NUMERICAL SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION USING RANDOMLY GENERATED FINITE GRIDS AND TWO-DIMENSIONAL FRACTIONAL-ORDER LEGENDRE FUNCTION

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Abstract

There are various methods to solve the physical life problem involving engineering, scientific and biological systems. It is found that numerical methods are approximate solutions. In this way, randomly generated finite difference grids achieve an approximation with fewer iterations. The idea of randomly generated grids in cartesian coordinates and polar form are compared with the exact, iterative method, uniform grids, and approximate solutions in a generalized expansion form of two-dimensional fractional-order Legendre functions. The most ideal and benchmarking method is the finite difference method over randomly generated grids on Cartesian coordinates, polar coordinates used for numerical solutions. This concept motivates the investigation of the effects of the randomly generated meshes. The two-dimensional equation is solved over randomly generated meshes to test

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randomly generated grids and the implementation. The feasibility of the numerical solution is analyzed by comparing simulation profiles.

**Keywords:** Partial differential equation, Finite difference method, Polar coordinates, Randomly generated grids, Uniform meshes, fractional-order Legendre functions.

### I. Introduction

The heat equation is the basic partial differential equation: time derivatives than Laplace and Poisson’s equation. Partial Differential Equation (PDE) is used to solve various engineering and physical problems [VIII, XXI]. The finite difference method is an iterative technique [XI, XXVI] so the numerical solution of the Heat equation of 2D partial differential equation through the numerical scheme is a finite difference method using grids or mesh [XXXVI]. Recently, several different methods, including Adomian’s decomposition method (ADM) [XXXI, IX], homotopy perturbation method (HPM) [I, XVI], variational iteration method (VIM) [X, XXII, XI], spectral methods [XXXVII], orthogonal polynomials method [VI], and wavelets method [XII, XXXI] have been presented and applied to solve FPDEs. The Mesh generation procedure is not unique. It is a practice, and the pattern varies from problem to problem. Mesh is usually called grids or nodes formed as desired and physical or engineering models [VII, XV, II, XVII, IV, V, XIV]. It is mostly found that mesh quality depends on the rate of convergence; if mesh quality is not good, the rate of convergence cannot reach some cases [XXXIII, XXXIV]. It is the difficult and challenging design of the most feasible grids for the specific model because of variability in boundary conditions and the domain structure. Meshes are divided into structured, unstructured and hybrid grids. A 2 dimension domain is filled with the design that is regular parts and easier to implement, compute and unstructured is usually use for the complex part in worse shaped and hybrid contain a combination of both grids that is structured and unstructured with quadrilateral and triangular shapes are often computed by Delaunay triangulation of given set of points [III]. Structured and unstructured meshes having different approach and advantages and the limitations. The inspiration is a new technique that is to “Generate random grids” and solve a twodimensional partial differential equation with the help of FDM. So, they hypothesized that “the randomly generated grids (meshes) improve the convergence of the numerical solution.” Mesh has different types like curvilinear, cartesian grid, rectilinear grid, triangular grids, domain decomposition method, and many others. These all types of grids found in the major two categories are known as structured and unstructured. Structured grids are used for regular shapes. The irregular shape we use unstructured grids or meshes with quadrilateral and triangular connectivity is regular or uniform and irregular or non-uniform.

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grids. A hybrid grid is a scheme for joining the unstructured and structured grids [XX-XXIX]. In the single strategy, both types were discussed and benefited

The meshes or the grids or nodes are designed recalled as per mathematical model structures. The physical problem and model were solved using computer technology to find various algorithms and solutions in less computational time. To highest convergence with less computational time, a partial differential equation of Heat equation solved, designed and random grids intended for ease and various problems. In this way, the numerical algorithm development for the partial differential equation, usually called PDEs, purely depends upon grids. The mesh generation already gained much consideration because of the applicability of physical problems, structural mechanics, CFD computational fluid dynamics and electromagnetism [XXXV-XXXVIII] Now the boundaries of shapes change in to meshes from finite difference methods and meshes can construct to the finite element meshes [XIC-XVII, XIX]. The technique of discretizing PDEs in a various technique proposed in literature. However, the numerical solution approach varies with the model as physical phenomenon to be simulate and the type of foremost equation and the computational domain of problem.

The methods are based on discretization of the differential equation by finite difference quotients, parameters of mesh. It is frequently observed that the finite difference method (FDM) operates on the regular step or equal step size or variable step grid size.

II. Numerical Methodology (Basic Partial differential equation (PDE’S)

Material and Methodology

Three different models and their conditions are defined below, finite discretization of the equations is presented. The Mesh generation process is described for regular (Uniform) and random step or spacing. Different meshes are generated on uniform spacing to test and analyze data collected in uniform grids and random grids viz. On each uniform sample size, different random meshes are generated. That is the random spacing generated.

Model Problem Specification

The two-dimensional partial differential governing equation with initial and boundary conditions is shown below.

\[ u_{xx} + u_{yy} + u_{zz} = f(x, y, z), x^2 + y^2 + z^2 < R \]  \hspace{1cm} (1)

Where \( u \) denote any physical phenomenon, the distribution of steady-state temperature in the metallic plate with constant applied temperature on the left, right, top, and boundaries as \( a_1 \)°C to \( b_1 \)°C. \( u \) is depending on the variable that depends upon \( x \) and \( y \) coordinates. Where \( f \) is forcing function with left, right, top and below limitations \( a_1, a_2, b_1 \text{ and } b_2 \) respectively.

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**Finite Difference Discretization of the Model**

The discretization of the model problem, given the problem, is well modeled, so there is no special treatment required for consistency and stability. The second-order central finite difference scheme is described in the following equation 02.

\[
\frac{u(i+1,j)-2u(i,j)+u(i-1,j)}{s^2} + \frac{u(i,j+1)-2u(i,j)+u(i,j-1)}{t^2} = f(i,j)
\]

\[\forall i, j \in N, i < n, j < m, \text{where } u(i, 1) = u(1, j) = a_1, u(m, j) = u(i, n) = a_2,\]

The Computational domain is discretized according to defined finite difference mesh scheme. The Fig.02, is a schematic of discretized with regular spacing.

**Generation of Uniform Finite Difference Grids and Random grids**

The regular spacing grids are generated by writing a code in MATLAB software. The results are implemented on each interior node by GS iterative method explicitly.

\[
u^{k+1}(i,j) = \frac{q^2(u^k(i + 1, j) - u^k(i - 1, j)) + r^2(u^k(i, j + 1) - u^k(i, j - 1))}{2(q^2 + r^2)}
\]

Where \(k\) and \(k + 1\) represent iterations and successive iterations, respectively.

The Uniform grids are generated with cell size, number of nodes, number of cell size average cell size and standard deviations recorded with MATLAB, using data, and Random grids by varying “nrand” command grids size, we generated the random grids that are shown below.
III. Dirichlet & Heat Problems in Polar Coordinates

The steady-state temperature in a circular Plate

The two-dimensional partial differential equation will be used by recalling the polar coordinates.

\[ x^2 + y^2 = r^2, \quad x = r \cos \theta, y = r \sin \theta \text{ and } \theta = \tan^{-1}(y/x). \]

The equation (1) became as

\[ \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = f(r, \theta) \quad (4) \]

Example 1. \( u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0, 0 < r < 1, 0 < \theta < 2\pi \)

(a) \( u(1, \theta) = \cos 2\theta \) (b) \( u(1, \theta) = \sin 2\theta \), \( u(r, \theta) \) is bounded.

Figure 2. Grids, Uniform grids and random grids

Figure 3. Cos2 theta and Sin2 theta with exact, approximate and random solutions

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Example 2. The numerical profile simulation of steady-state temperature is defined in cartesian and polar coordinates. Figure 4 shows the polar coordinates, cartesian solutions as the local solution profile where the solution varies from 10°C to 90°C from cold to hot temperature.

\[
\begin{align*}
    u_{xx}(x, y, z) + u_{yy}(x, y, z) + u_{zz}(x, y, z) &= f(x, y, z), \quad x^2 + y^2 + z^2 < R^3 \\
    \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= f(x, y, z), \quad a < x < b, \quad a < y < b \quad \text{and} \quad a < z < b,
\end{align*}
\]

with \(u(x, y, z) = 0\), \(u(a, y, z) = a_1\), \(u(x, a, z) = a_1\), \(u(x, y, a) = a_1\) & \(u(b, y, z) = b_1\), \(u(x, b, z) = b_1\), \(u(x, y, b) = b_1\) where \(a_1 = 10^0\)C and \(b_1 = 100^0\)C.

**Figure 4.** Numerical solution of Heat equation using Cartesian and Polar coordinates

Example 3. Fractional Calculus Theory: Some necessary definitions and Lemma of the Fractional calculus are listed here for our subsequent development.

**Definition 1.**

A real functions \(h(t), t > 0\) is said to be in the space \(C_\mu, \mu \in R\) if there exists a real number that is \(r > \mu\), such that \(h(t) = t^r h_1(t)\), where \(h_1(t) \in C(0, \infty)\), and it is said to be in the space \(C_\mu^n\), iff \(h^{(n)} \in C_\mu, n \in N\).

**Definition 2.**

The Riemann-Liouville fractional integral operator \((I^\alpha)\) of the order \(\alpha \geq 0\), if a function, \(f \in C_\mu, \mu \geq -1\) is defined as:

\[
\begin{align*}
    I^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \\
    I^0 f(t) &= f(t),
\end{align*}
\]

\(\text{(5)}\)

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Where $\Gamma(\alpha)$ is the well known Gamma Function, Some properties of the operator $D^\alpha$ of the order $\alpha \geq 0$, of a function.

**Definition 3.**

The fractional derivatives of $f(x)$ in Caputo, the sense is defined as

$$(D^\alpha f)(x) = \left\{ \begin{array}{ll} \frac{1}{\Gamma(m-\alpha)} \times \int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha+1}} d\xi, & (\alpha > 0, \xi - 1 < \alpha < \xi) \\ \frac{d^m f(x)}{dx^m}, & (\alpha = \xi) \end{array} \right.$$  

(6)

where $f: R \rightarrow R, x \rightarrow f(x)$ denotes a continuous (bit necessarily differentiable) function.

Lemma 1. Let $n - 1 < \alpha \leq n, n \in N, t > 0, h \in C^n_\mu, \mu \geq -1$.

Then

$$f^\alpha(D^\alpha h(t)) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0^+) \frac{t^k}{k!}$$  

(7)

The Fractional Legendre Functions, In this section, we are discussing the fractional-order Legendre functions, which were first proposed by Kazem et. al. the normalized function for FLF is

$$(x - x^1+\alpha)L^\alpha_{i-1}(x) + \alpha^2 \alpha(i + 1)x^{\alpha-1}L^\alpha_i(x) = 0, x \in (0,1)$$  

(8)

Which is singular Strum-Liouville problem. The fractional-order Legendre polynomials denoted by $FL^\alpha_i(x)$ are defined on $[0,1]$ can be determined with the aid of the following recurrence formulae.

$$FL^\alpha_0(x) = 1, FL^\alpha_1(x) = 2x^{\alpha-1}, FL^\alpha_{i+1}(x) = \frac{(2i+1)(2^{\alpha-1})}{i+1} FL^\alpha_i(x) - \frac{i}{i+1} FL^\alpha_{i-1}(x),$$  

(9)

and the analytical form of $FL^\alpha_i(x)$ of degree, $i$ is given

$$FL^\alpha_i(x) = \sum_{k=0}^{i} b_{k,i} x^{\alpha k}, b_{k,i} = \frac{(-1)^k i^{(i+k)}}{(i-k)!k!}$$  

(10)

Where $FL^\alpha_i(0) = (-1)^i$ and $FL^\alpha_i(1) = 1$. The orthogonality condition is

$$\int_0^1 FL^\alpha_i(x)FL^\alpha_m(x) \omega(x)dx = \frac{1}{(2n+1)\alpha} \delta_{im}$$  

(11)

Where $\omega(x) = x^{\alpha-1}$ is the weight function and $\delta$ is the Kronecker delta.

Example 5. Consider the one-dimensional linear on homogeneous fractional burger equation.

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{2t^{2-\beta}}{\Gamma(3 - \beta)} + 2x - 2, 0 < \beta \leq 1,$$

With the initial condition $u(x, 0) = x^2$ and exact solution being $u(x, t) = x^2 + t^2$. By employing the 2D-FLFs method, given figure below show the numerical results for $\beta = 0.25$ with $m = 3$, $m' = 9$ and $\beta = 0.5$ with $m = 3$, $m' = 5$, respectively.

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IV. Regression Fitting for Converging Iteration and Test of Statistical Significance (Uniform Versus Random Meshes)

As discussed in the previous section, the relationship between the converging iterations over uniform meshes and random meshes can predict the minimum iterations for a given Uniform mesh size. Thus, the relationships between the iterations of uniform mesh and random meshes' iterations are established for each sample by regression fitting (used polynomial interpolation method). The regression equation is an 8th-degree polynomial whose parameters lie within the 95% confidence interval. Also, the goodness of fit is presented in Figure 6.

![Figure 5](image1.png)

**Figure 5.** 2D-FLF at $\beta = 0.25$ with $m = 3$, $m' = 9$ and $\beta = 0.5$ with $m = 3$, $m' = 5$

![Figure 6](image2.png)

**Figure 6.** Regression fit for the four realizations of each mesh size

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Regression Parameters and Goodness of fit

\[
f_1(x) = p_1x^4 + p_2x^3 + p_3x^2 + p_4x + p_5 + p_6x^3 + p_7x^2 + p_8x + p_9,
\]

Coefficients (with 95% confidence bounds):

- \( p_1 = 9.763e-022 \), \(-6.203e-021, 8.156e-021\)
- \( p_2 = -1.273e-017 \), \(-1.095e-016, 8.408e-017\)
- \( p_3 = 6.779e-014 \), \(-4.7e-013, 6.056e-013\)
- \( p_4 = -1.901e-010 \), \(-1.779e-009, 1.399e-009\)
- \( p_5 = 3.018e-007 \), \(-2.384e-006, 2.988e-006\)
- \( p_6 = -0.0002717 \), \(-0.0002871, 0.0002328\)
- \( p_7 = 0.1312 \), \(-1.127, 1.489\)
- \( p_8 = -28.22 \), \(-358.1, 301.7\)
- \( p_9 = 2051 \), \(-2.339e+004, 2.749e+004\)

**Goodness of fit:**

- Sum of Square of Error: 6.96e+004
- R-square (Square of Coefficient of Correlation): 0.9937
- Root Mean Square Error: 263.8

V. Conclusion & Future work.

In this research, the two-dimensional partial differential equation's numerical solution with Dirichlet boundary conditions is solved using cartesian and polar coordinates with randomly generated grids. Random samples may provide faster convergence than uniform meshes. Random meshes have a better solution than uniform meshes. Interestingly, we mostly found results with fewer iterations on short computational time and less percentage error. However, specific random samples have good results in minimum iteration, with minimum time and less error at some places.

The work can be extended in other directions. Mostly, Engineering models have many applications, like air conditioning and cooling system. Financial engineering, medicine and biology system, gene regulatory network, fluid dynamics and Various scientific models will be discretized using randomly generated grids. The idea also can be extended to the finite volume method.

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**Novel Method:** To the best of the authors’ knowledge, the proposed method is Randomly generated grids introduced by Sanaullah Mastoi. The proposed method is also knowns as SM’s method that is numerical solution is based on finite difference method using randomly generated grids.

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Conflicts of Interest:

The authors declare that they have no conflicts of interest to report regarding the present study.

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