About Almost Geodesic Curves

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Abstract. We determine in $\mathbb{R}^n$ the form of curves $C$ for which also any image under an $(n-1)$-dimensional algebraic torus is an almost geodesic with respect to an affine connection $\nabla$ with constant coefficients and calculate the components of $\nabla$.

1. Introduction

This paper is a result following on from D. Betten researchs \cite{4} and our papers \cite{1, 8–10}.

The geodesics and almost geodesics play an important role in differential geometry. For this reason many geometricians study almost geodesic mappings (see \cite{3}, \cite{14}, \cite{17}). In \cite{12}, \cite{13} almost geodesic curves were considered in generalized Riemannian and Kählerian spaces. E. Beltrami \cite{6} has shown that a differentiable curve is a local geodesic with respect to an affine connection $\nabla$ precisely if it is a solution of an Abelian differential equation with coefficients which are functions of the components of $\nabla$. The investigation with systems of lines of 2-dimensional topological geometries was started in \cite{15}. The explicit calculation of the form of curves $C$ in the $n$-dimensional real space $\mathbb{R}^n$ which are geodesics or almost geodesics with respect to an affine connection $\nabla$ is not achievable even in the case if the components $\Gamma^j_{ij}$ of $\nabla$ are constant. But we did it. In \cite{2} the geodesics and special case of almost geodesics were considered. We supposed that with $C$ also all images of $C$ under a real $(n - 1)$-dimensional algebraic torus are also geodesics, respectively almost geodesics. This implies that the determination of $C$ becomes an algebraic problem (a problem of polynomial identities). Our model allows you to look at known things globally. In this paper we continue to study almost geodesic curves \cite{7}, \cite{16} and here we will consider other case.

We consider a curve $C$ homeomorphic to $\mathbb{R}$ which is a closed subset of $\mathbb{R}^n$ and has the form

$$C = (t, f_2(t), \ldots, f_n(t)), \ t \in \mathbb{R},$$

(1)

where $f_i(t): \mathbb{R} \to \mathbb{R}, i = 2, \ldots, n$, are three times differentiable non-constant functions. The system

$$\mathcal{X}(C) = \{(t + c_1, b_2f_2(t) + c_2, \ldots, b_nf_n(t) + c_n), t \in \mathbb{R}\},$$

where $b_i \neq 0, c_i \in \mathbb{R},$ is a set of imagines of $C$. 

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If every curve of \( x(C) \) is a geodesic with respect to an affine connection \( \nabla \) with constant coefficients \( \Gamma^h_{ij} \), then the derivatives \( f'_i(t) \) of the functions \( f_i(t) \) are solutions of the first order linear ordinary differential equations. If every curve of \( x(C) \) is an almost geodesic with respect to \( \nabla \), then the derivatives \( f'_i(t) \) are solutions of harmonic oscillator equations. If \( x(C) \) consists of Euclidean lines which are geodesics with respect to \( \nabla \), then at the most \( \Gamma^h_{ij} \) may be different from 0. In contrast to this if \( x(C) \) consists of Euclidean lines then there is huge quantity of non-trivial connections \( \nabla \) such that the lines of \( x(C) \) are almost geodesic with respect to \( \nabla \).

Since we apply results of differential geometry only for the \( n \)-dimensional space \( \mathbb{R}^n \), where global coordinates exist and the components \( \Gamma^h_{ij} \), \( h, i, j \in \{1, 2, \ldots, n\} \), of any affine connection \( \nabla \) can be written in unique way in these coordinates.

**Remark 1.1.** If coefficients \( \Gamma^h_{ij} \) of an affine connection \( \nabla \) are constants then there exist groups of affine movements.

**Remark 1.2.** It is possible to apply our model for other spaces, because a geodesic and an almost geodesic can be defined in other spaces in the same manner as in \( \mathbb{R}^n \) [11].

2. Almost geodesic curves

Let

\[
\ell = (t + c_1, b_2 f_2(t) + c_2, \ldots, b_n f_n(t) + c_n), \quad t \in \mathbb{R},
\]

be a curve of \( x(C) \). Then

\[
\ell = (1, b_2 f'_2(t), \ldots, b_n f'_n(t)), \quad \dot{\ell} = (0, b_2 f''_2(t), \ldots, b_n f''_n(t)).
\]

By an almost geodesic of an affine connection \( \nabla \) we mean a piecewise \( C^3 \)-curve \( \gamma: I \to \mathbb{R}^n \) satisfying

\[
\nabla_{\gamma'}(\nabla_{\gamma'}\gamma') = \varrho \cdot \gamma' + \sigma(t) \cdot \gamma'' + \sum_{i,j=1}^n \Gamma^h_{ij} \gamma^i \gamma^j.
\]

where \( \varrho, \sigma: I \to \mathbb{R} \) are continuous functions, \( I \subset \mathbb{R} \) is an open interval (cf. [16, p. 158], [7, p. 456]).

Using the components of \( \nabla \) the system of differential equations for almost geodesics has the form

\[
\ddot{\gamma}^i + \sum_{i,j,k=1}^n (\partial_i \Gamma^h_{ij} + \Gamma^h_{ij} \ddot{\gamma}^j + \sum_{i,j=1}^n \Gamma^h_{ij} \gamma^j \gamma^k + \sum_{i,j=1}^n \Gamma^h_{ij} \gamma^i \gamma^j \gamma^k + \sum_{i,j=1}^n \Gamma^h_{ij} \gamma^i \gamma^j \gamma^k = \varrho(t) \cdot \gamma^i + \sigma(t) \cdot (\gamma'' + \sum_{i,j=1}^n \Gamma^h_{ij} \gamma^i \gamma^j). \tag{2}
\]

A curve \( \ell \) of \( x(C) \) is an almost geodesic with respect to a connection \( \nabla \) with constant coefficients \( \{\Gamma^h_{ij}\} \) if and only if according to (2) we have

\[
\ddot{\ell}^i + \sum_{i,j,k=1}^n \Gamma^h_{ij} \ddot{\ell}^j \ell^k + 2 \sum_{i,j=1}^n \Gamma^h_{ij} \ell \dot{\ell}^j + \sum_{i,j=1}^n \Gamma^h_{ij} \dot{\ell} \dot{\ell}^j = \varrho(t) \cdot \dot{\ell}^i + \sigma(t) \cdot (\ddot{\ell} + \sum_{i,j=1}^n \Gamma^h_{ij} \ell \dot{\ell}^j). \tag{3}
\]

We rewrite the formula (3) for \( h = 1 \) and obtain the function \( \varrho(t) \). For \( h = 2, \ldots, n \) after substitution \( \varrho(t) \) in (3) we get

\[
\begin{align*}
&\sum_{i,j,k=1}^n (\Gamma^h_{ij} \ddot{\ell}^j \ell^k + 2 \sum_{i,j=1}^n \Gamma^h_{ij} \ell \dot{\ell}^j + \sum_{i,j=1}^n \Gamma^h_{ij} \dot{\ell} \dot{\ell}^j) = \varrho(t) \cdot \ddot{\ell}^i + \sigma(t) \cdot (\ddot{\ell} + \sum_{i,j=1}^n \Gamma^h_{ij} \ell \dot{\ell}^j), \\
&\sum_{i,j,k=1}^n (\Gamma^h_{ij} \ddot{\ell}^j \ell^k + 2 \sum_{i,j=1}^n \Gamma^h_{ij} \ell \dot{\ell}^j + \sum_{i,j=1}^n \Gamma^h_{ij} \dot{\ell} \dot{\ell}^j) = \varrho(t) \cdot \ddot{\ell}^i + \sigma(t) \cdot (\ddot{\ell} + \sum_{i,j=1}^n \Gamma^h_{ij} \ell \dot{\ell}^j).
\end{align*}
\]
\[ \sum_{i,j=2}^{n} (2\Gamma_{ij}^1 + \Gamma_{ij}^1) b_i f''(t) + \sum_{i,j=2}^{n} (2\Gamma_{ij}^1 + \Gamma_{ij}^1) b_i f_j(t) f'_i(t) - \]
\[ b_i f''_i(t) \left( \sum_{m=1}^{n} \Gamma_{m1}^m \Gamma_{11}^m + \sum_{i=2,m=1}^{n} (\Gamma_{m1}^m \Gamma_{11}^m + \Gamma_{i1}^m \Gamma_{11}^m + \Gamma_{1m}^m \Gamma_{11}^m) b_i f'_i(t) + \right. \]
\[ \sum_{i,j=2,m=1}^{n} (\Gamma_{m1}^m \Gamma_{11}^m + \Gamma_{i1}^m \Gamma_{11}^m + \Gamma_{1m}^m \Gamma_{11}^m) b_i f_j(t) f'_i(t) f'_j(t) + \]
\[ \sum_{i,j,k=2,m=1}^{n} \Gamma_{m1}^m \Gamma_{11}^m b_i b_j b_k f'_i(t) f'_j(t) f'_k(t) \]
\[ \left. \sum_{i,j=2}^{n} (2\Gamma_{ij}^1 + \Gamma_{ij}^1) b_i f''(t) + \sum_{i,j=2}^{n} (2\Gamma_{ij}^1 + \Gamma_{ij}^1) b_i f_j(t) f'_i(t) f'_j(t) \right) = \]
\[ \sigma(t) \cdot \left( \Gamma_{11}^1 - \Gamma_{11}^1 + 2 \sum_{i=2}^{n} (\Gamma_{li}^1 + \Gamma_{li}^1 - \Gamma_{i1}^1) b_i f'_i(t) + \sum_{i,j=2}^{n} (\Gamma_{ij}^1 - \Gamma_{ij}^1) b_i f_j(t) f'_i(t) f'_j(t) \right). \]

(4)

One can determine \( \sigma \) only if not all coefficients in (4) are zero. In [2] we treated the case that for \( h \geq 2 \) one has
\[ \Gamma_{li}^1 + \Gamma_{li}^1 = \Gamma_{i1}^1 + \Gamma_{i1}^1, \quad i \geq 1, \quad \text{and} \quad \Gamma_{ij}^1 = \Gamma_{ij}^1 \] for all \( i, j \geq 2 \).

Now let an \( \alpha \) and \( i_0, j_0 \) such that for these indices we have
\[ \Gamma_{11}^a \neq \Gamma_{11}^1 \text{ or } \Gamma_{10}^a = \Gamma_{10}^1, \quad i_0, j_0 \geq 2. \]

(5)

In this case the coefficient of \( \sigma \) is not identically zero, and we can compute \( \sigma \). Putting the expression of \( \sigma \) into relation (4) we obtain
\[ \left( \Gamma_{11}^1 - \Gamma_{11}^1 \right) + 2(\Gamma_{li}^1 + \Gamma_{li}^1 - \Gamma_{i1}^1) f'_i b_i + \left( \Gamma_{ij}^1 - \Gamma_{ij}^1 \right) f'_j b_j \right) \]
\[ = \left( T_{a111} + (f''' - T_{a111}) \right) b_a + \sum_{i=2}^{n} \left( S_{a111} f''_i + (2\Gamma_{li}^1 + \Gamma_{li}^1) f'_i b_i b_i \right) \]
\[ - \sum_{i,j=2}^{n} \left( S_{a111} f''_i + (2\Gamma_{li}^1 + \Gamma_{li}^1) f'_i b_i \right) b_i b_j + \sum_{i,j,k=2}^{n} T_{a1i} f''_i f''_j f''_k b_i b_j b_k = \left( \Gamma_{i1}^1 - \Gamma_{i1}^1 \right) + 2 \left( \Gamma_{i1}^1 + \Gamma_{i1}^1 - \Gamma_{i1}^1 \right) f'_i b_i + \sum_{i,j=2}^{n} \left( \Gamma_{ij}^1 - \Gamma_{ij}^1 \right) f'_i f'_j b_i b_j \right) \]
\[ = \left( T_{a111} + (f'''' - T_{a111}) \right) b_a + \left( S_{a111} f''_i + (2\Gamma_{li}^1 + \Gamma_{li}^1) f'_i \right) b_i b_i + \left( S_{a111} f''_i + (2\Gamma_{li}^1 + \Gamma_{li}^1) f'_i \right) b_i b_j \]
\[ + \sum_{i,j,k=2}^{n} T_{a1i} f''_i f''_j f''_k b_i b_j b_k \right) \]
where
\[ S_{ABCD} \stackrel{\text{def}}{=} \sum_{m=1}^{n} (\Gamma^A_{mD} \Gamma^m_{Be} + \Gamma^A_{mB} (\Gamma^m_{De} + \Gamma^m_{CD})), \]

\[ T_{ABCD} \stackrel{\text{def}}{=} \sum_{m=1}^{n} \Gamma^A_{mB} \Gamma^m_{CD}. \]

If \( n = 2 \), then \( h = \alpha = 2 \) and from (6) we obtain that any plane curve of the system \( X(\mathcal{C}) \), where \( \mathcal{C} \) has the form (1), is an almost geodesic if the affine connection \( \Gamma^h_{ij} \) satisfies the conditions (5). Hence we assume \( n \geq 3 \).

Now we consider the first case, when
\[ \Gamma^h_{11} = \Gamma^1_{11}, \text{ for all } 2 \leq h \leq n \text{ and } \Gamma^h_{11} + \Gamma^h_{1i} = \Gamma^1_{1i} + \Gamma^1_{1i}, \text{ for all } 2 \leq i \leq n, \]

but there exists an \( a \) and \( i_0, j_0 \) such that
\[ \Gamma^a_{i_0j_0} \neq \Gamma^1_{i_0j_0}. \]

Writing a system of equations and conditions which follow from (6) and using linear independence functions we get differential equations. Integrating them (see. [5]) we obtain the following

**Theorem 2.1.** Let \( C \) be a curve of the form (1) and \( \nabla \) be a connection with constant coefficients \( \{ \Gamma^h_{ij} \} \) satisfying relations (7), (8).

Then any curve \( \ell \) of \( \mathcal{X}(\mathcal{C}) \) is almost geodesic with respect to \( \nabla \) if and only if \( \ell \) is represented by the functions \( f_h, f_a, f_0 \) having the following forms

* \( f_h(t) = C_h e^{\alpha t} + D_h e^{2\alpha t} \), where \( C_h, D_h \in \mathbb{R} \) are not both zero and \( a^2_n - 4c_h > 0 \),

* \( f_h(t) = (C_h t + D_h) e^{2\alpha t} \), where \( C_h, D_h \in \mathbb{R} \) are not both zero and \( a^2_n - 4c_h = 0 \),

* \( f_h(t) = e^{a_0 t/2} (a_0 \cos \sqrt{\frac{a^2_n - 4c_h}{2}} t + D_h \sin \sqrt{\frac{a^2_n - 4c_h}{2}} t) \), where \( C_h, D_h \in \mathbb{R} \) are not both zero and \( a^2_n - 4c_h < 0 \)

with
\[
\alpha_h = 2\Gamma^h_{h3} + \Gamma^h_{1h}, \quad c_h = S_{011h} - T_{1111},
\]
\[
\lambda_1^h = \frac{-a_h - \sqrt{a^2_h - 4c_h}}{2}, \quad \lambda_2^h = \frac{-a_h + \sqrt{a^2_h - 4c_h}}{2};
\]

* \( f_a(t) = C_a t^2 + D_a t + E \), where \( C_a, D_a, E \in \mathbb{R} \), \( C_a, D_a \) are not both zero and \( \gamma_a = 0 \),

* \( f_a(t) = C_a e^{-\sqrt{\gamma_a t}} - D_a e^{-\sqrt{\gamma_a t}} \), where \( C_a, D_a \in \mathbb{R} \) are not both zero and \( \gamma_a < 0 \),

* \( f_a(t) = \tilde{C}_a \sin(\sqrt{\gamma_a t}) - \tilde{D}_a \cos(\sqrt{\gamma_a t}) \), where \( \tilde{C}_a, \tilde{D}_a \in \mathbb{R} \) are not both zero and \( \gamma_a > 0 \)

with
\[
\gamma_a = \frac{(\Gamma^a_{i_0j_0} - \Gamma^1_{i_0j_0})(T_{h_{aij}} + T_{h_{aji}} + T_{h_{iaj}} + T_{h_{aij}} + T_{h_{ija}} + T_{h_{ija}})}{\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji}} - T_{1111}.
\]

* \( f_h(t) = \tilde{C}_h e^{\alpha t/2} + \tilde{D}_h e^{2\alpha t} \), where \( \tilde{C}_h, \tilde{D}_h \in \mathbb{R} \) are not both zero and \( a^2_h - 4c_h > 0 \),

* \( f_h(t) = (\tilde{C}_h t + \tilde{D}_h) e^{2\alpha t} \), where \( \tilde{C}_h, \tilde{D}_h \in \mathbb{R} \) are not both zero and \( a^2_h - 4c_h = 0 \),

* \( f_h(t) = e^{-a_0 t/2} (a_0 \cos \sqrt{\frac{a^2_h - 4c_h}{2}} t + D_h \sin \sqrt{\frac{a^2_h - 4c_h}{2}} t) \), where \( \tilde{C}_h, \tilde{D}_h \in \mathbb{R} \) are not both zero and \( a^2_h - 4c_h < 0 \)
with

\[ a_{ab} = 2\Gamma^b_{a1} + \Gamma^b_{1a}, \]
\[ c_{bh} = \frac{(\Gamma^b_{ij} - \Gamma^i_{bj})(T_{h\alpha ij} + T_{hhi j} + T_{hij \alpha} + T_{hij \alpha})}{\Gamma^i_{ij} + \Gamma^j_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji}} + S_{b11h} - T_{111h}, \]
\[ \lambda^b_1 = \frac{-a_{ab} - \sqrt{a^2 - 4c_{bh}}}{2}, \quad \lambda^b_2 = \frac{-a_{ab} + \sqrt{a^2 - 4c_{bh}}}{2}. \]

The components \( \{\Gamma^i_{ij}\} \) of affine connection \( \nabla \) satisfy the following relations

\[ 2\Gamma^1_{ij} + \Gamma^2_{ij} = 0, \quad 2\Gamma^1_{ij} + \Gamma^3_{ij} = 0, \quad 2\Gamma^1_{ij} + \Gamma^4_{ij} = 0, \quad 2\Gamma^3_{ij} + \Gamma^4_{ij} = 0, \quad 2\Gamma^1_{ij} + \Gamma^h_{ij} = 0, \]
\[ \Gamma^i_{ij} + \Gamma^j_{ji} - \Gamma^k_{ij} - \Gamma^k_{ji} = 0 \text{ for } k = i, \alpha, \quad 2\Gamma^2_{ik} + \Gamma^3_{ki} - 2\Gamma^1_{ij} - \Gamma^1_{ji} = 0 \text{ for } k = i_0, h, \]
\[ (\Gamma^1_{ij} + \Gamma^2_{ji} - \Gamma^3_{ij} - \Gamma^3_{ji})S_{111h} = 0, \quad (\Gamma^1_{ij} + \Gamma^2_{ji} - \Gamma^3_{ij} - \Gamma^3_{ji})(2\Gamma^1_{ij} + \Gamma^1_{ij}) = 0, \]
\[ (\Gamma^1_{ij} + \Gamma^2_{ji} - \Gamma^3_{ij} - \Gamma^3_{ji})(2\Gamma^1_{ij} + \Gamma^1_{ij}) = 0, \]
\[ (\Gamma^1_{ij} - \Gamma^3_{ij} - \Gamma^2_{ji} - \Gamma^2_{ij})T_{a111} = 0, \]
\[ (\Gamma^1_{ij} - \Gamma^3_{ij} - \Gamma^2_{ji} - \Gamma^2_{ij})(S_{kk1} + S_{kk1} - S_{111}) + (\Gamma^1_{ij} + \Gamma^3_{ij} - \Gamma^2_{ki} - \Gamma^2_{ik})T_{a111} = 0 \text{ for } k = i_0, h, \]
\[ (\Gamma^1_{ij} + \Gamma^3_{ij} - \Gamma^2_{ij} - \Gamma^2_{ji})T_{1b_i jk} + (\Gamma^1_{ij} + \Gamma^3_{ij} - \Gamma^2_{ik} - \Gamma^2_{ki})T_{1b_i jk} = 0, \]
\[ S_{111h} = 2\Gamma^1_{h1} + \Gamma^1_{h1} = 0 \text{ if } f^h_{\neq} = 0, \quad S_{111h} = 2\Gamma^1_{h1} + \Gamma^1_{h1} \text{ if } f^h_{\neq} = D_h e^\frac{2\pi}{t}, \]
\[ (\Gamma^1_{ij} + \Gamma^3_{ij} - \Gamma^2_{ij} - \Gamma^2_{ji})(T_{ab_1hj} - S_{111b_1h}) + (\Gamma^1_{ab} + \Gamma^3_{ab} - \Gamma^2_{ab} - \Gamma^2_{ab})T_{ab_1hj} = 0, \]
\[ (\Gamma^1_{ab} + \Gamma^3_{ab} - \Gamma^2_{ab} - \Gamma^2_{ab})T_{ab_1hj} = 0, \]
\[ (\Gamma^1_{ab} + \Gamma^3_{ab} - \Gamma^2_{ab} - \Gamma^2_{ab})(S_{ab_1h}) = 0, \]
\[ (\Gamma^1_{ij} + \Gamma^3_{ij} - \Gamma^2_{ij} - \Gamma^2_{ji})(S_{ab_1h}) = 0, \]
\[ T_{a_1h_1} = S_{1b_1h_1} + S_{b_1h_1} = 0, \]
\[ T_{ab_1h} + T_{ab_1h} + T_{ab_2h} + T_{ab_2h} + T_{ab_2h} + T_{ab_2h} = 0, \]
\[ T_{kk_1j} + T_{kk_1j} + T_{kk_1j} + T_{kk_1j} + T_{kk_1j} - S_{kk_1j} - S_{111j} = 0 \text{ for } k = i_0, h, \alpha. \]
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