GEOMETRIC STABILITY
OF THE COULOMB ENERGY

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Abstract. The Coulomb energy of a charge distribution that is
uniformly distributed on some set is maximized (among sets of
given volume) by balls. It is shown here that near-maximizers are
close to balls.

1. Introduction and main result

The Coulomb energy of a charge distribution $f$ on $\mathbb{R}^3$ is — up to a
multiplicative physical constant — given by the singular integral

$$
\mathcal{E}(f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)f(y)}{|x-y|} \, dx \, dy.
$$

According to a classical inequality of F. Riesz, the energy of a positive
charge distribution increases under symmetric decreasing rearrange-
ment: If $f^*$ is radially decreasing and equimeasurable with $f$, then

(1) $$
\mathcal{E}(f) \leq \mathcal{E}(f^*).
$$

The physical reason is that symmetrization increases the interaction of
the charges by reducing the typical distance between them. Equality
holds only if the charge distribution is already radially decreasing about
some point in $\mathbb{R}^3$. Is this characterization of equality cases stable?
If the two sides of Eq. (1) almost agree, how close must $f$ be to a
translate of $f^*$?

We answer this question for charge distributions that are uniform
on some set $A \subset \mathbb{R}^3$ of finite volume. Let $A^*$ be the ball of the same
volume. With a slight abuse of notation, denote by

$$
\mathcal{E}(A) = \int_A \int_A \frac{1}{|x-y|} \, dx \, dy
$$

the Coulomb energy of the uniform charge distribution on $A$. 

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Theorem 1. There exists a constant $c > 0$ such that

$$
\frac{\mathcal{E}(A^*) - \mathcal{E}(A)}{\mathcal{E}(A^*)} \geq c \left( \inf_{\tau} \frac{\text{Vol}(\tau A \triangle A^*)}{2\text{Vol}(A)} \right)^2.
$$

for every $A \subset \mathbb{R}^3$ of finite positive volume. Here, $\tau$ runs over all translations in $\mathbb{R}^3$, and $\triangle$ denotes the symmetric difference.

The exponent 2 is best possible; it is achieved for sets constructed from the unit ball by removing an annulus whose outer boundary is the unit sphere and adding an annulus of the same volume whose inner boundary is the unit sphere.

Geometric stability results where a deficit (the deviation of a functional from its optimal value) controls some measure of asymmetry (the distance from the manifold of optimizers) have been established for many classical inequalities. The first results in that direction, due to Bonnesen in the 1920s, were quantitative improvements of the isoperimetric inequality for convex sets in the plane. Two papers from the early 1990s have inspired much recent progress. One is Hall’s work on the isoperimetric inequality in $\mathbb{R}^n$, where he proves stability and raises the question of optimal exponents [2]; the other is the result of Bianchi and Egneell on the stability of the Sobolev inequality for $||\nabla f||^2$ in dimension $n \geq 3$ [3]. We refer the interested reader to the surveys [4, 5].

Less is known for non-local functionals that involve convolutions, even though stability results for those have important applications in Mathematical Physics [6]. In many variational problems for integral functionals, one can show by compactness arguments that all optimizing sequences must converge — modulo the symmetries of the functional — to extremals [7], but bounds for the asymmetry in terms of the deficit are a different matter. Very recently, Christ has introduced tools from additive number theory to prove stability of Riesz’ rearrangement inequality in one dimension [8]. Figalli and Jerison have obtained stability results on the Brunn-Minkowski inequality for non-convex sets in $\mathbb{R}^n$ [9]. For the Coulomb energy, Guo conjectured that

$$\mathcal{E}(f^*) - \mathcal{E}(f) \geq c' \inf_{\tau} \mathcal{E}(f \circ \tau^\tau - f^*)$$

with some constant $c' > 0$. (No normalization is required in this inequality, because both sides scale in the same way.) Since the Coulomb kernel is positive definite, the right hand side can be viewed as the square of a distance. The relationship between Eqs. (2) and (3) with $f = \chi_A$ will be clarified by Lemma 2.

The proof of Theorem 1 consist of two parts. After some preliminaries, we use the reflection positivity of the functional and a lemma of
Fusco, Maggi, and Pratelli [10] to reduce the problem to sets that are symmetric under reflection at the coordinate hyperplanes. The second part of the proof consists of an estimate for the Newton potential of symmetric sets. At the end of the paper, we briefly discuss stability for other Riesz kernels and in higher dimensions.

2. Notation, and stability in higher dimensions

By the *volume* of a set $A \subset \mathbb{R}^n$, denoted $\text{Vol}(A)$, we mean its $n$-dimensional Lebesgue measure. The centered open ball of the same volume is denoted by $A^*$; its radius is called the *volume radius* of $A$, and denoted by $R_A$. The *Fraenkel asymmetry* of $A$ is defined by

$$\alpha(A) = \inf_{\tau} \frac{\text{Vol}((\tau A) \bigtriangleup A^*)}{2\text{Vol}(A)}.$$  

Further, $B_R$ stands for the open ball of radius $R$ centered at the origin, and $\omega_n$ for the volume of the unit ball. The uniform surface measure that is induced on the sphere $\partial B_r \subset \mathbb{R}^n$ by the ambient Lebesgue measure is denoted by $\sigma$.

We consider functionals of the form

$$\mathcal{E}(A) = \int_A \int_A |x - y|^{-\lambda} \, dx \, dy$$

with $n \geq 3$ and $\lambda \in [n - 2, n)$. (The classical Coulomb energy corresponds to the case $n = 3$ and $\lambda = 1$.) These functionals share the properties that they are reflection positive as well as positive definite (see [11]). Balls uniquely maximize them among sets of given volume [1]; balls are also the unique convex sets for which certain related overdetermined boundary-value problems have solutions [12]. By scaling,

$$\mathcal{E}(A) \leq \mathcal{E}(A^*) = \text{Constant} \cdot (\text{Vol}(A))^{2 - \frac{n}{\lambda}}.$$  

The *deficit* of $A$ is defined by

$$\delta(A) = \frac{\mathcal{E}(A^*) - \mathcal{E}(A)}{\mathcal{E}(A^*)}.$$  

Each of the functionals can be expressed in terms of the corresponding *Riesz potential*

$$\Phi_A(x) = \int_A |x - y|^{-\lambda} \, dy, \quad x \in \mathbb{R}^n$$

as $\mathcal{E}(A) = \int_A \Phi_A$. By the Hardy-Littlewood-Sobolev inequality, $\Phi_A$ lies in $L^p$ for every $p \geq n/\lambda$. It is subharmonic on $\mathbb{R}^n$ and smooth on the
complement of $A$, though discontinuities may occur on $\partial A$. The Riesz potential is the unique solution of the pseudodifferential equation

$$(-\Delta)^{\frac{\lambda}{2}} \Phi = \text{Constant} \cdot \mathcal{X}_A$$

that decays at infinity. The constant $c_{n,\lambda}$ can be computed with the help of the Fourier transform (see [13, Theorem 5.9]).

Riesz’ rearrangement inequality implies that

$$\int_E \Phi_A(x) \, dx \leq \int_{E^*} \Phi_{A^*}(x) \, dx \quad (9)$$

for every set $E \subset \mathbb{R}^n$ of finite volume (see [1] or [13, Theorem 3.6]). In particular, $\Phi_{A^*}$ is radially decreasing, and

$$\sup_x \Phi_A(x) \leq \Phi_{A^*}(0) = \int_{A^*} |y|^{-\lambda} \, dy = \frac{n\omega_n}{n-\lambda} R_{A^*}^{n-\lambda}. \quad (10)$$

Our proof of Theorem 1 fails in higher dimensions, because the crucial lower bound in Lemma 6 becomes negative. Nevertheless, we expect that the conclusion should hold — with the sharp exponent 2 and suitable constants $c_{n,\lambda}$ — for the entire family of functionals in Eq. (5) with $n \geq 3$ and $\lambda \in [n-2, n)$.

When $\lambda = n - 2$, we call $\mathcal{E}(A)$ the Coulomb energy and $\Phi_A$ the Newton potential associated with the uniform charge distribution on $A$. The Newton potential has many special properties related to Poisson’s equation

$$-\Delta \Phi_A = n(n-2)\omega_n \mathcal{X}_A;$$

in particular, it is harmonic on the complement of $A$, subharmonic on $\mathbb{R}^n$, and satisfies the Gauss law. For later use, we compute the Newton potential of the centered ball of radius $R$ as

$$\Phi_{B_R}(x) = \omega_n R^2 \cdot \begin{cases} \frac{n}{2} - \frac{n-2}{2} \left(\frac{|x|}{R}\right)^2, & |x| \leq R, \\ \left(\frac{|x|}{R}\right)^{(n-2)}, & |x| \geq R, \end{cases} \quad (11)$$

and its Coulomb energy as

$$\mathcal{E}(B_R) = \frac{2n}{n+2} \omega_n^2 R^{n+2} = \frac{4}{n+2} \text{Vol}(B_R) \cdot \Phi_{B_R}(0).$$

A remarkable fact is Talenti’s comparison principle, which says that the symmetric decreasing rearrangement of the Newton potential of a charge distribution is pointwise smaller than the potential resulting from first symmetrizing the charge distribution itself [14],

$$\Phi_A^*(x) \leq \Phi_{A^*}(x), \quad x \in \mathbb{R}^n. \quad (12)$$

The corresponding inequality holds between the gradients of these functions. Since $\int_E \Phi_A \leq \int_{E^*} (\Phi_A)^*$ for every set $E$ of finite volume (see [13],
Theorem 3.4], Eq. (12) is much stronger than Riesz’ inequality (9). We will use Talenti’s comparison principle to prove the following result.

**Theorem 2.** Let $E$ be defined by Eq. (5) on $\mathbb{R}^n$ with $\lambda = n - 2$. For each $n \geq 3$, there exists a constant $c_n$ such that

$$E(A^*) - E(A) \geq c_n \alpha(A)^{n+2}$$

for every $A \subset \mathbb{R}^n$ of finite positive volume.

Note that the conclusion for $n = 3$ is weaker than Theorem 1.

3. Preliminary estimates

Throughout this section, $A \subset \mathbb{R}^n$ is a set of finite positive volume, the functional $E(A)$ is given by Eq. (5) with $\lambda \in [0, n)$, and $\Phi_A$ is the corresponding Riesz potential. We start by sharpening the bound on the maximum of $\Phi_A$ from Eq. (10).

**Lemma 1.** If $A \subset \mathbb{R}^n$ has finite positive volume, then

$$\sup_{x \in \mathbb{R}^n} \Phi_A(x) \leq \Phi_{A^*}(0) \cdot \left(1 - \frac{\lambda(n - \lambda)}{n^2} \alpha(A)^2\right).$$

**Proof.** By scaling, we may take $A^*$ to be the unit ball. For $x \in \mathbb{R}^n$,

$$\Phi_{A^*}(0) - \Phi_A(x) = \int_{A^* \setminus (x - A)} |y|^{-\lambda} dy - \int_{(x - A) \setminus A^*} |y|^{-\lambda} dy.$$

If $\alpha(A) = \alpha$, then each of the two regions of integration has volume at least $\omega_n \alpha$. Since the kernel is radially decreasing, the first integral is minimized when $A^* \setminus (x - A)$ is an annulus whose outer boundary is the unit sphere, and the second integral is maximized when $(x - A) \setminus A^*$ is an annulus whose inner boundary is the unit sphere. Using annuli of the appropriate volume, we calculate in polar coordinates

$$\Phi_{A^*}(0) - \Phi_A(x) \geq n \omega_n \int_0^1 r^{n-1-\lambda} dr - n \omega_n \int_1^{(1+\alpha)^{1/n}} r^{n-1-\lambda} dr$$

$$= \frac{\lambda(n - \lambda)}{n^2} \Phi_{A^*}(0) \int_0^\alpha \int_{-s}^s (1 + t)^{-1-\frac{\lambda}{n}} dt ds,$$

where we have used the Fundamental Theorem of Calculus twice. By Jensen’s inequality, the value of the double integral exceeds $\alpha^2$. \qed

Lemma 1 is needed for the proof of Theorem 2. In the next lemma, we use a similar estimate to relate $\alpha(A)$ to the notion of asymmetry that appears in Guo’s conjecture (3). (This plays no role in the proofs of the main results.)
Lemma 2. There exist positive constants $c_{n,\lambda}$ and $C_{n,\lambda}$ such that
\[ c_{n,\lambda} \alpha(A)^4 \leq \inf_\tau \frac{\mathcal{E}(X_A \circ \tau^{-1} - X_{A^*})}{\mathcal{E}(A^*)} \leq C_{n,\lambda} \alpha(A)^{2 - \frac{2 \lambda}{n}} \]
for every $A \subset \mathbb{R}^n$ of finite positive volume.

Proof. Assume by scaling that $A^*$ is the unit ball, and set $\alpha = \alpha(A)$. For the first inequality, we translate $A$ such that the infimum in the middle term is assumed when $\tau$ is the identity. Since $\mathcal{E}$ extends to a positive definite quadratic form on $L^1 \cap L^\infty$, we can use Schwarz' inequality to obtain
\[
\mathcal{E}(X_A - X_{A^*}) \geq \int \int \frac{(X_{A^*}(x) - X_A(x))X_{A^*}(y)}{|x - y|^\lambda} \, dx \, dy
\]
\[
= \int_{A^* \setminus A} \Phi_{A^*}(x) \, dx - \int_{A \setminus A^*} \Phi_{A^*}(x) \, dx
\]
\[
\geq \int_{1-\alpha<|x|<1} \Phi_{A^*}(x) \, dx - \int_{1<|x|<1+\alpha} \Phi_{A^*}(x) \, dx
\]
\[
\geq \text{Constant} \cdot \alpha^2,
\]
where the constant depends on $n$ and $\lambda$. We have used that $\Phi_{A^*}$ is strictly radially decreasing to replace $A^* \setminus A$ and $A \setminus A^*$ with annuli. The last line follows since the gradient of $\Phi_{A^*}$ vanishes only at $x = 0$.

For the second inequality, we translate $A$ so that the infimum in Eq. (4) is assumed at the identity. The Hardy-Littlewood-Sobolev inequality implies that
\[
\inf_\tau \mathcal{E}(X_A \circ \tau^{-1} - X_{A^*}) \leq C_{n,\lambda} \|X_A - X_{A^*}\|_{L^2}^2 \leq C_{n,\lambda} \alpha^{2 - \frac{\lambda}{n}}.
\]

We need a few more lemmas for the proof of Theorem 1. The following integral representation will appear several times.

Lemma 3. Let $\rho(r)$ denote the volume radius of $A \cap B_r$. For any $R > 0$,
\[
\mathcal{E}(A^*) - \mathcal{E}(A) \geq \int_R^\infty \int_{A \cap \partial B_r} \left( \Phi_{(A \cap B_r)^*} \left|_{\partial B_{\rho(r)}} - \Phi_{A \cap B_r(x)} \right) \, d\sigma(x) \, dr.
\]

Proof. The functional can be written as
\[
\mathcal{E}(A) = 2 \int_{A \cap \partial B_R} \mathcal{X}_{||x||>|y|} |x - y|^{-\lambda} \, dy \, dx
\]
\[
= 2 \int_{A \cap B_R} \Phi_{A \cap B_{|x|}}(x) \, dx + 2 \int_{A \setminus B_R} \Phi_{A \cap B_{|x|}}(x) \, dx
\]
\[
= \mathcal{E}(A \cap B_R) + \int_R^\infty \int_{A \cap \partial B_r} \Phi_{A \cap B_r(x)} \, d\sigma(x) \, dr.
\]
Applying Eq. (14) to \( A^* \) with \( \rho(R) \) in place of \( R \), we see that

\[
\mathcal{E}(A^*) = \mathcal{E}(B_{\rho(R)}) + 2 \int_{\rho(R)}^{\infty} \Phi_{B_{\rho}} \left|_{|x|=\rho} \right. n\omega_n \rho^{n-1} \, d\rho \\
= \mathcal{E}((A \cap B_R)^*) + 2 \int_R^{\infty} \Phi_{(A \cap B_r)^*} \left|_{|x|=\rho(r)} \right. \sigma(A \cap \partial B_r) \, dr.
\]

In the first line, we have used that \( B_{\rho(R)} \subset A^* \). The Jacobian for the change of variables is determined by the relation

\[
n\omega_n \rho^{n-1} \, d\rho = \sigma(A \cap \partial B_r) \, dr.
\]

Since \( \mathcal{E}(A \cap B_R) \leq \mathcal{E}((A \cap B_R)^*) \) by Eq. (1), the claim follows upon subtracting Eq. (14).

The next lemma reduces the stability problem to bounded sets.

**Lemma 4.** For every \( n \geq 3 \) and \( \lambda \in [n-2, n) \) there are positive constants \( \alpha_{n,\lambda} \) and \( c_{n,\lambda} \) with the following property. Given a set \( A \subset \mathbb{R}^n \) of finite positive volume with \( \alpha_0 := \text{Vol}(A \triangle A^*)/(2\text{Vol}(A)) \leq \alpha_{n,\lambda} \), there exists a set \( \tilde{A} \) of the same volume such that

\[
\tilde{A} \subset (1 + c_{n,\lambda}\alpha_0^{1-\frac{1}{n}})A^*, \quad \frac{\text{Vol}(\tilde{A} \triangle A^*)}{2\text{Vol}(A)} = \alpha_0, \quad \delta(\tilde{A}) \leq \delta(A).
\]

If \( A \) is symmetric about the origin, then so is \( \tilde{A} \).

**Proof.** By scaling, we may assume that \( A^* \) is the unit ball, i.e., \( R_A = 1 \). Given \( R > 1 \), determine \( r > 1 \) such that

\[
\tilde{A} = (A \cap B_R) \cup (B_r \setminus A^*)
\]

has the same volume as \( A \). We assume that \( R \) is large enough, compared to \( \alpha_0 \), that \( r \leq R \). By construction, \( \text{Vol}(\tilde{A} \triangle A^*) = \text{Vol}(A \triangle A^*) \), and \( r \leq (1 + \alpha_0)^{1/n} \).

We want to choose \( R \) so that \( \mathcal{E}(\tilde{A}) \geq \mathcal{E}(A) \). It follows from Eq. (14) that

\[
\mathcal{E}(A) \leq \mathcal{E}(A \cap B_R) + 2\text{Vol}(A \setminus B_R) \cdot \sup_{|x| \geq R} \Phi_{A}(x).
\]

Since \( \Phi_A \leq \Phi_{A^*} + \Phi_{A \setminus A^*} \), Eq. (10) implies

\[
\Phi_A(x) \leq \Phi_{A^*}(x) + \frac{n\omega_n}{n-\lambda} \alpha_0^{1-\lambda/n}.
\]

Similarly, since \( \tilde{A} \cap A = A \cap B_R \) by construction,

\[
\mathcal{E}(\tilde{A}) = \int_{\tilde{A} \cap A} \Phi_{\tilde{A} \setminus A}(x) \, dx + \int_{\tilde{A} \setminus A} 2\Phi_{\tilde{A} \setminus A}(x) + \Phi_{\tilde{A} \setminus A}(x) \, dx \\
\geq \mathcal{E}(A \cap B_R) + 2\text{Vol}(A \setminus B_R) \cdot \inf_{|x| \leq r} \Phi_{\tilde{A}}(x) - \mathcal{E}(\tilde{A} \setminus A),
\]

The proof is complete. \( \square \)
and
\[ \Phi_{\tilde{A}}(x) \geq \Phi_{A^*}(x) - \frac{n\omega_n}{n-\lambda} \alpha_0^{1-\lambda/n}. \]

We estimate \( E(\tilde{A} \setminus A) \) by Eq. (6) and collect terms to obtain that
\[
\begin{align*}
E(\tilde{A}) - E(A) &\geq 2\text{Vol}(A \setminus B_R) \cdot \left( \Phi_{A^*} \big|_{|x|=(1+\alpha_0)^{1/n}} - \Phi_{A^*} \big|_{|x|=R} - \frac{n\omega_n}{n-\lambda} \alpha_0^{1-\lambda/n} - \text{Constant} \cdot \text{Vol}(A \setminus B_R)^{1-\lambda/n} \right).
\end{align*}
\]

We have used that \( \Phi_{A^*} \) is radially decreasing to replace \( r \) with \( (1 + \alpha_0)^{1/n} \). Since \( \Phi_{A^*} \) is a smooth, strictly radially decreasing function whose gradient does not vanish outside \( A^* \), there exists a constant \( c_{n,\lambda} \) such that the right hand side is positive for
\[ R = 1 + c_{n,\lambda} \alpha_0^{1-\lambda/n} \]
when \( \alpha_0 \) is sufficiently small. \( \square \)

We now introduce reflection symmetries to the problem. The basic construction is as follows. Given a hyperplane that bisects \( A \) into two parts of equal volume, the set \( A \) is replaced by the union of one of these parts with its mirror image. We refer to the two sets that can be obtained in this way as symmetrizations of \( A \) at the hyperplane. Clearly, the symmetrizations have the same volume as \( A \).

**FMP Symmetrization Lemma** [10, Theorem 2.1]. For every \( n \geq 1 \) there is a constant \( c_n \) with the following property. Given a set \( A \subseteq \mathbb{R}^n \) of finite positive volume, there exists a set \( \tilde{A} \) obtained by successive symmetrization of \( A \) at \( n \) orthogonal hyperplanes such that
\[ \alpha(\tilde{A}) \geq c_n \alpha(A). \]

**Lemma 5.** If \( \lambda \in [n-2,n) \), then the set constructed in the FMP lemma satisfies
\[ \delta(\tilde{A}) \leq 2^n \delta(A). \]

**Proof.** Consider the two possible symmetrizations \( A_+ \) and \( A_- \) of \( A \) at a single hyperplane. Since \( \lambda \in [n-2,n) \), the functional is reflection positive, meaning that
\[ \mathcal{E}(A_+) + \mathcal{E}(A_-) \geq 2\mathcal{E}(A), \]
see [11, Section 1.1]. Using that \( (A_+)^* = (A_-)^* = A^* \), we subtract both sides of the inequality from \( \mathcal{E}(A^*) \) to see that
\[ \delta(A_+) + \delta(A_-) \leq 2\delta(A), \]

and conclude that the symmetrized sets satisfy $\delta(A_{\pm}) \leq 2\delta(A)$. The claim follows by repeating the construction $n$ times.

We translate and rotate $\tilde{A}$ to a set that is symmetric at the coordinate hyperplanes, and thus symmetric under $x \mapsto -x$. Such sets have the useful property that their asymmetry is comparable to their symmetric difference from a centered ball \cite{10} Lemma 2.2. The following estimate for the potential is the key to the proof of Theorem 1.

**Lemma 6.** If $A \subset B_r$ is symmetric about the origin, then

$$\Phi_A(x) \leq \Phi_{B_r}(x) - (\sqrt{2}r)^{-\lambda} \text{Vol}(B_r \setminus A)$$

for all $x \in \partial B_r$.

**Proof.** Let $x \in \partial B_r$ be given. The function

$$f(y) = \frac{1}{2} (|y - x|^{-\lambda} + |y + x|^{-\lambda})$$

assumes its minimum at a point on $\partial B_r$ equidistant to $x$ and $-x$, and the minimum value is $(\sqrt{2}r)^{-\lambda}$. Since $A$ is symmetric,

$$\Phi_A(x) = \int_A |x - y|^{-\lambda} \, dy \geq (\sqrt{2}r)^{-\lambda} \text{Vol}(A).$$

The claim follows by replacing $A$ with $B_r \setminus A$. \qed

For the Newton potential of $A \subset B_r$, Lemma 6 implies that

$$\Phi_A^* \bigg|_{\partial A^*} - \inf_{\partial B_r} \Phi_A \geq \omega_n \left( R_A^2 - r^2 + \frac{r^n - R_A^n}{(\sqrt{2}r)^{n-2}} \right)$$

$$= \omega_n R_A^2 (-2 + n2^{1 - \frac{n}{2}}) (\frac{r}{R_A} - 1) + O\left(\frac{r}{R_A} - 1\right)^2$$

(15)

uniformly in $A$ as $\frac{r}{R_A} \to 1$. Note that the leading term is positive in dimension $n = 3$.

4. PROOF OF THE MAIN RESULTS

**Proof of Theorem 7.** We specialize to the case of the Coulomb energy in $\mathbb{R}^3$, where $\lambda = 1$. We want to find a constant $c > 0$ such that $\delta(A) \geq c\alpha(A)^2$ for all sets of finite positive volume $A \subset \mathbb{R}^3$. By scaling, we may assume that $\text{Vol}(A) = \omega_3 = 4\pi/3$, so that $A^*$ is the unit ball. Since $\alpha(A) \leq 1$ by definition, it suffices to prove the claim for $\alpha$ sufficiently small.

By Lemma 5 we may assume that $A$ is symmetric about the origin. Therefore, by \cite{10} Lemma 2.2,

$$\alpha_0 := \frac{\text{Vol}(A \triangle A^*)}{(2\omega_3)} \leq 3\alpha(A).$$
By Lemma 4 we may assume furthermore that $A$ lies in the ball of radius

$$R_0 = 1 + c_{3,1} \alpha_0^2,$$

provided that $3 \alpha(A) \leq \alpha_{3,1}$. We use Lemma 3 with $R = 1$ to see that

$$\mathcal{E}(A^*) - \mathcal{E}(A) \geq \int_1^{R_0} \int_{A \cap \partial B_r} \Phi_{(A \cap B_r^*)} - \Phi_{A \cap B_r}(x) \, d\sigma(x) \, dr,$$

where $\rho(r)$ is the volume radius of $A \cap B_r$. By Eq. (15), the integrand is bounded from below by

$$\Phi_{(A \cap B_r^*)} - \Phi_{A \cap B_r}(x) \geq \omega_3 \inf_{1 \leq r \leq R_0} \left\{ \rho(r)^2 - r^2 + \frac{r^3 - \rho(r)^3}{\sqrt{2r}} \right\}.$$

The function inside the braces can be written as a product

$$(r^3 - \rho(r)^3) \left( -\frac{r + \rho(r)}{r^2 + r \rho(r) + \rho^2(r)} + \frac{1}{\sqrt{2r}} \right).$$

Since the first factor is non-decreasing in $r$, it is bounded from below by $1 - \rho(1)^3 = \alpha_0$. This gives for the integral

$$\mathcal{E}(A^*) - \mathcal{E}(A) \geq \frac{4\pi}{3} \alpha_0^2 \cdot \inf_{(1-\alpha_0)^{1/3} \leq r \leq R_0} \left\{ -\frac{r + \rho}{r^2 + r \rho + \rho^2} + \frac{1}{\sqrt{2r}} \right\}.$$

The infimum is strictly positive for $\alpha_0$ sufficiently small. Using once more that $\alpha_0/3 \leq \alpha(A) \leq \alpha_0$ by the symmetry of $A$, the theorem follows.

The proof of Theorem 1 used that the Coulomb kernel $|x|^{-1}$ is symmetric decreasing and reflection positive, without taking advantage of the special properties of the Newton potential. Since all estimates used in the proof depend continuously on $\lambda$, the conclusion extends to nearby values.

**Corollary** Let $\mathcal{E}_\lambda$ be defined by Eq. (5) for $n = 3$ and $\lambda > 1$, and let $\delta_\lambda$ be the corresponding deficit given by Eq. (7). For every $\lambda$ sufficiently close to 1 there exists a constant $c_\lambda$ such that

$$\delta_\lambda(A) \geq c_\lambda \alpha(A)^2$$

for all $A \subset \mathbb{R}^3$.

Finally we turn to the Coulomb energy in dimension $n \geq 3$.

**Proof of Theorem 2** Let $n \geq 3$ and $\lambda = n - 2$. Assume, by scaling, that $A^*$ is the unit ball, and let $\alpha = \alpha(A)$ be the asymmetry of $A$. Since $\int_A \Phi_A \leq \int_{A^*} (\Phi_A)^*$,

$$\mathcal{E}(A^*) - \mathcal{E}(A) \geq \int_{A^*} \Phi_{A^*} - (\Phi_A)^* \, dx.$$
By Talenti’s comparison principle, the integrand is nonnegative. Moreover, by Lemma 1 and Eq. (11),
\[ \Phi_{A^*}(x) - (\Phi_A)^*(x) \geq \Phi_{A^*}(x) - \sup_y \Phi_A(y) \]
\[ \geq \frac{n - 2}{2} \omega_n \left( \frac{2\alpha^2}{n} - |x|^2 \right). \]
Integration yields
\[ \mathcal{E}(A^*) - \mathcal{E}(A) \geq \frac{n - 2}{2} \omega_n \int_{A^*} \left[ \frac{2\alpha^2}{n} - |x|^2 \right]_+ dx \]
\[ = \frac{n - 2}{2n} \mathcal{E}(A^*) \cdot \left( \frac{\sqrt{2\alpha}}{\sqrt{n}} \right)^{n+2}, \]
which proves Eq. (13) with \( c_n = (n - 2)2^{n/2}/n^{2+n/2} \). □

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