A new characterization of the dual polar graphs

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Abstract

In this paper we give a new characterization of the dual polar graphs, extending the work of Brouwer and Wilbrink on regular near polygons. Also as a consequence of our characterization we confirm a conjecture of the authors on non-bipartite distance-regular graphs with smallest eigenvalue at most \(-k/2\), where \(k\) is the valency of the distance-regular graph, in case of \(c_2 \geq 3\) and \(a_1 = 1\).

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1 Introduction

In this paper we study geometric distance-regular graphs. They were introduced by Godsil \cite{G} as a generalization of geometric strongly regular graphs. Among the examples are the Johnson graphs, the Hamming graphs, the Grassmann graphs, the bilinear forms graphs and the dual polar graphs. Koolen and Bang \cite{KB} and Bang, Dubickas, Koolen and Moulton \cite{BDKM} showed that for any integer \(m\) at least 2, there are only finitely many coconnected non-geometric distance-regular graphs with smallest eigenvalue at least \(-m\) and valency at least 3. An important class of geometric distance-regular graphs are the regular near 2\(D\)-gons. Brouwer and Wilbrink \cite{BW} (completed in De Bruyn \cite{D}) showed that the only thick regular 2\(D\)-gons with intersection number \(c_2\) at least three and \(D\) at least four are the dual polar graphs (see \cite{S} Theorem 9.11] and Theorem \cite{S} 2.6). In this paper we extend the classification of Brouwer and Wilbrink. We give the following characterization of the dual polar graphs \(\mathbb{^2A_{2D-1}}(\sqrt{q})\), \(B_D(q)\) and \(C_D(q)\):

\begin{itemize}
  \item \(\mathbb{^2A_{2D-1}}(\sqrt{q})\)
  \item \(B_D(q)\)
  \item \(C_D(q)\)
\end{itemize}
Theorem 1.1. Let $\Gamma$ be a non-bipartite geometric distance-regular graph with diameter $D \geq 4$ and $c_2 \neq 1$. Then the following hold:

i) If $a_i = c_i(a_1 + 1)$ ($i \leq 2$) and $c_3 = (a_1^2 + a_1 + 1)(a_1 + c_2 + 1)$, then $\Gamma$ is the dual polar graph $^2A_{2D-1}(\sqrt{q})$.

ii) If $a_i = c_i(a_1 + 1)$ ($i \leq 3$) and $c_4 = (a_1^2 + 2a_1 + 2)(c_3 - (a_1 + 1)^2)$, then $\Gamma$ is the dual polar graph $^2A_{2D-1}(\sqrt{q})$, $B_D(q)$ or $C_D(q)$.

The idea of the proof is an extension of the main idea used in Lang [13].

As a consequence of Theorem 1.1, we show the following:

Theorem 1.2. Let $\Gamma$ be a non-bipartite distance-regular graph with valency $k \geq 3$, diameter $D \geq 2$ and smallest eigenvalue $\theta_{\min} \leq -k/2$. If $a_1 = 1$ and $c_2 = 3$, then $\Gamma$ is one of the following graphs:

i) one of the two distance-regular graphs with intersection array $\{8, 6, 1; 1, 3, 8\}$ (see [2, p.386]),

ii) the Witt graph associated to $M_{24}$ with intersection array $\{30, 28, 24; 1, 3, 15\}$ (see [2, Section 11.4]),

iii) the dual polar graph $B_D(2)$.

This result shows that the following conjecture of the authors is correct if both $c_2 \geq 3$ and $a_1 = 1$ hold.

Conjecture. [14] When $D$ is large enough, a non-bipartite distance-regular graph with valency $k$, diameter $D$ and smallest eigenvalue $\theta_{\min} \leq -k/2$ is one of the following graphs:

i) the odd polygons,

ii) folded $(2D + 1)$-cubes,

iii) the odd graphs $O_k$,

iv) the Hamming graphs $H(D, 3)$,

v) the dual polar graphs $^2A_{2D-1}(2)$,

vi) the dual polar graphs $B_D(2)$.

Theorem 1.2 improves earlier results of the authors in [12] in which we showed that such a graph is the dual polar graph $^2A_{2D-1}$ when $a_1 = 1$ and $c_2 \geq 4$. In this paper, we first give a characterization of dual polar graphs. Then we show the conjecture is true with $a_1 = 1$ and $c_2 = 3$, where if $D \geq 4$, such a graph is $B_D(2)$.

This paper is organized as follows. In the next section, we give the definitions and some preliminary results. Then in Section 3 we give a characterization of dual polar graphs and prove Theorem 1.1. In the last section we give a proof of Theorem 1.2.
2 Preliminaries

For more background, see [2] and [15].

All the graphs considered in this paper are finite, undirected and simple. Let $\Gamma$ be a graph with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. Denote $x \sim y$ if the vertices $x,y \in V$ are adjacent. The distance $d(x,y) = d_\Gamma(x,y)$ between two vertices $x,y \in V(\Gamma)$ is the length of a shortest path connecting $x$ and $y$. The maximum distance between two vertices in $\Gamma$ is the diameter $D = D(\Gamma)$. We use $\Gamma_i(x)$ for the set of vertices at distance $i$ from $x$ and write, for the sake of simplicity, $\Gamma(x) := \Gamma_0(x)$. For two vertices $x,y \in V$, we denote $\Gamma_j(x,y) := \Gamma_i(x) \cap \Gamma_j(y)$. The valency of $x$ is the number $|\Gamma(x)|$ of vertices adjacent to it. A graph is regular with valency $k$ if the valency of each of its vertices is $k$. A graph is called bipartite if it has no odd cycle.

A connected graph $\Gamma$ with diameter $D$ is called distance-regular if there are integers $b_i,c_i (0 \leq i \leq D)$ such that for any two vertices $x,y \in V(\Gamma)$ with $d(x,y) = i$, there are exactly $c_i$ neighbors of $y$ in $\Gamma_{i-1}(x)$ and $b_i$ neighbors of $y$ in $\Gamma_{i+1}(x)$, where we define $b_D = c_0 = 0$. In particular, $\Gamma$ is a regular graph with valency $k := b_0$. We define $a_i := k - b_i - c_i (0 \leq i \leq D)$ for notational convenience. Note that $a_i = |\Gamma(y) \cap \Gamma_i(x)|$ holds for any two vertices $x,y$ with $d(x,y) = i (0 \leq i \leq D)$.

For a distance-regular graph $\Gamma$ and a vertex $x \in V(\Gamma)$, we denote $k_i := |\Gamma_i(x)|$ and $p_{ij}^k := |\{w \mid w \in \Gamma_i(x) \cap \Gamma_j(y)\}|$ for any $y \in \Gamma_h(x)$. It is easy to see that $k_i = b_0 b_1 \cdots b_{i-1}/(c_1 c_2 \cdots c_i)$ and hence it does not depend on $x$. The numbers $a_i, b_i$ and $c_i (0 \leq i \leq D)$ are called the intersection numbers, and the array $\{b_0,b_1,\ldots,b_{D-1};c_1,c_2,\ldots,c_D\}$ is called the intersection array of $\Gamma$.

Let $\Gamma$ be a distance-regular graph with $v$ vertices and diameter $D$. Let $A_i (0 \leq i \leq D)$ be the $(0,1)$-matrix whose rows and columns are indexed by the vertices of $\Gamma$ and the $(x,y)$-entry is 1 whenever $d(x,y) = i$ and 0 otherwise. We call $A_i$ the distance-$i$ matrix and $A := A_1$ the adjacency matrix of $\Gamma$. The eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$ of the graph $\Gamma$ are just the eigenvalues of its adjacency matrix $A$. We denote $m_i$ the multiplicity of $\theta_i$.

For each eigenvalue $\theta_i$ of $\Gamma$, let $U_i$ be a matrix with its columns forming an orthonormal basis for the eigenspace associated with $\theta_i$. And $E_i := U_i U_i^T$ is called the minimal idempotent associated with $\theta_i$, satisfying $E_i E_j = \delta_{ij} E_j$ and $AE_i = \theta_i E_i$, where $\delta_{ij}$ is the Kronecker delta. Note that $vE_0$ is the all-ones matrix $J$.

The set of distance matrices $\{A_0 = I, A_1, A_2, \ldots, A_D\}$ forms a basis of a commutative $\mathbb{R}$-algebra $A$, known as the Bose-Mesner algebra. The set of minimal idempotents $\{E_0 = \frac{1}{v} J, E_1, E_2, \ldots, E_D\}$ is another basis of $A$. There exist $(D+1) \times (D+1)$ matrices $P$ and $Q$ (see [2, p.45]), such that the following relations hold

$$A_i = \sum_{j=0}^{D} P_{ji} E_j \quad \text{and} \quad E_i = \frac{1}{v} \sum_{j=0}^{D} Q_{ji} A_j \quad (0 \leq i \leq D).$$  (1)

Note that $Q_{0i} = m_i$ (see [2, Lemma 2.2.1]).

Let $E_i := U_i U_i^T$ be the minimal idempotent associated with $\theta_i$, where the columns of $U_i$ form an orthonormal basis of the eigenspace associated with $\theta_i$. We denote the $x$-th row
of $\sqrt{v/m_i}U_i$ by $\hat{x}$. Note that $E_i \circ A_j = \frac{1}{d}Q_iA_j$, hence all the vectors $\hat{x}$ are unit vectors and the cosine of the angle between two vectors $\hat{x}$ and $\hat{y}$ is $u_j(\theta_i) := \frac{Q_i}{Q_0}$, where $d(x, y) = j$. The map $x \mapsto \hat{x}$ is called a normalized representation and the sequence $(u_j(\theta_i))_{j=0}^D$ is called the standard sequence of $\Gamma$, associated with $\theta_i$. As $AU_i = \theta_iU_i$, we have $\theta_i\hat{x} = \sum_{y \sim x} \hat{y}$, and hence the following holds:

$$
\begin{align*}
&\begin{cases}
c_j u_{j-1}(\theta_i) + a_j u_j(\theta_i) + b_j u_{j+1}(\theta_i) = \theta_i u_j(\theta_i) & (1 \leq j \leq D - 1), \\
c_D u_{D-1}(\theta_i) + a_D u_D(\theta_i) = \theta_D u_D(\theta_i),
\end{cases}

\end{align*}
$$

(2)

with $u_0(\theta_i) = 1$ and $u_1(\theta_i) = \frac{\theta_i}{k}$.

**Lemma 2.1.** [15, Theorem 2.8] Let $\Gamma$ be a distance-regular graph with diameter $D$ and $v$ vertices. Let $\theta$ be an eigenvalue of $\Gamma$ and $(u_i)_{i=0}^D$ be the standard sequence associated with $\theta$. Then the multiplicity $m(\theta)$ of $\theta$ as an eigenvalue of $\Gamma$ satisfies

$$
m(\theta) = \frac{v}{\sum_{i=0}^D k_i u_i^2}.
$$

**Lemma 2.2.** [2, Proposition 4.1.6] Let $\Gamma$ be a distance-regular graph with valency $k$ and diameter $D$. Then the following conditions hold:

1) $1 = c_1 \leq c_2 \leq \cdots \leq c_D$,

2) $k = b_0 \geq b_1 \geq \cdots \geq b_{D-1}$,

3) $k_i$’s $(1 \leq i \leq D)$ are positive integers,

4) the multiplicities are positive integers.

Let $\Pi = \{P_1, \ldots, P_t\}$ be a partition of the vertex set of a graph $\Gamma$. Let $f_{ij}$ $(1 \leq i, j \leq t)$ be the average number of neighbors in $P_j$ of a vertex in $P_i$. The partition $\Pi$ is called equitable if for all $1 \leq i, j \leq t$, every vertex in $P_i$ has exactly $f_{ij}$ neighbors in $P_j$.

The following result was first shown by Delsarte [4] for strongly regular graphs, and extended by Godsil [7] to distance-regular graphs.

**Lemma 2.3.** Let $\Gamma$ be a distance-regular graph with valency $k$ and smallest eigenvalue $\theta_{\min}$. Let $C$ be a clique in $\Gamma$ with $c$ vertices. Then $c \leq 1 - \frac{k}{\theta_{\min}}$.

A clique $C$ in a distance-regular graph $\Gamma$ that attains this Delsarte bound is called a Delsarte clique. A distance-regular graph is called geometric (with respect to $C$) if it contains a collection $C$ of Delsarte cliques such that each edge is contained in a unique $C \in C$.

Let $\Gamma$ be a connected graph with diameter $D$, and let $C$ be a subset of $V(\Gamma)$. For $i \geq 0$, let $C_i = \{x \in V \mid d(x, C) = i\}$, where $d(x, C) = \min\{d(x, c) \mid c \in C\}$. The covering radius of $C$, denoted by $\rho = \rho(C)$, is the maximum $i$ such that $C_i \neq \emptyset$. The subset (or code) $C$ is called completely regular if the distance partition $\Pi = \{C_i \mid i = 0, 1, \ldots, \rho\}$ is equitable.
Lemma 2.4. [15] Proposition 4.2] Let $\Gamma$ be a geometric distance-regular graph with diameter $D$ and smallest eigenvalue $\theta_{\text{min}}$. Then any $(a_1 + 2)$-clique $C$ in $\Gamma$ is a completely regular code with covering radius $D - 1$ and $\gamma_i u_i + (a_1 + 2 - \gamma_i) u_{i+1} = 0$ ($0 \leq i \leq D - 1$), where $(u_i)^D_{i=0}$ is the standard sequence associated with $\theta_{\text{min}}$ and $\gamma_i = |\Gamma_i(x) \cap C|$ for a vertex $x$ at distance $i$ from $C$.

Let $\Gamma$ be a geometric distance-regular graph with diameter $D$, then the parameter $\gamma_i := |\Gamma_i(x) \cap C|$ ($0 \leq i \leq D - 1$) is well-defined, where $C$ is any Delsarte clique and $x$ is any vertex at distance $i$ from $C$.

Lemma 2.5. Let $\Gamma$ be a geometric distance-regular graph with diameter $D$. Then $1 \leq \gamma_i \leq a_1 + 1$ and $\gamma_i$ is increasing ($0 \leq i \leq D - 1$).

Proof. Let $\theta = -\frac{k}{a_1 + 1}$ be the smallest eigenvalue of $\Gamma$ with associated standard sequence $(u_i)^D_{i=0}$. By definition, we see that $1 \leq \gamma_i \leq a_1 + 2$ ($0 \leq i \leq D - 1$). Note $u_0 = 1 = \gamma_0$, by Lemma 2.4 it follows that $(a_1 + 2 - \gamma_i) u_{i+1} \neq 0$, that is $u_{i+1} \neq 0$ and $\gamma_i \leq a_1 + 1$ ($0 \leq i \leq D - 1$). For any $0 \leq i \leq D - 2$, we can take two vertex $x,y$ with $d(x,C) = i + 1$, $d(x,y) = 1$ and $d(y,C) = i$. Then we see that $\Gamma_i(y) \cap C \subseteq \Gamma_{i+1}(x) \cap C$, that is $\gamma_i \leq \gamma_{i+1}$. □

Let $\Gamma$ be a geometric distance-regular graph with diameter $D \geq 2$, and let $\theta_{\text{min}}$ be the smallest eigenvalue of $\Gamma$ with associated standard sequence $(u_i)^D_{i=0}$ and normalized representation $x \mapsto \hat{x}$. For any $2 \leq j \leq D$, choose two vertices $x, y$ with $d(x,y) = j$ and define $F_j, C_j$ and $S^C_j$ with respect to $x, y$ as the following

$$F_j = \hat{x} - \hat{y}, \quad (4)$$
$$C_j = \sum_{x \in \Gamma_{j-1}(x,y)} \hat{z} - \sum_{w \in \Gamma_{j-1}(x,y)} \hat{w}, \quad (5)$$
$$S^C_j = \langle C_j, C_j \rangle \langle F_j, F_j \rangle - \langle C_j, F_j \rangle \langle C_j, F_j \rangle. \quad (6)$$

By definition, we have

$$\langle F_j, F_j \rangle = 2(u_0 - u_j), \quad (7)$$
$$\langle C_j, F_j \rangle = 2c_j(u_1 - u_{j-1}). \quad (8)$$

Moreover, if $\gamma_{j-1} = 1$, then

$$\langle C_j, C_j \rangle = 2c_j((u_0 + (c_j - 1)u_2) - (c_{i-1}u_{j-2} + (c_j - c_{j-1})u_j)), \quad (9)$$

Figure 1: Diagram of $F_j$ and $C_j$
and $S_j^C$ is independent of the choice of the representation. Note that $S_j^C \geq 0$ and equality holds if and only if $F_j$ and $C_j$ are linearly dependent.

A distance-regular graph $\Gamma$ is of order $(s, t)$ (for some integers $s$ and $t$) if it is locally the disjoint union of $t+1$ cliques of size $s$. A distance-regular graph $\Gamma$ of order $(s, t)$ with diameter $D$ is called a regular near $2D$-gon if $a_i = c_ia_1$ ($1 \leq i \leq D$), and such a graph is geometric. We call $\Gamma$ thick if $s \geq 2$.

**Theorem 2.6.** (c.f. [2, Theorem 6.6.1], [15, Theorem 9.11]) Let $\Gamma$ be a thick regular near $2D$-gon with diameter $D \geq 4$ and $c_2 \neq 1$. Then $\Gamma$ is a Hamming graph or a dual polar graph.

**Lemma 2.7.** Let $\Gamma$ be a geometric distance-regular graph with diameter $D \geq 2$. Then the following holds:

$$
\begin{align*}
\begin{cases}
    a_i = c_i \frac{a_1 + 1 - \gamma_{i-1}}{\gamma_{i-1}} + b_i \frac{\gamma_i - 1}{a_1 + 1 - (\gamma_i - 1)} & (1 \leq i \leq D - 1), \\
    a_D = c_D \frac{a_1 + 1 - \gamma_{D-1}}{\gamma_{D-1}}.
\end{cases}
\end{align*}
$$

Moreover, $\Gamma$ is a regular near $2D$-gon if and only if $\gamma_{D-1} = 1$.

**Proof.** Let $(u_i)_{i=0}^D$ be the standard sequence associated with the smallest eigenvalue $\theta_{\min} = -\frac{1}{a_1+1}$. Then we see that $1 \leq \gamma_i \leq a_1 + 1$ ($0 \leq i \leq D - 1$) by Lemma 2.5 and Lemma 2.4 implies

$$
\begin{align*}
\begin{cases}
    u_{i+1} = -\frac{\gamma_i}{a_1 + 2 - \gamma_i} u_i, \\
    u_{i-1} = -\frac{a_1 + 2 - \gamma_{i-1}}{\gamma_{i-1}} u_i & (1 \leq i \leq D - 1).
\end{cases}
\end{align*}
$$

Then substitute $\theta = -\frac{a_1 + b_i}{a_1 + 1}$ and Equation (11) into Equation (2), we obtain Equation (10). And we see that $\gamma_{D-1} = 1$ if and only if $a_i = c_ia_1$ ($0 \leq i \leq D$), that is $\Gamma$ is a regular near $2D$-gon. \hfill \Box

A subgraph $\Delta$ of $\Gamma$ is called strongly closed if $z \in V(\Delta)$ for all vertices $x, y \in V(\Delta)$ and $z \in V(\Gamma)$, such that $d_\Gamma(x, z) + d_\Gamma(z, y) \leq d_\Gamma(x, y) + 1$. A distance-regular graph $\Gamma$ with diameter $D$ is said to be $m$-bounded for some $m = 1, 2, \ldots, D$ if for all $i = 1, 2, \ldots, m$ and all vertices $x$ and $y$ at distance $i$ there exists a strongly closed subgraph $\Delta(x, y)$ with diameter $i$, containing $x$ and $y$ as vertices.

**Lemma 2.8.** (c.f. [9, Theorem 1.1]) Let $\Gamma$ be a non-bipartite geometric distance-regular graph with diameter $D \geq 3$ and $c_2 \neq 1$. Then the following are equivalent:

i) $\gamma_m = 1$,

ii) $\Gamma$ is $m$-bounded.
Proof. Assume \( \gamma_m = 1 \). Then \( \gamma_i = 1 \) (0 \( \leq i \leq m \)) and Lemma \([2, Proposition 11.3] \) implies \( a_i = c_ia_1 \) (1 \( \leq i \leq m \)). Since \( c_2 \neq 1 \), by \([2, Theorem 5.2.1] \), we see \( c_i \geq c_{i-1} + 1 \) (1 \( \leq i \leq m \)). Then \([15, Proposition 11.3] \) implies that \( \Gamma \) is \( m \)-bounded.

Now we assume \( \Gamma \) is \( m \)-bounded. Then \([15, Proposition 11.3] \) implies \( a_i = c_ia_1 \) (1 \( \leq i \leq m \)). It follows from Lemma \([2, Proposition 11.3] \) that \( \gamma_i = 1 \) (0 \( \leq i \leq m \)). \( \square \)

3 Characterization of dual polar graphs

In this section we will characterize the dual polar graphs.

Lemma 3.1. Let \( \Gamma \) be a non-bipartite geometric distance-regular graph with diameter \( D \geq 4 \), and let \( \theta \neq k \) be an eigenvalue of \( \Gamma \) with associated standard sequence \( (u_i)_{i=0} \). If there exists some \( j \) satisfying \( 3 \leq j \leq D - 1 \), such that \( S_j^C = 0 \), then the following hold:

\[
(u_i - u_{i-j+2}) = \frac{u_{i+1} - u_{i-j}}{u_0 - u_j}(u_{i+1} - u_{i-j+1}) \quad (j \leq i \leq D - 1).
\]

Proof. Take three vertices \( x, y, u \) with \( d(x, u) = i + 1 \), \( d(x, y) = j \) and \( d(y, u) = i - j + 1 \) (\( j \leq i \leq D - 1 \)). Consider a normalized representation \( w \mapsto \hat{w} \) associated with \( \theta \) and \( C_j, F_j \) with respect to \( x, y \).

By \([2, Proposition 4.4.7] \), we see that \( u_j \neq u_0 \) for 1 \( \leq j \leq D - 1 \) as the graph is not bipartite. It follows from Equation \([7] \) that \( F_j \) is a non-zero vector. Then we see \( S_j^C = 0 \) implies \( C_j \) and \( F_j \) are linearly dependent, that is, there exists a constant \( t_j \) such that \( C_j = t_jF_j \). Substitute Equation \([7] \) and \([8] \) into \( t_j = \frac{\langle C_j, F_j \rangle}{\langle F_j, F_j \rangle} \), we see that

\[
t_j = c_j \frac{u_{i+1} - u_{i-j}}{u_0 - u_j}.
\]

Substitute Equation \([13] \) \([14] \) and \([15] \) into \( \langle \hat{u}, C_j \rangle = t_j\langle \hat{u}, F_j \rangle \), the result follows. \( \square \)

Remark 3.2. i) When \( j = 2 \), then \( C_j \) is the zero vector and Equation \([12] \) always holds.

ii) The standard sequence \( (u_i)_{i=1}^D \) satisfies the following relations, and thus it is determined by the numbers \( u_0, u_1, \ldots, u_j \).

\[
\begin{cases}
u_{i+1} = (u_i - u_{i-j+2}) \frac{u_0 - u_j}{u_1 - u_{j-1}} + u_{i+1}, & \text{if } u_1 \neq u_{j-1}, \\
u_{i+1} = u_{i-j+3}, & \text{if } u_1 = u_{j-1}.
\end{cases}
\]
Lemma 3.3. (c.f. [5, Theorem 6.3]) Let $\Gamma$ be a non-bipartite geometric distance-regular graph with diameter $D \geq 4$ and $\gamma_{j-1} = 1$. Then the following hold:

i) If $j = 3$, then $c_3 \leq (a_1^2 + a_1 + 1)(a_1 + c_2 + 1)$ and equality holds if and only if $S_3^C = 0$.

ii) If $j = 4$, then $c_4 \geq (a_1^2 + 2a_1 + 2)(c_3 - (a_1 + 1)^2)$ and equality holds if and only if $S_4^C = 0$.

Proof. Let $(u_i)_{i=0}^{D}$ be the standard sequence associated with the smallest eigenvalue $\theta_{\min} = -\frac{k}{a_1 + 1}$. Since $\gamma_{j-1} = 1$, Lemma 2.4 implies that $u_i = (-\frac{1}{a_1 + 1})^i$ ($0 \leq i \leq j$).

By Equation (7) (8) and (9), when $j = 3$, we see that

$$S_3^C = \frac{4(a_1 + 2)^2 a_1 c_3}{(a_1 + 1)^6}((a_1^2 + a_1 + 1)(a_1 + c_2 + 1) - c_3). \quad (16)$$

Then i) follows.

Similarly, when $j = 4$, we have

$$S_4^C = \frac{4(a_1 + 2)^2 a_1^2 c_4}{(a_1 + 1)^8}(c_4 - (a_1^2 + 2a_1 + 2)(c_3 - (a_1 + 1)^2)). \quad (17)$$

And ii) follows. 

Lemma 3.4. Let $\Gamma$ be a non-bipartite geometric distance-regular graph with diameter $D \geq 4$. If there exists some $j$ satisfying $3 \leq j \leq D - 1$, such that $\gamma_{j-1} = 1$ and $S_j^C = 0$, then $\Gamma$ is a regular near $2D$-gon.

Proof. Let $(u_i)_{i=0}^{D}$ be the standard sequence associated with the smallest eigenvalue $\theta_{\min} = -\frac{k}{a_1 + 1}$. Since $\gamma_{j-1} = 1$, by Lemma 2.4 we see $u_i = (-\frac{1}{a_1 + 1})^i$ ($0 \leq i \leq j$). Remark 3.2 implies that $u_i = (-\frac{1}{a_1 + 1})^i$ ($0 \leq i \leq D$). Then we see $\gamma_i = 1$ ($0 \leq i \leq D - 1$) by Lemma 2.4. It follows from Lemma 2.7 that $\Gamma$ is a regular near $2D$-gon. 

Now we give a proof of Theorem 1.1.

Proof of Theorem 1.1. i) Since $a_i = c_i(a_1 + 1)$ $(i \leq 2)$, we see that $\gamma_2 = 1$ by Lemma 2.7. As $c_3 = (a_1^2 + a_1 + 1)(a_1 + c_2 + 1)$, we have $S_3^C = 0$ by Lemma 3.3 and $\Gamma$ is a regular near $2D$-gon by Lemma 3.4. Since $c_2 \neq 1$, we see that $\Gamma$ is a Hamming graph or a dual polar graph by Theorem 2.6. Then we check the intersection arrays (see [2, Section 9.2 and 9.4]) and the result follows.

ii) The result follows in the same way. 

Corollary 3.5. Let $\Gamma$ be a non-bipartite geometric distance-regular graph with diameter $D \geq 4$ and $c_2 \neq 1$. Then the following hold:

i) If $\gamma_2 = 1$ and $c_2 \geq (a_1 + 1)^2 + 1$, then $\Gamma$ is the dual polar graph $^2A_{2D-1}(\sqrt{q})$. 


ii) If \( \gamma_3 = 1 \), \( c_2 \geq a_1 + 2 \) and \( c_4 \leq (a_1 + 2)(a_1^2 + 2a_1 + 2) \), then \( \Gamma \) is the dual polar graph \( B_D(q) \) or \( C_D(q) \).

**Proof.** i) \( \gamma_2 = 1 \) implies that \( \Gamma \) is 2-bounded by Lemma 2.8. Then \( c_2^2 - c_2 - 1 \leq c_3 \leq (a_1^2 + a_1 + 1)(a_1 + c_2 + 1) \), where the inequalities are followed from \( \text{[10, Proposition 19]} \) and Lemma 3.3 respectively. We see \( c_2 \geq (a_1 + 1)^2 + 1 \) implies that \( c_3 = (a_1^2 + a_1 + 1)(a_1 + c_2 + 1) \) and the result follows from Theorem 1.1.

ii) \( \gamma_3 = 1 \) implies that \( \Gamma \) is 3-bounded by Lemma 2.8. Then \( c_3 \geq c_2^2 - c_2 + 1 \) by \( \text{[10, Proposition 19]} \). Together with Lemma 3.3, we see \( c_4 \geq (a_1^2 + 2a_2 + 2)(c_2^2 - c_2 + 1 - (a_1 + 1)^2) \). And \( c_2 \geq a_1 + 2 \) and \( c_4 \leq (a_1 + 2)(a_1^2 + 2a_1 + 2) \) implies that \( c_4 = (a_1^2 + 2a_1 + 2)(c_3 - (a_1 + 1)^2) \) and \( c_2 = a_1 + 2 \). The result follows from Theorem 1.1.

4 Proof of Theorem 1.2

**Proof of Theorem 1.2** As \( a_1 = 1 \) and the smallest eigenvalue \( \theta_{\text{min}} = -\frac{k}{2} \), we see that each triangle is a Delsarte clique, and \( \Gamma \) is geometric.

If the diameter \( D \leq 4 \), the result follows from \( \text{[13, Theorem 1.2]} \). So we may assume \( D \geq 5 \).

Note that \( 1 \leq \gamma_i \leq a_1 + 1 = 2 \) and \( \gamma_i \) is increasing \((0 \leq i \leq D - 1)\) by Lemma 2.5. If \( \gamma_{D-1} = 1 \), then \( \Gamma \) is a regular near \( 2D \)-gon by Lemma 2.7. Then by \( \text{[15, Theorem 9.11]} \), we see that \( \Gamma \) is a strongly closed subgraph with diameter 4. Then \( \Delta_4 \) is a regular near octagon with the same intersection number \( c_i \) and \( a_i \) \((i \leq 4)\) as \( \Gamma \) (especially \( c_3 = 3 \)). By \( \text{[15, Theorem 9.11]} \), we see that \( \Gamma \) is \( B_4(2) \). Then by Theorem 1.1, we see that \( \Gamma \) is the dual polar graph \( B_D(q) \). So we may assume \( j \leq 4 \). Let \( (u_i)_{i=0}^D \) be the standard sequence associated with \( \theta_{\text{min}} \). Then by Lemma 2.4, we see that

\[
\begin{align*}
\begin{cases}
u_i = (-1)^i 2^{-i}, & i \leq j, \\u_i = (-1)^{2j-i} 2^{-i}, & i \geq j.
\end{cases}
\end{align*}
\]

Since \( |u_i| \leq 1 \), we see that \( D \leq 2j \). Since \( j \leq 4 \), we see \( D \leq 8 \).

By Lemma 2.1, we see that the multiplicity \( m \) of \( \theta_{\text{min}} \) satisfies

\[
m = \frac{\sum_{i=0}^{D} k_i}{\sum_{i=0}^{D} k_i u_i^2} \leq \max_{0 \leq i \leq D} \frac{1}{u_i^2} = 4^j.
\]

By \( \text{[2, Theorem 4.4.4]} \), we have \( k \leq 2m \leq 2^{2j+1} \).

Let \( \Delta \) be a strongly closed subgraph of \( \Gamma \) with diameter \( j-1 \). Note that \( \Delta \) is determined by any pair of vertices in \( V(\Delta) \) at distance \( j-1 \). Choose two vertices \( x, y \in V(\Delta) \) with
\(d(x, y) = j - 2\). Denote \(S = \Gamma(y) \cap \Gamma_{j-1}(x)\) with \(|S| = b_{j-2}\) in the graph \(\Gamma\), and \(T = S \cap V(\Delta)\) with \(|T| = b'_{j-2}\), where \(b'\) denotes the corresponding intersection numbers of \(\Delta\). Then for any \(z \in S \cap V(\Delta)\), the strongly closed subgraph with diameter \(j - 1\) containing \(x, z\) is also \(\Delta\). It follows that \(S\) is partitioned by its intersection with strongly closed subgraphs containing \(x\) with diameter \(j - 1\), and

\[
b'_{j-2} | b_{j-2} = k - 2c_{j-2}.
\]

Now we consider \(5 \leq D \leq 8\). As \(\frac{D}{2} \leq j \leq 4\), the only possible \((j, D)\) are the following

\[L = \{(3, 5), (4, 5), (3, 6), (4, 6), (4, 7), (4, 8)\}.
\]

By Lemma 2.7, we see that

\[
\begin{cases}
  a_i = c_i, & i \leq j - 1, \\
  a_i = \frac{k}{2}, & i = j, \\
  a_i = b_i, & i \geq j + 1.
\end{cases}
\]

If \(j = D/2\), by Equation (18), we see that \(u_D = 1\) and \(\Gamma\) is antipodal by [2, Proposition 4.4.7]. Then \(b_i = c_{D-i}\), for \(i \neq D/2\) by [2, Proposition 4.2.2]. When \((j, D)\) is \((3, 6)\) or \((4, 8)\), by Equation (21), the intersection number of \(\Gamma\) is determined by \((c_3, c_4)\), respectively. If \(j \neq D/2\), then the intersection number of \(\Gamma\) is determined by \((c_3, \ldots, c_{D-1})\).

If \(j = 4\), as \(\Gamma\) is 3-bounded, \(\Gamma\) contains a strongly closed subgraph \(\Delta_3\) with diameter 3, which is a regular near hexagon. Then by [14, Theorem 1.2], the graph \(\Delta_3\) is \(B_D(2)\) or the Witt graph associated to \(M_{24}\), with \(c_3 = 7\) or 15, respectively.

Now we give a list of necessary conditions:

i) \(k \leq 2^{j+1}\).

ii) If \((j, D) = (3, 6)\), then \(4 \mid k - 2\) (Equation (20)) and the array is determined by \(c_3\).

iii) If \((j, D) = (4, 8)\), then \(c_3 = 7\) and \(8 \mid k - 6\), or \(c_3 = 15\) and \(24 \mid k - 6\). The array is determined by \((c_3, c_4)\).

iv) If \((j, D) = (3, 5)\), then \(4 \mid k - 2\) and the array is determined by \((c_3, c_4)\).

v) If \((j, D) = (4, 5), (4, 6)\) or \((4, 7)\), then \(c_3 = 7\) and \(8 \mid k - 6\), or \(c_3 = 15\) and \(24 \mid k - 6\). The array is determined by \((c_3, \ldots, c_{D-1})\).

For all possible pairs \((j, D) \in L\), no array \((k, b_1, \ldots, b_D; c_1, c_2, c_3, \ldots, c_D)\) satisfies the above necessary conditions and Lemma 2.2. And the result follows.

**Remark 4.1.**

i) The dual polar graphs \(B_D(2)\) and \(C_D(2)\) are isomorphic.

ii) See [12, Theorem 1.1] for a result similar to Corollary 3.4 i). And for the case \(c_2 = 5\) in Theorem 1.2 see [12, Theorem 1.2].

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