AN IMPLICIT NUMERICAL SCHEME FOR A CLASS OF BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider a class of backward doubly stochastic differential equations (BDSDE for short) with general terminal value and general random generator. Those BDSDEs do not involve any forward diffusion processes. By using the techniques of Malliavin calculus, we are able to establish the $L^p$-Hölder continuity of the solution pair. Then, an implicit numerical scheme for the BDSDE is proposed and the rate of convergence is obtained in the $L^p$-sense. As a by-product, we obtain an explicit representation of the process $Y$ in the solution pair to a linear BDSDE with random coefficients.

1. Introduction

Let $\{W_t\}_{0 \leq t \leq T}$ and $\{B_t\}_{0 \leq t \leq T}$ be two independent standard Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets. For each $t \in [0, T]$, we define

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^R,$$

where

$$\mathcal{F}_t^W = \sigma\{W_s, 0 \leq s \leq t\} \vee \mathcal{N} \quad \text{and} \quad \mathcal{F}_t^R = \sigma\{B_s - B_t, t \leq s \leq T\} \vee \mathcal{N}.$$

The purpose of this paper is to study an implicit numerical scheme for the following backward doubly stochastic differential equation (BDSDE for short)

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T g(Y_r)d\hat{B}_r - \int_t^T Z_r dW_r, \quad 0 \leq t \leq T,$$

where $\xi$ is a given terminal value, $f$ is a given (random) generator, $g$ is a deterministic function, and $\int_t^T g(Y_r)d\hat{B}_r$ denotes the backward Itô integral. BDSDEs were introduced by Pardoux and Peng in [16] as a generalization of the classical backward stochastic differential equations (BSDEs for short) considered in the pioneering paper [15] by these authors, in order to give a probabilistic representation of solutions to a class of systems of quasilinear stochastic partial differential equations.

There is an extensive literature on numerical schemes for BSDEs. Most of the works deal with the case where the terminal random variable $\xi$ is a functional of a forward diffusion process $\{X_t\}_{0 \leq t \leq T}$ and the generator $f$ is of the form $f(t, X_t, Y_t, Z_t)$, where $f$ is a deterministic function. Starting from the four-step numerical scheme considered by Ma, Protter and Yong in [13], many authors have contributed to this problem (see, for instance, [2, 3, 4, 5, 7, 10, 12]). In [17], Zhang introduced a discretization method based on the $L^2$-regularity of the process $Z$.

2010 Mathematics Subject Classification. 60H10; 60H07; 60H05.

Key words and phrases. Malliavin calculus, Backward doubly stochastic differential equations, explicit solution to linear bdsde, implicit scheme, Hölder continuity of the solution pairs, rate of convergence.

D. Nualart was supported by the NSF grant DMS1512891.
In [9] the present authors considered the case of a BSDE with a general terminal value $\xi$ which is twice differentiable in the sense of Malliavin calculus and the first and second Malliavin derivatives satisfy some integrability conditions and we also made similar assumptions for the generator $f$. In this general framework, we were able to obtain an estimate of the form $\mathbb{E}|Z_t - Z_s|^p \leq K|t - s|^{\frac{p}{2}}$ for any $p \geq 2$ and we applied this result to study the rate of convergence of different types of numerical schemes, including an implicit one.

Unlike the case of BSDEs, numerical schemes for BDSDEs have received much less attention. The presence of a backward Itô stochastic integral creates additional difficulties when deriving the path regularity of the process $Z$ and computing the rate of convergence of numerical schemes. In the present paper, we consider an implicit numerical scheme introduced by Bachouch, Ben Lasmar, Matoussi and Mnif in an unpublished note [1]. Under the general assumptions on the terminal variable $\xi$ and the generator $f$ considered in [9] we have been able to show the Hölder continuity of the process $Z$ and to derive a rate of convergence of the scheme (see the estimate (3.33)) in Theorem 3.9. The approach is similar to that developed in [9], however, there is a new significant difficulty. Unlike BSDE, linear BDSDEs do not have an explicit solution in exponential form and the desired representation of the process $Y$ (see formula (3.5)) cannot be deduced directly from Itô’s formula. We shall get around this difficulty by using Taylor expansion with some explicit computations. On the other hand, we need to assume that the function $g$ in Equation (1.1) is a deterministic functional of the process $Y$.

The paper is organized as follows. Section 2 contains some preliminaries on backward stochastic integrals and Malliavin calculus. The Malliavin calculus provides a representation of the random variable $Z_t$ as the derivative $D_t Y_t$, which is very useful to derive the Hölder continuity and other regularity properties of $Z$. Our main results are stated in Section 3 and Sections 4 to 7 are devoted to the proofs.

The results of this paper still hold if the Brownian motions are multidimensional, but we have restricted the presentation to the one-dimensional case for the sake of simplicity.

2. Notation and preliminaries

2.1. Backward and forward Itô integrals. Recall all the notations defined at the beginning of the previous section and for any $t \in [0,T]$ define

$$ \mathcal{G}_t = \mathcal{F}^W_t \vee \mathcal{F}^B_{0,t}. $$

Note that $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is not a filtration, while $\{\mathcal{G}_t\}_{0 \leq t \leq T}$ is a filtration.

We say that a stochastic process $\{u_t\}_{0 \leq t \leq T}$ is $\mathcal{G}$-adapted ($\mathcal{F}$-adapted, respectively), if $u_t$ is $\mathcal{G}_t$-measurable ($\mathcal{F}_t$-measurable, respectively) for all $t \in [0,T]$. Consider the following spaces of random variables and processes:

- $M^p$, for any $p \geq 2$, denotes the class of $L^p$-integrable random variables $F$ with a stochastic integral representation of the form

$$ F = \mathbb{E}(F|\mathcal{G}_0) + \int_0^T u_t dW_t, $$

where $u$ is a $\mathcal{G}$-adapted stochastic process satisfying $\sup_{0 \leq t \leq T} \mathbb{E}|u_t|^p < \infty$;
\( H^p_T([0, T]) \) (\( H^p_G([0, T]) \), respectively), for any \( p \geq 1 \), denotes the set of jointly measurable and \( \mathcal{F} \)-adapted (\( G \)-adapted, respectively) processes \( \{\varphi_t\}_{0 \leq t \leq T} \) satisfying
\[
\|\varphi\|_{H^p} = \left( \mathbb{E} \left( \int_0^T |\varphi_t|^2 dt \right)^{\frac{1}{p}} \right) < \infty;
\]

- \( S^p_T([0, T]) \) (\( S^p_G([0, T]) \), respectively), for any \( p \geq 1 \), denotes the set of all RCLL (right-continuous with left limits) \( \mathcal{F} \)-adapted (\( G \)-adapted, respectively) processes \( \{\varphi_t\}_{0 \leq t \leq T} \) satisfying
\[
\|\varphi\|_{S^p} = \left( \mathbb{E} \sup_{0 \leq t \leq T} |\varphi_t|^p \right)^{\frac{1}{p}} < \infty.
\]

The backward Itô integral is similar to the classical (forward) Itô integral, if we just reverse the time. Therefore, Itô’s formula and Itô’s isometry also hold for the backward Itô integral. In particular, Itô’s formula has the following form because of the backward integral (see Lemma 1.3 in [16]).

**Lemma 2.1.** Suppose that \( \beta, \gamma \) and \( \sigma \) are processes in \( H^2_T([0, T]) \). Let the \( \mathcal{F} \)-adapted process \( \alpha \) have the following form
\[
\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s d\widehat{B}_s + \int_0^t \sigma_s dW_s, \quad 0 \leq t \leq T.
\]
Then,
\[
\alpha_t^2 = \alpha_0^2 + 2 \int_0^t \alpha_s \beta_s ds + 2 \int_0^t \alpha_s \gamma_s d\widehat{B}_s + 2 \int_0^t \alpha_s \sigma_s dW_s - \int_0^t \gamma_s^2 ds + \int_0^t \sigma_s^2 ds.
\]
More generally, if \( f \in C^2(\mathbb{R}) \), then we have the following Itô’s formula
\[
f(\alpha_t) = f(\alpha_0) + \int_0^t f'(\alpha_s) \beta_s ds + \int_0^t f'(\alpha_s) \gamma_s d\widehat{B}_s + \int_0^t f'(\alpha_s) \sigma_s dW_s
\]
\[
- \frac{1}{2} \int_0^t f''(\alpha_s) \gamma_s^2 ds + \frac{1}{2} \int_0^t f''(\alpha_s) \sigma_s^2 ds,
\]
for \( t \in [0, T] \).

## 2.2. Malliavin calculus with respect to the Brownian motion \( W \).

In this subsection, we present some preliminaries on Malliavin calculus and we refer the reader to the books [8] and [14] for more details.

Let \( \mathbf{H} = L^2([0, T]) \) be the separable Hilbert space of all square integrable real-valued functions on the interval \([0, T]\) with scalar product denoted by \( \langle \cdot, \cdot \rangle_{\mathbf{H}} \). The norm of an element \( h \in \mathbf{H} \) will be denoted by \( \|h\|_{\mathbf{H}} \). For any \( h \in \mathbf{H} \) we put \( W(h) = \int_0^T h(t) dW_t \) and \( B(h) = \int_0^T h(t) dB_t \).

For any \( m, n \in \mathbb{N} \), we denote by \( C^\infty_p(\mathbb{R}^{m+n}) \) the set of all infinitely differentiable functions \( g : \mathbb{R}^{m+n} \to \mathbb{R} \) such that \( g \) and all of its partial derivatives have polynomial growth. We make use of the notation \( \partial_i g = \frac{\partial g}{\partial x_i} \) whenever \( g \in C^1(\mathbb{R}^{m+n}) \).
Let $\mathcal{S}$ denote the class of smooth and cylindrical random variables such that a random variable $F \in \mathcal{S}$ has the form

$$F = g(W(h_1), \ldots, W(h_m), B(k_1), \ldots, B(k_n)),$$

where $g$ belongs to $C^\infty_p(\mathbb{R}^{m+n})$, $h_1, \ldots, h_m$ and $k_1, \ldots, k_n$ are in $\mathbb{H}$, and $m, n \in \mathbb{N}$.

For a smooth and cylindrical random variable $F$ of the form (2.1), its Malliavin derivative with respect to $W$ is the $\mathbb{H}$-valued random variable given by

$$D_tF = \sum_{i=1}^m \partial_i g(W(h_1), \ldots, W(h_m), B(k_1), \ldots, B(k_n))h_i(t), \quad t \in [0, T].$$

For any $p \geq 1$ we will denote the domain of $D$ in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$, meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth and cylindrical random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = (\mathbb{E}|F|^p + \mathbb{E}\|DF\|_\mathbb{H}^p)^{\frac{1}{p}}.$$

We can define the iteration of the operator $D$ in such a way that for a smooth and cylindrical random variable $F$, the iterated derivative $D^kF$ is a random variable with values in $\mathbb{H}^\otimes k$. For every $p \geq 1$ and any natural number $k \geq 1$ we introduce the seminorm on $\mathcal{S}$ defined by

$$\|F\|_{k,p} = \left(\mathbb{E}|F|^p + \sum_{j=1}^{k} \mathbb{E}\|D^jF\|_{\mathbb{H}^\otimes j}^p\right)^{\frac{1}{p}}.$$

We will denote by $\mathbb{D}^{k,p}$ the completion of the family of smooth and cylindrical random variables $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$.

Let $\mu$ be the Lebesgue measure on $[0, T]$. For any $k \geq 1$ and $F \in \mathbb{D}^{k,p}$, the derivative

$$D^kF = \{D^k_{t_1,\ldots,t_k}F, \; t_i \in [0, T], \; i = 1, \ldots, k\},$$

is a measurable function on the product space $[0, T]^k \times \Omega$, which is defined a.e. with respect to the measure $\mu^k \times \mathbb{P}$.

We denote by $\mathbb{L}^{1,p}_a$ the set of real-valued jointly measurable processes $u = \{u_t\}_{0 \leq t \leq T}$ such that

(i): For each $t \in [0, T]$, $u_t$ is $\mathcal{F}_t$-measurable.
(ii): For almost all $t \in [0, T]$, $u_t \in \mathbb{D}^{1,p}$.
(iii): $\mathbb{E}\left(\int_0^T |u_t|^2 dt\right)^{\frac{p}{2}} + \left(\int_0^T \int_0^T |D_gu_t|^2 d\theta dt\right)^{\frac{p}{2}} < \infty$.

3. Main results

In this section, we will give a summary of main results whose proofs will be provided in subsequent sections.

3.1. Estimates on the solutions of BDSDEs. We assume that the generator in the BDSDE (1.1) is a jointly measurable function $f : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega \to \mathbb{R}$, such that, for each fixed pair $(y, z) \in \mathbb{R}^2$, $f(t, y, z)$ is $\mathcal{F}_t$-measurable for all $t \in [0, T]$. We suppose also that the terminal value $\xi$ is an $\mathcal{F}_T$-measurable random variable.
**Definition 3.1.** A solution to the BDSDE (1.1) is a pair of $\mathcal{F}$-adapted processes $(Y, Z)$ such that: $\int_0^T |Z_t|^2 dt < \infty$, $\int_0^T |g(Y_s)|^2 ds < \infty$, $\int_0^T |f(t, Y_t, Z_t)| dt < \infty$, a.s., and

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T g(Y_s)dB_s - \int_t^T Z_r dW_r, \quad 0 \leq t \leq T.$$

Theorem 3.1. Fix $L$ for all $y \leq S$ and $H$. Under the assumptions in Theorem 3.2.

Corollary 3.2. Let $(Y, Z) \in S \times H([0, T])$ be the unique solution pair to the BDSDE (1.1). Moreover, we have the following estimate for the solution

$$\mathbb{E} \sup_{0 \leq t \leq T} |Y_t|^q + \mathbb{E} \left( \int_0^T |Z_t|^2 dt \right)^{\frac{q}{2}} \leq K \left( \mathbb{E} |\xi|^q + \mathbb{E} \left( \int_0^T |f(t, 0, 0)|^2 dt \right)^{\frac{q}{2}} + |g(0)|^q \right),$$  \hspace{1cm} (3.1)

where $K$ is a constant depending only on $L$, $q$ and $T$.

Corollary 3.2. Under the assumptions in Theorem 3.1, let $(Y, Z) \in S \times H([0, T])$ be the unique solution pair to the BDSDE (1.1). If $\sup_{0 \leq t \leq T} \mathbb{E} |Z_t|^q < \infty$, then there exists a constant $C$, depending on $L$, $q$, $T$ and the quantity appearing in the right-hand side of (3.1), such that, for any $s, t \in [0, T]$,

$$\mathbb{E} |Y_t - Y_s|^q \leq C |t - s|^{\frac{q}{2}}.$$  \hspace{1cm} (3.2)

3.2. Linear BDSDEs. As we will see later, the component $Z$ of the solution of a given BDSDE can be represented in terms of the Malliavin derivative of the solution $Y$, which satisfies a linear BDSDE with random coefficients. In order to describe the properties of $Z$ we first study a class of linear BDSDEs.

We consider the following linear BDSDE:

$$Y_t = \xi + \int_t^T (\alpha_s Y_s + \beta_s Z_s + f_s) ds + \int_t^T \gamma_s Y_s dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \hspace{1cm} (3.3)$$

where the processes $\alpha, \beta, \gamma$ and $f$ are jointly measurable and $\mathcal{F}$-adapted.

We impose the following boundedness condition on the coefficients.

**(H1):** The processes $\{\alpha_t\}_{0 \leq t \leq T}$, $\{\beta_t\}_{0 \leq t \leq T}$ and $\{\gamma_t\}_{0 \leq t \leq T}$ are uniformly bounded, namely, there exists a constant $L > 0$ such that

$$|\alpha_t| + |\beta_t| + |\gamma_t| \leq L, \quad a.s. \mu \times \mathbb{P} \text{ on } [0, T] \times \Omega.$$
Under the condition \((H1)\), we define a process \(\rho\) by
\[
\rho_t = \exp \left\{ \int_0^t \beta_s dW_s + \int_0^t \gamma_s dB_s + \int_0^t \left( \alpha_s - \frac{1}{2} \beta_s^2 - \frac{1}{2} \gamma_s^2 \right) ds \right\}, \quad 0 \leq t \leq T. \tag{3.4}
\]
Note that \(\rho\) is not \(\mathcal{F}_T\)-measurable but \(\mathcal{G}_T\)-measurable and that \(\rho\) has a continuous version. In fact, \(\rho\) does not satisfy any stochastic differential equation and hence Itô’s formula cannot be applied to the process \(\rho\). However, we are still able to prove the following result.

**Theorem 3.3.** Let \(\xi \in L^2(\Omega)\) and \(f \in H^2_\mathcal{F}([0,T])\). Assume that the processes \(\alpha, \beta\) and \(\gamma\) satisfy condition \((H1)\). Let \(\rho\) be defined in (3.4). Then, there exists a unique solution \((Y, Z) \in S^2_{\mathcal{F}}([0,T]) \times H^2_{\mathcal{F}}([0,T])\) to (3.3) and the following equation holds
\[
Y_t \rho_t = \xi \rho_T + \int_t^T \rho_s f_s ds - \int_t^T (\rho_s Z_s + Y_s \rho_s \beta_s) dW_s, \quad 0 \leq t \leq T. \tag{3.5}
\]

As a consequence, we have the following representation for \(Y\):
\[
Y_t = \rho_t^{-1} \mathbb{E} \left( \xi \rho_T + \int_t^T \rho_s f_s ds \bigg| \mathcal{G}_t \right). \tag{3.6}
\]

The following result on the moment estimate of the increment of \(Y\) in the linear BDSDE (3.3) will play a critical role in the proof of our main result in this paper.

**Theorem 3.4.** Let \(q > p \geq 2\) and let \(\xi \in L^q(\Omega)\) and \(f \in H^q_\mathcal{F}([0,T])\). Assume that the processes \(\alpha, \beta\) and \(\gamma\) satisfy the condition \((H1)\) and that the random variables \(\xi \rho_T\) and \(\int_0^T \rho_t f_t dt\) belong to \(M^q\), where the process \(\rho\) is defined in (3.4). Then the linear BDSDE (3.3) has a unique solution \((Y, Z)\), and there exists a constant \(K > 0\) such that
\[
\mathbb{E}|Y_t - Y_s|^p \leq K|t - s|^{\frac{q}{2}}, \tag{3.7}
\]
for all \(s, t \in [0,T]\).

### 3.3. The Malliavin calculus for BDSDEs and the path regularity of \(Z\).

In this subsection, we consider the Malliavin calculus for the BDSDE (1.1). First, we make the following assumptions on the terminal value \(\xi\) and generator \(f\).

**Assumption (A):** Fix \(2 \leq p < \frac{q}{2}\).

(i) \(\xi \in D^{2,q}\), and there exists \(L > 0\), such that for all \(\theta, \theta' \in [0,T]\),
\[
\mathbb{E}|D_\theta \xi - D_{\theta'} \xi|^p \leq L|\theta - \theta'|^{\frac{q}{2}}, \tag{3.8}
\]
\[
\sup_{0 \leq \theta \leq T} \mathbb{E}|D_\theta \xi|^q < \infty, \tag{3.9}
\]
and
\[
\sup_{0 \leq \theta \leq T} \sup_{0 \leq u \leq T} \mathbb{E}|D_u D_\theta \xi|^q < \infty. \tag{3.10}
\]

(ii) The generator \(f(t, y, z)\) has continuous and uniformly bounded first and second order partial derivatives with respect to \(y\) and \(z\), and \(f(\cdot, 0, 0) \in H^q_\mathcal{F}([0,T])\).

(iii) The function \(g\) has continuous and bounded first and second order derivatives \(g'\) and \(g''\) respectively.
(iv) Assume that $\xi$ and $f$ satisfy the above conditions (i) and (ii). Let $(Y, Z)$ be the unique solution to (1.1) with terminal value $\xi$ and generator $f$. For each $(y, z) \in \mathbb{R} \times \mathbb{R}$, $f(\cdot, y, z)$, $\partial_y f(\cdot, y, z)$, and $\partial_z f(\cdot, y, z)$ belong to $L_{\alpha}^{1,q}$, and the corresponding Malliavin derivatives $Df(\cdot, y, z)$, $D\partial_y f(\cdot, y, z)$, and $D\partial_z f(\cdot, y, z)$ satisfy

$$\sup_{0 \leq \theta \leq T} \mathbb{E} \left( \int_{\theta}^{T} |D\theta f(t, Y_t, Z_t)|^2 dt \right)^{\frac{q}{2}} < \infty, \quad (3.11)$$

$$\sup_{0 \leq \theta \leq T} \mathbb{E} \left( \int_{\theta}^{T} |D_{\theta} \partial_y f(t, Y_t, Z_t)|^2 dt \right)^{\frac{q}{2}} < \infty, \quad (3.12)$$

$$\sup_{0 \leq \theta \leq T} \mathbb{E} \left( \int_{\theta}^{T} |D_{\theta} \partial_z f(t, Y_t, Z_t)|^2 dt \right)^{\frac{q}{2}} < \infty, \quad (3.13)$$

and there exists $L > 0$ such that for any $t \in (0, T]$, and for any $0 \leq \theta, \theta' \leq t \leq T$

$$\mathbb{E} \left( \int_{t}^{T} \left| D\theta f(r, Y_r, Z_r) - D\theta' f(r, Y_r, Z_r) \right|^2 dr \right)^{\frac{q}{2}} \leq L|\theta - \theta'|^{\frac{q}{2}}. \quad (3.14)$$

For each $\theta \in [0, T]$, and each pair of $(y, z)$, $D\theta f(\cdot, y, z) \in L_{\alpha}^{1,q}$ and it has continuous partial derivatives with respect to $y, z$, which are denoted by $\partial_y D\theta f(t, y, z)$ and $\partial_z D\theta f(t, y, z)$, and the Malliavin derivative $D_u D\theta f(t, y, z)$ satisfies

$$\sup_{0 \leq \theta \leq T} \sup_{0 \leq u \leq T} \mathbb{E} \left( \int_{\theta \vee u}^{T} \left| D_u D\theta f(t, Y_t, Z_t) \right|^2 dt \right)^{\frac{q}{2}} < \infty. \quad (3.15)$$

The following property is easy to check and we omit the proof.

**Remark 3.5.** Conditions (3.12) and (3.13) imply

\[
\begin{align*}
\sup_{0 \leq \theta \leq T} \mathbb{E} \left( \int_{\theta}^{T} |\partial_y D\theta f(t, Y_t, Z_t)|^2 dt \right)^{\frac{q}{2}} &< \infty, \\
\sup_{0 \leq \theta \leq T} \mathbb{E} \left( \int_{\theta}^{T} |\partial_z D\theta f(t, Y_t, Z_t)|^2 dt \right)^{\frac{q}{2}} &< \infty.
\end{align*}
\]

We refer to [9, Section 2.4] for several examples where Assumption (A) is satisfied, including the cases where $\xi$ is a multiple stochastic integral, a twice Fréchet differentiable function of $W$ and a nonnecessarily Lipschitz function of the trajectories of a forward diffusion.

The following is the main result in this subsection.

**Theorem 3.6.** Let Assumption (A) be satisfied.

(a) Suppose that $(Y, Z)$ is the unique solution pair in $S_{F}^{\alpha}(0, T] \times H_{F}^{q}(0, T]$ to the BDSDE (1.1). Then, $Y$ and $Z$ are in $L_{\alpha}^{1,q}$ and there exists a version of the Malliavin derivatives $\{(D\theta Y_t, D\theta Z_t)\}_{0 \leq \theta \leq T}$ of the solution pair that satisfies the following linear
BDSDE:

\[ D_\theta Y_t = D_\theta \xi + \int_t^T \left[ \partial_y f(r, Y_r, Z_r) D_\theta Y_r + \partial_z f(r, Y_r, Z_r) D_\theta Z_r + D_\theta f(r, Y_r, Z_r) \right] dr \]
\[ + \int_t^T g'(Y_r) D_\theta Y_r d\overline{B}_r - \int_t^T D_\theta Z_r dW_r, \quad 0 \leq \theta \leq t \leq T; \]  
\[ D_\theta Y_t = 0, D_\theta Z_t = 0, \quad 0 \leq t < \theta \leq T. \]  

Moreover, \( \{D_\theta Y_t\}_{0 \leq t \leq T} \) defined by (3.18) gives a version of \( \{Z_t\}_{0 \leq t \leq T} \), namely, \( \mu \times P \) a.e.

\[ Z_t = D_\theta Y_t. \]  

(b) There exists a constant \( K > 0 \), such that, for all \( s, t \in [0, T] \),
\[ \mathbb{E}|Z_t - Z_s|^p \leq K|t - s|^\gamma. \]  

**Remark 3.7.** From Theorem 3.6 we know that \( \{(D_\theta Y_t, D_\theta Z_t)\}_{0 \leq \theta \leq t \leq T} \) satisfies Equations (3.18) and (3.19) and \( Z_t = D_\theta Y_t, \mu \times P \) a.e. Moreover, since (3.9) and (3.11) hold, we can apply the estimate (3.1) in Theorem 3.1 to the linear BDSDE (3.18)-(3.19) and deduce \( \sup_{0 \leq t \leq T} \mathbb{E}|Z_t|^q < \infty \). Therefore, by Corollary 3.2, the process \( Y \) satisfies the inequality (3.2). By Kolmogorov’s continuity criterion, this implies that \( Y \) has Hölder continuous trajectories of order \( \gamma \) for any \( \gamma < \frac{1}{2} - \frac{1}{q} \).

### 3.4. An implicit numerical scheme for (1.1).

In this subsection, we consider an implicit numerical scheme for the BDSDE (1.1). By using the path regularity of the process \( Z \) in (3.21), we are able to give an estimate on the error in \( L^p \) sense.

We will need the following fixed point result in the construction of our Euler scheme for the BDSDE (1.1). Its proof is easy to obtain, so we omit it here.

**Remark 3.8.** Let \( f \) satisfy (ii) in Assumption (A), and let \( h > 0 \) be a constant with \( hL < 1 \), where \( L \) is the Lipschitz constant for \( f \). For any given \( \eta, z \in \mathbb{R} \) and \( t \in [0, T] \), there exists a unique \( y = y(\omega) \), such that,
\[ y = \eta + h f(t, y, z), \quad a.s.. \]

Let \( \pi = \{0 = t_0 < t_1 < \cdots < t_n = T\} \) be a partition of the interval \( [0, T] \) and \( |\pi| = \max_{0 \leq i \leq n-1} |t_{i+1} - t_i| \). Denote \( \Delta_i = t_{i+1} - t_i, \Delta B_i = B_{t_{i+1}} - B_{t_i} \) and \( \Delta W_i = W_{t_{i+1}} - W_{t_i}, \quad 0 \leq i \leq n - 1 \). We also assume that
\[ |\pi|L < 1, \]  
where \( L \) is the Lipschitz constant of the generator \( f \).

From the BDSDE (1.1), we know that, when \( t \in [t_i, t_{i+1}] \),
\[ Y_t = Y_{t_{i+1}} + \int_t^{t_{i+1}} f(r, Y_r, Z_r) dr + \int_t^{t_{i+1}} g(Y_r) d\overline{B}_r - \int_t^{t_{i+1}} Z_r dW_r. \]  

We consider a numerical scheme similar to that introduced in [1]. For this scheme, we are able to achieve an estimate on the error in \( p \)-th moment, which is better than the estimates existing in the literature.
The numerical scheme we consider is as follows:

\[ Y_{t_i}^\pi = \xi^\pi, \quad Z_{t_i}^\pi = 0, \quad (3.24) \]

and for \( i = n-1, n-2, \ldots, 1, 0, \) we define \( Y_{t_i}^\pi \) as follows

\[
Z_{t_i}^\pi = \frac{1}{\Delta_i} \mathbb{E} \left( Y_{t_{i+1}}^\pi \Delta W_i + g(Y_{t_{i+1}}^\pi) \Delta B_i | \mathcal{F}_{t_i} \right), \quad (3.25)
\]

\[
Y_{t_i}^\pi = \mathbb{E} \left( Y_{t_{i+1}}^\pi + g(Y_{t_{i+1}}^\pi) \Delta B_i | \mathcal{F}_{t_i} \right) + f(t, Y_{t_i}^\pi, Z_{t_i}^\pi) \Delta_i, \quad (3.26)
\]

where \( \xi^\pi \in L^p(\Omega) \) is an approximation of the terminal condition \( \xi. \) Then Remark 3.8, (3.25) and (3.26) lead to a backward recursive formula for the sequence \( \{Y_{t_i}^\pi, Z_{t_i}^\pi\}_{0 \leq i \leq n}. \)

Next, for each partition \( \pi, \) we introduce \( (Y^{1,\pi}, Z^{1,\pi}) \) and give a connection between the approximation solution \( (Y^\pi, Z^\pi) \) and the new defined approximation \( (Y^{1,\pi}, Z^{1,\pi}). \) More precisely, we proceed as follows. Once \( Y_{t_{i+1}}^\pi \) and \( Z_{t_{i+1}}^\pi, \) which are \( \mathcal{F}_{t_{i+1}} \)-measurable, are defined, then, for \( t \in [t_i, t_{i+1}], \) we set

\[
Y_t^{1,\pi} = \mathbb{E} \left( Y_{t_{i+1}}^\pi + g(Y_{t_{i+1}}^\pi) \Delta B_i | \mathcal{F}_t \cup \mathcal{F}_t^{1,\pi} \right).
\]

By the stochastic integral representation, we have

\[
Y_t^{1,\pi} = Y_{t_{i+1}}^\pi + g(Y_{t_{i+1}}^\pi) \Delta B_i - \int_t^{t_{i+1}} Z_r^{1,\pi} dW_r, \quad t \in [t_i, t_{i+1}], \quad (3.27)
\]

where \( Y_t^{1,\pi} \) and \( Z_t^{1,\pi} \) are \( \mathcal{F}_t \cup \mathcal{F}_t^{1,\pi} \)-measurable for all \( t \in [t_i, t_{i+1}]. \) In particular, at the endpoint \( t = t_i, \) it holds that

\[
Y_{t_i}^{1,\pi} = \mathbb{E} \left( Y_{t_{i+1}}^\pi + g(Y_{t_{i+1}}^\pi) \Delta B_i | \mathcal{F}_{t_i} \right). \quad (3.28)
\]

Hence from (3.25) and (3.27) (with \( t = t_i \)) it follows

\[
Z_{t_i}^\pi = \frac{1}{\Delta_i} \mathbb{E} \left( \left[ Y_{t_{i+1}}^\pi + g(Y_{t_{i+1}}^\pi) \Delta B_i \right] \Delta W_i | \mathcal{F}_{t_i} \right) = \frac{1}{\Delta_i} \mathbb{E} \left( \left[ Y_{t_i}^{1,\pi} + \int_{t_i}^{t_{i+1}} Z_r^{1,\pi} dW_r \right] \Delta W_i | \mathcal{F}_{t_i} \right)
\]

\[
= \frac{1}{\Delta_i} \mathbb{E} \left( \int_{t_i}^{t_{i+1}} Z_r^{1,\pi} dW_r | \mathcal{F}_{t_i} \right). \quad (3.29)
\]

Then, by (3.26) and (3.28), the connection between \( Y_{t_i}^\pi \) and \( Y_{t_i}^{1,\pi} \) is given by

\[
Y_{t_i}^\pi = Y_{t_i}^{1,\pi} + f(t, Y_{t_i}^{1,\pi}, Z_{t_i}^{1,\pi}) \Delta_i. \quad (3.30)
\]

Thus, from (3.27) with \( t = t_i \) and (3.30), we have

\[
Y_{t_i}^\pi = Y_{t_{i+1}}^\pi + f(t, Y_{t_i}^\pi, Z_{t_i}^\pi) \Delta_i + g(Y_{t_{i+1}}^\pi) \Delta B_i - \int_{t_i}^{t_{i+1}} Z_r^{1,\pi} dW_r, \quad i = n-1, \ldots, 0. \quad (3.31)
\]
Theorem 3.9. Let Assumption (A) be satisfied, and let the partition \( \pi \) satisfy (3.22). Consider the approximation scheme (3.24)-(3.25). Assume that \( \xi^n \in L^p(\Omega) \) and that there exists a constant \( L_1 > 0 \) such that

\[
|f(t_2, y, z) - f(t_1, y, z)| \leq L_1|t_2 - t_1|^{\frac{1}{2}}, \; a.s.
\]

for all \( t_1, t_2 \in [0, T] \) and \( y, z \in \mathbb{R} \). Then, there are positive constants \( K \) and \( \delta \), independent of the partition \( \pi \), such that, if \( |\pi| < \delta \), then

\[
\mathbb{E} \max_{0 \leq i \leq n-1} |Y_{t_i} - Y_{t_i}^\pi|^p + \mathbb{E} \left( \int_0^T |Z_t - Z_t^1, \pi|^2 dt \right)^{\frac{p}{2}} \leq K \left( \mathbb{E}|\xi - \xi^n|^p + |\pi|^\frac{p}{2} \right). \tag{3.33}
\]

4. Proofs of Theorem 3.1 and Corollary 3.2

Proof of Theorem 3.1. The proof of the existence and uniqueness of a solution \( (Y, Z) \in S_{\mathcal{F}, \mathcal{G}}^q([0, T]) \times \mathcal{H}_{\mathcal{F}}^q([0, T]) \) can be found in [16, Theorem 1.1].

Let us show the estimate (3.1). We first consider the BDSDE (1.1) on a fixed interval \([a, b]\) for any \( 0 \leq a < b \leq T \). From (1.1) we get, for any \( t \in [a, b] \),

\[
Y_t = \mathbb{E} \left( Y_b + \int_a^b f(r, Y_r, Z_r) dr + \int_a^b g(Y_r) d\tilde{B}_r \right) |_{G_t} + \mathbb{E} \left( \int_a^b g(Y_r) d\tilde{B}_r \right) |_{G_t}.
\]

The above conditional expectation and \( \int_a^b g(Y_r) d\tilde{B}_r \) are martingales if they are considered as processes indexed by \( t \) for \( t \in [a, b] \). Using Doob’s maximal inequality, the Burkholder-Davis-Gundy inequality, the isometry property for Itô’s integral and the Lipschitz conditions on \( f \) and \( g \), we have

\[
\mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q \leq C\mathbb{E}|Y_b|^q + C\mathbb{E} \left( \int_a^b |f(r, Y_r, Z_r)| dr \right)^q + C\mathbb{E} \left( \int_a^b g(Y_r) d\tilde{B}_r \right)^q
\]

\[
+ C\mathbb{E} \left( \int_a^b |g(Y_r)|^2 dr \right)^{\frac{q}{2}}
\]

\[
\leq C\mathbb{E}|Y_b|^q + C(b - a)^{\frac{q}{2}} \mathbb{E} \left( \int_a^b |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}} + C(b - a)^{\frac{q}{2}} \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q
\]

\[
+ C(b - a)^{\frac{q}{2}} \mathbb{E} \left( \int_a^b |Z_r|^2 dr \right)^{\frac{q}{2}} + C\mathbb{E} \left( \int_a^b |g(Y_r)|^2 dr \right)^{\frac{q}{2}}
\]

\[
\leq C\mathbb{E}|Y_b|^q + C(b - a)^{\frac{q}{2}} \mathbb{E} \left( \int_a^b |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}} + C(b - a)^{\frac{q}{2}} \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q
\]

\[
+ C(b - a)^{\frac{q}{2}} \mathbb{E} \left( \int_a^b |Z_r|^2 dr \right)^{\frac{q}{2}} + C(b - a)^{\frac{q}{2}} |g(0)|^q + C(b - a)^{\frac{q}{2}} \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q,
\]
where $C$, in the above inequalities and in the sequel, is a generic constant independent of $a$ and $b$, which may vary from line to line. Thus, we have

$$
\mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q \leq C \mathbb{E}|Y_b|^q + C(b - a)^{\frac{2}{q}} \mathbb{E} \left( \int_a^b |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}} + C(b - a)^{\frac{2}{q}} \mathbb{E} \left( \int_a^b |Z_r|^2 dr \right)^{\frac{q}{2}} + C(b - a)^{\frac{2}{q}} |g(0)|^q. \tag{4.1}
$$

By the Burholder-Davis-Gundy inequality, one has

$$
\mathbb{E} \left( \int_a^b |Z_r|^2 dr \right)^{\frac{q}{2}} \leq c_q \mathbb{E} \left| \int_a^b Z_r dW_r \right|^q, \tag{4.2}
$$

for some positive constant $c_q$ depending only on $q$. From (1.1), one also has

$$
\int_a^b Z_r dW_r = Y_b - Y_a + \int_a^b f(r, Y_r, Z_r) dr + \int_a^b g(Y_r) d\tilde{B}_r. \tag{4.3}
$$

From (4.2), (4.3), the isometry property of Itô’s integral and the Lipschitz conditions on $f$ and $g$, one can write

$$
\mathbb{E} \left( \int_a^b |Z_r|^2 dr \right)^{\frac{q}{2}} \leq C \mathbb{E}|Y_b|^q + C \mathbb{E}|Y_a|^q + C \mathbb{E} \left( \int_a^b |f(r, Y_r, Z_r)| dr \right)^q + C \mathbb{E} \left| \int_a^b g(Y_r) d\tilde{B}_r \right|^q
$$

$$
\leq C \mathbb{E}|Y_b|^q + C \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q + C(b - a)^{\frac{2}{q}} \mathbb{E} \left( \int_a^b |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}}
$$

$$
+ C(b - a)^{\frac{2}{q}} \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q + C(b - a)^{\frac{2}{q}} \mathbb{E} \left( \int_a^b |Z_r|^2 dr \right)^{\frac{q}{2}}
$$

$$
+ C(b - a)^{\frac{2}{q}} |g(0)|^q + C_1(b - a)^{\frac{2}{q}} \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q, \tag{4.4}
$$

where $C_1$ is a positive constant independent of $a$ and $b$. If $C_1(b - a)^{\frac{2}{q}} < \frac{1}{2}$, then from (4.4) we have

$$
\mathbb{E} \left( \int_a^b |Z_r|^2 dr \right)^{\frac{q}{2}} \leq 2C_1 \mathbb{E}|Y_b|^q + 2C_1 \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q + 2C_1(b - a)^{\frac{2}{q}} \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q
$$

$$
+ 2C_1(b - a)^{\frac{2}{q}} \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q + 2C_1(b - a)^{\frac{2}{q}} \mathbb{E} \left( \int_a^b |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}}
$$

$$
+ 2C_1(b - a)^{\frac{2}{q}} |g(0)|^q. \tag{4.5}
$$
Substituting (4.5) into (4.1) yields
\[
\mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q \leq C_2 (1 + (b - a) \frac{q}{2}) \mathbb{E} |Y_b|^q + C_2 (b - a) \frac{q}{2} \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q \\
+ C_2 (b - a) \frac{q}{2} \mathbb{E} \left( \int_a^b |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}} + C_2 (b - a) \frac{q}{2} |g(0)|^q,
\]
for some positive constant $C_2$ independent of $a$ and $b$.

If $C_2 (b - a) \frac{q}{2} < \frac{1}{2}$, then we have
\[
\mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q \leq 2C_2 (1 + (b - a) \frac{q}{2}) \mathbb{E} |Y_b|^q + 2C_2 (b - a) \frac{q}{2} \mathbb{E} \left( \int_a^b |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}} \\
+ 2C_2 (b - a) \frac{q}{2} |g(0)|^q.
\]

Denote
\[
\Theta_{a, b, q} := \mathbb{E} \sup_{a \leq t \leq b} |Y_t|^q + \mathbb{E} \left( \int_a^b |Z_r|^2 dr \right)^{\frac{q}{2}}.
\]
We choose a positive constant $\delta$ such that
\[
C_1 \delta^2 < \frac{1}{2}, \quad C_2 \delta^2 < \frac{1}{2}.
\]
If $b - a \leq \delta$, then from (4.5) and (4.7) it follows that there exists a positive constant $C_3$ independent of $a$ and $b$ such that
\[
\Theta_{a, b, q} \leq C_3 \left( \mathbb{E} |Y_b|^q + \mathbb{E} \left( \int_a^b |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}} + |g(0)|^q \right).
\]

Now, let $l = \lceil \frac{T}{\delta} \rceil + 1$ and $t_i = \frac{iT}{l}$ for $i = 0, 1, \ldots, l$. By (4.8) we have on the interval $[t_{i-1}, t_i]$
\[
\Theta_{t_{i-1}, t_i, q} \leq C_3 \left( \mathbb{E} |\xi|^q + \mathbb{E} \left( \int_{t_{i-1}}^{t_i} |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}} + |g(0)|^q \right).
\]

On the interval $[t_{i-2}, t_{i-1}]$, we have in a similar way
\[
\Theta_{t_{i-2}, t_{i-1}, q} \leq C_3 \left( \mathbb{E} |Y_{t_{i-1}}|^q + \mathbb{E} \left( \int_{t_{i-2}}^{t_{i-1}} |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}} + |g(0)|^q \right) \\
\leq C_3^2 \left( \mathbb{E} |\xi|^q + \mathbb{E} \left( \int_{t_{i-2}}^{t_i} |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}} + |g(0)|^q \right) \\
+ C_3 \left( \mathbb{E} \left( \int_{t_{i-2}}^{t_{i-1}} |f(r, 0, 0)|^2 dr \right)^{\frac{q}{2}} + |g(0)|^q \right).
\]
Or
\[
\Theta_{t_{i-2},t_{i-1},q} \leq C_3^2\mathbb{E}|\xi|^q + C_3^2\mathbb{E}\left(\int_{t_{i-1}}^{t_i} |f(r, 0, 0)|^2 dr\right)^{\frac{q}{2}} + C_3\mathbb{E}\left(\int_{t_{i-2}}^{t_{i-1}} |f(r, 0, 0)|^2 dr\right)^{\frac{q}{2}} + (C_3^2 + C_3)|g(0)|^q.
\]

By induction, for \( i = 1, 2, \ldots, l \), one can write
\[
\Theta_{t_{i-1},t_{i-1+1},q} \leq C_3^i\mathbb{E}|\xi|^q + \sum_{j=1}^{i} C_3^{i+1-j}\mathbb{E}\left(\int_{t_{i-j}}^{t_{i-j+1}} |f(r, 0, 0)|^2 dr\right)^{\frac{q}{2}} + \sum_{j=1}^{i} C_3^j|g(0)|^q.
\]

As a consequence, we have
\[
\Theta_{0,T,q} \leq \sum_{i=1}^{l} \mathbb{E}\sup_{t_{i-1} \leq t \leq t_{i-1+1}} |Y_t|^q + l^{\frac{q}{2}} \sum_{i=1}^{l} \mathbb{E}\left(\int_{t_{i-1}}^{t_{i-1+1}} |Z_r|^2 dr\right)^{\frac{q}{2}}
\leq C \left(\sum_{i=1}^{l} C_3^i\mathbb{E}|\xi|^q + \sum_{j=1}^{l} C_3^j|g(0)|^q\right)
\leq C \left(\sum_{i=1}^{l} C_3^i\mathbb{E}|\xi|^q + \sum_{i=1}^{l} \sum_{j=1}^{l} C_3^j|g(0)|^q\right)
\leq C \left(\sum_{i=1}^{l} C_3^i\mathbb{E}|\xi|^q + \sum_{i=1}^{l} \sum_{j=1}^{l} C_3^j|g(0)|^q\right)
\leq C \left(\sum_{i=1}^{l} C_3^i\mathbb{E}|\xi|^q + \sum_{i=1}^{l} \mathbb{E}\left(\int_{t_{i-j}}^{t_{i-j+1}} |f(r, 0, 0)|^2 dr\right)^{\frac{q}{2}} + l|g(0)|^q\right)
\leq C \left(\sum_{i=1}^{l} C_3^i\mathbb{E}|\xi|^q + \mathbb{E}\left(\int_{0}^{T} |f(r, 0, 0)|^2 dr\right)^{\frac{q}{2}} + |g(0)|^q\right)
\leq K \left(\mathbb{E}|\xi|^q + \mathbb{E}\left(\int_{0}^{T} |f(r, 0, 0)|^2 dr\right)^{\frac{q}{2}} + |g(0)|^q\right),
\]

which is the estimate (3.1).

\[
\square
\]

**Proof of Corollary 3.2.** Without loss of generality we assume \( 0 \leq s \leq t \leq T \). Let \( C > 0 \) denote a generic constant independent of \( s \) and \( t \) but depending only on \( L, q, T \) and the
quantity \( \mathbb{E}|\xi|^q + \mathbb{E} \left( \int_0^T |f(r,0,0)|^2 dr \right)^{\frac{q}{2}} + |g(0)|^q \), which may vary from line to line. Since
\[ Y_s = Y_t + \int_s^t f(r, Y_r, Z_r) dr + \int_s^t g(Y_s) d\widehat{B}_r - \int_s^t Z_r dW_r, \]
we have, by the Lipschitz condition on \( f \) and \( g \),
\[
\mathbb{E}|Y_t - Y_s|^q = \mathbb{E} \left| \int_s^t f(r, Y_r, Z_r) dr + \int_s^t g(Y_s) d\widehat{B}_r - \int_s^t Z_r dW_r \right|^q \\
\leq 3^{q-1} \left( \mathbb{E} \left| \int_s^t f(r, Y_r, Z_r) dr \right|^q + \mathbb{E} \left| \int_s^t g(Y_s) d\widehat{B}_r \right|^q + \mathbb{E} \left| \int_s^t Z_r dW_r \right|^q \right) \\
\leq C \left( |t-s|^2 \mathbb{E} \left( \int_s^t |f(r, Y_r, Z_r)|^2 dr \right)^{\frac{q}{2}} + \mathbb{E} \left( \int_s^t |g(Y_s)|^2 dr \right)^{\frac{q}{2}} + \mathbb{E} \left( \int_s^t |Z_r|^2 dr \right)^{\frac{q}{2}} \right) \\
\leq C \left\{ |t-s|^2 \mathbb{E} \left( \int_s^t |Y_r|^2 dr \right)^{\frac{q}{2}} + \mathbb{E} \left( \int_s^t |Z_r|^2 dr \right)^{\frac{q}{2}} + \mathbb{E} \left( \int_s^t |f(r,0,0)|^2 dr \right)^{\frac{q}{2}} \right\} \\
+ |t-s|^{\frac{q}{2}} \left( \mathbb{E} \sup_{0 \leq r \leq T} |Y_r|^q \right) + |t-s|^{\frac{q}{2}} \sup_{0 \leq r \leq T} \mathbb{E}|Z_r|^q \\
\leq C|t-s|^{\frac{q}{2}}.
\]
The proof is completed. \( \square \)

5. Proof of the results for linear BDSDEs

We first study the properties of the process \( \rho \) defined in (3.4). The following lemmas will be needed to prove Theorems 3.3 and 3.4.

**Lemma 5.1.** Let the processes \( \alpha, \beta \) and \( \gamma \) satisfy the condition \((\text{H1})\), and let \( \rho \) be defined by (3.4). Then the following properties are true:

(a) For any \( r \in \mathbb{R} \) we have \( \mathbb{E} \left( \sup_{0 \leq t \leq T} \rho_t^r \right) < \infty; \)

(b) For any \( r > 0 \) and \( 0 \leq s \leq t \leq T \), we have
\[
\mathbb{E}|\rho_t - \rho_s|^r \leq C|t-s|^{\frac{r}{2}},
\]
where \( C \) is a positive constant which is independent of \( s \) and \( t \).

**Proof.** The proof of the part (a) is analogous to that of Lemma 2.4 in [9]. The only difference is the backward integral with respect to \( d\widehat{B}_s \). For this we just need to reverse the time from \( T \) to 0. We omit the details here.

Part (b): Using Taylor’s expansion for the function \( h(x) = e^x \) up to the first order, we have
\[
\rho_t - \rho_s = \rho_s e^\eta \left( \int_s^t \gamma_r d\widehat{B}_r + \int_s^t \beta_r dW_r + \int_s^t \left( \alpha_r - \frac{1}{2} \alpha_r^2 \right) dr \right) = \rho_s \delta_{s,t} e^\eta, \quad (5.1)
\]
for some random variable $\eta$ between 0 and $\delta_{s,t}$, where

$$\delta_{s,t} = \int_s^t \gamma_r d\overline{B}_r + \int_s^t \beta_r dW_r + \int_s^t \left(\alpha_r - \frac{1}{2} \beta_r^2 - \frac{1}{2} \gamma_r^2\right) dr.$$

Using part (a) and Hölder’s inequality we obtain, for any $p > 0$,

$$\mathbb{E} e^{\rho_t} = \mathbb{E} \left( e^{\rho_t |\eta|_{\mathbb{P}}} \right) + \mathbb{E} \left( e^{\rho_t |\eta|_{\mathbb{Q}}} \right) \leq 1 + \mathbb{E} \left( \sup_{0 \leq t \leq T} \rho_t^p \sup_{0 \leq s \leq T} \rho_s^p \right) \leq 1 + \mathbb{E} \left( \sup_{0 \leq t \leq T} \rho_t^p \sup_{0 \leq s \leq T} \rho_s^p \right) < \infty. \quad (5.1)$$

Thus, by (5.1), (5.2), part (a), Hölder’s inequality, Burkholder-Davis-Gundy inequality and the boundedness of the processes $\alpha$, $\beta$ and $\gamma$, we can show the result in part (b) as follows

$$\mathbb{E} |\rho_t - \rho_s|^r \leq \left( \mathbb{E} (\rho_s^2 e^{2\rho_t}) \right)^{\frac{r}{2}} \times \left( \mathbb{E} \left| \int_s^t \gamma_r d\overline{B}_r + \int_s^t \beta_r dW_r + \int_s^t \left(\alpha_r - \frac{1}{2} \beta_r^2 - \frac{1}{2} \gamma_r^2\right) dr \right|^2 \right)^{\frac{1}{2}} \leq C \left( \mathbb{E} \left| \int_s^t \gamma_r d\overline{B}_r \right|^2 + \mathbb{E} \left| \int_s^t \beta_r dW_r \right|^2 + |t - s|^{2r} \right)^{\frac{1}{2}} \leq C \left( \mathbb{E} \left| \gamma_r dr \right|^2 + \mathbb{E} \left| \beta_r^2 dr \right|^r + |t - s|^{2r} \right)^{\frac{1}{2}} \leq C |t - s|^\frac{r}{2},$$

where $C$ is a generic constant independent of $s$ and $t$. \hfill \Box

Part (b) of Lemma 5.1 implies that for any $0 < \varepsilon < \frac{1}{2}$ there exists a random variable $G_\varepsilon$ which has moments of all orders, such that, for any $s, t \in [0, T],$

$$|\rho_t - \rho_s| \leq G_\varepsilon |t - s|^\varepsilon. \quad (5.3)$$

**Proof of Theorem 3.3.** The existence and uniqueness of a solution $(Y, Z)$ to equation (3.3) follows from Theorem 3.1. Moreover, the solution pair $(Y, Z)$ satisfies the estimate (3.1).

Now we are going to show formula (3.5). Since the process $\rho$ does not satisfy a stochastic differential equation, this formula cannot be deduced from a version of Itô’s formula for forward and backward stochastic integrals and we need to show this formula by a suitable approximation argument. For any $t < T$, we introduce a sequence of partitions $\pi^n = \{t = t_0^n < t_1^n < \cdots < t_n^n = T\}$ such that

$$\lim_{n \to \infty} |\pi^n| = 0,$$

and there exists a constant $K$ such that for all $n$,

$$n|\pi^n| \leq K. \quad (5.4)$$

where $|\pi^n| = \max_{0 \leq i \leq n - 1} \{t_{i+1}^n - t_i^n\}$. To simplify the notation we omit the superindex in the partition points and we simply write $t_i^n = t_i$. 


Consider the decomposition
\[ Y_T \rho_T - Y_t \rho_t = \sum_{i=0}^{n-1} (Y_{t+i} \rho_{t+i} - Y_t \rho_t) \]
\[ = \sum_{i=0}^{n-1} ((Y_{t+i} - Y_t) \rho_t + Y_{t+i} (\rho_{t+i} - \rho_t)) = A_n + B_n. \] (5.5)

For the term \( A_n \) we can write
\[ A_n = \sum_{i=0}^{n-1} \rho_t \left( \int_{t_i}^{t_{i+1}} (-f_r - \alpha_Y Y_r - \beta_r Z_r) dr - \int_{t_i}^{t_{i+1}} \gamma_r Y_r d\tilde{B}_r + \int_{t_i}^{t_{i+1}} Z_r dW_r \right) \]
\[ = A_{n,0} + A_{n,1}, \] (5.6)

where
\[ A_{n,0} = -\sum_{i=0}^{n-1} \rho_t \int_{t_i}^{t_{i+1}} \gamma_r Y_r d\tilde{B}_r, \] (5.7)

and
\[ A_{n,1} = \sum_{i=0}^{n-1} \rho_t \left( \int_{t_i}^{t_{i+1}} (-f_r - \alpha_Y Y_r - \beta_r Z_r) dr + \int_{t_i}^{t_{i+1}} Z_r dW_r \right). \] (5.8)

As we shall see that \( A_{n,0} \) will be canceled by a term to be introduced later. As for \( A_{n,1} \) it holds that
\[ \lim_{n \to \infty} A_{n,1} = \int_t^T (-f_r - \alpha_Y Y_r - \beta_r Z_r) \rho_r dr + \int_t^T Z_r \rho_r dW_r, \text{ in } L^1(\Omega). \] (5.9)

Indeed, the convergence of the Lebesgue integral follows from the continuity of \( \rho \) (see (5.3)) and the integrability properties of the processes \( f, \alpha Y \) and \( \beta Z \). For the stochastic integral, taking into account that \( \rho \) is \( \mathcal{G} \)-adapted, the process \( Z \) is in \( H^2_{\mathcal{G}}([0,T]) \) due to (3.1). Using the Burkholder-Davis-Gundy inequality and inequality (5.3), we obtain
\[ \mathbb{E} \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} Z_r (\rho_{t_i} - \rho_r) dW_r \right| \leq \mathbb{E} \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} Z_r^2 (\rho_{t_i} - \rho_r)^2 dr \right|^{\frac{1}{2}} \]
\[ \leq \left( \mathbb{E} \int_0^T Z_r^2 dr \right)^{\frac{1}{2}} (\mathbb{E} \delta_{\varepsilon})^{\frac{1}{2}} |n|^{\varepsilon}, \]

for any \( \varepsilon \in (0, \frac{1}{2}) \). This proves (5.9).

Next, let us consider the term \( B_n \). Using Taylor’s expansion up to the second order, \( \rho_{t_{i+1}} - \rho_t \) can be decomposed as follows
\[ \rho_{t_{i+1}} - \rho_t = \rho_t \left( \Psi_i + \frac{1}{2} \Psi_i^2 + R_i \right), \]
where
\[ \Psi_i = \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r + \int_{t_i}^{t_{i+1}} \beta_r dW_r + \int_{t_i}^{t_{i+1}} \left( \alpha_r - \frac{1}{2} \beta_r^2 - \frac{1}{2} \gamma_r^2 \right) dr \]
and the residual term $R_i$ has the form $R_i = \frac{1}{\eta} \Psi_i^2 e^\eta$, with $\eta$ between 0 and $\Psi_i$. It is easy to show that
\[
\sum_{i=0}^{n-1} Y_{t_{i+1}} \rho_t \left( \frac{1}{2} \left( \int_{t_i}^{t_{i+1}} \left( \alpha_r - \frac{1}{2} \beta_r^2 - \frac{1}{2} \gamma_r^2 \right) dr \right)^2 + \left( \int_{t_i}^{t_{i+1}} \left( \alpha_r - \frac{1}{2} \beta_r^2 - \frac{1}{2} \gamma_r^2 \right) dr \right) \left( \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r + \int_{t_i}^{t_{i+1}} \beta_r dW_r \right) + R_i \right)
\]
converges in probability to zero as $n$ tends to infinity. Therefore,
\[
\lim_{n \to \infty} B_n = \lim_{n \to \infty} B'_n
\]
in probability, where
\[
B'_n = \sum_{i=0}^{n-1} Y_{t_{i+1}} \rho_t \Psi_t + \frac{1}{2} \sum_{i=0}^{n-1} Y_{t_{i+1}} \rho_t \left( \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r + \int_{t_i}^{t_{i+1}} \beta_r dW_r \right)^2
\]
\[
= \sum_{i=0}^{n-1} Y_{t_{i+1}} \rho_t \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r + \sum_{i=0}^{n-1} Y_{t_{i+1}} \rho_t \int_{t_i}^{t_{i+1}} \beta_r dW_r + \sum_{i=0}^{n-1} (Y_{t_{i+1}} - Y_{t_i}) \rho_t \int_{t_i}^{t_{i+1}} \beta_r dW_r
\]
\[
+ \sum_{i=0}^{n-1} Y_{t_{i+1}} \rho_t \int_{t_i}^{t_{i+1}} \left( \alpha_r - \frac{1}{2} \beta_r^2 \right) dr + \frac{1}{2} \sum_{i=0}^{n-1} Y_{t_{i+1}} \rho_t \left[ \left( \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r \right)^2 - \int_{t_i}^{t_{i+1}} \gamma_r^2 dr \right]
\]
\[
+ \frac{1}{2} \sum_{i=0}^{n-1} Y_{t_{i+1}} \rho_t \left( \int_{t_i}^{t_{i+1}} \beta_r dW_r \right)^2 + \frac{1}{2} \sum_{i=0}^{n-1} (Y_{t_{i+1}} - Y_{t_i}) \rho_t \left( \int_{t_i}^{t_{i+1}} \beta_r dW_r \right)^2
\]
\[
= B_{n,1} + B_{n,2} + B_{n,3} + B_{n,4} + B_{n,5} + B_{n,6} + B_{n,7} + B_{n,8}.
\]
We are going to analyze the asymptotic behavior of each term in the above decomposition. First, one can easily show that the following limits hold true:
\[
\lim_{n \to \infty} B_{n,2} = \int_t^T \beta_r Y_r \rho_r dW_r, \quad \text{in } L^1(\Omega),
\]
\[
\lim_{n \to \infty} B_{n,4} = \int_t^T \left( \alpha_r Y_r \rho_r - \frac{1}{2} \beta_r^2 Y_r \rho_r \right) dr, \quad \text{a.s.,}
\]
\[
\lim_{n \to \infty} B_{n,6} = \frac{1}{2} \int_t^T \beta_r^2 Y_r \rho_r dr, \quad \text{in } L^1(\Omega).
\]
Using the fact that
\[
\mathbb{E} \left( \max_{0 \leq i \leq n-1} |Y_{t_{i+1}} - Y_{t_i}|^2 \right) \to 0, \quad \text{as } n \to \infty,
\]
which can be proved by the estimate \((3.1)\) and the dominated convergence theorem, and also using part (a) in Lemma 5.1, we can show that

\[
\mathbb{E}|B_{n,7}| \leq \frac{1}{2} \mathbb{E} \left( \max_{0 \leq i \leq n-1} |Y_{t_{i+1}} - Y_{t_i}| \sup_{0 \leq t \leq T} \rho_t \int_{t_i}^{t_{i+1}} \beta_r dW_r \right)^2 \leq \frac{1}{2} \left( \mathbb{E} \left( \max_{0 \leq i \leq n-1} |Y_{t_{i+1}} - Y_{t_i}|^2 \right) \right)^{\frac{1}{2}} \left( \mathbb{E} \sup_{0 \leq t \leq T} \rho_t^4 \right)^{\frac{1}{2}} \times \left( \mathbb{E} \left[ \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} \beta_r dW_r \right)^2 \right] \right)^{\frac{1}{2}} \to 0,
\]

(5.14)

as \(n \to \infty\), since \(\lim_{n \to \infty} \mathbb{E} \left[ \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} \beta_r dW_r \right)^2 \right]^4 = \mathbb{E} \left( \int_0^T \beta_r^2 dr \right)^4\).

The term \(B_{n,1}\) can be further decomposed as follows

\[
B_{n,1} = \sum_{i=0}^{n-1} \rho_i \int_{t_i}^{t_{i+1}} \gamma_r Y_r \tilde{d}B_r + \sum_{i=0}^{n-1} \rho_i \left[ Y_{t_{i+1}} \int_{t_i}^{t_{i+1}} \gamma_r \tilde{d}B_r - Y_{t_i} \int_{t_i}^{t_{i+1}} \gamma_r \tilde{d}B_r \right]
= -A_{n,0} + B_{n,1,1},
\]

(5.15)

where \(A_{n,0}\) is defined by \((5.7)\), which is canceled with the term in \((5.6)\). Next, we will show the following two limits hold in \(L^1(\Omega)\):

\[
\lim_{n \to \infty} B_{n,1,1} = \lim_{n \to \infty} \sum_{i=0}^{n-1} \rho_i \left[ Y_{t_{i+1}} \int_{t_i}^{t_{i+1}} \gamma_r \tilde{d}B_r - Y_{t_i} \int_{t_i}^{t_{i+1}} \gamma_r \tilde{d}B_r \right] = 0,
\]

(5.16)

and

\[
\lim_{n \to \infty} B_{n,5} = \lim_{n \to \infty} \frac{1}{2} \sum_{i=0}^{n-1} Y_{t_{i+1}} \rho_i \left[ \left( \int_{t_i}^{t_{i+1}} \gamma_r \tilde{d}B_r \right)^2 - \int_{t_i}^{t_{i+1}} \gamma_r^2 dr \right] = 0.
\]

(5.17)

Proof of the convergence \((5.16)\): For any positive integer \(m < n\), consider the partition \(\pi^m = \{t = t_0^m < t_1^m < \cdots < t_n^m = T\}\). Define for each \(i = 0, 1, \ldots, n\),

\[
\tau_i = \max\{t_j^m : t_j^m \leq t_i\} \text{ and } \sigma_i = \min\{t_j^m : t_j^m > t_i\}.
\]
Then, we can rewrite $B_{n,1,1}$ as follows

$$B_{n,1,1} = \sum_{i=0}^{n-1} \rho_i \left[ Y_{t_{i+1}} \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r - \int_{t_i}^{t_{i+1}} \gamma_r Y_r d\tilde{B}_r \right]$$

$$+ \sum_{i=0}^{n-1} (\rho_i - \rho_{i+1}) \left[ Y_{t_{i+1}} \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r - \int_{t_i}^{t_{i+1}} \gamma_r Y_r d\tilde{B}_r \right]$$

$$= \sum_{j=0}^{m-1} \sum_{i,t_j^m \leq t_i < t_j^m+1} \rho_{j} \left[ Y_{t_{i+1}} \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r - \int_{t_i}^{t_{i+1}} \gamma_r Y_r d\tilde{B}_r \right]$$

$$+ \sum_{i=0}^{n-1} (\rho_i - \rho_{i+1}) \left[ Y_{t_{i+1}} \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r - \int_{t_i}^{t_{i+1}} \gamma_r Y_r d\tilde{B}_r \right]$$

$$=: B_{n,m,1} + B_{n,m,2}. \quad (5.18)$$

If $m$ is fixed, then for each $j = 0, 1, \ldots, m - 1$ we have

$$\lim_{n \to \infty} \sum_{i,t_j^m \leq t_i < t_j^m+1} \left[ Y_{t_{i+1}} \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r - \int_{t_i}^{t_{i+1}} \gamma_r Y_r d\tilde{B}_r \right] = 0,$$

in $L^2(\Omega)$, which implies

$$\lim_{n \to \infty} B_{n,m,1} = 0, \quad (5.19)$$

in $L^2(\Omega)$, for each fixed $m$.

For the term $B_{n,m,2}$, using Hölder’s inequality, the estimate (5.3) and the isometry of backward Itô stochastic integrals, we can write

$$\mathbb{E}|B_{n,m,2}| \leq \mathbb{E} \left( \max_{0 \leq i \leq n-1} \left| \rho_i - \rho_{i+1} \right| \sum_{i=0}^{n-1} \left| Y_{t_{i+1}} \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r - \int_{t_i}^{t_{i+1}} \gamma_r Y_r d\tilde{B}_r \right| \right)$$

$$\leq \left( \mathbb{E} \max_{0 \leq i \leq n-1} \left| \rho_i - \rho_{i+1} \right|^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \left( \sum_{i=0}^{n-1} \left| Y_{t_{i+1}} \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r - \int_{t_i}^{t_{i+1}} \gamma_r Y_r d\tilde{B}_r \right|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$\leq |\pi|^\frac{\varepsilon}{2} \left( \mathbb{E} \left( \sum_{i=0}^{n-1} \left| Y_{t_{i+1}} \int_{t_i}^{t_{i+1}} \gamma_r d\tilde{B}_r - \int_{t_i}^{t_{i+1}} \gamma_r Y_r d\tilde{B}_r \right|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$= |\pi|^\frac{\varepsilon}{2} \left( \mathbb{E} \left( \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left| \gamma_r (Y_{t_{i+1}} - Y_r) \right|^2 dr \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$\leq L |\pi|^\frac{\varepsilon}{2} \left( \mathbb{E} \left( \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Y_{t_{i+1}} - Y_r|^2 dr \right)^{\frac{1}{2}} \right). \quad (5.20)$$
We are going to make use of the following estimate for the linear equation (3.3) over the interval \([s, t] \subseteq [0, T]\)

\[
\mathbb{E}|Y_t - Y_s|^2 \leq C \left( |t - s| + \int_s^t |Z_r|^2 dr \right), \quad 0 \leq s \leq t \leq T, \quad (5.21)
\]

which can be easily proved by the boundedness of the coefficients \(\alpha, \beta\) and \(\gamma\), and the estimate (3.1) for the linear equation (3.3). From (5.20), (5.21) and (5.4), we obtain

\[
\mathbb{E}|B_{n,m,2}| \leq C|\pi|^\varepsilon (\mathbb{E}G_\varepsilon^2)^{\frac{1}{2}} \left( n \left( |\pi^n| + \mathbb{E} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{r}^{t_{i+1}} |Z_u|^2 dudr \right) \right) \frac{1}{2}
\]

\[
\leq C|\pi|^\varepsilon (\mathbb{E}G_\varepsilon^2)^{\frac{1}{2}} \left( n |\pi^n| \left( 1 + \mathbb{E} \int_T^t |Z_u|^2 du \right) \right) \frac{1}{2}
\]

\[
\leq C|\pi|^\varepsilon (\mathbb{E}G_\varepsilon^2)^{\frac{1}{2}} \left( 1 + \mathbb{E} \int_T^t |Z_u|^2 du \right)^{\frac{1}{2}}, \quad (5.22)
\]

which converges to 0 as \(m \to \infty\), uniformly in \(n\). Thus, the limit (5.16) follows from (5.18), (5.19) and (5.22).

**Proof of the convergence (5.17):** Similarly, we write \(B_{n,5}\) as

\[
B_{n,5} = \frac{1}{2} \sum_{i=0}^{n-1} Y_{\pi_{i+1}} \rho_{\pi_i} \left[ \left( \int_{t_i}^{t_{i+1}} \gamma_r d\widehat{B}_r \right)^2 - \int_{t_i}^{t_{i+1}} \gamma_r^2 dr \right] + \frac{1}{2} \sum_{i=0}^{n-1} \left( Y_{\pi_{i+1}} \rho_{\pi_i} - Y_{\pi_{i+1}} \rho_{\pi_i} \right) \left[ \left( \int_{t_i}^{t_{i+1}} \gamma_r d\widehat{B}_r \right)^2 - \int_{t_i}^{t_{i+1}} \gamma_r^2 dr \right]
\]

\[
= \frac{1}{2} \sum_{j=0}^{m-1} \rho_j Y_{\pi_{i+1}} \sum_{t_i \leq t_j < t_{i+1}} \left[ \left( \int_{t_i}^{t_{i+1}} \gamma_r d\widehat{B}_r \right)^2 - \int_{t_i}^{t_{i+1}} \gamma_r^2 dr \right] + \frac{1}{2} \sum_{i=0}^{n-1} \left( Y_{\pi_{i+1}} \rho_{\pi_i} - Y_{\pi_{i+1}} \rho_{\pi_i} \right) \left[ \left( \int_{t_i}^{t_{i+1}} \gamma_r d\widehat{B}_r \right)^2 - \int_{t_i}^{t_{i+1}} \gamma_r^2 dr \right]
\]

\[
= B_{n,m,5,1} + B_{n,m,5,2}. \quad (5.23)
\]

For any fixed \(m\), we have, for each \(j = 0, 1, \ldots, m - 1\),

\[
\lim_{n \to \infty} \sum_{i, t_i \leq t_j < t_{i+1}} \left[ \left( \int_{t_i}^{t_{i+1}} \gamma_r d\widehat{B}_r \right)^2 - \int_{t_i}^{t_{i+1}} \gamma_r^2 dr \right] = 0,
\]

in \(L^2(\Omega)\), which implies

\[
\lim_{n \to \infty} B_{n,m,5,1} = 0 \quad (5.24)
\]

in \(L^1(\Omega)\), for each fixed \(m\).
For the term $B_{n,m,5,2}$, applying Cauchy-Schwarz inequality, we can write

$$
\mathbb{E}[B_{n,m,5,2}] \leq \frac{1}{2} \mathbb{E} \left( \max_{0 \leq i \leq n-1} \left\{ |Y_{t_{i+1}}| |\rho_{t_{i}} - \rho_{t_{i+1}}| + |Y_{t_{i+1}} - Y_{\sigma_{t_{i+1}}}||\rho_{t_{i}}| \right\} \right)
\times \sum_{i=0}^{n-1} \left| \left( \int_{t_i}^{t_{i+1}} \gamma_{t} d\hat{B}_{t} \right)^2 - \int_{t_i}^{t_{i+1}} \gamma_{t}^2 dt \right|
\leq \frac{1}{2} \left( \mathbb{E} \left( \max_{0 \leq i \leq n-1} \left\{ |Y_{t_{i+1}}| |\rho_{t_{i}} - \rho_{t_{i+1}}| + |Y_{t_{i+1}} - Y_{\sigma_{t_{i+1}}}||\rho_{t_{i}}| \right\} \right)^2 \right)^{\frac{1}{2}}
\times \left( \mathbb{E} \sum_{i=0}^{n-1} \left[ \left( \int_{t_i}^{t_{i+1}} \gamma_{t} d\hat{B}_{t} \right)^2 - \int_{t_i}^{t_{i+1}} \gamma_{t}^2 dt \right]^2 \right)^{\frac{1}{2}}
=: \frac{1}{2} (B_{n,m,5,3} \cdot B_{n,m,5,4})
$$

Clearly, the term $B_{n,m,5,4}$ is uniformly bounded by a constant depending only on the constant in Burkholder’s inequality and the bound of the process $\gamma$. On the other hand, $B_{n,m,5,3}$ converges to zero as $m$ tends to infinity, uniformly in $n$. Hence, we proved that

$$
\lim_{m \to \infty} \mathbb{E}[B_{n,m,5,2}] = 0,
$$

(5.25)

uniformly in $n$. Therefore, the limit (5.17) can be proved by (5.23), (5.24) and (5.25).

Finally, we consider the remaining terms $B_{n,3}$ and $B_{n,8}$ together. Using the equation satisfied by $Y$ we can write

$$
B_{n,3} + B_{n,8} = \sum_{i=0}^{n-1} \rho_{t_{i}} \left( \int_{t_{i}}^{t_{i+1}} (-f_{r} - \alpha_{r} Y_{r} - \beta_{r} Z_{r}) dr + \int_{t_{i}}^{t_{i+1}} Z_{r} dW_{r} \right) \int_{t_{i}}^{t_{i+1}} \beta_{r} dW_{r}
+ \sum_{i=0}^{n-1} \rho_{t_{i}} \left[ Y_{t_{i+1}} \int_{t_{i}}^{t_{i+1}} \gamma_{r} d\hat{B}_{r} - \int_{t_{i}}^{t_{i+1}} \gamma_{r} Y_{r} d\hat{B}_{r} \right] \int_{t_{i}}^{t_{i+1}} \beta_{r} dW_{r}
=: B_{n,3,1} + B_{n,3,2}.
$$

(5.26)

From the adaptedness of the process $\rho$ to the filtration $\mathcal{G}$ and the classical Itô calculus, together with the estimates proved in Lemma 5.1, we can show that

$$
\lim_{n \to \infty} B_{n,3,1} = \int_{t}^{T} \beta_{r} \rho_{r} Z_{r} dr, \text{ in } L^{1}(\Omega).
$$

(5.27)

For the term $B_{n,3,2}$, we have the estimate

$$
|B_{n,3,2}| \leq \sup_{0 \leq t \leq T} \rho_{t} \left( \sum_{i=0}^{n-1} \left( \int_{t_{i}}^{t_{i+1}} \gamma_{r} (Y_{t_{i+1}} - Y_{r}) d\hat{B}_{r} \right)^2 \right)^{\frac{1}{2}} \left( \sum_{i=0}^{n-1} \left( \int_{t_{i}}^{t_{i+1}} \beta_{r} dW_{r} \right)^2 \right)^{\frac{1}{2}}.
$$
The factors $\sup_{0 \leq t \leq T} \rho_t$ and $\left( \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} \beta_s dW_s \right)^2 \right)^{\frac{1}{2}}$ have moments of all orders uniformly bounded in $n$. Therefore, in order to show that

$$\lim_{n \to \infty} E|B_{n,3,2}| = 0, \quad (5.28)$$

it suffices to prove that

$$\lim_{n \to \infty} E \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} \gamma_r(Y_{t_{i+1}} - Y_r) d\widehat{B}_r \right)^2 = 0,$$

which follows from the following estimate

$$\mathbb{E} \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} \gamma_r(Y_{t_{i+1}} - Y_r) d\widehat{B}_r \right)^2 = \mathbb{E} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \gamma_r^2(Y_{t_{i+1}} - Y_r)^2 dr \leq L^2 T \mathbb{E} \left( \sup_{|r-s| \leq \pi n} |Y_s - Y_r|^2 \right).$$

From (5.26), (5.27) and (5.28), we deduce that

$$B_{n,3} + B_{n,8} \to \int_t^T \beta_r \rho_r Z_r dr, \text{ in } L^1(\Omega), \text{ as } n \to \infty. \quad (5.29)$$

Therefore, from (5.5)-(5.17) and (5.29), we can prove (3.5), and hence, (3.6). \hfill □

**Proof of Theorem 3.4.** The existence and uniqueness of a solution has been shown in Theorem 3.1, where the estimate (3.1) is also proved. On the other hand, we have proved the following explicit representation for the process $Y$ (see (3.6))

$$Y_t = \rho_t^{-1} \mathbb{E} \left( \xi \rho_T + \int_t^T \rho_s f_s ds \, \big| \mathcal{G}_t \right) = \mathbb{E} \left( \xi_{t,t} + \int_t^T \rho_{t,r} f_r dr \, \big| \mathcal{G}_t \right),$$

where $\rho_{t,r} = \rho_t^{-1} \rho_r$ for any $0 \leq t \leq r \leq T$. For any $t \in [0, T]$, denote

$$\delta_t = \rho_t^{-1} = \exp \left\{ - \int_0^t \beta_s dW_s - \int_0^t \gamma_s d\widehat{B}_s - \int_0^t \left( \alpha_s - \frac{1}{2} \beta_s^2 - \frac{1}{2} \gamma_s^2 \right) ds \right\}.$$

Then, $\rho_{t,r} = \delta_t \rho_r$ with $0 \leq t \leq r \leq T$.

For any $0 \leq s \leq t \leq T$ and any positive number $r \geq 1$, applying Lemma 5.1 to the process $\{\delta_t\}_{0 \leq t \leq T}$, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \delta_t^r \leq C,$$

and

$$\mathbb{E} |\delta_t - \delta_s|^r \leq C(t - s)^{\frac{r}{2}},$$

where $C$ is a positive constant depending only on $L$, $T$ and $r$.

Once we obtain the representation (3.6) and the above two estimates, the proof of the estimate (3.7) will be analogous to the proof of a similar estimate in Theorem 2.3 of [9]. \hfill □
6. Proof of Theorem 3.6

Proof of Theorem 3.6. Part (a): The existence and uniqueness of the solution \((Y, Z) \in S^2_T([0,T]) \times H^2_T([0,T])\) are obtained in Theorem 3.1. We first show that \(Y, Z \in \mathbb{L}^{1,2}_a\) following a recursive argument similar to that used in the proof of Proposition 5.3 in [11].

Let \(Y^0_t = 0\) and \(Z^0_t = 0\) for all \(t \in [0, T]\). It is obvious that \((Y^0, Z^0)\) is in \(\mathbb{L}^{1,2}_a\). We define a sequence of \(\{(Y^n, Z^n)\}_{n=0}^{\infty}\) as follows:

\[
Y^{n+1}_t = \xi + \int_t^T f(r, Y^n_r, Z^n_r)dr + \int_t^T g(Y^n_r)d\hat{B}_r - \int_t^T Z^{n+1}_r dW_r, \quad 0 \leq t \leq T.
\]

In the proof of Theorem 1.1 in [16], the convergence of the sequence \(\{(Y^n, Z^n)\}_{n=0}^{\infty}\) to \((Y, Z)\) in the norm of \(S^2_T([0,T]) \times H^2_T([0,T])\) is established.

If \((Y^n, Z^n)\) is in \(\mathbb{L}^{1,2}_a\), then \(\xi + \int_t^T f(r, Y^n_r, Z^n_r)dr + \int_t^T g(Y^n_r)d\hat{B}_r\) is in \(\mathbb{D}^{1,2}\), and hence

\[
Y^{n+1}_t = \mathbb{E}\left( \xi + \int_t^T f(r, Y^n_r, Z^n_r)dr + \int_t^T g(Y^n_r)d\hat{B}_r \bigg| \mathcal{G}_t \right)
\]

is in \(\mathbb{D}^{1,2}\), for all \(t \in [0, T]\). Note that \(D_\theta Y^{n+1}_t = 0\) if \(0 \leq t < \theta \leq T\). Since

\[
\xi + \int_t^T f(r, Y^n_r, Z^n_r)dr + \int_t^T g(Y^n_r)d\hat{B}_r - Y^{n+1}_t = \int_t^T Z^{n+1}_r dW_r,
\]

it follows from Lemma 5.1 in [11] that \(Z^{n+1}\) is in \(\mathbb{L}^{1,2}_a\), and furthermore, \(D_\theta Z^{n+1}_t = 0\) if \(0 \leq t < \theta \leq T\).

Then, the Malliavin derivatives of \(Y^{n+1}\) and \(Z^{n+1}\) satisfy the following equation

\[
D_\theta Y^{n+1}_t = D_\theta \xi + \int_t^T \left[ \partial_y f(r, Y^n_r, Z^n_r)D_\theta Y^n_r + \partial_z f(r, Y^n_r, Z^n_r)D_\theta Z^n_r + D_\theta f(r, Y^n_r, Z^n_r) \right]dr
\]

\[
+ \int_t^T g'(Y^n_r)D_\theta Y^n_r d\hat{B}_r - \int_t^T D_\theta Z^{n+1}_r dW_r, \quad 0 \leq \theta \leq t \leq T;
\]

\[
D_\theta Y^{n+1}_t = 0, \quad D_\theta Z^{n+1}_t = 0, \quad 0 \leq t < \theta \leq T.
\]

From the convergence of the sequence \(\{(Y^n, Z^n)\}_{n=0}^{\infty}\) to \((Y, Z)\) and the Assumption (A), by using techniques similar to those in the proof of Proposition 5.3 in [11], we can prove that the sequence \(\{(D_\theta Y^n, D_\theta Z^n)\}_{n=0}^{\infty}\) converges to \((D_\theta Y, D_\theta Z)\), where \((D_\theta Y, D_\theta Z)\) is the unique solution to (3.18) and (3.19). Note that

\[
\int_0^T (\|D_\theta Y\|^2_{S^2} + \|D_\theta Z\|^2_{H^2})d\theta < \infty
\]

follows from the estimate in Theorem 3.6 with \(q = 2\). Hence, \((Y, Z)\) is in \(\mathbb{L}^{1,2}_a\).

Furthermore, from conditions (3.9) and (3.11) and the estimate in Theorem 3.1, we obtain

\[
\sup_{0 \leq \theta \leq T} \left\{ \mathbb{E} \sup_{0 \leq t \leq T} |D_\theta Y_t|^q + \mathbb{E} \left( \int_0^T |D_\theta Z_t|^2 dt \right)^{\frac{q}{2}} \right\} < \infty. \tag{6.1}
\]

Hence, by Proposition 1.5.5 in [14], \(Y\) and \(Z\) belong to \(\mathbb{L}^{1,q}_a\). The representation (3.20) can be proved by using the same technique in the proof of Proposition 5.3 in [11].
Part (b): Let $0 \leq s \leq t \leq T$. In this proof, $C > 0$ will be a constant independent of $s$ and $t$, and may vary from line to line.

By the representation (3.20), we have

$$ Z_t - Z_s = D_t Y_t - D_s Y_s = (D_t Y_t - D_s Y_t) + (D_s Y_t - D_s Y_s). \quad (6.2) $$

From Theorem 3.1 and Equation (3.18) for $\theta = s$ and $\theta' = t$ respectively, we obtain, using conditions (3.8) and (3.14),

$$ \mathbb{E}|D_t Y_t - D_s Y_t|^p + \mathbb{E} \left( \int_t^T |D_r Z_r - D_s Z_r|^2 dr \right)^{\frac{p}{2}} \leq C \left[ \mathbb{E}|D_t \xi - D_s \xi|^p + \mathbb{E} \left( \int_t^T |D_r f(r, Y_r, Z_r) - D_s f(r, Y_r, Z_r)|^2 dr \right)^{\frac{p}{2}} \right] \leq C|t - s|^{\frac{p}{2}}. \quad (6.3) $$

Denote $\alpha_u = \partial_u f(u, Y_u, Z_u)$, $\beta_u = \partial_z f(u, Y_u, Z_u)$ and $\gamma_u = g'(Y_u)$ for all $u \in [0, T]$. Then, by Assumption (A) (ii) and (iii), the processes $\alpha$, $\beta$ and $\gamma$ satisfy condition (H1), and from (3.18) we have for $r \in [s, T]$

$$ D_s Y_r = D_s \xi + \int_s^T [\alpha_u D_u Y_u + \beta_u D_u Z_u + D_s f(u, Y_u, Z_u)] du + \int_s^T \gamma_u D_s Y_u d\overline{B}_r - \int_s^T D_s Z_u dW_u. $$

Next, we are going to apply Theorem 3.4 to the above linear BDSDE to estimate $\mathbb{E}|D_t Y_t - D_s Y_t|^p$. Fix $p'$ with $p < p < \frac{p}{2}$ (notice that $p' < \frac{p}{2}$ is equivalent to $\frac{p}{q-p'} < 1$). From conditions (3.9) and (3.11), it is obvious that $D_s \xi \in L^q(\Omega) \subset L^{p'}(\Omega)$ and $D_s f(\cdot, Y, Z) \in H^{q'}_T([0, T]) \subset H^p_T([0, T])$ for any $s \in [0, T]$.

Recall that the random variable $\rho$ defined in (3.4):

$$ \rho_r = \exp \left\{ \int_0^r \beta_u dW_u + \int_0^r \gamma_u d\overline{B}_r + \int_0^r \left( \alpha_u - \frac{1}{2} \beta^2_u - \frac{1}{2} \gamma^2_u \right) du \right\}, $$

is $\mathcal{G}_r$-measurable.

For any $0 \leq \theta \leq r \leq T$, let us compute

$$ D_\theta \rho_r = \rho_r \left\{ \int_\theta^r [\partial_{yz} f(u, Y_u, Z_u) D_\theta Y_u + \partial_{zz} f(u, Y_u, Z_u) D_\theta Z_u + D_\theta \partial_z f(u, Y_u, Z_u)] dW_u \right. $$

$$ + \partial_z f(\theta, Y_\theta, Z_\theta) \int_\theta^r g''(Y_u) D_\theta Y_u d\overline{B}_u $$

$$ + \int_\theta^r (\partial_{yy} f(u, Y_u, Z_u) - \partial_{yz} f(u, Y_u, Z_u) \beta_u + g'(Y_u) g''(Y_u)) D_\theta Y_u du $$

$$ + \int_\theta^r (\partial_{yz} f(u, Y_u, Z_u) - \partial_{zz} f(u, Y_u, Z_u) \beta_u) D_\theta Z_u du $$

$$ + \int_\theta^r (D_\theta \partial_y f(u, Y_u, Z_u) - \beta_u D_\theta \partial_z f(u, Y_u, Z_u)) du \right\}. $$

By the boundedness of the first and second order partial derivatives of $f$ with respect to $y$ and $z$, the boundedness of $g'$ and $g''$, (3.12), (3.13), (6.1), Lemma 5.1, the Hölder inequality
and the Burkholder-Davis-Gundy inequality, it is easy to show that for any $p'' < q$,
\[
\sup_{0 \leq \theta \leq T} \sup_{\theta \leq r \leq T} |D_{\theta} \rho_r|^p'' < \infty. \tag{6.4}
\]
Then, the method to show $\rho_T D_s \xi \in M^{p'}$ and $\int_s^T \rho_u D_s f(u, Y_u, Z_u) du \in M^{p'}$ is exactly the same as that in the proof of Theorem 2.6 in [9]. Together with Theorem 3.4, we are able to show
\[
E|D_s Y_t - D_s Y_s|^p \leq C|t - s|^\frac{p}{2}, \tag{6.5}
\]
for all $s, t \in [0, T]$. Combining (6.5) with (6.2) and (6.3), we obtain that there is a constant $K > 0$ independent of $s$ and $t$, such that,
\[
E|Z_t - Z_s|^p \leq K|t - s|^\frac{p}{2},
\]
for all $s, t \in [0, T]$. This completes the proof of Theorem 3.6. \qed

7. Proof of Theorem 3.9

We start with a comparison between the values of the process $Y$ and the approximation $Y^\pi$ at the partition points. It follows from (3.23) and (3.31) that, for $i = n - 1, n - 2, \ldots, 1, 0$,
\[
Y_{t_i} - Y^\pi_{t_i} = Y_{t_{i+1}} - Y^\pi_{t_{i+1}} + \int_{t_i}^{t_{i+1}} [f(r, Y_r, Z_r) - f(t_i, Y^\pi_{t_i}, Z^\pi_{t_i})] dr
\]
\[
+ \int_{t_i}^{t_{i+1}} [g(Y_r) - g(Y^\pi_{t_i})] d\hat{B}_r - \int_{t_i}^{t_{i+1}} [Z_r - Z^1_{r, \pi}] dW_r
\]
\[
= Y_{t_{i+1}} - Y^\pi_{t_{i+1}} + [f(t_i, Y_{t_i}, Z_{t_i}) - f(t_i, Y^\pi_{t_i}, Z^\pi_{t_i})] \Delta_i + [g(Y_{t_{i+1}}) - g(Y^\pi_{t_{i+1}})] \Delta B_i
\]
\[
+ \int_{t_i}^{t_{i+1}} [f(r, Y_r, Z_r) - f(t_i, Y_{t_i}, Z_{t_i})] dr + \int_{t_i}^{t_{i+1}} [g(Y_r) - g(Y^\pi_{t_i})] d\hat{B}_r
\]
\[
- \int_{t_i}^{t_{i+1}} [Z_r - Z^1_{r, \pi}] dW_r.
\]
This can be written as
\[
Y_{t_i} - Y^\pi_{t_i} = \xi - \xi^\pi + \sum_{j=1}^{n-1} [f(t_j, Y_{t_j}, Z_{t_j}) - f(t_j, Y^\pi_{t_j}, Z^\pi_{t_j})] \Delta_j + \sum_{j=1}^{n-1} [g(Y_{t_{j+1}}) - g(Y^\pi_{t_{j+1}})] \Delta B_j
\]
\[
+ \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} [f(r, Y_r, Z_r) - f(t_j, Y_{t_j}, Z_{t_j})] dr
\]
\[
+ \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} [g(Y_r) - g(Y^\pi_{t_j})] d\hat{B}_r - \int_{t_i}^{T} [Z_r - Z^1_{r, \pi}] dW_r
\]
\[
= \xi - \xi^\pi + \sum_{j=1}^{n-1} [f(t_j, Y_{t_j}, Z_{t_j}) - f(t_j, Y^\pi_{t_j}, Z^\pi_{t_j})] \Delta_j + \sum_{j=1}^{n-1} [g(Y_{t_{j+1}}) - g(Y^\pi_{t_{j+1}})] \Delta B_j
\]
\[
- \int_{t_i}^{T} [Z_r - Z^1_{r, \pi}] dW_r + R^\pi_{t_i} + G^\pi_{t_i}, \tag{7.1}
\]
where
\[ R_{t_i}^\pi = \sum_{j=i}^{n-1} \int_{t_j}^{t_{j+1}} [f(r, Y_r, Z_r) - f(t_j, Y_{t_j}, Z_{t_j})] dr, \quad (7.2) \]
and
\[ G_{t_i}^\pi = \sum_{j=i}^{n-1} \int_{t_j}^{t_{j+1}} [g(Y_r) - g(Y_{t_{j+1}})] d\widetilde{B}_r, \quad (7.3) \]

The following two lemmas will be needed in the proof of Theorem 3.9.

**Lemma 7.1.** Let all the conditions in Theorem 3.9 be satisfied, and let \( R^\pi \) and \( G^\pi \) be defined in (7.2) and (7.3) respectively. Then, the following estimates hold
\[
\mathbb{E} \max_{0 \leq t \leq n-1} \left( |R_{t_i}^\pi|^p + |G_{t_i}^\pi|^p \right) \leq K|\pi|^2, \quad (7.4) \\
\mathbb{E} \max_{0 \leq t \leq n-1} \left( \mathbb{E} \left( |R_{t_i}^\pi|^p \mid \mathcal{G}_{t_i} \right) \right) + \mathbb{E} \max_{0 \leq t \leq n-1} \left( \mathbb{E} \left( |G_{t_i}^\pi|^p \mid \mathcal{G}_{t_i} \right) \right) \leq K|\pi|^2, \quad (7.5) 
\]
for some constant \( K > 0 \).

**Proof.** In this proof, let \( C > 0 \) be a generic constant depending only on \( T \), \( p \) and the constants appearing in the assumptions in Theorem 3.9.

Define two functions \( \{t_1(r)\}_{0 \leq r \leq T} \) and \( \{t_2(r)\}_{0 \leq r \leq T} \) by
\[
t_1(r) = \begin{cases} 
0 & \text{if } r = 0, \\
\frac{t_{i+1} - t_i}{T} & \text{if } t_i < r \leq t_{i+1}, \ i = 0, \ldots, n - 1, 
\end{cases}
\]
and
\[
t_2(r) = \begin{cases} 
T & \text{if } r = T, \\
t_{j+1} & \text{if } t_j \leq r < t_{j+1}, \ j = n - 1, \ldots, 0.
\end{cases}
\]

From (7.2), (3.32), Hölder’s inequality, the Lipschitz condition on \( f \), Corollary 3.2 and Theorem 3.6 (b), we obtain
\[
\mathbb{E} \left[ \max_{0 \leq t \leq n-1} |R_{t_i}^\pi|^p \right] \leq \mathbb{E} \sup_{0 \leq t \leq n-1} \left( \sum_{j=i}^{n-1} \int_{t_j}^{t_{j+1}} |f(r, Y_r, Z_r) - f(t_j, Y_{t_j}, Z_{t_j})| dr \right)^p \\
= \mathbb{E} \sup_{0 \leq t \leq n-1} \left( \int_{t_i}^{T} |f(r, Y_r, Z_r) - f(t_1(r), Y_{t_1(r)}, Z_{t_1(r)})| dr \right)^p \\
\leq \mathbb{E} \left( \int_{0}^{T} |f(r, Y_r, Z_r) - f(t_1(r), Y_{t_1(r)}, Z_{t_1(r)})| dr \right)^p \\
\leq T^{p-1} \mathbb{E} \int_{0}^{T} |f(r, Y_r, Z_r) - f(t_1(r), Y_{t_1(r)}, Z_{t_1(r)})|^p dr \leq C|\pi|^{\frac{2}{p}}. \quad (7.6)
\]
By Burkholder-Davis-Gundy’s inequality, condition (iii) in Assumption (A), Corollary 3.2 and Theorem 3.6, we have
\[
\mathbb{E} \max_{0 \leq i \leq n-1} \left| G_{t_i}^n \right|^p = \mathbb{E} \sup_{0 \leq i \leq n-1} \left| \int_{t_i}^T [g(Y_r) - g(Y_{t_2(r)})]d\tilde{B}_r \right|^p \\
\leq C \mathbb{E} \left( \int_0^T \left| g(Y_r) - g(Y_{t_2(r)}) \right|^2 dr \right)^{\frac{p}{2}} \\
\leq CT^{-\frac{p}{2}} \mathbb{E} \int_0^T \left| g(Y_r) - g(Y_{t_2(r)}) \right|^p dr \leq C \pi^{\frac{p}{2}}. \quad (7.7)
\]

Then, the estimate (7.4) follows from (7.6) and (7.7).

Let us now turn to the proof of (7.5). By Doob’s maximal inequality, Hölder’s inequality, (3.32), the Lipschitz condition on $f$, Corollary 3.2 and Theorem 3.6 (b), we get
\[
\mathbb{E} \max_{0 \leq i \leq n-1} \left( \mathbb{E} \left( |R_{t_i}^n| \mid \mathcal{G}_{t_i} \right) \right)^p \\
\leq \mathbb{E} \left( \max_{0 \leq i \leq n-1} \mathbb{E} \left( \int_{t_i}^T |f(r, Y_r, Z_r) - f(t_1(r), Y_{t_1(r)} Z_{t_1(r)})| dr \mid \mathcal{G}_{t_i} \right) \right)^p \\
\leq \mathbb{E} \left( \max_{0 \leq i \leq n-1} \mathbb{E} \left( \int_0^T |f(r, Y_r, Z_r) - f(t_1(r), Y_{t_1(r)} Z_{t_1(r)})| dr \mid \mathcal{G}_{t_i} \right) \right)^p \\
\leq C \mathbb{E} \left( \int_0^T |f(r, Y_r, Z_r) - f(t_1(r), Y_{t_1(r)} Z_{t_1(r)})| dr \right)^p \\
\leq CT^{p-1} \mathbb{E} \int_0^T \left| f(r, Y_r, Z_r) - f(t_1(r), Y_{t_1(r)} Z_{t_1(r)}) \right|^p dr \leq CT^p |\pi|^\frac{p}{2}. \quad (7.8)
\]

This proves the desired bound for the first summand in (7.5). For the second summand, we can write
\[
\mathbb{E} \max_{0 \leq i \leq n-1} \left[ \mathbb{E} \left( G_{t_i}^n \mid \mathcal{G}_{t_i} \right) \right]^p \\
= \mathbb{E} \max_{0 \leq i \leq n-1} \left[ \mathbb{E} \left( \int_{t_i}^T [g(Y_r) - g(Y_{t_2(r)})]d\tilde{B}_r \mid \mathcal{G}_{t_i} \right) \right]^p \\
= \mathbb{E} \max_{0 \leq i \leq n-1} \left[ \mathbb{E} \left( \int_0^{t_i} [g(Y_r) - g(Y_{t_2(r)})]d\tilde{B}_r \mid \mathcal{G}_{t_i} \right) - \int_0^{t_i} [g(Y_r) - g(Y_{t_2(r)})]d\tilde{B}_r \right]^p \\
\leq 2^{p-1} \left[ \mathbb{E} \max_{0 \leq i \leq n-1} \left[ \mathbb{E} \left( \int_0^T [g(Y_r) - g(Y_{t_2(r)})]d\tilde{B}_r \mid \mathcal{G}_{t_i} \right) \right]^p \\
+ \mathbb{E} \max_{0 \leq i \leq n-1} \left[ \int_0^{t_i} [g(Y_r) - g(Y_{t_2(r)})]d\tilde{B}_r \right]^p \right] = 2^{p-1} [A_1 + A_2]. \quad (7.9)
\]

Using Doob’s maximal inequality, we obtain
\[
A_1 \leq c_p \mathbb{E} \left| \int_0^T [g(Y_r) - g(Y_{t_2(r)})]d\tilde{B}_r \right|^p, \quad (7.10)
\]
and

\[ A_2 \leq \mathbb{E} \max_{0 \leq i \leq n-1} \left| \int_0^T [g(Y_r) - g(Y_{t_i(r)})] d\tilde{X}_r - \int_{t_i}^T [g(Y_r) - g(Y_{t_i(r)})] d\tilde{X}_r \right|^p \]

\[ \leq c_p^\prime \mathbb{E} \left| \int_0^T [g(Y_r) - g(Y_{t_i(r)})] d\tilde{X}_r \right|^p. \] (7.11)

Substituting (7.10) and (7.11) into (7.9) and applying Burkholder-Davis-Gundy’s inequality, Hölder’s inequality, (iii) in Assumption (A), Corollary 3.2 and Theorem 3.6 (b), we obtain

\[ \mathbb{E} \max_{0 \leq i \leq n-1} |\mathbb{E} (G^\pi_{t_i}| \mathcal{G}_{t_i})|^p \leq C \mathbb{E} \left( \int_0^T |g(Y_r) - g(Y_{t_i(r)})|^2 dr \right)^\frac{p}{2} \]

\[ \leq CT^{\frac{p-2}{4}} \mathbb{E} \int_0^T |Y_r - Y_{t_i(r)}|^p dr \leq CT^{\frac{p}{4}} |\pi|^{\frac{p}{2}}. \] (7.12)

Therefore, (7.8) and (7.12) yield the desired inequality. \( \square \)

Finally, let us give the proof of our main result on the rate of convergence of the implicit numerical scheme.

**Proof of Theorem 3.9.** In this proof, let \( C > 0 \) be a generic constant depending only on \( T \), \( p \) and all the constants appearing in the assumptions in this theorem, and not depending on the partition \( \pi \).

Note that both \( Y_t \) and \( Y^{\pi}_t \) are \( \mathcal{F}_t \subseteq \mathcal{G}_t \)-measurable for all \( i = n, n-1, \ldots, 0 \) and that \( Z^{1,\pi}_t \) is \( \mathcal{G}_t \)-measurable for all \( t \in [0, T] \). By (7.1) we obtain, for \( i = n-1, \ldots, 1, 0 \),

\[ Y_{t_i} - Y^{\pi}_{t_i} = \mathbb{E} (\xi - \xi^{\pi}| \mathcal{G}_{t_i}) + \mathbb{E} \left( \sum_{j=1}^{n-1} [f(t_j, Y_{t_j}, Z_{t_j}) - f(t_j, Y^{\pi}_{t_j}, Z^{\pi}_{t_j})] \Delta_j \right| \mathcal{G}_{t_i}) \]

\[ + \mathbb{E} \left( \sum_{j=1}^{n-1} [g(Y_{t_{j+1}}) - g(Y^{\pi}_{t_{j+1}})] \Delta B_j \right| \mathcal{G}_{t_i}) + \mathbb{E} (R^\pi_{t_i}| \mathcal{G}_{t_i}) + \mathbb{E} (G^\pi_{t_i}| \mathcal{G}_{t_i}). \] (7.13)

To simplify the notation we denote, for \( i = n, n-1, \ldots, 0 \),

\[ \delta Y^{\pi}_{t_i} = Y_{t_i} - Y^{\pi}_{t_i}, \quad \delta Z^{\pi}_{t_i} = Z_{t_i} - Z^{\pi}_{t_i}, \]

\[ \hat{f}^{\pi}_{t_i} = f(t_i, Y_{t_i}, Z_{t_i}) - f(t_i, Y^{\pi}_{t_i}, Z^{\pi}_{t_i}), \quad \hat{g}^{\pi}_{t_i} = g(Y_{t_i}) - g(Y^{\pi}_{t_i}), \]

and

\[ \delta Z^{1,\pi}_t = Z_t - Z^{1,\pi}_t, \]

for any \( t \in [0, T] \). By convention, \( \delta Y^{\pi}_{t_n} = \xi - \xi^{\pi} \) and \( \delta Z^{\pi}_{t_n} = 0 \). Then, we can rewrite (7.13) as

\[ \delta Y^{\pi}_{t_i} = \mathbb{E} (\xi - \xi^{\pi}| \mathcal{G}_{t_i}) + \mathbb{E} \left( \sum_{j=1}^{n-1} \hat{f}^{\pi}_{t_j} \Delta_j \right| \mathcal{G}_{t_i}) + \mathbb{E} \left( \sum_{j=1}^{n-1} \hat{g}^{\pi}_{t_{j+1}} \Delta B_j \right| \mathcal{G}_{t_i}) \]

\[ + \mathbb{E} (R^\pi_{t_i}| \mathcal{G}_{t_i}) + \mathbb{E} (G^\pi_{t_i}| \mathcal{G}_{t_i}). \] (7.14)
Thus, for $k = n - 1, \ldots, 0,$

\[
\max_{k \leq i \leq n} |\delta Y^\pi_{t_i}| \leq \max_{0 \leq i \leq n} \mathbb{E} \left( |\xi - \xi^\pi| \big| G_{t_i} \right) + \max_{k \leq i \leq n} \mathbb{E} \left( \sum_{j=k}^{n-1} |f_j^\pi| \Delta_j \big| G_{t_i} \right) \\
+ \max_{k \leq i \leq n} \mathbb{E} \left( \sum_{j=i}^{n-1} \bar{g}_{t_{j+1}}^\pi \Delta B_j \big| G_{t_i} \right) + \max_{0 \leq i \leq n} \mathbb{E} \left( |R_j^\pi| \big| G_{t_i} \right) \\
+ \max_{0 \leq i \leq n} \mathbb{E} \left( G_{t_i}^n \big| G_{t_i} \right). \tag{7.15}
\]

Then, by Doob’s maximal inequality, the Lipschitz condition on $f$, (3.29) and Lemma 7.1, we are able to show the following estimate:

\[
\mathbb{E} \max_{k \leq i \leq n} |\delta Y^\pi_{t_i}|^p \\
\leq C \left[ \mathbb{E} |\xi - \xi^\pi|^p + |\pi|^p \mathbb{E} \left( \sum_{j=k}^{n-1} |f_j^\pi| \Delta_j \right)^p + \mathbb{E} \max_{k \leq i \leq n} \left( \sum_{j=i}^{n-1} \bar{g}_{t_{j+1}}^\pi \Delta B_j \right)^p \right] \\
\leq C \left[ \mathbb{E} |\xi - \xi^\pi|^p + |\pi|^p \mathbb{E} \left( \sum_{j=k}^{n-1} |\delta Y^\pi_{t_j}| + |\delta Z^\pi_{t_j}| \right) \Delta_j^p + \mathbb{E} \max_{k \leq i \leq n} \left( \sum_{j=i}^{n-1} \bar{g}_{t_{j+1}}^\pi \Delta B_j \right)^p \right] \\
\leq C \left[ \mathbb{E} |\xi - \xi^\pi|^p + |\pi|^p \right] + C(T - t_k)^p \mathbb{E} \max_{k \leq i \leq n} |\delta Y^\pi_{t_i}|^p \\
+ C \mathbb{E} \left( \sum_{j=k}^{n-1} Z_{t_j} \Delta_j - \mathbb{E} \left( \int_{t_j}^{t_{j+1}} Z_{r}^{1, \pi} dr \big| F_{t_j} \right) \right)^p + C \mathbb{E} \max_{k \leq i \leq n} \mathbb{E} \left( \sum_{j=i}^{n-1} \bar{g}_{t_{j+1}}^\pi \Delta B_j \big| G_{t_i} \right)^p. \tag{7.16}
\]

Next, we will estimate the last two terms on the right-hand side of the above inequality. From the integral representation (3.27), we know that $Z_{r}^{1, \pi}$ is independent of $F_{0,t_j}$ for all $r \in [t_j, t_{j+1}], \; j = n - 1, \ldots, 0$. Thus, it holds that

\[
\mathbb{E} \left( \int_{t_j}^{t_{j+1}} Z_{r}^{1, \pi} dr \big| F_{t_j} \right) = \mathbb{E} \left( \int_{t_j}^{t_{j+1}} Z_{r}^{1, \pi} dr \big| F_{t_j} \vee F_{0,t_j}^B \right) = \mathbb{E} \left( \int_{t_j}^{t_{j+1}} Z_{r}^{1, \pi} dr \big| G_{t_j} \right).
\]

Then, we get

\[
\mathbb{E} \left( \sum_{j=k}^{n-1} Z_{t_j} \Delta_j - \mathbb{E} \left( \int_{t_j}^{t_{j+1}} Z_{r}^{1, \pi} dr \big| F_{t_j} \right) \right)^p \leq 2^{p-1} \left[ \mathbb{E} \left( \sum_{j=k}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E} \left( |Z_{t_j} - Z_{r}| \big| G_{t_j} \right) dr \right)^p + \mathbb{E} \left( \sum_{j=k}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E} \left( |Z_{r} - Z_{r}^{1, \pi}| \big| G_{t_j} \right) dr \right)^p \right] =: 2^{p-1}[I_1 + I_2]. \tag{7.17}
\]
Hölder’s and Jessen’s inequalities and (3.21) yield
\[
I_1 \leq \mathbb{E} \left( \sum_{j=k}^{n-1} \Delta_j \left( \int_{t_j}^{t_{j+1}} (\mathbb{E} (|Z_{t_j} - Z_r| \mid \mathcal{G}_{t_j}))^p \, dr \right) \right)^{\frac{1}{p}} \\
\leq (T - t_k)^{p-1} \mathbb{E} \sum_{j=k}^{n-1} \int_{t_j}^{t_{j+1}} (\mathbb{E} (|Z_{t_j} - Z_r| \mid \mathcal{G}_{t_j}))^p \, dr \\
\leq (T - t_k)^{p-1} \mathbb{E} \sum_{j=k}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E} (|Z_{t_j} - Z_r|^p \mid \mathcal{G}_{t_j}) \, dr \\
= (T - t_k)^{p-1} \mathbb{E} \sum_{j=k}^{n-1} \int_{t_j}^{t_{j+1}} |Z_{t_j} - Z_r|^p \, dr \leq C |\pi|^\frac{p}{2}. \tag{7.18}
\]

Applying Theorem 1.1 in [6] and Hölder’s inequality, we obtain
\[
I_2 \leq C \mathbb{E} \left( \sum_{j=k}^{n-1} \int_{t_j}^{t_{j+1}} |Z_r - Z_r^{1,\pi}| \, dr \right)^p = C \mathbb{E} \left( \int_{t_k}^{T} |Z_r - Z_r^{1,\pi}| \, dr \right)^p \\
\leq C (T - t_k)^{\frac{p}{2}} \mathbb{E} \left( \int_{t_k}^{T} |Z_r - Z_r^{1,\pi}|^2 \, dr \right)^{\frac{p}{2}}. \tag{7.19}
\]

To estimate the last term on the right-hand side of (7.16), adopting the notation \( t_2(r) \) introduced in the proof of Lemma 7.1, we can write
\[
\mathbb{E} \max_{k \leq i \leq n} \left| \mathbb{E} \left( \sum_{j=i}^{n} \tilde{g}_{t_{j+1}}^\pi \Delta B_j \mid \mathcal{G}_{t_i} \right) \right|^p \\
= \mathbb{E} \max_{k \leq i \leq n-1} \left| \mathbb{E} \left( \sum_{j=i}^{n-1} [g(Y_{t_{j+1}}) - g(Y_{t_{j+1}}^\pi)] \Delta B_j \mid \mathcal{G}_{t_i} \right) \right|^p \\
= \mathbb{E} \max_{k \leq i \leq n-1} \left| \mathbb{E} \left( \int_{t_k}^{T} [g(Y_{t_2(r)}) - g(Y_{t_2(r)}^\pi)] \hat{e}_B(r) \mid \mathcal{G}_{t_i} \right) \right|^p \\
\leq 2^{p-1} \left[ \mathbb{E} \max_{k \leq i \leq n-1} \left| \mathbb{E} \left( \int_{t_k}^{T} [g(Y_{t_2(r)}) - g(Y_{t_2(r)}^\pi)] \hat{e}_B(r) \mid \mathcal{G}_{t_i} \right) \right|^p \\
+ \mathbb{E} \max_{k \leq i \leq n-1} \left( \int_{t_k}^{t_i} [g(Y_{t_2(r)}) - g(Y_{t_2(r)}^\pi)] \hat{e}_B(r) \right)^p \right] = 2^{p-1} [B_1 + B_2]. \tag{7.20}
\]

Using Doob’s maximal inequality, we obtain
\[
B_1 \leq c_p \mathbb{E} \left| \int_{t_k}^{T} [g(Y_{t_2(r)}) - g(Y_{t_2(r)}^\pi)] \hat{e}_B(r) \right|^p \tag{7.21}
\]
and
\[
B_2 = \mathbb{E} \max_{k \leq i \leq n-1} \left| \int_{t_k}^{T} [g(Y_{t_{2(r)}}) - g(Y_{t_{2(r)}}^\pi)] d\tilde{B}_r - \int_{t_i}^{T} [g(Y_{t_{2(r)}}) - g(Y_{t_{2(r)}}^\pi)] d\tilde{B}_r \right|^p
\]
\[
\leq c_p \mathbb{E} \int_{t_k}^{T} [g(Y_{t_{2(r)}}) - g(Y_{t_{2(r)}}^\pi)] d\tilde{B}_r \left|^p .
\]  

(7.22)

Substituting (7.21) and (7.22) into (7.20) and applying Burkholder-Davis-Gundy’s inequality, Hölder’s inequality and \((iii)\) in Assumption (A), we obtain
\[
\mathbb{E} \max_{k \leq i \leq n} \left| \sum_{j=i}^{n-1} \tilde{g}_{t_j+1} \Delta B_j \right| \left| G_{t_i} \right|^p \]  
\[
\leq C \mathbb{E} \left( \int_{t_k}^{T} |g(Y_{t_{2(r)}}) - g(Y_{t_{2(r)}}^\pi)|^2 dr \right)^{\frac{p}{2}} \leq C(T - t_k)^{\frac{p}{2}} \mathbb{E} \max_{k \leq i \leq n-1} |\delta Y_{t_{i+1}}^\pi|^p
\]
\[
\leq C(T - t_k)^{\frac{p}{2}} \mathbb{E} \max_{k \leq i \leq n} |\delta Y_{t_i}^\pi|^p .
\]  

(7.24)

From (7.16), (7.17), (7.18), (7.19) and (7.24), it follows
\[
\mathbb{E} \max_{k \leq i \leq n} |\delta Y_{t_i}^\pi|^p \leq C \left[ \mathbb{E} |\xi - \xi^\pi|^p + |\pi|^{\frac{p}{2}} \right] + C(T - t_k)^{\frac{p}{2}} \mathbb{E} \max_{k \leq i \leq n} |\delta Y_{t_i}^\pi|^p
\]
\[
+ C(T - t_k)^{\frac{p}{2}} \mathbb{E} \left( \int_{t_k}^{T} |Z_r - Z_{r,1}^\pi|^2 dr \right)^{\frac{p}{2}}.
\]  

(7.25)

By Burkholder-Davis-Gundy’s inequality, we have
\[
\mathbb{E} \left( \int_{t_k}^{T} |Z_r - Z_{r,1}^\pi|^2 dr \right)^{\frac{p}{2}} \leq c_p \mathbb{E} \int_{t_k}^{T} (Z_r - Z_{r,1}^\pi) dW_r \left|^p .
\]  

(7.26)

From (7.1), we obtain
\[
\int_{t_k}^{T} (Z_r - Z_{r,1}^\pi) dW_r = -\delta Y_{t_k}^\pi + \xi - \xi^\pi + \sum_{i=k}^{n-1} \tilde{g}_{t_i} \Delta i + \sum_{i=k}^{n-1} \tilde{g}_{t_{i+1}} \Delta B_i + R_{t_k}^\pi + G_{t_k}^\pi .
\]  

(7.27)

Then, it follows from (7.26), (7.27), Lemma 7.1 and the arguments used in the proof of (7.25), that there exists a constant $C_1 > 0$ independent of the partition $\pi$ such that
\[
\mathbb{E} \left( \int_{t_k}^{T} |Z_r - Z_{r,1}^\pi|^2 dr \right)^{\frac{p}{2}} \leq C_1 \left[ \mathbb{E} |\xi - \xi^\pi|^p + |\pi|^{\frac{p}{2}} \right] + C_1(T - t_k)^{\frac{p}{2}} \mathbb{E} \max_{k \leq i \leq n} |\delta Y_{t_i}^\pi|^p
\]
\[
+ C_1(T - t_k)^{\frac{p}{2}} \mathbb{E} \left( \int_{t_k}^{T} |Z_r - Z_{r,1}^\pi|^2 dr \right)^{\frac{p}{2}}.
\]  

If $C_1(T - t_k)^{\frac{p}{2}} < \frac{1}{2}$, then we get
\[
\mathbb{E} \left( \int_{t_k}^{T} |Z_r - Z_{r,1}^\pi|^2 dr \right)^{\frac{p}{2}} \leq 2C_1 \left[ \mathbb{E} |\xi - \xi^\pi|^p + |\pi|^{\frac{p}{2}} \right] + 2C_1(T - t_k)^{\frac{p}{2}} \mathbb{E} \max_{k \leq i \leq n} |\delta Y_{t_i}^\pi|^p .
\]  

(7.28)
Substituting (7.28) into (7.25), we can find a constant $C_2 > 0$ independent of the partition $\pi$ such that
\[
\mathbb{E} \max_{k \leq t \leq n} |Y_{t_i}^{\pi}|^p \leq C_2 \left[ \mathbb{E} |\xi - \xi^\pi|^p + |\pi|^{\frac{p}{2}} \right] + C_2 (T - t_k)^{\frac{p}{2}} \mathbb{E} \max_{k \leq t \leq n} |Y_{t_i}^{\pi}|^p. \tag{7.29}
\]
Fix a positive constant $\delta$ independent of the partition $\pi$ such that
\[
C_1 (3\delta)^{\frac{p}{2}} < \frac{1}{2}, \quad C_2 (3\delta)^{\frac{p}{2}} < \frac{1}{2}, \quad 2\delta < T.
\]
Denote $l = \left[ \frac{T}{2\delta} \right]$. Then $l \geq 1$ is an integer independent of the partition $\pi$. If $|\pi| < \delta$, then for the partition we choose $n - 1 > i_1 > \cdots > i_l \geq 0$ such that $T - 2\delta \in (t_{i_l-1}, t_{i_l})$, $T - 4\delta \in (t_{i_l-1}, t_{i_l})$, $\ldots$, $T - 2\delta l \in [0, t_{i_l})$ (with $t_{-1} = 0$). For simplicity, we denote $t_{i_0} = T$ and $t_{i_{i+1}} = 0$. Each interval $[t_{i_j}, t_{i_{j+1}}]$, $j = 0, 1, \ldots, l$, has length less than $3\delta$, that is, $|t_{i_j} - t_{i_{j+1}}| < 3\delta$.

On each interval $[t_{i_j}, t_{i_{j+1}}]$, $j = 0, 1, \ldots, l$, we carry out the same analysis as that in (7.16)-(7.29), and in this way we obtain
\[
\mathbb{E} \left( \int_{t_{i_j}}^{t_{i_{j+1}}} |Z_r - Z_r^{1, \pi}|^2 \, dr \right)^{\frac{p}{2}} \leq 2C_1 \mathbb{E} |Y_{t_{i_j}}^{\pi}|^p + 2C_1 |\pi|^{\frac{p}{2}} + 2C_1 (t_{i_j} - t_{i_{j+1}})^{\frac{p}{2}} \mathbb{E} \max_{t_{i_{j+1}} \leq t \leq t_{i_j}} |Y_{t_i}^{\pi}|^p
\]
and
\[
\mathbb{E} \max_{i_{j+1} \leq t \leq i_j} |Y_{t_i}^{\pi}|^p \leq C_2 \left[ \mathbb{E} |Y_{t_{i_j}}^{\pi}|^p + |\pi|^{\frac{p}{2}} \right] + C_2 (t_{i_j} - t_{i_{j+1}})^{\frac{p}{2}} \mathbb{E} \max_{i_{j+1} \leq t \leq i_j} |Y_{t_i}^{\pi}|^p
\]
\[
\leq C_2 \left[ \mathbb{E} |Y_{t_{i_j}}^{\pi}|^p + |\pi|^{\frac{p}{2}} \right] + \frac{1}{2} \mathbb{E} \max_{t_{i_{j+1}} \leq t \leq t_{i_j}} |Y_{t_i}^{\pi}|^p.
\]
Hence,
\[
\mathbb{E} \max_{i_{j+1} \leq t \leq i_j} |Y_{t_i}^{\pi}|^p \leq 2C_2 \left[ \mathbb{E} |Y_{t_{i_j}}^{\pi}|^p + |\pi|^{\frac{p}{2}} \right].
\]
By recurrence, we have
\[
\mathbb{E} \max_{i_{j+1} \leq t \leq i_j} |Y_{t_i}^{\pi}|^p \leq (2C_2)^j \frac{1}{2} \mathbb{E} |\xi - \xi^\pi|^p + 2C_2 \sum_{j=1}^l \frac{2C_2}{1 - 2C_2} \left[ \mathbb{E} |Y_{t_{i_j}}^{\pi}|^p + |\pi|^{\frac{p}{2}} \right]
\]
\[
\leq (2C_2)^l \frac{1}{2} \mathbb{E} |\xi - \xi^\pi|^p + \frac{2C_2 (1 - (2C_2)^l)}{1 - 2C_2} |\pi|^{\frac{p}{2}}.
\]
Thus, taking $K_1 = (l + 1)^p \max \left\{ (2C_2)^{l+1}, \frac{2C_2 (1 - (2C_2)^{l+1})}{1 - 2C_2} \right\}$, we have the following estimate
\[
\mathbb{E} \max_{0 \leq t \leq n} |Y_{t_i}^{\pi}|^p \leq (l + 1)^p \sum_{j=0}^{l-1} \mathbb{E} \max_{i_{j+1} \leq t \leq i_j} |Y_{t_i}^{\pi}|^p \leq K_1 \left[ \mathbb{E} |\xi - \xi^\pi|^p + |\pi|^{\frac{p}{2}} \right]. \tag{7.31}
\]
Plugging (7.31) into (7.30), we obtain
\[
\mathbb{E} \left( \int_{t_{i_{j+1}}}^{t_{i_j}} |Z_r - Z_r^{1, \pi}|^2 \, dr \right)^{\frac{p}{2}} \leq 2C_1 |\pi|^{\frac{p}{2}} + 2C_1 K_1 (1 + T^{\frac{p}{2}}) \left[ \mathbb{E} |\xi - \xi^\pi|^p + |\pi|^{\frac{p}{2}} \right].
\]
Then, by taking $K_2 = (l + 1)\frac{p}{2}(2C_1 + 2C_1K_1(1 + T^\frac{p}{2}))$, we have

$$
\mathbb{E}\left(\int_0^T |Z_r - Z_r^1|^{p}\ dr\right)^{\frac{p}{2}} \leq (l + 1)\frac{p}{2} \sum_{i=0}^l \mathbb{E}\left(\int_{t_{i+1}}^{t_i} |Z_r - Z_r^1|^{p}\ dr\right)^{\frac{p}{2}}
$$

$$
\leq (l + 1)\frac{p}{2} \left(2C_1|\pi|^p + 2C_1K_1(1 + T^\frac{p}{2})\left[\mathbb{E}|\xi - \xi^p|^p + |\pi|^\frac{p}{2}\right]\right)
$$

$$
\leq K_2 \left[\mathbb{E}|\xi - \xi^p|^p + |\pi|^\frac{p}{2}\right].
$$

Therefore, by taking $K = K_1 + K_2$ and adding (7.31) and (7.32), we deduce (3.33). □

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