COUNTING USING HALL ALGEBRAS III. QUIVERS WITH POTENTIALS

JIARUI FEI

Abstract. For a quiver with potential, we can associate a vanishing cycle to each representation space. If there is a nice torus action on the potential, the vanishing cycles can be expressed in terms of truncated Jacobian algebras. We study how these vanishing cycles change under the mutation of Derksen-Weyman-Zelevinsky. The wall-crossing formula leads to a categorification of quantum cluster algebras under some assumption. This is a special case of A. Efimov’s result, but our approach is more concrete and down-to-earth. We also obtain a counting formula relating the representation Grassmannians under sink-source reflections.

Introduction

We continue our development on algorithms to count the points of varieties related to quiver representations. In this note, we focus on the quivers with potentials. A potential $W$ on a quiver $Q$ is just a linear combination of oriented cycles of $Q$. It can be viewed as a function on certain noncommutative space attached to $Q$. When composing with the usual trace function, it becomes a regular function $\omega$ on each representation spaces $\text{Rep}_\alpha(Q)$. This function further descends to various moduli spaces.

Let $f$ be a regular function on a complex variety $X$. Consider the scheme theoretic degeneracy locus $\{df = 0\}$ of $f$. Behrend, Bryan and Szendrői define in $^{[3]}$ a class $[\varphi_f(X)] \in K_0(\text{Var}_\mathbb{C})[\mathbb{L}^{-1}]$ associated to each such locus, essentially given by the motivic Milnor fibre of the map $f$. When $X$ admits a suitable torus action, this class can be expressed as $^{[4]}$ $[\varphi_f(X)] = [f^{-1}(0)] - [f^{-1}(1)]$.

Deligne’s mixed Hodge structure on compactly supported cohomology gives rise to the $E$-polynomial homomorphism $E : K_0(\text{Var}_\mathbb{C}) \to \mathbb{Z}[x, y]$ given by $E([Y]; x, y) = \sum_{p,q} x^p y^q \sum (-1)^i \dim H_{p,q}(H_i^c(Y, \mathbb{Q}))$.

Following N. Katz $^{[11]}$ Appendix, we say that an element $\gamma \in K_0(\text{Var}_\mathbb{C})$ is fibrewise polynomial-count, if for every finite field $k$, and every ring homomorphism $\phi : \mathbb{C} \to k$.

2010 Mathematics Subject Classification. 16G20; Secondary 16G10, 14N10, 13F60.

Key words and phrases. Quiver Representation, Quiver with Potential, Hall Algebra, Donaldson-Thomas Invariants, Vanishing Cycle, Virtual Motive, Moduli space, Quiver Grassmannian, Quantum Cluster Algebra, Cluster Character, Quantum Dilogarithm, Wall-Crossing, BB-Tilting, Mutation, Jacobian Algebra, Polynomial-count.

$^1$This definition of $[\varphi_f(X)]$ differs from the original one by a negative sign.
Let $k$, the element $\gamma \in K_0(\text{Var}_k)$ deduced from $\gamma$ by extension of scalars is polynomial-count (but we allow its counting polynomial to vary with $\phi$). It is known ([11, Theorem 6.1.2, 6.1.4]) that

**Lemma 0.1.** If $\gamma \in K_0(\text{Var}_k)$ is fibrewise polynomial-count, then all counting polynomials coincide as one $P_\gamma(t) \in \mathbb{Z}[t]$ with $E(\gamma; x, y) = P_\gamma(xy)$.

In this notes, we follow an algebraic approach to compute $P_\gamma(t)$ directly in the above quiver setting. Namely, $\gamma$ is the class $[\varphi_\omega]$, defined by the regular function $\omega$ on representation spaces. In fact, we will work with the generating series of virtual class $[\varphi_\omega](X)_{\text{vir}} := q^{-\frac{1}{2} \dim x}$ for all dimension vectors:

$$\mathcal{V}(Q, W) := \sum_\alpha \frac{|\varphi_\omega(\text{Rep}_\alpha(Q))|_{\text{vir}} x^\alpha}{|GL_\alpha|_{\text{vir}}}. $$

Our main results contain two wall-crossing formula – one for the ordinary $\mathcal{V}(Q, W)$, the other for the one with a framing stability $\mu_\infty$:

$$\mathcal{T}(Q, W) := \sum_\beta |\varphi_\omega(\text{Mod}^\mu_{(1, \beta)}(\bar{Q}))|_{\text{vir}} x^{(1, \beta)}, $$

where $\text{Mod}^\mu_{(1, \beta)}(Q)$ denotes the GIT moduli of $\alpha$-dimensional $\mu$-stable representations. To state them, let $\mu_k$ be the mutation of $(Q, W)$ in the sense of Derksen, Weyman, and Zelevinsky [7]. Let $\mathcal{V} := \mathcal{V}(\mu_k(Q, W))$ and $E_k := \exp_q(\frac{q^{1/2}}{q-1}x_k)$, then under the technique condition $\odot$ of Section [3]

**Theorem 0.2.** We have that

\begin{align}
(0.1) \quad E_k^\prime \Phi_k^\prime(\mathcal{V}E_k^{-1}) & = \mathcal{V} = \Phi_k^{\prime}(E_k^{-1}\mathcal{V})E_k^{\prime}; \\
(0.2) \quad (E_k^\prime)^{-1}\Phi_k(T)E_k^\prime = T' = \Phi_k^{\prime}(E_k^\prime TE_k^{-1}).
\end{align}

Here, multiplications are performed in appropriate completed quantum Laurent series algebras. $\Phi_k^\prime$ and $\Phi_k$ are certain linear monomial change of variables.

There are two main ingredients in the proof. One is a construction of Mozgovoy, which relates the Hall algebra of a quiver to the quantum Laurent series ring. The other is a ‘dimension reduction’ technique used by A. Morrison and K. Nagao. The equation (0.1) already appeared in [20] and [14], but we put it in a right assumption.

Nagao also suggested in [20] that this theory can be used to study quantum cluster algebra. We follow his suggestion and use (0.2) to categorify quantum cluster algebras under the assumption of existence of certain potentials. If a cluster algebra has such a categorification, then its strong positivity will be implied by a result of [9] on the purity of the vanishing cycles.

Let $\Lambda$ and $B_p$ be two skew-symmetric matrices of size $m \times m$ and $n \times n$ ($m \geq n$). We extend $B_p$ from the right to an $n \times m$ matrix $B$. We assume $\Lambda$ and $B$ are unitarily compatible, that is, $B\Lambda = [-I_n, 0]$. We can associate $B$ a quiver $Q$ without loops and 2-cycles with $b_{ij} = |\text{arrows } j \to i| - |\text{arrows } i \to j|$. We endow $Q$ with some potential $W$ of nice properties.

Let $\mu_k$, be a sequence of mutations, and set $(Q_t, W_t) = \mu_k(Q, W)$. Let $x^g (g \in \mathbb{Z}_{\geq 0}^{m, n})$ be some initial cluster monomial in the quantum Laurent polynomial ring $X_A$. We extend QP $(Q_t^g, W_t)$ from $(Q_t, W_t)$ by adding a new vertex $\infty$ and $q_t$ new arrows from $i$ to $\infty$. We apply the inverse of $\mu_k$ to $(Q_t^g, W_t)$, and obtain a QP $(Q^g, W^g) := \mu_k^{-1}(Q_t^g, W_t)$. Let $B$ be the extended $B$-matrix of $Q^g$.
Theorem 0.3. The mutated cluster monomial $X_t(g) := \mu_k(x^g)$ is equal to
$$\sum_{\beta} \varphi_{\omega}(\text{Mod}^{\mu_{(1, \beta)}(g)}(Q^g))|_{\text{vir}} x^{(1, \beta)} B.$$ 

This result may just be a special case of [9], where A. Efimov assumed a much weaker condition on the potential $W$. However, our approach and result are more down-to-earth and computable. It depends only on [3] rather than [16].

The second application is on the generating series counting the quiver representation Grassmannian. Let $s$ be a sink of $Q$, and $M$ be a representation of $Q$. We assume that $M$ does not contain the simple representation $S_s$ as a direct summand.

Let $T(M) := \sum_{\beta} q^{-\frac{1}{2} \langle \text{M} - \beta, \beta \rangle} |\text{Gr}^\beta(M)| x^{(1, \beta)},$

where $\text{Gr}^\beta(M)$ is the variety parameterizing $\beta$-dimensional quotient representations of $M$.

Theorem 0.4. $T(M)$ and $T(\mu_s M)$ are also related via (0.2). In particular, if $M$ is polynomial-count, that is, all its Grassmannians $\text{Gr}^\beta(M)$ are polynomial-count, then so are all reflection equivalent classes of $M$.

This note is organized as follows. In Section 1, we recall some basics about quiver representations and their Hall algebras. In Section 2, we recall the concept of quiver with potential and a cut, and its associated algebras. In Section 3, we recall a construction of Mozgovoy and the dimension reduction technique (Lemma 3.3). In Section 4, we recall the mutation of QP with a cut, and set up the key assumption for our main results. When this assumption holds is illustrated in Corollary 4.3, whose proof will be given in the appendix. In Section 5, we prove our first two main results – Theorem 5.2 and Theorem 5.3. In Section 6, we prove our third main result, Theorem 6.8, on a categorification of quantum cluster algebras. In Section 7, we prove our last main result, Theorem 7.4, on the representation Grassmannians under reflections.

Notations and Conventions.

- All modules are right modules and all vectors are row vectors.
- For an arrow $a$, $ta$ is the tail of $a$ and $ha$ is the head of $a$.
- For any representation $M$, we use $\overline{M}$ to denote its dimension vector.
- $S_i$ is the simple module at the vertex $i$, and $P_i$ is its projective cover.
- Superscript $*$ is the trivial dual $\text{Hom}_k(-, k)$.

1. Basics on Quivers and their Hall Algebras

From now on, we assume our base field $k = \mathbb{F}_q$ to be the finite field with $q$ elements. Let $Q$ be a finite quiver with the set of vertices $Q_0$ and the set of arrows $Q_1$. We write $\langle -, - \rangle$ for the usual Euler form of $Q$. We also have the antisymmetric form $\langle - , - \rangle$ associated to $Q$. The matrix of $\langle -, - \rangle$ denoted by $B$ is given by

$$b_{ij} = |\text{arrows } j \rightarrow i| - |\text{arrows } i \rightarrow j|.$$ (1.1)

For any three $kQ$-modules $U, V$ and $W$ with dimension vector $\beta, \gamma$ and $\alpha = \beta + \gamma$, the Hall number $F^W_{UV}$ counts the number of subrepresentations $S$ of $W$ such that $S \cong V$ and $W/S \cong U$. We denote $a_W := |\text{Aut}_Q(W)|$. Let $H(Q)$ be the space of all formal (infinite) linear combinations of isomorphism classes of $kQ$-modules.
Lemma 1.1. [22] Proposition 1.1] The completed Hall algebra $H(Q)$ is the associative algebra with multiplication

$$[U][V] := \sum_{[W]} F^W_{UV}[W],$$

and unit $\eta : k \mapsto k[0]$.

Let $C$ be a subcategory of $\text{mod} kQ$. We denote $\chi(C) := \sum_{M \in C} [M]$. We use the shorthand $\chi$ and $\chi_\alpha$ for $(\chi(\text{mod} kQ))$ and $(\chi(\text{mod}_\alpha kQ))$. Let $(T, \mathcal{F})$ be a torsion pair ([22] Definition VI.1.1]) in $\text{mod} kQ$, then for any $M \in \text{mod} kQ$, there exists a short exact sequence $0 \to L \to M \to N \to 0$ with $L$ unique in $T$ and $N$ unique in $\mathcal{F}$. In terms of the Hall algebra, this says that

Lemma 1.2. $\chi = \chi(\mathcal{F})\chi(T)$.

A weight $\sigma$ is an integral linear functional on $\mathbb{Z}^{Q_0}$. A slope function $\mu$ is a quotient of two weights $\sigma/\theta$ with $\theta(\alpha) > 0$ for any non-zero dimension vector $\alpha$.

Definition 1.3. A representation $M$ is called $\mu$-semi-stable (resp. $\mu$-stable) if $\mu(L) \leq \mu(M)$ (resp. $\mu(L) < \mu(M)$) for every non-trivial subrepresentation $L \subset M$.

We denote by $\text{Rep}_\mu^0(Q)$ the variety of $\alpha$-dimensional $\mu$-semistable representations of $Q$. By the standard GIT construction [15], there is a categorical quotient $q : \text{Rep}_\mu^0(Q) \to \text{Mod}_\mu^0(Q)$ and its restriction to the stable representations $\text{Rep}_\mu^{\text{st}}(Q)$ is a geometric quotient.

A slope function $\mu$ is called coprime to $\alpha$ if $\mu(\gamma) \neq \mu(\alpha)$ for any $\gamma < \alpha$. So if $\mu$ is coprime to $\alpha$, then there is no strictly semistable (semistable but not stable) representation of dimension $\alpha$. In this case, $\text{Mod}_\mu^0(A)$ must be a geometric quotient.

Lemma 1.4. [21] Proposition 2.5] Harder-Narasimhan filtration: Every representation $M$ has a unique filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} \subset M_m = M$$

such that $N_i = M_i/M_{i+1}$ is $\mu$-semi-stable and $\mu(N_i) > \mu(N_{i+1})$.

We fix a slope function $\mu$. For a dimension vector $\alpha$, let $\chi_\alpha^\mu = \sum_{M \in \text{mod}_\mu^0(Q)} [M]$. The existence of the Harder-Narasimhan filtration yields the following identity in the Hall algebra $H(Q)$.

Lemma 1.5. [21] Proposition 4.8]

$$\chi_\alpha = \sum \chi_\alpha^{\mu_1} \cdots \chi_\alpha^{\mu_s},$$

where the sum running over all decomposition $\alpha_1 + \cdots + \alpha_s = \alpha$ of $\alpha$ into non-zero dimension vectors such that $\mu(\alpha_1) < \cdots < \mu(\alpha_s)$. In particular, solving recursively for $\chi_\alpha^\mu$, we get

(1.2)

$$\chi_\alpha^\mu = \sum_s (-1)^{s-1} \chi_{\alpha_1} \cdots \chi_{\alpha_s},$$

where the sum runs over all decomposition $\alpha_1 + \cdots + \alpha_s = \alpha$ of $\alpha$ into non-zero dimension vectors such that $\mu(\sum_{i=1}^k \alpha_i) < \mu(\alpha)$ for $k < s$.

Lemma 1.6. [21] Lemma 6.1] Let $X(Q)$ be the quantum Laurent series algebra in $|Q_0|$ variables with multiplication given by $x^\beta x^\gamma = q^{\frac{1}{2}((\beta, \gamma))} x^{\beta + \gamma}$. Then $f : [M] \mapsto q^{\frac{1}{2}(\alpha, \alpha)} \frac{x^\alpha}{a_M}$ is an algebra homomorphism $H(Q) \to X(Q)$. 

It is clear that for algebraic cuts, the form \( \omega(\alpha, \alpha) \) is defined by \( \omega(\alpha, \alpha) = \text{tr}(\alpha) \). If \( kQ/\partial W \) is finite dimensional, then the completion is unnecessary.

Let \( \omega : \text{Rep}(Q) \to k \) be the trace function corresponding to the potential \( W \) defined by \( M \mapsto \text{tr}(W(M)) \). Note that \( \omega \) is an additive function, so it in fact descends to the Grothendieck group: \( \omega : k_0(\text{Rep}(Q)) \to k \). By abuse of notation, we also use \( \omega \) for the same trace function defined on the affine representation varieties and moduli spaces. The following lemma is well-known.

**Lemma 2.1.** A representation of \( Q \) a representation of \( J(Q, W) \) if and only if it is in the degeneracy locus of \( \omega \).

Following [12], we define

**Definition 2.2.** A cut of a QP is a subset \( C \subset Q_1 \) such that the potential \( W \) is homogenous of degree 1 for the weight function \( \omega : Q_1 \to \mathbb{N} \) defined by \( \omega(a) = 1 \) for \( a \in C \) and zero otherwise.

Note that this weight function defines a \( k^* \)-action on \( \text{Rep}_\alpha(Q) \) such that the trace function is equivariant: \( \omega(tM) = t\omega(M) \).

**Definition 2.3.** The algebra \( J(Q, W; C) \) associated to a QP \( (Q, W) \) with a cut \( C \) is the quotient algebra of the Jacobian algebra \( J(Q, W) \) by the ideal generated by \( \partial_C W \).

Note that \( J(Q, W; C) \) is isomorphic to degree zero part of \( J(Q, W) \). We denote by \( Q_C \) the subquiver \( (Q_0, Q_1 \setminus C) \) of \( Q \) and by \( \langle \partial_C W \rangle \) the ideal \( \langle \partial_C W \mid c \in C \rangle \). It is clear that \( J(Q, W; C) \) can also be presented as \( kQ_C / \langle \partial_C W \rangle \).

We put
\[
\langle \alpha, \beta \rangle_C = \sum_{c \in C} \alpha(tc)\beta hc, \\
\langle \alpha, \beta \rangle_{Jc} = \langle \alpha, \beta \rangle_Q + \langle \alpha, \beta \rangle_C + \langle \beta, \alpha \rangle_C.
\]

**Definition 2.4 (12).** A cut is called algebraic if

1. \( J(Q, W; C) \) is a finite dimensional \( k \)-algebra of global dimension 2,
2. \( \langle \partial_c W \rangle_{c \in C} \) is a minimal set of generators of the ideal \( \langle \partial_C W \rangle \) in \( kQ \).

It is clear that for algebraic cuts, the form \( \langle -, - \rangle_{Jc} \) is exactly the Euler form of \( J(Q, W; C) \). From now on, we assume that all cuts are algebraic.
Conversely, given any \( k \)-algebra \( A \) presented by a quiver \( Q \) with a minimal set of relations \( \{r_1, r_2, \ldots, r_l\} \), we can associate with it a QP \((\overline{Q}, W)\) with a cut \( C \) as follows: \( \overline{Q}_0 = Q_0, \overline{Q}_1 = Q_1 \) with \( C = \{c_i : h(r_i) \rightarrow t(r_i)\}_{i=1..l} \), and \( W = \sum_{i=1}^l c_i r_i \). If \( A \) has global dimension 2, it is known \([12, \text{Theorem 6.10}]\) that \( J(\overline{Q}, W) \) is isomorphic to the algebra \( \Pi_3(A) := \prod_{i \geq 0} \text{Ext}^2_A(DA, A)^{\otimes A} \). In particular, \( J(\overline{Q}, W) \) does not depend on the minimal set of relations that we chose. It is clear that (see also \([12, \text{Proposition 3.3}]\))

**Proposition 2.5.**

1. \( J(\overline{Q}, W; C) \cong A \).
2. Let \( C \) be an algebraic cut of \((Q, W)\) and \( A = J(Q, W; C) \), then \((Q, W)\) and \((\overline{Q}, W)\) are isomorphic QPs.

3. A Construction of Mozgovoy

Let \( X \) be a quasiprojective variety with a \( k^* \)-action, and \( \omega \) a regular function on \( X \). We assume that \( \omega \) is equivariant with respect to a primitive character. We set

\[
|\varphi_\omega(X)| = |\omega^{-1}(0)| - |\omega^{-1}(1)|.
\]

**Remark 3.1.** If \( X \) is a complex variety, and we further assume that \( \lim_{t \to 0} tx \) exist for all \( x \in X \), then according to \([3, \text{Proposition 1.11}]\), \( [\omega^{-1}(1)] \in K_0(\text{Var}_C) \) is the nearby fibre of \( \omega \). Its difference with the central fibre \( \omega^{-1}(0) \) defines a class called the (absolute) vanishing cycle of \( \omega \) on \( X \):

\[
[\varphi_\omega(X)] := [\omega^{-1}(0)] - [\omega^{-1}(1)].
\]

Due to the torus action, we have that

\[
(q - 1)|\omega^{-1}(1)| = |X| - |\omega^{-1}(0)|,
\]

so

\[(3.1) \quad |\varphi_\omega(X)| = \frac{|\omega^{-1}(0)| - |X|}{q - 1}.
\]

We denote the \( q \)-shifted \( q^{\frac{\dim Q}{2}}|\varphi_\omega(X)| \) by \( |\varphi_\omega(X)|_{\text{vir}} \).

Let \( \omega : \text{Rep}_\alpha(Q) \rightarrow k \) be the trace function corresponding to the potential \( W \). For \( h = \sum c_M[M] \in H(Q) \), we define \( h_0 := \sum_{|M|=0} c_M[M] \). Such an \( h \) is called equivariant if \( c_M = c_M \) for any \( t \in k^* \). Let \( H_\text{eq}(Q) \) be the subalgebra of \( H(Q) \) consisting of equivariant elements.

**Lemma 3.2.** \([19, \text{Proposition 5.2}]\) The map \( \int_\omega : H_\text{eq}(Q) \rightarrow X(Q) \) defined by

\[
h \mapsto \frac{q \int h_0 - \int h}{q - 1}
\]

is an algebra morphism.

Note that if \( W \) is trivial, then \( \int_\omega = \int \). We see from (3.1) that

\[
u_\alpha := \int_\omega \chi_\alpha = \frac{|\varphi_\omega(\text{Rep}_\alpha(Q))|_{\text{vir}}}{|\text{GL}_\alpha|} x^\alpha.
\]

We denote the generating series \( \int_\omega \chi \) by \( \forall(Q, W) := \sum_\alpha v_\alpha x^\alpha \).

**Lemma 3.3.** \([20, \text{Theorem 4.1}]\) \( |\varphi_\omega(\text{Rep}_\alpha(Q))| = q^{(\alpha, \alpha)C} |\text{Rep}_\alpha(J(Q, W; C))| \). So

\[
v_\alpha = q^{\frac{1}{2}(\alpha, \alpha)_C} \frac{|\text{Rep}_\alpha(J(Q, W; C))|}{|\text{GL}_\alpha|}.
\]
Lemma 3.7. If \( \text{same as that in \cite{20}}. \) For readers’ convenience, we copy the proof here.

So \( \pi : \text{Rep} \) A pair \((\alpha, \mu)\) statement does not hold for \( \text{Rep} \) A algebra \( \ast \) where the summation \( \omega \) vanishing cycle of \( \omega \) verify is that \( \lim_{t \to 0} tx \) exists in \( \text{Mod} \) for any \( x \in \text{Mod} \). However, similar statement does not hold for \( \text{Rep} \) in general.

Apply the Hall character \( \int_{\omega} \) to the identity \( \cite{12} \), then we obtain \( \cite{19} \) Theorem 5.7:

**Proposition 3.4.**

\[
\frac{|\varphi_{\omega}(\text{Rep}_\alpha^\mu(Q))|_{\text{vir}}}{|GL_\alpha|_{\text{vir}}} = \sum_{s} (-1)^{s-1} q^{\frac{s}{2} \sum_{j>\alpha} (\alpha_i, \alpha_j)} \prod_{k=1}^{s} v_{\alpha_k}(q),
\]

where the summation \( s \) is the same as in Lemma \( \cite{13} \).

We denote by \( v_\alpha^\mu(q) \) the above rational function in \( q^{\frac{s}{2}} \). Following \( \cite{10} \), we say an algebra \( A \) is polynomial-count if each \( \text{Rep}_\alpha(A) \) is polynomial-count.

**Corollary 3.5.** Assume that the GIT quotient \( \text{Mod}_\alpha^\mu(Q) \) is geometric. If \( J(Q, W; C) \) is polynomial-count, then so is \( \varphi_{\omega}(\text{Mod}_\alpha^\mu(Q)) \).

**Definition 3.6.** A pair \((\alpha, \mu)\) is called numb to a cut \( C \) on \( Q \) if the vector bundle \( \pi : \text{Rep}_\alpha(Q) \to \text{Rep}_\alpha(Q_C) \) restricts to \( \mu \)-semistable representations.

Later we will need the following generalization of Lemma \( \cite{3.3} \). The proof is the same as that in \( \cite{20} \). For readers’ convenience, we copy the proof here.

**Lemma 3.7.** If \( \mu \) is numb to \( C \), then \( |\varphi_{\omega}(\text{Rep}_\alpha^\mu(Q))| = q^{(\alpha, \alpha)_C} |\text{Rep}_\alpha^\mu(J(Q, W; C))|. \) So \( v_\alpha^\mu = q^{\frac{s}{2} (\alpha, \alpha)_C} |\text{Rep}_\alpha^\mu(J(Q, W; C))| / |GL_\alpha| \).

**Proof.** By assumption, \( \pi : \text{Rep}_\alpha^\mu(Q) \to \text{Rep}_\alpha^\mu(Q_C) \) is a vector bundle of rank \( d = (\alpha, \alpha)_C \). The restriction of \( \omega \) to the fibre \( \pi^{-1}(M) \) is zero if \( M \in \text{Rep}_\alpha^\mu(J(Q, W; C)) \), and is a non-zero linear function if \( x \notin \text{Rep}_\alpha^\mu(J(Q, W; C)). \) Hence \( |\omega^{-1}(0)| = q^d (|\text{Rep}_\alpha^\mu(J(Q, W; C))| - |\text{Rep}_\alpha^\mu(J(Q, W; C))|). \) By \( \cite{3.1} \),

\[
|\varphi_{\omega}(\text{Rep}_\alpha^\mu(Q))| = \frac{q |\omega^{-1}(0)| - |\text{Rep}_\alpha^\mu(Q)|}{q - 1},
\]

\[
= q^{(\alpha, \alpha)_C} |\text{Rep}_\alpha^\mu(J(Q, W; C))|. \]

\( \square \)

Let \( \{e_i\}_i \) be the standard basis of \( \mathbb{Z}^{Q_0} \). The \( k \)-th (absolute) framing stability \( \mu_k \) is the slope function given by \( e^+_i / d \), where \( d(\alpha) = \sum \alpha \). It is not hard to see that if all arrows in \( C \) end in \( k \), then \((\alpha, \mu_k) \) with \( \alpha_k = 1 \) is numb to \( C \).

4. Mutation of Quivers with Potentials

The key notion in \( \cite{2} \) is the definition of mutation \( \mu_k \) of a quiver with potentials at some vertex \( k \in Q_0 \). Let us briefly recall it. The first step is to define the following new quiver with potential \( \tilde{\mu}_k(Q, W) = (\tilde{Q}, \tilde{W}). \) We put \( Q_0 = Q_0 \) and \( Q_1 \) is the union of three different kinds

- all arrows of \( Q \) not incident to \( k \),
Theorem 4.3. Assume \( J \) and \( J \) that trivial if

Corollary 4.4. By Lemma 4.2, the above (4.1) all arrows starting (resp. outgoing arrow b) at k.

The new potential is given by

\[
\tilde{W} := [W] + \sum_{ha = tb = k} b^* a^*[ab],
\]

where \([W]\) is obtained by substituting \([ab]\) for each words \(ab\) occurring in \(W\).

Let \( A \) be the algebra \( J(Q,W;C) \) and \( T \) be the representation \( kA/P_k \oplus T_k \), where \( T_k := \tau^{-1}S_k \). Clearly, \( T_k \) can be presented as \( P_k \frac{(a)_{a}}{ha = k} \bigoplus P_a \rightarrow T_k \rightarrow 0 \).

Lemma 4.1. \([17] \) Corollary 2.2.b] \( T \) is a tilting module iff. \( \alpha := (a)_a \) is injective.

In this case, \( T \) is called the BB-tilting module at \( k \). The dual notion of \( T \) is the BB-cotilting module \( T^\vee = kA^*/I_k \oplus \tau S_k \). What we desire is the following nice situation:

\[
\text{\textcircled{\ominus} There is an algebraic cut} \tilde{C} \text{ on} (\tilde{Q},\tilde{W}) \text{ such that} J(Q,W;C) \text{ and} J(\tilde{Q},\tilde{W};\tilde{C}) \text{ are tilting equivalent via the functor} \text{Hom}_A(T,\cdot) \text{ or} \text{Hom}_A(\cdot,T^\vee).}
\]

In general, the existence of another cut on \( W \) is not guaranteed. However, if we assume that

\[ (4.1) \quad \text{all arrows ending in} \ k \text{ do not belong to a cut} \ C, \]

then we can assign a new cut \( \tilde{C} \) containing all

- \( c \in C \) if \( tc \neq k \),
- arrows \( b^* \) if \( b \notin C \),
- composite arrows \( [ab] \) with \( b \in C \).

This definition is related to the graded right mutation defined in \([1]\). There is a graded version of splitting theorem (\([7] \) Theorem 4.6). Applied to QP with a cut, we have

Lemma 4.2. \( (Q,W,C) \) is graded right-equivalent to the direct sum \( (Q_{\text{red}},W_{\text{red}},C_{\text{red}}) \oplus (Q_{\text{triv}},W_{\text{triv}},C_{\text{triv}}) \), where \( (Q_{\text{red}},W_{\text{red}},C_{\text{red}}) \) is reduced and \( (Q_{\text{triv}},W_{\text{triv}},C_{\text{triv}}) \) is trivial, both unique up to graded right-equivalence.

We denote the reduced part of \((\tilde{Q},\tilde{W},\tilde{C})\) by \( \mu_k(Q,W,C) := (Q',W',C') \).

Theorem 4.3. Assume \( C \) is an cut satisfying Definition \([22] \) (2) and \((4.1)\), and that \( \text{Ext}^3_A(S_i,S_k) = 0 \) for any \( i \neq k \). Then \( J(Q,W;C) \) is tilting equivalent to \( J(\tilde{Q},\tilde{W};\tilde{C}) \) via \( \text{Hom}_A(T,\cdot) \).

Corollary 4.4. If \( C \) is an algebraic cut satisfying \((4.1)\), then \( C' \) is also algebraic, and \( J(Q,W;C) \) is tilting equivalent to \( J(\tilde{Q},\tilde{W};\tilde{C}) \) via \( \text{Hom}_A(T,\cdot) \).

These slightly generalize the main results of \([18]\). We will prove them in the appendix. By Lemma \([42]\) the above \( J(\tilde{Q},\tilde{W};\tilde{C}) \) can be replaced by \( J(Q',W';C') \).

If we want to work with the assumption dual to \((4.1)\), that is, all arrows starting with \( k \) do not belong to \( C \), then we should take the functor \( \text{Hom}_A(\cdot,T^\vee) \).
The equivalence $\text{Hom}_A(T, -)$ induces a map $\phi_k$ in the corresponding $K_0$-group

$$\phi_k([S_i]) = \begin{cases} [S'_i] & i \neq k, \\ -[S'_k] + \sum_{a=b=k} [S'_{ka}] & i = k; \end{cases}$$

and its dual $\text{Hom}_A(-, T')$ induces $\phi_k'$ given by

$$\phi_k'([S_i]) = \begin{cases} [S'_i] & i \neq k, \\ -[S'_{ki}] + \sum_{b=k} [S'_{hb}] & i = k. \end{cases}$$

By slight abuse of notation, we also write $\phi_k$ and $\phi_k'$ for the corresponding linear isometries on $Z^{Q_k}$. Due to the equivalence, we have that $(\alpha, \beta)_{J_C} = (\phi_k \alpha, \phi_k \beta)_{J_C'}$. Moreover, it is easy to verify that $(\alpha, \beta) = (\phi_k \alpha, \phi_k \beta)'$, or equivalently, $B' = \phi_k B \phi_k^T$, where $(-, -)'$ is the antisymmetric form of $Q'$.

We denote

$$\text{mod}(A)_{k} := \{ M \in \text{mod} A \mid \text{Hom}_A(S_k, M) = 0 \},$$
$$\text{mod}(A)^k := \{ M \in \text{mod} A \mid \text{Hom}_A(M, S_k) = 0 \}. $$

Note that under the assumption $\ominus$, $\text{mod}(J(Q, W; C))^k$ (resp. $\text{mod}(J(Q', W'; C'))_k$) is the torsion (resp. torsion-free) class determined by the tilting module $T$. So

$$\text{mod}(J(Q, W; C))^k \cong \text{mod}(J(Q, W; C))_k. $$

In particular, for $\alpha' = \phi_k(\alpha)$ we have that

$$\frac{\left| \text{Rep}_\alpha(J(Q, W; C))^k \right|}{\left| \text{GL}_\alpha \right|} = \frac{\left| \text{Rep}_{\alpha'}(J(Q', W'; C'))_k \right|}{\left| \text{GL}_{\alpha'} \right|}.$$  

5. Wall-crossing Formula

**Definition 5.1.** We denote $E_k := \int_\omega \chi((S_k)) = \sum_{\alpha} \frac{q^{\alpha^2/2}}{\left| \text{GL}_\alpha \right|} x_\alpha^n = \exp_q \left( \frac{q^{1/2}}{1-q} x_k \right)$. Let $V := V(Q, W) \in X(Q)$, and $V' := V(\mu_k(Q, W)) \in X(Q')$. Let $\Phi_k$ (resp. $\Phi_k'$) be the ring homomorphism $X(Q) \to X(Q')$ defined by $x^\alpha \mapsto (x')^{\phi_k \alpha}$ (resp. $(x')^{\phi_k' \alpha}$).

**Theorem 5.2.** Assuming the condition $\ominus$, we have that

$$E_k' \Phi_k'(\forall E_k^{-1}) = V' = \Phi_k(E_k^{-1} \forall V)E_k'. $$

**Proof.** We apply the character $\int_\omega$ to the torsion-pair identity in $H(Q)$

$$\chi(\text{mod}(Q)_{k}) \chi((S_k)) = \chi = \chi((S_k)) \chi(\text{mod}(Q)^k).$$

We get

$$E_k^{-1} V = \int_\omega \chi(\text{mod}(Q)^k)$$

$$= \sum_{\alpha} \langle \alpha, \alpha \rangle_{J_C} \frac{\left| \text{Rep}_\alpha(J(Q, W; C))^k \right|}{\left| \text{GL}_\alpha \right|} x_\alpha^n, \quad \text{(similar to Lemma 3.7)}$$

$$= \sum_{\alpha} \langle \phi_k(\alpha), \phi_k(\alpha) \rangle_{J_C} \frac{\left| \text{Rep}_{\phi_k(\alpha)}(J(Q', W'; C'))_k \right|}{\left| \text{GL}_{\phi_k(\alpha)} \right|} x_\alpha^n, \quad \text{(4.3)}$$
Similarly
\[ \mathbb{V}'E_k^{-1} = \sum_{\alpha} \langle \alpha, \alpha \rangle_{J_C} \frac{|\text{Rep}_\alpha(J(Q', W'; C'))_k|}{|\text{GL}_\alpha|} x^\alpha, \]
Hence
\[ \mathbb{V}' = \Phi_k(E_k^{-1} \mathbb{V})E_k'. \]

The other half is similar. □

**Framing.** We freeze a vertex \( \infty \) of \( Q \), that is, we do not allow to mutate at \( \infty \). Let \( \text{mod}_0(Q) \) be all modules supported outside \( \infty \). Note that \( \text{mod}_0(Q) \) is an exact subcategory of \( \text{mod}(Q) \). In particular, it is a torsion-free class, and let \( T_0(Q) \) be its corresponding torsion class.

Let \( T := \int_\omega \chi(T_0(Q)) \) and \( \mathbb{V}_0 := \int_\omega \chi(\text{mod}_0(Q)) \). It follows from the torsion pair identity that
\[ T' = (E_k')^{-1} \Phi_k(T)E_k'. \]

Using another relation, we obtain that
\[ T' = \Phi_k'(E_kTE_k^{-1}). \]

We can also treat \( \text{mod}_0(Q) \) as a torsion class, and work with its torsion-free class \( F_0 \). If we set \( F = \int_\omega F_0 \), then \( F = \mathbb{V}T_0^{-1} \), and we have the dual formula
\[ E_k' \Phi_k'(F)(E_k')^{-1} = F' = \Phi_k'(E_k^{-1} F_k). \]

Consider the subcategory of \( T_0^1(Q) \) of \( T_0(Q) \), which contains all representations having dimension one at the vertex \( \infty \). It is well-known that the class \( T_0^1(Q) \) are determined by the framing stability \( \mu_\infty \). Let
\[ T(Q, W) := (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \int_\omega \chi(T_0^1(Q)). \]

Since the dimension vector \((1, \beta)\) is coprime to the slope function \( \mu_\infty \), the moduli space \( \text{Mod}^{\mu_\infty}_{(1, \beta)}(Q) \) is a geometric quotient, and thus we have
\[ T(Q, W) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \left| \frac{\varphi_\omega(\text{Rep}^{\mu_\infty}_{(1, \beta)}(Q))_{\text{vir}}}{|\text{GL}_{(1, \beta)}|_{\text{vir}} x^{(1, \beta)}} \right. \]
\[ = \sum_{\beta} \left| \frac{\varphi_\omega(\text{Mod}^{\mu_\infty}_{(1, \beta)}(Q))_{\text{vir}}}{x^{(1, \beta)}} \right. . \]

Clearly, we also have that
Theorem 5.3.  

\[
(5.1) \quad (E'_k)^{-1} \Phi_k(T)E'_k = T' = \Phi'_k(E_kTE_k^{-1}).
\]

In the next section, by abuse of notation, we will write $\mu_k$ for the operator $\text{Ad}^{-1}(E'_k) \circ \Phi_k$, and $\mu'_k$ for the operator $\Phi'_k \circ \text{Ad}(E_k)$.

6. Application to Cluster Algebras

Let $\Lambda$ and $B_p$ be two skew-symmetric matrices of size $m \times m$ and $n \times n$ ($m \geq n$). We extend $B_p$ from the right to an $n \times m$ matrix $B$. We assume $\Lambda$ and $B$ are unitally compatible, that is, $BA = [-I_n, 0]$.

We can associate $B$ a quiver $Q$ without loops and 2-cycles satisfying (1.1). The vertices $n + 1 \leq v \leq m$ are frozen vertices. We denote by $Q_p$ the principal part of $Q$, that is, the subquiver of $Q$ by forgetting all frozen vertices.

Let $X_\Lambda$ be the quantum Laurent polynomial ring in $m$ variables with $x^\alpha x^\beta = q^{\frac{1}{2} \Lambda(\alpha, \beta)} x^{\alpha + \beta}$. It is contained in its skew-field of fractions $F$.

Definition 6.1. A toric frame is a map $X: \mathbb{Z}_m^m \to F$, such that $X(c) = \rho(x^{\eta(c)})$ for some automorphism $\rho$ of the skew-field $F$, and some automorphism $\eta$ of the lattice $\mathbb{Z}_m$.

Let $\{e_i\}_{1 \leq i \leq m}$ be the standard basis of $\mathbb{Z}_m$. We also denote by $\phi_k$ the matrix of the linear isometry $[1,2]$, and by $\phi_k^p$ its restriction on the principal part $Q_p$. For any integer $b$, we write $[b]_+$ for $\max(0, b)$.

Definition 6.2. A seed is a triple $(\Lambda, B, X)$ such that $X(g)X(h) = q^{\frac{1}{2} \Lambda(\alpha, \beta)} X(g+h)$ for all $g, h \in \mathbb{Z}_m^m$. The mutation of $(\Lambda, B, X)$ at $k$ is a new triple $(\Lambda', B', X') = \mu_k(\Lambda, B, X)$ defined by

\[
(6.1) \quad (\Lambda', B') = (\phi_k^T \Lambda \phi_k, \phi_k^p B \phi_k^T),
\]

and $X'$ is determined by the following exchange relation

\[
(6.2) \quad X'(e_k) = X(\sum 1 \leq j \leq m [b_{jk}]_+ e_j - e_k) + X(\sum 1 \leq j \leq m [b_{kj}]_+ e_j - e_k),
\]

\[
(6.3) \quad X'(e_j) = X(e_j) \quad 1 \leq j \leq m, j \neq k.
\]

Since $\phi_k = \phi_k^{-1}$, we see that $(\Lambda', B')$ is also unitally compatible. It was constructed explicitly in [3] Proposition 4.2 the automorphism $\rho$ for $X'$. One should notice that the mutation $\mu_k$ is an involution.

Let $T_n$ be the $n$-regular tree with root $t_0$. There is a unique way of associating a seed $(\Lambda_t, B_t, X_t)$ for each vertex $t \in T_n$ such that

1. $(\Lambda_{t_0}, B_{t_0}, X_{t_0}) = (\Lambda, B, X_\Lambda)$.
2. if $t$ and $t'$ are linked by an edge $k$, then the seed $(\Lambda_{t'}, B_{t'}, X_{t'})$ is obtained from $(\Lambda_t, B_t, X_t)$ by the mutation at $k$.

Definition 6.3. The quantum cluster algebra $C(\Lambda, B)$ with initial seeds $(\Lambda, B, X_\Lambda)$ is the $k$-subalgebra of $F$ generated by all cluster variables $X_t(e_i)$ ($0 \leq i \leq n$), coefficients $X_t(e_i)$ and $X_t(e_i)^{-1}$ ($n + 1 \leq i \leq m$).
Recall the operators \( \mu_k = \text{Ad}^{-1}(E_k') \circ \Phi_k \) and \( \mu_k' = \Phi_k' \circ \text{Ad}(E_k) \). Here \( E(y) = \exp_{\frac{q}{2}}(y) \) can also be written as the formal product

\[
\prod_{i=0}^{\infty} (1 + q^{\frac{i}{2}} y)^{-1}.
\]

So its inverse is the \( q \)-Pochhammer symbol \((\frac{q}{2}y; q)_\infty\). It satisfies

\[
E(q^{-1} y) = (1 + q^{\frac{i}{2}} y^{\pm 1} E(y)
\]

from the fact that \( y_i E(y_k) = E(q^{b_{ik}} y_k)y_i \) and \( E(y_k)^{-1} E(q^y y_k) = \prod_{i=1}^{b} (1 + q^{\frac{i}{2}} y_k) \), we can easily deduce the following \( Y \)-seeds mutation formula

**Lemma 6.4.** \([14, (4.11)]\)

\[
(6.4) \quad \mu_k'(y_i) = \mu_k(y_i) = \begin{cases} 
\frac{y_k^{-1}}{y^{a_i + \frac{1}{2}} y_k} & \quad (i = k), \\
\prod_{i=1}^{b} (1 + q^{\text{sgn}(b_{ik})(i - \frac{1}{2}) y_k})^{\text{sgn}(b_{ik})} & \quad \text{otherwise}.
\end{cases}
\]

We consider the lattice map \( \mathbb{Z}^n \to \mathbb{Z}^m, \beta \mapsto \beta B \). This map induces a operator \( b : Y_{(Q_k)} \to X_{\Lambda}, y^\beta \mapsto x^\beta B \). By the unital compatibility of \( \Lambda \) and \( B \), we have that \( \alpha B \beta^T = \Lambda(\alpha B, \beta B) \). So we conclude that

**Lemma 6.5.** The operator \( b \) is an algebra homomorphism.

Let \( k_s := (k_1, k_2, \ldots, k_s) \) be a sequence of edges connecting \( t_0 \) and \( t_s \). We write \( \mu_{k_s} \) for the sequence of mutation \( \mu_{k_0} \cdots \mu_{k_{s-1}} \mu_{k_s} \). For simplicity, we write \( B_t \) for \( B_{t_s} \) and \( X_t \) for \( X_{t_s} \). The next lemma says that the operator \( b \) is compatible with mutations.

**Lemma 6.6.** \( b \circ \mu_{k_s}^{-1}(y^\beta) = X_s(\beta B_s) \) for any \( \beta \in \mathbb{Z}^n \).

**Proof.** Using the unitaly compatibility of \( \Lambda \) and \( B \), this is clearly reduced to prove for \( \beta = e_s \). We prove by induction on \( s \). For \( s = 0 \), it is trivial. Suppose that it is true for \( s \), then

\[
b \circ \mu_{k_{s+1}}^{-1}(y^{e_{s+1}}) = b \circ \mu_{k_{s}}^{-1}(\mu_{k_{s+1}}^{-1}(y^{e_{s+1}})),
\]

\[
= b \circ \mu_{k_{s}}^{-1}(y^{e_{s+1} + [b_{k_{s+1}}] + e_{k_{s}}}) \prod_{i=1}^{b_{k_{s+1}}+1} (1 + q^{\text{sgn}(b_{ik_{s+1}})(i - \frac{1}{2}) y_k})^{\text{sgn}(b_{ik_{s+1}})},
\]

\[
= X_s(e_s B_s + [b_{k_{s+1}}] + e_k B_s) Y_{s+1} \prod_{i=1}^{b_{k_{s+1}}} (1 + q^{\text{sgn}(b_{ik_{s+1}})(i - \frac{1}{2}) X_s(e_k B_s)})^{\text{sgn}(b_{ik_{s+1}})}.
\]
On the other hand, 
\[ X_{s+1}(e_t B_{s+1}) = X_{s+1}(\sum_j b_{ij}^{s+1} e_j), \]
\[ = \Lambda_{s+1}(e_t B_{s+1} - b_{ik}^s e_k, b_{ik}^s e_k) X_{s+1}(\sum_{j \neq k} b_{ij}^{s+1} e_j) X_{s+1}(b_{ik}^{s+1} e_k), \]
\[ = X_s(\sum_{j \neq k} b_{ij}^{s+1} e_j) \cdot (X_s(\sum_{1 \leq j \leq m} [b_{ij}^k]_+ + e_j - e_k) + X_s(\sum_{1 \leq j \leq m} [b_{ij}^k]_+ + e_j - e_k)) b_{s+1}^e, \]
\[ = X_s(\sum_{j \neq k} b_{ij}^{s+1} e_j) \cdot (X_s(\sum_{1 \leq j \leq m} [b_{ij}^k]_+ + e_j - e_k)(1 + q^{[e_k B_s]} X_s(e_k B_s))) b_{s+1}^e, \]
\[ = X_s(\sum_{j \neq k} b_{ij}^{s+1} e_j) \cdot (X_s(\sum_{1 \leq j \leq m} [b_{ij}^k]_+ + e_j - e_k)(1 + q^{[e_k B_s]} X_s(e_k B_s))) b_{s+1}^e, \]
\[ = X_s(\sum_{j \neq k} b_{ij}^{s+1} e_j) \cdot (X_s(\sum_{1 \leq j \leq m} [b_{ij}^k]_+ + e_j - e_k)(1 + q^{[e_k B_s]} X_s(e_k B_s))) b_{s+1}^e, \]
\[ = X_s(\sum_{j \neq k} b_{ij}^{s+1} e_j) \cdot (X_s(\sum_{1 \leq j \leq m} [b_{ij}^k]_+ + e_j - e_k)(1 + q^{[e_k B_s]} X_s(e_k B_s))) b_{s+1}^e, \]
\[ = X_s(\sum_{j \neq k} b_{ij}^{s+1} e_j) \cdot (X_s(\sum_{1 \leq j \leq m} [b_{ij}^k]_+ + e_j - e_k)(1 + q^{[e_k B_s]} X_s(e_k B_s))) b_{s+1}^e, \]
\[ = X_s(\sum_{j \neq k} b_{ij}^{s+1} e_j) \cdot (X_s(\sum_{1 \leq j \leq m} [b_{ij}^k]_+ + e_j - e_k)(1 + q^{[e_k B_s]} X_s(e_k B_s))) b_{s+1}^e. \]

For any \( t \in \mathbb{T}_n \), there is a unique sequence of edges \( k_t \) connecting \( t_0 \) and \( t \). Let \( W \) be some non-degenerate (Definition 7.2, Proposition 7.3) potential of \( Q \), and set \( (Q_t, W_t) = \mu_{k_t}(Q, W) \). We shall assume the following condition for \( W \):

For any \( t \in \mathbb{T}_n \), and any \( k \in Q_t \), the assumption \( \otimes \) holds for \( (Q_t, W_t) \).

We do not know if such a potential exists for any \( B \)-matrix.

To give another definition of \( X_t(g) \) for \( g \in \mathbb{Z}_{\geq 0}^m \), we consider the extended QP \( (Q^e_t, W_t) \) from \( (Q_t, W_t) \) by adding a new vertex \( \infty \) and \( y \), new arrows from \( y \) to \( \infty \).

We apply the inverse of \( \mu_{k_t} \) to \( (Q^e_t, W_t) \), and obtain a QP \( (Q^g, W^g) := \mu_{k_t}^{-1}(Q^e_t, W_t) \).

We freeze the same set of vertices of \( Q^g \) as that of \( Q \), and let \( \hat{B} \) be the extended \( B \)-matrix of \( Q^g \). Although the extended vertex \( \infty \) is not frozen, we will never perform mutation at \( \infty \) henceforth.

**Definition 6.7.** For any \( g \in \mathbb{Z}_{\geq 0}^m \), we define \( X_t(g) = b(T(\mu_{k_t}^{-1}(Q^e_t, W_t))) \).

By Theorem 5.3, \( T(\mu_{k_t}^{-1}(Q^e_t, W_t)) = \mu_{k_t}^{-1}((q^{1/2} - q^{-1/2})y_{\infty}^{1/2} - q_{\infty}^{1/2}) = \mu_{k_t}^{-1}(y_{\infty}) \).

**Theorem 6.8.** Definition 6.7 defines the quantum cluster algebra \( \mathcal{C}(\Lambda, B) \). In particular, We have the following quantum cluster character

\[ X_t(g) = \sum_{\beta} |\varphi_{\omega_s}(\text{Mod}_{(1, \beta)}(Q^g))|_{\text{vir}} x^{(1, \beta)B}. \]

**Proof.** We trivially extend \( \Lambda \) to \( \hat{\Lambda} := \begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix} \), and thus we have the natural embedding \( X_\Lambda \hookrightarrow X_{\hat{\Lambda}} \). Note that \( \hat{B} \hat{\Lambda} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{B} \end{pmatrix} \), so they are not unitally compatible. We say that they are unitally compatible on the principal part. However, it still makes perfect sense if we define \( \hat{B}' \hat{\Lambda}' \) by \( \begin{pmatrix} 0 & 0 \\ 0 & \hat{B}' \end{pmatrix} \) and \( \hat{X}'(e_i) \) by the relation \( \begin{pmatrix} 0 & 1 \\ 0 & \hat{B}' \end{pmatrix} \). Clearly, for any \( k \neq \infty \), \( (\hat{B}', \hat{\Lambda}') \) is also unitally compatible on the principal part.
This is all we need for the following analogue (6.5) of Lemma 6.6 holds: For each \( t \in \mathcal{T}_n \), we associate as before \( \tilde{X}_t \), then
\[
(6.5) \quad b \circ \mu_k^{-1}(y^\beta) = \tilde{X}_t(\beta \tilde{B}_t).
\]
Moreover, \( \tilde{\Lambda}' \) extends \( \Lambda' \) in the same way: \( \tilde{\Lambda}' := \begin{pmatrix} 0 & 0 \\ 0 & \Lambda' \end{pmatrix} \) so that we have the natural embedding \( X_t(Z^m) \hookrightarrow \tilde{X}_t(Z^{m+1}) \) for each \( t \in \mathcal{T}_n \). Hence,
\[
\quad b \circ \mu_k^{-1}(y_{\infty}) = \tilde{X}_t(e_{\infty} \tilde{B}_t) = X_t(g).
\]
\[\square\]

Remark 6.9. We can view \((1, \beta) \tilde{B}\) as \( \beta B - g \), where \( g \) is the extended (dual) \( g \)-vector corresponding to the mutated cluster monomial.

Example 6.10. Consider quiver

\[
\begin{array}{c}
1 \\
\downarrow \downarrow \\
1 & \downarrow \\
\downarrow & \downarrow \\
1 & \downarrow \\
\end{array}
\]

with potential \( abc \). We perform a sequence of mutations \( \{1, 2, 3, 1\} \), and obtain the quiver

\[
\begin{array}{c}
1 \\
\downarrow \downarrow \\
1 & \downarrow \\
\downarrow & \downarrow \\
1 & \downarrow \\
\end{array}
\]

with the same potential. We choose \( c \) as the cut. It is easy to count the vanishing cycles for each dimension vector. For example, for \( \beta = (1, 1, 1) \),
\[
|\varphi_{\omega s}(\text{Mod}^{\infty}(Q^g)))| = q(2q + 2).
\]

Note that \( 2q + 2 \) counts neither the representation Grassmannian of \( P_1 \oplus P_2 \oplus P_3 \) of the algebra

\[
\begin{array}{c}
1 \\
\downarrow \downarrow \\
1 & \downarrow \\
\downarrow & \downarrow \\
1 & \downarrow \\
\end{array}
\]

nor that of the algebra

\[
\begin{array}{c}
1 \\
\downarrow \downarrow \\
1 & \downarrow \\
\downarrow & \downarrow \\
1 & \downarrow \\
\end{array}
\]

In particular, the condition ‘numb’ cannot be removed from Lemma 6.7.

7. Application: Representation Grassmannians and Reflections

Let \( s \) be a sink of \( Q \), and \( M \) be a representation of \( Q \). We assume that \( M \) does not contain the simple representation \( S_s \) as a direct summand. Let
\[
T(M) := \sum q^{-\frac{1}{2}(M^{-\beta};\beta)}|\text{Gr}^\beta(M)|_{x^{(1,\beta)}}.
\]

We want to compare \( T(M) \) with \( T(\mu_s(M)) \).
We first consider the extension $A = kQ[M] := \left( \frac{kQ}{M} \right)$ of $Q$ by $M$. This is an algebra of global dimension two, so we can complete it to a QP $(Q, W)$ with a cut $C$ such that $J(Q, W; C) = A$. We freeze the extended vertex $\infty$ of $A$, then

**Lemma 7.1.** $T(Q, W) = T(M)$.

**Proof.** Since all arrows in $C$ end in $\infty$, by Lemma 3.7

$$T(Q, W) = \int q \chi(T_0^1(Q)) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{\alpha} q^{\frac{1}{2}} q^{-\frac{1}{2}} (M - \beta, \beta) |\text{Rep}_{(1, \beta)}(Q[M])| |GL_{(1, \beta)}| x^{(1, \beta)},$$

where $\langle - , - \rangle_{Q[M]}$ is the Euler form of $kQ[M]$, and $\mu_{\infty}$ is the framing stability. $	ext{Rep}_{(1, \beta)}(Q[M])$ can be identified with

$$\{(N, f) \in \text{Rep}_{\beta}(Q) \times \text{Hom}(M, \kappa^\beta) \mid f \in \text{Hom}_Q(M, N) \text{ is surjective}\}.$$ 

So the quotient $\text{Rep}_{(1, \beta)}(Q[M]) / \text{GL}_\beta$ is the representation Grassmannian $Gr^\beta(M)$, and thus

$$T(Q, W) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{\beta} q^{\frac{1}{2}} q^{-\frac{1}{2}} (M - \beta, \beta) |\text{Rep}_{(1, \beta)}(Q[M])| |GL_{(1, \beta)}| x^{(1, \beta)},$$

$$= \sum_{\beta} q^{-\frac{1}{2}} (M - \beta, \beta) |\text{Gr}^\beta(M)| x^{(1, \beta)}.$$ 

**Remark 7.2.** The analogous statement does not hold for quivers with potentials in general (see Example 6.10).

For any $s \in Q_0$ (not necessary a sink), the cut $C$ of $Q$ satisfies the conditions in Corollary 7.3, so we get another algebra $A' = \text{End}_A(T) = J(Q, W'; C')$.

**Lemma 7.3.** The algebra $A'$ is extended from $Q' := \mu_s(Q)$ by $M' := \mu_s(M)$.

**Proof.** Let $\infty$ be the extended vertex of $Q[M]$, and $P_\infty := kA/P_\infty$. Then

$$\text{End}_A(P_\infty) = kQ$$

and $\text{Hom}_A(P_\infty, P_\infty) = M$. Since there is no incoming arrows to $\infty$ for $Q[M]$, we have that $P_\infty \cong kQ$. Let $T = kA/P_\infty$, be the BB-tilting module of $A$ at $s$, and $T_\infty : = T/P_\infty$. We need to show that

$$\text{End}_A(T_\infty) = k\mu_s(Q)$$

and $\text{Hom}_A(T_\infty, P_\infty) = \mu_s(M)$.

Let $T' = kQ/P_s \oplus T_s'$ be the BB-tilting module of $kQ$ at $s$. Since $M$ does not contain $S_s$ as a direct summand the quiver of $Q[M]$ has no arrow from $\infty$ to $s$. So $T_s = T'_s$, and thus $T_\infty = T'$. Hence, $\text{End}_A(T_\infty) = k\mu_s(Q)$. For the second one, we consider

$$\mu_s(M) = \text{Hom}_Q(T', \text{Hom}_A(P_\infty, P_\infty)),$$

$$= \text{Hom}_A(T' \otimes_{kQ} P_\infty, P_\infty),$$

$$= \text{Hom}_A(T_\infty, P_\infty).$$

**Theorem 7.4.** $T(M)$ and $T(M')$ are related via (8.1). In particular, if $M$ is polynomial-count, that is, all its Grassmannians $Gr^\beta(M)$ are polynomial-count, then so are all reflection equivalent classes of $M$. 

□
Example 7.5. Consider the following quiver

\[
\begin{array}{c}
2 \\
\downarrow \\
1 \\
\downarrow \\
3
\end{array}
\]

with potential \( W = \sum_{i=1}^{3} \sum_{j=1}^{3} (-1)^{ij} a_{ij} b_{jk} c_{ki} \) and cut \( C = \{ c_1, c_2, c_3 \} \). Then the algebra \( J(Q, W; C) \) is extended from the quiver \( 2 \xrightarrow{b_1,b_2,b_3} 3 \) by a representation \( M \) of dimension \((3,6)\) (see \[10\] Example 7.8). To compute \( T(M) \), we can first compute \( T(\mu_3(M)) \), where \( \mu_3(M) \) can be presented by the following base diagram.

The black (resp. white) dots are a basis at vertex 3 (resp. vertex 2); The letter on an arrow represents the identity map on the arrow of the same letter.

\[
T(\mu_3(M)) = 1 + x^{(1,0,3)} + x^{(1,3,3)} + 3(x^{(1,0,1)} + x^{(1,0,2)} + x^{(1,1,3)} + x^{(1,2,3)}) + 3q^2 x^{(1,1,2)}.
\]

Here \( [n] \) is the quantum number \( \frac{q^n - 1}{q - 1} \). Using Theorem 4.3 we find that

\[
T(M) = 1 + x^{(1,3,0)} + x^{(1,3,6)} + 3(x^{(1,1,0)} + x^{(1,2,0)} + x^{(1,2,3)}) + 3q^2 x^{(1,1,1)}
\]

\[
+ 3[3][3](x^{(1,2,1)} + x^{(1,3,2)}) + (q^2 + q^3)[3](x^{(1,3,1)} + x^{(1,3,3)})
\]

\[
+ (q - 1 + q^{-1})[3][5](x^{(1,3,2)} + x^{(1,3,4)}) + (q - 1 + q^{-1})[5][5] x^{(1,3,3)}.
\]

Employing the methods developed in \[10\], we can compute \( |\varphi_\omega(\text{Mod}^n_\omega(Q))| \) and \( |\text{Mod}^n_\omega(J(Q, W; C))| \) for all \( \alpha \) with \( \alpha_3 = 1 \) and generic \( \mu \).

8. APPENDIX: PROOF OF THEOREM 4.3

Theorem 4.3 generalizes the main result of \[15\] from APR-tilting modules to BB-tilting modules. We slightly simplify its proof as well.

Lemma 8.1. \[4\] Proposition 3.3] Let \( Q \) be a finite quiver and \( A \) be a finite dimensional basic algebra. Let \( R \) be a set of relations, then the following are equivalent

1. \( A \) can be presented as \( \bar{k}Q/(R) \).
2. There is an algebra homomorphism \( \pi : \bar{k}Q \rightarrow A \) such that the sequence

\[
\bigoplus_{tr=i} \pi(e_{hr}) A \xrightarrow{\pi(a_{-1} r)} \bigoplus_{ta=i} \pi(e_{ha}) A \xrightarrow{\pi(a)} \text{rad}(\pi(e_i) A) \rightarrow 0
\]

is exact.

We set \( P_{in} = \bigoplus_{ha=k} P_{ta} \) and \( P_{out} = \bigoplus_{tb=k} P_{hb} \). Recall that the BB-tilting module \( T_k := \tau^{-1} S_k \) can be presented as

\[
0 \rightarrow P_k \xrightarrow{\alpha} P_{in} \xrightarrow{g} T_k \rightarrow 0,
\]

where \( \alpha := (a)_a \) and \( g := a(g_a) \).
Using the presentation $\widehat{kQ_C}/(\partial_C W)$ of $J(Q, W; C)$. We have for $i \neq k$

\begin{equation}
\cdots \to P' \to \bigoplus_{h = i, c \in C} P_{tc} \xrightarrow{\partial_{cb}} \bigoplus_{tb = i} P_{hb} \xrightarrow{\beta} P_i \to S_i \to 0,
\end{equation}

where $\beta := b(b)$ and $\partial_{cb} := c(\partial_c \partial_b W)_b$. Since the cut $C$ satisfies Definition 2.3(2), the first three terms are part of the minimal projective resolution of $S_i$. We assume that the projective $P'$ is minimal as well.

This fits in the following commutative diagram

\[0 \to c_{ki} P_k \xrightarrow{\partial \alpha} c_{kT} \xrightarrow{g} c_{kT} \to 0\]

\[P' \xrightarrow{\iota} P \xrightarrow{\partial_{cb}} P_{hb} \xrightarrow{\beta} P_i \to S_i\]

Here $c_{ki} = |C \cap Q(k, i)|$, and the first row is a direct sum of $c_{ki}$ copies of $\ell_{ai}$. The map $\iota$ is the natural embedding, the map $\partial_{cb}$ is given by the matrix $a_{c,b}(\partial_a \partial_b W)_b$, and $f$ is induced from $\partial_{acb}$. We then take the mapping cone of the above diagram, and cancel out the last term $c_{kT}$. We end up with

\begin{equation}
P' \xrightarrow{h} c_{ki} P_{in} \bigoplus_{h = i, c \neq k} P_{tc} \xrightarrow{g' := (\beta \circ a \partial_{acb})} c_{ki} T_k \bigoplus_{tb = i} P_{hb} \xrightarrow{f' := (\beta \circ f)} P_i \to S_i\end{equation}

Let $\widehat{\mathcal{Q}_C}$ be the quiver obtained from $\mathcal{Q}$ by forgetting all arrows in $\mathcal{C}$. To apply Lemma 8.1, we construct an algebra homomorphism $\pi : \widehat{\mathcal{Q}_C} \to \operatorname{End}_A(T)$ as follows. For any direct summands $T_i, T_j$ of $T$, we will view $\operatorname{Hom}_A(T_i, T_j)$ under the natural embedding into $\operatorname{Hom}_A(T, T)$. Let $\operatorname{id}_i$ be the identity map in $\operatorname{Hom}_A(T_i, T_i)$.

We define

1. $\pi(e_i) = \operatorname{id}_i$,
2. $\pi(a) = a \in \operatorname{Hom}_A(P_i, P_j)$ for $i, j \neq k$,
3. $\pi(a^*) = g_a \in \operatorname{Hom}_A(P_{ta}, T_k)$,
   \[\pi(b^*) = -f_c \in \operatorname{Hom}_A(T_k, P_{hc})\text{ for } b \in C,
   \pi([ab]) = ab \in \operatorname{Hom}_A(P_{ha}, P_{hb})\text{ for } b \notin C.

Recall that $\mathcal{W} := [W] + \sum_{ha = tb = k} b^* a^* [ab]$, and $\mathcal{C}$ contains all

- $c \in C$ if $tc \neq k$,
- arrows $b^*$ if $b \notin C$,
- composite arrows $[ab]$ with $b \in C$.

So the corresponding relations $\partial_C \mathcal{W}$ are given by

- $R_0 = \{\partial_c [W]\}_{tc \neq k}$,
- $R_1 = \{a^*[ab]\}_{b \notin C}$,
- $R_2 = \{\partial_{bc} W + b^* a^*\}_{b \in C}$.

Hence, Theorem 4.3 is the consequence of the following two lemmas.

**Lemma 8.2.** We have the following exact sequence

\[\operatorname{Hom}_A(T, P_{out}) \xrightarrow{r_1} \operatorname{Hom}_A(T, P_{in}) \xrightarrow{g} \operatorname{rad}(\operatorname{Hom}_A(T, T_k)) \to 0,\]

where $r_1$ is the matrix $b \{ba\}_a$.  

Proof. We apply $\text{Hom}_A(T, -)$ to the exact sequence (8.1), and get

$$0 \to \text{Hom}_A(T, P_k) \xrightarrow{\alpha} \text{Hom}_A(T, P_{m}) \xrightarrow{\alpha g} \text{Hom}_A(T, T_k) \to \text{Ext}^1_A(T, P_k) \to \text{Ext}^1_A(T, P_{m}).$$

The last term $\text{Ext}^1_A(T, P_m)$ vanishes because the first map below is surjective

$$\text{Hom}_A(P_m, P_m) \to \text{Hom}_A(P_k, P_m) \to \text{Ext}^1_A(T_k, P_m) \to 0.$$

Next, $\text{Ext}^1_A(T, P_k)$ is one-dimensional because of the following exact sequence

$$\text{Hom}_A(P_m, P_k) \to \text{Hom}_A(P_k, P_k) \to \text{Ext}^1_A(T_k, P_k) \to 0.$$

Finally, we claim the image of $\text{Ext}^1_A(T, P_m)$ has no cokernel of $\text{Hom}_A(T, \text{out})$. It suffices to show that $\text{Hom}_A(T, P_{out}) \xrightarrow{\alpha \beta} \text{Hom}_A(T, P_k)$ is surjective. But the cokernel of $\alpha \beta$ is $\text{Hom}_A(T, S_k) = 0$.

Applying $\text{Hom}_A(T, -)$ to (8.3), we get the complex

$$\text{Hom}_A(T, c_{ki} P_m \oplus \bigoplus_{hc=i, tc \neq k} P_{tc}) \xrightarrow{\alpha g'} \text{Hom}_A(T, c_{ki} T_k \oplus \bigoplus_{tb=i} P_{hb}) \xrightarrow{\varphi f'} \text{Hom}_A(T, P_i) \to \text{Hom}_A(T, S_i)$$

Lemma 8.3. This complex is exact and induces

$$\text{Hom}_A(T, c_{ki} P_m \oplus \bigoplus_{hc=i, tc \neq k} P_{tc}) \to \text{Hom}_A(T, c_{ki} T_k \oplus \bigoplus_{tb=i} P_{hb}) \to \text{rad}(\text{Hom}_A(T, P_i)).$$

Proof. We first show that the complex is exact at $\text{Hom}_A(T, c_{ki} T_k \oplus \bigoplus_{tb=i} P_{hb})$. We apply $\text{Hom}_A(T, -)$ to the exact sequence

$$0 \to \text{Im} h \to c_{ki} P_m \oplus \bigoplus_{hc=i} P_{tc} \to \text{Im} g' \to 0,$$

and get

$$\text{Hom}_A(T, c_{ki} P_m \oplus \bigoplus_{hc=i, tc \neq k} P_{tc}) \to \text{Hom}_A(T, \text{Im} g') \to \text{Ext}^1_A(T, \text{Im} h).$$

If $\varphi \in \text{Hom}_A(T, c_{ki} T_k \oplus \bigoplus_{tb=i} P_{hb})$ such that $\varphi f' = 0$, then $\varphi(T) \subseteq \text{Im} g'$. So it suffices to show that $\text{Ext}^1_A(T, \text{Im} h) = 0$. The condition $\text{Ext}^3_A(S_i, S_k)$ implies that $P'$ has no $P_k$ as its summands. So $\text{Ext}^1_A(T, P') = 0$, and thus

$$0 \to \text{Ext}^1_A(T, \text{Im} h) \to \text{Ext}^2_A(T, \text{Ker} h) = 0.$$

Since $c_{ki} P_m \oplus \bigoplus_{hc=i, tc \neq k} P_{tc}$ has no $P_k$ as its direct summands, for the same reason the complex is exact at $\text{Hom}_A(T, P_i)$.

We remain to show that the cokernel of $\varphi f'$ is one-dimensional. Let $\Omega S_i$ be the first syzygy of $S_i$. We apply $\text{Hom}_A(T, -)$ to

$$0 \to \Omega S_i \xrightarrow{f''} P_i \to S_i \to 0,$$

and obtain

$$\text{Hom}_A(T, \Omega S_i) \xrightarrow{\varphi f''} \text{Hom}_A(T, P_i) \to \text{Hom}_A(T, S_i) \to \text{Ext}^1_A(T, \Omega S_i) \to \text{Ext}^1_A(T, P_i) = 0.$$

Since $\text{Ext}^1_A(T, c_{ki} P_m \oplus \bigoplus_{hc=i, tc \neq k} P_{tc})$ vanishes, the cokernel of $\varphi f'$ is the same as that of $f''$. By applying $\text{Hom}_A(S_i, -)$ to (8.1), we see that

$$\text{Hom}_A(T, S_i) = \text{Hom}_A(T_k, S_i) \oplus k \cong \text{Ext}^1_A(S_i, S_k)^* \oplus k.$$
In the meanwhile,
\[ \text{Ext}_A^1(T, \Omega S_i) = \text{Ext}_A^1(\tau^{-1} S_k, \Omega S_i) = \overline{\text{Hom}}_A(\Omega S_i, S_k)^* = \text{Hom}_A(\Omega S_i, S_k)^*, \]
\[ 0 = \text{Hom}_A(P, S_k) \rightarrow \text{Hom}_A(\Omega S_i, S_k) \rightarrow \text{Ext}_A^1(S_i, S_k) \rightarrow \text{Ext}_A^1(P, S_k) = 0. \]
So
\[ \text{Ext}_A^1(T, \Omega S_i) = \text{Ext}_A^1(S_i, S_k)^*. \]
Together with (8.4), we conclude that the cokernel of \( \circ f' \) is \( k \).

\[ \square \]

Finally, we prove Corollary 4.4.

**Proof.** Since the cut satisfies (4.1), there is no relation starting from \( S_k \). So \( S_k \) has projective dimension one, and we have that \( 0 \rightarrow P_{\text{out}} \rightarrow P_k \rightarrow S_k \rightarrow 0 \). Hence \( \text{Hom}_A(T, P_{\text{out}}) = \text{Hom}_A(T, P_k) \), and the map \( \circ r_1 \) in Lemma 8.2 is in fact injective. Now \( J(\hat{Q}, \hat{W}; C) \) has global dimension 2, so \( P' \) in (8.3) is zero, and thus the map \( \circ g' \) of Lemma 8.3 is injective. We conclude that \( J(\tilde{Q}, \tilde{W}; \tilde{C}) \) has global dimension 2 as well. The two resolutions of Lemma 8.2 and 8.3 also imply that \( \{ \partial_c \tilde{W} \}_{c \in C} \) is a minimal set of generators in \( \langle \partial_c \tilde{W} \rangle \).

\[ \square \]

**Acknowledgement**

The author thanks Mathematical Science Research Institute in Berkeley (MSRI) for its hospitality and support during the research program Cluster Algebras of Fall 2012 when most of results are obtained. He also wants to thank Fan Qin for patiently answering several questions, and Professor Bernhard Keller for his encouragement.

**References**

1. C. Amiot, S. Oppermann, *Cluster equivalence and graded derived equivalence*, arXiv:1003.4916.
2. I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras*, London Mathematical Society Student Texts 65, Cambridge University Press, 2006.
3. K. Behrend, J. Bryan, B. Szendrői, *Motivic degree zero Donaldson-Thomas invariants*, Invent. Math. 192 (2013), no. 1, 111–160.
4. A. B. Buan, O. Iyama, I. Reiten, D. Smith, *Mutation of cluster-tilting objects and potentials*, Amer. J. Math. 133 (2011), no. 4, 835–887.
5. A. Berenstein, A. Zelevinsky, *Quantum cluster algebras*, Adv. Math. 195 (2005), no. 2, 405–455.
6. B. Davison, D. Maulik, J. Schuemann, B. Szendrői, *Purity for graded potentials and cluster positivity*, Preprint.
7. H. Derksen, J. Weyman, A. Zelevinsky, *Quivers with potentials and their representations I. Mutations*, Selecta Math. (N.S.) 14 (2008), no. 1, 59–119.
8. H. Derksen, J. Weyman, A. Zelevinsky, *Quivers with potentials and their representations II. Applications to cluster algebras*, J. Amer. Math. Soc. 23 (2010), No. 3, 749–790.
9. A. I. Efimov, *Quantum cluster variables via vanishing cycles*, arXiv:1112.3601.
10. J. Fei, *Counting using Hall algebras II. Extensions from quivers*, arXiv:1302.1885.
11. T. Hausel, F. Rodriguez-Villegas *Mixed Hodge polynomials of character varieties*, Invent. Math. 174 (2008), no. 3, 555–624.
12. M. Herschend, O. Iyama *Selfinjective quivers with potential and 2-representation-finite algebras*, Compos. Math. 147 (2011), no. 6, 1885–1920.
13. B. Keller *Deformed Calabi-Yau Completions*, J. Reine Angew. Math. 654 (2011), 125–180.
14. B. Keller, *On cluster theory and quantum dilogarithm identities*, Representations of algebras and related topics, 85–116, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011.
15. A.D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) 45 (1994), no. 180, 515–530.
16. M. Kontsevich, Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, arXiv:1006.2706.

17. S. Ladkani, *Perverse equivalences, BB-tilting, mutations and applications*, arXiv:1001.4765.

18. Y. Mizuno, *APR tilting modules and graded quivers with potential*, arXiv:1112.4266.

19. S. Mozgovoy, *On the motivic Donaldson-Thomas invariants of quivers with potentials*, arXiv:1103.2902.

20. K. Nagao, *Wall-crossing of the motivic Donaldson-Thomas invariants*, arXiv:1103.2922.

21. M. Reineke, *The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli*, Invent. Math. 152 (2003), no. 2, 349–368.

22. O. Schiffmann, *Lectures on Hall algebras*, arXiv:0611617.