NON-DEGENERATE JUMP OF MILNOR NUMBERS OF SURFACE SINGULARITIES

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Abstract. The jump of the Milnor number of an isolated singularity \( f_0 \) is the minimal non-zero difference between the Milnor numbers of \( f_0 \) and one of its deformations \( p f_s q \). We give a formula for the jump in some class of surface singularities in the case deformations are non-degenerate.

1. Introduction

Let \( f_0 : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) be an (isolated) singularity, i.e. let \( f_0 \) be a germ at 0 of a holomorphic function having an isolated critical point at \( 0 \in \mathbb{C}^n \), and \( 0 \in \mathbb{C} \) as the corresponding critical value. More specifically, there exists a representative \( \hat{f}_0 : U \rightarrow \mathbb{C} \) of \( f_0 \) holomorphic in an open neighborhood \( U \) of the point \( 0 \) such that:

- \( \hat{f}_0(0) = 0 \),
- \( \nabla \hat{f}_0(0) = 0 \),
- \( \nabla \hat{f}_0(z) \neq 0 \) for \( z \in U \setminus \{0\} \),

where for a holomorphic function \( f \) we put \( \nabla f := (\partial f / \partial z_1, \ldots, \partial f / \partial z_n) \).

In the sequel we will identify germs of functions with their representatives or the corresponding convergent power series. The ring of germs of holomorphic functions of \( n \) variables will be denoted by \( \mathcal{O}_n \).

A deformation of the singularity \( f_0 \) is a germ of a holomorphic function \( f = f(s, z) : (\mathbb{C} \times \mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) such that:

- \( f(0, z) = f_0(z) \),
- \( f(s, 0) = 0 \),

The deformation \( f(s, z) \) of the singularity \( f_0 \) will also be treated as a family \( \{ f_s \} \) of germs, putting \( f_s(z) := f(s, z) \). Since \( f_0 \) is an isolated singularity, \( f_s \) has also isolated singularities near the origin, for sufficiently small \( s \) [GLS07, Theorem 2.6 in Chap. 1].

Remark. Notice that in the deformation \( \{ f_s \} \) there can occur in particular smooth germs, that is germs satisfying \( \nabla f_s(0) \neq 0 \). In this context, the symbol \( \nabla f_s \) will always denote \( \nabla_z f_s(z) \).

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By the above assumptions it follows that, for every sufficiently small \( s \), one can define a (finite) number \( \mu_s \) as the Milnor number of \( f_s \), namely

\[
\mu_s := \mu(f_s) = \dim_\mathbb{C} \mathcal{O}_n/(\mathcal{V} f_s) = \mu\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right),
\]

where the symbol \( \mu(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}) \) denotes intersection multiplicity of the ideal \( (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}) \in \mathcal{O}_n \).

Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities [GLS07, Theorem 2.57 in Chap. II], there exists an open neighborhood \( S \) of the point \( 0 \in \mathbb{C} \) such that

- \( \mu_s = \text{const. for } s \in S \setminus \{0\}, \)
- \( \mu_0 \geq \mu_s \) for \( s \in S \).

The (constant) difference \( \mu_0 - \mu_s \) for \( s \in S \setminus \{0\} \) will be called the jump of the deformation \( (f_s) \) and denoted by \( \lambda((f_s)) \). The smallest nonzero value among all the jumps of deformations of the singularity \( f_0 \) (such a value exists because one can always consider a deformation of \( f_0 \) built of smooth germs and then for it it is \( \mu_s = 0 \); cf. Remark 1) will be called the jump (of the Milnor number) of the singularity \( f_0 \) and denoted by \( \lambda(f_0) \).

The first general result concerning the jump was S. Gusein-Zade’s [GZ93], who proved that there exist singularities \( f_0 \) for which \( \lambda(f_0) > 1 \) and that for irreducible plane curve singularities it holds \( \lambda(f_0) = 1 \). In [BK14] the authors proved that \( \lambda(f_0) \) is not a topological invariant of \( f_0 \) but it is an invariant of the stable equivalence. The computation of \( \lambda(f_0) \) for a specific reducible singularity (or for a class of reducible singularities) is not an easy task. It is related to the problem of adjacency of classes of singularities. Only for a few classes of singularities we know the exact value of \( \lambda(f_0) \). For plane curve singularities \( (n = 2) \) we have (see [AGZV85] for terminology):

- for the one-modal family of singularities in the \( X_9 \) class, that is singularities of the form
  \[
  f_0^a(x, y) := x^4 + y^4 + ax^2 y^2, \quad a \in \mathbb{C}, \quad a^2 \neq 4,
  \]
  we have \( \lambda(f_0^a) = 2 \) ([BK14]),

- for the two-modal family of singularities in the \( W_{1,0} \) class, that is singularities of the form
  \[
  f_0^{a,b}(x, y) := x^4 + y^6 + (a + by)x^2 y^3, \quad a, b \in \mathbb{C}, \quad a^2 \neq 4,
  \]
  we have
  \[
  \lambda(f_0^{a,b}) = \begin{cases} 1, & \text{if } a = 0 \ (\text{[BK14])} \\ \geq 2, & \text{for generic } a, b \ (\text{[GZ93])}. \end{cases}
  \]

- for specific homogenous singularities \( f_0^d(x, y) := x^d + y^d, d \geq 2 \), we have \( \lambda(f_0^d) = \left\lfloor \frac{d}{2} \right\rfloor \) ([BK14]),

- for homogeneous singularities of degree \( d \) with generic coefficients \( f_0 \) we have \( \lambda(f_0) < \left\lfloor \frac{d}{2} \right\rfloor \) ([BK14]).
In the present paper we consider a weaker problem: compute the jump $\lambda^\text{nd}(f_0)$ of $f_0$ over all non-degenerate deformations of $f_0$ (i.e. the $f_s$ in the deformations $(f_s)$ of $f_0$ are non-degenerate singularities). Clearly, we always have $\lambda(f_0) \leq \lambda^\text{nd}(f_0)$. Up to now, this problem has been studied only for plane curve singularities.

- A. Bodin ([Bod07]) gave a formula for $\lambda^\text{nd}(f_0)$ for $f_0$ convenient with its Newton polygon reduced to one segment,
- J. Walewska in [Wal13] generalized Bodin’s results to the non-convenient case,
- the authors ([BKW14]) calculated all possible Milnor numbers of all non-degenerate deformations of homogenous singularities,
- J. Walewska ([Wal10]) proved that the second non-degenerate jump of $f_0$ satisfying Bodin’s assumptions is equal to 1.

In this paper we want to pass to surface singularities ($n = 3$). We give a formula (more precisely: a simple algorithm) for $\lambda^\text{nd}(f_0)$ in the case where $f_0$ is non-degenerate, convenient and has its Newton diagram reduced to one triangle, (see Figure 1) i.e. $f_0$ of the form

$$f_0(x,y,z) = ax^p + by^q + cz^r + \ldots \quad (p, q, r \geq 2, \ abc \neq 0).$$

Moreover, for simplicity reasons, we will only consider the case of $p, q, r$ being pairwise coprime integers. The general case of arbitrary $p, q, r$ will be the topic of a next paper.

2. Non-degenerate singularities

In this Section we recall the notion of non-degenerate singularities. We restrict ourselves to surface singularities. All notions can easily be generalized to higher dimensions. Let $f_0(x,y,z) := \sum_{i,j,k \in \mathbb{N}} a_{ijk} x^i y^j z^k$, be a singularity. Let $\text{supp}(f_0) := \{(i,j,k) \in \mathbb{N}^3 : a_{ijk} \neq 0\}$ be the support of $f_0$. The Newton polyhedron $\Gamma_+(f_0)$ of $f_0$ is the convex hull of the set

$$\bigcup_{(i,j,k) \in \text{supp}(f_0)} (i,j,k) + \mathbb{R}_+^3,$$

where $\mathbb{R}_+^3$ is the closed octant of $\mathbb{R}^3$ consisting of points with nonnegative coordinates. The boundary (in $\mathbb{R}^3$) of $\Gamma_+(f_0)$ is an unbounded polyhedron with a finite
number of 2-dimensional faces, which are (not necessarily compact) polygons. The singularity $f_0$ is called convenient if $\Gamma_+(f_0)$ has some points in common with all three coordinate axes in $\mathbb{R}^3$. The set of compact faces (of all dimensions) of $\Gamma_+(f_0)$ constitutes the Newton diagram of $f_0$ and is denoted by $\Gamma(f_0)$. For each face $S \in \Gamma(f_0)$ we define a weighted homogeneous polynomial

$$(f_0)_S := \sum_{(i,j,k) \in S} a_{i,j,k} x^i y^j z^k.$$ 

We call the singularity $f_0$ non-degenerate on $S \in \Gamma(f_0)$ if the system of equations

$$\frac{\partial (f_0)_S}{\partial x}(x, y, z) = 0, \quad \frac{\partial (f_0)_S}{\partial y}(x, y, z) = 0, \quad \frac{\partial (f_0)_S}{\partial z}(x, y, z) = 0$$

has no solutions in $\mathbb{C}^3$; $f_0$ is non-degenerate (in the Kouchnirenko sense) if $f_0$ is non-degenerate on every face $S \in \Gamma(f_0)$.

Assume now that $f_0$ is convenient. We introduce the following notation:

- $\Gamma_-(f_0)$ – the compact polyhedron bounded by $\Gamma(f_0)$ and the three coordinate planes (labeled in a self-explanatory way as OXY, OXZ, OYZ); in other words, $\Gamma_-(f_0) := \mathbb{R}^3_+ \setminus \Gamma_+(f_0)$,
- $V$ – the volume of $\Gamma_-(f_0)$,
- $P_1, P_2, P_3$ – the areas of the two-dimensional faces of $\Gamma_-(f_0)$ lying in the planes OXY, OXZ, OYZ, respectively; e.g. $P_1$ is the area of the set $\Gamma_-(f_0) \cap$ OXY,
- $W_1, W_2, W_3$ – the lengths of the edges (= one-dimensional faces) of $\Gamma_-(f_0)$ lying in the axes OX, OY, OZ, respectively (see Figure 2).

**Figure 2.** Geometric meaning of volume $V$, areas $P_i$ and lengths $W_j$.

We define the Newton number $\nu(f_0)$ of $f_0$ by

$$\nu(f_0) := 3!\frac{V}{2!} - 2!(P_1 + P_2 + P_3) + 1!(W_1 + W_2 + W_3) - 1.$$ 

The importance of $\nu(f_0)$ has its source in the celebrated Kouchnirenko theorem:

**Theorem 2.1** ([Kou76]). If $f_0$ is a convenient singularity, then

1. $\mu(f_0) \geq \nu(f_0)$,
2. if $f_0$ is non-degenerate then $\mu(f_0) = \nu(f_0)$.

**Remark 1.** The Kouchnirenko theorem is true in any dimension [Kou76].
3. Non-degenerate jump of Milnor numbers of singularities

Let \( f_0 \in \mathcal{O}_3 \) be a singularity. A deformation \( (f_s) \) of \( f_0 \) is called \textit{non-degenerate} if \( f_s \) is non-degenerate for \( s \neq 0 \). The set of all non-degenerate deformations of the singularity \( f_0 \) will be denoted by \( D_{\text{nd}}(f_0) \). \textit{Non-degenerate jump} \( \lambda_{\text{nd}}(f_0) \) of the singularity \( f_0 \) is the minimal of non-zero jumps over all non-degenerate deformations of \( f_0 \), which means

\[
\lambda_{\text{nd}}(f_0) := \min_{(f_s) \in D_{\text{nd}}(f_0)} \lambda((f_s)),
\]

where by \( D_{\text{nd}}^0(f_0) \) we denote all the non-degenerate deformations \( (f_s) \) of \( f_0 \) for which \( \lambda((f_s)) \neq 0 \).

Obviously

**Proposition 3.1.** For each singularity \( f_0 \) we have the inequality

\[
\lambda(f_0) \leq \lambda_{\text{nd}}(f_0).
\]

In investigations concerning \( \lambda_{\text{nd}}(f_0) \) we may restrict our attention to non-degenerate \( f_0 \) because the non-degenerate jump for degenerate singularities can be found using the proposition below (cf. [Bod07, Lemma 5]). Let \( f_0^{\text{nd}} \) denote any non-degenerate singularity for which \( \Gamma(f_0) = \Gamma(f_0^{\text{nd}}) \). Such singularities always exist.

**Proposition 3.2.** If \( f_0 \) is degenerate then

\[
\lambda_{\text{nd}}(f_0) = \begin{cases} 
\mu(f_0) - \mu(f_0^{\text{nd}}), & \text{if } \mu(f_0) - \mu(f_0^{\text{nd}}) > 0 \\
\lambda_{\text{nd}}(f_0^{\text{nd}}), & \text{if } \mu(f_0) - \mu(f_0^{\text{nd}}) = 0.
\end{cases}
\]

**Proof.** This follows from the fact that a generic small perturbation of coefficients of these monomials of \( f_0 \) which correspond to points belonging to \( \bigcup \Gamma(f_0) \) (which are finite in number) give us non-degenerate singularities with the same Newton polyhedron as \( f_0 \).

**Remark 2.** By the Płoski theorem ([Pło90, Lemma 2.2], [Pło99, Theorem 1.1]), for degenerate plane curve singularities (\( n = 2 \)) the second possibility in Proposition 3.2 is excluded.

A crucial rôle in the search for the formula for \( \lambda_{\text{nd}}(f_0) \) will be played by the monotonicity of the Newton number with respect to the Newton polyhedron. Namely, J. Gwoździewicz [Gwo08] and M. Furuya [Fur04] proved:

**Theorem 3.3** (Monotonicity Theorem). Let \( f_0, \tilde{f}_0 \in \mathcal{O}_n \) be two convenient singularities such that \( \Gamma_+(f_0) \subseteq \Gamma_+(\tilde{f}_0) \). Then \( \nu(f_0) \geq \nu(\tilde{f}_0) \).

By this theorem the problem of calculation of \( \lambda_{\text{nd}}(f_0) \) can be reduced to a purely combinatorial one. Namely, we define specific deformations of a convenient and non-degenerate singularity \( f_0 \in \mathcal{O}_n \). Denote by \( J \) the set of integer points \( i = (i_1, \ldots, i_n) \neq 0 \) lying in the closed domain bounded by coordinate hyperplanes \( \{z_i = 0\} \) and the Newton diagram; in other words \( J := \Gamma_+(f_0) \cap \mathbb{Z}^n \). Obviously, \( J \) is a finite set. For \( i = (i_1, \ldots, i_n) \in J \) we define the deformation \( (f^I_i)_{x \in \mathbb{C}} \) of \( f_0 \) by the formula

\[
f^I_i(z_1, \ldots, z_n) := f_0(z_1, \ldots, z_n) + z_i^1 \ldots z_i^n.
\]
Proposition 3.4. For every \( i \in J \) the deformation \((f_s^i)\) of \( f_0 \) is convenient and non-degenerate for all sufficiently small \(|s|\).

Proof. See [Kou76] or [Oka79, Appendix]. \( \square \)

Combining the Monotonicity Theorem with the above proposition we reach the conclusion that in order to find \( \lambda^{nd}(f_0) \) it is enough to consider only the non-degenerate deformations of the type \((f_s^i)\).

Theorem 3.5. If \( f_0 \) is a convenient and non-degenerate singularity, then

\[
\lambda^{nd}(f_0) = \min_{i \in J_0} \lambda(f_s^i)
\]

where \( J_0 \subset J \) is the set of those \( i \in J \) for which \( \lambda^{nd}(f_s^i) > 0 \).

Proof. By the Kouchnirenko theorem it suffices to consider non-degenerate deformations of \( f_0 \) of the form

\[
(f_0(z_1,\ldots,z_n) = f_0(z_1,\ldots,z_n) + \sum_{i \in J} a_i(s)z^i,
\]

where \( a_i(s) \) are holomorphic at \( 0 \in \mathbb{C} \) and \( a_i(0) = 0 \). Then by the Monotonicity Theorem we may restrict the scope of deformations \((\ast)\) to deformations with only one term added i.e. the deformations \((f_s^i)\) for \( i \in J_0 \). \( \square \)

Corollary 3.6. If \( f_0 \) and \( \tilde{f}_0 \) are non-degenerate and convenient singularities and \( \Gamma(f_0) = \Gamma(\tilde{f}_0) \) then \( \lambda^{nd}(f_0) = \lambda^{nd}(\tilde{f}_0) \).

4. An algorithm for \( \lambda^{nd}(f_0) \) in the case of one face Newton diagram of surface singularities

In this Section we give a simple algorithm for calculating \( \lambda^{nd}(f_0) \) provided that \( f_0 \in \mathcal{O}_3 \) is a convenient and non-degenerate singularity with one two-dimensional face of its Newton diagram. Let \( p,q,r \) be the first (i.e. nearest to the origin) points of \( \Gamma_+(f_0) \) lying on the axes \( OX, OY \) and \( OZ \), respectively. Then by Corollary 3.6 we may assume that

\[ f_0(x,y,z) = x^p + y^q + z^r, \quad p,q,r \geq 2. \]

By formula \((\ast)\) we have \( \mu(f_0) = (p-1)(q-1)(r-1) \). Moreover, without loss of generality we may also assume that

\[ p \geq q \geq r. \]

Additionally, we demand that \( p,q,r \) are pairwise coprime

\[ \text{GCD}(p,q) = \text{GCD}(p,r) = \text{GCD}(q,r) = 1. \]

By Theorem 3.5 we have to compare the jumps of deformations \((f_s^i)_{s \in \mathbb{C}}\), where \( i \in J \), i.e. \( i \) are integer points lying in the octant of \( \mathbb{R}^3 \) under the triangle with vertices \((p,0,0), (0,q,0), (0,0,r)\) (see Figure 1).
I. First we consider points in \( J \) lying on the axes. Using formula \((\circ)\) and assumption \((\dagger)\) we easily check that the axes-jump is realized by the deformation \( \left( f_s^{(p-1,0,0)} \right) \), i.e.
\[
f_s^{(p-1,0,0)}(x, y, z) = x^p + y^q + z^r + sx^{p-1},
\]
and the jump is equal to \((q-1)(r-1)\).

II. Now we consider points in \( J \) lying in coordinate planes. By the results of Bodin [Bod07] and Walewska [Wal10] we easily check that the minimal jumps on respective planes are realized by

i. the deformation \( \left( f_s^{(b, q-a, 0)} \right) \), where \( a_1, b_1 \in \mathbb{Z} \) are such that \( a_1 p - b_1 q = 1 \) and \( 0 < a_1 < q, b_1 > 0 \); this delivers the OXY-jump equal to \((r-1)\),

ii. the deformation \( \left( f_s^{(0, b_2, r-a_2)} \right) \), where \( a_2, b_2 \in \mathbb{Z} \) are such that \( a_2 q - b_2 r = 1 \) and \( 0 < a_2 < r, b_2 > 0 \); this delivers the OYZ-jump equal to \((p-1)\),

iii. the deformation \( \left( f_s^{(b_3, 0, p-a_3)} \right) \), where \( a_3, b_3 \in \mathbb{Z} \) are such that \( a_3 p - b_3 r = 1 \) and \( 0 < a_3 < p, b_3 > 0 \); this delivers the OXZ-jump equal to \((q-1)\).

The above considerations imply that the jump realized by the points lying either in coordinate planes or on axes is equal to \((r-1)\).

III. Let us pass to the deformations \( \left( f_s^i \right) \) for which the point \( i \) lies in the interior of the tetrahedron with vertices \( (0,0,0), (p,0,0), (0,q,0), (0,0,r) \). Any such point \((a, \beta, \gamma)\) satisfies the conditions:

(A) \( 0 < a < p, 0 < \beta < q, 0 < \gamma < r \),

(B) \( \frac{a}{p} + \frac{\beta}{q} + \frac{\gamma}{r} < 1 \) or equivalently \( aqr + bpr + \gamma pq < pqr \).

Moreover, the jump of the deformation \( \left( f_s^{(a, \beta, \gamma)} \right) \) is equal to \(6\) times the volume of the tetrahedron with vertices \((p,0,0), (0,q,0), (0,0,r), (a, \beta, \gamma)\) i.e.
\[
pqr - aqr - bpr - \gamma pq.
\]

Thus, we have reduced our original problem to a number theoretic one.

**Problem.** Given pairwise coprime integers \( p > q > r \) greater than \( 1 \). Find positive integers \( a, \beta, \gamma \) satisfying (A) and (B) for which the expression \( pqr - aqr - bpr - \gamma pq \) attains its positive minimum.

In order to solve it, first notice that \( \text{GCD}(qr, pr, pq) = 1 \). Consequently, there are integers \( a, b, c \) such that
\[
aqr + bpr + cpq = 1.
\]

They can be obtained by the Euclid algorithm using the well-known associativity law: for any integers \( u, v, w \) we have \( \text{GCD}(u, v, w) = \text{GCD}(\text{GCD}(u, v), w) \). Notice that in any identity of the type \((\ddagger)\) it holds \( abc \neq 0 \). If we write \( a = a'p + a'' \), \( 0 \leq a'' < p \), then, by abuse of notation, we obtain yet another identity \( aqr + bpr + cpq = 1 \), but now \( 0 < a < p \). Next, we write \( b = b'q - b'' \), \( 0 < b'' < q \), and we use it to obtain a similar identity \( aqr - bpr + cpq = 1 \) in which \( 0 < a < p \) and \( 0 < b < q \). Notice that then \( 0 < c < r \). In fact, \( |cpq| = |1 - aqr + bpr| \leq 1 + r|bp - aq| \leq 1 + r(pq - p - q) = pqr - pr - qr + 1 < pqr \). Thus, finally we have obtained the identity
\[\text{[?]}\]
Theorem 4.1. Let \( f_0 \in O_3 \) be a convenient and non-degenerate singularity with only one two-dimensional face in its Newton diagram. Assume that the vertices \((p,0,0), (0,q,0), (0,0,r)\) of this face are such that \( p \geq q \geq r \geq 2 \) and the numbers \( p, q, r \) are pairwise coprime. Then

\[
\lambda^{nd}(f_0) = \begin{cases} 
    i_0 & \text{if there exist integers } a, b, c \text{ such that} \\
    aqr - bpr + cpq = i_0, \ 1 \leq i_0 \leq r - 2, \\
    0 < a < p, \ 0 < -b < q, \ 0 < -c < r, \ i_0 - \text{minimal} \\
    r - 1 & \text{otherwise.}
\end{cases}
\]

Moreover, \( i_0 \) can be found algorithmically using only Euclid’s algorithm.
Corollary 4.2. Under the assumptions of Theorem 4.1, if $r \geq 2$ then $\lambda^{nd}(f_0) = 1$.

Example. For $f_0(x,y,z) := x^{11} + y^6 + z^5$ we have $p = 11$, $q = 6$, $r = 5$ and
\begin{align*}
7 \cdot qr - 5 \cdot pr + 1 \cdot pq &= 1 \quad \text{-- does not satisfy the conditions in the theorem} \\
3 \cdot qr - 4 \cdot pr + 2 \cdot pq &= 2 \quad \text{-- does not satisfy the conditions in the theorem} \\
10 \cdot qr - 3 \cdot pr - 2 \cdot pq &= 3 \quad \text{-- do satisfy the conditions in the theorem.}
\end{align*}

Hence, $\lambda^{nd}(f_0) = 3$. This jump is realized by the deformation $f^{(1,3,2)}_5(x,y,z) := x^{11} + y^6 + z^5 + sxy^3z^2$. The minimal jump realized by the points lying either in coordinate planes or on axes in equal to $r - 1 = 4$.

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