Dynamical polarizability, screening and plasmons in one, two and three dimensional massive Dirac systems

Anmol Thakur¹, Rashi Sachdeva² and Amit Agarwal¹

¹ Department of Physics, Indian Institute of Technology Kanpur, Kanpur 208016, India
² Quantum Systems Unit, Okinawa Institute of Science and Technology Graduate University, Okinawa 904-0495, Japan

E-mail: amitag@iitk.ac.in

Received 10 November 2016, revised 3 January 2017
Accepted for publication 9 January 2017
Published 31 January 2017

Abstract
We study the density–density response function of a collection of charged massive Dirac particles and present analytical expressions for the dynamical polarization function in one, two and three dimensions. The polarization function is then used to find the dispersion of the plasmon modes, and electrostatic screening of Coulomb interactions within the random phase approximation. We find that for massive Dirac systems, the oscillating screened potential (or density) decays as \( r^{-2} \) and \( r^{-3} \) in two and three dimensions respectively, and as \( r^{-1} \) for one dimensional non-interacting systems. However for massless Dirac systems there is no electrostatic screening or Friedel oscillation in one dimension, and the oscillating screened potential decays as \( r^{-3} \) and \( r^{-4} \), in two and three dimensions respectively. Our analytical results for the polarization function will be useful for exploring the physics of massive and massless Dirac electrons in different experimental systems with varying dimensionality.

Keywords: massive Dirac systems, plasmons, Friedel oscillations, dynamical polarization function

(Some figures may appear in colour only in the online journal)
and validating a similar study for gapless Dirac systems [40]. The aim of the present article is to go beyond the long wavelength approximation for massive Dirac systems to calculate the exact polarization function and use it to study charge screening and plasmons in all three dimensions. This allows us to calculate the exact plasmon dispersion for a given form of interaction, and the scaling of the Friedel oscillations due to a charged impurity at large distances. We find that compared to the plasmon dispersion in the long wavelength limit, the exact plasmon dispersion gets damped generally at a higher wave-vector. Additionally using the exact polarization function in the static limit, we find that the Friedel oscillations in a massive Dirac system decay as \( r^{-2} \) and \( r^{-3} \) in 2d and 3d respectively similar to the case of parabolic dispersion and unlike the massless Dirac case where the corresponding Friedel oscillations decay as \( r^{-3} \) and \( r^{-4} \).

The paper is organized as follows. In section 2 we introduce the general form of the dynamic polarization function for massive and massless Dirac systems, and the random phase approximation (RPA) approach for calculating dispersion of the collective density excitations. In sections 3–5, we present the analytical expressions for the dynamic polarization function of massive Dirac materials, its static limit and Friedel oscillations, charge plasmons, and the gapless limit, in 1d, 2d and 3d respectively. Finally, we discuss the non-relativistic limit of the polarization function in section 6, and summarize our findings in section 7.

2. Polarization function

We consider a massive (gapped) Dirac plasma in 1d, 2d, 3d and analytically calculate the finite frequency and finite wave-vector density–density response function. The polarization function for a massive Dirac material is given by \([33, 39]\)

\[
\Pi(q, \omega) = \frac{g}{L^d} \sum_{\mathbf{k}, \lambda, \lambda'} F_{k, \lambda}(\mathbf{k}, \mathbf{k}') \frac{n_F(\epsilon_{\mathbf{k}}) - n_F(\epsilon_{\mathbf{k}'})}{\hbar \omega + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'} + i\eta},
\]

(1)

where \(L^d\) is the system volume for a d dimensional system, \(\omega\) is the angular frequency, \(q\) is the wave vector, \(\eta \rightarrow 0\), \(g = gg\) is the degeneracy due to spin, and \(\xi \rightarrow 0\) is the degree of freedom. In equation (1), \(\mathbf{k}' = \mathbf{k} + \mathbf{q}, \lambda, \lambda' = \pm 1\) denotes the conduction (particle) and valence (hole) bands, \(E_k = \hbar v_F |\mathbf{k}|^2 + (\Delta/\hbar v_F)^2\) with \(2\Delta\) being the energy gap, \(\eta\) is the width of the distribution \(\mathcal{F}_{k, \lambda}(\mathbf{k}, \mathbf{k}') = 1 + \lambda [\mathbf{k} \cdot \mathbf{k}' + \Delta^2]/(\hbar E_k\epsilon_{\mathbf{k}'}\epsilon_{\mathbf{k}})\) is the overlap function. Note that we have defined \(\xi \equiv x/\hbar v_F\), which will also be used later in the manuscript. The factor \(g = gg\) is the degeneracy factor comprising of the spin degeneracy factor \(2 (=2)\), and the valley (or pseudo spin) degeneracy factor \(g = 2\) for graphene and other Dirac materials with honeycomb lattice structure. Given the general relation \(\Pi(q, -\omega) = \Pi(q, \omega)^*\) as evident from equation (1), and the fact that the polarization function depends only on the absolute value of the Fermi energy \(\mu\), we present results only for \(\mu > 0\) and \(\omega > 0\). Furthermore, we work at zero temperature so that the Fermi functions can be replaced by Heaviside theta functions, i.e. \(n_F(x) = \theta(\mu - x)\).

Depending upon the placement of the Fermi energy \(\mu\), with respect to the energy dispersion, we can split our polarization function into two parts, namely the intrinsic \((\mu < \Delta)\) and extrinsic polarization \((\mu > \Delta)\). More explicitly, equation (1) can be expressed as

\[
\Pi(q, \omega) = -\chi^\perp_{\mu}(q, \omega) + \chi^\parallel_{\mu}(q, \omega),
\]

(2)

where \(\Pi_0(q, \omega) = -\chi^\perp_{\mu}(q, \omega)\) denotes the intrinsic \((\mu < \Delta)\) part of the polarization function, and \(\Pi(q, \omega) = \chi^\parallel_{\mu}(q, \omega) + \chi^\perp_{\mu}(q, \omega)\) denotes the extrinsic part of the polarization function. In equation (2), we have defined

\[
\chi^\pm_{\mu}(q, \omega) = -\frac{g}{(2\pi)^d} \int d^dk \theta(D^2 - \Delta^2 - k^2) \times \left(1 \pm \frac{\mathbf{k} \cdot \mathbf{k}' + \Delta^2}{E_k E_{k'}}\right) \left[\frac{E_k \mp E_{k'}}{(\hbar \omega + i\eta)^2 - (E_k \mp E_{k'})^2}\right] .
\]

(3)

Here the upper and lower signs correspond to intraband and interband transitions respectively and the parameter \(D\) defines the integration limits via the Heaviside \(\theta\) function. To proceed further for calculating the polarization function, we split the three components of the polarization function in equation (2), into the real and imaginary parts using the real line version of the Sokhotski–Plemelj theorem: \(\frac{1}{\pm i\omega} = \mathcal{P}\left(\frac{1}{i\omega}\right) \mp i\pi\delta(y)\) and then do the wave-vector integration.

Once the non-interacting density–density response function is known, the collective density excitations (plasmon modes) are given by the poles of the interacting density–density response function. Within the RPA, the poles of the interacting response function, coincide with the zeros of the complex longitudinal ‘dielectric function’ \(\epsilon(q, \omega)\), which is given by

\[
\epsilon(q, \omega) = 1 - V_q \Pi(q, \omega) = 0,
\]

(4)

where \(V_q\) is the Fourier transform of the relevant bare electron–electron (or Coulomb) interaction, and \(\Pi(q, \omega)\) is the non-interacting polarizability of the system, given by equation (1). The Fourier transform of the unscreened Coulomb interaction \(V(r) = e^2/|qr|\), in the appropriate \(d\)-dimensional space is given by

\[
V_q = \begin{cases} 
\frac{4\pi e^2}{\kappa q^2} & d = 3, \\
\frac{2\pi e^2}{\kappa q} & d = 2, \\
\frac{2\pi e^2}{\kappa} & K_0(qa) & d = 1,
\end{cases}
\]

(5a)

(5b)

(5c)

where \(\kappa\) is the background material dependent dielectric constant, and \(K_0\) denotes the zeroth order modified Bessel function of the second kind. Note that in 1d, the length scale \(\kappa a\) characterizes the lateral confinement size (say radius of the 1d ribbon), and \(V_q \approx -2e^2 \ln(qa)/\kappa\) for \(qa \ll 1\) [41]. We note here that an alternate form of the 1d effective Coulomb potential,
as specified in [2], can also be used for the 1D Coulomb potential, and it also yields the same asymptotic form as equation (5c) in the $q \alpha \ll 1$ limit.

The static limit of the polarization function is also useful for determining the screened Coulomb potential of a charged impurity. Consider a localized charged impurity which has a vacuum potential $\phi_{\text{vac}}(r)$ in real space or $\phi_{\text{vac}}(q)$ in the momentum space. The potential of the charged impurity, when it is embedded at the origin in the electron liquid, gets modified into

$$\phi(r) = \int dq \, e^{iq \cdot r} \frac{\phi_{\text{vac}}(q)}{\epsilon(q, 0)} e^{-qs},$$

where $s \to 0$. Now in the asymptotic limit of $r \to \infty$, the integrand in equation (6) is highly oscillatory for any wave-vector, and thus the asymptotic behaviour of the screened potential is determined by the non-analyticity of the static dielectric function, generally at $q = 2k_F$, which leads to Friedel oscillations in the electron gas with a power law decay [2, 42–44]. A similar behaviour is also reflected in the charge density, and has been demonstrated experimentally [45–48].

In a realistic experimental situation, in place of equation (5), a screened Coulomb interaction may have to be considered. Any such screened potential will change the Friedel oscillations quantitatively while keeping their qualitative nature intact. However the plasmon dispersion in presence of a screened potential is likely to change qualitatively (in terms of the long wavelength $q$ dependence) as well as quantitatively.

In the next few sections we calculate the polarization function, long wavelength plasmon dispersion, static dielectric function and Friedel oscillations for 1d, 2d and 3d massive Dirac systems. Note that till now, we have explicitly written all the quantities in their usual form with their respective dimensions. However, in what follows we transform all quantities with dimension of energy, i.e. $\hbar \omega, \Delta, \mu$ etc, to $\omega, \Delta, \mu$ etc, to have the dimension of $1/L$ by dividing them with $\hbar v_F$. Thus, we define the following equivalent variables: $\tilde{\omega} \equiv \omega/\hbar v_F$, $\tilde{\Delta} \equiv \Delta/(\hbar v_F)$, $\tilde{\mu} \equiv \mu/(\hbar v_F)$ and $\tilde{E}_k \equiv E_k/(\hbar v_F)$, with the dimension of a wave-vector.

### 3. Polarization function in 1d massive Dirac materials

In this section, we calculate and present the full analytical expression of the dynamical polarization function for 1d massive Dirac materials. We treat the intrinsic and extrinsic components of the polarization function, separately as specified by equation (2), and determine the real and imaginary parts of each component using the Sokhotski–Plemelj theorem. As a consistency check, we note that the explicit analytical forms of the real and imaginary part of the polarization function presented below, have been verified against direct numerical integration of equations (2) and (3).

Let us first consider the imaginary part of the polarization function. Performing the wave-vector integration in equation (3) and introducing the following notation to express our results in a compact form.

$$\Im \Pi^{(1d)}(q, \tilde{\omega}) = \frac{2 \alpha(q, \tilde{\omega})}{\hbar v_F} \theta(\omega^2 - q^2 - 4 \tilde{\Delta}^2),$$

where $\omega = \sqrt{\tilde{\mu}^2 - \Delta^2}$ is the Fermi wave-vector, and we have defined the function $\gamma(x) \equiv \sqrt{x^2 + \tilde{\Delta}^2}$. Here regions 1A and 2A, denote the regions of the $\omega - q$ space in which only the

![Figure 1. Different regions in the $\omega - q$ plane used to define the polarization function for (a) massless Dirac systems and (b) massive Dirac systems, in all three dimensions (see equation (10)).](image-url)}

Here $\Delta = 0.6\mu$. The 4B region in panel (b), is a minuscule region enclosed by the upper boundary of 5B region and the lower boundary of the 2B region, whose visibility depends on $\Delta$. 

$$\gamma(x) = \frac{1 - 4 \tilde{\Delta}^2}{\omega^2 - q^2},$$

$\alpha(q, \tilde{\omega}) \equiv \frac{\Delta^2 q^2}{\hbar v_F}.$

We obtain the imaginary part of the intrinsic polarization function to be

$$\Im \Pi^{(1d)}_{\text{int}}(q, \tilde{\omega}) = \frac{\hbar v_F}{\omega} \theta(\omega^2 - q^2),$$

As a consistency check we note that equation (8) is consistent with the imaginary part of the polarization function derived in the context of the Schwinger model in 1 + 1 dimensions in [49].

A similar calculation for the imaginary part of the extrinsic polarization function yields,

$$\Im \Pi^{(1d)}_{\text{ext}} = \frac{1}{\hbar v_F} \begin{cases} -\alpha(q, \tilde{\omega}), & 2A \\ \alpha(q, \tilde{\omega}), & 2B \\ 2\alpha(q, \tilde{\omega}), & 1B \\ 0, & 1A, 3A, 3B, 4A, 4B, 5B. \end{cases}$$

In equation (9), we have defined the following regions in the $(\tilde{\omega}, q)$ plane (see figure 1):

1A : $0 < \tilde{\omega} < \tilde{\mu} - \gamma(k_F - q)$
2A : $\tilde{\mu} + \gamma(k_F - q) < \tilde{\omega} < \tilde{\mu} + \gamma(k_F + q)$
3A : $\tilde{\omega} < -\tilde{\mu} + \gamma(q - k_F)$
4A : $-\tilde{\mu} + \gamma(k_F + q) < \tilde{\omega} < q$
1B : $q < 2k_F$ and $q^2 + 4\tilde{\Delta}^2 < \tilde{\omega} < \tilde{\mu} + \gamma(k_F - q)$
2B : $\tilde{\mu} + \gamma(k_F - q) < \tilde{\omega} < \tilde{\mu} + \gamma(k_F + q)$
3B : $\tilde{\omega} < \tilde{\mu} + \gamma(k_F + q)$
4B : $q < 2k_F$ and $q^2 + 4\tilde{\Delta}^2 < \tilde{\omega} < \til{\mu} + \gamma(k_F - q)$
5B : $q < \til{\omega} < \sqrt{q^2 + 4\til{\Delta}^2}$

(10)
intrasubband single p-h transitions are allowed, and region 2B denotes the regions where only inter-band single p-h transitions are allowed. In all other regions there are no single p-h excitations. Of these the region 1A is very interesting because in 1d, there are no single p-h excitations in the 1A region as a consequence of the restricted phase space for low energy ($\omega \to 0$, and finite $q$) scattering in 1d. In fact it is this absence of low energy finite $q$ single particle excitation in 1d, which leads to the Luttinger liquid behaviour.

For calculating the real part of the polarization function we rewrite the integral in equation (3) as

$$\Re e \chi_D^{\pm} = -\frac{g}{\pi} \int_{-\beta}^\beta dk \, F_{\pm}(k, k') \left( \frac{E_k + E_{k'}}{\omega^2 - (E_k + E_{k'})^2} \right), \quad (11)$$

where $\beta \equiv \sqrt{D^2 - \Delta^2}$.

Performing the integration in equation (11) we obtain the real component of the intrinsic polarization function in 1d to be,

$$\Re e \Pi_0^{(1d)} = \frac{g}{2\pi \hbar v_F} h(q, \omega) \left( \omega^2 - q^2 + g(q, \omega) \right), \quad (12)$$

where we have defined the following functions

$$h(q, \omega) \equiv \frac{2q^2}{\omega^2 - q^2}, \quad (13)$$

$$g(q, \omega) \equiv \begin{cases} \frac{2\Delta^2}{x_0} & \text{1A, 2A, 3A, 4A} \\ \log \left[ \frac{1 + x_0}{1 - x_0} \right] & \text{1B, 2B, 3B, 4B}, \end{cases}$$

Note that in the 5B region, $x_0$ is imaginary ($x_0 = i|x_0|$), and thus $g(q, \omega)$ is real valued. Additionally we have also checked numerically that $\Pi_0(q, \omega)$ specified by equations (8) and (12) satisfy the Kramers–Kronig relations.

The real component of the extrinsic part of the polarization function is given by

$$\Re e \Pi_1^{(1d)} = -\frac{g}{2\pi \hbar v_F} \left[ f_1 \theta(\omega^2 - q^2 - 4\Delta^2) \\
+ f_2 \theta(q^2 + 4\Delta^2 - \omega^2) \theta(\omega - q) + f_3 \theta(q - \omega) \right], \quad (14)$$

where we have defined the following functions,

$$f_1 \equiv \frac{h_1 \Delta^2}{x_0} \log \left[ \frac{-a_1(k_0)a_1(k_0 - q)b_1(k_0)b_3(-k_0)}{a_1(-k_0 - q)b_3(k_0)a_1(-k_0)b_2(-k_0)} \right],$$

$$f_2 \equiv \frac{h_2 \Delta^2}{x_0} i \Im \left[ \log c(k_0) - \log c(-k_0) \right], \quad (15)$$

$$f_3 \equiv \frac{h_3 \Delta^2}{x_0} \log \left[ \frac{-a_1(k_0)a_1(-k_0 - q)b_3(-k_0)b_2(-k_0)}{a_1(-k_0)b_3(k_0)a_1(k_0 - q)b_2(k_0)} \right],$$

which in turn use the following:

$$a_0(k) \equiv \tilde{\omega}x_0 + 2k + q,$$

$$b_1(k) \equiv 2\Delta^2 - k(\tilde{\omega}x_0 + q) - \tilde{E}_k(\omega + qx_0),$$

$$b_2(k) \equiv 2\Delta^2 + k(\tilde{\omega}x_0 - q) - \tilde{E}_k(\omega - qx_0),$$

$$b_3(k) \equiv 2\Delta^2 - k(q + \tilde{\omega}x_0) + \tilde{E}_k(\omega + qx_0),$$

$$b_4(k) \equiv 2\Delta^2 - k(q - \tilde{\omega}x_0) + \tilde{E}_k(\omega - qx_0),$$

$$c(k) \equiv b_1(k)b_2(k).$$

Note that in the function $f_2$, $x_0$ is imaginary, hence the overall output of $f_2$ is real.

Having obtained the real and imaginary part of the 1d polarization function, let us consider various limiting cases, starting with the static case ($\omega \to 0$) in the next section.

3.1. Static dielectric function and Friedel oscillations

In the static limit, i.e. $\omega = 0$ case for 1d massive Dirac systems the intrinsic part of the Lindhard function reduces to

$$\Pi_0^{(1d)} = \frac{-g}{\pi \hbar v_F} \left[ 1 + \frac{2\Delta^2}{q\sqrt{q^2 + 4\Delta^2}} \log \left( \frac{\sqrt{q^2 + 4\Delta^2} - q}{\sqrt{q^2 + 4\Delta^2} + q} \right) \right], \quad (17)$$

and the extrinsic part is given by

$$\Pi_1^{(1d)} = -\frac{g}{\pi \hbar v_F} \frac{2\Delta^2}{q\sqrt{q^2 + 4\Delta^2}} \log \left( \frac{(q + 2k_F)\beta(k_F)}{|q - 2k_F|\beta(-k_F)} \right), \quad (18)$$

where $\beta(k_F) = \sqrt{k_F^2 + \Delta^2}(q^2 + 4\Delta^2)$. Note that $\Pi_1^{(1d)}$ in equation (18) diverges logarithmically as $q \to 2k_F$, similar to the case of 1d parabolic systems [2]. In the limiting case of $q \to 0$, equations (17) and (18) lead to

$$\Pi^{(1d)}(q \to 0, 0) = -\frac{g}{\pi \hbar v_F k_F} \left( 1 + \frac{k_F^2}{\Delta^2 + \Delta \sqrt{k_F^2 + \Delta^2}} \right) = -g\mu/(\pi \hbar v_F k_F). \quad (19)$$

Note that $g\mu/(\pi \hbar v_F k_F)$ is the density of states of a Dirac system in 1d at the Fermi energy.

In general the static dielectric function can also be used to calculate the Friedel oscillations as shown in appendix. However in 1d the RPA calculation of the Friedel oscillation breaks down on account of the logarithmic divergence of $\Pi_1^{(1d)}$ in equation (18). This highlights the failure of RPA or a mean field like calculation to describe a finite wavevector ($q = 2k_F$) phenomena like the Friedel oscillations for 1d interacting electrons. Interacting 1d electron liquids are more appropriately described by the Luttinger liquid theory which treats the Coulomb interactions in a exact way (barring some anomalous terms). Using Luttinger liquid theory it has been shown that the asymptotic decay of the Friedel oscillation in the density profile with distance from the impurity has a power law decay, with an exponent which depends on
3.2. Plasmons

The polarization function in the dynamic long wavelength limit \( q \to 0 \) first and then \( \omega \to 0 \), i.e. with \( \omega > q \) fixed, which is useful to obtain the long wavelength plasmon dispersion, can be obtained from equations (12)–(14), and it is given by

\[
\Pi^{(1d)}(q \to 0, \omega) = \frac{g}{\pi \hbar q} \frac{k_F}{\sqrt{q_F^2 + \Delta^2}} \frac{q^2}{\omega^2} + O(q^4/\omega^4). \tag{20}
\]

As a consequence the long wavelength limit of the plasmon dispersion is given by

\[
\omega^{(1d)} = \sqrt{\frac{2g^2e^2}{\pi \hbar q} \sqrt{\hbar q v_F k_F(q_F) / \mu} + O(q^3)}, \tag{21}
\]

which is consistent with the results of [39]. The plasmon dispersion is displayed in the backdrop of the p-h continuum (i.e. where the imaginary contribution of polarization function is non zero) in figure 2(b). At \( T = 0 \), the plasmon mode remains undamped for a wide range of \( q \) and \( \omega \), and then it becomes damped as it enters the p-h region (blue shaded region of figure 2(b)) and decays by creating single p-h excitations.

3.3. Massless 1d electrons (\( \Delta \to 0 \))

In the \( \Delta \to 0 \) limit in 1d, the total polarization function reduces to

\[
\Pi^{(1d)}(q, \omega, \Delta = 0) = \frac{g}{\pi \hbar q} \frac{q^2}{(\omega + i\eta)^2 - q^2}, \tag{22}
\]

where the whole contribution arises from the intrinsic part. This can also be deduced from equations (12)–(14), where all the terms proportional to \( \Delta \) vanish, and only the \( h(q, \omega) \) term in equation (12) contributes. Note that in this case, the imaginary part of the polarization function is non-zero only around the line \( \omega = q \), i.e.

\[
3m \Pi^{(1d)}(q, \omega, \Delta = 0) = \frac{g}{\pi \hbar q} q^2 \delta(\omega - q), \tag{23}
\]

as shown in figure 2(a). This has two remarkable consequences. (1) since there is no non-analyticity in the static response function (see figure 2(c)), there are no Friedel oscillations in the density profile in the vicinity of a scatterer for the non-interacting case in 1d Dirac system. In general the RPA or the mean field description fails to capture Friedel oscillations in 1d, whose asymptotic behaviour depends on the strength of electron–electron interaction, and only in the non-interacting case does one recover the \( r^{-1} \) decay.

Figure 2. (a) \( \Im \Pi(q, \omega) \) in the \( \omega - q \) plane, for a 1d massless Dirac system along with the exact plasmon dispersion—see equation (24). Note that the \( q - \omega \) space for p-h excitations is only a straight line due to conservation of chirality and the system being 1d. (b) \( \Im \Pi(q, \omega) \) for a massive 1d Dirac system along with the long wavelength plasmon dispersion—see equation (21). (c) and (d) \( \Re \Pi(q, \omega) \) for \( \Delta = 0 \) and \( \Delta = 0.6\mu \), respectively. In panels (a)–(d) the polarization function is in units of \( g/(\hbar q) \). For calculating the plasmon dispersion in panels (a) and (b) we have used \( a = 0.02k_F \), and \( 8\pi e^2/(\pi \hbar q) = 0.6 \). (e) The static \( \Pi(q, \omega = 0) \) versus \( q \), for the massless and massive Dirac system in 1d. Note that there is no discontinuity in \( \epsilon(q, 0) \) for massless Dirac system in 1d and as a consequence there are no Friedel oscillations. However in the massive case, there is a logarithmic singularity at \( q = 2k_F \), which is marked by the thin vertical line. (f) Asymptotic behaviour of the Friedel oscillations in the density profile in the vicinity of a scatterer for the non-interacting case in 1d Dirac system. In general the RPA or the mean field description fails to capture Friedel oscillations in 1d, whose asymptotic behaviour depends on the strength of electron–electron interaction, and only in the non-interacting case does one recover the \( r^{-1} \) decay.

Backscattering in massless Dirac systems is the phenomena of perfect Klein tunneling [50, 51].

The interaction strength [55]. Only in the non-interacting limit, does one recover the \( r^{-1} \) decay of the Friedel oscillations as shown in figure 2(f). Generally repulsive interactions tend to slow down the decay of the Friedel oscillations in 1d systems. However, for the case of a gapless system in 1d, with chiral electrons, Friedel Oscillations vanishes completely. This is a direct consequence of the fact that chirality (eigenstate of the \( \gamma_5 + k \) operator), in a gapless Dirac system is conserved even in the presence of Coulomb interaction. This is equivalent to saying that perfect backscattering is forbidden in Dirac systems, and as a consequence the interference between the forward and backward propagating disturbance, which leads to Friedel oscillation in 1d, does not occur in 1d massless Dirac systems. Another interesting and well known effect related to the conservation of chirality in 1d and lack of perfect backscattering in massless Dirac systems is the phenomena of perfect Klein tunneling [50, 51].

For the case of massless Dirac fermions in 1d, we can easily obtain the exact plasmon dispersion for a given form of the electrostatic potential \( V_q \) to be

\[
\omega^{(1d)}(\Delta \to 0) = \hbar q \sqrt{1 + \frac{g}{\pi \hbar q} V_q^{1/2}.} \tag{24}
\]
Note that since the p-h continuum is almost non-existent in 1d gapless Dirac systems (except along the \( \omega = v_F q \) line as shown in figure 2(a)), the plasmon dispersion will be practically undamped with a very high quality factor.

Before proceeding to the case of 2d massive Dirac Fermions, we note that RPA or the treatment of the electron–electron interaction at the mean field level retaining only Hartree contribution, works much better in higher dimensions as opposed to 1d, where the Luttinger liquid theory [52] works well to describe the interacting itinerant electrons. In fact we already discussed in section 3.1 that RPA fails to describe the Friedel oscillations (generally a large wave-vector phenomena involving \( q = 2k_F \)) for 1d interacting electrons. However RPA still gives a good estimate of the plasmon dispersion in the 1d parabolic case, with the corresponding experimental results in [53]. More importantly, Luttinger liquid theory is based on the linearized spectrum of the p-h continuum around the Fermi point, and consequently the plasmons within the Luttinger liquid will always be damped. However, within RPA, the plasmons can also lie outside the regime of p-h continuum, and will be completely undamped in that regime [39–41, 54].

4. Polarization function of 2d massive Dirac systems

Having studied the 1d massive Dirac case, we now focus on the 2d case of gapped Dirac fermions. The polarization function for this case has already been derived in [33], and we reproduce the results here for completeness. Let us first consider the intrinsic part, \( \Pi^{(2d)}(q, \tilde{\omega}) \), whose real and imaginary parts can be calculated following [33], and combined to obtain,

\[
\Pi^{(2d)}_0(q, \tilde{\omega}) = \frac{-gq^2}{8\pi\hbar v_F (q^2 - \tilde{\omega}^2)^2} \left( 2\Delta + \frac{q^2 - \tilde{\omega}^2 - 4\Delta^2}{\sqrt{q^2 - \tilde{\omega}^2}} \right) \times \arcsin \left( \frac{q^2 - \tilde{\omega}^2}{\sqrt{q^2 - \tilde{\omega}^2 + 4\Delta^2}} \right)
\]

(25)

For evaluating the extrinsic part of the polarization function (for \( \mu > \Delta \)) we split the \( \chi_\mu \) integrals of equation (3) into real and imaginary parts using the Sokhotski–Plemelj theorem. Thereafter doing some algebraic simplifications and by introducing the following notation,

\[
f(q, \tilde{\omega}) = \frac{-gq^2}{16\pi\sqrt{|q^2 - \tilde{\omega}^2|}},
\]

\[
G_\gamma(x) = x_0 \sqrt{x^2 - x_0^2} - (2 - x_0^2) \arccos(x/x_0),
\]

\[
G_\delta(x) = x_0 \sqrt{x^2 - x_0^2} - (2 - x_0^2) \arcsin(x/\sqrt{-x_0^2}),
\]

where \( x_0 \) is defined in equation (7), we find the real part of the extrinsic polarization function to be

\[
\text{Re} \, \Pi^{(2d)}_{1}(q, \tilde{\omega}) = \frac{-g(\tilde{\mu} - \tilde{\Delta})}{2\pi\hbar v_F} - \frac{f(q, \tilde{\omega})}{\hbar v_F},
\]

(27)

where the various regions in the \( \tilde{\omega} - q \) plane have been defined in equation (10) and marked in figure 1(b). A similar calculation for the imaginary part of the extrinsic polarization function leads to
\[ \Im m \Pi^{(2d)}_0(q, \omega) = \frac{f(q, \omega)}{\hbar v_F} \]

\[ G_\omega = \frac{2\bar{\mu} + \bar{\omega}}{q} - \frac{2\bar{\mu} - \bar{\omega}}{q}, \quad \text{1A} \]

\[ G_- = \frac{2\bar{\mu} + \bar{\omega}}{q} - \frac{2\bar{\mu} - \bar{\omega}}{q}, \quad \text{2A} \]

\[ \pi(2 - x_0^2), \quad \text{1B} \]

\[ -\pi(2 - x_0^2), \quad \text{2B} \]

\[ 0, \quad \text{3A, 3B, 4A, 4B, 5B} \]

Note that the real and imaginary part of the intrinsic polarization function (see equation (25)) can also be split region-wise as

\[ \Re e \Pi^{(2d)}_0(q, \omega) = -\frac{g\Delta}{2\pi \hbar v_F} + \frac{f(q, \omega)}{\hbar v_F} \]

\[ \times \left\{ \begin{array}{l}
G_- = \frac{2\bar{\Delta} + \bar{\omega}}{q} - \frac{2\bar{\Delta} - \bar{\omega}}{q}, \\
G_0 = \frac{2\bar{\Delta} + \bar{\omega}}{q} + \frac{2\bar{\Delta} - \bar{\omega}}{q}, \\
G_+ = \frac{2\bar{\Delta} - \bar{\omega}}{q} - \frac{2\bar{\Delta} + \bar{\omega}}{q}, \\
0, 
\end{array} \right. \quad \text{1B, 2B, 3B, 4B} \]

where the \( G_-, G_0, \) and \( G_+ \) functions are defined in equation (26), and

Adding equation (27) to equation (29), and equation (28) to equation (30) respectively, gives the total polarization function which is identical to that reported in [33]. In the gapless limit (\( \Delta \to 0 \)), the polarization function reduces to the well known polarization function of graphene [22, 23]. We now use these analytical expressions to study the static limit, and calculate the plasmon dispersion and its long wavelength limit in subsequent sections.

4.1 Static dielectric function and Friedel oscillations

The intrinsic part of the Lindhard function for \( \mu < \Delta \) in the static limit (\( \omega = 0 \)) can be obtained from equation (25), and it is given by

\[ \Pi^{(2d)}_0(q, 0) = -\frac{g}{8\pi \hbar v_F} \left( \frac{2\bar{\Delta} + q^2 - 4\bar{\Delta}^2}{q} \arcsin \frac{q}{\sqrt{q^2 + 4\bar{\Delta}^2}} \right) \quad \text{(31)} \]

The static limit for the extrinsic polarization function (\( \mu > \Delta \)) can be obtained using equations (27)–(28), and added to equation (31) to obtain the total static polarization,

\[ \Pi^{(2d)}(q, 0) = -\frac{g\bar{\mu}}{2\pi \hbar v_F} \left[ 1 - \theta(q - 2k_F) \sqrt{\frac{q^2 - 4k_F^2}{2q}} \right] - \frac{q^2 - 4\bar{\Delta}^2}{4q\bar{\mu}} \arctan \frac{\sqrt{q^2 - 4k_F^2}}{2k_F}. \]

For a localized charged impurity placed in a 2d Dirac electron liquid, the oscillating screened potential far away from the impurity location, is given by (see appendix),

\[ \phi(r) \approx A^{(2)}(\mu, k_F) \sin(2k_F r) \frac{(2k_F)^3}{\hbar v_F} \left( 1 - \frac{k_F^2 - \bar{\Delta}^2}{\mu^2} \right). \]

where \( A^{(2)} \) is a constant given by equation (A.2). The screened potential in the 2d massive Dirac case, decays asymptotically as \( r^{-2} \), similar to the case of a parabolic 2d electron gas, and unlike that in 2d gapless Dirac electron gas, in which the decay rate is proportional to \( r^{-3} \) [22]. This is also evident from equation (33) in which the \( r^{-2} \) dependence arises from the first order term in the expansion of \( \arctan(\sqrt{q - 2k_F}) \), and in which the right hand side vanishes for the gapless case i.e. \( \Delta \to 0 \). Following the methodology of appendix, it is easy to show that the main contribution in the gapless case arises from the second order term in expansion of \( \arctan(\sqrt{q - 2k_F}) \) and in this case the screened potential in the asymptotic limit is given by [11, 33]

\[ \phi(r) \approx A^{(2)}_{\Delta=0} \frac{g\bar{\mu}k_F \cos(2k_F r)}{\hbar v_F} \left( 1 - \frac{k_F^2 - \bar{\Delta}^2}{\mu^2} \right). \]

where \( A^{(2)}_{\Delta=0} \) is a constant given by equation (A.2) with the substitution \( \Delta = 0 \). The different scaling of the screening potential is a consequence of the fact that in the gapless case the polarization function at \( q = 2k_F \) has a discontinuity in second derivative, while in the gapped case the polarization function has a discontinuity in the first derivative at \( q = 2k_F \).

4.2. Plasmons

To calculate the frequency of the collective density excitations i.e. the plasmon dispersion, we solve equation (4) numerically. The numerically calculated plasmon dispersion is displayed in figure 3(b), via solid black line, on top of the imaginary part of the polarization function which depicts various regions of the p-h continuum. Note that the imaginary part of the polarization function is non-zero in regions 1A, 2A, 2B and 4B of figure 1(b), and hence these regions form the single particle excitation spectrum, as marked by shaded area in figure 3(c). In the dynamic long wavelength limit, \( q \to 0 \) first and then \( \omega \to 0 \), i.e. with \( \omega > q \) fixed, the polarization function is given by
\[ \Pi^{(2d)}(q \to 0, \omega) = \frac{g}{2\pi \hbar v_F} \frac{k_F^2}{\sqrt{k_F^2 + \Delta^2}} q^2 + C q^4 / \omega^4. \] (35)

Using the above expression for the polarization function, we find the long wavelength plasmon dispersion to be

\[ \omega^{(2d)}_p(q \to 0) = \sqrt{\frac{ge^2 \mu q}{2\pi \hbar^2}} \left( 1 - \frac{\Delta^2}{\mu^2} \right)^{1/2}. \] (36)

Note that similar to the case for massless Dirac fermions, and parabolic dispersion systems in two dimensions, the plasmon dispersion in the long wavelength limit is \( q^{1/2} \). This is a consequence of the charge continuity equation, and the form of the bare Coulomb interaction in the momentum space in the long wavelength limit in 2d. However the density dependent pre-factor in all the three cases are different, and depend on the details of the electronic dispersion relation, doping etc of the system considered.

### 4.3. Massless 2d electrons, \( \Delta \to 0 \)

In the massless limit (\( \Delta \to 0 \)) for Dirac fermions in 2d, \( x_0 \) becomes equal to 1, hence regions 4A, 4B, and 5B vanish (see figure 1(a)), and the real and imaginary parts of the total polarization function can be combined to obtain [33],

\[ \Pi^{(2d)}(q, \omega) = \frac{-\mu}{2k_F} + F(q, \omega)[G_1(\omega_I) + G_0(\omega_I)], \] (37)

where \( N_0 = gk_F/(\pi \hbar v_F) \) is the density of states at the Fermi surface of a 2d gapless system, and we have defined the dimensionless variables as \( \nu_\pm = 2v_Fq \pm \omega/Iq \). Additionally in equation (37), we have used the following functions,

\[ F(q, \omega) = \frac{q^2}{16k_F^2 \sqrt{q^2 - \omega^2}}, \] (38)

and

\[ G_A(z) = \zeta(1 - 1 + i \cosh^{-1}(z)). \] (39)

The polarization function in the static limit, \( \omega \to 0 \), is displayed in figure 3(e), and it remains constant at \( 2k_F^2/\mu^2 \) for \( k < 2k_F \). The \( q \to 0 \) limit of the plasmon dispersion is easily obtained from equation (36), by setting \( \Delta \to 0 \). The exact plasmon dispersion calculated numerically, and the long wavelength results, are shown in figure 3(a), in black and green solid lines, respectively. Clearly the exact plasmon dispersion enters the p-h continuum at a larger \( q \) (and lower \( \omega \)), as compared to the long wavelength result. In figure 3(b), the exact plasmon dispersion becomes almost parallel to the p-h boundary and enters the p-h continuum at a much larger \( q \) and \( \omega \) value, as opposed to the long wavelength result.

### 5. Polarization function of 3d massive Dirac systems

In this section, we calculate the polarization function for 3d massive Dirac materials. Similar to 1d and 2d calculations, we evaluate the intrinsic and extrinsic parts of the polarization function separately. The general form of intrinsic \( \Pi_0^{(3d)}(q, \omega) \) and extrinsic \( \Pi_1^{(3d)}(q, \omega) \) polarization function following from equation (3) and after integration over azimuthal angle \( \phi \) and upon simplifying the \( \theta \) integration is given by

\[ \Pi_0^{(3d)}(q, \omega) = I(q, \omega) + I(q, -\omega), \] (40)

\[ \Pi_1^{(3d)}(q, \omega) = J(q, \omega) + J(q, -\omega), \] (41)

where
In equations (42) and (43), we have $\omega \rightarrow \omega + i\eta$, $l_1 = \sqrt{|k - q|^2 + \Delta^2}$, $l_2 = \sqrt{(k + q)^2 + \Delta^2}$, $\Lambda$ is the large momentum cut off and $k'$ originally was $E_{k+q}$. Calculating the imaginary part of the intrinsic polarization function using the Sokhotski–Plemelj theorem, we obtain

$$\Im m \Pi_0^{(3d)} = \frac{gq^2}{48\pi\hbar v_F} \delta(\omega^2 - q^2 - 4\Delta^2),$$

(44)

where $x_0$ is specified by equation (7). Similarly the imaginary component of the extrinsic polarization function is given by

$$\Im m \, \Pi_0^{(3d)}(q, \omega) = \frac{g}{16\pi^2q\hbar v_F} \int_0^{k_F} \frac{kq}{E_k} \left\{ 2kq + 3E_k\sqrt{(k - q)^2 + \Delta^2} - 3E_k\sqrt{(k + q)^2 + \Delta^2} + (2E_k + \omega)^2 - q^2 \right\} \log \left[ \frac{E_k + \sqrt{(k + q)^2 + \Delta^2 + \omega}}{E_k + \sqrt{(k - q)^2 + \Delta^2 + \omega}} \right] + \omega \rightarrow -\omega,$$

(47)

$$\Im m \, \Pi_1^{(3d)}(q, \omega) = \frac{g}{16\pi^2q\hbar v_F} \int_0^{k_F} \frac{kq}{E_k} \left\{ -4kq + [(2E_k + \omega)^2 - q^2] \log \left[ \frac{(E_k + \omega)^2 - (k - q)^2 + \Delta^2}{(E_k + \omega)^2 - (k + q)^2 + \Delta^2} \right] \right\} + \omega \rightarrow -\omega.$$  

(48)

While the integrals in equations (47) and (48) can be done analytically, the resulting expressions are very cumbersome, and do not offer any useful insight. Additionally, we find it easier to do these integrals numerically and use the numerical results to obtain the plasmon dispersion and other limiting cases thereafter. We note that in the limiting case of massless Dirac systems i.e. $\Delta = 0$ implying $x_0 = 1$, equations (47) and (48) reproduce the known result for the real part of the polarization function for a gapless 3d Dirac system [30].

5.1 Static dielectric function and Friedel oscillations

In the limiting case of $\omega = 0$ at finite $q$, the imaginary part of the static polarization vanishes, and it reduces to

$$\Pi_0^{(3d)}(q, \omega = 0) = \frac{g}{48\pi^2\hbar v_F q} \left[ 8q^2\Delta^2 + 4q^4 \log \frac{\Delta}{2\Lambda} + \frac{2(q^4 + 2q^2\Delta^2 - 8\Delta^4)}{\sqrt{q^2 + 4\Delta^2}} \log \frac{q + \sqrt{q^2 + 4\Delta^2}}{q - \sqrt{q^2 + 4\Delta^2}} \right],$$

(49)

$$\Pi_1^{(3d)}(q, \omega = 0) = \frac{g}{24\pi^2\hbar v_F q} \left[ -8k_Fq\mu + \left( \frac{\mu(4k_F^2 - 3q^2 + 4\Delta^2) + q^4 + 2q^2\Delta^2 - 8\Delta^4}{\sqrt{q^2 + 4\Delta^2}} \log \frac{q - 2k_F}{q + 2k_F} \right) \left. \right] \right] + 2q^3 \log \frac{k_F + \mu}{\Delta} + \frac{q^4 + 2q^2\Delta^2 - 8\Delta^4}{\sqrt{q^2 + 4\Delta^2}} \log \frac{-k_Fq + 2\Delta^2 + \mu\sqrt{q^2 + 4\Delta^2}}{k_Fq + 2\Delta^2 + \mu\sqrt{q^2 + 4\Delta^2}}.$$  

(50)
The oscillating behaviour (Friedel oscillations) of the screened potential in 3d gapped Dirac system, obtained numerically from equations (49), (50) and (A.1), is displayed in figure 4(f) (red curve) and the \( r^{-3} \) decay is evident. The \( r^{-3} \) decay can also be inferred from the fact that the static extrinsic polarization function in equation (50), has a second derivative discontinuity at \( q = 2k_F \), from which we can estimate the asymptotic form of screened potential to be

\[
\phi(r) \approx \frac{\sin(2k_F r)}{r^3}.
\]

In gapless 3d system the screened potential can be calculated analytically using equation (A.1), and it is given by

\[
\phi(r) \approx A^{(3)} \frac{2g k_F^3}{\hbar v_F} \frac{\sin(2k_F r)}{(2k_F r)^3},
\]

where \( A^{(3)} \) is a constant given by equation (A.2). Note that equation (52) is a reproduction of equation (19) of [29]. The faster \( r^{-3} \) decay in the Friedel oscillation of gapless 3d Dirac systems can be traced back to the fact that the static polarization function at \( q = 2k_F \) has a discontinuity in third derivative as compared to second derivative discontinuity at \( q = 2k_F \) for gapped static response function.

5.2. Plasmons

The exact plasmon dispersion for the 3d massive Dirac system is obtained by numerically solving equation (4) and is displayed in figure 4(b), over the background of the \( 2m \Pi_0(q, \omega) \), which indicates the p-h continuum. The polarization function in the dynamic long wavelength limit \( q \to 0 \) first and then \( \omega \to 0 \), i.e. with \( \omega > q \) fixed, is given by

\[
\Pi^{(3d)}(q \to 0, \omega) = \frac{g}{6\pi^2 \hbar v_F} \frac{k_F^3}{\sqrt{k_F^2 + \Delta^2}} \frac{q^2}{\omega^2} + O(q^4/\omega^4).
\]

Using this expression for polarization function at small energies and momenta, we find the plasmon dispersion at long wavelength to be

\[
\omega^{(pl)}_{\omega = 0} = \frac{2g^2 \mu^2}{3 \pi \epsilon_0 \hbar^3 v_F} \left( 1 - \frac{\Delta^2}{\mu^2} \right)^{3/4},
\]

independent of \( q \). This is consistent with the case of systems with parabolic dispersion in 3d, and massless Dirac fermions in 3d, as expected from the charge continuity equation and the form of the bare Coulomb interaction in momentum space in 3d. Note however, that at finite \( q \), the plasmon dispersion develops a dependence on \( q \), as evident from the exact numerical solution, displayed in figures 4(a) and (b). Note that as opposed to the long wavelength result, in both panels (a) and (b) of figure 4, the exact plasmon dispersion enters the p-h continuum and gets damped at a much larger \( q \) and \( \omega \) values.

5.3. Massless 3d electrons, \( \Delta \to 0 \)

In the massless limit for 3d polarization function, the real and imaginary parts of the intrinsic polarization can be obtained as a limiting case of equations (44) and (47), and are given by

\[
\Re \Pi_0^{(3d)}(q, \omega) = -\frac{gq^2}{24\pi^2 \hbar v_F} \log \left( \frac{4\Lambda^2}{q^2 - \Delta^2} \right),
\]

\[
\Im \Pi_0^{(3d)}(q, \omega) = -\frac{gq^2}{24\pi^2 \hbar v_F} \theta(\omega - q).
\]

The real part of the extrinsic polarization function is given by

\[
\Re \Pi_{\omega}^{(3d)}(q, \omega) = \frac{gq^2}{8\pi^2 \hbar v_F} \left[ \frac{8\mu^2}{3q^2} + \frac{G(q, \omega)H(q, \bar{\omega})}{q^2} 
+ \frac{G(-q, \omega)H(-q, \bar{\omega}) + G(q, -\omega)H(q, -\bar{\omega})}{q^2} \right].
\]

where \( G \) is specified by equation (46), and \( H \) is defined as

\[
H(q, \omega) = \log \left| \frac{2\mu + \omega - q}{q - \bar{\omega}} \right|.
\]

Similarly the imaginary part of the extrinsic polarization function can be obtained and it is given by

\[
\Im \Pi^{(3d)}(q, \omega) = \left\{ \begin{array}{ll}
G(q, \omega) - G(q, -\omega), & 1A \\
G(q, \omega), & 2A \\
-\frac{gq^2}{3}, & 1B \\
-G(-q, -\omega), & 2B \\
0, & 3A, 3B, 4A, 4B, 5B.
\end{array} \right.
\]

These expressions are consistent with the existing results of [30] for 3d massless Dirac fermions. Further in the static limit i.e. \( \omega \to 0 \), the imaginary part of the response function vanishes and equations (55)–(57) reduces to

\[
\Pi_{\omega}^{(3d)}(q, \omega = 0) = -\frac{gq^2}{12\pi^2 \hbar v_F} \log \left( \frac{2\Lambda^2}{q^2} \right),
\]

and

\[
\Pi_{\omega}^{(3d)}(q, \omega = 0) = -\frac{gq^2}{8\pi^2 \hbar v_F} \left[ \frac{8\mu^2}{3q^2} - \frac{1}{6q^3} \left( 2q^3 \log \left( \frac{4\Lambda^2}{q^2} \right) - \frac{4\mu^2}{q^2} \right) \right] \\
+ \left( 8\mu^3 - 6\lambda q^2 \right) \log \left( \frac{2\mu - q}{2\mu + q} \right).
\]

Note that unlike the massive case, in equation (61) the third derivative of the polarization function has a discontinuity at \( q = 2k_F \) and this leads to a \( r^{-4} \) decay in the Friedel oscillations, as mentioned earlier and depicted in figure 4(f) via the blue curve. The long-wavelength plasmon dispersion for the gapless 3d Dirac system can be obtained by taking the \( \Delta \to 0 \) limit, in equation (54).

6. The non-relativistic limit of the massive Dirac polarization function

Note that while we have focussed on the polarization function for relativistic systems (\( \Delta \to 0 \), and finite \( \Delta \)) in this paper, the opposite limit of a dominant mass term is also very
To summarize, we have calculated the exact one-loop polarization function for massive as well as massless Dirac systems in 1d, 2d and 3d. The calculated polarization function is then used to obtain the exact plasmon dispersion, and the asymptotic form of the screened potential of a localized impurity. Both of these cannot be deduced from the long wavelength limit of the polarization function. The exact plasmon dispersion deviates from the long wavelength limit result at larger $q$, and it enters the p-h continuum generally at a larger $q$. Additionally using the exact polarization function in the static limit, we find that the Friedel oscillations in a massive Dirac system decay as $r^{-2}$ and $r^{-3}$ in 2d and 3d respectively similar to the case of parabolic dispersion and unlike the massless Dirac case where the corresponding Friedel oscillations decay as $r^{-3}$ and $r^{-4}$.

For the case of 1d massless Dirac fermions, the p-h continuum exists only along the $\omega = \nu_{F}q$ line, and additionally there are no Friedel oscillations. Both of these are a consequence of the fact that perfect backscattering or mixing of different chirality fermions in one dimension is forbidden. Another well known consequence of this forbidden backscattering or conservation of chirality is Klein tunneling [50] in massless Dirac systems such as graphene. For non-interacting massive Dirac fermions in 1d, the density profile in vicinity of a static scatterer decays as $1/r$. However for interacting 1d massive electron liquid, RPA fails to correctly describe the Friedel oscillations, which is a finite wave-vector phenomena. The correct behaviour of the Friedel oscillations as predicted by Luttinger liquid theory decays as $r^{-\alpha}$ with $\alpha < 1$ for repulsive electron–electron interactions.

For the plasmon dispersion in massive Dirac systems, calculated both numerically and analytically in the long wavelength limit, we find that while the long wavelength limit behaviour in terms of $q$-dependence is purely determined by the charge continuity equation and dimensionality of the system as expected, the density dependence of the plasmon dispersion is governed by the details of the dispersion, doping etc.

We hope that our analytical results for the polarization function, plasmon dispersion and Friedel oscillations, will be useful for exploring the physics of massive and massless Dirac electrons in different experimental systems with varying dimensionality.

A useful extension of our work will be to include finite temperature effects in the calculation of the polarization function. A similar study has been done for the case of massless [56, 57] and massive Dirac systems [58] in 2d. In general we expect the Friedel oscillations to be smeared out depending on the temperature (as compared to Fermi energy), and the plasmon dispersion to change quantitatively while retaining all the qualitative features.

Acknowledgments

We thank T K Ghosh for stimulating discussions. A gratefully acknowledges funding from the INSPIRE Faculty Award by DST (Govt. of India). We sincerely thank the referees for
independently checking and correcting several of the results in this manuscript.

**Appendix. Calculation of Friedel oscillations**

The homogeneous or the long wavelength, \( q \to 0 \), behaviour of the static dielectric function, gives rise to the Thomas–Fermi screening which beautifully explains how the singularities of the long range Coulomb repulsion are ‘regularized’ by screening. However Thomas–Fermi screening fails to adequately describe the response of the electron gas, to short range perturbations, for which the non-analyticity of the static dielectric function at finite wave-vector, typically at \( q = 2k_F \), need to be accounted for \([42–44]\). To obtain the asymptotic behaviour of the screened potential from equation (6) we follow \([42]\) and make use of the Riemann–Lebesgue lemma, which states that if a function oscillates rapidly around zero then its integral is small and the principal contribution to the integral is determined by the behaviour of the integrand in the vicinity of its non-analytic points. In equation (6), since \( \epsilon(q, 0) \) is non-analytic at \( q = 2k_F \) in all three dimensions for Dirac systems, equation (6) can be reduced to the following asymptotic form after doing the angular integration,

\[
\phi(r) \approx A^{(d)} \int \Delta \Pi(x) S(x, r) e^{-sx},
\]

(A.1)

where \( \Delta \Pi(x) = \Pi(2k_F + x) - \Pi(2k_F) \), is the increment of the static polarization function near the singular point \( q = 2k_F \) and \( s \to 0 \). In equation (A.1), we have defined \( A^{(d)}(q) \) to be a dimension dependent constant for \( d \geq 3 \) as

\[
A^{(d)} = \frac{k_F^{d-1} \phi c_d(2k_F)V_{q}^{(d)} q^{-2d}e^{-2k_Fr}}{[1 - V_{q}^{(d)} q^{-2d} \Pi(2k_F)]^2},
\]

(A.2)

where \( V_{q}^{(d)} \) is the Coulomb potential in \( d \) dimensions in momentum space (see equation (5)), and

\[
S(x, r) = 4\pi \left\{ \frac{\sin[(2k_F + x)r]}{2k_Fr}, \quad d = 3 \right. \\
\left. = 2\sqrt{2\pi} \cos[(2k_F + x)r - \pi/4], \quad d = 2 \right. \\
= 2 \cos[(2k_F + x)r], \quad d = 1.
\]

(A.3)

Note that in some cases where equation (A.1) cannot be solved analytically, its asymptotic behaviour for \( r \to \infty \), can be inferred simply from the lowest power of \( x \) in the series expansion of \( \Delta \Pi(x) \) around \( x = 0 \).

As evident from equation (A.2), \( A^{(d)} \) depends on the value of static polarization function at the \( \omega = 0 \) p-h boundary at \( q = 2k_F \), and it vanishes if the static polarization function at \( q = 2k_F \) diverges, as in the case of 1d systems (see equation (18) for Dirac systems and \([2]\) for parabolic systems). This is an example of the breakdown of Fermi liquid theory and mean field (RPA) based calculations to describe finite wave-vector phenomena in 1d. One dimensional interacting electron liquid, is more appropriately described by the Luttinger liquid theory, and it has been shown that the asymptotic decay of the Friedel oscillations in 1d systems, has a power law decay where the decay exponent is dependent on the strength of the electron–electron interactions. Typically repulsive electron–electron interactions slow down the decay of the Friedel oscillations as opposed to the non-interacting 1d case, where the Friedel oscillations decay as \( r^{-1} \) \([2, 55]\).

**References**

[1] Pines D and Noziéres P 1966 *The Theory of Quantum Liquids* (New York: Benjamin)

[2] Giuliani G F and Vignale G 2005 *Quantum Theory of the Electron Liquid* (Cambridge: Cambridge University Press)

[3] Maier S A 2007 *Plasmonics: Fundamentals and Applications* (New York: Springer)

[4] Tame M S, McEnery K R, Özdemir S K, Lee J, Maier S A and Kim M S 2013 Quantum plasmonics *Nat. Phys.* 9 329

[5] Ando T, Fowler A B and Stern F 1982 Electronic properties of two-dimensional systems *Rev. Mod. Phys.* 54 437

[6] Agarwal A, Chesi S, Jungwirth T, Sinova J, Vignale G and Polini M 2011 Plasmon mass and Drake weight in strongly spin–orbit-coupled two-dimensional electron gases *Phys. Rev. B* 83 115135

[7] Agarwal A, Polini M, Vignale G and Flatté M E 2014 Persistent spin oscillations in a spin–orbit-coupled superconductor *Phys. Rev. B* 90 155409

[8] Brey L, Fertig H A and Das Sarma S 2007 Diluted graphene antiferromagnet *Phys. Rev. Lett.* 99 116802

[9] Schliemann J 2010 Dielectric function of the semiconductor hole gas *Europhys. Lett.* 91 67004

[10] Schliemann J 2011 Dielectric function of the semiconductor hole liquid: full frequency and wave-vector dependence *Phys. Rev. B* 84 155201

[11] Scholz A, Stauber T and Schliemann J 2012 Dielectric function, screening, and plasmons of graphene in the presence of spin–orbit interactions *Phys. Rev. B* 86 195424

[12] Ozbay E 2006 *Plasmonics: merging photonics and electronics at nanoscale dimensions* Science 311 189

[13] Marinica D C, Zatapa M, Nordlander P, Kazansky A K, Echenique P M, Aizpurua J and Borisov A G 2015 Active quantum plasmonics *Sci. Adv.* 1 e1501095

[14] Meinzer N, Barnes W L and Hooper I R 2014 Plasmonic meta-atoms and metasurfaces *Nat. Photon.* 8 889

[15] Lu J, Geng B, Horng J, Girit C, Martin M, Hao Z, Bechtel H A, Liang X, Zettl A, Ron Shen Y and Wang F 2011 Graphene plasmonics for tunable terahertz metamaterials *Nat. Nanotechnol.* 6 630

[16] Grigorenko A N, Polini M and Novoselov K S 2012 Graphene plasmonics *Nat. Photon.* 6 749

[17] Stauber T 2014 Plasmonics in Dirac systems: from graphene to topological insulators *J. Phys.: Condens. Matter* 26 123201

[18] Koppens F H L, Chang D E and Garcia de Abajo F J 2011 Graphene plasmonics: a platform for strong light–matter interactions *Nano Lett.* 11 3370

[19] Garcia de Abajo F J 2014 Graphene plasmonics: challenges and opportunities *ACS Photonics* 1 1135

[20] Weick G, Woollacott C, Barnes W L, Hess O and Mariani E 2013 Dirac-like plasmons in honeycomb lattices of metallic nanoparticles *Phys. Rev. Lett.* 110 106801

[21] Downing C A and Weick G 2016 Topological collective plasmons in biparticle chains of metallic nanoparticles (arXiv:1611.03249)

[22] Wunsch B, Stauber T, Sols F and Guinea F 2006 Dynamical polarization of graphene at finite doping *New J. Phys.* 8 318
[23] Hwang E H and Das Sarma S 2007 Dielectric function, screening, and plasmons in two-dimensional graphene Phys. Rev. B 75 205418
[24] Barlas Y, Perez-Barnea T, Polini M, Asgari R and MacDonald A H 2007 Chirality and correlations in graphene Phys. Rev. Lett. 98 236601
Polini M, Asgari R, Borghi G, Barlas Y, Perez-Barnea T and MacDonald A H 2008 Plasmons and the spectral function of graphene Phys. Rev. B 77 081411
[25] Jablan M, Buljan H and Soljacic M 2009 Plasmonics in graphene: a way to measure spin polarization Phys. Rev. B 91 245407
[26] Di Pietro P et al 2013 Observation of Dirac plasmons in a topological insulator Nat. Nanotechnol. 8 556
[27] Laghu S, Chung S B, Qi X-L and Zhang S-C 2010 Collective modes of a helical liquid Phys. Rev. Lett. 104 116401
[28] Lv M and Zhang S-C 2013 Dielectric function, Friedel oscillations and plasmons in Weyl semimetals Int. J. Mod. Phys. B 27 1350177
[29] Zhou J, Zhang H-R and Xiao D 2015 Plasmon mode as a detection of the chiral anomaly in Weyl semimetals Phys. Rev. B 91 245407
[30] Panfilov I, Burkov A A and Pesin D A 2014 Density response in Weyl metals Phys. Rev. B 89 245103
[31] Hofmann J and Das Sarma S 2015 Plasmon signature in Dirac–Weyl liquids Phys. Rev. B 91 241108
[32] Pyatkovskiy P K 2009 Dynamical polarization, screening, and plasmons in gapped graphene J. Phys.: Condens. Matter 21 025306
[33] Ramezanali M R, Vazifeh M M, Asgari R, Polini M and Barlas Y 2015 Observation of Dirac plasmons in a topological insulator Phys. Rev. B 91 035114
[34] Tabert C J and Nicol E J 2014 Dynamical polarization function, plasmons, and screening in silicone and other buckled honeycomb lattices Phys. Rev. B 89 195410
[35] Chang H R, Zhou J, Zhang H and Yao Y 2014 Probing the topological phase transition via density oscillations in silicone and germanene Phys. Rev. B 89 201411
[36] Dupper B V, Vasiliopoulos P and Peeters F M 2014 Spin and valley polarization of plasmons in silicone due to external fields Phys. Rev. B 90 035142
[37] Scholz A and Schliemann J 2011 Dynamical current–current susceptibility of gapped graphene Phys. Rev. B 83 235409
[38] Sachdeva R, Thakur A, Vignale G and Agarwal A 2015 Plasmon modes of a massive Dirac plasma, and their superlattices Phys. Rev. B 91 205426
[39] Das Sarma S and Hwang E H 2009 Collective modes of the massless Dirac plasma Phys. Rev. Lett. 102 206412
[40] Das Sarma S and Lu W-Y 1985 Screening and elementary excitations in narrow-channel semiconductor microstructures Phys. Rev. B 32 1401
[41] Badalyan S M, Matos-Abiague A, Vignale G and Fabian J 2010 Beating of Friedel oscillations induced by spin–orbit interaction Phys. Rev. B 81 205314
[42] Chen G-H and Raikh M E 1999 Small-q anomaly in the dielectric function and high-temperature oscillations of the screening potential in a two-dimensional electron gas with spin–orbit coupling Phys. Rev. B 59 5090
[43] Simon G E and Giuliani G F 2005 Friedel oscillations in a Fermi liquid Phys. Rev. B 72 045127
[44] Crommie M F, Lutz C P and Eigler D M 1993 Imaging standing waves in a two-dimensional electron gas Nature 363 524–7
[45] Hasegawa Y and Avouris Ph 1993 Direct observation of standing wave formation at surface steps using scanning tunneling spectroscopy Phys. Rev. Lett. 71 1071
[46] Sprunger P T, Petersen L, Plummer E W, Lie gsaard E and Besenbacher F 2001 Giant Friedel oscillations on the beryllium(000 1) surface Science 307 1764
[47] Topinka M A, LeRoy B J, Westervelt R M, Shaw S E J, Fleischmann R, Heller E J, Maranowski K D and Gossard A C 2001 Coherent branched flow in a two-dimensional electron gas Nature 410 183
[48] Bai A and Pilon E 1991 On the axial anomaly at finite temperature in the Schwinger model Z. Phys. C 52 339
[49] Katnelson M I, Novoselov K S and Geim A K 2006 Chiral tunnelling and the Klein paradox in graphene Nat. Phys. 2 620
[50] Klein O 1929 Die Reflexion von Elektronen an einem Potentialfluss der relativistischen Dynamik von Dirac Z. Phys. 53 157
[51] Giamarchi T 2003 Quantum Physics in One Dimension (New York: Oxford University Press)
[52] Gori A R, Pinczuk A, Weiner J S, Calleja J M, Dennis B S, Pfeiffer L N and West K W 1991 One-dimensional plasmon dispersion and dispersionless intersubband excitations in GaAs quantum wires Phys. Rev. Lett. 67 3298
[53] Li Q P and Das Sarma S 1991 Elementary excitation spectrum of one-dimensional electron systems in confined semiconductor structures: zero magnetic field Phys. Rev. B 43 11768
[54] Egger R and Grabert H 1995 Friedel oscillations for interacting fermions in one dimension Phys. Rev. Lett. 75 3505
[55] Sarma S D and Li Q 2013 Intrinsic plasmons in two-dimensional Dirac materials Phys. Rev. B 87 235418
[56] Ramezani M R, Vazifeh M M, Asgari R, Polini M and MacDonald A H 2009 Finite-temperature screening and the specific heat of doped graphene sheets J. Phys. A: Math. Theor. 42 214015
[57] Patel D K, Ashraf S S Z and Sharma A C 2015 Finite temperature dynamical polarization and plasmons in gapped graphene Phys. Status Solidi B 252 1817–26