Monotonicity-based inversion of fractional semilinear elliptic equations with power type nonlinearities

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Abstract
We investigate the monotonicity method for fractional semilinear elliptic equations with power type nonlinearities. We prove that if-and-only-if monotonicity relations between coefficients and derivatives of the Dirichlet-to-Neumann map hold. Based on the strong monotonicity relations, we study a constructive global uniqueness for coefficients and inclusion detection for the fractional Calderón type inverse problem. Meanwhile, we can also derive the Lipschitz stability with finitely many measurements. The results hold for any $n \geq 1$.

Mathematics Subject Classification 35R30 · 26A33 · 35J61

Contents

1 Introduction ............................................... 2
2 Preliminaries ............................................... 7
   2.1 Function spaces ........................................... 7
   2.2 The exterior dirichlet problem ............................. 8
3 Monotonicity and localized potentials .................................. 13
   3.1 Monotonicity relations ....................................... 13
   3.2 Localized potentials for the fractional laplacian ............... 15
4 Converse monotonicity, uniqueness, and inclusion detection .............. 17
   4.1 Converse monotonicity and the fractional calderón problem ... 17
   4.2 A monotonicity-based reconstruction formula ................... 19
   4.3 Inclusion detection by the monotonicity test ................. 20
5 Lipschitz stability with finitely many measurements .......................... 22
Appendix A. The $L^p$-estimate for the fractional laplacian .................. 24
Appendix B. The maximum principle ................................... 27
References .................................................. 27

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1 Introduction

In this work, we extend the monotonicity method [27, 28] to the case of fractional semilinear elliptic equations with power type nonlinearities. The mathematical formulation is given as follows. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$-boundary $\partial \Omega$, for $n \geq 1$. For $0 < s < 1$, and any $m \geq 2, m \in \mathbb{N}$. Let $q \in L^\infty(\Omega)$ be a potential, then we consider the Dirichlet problem for the fractional semilinear elliptic equation with power type nonlinearities

$$ (-\Delta)^s u + qu^m = 0 \quad \text{in } \Omega,$$

$$ u = f \quad \text{in } \Omega_e := \mathbb{R}^n \setminus \overline{\Omega}. \quad \tag{1.1} $$

The well-posedness of (1.1) holds for any sufficiently small exterior data $f$ in an appropriate function space, which will be demonstrated in Sect. 2. Here the fractional Laplacian $(-\Delta)^s$ is defined via the integral representation

$$ (-\Delta)^s u = c_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy, \quad \tag{1.2} $$

for $u \in H^s(\mathbb{R}^n)$, where P.V. denotes the principal value and

$$ c_{n,s} = \frac{\Gamma\left(\frac{n}{2} + s\right)}{\Gamma(-s)} \frac{4^s}{\pi^{n/2}} \quad \tag{1.3} $$

is a constant that was explicitly calculated in [11]. Here $H^s(\mathbb{R}^n)$ is the fractional Sobolev space, which will be introduced in Sect. 2. On one hand, we want to emphasize that the regularity condition of $\partial \Omega \in C^{1,1}$ is needed due to the well-posedness and suitable $L^p$-estimates for the fractional Laplacian. On the other hand, it is worth mentioning that in the study of fractional inverse problems for linear equations, in general we do not need any regularity assumption on the domain.

Let us discuss regularity issues for nonlinear partial differential equations. For example, consider a semilinear fractional elliptic equation

$$ (-\Delta)^s u + qu^2 = 0, $$

if a solution $u$ belongs to some space $\mathcal{H}$, we require $u^2 = u \cdot u \in \mathcal{H}$, which is the product of solutions. Therefore, it is natural to seek for solution spaces, which possess the (Banach) algebraic structure. For the linear counterpart (one may consider $(-\Delta)^s u + qu = 0$ as an example), it suffices to show the well-posedness with respect to appropriate Sobolev spaces by using the Lax-Milgram theorem. On the other hand, for nonlinear models, a possible candidate for solution spaces could be Hölder continuous spaces, since $\|\phi \psi\|_{C^{k,\alpha}} \leq C \|\phi\|_{C^{k,\alpha}} \|\psi\|_{C^{k,\alpha}}$ holds, for some $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0, 1)$, where $C > 0$ is a constant independent of $\phi$ and $\psi$.

In this article, we study the fractional type Calderón problem for the equation (1.1) of reconstruction an unknown potential $q$ from the (exterior) Dirichlet-to-Neumann (DN) map $\Lambda_q$:

$$ \Lambda_q : H^s(\Omega_e) \to H^s(\Omega_e)^*, \quad f \mapsto (-\Delta)^s u|_{\Omega_e}, $$

for any sufficiently small exterior data $f \in H^s(\Omega_e)$, where $u \in H^s(\mathbb{R}^n)$ is the solution of (1.1) and $H^s(\Omega_e)^*$ is the dual space of $H^s(\Omega_e)$ (see (2.5) for the rigorous definition of the DN map in Sect. 2). For the sake of self-containedness, we will recall the well-posedness of (1.1) under the smallness condition of exterior data, which implies that the DN map $\Lambda_q$ of (1.1) is well-defined in the exterior domain $\Omega_e$ (see Sect. 2).
• Previous literature.

The (space) fractional Calderón problem was first proposed by Ghosh-Salo-Uhlmann [23], and related fractional inverse problems have been investigated by many researchers recently, such as [7, 8, 19, 20, 27, 28, 41, 45, 46, 50, 58] and references therein. The key ingredients in the fractional inverse problems are the strong uniqueness (Proposition 3.3) and the Runge approximation (Theorem 3.2) in $L^p(\Omega)$, for $p > 1$. In [23], the authors have discovered any $L^2$ function can be approximated by solutions of the fractional Schrödinger equation in the same domain. Based on these properties, many researchers have developed the fractional type Calderón problem with partial data, monotonicity-based inversion formula and simultaneously recovering problems. We want to emphasize that Proposition 3.3 and Theorem 3.2 are not true for the case $s = 1$, which are the main differences while studying the local and nonlocal inverse problems.

The research of fractional semilinear Schrödinger equations arises in the quantum effects in Bose-Einstein Condensation [65]. In the ideal boson systems, the Gross-Pitaevskii equations characterizes condensation of weakly interacting boson atoms at a low temperature, wherever the probability density of quantum particles is conserved. Moreover, in the inhomogeneous media with long-range or nonlocal interactions between particles, this yields the density profile no longer retains its shape as in the classical Gross-Pitaevskii equations. This dynamics can be described by the fractional Gross-Pitaevskii equation, regarded as the fractional semilinear Schrödinger equation, in which the turbulence and decoherence emerge. It was investigated in [44] that the turbulence appears from the nonlocal property of the fractional Laplacian; while the local nonlinearity helps maintain coherence of the density profile. Meanwhile, the simulations in [44] indicate that the nonlinearity helps to preserve the shape of the ground state profile, delaying or averting the percolation to high frequencies.

In general, it is known that the nonlinear and nonlocal problems are harder than their local counterparts for forward mathematical problems. For the local case, i.e., $s = 1$, one can consider analogous inverse boundary value problems for the semilinear elliptic equation $\Delta u + a(x, u) = 0$ in $\Omega$ with $u = f$ on $\partial \Omega$. Similar inverse problems are recently treated in the independent works [13, 49]. By using the knowledge of the corresponding DN map, the authors [13, 49] have introduced the higher order linearization method, to investigate that unknown coefficients can be uniquely determined by its associated DN map (on the boundary). In addition, [42, 43, 48] have extended the unique determination results into the partial data setup, and the key ingredient is also relied on the higher order linearization.

• Higher order linearization.

Let us explain the ideas of the higher order linearization method within more details. For example, let $\epsilon_1, \epsilon_2$ be parameters such that $|\epsilon_1|, |\epsilon_2|$ are sufficiently small. Consider the semilinear elliptic equation with power type nonlinearities

$$\Delta u + q u^2 = 0 \text{ in } \Omega \quad \text{and} \quad u = \epsilon_1 g_1 + \epsilon_2 g_2 \text{ on } \partial \Omega,$$

where $q = q(x)$ is an unknown coefficient. When $\epsilon_1 = \epsilon_2 = 0$, $u \equiv 0$ in $\Omega$ is the solution of (1.4). Hence, by differentiating (1.4) with respect to $\epsilon_\ell$, the first linearization will make the unknown coefficients $q$ disappear, so that one obtains $\Delta (\partial_{\epsilon_\ell} |_{\epsilon_1 = \epsilon_2 = 0} u) = 0$ in $\Omega$, which is the Laplace equation, for $\ell = 1, 2$. Furthermore, by direct computations, the second linearization of (1.4) yields that $\Delta (\partial_{\epsilon_1^2} |_{\epsilon_1 = \epsilon_2 = 0} u) + 2q (\partial_{\epsilon_1} |_{\epsilon_1 = \epsilon_2 = 0} u)(\partial_{\epsilon_2} |_{\epsilon_1 = \epsilon_2 = 0} u) = 0$ in $\Omega$.

In order to determine the unknown $q$, one can apply the density property of the scalar products of harmonic functions (see [6] for the full data case and [12] for the partial data

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2 For the local case (i.e., $s = 1$), any solution of Schrödinger equation in a smaller domain can be approximated by solutions of the same equation in a larger domain.
case). With these density results of harmonic functions at hand, one can even study the partial data and simultaneously recovering inverse problems (see [42, 43, 48]), which are still open in their linear counterparts. In short, nonlinearities might help us to study related inverse problems, and we also refer readers to [13, 42, 43, 47–49, 51] for more related works.

Very recently, Lai and myself [46] have studied related inverse problems for fractional semilinear elliptic equations. We can recover unknown coefficients and obstacles by using the higher order linearization method, where we have simply used a single (small) parameter $\epsilon$ instead of multiple (small) parameters shown above. On one hand, when $|\epsilon|$ is sufficiently tiny, we can derive the well-posedness for the exterior problem

$$( -\Delta)^s u + a(x, u) = 0 \text{ in } \Omega \quad \text{and} \quad u = f \text{ in } \Omega_\epsilon.$$ 

On the other hand, by differentiating with respect to $\epsilon$ any times, we can apply the higher order linearization method to study our inverse problems. In fact, in [46], we only need to utilize a single exterior measurement to recover coefficient and obstacle simultaneously.

- **Main results.**

The goal of this work is to study related fractional inverse problems for (1.1) via monotonicity tests combining with the higher order linearization method. Let us formulate the if-and-only-if monotonicity relations in the following. For any potentials $q_1, q_2 \in L^\infty(\Omega)$, we will use the monotonicity arguments and localized potentials for the linearized equations to show that

$$q_1 \leq q_2 \quad \text{if and only if} \quad (D^m \Lambda_{q_1})_0 \leq (D^m \Lambda_{q_2})_0,$$

where $m \in \mathbb{N}$ is the integer of the fractional elliptic equation (1.1) with power type nonlinearities $q_j(x)u^m$, and $(D^m \Lambda_{q_j})_0$ denotes the $m$-th order derivative of the DN map $\Lambda_{q_j}$ evaluated at the 0 exterior data, for $j = 1, 2$. Here $q_1 \leq q_2$ means that $q_1(x) \leq q_2(x)$ for almost everywhere (a.e.) $x \in \Omega$.

In this work, the inequality $(D^m \Lambda_{q_1})_0 \leq (D^m \Lambda_{q_2})_0$ in (1.5) is denoted in the sense that

$$\left( (D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0 \right) (g, \ldots, g, h) \leq 0,$$

for any $g \in C_0^{2,s}(\Omega_\epsilon)$ and for some suitable $h \in C_0^{2,s}(\Omega_\epsilon)$, where $0 < s < 1$ is the same fractional exponent as $(-\Delta)^s$. The exterior data $h \in C_0^{2,s}(\Omega_\epsilon)$ would be chosen differently when the integer number $m$ is even or $m$ is odd. We will give more detailed discussions in Sect. 3. Here $(D^m \Lambda_{q})_0$ can be computed directly from

$$(D^m \Lambda_{q})_0(g, \ldots, g)|_{\Omega_\epsilon} = \partial_{\epsilon}^m \bigg|_{\epsilon = 0} (-\Delta)^s u_{\epsilon g}|_{\Omega_\epsilon},$$

where $u_{\epsilon g} \in H^s(\mathbb{R}^n)$ is the solution of

$$(-\Delta)^s u + q u^m = 0 \text{ in } \Omega \quad \text{with} \quad u = \epsilon g \text{ in } \Omega_\epsilon.$$ 

We will characterize the preceding discussions in Sect. 2 with more details. In addition, once we know the information of exterior measurements $\Lambda_q$, then we can determine the $m$-th order derivative of $\Lambda_q$.

The first main result in this work is that the if-and-only-if monotonicity relations (1.5) yield a constructive uniqueness proof of the potential $q(x)$ by knowing the knowledge of the $m$-th order derivative of the DN map $\Lambda_q$. In the rest of this paper, we will assume that the exterior data $f$ of (1.1) is small enough, such that the DN map $\Lambda_q$ always exists. Then the first main result in this paper is stated as follows.
Theorem 1.1 (The if-and-only-if monotonicity relations) Consider $\Omega \subset \mathbb{R}^n$, $n \geq 1$ to be a bounded domain with $C^{1,1}$ boundary $\partial \Omega$, and $0 < s < 1$. Let $q_1, q_2 \in L^\infty(\Omega)$, $m \geq 2$ and $m \in \mathbb{N}$. Let $\Lambda_{q_j}$ be the DN maps of the semilinear elliptic equations $(-\Delta)^s u + q_j u^m = 0$ in $\Omega$, for $j = 1, 2$. Then we have

$$q_1 \geq q_2 \text{ a.e. in } \Omega \text{ if and only if } (D^m_0 \Lambda_{q_1})_0 \geq (D^m_0 \Lambda_{q_2})_0.$$  \hspace{1cm} (1.7)

Remark 1.1 It is worth mentioning that

(a) When $s = 1$, i.e., for the local case, one can only expect that the monotonicity relations of potentials will imply the monotonicity relations of the corresponding DN maps. It is hard to show a converse statement to be true of the monotonicity formula. Fortunately, with the aids of the strong uniqueness of the fractional Laplacian, we are able to prove Theorem 1.1 (see Sect. 4), which is similar to the works [27, 28].

(b) It is natural to consider the $m$-th order derivative of DN map $(D^m \Lambda_{q_j})_0$ instead of the original DN map $\Lambda_{q_j}$. Due to the well-posedness, one can trace the information of $(D^k \Lambda_{q_j})_0$ for all $k \in \mathbb{N}$, and one cannot see any differences of $(D^k \Lambda_{q_j})_0$ for any $k = 0, 1, \cdots, m-1$ (see Sect. 2).
The following theorem shows that the support of \( q - q_0 \) can be found by shrinking closed set. We also refer readers to \([22, 27, 28, 38]\) for the linear cases.

**Theorem 1.2** (Unknown inclusion detection) Consider \( \Omega \subset \mathbb{R}^n, n \geq 1 \) to be a bounded domain with \( C^{1,1} \) boundary \( \partial \Omega \), and \( 0 < s < 1 \). Let \( q_1, q_2 \in L^\infty(\Omega), m \geq 2 \) and \( m \in \mathbb{N} \). Let \( \Lambda_{q_j} \) be the DN maps of the semilinear elliptic equations \((-\Delta)^s u + q_j u^m = 0 \in \Omega \) for \( j = 1, 2 \). For each closed subset \( C \subseteq \Omega \),

\[
\supp(q - q_0) \subseteq C, \text{ if and only if } \exists \alpha > 0 : -\alpha T_C \leq (D^m \Lambda_{q_0})_0 - (D^m \Lambda_{q_0})_0 \leq \alpha T_C.
\]

Thus,

\[
\supp(q - q_0) = \bigcap \{ C \subseteq \Omega \text{ closed} : \exists \alpha > 0 : -\alpha T_C \leq (D^m \Lambda_{q_0})_0 - (D^m \Lambda_{q_0})_0 \leq \alpha T_C \}.
\]

Note that Theorem 1.2 is not a deterministic result, but it is a reconstruction result. The proof of Theorem 1.2 can be regarded as an application of Theorem 1.1. Via the monotonicity tests, we can give a reconstruction algorithm by utilizing the testing operator \( T_M \) in Sect. 4.

The last main contribution of this article is about the Lipschitz stability of the fractional Calderón problem with finitely many measurements. The Lipschitz stability with finitely many measurements has been studied by in various mathematical settings, we refer readers to \([28, 33, 57, 59]\) and references therein for more detailed descriptions. In this work, we only consider the case that the set \( Q \subset L^\infty(\Omega) \) is a finite-dimensional subspace of piecewise analytic functions, and

\[
Q_\lambda := \left\{ q \in Q : \| q \|_{L^\infty(\Omega)} \leq \lambda \right\},
\]

for some constant \( \lambda > 0 \).

**Theorem 1.3** Consider \( \Omega \subset \mathbb{R}^n, n \geq 1 \) to be a bounded domain with \( C^{1,1} \) boundary \( \partial \Omega \), and \( 0 < s < 1 \). Let \( q_1, q_2 \in L^\infty(\Omega), m \geq 2 \) and \( m \in \mathbb{N} \). Let \( \Lambda_{q_j} \) be the DN maps of the semilinear elliptic equations \((-\Delta)^s u + q_j u^m = 0 \in \Omega \) for \( j = 1, 2 \). Then there exists a constant \( c_0 > 0 \) such that

\[
\|(D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0\|_{\ast} \geq c_0 \| q_1 - q_2 \|_{L^\infty(\Omega)},
\]

for any \( q_1, q_2 \in Q_\lambda \).

**Remark 1.2** The operator norm \( \| \cdot \|_{\ast} \) for the \( m \)-th order derivative DN map is defined by

\[
\| A \|_{\ast} = \sup \left\{ |\langle A(g, \ldots, g), h \rangle| : g, h \in C_0^{2,s}(\Omega_e), \| g \|_{H^s} = \| h \|_{H^s} = 1 \right\}.
\]

One can show that a sufficiently high number of the exterior DN maps uniquely determines a potential in \( Q_\lambda \) and prove a Lipschitz stability result for the equation (1.1). In order to formulate the result, let us denote the orthogonal projection operators from \( H^s(\Omega_e) \) to a subspace \( H \) by \( P_H \), i.e. \( P_H \) is the linear operator with

\[
P_H : H^s(\Omega_e) \to H, \quad P_H g := \begin{cases} g & \text{if } g \in H, \\ 0 & \text{if } g \in H^\perp \subseteq H^s(\Omega_e). \end{cases}
\]

\( P_H' : H^s \to H^s(\Omega_e)^\ast \) stands for the dual operator of \( P_H \).

**Theorem 1.4** Let \( \Omega \subset \mathbb{R}^n, n \geq 1 \) be a bounded domain with \( C^{1,1} \) boundary \( \partial \Omega \), and \( 0 < s < 1 \). Let \( m \geq 2, m \in \mathbb{N} \). Let \( q_1, q_2 \in L^\infty(\Omega) \), and \( \Lambda_{q_j} \) be the DN maps of the
Monotonicity inversion of fractional semilinear elliptic equations

For every sequence of subspaces

\[ H_1 \subseteq H_2 \subseteq H_3 \subseteq \ldots \subseteq H^s(\Omega_\epsilon), \quad \text{and} \quad \bigcup_{\ell \in \mathbb{N}} H_\ell = H^s(\Omega_\epsilon), \]

there exists \( k \in \mathbb{N} \), and \( c > 0 \), so that

\[ \| P'_{H_\ell} \left( (D^m \Lambda q_2)_0 - (D^m \Lambda q_1)_0 \right) P_{H_\ell} \|_* \geq c \| q_2 - q_1 \|_{L^\infty(\Omega)} \]  

(1.10)

for all \( q_1, q_2 \in Q_\lambda \) and all \( l \geq k \).

The article is structured as follows. In Sect. 2, we offer preliminary results for function space (fractional Sobolev spaces and Hölder spaces). We also proved the well-posedness of (1.1), i.e., there exists a unique solution \( u \) of (1.1), whenever the exterior Dirichlet data \( f \) are sufficiently small. In Sect. 3, we derive the monotonicity relations between potentials and its corresponding \( m \)-th order derivative \( D^m \) maps. By combining with the monotonicity relations and localized potentials, we can prove the converse monotonicity relations in Sect. 4, so that we can prove our main results Theorem 1.1 and Theorem 1.2. We prove the Lipschitz stability results in Sect. 5. Finally, we recall some known results that the \( L^p \)-type estimates of solutions, and the maximum principle of the fractional Laplacian in Appendix A and Appendix B, respectively.

2 Preliminaries

In this section, we introduce function spaces and well-posedness of the Dirichlet problem (1.1). The well-posedness of \(-\Delta^s u + a(x, u) = 0\) has been proved in [46], and we simply apply the result when \( a(x, u) = q(x)u^m \), when \( q \in L^\infty(\Omega) \) and for \( m \geq 2, m \in \mathbb{N} \). Let us recall several function spaces which we will use in the rest of the paper.

2.1 Function spaces

Recalling the definition Hölder spaces as follows. Let \( D \subset \mathbb{R}^n \) be an open set, \( k \in \mathbb{N} \cup \{0\} \) and \( 0 < \alpha < 1 \), then the space \( C^{k,\alpha}(D) \) is defined by

\[ C^{k,\alpha}(D) := \left\{ f : D \to \mathbb{R} : \| f \|_{C^{k,\alpha}(D)} < \infty \right\}. \]

The norm \( \| \cdot \|_{C^{k,\alpha}(D)} \) is given by

\[ \| f \|_{C^{k,\alpha}(D)} := \sum_{|\beta| \leq k} \| \partial^\beta f \|_{L^\infty(D)} + \sum_{|\beta| = k} \sup_{x, y \in D, x \neq y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x - y|^\alpha} \]

\[ = \sum_{|\beta| \leq k} \| \partial^\beta f \|_{L^\infty(D)} + \sum_{|\beta| = k} \| \partial^\beta f \|_{C^\alpha(D)} \]

where \( \beta = (\beta_1, \ldots, \beta_n) \) is a multi-index with \( \beta_i \in \mathbb{N} \cup \{0\} \) and \( |\beta| = \beta_1 + \ldots + \beta_n \). Here \( [\partial^\beta f]_{C^\alpha(D)} \) is denoted as the seminorm of \( C^{0,\alpha}(D) \). Furthermore, we also denote the space

\[ C^{k,\alpha}_0(D) := \text{closure of } C^\infty_c(D) \text{ in } C^{k,\alpha}(D). \]

We also denote \( C^\alpha(D) \equiv C^{0,\alpha}(D) \) when \( k = 0 \).
We next remind readers in the context of fractional Sobolev spaces. Given $0 < s < 1$, the $L^2$-based fractional Sobolev space is $H^s(\mathbb{R}^n) := W^{s,2}(\mathbb{R}^n)$ with the norm
\[
\|u\|_{H^s(\mathbb{R}^n)}^2 = \|u\|_{L^2(\mathbb{R}^n)}^2 + \|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}^2.
\]
Furthermore, via the Parseval identity, the semi-norm $\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}^2$ can be rewritten as
\[
\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}^2 = ((-\Delta)^s u, u)_{\mathbb{R}^n},
\]
where $(-\Delta)^s$ is the fractional Laplacian (1.2).

Let $D \subset \mathbb{R}^n$ be an open set and $a \in \mathbb{R}$, then we denote the following Sobolev spaces,
\[
H^a(D) := \{u \mid_D : u \in H^a(\mathbb{R}^n)\},
\]
\[
\widetilde{H}^a(D) := \text{closure of } C_c^{\infty}(D) \text{ in } H^a(\mathbb{R}^n),
\]
\[
H_0^a(D) := \text{closure of } C_c^{\infty}(D) \text{ in } H^a(D),
\]
and
\[
H^a_D := \{u \in H^a(\mathbb{R}^n) : \text{supp}(u) \subset \overline{D}\}.
\]

The fractional Sobolev space $H^a(D)$ is complete under the norm
\[
\|u\|_{H^a(D)} := \inf \left\{ \|v\|_{H^a(\mathbb{R}^n)} : v \in H^a(\mathbb{R}^n) \text{ and } v|_D = u \right\}.
\]

Moreover, when $D$ is a Lipschitz domain, the dual spaces can be expressed as
\[
\left(H^a_D(\mathbb{R}^n)\right)^* = H^{-a}(D), \quad \text{and} \quad (H^s(D))^* = H^{-s}_D(\mathbb{R}^n).
\]

If reader are interested in the properties of fractional Sobolev spaces, we refer readers to the references [11, 52].

### 2.2 The exterior dirichlet problem

For $m \geq 2$, $m \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1.1}$ boundary for $n \geq 1$, and let $q(x) \in L^\infty(\Omega)$. Let us prove the well-posedness for the exterior Dirichlet problem
\[
\begin{aligned}
(-\Delta)^s u + qu^m &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{in } \Omega_e,
\end{aligned}
\]
under the condition that $\|f\|_{C^{2,s}_0(\Omega_e)}$ is sufficiently small.

**Proposition 2.1** (Well-posedness) Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with $C^{1.1}$ boundary $\partial \Omega$, and $0 < s < 1$. Suppose that $q = q(x) \in L^\infty(\Omega)$, $m \geq 2$ and $m \in \mathbb{N}$. Then there exists $\varepsilon > 0$ such that when
\[
f \in E_\varepsilon := \left\{ f \in C^{2,s}_0(\Omega_e) : \|f\|_{C^{2,s}_0(\Omega_e)} \leq \varepsilon \right\},
\]
the boundary value problem (2.1) has a unique solution $u_f \in C^s(\mathbb{R}^n)$ within the class
\[
\left\{ w \in C^s(\overline{\Omega}) : \|w\|_{C^s(\overline{\Omega})} \leq C\varepsilon \right\}.
\]
Furthermore, the following estimate holds
\[
\|u_f\|_{C^s(\mathbb{R}^n)} \leq C\|f\|_{C^{2,s}_0(\Omega_e)},
\]
where $C$ is a constant independent of $f$. 

\[\square\]
for some constant $C > 0$, independent of $u_f$ and $f$. Furthermore, there exist $\mathcal{C}^\infty$ Fréchet differentiable maps

$$S : \mathcal{E}_\epsilon \to \mathcal{C}^s(\overline{\Omega}), \quad f \mapsto u_f,$$

$$\Lambda_q : \mathcal{E}_\epsilon \to H^s(\Omega_e)^*, \quad f \mapsto (-\Delta)^s u_f|_{\Omega_e}.$$

**Proof** The proof has been shown in [46] for the case $(-\Delta)^s u + a(x, u) = 0$ in $\Omega$, with $u = f$ in $\Omega_e$, where $a(x, u)$ is analytic in $u$ and Hölder in $x \in \Omega$ such that $a(x, 0) = 0$ and $\partial_u a(x, u) \geq 0$. In particular, by choosing $a(x, u) = q(x)u^m$ with some $m \geq 2$, $m \in \mathbb{N}$, one has $a(x, 0) = \partial_u a(x, 0) = 0$ for $x \in \Omega$. Therefore, we also have the well-posedness at hand immediately.

**Remark 2.2** Via Proposition 2.1, as a matter of fact, one can that the solution $u$ of (2.1) is in $\mathcal{C}^s_2(\mathbb{R}^n)$, whenever the exterior data $f|_{\Omega_e}$ is sufficiently small and smooth. Moreover, we can derive that $u \in H^s(\mathbb{R}^n)$, by the following straightforward computations. Consider the function $w = u - f$, where $u$ is the solution of (2.1) and $f \in C^2_0(\Omega_e) \subset C^2(\mathbb{R}^n)$. Then $w$ is the solution of

$$\begin{cases}
(-\Delta)^s w + w = -qu^m + u - f - (-\Delta)^s f & \text{in } \Omega, \\
w = 0 & \text{in } \Omega_e. 
\end{cases} \quad (2.4)$$

By multiplying $w$ to (2.4) and integrating over $\mathbb{R}^n$, we can derive that $w \in H^s(\mathbb{R}^n)$, where we have utilized the estimate (2.3). Hence, $u = w + f \in H^s(\mathbb{R}^n)$.

We can define the DN map rigorously as follows.

**Proposition 2.3** (The DN map) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary $\partial \Omega$ for $n \geq 1$, $0 < s < 1$ and let $q \in L^\infty(\Omega)$. Define

$$\langle \Lambda_q f, \varphi \rangle := \int_{\mathbb{R}^n} (-\Delta)^{s/2} u_f (-\Delta)^{s/2} \varphi \, dx + \int_{\Omega} q u_f^m \varphi \, dx, \quad (2.5)$$

for $f, \varphi \in C^2_0(\mathbb{R}^n)$. Then the DN map

$$\Lambda_q : H^s(\Omega_e) \to H^s(\Omega_e)^*$$

is bounded, and

$$\Lambda_q f|_{\Omega_e} = (-\Delta)^s u_f|_{\Omega_e}, \quad (2.6)$$

where the function $u_f \in C^s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ is the solution of (2.1) with the sufficiently small exterior data $f \in C^2_0(\Omega_e)$.

**Proof** Notice that (2.5) is not a bilinear form, since $qu^m_f$ is not a linear function, for $m \geq 2$. By using the Parseval identity, we have that

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} u_f (-\Delta)^{s/2} \varphi \, dx + \int_{\Omega} q u_f^m \varphi \, dx$$

$$= \int_{\mathbb{R}^n} (-\Delta)^s u_f \varphi \, dx + \int_{\Omega} q u_f^m \varphi \, dx$$

$$= \int_{\Omega_e} (-\Delta)^s u_f \varphi \, dx,$$
where we have utilized that \( u_f \in H^s(\mathbb{R}^n) \) is the solution of (2.1) shown as in Remark 2.2, and
\[
\int_{\mathbb{R}^n} (-\Delta)^s u_f \varphi \, dx = \int_{\Omega} (-\Delta)^s u_f \varphi \, dx + \int_{\Omega_e} (-\Delta)^s u_f \varphi \, dx.
\]
The preceding identity was justified in [23]. Since \( \varphi \in C_0^{2,s}(\mathbb{R}^n) \subseteq H^s(\mathbb{R}^n) \) is arbitrary, by the duality argument, then we prove the proposition.

Notice that for the nonlinearities \( q(x)u^m \), the higher order linearizations of the exterior DN map \( \Lambda_q \) is particularly simple (see [49, Section 2] for the local case \( s = 1 \)). It is slightly different from the earlier work [49], which adapts multiple small parameters to do the higher order linearization. Instead, we use the ideas from [46], via a single \( \epsilon \) parameter to do the higher order linearization for fractional semilinear equations. Let \( \epsilon > 0 \) be a sufficiently small number, and \( g \in C_0^{2,s}(\Omega_e) \). The next proposition demonstrates that we may differentiate the fractional semilinear equation
\[
\begin{cases}
(-\Delta)^s u + q(x)u^m = 0 & \text{in } \Omega, \\
u = \epsilon g & \text{in } \Omega_e,
\end{cases}
\]formally in the \( \epsilon \) variable to have equations corresponding to first linearization and \( m \)-th linearization
\[
\begin{cases}
(-\Delta)^s v_g = 0 & \text{in } \Omega, \\
v_g = g & \text{in } \Omega_e,
\end{cases}
\]and
\[
\begin{cases}
(-\Delta)^s w = -(m!)q(v_g)^{m+1} & \text{in } \Omega, \\
w = 0 & \text{in } \Omega_e,
\end{cases}
\]respectively. We call the solution \( v_g \) of the fractional Laplacian equation (2.8) to be \( s \)-harmonic in the rest of paper.

The DN map of the solution \( w \) of (2.9) is the \( m \)-th linearization of the DN map of (2.7). Let
\[
(D^k T)_x(y_1, \ldots, y_k)
\]
denote the \( k \)-th derivative at \( x \) of a mapping \( T \) between Banach spaces, which can be regarded as a symmetric \( k \)-linear form acting on \( (y_1, \ldots, y_k) \). We refer to [34, Section 1.1], where the notation \( T^{(k)}(x; y_1, \ldots, y_k) \) is used instead of \( (D^k T)_x(y_1, \ldots, y_k) \).

For \( f \in C_0^{2,s}(\Omega_e) \) with \( \|f\|_{C_0^{2,s}(\Omega_e)} \) to be sufficiently small. By using the notation of \( s \)-harmonic functions \( v_g \) given by (2.8), we have the following result.

**Proposition 2.4** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^{1,1} \) boundary \( \partial \Omega \) for \( n \geq 1 \), \( 0 < s < 1 \) and let \( q \in L^\infty(\Omega) \). Let \( \Lambda_q \) be the DN map for the fractional semilinear elliptic equation
\[
(-\Delta)^s u + qu^m = 0 \text{ in } \Omega,
\]where \( m \in \mathbb{N} \) and \( m \geq 2 \). The first linearization \((D\Lambda_q)_0\) of \( \Lambda_q \) at \( g = 0 \) is the DN map of the fractional Laplacian (2.8) such that
\[
(D\Lambda_q)_0 : H^s(\Omega_e) \to H^s(\Omega_e)^*, \quad g \mapsto (-\Delta)^s v_g|_{\Omega_e}.
\]
The higher order linearizations \((D^j \Lambda(q))_0\) are identically zero for \(2 \leq j \leq m - 1\).

The \(m\)-th linearization \((D^m \Lambda_q)_0\) of \(\Lambda_q\) at \(g = 0\) can be characterized by

\[
\int_{\Omega_e} (D^m \Lambda_q)_0(g, \ldots, g) h \, dx = (m!) \int_{\Omega} q(v_g)^m v_h \, dx,
\]

where \(v_g\) and \(v_h\) are \(s\)-harmonic in \(\Omega\) with the exterior value \(v_g = g\) and \(v_h = h\) in \(\Omega_e\), respectively.

**Remark 2.5** We point out:

(a) Even though the original DN map \(\Lambda_q\) depends on \(q\) non-linearly, it is worth emphasizing that the integral identity (2.11) implies that the \(m\)-th order derivative of \(\Lambda_q\) depends linearly on \(q\).

(b) Proposition 2.4 plays an essential role to prove our main results of this article.

**Proof of Proposition 2.4** Via Proposition 2.1, the DN map \(\Lambda_q(f) = (-\Delta)^s(Sf)|_{\Omega_e}\) is well defined for sufficiently small exterior data \(f\), where \(S : f \mapsto u_f\) is the solution operator for the equation (2.10). In order to compute the derivatives of \(\Lambda_q\) at \(0\), it suffices to consider the derivatives of \(S\). Furthermore, by using Proposition 2.1, the maps

\[
S : \mathcal{E}_\delta \to C^s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n), \quad f \mapsto u_f,
\]

\[
\Lambda_q : \mathcal{E}_\delta \to H^s(\Omega_e)^*, \quad f \mapsto (-\Delta)^s u_f|_{\Omega_e}
\]

are \(C^\infty\) Fréchet differentiable mappings, where \(\mathcal{E}_\delta\) is the set defined by (2.2) to denote the set of small exterior data.

Let us write \(f = f(x; \epsilon) := \epsilon g(x) \in C^{2,s}_0(\Omega_e)\), then the function \(u_{\epsilon g} = S(\epsilon g) \in C^s(\bar{\Omega})\) depends smoothly on the small parameter \(\epsilon\). By applying \(\frac{\partial^m}{\partial \epsilon^m}\bigg|_{\epsilon = 0}\) to the Taylor’s formula for \(C^\infty\) Fréchet differentiable mappings (see e.g. [34, Equation 1.1.7])

\[
S(f) = \sum_{j=0}^{k} \frac{(D^j S)_0(f, \ldots, f)}{j!} + \int_0^1 \frac{(D^{k+1} S)_0(f, \ldots, f)}{k!} (1 - t)^k \, dt
\]

implies that \((D^k S)_0\) may be computed using the formula

\[
(D^k S)_0(f, \ldots, f) = \frac{\partial^m}{\partial \epsilon^m} u_f\bigg|_{\epsilon = 0}.
\]

Moreover, since \(u_f\) is smooth in the \(\epsilon\) variables and the fractional Laplacian \((-\Delta)^s\) is linear, one may differentiate the equation

\[
(-\Delta)^s u_f + q u_f^m = 0 \text{ in } \Omega, \quad u_f = f \text{ in } \Omega_e
\]

with respect to the \(\epsilon\) variable.

For the first linearization \(k = 1\) with \(u = u_{\epsilon g}\), we have \(u_0 = 0\) in \(\mathbb{R}^n\) and \(m \geq 2\), the derivative of (2.12) in \(\epsilon\) evaluated at \(\epsilon = 0\) satisfies

\[
(-\Delta)^s \left( \partial\bigg|_{\epsilon = 0} u_f \right) = 0 \text{ in } \Omega, \quad \partial u_{\epsilon = 0} = g \text{ in } \Omega_e.
\]

Thus the first linearization of the map \(S\) at \(f = 0\) \((f = \epsilon g\) with \(\epsilon = 0\) is

\[
(DS)_0(g) = \partial|_{\epsilon = 0} u_{\epsilon g} = v_g, \quad \text{for } g \in C^{2,s}_0(\Omega_e),
\]

where \(v_g\) is \(s\)-harmonic in \(\Omega\) with \(v_g = g\) in \(\Omega_e\).
For $2 \leq k \leq m - 1$, applying the $k$-th order derivatives $\partial^k \epsilon |_{\epsilon=0}$ to (2.12) gives that

$$(-\Delta)^s \left( \partial^k \epsilon |_{\epsilon=0} u_f \right) = 0 \text{ in } \Omega,$$

$$\partial^k \epsilon |_{\epsilon=0} u_f = 0 \text{ in } \Omega_e,$$

since $\partial^k \epsilon (q(x)u^m_f)$ is a sum of terms containing positive powers of the solution $u_f$, which are equal to zero whenever $\epsilon = 0$. The uniqueness of solutions for the fractional Laplace equation implies that

$$\begin{align*}
(D^k S)_{0, k \text{-linear form}} (g, \ldots, g) = 0, & \quad \text{for } 2 \leq k \leq m - 1.
\end{align*}$$

More precisely, we have used the fact that any $s$-harmonic function with 0 exterior data is zero in $\mathbb{R}^n$.

When $k = m$, the only nonzero term in the expansion of $\partial^m \epsilon |_{\epsilon=0} (q(x)u^m_f)$ does not contain second or higher order derivatives of $u_f$ with respect to $\epsilon$. The nonzero term after inserting $\epsilon = 0$ is

$$q(x)(m!) (\partial \epsilon |_{\epsilon=0} u_f)^m = q(x)(m!)(v_g)^m.$$

Hence, the function

$$w := (D^m S)_{0, g, \ldots, g} = \partial^m \epsilon |_{\epsilon=0} u_f \text{ in } \mathbb{R}^n$$

solves

$$(-\Delta)^s w + q(x)(m!)(v_g)^m = 0 \text{ in } \Omega,$$

(2.13)

with zero exterior data in $\Omega_e$.

By linearity we have

$$(D^k \Lambda_q)_{0, \Omega_e} = (-\Delta)^s (D^k S)_{0, \Omega_e}.$$

The claims for derivatives of DN map $(D^k \Lambda_q)_{0}$ when $1 \leq k \leq m - 1$ follow immediately. For $k = m$ we observe that $(D^m \Lambda_q)_{0, g, \ldots, g} = (-\Delta)^s w |_{\Omega_e}$ satisfies

$$\begin{align*}
\int_{\Omega_e} ((-\Delta)^s w)h \, dx &= \int_{\mathbb{R}^n} (-\Delta)^s w v_h \, dx + m! \int_{\Omega} q v^m v_h \, dx \\
&= m! \int_{\Omega} q v^m v_h \, dx,
\end{align*}$$

(2.14)

where $v_h$ is $s$-harmonic in $\Omega$ with $v_h = h$ in $\Omega_e$. Finally, we have used that $\int_{\mathbb{R}^n} (-\Delta)^s w h \, dx = 0$ in (2.14) due to the Parseval’s identity that

$$\begin{align*}
\int_{\mathbb{R}^n} (-\Delta)^s w h \, dx &= \int_{\mathbb{R}^n} w(-\Delta)^s h \, dx = \int_{\Omega} w(-\Delta)^s h \, dx + \int_{\Omega_e} w(-\Delta)^s h \, dx = 0.
\end{align*}$$

Thus, the proposition follows by using (2.13).

For the sake of convenience, in the rest of this paper, let us utilize the pairing notation

$$\langle (D^m \Lambda_q)_{0, g, \ldots, g}, h \rangle = \int_{\Omega_e} (D^m \Lambda_q)_{0, g, \ldots, g} h \, dx,$$
where $(D^m \Lambda_{q_j})_0 : H^s(\Omega_e)^m \to H^s(\Omega_e)^*$ is regarded as an $m$-form acting on an $m$-vector valued function $(g, \ldots, g)$.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with $C^{1,1}$ boundary $\partial \Omega$, and $0 < s < 1$. Let $q_1, q_2 \in L^\infty(\Omega)$, and $\Lambda_{q_j}$ be the DN maps of the semilinear elliptic equations $(-\Delta)^s u + q_j u^m = 0$ in $\Omega$ for $j = 1, 2$. As we mentioned in Proposition 2.4, there are no information of $(D^k \Lambda_{q_j})_0$ for any $k = 2, \ldots, m - 1$. Let us look at the case $k = 0$ and $k = 1$. For $k = 0$, we have that

$$(D^0 \Lambda_{q_j})_0 = \Lambda_{q_j}(\epsilon g)|_{\epsilon = 0} = 0, \quad \text{for } j = 1, 2,$$

due to the well-posedness of (2.1). Meanwhile, for the case $k = 1$, the map $(D^1 \Lambda_{q_j})_0$ denotes the DN map of the fractional Laplacian equation (2.8), for $j = 1, 2$, which has no unknown coefficients in the equation (2.8). Hence, we must have

$$(D^1 \Lambda_{q_1})_0 = (D^1 \Lambda_{q_2})_0.$$

Therefore, in order to understand the relations of the DN maps $\Lambda_{q_j}$, one can obtain the information of the $m$-th order derivative $(D^m \Lambda_{q_j})_0$ of the DN map $\Lambda_{q_j}$, for $j = 1, 2$.

### 3 Monotonicity and localized potentials

In this section, we show monotonicity relations between potentials $q$ and their corresponding DN maps, and we demonstrate how to control the energy terms in the monotonicity formulas with the localized potentials of the fractional Laplacian.

#### 3.1 Monotonicity relations

We study the monotonicity relations between the $m$-th order derivative of DN maps and the potentials via the following integral identity. Let us define the energy inequalities of the $m$-th order derivative of DN maps:

**Definition 3.1** Consider $\Omega \subset \mathbb{R}^n$, $n \geq 1$ to be a bounded domain with $C^{1,1}$ boundary $\partial \Omega$, and $0 < s < 1$. Let $q_1, q_2 \in L^\infty(\Omega)$, $m \geq 2$ and $m \in \mathbb{N}$. Let $\Lambda_{q_j}$ be the DN maps of the semilinear elliptic equations $(-\Delta)^s u + q_j u^m = 0$ in $\Omega$ for $j = 1, 2$. Then the inequality $(D^m \Lambda_{q_1})_0 \geq (D^m \Lambda_{q_2})_0$ can be defined as follows:

(a) When $m$ is odd, $(D^m \Lambda_{q_1})_0 \geq (D^m \Lambda_{q_2})_0$ is denoted by

$$\langle [(D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0], (g, \ldots, g) \rangle \geq 0,$$

for any $g \in C^{2,s}_0(\Omega_e)$. (3.1)

(b) When $m$ is even, $(D^m \Lambda_{q_1})_0 \geq (D^m \Lambda_{q_2})_0$ is denoted by

$$\langle [(D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0], (g, \ldots, g), h \rangle \geq 0,$$

for any $g, h \in C^{2,s}_0(\Omega_e)$ with $h \geq 0$. (3.2)

We next demonstrate the monotonicity relations between potentials and the $m$-th order derivatives of the DN map. Due the particular structure of the power type nonlinearities, the integral identity will imply the monotonicity formulas directly, which is a more straightforward result than its linear counterpart.
Theorem 3.1 (Monotonicity relations) Consider $\Omega \subset \mathbb{R}^n$, $n \geq 1$ to be a bounded domain with $C^{1,1}$ boundary $\partial \Omega$, and $0 < s < 1$. Let $q_1, q_2 \in L^\infty(\Omega)$, $m \geq 2$ and $m \in \mathbb{N}$. Let $\Lambda_{q_j}$ be the DN maps of the semilinear elliptic equations $(-\Delta)^{s}u + q_j u^m = 0$ in $\Omega$ for $j = 1, 2$. Then

(a) We have the integral identity

$$\langle [(D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0] (g, \ldots, g), h \rangle = (m!) \int_\Omega (q_1 - q_2) (v_g)^m v_h \, dx$$

(3.3)

where $v_g$ and $v_h$ are $s$-harmonic in $\Omega$ with $v_g = g$ and $v_h = h$ in $\Omega_e$, respectively, for $g, h \in C^{2,s}_0(\Omega_e)$.

(b) We have the monotonicity relation

$$q_1 \geq q_2 \text{ in } \Omega \quad \text{implies that} \quad (D^m \Lambda_{q_1})_0 \geq (D^m \Lambda_{q_2})_0.$$

**Proof** For (a), the proof is a simple application of Proposition 2.4. Via (2.11), one has

$$\langle (D^m \Lambda_{q_j})_0 (g, \ldots, g), h \rangle = \int_{\Omega_e} (D^m \Lambda_{q_j})_0 (g, \ldots, g) h \, dx = (m!) \int_\Omega q_j (v_g)^m v_h \, dx,$$

for $j = 1, 2$. By subtracting the preceding identity with $j = 1$ and $j = 2$, we have the desired identity (3.3).

For (b), we first show the case when $m$ is odd. Let us take $h = g \in C^{2,s}_0(\Omega_e)$, then the uniqueness of the fractional Laplacian implies that $v_g = v_h$ in $\Omega$. By plugging $q_1 - q_2 \geq 0$ in $\Omega$ into (3.3), we must have

$$\langle [(D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0] (g, \ldots, g), g \rangle = (m!) \int_\Omega (q_1 - q_2) (v_g)^{m+1} \, dx \geq 0,$$

where we have used that $m$ is odd so that $(v_g)^{m+1} = |v_g|^{m+1} \geq 0$ in $\Omega$. This satisfies (3.1) so that $(D^m \Lambda_{q_1})_0 \geq (D^m \Lambda_{q_2})_0$.

When $m$ is even, we take $h \in C^{2,s}_0(\Omega_e)$ with $h \geq 0$. Note that $v_h$ is $s$-harmonic in $\Omega$ with $v_h = h \geq 0$ in $\Omega_e$, then the maximum principle for the fractional Laplacian yields that $v_h \geq 0$ in $\Omega$ (for example, see [54]). By plugging $q_1 - q_2 \geq 0$ in $\Omega$ into (3.3), we must have

$$\langle [(D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0] (g, \ldots, g), h \rangle = (m!) \int_\Omega (q_1 - q_2) (v_g)^m v_h \, dx \geq 0,$$

where we have used that $m$ is even so that $(v_g)^m = |v_g|^m \geq 0$ in $\Omega$ and $v_h \geq 0$ in $\Omega$. This satisfies (3.2) so that $(D^m \Lambda_{q_1})_0 \geq (D^m \Lambda_{q_2})_0$. This completes the proof.

**Remark 3.2** From Theorem 3.1, we have:

(a) In the proof of part (b) of Theorem 3.1, one can see that why we need to choose different $s$-harmonic function $v_h$ in (3.3) so that (3.1) and (3.2) have correct sign conditions. In particular, when $h \leq 0$ in $\Omega_e$, the maximum principle (see Appendix B) yields that $(D^m \Lambda_{q_1})_0 \leq (D^m \Lambda_{q_2})_0$, provided that $q_1 \geq q_2$ in $\Omega$. However, for general $h \in C^{2,s}_0(\Omega_e)$, we do not know the sign condition of the $s$-harmonic function $v_h$ in $\Omega$ so that we cannot have the monotonicity relation as in Theorem 3.1 (b).

(b) In particular, when $m = 1$, i.e., for the (linear) fractional Schrödinger equation, one can adapt (3.2) as the monotonicity assumption. One can see that if we do the "linearization" to the fractional Schrödinger equation, then the "linearized" equation is also the same fractional Schrödinger equation. The monotonicity relations were derived in the works [27, 28].
In the semilinear case, the monotonicity relation between potentials and $m$-th order derivative of DN maps is equivalent to the integral identity (3.3), which makes the monotonicity tests be easier for the fractional semilinear elliptic equation than their linear counterparts.

3.2 Localized potentials for the fractional laplacian

We demonstrate the existence of localized potentials for $s$-harmonic functions. For the fractional Laplacian, the existence of localized potentials is a simple consequence of the strong uniqueness and Runge approximation, which was demonstrated by [23]. In this work, we use slightly different settings. For the sake of completeness, let us state the the strong uniqueness, Runge approximation, and localized potentials as follows.

Proposition 3.3 (Strong uniqueness) For $n \geq 1$, $0 < s < 1$, let $\psi \in L^p(\mathbb{R}^n)$ for some $1 < p < 2$ satisfy both $\psi$ and $(-\Delta)^sv$ vanish in the same arbitrary non-empty open set in $\mathbb{R}^n$, then $\psi \equiv 0$ in $\mathbb{R}^n$.

The preceding proposition was shown in the proof of [23, Theorem 1.2] for the case $\psi \in H^a(\mathbb{R}^n)$ for some $a \in \mathbb{R}$. In particular, Proposition 3.3 was recently proved by Covi-Mönkkönen-Railo [9, Corollary 4.5].

We next prove the Runge approximation, and the mathematical settings are slightly different from [23]. In [23], the authors proved any $L^2$ functions can be approximated by solutions of the fractional Schrödinger equation. In this work, our aim is only to demonstrate that any $L^a$-integrable functions for $a > 1$, can be approximated by a sequence of $s$-harmonic functions.

Theorem 3.2 (Runge approximation for the fractional Laplacian) For $n \geq 1$, $0 < s < 1$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary $\partial \Omega$, and $O \Subset \Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ be open. Let $m \geq 2$, $m \in \mathbb{N}$. Given an arbitrary $a > 1$, for every $\phi \in L^a(\Omega)$ there exists a sequence $u_k \in C^\infty_c(O)$, so that the corresponding solutions $v_k \in H^s(\mathbb{R}^n)$ to

$$(-\Delta)^sv_k = 0 \text{ in } \Omega, \quad u^k = g^k \text{ in } \Omega_e,$$

satisfy that $v_k|_\Omega \to \phi$ in $L^a(\Omega)$ as $k \to \infty$.

Proof The idea of the proof is similar to the proof of [23, Theorem 1.3], but we will use the fact that if $v$ is the solution of $(-\Delta)^sv = 0$ in $\Omega$ with $v = g \in C^{2,s}_0(\Omega_e)$, then the well-posedness yields that $v \in H^s(\mathbb{R}^n)$. Furthermore, by using the global Hölder estimate [55, Proposition 1.1], one has $v \in C^s(\mathbb{R}^n)$.

In order to prove the theorem, let us consider the set

$$\mathbb{D} = \{ v_g|_\Omega : g \in C_c^\infty(O) \},$$

where $v_g \in H^s(\mathbb{R}^n)$ is the unique solution of

$$\begin{cases} (-\Delta)^sv_g = 0 & \text{in } \Omega, \\ v_g = g & \text{in } \Omega_e, \end{cases} \quad (3.4)$$

with $g \in C^{2,s}_0(\Omega_e)$. Then $\mathbb{D}$ is dense in $L^a(\Omega)$. Via [55, Proposition 1.1], it is easy to see that $\mathbb{D} \subset C^s(\overline{\Omega})$ which implies $\mathbb{D} \subset L^a(\Omega)$, for all $a > 1$. By the Hahn-Banach theorem, it suffices to show that for any function $\phi \in L'(\Omega)$ satisfying $\int_{\Omega} \phi v_g \, dx = 0$ for any $v \in \mathbb{D}$, where $\frac{1}{r} + \frac{1}{a} = 1$, then $\phi \equiv 0$.
Let $\varphi$ be a such function, which means $\varphi$ satisfies
\[ \int_{\Omega} \varphi v_g \, dx = 0, \quad \text{for any } g \in C^\infty_c(O). \tag{3.5} \]
Next, let $\phi$ be the solution of
\[ \begin{cases} (-\Delta)^s \phi = \varphi & \text{in } \Omega, \\ \phi = 0 & \text{in } \Omega_e. \end{cases} \]

By using the $L^p$ estimate for the fractional Laplacian (see Proposition A.3 and Remark A.4), we know that $\phi \in L^p(\Omega)$ for some $p \in (1, 2)$ since $\varphi \in L^r(\Omega)$ for some $r > 1$.

We next claim that for any $g \in C^\infty_c(\Omega)$, the following relation
\[ \int_{\Omega} \varphi v_g \, dx = -\int_{\mathbb{R}^n} (-\Delta)^{s/2} \phi (-\Delta)^{s/2} g \, dx \tag{3.6} \]
holds. In other words, $\int_{\mathbb{R}^n} (-\Delta)^{s/2} \phi (-\Delta)^{s/2} w \, dx = \int_{\Omega} \varphi w \, dx$ for any $w \in \mathbb{D} \subset L^q(\Omega)$. In order to prove (3.6), let $g \in C^{2s}(O)$, and $v_g$ be the solution of (3.4). Then by [55, Proposition 1.1], we have $v_g \in C^s(\mathbb{R}^n)$ with $v_g - g \in C^s_c(\Omega)$ and
\[ \int_{\Omega} \varphi v_g \, dx = \int_{\Omega} \varphi (v_g - g) \, dx 
= \int_{\mathbb{R}^n} (-\Delta)^{s/2} \phi (-\Delta)^{s/2} v_g - g) \, dx 
= -\int_{\mathbb{R}^n} (-\Delta)^{s/2} \phi (-\Delta)^{s/2} g \, dx, \]
where we have utilized that $v_g$ is $s$-harmonic in $\Omega$ and $\phi = 0$ in $\Omega_e$.

Hence, (3.5) and (3.6) yield that imply that
\[ \int_{\mathbb{R}^n} (-\Delta)^{s/2} \phi (-\Delta)^{s/2} g \, dx = 0, \quad \text{for any } g \in C^\infty_c(O). \]
Moreover, we know that $g|_{\Omega} = 0$ due to $g \in C^\infty_c(O)$, then the Parseval’s identity infers that
\[ \int_{\mathbb{R}^n} (-\Delta)^{s} \phi g \, dx = 0, \quad \text{for any } g \in C^\infty_c(O). \]
In the end, we know that $\phi \in L^p(\Omega)$ with $\phi = 0$ in $\Omega_e$, which satisfies $\phi \in L^p(\mathbb{R}^n)$ for some $p \in (1, 2)$, and
\[ \phi|_{O} = (-\Delta)^{s} \phi|_{O} = 0. \]
By applying Proposition 3.3, we obtain $\phi \equiv 0$ in $\mathbb{R}^n$ so that $v \equiv 0$ as desired. \hfill \Box

Based on the Runge approximation, one can obtain the existence of the localized potentials immediately.

**Corollary 3.4 (Localized potentials)** For $n \geq 1$, let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary $\partial \Omega$, $0 < s < 1$, and $O \subseteq \Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ be an arbitrary open set. For any $a > 1$ and every measurable set $M \subseteq \Omega$, there exists a sequence $g^k \in C^\infty_c(O)$, so that the corresponding solutions $v^k \in H^s(\mathbb{R}^n)$ of
\[ (-\Delta)^s v^k = 0 \quad \text{in } \Omega, \quad v^k|_{\Omega_e} = g^k, \quad \text{for all } k \in \mathbb{N} \tag{3.7} \]

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satisfy that
\[ \int_M |v^k|^a \, dx \to \infty \quad \text{and} \quad \int_{\Omega\setminus M} |v^k|^a \, dx \to 0 \quad \text{as } k \to \infty. \]

**Proof** The proof is based on the Runge approximation (Theorem 3.2) and the normalization argument. By Theorem 3.2, there exists a sequence \( \tilde{g}^k \in C^{2,\alpha}_0(\Omega_e) \) so that the corresponding solutions \( \tilde{v}^k|_\Omega \) converge to \( \left( \frac{1}{|M|} \right)^{\frac{1}{a}} \chi_M \) in \( L^a(\Omega) \), where \( |M| \) denotes the Lebesgue measure of the measurable set \( M \). This implies that
\[
\|\tilde{v}^k\|_{L^a(M)} = \int_M |\tilde{v}^k|^a \, dx \to 1, \quad \text{and} \quad \|\tilde{v}^k\|_{L^a(\Omega\setminus M)} = \int_{\Omega\setminus M} |\tilde{v}^k|^a \, dx \to 0,
\]
as \( k \to \infty \).

Without loss of generality, we can assume for all \( k \in \mathbb{N} \) that \( \tilde{v}^k \neq 0 \), so that \( \|\tilde{v}^k\|_{L^a(\Omega\setminus M)} > 0 \) follows due to the strong uniqueness of the fractional Laplacian (Proposition 3.3). Assume that the normalized exterior data
\[
g^k := \frac{\tilde{g}^k}{\|\tilde{v}^k\|_{L^a(\Omega\setminus M)}} \in C^{2,\alpha}_0(\Omega_e),
\]
then the sequence of corresponding solutions \( v^k \in C^s(\mathbb{R}^n) \) of (3.7) has the desired property that
\[
\|v^k\|_{L^a(M)} = \frac{\|\tilde{v}^k\|_{L^a(M)}}{\|\tilde{v}^k\|_{L^a(\Omega\setminus M)}} \to \infty, \quad \text{and} \quad \|v^k\|_{L^a(\Omega\setminus M)} = \|\tilde{v}^k\|_{L^a(\Omega\setminus M)} \to 0, \quad (3.8)
\]
as \( k \to \infty \), where we have used the exponent \( a > 1 \).

**Remark 3.5** The construction of the localized potentials for is based on the Runge approximation for the fractional Laplacian, which is a linear fractional differential equation. Notice that one might be able to study the approximation property for the fractional semilinear elliptic equation \( (-\Delta)^s u + q u^m = 0 \) for \( m \geq 2 \), \( m \in \mathbb{N} \), however, one cannot expect the existence of the localized potential for fractional semilinear equations. The reason is due to the well-posedness (Proposition 2.1), which requires sufficiently small exterior data, such that the solution is small as well. Therefore, the well-posedness for the fractional semilinear elliptic equation (1.1) is an obstruction to construct the energy concentration on any (positive) measurable region inside a given domain. This implies that the \( L^a \)-norm of the normalized solution (see (3.8)) can be arbitrarily large in some region is impossible.

4 Converse monotonicity, uniqueness, and inclusion detection

This section consists the proof of the first main result of the work. With the localized potentials (3.8) and the integral identity (3.3) at hand, we can extend Theorem 3.1 to an if-and-only-if statement.

4.1 Converse monotonicity and the fractional calderón problem

Let us prove the if-and-only-if monotonicity relation between the potential and the \( m \)-th order derivative of the DN map.
Proof of Theorem 1.1 Via Theorem 3.1, \( q_1 \geq q_2 \) a.e. in \( \Omega \) implies \( (D_0^m \Lambda_{q_1})_0 \geq (D_0^m \Lambda_{q_2})_0 \) (in the sense of Definition 3.1). The conclusion holds if we can show that \( (D_0^m \Lambda_{q_1})_0 \geq (D_0^m \Lambda_{q_2})_0 \) implies \( q_1 \geq q_2 \) a.e. in \( \Omega \).

Suppose that \( (D_0^m \Lambda_{q_1})_0 \geq (D_0^m \Lambda_{q_2})_0 \) holds, then the integral identity (3.3) yields that
\[
\int_\Omega (q_1 - q_2) v_g^m v_h \, dx \geq 0, \tag{4.1}
\]
where \( v_g = v_h \) if \( m \) is odd and \( v_h \geq 0 \) if \( m \) is even (see Definition 3.1 and Theorem 3.1). In order to show that \( q_1 \geq q_2 \) a.e. in \( \Omega \), we prove it by a standard contradiction argument. Suppose that there exists a constant \( \delta > 0 \) and a positive measurable subset \( M \subset \Omega \) such that \( q_2 - q_1 \geq \delta > 0 \) in \( M \). By applying the localized potentials from Corollary 3.4 for an appropriate exponent \( a > 1 \), which will be determined later. Hence, there must exist a sequence \( \{g^k\} \) such that the corresponding \( s \)-harmonic functions \( v^k \) with \( v^k = g^k \) in \( \Omega \) satisfy
\[
\int_M |v^k|^a \, dx \to \infty \quad \text{and} \quad \int_{\Omega \setminus M} |v^k|^a \, dx \to 0, \tag{4.2}
\]
as \( k \to \infty \).

Combine with (4.1), then we have:

(a) When \( m \) is odd, we take the \( s \)-harmonic functions \( v_g = v_h \) to be the localized potentials \( \{v^k\} \) into (3.3) such that
\[
0 \leq \int_\Omega (q_1 - q_2)|v^k|^{m+1} \, dx
\leq -\delta \int_M |v^k|^{m+1} \, dx + \|q_1 - q_2\|_{L^\infty(\Omega)} \int_{\Omega \setminus M} |v^k|^{m+1} \, dx
\to -\infty,
\]
as \( k \to \infty \), where we have utilized (4.2) as the exponent \( a = m + 1 \), then

(b) When \( m \) is even, we need to use the other monotonicity definition 3.2. In this case, we choose the exterior data \( h \in C^{2,\delta}_0(\Omega_e) \), \( h \geq 0 \) and \( h \neq 0 \). Then by the maximum principle (Proposition B.1) in Appendix B, we must have \( v_h > 0 \) in \( \Omega \). Meanwhile, by using the global \( C^s \) estimate for the solution to the fractional Laplacian, we have \( v_h \in C^s(\mathbb{R}^n) \) whenever \( h \in C^{2,\delta}_0(\Omega_e) \). Thus, by the continuity of \( v_h \), there must exists a constant \( c_h > 0 \) such that \( v_h \geq c_h > 0 \) in \( \Omega_e \).

Now, let us plug the \( s \)-harmonic functions \( v_g \) to be the localized potentials \( \{v^k\} \) and \( v_h > 0 \) into (3.3) such that
\[
0 \leq \int_\Omega (q_1 - q_2)|v^k|^m v_h \, dx
\leq -\delta c_h \int_M |v^k|^m \, dx + \|q_1 - q_2\|_{L^\infty(\Omega)} \|v_h\|_{L^\infty(\Omega)} \int_{\Omega \setminus M} |v^k|^m \, dx
\to -\infty,
\]
as \( k \to \infty \).
The preceding arguments yield a contradiction. This implies that that \( q_1 \geq q_2 \) in \( \Omega \) in both cases (a) and (b). Therefore, we conclude the if-and-only-if monotonicity relations (1.7) holds.

**Corollary 4.1** Let \( \Omega \subset \mathbb{R}^n, n \geq 1 \) be a bounded domain with \( C^{1,1} \) boundary \( \partial \Omega \), and \( 0 < s < 1 \). Let \( m \geq 2, m \in \mathbb{N} \). Let \( q_1, q_2 \in L^\infty(\Omega) \), and \( \Lambda_{q_j} \) be the DN maps of the semilinear elliptic equations \((\Delta)^s u + q_j u^m = 0 \text{ in } \Omega \) for \( j = 1, 2 \). Then we have

\[ q_1 = q_2 \text{ in } \Omega \quad \text{if and only if} \quad (D^m_0 \Lambda_{q_1})_0 = (D^m_0 \Lambda_{q_2})_0. \]

**Proof** The results follows immediately from Theorem 1.1. \( \square \)

**Remark 4.2** We want to point out that:

(a) The if-and-only-if monotonicity relations has been shown by Theorem 1.1 for general potentials \( q_1, q_2 \in L^\infty(\Omega) \), without any sign constraints. For the (linear) fractional Schrödinger equation, the monotonicity relations can be proved by using the Lowner order (see [28, 35]), which involves more functional analysis techniques in the arguments. We also refer readers to the further study [36] for the local case.

(b) Corollary 4.1 is derived via the monotonicity method (Theorem 1.1). In fact, in order to determine \( q_1 = q_2 \) in \( \Omega \), one can only consider the condition of the original DN maps \( \Lambda_{q_1} = \Lambda_{q_2} \) in the exterior domain. The proof is based on the higher order linearization and the Runge approximation for the fractional Laplacian, which needs to prove \((D^m_0 \Lambda_{q_1})_0 = (D^m_0 \Lambda_{q_2})_0\) by assuming \( \Lambda_{q_1} = \Lambda_{q_2} \). For more details in different approaches, we refer the reader to [46].

### 4.2 A monotonicity-based reconstruction formula

In the end of this section, let us demonstrate a proof of the constructive uniqueness for the potential \( q \in L^\infty(\Omega) \) of the fractional semilinear elliptic equation (1.1). Inspired by [27, 28], we will show that the potential \( q \in L^\infty(\Omega) \) can be reconstructed from the DN map \( \Lambda_q \) by testing \( \Lambda \psi \), where \( \psi \) is a simple function.

To this end, let \( M \) be a measurable set, and \( M \) is called a density one set if it is non-empty, measurable and has Lebesgue density 1 for all \( x \in M \). The set of density one simple functions is defined by

\[ \Sigma := \left\{ \psi = \sum_{j=1}^m a_j \chi_{M_j} : a_j \in \mathbb{R}, \ M_j \subseteq \Omega \text{ is a density one set} \right\}. \]

Notice that every simple function agrees with a density one simple function almost everywhere due to the Lebesgue’s density theorem. For our purposes, it is important to control the values on measure zero sets since these values might still affect the supremum when the supremum is taken over uncountably many functions.

We have the following constructive global uniqueness result.

**Theorem 4.1** Let \( n \geq 1, \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^{1,1} \) boundary \( \partial \Omega \), and \( s \in (0, 1) \). Let \( q \in L^\infty(\Omega) \) and \( \Lambda_q \) be the DN maps of the semilinear elliptic equations \((\Delta)^s u + q u^m = 0 \text{ in } \Omega \), where \( m \geq 2, m \in \mathbb{N} \). A potential \( q = q(x) \) can be uniquely recovered by \((D^m \Lambda_q)_0\) via the following formula

\[
q(x) = \sup \left\{ \psi(x) : \psi \in \Sigma, \ D^m \Lambda \psi)_0 \leq (D^m \Lambda \psi)_0 \right\} + \inf \left\{ \psi(x) : \psi \in \Sigma, \ (D^m \Lambda \psi)_0 \geq (D^m \Lambda \psi)_0 \right\},
\]

(4.3)

for all \( x \in \Omega \).
Remark 4.3  For the local case \( s = 1 \) and \( m = 2 \), the reconstruction formula for the potential \( q(x) \) has been studied in \([49, Corollary 3.1]\)^3. The reconstruction formula was using the known the Calderón exponential solutions \([6]\) for the Laplace equation.

To prove Theorem 4.1, let us adapt the following lemma which was shown in \([28, Lemma 4.4]\).

Lemma 4.4  (Simple function approximation) For any function \( q \in L^\infty(\Omega) \), and \( x \in \Omega \) a.e., we have that
\[
\max\{q(x), 0\} = \sup\{\psi(x) : \psi \in \Sigma \text{ with } \psi \leq q\}.
\]

With the preceding lemma at hand, we can prove Theorem 4.1.

Proof of Theorem 4.1  Via Lemma 4.4 and Theorem 1.1, the potential \( q \in L^\infty(\Omega) \) can be reconstructed by
\[
q(x) = \max\{q(x), 0\} - \max\{-q(x), 0\}
= \sup\{\psi(x) : \psi \in \Sigma, \psi \leq q\} - \sup\{\psi(x) : \psi \in \Sigma, \psi \leq -q\}
= \sup\{\psi(x) : \psi \in \Sigma, \psi \leq q\} + \inf\{\psi(x) : \psi \in \Sigma, \psi \geq 0\}
= \sup\{\psi(x) : \psi \in \Sigma, (D^m \Lambda \psi)_0 \leq (D^m \Lambda q)_0\}
+ \inf\{\psi(x) : \psi \in \Sigma, (D^m \Lambda \psi)_0 \geq (D^m \Lambda q)_0\},
\]
for almost everywhere \( x \in \Omega \). This shows (4.3) holds for almost everywhere \( x \in \Omega \). This completes the proof. \( \square \)

4.3 Inclusion detection by the monotonicity test

In this subsection, we will prove the second main result of this paper. The proof is also based on the if-and-only-if monotonicity relations (Theorem 1.1), which can be regarded as an application of the converse monotonicity relation. Recall that the testing operator \( T_M : H^s(\Omega_\epsilon)^m \to H^s(\Omega)^* \) is defined by
\[
\langle (T_M)(g, \ldots, g), h \rangle = \int_M v_g^m v_h \, dx,
\]
where \( v_g \) and \( v_h \) are \( s \)-harmonic in \( \Omega \) with \( v_g = g \) and \( v_h = h \) in \( \Omega_\epsilon \), respectively.

Proof of Theorem 1.2  Let \( \text{supp}(q - q_0) \subset C \), then there must exist some (large) constant \( \alpha > 0 \) such that
\[
-\alpha \chi_C \leq q - q_0 \leq \alpha \chi_C. \quad (4.4)
\]
By using Theorem 1.1, we know that (4.4) is equivalent to
\[
(D^m \Lambda_{q_0 - \alpha \chi_C})_0 \leq (D^m \Lambda q)_0 \leq (D^m \Lambda_{q_0 + \alpha \chi_C})_0. \quad (4.5)
\]

^3 In fact, the reconstruction formula in \([49, Corollary 3.1]\) also holds for general \( m \geq 2 \) with \( m \in \mathbb{N} \) in any bounded Euclidean domain \( \Omega \subset \mathbb{R}^n \).
Furthermore, via the identity for the \( m \)-th order derivative of the DN map (2.11), the elements \((D^m \Lambda_{q_0 \pm \alpha \chi_C})_0\) in (4.5) can be written as
\[
\langle (D^m \Lambda_{q_0 \pm \alpha \chi_C})_0 (g, \ldots, g), h \rangle = \int_{\Omega} (q_0 \pm \alpha \chi_C) v^m g h \, dx = \int_{\Omega} q_0 v^m g h \, dx \pm \alpha \int_{C} v^m g h \, dx = \langle (D^m \Lambda_{q_0})_0 (g, \ldots, g), h \rangle \pm \alpha \langle TM (g, \ldots, g), h \rangle,
\]
where we have used the definition (1.8). Combining (4.5) and (4.6), one obtains
\[
(D^m \Lambda_{q_0})_0 - \alpha T_C \leq (D^m \Lambda_{q_0})_0 \leq (D^m \Lambda_{q_0})_0 + \alpha T_C,
\]
provided that the condition (4.4) holds.

We next prove the converse part that if there exists some \( \alpha > 0 \) such that (4.7) holds, then we must have \( \text{supp}(q - q_0) \subseteq C \). Suppose (4.7) holds, then Theorem 1.1 implies that
\[ -\alpha \chi_C \leq q - q_0 \leq \alpha \chi_C. \]
The above inequality already shows that \( q - q_0 = 0 \) in \( \Omega \setminus C \), which infers that \( \text{supp}(q - q_0) \subseteq C \) as desired. Hence, the assertion is proved by the monotonicity test. \( \square \)

Note that in the statement of Theorem 1.2, we do not need to assume the definite case, i.e., either \( q \geq q_0 \) or \( q \leq q_0 \) in \( \Omega \). We will demonstrate that it is enough to test open sets to reconstruct the inner support for either \( q \geq q_0 \) or \( q \leq q_0 \).

**Definition 4.5** The inner support \( \text{inn supp}(\phi) \) of a measurable function \( \phi : \Omega \to \mathbb{R} \) is the union of all open sets \( U \subseteq \Omega \), for which the essential infimum \( \text{ess inf}_{x \in U} |\phi(x)| \) is positive.

**Theorem 4.2** Let \( q_0, q \in L^\infty(\Omega) \) be potentials. For the definite case, we have:

(a) Let \( q \leq q_0 \). For every open set \( B \subseteq \Omega \) and every \( \alpha > 0 \)
\[
q \leq q_0 - \alpha \chi_B \implies (D^m \Lambda_q)_0 \leq (D^m \Lambda_{q_0})_0 - \alpha T_B \implies B \subseteq \text{supp}(q - q_0).
\]
Thus,
\[
\text{inn supp}(q - q_0) \subseteq \bigcup \{ B \subseteq \Omega \text{ open ball} : \exists \alpha > 0 : (D^m \Lambda_q)_0 \leq (D^m \Lambda_{q_0})_0 - \alpha T_B \} \subseteq \text{supp}(q - q_0).
\]

(b) Let \( q \geq q_0 \). For every open set \( B \subseteq \Omega \) and every \( \alpha > 0 \)
\[
q \geq q_0 + \alpha \chi_B \implies (D^m \Lambda_q) \geq (D^m \Lambda_{q_0})_0 + \alpha T_B,
\]
and
\[
(D^m \Lambda_q)_0 \geq (D^m \Lambda_{q_0})_0 + \alpha T_B \implies q \geq q_0 + \alpha \chi_B.
\]
Thus,
\[
\text{inn supp}(q - q_0) = \bigcup \{ B \subseteq \Omega \text{ open ball} : \exists \alpha > 0 : (D^m \Lambda_q)_0 \geq (D^m \Lambda_{q_0})_0 + \alpha T_B \}.
\]
Theorem 1.1, the inequality (4.11) must yield
With the localized potential for the fractional Laplacian at hand (similar to the proof of Theorem 1.2), we have that

\[(D^m \Lambda q)_0 - (D^m \Lambda q_0)_0 \leq -\alpha T_B.\]

Moreover, if \((D^m \Lambda q)_0 \leq (D^m \Lambda q_0)_0 \leq -\alpha T_B\), by Theorem 1.1 and Theorem 1.2 again, that there exists \(c > 0\) with

\[\alpha T_B \leq (D^m \Lambda q_0)_0 - (D^m \Lambda q)_0 = \int_\Omega (q_0 - q)v^m g v_h \, dx,
\]
which implies

\[\int_\Omega (\alpha \chi_B v^m g v_h - \|q - q_0\|_{L^\infty(\Omega)} \chi_{\text{supp}(q - q_0)} v^m g v_h) \, dx \leq 0. \tag{4.11}\]

With the localized potential for the fractional Laplacian at hand (similar to the proof of Theorem 1.1), the inequality (4.11) must yield

\[\alpha \chi_B \leq \|q - q_0\|_{L^\infty(\Omega)} \chi_{\text{supp}(q - q_0)}.\]

(b) The results (4.9) and (4.10) are simple applications of Theorem 1.1. \(\Box\)

5 Lipschitz stability with finitely many measurements

In the last section of this paper, we prove Theorem 1.4, and the ideas of the proof are from [26, 28].

Proof of Theorem 1.3 Let us divide the proof into several steps.

Step 1. Fundamental estimates

For \(q_1 \neq q_2\) in \(\Omega\) with \(q_1, q_2 \in \mathcal{Q}\), we want to show that

\[
\frac{\|D^m \Lambda_{q_1}\|_0 - D^m \Lambda_{q_2}\|_0\|_*}{\|q_1 - q_2\|_{L^\infty(\Omega)}} \geq \inf_{\kappa \in \mathcal{K}} \sup_{g, h \in C^2_0(\Omega_e), \|g\|_{H^s} = \|h\|_{H^s} = 1} \Phi(\kappa, g, h), \tag{5.1}
\]
where \(\Phi : \mathcal{K} \times C^2_0(\Omega_e) \times C^2_0(\Omega_e)\) is given by

\[
\Phi(\kappa, g, h) := \max \{(D^m \Lambda_{\kappa})_0(g, \ldots, g, g)\},
\]
and \(\mathcal{K} = \{\kappa \in \text{span} \mathcal{Q} : \|\kappa\|_{L^\infty(\Omega)} = 1\}\) is a finite-dimensional subspace of \(L^\infty(\Omega)\). By the definition of \(\|\cdot\|_*\), we have

\[
\|(D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0\|_* = \sup_{g, h \in C^2_0(\Omega_e), \|g\|_{H^s} = \|h\|_{H^s} = 1} \langle (D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0(g, \ldots, g, h) \rangle
\]
and

\[
\left|\langle (D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0(g, \ldots, g, h) \rangle\right|
\]

\[
= \max \{(D^m \Lambda_{q_1})_0 - (D^m \Lambda_{q_2})_0(g, \ldots, g, h), \langle (D^m \Lambda_{q_2})_0 - (D^m \Lambda_{q_1})_0(g, \ldots, g, h) \rangle\}
\]

\[
= \|q_1 - q_2\|_{L^\infty(\Omega)} \max \left\{\int_\Omega \frac{q_1 - q_2}{\|q_1 - q_2\|_{L^\infty(\Omega)}} v^m g v_h \, dx, \int_\Omega \frac{q_2 - q_1}{\|q_1 - q_2\|_{L^\infty(\Omega)}} v^m g v_h \, dx\right\}
\]

\[
= \|q_1 - q_2\|_{L^\infty(\Omega)} \Phi \left(\frac{q_1 - q_2}{\|q_1 - q_2\|_{L^\infty(\Omega)}}, g, h\right),
\]

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where we have utilized the integral identity (3.3) and the linearity of the \(m\)-th order derivative of the DN map \(\Lambda_{q}\) in the above computations. Therefore, the claim (5.1) must hold.

**Step 2. Positive lower bound of \(\Phi\)**

We next show that there exists \(\widehat{\kappa} \in \mathcal{K}\) such that

\[
\inf_{\kappa \in \mathcal{K}} \sup_{g, h \in C_{0}^{2,\infty}(\Omega_{\ast}), \|g\|_{H^{s}} = \|h\|_{H^{s}} = 1} \Phi(\kappa, g, h) = \sup_{g, h \in C_{0}^{2,\infty}(\Omega_{\ast}), \|g\|_{H^{s}} = \|h\|_{H^{s}} = 1} \Phi(\widehat{\kappa}, g, h).
\]

The fact directly follows by the smoothness of the DN map (see Sect. 2) such that the function

\[
\kappa \mapsto \sup_{g, h \in C_{0}^{2,\infty}(\Omega_{\ast}), \|g\|_{H^{s}} = \|h\|_{H^{s}} = 1} \Phi(\kappa, g, h)
\]

is lower semicontinuous and its minimum can be achieved over the compact set \(\mathcal{K}\) (a finite dimensional subspace of \(L^{\infty}(\Omega)\)).

Finally, since \(q_{1} - q_{2} \neq 0\) in \(\Omega\), by applying the localized potentials for \(s\)-harmonic functions (Corollary 3.4), there must exist \(g, h \in H^{s}(\Omega_{\ast})\) such that

\[
\text{either } \int_{\Omega} \kappa v_{g}^{m} v_{h} \, dx > 0 \quad \text{or} \quad \int_{\Omega} \kappa v_{g}^{m} v_{h} \, dx < 0,
\]

where we have utilized the fact that \(\kappa \in \text{span}\mathcal{Q}\). Hence, we can obtain

\[
c_{0} := \sup_{g, h \in C_{0}^{2,\infty}(\Omega_{\ast}), \|g\|_{H^{s}} = \|h\|_{H^{s}} = 1} \Phi(\kappa, g, h) > 0, \quad \text{for any } \kappa \in \mathcal{K},
\]

which completes the proof. \(\square\)

It remains to prove our last theorem in the paper.

**Proof of Theorem 1.4** By using

\[
\left\| P'_{H_{\ell}} \left( (D^{m}\Lambda_{q_{2}})_{0} - (D^{m}\Lambda_{q_{1}})_{0} \right) P_{H_{\ell}} \right\|_{*} = \sup_{g, h \in H_{\ell}} \left| \left\langle \left( (D^{m}\Lambda_{q_{2}})_{0} - (D^{m}\Lambda_{q_{1}})_{0} \right) (g, \ldots, g), h \right\rangle \right|,
\]

and applying the preceding arguments, for any \(\ell \in \mathbb{N}\), there exists \(\kappa_{\ell} \in \mathcal{K}\) such that

\[
\left\| P'_{H_{\ell}} \left( (D^{m}\Lambda_{q_{2}})_{0} - (D^{m}\Lambda_{q_{1}})_{0} \right) P_{H_{\ell}} \right\|_{*} \geq \sup_{g, h \in H_{\ell}, \|g\|_{H^{s}} = \|h\|_{H^{s}} = 1} \Phi(\kappa_{\ell}, g, h).
\]

Notice that the right hand side of (5.3) is monotonically increasing in \(\ell \in \mathbb{N}\), since \(H_{1} \subseteq H_{2} \subseteq \ldots \subseteq H_{\ell} \subseteq \ldots \subseteq H^{s}(\Omega_{\ast})\). Therefore, Theorem 1.4 holds if we can prove that there is \(\ell \in \mathbb{N}\) such that

\[
\sup_{g, h \in H_{\ell}, \|g\|_{H^{s}} = \|h\|_{H^{s}} = 1} \Phi(\kappa, g, h) > 0, \quad \text{for all } \kappa \in \mathcal{K}.
\]

We prove the claim (5.4) by a contradiction argument, i.e., there must exist a sequence \((\kappa_{\ell})_{\ell \in \mathbb{N}} \subset \mathcal{K}\) such that

\[
\sup_{g, h \in H_{\ell}, \|g\|_{H^{s}} = \|h\|_{H^{s}} = 1} \Phi(\kappa_{\ell}, g, h) \leq 0, \quad \text{for } \ell \geq m.
\]
for any \( m \in \mathbb{N} \). After passing a subsequence (if necessary), by the compactness (due to the finite dimensional assumption of \( Q \)), we can assure that there exists an element \( \kappa_\infty \in \mathcal{K} \) such that
\[
\lim_{\ell \to \infty} \kappa_\ell = \kappa_\infty
\]
and
\[
\sup_{g, h \in H_m, \|g\|_{H^s} = \|h\|_{H^s} = 1} \Phi(\kappa_\ell, g, h) \leq \lim_{\ell \to \infty} \sup_{g, h \in H_m, \|g\|_{H^s} = \|h\|_{H^s} = 1} \Phi(\kappa_\ell, g, h) = 0,
\]
where we have utilized the lower semicontinuous of the function
\[
\kappa \to \sup_{g, h \in H} \Phi(\kappa, g, h).
\]
On the other hand, by the continuity, we must have
\[
\Phi(\kappa_\infty, g, h) \leq 0, \quad \text{for all } g, h \in \bigcup_{m \in \mathbb{N}} H_m = H^s(\Omega_e),
\]
which contradicts to (5.1) in the previous proof. This proves that (5.4) must hold for some \( \ell \in \mathbb{N} \) as desired.

\[\square\]

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Appendix A. The \( L^p \)-estimate for the fractional laplacian

Let us review the other estimates for solutions to the fractional Laplacian: The \( L^p \) estimate. Before doing so, let us recall some fundamental properties for the Riesz potential.

Proposition A.1 (Riesz potential) For \( 0 < s < 1 \) with \( n > 2s \). Let \( V \) and \( F \) satisfy
\[
V = (-\Delta)^{-s} F \text{ in } \mathbb{R}^n,
\]
in the sense that \( V \) is the Riesz potential of order \( 2s \) of the function \( F \).

(a) If \( F \in L^1(\mathbb{R}^n) \), then there exists a constant \( C > 0 \) depending only on \( n \) and \( s \) such that
\[
\|V\|_{L^p_w(\mathbb{R}^n)} \leq C \|F\|_{L^1(\mathbb{R}^n)},
\]
where \( L^p_w \) denotes the weak-\( L^p \) norm and \( p = \frac{n}{n - 2s} \).

(b) For \( r \in (1, \frac{n}{2s}) \), \( F \in L^r(\mathbb{R}^n) \), then there exists a constant \( C > 0 \) depending only on \( n \), \( s \), and \( r \) such that
\[
\|V\|_{L^p(\mathbb{R}^n)} \leq C \|F\|_{L^r(\mathbb{R}^n)},
\]
where \( p = \frac{nr}{n - 2rs} \).

(c) For \( r \in (\frac{n}{2s}, \infty) \), then there exists a constant \( C > 0 \) depending only on \( n \), \( s \), and \( r \) such that
\[
[u]_{C^s(\mathbb{R}^n)} \leq C \|F\|_{L^r(\mathbb{R}^n)},
\]
where \( \alpha = 2s - \frac{n}{p} \) and \( [u]_{C^s(\mathbb{R}^n)} \) is the seminorm given in Sect. 2.
**Proof** Parts (a) and (b) are classical results for the Riesz potential, and the proof can be found in Stein’s book [61, Chapter V]. For (c), we refer readers to [61, p.164] and [16]. □

Furthermore, we have the following Hölder estimate for the fractional Laplacian, which was shown in [56, Proposition 1.7]. We state the result in the following proposition and the proof can be found in [56].

**Proposition A.2** *(Cβ-estimate)* For $n \geq 1$, $0 < s < 1$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary $\partial \Omega$. Let $h \in C^\alpha(\Omega_e)$ for some $\alpha \in (0, 1)$. Let $w$ be the solution of

\[
\begin{aligned}
(-\Delta)^s w &= 0 \quad \text{in } \Omega, \\
w &= h \quad \text{in } \Omega_e.
\end{aligned}
\]

Then the solution $w \in C^\beta(\mathbb{R}^n)$, where $\beta = \min\{s, \alpha\}$, and

$$
\|w\|_{C^\beta(\mathbb{R}^n)} \leq C \|h\|_{C^\alpha(\Omega_e)},
$$

for some constant $C > 0$ depending only on $\Omega$, $\alpha$, and $s$.

The following proposition was also proved in [56], which is an important result in the proof of our Runge approximation (Theorem 3.2). We state the result and prove it for the sake of completeness. The proof is based on the preceding properties of the Riesz potential, the maximum principle for the fractional Laplacian and the $C^\beta$-estimate (Proposition A.2).

**Proposition A.3** For $n \geq 1$, $0 < s < 1$, let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary $\partial \Omega$. For $F \in L^r(\Omega)$, let $v$ be the solution of

\[
\begin{aligned}
(-\Delta)^s v &= F \quad \text{in } \Omega, \\
v &= 0 \quad \text{in } \Omega_e.
\end{aligned}
\]

(A.1)

then we have:

(a) Let $n > 2s$, $r = 1$, and $p \in [1, \frac{n}{n-2s})$ be an arbitrary number, then there exists a constant $C > 0$ independent of $v$ and $F$ such that

$$
\|v\|_{L^p(\Omega)} \leq C \|F\|_{L^1(\Omega)}.
$$

(b) Let $n > 2s$, $r \in (1, \frac{n}{2s})$ and $p = \frac{nr}{n-2sr}$, then there exists a constant $C > 0$ independent of $v$ and $F$ such that

$$
\|v\|_{L^p(\Omega)} \leq C \|F\|_{L^r(\Omega)}.
$$

(c) Let $n > 2s$, $r \in (\frac{n}{2s}, \infty)$, and $\beta = \min\{s, 2s - \frac{n}{r}\}$, then there exists a constant $C > 0$ independent of $v$ and $F$ such that

$$
\|v\|_{C^\beta(\Omega)} \leq C \|F\|_{L^r(\Omega)}.
$$

(d) Let $n = 1$, $s \in [\frac{1}{2}, 1)$, $r \geq 1$, and any $p < \infty$, then there exists a constant $C > 0$ independent of $v$ and $F$ such that

$$
\|v\|_{L^p(\Omega)} \leq C \|F\|_{L^r(\Omega)}.
$$
**Proof** (a) Let us extend the function $F$ by 0 outside $\Omega$, and we still denote the function as $F$. Let $V$ be the solution of

$$(-\Delta)^s V = |F| \text{ in } \mathbb{R}^n,$$

so that $V = (-\Delta)^{-s}|F|$ in $\mathbb{R}^n$, where $(-\Delta)^{-s}|F|$ is the Riesz potential of $|F|$. By the definition of the Riesz potential, we have $V \geq 0$ in $\Omega_e$. Via the maximum principle, we obtain that $|v| \leq V$ in $\Omega$. By applying Proposition A.1, one can see that

$$\|v\|_{L^q(\Omega)} \leq \|V\|_{L^q(\Omega)} \leq C\|F\|_{L^1(\Omega)},$$

if $F \in L^1(\Omega)$ and for some constant $C > 0$ independent of $v$ and $F$. Thus, one has

$$\|v\|_{L^r(\Omega)} \leq C\|F\|_{L^1(\Omega)},$$

for some constant $C > 0$ independent of $v$ and $F$. This proves (a).

(b) Similarly, the proof of (b) can be completed by using the result (b) in Proposition A.1 and the maximum principle for the fractional Laplacian as before. Furthermore, when $r = \frac{n}{2s}$, it is easy to see that $F \in L^r(\Omega) \subset L^{\tilde{r}}(\Omega)$, for any $\tilde{r} \in [1, r]$ (since $\Omega$ is bounded). We still have the $L^p$ estimate for the solution in the borderline case $r = \frac{n}{2s}$.

(c) Let us write $v = \tilde{v} + w$, where $\tilde{v}$ and $w$ are given by

$$\tilde{v} = (-\Delta)^{-s} F \text{ in } \mathbb{R}^n,$$

and

$$\left\{ \begin{array}{ll} (-\Delta)^s w = 0 & \text{in } \Omega, \\ w = \tilde{v} & \text{in } \Omega_e. \end{array} \right. \quad (A.3)$$

By using (A.2) and Proposition A.1 (c), there exists a constant $C > 0$ depending only on $n$, $s$, and $r$ such that

$$[\tilde{v}]_{C^\alpha(\mathbb{R}^n)} \leq C\|F\|_{L^r(\mathbb{R}^n)}, \quad \text{where } \alpha = 2s - \frac{n}{r}. $$

Since $\Omega$ is bounded and $F$ is compactly supported, one has $\tilde{v}$ decays at infinity. This implies that

$$\|\tilde{v}\|_{C^\alpha(\mathbb{R}^n)} \leq C\|F\|_{L^r(\mathbb{R}^n)}, \quad \text{where } \alpha = 2s - \frac{n}{r}, \quad (A.4)$$

for some constant $C > 0$ depending only on $n$, $s$, $r$ and $\Omega$.

On the other hand, we can apply Proposition A.2 to derive the Hölder estimate for the solution $w$ of (A.3) that

$$\|w\|_{C^\beta(\mathbb{R}^n)} \leq C\|\tilde{v}\|_{C^\alpha(\Omega_e)}, \quad (A.5)$$

for some constant $C > 0$ depending only on $\Omega$, $\alpha$, and $s$, where

$$\beta = \min\{\alpha, s\} = \min\left\{s, 2s - \frac{n}{r}\right\}. $$

Combining with (A.4) and (A.5), we can obtain the Hölder estimate for the solution $v = \tilde{v} + w$ such that (c) holds. Moreover, since $v \in C^\beta(\Omega)$ with $v = 0$ in $\Omega_e$, we must have $v \in L^p(\mathbb{R}^n)$ for any $p \geq 1$.

(d) Notice that for $s < \frac{1}{2}$, we have $n = 1 > 2s$ automatically, such that the case (d) holds by applying the results either (a) or (b). On the other hand, for the case $1 = n \leq 2s$, this implies that $\frac{1}{2} \leq s < 1$. Under this situation, any bounded domain is of the form
Ω = (a, b) ⊂ ℝ. By [2], the Green function G(x, y) for the exterior value problem (A.1) is explicit. Furthermore, G(·, y) ∈ L^∞(Ω) when s > 1/2 and G(x, y) ∈ L^r(Ω) for any r < ∞ when s = 1/2. Therefore, one has

||v||_{L^\infty(\Omega)} \leq C ||F||_{L^1(\Omega)},

for some constant C > 0 independent of v and F, where n < 2s. For the case n = 2s, we have either

||v||_{L^p(\Omega)} \leq C ||F||_{L^1(\Omega)}, \quad \text{for all } p < \infty,

or

||v||_{L^\infty(\Omega)} \leq C ||F||_{L^r(\Omega)}, \quad \text{for } r > 1,

for some constant C > 0 independent of v and F. □

Remark A.4 From the L^p estimate of s-harmonic functions, we have:

(a) No matter what exponent r ≥ 1 and what space dimension n are, for any F ∈ L^r(Ω) with Ω ⊂ ℝ^n in the statement of Proposition A.3, then we can always conclude that the solution v of (A.1) must belong to L^p(ℝ^n), for some p > 1.

(b) Moreover, since the domain Ω is bounded in ℝ^n, then we can confine the exponent p in the region p ∈ (1, 2). The condition p ∈ (1, 2) plays an essential role in order to prove Proposition 3.3 (see [9, Section 4] for more detailed discussions about the strong uniqueness of the s-harmonic function). Meanwhile, we also need to use the L^p-estimate to prove the Runge approximation via the strong uniqueness for the fractional Laplacian in Sect. 3.

Appendix B. The maximum principle

We review the known maximum principle for the fractional Laplacian in the end of this work. These results were shown in [5, 54] for the fractional Laplacian equation and [45, 46] for the fractional Schrödinger equation. For the sake of convenience, we state the results as follows.

Proposition B.1 (The maximum principle) Let Ω ⊂ ℝ^n, n ≥ 1 be a bounded domain with Lipschitz boundary ∂Ω, and 0 < s < 1. Let v ∈ H^s(ℝ^n) be the unique solution of

\[\begin{cases} 
(-\Delta)^s v = F & \text{in } \Omega, \\
v = g & \text{in } \Omega_e. 
\end{cases}\]

Suppose that 0 ≤ F ∈ L^\infty(\Omega) in Ω and 0 ≤ g ∈ L^\infty(\Omega_e) in Ω_e. Then v ≥ 0 in Ω. Moreover, if g ≠ 0 in Ω_e, then v > 0 in Ω.

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