Hypersymplectic 4-manifolds, the $G_2$-Laplacian flow and extension assuming bounded scalar curvature

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Abstract

A hypersymplectic structure on a 4-manifold $X$ is a triple $\omega$ of symplectic forms which at every point span a maximal positive-definite subspace of $\Lambda^2$ for the wedge product. This article is motivated by a conjecture of Donaldson: when $X$ is compact $\omega$ can be deformed through cohomologous hypersymplectic structures to a hyperkähler triple. We approach this via a link with $G_2$-geometry. A hypersymplectic structure $\omega$ on a compact manifold $X$ defines a natural $G_2$-structure $\phi$ on $X \times T^3$ which has vanishing torsion precisely when $\omega$ is a hyperkähler triple. We study the $G_2$-Laplacian flow starting from $\phi$, which we interpret as a flow of hypersymplectic structures. Our main result is that the flow extends as long as the scalar curvature of the corresponding $G_2$-structure remains bounded. An application of our result is a lower bound for the maximal existence time of the flow, in terms of weak bounds on the initial data (and with no assumption that scalar curvature is bounded along the flow).

1 Introduction

Let $X$ be an oriented 4-manifold. A hypersymplectic structure on $X$ is a triple $\omega = (\omega_1, \omega_2, \omega_3)$ of closed 2-forms which at every point span a maximal positive-definite subspace of $\Lambda^2$ for the wedge product. In particular, each $\omega_i$ is symplectic. Hypersymplectic structures were introduced by Donaldson [8] and they play an important role in the programme he has introduced to study the adiabatic limit of $G_2$-manifolds [9]. A special case of this definition appears in earlier work of Geiges [12] which studies both pairs and triples of symplectic forms with $\omega_i \wedge \omega_j = 0$ for $i \neq j$.

Hypersymplectic structures have also appeared in the work of Madsen [18] and Madsen–Swann [20] on reductions of metrics with special holonomy.

The simplest example of a hypersymplectic structure is the triple of Kähler forms of a hyperkähler metric. The main motivation for this article is a conjecture of Donaldson that on a compact 4-manifold and up to isotopy this is the only example.

Conjecture 1.1 (Donaldson [8]). Let $(X, \omega)$ be a compact 4-manifold with a hypersymplectic structure. Suppose moreover that $\int \omega_i \wedge \omega_j = \delta_{ij}$. Then there exists a deformation of $\omega$ through cohomologous hypersymplectic structures to a hyperkähler triple.

(Note that one can always apply a constant linear transformation to a given hypersymplectic structure to ensure $\int \omega_i \wedge \omega_j = \delta_{ij}$.)

Donaldson’s conjecture can be seen as a special case of an important folklore conjecture in 4-dimensional symplectic geometry: if $(X, \omega)$ is a compact symplectic 4-manifold with $c_1 = 0$ and $b_+ = 3$, then there is a compatible complex structure on $X$ making it into a hyperkähler manifold. Now given a hypersymplectic structure $\omega$ on $X$, we have $c_1(X, \omega_1) = 0$ and $b_+ = 3$. To see this, consider the conformal structure on $X$ for which $\Lambda^+ = \langle \omega_1 \rangle$; this determines an almost complex structure compatible with $\omega_1$ and whose canonical bundle is isomorphic to the sub-bundle of $\Lambda^+$.
which is orthogonal to $\omega_1$. The form $\omega_2$, say, projects to give a nowhere vanishing section of the canonical bundle, showing that $c_1 = 0$. Since the $\omega_i$ are independent (at every point even), it follows that $b_+ \geq 3$ and a theorem of Bauer then shows that in fact $b_+ = 3$ [2, Corollary 1.2]. Assuming Conjecture 1.1, Moser’s theorem then proves the folklore conjecture for $(X, \omega_1)$.

This article investigates a geometric flow which is designed to deform a given hypersymplectic structure towards a hyperkähler one. The flow in question is a dimensional reduction of the $G_2$-Laplacian flow, which we now describe. We begin with a rapid overview of $G_2$-structures, to fix notation. Let $M$ be a 7-manifold and $\phi$ a 3-form on $M$. There is a symmetric bilinear form $B_\phi$ on $TM$ with values in $\Lambda^2$ given by

$$B_\phi(u, v) = \frac{1}{6} \epsilon_{uv\phi} \wedge \epsilon_{v\phi} \wedge \phi$$

When $B_\phi$ is definite $\phi$ is called a $G_2$-structure. In this case there is a unique Riemannian metric $g_\phi$ and orientation with the property that $g_\phi \otimes \text{dvol}_{g_\phi} = B_\phi$. If $\nabla \phi = 0$ (where $\nabla$ is the Levi-Civita connection of $g_\phi$) then $g_\phi$ has holonomy group contained in $G_2$. When this happens we say $\phi$ defines a $G_2$-metric.

A $G_2$-structure is called closed when $d\phi = 0$. A central question is to decide, given a closed $G_2$-structure $\phi$, whether or not there exists a genuine $G_2$-metric in the cohomology class $[\phi]$. To this end, Hitchin [13] studied the total volume functional $V(\phi) = \int M \text{dvol}_{g_\phi}$ restricted to $[\phi]$. He proved that the critical points are precisely those $\phi$ defining $G_2$-metrics. More precisely, fix a background closed $G_2$-structure $\phi_0$, let

$$\mathcal{H}_{\phi_0} = \{ \eta \in \Omega^2(M) : \phi_0 + d\eta \text{ is a } G_2\text{-structure}\} \subset \Omega^2(M)$$

and define a Riemannian metric on $\mathcal{H}_{\phi_0}$ by

$$(\eta_1, \eta_2)_\phi = \int_M \eta_1 \wedge *_{\phi_0} \eta_2$$

for all $\eta \in \mathcal{H}_{\phi_0}$ and $\eta_1, \eta_2 \in T_\eta \mathcal{H}_{\phi_0} = \Omega^2(M)$. Hitchin considered the functional on $\mathcal{H}_{\phi_0}$, defined by $\widetilde{V}(\eta) = \int M \text{dvol}_{g_{\phi_0} + \eta}$. With respect to the metric (1), $\text{grad} \widetilde{V}(\eta) = d^* \eta_0 + *_{\phi_0} (\phi_0 + d\eta)$ by a simple calculation. The gradient flow of $\widetilde{V}$ on $\mathcal{H}_{\phi_0}$ descends to a flow of closed $G_2$-structures (i.e. by setting $\phi = \phi_0 + d\eta$), evolving according to

$$\partial_t \phi = \Delta_\phi \phi$$

where $\Delta_\phi$ is the Hodge Laplacian of $g_\phi$. This flow, called the $G_2$-Laplacian flow, was also independently introduced by Bryant in [3]. Bryant–Xu proved short-time existence and uniqueness of the flow on compact manifolds in [1].

We now relate this to hypersymplectic structures (following Donaldson’s article [9]). Given a hypersymplectic structure $\omega$ on $X$ we consider the following 3-form $\phi$ on the 7-manifold $M = X \times T^3$. We use angular coordinates $t^1, t^2, t^3$ on $T^3 = S^1 \times S^1 \times S^1$ and define

$$\phi = dt^{123} - dt^1 \wedge \omega_1 - dt^2 \wedge \omega_2 - dt^3 \wedge \omega_3$$

One checks that $\phi$ is a closed $G_2$-structure precisely because $\omega$ is hypersymplectic (see Lemma 2.2 below). The $G_2$-Laplacian flow for $\phi$ descends to a flow of hypersymplectic structures on $X$ which we describe next.

As we have already mentioned, a hypersymplectic structure $\omega$ determines a conformal structure on $X$, for which $\Lambda^+ = (\omega_i)$. Given any choice of volume form $\mu$, we obtain a Riemannian metric in this conformal class and using this we can define the $3 \times 3$ symmetric matrix valued function

$$Q_{ij} = \frac{\omega_i \wedge \omega_j}{2\mu}$$
There is a unique volume form $\mu$ for which $\det Q = 1$ and, with this convention, $\omega$ defines a Riemannian metric $g_\omega$ on $X$. The triple $\omega$ is a hyperkähler triple precisely when $Q = 1$ (see, for instance, [8] §1]). When $Q$ is a constant matrix, then consider $\theta = Q^{1/2}\omega$, i.e., $\theta_i = \sum_{j=1}^3(Q^{-1})_{ij}\omega_j$ where $Q^{-1}$ is the positive square root of $Q^{-1}$. The new triple $\theta$ is a hyperkähler triple, and the Riemannian metrics $g_\omega = g_\theta$ are the same. In general, the matrix $Q$ and metric $g_\omega$ are related to the 7-dimensional metric $g_\phi$ via

$$g_\phi = g_\omega \oplus \sum_{i,j=1}^3 Q_{ij}d\theta^id\theta^j$$

with respect to the natural splitting $TM \cong TX \oplus T(T^3)$. (These claims are all proved in §2.1.)

With this in hand we can now describe the $G_2$-Laplacian flow on $X \times T^3$ in terms of $\omega$. If the flow begins with initial condition $\phi$ of the form $\theta$, then $\phi(t)$ is of the same form for all $t$ (Lemma 2.8 below) and corresponds to a flow of hypersymplectic structures which evolve according to the equation

$$\partial_t \omega = d(Q^*\omega^{-1})$$  \hspace{1cm} (4)

(The notation used here is that if $S_{ij}$ is a $3 \times 3$-matrix and $\omega$ is a triple of forms, then $S\omega$ denotes the triple $(S\omega)_i = S_{ij}\omega_j$.) We call $\phi$ the hypersymplectic flow and it is the focus of this article.

To put our main result in context, we first recall what is known about the $G_2$-Laplacian flow in general. The torsion of a closed $G_2$-structure $\phi$ is the 2-form $T = -\frac{1}{2}d^*\phi$, which vanishes precisely when $\phi$ determines a genuine $G_2$-metric. Lotay–Wei proved an extension theorem [17] Theorem 1.6], based on Shi-type estimates involving both torsion and curvature, more precisely using the quantity

$$\Lambda(\phi) = \sup_M (|\text{Rm}(g_\phi)|^2 + |\nabla T|^2)^{1/2}$$

Lotay–Wei’s extension theorem is:

**Theorem 1.2** (Lotay–Wei [17]). Let $\phi(t)$ be a solution of the $G_2$-Laplacian flow on a compact 7-manifold and time interval $t \in [0,s)$ with $s < \infty$. If $|\Delta_\phi\phi|_\phi$ is bounded on $[0,s)$ then the flow extends beyond $t = s$, to the time interval $[0,s + \epsilon)$ for some $\epsilon > 0$.

This is, of course, the $G_2$ analogue of Sösum’s theorem that Ricci flow exists as long as the Ricci curvature is bounded [22]. The main result of this article is that for the hypersymplectic flow, one can replace $|\Delta_\phi\phi|_\phi$ by the a priori weaker quantity $|T|$:

**Theorem 1.3.** Let $\omega(t)$ be a solution of the hypersymplectic flow on a compact 4-manifold $X$ and time interval $t \in [0,s)$ with $s < \infty$. If $|T|$ is bounded on $[0,s)$ then the flow extends beyond $t = s$ to the time interval $[0,s + \epsilon)$ for some $\epsilon > 0$.

A calculation of Bryant [3] says that the scalar curvature of a closed $G_2$-structure $\phi$ is $R(g_\phi) = -|T|^2$. Our result says that given a hypersymplectic structure $\omega(0)$, the $G_2$-Laplacian flow on $X \times T^3$ starting from $\phi(0)$ given by (3) can be continued for as long as the scalar curvature of the metric $g_{\phi(t)}$ remains bounded. It is interesting to note that this is much stronger than what is currently known for the Ricci flow (cf. the works [11] [24] [25] [7] [26] for the study of Ricci flow under bounded scalar curvature).

A consequence of Theorem 1.3 is a lower bound for the existence time of a hypersymplectic flow, purely in terms of a $C^1$ bound on $Q$ at $t = 0$. We do not assume here that the flow has bounded torsion. We give a loose statement of the result here, see Theorem 6.3 for the optimal statement.

**Theorem 1.4.** Let $K > 0$. There exists a constant $\epsilon > 0$ depending only on $K$, such that whenever $\omega(0)$ is a hypersymplectic structure with $\|Q\|_{C^1} \leq K$ then the hypersymplectic flow $\omega(t)$ starting at $\omega(0)$ exists for all $t \in [0,\epsilon]$.
We now give a brief outline of the proof of Theorem 1.3, the details of which occupy the rest of the paper. A key point is that when \( \omega(t) \) solves the hypersymplectic flow, the corresponding metric \( g(t) \) and positive definite matrix \( \tilde{Q}(t) \) solve a version of the coupled harmonic–Ricci flow (introduced by Buzano (né Müller) in [19]) with additional “forcing” terms involving \( T \). The proof exploits this by combining ideas from Ricci flow and the harmonic map flow.

We assume for a contradiction that the flow does not extend. We consider a parabolic rescaling of \( \omega(t) \) as \( t \) approaches the maximal time and, using Lotay–Wei’s estimates on \( \Lambda(\phi) \), take a limit. To do this, we show that the bound on \( |T| \) implies the flows are noncollapsed. We use here a recent result of Chen [6, Theorem 1.1] which generalises Perelman’s \( \kappa \)-noncollapsing theorem for the Ricci flow [21] to a metric flow \( g(t) \) for which \( \partial_t g \) is a bounded distance from \(-2\operatorname{Ric}\). To prove the hypotheses of Chen’s theorem are satisfied, we use the maximum principle, in the style of the harmonic map flow, to control \( dQ \), and show that bounds on torsion imply that the flow is uniformly noncollapsed, in finite time.

The next step, and the real crux of the argument, is to show that the limit is an asymptotically locally Euclidean (ALE) gravitational instanton. This sort of conclusion is currently out of reach in Ricci flow, and this explains the big difference between what is known there and what we are able to prove for the hypersymplectic flow. There are two separate things to show: proving that the limit is hyperkähler and proving that it has finite energy, i.e. finite \( L^2 \)-norm of curvature. (The fact that the limit is ALE follows from the uniform noncollapsing, which ensures Euclidean volume growth in the limit.)

To achieve the first part, we first bound the \( C^2 \)-norm of \( \omega \) in terms of bounds of \( dQ \) and Riemannian curvature. This enables us to take a limit of the rescaled hypersymplectic structures, giving a hyperkähler triple on the limit. The proof that the limit has finite energy is quite delicate. We show that the bound on \( |T| \) gives a bound on the energy along the hypersymplectic flow (in finite time). By scaling invariance, this translates to an energy bound in the limit. Our argument here is inspired by a result of Simon [23] which shows that a bound on scalar curvature of a compact 4-dimensional Ricci flow implies a bound on the energy in finite time. Simon’s proof involves integral estimates of two different functions, the “bad” term of one being cancelled by the “good” term of the other. Here the calculations are more complicated, with additional bad terms appearing when Simon’s two quantities are considered. We are able to complete the proof by a judicious choice of two additional functions which generate the required good terms.

To complete the proof, we invoke Kronheimer’s classification of ALE gravitational instantons [16] from which it follows that the limit contains a 2-sphere which is holomorphic for one of its hyperkähler complex structures. From this, and the fact that the hyperkähler triple is a scaling limit of the hypersymplectic structures, we find a contradiction using a topological argument. The main idea is that, since there is a \( J \)-holomorphic 2-sphere \( S_\infty \) in the limit, as we approach the singular time, we can find a sequence \( S_i \) of symplectic 2-spheres, for one of the symplectic forms, \( \omega_1 \) say, which converge to \( S_\infty \). We are able to show that the homology class \([S_i]\) lies in a finite set and so the a priori positive sequence \( \int_{S_i} \omega_1 \) has a strictly positive lower bound. Meanwhile, since the areas of the \( S_i \) converge to a finite limit after rescaling, we must have \( \int_{S_i} \omega_1 \to 0 \), which is a contradiction.

The article is organized as follows. In §2 we show short-time existence and uniqueness for the hypersymplectic flow [4]. This follows from the analogous result for the \( G_2 \)-Laplacian flow, together with a calculation of the corresponding flow on the 4-manifold. In §3 we give a series of identities relating various geometric quantities on \( (X, g_\omega) \) and \( (X \times \mathbb{T}^3, g_\phi) \). In §4 we derive the necessary evolution equations. Some can be deduced directly from the known equations for the \( G_2 \)-Laplacian flow, others have purely 4-dimensional derivations. We also prove the required maximum principles here. §5 is the technical heart of the paper, proving that the \( L^2 \)-norm of the curvature of \( g_\omega \) is also bounded in finite time. §6 then assembles all the parts of the proof of Theorems 1.3 and 1.4.
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2 Preliminary definitions and short time existence

The main result of this section is:

Proposition 2.1. Let \( \omega \) be a hypersymplectic structure on a compact 4-manifold \( X \). Then there exists a unique short time solution \( \omega(t) \) to the hypersymplectic flow \( \Box \) starting at \( \omega \).

This will follow easily from the analogous result of Bryant–Xu for the \( G_2 \)-Laplacian flow. Along the way we make explicit the relationship between the natural metric \( g_\omega \) on \( X \) induced by a hypersymplectic structure \( \omega \) and the metric \( g_\phi \) on \( X \times T^3 \) induced by the \( G_2 \)-structure \( \phi \) given by (3).

2.1 Relating \( g_\omega \) and \( g_\phi \)

Lemma 2.2. Given a triple \( \omega \) of 2-forms on \( X \), let \( \phi \) be the 3-form on \( X \times T^3 \) defined by (3). Then \( \phi \) is a \( G_2 \)-structure if and only if \( \langle \omega \rangle \subset \Lambda^2(T^*X) \) is a maximal definite subspace for the wedge product.

Proof. Let \( a = (a_1, a_2, a_3) \in \mathbb{R}^3 \) and write \( u = a_1 \partial_1 + a_2 \partial_2 + a_3 \partial_3 \), where \( \partial_i \) are the coordinate vector fields on \( T^3 \). Then

\[
B_\phi(u, u) = \frac{1}{6} t_u \phi \wedge t_u \phi \wedge \phi = \frac{1}{2} dt^{123} \wedge \sum_{i,j=1}^3 a_ia_j \omega_i \wedge \omega_j
\]

If \( \phi \) is a \( G_2 \)-form then \( B_\phi(u, u) \neq 0 \) whenever \( a \neq 0 \) and hence \( \langle \omega \rangle \) is a maximal definite subspace for the wedge product.

Conversely, if \( \langle \omega \rangle \) is a maximal definite space for the wedge product, \( B_\phi \) is definite on vectors tangent to the \( T^3 \) factor. Meanwhile, if \( u \) is tangent to \( T^3 \), and \( v, w \) are tangent to \( X \), then one checks that \( B_\phi(u, v) = 0 \) whilst \( B_\phi(v, w) = dt^{123} \wedge \beta_\omega(v, w) \) where

\[
\beta_\omega(v, w) = \frac{1}{6} \sum_{i,j,k=1}^3 \epsilon^{ijk} t_u \omega_i \wedge t_u \omega_j \wedge \omega_k
\]

(Here \( \epsilon^{ijk} \) is the sign of the permutation \( (i, j, k) \).) We must check that \( \beta_\omega \) is definite. To do so, we recall a standard fact from 4-dimensional linear algebra. Let \( V \) be a 4-dimensional real vector space. Conformal structures on \( V \), i.e., inner products taken up to scale, are in one-to-one correspondence with the open set \( C \subset \text{Gr}(3, \Lambda^2 V^*) \) consisting of all 3-dimensional subspaces of \( \Lambda^2 V^* \) on which the wedge product is positive definite. The correspondence is given by \( [g] \mapsto \Lambda^+_g \).

With this in hand, let \( g \) be the unique inner product with \( \Lambda^+_g = \langle \omega \rangle \) and with volume form given by \( \mu = \frac{1}{2} (\det(\omega_i \wedge \omega_j))^\frac{1}{2} \). Let \( \theta_1, \theta_2, \theta_3 \) be a basis for \( \langle \omega \rangle \) which diagonalises the wedge
product, i.e., such that \( \theta_i \wedge \theta_j = 2\delta_{ij}\mu \) for the volume form \( \mu \). We can write \( \omega_i = A_{ij}\theta_j \) in this basis. Then
\[
\beta_\omega(v, w) = \frac{1}{6} \sum_{i,j,k,p,q,r} \epsilon^{ijk} A_{ip} A_{jq} A_{kr} \epsilon_{pq} \theta_p \wedge \epsilon_{r} \theta_q \wedge \theta_r = (\det A) \beta_\omega(v, w)
\]
where \( \beta_\omega \) is given by the same formula as \( \beta_\omega \) but with \( \theta_i \) in place of \( \omega_i \). Take any orthonormal basis \( \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \) of \( V \) with respect to \( g \), and write down the standard basis of \( \Lambda^4_{[\omega]} = \langle \omega_i \rangle \):
\[
\tilde{\theta}_1 = \tilde{e}_1 \wedge \tilde{e}_2 + \tilde{e}_3 \wedge \tilde{e}_4 \\
\tilde{\theta}_2 = \tilde{e}_1 \wedge \tilde{e}_3 + \tilde{e}_4 \wedge \tilde{e}_2 \\
\tilde{\theta}_3 = \tilde{e}_1 \wedge \tilde{e}_4 + \tilde{e}_2 \wedge \tilde{e}_3
\]
Because \( \tilde{\theta}_i \wedge \tilde{\theta}_j = 2\delta_{ij}\mu \), there is an element \( F \in SO(\Lambda^+_{[\omega]}; g) \) such that \( F(\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3) = (\theta_1, \theta_2, \theta_3) \). We next use another standard fact, that the natural map
\[
SO(V, g) \rightarrow SO(\Lambda^+_{[\omega]}; g)
\]
is surjective. Let \( \tilde{F} \) be a pre-image of \( F \), and let \( e^i = \tilde{F}(\tilde{e}^i) \) for \( i = 1, 2, 3, 4 \). This gives a basis for which the \( \theta_i \)'s are standard; i.e., we have an orthonormal coframe \( e^1, e^2, e^3, e^4 \) such that \( \theta_1 = e^1 \wedge e^2 + e^3 \wedge e^4 \) etc. Using this basis, it is simple to calculate directly that \( \beta_\omega(v, w) = |v|^2 e^{1234} \) where \( |v|^2 \) is computed in the metric \( g \). It follows that \( \beta_\omega \) is also definite, which completes the proof.

**Definition 2.3.**

1. Given a hypersymplectic structure \( \omega \), the above proof shows that \( \beta_\omega \) defines a \( \Lambda^4 \)-valued definite bilinear form on \( TX \). This defines a conformal structure on \( \tilde{X} \) and, just as for \( G_2 \)-structures, there is a unique volume form \( \mu \) on \( X \) for which the resulting metric \( g_\omega \) in this conformal class satisfies \( g_\omega \otimes \mu = \beta_\omega \). We call \( g_\omega \) and \( \mu \) the metric and volume form induced by \( \omega \).

2. Write \( Q: X \rightarrow S^2 \mathbb{R}^3 \) for the symmetric \( 3 \times 3 \) matrix-valued function defined by
\[
Q_{ij} = \frac{\omega_i \wedge \omega_j}{2\mu}
\]
(where \( \mu \) is the volume form induced by \( \omega \)). \( Q \) will play a central role throughout this article.

**Lemma 2.4.** The 2-forms \( \omega_1, \omega_2, \omega_3 \) are self-dual with respect to the metric \( g_\omega \) and \( \det Q = 1 \).

**Proof.** This is essentially contained in the proof of Lemma 2.2. Let \( \theta_1, \theta_2, \theta_3 \) be a basis for \( \langle \omega_i \rangle \) with \( \theta_i \wedge \theta_j = 2\delta_{ij}\mu \), where \( \mu \) is the volume form of \( g_\omega \). The triple \( \theta_1, \theta_2, \theta_3 \) satisfy the pointwise conditions of a hyperkähler triple (although they certainly need not be closed). We see from (7) and the discussion immediately afterwards that \( g_\omega \) is the resulting metric defined in a standard fashion from this “quaternionic” triple \( \theta_1, \theta_2, \theta_3 \). It follows firstly that \( \Lambda^+ = \langle \theta_i \rangle = \langle \omega_i \rangle \). Moreover, if we write \( \omega_i = A_{ij}\theta_j \), then (7) implies that \( \det A = 1 \). Since \( Q_{ij} = A_{ip}A_{pj} \) we have \( \det Q = 1 \) as well.

**Lemma 2.5.** Let \( \omega \) be a hypersymplectic structure on \( X \) and \( \phi \) the corresponding \( G_2 \)-structure on \( X \times \mathbb{T}^3 \) given by (6). Then, with respect to the natural splitting \( T(X \times \mathbb{T}^3) = TX \oplus T\mathbb{T}^3 \), we have
\[
dvol_{g_\phi} = \mu \wedge d\mathbb{t}^{123}
\]
\[
g_\phi = g_\omega \oplus \sum_{i,j=1}^3 Q_{ij} dt^i dt^j
\]
Proof. From the proof of Lemma 2.2 in particular equations 5 and 6, we have

\[ B_\phi = \left( g_\omega + \sum_{i,j=1}^{3} Q_{ij} dt^i dt^j \right) \otimes (\mu \wedge dt^{123}) \]

The non-degenerate bilinear form \( B_\phi \) together with the volume form \( \mu \wedge dt^{123} \) determines a Riemannian metric \( g_\omega \oplus \sum_{i,j=1}^{3} Q_{ij} dt^i dt^j \), whose metric volume form is precisely \( \mu \wedge dt^{123} \) because \( \det Q = 1 \). By the definition of the induced Riemannian metric from a G\(_2\)-form, this metric is \( g_\phi \).

\[ \square \]

2.2 From the \( G_2 \)-Laplacian flow to the hypersymplectic flow

The goal of this section is to prove that if the \( G_2 \)-Laplacian flow \( \phi(t) \) starts from \( \phi(0) \) of the form of (3), then it remains of the same form, thus giving rise to a flow of hypersymplectic structures on \( X \).

**Lemma 2.6.** Let \( \phi(t) \) be the \( G_2 \)-Laplacian flow on \( X \times T^3 \) with initial condition \( \phi(0) \) given by (3), where \( \omega \) is a hypersymplectic structure. Then \( \phi(t) \) is \( T^3 \)-invariant for as long as it exists.

**Proof.** This is immediate from the uniqueness part of Bryant–Xu’s theorem [4, Theorem 0.1]: since the initial data \( \phi(0) \) is \( T^3 \)-invariant, so is the ensuing flow. \( \square \)

By \( T^3 \)-invariance, the 3-form \( \phi(t) \) on \( X \times T^3 \) necessarily has the shape

\[ \phi(t) = A dt^{123} + B_1 \wedge dt^{23} + B_2 \wedge dt^{31} + B_3 \wedge dt^{12} - dt^1 \wedge C_1 - dt^2 \wedge C_2 - dt^3 \wedge C_3 + D \]

where \( A(t) \in \Omega^0(X) \), \( B_i(t) \in \Omega^1(X) \), \( C_i(t) \in \Omega^2(X) \) and \( D(t) \in \Omega^3(X) \) are paths of differential forms on \( X \). Moreover, since \( d\phi(t) = 0 \), it follows that these forms on \( X \) are all closed.

**Lemma 2.7.** \( A(t) = 1 \).

**Proof.** We know that \( dA(t) = 0 \), i.e., that for every \( t \), \( A(t) \) is constant. Moreover,

\[ \int_{T^3} \phi(t) = (2\pi)^3 A(t) \]

Since \( \partial_t \phi \) is exact, this integral is independent of \( t \), and so \( A(t) = A(0) = 1 \). \( \square \)

Next we consider the involution \( \vartheta \): \( X \times T^3 \to X \times T^3 \) given by \( \vartheta(p,t) \mapsto (p,-t) \) and write \( \vartheta(t) = -\vartheta^* \phi(t) \).

**Lemma 2.8.** For all \( t \), \( \vartheta(t) = \phi(t) \). Hence \( B_i(t) = 0 = D(t) \) vanish identically, and \( \phi(t) \) remains of the form (3) for as long as it exists, for a closed triple \( \omega(t) \) of 2-forms on \( X \).

**Proof.** We will prove that \( \vartheta(t) \) solves the \( G_2 \)-Laplacian flow. Since \( \vartheta(0) = \phi(0) \) the lemma then follows from the uniqueness part of Bryant–Xu’s theorem. The minus sign in the definition \( \vartheta = -\vartheta^* \phi \), together with the fact that \( \vartheta \) is orientation reversing, means that \( \vartheta \) is a \( G_2 \)-structure inducing the same orientation as \( \phi \). One checks that \( g_\vartheta = \vartheta^* g_\phi \). Now given any metric \( g \) and any diffeomorphism \( \psi \), the Hodge–Laplacians of \( g \) and \( \psi^* g \) are related by \( \Delta_{\psi^* g}(\psi^* \alpha) = \psi^* (\Delta_\alpha) \). It follows that \( \vartheta \) solves the \( G_2 \)-Laplacian flow:

\[ \Delta_{\vartheta^* g_\phi} (\vartheta^* \phi) = -\vartheta^* (\Delta_\phi \phi) = -\vartheta^* \partial_t \phi = \partial_t \phi \]

It remains to derive the evolution equation (4) for the hypersymplectic flow. This will complete the proof of Proposition 2.1.
Lemma 2.9. Let \( \omega \) be a hypersymplectic structure on \( X \) and \( \phi \) the associated \( G_2 \)-structure on \( X \times T^3 \) defined in \([3]\). Then
\[
\Delta_\phi \phi = - \sum_{i,p,q} dt^i \wedge d(Q_{ip}d^*_t(Q^{pq}\omega_q))
\]
where \( d^*_t \) is the adjoint of \( d \) on \((X, g_\omega)\) and \( Q^{pq} \) is the inverse matrix to \( Q_{pq} \). It follows that if \(\omega(t)\) is the flow of hypersymplectic structures corresponding to the \( G_2 \)-Laplacian flow \( \phi(t) \) on \( X \times T^3 \), with \( \phi(t) \) and \( \omega(t) \) related by \([3]\), then
\[
\partial_t \omega = d (Q d^* (Q^{-1} \omega))
\]

Proof. We write \(*_3, *_4 \) and \(*_7 \) for the Hodge stars associated to the metrics \( Q_{ij} dt^i \otimes dt^j \) on \( T^3 \), \( g_\omega \) on \( X \) and \( g_\phi \) on \( X \times T^3 \) respectively. We write \( \hat{\omega}_t = dt^2 \) etc. Then \( *_3 dt^i = Q^{ij} \hat{\omega}^j \) and \( *_3 dt^i = Q_{ij} dt^j \). From this we have
\[
*_7 \phi = \mu - \sum_i (*_7 (dt^i \wedge \omega_i)) = \mu - \sum_i *_3 dt^i \wedge *_4 \omega_i = \mu - \sum_i \hat{\omega}_t \wedge Q^{ij} \omega_i
\]
Now \( d^*_7 \phi = -*_7 d* \phi \) which is
\[
d^*_7 \phi = *_7 \left( \sum_{i,j} \hat{\omega}_t \wedge d(Q^{ij} \omega_i) \right) = \sum_{i,j,k} Q_{jk} dt^k \wedge *_4 d(Q^{ij} \omega_i) \tag{8}
\]
and hence \( \Delta_\phi \phi = dd^*_7 \phi \) is given by
\[
\Delta_\phi \phi = \sum_{i,j,k} dt^k \wedge d(Q_{jk} *_4 d *_4 (Q^{ij} \omega_i)) = - \sum_{i,p,q} dt^i \wedge d(Q_{ip} d^*_t(Q^{pq} \omega_q))
\]
as claimed (where in the last line we have used the fact that \( Q \) is symmetric to reorder the indices). \( \square \)

3 Identities and inequalities

In this section we derive various identities relating the geometries of \( g_\phi \) and \( g_\omega \). We will then use these identities to show how geometric quantities can be controlled by \( Q \) and its derivatives. At this stage we consider a single hypersymplectic structure, i.e., not evolving with time.

3.1 Notation

It will be convenient to think of \( Q \) as a map from \( X \) into the space \( \mathcal{P} \) of all positive-definite symmetric \( 3 \times 3 \)-matrices, considered with its non-positively curved symmetric metric. We will state various formulae for geometric objects on \( \mathcal{P} \) here, all of which are standard. For want of a concise reference, however, we have given proofs in Appendix A. There is a natural affine structure on \( \mathcal{P} \), since it is an open subset of the vector space \( S^2(\mathbb{R}^3) \) of symmetric 3-by-3 matrices. This gives a canonical trivialisation of the tangent bundle \( \mathcal{T}\mathcal{P} \cong \mathcal{P} \times S^2(\mathbb{R}^3) \) under which a fixed symmetric matrix \( A \in S^2(\mathbb{R}^3) \) determines a global vector field on \( \mathcal{P} \), which we also denote by \( A \). The symmetric Riemannian metric on \( \mathcal{P} \) is given on \( T_0 \mathcal{P} \) by the formula
\[
\langle A, B \rangle_Q = \text{Tr}(Q^{-1} AQ^{-1} B) \tag{9}
\]
We write \( \nabla \) for the Levi-Civita connection of \( \mathcal{P} \). (The hat is to distinguish it from the flat connection coming from the affine trivialisation of \( T\mathcal{P} \).) Explicitly, if we identify a pair \( A, B \) of
This follows from the above formula (10) for the Levi-Civita connection of this to the Hessian of the individual components.

\( \hat{\nabla}_A B = -\frac{1}{2} A Q^{-1} B - \frac{1}{2} B Q^{-1} A \) (10)

(Note that whilst \( A \) and \( B \) are affine, \( \hat{\nabla}_A B \) is not.) To see that this is indeed the Levi-Civita connection it suffices to note that it preserves the metric and that it is symmetric in \( A \) and \( B \), which is equivalent to being torsion free (see Lemma A.1).

We use the same notation \( \nabla \) for the induced connection on tensors over \( X \) with values in \( Q^* TP \) obtained from the Levi-Civita connections on \( (X, g_\parallel) \) and \( P \). So, for example, the Hessian of \( Q \) is given by \( \nabla dQ \). This is a section of \( T^* X \otimes T^* X \otimes Q^* TP \cong T^* X \otimes T^* X \otimes S^2(\mathbb{R}^3) \). We can relate this to the Hessian of the individual components \( Q_{ij} \) of \( Q \) via

\[
(\nabla dQ)_{ij} = \nabla(dQ_{ij}) - \frac{1}{2} Q^{pq} (dQ_{ip} \otimes dQ_{qj} + dQ_{qj} \otimes dQ_{ip})
\]

(11)

This follows from the above formula (10) for the Levi-Civita connection of \( P \). See Lemma A.2 for the details.

Given a tensor \( \xi \) over \( X \) with values in \( Q^* TP \), there are two possible ways to take its norm. We write \( |\xi|_Q \) for the norm of \( \xi \) computed with respect to the Riemannian metric \( g_\parallel \) on \( X \) and the symmetric metric on \( P \). The subscript \( Q \) is meant to evoke the definition (9) of the symmetric metric on \( P \). The alternative, which we denote simply by \( |\xi| \) is to treat \( \xi \) as a tensor with values in \( S^2 \mathbb{R}^3 \) and use the fixed innerproduct \( (A, B) = \text{Tr}(AB) \) on the matrix factor. We give two examples as illustrations:

\[
|dQ|_Q^2 = Q^{ij}(\nabla_i Q_{jk})Q^{kl}(\nabla^l Q_{hi})
\]

\[
|\nabla dQ|_Q^2 = Q^{ij}(\nabla dQ)_{jk}, Q^{kl}(\nabla dQ)_{hi})g_\parallel
\]

We write \( \hat{\Delta} Q = \text{Tr}_{g_\parallel} (\hat{\nabla} dQ) \) for the Laplacian of the map \( Q : X \to P \) of Riemannian manifolds, where we use the metric \( g_\parallel \) on \( X \) and the symmetric metric on \( P \). This is the Laplacian which appears in the harmonic map equation (sometimes called the “tension field”). Again, the “hat” notation is to differentiate this from the Laplacian of \( Q \) thought of simply as a map to the affine space of all matrices, i.e., the ordinary Laplacian taken component-by-component, \( (\Delta Q)_{ij} = \Delta(Q_{ij}) \). The two Laplacians are related by

\[
(\hat{\Delta} Q)_{ij} = \Delta(Q_{ij}) - Q^{pq} (dQ_{ip}, dQ_{jq})_g_\parallel
\]

This is proved in Lemma A.3.

We will frequently move back and forth between the 4-dimensional hypersymplectic structure and the corresponding 7-dimensional \( G_2 \)-structure. For clarity of notations, we will use bold symbols for those quantities associated to the 7-dimensional Riemannian manifold \( (X \times T^3, g_\parallel) \) and normal symbols for those associated to the 4-dimensional Riemannian manifold \( (X, g_\parallel) \). So, for example, \( R \) denotes the scalar curvature of \( g_\parallel \) whilst \( R \) denotes the scalar curvature of \( g_\parallel \). To remain consistent with this convention, we denote the torsion 2-form of \( \phi \) by \( T \).

Finally, when working in abstract index notation, we will use Roman indices \( i, j, k, \ldots \) to refer to the \( T^3 \) directions and Greek indices \( \alpha, \beta, \gamma, \ldots \) to refer to the \( X \) directions. So, for example, the 7-dimensional and 4-dimensional metrics are related by the equations \( g_{ij} = Q_{ij}, g_{\alpha\beta} = g_{\alpha\beta} \) and \( g_{i\alpha} = 0 \).

### 3.2 The curvature tensors in 4 and 7 dimensions

In this subsection we explain how the curvature tensor of \( g_\parallel \) on \( X \times T^3 \) is made up of that of \( g_\parallel \) on \( X \) and terms involving \( Q \) and its first and second derivatives.
The first step is to compute the Christoffel symbols, $\Gamma$ of $g_\phi$. The following is a simple calculation, the result of which we simply state.

**Lemma 3.1.** We have the following formulae for the Christoffel symbols of $g_\phi$:

\[
\begin{align*}
\Gamma^k_{ij} &= 0 \\
\Gamma^\gamma_{ij} &= -\frac{1}{2} g^{\gamma\alpha} \nabla_\alpha Q_{ij} \\
\Gamma^k_{i\beta} &= \frac{1}{2} Q^{ki} \nabla_\beta Q_{il} \\
\Gamma^k_{\alpha\beta} &= 0 \\
\Gamma^\gamma_{\alpha\beta} &= \Gamma^\gamma_{\alpha\beta} \\
\Gamma^k_{\alpha\beta} &= 0 \\
\Gamma^k_{\gamma\beta} &= 1
\end{align*}
\]

From here one can directly compute the components of the curvature tensor of $g_\phi$.

**Lemma 3.2.** The components of the curvature tensor of $g_\phi$ are given by

\[
\begin{align*}
R_{ijk}^l &= \frac{1}{4} (\nabla^j Q_{ik}) Q^{lp} (\nabla^p Q_{jk}) - \frac{1}{4} (\nabla^j Q_{jk}) Q^{lp} (\nabla^p Q_{il}) \\
R_{ij\alpha}^\beta &= 0 \\
R_{ij\beta}^\alpha &= \frac{1}{4} (\nabla^\beta Q_{jk}) Q^{kl} (\nabla^l Q_{i\alpha}) - \frac{1}{4} (\nabla^\beta Q_{ij}) Q^{kl} (\nabla^l Q_{k\alpha}) \\
R_{i\beta\gamma}^k &= 0 \\
R_{\alpha\beta\gamma}^k &= 1
\end{align*}
\]

and other omitted components are obtained by symmetries of Riemannian curvature tensor. (We recall that whenever $\hat{\nabla}$ appears, it refers to the components of the covariant derivatives of the corresponding tensor on $X$ with values in $Q^*T_P$ using the coupled connection.)

**Proof.** We give the calculation for $R_{j\beta\gamma}^k$ and surpress the others. In the first line of the following, the sum over the index $z$ is over both the $X$ and the $T^3$ directions.

\[
\begin{align*}
R_{j\beta\gamma}^k &= \partial_j \Gamma^\gamma_{\beta k} - \partial_j \Gamma^\gamma_{\beta z} - \Gamma_{\beta k}^z \Gamma^z_{j\gamma} + \Gamma_{\beta k}^z \Gamma^z_{j\gamma} \\
&= \frac{1}{2} \partial_\beta (\nabla^\gamma Q_{jk}) + \frac{1}{2} (\nabla^\gamma Q_{jk}) \Gamma_{\alpha\beta}^\gamma - \frac{1}{4} Q^{pq} (\nabla_\beta Q_{kp}) (\nabla^\gamma Q_{qj}) \\
&= \frac{1}{2} \nabla_\beta \nabla^\gamma Q_{jk} - \frac{1}{4} Q^{pq} (\nabla_\beta Q_{kp}) (\nabla^\gamma Q_{qj})
\end{align*}
\]

Now recall the formula (11) for the Hessian $\hat{\nabla} dQ$ of $Q$ with respect to the symmetric metric on $P$. It rearranges to give

\[
\nabla^\gamma \nabla_\beta Q_{jk} = \hat{\nabla}^\gamma \nabla_\beta Q_{jk} + \frac{1}{2} (\nabla_\beta Q_{jp}) Q^{pq} (\nabla^\gamma Q_{qj}) + \frac{1}{2} (\nabla_\beta Q_{kp}) Q^{pq} (\nabla^\gamma Q_{qj})
\]

Substituting this in for $\nabla_\beta \nabla^\gamma Q_{jk} = \nabla^\gamma \nabla_\beta Q_{jk}$ gives the claimed formula. \hfill \Box

**Corollary 3.3.** For any $\alpha_1, \ldots, \alpha_m$ we have

\[
\nabla_{\alpha_1} \cdots \nabla_{\alpha_m} R_{\alpha\beta\gamma}^\mu = \nabla_{\alpha_1} \cdots \nabla_{\alpha_m} R_{\alpha\beta\gamma}^\mu
\]

and hence $|\nabla^m R_m|^2 \leq |\nabla^m R_m|^2$. 

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At various times from now on, we will diagonalize \( Q \) at a given point on \( X \), in order to streamline some calculations. More precisely, at any given point \( p \in X \), we choose an orthogonal matrix \( P \) such that \( PQ|_p P^t = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \). At the fixed point \( p \), the triple \( P \) is another hypersymplectic structure defining the Riemannian metric \( g_{ij} \otimes (PQ|_p P^t)_{ij} dt^i dt^j \), which is isometric at \( p \) to the original Riemannian metric \( g_{ij} \otimes Q_{ij} dt^i dt^j \). Thus, at \( p \) all the geometric quantities can be calculated assuming \( Q \) is diagonal, with \( Q = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \).

**Lemma 3.4.**

\[
|\nabla dQ|_Q^2 \leq \frac{5}{4} (|Rm|^2 + |dQ|_Q^2)
\]

**Proof.** According to the fifth identity in Lemma 3.2

\[
\hat{\nabla}^\alpha \nabla_\beta Q_{jk} = 2R_{j\beta k\alpha} - \frac{1}{2} (\nabla_\beta QQ^{-1} \nabla^\alpha Q)_{jk}
\]

From this we have

\[
|\nabla dQ|_Q^2 = \left( 2R_{j\beta k\alpha} \right)^2 - \frac{1}{2} |(\nabla_\beta QQ^{-1} \nabla^\alpha Q)_{jk}|^2
\]

Now diagonalizing \( Q \) at a point and using normal coordinates \( x_\alpha \) at this point for the metric \( g_{ij} \), we obtain:

\[
|\nabla dQ|_Q^2 = 4R_{j\beta k\alpha} R_{p \gamma q \alpha} Q^{jp} Q^{kq} - 2 \sum_{j,k,\alpha,\beta} \left( \lambda_j^{-\frac{1}{2}} \lambda_k^{-\frac{1}{2}} R_{j\beta k\alpha} \right) \left( (\nabla_\beta QQ^{-1} \nabla^\alpha Q)_{jk} \lambda_j^{-\frac{1}{2}} \lambda_k^{-\frac{1}{2}} \right)
\]

\[
+ \frac{1}{4} \text{Tr} \left( Q^{-1} \nabla_\beta QQ^{-1} \nabla_\alpha QQ^{-1} \nabla^\alpha Q \right)
\]

\[
\leq |Rm|^2 + \sum_{j,k,\alpha,\beta} \left( \lambda_j^{-\frac{1}{2}} \lambda_k^{-\frac{1}{2}} |R_{j\beta k\alpha}|^2 + \lambda_j^{-\frac{1}{2}} \lambda_k^{-\frac{1}{2}} \sum_p |\nabla_\beta Q_{jp} \lambda_j^{-\frac{1}{2}} \lambda_k^{-\frac{1}{2}} Q_{pk}|^2 \right)
\]

\[
+ \frac{1}{4} \text{Tr} \left( Q^{-1} \nabla_\beta QQ^{-1} \nabla_\alpha QQ^{-1} \nabla^\alpha Q \right)
\]

provided \( \lambda_j^{-\frac{1}{2}} \lambda_k^{-\frac{1}{2}} \) and is an application of Cauchy–Schwarz.

The term \( |Rm|^2 \) appears here because of the inequality

\[
|Rm|^2 \geq R_{j \beta k} R_{p \gamma q \alpha} Q^{jp} Q^{kq} + R_{j \alpha \beta} R_{p \beta q \alpha} Q^{jp} Q^{kq}
\]

\[
+ R_{j \beta k} R_{p \beta q \alpha} Q^{jp} Q^{kq} + R_{j \beta k} R_{p \beta q \alpha} Q^{jp} Q^{kq}
\]

\[
= 4R_{j \beta k} R_{p \beta q \alpha} Q^{jp} Q^{kq}
\]

(13)

where on the right hand side we have simply not included some of the positive terms which would appear in the norm, such as \( R_{\alpha \beta \gamma \delta} \). Meanwhile, the estimation of the term involving the sum over \( j, k, \alpha, \beta \) is an application of Cauchy–Schwarz.

Now, recalling that \( Q \) is diagonal and \( g_{\alpha \beta} = \delta_{\alpha \beta} \) at the point at which we’re working, the same inequality (13) gives

\[
\sum_{j,k,\alpha,\beta} \lambda_j^{-1} \lambda_k^{-1} |R_{j\beta k\alpha}|^2 \leq \frac{1}{4} |Rm|^2
\]

(14)
Meanwhile, by Cauchy–Schwarz,
\[
\left|\sum_p \nabla^\beta Q_{jp} \lambda_p^{-1} \nabla^\alpha Q_{pk}\right|^2 = \left|\sum_p (\nabla^\beta Q_{jp} \lambda_p^{-\frac{1}{2}}) (\lambda_p^{-\frac{1}{2}} \nabla^\alpha Q_{pk})\right|^2 \\
\leq \sum_p \lambda_p^{-1} |\nabla^\beta Q_{jp}|^2 \sum_q \lambda_q^{-1} |\nabla^\alpha Q_{pk}|^2
\]
Also,
\[
|dQ|_Q^2 = \text{Tr}(Q^{-1} \nabla^\alpha QQ^{-1} \nabla_\alpha Q) = \sum_{i,j,\alpha} \lambda_i^{-1} \lambda_j^{-1} |\nabla^\alpha Q_{ij}|^2
\]
From this we have that
\[
\sum_{j,k,\alpha,\beta,p} \left|\sum_p \nabla^\beta Q_{jp} \lambda_p^{-1} \nabla^\alpha Q_{pk}\right|^2 \leq \sum_{j,k,p,q,\alpha,\beta} \lambda_j^{-1} \lambda_k^{-1} \lambda_p^{-1} \lambda_q^{-1} |\nabla^\alpha Q_{jp}|^2 |\nabla^\beta Q_{pk}|^2 = |dQ|_Q^4 \tag{15}
\]
Putting (14) and (15) in (12) we obtain
\[
|\hat{\nabla}dQ|_Q^2 \leq \frac{5}{4} |\hat{\nabla} Rm|_Q^2 + |dQ|_Q^4 + \frac{1}{4} \text{Tr}(Q^{-1} \nabla^\beta QQ^{-1} \nabla_\beta QQ^{-1} \nabla^\alpha QQ^{-1} \nabla_\alpha Q) \tag{16}
\]
Finally we use the fact that for any positive definite matrix $A$, $\text{Tr}(A^2) \leq (\text{Tr} A)^2$. This gives
\[
\text{Tr}(Q^{-1} \nabla^\beta QQ^{-1} \nabla_\beta QQ^{-1} \nabla^\alpha QQ^{-1} \nabla_\alpha Q) = \text{Tr} \left( Q^{-\frac{1}{2}} (\nabla^\beta Q) Q^{-1} (\nabla_\beta Q) Q^{-\frac{1}{2}} \right)^2 \\
\leq \left[ \text{Tr} \left( Q^{-\frac{1}{2}} (\nabla^\beta Q) Q^{-1} (\nabla_\beta Q) Q^{-\frac{1}{2}} \right)^2 \right]^2 \\
= |dQ|_Q^4
\]
Together with (16), this completes the proof. □

**Lemma 3.5.** The components of the Ricci tensor of $g_\phi$ are:

\[
R_{ij} = -\frac{1}{2} (\Delta Q)_{ij} \\
R_{i\alpha} = 0 \\
R_{\alpha\beta} = \frac{1}{4} \text{Tr}(Q^{-1} \nabla Q \otimes Q^{-1} \nabla Q)_{\alpha\beta}
\]

The scalar curvature of $g_\phi$ is

\[
R = R - \frac{1}{4} |dQ|_Q^2
\]

**Proof.** These formulae are obtained by direct calculation from the components of the full curvature tensor. We give the first as an example and suppress the details of the other calculations.

\[
R_{jk} = R_{ijk}^i + R_{\alpha jk}^\alpha \\
= \frac{1}{4} (\nabla^\alpha Q_{ijk}) \nabla^p Q_{pj} - \frac{1}{4} (\nabla^\alpha Q_{jk}) \nabla^p Q_{pi} - \frac{1}{2} \hat{\nabla}^\alpha \nabla_\alpha Q_{jk} - \frac{1}{4} (\nabla_\alpha Q_{jp}) \nabla^\alpha \nabla^\alpha Q_{lk} \\
= -\frac{1}{2} (\Delta Q)_{jk}
\]
since the first and last terms in the second line cancel and the second vanishes because $\det Q = 1$, which implies that $\text{Tr}(Q^{-1} \nabla Q) = 0$. □
3.3 The Levi-Civita connection on $\Lambda^+$ and the torsion 2-form

The triple $\omega_1, \omega_2, \omega_3$ give a framing of the bundle $\Lambda^+ \to X$ of self-dual 2-forms. In this framing, the Levi-Civita connection of $\Lambda^+$ is given by a matrix $a_{ij}$ of 1-forms:

$$\nabla \omega_i = a_{ij} \otimes \omega_j \quad (17)$$

We make a comment here on our notation: to lighten equations, the Einstein summation convention of pairing superscripts and subscripts has been dropped from here on in; two identical Roman indices (regardless of upper or lower position) are considered to be summed over unless explicitly stated otherwise. We continue to use the notation $Q_{ij}$ for the components of the inverse matrix $Q^{-1}$.

In this subsection we will explain how to determine the $a_{ij}$’s in terms of the torsion 2-form $T$ of the $G_2$-structure. This will be important in the following subsection, when we control $\nabla \omega$, $\nabla^2 \omega$ and $\nabla T$.

To begin we give a purely 4-dimensional description of $T$. We let $E_i : \Lambda^1 \to \Lambda^1$ be the operator defined by

$$E_i(\alpha) = -\ast_4 (\alpha \wedge \omega_i) \quad (18)$$

When the triple $\omega$ is hyperkähler, the $E_i$’s are simply the hyperkähler complex structures. In general, $E_i$ is skew-adjoint and $E_i^2 = -Q_{ii}$ (no summation here). With this in hand, we can describe the torsion 2-form. Define a triple of 1-forms $\tau = (\tau_1, \tau_2, \tau_3)$ by

$$\tau_i = -E_k(dQQ^{-1})_{ik} \quad (19)$$

**Lemma 3.6.** Given a hypersymplectic structure $\omega$ on $X$, the torsion 2-form $T = -\frac{1}{2} \ast d\phi$ of the corresponding $G_2$-structure $\phi$ on $X \times T^3$, defined as in (3), is

$$T = -\frac{1}{2} dt^i \wedge \tau_i \quad (20)$$

**Proof.** Beginning from (8) we have that

$$d\ast \phi = Q_{jk} dt^k \wedge \ast_4 d(Q^{ij} \omega_i) = -dt^k \wedge E_i(dQQ^{-1} Q)_{ik} = dt^k \wedge \tau_k$$

which proves (20). $\square$

The main result of this subsection is the following.

**Proposition 3.7.** The Levi-Civita connection matrix is given by

$$a_{ij} = \frac{1}{2}(dQQ^{-1})_{ij} + (XQ^{-1})_{ij} \quad (21)$$

where $X$ is the matrix of 1-forms given by

$$X_{ij} = \frac{1}{2} \varepsilon_{ijk} Q^{kl} \tau_l$$

(Notice that, because the trivialisation $\omega_1, \omega_2, \omega_3$ of $\Lambda^+$ is not necessarily orthonormal, the connection matrix $a_{ij}$ is not skew in $i, j$.) The proof will follow from a series of lemmas.

**Lemma 3.8.** The connection matrix $a_{ij}$ is uniquely determined by the two equations

$$aQ + Qa^t = dQ \quad (22)$$
$$E_j a_{ij} = 0 \quad (23)$$
Proof. Just as for an affine connection on $T^*X$, it is possible to define the torsion of a connection $D$ on the bundle $\Lambda^+$: let $t(D) \in \Gamma(\Lambda^3 \otimes (\Lambda^+)^*)$ be given by
\[
t(D)(\xi) = (s \circ D - d)(\xi)
\]
where $s: \Lambda^1 \otimes \Lambda^+ \to \Lambda^3$ is skew-symmetrisation. Just as the Levi-Civita connection on $T^*X$ is the unique metric torsion-free connection on $T^*X$, so the Levi-Civita connection on $\Lambda^+$ is the unique metric-compatible torsion-free connection on $\Lambda^+$. (This is a classical fact in 4-dimensional Riemannian geometry, but which is not so simple to track down in the literature. The details are explained, for example, immediately after Lemma 2.2 of [14] and Proposition 2.3 therein).

Now, in the trivialisation given by the triple $\omega$, being metric-compatible is equivalent to (22). Meanwhile, since $d\omega_i = 0$, the torsion-free condition is equivalent to the equation $a_{ij} \wedge \omega_j = 0$. Applying the Hodge star, we see that this is equivalent to (23).

Lemma 3.9 ($Q$-twisted quaternion relations). The operators $E_1, E_2, E_3$ satisfy
\[
E_i E_j = -\epsilon_{ijk} Q^{kp} E_p - Q_{ij} \tag{24}
\]
\[
E_i E_j E_k = -Q_{ij} E_k - Q_{jk} E_i + Q_{ik} E_j + \epsilon_{ijk} \tag{25}
\]

Proof. We begin with (24). Let $Q^{-1/2}$ denote the positive definite square root of $Q^{-1}$. Write $\theta_i = (Q^{-1/2})_{ij} \omega_j$. Then algebraically, $\theta$ is a hyperkähler triple. In other words, if we define $J_i(\alpha) = -* (\alpha \wedge \theta_i)$ then the $J_i$’s satisfy the quaternion relations $J_i J_j = -\epsilon_{ijk} J_k - \delta_{ij}$. (The sign in front of the first term here is because we consider operators on covectors; the dual operators on tangent vectors satisfy $J_i J_j = \epsilon_{ijk} J_k - \delta_{ij}$.) Now, since $\omega_i = (Q^{1/2})_{ij} \theta_j$ we have that $E_i = (Q^{1/2})_{ij} J_j$ and so
\[
E_i E_j = (Q^{1/2})_{ij} (Q^{1/2})_{jq} J_p J_q = (Q^{1/2})_{ip} (Q^{1/2})_{jq} (-\epsilon_{pqr} J_r - \delta_{pq})
\]

Now for any $3 \times 3$ matrix $A$ we have $\epsilon_{ijk} A_{ip} A_{jq} A_{kr} = \epsilon_{pqr} \det A$. This implies the identity
\[
\epsilon_{ijk} A_{ip} A_{jq} = \epsilon_{pqr} (A^{-1})_{rk} \det A \tag{26}
\]

Since $\det(Q^{1/2}) = 1$ we have $\epsilon_{ijk} (Q^{1/2})_{ip} (Q^{1/2})_{jq} = \epsilon_{ijt} (Q^{-1/2})_{tr}$. This means that
\[
E_i E_j = -\epsilon_{ijt} (Q^{-1/2})_{tr} J_r - Q_{ij}
\]

Now $J_r = (Q^{-1/2})_{rs} E_s$ gives (24).

Equation (25) follows from two applications of (24) and the identity $\epsilon_{ipr} Q^{qs} Q^{rs} = \epsilon_{prq} Q_{ij}$, following from (26), which holds since $\det Q = 1$. We suppress the details.

Lemma 3.10. $\epsilon_{ijk} E_j \tau_k = (Q^{-1} r)_i$

Proof. We compute:
\[
\epsilon_{ijk} E_j \tau_k = -\epsilon_{ijk} E_j E_r (dQQ^{-1})_{kr}
\]
\[
= -\epsilon_{ijk} (-\epsilon_{jpr} Q^{qs} E_q - Q_{jr}) (dQQ^{-1})_{kr}
\]
\[
= \epsilon_{ijk} \epsilon_{jpr} Q^{qs} E_q (dQQ^{-1})_{kr}
\]
since the term we have dropped comes from summing $\epsilon_{ijk}$ against $Q_{jr} (dQQ^{-1})_{kr} = dQ_{kj}$ which is symmetric in $j, k$. It is easy to verify the following identity:
\[
\epsilon_{ijk} \epsilon_{abk} = \delta_{ia} \delta_{jb} - \delta_{ib} \delta_{ja} \tag{27}
\]
By (27),
\[
\epsilon_{ijk}\epsilon_{jqp}Q^{pq}E_q(dQQ^{-1})_{kr} = -(\delta_{kp}\delta_{ir} - \delta_{kr}\delta_{pi})Q^{pq}E_q(dQQ^{-1})_{kr}
\]
\[
= -Q^{-1}_{kq}E_q(dQQ^{-1})_{ki}
\]
where the term we have dropped involves Tr(dQQ^{-1}) which vanishes since det Q = 1. Continuing,
\[
-Q^{kq}E_q(dQQ^{-1})_{ki} = -E_q(Q^{-1}dQQ^{-1})_{iq}
\]
\[
= -E_q(Q^{-1}dQQ^{-1})_{iq}
\]
\[
= (Q^{-1}r)_i.
\]

**Remark 3.11.** For a general 7-manifold with $G_2$-structure $(M, \phi)$ there is a decomposition of $\Lambda^2(M)$ defined in terms of $\phi$ and $*\phi$ as follows (cf. for instance [3, Equation 2.14]):
\[
\Lambda^1_{14} = \{ T \in \Lambda^2(M) : T \wedge \phi = -*\phi T \}
\]
\[
\Lambda^2_{14} = \{ S \in \Lambda^2(M) : S \wedge \phi = 2 *\phi S \}
\]
When $d\phi = 0$, it is known that the torsion $-\frac{1}{2}d^*\alpha$ automatically lies in the sub-bundle $\Lambda^2_{14}$. In our situation, where $M = X \times T^3$, there is a further decomposition of $\Lambda^2$, according to the number of components in the $T^3$-directions:
\[
\Lambda^2_{14} = \Lambda^2_{14(0,3)} \oplus \Lambda^2_{14(1,8)} \oplus \Lambda^2_{14(2,3)}
\]
\[
\Lambda^2_{7} = \Lambda^2_{7(1,4)} \oplus \Lambda^2_{7(2,3)}
\]
where
\[
\Lambda^2_{14(0,3)} = \Lambda^- (X)
\]
\[
\Lambda^2_{14(1,8)} = \left\{ \sum_{i=1}^{3} dt^i \wedge \sigma_i : (Q^{-1}\sigma)_i = \epsilon_{ijk}E_j\sigma_k, \text{ where } \sigma_i \in \Lambda^1(X) \text{ for } i = 1, 2, 3 \right\}
\]
\[
\Lambda^2_{14(2,3)} = \text{Span} \left\{ 2\tilde{dt}^k + \omega_k : k = 1, 2, 3 \right\}
\]
\[
\Lambda^2_{7(1,4)} = \left\{ \sum_{i=1}^{3} dt^i \wedge \sigma_i : (Q^{-1}\sigma)_i = -\frac{1}{2}\epsilon_{ijk}E_j\sigma_k, \text{ where } \sigma_i \in \Lambda^1(X) \text{ for } i = 1, 2, 3 \right\}
\]
\[
\Lambda^2_{7(2,3)} = \text{Span} \left\{ \tilde{dt}^k - \omega_k | k = 1, 2, 3 \right\}
\]

Lemma 3.10 says that, in our more symmetric situation, not only does the torsion lie in $\Lambda^2_{14}$, but it is actually constrained to lie in the even smaller sub-bundle $\Lambda^2_{14(1,8)}$. This point might be useful in the future studies.

**Proof of Proposition 3.7** Set $X_{ij} = \frac{1}{2}\epsilon_{ijk}Q^{kl}\tau_l$. Recall that we must show that
\[
\alpha = \frac{1}{2}dQQ^{-1} + XQ^{-1}
\]
satisfies (22) and (23). Equation (22) is immediate, since $X$ is skew-symmetric. To prove (23), we compute:
\[
(XQ^{-1})_{ij} = \frac{1}{2}\epsilon_{ijk}Q^{kl}Q^{pq} = \frac{1}{2}\epsilon_{ijkl}Q_{ir}\tau_l
\]
\[
(28)
\]
\[\text{The first index in the parentheses denotes the number of component } dt^i \text{ in the 2-covectors in this subspace, and the second index denotes the dimension of this subspace.}\]
(using $\det Q = 1$) from which it follows that
\[
E_j(X Q^{-1})_{ij} = \frac{1}{2} \epsilon_{rij} Q_{ir} E_j \tau_r = \frac{1}{2} \tau_i
\]
by Lemma 3.10. Now
\[
E_j a_{ij} = \frac{1}{2} E_j (dQQ^{-1})_{ij} + \frac{1}{2} \tau_j = -\frac{1}{2} \tau_j + \frac{1}{2} \tau = 0
\]
as claimed. \hfill \square

### 3.4 Bounds on the derivatives of $\omega$ and of torsion

The aim of this subsection is to prove bounds on $\nabla^2 \omega$, $\nabla^3 \omega$, $\nabla T$, and $\nabla T^*$ purely in terms of $Q$ and its derivatives. These inequalities and constants are useful for the various quantities that one wishes to control. The key point is that at each stage the control is purely in terms of $\text{Tr} Q$, $|dQ|$ and $|\nabla dQ|$. We begin with a lemma which will allow us to pass between various matrix norms.

**Lemma 3.12.** Let $A$ be a $3 \times 3$ symmetric matrix of tensors. Then
1. \[
\frac{3}{(\text{Tr} Q)^2} |A|_Q \leq |A| \leq (\text{Tr} Q) |A|_Q
\]
2. \[
|AQ^{-1}|^2 \leq \frac{1}{3} (\text{Tr} Q)^3 |A|_Q^2
\]

**Proof.** These are pointwise estimates and so we can assume that $Q_{ij} = \lambda_i \delta_{ij}$ (no summation on $i$) is diagonal at the point of interest. Recall that $\det Q = \lambda_1 \lambda_2 \lambda_3 = 1$, a fact which we will use frequently and without further notice. We have
\[
|A|_Q^2 = \lambda_1 \lambda_2 \lambda_3 = \sum_{i,j} \lambda_i^{-1} \lambda_j^{-1} |A_{ij}|^2 \geq (\text{Tr} Q)^{-2} \sum_{i,j} |A_{ij}|^2 = (\text{Tr} Q)^{-2} |A|^2
\]
The inequality in the middle is because for any $i = 1, 2, 3$, $\text{Tr} Q \geq \lambda_i$ and so for any $i, j$, $\text{Tr}(Q^2) \geq \lambda_i \lambda_j$. (Here and throughout the proof the inner product $(\cdot, \cdot)$ and norm $|\cdot|$ are measured using $g_{ij}$, but the symbol “$\det$” is dropped to keep the notation clean while hopefully causing no confusion.)

This proves the upper bound on $|A|$. For the lower bound, we will use the inequality
\[
(\text{Tr} Q)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) \geq 3(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1)
\] (29)

Now,
\[
|A|_Q^2 = \sum_{i} \frac{1}{\lambda_i} |A_{i1}|^2 + \sum_{i,j,k} \frac{1}{\lambda_i \lambda_j \lambda_k} |A_{ijk}|^2
\]
\[
= \left( \frac{\lambda_2 \lambda_3}{\lambda_1} |A_{11}|^2 + \frac{\lambda_3 \lambda_1}{\lambda_2} |A_{12}|^2 + \frac{\lambda_1 \lambda_2}{\lambda_3} |A_{13}|^2 \right) + 2 \lambda_1 |A_{23}|^2 + 2 \lambda_2 |A_{31}|^2 + 2 \lambda_3 |A_{21}|^2
\]
\[
\leq \left( \frac{\lambda_2 \lambda_3}{\lambda_1} + \frac{\lambda_3 \lambda_1}{\lambda_2} + \frac{\lambda_1 \lambda_2}{\lambda_3} + 2 \lambda_1 + 2 \lambda_2 + 2 \lambda_3 \right) \sum_{i,j} |A_{ij}|^2
\]
\[
= (\lambda_2 \lambda_3 + \lambda_3 \lambda_1 + \lambda_1 \lambda_2) \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \sum_{i,j} |A_{ij}|^2
\]
\[
= (\lambda_2 \lambda_3 + \lambda_3 \lambda_1 + \lambda_1 \lambda_2)^2 \sum_{i,j} |A_{ij}|^2
\]
\[
\leq \frac{1}{9} (\text{Tr} Q)^4 |A|^2
\]
This proves the first pair of inequalities. To prove part 2, we note that
\[ |AQ^{-1}|^2 = \sum_{i,j} |(AQ^{-1})_{ij}|^2 = \sum_{i,j} \lambda_{ij}^{-2} |A_{ij}|^2 \]
\[ = \sum_i \frac{1}{\lambda_i^2} |A_{ii}|^2 + \sum_{j<k} (\lambda^{-2}_j + \lambda^{-2}_k) |A_{jk}|^2 \]
\[ = \sum_i \frac{1}{\lambda_i^2} |A_{ii}|^2 + \sum_{j<k} \left( \frac{\lambda_j}{\lambda_k} + \frac{\lambda_k}{\lambda_j} \right) \frac{1}{\lambda_j \lambda_k} |A_{jk}|^2 \]
\[ \leq \sum_i \frac{1}{\lambda_i^2} |A_{ii}|^2 + (\text{Tr } Q)(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) \sum_{j<k} \frac{1}{\lambda_j \lambda_k} |A_{jk}|^2 \]
\[ \leq \sum_i \frac{1}{\lambda_i^2} |A_{ii}|^2 + \frac{1}{3}(\text{Tr } Q)^3 \sum_{j<k} \frac{1}{\lambda_j \lambda_k} |A_{jk}|^2 \]
\[ \leq \frac{1}{3}(\text{Tr } Q)^3 |A|^2_Q \]

To obtain the first inequality here we use that for any \( j \neq k \),
\[ (\text{Tr } Q)(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) = (\lambda_1 + \lambda_2 + \lambda_3)(\lambda^{-1}_1 + \lambda^{-1}_2 + \lambda^{-1}_3) \geq \frac{\lambda_j}{\lambda_k} + \frac{\lambda_k}{\lambda_j} \]

For the second inequality, we use (29), and for the last inequality we use \( \frac{1}{3}(\text{Tr } Q)^3 \geq 1 \) (which holds since \( \det Q = 1 \)).

**Lemma 3.13.** We have the following bounds:

1. \( |Z|^2 \leq 2(\text{Tr } Q)|T|^2 \).
2. \( |T|^2 \leq \frac{3}{2}|dQ|^2 \).

**Proof.** The calculation is pointwise, so we assume that \( Q_{ij} = \lambda_j \delta_{ij} \) at the point of interest. From (20), we have
\[ |T|^2 = \frac{1}{2} Q^{ij} \langle \tau_i, \tau_j \rangle = \frac{1}{2} \sum_i \lambda_i^{-1} |\tau_i|^2 \geq \frac{1}{2 \text{Tr } Q} |Z|^2 \]
which proves the first inequality. For the second we compute, using \( E^*_j E_j = -E^3_j = \lambda_j \),
\[ |\tau_i|^2 = \left| \sum_j E_j (dQQ^{-1})_{ij} \right|^2 \leq 3 \sum_j |E_j (dQQ^{-1})_{ij}|^2 = 3 \sum_j \lambda_j |(dQQ^{-1})_{ij}|^2 = 3 \sum_j \lambda_j^{-1} |dQ_{ij}|^2 \]
Therefore,
\[ |T|^2 = \frac{1}{2} \sum_i \lambda_i^{-1} |\tau_i|^2 \leq \frac{3}{2} \sum_{i,j} \lambda_i^{-1} \lambda_j^{-1} |dQ_{ij}|^2 = \frac{3}{2}|dQ|^2_Q \]

**Lemma 3.14.** For any hypersymplectic structure \( \omega \),
\[ |\nabla \omega| \leq 13(\text{Tr } Q)^2|dQ|_Q \]

**Proof.** From (21) we have
\[ |a_{ij}| \leq \frac{1}{2} \left| \langle dQQ^{-1} \rangle_{ij} | + |(XQ^{-1})_{ij} \right| \]
\[ \leq \frac{1}{2\sqrt{3}} (\text{Tr } Q)^{3/2}|dQ|_Q + \frac{1}{2} |e_{pjq} Q_{ip} \tau_q| \]

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by Lemma 3.12 for the first term, and (28) for the second. For fixed $i, j$, $\varepsilon_{pjq}Q_{ip}Q_{q}$ is a sum of two terms of the form $Q_{ip}Q_{q}$. Since $|Q_{ip}|^2 \leq \text{Tr}(Q^2) \leq (\text{Tr} Q)^2$, we have $|\varepsilon_{pjq}Q_{ip}Q_{q}| \leq 2(\text{Tr} Q)|\tau|$. It follows that

$$|a_{ij}| \leq \frac{1}{2\sqrt{3}}(\text{Tr} Q)^{3/2} |dQ|_Q + (\text{Tr} Q)|\tau| \leq 3(\text{Tr} Q)^{3/2} |dQ|_Q$$

(30)

where we have used Lemma 3.13 which gives $|\tau| \leq \sqrt{3}(\text{Tr} Q)^{1/2} |dQ|_Q$. Hence

$$|\nabla \omega|^2 = \sum_{i,j,k} (a_{ij} \otimes \omega_j, a_{ik} \otimes \omega_k)$$

$$= 2 \sum_{i,j,k} Q_{jk} (a_{ij}, a_{ik})$$

$$\leq 2 \text{Tr} Q \sum_{i,j} |a_{ij}|^2$$

$$\leq 162(\text{Tr} Q)^4 |dQ|_Q^2$$

We next control $\nabla \tau$. We begin with a lemma.

**Lemma 3.15.** For all $\alpha \in \Lambda^1$, we have $(\nabla E_i)(\alpha) = a_{ij} \otimes E_j(\alpha)$.

**Proof.** The Levi-Civita connection commutes with the Hodge star. From this we see

$$\nabla(E_i(\alpha)) = -\nabla(*((\nabla \omega) \wedge \omega_i + \alpha \wedge \nabla \omega_i)) = E_i(\nabla \omega) + a_{ij} \otimes E_j(\alpha)$$

This gives the result in view of $(\nabla E_i)(\alpha) = \nabla(E_i(\alpha)) - E_i(\nabla \omega)$. \hfill $\square$

**Lemma 3.16.** For any hypersymplectic structure $\omega$,

$$|\nabla \tau| \leq 8(\text{Tr} Q)^2 \left( |\nabla dQ|_Q + (\text{Tr} Q)^5 |dQ|_Q^2 \right)$$

**Proof.** By definition of $\tau$, we have

$$\nabla \tau = -(\nabla E_j)(dQQ^{-1})_{ij} - E_j(\nabla dQQ^{-1})_{ij} + E_j(dQQ^{-1} \otimes dQQ^{-1})_{ij}$$

We now bound each term separately, beginning with the first. By the formula in Lemma 3.15

$$|(\nabla E_j)(dQQ^{-1})_{ij}|^2 \leq 9 \sum_{j,k} |a_{jk}|^2 |E_k(dQQ^{-1})_{ij}|^2$$

$$\leq 81(\text{Tr} Q)^4 |dQ|_Q^2 \sum_{j,k} |(dQQ^{-1})_{ij}|^2$$

$$\leq 81(\text{Tr} Q)^7 |dQ|_Q^4$$

where in the first line we have used Cauchy-Schwarz inequality. In the second line we have used (28) to bound $|a_{ijk}|^2$ and the fact that $|E_k(\alpha)|^2 \leq (\text{Tr} Q)|\alpha|^2$. And here we have written the sum explicitly since repeated indices have disappeared. Then in the final line we apply part 2 of Lemma 3.12.

For the second term,

$$|E_j(\nabla dQQ^{-1})_{ij}|^2 \leq 3 \text{Tr} Q \sum_{j} |(\nabla dQQ^{-1})_{ij}|^2$$

$$\leq 3(\text{Tr} Q)^4 |\nabla dQ|_Q^2$$
(by Lemma 3.12). We recall equation [11] that
\[(\nabla dQ)_{ij} = (\nabla dQ)_{ij} + \frac{1}{2} Q^p q (dQ_{ip} \otimes dQ_{qj} + dQ_{qj} \otimes dQ_{ip})\]
It follows that
\[|\nabla dQ - \nabla dQ|^2 = \sum_{i,j} \left( |(dQQ^{-1} \otimes dQ)_{ij} + (dQQ^{-1} \otimes dQ)_{ji}|^2 \right) \leq \sum_{i,j} |(dQQ^{-1} \otimes dQ)_{ij}|^2 \]
\[= \sum_{i,j} \left( \sum_{p,q} Q^p q (dQ_{ip} \otimes dQ_{qj}) \right)^2 \leq 9 \sum_{p,q} (Q^p q)^2 \sum_{i,j,k,l} |dQ_{ik} \otimes dQ_{lj}|^2 \]
\[\leq (\text{Tr} Q)^4 |dQ|^4\]
where in the fourth line inequality we’ve used the Cauchy-Schwarz inequality, which gives the coefficient 9, and for the last inequality we use
\[\text{Tr}(Q^{-1})^2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} = (\lambda_2 \lambda_3)^2 + (\lambda_3 \lambda_1)^2 + (\lambda_1 \lambda_2)^2 \leq (\lambda_2 \lambda_3 + \lambda_3 \lambda_1 + \lambda_1 \lambda_2)^2 \leq \frac{1}{9} (\text{Tr} Q)^4\]
by virtue of [29].
Converting the left hand side with Lemma 3.12 gives
\[|\nabla dQ - \nabla dQ|_Q^2 \leq (\text{Tr} Q)^2 |\nabla dQ - \nabla dQ|_Q^2 \leq (\text{Tr} Q)^2 (\text{Tr} Q)^4 |dQ|_Q^4 = (\text{Tr} Q)^4 |dQ|_Q^4\]
and so (by the elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$ applied to the matrices components)
\[|\nabla dQ|_Q^2 \leq 2(\nabla dQ - \nabla dQ|_Q^2 + |\nabla dQ|_Q^2) \leq 2 |\nabla dQ|_Q^2 + 2(\text{Tr} Q)^{10} |dQ|_Q^4\]  
(31)
This implies for the second term in $\nabla \tau_i$, that
\[|E_j (\nabla dQQ^{-1})_{ij}|^2 \leq 3(\text{Tr} Q)^4 (2 |\nabla dQ|_Q^2 + 2(\text{Tr} Q)^{10} |dQ|_Q^4)\]
Finally we come to the third term:
\[|E_j (dQQ^{-1} \otimes dQQ^{-1})_{ij}|^2 \leq 3 \text{Tr} Q \sum_j |(dQQ^{-1} \otimes dQQ^{-1})_{ij}|^2 \]
\[\leq 3 \text{Tr} Q \sum_{j,p,q} |(dQQ^{-1})_{ij}|^2 |(dQQ^{-1})_{jq}|^2 \]
\[\leq 3 \text{Tr} Q |dQQ^{-1}|^4 \]
\[\leq \frac{1}{3} (\text{Tr} Q)^7 |dQ|_Q^4\]
Putting the pieces together, we have
\[|\nabla \tau_i|^2 \leq 3 \left( |(\nabla E_j)(dQQ^{-1})_{ij}|^2 + |E_j (\nabla dQQ^{-1})_{ij}|^2 + |E_j (dQQ^{-1} \otimes dQQ^{-1})_{ij}|^2 \right) \]
\[\leq 18(\text{Tr} Q)^4 |\nabla dQ|_Q^4 + (244(\text{Tr} Q)^7 + 18(\text{Tr} Q)^{14}) |dQ|_Q^4\]
\[\leq 19(\text{Tr} Q)^4 |\nabla dQ|_Q^4 + (\text{Tr} Q)^{10} |dQ|_Q^4\]
again by using $\text{Tr} Q \geq 3$. The result stated in the lemma follows. 
\[\square\]
Lemma 3.17. For any hypersymplectic structure $\omega$, we have

$$|\nabla T| \leq 9(\text{Tr}Q)^8 \left( |\nabla dQ| + |dQ|^2 \right)$$

Proof. Since $T = -\frac{1}{2} dt^i \wedge \tau_i$ and since (by explicit calculation using Christoffel symbols from Lemma 3.1)

$$\nabla dt^i = -\frac{1}{2} dt^i \otimes (dQQ^{-1})_{ij} - \frac{1}{2} (dQQ^{-1})_{ij} \otimes dt^j$$

$$\nabla \tau_i = \nabla \tau_i + \frac{1}{2} \tau_i (\nabla Q_{jk}) dt^j \otimes dt^k$$

we have that

$$\nabla T = -\frac{1}{2} \nabla (dt^i \otimes \tau_i - \tau_i \otimes dt^i)$$

$$= -\frac{1}{2} (dt^i \otimes \nabla \tau_i - \nabla \tau_i \otimes dt^i)$$

$$+ \frac{1}{4} ((dQQ^{-1})_{ij} \otimes dt^i \otimes \tau_i + dt^i \otimes (dQQ^{-1})_{ij} \otimes \tau_i)$$

$$- \frac{1}{4} (\tau_i \otimes (dQQ^{-1})_{ij} \otimes dt^i + \tau_i \otimes dt^i \otimes (dQQ^{-1})_{ij})$$

$$- \frac{1}{2} \tau_i (\nabla Q_{jk}) (dt^i \otimes dt^j \otimes dt^k - dt^i \otimes dt^k \otimes dt^j)$$

From this we deduce that

$$|\nabla T| \leq \sum_i |dt^i||\nabla \tau_i| + \sum_{i,j} |dt^i||\tau_j||(dQQ^{-1})_{ij}| + \sum_{i,j,k} |\tau_i (\nabla Q_{jk})||dt^i||dt^j||dt^k|$$

$$\leq \text{Tr}Q|\nabla \tau| + 3 \text{Tr}Q|\tau||dQQ^{-1}| + 9(\text{Tr}Q)^3|\tau||dQ|$$

$$\leq 8(\text{Tr}Q)^8 \left( |\nabla dQ| + |dQ|^2 \right) + 3(\text{Tr}Q)^3|dQ|^2 + 9.3(\text{Tr}Q)^9/2|dQ|^2$$

$$\leq 9(\text{Tr}Q)^8 \left( |\nabla dQ| + |dQ|^2 \right)$$

where in the second line we have used $|dt|^2 = Q^{ii} \leq (\text{Tr}Q)^2$ (again because of $\text{det}Q = 1$) and in the third line we have used Lemmas 3.12, 3.13 and 3.16. In the last line (and all other places where two terms involving different powers of $\text{Tr}Q$ are replaced by the bigger power), we have used $\text{Tr}Q \geq 3$. 

Lemma 3.18. For any hypersymplectic structure $\omega$,

$$|\nabla^2 \omega| \leq 100(\text{Tr}Q)^9 \left( |\nabla dQ| + |dQ|^2 \right)$$

Proof. We compute, using the expressions (21) and (28) for $a_{ij}$,

$$|\nabla^2 \omega|^2 = |\nabla a_{ij} \otimes \omega_j + a_{ij} \otimes a_{jk} \otimes \omega_k|^2$$

$$\leq \frac{1}{2} \nabla (dQQ^{-1})_{ij} \otimes \omega_j + \frac{1}{2} \tau_{ji} \nabla (Q_{tr}\tau_i) \otimes \omega_j + a_{ij} \otimes a_{jk} \otimes \omega_k|^2$$

$$\leq \frac{3}{4} |\nabla (dQQ^{-1})_{ij} \otimes \omega_j|^2 + \frac{3}{4} |\tau_{ji} \nabla (Q_{tr}\tau_i) \otimes \omega_j|^2 + 3 |a_{ij} \otimes a_{jk} \otimes \omega_k|^2$$
The first term here is bounded by
\[
\frac{3}{4} \sum_j |\nabla (dQQ^{-1})_{ij}|^2 \sum_k |\omega_k|^2
\]
\[
= \frac{3}{2} \text{Tr} \sum_j \left| \left( (\nabla dQ)Q^{-1} - dQQ^{-1} \otimes dQQ^{-1} \right)_{ij} \right|^2
\]
\[
\leq 3 \text{Tr} \sum_j \left| \left( (\nabla dQ)Q^{-1} \right)_{ij} \right|^2 + 3 \text{Tr} \sum_j \left| \sum_p (dQQ^{-1})_{ip} \otimes (dQQ^{-1})_{pj} \right|^2
\]
\[
\leq 3 \text{Tr} \sum_j \left| \left( (\nabla dQ)Q^{-1} \right)_{ij} \right|^2 + 9 \text{Tr} \sum_{j,p} \left| (dQQ^{-1})_{ij} \right|^2 \sum_q \left| (dQQ^{-1})_{ij} \right|^2
\]
\[
\leq (\text{Tr} Q)^4 |\nabla dQ|_Q^2 + (\text{Tr} Q)^7 |dQ|_Q^4
\]
\[
\leq (\text{Tr} Q)^4 |\nabla dQ|_Q^2 + 2(\text{Tr} Q)^{10} |dQ|_Q^4 + (\text{Tr} Q)^7 |dQ|_Q^4
\]
where in the first inequality we use Cauchy-Schwarz inequality, we use the equation \( \sum_k |\omega_k|^2 = 2Q_{kk} = 2 \text{Tr} Q \) for the second equality in the second line, we use \( |\alpha + \beta|^2 \leq 2(\|\alpha\|^2 + \|\beta\|^2) \) for the third inequality, and Cauchy-Schwarz inequality again for the fourth inequality. To get the inequality in the fifth line, we applied part 2 of Lemma \[3.12\] to both terms of the previous line. The last inequality uses the inequality \[31\] in the proof of Lemma \[3.16\].

The second term is bounded by
\[
\leq \frac{3}{4} \sum_j |\epsilon_{rj} \nabla (Q_{ir} \tau_i)|^2 \sum_k |\omega_k|^2
\]
\[
= \frac{3}{2} \text{Tr} \sum_j |\epsilon_{rj} \nabla (Q_{ir} \tau_i)|^2
\]
\[
= \frac{3}{2} \text{Tr} \sum_j \left| \sum_{r,l \neq j; r < l} \nabla (Q_{ir} \tau_i) - \nabla (Q_{il} \tau_i) \right|^2
\]
\[
\leq 3 \text{Tr} \sum_j \sum_{r,l \neq j; r < l} \left| (\nabla (Q_{ir} \tau_i))^2 + (\nabla (Q_{il} \tau_i))^2 \right|
\]
\[
\leq 3 \text{Tr} \sum_j \sum_{r,l \neq j; r < l} |\nabla (Q_{ir} \tau_i)|^2
\]
\[
\leq \left( \sum_{j,l,r} |dQ_{ir}|^2 |\tau_i|^2 + (\text{Tr} Q)^2 |\nabla \tau_i|^2 \right)
\]
\[
\leq 18(\text{Tr} Q)^3 |dQ|_Q^2 |\tau|^2 + 54(\text{Tr} Q)^3 |\nabla \tau|^2
\]
where the first, the fourth and the sixth line are obtained by Cauchy-Schwarz inequality. For the first term of the last inequality we have used part 1 of Lemma \[3.12\].

Finally, the third term is
\[
3 \left( \sum_{j,k} |a_{ij}||a_{jk}||\omega_k| \right)^2 \leq 3 \left( \sum_{j,k} (3(\text{Tr} Q)^{3/2} |dQ|_Q \omega_k)^2 \right)^2
\]
\[
= 3 \left( \sum_k 3^3 (\text{Tr} Q)^3 |dQ|_Q^2 |\omega_k| \right)^2
\]
\[
\leq 2 \cdot 3^8 (\text{Tr} Q)^7 |dQ|_Q^4
\]
where the first inequality is obtained from \[30\].
To complete the proof it suffices to combine these inequalities with those of Lemmas 3.13, 3.16 (notice that during the calculation, we were repeatedly using Tr $Q \geq 3$ to change $(\text{Tr} \, Q)^a$ to a rougher bound $3^{a-b}(\text{Tr} \, Q)^b$ for any $a < b$).

\section{Evolution equations}

In this section, we derive the evolution equations satisfied by $\text{Tr} \, Q$ and $|dQ|^2_Q$, in order to apply the maximum principle. We begin with the evolution equations satisfied by $Q$ and $g_\omega$. Lotay and Wei have computed the evolution of $g_{ab}$ under the general $G_2$-Laplacian flow for a closed $G_2$-structure $\phi(t)$. We state their result here:

**Proposition 4.1** (Lotay–Wei \cite{17}, equation (3.6)). When a closed $G_2$-structure $\phi(t)$ evolves according to the $G_2$-Laplacian flow, the Riemannian metric $g(t) = g_\phi(t)$ satisfies

$$\partial_t g_{ab} = -2\text{Ric}_{ab} - \frac{2}{3}|T|^2 g_{ab} - 4T_a^bT_{cb}$$  \hspace{1cm} (32)

Given the decomposition of $g$ and $\text{Ric}$ (see Lemma 3.5) this leads quickly to the 4-dimensional evolution equations. To write the equations, we first recall that $\det Q = 1$. Now the submanifold $\mathcal{S} = \{Q \in \mathcal{P} : \det Q = 1\}$ is totally geodesic inside $\mathcal{P}$ and thus is also non-positively curved.

This property will be used crucially later on. This implies that $\Delta Q \in T_\mathcal{Q}\mathcal{S}$. Write $\text{pr}_Q : T_\mathcal{Q}\mathcal{P} \to T_\mathcal{Q}\mathcal{S}$ for the orthogonal projection. Explicitly, identifying $T_\mathcal{Q}\mathcal{P}$ with symmetric matrices, we have $T_\mathcal{Q}\mathcal{S} = \{ A : \text{Tr}(Q^{-1}A) = 0 \}$ and $\text{pr}_Q(A) = A - \frac{1}{3}\text{Tr}(Q^{-1}A)Q$. With this in hand we can state the evolution equations. They follow from direct manipulation of (32) and so we suppress the details.

**Corollary 4.2.** When $\omega(t)$ satisfies the hypersymplectic flow, $Q$ and $g_\omega$ evolve according to

$$\partial_t Q = \Delta Q + \text{pr}_Q \langle \tau, \tau \rangle$$

$$\partial_t g = -2\text{Ric} + \frac{1}{2}(dQ \otimes dQ)_Q + \text{Tr}(Q^{-1}\tau \otimes \tau) - \frac{2}{3}|T|^2 g$$

Here, $\langle \tau, \tau \rangle$ is the symmetric matrix with $(i,j)$-element $\langle \tau_i, \tau_j \rangle$, whilst

$$\langle dQ \otimes dQ \rangle_Q (u, v) = Q^{ij}\nabla_u Q_{jk}Q^{kl}\nabla_v Q_{li}$$

$$\text{Tr}(Q^{-1}\tau \otimes \tau)(u, v) = Q^{ij}\tau_i(u)\tau_j(v)$$

**Remark 4.3.** If one ignores the terms involving torsion, we have precisely the harmonic-Ricci flow studied in \cite{19}, in which the Ricci flow for $g$ and harmonic map flow for $Q$ are coupled.

We now consider the heat operator acting on $\text{Tr} \, Q$.

**Proposition 4.4.** Under the hypersymplectic flow,

$$(\partial_t - \Delta) \text{Tr} \, Q \leq \frac{5}{3}|T|^2 \text{Tr} \, Q$$

It follows that if $|T|$ is bounded for $t \in [0,s)$ with $s < \infty$ then $\text{Tr} \, Q$ is also bounded for $t \in [0,s)$.

**Proof.** We have

$$\partial_t Q = \Delta Q + \text{pr}_Q \langle \tau, \tau \rangle = \Delta Q - \langle dQ, Q^{-1}dQ \rangle + \text{pr}_Q \langle \tau, \tau \rangle$$

It follows that

$$(\partial_t - \Delta) \text{Tr} \, Q = -\text{Tr} \langle dQ, Q^{-1}dQ \rangle + |\zeta|^2 - \frac{1}{3}|T|^2 \text{Tr} \, Q$$

The stated inequality now follows from $\text{Tr} \langle dQ, Q^{-1}dQ \rangle \geq 0$ and the bound on $|\zeta|^2$ in Lemma 3.13. Finally, the fact that a bound on $|T|$ implies a bound on $\text{Tr} \, Q$ now follows from the maximum principle.

\[\blacksquare\]
Next we control the heat operator acting on $|dQ|^2_Q$.

**Proposition 4.5.** There is a constant $C$ such that when $\omega(t)$ evolves according to the hypersymplectic flow, we have

\[
(\partial_t - \Delta)|dQ|^2_Q \leq -|\bar{\nabla}dQ|^2_Q - \frac{1}{16}|dQ|^4_Q + C(\text{Tr} Q)^9|T|^2|dQ|^2_Q
\]

It follows that if $|T|$ is bounded for $t \in [0,s]$ with $s < \infty$ then $|dQ|^2_Q$ is also bounded for $t \in [0,s)$.

**Proof.** A standard calculation from the theory of harmonic maps (first carried out in [10]) gives

\[
\frac{1}{2}\Delta|dQ|^2_Q = |\bar{\nabla}dQ|^2_Q + g^{\alpha\beta} \left\langle \bar{\nabla}_\alpha \Delta Q, \bar{\nabla}_\beta Q \right\rangle_Q + R^{\alpha\beta} \left\langle \nabla_\alpha Q, \nabla_\beta Q \right\rangle_Q - K_P
\]

where $K_P$ is a term involving the sectional curvature of $\mathcal{P}$:

\[
K_P = \text{Rm}_P(\nabla_\alpha Q, \nabla_\beta Q, \nabla^\alpha Q, \nabla^\beta Q)
\]

Since $\mathcal{P}$ is non-positively curved $K_P \leq 0$ and so we can safely discard this term. (The fact that $\mathcal{P} \cong \text{GL}_+ (3, \mathbb{R})/\text{SO}(3)$ is non-positively curved follows from the general theory of symmetric spaces. Alternatively one can calculate directly from the definition of the Levi-Civita connection given at the start of §3.)

Meanwhile, writing momentarily $\partial_t Q = V$ and $\partial_t g_Q = N$, we have

\[
\frac{1}{2} \partial_t (|dQ|^2_Q) = g^{\alpha\beta} \left\langle \bar{\nabla}_\alpha V, \bar{\nabla}_\beta Q \right\rangle_Q - \frac{1}{2} N^{\alpha\beta} \left\langle \nabla_\alpha Q, \nabla_\beta Q \right\rangle_Q
\]

(34)

We expand the first term. To begin,

\[
\bar{\nabla}V = \bar{\nabla}\Delta Q + \bar{\nabla}\text{pr}_Q \langle \tau, \tau \rangle = \bar{\nabla}\Delta Q + \text{pr}_Q \bar{\nabla} \langle \tau, \tau \rangle
\]

since $S$ is totally geodesic. Moreover, $\nabla Q \in T_Q S$ so,

\[
\left\langle \text{pr}_Q \bar{\nabla} \langle \tau, \tau \rangle, \nabla Q \right\rangle_Q = \left\langle \bar{\nabla} \langle \tau, \tau \rangle, \nabla Q \right\rangle_Q = \langle \nabla \langle \tau, \tau \rangle - \langle \tau, \tau \rangle Q^{-1} \nabla Q, \nabla Q \rangle_Q
\]

This means that the first term of (34) is

\[
g^{\alpha\beta} \left\langle \bar{\nabla}_\alpha V, \bar{\nabla}_\beta Q \right\rangle_Q = g^{\alpha\beta} \left\langle \bar{\nabla}_\alpha \Delta Q + 2 \langle \nabla_\alpha \tau, \tau \rangle - \langle \tau, \tau \rangle Q^{-1} \nabla_\alpha Q, \nabla_\beta Q \right\rangle_Q
\]

(35)

Notice that (just as in the harmonic map flow) the first term of (35) cancels the corresponding term in (33). We claim that the last term of (35) is nonpositive. To see this, write $M_\alpha = \langle \tau, \tau \rangle^{1/2} Q^{-1/2} \nabla_\alpha QQ^{-1/2}$, then this term can be written as $-\sum_\alpha \text{Tr}(M_\alpha M_\alpha^t) \leq 0$.

The second term in (34), $-\frac{1}{2} N^{\alpha\beta} \left\langle \nabla_\alpha Q, \nabla_\beta Q \right\rangle_Q$, is

\[
R^{\alpha\beta} \left\langle \nabla_\alpha Q, \nabla_\beta Q \right\rangle_Q - \frac{1}{4} \langle \nabla Q, \nabla Q \rangle_Q^2 - \frac{1}{2} Q^{ij} \langle \tau_i (\nabla Q), \tau_j (\nabla Q) \rangle_Q + \frac{1}{3} |T|^2 |dQ|^2_Q
\]

(36)

Just as for the coupled harmonic-Ricci flow, the Ricci term here cancels the corresponding term in
For the second term we bound as

\[ |\langle \nabla Q, \nabla Q \rangle_Q|^2 = \sum_{\alpha,\beta} |\langle \nabla^\alpha Q, \nabla^\beta Q \rangle_Q|^2 \]
\[ \geq \sum_{\alpha} |\langle \nabla^\alpha Q, \nabla^\alpha Q \rangle_Q|^2 \]
\[ \geq \frac{1}{4} \left( \sum_{\alpha} \langle \nabla^\alpha Q, \nabla^\alpha Q \rangle_Q \right)^2 \]
\[ = \frac{1}{4} |dQ|^4_Q \]

The third term in (36) is nonpositive and so we can discard it.

Putting this together, we obtain

\[
(\partial_t - \Delta) |dQ|^2_Q \leq -2|\hat{\nabla} dQ|^2_Q - \frac{1}{8}|dQ|^4_Q + 4 \langle \langle \nabla^\alpha \tau, \tau \rangle, \nabla^\alpha Q \rangle_Q + \frac{2}{3}|T|^2 |dQ|^2_Q \tag{37}
\]

It remains to control the third term in (37). Using again Lemma 3.12, we have

\[
\langle \langle \nabla^\alpha \tau, \tau \rangle, \nabla^\alpha Q \rangle_Q \leq C (|\nabla^\alpha Q|_Q |\nabla^\alpha Q|_Q \leq C (\text{Tr} Q)^{19/2} (|\nabla dQ|_Q + |dQ|^2_Q) |T||dQ|_Q
\]

for some computable absolute constant \(C\) varying from line to line. We now invoke the bounds on \(\tau\) and \(\nabla \tau\) derived in Lemmas 3.13 and 3.16. This gives, for some computable absolute constant \(C\),

\[
\langle \langle \nabla^\alpha \tau, \tau \rangle, \nabla^\alpha Q \rangle_Q \leq C (\text{Tr} Q)^{19/2} \left( |\nabla dQ|_Q + |dQ|^2_Q \right) |T||dQ|_Q
\]

To complete the proof, we apply the “Peter–Paul” inequality, \(2ab \leq ea^2 + \epsilon^{-1}b^2\). This gives

\[
C (\text{Tr} Q)^{19/2} |\nabla dQ|^2_Q |T||dQ|_Q \leq \frac{1}{4} |\nabla dQ|^4_Q + C^2 (\text{Tr} Q)^{23} |T|^2 |dQ|^2_Q
\]
\[
C (\text{Tr} Q)^{19/2} |T||dQ|^3_Q \leq \frac{1}{64} |dQ|^4_Q + 16C^2 (\text{Tr} Q)^{19} |T|^2 |dQ|^2_Q
\]

This means that

\[
\langle \langle \nabla^\alpha \tau, \tau \rangle, \nabla^\alpha Q \rangle_Q \leq \frac{1}{4} |\nabla dQ|^4_Q + \frac{1}{64} |dQ|^4_Q + 17C^2 (\text{Tr} Q)^{19} |T|^2 |dQ|^2_Q
\]

Substituting this into (37) completes the proof of the heat inequality. The conclusion that a bound on \(|T|\) yields a bound on \(|dQ|_Q\) (in finite time) then follows from the maximum principle (given that \(\text{Tr} Q\) is also bounded, by Proposition 4.4).

5 Control of the \(L^2\)-norm of curvature

5.1 Overview of the proof

In this section we will show that a uniform bound on \(|T|\) along the hypersymplectic flow \(\omega(t)\) implies a uniform bound on the energy of the metric \(g_{\omega(t)}\), i.e., the \(L^2\)-norm of its curvature, at least for finite time. The inspiration for this is an article of Simon [23] which proves an analogous result for the 4-dimensional Ricci flow.
Theorem 5.1 (Simon [23]). Let \( g(t) \) be a solution to Ricci flow on a compact 4-manifold and time interval \( t \in [0, s) \) with \( s < \infty \). Suppose that for all \( t \), \(|R| \leq \beta/2\). Then there is a constant \( C \) depending only on \( \beta, s \) and the initial data such that

1. For all \( t \in [0, s) \),
\[
\int |Rm|^2 < C
\]

2. 
\[
\int_0^s dt \int |\text{Ric}|^4 + |\nabla \text{Ric}|^2 < C
\]

We begin by briefly reviewing Simon’s argument, before going on to explain the additional complications which arise in our situation. There are three key ingredients:

1. In dimension 4, the Chern–Gauss–Bonnet theorem says that for any compact Riemannian 4-manifold \( X \),
\[
32\pi^2 \chi(X) = \int |Rm|^2 - 4|Ric|^2 + R^2
\] (38)

2. There is a constant \( C > 0 \) such that along the Ricci flow,
\[
\frac{d}{dt} \int |\text{Ric}|^2 \leq \int -|\nabla \text{Ric}|^2 + 2C|\text{Rm}||\text{Ric}|^2
\] (39)

3. Assume that \(|R| \leq \beta/2\) along the Ricci flow. Then there is a constant \( C > 0 \) (depending on \( \beta \)) such that along the Ricci flow,
\[
\frac{d}{dt} \int \frac{|\text{Ric}|^2}{R + \beta} \leq \int -\frac{4}{\beta^2} |\text{Ric}|^4 + C|\text{Rm}||\text{Ric}|^2 + |\nabla \text{Ric}|^2
\] (40)

From here Simon argues as follows. Write \( I \) for the integrand in the Chern–Gauss–Bonnet formula (38). Let \( \epsilon > 0 \). In what follows, \( K \) denotes a positive constant which may depend on \( \epsilon \) and may change from line to line, but is otherwise absolute. We have
\[
2C|\text{Rm}||\text{Ric}|^2 \leq \epsilon |\text{Ric}|^4 + K|\text{Rm}|^2
\]
\[
\leq \epsilon |\text{Ric}|^4 + K(I + 4|Ric|^2)
\]
\[
\leq 2\epsilon |\text{Ric}|^4 + KI + K
\]

It follows from (39) that
\[
\frac{d}{dt} \int |\text{Ric}|^2 \leq \int -|\nabla \text{Ric}|^2 + 2\epsilon |\text{Ric}|^4 + K
\]

Similarly, starting with (40) we do the same, this time absorbing the positive \(|\text{Ric}|^4\) term into the negative \(-\frac{4}{\beta^2}|\text{Ric}|^4\). This means that there is a constant \( K \) such that
\[
\frac{d}{dt} \int \frac{|\text{Ric}|^2}{R + \beta} \leq \int -\frac{3}{\beta^2} |\text{Ric}|^4 + |\nabla \text{Ric}|^2 + K
\]

Now putting these two inequalities together, with a choice of small enough \( \epsilon > 0 \), gives a constant \( K \) such that
\[
\frac{d}{dt} \int |\text{Ric}|^2 + \frac{|\text{Ric}|^2}{2(R + \beta)} \leq \int -\frac{1}{\beta^2} |\text{Ric}|^4 - \frac{1}{2}|\nabla \text{Ric}|^2 + K
\]

Integrating this over \( t \in [0, s) \) proves a uniform bound on \( \int |\text{Ric}|^2 \) as well as the second part of Simon’s Theorem 5.1. The Chern–Gauss–Bonnet theorem then turns the bound on \( \int |\text{Ric}|^2 \) into a uniform bound on \( \int |Rm|^2 \), completing the proof.

Our main result in this section is an analogue of Simon’s theorem for the hypersymplectic flow:
**Theorem 5.2** (L²-estimate of 7-dimensional Ricci curvature). Let $\omega(t)$ be a solution to the hypersymplectic flow on a compact 4-manifold $X$ and time interval $[0,s)$ with $s < \infty$. Suppose that $|T|^2 \leq \beta/2$ at all times. Then there exists a constant $C$ which depends only on $\beta$, $s$ and the initial data such that

1. For all $t \in [0,s)$,
   $$\int_{X \times T^3} |\text{Ric}|^2 \mu \leq C$$

2. $$\int_0^s dt \int_{X \times T^3} (|\text{Ric}|^4 + |\nabla \text{Ric}|^2 + |\text{Ric}|^2 |\hat{\nabla} dQ|_Q^2 + |\hat{\nabla} dQ|_Q^2)^2 \mu \leq C$$

Before we prove this theorem, we first discuss some consequences. Using the relationship between the 4- and 7-dimensional curvature tensors (Lemma 3.5), we have

$$|\text{Ric}|^2 \leq 2|\text{Ric}|^2 + \frac{1}{8}|dQ|^2$$

Now $|dQ|^2$ is bounded along the flow (by Proposition 4.3), and the total volume is bounded by a constant depending only on the cohomology classes of $\omega_1, \omega_2, \omega_3$ (see Definition 2.3):

$$\int \mu \leq \frac{1}{3} \int (\text{Tr} \ Q) \mu = \frac{1}{6} \int (\omega_1^2 + \omega_2^2 + \omega_3^2)$$

It follows that the $L^2$-bound on $\text{Ric}$ gives an $L^2$-bound on the 4-manifold Ricci curvature $\text{Ric}$ and hence, by Chern–Gauss–Bonnet formula (38) for $X$, on $\text{Rm}$ as well. We can similarly translate the second part of Theorem 5.2 into 4-dimensional quantities. The result is as follows.

**Corollary 5.3** (4-dimensional Energy bound). Under the same assumptions as in Theorem 5.2, there exists a constant $C$, again depending only on $\beta$, $s$ and the initial data such that

1. For all $t \in [0,s)$,
   $$\int_X |\text{Rm}|^2 \leq C$$

2. $$\int_0^s dt \int_X |\text{Ric}|^4 + |\hat{\Delta} Q|_Q^2 + |\nabla Q|^2 + |\hat{\nabla} dQ|_Q^2 + |\text{Ric}|^2 |\hat{\nabla} dQ|_Q^2 + |\hat{\Delta} Q|_Q^2 |\hat{\nabla} dQ|_Q^2 \leq C$$

We now give the proof of Theorem 5.2 along the lines of Simon’s argument. The crux is to find a series of differential inequalities which play the role of (39) and (40). We will state the inequalities here and defer their proofs until the subsequent sections. Note that in the following proposition, all integrals are over the 7-manifold $X \times T^3$, and we have suppressed the volume form $\mu$ (which must be remembered when taking the time derivative, of course!).

**Proposition 5.4.** Let $\omega(t)$ be a solution to the hypersymplectic flow on a compact 4-manifold and time interval $t \in [0,s)$. Suppose moreover that $|T|^2 \leq \beta/2$ for all $t$. Then there exists a constant $C$, depending only on $\beta$, $s$ and the initial data, such that for all $t \in [0,s)$,

1. $\frac{d}{dt} \int |\text{Ric}|^2 \leq \int -|\nabla \text{Ric}|^2 + C \left(|\text{Rm}| |\text{Ric}|^2 + |\text{Ric}|^2 + |\hat{\nabla} dQ|_Q^2 + 1\right)$

2. $\frac{d}{dt} \int \frac{|\text{Ric}|^2}{|T|^2} \leq \int -\frac{4}{\beta^2} |\text{Ric}|^4 + |\nabla Q|^2 + C \left(|\text{Rm}| |\text{Ric}|^2 + |\text{Ric}|^2 |\nabla dQ|_Q^2 + |\text{Ric}|^2 + |\nabla dQ|_Q^2 + 1\right)$

3. $\frac{d}{dt} \int |\text{Ric}|^2 |dQ|_Q^2 \leq \int -\frac{1}{2} |\text{Ric}|^2 |\nabla dQ|_Q^2 + C \left(|\text{Rm}| |\text{Ric}|^2 + |\nabla Q|^2 + |\nabla \text{Ric}|^2 + |\text{Ric}|^2 + 1\right)$
Now that the integrand on the right hand side is pulled back from the 4-manifold theorem over fibres have unit area, the right hand side is equal to the integral over \(X\) with 5.4, applying Peter–Paul inequality to the term [3.5] gives an explicit expression for \(Rm\) in terms of \(Q, dQ, \nabla dQ\) and \(Rm\). Since \(\text{Tr} Q\) and \(|dQ|_Q\) are bounded Propositions 4.4 and 4.5, it follows that

\[
|Rm|^2 \leq C \left( |Rm|^2 + |\nabla dQ|_Q^2 + 1 \right)
\]

We integrate this over \(X \times \mathbb{T}^3\):

\[
\int |Rm|^2 \leq C \int \left( |Rm|^2 + |\nabla dQ|_Q^2 + 1 \right)
\]

Now that the integrand on the right hand side is pulled back from the 4-manifold \(X\). Since the \(\mathbb{T}^3\)-fibres have unit area, the right hand side is equal to the integral over \(X\). By Chern–Gauss–Bonnet theorem over \(X\) we have

\[
\int_X |Rm|^2 \leq C \int_X (|Ric|^2 + 1)
\]

We now apply Lemma 3.5 which gives the \(X\)-directional components of \(Ric\) in terms of \(Ric\), \(Q\) and \(dQ\). This and the boundedness of \(Q\) and \(dQ\) gives that

\[
|Ric|^2 \leq 2|Ric|^2 + C
\]

and from here, (42) follows.

With this in hand, we are in a position to emulate Simon’s argument. From part 2 of Proposition 5.4, applying Peter–Paul inequality to the term \(|Rm||Ric|^2\) and absorbing the resulting small multiple of \(|Ric|^4\) into the term \(-\frac{1}{3}\beta |Ric|^4\), we have

\[
\frac{d}{dt} \int \frac{|Ric|^2}{R + \beta} \leq \int -\frac{3}{\beta^2} |Ric|^4 + |\nabla Ric|^2 + C \left( |Rm|^2 + |Ric|^2 |\nabla dQ|_Q^2 + |Ric|^2 + |\nabla dQ|_Q^2 + 1 \right)
\]

Applying (42) to replace the \(|Rm|^2\) term, we get

\[
\frac{d}{dt} \int \frac{|Ric|^2}{R + \beta} \leq \int -\frac{3}{\beta^2} |Ric|^4 + |\nabla Ric|^2 + C \left( |\nabla dQ|_Q^2 + |Ric|^2 |\nabla dQ|_Q^2 + |Ric|^2 + 1 \right)
\]

We can also absorb \(C|Ric|^2\) into the term \(-\frac{1}{3\beta} |Ric|^4\), giving

\[
\frac{d}{dt} \int \frac{|Ric|^2}{R + \beta} \leq \int -\frac{2}{\beta^2} |Ric|^4 + |\nabla Ric|^2 + C \left( |\nabla dQ|_Q^2 + |Ric|^2 |\nabla dQ|_Q^2 + 1 \right)
\]

We next add a suitable multiple of inequality 3 from Proposition 5.4 to deal with the \(|Ric|^2|\nabla dQ|_Q^2\) term. This will introduce more terms of the form \(|Rm||Ric|^2\) and \(|Ric|^2\) which we can deal with exactly as before, via (42) and absorb them into the term \(-\frac{1}{\beta^2} |Ric|^4\). This means that, for an appropriate choice of \(A_1 > 0\), we have

\[
\frac{d}{dt} \int \frac{|Ric|^2}{R + \beta} + A_1|Ric|^2 |dQ|_Q^2 \leq \int -\frac{1}{\beta^2} |Ric|^4 - |Ric|^2 |\nabla dQ|_Q^2 + C \left( |\nabla dQ|_Q^2 + |\nabla Ric|^2 + 1 \right)
\]

4. \(\frac{1}{4} \int |dQ|_Q^2 \leq \int -|\nabla dQ|_Q^2 + C\)

Proof of Theorem 5.2, assuming Proposition 5.4. Throughout, \(C\) denotes a constant which depends only on \(\beta\), \(s\) and the initial data \(\omega(0)\), but which may change from line to line.

We begin by proving that there exists a constant \(C\) such that

\[
\int |Rm|^2 \leq C \int \left( |\nabla dQ|_Q^2 + |Ric|^2 + 1 \right)
\]

(42)

We integrate this over \(X \times \mathbb{T}^3\):

\[
\int |Rm|^2 \leq C \int \left( |Rm|^2 + |\nabla dQ|_Q^2 + 1 \right)
\]

\[
\int_X |Rm|^2 \leq C \int_X (|Ric|^2 + 1)
\]

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Next we add a suitable multiple of inequality 1 from Proposition 5.4 to deal with the $|\nabla \text{Ric}|^2$ term. This again introduces more terms of the form $|\text{Rm}|\text{Ric}^2$ and $|\text{Ric}|^2$ which we deal with as before, again via (42). This means that, for an appropriate choice of $A_2 > 0$ we have

$$\frac{d}{dt} \int \frac{|\text{Ric}|^2}{R + \beta} + A_1 |\text{Ric}|^2|dQ|_Q^2 + A_2 |\text{Ric}|^2 \leq \int -\frac{1}{2\beta^2} |\text{Ric}|^4 - |\text{Ric}|^2|\nabla dQ|_Q^2 - |\nabla \text{Ric}|^2 + C$$

Finally, we take care of the Hessian term via inequality 4 from Proposition 5.4. For a suitable choice of $A_3 > 0$, we have

$$\frac{d}{dt} \int \frac{|\text{Ric}|^2}{R + \beta} + A_1 |\text{Ric}|^2|dQ|_Q^2 + A_2 |\text{Ric}|^2 + A_3 |dQ|_Q^2 \leq \int -\frac{1}{3\beta^2} |\text{Ric}|^4 - |\text{Ric}|^2|\nabla dQ|_Q^2 - |\nabla \text{Ric}|^2 - |\nabla dQ|_Q^2 + C$$

We now pick $t \in [0, s)$ and integrate (43) from 0 to $t$. Recalling that the total volume is uniformly bounded, by (41), this gives

$$\left( \int_0^t dt \int \frac{1}{12|\beta|^2} |\text{Ric}|^4 + |\text{Ric}|^2|\nabla dQ|_Q^2 + |\nabla \text{Ric}|^2 + |\nabla dQ|_Q^2 \right)$$

$$+ \left( \int \frac{|\text{Ric}|^2}{R + \beta} + A_1 |\text{Ric}|^2|dQ|_Q^2 + A_2 |\text{Ric}|^2 + A_3 |dQ|_Q^2 \right) \leq C$$

Every term on the left hand side is positive. Dropping all but the $A_2 |\text{Ric}|^2$ term proves part 1 of Theorem 5.2 whilst dropping the purely spatial integral and letting $t \to s$ proves part 2.

We now give the proofs of the inequalities in Proposition 5.4. At times the formulae involved may appear intimidating on the page, but the arguments involve nothing more than integration by parts and careful bookkeeping.

### 5.2 Evolution of Ricci curvature under the $G_2$-Laplacian flow

The first two inequalities in Proposition 5.4 will follow from more general inequalities which hold for an arbitrary $G_2$-Laplacian flow of closed $G_2$-structures. We begin by recalling the 7-dimensional evolution equations for the volume $\mu$, scalar curvature $R$ (which is equal to $-|T|^2$ by [17, Corollary 2.5]), and Ricci tensor $\text{Ric}$ derived by Lotay–Wei [17].

$$\partial_t \mu = \frac{2}{3} |T|^2 \mu$$

$$\partial_t R = \Delta R - 4 \nabla^b \nabla^a (T^c_a T_{cb}) + 2 |\text{Ric}|^2 - \frac{2}{3} R^2 + 4 R^{ab} T^c_a T_{cb}$$

$$\partial_t T_{ab} = \Delta_L \left( \frac{2}{3} |T|^2 + 2 T^c_a T_{cb} \right)$$

$$- \frac{1}{3} \nabla_a \nabla_b |T|^2 - 2 \left( \nabla_a \nabla^c (T^d_c T_{db}) + \nabla_a \nabla^c (T^d_c T_{da}) \right)$$

where $\Delta_L \eta_{ab} = \Delta \eta_{ab} - R^c_{db} \eta_{ac} - R^c_{db} \eta_{ac} + 2 R_{cab} \eta^{dc}$ is the Lichnerowicz Laplacian.
Using (46) one finds the heat equation satisfied by $|\text{Ric}|^2$:

$$(\partial_t - \Delta)|\text{Ric}|^2 = -2|\nabla \text{Ric}|^2 + 4\text{R}_{abcd}\text{R}^{ab}\text{R}^{cd} + 4\text{R}^{ab}\Delta (T_a^c T_b^d)$$

$$+ \frac{2}{3}\Delta |T|^2 - 4\text{R}^{ab} (\nabla_a \nabla^c (T_c^d T_{db}) + \nabla_b \nabla^c (T_c^d T_{da}))$$

$$- \frac{2}{3}\text{R}^{ab} \nabla_a \nabla_b |T|^2 + 8\text{R}_{abcd}\text{R}^{ab}\text{R}^{cd} T_e^d + \frac{4}{3}|\text{Ric}|^2 |T|^2$$  \hspace{1em} (47)

We will also need the following inequality, satisfied by any closed $G_2$-structure. Since $|\text{Ric}|^2 \geq \frac{1}{7}|\text{R}|^2$, we have

$$|T|^2 \leq \sqrt{\frac{1}{7}}|\text{Ric}|$$  \hspace{1em} (48)

We now turn to our differential inequalities. Note that in the lemmas below, the volume form $\mu$ is omitted to ease the notation, but during the calculation of the time derivatives we have to take it into consideration.

**Lemma 5.5.** Let $\phi(t)$ be a path of closed $G_2$-structures solving the $G_2$-Laplacian flow. Then

$$\frac{d}{dt} \int |\text{Ric}|^2 \leq \int (-\nabla |\text{Ric}|^2 + 28\text{Rm} |\text{Ric}|^2 + 26|T|^2 |\nabla T|^2 + 4|\text{Ric}|^2 |T|^2)$$  \hspace{1em} (49)

**Proof.** From (44), (47), integration by parts and the contracted Bianchi identity (i.e. $\nabla_a \text{R}^{ab} = \frac{1}{2} \nabla^b \text{R}$) we get

$$\frac{d}{dt} \int |\text{Ric}|^2 = \int (\partial_t - \Delta) |\text{Ric}|^2 + \frac{2}{3} |\text{Ric}|^2 |T|^2$$

$$\hspace{1em} = \int -2|\nabla \text{Ric}|^2 + 4\text{R}_{abcd}\text{R}^{ab}\text{R}^{dc} - 4\nabla \text{R}^{ab}\nabla_c (T_a^c T_{db})$$

$$\hspace{2em} - \frac{1}{3}\nabla^c \text{R}^{ab} |T|^2 + 4\nabla^b \text{R}^{ab} |T|^2 (T_c^d T_{db}) + 8\text{R}_{abcd}\text{R}^{ab}\text{R}^{cd} T_e^d + 2|\text{Ric}|^2 |T|^2$$

Next we apply Kato’s inequality, $|\nabla |T|| \leq |\nabla T|$ and the identity $\nabla \text{R} = -2|T| \nabla |T|$ to obtain

$$\frac{d}{dt} \int |\text{Ric}|^2 \leq \int -2|\nabla \text{Ric}|^2 + 4|\text{Rm}||\text{Ric}|^2 + 8|\nabla \text{Ric}||T||\nabla T| + \frac{4}{3}|T|^2 |\nabla T|^2$$

$$\hspace{2em} + 8|T|^2 |\nabla T|^2 + 8|\text{Rm}||\text{Ric}|^2 + 2|\text{Ric}|^2 |T|^2$$

$$\hspace{1em} \leq \int -|\nabla \text{Ric}|^2 + (4 + 8\sqrt{7}) |\text{Rm}||\text{Ric}|^2 + \frac{76}{3} |T|^2 |\nabla T|^2 + 2|\text{Ric}|^2 |T|^2$$

where in the last step we have used Cauchy–Schwarz inequality $(8|\nabla T||T||\nabla T| \leq |\nabla T|^2 + 16|T|^2 |\nabla T|^2)$ and (48). The stated inequality now follows (by replacing coefficients by larger integers). \hfill $\square$

**Corollary 5.6.** Let $\omega(t)$ be a solution to the hypersymplectic flow on a compact 4-manifold and time interval $t \in [0, s)$ where $s < \infty$. Suppose moreover that $|T|^2 \leq \beta/2$ for all $t$. Then there exists a constant $C$, depending only on $\beta$, $s$ and the initial data, such that for all $t \in [0, s)$

$$\frac{d}{dt} \int |\text{Ric}|^2 \leq \int -|\nabla \text{Ric}|^2 + C \left( |\text{Rm}||\text{Ric}|^2 + |\text{Ric}|^2 + |\nabla dQ|^2 + 1 \right)$$

**Proof.** By Proposition 4.4 Tr $Q$ is bounded. By Proposition 4.5 $|dQ|_Q$ is bounded. It then follows from Lemma 5.17 that

$$|\nabla T| \leq C \left( |\nabla dQ|_Q + 1 \right)$$

The result now follows by substituting these bounds into (49). \hfill $\square$
We now turn to the second inequality in Proposition 5.4.

**Lemma 5.7.** Let \( \phi(t) \) be a path of closed \( G_2 \)-structures solving the \( G_2 \)-Laplacian flow on \([0, s]\) (here \( s \) is possibly \( \infty \)), and assume moreover that there exists \( \beta > 0 \) such that for all \( t \in [0, s) \) we have \( R + \beta > 0 \). Then for all \( t \in [0, s) \)

\[
\frac{d}{dt} \int \frac{|{\text{Ric}}|^2}{R + \beta} \leq \int \frac{|{\text{Ric}}|^4}{(R + \beta)^2} + \frac{4 + 8\sqrt{7}}{R + \beta} |{\text{Rm}}| |{\text{Ric}}|^2 + \left( \frac{2|T|^2}{R + \beta} + \frac{14|T|^4}{3(R + \beta)^2} \right) |{\text{Ric}}|^2 + |{\nabla {\text{Ric}}}|^2 \\
+ \left( \frac{32|T|^2}{(R + \beta)^2} + \frac{16|T|^2}{(R + \beta)^3} + \frac{104\sqrt{7}}{3(R + \beta)^2} \right) |{\text{Ric}}|^2 |{\nabla T}|^2 \\
+ \left( \frac{32 + 8\beta/3}{R + \beta} \right) |{\text{Ric}}|^2 |{\nabla T}|^2
\]

**Proof.** We compute:

\[
\frac{d}{dt} \int \frac{|{\text{Ric}}|^2}{R + \beta} = \int (\partial_t - \Delta) \frac{|{\text{Ric}}|^2}{R + \beta} + \frac{2|{\text{Ric}}|^2|T|^2}{3(R + \beta)} \\
= \int \frac{(\partial_t - \Delta)|{\text{Ric}}|^2}{R + \beta} + \frac{2(\nabla |{\text{Ric}}|^2, \nabla R)}{(R + \beta)^2} - \frac{2|Ric|^2|\nabla R|^2}{(R + \beta)^3} \\
= \frac{|{\text{Ric}}|^2(\partial_t - \Delta)R}{(R + \beta)^2} + \frac{2|Ric|^2|T|^2}{3(R + \beta)}
\]

Now substituting in the evolution equations (45) and (47), and completing the square on the resulting \(|{\nabla {\text{Ric}}}|^2\) term, this is equal to

\[
\int -\frac{2}{R + \beta} \nabla {\text{Ric}} - \nabla R \otimes {\text{Ric}} + \frac{2|Ric|^2|T|^2}{3(R + \beta)} \\
+ \frac{1}{R + \beta} \left\{ 4R_{abcd}R^{ab}R^{dc} + 4R^{ab}\Delta(T_a^cT_{cb}) + \frac{2}{3}R \Delta|T|^2 + 8R_{cdef}R^{cdef}T_e^d + \frac{4}{3}|Ric|^2|T|^2 \right\} \\
- 4R^{ab}(\nabla_a \nabla^c(T_e^dT_{eb}) + \nabla_b \nabla^c(T_e^dT_{da})) \right\} \\
- \frac{2}{3}R + 4R^{ab}T_a^cT_{cb}
\]

Now we gather terms and integrate by parts:

\[
= \int -\frac{2}{R + \beta} \nabla {\text{Ric}} - \nabla R \otimes {\text{Ric}} + \frac{2|Ric|^4}{(R + \beta)^2} + \frac{2|Ric|^2|T|^2}{R + \beta} + \frac{2|Ric|^2|T|^2}{3(R + \beta)^2} \\
+ \frac{4R_{abcd}R^{ab}R^{dc}}{R + \beta} + \frac{8R_{cdef}R^{cdef}T_e^d}{R + \beta} - \frac{4|Ric|^2R^{ab}T_a^cT_{cb}}{(R + \beta)^2} \\
- 4\nabla^c \left( \frac{R^{ab}}{R + \beta} \right) \nabla_a(T_e^dT_{db}) - \frac{2}{3} \nabla_e \left( \frac{R}{R + \beta} \right) \nabla^c|T|^2 + 8\nabla_a \left( \frac{R^{ab}}{R + \beta} \right) \nabla^c(T_e^dT_{eb}) \\
+ \frac{2}{3} \nabla_a \left( \frac{R^{ab}}{R + \beta} \right) \nabla_e|T|^2 - 4\nabla^b \left( \frac{|Ric|^2}{(R + \beta)^2} \right) \nabla^a(T_a^cT_{cb})
\]

Next we bound this by the norms of various terms, also applying the contracted Bianchi identity,
Kato’s inequality, and \( [45] \) as appropriate. This means that
\[
\frac{d}{dt} \int \frac{\text{|Ric|}^2}{R + \beta} \leq -\frac{2\text{Ric}^2}{R + \beta} + 2\text{Ric}^2\text{T}^2 + 2\text{Ric}^2\text{R}^2 + \frac{(4 + 8\sqrt{7})\text{Rm}\text{Ric}^2}{R + \beta} + 4\text{Ric}^3\text{T}^2
\]
\[
+ \left( \frac{8|\nabla\text{Ric}|}{3(R + \beta)} + \frac{8|\text{Ric}|\nabla\text{Ric}|}{3(R + \beta)^2} \right)|\text{T}|\text{V} \text{T}| + \frac{8|\text{Ric}|\nabla\text{Ric}|}{3(R + \beta)^2} + \left( \frac{8|\text{Ric}|\nabla\text{Ric}|}{3(R + \beta)^2} \right)|\text{T}|\text{V} \text{T}|
\]
\[
+ \left( \frac{|\nabla\text{Ric}|}{3(R + \beta)} + \frac{2|\text{Ric}|\nabla\text{Ric}|}{3(R + \beta)^2} \right)|\text{T}|\text{V} \text{T}| + \left( \frac{8|\text{Ric}|\nabla\text{Ric}|}{(R + \beta)^2} + \frac{8|\text{Ric}|\nabla\text{Ric}|}{(R + \beta)^2} \right)|\text{T}|\text{V} \text{T}|
\]
\[
\leq -\frac{2|\text{Ric}|}{(R + \beta)^2} + \frac{2|\text{Ric}|\text{T}^2}{3(R + \beta)^2} + \frac{2|\text{Ric}|\text{R}^2}{3(R + \beta)^2} + \frac{(4 + 8\sqrt{7})\text{Rm}|\text{Ric}|^2}{(R + \beta)^2} + 4|\text{Ric}|^3\text{T}^2
\]
\[
+ \frac{8|\nabla\text{Ric}|\text{T}|\text{V} \text{T}|}{(R + \beta)^2} + \frac{104|\text{Ric}|\text{T}^2\text{V} \text{T}|^2}{3(R + \beta)^2} + 8|\text{T}|^2\text{V} \text{T}|^2 + \frac{28|\text{T}|^2\text{V} \text{T}|}{3(R + \beta)^2}
\]
\[
+ \frac{16|\text{Ric}|^2\text{T}^2\text{V} \text{T}|^2}{(R + \beta)^2}
\]
\[\text{(50)}\]

Some steps taken in reaching this inequality may need extra explanation. The “coclosed condition” \( d^* \text{T} = 0 \) \([17\text{ Equation (2.15)}]\), (which reads as \( \nabla^* \text{T} = 0 \) in index notation) implies
\[
\nabla^c(\text{T}^d_e \text{T}_{eb}) = \nabla^c_\text{T}^d_e \text{T}_{eb} + \text{T}^d_e \nabla^c \text{T}_{eb} = \text{T}^d_e \nabla^c \text{T}_{eb}
\]

From this we have that
\[
8\nabla_a \left( \frac{\text{R}^a_b}{R + \beta} \right) \nabla^c(\text{T}^d_e \text{T}_{eb}) = 4 \frac{\nabla^b \text{R}}{R + \beta} \text{T}^d_e \nabla^c \text{T}_{eb} - 8 \frac{\text{R}^a_b \nabla_a \text{R}}{(R + \beta)^2} \text{T}^d_e \nabla^c \text{T}_{eb}
\]
\[
= 4 \frac{\nabla^b \text{R}}{R + \beta} \text{T} \cap \text{Ric} \nabla \text{R} - 8 \frac{\text{R}^a_b \nabla_a \text{R}}{(R + \beta)^2} \text{T} \cap \text{Ric} \nabla \text{R} \cap \nabla \text{R}
\]
\[
\leq 4 \frac{\nabla \text{R}}{R + \beta} |\text{T}| |\nabla \text{T}| + 8 \frac{|\nabla \text{R}| |\text{Ric}|}{(R + \beta)^2} |\text{T}| |\nabla \text{T}|
\]

Similarly \( \nabla^a(\text{T}^c_a \text{T}_{cb}) = \text{T}^c_a \nabla^a \text{T}_{cb} \) implies
\[
-4 \nabla_b \left( \frac{|\text{Ric}|^2}{(R + \beta)^2} \right) \nabla^a(\text{T}^c_a \text{T}_{cb}) = -4 \frac{\nabla^b |\text{Ric}|^2}{(R + \beta)^2} \text{T}^c_a \nabla^a \text{T}_{cb} + 8 \frac{|\text{Ric}|^2 \nabla^b \text{R}}{(R + \beta)^3} \text{T}^c_a \nabla^a \text{T}_{cb}
\]
\[
= -4 \frac{\nabla^b |\text{Ric}|^2}{(R + \beta)^2} \text{T} \cap \nabla |\text{Ric}|^2 \nabla \text{T} + 8 \frac{|\text{Ric}|^2 \nabla \text{R}}{(R + \beta)^3} \text{T} \cap \nabla \text{R} \cap \nabla \text{T}
\]
\[
\leq 4 \frac{|\nabla \text{R}|^2}{(R + \beta)^2} |\text{T}| |\nabla \text{T}| + 8 \frac{|\text{Ric}|^2 |\nabla \text{R}|}{(R + \beta)^2} |\text{T}| |\nabla \text{T}|
\]
\[
\leq 8 \frac{|\text{Ric}|^2 |\nabla \text{R}|}{(R + \beta)^2} |\text{T}| |\nabla \text{T}| + 8 \frac{|\nabla \text{R}| |\text{Ric}|}{(R + \beta)^2} |\text{T}| |\nabla \text{T}|
\]

This gives the last two terms in the second and the third line. In the derivation of inequality \[50\] we also use the inequality
\[
-\frac{2}{3} \nabla_c \left( \frac{\text{R}}{R + \beta} \right) |\nabla \text{T}|^2 = \frac{2 \beta}{3} \nabla_c |\nabla \text{T}|^2 |\nabla \text{T}|^2 \leq \frac{8\beta |\nabla \text{T}|^2 |\nabla \text{T}|^2}{3(R + \beta)^2}
\]
to get the middle term in the second line. Meanwhile, the term with coefficient \[\frac{104}{3}\] comes from the combination of the following three terms:
\[
\frac{8|\nabla \text{R}|}{(R + \beta)^2} |\text{T}| |\nabla \text{T}|, \quad \frac{8|\nabla \text{R}|}{(R + \beta)^2} |\text{T}| |\nabla \text{T}| \quad \text{and} \quad \frac{2|\text{Ric}| |\nabla \text{R}|}{3(R + \beta)^2} |\nabla \text{T}|^2
\]
together with $|\nabla R| = |\nabla |T|^2| \leq 2|T||\nabla T|$.

We now use Cauchy–Schwarz to isolate the $|\nabla \text{Ric}|$ term. This implies that the right hand side of inequality [49] is bounded by

$$\leq \int \frac{-2|\text{Ric}|^4}{(R + \beta)^2} + \frac{2|\text{Ric}|^2|T|^2}{R + \beta} + \frac{2|\text{Ric}|^2R^2}{3(R + \beta)} + \frac{(4 + 8\sqrt{7})|\text{Rm}||\text{Ric}|^2}{R + \beta} + \frac{4|\text{Ric}|^3|T|^2}{(R + \beta)^2}$$

$$+ |\nabla \text{Ric}|^2 + \left(\frac{32}{3(R + \beta)^2} + \frac{8\beta}{3(R + \beta)^2} + \frac{28}{3(R + \beta)}\right)|T|^2|\nabla T|^2$$

$$+ \left(\frac{104\sqrt{7}}{3(R + \beta)^2} + \frac{32}|T|^2}{R + \beta} + \frac{16|T|^2}{R + \beta^2}\right)|\text{Ric}|^2|\nabla T|^2$$

where we have used [48] which shows

$$\frac{104|\text{Ric}||T|^2|\nabla T|^2}{3(R + \beta)^2} \leq \frac{104\sqrt{7}|\text{Ric}|^2|\nabla T|^2}{3(R + \beta)^2}$$

Finally, we absorb the positive $|\text{Ric}|^3$ term into the negative $|\text{Ric}|^4$ term using Cauchy–Schwarz inequality, at the expense of introducing a term with $|\text{Ric}|^2$ in it. This gives the desired bound

$$\leq \int \frac{-|\text{Ric}|^4}{(R + \beta)^2} + \frac{(4 + 8\sqrt{7})|\text{Rm}||\text{Ric}|^2}{R + \beta} + \left(\frac{2|T|^2}{R + \beta} + \frac{14|T|^4}{3(R + \beta)^2}\right)|\text{Ric}|^2$$

$$+ \left(\frac{32 + 8\beta}{3(R + \beta)^2} + \frac{28}{3(R + \beta)}\right)|\nabla T|^2 + \left(\frac{104\sqrt{7}}{3(R + \beta)^2} + \frac{32}{R + \beta} + \frac{16}{R + \beta^2}\right)|\text{Ric}|^2|\nabla T|^2$$

$$+ |\nabla \text{Ric}|^2$$

Corollary 5.8. Let $\omega(t)$ be a solution to the hypersymplectic flow on a compact 4-manifold and time interval $t \in [0, s]$ where $s < \infty$. Suppose moreover that $|T|^2 \leq \beta/2$ for all $t \in [0, s]$ (for some constant $\beta > 0$). Then there exists a constant $C$, depending only on $\beta$, $s$ and the initial data $\omega(0)$, such that for all $t \in [0, s)$

$$\frac{d}{dt} \int \frac{|\text{Ric}|^2}{R + \beta} \leq \int -\frac{4}{\beta^2} |\text{Ric}|^4 + |\nabla \text{Ric}|^2 + C \left(|\text{Rm}||\text{Ric}|^2 + |\text{Ric}|^2|\text{d}Q|_Q^2 + |\text{Ric}|^2 + |\nabla \text{d}Q|_Q^2 + 1\right)$$

Proof. This follows from the inequality in Lemma 5.7 by substituting the bounds $R + \beta = |T|^2 + \beta \geq \beta/2$, and $|\nabla T|^2 \leq C(|\nabla dQ|_Q^2 + 1)$ (see the inequality in the proof of Corollary 5.6). □

5.3 Evolution of integral quantities in 4-dimension

We move now to the evolution of $\int |\text{d}Q|_Q^2$. Proposition 4.5 shows that there is a constant $C$ (depending only on $\text{Tr } Q$) such that

$$(\partial_t - \Delta)|\text{d}Q|_Q^2 \leq -|\nabla \text{d}Q|_Q^2 + C|T|^2|\text{d}Q|_Q^2$$

(51)

Lemma 5.9. Let $\omega(t)$ be a solution to the hypersymplectic flow on $[0, s)$ with $s < \infty$. Let $c > 0$, suppose $\text{Tr } Q < c$ for all $t \in [0, s)$. Then there is a constant $C$ depending only on $c$ such that

$$\frac{d}{dt} \int |\text{d}Q|_Q^2 \leq \int \left(-|\nabla \text{d}Q|_Q^2 + C|T|^2|\text{d}Q|_Q^2\right)$$

In particular, this gives the fourth inequality of Proposition 5.4.
Proof. Directly from \cite{[51]}, we have
\[ \frac{d}{dt} \int |dQ|^2_Q = \int (\partial_t - \Delta) |dQ|^2_Q + \frac{2}{3} |dQ|^2_Q |T|^2 \leq \int -|\nabla dQ|^2 + \left( C + \frac{2}{3} \right) |T|^2 |dQ|^2_Q. \]

It remains to prove the third inequality of Proposition \cite{[54]}

**Lemma 5.10.** Let $\omega(t)$ be a solution to the hypersymplectic flow on a compact 4-manifold and time interval $t \in [0, s)$. Suppose moreover that $|T|^2 \leq \beta/2$ for all $t \in [0, s)$. Then there exists a constant $C$, depending only on $\beta$, $s$ and the initial data, such that for all $t \in [0, s)$
\[ \frac{d}{dt} \int |\text{Ric}|^2 |dQ|^2_Q \leq \int -\frac{1}{2} |\text{Ric}|^2 |\nabla dQ|^2_Q + C \left( |\text{Rm}| |\text{Ric}|^2 + |\nabla dQ|^2_Q + |\nabla \text{Ric}|^2 + |\text{Ric}|^2 + 1 \right) \]

**Proof.** We begin the computation by substituting (47) and (51) (notice $\Delta$ and $\Delta$ coincide for $|dQ|^2_Q$ since it is $T^3$-invariant):
\[ \frac{d}{dt} \int |\text{Ric}|^2 |dQ|^2_Q = \int |dQ|^2_Q (\partial_t - \Delta) |\text{Ric}|^2 + |\text{Ric}|^2 (\partial_t - \Delta) |dQ|^2_Q \]
\[ - 2 \langle |\nabla |\text{Ric}|^2, \nabla |dQ|^2_Q \rangle + \frac{2}{3} |\text{Ric}|^2 |dQ|^2_Q |T|^2 \]
\[ \leq \int |dQ|^2_Q \left\{ -2 |\text{Ric}|^2 + 4 R_{abcd} R^{ab} R^{cd} + 4 R^{ab} \Delta (T^{ce} T^{cd}) \right\} \]
\[ + 2 |\text{Ric}|^2 \left( -|\nabla dQ|^2_Q + C |T|^2 |dQ|^2_Q \right) \]
\[ + 8 |dQ|^2_Q |\nabla \text{Ric}| |\nabla dQ|^2_Q + \frac{2}{3} |\text{Ric}|^2 |dQ|^2_Q |T|^2 \]

where we have used Kato’s inequality for the tensors $\text{Ric}$ and $dQ$ (and treating them as sections of suitable bundles) to deal with the term
\[ -2 \langle |\nabla |\text{Ric}|^2, \nabla |dQ|^2_Q \rangle \leq 2 |\nabla |\text{Ric}|^2| |\nabla dQ|^2_Q \]
\[ = 8 |\text{Ric}| |\nabla \text{Ric}| |dQ|^2_Q |\nabla dQ|^2_Q \]
\[ \leq 8 |\text{Ric}| |\nabla dQ|^2_Q |dQ|^2_Q \]

Notice that to get the second line equality we use the fact that $\nabla |dQ|^2_Q = \nabla |dQ|^2_Q$ because $|dQ|^2_Q$ is $T^3$-invariant. Then we integrate by parts to bound this by
\[ \leq \int (4 + 8 \sqrt{7}) |dQ|^2_Q |\text{Rm}| |\text{Ric}|^2 + (2 + C) |\text{Ric}|^2 |dQ|^2_Q |T|^2 \]
\[ + 8 \left( |\nabla |dQ|^2_Q |\text{Ric}| + |dQ|^2_Q \nabla \text{Ric}| \right) |T|^2 |\nabla T| + \frac{2}{3} \left( |\nabla |dQ|^2_Q |\text{Ric}| + |dQ|^2_Q \nabla \text{Ric}| \right) |\nabla |T|^2 | \]
\[ + 8 \left( |\nabla |dQ|^2_Q |\text{Ric}| + \frac{1}{2} |dQ|^2_Q \nabla \text{Ric}| \right) |T|^2 |\nabla T| + \frac{2}{3} \left( |\nabla |dQ|^2_Q |\text{Ric}| + \frac{1}{2} |dQ|^2_Q \nabla \text{Ric}| \right) |\nabla |T|^2 | \]
\[ + 8 |dQ|^2_Q |\nabla \text{Ric}| |\nabla dQ|^2_Q - |\nabla |dQ|^2_Q |\nabla \text{Ric}|^2 \]
\[ \leq \int (4 + 8 \sqrt{7}) |dQ|^2_Q |\text{Rm}| |\text{Ric}|^2 + (2 + C) |\text{Ric}|^2 |dQ|^2_Q |T|^2 \]
\[ + 104 \frac{1}{3} |dQ|^2_Q |T|^2 |\nabla dQ|^2_Q |\text{Ric}| |\nabla T| + \frac{8}{3} |dQ|^2_Q |T|^2 |\nabla dQ|^2_Q |\nabla T| \]
\[ + 12 |dQ|^2_Q |T|^2 |\nabla T|^2 + 8 |dQ|^2_Q |\nabla \text{Ric}| |T|^2 |\nabla T| \]
\[ + 8 |dQ|^2_Q |\nabla \text{Ric}| |\nabla dQ|^2_Q - |\nabla |dQ|^2_Q |\nabla \text{Ric}|^2 \]

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Next we apply Cauchy–Schwarz inequality in various places, giving
\[
\leq \int (4 + 8 \sqrt{7}|dQ|_Q^2|Rm||Ric|^2 + (2 + C)|Ric|^2|dQ|_Q^2|T|^2
\]
\[
+ \left( \frac{1}{4}|Ric|^2|\nabla dQ|_Q^2 + \left( \frac{104}{3} \right)^2 |T|^2|dQ|_Q^2|\nabla T|^2 \right)
\]
\[
+ \left( \frac{4}{3}|T|^4|\nabla dQ|_Q^2 + \frac{4}{3}|dQ|_Q^2|T|^2|\nabla T|^2 \right)
\]
\[
+ 12|dQ|_Q^2|\nabla T|^2 + (2|dQ|_Q^2|\nabla Ric|^2 + 8|dQ|_Q^2|T|^2|\nabla T|^2)
\]
\[
+ \left( \frac{1}{4}|Ric|^2|\nabla dQ|_Q^2 + 64|dQ|_Q^2|\nabla Ric|^2 \right) - |Ric|^2|\nabla dQ|_Q^2 - 2|dQ|_Q^2|\nabla Ric|^2
\]
Collecting the pieces, we see we have isolated the negative term \(-\frac{1}{2}|Ric|^2|\nabla dQ|_Q^2\). The result now follows from the bounds \(|dQ|_Q < C\) and \(|\nabla T|^2 \leq C(|\nabla dQ|_Q^2 + 1)\) which follow from the assumed bound on \(|T|\).

This lemma completes the proof of Proposition 5.4 and hence the proof of Theorem 5.2.

6 The hypersymplectic flow extends as long as torsion is bounded

We now turn to the proof of our main result, Theorem 1.3. A key ingredient is the recent result of Chen [6, Thm. 1.1], generalizing Perelman’s noncollapsing theorem for Ricci flow [21] (see also the discussion in [15, section 13]). We begin by recalling the notion of \(\kappa\)-noncollapsing introduced by Perelman.

**Definition 6.1.** An \(n\)-dimensional Riemannian manifold \((M, g)\) is said to be \(\kappa\)-noncollapsed relative to an upper bound of scalar curvature on the scale \(\rho\) if for all \(B_\rho(x, r) \subset M\) with \(r < \rho\),

\[
\text{if } \sup_{B(x, r)} R \leq \frac{1}{r^2} \text{ then } \text{vol}_g (B_\rho (x, r)) \geq \kappa r^n
\]

We now consider an arbitrary path of Riemannian metrics \(g(t)\) and write

\[
\partial_t g = -2Ric + P
\]

where \(P\) measures the failure of \(g(t)\) to be a solution of Ricci flow.

**Theorem 6.2** (Chen [6], cf. Perelman [15]). If \(|P|\) is bounded along the flow (52) for \(t \in [0, s)\) with \(s < \infty\), then there exists \(\kappa > 0\) such that for all \(t \in [0, s)\), \(g(t)\) is \(\kappa\)-noncollapsing relative to an upper bound of scalar curvature on the scale \(\rho = \sqrt{s}\).

With this in hand, we now give the proof of Theorem 1.3. We assume that \(\omega(t)\) is a solution of the hypersymplectic flow for \(t \in [0, s)\), with \(s < \infty\), that \(|T| < C\) for all \(t \in [0, s)\) and that, for a contradiction, the flow does not extend to \(t = s\).

Write \(g(t) = g_\omega(t)\). By Propositions 4.4 and 4.5, \(\text{Tr} Q\) and \(|dQ|_Q\) are bounded along the flow, and hence by Lemma 3.13 \(|\tau|\) is also bounded. From this and the evolution equation for \(g_\omega\) in Corollary 4.2 we see that the “error term”

\[
P := \partial_t g + 2Ric = \frac{1}{2} \langle dQ, dQ \rangle_Q + \text{Tr}(Q^{-1} \tau \otimes \tau) - \frac{2}{3} |T|^2 g
\]

is bounded along the flow. Moreover, by Lemma 3.5, \(R = \frac{1}{4}|dQ|_Q^2 - |T|^2\) and so \(R\) is also uniformly bounded. Theorem 6.2 then implies that there exists \(\kappa > 0\) and \(\rho > 0\) such that for any \(p \in X\), \(t \in [0, s)\) and \(r \leq \rho\),

\[
\text{vol}_{g(t)} B_\rho (x, r) \geq \kappa r^4
\]
Write
\[ \Lambda(x, t) = (|\mathbf{Rm}|^2(x, t) + |\nabla \mathbf{T}|^2(x, t))^{1/2} \]
Since we have assumed that the flow does not extend to \( t = s \), it follows from Lotay–Wei’s extension result \cite{lotay2013finite} Theorem 1.3] that there is a sequence \((p_k, t_k) \in M \times [0, s)\) such that \( t_k \to s \) and \( \Lambda(p_k, t_k) \to \infty \). Moreover, we can choose \((p_k, t_k)\) so that

\[ \Lambda(p_k, t_k) = \sup \{ \Lambda(x, t) : x \in M, t \in [0, t_k] \} \]

To lighten the notation we write \( \Lambda_k := \Lambda(p_k, t_k) \). We define a sequence of flows by parabolic rescaling; for \( t \in [-\Lambda_k t_k, 0] \) we set

\[ \phi^{(k)}(t) = \Lambda_k \phi \left( \Lambda_k^{-1} t + t_k \right) \]

\[ \omega^{(k)}(t) = \Lambda_k \omega \left( \Lambda_k^{-1} t + t_k \right) \]

\[ g^{(k)}(t) = \Lambda_k g \left( \Lambda_k^{-1} t + t_k \right) \]

Here \( \phi^{(k)}(t) \) is a sequence of \( G_2 \)-Laplacian flows corresponding to the sequence \( \omega^{(k)}(t) \) of hypersymplectic flows, with induced metrics \( g^{(k)}(t) \) on \( X \times \mathbb{T}^3 \). Write \( \Lambda^{(k)}(x, t) \) for the quantity analogous to \( \Lambda(x, t) \), but defined using \( \phi^{(k)}(t) \). By construction,

\[ \Lambda^{(k)}(x, t) = \Lambda_k^{-1} \Lambda(x, \Lambda_k^{-1} t + t_k) \leq 1 \]

for any \((x, t) \in X \times [-\Lambda_k t_k, 0]\). Now the Shi-type estimates proved by Lotay–Wei \cite{lotay2013finite} tell us that for any \( A > 0, l \in \mathbb{N} \), there exists \( c_{l, A} \) such that

\[ \sup_{X \times [-A, 0]} \left( |\nabla^{l+1} \mathbf{T}^{(k)}| + |\nabla^l \mathbf{Rm}(g^{(k)})| \right) \leq c_{l, A} \]

It follows from Corollary \ref{corollary3.3} and Lemmas \ref{lem3.4} and \ref{lem3.18} and Lemma \ref{lemma3.4} that \(|\nabla \omega^{(k)}_i|, |\nabla^2 \omega^{(k)}_i|\) and \(|\nabla^l \mathbf{Rm}(g^{(k)})|\) are uniformly bounded for any choice of \( l \).

By Cheeger–Gromov–Taylor \cite{cheeger1992structure} Theorem 4.7, the volume lower bound \cite{de1983structure} and uniform curvature bound imply that the injectivity radius of \( g^{(k)}(0) \) is uniformly bounded below away from zero. It then follows from Cheeger–Gromov compactness theorem that we can take a limit: there exists a pointed Riemannian 4-manifold \((X^\infty, g^\infty, p_\infty)\) such that

\[ (X, g^{(k)}(0), p_k) \overset{\text{Cheeger–Gromov}}{\to} (X^\infty, g^{(\infty)}(0), p_\infty) \]

In other words, there is an exhaustion \( X^\infty = \bigcup_{k=1}^\infty \Omega_k \) by nested neighborhoods of \( p_\infty \) and a sequence of (pointed) diffeomorphisms into \( X \), i.e. \( f_k : \Omega_k \to X \) with \( f_k(p_\infty) = p_k \), such that on any fixed compact subset \( K \subset X^\infty \), we have \( f_k \circ g^{(k)}(0) \to g^\infty \).

It follows from Proposition \ref{proposition4.4} and Lemmas \ref{lemma3.3}, \ref{lem3.4} and \ref{lem3.18} that we have a uniform \( C^2 \)-bound on \( \omega \). By the Arzelà–Ascoli theorem, we pass to a subsequence for which \( f_k^* \omega^{(k)}(0) \) converges in \( C^{1,\alpha} \) to a limiting triple of closed self-dual 2-forms \( \omega^\infty \). Write \( Q^\infty = \frac{1}{2} \langle \omega^\infty, \omega^\infty \rangle \) for the matrix of pointwise inner product. Now \( Q^\infty \) is the limit (under the diffeomorphisms \( f_k \)) of the sequence \( Q^{(k)}(t) \) of matrices defined by \( \omega^{(k)}(t) \). But \( Q \) is a scale-invariant quantity and so \( Q^{(k)}(t) = Q(t) \), the lowest eigenvalue of which is uniformly bounded below away from zero. It follows that \( Q^\infty \) is positive definite and so \( \omega^\infty \) is a hypersymplectic structure determining the metric \( g^\infty \).

We now compute

\[ |dQ^\infty|^2_{Q^\infty} = \lim_{k \to \infty} |dQ^{(k)}|^2_{Q^{(k)}(0)} = \lim_{k \to \infty} \Lambda_k^{-1} |dQ^2_{Q^{(k)}}(t_k) = 0 \]

where the middle norm is taken with respect to \( g^{(k)}(0) \), the right hand norm with \( g(t_k) \) and we have used the facts that \( Q^{(k)} = Q \) and that \( |dQ^2|_{Q^2} \) is uniformly bounded. It follows that \( g^\infty \) is hyperkähler and \( \omega^\infty \) is a constant rotation of a hyperkähler triple of 2-forms.
The uniform bound of the scale invariant energy \( \int |Rm|^2 \) along the flow (Corollary 5.3) shows that
\[
\int_{X} |Rm(g^\infty)|^2 < \infty
\]
which means that \((X^\infty, g^\infty)\) is a gravitational instanton. Moreover, since
\[
|Rm(g^\infty)|(p_\infty, 0) = \Lambda_{g^\infty} = \lim_{k \to \infty} \Lambda_k^{-1} \Lambda_{g(t_k)}|p_k, t_k| = 1
\]
we see that \((X^\infty, g^\infty)\) is a non-trivial gravitational instanton.

The volume ratio lower bound \([53]\) implies \((X^\infty, g^\infty)\) has Euclidean volume growth and so, by a theorem of Bando–Kasue–Nakajima \([1]\), \((X^\infty, g^\infty)\) is asymptotically locally Euclidean. We now invoke Kronheimer’s classification of asymptotically locally Euclidean gravitational instantons \([16]\), which implies that there is at least one 2-sphere \(S \subset X^\infty\) which is holomorphic for one of the hyperkähler complex structures \(J\). Without loss of generality we assume this \(J\) corresponds to the Kähler form \(\omega^\infty\) (otherwise we apply a linear transformation, constant in space and time, to the triples). We also note that such a 2-sphere automatically has \(|S|^2 = -2\). (This follows from the adjunction formula and the fact that the canonical bundle of a hyperkähler manifold is trivial.)

Recall the diffeomorphisms \(f_k : \Omega_k \to X\) in the Cheeger–Gromov convergence. For any \(i = 1, 2, 3\), we have
\[
\int_S \omega_i^\infty = \lim_{k \to \infty} \int_S f_k^* \omega_i^{(k)}(0) = \lim_{k \to \infty} \int_{f_k(S)} \omega_i^{(k)}(0) = \lim_{k \to \infty} \Lambda_k \langle [\omega_i], [f_k(S)] \rangle
\]
It follows that \(\langle [\omega_i], [f_k(S)] \rangle \to 0\). Meanwhile, since \(S\) is a symplectic submanifold for \(\omega^\infty\), it follows that it is symplectic for \(f_k^* \omega_i^{(k)}\) for all large \(k\). For these values of \(k\) we have \(\langle [\omega_i], [f_k(S)] \rangle > 0\).

We will now deduce a contradiction. Note that \(b_+ = 3\) and the \([\omega_1], [\omega_2], [\omega_3]\) span a maximal positive-definite subspace of \(H^2(X, \mathbb{R})\) for the cup product (as is explained in \([4]\) straight after Conjecture 1.1). Set
\[
H_2^- = \{ Z \in H_2(X, \mathbb{R}) : \langle [\omega_i], Z \rangle = 0 \text{ for } i = 1, 2, 3 \}
\]
By Poincaré duality this is a maximal negative-definite subspace of \(H_2(X, \mathbb{R})\) for the intersection form. Its orthogonal complement \(H_2^+ \subset H^2(X, \mathbb{R})\) (defined via the intersection form) is a maximal positive-definite subspace. Now write \([f_k(S)] = P_k + N_k\) with \(P_k \in H_2^+\) and \(N_k \in H_2^-\). Since \(\langle [\omega_i], [f_k(S)] \rangle \to 0\) we see that \(P_k \to 0\). Since \([f_k(S)]^2 = -2\) we deduce that \(N_k^2 \to -2\) and so \(N_k\) is bounded (because the intersection form is a negative definite quadratic form on \(H_2^+\)). It follows that \([f_k(S)]\) lies in a bounded set in \(H_2^+\). Crucially, it also lies in the integral lattice \(H_2(X, \mathbb{Z})\) and hence it can take at most finitely many different values. Now \(\langle [\omega_i], [f_k(S)] \rangle\) has a smallest non-zero value on this finite set, \(v > 0\) say. It follows that for all large \(k\), \(\langle [\omega_i], [f_k(S)] \rangle \geq v > 0\) which contradicts the fact that this same sequence converges to zero. This contradiction completes the proof of Theorem 1.3.

We finish the article with an application of Theorem 1.3 giving an interesting estimate for the maximal existence time of a hypersymplectic flow on a compact hypersymplectic manifold, in terms of some quite weak bounds on the initial data (compared, for example, with the “doubling-time estimate” of \([17]\) Proposition 4.1]).

**Theorem 6.3.** There exists a computable absolute constant \(C > 0\) such that any hypersymplectic flow \(\omega(t)\) exists on the interval \([0, s]\) for
\[
\frac{1}{CK^{19}A} \leq s
\]
where \(K = \sup \text{Tr} Q(0)\) and \(A = \sup |dQ(t)|^2_Q(0)\) are the quantities associated with \(\omega(0)\).
Proof. By Propositions 4.4 and 4.5, there is a constant $C$ such that
\[
(\partial_t - \Delta) |dQ|_Q^2 \leq C (\text{Tr}\, Q)^{19} |T|^2 |dQ|_Q^2
\]
\[
(\partial_t - \Delta) \text{Tr}\, Q \leq \frac{5}{3} |T|^2 \text{Tr}\, Q
\]

Let
\[
f(t) = \sup |dQ|_Q^2 (t)
\]
\[
f_1(t) = \sup \text{Tr}\, Q (t)
\]
\[
f_2(t) = \sup |T|^2 (t)
\]

The heat inequalities and the maximum principle give
\[
f' \leq Cf_1^{19} f_2 f
\]
\[
(f_1^{19})' \leq \frac{95}{3} f_2 f_1^{19}
\]
from which it follows that
\[
(f_1^{19})' = (f_1^{19})' f + f_1^{19} f' \leq \frac{95}{3} f_1^{19} f_2 f + Cf_1^{38} f_2 f \leq C'(f_1^{19} f)^2
\]

where in the last step we have used the inequality $f_2 \leq \frac{3}{2} f$ by part 2 of Lemma 3.13. The consequence is that
\[
f_1^{19}(t)f(t) \leq \frac{1}{(f_1^{19} f)^{-1}(0) - Ct}
\]

Now, assume for a contradiction that the maximal time of the flow is $s \leq \frac{1}{2C'K^{19}A}$. Then the above bound shows that
\[
\sup_{[0,s]} f(t) < \infty
\]

and hence $|T|^2 \leq \frac{3}{2} |dQ|_Q^2$ (by part 2 of Lemma 3.13) is uniformly bounded on $[0,s)$. But, by Theorem 1.3 this contradicts the fact that $s$ is maximal. \hfill \Box

A Formulae for the symmetric metric on $\mathcal{P}$

This appendix gives the derivation of the Levi-Civita connection for the symmetric metric on $\mathcal{P}$, and also the formulae for the Laplacian and the Hessian of the map $Q: X \to \mathcal{P}$ from a Riemannian manifold $(X,g)$.

Lemma A.1. Let $\mathcal{P}$ denote the space of positive definite symmetric $n \times n$ real matrices, with its complete symmetric Riemannian metric, and let $S^2(\mathbb{R}^n)$ denote the space of $n \times n$ real symmetric matrices. Since $\mathcal{P} \subset S^2(\mathbb{R}^n)$ is an open set in the vector space of symmetric matrices, we can identify $\mathcal{P} \cong \mathcal{P} \times S^2(\mathbb{R}^n)$. In this trivialisation, the Levi-Civita connection is given on affine vector fields $A,B \in S^2(\mathbb{R}^n)$ by
\[
(\nabla_A B)(P) = -\frac{1}{2} AP^{-1} B - \frac{1}{2} BP^{-1} A
\]

Proof. Affine vector fields commute. Since the above expression for $\nabla_A B$ is symmetric in $A,B$ it follows that the connection is torsion free. It remains to check that it is metric. We compute
\[
A \cdot \langle B, C \rangle = A \cdot \text{Tr} (P^{-1} BP^{-1} C)
\]
\[
 = - \text{Tr} (P^{-1} AP^{-1} BP^{-1} C) - \text{Tr} (P^{-1} BP^{-1} AP^{-1} C)
\]
Meanwhile,
\[
\langle \nabla A, B, C \rangle + \langle A, \nabla B, C \rangle = -\frac{1}{2} \text{Tr}(P^{-1}AP^{-1}BP^{-1}C) - \frac{1}{2} \text{Tr}(P^{-1}BP^{-1}AP^{-1}C) - \frac{1}{2} \text{Tr}(P^{-1}AP^{-1}BP^{-1}C) - \frac{1}{2} \text{Tr}(P^{-1}AP^{-1}CP^{-1}B) = -\text{Tr} \left( P^{-1}AP^{-1}BP^{-1}C \right) - \text{Tr} \left( P^{-1}BP^{-1}AP^{-1}C \right)
\]

where we have used cyclic symmetry of the trace in the last line.

Fix a smooth map \(Q: X \to P\) and let \(V\) denote a tensor on \(X\) with values in \(Q^*TP\). We can identify \(V\) with a tensor with values in the fixed vector space \(S^2(\mathbb{R}^n)\) of symmetric matrices. We now have two different ways to differentiate \(V\). We write \(\nabla V\) for the covariant derivative (using the Levi-Civita connection of \(g\)) of this tensor thought of simply as vector valued, and we write \(\hat{\nabla} V\) for the covariant derivative of \(V\) defined using the Levi-Civita connection of \(g\) and of \(P\) pulled back via \(Q\) to the bundle \(Q^*TP\).

**Lemma A.2.** Let \((X, g)\) be a Riemannian manifold and \(Q: X \to P\) a smooth map. The Hessian \(\nabla dQ\) of \(Q\) determined by the symmetric space metric on \(P\) is related to the Hessian \(\nabla dQ\) of \(Q\) determined by the flat affine structure on \(P\) by the formula
\[
(\hat{\nabla} dQ)_{ij} = \nabla (dQ)_{ij} - \frac{1}{2} \epsilon^{pq} (dQ_{ip} \otimes dQ_{qj} + dQ_{qj} \otimes dQ_{ip})
\]

**Proof.** Write \(dQ = \sum \alpha_i \otimes M_i\) where the \(\alpha_i\) are 1-forms on \(X\) and \(M_i: X \to S^2(\mathbb{R}^n)\) take values in symmetric matrices. On the one hand,
\[
\nabla dQ = \sum \nabla \alpha_i \otimes M_i + \alpha_i \otimes dM_i
\]

On the other hand, by Lemma [A.1]
\[
\hat{\nabla} dQ = \sum \nabla \alpha_i \otimes M_i + \alpha_i \otimes \hat{\nabla} M_i
\]
\[
= \sum \nabla \alpha_i \otimes M_i + \alpha_i \otimes \left( dM_i - \frac{1}{2} dQQ^{-1}M_i - \frac{1}{2} M_iQ^{-1}dQ \right)
\]

**Lemma A.3.** Let \((X, g)\) be a Riemannian manifold and \(Q: X \to P\) a smooth map. The Laplacian \(\Delta Q\) determined by the symmetric space metric on \(P\) is related to the Laplacian \(\Delta Q\) (the analysts’ convention) determined by the flat affine structure on \(P\) by the formula
\[
\Delta Q = \Delta Q - \langle dQQ^{-1}, dQ \rangle_g
\]

**Proof.** This follows from taking the trace via \(g\) of the two different Hessians in Lemma [A.2]

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