MINIMAL MODELS FOR GRAPH-RELATED (HYPER)OPERADS

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ABSTRACT. We construct explicit minimal models for the (hyper)operads governing modular, cyclic and ordinary operads, and wheeled properads, respectively. Algebras for these models are homotopy versions of the corresponding structures.

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INTRODUCTION

The fundamental feature of Batanin-Markl’s theory of operadic categories [4] is that the objects under study are viewed as algebras over (generalized) operads in a specific operadic category, cf. also the introduction to [2]. Thus, for instance, ordinary operads arise as algebras over the terminal operad $1_{\mathcal{R}_{\mathcal{R}_{\mathcal{R}_{\mathcal{R}_{\mathcal{R}}}}}}$ in the operadic category $\mathcal{R}_{\mathcal{R}_{\mathcal{R}_{\mathcal{R}_{\mathcal{R}}}}}$ of rooted trees, modular operads are algebras over the terminal operad $1_{\mathcal{G}_{\mathcal{G}_{\mathcal{G}_{\mathcal{G}_{\mathcal{G}}}}}}$ in the operadic category $\mathcal{G}_{\mathcal{G}_{\mathcal{G}_{\mathcal{G}_{\mathcal{G}}}}}$ of genus-graded connected graphs, &c.

Our aim is to construct explicit minimal models for the (hyper)operads governing modular, cyclic and ordinary operads, and wheeled properads. We believe that the methods developed here can be easily modified to obtain minimal models for operads governing other common
operad- or PROP-like structures. According to general philosophy [21], algebras for these models describe strongly homotopy versions of the corresponding objects whose salient feature is the transfer property over weak homotopy equivalences. This might be compared to the following classical situation.

Associative algebras are algebras over the non-Σ operad \( \mathcal{A}_{\text{ss}} \). Algebras over the minimal model of \( \mathcal{A}_{\text{ss}} \) are Stasheff’s strongly homotopy associative algebras, also called \( A_\infty \)-algebras, cf. [23, Example 4.8]. This situation fits well into the framework of the current article, since \( \mathcal{A}_{\text{ss}} \) is the terminal non-Σ operad or, which is the same, the terminal operad in the operadic category of finite ordered sets and their order-preserving epimorphisms.

The case of strongly homotopy cyclic operads was treated by the third author in [28], while modular operads were addressed by B. Ward in [34]. Both articles use the language of colored operads while the operadic category lingo used here is, as we believe, more concise and efficient, after the necessary preparatory material developed in [2, 3] has been available.

**Comparison with other approaches and the context.** One of the major challenges of the theory of algebraic operads is to understand their strongly homotopy versions, also called \( \infty \)-operads or higher homotopy operads, and the related deformation theory. All approaches known to us are based on the interpretation of the operads in question as algebras over a specific ‘hyperoperad.’ The strongly homotopy versions then appear as algebras over a cofibrant, in some cases even minimal, resolution of that hyperoperad. To construct the required resolutions, one tries to mimics the methods of the theory of ‘classical’ algebraic operads.

Below we give a brief account of the approaches preceding the present article, and compare them to ours. The flavor of \( \infty \)-operads in spaces, see e.g. [18, 27], is quite different, so we do not discuss them here.

The first work that systematically treated operads as algebras over a ‘hyperoperad’ was the 2003 preprint by P. van der Laan [16] who interpreted nonsymmetric (non-Σ) operads as algebras over a colored operad \( \text{nsOp}_{\mathbb{N}} \), with natural numbers \( \mathbb{N} \) as the set of colors. More generally, nonsymmetric \( \mathcal{C} \)-colored operads are algebras over a colored operad \( \text{nsOp}_{\mathcal{C}} \) with the colors

\[
\mathcal{C}^+ = \left\{ \left( \begin{array}{c} c \\ c_1, \ldots, c_n \end{array} \right) \right\}, \quad c, c_1, \ldots, c_n \in \mathcal{C}, \ n \in \mathbb{N},
\]

where \( c_1, \ldots, c_n \) are the input colors, and \( c \) the color of the output. In turn, nonsymmetric \( \mathcal{C}^+ \)-colored operads are algebras over a \( \mathcal{C}^{++} \)-colored operad \( \text{nsOp}_{\mathcal{C}^+} \), where

\[
\mathcal{C}^{++} = \left\{ \left( \begin{array}{c} c^+ \\ c_1^+, \ldots, c_n^+ \end{array} \right) \right\}, \quad c^+, c_1^+, \ldots, c_n^+ \in \mathcal{C}^+, \ n \in \mathbb{N},
\]

and so on. Van der Laan proved that \( \text{nsOp}_{\mathbb{N}} \) is quadratic Koszul, with the binary generators

\[
a_i \in \text{nsOp} \left( \binom{m + n - 1}{m, n} \right), \quad m, n \in \mathbb{N}, \ 1 \leq i \leq m,
\]
representing the partial compositions

\[ \circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1), \ m, n \in \mathbb{N}, \ 1 \leq i \leq m. \]

The operad nsOpC with an arbitrary set of colors admits a similar presentation. Since nsOpC is quadratic Koszul, it has a nice canonical resolution whose algebras are strongly homotopy non-symmetric C-colored operads. An independent combinatorial description of this resolution was given by the third author in [28].

The case of symmetric operads is dramatically different. They are algebras over an \( \mathbb{N} \)-colored operad Op generated, along with the quadratic generators (1), also by the linear ones

\[ g_\sigma \in Op \left( \frac{n}{n} \right), \ n \in \mathbb{N}, \ \sigma \in \Sigma_n \setminus \{1_n\}, \]

encoding the symmetric group action. A cofibrant resolution of Op was described, using the curved Koszul duality, by M. Dehling and B. Vallette in the fascinating article [8]. Strongly homotopy symmetric operads appearing in this way involve also resolutions of the symmetric group action.

Since our main applications, such as modular operads or wheeled PROPs, live over a field of characteristic zero, we do not want to touch the actions of the symmetric groups, but have them hidden in the toolbox. Good analogy is the Koszul duality theory for algebraic operads [25, Chapter 3], where the symmetric group actions enter the picture already at the level of the generating collections.

Group actions can be incorporated with the use of (hyper)operads whose colors are objects of groupoids. For instance, such a groupoid-colored operad governing symmetric operads does not require generators (2), since the symmetric group actions appear as the morphism spaces

\[ \text{Hom}_{\mathbb{N}}(m, n) = \begin{cases} \emptyset & \text{if } m \neq n, \\ \Sigma_m & \text{if } m = n. \end{cases} \]

between the colors in \( \mathbb{N} \). Groupoid-colored operads are equivalent to Feynman categories of Kaufman and Ward [1, 15] and, indeed, a foundation of Koszul duality for operads in the context of Feynman categories has been developed in [15, 33, 34].

In our approach, the ‘hyperoperads’ are operads, in the generalized sense of [4, Definition 1.11], over certain operadic categories of graphs which already contain the symmetric group actions as particular automorphisms. Having the symmetric group actions swept under the carpet, our setup is analogous to the classical theory of algebraic operads in characteristic zero.

Let us explain how symmetric operads used above as an example are treated in our approach. They appear as algebras for the terminal operad 1_{\text{RTr}} over the operadic category \text{RTr} [nests.tex]
of rooted trees, see Subsection 3.5 for details. The vertices of the trees in $\mathbb{R}Tr$ are linearly ordered, and also the incoming edges of each vertex and the legs are linearly ordered by the local resp. global orders, cf. Figure 1.

The orders determine which operation a given rooted tree represents. For instance, the tree in Figure 1 represents the operation that to elements $a_1, a_2 \in \mathcal{P}(2)$ and $a_3 \in \mathcal{P}(3)$ of a symmetric operad $\mathcal{P}$ assigns the operation in $\mathcal{P}(5)$ acting on the ‘variables’ $x_1, \ldots, x_5$ as $a_2(a_3(x_4, x_1, x_2), a_1(x_5, x_3))$. The category $\mathbb{R}Tr$ contains $2! \times 2! \times 3!$ rooted trees of the same shape, with the same global order, with the same order of the vertices, but with possibly different local orders. All these trees are related by ‘local isomorphisms’ which incorporate the symmetric groups actions to our approach. A schematic picture of a configuration of ‘pancakes’ describing operations of cyclic operads can be found in [11, pages 95-96].

**Warning.** As in the approach based on colored operads, the objects we study appear in our approach as algebras for a certain, in most but not all cases terminal, ‘hyperoperad’ over a suitable operadic category, though they themselves need not be operads in the sense of [4]. Thus, for instance, there is no operadic category having cyclic operads as its operads, but cyclic operads are algebras for the terminal ‘hyperoperad’ over the operadic category of trees.

**The models.** Here we point to the places where the advertised constructions can be found.

- The minimal model $\mathcal{M}_{gg\text{grc}}$ of the operad $1_{gg\text{grc}}$ governing modular operads is constructed in Subsection 3.2. Algebras for this minimal model are strongly homotopy modular operads.
- The minimal model $\mathcal{M}_{Tr}$ of the operad $1_{Tr}$ governing cyclic operads is constructed in Subsection 3.3. Algebras for this minimal model are strongly homotopy cyclic operads.
- The minimal model $\mathcal{M}_{whe}$ of the operad $1_{whe}$ governing wheeled properads is constructed in Subsection 3.4. Algebras for this minimal model are strongly homotopy wheeled properads.
There are two operadic categories such that the algebras for their terminal operads are ordinary operads – the category \( \mathcal{RTr} \) of rooted trees and its full subcategory \( \mathcal{SRTTr} \) of strongly rooted trees. The minimal models \( \mathcal{M}_{\mathcal{RTr}} \) resp. \( \mathcal{M}_{\mathcal{SRTTr}} \) of the corresponding terminal operads \( 1_{\mathcal{RTr}} \) resp. \( 1_{\mathcal{SRTTr}} \) are constructed in Subsections 3.5 resp. 3.6. Both \( \mathcal{M}_{\mathcal{RTr}} \) and \( \mathcal{M}_{\mathcal{SRTTr}} \) have the same algebras, namely strongly homotopy ordinary operads. The reason why we consider two categories governing the same structures is explained below.

Methods used. We begin with the particular case of the operadic category \( \mathcal{Grc} \) of connected graphs. Algebras for the terminal operad \( 1_{\mathcal{Grc}} \) in that category are modular operads without the genus grading. We explicitly define, in Section 2, a minimal \( \mathcal{Grc} \)-operad \( \mathcal{M}_{\mathcal{Grc}} = (F(D), \partial) \) and a map \( \mathcal{M}_{\mathcal{Grc}} \xrightarrow{\rho} 1_{\mathcal{Grc}} \) of differential graded \( \mathcal{Grc} \)-operads. Theorem 14 states that \( \rho \) is a level-wise homological isomorphism, meaning that \( \mathcal{M}_{\mathcal{Grc}} \) is a minimal model of \( 1_{\mathcal{Grc}} \). Proof of Theorem 14 is a combination of the following facts.

On one hand, using the apparatus developed in [3], we describe, in Subsection 1.2, the piece \( F(D)(\Gamma), \Gamma \in \mathcal{Grc} \), of the free operad \( F(D) \) as a colimit over the poset \( g\mathcal{Tr}(\Gamma) \) of graph-trees associated to \( \Gamma \), which are abstract trees whose vertices are decorated by graphs from \( \mathcal{Grc} \) and which fulfill suitable compatibility conditions involving \( \Gamma \).

On the other hand, to each \( \Gamma \in \mathcal{Grc} \) we associate, in Subsection 2.3, a hypergraph \( H_{\Gamma} \) and to that hypergraph a poset \( \mathcal{A}(H_{\Gamma}) \) of its constructs, which are certain abstract trees with vertices decorated by subsets of the set of internal edges of \( \Gamma \). We prove, in Proposition 18, that the poset \( g\mathcal{Tr}(\Gamma) \) is order-isomorphic to the poset \( \mathcal{A}(H_{\Gamma}) \).

Lemma 3 asserts that \( \mathcal{A}(H_{\Gamma}) \) is in turn order-isomorphic to the face lattice of a certain convex polytope \( C(H_{\Gamma}, \Pi) \). The construction of this polytope is a generalization of the construction from [7] and has an interesting interpretation in terms of game theory. This game theoretic interpretation is not a central theme of our paper but we decided to include a brief description of this topic because it opens up some new perspectives on operad theory and can also be useful in calculations.

Finally, using the ‘ingenious’ Lemma 21 we show that the faces of \( C(H_{\Gamma}, \Pi) \) can be oriented so that the cellular chain complex of \( C(H_{\Gamma}, \Pi) \) is isomorphic, as a differential graded vector space, to \( (F(D)(\Gamma), \partial) \). Since \( C(H_{\Gamma}, \Pi) \) is acyclic in positive dimension, the same must be true for \( (F(D)(\Gamma), \partial) \). It remains to show that \( \rho \) induces an isomorphism of degree 0 homology, but this is simple. The conclusion is that \( \mathcal{M}_{\mathcal{Grc}} \) is indeed a minimal model of \( 1_{\mathcal{Grc}} \).

In constructing the minimal models of the terminal operads \( 1_{\text{ggGrc}}, 1_{\text{Tr}} \) and \( 1_{\text{Whe}} \) in the operadic categories \( \text{ggGrc} \) of genus-graded connected graphs, \( \text{Tr} \) of trees and \( \text{Whe} \) of ordered (‘wheeled’) connected graphs, respectively, we use the fact observed in [2, Section 4] that these categories are discrete operadic opfibrations over \( \mathcal{Grc} \). Their minimal models are then, thanks to Corollary 28, the restrictions of the minimal model for \( 1_{\mathcal{Grc}} \) along the corresponding opfibration map.
The situation of the terminal operad $1_{\text{RTr}}$ in the operadic category $\text{RTr}$ of rooted trees is different, since this category is not an opfibration over $\text{Grc}$. It is, however, a discrete operadic fibration with finite fibers, so Corollary 28 of Section 3 applies as well.

We finally introduce a full subcategory $\text{SRTr} \subset \text{RTr}$ consisting of strongly rooted trees. The algebras for the terminal $\text{SRTr}$-operad $1_{\text{SRTr}}$ are the same as $1_{\text{RTr}}$-algebras, i.e. ordinary operads. We consider this subcategory since it is the most economic description of ordinary operads. Although it is neither a fibration, nor an opfibration over $\text{Grc}$, we show in Subsection 3.5 that the minimal model for $1_{\text{SRTr}}$ can be obtained by a straightforward modification of the construction of the minimal model for $1_{\text{Grc}}$ given in Section 2.

**Limitations and generalizations.** Minimal models studied via the methods developed in the present work appear as the cellular chain complexes of sequences of contractible polytopes. Since the homology of such complexes is a one-dimensional vector space sitting in degree 0, our approach clearly applies only to (hyper)operads that are terminal in an appropriate category of (hyper)operads. This limitation however still leaves room for the study of structures such as non-symmetric modular operads and modular hybrids introduced in [12], dioperads [13], and a couple of others not addressed in the present article.

An interesting situation occurs for terminal (hyper)operads that are quadratic but not Koszul self-dual. This is the case, e.g., of the operad $1_{\text{ggGrc}}$ governing modular operads. The proof of [3, Theorem 9.6] establishing the Koszulity of $1_{\text{ggGrc}}$ uses the explicit minimal model $\mathcal{M}_{\text{ggGrc}}$ constructed in the present paper. The dual dg operad $\mathcal{D}(1_{\text{ggGrc}})$ then in turn provides an explicit minimal model for the Koszul dual of $1_{\text{ggGrc}}$, which is the non-terminal operad $R_{\text{ggGrc}}$ whose algebras are odd modular operads. See Sections 5 and 9 of [3] for the terminology and definitions. The methods of the present paper may therefore lead to explicit minimal models for some non-terminal (hyper)operads as well.

**Applications.** As explained e.g. in [20] or [22], an explicit minimal model of a traditional operad $\mathcal{P}$ leads to an explicit $L_\infty$ (= strongly homotopy Lie) algebra which, via the related simplicial Maurer-Cartan space, provides full information about the moduli space of deformations of $\mathcal{P}$-algebras. We believe that the same is true also in the generalized context of this paper.

In particular, the constructions presented here should provide understanding of deformations of modular, cyclic and traditional operads, as well as wheeled properads, including the associated cohomology theory and higher homotopy operations analogous to the Massey products for modular operads constructed in [34]. Since all minimal models constructed here possess quadratic differentials, the governing $L_\infty$-algebra is actually just the `ordinary’ differential graded Lie algebra, which makes the related theory conceptually very simple.
Plan of the paper. In Section 1 we recall necessary facts about hypergraph polytopes, and free operads in operadic categories. Section 2 is devoted to the construction of the minimal model for the terminal $\mathbf{Gr}_{\mathbf{c}}$-operad, and presentation of the necessary preparatory material. Section 3 addresses minimal models for terminal operads in the operadic categories of genus-graded graphs, trees, wheeled graphs and strongly rooted trees.

Conventions. All algebraic objects will be considered over a field $k$ of characteristic zero. By $|X|$ we denote either the cardinality if $X$ is a finite set, or the geometric realization if $X$ is a graph. If not specified otherwise, (hyper)operads featured here will live in the monoidal category of differential graded $k$-vector spaces. The terminal operad in a given operadic category is the one all of whose components equal $k$ and whose structure operations are the identities. These operads are linearizations of the corresponding terminal set-operads, which hopefully justifies our relaxed terminology.

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1. Recollections

This section contains a preparatory material regarding hypergraph polytopes and operadic categories. The basic references are [7, 28] for the former and [2, 3, 4] for the latter.

1.1. Hypergraph polytopes. They are abstract polytopes whose geometric realization can be obtained by truncating the vertices, edges and other faces of simplices, in any finite dimension. In particular, the family of $n$-dimensional hypergraph polytopes consists of an interval of simple polytopes starting with the $n$-simplex and ending with the $n$-dimensional permutahedron.

Hypergraph terminology. For a set $H$ let $\mathcal{P}(H)$ be its power-set.

A hypergraph is a pair $H = (H, \mathbf{H})$ of a finite set $H$ of vertices and a subset $\mathbf{H} \subseteq \mathcal{P}(H) \setminus \emptyset$ of hyperedges, such that, for all $x \in H$, $\{x\} \in \mathbf{H}$ (note that this property implies that $\bigcup \mathbf{H} = H$ and justifies the convention to use the bold letter $\mathbf{H}$ for both the hypergraph itself and its set of hyperedges). A hypergraph $\mathbf{H}$ is connected if there are no non-trivial partitions $H = H_1 \cup H_2$, such that

$$H = \{X \in \mathbf{H} \mid X \subseteq H_1\} \cup \{Y \in \mathbf{H} \mid Y \subseteq H_2\}.$$ 

A hypergraph $\mathbf{H}$ is saturated when, for every $X, Y \in \mathbf{H}$ such that $X \cap Y \neq \emptyset$, we have that $X \cup Y \in \mathbf{H}$. Every hypergraph can be saturated by adding the missing (unions of) hyperedges. Let us introduce the notation

$$H_X := \{Z \in \mathbf{H} \mid Z \subseteq X\},$$
for a hypergraph $H$ and $X \subseteq H$. The \textit{saturation} of $H$ is then formally defined as the hypergraph
\[
Sat(H) := \{X \mid \emptyset \subsetneq X \subseteq H \text{ and } H_X \text{ is connected}\}.
\]

For a hypergraph $H$ and $X \subseteq H$, we also set
\[
H \setminus X := H \backslash H_X.
\]
Observe that for each finite hypergraph there exists a partition $H = H_1 \cup \ldots \cup H_m$, such that each hypergraph $H_{H_i}$ is connected and $H = \bigcup (H_{H_i})$. The $H_{H_i}$’s are called the \textit{connected components} of $H$. We shall write $H_i$ for $H_{H_i}$. We shall use the notation
\[
H \setminus X \leadsto H_1, \ldots, H_n
\]
to indicate that $H_1, \ldots, H_n$ are the connected components of $H \setminus X$.

\textit{Abstract polytope of a hypergraph.} We next recall from \cite{7} the definition of the abstract polytope $A(H) = (A(H) \cup \{\emptyset\}, \leq_H)$ associated to a connected hypergraph $H$.

The elements of the set $A(H)$, to which we refer as the \textit{constructs} of $H$, are the non-planar, vertex-decorated rooted trees defined recursively as follows.

(C0) If $H$ is the empty hypergraph, then $A(H) = \{\emptyset\}$, i.e. $A(H)$ is the singleton poset containing $\emptyset$.

Otherwise, let $\emptyset \neq X \subseteq H$ be a subset of the set of vertices of $H$.

(C1) If $X = H$, then the abstract rooted tree with a single vertex labeled by $X$ and without any inputs, is a construct of $H$; we denote it by $H$.

(C2) If $X \subsetneq H$, if $H \setminus X \leadsto H_1, \ldots, H_n$, and if $C_1, \ldots, C_n$ are constructs of $H_1, \ldots, H_n$, respectively, then the tree whose root vertex is decorated by $X$ and that has $n$ inputs, on which the respective $C_i$’s are grafted, is a construct of $H$; we denote it by $X\{C_1, \ldots, C_n\}$.

In what follows, we shall refer to the vertices of constructs by the sets decorating them, since they are a fortiori all distinct. The notation $C : H$ will mean that $C$ is a construct of $H$.

The partial order $\leq_H$ on non-empty constructs is generated by the edge-contraction:
\[
Y\{X\{C_1, \ldots, C_m\}, C_2, \ldots, C_n\} \leq_H (Y \cup X)\{C_1, \ldots, C_m, C_2, \ldots, C_n\}
\]
and the relation
\[
\text{if } C'_1 \leq_{H_i} C''_1 \text{ then } X\{C'_1, \ldots, C_n\} \leq_H X\{C''_1, \ldots, C_n\}.
\]
In addition, for each construct $C$ of $H$, we have that $\emptyset \leq_H C$.

The faces of $A(H)$ are ranked by integers ranging from $-1$ to $|H| - 1$. The face $\emptyset$ is the unique face of rank $-1$, whereas the rank of a construct $C$ is $|H| - |\text{vert}(C)|$. In particular, constructs whose vertices are all decorated with singletons are faces of rank 0, whereas the construct $H$ is the unique face of rank $|H| - 1$.

Convex realization of $A(H)$ as core of a game. We recall some terminology of cooperative game theory [30]. Let $H$ be a finite set. A cooperative game of $n = |H|$ players is a function

$$\Pi : \mathcal{P}(H) \setminus \emptyset \to \mathbb{R}_{\geq 0}.$$ 

A classical interpretation of such a game is that every nonempty subset $I = \{i_1, \ldots, i_k\} \in \mathcal{P}(H) \setminus \emptyset$ (called coalition of players) has certain utility $\Pi(I)$ (blocking power of the coalition) in its disposition which can be distributed among members of $I$. An outcome of such a distribution is a real valued vector $(x_{i_1}, \ldots, x_{i_k}) \in \mathbb{R}_{\geq 0}^{|I|}$ such that $\sum_{i \in I} x_i \leq \Pi(I)$. Let

$$\pi_I^+ := \{(x_1, \ldots, x_n) \in \mathbb{R}_{\geq 0}^n \mid \sum_{i \in I} x_i \geq \Pi(I)\}$$

and

$$\pi_I := \{(x_1, \ldots, x_n) \in \mathbb{R}_{\geq 0}^n \mid \sum_{i \in I} x_i = \Pi(I)\}.$$ 

Then the core of the game is defined as a convex set

$$\mathcal{C}(\Pi) := \bigcap_{I \in \mathcal{P}(H) \setminus \{H\}} \pi_I^+ \cap \pi_H.$$ 

The vectors of this set can be interpreted as some sort of stable distributions among players, where there is no intention among players to form a smaller coalition which can deliver a better distribution of utility for its members, that is, no smaller coalition would wish to block such a distribution.

A cooperative game is called convex if the inequality

$$\Pi(X \cup Y) \geq \Pi(X) + \Pi(Y) - \Pi(X \cap Y)$$

holds for all $X, Y \in \mathcal{P}(H) \setminus \emptyset$. For a strictly convex game (also known as upper supermodular function in lattice theory) this inequality holds strictly for all $X, Y$ such that $X$ is not a subset of $Y$ or $Y$ is not a subset of $X$. A classical result of Shapley [30] is that the core of a convex game is nonempty and, moreover, the core of a strictly convex game is an $(n - 1)$-dimensional convex polytope which is combinatorially equivalent to the permutahedron on $n$ letters.

Example 1. Let $G$ be a cooperative game given by the function $G(I) = 3^{|I|}$ for a coalition $I \subset H$. It is not hard to check that this is a strictly convex game. In fact, this was checked implicitly by Došen and Petrić in [10, Lemma 9.1].
Example 2. Let $L$ be a cooperative game given by the function

$$L(I) = 1 + 2 + \cdots + |I| = |I|(|I| + 1)/2.$$  

This is again a strictly convex game which is easy to check. The core of this game is the classical convex realization of the permutahedron $P_n$ as the convex hull of $\{\sigma(1, \ldots, n) \mid \sigma \in S_n\}$.

The reader is referred to [30] for a zoo of various examples of convex games.

Let now $H = (H, H)$ be a hypergraph. An $H$-cooperative game is a cooperative game $\Pi$ of $|H|$-players. We call such a game (strictly) convex if $\Pi$ is (strictly) convex game. We now want to adapt the concept of the core of such a game to take into account the hypergraph structure. Namely, we forcibly (that is by law) forbid to form coalitions which are not the hyperedges of the saturation of $H$. That is

$$C(H, \Pi) := \bigcap_{Y \in \text{Sat}(H) \setminus \{H\}} \pi^+_I \cap \pi_H.$$  

Another way to say it is that we form a new game in which all coalitions which are not the hyperedges of the saturation of $H$ have blocking power 0 but the blocking power of other coalitions remains the same. Obviously the core of such a game is precisely $C(H, \Pi)$.

Lemma 3. Let $\Pi$ be a strictly convex $H$-cooperative game. The poset $A(H)$ is order-isomorphic to the face lattice of a convex polytope $C(H, \Pi)$ obtained as a truncation of the $(|H| - 1)$-dimensional simplex. In particular, $A(H)$ is an abstract polytope of rank $|H| - 1$.

Proof. The order-isomorphism between the poset of constructs of $H$ and the poset of geometric faces of $C(H, \mathcal{G})$, where $\mathcal{G}$ is the game from the Example 1 is defined in [7, Section 3.3] (this polytope was denoted $\mathcal{G}(H)$ there). The fact that $A(H)$ is an abstract polytope follows in this particular case from Lemma 9.1 of [10]. It is however not hard to check that the arguments of [7] and [10] work for any strictly convex game $\Pi$ instead of $\mathcal{G}$. The lemma can also be deduced from the classical combinatorial description of the core by Shapely [30].  

Remark 4. One can use the game $L$ from Example 2 to get Loday’s realization of the associahedron and its generalizations, see [6, 17] for a survey and comparison of different convex realizations of generalized associahedra and other polytopes, and also [5, 9] or [31] for earlier sources addressing the problem of geometric realization of polytopes.

1.2. Free operads in the operadic category of graphs. The basic operadic category in this section will be the category $\mathbf{Grc}$ of connected ordered graphs introduced in [2, Section 3] and Example 5.7 loc. cit. to which we also refer for terminology and notation. Results for other categories of graphs will be straightforward modifications of this situation. Recall that the adjective ordered means that the (finite) set of vertices of $\Gamma$ is (linearly) ordered, as well as are the (finite) sets of half-edges adjacent to each vertex of $\Gamma$, and that also the (finite)
set of legs of $\Gamma$ is ordered. To simplify the terminology, by a graph we always mean in this section an object of $\text{Gr}_c$. As the first step in describing the component $F(E)(\Gamma)$, $\Gamma \in \text{Gr}_c$, of the free operad $F(E)$ generated by a 1-connected collection $E$ we identify, in Theorem 7 below, the set $\pi_0(\text{lTw}(\Gamma))$ of connected components of the groupoid $\text{lTw}(\Gamma)$ of labelled towers [3, Section 3] with a certain class of trees defined below. Recall that we work with a skeletal version of the category of finite ordered sets, therefore two arbitrary order-isomorphic finite sets are the same.

Recall that a map of graphs is a quasibijection if all its fibers are trivial, i.e. are corollas whose local and global orders agree [2, Section 3]. By [2, Lemma 3.15], all quasibijections in $\text{Gr}_c$ are isomorphisms. A map of graphs is called order-preserving if the induced map of vertices preserves the orders. An order-preserving map is elementary if all its fibers are trivial except precisely one which is required to have at least one internal edge.

Before we continue, we introduce a particular class of maps between graphs, called canonical contractions (or cc’s for short) of a subgraph. The informal definition is the following.

Let $\Gamma \subset \Gamma'$ be a subgraph and $\Gamma''$ be obtained from $\Gamma'$ by contracting all internal edges of $\Gamma$ into a vertex. The canonical contraction $\pi : \Gamma' \to \Gamma''$ is then the ‘obvious projection.’ We however need to specify labellings and orders of the vertices and flags of $\Gamma$ and $\Gamma''$, so a more formal definition is needed.

Assume that $\Gamma' = (V', F') \in \text{Gr}_c$ is a graph with the set of vertices $V'$, the set of flags $F'$ and the structure map $g' : F' \to V'$, see [2, Definition 3.1]. Choose a nonempty subset $V \subset V'$ and a nonempty set $E$ of edges of $\Gamma'$ formed by the half-edges in $g'^{-1}(V) \subset F'$ such that the subgraph of $\Gamma'$ spanned by $E$ is connected. Let us denote by $V'/V$ the ordered set

$$V'/V := (V' \setminus V) \cup \{\text{min}(V)\};$$

the notation being justified by the canonical set-isomorphism of $V/V'$ as above with the set-theoretic quotient $V'$ by the subset $V$. Let finally $V'' := V'/V$ and

$$\phi : V' \to V'' = V'/V$$

be the ‘projection’ that is the identity on $V' \setminus V$ while it sends all elements of $V$ to $\text{min}(V)$.

We construct $\Gamma''$ as the graph whose set of vertices is $V''$ and whose set of flags is $F'' := F' \setminus E$. The defining map $g'' : F'' \to V''$ is the restriction of the composite $\phi \circ g'$, as in

$$\begin{array}{ccc}
F' & \xrightarrow{\psi} & F'' := F' \setminus E \\
\downarrow{g'} & & \downarrow{g''} \\
V' & \xrightarrow{\phi} & V''.
\end{array}$$
The involution \(\sigma'' : F'' \to F''\) is the restriction of the involution \(\sigma' : F' \to F'\) of \(\Gamma'\). The map \(g''\) defined by (1) is however not order-preserving as required by the definition of a graph. We therefore reorder \(F''\) by imposing the lexicographic order requiring that, for \(a, b \in F''\),
\[
a < b \text{ if and only if } \begin{cases} g''(a) < g''(b) \text{ in } V'', & \text{or} \\ g''(a) = g''(b) \text{ and } a < b \text{ in } F''. \end{cases}
\]
This formula obviously does not change the local orders of flags in \(F''\) around a given vertex.

We finally define the cc \(\pi : \Gamma' \to \Gamma''\) as the couple \((\psi, \phi)\) with \(\psi : F'' \hookrightarrow F'\) the inclusion. The unique nontrivial fiber of \(\pi\) is the graph \(\Gamma\) given by the restriction \(F \xrightarrow{g} V\) of \(g'\) to \(F := g'^{-1}(V)\) whose involution is trivial everywhere except for the flags forming the edges in \(E\), in which case it coincides with the involution of \(\Gamma'\). A simple example of a canonical contraction can be found in Figure 11 below.

We may sometimes loosely denote \(\Gamma'' := \Gamma' / \Gamma\). Canonical contractions in the above sense are modifications of pure contractions of [2, Definition 3.5] in that here we do not require the map of vertices to be order-preserving, which is compensated by introducing the lexicographic order on the flags of \(\Gamma''\). Canonical contractions are close to elementary morphisms in that they have precisely one nontrivial fiber with at least one nontrivial internal edge, but they need not be order-preserving. For the purposes of the proofs in this article only, we will call such morphisms pre-elementary. By definition, a pre-elementary morphism is elementary if and only if it is order-preserving. Canonical contractions provide representatives of morphisms with the property specified in the following lemma.

**Lemma 5.** Let \(\tau : \Delta' \to \Delta''\) be a map in \(\text{GrC}\) whose all fibers are corollas except precisely one (which thus has at least one internal edge). Then there exists a unique canonical contraction \(\pi : \Delta' \to \Delta\) and a unique isomorphism \(\sigma : \Delta'' \to \Delta\) such that the diagram

\[
\begin{array}{ccc}
\Delta' & \xrightarrow{\tau} & \Delta'' \\
\downarrow{\cc} & & \downarrow{\sigma} \\
\Delta & & \Delta
\end{array}
\]

commutes.

**Proof.** Assume that \(\Delta' = (V', F')\), \(\Delta'' = (V'', F'')\), and that \(\tau\) is given by the pair \((\phi, \psi)\) of maps in the diagram

\[
\begin{array}{ccc}
F' & \xrightarrow{\psi} & F'' \\
\downarrow{g'} & & \downarrow{g''} \\
V' & \xrightarrow{\phi} & V''.
\end{array}
\]

Let the only fiber of \(\tau\) which is not a corolla be the one over some \(x \in V''\). Then the canonical contraction \(\pi\) in the lemma is given by the data \(V := \phi^{-1}\{x\} \subset V'\) and \(E := F' \setminus F''\). It is simple to check that there exists a unique isomorphism \(\sigma\), symbolically expressed as...
\(\tau^{-1}\pi\), making diagram (5) commutative and that the canonical projection \(\pi\) for which such an isomorphism exists is unique as well. \(\Box\)

Let us return to the main topics of this section. A *graph-labelled tree*, or *graph-tree* for short, is a rooted tree \(T\) such that the union of the sets of input leaves and of internal edges is labelled by a finite ordered set \(V\) subject to the condition that an internal edge \(e\) of \(T\) is labelled by the minimum of the labels of the input leaves of the subtree of \(T\) ‘below’ \(e\), i.e. of the maximal subtree of \(T\) away from the root of \(T\) whose root vertex is \(e\). Moreover, vertices of a graph tree \(T\) are labelled by graphs in \(\text{Grc}\). This labelling shall satisfy two conditions. 

*Compatibility 1.* The ordered set of vertices of \(\Gamma_u\) labelling a vertex \(u\) of \(T\) equals the ordered set of the labels of the input edges of \(u\).

*Compatibility 2.* Let \(e\) be an internal edge of \(T\) pointing from (the vertex labelled by) \(\Gamma_u\) to (the vertex labelled by) \(\Gamma_v\). Then the ordered set of the half-edges of \(\Gamma_v\) adjacent to its vertex corresponding to \(e\) is the same as the ordered set of the legs of \(\Gamma_u\).

Since we are going to study free operads generated by 1-connected collections only, we assume that the graphs labelling the vertices of a graph-tree have at least one internal edge.

![diagram](image)

**Figure 2.** A graph-tree.

**Example 6.** A portrait of a graph-tree is given in Figure 2. The set \(V\) equals in this case to \(\{a, b, c, d, e, f\}\) with some (linear) order. The graph \(\Gamma_4\) has three vertices labelled by the elements of the subset

\[
\{ \min\{a, b, c\}, \min\{d\} = d, \min\{e, f\} \} \subset \{a, b, c, d, e, f\}
\]

with the induced linear order. The graph \(\Gamma_5\) has only one vertex labelled by \(d\).
Let $e$ be an internal edge of a graph-tree $T$ pointing from $\Gamma_u$ to $\Gamma_v$. Then the tree $T/e$ obtained by contracting the edge $e$ has an induced structure of a graph-tree given as follows. The leaves and internal edges of $T/e$ bear the same labels as they did in $T$. Also the vertices of $T/e$ except of the one, say $x$, created by the collapse of $e$, are labelled by the same graphs as in $T$. Finally, the vertex $x$ is labelled by the graph $\Gamma_x$ given by the vertex insertion of $\Gamma_u$ into the vertex of $\Gamma_v$ labelled by $e$. Since, by Compatibility 2, the ordered set of legs of $\Gamma_u$ is the same as the ordered set of the half-edges adjacent to the vertex of $\Gamma_v$ labelled by $e$, the vertex insertion is uniquely and well-defined. One clearly has

$$\text{vert}(\Gamma_x) = (\text{vert}(\Gamma_v) \setminus \{\text{the vertex labelled by } e\}) \cup \text{vert}(\Gamma_u),$$

where the union in the right hand side is disjoint thanks to Compatibility 1. The set $\text{vert}(\Gamma_x)$ bears an order induced from the inclusion $\text{vert}(\Gamma_x) \subset V$.

Repeating the collapsings described above we finally obtain a graph-tree with one vertex (i.e. a rooted corolla) whose only vertex is labelled by some graph $\Gamma \in \text{GrT}$ with the ordered set of vertices $V$. We denote the graph $\Gamma$ thus obtained, which clearly does not depend on the order in which we contracted the edges of $T$, by $\text{gr}(T)$.

**Theorem 7.** The set of connected components of the groupoid $\text{Tw}(\Gamma)$ is canonically isomorphic to the set $\text{gTr}(\Gamma)$ of graph-trees $T$ with $\text{gr}(T) = \Gamma$.

![Figure 3. Introducing levels to the tree in Figure 2. The labels of its leaves and edges are the same as in Figure 2](figure3.png)
Proof. Recall from [3] Section 3] that the objects of $1\text{Tw}(\Gamma)$ are labelled towers
\begin{equation}
\mathcal{F} = (\mathcal{F}, \ell) : \Gamma \xrightarrow{\ell} \Delta_0 \xrightarrow{\tau_1} \Delta_1 \xrightarrow{\tau_2} \Delta_2 \xrightarrow{\tau_3} \cdots \xrightarrow{\tau_{k-1}} \Delta_{k-1},
\end{equation}
where $\Delta_0, \ldots, \Delta_{k-1}$ are graphs in $\text{GrC}$, $\ell$ an isomorphism, and $\tau_1, \ldots, \tau_{k-1}$ elementary maps.

We will construct a map
\[ A : g\text{Tr}(\Gamma) \longrightarrow \pi_0(1\text{Tw}(\Gamma)) \]
of sets as follows. Assume that $T \in g\text{Tr}(\Gamma)$ is a graph-tree with $k$ vertices. We distribute the vertices of $T$ to levels such that each level contains precisely one vertex, see Figure [3] for an example. Let $T_{i-1}$, $1 \leq i \leq k$, be the graph-tree obtained from $T$ by truncating everything above level $i$, level $i$ included, see Figure [3] again. Denote $\tilde{\Delta}_{i-1} := \text{gr}(T_{i-1})$, $1 \leq i \leq k$. Notice that $\tilde{\Delta}_0 = \Gamma$ by definition. One thus obtains a sequence of pre-elementary maps
\begin{equation}
\Gamma \xrightarrow{1} \tilde{\Delta}_0 \xrightarrow{\tilde{\tau}_1} \tilde{\Delta}_1 \xrightarrow{\tilde{\tau}_2} \tilde{\Delta}_2 \xrightarrow{\tilde{\tau}_3} \cdots \xrightarrow{\tilde{\tau}_{i-1}} \tilde{\Delta}_{i-1}
\end{equation}
in which the map $\tilde{\tau}_i : \tilde{\Delta}_{i-1} \rightarrow \tilde{\Delta}_i$, $1 \leq i \leq k - 1$, is defined as follows. Let $u$ be the only vertex on the $i$th level and $e$ its out-going edge. Then $\tilde{\tau}_i$ is the map that contracts the subgraph $\Gamma_u$ of $\tilde{\Delta}_{i-1}$ into the vertex of $e$ labelled by $e$. In other words, $\tilde{\tau}_i$ is the canonical contraction $\tilde{\Delta}_{i-1} \rightarrow \tilde{\Delta}_i = \tilde{\Delta}_{i-1}/\Gamma_u$ and thus a pre-elementary map.

For instance, in the situation of Figure [3] the graph $\tilde{\Delta}_0$ has vertices $\{a, b, c, d, e, f\}$ and
\[ \text{vert}((\tilde{\Delta}_1)) = \{a, b, c, d, \text{min}\{e, f\}\}. \]
The map $\tilde{\tau}_1$ contacts the subgraph $\Gamma_3$ of $\tilde{\Delta}_0$ into the vertex $\text{min}\{e, f\}$ of $\tilde{\Delta}_1$. Likewise,
\[ \text{vert}(\tilde{\Delta}_2) = \{a, \text{min}\{b, c\}, d, \text{min}\{e, f\}\} \]
and $\tilde{\tau}_2$ contacts the subgraph $\Gamma_1$ into the vertex $\text{min}\{b, c\}$ of $\tilde{\Delta}_2$. Notice that $\tilde{\tau}_1$, resp. $\tilde{\tau}_2$ is order-preserving if and only if $\{e, f\}$ resp. $\{b, c\}$ is an interval. This shows that $\tilde{\tau}_i$’s in (7) need not in general be elementary, i.e. preserving the orders of the set of vertices.

Out next task will be to modify the tower (7) into a tower as in (6) with all $\tau_i$’s elementary. To do so we use the fact that the category $\text{GrC}$ is factorizable [2] Lemma 3.16], meaning that each morphism can be written as $\phi \sigma$, where $\phi$ is order-preserving and $\sigma$ a quasibijection. It also follows from [2] Lemma 2.1] resp [2] Lemma 2.2] that, if $\sigma$ is a quasibijection and $\psi$ a pre-elementary map, then both $\sigma \psi$ and $\psi \sigma$ are pre-elementary as well. Recall also that in $\text{GrC}$ all quasibijections are invertible and their inverses are quasibijections again.

The process of modification is described in Figure [4]. We start at the bottom, by putting $\Delta_{k-1} := \tilde{\Delta}_{k-1}$ and decomposing $\tilde{\tau}_{k-1}$ into a quasibijection $\sigma_{k-2}$ followed by an order-preserving $\tau_{k-1}$. By the above remarks, $\tau_{k-1} = \tilde{\tau}_{k-1} \sigma_{k-1}^{-1}$ is pre-elementary and, since it is order-preserving, it is elementary. Now decompose $\sigma_{k-2} \tilde{\tau}_{k-2}$ as a quasibijection $\sigma_{k-3}$ followed by an order preserving $\tau_{k-2}$. By the same reasoning, $\tau_{k-2} = \sigma_{k-2} \tilde{\tau}_{k-2} \sigma_{k-3}^{-1}$ is elementary. We go all the way up, ending with $\ell := \sigma_0$. 

Figure 4. Replacing (7) by a labelled tower of elementary maps (left); independence of the choices of factorizations (center); replacing (6) by a tower of canonical contractions (right).

The actual value of $a(T)$ might depend on the choice of factorizations but, as the diagram at the center of Figure 4 shows, the results are related by an isomorphism of the 1st type in the sense of [3, Section 3] whose definition we recall in Figure 5. Indeed, take $\sigma_{k-1} := \mathbb{I}$ and $\sigma_i := \sigma_i'' \sigma_i'^{-1}$ for $0 \leq i \leq k - 2$ in that figure. The value of $a(T)$ might also depend on the choice of levels of $T$, but any two such values are related by an isomorphism of the 2nd type in the sense of [3, Section 3]. The connected component of $a(T)$ therefore does not depend on the choices, so we may define $A(T) := \pi_0(a(T)) \in \pi_0(1\text{Tw}(\Gamma))$.

Let us proceed to the construction of the inverse $B : \pi_0(1\text{Tw}(\Gamma)) \to g\text{Tr}(\Gamma)$ of $A$. Suppose that we are given a labelled tower $\mathcal{T}$ as in (6). Our strategy will be to modify it into the form where $\ell$ is the identity and the remaining maps are canonical contractions. We start by absorbing $\ell$ into $\tau_1$ in (6) by replacing it with

$$(8) \quad \Gamma \xrightarrow{\mathbb{I}} \tilde{\Delta}_0 \xrightarrow{\tilde{\tau}_1} \Delta_1 \xrightarrow{\tau_2} \Delta_2 \xrightarrow{\tau_3} \cdots \xrightarrow{\tau_{k-1}} \Delta_{k-1}$$

where $\tilde{\Delta}_0 := \Gamma$ and $\tilde{\tau}_1 := \tau_1 \circ \ell$. Notice that all morphisms in (8) satisfy the hypothesis of Lemma 5. The next steps are illustrated by the diagram on the right of Figure 7. In that diagram, $\tilde{\tau}_1$ is the canonical projection obtained by taking, in Lemma 5, $\Delta' = \tilde{\Delta}_0$, $\Delta'' = \Delta_1$ and $\tau = \tilde{\tau}_1$. Now, the composition $\tau_2 \sigma_1^{-1}$ satisfies the assumptions of Lemma 5 and $\tilde{\tau}_2$ is the
canonical projection obtained from that lemma by taking $\Delta' = \Delta_1$, $\Delta'' = \Delta_2$ and $\tau = \tau_2\tau_1^{-1}$. We then continue all the way down till we eventually construct the canonical projection $\tilde{\tau}_{k-1}$.

We thus modified the labelled tower $\mathcal{T}$ in (6) to the tower (7) in which all $\tilde{\tau}_i$’s are canonical contractions. We will say that such a tower has the canonical form. Denote by $V_i$ the set of vertices of $\tilde{\Delta}_i$ in (7), $0 \leq i \leq k-1$. It follows from the definition of canonical contractions that $V_0 \supset V_1 \supset \cdots \supset V_{k-1}$. Moreover, each $V_i$ contains a distinguished element $x_i$ over which the unique nontrivial fiber of $\tau_i$ lives. We extend the notation by putting $V_k := \{\ast\}$, the one-point set, and $x_k := \ast$. The vertex parts of $\tau_i$’s give rise to the sequence

$$V_0 \xrightarrow{\phi_1} V_1 \xrightarrow{\phi_2} V_2 \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_{k-1}} V_{k-1} \xrightarrow{\phi_k} V_k = \{\ast\}$$

of epimorphisms with the property that $\min(\phi_1^{-1} \circ \cdots \circ \phi_i^{-1}(x_i)) = x_i$, $1 \leq i \leq k-1$. Such a sequence of epimorphisms of finite ordered sets determines in the standard manner a rooted tree with levels, with its leaves labelled by $V_0$, with the root $\ast$ and the remaining vertices $x_1, \ldots, x_{k-1}$. Forgetting the levels, decorating the root by $\tilde{\Delta}_0 = \Gamma$ and $x_i$ by the fiber $\Gamma_i$ of $\tau_i$, $1 \leq i \leq k-1$, leads to a graph-tree $B(\mathcal{T}) \in \text{gTr}(\Gamma)$.

The reason why $B(\mathcal{T})$ is well-defined, that is, $B(\mathcal{T}') = B(\mathcal{T}'')$ if $\mathcal{T}'$ and $\mathcal{T}''$ are isomorphic labelled towers, is that for isomorphisms of the second type, see [3, Section 3] for terminology, the difference disappears after forgetting the levels of the tree corresponding to (6), while it is not difficult to see that the canonical forms of labelled towers related by an isomorphisms of the first type are the same.

It is clear that $(B \circ A)(T) = T$ for $T \in \text{gTr}(\Gamma)$. Given a labelled tower $\mathcal{T} \in \text{1Tw}(\Gamma)$, the concrete form of the tower $a(B(\mathcal{T})) \in \text{1Tw}(\Gamma)$ representing $(A \circ B)(\mathcal{T}) \in \pi_0(\text{1Tw}(\Gamma))$.
depends on the choice of levels for the tree $B(\mathcal{J})$. But any two such towers are related by a type two isomorphism. Since modifying a tower into its canonical form does not change its isomorphism class, we established that $B$ is also a left inverse of $A$. □

The set $g\text{Tr}(\Gamma)$ and therefore also the set $\pi_0(1\text{Tw}(\Gamma))$ of connected components of the category $1\text{Tw}(\Gamma)$ has a natural poset structure induced by the relation $T \prec T/e$ for a graph-tree $T \in g\text{Tr}(\Gamma)$ and its edge $e$. Its categorical origin is the following.

Let us denote, only for the purpose of this explanation, by $\mathcal{C}$ the category whose objects are the same as the objects of $1\text{Tw}(\Gamma)$, i.e. the labelled towers $\mathcal{J}$ as in (6). We postulate that there is a unique morphism $\mathcal{J} \to S$, $\mathcal{J} \neq S$, in $\mathcal{C}$ if and only if $S$ is obtained from $\mathcal{J}$ by composing two or more adjacent morphisms $\tau_i$’s that have mutually joint fibers, in the sense of [2, Definition 5.4]. The only other morphisms in $\mathcal{C}$ are the identities.

We denote by $1\text{Tw}(\Gamma) \int \mathcal{C}$ the category with the objects of $1\text{Tw}(\Gamma)$ whose morphisms are formal compositions of a morphism of $1\text{Tw}(\Gamma)$ with a morphism of $\mathcal{C}$. The poset $(\pi_0(1\text{Tw}(\Gamma)), \prec)$ considered in the standard manner as a category is then canonically isomorphic to the pushout in $\text{Cat}$ of the diagram

$$
\begin{array}{ccc}
1\text{Tw}(\Gamma) & \xrightarrow{\pi_0(1\text{Tw}(\Gamma))} & 1\text{Tw}(\Gamma) \int \mathcal{C} \\
\downarrow & & \downarrow \\
\pi_0(1\text{Tw}(\Gamma)) & & 
\end{array}
$$

in which $\pi_0(1\text{Tw}(\Gamma))$ is taken as a discrete category.

We are finally going to give an explicit formula for the free $\text{GrC}$-operad $F(E)$ generated by a 1-connected collection $E$ evaluated at a graph $\Gamma$. Recall that $E$ is a representation, in the category of graded vector spaces, of the groupoid $\text{QVrt}(e)$ whose objects are graphs in $\text{GrC}$ and morphisms are virtual isomorphisms which are, in this specific case, isomorphisms of graphs which need not respect the orders of the legs. The 1-connectivity means that $E(\Gamma) \neq 0$ implies that $\Gamma \in \text{GrC}$ has at least one internal edge.

**Warning 1.** Let us consider the classical free non-$\Sigma$ operad $F(E) = \{F(E)(n)\}_{n \geq 1}$ generated by a collection $E$ of graded vector spaces. A common mistake is to assume that the elements of $F(E)$ are (represented by) trees with vertices decorated by elements of $E$. This is true only when $E$ is concentrated in even degrees. Otherwise we need one more piece of information, namely a choice of levels of the underlying tree.

Assume for instance that $s \in E(2)$ is a degree 1 generator. The leftmost tree in Figure 6 represents $(s \circ_2 s) \circ_1 s \in F(E)(4)$ while the middle one $(s \circ_1 s) \circ_3 s$ in the same piece of $F(E)$. By the parallel associativity of the $\circ_i$-operations

$$
(s \circ_2 s) \circ_1 s = -(s \circ_1 s) \circ_3 s,
$$
thus the two decorated trees represent different elements. If we do not specify the levels in the rightmost tree in Figure 6, we do not know to which one we refer to. The same caution is necessary also in case of free Grc-operads.

Let us return to our description of the free operad $F(E)$. For a graph-tree $T$ we denote by $\text{Lev}(T)$ the chaotic groupoid whose objects are all possible arrangements of levels of $T$. For a given $\lambda \in \text{Lev}(T)$, let $\Gamma_i$, $1 \leq i \leq k-1$, be the fiber of $\tau_i$ in the tower \((\ref{tower})\) associated to $T$ with levels $\lambda$. Here ‘chaotic’ means that the category $\text{Lev}(T)$ has a unique morphism $\lambda' \to \lambda''$ for any $\lambda', \lambda'' \in \text{Lev}(T)$; this morphism is necessarily an isomorphism. We extend the notation by $\Gamma_k := \Delta_k$. For a 1-connected collection we define
\begin{equation}
E(T, \lambda) := E(\Gamma_1) \otimes \cdots \otimes E(\Gamma_k).
\end{equation}
For different $\lambda$'s this expression differs only by the order of the factors, so we may, using the commutativity constraint for graded vector spaces, promote formula \((\ref{E(T, \lambda)})\) into a functor
\begin{equation}
E : \text{Lev}(T) \longrightarrow \text{Vec}
\end{equation}
into the category of graded vector spaces.

**Theorem 8.** Given a 1-connected collection $E$, one has the following description of the arity $\Gamma$ piece of the free operad $\mathbb{F}(E)$:
\begin{equation}
\mathbb{F}(E)(\Gamma) \cong \begin{cases} 
\bigoplus_{T \in \text{gTr}(\Gamma)} \text{colim} \ E(T, \lambda) & \text{if } \Gamma \text{ has at least one internal edge, and} \\
\mathbb{F} & \text{if } \Gamma \text{ has no internal edges.}
\end{cases}
\end{equation}

**Proof.** The statement is proved by applying the formulas of \cite{3} Section 3 to the particular case of Grc. Notice that $\Gamma$ has no internal edges if and only if $\text{gTr}(\Gamma) = \emptyset$. \hfill \qed

The reader may wonder how the formula in \((\ref{mathbb{F}})\) reflects any relation of an algebra between operations corresponding to the same underlying

Let us describe the operad structure of $\mathbb{F}(E)$ given in \((\ref{mathbb{F}})\). Recall first that the local terminal objects in the category Grc are ordered graphs with no internal edges, i.e. ordered corollas. The operad $\mathbb{F}(E)$ is strictly unital in the sense of \cite{2} Definition 6.2], with the transformation $\eta$ in \cite{2} eqn. (44)] given by the defining identity
\[ \mathbb{F}(E)(\Gamma) = \mathbb{F} \text{ if } \Gamma \text{ is local terminal.} \]
We describe next the action of the groupoid $\mathcal{QVrt}(e)$ generated by local isomorphisms, local reorderings and morphisms changing the global orders of legs of graphs. Let us start with the latter.

Let $T \in \mathcal{gTr}(\Gamma)$ be a graph tree and $\vartheta : \Gamma \to \Upsilon$ be an isomorphism changing the global orders of the legs. In other words, the graph $\Upsilon$ differs from $\Gamma$ only by the order of its legs. Since the legs of $\Gamma$ are the same as the legs of the graph $\Gamma_1$ decorating the root of $T$, one also has the induced isomorphism $\vartheta_1 : \Gamma_1 \to \Delta_1 \in \mathcal{QVrt}(e)$, where $\Delta_1$ is obtained from $\Gamma_1$ by reordering its legs according to $\vartheta$.

We define $S \in \mathcal{gTr}(\Upsilon)$ to be the graph-tree whose underlying tree is the same as the underlying tree of $T$, its edges have the same decorations as the corresponding edges in $T$, and also the vertices have the same decorations as in $T$ except for the root vertex of $S$ which is decorated by $\Delta_1$. If $T$ has levels $\lambda \in \text{Lev}(T)$, we equip $S$ with the same levels. One then has the action

$$E(T, \lambda) = E(\Gamma_1) \otimes \cdots \otimes E(\Gamma_k) \xrightarrow{E(\vartheta_1) \otimes 1^{\otimes k-1}} E(\Delta_1) \otimes \cdots \otimes E(\Gamma_k) = E(S, \lambda)$$

induced by the $\mathcal{QVrt}(e)$-action $E(\vartheta_1) : E(\Gamma_1) \to E(\Delta_1)$ on the generating collection. The above actions assemble into an action $\mathbb{F}(E)(\Gamma) \to \mathbb{F}(E)(\Upsilon)$ on the colimits (12).

The actions of local isomorphisms and local reorderings are defined similarly, so we can be brief. Given $T \in \mathcal{gTr}(\Gamma)$, a local reordering of $\Gamma$ induces in the obvious way local reorderings of the graphs decorating the vertices of $T$, and therefore also on the products (10). The reader may have a look at the proof of Proposition 5.10 in [3] for a detailed description of the action of the groupoid of local isomorphisms. The example presented in Figure 1 should also be helpful.

Local isomorphisms act by reorderings of the set $V$ of vertices of $\Gamma$. Note that, by the definition of a graph-tree, the set $V$ and its order determine the labels of the edges of $T$, so a reordering of $V$ may change the labels of the edges of $T$. Thus, according to Compatibility 1 for graph-trees, it induces local isomorphisms of the graphs decorating the vertices of $T$ which in turn act on the products (10).

Let us finally attend to the operad composition. That is, for an elementary morphism $F \triangleright \Gamma \xrightarrow{\phi} \Upsilon$ in $\mathcal{Grc}$, we must describe a map

$$o_\phi : \mathbb{F}(E)(\Upsilon) \otimes \mathbb{F}(E)(F) \longrightarrow \mathbb{F}(E)(\Gamma).$$

Given such a $\phi$, one can find as in the previous pages a canonical contraction $\hat{F} \triangleright \Gamma \xrightarrow{\hat{\phi}} \hat{\Upsilon}$ and an isomorphism $\sigma : \Upsilon \to \hat{\Upsilon}$ in the commutative diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\phi} & \Upsilon \\
\downarrow{\phi} & & \downarrow{\sigma} \\
\hat{\Gamma} & \xrightarrow{\hat{\phi}} & \hat{\Upsilon}.
\end{array}
\]
Using the equivariance [3, eqn. (21)] with $\omega = 1$, $\phi' = \phi$ and $\phi'' = \hat{\phi}$, we see that $\circ_\phi$ is uniquely determined by $\circ_{\hat{\phi}}$. So we may assume that $\phi$ in (13) is a canonical contraction.

Let $S \in g\text{Tr}(T), R \in g\text{Tr}(F), \lambda' \in \text{Lev}(S)$ and $\lambda'' \in \text{Lev}(R)$. Let also $x \in \text{vert}(\Upsilon)$ be the vertex over which the unique nontrivial fiber of $\phi$ lives. We define $T \in g\text{Tr}(\Gamma)$ as the graph-tree whose underlying tree is obtained by grafting the root of the underlying tree of $R$ to the leg of the underlying tree of $S$ labelled by $x$. The decorations of $T$ is inherited from the decorations of its graph-subtrees $S$ and $R$. It is simple to check that, since $\phi$ is a cc, $T$ is indeed a graph-tree.

We finally define $\lambda = \lambda' \circ_\phi \lambda'' \in \text{Lev}(T)$ by postulating that all vertices of $R$ are below the vertices of $S$ and that the restriction of $\lambda$ to the subtrees $S$ resp. $R$ is $\lambda'$ resp. $\lambda''$. The map (13) is then the colimit of the obvious canonical isomorphisms

$$E(T, \lambda) \cong E(S, \lambda') \otimes E(R, \lambda'').$$

**Remark 9.** When the generating collection is evenly graded, the elements of the product (11) represents the same elements of $\mathcal{F}(E)(\Gamma)$ for an arbitrary choice of $\lambda$, thus (12) can be replaced by a more friendly formula

$$\mathcal{F}(E)(\Gamma) \cong \bigoplus_{T \in g\text{Tr}(\Gamma)} \bigotimes_{v \in \text{vert}(T)} E(\Gamma_v).$$

As illustrated in Warning 1, this simplification is not possible for general collections. Yet, since the input edges of each graph-tree are ordered, there exists a preferred choice of the levels specified by the following lexicographic rule.

Assume that $a < b$ are (the labels of) two input edges of a vertex $v \in \text{vert}(T)$. Then all levels of the subtree of $T$ with the root $a$ are below the levels of the subtree with the root $b$. Denoting by $\lambda_{\text{lex}}$ the above arrangement, then

$$\mathcal{F}(E)(\Gamma) \cong \bigoplus_{T \in g\text{Tr}(\Gamma)} E(T, \lambda_{\text{lex}}).$$

One must however keep in mind that the combination $\lambda'_{\text{lex}} \circ_\phi \lambda''_{\text{lex}}$ of two lexicographic arrangements may not be lexicographic. Thus, if we want to use (14) the operadic composition based on the isomorphism

$$E(T, \lambda'_{\text{lex}} \circ_\phi \lambda''_{\text{lex}}) \cong E(S, \lambda'_{\text{lex}}) \otimes E(R, \lambda''_{\text{lex}})$$

must be followed by bringing the result back into the preferred form.

### 2. Minimal model for $1_{\text{GrC}}$

The aim of this section is to construct an explicit minimal model of the terminal $\text{GrC}$-operad $1_{\text{GrC}}$ governing non-genus graded modular operads.
2.1. Free operads and derivations. Free $\text{Grc}$-operads are graded,
\[
\mathbb{F}(E)(\Gamma) = \bigoplus_{n \geq 0} \mathbb{F}^n(E)(\Gamma), \quad \Gamma \in \text{Grc},
\]
where $\mathbb{F}(E)^0(\Gamma) = k$ and the higher pieces are given by the modification of (12):
\[
\mathbb{F}^n(E)(\Gamma) \cong \bigoplus_{T \in g\text{Tr}^n(\Gamma)} \text{colim}_{\lambda \in \text{Lev}(T)} E(T, \lambda),
\]
in which $g\text{Tr}^n(\Gamma)$ is, for $n \geq 1$, the subset of $g\text{Tr}(\Gamma)$ consisting of graph-trees $T$ with exactly $n$ vertices. Clearly $\mathbb{F}^1(E)(\Gamma) \cong E(\Gamma)$. To describe $\mathbb{F}^2(E)(\Gamma)$, we realize that there is precisely one way to introduce levels into a graph-tree $T \in g\text{Tr}^2(\Gamma)$, so (15) takes the form
\[
\mathbb{F}^2(E)(\Gamma) \cong \bigoplus_{T \in g\text{Tr}^2(\Gamma)} E(\Gamma_v) \otimes E(\Gamma_u),
\]
where $\Gamma_v$ (resp. $\Gamma_u$) is the graph decorating the vertex $v$ at the top level of $T$ (resp. the vertex $u$ at the bottom level of $T$). We also have the obvious Definition 10. A degree $s$ linear map $\varpi : \mathbb{F}(E) \rightarrow \mathbb{F}(E)$ of collections is a degree $s$ derivation if
\[
\varpi \circ \phi = \phi(\varpi \otimes 1) + \phi(1 \otimes \varpi),
\]
for every elementary morphism $F \triangleright \Gamma \xrightarrow{\phi} \Upsilon$ and $\circ \phi$ as in (13).

As expected, every derivation $\varpi$ is determined by its restriction $\varpi|_E : E = \mathbb{F}^1(E) \rightarrow \mathbb{F}(E)$, and every such a map extends to a derivation.

Remark 11. Given a linear map $\omega : E \rightarrow \mathbb{F}(E)$, its extension $\varpi : \mathbb{F}(E) \rightarrow \mathbb{F}(E)$ into a derivation is obtained by subsequent applications of $\omega$ to the factors $E(\Gamma_i), 1 \leq i \leq k$, of $E(T, \lambda)$ in (10), replacing each of these factors by its $\omega$-image.

2.2. Minimal models. They came to life, for dg associative commutative resp. dg Lie algebras, as the Sullivan resp. Quillen minimal models of rational homotopy types, see [32] and citations therein. Minimal models for (classical) operads were introduced and studied in [23], while minimal models for (hyper)operads governing permutads were treated in [26]. Below we give a definition for $\text{Grc}$-operads, definitions for other types of (hyper)operads featuring in this paper are obvious modifications and we will thus not spell them out explicitly.

Definition 12. The minimal model of a dg $\text{Grc}$-operad $P$ is dg $\text{Grc}$-operad $M$ together with a dg $\text{Grc}$-operad morphism $\rho : M \rightarrow P$, such that

(i) the component $\rho(\Gamma) : M(\Gamma) \rightarrow P(\Gamma)$ of $\rho$ is a homology isomorphism of dg vector spaces for each $\Gamma \in \text{Grc}$, and

(ii) the underlying non-dg $\text{Grc}$-operad of $M$ is free, and the differential $\partial$ of $M$ has no constant and linear terms (the minimality condition).
One can prove, adapting the proof of Theorem II.3.127 in [25], that minimal models are unique up to isomorphism. Our construction of the minimal model for $1_{\mathcal{Grc}}$ begins by describing its generating 1-connected collection. For a vector space $A$ of dimension $k$, we denote by $\det(A) := \Lambda^k(A)$ the top-dimensional piece of its Grassmann algebra. If $S$ is a non-empty finite set, we let $\det(S)$ to be the determinant of the vector space spanned by $S$. Given two finite sets $S_1 = \{e_1^1, \ldots, e_{a}^1\}, S_2 = \{e_1^2, \ldots, e_{b}^2\}$, we define

$$\omega_{S_1, S_2} : \det(S_1 \sqcup S_2) \rightarrow \det(S_1) \otimes \det(S_2).$$

by

$$\omega_{S_1, S_2}(e_1^1 \wedge \cdots \wedge e_{a}^1 \wedge e_1^2 \wedge \cdots \wedge e_{b}^2) := (e_1^1 \wedge \cdots \wedge e_{a}^1) \otimes (e_1^2 \wedge \cdots \wedge e_{b}^2).$$

Let, for $\Gamma \in \mathcal{Grc}$, $edg(\Gamma)$ denote the set of its internal edges, and $\det(\Gamma) := \det(edg(\Gamma))$. With this notation, the generating collection of the minimal model for $1_{\mathcal{Grc}}$ is defined as the one-dimensional vector space

$$D(\Gamma) := \det(\Gamma), \quad \Gamma \in \mathcal{Grc},$$

placed in degree $|\Gamma| := \text{card}(edg(\Gamma)) - 1$ if $\Gamma$ has at least one internal edge, while $D(\Gamma) := 0$ if $\Gamma$ is a corolla. Notice that for $\Gamma$ with exactly one internal edge, $\det(\Gamma)$ is canonically isomorphic to $k$.

The degree $-1$ differential $\partial$ will be determined by its restriction (denoted by the same symbol)

$$\partial : D \rightarrow \mathcal{F}^2(D) \subset \mathcal{F}(D)$$

as follows. Given $T \in g\text{Tr}^2(\Gamma)$, let $\Gamma_v, \Gamma_u \in \mathcal{Grc}$ have the same meaning as in (16), and $E_v := edg(\Gamma_v), E_u := edg(\Gamma_u)$. For $\mu \in D(\Gamma) = \det(\Gamma)$ we put

$$\partial_T(\mu) := \bigoplus_{T \in g\text{Tr}^2(\Gamma)} \partial_T(\mu),$$

where

$$\partial_T(\mu) := (-1)^{|\Gamma|} \omega_{E_v, E_u}(\mu) \in D(\Gamma_v) \otimes D(\Gamma_u) \subset \mathcal{F}^2(D)(\Gamma).$$

**Lemma 13.** The derivation $\partial$ defined above is a differential, i.e. $\partial^2 = 0$.

**Proof.** It is simple to see that $\partial^2$ is a derivation as well, so it suffices only to verify that $\partial^2$ vanishes on the generating collection. We leave this as an exercise to the reader. \qed

Let $\rho : \mathcal{F}(D) \rightarrow 1_{\mathcal{Grc}}$ be the unique map of $\mathcal{Grc}$-operads whose restriction $\rho|_{D(\Gamma)}$ is, for $\Gamma \in \mathcal{Grc}$, given by

$$\rho|_{D(\Gamma)} := \begin{cases} \mathbb{1}_k : D(\Gamma) = k \rightarrow k = 1_{\mathcal{Grc}}(\Gamma), & \text{if } |edg(\Gamma)| = 1, \text{ while} \\ 0, & \text{if } |edg(\Gamma)| \geq 2. \end{cases}$$

Having all this, we formulate:
Theorem 14. The object \( \mathcal{M}_{Grc} := (\xi(D), \partial) \xrightarrow{\rho} (1_{Grc}, \partial = 0) \) is a minimal model of the terminal \( Grc \)-operad.

The rest of this section is devoted to the proof of Theorem 14 and of the necessary auxiliary material.

2.3. **Constructs represent graph-trees.** The material of this subsection is based on modification and generalization of [28]. We start by associating to each object \( \Gamma \) of \( Grc \) a hypergraph \( H_\Gamma \) defined as follows: the vertices of \( H_\Gamma \) are the internal edges of \( \Gamma \) and two vertices are connected by an edge in \( H_\Gamma \) whenever, as edges of \( \Gamma \), they share a common vertex. Observe that the leaves of \( \Gamma \) play no role in the definition of \( H_\Gamma \).

**Example 15.** Here is an example of the association of a hypergraph to a graph:

\[
\Gamma = \begin{array}{c}
\begin{array}{c}
\circ \circ \circ \\
2 \quad 3 \\
\circ \circ \circ
\end{array}
\end{array}
\]

\[
x \quad y \quad z \\
u \quad v \quad w
\]

\[
H_\Gamma = \begin{array}{c}
\begin{array}{c}
\circ \circ \circ \\
x \quad y \quad z \\
u \quad v \quad w
\end{array}
\end{array}
\]

Assume that \( \Gamma = (V, F) \) is a graph with the structure map \( g : F \rightarrow V \). Choose a subset \( V' \subset V \) and a subset \( E' \) of edges of \( \Gamma \) formed by the half-edges in \( g^{-1}(V') \subset F \) such that the subgraph of \( \Gamma \) spanned by \( E' \) is connected. Let \( \Gamma' \) be the graph \( \Gamma' = (V', E') \) with \( F' := g^{-1}(V') \), with the structure map \( g' : F' \rightarrow V' \) given by the restriction of \( g \), and the involution which coincides with the involution of \( \Gamma \) on the half-edges forming the edges in \( E' \), and which is trivial on the remaining half-edges of \( \Gamma' \).

To simplify the terminology, we will still call \( \Gamma' \) a subgraph of \( \Gamma \) determined by the set of edges \( E' \) though, formally speaking, \( \Gamma' \) is obtained from an actual subgraph of \( \Gamma \) by cutting some of its edges in two half-edges. For example, the ‘subgraph’ of the graph \( \Gamma \) from Example 15 determined by the internal edge \( x \) is

\[
\begin{array}{c}
\begin{array}{c}
\circ \circ \\
2 \quad 3
\end{array}
\end{array}
\]

**Lemma 16.** The connected subgraphs, in the above relaxed sense, of a graph \( \Gamma \) that have at least one internal edge are in one-to-one correspondence with the connected subsets of \( H_\Gamma \), i.e. with the non-empty subsets \( X \) of vertices of \( H_\Gamma \) such that the hypergraph \( (H_\Gamma)_X \) is connected.

[October 17, 2022]
**Remark 17.** Thanks to Lemma [16] for a graph $\Gamma$ and $\emptyset \neq X \subseteq \text{edg}(\Gamma)$, we can index the connected components of $H_\Gamma \setminus X$ by the corresponding subgraphs of $\Gamma$, by writing

$$H_\Gamma \setminus X \sim H_{\Gamma_1}, \ldots, H_{\Gamma_n}.$$ 

Observe that the subgraphs $\Gamma_1, \ldots, \Gamma_n$ of $\Gamma$ do not in general make a decomposition of $\Gamma$, in the sense that the removal of the edges from the set $X$ may result in a number of subgraphs of $\Gamma$ reduced to a corolla without internal edges.

**Proposition 18.** There exists a natural isomorphism $\alpha_{\Gamma} : A(H_\Gamma) \xrightarrow{\cong} \text{gTr}(\Gamma)$ between the abstract polytope $A(H_\Gamma)$ of constructs of the hypergraph $H_\Gamma$ and the poset $\text{gTr}(\Gamma)$ of graph-trees such that $\text{gr}(T) = \Gamma$.

**Proof.** We define the announced one-to-one correspondence $\alpha_{\Gamma}$ between constructs $C : H_\Gamma$ and graph-trees $T \in \text{gTr}(\Gamma)$ by induction on the number of vertices of $C$. If $C$ is the maximal construct $\text{edg}(\Gamma) : H_\Gamma$, then $\alpha_{\Gamma}(T)$ is the planar rooted corolla

$$\alpha_{\Gamma}(C) = \Gamma \xrightarrow{\text{root}} \ldots$$

with the vertex decorated by $\Gamma$ and legs labelled by the ordered set $\text{vert}(\Gamma)$.

Suppose that $C = X\{C_1, \ldots, C_p\}, \ X \subset \text{edg}(\Gamma), H_\Gamma \setminus X \sim H_1, \ldots, H_p$ and $C_i : H_i$ for $1 \leq i \leq p$. By Lemma [16] there are connected subgraphs $\Gamma_i$ of $\Gamma$ such that $H_i = H_{\Gamma_i}$. There, moreover, exists a graph $\Gamma_X \in \text{GrC}$ such that $\Gamma_1, \ldots, \Gamma_p$ are the fibers of the iterated canonical contraction $\Gamma \rightarrow \Gamma_X$. This understood, we are in the situation when $H_\Gamma \setminus X \sim H_{\Gamma_1}, \ldots, H_{\Gamma_p}$ and $C_i : H_{\Gamma_i}, 1 \leq i \leq p$.

The root vertex of the graph-tree $\alpha_{\Gamma}(C)$ will be decorated by $\Gamma_X$. We already have, by induction, the graph-trees $\alpha_{\Gamma_i}(C_i)$, and each of these trees is connected with the root of $\alpha_{\Gamma}(C)$ by the edge bearing the label of the vertex of $\Gamma_X$ to which $\Gamma_i$ has been contracted. We believe that Figure 7 makes this construction clear. The inductive step is finished by joining to the root of the graph-tree $\alpha_{\Gamma}(C)$ the legs indexed by the remaining vertices of $\Gamma_X$.

The inverse of $\alpha_{\Gamma}$ is defined by extracting the construct from a graph-tree $T$ in the following way. First, remove all the leaves of $T$ and then, for each vertex of $T$, replace the graph that decorates that vertex by the maximal construct of its associated hypergraph. In more detail, assume that $T \in \text{gTr}(\Gamma), \ \Gamma \in \text{GrC}$. The underlying rooted tree of the construct $\alpha_{\Gamma}^{-1}(T)$ is obtained from the underlying tree of $T$ by removing its legs. The vertex of $\alpha_{\Gamma}^{-1}(T)$ corresponding to a vertex $v \in \text{vert}(T)$ decorated by $\Gamma_v \in \text{GrC}$ is decorated by the set $\text{edg}(\Gamma_v) \subset \text{edg}(\Gamma)$ of edges of $\Gamma_v$. 

[nests.tex] [October 17, 2022]
There is the following inductive, alternative construction of $\alpha_\Gamma^{-1}(T)$ that leads manifestly to a construct of $H_\Gamma$. Assume that $e_1, \ldots, e_s \in V$ are the labels of the incoming internal edges of a vertex $v \in \text{vert}(T)$, and that $v_1, \ldots, v_s \in \text{vert}(T)$ are the initial vertices of these edges. Further, let $T_i$ be the maximal rooted graph-subtree of $T$ with the root $v_i$ and $\Gamma_i := \text{gr}(T_i)$, $1 \leq i \leq s$. Then the corresponding subtree of $\alpha_\Gamma^{-1}(T)$ is the construct

$$\text{edg}(\Gamma_v)\{\alpha_{\Gamma_1}^{-1}(T_1), \ldots, \alpha_{\Gamma_s}^{-1}(T_s)\}.$$
Notice that the construct $\alpha_{\Gamma}^{-1}(T)$ inherits the planar structure of $T$. It is easy to verify that the correspondence

$$g\text{Tr}(\Gamma) \ni T \leftrightarrow \alpha_{\Gamma}(T) \in \{C \mid C : H_{\Gamma}\}$$

(20)

preserves the poset structures. □

**Example 19.** For the graph $\Gamma$ from Example 15, the graph-tree $\alpha_{\Gamma}(C)$ associated to the construct $C = \{x, y\}\{\{u, v, z\}\}$ of the hypergraph $H_{\Gamma}$ is shown in Figure 8.

For an object $\Gamma$ of $\mathcal{GrC}$ and a construct $C : H_{\Gamma}$, let $\text{Lev}(C)$ denote the chaotic groupoid whose objects are all possible arrangements of levels of $C$, whereby a level of a construct is defined analogously as the one of a graph tree. It is clear that the correspondence (20) defines a canonical isomorphism between $\text{Lev}(C)$ and $\text{Lev}(\alpha_{\Gamma}(C))$, thus each 1-connected collection $E$ promotes into a functor $E : \text{Lev}(C) \rightarrow \text{Vec}$ in the diagram

$$\text{Vec} \xrightarrow{E} \text{Lev}(C) \xleftarrow{\cong} \text{Lev}(\alpha_{\Gamma}(C))$$

where the vertical up-going arrow is (11). The following reformulation of Theorem 8 is a direct consequence of Proposition 18.

**Theorem 20.** For a 1-connected collection $E$, the arity $\Gamma$ piece of the free operad $\mathbb{F}(E)$ is given by

$$\mathbb{F}(E)(\Gamma) \cong \begin{cases} \bigoplus \colim_{C : H_{\Gamma}} E(C, \varsigma) & \text{if } \Gamma \text{ has at least one internal edge, and} \\ k & \text{if } \Gamma \text{ has no internal edges.} \end{cases}$$

(21)

2.4. **A chain complex.** In this subsection we recall a chain complex associated to a convex polyhedron featuring in Lemma 21 below. Let therefore $K$ be such an $n$-dimensional polyhedron realized as the convex hull of finitely many points in $\mathbb{R}^n$. Each $k$-dimensional face $e$ of $K$, $0 \leq k \leq n$, is then embedded canonically into a $k$-dimensional affine subspace $\mathbb{A}_e$ of $\mathbb{R}^n$, namely into the span of its vertices. By an orientation of $e$ we understand an orientation of each face has been specified.

Assume that $a$ is a codimension one subface of $e$ and that the dimension of $a$ is $\geq 1$. Clearly $\mathbb{A}_a$ divides $\mathbb{A}_e$ into two half-spaces. Denote by $\mathbb{A}_e^a \subset \mathbb{A}_e$ the one having non-empty intersection with $K$. Let the orientation of $a$ be given by linearly independent vectors $(v_1, \ldots, v_{k-1})$ in $\mathbb{A}_e$. We say that an orientation of $a$ is compatible with the orientation of $e$ if the frame
Figure 9. Configurations of $a$ and $e$ (left) and $a, e', e''$ and $h$ (right).

$(v_1, \ldots, v_{k-1}, n)$ in $A^a_e$, where $n$ is a vector normal to $A_a \subset A^a_e$, defines the orientation of $e$, cf. Figure 9 (left) where $k = 2$. A modification of this notion to $0$-dimensional $a$‘s is obvious.

We assign to $K$ a chain complex $(C_\ast(K), \partial)$ of free abelian groups whose $k$th piece $C_k(K)$ is generated by $k$-dimensional faces of $K$. The value of the differential on a $k$-dimensional generator $\lambda$ is defined by

$$\partial(\lambda) = \sum \eta^\delta_\lambda \cdot \delta,$$

where $\delta$ runs over all codimension one faces of $\lambda$ and

$$\eta^\delta_\lambda := \begin{cases} +1, & \text{if } \delta \text{ is oriented compatibly with } \lambda, \\ -1, & \text{otherwise}. \end{cases}$$

It follows from standards methods of algebraic topology that $(C_\ast(K), \partial)$ is acyclic in positive dimensions while its 0th homology equals $\mathbb{Z}$.

2.5. An ingenious lemma. Let $L = (L, \prec)$ be the face poset of an $n$-dimensional polyhedron $K$, ordered by the inclusion. Assume that $K$ is such that the following ‘diamond’ condition is satisfied.

Diamond. Let $0 < k < n$ and let $a$ be a $(k-1)$-dimensional face of $K$ which is a common boundary of two $k$-dimensional faces $e', e''$. Then there exists a $(k+1)$-dimensional face $h$ with $e', e'' \prec h$, diagrammatically

$$\text{hence the name. It follows from the properties of abstract polytopes that $e'$ and $e''$ are the only faces in the interval $[a, h]$. The diamond condition guarantees that the inductive construction of an orientation in the proof of Lemma 21 below is independent of the choices.}$$
The diamond condition need not be satisfied by a general polytope. An example is the pyramid, with $e', e''$ a pair of opposite 1-dimensional edges meeting at the apex.

Assume that $(C_*(L), \partial)$ is a chain complex such that each $C_k(L)$ is the free abelian group generated by $k$-dimensional elements of $L$, $0 \leq k \leq n$. Suppose moreover that, for each $\lambda \in L$, $\partial(\lambda)$ is of the form

$$\partial(\lambda) = \sum \eta^\delta_{\lambda} \cdot \delta,$$

where $\eta^\delta_{\lambda} \in \{-1, +1\}$ and $\delta$ runs over all codimension one faces of $\lambda$. Then one has:

**Lemma 21.** The faces of $K$ could be oriented so that $(C_*(L), \partial)$ is the chain complex $(C_*(K), \partial)$ recalled in Subsection 2.4.

**Proof.** The lemma will be proved by downward induction on the dimension of the faces of $K$. We start by choosing an orientation of the unique $n$-dimensional face of $K$ arbitrarily.

Assume that we have oriented all faces of $K$ of dimensions $\geq k$ for some $n > k \geq 0$. Let $a$ be a $(k-1)$ dimensional face of $K$, and choose some $k$ dimensional face $e$ such that $a \prec e$. This is always possible, since otherwise the face $a$ would be maximal, which contradicts the properties of a polytope. If $a$ occurs in $\partial(e)$ with the $+1$ sign, we equip it with the compatible orientation, if it occurs with the $-1$ sign, we equip it with the orientation opposite to the compatible one. We need to show that this recipe does not depend on the choice of $e$.

Assume therefore that $e'$ and $e''$ are two faces of $K$ with the properties described above. Let $h$ be a cell required by the diamond property. Then

$$\partial(h) = \eta' \cdot e' + \eta'' \cdot e'' + \text{other terms}, \quad \eta', \eta'' \in \{-1, +1\},$$

$$\partial(e') = \varepsilon' \cdot a + \text{other terms}, \quad \varepsilon' \in \{-1, +1\}, \quad \text{and}$$

$$\partial(e'') = \varepsilon'' \cdot a + \text{other terms}, \quad \varepsilon'' \in \{-1, +1\}.$$

The condition $\partial^2(h) = 0$ together with the fact that $e'$ and $e''$ are the only faces in the interval $[a, h]$ imply

$$\eta' \varepsilon' + \eta'' \varepsilon'' = 0.$$

The configuration of the relevant cells is indicated in Figure 9 (right) which shows a section of $h$ with a hyperplane orthogonal to $A_a$.

Assume e.g. that $\eta' = \eta'' = 1$. Then both $e'$ and $e''$ have the orientation compatible with the orientation of $h$. By (22) one has $\varepsilon' = -\varepsilon''$; assume for instance that $\varepsilon' = 1, \varepsilon'' = -1$. Then $a$ gets from $e'$ the compatible orientation, and from $e''$ the orientation opposite to the compatible one. It easily follows from the local geometry of the section in Figure 9 that these two orientations of $a$ are the same. The remaining cases can be analyzed similarly. □
2.6. Splits and collapses. The proof of Lemma 22 below relies on the actions of splitting the vertices and collapsing the edges of constructs of a hypergraph $H$. We formalize the corresponding constructions below. Let $C : H$.

Splitting the vertices of $C$. Let $V \in \text{vert}(C)$ be such that $|V| \geq 2$. Let $H - V$ be the hypergraph defined by

$$H - V := \{X \setminus V \mid X \in \text{Sat}(H)\} \setminus \{\emptyset\}.$$ 

Observe that, in general, $H - V \neq H \setminus V$. For example, for the hypergraph $H_T$ from Example 15 we have that $H_T - \{x, y\}$ is the complete graph on the vertex set $\{z, u, v\}$, whereas $H_T \setminus \{x, y\}$ can be obtained from $H_T - \{x, y\}$ by removing the edge $\{u, z\}$ and, hence, is a linear graph. Let $\{X, Y\}$ be a partition of $V$ such that the tree $X\{Y\}$ is a construct of $H - V$.

We define the construct $C[X\{Y\}/X \cup Y] : H$, obtained from $C$ by splitting the vertex $V$ into the edge $X\{Y\}$, by induction on the number of vertices of $C$, as follows. If $C = H$, we set $C[X\{Y\}/X \cup Y] := X\{Y\}$.

Suppose that, for $Z \subset H$, $C = Z\{C_1, \ldots, C_p\}$, $H \setminus Z \simeq H_1, \ldots, H_p$ and $C_i : H_i$. If there exists an index $i$, $1 \leq i \leq p$, such that $V \in \text{vert}(C_i)$, we define

$$C[X\{Y\}/X \cup Y] := Z\{C_1, \ldots, C_{i-1}, C_i[X\{Y\}/X \cup Y], C_{i+1}, \ldots, C_p\}.$$ 

Assume that $V = Z$ and let $\{i_1, \ldots, i_q\} \cup \{j_1, \ldots, j_r\}$ be the partition of the set $\{1, \ldots, p\}$ such that the hypergraphs $H_{i_1}$, for $1 \leq s \leq q$, contain a vertex adjacent to some vertex of $Y$, while the hypergraphs $H_{i_t}$, for $1 \leq t \leq q$, have no vertices adjacent to a vertex of $Y$. We define

$$C[X\{Y\}/X \cup Y] := X\{Y, C_{i_1}, \ldots, C_{i_q}, C_{j_1}, \ldots, C_{j_r}\}.$$ 

If, exceptionally, $\{i_1, \ldots, i_q\} = \emptyset$ resp. $\{j_1, \ldots, j_r\} = \emptyset$, we set

$$C[X\{Y\}/X \cup Y] := X\{Y, C_1, \ldots, C_p\}$$ 

resp. $C[X\{Y\}/X \cup Y] := X\{Y, C_1, \ldots, C_p\}$.

The proof that the non-planar rooted tree $C[X\{Y\}/X \cup Y]$ is indeed a construct of $H$ goes easily by induction on the number of vertices of $C$, the only interesting case being $C = Z\{C_1, \ldots, C_p\}$. In that case, the argument is based on the fact that the set of vertices $Y \cup \bigcup_{i \in \{i_1, \ldots, i_q\}} \text{vert}(H_i)$ determines a connected component $H'$ of $H$ and, furthermore, that $Y\{C_{i_1}, \ldots, C_{i_q}\} : H'$.

Collapsing the edges of $C$. One can similarly define the construct $C[X \cup Y/X\{Y\}] : H$, obtained from $C$ by collapsing the edge $X\{Y\}$ into the vertex $X \cup Y$.

Lemma 22. The polytope $C(H, \Pi)$ that realizes the abstract polytope $A(H)$ (see Lemma 13) of a hypergraph $H$ satisfies the diamond property.
Proof. We prove the lemma by constructing, for each construct $C : H$ of rank $k - 1$ for which there exist constructs $C'$ and $C''$ of rank $k$ such that

$$C \preceq_H C' \quad \text{and} \quad C \preceq_H C'', \tag{23}$$

a construct $D : H$ of rank $k + 1$ such that $C' \preceq_H D$ and $C'' \preceq_H D$.

By definition of the partial order $\preceq_H$ of $\mathcal{A}(H)$, the relations (23), together with the fact that the rank of $C$ differs by 1 from the rank of $C'$ and $C''$, mean that there exists a vertex $X \cup Y$ of $C'$ and a vertex $U \cup V$ of $C''$, such that

$$C = C'[X \{Y \}/X \cup Y] = C''[U \{V \}/U \cup V].$$

As vertices of $C$, the sets $X$, $Y$, $U$ and $V$ satisfy one of the following relations: they can either be mutually disjoint, or it can be the case that $X = U$ and $Y \cap V = \emptyset$, or it can be the case that $Y = U$ and $X \cap V = \emptyset$, plus the ‘mirror’ reflection of the last case, namely $X = V$ and $U \cap V = \emptyset$.

It is easily seen that other possible relations are forbidden. For example, the relation $Y = V$ would imply that $C$ is not a rooted tree. Depending on the mutual relation of the vertices $X$, $Y$, $U$ and $V$ of $C$, the above equality implies that the action of collapsing a particular edge of $C'$ and a particular edge of $C''$ leads to the same construct. Indeed, if $X$, $Y$, $U$ and $V$ are mutually disjoint, then

$$C'[U \cup V/U \{V \}] = C''[X \cup Y/X \{Y \}],$$

if $X = U$ and $Y \cap V = \emptyset$, then

$$C'[(X \cup Y) \cup V/(X \cup Y)\{V \}] = C''[(X \cup V) \cup Y/(X \cup V)\{Y \}],$$

and if $Y = U$ and $X \cap V = \emptyset$, then

$$C'[(X \cup Y) \cup V/(X \cup Y)\{V \}] = C''[X \cup (Y \cup V)/X \{Y \cup V \}].$$

We define $D$ to be precisely the construct obtained from $C'$ (or, equivalently, from $C''$) by such a collapse. The three diamonds corresponding to the three possible constructions of $D$ can be pictured respectively as follows:

\[\text{nests.tex}\]
where we only display the edges involved in the construction. By definition, the construct $D$ satisfies the required properties. 

2.7. Proof of Theorem 14. We establish first that $\mathfrak{M}_{\text{grc}}$ is acyclic in positive dimensions and that $H_0(\mathfrak{M}_{\text{grc}}) \cong \mathbb{k}$. By Proposition 13 each construct $C : H_\Gamma$ is, for $\Gamma \in \text{grc}$ with at least one internal edge, of the form $\alpha_\Gamma(T)$ for some graph-tree $T \in \text{gTr}(\Gamma)$. It is therefore supported by a rooted planar tree, so we may introduce the lexicographic arrangement $\varsigma_{\text{lex}}$ of levels of its underlying tree. Consequently we get from (21) an analog

\[ \mathbb{F}(E)(\Gamma) \cong \bigoplus_{C : H_\Gamma} E(C, \varsigma_{\text{lex}}) \]

of formula (14).

The case which interests us is when $E$ is the collection $D$ in (17) generating $\mathfrak{M}_{\text{grc}}$. A vertex $v$ of $C$ is decorated by a subset $X_v \subset \text{edg}(\Gamma)$, thus it contributes to $D(C, \varsigma_{\text{lex}})$ by the multiplicative factor $\det(X_v)$. Let us fix an order of $\text{edg}(\Gamma)$. Then each $X_v$ bears an induced order, hence $\det(X_v)$ has a preferred basis element $x_1 \wedge \cdots \wedge x_r \in \det(X_v)$, $x_1 < \cdots < x_r$, $X_v = \{x_1, \ldots, x_r\}$, so it is canonically isomorphic to $\mathbb{k}$ placed, according to our conventions, in degree $|X_v| - 1$. Combining the above facts, we arrive at the canonical isomorphism

\[ \mathbb{F}(D)(\Gamma) \cong \bigoplus_{C : H_\Gamma} \text{Span}(\{e_C\}), \]

where $\text{Span}(\{e_C\})$ is the vector space spanned by a generator $e_C$ placed in degree that equals the rank of $C$, which in this case equals $|\text{edg}(\Gamma)| - |\text{vert}(C)|$.

The differential $\partial$ of the minimal model transfers, via isomorphism (24), into a differential denoted by the same symbol of the graded vector space at the right hand side of (24). It is straightforward to verify that the transferred differential has the form required by Lemma 21, i.e.

\[ \partial(e_C) = \sum \eta_F e_F, \]

where $\eta_F \in \{-1, +1\}$ and $F$ runs over all $F : H_\Gamma$ such that $\text{grad}(F) = \text{grad}(C) - 1$.

Remark 23. It is possible to establish the explicit values of the coefficients $\eta_F$ in (25), but the ingenuity of Lemma 21 makes it unnecessary.

Now we invoke that the poset $\mathcal{A}(H_\Gamma)$ of constructs of $H_\Gamma$ is, by Lemma 8 the poset of faces of a convex polytope $K$ which moreover fulfills the diamond property by Lemma 22. By Lemma 21, the cells of $K$ can be oriented so that

\[ \left( \bigoplus_{C : H_\Gamma} \text{Span}(\{e_C\}), \partial \right) \]
is the cell complex $C_\ast(K)$. It is thus acyclic in positive dimension, and so is $(F(D)(\Gamma), \partial) = \mathcal{M}_{\text{Grc}}(\Gamma)$, for each $\Gamma \in \text{Grc}$. By the same reasoning,

$$H_0(\mathcal{M}_{\text{Grc}})(\Gamma) \cong \mathbb{k} \quad \text{for each} \quad \Gamma \in \text{Grc}. \tag{26}$$

The next step is to prove that the operad morphism $\rho : F(D) \to 1_{\text{Grc}}$ commutes with the differentials, which clearly amounts to proving that $\rho(\partial x) = 0$ for each degree 1 element $\mu \in F(D)(\Gamma)_1$. By the derivation property of $\partial$, it is in fact enough to address only the case when $\mu$ is a generator of degree 1, i.e. an element of $D(\Gamma) = \det(\text{edg}(\Gamma))$ with $\Gamma$ having exactly two internal edges.

Let thus $\Gamma$ be such a graph and $a, b$ its two internal vertices. There are precisely two graph-trees $T', T'' \in g\text{Tr}(\Gamma)$, both with two vertices and one internal edge. The root vertex of $T'$ is decorated by some graph $\Gamma'_v$ with the only internal edge $a$, and the other vertex of $T'$ by $\Gamma'_u$ with the only internal edge $b$. The graph-tree $T''$ has similar decorations $\Gamma''_v$ and $\Gamma''_u$, but this time $\text{edg}(\Gamma''_v) = \{b\}$ and $\text{edg}(\Gamma''_u) = \{a\}$. For a generator $\mu := a \wedge b \in D(\Gamma) = \det(\{a, b\})$ formula (18a) gives

$$\partial(a \wedge b) = a \otimes b - b \otimes a \in (D(\Gamma'_v) \otimes D(\Gamma'_u)) \oplus (D(\Gamma''_v) \otimes D(\Gamma''_u)) \subset F^2(D)(\Gamma).$$

By the definition (19) of the morphism $\rho$,

$$\rho(\partial(a \wedge b)) = \rho(a \otimes b - b \otimes a) = 1 \cdot 1 - 1 \cdot 1 = 0$$

as required.

The last issue that has to be established is that $\rho$ induces an isomorphism

$$H_0(\rho) : H_0(\mathcal{M}_{\text{Grc}}) \xrightarrow{\cong} 1_{\text{Grc}}.$$ 

To this end, in view of (26), it is enough to prove that

$$H_0(\rho)(\Gamma) : H_0(\mathcal{M}_{\text{Grc}})(\Gamma) \to 1_{\text{Grc}}(\Gamma) = \mathbb{k}$$

is nonzero for each $\Gamma \in \text{Grc}$. Equation (15) readily gives

$$F(D)(\Gamma)_0 \cong \bigoplus_{T \in g\text{Tr}_0(\Gamma)} \text{colim}_{\lambda \in \text{Lev}(T)} D(T, \lambda), \tag{27}$$

in which $g\text{Tr}_0(\Gamma)$ is the subset of $g\text{Tr}(\Gamma)$ consisting of graph-trees for which each decorating graph $\Gamma_v, v \in \text{vert}(\Gamma)$, has exactly one internal edge. For such a graph, $D(\Gamma_v) = \det(\text{edg}(\Gamma_v))$ is canonically isomorphic to $\mathbb{k}$ placed in degree 0. The groupoid $\text{Lev}(T)$ therefore acts trivially on $D(T, \lambda)$ which is canonically isomorphic to $\mathbb{k}$, so (27) leads to

$$F(D)(\Gamma)_0 \cong \text{Span}(g\text{Tr}_0(\Gamma)), \tag{28}$$

in which each $T \in g\text{Tr}_0(\Gamma)$ corresponds to a vertex of the polytope $K$ associated to $A(H_T)$ and therefore represents a cycle that linearly generates $H_0(\mathcal{M}_{\text{Grc}})$. We will show that $\rho(T) \neq 0.$

[\text{nests.tex}]
Under isomorphism (28), each $T$ is an operadic composition of graph trees in $\text{gTr}^1_0(\Gamma)$, i.e. graph trees whose underlying tree has one vertex which is decorated by a graph with one internal edge. By (19), $\rho(S) = 1 \in k$ for $S \in \text{gTr}^1_0(\Gamma)$. Since all operadic compositions in $1_\text{Grc}$ are the identities $1 : k \otimes k \to k$, $\rho(T) = 1$ for the composite $T$ as well. This finishes the proof of Theorem 14.

3. Other cases

As the diagram in Figure 10 taken from [2] teaches us, many operadic categories of interest are obtained from the basic category $\text{Grc}$ of ordered connected graphs by iterated discrete operadic fibrations or opfibrations. This is in particular true for the category $\text{ggGrc}$ of genus-graded graphs, the category $\text{Tr}$ of trees, and the category $\text{Whe}$ of wheeled graphs; they all are discrete operadic opfibrations over $\text{Grc}$. Moreover, the inclusion $\text{RTr} \hookrightarrow \text{Grc}$ of the operadic category of rooted trees is a discrete operadic fibration with finite fibers. Corollary 28 of Subsection 3.1 below states that the restrictions along discrete operadic opfibrations or fibrations with finite fibers preserve minimal models of the terminal operads. Therefore the minimal models of the terminal operads in the above mentioned categories are suitable restrictions of the minimal model $\mathfrak{M}_{\text{Grc}}$ of the terminal $\text{Grc}$-operad constructed in Section 2. We close this section by describing the minimal model of the terminal operad in the category $\text{SRTr}$ of strongly rooted trees.

3.1. Operadic (op)fibrations and minimal models. The following material uses the terminology of [2, 3]. All operadic categories in this subsection will be factorizable, graded, and such that all quasibijections are invertible, the blow up and unique fiber axioms are fulfilled, and a morphism is an isomorphisms if it is of grade 0. These assumptions are fulfilled by all operadic categories discussed in the present paper.

Assume that $O$ is such an operadic category. As argued in [3, Section 3], one has the natural forgetful functor $\mathcal{U}_O : O-\text{Oper}^1_1 \to O-\text{Coll}^1_1$ from the category of 1-connected strictly unital Markl’s $O$-operads with values in a symmetric monoidal category $V$ to the category...
of 1-connected $O$-collections in $V$. Its left adjoint $F_0 : O{-}\text{Coll}^\y V \to O{-}\text{Oper}^\y V$ is the free operad functor.

Each strict operadic functor $p : O \to P$ induces the restriction $p^* : P{-}\text{Oper}^\y V \to O{-}\text{Oper}^\y V$ acting on objects by the formula

$$p^*(\mathcal{P})(t) := \mathcal{P}(p(t)), \quad \mathcal{P} \in P{-}\text{Oper}^\y V, \quad t \in O.$$  

The restriction $p^*$ may or may not have a right adjoint $p_* : O{-}\text{Oper}^\y V \to P{-}\text{Oper}^\y V$ and even if it exists its form may not be simple unless $p$ has some special properties.

Recall the following general categorical definition. Assume we are given a commutative diagram of right adjoints

$$p^* \quad \text{in which } p^* \text{ and } q^* \text{ are also left adjoints. These functors can be organized into the following diagram of adjunctions}

$$q^* p^* \leftrightarrow q^* v^* \quad \text{and}

$$q^* \quad \text{is an isomorphism. Symmetrically, (30) is a left Beck-Chevalley square if the composite}

$$q^* u^* \rightarrow q^* v^* p^* p_1 = q^* v^* p_1 \rightarrow v^* p_1

$$is an isomorphism, cf. [19].

**Lemma 24.** The following two conditions are equivalent:

(i) the mate $q_* u^* \leftrightarrow q_* v^* p^* p_1 = q_* v^* p_1$ is an isomorphism and

(ii) the square (30) is a right Beck-Chevalley square.

If $p_1$ is also a right adjoint to $p^*$ (that is, $p_1 \cong p_*$) and $q_1$ is a right adjoint to $q^*$ then (30) is a right Beck-Chevalley square if and only if it is a left Beck-Chevalley square.
Proof. Condition (i) just says that the right adjoints commute up to isomorphism. It follows
that the left adjoints commute up to isomorphism as well, which is the right Beck-Chevalley
condition (ii). The converse is clearly true as well.

If \( p_i \) is also a right adjoint to \( p^* \) and \( q_i \) is a right adjoint to \( q^* \) then obviously the left
Beck-Chevalley condition is again about commutation of right adjoints, hence their left
adjoints commute and the right Beck-Chevalley condition holds. The inverse implication is
similar.

\[ \square \]

Remark 25. It was pointed to us by our anonymous referee that in [33, Lemma 7.10] an
analogue of our Lemma 24 is given under the so called ‘Wirthmüller context’ for the six
operations formalism (the existence of \( p_* \) is a sufficient condition). The referee also asked
which morphisms between operadic categories may induce the ‘Grothendieck context.’ The
existence of such a context would provide an alternative condition for the preservation of
minimal models by the restriction functor \( p^* \). We do not have an immediate answer but
we are grateful to our referee for raising this interesting question, which certainly deserves
further study.

In the following proposition, \( p^*: P-\text{Oper}_1^\mathcal{V} \to O-\text{Oper}_1^\mathcal{V} \) is the restriction functor defined
by (29) and \( p_0^*: P-\text{Coll}_1^\mathcal{V} \to O-\text{Coll}_1^\mathcal{V} \) is the obvious similar restriction between the cate-
gories of collections.

Proposition 26. The square

\[ \begin{array}{ccc}
O-\text{Oper}_1^\mathcal{V} & \xrightarrow{p^*} & P-\text{Oper}_1^\mathcal{V} \\
\downarrow & & \downarrow \\
O-\text{Coll}_1^\mathcal{V} & \xleftarrow{p_0^*} & P-\text{Coll}_1^\mathcal{V}
\end{array} \]

is a right Beck-Chevalley square provided any of the two following conditions hold:

(i) \( p \) is a discrete operadic opfibration and \( \mathcal{V} \) a cocomplete symmetric monoidal category;
(ii) \( p \) is a discrete operadic fibration with finite fibers and \( \mathcal{V} \) an additive cocomplete sym-
metric monoidal category.

Proof. The right adjoint \( (p_0)_*: O-\text{Coll}_1^\mathcal{V} \to P-\text{Coll}_1^\mathcal{V} \) to the restriction \( p_0^*: P-\text{Coll}_1^\mathcal{V} \to O-\text{Coll}_1^\mathcal{V} \)
is given on objects by

\[ (p_0)_*(E)(T) := \prod_{p(t)=T} E(t), \quad E \in O-\text{Coll}_1^\mathcal{V}, \quad T \in P. \]
Assume that $p : 0 \to \mathcal{P}$ is a discrete operadic opfibration. By dualizing [1, Theorem 2.4] one verifies that the right adjoint $p_* : \text{0-Oper}_1 \to \text{P-Oper}_1$ is defined on objects by

$$(33b) \quad p_*(O)(T) := \prod_{p(t) = T} O(t), \quad O \in \text{0-Oper}_1, \quad T \in \mathcal{P}.$$ 

Comparing (33a) with (33b) we see that $(p_0)_! \circ U = U \circ p_*$, which is condition (i) of Lemma 24. Thus (32) is right Beck-Chevalley by the same lemma. This finishes the proof of the case of a discrete opfibration.

Let us assume that $p : 0 \to \mathcal{P}$ is a discrete operadic fibration with finite fibers. We want to verify the assumptions of the second part of Lemma 24, i.e. to check that $(p_0)_!$ is a right adjoint to $p_0^*$ and that $p_!$ is a right adjoint to $p^*$.

It is clear that $(p_0)_!$ is for an arbitrary $p : 0 \to \mathcal{P}$ given on objects by the formula

$$(p_0)_!(E)(T) := \bigoplus_{p(t) = T} E(t), \quad E \in \text{0-Coll}_1^\mathcal{V}, \quad T \in \mathcal{P}.$$ 

Since $\mathcal{V}$ is additive and $p$ has finite fibers, this functor coincides with the right adjoint $(p_0)_*$ described in (33a). On the other hand, [1, Theorem 2.4] gives the following formula for the underlying collection of $p_!(O)$:

$$p_!(O)(T) := \bigoplus_{p(t) = T} O(t), \quad O \in \text{0-Oper}_1^\mathcal{V}, \quad T \in \mathcal{P}.$$ 

It is not hard to see, using the additivity of $\mathcal{V}$ and the finiteness of the fibers of $p$, that this formula describes also a right adjoint to $p^*$, which completes the proof for operadic fibrations.

In the rest of this section, the coefficient category $\mathcal{V}$ will be that of differential graded vector spaces. It clearly satisfies all assumptions required in Proposition 26.

**Proposition 27.** Assume that (32) is a right Beck-Chevalley square and $\rho : \mathcal{M}_p \to 1_p$ is the minimal model of the terminal $\mathcal{P}$-operad $1_p$. Then

$$\mathcal{M}_0 := p^*(\mathcal{M}_p) \xrightarrow{p^*(\rho)} p^*(1_p) = 1_0$$

is the minimal model of the terminal $\text{0-operad}\ 1_0$.

**Proof.** It is clear that $p^*(1_p) = 1_0$. Let $\mathcal{M}_p = (F_p(E_p), \partial_p)$. Diagram (32) is, by definition, a right Beck-Chevalley square if $p^* F_p \cong F_0 p_0^*$. In particular,

$$p^*(F_p(E_p)) \cong F_0(p_0^*(E_p)),$$

thus $p^*(\mathcal{M}_p)$ is the free operad generated by the collection $E_0 := p_0^*(E_p)$. It is easy to verify that $p^*$ brings derivations to derivations and differentials to differentials. We therefore conclude that

$$p^*(\mathcal{M}_p) \cong (F_0(E_0), \partial_0).$$
where the minimality of $\partial_0$ can also be established easily.

It remains to prove that $p^*(\rho)$ induces a component-wise isomorphism of homology. This however follows immediately from the definition of the restriction functor requiring that

$$p^*(\rho)(t) = \rho(p(t)) : \mathcal{M}_p(p(t)) \to 1_p(p(t)) = k, \quad t \in O,$$

where $\rho(p(t))$ is a homology isomorphism since $\rho : \mathcal{M}_p \to 1_p$ is the minimal model of $1_p$ by assumption. \hfill \Box

**Corollary 28.** Let $p : O \to P$ be either a discrete operadic opfibration, or a discrete operadic fibration with finite fibers, and $\rho : \mathcal{M}_p \to 1_p$ the minimal model of the terminal $P$-operad. Then

$$\mathcal{M}_O := p^*(\mathcal{M}_p) \xrightarrow{p^*}(\rho) p^*(1_p) = 1_O$$

is the minimal model of the terminal $O$-operad.

**Remark 29.** The assumptions and conclusion of Corollary 28 were verified in the context of operadic categories related to permutads in [26].

### 3.2. Minimal model for $1_{ggGrc}$.

The operadic category $ggGrc$ consists of graphs $\Gamma \in Grc$ equipped with a genus grading, which is a non-negative integer $g(v) \in \mathbb{N}$ specified for each $v \in \text{vert}(\Gamma)$. The genus of the entire graph $\Gamma$ is defined by

$$g(\Gamma) := \sum_{v \in \text{vert}(\Gamma)} g(v) + \dim(H^1(|\Gamma|; \mathbb{Z})).$$

where $|\Gamma|$ is the obvious geometric realization of $\Gamma$. As shown in [3, Section 5], algebras for $1_{ggGrc}$ are modular operads introduced in [14].

Assume that $\Gamma \in ggGrc$ and that $T \in \text{Tr}(\Gamma)$ is a graph-tree. Then there exists a unique genus grading of each of the graphs $\Gamma_v$ decorating the vertices of $T$ subject, along with the compatibilities required in Subsection 1.2, also to:

*Genus compatibility.* Let $e$ be an internal edge of $T$ pointing from the vertex labelled by $\Gamma_u$ to the vertex labelled by $\Gamma_v$. By Compatibility 1, $e$ is also (the label of) a vertex of $\Gamma_v$. With this convention in mind we require that

$$g(e) = g(\Gamma_u).$$

In words, the vertex of $\Gamma_v$ to which $\Gamma_u$ is contracted bears the genus $g(\Gamma_u)$.

The statement can be verified directly, which we leave as an exercise to the reader. It can also be established by inductive applications of

**Lemma 30.** Let $\phi : \Gamma \to \Gamma''$ be an elementary morphism in $Grc$ with fiber $\Gamma'$, in shorthand

$$\Gamma' \rightrightarrows \Gamma \xrightarrow{\phi} \Gamma''.$$

Assume moreover that $\Gamma$ bears a genus grading. Then there are unique genus gradings of $\Gamma'$ and $\Gamma''$ such that (34) becomes a diagram, in $ggGrc$, of an elementary map and its fiber.
Proof. A consequence of the fact that the obvious projection $p : ggGrc \to Grc$ is a discrete operadic opfibration, though it can also be verified directly. □

For $\Gamma \in ggGrc$ having at least one internal edge and for a 1-connected $ggGrc$-collection $E$, the right hand side of

\[(35) \quad F_{gg}(E)(\Gamma) := \bigoplus_{T \in gTr(\Gamma)} \colim_{\lambda \in \Lev(T)} E(T, \lambda),\]

makes sense because, as explained above, each of the graphs $\Gamma_i, 1 \leq i \leq k$, in (10) where $E(T, \lambda)$ was defined, bears a unique genus grading induced by the genus grading of $\Gamma$.

Let $p : ggGrc \to Grc$ be as before the canonical projection that forgets the genus grading, and $p^* : Grc-\text{Oper}_1 \to ggGrc-\text{Oper}_1$ resp. $\hat{p}^* : Grc-\text{Coll}_1 \to ggGrc-\text{Coll}_1$ the induced restrictions. The values of the $ggGrc$-collection $D_{gg} \in ggGrc-\text{Coll}_1$ given by

$$D_{gg}(\Gamma) := \text{det}(\Gamma), \quad \Gamma \in ggGrc,$$

do not depend on the genus grading, thus $D_{gg} = p^*_0(D)$, where $D \in Grc-\text{Coll}_1$ is as in (17). For the same reasons

$$F_{gg}(D_{gg}) = p^*F(D),$$

so, since $p : ggGrc \to Grc$ is a discrete operadic opfibration, $F_{gg}(D_{gg})$ defined by (35) with $E = D_{gg}$ represents the free $ggGrc$-operad on $D_{gg}$ by Proposition 26. The differential $\partial$ on $F_{gg}(D_{gg})$ is given by an obvious analog of (18b).

As expected, we define $\rho : F_{gg}(D_{gg}) \to 1_{ggGrc}$ as the unique map of $ggGrc$-operads whose restriction $\rho|_{D_{gg}(\Gamma)}$ is, for $\Gamma \in ggGrc$, given by a modification of (19), namely by

$$\rho|_{D(\Gamma)} := \begin{cases} 1 : D(\Gamma) = k \to k = 1_{ggGrc}(\Gamma), & \text{if } |\text{edg}(\Gamma)| = 1, \text{ while} \\ 0, & \text{if } |\text{edg}(\Gamma)| \geq 2.\end{cases}$$

**Theorem 31.** The object $\mathcal{M}_{ggGrc} = (F_{gg}(D_{gg}), \partial) \xrightarrow{\rho} (1_{ggGrc}, \partial = 0)$ is a minimal model of the terminal $ggGrc$-operad $1_{ggGrc}$.

Proof. A consequence of Corollary 28 though the acyclicity of $\mathcal{M}_{ggGrc}$ in positive dimensions follows directly from the acyclicity of $\mathcal{M}_{Grc}$ proven in Subsection 2.7 thanks to the isomorphism

$$\mathcal{M}_{ggGrc}(\Gamma) \cong \mathcal{M}_{Grc}(\hat{\Gamma}), \quad \Gamma \in ggGrc,$$

of dg vector spaces, where $\hat{\Gamma} \in Grc$ is $\Gamma$ stripped of the genus grading. □
3.3. **Minimal model for $1_{\mathbf{Tr}}$.** Let $\mathbf{Tr} \subset \mathbf{Grc}$ be the full subcategory of contractible, i.e. simply connected graphs. Algebras over the terminal $\mathbf{Tr}$-operad $1_{\mathbf{Tr}}$ are cyclic operads. Although it was not stated in [2], the inclusion $p : \mathbf{Tr} \hookrightarrow \mathbf{Grc}$ is a discrete operadic opfibration as well, we thus are still in the comfortable situation of Subsection 3.1. Also an analog of Lemma 30 is obvious: if $\Gamma \in \mathbf{Grc}$ is contractible, then $\Gamma'$, as a connected subgraph of $\Gamma$, is contractible too, and so is the quotient $\Gamma''$. The minimal model for $1_{\mathbf{Tr}}$ can therefore be constructed by mimicking the methods of Subsection 3.2, so we will be telegraphic.

For a graph $\Gamma \in \mathbf{Tr}$ having at least one internal edge and a 1-connected $\mathbf{Tr}$-collection $E$, the expression in the right hand side of

\[(36) \quad F_{\mathbf{Tr}}(E)(\Gamma) := \bigoplus_{T \in g_{\mathbf{Tr}}(\Gamma)} \operatorname{colim}_{\lambda \in \lambda(T)} E(T, \lambda)\]

makes sense, since each of the graphs $\Gamma_1, \ldots, \Gamma_k$ in the definition (10) of $E(T, \lambda)$ is connected.

Let $D_{\mathbf{Tr}} \in \mathbf{Tr} \text{-Coll}_1$ be the collection with $D_{\mathbf{Tr}}(\Gamma) := \det(\Gamma)$, $\Gamma \in \mathbf{Tr}$.

For $D_{\mathbf{Tr}}$ in place of $E$, formula (36) describes the pieces of the free operad $F_{\mathbf{Tr}}(D_{\mathbf{Tr}})$. The differential $\partial$ on $F_{\mathbf{Tr}}(D_{\mathbf{Tr}})$ is given by an obvious modification of formula (18b). Also the definition of $\rho : F_{\mathbf{Tr}}(D_{\mathbf{Tr}}) \rightarrow 1_{\mathbf{Tr}}$ is the expected one. We have

**Theorem 32.** The object $\mathcal{M}_{\mathbf{Tr}} = (F_{\mathbf{Tr}}(D_{\mathbf{Tr}}), \partial) \xrightarrow{\rho} (1_{\mathbf{Tr}}, \partial = 0)$ is a minimal model of the terminal $\mathbf{Tr}$-operad $1_{\mathbf{Tr}}$.

**Proof.** Verbatim modification of the proof of Theorem 31. \hfill $\square$

3.4. **Minimal model for $1_{\mathbf{Whe}}$.** We say, following [2, Example 4.19], that an ordered connected graph $\Gamma \in \mathbf{Gr}$ is **oriented** if

(i) each internal edge if $\Gamma$ is oriented, meaning that one of the half-edges forming this edge is marked as the input one, and the other as the output, and

(ii) also the legs of $\Gamma$ are marked as either input or output ones.

Oriented ordered graphs form an operadic category $\mathbf{Whe}$. Algebras for the terminal $\mathbf{Whe}$-operad $1_{\mathbf{Whe}}$ are wheeled properads introduced in [24]. As noted in Example 2.19 loc. cit., the functor $p : \mathbf{Whe} \rightarrow \mathbf{Grc}$ that forgets the orientation is a discrete operadic opfibration, thus the constructions of the previous two subsections, including the description of the minimal model for $1_{\mathbf{Whe}}$, translate verbatim. We leave the details to the reader.

[October 17, 2022]
3.5. **Minimal model for $1_{\mathbb{R}Tr}$**. We will call the leg of $\Gamma \in \mathbb{Tr}$, minimal in the global order, the *root* of $\Gamma$. Let us orient edges of $\Gamma \in \mathbb{Tr}$ so that they point to the root. We say that $\Gamma$ is *rooted* if the outgoing half-edge of each vertex is the smallest in the local order at that vertex. In [2] we considered the full subcategory $\mathbb{R}Tr$ of $\mathbb{Tr}$ consisting of rooted trees and identified algebras over the terminal $\mathbb{R}Tr$ operad $1_{\mathbb{R}Tr}$ with ordinary, classical operads. The inclusion $p : \mathbb{R}Tr \hookrightarrow \mathbb{Tr}$ is, however, a discrete operadic *fibration*, not an opfibration, cf. [2, Example 4.9]. Nevertheless, the fibers of $p$ are finite, being either empty or an one-point set, thus Corollary 28 applies, so we can construct an explicit minimal model for $1_{\mathbb{R}Tr}$ by obvious modifications of the methods used in the previous subsections.

**Example 33.** Figure [11] illustrates the failure of Lemma 30 for $\mathbb{Tr}$ in place of $\mathbb{Grc}$ and $\mathbb{R}Tr$ in place of $\mathbb{ggGrc}$. The graph $\Gamma$ in that figure has vertices (indexed by) $\{1, 2, 3\}$ and half-edges $\{1, 2, 3, 4, 5, 6\}$, the graph $\Gamma''$ has vertices $\{1, 2\}$ and half-edges $\{1, 2, 3, 4\}$. The map $\phi : \Gamma \to \Gamma''$ sends the vertices 1 and 3 of $\Gamma$ to the vertex 1 (the fat one) of $\Gamma''$, and the vertex 2 of $\Gamma$ to the vertex of $\Gamma''$ with the same label. The labels in the circles indicate the global orders. While $\Gamma$ is rooted, $\Gamma''$ is not, although $\phi$ is even a canonical contraction.

![Figure 11. Failure of Lemma 30: $\Gamma''$ is not rooted whereas $\Gamma$ is.](image)

3.6. **Minimal model for $1_{\mathbb{S}Tr}$**. It turns out that the operadic category $\mathbb{R}Tr$ contains much smaller subcategory which still captures the classical operads in the same way $\mathbb{R}Tr$ does. It is defined as follows. We say that a rooted tree $\Gamma \in \mathbb{R}Tr$ is *strongly rooted*, if the order of its set $V$ of vertices is compatible with the rooted structure. By this we mean that, if $v \in V$ lies on the path connecting $u \in V$ with the root, then $v < u$ in $V$. We denote by $\mathbb{S}Tr \subset \mathbb{R}Tr$ the full subcategory of strongly rooted trees. It is easy to show that all fibers of a map $\phi : \Gamma' \to \Gamma''$ between strongly rooted trees are strongly rooted, and also that all rooted corollas are clearly strongly rooted. Consequently, $\mathbb{S}Tr$ is an operadic category.

We claim that algebras over the terminal $\mathbb{S}Tr$-operad $1_{\mathbb{S}Tr}$ are the same as $1_{\mathbb{R}Tr}$-algebras, i.e. that they are ordinary operads. This might sound surprising, since $\mathbb{S}Tr$ has less objects than $\mathbb{R}Tr$, therefore $1_{\mathbb{S}Tr}$-algebras have less operations than $1_{\mathbb{R}Tr}$-algebras. Each operation of a $1_{\mathbb{R}Tr}$-algebra can however be obtained from an operation of a $1_{\mathbb{S}Tr}$-algebra via certain permutation of inputs, since each rooted tree is isomorphic with a strongly rooted tree, by a local isomorphism.

[nests.tex]
Example 34. Consider the rooted trees in Figure 12. The left one belongs to $\text{SRTr}$ and represents the operation

$$
\mathcal{O}_{\Gamma'} : P(3) \otimes P(2) \longrightarrow P(4)
$$
given by $\mathcal{O}_{\Gamma'}(x \otimes y) = x \circ_2 y$, where $\circ_2$ is the standard $\circ$-operation in a unital operad $P$, while

$$
\mathcal{O}_{\Gamma''} : P(2) \otimes P(3) \longrightarrow P(4)
$$
is given by $\mathcal{O}_{\Gamma''}(a \otimes b) = b \circ_2 a$. Thus $\mathcal{O}_{\Gamma''} = \mathcal{O}_{\Gamma'} \circ \sigma$ with $\sigma \in \Sigma_2$ the transposition.

Neither the inclusion $\text{SRTr} \hookrightarrow \text{RTr}$, nor the composite $\text{SRTr} \hookrightarrow \text{RTr} \hookrightarrow \text{Tr}$ is a fibration or opfibration, but the category $\text{SRTr}$ is, unlike $\text{RTr}$, closed under canonical contractions. It can indeed be easily verified that, if $\Gamma' \in \text{SRTr}$ and if $\pi : \Gamma' \to \Gamma''$ is the canonical contraction, then $\Gamma''$ and also the fiber of $\pi$ belongs to $\text{SRTr}$. The methods developed in Subsection 1.2 can therefore be used with $\text{SRTr}$ in place of $\text{Grc}$. Namely, each tower $[6]$ in $\text{SRTr}$ can be brought into the canonical form where $\ell = \Pi_{\Gamma}$ and all $\tau$’s are canonical contractions, and as such be represented by a graph tree in $g_{\text{Tr}}(\Gamma)$. The right hand side of formula (12) then, for $\Gamma \in \text{SRTr}$ and $E \in \text{SRTr}\text{-Coll}_1$, expresses the component of the free $\text{SRTr}$-operad $\mathbb{F}_{\text{SRTr}}(E)$.

Our description of a minimal model for $1_{\text{SRTr}}$ is the expected one. We define the collection $D_{\text{SRTr}} \in \text{SRTr}\text{-Coll}_1$ by

$$
D_{\text{SRTr}}(\Gamma) := \det(\Gamma), \quad \Gamma \in \text{SRTr},
$$
and the differential $\partial$ on the free operad $\mathbb{F}_{\text{SRTr}}(D_{\text{SRTr}})$ whose components are

$$(37) \quad \mathbb{F}_{\text{SRTr}}(D_{\text{SRTr}})(\Gamma) := \bigoplus_{T \in g_{\text{Tr}}(\Gamma)} \colim_{\lambda \in \text{Lev}(T)} D_{\text{SRTr}}(T, \lambda)
$$
by the verbatim version of formula (18b). The morphism $\rho : \mathbb{F}_{\text{SRTr}}(D_{\text{SRTr}}) \to 1_{\text{SRTr}}$ is given by an obvious analog of (19). One has

**Theorem 35.** The object $M_{\text{SRTr}} := (\mathbb{F}_{\text{SRTr}}(D_{\text{SRTr}}), \partial) \xrightarrow{\rho} (1_{\text{SRTr}}, \partial = 0)$ is a minimal model of the terminal $\text{SRTr}$-operad.
Proof. The only possibly nontrivial issue is the acyclicity $\mathcal{M}_{\text{SRTr}}$ in positive dimensions. Comparing the formula

$$F(D)(\Gamma) := \bigoplus_{T \in \text{gTr}(\Gamma)} \text{colim}_{\lambda \in \text{Lev}(T)} D(T, \lambda)$$

defining the component of the minimal model $\mathcal{M}_{\text{Grc}}$ for $1_{\text{Grc}}$ with (37) we notice the equality

$$(F_{\text{SRTr}}(D_{\text{SRTr}})(\Gamma), \partial) = (F(D)(\Gamma), \partial)$$

for $\Gamma \in \text{SRTr}$. In other words

$$\mathcal{M}_{\text{SRTr}}(\Gamma) = \mathcal{M}_{\text{Grc}}(\Gamma), \quad \text{for } \Gamma \in \text{SRTr} \subset \text{Grc}.$$ 

The acyclicity of $\mathcal{M}_{\text{SRTr}}$ thus follows from the acyclicity of $\mathcal{M}_{\text{Grc}}$ established in the proof of Theorem 14. □

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