Phase diagram of Landau-Zener phenomena in coupled one-dimensional Bose quantum fluids

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We study stationary and dynamical properties of the many-body Landau-Zener dynamics of a Bose quantum fluid confined in two coupled one-dimensional chains, using a many-body generalization recently reported \cite{Y.-A. Chen et al.}, within the decoupling approximation and the one-level band scheme. The energy spectrum evidences the structure of the avoided level crossings as a function of the on-site inter particle interaction strength. On the dynamical side, a phase diagram of the transfer efficiency across ground-state and inverse sweeps is presented. A totally different scenario with respect to the original single-particle Landau-Zener scheme is found for ground-state sweeps, in which a breakdown of the adiabatic region emerges as the sweep rate decreases. On the contrary, the transfer efficiency across inverse sweeps reveals consistent results with the single-particle Landau-Zener predictions. In the strong coupling regime, we find that there is a critical value of the on-site interaction for which the transfer of particles starts to vanish independently of the sweep rate. Our results are in qualitative agreement with those of the experimental counterpart.

I. INTRODUCTION.

The original Landau-Zener (LZ) problem involves a two-level system whose energy separation varies as a linear function of time. These two states can be associated to the lowest states of a single-particle in a double-well potential. As it is well known \cite{1, 2}, the transition probability between the two energy states is the well known LZ formula \( P_{\text{LZ}} = \exp\left(-\frac{2\pi J^2}{\hbar \alpha} \right) \) written in terms of the tunnelling-coupling parameter \( J \) and the time-dependent detunning \( \Delta = \alpha t \), accounting for the probability that a particle initially in the bottom well at \( t = -\infty \) reaches at \( t = \infty \) the opposite well. Recent experiments have demonstrated the many body generalization of the LZ phenomena by loading an ultracold Bose gas into a pair of coupled one-dimensional chains where a controllable inter-chain sweep leads to the observation of both, avoided crossings and breakdown of adiabatic inter-chain transfer \cite{3}.

Before the experimental realization of the many body LZ generalization, several theoretical descriptions of a Bose-Einstein condensate (BEC) confined in a double-well potential have addressed the LZ problem within two intrinsically different approaches. We make reference to the two-mode mean-field and Bose-Hubbard schemes inherited from the Gross-Pitaevskii and full quantum approaches \cite{4, 5}. As a result of incorporating the linear variation in time between the two levels, all of those treatments suggested a breakdown of the adiabatic limit, that is, that the LZ transition probability does not vanish even in the adiabatic limit. For example, the non-linear two-level system shows that the mean-field interactions among particles tend to increase the tunnelling probability and that there exists a critical value of the interaction strength beyond which the transition probability becomes nonzero even in the adiabatic limit \cite{5}. On the other hand, a stationary phase approximation leads to a characteristic scaling or power law for the critical behavior that occurs as the nonlinear parameter equals the gap of avoided crossing energy levels \cite{6}. Regarding the asymmetric LZ tunnelling in a periodic potential, Jonas-Lasinio et. al. \cite{9} found that the tunnelling rates for the two directions of tunnelling are not the same. Tunnelling from the ground state to the excited state is enhanced by the nonlinearity, where as in the opposite direction it is suppressed. Even more, a LZ formula has been derived for the two mode many-particle scenario \cite{7}.

Here, we are devoted to the description of the many body LZ generalization described in the first paragraph. That is, we are interested in describing a BEC confined in two-coupled 1D chains where the potential depths, defining the lattice sites, are linearly modified in time such that a sweep from an initial energy difference \(-\Delta\) to a final one \(\Delta\) is achieved. To analyze such a system, we shall use a model Hamiltonian that incorporates both, the full quantum frame of the Bose-Hubbard Hamiltonian \cite{10} and a decoupling approximation scheme \cite{11, 12} that involves certain superfluid order parameters. These are \(\psi_{\nu} = \langle b_{\nu}^\dagger \rangle = \langle b_{\nu} \rangle\), where \(b_{\nu}^\dagger\) and \(b_{\nu}\), \(\nu \in \{L, R\}\) create and annihilate particles in lattice sites \(i\) and side chain \(\nu \in L, R\). The average is the expectation value in the investigated states. A previous theoretical work has employed the time-dependent density matrix renormalization group method \cite{13} where, with a small number of links in the chains, qualitative agreement with the experiment was found. Our model simulates two coupled infinite chains with a local site Hamiltonian written in terms of the on-site non-perturbative inter-particle interaction strength \(U\), the time-dependent detunning \(\Delta\) and the intra- and inter-chain coupling energies \(J^\parallel\) and \(J^\perp\). The model satisfactorily describes qualitatively the main features of the experiment: the transfer efficiency across...
ground-state sweeps, that in turn exhibits a breakdown of adiabaticity as the sweep rate decreases, and the transfer efficiency across inverse sweeps, that actually show consistent results with the single-particle Landau-Zener scenario. Regarding the single-particle LZ result the model Hamiltonian allows us to recover the result for a single dimer. In addition, since the number of particles per lattice sites in the actual experiments is not too large, it is possible to perform accurate numerical simulations of the system. Such a analysis gives rise to a richer spectrum with respect to the single dimer considered in [3].

The manuscript is organized in VI sections. In section II we introduce the model Hamiltonian that provides a suitable description to the experiment. We also justify the use of the decoupling approximation and the one band scheme within the full quantum and mean-field approximation schemes. In section III we characterize the stationary states as a function of the both, the inter-particle interaction strength $U$ and the the parameter characterizing the tilt $\Delta$. In particular, we concentrate in determining the energy spectrum for the model Hamiltonian and the behaviour of the superfluid order parameters $\psi_i$. An exhaustive exploration of the parameter space allows us to encode the dynamical behavior of the of ground-state and inverse sweeps in a phase diagram. The latter are presented in sections IV and V respectively. Finally, in section VI a summary of the main results is given.

II. THE MODEL, DECOUPLING APPROXIMATION IN THE ONE-LEVEL BAND PICTURE

The physical system that we want to describe is a finite collection of interacting Bose atoms initially placed in one of the sides of a composed pairwise coupled lattice, see Fig. 1 and its subsequent evolution when the pairwise potential depths are linearly modified in time. As schematically shown, the transport of particles in $x$ and $z$ directions occurs as a consequence of the intrinsic tunneling among the pairwise coupled sites, but it is also affected by the external variation of the well potential depths with time in the form of a sweep from an initial energy difference $\Delta_i = -\Delta$ to a final one $\Delta_f = \Delta$. The parameters responsible for such a transport are the on-site inter-particle interaction strength $U$, the intra-chain coupling energy $J^\parallel$ ($z$ direction), the inter-chain coupling energy $J^\perp$ ($x$ direction) and the time-dependent parameter characterizing the tilt between the two wells of each site in the chain, namely, the detuning parameter $\Delta$.

The purpose of the present paper is to fully characterize stationary and dynamical properties of the system described above. To proceed, we will consider the experimental realization reported in Ref. [3] where the transfer of particles in $x$ and $z$ directions is studied in what it is called the ground-state and inverse sweeps, the former accounting for a sweep starting from the ground state, namely, where all the particles initiate in the bottom well with lower energy, and the latter where the initially filled sites are the ones with higher energy. This scheme leads us to answer several questions, on one side those related to the stationary properties, such as, the energy spectrum, and on the other side, aspects related to the dynamical behavior of the atomic population in the wells. Of special interest in this regard is the determination of the phase diagram encoding the dependence of the transfer efficiency on the parameters characterizing the dynamics.

Let us now turn our attention to the model Hamiltonian of the system. Since the particles are not allowed to tunnel between left and right wells of different sites $i$, movements in $x$ and $z$ directions must be considered separately. Therefore, we consider the dynamics along the sites and their counterpart between the left and right chains.

Regarding the two-well geometry and its intrinsic asymmetry supplied by the time-dependent parameter $\Delta$, we should first argue about the level picture scheme where our system is appropriately described. Due to the variation of such a parameter not just the first band but excited levels might play a significant role. As established by Dounas-Frazer et. al. [10] the transport phenomena of a Bose-Einstein condensate in an asymmetric double well leads to three new energy scales with respect to the symmetric trap and one-band assumptions. These are the hopping, tilt and the energy gap between the first and second bands. The task of including at least a two-band picture (that is, four levels per site) leads to a Hilbert space of size $(N + 1)(N + 2)(N + 3)/6$ with $N$ the number of atoms per site. However, since the LZ phenomena involves the presence of just two states, we should restrict ourselves to the first band of the time-varying tilted double-well potential. This can be done by appropriately taking the formal bounds where a one-band picture is well justified, which primarily assumes that the interaction energy is much smaller that the energy band difference. In the appendix we present a study

![FIG. 1. (color online) Schematic representation of the double-well trap chain potential. The coupling energies are intra-chain $J^\parallel$ ($z$ direction) and inter-chain $J^\perp$ ($x$ direction). The time-dependent parameter $\Delta$ characterizes the tilt between the two wells of each site in the chain.](image-url)
of the single-particle energy levels in a double-well potential as a function of the tilt $\Delta$. To fulfill the condition that the inter-particle energy interaction is much smaller than the energy gap between the first and second bands we shall confine ourselves to values of $\Delta$ for which the bands are well separated and thus the condition with the inter-particle interaction will be considered accordingly.

Starting from the Bose-Hubbard Hamiltonian of an optical lattice \cite{23}, one can write down the Hamiltonian of atoms in the two-well chain as,

$$\mathcal{H} = \mathcal{H}_i^{\parallel} + \sum_i \mathcal{H}_i^{\perp} + \sum_i \mathcal{H}_i^U$$

where the labels $\parallel$ and $\perp$ denote movements in $z$ and $x$ directions respectively. $\mathcal{H}_i^{\parallel}$ represents the tunnelling along the chains, while $\mathcal{H}_i^{\perp}$ takes into account the local two-level structure at sites $i$. $\mathcal{H}_i^U$ corresponds to the on-site interaction energy.

To consider the participation of the lattice sites, that is the transport in the $z$ direction, we use the decoupling approximation where order parameters $\psi_\nu = \langle \hat{b}_{i,\nu}\rangle = \langle \hat{b}_{i,\nu}^\dagger \rangle$, $\nu \in \{L, R\}$ are introduced to account for the expectation value of the tunnelling of particles between left and right wells, disregarding the position of the site $i$ within the chain. We consider negligible the average the number of particles per site on left and right sides, ($z$ direction is accounted for in a mean-field fashion, the tunnelling among left and right sides ($x$ direction) is fully quantum described). The assumption of the mean-field approach in this case is well justified in the sense that in the experimental situation near the centre of the trap, the average the number of particles per site on left and right sides, is the same along the chains. As we shall see in the next sections, this treatment allows us to study up to $N = 8$ particles per site.

III. STATIONARY STATES

In this section we study the stationary states of Hamiltonian (3) for a given number of particles, as a function of the interaction strength $U$ and the parameter characterizing the tilt $\Delta$. In particular, we solve the time-independent Schrödinger equation for the model Hamiltonian to determine the energy spectrum as a function of the parameters $U$ and $\Delta$. It is important to point out that in our calculations both, the parameters $U$ and $\Delta$ are referred to the tunnelling coupling parameter $J = J^\parallel = J^\perp$. Here and henceforth we use $\bar{U} = U / J$ and $\bar{\Delta} = \Delta / J$.

To determine the entire energy spectrum of Hamiltonian (3) we have to find the value of the order parameters $\psi_\nu$ ($\nu = L, R$). We proceed as follows. For fixed values of $\bar{U}$ and $\bar{\Delta}$ we obtain by means of a variational procedure the values of $\psi_\nu$ ($\nu = L, R$) \cite{15} that minimize the ground state energy $E_0(\bar{U}, \bar{\Delta}, \psi_L, \psi_R)$. In other words, we search for the lowest eigenvalue of the Hamiltonian $\mathcal{H}_i^{\text{loc}}(\bar{U}, \bar{\Delta}, \psi_L, \psi_R)$ by minimizing with respect to $\psi_\nu$ ($\nu = L, R$). Once those optimal values have been de-
been rescaled with respect to \( \tilde{E} \) different values of \( \tilde{\Delta} \). The circles indicate how tunnelling resonances are crossed and refer to the number of particles in the right well, while the circles of a single double-well. The right figures of the panel (Fig.1) that is, they correspond to the energy spectrum of Hamiltonian \( \tilde{H} \) for \( N = 6 \).

For illustration purposes, in Fig. 4 we show the energy spectrum \( \tilde{\Delta} \) for \( U = U/J > > 1 \) split into \( 2N - 1 \) avoided level crossings. Regarding the number of avoided crossing and its dependence with the number of particles \( N \) in the limit of non-interacting bosons, the single-particle \( \text{LZ} \) result is recovered for weaker interactions when the transport in \( x \) and \( z \) axis is considered. It is important to emphasize that the fact of considering the combined transport in the parallel and perpendicular directions, in general gives rise to a richness spectrum with respect to the single dimer. Even more, leads to predict additional tunnelling resonances with respect to the single dimer.

### IV. PHASE DIAGRAM

The \( \text{LZ} \) dynamics from \( \tilde{\Delta} = -\alpha T_{\text{max}} \) to \( \tilde{\Delta} = \alpha T_{\text{max}} \) for a given value of \( \alpha \) are obtained by evolving in time \( (\tilde{\Delta} = \alpha t) \) the initial state \( |\Psi(0)\rangle \) associated to the ground state energy \( \tilde{E}_0(U, \tilde{\Delta}) \). We have that, for a given value of \( \alpha \) one should follow the evolution of the state where all the particles initiate in the left chain with \( \tilde{\Delta} < 0 \), such that the filled chain is the one with lower energy. Experimentally, this process is identified as a ground-state sweep. The evolution is given by [17],

\[
|\Psi(t_{i+1})\rangle \approx e^{-iH(U, \tilde{\Delta})\delta t}|\Psi(t_i)\rangle
\]

where \( H \) is the local Hamiltonian \( \tilde{H} \) and \( \hbar = 1 \) is assumed. The initial time \( t_0 \) corresponds to \( T_{\text{max}} = -\Delta_0/\alpha \) and the temporal step \( \delta t \) that we select, \( \delta t = 0.2 \), produces qualitatively similar results than \( \delta t \leq 0.2 \). It is important to emphasize that the minimization procedure described above (section II) is performed at each temporal step \( \delta t \) to take into account the variation of \( \tilde{\Delta} \) at each time interval \( \delta t \). Consequently, the ground state \( |\Psi(t_i)\rangle \) is updated at each temporal step and serves as a seed for the subsequent time. The full dynamics, for a given rate \( \alpha \) and inter particle interaction strength \( \tilde{U} \), ends when the time \( t_{i+1} \) reaches its maximum value \( T_{\text{max}} = \tilde{\Delta}_0/\alpha \). To illustrate the evolution in time of the population on the right side, in Fig. 5(a,b,c) we plot \( n_R(t) \) for various rates \( \alpha \), \( \tilde{U} = 0.75 \) and \( N = 6 \).
time is rescaled with respect to $T_{\text{max}}$, $\tilde{t} = t/T_{\text{max}}$ with $\Delta_0 = 20$.

For $N = 6$ we investigated the many-body dynamics for sweep rates in the interval $2\pi/\alpha \in [0.1, 10]$ and $\tilde{U} \in [0, 10]$ [19]. To condense the information of the LZ dynamics, as a function of the interaction strength $\tilde{U}$ and the sweep rate $\alpha$, we show in a phase diagram the final state that the system reaches as the initial state is evolved in the interval $[-T_{\text{max}}, T_{\text{max}}]$. In Fig. 5(d) we show a density plot with the normalized transfer efficiency on the right side at $t = T_{\text{max}}$. To complementary visualize the dependence of $n_R$ on $\tilde{U}$ in Fig. 5(e) we present several contour plots of the normalized transfer efficiency as a function of the sweep rate $\alpha$, in the interval $\tilde{U} \in [0, 3]$.

From Fig. 5(d) one can observe how the influence of both, the parameter characterizing the interaction among particles and the sweep rate $\alpha$, lead the system to well defined quantum phases from minimum to maximal values of the transfer efficiency $n_R$. As can be observed from this figure, the effect of having $N > 1$ induces the opposite behavior with respect to the original LZ phenomena when the sweep rate $\alpha$ decreases. That is, the transfer of particles does not vanish as the sweep rate tends to zero, on the contrary, when the sweep rate $\alpha$ takes smaller values, the total population initially placed on the left side is transferred almost completely to the right chain. In other words, the main effect of having an interacting many-body system is the breakdown of adiabaticity.

Regarding the role of the on-site interactions, one can observe that the transfer efficiency diminishes as $\tilde{U}$ increases. This result can be understood in the light of the well known prediction for bosonic Josephson junctions [20] where the transport of particles diminishes as the interaction strength increases. We observe that for a given value of the on-site inter-particle interaction the effect of decreasing the sweep rate $\alpha$, is turned from detrimental into favourable for the transfer efficiency in the $N$-body LZ scheme [see Fig. 5(d) and (e)].

To illustrate the dependence of the many-body dynamics on the initial value of the tilt $-\Delta_0$ and on the number of particles, we performed calculations for $\Delta_0 = 10$ and 100 and $N = 2$ and 8 particles per site. The results are shown in the panel of Fig. 6. As above, we plot the normalized transfer efficiency in a density plot for $2\pi/\alpha \in [0.1, 10.0]$ and $\tilde{U} N/\Delta_0 \in [0, 3]$. The phase diagrams in Fig. 6 evidence the impact of the initial tilt $-\Delta_0$.
ψ to determine the variational order parameters ψ as before, by (4), and the same minimization procedure
population on the left side. The time evolution is given (max(
the of the atoms transfer to the right side of the chain
and MI phases, give rise to reach a saturation value for
in time the presence and the crossing between superfluid
to the variation of ψ that the transfer efficiency is very sensitive with respect
the relation with the order parameters [Fig. 3], we find
for which the atoms remain evenly distributed. Due to
its a critical value of the on-site interaction ˜U
bias grows. From our simulations we find that there ex-
utility becomes larger as the value of time dependent energy
That is, the region exhibiting breakdown of adiabatic-
N for either ˜∆ = 2 and ˜∆ = 8 (c,d). As can be observed from
the experimental counterpart, the transferred population as a
comparison purposes with the results presented in Fig.
6, we plot in Fig. 7 the phase diagram of
FIG. 6. (color on-line) Density plot of the normalized transfer
efficiency nR of ground-state sweeps as a function of dimensionless parameters ˜U and α at t = T_{max} = ˜Δ_{0}/ α for N = 2
(a,b) and, N = 8 (c,d). - ˜Δ_{0} is the initial value of the tilt.
Parameters are: ˜Δ_{0} = 10 (a,c) and ˜Δ_{0} = 100 (b,d).

on the final state that the system reaches after a com-
plete sweep. The region of maximum transfer is reduced
for either N = 2 and N = 8 as ˜Δ_{0} goes from 10 to 100.
That is, the region exhibiting breakdown of adiabi-
city becomes larger as the value of time dependent energy
bias grows. From our simulations we find that there ex-
ts a critical value of the on-site interaction ˜U_{c} ∼ ˜Δ_{0}/N
for which the atoms remain evenly distributed. Due to
the relation with the order parameters [Fig. 3], we find
that the transfer efficiency is very sensitive with respect
to the variation of ψ_{L} and ψ_{R}. As the system evolves
in time the presence and the crossing between superfluid
and MI phases, give rise to reach a saturation value for
the of the atoms transfer to the right side of the chain
(max(nR) ≲ 0.5 for ˜U ≳ ˜U_{c}).

FIG. 7. (color on-line) Density plot of normalized transfer
efficiency nR of inverse sweeps as a function of dimensionless
parameters ˜U and α for 2π/α ∈ [0,10.0] and ˜UN/˜Δ_{0} ∈ [0,3]. Parameters are: ˜Δ_{0} = 10 (a), ˜Δ_{0} = 100 (b), N = 2.

\[ \hat{\Delta} = \alpha T_{\text{max}} = \hat{\Delta}_{0} \] to \[ \hat{\Delta} = -\alpha T_{\text{max}} = -\hat{\Delta}_{0} \] for a given
sweep rate α.

To obtain the phase diagram for the inverse sweep
scheme we explore sweep rates in the interval 2π/α ∈ [0.1,10.0] for N = 2, and set ˜Δ_{0} = 10 and ˜Δ_{0} = 100. For
comparison purposes with the results presented in Fig.
6 we plot in Fig. 7 the phase diagram of nR in terms of
˜UN/˜Δ_{0}. In this case consistent results with the single-
particle LZ scenario are found: the maximum of transfer
efficiency is observed as the sweep rate increases. The
effect of increasing the value of the tilt ˜Δ_{0} from 10 to
100 gives rise to a reduction of the region of maximum
transfer. That is, it leads to similar qualitative features
than those found for the ground state sweeps. The role
of the inter particle interaction strength is perceived as
in the ground-state scheme, that is, increasing the value
of ˜U destroys localisation between left and right chains
(nR ≈ nL ≲ 0.5).

To compare the dynamical behavior between the
ground-state and inverse sweeps we study, as in the
experimental counterpart, the transferred population as a
function of the final value of the tilt ˜Δ_{f}, for N = 2, a
fixed sweep rate α and fixed interaction ˜U. As before,
the initial state for the ground-state (inverse) sweep cor-
responds to the total particle population localized in the
bottom well with lower (higher) energy. However, it is
important to point out that in this case the sweeps are
carried out from ˜Δ_{i} to ˜Δ_{f}, that is, the sweeps are not
accomplished from −| ˜Δ_{i}| to | ˜Δ_{i}| as in section IV. As ini-
tial condition for each sweep we consider ˜Δ_{i} = 20 and
˜Δ_{i} = −20 for ground-state and inverse sweeps respec-
tively. Fig. 8 corresponds to ground-state (a,c) [inverse
(b,d)] sweeps. In Fig. 8 (a) [(b)] we plot nR for ground-
state [inverse] sweeps as a function of ˜Δ_{f} for ˜U = 1.5
and several values of 2π/α. In Fig. 8 (c) [(d)] we plot nR
for ground-state [inverse] sweeps as a function of ˜Δ_{f} for
2π/α = 2.0[1.0] and ˜U ∈ [0, 10]. As can be observed from

V. MANY-BODY DYNAMICS:
GROUND-STATE VS INVERSE SWEEPS

Following the experimental realization of LZ many-
body dynamics [3], we extend our study to investigate
the dynamics across inverse sweeps. Namely, the initial
condition corresponds to ˜Δ > 0 with the total particle
population on the left side. The time evolution is given
as before, by \[ \psi \], and the same minimization procedure
to determine the variational order parameters ψ_{L} and
ψ_{R} at each temporal step δt must be carried out from
VI. FINAL REMARKS

We have studied the dynamical and stationary properties of an interacting Bose quantum fluid confined in two coupled one-dimensional chains. Such a system has been realized experimentally [3] and represents a generalization of the single-particle Landau-Zener dynamics. The competition terms that determine the evolution in time from a given initial state depend on the inter particle interaction strength $U$, the intra-chain coupling energy $J^\parallel$ ($z$ direction), the inter-chain coupling energy $J^\perp$ ($x$ direction) and the time-dependent parameter characterizing the tilt between the two wells of each site in the chain, that is, the detuning parameter $\Delta$.

To investigate the dynamics of the generalized LZ realization, we work in the decoupling approximation and the one-level band scheme, and propose a model lattice site Hamiltonian that represents two coupled infinite chains. Our full many-body model is written in terms of the order parameters $\psi_\nu$, $\nu \in \{L,R\}$ that account for the superfluid component of the particle population. We exhaustively explore the space of parameters $\{U, \Delta\}$ to determine by variational means the order parameters $\psi_\nu$, $\nu \in \{L,R\}$ for the ground state energy $E_0(U, \Delta, \psi_L, \psi_R)$. Then, we diagonalized the Hamiltonian to obtain the energy spectrum. Concerning the stationary state properties, since the number of particles per lattice sites in the actual experiments is not too large, we were able to perform accurate numerical simulations of the system to fully characterize the entire energy spectrum. Such calculations lead to predict a richer spectrum with respect to that performed for a single dimer, that is, when the movement along $z$ axis is neglected [3]. We found that as $U$ grows in magnitude, a more complex structure of the energy spectrum with respect to the double-well system is developed in correlation with the suppression of the superfluid component in the system, thus promoting self-trapping behavior. In addition our model allow us to recover the single-particle LZ result for a single dimer.

Regarding the dynamical evolution of the LZ many body generalization, we concentrate in studying ground-state and inverse sweeps in order to compare the predictions of our model with the experimental realization. The ground-state and inverse sweeps dynamics account for a sweep from a state where all the particles initiate in the bottom well with lower energy (left well) and the counterpart where the filled sites are the ones with higher energy, respectively. In a phase diagram in terms of the interaction $U$ and the sweep rate $\alpha$ we condense the normalized final transfer of atoms. A totally different scenario with respect to the original single-particle Landau-Zener scheme was found for ground-state sweeps. That is, the transfer of particles does not vanish as the sweep rate tends to zero, on the contrary, when the sweep rate $\alpha$ takes smaller values, the total population initially placed on the left side is transferred almost completely to the right chain. A different situation occurs in the study of the transfer efficiency for inverse sweeps, this reveals consistent results with the single-particle Landau-Zener phenomena predictions. As the system evolves in time depending on the interaction, the system can transit different phases (superfluid or insulating) multiple times, see Fig. 2. We identified a a critical value of the interaction $U_c$, for which $U \gtrsim U_c$, the system is in a crossover from insulating states ($n_R \neq n_L$) with $\psi_{RL} = 0$ to a superfluid states ($n_R \approx n_L \lesssim 0.5$) with $\psi_{RL} \neq 0$. It is important to emphasize that all of our results are in qualitative agreement with the experimental realization. In addition, our analysis provides insight to the region where the interaction is non-perturbative.

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FIG. 8. (color on-line) Normalized transfer efficiency $n_R$ for ground-state (a) [inverse (b)] sweeps as a function of the final tilt $\Delta_f$, for $U = 1.5$ and $2\pi/\alpha \in [0.1, 3.0]$. Normalized transfer efficiency $n_R$ as a function of the final tilt $\Delta_f$, for $2\pi/\alpha = 2.0$, $\{2\pi/\alpha = 1.0\}$ in the ground-state (c) [inverse (d)] case and $U \in [0, 10]$. The number of particles is $N = 2$. Dimensionless parameters are: $n_R$, $\Delta_f$, $\tilde{U}$ and $\alpha$. These figures the ground state sweeps show that the normalized population transferred increases from zero to a maximum $\max(n_R) < 1$ for $2\pi/\alpha \lesssim 1$ while the behavior for the inverse sweeps shows that the population transferred is partial, reaching a maximum value below one. The effect of increasing interaction for a fixed sweep rate, is to suppress the maximum population transfer. Comparison with the experimental results show qualitative agreement. See Fig. 3 of Ref. [3].
FIG. 9. (color on-line) Energy spectrum of a single-particle in a double well potential. The asymmetry is determined by the value of the parameter \( c \) in (A.1). In these plots \( c = 0, 0.04 \) and 0.08.

appendix: Single particle in a tilted double well

To establish appropriate bounds for the values of the parameters that determine the dynamics of the Bose gas in the coupled one-dimensional chains, that is, their practical use in relation to the proposed model Hamiltonian \( \text{(A.1)} \), we analyze here the eigensystem of a single particle confined in an asymmetric double well. Although a previous study has provided quantitative criteria regarding the energy level scheme where a many-particle system can be properly described for those potentials \([10]\), the intrinsic nature of the Landau-Zener phenomena, as well as, the proposed model require to justify the plausibility of the chosen parameters.

Let us consider the potential

\[
V(x) = cx - x^2 + dx^4, \tag{A.1}
\]

where \( d = 0.05 \) and the parameter \( c \) incorporates the asymmetry. In Fig. \( 9 \) we show the energy spectrum, as well as, the single-particle wave functions corresponding to different values of \( c \). From this figure we observe several features. As expected, the energy levels are arranged in bands of two levels. The energy difference between bands becomes similar to the energy difference in bands as the asymmetry in the potential grows. That is, as the asymmetry in the double-well potential, represented by the parameter \( \Delta = \alpha t \) changes with time \( t \), the energy spacing between bands starts to match the energy difference among levels in the bands. This fact imposes a restriction on the possible values of the \( \Delta \) parameter. In principle, one could think that the suitable range for values of \( \Delta \) is severely limited. However, we recall that in our calculations all the parameters are referred to the tunnelling coupling parameter \( J = J^\parallel = J^\perp \), which is related to the energy difference of the two first energy levels at \( t = 0 \) (\( c = 0 \)) as \( J = (\epsilon_1 - \epsilon_0)/2 \) \([21]\), being \( \epsilon_0 \) and \( \epsilon_1 \) the ground state and the first excited state energies of the single particle. Therefore, by restricting ourselves to those values of \( \Delta = \Delta/J << \hbar \omega/J \) and \( \bar{U} = U/J << \hbar \omega/J \), being \( \hbar \omega \approx (\epsilon_1 + \epsilon_2 - \epsilon_1 - \epsilon_0)/2 \) the energy difference among bands, we can validate the one-band scheme to determine the dynamics of the proposed model Hamiltonian. For the chosen double-well potential \( \text{(A.1)} \), we found \( \hbar \omega/J \approx 1.2 \times 10^6 \) and consequently \( \Delta_{\text{max}}/J \approx 1.2 \times 10^6 \). Thus, for values of \( \Delta < \Delta_{\text{max}}/J \) the choice of parameters in the simulations and our model are well justified.

References

[1] L.D. Landau, Phys. Z. Sowjetunion 2, 46 (1932).
[2] C.M. Zener, Proc. R. Soc. London A 137, 696 (1932).
[3] Y. Chen, S.D. Huber, S. Trotsky, I. Bloch and E. Altman, Nature Phys. 7, 61-67 (2011).
[4] O. Zobay and B.M. Garraway, Phys. Rev. A 61, 033603 (2000).
[5] B. Wu and Q. Niu, Phys. Rev. A 61, 023402 (2000).
[6] B. Wu and Q. Niu, New. J. Phys. 5, 104 (2003).
[7] D. Witthaut, E.M. Graefe, H.J. Korsch, Phys. Rev. A 73, 063609 (2006).
[8] J. Liu, L. Fu, B. Y. Ou, S. G. Chen, D. I. Choi, B. Wu, Q. Niu, Phys. Rev. A 66, 023404 (2002).
[9] M. Jona-Lasinio, O. Morsch, M. Cristiani, N. Malossi, J. H. Müller, E. Courtade, M. Anderlini, and E. Arimondo, Phys. Rev. Lett. 91, 230406 (2003).
[10] D. R. Dounas-Frazer, A. M. Hermundstad, and L. D. Carr, Phys. Rev. Lett. 99, 200402 (2007).
[11] D. van Oosten, P. van der Straten, and H.T. C. Stoof, Phys. Rev. A 63, 053601 (2001).
[12] M.P.A. Fisher, P.B. Weichman, G. Grinstein and D.S. Fisher, Phys. Rev. B 40, 546 (1989).
[13] C. Kasztelan, S. Trotzky, Y.-A. Chen, I. Bloch, I. P. McCulloch, U. Schollwöck and G. Orso, Phys. Rev. Lett. 106, 155302 (2011).
[14] I. Danshita, J.E. Williams, C.A.R. Sá de Melo, and C.W. Clark, Phys. Rev. A 76, 043606 (2007).
[15] We use the Nelder-Mead simplex method to find the minimum of the surface defined by \( U/J \) and \( \Delta/J \) for the lowest eigenvalue of the Hamiltonian \( \text{(A.1)} \) with fixed \( N \).
[16] S. F. Caballero-Benitez and E. A. Ostrovskaya, J. Phys. B: At. Mol. Opt. Phys. 44, 135301 (2011).
[17] The approximation of the time evolution is accurate to \( O(\delta t^2) \).
[18] T. Venumadhav, Masudul Haque and R. Moessner, Phys. Rev. B 81, 054305 (2010).
[19] The interval of study in \( 2\pi/\alpha \) and \( \bar{U} = 10/2^\alpha \), that is, we explore a grid of size \( 2^\alpha \times 2^\beta \).
[20] S. F. Caballero Benitez, V. Romero-Rochín and R. Pare-
[20] J. Phys. B: At. Mol. Opt. Phys. 43, 115301 (2010).
[21] R. Paredes, Phys. Rev. A 73, 033616 (2006).