ON \( n \)-HEREDITARY ALGEBRAS AND \( n \)-SLICE ALGEBRAS

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Abstract. In this paper we show that acyclic \( n \)-slice algebras are exactly acyclic \( n \)-hereditary algebras whose \((n + 1)\)-preprojective algebras are \((q + 1, n + 1)\)-Koszul. We also list the equivalent triangulated categories arising from the algebra constructions related to an \( n \)-slice algebra. We show that higher slice algebras of finite type appear in pairs and they share the Auslander-Reiten quiver for their higher preprojective components.

1. Introduction

The representation theory of hereditary algebras has been one of the central subjects in modern representation theory of algebras. In recent years, \( n \)-hereditary algebras, including \( n \)-representation finite and \( n \)-representation infinite algebras, are introduced and studied as generalization of hereditary algebras in higher Auslander-Reiten theory \((22, 23, 25, 26, 21)\). Graded self-injective algebras bear certain higher representation theory feature \([17]\), its bound quiver is a stable \( n \)-translation quiver with natural \( n \)-translation \( \tau \) induced by Nakayama functor \([11, 12]\). An \( n \)-slice is the quadratic dual of a complete \( \tau \)-slice in such quiver, and an \( n \)-slice algebra is an algebra whose bound quiver is an \( n \)-slice and whose \((n + 1)\)-preprojective algebras is \((q + 1, n + 1)\)-Koszul \([19]\).

In this paper, we first prove that acyclic \( n \)-hereditary algebras whose \((n + 1)\)-preprojective algebras are \((q + 1, n + 1)\)-Koszul are exactly acyclic \( n \)-slice algebras (Theorem 3.3 and Theorem 4.2). Recall that for the hereditary algebras, their preprojective algebras are \((q + 1, 2)\)-Koszul algebras, so they are exactly 1-slice algebras, and \( q + 3 \) is exactly the Coxeter number of the underlying Dynkin diagram of its quiver when the algebra is of finite type \([3]\). For an algebra defined by an \( n \)-translation quiver, being \((n + 1, q + 1)\)-Koszul ensures its quadratic dual has \( n \)-almost split sequence in the category of finitely generated projective modules under mild condition \([12]\).

The hereditary algebras are classified as of representation finite, tame and wild types. For \( n \)-hereditary algebras, they are divided as \( n \)-representation-finite and \( n \)-representation-infinite, and tame ones is defined inside \( n \)-representation-infinite \([21]\). A finite-tame-wild trichotomy for \( n \)-slice algebra using its Coxeter index and its spectral radius is proposed in \([19]\), which is consistent with \([21]\) for the finite-infinite dichotomy, but inconsistent for the tame case. The tameness in \([21]\) is defined by the Noetherian \((n + 1)\)-preprojective algebra and the tameness in \([19]\) is defined by the finiteness of the Gelfand-Kirillov dimension of its \((n + 1)\)-preprojective algebra, or equivalently, by the finite complexity of the trivial extension of its quadratic dual. It is proved that the tameness defined in \([21]\) is the tameness defined in \([19]\), and the converse is conjectured to be true in \([19]\).

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On the other hand, \( n \)-slice algebras of infinite type are exactly the quasi \( n \)-Fano algebras whose \((n+1)\)-preprojective algebras are Koszul (or \((\infty, n+1)\)-Koszul, see Theorem 4.4).

Apart from the \((q+1, n+1)\)-Koszulity of its \((n+1)\)-preprojective algebra, an \( n \)-slice algebra is defined using quiver with relations. In fact, an \( n \)-slice is quadratic dual of \( n \)-properly-graded quiver \( Q \) (see Section 2 for definition), obtained by taking complete \( \tau \)-slice in the infinite stable \( n \)-translation quiver \( Z_{|n-1}Q \) (see Section 5 for definition), a generalization of \( ZQ \). So \( n \)-slice algebras provide a combinatoric approach to \( n \)-hereditary algebra, and this may leads to more direct generalization of the representation theory of hereditary algebras. For example, for an \( n \)-slice algebra with bound quiver \( Q \), the Auslander-Reiten quiver of the \( n \)-preprojective and \( n \)-preinjective components are described using the quiver \( Z_{|n-1}Q \) (see [10]), and the \( n \)-APR tilts are realized by a \( \tau \)-mutation of the \( n \)-slice in \( Z_{|n-1}Q \) (see [18]). As a consequence, \( n \)-slice algebras and their types (finite, tame and wild) are invariant under APR tilts. An \((n+1)\)-slice algebra can be constructed from an \( n \)-slice using returning arrow constructions of such quivers, see [15] for certain finite type ones and [14] for infinite type.

Start from an \( n \)-slice algebra \( \Gamma \) with bound quiver \( Q^{\perp} \), write \( \Lambda = \Gamma^{\dagger,op} \) for its quadratic dual. We revisit variants of constructions of quivers, such as the returning arrow quiver \( \tilde{Q} \), the quivers \( Z_{c_{\tilde{Q}}} \) and \( Z_{|n-1}Q \) related to \( Q \), and algebras related to these quivers in Section 5. It is proved in [13] that certain twisted trivial extension \( \tilde{\Lambda} \) of the quadratic dual \( \Lambda \) of \( \Gamma \) is isomorphic to the quadratic dual of the \((n+1)\)-preprojective algebra \( \tilde{\Gamma} = \Pi(\Gamma) \) of \( \Gamma \) (See [3] in Section 2). One also constructs the Beilinson-Green algebras \( \Lambda^{N} \) of \( \Lambda \) and \( \Gamma^{N} \) of \( \Gamma \), the smash products \( \tilde{\Lambda}^{N} \) of \( \tilde{\Lambda} \) and \( \tilde{\Gamma}^{N} \) of \( \tilde{\Gamma} \), and some other related algebras. We listed in Section 5 the equivalent triangulate categories related to the derived category of an \( n \)-slice algebra arising from these algebras constructions. For the \( n \)-slice algebra of infinite type, its \((n+1)\)-preprojective algebra is an AS-regular algebra and this can be regarded as an algebra version of common generalization of Beilinson correspondence [4] and Bernstein-Gelfand-Gelfand correspondence [5] (see Picture (17) in Section 5). The quiver constructions are used in the last part of the paper to study the higher preprojective components of \( n \)-slice algebras of finite type.

Auslander-Reiten quivers are important tools in studying the representation theory of hereditary algebra of finite type. By using hammocks [7, 33], one obtains the Auslander-Reiten quiver of the path algebra \( kQ \) of finite type from the quiver \( ZQ \) by attaching to each vertex of \( Q \) the hammock starting at it. This has a higher representation theoretic interpretation: higher slice algebras of finite type appear in pairs (Theorem 6.4) and they share the same quiver as their Auslander-Reiten quiver of higher preprojective (higher preinjective) components (Theorem 7.6). We introduce double translation quiver and double slice to characterize these quivers: the Auslander-Reiten quiver of the \( n \)-preprojective component of an \( n \)-slice algebra \( \Gamma \) of finite type of Coxeter index \( q+1 \) is the opposite quiver of a double slice which is obtained by connecting the quiver \( Q^{\perp} \) of \( \Gamma \) and the quiver \( Q^{c\perp} \) of its companion \( \Gamma^{c} \), a \( q \)-slice algebra of finite type of Coxeter index \( n+1 \). Double slices provided higher representation theoretic analog of the constructions in [7, 33]. In fact, double slice has been used in [15] (called cuboid truncation there) to give a iterated construction of higher representation-finite algebra of type \( A \).
The paper is organized as follows. In Section 2, we recall notions and some known results needed in the paper. We prove that acyclic \(n\)-slice algebras are exactly acyclic \(n\)-hereditary algebras whose \((n+1)\)-preprojective algebras are \((q+1,n+1)\)-Koszul in the next two sections. In Section 5, we recall algebra constructions related to an \(n\)-slice algebra and list the equivalent triangulate categories arising from these constructions. In Section 6, we prove that higher slice algebras of finite type appear in pairs, we also introduce double translation quivers and double slices and discuss their properties. We show in the last section that the pair of the higher slice algebras of finite type share the Auslander-Reiten quiver for their higher preprojective components, with an example for illustrations.

2. Preliminary

In this paper, we assume that \(k\) is a field, and \(\Lambda = \Lambda_0 + \Lambda_1 + \cdots\) is a graded algebra over \(k\) with \(\Lambda_0\) a direct sum of copies of \(k\) such that \(\Lambda\) is generated by \(\Lambda_0\) and \(\Lambda_1\). Such algebra is determined by a bound quiver \(Q = (Q_0, Q_1, \rho)\) [12].

Recall that a bound quiver \(Q = (Q_0, Q_1, \rho)\) is a quiver with \(Q_0\) the set of vertices, \(Q_1\) the set of arrows and \(\rho\) a set of relations. The arrow set \(Q_1\) is usually defined with two maps \(s, t\) from \(Q_1\) to \(Q_0\) to assign an arrow \(\alpha\) its starting vertex \(s(\alpha)\) and its ending vertex \(t(\alpha)\). Write \(Q_i\) for the set of paths of length \(i\) in the quiver \(Q_i\), and write \(kQ_i\) for space spanned by \(Q_i\). Let \(kQ = \bigoplus_{i \geq 0} kQ_i\) be the path algebra of the quiver \(Q\) over \(k\). We also write \(s(p)\) for the starting vertex of a path \(p\) and \(t(p)\) for the terminating vertex of \(p\). Write \(s(x) = i\) if \(x\) is a linear combination of paths starting at vertex \(i\), and write \(t(x) = j\) if \(x\) is a linear combination of paths ending at vertex \(j\). The relation set \(\rho\) is a set of linear combinations of paths of length \(\geq 2\). We may assume that the paths appearing in each of the element in \(\rho\) have the same length, since we deal with graded algebra. Through this paper, we assume that the relations are normalized such that each element in \(\rho\) is a linear combination of paths of the same length starting at the same vertex and ending at the same vertex. Conventionally, modules are assumed to be finitely generated left module in this paper.

Let \(\Lambda_0 = \bigoplus_{i \in Q_0} k_i\), with \(k_i \simeq k\) as algebras, and let \(e_i\) be the image of the identity in the \(i\)th copy of \(k\) under the canonical embedding. Then \(\{e_i | i \in Q_0\}\) is a complete set of orthogonal primitive idempotents in \(\Lambda_0\) and \(\Lambda_1 = \bigoplus_{i,j \in Q_0} e_j \delta_{1i} e_i\) as \(\Lambda_0\)-bimodules. Fix a basis \(Q_1^i_j\) of \(e_j \Lambda_1 e_i\) for any pair \(i, j \in Q_0\), take the elements of \(Q_1^i_j\) as arrows from \(i\) to \(j\), and let \(Q_1 = \cup_{(i,j) \in Q_0 \times Q_0} Q_1^i_j\). Thus \(\Lambda \simeq kQ/(\rho)\) as algebras for some relation set \(\rho\), and \(\Lambda_0 \simeq kQ_1/(\rho \cap kQ_1)\) as \(\Lambda_0\)-bimodules. A path whose image is nonzero in \(\Lambda\) is called a bound path.

A bound quiver \(Q = (Q_0, Q_1, \rho)\) is called quadratic if \(\rho\) is a set of linear combination of paths of length 2. In this case, \(\Lambda = kQ/(\rho)\) is called a quadratic algebra.

Regard the idempotents \(e_i\) as the linear functions on \(\bigoplus_{i \in Q_0} k e_i\) such that \(e_i(e_j) = \delta_{i,j}\), for each \(i, j \in Q_0\), arrows from \(i\) to \(j\) are also regarded as the linear functions on \(e_j kQ_1 e_i\) such that for any arrows \(\alpha, \beta\), we have \(\alpha(\beta) = \delta_{\alpha, \beta}\). By defining \(\alpha_1 \cdots \alpha_r(\beta_1 \cdots \beta_s) = \alpha_1(\beta_1) \cdots \alpha_r(\beta_s)\), \(e_j kQ_1 e_i\) is identified with its dual space for each \(r\) and each pair \(i, j\) of vertices, and the set of paths of length \(r\) is regarded as the dual basis to itself in \(e_j kQ_r e_i\). Take a spanning set \(\rho_{i,j}^r\) of orthogonal subspace

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of the set $e_jkpe_i$ in the space $e_jkQ_2e_i$ for each pair $i,j \in Q_0$, and set
\[
\rho^\perp = \bigcup_{i,j \in Q_0} \rho_{i,j}^\perp.
\] (1)

The quadratic dual quiver of $Q$ is defined as the bound quiver $Q^\perp = (Q_0, Q_1, \rho^\perp)$, and the quadratic dual algebra of $\Lambda$ is the algebra $\Lambda^{\text{op}} \simeq kQ/(\rho^\perp)$ defined by the bound quiver $Q^\perp$ (see [13]).

Recall that a bound quiver $Q = (Q_0, Q_1, \rho)$ is called $n$-properly-graded if all the maximal bound paths have the same length $n$. The graded algebra $\Lambda = kQ/(\rho)$ defined by an $n$-properly-graded quiver is called an $n$-properly-graded algebra.

If an $n$-properly-graded quiver $Q$ is quadratic, its quadratic dual $Q^\perp$ is called an $n$-slice.

Let $Q = (Q_0, Q_1, \rho)$ be an acyclic bound quiver without bound path of infinite length. Let $\Lambda$ be the algebra defined by $Q$. The returning arrow quiver $\hat{Q}$ is the quiver with the same vertex set $\hat{Q}_0 = Q_0$ as $Q$ and arrow set $\hat{Q}_1 = Q_1 \cup Q_{1,\Lambda}$, with $Q_{1,\Lambda} = \{ t_p : t(p) \to s(p) | p \in M \}$, where $M$ is a basis for $\text{soc}_\Lambda \Lambda$. If $Q$ is an $n$-properly-graded quiver, $\Lambda$ is a maximal linearly independent set of maximal bound paths of $Q$ and the relation set $\hat{\rho}$ will be discussed in this case.

By [13], there is an epimorphism $\phi$ from the free $\Lambda$-bimodule $M_M$ generated by $Q_{1,\Lambda}$ to $DA$ if $Q$ is $n$-properly-graded quiver, take $\rho_M$ a generating set of $\ker \phi$.

Recall that the trivial extension $\Lambda \ltimes M$ of an algebra $\Lambda$ by an $\Lambda$-bimodule $M$ is the algebra defined on the vector spaces $\Lambda \oplus M$ with the multiplication defined by $(a,x)(b,y) = (ab, ay + xb)$ for $a,b \in \Lambda$ and $x,y \in M$. $\Delta \Lambda = \Lambda \ltimes DA$ is called the trivial extension of $\Lambda$. Let $\sigma$ be a graded automorphism of $\Lambda$, that is, an automorphism that preserves the degree of homogeneous element. Let $M$ be an $\Lambda$-bimodule. Define the twist $M^\sigma$ of $M$ as the bimodule with $M$ as the vector space. The left multiplication is the same as $M$, and the right multiplication is twisted by $\sigma$, that is, defined by $x \cdot b = x\sigma(b)$ for all $x \in M^\sigma$ and $b \in \Lambda$. Define the twisted trivial extension $\Delta_{\sigma} \Lambda = \Lambda \ltimes D\Lambda^\sigma$ to be the trivial extension of $\Lambda$ by the twisted $\Lambda$-bimodule $D\Lambda^\sigma$ if $Q$ is $n$-properly-graded quiver. Write $\rho_{M,\sigma} = \{ \sum x_p t_p \beta_p \sigma(y_p, t) | \sum x_p t_p \beta_p y_p, t \in \rho_M \}$, then $\rho_{M,\sigma}$ is a relation set for the bimodule $D\Lambda^\sigma$. Then $\rho_{M,\sigma}$ is a relation set for the bimodule $D\Lambda^\sigma$ if $Q$ is $n$-properly-graded quiver. Write $Q_{2,\Lambda}$ for the set of paths of length 2 formed by arrows in $Q_{1,\Lambda}$. By Proposition 2.2 of [9] and Theorem 3.1 and Corollary 3.3 of [13], we have the following results.

Proposition 2.1. If $\Lambda$ is an algebra with acyclic bound quiver $Q$, then $\Delta \Lambda \simeq k\hat{Q}/(\hat{\rho})$ and $\Delta_{\sigma} \Lambda \simeq k\hat{Q}/(\hat{\rho}^\sigma)$.

If $Q$ is an $n$-properly-graded quiver, then

$$\hat{\rho} = \rho \cup \rho_M \cup Q_{2,\Lambda} \text{ and } \hat{\rho}^\sigma = \rho \cup \rho_{M,\sigma} \cup Q_{2,\Lambda}.$$  

Note that $\hat{Q}$ is a stable $n$-translation quiver introduced in [11] [12]. The bound quivers of graded self-injective algebras are exactly the stable $n$-translation quivers, with the Nakayama permutation $\tau$ as its $n$-translation [12] (called stable bounded quiver of Loewy length $n + 2$ in [11]).

Write $\Pi(\Gamma)$ for the $(n+1)$-preprojective algebra of $\Gamma$, the following result is proved in [13].
Theorem 2.2. Assume that \( \Lambda \) is a Koszul \( n \)-properly-graded algebra and \( \Delta \Lambda \) is quadratic. Then

\[
\Pi(\Lambda^{1,op}) \simeq (\Delta_{\nu}\Lambda)^{1,op},
\]

for the graded automorphism \( \nu \) of \( \Lambda \) which sending an arrow \( \alpha \) to \((-1)^n\alpha\).

So we have that \( \Pi(\Lambda^{1,op}) \simeq k\tilde{Q}/(\tilde{\rho}_q, \perp) \).

Recall that a graded algebra \( \tilde{\Lambda} = \sum_{t \geq 0} \tilde{\Lambda}_t \) is called a \((p, q)\)-Koszul algebra if \( \tilde{\Lambda}_t = 0 \) for \( t > p \) and \( \tilde{\Lambda}_0 \) has a graded projective resolution

\[
\cdots \rightarrow P^t f_q \rightarrow \cdots \rightarrow P^1 f_1 \rightarrow P^0 f_0 \rightarrow \tilde{\Lambda}_0 \rightarrow 0,
\]

such that \( P^t \) is generated by its degree \( t \) part for \( t \leq q \) and \( \ker f_q \) is concentrated in degree \( q + p \) [12]. The concept of \((p, q)\)-Koszul unifies almost Koszul and Koszul by allowing \( p, q \) to be infinite: a \((p, q)\)-Koszul algebra is Koszul when one of \( p, q \) is infinite [3]. Note that a \((p, q)\)-Koszul algebra is quadratic.

A graded algebra \( \tilde{\Lambda} \) defined by an \( n \)-translation quiver \( \tilde{Q} \) is called an \( n \)-translation algebra, if there is a \( q \geq 1 \) or \( q = \infty \) such that \( \tilde{\Lambda} \) is an \((n + 1, q + 1)\)-Koszul algebra [12]. Conventionally, we take \( q + 1 = \infty \) when \( q = \infty \). A stable \( n \)-translation algebra \( \tilde{\Lambda} \) is a \((n + 1, q + 1)\)-Koszul self-injective algebra for some \( q \geq 1 \) or \( q = \infty \), and we call \( q + 1 \) the Coxeter index of \( \tilde{\Lambda} \).

Let \( \Lambda \) be the \( n \)-properly-graded algebra defined by \( Q \), if the trivial extension \( \Delta \Lambda \) is a stable \( n \)-translation algebra, the quadratic dual algebra \( \Gamma = kQ/(\rho^\perp) \) defined by the \( n \)-slice \( Q^\perp \) is called an \( n \)-slice algebra.

Theorem 2.2 tells us that

\[
\Pi(\Gamma) \simeq (\Delta_{\nu}(\Gamma^{1,op}))^{1,op}
\]

for an \( n \)-slice algebra \( \Gamma \).

The classification of \( n \)-slice algebra are discussed in [19], where the Loewy matrix of its quadratic dual plays an important role.

Let \( \tilde{\Gamma} = \Pi(\Gamma) \) be the \((n + 1)\)-preprojective algebra of an \( n \)-slice algebra \( \Gamma \). Recall that \( \Gamma \) is of finite type if \( \tilde{\Gamma} \) is finite dimensional, of tame type if \( \tilde{\Gamma} \) is of finite Gelfand-Kirillov dimension (not zero), and of wild type if \( \tilde{\Gamma} \) is of infinite Gelfand-Kirillov dimension. Let \( \tilde{\Lambda} \) be the quadratic dual of \( \Gamma \). This trichotomy of \( n \)-slice algebra \( \Gamma \) is also characterized by using the Coxeter index and the Loewy matrix of \( \tilde{\Lambda} \), see Theorem 3.7 of [19].

For an \( n \)-slice algebra \( \Gamma \), its quadratic dual \( \Lambda \) is an \( n \)-properly-graded algebra. Let \( \tilde{\Lambda} \) be its twisted trivial extension, it is a \((n + 1, q + 1)\)-Koszul self-injective algebra of Loewy length \( n + 2 \). Write \( L = L_{\tilde{\Lambda}} \) for the Loewy matrix of \( \tilde{\Lambda} \) [17], we have following characterization of finite-tame-wild trichonomy using \( \tilde{\Lambda}(\text{Theorem 3.7 of [19]}).

Theorem 2.3. Let \( \Gamma \) be an \( n \)-slice algebra with Coxeter index \( q + 1 \).

1. \( \Gamma \) is of finite type if and only if \( q + 1 \) is finite.
2. \( \Gamma \) is of tame type if and only if \( \tilde{\Lambda} \) is of finite complexity and not periodic, if and only if \( q \) is infinite and the spectral radius of \( L \) is 1.
3. \( \Gamma \) is of wild type if and only if \( \tilde{\Lambda} \) is of infinite complexity, if and only if \( q \) is infinite and the spectral radius of \( L \) is larger than 1.
3. Twisted trivial extensions and higher preprojective algebras

Let $\Gamma$ be a finite dimensional algebra of global dimension $\leq n$, let $\tilde{\Gamma} = \Pi(\Gamma)$ be the $(n+1)$-preprojective algebra of $\Gamma$. If $\Gamma$ is quadratic, write $\Lambda = \Gamma^{!,op}$ for its quadratic dual.

Assume that $\Lambda$ is an acyclic $n$-properly-graded algebra, consider the twisted trivial extension $\Lambda$ of $\Lambda$ with respect to an automorphism $\sigma$. With the returning arrow as degree 1 element, $\Delta_{\sigma} \Lambda$ are naturally a graded algebras extending the gradation of $\Lambda$ as its first gradation. With the algebra $\Lambda$ as degree 0 component and $DA^\sigma$ as degree 1 component, $\Delta_{\sigma} \Lambda$ is endowed with a second gradation and is made a bigraded algebra graded by $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. This is equivalent to take the gradation of the tensor algebra $T_{\Gamma}(\operatorname{Ext}_{\Gamma}^n(D\Gamma, \Gamma))$ as the second gradation, and $\tilde{\Gamma}$ is a bigraded algebra.

As the algebra defined by the returning arrow quiver $\tilde{Q}^\sigma$, the second gradation is defined by endowing the old arrows in $Q$ with degree 0 and the returning arrows with degree 1. We have the decomposition $\Delta_{\sigma} \Lambda = \sum_t \Delta_{\sigma} \Lambda_{t,-}$ with respect to the first gradation and $\Delta_{\sigma} \Lambda = \sum_t \Delta_{\sigma} \Lambda_{-.t}$ with respect to the second gradation. Write $()_{-,-}$ for the shift with respect to the first gradation and $(,)_{-,-}$ for the shift with respect to the second gradation.

We have that
\[
\Delta_{\sigma} \Lambda_{t,-} = \Lambda_t + D\Lambda_{n+1-t},
\]
with the convention that $D\Lambda_{n+1} = 0$. So
\[
DA_{1,-} = kQ_{1, M} = \sum_{p \in M} k\beta_p = D\Lambda_n,
\]
and
\[
DA_{t,-} = \sum_{s=0}^{n} \Lambda_{s}DA_{n} \Lambda_{t,-s} \quad \text{and} \quad DA_{-.1} = DA,
\]
that is $DA$ is concentrated in degree $(-, 1)$ with respect to the second grading. Especially
\[
\Delta_{\sigma} \Lambda_{-.0} = \Lambda \quad \text{and} \quad \Delta_{\sigma} \Lambda_{-.1} = DA.
\]

Now assume that $\Lambda^{\tilde{\sigma}} = \Delta_{\sigma} \Lambda$ is quadratic, write $\tilde{\Gamma}^{\tilde{\sigma}} = \Lambda^{\sigma,!,op}$ for its quadratic dual. Then since $\tilde{\Gamma}^{\tilde{\sigma}} \simeq k\tilde{Q}^{\tilde{\sigma}} / (\tilde{\sigma}^{\tilde{\sigma}})$, the second gradation of $\Lambda^{\tilde{\sigma}}$ induces a second gradation on $\tilde{\Gamma}^{\tilde{\sigma}}$, by endowing the arrows in $Q_1$ with degree 0 and the arrow in $Q_{1, M}$ with degree 1. In this case, we obviously have that
\[
\tilde{\Gamma}^{\tilde{\sigma}}_{-.0} = \Gamma.
\]

Assume that $\tilde{\Gamma}$ is a $(q+1, n+1)$-Koszul algebra with respect to the first gradation, it is quadratic with this gradation. The bigradation of $\tilde{\Gamma}$ is coherent with the bigradation on its quadratic dual $\Lambda = \tilde{\Gamma}^{!,op}$, and $\Lambda$ is an $(n+1, q+1)$-Koszul algebra with respect to the first gradation. So we have a Koszul complex
\[
\tilde{\Gamma} \otimes D\Lambda_{(n+1, *)} \xrightarrow{f_{n+1}} \tilde{\Gamma} \otimes D\Lambda_{(n, *)} \longrightarrow \cdots \longrightarrow \tilde{\Gamma} \otimes D\Lambda_{(1, *)} \xrightarrow{f_1} \tilde{\Gamma} \otimes D\Lambda_{(0, *)} \quad (9)
\]
with respect to the first gradation.

Lemma 3.1. If $\tilde{\Gamma}$ is $(q+1, n+1)$-Koszul, then $\Gamma$ is Koszul.

Proof. Since $\tilde{\Gamma}$ is $(q+1, n+1)$-Koszul, we have that (9) is exact except for the first and the last term. The cokernel of $f_1$ is $\tilde{\Gamma}_0 = \Gamma_0$, kernel of $f_{n+1}$ is concentrated at
degree \((n + 1 + q, \ast)\) if \(q\) is finite, and 0 otherwise. By taking the components with the second degree 0, we get

\[
\bar{\Gamma} \otimes D\tilde{\Lambda}_{(n+1,0)} \xrightarrow{f_{n+1}} \bar{\Gamma} \otimes D\tilde{\Lambda}_{(n,0)} \rightarrow \cdots \rightarrow \bar{\Gamma} \otimes D\tilde{\Lambda}_{(1,0)} \xrightarrow{f_1} \bar{\Gamma} \otimes D\tilde{\Lambda}_{(0,0)} \rightarrow \Gamma_0 \rightarrow 0.
\]

(10)

Note that \(\tilde{\Lambda}_{(n+1,0)} = 0\), and \(\bar{\Gamma} \otimes D\tilde{\Lambda}_{(t,0)}\) is generated in degree \((t,0)\) for \(0 \leq t \leq n\), and \(\bar{\Gamma}_{(0,0)} = \Gamma_0\). So we get an exact sequence

\[
0 \rightarrow \Gamma \otimes D\tilde{\Lambda}_{(n,0)} \rightarrow \cdots \rightarrow \Gamma \otimes D\tilde{\Lambda}_{(1,0)} \rightarrow \Gamma \otimes D\tilde{\Lambda}_{(0,0)} \rightarrow \Gamma_0 \rightarrow 0,
\]

(11)

which is a projective resolution of \(\Gamma_0\) with the \(t\)th term \(\Gamma \otimes D\tilde{\Lambda}_{(t,0)}\) generated in degree \(t\), so \(\Gamma\) is Koszul.

\[\square\]

**Proposition 3.2.** Let \(\Lambda\) be an acyclic finite dimensional Koszul algebra. If its twisted trivial extension \(\tilde{\Lambda}\) is \((n+1, q+1)\)-Koszul, then \(\Lambda\) is an \(n\)-properly-graded algebra.

**Proof.** Let \(Q\) be the bound quiver of \(\Lambda\) and let \(\tilde{Q}\) be the bound quiver of \(\tilde{\Lambda}\). Then \(\tilde{Q}\) is the returning arrow quiver of \(Q\).

Now \(\tilde{\Lambda}\) is \((n+1, q+1)\)-Koszul, so it is a \(n\)-translation quiver with identity as \(n\)-translation, hence is \((n+1)\)-properly graded, and \(\tilde{\Lambda}\) is an \(n\)-translation algebra.

Let \(p = \alpha_r \cdots \alpha_1\) be a bound path in \(Q\) of maximal length with arrows \(\alpha_t : i_{t-1} \rightarrow i_t\) and \(i = i_0, j = i_r\), it is a bound path in \(\tilde{Q}\). So there is a bound path \(p' = \gamma_{n+1-r} \cdots \gamma_1\) such that \(p'p\) is a bound path in \(Q\) from \(j\) to \(j\).

Note that we have that \((DA)^2 = 0\), we have only one returning arrow in \(p'\), say \(\gamma_n = \beta_q\) in \(p'\). If \(r < n\), then \(p' = p_1\beta_qp_2\) for some path \(p_1, p_2\) in \(Q\) and now we have that \(0 \neq p'p = p_1\beta_qp_2p_1 \neq 0\) and \(p_2p_1 \neq 0\), this contradicts the choice of \(p\).

So \(r = n\) and \(Q\) is \(n\)-properly-graded.

\[\square\]

**Theorem 3.3.** Let \(\Gamma\) be an acyclic \(n\)-hereditary algebra. If its \((n+1)\)-preprojective algebra \(\tilde{\Gamma} = \Pi(\Gamma)\) is \((q+1, n+1)\)-Koszul, then \(\Gamma\) is an \(n\)-slice algebra.

**Proof.** Let \(\Gamma\) be an \(n\)-hereditary algebra with bound quiver \(Q^\perp\). Let \(\tilde{\Gamma} = \Pi(\Gamma)\) be the \((n+1)\)-preprojective algebra of \(\Gamma\).

Let \(\Lambda = \tilde{\Gamma}^\perp \otimes\) with bound quiver \(Q\), then by Theorem C of [10], we have the isomorphism \(\tilde{\Gamma}^\perp \otimes \tilde{\Lambda}^\sigma \simeq \tilde{\Lambda}^\sigma\) of for some twisted trivial extension \(\tilde{\Lambda}^\sigma\) of \(\Lambda\). Since \(\tilde{\Gamma}\) is \((q+1, n+1)\)-Koszul, \(\tilde{\Lambda}^\sigma\) is \((n+1, q+1)\)-Koszul, so it is an \(n\)-translation algebra. Thus \(\tilde{\Lambda}\) is also an \(n\)-translation algebra. By Proposition 3.2, \(Q\) is \(n\)-properly-graded quiver, and \(Q^\perp\) is an \(n\)-slice.

This prove that \(\Gamma\) is an \(n\)-slice algebra.

\[\square\]

4. \(n\)-HEREDITARY ALGEBRAS AND \(n\)-SLICE ALGEBRAS

In this section we prove that an \(n\)-slice algebra is \(n\)-hereditary by showing that its \((n+1)\)-preprojective algebra is \((n+1)\)-Calabi-Yau or stably \((n+1)\)-Calabi-Yau.

We first study the Nakayama automorphism twisted trivial extension of an \((n+1)\)-properly-graded algebra.

Let \(\Lambda\) be an \(n\)-properly-graded algebra with acyclic bound quiver \(Q\). By Proposition 2.1 the bound quiver of the trivial extension \(\Delta\Lambda\) of \(\Lambda\) is the returning arrow
quiver $\tilde{Q}$ of $Q$. Let $\{p^*|p \in \mathcal{M}\}$ be the dual basis of $\mathcal{M}$ in $D\Lambda_n$. Note that for each bound path $q = \alpha_r \cdots \alpha_1$ in $\Delta \Lambda$, write

$$\zeta_q = \begin{cases} 0 & \alpha_t \in Q_1, 1 \leq t \leq r \\ p^* & \alpha_{r_0} = \beta_p^* \in Q_{1,\mathcal{M}}, 1 \leq r_0 \leq r \end{cases}$$

the map

$$\mu : q \to \zeta_q(\alpha_{r_0-1} \cdots \alpha_1 \alpha_r \cdots \alpha_{r_0+1})$$

(12)
defines a linear map on $\Delta \Lambda$. Set

$$(x, y) = \mu(xy),$$

(13)
this is a non-degenerate bilinear form on $\Delta \Lambda$. The algebra $\Delta \Lambda$ is symmetric, so the Nakayama automorphism $\omega$ of $\Delta \Lambda$ is the identity, and we have that

$$(x, y) = (y, x).$$

(14)

Now we consider the Nakayama automorphism of $\Delta \sigma \Lambda$, that is, the automorphism $\omega$ of $\Delta \sigma \Lambda$, such that $\sigma(a, b) = (b, \omega(a))$, with respect to the non-degenerate bilinear form above. Since $\sigma$ is an automorphism on $\Lambda$, the map $\omega$ sending $p^* \sigma$ defines an automorphism on the vector space $D\Lambda_n$, which is isomorphic to the subspace $kQ_{1,\mathcal{M}}$.

**Proposition 4.1.** Let $\Lambda$ be an acyclic algebra and let $\sigma$ be a graded automorphism of $\Lambda$. Then the Nakayama automorphism of $\Delta \sigma \Lambda$ is the automorphism defined by $\omega(\beta_p) = p^* \sigma$ on $Q_{1,\mathcal{M}}$, $\omega(e_i) = e_i$ for $i \in Q_0$ and $\omega = \sigma^{-1}$ on $Q_1$.

**Proof.** If $q = e_i$ is a primitive idempotent and $q'$ be a bound path in $\tilde{Q}$, clearly we have that $(q, q') \neq 0$ if $q'$ is a cyclic bound path in $\tilde{Q}$ starting and ending at the vertex $i$. Thus we have $(e_i, q') = (q', e_i)$ and $\omega(e_i) = e_i$.

Now assume that $q$ is an arrow, and let $q'$ be a bound path in $Q$.

If $q = \beta_p^*$ is a returning arrow, then

$$(q, q') = \mu(\beta_p \cdot q') = \mu(\beta_p \sigma(q')) = p^* \sigma(q') = (p^* \sigma)(q'),$$

while

$$(q', q) = \mu(q' \beta_p^*) = p^*(q').$$

Since $\omega(\beta_p)$ is the isomorphic image of $p^* \sigma$ in $kQ_{1,\mathcal{M}}$,

$$(q', \omega(q)) = \mu(q' \omega(\beta_p)) = (p^* \sigma)(q').$$

Assume that $q = \alpha$ is an arrow in $Q$. If $q'$ is in $Q$, then $(q, q') = 0 = (q', q)$. Now let $q' = u \beta_p v$, then

$$(q, q') = \mu(\alpha u \beta_p v) = p^*(v \alpha u)$$

and

$$(q', \sigma^{-1}(q)) = \mu(u \beta_p v \cdot \sigma^{-1}(\alpha)) = \mu(u \beta_p v \sigma(\sigma^{-1}(\alpha))) = p^*(v \alpha u) = (q, q').$$

Define $\omega(\alpha) = \sigma^{-1}(\alpha)$ for $\alpha \in Q_1$, then $\omega$ defines an automorphism of the space $k\tilde{Q}_1$. 
Now extend $\omega$ to an algebra automorphism on $\Delta_\sigma \Lambda = k\tilde{Q}/(\tilde{p})$. For bound paths $u, u', v$ in $Q$ and arrow $\beta_p$ in $Q_{1, M}$, we have that

\[(u', u\beta_p v) = \mu(u' u \beta_p v) = p^* (v u' u) = \mu(u \beta_p v, \sigma^{-1}(u')) = (u \beta_p v, \sigma^{-1}(u')) = (u \beta_p v, \omega(u')).\]

So we get that $\omega$ is the Nakayama automorphism of $\Delta_\sigma \Lambda$. \hfill \Box

Calabi-Yau category is introduced in [29] (see [28]). Let $\mathcal{T}$ be a $k$-linear triangulated category with finite dimensional Hom spaces. The category $\mathcal{T}$ is said to be $d$-Calabi-Yau if for all objects $X, Y$ in $\mathcal{T}$, there exists an isomorphism $\text{Hom}_\mathcal{T}(X, Y) \simeq D\text{Hom}_\mathcal{T}(Y, X[1])$ functorial in $X$ and $Y$.

Recall that a graded algebra $\tilde{\Gamma}$ is called an bimodule $(d + 1)$-Calabi-Yau algebra of Gorenstein parameter $p$ if it is an homologically smooth algebra satisfying

\[R\text{Hom}_{\text{grmod}_M}(\tilde{\Gamma}, \tilde{\Gamma}^e)[d + 1] \simeq \tilde{\Gamma}(p)\]

in $\mathcal{D}^b(\tilde{\Gamma}^e)$.

A graded algebra $\tilde{\Gamma}$ is called bimodule stably $d$-Calabi-Yau of Gorenstein parameter $p$ if there is an isomorphism

\[R\text{Hom}_{\text{grmod}_M}(\tilde{\Gamma}, \tilde{\Gamma}^e)[d + 1] \simeq \tilde{\Gamma}(p)\]

in $\text{grmod}_{CM}(\tilde{\Gamma}^e)$, where $\text{grmod}_{CM}(\tilde{\Gamma}^e)$ is the stable category of graded Cohen-Macaulay $\tilde{\Gamma}^e$-modules.

**Theorem 4.2.** An acyclic $n$-slice algebra is an $n$-hereditary algebra.

**Proof.** Let $\Gamma$ be an $n$-slice algebra and let $\Lambda$ be its quadratic dual. Then $\Lambda$ is an $n$-properly graded algebra whose twisted trivial extension $\Delta_\sigma \Lambda$ are stable $n$-translation algebra for all graded automorphism $\sigma$ of $\Lambda$.

So there is an $q$ such that $\Delta_\sigma \Lambda$ is an $(n + 1, q + 1)$-Koszul self-injective algebra. Let $\epsilon$ be the automorphism of $\Lambda$ sending arrows to its scalar by $(-1)$. Take $\sigma = \epsilon^n$, it is the automorphism of $\Lambda$ sending arrows to its scalar by $(-1)^n$, then by Theorem 5.3 of [13], $\Pi(\Gamma) = \Delta_\sigma \Lambda_{1, op}$.

We have $\omega(\beta_p x) = p^* u^n x = (1)^n \beta_p x \sigma^{-1}(x)$, so $\omega(\beta_p) = (1)^n \beta_p$. This shows that the Nakayama automorphism $\omega$ of $\Delta_\sigma \Lambda$ maps $x$ to $(-1)^{\text{deg} x} x$.

If $q$ is infinite, then $\Delta_\sigma \Lambda$ is Koszul of with $(n + 1)$th power of its radical zero, and $\Gamma = \Pi(\Gamma)$ is of global dimension $n + 1$. Since $(-1)^n = (1)^{n+2}$, by Theorem 4.2 of [32], the quadratic dual $\tilde{\Gamma} = \Delta_\sigma \Lambda_{1, op}$ is Calabi-Yau.

By [33], we have $\tilde{\Gamma}_{-0} = \Gamma$ so

\[R\text{Hom}_{\text{grmod}_M}(\tilde{\Gamma}, \tilde{\Gamma}^e)[n + 1] \simeq \tilde{\Gamma}(1)_{-0}^e \tag{15}\]

in $\mathcal{D}^b(\tilde{\Gamma}^e)$, and $\Gamma$ is $n$-representation-infinite, by Theorem 3.1 of [1].

If $q$ is finite, then $\Delta_\sigma \Lambda$ is an$(n + 1, q + 1)$-Koszul self-injective algebra, it is Frobenius of periodic type. So by Theorem 3.4 and Corollary 4.2 of [31], $\tilde{\Gamma} = \Delta_\sigma \Lambda_{1, op}$ is stably Calabi-Yau. By [33], we have $\tilde{\Gamma}_{-0} = \Gamma$ so

\[R\text{Hom}_{\text{grmod}_M}(\tilde{\Gamma}, \tilde{\Gamma}^e)[n + 1] \simeq \tilde{\Gamma}(1)_{-0} \tag{16}\]

in $\text{grmod}_{CM}(\tilde{\Gamma}^e)$, and $\Gamma$ is $n$-representation-finite, by Theorem 3.2 of [1].
This proves that an acyclic $n$-slice algebra $\Gamma$ is $n$-hereditary algebra and $\widetilde{\Gamma}$ is its $(n+1)$-preprojective algebra for $\sigma = e^{n+2}$.

Combine Theorem 5.3 and 4.2 we prove that

**Theorem 4.3.** An acyclic $n$-hereditary algebra is $n$-slice algebra if and only if its $(n+1)$-preprojective algebra is $(q+1, n+1)$-Koszul for some $q$.

In fact, $n$-slice algebras of infinite type are exactly the $n$-Fano algebras with Koszul $(n+1)$-preprojective algebras, as is proved in the next theorem.

**Theorem 4.4.** An acyclic algebra $\Gamma$ is quasi $n$-Fano algebra with Koszul $(n+1)$-preprojective algebra if and only if it is an $n$-slice algebra of infinite type.

**Proof.** By Theorem 5.3 we need only to prove that quasi $n$-Fano algebra with Koszul $(n+1)$-preprojective algebra is an $n$-slice algebra of infinite type.

If $\Gamma$ is quasi $n$-Fano algebra, then by Theorem 4.2 of [31], its $(n+1)$-preprojective algebra $\tilde{\Gamma}$ is AS-regular of dimension $n+1$. Since $\tilde{\Gamma}$ is Koszul, by Theorem 5.1 of [30], its quadratic dual $\Lambda = \tilde{\Gamma}^{\text{op}}$ is Koszul self-injective algebra of Loewy length $n+1$, that is, $\Lambda$ is a stable $n$-translation algebra. By Proposition 5.2 $\Gamma$ is an $n$-slice algebra. By Theorem 3.2 of [19], $\Gamma$ is an $n$-slice algebra of infinite type.

Since an $n$-representation-infinite algebra is extremely Fano, so a quasi $n$-Fano algebra with Koszul $(n+1)$-preprojective algebra is extremely Fano.

5. Algebras and triangulated categories associated to an $n$-slice algebra

Start with an $n$-slice algebra $\Gamma$ with bound quiver $Q^{-}$ and its quadratic dual, the $n$-properly-graded algebra $\Lambda = \Gamma^{\oplus}$ with bound quiver $Q$. In Section 2 we have constructed the returning arrow quivers $\overline{Q^{-}}$ and $\overline{Q}$, the $(n+1)$-preprojective algebra $\tilde{\Gamma} = \Pi(\Gamma)$ and its quadratic dual $\tilde{\Lambda} = \Pi(\Gamma)^{\text{op}}$, which is a twisted trivial extension of $\Lambda$. In the previous sections, we see that we have two series of algebras parameterized by the graded automorphisms related to the returning quiver. Though only with the automorphism $\sigma = e^{n+2}$, one gets Calabi-Yau, it seems the representation theory are independent of the automorphism. Now we recall some other constructions of quivers and algebras related to an $n$-slice algebra.

Given a finite stable $n$-translation quiver $\overline{Q}$ with $n$-translation $\tau$, we construct an infinite acyclic stable $n$-translation quiver

\[ Z_{0}\overline{Q} = Z_{s_{-}}\overline{Q} = (Z_{s_{-}}\overline{Q}_{0}, Z_{s_{-}}\overline{Q}_{1}, p_{Z_{s_{-}}\overline{Q}}) \]

as follows (denoted by $\overline{Q}$ and called separated directed quiver in [11] ), with the vertex set

\[ Z_{s_{-}}\overline{Q}_{0} = \{(i, n) | i \in Q_{0}, n \in \mathbb{Z}\}, \]

and the arrow set

\[ Z_{s_{-}}\overline{Q}_{1} = \{(\alpha, n) : (i, n) \to (j, n + 1) | \alpha : i \to j \in Q_{1}, n \in \mathbb{Z}\}. \]

If $p = \alpha_{s} \cdots \alpha_{1}$ is a path in $\overline{Q}$, define $p[m] = (\alpha_{s}, m + s - 1) \cdots (\alpha_{1}, m)$, and for an element $z = \sum pt$, with each $pt$ a path in $\overline{Q}$, $\alpha_{t} \in k$, define $z[m] = \sum \alpha_{t}pt[m]$ for each $m \in \mathbb{Z}$. Define relations

\[ p_{Z_{s_{-}}\overline{Q}} = \{\zeta[m] | \zeta \in \tilde{\rho}, m \in \mathbb{Z}\}, \]
here $\zeta[m] = \sum_{t} a_t p_t [m]$ for each $\zeta = \sum_{t} a_t p_t \in \tilde{\rho}$. By [11], it is the bound quiver of the smash product $\tilde{\Lambda}^N = \tilde{\Lambda}^{\# \ast, k\mathbb{Z}^*}$ with respect to the first grading of $\tilde{\Lambda}$.

The quiver $Z_0\tilde{Q}$ is a locally finite bound quiver if $Q$ is so. By setting $\tau(i, t) = (\tau i, t - n - 1)$, $Z_{0, -}\tilde{Q}$ becomes a stable $n$-translation quiver [11]. Clearly it is an acyclic infinite $n$-translation quiver.

A quiver $Q$ is called nicely-graded if there is a map $d$ from $Q_0$ to $\mathbb{Z}$ such that $d(j) = d(i) + 1$ for any arrow $\alpha : i \to j$. Clearly, a nicely graded quiver is acyclic.

The following properties of $Z_0\tilde{Q}$ are in Proposition 2.1 of [19].

**Proposition 5.1.**

1. Let $d$ be the greatest common divisor of the length of cycles in $Q$, then $Z_0\tilde{Q}$ has $d$ connected components.
2. All the connected components of $Z_0\tilde{Q}$ are isomorphic.
3. Each connected component of $Z_0\tilde{Q}$ is nicely graded quiver.

Now assume that $Q$ is an $n$-properly-graded quiver, let $\tilde{Q}$ be its returning arrow quiver. Conventionally, we assume that $\tilde{Q}$ is quadratic. Now we have another infinite acyclic stable $n$-translation quiver

$$Z|_{n-1} Q = Z_{-}, \tilde{Q} = (Z_{-}, \tilde{Q} Q_0, Z_{-}, \tilde{Q} 1, \rho Z_{-}, \tilde{Q})$$

related to $\tilde{Q}$ with respect to the returning arrow grading, as the bound quiver with vertex set

$$(Z_{-}, \tilde{Q})_0 = \{(u, t)|u \in \tilde{Q}_0, t \in \mathbb{Z}\},$$

the arrow set

$$(Z_{-}, \tilde{Q})_1 = Z \times Q_1 \cup Z \times Q_{1, \mathcal{M}}$$

$$= \{(\alpha, t) : (i, t) \to (i', t)|\alpha : i \to i' \in Q_1, t \in \mathbb{Z}\}$$

$$\cup\{ (\beta p, t) : (i', t) \to (i, t + 1)|p \in \mathcal{M}, s(p) = i, t(p) = i'\}$$

and the relation set

$$\rho_{Z_{-}, \tilde{Q}} = \rho_{\tilde{Q}} \cup Z_{Q, \mathcal{M}} \cup Z_{\rho_{\mathcal{M}}},$$

where

$$Z_{\rho} = \{ \sum_s a_s (\alpha_s, t)(\alpha'_s, t)|\sum_s a_s \alpha_s \alpha'_s \in \rho, t \in \mathbb{Z}\},$$

$$Z_{Q, \mathcal{M}} = \{ (\beta_{p', t + 1})(\beta_{p, t})|\beta_{p', t + 1}, \beta_{p, t} \in Q_{1, \mathcal{M}}, t \in \mathbb{Z}\}$$

and

$$Z_{\rho_{\mathcal{M}}} = \{ \sum_s a_s (\beta_{p', t + 1})(\beta_{p, t}) + \sum_s b_s (\alpha_s, t + 1)(\beta_{p', t})|\sum_s a_s \beta_{p', t}, \alpha'_s + \sum_s b_s \alpha_s \beta_{p, t} \in \rho_{\mathcal{M}}, t \in \mathbb{Z}\}.$$
A complete $\tau$-slice in an acyclic stable $n$-translation quiver is a full convex subquiver which intersects each $\tau$-orbit exactly once [11]. We usually take a complete $\tau$-slice as a bound subquiver. An algebra defined by a complete $\tau$-slice is called a $\tau$-slice algebra.

An $n$-properly-graded quiver $Q$ is a $\tau$-slice in $\mathbb{Z}_{n-1} Q$, so an $n$-slice is a quadratic dual of some $\tau$-slice. So we have the following result justifying the name of $n$-slice algebra.

**Proposition 5.2.** An $n$-slice algebra is the quadratic dual of a $\tau$-slice algebra.

Consider an returning arrow quiver $\tilde{Q}$, it is known that the full bound subquiver $\mathbb{Z} \cdot \tilde{Q}[0,n]$ of $\mathbb{Z} \cdot \tilde{Q}$ with vertex set $\{(i,t)|i \in \tilde{Q}_0, 0 \leq t \leq n\}$ is a $\tau$-slice in $\mathbb{Z} \cdot \tilde{Q}$, so an $n$-slice is a quadratic dual of some $\tau$-slice. So we have the following result justifying the name of $n$-slice algebra.

Write $Q_N = \mathbb{Z} \cdot \tilde{Q}[0,n]$, and let $\Lambda_N$ be the algebra defined by $Q_N$, then $\Lambda_N^\# = \Lambda^\# \cdot \Delta(\Lambda_N)$ is the repetitive algebra of $\Lambda_N$, by Theorem 5.12 of [11], and we write $\tilde{\Lambda}_N$ is the Beilinson-Green algebra of $\tilde{\Lambda}$ defined in [8](called Beilinson algebra there).

Write $\Gamma_N = (\Lambda_N)^{!,op}$ for the quadratic dual of $\Lambda_N$, and write $\Gamma_N^\#$ for the repetitive algebra of $\Gamma_N$. Write $\Lambda = \tilde{\Lambda}^\# \cdot \Delta(\Lambda_N)$, similar to the argument in [11], we also have $\Lambda^\#$ is the repetitive algebra of $\Lambda$.

The quiver $\mathbb{Z}_{n-1} Q$ are important in studying the higher preprojective and preinjective components (see [16]).

The representation theory of the $n$-slice algebra and of the algebras constructed above are also related.

The algebras $\Gamma$ and $\Lambda$ are Koszul dual by Section 2 of [6], and there is an equivalence between their derived categories of finite generated graded modules as triangulated categories. This equivalence extends to certain triangulated categories related to the algebras constructed.

For an AS-regular algebra $\tilde{\Gamma}$, denote by $\text{qgr}\tilde{\Gamma}$ the non-commutative projective scheme of $\tilde{\Gamma}$ (see [2]). The following theorem is the generalization of a theorem stated and proved in [19] for the triangulated categories related to $n$-slice algebras obtained from a McKay quiver.

**Theorem 5.3.** Let $\Gamma$ be an $n$-slice algebra, and let $\Lambda$ be its quadratic dual. Then the following categories are equivalent as triangulated categories:

1. the bounded derived category $\mathcal{D}^b(\Gamma_N)$ of the finitely generated $\Gamma_N$-modules;
2. the bounded derived category $\mathcal{D}^b(\Lambda_N)$ of the finitely generated $\Lambda_N$-modules;
3. the stable category $\text{grmod}_{-,\tilde{\Lambda}} \tilde{\Lambda}$ of finitely generated graded $\tilde{\Lambda}$-modules with respect to the first grading;
4. the stable category $\text{grmod}_{-\Delta(\Lambda_N)} \Lambda$ of finitely generated graded $\Delta(\Lambda_N)$-modules with respect to the first grading;
5. the stable category $\text{grmod}_{-\Delta(\Gamma_N)} \Gamma$ of finitely generated graded $\Delta(\Gamma_N)$-modules with respect to the first grading;
6. the stable category $\text{mod} \Lambda_N$ of finitely generated $\Lambda_N$-modules;
7. the stable category $\text{mod} \Gamma_N$ of finitely generated $\Gamma_N$-modules.

If $\Gamma$ is of infinite type, then they are also equivalent to the following triangulated category.

8. the bounded derived category $\mathcal{D}^b(\text{qgr}\tilde{\Gamma})$ of the non-commutative projective scheme of $\tilde{\Gamma}$.
Proof. By Propositions 2.6 and 6.5 of [13], we have $\Gamma^N \cong \Lambda^{N,1,op}$ is Koszul. By Proposition 2.5, and Corollary 2.4 of [13], we have that $D^b(\text{mod} \Lambda^N)$ and $D^b(\text{mod} \Lambda^N)$ are equivalent to the orbit categories of $D^b(\text{grmod} \Lambda^N)$ and to the orbit categories of $D^b(\text{grmod} \Lambda^N)$, respectively, using the proof Theorem 2.12.1 of [6] (see the arguments before Theorem 6.7 of [13]). So by Theorem 2.12.6 of [6], $D^b(\text{mod} \Gamma^N)$ and $D^b(\text{mod} \Lambda^N)$ are equivalent as triangulated categories. This proves the equivalence of (1) and (2).

Since $\Lambda^N$ is the Beilinson-Green algebra of $\tilde{\Lambda}$, we have that $\text{grmod}_{\ast -} \tilde{\Lambda}$ and $\text{grmod}_{\ast -} \Delta(\Lambda^N)$, are equivalent as triangulated categories. This gives the equivalence of (3) and (4).

By Corollary 1.2 of [8], we have that $\text{grmod}_{\ast -} \tilde{\Lambda}$ and $\text{grmod}_{\ast -} \Delta(\Lambda^N)$, are equivalent as triangulated categories. This proves the equivalence of (3) and (4). The equivalence of (5) and (1) follows similarly.

The equivalences of (2) and (6) and of (1) and (7) are direct consequence of Theorem II.4.9 of [20].

If $\Gamma$ is of infinite type, then $\tilde{\Gamma}$ is an AS-regular algebra. By Theorem 4.14 of [31], we have that $D^b(\text{qgr} \tilde{\Gamma})$ is equivalent to $D^b(\text{gr} \Gamma^N)$ as triangulated categories. This is the equivalence of (1) and (8) for $n$-slice algebra of infinite type. □

We remark that the equivalence of (1) and (8) can be regarded as a generalization of Beilinson correspondence and the equivalence of (3) and (8) can be regarded as a generalization of Berstein-Gelfand-Gelfand correspondence [4, 5]. So we have the following picture for the equivalences of the triangulated categories in Theorem 5.3 for infinite types:

\[ \cdots \]

(17)

Similar to the above, we also have the following equivalent triangulated categories, using the second grading of $\tilde{\Gamma}$. Now we have the algebras $\hat{\Lambda} = \hat{\Lambda}_{\# -}Z^*$ with the bound quiver $\mathbb{Z}_{n-1}Q$ and $\hat{\Gamma} = \hat{\Gamma}_{\# -}Z^*$ with the bound quiver $\mathbb{Z}_{n-1}Q^\perp$.

**Theorem 5.4.** Let $\Gamma$ be a nicely-graded $n$-slice algebra, and let $\Lambda$ be its quadratic dual. Then the following triangulated categories are equivalent

1. the bounded derived category $D^b(\Gamma)$ of the finitely generated $\Gamma$-modules;
2. the bounded derived category $D^b(\Lambda)$ of the finitely generated $\Lambda$-modules;
3. the stable category $\text{grmod}_{\ast -} \tilde{\Lambda}$ of finitely generated graded $\tilde{\Lambda}$-modules with respect to the second grading;
4. the stable category $\text{grmod}_{\ast -} \Delta(\Gamma)$ of finitely generated graded $\Delta(\Gamma)$-modules with respect to the second grading;
5. the stable category $\text{mod} \hat{\Lambda}$ of finitely generated $\hat{\Lambda}$-modules;
6. the stable category $\text{mod} \hat{\Gamma}$ of finitely generated $\hat{\Gamma}$-modules.
By Theorems 5.3 and 5.4 we can approach the representation theory of a \(n\)-slice algebra from different point of views, if it is of infinite type, such approaches also have a non-commutative interpretation as in (8). In [21], the representation theory of \(n\)-representation-infinite algebra is studied via the category \(D^n(qgr\Gamma)\).

In general, \(\Gamma\) and \(\Gamma^N\) are not well related, see the example 4.7 in [14]. The algebra \(\Gamma^N\) is always nicely-graded. When \(\Gamma\) is nicely-graded, \(\Gamma\) is obtained by a sequence of \(n\)-APR tilts from any indecomposable summand of \(\Gamma^N\). So it is natural to study nicely-graded \(n\)-slice algebras when dealing with the representation theory.

6. Pairs of higher slice algebras of finite type

Assume that \(\Gamma\) is an \(n\)-slice algebra of finite type with Coxeter index \(q + 1\), it is an \(n\)-representation-finite algebra with \((q + 1, n + 1)\)-Koszul \((n + 1)\)-preprojective algebra \(\tilde{\Gamma}\). Let \(\Lambda\) be the quadratic dual of \(\Gamma\) and let \(\hat{\Lambda} = \Delta_{\sigma}\Lambda\) be the twisted trivial extension of \(\Lambda\). Let \(Q = (Q_0, Q_1, \rho)\) be the bound quiver of \(\Lambda\), \(Q^\perp = (Q_0, Q_1, \rho^\perp)\) be the bound quiver of \(\Gamma\), then \(Q\) is an \(n\)-properly-graded quiver and \(Q^\perp\) is an \(n\)-slice. Now we have the returning arrow quiver \(\hat{Q}\) for \(\hat{\Lambda}\), its quadratic dual quiver \(\hat{Q}^\perp\) for \(\hat{\Gamma}\), the infinite quivers \(Z\hat{Q}\) for the smash product \(\hat{\Lambda}^\perp = \hat{\Lambda}\# Z^*\) and \(Z|_{n-1}\hat{Q}\) for the smash product \(\hat{\Lambda} = \hat{\Lambda}\# Z^*\).

We now show that higher slice algebras of finite type appear in pairs.

**Theorem 6.1.** Assume that \(\Gamma\) is an acyclic \(n\)-slice algebra of finite type with Coxeter index \(q + 1\). Then there is a \(q\)-slice algebra \(\Gamma'\) with Coxeter index \(n + 1\) such that their repetitive algebras are quadratic dual.

The \(q\)-slice algebra \(\Gamma'\) is called a companion of \(\Gamma\).

**Proof.** Let \(\Gamma\) be an acyclic \(n\)-slice algebra with bound quiver \(Q^\perp\), and let \(\Lambda\) be the quadratic dual of \(\Gamma\), with bound quiver \(Q\). Then \(\hat{\Lambda}\) and \(\hat{\Lambda}\) are both stable \(n\)-translation algebras with bound quiver the returning arrow quiver \(\hat{Q}\) and the covering quiver \(Z|_{n-1}\hat{Q}\), and both are \((q + 1, n + 1)\)-Koszul. Now consider their respectively quadratic duals \(\hat{\Gamma} = \hat{\Lambda}\) and \(\hat{\Gamma} = \hat{\Lambda}\), their bound quivers are respectively \(\hat{Q}^\perp\) and \(Z|_{n-1}\hat{Q}^\perp\).

By Theorem 6.6 of [13], \(\hat{\Lambda}\) is the \((n + 1)\)-preprojective algebra of an \(n\)-representation-finite algebra \(\Gamma\). So by corollary 3.4 of [24], \(\hat{\Lambda}\) is self-injective, and by Proposition 3.4 of [3], it is \((q + 1, n + 1)\)-Koszul. So by Theorem 5.3 of [12], \(\Lambda\) is also \((q + 1, n + 1)\)-Koszul self-injective. This implies that \(\Lambda\) is a \(q\)-translation algebra and \(Z|_{n-1}\hat{Q}^\perp\) is a \(q\)-translation quiver with \(q\)-translation \(\tau_\perp\).

Now let \(Q'\) be a complete \(\tau_\perp\)-slice of \(Z|_{n-1}\hat{Q}^\perp\), then it is a \(q\)-properly-graded quiver by Lemma 6.1 of [13] and we have that \(Z|_{n-1}\hat{Q}^\perp \simeq Z|_{q-1}Q'\) by Proposition 3.5 of [16]. Let \(\Lambda'\) be the algebra defined by \(Q'\), it is a \(q\)-properly-graded algebra. Similar to the proof of Theorem 6.12 of [14], \(Z|_{n-1}Q^\perp\) is the bound quiver of its repetitive algebra \(\Lambda'\), which is \((q + 1, n + 1)\)-Koszul, so \(\Gamma' = \Lambda'\) is a \(q\)-slice algebra of finite type with Coxeter index \(n + 1\).

This proves the theorem. \(\square\)

The algebra \(\Gamma'\) is not unique in general, but there are only finite many of them up to isomorphism. In fact, let \(Q'\) be a complete \(\tau_\perp\)-slice of \(Z|_{n-1}Q^\perp\), then any complete \(\tau_\perp\)-slice \(Q''\) of \(Z|_{n-1}Q^\perp\) has the same number, say \(r\), of vertices as \(Q'\). Since \(\tau_\perp\) is an automorphism of \(Z|_{n-1}Q^\perp\), for any complete \(\tau_\perp\)-slice \(Q''\) of \(Z|_{n-1}Q^\perp\), by shifting with \(\tau_\perp^r\) if necessary, has a common vertex with \(Q'\). Thus for any vertex.
i of $Q'$, there is either a path of length no longer than $2r$ from some vertex $j$ in $Q'$ to $i$, or a path of length no longer than $2r$ from $i$ to some vertex $j$ in $Q'$. So the vertex set of the non-isomorphic complete $\tau_\perp$-slices of $\mathbb{Z}_{|n-1}Q^\perp$ can be chosen from the set $V$ of vertices of the paths of length $2r$ starting or ending at a vertex in $Q'$, and $V$ is a finite set. This tells us that there are only finite many non-isomorphic complete $\tau_\perp$-slices of $\mathbb{Z}_{|n-1}Q$.

As a corollary of Theorem 6.1, we have the following result on their bound quivers from its proof.

**Corollary 6.2.** Let $Q^\perp$ be the bound quiver of an $n$-slice algebra $\Gamma$ of finite type with Coxeter index $q+1$ and let $Q'^\perp$ be the bound quiver of its companion $\Gamma'$. Then $\mathbb{Z}_{n-1}Q \simeq \mathbb{Z}_{q-1}Q'^\perp$

Now assume that $\Gamma$ is nicely-graded and connected, then $\mathbb{Z}_{n-1}Q$ is a connected component of $\mathbb{Z}_Q$, while $\mathbb{Z}_{|q-1}Q'$ is a connected component of $\mathbb{Z}_Q$. Thus the connected components of $\mathbb{Z}_Q$ and of $\mathbb{Z}_Q'$ are all the same as quivers.

We see from the proof Theorem 6.1 that $Q^\perp$ is a $q$-translation quiver with $q$-translation $\tau_\perpQ$. Write the $n$-translation of the $n$-translation quiver $\mathbb{Z}_Q$ as $\tau$ and the $q$-translation of the $q$-translation quiver $\mathbb{Z}_Q'\perp$ as $\tau_\perp$. With the index of the quiver $\mathbb{Z}_Q$, we have that $\tau(i, s) = (\tau i, s - n - 1)$ and $\tau_\perp(i, s) = (\tau_\perp i, s - q - 1)$, by our construction.

Now we study higher translation quivers related to a pair of higher slice algebras of finite type.

A quadratic quiver $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, \overline{p})$ is called a double translation quiver of type $(n, q)$ if $\overline{Q}$ is a stable $n$-translation quiver and $\overline{Q}^\perp$ is a stable $q$-translation quiver. We obviously have the following proposition, restates Corollary 6.2.

**Proposition 6.3.** Let $\Gamma$ be an $n$-slice algebra of finite type with Coxeter index $q+1$ with bound quiver $Q^\perp$. Then $\mathbb{Z}_{|n-1}Q$ is a double translation quiver of type $(n, q)$.

If $\overline{Q}$ is a double translation quiver of type $(n, q)$, the algebra $\overline{\mathbb{A}}$ defined by the bound quiver $\overline{Q}$ is an $n$-translation algebra and the algebra $\overline{\mathbb{A}}^\perp$ defined by the bound quiver $\overline{Q}^\perp$ is a stable $q$-translation algebra. So the maximal bound paths of $\overline{Q}$ define an $n$-translation $\tau$ on $\overline{Q}$ and the maximal bound paths of $\overline{Q}^\perp$ define an $q$-translation $\tau_\perp$. Both defines permutations on the vertex set of $\overline{Q}$, and each of them induces an automorphism on the quiver $\overline{Q}$. We obviously have the following commutative relation for them.

**Lemma 6.4.**

$$\tau_\perp \tau = \tau \tau_\perp.$$

Let $Q$ be a complete $\tau$-slice in $\overline{Q}$, then $\overline{Q} = \mathbb{Z}_{|n-1}Q$. So all the double translation quivers of type $(n, q)$ are of this form. As a corollary, we get the following result.

**Lemma 6.5.** If $Q$ is a complete $\tau$-slice of $\overline{Q} = \mathbb{Z}_{|n-1}Q$, then $\tau^t Q$ is a complete $\tau$-slice for all the integer $t$.

If $Q'$ is a complete $\tau_\perp$-slice of $\mathbb{Z}_{|n-1}Q^\perp$, then $\tau^t Q'$ is a complete $\tau_\perp$-slice for all the integer $t$.

As a quiver, we have $\mathbb{Z}_{|n-1}Q = \mathbb{Z}_{|q-1}Q'$. 
A full convex subquiver $D$ of a double translation quiver $\overline{Q}$ of type $(n, q)$ is called a **double slice** if it is maximal with the property that for any vertex $\bar{i}$ in $D$, at most one of $\tau^{-1}\bar{i}$ and $\tau_{-1}\bar{i}$ is in $D$.

We remark that if $\bar{i}$, $\tau^{-1}\bar{i} \in D$ and there is a bound path of length $q + 1$ from $\tau^{-1}\bar{i}$ to $\bar{j}$ in $\overline{Q}^-$, then $\bar{j} \notin D$. If not we have $\tau_+ \bar{j} = \bar{i}$ since $\tau_+ \bar{j} = \tau^{-1}\bar{i}$ and thus $\tau \bar{j} \in D$, which contradicts the definition of double slice.

We remark that a double slice is an maximal $\tau$-mature bound subquiver, in the term of $\tau$-maturity defined in [19].

If $\Gamma$ is an $n$-slice algebra of Coxeter index $q + 1$ with bound quiver $Q^\perp$, then $\mathbb{Z}_0 \overline{Q}$ is a double translation quiver of type $(n, q)$, and $\mathbb{Z}_0 \overline{Q}[h, h + n + q + 1]$ is a double slice for any $h$.

Now let $\overline{Q}$ be nicely-graded stable double translation quiver of type $(n, q)$, with $n$-translation $\tau$ and $q$-translation $\tau_\perp$. If $S$ is a complete $\tau$-slice in $\overline{Q}$, let $D(S^\perp)$ be the full subquiver formed by $S$ and all the $\tau_\perp$-hammocks $H^\perp$ starting at each vertex $\bar{i}$ of $S$, let $D(-S)$ be the full subquiver formed by $S$ and all the $\tau_\perp$-hammocks $H^\perp_1$ ending at each vertex $\bar{i}$ of $S$. In $D(S^\perp)$, we have two complete $\tau$-slices, $S$ and $\tau_{-1}S$, and in $D(-S)$, we have two $\tau$-slices, $S$ and $\tau_\perp S$. They are the union of $\tau_\perp$-hammocks connecting the vertices these two $\tau$-slices. Note that each $\tau$-hammock is convex, so they are also convex.

**Proposition 6.6.** $D(S^\perp)$ and $D(-S)$ are convex.

We clearly have that $D(S^\perp) = D(-\tau_{-1}S)$. Write $-\tau_{\perp}S$ for the full subquiver of $D(S^\perp)$ with vertex set $D(S^\perp) \setminus S_0$, and write $\tau_{\perp}S$ for the full subquiver of $D(-S)$ with vertex set $D(-S) \setminus S_0$.

**Proposition 6.7.** The full subquiver $D(S^\perp)$ is formed by a complete $\tau$-slice $S$, and a $\tau_{\perp}$-slice $-\tau_{\perp}S$, connected with arrows from non-source vertices of $S$ to non-sink vertices $-\tau_{\perp}S$.

The full subquiver $D(-S)$ is formed by a complete $\tau$-slice $S$, and a complete $\tau_{\perp}$-slice $\tau_{\perp}S$, connected with arrows from non-source vertices of $\tau_{\perp}S$ to non-sink vertices $S$.

**Proof.** We prove the first assertion, the second follows dually.

We prove that $-\tau_{\perp}S$ forms a $\tau_{\perp}$-slice in $\mathbb{Z}_{n-1}Q^\perp$. If there is a vertex $\bar{j}$ such that the $\tau_{\perp}$-orbit of $\bar{j}$ has no intersection with $-\tau_{\perp}S$, we may choose $\bar{j}$ such that there is an arrow $\alpha : \bar{j} \rightarrow \bar{i}$ such that $\bar{i}$ is in $-\tau_{\perp}S$. But this force $\bar{j}$ in $S$, so $\tau_{-1}\bar{j}$ is in $-\tau_{\perp}S$, this is a contradiction. So $-\tau_{\perp}S$ intersect each $\tau_{\perp}$-orbit by the construction, $-\tau_{\perp}S$ intersects each $\tau_{\perp}$-orbit at most once.

Now $-\tau_{\perp}S$ is obtained as a union of $\tau_{\perp}$-hammocks with their sources removed, it is convex since each of these source removed $\tau_{\perp}$-hammocks are convex, and all the removed sources form a convex set which preceding $-\tau_{\perp}S$.

This proves that $-\tau_{\perp}S$ forms a $\tau_{\perp}$-slice in $\mathbb{Z}_{n-1}Q^\perp$. □

**Proposition 6.8.** $D(S^\perp)$ and $D(-S)$ are double slices.

**Proof.** We prove that $D(S^\perp)$ is a double slice. The other assertion follows from duality.

For each vertex $\bar{i}$ in $D(S^\perp)$, if $\bar{i}$ and $\tau_{-1}\bar{i}$ are in $D(S^\perp)$, then by the construction $\bar{i}$ is in $S$, so $\tau \bar{i}$ is not $S$ and hence not in $D(S^\perp)$. 

Now $\tau^{-1}_j S$ also forms a complete $\tau$-slice isomorphic to $S$. If $\bar{j}$ is a vertex not in $D(S+)$ such that there is an arrow from $\bar{j}$ to a vertex $\bar{i}'$ in $S$, then we have $\tau^{-1}_j \bar{j}$ is in $S$ and $\tau^{-1}_j \tau^{-1}_i \bar{j}$ in $D(S+)$. If $\bar{j}$ is a vertex not in $D(S+)$ such that there is an arrow from $\bar{i}'$ in $\tau^{-1}_j S$ to $\bar{j}$, then we have $\tau_j \bar{j}$ in $\tau^{-1}_j S$ and $\tau_1 \tau_j \bar{j}$ in $S$.

This shows that $D(S+)$ is maximal, and hence a double slice. 

Note that the vertex set of $\tau(-S_-)$ is also in $D(S)$ and it is a complete $\tau_-S$-slice in $\mathbb{Z}_{n-1}Q^\perp$. Regard as a double translation quiver of type $(q,n)$, we also construct the double slice $D(\tau(-S_-))$ and $D(\tau^{-1}(S_+))$ in $\mathbb{Z}_{n-1}Q^\perp$, and obviously have the following.

**Proposition 6.9.** $D(\tau(-S_-)) = D(S+)^\perp$, and $D(\tau^{-1}(S_+)) = D(-S)^\perp$.

Let $Q^\perp$ be the bound quiver of the $n$-slice algebra $R$, then $\mathbb{Z}_{n-1}Q$ is a double translation quiver of type $(n,q)$, and $S = Q$ is a $\tau$-slice. The algebra $R^{-1}$ defined by $\tau(-S_-)$ and $R^{\perp-1}$ defined by $\tau^{-1}(S_+)$ are companions of $R$. Called the right and left companion of $R$, respectively. We usually take the right companion of $R$ as default one and write it as $I^c$.

Recall that the complete $\tau$-slices are related by the $\tau$-mutations defined as follow. If $i$ is a sink of a complete $\tau$-slice $S$, we define the $\tau$-mutation $s_i S$ of $S$ at $i$ as the full bound subquiver of $\overline{Q}$ obtained by replacing the vertex $i$ by its $n$-translation $\tau i$. If $i$ is a source of a complete $\tau$-slices $S$, define the $\tau$-mutation $s^\perp_i S$ of $S$ at $i$ as the full bound subquiver in $\overline{Q}$ obtained by replacing the vertex $i$ by its inverse $n$-translation $\tau^{-1} i$.

Double slices are also related by mutations. The mutation $s^\perp_i D$ of double slice $D$ with respect to a source $i$ is obtained by removing the vertex $i$ and add the vertex $\tau^{-1}_i s_i D$ and the arrows to $\tau^{-1}_i \tau^{-1} i$ in $\overline{Q}$; the mutation $s_i D$ of double slice $D$ with respect to a sink $i$ is obtained by removing the vertex $i$ and add the vertex $\tau \tau^{-1}_i i$ and the arrows from $\tau \tau^{-1}_i i$ in $\overline{Q}$.

**Proposition 6.10.** The $\tau$-mutation $s^\perp_i$ on $S$ induces a mutation $\bar{s}^\perp_i$ on $D(S+)$; The $\tau$-mutation $s_i$ on $S$ induces a mutation $\bar{s}_i$ on $D(-S)$.

**Proof.** We prove the first assertion, the second follows from duality.

By apply $\tau$-mutation $s^\perp_i S$ on $S$ to get $s^\perp_i S$, we remove the source $i$ from $S$ and add $\tau^{-1}_i$ as a sink of $s^\perp_i S$.

Since $n, q \geq 2$, if there are arrow $i \rightarrow j, s = 1, \ldots, h$ for some $h$, then $j$ is in $S$ and all the vertices of the $\tau_1$-hammocks $H^j$ except $i$ is contained the union of these $H^j$. Similarly, for the arrows $j' \rightarrow \tau^{-1}_i i$, all the vertices of the $\tau_1$-hammocks $H^{\tau^{-1}_i}$ except $\tau^{-1}_i \tau^{-1} \bar{i}$ are contained the union of these $H^{\tau^{-1}_i}$. This proves $s^\perp D(S+) = D(s^\perp S+).$ 

By Lemma 6.5 of [11] and the above Proposition, we have the following result.

**Proposition 6.11.** Let $D$ and $D'$ be two double slice of a stable double translation quiver. Then there is a sequences of mutations $s_1, \ldots, s_l$ such that $s_1 \cdots s_l D = D'$.

Now assume that $Q$ is nicely-graded and connected, then $\mathbb{Z}_{n-1}Q$ and $\mathbb{Z}_{n-1}Q^\perp$ are both nicely graded and connected. So these quivers are connected component of $\mathbb{Z}_n Q$. Using the index of the vertices of $\mathbb{Z}_n Q$ for the vertices of $\mathbb{Z}_{n-1}Q$ and $\mathbb{Z}_{n-1}Q^\perp$. Now for any $t$, $\mathbb{Z}_n Q[t, t + q + n + 1]$ is a double slice of $\mathbb{Z}_n Q$, and we can take $D^0 = \mathbb{Z}_n Q[-n, q + 1]$, then any double slice can be obtained from $D^0$ by a sequences of mutations by Proposition 6.11.
7. The Higher Preprojective Components

Now we study the representations of the pair of higher slice algebras of finite type, showing that they share the same quiver as the Auslander-Reiten quiver for their higher preprojective components.

For a finite dimensional algebra $\Gamma$ of global dimension $n$, the $n$-Auslander-Reiten translations of $\Gamma$ modules are introduced by Iyama (see [22] [23]),

$$\tau_n^{-1} = \text{Tr } D\Omega^{1-n} = \text{Ext}_n^\Gamma(D-, \Gamma) \quad \text{and} \quad \tau_n = D\text{Tr } \Omega^{n-1} = D\text{Ext}_n^\Gamma(-, \Gamma),$$

with the convention that $\tau_0 = \tau_n^{-1} = 1$.

Modules in

$$\mathcal{M}^+(\Gamma) = \{\tau_n^t \Gamma | t \geq 0\} \quad \text{and} \quad \mathcal{M}^-(\Gamma) = \{\tau_n^{-t} \Gamma | t \geq 0\}$$

are called $n$-preinjective modules of $\Gamma$ and $n$-preprojective modules of $\Gamma$, respectively (see [21]). These modules are natural generalization of preinjective and preprojective modules, respectively, thus these classes are called $n$-preprojective component and $n$-preinjective component, respectively.

Now assume that $\Gamma$ is an acyclic $n$-slice algebra of finite type with bound quiver $Q$, then $\Gamma$ is $n$-representation-finite. By Proposition 1.12 of [25], we have that $\mathcal{M}^+(\Gamma) = \mathcal{M}^-(\Gamma)$, which we write as $\mathcal{M}(\Gamma)$. By Proposition 2.3 of [26], $\mathcal{M}(\Gamma)$ is an $n$-cluster tilting subcategory and by Theorem 3.3.1 of [22], $\mathcal{M}(\Gamma)$ has $n$-almost split sequences.

Assume that $\Gamma$ is nicely-graded and connected, so $Q$ is nicely-graded connected quiver and each connected components of $\mathbb{Z}_nQ$ is isomorphic to $\mathbb{Z}_{n-1}Q$, which is an $n$-translation quiver, write its $n$-translation as $\tau$.

Let $\overline{\Lambda}$ be the algebra defined by the bound quiver $\mathbb{Z}_{n-1}Q$ and let $\overline{T}$ be the algebra defined by the bound quiver $\mathbb{Z}_{n-1}Q^\perp$. Since $\overline{\Lambda}$ is an $n$-translation algebra, $\overline{\Lambda}$ is an $n$-translation algebra with Coxeter index $q+1$, by Theorem 5.3 of [12]. So $\overline{T}$ is $(q+1, n+1)$-Koszul. By Corollary [22], $\mathbb{Z}_{n-1}Q^\perp$ is a $q$-translation quiver, write $\tau_e$ for its $q$-translation. Identify $\overline{Q}$ with the complete $\tau$-slice $\mathbb{Z}_{n-1}Q[0]$ of $\mathbb{Z}_{n-1}Q$ and write $i$ for the vertex $(i,0)$ in $Q$ when needed.

Let $d$ be a function $d: \mathbb{Z}_{n-1}Q \to \mathbb{Z}$ such that $d(j, t') = d(i, t) + 1$ if there is an arrow from $(i, t)$ to $(j, t')$, we may assume that $\max\{d(i) | i \in Q_0\} = 0$. Fix a vertex $(i, t)$ in $\mathbb{Z}_{n-1}Q$, set

$$\varphi_{(i, t)}(j, t') = (t - t')(n + 1) + d(i) - d(j).$$

Since $\mathbb{Z}_{n-1}Q$ is nicely-graded, a $\tau$-hammock $H_{(i, t)}$ in $\mathbb{Z}_{n-1}Q$ ending at a vertex $(i, t)$ is identified with the full subquiver of $\mathbb{Z}_{n-1}Q$ with vertex set

$$H_{(i, t), 0} = \{(j, t') | (j, t') \in \mathbb{Z}_{n-1}Q_0, \exists \text{ path } 0 \neq p \in \overline{\Lambda}_{(i, t)\rightarrow (j, t')}, s(p) = (j, t'), t(p) = (i, t)\}.$$

The hammock function $\mu_{(i, t)}: H_{(i, t), 0} \to \mathbb{Z}$ is an integral map on the vertices defined by

$$\mu_{(i, t)}(j, t') = \dim_k e_{(i, t)} \overline{\Lambda}_{(i, t)\rightarrow (j, t')} e_{(j, t')}.$$

For each $(i, t)$ in $\mathbb{Z}_{n-1}Q$, we have a Koszul complex

$$\mathcal{M}_{(i, t)}: M_{n+1} = \overline{\Lambda} e_{(i, t)} \xrightarrow{f_{n+1}} \cdots \xrightarrow{f_1} M_e \rightarrow \cdots \xrightarrow{f_2} M_0 = \overline{\Lambda} e_{(i, t-1)} \quad (18)$$
in add $\Gamma$, with

$$ M_r = \bigoplus_{(j,r) \in \Lambda, (i,t) \in D} (\Gamma e_{(j,t)})^{\mu_{(i,r)}}(j,t) $$

for $0 \leq r \leq n + 1$. This is the projective resolution of the simple $\Gamma$-module $\Gamma_0 e_{(i,t-1)}$ (Proposition 2.1 of [16], see also Proposition 7.4 of [12]). Since $\Gamma$ is $(q + 1, n + 1)$-Koszul algebra, we have that

$$ \text{Ker } f_{n+1} \simeq \Gamma_0 e_{\tau_i^{-1}(i,t)} $$

the simple $\Gamma$-module $\Gamma e_{\tau_i^{-1}(i,t)}$ corresponding to the vertex $\tau_i^{-1}(i,t)$.

It is obvious that, regarding as a bound quiver, each convex full subquiver $T$ of $\mathbb{Z}_n Q$ is an $n$-translation quiver and it defines an $n$-translation algebra $\Lambda(T)$ with Coxeter number $q + 1$ (so it is an $(n + 1, q + 1)$-algebra). Write its quadratic dual as $\Gamma(T)$. Then

$$ \Lambda(T) = e_T \bar{X} e_T \text{ and } \Gamma(T) = e_T \Gamma e_T $$

for the idempotent $e_T = \sum_{(i,t) \in T} e_{(i,t)}$. Let $T$ be a convex full subquiver with $Q$ a terminal complete $\tau$-slice, then add $\Gamma$ is full subcategory of

$$ \mathcal{G}(T) = \text{add } \Gamma(T) \simeq \text{add } \{ \Gamma e_{(i,t)} | (i,t) \in T_0 \}. $$

By Lemma 7.1 of [12], we have the following Lemma.

**Lemma 7.1.** If $H_{i,t}$ is in $T$, then $e_T \mathcal{M}_{i,t}$ is an $n$-almost split sequence in $\mathcal{G}(T)$ if and only if $\text{Ker } f_{n+1}|T = 0$.

Let $D = D(-Q)$ be the double slice with $Q$ a terminal complete $\tau$-slice, then $D^{op}$ is a double slice in $\mathbb{Z}_n Q^{op}$. So we have that $Q^{op}$ is embedded into $D^{op}$ as an initial complete $\tau$-slice.

Let $\mathcal{G}(D)$ be the bound path category defined by the bound quiver $D$ with relation set the restriction of $\rho_{\mathbb{Z}_n Q}$. We have that

$$ \mathcal{G}(D) \simeq \text{add } \{ \Gamma e_{(j,r)} | (j,r) \in D_0 \} \simeq \text{add } \{ e_D \Gamma e_{(j,r)} | (j,r) \in D_0 \} $$

by Proposition 2.6 of [16]. Since $D$ is a double slice, if $(i, -t)$ and $(i, -t - 1)$, are in $D$, then $\tau_i^{-1}(i, -t)$ is not in $D$ and $\text{Ker } f_{n+1}(j=0 \in e_D \mathcal{M}_{i, -t})$, by Lemma 7.1 and Theorem 7.5 of [12], we have the following Lemma.

**Lemma 7.2.** $\mathcal{G}(D)$ has an $n$-almost split sequence.

If both $(i, -t)$ and $(i, -t - 1)$ are in $D$, then $e_D \mathcal{M}_{(i, -t)}$ is an $n$-almost split sequence in $\mathcal{G}(D)$.

Let $e = e_Q$, then $\Gamma = e \Gamma e$. Write $Q(\mathcal{M}(\Gamma))$ for the full subquiver of $\mathbb{Z}_n Q^{op}$ with vertex set

$$ Q_0(\mathcal{M}(\Gamma)) = \{(j, -r) \in \mathbb{Z}_n Q^{op} | \tau_{n-r}^r \Gamma e_j \neq 0 \} $$

for $j \in Q_0^{op}$.

**Lemma 7.3.** The bound subquiver $Q(\mathcal{P}(\Gamma))$ of $\mathbb{Z}_n Q^{op}$ is convex.

**Proof.** Write $Q(\mathcal{P}(\Gamma)) = Q$ for the full subquiver of $Q(\mathcal{M}(\Gamma))$ with vertex set

$$ Q_0(\mathcal{P}(\Gamma)) = \{(j, 0) \in \mathbb{Z}_n Q_0^{op} | j \in Q_0^{op} \} $$

and $Q(\mathcal{I}(\Gamma))$ for the full subquiver of $Q(\mathcal{M}(\Gamma))$ with vertex set

$$ Q_0(\mathcal{I}(\Gamma)) = \{(j, -r) \in Q_0(\mathcal{M}(\Gamma)) | \tau_{n-r}^r \Gamma e_j = 0 \}. $$
By Proposition 1.12 of [25], $Q(P(\Gamma))$ and $Q(I(\Gamma))$ are isomorphic, and they form complete $\tau$-slices. By definition, the vertices in $Q(M(\Gamma))$ are exactly vertices lying between $Q(P(\Gamma))$ and $Q(Z(\Gamma))$, so $Q(M(\Gamma))$ is convex. □

The vertices of $\mathbb{Z}|_{n-1}Q$ is totally ordered such that $(i, t) < (j, t')$ if $t < t'$, or $t = t'$ and $d(i) < d(j)$. For a convex subquiver $T$, write $T(i, t)$ the subquiver of $T$ consisting of the vertices $(j, t')$ with $(j, t') > (i, t)$, and $T[i, t]$ the subquiver of $T$ consisting of the vertices $(j, t')$ with $(j, t') \geq (i, t)$. Let $(i_0, 0)$ be the element in $Q$ which is maximal with respect to this order.

Now we show that the subquivers $Q(M(\Gamma))$ and $D^{op}$ are the same.

**Lemma 7.4.** $D^{op} = Q(M(\Gamma))$

**Proof.** We prove that they have the same vertex set, that is $D_0 = Q_0(M(\Gamma))$.

Note that $G(D)$ is an Orlov category, so $\text{Hom}_{G(D)}(\tau e_{(i, t)}, \bar{\tau} e_{(j, t')}) = 0$ whenever $(j, t') > (i, t)$. By Proposition 5.3 of [16], $M(\Gamma)$ is an Orlov category, with respect to the order defined above. Set

$$G(D)(j, -r) = \text{add}\{\tau_1 e_{(i, t)} | (i, t) \in D_0(j, -r)\},$$

$$G(D)[j, -r] = \text{add}\{\tau_1 e_{(i, t)} | (i, t) \in D_0(j, -r)\},$$

and

$$M(\Gamma)(j, -r) = \text{add}\{\tau_1^{-1} e_{(i, t)} | (i, t) \in Q_0(M(\Gamma))(j, -r)\}.$$

Then

$$D^{op}(i_0, -1) = Q^{op} = Q(M(\Gamma))(i_0, -1),$$

$$G(D)(i_0, -1) \simeq M(\Gamma)(i_0, -1) \simeq \text{add}\Gamma,$$

and we have for each $j \in Q_0$,

$$e\bar{\tau} e_{(j, 0)} = \Gamma e_j.$$

Now assume for a vertex $(i, -r - 1)$,

$$G(D)(i, -r - 1) \simeq M(\Gamma)(i, -r - 1)$$

and for each $(j, -t) \in D_0(i, -r - 1)$,

$$e\bar{\tau} e_{(j, -t)} \simeq \tau_1^{-r - 1} \Gamma e_j.$$

If $(i, -r), (i, -r - 1) \in D_0$, then $eD M_{(i, -r)}$ is an $n$-almost split sequence in $G(D)$, by Lemma [22]. So $eD M_{(i, -r)}$ is a sink sequence in $G(D)[i, -r - 1]$, hence also a sink sequence in $M(\Gamma)[i, -r - 1]$. Since $M(\Gamma)$ is Orlov, it is a sink sequence in $M(\Gamma)$.

By Proposition 3.3 of [22], $eD M_{(i, -r)}$ is an $n$-almost split sequence in $M(\Gamma)$ and by Theorem 3.3.1 of [22],

$$e\bar{\tau} e_{(i, -r - 1)} \simeq \tau_1^{-r - 1} \Gamma e_{(i, -r)} \simeq \tau_1^{-r - 1} \Gamma e_i.$$

So $G(D)[i, -r - 1]$ is equivalent to $M(\Gamma)[i, -r - 1]$. Inductively, we see

$$D_0 \subset Q_0(M(\Gamma))$$

and

$$G(D) \simeq \text{add}\{\tau_1 e_{(i, t)} | (i, t) \in D_0\}$$

is regarded as a full subcategory of

$$M(\Gamma) \simeq \text{add}\{\tau_1 e_{(i, t)} | (i, t) \in Q_0(M(\Gamma))\}.$$

If $(i, -r - 1) \notin D_0$, we have both $\tau_1^{-1} (i, -r - 1) = (i, -r)$ and $\tau_1 (i, -r) = (i', -t')$ are in $D$, since $D$ is double slice.
Since $M_{h(-r)}$ is a sink sequence in $\text{add } \Gamma$, it is a sink sequence in $\text{add } \{G(D), \Gamma e_{(i,-r-1)}\}$, since $\text{add } \Gamma$ is Orlov.

If $(i, -r-1) \in Q_0(\mathcal{M}(\Gamma))$ then $\tau_{n-r-1} \Gamma e_i \neq 0$, $eM_{h(-r)}$ is a sink sequence and hence an $n$-almost split sequence in $\mathcal{M}(\Gamma)$, by Proposition 3.3 of [22] and 

$$e\Gamma e_{(i,-r-1)} \cong \tau_{n-r-1} \Gamma e_i.$$

So $(e_D + e_{(i,-r-1)})M_{h(-r)}$ is also a source sequence in $\mathcal{G}(D \cup (i, -r-1))$, where $D \cup (i, -r-1)$ is the full bound subquiver of $Z|_{n-1}Q$ with vertex set $D_0 \cup \{(i, -r-1)\}$. But as the complex of $\Gamma(D \cup (i, -r-1))$, $\text{Ker } f_{n+1} \mid \text{R}_D(i, -r-1)) \cong \Gamma_0 e_{\tau_{n-r-1}}(i, -r)$ which is not zero since $\tau_{n-r-1}(i, -r)$ is in $D$. This contradicts Lemma 7.1.

So $(i, -r-1) \notin Q_0(\mathcal{M}(\Gamma))$. This proves $D^{\text{op}} = Q(\mathcal{M}(\Gamma))$. \hfill $\Box$

As a corollary of Lemma 7.4 we obtain the following theorem.

**Lemma 7.5.** We have the equivalence $\mathcal{G}(D) \cong \mathcal{M}(\Gamma)$.

**Remark.** From the proof of Lemma 7.4, we see the assumption of “$r$-mature” in Theorem 5.6 of [16] can be removed.

As an immediate consequence of Lemma 7.5, we have the following result on the Auslander-Reiten quiver of an $n$-slice algebra of finite type.

**Theorem 7.6.** Let $\Gamma$ be an $n$-slice algebra of finite type with nicely-graded bound quiver $Q^\perp$. Then the Auslander-Reiten quiver of its $n$-preprojective component of $\Gamma$ is the opposite quiver of a double slice in $Z|_{n-1}Q$.

More precise, we have

**Theorem 7.7.** Let $\Gamma$ be an $n$-slice algebra of finite type with nicely-graded bound quiver $Q^\perp$ and $q + 1$ as its Coxeter index and let $Q$ be the Auslander-Reiten quiver of its $n$-preprojective modules. Then

1. There is a $q$-slice algebra $\Gamma^c$ of finite type such that $Q$ is the Auslander-Reiten quiver of the $q$-preprojective component of $\Gamma^c$.
2. $Q = D(Q)^{\text{op}}$.
3. As a quiver $Q$ is the opposite quiver of a quiver obtained by connecting the quiver $Q^\perp$ of $\Gamma$ and the quiver $Q^{\perp \perp}$ of $\Gamma^c$ by some arrows.

Note that the category of $n$-preprojective modules of an $n$-representation-finite algebra is an $n$-cluster tilted subcategory Proposition 2.6 of [22]. We have the following result.

**Theorem 7.8.** Let $\Gamma$ be an $n$-slice algebra of finite type with Coxeter index $q + 1$, and let $\Gamma^c$ be its companion. Then the $n$-cluster tilted subcategory of $\Gamma$ and the $q$-cluster tilted subcategory of $\Gamma^c$ are quadratic duals.

As a corollaries, we have the following results for hereditary algebras, using Corollary 4.3 of [23].

**Corollary 7.9.** Let $\Gamma$ be an hereditary algebra of finite type and let $h$ be Coxeter number of its quiver. Then there is an $(h-3)$-slice algebra $\Gamma^c$ of finite type such that the module category of $\Gamma$ is the quadratic dual of the $(h-3)$-cluster tilted subcategory of $\Gamma^c$-modules.
Using the higher Auslander algebra introduced in [24], one gets the following Corollary.

**Corollary 7.10.** For each \( n \)-Auslander algebra \( A \) of an \( n \)-slice algebra of finite type, there exist a \( q \) and a \( q \)-Auslander algebra \( A^c \) of a \( q \)-slice algebra of finite type such that \( A \) and \( A^c \) are quadratic dual one another.

It is interesting to know if Theorem 7.8 holds for higher cluster tilting subcategories of finite type in general?

Higher slice algebras of finite type appear in pairs, the following is an example.

**Example 7.11.** The following example was presented in [18] to illustrates the \( \tau \)-hammocks and \( \tau \)-mutations for \( n \)-slice algebra of finite type. Now we consider its dual higher slice algebra. The Auslander algebra \( \Gamma \) of the path algebra of type \( A_3 \) with linear orientation, is a 2-representation-finite algebra, given by the quiver \( Q \):

\[
\begin{array}{c}
6 \circ \quad 5 \circ \quad 3 \circ \\
\quad \downarrow \quad \quad \downarrow \\
4 \circ \quad 2 \circ \\
\quad \downarrow \\
1 \circ 
\end{array}
\]

with \( \alpha_i \) for the vertical arrow ending at vertex \( i \), and \( \beta \) for the horizontal arrow ending at vertex \( i \). The the relation set \( \rho = \{ \alpha \alpha, \beta \beta, \alpha \beta - \beta \alpha \} \) and \( \rho^\perp = \{ \alpha \beta + \beta \alpha \} \cup \{ \alpha_1 \beta_2, \alpha_4 \beta_5 \} \). The returning arrow quiver \( \tilde{Q} \),

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \\
\quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\circ \quad \circ \quad \circ \\
\quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\circ \quad \circ \quad \circ 
\end{array}
\]

is obtained from \( Q \) by adding the arrow \( \gamma_i \) going up left to vertex \( i \). The relation set \( \tilde{\rho} = \{ \alpha \alpha, \beta \beta, \gamma \gamma, \alpha \beta - \beta \alpha, \alpha \gamma - \gamma \alpha, \beta \gamma - \gamma \beta \} \), and its dual relation set \( \tilde{\rho}^\perp = \{ \alpha \beta + \beta \alpha, \alpha \gamma + \gamma \alpha, \beta \gamma + \gamma \beta \} \cup \{ \alpha_1 \beta_2, \alpha_4 \beta_5, \beta_2 \gamma_4, \beta_3 \gamma_5, \gamma_5 \alpha_2, \gamma_6 \alpha_4 \} \). For each rhombus formed by two paths of length 2, we have commutative relations in both relation sets. We have zero relations for paths of length 2 of the same type (going in the same direction in our presentation) in \( \tilde{\rho} \), while zero relations in \( \tilde{\rho}^\perp \) are the complements in the cyclic paths of length 3 of the arrows \( \alpha_1, \alpha_2, \beta_3, \beta_5, \gamma_4, \gamma_6 \).

With the relation set \( \tilde{\rho} \), \( \tilde{Q} \) becomes a 2-translation quiver with the trivial translation, and with quadratic dual relation \( \tilde{\rho}^\perp \), \( \tilde{Q} \) becomes a 1-translation quiver with translation defined by \( \tau 1 = 3, \tau 2 = 5, \tau 3 = 6, \tau 4 = 2, \tau 5 = 4, \tau 6 = 1 \).

The 3 connected components of \( \mathbb{Z}_6 \tilde{Q} \) look the same, they are isomorphic to the quiver \( \mathbb{Z} \tilde{Q} \) as follows.
The \( \tau \)-hammocks are as follows.

The homogeneous 2-slice quivers of finite type obtained by take connected component in \( \mathbb{Z}_6\hat{Q}^1 \):

\[
\begin{array}{ccc}
  & S_1 & S_2 & S_3 \\
\end{array}
\]

They all produce isomorphic algebras. Other 2-slice quivers can be obtained by taking \( \tau \)-mutations on homogeneous ones with \( S_4 = s^{(2,0)}S_1 \), \( S_5 = s^{(1,0)}S_2 \) and \( S_6 = s^{(1,1)}s^{(2,0)}S_1 \).
The $\tau_\perp$-hammocks are as follows.

The dual one is a 1-slice algebra of finite type. Here are the homogeneous 1-slice quivers of finite type:

They are isomorphic 1-slices in $\mathcal{Q}$. Using $\tau_\perp$-mutation, we get all the 1-slice quivers in $\mathcal{Q}$. The non-isomorphic ones are $T_4 = s^{(6,0)}T_1$, $T_5 = s^{(2,0)}T_1$ and $T_6 = s^{(1,1)}s^{(2,0)}T_1$.

Double slices are as following.
Double slices with respect to $\tau_{\bot}$.
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ON \( n \)-HEREDITARY ALGEBRAS AND \( n \)-SLICE ALGEBRAS

REFERENCES

[1] C. Amiot, Preprojective algebras and Calabi-Yau duality. preprint arXiv:1404.4764.
[2] M. Artin, and J. Zhang, Noncommutative Projective Schemes, Adv. in Math., 109 (1994), 228-287.
[3] S. Brenner, M. C. R. Butler A. D. King, Periodic algebras which are almost Koszul. Algebr. Represent. Theory 5 (4) (2002) 331–367.
[4] A. A. Beilinson, Coherent sheaves on \( \mathbb{P}^n \) and problems of linear algebra. Funct. Anal. Appl. 12 (1978), 214–216.
[5] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, Algebraic vector bundles on \( \mathbb{P}^n \) and problems of linear algebras, Funct and its Appl., 12 (1978), 212–214.
[6] A. Beilinson, V. Ginsberg, W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996) 473–527.
[7] S. Brenner, A combinatorial characterisation of finite Auslander-Reiten quivers. In Representation Theory I. Lecture Notes in Mathematics 1177, Springer-Verlag, Berlin-Heidelberg, 1986, 13–49.
[8] X. Chen, Graded self-injective algebras “are” trivial extensions, J. Algebra, 322 (2009), 2601–2606.
[9] E. A. Fernández and M. I. Platzeck, Presentations of Trivial Extensions of Finite Dimensional Algebras and a Theorem of Sheila Brenner, J. Algebra 249, 326–344(2002)
[10] J. Grant and O. Iyama, Higher preprojective algebras, Koszul algebras, and superpotentials. Compositio Mathematica, 156(2020) 2588-2627.
[11] J. Y. Guo, Coverings and truncations of graded self-injective algebras, J. Algebra, 355 (2012), 9–34.
[12] J. Y. Guo, On \( n \)-translation algebras, J. Algebra 453 (2016) 400–428.
[13] J. Y. Guo, On trivial extensions and higher preprojective algebras, J. Algebra 547 (2020) 379–397.
[14] J. Y. Guo, Y. Hu and D. Luo, Multi-layer quivers and higher slice algebras, preprint.
[15] Guo, J. Y., Luo, D., On \( n \)-cubic Pyramid Algebras. Algebr. Represent. Theory 19(2016), 991–1016
[16] J. Y. Guo, X. Lu, Y. Hu and D. Luo, ZQ type constructions in higher representation theory, arXiv:1908.06540
[17] J. Y. Guo and Q. Wu, Loewy matrix, Koszul cone and applications, Comm. Algebra, 28 (2000), 925–941.
[18] J. Y. Guo and C. Xiao, \( n \)-APR tilting and \( \tau \)-mutations. J. Algebr. Comb. (2021) doi:10.1007/s10801-021-01015-z.
[19] J. Y. Guo, C. Xiao, X. Lu, On \( n \)-slice algebras and related algebras, ERA 29(2021), 2687-2718 doi:10.3934/era.2021009
[20] D. Happel, Triangulated categories in the representation theory of finite-dimensional algebras, London Math. Soc. Lecture Note Ser., vol. 119. Cambridge University Press, Cambridge, 1988.
[21] M. Herschend, O. Iyama, S. Oppermann, \( n \)-representation infinite algebras, Adv. Math. 252 (2014) 292–342.
[22] O. Iyama, Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories, Adv. Math. 210 (2007), 22–50.
[23] O. Iyama, Auslander correspondence, Adv. Math. 210 (2007) 51–82.
[24] O. Iyama, Auslander-Reiten theory revisited, in: Trends in Representation Theory of Algebras and Related Topics, European Mathematical Society, 2008, pp. 349-398.
[25] O. Iyama, Cluster tilting for higher Auslander algebras, Adv. Math. 226 (2011) 1–61.
[26] O. Iyama, S. Oppermann, \( n \)-representation-finite algebras and \( n \)-APR tilting, Trans. Amer. Math. Soc. 363 (2011) 6575–6614.
[27] O. Iyama, S. Oppermann, Stable categories of higher preprojective algebras, Adv. Math. 244 (2013) 23–68.
[28] B. Keller, Calabi-Yau triangulated categories, in: A. Skowroński (Ed.), Trends in Representation Theory of Algebras and Related Topics, in: EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 467–489.
[29] M. Kontsevich, Triangulated categories and geometry, Course at the École Normale Supérieure, Paris, Notes taken by J. Bellaïche, J.F. Dat, I. Marin, G. Racinet, and H. Randriambolona, 1998.

[30] R. Martínez-Villa, Graded, selfinjective and Koszul algebras, J. Algebra 215 (1999) 34–72.

[31] H. Minamoto, I. Mori, Structures of AS-regular algebras, Adv. Math. 226 (2011) 4061–4095.

[32] M. Reyes, D. Rogalski and J.J. Zhang, Slew Calabi-Yau triangulated categories and Frobenius Ext-algebras, Trans. Amer. Math. Soc. 369 (2017), 309–340.

[33] C. M. Ringel, D Vossieck, Hammocks. Proc. London Math. Soc. (3) 54 (1987), 216–246.

[34] X. Yu, Almost Koszul algebras and stably Calabi–Yau algebras, J. Pure Appl. Algebra 216(2012), 337-354.

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