UNIQUENESS OF SOLUTIONS TO THE 3D QUINTIC GROSS-PITAEVSKII HIERARCHY

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ABSTRACT. In this paper, we study solutions to the three-dimensional quintic Gross-Pitaevskii hierarchy. We prove unconditional uniqueness among all small solutions in the critical space \( \mathcal{F}^1 \) (which corresponds to \( H^1 \) on the NLS level). With slight modifications to the proof, we also prove unconditional uniqueness of solutions to the Hartree hierarchy without a smallness condition. Our proof uses the quantum de Finetti theorem, and is an extension of the work by Chen-Hainzl-Pavlovic-Seiringer \[6\], and our previous work \[32\].

1. Introduction

1.1. Statement of the main result. In this paper, we establish uniqueness of small solutions to the three-dimensional quintic Gross-Pitaevskii (GP) hierarchy in the scaling-critical Sobolev type space.

The 3d quintic GP hierarchy is an infinite system of coupled linear equations

\[
i\partial_t \gamma^{(k)} = (-\Delta_{x_k} + \Delta_{x_k'}) \gamma^{(k)} + \lambda \sum_{j=1}^{k} B_{j,k+1,k+2} \gamma^{(k+2)}, \quad k \in \mathbb{N},
\]

where \( \gamma^{(k)} = \gamma^{(k)}(t, x_k; x_k') : [0, T) \times \mathbb{R}^{3k} \times \mathbb{R}^{3k} \to \mathbb{C} \), the underlined variables \( x_k \) and \( x_k' \) denote \( k \)-tuples of spacial variables, i.e., \( x_k = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^{3k} \) and \( x_k' = (x_1', x_2', \cdots, x_k') \in \mathbb{R}^{3k} \), and the Laplacians are given by \( \Delta_{x_k} := \sum_{j=1}^{k} \Delta_{x_j} \) and \( \Delta_{x_k'} := \sum_{j=1}^{k} \Delta_{x_j'} \). We assume that for each \( k \in \mathbb{N} \), \( \gamma^{(k)} \) is a symmetric marginal density matrix such that

\[
\gamma^{(k)}(t, x_k; x_k') = \gamma^{(k)}(t, x_k'; x_k)
\]

and

\[
\gamma^{(k)}(t, x_{\sigma(1)}, \cdots, x_{\sigma(k)}; x'_{\sigma'(1)}, \cdots, x'_{\sigma'(k)}) = \gamma^{(k)}(t, x_k'; x_k)
\]

for any permutations \( \sigma \) and \( \sigma' \) on \( \{1, 2, \cdots, k\} \). The contraction operator \( B_{j,k+1,k+2} \) is defined by

\[
B_{j,k+1,k+2} \gamma^{(k+2)}(t, x_k; x_k')
= \int dx_{k+1} dx_{k+2} \delta(x_j - x_{k+1}) \delta(x_j - x_{k+2}) \gamma^{(k+2)}(t, x_{k+1}, x_{k+2}; x_{k+1}', x_{k+2}')
\]

\[
- \int dx_{k+1} dx_{k+2} \delta(x_j' - x_{k+1}) \delta(x_j' - x_{k+2}) \gamma^{(k+2)}(t, x_{k+1}, x_{k+2}; x_{k+1}', x_{k+2}')
\]

\[
= \gamma^{(k+2)}(t, x_k, x_j; x_{k+1}, x_{k+2}; x_k', x_j', x_{k+1}', x_{k+2}') - \gamma^{(k+2)}(t, x_k, x_j; x_{k+1}, x_{k+2}; x_k', x_j', x_{k+1}', x_{k+2}').
\]

The coupling constant is either \(-1\) or \(+1\). We call the GP hierarchy \((1.1)\) defocusing if \( \lambda = 1 \), and focusing if \( \lambda = -1 \).

To define solutions to the GP hierarchy, we introduce the following definitions (see also \[6\][23][24][25][26]). For \( s \geq 0 \), we define the homogeneous Sobolev space \( \mathcal{F}^s \) for sequences by

\[
\mathcal{F}^s := \left\{ \gamma^{(k)} \in \mathbb{N}^k : \text{Tr} \left( |R^{(k,s)} \gamma^{(k)}| \right) < M^{2k} \right\}
\]

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where
\[ R^{(k,s)} := \prod_{j=1}^{k} (-\Delta_{x_j})^{\frac{s}{2}} (-\Delta_{x_j})^{\frac{s}{2}}. \]

Similarly, we define the inhomogeneous Sobolev space \( \dot{S}^s \) for sequences by
\[ \dot{S}^s := \left\{ \{\gamma^{(k)}\}_{k \in \mathbb{N}} : \text{Tr} \left( |S^{(k,s)}(\gamma^{(k)})| \right) < M^{2k} \text{ for some constant } M < \infty \right\} \]
where
\[ S^{(k,s)} := \prod_{j=1}^{k} (1 - \Delta_{x_j})^{\frac{s}{2}} (1 - \Delta_{x_j})^{\frac{s}{2}}. \]

A sequence \( \{\gamma^{(k)}(t)\}_{k \in \mathbb{N}} \) is called a mild solution in \( L^\infty_{t \in [0,T)} \dot{S}^s \) (or \( L^\infty_{t \in [0,T)} S^s \)) to the quintic GP hierarchy if it solves the hierarchy of the integral equations
\[ \gamma^{(k)}(t) = U^{(k)}(t) \gamma^{(k)}(0) + i \sum_{j=1}^{k} \int_{0}^{t} U^{(k)}(t-s) B_{j;k+1,k+2} \gamma^{(k+2)}(s) ds, \quad \forall k \in \mathbb{N}, \]
where \( U^{(k)}(t) := e^{it\Delta_{\mathbb{R}^k} - \Delta_{\mathbb{R}^k}} \) is the free evolution operator. A sequence \( \{\gamma^{(k)}\}_{k \in \mathbb{N}} \) is called admissible if for each \( k \in \mathbb{N} \) and \( t \in [0,T) \), \( \gamma^{(k)} \) is a non-negative trace class operator on \( L^2_{\text{sym}}(\mathbb{R}^{3k} \times \mathbb{R}^{3k}) \) (subset of \( L^2 \) functions that satisfy (1.3)) and
\[ \gamma^{(k)} = \text{Tr}_{k+1}(\gamma^{(k+1)}) = \int_{\mathbb{R}^3} dx_{k+1} \gamma^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}_k, x_{k+1}). \]

We call a sequence \( \{\gamma^{(k)}\}_{k \in \mathbb{N}} \) a limiting hierarchy if there is a sequence \( \{\gamma^{(N)}_{\mathbb{N}}\}_{N \in \mathbb{N}} \) of non-negative density matrices on \( L^2_{\text{sym}}(\mathbb{R}^{3N} \times \mathbb{R}^{3N}) \) with \( \text{Tr}(\gamma^{(N)}_{\mathbb{N}}) = 1 \) such that \( \gamma^{(k)} \) is the weak-* limit of the \( k \)-particle marginals of \( \gamma^{(N)}_{\mathbb{N}} \) in the trace class on \( L^2_{\text{sym}}(\mathbb{R}^{3k} \times \mathbb{R}^{3k}) \), that is,
\[ \gamma^{(k)}_{\mathbb{N}} := \text{Tr}_{k+1,\ldots,N}(\gamma^{(N)}_{\mathbb{N}}) \]
\[ = \int_{\mathbb{R}^{(N-k)}} dx_{k+1} \cdots dx_N \gamma^{(N)}_{\mathbb{N}}(\mathbf{x}_k, x_{k+1}, \ldots, x_N; \mathbf{x}_k, x_{k+1}, \ldots, x_N) \]
\[ \text{as } N \to \infty. \]  

In this paper, we consider mild solutions to the GP hierarchy [1,1] that are admissible or limiting hierarchies. Such mild solutions are physically relevant in the theory of derivation of the nonlinear Schrödinger equation (NLS) from the many body linear Schrödinger equation (see Section 1.2).

We now state our main result.

**Theorem 1.1** (Uniqueness of small solutions to the quintic GP hierarchy). Suppose that \( \{\gamma^{(k)}(t)\}_{k \in \mathbb{N}} \) is a mild solution to \( L^\infty_{t \in [0,T)} \dot{S}^1 \) to the quintic GP hierarchy (1.1) with initial data \( \{\gamma^{(k)}(0)\}_{k \in \mathbb{N}} \), which is either admissible or a limiting hierarchy for each \( t \). If there exists a sufficiently small \( M > 0 \) such that \( \text{Tr} \left( |R^{(k,1)}(\gamma^{(k)})| \right) < M^{2k} \) for all \( k \in \mathbb{N} \) and \( t \in [0,T) \), then \( \{\gamma^{(k)}(t)\}_{k \in \mathbb{N}} \) is the only such solution for the given initial data.

The quintic GP hierarchy is closely related to the quintic NLS via factorized functions. Indeed, one can check that if \( \phi_t \) is a solution to the quintic NLS
\[ i \partial_t \phi_t = (-\Delta) \phi_t + \lambda |\phi_t|^4 \phi_t, \quad (1.10) \]
then a sequence of factorized functions,
\[ \gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') = \langle |\phi_t \rangle \langle \phi_t | \rangle^{\otimes k} := \prod_{j=1}^{k} \phi_t(x_j) \overline{\phi_t(x_j')}, \quad (1.11) \]
solves the GP hierarchy \([1,1]\). In this sense, proving uniqueness for the GP hierarchy is more difficult than it is for the quintic NLS.

The quintic GP hierarchy was studied by T. Chen and Pavlović [8] for the derivation of the quintic NLS as the Gross-Pitaevskii field limit of a non-relativistic Bose gas with 3-particle interactions. As a part of their analysis, the authors derived the quintic GP hierarchy from the linear Schrödinger equation, and they proved (conditional) uniqueness of solutions to the quintic GP hierarchy in an energy space, that is, a Sobolev type space of order 1, in one and two dimensions. However, derivation and uniqueness for the GP hierarchy are open in three dimensions.

Theorem 1.1 provides an answer for the uniqueness problem under a smallness assumption. We remark that the 3d quintic GP hierarchy is scaling-critical in \(\mathcal{H}^1\), and that even with our smallness assumption, our theorem is the first uniqueness theorem for the cubic or quintic GP hierarchy in a scaling-critical space. We recall that in the NLS theory, uniqueness of solutions that belong only to the space \(C([0,T];H^s)\) is called unconditional, while uniqueness in the intersection of \(C([0,T];H^s)\) and an auxiliary space such as \(L^q([0,T];W^{a,p}_r)\) is called conditional. In this sense, uniqueness in Theorem 1.1 is unconditional except that solutions are either admissible or limiting hierarchies, and that they are small.

It remains an open problem to remove the smallness assumption. In the case of the 3d quintic NLS, it is known that solutions are unique in the space \(H^s\) for \(s \geq 1\), without a smallness assumption [5,21,34]. However, the proof of unconditional uniqueness in the scaling-critical case \(s = 1\) differs from the proof in the subcritical case \(s > 1\). In the case of the 3d quintic GP hierarchy, we also expect that an approach different from the one that we use in the scaling-subcritical case is needed to remove the smallness assumption in the scaling-critical case. Currently, the main obstacle to removing the smallness assumption for solutions to the 3d quintic GP hierarchy in the scaling-critical case is the generally infinite cardinality of the support of the measure \(\mu\) in the statement of the quantum de Finetti theorem, Theorem 2.1 (see Remark 5.1).

To compare scaling-critical and subcritical regimes, we provide a uniqueness theorem for the 3d quintic Hartree hierarchy. The 3d quintic Hartree hierarchy is also an infinite hierarchy as \([1,1]\). However the contraction operator \(B_{j,k+1,k+2}\) in \([1,4]\) is replaced by

\[
B_{j,k+1,k+2}(t,x_k,x_k') := \int dx_{k+1}dx_{k+2}V(x_j - x_{k+1},x_j - x_{k+2})\gamma^{(k+2)}(t,x_k,x_{k+1},x_{k+2};x_k',x_{k+1},x_{k+2})
\]

\[\quad - \int dx_{k+1}dx_{k+2}V(x_j' - x_{k+1},x_j' - x_{k+2})\gamma^{(k+2)}(t,x_k,x_{k+1},x_{k+2};x_k',x_{k+1},x_{k+2}).\]

Note that the 3d quintic Hartree equation is subcritical in \(L^r_{L^2([0,T])}\) if the three-particle interaction potential \(V\) is less singular than the product of delta functions. This is, if \(V(\cdot,\cdot) \in L^r_{x,y}(\mathbb{R}^3 \times \mathbb{R}^3)\) for some \(r > 1\). In this case, we can show unconditional uniqueness for the 3d quintic Hartree hierarchy without a smallness assumption.

**Theorem 1.2** (Unconditional uniqueness for the quintic Hartree hierarchy). Suppose that \(V(\cdot,\cdot) \in L^r_{x,y}(\mathbb{R}^3 \times \mathbb{R}^3)\) for some \(r > 1\). Let \(\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}} \in \mathcal{H}^1\) be a mild solution to the quintic Hartree hierarchy with initial data \(\{\gamma^{(k)}(0)\}_{k \in \mathbb{N}}\), which is either admissible or a limiting hierarchy for each \(t\). If there exists \(M > 0\) such that \(\text{Tr}(\|S^{(k)}\|) < M^{2k}\) for all \(t \in [0,T]\), then \(\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}\) is the only such solution for the given initial data.

1.2. Related works. The background work in this line goes back to the derivation of Schrödinger type equations from the many-body quantum systems. In the pioneering works by Hepp [31], Spohn [41] and in a series of more recent breakthroughs by Erdös, Schlein and Yau [23,24,25,26], the authors derived the cubic NLS in \(\mathbb{R}^3\). A major ingredient in this derivation is the establishment of the unconditional uniqueness of solutions to the corresponding GP hierarchy. The proof of
uniqueness by Erdős-Schlein-Yau requires sophisticated Feynman graph expansions. For the one-
dimensional case, we refer Adami, Golse and Teta \[1\].

Later, Klainerman and Machedon \[36\] rephrased the Feynman graph extensions as a board game
argument to provide an alternative approach to prove uniqueness of solutions. However, the result
in \[36\] is conditional in that the solutions satisfy an a-priori space-time bound assumption. In
Kirkpatrick, Schlein and Staffilani \[35\], the authors derived the cubic NLS on $\mathbb{R}^2$ and $\mathbb{T}^2$
by proving that solutions to the GP hierarchy, obtained in the limits, obey such a space-time bound.
This approach has been adapted in various setting for the derivation of the NLS \[8, 10, 11, 13, 14, 15,
16, 17, 18, 19, 43\]. Inspired by this method, the Cauchy problem for the GP hierarchy is studied
in \[7, 9, 11, 27\].

A recent new proof on the unconditional uniqueness of 3d cubic GP hierarchy was given by
T.Chen, Hainzl, Pavlović and Seringer \[6\] using the quantum de Finetti theorem. The quantum de
Finetti theorem is a quantum analogue of the Hewitt-Savage theorem in probability theory. The
strong version of the quantum de Finetti theorem (see (2.1)) asserts that an infinite sequence of
admissible marginal density matrices can be expressed as an average over factorized states. How-
ever, for each $t$, the limiting hierarchies of density matrices do not necessarily satisfy admissibility.
In this case, one uses the weak version of the de Finetti theorem (see (2.2)). This is necessary when
working with the BBGKY hierarchy approach for the derivation of NLS as in \[23, 24, 25, 26\], where
one starts with a finite BBGKY hierarchy of $N$ equations for the bosonic $N$-particle system (see (2.1) in \[24\]). In this case, the GP hierarchy of equations is obtained by taking $N \to \infty$ in the
finite hierarchy. As a part of the derivation, one proves that the weak-* limit of solutions $\gamma_N^{(k)}$
to the BBGKY hierarchy solve the infinite GP hierarchy.

By taking advantage of the quantum de Finetti theorems that give an alternative factorized
formula for the solutions to the hierarchy, the authors of \[32\] established unconditional uniqueness
for cubic GP hierarchy at the same regularity level of the corresponding NLS. Others have also used
the de Finetti theorem to prove unconditional uniqueness for GP hierarchies in various settings.
In \[40\], V. Sohinger adapted the method from \[6\] to cubic GP hierarchy in a periodic setting.
In \[20\], X. Chen-Smith studied a Chen-Simon-Schrödinger hierarchy.

Not only the derivations of NLS type equations, it is also interesting to explore the rate of
convergence from the many-body system. In \[39\], Rodnianski and Schlein first addressed this
question for the nonlinear Hartree equation, based on the Fock space technique in Hepp \[31\].
Subsequently, many authors have investigated the convergence rate \[4, 12, 28, 29, 30, 38\], and it is a
very active research topic in this field.

1.3. **Strategy of the proof.** We prove Theorem 1.1 and Theorem 1.2 in the framework of
Chen-Hainzl-Pavlović-Seringer \[6\]. Due to the linearity of the hierarchy, it suffices to show that solutions
having a zero initial are the zero solution. In our proof, we iterate the Duhamel formula
(1.7) with zero initial data $n$ times, resulting in a number of terms that grows factorially in $n$.
We reduce the number of terms by the Erdős-Schlein-Yau combinatorial argument in Klainerman-
Machedon’s formulation \[36\]. The quintic version of this combinatoric reduction was used by
Chen-Pavlovic in \[8\]. We use it for the 3d quintic GP and Hartee hierarchies without modification.
Next, we apply the quantum de Finetti theorem to write each term as an integral sum of factorized
states, and reorganize them using a tree-graph structure (see Figure 1 below) which extends the
tree-graph in Chen-Hainzl-Pavlović-Serirger \[6\]. Then, we iteratively estimate the $n$ integrals. In
each step, we apply our multilinear estimates, which can be found in Appendix A. Finally, we send
$n \to \infty$ and find that solutions having zero initial data must be the zero solution.

In our previous work \[32\], we proved unconditional uniqueness for the cubic GP hierarchy in
a low regularity setting, using a similar approach. In \[32\], our key ingredients were the trilinear
estimates (2.19), (2.21) and (2.23) in Lemma 2.6. These estimates are based on the dispersive
estimates
\[ \|e^{it\Delta}f\|_{L^p(\mathbb{R}^d)} \lesssim |t|^{-d\left(\frac{1}{p} - \frac{1}{2}\right)}\|f\|_{L^q(\mathbb{R}^d)}, \quad p \geq 2, \tag{1.13} \]
and negative order Sobolev norm estimates (Lemma A.3 in [32]). In the proof, we applied these estimates to the reorganized integrals iteratively together with multilinear estimates based on Strichartz estimates ((2.20), (2.22) and (2.24) in Lemma 2.6). We remark that the use of dispersive estimates is crucial in obtaining the optimal subcritical low regularity uniqueness theorem in \(\mathbb{R}^d\) for \(d \geq 3\). The dispersive estimates don’t work in the scaling-critical space, however. Roughly speaking, this is due to the failure of integrability (in time) of the bound in (1.13). For instance, if one tries to prove uniqueness for the 3d quintic GP hierarchy in \(L_{t \in [0,T]}^\infty H^1\) by the same approach, one should choose \(p = 6\) for the multilinear estimate. Then, the bound in (1.13) is not integrable in time.

In the present work, instead of using dispersive estimates, we use multilinear estimates (Proposition A.1 and Propositions A.3) that are based on Strichartz estimates and a negative order Sobolev norm bound. In the case of the Hartree hierarchy, we also make use of a convolution estimates of W. Beckner [3].

1.4. Notation. In order to prove Theorem 1.1 and Theorem 1.2 at the same time, we define
\[
V_{x}(y,z) := \begin{cases} 
V(y,z), & \text{for the Hartree hierarchy.} \\
\lambda \delta(y)\delta(z), & \text{for the GP hierarchy.} 
\end{cases} \tag{1.14}
\]
With this notation, we can now combine definitions (1.4) and (1.12) of \(B_{j,k+1,k+2}\) for the GP hierarchy and the Hartree hierarchy, respectively, as follows.
\[
B_{j,k+1,k+2}(t, x_k, x'_k) = \int dx_{k+1}dx_{k+2} V_{x}(x_j - x_{k+1}, x_j - x_{k+2})\gamma^{(k+2)}(t, x_k, x_{k+1+k+2}) 
- \int dx_{k+1}dx_{k+2} V_{x}(x_j' - x_{k+1}, x_j' - x_{k+2})\gamma^{(k+2)}(t, x_k, x_{k+1+k+2}). \tag{1.15}
\]
For notational convenience, with abuse of notation, when \(V_{x}(y,z) = \lambda \delta(y)\delta(z)\), we formally write \(\|V_{x}\|_{L^1_{t,y,z}} = \lambda\).

1.5. Organization of the paper. This paper is organized as follows. In section 2 we present the road map for the proof of the main theorems and reduce the the main theorems to Proposition 2.1. We illustrate with an example how to factorize solutions in section 3. In section 4, we introduce tree graphs to illustrate our decomposition of each factor, and present properties of the associated kernels. The proof of Proposition 2.1 occupies section 5. In appendix A, we prove several multilinear estimates that we use section 5.

2. Outline of the Proof
We describe the strategy to prove uniqueness in more detail.

2.1. Setup. Let \(\{\gamma_1^{(k)}(t)\}_{k \in \mathbb{N}}\) and \(\{\gamma_2^{(k)}(t)\}_{k \in \mathbb{N}}\) be two mild solutions in \(L_{t \in [0,T]}^\infty H^1\) that solve (1.7) with the same initial data, and are either admissible or limiting hierarchies. To prove uniqueness, we will show that their difference \(\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}\), given by
\[
\gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t), \quad k \in \mathbb{N}, \tag{2.1}
\]
is zero. Indeed, it suffices to prove the following.
Proposition 2.1. For $A > 1 > a > 0$, we define the medium frequency cut-offs by $P_{a,A,j} = 1_{|a - \Delta_j| \ll A}$ and $P'_{a,A,j} = 1_{|a - \Delta_j| \ll A}$, where $1_{|a - |x| \ll A} = 1$ if $a \leq |x| \leq A$ and $1_{a \ll |x| \ll A} = 0$ otherwise. Let

$$P^{(k)}_{a,A} = \prod_{j=1}^{k} P_{a,A,j} P'_{a,A,j}. \quad (2.2)$$

(i) Suppose that $\{\gamma_1^{(k)}(t)\}_{k \in \mathbb{N}}$ and $\{\gamma_2^{(k)}(t)\}_{k \in \mathbb{N}}$ are mild solutions to the quintic GP hierarchy \((1.1)\) with the same initial data, and they obey the hypothesis in Theorem 1.1. Then, for any $A > 1 > a > 0$,

$$\text{Tr}(|P^{(k)}_{a,A} \gamma^{(k)}(t)|) = 0, \quad \forall k \in \mathbb{N}, \quad (2.3)$$

where $\gamma^{(k)}(t) = \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t)$.

(ii) Suppose that $V(\cdot, \cdot) \in L^r_{x,g}(\mathbb{R}^3 \times \mathbb{R}^3)$ for some $r > 1$. We assume that $\{\gamma_1^{(k)}(t)\}_{k \in \mathbb{N}}$ and $\{\gamma_2^{(k)}(t)\}_{k \in \mathbb{N}}$ are mild solutions to the quintic Hartree hierarchy with the same initial data, and they obey the hypothesis in Theorem 1.2. Then, \((2.3)\) holds for all $A > 1 > a > 0$.

Proof of Theorem 1.1 and 1.2, assuming Proposition 2.1. We just prove Theorem 1.1 from Proposition 2.1 (i), since Theorem 1.2 follows by the same argument. Fix $k \in \mathbb{N}$ and $t \in [0, T)$. Then, by density, for any $\epsilon > 0$, there exists a finite-rank operator

$$\gamma^{(k)}_\epsilon(t) = \sum_{\ell=1}^{L_x} \lambda_\ell \left| |\nabla x_1|^{-1} \cdots |\nabla x_k|^{-1} \phi_\ell(t) \right| |\nabla x_1|^{-1} \cdots |\nabla x_k|^{-1} \phi_\ell(t),$$

with $\phi_\ell(t) \in L^2_{a,k}$, such that $\text{Tr}(\left| R^{(k,1)}(\gamma^{(k)}(t) - \gamma^{(k)}(t)) \right|) \leq \epsilon$. Therefore, by the triangle inequalities, the choice of $\gamma^{(k)}_\epsilon$ and \((2.3)\), we get

$$\text{Tr}(\left| R^{(k,1)}(\gamma^{(k)}(t)) \right|) \leq \text{Tr}(\left| R^{(k,1)}(P^{(k)}_{a,A} \gamma^{(k)}(t)) \right|) + \text{Tr}(\left| R^{(k,1)}(1 - P^{(k)}_{a,A}) \gamma^{(k)}(t) \right|)$$

$$\leq A^{2k} \text{Tr}(\left| P^{(k)}_{a,A} \gamma^{(k)}(t) \right|) + \text{Tr}(\left| R^{(k,1)}(1 - P^{(k)}_{a,A}) \gamma^{(k)}(t) \right|) + \epsilon$$

$$\leq 0 + \sum_{\ell=1}^{L_x} \lambda_\ell \left| \text{Tr}(\left| \phi_\ell(t) \phi_\ell(t) \right| - \sum_{j=1}^{k} |P_{a,A,j} \phi_\ell(t) \phi_\ell(t)| P'_{a,A,j} \phi_\ell(t)| \right|$$

Expanding

$$\left| \phi_\ell(t) \phi_\ell(t) \phi_\ell(t) \right| - \sum_{j=1}^{k} \left| P_{a,A,j} \phi_\ell(t) \phi_\ell(t) P'_{a,A,j} \phi_\ell(t) \right|$$

$$= \left| (1 - P_{a,A,j}) \phi_\ell(t) \phi_\ell(t) \right| + \left| P_{a,A,j} \phi_\ell(t) \phi_\ell(t) (1 - P'_{a,A,j}) \phi_\ell(t) \right| + \cdots$$

$$+ \left| P_{a,A,j} \phi_\ell(t) \phi_\ell(t) \right| \left| \prod_{j=1}^{k-1} P'_{a,A,j} (1 - P'_{a,A,k}) \phi_\ell(t) \right|$$

and using the identity $\text{Tr}(\left| \phi \phi \psi \right|) = \| \phi \|_{L^2} \| \psi \|_{L^2}$ (which follows from $\| \phi \|_{L^2}^2 = \| \psi \phi \phi \psi \| = \| \phi \|_{L^2}^2 \| \psi \phi \|_{L^2}^2$, we prove that

$$\sum_{\ell=1}^{L_x} \lambda_\ell \left| \text{Tr}(\left| \phi_\ell(t) \phi_\ell(t) \phi_\ell(t) \right| - \sum_{j=1}^{k} \left| P_{a,A,j} \phi_\ell(t) \phi_\ell(t) P'_{a,A,j} \phi_\ell(t) \right|) \right|$$

$$\leq \sum_{\ell=1}^{L_x} \lambda_\ell \left\{ \| (1 - P_{a,A,j}) \phi_\ell(t) \|_{L^2_{a,k}} \| \phi_\ell(t) \|_{L^2_{a,k}} + \cdots \right\}$$
For notational convenience, we introduce the following notation:

\[ r \]

Here, for each \( \gamma \), the difference \( R^{(k,1)}_\gamma(t) \) is zero, in later sections, we write \( \gamma^{(k)}(t) \) as

\[ \gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{A}_{k,n}} \int_{D_{\sigma,t}} J^k(\ell_n; \sigma) dt_{\ell_n} \]

(see (2.10)), and decompose each \( J^k(\ell_n; \sigma) \) into a product of one distinguished factor and \((k - 1)\) regular factors, each of which is a one-particle kernel (see Section 4). Then, we recursively apply (A.1) to the distinguished factor, and (A.2) to the regular factors. To this end, we should take the \( H^{-1} \) norm for the distinguished factor and take the \( H^1 \) norm for the regular ones. We note that if one deals with \( \text{Tr}(|R^{(k,1)}_\gamma(t)|) \), one cannot give the \( H^{-1} \) norm to the distinguished factor. However, if one consider \( \text{Tr}(|P^{(k)}_A \gamma^{(k)}|) \), one can freely give either the \( H^1 \) norm or the \( H^{-1} \) norm to each factor, since \( \|P_{A,j}f\|_{L^2_{\tilde{J}^\gamma_{\tilde{x}}}^1} \) is comparable with both \( \|P_{A,j}f\|_{H^1_{\tilde{J}^\gamma_{\tilde{x}}}^1} \) and \( \|P_{A,j}f\|_{H^{-1}_{\tilde{J}^\gamma_{\tilde{x}}}^1} \) (up to constant multiple depending on \( a \) and \( A \)).

2.2. Duhamel expansion. To show (2.3), we first generate a Duhamel expansion as follows. By linearity, the difference \( \{\gamma^{(k)}(t)\}_{k \in \mathbb{N}} \), with \( \gamma^{(k)}(t) = \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t) \), also solves the GP (or Hartree) hierarchy with zero initial data. Hence, for each \( k \in \mathbb{N} \), \( \gamma^{(k)}(t) \) solves

\[ \gamma^{(k)}(t) = i\lambda \sum_{j=1}^{k} \int_{0}^{t} U^{(k)}(t-t_1)B_{j,k+2}(t_1)dt_1. \]  

(2.4)

Fix \( k \in \mathbb{N} \). Iterating the integral equation (2.4) \((n-1)\) times, we write

\[ \gamma^{(k)}(t) = (i\lambda)^{n} \int_{t_n \leq \cdots \leq t_1 \leq t} U^{(k)}(t-t_1)B_{k+2}\cdots U^{(k+2n-2)}(t_{n-1} - t_n)B_{k+2n}\gamma^{(k+2n)}(t_n)dt_1 \cdots dt_n. \]

(2.5)

Here, for each \( r \geq 1 \), the combined contraction operator is the sum of \( k + 2(r-1) \) many operators,

\[ B_{k+2r} := \sum_{j=1}^{k+2(r-1)} B_{j,k+2r-1,k+2r}. \]

For notational convenience, we introduce the following notation.

\[ U_{j,j'}^{(i)} := U^{(i)}(t_j - t_j'), \]

\[ \ell_n := (t, t_1, \cdots, t_n), \quad t_0 = t, \]

\[ J^k(\ell_n) := U_{0,1}^{(k)}B_{k+2}U_{1,2}^{(k+2)}B_{k+4}\cdots U_{n-1,n}^{(k+2n-2)}B_{k+2n}\gamma^{(k+2n)}(t_n). \]

Then \( \gamma^{(k)}(t) \) in (2.5) can be expressed in a compact form as

\[ \gamma^{(k)}(t) = (i\lambda)^n \int_{t_n \leq \cdots \leq t_1 \leq t} J^k(\ell_n)dt_{\ell_n}. \]

(2.6)

One may have observed that for fixed \( k \), the number of terms in \( J^k(\ell_n) \) is \( k(k+2)\cdots(k+2n-2) \sim O((2n)!)) \). This factorial growth on the number of Duhamel expansion terms is the first difficulty before we proceed with the proof of proposition 2.1. As a preparation, we will present a summary
of the combinatorial reduction process in section 2.3 to reduce $J^k(t_n)$ into a smaller number of terms that we can control.

2.3. Combinatorial reduction. In the celebrated works [23][24][25][26], Erdős-Schlein-Yau developed a sophisticated combinatorial arguments to reduce the number of Duhamel terms. Later, Klainerman and Machedon [36] rephrased this as a board game, which was extended to the quintic GP hierarchy by Chen-Pavlović in [8]. Since we will use the same arguments, we only present the notation and key reduction steps in this section. We refer the readers to [8] for the proofs of the related lemmas and theorems.

Let $\sigma$ be a map from $\{k+1, k+2, \cdots, k+2n-1\}$ to $\{1, 2, 3, \cdots, k+2n-2\}$ such that $\sigma(2) = 1$ and $\sigma(j) < j$ for all $j$. $\mathcal{M}_{k,n}$ denotes the set of all such mappings. Then we have that

$$J^k(t_n) = \sum_{\sigma \in \mathcal{M}_{k,n}} J^k(t_n; \sigma),$$

where

$$J^k(t_n; \sigma) = U^{(k)}_{0,1} B_{\sigma(k+1); k+1, k+2} U^{(k+2)}_{1,2} \cdots U^{(k+2n-2)}_{n-1,n} B_{\sigma(k+2n-1); k+2n-1, k+2n} (\gamma^{(k+2n)}(t_n))$$

is a basic term in $J^k(t_n)$.

Next, for each $\sigma \in \mathcal{M}_{k,n}$ there is a $(k+2n-1) \times n$ matrix corresponding to it. This matrix can be reduced to a special upper echelon matrix that corresponds to $\sigma_s$ via finite many so called acceptable moves. This transformation defines an equivalence relation among all the maps in $\mathcal{M}_{k,n}$. If $\sigma$ and $\sigma_s$ are equivalent, we denote this equivalence by $\sigma \sim \sigma_s$. From each equivalence classes, we pick one map that corresponds to a special upper echelon matrix, denote it by $\sigma_s$. Theorem 7.4 in [8] confirms that there is a subset $D_{\sigma_s, t} \subset [0, t]^n$, such that

$$\sum_{\sigma \sim \sigma_s} \int_0^t \cdots \int_0^{t_{n-1}} J^k(t_n; \sigma) dt_1 \cdots dt_n = \int_{D_{\sigma_s, t}} J^k(t_n; \sigma_s) dt_1 \cdots dt_n.$$

Hence we have a new formula for $\gamma^{(k)}(t)$

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}^*_{k,n}} \int_{D_{\sigma, t}} J^k(t_n; \sigma) dt_n,$$

where $\mathcal{M}^*_{k,n}$ is the union of all maps that correspond to special upper echelon matrices. By Lemma 7.3 of [8],

$$\#(\mathcal{M}^*_{k,n}) \leq 2^{k+3n-2}$$

2.4. Quantum de Finetti theorem. After decomposing $\gamma^{(k)}$ into a sum, we use the quantum de Finetti theorems to express each term in a factorized form. The quantum de Finetti theorem has a strong and weak version, and pertains to to bosonic density matrices that are either admissible or obtained as a weak-* limit, respectively. We state both the strong and weak versions [37] below to be used in section 2.3.

**Theorem 2.1** (Strong quantum de Finetti theorem). If a sequence $\{\gamma^{(k)}\}_{k \in \mathbb{N}}$ of bosonic density matrices on $L^2_{sym}(\mathbb{R}^3)$ is admissible, then there exists a unique Borel probability measure $\mu$, supported on the unit sphere $S \subset L^2(\mathbb{R}^3)$ and invariant under multiplication of $\phi \in L^2(\mathbb{R}^3)$ by complex numbers of modulus one, such that

$$\gamma^{(k)} = \int d\mu(\phi) (|\phi \times \phi|)^{\otimes k}, \quad k \in \mathbb{N}.$$

The multiplier $2^{k+3n-2}$ is affordable to us, since it can be absorbed by $(CM)^n$ for sufficiently small $M > 0$. 

Theorem 2.2 (Weak quantum de Finetti theorem). If a sequence \( \{ \gamma^{(k)} \}_{k \in \mathbb{N}} \) of bosonic density matrices on \( L^2_{sym}(\mathbb{R}^{3k}) \) is a limiting hierarchy, then there exists a unique Borel probability measure \( \mu \), supported on the unit ball \( \mathcal{B} \subset L^2(\mathbb{R}^3) \) and invariant under multiplication of \( \phi \in L^2(\mathbb{R}^3) \) by complex numbers of modulus one, such that (2.12) holds.

There are different formulations of these theorems that are used in different settings. The formulation for density matrices was presented in a paper Lewin, Nam and Rougerie [37], and in a paper by Ammari and Nier [2]. For additional results related the de Finetti theorems, we refer the reader to Diaconis and Freedman [22], Hudson and Moody [33], and Stormer [42].

Applying the strong or the weak quantum de Finetti theorem to \( \gamma_1^{(k)}(t) \) and \( \gamma_2^{(k)}(t) \), we write

\[
\gamma_i^{(k)}(t) = \int dt \tilde{\mu}_t^{(i)}(|\phi \rangle \langle \phi|)^{\otimes k}, \quad i = 1, 2.
\]

Then, by linearity,

\[
\gamma^{(k)}(t) = \int dt \tilde{\mu}_t(|\phi \rangle \langle \phi|)^{\otimes k}, \quad \forall k \in \mathbb{N},
\]

where \( \tilde{\mu}_t = \mu^{(1)}_t - \mu^{(2)}_t \). Plugging (2.13) into \( J^k(t_n; \sigma) \) in the reduced Duhamel expansion (2.8), we obtain

\[
\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}_{k,n}} \int_{D_{\sigma,t}} dt \int dt \tilde{\nu}_n(\phi)J^k(t_n; \sigma).
\]

where

\[
J^k(t_n; \sigma) = U_{0,1}^{(k)} B_{\sigma(k+1):k+1,k+2} U_{1,2}^{(k+2)} \cdots U_{n-1,n}^{(k+2n-2)} B_{\sigma(k+2n-1):k+2n-1,k+2n}(|\phi \rangle \langle \phi|)^{(k+2n)}.
\]

We remark that \( J^k(t_n; \sigma) = J^k(t_n; \sigma; \bar{x}_k; \bar{x}'_k) \) depends on \( \bar{x}_k, \bar{x}'_k \). We omit the spatial variables for simplicity. We note that each factor in

\[
(|\phi \rangle \langle \phi|)^{(k+2n)}(\bar{x}_{k+2n}; \bar{x}'_{k+2n}) = \prod_{i=1}^{k+2n} (|\phi \rangle \langle \phi|)(x_i; x'_i)
\]

is a one-particle kernel, and that we can further decompose \( J^k(t_n; \sigma) \) as

\[
J^k(t, t_1, \ldots, t_n; \sigma; \bar{x}_k; \bar{x}'_k) = \prod_{j=1}^k J_j^k(t, t_{ij,1}, \ldots, t_{ij,n}; \sigma; x_j; x'_j).
\]

To better explain the reduction procedure, we present an example in section 3 and then go back to the general case in section 4.

3. Example Factorization

Consider \( k = 2, n = 4 \), and \( \rho \) a permutation of \( \{1, 2, \ldots, n\} \). The map \( \sigma \) is represented by the following upper echelon matrix (each highlighted entry in a row is to the left of each highlighted entry in a lower row)

\[
\begin{pmatrix}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 3 & 4 & 2 \\
2 & 3 & 4 & 1 \\
0 & 1 & 2 & 3
\end{array}
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{pmatrix}
\]

(3.1)
Then, we have

\[ J^2(\ell_4; \sigma) = U^{(2)}_{0,1}B_{1;3,4}U^{(4)}_{1;2}B_{2;5,6}U^{(6)}_{2;3}B_{4;7,8}U^{(8)}_{3,4}B_{4,9,10}. \] (3.2)

We will organize the terms in expansion of \( J^2(\ell_4; \sigma) \) into two one-particle density matrices by examining the effect of the contraction operators starting with the last one on the RHS of (3.2). We denote each factor in the last term \( (|\phi\rangle \langle \phi|)^{\otimes 10} \) by \( u_i \), ordered by increasing index \( i \), so that \( (|\phi\rangle \langle \phi|)^{\otimes 10} = \otimes_{i=1}^{10} u_i \).

First of all, in (3.2), the last interaction operator \( B_{4,9,10} \) contracts the factor \( u_4, u_9 \) and \( u_{10} \), and leaves all other factors unchanged.

\[ B_{4,9,10}(\otimes_{i=1}^{10} u_i) = u_1 \otimes u_2 \otimes u_3 \otimes \Theta_4 \otimes u_5 \cdots \otimes u_8, \] (3.3)

where

\[ \Theta_4 := B_{1;2,3}(u_4 \otimes u_9 \otimes u_{10}). \]

The index \( \alpha \) in \( \Theta_\alpha \) associates \( \Theta_\alpha \) to the \( \alpha \)-th interaction operator from the left in (3.2). Since we only run the expansion to the \( n \)-th level, we have \( 1 \leq \alpha \leq n \). In this specific case, \( n = 4 \), and the 4th interaction operator is \( B_{4,9,10} \).

Next, \( B_{4,7,8} \) contracts \( U^{(8)}_{3,4}\Theta_4, U^{(8)}_{3,4}u_7 \) and \( U^{(8)}_{3,4}u_8 \).

\[ B_{4,7,8}U^{(8)}_{3,4}(3.3) = (U^{(3)}_{3,4}(u_1 \otimes u_2 \otimes u_3)) \otimes \Theta_3 \otimes (U^{(2)}_{3,4}(u_5 \otimes u_6)), \] (3.4)

where

\[ \Theta_3 := B_{1;2,3}(U^{(1)}_{3,4}\Theta_4) \otimes (U^{(1)}_{3,4}u_7) \otimes (U^{(1)}_{3,4}u_8). \]

Then, by the semigroup property, \( U^{(i)}_{2,3}U^{(i)}_{3,4} = U^{(i)}_{2,4} \). The operator \( B_{2,5,6} \) contracts \( U^{(1)}_{2,4}u_2, U^{(1)}_{2,4}u_5 \) and \( U^{(1)}_{2,4}u_6 \), which correspond to the 2nd, 5th, and 6th factors in (3.4). The other factors are left invariant.

\[ B_{2,5,6}U^{(6)}_{2,3}(3.4) = (U^{(1)}_{2,4}u_1) \otimes \Theta_2 \otimes (U^{(1)}_{2,4}u_3) \otimes (U^{(1)}_{2,3}\Theta_3), \] (3.5)

where

\[ \Theta_2 = B_{1;2,3}(U^{(3)}_{2,4}(u_2 \otimes u_5 \otimes u_6)). \]

Finally, \( B_{1,3,4} \) contracts \( U^{(1)}_{1,4}u_1, U^{(1)}_{1,4}u_3, \) and \( U^{(1)}_{1,3}\Theta_3 \) and leaves other factors unchanged.

\[ B_{1,3,4}U^{(4)}_{1,2}(3.5) = \Theta_1 \otimes (U^{(1)}_{1,2}\Theta_2), \] (3.6)

where

\[ \Theta_1 = B_{1;2,3}(U^{(1)}_{1,4}u_1) \otimes (U^{(1)}_{1,4}u_3) \otimes (U^{(1)}_{1,3}\Theta_3)). \]

Therefore, \( J^2 \) can be factorized as

\[ J^2 = (U^{(1)}_{0,1}\Theta_1) \otimes (U^{(1)}_{0,2}\Theta_2) := J^1_1 \otimes J^1_2. \] (3.7)

Now \( J^2 \) in (3.7) has two factors \( J^1_j \) (note \( j \leq k = 2 \)), which are 1-particle matrices. The reason we have such a decomposition is that \( B_{\sigma_i(r),r,r+1} \) only affects three \( u_i \) each time, and as the contraction processes, all the \( u_i \) might be divided into different groups.

For \( j = 1 \), after replacing back \( u_i = |\phi\rangle \langle \phi|, \) \( i \leq k + 2n = 10 \), we have

\[ J^1_1 = U^{(1)}_{0,1}B_{1,2,3}U^{(2)}_{1,3}B_{3,4,5}U^{(3)}_{3,4}B_{3,6,7}(|\phi\rangle \langle \phi|)^{\otimes 7} \] (3.8)
where we relabel the index in operators $B_{\sigma(r);r,r+1}$ such that the interaction operators in (3.8) correspond to $B_{1;3,4}, B_{4;7,8}, B_{4;9,10}$ respectively, and leave the connectivity structure among them unchanged. The labeling of function $\sigma_1$ (see the notation in (2.16)) takes values $\sigma_1(2) = 1, \sigma_1(4) = 3,$ and $\sigma_1(6) = 3$.

For $j = 2$, we perform the relabeling in the same spirit find that

$$J^1_2 = U^{(1)}_{0,2} B_{1;2,3} U^{(3)}_{2,4} (|\phi\rangle\langle\phi|)^{\otimes 3},$$

where $\sigma_2(2) = 1$.

We note that for any $\ell < \ell'$, the interaction operators $B_{\sigma(\ell);\ell,\ell+1}$ and $B_{\sigma(\ell');\ell',\ell'+1}$ in $J^2$ (which are highlighted in (3.1)) belong to the same factor $J^1_1$ if either $\sigma(\ell) = \sigma(\ell')$ or $\sigma(\ell') = \ell$. In such cases, we consider them as being connected. This connectivity structure is exactly the key point of the Duhamel terms that we want to illustrate using tree graphs. We include the detailed definitions and descriptions in section 4.

We further note that each $\sigma_j$ can be viewed as the restriction of $\sigma$ to $J^1_j$. We call factors that have a free propagator applied to each $\phi$ (like $J^2_2$) regular, and factors that have the contractions of $(|\phi\rangle\langle\phi|)^{\otimes 3}$ without free propagator in between (like $J^1_1$) distinguished.

4. Tree graphs for the general case

4.1. The tree graphs. We begin by recalling that, from (2.15), $J^k$ is given by

$$J^k(t_n;\sigma) = U^{(k)}_{0,1} B_{\sigma(k+1);k+1,k+2} U^{(k+2)}_{1,2} \cdots U^{(k+2n-2)}_{n-1,n} B_{\sigma(k+2n-1);k+2n-1,k+2n} (|\phi\rangle\langle\phi|)^{\otimes 2n}.$$

where

$$(|\phi\rangle\langle\phi|)^{\otimes 2n} (x_1; x_2, \ldots, x_{2n}) = \prod_{i=1}^{2n} (|\phi\rangle\langle\phi|)(x_i; x_i')$$

is a product of one-particle kernels. Contracting by interaction operators $B_{\sigma(r);r,r+1}$, we decompose

$$J^k(t, t_1, \ldots, t_n; \sigma; x_k; x'_k) = \prod_{j=1}^k J^1_j(t, t_{\ell_j,1}, \ldots, t_{\ell_j,m_j}; \sigma; x_j; x'_j)$$

into a product of one-particle kernels $J^1_j$. We associate to this decomposition $k$ disjoint tree graphs $\tau_1, \tau_2, \ldots, \tau_k$. These graphs appear as skeleton graphs in (23, 24, 25, 26). As in [6, 32], we assign root, internal, and leaf vertices to each tree $\tau_j$.

- A root vertex labeled as $W_j$, $j = 1, 2, \ldots, k$, to represent $J^1_j(x_j; x'_j)$.
- An internal vertex labeled by $v_\ell$, $\ell = 1, 2, \ldots, n$, corresponding to $B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell}$ and attached to the time variable $t_\ell$.
- A leaf vertex $u_i$, $i = 1, 2, \ldots, k + 2n$, representing each factor $(|\phi\rangle\langle\phi|)(x_i; x'_i)$.

Next, we connect the vertices with edges, as described below.

- If $v_\ell$ is the smallest value of $\ell$ such that $\sigma(k + 2\ell - 1) = j$, then we connect $v_\ell$ to the root vertex $W_j$ and write $W_j \sim v_\ell$ (or equivalently $W_j \sim B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell}$). If there is no internal vertex connected to a root vertex $W_j$, then we connect $W_j$ to the leaf $u_j$, and write $W_j \sim u_j$.
- For any $1 < \ell \leq n$, if $3\ell > l$ such that $\sigma(k+2\ell-1) = \sigma(k+2\ell'-1)$ or $\sigma(k+2\ell'-1) = k+2\ell-1$, then we connect $v_\ell$ and $v_{\ell'}$ and write $v_\ell \sim v_{\ell'}$ (or equivalently $B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell} \sim B_{\sigma(k+2\ell'-1);k+2\ell'-1,k+2\ell'}$). In this case, we call $v_\ell$ the parent vertex of $v_{\ell'}$, and $v_{\ell'}$ the child
We call the tree vertex, and each internal vertex has exactly three child vertices (which can be either internal and connected to ).

Clearly, there are notation, we refer to the vertex with vertices with .

The distinguished one particle kernel is the form.

A sample tree graph is given in Figure 1, for . Each tree corresponds to a one-particle kernel in the example in section where .

We remark that it follows from the construction above that each root vertex has only one child vertex, and each internal vertex has exactly three child vertices (which can be either internal and leaf). We call the tree distinguished if and regular if . The three leaves connected to are called distinguished leaf vertices, and all other leaves are called regular leaf vertices. Clearly, there are regular trees and one distinguished tree in each tree graph.

A sample tree graph is given in Figure 1 for as in equation (3.2). Each tree has root vertex , for . The leaf vertices and the internal vertices are distinguished. is the distinguished tree, and is drawn with thick edges. Tree with vertices is the regular tree, and is drawn with thin edges.

4.2. The distinguished one particle kernel . Let denote the distinguished tree graph. It has internal vertices and leaf vertices . We enumerate the internal vertices with and the leaf vertices with . To simplify notation, we refer to the vertex by its label . We observe that has the form

\[
J_j^1(t, t_{\ell,1},\ldots,t_{\ell,j,m_j};\sigma_j) = U^{(1)}(t-t_1)\cdots U^{(1)}(t_{\ell,1,1}-t_{\ell,1})B_{\sigma_j(2);2,3}\cdots \]

\[
\cdots B_{\sigma_j(2\alpha-2);2\alpha-2,2\alpha-1}U^{(2\alpha-1)}(t_{\ell,\alpha,1}-t_{\ell,\alpha,1})\cdots U^{(2\alpha-1)}(t_{\ell,\alpha,1}-t_{\ell,\alpha})B_{\sigma_j(2\alpha);2\alpha,2\alpha+1}\cdots \]

\[
\cdots U^{(2m_j-1)}(t_{\ell,j,m_j}-t_{\ell,j,m_j})B_{\sigma_j(2m_j),2m_j,2m_j+1}(\phi)\delta(2m_j+1).\]

\[\text{Figure 1. An example tree graph for } J^k. \text{ It is a disjoint union of two trees } \tau_1 \text{ and } \tau_2 \text{ with root vertices } W_1 \text{ and } W_2, \text{ respectively. Each tree corresponds to a one-particle kernel in the example in section 3 where } k = 2 \text{ and } n = 4.\]
By the semigroup property
\[ U^{(\alpha)}(t)U^{(\alpha)}(s) = U^{(\alpha)}(t + s), \]
and the fact that \( \sigma_j(2) = 1 \), \([4,2]\) reduces to
\[ J_j^1(t, t_{\ell_j,1}, \ldots, t_{\ell_j,m_j}; \sigma_j) = U^{(1)}(t - t_{\ell_j,1})B_{1;2,3} \cdots \]
\[ \cdots B_{\sigma_j(2\alpha-2);2\alpha-2,2\alpha-1}U^{(2\alpha-1)}(t_{\ell_j,\alpha-1} - t_{\ell_j,\alpha})B_{\sigma_j(2\alpha);2\alpha,2\alpha+1} \cdots \]
\[ \cdots U^{(2m_j-1)}(t_{\ell_j,m_j-1} - t_{\ell_j,m_j})B_{\sigma_j(2m_j);2m_j,2m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(2m_j+1)}, \]
where \( \ell_j,m_j = n \).

### 4.3. Definition of the kernels \( \Theta_\alpha \) at the vertices of the distinguished tree graph.

In this section, we proceed as in \([6]\), and recursively assign a kernel \( \Theta_\alpha \) to each vertex \( \alpha \) of the distinguished tree graph. The kernels at the vertices of the regular tree graph are defined similarly. We begin by assigning the kernel
\[ \Theta_\alpha(x; x') := \phi(x)\overline{\phi}(x') \]
to the leaf vertex with label \( \alpha \in \{m_j + 1, \ldots, 3m_j + 1\} \).

Next, we determine \( \Theta_{m_j} \) at the distinguished vertex \( \alpha = m_j \) from the term on the last line of \((4.3)\), given by
\[ B_{\sigma_j(2m_j);2m_j,2m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(2m_j+1)} = (|\phi\rangle\langle\phi|)^{\otimes(\sigma_j(2m_j)-1)} \otimes \Theta_{m_j} \otimes (|\phi\rangle\langle\phi|)^{\otimes(2m_j+1-\sigma_j(2m_j)-2)} \]
where
\[ \Theta_{m_j}(x; x') := \tilde{\psi}(x)\overline{\phi}(x') - \phi(x)\tilde{\psi}(x') \]
with \( \tilde{\psi} := |\phi|^4\phi \). It is obtained from contracting three copies of \( |\phi\rangle\langle\phi| \) at the three leaf vertices \( \kappa_-(m_j), \kappa(m_j), \kappa_+(m_j) \) which have \( m_j \) as their parent vertex.

Now we are ready to begin the induction. Let \( \alpha \in \{1, \ldots, m_j - 1\} \). Suppose that the kernels \( \Theta_{\alpha'} \) have been determined for all \( \alpha' > \alpha \). We let \( \kappa_-(\alpha), \kappa(\alpha), \kappa_+(\alpha) \) label the three child vertices (of internal or leaf type) of \( \alpha \). Since \( \Theta_{\kappa_-(\alpha)}, \Theta_{\kappa(\alpha)}, \) and \( \Theta_{\kappa_+(\alpha)} \) have already been determined, we can now define
\[ \Theta_\alpha(x; x') \]
\[ = B_{1;2,3}((U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)})\Theta_{\kappa_-(\alpha)}) \otimes (U^{(1)}(t_\alpha - t_{\kappa(\alpha)})\Theta_{\kappa(\alpha)}) \otimes (U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)}))(x; x'). \]

The induction ends when we obtain the kernel \( \Theta_1 \) at \( \alpha = 1 \).

### 4.4. Key properties of the kernels \( \Theta_\alpha \).

As in \([6]\), we observe that the kernels \( \Theta_\alpha \) satisfy the following properties.

- \( \Theta_\alpha \) can be written as a sum of differences of factorized kernels
\[ \Theta_\alpha(x; x') = \sum_{\beta_\alpha} c_{\beta_\alpha}^{\alpha} x_{\beta_\alpha}(x)\overline{\psi}_{\beta_\alpha}(x') \]
with at most \( 2^{m_j-\alpha} \) nonzero coefficients \( c_{\beta_\alpha}^{\alpha} \in \{1, -1\} \).
The product $\chi_{\beta_\alpha}(x)\overline{\psi}_{\beta_\alpha}(x')$ in (4.5) above is either of the form

$$\chi_{\beta_\alpha}(x)\overline{\psi}_{\beta_\alpha}(x') = (U_{\alpha;\kappa_-(\alpha)}\chi_{\beta_{\kappa_-(\alpha)}}(x)(U_{\alpha;\kappa_-(\alpha)}\psi_{\beta_{\kappa_-(\alpha)}}(x'))$$

or

$$\chi_{\beta_\alpha}(x)\overline{\psi}_{\beta_\alpha}(x') = (U_{\alpha;\kappa_-(\alpha)}\chi_{\beta_{\kappa_-(\alpha)}}(x)(U_{\alpha;\kappa_-(\alpha)}\psi_{\beta_{\kappa_-(\alpha)}}(x'))$$

for some values of $\beta_{\kappa_-(\alpha)}, \beta_{\kappa_+(\alpha)}, \beta_{\kappa_+(\alpha)}$ that depend on $\beta_\alpha$. The trilinear operator $A$ defines as

$$A[V_x, f, g](x) := \int \int V_x(x - y_1, x - y_2)f(y_1)g(y_2) dy_1 dy_2. \quad (4.8)$$

Observe that above, the function $\chi_{\beta_\alpha}$ is either of the quintic form

$$\chi_{\beta_\alpha}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi_{\beta_{\kappa_-(\alpha)}}(x)$$

or the linear form

$$\chi_{\beta_\alpha}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi_{\beta_{\kappa_-(\alpha)}}(x). \quad (4.11)$$

Accordingly, $\psi_{\beta_\alpha}^\alpha$ respectively is either of linear or quintic form, and the product $\chi_{\beta_\alpha}(x)\overline{\psi}_{\beta_\alpha}(x')$ always has sextic form (4.6) or (4.7).

- We call the functions $\chi_{\beta_\alpha}^\alpha, \psi_{\beta_\alpha}^\alpha$ in the sum (4.5) **distinguished** if they are a function of $|\phi|^4\phi$. In the product on the right hand side of (4.6), respectively (4.7), at most one of the six factors is distinguished. Indeed, this is true for all regular leaf vertices, and for the distinguished vertex (4.4). By induction along decreasing values of $\alpha$, it is also true for the internal vertices.

As in (6), we make the following assumption, which simplifies the notation without loss of generality.

**Hypothesis 4.1.** We assume that only the functions $\psi_{\beta_\alpha}^\alpha$ and $(\psi_{\beta_\alpha}^{\kappa_+(1)})^q$ are distinguished, where we define

$$\kappa_+(1) := \kappa_+(\kappa_+(\ldots(\kappa_+(1)) \ldots)).$$

...
5. Proof of Proposition 2.1

In this section, we prove Proposition 2.1. To simplify notation, we denote the time variable $t_{j,\alpha}$ by $t_{\alpha}$. We denote the subtree of $\tau_j$ with root at the vertex $\alpha$ by $\tau_{j,\alpha}$, and let

$$
\int \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] := \int_{[0,T)^{d_{\alpha}}} \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right]
$$

be integration with respect to all time variables attached to the internal and root vertices of the subtree $\tau_{j,\alpha}$. Here, the total number of internal and root vertices of the tree $\tau_{j,\alpha}$ is denoted by $d_{\alpha}$.

**Lemma 5.1.** For $V_\infty \in L^{\frac{1}{1+\epsilon}}(\mathbb{R}^0)$ with small $\epsilon \geq 0$ (or $V_\infty(y, z) = \lambda \delta_0(y) \delta_0(z)$ with $\epsilon = 0$ and $\|V_\infty\|_{L^1} := \lambda$), we have the $\dot{H}^{-1}$ bound

$$
\int \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \left\| \psi_{\beta_\alpha}^\alpha \right\|_{\dot{H}^{-1}} \left\| \chi_{\beta_\alpha}^\alpha \right\|_{\dot{H}^1} 
\leq C T^{3\epsilon} \left\| V_\infty \right\|_{L^{\frac{1}{1+\epsilon}}} \int \left[ \prod_{\alpha' \in \tau_{j,\kappa_{-\alpha}}} dt_{\alpha'} \right] \left\| \psi_{\beta_{\kappa_{-\alpha}}}^\kappa \right\|_{\dot{H}^{-1}} 
\left\| \chi_{\beta_{\kappa_{-\alpha}}}^\kappa \right\|_{\dot{H}^1} 
\cdot \int \left[ \prod_{\alpha' \in \tau_{j,\kappa_{+\alpha}}} dt_{\alpha'} \right] \left\| \psi_{\beta_{\kappa_{+\alpha}}}^\kappa \right\|_{\dot{H}^{-1}} 
\left\| \chi_{\beta_{\kappa_{+\alpha}}}^\kappa \right\|_{\dot{H}^1}
$$

and the $\dot{H}^1$ bound

$$
\int \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \left\| \psi_{\beta_\alpha}^\alpha \right\|_{\dot{H}^1} \left\| \chi_{\beta_\alpha}^\alpha \right\|_{\dot{H}^1} 
\leq C T^{3\epsilon} \left\| V_\infty \right\|_{L^{\frac{1}{1+\epsilon}}} \int \left[ \prod_{\alpha' \in \tau_{j,\kappa_{-\alpha}}} dt_{\alpha'} \right] \left\| \psi_{\beta_{\kappa_{-\alpha}}}^\kappa \right\|_{\dot{H}^1} 
\left\| \chi_{\beta_{\kappa_{-\alpha}}}^\kappa \right\|_{\dot{H}^1} 
\cdot \int \left[ \prod_{\alpha' \in \tau_{j,\kappa_{+\alpha}}} dt_{\alpha'} \right] \left\| \psi_{\beta_{\kappa_{+\alpha}}}^\kappa \right\|_{\dot{H}^1} 
\left\| \chi_{\beta_{\kappa_{+\alpha}}}^\kappa \right\|_{\dot{H}^1}.
$$

**Proof.** To prove (5.1), we apply the bound (A.3) (or (A.1)) to (4.6) and (4.7) and obtain

$$
\int \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \left\| \psi_{\beta_\alpha}^\alpha \right\|_{\dot{H}^{-1}} \left\| \chi_{\beta_\alpha}^\alpha \right\|_{\dot{H}^1} 
\leq C T^{3\epsilon} \left\| V_\infty \right\|_{L^{\frac{1}{1+\epsilon}}} \int_{[0,T)^{d_{\alpha}}} \left[ \prod_{\alpha' \in \tau_{j,\kappa_{-\alpha}}} \right] \left[ \prod_{\alpha' \in \tau_{j,\kappa_{+\alpha}}} dt_{\alpha'} \right] \left\| \psi_{\beta_{\kappa_{-\alpha}}}^\kappa \right\|_{\dot{H}^{-1}} 
\left\| \chi_{\beta_{\kappa_{-\alpha}}}^\kappa \right\|_{\dot{H}^1} 
\cdot \left\| \psi_{\beta_{\kappa_{+\alpha}}}^\kappa \right\|_{\dot{H}^{-1}} 
\left\| \chi_{\beta_{\kappa_{+\alpha}}}^\kappa \right\|_{\dot{H}^1} 
\left\| \psi_{\beta_{\kappa_{-\alpha}}}^\kappa \right\|_{\dot{H}^{-1}} 
\left\| \chi_{\beta_{\kappa_{+\alpha}}}^\kappa \right\|_{\dot{H}^1}.
$$
Proposition 5.2. For the distinguished tree $\tau_j$, we have the bound
\[
\int_{[0,T]^{m_j-1}} dt_1 \ldots dt_{m_j-1} \text{Tr} \left( \left| P_{a,A}^{(1)} J_j^1(t,t_1,\ldots,t_{m_j};\sigma_j) \right| \right) \leq \sqrt{A/a} 2^{m_j} C^{m_j-1} T^{3\epsilon(m_j-1)} V \| t_{m_j-1} \|_{H^{1/2}} \| \phi \|_{H^1}^{4 m_j-3} \| A[V_x,|\phi|^2,|\phi|^2] \phi \|_{H^{-1}}. \quad (5.3)
\]

Proof.
\[
\int_{[0,T]^{m_j-1}} dt_1 \ldots dt_{m_j-1} \text{Tr} \left( \left| P_{a,A}^{(1)} J_j^0(t,t_1,\ldots,t_{m_j};\sigma_j) \right| \right) = \sum_{\beta_1} \int_{[0,T]^{m_j-1}} dt_1 \ldots dt_{m_j-1} \| P_{a,A} \psi_{\beta_1}^1 \|_{L^2} \| P_{a,A}^1 \chi_{\beta_1}^1 \|_{L^2} \leq \sqrt{A/a} \sum_{\beta_1} \int_{[0,T]^{m_j-1}} dt_1 \ldots dt_{m_j-1} \| \psi_{\beta_1}^1 \|_{H^{-1}} \| \chi_{\beta_1}^1 \|_{H^1} \leq \sqrt{A/a} \sum_{\beta_1} \left( C T^{3\epsilon} \| V_x \|_{L^{1/2}} \int_{[0,T]^{d_{\kappa_1}(a)}} dt_\alpha' \| \chi_{\beta_1}^\kappa(\alpha) \|_{H^1} \| \psi_{\beta_1}^\kappa(\alpha) \|_{H^{-1}} \right) (5.4)
\]
Proof.

From (2.14) and (2.11), we have

\[ \int_{[0,T)} d\tau \left[ \prod_{\alpha' \in \tau_{j,\kappa(\alpha)}} dt_{\alpha'} \right] \| X_{\beta_{\kappa(\alpha)}} \|_{H^1} \| \psi_{\beta_{\kappa(\alpha)}} \|_{H^1} \]  

(5.5)

\[ \int_{[0,T)} d\tau \left[ \prod_{\alpha' \in \tau_{j,\kappa(\alpha)}} dt_{\alpha'} \right] \| X_{\beta_{\kappa(\alpha)}} \|_{H^1} \| \psi_{\beta_{\kappa(\alpha)}} \|_{H^1} \]  

(5.6)

In the last step, we performed the \( t_1 \) integral using (5.1). Now, to bound (5.4) and (5.5), we iterate the \( H^1 \) bound (5.2). To bound (5.6), we iterate both (5.1) and (5.2). This establishes (5.3). \( \Box \)

Proposition 5.3. For the regular tree \( \tau_j \), we have the bound

\[ \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \text{Tr} \left( \left| P_{a,A,1} J_1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \leq \frac{2^{m_j} C m_j T^{3m_j}}{a} \| \phi \|_{H^1}^{4m_j+1}. \]  

(5.7)

Proof.

\[ \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \text{Tr} \left( \left| P_{a,A,1} J_1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \]

\[ = \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \text{Tr} \left( \left| P_{a,A} U^{(1)}(t - t_1) I_1 \right| \right) \]

\[ \leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} P_{a,A,1} \psi_{\beta_1}^1 \| L_2^2 \| P_{a,A,1} \chi_{\beta_1}^1 \| L_2^1 \]

\[ \leq \frac{1}{a} \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \| \psi_{\beta_1}^1 \|_{H^1} \| \chi_{\beta_1}^1 \|_{H^1} \]

From here, we iterate the \( H^1 \) bound (5.2) to obtain (5.7). \( \Box \)

Lemma 5.4. Suppose that \( V_{\infty} \in L^{1+\epsilon} \). Then

\[ \| A[V_{\infty}, \phi^2, \phi^2] \phi \|_{H^{-1}} \leq \begin{cases} \| V_{\infty} \|_{L^{1+\epsilon}} \| \phi \|_{H^1}^5, & \text{if } \epsilon = 0, \\ \| V_{\infty} \|_{L^{1+\epsilon}} \| \phi \|_{H^1}^5, & \text{if } \epsilon > 0. \end{cases} \]

Notice that when \( \epsilon > 0 \), we measure the norm of \( \phi \) in the non-homogeneous Sobolev space \( H^1 \).

Proof. By Strichartz estimates, Sobolev embedding, and Theorem A.1 we have

\[ \| A[V_{\infty}, \phi^2, \phi^2] \phi \|_{H^{-1}} \leq \| A[V_{\infty}, \phi^2, \phi^2] \phi \|_{L^\infty}^5 \]

\[ \leq \| A[V_{\infty}, \phi^2, \phi^2] \phi \|_{L^2}^5 \| \phi \|_{L^6} \]

\[ \leq \| V_{\infty} \|_{L^{1+\epsilon}} \| \phi \|_{L^2}^5 \| \phi \|_{L^6} \]

\[ = \| V_{\infty} \|_{L^{1+\epsilon}} \| \phi \|_{L^6}^5 \]

\[ \leq \begin{cases} \| V_{\infty} \|_{L^{1+\epsilon}} \| \phi \|_{H^1}^5, & \text{if } \epsilon = 0, \\ \| V_{\infty} \|_{L^{1+\epsilon}} \| \phi \|_{H^1}^5, & \text{if } \epsilon > 0. \end{cases} \]

\( \Box \)

We are now ready to conclude the proof of Proposition 2.1

Proof of Proposition 2.1. From (2.14) and (2.11), we have

\[ \text{Tr} \left| P_{a,A}^{(k)} \right| \leq 2 \cdot 2^{k+3n-2} \sup_{\sigma \in M_{k,n}} \sup_{i=1,2} \int_{[0,T)^n} dt_{i,n} \int d\mu_{t_n}(\phi) \text{Tr}(\left| P_{a,A} J^k(t_n; \sigma) \right|). \]  

(5.8)
Recall from (4.1) that $J^k$ can be decomposed into a product of $k$ one-particle kernels

$$J^k(t, t_1, \ldots, t_n; \sigma) = \prod_{j=1}^{k} J^1_j(t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; \sigma_j),$$

where only one of the factors $J^1_j$ distinguished. It now follows from Propositions 5.2 and 5.3 that

$$\int_{[0,T]^{n-1}} dt_1 \cdots dt_{n-1} \text{Tr}\left(\left| P_{a,A}^k J^k(t, t_1, \ldots, t_n; \sigma) \right| \right) = \int_{[0,T]^{n-1}} dt_1 \cdots dt_{n-1} \prod_{j=1}^{k} \text{Tr}\left(\left| P_{a,A}^{(1)} J^1_j(t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; \sigma_j) \right| \right) \leq (\max\{\sqrt{A/a}, 1/a\})^k \frac{2^n}{C} \frac{C^n}{T^{3k(n-1)}} \|V\|^{-1} \|\phi\|_{H^1}^{4(k+n)-5} \|A[V_{\infty}, |\phi|, |\phi|^2]\|_{H^{-1}}^{-1} \tag{5.9}$$

$$\leq (\max\{\sqrt{A/a}, 1/a\})^k \left(\frac{C}{T^{3k}}\|V\|^{-1} \|\phi\|_{H^1}^{4(k+n)-5} \|A[V_{\infty}, |\phi|, |\phi|^2]\|_{H^{-1}}^{-1} \right) \times \int_0^T dt_n \int d\mu_{n}^{(i)}(\phi) \|\phi\|_{H^1}^{4(k+n)-5} \|A[V_{\infty}, |\phi|, |\phi|^2]\|_{H^{-1}}^{-1}.$$

For Proposition 2.1 (i), we note that by the assumption (from Theorem 1.1), $\mu^{(1)}$ and $\mu^{(2)}$ are supported in $\{\phi \in L^2 : \|\phi\|_{H^1} < M\}$. Indeed, since

$$\text{Tr}(|R^{(k,1)}(\gamma^{(1)}(\lambda))|) = \int d\mu^{(i)}(\phi) \|\phi\|_{H^1}^{2k} < M^{2k}, \quad \forall k \in \mathbb{N},$$

it follows from the Chebyshev's inequality that if $\lambda > M$, then

$$\mu^{(i)}\left(\{\phi \in L^2 : \|\phi\|_{H^1} > \lambda\}\right) \leq \frac{1}{\lambda^{2k}} \int d\mu(\phi) \|\phi\|_{H^1}^{2k} \leq \frac{M^{2k}}{\lambda^{2k}} \to 0 \quad \text{as } k \to \infty.$$

Therefore, by (5.9) with $\epsilon = 0$ and Lemma 5.4, we obtain

$$\int_{[0,T]^n} dt_n \int d\mu_{n}^{(i)}(\phi) \text{Tr}\left(|P_{a,A}^k J^k(t_n; \sigma)\right) \leq (\max\{\sqrt{A/a}, 1/a\})^k \left(\frac{C}{T^{3k}}\|V\|^{-1} \|\phi\|_{H^1}^{4(k+n)} \right) \int_0^T dt_n \int d\mu_{n}^{(i)}(\phi) \|\phi\|_{H^1}^{4(k+n)} \leq (\max\{\sqrt{A/a}, 1/a\})^k \left(\frac{C}{T^{3k}}\|V\|^{-1} \|\phi\|_{H^1}^{4(k+n)} \right) TM^{4(k+n)}.$$

Inserting this bound to (5.8), we prove that

$$\text{Tr}|P_{a,A}^k| \to 0 \text{ as } n \to \infty \tag{5.10}$$

provided that $M$ is sufficiently small. Thus, we conclude that $\text{Tr}|P_{a,A}^k(\gamma^{(k)})| = 0$.

For Proposition 2.1 (ii), as above, we observe that $\mu^{(1)}$ and $\mu^{(2)}$ are supported in $\{\phi \in L^2 : \|\phi\|_{H^1} < M\}$. Then, by (5.9) with $\epsilon > 0$ and Lemma 5.4, we get

$$\int_{[0,T]^n} dt_n \int d\mu_{n}^{(i)}(\phi) \text{Tr}\left(|P_{a,A}^k J^k(t_n; \sigma)\right) \leq (\max\{\sqrt{A/a}, 1/a\})^k \left(\frac{C}{T^{3k}}\|V\|^{-1} \|\phi\|_{H^1}^{4(k+n)} \right) \int_0^T dt_n \int d\mu_{n}^{(i)}(\phi) \|\phi\|_{H^1}^{4(k+n)} \leq (\max\{\sqrt{A/a}, 1/a\})^k \left(\frac{C}{T^{3k}}\|V\|^{-1} \|\phi\|_{H^1}^{4(k+n)} \right) TM^{4(k+n)}.$$
Therefore, if $T$ is sufficiently small, going back to (5.8), we prove that $\text{Tr} | P_{a,\lambda}^{(k)} | \to 0$ as $n \to \infty$.

**Remark 5.1.** For convergence to zero in (5.10), we impose the condition that $M$ is sufficiently small. The main difficulty to remove this smallness assumption is that contrary to the proof of the unconditional uniqueness for the NLS (see §16 in [27] for instance), one cannot make $\text{Tr} (| P_{a,\lambda}^{(k)} J^k (t_n; \sigma) |)$ in the integral (5.8) small uniformly in $\phi$ by choosing small $T > 0$, since the measure $\mu_{n}^{(i)}$ can be supported on a set of infinitely many $\phi$’s.

## Appendix A. Multilinear Estimates

In this section, we present the key multilinear estimates that we will use to prove our main theorems. For the GP hierarchy, our key estimates are in Proposition A.1. The key estimates for the Hartree hierarchy are in Propositions A.3

**Proposition A.1 (Multilinear estimates for GP).**

\[
\| (e^{it\Delta} f_1)(e^{it\Delta} f_2)(e^{it\Delta} f_3)(e^{it\Delta} f_4)(e^{it\Delta} f_5) \|_{L_t^1 H_x^{-\frac{1}{4}}} \lesssim \| f_1 \|_{\dot{H}^{-\frac{1}{4}}} \prod_{j=2}^5 \| f_j \|_{\dot{H}^1},
\]

(A.1)

\[
\| (e^{it\Delta} f_1)(e^{it\Delta} f_2)(e^{it\Delta} f_3)(e^{it\Delta} f_4)(e^{it\Delta} f_5) \|_{L_t^1 W_x^{-\frac{1}{6}}} \lesssim \prod_{j=1}^5 \| f_j \|_{\dot{H}^1}.\]

(A.2)

For the proof, we need

**Lemma A.2 (Negative Sobolev norm estimate).**

\[
\| fg \|_{\dot{H}^{-1}} \leq \| f \|_{\dot{W}^{-1,6}} \| g \|_{\dot{W}^{1,\frac{3}{2}}}.\]

Proof. We prove the lemma by the standard duality argument, the product rule and the Sobolev inequality.

\[
\int fg \overline{h} \, dx \lesssim \| f \|_{\dot{W}^{-1,6}} \| g \|_{\dot{W}^{1,\frac{3}{2}}} \lesssim \| f \|_{\dot{W}^{-1,6}} \left( \| g \|_{L^3} \| h \|_{\dot{H}^1} + \| g \|_{\dot{W}^{1,\frac{3}{2}}} \| h \|_{L^6} \right) \lesssim \| f \|_{\dot{W}^{-1,6}} \| g \|_{\dot{W}^{1,\frac{3}{2}}} \| h \|_{\dot{H}^1}.\]

\[\square\]

Proof. By Lemma A.2 Sobolev embedding and Strichartz estimates, we prove that

\[
\| (e^{it\Delta} f_1)(e^{it\Delta} f_2)(e^{it\Delta} f_3)(e^{it\Delta} f_4)(e^{it\Delta} f_5) \|_{L_t^1 H_x^{-\frac{1}{4}}} \leq \| e^{it\Delta} f_1 \|_{L_t^2 W_x^{-\frac{1}{6}}} \prod_{j=2}^5 \| e^{it\Delta} f_j \|_{L_t^\infty W_x^{-\frac{1}{2}}} \lesssim \| f_1 \|_{\dot{H}^{-1}} \left( \| e^{it\Delta} f_2 \|_{L_t^2 W_x^{-\frac{1}{6}}} \prod_{j=3}^5 \| e^{it\Delta} f_j \|_{L_t^\infty L_x^6} + \text{three similar terms (by the product rule)} \right) \lesssim \| f_1 \|_{\dot{H}^{-1}} \prod_{j=2}^5 \| f_j \|_{\dot{H}^1}
\]

and

\[
\| (e^{it\Delta} f_1)(e^{it\Delta} f_2)(e^{it\Delta} f_3)(e^{it\Delta} f_4)(e^{it\Delta} f_5) \|_{L_t^1 W_x^{-\frac{1}{6}}} \lesssim \prod_{j=1}^5 \| f_j \|_{\dot{H}^1},
\]

\[\square\]
Proof of Theorem A.1.

We note that Theorem A.1 also holds for

\[ L^2 W^1 < 2 \]

As an analogue of Proposition A.1, we prove:

Recall the definition of the the trilinear operator \( A \) in (4.8)

\[
A[V_x, f, g](x) := \int \int V_x(x - y_1, x - y_2) f(y_1) g(y_2) dy_1 dy_2.
\]

As an analogue of Proposition A.1 we prove:

**Proposition A.3 (Multilinear estimates for Hartree).** Let \( \epsilon \geq 0 \). Then, we have

\[
\| A[V_x, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5) \|_{L^1_t H^{-1}} \leq T^{3\epsilon} \| V_x \|_{L^{1/2}_{t\epsilon}} \prod_{\ell=1}^5 \| f_\ell \|_{H^1}. \quad \forall \ell = 1, \ldots, 5
\]

and

\[
\| A[V_x, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5) \|_{L^1_t L^1_x} \leq T^{3\epsilon} \| V_x \|_{L^{1/2}_{t\epsilon}} \prod_{\ell=1}^5 \| f_\ell \|_{H^1}. \quad \forall \ell = 1, \ldots, 5
\]

We recall the convolution estimates in Beckner [3].

**Theorem A.1.** For \( 1 < p < q < \infty, 1 < s_k < p'/q', k = 1, 2 \) and \( 1/q + 2/p' = \sum 1/s_k, 2 < p'/q' \),

\[
\| A[V_x, f, g] \|_{L^q(\mathbb{R}^d)} \leq \| V_x \|_{L^p(\mathbb{R}^d)} \| f \|_{L^{s_1}(\mathbb{R}^d)} \| g \|_{L^{s_2}(\mathbb{R}^d)}.
\]

We note that Theorem A.1 also holds for \( p = 1 \). Indeed, by the change of variables \( (x - y, x - z) \rightarrow (y, z) \), Minkowski’s inequality, and Hölder’s inequality, we have

\[
\| A[V_x, f, g] \|_{L^q} = \left\| \int \int V_x(y, z) f(x - y)g(x - z) dy dz \right\|_{L^q} \leq \int \int \left| V_x(y, z) \right| \| f(x - y)g(x - z) \|_{L^q} dy dz
\]

\[
\leq \int \int \left| V_x(y, z) \right| \| f(x - y) \|_{L^{s_1}} \| g(x - z) \|_{L^{s_2}} dy dz
\]

\[
= \| V_x \|_{L^1} \| f \|_{L^{s_1}} \| g \|_{L^{s_2}}.
\]

**Proof of A.4.** For \( j \in \{1, 2, 3\} \), we have

\[
\| \partial_j \left[ A[V_x, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5) \right] \|_{L^1_t L^2_x} \leq \| A[V_x, (\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5) \|_{L^1_t L^2_x} + \text{four similar terms (by the product rule)}
\]

\[
=: I_1 + I_2 + I_3 + I_4 + I_5.
\]
By Theorem A.1, Strichartz estimates, and Sobolev embedding,
\[ I_1 \leq \left\| A[V_\xi, (\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \right\|_{L^2_x} \left\| (e^{it\Delta} f_5) \right\|_{L^3_t} \]
\[ \leq \left\| V_\xi \right\|_{L^{1+\varepsilon}_x} \left\| \partial_j e^{it\Delta} f_1 e^{it\Delta} f_2 \right\|_{L^\frac{1}{2+\varepsilon}_x L^\frac{6}{3} T} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^6_x} \left\| (e^{it\Delta} f_5) \right\|_{L^3_t} \]
\[ \leq T^{3k} \left\| V_\xi \right\|_{L^{1+\varepsilon}_x} \left\| \partial_j e^{it\Delta} f_1 \right\|_{L^{\frac{1}{2+\varepsilon}}_t L^\frac{6}{3} T} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^6_x} \left\| \partial_j e^{it\Delta} f_5 \right\|_{L^6_x} \]
\[ \leq T^{3k} \left\| V_\xi \right\|_{L^{1+\varepsilon}_x} \left\| f_\ell \right\|_{H^1} \]
and similarly for \( k \in \{2, 3, 4\} \). For \( k = 5 \), we have
\[ I_5 \leq \left\| A[V_\xi, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \right\|_{L^2_x} \left\| \partial_j e^{it\Delta} f_5 \right\|_{L^6_x} \]
\[ \leq \left\| V_\xi \right\|_{L^{1+\varepsilon}_x} \left\| e^{it\Delta} f_1 e^{it\Delta} f_2 \right\|_{L^{\frac{1}{2+\varepsilon}}_x L^\frac{6}{3} T} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^6_x} \left\| \partial_j e^{it\Delta} f_5 \right\|_{L^6_x} \]
\[ \leq T^{3k} \left\| V_\xi \right\|_{L^{1+\varepsilon}_x} \left\| e^{it\Delta} f_1 \right\|_{L^{\frac{1}{2+\varepsilon}}_x L^\frac{6}{3} T} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^6_x} \left\| \partial_j e^{it\Delta} f_5 \right\|_{L^6_x} \]
\[ \leq T^{3k} \left\| V_\xi \right\|_{L^{1+\varepsilon}_x} \left\| f_\ell \right\|_{H^1}. \]

Before we proceed to the proof of (A.3), we define \( \{P_1, P_2, P_3\} \) to be a conic decomposition of \( \mathbb{R}^3 \). That is, \( P_j \) is a Fourier multiplier with symbol \( p_j : \mathbb{R}^3 \to [0, 1] \) such that for \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \),
\[ p_j(\xi) = 1 \text{ for } \xi_j^2 \geq 2 \sum_{j \neq j} \xi_j^2, \]
\[ p_j(\xi) = 0 \text{ for } \xi_j^2 \leq \frac{1}{2} \sum_{j \neq j} \xi_j^2, \] and
\[ \sum_j p_j(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^3. \]
Observe that \( |\xi_j| \sim |\xi| \) on the support of \( p_j \).

**Proof of (A.3) when \( m = 5 \).** For \( h \in H^1(\mathbb{R}^3) \), we have
\[ \int A[V_\xi, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)](x)(e^{it\Delta} f_5)(x) \overline{h}(x) \, dx \]
\[ = \sum_{j=1}^{3} \int \int V_\xi(y, z) \partial_j \left( e^{it\Delta} f_1 e^{it\Delta} f_2)(x - y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x - z) \overline{h}(x) \right) \]
\[ \times (\partial_j^{-1} P_j e^{it\Delta} f_5)(x) \, dy \, dz \, dx \]
\[ = \sum_{j=1}^3 \int \int \int V_{\infty}(y, z)(\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2)(x-y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) \]
\[ \times \mathcal{H}(x)(\partial_j^{-1} P_j e^{it\Delta} f_5)(x) \, dy \, dz \, dx \]
\[ =: I_1 + I_2 + I_3 + I_4 + I_5. \]

By duality, it now suffices to show that
\[ \| I_k \|_{L^1_t} \lesssim \left\| T^{3\epsilon} \| V_{\infty} \|_{L^{\frac{1}{1+\epsilon}}_{t,x}} f_5 \|_{\dot{H}^{-1}} \left( \prod_{\ell=1}^4 \| f_\ell \|_{\dot{H}^\epsilon} \right) \right\|_{\dot{H}^1} \]  

(A.6) holds for \( k \in \{1, 2, 3, 4, 5\} \). By Theorem [A.1], Strichartz estimates, and Sobolev embedding, we have
\[ \| I_1 \|_{L^1_t} \leq \sum_{j=1}^3 \left\| \int \int V_{\infty}(y, z)(\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2)(x-y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) \, dy \, dz \right\|_{L^3_x} \]
\[ \times \left\| \partial_j^{-1} P_j e^{it\Delta} f_5 \right\|_{L^\infty_t} \| h \|_{L^6_x} \right\|_{L^1_t} \]
\[ \lesssim \sum_{j=1}^3 \left\| V_{\infty} \right\|_{L^{\frac{1}{1+\epsilon}}} \left\| e^{it\Delta} f_1 e^{it\Delta} f_2 \right\|_{L^{3\epsilon}_{t,x}} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^1_t} \left\| \partial_j^{-1} P_j e^{it\Delta} f_5 \right\|_{L^\infty_t} \| h \|_{L^6_x} \right\|_{L^1_t} \]
\[ \lesssim \sum_{j=1}^3 T^{3\epsilon} \left\| V_{\infty} \right\|_{L^{\frac{1}{1+\epsilon}}} \left\| e^{it\Delta} f_1 \right\|_{L^2_{t,x}} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^1_t} \left\| \partial_j^{-1} P_j e^{it\Delta} f_5 \right\|_{L^\infty_t} \| h \|_{L^6_x} \right\|_{L^1_t} \]
\[ \lesssim \sum_{j=1}^3 \left\| V_{\infty} \right\|_{L^{\frac{1}{1+\epsilon}}} \left\| e^{it\Delta} f_1 \right\|_{L^2_{t,x}} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^1_t} \left\| \partial_j^{-1} P_j e^{it\Delta} f_5 \right\|_{L^\infty_t} \| h \|_{L^6_x} \right\|_{L^1_t} \]
\[ \lesssim \sum_{j=1}^3 \left\| V_{\infty} \right\|_{L^{\frac{1}{1+\epsilon}}} \left\| e^{it\Delta} f_1 \right\|_{L^2_{t,x}} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^1_t} \left\| \partial_j^{-1} P_j e^{it\Delta} f_5 \right\|_{L^\infty_t} \| h \|_{L^6_x} \right\|_{L^1_t} \]
\[ \lesssim \sum_{j=1}^3 T^{3\epsilon} \left\| V_{\infty} \right\|_{L^{\frac{1}{1+\epsilon}}} \left\| e^{it\Delta} f_1 \right\|_{L^2_{t,x}} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^1_t} \left\| \partial_j^{-1} P_j e^{it\Delta} f_5 \right\|_{L^\infty_t} \| h \|_{L^6_x} \right\|_{L^1_t} \]
\[ \lesssim \sum_{j=1}^3 T^{3\epsilon} \left\| V_{\infty} \right\|_{L^{\frac{1}{1+\epsilon}}} \left\| e^{it\Delta} f_1 \right\|_{L^2_{t,x}} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^1_t} \left\| \partial_j^{-1} P_j e^{it\Delta} f_5 \right\|_{L^\infty_t} \| h \|_{L^6_x} \right\|_{L^1_t} \]
\[ \lesssim \sum_{j=1}^3 T^{3\epsilon} \left\| V_{\infty} \right\|_{L^{\frac{1}{1+\epsilon}}} \left\| e^{it\Delta} f_1 \right\|_{L^2_{t,x}} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^1_t} \left\| \partial_j^{-1} P_j e^{it\Delta} f_5 \right\|_{L^\infty_t} \| h \|_{L^6_x} \right\|_{L^1_t} \]
Proof of (A.3) when \(m \neq 5\). We present the proof for \(m = 1\), and note that the proof for \(m \in \{2, 3, 4\}\) is similar. i.e. we show that

\[
\left\| A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \right\|_{L^1_t H^{-1}_x} \lesssim T^{3\epsilon}\|V_\infty\|_{L^{1,1}_t} \|f_1\|_{H^{-1}} \prod_{\ell=2}^5 \|f_\ell\|_{H^1}.
\]

For \(h \in \dot{H}^1(\mathbb{R}^3)\), we have

\[
\int A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)](x)(e^{it\Delta} f_5)(x)\overline{h}(x) \, dx
\]

\[
= \sum_{j=1}^3 \int \int \int V_\infty(y, z)(\partial^{-1}_j P_j e^{it\Delta} f_1)(y - z)
\times \partial_j \left[ (e^{it\Delta} f_2)(x - y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x - z)(e^{it\Delta} f_5)(x)\overline{h}(x) \right] \, dy \, dz \, dx
\]

\[
= \sum_{j=1}^3 \int \int \int V_\infty(y, z)(\partial^{-1}_j P_j e^{it\Delta} f_1 \cdot \partial_j e^{it\Delta} f_2)(y - z)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x - z)
\times (e^{it\Delta} f_5)(x)\overline{h}(x) \, dy \, dz \, dx
\]

+ four similar terms (by the product rule)

\[=: I_1 + I_2 + I_3 + I_4 + I_5.\]

By duality, it now suffices to show that

\[
\|I_k\|_{L^1_t} \lesssim T^{3\epsilon}\|V_\infty\|_{L^{1,1}_t} \|f_1\|_{H^{-1}} \left( \prod_{\ell=2}^5 \|f_\ell\|_{H^1} \right) \|h\|_{H^1}
\]

(A.7)

holds for \(k \in \{1, 2, 3, 4, 5\}\). By Theorem (A.1) Strichartz estimates, and Sobolev embedding, we have

\[
\|I_1\|_{L^1_t} \leq \sum_{j=1}^3 \left\| \int \int V_\infty(y, z)(\partial^{-1}_j P_j e^{it\Delta} f_1 \cdot \partial_j e^{it\Delta} f_2)(y - z)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x - z) \, dy \, dz \right\|^{\frac{3}{2}}_{L^2_t}
\]

\[
\times \left\| e^{it\Delta} f_5 \right\|_{L^6_t} \left\| h \right\|_{L^6_t}
\]

\[
\lesssim \sum_{j=1}^3 \|V_\infty\|_{L^{1,1}_t} \|\partial^{-1}_j P_j e^{it\Delta} f_1 \cdot \partial_j e^{it\Delta} f_2\|_{L^{1,1}_t} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^3} \left\| e^{it\Delta} f_5 \right\|_{L^6_t} \left\| h \right\|_{L^6_t}
\]

\[
\lesssim \sum_{j=1}^3 T^{3\epsilon}\|V_\infty\|_{L^{1,1}_t} \|\partial^{-1}_j P_j e^{it\Delta} f_1\|_{L^{2,\infty}_t} \left\| \partial_j e^{it\Delta} f_2\right\|_{L^{2,\infty}_t} \left\| e^{it\Delta} f_3 e^{it\Delta} f_4 \right\|_{L^3} \left\| e^{it\Delta} f_5 \right\|_{L^6_t} \left\| h \right\|_{L^6_t}
\]

\[
\lesssim T^{3\epsilon}\|V_\infty\|_{L^{1,1}_t} \|f_1\|_{H^{-1}} \left( \prod_{\ell=3}^5 \|f_\ell\|_{H^1} \right) \|h\|_{H^1},
\]

and similarly, (A.7) holds for \(k \in \{2, 3, 4\}\). Finally, we bound \(I_5\) by

\[
\sum_{j=1}^3 \left\| \int \int V_\infty(y, z)(\partial^{-1}_j P_j e^{it\Delta} f_1 \cdot e^{it\Delta} f_2)(y - z)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x - z) \, dy \, dz \right\|_{L^1_t L^2_x}
\]

\[
\times \left\| e^{it\Delta} f_5 \right\|_{L^6_t L^6_x} \left\| \partial_j h \right\|_{L^6_x}
\]

\[23\]
\begin{equation}
\leq \sum_{j=1}^{3} \| V_{\infty} \|_{L^1_{t} \supset L^1_{\infty}} \left\| \partial_j^{-1} P_j e^{it\Delta} f_1 \cdot e^{it\Delta} f_2 \right\|_{L^1_{t} \supset L^1_{\infty}} \| e^{it\Delta} f_3 e^{it\Delta} f_4 \|_{L^3_{t} L^3_{\infty}} \| e^{it\Delta} f_5 \|_{L^5_{t} L^5_{\infty}} \| \partial_j h \|_{L^2_{t} L^2_{\infty}}
\end{equation}

\begin{equation}
\leq \sum_{j=1}^{3} \| V_{\infty} \|_{L^1_{t} \supset L^1_{\infty}} \left\| \partial_j^{-1} P_j e^{it\Delta} f_1 \right\|_{L^1_{t} \supset L^1_{\infty}} \frac{4}{\ell = 2} \| e^{it\Delta} f_2 \|_{L^6_{t} L^6_{\infty}} \| e^{it\Delta} f_3 \|_{L^6_{t} L^6_{\infty}} \| \partial_j h \|_{L^2_{t} L^2_{\infty}}
\end{equation}

\begin{equation}
\leq \sum_{j=1}^{3} T^{3k} \| V_{\infty} \|_{L^1_{t} \supset L^1_{\infty}} \left\| \partial_j^{-1} P_j e^{it\Delta} f_1 \right\|_{L^1_{t} \supset L^1_{\infty}} \frac{4}{\ell = 2} \| e^{i\omega \Delta} f_2 \|_{L^6_{t} L^6_{\infty}} \| e^{it\Delta} f_3 \|_{L^6_{t} L^6_{\infty}} \| \partial_j h \|_{L^2_{t} L^2_{\infty}}
\end{equation}

\begin{equation}
\leq T^{3k} \| V_{\infty} \|_{L^1_{t} \supset L^1_{\infty}} \left\| f_1 \right\|_{H^{-1}} \left( \prod_{\ell = 2}^{5} \| f_\ell \|_{H^1} \right) \| h \|_{H^1} \cdot \square
\end{equation}

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