin-flavor group SU(8), the levels of the multiplets, each of which has a different number of heavy quarks, belong to different reps in this construction and thus are a priori unrelated.

In principle, any symmetry may be used to describe the heavy baryons as long as it contains operators in all allowed reps that contribute to physical quantities; the relation between two such symmetries is simply a basis transformation. For example, Jenkins [4] considers the heavy baryons in the large-\(N_c\) QCD expansion. It is only in assuming that operators in certain reps give smaller contributions than others to physical quantities, that one obtains relations between observables, and since different symmetries organize the same space of operators in different manners, distinct predictions arise.

In Sec. 2 we review current experimental knowledge of heavy baryon masses. Section 3 presents the tensor formalism for baryons in SU(2) × SU(6) and explains how the relevant Clebsch–Gordan coefficients are obtained. In Sec. 4 we describe the phenomenological application of the symmetry and compare the results to experiment. Section 5 compares the consequences of this symmetry to spin-flavor SU(8), large-\(N_c\) QCD, and other recent work. Section 6 briefly discusses other applications of the formalism and concludes. The Clebsch–Gordan coefficients and mass operator decompositions appear in the Appendix.
II. STATUS OF EXPERIMENTAL RESULTS

Let us briefly review the state of experimental measurements, both to indicate the level of completeness of the multiplets and to fix notation for degenerate states. See [3] for a geometrical picture of the weight space. The $Q = 0$ levels of the multiplets are the well-known SU(3) 8 for spin-1/2 and 10 for spin-3/2, whereas none of the $Q = 2$ or $Q = 3$ baryons have yet been observed. Signals for almost all of the $Q = 1$ charmed baryons have been seen, excepting the $\Omega^*_c$ and some of the distinct isospin states. For the $Q = 1$ bottom baryons, experimental uncertainties on $\Lambda_b$ mass measurements are rapidly decreasing, preliminary values for the $\Sigma_b$ and $\Sigma^*_b$ masses exist, and evidence for the $\Xi_b$ baryon has been presented [5], although mass measurements have not yet appeared. The data is summarized in Table I.

Note that two distinct $\Xi_Q$ baryons occupy the same sites in the multiplets, and thus mixing terms between them occur. We use the notation recently favored by experimental groups, that $\Xi'_Q$ and $\Xi_Q$ respectively indicate the sextet and antitriplet states of SU(3). In sextets (antitriplets), the two light quarks are symmetric (antisymmetric) under exchange of indices, and thus are in a relative spin-1 (spin-0) state. One expects $\Xi'_Q > \Xi_Q$ from a simple quark model-inspired analysis of the spin-spin coupling in each baryon, in which aligned spins repel and anti-aligned spins attract. Another notation for these particles [14] is to use a subscript 1,2 to indicate the light quarks in a $\bar{3}$ (6).

III. TENSOR ANALYSIS IN SU(2) $\times$ SU(6)

The analysis presented here of SU(2) $\times$ SU(6) group theory by means of tensors closely parallels that in Ref. [I] (hereafter called (I)). We begin by defining appropriately symmetrized tensors to keep track of quark flavor and spin indices. In essence, this construction encapsulates all of the group-theoretical information and thus provides a route for obtaining all the relevant Clebsch–Gordan coefficients. Although the manipulations that follow apply to bottom as well as charmed baryons, we present the results for the latter; the correspond-
ing results for the $b$, $bb$, and $bbb$-baryons are obtained by subtracting one, two, and three units from the charge superscript of $c$, $cc$, and $ccc$-baryons, respectively.

We begin by noting that light quarks in SU(6) transform according to the fundamental 6 rep. For ground-state baryons with one heavy quark, the two light quarks are completely symmetric with respect to exchange of (spin × flavor) indices, owing to the antisymmetry of the baryon wavefunction under color. The light diquark is then in the symmetric SU(6) rep from $(6 \otimes 6)$, which is the 21. For ground-state baryons with two heavy quarks, the light quark is necessarily in a 6. The analogous wavefunction for the $Q = 3$ baryon $\Omega_{ccc}^{+++}$ is trivial. In (I) we considered baryons with no heavy quarks, for which the completely symmetric SU(6) rep is the 56.

The tensor completely symmetric under the exchange of paired spin and flavor indices for the light diquark system (a 21 rep of SU(6)), which leads to the $Q = 1$ baryon tensor, may be represented by

$$ B^{ai,bj} \equiv \chi^{ij} B^{ab} + \frac{1}{2} \epsilon^{ij}_{\ abc} B_{c}, $$

where $\epsilon^{ij}$ and $\epsilon^{abc}$ are Levi-Civita tensors, $\chi^{ij}$ represents the symmetric spin-one tensor for the light diquark,

$$ \chi^{ij} = \begin{pmatrix} |1, +1 > & \frac{1}{\sqrt{2}} |1, 0 > \\ \frac{1}{\sqrt{2}} |1, 0 > & |1, -1 > \end{pmatrix}, $$

and the SU(3) baryon tensors $B$ are constructed as follows: The 6 is assigned the entries of the symmetric tensor $B^{ab}$ by

$$ B^{11} = \Sigma_{c}^{++}, \ B^{12} = \frac{1}{\sqrt{2}} \Sigma_{c}^{+}, \ B^{22} = \Sigma_{c}^{0}, $$
$$ B^{13} = \frac{1}{\sqrt{2}} \Xi_{c}^{+++}, \ B^{23} = \frac{1}{\sqrt{2}} \Xi_{c}^{0}, \ B^{33} = \Omega_{c}^{0}, $$

and the 3 is assigned the entries of the tensor $B_{c}$, in a particular phase convention [13], according to

$$ B_{1} = \Xi_{c}^{0}, \ B_{2} = -\Xi_{c}^{+}, \ B_{3} = \Lambda_{c}^{+}. $$
Because these tensors describe diquarks rather than the full heavy baryon, the identity of the baryon only becomes fixed when the spin of the heavy quark is included. In particular, the entries of the $6$ should be taken to represent either spin-1/2 or spin-3/2 baryons, depending upon the spin of the heavy quark. Then the full baryon tensor wavefunction, including the heavy quark spinor $\chi^k$ with $\chi^{1,2} = \uparrow, \downarrow$, is

$$B_{a_i,bj;}^k \equiv B_{a_i,bj}^k \chi^k.$$

(3.5)

Lastly, the factor $1/2$ in Eq. (3.1) is determined by the singlet normalization that

$$\overline{B}_{a_i,bj;} B_{a_i,bj;}$$

produces all bilinears for each baryon field in a given spin state appearing only with its conjugate, and each with coefficient unity.

The tensor rep for baryons consisting of two heavy quarks and one light quark is much simpler. The light quark piece ($6$ of SU(6)) is just

$$B_{a}^i = B^a \chi^i,$$

(3.6)

where $\chi^i$ is its spin, with $\chi^{1,2} = \uparrow, \downarrow$, and the $3$ of SU(3) is assigned the entries of the tensor $B^a$ according to

$$B^1 = \Xi^{++}_{cc}, \ B^2 = \Xi^{+}_{cc}, \ B^3 = \Omega^{+}_{cc}.$$

(3.7)

As before, these components may refer to either spin-1/2 or spin-3/2 baryons, depending upon the spin state of the two heavy quarks. The full baryon tensor is then

$$B_{a_i;jk} = B_{a_i}^j \chi^{jk},$$

(3.8)

where $\chi^{jk}$ is the (symmetric) spin tensor for the two heavy quarks and has the same form as Eq. (3.2).

The SU(2) × SU(6) decomposition using these tensors now follows from the same methods as in (I): Baryon mass terms are bilinears with $J_3 = I_3 = Y = 0$ and total spin $J = 0$, so the SU(6) $1$ and $35$ combinations are obtained by computing the bilinear expressions

$$\overline{B}_{a_i,bj;} \overline{T}_{bj}^c T_{c^l} J_k^m B_{a_i,c^l;m},$$

(3.9)
for $Q = 1$, and

$$B^{a;i;k} T_{a}^{b \ell} J_{k}^{m} B_{b \ell;jm},$$

(3.10)

for $Q = 2$, where $T$ are light-quark spin-flavor generators and $J$ are heavy-quark spin generators. It is enough to compute explicitly the tensor for $T \otimes J = 1 \otimes 1$, $T^{3} \otimes 1$, $T^{8} \otimes 1$, $J_{3} \otimes J_{3}$, $(T^{3} J_{3}) \otimes j_{3}$, and $(T^{8} J_{3}) \otimes j_{3}$, because these give the SU(6) 1 and 35, and the 405 for $Q = 1$ may be found by orthogonality.

In the Appendix we begin by computing the combinations of bilinears (“chiral coefficients”) transforming under particular SU(3) and isospin reps, and combine them into SU(6) chiral coefficients by means of the tensor methods just outlined. The only complication is that one must take care to project out the appropriate components of heavy quark spin to obtain baryons with the desired total spin. Chiral coefficients not involving a spin flip of the heavy quark (heavy and light quark bilinears each with $j = 0$) are labeled $X$; those with a spin flip (heavy and light quark bilinears each with $j = 1$), and therefore suppressed in the infinite quark mass limit, are labeled $Y$. Similarly, the mass combinations associated with each chiral coefficient are labeled with the corresponding lower-case letter $x$ or $y$.

There are 18 distinct mass combinations for the $Q = 1$ baryons and 6 for $Q = 2$, and these numbers are borne out by the mass combinations listed in Eqs. (A16–A17). The former number is obtained by noticing that, in addition to two sextets and one antitriplet, there is a mixing parameter between each member of the $\bar{3}$ and the state in the spin-1/2 6 with the same weight.

**IV. PHENOMENOLOGICAL CONSEQUENCES**

In order to estimate the expected sizes of mass combination coefficients $x$ and $y$ listed in the Appendix, one must make some assumptions regarding the pattern of symmetry breaking; this analysis is similar to that in Ref. [10]. First note that all combinations in Eqs. (A16–A17) have zero net baryon and charm number except for the overall singlet term.
$x_{1,0}^1$, and therefore vanish in the SU(2) × SU(6) limit. Then the amount by which each combination deviates from zero is determined by the finding its overall scale and factors associated with symmetry breaking. To accomplish this, the combination is set to zero in the form $lhs = rhs$, where $lhs$ and $rhs$ are combinations of baryon masses with positive coefficients. Dividing by one-half of the sum of the numerical coefficients on either side ($\equiv k/2$) gives a scale-independent result, and the magnitude of the combination is set by the typical uncharmed baryon mass $\Lambda_\chi \approx 1$ GeV (uncharmed because the full combination has net zero charm number). Before including explicit symmetry breaking, the combination naively satisfies $|x_N^{RI}| \lesssim k/2$, although in this expression we neglect an unknown coefficient of order unity that may make precise value of the combination correspondingly larger or smaller than this estimate. The factor of 1/2 places the uncertainty in the combination symmetrically about zero.

For the light diquark, symmetry breaking appears in the adjoint 35 rep of SU(6) in the form of relative spin flips, SU(3) breaking, and $I = 1$ isospin breaking, the latter two being respectively parametrized by $\epsilon$ and $\epsilon'$. $I = 2$ isospin breaking, parametrized by $\epsilon''$, first appears in the 405, with its leading contribution arising from electromagnetic effects. Since spin operators may only appear in pairs in mass bilinears, light-quark spin flips first appear in the 405 and are parametrized by $\delta$, and spin flips involving the heavy quark are parametrized by a factor $\theta \equiv \Lambda_{QCD}/m_c$. For example, the combination $x_{405}^{8,0}$ has an anticipated size of $12\Lambda_\chi \delta \epsilon$, with the scale from the arguments of the previous paragraph, and two factors of SU(6) breaking because the 405 requires a product of two 35’s, which must include SU(3) breaking because the SU(3) content is $I = 0$ octet. The anticipated magnitudes of the combinations are listed in Table II.

The set of baryon mass differences is thus reduced to the scale $\Lambda_\chi$ and the dimensionless parameters $\delta$, $\epsilon$, $\theta$, $\epsilon'$, and $\epsilon''$. We estimate $\Lambda_\chi \approx 1$ GeV, $\delta \approx \epsilon \approx 0.3$, $\theta \approx 0.2$, $\epsilon' \approx 0.005$, and $\epsilon'' \approx 0.001$. This value for $\delta$ comes from the observation that the spin-flip operator explains the fractional difference of light octet and decuplet baryons, $\epsilon$ and $\epsilon'$ respectively

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arise from $m_s/\Lambda_{\chi}$ and $(m_u - m_d)/\Lambda_{\chi}$ effects, and $\epsilon''$ arises from noting that $I = 2$ effects occur in electromagnetic terms of $O(\alpha \Lambda_{\chi}/4\pi)$. These values should be taken as indicative rather than definitive, but the basic pattern should remain.

From Table [I] it is clear that one should focus upon the combinations associated with the largest reps, where the estimated magnitudes are smallest. Hyperfine ($y$) combinations are also highly suppressed. One caveat is that when $x$’s or $y$’s are combined, the numerical coefficient and suppressions like those in Table [I] must be recomputed for the particular combination. For the moment, let us assume that mass mixings are negligible, so that the observed states are SU(3) eigenstates, although we will see that this may not be true for $\Xi'_c$ and $\Xi_c$.

We begin by considering a combination to which Savage [14] computed SU(3) chiral loop corrections, $y_{405}^{27.0}$. Taking the experimental numbers in Table [I] at face value, we predict

$$\Omega_c^* = 2790 \pm 31(\pm 36) \text{ MeV}, \hspace{2cm} (4.1)$$

where the first error is from experimental uncertainties and the second follows from analysis as in Table [I]. As for the poorly-known $\Xi'_c$ mass, we may either check the measured value as it appears in $\frac{1}{5}(y_{35}^{8.0} + 2y_{405}^{27.0}) \lesssim \Lambda_{\chi} \theta \epsilon \approx 60 \text{ MeV}$:

$$(\Sigma_c^* - \Sigma_c) - (\Xi_c^* - \Xi'_c) = -7 \pm 17 \text{ MeV}, \hspace{2cm} (4.2)$$

which is certainly consistent with our estimates, or use it to predict the $\Xi'_c$ mass:

$$\Xi'_c = 2567 \pm 7(\pm 60) \text{ MeV}. \hspace{2cm} (4.3)$$

From this example we see that, as one is forced to employ less-suppressed combinations (here $y_{35}^{8.0}$), the theoretical uncertainty becomes too large to make very useful predictions for individual masses. Particular models can do much better, of course, because in such cases the dynamical assumptions are much more restrictive. For the remainder of the $I = 0$ combinations, we seek to demonstrate that the values associated with each combination in Table [I] are reasonable by eliminating from each the unknown $\Omega_c^*$ and poorly-known $\Xi'_c$.
masses. These results are presented in Table III. The first line in the table exhibits the size of the overall singlet of the $C = 1$ multiplet, $M_c = 2558 \pm 2 \text{(expt.)} + 3 \text{(theor.)}$ MeV, whereas comparing the second and fifth columns in the succeeding lines indicates that the symmetry with our choice of expansion parameters is reliable, although the experimental smallness of the final $I = 0$ combination is notable.

The known $I = 1$ mass combinations are $\Sigma_{c1} \equiv (\Sigma_{c}^{++} - \Sigma_{c}^{0})$, $\Xi_{c1} \equiv (\Xi_{c}^{+} - \Xi_{c}^{0})$, and $\Xi_{c1}^* \equiv (\Xi_{c}^{*+} - \Xi_{c}^{0})$, while $\Sigma_{c1}^* \equiv (\Sigma_{c}^{*++} - \Sigma_{c}^{*0})$ and $\Xi_{c1}' \equiv (\Xi_{c}^{*'++} - \Xi_{c}^{*0})$ are unknown. The predictions for the latter are

$$\Sigma_{c1}^* = \Sigma_{c1} + \frac{1}{5}(2y_{35}^{8,1} + y_{405}^{27,1}) \quad = 0.7 \mp 0.4 \pm (1) \text{ MeV}, \quad (4.4)$$

$$\Xi_{c1}' = \Xi_{c1} - \frac{1}{5}(y_{35}^{8,1} - 2y_{405}^{27,1}) \quad = -5.2 \mp 2.2 \pm (1) \text{ MeV}, \quad (4.5)$$

where in both cases the scale of the theoretical uncertainty is $\Lambda_x \theta \epsilon' \approx 1$ MeV. The results from eliminating $\Sigma_{c1}$ and $\Xi_{c1}'$ from the other combinations appear in Table III: the experimental smallness of the first $I = 1$ combination in the table and the largeness of the second compared to estimates may be related to the problem of $\Xi'_{c} - \Xi_{c}$ mixing, since both contain $\Xi_{c1}$, whereas the combination of the two eliminating $\Xi_{c1}$ is of expected size.

The analysis for $I = 2$ mass combinations predicts

$$(\Sigma_{c}^{*++} - 2\Sigma_{c}^{*+} + \Sigma_{c}^{0}) = (\Sigma_{c}^{++} - 2\Sigma_{c}^{+} + \Sigma_{c}^{0}) + y_{405}^{27,2} \quad = -2.1 \pm 1.3 \pm (0.4) \text{ MeV}, \quad (4.6)$$

while eliminating the $\Sigma_{c}^*$ combination gives the final line in Table III.

Finally, we consider $\Sigma_{c}^+ - \Lambda_{c}^+$ and $\Xi'_{c} - \Xi$ mixings. Performing the diagonalization of a $2 \times 2$ mass matrix with diagonal entries $\mu_{1,2}$ and mixing $\nu$ gives the eigenvalues

$$\mu_{1,2}^{\text{phys}} = \frac{\mu_1 + \mu_2}{2} \pm \sqrt{\left(\frac{\mu_1 - \mu_2}{2}\right)^2 + \nu^2}. \quad (4.7)$$

When the mixing $\nu$ is pure $I = 1$, it is not only suppressed by $\epsilon'$, but much more so through the Pythagorean sum. Thus the mixing $\gamma$ as defined in the Appendix has very little effect on
the masses of $\Sigma_c^+$ or $\Lambda_c^+$, as was the case for $\beta$ with $\Sigma^0$ and $\Lambda$ in (I). On the other hand, $\Xi'_c - \Xi_c$ has both $I = 1$ and $I = 0$ mixings, proportional to $(\delta_+ \pm \delta_0)$. The $I = 0$ mixing parameters may be as large as 60 MeV, so this mixing may contribute tens of MeV to the splitting between $\Xi'_c$ and $\Xi_c$. In fact, since one-half the observed splitting between the physical $\Xi'_c$ and $\Xi_c$ is about 45 MeV, it is quite possible that the pure $6 \; \Xi'_c$ and pure $3 \; \Xi_c$ could be degenerate in mass. Certainly a more detailed analysis of $\Xi_c$ masses must either take this mixing into account or else explain why it is suppressed below its natural size.

V. OTHER SCHEMES FOR HEAVY BARYON MASSES

A. Relationship to SU(8)

The discovery of charm in 1974 prompted much analysis of hadronic properties in terms of an assumed four-flavor symmetry SU(4) among the quark flavors $(u, d, s, c)$ then known. SU(4) was naturally extended to the spin-flavor symmetry SU(8), in analogy with the spin-flavor symmetry SU(6) among the three light quark flavors studied extensively in the 1960s. However, the four-flavor symmetries are badly broken because the charm quark is not only much heavier than $u, d, s$, but also the QCD scale $\Lambda_{QCD} \lesssim 1 \text{ GeV}$. Thus one should not take SU(8) seriously as an accurate description of physical quantities, but as a mathematical question it is nonetheless interesting to compare its predictions to those of its subgroup SU(2) x SU(6) considered in this work, because the predictions of SU(8) analysis exist in the literature.

In analogy to the 56 rep of SU(6), one considers the baryons of SU(8) to fill the completely symmetric rep with Dynkin symbol $(3,0,0,0,0,0,0) = 120$. We now recapitulate the arguments in (I) for SU(6). All group-theoretical relations between static baryon quantities (masses, magnetic moments, etc.) are obtained by decomposing the bilinear product

$$120 \otimes 120 = 1 \oplus 63 \oplus 1232 \oplus 13104$$

$$= (0, 0, 0, 0, 0, 0, 0) \oplus (1, 0, 0, 0, 0, 0, 1) \oplus (2, 0, 0, 0, 0, 1, 0) \oplus$$
One now argues that symmetry-breaking effects among the members of the $120$, due to quark masses, charges, or relative spin flips, occur in the adjoint rep $63$. Since it possesses one fundamental and one antifundamental quark index, the adjoint is called a one-body operator. But now note that the most general two-body operator transforms under the reps in $63 \otimes 63$, of which the largest is the $1232$; thus the largest rep in $\mathbf{120} \otimes \mathbf{120}$, the $\mathbf{13104}$, does not appear when three-body operators are neglected. The neglect of three-body operators was precisely the assumption made by Franklin [17] when deriving charmed baryon mass relations, so his results correspond in group-theoretical terms to the neglect of mass operators transforming under the $\mathbf{13104}$ rep of SU(8).

To count the number of mass relations provided by this Ansatz, one decomposes the $\mathbf{13104}$ into spin-flavor SU(2) $\times$ SU(4) reps $(J, R)$ and counts the number of bilinear states among these with $J = 0$ ($J = 1$ for magnetic moment relations, and so on), $I_3 = 0$, $Y = 0$, and heavy quark number $Q = 0$. This gives the mass operators in the $\mathbf{13104}$, which by the Ansatz have vanishing matrix elements and thus produce relations among the baryon masses. It turns out that the decomposition of the $\mathbf{13104}$ with $J = 0$ gives 32 such states, of which 4 are removed by the imposition of time reversal invariance (see Appendix). Ten of these are the three-body SU(6) $\mathbf{2695}$ relations exhibited in (I), so there must be 18 relations involving charmed baryons. Ref. [17] exhibits 14 such relations; when including the 6-3 mixing terms $\gamma$ and $\delta_{+0}$ defined in the Appendix, we find three more relations by applying his method to the $\mathbf{2695}$ relation involving the $\Lambda-\Sigma^0$ mixing $\beta$ from (I),

\begin{align}
4\sqrt{3}\gamma &= (\Sigma_c^{++} - \Sigma_c^0) + (\Xi_c^{++} - \Xi_c^+ - \Xi_c^-) - (\Sigma_c^{++} - \Sigma_c^0) + (\Xi_c^{++} - \Xi_c^-), \\
4\sqrt{3}(\delta_0 - \delta_+) &= (\Sigma_c^{++} - \Sigma_c^0) + (\Xi_c^{++} - \Xi_c^+) - (\Sigma_c^{++} - \Sigma_c^0) + (\Xi_c^{++} - \Xi_c^+), \\
4\sqrt{3}(\delta_0 + \delta_+) &= 2(\Omega_c - \Omega_c^*) + 2(\Omega_{cc} - \Omega_{cc}^*) + (\Sigma_c^{++} + \Sigma_c^{*+}) - (\Sigma_c^0 + \Sigma_c^{++}) \\
&+ (\Xi_c^{++} + \Xi_c^{*+}) - (\Xi_c^0 + \Xi_c^{++}).
\end{align}

An 18th independent relation derivable from his method but not appearing in [17] is taken
to be

$$(\Sigma^{*+} - \Sigma^{*-}) - 2(\Xi^{*0} - \Xi^{*+}) = (\Sigma_c^{*++} - \Sigma_c^{*0}) - 2(\Xi_c^{*+} - \Xi_c^{*0}). \quad (5.3)$$

The interesting question is how many of these relations turn out to be supported in SU(2) × SU(6). Recall that SU(2) × SU(6) decouples sectors with different values of $Q$, and so we must look for SU(8) relations in which only baryons with the same number of heavy quarks occur. Direct manipulation shows that the only three-body SU(8) mass relations with a single value of $Q$ are: with $Q = 0$, the ten listed in (I) arising from SU(6) analysis, and with $Q = 1$, using the notation in Eq. (A16), the five relations

$$
\sqrt{3}y_{405}^{8,0} = -\frac{2}{5}y_{35}^{8,0}, \quad y_{405}^{27,0} = 0, \\
\frac{1}{\sqrt{3}}y_{405}^{8,1} = \frac{1}{5}y_{35}^{8,1}, \quad y_{405}^{27,1} = 0, \\
y_{405}^{27,2} = 0. \quad (5.4)
$$

None of the SU(8) relations can be written solely in terms of $Q = 2$ or $Q = 3$ baryons. Note that all of the $Q = 1$ SU(8) relations involve hyperfine splittings or mixing terms.

B. Relationship to Large-$N_c$

The large-$N_c$ QCD expansion for baryons possesses an approximate contracted spin-flavor symmetry [18]; for $N_f$ light flavors, this symmetry is very similar, although not identical, to SU(2$N_f$). The analysis of the light baryon masses in large-$N_c$ was performed in Ref. [16].

The physical baryons known to experiment are taken to occupy small corners of the large baryon reps allowed in large-$N_c$, where $J$, $I$, and $Y$ are all $O(1)$, not $O(N_c)$, and this Ansatz leads to substantial predictive power. One proceeds by writing down all allowed 0-, 1-, 2-, and 3-body spin, flavor, and spin-flavor operators between the quarks, making use of identities [19] that reduce their number to an easily manageable set spanning the physical baryon masses. In general, the higher-body operators tend to have more $N_c$ suppressions than lower-body operators. Some of the operators are accompanied by explicit factors of
1/$N_c$, whereas others produce factors of $N_c$ when acting upon the physical baryons. One may also include explicit factors of SU(3) and isospin breaking.

The case of heavy baryons was recently considered by Jenkins [4]; the main difference from the light baryon case is that one must also include the heavy quark spin operator in the analysis. Note the similarities to this study: In both cases, the basic one-body operators are spin, flavor, and spin-flavor operators, and a separate heavy-quark spin operator; and more suppressed contributions occur with higher-body operators. The two methods thus lead to very similar results. For example, pure 2-body light-quark operators for $Q = 1$ baryons in large-$N_c$ transform the same way as operators in the SU(6) 405, so in both analyses the combinations $y_{405}^{27,1}$ in Eq. (A16) are highly suppressed. A major theoretical difference is that the suppression of light-quark spin-flips in this work is suppressed by the phenomenological parameter $\delta$, whereas large-$N_c$ includes an explicit factor of 1/$N_c$ with each light-quark spin $J^i$, so this suppression is more natural in large-$N_c$.

C. Comparison to Other Analyses

Here we focus briefly on two recent works on understanding the heavy baryon spectrum. First, Zalewska and Zalewski [20] propose a “simple-minded” pattern for heavy baryon masses based on the following three rules: i) equal spacing between sextet isomultiplets, ii) equal spacing between corresponding spin-1/2 baryons containing $c$ and $b$ quarks, and iii) hyperfine splittings inversely proportional to the heavy quark mass. In terms of the mass combinations presented here, these rules correspond to i) $x_N^{27,0} = y_N^{27,0} = 0$, because the SU(3) singlets do not split masses and octets alone produce equal-spacing, and $y_{55}^{8,0} = 0$, because the equal-spacing coefficients cancel; ii) $x_N^{R,0}(c) = x_N^{R,0}(b)$ for $R \neq 1$, for which the net coefficients of spin-1/2 and spin-3/2 baryons are separately zero, as well as a nontrivial relation

\[
(x_{405}^{1,0}(b) - x_{405}^{1,0}(c)) = +2(y_{35}^{1,0}(b) - y_{35}^{1,0}(c)).
\]  

(5.5)
Finally, iii) corresponds to \( y_N^{RJ}(c)/y_N^{RJ}(b) = m_b/m_c \). To obtain direct relations between \( b \) and \( c \) mass combinations here would require the expansion to a symmetry group including heavy flavor, such as \( \text{SU}(2) \times \text{SU}(2) \times \text{SU}(6) \), or better still, \( \text{SU}(4) \times \text{SU}(6) \), which incorporates the full heavy quark spin-flavor symmetry.

The third rule, which arises from heavy-quark flavor symmetry, is called into question by recent data (see Table I) for the ratio \( (\Sigma^*_b - \Sigma_b)/(\Sigma^*_c - \Sigma_c) \approx 0.73 \pm 0.13 \), which is rather different from the expected \( m_c/m_b \approx 1/3 \). Based on this observation and the purported failure of an equal-spacing relation

\[
\Sigma_c - \Lambda_c - \Xi^*_c + \Xi_c = 0,
\]

by as much as 80 MeV (the bar indicates a spin average over corresponding spin-1/2 and spin-3/2 states), Falk [21] proposes a new identification of the heavy baryons in which the experimentally-observed states identified as \( \Sigma_Q \) and \( \Sigma^*_Q \) are actually \( \Sigma^*_Q \) and the orbital excitation \( \Sigma^*_{Q(0)} \), while the true \( \Sigma_Q \) decays radiatively and has not yet been observed. While the magnitude of the hyperfine ratio might pose difficulties for heavy quark theory if it persists, the problem of Eq. (5.6) is less troubling. Its magnitude is given by

\[
\frac{1}{15} (x_{405}^{8.0} + 2x_{405}^{27.0}) \Lambda_\chi \delta \epsilon \approx 90 \text{ MeV},
\]

which by our estimates can easily accommodate the experimental value. Moreover, recall that the identification of the physical \( \Xi_Q \) and \( \Xi'_Q \) with pure \( 3 \) and \( 6 \) states is obfuscated by a mixing parameter that \textit{a priori} leads to mass shifts as large as tens of MeV.

\[\text{VI. CONCLUSIONS}\]

The symmetry group \( \text{SU}(2) \times \text{SU}(6) \) provides a natural organization for calculating quantities relevant to the heavy baryons. By construction, it is designed to allow both a heavy-quark expansion and a light-quark spin-flavor expansion. In this paper we exhibited the group-theoretical features of the symmetry by explicitly constructing the tensor
rep of ground-state baryons, and applied this to the mass spectrum. We found that the observed particles tend to fit well into multiplets of this symmetry when natural values for symmetry-breaking parameters are assumed, although the possibility of large $\Xi'_c-\Xi_c$ mixings may complicate the spectroscopy.

Other computations, for example comparing decays of different heavy baryons, may be performed using this symmetry. By projecting out bilinears of total spin $J = 1, 2, \text{or} 3$, one may examine the structure of magnetic dipole, electric quadrupole, or magnetic octupole moments of the baryons, but it is questionable whether any of these quantities will be measured in the near future, owing to the short lifetimes of heavy baryons. Nevertheless, channeling of short-lived particles through bent crystals, in which very large effective magnetic fields are possible, may make such measurements feasible\(^2\).\(^2\)

Another direction involves a similar tensor analysis for the orbitally-excited baryons, beginning with the observed $\Lambda^*_{c1}$ and $\Lambda^*_{c2}$. In this case, the tensor (3.1) is modified to include $\ell = 1$ by the additional product of a tensor transforming like the spherical harmonic $Y_1^m$.

Furthermore, generalization of the symmetry group $\text{SU}(2) \times \text{SU}(6)$ to $\text{SU}(4) \times \text{SU}(6)$, where the $\text{SU}(4)$ is the full spin-flavor group of $b$ and $c$ quarks in heavy quark effective theory, may yield some interesting results. The phenomenological analysis then is supplemented by the additional expansion parameter $m_c/m_b$. In particular, one can test whether the experimental ratio of $\Sigma_b$ to $\Sigma_c$ hyperfine splittings remains much larger than $m_c/m_b$ when corrections of natural size are taken into account, analogous to the analysis in Sec. 4.

In its own right, this work provides a framework for comparing mass calculations through potential models or lattice simulations to what is expected based on considerations of a physically-motivated approximate symmetry. As additional heavy baryons are observed and their masses are measured with decreasing uncertainties, one will gain greater insight into the symmetries obeyed — or not obeyed — in the interactions between heavy and light quarks.
Acknowledgments

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REFERENCES

[1] R. F. Lebed, Phys. Rev. D 51 (1995) 5039.

[2] J. G. Körner, Mainz Report No. MZ-TH/96-04 [hep-ph/9602243] (unpublished).

[3] L. Montanet et al. (Particle Data Group), Phys. Rev. D 50 (1994) 1173. The baryon multiplets appear on p. 1782, the naming scheme on p. 1324.

[4] E. Jenkins, U. C. San Diego Report No. UCSD/PTH 96-08 (unpublished).

[5] ALEPH Collaboration, Report No. EPS-0406, contribution to the International Europhysics Conference on High Energy Physics, Brussels, Belgium, July 1995; P. Abreu et al. (DELPHI Collaboration), Report No. CERN-PPE/95-29 (unpublished).

[6] R. Werding, for the WA89 Collaboration, Report No. PRINT-94-0198, presented at the 27th International Conference on High Energy Physics, Glasgow, Scotland, July 1994.

[7] P. L. Frabetti et al. (E687 Collaboration), Phys. Lett. B 338 (1994) 106.

[8] V. V. Ammosov et al. (SKAT Collaboration), Pis’ma Zh. Eksper. Teor. Fiz. 58 (1993) 241 [JETP Lett. 58 (1993) 247.

[9] P. Avery et al. (CLEO Collaboration), Report No. CLNS 95/1352 (unpublished); L. Gibbons et al. (CLEO Collaboration), Report No. CLNS 96/1394 (unpublished).

[10] CDF B Physics Group, HyperText Document http://www-cdf.fnal.gov/physics/new/bottom/cdf3352/cdf3352.html (unpublished).

[11] D. Buskulic et al. (ALEPH Collaboration), Report No. CERN-PPE/96-28 (unpublished).

[12] P. Abreu et al. (DELPHI Collaboration), Report No. CERN-PPE/96-16 (unpublished).

[13] M. Feindt et al. (DELPHI Collaboration), Report No. DELPHI 95-107, EPS-0565, Contribution to the International Europhysics Conference on High Energy Physics, Brussels, Belgium, July 1995. To obtain absolute $\Sigma_b$ masses, we have added the results of this paper.
(Σ_b^{(*)} - Λ_b) to the (somewhat high) DELPHI Λ_b mass measurement, Ref. [12] and assumed that the systematic uncertainties in both experiments have the same sources.

[14] M. J. Savage, Phys. Lett. B 359 (1995) 189.

[15] This is the phase convention for conjugate reps that gives the fundamental quark triplet as (u, d, s), but the fundamental antitriplet as (¯¯u, ¯¯¯d, ¯¯¯s); it guarantees that all raising and lowering operators produce positive coefficients. Because mass operators are quadratic bilinears, in our case it has no effect except to redefine the sign of Ξ^+_c mixing terms.

[16] E. Jenkins and R. F. Lebed, Phys. Rev. D 52 (1995) 282.

[17] J. Franklin, Phys. Rev. D 12 (1975) 2077. Note that his notation for Ξ'_Q and Ξ_Q is opposite to that used in this work.

[18] R. F. Dashen, E. Jenkins, and A. V. Manohar, Phys. Rev. D 49 (1994) 4713.

[19] R. F. Dashen, E. Jenkins, and A. V. Manohar, Phys. Rev. D 51 (1995) 3697.

[20] A. Zalewska and K. Zalewski, INP-TH [hep-ph/9603295] (unpublished).

[21] A. F. Falk, Johns Hopkins Report No. JHU-TIPAC-96007 [hep-ph/9603389] (unpublished).

[22] D. Chen et al. (E761 Collaboration), Phys. Rev. Lett. 69 (1992) 3286.

[23] One useful collection of SU(3) Clebsch–Gordan coefficients is: T. A. Kaeding, Lawrence Berkeley Laboratory Report No. LBL-36659 and University of California Report No. UCB-PTH-95/01 [nucl-th/9502037] (unpublished).

**APPENDIX: DECOMPOSITION OF MASS OPERATORS**

We begin by defining the “chiral coefficients” of SU(3), which are combinations of baryon bilinears transforming under distinct reps of SU(3) and isospin. The relation between the
ordinary bilinears and chiral coefficients is simply a basis change, so they are related by an orthogonal transformation. For definiteness of notation for isospin states, we use charmed baryon labels. Starting with $Q = 1$ baryons, define

$$M_{66} = c_{66} a_{66},$$
$$M_{63} = c_{63} b_{63},$$
$$M_{36} = c_{36} \bar{b}_{36},$$
$$M_{33} = c_{33} c_{33},$$

(A1)

where

$$M_{T}^{66} = (\sum_{c}^{+} \Sigma_{c}^{+}, \sum_{c}^{+} \Sigma_{c}^{+}, \sum_{c}^{0} \Sigma_{c}^{0}, \Xi_{c}^{+} \Xi_{c}^{+}, \Xi_{c}^{0} \Xi_{c}^{0}, \Omega_{c}^{0} \Omega_{c}^{0}),$$
$$M_{T}^{63} = (\sum_{c}^{+} \Lambda_{c}^{+}, \Xi_{c}^{+} \Xi_{c}^{+}, \Xi_{c}^{0} \Xi_{c}^{0}),$$
$$M_{T}^{36} = M_{T}^{63},$$
$$M_{T}^{33} = (\sum_{c}^{+} \Lambda_{c}^{+}, \Xi_{c}^{+} \Xi_{c}^{+}, \Xi_{c}^{0} \Xi_{c}^{0}),$$

(A2)

with an analogous $M_{T}^{66}$ for spin-$3/2$ states, and

$$a_{T}^{66} = (a_{6}^{0}, a_{6}^{8}, a_{6}^{1}, a_{6}^{27}, a_{6}^{27}, a_{6}^{27}),$$
$$b_{T}^{63} = (b_{6}^{0}, b_{6}^{8}, b_{6}^{1}, b_{6}^{27}),$$
$$\bar{b}_{T}^{36} = b_{T}^{36},$$
$$c_{T}^{33} = (c_{3}^{0}, c_{3}^{8}, c_{3}^{8}),$$

(A3)

with upper (lower) indices indicating SU(3) (isospin) reps. Then we compute

$$C_{66} = \begin{pmatrix}
\frac{1}{\sqrt{6}} + \frac{\sqrt{2}}{15} & + \frac{\sqrt{2}}{5} & + \frac{1}{\sqrt{30}} & + \frac{1}{\sqrt{10}} & + \frac{1}{\sqrt{6}} \\
+ \frac{1}{\sqrt{6}} & + \frac{\sqrt{2}}{15} & 0 & + \frac{1}{\sqrt{30}} & 0 & - \frac{\sqrt{2}}{3} \\
+ \frac{1}{\sqrt{6}} & + \frac{\sqrt{2}}{15} & - \sqrt{\frac{2}{5}} & + \frac{1}{\sqrt{30}} & - \frac{1}{\sqrt{10}} & + \frac{1}{\sqrt{6}} \\
+ \frac{1}{\sqrt{6}} & - \frac{1}{\sqrt{30}} & + \frac{1}{\sqrt{10}} & - \frac{3}{10} & - \frac{\sqrt{2}}{5} & 0 \\
+ \frac{1}{\sqrt{6}} & - \frac{1}{\sqrt{30}} & - \frac{1}{\sqrt{10}} & - \frac{3}{10} & + \frac{\sqrt{2}}{5} & 0 \\
+ \frac{1}{\sqrt{6}} & - 2 \frac{\sqrt{2}}{15} & 0 & + \frac{2}{10} & 0 & 0
\end{pmatrix},$$

(A4)
\[
C_{\bar{6}6} = \begin{pmatrix}
0 & +\sqrt{\frac{2}{3}} + \frac{1}{\sqrt{3}} \\
+\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \\
+\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}
\end{pmatrix},
\]
(A5)

\[
C_{3\bar{3}} = \begin{pmatrix}
+\frac{1}{\sqrt{3}} & +\sqrt{\frac{2}{3}} & 0 \\
+\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{2}} \\
+\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\]
(A6)

For the states with \( Q = 2 \),

\[
M_{33} = C_{3\bar{3}} d_{33},
\]
(A7)

where we may choose the phase convention

\[
C_{3\bar{3}} = C_{3\bar{3}},
\]
(A8)

with

\[
M_{33}^T = (\Omega_{cc}^+ \Omega_{cc}, \Xi_{cc}^+ \Xi_{cc}, \Xi_{cc}^{++} \Xi_{cc}^{++}),
\]
(A9)

and an analogous \( M_{33}^T \) for spin-3/2 states, and

\[
d_{33}^T = (d_1^1, d_8^1, d_1^0).
\]
(A10)

Using the notation that the SU(3) chiral coefficients for the spin-1/2 and spin-3/2 sextet are \( a_I^R \) and \( a_I^{R'} \) respectively, the total \( J = 0 \) (mass) chiral coefficients of SU(2) \( \times \) SU(6) are given, with the SU(6) rep in the lower index and SU(3) and isospin in the upper indices, by

\[
\begin{pmatrix}
X_{35}^{1.0} \\
X_{405}^{1.0} \\
Y_{35}^{1.0}
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix}
\frac{1}{2} \sqrt{2} & +\sqrt{\frac{5}{2}} & 0 & +\frac{1}{4} \\
+\frac{1}{2} \sqrt{6} & +\frac{1}{4} \sqrt{3} & 0 & -\frac{\sqrt{15}}{4} \\
+\frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 & 0
\end{pmatrix}
\end{pmatrix} \begin{pmatrix}
a_0^1 \\
a_0^1 \\
c_0^1
\end{pmatrix},
\]
(A11)

\[
\begin{pmatrix}
X_{35}^{8.1} \\
X_{405}^{8.1} \\
Y_{35}^{8.1} \\
Y_{405}^{8.1}
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix}
+\frac{1}{2} \sqrt{2} & +\sqrt{\frac{5}{2}} & 0 & +\frac{1}{4} \\
+\frac{1}{2} \sqrt{6} & +\frac{1}{4} \sqrt{3} & 0 & -\frac{\sqrt{15}}{4} \\
+\frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 & 0
\end{pmatrix}
\end{pmatrix} \begin{pmatrix}
a_1^8 \\
a_1^8 \\
c_1^8 \\
\frac{1}{\sqrt{2}} (b_1^8 + \bar{b}_1^8)
\end{pmatrix},
\]
(A12)
\[ Y_{405}^{10+,1} = + \frac{1}{\sqrt{2}} (b_{10}^I + \bar{b}_{10}^I), \]  \hspace{1cm} (A13)

\[
\begin{pmatrix}
X_{405}^{27,I} \\
Y_{405}^{27,I}
\end{pmatrix} = \begin{pmatrix}
\sqrt{\frac{2}{3}} & - \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}
\end{pmatrix} \begin{pmatrix}
a_I^{27} \\
a_I^{27}
\end{pmatrix}. \hspace{1cm} (A14)
\]

\(X \) and \(Y \) are defined as chiral coefficients of SU(2) \(\times\) SU(6) in which the heavy-quark bilinear, as well as the light-quark bilinear, has \(J = 0\) or \(1\), respectively; these can be thought of as sets of bilinears that preserve or flip the heavy quark spin, and so the latter are suppressed operators in the infinite mass limit. In adding chiral coefficients to their conjugates (e.g. \(\frac{i}{\sqrt{2}}(b_{I}^8 + \bar{b}_{I}^8)\)) we are imposing time-reversal invariance (TRI) of the strong and electromagnetic Lagrangian, which is responsible for the baryon masses. Hermiticity alone would permit combinations like \(\frac{i}{\sqrt{2}}(b_{I}^8 - \bar{b}_{I}^8)\), but these violate TRI.

For the states with two heavy quarks,
\[
X_{N}^{R,I} = + \frac{1}{\sqrt{3}}(\sqrt{2}d_{I}^{R} + d_{I}^{R}), \hspace{1cm} Y_{N}^{R,I} = + \frac{1}{\sqrt{3}}(d_{I}^{R} - \sqrt{2}d_{I}^{R}). \hspace{1cm} (A15)
\]

Presenting the Clebsch–Gordan coefficients in this factorized way is equivalent to generating the isoscalar factors of SU(6) \(\supset\) SU(3) \(\times\) SU(2) for the product \(2\mathbf{1} \otimes 2\mathbf{1}\) for \(Q = 1\) states, or \(\mathbf{6} \otimes \mathbf{6}\) for \(Q = 2\) states. The analogous statement in (I) is that the isoscalar factors for \(\mathbf{56} \otimes \mathbf{56}\) were generated.

Finally, we extract the baryon mass combinations that form the coefficients of operators transforming under particular reps of SU(2) \(\times\) SU(6). Starting with the chiral coefficients \(X \) and \(Y \), we read off the numerical coefficient of each baryon bilinear. To obtain the correct numerical coefficient in the corresponding mass combination, we use the Wigner–Eckart theorem to remove the relevant spin-SU(2) Clebsch–Gordan coefficient included in creating a total \(J = 0\) bilinear, which is \(1/2\) for spin-3/2, \(1/\sqrt{2}\) for spin-1/2; in short, one multiplies each spin-3/2 coefficient by \(\sqrt{2}\). The mass combination is denoted with a lower-case letter.

In the following expressions, baryon masses are designated with their symbols, and \(\gamma, \delta_+, \) and \(\delta_0\) denote the coefficients of \((\Sigma_c^+ + \Lambda_c^+ + h.c.)\), \((\Xi_c^+ + \Xi_c^0 + h.c.)\), and \((\Xi_c^0 + \Xi_c^0 + h.c.)\), respectively. Baryon masses without isospin labels denote an average over all isospin channels.
Likewise, for the doubly-charmed baryons, we obtain
\begin{align}
x_{1}^{1.0} &= 2(\Sigma_{c}^{*} + 2\Xi_{c}^{*} + \Omega_{c}^{*}) + (3\Sigma_{c} + 2\Xi_{c}^{'} + \Omega_{c}) + (\Lambda_{c} + 2\Xi_{c}), \\
x_{405}^{1.0} &= 2(\Sigma_{c}^{*} + 2\Xi_{c}^{*} + \Omega_{c}^{*}) + (3\Sigma_{c} + 2\Xi_{c}^{'} + \Omega_{c}) - 6(\Lambda_{c} + 2\Xi_{c}), \\
y_{35}^{1.0} &= 3(\Sigma_{c}^{*} - \Sigma_{c}) + 2(\Xi_{c}^{*} - \Xi_{c}^{'}) + (\Omega_{c}^{*} - \Omega_{c}), \\
x_{35}^{8.0} &= 2(3\Sigma_{c}^{*} - \Xi_{c}^{*} - 2\Omega_{c}^{*}) + (3\Sigma_{c} - \Xi_{c}^{'} - 2\Omega_{c}) + (\Lambda_{c} - \Xi_{c}), \\
x_{35}^{8.0} &= 2[2(\Sigma_{c}^{*} + \Xi_{c}^{0}) + (\Xi_{c}^{*} + \Xi_{c}^{0})] + [2(\Sigma_{c}^{*} + \Xi_{c}^{0}) + (\Xi_{c}^{*} + \Xi_{c}^{0})] + (\Xi_{c}^{*} - \Xi_{c}^{0}), \\
x_{405}^{8.0} &= 2(3\Sigma_{c}^{*} - \Xi_{c}^{*} - 2\Omega_{c}^{*}) + (3\Sigma_{c} - \Xi_{c}^{'} - 2\Omega_{c}) - 15(\Lambda_{c} - \Xi_{c}), \\
x_{405}^{8.0} &= 3(\Sigma_{c}^{*} - \Sigma_{c}) - (\Xi_{c}^{*} - \Xi_{c}^{'}) - 2(\Omega_{c}^{*} - \Omega_{c}), \\
y_{35}^{8.1} &= 2[(\Sigma_{c}^{*} - \Xi_{c}^{0}) - (\Sigma_{c}^{*} - \Xi_{c}^{0})] + [(\Xi_{c}^{*} + \Xi_{c}^{0}) - (\Xi_{c}^{*} + \Xi_{c}^{0})], \\
y_{405}^{8.0} &= \delta_{+} + \delta_{0}, \\
y_{405}^{10.1} &= \gamma - \delta_{+} + \delta_{0}, \\
y_{405}^{8.1} &= 2\gamma + \delta_{+} - \delta_{0}, \\
x_{405}^{27.0} &= 2(\Sigma_{c}^{*} - 2\Xi_{c}^{*} + \Omega_{c}^{*}) + (\Sigma_{c} - 2\Xi_{c}^{'} + \Omega_{c}), \\
x_{405}^{27.1} &= 2(\Sigma_{c}^{*} - \Sigma_{c}^{0}) - 2(\Xi_{c}^{*} - \Xi_{c}^{0}) + [(\Sigma_{c}^{*} - \Sigma_{c}^{0}) - 2(\Xi_{c}^{*} - \Xi_{c}^{0})], \\
x_{405}^{27.2} &= 2(\Sigma_{c}^{*} + 2\Sigma_{c}^{*} + \Sigma_{c}^{0}) + (\Sigma_{c}^{*} - 2\Sigma_{c}^{*} + \Sigma_{c}^{0}), \\
y_{405}^{27.0} &= (\Sigma_{c}^{*} - \Sigma_{c}) - 2(\Xi_{c}^{*} - \Xi_{c}^{'}) + (\Omega_{c}^{*} - \Omega_{c}), \\
y_{405}^{27.1} &= [(\Sigma_{c}^{*} - \Sigma_{c}^{0}) - (\Sigma_{c}^{*} - \Sigma_{c}^{0})] - 2[(\Xi_{c}^{*} - \Xi_{c}^{0}) - (\Xi_{c}^{*} - \Xi_{c}^{0})], \\
y_{405}^{27.2} &= (\Sigma_{c}^{*} + 2\Sigma_{c}^{*} + \Sigma_{c}^{0}) - (\Sigma_{c}^{*} - 2\Sigma_{c}^{*} + \Sigma_{c}^{0}).
\end{align}

Likewise, for the doubly-charmed baryons, we obtain
\begin{align}
x_{1}^{1.0} &= 2(\Omega_{cc}^{*} + 2\Xi_{cc}^{*}) + (\Omega_{cc} + 2\Xi_{cc}), \\
x_{35}^{8.0} &= 2(\Omega_{cc}^{*} - \Xi_{cc}^{*}) + (\Omega_{cc} - \Xi_{cc}), \\
x_{35}^{8.1} &= 2(\Xi_{cc}^{*} - \Xi_{cc}^{*}) + (\Xi_{cc}^{*} - \Xi_{cc}^{*}), \\
y_{35}^{1.0} &= (\Omega_{cc}^{*} - \Omega_{cc}) + 2(\Xi_{cc}^{*} - \Xi_{cc}), \\
y_{35}^{8.0} &= (\Omega_{cc}^{*} - \Omega_{cc}) - (\Xi_{cc}^{*} - \Xi_{cc}), \\
y_{35}^{8.1} &= (\Xi_{cc}^{*} - \Xi_{cc}^{*}) - (\Xi_{cc}^{*} - \Xi_{cc}^{*}).
\end{align}
### TABLE I. Current experimental values for the heavy baryon masses.

| State   | Mass (MeV)        | Ref. |
|---------|-------------------|------|
| $\Lambda_c$ | 2285.1 ± 0.6         | 3    |
| $\Sigma_c^{++}$ | 2453.1 ± 0.6         | 3    |
| $\Sigma_c^+$ | 2453.8 ± 0.9         | 4    |
| $\Sigma_c^0$ | 2452.4 ± 0.7         | 3    |
| $\Xi_c^+$ | 2465.1 ± 1.6         | 5    |
| $\Xi_c^0$ | 2470.3 ± 1.8         | 5    |
| $\Xi_c^{'+}$ | 2560 ± 15           | 6    |
| $\Omega_c^0$ | 2699.9 ± 1.5 ± 2.5  | 7    |
| $\Sigma_c^{*++}$ | 2530 ± 5 ±5         | 5    |
| $\Xi_c^{*+}$ | 2644.6 ± 2.3         | 5    |
| $\Xi_c^{*0}$ | 2642.8 ± 2.2         | 5    |
| $\Lambda_b$ | 5641 ± 50           | 8    |
|          | 5623 ± 5 ± 4        | 10   |
|          | 5614 ± 21 ± 4       | 11   |
|          | 5668 ± 16 ± 8       | 12   |
| $\Sigma_b^{\pm}$ | 5841 ± 16 ± 8       | 13   |
| $\Sigma_b^{*\pm}$ | 5897 ± 16 ± 8       | 13   |
| Combination | Est. mag. | (MeV) | Combination | Est. mag. | (MeV) |
|-------------|----------|-------|-------------|----------|-------|
| $Q = 1$     |          |       |             |          |       |
| $x_{405}^{1.0}$ | $9\Lambda\chi\delta$ | 2700 | $y_{405}^{10+10.1}$ | $3\Lambda\chi\epsilon^2\epsilon'$ | 0.27 |
| $y_{35}^{1.0}$ | $3\Lambda\chi\theta$ | 600  | $y_{405}^{8.1}$ | $4\Lambda\chi\epsilon'$ | 4 |
| $x_{35}^{8.0}$ | $5\Lambda\chi\epsilon$ | 1500 | $x_{405}^{27.0}$ | $3\Lambda\chi\epsilon^2$ | 270 |
| $x_{405}^{8.1}$ | $5\Lambda\chi\epsilon'$ | 25   | $x_{405}^{27.1}$ | $\frac{9}{2}\Lambda\chi\epsilon\epsilon'$ | 6.8 |
| $x_{405}^{8.0}$ | $12\Lambda\chi\delta\epsilon$ | 1080 | $x_{405}^{27.2}$ | $3\Lambda\chi\epsilon''$ | 3 |
| $x_{405}^{8.1}$ | $12\Lambda\chi\delta\epsilon'$ | 18   | $y_{405}^{27.0}$ | $2\Lambda\chi\epsilon^2$ | 36 |
| $y_{35}^{8.0}$ | $3\Lambda\chi\theta\epsilon$ | 180  | $y_{405}^{27.1}$ | $3\Lambda\chi\theta\epsilon'$ | 0.9 |
| $y_{35}^{8.1}$ | $3\Lambda\chi\theta\epsilon'$ | 3    | $y_{405}^{27.2}$ | $2\Lambda\chi\theta\epsilon''$ | 0.4 |
| $y_{405}^{8.0}$ | $2\Lambda\chi\theta\epsilon$ | 120  |             |          |       |
| $Q = 2$     |          |       |             |          |       |
| $x_{35}^{8.0}$ | $\frac{3}{2}\Lambda\chi\epsilon$ | 450  | $y_{35}^{8.0}$ | $\Lambda\chi\theta\epsilon$ | 60 |
| $x_{35}^{8.1}$ | $\frac{3}{2}\Lambda\chi\epsilon'$ | 7.5  | $y_{35}^{8.1}$ | $\Lambda\chi\theta\epsilon'$ | 1 |
| $y_{35}^{1.0}$ | $\frac{3}{2}\Lambda\chi\theta$ | 300  |             |          |       |
TABLE III. Mass combinations versus experimental values.

| Masses | Exp. (MeV) | Combination | Est. mag. (MeV) |
|--------|------------|-------------|----------------|
| $\frac{1}{\pi}(6\Sigma^*_c+6\Xi^*_c+3\Sigma_c+3\Omega_c+\Lambda_c+2\Xi_c)$ | $2558 \pm 2$ | $\frac{1}{10}(5x_{1}^{1,0} + 2y_{35}^{8,0} - 6y_{405}^{27,0})$ | $M_c \pm \frac{3}{30} \Lambda_c \theta e \ M_c \pm 3$ |
| $2\Sigma^*_c + 2\Xi^*_c + \Sigma_c + \Omega_c - 2\Lambda_c - 4\Xi_c$ | $1059 \pm 16$ | $\frac{1}{10}(5x_{405}^{1,0} + 2y_{35}^{8,0} - 6y_{405}^{27,0})$ | $3\Lambda_c \delta \ 900$ |
| $\Sigma^*_c - \Sigma_c$ | $77 \pm 7$ | $\frac{1}{30}(5y_{35}^{1,0} + 4y_{35}^{8,0} + 3y_{405}^{27,0})$ | $\frac{1}{2} \Lambda_c \theta \ 100$ |
| $3\Sigma^*_c - 3\Xi^*_c + 6\Sigma_c - 6\Omega_c + \Lambda_c - \Xi_c$ | $-2000 \pm 30$ | $\frac{1}{3}(5y_{35}^{8,0} - 7y_{35}^{8,0} + 6y_{405}^{27,0})$ | $5\Lambda_c e \ 1500$ |
| $\Sigma^*_c - \Xi^*_c + 2\Sigma_c - 2\Omega_c - 5\Lambda_c + 5\Xi_c$ | $306 \pm 12$ | $\frac{1}{10}(5x_{405}^{8,0} - 7y_{35}^{8,0} + 6y_{405}^{27,0})$ | $4\Lambda_c \delta e \ 360$ |
| $2\Sigma^*_c - 2\Xi^*_c - \Sigma_c + \Omega_c$ | $19 \pm 15$ | $\frac{1}{15}(5x_{405}^{27,0} + 6y_{35}^{8,0} + 2y_{405}^{27,0})$ | $\frac{3}{2} \Lambda_c e^2 \ 135$ |
| $3\Xi^*_c + 6\Sigma_1 + \Xi_1$ | $2.9 \pm 8.5$ | $\frac{1}{15}(5x_{35}^{8,1} - 7y_{35}^{8,1} - 6y_{405}^{27,1})$ | $5\Lambda_c e' \ 25$ |
| $\Xi^*_c + 2\Sigma_1 - 5\Xi_1$ | $29 \pm 11$ | $\frac{1}{15}(5x_{405}^{8,1} - 7y_{35}^{8,1} - 6y_{405}^{27,1})$ | $4\Lambda_c \delta e' \ 6$ |
| $\Sigma_1 - 2\Xi_1$ | $-1.9 \pm 5.2$ | $\frac{1}{15}(5x_{405}^{27,1} - 6y_{35}^{8,1} + 2y_{405}^{27,1})$ | $\frac{3}{2} \Lambda_c e e' \ 2.3$ |
| $\Sigma^+_c - 2\Sigma^+_c + \Sigma^0_c$ | $-2.1 \pm 1.3$ | $\frac{1}{3}(x_{405}^{27,2} - 2y_{405}^{27,2})$ | $\Lambda_c e'' \ 1$ |