SPECTRAL THEORY OF ELLIPTIC OPERATORS IN EXTERIOR DOMAINS

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ABSTRACT. We consider various closed (and self-adjoint) extensions of elliptic differential expressions of the type \( A = \sum_{0 \leq |\alpha|, |\beta| \leq m} (-1)^n D^\alpha a_{\alpha, \beta}(x) D^\beta, \ a_{\alpha, \beta}(\cdot) \in C^\infty(\overline{\Omega}), \) on smooth (bounded or unbounded) domains \( \Omega \) in \( \mathbb{R}^n \) with compact boundary \( \partial \Omega \). Using the concept of boundary triples and operator-valued Weyl–Titchmarsh functions, we prove various trace ideal properties of powers of resolvent differences of these closed realizations of \( A \) and derive estimates on eigenvalues of certain self-adjoint realizations in spectral gaps of the Dirichlet realization.

Our results extend classical theorems due to Vishik, Povzner, Birman, and Grubb.

1. Introduction

Let \( \Omega \) be an open domain in \( \mathbb{R}^n \) (bounded or unbounded) with compact boundary \( \partial \Omega \). Throughout we assume that \( \partial \Omega \) is an \((n-1)\)-dimensional (not necessarily connected) \( C^\infty \)-manifold. Let \( A \) be the differential expression

\[
A = \sum_{0 \leq |\alpha|, |\beta| \leq m} (-1)^n D^\alpha a_{\alpha, \beta}(x) D^\beta, \quad a_{\alpha, \beta}(\cdot) \in C^\infty(\overline{\Omega}),
\]

where \( \text{ord}(A) = 2m \), which is elliptic in \( \overline{\Omega} \). Moreover, we assume that \( A \) is properly elliptic in \( \overline{\Omega} \) (which is automatically satisfied if either \( n > 2 \) or the symbol of \( A \) is real, cf. [11]). In addition to (1.1) we consider its formal adjoint \( A^T = \sum_{0 \leq |\alpha|, |\beta| \leq m} (-1)^n D^\alpha a_{\alpha, \beta}(x) D^\beta, \) which is also properly elliptic in \( \overline{\Omega} \) (cf. [11]).

Denote by \( A = A_{\min} (A^T = A_{\min}^T) \) the minimal operator associated in \( L^2(\Omega) \) with the differential expression \( A \) (resp., \( A^T \)), that is, the closure of \( A \) defined on \( C_0^\infty(\Omega) \). The maximal operators \( A_{\max} \) and \( A_{\max}^T \) are then defined by \( A_{\max} = (A_{\min}^T)^* = (A^T)^* \) and \( A_{\max}^T = (A_{\min}^T)^* = A^* \), respectively. We emphasize that \( H^{2m}(\Omega) \subset \text{dom}(A_{\max}) \subset H^{2m}_0(\Omega), \) while \( \text{dom}(A_{\max}) \neq H^{2m}(\Omega) \).

After the pioneering work by Vishik [15], nonlocal boundary value problems of the form \( A_{\max} u = f, (\partial u/\partial n - Ku) \mid \partial \Omega = 0 \) for elliptic operators (1.1) (with \( m = 1 \)) in bounded domains were considered by numerous authors (see, e.g., [3, 9] and the references therein). Vishik was the first to consider these problems in the framework of extension theory of dual pairs of operators. Starting with a formula for the domain \( \text{dom}(A_{\max}) \) of \( A_{\max} \), he applied it to an appropriate regularization of the classical Green’s formula, using the Calderon operator. The latter allowed him to extend the Green’s formula from \( H^{2m}(\Omega) \) to \( \text{dom}(A_{\max}) \). The next fundamental contribution to the subject was made by Grubb [9]. Using the theory of Lions and Magenes [11], Grubb substantially extended and completed the results of [15]. In particular, Grubb obtained the (regularized) Green’s formula which (in the special case \( m = 1 \)) reads as follows:

\[
(A_{\max} u, v)_{L^2(\Omega)} - (u, A_{\max}^T v)_{L^2(\Omega)} = (\tilde{\Gamma}_{\Omega,1} u, \tilde{\Gamma}_{\Omega,0}^T v)_{1/2,-1/2} - (\tilde{\Gamma}_{\Omega,0} u, \tilde{\Gamma}_{\Omega,1}^T v)_{-1/2,1/2}.
\]

Here \( (\cdot, \cdot)_{k,-k} \) denotes the duality pairing between \( H^k(\partial \Omega) \) and \( H^{-k}(\partial \Omega) \), \( u \in \text{dom}(A_{\max}), v \in \text{dom}(A_{\max}^T) \), and \( \tilde{\Gamma}_{\Omega,0}, \tilde{\Gamma}_{\Omega,1}, \tilde{\Gamma}_{\Omega,1}^T \) and \( \tilde{\Gamma}_{\Omega,1} \) are regularized trace operators, having the properties

\[
\tilde{\Gamma}_{\Omega,1} : D(A_{\max}) \to H^{1/2}(\partial \Omega), \quad \tilde{\Gamma}_{\Omega,0}^T : D(A_{\max}) \to H^{-1/2}(\partial \Omega), \quad \text{ran}(\tilde{\Gamma}_{\Omega,1}, \tilde{\Gamma}_{\Omega,0}^T) = H^{1/2}(\partial \Omega) \times H^{-1/2}(\partial \Omega).
\]

Later, we will use a somewhat different approach (cf. Proposition 2.7).

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On the other hand, during the past three decades a new approach to the extension theory, based on the concept of a boundary triple and the corresponding operator-valued Weyl–Titchmarsh function, was developed in [7] (cf. the references therein for the symmetric case) and in [13] (in the case of dual pairs). In this paper we apply some results and technique from [7] and [13] to elliptic operators on unbounded domains. The most important ingredients from the elliptic theory we need are the regularized Green’s formula and a priori coercivity-type estimates for the elliptic realizations \( A_B \) of \( A \) (see (2.2) below). To obtain the latter on unbounded domains one needs additional restrictions on the coefficients of \( A \), since an elliptic realization is not necessarily coercive. Here we restrict ourselves to the case of bounded coefficients \( a_{\alpha,\beta}(\cdot) \). Using the formalism of boundary triples and the corresponding operator-valued Weyl–Titchmarsh functions in [7, 13], we investigate the resolvent difference of two realizations and complement the results of Povzner [14], Birman [5], and Grubb [10] in this direction.

In addition, assuming \( A_{\min} > 0 \), we compute the number of negative eigenvalues of a realization \( A_K \) and the number of eigenvalues of \( A_K \) within spectral gaps of the Dirichlet realization \( \tilde{A}_{\gamma_D} \), where \( \gamma_D = \{ \gamma_j \}_{j=0}^{m-1} \).

**Notations.** \( \mathcal{S} \) and \( \mathcal{H} \) represent complex, separable Hilbert spaces; \( \mathcal{B}(\mathcal{H}), \mathcal{B}_2(\mathcal{H}), \) and \( \mathcal{C}(\mathcal{H}) \) denote the sets of bounded, compact, and closed linear operators in \( \mathcal{H} \); \( \text{dom}(\cdot), \text{ran}(\cdot), \) and \( \ker(\cdot) \) denote the domain, range, and kernel of a linear operator, \( \rho(\cdot) \) and \( \sigma(\cdot) \) stand for the resolvent set and spectrum of a linear operator.

As usual, \( C^\infty(\Omega) \) denotes the set of infinity differentiable functions in the domain \( \Omega \), \( C^\infty_0(\Omega) \) the subset of \( C^\infty(\Omega) \)-functions of compact support in \( \Omega \), \( C(\Omega) \) the set of uniformly continuous functions in \( \Omega \), \( C_0(\Omega) = C(\Omega) \cap L^\infty(\Omega) \), \( C_0^\infty(\Omega) \) the set of uniformly continuous functions in \( \Omega \), \( C_0(\Omega) = C(\Omega) \cap L^\infty(\Omega) \), \( H^s(\Omega) \) the usual Sobolev spaces.

## 2. Dual pairs, boundary triples, and operator-valued Weyl–Titchmarsh functions

**Definition 2.1.** Let \( A \) and \( A^\top \) be densely defined (not necessarily closed) linear operators in \( \mathcal{S} \). Then \( A \) and \( A^\top \) form a dual pair \( \{ A, A^\top \} \) in \( \mathcal{S} \) if \( (Af, g) = (f, A^\top g) \) for all \( f \in \text{dom}(A) \), \( g \in \text{dom}(A^\top) \). An operator \( \tilde{A} \) is called a proper extension of the dual pair \( \{ A, A^\top \} \), and we write \( \tilde{A} \in \text{Ext}\{ A, A^\top \} \), if \( A \subsetneq \tilde{A} \subsetneq (A^\top)^* \).

**Definition 2.2.** (cf. [12], [13]) (i) Let \( \mathcal{S}, \mathcal{H}_0, \) and \( \mathcal{H}_1 \) be complex, separable Hilbert spaces and\[ \Gamma^\top = \begin{pmatrix} \Gamma_0^\top \\ \Gamma_1^\top \end{pmatrix} : \text{dom}((A^\top)^*) \rightarrow \mathcal{H}_0 \oplus \mathcal{H}_1 \]and\[ \Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : \text{dom}(A^*) \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_0 \]be linear mappings. Then \( \Pi = \{ \mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma^\top, \Gamma \} \) is called a boundary triple for the dual pair \( \{ A, A^\top \} \) if \( \Gamma^\top \) and \( \Gamma \) are surjective and the Green’s identity holds:

\[
((A^\top)^* f, g)_{\mathcal{S}} - (f, A^\top g)_{\mathcal{S}} = (\Gamma_0^\top f, \Gamma_0 g)_{\mathcal{H}_0} - (\Gamma_1^\top f, \Gamma_1 g)_{\mathcal{H}_0}, \quad f \in \text{dom}((A^\top)^*), \ g \in \text{dom}(A^*).
\]

We set \( A_0 = (A^\top)^* \upharpoonright \ker(\Gamma_0^\top) \) and \( A_0^\top = A^\top \upharpoonright \ker(\Gamma_0) \).

(ii) The operator-valued function \( M_{\Omega}(z) \) defined by

\[
\Gamma_0^\top f_z = M_{\Omega}(z) \Gamma_0^\top f_z, \quad f_z \in \ker((A^\top)^* - z), \quad z \in \rho(A_0),
\]

is called the Weyl–Titchmarsh function corresponding to the boundary triple \( \Pi \).

Due to Green’s identity, \( (A_{\min} u, v)_{L^2(\Omega)} = (u, A_{\min}^\top v)_{L^2(\Omega)}, \ u, v \in C_0^\infty(\Omega), \) the operators \( A \) and \( A^\top \) form a dual pair \( \{ A, A^\top \} \) of elliptic operators in \( L^2(\Omega) \). Any proper extension \( \tilde{A} \in \text{Ext}\{ A, A^\top \} \) of \( \{ A, A^\top \} \) is called a realization of \( A \). Clearly, any realization \( \tilde{A} \) of \( A \) is closable. We equip \( \text{dom}(A_{\max}) \) and \( \text{dom}(A_{\max}^\top) \) with the corresponding graph norms. It is known (cf. [4, 11]) that if a domain \( \Omega \) is bounded, then \( \text{dom}(A_{\min}) = \text{dom}(A_{\min}^\top) = H^2_{\Omega}(\Omega) \), where the norms in \( \text{dom}(A_{\min}) \) and \( H^2_{\Omega}(\Omega) \) are equivalent. Denote by \( \gamma_j \) the mappings \( \gamma_j : C^\infty(\overline{\Omega}) \rightarrow C^\infty(\partial \Omega), \gamma_j u = \gamma_{0j}(\partial_j u/\partial n^j) = \partial_j u/\partial n^j \mid \partial \Omega, \ 1 \leq j \leq m - 1, \ \gamma_0 u = u \mid \partial \Omega, \) where \( n \) stands for the interior normal to \( \partial \Omega \). Next we introduce the boundary operators \( B_j \) as

\[
B_j u = \sum_{0 \leq j \leq m_j} b_{j, \gamma_{0j}}(D^j u), \quad b_{j, \gamma_{0j}}(\cdot) \in C^\infty(\partial \Omega), \quad \text{ord}(B_j) = m_j \leq 2m - 1. \tag{2.1}
\]

Here \( B_j : C^\infty(\overline{\Omega}) \rightarrow C^\infty(\partial \Omega) \) will eventually be extended to appropriate Sobolev spaces \( H^s(\Omega) \) and in some cases to \( \mathcal{D}(A_{\max}) \). \( B_j \) in (2.1) can also be rewritten as \( B_j u = b_j \gamma_{0j} u + \sum_{0 \leq k \leq m_j - 1} T_{j,k} \gamma_{0j} u \), where \( b_j(\cdot) \in C^\infty(\partial \Omega) \) and \( T_{j,k} \) are tangential differential operators in \( \partial \Omega \) of orders \( \text{ord}(T_{j,k}) \leq m_j - k \) with \( C^\infty(\partial \Omega) \)-coefficients.
With any elliptic operator $A$ (1.1) and a system $B = \{B_j\}_{j=1}^{m-1}$ we associate the operator $\hat{A}_B$ defined by

$$\hat{A}_B = A_{\max} \upharpoonright \text{dom}(\hat{A}_B), \quad \text{dom}(\hat{A}_B) = H^{2m}(\Omega) = \{u \in H^{2m}(\Omega) \mid Bu = 0\}. \quad (2.2)$$

Our considerations are based on [11, Thm. 2.2.1]. According to this result, for any elliptic differential expression $A$ in (1.1) and any normal system $\{B_j\}_{j=0}^{m-1}$ on $\partial \Omega$ given by (2.1), there exists a system of boundary operators $\{C_j\}_{j=0}^{m-1}$, $\text{ord}(C_j) = j_j \leq 2m - 1$, such that the system $\{B_0, \ldots, B_{m-1}, C_0, \ldots, C_{m-1}\}$ is a Dirichlet system of order $2m$ and another Dirichlet system of boundary operators $\{B_j^\top\}_{j=0}^{m-1} \cup \{C_j^\top\}_{j=0}^{m-1}$, such that the following Green’s formula hold

$$(Au, v)_{L^2(\Omega)} - (u, A^\top v)_{L^2(\Omega)} = \sum_{0 \leq j \leq m-1} [(C_j u, B_j^\top v)_{L^2(\partial\Omega)} - (B_j u, C_j^\top v)_{L^2(\partial\Omega)}], \quad u, v \in H^{2m}(\Omega). \quad (2.3)$$

Next, following [9] and [11], we introduce the spaces $D^*_A(\Omega) = \{u \in H^s(\Omega) \mid Au \in H^0(\Omega), \, s \in \mathbb{R}\}$, provided with the graph norm $\|u\|_{D^*_A(\Omega)} = (\|u\|_s^2 + \|Au\|_0^2)^{1/2}$. Clearly, $D^*_A(\Omega) = \text{dom}(A)$ and $D^*_A(\Omega) \hookrightarrow H^s(\Omega)$.

**Definition 2.3.**

(i) The operator $\hat{A}_B$ defined by (2.2) is called elliptic and is put in the class $\text{Ell}(A)$ if $A$ is properly elliptic on $\Omega$ and the system $\{B_j\}_{j=0}^{m-1}$ is normal and satisfies the covering condition (cf. [11, Sects. 2.1.1–2.1.4]) at any point of the boundary $\partial\Omega$.

(ii) The operator $\hat{A}_B$ is called coercive in $H^s(\Omega)$ with $s \geq 2m$ if the a priori estimate (2.25) in [2] (cf. also [11, Sect. 2.9.6]) holds.

We note that $\hat{A}_B$ is a closed operator if $B$ satisfies the covering condition (cf. [2, Sect. 6.5], [11, Thm. 2.8.4]).

If $\Omega$ is bounded, then any elliptic differential expression $A$ with $C(\overline{\Omega})$-coefficients is uniformly elliptic in $\overline{\Omega}$.

In this case, $\hat{A}_B \in \text{Ell}(A)$ if and only if $\hat{A}_B$ is coercive in $H^{2m}(\Omega)$ (see [1], [11, Sect. 2.9.6]). If $\Omega$ is unbounded, then the condition $\hat{A}_B \in \text{Ell}(A)$ is still necessary for coerciveness in $H^{2m}(\Omega)$, though, it is no longer sufficient without additional assumptions on $A$.

**Hypothesis 2.4.** Assume that $A$ is a uniformly elliptic operator, $a_{\alpha,\beta}(\cdot) \in C_b(\Omega)$ for $|\alpha| + |\beta| \leq 2m$ and $a_{\alpha,\beta}(\cdot) \in C_{ub}(\Omega)$ for $|\alpha| + |\beta| = 2m$.

**Proposition 2.5.** Assume Hypothesis 2.4, $\hat{A}_B \in \text{Ell}(A)$, and $0 \in \rho(\hat{A}_B)$. Then for any $s \in \mathbb{R}$, the mappings $B$ and $B^\top$, isomorphically map $Z^*_A(\Omega) = \{u \in D^*_A(\Omega) \mid A_{\max} u = 0\}$ and $Z^*_{A\top}(\Omega) = \{u \in D^*_A(\Omega) \mid A^\top_{\max} u = 0\}$ isomorphically onto $\Pi^m_{j=1} H^{s-m_j-(1/2)}(\partial\Omega)$ and onto $\Pi^m_{j=1} H^{s-2m_j+m_j+(1/2)}(\partial\Omega)$, respectively.

**Definition 2.6.** ([9, 15]) (i) Under the assumptions of Proposition 2.5, let $\varphi \in \Pi^m_{j=0} H^{s-m_j-(1/2)}(\partial\Omega)$, $s \in \mathbb{R}$. Then one defines $P(z)\varphi$ to be the unique $u \in Z^*_A - \mathbb{I}_{L^2(\partial\Omega)}(\Omega)$ satisfying $Bu = \varphi$.

(ii) The Calderon operator $\Lambda(z)$ is defined by

$$\Lambda(z) : \Pi^m_{j=0} H^{s-m_j-(1/2)}(\partial\Omega) \rightarrow \Pi^m_{j=0} H^{s-m_j-(1/2)}(\partial\Omega), \quad \Lambda(z) \varphi = CP(z)\varphi. \quad (2.4)$$

(iii) Similarly, let $\psi \in \Pi^m_{j=0} H^{s-2m_j+m_j+(1/2)}(\partial\Omega)$. Then $P(z)\psi$ is defined to be the unique solution in $Z^*_A - \mathbb{I}_{L^2(\partial\Omega)}(\Omega)$ of $B^\top u = \psi$ and the Calderon operator $\Lambda(z)^\top$ is defined as $\Lambda(z)^\top \psi = C^\top P_z^\top \psi$.

Let $\Delta_{\partial\Omega}$ be the Laplace-Beltrami operator in $L^2(\partial\Omega)$, $-\Delta_{\partial\Omega} = -\mathbb{I}_{L^2(\partial\Omega)}$. Then $-\Delta_{\partial\Omega,1} = -\Delta_{\partial\Omega,1} \geq \mathbb{I}_{L^2(\partial\Omega)}$. Moreover, $(-\Delta_{\partial\Omega,1} = -\Delta_{\partial\Omega,1}^s)$ isometrically maps $H^0(\partial\Omega)$ onto $H^s(\partial\Omega)$, $s \in \mathbb{R}$. Next, we introduce the diagonal $m \times m$ operator matrices $-\Delta_{\partial\Omega,1,m}$ and $-\Delta_{\partial\Omega,1,\mu}$ with the $(j,j)$-th entry $(-\Delta_{\partial\Omega,1})(m_j/2)+(1/4)$ (resp., $(-\Delta_{\partial\Omega,1})(m_j/2)-(1/4)$).

**Proposition 2.7.** Assume Hypothesis 2.4, $\hat{A}_B \in \text{Ell}(A)$, and $0 \in \rho(\hat{A}_B)$. Set

$$\Gamma_{0,0} u = (-\Delta_{\partial\Omega,1,m})^{-1} B u, \quad \Gamma_{0,1} u = (-\Delta_{\partial\Omega,1,\mu}) (Cu - \Lambda(0) B u), \quad u \in \text{dom}(A_{\max}), \quad (2.5)$$

$$\Gamma^\top_{1,0} v = (-\Delta_{\partial\Omega,1,\mu})^{-1} B^\top v, \quad \Gamma^\top_{1,1} v = (-\Delta_{\partial\Omega,1,m}) (C^\top v - \Lambda(0)^\top B^\top v), \quad v \in \text{dom}(A^\top_{\max}). \quad (2.6)$$
Then the following holds:

(i) \( \Pi = \{ \mathcal{H}_0 \Omega \oplus \mathcal{H}_0, \Gamma_0, \Gamma_0^\top \} \), with

\[
\mathcal{H}_0 = \Pi_{j=0}^{m-1} H^0(\partial \Omega) = \Pi_{j=0}^{m-1} L^2(\partial \Omega), \quad \Gamma_0 = (\Gamma_{0,0}, \Gamma_{0,1}), \quad \Gamma_0^\top = (\Gamma_{0,0}^\top, \Gamma_{0,1}^\top),
\]

forms a boundary triple for the dual pair \( (A, A^\top) \) of elliptic operators in \( L^2(\Omega) \). In particular, the following Green's formula holds

\[
(A_{\max} u, v)_{L^2(\Omega)} - (u, A_{\max}^\top v)_{L^2(\Omega)} = (\Gamma_{0,1} u \Gamma_{0,0}^\top v)_{\mathcal{H}_0} - (\Gamma_{0,0} u \Gamma_{0,1}^\top v)_{\mathcal{H}_0}, \quad u \in \text{dom}(A_{\max}), v \in \text{dom}(A_{\max}^\top).
\]

(ii) The corresponding operator-valued Weyl–Titchmarsh function is given by

\[
M_{\Omega,\Pi}(z) = (-\Delta_{\mathcal{H}_{0,1},\mu})(\Lambda(z) - \Lambda(0))(-\Delta_{\mathcal{H}_{0,1},\nu}), \quad z \in \rho(\hat{A}_B).
\]

In the context of operator-valued Weyl–Titchmarsh functions and elliptic partial differential operators we also refer to the recent preprint [6] (and the references cited therein).

**Definition 2.8.** For any operator \( K : \text{dom}(K) \to \Pi_{j=0}^{m-1} H^{-m_j}((\partial \Omega), \text{dom}(K) \subseteq \Pi_{j=0}^{m-1} H^{-m_j}((\partial \Omega)), \) we set

\[
A_K = A_{\max} \cup \text{dom}(A_K), \quad \text{dom}(A_K) = \{ u \in \text{dom}(A_{\max}) | Bu \in \text{dom}(K), Cu = KBu \}.
\]

**Definition 2.9.** Define \( S_p(\Sigma) = \{ T \in B_{\infty}(\Sigma) | s_j(T) = O(j^{-1/p}) \text{ as } j \to \infty \}, \) \( p > 0, \) where \( s_j(T), j \in \mathbb{N}, \) denote the singular values of \( T \) (i.e., the eigenvalues of \( (T^*T)^{1/2} \) ordered in decreasing magnitude, counting multiplicity).

**Theorem 2.10.** Assume the conditions of Proposition 2.7 and suppose that \( 0 \in \rho(\hat{A}_C) \) and \( K \in \mathcal{C}(\mathcal{H}_0) \). Then:

(i) For any realization \( A_K \in \mathcal{C}(L^2(\Omega)) \) of the form (2.7), satisfying \( \rho(A_K) \cap \rho(\hat{A}_B) \neq \emptyset \), the following holds,

\[
[(A_K - zI_{L^2(\Omega)})^{-\ell} - (\hat{A}_B - zI_{L^2(\Omega)})^{-\ell}] \in S_{\frac{1}{m-\ell}}(L^2(\Omega)), \quad z \in \rho(A_K) \cap \rho(\hat{A}_B), \quad \ell \in \mathbb{N}.
\]

(ii) If \( B = \{ B_j \}_{j=0}^{m-1} \) is a Dirichlet system, \( K \in \mathcal{B}(\mathcal{H}_0), \) and \( \rho(A_K) \cap \rho(\hat{A}_B) \neq \emptyset, \) one has

\[
[(A_K - zI_{L^2(\Omega)})^{-1} - (\hat{A}_B - zI_{L^2(\Omega)})^{-1}] \in S_{\frac{1}{m-1}}(L^2(\Omega)), \quad z \in \rho(A_K) \cap \rho(\hat{A}_B).
\]

Combining Weyl's theorem with Theorem 2.10 one obtains the following result:

**Corollary 2.11.** Assume the conditions of Theorem 2.10. Then, \( \sigma_{\text{ess}}(A_K) = \sigma_{\text{ess}}(\hat{A}_B). \)

In the context of elliptic realizations \( \hat{A}_G \in \text{Ell}(A), \) we have the following stronger result:

**Theorem 2.12.** Suppose that the conditions of Proposition 2.7 are satisfied and \( \hat{A}_G \in \text{Ell}(A), \) that is, \( \hat{A}_G \) is the elliptic realization of \( A \) with \( G = \{ G_j \}_{j=0}^{m-1}. \) Then for any \( \ell \in \mathbb{N}, \)

\[
[(\hat{A}_G - zI_{L^2(\Omega)})^{-\ell} - (\hat{A}_B - zI_{L^2(\Omega)})^{-\ell}] \in S_{\frac{1}{m-\ell}}(L^2(\Omega)), \quad z \in \rho(\hat{A}_G) \cap \rho(\hat{A}_B).
\]

3. The formally self-adjoint case, nonnegative elliptic operators, and eigenvalues in gaps

Let \( A \) be a formally self-adjoint elliptic differential expression of the form (1.1), that is, \( A = A^\top \) or equivalently, \( a_{p,\rho} = \overline{a_{\rho,\rho}} \in C^\infty(\Omega). \) In this case \( A = A_{\min} = A_{\min}^\top = A^\top, \) that is, \( A \) is symmetric, and \( A_{\max} = (A_{\min}^\top)^* = A^*. \) If a normal system \( \{ B_j \}_{j=0}^{m-1} \) is chosen to be formally self-adjoint, that is, \( \hat{A}_B = (\hat{A}_B)^*, \) then a system \( \{ C_j \}_{j=0}^{m-1} \) can be chosen formally self-adjoint too. In this case \( B_j^\top = B_j \) and \( C_j^\top = C_j. \) Moreover, in this case, \( \mu_j = \text{ord}(C_j) = \text{ord}(C_j^\top) = 2m - 1 - m_j. \) It follows that \( \Delta_{\mathcal{H}_{0,1},\mu} = \Delta_{\mathcal{H}_{0,1},\nu}. \) Hence, Proposition 2.7 yields the following result:

**Proposition 3.1.** Let \( A \) be a formally symmetric elliptic differential expression and assume that \( \hat{A}_B \) and \( \hat{A}_C \) are self-adjoint. In addition, assume the conditions of Proposition 2.7 are satisfied. Then:

(i) \( \Pi = \{ \mathcal{H}_{0,\Omega}, \Gamma_0, \Gamma_1 \} \) with \( \mathcal{H}_{0,\Omega} = \Pi_{j=0}^{m-1} L^2(\partial \Omega), \) and \( \Gamma_0, \Gamma_1, \Omega \) defined by (2.5), forms a boundary triple for the operator \( A^*. \) In particular, the following Green's formula holds

\[
(A_{\max} u, v)_{L^2(\Omega)} - (u, A_{\max}^\top v)_{L^2(\Omega)} = (\Gamma_{0,1} u, \Gamma_{0,0}^\top v)_{\mathcal{H}_{0,\Omega}} - (\Gamma_{0,0} u, \Gamma_{0,1}^\top v)_{\mathcal{H}_{0,\Omega}}, \quad u, v \in D(A_{\max}).
\]
Proposition 3.3. The corresponding Weyl-Titchmarsh operator is given by $M_{\Omega,1}(z) = (-\Delta_{\Omega,1,m},(\Lambda(z) - \Lambda(0))(-\Delta_{\Omega,1,m})$.

For any self-adjoint operator $T = T^* \in C(\mathfrak{f})$ with associated family of spectral projections $E_T(\cdot)$, we set $\kappa(\alpha,\beta)(T) = \text{dim}(E_T((\alpha,\beta)\mathfrak{f}))$, $-\infty \leq \alpha < \beta$ (these numbers may of course be infinite).

**Theorem 3.2.** Suppose that $A > 0$ is a positive definite elliptic operator, and $\Pi = \{\mathcal{H}_{\Omega},\Gamma_0,\Gamma_1\}$ is the boundary triple for $A^*$ in Proposition 3.1 with $A_0 := A^* \mid \ker(\Gamma_0) = \tilde{A}_{\gamma D}$, the Dirichlet realization of $A$. Assume also that the operator $\tilde{A}_C > 0$ is positive definite, $0 \in \rho(\tilde{A}_C)$. Let $K$ be a densely defined (not necessarily closed) symmetric operator in $\mathcal{H}_{\Omega}$ and $A_K$ the corresponding extension defined by (2.7). Then:

(i) The Calderon operator $\Lambda(0)$ is self-adjoint and negative definite, $\Lambda(0) < 0$.

(ii) If $K$ is $\Lambda(0)$-bounded with bound strictly less than one, then $A_K$ is symmetric (but not necessarily closed).

If in addition, $\text{ran}(K - \Lambda(0))$ is closed, then so is $A_K$, that is, $A_K \in C(L^2(\Omega))$.

(iii) If $K$ is $\Lambda(0)$-compact and self-adjoint, then $A_K$ is self-adjoint $A_K = (A_K)^*$, $\kappa(-\infty,0)(A_K) < \infty$, and

$$\kappa(-\infty,0)(A_K) = \kappa(-\infty,0)(I_{L^2(\Omega)} + (-\Lambda(0))^{-1/2})K(-\Lambda(0))^{-1/2}. \quad (3.1)$$

(iv) If $K$ is $\Lambda(0)$-compact and sectorial (resp., m-sectorial) with vertex $\zeta$ and semi-angle $\omega \in [0, \pi/2)$, then $A_K$ is sectorial (resp., m-sectorial) with vertex $\zeta$ and semi-angle $\omega$ too.

**Proposition 3.3.** Let $A$ be formally self-adjoint and assume the conditions of Proposition 3.1. Assume also that $A_K = (A_K)^*$ is a self-adjoint extension of the form (2.7) with $K \in C(\mathcal{H})$. Then the absolutely continuous parts $A_{K,ac}$ and $\tilde{A}_{B,ac}$ of $A_K$ and $\tilde{A}_B$, respectively, are unitarily equivalent. In particular, $\sigma_{ac}(A_K) = \sigma_{ac}(\tilde{A}_B)$.

**Proposition 3.4.** Suppose that $A = A_{\min}$ is symmetric, and let $\tilde{A}_B = \tilde{A}_{\gamma D}$ be the Dirichlet realization of $A$. Assume the conditions of Theorem 2.12 to be satisfied and that $\tilde{A}_G = (\tilde{A}_G)^*$ is an elliptic realization of $A$ with $G = \{G_j\}_{0}^{n-1}$. Then the absolutely continuous parts $\tilde{A}_{G,ac}$ and $\tilde{A}_{G,ac}$ of $\tilde{A}_G$ and $\tilde{A}_{\gamma D}$, respectively, are unitarily equivalent. In particular, $\sigma_{ac}(\tilde{A}_G) = \sigma_{ac}(\tilde{A}_{\gamma D})$.

Finally, we turn to eigenvalues in spectral gaps:

**Definition 3.5.** Let $A$ be a symmetric operator in $\mathcal{H}$. Then $(\alpha,\beta)$, $-\infty < \alpha < \beta < \infty$, is called a gap of $A$ if $\|2A - (\alpha + \beta)I\|_{\mathcal{H}} \geq (\beta - \alpha)$ for all $f \in \text{dom}(A)$.

By Corollary 2.11, $\sigma_{\text{ess}}(A_K) = \sigma_{\text{ess}}(\tilde{A}_B)$. Therefore, in the gaps of $\tilde{A}_B$, the point spectrum of $A_K$ can possibly accumulate at most at the endpoints of the gaps. Next, we actually show that $\sigma_{p}(A_K)$ cannot accumulate at the left end point of any gap:

**Theorem 3.6.** Suppose that the conditions of Theorem 3.2 are satisfied, and that $K$ is a symmetric $\Lambda(0)$-compact operator in $\mathcal{H}_{\Omega}$. In addition, let $(\alpha,\beta)$ be a finite gap of $A_0 = \tilde{A}_{\gamma D}$ and introduce $T_0(z) = \Lambda(z) - \Lambda(0)$.

Then:

(i) $T(z) = T_0(z) \in B_{\infty}(\mathcal{H}_{\Omega})$ for all $z \in \rho(\tilde{A}_{\gamma D})$.

(ii) There exists $\varepsilon_0 \in (0,(\beta - \alpha)/2)$ such that $E_{A_K}((\alpha,\alpha + \varepsilon_0)) = 0$, hence $\kappa(\alpha,\beta - \varepsilon)(A_K) = \text{dim}(E_{A_K}((\alpha,\beta - \varepsilon))) < \infty$ for any $\varepsilon \in (0,\beta - \alpha)$. Moreover, for any $\varepsilon \in (0,\varepsilon_0)$ the following equality holds (with $\Lambda := \Lambda(0)$):

$$\kappa(\alpha,\beta - \varepsilon)(A_K) = \kappa(-\infty,0)(I_{\mathcal{H}_{\Omega}} + (\Lambda + T(\beta - \varepsilon))^{-1/2} - \kappa(-\infty,0)(I_{\mathcal{H}_{\Omega}} + (\Lambda + T(\alpha + \varepsilon))^{-1/2}).$$

**Remark 3.7.** For Robin-type realizations $[\partial u/\partial n - \sigma u] \mid \partial \Omega = 0$, $\sigma \in L^\infty(\partial \Omega)$, of Schrödinger operators $-\Delta + q$ on exterior domains $\Omega \subset \mathbb{R}^3$, the estimate (2.9) (with $\ell = 1$) goes back to the pioneering work by Povzner [14]. For Robin realizations $A_\sigma$ of a second-order elliptic operator $A = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}a_{j,k}(x)x_j + q(x)$, with $q \geq 1$, and $\sum_{j,k=1}^{n} \xi_j a_{j,k}(x)\xi_k > 0$ for all $\{x, \xi \in \Omega \times (\mathbb{R}^n \setminus \{0\})\}$,

$$A_\sigma = A_{\max} \mid \text{dom}(A_\sigma), \text{dom}(A_\sigma) = \{u \in H^2(\Omega) \mid [\partial u/\partial n - \sigma u] \mid \partial \Omega = 0\}, \partial/\partial n = \sum_{j,k=1}^{n} a_{j,k}(x) \cos(x_j, x_j) \frac{\partial}{\partial x_k},$$

$\sigma \in L^\infty(\partial \Omega)$, the estimate (2.9) was obtained by Birman [5]. Moreover, in [5, Thm. 6.6] it is also proved that $\kappa_{(-\infty,0)}(A_\sigma) < \infty$. Thus, for $m = 1$ and $A_K = A_\sigma$, equality (3.1) with $K$ being a multiplication operator, $K : u \mapsto \sigma u$, yields a stronger result as it describes the actual number of eigenvalues in the gap $(-\infty,0)$. 


For positive elliptic realizations $\hat{A}_G$ of a nonnegative elliptic operator $A$ of order $2m$ in a bounded domain $\Omega \subset \mathbb{R}^n$, the estimate (2.9) is implied by a sharp estimate due to Grubb [10, eq. (3.22)].

Detailed proofs of these results will appear in [8].

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