ON THE INFLUENCE OF THE COUPLING ON THE DYNAMICS
OF SINGLE-OBSERVED CASCADE SYSTEMS OF PDE’S

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ABSTRACT. We consider single-observed cascade systems of hyperbolic equations. We first consider the class of bounded operators that satisfy a non negativity property \((NNP)\). Within this class, we give a necessary and sufficient condition for observability of the cascade system by a single observation. We further show that if the coupling operator does not satisfy \((NNP)\) (contrarily to [5], or also e.g. [3, 4] for symmetrically coupled systems), the usual observability inequality through a single component may still occur in a general framework, under some smallness conditions, but it may also be violated. When the coupling operator is a multiplication operator, \((NNP)\) is violated whenever the coupling coefficient changes sign in the spatial domain. We give explicit constructive examples of such coupling operators for which unique continuation may fail for an infinite dimensional set of initial data, that we characterize explicitly. We also exhibit examples of couplings and initial data for which the observability inequality holds but in weaker norms. These examples extend to parabolic systems. Finally, we show that the two-level energy method [1, 2] which involves different levels of energies for the observed and unobserved component, may involve the same levels of energies of these respective components, if the differential order of the coupling is higher (operating here through velocities instead of displacements). We further give an application to controlled systems coupled in velocities. This shows that the answer to observability and unique continuation questions for single-observed cascade systems is much more involved in the case of coupling operators that violate \((NNP)\) or of higher order coupling operators, and that the mathematical properties of the coupling operator greatly influence the dynamics of the observed system even though it operates through lower order differential terms. We indicate several extensions and future directions of research.

1. Introduction. We consider controlled cascade systems of coupled reversible hyperbolic equations. Such systems arise naturally when studying insensitizing controllability for a single scalar equation (see e.g. [23, 28, 29, 14, 26, 27, 5, 7]) or simultaneous control (see e.g. [22, 7]).

Coupled systems have many concrete and potential applications. Controlling them at lower cost or through a reduced number of controls (acting on certain components of the devices) is a challenging issue. In this spirit, we shall consider either locally distributed or boundary controlled cascade systems. Moreover, we
focus on the study of single-actionned controlled cascade systems, that is we consider the situation for which a single control acts on the system to drive the full state from its initial state to a desired final state (here the equilibrium, without loss of generality). Hence, if controllability holds for the full system, the component which is not directly controlled, is indeed indirectly controlled thanks to the coupling operator. This is also called indirect controllability. Other interesting notions of controllability such as approximate controllability, partial controllability, or exact synchronization can be considered. Approximate controllability is equivalent by duality to a unique continuation property for the dual problem. Partial observability (introduced by J.-L. Lions [22]) requires to drive back to equilibrium the component which is controlled whereas the initial data of the other component is assumed to vanish. The notion of exact synchronization has been first studied for finite dimensional systems (see e.g. [16, 31] and the references therein), and extended later on for infinite dimensional systems (see e.g. [18, 20, 21]). Exact synchronization at time $T > 0$ requires that all the components reach and keep a common value after the time $T$. The property of indirect controllability (and by duality that of indirect observability) which is studied here, is a more demanding property—and therefore, harder to prove—than the properties of partial controllability (see the introduction of [2]). It is also a stronger property than the properties of approximate controllability, or of exact synchronization.

In this paper, we are mainly interested by hyperbolic systems. However, we give some extension to parabolic systems. Indirect controllability results for parabolic systems are also of great interest (see [17, 9] and [8] for a survey and further references in this direction). We also refer to [24] for some results on approximate controllability for parabolic systems and also to [11] for negative and positive unique continuation results for one-dimensional parabolic systems with locally distributed controls. We refer to [3, 25, 4, 5] for positive results on indirect controllability of parabolic systems thanks to the transmutation method.

Let us first describe the abstract framework, and point out that the scope is general and may model different and complex mechanical systems (see e.g. [2] for the same abstract framework with applications to Kirchhoff plates, the linear elastodynamic system, plate system...). Our scope is to focus on interesting mathematical features, which may be common to several models. One way to capture them is to first look at the problem on a larger point of view. We shall first recall some recent results on necessary and sufficient conditions for the control of cascade systems in this abstract framework. We shall then analyze through significative examples, how the possibility to identify/observe (and by duality control) is affected when the assumptions on the coupling operator are modified.

The abstract framework is as follows. We consider the abstract coupled cascade system

$$\begin{align*}
y''_1 + Ay_1 + C^*y_2 &= 0, \\
y''_2 + Ay_2 &= Bv, \\
(y_i, y'_i)(0) &= (y^0_i, y^1_i) \text{ for } i = 1, 2,
\end{align*} \tag{1}$$

where $H$ is an Hilbert space with norm $|\cdot|$ and scalar product $\langle \cdot, \cdot \rangle$ and the coupling operator $C$ is bounded in $H$, $A$ satisfies

$$\begin{align*}
(A) & \quad A : D(A) \subset H \mapsto H, A^* = A, \\
& \quad \exists \omega > 0, |Au| \geq \omega |u| \quad \forall u \in D(A), \\
& \quad A \text{ has a compact resolvent}.
\end{align*}$$
Furthermore, $B \in \mathcal{L}(G; H)$ (bounded control operator) or $B \in \mathcal{L}(G, H'_2)$ (unbounded control operator), where $G$ is a given Hilbert space equipped with the norm $||.||_G$ and identified with its dual space, whereas $H_2 = D(A)$, $H'_2$ denoting the dual space of $H_2$ with respect to the pivot space $H$ (these definitions will be set precisely in the next section). It is well-known that exact controllability results can be deduced by duality, from observability inequalities for the dual problem (see e.g. [22]). The dual problem reads as follows

\[
\begin{align*}
\begin{cases}
  u''_1 + Au_1 &= 0, \\
u''_2 + Au_2 + Cu_1 &= 0, \\
(u_i, u'_i)(0) &= (u_i^0, u'_i) \text{ for } i = 1, 2,
\end{cases}
\end{align*}
\]  

(2)

Thus, we shall focus mainly on the dual problem, the subsequent controllability or non controllability results being deduced by a standard procedure (see [5] for abstract control results and applications to examples of PDE’s).

The main purpose of this paper is to study the influence of the coupling operator $C$ on the dynamics of the homogeneous system (2). More precisely, we shall examine through abstract results and significant examples the influence of $C$ on the unique continuation and quantitative unique continuation properties (i.e. observability estimates) associated to the adjoint of the control operator $B$. We shall mainly focus on the influence of the following properties

1. coercivity/ non coercivity of the operator $C$ in suitable spaces,
2. the order of $C$ (here we shall see the difference if $C$ operates through the displacement or the velocity).

In the case for which $A = -\Delta$, $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, $H = L^2(\Omega)$, $Cu = cu$ for every $u \in H$ where $c \in L^\infty(\Omega)$, we recover as a particular case

\[
\begin{align*}
\begin{cases}
y_{1,t}(t, x) - \Delta y_1(t, x) + c(x)y_2(t, x) &= 0, & (t, x) \in (0, T) \times \Omega, \\
y_{2,t}(t, x) - \Delta y_2(t, x) &= 0, & (t, x) \in (0, T) \times \Omega \\
y_1 &= 0 \text{ in } (0, T) \times \Gamma, & y_2 = b(x)v \text{ in } (0, T) \times \Gamma, \\
(y_i, y_{i,t})(0) &= (y_i^0, y'_i), & \text{ in } \Omega, \text{ for } i = 1, 2,
\end{cases}
\end{align*}
\]  

(3)

in the case of boundary control. Here $\Omega$ is a bounded open set in $\mathbb{R}^d$ with a sufficiently smooth boundary $\Gamma$, the initial data $(y_i^0, y'_i)$, $i = 1, 2$ are given in a suitable energy space, the functions $c$ and $b$ are respectively the coupling coefficient and the control coefficient, whereas $v$ is the control.

In the case for which, for instance $A = \Delta^2$ with $D(A) = H^4(\Omega) \cap H^2_0(\Omega)$, $H = L^2(\Omega)$, $Cu = cu$ for every $u \in H$ where $c \in L^\infty(\Omega)$, we recover as a particular case, a coupled plate model

\[
\begin{align*}
\begin{cases}
y_{1,tt}(t, x) + \Delta^2 y_1(t, x) + c(x)y_2(t, x) &= 0, & (t, x) \in (0, T) \times \Omega, \\
y_{2,tt}(t, x) + \Delta^2 y_2(t, x) = b(x)v(t, x), & (t, x) \in (0, T) \times \Omega \\
y_1 &= \frac{\partial y_1}{\partial \nu} = 0 \text{ in } (0, T) \times \Gamma, & y_2 = \frac{\partial y_2}{\partial \nu} = 0, \text{ in } (0, T) \times \Gamma, \\
(y_i, y_{i,t})(0) &= (y_i^0, y'_i), & \text{ in } \Omega, \text{ for } i = 1, 2,
\end{cases}
\end{align*}
\]  

(4)

in the case of locally distributed control. One may also consider other types of boundary conditions, such as for instance $u_i = \Delta u_i = 0$ in $(0, T) \times \Gamma$ in the above model, or Neumann or Robin boundary conditions for the wave equation. We can in
the same way describe coupled systems of the linear elastodynamic system, Euler-Bernoulli beams, systems with variable coefficients, ... , so that the scope goes beyond the usual wave equation describing the deformations of an elastic membrane.

Let us describe our goals on a concrete example. The dual problem of (3) is given by

\[
\begin{align*}
  &u_{1,tt} - \Delta u_1 = 0, \quad (t, x) \in (0, T) \times \Omega, \\
  &u_{2,tt} - \Delta u_2 + c(.)u_1 = 0, \quad (t, x) \in (0, T) \times \Omega, \\
  &u_i = 0, \quad \text{in } (0, T) \times \Gamma \text{ for } i = 1, 2, \\
  &(u_i, u'_i)(0, \cdot) = (u^0_i, u^1_i)(\cdot) \quad \text{in } \Omega, \quad \text{for } i = 1, 2,
\end{align*}
\]

In insensitizing issues, the coupling coefficient \( c \) keeps a constant sign in \( \Omega \) and is bounded away from zero by a strictly positive constant in a subregion \( O \) of \( \Omega \). In the class of coupling coefficients \( c \) satisfying these positivity conditions and some further smoothness assumptions, we give in [5], a necessary and sufficient condition for the observability of (5) (we shall recall this condition in a more general abstract framework below). Hence we give a characterization of cascade systems of two equations which are observable by a single observation in this class of coupling coefficients.

An important question which rises then is to determine whether the conditions on \( c \) can be relaxed.

1. In particular, does observability hold for (5) when \( c \) changes sign within \( \Omega \) and in which functional spaces?
2. Moreover, if observability does not hold in these spaces, does a weaker observability hold?
3. And if not, does unique continuation property hold?
4. If unique continuation fails to occur, is it possible to characterize the set of initial data for which unique continuation does not hold, or at least to give counterexamples?

More generally, we are interested on the influence of the coupling operator on the dynamics of the controlled/observed system. We shall see that if the coupling operates through the velocities, observability of both components through a single observation holds under some partial coercitivity assumptions on the coupling and involves the same levels of energies for both components. Hence the properties of the coupling operator strongly influence the dynamics of the single-controlled coupled system. The purpose of this paper is to provide significant examples of this influence and to prove positive and negative results of sign changing coupling coefficients \( c \), for which each of the following situation can occur:

- Positive observability inequalities in the expected functional spaces (the same than for \( c \) satisfying some positivity property).
- Examples of non unique continuation results for an infinite dimensional set of initial data.
- Examples of coupling coefficients for which weaker observability estimates hold.

This shows that challenging questions remain for sign-changing coefficients and that some refined conditions should be identified to characterize coupling coefficients for which observability in the expected spaces holds, or holds in weaker spaces or for which unique continuation does not hold. In this latter case, a further analysis is to characterize the set of initial data for which unique continuation does not hold.
Another point we would like to stress is the influence of the coupling on the whole
dynamics of the system. For this, we will analyze cascade systems coupled through
velocities and show which results are available.

2. Preliminaries and abstract set-up. We shall first recall the general abstract
setting and our results in the case of partially coercive coupling operators.

We shall need the following notation. We set \( H_k = D(A^{k/2}) \) for \( k \in \mathbb{N} \), with the
convention \( H_0 = H \). The set \( H_k \) equipped with the norm \( |·|_k \) defined by \( |A^{k/2}·| \)
and the associated scalar product, is a Hilbert space. We denote by \( H_{-k} \) the dual
space of \( H_k \) with the pivot space \( H \). We equip \( H_{-k} \) with the norm \( |·|_{-k} = |A^{-k/2}·| \).

We define the energy space associated to (2) by \( \mathcal{H} = (H_1)^2 \times (H_0)^2 \).

The system (2) can then be reformulated as the first order abstract system
\[
\begin{align*}
U'(t) &= AU,
U(0) &= U^0 = (u_1^0, u_2^0, u_1^1, u_2^1),
\end{align*}
\]
where \( U = (u_1, u_2, v_1, v_2) \) and \( A \) is the unbounded operator in \( \mathcal{H} \) with domain
\( D(A) = H_2^2 \times H_1^2 \) defined by
\[
AU = (v_1, v_2, -Au_1, -Au_2 - Cu_1).
\]

Remark 1. Note that the relation \( U' = AU \) implies in particular that \( v_i = u_i' \) for
\( i = 1, 2 \). This is standard to switch from a second order equation in time, to a first
order one. We make use of these relations in the sequel without further detailing this.

Using semigroup theory (see e.g. [30] in the context of control theory), it is easy
to establish the well-posedness of the abstract system (6) for initial data \( U_0 \in \mathcal{H} \).
Moreover for initial data \( U^0 \in D(A) \), the solution \( U \) of (6) is such that \( U \in \mathcal{C}((0,T]; D(A)) \cap \mathcal{C}^1((0,T]; \mathcal{H}) \). Moreover, assuming that \( C \in \mathcal{L}(H_{k-1}) \) for \( k \in \mathbb{Z}^+ \),
the problem (6) (and similarly (2)) is well-posed in \( H_k^2 \times H_{k-1}^2 \), that is if the initial
data are in \( H_k^2 \times H_{k-1}^2 \), then the solution \( U \) of (6) (and similarly that of (2)) is in
\( \mathcal{C}((0,T]; H_k^2 \times H_{k-1}^2) \). For a solution \( U = (u_1, u_2, v_1, v_2) \) of (2), we have \( v_i = u_i' \) for
\( i = 1, 2 \). For such a solution, we set
\[
U_i = (u_i, u_i') \text{ for } i = 1, 2. \tag{8}
\]
For \( U_i \in H_k \times H_{k-1} \), we define the energies of level \( k \) as
\[
e_k(U_i)(t) = \frac{1}{2} \left( |A^{k/2}u_i|^2 + |A^{(k-1)/2}u_i'|^2 \right), \quad k \in \mathbb{Z}, \quad i = 1, 2. \tag{9}
\]

For more general set of vector-valued functions \( t \mapsto V(t) = (v_1(t), v_2(t)) \in H_k \times
H_{k-1} \) for convenience, we define \( e_k(V)(t) \) as above without further recalling this in
the sequel. Moreover we denote by \( G \) a given Hilbert space with norm \( ||·||_G \) and
scalar product \( ⟨·,·⟩_G \). The space \( G \) will be identified to its dual space in the sequel.

We set \( X = H \times H_1 \times H_{-1} \times H \).

In [5], we prove the following results

**Theorem 2.1.** Assume (A). We make the following assumptions

- (i) The coupling operator \( C \) satisfies the following regularity and partial coercivity properties

\[
\begin{align*}
&(B1) \quad \{ \begin{array}{l}
C^* \in \mathcal{L}(H_k) \text{ for } k \in \{0,1\},
||C|| = \beta, ||Cw||^2 \leq \beta⟨Cw,w⟩ \quad \forall \ w \in H,
\end{array} \}
\end{align*}
\]
and the following observability inequality on C

\[
\begin{aligned}
\exists \, T_C > 0, \forall \, T > T_C, & \exists \, R_C(T) > 0 \text{ such that} \\
\forall \, (w^0, w^1) \in H_1 \times H & \text{ the solution } w \text{ of} \\
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} w + A w = 0, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\
\int_0^T |C w'|^2 \, dt \geq R_C(T)e_1(W)(0),
\end{array} \right.
\end{aligned}
\]

where \( W = (w, w'). \)

- (ii)-(a) The observability operator \( B^* \) satisfies the following admissibility property

\[
\begin{aligned}
B^* \in \mathcal{L}(H_2 \times H; G), \\
\forall \, T > 0 \exists \, C_T > 0, \text{ such that for all } (w^0, w^1) \in H_1 \times H \\
\text{ and all } f \in L^2([0, T]; H), \text{ the solution } w \text{ of} \\
w'' + A w = f, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\
\int_0^T \|B^*(w, w')\|_G^2 \, dt \\
\leq C_T \left( e_1(W)(0) + e_1(W)(T) + \int_0^T e_1(W)(t) \, dt + \int_0^T |f|^2 \, dt \right),
\end{aligned}
\]

where \( W = (w, w'). \)

- (ii)-(b) \( B^* \) satisfies the following observability inequality

\[
\begin{aligned}
\exists \, T_0 > 0, \forall \, T > T_0, & \exists \, C_1(T) > 0 \text{ such that} \\
\forall \, (w^0, w^1) \in H_1 \times H & \text{ the solution } w \text{ of} \\
w'' + A w = 0, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\
\int_0^T \|B^*(w, w')\|_G^2 \, dt \geq C_1(T)e_1(W)(0).
\end{aligned}
\]

Then for all \( T > 0 \), there exists a constant \( C(T) > 0 \) such that for all initial data \( U^0 \in X \), the solution of (2) satisfies the following direct inequality

\[
\int_0^T \|B^* U_2\|_G^2 \, dt \leq C(T) \left( e_0(U_1)(0) + e_1(U_2)(0) \right).
\]

Moreover there exists \( T^* > 0 \) such that for all \( T > T^* \), and all initial data \( U^0 \in X \), the solution of (2) satisfies the observability estimates

\[
\begin{aligned}
d_1(T) \int_0^T \|B^* U_2\|_G^2 \geq e_0(U_1)(0), \\
d_2(T) \int_0^T \|B^* U_2\|_G^2 \geq e_1(U_2)(0),
\end{aligned}
\]

where the constants \( d_i(T) > 0 \) depend on \( T \) (and on the coupling operator \( C \)).

**Remark 2.** It is important to notice the above decoupled structure for the observability inequality, which is more precise than the coupled observability inequality (12) below, with different asymptotic behaviors of the constants \( d_1(T) \) and \( d_2(T) \) with respect to the time \( T \) (see (26) below and also [5]). These asymptotic behaviors are important in the two and multi-levels every methods and also for the generalization of the above result to the control of bi-diagonal cascade systems of \( n \)-equations with \( n > 2 \) by a single control (see [7]). As explained in [7], this decoupled form is due to the specific form of cascade systems, and do not hold true for instance for symmetrically coupled systems. Notice also that this specific property is more precise than the one derived in [15] by contradiction arguments.

We also prove in [5] (see also [6]) the following result.
Theorem 2.2 (Necessary and sufficient conditions). Assume (A), (B1) and (C1). Then there exists $T^* > 0$ such that for all $T > T^*$, the following property holds
\[
(\text{OBS}) \begin{cases} 
\exists \ C(T) > 0 \text{ such that } \forall \ U^0 \in X \text{ the solution of } (2) \text{ satisfies } \\
\eta_0(U_1)(0) + \eta_1(U_2)(0) \leq C(T) \int_0^T \|\mathbf{B}^* U_2\|_{G}^2 \, dt.
\end{cases}
\] (12)
if and only if (B2) and (C2) hold.

Remark 3. Notice that if $H = L^2(\Omega)$ and if $C$ is defined as $C u = c(\cdot) w$ for $u \in H$, the assumption (B1) implies some smoothness of $c$ and that $c \geq 0$ in $\Omega$. This assumption, together with assumption (B2) also require that $c > 0$ on at least an open subset that may satisfy geometric assumptions. This is these two properties that we define as a partial coercivity assumption on $C$.

Remark 4. In (B1), the smoothness assumption $C^* \in \mathcal{L}(\mathcal{H}_1)$, can be suppressed both in Theorems (2.1) and (2.2). It is indeed only required for deducing an exact controllability result for the single-controlled system (1). This can be easily checked in the proof of Theorem (2.1) in [5], but is also proved in the present paper, thanks to Theorem (4.1) and Remark (15) below. Hence, these two Theorems still hold true if the assumption (B1) is replaced by a non negativity property (see Remark (10))

\[
C \in \mathcal{L}(H), \\
\|C\| = \beta, |C w|^2 \leq \beta \langle C w, w \rangle \ \forall \ w \in H.
\]

Remark 5. These results apply to many various physical situations, as previously recalled. Notice also that the above necessary and sufficient conditions are very general and that many results (on the admissibility (C1) and observability inequalities (B2) and (C2)) are already known in the literature on the single free equation $w'' + A w = 0$. Moreover we may not only rely on the so-called Geometric Control Condition [10] (denoted by (GCC)) to check that (B2) and (C2) hold but also on all the other results based on non-harmonic analysis, multipliers, moment methods and valid for instance when (GCC) is not a necessary condition. So we may rely on all these results to guarantee that (B2) and (C2) hold ((C1) is easier to check and is proved generally by means of the multiplier method).

3. Positive observability estimates for examples of sign changing coupling coefficients.

Theorem 3.1. Assume that $A$ satisfies (A) and that the observability operator $\mathbf{B}^*$ satisfies (C1) — (C2). Let $C_1$ and $C_2$ be two bounded operators in $H$ satisfying the assumptions (B1) and (B2) and define the bounded operator $C$ by $C = C_1 - \varepsilon C_2$ where $\varepsilon > 0$. Then there exists $\varepsilon_0 > 0$ and $T^* > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, all $T > T^*$, and for all $U^0 \in X$ the solution of (2) satisfies
\[
\eta_{1,\varepsilon} v_0(U_1)(0) + \eta_{2,\varepsilon} v_1(U_2)(0) \leq \int_0^T \|\mathbf{B}^* U_2\|_{G}^2 \, dt
\] (13)
where the constants $\eta_{i,\varepsilon} > 0$ for $i = 1, 2$ depend on $T$ but not on $U^0$.

Proof. We consider $\varepsilon \in (0, 1)$. Let us set $\beta_i = \|C_i\|$ for $i = 1, 2$ and set $T_{C_i} = T_i$ and $R_{C_i}(T) = R_i(T)$ for $i = 1, 2$ in assumption (B2). Let $(u_1^0, u_2^0, u_1^0, u_2^0) \in X$ be given and $(u_1, u_2)$ be the solution of (2) with these initial data and $C = C_1 - \varepsilon C_2$. 

Then we have $u_2 = v_2 - \varepsilon z_2$ where $(u_1, v_2)$ and $(u_1, z_2)$ solve respectively

\[
\begin{cases}
  u''_1 + Au_1 = 0, \\
  v''_2 + Av_2 + C_1u_1 = 0,
\end{cases}
\]

and

\[
\begin{cases}
  u''_1 + Au_1 = 0, \\
  z''_2 + A z_2 + C_2 u_1 = 0,
\end{cases}
\]

(14) and (15).

We set $V_2 = (v_2, v'_2)$ and $Z_2 = (z_2, z'_2)$. We have

\[
\int_0^T ||B^* U_2||^2_G dt \geq (1 - \varepsilon) \int_0^T ||B^* V_2||^2_G dt - \varepsilon(1 - \varepsilon) \int_0^T ||B^* Z_2||^2_G dt. 
\]

(16)

On the other hand, since the coupling operators $C_i$ for $i = 1, 2$ satisfy the assumptions $(B1) - (B2)$ and $A, B^*$ satisfy the assumptions of Theorem (2.1), we can respectively apply this Theorem to (14) and (15). Hence thanks to (10) written for $Z_2$, there exists $d_1(T) > 0$ such that

\[
\int_0^T ||B^* Z_2||^2_G dt \leq d_1(T) \left(c_0(U_1)(0)\right),
\]

(17)

since $Z_2(0) = (0, 0)$. Moreover there exists $T^* > 0$ such that for all $T > T^*$, and all initial data $U^0 \in X$, the solution of (14) satisfies the observability estimates

\[
\begin{cases}
  \int_0^T ||B^* V_2||^2_G \geq 2c_1(T)c_0(U_1)(0), \\
  \int_0^T ||B^* V_2||^2_G \geq 2c_2(T)c_1(U_2)(0),
\end{cases}
\]

(18)

since $V_2(0) = U_2(0)$. Using these inequalities in (16), we obtain

\[
\int_0^T ||B^* U_2||^2_G dt \geq (1 - \varepsilon)c_1(T)c_0(U_1)(0) + (1 - \varepsilon)c_2(T)c_1(U_2)(0) - \varepsilon(1 - \varepsilon)d_1(T)c_0(U_1)(0).
\]

Hence, we have

\[
\int_0^T ||B^* U_2||^2_G dt \geq (1 - \varepsilon) \left(c_1(T) - \varepsilon d_1(T)\right)e_0(U_1)(0) + (1 - \varepsilon)c_2(T)e_1(U_2)(0). 
\]

(19)

Thus, we obtain the desired result with $0 < \varepsilon < \min(1, \frac{c_1(T)}{d_1(T)})$, and $\eta_{1, \varepsilon} = (1 - \varepsilon) \left(c_1(T) - \varepsilon d_1(T)\right)$. □

**Remark 6.** The above results can be applied to various PDE’s (see [2]) and to locally distributed as well as boundary observability (see [5]).

**Remark 7.** Notice that this Theorem can be easily generalized to coupling operators $C$ of the form $C = C_1 - \sum_{i=2}^{K} \varepsilon_i C_i$ where the operators $C_i$, $i = 1, \ldots, K$ are partially coercive operators, and the $\varepsilon_i$ are chosen sufficiently small.
Example 1. Set $\Omega = (0, 1)$. Let $H = L^2(\Omega)$, $A = -\Delta$ with $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. Let $C_i$ be the operators defined by $C_i u = c_i u$ for $i = 1, 2$ and $u \in H$, where $c_i$ are sufficiently smooth coefficients defined on $\overline{\Omega}$, satisfying $c_i \geq 0$ on $\Omega$. Let $0 \leq a_i < b_i \leq 1$ for $i = 1, 2$ be such that $[a_1, b_1] \cap [a_2, b_2] = \emptyset$, and $c_1, c_2$ be such that $c_i > 0$ on $(a_i, b_i)$ and vanishes on $[0, 1] \setminus (a_i, b_i)$ for $i = 1, 2$ and set $c = c_1 - \varepsilon c_2$ where $\varepsilon > 0$ is sufficiently small so that Theorem (3.1) holds, whereas $c > 0$ on $(a_1, b_1)$ and $c < 0$ on $(a_2, b_2)$. We can generalize to coefficients $c$ that change sign several times on $\Omega$, using Remark (7).

Example 2. We can extend this construction to higher dimensions. Let us assume that $\Omega = B(0, R)$, the ball of center 0 and radius $R > 0$ in $\mathbb{R}^n$. We denote its boundary $\Gamma$. Let $H$, $A$, $C_i$ be defined as above where $c_i$ are sufficiently smooth coefficients defined on $\overline{\Omega}$, satisfying $c_i \geq 0$ on $\Omega$. We assume that $\{x \in \Omega, c_i(x) > 0\}$ satisfies (GCC) [10] and that $\text{supp}\{c_1\} \cap \text{supp}\{c_2\} = \emptyset$. We set $c = c_1 - \varepsilon c_2$ where $\varepsilon > 0$ is sufficiently small so that Theorem (3.1) holds. Then, $c$ changes sign in $\Omega$ since $c > 0$ in $\{x \in \Omega, c_1(x) > 0\}$ whereas $c(x) < 0$ in $\{x \in \Omega, c_2(x) > 0\}$. Hence observability of the system through the single observation on $u_2$ holds, even if the coupling coefficient $c$ changes sign in $\Omega$.

Notice that this construction is always possible. It is sufficient for this to choose $c_1$ such that $\{x \in \Omega, c_1(x) > 0\}$ is a neighborhood of $\Gamma(x^0) = \{x \in \Gamma, (x-x^0) \cdot \nu(x) > 0\}$ where $\nu$ is the unit exterior normal to $\Gamma$, where $x^0 \in \mathbb{R}^n \setminus \overline{\Omega}$ is given, and such that $\{x \in \Omega, c_2(x) > 0\}$ is a neighborhood of any radius of the ball $\Omega$ that does not meet $\text{supp}\{c_1\}$.

Remark 8. Hence Theorem (3.1) relaxes the partial coercitivity assumption on the coupling operator $C$, since the single-observed cascade system may still be observable by a single observation even though the coupling operator is not partially coercive.

4. Cascade systems coupled in velocity. We keep the notation of Section 1 and consider a cascade system coupled in velocity, that is

$$\begin{aligned}
\begin{cases}
 u''_1 + Au_1 = 0, \\
u''_2 + Au_2 + Cu'_1 = 0, \\
u_1(0), u'_1(0) = (u^n_1, u^n_2) \\
u_2(0), u'_2(0) = (u^n_0, u^n_1, u^n_2)
\end{cases}
\tag{20}
\end{aligned}$$

The system (20) can then be reformulated as the first order abstract system

$$\begin{aligned}
\begin{cases}
 U' = AU, \\
U(0) = U^0 = (u^0_1, u^0_2, u'_1, u'_2)
\end{cases}
\tag{21}
\end{aligned}$$

where $U = (u_1, u_2, v_1, v_2)$ and $A$ is the unbounded operator in $\mathcal{H}$ with domain $D(A) = H^2_2 \times H^2_1$ defined by

$$AU = (v_1, v_2, -Av_1, -Av_2 - Cv_1).$$

Using semigroup theory, it is easy to establish the well-posedness of the abstract system (21) for initial data $U^0 \in \mathcal{H}$. We now assume that the coupling operator satisfies the following non negativity property (see Remark (10) below)

$$\text{NNP} \begin{cases}
 C^* \in \mathcal{L}(\mathcal{H}), \\
 ||C|| = \beta, |Cw|^2 \leq \beta(Cw, w) \quad \forall w \in \mathcal{H},
\end{cases}$$

instead of assumption (B1). Then we have the following result.
Theorem 4.1. Assume (A), (NNP) and (C1). Then there exists $T^* > 0$ such that for all $T > T^*$, the following property holds

$$
\exists C(T) > 0 \text{ such that for } U^0 \in X \text{ the solution of (20) satisfies } e_1(U_1)(0) + e_1(U_2)(0) \leq C(T) \int_0^T ||B^*P||_G^2 dt.
$$

(24)

if and only if (B2) and (C2) hold. Moreover if (B2) and (C2) hold, we have the following decoupled observability inequalities: for all $T > T^*$, and all initial data $U^0 \in X$, the solution of (20) satisfies the observability estimates

$$
\begin{cases}
  d_1(T) \int_0^T ||B^*P||_G^2 \geq e_1(U_1)(0), \\
  d_2(T) \int_0^T ||B^*P||_G^2 \geq e_1(U_2)(0),
\end{cases}
$$

(25)

where

$$
d_1(T) \leq \frac{K}{T^3}, \quad d_2(T) \leq \frac{K}{T},
$$

(26)

and where $K$ is constant which does not depend on $T$.

Remark 9. We have, as before for cascade systems coupled in displacements (see Remark (2)), a decoupled structure for the observability inequality, more precise than the coupled observability inequality (24).

Remark 10. Note that when $H = L^2(\Omega)$ and $C = c(\cdot)I$ where $c \in L^\infty(\Omega)$ is the coupling coefficient, then (NNP) implies that $c \geq 0$ a.e. in $\Omega$. Conversely if $c \in L^\infty(\Omega)$ and $c \geq 0$ a.e. in $\Omega$, then (NNP) holds. This is the reason why, we named this property a non negativity property. Note also that the sign condition on $c$ has been used as part of the various sufficient conditions proved in the literature on the subject (see e.g. [2, 4, 5, 7, 15, 26, 25]), it holds true for the application to insensitizing control (see [5]). Let us further mention that the above results are still valid if $C$ satisfies the corresponding non positive property (reversing the inequality in (NNP)).

Remark 11. Theorem (4.1) states a necessary and sufficient condition in the class of bounded coupling operators that satisfy the above non negative property. It fully answers the single-observed (resp. single-controlled) problem for cascade systems of two equations in this class of coupling operators.

The proof of Theorem (4.1) will follow that of Theorem (2.1) which holds for a coupling acting on $u_1$ (the displacement) (see [5]). For the sake of completeness, we shall develop it and indicate in Remark (15) how this result can also be deduced directly from Theorem (2.1) and vice versa. This Remark also proves that the smoothness assumption $C^* \in {\mathcal L}(H_1)$ in (B1) on the coupling operator is not necessary (as explained in Remark (4)).

We recall the following results.

Lemma 4.2 ([4], Lemma 3.3, pp. 14). We assume the hypotheses (A), (C1) and (C2). Then, there exist constants $\eta_0 > 0$ and $\alpha_0 > 0$ such that for all $T > T_0$, and for any solution $P = (p, p')$ of the nonhomogeneous equation

$$
p'' + Ap = f \in L^2([0,T]; H),
$$

(27)

the following uniform observability estimate holds

$$
\eta_0 \int_0^T ||B^*P||_G^2 dt \geq \int_0^T e_1(P)(t) dt - \alpha_0 \int_0^T |f|^2 dt.
$$

(28)
In a similar way if \( (A) \) and \( (B2) \) hold then there exist \( \gamma_0 > 0 \) and \( \delta_0 > 0 \) such that for all \( T > T_0 \), and for any solution \( P = (p, p') \) of (27), the following uniform observability estimate holds

\[
\gamma_0 \int_0^T |Cp'|^2 \, dt \geq \int_0^T e_1(P)(t) \, dt - \delta_0 \int_0^T |f|^2 \, dt. \tag{29}
\]

**Remark 12.** We follow the presentation of [5] rather than the original Lemma 3.3 in [4] for the sake of the presentation and without loss of generality.

We deduce from this lemma the following corollary.

**Corollary 1.** Assume that \( (A) \) and \( (B2) \) hold, then there exists a constant \( \gamma_0 > 0 \) such that for all \( T > T_0 \) and for any solution \( V = (v, v') \) of

\[
\begin{align*}
(v'' + Av &= 0, \\
(v, v')(0) &= (v^0, v^1),
\end{align*}
\tag{30}
\]

the following observability inequality holds

\[
T e_1(V)(0) \leq \gamma_0 \int_0^T |Cv'|^2 \, dt. \tag{31}
\]

**Remark 13.** The main point in the above results is that the constants \( \eta_0, \alpha_0, \gamma_0, \delta_0 \) are uniform with respect to \( T \) (this has to be proved and is not contained in the assumptions \( (B2) \) and \( (C2) \)).

We shall now prove intermediate estimates to prove the final observability estimates.

**Lemma 4.3.** (Admissibility property) Assume the hypotheses \( (A), (C1) \) and that \( C \in \mathcal{L}(H) \), then for all \( T > 0 \), there exists a constant \( C(T) > 0 \) such that for all initial data \( U^0 \in H \), the solution of (20) satisfies the following direct inequality

\[
\int_0^T ||B^* U_2||_G^2 \, dt \leq C(T) \left( e_1(U_1)(0) + e_1(U_2)(0) \right). \tag{32}
\]

**Remark 14.** This Lemma establishes a hidden regularity property of the solutions by density arguments and extension, namely that for all \( U^0 \in X \), \( B^* U_2 \in L^2([0, T]; G) \).

**Lemma 4.4.** We assume \( (A) \). Let \( U^0 \in D(A) \). Then the solution of (20) satisfies the estimate

\[
\int_0^T \langle Cu'_1, u'_1 \rangle \, dt \leq 2\eta e_1(U_1)(0) + \frac{1}{\eta} \left( e_1(U_2)(T) + e_1(U_2)(0) \right), \forall \eta > 0. \tag{33}
\]

**Proof.** Since \( (u_1, u_2) \) is a solution of (20), we have

\[
\int_0^T \left( \langle u''_1 + Au_1, u'_2 \rangle + \langle u''_2 + Au_2 + Cu'_1, u'_1 \rangle \right) \, dt = 0,
\]

so that

\[
\int_0^T \langle Cu'_1, u'_1 \rangle \, dt = - \left[ \langle u'_1, u'_2 \rangle \right]_0^T - \left[ \langle A^{1/2}u_1, A^{1/2}u_2 \rangle \right]_0^T. \tag{34}
\]

We estimate the right hand side of (34) as follows. We set

\[
J(t) = \langle u'_1, u'_2 \rangle + \langle A^{1/2}u_1, A^{1/2}u_2 \rangle.
\]
Then we have

\[ |J(t)| \leq \frac{\eta}{2} \left( |u_1'|^2 + |A^{1/2} u_1|^2 \right) + \frac{1}{2\eta} \left( |A^{1/2} u_2|^2 + |u_2'|^2 \right). \]

Since the energy \( e_1(U_1) \) is conserved, we obtain the desired estimate. \( \square \)

**Corollary 2.** We assume \((A), (N NP), \) and \((B2) - (C2)\). Let \( U^0 \in D(A) \). Then the solution of \((20)\) satisfies

\[ \int_0^T \langle Cu_1', u_1' \rangle \, dt \leq \frac{8\beta \gamma_0}{T} (e_1(U_2)(T) + e_1(U_2)(0)). \]  

(35)

**Proof.** Thanks to the hypothesis \((B2)\) for \((u_1, u_1')\) and to Lemma \((4.2)\), \((31)\) holds for \( V = U_1 \). Thus, thanks to the hypotheses \((N NP)\) and \((B2)\) and to \((31)\) together with \((33)\) with the choice \( \eta = \frac{T}{16\beta \gamma_0} \), we obtain

\[ e_1(U_1)(0) \leq \frac{8\beta^2 \gamma_0^2}{T^2} (e_1(U_2)(T) + e_1(U_2)(0)) \].

Using this estimate in \((33)\) with the above choice of \( \eta \), we get \((35)\). \( \square \)

**Lemma 4.5.** Assume the hypotheses of Corollary \((2)\). Then,

\[ (e_1(U_2)(T) + e_1(U_2)(0)) \leq c_1 e_1(U_2)(0) + \frac{c_2 \beta^2 \gamma_0}{T} \int_0^T e_1(U_2) \, dt, \]  

(36)

and,

\[ \int_0^T \langle Cu_1', u_1' \rangle \, dt \leq \frac{c_3 \beta \gamma_0}{T} e_1(U_2)(0) + \frac{c_4 \beta^3 \gamma_0^2}{T^2} \int_0^T e_1(U_2) \, dt, \]  

(37)

where \( c_1, c_2, c_3, c_4 \) are positive numerical constants.

**Proof.** Let \((u_1, u_2)\) be a solution of \((20)\), then we have

\[ \langle u_2'' + Au_2 + Cu_1', u_2' \rangle = 0, \]

so that

\[ e_1'(U_2)(t) = -\langle Cu_1', u_2' \rangle(t). \]  

(38)

Integrating this relation between 0 and \( T \), we obtain

\[ e_1(U_2)(T) + e_1(U_2)(0) \leq 2e_1(U_2)(0) + \frac{T}{16\beta \gamma_0} \int_0^T \langle Cu_1', u_1' \rangle \, dt + \frac{8\beta^2 \gamma_0}{T} \int_0^T e_1(U_2) \, dt. \]

We use \((35)\) in this last estimate. This gives \((36)\). Using \((36)\) in \((35)\), we obtain \((37)\). \( \square \)

**Lemma 4.6.** Assume the hypotheses of Corollary \((2)\). Then,

\[ \int_0^T e_1(U_2) \, dt \geq M T e_1(U_2)(0), \]  

(39)

where \( M \) is defined by \((43)\) and depends only on \( \beta, \gamma_0 \).

**Proof.** We set

\[ a = \frac{c_3 \beta^2 \gamma_0}{2}, \]  

(40)

and

\[ b = \frac{c_4 \beta^3 \gamma_0^2}{2}. \]  

(41)

We define

\[ \nu = (a + \sqrt{a^2 + a + b}) \]  

(42)
We integrate twice the two sides of (38), first between 0 and \(s\) and then between 0 and \(T\). This gives
\[
\int_0^T e_1(U_2) \, dt = Te_1(U_2)(0) - \int_0^T (T - t) \langle Cu_1', u_2' \rangle \, dt.
\]
Using Young’s inequality on the second term of the right hand side in the above relation, together with (23) in assumption \((NNP)\) and the definition of \(e_1(U_2)\), we deduce that
\[
(1 + \nu) \int_0^T e_1(U_2) \, dt \geq Te_1(U_2)(0) - \frac{\beta T^2}{2\nu} \int_0^T \langle Cu_1', u_2' \rangle \, dt.
\]
Using (37) in this last estimate and the definition of \(a, b\) and \(\nu\), we obtain (39) where
\[
M = \sqrt{a^2 + a + b} \left( a + \sqrt{a^2 + a + b} \right) + a + 2b.
\]

Lemma 4.7. Assume the hypotheses of Corollary (2). We set
\[
T_1 = \sqrt{2c_4\alpha_0\beta_0^2\gamma_0}, \quad T_2 = \sqrt{2c_3\alpha_0\beta_2^2\gamma_0} / \sqrt{M}, \quad T_3 = \max\left( T_1, T_2 \right), \quad T^* = \max\left( T_0, T_C, T_3 \right),
\]
where \(T_C\) is given in \((B2)\), and \(T_0\) in \((C2)\). Then for all \(T > T^*\), we have
\[
\eta_0 \int_0^T \|B^* U_2\|_G^2 \, dt \geq \frac{M}{T} \left( T_2^2 - T_3^2 \right) e_1(U_2)(0),
\]
and
\[
\int_0^T e_1(U_2) \, dt \leq \eta_0 \frac{T^2}{T_2^2 - T_3^2} \int_0^T \|B^* U_2\|_G^2 \, dt.
\]

Proof. Applying Lemma (4.2) for the equation satisfied by \(u_2\) in (20), we deduce that (28) holds, that is
\[
\eta_0 \int_0^T \|B^* U_2\|_G^2 \, dt \geq \int_0^T e_1(U_2) \, dt - \alpha_0 \beta \int_0^T \langle Cu_1', u_1' \rangle \, dt.
\]
We use (37) and (39) in this last inequality. Then for all \(T > T^*\), we obtain (45). Using in a similar way, (37) and (39) in (47), together with the definitions of \(T_1\), \(T_2\), and \(T_3\), and (45) in the resulting equation, we obtain (46). Hence, we proved that we can reconstruct the initial data of the second component of the state, which is coupled to the first component, from the observation of this second component. We now have to prove that we can reconstruct the energy of the first component from the observation of the second component.

Lemma 4.8. Assume the hypotheses of Theorem (2). We define \(T^*\) as in (44). Then for all \(T > T^*\), we have
\[
e_1(U_1)(0) \leq \frac{\eta_0 \gamma_0^2 \beta^2}{(T^2 - T_3^2)T} \left[ \frac{c_3}{M} + c_4 \beta^2 \gamma_0 \right] \int_0^T \|B^* U_2\|_G^2 \, dt.
\]
Proof. Using (46) and (37), and since $T > T^*$, we obtain
\[
\int_0^T \langle Cu_1', u_1' \rangle \, dt \leq \left( \frac{c_3 \gamma_0 \beta}{\gamma_0} \right) T \int_0^T \langle Cu_2', u_2' \rangle \, dt.
\]
(49)
Thanks to the uniform observability inequality (31), and to (NNP), we deduce that
\[
e_1(U_1)(0) \leq \frac{\gamma_0 \beta}{T} \int_0^T \langle Cu_1', u_1' \rangle \, dt.
\]
(50)
Inserting (45) in (49) and using (50) we derive (48).

Proof of Theorem (4.1). Thus, thanks to (45) and (48), the sufficient part of Theorem (4.1) is proved. The proof of the necessary part can easily be adapted from that of Theorem 2.8 in [5].

Remark 15. Theorem (4.1) (resp. its sufficient part) and Theorem (2.2) (resp. Theorem (2.1)) can be deduced from each other. This is due to the following property. If $(u_1, u_2)$ solves the cascade system coupled in velocity (20), that is
\[
\begin{align*}
u_1'' + A u_1 &= 0, \\
u_2'' + A u_2 + C u_1 &= 0, \\
(u_i, u_i')(0) &= (u_i^0, u_i^1) \quad \text{for } i = 1, 2,
\end{align*}
\]
then setting $v_1 = u_1', (v_1, u_2)$ solves the cascade system coupled in displacement (2), that is
\[
\begin{align*}
u_1'' + A v_1 &= 0, \\
u_2'' + A u_2 + C v_1 &= 0, \\
(v_1, v_1', u_2(0), u_2')(0) &= (u_1^1, -A u_1^0, u_2^0, u_2^1).
\end{align*}
\]
Noticing further that $e_1(U_1)(0) = e_0(V_1)$, we easily conclude.

Hence we deduce the following Corollary, that we formulate explicitly for the sake of completeness.

Corollary 3. Assume (A), (NNP) and (C1). Then there exists $T^* > 0$ such that for all $T > T^*$, the observability inequality (12) holds, if and only if (B2) and (C2) hold. Moreover if (B2) and (C2) hold, then for all $T > T^*$, and all initial data $U^0 \in X$, the solution of (2) satisfies the decoupled observability estimates (11).

Remark 16. Theorem (4.1) can be as well deduced from the above Corollary as indicated in Remark (15).

Remark 17. We can of course generalize Theorem (4.1) to non partially coercive coupling operators as in Theorem (3.1), under similar smallness conditions.

Let us now consider the single-controlled system coupled in velocity
\[
\begin{align*}
y_1'' + A y_1 - C^* y_2 &= 0, \\
y_2'' + A y_2 &= B v, \\
(y_i, y_i')(0) &= (y_i^0, y_i^1) \quad \text{for } i = 1, 2,
\end{align*}
\]
(51)
where either $B \in \mathcal{L}(G; H)$ (bounded control operator) or $B \in \mathcal{L}(G, H_2')$ (unbounded control operator). The solutions of this systems are defined by transposition (see e.g. [22, 5]).
By standard duality arguments, we deduce the following exact controllability result.

**Theorem 4.9.** Assume the hypotheses $(A)$, $(N_{NP})$ and $(B2) - (C2)$. We set
\[ X_{-1}^* = H_1 \times H_1 \times H \times H, \]
and
\[ X^*_1 = H \times H \times H_{-1} \times H_{-1}. \]
We define $T^* > 0$ as in Theorem (4.1). We have the following properties.

- (i) Let $B^*(w, w') = B^* w'$ where $B \in \mathcal{L}(G, H)$ is such that $(C1) - (C2)$ holds. Assume moreover that $C^* \in \mathcal{L}(H_1)$. Then, for all $T > T^*$, and all $Y_0 \in X_{-1}^*$, there exists a control function $v \in L^2((0, T); G)$ such that the solution $Y = (y_1, y_2, y'_1, y'_2)$ of (51) satisfies $Y(T) = 0$.

- (ii) Let $B^*(w, w') = B* w'$ where $B \in \mathcal{L}(G, H'_2)$ is such that $(C1) - (C2)$ holds. Then, for all $T > T^*$, and all $Y_0 \in X^*_1$, there exists a control function $v \in L^2((0, T); G)$ such that the solution $Y = (y_1, y_2, y'_1, y'_2)$ of (51) satisfies $Y(T) = 0$.

**Proof.** We first consider the case (i). Let $Y^0 = (y_1^0, y_2^0, y'_1^0, y'_2^0) \in X_{-1}^*$. We consider the bilinear form $\Lambda$ on $X_{-1} := H \times H \times H_{-1} \times H_{-1}$, defined by
\[ \Lambda(W^T, \tilde{W}^T) = \int_0^T \langle B^* w_2, B^* \tilde{w}'_2 \rangle_G \, dt, \forall W^T, \tilde{W}^T \in X_{-1}, \tag{52} \]
and the linear form on $X_{-1}$ defined for all $W^T \in X_{-1}$ by
\[ \mathcal{L}(W^T) = \langle y'_1, w_1(0) \rangle_{H_{-1}} - \langle y'_2, w_2(0) \rangle_{H_{-1}}, \forall W^T \in X_{-1}, \tag{53} \]
where $W = (w_1, w_2, w'_1, w'_2)$ and $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}'_1, \tilde{w}'_2)$ are respectively solutions of
\[
\begin{cases}
  w''_1 + Aw_1 = 0, \\
  w''_2 + Aw_2 + Cw'_1 = 0, \\
  W_{|t=T} = W^T.
\end{cases} \tag{54}
\]
and
\[
\begin{cases}
  \tilde{w}''_1 + A\tilde{w}_1 = 0, \\
  \tilde{w}''_2 + A\tilde{w}_2 + C\tilde{w}'_1 = 0, \\
  \tilde{W}_{|t=T} = \tilde{W}^T.
\end{cases} \tag{55}
\]
We set $Z = A^{-1}W$. From the usual energy estimates for the time reverse problem for $Z$, and the conservation of $e_1(Z_1)$ through time we have
\[ e_1(Z_1)(0) + e_1(Z_2)(0) \leq C_0(e_1(Z_1)(T) + e_1(Z_2)(T)), \tag{56} \]
where $C_0$ is a positive constant. On the other hand, we have
\[ \Lambda(W^T, W^T) = \int_0^T \|B^* w_2\|^2_G \, dt = \int_0^T \|B^* Z_2\|^2_G \, dt, \]
so that thanks respectively to the admissibility inequality (32) and to the observability inequality (24) with $T$ replacing $0$, for $T > T^*$, we have
\[ C_1 \left( e_1(Z_1)(T) + e_1(Z_2)(T) \right) \leq \Lambda(W^T, W^T) \leq C_2 \left( e_1(Z_1)(T) + e_1(Z_2)(T) \right), \]
and
\[ C_1 \left( e_1(Z_1)(T) + e_1(Z_2)(T) \right) \leq \Lambda(W^T, W^T) \leq C_2 \left( e_1(Z_1)(T) + e_1(Z_2)(T) \right). \]
where $C_1$ and $C_2$ are positive constants. Now since $Z = A^{-1}W$, one can easily prove that there exists positive constants $D_1$ and $D_2$ such that

$$D_1 \left( e_1(Z_1)(T) + e_1(Z_2)(T) \right)$$

$$\leq e_0(W_1)(T) + e_0(W_2)(T) \leq D_2 \left( e_1(Z_1)(T) + e_1(Z_2)(T) \right),$$

This proves that $\Lambda$ is continuous and coercive on $X_{-1}$. We prove in a similar way that $\mathcal{L}$ is continuous on $X_{-1}$. Hence, thanks to Lax-Milgram Lemma, there exists a unique $W^T \in X_{-1}$ such that

$$\Lambda(W^T, \tilde{W}^T) = -\mathcal{L}(\tilde{W}^T), \quad \forall \tilde{W}^T \in X_{-1}.$$  \tag{57}

We set $v = B^*w_2$. Then, thanks to the hidden regularity property due to (32), we have $v \in L^2([0,T];G)$. Thus, we have by definition of the solution of (1) by transposition

$$\int_0^T \langle v, B^*\tilde{w}_2 \rangle_G dt = \langle y'_1(T), \tilde{w}_1(T) \rangle_{H,H} - \langle y_1(T), \tilde{w}_1'(T) \rangle_{H_1,H_{-1}}$$

$$+ \langle y'_2(T), \tilde{w}_2(T) \rangle_{H,H} - \langle y_2(T), \tilde{w}_2'(T) \rangle_{H_1,H_{-1}}$$

$$- \langle C^*y_2(T), \tilde{w}_1(T) \rangle_{H,H} - \mathcal{L}(\tilde{W}^T), \forall \tilde{W}^T \in X_{-1}.$$  \tag{58}

On the other hand, we have

$$\int_0^T \langle v, B^*\tilde{w}_2 \rangle_G dt = \Lambda(W^T, \tilde{W}^T) = -\mathcal{L}(\tilde{W}^T),$$

so that, we deduce from these two relations that $Y(T) = (y_1, y_2, y'_1, y'_2)(T) = 0$.

Assume now that (ii) holds. Let $Y_0 \in X^*_1$. We consider on $X_1 := H_1 \times H_1 \times H \times H$ the bilinear form

$$\Lambda(U^T, \tilde{U}^T) = \int_0^T \langle B^*u_2, B^*\tilde{w}_2 \rangle_G dt, \forall U^T, \tilde{U}^T \in X_1,$$  \tag{59}

and the linear form on $X_1$ defined by

$$\mathcal{L}(U^T) = \langle y_1^1, u_1(0) \rangle_{H_{-1},H_{1}}, - \langle y'_1^0, u'_1(0) \rangle_{H,H}$$

$$+ \langle y_2^1, u_2(0) \rangle_{H_{-1},H_{1}}, - \langle y'_2^0, u'_2(0) \rangle_{H,H} - \langle C^*y'_2^0, w_1(0) \rangle_{H,H}, \forall U^T \in X_1.$$  \tag{60}

Thanks respectively to the admissibility inequality (32) and to the observability inequality (24), $\Lambda$ is continuous and coercive on $X_1$ for $T > T^*$. On the other hand $\mathcal{L}$ is continuous on $X_1$ thanks to (56) with $U$ replacing $Z$. Hence, thanks to Lax-Milgram Lemma, there exists a unique $U^T \in X_1$ such that

$$\Lambda(U^T, \tilde{U}^T) = -\mathcal{L}(\tilde{U}^T), \quad \forall \tilde{U}^T \in X_1.$$  \tag{61}

We set $v = B^*u_2$. We deduce as for the case (i) that $Y(T) = 0$. \hfill \square

5. **Non-unique continuation results.** We focus in this section on the unique continuation property, that we shall denote by $(UCP)$ in the sequel. We recall that the unique continuation property is said to hold for the system (2) and the observation operator $B^*$, if any solution $(u_1, u_2)$ of (2) such that $B^*(u_2, u'_2) \equiv 0$ satisfies $(u_1, u_2) \equiv (0, 0)$. This is a weaker property than the observability property. By duality, the unique continuation property implies approximate controllability, whereas the observability property implies the exact controllability for the control problem (1) (both in suitable functional spaces).
Let us give some mathematical motivations for this section. In Corollary (3), we give a necessary and sufficient condition, in the class of non negative bounded coupling operators (see (1NPP)) for positive observability results for single-observed cascade systems of two equations. In particular, these results apply to the following dual cascade system of one-dimensional wave equations

\[
\begin{aligned}
    u_{1,tt} - u_{1,xx} &= 0, \quad 0 < t < T, \quad 0 < x < \pi, \\
    u_{2,tt} - u_{2,xx} + c(x)u_1 &= 0, \quad 0 < t < T, \quad 0 < x < \pi, \\
    u_i(t,0) &= u_i(t,\pi) = 0, \quad \text{for } i = 1,2, \quad 0 < t < T, \\
    u_i(0,x) &= u_i^0(x), \quad u_{i,t}(0,x) = u_{i1}^1(x), \quad \text{for } i = 1,2, \quad 0 < x < \pi,
\end{aligned}
\]  

and the following boundary observation operator

\[
B^*u_2 = u_{2,x}(.,0).
\]  

Note that (62) is well-posed for any initial data \((u_1^0, u_2^0) \in L^2(0,\pi) \times H^{-1}(0,\pi)\) and \((u_2^0, u_2^1) \in H^1_0(0,\pi) \times L^2(0,\pi)\), that is: there is a unique solution \((u_1, u_2)\) such that \(u_1 \in \left(C^0([0,T]; L^2(0,\pi)) \cap C^1([0,T]; H^{-1}(0,\pi))\right)\), \(u_2 \in \left(C^0([0,T]; H^1_0(0,\pi)) \cap C^1([0,T]; L^2(0,\pi))\right)\).

Thanks to Corollary (3), we prove that in the class of coefficients \(c\) such that

\[c \in L^\infty(0,\pi),\]
\[c \text{ keeps a constant sign in } (0,\pi),\]

there exists \(T^* > 0\) such that for all \(T > T^*\), the following observability estimates hold

\[
\begin{aligned}
    \int_0^T |u_{2,x}(t,0)|^2 dt &\geq C_1(T) \left\| (u_1^0, u_1^1) \right\|_{L^2(0,\pi) \times H^{-1}(0,\pi)}^2, \\
    \int_0^T |u_{2,x}(t,0)|^2 dt &\geq C_2(T) \left\| (u_2^0, u_2^1) \right\|_{H^1_0(0,\pi) \times L^2(0,\pi)}^2,
\end{aligned}
\]  

for all \((u_1^0, u_1^1) \in L^2(0,\pi) \times H^{-1}(\Omega)\) and all \((u_2^0, u_2^1) \in H^1_0(0,\pi) \times L^2(0,\pi)\), where \(C_1(T), C_2(T)\) are positive constants.

We also proved in Theorem (3.1) that this non negativity property can be relaxed under some smallness condition.

Therefore, a further challenging question, is to determine whether this result always hold true for more general coupling operators, that is operators that are neither non negative or non positive and that do not satisfy this smallness property. The purpose of this section is to show that once we consider coupling coefficients that may change sign with no smallness conditions (such as the conditions in section 3), the situation is much more complex. Hence we reverse our point of view, and now look for negative results in this direction. Also, since the situation is more complex, it is natural to work on the weaker property \((UCP)\), and to look for negative examples for \((UCP)\), leading to negative examples for the observability property.

Our main concern in the present paper, is to answer the following questions:

1. Does there exist situations for which \(c\) changes sign in \((0,\pi)\) and for which \((UCP)\) does not hold?
2. Does the observability estimate (65) still hold for coefficients \(c\) that violate (64), in particular for coefficients that are changing sign within \((0,\pi)\)?
3. If this observability estimate does not hold, does it hold in weaker spaces, at least for certain initial data?
We have already shown in section 3 that the answer to the above question 2 is positive (see Theorem (3.1), and also Remark (17)). We provide in this section, several examples of coupling coefficients \( c \) and operators \( A \) showing that the answer to the above question 1 is positive. Examples showing that the answer to question 3 may be positive are given in section 6.

Our first result shows that there exist smooth coupling coefficients \( c \) which change sign within \((0, \pi)\) and for which, there is non unique continuation.

We start with a first example of coupling coefficients for which unique continuation does not hold.

**Proposition 1.** Let \( k_0 \in \mathbb{N}^* \) be given and let us assume that the coefficient \( c \) is given by
\[
c(x) = 8k_0^3 \left(3 - 4 \sin^2(k_0 x)\right) \quad x \in [0, \pi].
\] (66)
Then there exist non-vanishing initial data \((u_0^1, u_1^0, u_1^1, u_2^1) \in L^2(0, \pi) \times H^1(0, \pi) \times H^{-1}(0, \pi) \times L^2(0, \pi), i = 1, 2,\) for which the solution of (62) satisfies \( B^* u_2(t, 0) = 0 \) for all \( t \in (0, T) \).

**Proof.** Let \((\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}\) be arbitrary. Let us consider the following set of initial data
\[
(u_0^1, u_1^1)(x) = (\alpha \sin(k_0 x), k_0 \beta \sin(k_0 x)),
\]
\[
(u_2^0, u_2^1)(x) = (4k_0 \alpha \sin^3(k_0 x), 4k_0 \beta \sin(k_0 x)^3).
\]
for \( x \in [0, \pi] \). Then one can easily check that the couple of functions defined by
\[
(u_1, u_2)(t, x) = 
\left((\alpha \cos(k_0 t) + \beta \sin(k_0 t)) \sin(k_0 x), 4k_0 (\alpha \cos(k_0 t) + \beta \sin(k_0 t)) \sin^3(k_0 x)\right)
\]
is in \( C^0([0, T]; H^1_0(0, \pi) \times L^2(0, \pi))^2 \) and solves (62). Hence by uniqueness, \((u_1, u_2)\) is the solution of (62). Moreover \( B^* u_2 = 0 \). Therefore, there is not unique continuation for (62) when \( c \) is given by (66).

**Remark 18.** Note that for \( k_0 = 1 \), the function \( c \) given by (66) changes sign four times within \((0, \pi)\) and it is positive and negative on open nonempty subintervals of \((0, \pi)\).

**Remark 19.** Non unique continuation also holds for any \( c \) of the form
\[
c(x) = m_0 \left((2m_0^2 - 4k_0^2) \cos(m_0 x) + 2m_0 k_0 \cos(k_0 x) \sin(m_0 x) \right) / \sin(k_0 x),
\]
where \( m_0 = pk_0 \) with \( p \geq 2 \), so that \( c \) is well-defined and smooth on \([0, \pi] \). The solution which does not satisfy the unique continuation property can then easily be deduced (this is left to the reader).

In the above two examples, the given set of initial data for which unique continuation does not hold is finite dimensional. We now give examples of coupling coefficients for which the set of initial data such that the unique continuation property does not hold, is infinite dimensional (and we characterize it).

**Theorem 5.1.** Assume that the coupling coefficient is given by
\[
c(x) = \cos(m_0 x), \quad x \in (0, \pi), \quad \text{where} \quad m_0 = 2p_0 + 1, \quad p_0 \in \mathbb{N}.
\] (67)
Define
\[ I = \left\{ (u_0^0, u_0^1, u_1^1, u_2^0) \mid u_0^0(x) = \sum_{k=1}^{\infty} a_k \sin(kx), u_1^0(x) = \sum_{k=1}^{\infty} kb_k \sin(kx), \right\} \]
\[ u_2^0(x) = \sum_{k=1}^{m_0-1} \left( \frac{a_k}{4k^2 - m_0^2} + \frac{a_{m_0+k}}{2m_0(2k + m_0)} - \frac{a_{m_0-k}}{2m_0(m_0 - 2k)} \right) \sin(kx) + \sum_{k=m_0+1}^{\infty} \left( \frac{a_k}{4k^2 - m_0^2} + \frac{a_{m_0+k}}{2m_0(2k + m_0)} - \frac{a_{m_0-k}}{2m_0(m_0 - 2k)} \right) \sin(kx) \]
\[ + \left( a_{m_0} + \frac{a_{m_0+2}}{2} \right) \frac{\sin(m_0x)}{3m_0^2}, \quad x \in (0, \pi), \]
\[ u_1^2(x) = \sum_{k=1}^{m_0-1} \left( \frac{kb_k}{4k^2 - m_0^2} + \frac{(k + m_0)b_{m_0+k}}{2m_0(2k + m_0)} - \frac{(m_0 - k)b_{m_0-k}}{2m_0(m_0 - 2k)} \right) \sin(kx) + \sum_{k=m_0+1}^{\infty} \left( \frac{kb_k}{4k^2 - m_0^2} + \frac{(k + m_0)b_{m_0+k}}{2m_0(2k + m_0)} - \frac{(m_0 - k)b_{m_0-k}}{2m_0(m_0 - 2k)} \right) \sin(kx) \]
\[ + \left( b_{m_0} + b_{m_0+2} \right) \frac{\sin(m_0x)}{3m_0}, \quad x \in (0, \pi), \]
\[ \sum_{k=1}^{\infty} a_k^2 < \infty, \quad \sum_{k=1}^{\infty} b_k^2 < \infty \].

Then \( I \) is a set of infinite dimension such that
\[ I \subset L^2(0, \pi) \times H^1_0(0, \pi) \times H^{-1}(0, \pi) \times L^2(0, \pi), \]
and for all \((u_1^0, u_0^0, u_1^1, u_2^0) \in I\), the solution of (62) satisfies
\[ B^*u_2(., 0) \equiv 0 \text{ on } (0, T), \]
where \( B^* \) is defined in (63). Moreover \( I \setminus \{(0, 0, 0, 0)\} \) is the set of initial data for which unique continuation does not hold.

**Remark 20.** The main interest of this choice of coupling coefficient is that it shows how this coefficient interacts with elementary waves composing the solution \( u_1 \) of the free wave equation, to generate non-unique continuation properties, modifying by this way the dynamics of the whole system. This interaction occurs at all frequencies, and in particular at high frequency. Notice that if \( c(x) = \cos(x/2) \) (in this case \( m_0 \) is no longer an integer), then \( c \geq 0 \) on \( \Omega \) and \( c > 0 \) in a non-empty subset of \( \Omega \), hence Theorem (2.2) can be applied, so that observability estimates hold in the expected space. So the properties of the coupling coefficient are important to determine whether if they interact with elementary waves or not.

**Remark 21.** For \( m_0 = 1 \), we have \( c(x) = \cos(x) \), so that \( c(x) \geq 0 \) for \( x \in [0, \frac{\pi}{2}] \) and \( c(x) \leq 0 \) for \( x \in [\frac{\pi}{2}, \pi] \).

**Remark 22.** The above result extends to even integers \( m_0 \), one has to remove from all the above sums over \( k \), the case \( k = \frac{m_0}{2} \), so that a similar non-unique continuation result holds.

**Proof.** of Theorem (5.1). We recall that
\[ u_1^0 \in L^2(0, \pi) \iff u_1^0(x) = \sum_{k=1}^{\infty} a_k \sin(kx) \forall x \in (0, \pi), \text{ with } \sum_{k=1}^{\infty} |a_k|^2 < \infty. \]
Moreover, the unique solution of
\[ u_1(t,0) = u_1(t,\pi) = 0, \quad 0 < t < T, \]
\[ u_0(x) = \sum_{k=1}^{\infty} a_k \sin(kx) \forall x \in (0,\pi) , \quad u_1(x) = \sum_{k=1}^{\infty} k b_k \sin(kx) \forall x \in (0,\pi) , \]

where \( \sum_{k=1}^{\infty} |a_k|^2 < \infty \) and \( \sum_{k=1}^{\infty} |b_k|^2 < \infty \), is then given by
\[ u_1(t,x) = \sum_{k=1}^{\infty} (a_k \cos(kt) + b_k \sin(kt)) \sin(kx) , \quad t \in [0,T], x \in (0,\pi) . \] (70)

Here we use the fact that the functions \( e_k(\cdot) = \sin(k \cdot) \) for \( k \in \mathbb{N}^* \) are the eigenfunctions of the operator \( A = -\frac{d^2}{dx^2} \) with domain \( D(A) = H^2(0,\pi) \cap H_0^1(0,\pi) \) associated to the eigenvalues \( \lambda_k = k^2. \) This family of eigenfunctions is an orthogonal basis of \( L^2(0,\pi) \) and the usual results of spectral analysis are available.

Replacing \( u_1 \) by the expression given in (70), the equation in \( u_2 \) becomes
\[
\begin{cases}
    u_{2,tt} - u_{2,xx} + 
    \cos(m_0x) \sum_{k=1}^{\infty} (a_k \cos(kt) + b_k \sin(kt)) \sin(kx) = 0 \text{ in } (0,T) \times (0,\pi), \\
    u_2(t,0) = u_2(t,\pi) = 0, \quad \text{for } 0 < t < T, \\
    u_2(0,x) = u_2^0(x), \quad u_2(t,0,x) = u_2^1(x), \quad \text{for } 0 < x < \pi .
\end{cases}
\] (71)

We set
\[ \phi_k(t) = a_k \cos(kt) + b_k \sin(kt) \] (72)

Using the properties of the \textit{sine} and \textit{cosine} functions, and since \( m_0 \) has been assumed to be an odd integer, we have
\[ u_2(t,x) = \sum_{k=1}^{\infty} (a_k \cos(kt) + \beta_k \sin(kt)) \sin(kx) + \sum_{k=1}^{\infty} \frac{\phi_k(t)}{2m_0} \left[ \frac{\sin((k-m_0)x)}{2k-m_0} - \frac{\sin((k+m_0)x)}{2k+m_0} \right] , \] (73)

where the \( \alpha_k \) and \( \beta_k \) will be determined later on. We deduce from this relation
\[ u_{2,x}(t,0) = \sum_{k=1}^{\infty} \left[ k \left( \alpha_k - \frac{a_k}{4k^2-m_0^2} \right) \cos(kt) + k \left( \beta_k - \frac{b_k}{4k^2-m_0^2} \right) \sin(kt) \right] . \]

Hence, we have
\[ u_{2,x}(t,0) = 0 \iff \alpha_k = \frac{a_k}{4k^2-m_0^2} \text{ and } \beta_k = \frac{b_k}{4k^2-m_0^2} . \] (74)
Reindexing appropriately the last sum in (73), we obtain
\[ u_2(t, x) = \sum_{k=1}^{m_0-1} \left[ \alpha_k \cos(kt) + \beta_k \sin(kt) + \frac{1}{2m_0} \left( \phi_{m_0+k}(t) - \phi_{m_0-k}(t) \right) \right] \sin(kx) \]
\[ + \sum_{k=m_0+1}^{\infty} \left[ \alpha_k \cos(kt) + \beta_k \sin(kt) + \frac{1}{2m_0} \left( \phi_{m_0+k}(t) - \phi_{m_0-k}(t) \right) \right] \sin(kx) \]
\[ + \left[ \alpha_{m_0} \cos(m_0 t) + \beta_{m_0} \sin(m_0 t) + \frac{\phi_{2m_0}(t)}{6m_0^5} \right] \sin(m_0 x) . \]

Hence \( u_{2,x}(t, 0) = 0 \) holds if and only if
\[ u_2^{0}(x) = \sum_{k=1}^{m_0-1} \left[ \frac{a_k}{4k^2 - m_0^2} + \frac{1}{2m_0} \left( \frac{a_{m_0+k}}{m_0 + 2k} - \frac{a_{m_0-k}}{m_0 - 2k} \right) \right] \sin(kx) \]
\[ + \sum_{k=m_0+1}^{\infty} \left[ \frac{a_k}{4k^2 - m_0^2} + \frac{1}{2m_0} \left( \frac{a_{m_0+k}}{m_0 + 2k} - \frac{a_{m_0-k}}{2k - m_0} \right) \right] \sin(kx) \]
\[ + \left[ \frac{a_{m_0}}{3m_0^2} + \frac{a_{2m_0}}{6m_0} \right] \sin(m_0 x) , \]

and
\[ u_2^{1}(x) = \sum_{k=1}^{m_0-1} \left[ \frac{kb_k}{4k^2 - m_0^2} + \frac{1}{2m_0} \left( \frac{(m_0 + k)b_{m_0+k}}{m_0 + 2k} - \frac{(m_0 - k)b_{m_0-k}}{m_0 - 2k} \right) \right] \sin(kx) \]
\[ + \sum_{k=m_0+1}^{\infty} \left[ \frac{kb_k}{4k^2 - m_0^2} + \frac{1}{2m_0} \left( \frac{(m_0 + k)b_{m_0+k}}{m_0 + 2k} - \frac{(k-m_0)b_{m_0-k}}{2k - m_0} \right) \right] \sin(kx) \]
\[ + \left[ \frac{b_{m_0}}{3m_0} + \frac{b_{2m_0}}{3m_0} \right] \sin(m_0 x) , \]

Hence we recover (68). Setting
\[ c_k = \left[ \frac{a_k}{4k^2 - m_0^2} + \frac{1}{2m_0} \left( \frac{a_{m_0+k}}{m_0 + 2k} - \frac{a_{m_0-k}}{2k - m_0} \right) \right] , \]
for \( k \geq m_0 + 1 \), and noticing that \( \sum_{k=1}^{\infty} |a_k|^2 < \infty \), we check that
\[ \sum_{k \geq m_0+1} k^2 c_k^2 < \infty \]

This proves that \( u_2^0 \in H_0^1(0, \pi) \). Setting
\[ d_k = \left[ \frac{kb_k}{4k^2 - m_0^2} + \frac{1}{2m_0} \left( \frac{(m_0 + k)b_{m_0+k}}{m_0 + 2k} - \frac{(k-m_0)b_{m_0-k}}{2k - m_0} \right) \right] , \]
for \( k \geq m_0 + 1 \), and noticing that \( \sum_{k=1}^{\infty} |b_k|^2 < \infty \), we check that
\[ \sum_{k \geq m_0+1} d_k^2 < \infty . \]

This proves that \( u_2^1 \in L^2(0, \pi) \), which concludes the proof. \( \Box \)

**Remark 23.** We can generalize this Theorem to coefficients \( c \) which are finite linear combination of cosine functions. For the sake of simplicity, we just give the example of such coefficients in the case of odd integers in the arguments of the
involved cosine function (see Remark (22) for the extension to the general case of odd and even arguments). Assume that the coupling coefficient is given by

$$c(x) = \sum_{i=0}^{L} \gamma_i \cos((2i+1)x), \; x \in (0, \pi), \; \text{where} \; \gamma_i \in \mathbb{R}. \quad (75)$$

Then we can characterize explicitly the set of initial data for which the unique continuation property does not hold and this set if infinite dimensional (the details are left to the reader).

We extend these results to coupled cascade systems of plates, that is

$$\begin{align*}
&u_{1,tt} + u_{1,xxxx} = 0, \quad 0 < t < T, 0 < x < \pi, \\
&u_{2,tt} + u_{2,xxxx} + c(x)u_1 = 0, \quad 0 < t < T, 0 < x < \pi, \\
&u_i(t,0) = u_i(t,\pi) = 0 \quad \text{for} \; i = 1, 2, 0 < t < T, \\
&u_i(x,0) = u_i^0(x), \; u_i,tt(0, x) = u_i^1(x), \quad \text{for} \; i = 1, 2, 0 < x < \pi,
\end{align*} \quad (76)$$

for the same form of the coupling coefficient $c$. We set $A = \Delta^2$ with $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ and $H = L^2(\Omega)$.

**Theorem 5.2.** Assume that the coupling coefficient is given by (67), and define

$$\mathcal{T} = \left\{ (u_1^0, u_2^0, u_1^1, u_2^1), u_1^0(x) = \sum_{k=1}^{\infty} a_k \sin(kx), u_1^1(x) = \sum_{k=1}^{\infty} k^2 b_k \sin(kx), \\
 u_2^0(x) = \sum_{k=1}^{\infty} \frac{a_k (3m_0^2 - 2k^2)}{16k^6 - 4k^4m_0^2 + 4k^2m_0^4 - m_0^6} \sin(kx) - \sum_{k=1}^{\infty} \frac{2m_0}{4k^3 + 6k^2m_0 + 4km_0^2 + m_0^3} \left[ \frac{\sin((k + m_0)x)}{4k^3 - 6k^2m_0 + 4km_0^2 - m_0^3} - \frac{\sin((k - m_0)x)}{4k^3 - 6k^2m_0 + 4km_0^2 - m_0^3} \right], x \in (0, \pi), \\
 u_2^1(x) = \sum_{k=1}^{\infty} \frac{k^2 b_k (3m_0^2 - 2k^2)}{16k^6 - 4k^4m_0^2 + 4k^2m_0^4 - m_0^6} \sin(kx) - \sum_{k=1}^{\infty} \frac{2m_0}{4k^3 + 6k^2m_0 + 4km_0^2 + m_0^3} \left[ \frac{\sin((k + m_0)x)}{4k^3 - 6k^2m_0 + 4km_0^2 - m_0^3} - \frac{\sin((k - m_0)x)}{4k^3 - 6k^2m_0 + 4km_0^2 - m_0^3} \right], x \in (0, \pi), \right\} \quad (77)$$

Then, $\mathcal{T}$ is a set of infinite dimension such that

$$\mathcal{T} \subset L^2(0, \pi) \times D(A^{-1/2}) \times D(A^{1/2}) \times L^2(0, \pi),$$

and for all $(u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{T}$, the solution of (76) satisfies

$$B^*u_2(_,0) \equiv 0 \text{ on } (0, T), \quad (78)$$

where $B^*$ is defined in (63). Moreover $\mathcal{T}\backslash\{(0,0,0,0)\}$ is the set of initial data for which unique continuation does not hold.
Proof. We proceed as for the proof of Theorem (5.1). We recall that the unique solution of
\[
\begin{align*}
    u_{1,t} + u_{1,xxxx} &= 0, \quad 0 < t < T, 0 < x < \pi, \\
    u_1(t, 0) &= u_{1,x}(t, 0) = u_1(t, \pi) = u_{1,x}(t, \pi) = 0, \quad 0 < t < T, \\
    u_1^0(x) &= \sum_{k=1}^{\infty} a_k \sin(kx) \forall x \in (0, \pi), \quad u_1^1(x) = \sum_{k=1}^{\infty} k^2 b_k \sin(kx) \forall x \in (0, \pi),
\end{align*}
\]
where \(\sum_{k=1}^{\infty} |a_k|^2 < \infty\) and \(\sum_{k=1}^{\infty} |b_k|^2 < \infty\), is then given by
\[
    u_1(t, x) = \sum_{k=1}^{\infty} (a_k \cos(k^2t) + b_k \sin(k^2t)) \sin(kx), \quad t \in [0, T], x \in (0, \pi).
\]

Here we use the fact that the functions \(e_k(\cdot) = \sin(k \cdot)\) for \(k \in \mathbb{N}^*\) are the eigenfunctions of the operator \(A = \frac{d^4}{dx^4}\) with domain \(D(A) = H^4(0, \pi) \cap H^1_0(0, \pi)\) associated to the eigenvalues \(\lambda_k = k^4\). This family of eigenfunctions is an orthogonal basis of \(L^2(0, \pi)\) and the usual results of spectral analysis are available.

Replacing \(u_1\) by the expression given in (79), the equation in \(u_2\) becomes
\[
\begin{align*}
    \left \{ \begin{array}{l}
    u_{2,t} + u_{2,xxxx} + \\
    \cos(m_0 x) \sum_{k=1}^{\infty} (a_k \cos(k^2t) + b_k \sin(k^2t)) \sin(kx) = 0, \quad 0 < t < T, 0 < x < \pi, \\
    u_2(t, 0) = u_{2,x}(t, 0) = u_2(t, \pi) = u_{2,x}(t, \pi) = 0, \quad \text{for } 0 < t < T, \\
    u_2(0, x) = u_2^0(x), u_2(t, 0, x) = u_2^1(x), \quad \text{for } 0 < x < \pi.
\end{array} \right. 
\end{align*}
\]

We set
\[
    \psi_k(t) = a_k \cos(k^2t) + b_k \sin(k^2t)
\]
Using the properties of the sine and cosine functions, and since \(m_0\) has been assumed to be an odd integer, we have
\[
    u_2(t, x) = \sum_{k=1}^{\infty} (\alpha_k \cos(k^2t) + \beta_k \sin(k^2t)) \sin(kx)
\]
\[
+ \sum_{k=1}^{\infty} \frac{\psi_k(t)}{2m_0} \left[ \frac{\sin((k - m_0)x)}{4k^3 - 6k^2m_0 + 4km_0^2 - m_0^3} - \frac{\sin((k + m_0)x)}{4k^3 + 6k^2m_0 + 4km_0^2 + m_0^3} \right],
\]
where the \(\alpha_k\) and \(\beta_k\) will be determined later on. We deduce from this relation
\[
    u_{2,x}(t, 0) = \sum_{k=1}^{\infty} \left[ k \left( \alpha_k + \frac{a_k(2k^2 - 3m_0^2)}{16k^6 - 4k^4m_0^2 + 4k^2m_0^4 - m_0^6} \right) \cos(k^2t) \right]
\]
\[
+ k \left( \beta_k + \frac{b_k(2k^2 - 3m_0^2)}{16k^6 - 4k^4m_0^2 + 4k^2m_0^4 - m_0^6} \right) \sin(k^2t) \right].
\]
Hence, we have
\[
    u_{2,x}(t, 0) = 0 \iff \alpha_k = \frac{a_k(3m_0^2 - 2k^2)}{16k^6 - 4k^4m_0^2 + 4k^2m_0^4 - m_0^6} \quad \text{and} \quad \beta_k = \frac{b_k(3m_0^2 - 2k^2)}{16k^6 - 4k^4m_0^2 + 4k^2m_0^4 - m_0^6}.
\]
We check easily that the assumptions \(\sum_{k=1}^{\infty} a_k^2 < \infty\) and \(\sum_{k=1}^{\infty} b_k^2 < \infty\) on the coefficients \(a_k\) and \(b_k\) imply that \((u_1^0, u_1^1) \in H \times D(A^{-1/2})\), and \((u_2^0, u_2^1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times L^2(\Omega) = D(A^{1/2}) \times H)\).
6. Examples of sign changing coupling coefficients and observability estimates in weaker spaces. Our second result shows that there exist smooth coupling coefficients \( c \) which changes sign within \((0, \pi)\) and for which, there exist initial data such that the observability estimate (65) does not hold, whereas a weaker observability estimate hold.

**Theorem 6.1.** Let \( k \in \mathbb{N}^* \) be given and let us assume that the coefficient \( c \) is given by

\[
c(x) = \cos(m_0 x) \quad x \in [0, \pi],
\]

where \( m_0 \in \mathbb{N}^* \). Then there exists \( T_* > 0 \) and an infinite dimensional set of non-vanishing initial data \( (u_1^0, u_2^0, u_1^1, u_2^1) \in L^2(0, \pi) \times H_0^1(0, \pi) \times H^{-1}(0, \pi) \times L^2(0, \pi) \) such that for all \( T \geq T_* \), the solution \( u \) of (62) satisfies the weaker observability estimate

\[
\begin{align*}
\int_0^T \sum |u_{2,x}(t, 0)|^2 \, dt &\geq C_3(T) \left( \sum |u_{1}^0|^2 \right)^2_{H^{-1}(0, \pi) \times H^{-2}(0, \pi)}, \\
\int_0^T \sum |u_{2,x}(t, 0)|^2 \, dt &\geq C_4(T) \left( \sum |u_{1}^0|^2 \right)^2_{L^2(0, \pi) \times H^{-1}(0, \pi)},
\end{align*}
\]

where \( C_3(T), C_4(T) \) are positive constants.

**Proof.** We consider initial data given by

\[
u_1^0(x) = \sum_{k=1}^{\infty} a_k \sin(kx) \quad x \in (0, \pi),
\]

\[
u_1^1(x) = \sum_{k=1}^{\infty} kb_k \sin(kx) \quad x \in (0, \pi),
\]

and

\[
u_2^0(x) = \sum_{k=1}^{m_0-1} \left[ \frac{1}{2m_0} \left( \frac{a_{m_0+k}}{m_0 + 2k} - \frac{a_{m_0-k}}{m_0 - 2k} \right) \right] \sin(kx) + \sum_{k=m_0+1}^{\infty} \left[ \frac{1}{2m_0} \left( \frac{a_{m_0+k}}{m_0 + 2k} - \frac{a_{m_0-k}}{2k - m_0} \right) \right] \sin(kx) + \left[ \frac{a_{2m_0}}{6m_0^2} \right] \sin(m_0x),
\]

and

\[
u_2^1(x) = \sum_{k=1}^{m_0-1} \left[ \frac{1}{2m_0} \left( \frac{(m_0 + k)b_{m_0+k}}{m_0 + 2k} - \frac{(m_0 - k)b_{m_0-k}}{m_0 - 2k} \right) \right] \sin(kx) + \sum_{k=m_0+1}^{\infty} \left[ \frac{1}{2m_0} \left( \frac{(m_0 + k)b_{m_0+k}}{m_0 + 2k} - \frac{(k-m_0)b_{k-m_0}}{2k - m_0} \right) \right] \sin(kx) + \left[ \frac{b_{2m_0}}{3m_0} \right] \sin(m_0x),
\]

where \( \sum_{k=1}^{\infty} a_k^2 < \infty \) and \( \sum_{k=1}^{\infty} b_k^2 < \infty \). Thanks to the proof of Theorem (5.1), we know that \( (u_1^0, u_1^1) \in L^2(0, \pi) \times H^{-1}(0, \pi) \) and \( (u_2^0, u_2^1) \in H_0^1(0, \pi) \times L^2(0, \pi) \).

Moreover, we have

\[
u_{2,x}(t, 0) = - \sum_{k=1}^{\infty} \left[ k \left( \frac{ak}{4k^2 - m_0^2} \right) \cos(kt) + k \left( \frac{b_k}{4k^2 - m_0^2} \right) \sin(kt) \right].
\]

Assume that \( T \geq 2q\pi \) where \( q \in \mathbb{N}^* \). We have

\[
\int_0^T \sum |u_{2,x}(t, 0)|^2 \, dt \geq \int_0^{2q\pi} \sum |u_{2,x}(t, 0)|^2 \, dt = q\pi \sum_{k=1}^{\infty} \frac{k^2}{(4k^2 - m_0^2)^2} (|a_k|^2 + |b_k|^2). \]
Noticing that
\[ \sum_{k=1}^{\infty} \frac{a_k^2}{k^2} = ||u_1^0||^2_{H^{-1}(0, \pi)} \cdot \sum_{k=1}^{\infty} \frac{k^2b_k^2}{k^4} = ||u_1^1||^2_{H^{-2}(0, \pi)}, \]
we conclude that
\[ \int_0^T \left| u_{2,x}(t, 0) \right|^2 dt \geq C \left( ||u_1^0||^2_{L^2(0, \pi)} + ||u_1^1||^2_{H^{-2}(0, \pi)} \right). \]

We also easily check the
\[ \int_0^T \left| u_{2,x}(t, 0) \right|^2 dt \geq C \left( ||u_2^0||^2_{L^2(0, \pi)} + ||u_2^1||^2_{L^2(0, \pi)} \right). \]

\[ \square \]

**Remark 24.** One can note that the above proof follows that of Theorem (5.1), by taking now in the proof the coefficients \( \alpha_k = \beta_k = 0 \) for all \( k \in \mathbb{N}^* \).

Hence, we can draw some conclusions giving some positive answers to the questions 1 and 3 stated at the beginning of Section 5. We first proved in Section 5 that for couplings of the form \( \cos(m_0 x) \) with \( m_0 \in \mathbb{N}^* \), there exist initial data for which unique continuation does not hold, whereas we proved in this Section that there exist initial data for which weaker observability estimates hold.

7. Examples of non-unique continuation for parabolic cascade systems.

We consider the coupled cascade system of heat equations
\[ \begin{cases} 
 u_{1,t} - u_{1,xx} = 0, & 0 < t < T, 0 < x < \pi, \\
 u_{2,t} - u_{2,xx} + c(x)u_1 = 0, & 0 < t < T, 0 < x < \pi, \\
 u_i(t, 0) = u_i(t, \pi) = 0, & \text{for } i = 1, 2, 0 < t < T, \\
 u_i(0, x) = u_i^0(x), & \text{for } i = 1, 2, 0 < x < \pi. 
\end{cases} \]

(86)

The boundary observation operator under consideration is
\[ B^* u_2 = u_{2,x}(. , 0). \]

**Theorem 7.1.** Assume that the coupling coefficient is given by (67). Define
\[ K = \{(u_1^0, u_2^0), u_1^0(x) = \sum_{k=1}^{\infty} a_k \sin(kx), \]
\[ u_2^0(x) = \sum_{k=1}^{m_0-1} \left( \frac{a_k}{4k^2 - m_0^2} + \frac{a_{m_0+k}}{2m_0(2k+m_0)} - \frac{a_{m_0-k}}{2m_0(m_0-2k)} \right) \sin(kx) \]
\[ + \sum_{k=m_0+1}^{\infty} \left( \frac{a_k}{4k^2 - m_0^2} + \frac{a_{m_0+k}}{2m_0(2k+m_0)} - \frac{a_{k-m_0}}{2m_0(2k-m_0)} \right) \sin(kx) \]
\[ + (a_{m_0} + \frac{a_{2m_0}}{2}) \frac{\sin(m_0 x)}{3m_0^2}, x \in (0, \pi), \sum_{k=1}^{\infty} a_k^2 < \infty \}. \]

Then \( K \) is a set of infinite dimension such that
\[ K \subset L^2(0, \pi) \times L^2(0, \pi), \]
and for all \((u_1^0, u_2^0, u_1^1, u_2^1) \in K \), the solution of (86) satisfies
\[ B^* u_2(t, 0) = u_{2,x}(. , 0) \equiv 0 \text{ on } (0, T). \]

(88)
Moreover \( K \setminus \{(0, 0, 0)\} \) is the set of initial data for which unique continuation does not hold.

Proof. We recall that
\[
    u_1^0 \in L^2(0, \pi) \iff u_1^0(x) = \sum_{k=1}^{\infty} a_k \sin(kx) \forall x \in (0, \pi), \text{ with } \sum_{k=1}^{\infty} |a_k|^2 < \infty.
\]

Moreover, the unique solution of
\[
    \begin{cases}
        u_{1,t} - u_{1,xx} = 0, & 0 < t < T, 0 < x < \pi, \\
        u_1(t, 0) = u_1(t, \pi) = 0, & 0 < t < T, \\
        u_1^0(x) = \sum_{k=1}^{\infty} a_k \sin(kx) \forall x \in (0, \pi), \forall x \in (0, \pi),
    \end{cases}
\]
where \( \sum_{k=1}^{\infty} |a_k|^2 < \infty \), is then given by
\[
    u_1(t, x) = \sum_{k=1}^{\infty} a_k e^{-k^2 t} \sin(kx), t \in [0, T], x \in (0, \pi).
\] (89)

Hence the equation in \( u_2 \) becomes
\[
    \begin{cases}
        u_{2,t} - u_{2,xx} + \cos(m_0 x) \sum_{k=1}^{\infty} a_k e^{-k^2 t} \sin(kx) = 0, & 0 < t < T, 0 < x < \pi, \\
        u_2(t, 0) = u_2(t, \pi) = 0, & 0 < t < T, \\
        u_2(0, x) = u_2^0(x), & 0 < x < \pi.
    \end{cases}
\] (90)

Proceeding as in the proof of Theorem (5.1), we obtain
\[
    u_2(t, x) = \sum_{k=1}^{\infty} (\alpha_k e^{-k^2 t}) \sin(kx) + \sum_{k=1}^{\infty} \frac{\alpha_k e^{-k^2 t}}{2m_0} \left[ \frac{\sin((k - m_0)x)}{2k - m_0} - \frac{\sin((k + m_0)x)}{2k + m_0} \right],
\]
where the \( \alpha_k \) will be determined later on. We deduce from this relation
\[
    u_{2,x}(t, 0) = \sum_{k=1}^{\infty} \left[ k \left( \alpha_k - \frac{\alpha_k}{4k^2 - m_0^2} \right) e^{-k^2 t} \right].
\]
Hence, we have
\[
    u_{2,x}(t, 0) = 0 \iff \alpha_k = \frac{\alpha_k}{4k^2 - m_0^2}.
\] (91)

8. Further extensions to multi-dimensional domains and conclusion. We assume that the space dimension is 3 and we consider \( \Omega = B(0, R_1 + \pi) \setminus B(0, R_1) \) in \( \mathbb{R}^3 \), where \( R_1 > 0 \) and \( B(0, R) \) stands for the ball of center 0 and radius \( R > 0 \).

We consider the coupled cascade wave system
\[
    \begin{cases}
        u_{1,t} - \Delta u_1 = 0, & (t, x) \in (0, T) \times \Omega, \\
        u_{2,t} - \Delta u_2 + d(\cdot)u_1 = 0, & (t, x) \in (0, T) \times \Omega, \\
        u_i = 0, & \text{in } (0, T) \times \Gamma \text{ for } i = 1, 2, \\
        (u_{i, t}^0, u_{i}^0)(0, \cdot) = (u_{i, t}^0, u_{i}^0)(\cdot) \text{ in } \Omega, & \text{for } i = 1, 2,
    \end{cases}
\] (92)

for initial data and coupling coefficients \( d \) that depend only on the radial component, that is such that \( u_1^0(\cdot) = u_1^0(r) \) and \( u_2^0(\cdot) = u_2^0(r) \) in \( \Omega \) for \( i = 1, 2 \), where \( (u_1^0, u_1^0) \in \mathbb{R}^2 \).
Thus, non-unique continuation holds for examples of multidimensional domains.

We further set
\[ x = r - R_1, c(x) = d(x + R_1), w_i(t, x) = v_i(t, x + R_1), t \in (0, T), x \in (0, \pi). \]

Then \((w_1, w_2)\) solves (62) and the observation operator is transformed in an observation of the trace of the derivative with respect to \(x\) of \(w_2\), at \(x = \pi\). The condition on \(\alpha_k\) and \(\beta_k\) giving the expression of \(w_2\) in (73) for non-continuation to occur, that is \(w_2(t, \pi) = 0\) holds if and only if
\[
\alpha_k = (-1)^{m_0} \frac{a_k}{4k^2 - m_0^2}, \quad \beta_k = (-1)^{m_0} \frac{b_k}{4k^2 - m_0^2}.
\]

Thus, non unique continuation holds for examples of multidimensional domains.

The above results can be extended to other boundary conditions, such as Neumann boundary conditions. If for instance, one considers Neumann boundary conditions for the wave equation
\[
\begin{align*}
&u_{1,tt} - u_{1,xx} = 0, \quad 0 < t < T, 0 < x < \pi, \\
&u_{2,tt} - u_{2,xx} + c(x)u_1 = 0, \quad 0 < t < T, 0 < x < \pi,
\end{align*}
\]
then the eigenfunctions of the free wave equation with these conditions will be cosine functions and we can perform the same type of analysis for non-unique continuation properties.

Other examples of non-unique continuation can be built. The above analysis relies on the explicit expression of the eigenfunctions \(e_k\) of the operator \(A\) involved in the free abstract wave equation \(u'' + Au = 0\), and the fact that the action of the coupling operator on these eigenfunctions, that is
\[ Ce_k, \]
may be rewritten, under certain hypotheses on \(C\) (being a multiplicative operator, with particular form of the coupling coefficient) as a finite linear combination of the eigenfunctions, that is
\[ Ce_k = \sum_{j \in J_k} \alpha_j e_j, \quad \forall k \in \mathbb{N}^*, \]
where \(J_k\) is a finite subset of \(\mathbb{N}^*\). An analysis to more general situations for which the eigenfunctions and eigenvalues are not known, but still satisfy some nice properties, may be performed, using implicit expressions and more refined spectral analysis.
Our results also generalize to cascade systems involving more equations and have applications on simultaneous control as presented in [7].

The present paper is an exploratory paper. Its scope is to give a view of the variety of situations that occur if the assumption of non negative property (see (NNP)) of the coupling operator is removed, or if this operator is of higher order. Its point of view is also to build intuitions on simple but clever phenomenon, so that through rich enough examples one can progress to establish general statements.

The goal is also, as in our previous works, to build flexible and robust methods which apply to various models and can be extended as the two-level energy method, because they capture intrinsic phenomenon. We have already shown in our first papers on indirect control [1, 2] that it was possible to extend the partial controllability (and by duality partial observability) results stated by J.-L. Lions in [22] to positive indirect controllability results. This intuition was first build on examples announced in [1]. These ideas and results have inspired several subsequent works (see e.g. [25, 4, 15]). One of our goal beyond the results stated above, is to suggest several directions of research:

- The above analysis can be further extended to other multidimensional domains such as squares in the plane (or N-dimensional intervals), provided that the spectral problem can be analyzed and that one can determine suitable coupling coefficients adapted to properties of the associated eigenfunctions.
- Further analysis to the case of infinite sums of linear combination of cosine functions in (75), in one-dimensional framework, should be performed to generalize non-unique continuation results to more general coupling functions that may be decomposed as such sums.
- Build other significative examples together with abstract results as in the present paper to get an insight of general statements which may hold when one authorizes non partially coercive coupling operators. In particular, the next most challenging open questions in these directions are, to our point of view:

  - Can we characterize under which necessary and sufficient condition, the unique continuation property holds, this for any bounded coupling operator (i.e. in particular that does not satisfy a sign property)?
  - If the unique continuation property holds for general such coupling operators, can we, in a similar way, characterize under which necessary and sufficient condition the observability inequality holds and in which spaces?
- We have already explored in [2], the fact to combine equations with different speeds of propagation or different differential operators in the case of symmetrically coupled systems. It is a challenging question to extend such analysis to cascade systems at least, in the case of non partially coercive coefficients.
- Symmetrically coupled single-controlled systems are slightly more complex to analyze. This is due to the fact that they are more coupled than cascade systems as explained in [7], in particular the coupling operates on each equation and not on a single one as in the cascade case. It would be interesting to extend the present results to symmetrically coupled systems.
- Many potential applications involve nonlinear coupling effects, and interesting properties due to the nonlinear coupling effects may occur as studied in [12, 13]. A challenging question is to consider nonlinear either partially coercive or non partially coercive couplings.
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