On certain quasi-local spin-angular momentum expressions for small spheres

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The Ludvigsen–Vickers and two recently suggested quasi-local spin-angular momentum expressions, based on holomorphic and anti-holomorphic spinor fields, are calculated for small spheres of radius \( r \) about a point \( o \). It is shown that, apart from the sign in the case of anti-holomorphic spinors in non-vacuum, the leading terms of all these expressions coincide. In non-vacuum spacetimes this common leading term is of order \( r^6 \), and it is the product of the contraction of the energy-momentum tensor and an average of the approximate boost-rotation Killing vector that vanishes at \( o \) and of the 3-volume of the ball of radius \( r \). In vacuum spacetimes the leading term is of order \( r^6 \), and the factor of proportionality is the contraction of the Bel–Robinson tensor and an other average of the same approximate boost-rotation Killing vector.

1. Introduction

In the classical theory of matter fields the energy-momentum and angular momentum density are described by the symmetric energy-momentum tensor \( T^{ab} \). For Killing vectors \( K^a \) the current \( T^{ab}K_b \) is conserved, and it is interpreted as a component of the energy-momentum or angular momentum current density, depending on the nature of the Killing field. E.g. the translation- and rotation Killing vectors (or rather 1-forms) of the Minkowski spacetime are given by \( K^a := \nabla_a x^a \) and by \( K^{ab} := x^a K_b^b - x^b K_a^a \), where \( x^a \) are the standard Descartes coordinates, and the corresponding currents are the four conserved energy-momentum and six conserved total (i.e. orbital- and spin-) angular momentum currents, respectively. By converting the name indices of the Killing 1-forms to spinor name indices the translations form a constant Hermitian matrix valued 1-form \( K^{AB} \), and the rotations can be decomposed as \( K^{AB} := \varepsilon^{AB} K^{c} e_c + \varepsilon^{AB} K^{c} e_c \), the sum of the anti-self-dual and self-dual rotations, where e.g. the anti-self-dual rotations are given explicitly by \( K^{AB} := x^a (A_b B_a) B^{B} \). If \( \Sigma \) is any smooth spacelike hypersurface with fixed smooth 2-boundary \( \partial \Sigma \), \( T^{ab} \) is its future directed unit normal and \( d\Sigma \) is the induced volume element, then \( P_{\Sigma AB}^{\mathbf{AB}} := \int_{\Sigma} T^{ab} t_a K_b^a d\Sigma \) and \( J_{\Sigma}^{\mathbf{AB}} := \int_{\Sigma} T^{ab} t_a K_b^a d\Sigma \) depend only on the boundary \( \partial \Sigma \) (and are independent of the actual \( \Sigma \) defining the homology between \( \partial \Sigma \) and zero), and hence may be interpreted as the quasi-local energy-momentum and angular momentum (with respect to the origin \( o \) of the Descartes coordinate system) of the matter fields associated with \( \partial \Sigma \), respectively. Let \( r \) be a positive number and \( \Sigma := \{ (t, x, y, z) | t = r \geq \sqrt{x^2 + y^2 + z^2} \} \), i.e. the piece of the spacelike hyperplane \( t = r \) that is bounded by its intersection with the future null cone \( \mathcal{N}_o \) of the origin \( o \), and let \( \Sigma_r := \partial \Sigma \). Then the quasi-local energy-momentum and angular momentum can be computed for the sphere \( \Sigma_r \) of radius \( r \) by taking the flux integrals of the conserved currents on \( \Sigma \). But since \( P_{\Sigma AB}^{\mathbf{AB}} \) and \( J_{\Sigma}^{\mathbf{AB}} \) depend only on \( \Sigma_r \) and the origin \( o \) is a regular point, they equal to the flux integral of the conserved currents on the null cone \( \mathcal{N}_o \) between the vertex \( o \) and \( \Sigma_r \). (For the technical details, e.g. the volume element on the null \( \mathcal{N}_o \), see Section 3 below.) For sufficiently small \( r \) the quasi-local energy-momentum is \( \frac{4}{3} \pi r^3 (T^{ab} t_a K_b^a) \), the product of the 3-volume of \( \Sigma \) and the value of \( T^{ab} t_a K_b^a \) at the origin. To compute the angular momentum too, it seems useful to introduce spinors. Let \( \mathcal{L}^A := \{ O^A, I^A \} \)
be the constant normalized spin frame field associated with the Descartes coordinates, \( E_A^\Sigma := \{-I_A, O_A\} \) its dual basis, and \( e_A^A := \{o^A, t^A\}, \bar{e}_A^\Sigma := \{-t_A, o_A\} \) the pair of standard dual spin frames on the null cone \( \mathcal{N}_o \).

If \( \zeta := \exp i \cot \frac{\gamma}{2} \), the complex stereographic coordinates (and hence \( r, \zeta, \bar{\zeta} \) form a coordinate system on \( \mathcal{N}_o \setminus \{o\} \)), then \( o^A(r, \zeta, \bar{\zeta}) = i \sqrt{2}(1 + \zeta \bar{\zeta})^{-\frac{1}{2}}(\zeta O^A + t^A) =: X^A e_A^\Sigma \), and hence \( X^A = \bar{e}_A^\Sigma o^A \). On the null cone the Hermitian matrix \( x^{AB'} \) of the Descartes coordinates becomes \( r \)-times the dyadic product of the ‘spinor coordinates’: \( x^{AB'} = r X^A \bar{X}^{B'} \). Thus on the null cone the anti-self-dual rotation Killing field becomes

\[
K^A_{BC} = ro^A \bar{e}_C^B \bar{e}_D^C \bar{e}_E^D \bar{e}_F^E,
\]

and, with accuracy \( r^4 \), the quasi-local anti-self-dual angular momentum for small spheres of radius \( r \) is

\[
J^{AB} = \frac{1}{4} r^4 \left( T^{ab} e_C^b \left( e_D^C \bar{e}_B^D \right) \right) |_o \oint_\$ \bar{o}_A o_{AB'} o_{BC} \bar{o}_D d\$ = \frac{4\pi}{3} r^4 T_{A' B'B} t^{A' A'' B' B''} e_{E}^{B'} e_{E}^{F} e_{C}^{F} |_o.
\]

Here \( \$ \) is the unit sphere and \( d\$ := -2i(1+\zeta \bar{\zeta})^{-2}d\zeta d\bar{\zeta} \), the 2-surface element on \( \$ \). Therefore the quasi-local angular momentum for small sphere of radius \( r \) is \( \frac{4\pi}{3} r^3 \) times the contraction of the energy-momentum tensor at \( o \) and an average of the anti-self-dual rotation Killing vector that vanishes at \( o \).

As a consequence of the diffeomorphism invariance of the geometric theories of gravity every vector field generates a conserved current (and its flux a conserved quantity), which can always be derived from a globally defined superpotential 2-form, independently of the homological structure of the spacetime [1]. In the case of energy-momentum in Einstein’s theory this superpotential appears to be the Nester–Witten 2-form \( u(\lambda, \bar{\mu})_{ab} \), associated with any pair of spinor fields \( \lambda_A \) and \( \mu_A \) [2,3], because of the following reasons.

First, this 2-form defines a 2-form on the spin frame bundle, and hence on the bundle of orthonormal frames over the spacetime manifold too, which 2-form extends uniquely to the bundle of linear frames \( L(M) \).

The superpotentials of the various classical energy-momentum pseudotensors (e.g. the Einstein, Bergmann, Landau-Lifshitz and the tetrad–Møller pseudotensors) are just the pull backs of various forms of this 2-form along various local cross sections of \( L(M) \) [4,5]. In fact, the exterior derivative of the Nester–Witten form, known as Sparling’s equation, looks like the Noether identity:

\[
\nabla [u(\lambda, \bar{\mu})_{abc}] = \Gamma(\lambda, \bar{\mu})_{abc} - \frac{1}{2} \lambda_D \bar{\mu}_D G^{de} \frac{1}{3} \varepsilon_{abc},
\]

where \( G_{ab} \) is the Einstein tensor and \( \Gamma(\lambda, \bar{\mu})_{abc} \), the so-called Sparling 3-form, is a quadratic expression of the derivatives of the spinor fields [6,7]. Second, both the ADM and Bondi–Sachs four-momenta can be written as the 2-surface integral of the Nester–Witten 2-form at spacelike and null infinity, where the spinor fields are chosen to be the constant or the asymptotic spinors there, respectively. These spinors may also be interpreted as the spinor constituents of the asymptotic translations at infinity. The simplest proof of the positivity of the ADM and Bondi–Sachs masses is probably that based on the use of the Nester–Witten 2-form [2,3,8-13]. Finally, the integral of the Nester–Witten 2-form for a spacelike topological 2-sphere \( \$ \) in the spacetime can be used to define energy-momentum, a manifest Lorentz covariant quantity (and not only energy, or energy and linear momentum separately), at the quasi-local level too. The only question is how to choose the two spinor fields \( \lambda_A \) and \( \mu_A \). Ludvigsen and Vickers [14] suggested a rule of transportation of the asymptotic spinors from infinity back to \( \$ \) if the spacetime is asymptotically flat at future null infinity and \( \$ \) can be connected with the future null infinity by a smooth null hypersurface. If the 2-surface \( \$ \) is a smooth spacelike cut of the future null cone of a point \( o \in M \), then the two independent point spinors at \( o \) can be transported to \( \$ \) by the same law of transportation [15], too. Dougan and Mason [16] suggested to choose anti-holomorphic or holomorphic spinor fields, which were shown to form two complex dimensional vector spaces in the generic case. Their constructions are genuine quasi-local in the sense that they can be applied in any spacetime for generic spacelike 2-surfaces which are homeomorphic to topological 2-spheres. The properties of these suggestions have been studied in a number of situations [14-20]. In particular, the Dougan–Mason energy-momentum is null if and only if the domain of dependence \( D(\Sigma) \) of a spacelike
hypersurface $\Sigma$ with smooth 2-boundary $\$ is a \textit{pp}-wave geometry and the matter is pure radiation, and the energy-momentum is vanishing if and only if $D(\Sigma)$ is flat [18-20]. Furthermore, they have been calculated for small spheres of radius $r$ [15,17]. All the three expressions coincide in the leading order: in presence of matter they are $\frac{1}{3}\pi r^3 T^{ab}b_{ab}$, in complete agreement with the expectation, whilst in vacuum (or if at least an open neighbourhood of the point $o$ is Ricci-flat) they are $\frac{1}{10}G r^5 T^{abcd}b_{ab}b_{cd}$, where $T^{abcd}$ is the Bel–Robinson tensor and $G$ is Newton’s gravitational constant. The Dougan–Mason energy-momenta have also been calculated with accuracy $r^6$, and the two constructions deviate in this order.

Because of the lack of any geometric meaning of the coordinates, the notion of angular momentum in general relativity is a more delicate problem, and it is argued e.g. in [21] that angular momentum should be connected with the internal Lorentz rotations of the theory, and hence should be analogous to the spin. In fact, general relativity has a Yang–Mills formulation (see e.g. [22,23]), in which Bramson derived the conserved Noether current corresponding to the Lorentz gauge symmetry and its superpotential 2-form [24] (and which has been rediscovered recently [25]). The Bramson spin superpotential 2-form, denoted here by $w(\lambda,\mu)_{ab}$, is well defined for any pair of spinor fields. Bramson used the asymptotic twistor equation to specify these spinor fields at future null infinity, and obtained an expression for the \textit{global} spin-angular momentum by taking its integral on a spherical cut of null infinity. Thus to have a reasonable \textit{quasi-local} spin-angular momentum associated with the 2-surface $\$, the two spinor fields $\lambda_A$ and $\mu_A$ must be specified there. For them it seems natural to use the spinor fields of the quasi-local energy-momentum. In fact, Ludvigsen and Vickers defined their quasi-local spin-angular momentum [14] as the integral of the Bramson superpotential with the spinor fields that they used in their energy-momentum expression. Recently it was suggested to use the anti-holomorphic or the holomorphic spinor fields in Bramson’s superpotential, which yield genuine \textit{quasi-local} spin-angular momentum expressions [20]. (Dougan and Mason suggested quasi-local angular momentum expressions, too, by using the spinor parts of the holomorphic or anti-holomorphic local twistors on $\$ in the Nester–Witten 2-form [16]. In the present paper, however, their suggestion will not be investigated.) However we think that it is not enough to give a definition, but it should be clarified in various situations whether the new angular momentum expression really has the expected properties of the angular momentum, or, more generally, whether Bramson’s superpotential serves an appropriate framework for finding the (quasi-local) measure of the angular momentum of gravity. In fact, in [20] the energy-momentum and spin-angular momentum expressions based on the anti-holomorphic unprimed spinors were calculated for finite axi-symmetric \textit{pp}-wave Cauchy developments, and the energy-momentum was shown to be an eigenvector of the spin-angular momentum tensor. Therefore the null energy-momentum and the Pauli–Lubanski spin are proportional, a reasonable property that is shared by zero-rest-mass radiative matter fields in Minkowski spacetime.

In the present paper the quasi-local Bramson spin-angular momenta are calculated for small spheres of radius $r$ for the Ludvigsen–Vickers and for the holomorphic and anti-holomorphic spinor fields. Since the angular momentum is the \textit{momentum} of the energy-momentum, the leading orders are expected to be higher than those for the energy-momentum, namely $r^4$ for non-vacuum and $r^6$ for vacuum spacetimes. In fact, for non-vacuum spacetimes any reasonable \textit{total} angular momentum expression can be expected to yield (1.2) as the leading term, simply because the energy-momentum tensor plays the role of the source for the gravity. The spin-angular momentum is a part of the total angular momentum, thus the same $r$-dependence is expected for the leading orders of the spin-angular momentum too. The prototypes of the small sphere calculations are those of Horowitz and Schmidt [26] for the Hawking energy [27] and of Kelly, Tod and Woodhouse [28] for the Penrose mass [29]. Recently Brown, Lau and York [30] calculated the Brown–York
energy [31] for small spheres and increased the accuracy of some of the spin coefficients.* Although the order of accuracy of these calculations is enough to calculate the quasi-local energy-momentum even in order $r^6$ (for the Dougan–Mason energy-momentum see [17]), the order of accuracy must be increased for the calculation of angular momenta. All the spin coefficients are needed with accuracy $r^3$. Thus first we review the main points of the framework of the calculations and present the spin coefficients and curvature components with $r^3$ accuracy. Section three is a quick review of the Nester–Witten and Bramson superpotentials, some of their properties and those forms of their integrals that we use. Since, as far as we know, the Ludvigsen–Vickers energy-momentum has not been calculated in vacuum in $r^6$ order, in Section 4 first we calculate this and compare with those of Dougan and Mason. Then the angular momenta will be calculated for the Ludvigsen–Vickers, the holomorphic and the anti-holomorphic spinors, both in non-vacuum and vacuum spacetimes. We will see that our expectation was, in fact, correct: In the non-vacuum case we recover (1.2) in all the three constructions, but with opposite sign for the anti-holomorphic spinors. In the vacuum case all the three construction give the same leading term. It is of order $r^6$ and the factor of proportionality is the contraction of the Bel–Robinson tensor and an other average of the approximate Killing vector (1.1). The results will be discussed and summarized in Section 5.

Our conventions and notations are mostly those of [32]. In particular, we use the abstract index notations, and only the boldface and underlined indices take numerical values, e.g. $a = 0,...,3$, $A = 0,1$ and $\mathcal{A} = 0,1$. The spacetime signature is $-2$, the curvature- and Ricci tensors and the curvature scalar are defined by $-R^a_{bcd}X^bY^cZ^d := \nabla_Y(\nabla_ZX^a) - \nabla_Z(\nabla_YX^a) - \nabla_{[Y,Z]}X^a$, $R_{ab} := R^c_{acb}$ and $R := R_{ab}g^{ab}$, respectively, and hence Einstein’s equations take the form $G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab} = -8\pi GT_{ab}$. On the other hand we use the GHP formalism in its original form [33], and we refer e.g. to equation (2.21) of [33] as (GHP2.21).

2. Small spheres

First, mostly to fix the notations and ensure the coherence and readability of the paper, we summarize the geometric framework and philosophy of the small sphere calculations of [26] and especially of [28]. Since in this approximation the GHP equations have a hierarchical structure, they can be integrated with arbitrary high accuracy. In the second half of this section we integrate the GHP equations with the accuracy needed in Section 3, namely with $r^3$.

Let $o \in M$ be an arbitrary point, $\{l^a, x^a, y^a, z^a\}$ be an orthonormal basis in $T_oM$ with $l^a$ future pointing, and parametrize the future celestial sphere at $o$ by $l^a(\theta, \phi) := t^a + x^a \sin \theta \cos \phi + y^a \sin \theta \sin \phi + z^a \cos \theta$. They are precisely those null vectors at $o$ whose scalar product with $t^a$ is one. Let $N_o := \partial l^+(o)$, the ‘future null cone’ of $o$, let $l^a$ denote the tangent of its null geodesic generators coinciding with $l^a(\theta, \phi)$ at $o$ and let $r$ be the affine parameter along the integral curves of $l^a$, $l^a_t r = 1$, and $r(o) = 0$. Then in an open neighbourhood of $o$ the set $N_o - \{o\}$ is a smooth null hypersurface and $S_r := \{p \in N_o | r(p) = r\}$ is a smooth spacelike 2-surface and homeomorphic to $S^2$. $S_r$ is called a small sphere of radius $r$. $(r, \theta, \phi)$, or equivalently $(r, \zeta, \bar{\zeta})$, forms a coordinate system on the smooth part of $N_o$, where $\zeta := \exp i \phi \cot \frac{\theta}{2}$. Let us complete $l^a$ to a complex null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ such that $n^a$ is orthogonal to, and the complex null vectors $m^a$, $\bar{m}^a$ are tangent to the spheres of constant radius $S_r$. If $z^A := (o^A, \iota^A)$ is the corresponding normalized spinor dyad on $N_o$ such that $l^a \nabla a o^B = 0$, then, apart from constant transformations, the complex null

* For small spheres the Hawking and the Brown–York energies give the same result in the leading orders, namely $\frac{2}{3r^2}T_{ab}t^at^b$ in non-vacuum and $\frac{2}{3r^6}T_{abcd}t^at^b t^ct^d$ in vacuum. Thus, assuming that the weak energy condition is satisfied, the Ludvigsen–Vickers, the Dougan–Mason, the Hawking and the Brown–York energies are all positive for small spheres both in non-vacuum and vacuum spacetimes, which property can be interpreted as some form of the local positivity of the quasi-local energy.
tetrad (and hence the spinor dyad too) becomes fixed. The dual spin frame is $\varepsilon_A^D := \{-\varepsilon_A, o_A\}$. The basis spinors $\{o^A(r, \zeta, \bar{\zeta}), \varepsilon^A(r, \zeta, \bar{\zeta})\}$ (and the spinor components in this basis) have direction dependent limits at the vertex $o$. In fact, if $E_A^E := \{O^A, I^A\}$ is the normalized spinor dyad at $o$ associated with the orthonormal basis $\{e^a, x^a, y^a, z^a\}$ (which we call DesCartes spinors), then at $o$

$$o^A(0, \zeta, \bar{\zeta}) = \frac{-i\sqrt{2}}{\sqrt{1+\zeta\bar{\zeta}}}(O^A + I^A), \quad \varepsilon^A(0, \zeta, \bar{\zeta}) = \frac{i\sqrt{2}}{\sqrt{1+\zeta\bar{\zeta}}}(O^A - \bar{\zeta}I^A). \quad (2.1)$$

In the coordinate system $(r, \zeta, \bar{\zeta})$ the covariant derivative direction operators take the form $D := t^a\nabla_a = (\partial/\partial r), \delta := m^a\nabla_a = P(\partial/\partial r) + Q(\partial/\partial r)$, and $\delta := \bar{m}^a\nabla_a$, where $P$ and $Q$ are functions of the coordinates. The choice of the spinor dyad above yields the following relations on the GHP spin coefficients $\kappa = \varepsilon = \rho = \tau + \bar{\beta} - \beta = \rho' - \bar{\beta}' = \tau' - \beta' + \bar{\beta} = 0$, and expressions for the GHP differential operators: $\partial f = Df$, $\partial f = (\delta - (p-q)\beta - q\tau)f$ and $\partial f = (\delta + (p-q)\bar{\beta} - p\bar{\tau})f$, where $f$ has type $(p, q)$. In flat spacetime the relevant spin coefficients and the functions $P$ and $Q$ are $\kappa = \varepsilon = \tau = \tau' = \sigma = \sigma' = 0, \rho = -\frac{1}{r}, \rho' = \frac{1}{r}, \beta = -\frac{1}{\sqrt{2}}\zeta$, and $P = \frac{1}{\sqrt{2r}}(1 + \zeta\bar{\zeta})$ and $Q = 0$. Thus in flat spacetime $\delta$ reduces to $\frac{1}{r}\partial_0$, where $\partial_0 := \frac{1}{\sqrt{2}}(1 + \zeta\bar{\zeta})$, and e.g. $\partial f$ reduces to $\frac{1}{r}\partial_0 f$, where $\partial_0 f := \partial_0 f + \frac{1}{2\sqrt{2}(p-q)}f = \frac{1}{\sqrt{2}}(1 + \zeta\bar{\zeta})^{-s}\partial_0((1 + \zeta\bar{\zeta})^s f)$ with $s := \frac{1}{2}(p-q)$, the spin weight of $f$. The Ricci identities (GHP2.21-24) and the primed version of (GHP2.25) and (GHP2.26), and the commutators (GHP2.31) and (GHP2.32) show that in general the deviation of the non-vanishing spin coefficients from these values is of order $O(r)$, e.g. $\rho = -\frac{1}{r} + O(r), \sigma = O(r)$, and similarly $P = \frac{1}{\sqrt{2r}}(1 + \zeta\bar{\zeta}) + O(r)$ and $Q = O(r)$. Thus the flat space values of these quantities can be used as the initial data in finding the approximate solutions of accuracy $r^k, k \geq 1$, of the GHP equations. The spin coefficients $\kappa'$ and $\varepsilon'$ and the operator $\partial'$ do not play any role in the small sphere calculations; and, in fact, the geometry of $\mathcal{N}_0$ does not determine them.

To find these solutions with accuracy $r^3$, first let us expand the spin coefficients and the functions $P$ and $Q$ as their flat space value plus polynomials up to third order in $r$, e.g. $\rho = -\frac{1}{r} + r\rho^{(1)} + r^2\rho^{(2)} + r^3\rho^{(3)} + O(r^4)$ and $P = \frac{1}{\sqrt{2r}}(1 + \zeta\bar{\zeta}) + O(r^3) + r^2P^{(2)} + r^3P^{(3)} + O(r^4)$, where the expansion coefficients are functions of $\zeta$ and $\bar{\zeta}$. Similarly, let us expand the curvature components retaining their first three non-trivial expansion coefficients, too, i.e. for example $\nu_0 = \nu_0^{(0)} + r\nu_0^{(1)} + r^2\nu_0^{(2)} + O(r^3)$. (At this point we should note that the different expansion coefficients have different $(p, q)$ types. For example $\nu_0^{(0)}, \nu_0^{(1)}$ and $\nu_0^{(2)}$ are of type $(4,0)$, $(5,1)$ and $(6,2)$, respectively. Thus to save the extra care in using the GHP formalism we use the operators $\nu_0^{(1)}$ and $\nu_0^{(2)}$. Then substituting the spin coefficients and curvature components into the GHP commutator (GHP2.31) (applied to the type $(0,0)$ functions $\zeta$ and $\bar{\zeta}$) and the Ricci identities (GHP2.22-24), we get

$$P = \frac{1}{\sqrt{2}}(1 + \zeta\bar{\zeta})\left\{\frac{1}{r} + \frac{1}{6}\nu_0^{(0)} + \frac{1}{12}\nu_0^{(1)} + \frac{1}{20}\nu_0^{(2)} + \frac{7}{360}(\nu_0^{(0)})^2 + \frac{1}{20}\nu_0^{(2)}\right\} + O(r^4), \quad (2.2a)$$

$$Q = \frac{1}{\sqrt{2}}(1 + \zeta\bar{\zeta})\left\{\frac{1}{6}\nu_0^{(0)} + \frac{1}{12}\nu_0^{(1)} + \frac{1}{20}\nu_0^{(2)} + \frac{7}{90}\nu_0^{(0)} + \frac{7}{120}\nu_0^{(2)}\right\} + O(r^4), \quad (2.2b)$$

$$\rho = \frac{1}{\sqrt{2}}(1 + \zeta\bar{\zeta})\left\{\frac{1}{6}\nu_0^{(0)} + \frac{1}{12}\nu_0^{(1)} + \frac{1}{20}\nu_0^{(2)} + \frac{7}{90}\nu_0^{(0)} + \frac{7}{120}\nu_0^{(2)}\right\} + O(r^4), \quad (2.2c)$$

$$\sigma = \frac{1}{r}\left(\frac{1}{3}\nu_0^{(0)} + \frac{1}{4}\nu_0^{(1)} + \frac{1}{4}\nu_0^{(2)} + \frac{1}{45}\nu_0^{(0)} + \frac{1}{45}\nu_0^{(2)}\right) + O(r^4), \quad (2.3a)$$

$$\tau = \frac{1}{r}\left(\frac{1}{3}\nu_0^{(0)} + \frac{1}{4}\nu_0^{(1)} + \frac{1}{4}\nu_0^{(2)} + \frac{1}{45}\nu_0^{(0)} + \frac{1}{45}\nu_0^{(2)}\right) + O(r^4). \quad (2.3b)$$

Then from the commutator (GHP2.32) (applied either to $\zeta$ or to $\bar{\zeta}$) the spin coefficient $\beta$ can be expressed by the expansion coefficients of the curvature and their $\nu_0$- and $\nu_0^\prime$-derivatives. To express the derivatives of the
curvature components by the curvature components themselves, use the GHP Bianchi identity (GHP2.33). We get

\[ \beta = -\frac{1}{2\sqrt{2}}\zeta + \frac{1}{2}\left(\psi_1^{(0)} + \frac{1}{6\sqrt{2}}(\zeta\psi_0^{(0)} - \zeta\phi_0^{(0)})\right) + \frac{2}{3}r^2\left(\psi_1^{(1)} + \frac{1}{18}(\zeta\psi_0^{(1)} - \zeta\phi_0^{(1)})\right) + \frac{1}{4}r^3\left(\psi_1^{(2)} + \frac{1}{10\sqrt{2}}(\zeta\psi_0^{(2)} - \zeta\phi_0^{(2)})\right) + \frac{1}{18}(\psi_0^{(0)}\psi_1^{(1)} + 4\psi_0^{(0)}\phi_0^{(1)} + 3\phi_0^{(0)}\psi_0^{(0)}) - \frac{7}{180\sqrt{2}}(\zeta(\phi_0^{(0)}\phi_0^{(1)} - \phi_0^{(0)}\phi_0^{(1)}) - O(r^4). \] (2.3d)

Then the Ricci identity (GHP2.21) doesn’t give anything new. To get the spin coefficients \( \rho' \) and \( \sigma' \) let us use the primed version of (GHP2.25) and (GHP2.26). They will be expressions of the expansion coefficients of the curvature and of their \( \partial \delta \)- and \( \partial \delta \)-derivatives. Some, but not all, of the derivatives of the curvature can be expressed by the curvature components themselves. First observe that the \( \partial \delta \)- and \( \partial \delta \)-derivatives of the zeroth order curvature components, e.g. of \( \psi_1^{(0)} \) and of \( \phi_0^{(0)} \), can be calculated from their definition using (2.1) above. (Geometrically, these formulae are not Bianchi identities, because they don’t contain the derivatives of the curvature itself.) As a result \( \rho'^{(1)} \) and \( \sigma'^{(1)} \) become algebraic expressions of the curvature:

\[ \rho' = \frac{1}{2r} - \frac{1}{6}r\left(3\psi_2^{(0)} + 3\psi_2^{(0)} + 2\phi_1^{(1)} - \phi_0^{(0)} + 6A^{(0)}\right) + O(r^2), \] (2.3e')

\[ \sigma' = \frac{1}{6}r^2\left(\psi_0^{(1)} - 4\phi_2^{(0)}\right) + O(r^2). \] (2.3f')

To determine the higher order terms let us use the difference of the Bianchi identities (GHP2.34) and (GHP2.37), from which \( \partial \delta \psi_1^{(1)} \) and \( \partial \delta \psi_1^{(2)} \) can be expressed by the curvature components and by \( \partial \delta \psi_1^{(1)} \) and \( \partial \delta \psi_1^{(2)} \), respectively. Then, using the definitions and (2.1) above, the remaining derivatives can be expressed by the \( 0 \)-components of the (first and second) derivatives of the Weyl spinor and by the spinor components of certain irreducible parts of the derivatives of the Ricci spinor. In general these formulae are rather complicated and we do not need the general expressions, we concentrate only on the vacuum case. In vacuum we get

\[ \rho' = \frac{1}{2r} - \frac{1}{2}r\left(\psi_2^{(0)} + \phi_2^{(0)}\right) - \frac{1}{3}r^2\left(\psi_1^{(0)} - \phi_0^{(0)}\right) - \frac{1}{4}r^3\left( \frac{1}{4}(\psi_2^{(0)} + \phi_2^{(0)}) - \frac{1}{30}\psi_0^{(0)}\phi_0^{(0)} + \frac{7}{90}\psi_1^{(0)}\phi_1^{(0)}\right) + O(r^4), \] (2.3e'')

\[ \sigma' = \frac{1}{6}r^2\left(\psi_0^{(0)} - 4\phi_2^{(0)}\right) + r^2\left(\frac{1}{8}\psi_0^{(1)} - \frac{1}{12}d^e(\nabla_e \psi)^{e'} + \frac{1}{10}\phi_0^{(1)}\phi_0^{(1)} - \frac{1}{20}d^e(\nabla_e \psi)_{e'} - \frac{1}{10}\phi_2^{(0)}\psi_0^{(0)} + \frac{1}{15}\psi_2^{(0)}\phi_0^{(0)} - \frac{11}{90}(\phi_0^{(0)}\phi_0^{(0)})^2\right) + O(r^4). \] (2.3f'')

Here we used the notation \( (\nabla_e \psi)_k \) and \( (\nabla_e \nabla_f \psi)_k \) for the \( \nabla_e \)- and \( \nabla_e \nabla_f \)-derivative of the Weyl spinor at \( o \), respectively, contracted with \( k e^A \) and \( (4 - k) \sigma^A \) spinors. The remaining Ricci identity, namely the primed version of (GHP2.21), and the Bianchi identity (GHP2.34) do not restrict these components of the derivatives of the Weyl spinor further.

By calculating the determinant of the induced 2-metric on \$s_r \$, one gets the 2-area element on \$s_r \$. It is \( d\$_r := -i\mu_0 m_\mu = -i(P\bar{P} - Q\bar{Q})^{-1}d\zeta \wedge d\bar{\zeta} \). Then \( D(d\$_r) = -2\rho d\$_r \), and hence if \( d\$_r := \lim_{r\to 0}(\frac{1}{4\sqrt{2}}d\$_r) = -2i(1 + \zeta\bar{\zeta})^{-2}d\zeta \wedge d\bar{\zeta} \), the area element of the unit metric sphere, then
\[ \mathrm{d}s_r = r^2 \left( 1 - \frac{1}{3} r \rho_0^2 + \frac{1}{6} r \rho_0^4 - \frac{1}{90} r^4 \left( \psi_0^2 \psi_0^0 - 4 \phi_0^2 \psi_0^0 + 9 \phi_0^2 \right) + O(r^5) \right) \mathrm{d}s. \] (2.4)

Thus in non-vacuum spacetimes \( \mathrm{d}s_r = r^2 \mathrm{d}s + O(r^4) \), whilst in vacuum \( \mathrm{d}s_r = r^2 \mathrm{d}s + O(r^6) \).

Finally, by \( t^A t^A = \frac{1}{2} \phi^A \phi^A + t^A t^A \) the primed spin vectors at \( \phi \) can be expressed by the unprimed ones: \( \tilde{\phi}_A = 2t_A C \tilde{C} \) and \( \tilde{t}_A = t^A C \tilde{C} \). Thus, combining these with Lemma (4.15.86) of [33], we get

\[
\oint_S o_A (\zeta, \tilde{\zeta}) \ldots o_A (\zeta, \tilde{\zeta}) \partial_A (\zeta, \tilde{\zeta}) \ldots \partial_A (\zeta, \tilde{\zeta}) t^{B_1} (\zeta, \tilde{\zeta}) \ldots t^{B_m} (\zeta, \tilde{\zeta}) t^{B_1'} (\zeta, \tilde{\zeta}) \ldots t^{B_n'} (\zeta, \tilde{\zeta}) \mathrm{d}s =
\]

\[
= 2^k t_A C_1 \ldots t_A C_l t^{B_1}_D_1 \ldots t^{B_n}_D_n \oint_S o_A \ldots o_A o_{D_1} \ldots o_{D_n} t^{B_1} \ldots t^{B_m} t^{C_1} \ldots t^{C_l} \mathrm{d}s =
\]

\[
= \begin{cases} 
\frac{4r}{k+n+1} 2^k t_A B_{m+1} \ldots t_A B_{m+l} t^{B_1}_A B_{m+1} \ldots t^{B_n}_A B_{m+n} \delta_{B_1 A} \ldots \delta_{B_{m+n} A}, & \text{if } k + n = l + m; \\
0, & \text{otherwise.}
\end{cases}
\] (2.5)

All the quasi-local energy-momentum and spin-angular momentum expressions for small spheres reduce to integrals of this type.

3. The Nester–Witten and Bramson superpotentials

First recall that for any pair of spinor fields \( \lambda_A \) and \( \mu_B \) the Nester–Witten and Bramson 2-forms are defined by

\[
u(\lambda, \mu)_{ab} := \frac{1}{2} (\bar{\mu}_A \nabla_{AB} \lambda_A - \bar{\mu}_B \nabla_{AB} \lambda_B),
\]

\[
u(\lambda, \mu)_{ab} := - i \lambda_A (\lambda_B \varepsilon_{AB})
\] (3.1), (3.2)

If \( S \) is any orientable spacelike 2-surface in \( M \) then their integral on \( S \) defines a Hermitian and a symmetric bilinear map, respectively, from \( C^\infty(S, S_A) \), the (infinite dimensional) space of the smooth covariant spinor fields on \( S \), to the field of complex numbers. Thus if \( \lambda_A^1, \mu_B = 0, 1 \), is any pair of spinor fields on \( S \) then under the transformation of these spinor fields by \( \text{constant } SL(2, \mathbb{C}) \) matrices the integrals

\[
\begin{align*}
P^{AB'} := & \frac{1}{4\pi G} \oint_S u(\lambda_A, \lambda_B')_{ab}, \\
J^{AB} := & \frac{1}{8\pi G} \oint_S w(\lambda_A, \lambda_B)_{ab}
\end{align*}
\]

(3.3), (3.4)

transform as a Hermitian and a symmetric spinor, respectively. Since the 2-forms (3.1) and (3.2) are superpotentials for the energy-momentum and spin-angular momentum, for appropriately chosen spinor fields \( \lambda_A \) these integrals are intended to define the quasi-local energy-momentum and the (anti-self-dual) spin-angular momentum, associated with the 2-surface \( S \), of the gravity plus matter system, respectively. Every two dimensional subspace of \( C^\infty(S, S_A) \) with some \( SL(2, \mathbb{C}) \) structure gives a potential definition for the quasi-local energy-momentum and spin-angular momentum. If one of the spinor fields in the arguments of the Nester–Witten 2-form, e.g. \( \lambda_A^1 \), is constant on \( S \) in the sense \( m^e \nabla_e \lambda_A = 0 \) and \( m^e \nabla_e \lambda_A = 0 \) (whenever the spacetime geometry is considerably restricted [18–20]), then \( u(\lambda^0, \lambda^0)_{ab} \) is exact, and hence the 00', 01' and 10' components of \( P^{AA'} \) are vanishing, and this \( P^{AA'} \) is null with respect to the \( SL(2, \mathbb{C}) \) structure above. For this \( P^{AA'} \) and the spin-angular momentum tensor \( J^{AA'BB'} := \varepsilon_{AB} J^{AA'B'} + \varepsilon_{AB'} J^{AB} \) we have

\[
P^{AA'} J^{AA'BB'} = (J^{01'} + \delta^{01'}) P^{BB'} + (\delta_0^1 \delta_0^1 \psi_0^0 + \delta_1^0 \delta_0^0 \psi_0^0) P^{11'}.
\]

Thus the null \( P^{AA'} \) is an eigenvector of \( J^{AA'BB'} \) iff \( J^{00} \) is vanishing; i.e. if and only if the pull back of \( w(\lambda^0, \lambda^0)_{ab} \) to \( S \) is exact. However in general, without additional restrictions on \( \lambda_A^1 \) too, that is not exact. Mathematically, \( J^{AB} \) is a measure on \( S \) of the
non-integrability of the vector basis $E_{A A'}^A := \lambda_A^A \lambda_{A'}^{A'}$. If both $\lambda_A^A$ are constant on $\mathcal{S}$, e.g. the restriction to $\mathcal{S}$ of the two covariantly constant spinor fields in Minkowski spacetime, then $w(\lambda^A, \lambda^B)_{ab}$ are exact and hence $J^{AB}$ is vanishing. Thus $J^{AA'BB'}$ is a measure how much the actual vector basis $E_{A A'}^A$ is ‘distorted’ relative to the constant basis of the Minkowski spacetime.

Next let us concentrate on the small spheres $\mathcal{S}_r$ of radius $r$ about $o$, and denote the corresponding integrals (3.3) and (3.4) by $P_r^{AB'}$ and $J_r^{AB}$, respectively. Although these quantities can be calculated directly using the definitions (3.3) and (3.4), the formula (2.4) and the expansions of the spinor fields given explicitly in the next section, we can follow the philosophy of [26,15,17] by converting the 2-surface integrals into integrals on the light cone too: Since for the Ludvigsen–Vickers, the anti-holomorphic and the holomorphic spinor fields $\lim_{r \to 0} P_r^{AB'} = 0$ and $\lim_{r \to 0} J_r^{AB} = 0$, by the Stokes theorem these integrals can be calculated as the integrals of the exterior derivative of the superpotential 2-forms on $N_o^r$, the portion of the null cone $N_o$ between the vertex and $\mathcal{S}_r$, too. To do the small sphere calculations in this way one must have a volume 3-form on the null hypersurface $N_o$. It is chosen to be $\varepsilon_{abc} := -i3! n_{a} m_{b} \tilde{m}_{c}$, because, for the naturally defined real tri-vector $\epsilon_{abc} := i3! n_{a} m_{b} \tilde{m}_{c}$ on $N_o - \{o\}$ and area 2-forms $\varepsilon_{ab} := -i2 m_{a} \tilde{m}_{b}$ on the spacelike 2-surfaces $\mathcal{S}_r$ one has $\epsilon_{ijk} \varepsilon_{abc} = -\delta_{abc}$, $\varepsilon_{bc} = 1 r^a \varepsilon_{abc}$ and $\varepsilon_{abc} = 3 n_{a} \tilde{m}_{b} \tilde{c}_{c}$, and hence the volume element 3-form on $N_o$ is $\frac{1}{3!} \varepsilon_{abc} = \frac{1}{6} \varepsilon_{[abc]} = d\mathcal{S}_r \wedge dr$. Then the integral (3.3) takes the form

$$P_{r}^{AB'} = \frac{1}{4\pi G} \int_{N_o^r} \nabla_{\mathcal{S}_r} w(\lambda^A, \lambda^B)_{bc} = \frac{1}{4\pi G} \int_{0}^{r} \int_{\mathcal{S}_r} \left\{ (D\lambda^A_B) o^B + (D\lambda^B_A) o^A - \right. \left. \bar{\delta} \lambda^A_B o^B - \bar{\delta} \lambda^B_A o^A - \frac{1}{2} \lambda^A_B \lambda^{B'}_A G^{AF} o_F o_F' \right\} d\mathcal{S}_r \wedge dr',$$

where we used the Sparling equation. Similarly, (3.4) can be written as

$$J_{r}^{AB} = \frac{1}{8\pi G} \int_{N_o^r} \nabla_{\mathcal{S}_r} w(\lambda^A, \lambda^B)_{bc} = \frac{1}{8\pi G} \int_{0}^{r} \int_{\mathcal{S}_r} \left\{ D(\lambda^A_B o^A_B + o^{A_B}_B) - 2\bar{\delta}(\lambda^A_B o^A_B) \right\} d\mathcal{S}_r \wedge dr.'$$

For small enough $r$ these integrals can be expanded as a power series of $r$. To calculate these integrals with accuracy $r^k$, by (2.4) we need to calculate their integrand with accuracy $r^{(k-3)}$, and hence, because of the operator $D$, we need to calculate the spinor fields with accuracy $r^{(k-2)}$. However, as we will see explicitly, to compute (3.5) with accuracy $r^k$ it will be enough to calculate the spinor fields with accuracy $r^{(k-3)}$. This is due to the special nature of the Nester–Witten form.

4. Quasi-local spin-angular momenta for small spheres

4.1 The Ludvigsen–Vickers spinors

First let us recall the definition of the Ludvigsen–Vickers spinors in the small sphere context [15]. They are the point spinors transported from $o$ to $\mathcal{S}_r$ along $N_o$ by the propagation laws $(D\lambda^A) o^A = D\lambda^A = 0$, $(\bar{\delta} \lambda^A) o^A = \partial^A \lambda^A + \rho \lambda^A = 0$, where the spinor components are defined by $\lambda_A \eta_A - \lambda_{0_A} := \lambda_A^{A A' A'} := \lambda_A$, and the two initial values for $\lambda^A$ at $o$ are the 0-components of the Descartes spinors $\epsilon_{A}^{A A'} = \{-I_{A}, O_{A}\}$: $\lambda^{0}_A(0) := \epsilon_{A}^{A A'} o^A(0) = i \sqrt{2}(1 + \zeta)^{-\frac{1}{2}}$ and $\lambda_{0}^A(0) := \epsilon_{A}^{A A'} o^A(0) = i \sqrt{2}(1 + \zeta)^{-\frac{1}{2}}$. The first of the propagation laws implies that the spinor components $\lambda^{0}_A$ are independent of the affine parameter $r$, thus $\lambda^{0}_A$ and $\lambda_A$ on $N_o$ are given explicitly by these expressions. (If a spinor has a name index, too, then its spinor components in
the spin frame \( \{ \varepsilon^A \} \) will be written as subscripts and its name index as a superscript. Thus, for example, \( \lambda^0 \) is the 1-component of the zeroth spinor, \( \lambda^1 \) is the 0-component of the first spinor, \( \lambda^1 \) denotes the holomorphic energy-momentum for small spheres. Since the Ludvigsen–Vickers spinors are completely determined by the initial values, i.e. the Descartes spinors \( E^A \) at \( o \), the metric \( \varepsilon^{AB} \) on the space of point spinors at \( o \) determines an \( SL(2, \mathbb{C}) \) structure on the space of the Ludvigsen–Vickers spinors on \( \mathcal{S}_r \), too. It is this \( SL(2, \mathbb{C}) \) structure that should be used to define the quasi-local mass as the length of \( P^{AB'} \). In general the pointwise scalar product of the Ludvigsen–Vickers spinors, e.g. that for the basis spinors \( \varepsilon^{AB} \lambda^A_B \), is not constant on \( \mathcal{S}_r \).

Following the general prescription of \([26,28,15,17]\), let us expand the 1-components as a power series of \( r \). But since the operator \( D \) in (3.5) and (3.6) reduces the power of \( r \) to have \( r^p \) accurate integrals, the spinor fields must be calculated with accuracy \( r^4 \): 

\[
\lambda^A_1 = \lambda^A_1(0) + r \lambda^A_1(1) + r^2 \lambda^A_1(2) + r^3 \lambda^A_1(3) + r^4 \lambda^A_1(4) + O(r^5). 
\]

Substituting this expansion into the second of the propagation laws and using (2.2) and (2.3) we get

\[
\lambda^A_1(0) = 0, \\
\lambda^A_1(4) = \frac{1}{4} \left( \psi_0^2 - \frac{1}{4} \psi_1^2 - \frac{19}{8} \psi_0^2 \psi_0^0 + \frac{11}{4} \psi_1^0 \psi_1^0 + \frac{5}{12} \psi_0^2 \psi_1^0 + \frac{3}{2} \psi_1^0 \right) \lambda^A_1(0). 
\]  

Thus \( \lambda^A_1(0) \) are just the 1-components of the Descartes spinors: \( \lambda^A_1(0) = \mathcal{E}^A_1 t^A \). Therefore the spinor components \( \lambda^A_0 \) and \( \lambda^A_1(0) \) satisfy \( \partial \partial \lambda^A_0 = 0, \) \( \partial \lambda^A_1(0) + \frac{9}{4} \lambda^A_0 \) and \( \partial \partial \lambda^A_1(0) = 0, \) too. The pointwise symplectic scalar product of the basis Ludvigsen–Vickers spinors is \( \varepsilon^{AB} \lambda^A_1 \lambda^B_1 = \varepsilon^{AB}(1 + \frac{1}{2} r^2 \phi_0^2 + \frac{1}{4} r^3 \phi_1^1 + \frac{1}{4} r^4 (\phi_0^2 + \frac{1}{2} \phi_0^2 \psi_0^0 + \frac{1}{2} \psi_0^0 \psi_0^1) + O(r^5)) \); i.e. the natural \( SL(2, \mathbb{C}) \) structure cannot be realized by the pointwise scalar product even in vacuum spacetimes.

Before calculating the Ludvigsen–Vickers spin-angular momentum, for the sake of completeness let us calculate their energy-momentum with accuracy \( r^6 \) in vacuum spacetimes. As a consequence of the propagation laws, all the terms except the fifth in the integrand on the right hand side of (3.5) vanish. But, because of \( g \partial \lambda^A_0 = 0 \), the integrand will be of order \( r^2 \), therefore to have \( r^6 \) accurate integral, by (2.4) it is enough to approximate \( dS_r \) by \( r^2 dS \). Finally, using (2.5), we get

\[
P^{AB'}_r = \frac{1}{10G} r^5 T^a_{\text{bcat}} t^b c^d \varepsilon^A_1 t^d_1 \mathcal{E}^A_1 \mathcal{E}^A_1 + \frac{4}{45G} r^6 t \left( \nabla c T^a_{\text{bcat}} \right) t^b c^d \varepsilon^A_1 \mathcal{E}^A_1 + O(r^7), 
\]  

where \( K^{AB'}_a := \mathcal{E}^A_1 \mathcal{E}^B_1 t^A_1 \) may be interpreted as a translation at \( o \) in the \( AB' \) direction, and \( T_{\text{abce}} := \psi_{ABCD} \mathcal{E}^B_1 C' \mathcal{E}^D_1 \), the Bel–Robinson tensor at \( o \). Thus the Ludvigsen–Vickers energy-momentum coincides with the Dougan–Mason energy-momentum based on the holomorphic unprimed spinors \([17]\) even in sixth order. It is known \([17]\), on the other hand, that at future null infinity it is the expression based on the anti-holomorphic rather than the holomorphic unprimed spinors that coincides with the Bondi–Sachs (and hence the Ludvigsen–Vickers) energy-momentum. Therefore the Ludvigsen–Vickers energy-momentum is interpolating between the anti-holomorphic Dougan–Mason energy-momentum for large spheres near the future null infinity and the holomorphic Dougan–Mason energy-momentum for small spheres.

Next let us calculate the Ludvigsen–Vickers spin-angular momentum in non-vacuum spacetimes with accuracy \( r^4 \). Since by (4.1.1) the integrand on the right hand side of (3.6) is of order \( r \), we may write \( dS_r = r^2 dS \), and, using Einstein’s equations, a straightforward calculation yields
where $T^{ab}$ is the energy-momentum tensor at the vertex $o$, and the symmetric complex matrix valued 1-form $K_{AB}^o$ is given by (1.1). Therefore the Ludvigsen–Vickers spin-angular momentum for small spheres in non-vacuum spacetimes gives precisely the expected result (1.2): the leading order is $r^4$, and the factor of proportionality is the contraction of the energy-momentum tensor and the average of the null tangent of the light cone of $o$ and $K_{AB}^o$, a 1-form field that can be interpreted as the approximate anti-self-dual rotation Killing 1-form that vanishes at $o$. We would like to stress that the energy-momentum tensor appears here in a rather non-trivial way, in contrast to the $r^3$ order calculations of the quasi-local energy-momentum, where the energy-momentum tensor was present explicitly in the exact expressions just because of the Sparling equation. The integrand of (3.6) is still of order $r$ in vacuum, thus to calculate the Ludvigsen–Vickers spin-angular momentum in vacuum with accuracy $r^6$ it is still enough to write $d\mathcal{S}_r = r^2 d\mathcal{S}$ by (2.4). But then the integrand should be calculated with accuracy $r^3$. A direct calculation yields

$$J^A_{r AB} = \frac{1}{144\pi G} r^5 T^{abcd} \int_{\mathcal{S}} l_a l_b l_c K^d_{AB} d\mathcal{S} + O(r^7) = \frac{4}{45 G} r^6 T_{AA'BB'C'D'D'} t^{AB} t^{CC'} t^{DD'} \varepsilon_{E F} \varepsilon_{E F} + O(r^7).$$

(4.1.4)

Therefore in vacuum the leading term is of order $r^6$, and the structure of this expression is the same that of (4.1.3) with the Bel–Robinson tensor of the gravitational ‘field’, instead of the energy-momentum tensor of the matter fields.

4.2 The holomorphic spinors

Recall that a spinor field $\lambda_A$ on a spacelike 2-surface $\mathcal{S}$ is called holomorphic if $\tilde{m}^b \nabla_b \lambda_A = 0$, which in the GHP formalism is equivalent to $\partial' \lambda_1 + \sigma' \lambda_0 = 0$ and $\partial' \lambda_0 + \rho \lambda_1 = 0$. If $\mathcal{S}$ is homeomorphic to $S^2$ then there are at least two, and for metric spheres there are precisely two linearly independent holomorphic spinor fields $\lambda^A_A$, $A = 0, 1$, for which $\varepsilon^{AB} \lambda^A_A \lambda^B_B$ is always constant on $\mathcal{S}$. Apart from certain exceptional 2-surfaces (e.g. future marginally trapped surfaces, i.e. for which the GHP spin coefficient $\rho$ is zero) the two spinor fields can be chosen to be normalized to the Levi-Civita symbol: $\varepsilon^{AB} \lambda^A_A \lambda^B_B = \varepsilon^{AB}$. For small perturbations of metric spheres, e.g. actually for small spheres, the number of the holomorphic spinor fields is still two, and they span the spin space at each point of $\mathcal{S}$ (see e.g. [19]). Therefore the pointwise symplectic scalar product defines a natural $SL(2, \mathbb{C})$ structure on the space of holomorphic spinor fields.

Following the general prescription of the small sphere calculations let us expand the components of the holomorphic spinor fields: $\lambda^A_A = \lambda^A_A (0) + \ldots + r^4 \lambda^A_A (4) + O(r^5)$. Then the condition of holomorphy above yields a hierarchical system of inhomogenous linear partial differential equations for these components:

$$0 \partial' \lambda_0^{(k)} - \lambda_1^{(k)} = \sum_{l=0}^{k-2} \left( C^A_l \lambda^A_A (l) + D^A_l \frac{\partial \lambda^A_A (l)}{\partial \zeta} + E^A_l \frac{\partial \lambda^A_A (l)}{\partial \zeta} \right),$$

$$0 \partial' \lambda_1^{(k)} = \sum_{l=0}^{k-2} \left( F^A_l \lambda^A_A (l) + G^A_l \frac{\partial \lambda^A_A (l)}{\partial \zeta} + H^A_l \frac{\partial \lambda^A_A (l)}{\partial \zeta} \right), \quad k = 0, \ldots, 4$$

(4.2.1)

where $C^A_l, \ldots, H^A_l$ are explicit expressions of the expansion coefficients of the GHP spin coefficients and the functions $P$ and $Q$; and they are vanishing in flat spacetime. Its general solution can be written as the sum of two spinor fields. The first, as we will see explicitly, is determined completely by the zeroth order holomorphic spinor fields, which turns out to be a linear combination of the Descartes spinors at $o$, and,
in addition, its first order term is vanishing. Thus there are two such independent spinor fields, and they have the structure $\lambda_A = a_\mathbb{A}\xi_A^\mathbb{A} + r^2\lambda_A^{(2)} + r^3\lambda_A^{(3)} + ...$, where $Nester-Witten$, $\lambda_A^{(2)}$, $\lambda_A^{(3)}$, ... are determined by $a_\mathbb{A}\xi_A^\mathbb{A}$ (see below). They are particular solutions of the inhomogenous system (4.2.1). The second is the general solution of the homogenious equations and turns out to be the sum of arbitrary complex combination of the Cartes spinors in each order, i.e. it has the form $r\xi_A^{(1)} + r^2\lambda_A^{(2)} + r^3\lambda_A^{(3)} + ...$. Hence the $r^k$ accurate homogenous solutions form a $2k$ dimensional vector space. However, as noted in [17] (and realized first in the case of 2-surface twistors by Kelly, Tod and Woodhouse [28]), these ‘spurious’ solutions correspond to the lack of any canonical isomorphism between the space of holomorphic spinor fields on $\mathcal{S}_r$ and $\mathcal{S}_{r'}$ with different radii $r$ and $r'$; or, in other words, between the space of holomorphic spinor fields on $\mathcal{S}_r$ and the space of point spinors at $o$. Hence they should be ‘gauge solutions’, and neither the integral of the Nester–Witten nor that of the Bramson 2-form should be sensitive to them. (This issue has not been discussed even for the Nester–Witten form.) To check this, let $\lambda_A$ be any spinor field whose zeroth and first order coefficients in its power series expansion are constant linear combinations of the particular solutions $\lambda_A^{(0)} = a_\mathbb{A}\xi_A^\mathbb{A} + rb_\mathbb{A}\xi_A^\mathbb{A} + r^2\lambda_A^{(2)} + ...$, let $\gamma_A = r^kA\xi_A^\mathbb{A}$, a $k$th order gauge solution, $k = 1, 2, ...$, and determine the leading order in the integral of $u(\lambda, \bar{\gamma})_{ab}$ and of $w(\lambda, \gamma)_{ab}$ on $\mathcal{S}_r$. (\lambda_A has the structure of a particular solution for $b_\mathbb{A} = 0$ and of another gauge solution for $a_\mathbb{A} = 0$. However, we don’t impose any further condition on $\lambda_A$.) Then the leading orders in these integrals are

$$\oint_{\mathcal{S}_r} u(\lambda, \bar{\gamma})_{ab} = \begin{cases} O(r^{k+3}) & \text{in non-vacuum,} \\ O(r^{k+5}) & \text{in vacuum;} \end{cases} \quad (4.2.2)$$

$$\oint_{\mathcal{S}_r} w(\lambda, \gamma)_{ab} = \begin{cases} O(r^{k+4}) & \text{in non-vacuum,} \\ O(r^{k+6}) & \text{in vacuum.} \end{cases}$$

(To calculate these integrals explicitly we need to use the full (2.4).) Thus the expected leading terms, namely the $r^3$ and $r^4$ order terms in non-vacuum and the $r^5$ and $r^6$ terms in vacuum spacetimes for the integral of the Nester–Witten and Bramson 2-forms, respectively, are in fact not sensitive to the spurious solutions, i.e. they are really ‘gauge solutions’. However, without further conditions on the spinor fields $\lambda_A$, the next orders are already sensitive to them. But since it is only the first order gauge solutions that yield ambiguity in the next non-trivial orders by (4.2.2), it seems natural to impose the gauge condition that the first order term in the spinor fields $\lambda_A$ be vanishing; i.e. we exclude first order gauge solutions by hand.

Note that although in general, without additional structures, it is meaningless to speak about “how much homogenous solution is contained in the particular solution $\lambda_A^{(0)}$”, because of the specific structure of the particular solutions (given below) this gauge condition is well defined. With this gauge condition the first two non-vanishing orders of the energy-momentum and spin-angular momentum will be independent of the emerging gauge solutions. The gauge independence of higher order terms cannot be ensured by similar, additional gauge conditions.

The quotient of the space of the holomorphic spinor fields on $\mathcal{S}_r$ and that of the gauge solutions form in fact a two complex dimensional vector space. Up to the second order in arbitrary spacetime we can choose the following particular solutions:

$$\lambda_0^\mathbb{A} = \xi_\mathbb{A} A + r^2\left(2\xi_0^{(0)} + \xi_{12}^{(0)}\right) + r^3\left(\xi_1^{(0)} + \xi_{22}^{(0)} - \frac{1}{4}\tilde{\phi}_{00}^{(0)}\right) + O(r^q),$$

$$\lambda_1^\mathbb{A} = \xi_\mathbb{A} A + r^2\left(\xi_1^{(0)} + \xi_{22}^{(0)} - \frac{1}{4}\tilde{\phi}_{00}^{(0)}\right) + r^3\left(2\tilde{\phi}_{10}^{(0)} + \frac{1}{2}\psi_i^{(0)}\right) + O(r^q).$$

(4.2.3a)

In the quotient space these form a basis which is normalized in the sense $\xi^{AB}\lambda_A^\mathbb{A}\lambda_B^\mathbb{A} = \xi^{AB}(1 + O(r^q))$. In vacuum these reduce to Dougan’s solutions [17], and substituting them into (3.5) we recover the Dougan–Mason energy-momentum with $r^5$ accuracy. (Having imposed the gauge condition above, by (4.2.3) one
could compute the Dougan–Mason energy-momentum in non-vacuum with \( r^4 \) accuracy. However we don’t expect anything interesting in this order.) Substituting \((4.2.3a)\) into \((3.6)\) for the self-dual spin-angular momentum we get the expected result \((1.2)\).

To calculate the holomorphic Dougan–Mason energy-momentum and the holomorphic spin-angular momentum in \( r^6 \) order, we need to know the holomorphic spinor fields with \( r^3 \) and \( r^4 \) accuracy, respectively. In vacuum they are

\[
\lambda^A_0 = o^A \mathcal{E}^A_A - \frac{1}{18} r^4 \left( \frac{1}{8} \psi^1(0) \bar{\psi}^0(0) + 3 \psi^2(0) \bar{\psi}^0(0) + 4 \psi^3(0) \bar{\psi}^0(0) + 2 \psi^4(0) \bar{\psi}^0(0) \right) o^A \mathcal{E}^A_A + \frac{1}{9} r^4 \left( \psi^1(0) \bar{\psi}^1(0) + 3 \psi^2(0) \bar{\psi}^2(0) + 4 \psi^3(0) \bar{\psi}^3(0) + 2 \psi^4(0) \bar{\psi}^4(0) \right) \lambda^A \mathcal{E}^A_A + O(r^5),
\]

\[
\lambda^A_1 = i^A \mathcal{E}^A_A + \frac{1}{12} r^2 \psi^1(0) o^A \mathcal{E}^A_A + \frac{1}{10} r^4 \left( \frac{1}{4} \mathcal{E}^A_A + \frac{1}{3} \psi^1(0) \bar{\psi}^0(0) \right) o^A \mathcal{E}^A_A + \frac{1}{18} r^4 \left( \frac{1}{8} \psi^1(0) \bar{\psi}^0(0) + 3 \psi^2(0) \bar{\psi}^0(0) + 4 \psi^3(0) \bar{\psi}^0(0) + 2 \psi^4(0) \bar{\psi}^0(0) \right) i^A \mathcal{E}^A_A + O(r^5).
\]

These are normalized with accuracy \( r^4 \): \( \varepsilon^{AB} \lambda^A_0 \lambda^B_0 = \varepsilon^{AB}(1 + O(r^5)) \). Substituting its \( r^3 \) accurate part into \((3.5)\) we recover the result of \([17]\), i.e. \((4.1.2)\). Finally, substituting \((4.2.3b)\) into \((3.6)\), we obtain \((4.1.4)\).

Therefore the Ludvigsen–Vickers and the holomorphic expressions yield the same results even in \( r^6 \) order both for the energy-momentum and the spin-angular momentum.

### 4.3 The anti-holomorphic spinors

A spinor field \( \lambda_A \) on \$ is called anti-holomorphic if \( m^\alpha \nabla_\alpha \lambda_A = 0 \), which in the GHP formalism is equivalent to \( \partial \lambda_1 + \rho \lambda_0 = 0 \) and \( \partial \lambda_0 + \sigma \lambda_1 = 0 \). The philosophy and the calculations are quite similar to those in the holomorphic case, thus we present only the results. Apart from the ‘spurious’ solutions, for the independent, normalized anti-holomorphic spinor fields with accuracy \( r^2 \) in an arbitrary spacetime we can choose

\[
\lambda^A_0 = o^A \mathcal{E}^A_A + r^2 \left( \psi^1(0) \bar{\lambda}^A - (\psi^2(0) - \Lambda(0)) o^A \right) \mathcal{E}^A_A + O(r^3),
\]

\[
\lambda^A_1 = i^A \mathcal{E}^A_A + r^2 \left( \psi^1(0) - \Lambda(0) \right) o^A \mathcal{E}^A_A + O(r^3).
\]

Substituting these into \((3.6)\) for the anti-self-dual spin-angular momentum we get

\[
J^A_{\mathcal{E}_E} = \frac{4\pi}{3} r^4 T_{AA'B'B'} T^{AA'} l^{B'E} c^{(A} \mathcal{E}^{B)}_{E} + O(r^5),
\]

i.e. that for the Ludvigsen–Vickers and the holomorphic spinors but with opposite sign. A pair of normalized holomorphic spinor fields in vacuum with accuracy \( r^4 \) is
\[
\lambda_0^A = o^A \mathcal{E}_A^A + r^2 \left( \psi_1(0) \mathcal{L}^A - \psi_2(0) o^A \right) \mathcal{E}_A^A + \frac{1}{3} r^3 \left( \left( \psi_1(1) + 2 \mathcal{T} \left( \nabla_e \psi \right)_1 \right) \mathcal{L}^A - \left( \psi_2(1) + 2 \mathcal{T} \left( \nabla_e \psi \right)_2 \right) o^A \right) \mathcal{E}_A^A + \\
+ r^4 \left( \frac{1}{12} \left( \mathcal{T}^2 + 2 \mathcal{T} \mathcal{T} + 4 \mathcal{T} \mathcal{T} \right) \left( \nabla_e \nabla_f \psi \right)_1 - \frac{4}{9} \psi_1(0) \psi_3(0) + \frac{1}{3} \left( \psi_2(0) \right)^2 \right) o^A \mathcal{E}_A^A + \\
+ r^4 \left( \frac{1}{12} \left( \mathcal{T}^2 + 2 \mathcal{T} \mathcal{T} + 4 \mathcal{T} \mathcal{T} \right) \left( \nabla_e \nabla_f \psi \right)_1 - \frac{5}{36} \psi_0(0) \psi_3(0) + \frac{1}{4} \psi_1(0) \psi_2(0) \right) \mathcal{L}^A \mathcal{E}_A^A + O(r^5),
\]

Substituting its \( r^3 \) accurate part into (3.5) we recover the result of [17], namely (4.1.2) with the numerical coefficient \( \frac{1}{9} \) instead of \( \frac{4}{36} \) in the second term. (The \( r^3 \) accurate solution wasn’t given in [17].) Finally, substituting (4.3.1.1) into (3.6) we get (4.1.4).

5. Discussion and conclusions

The integral \( J^{AB} \) of the Bramson superpotential with the independent holomorphic or anti-holomorphic spinor fields on orientable 2-surfaces in arbitrary spacetimes and with the Ludvigsen–Vickers spinors in certain asymptotically flat spacetimes are well defined quasi-local observables of the gravity plus matter system. Their usefulness and interpretation as the spin-angular momentum, however, depends on their properties in specific situations, e.g. in the case of small spheres. To determine the general structure of the small sphere expression of such an integral one can follow the argument of Horowitz and Schmidt [26], applied originally to the Hawking energy: A simple dimension analysis shows that the number of derivatives of the metric in the coefficient of the \( r^k \) term in the power series expansion of the quasi-local spin-angular momentum expressions is \( k - 2 \). But the lowest order tensorial quantities as expansion coefficients appear for \( k = 4 \) in non-vacuum and for \( k = 6 \) in vacuum spacetimes, and, apart from numerical coefficients, these are the energy-momentum and the Bel–Robinson tensor, respectively, contracted with the unit vector \( r^a \) and the generator of the physical quantity in question. The numerical coefficients may be different in the different constructions.

The present calculations for the specific constructions confirm this ‘universal’ structure of the small sphere expansions and the interpretation of \( J^{AA'BB'} \) as the gravitational angular momentum. Moreover, apart from the sign difference between the Ludvigsen–Vickers and holomorphic constructions on the one hand and the anti-holomorphic on the other in non-vacuum spacetimes, the results in the leading order are exactly the same in all the three constructions, and it is the expected result (1.2). The spin-angular momenta coincide in vacuum in the sixth order, and the factor of proportionality is just the Bel–Robinson tensor contracted with the unit vector \( r^a \) and an average of the approximate boost-rotation Killing vector that vanishes at the origin. We stress that the approximate boost-rotation Killing vector \textit{appears} at the end of the calculations, that is not put in the exact formulae by hand, in contrast to the translations in the small sphere calculations of the quasi-local energy-momentum. This result shows up the proper interpretation of the Bel–Robinson
tensor as a quantity analogous to the energy-momentum tensor of the matter fields: it gives the main contribution to the gravitational self-energy-momentum, as has already been pointed out \cite{26,28,15,17,30}, and angular momentum at the quasi-local level. It might be worth noting that our previous result \cite{20} that the null energy-momentum $P^{AA'}$ of an (axi-symmetric) domain of dependence $D(\Sigma)$ is an eigenvector of the angular momentum tensor $J^{AA'BB'}$ can easily be recovered in the small sphere approximation: If $\Sigma_r$ is any smooth hypersurface whose boundary is $\$_r$, the dominant energy condition is satisfied and $P^{AA'}_r$ is null, e.g. $P^{AA'}_r = \delta_r^A \delta_r^{A'} P^{AA'}_r$, then $D(\Sigma_r)$ is known to be a pp-wave geometry and the matter is pure radiation \cite{18,19}. Then by $\alpha \in D(\Sigma_r)$ the energy-momentum and the Bel–Robinson tensor have the form $T_{AA'BB'} = |\varphi|^2 \lambda_A^0 \lambda_{A'}^0 \lambda_B^0 \lambda_{B'}^0$ and $T_{AA'BB'CC'DD'} = |\psi|^2 \lambda_A^0 \bar{\lambda}_{A'^0} \lambda_B^0 \bar{\lambda}_{B'^0} \lambda_C^0 \bar{\lambda}_{C'^0} \lambda_D^0 \bar{\lambda}_{D'^0}$, respectively. Now it is easy to compute $P^{AA'}_r$ and, using (4.1.3) and (4.1.4), the angular momentum tensor $J^{AA'BB'}_r$ explicitly. Finally, one can check that $P^{AA'}_r$ is, in fact, an eigenvector of $J^{AA'BB'}_r$. The quasi-local Pauli–Lubanski spin vector associated with the small sphere $\$_r$, defined by $S^{AA'} := \frac{1}{2} \varepsilon^{AA'BB'CC'DD'} \bar{P}^{BB'}JJ^{CC'DD'}$, is however vanishing in the expected leading orders, i.e. in $r^7$ and $r^{11}$, respectively.

Since the angular momentum is not expected to have any definite sign (unlike the energy), the minus sign in (4.3.2) doesn’t mean that the expression based on the anti-holomorphic spinors would be ‘non-physical’. The relative sign difference between the holomorphic and anti-holomorphic expressions shows only that, roughly speaking, the vector basis built from the holomorphic and anti-holomorphic spin frames are distorted relative to the constant basis by the curvature just in the opposite direction in $r^2$ order.

Having imposed the gauge condition one can calculate the $r^5$ and $r^7$ order corrections. In order $r^5$ we don’t expect anything interesting. The $r^7$ accurate calculations for the spin-angular momentum would however be more interesting, because the holomorphic and anti-holomorphic constructions are expected to be different, but probably the Ludvigsen–Vickers and the holomorphic constructions are still coincide in this order. But these calculations would be much more complicated because the spin coefficients and the spinor fields would have to be calculated with accuracy $r^4$ and $r^5$, respectively.

In non-vacuum the result, apart from the sign in the anti-holomorphic case, is precisely the expected total quasi-local angular momentum of the matter fields. Therefore it is natural to think of the integral $J^{AB}$ with appropriately chosen spinor fields as the sum of the total angular momentum of the matter fields and the angular momentum of the gravity itself. But then the question arises whether that represents the total angular momentum of the gravitational field too, as claimed by some authors, or, in accordance with the expectation of Bramson, only its spin part; and in the latter case whether $J^{AB}$ should be completed by some quasi-local orbital angular momentum or not. Although the results of the present paper show that in the small sphere context the Bramson superpotential serves an appropriate framework for defining the gravitational angular momentum, in other situations it may yield wrong results. A general analysis shows that such an orbital term would have to be the integral of the momentum of the Nester–Witten superpotential \cite{34}, but it is not quite clear how this moment would have to be defined. To answer these questions one should consider other situations, e.g. large spheres near spacelike and/or null infinity, stationary axisymmetric systems etc, the subject of a future paper.

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