Quantum teleportation using cluster states

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A protocol of quantum communication is proposed in terms of the multi-qubit quantum teleportation through cluster states (Phys. Rev. Lett. 86, 910 (2001)). Extending the cluster state based quantum teleportation on the basic unit of three qubits (or qudits), the corresponding multi-qubit network is constructed for both the qubits and qudits (multi-level) cases. The classical information costs to complete this communication task is also analyzed. It is also shown that this quantum communication protocol can be implemented in the spin-spin system on lattices.

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Quantum communication, a way of communication based on various features of quantum coherence (such as quantum entanglement and quantum non-cloning, which undoubtedly have no counterparts in classical worlds), are more reliable and efficient than any classical communication method. Quantum teleportation is one of the most remarkable protocols among those tasks of quantum communication, which completes the striking task of remotely preparing an unknown quantum state of a particle without sending this particle itself. Since proposed by Bennett et. al, quantum teleportation has attracted much attention from both experimentalists and theoreticians. Appropriate choice and measurements of entangled states are crucial to the implementation of those quantum communication protocols.

Recently, a novel kind of entangled states - cluster states was introduced with their remarkable properties such as maximal connectedness and persistency of entanglement. They can be used to build a one-way universal quantum computer associated with only single-qubit measurements. It was also pointed out that those computers can be implemented physically since the creation of the cluster states needs only the Ising-type interactions, which can be easily found in various condensed matter systems with proper spin-spin coupling on lattices and even in the cold atom system in optical lattices. Very recently, with an intrinsically algebraic consideration, we generalized the concept of cluster states to the more general case with qudits (multi-level systems) and then demonstrate their quantum correlation features and the fundamental computational properties in many aspects.

In this letter the cluster state is chosen as the source of entanglement to propose a novel protocol of quantum communication. The essence and functioning of this quantum teleportation scheme using the cluster state can be viewed as an identity quantum logic gate, which was originally used in the scheme of measurement-based one-way quantum computers only based on the cluster state and the corresponding single quantum measurements. The advantage of this implementation is its scalability: a basic “unit” can be easily constructed for a smallest (three-qubit) quantum logic network and then this unit can be naturally enlarged to form a more-qubit network. Moreover, this protocol can be realized in the spin-spin system on lattices, which physically guarantees the scalability and implies that the communication can be carried out from certain lattice sites to those at a long distance. With those considerations we will explicitly build the quantum network for a quantum teleportation of N-qubits and N-qudits. We also analyze the entanglement and classical information costs to complete the teleportation task on a cluster.

We first view that the original procedure of quantum teleportation protocol mainly depends on quantum measurement with respect to four Bell states, the complete set of maximally entangled state of two particles. It was discovered that the Bell states can be obtained by a proper interaction from certain factorized states. In fact, using the Pauli matrices \(\sigma_x(s = x, y, z)\) defined for qubit states \(|0\rangle\) and \(|1\rangle\), we write the tensor product operator \(S_{ab} = \frac{1}{2}(I + \sigma_x^a + \sigma_x^b - \sigma_z^a \otimes \sigma_z^b)\) acting on particles \(a\) and \(b\). Experimentally, this can be easily realized as a time evolution governed by the Ising-type interaction Hamiltonian, i.e. \(H = \hbar g \sum_{a=1}^{4} \sigma_z^a \sigma_z^b\) for a spin-lattice systems or a cold atom system in optical lattices. Define \(|\pm_s\rangle = \frac{1}{\sqrt{2}}(|0\rangle_s \pm |1\rangle_s)(s = 1, 2, 3, \ldots)\). It is easy to check that \(S_{ab}^+\) transfer four factorized states \(|\pm_h \otimes |\mp_h\rangle\) to four Bell basis vectors \(|B_{\pm 1}(ab)\rangle = \frac{1}{\sqrt{2}}(|0\rangle_a|\mp_b \pm |1\rangle_a|\pm_b\rangle\) and \(|B_{\pm 0}(ab)\rangle = \frac{1}{\sqrt{2}}(|0\rangle_a|\mp_b \pm |1\rangle_a|\pm_b\rangle\). Therefore, for a given two particle wave function \(|\Phi\rangle_{ab}\), the quantum measurements on its unitary transformation \(S_{ab}^+ |\Phi\rangle_{ab}\) with respect to four states \(|\pm_h \otimes |\mp_h\rangle\) and \(|\pm_h \otimes |\mp_h\rangle\) are exactly equivalent to a Bell-measurement with respect to \(|B_{\pm 1}(ab)\rangle\) and \(|B_{\pm 0}(ab)\rangle\). In this sense the results of the single particle measurements on \(|\pm_h\rangle\) and \(|\pm_h\rangle\) will determine the Bell-measurements on \(|\Phi\rangle_{ab}\). We call this
“two-step Bell-measurement”.

With the above observations we now turn to consider how to teleport a state of a single particle located in site 1 of the lattice to another particle located in site 3. Since the controlled evolution governed by the above mentioned “Ising” type interaction plus the single particle measurements can realize a two-step Bell-measurement, a quantum teleportation can be naturally viewed as an identity quantum logic gate on the one way quantum computer. To be more concrete, we consider the simplest quantum logic network which can implement the identity gate shown in Fig. 1. There is a linear array of three sites on lattice and three spins attached on these sites interact with each other through the “Ising” type interaction. For this physical system we can propose our protocol of quantum teleportation using cluster states in three steps:

![Cluster diagram](image)

FIG. 1: The cluster for quantum teleportation of one qubit state. The circle with the number n denotes the n-th qubit, and the two qubits connected by a line are neighbours. The green, blue and red colors are used to mark the input, body and output parts of the cluster respectively.

I. Initially, Alice holds two particles 1 and 2 and let the particle 1 be in an unknown state \( |\psi\rangle_{in} = |\sigma(\alpha, \beta)\rangle = \alpha|0\rangle_1 + \beta|1\rangle_1 \) (where \( |\alpha|^2 + |\beta|^2 = 1 \)) and the particle 2 in the state \( |+\rangle_2 \). Bob holds particle 3 in the state \( |+\rangle_3 \). After an evolution derived by the Ising type interaction for a length time \( t \) (governed by the equation \( \pi = \int g(t)dt \)), the unitary transformation \( S = S_1^{12}S_2^{23} \) on the initial product state \( |\Psi(0)\rangle = |\pi_1(\alpha, \beta)\rangle \otimes |+\rangle_2 \otimes |+\rangle_3 \) will result in a three particle entangled state \( |\Psi(t)\rangle \). It is noticed that, since \( S_1^{12} \) commutes with \( S_2^{23} \), the unitary transformation \( S \) can be decomposed into two independent sub-steps.

II. Alice measure \( \sigma_1 \) and \( \sigma_2 \) for her particles 1 and 2 in the single-particle basis \( |\pm\rangle \) respectively and then tell Bob which \( (x_1, x_2) \) of the four possible results about \( \frac{1}{\sqrt{2}}(\sigma_2^1 + 1) \) and \( \frac{1}{\sqrt{2}}(\sigma_2^2 + 1) \), where \( x_1, x_2 \in \{0, 1\} \sim (+, -) \). In this sense the total wave function \( |\Psi(t)\rangle \) collapses onto

\[
|\Psi_3(t)\rangle \propto (x_1, x_2)|S_1^{12}S_2^{23}|\Psi(0)\rangle = (B_{x_1}B_{x_2})|S_2^{23}|\Psi(0)\rangle.
\]

That is just the projection of \( |\Phi(t)\rangle = S_3^{23}|\Psi(0)\rangle \) onto the Bell basis \( |\Phi_3\rangle = S_3^{12}|x_1, x_2\rangle \), where the role of \( S_3^{23} \) (which transforms the factorized state \( |+\rangle_2 \otimes |+\rangle_3 \) to a maximally entangled state \( \frac{1}{\sqrt{2}}(|0\rangle_2|+\rangle_3 + |1\rangle_2|+\rangle_3) \)) is to distribute an entanglement source to Alice and Bob.

III. According to this result \( (x_1, x_2) \) Bob make an \( (\alpha, \beta) \)-independent unitary transformation

\[
U_{x_1} = (\sigma_3^{(3)}x_2^1 + 1)(\sigma_3^{(3)}x_1^1 + 1)
\]

on his particle 3 to obtain unknown state \( |\pi_3(\alpha, \beta)\rangle = \alpha|0\rangle_3 + \beta|1\rangle_3 \) for particle 3 exactly. In fact, according to the “one way quantum computer” theory, we have the measurement induced output \( |\psi\rangle_{out} = U_{x_1}^\dagger |\psi\rangle_{in} \).

Here, \( U_{x_1} \) is the extra rotation we should carry out in order to finish our task of teleportation. Since we measure particles 1 and 2 in \( \sigma_3 \) basis, \( U_{x_1} \) must belong to the Pauli algebra generated by \( \sigma_3^{(3)}(s = 1, 2, 3) \) and thus has the form \( U_{x_1} = (\sigma_3^{(3)}x_1^1 + 1)(\sigma_3^{(3)}x_2^1 + 1) \), which is determined by the outcomes of measurement results \((+, -)\) in \( \sigma_3 \) basis. Thus a one-way unitary transformation \( U_{x_1}^\dagger \) is sufficient to transform particle 3 to \( |\pi_3(\alpha, \beta)\rangle \sim |\psi\rangle_{in} \).

It is noticed that the steps I and II together implement an identity quantum logic gate on the cluster. However, it should be emphasized that the quantum teleportation cannot be implemented only using cluster state and one-qubit measurement. Indeed, the step III must be carried out, i.e. we should make a unitary transformation on the output particle 3 to ensure that the state of particle 3 is exactly that of particle 1. This is the main difference between quantum teleportation and quantum computation based on cluster state. In fact, in the measurement-based quantum computation, we only need the output \( |\psi\rangle_{out} = U_{x_1}^\dagger |\psi\rangle_{in} \) (other than \( |\psi\rangle_{in} \)) itself to readout the result of computing. Finally we come to the conclusion that, to implement quantum teleportation on a cluster is equivalent to realize an identity quantum logic gate on the same cluster. Thus, our task remains to construct a quantum logic network to implement an identity quantum logic gate and find the explicit form of the unitary transformation \( U_{x_1} \) in terms of the outcomes of the measurements.

Based on the detailed analysis above, it is straightforward to suggest a general protocol for cluster-based quantum teleportation to transport multi-qubit information between the two sets of qubits located on the lattice sites with a long distance. Essentially, it is a general identity quantum logic gate realized with a quantum network formed by a cluster plus the appropriate single particle measurements (as Fig.2). This cluster-based quantum network can be formally divided into three parts, the input (I) unit with \( N \) qubits, the output unit (O) with \( N \) qubits and the connection block (B)-the body part. When we input a \( N \) qubit state \( |\psi\rangle_{in} \), the output \( |\psi\rangle_{out} = U_{x_1}^\dagger |\psi\rangle_{in} \) is induced by the single qubit measurements performed simultaneously on all sites of I and B. This seems simple, but one should pay attention to the corresponding problem of information cost. As well-known, to teleport an arbitrary \( N \)-qubits states, at least \( N \) ebits (\( N \) entanglement pairs) and \( 2N \) classical bits of information must be used. Can we reach this tightly lower bound using an entangled cluster state? The answer is affirmative and we will also show how to build the quantum logic network to realize such quantum communication with the tightly lower bound of cost of information as follows. A one-dimensional chain cluster is sufficient to realize our entanglement sources. In fact, we can use ordered indices to label qubits in the one-dimensional cluster of \( M \) (even number) qubits. Therein the qubits with odd and even indices (odd- and even-bit) can form
two subsystems with $M/2$ qubits respectively. Then a maximally entangled state can form between these two subsystems. This conclusion can be reached by observing that, by measuring all the qubits with odd indices in the chain cluster with respect to the $\sigma_z$-basis, all the qubits with even indices will be projected onto the $\sigma_x$-basis. Then there is a one-to-one correspondence between the measurement result and the projection result. Thus, we will have $N$-bits of entanglement if we use a one-dimensional cluster state of $2N$-qubits and give the odd-qubits to Alice and the even-qubits to Bob.

With above general consideration for the information cost in the cluster-based teleportation of $N$-qubits, we can embark on constructing a proper cluster to implement quantum teleportation of unknown $N$ qubit state. Without loss of generality, we consider the quantum teleportation of 2-qubit state on the six qubit cluster as shown in Fig. 2 (a).

\begin{align}
\sigma_z^{(1)}\sigma_z^{(3)}|\phi\rangle = |\phi\rangle, \\
\sigma_z^{(2)}\sigma_z^{(4)}|\phi\rangle = |\phi\rangle, \\
\sigma_z^{(3)}\sigma_z^{(5)}|\phi\rangle = |\phi\rangle, \\
\sigma_z^{(1)}\sigma_z^{(2)}|\phi\rangle = |\phi\rangle, \\
\sigma_z^{(1)}\sigma_z^{(4)}|\phi\rangle = |\phi\rangle, \\
\sigma_z^{(2)}\sigma_z^{(5)}|\phi\rangle = |\phi\rangle. 
\end{align}

The above system of equation can be simplified as

\begin{align}
\sigma_z^{(1)}\sigma_z^{(5)}|\phi\rangle &= |\phi\rangle, \\
\sigma_z^{(2)}\sigma_z^{(6)}|\phi\rangle &= |\phi\rangle, \\
\sigma_z^{(1)}\sigma_z^{(3)}\sigma_z^{(4)}|\phi\rangle &= |\phi\rangle, \\
\sigma_z^{(2)}\sigma_z^{(3)}\sigma_z^{(4)}|\phi\rangle &= |\phi\rangle. 
\end{align}

For the measurement pattern $\{\sigma_z^{(1)}, \sigma_z^{(2)}, \sigma_z^{(3)}, \sigma_z^{(4)}\}$, we obtain the measurement result $(s_1, s_2, s_3, s_4)$. Based on the theorem 1 in Ref. [15], we conclude that the realized unitary transformation in the two particle Hilbert space of qubit 5 and 6 is an identity $I$. Correspondingly the extra rotation $U_\Sigma$ is

$$U_\Sigma = (\sigma_z^{(5)})^{s_1}(\sigma_z^{(6)})^{s_2}(\sigma_z^{(5)})^{s_3}(\sigma_z^{(6)})^{s_4+s_5}. $$

After obtaining the measurement results through classical channels, one needs to perform the unitary gate $U_\Sigma$ in the output unit. Then the input state is determinately teleported to the output unit.

With the aid of the connecting natures of the cluster state, the above quantum teleportation protocol for 2-qubit state can be generalized directly to the case with $N$-qubit state. Along the vertical direction, we extend the above cluster to that with $3N$ qubits, shown as in Fig. 2 (c). After measuring Pauli operator $\sigma_x$ in the input unit and the connection block of the cluster, thereafter performing the proper unitary transformation, depending on the former measurement results, the quantum teleportation of $N$-qubit state is completed from the input unit to output part. Another way to extend the protocol is to quantum teleport a state to a longer distance through many medial qubits. Contrast to the above extension along the vertical direction, we extend the cluster along the horizontal direction as in Fig. 2 (b). This expansion might be understood as connection of the basic units of quantum teleportation in sequence. Thus, to complete such quantum teleportation, we also need to measure $\sigma_x$ for all the qubits in the input unit and the body part of the cluster, and then to perform a measurement-dependent unitary transformation in the output part. It must be emphasized that only the collective measurement results (more precisely, the sums of the measured values of $\sigma_x$ in every horizontal line) can influence the final unitary transformation in the output part. Accordingly, the classical information cost to implement this teleportation protocol for every one qubit state is two bits, which is exactly the same as that in the original protocol suggested by Bennett et al.

Extending the cluster in both vertical and horizontal directions, we obtain a two dimensional lattice array of qubits as demonstrated in Fig. 2 (d). Obviously, the same procedure as above can complete quantum teleportation on this lattice. In addition, if provided with a large lattice of spins, we can chose the proper cluster by measuring $\sigma_z$ to delete the redundant qubits from the lattice. Then, as long as the cluster for quantum teleportation

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{The clusters for quantum teleportation of qubit states. The circle with the number $n$ denotes the $n$-th qubit, and the two qubits connected by a line are neighbors. The green, blue and red colors are used to mark the input, body and output parts of the cluster respectively. (a)The cluster to quantum teleport the state of qubits 1 and 2 to that of 3 and 4; (b)The horizontal expansion of the cluster in (a);(c)The vertical expansion of the cluster in (a);(d)The lattice for general quantum teleportation.}
\end{figure}
can be formed in the lattice, we can teleport the state of the input qubits to the output qubits, whichever sites they locate.

So far our investigation has centered in how to teleport the state of qubits using cluster state. Then a natural question arises: Can we quantum teleport the state of qudits (d-level system) in a similar way? To teleport the state of qudits, it is indispensable to generalize the concept of cluster state of qubits for qudits. Most recently, we have just carried out this generalization successfully \[15\] and proved two useful theorems parallel to that in Ref. \[16\]. With this generalization we can parallel propose the protocol of qudit based quantum teleportation.

Let us begin with the simplest case—the quantum teleportation of one qudit state. We take a cluster of three qudits as Fig.1. The higher dimensional cluster state \[16\] is defined by

\[
    X_1^iX_2^j|\phi\rangle_C = |\phi\rangle_C, \quad Z_1X_2^j|\phi\rangle_C = |\phi\rangle_C, \quad Z_2X_1^i|\phi\rangle_C = |\phi\rangle_C, \quad (14) \quad (15) \quad (16)
\]

where the generalized Pauli operators \( Z \) and \( X \) are the generators of quantum plane algebra with \( q^d = 1 \). The \( Z \)-diagonal representation of \( Z \) and \( X \) given by

\[
    Z \equiv \sum_{k_0}^{d-1} |k\rangle q_d^k \langle k|, \quad (17) \\
    X \equiv \sum_{k=0}^{d-1} |k\rangle \langle k+1|, \quad (18)
\]

for \( q_d = e^{i2\pi} \). From Eqs. \[14\] \[15\] \[16\] we obtain

\[
    X_1X_2^i|\phi\rangle_C = |\phi\rangle_C, \quad (19)
\]

When we obtain the measurement results \( \{s_1, s_2\} \) for the measurement pattern \( \{X_1, X_2\} \), the fundamental theorem 2 proved in Ref. \[16\] determines the unitary transformation on the third qudit \( U_{U\Sigma} \). Here, \( U \) is defined by

\[
    UX_3U^\dagger = X_3^1, \quad UZ_3U^\dagger = Z_3^1, \quad (21)
\]

and \( U_{\Sigma} = Z_3^{-s_1}X_3^{s_2} \). To complete this quantum teleportation, one needs to perform a unitary transformation \( U_{U\Sigma}^\dagger U \) on the third qudit. Therefore, in the same way as in the qubit case, the above one-qudit quantum teleportation scheme can be extended to the quantum teleportation of \( N \)-qudit state, and even be extended to the quantum teleportation of qudit states through a large two dimensional lattice.

To sum up, we have universally proposed a novel protocol for quantum teleportation of qubits and qudits based on cluster states. According to the analysis about the properties of entanglement resources, a cluster composed of 3\( N \) qubits (or qudits) suffices to teleport a \( N \) qubit (or qudit) state. When the cluster is extended to form a two or higher dimensional lattice, this protocol of quantum teleportation can still work well.

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