Equilibrium Fluctuations for a Non Gradient Energy Conserving Stochastic Model

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Abstract: In this paper we study the equilibrium energy fluctuation field of a one-dimensional reversible non gradient model. We prove that the limit fluctuation process is governed by a generalized Ornstein-Uhlenbeck process, whose covariances are given in terms of the diffusion coefficient.

The fact that the conserved, quantity (energy) is not a linear functional of the coordinates of the system: introduces new difficulties of a geometric nature when adapting the non gradient method introduced by Varadhan.

1. Introduction

In recent works, a microscopic model for heat conduction in solids has been considered (c.f. [3,4,10]). In this model nearest neighbor atoms interact as coupled oscillators forced by an additive noise which exchange momenta between nearest neighbors.

More precisely, in the case of periodic boundary conditions, atoms are labeled by \( x \in \mathbb{T}_N = \{1, \ldots, N\} \). The configuration space is defined by \( \Omega^N = (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N} \), where for a typical element \( (p_x, r_x)_{x \in \mathbb{T}_N} \in \Omega^N \), \( r_x \) represents the distance between particles \( x \) and \( x + 1 \), and \( p_x \) the velocity of the particle \( x \). The formal generator of the system reads as \( \mathcal{L}_N = \mathcal{A}_N + \mathcal{S}_N \), where

\[
\mathcal{A}_N = \sum_{x \in \mathbb{T}_N} \{(p_{x+1} - p_x)\partial_{r_x} + (r_x - r_{x-1})\partial_{p_x}\}, \quad (1.1)
\]

and

\[
\mathcal{S}_N = \frac{1}{2} \sum_{x \in \mathbb{T}_N} X_{x,x+1}[X_{x,x+1}], \quad (1.2)
\]

with \( X_{x,x+1} = p_{x+1}\partial_{p_x} - p_x\partial_{p_{x+1}} \). Here \( \mathcal{A}_N \) is the Liouville operator of a chain of interacting harmonic oscillators and \( \mathcal{S}_N \) is the noise operator.
In this work we focus on the noise operator $\mathcal{S}_N$, which acts only on velocities. Therefore, we restrict the configuration space to $\mathbb{R}^{\mathbb{T}_N}$. The total energy of the configuration $(p_x)_{x \in \mathbb{T}_N}$ is defined by

$$E = \sum_{x \in \mathbb{T}_N} p_x^2. \quad (1.3)$$

It is easy to check that $\mathcal{S}_N(E) = 0$, i.e. total energy is constant in time.

The generator $\mathcal{S}_N$ defines a diffusion process with Gaussian invariant measures $\{\nu^N_\beta\}_{\beta > 0}$ [see (2.2)]. This process is not ergodic with respect to these measures. In fact, for all $T > 0$ the hyperspheres $p_1^2 + \cdots + p_N^2 = NT$ of average kinetic energy $T$ are invariant sets. Nevertheless, the restriction of the diffusion to each of these hyperspheres is non degenerate and ergodic.

In analogy to [24] (see also [20]), where Varadhan introduced the non gradient method, we introduce inhomogeneities into the diffusion generated by (1.2) through a differentiable function $a(r, s)$ satisfying $0 \leq a(r, s) \leq C < \infty$ and having bounded continuous first derivatives [see (2.1)]. As a result, the introduction of the function $a(r, s)$ breaks the gradient structure of this diffusion.

The main result of this work is the convergence of the energy fluctuation field defined in (2.6) to a generalized Ornstein-Uhlenbeck process, when the process is at equilibrium. The covariances characterizing this generalized process are given in terms of the diffusion coefficient $\hat{a}(\beta)$ [see (2.7)]. This diffusion coefficient is given in terms of a variational formula which is equivalent to the Green-Kubo formula [c.f. [22] p.180]. The main task of this work is to rigorously establish this variational formula.

In order to study the equilibrium fluctuations of interacting particle systems, Brox and Rost [5] introduced the Boltzmann-Gibbs principle and proved its validity for attractive zero range processes. Chang and Yau [8] proposed an alternative method to prove the Boltzmann-Gibbs principle for gradient systems. This approach was extended to non gradient systems by Lu [17] and Sellami [21].

In what follows we describe the main features of the model we consider.

The model is non gradient. This difficulty has already appeared in the work of Bernardin [3], where a hydrodynamic limit is derived for a system with two conserved quantities (total deformation and total energy). The energy current is not the gradient of a local function. To overcome this problem, an exact fluctuation-dissipation relation is obtained; that is, the current is written as a gradient plus a fluctuation term. On the other hand, in [10] Fritz et al studied the equilibrium fluctuations for the model given in [3]. The exact fluctuation-dissipation relation mentioned above plays a central role in the proofs of the hydrodynamic limit and the equilibrium fluctuations.

Systems for which there exists an exact fluctuation-dissipation relation are called almost gradient systems. For these kind of systems, the minimizer in the variational formula of the diffusion coefficient can be found explicitly. In our setting we do not have such an exact relation, so we use the non gradient Varadhan’s approach.

The only conserved quantity (total energy) is not a linear function of the coordinates of the system. In other words, the invariant surfaces are not hyperplanes. Specifically, in our case invariant surfaces are hyperspheres. Therefore, more geometry is presented and some standard arguments become more subtle.

For instance, in the non gradient Varadhan’s method, it is central to have a characterization of the space over which the infimum in the variational problem defining