Stochastic Resonance in Two Dimensional Landau Ginzburg Equation

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We study the mechanism of stochastic resonance in a two dimensional Landau Ginzburg equation perturbed by a white noise. We shortly review how to renormalize the equation in order to avoid ultraviolet divergences. Next we show that the renormalization amplifies the effect of the small periodic perturbation in the system. We finally argue that stochastic resonance can be used to highlight the effect of renormalization in spatially extended system with a bistable equilibrium.

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About twenty years ago, in studying the possible causes of the ice ages\(^1\)\(^2\), we introduced the concept of stochastic resonance\(^3\). This process (hereafter, SR) requires that a physical system has its fundamental symmetry broken in such a way that its phase space is not homogeneous. Thereby, the time spent in different regions of the phase space is not uniformly distributed. If the system is stochastically forced, the probability of large fluctuation can be periodically locked to a small external periodic forcing. The relevant implications of the SR are the generality of its set up and the limited band, in the parameter space, wherein the mechanism occurs. Furthermore, we notice that the mechanism only requires that the system has large deviations properties and, therefore, the restriction to systems stochastically perturbed, is just a case amenable to explicit evaluation of the resonance range in the parameter space. In practice, a stochastic forcing is required only for systems of gradient type (i.e., when a Lyapunov function exists surely) or whenever single point attractors exist.

At the time of the publication above mentioned, we had doubts whether the mathematical process leading to stochastic resonance were a stochastic process at all. Fortunately in\(^4\)\(^5\)\(^6\), has proved that a wide class of stochastic process obeying to a SR mechanism exists. Therefore we feel confident that, at least in a finite dimensional space, the SR is a well posed mathematical problem. When we consider stochastically perturbed infinite dimensional systems the problem may lead to a not consistent mathematics since the equations for the backward and the forward probability density function may not exist. Moreover, when a white noise in space and time is considered, the resulting equations are plagued by divergent correlation functions. In a series of papers\(^7\)\(^8\)\(^9\), it was proved that these problems could be overcome by the method of the renormalisation theory. On the ground of these rigorous, mathematical results, in the present paper we wish to study the consequences of applying an external periodic (in time) forcing to a simple infinite dimensional stochastic equation.

In particular, we discuss the SR phenomenon in two dimensional Landau Ginzburg equation stochastically perturbed by a white noise:

\[
\frac{\partial \phi}{\partial t} = \nu \Delta \phi - \frac{\partial V(\phi)}{\partial \phi} + \epsilon \frac{dW}{dt}
\]

where

\[
V(\phi) = -\frac{1}{2} m \phi^2 - \frac{1}{4} \phi^4
\]

and \(dW/dt\) is a white noise delta correlated both in space and time. The field \(\phi\) is assumed to be real and defined on the torus \([0, L] X [0, L], L = 2\pi\), with periodic boundary conditions. Notice that the previous equation describes mean field theories nearby critical points, therefore, the use of white noise in space and time is mandatory when we do not want to introduce unknown physical scales. Moreover, unless \(m < 0\), the theory has no broken symmetries.

As we mentioned, the phenomenon of SR has been widely studied in literature since it was introduced\(^12\)\(^13\)\(^14\), for a review see\(^10\). The study of SR in one space dimension, with a broken symmetry, has been presented in\(^11\). In this case, the stochastic equation does not show any particular novelty. As we shall see, the extension to more spatial dimension leads to interesting consequences. In particular we will consider their effects on the SR.

As it has been discussed in\(^12\)\(^16\), that equation\(^11\) has no meaning in more than one space dimension unless it is renormalized. In \(D = 2\), the renormalization amounts to change any non linear term in\(^11\) by its Wick product, a rather well known procedure in quantum field theory. By using the Wick product, equations\(^11\) and\(^2\) become:

\[
\frac{\partial \phi_\Lambda}{\partial t} = \nu \Delta \phi_\Lambda - m \phi_\Lambda - g \phi^3_\Lambda + 3gE_m(\Lambda) \phi_\Lambda + \epsilon \frac{dW_\Lambda}{dt}
\]

In eq.\(^3\) we have introduced the cutoff field \(\phi_\Lambda\), where \(\Lambda\) is an ultraviolet cutoff in the Fourier space. Namely by denoting \(\Phi(\vec{k})\) the Fourier transform of \(\phi(x)\) we have:

\[
\phi_\Lambda = \int_{|\vec{k}| < \Lambda} \Phi(\vec{k}) e^{i\vec{k}\vec{x}} \, d\vec{k}
\]

where
and an analogous definition has been used for $\frac{dW}{dt}$. In the quantity $E_m(\Lambda)$ is the expectation value of second order moment of the field $Z_\Lambda$ satisfying the linear stochastic differential equation:

$$\partial Z_\Lambda = \nu \Delta Z_\Lambda - m Z_\Lambda + \epsilon \frac{dW_\Lambda}{dt} \tag{5}$$

In two space dimensions one has:

$$E_m(\Lambda) = C \frac{\epsilon^2}{\nu} \ln \left[ \frac{\mu^2 \Lambda^2 + m}{m} \right] \tag{6}$$

where $C$ is a constant order one. The value of $\rho$ is determined by the following set of equations:

$$\rho = -m + 3g [E_m(\Lambda) - E_\alpha(\Lambda)] \tag{9}$$

$$\alpha = \frac{\partial^2 V_{eff}(\phi)}{\partial \phi^2} \text{ at } \phi = \phi_e \tag{10}$$

where $\phi_e$ is the equilibria of the effective potential, i.e. $\phi_e$ is determined by solving the equation:

$$\frac{\partial V_{eff}}{\partial \phi} = 0 \tag{11}$$

The set of equations (9, 10, 11) have a clear physical meaning, shortly reviewed as follows. First of all, let us note that $\rho$ is independent of $\Lambda$ in the limit $\Lambda \rightarrow \infty$ because of eq. (6). Eq. (9) is a redefinition of the Wick product in terms of a new constant $\phi_\Lambda$. The physical idea of eq. (10) is that, the counter term introduced in order to renormalize the Landau Ginzburg equation, produces a double well potential. The position of the new minimum, determined by $\rho$, must be consistent with the fluctuations around the minimum itself.

Using equations (10, 11, 12) one gets a nonlinear equation for the variable $x = \frac{\phi_\Lambda}{m}$, namely

$$x = -2 + 2K \log x \tag{12}$$

where $K = \frac{3gC}{m\nu}$. Let us remark, for future purpose, that it is possible to show that a solution of eq. (12) always satisfies $x > 2K$. As discussed in [12], the effective potential describes rather well the statistical properties of the space average field of eq. (11).

In general we can consider the variables $\Phi(l)$ defined as the average in space of $\phi(x)$ on a box of side $l$. For $l < L = 2\pi$, the statistical properties of $\Phi(l)$ are no longer described by the same value of $\rho$ because of the increasing effect, at small scales, of the stochastic fluctuations. In particular, for $l \rightarrow 0$, one finds that the statistical properties of $\Phi(l)$ are described by a single well potential centered around $\Phi(l) = 0$.

We want now to discuss the solutions of eq. (11) when a small periodic perturbation is added to the system and in particular we want to discuss the effect of renormalization of the mechanism on the stochastic resonance.

To this aim, following the original discussion given in [3] we consider the case of a time independent constant $A$ added to the r.h.s. of (11). The crucial point is the computation of the effective potential $V_{eff}$ previously introduced. Equation (10) are unchanged while equations (11, 12) become:

$$\alpha = -\rho + 3g \phi_e^2 \tag{13}$$

$$\rho \phi_e - g \phi_e^3 + A = 0 \tag{14}$$

The effect of the constant perturbation is to change the equilibrium solution of the effective potential and therefore to change the fluctuations around the equilibrium solution, which in turn change the value of $\rho$. The latter effect is due only to renormalization.

In order to understand qualitatively the contribution due to renormalization, one can compute a perturbative solution of the set of equations (6, 9, 10) in power of $A$. At first order one obtains:

$$\rho_{1,2} = \rho_0 + \frac{6g \phi_e \omega H A}{1 - 2H 2\rho_0} \tag{15}$$

$$\phi_{e1,2} = \phi_{e0} + \frac{1 + H A}{1 - 2H 2\rho_0} \tag{16}$$

where $H = \frac{3gC}{\rho_0}$ (with $\omega_0 = 2\rho_0$) and all quantities with the index 0 refer to values computed for $A = 0$. The previous definition of $H$ implies $H = \frac{K}{\nu} < \frac{1}{2}$ because of the discussion on the solution of eq. (12). Let us remark that we obtain two different values of $\rho$ and $\phi_e$ because two different values of $\phi_{e0}$ are possible, one negative and the other one positive. Thus there exists two possible effective potential, each one describing the two well respectively.

By using equations (15) and (16) is now possible to compute the effective potential difference between the two stable solutions and the unstable solution. One gets:

$$\Delta V_{1,2} = -\frac{\rho^2}{4g} + A_{R}\phi_{e0} \tag{17}$$

$$A_R = A(1 + \frac{3H}{2(1 - 2H)}) \tag{18}$$
In the limit of $\epsilon \to 0$ we have $H \to 0$ and the effect of renormalization disappears. For finite value of the noise and for $g$ large enough the renormalization amplifies the asymmetry of the double well potential. Therefore, we should expect that, in two dimensions Landau Ginzburg equation, the mechanism of SR acts for smaller value of the forcing amplitude with respect to the one estimated by looking at the equilibrium probability distribution with no periodic forcing.

In order to test the renormalization effect on SR, we have performed a numerical simulation on square lattice of 16x16 points. The parameters of our numerical simulation are $g=220$, $\nu=0.1$, $m=0.1$, $\epsilon=0.1$. On the lattice, the quantity $E_m(\Lambda)$, defined in eq. (10), is given by:

$$E_m(a) = \frac{\sigma^2}{2} S(m, \nu, q, p)$$  \hspace{1cm} (19)

where

$$S(m, \nu, q, p) = ma^2 + 4\nu - 2\nu \cos(2\pi q/N) - 2\cos(2\pi p/N)$$

In equation (19) we introduce the lattice mesh $a = 2\pi/N$. It follows that $\Lambda = a^{-1}$.

For different values of $A$, by applying Newton method, we have computed the solution of equations (9,13,14), using (19). It turns out that, with our numerical parameters, $\rho_0 = 4.9$, and $A_R \sim 2A$, i.e, the external forcing is amplified by almost a factor 2.

We have performed a long time integration of (1) with $A = 0$. The equilibrium probability distribution of $\Phi(L)$ has been found to be bimodal with maxima centered at $\pm \sqrt{\frac{2\nu}{g}}$ and the average transition time $\tau$ between the two equilibrium $\tau = 9$ in time units of $m^{-1}$. Next, using the analytical theory of SR discussed in [2], we have computed, by taking into account the amplification of the renormalization, the value of $A$ for which the system should show a clear SR. We have found $A = 0.1$. Finally we have applied a periodic perturbation on the r.h.s. of eq. (1) with amplitude $A = 0.1$ and period equal to $2\tau = 18$, i.e. two times the average transition time. We have performed a long time integration corresponding to 2048 time units. In figure (1) we show the power spectrum of the average field $\Phi(L)$. As one can clearly seen, there is an extremely well defined peak at frequency 113 which corresponds to a period of 18 time units.

In order to show the amplification effect on SR due to renormalization, we have numerically integrated the zero dimensional stochastic differential equation:

$$\frac{dx}{dt} = \rho_0 x - gx^3 + A \cos\left(\frac{2\pi}{18}\right) + \sigma \frac{dW}{dt}$$  \hspace{1cm} (20)

where $A = 0.1$ as before and the variance $\sigma$ of the white noise has been tuned in order to reproduce an average transition time of 9 ($\sigma = 0.0165$) for $A = 0$. With this choice of $\sigma$ we have been able to reproduce the equilibrium probability distribution of $\Phi(L)$ at $A = 0$ for the 2D Landau Ginzburg equation. As for the previous case, we have integrated eq. (20) for 2048 time units. In figure (2) we show the power spectrum of $x$. At variance with figure (1), only a small effect of the periodic forcing is felt by the system, i.e. the system does not exhibit any SR.

The comparison between figures (1) and (2), indicates clearly that the renormalization effect is acting as an amplifier of the external forcing, as previously discussed in the framework of a perturbation theory.

It is quite interesting to compute the statistics of the transition times between the minima at different scales as described by the variables $\Phi(l)$. In absence of forcing the probability distribution of the transition time for $\Phi(L)$ is exponential [2]. In the stochastic resonance the proba-
FIG. 3: Probability distribution of the transition time for $\Phi(L)$, continuos curve, and $\Phi(L/16)$, continuos curve with symbols.

probability distribution of the transition time for $\Phi(L)$ should show a well defined maximum at 9 time units. On the other hand, if we consider the transition time for $\Phi(l)$ with $l \ll L$, then we expect that the forcing is not able to produce any SR. This is due to the fact that the parameter describing the effective potential for small $l$ are changed and, in particular, the average transition time between the minima becomes much smaller $^{12}$. This effect is indeed observed in our numerical simulation. In fig. (2) we show the probability distribution of the transition time for $\Phi(L)$ and $\Phi(L/16)$. As predicted, for the small scale, the probability distribution of the transition time is exponential. We can therefore reach the conclusion that, because of the renormalization, the SR mechanism becomes scale dependant, a feature not previously observed.

In this letter we have shown that renormalization can amplify the SR in two dimensional Landau Ginzburg equation. We want to remark that this effect can also be used to test, in a given experimental situation, whether or not a renormalization mechanism is acting in the system. In this case SR can be used as a tool to measure quantitatively the effect, if any, of renormalization in spatially exntended systems. We remark that the present effect may be not restricted to system of gradient type and, in principle, could be detected in spatially extended deterministic systems.

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