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G\text{-IN Variant Quasimorphisms and SYMPLECTIC GEOMETRY OF SURFACES}}

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Abstract. Let \(\hat{G}\) be a group and \(G\) its normal subgroup. In this paper, we study \(\hat{G}\)-invariant quasimorphisms on \(G\) which appear in symplectic geometry and 2-dimensional topology. As its application, we prove the non-existence of a section of the flux homomorphism on closed surfaces with higher genus. We also prove that Py’s Calabi quasimorphism and Entov–Polterovich’s partial Calabi quasimorphism cannot be extended to the group of symplectomorphism as partial quasimorphisms.

1. Introduction

In this paper, we introduce and study the notions of \(\hat{G}\)-invariant quasimorphism and \((\hat{G},G)\)-commutator length. Many examples in this paper come from the symplectic geometry. See Section 5 for notions in the symplectic geometry.

1.1. \(\hat{G}\)-invariant quasimorphism. A real-valued function \(\phi\) on a group \(G\) is a quasimorphism if there exists a constant \(C\) such that

\[ |\phi(gh) - \phi(g) - \phi(h)| \leq C \]

for all \(g, h \in G\). Such the smallest \(C\) is called the defect of \(\phi\) and denoted by \(D(\phi)\). A quasimorphism \(\phi\) on \(G\) is homogeneous if \(\phi(g^n) = n\phi(g)\) for all \(g \in G\) and \(n \in \mathbb{Z}\).

The main object we consider in this paper is \(\hat{G}\)-invariant quasimorphism.

Definition 1.1. For a group \(\hat{G}\) and its normal subgroup \(G\), we say that a quasimorphism \(\phi: G \to \mathbb{R}\) on \(G\) is \(\hat{G}\)-invariant if \(\phi(\hat{g}gg^{-1}) = \phi(g)\) for all \(\hat{g} \in \hat{G}\) and \(g \in G\).

Quasimorphisms appear in various situations as in dynamical systems as the rotation number, in symplectic topology as spectral invariants, in geometric group theory as a characterization of non-positively curved groups, in the theory of bounded cohomology and so on. \(\hat{G}\)-invariant quasimorphisms on \(G\) also appear in several contexts. For example,

- Let \((M,\omega)\) be a symplectic manifold. \(\hat{G}\) is the identity component of the group \(\text{Symp}_0(M,\omega)\) of symplectomorphisms and \(G\) is the group \(\text{Ham}(M,\omega)\) of Hamiltonian diffeomorphisms [EP03, GG, Py, Bra, BKS, FOOO, et al.].

- \(\hat{G}\) is the mapping class group \(\mathcal{M}(\Sigma)\) of a compact oriented surface \(\Sigma\) with non-empty boundary and \(G\) is the Torelli group \(\mathcal{I}(\Sigma)\) of \(\Sigma\) or the Johnson kernel \(\mathcal{K}(\Sigma)\) of \(\Sigma\) [CHH].
1.2. Bavard-type duality theorem. For a group $G$, $\text{cl}_G$ denotes the commutator length on $[G,G]$ and the stable commutator length $\text{scl}_G$ is defined by $\text{scl}_G(x) = \lim_{n \to \infty} \text{cl}_G(x^n)/n$ for $x \in [G,G]$. The following Bavard duality theorem, which relates quasimorphisms and stable commutator length (scl), is one of the most fundamental results in the theory of quasimorphism.

Theorem 1.2 ($\text{[Bav]}$). Let $G$ be a group. For any $x \in [G,G]$,

$$\text{scl}_G(x) = \sup_{\phi} \frac{1}{2} \frac{\phi(x)}{D(\phi)},$$

where the supremum is taken over all homogeneous quasimorphisms on $G$.

We will show a Bavard-type duality for $\hat{G}$-invariant quasimorphisms and a variant of commutator length. We define an element of the form $[\hat{g}, g]$, where $\hat{g} \in \hat{G}$ and $g \in G$, as a $(\hat{G}, G)$-commutator. We define the $(\hat{G}, G)$-commutator subgroup $[\hat{G}, G]$ and the $(\hat{G}, G)$-commutator length $\text{cl}_{\hat{G}, G}$ in the same way as the ordinary ones (see Section 2.1).

Theorem 1.3. Assume that $G = [\hat{G}, G]$. For any $x \in [\hat{G}, G]$,

$$\text{scl}_{\hat{G}, G}(x) = \sup_{\phi} \frac{1}{2} \frac{\phi(x)}{D(\phi)},$$

where the supremum is taken over all $\hat{G}$-invariant homogeneous quasimorphisms on $G = [\hat{G}, G]$.

Since $G$ is a normal subgroup of $\hat{G}$, we have $[G,G] < [\hat{G}, G] < G$. Thus, we note that $G = [\hat{G}, G]$ if $G$ is perfect i.e. $G = [G,G]$.

1.3. Comparison with the ordinary commutator length. We say that two functions $\nu$ and $\mu$ are equivalent if there are positive constants $C_1$ and $C_2$ such that $C_1 \mu \leq \nu \leq C_2 \mu$. In $\text{[CZ]}$, Calegari and Zhuang gave a concept of $W$-length generalizes the commutator length. They proved that the stabilization of some $W$-lengths are equivalent to the stable commutator length $\text{[CZ, Corollary 3.25]}$. In this paper, we consider a similar problem for our situation. Namely, we compare our norm $\text{cl}_{\hat{G}, G}$ with the norms $\text{cl}_{\hat{G}}$ or $\text{cl}_G$.

We can prove that the stabilizations of $\text{cl}_{\hat{G}, G}$ and $\text{cl}_{\hat{G}}$ are equivalent in the following situation.

Proposition 1.4. Let $G$ be a normal subgroup of a group $\hat{G}$. Assume that $G = [\hat{G}, G]$. If there exists a section homomorphism of the quotient map $q: \hat{G} \to \hat{G}/G$ i.e. there is a group homomorphism $s: \hat{G}/G \to \hat{G}$ such that $q \circ s = \text{id}$, then

$$\text{scl}_G(x) \leq \text{scl}_{\hat{G}, G}(x) \leq 2 \text{scl}_{\hat{G}}(x)$$

for any $x \in [\hat{G}, G]$.

Because we use Theorem 1.3 to prove Proposition 1.4, the authors do not know whether $\text{cl}_{\hat{G}, G}$ and $\text{cl}_{\hat{G}}$ (not stabilized) are equivalent or not.

Example 1.5. Let $\hat{G}$ be the braid group $B_n$ of $n$ strands and $G$ its commutator subgroup $[B_n, B_n]$. For any integer $n > 4$, $G$ is a perfect group $\text{[GL]}$, especially $G = [\hat{G}, G]$. It is known that $\hat{G}/G \cong \mathbb{Z}$ and the abelianization map $\hat{G} \to \hat{G}/G$ is given by the index sum homomorphism $\hat{G} \to \mathbb{Z}$ defined by $\sigma_i \mapsto 1$ for $i = 1, 2, \ldots, n-1$, where
σ_i is the i-th Artin generator. Since there is a section homomorphism s: Z → G, the pair (G, G) satisfies the assumptions of Proposition 1.4 if n > 4.

**Example 1.6.** Let (M, ω) be an exact symplectic manifold. Let G be the group Ham(M, ω) of Hamiltonian diffeomorphisms and G the commutator subgroup of Ham(M, ω). Let Cal: Ham(M, ω) → R denote the Calabi homomorphism.

It is known that G/G ≃ R and the abelianization map G → G/G is given by the Calabi homomorphism. We can take a time-independent Hamiltonian function H: M → R such that Cal(H) = 1 (for instance, consider a function supported on a Darboux ball). Then, the map s: R → Ham(M, ω) defined by s(t) = ϕ_H is a section homomorphism of Cal. Since it is known that G is a perfect group ([Ban78]), the pair (G, G) satisfies the assumptions of Proposition 1.4.

**Example 1.7.** Let T^2 be a 2-dimensional torus and ω a symplectic form on T^2. Let G be the identity component Symp_0(T^2, ω) of the group of symplectomorphisms of (T^2, ω) and G the group Ham(T^2, ω) of Hamiltonian diffeomorphisms of (T^2, ω).

Since there exists a section homomorphism of the (descended) flux homomorphism Flux_ω: Symp_0(T^2, ω) → H^1(T^2; R)/H^1(T^2; Z), Ker(Flux_ω) = G and G is known to be perfect [Ban78], G and G satisfy the assumption of Proposition 1.4. Thus scl_{G, G} and scl_{G} are equivalent.

However, in the following example, scl_{G, G}(x) and scl_{G} are not equivalent.

**Theorem 1.8.** Let Σ be a closed orientable surface whose genus is larger than one and ω a symplectic form on Σ. Set G = Symp_0(Σ, ω) and G = Ham(Σ, ω). Then, there exists f ∈ G such that scl_{G, G}(f) > 0 and scl_{G}(f) = 0.

By Proposition 1.4, Theorem 1.8 gives a negative answer for the following version (Nielsen) realization problem by symplectomorphisms.

**Corollary 1.9.** Let Σ be a closed orientable surface whose genus is larger than one and ω a symplectic form on Σ. Then, there is no section homomorphism of the flux homomorphism Flux_ω: Symp_0(Σ, ω) → H^1(Σ; R).

For various versions of (Nielsen) realization problems by diffeomorphisms, [MT] is a good survey.

Corollary 1.9 is slightly surprising because the following proposition is essentially proved by Fathi.

**Proposition 1.10 ([F]).** Let M be an n-dimensional closed manifold and Ω a volume form on M. Suppose that n ≥ 3 and there is a basis of H_1(M; R) which is represented by embedded curves having tubular neighborhoods. Then, there is a section homomorphism of the flux homomorphism Flux_Ω: Diff_0(M, Ω) → H^{n-1}(M; R).

Note that for a closed orientable surface Σ whose genus is larger than 1 and a symplectic form ω, Diff_0(Σ, ω) = Symp_0(Σ, ω) = Ham(Σ, ω). Also note that the symplectic flux homomorphism corresponds to the volume flux homomorphism when the dimension of the manifold is two. Thus, Corollary 1.9 shows that Proposition 1.10 does not hold when n = 2.

We have the following geometric interpretation of Corollary 1.9. For a vector field X on a manifold, let L_X and i_X denote the Lie derivative and the interior product with respect to X, respectively.
Corollary 1.11. Let $\Sigma$ be a closed orientable surface whose genus is larger than one and $\omega$ a symplectic form on $\Sigma$. There are no smooth vector fields $X_1, \ldots, X_{2g}$ on $\Sigma$ satisfying the following conditions.

(1) $L_{X_i} \omega = 0$,
(2) $\{[X_1 \omega], \ldots, [X_{2g} \omega]\}$ is a basis of $H^1(\Sigma; \mathbb{R})$,
(3) $[X_i, X_j] = 0$ for any $i, j$.

We also provide examples of $G$, $\hat{G}$ and $\alpha \in [\hat{G}, G]$ such that $\text{scl}_{\hat{G}, G}(\alpha) = 0$ and $\text{scl}_G(\alpha) > 0$ (see Proposition 4.1).

1.4. Extension problem of (partial) quasimorphisms. It is a quite natural problem whether a homogeneous quasimorphism $\phi$ on $G$ can be extended as a homogeneous quasimorphism on $\hat{G}$. It is known that every homogeneous quasimorphism on $\hat{G}$ is $\hat{G}$-invariant ([Cale]). Thus, we see that $\hat{G}$-invariance is necessary to extend $\phi: G \to \mathbb{R}$ to a homogeneous quasimorphism on $\hat{G}$. Shtern and the first author also studied a similar topic [Sh, Ka18].

First, we provide a sufficient condition of quasimorphisms to extend. It also follows from the result of Shtern [Sh, Theorem 3]. However, we provide an estimate of defect in order to prove Proposition 1.12.

Proposition 1.12. Let $G$ be a normal subgroup of a group $\hat{G}$. If there exists a section homomorphism $s: \hat{G}/G \to \hat{G}$ of the quotient homomorphism $\hat{G} \to \hat{G}/G$, then for any homogeneous $\hat{G}$-invariant quasimorphism $\phi$ on $G$, there exists a homogeneous quasimorphism $\hat{\phi}$ on $\hat{G}$ such that $\hat{\phi}|_G = \phi$ and $D(\hat{\phi}) \leq 2D(\phi)$.

Shtern [Sh, Example 1] provided an example of $\hat{G}$-invariant homomorphism on $G$ which cannot be extended to $\hat{G}$ as a quasimorphism when $\hat{G}$ is the Heisenberg group and $G$ is the commutator subgroup of $\hat{G}$. In this paper, we provide examples of $\hat{G}$-invariant “partial quasimorphisms” on $G$ which cannot be extended to $\hat{G}$ as a “partial quasimorphism” when $\hat{G}$ is the identity component of the group of sympletomorphism of surfaces and $G$ is the commutator subgroup of $\hat{G}$.

To explain our obstructive result, we prepare some notions on “partial quasimorphisms”. Burago, Ivanov and Polterovich defined the notion of conjugation-invariant norm.

Definition 1.13 ([BIP]). Let $G$ be a group. A function $\nu: G \to \mathbb{R}$ is called a conjugation-invariant norm on $G$ if $\nu$ satisfies the following axioms:

(1) $\nu(1) = 0$;
(2) $\nu(f) = \nu(f^{-1})$ for every $f \in G$;
(3) $\nu(fg) \leq \nu(f) + \nu(g)$ for every $f, g \in G$;
(4) $\nu(f) = \nu(gfg^{-1})$ for every $f, g \in G$;
(5) $\nu(f) > 0$ for every $f \neq 1 \in G$.

Example 1.14. We define a function $\nu_0: G \to \mathbb{R}$ by

$$
\nu_0(g) = \begin{cases} 
0 & (g = 1), \\
1 & \text{(otherwise)}.
\end{cases}
$$

Then, $\nu_0$ is a conjugation-invariant norm.
Example 1.15. Let $G$ be a group and $H$ a subgroup of $G$. We define the fragmentation norm $\nu_H$ with respect to $H$ by for an element $f$ of $G$,
$$\nu_H(f) = \min\{k; \exists g_1, \ldots, g_k \in G, \exists h_1, \ldots, h_k \in H \text{ such that } f = g_1 h_1 g_1^{-1} \cdots g_k h_k g_k^{-1}\}.$$ 
If there is no such decomposition of $f$, we set $\nu_H(f) = +\infty$. If $\nu(f) < +\infty$ for any $f \in G$, $\nu$ is a conjugation-invariant norm.

In [EP06], Entov and Polterovich essentially considered a concept of “partial quasimorphism” (relative quasimorphism, norm-controlled quasimorphism).

Definition 1.16. Let $G$ be a group and $\nu$ a conjugation-invariant norm on $G$. A function $\phi: G \to \mathbb{R}$ is called a $\nu$-quasimorphism (quasimorphism relative to $\nu$ or quasimorphism controlled by $\nu$) if there exists a positive number $C$ such that for any elements $f, g \in G$,
$$|\phi(fg) - \phi(f) - \phi(g)| < C \min\{\nu(f), \nu(g)\}.$$ 
The infimum of such $C$ is called the defect of $\phi$ and let $D(\phi)$ denote the defect of $\phi$. $\phi$ is called semi-homogeneous if $\phi(f^n) = n\phi(f)$ for any element $f$ of $G$ and any non-negative integer $n$.

Definition 1.17. Let $G$ be a normal subgroup of a group $\hat{G}$ and $\nu: G \to \mathbb{R}$ a conjugation-invariant norm on $G$. A semi-homogeneous $\nu$-quasimorphism $\mu$ on $G$ is called extendable to $\hat{G}$ if there are a conjugation-invariant norm $\hat{\nu}$ on $\hat{G}$ and a semi-homogeneous $\hat{\nu}$-quasimorphism $\hat{\mu}$ on $\hat{G}$ such that $\hat{\mu}(g) = \mu(g)$ for any $g \in G$. A homogeneous quasimorphism $\mu$ on $G$ is called non-extendable to $\hat{G}$ otherwise.

We give a convenient lemma for proving non-extendability.

Lemma 1.18. Let $\mu$ be a semi-homogeneous $\hat{G}$-invariant $\nu$-quasimorphism on $G$. Let $f, g$ be elements of $\hat{G}$ satisfying
- $(fgf^{-1})g^{-1} = g^{-1}(fgf^{-1})$,
- $[f, g] \in G$,
- $\mu([f, g]) \neq 0$.

Then, $\mu$ is non-extendable to $\hat{G}$.

Here, we provide some applications of Lemma 1.18.

For a closed orientable surface $\Sigma$ whose genus is larger than one and a symplectic form $\omega$ on $\Sigma$, Py constructed a Calabi quasimorphism $\mu_P: \text{Ham}(\Sigma, \omega) \to \mathbb{R}$ called Py’s Calabi quasimorphism [Py]. Py’s Calabi quasimorphism $\mu_P$ is known to be a $\text{Symp}_0(\Sigma, \omega)$-invariant quasimorphism.

Theorem 1.19. Let $\Sigma$ be a closed orientable surface whose genus is larger than one and $\omega$ a symplectic form on $\Sigma$. Then, Py’s Calabi quasimorphism $\mu_P: \text{Ham}(\Sigma, \omega) \to \mathbb{R}$ is non-extendable to $\text{Symp}_0(\Sigma, \omega)$.

Since any quasimorphism is a $\nu_0$-quasimorphism, Proposition 1.12 and Theorem 1.19 give another proof of Corollary 1.17.

Theorem 1.19 has the following corollary. To explain it, we introduce some notions. For a group $G$, let $Q(G)$ denote the real linear space of homogeneous quasimorphism on $G$. For a closed orientable surface $\Sigma$ whose genus is larger than one, let $B_n(\Sigma)$ denote the full braid group on $n$ strings on $\Sigma$. For a symplectic form $\omega$ on $\Sigma$, Brandenbursky [Bra] constructed a liner map $\Gamma_n: Q(B_n(\Sigma)) \to Q(\text{Ham}(\Sigma, \omega))$ by generalizing Gambaudo-Ghys’ idea [GG].
Generalizing and sophisticating Ishida’s idea [1], Brandenbursky proved that the image \( \text{Im}(\Gamma_2) \) of \( \Gamma_2 \) contains infinitely many \( \text{Symp}_0(\Sigma, \omega) \)-invariant Calabi quasimorphisms. Thus, it is a natural problem whether Py’s Calabi quasimorphism \( \mu_P \) can be constructed by Brandenbursky’s method or not. However, all elements of \( \text{Im}(\Gamma_n) \) are known to be extendable to \( \text{Symp}_0(\Sigma, \omega) \). Hence, we obtain the following corollary of Theorem 1.19.

**Corollary 1.20.** Let \( \Sigma \) be a closed orientable surface whose genus is larger than one and \( \omega \) a symplectic form on \( \Sigma \). Then, \( \mu_P \notin \text{Im}(\Gamma_n) \) for any \( n \geq 2 \).

We provide another example of non-extendable partial quasimorphism. For a closed orientable surface \( \Sigma \) and a symplectic form \( \omega \) on \( \Sigma \), Entov and Polterovich constructed a partial quasimorphism \( \mu_{\text{EP}} : \text{Ham}(\Sigma, \omega) \rightarrow \mathbb{R} \) as the asymptotization of the Oh-Schwarz spectral invariant (EP06). \( \mu_{\text{EP}} \) is a semi-homogeneous \( \nu_{\text{Ham}(U)} \)-quasimorphism for any displaceable open subset \( U \) of \( M \). (Note that we regard \( \text{Ham}(U, \omega) \) as a subgroup of \( \text{Ham}(M, \omega) \))

**Theorem 1.21.** Let \( \Sigma \) be a closed orientable surface with positive genus and \( \omega \) a symplectic form on \( \Sigma \). Then, Entov–Polterovich’s partial Calabi quasimorphism \( \mu_{\text{EP}} : \text{Ham}(\Sigma, \omega) \rightarrow \mathbb{R} \) is non-extendable to \( \text{Symp}_0(\Sigma, \omega) \).

Theorem 1.21 is interesting because of the following reason. As we noted in Example 1.17, Flux:\( \text{Symp}_0(T^2, \omega) \rightarrow H^1(T^2; \mathbb{R})/H^1(T^2; \mathbb{Z}) \) has a section homomorphism. Thus, Theorem 1.21 shows that the same statement as Proposition 1.12 does not hold for partial quasimorphisms.

## 2. \( \hat{G} \)-invariant Bavard duality

### 2.1. \((\hat{G}, G)\)-commutator length

We recall that a \((\hat{G}, G)\)-commutator is an element \([\hat{g}, g] \) with \( \hat{g} \in \hat{G} \) and \( g \in G \). Let \([\hat{G}, G] \) denote the subgroup of \( G \) generated by \((\hat{G}, G)\)-commutators. For \( x \in [\hat{G}, G] \) we define the \((\hat{G}, G)\)-commutator length \( c_{\hat{G}, G}(x) \) of \( x \) by the smallest number of \((\hat{G}, G)\)-commutators whose product is equal to \( x \). Since \( c_{\hat{G}, G} \) is subadditive, the limit \( \lim_{n \to \infty} c_{\hat{G}, G}(x^n)/n \) exists.

**Lemma 2.1.** Let \( \phi \) be a \( \hat{G} \)-invariant homogeneous quasimorphism on \( G \). For any \( x \in [\hat{G}, G] \),

\[
\text{scl}_{\hat{G}, G}(x) \geq \frac{1}{2} \frac{|\phi(x)|}{D(\phi)}.
\]

**Proof.** Note that \( |\phi([\hat{g}, g])| = |\phi(\hat{g}g\hat{g}^{-1}) - \phi(g^{-1})| \leq D(\phi) \) for any \((\hat{G}, G)\)-commutator \([\hat{g}, g] \in [\hat{G}, G] \). If \( x^n \) is a product of \((\hat{G}, G)\)-commutators \( c_1, \ldots, c_m \), then we obtain an inequality

\[
n|\phi(x)| = |\phi(x^n)| \leq (m - 1)D(\phi) + \sum_{k=1}^{k=m} |\phi(c_k)| < 2mD(\phi).
\]

and the lemma follows from it. \( \square \)

### 2.2. Proof of the duality theorem

Now we give a proof of Theorem 1.3. For proving the equality, it is sufficient to prove the inequalities in both directions. One side follows from Lemma 2.1; thus we prove the other side. For this purpose, we use the strategy of Calegari–Zhuang CZ (see also Ka17). Some parts of the proof go through in the same way as the arguments in Ka17. Moreover, some parts are
much easier than the ones in [Ka17] because a technical lemma corresponding to [Ka17] Lemma 2.6] follows immediately in our situation. Thus, we often omit such parts of the proof.

Set $\Gamma = \hat{G}, G$ and define a set

$$A_\Gamma = \bigcup_{k=0}^\infty (\Gamma \times \mathbb{R})^k.$$ 

Let $x_{s_1} \cdots x_{s_k}$ denote elements of $A_\Gamma$, where $x_1, \ldots, x_k \in \Gamma$ and $s_1, \ldots, s_k \in \mathbb{R}$. We define a function $\| \cdot \|_\Gamma: A_\Gamma \to \mathbb{R} \geq 0$ by

$$\|x_{s_1} \cdots x_{s_k}\|_\Gamma = \lim_{n \to \infty} \frac{1}{n} \text{cl}_{\hat{G}, G}(x_{\lfloor s_1rn \rfloor} \cdots x_{\lfloor s_krn \rfloor}),$$

where $\lfloor t \rfloor$ is the integer part of $t \in \mathbb{R}$. The function $\| \cdot \|_\Gamma: A_\Gamma \to \mathbb{R} \geq 0$ is well-defined [Ka17, Proposition 2.1].

We define some operation on $A_\Gamma$. For elements $x = x_{s_1} \cdots x_{s_k}, y = y_{t_1} \cdots y_{t_l}$ of $A_\Gamma$ and a real number $\lambda$, we define $x \star y$, $\bar{x}$, and $x^{(\lambda)}$ by

$$x \star y = x_{s_1} \cdots x_{s_k} y_{t_1} \cdots y_{t_l}, \quad \bar{x} = x_{-s_1} \cdots x_{-s_k}, \quad \text{and} \quad x^{(\lambda)} = x_{\lambda s_1} \cdots x_{\lambda s_k}.$$

We define the equivalence relation $\sim$ on $A_\Gamma$ by $x \sim y$ if and only if $\|xy\|_\Gamma = 0$ for $x, y \in A_\Gamma$. Let $A$ denote the quotient set $A_\Gamma/\sim$. The function $\| \cdot \|_\Gamma: A_\Gamma \to \mathbb{R} \geq 0$ on $A_\Gamma$ induces the function $\| \cdot \|: A \to \mathbb{R} \geq 0$ on $A$. Let $[x] \in A$ denote the equivalence class of $x \in A_\Gamma$. For $x = [x], y = [y]$ in $A$ and a real number $\lambda$, we define $x + y$ and $\lambda x$ by

$$x + y = [x \star y] \quad \text{and} \quad \lambda x = [x^{(\lambda)}].$$

These operators are well-defined [Ka17, Proposition 2.2] and $(A, \| \cdot \|)$ is a normed vector space [Ka17, Proposition 2.3]. By the Hahn-Banach theorem, we obtain the following proposition.

**Proposition 2.2.** For any $x \in A$,

$$\|x\| = \sup_{\tilde{\phi} \in A^*} \frac{\tilde{\phi}(x)}{\|\tilde{\phi}\|_*},$$

where $A^*$ is the dual space of $A$ and $\| \cdot \|_*$ is the dual norm on $A^*$.

On the other hand, we can construct a $\hat{G}$-invariant quasimorphism in the following way.

**Proposition 2.3.** For $\tilde{\phi} \in A^*$, the function $\phi: \Gamma \to \mathbb{R}$ defined by $\phi(x) = \tilde{\phi}([x^1])$ is a $\hat{G}$-invariant homogeneous quasimorphism.

**Proof.** (\phi \text{ is a quasimorphism})

For any $x, y \in \Gamma$,
\begin{align*}
& |\phi(xy) - \phi(x) - \phi(y)| \\
& = |\hat{\phi}((xy)^1)) - \hat{\phi}([x^1]) - \hat{\phi}([y^1])| \\
& = |\hat{\phi}((xy)^1) + (-1)[x^1] + (-1)[y^1])| \\
& \leq \|\hat{\phi}\|^* \|(xy)^1 \ast x^{-1} \ast y^{-1}\|_G \\
& = \|\hat{\phi}\|^* \cdot \lim_{n \to \infty} \frac{1}{n} \mathrm{cl}_{\hat{G}, G}((xy)^n x^{-n} y^{-n}).
\end{align*}

Since \((xy)^2n x^{-2n} y^{-2n}\) is a product of \(n\) commutators (see [Cale] Lemma 2.24 for example),
\[
\lim_{n \to \infty} \frac{1}{n} \mathrm{cl}_{\hat{G}, G}((xy)^n x^{-n} y^{-n}) \leq \frac{1}{2}.
\]
Hence
\[
|\phi(xy) - \phi(x) - \phi(y)| \leq \frac{1}{2} \|\hat{\phi}\|^*.
\]

\begin{itemize}
  \item (\(\phi\) is homogeneous)
  Since \((x^n)^1 \sim x^n\) for any \(x \in \hat{G}, G\) and any integer \(n\),
  \[
  \phi(x^n) = \hat{\phi}((x^n)^1) = \hat{\phi}(x^n) = \hat{\phi}(n[x^1]).
  \]
  for any \(x \in \Gamma\) and any integer \(n\). Since \(\hat{\phi}: A \to \mathbb{R}\) is a linear map,
  \[
  \hat{\phi}(n[x^1]) = n\hat{\phi}([x^1]) = n\phi(x).
  \]
  for any \(x \in \Gamma\) and any integer \(n\). Hence \(\phi\) is homogeneous.
  \item (\(\phi\) is \(\hat{G}\)-invariant)
  For any \(\hat{g} \in \hat{G}\) and any \(x \in \Gamma \subset G\),
  \[
  |\phi(\hat{g} x \hat{g}^{-1}) - \phi(x)| \\
  = |\hat{\phi}((\hat{g} x \hat{g}^{-1})^1)) - \hat{\phi}([x^1])| \\
  = |\hat{\phi}((\hat{g} x \hat{g}^{-1})^1) + (-1)[x^1]| \\
  \leq \|\hat{\phi}\|^* \|(\hat{g} x \hat{g}^{-1})^1 \ast x^{-1}\|_G \\
  = \|\hat{\phi}\|^* \cdot \lim_{n \to \infty} \frac{1}{n} \mathrm{cl}_{\hat{G}, G}((\hat{g} x \hat{g}^{-1})^n x^{-n}) \\
  = \|\hat{\phi}\|^* \cdot \lim_{n \to \infty} \frac{1}{n} \mathrm{cl}_{\hat{G}, G}([\hat{g}, x^n]) \\
  = 0.\]
\end{itemize}

As a corollary of Propositions 2.2 and 2.3, we have the following proposition.

**Proposition 2.4.** For any \(x \in \hat{G}, G\),
\[
\mathrm{scl}_{\hat{G}, G}(x) \leq \sup_{\phi} \frac{1}{2} \frac{|\phi(x)|}{D(\phi)},
\]
where the supremum is taken over all \(\hat{G}\)-invariant homogeneous quasimorphisms on \([\hat{G}, G]\).
Proof. By Proposition 2.2 and 2.3 since \( D(\phi) \leq \frac{1}{2} \| \phi \| \),

\[
\text{scl}_{\hat{G},G}(x) = \| x^1 \| = \sup_{\hat{\phi} \in A^*} \frac{\hat{\phi}(x^1)}{\| \hat{\phi} \|} \leq \frac{1}{2} D(\phi). \quad \Box
\]

Theorem 1.3 follows from Lemma 2.1 and Proposition 2.4.

3. Extension of Quasimorphism

For any quasimorphism \( \phi \) on a group \( G \), we can obtain a homogeneous quasimorphism \( \tilde{\phi} \) by \( \tilde{\phi}(g) = \lim_{n \to \infty} \frac{\phi(g^n)}{n} \) for \( g \in G \). We refer to \( \tilde{\phi} \) as the homogenization of \( \phi \).

Proof of Proposition 1.12. Let \( \pi : \hat{G} \to \hat{G}/G \) be the natural projection. For \( \hat{g} \in \hat{G} \), we set \( q_{\hat{g}} = s(\pi(\hat{g})) \) and \( g_{\hat{g}} = q_{\hat{g}}^{-1} \hat{g} \in G \). We define the function \( \phi' : \hat{G} \to \mathbb{R} \) by \( \phi'(\hat{g}) = \phi(g_{\hat{g}}) \). Since \( s \circ \pi \) is a homomorphism, \( q_{\hat{g}_1 \hat{g}_2} = q_{\hat{g}_1} q_{\hat{g}_2} \) for \( \hat{g}_1, \hat{g}_2 \in \hat{G} \). Thus

\[
\begin{align*}
|\phi'(\hat{g}_1 \hat{g}_2) - \phi'(\hat{g}_1) - \phi'(\hat{g}_2)| &= |\phi(g_{\hat{g}_1 \hat{g}_2}) - \phi(g_{\hat{g}_1}) - \phi(g_{\hat{g}_2})| \\
&= |\phi(q_{\hat{g}_2}^{-1} q_{\hat{g}_1}^{-1} \hat{g}_1 \hat{g}_2) - \phi(q_{\hat{g}_2}^{-1} \hat{g}_2) - \phi(q_{\hat{g}_1}^{-1} \hat{g}_1) - \phi(q_{\hat{g}_2}^{-1} \hat{g}_2)| \\
&= |\phi(q_{\hat{g}_2}^{-1} \hat{g}_1 q_{\hat{g}_2}^{-1} \hat{g}_2) - \phi(q_{\hat{g}_1}^{-1} \hat{g}_1) - \phi(q_{\hat{g}_2}^{-1} \hat{g}_2) - \phi(q_{\hat{g}_2}^{-1} \hat{g}_2)| \\
&\leq D(\phi).
\end{align*}
\]

The homogenization \( \tilde{\phi} \) of \( \phi' \) is also an extension of \( \phi \) and \( D(\tilde{\phi}) \leq 2D(\phi') \) ([Cal], Corollary 2.59). \( \Box \)

Lemma 1.18 immediately follows from the following lemma.

Lemma 3.1. Let \( \nu \) be a conjugation-invariant norm on a group \( \hat{G} \), \( \mu \) a semi-homogeneous \( \nu \)-quasimorphism on a group \( \hat{G} \) and \( f, g \) elements of \( \hat{G} \) satisfying \((fg^{-1})g^{-1} = g^{-1}(fg^{-1})\). Then, \( \mu([f, g]) = 0 \).

To prove Lemma 3.1 we use the following lemma essentially proved in [MVZ] Theorem 1.3 and [KO] Lemma 3.17.

Lemma 3.2. Let \( \nu \) be a conjugation-invariant norm on a group \( \hat{G} \), \( \mu \) a semi-homogeneous \( \nu \)-quasimorphism on a group \( \hat{G} \). Then, \( \mu(gfg^{-1}) = \mu(f) \).

Proof. By the definitions of partial quasimorphism and conjugation-invariant norm, for any positive integer \( k \),

\[
\begin{align*}
\mu(f^k) &\leq \mu(g) + \mu(g^{-1}f^kg) + \mu(g^{-1}) + C \cdot \nu(g) + C \cdot \nu(g^{-1}), \\
\mu(g^{-1}f^kg) &\leq \mu(g^{-1}) + \mu(f^k) + \mu(g) + C \cdot \nu(g^{-1}) + C \cdot \nu(g).
\end{align*}
\]

Thus,

\[
\begin{align*}
\mu(f^k) - \mu(g) - \mu(g^{-1}) - C \cdot \nu(g) - C \cdot \nu(g^{-1}) \\
&\leq \mu(g^{-1}f^kg) - \mu(g) - \mu(g^{-1}) + C \cdot \nu(g) + C \cdot \nu(g^{-1})
\end{align*}
\]

Since \( \mu \) is semi-homogeneous, \( \mu(f^k) = k\mu(f) \) and \( \mu(g^{-1}f^kg) = \mu((g^{-1}fg)^k) = k\mu(g^{-1}fg) \) for any positive integer \( k \). Therefore, by dividing the above inequality by \( k \) and passing to the limit as \( k \to \infty \), we obtain \( \mu(gfg^{-1}) = \mu(f) \). \( \Box \)
Proof of Lemma 3.7. By \( \mu(gg^{-1}) = \mu(f) \), \([f, g]^n = [f, g^n] \) for any integer \( n \). Thus, since \( \hat{\mu} \) is semi-homogeneous, for any positive integer \( n \),

\[
n\hat{\mu}([f, g]) = \hat{\mu}([f, g]^n) = \hat{\mu}([f, g^n]) = \hat{\mu}(fg^nf^{-1}g^{-n}).
\]

Thus, by Lemma 3.2,

\[
-C \cdot \nu(f) = \hat{\mu}(f) - \hat{\mu}(f) - C \cdot \nu(f) = \hat{\mu}(f) - \hat{\mu}((g^nf^{-1}g^{-n})^{-1}) - C \cdot \nu(f) \\
\leq \hat{\mu}(fg^nfg^{-n}) \\
\leq \hat{\mu}(f) + \hat{\mu}(g^nfg^{-n}) + C \cdot \nu(f) \\
= \hat{\mu}(f + \hat{\mu}(f^{-1}) + C \cdot \nu(f).
\]

Set

\[
R = \max \{|\hat{\mu}(f) + \hat{\mu}(f^{-1}) + C \cdot \nu(f)|, |C \cdot \nu(f)|\}.
\]

Then, by \( n\hat{\mu}([f, g]) = \hat{\mu}(fg^nfg^{-n}) \), \(|\hat{\mu}([f, g])| < \frac{2}{n} \) for any positive integer \( n \). Hence, \( \hat{\mu}([f, g]) = 0 \). \( \square \)

4. Comparison of Commutator Lengths

We compare the \((\hat{G}, G)\)-commutator length \( cl_{\hat{G}, G} \) with the ordinary commutator lengths \( cl_G \) of \( G \) and \( cl_{\hat{G}} \) of \( \hat{G} \). By the definitions, \( cl_G \leq cl_{\hat{G}, G} \) on \([\hat{G}, G]\), and \( cl_{\hat{G}, G} \leq cl_G \) on \([G, G]\).

4.1. \( cl_{\hat{G}, G} \) vs \( cl_{\hat{G}} \). Now we prove Proposition 1.4 which states that \( scl_{\hat{G}, G} \) and \( scl_{\hat{G}} \) are equivalent if there exists a section homomorphism.

Proof of Proposition 1.4. The inequality \( scl_{\hat{G}}(x) \leq scl_{\hat{G}, G}(x) \) immediately follows from the definitions of norms. Thus, we prove \( scl_{\hat{G}, G}(x) \leq 2scl_{\hat{G}}(x) \) from now.

By Theorem 1.3, for any \( \epsilon > 0 \), there exists a \( \hat{G} \)-invariant homogeneous quasi-morphism \( \hat{\phi} \) such that

\[
scl_{\hat{G}, G}(x) - \epsilon \leq \frac{1}{2}\frac{\phi(x)}{D(\hat{\phi})}.
\]

By Proposition 1.12 there exists an extension \( \hat{\phi} \) of \( \phi \) which is homogeneous and \( D(\hat{\phi}) \leq 2D(\phi) \). Therefore,

\[
\frac{1}{2}\frac{\phi(x)}{D(\hat{\phi})} \leq \frac{\hat{\phi}(x)}{D(\hat{\phi})} \leq 2scl_{\hat{G}}(x).
\]

Since \( \epsilon \) can be taken arbitrary small, we have finished the proof. \( \square \)

4.2. \( cl_{\hat{G}, G} \) vs \( cl_G \). We give an example of a pair \((\hat{G}, G)\) of groups such that \( scl_{\hat{G}, G} \) and \( scl_G \) are not equivalent even if the quotient group \( \hat{G}/G \) is a finite group.

Let \( B_3 \) and \( P_3 \) denote the braid group and the pure braid group on 3 strands, respectively. Set \( \Delta = \sigma_1\sigma_2\sigma_3 = \sigma_3\sigma_1\sigma_2 \), where \( \sigma_1 \) and \( \sigma_2 \) are the Artin generators. Note that \( \Delta^2 \) is the full twist. Set \( x = \sigma_1^2 \), \( y = \sigma_2^2 \) and \( z = \Delta^2 \). Then \( P_3 \) has a presentation

\[
P_3 = \langle x, y, z \mid xz = zx, yz = zy \rangle \cong F_2 \times \mathbb{Z}.
\]
**Proposition 4.1.** For \( \hat{G} = B_3 \) and \( G = P_3 \), there exists an element \( \alpha \in [G, G] \) such that \( \text{scl}_{\hat{G}, G}(\alpha) = 0 \) and \( \text{scl}_G(\alpha) > 0 \).

To prove Proposition 4.1, we use the Brooks’ counting quasimorphism on free groups [Bro]. Let \( F_2 = \langle x, y \rangle \) be a free group of rank 2 and \( w \) be a reduced word in \( \{x^{\pm 1}, y^{\pm 1}\} \). A counting function \( c_w: F_2 \to \mathbb{Z} \) is defined by \( c_w(g) \) as the maximal number of disjoint copies of \( w \) in the reduced representative of \( g \in F_2 \). A counting quasimorphism is a function of the form

\[
\hat{h}_w(g) = c_w(g) - c_{w^{-1}}(g).
\]

**Proof of Proposition 4.1.** We get \( \hat{h}_w \circ \text{pr}_1 \), where \( w = x y x^{-1} y^{-1} \) and \( \text{pr}_1: P_3 \cong F_2 \times \mathbb{Z} \to F_2 \) is the first projection homomorphism. Since \( c_w([x, y]^n) = n \) and \( c_{w^{-1}}([x, y]^n) = 0 \),

\[
\hat{\phi}(\alpha) = \hat{h}_w([x, y]) = 1
\]

(it also says that \( \hat{\phi} \) is not a homomorphism). Therefore, by Theorem 1.2

\[
\text{scl}_G(\alpha) \geq \frac{1}{2} \frac{1}{D(\hat{\phi})} > 0.
\]

5. Applications to symplectic geometry

First, we prepare notions in symplectic geometry and the flux homomorphism. For a more precise description, refer to [Ban78, P01] for examples.

Let \((M, \omega)\) be a symplectic manifold. Let \( \text{Symp}(M, \omega) \) denote the group of symplectomorphisms with compact support and \( \text{Symp}_0(M, \omega) \) denote the identity component of \( \text{Symp}(M, \omega) \). Here, we consider the \( C^\infty \)-topology on \( \text{Symp}(M, \omega) \).

For a Hamiltonian function \( H: M \to \mathbb{R} \) with compact support, we define the **Hamiltonian vector field** \( X_H \) associated with \( H \) by

\[
\omega(X_H, V) = -dH(V) \text{ for any } V \in \mathcal{X}(M),
\]

where \( \mathcal{X}(M) \) denotes the set of smooth vector fields on \( M \).

Let \( S^1 \) denote \( \mathbb{R}/\mathbb{Z} \). For a (time-dependent) Hamiltonian function \( H: S^1 \times M \to \mathbb{R} \) with compact support and for \( t \in S^1 \), we define a function \( H_t: M \to \mathbb{R} \) by \( H_t(x) = H(t, x) \). Let \( X'_{H_t} \) denote the Hamiltonian vector field associated with \( H_t \) by \( \{d\mathcal{L}_{X'_{H_t}}\} \) denote the isotopy generated by \( X'_{H_t} \) such that \( \varphi^0 = \text{id} \). Let \( \varphi_H \) denote \( \varphi_{H_t} \) and \( \varphi_H \) is called the **Hamiltonian diffeomorphism generated by** \( H \). For a symplectic manifold \((M, \omega)\), we define the group of Hamiltonian diffeomorphisms by

\[
\text{Ham}(M, \omega) = \{ \varphi \in \text{Diff}(M) \mid \exists H \in C^\infty(S^1 \times M) \text{ such that } \varphi = \varphi_H \}.
\]

We note that \( \text{Ham}(M, \omega) \) is a normal subgroup of \( \text{Symp}_0(M, \omega) \).

Let \( X \) be a subset of a symplectic manifold \((M, \omega)\). \( X \) is **displaceable** if there exists a Hamiltonian function \( H: S^1 \times M \to \mathbb{R} \) such that \( \varphi_H(X) \cap \bar{X} = \emptyset \), where \( \bar{X} \) is the topological closure of \( X \).
For an exact symplectic manifold $(M, \omega)$, we recall that the **Calabi homomorphism** is a function $\text{Cal}_M: \text{Ham}(M, \omega) \to \mathbb{R}$ defined by

$$\text{Cal}_M(\varphi_t) = \int_0^1 \int_M F_t \omega^n \, dt.$$ 

The Calabi homomorphism is known to be well-defined and a group homomorphism (see [Cala], [Ban78], [Ban97] and [MS]).

**Definition 5.1.** Let $\mu: \text{Ham}(M, \omega) \to \mathbb{R}$ be a homogeneous quasimorphism. An open subset $U$ of $M$ has the Calabi property with respect to $\mu$ if $\omega|_U$ is exact and the restriction of $\mu$ to $\text{Ham}(U, \omega)$ coincides with the Calabi homomorphism $\text{Cal}_U$.

In terms of subadditive invariants, the Calabi property corresponds to the asymptotically vanishing spectrum condition in [KO, Definition 3.5].

**Definition 5.2 ([EP03] [PR]).** A **Calabi quasimorphism** is a homogeneous quasimorphism $\mu: \text{Ham}(M, \omega) \to \mathbb{R}$ such that any displaceable open subset of $M$ has the Calabi property with respect to $U$.

Here, we introduce the notion of the (volume) flux homomorphism. Let $M$ be an $n$-dimensional manifold and $\Omega$ a volume form on $M$. Let $\text{Diff}(M, \Omega)$ denote the group of diffeomorphisms preserving $\Omega$ with compact support, $\text{Diff}_0(M, \Omega)$ denote the identity component of $\text{Diff}(M, \Omega)$ and $\tilde{\text{Diff}}(M, \Omega)$ denote the universal covering of $\text{Diff}(M, \Omega)$. We define the (volume) flux homomorphism $\text{Flux}_\Omega: \text{Diff}(M, \Omega) \to H^{-1}_c(M, \mathbb{R})$ by

$$\text{Flux}_\Omega([\psi^t]_{t \in [0,1]}) = \int_0^1 [\iota_{X_t}, \Omega] \, dt,$$

where $\{\psi^t\}_{t \in [0,1]}$ is a path in $\text{Diff}_0(M, \Omega)$ with $\psi^0 = 1$ and $[\psi^t]_{t \in [0,1]}$ is the element of the universal covering $\tilde{\text{Diff}}(M, \Omega)$ represented by the path $\{\psi^t\}_{t \in [0,1]}$. It is known that $\text{Flux}_\Omega$ is a well-defined homomorphism. For a more precise description, refer to [Ban97] Section 3 for example.

If $(M, \omega)$ is a 2-dimensional symplectic manifold, then it is known that $\text{Ker}(\text{Flux}_\omega) = \text{Ham}(M, \omega)$ (note that a symplectic form is a volume form on a surface).

Let $\Sigma$ be a closed orientable surface with positive genus and $\omega$ a symplectic form on $\Sigma$. In order to prove Theorems 1.8, 1.19 and 1.21 we prepare $f_0, g_0 \in \text{Symp}_0(\Sigma, \omega)$ by the following way.

Since the genus of $\Sigma$ is positive, we can take a non-separating simple closed curve $C$ in $\Sigma$. Then, there are a positive number $r$ and a symplectic embedding $\iota: (-1, 1) \times \mathbb{R}/r\mathbb{Z} \to \Sigma$ such that $\iota([0] \times \mathbb{R}/r\mathbb{Z}) = C$. Here, the symplectic form on $(-1, 1) \times \mathbb{R}/r\mathbb{Z}$ is defined by $dx \wedge dy$, where $(x, y)$ is the coordinate on $(-1, 1) \times \mathbb{R}/r\mathbb{Z}$.

Let $\epsilon \in (0, 1)$ and $\chi: (-1,1) \to [0,1]$ be a function satisfying the following conditions.

- $\chi(x) = 0$ for any $x \in (-1, -1+\epsilon) \cup (1-\epsilon, 1)$,
- $\chi(x) + \chi(1+x) = 1$ for any $x \in (-1, 0)$.

By the above conditions, we see that $\chi(x) = 1$ for any $x \in (-\epsilon, \epsilon)$. Define a function $F: \Sigma \to \mathbb{R}$ by

$$F(z) = \begin{cases} 
\chi(x) & \text{if } z = \iota(x, y) \text{ for some } (x, y) \in (-1,1) \times \mathbb{R}/r\mathbb{Z}, \\
0 & \text{if } z \notin \text{Im}(\iota).
\end{cases}$$
Since $C$ is non-separating, $\Sigma \setminus \text{Im}(\iota)$ is path-connected. Thus, there exists $g_0 \in \text{Symp}_0(\Sigma, \omega)$ such that $g_0(\iota(x, y)) = \iota(x + 1, y)$ for any $(x, y) \in (-1, 0) \times \mathbb{R}/r\mathbb{Z}$.

Define a map $f_0: \Sigma \to \Sigma$ by

$$f_0(z) = \begin{cases} \varphi_F(z) & \text{(if } z \in \iota((-1, 0) \times \mathbb{R}/r\mathbb{Z})) \\ z & \text{(otherwise)}. \end{cases}$$

Since $f_0(z) = z$ for any $z \in \iota((-1, -1 + \epsilon) \cup (-\epsilon, \epsilon)) \times \mathbb{R}/r\mathbb{Z})$, $f_0$ is well-defined and $f_0 \in \text{Symp}_0(\Sigma, \omega)$. Since $\chi(x) + \chi(1 + x) = 1$ for any $x \in (-1, 0)$, by the definition of $g_0$,

$$g_0 f_0^{-1} g_0^{-1}(z) = \begin{cases} \varphi_F(z) & \text{(if } z \in \iota((0, 1) \times \mathbb{R}/r\mathbb{Z})) \\ z & \text{(otherwise)}. \end{cases}$$

Thus, we obtain $\varphi_F = f_0 g_0 f_0^{-1} g_0^{-1}$. Since $\text{Supp}(f_0) \subset \iota((-1, 0) \times \mathbb{R}/r\mathbb{Z})$ and $\text{Supp}(g_0 f_0^{-1} g_0^{-1}) \subset \iota((0, 1) \times \mathbb{R}/r\mathbb{Z})$, $f_0 g_0 f_0^{-1} g_0^{-1} = (g_0 f_0^{-1} g_0^{-1}) f_0$.

To prove Proposition 5.3 and Theorem 1.19 we use the following properties of Py’s Calabi quasimorphism.

**Proposition 5.3 [Py].** Let $\Sigma$ be a closed orientable surface whose genus is larger than one, $\omega$ a symplectic form on $\Sigma$ and $U$ an open subset of $\Sigma$ which is homeomorphic to an annulus. Then $U$ has the Calabi property with respect to Py’s Calabi quasimorphism $\mu_P$.

**Proof of Theorem 1.18** By the definition of $F$, $\int_\Sigma F \omega > 0$. By Proposition 5.3 $\text{Im}(\iota)$ has the Calabi property with respect to $\mu_P$. Since $\varphi_F = f_0 g_0 f_0^{-1} g_0^{-1}$ and $\text{Supp}(F) \subset \text{Im}(\iota)$,

$$\mu_P([f_0, g_0]) = \mu_P(\varphi_F) = \int_\Sigma F \omega > 0.$$ 

Thus, by Theorem 1.18 $\text{scl}_{\hat{G}}(\hat{G}, (f_0, g_0)) > 0$.

On the other hand, by $f_0 (g_0 f_0^{-1} g_0^{-1}) = (g_0 f_0^{-1} g_0^{-1}) f_0$, for any integer $n$,

$$[f_0, g_0]^n = (f_0 (g_0 f_0^{-1} g_0^{-1}))^n = f_0^n (g_0 f_0^{-1} g_0^{-1})^n = f_0^n (g_0 f_0^{-n} g_0^{-1}) = [f_0^n, g_0].$$

Thus,

$$\text{cl}_{\hat{G}}(\{f_0, g_0\}) = \text{cl}_{\hat{G}}([f_0^n, g_0]) \leq 1$$

for any integer $n$. Hence, $\text{scl}_{\hat{G}}(f_0, g_0) = 0$. \hfill $\Box$

**Proof of Corollary 1.17** To prove by contradiction, we suppose there exist vector fields $X_1, \ldots, X_2g$ satisfying the conditions.

Let $\varphi_i^t$ denote the time-$t$ map of the flow generated by $X_i$. Set $\alpha_i = [\iota X_i, \omega] \in H^1(\Sigma; \mathbb{R})$ for $i = 1, 2g$. Define a map $s: H^1(\Sigma; \mathbb{R}) \to \text{Symp}_0(\Sigma; \mathbb{R})$ by

$$s(t_1 \alpha_1 + t_2 \alpha_2 + \cdots + t_{2g} \alpha_{2g}) = \varphi_1^{t_1} \circ \varphi_2^{t_2} \circ \cdots \circ \varphi_{2g}^{t_{2g}}.$$

Since $L_{X_i, \omega} = 0$ for any $i$, $\varphi_1^{t_1} \circ \varphi_2^{t_2} \circ \cdots \circ \varphi_{2g}^{t_{2g}} \in \text{Symp}_0(\Sigma; \mathbb{R})$. Since $\alpha_i$ is a basis, $s$ is well-defined. Since $[X_i, X_j] = 0$ for any $i, j$, $s$ is a homomorphism. By the definition of the flux homomorphism, $s$ is a section. It contradicts Corollary 1.17 \hfill $\Box$

**Proof of Theorem 1.21** As we showed in the proof of Theorem 1.18 $\mu_P([f_0, g_0]) > 0$. Thus, by Lemma 1.13 $\mu_P$ is non-extendable to $\hat{G}$. \hfill $\Box$

To prove Theorem 1.21 we introduce the following property of Entov–Polterovich’s partial Calabi quasimorphism $\mu_{EP}$. This is a corollary of “heaviness” of $C$ in the sense of [EP09].
Proposition 5.4 (EP09 Example 1.18). For the above Hamiltonian function \( F: \Sigma \to \mathbb{R} \),
\[
\mu_{EP}(\varphi_F) = \int_{\Sigma} F\omega - \int_{\Sigma} \omega.
\]

Proof of Theorem 1.21. Set \( \hat{G} = \text{Symp}_0(\Sigma, \omega) \) and \( G = \text{Ham}(\Sigma, \omega) \). By Proposition 5.4,
\[
\mu_{EP}(\varphi_F) = \int_{\Sigma} F\omega - \int_{\Sigma} \omega < 0.
\]
By \([f_0, g_0] = \varphi_F\) and Lemma 1.18, \( \mu_{EP} \) is non-extendable to \( \hat{G} \). \( \square \)

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