Enhanced Error Estimates for Augmented Subspace Method

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Abstract

In this paper, some enhanced error estimates are derived for the augmented subspace methods which are designed for solving eigenvalue problems. We will show that the augmented subspace methods have the second order convergence rate which is better than the existing results. These sharper estimates provide a new dependence of convergence rate on the coarse spaces in augmented subspace methods. These new results are also validated by some numerical examples.

Keywords. eigenvalue problem, augmented subspace method, enhanced error estimate, finite element method.

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1 Introduction

One of the fundamental problems in modern science and engineering society is to solve large-scale eigenvalue problems. This is always a very difficult task to solve high-dimensional eigenvalue problems which come from practical physical and chemical sciences. Compared with linear boundary value problems, there are no so many efficient numerical methods for solving eigenvalue problems with optimal complexity. Solving large-scale eigenvalue problems poses significant challenges for scientific computing. In order to solve large sparse eigenvalue problems, there have developed eigensolvers such as Krylov subspace type methods (Implicitly Restarted Lanczos/Arnoldi Method (IRLM/IRAM) [18]), the Preconditioned INVerse ITeration (PINVIT) method [5, 11, 13], the Locally Optimal Block Preconditioned Conjugate Gradient (LOBPCG) method [14, 15], and the Jacobi-Davidson-type techniques [4]. All these popular methods include the orthogonalization steps during computing Rayleigh-Ritz problems which are always the bottlenecks for designing efficient parallel schemes for determining relatively many eigenpairs. Recently, a type of multilevel correction method is proposed for solving eigenvalue problems in [8, 12, 16, 20, 21, 22, 23]. In this multilevel correction scheme, there exists an augmented subspace which is constructed with the help of the low dimensional finite element space defined on the coarse grid. Based on this special...
augmented subspace, we have designed some efficient numerical methods for solving eigenvalue problems and nonlinear equations. This type of augmented subspace methods only need a low dimension finite element space on the coarse mesh and the final finite element space on the finest mesh. This method can also work even the coarse and finest mesh has no nested property which is an extension of the multilevel correction method. The application of this augmented subspace can transform the solution of the eigenvalue problem on the final level of mesh can be reduced to the solution of boundary value problems on the final level of mesh and the solution of the eigenvalue problem on the low dimensional augmented subspace. The multilevel correction method and augmented subspace method give the ways to construct the multigrid method for eigenvalue problems. More important, we can design an eigenpair-wise parallel eigensolver for the eigenvalue problems based on the augmented subspace. This type of parallel method avoids doing orthogonalization and inner-products in the high dimensional space which account for a large portion of the wall time in the parallel computation. For more information, please refer to [23].

The aim of this paper is to give new and sharper error estimates for the augmented subspace method. These new error estimates provide new investigations between the augmented subspace method and the two-grid method [24]. Roughly speaking, we will give the following error estimate for the augmented subspace method

\[ \| \bar{u}_h - u_h^{(l+1)} \|_a \lesssim \eta^2_h(V_H) \| \bar{u}_h - u_h^{(l)} \|_a, \]

which is sharper than the existed results included in [16, 20, 21, 22, 23]. This estimate also shows the dependence of the convergence rate for the augmented subspace method on the low dimensional space \( V_H \).

An outline of the paper goes as follows. In Section 2, we introduce the finite element method for the eigenvalue problem and the corresponding error estimates. The augmented subspace method and some enhanced error estimates will be given in Section 3 which is the main part of this paper. In section 4, two choices of the coarse spaces are discussed and some numerical examples are provided to validate the enhanced results in this paper. Some concluding remarks are given in the last section.

2 Discretization by finite element method

In this section, we introduce some notation and error estimates of the finite element approximation for eigenvalue problems. In this paper, the letter \( C \) (with or without subscripts) denotes a generic positive constant which may be different at different occurrences. For convenience, the symbols \( \lesssim, \gtrsim \) and \( \approx \) will be used in this paper. That \( x_1 \lesssim y_1, x_2 \gtrsim y_2 \) and \( x_3 \approx y_3 \), mean that \( x_1 \leq C_1 y_1, x_2 \geq c_2 y_2 \) and \( c_3 x_3 \leq y_3 \leq C_3 x_3 \) for some constants \( C_1, c_2, c_3 \) and \( C_3 \) that are independent of mesh sizes.

For generality, let \( V \) and \( W \) denote two Hilbert spaces and \( V \subset W \). Then let \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) be two positive definite symmetric bilinear forms on \( V \times V \) and \( W \times W \), respectively. Furthermore, based on the bilinear form \( a(\cdot, \cdot) \), we can define the norm on the space \( V \) as follows

\[ \| v \|_a = \sqrt{a(v,v)}, \quad \forall v \in V. \quad (2.1) \]

Similarly, we can define the norm \( \| \cdot \|_b \) by the bilinear form \( b(\cdot, \cdot) \) on the space \( W \)

\[ \| w \|_b = \sqrt{b(w,w)}, \quad \forall w \in W. \quad (2.2) \]

In this paper, we assume that the norm \( \| \cdot \|_a \) is relatively compact with respect to the norm \( \| \cdot \|_b \) [10].
In our methodology description, we are concerned with the following general eigenvalue problem: Find \((\lambda, u) \in \mathcal{R} \times V\) such that \(a(u, v) = 1\) and
\[
a(u, v) = \lambda b(u, v), \quad \forall v \in V. \tag{2.3}
\]
It is well known that the eigenvalue problem (2.3) has an eigenvalue sequence \(\{\lambda_j\}\) (cf. [2, 7]):
\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k \to \infty} \lambda_k = \infty,
\]
and associated eigenfunctions
\[
u_1, \nu_2, \ldots, \nu_k, \ldots,
\]
where \(a(\nu_i, \nu_j) = \delta_{ij}\) (\(\delta_{ij}\) denotes the Kronecker function). In the sequence \(\{\lambda_j\}\), the \(\lambda_j\) are repeated according to their geometric multiplicity.

Now, let us define the finite dimensional subspace approximations of the problem (2.3). For generality, let \(V_h\) denote some type of finite dimensional subspace of the Hilbert space \(V\). It is well known that the finite element method is the widest used way to build the subspace \(V_h\). For easy understanding and as an example, we use the finite element method to build the space \(V_h\). Based on the mesh \(T_h\), we can construct a finite element space denoted by \(V_h \subset V\). For simplicity, we set \(V_h\) as the Lagrange type finite element space which is defined as follows
\[
V_h = \{ v_h \in C(\Omega) \mid v_h|_K \in P_k, \; \forall K \in T_h \} \cap H^1_0(\Omega), \tag{2.4}
\]
where \(P_k\) denotes the polynomial space of degree at most \(k\).

Then, we can define the standard finite element scheme for eigenvalue problem (2.3): Find \((\bar{\lambda}_h, \bar{u}_h) \in \mathcal{R} \times V_h\) such that \(a(\bar{u}_h, v_h) = 1\) and
\[
a(\bar{u}_h, v_h) = \bar{\lambda}_h b(\bar{u}_h, v_h), \quad \forall v_h \in V_h. \tag{2.5}
\]
It is well known that \(V_h \subset V\) is a family of finite-dimensional spaces that satisfy the following assumption: For any \(w \in V\)
\[
\lim_{h \to 0} \inf_{v_h \in V_h} \| w - v_h \|_a = 0. \tag{2.6}
\]
From [2, 3], the discrete eigenvalue problem (2.5) has eigenvalues:
\[
0 < \bar{\lambda}_{1,h} \leq \bar{\lambda}_{2,h} \leq \cdots \leq \bar{\lambda}_{k,h} \leq \cdots \leq \bar{\lambda}_{N_h,h},
\]
and corresponding eigenfunctions
\[
\bar{u}_{1,h}, \bar{u}_{2,h}, \ldots, \bar{u}_{k,h}, \ldots, \bar{u}_{N_h,h},
\]
where \(a(\bar{u}_{i,h}, \bar{u}_{j,h}) = \delta_{ij}, 1 \leq i, j \leq N_h\) (\(N_h\) is the dimension of the finite element space \(V_h\)). From the min-max principle [2, 3], the eigenvalues of (2.5) provide upper bounds for the first \(N_h\) eigenvalues of (2.3)
\[
\lambda_i \leq \bar{\lambda}_{i,h}, \quad 1 \leq i \leq N_h. \tag{2.8}
\]
For the following analysis in this paper, we define \(\mu_i = 1/\lambda_i\) for \(i = 1, 2, \cdots\), and \(\bar{\mu}_i = 1/\bar{\lambda}_{i,h}\) for \(i = 1, \cdots, N_h\). In order to measure the error of the finite element space to the desired function, we define the following notation
\[
\delta(w, V_h) = \inf_{v_h \in V_h} \| w - v_h \|_a, \quad \text{for } w \in V. \tag{2.9}
\]
In this paper, we also need the following quantity for error analysis:

\[ \eta_h(V_h) = \sup_{f \in W} \inf_{v_h \in V_h} \|Tf - v_h\|_a, \]  

(2.10)

where \( T : W \rightarrow V \) is defined as

\[ a(Tf, v) = b(f, v), \quad \forall v \in V \quad \text{for} \quad f \in W. \]  

(2.11)

In order to understand the method more clearly, we state the error estimate for the eigenpair approximation by the finite element method. For this aim, we define the finite element projection \( P_h : V \rightarrow V_h \) as follows

\[ a(P_h w, v_h) = a(w, v_h), \quad \forall v_h \in V_h \quad \text{for} \quad w \in V. \]  

(2.12)

It is obvious that the finite element projection operator \( P_h \) has following error estimates.

**Lemma 2.1.** For any function \( w \in V \), the finite element projection operator \( P_h \) has following error estimates

\[ \|w - P_h w\|_a = \inf_{w_h \in V_h} \|w - w_h\|_a = \delta(w, V_h), \]  

(2.13)

\[ \|w - P_h w\|_b \leq \eta_h(V_h)\|w - P_h w\|_a. \]  

(2.14)

Before stating error estimates of the subspace projection method, we introduce a lemma which comes from [19]. For completeness, a proof is provided here.

**Lemma 2.2.** ([19, Lemma 6.4]) For any eigenpair \( (\lambda, u) \) of (2.3), the following equality holds

\[ (\bar{\lambda}_{j,h} - \lambda)b(P_h u, \bar{u}_{j,h}) = \lambda b(u - P_h u, \bar{u}_{j,h}), \quad j = 1, \cdots, N_h. \]

Proof. Since \(-\lambda b(P_h u, \bar{u}_{j,h})\) appears on both sides, we only need to prove that

\[ \bar{\lambda}_{j,h}b(P_h u, \bar{u}_{j,h}) = \lambda b(u, \bar{u}_{j,h}). \]

From (2.3), (2.5) and (2.12), the following equalities hold

\[ \bar{\lambda}_{j,h}b(P_h u, \bar{u}_{j,h}) = a(P_h u, \bar{u}_{j,h}) = a(u, \bar{u}_{j,h}) = \lambda b(u, \bar{u}_{j,h}). \]

Then the proof is complete.

The following lemma has already been presented in [22] which gives the error estimates for the one eigenpair approximation. This lemma will be used for analyzing the error estimates for the augmented subspace method for only one eigenpair. For the proof, please refer to [22].

**Lemma 2.3.** ([22, Lemma 3.3]) Let \( (\lambda, u) \) denote an exact eigenpair of the eigenvalue problem (2.3). Assume the eigenpair approximation \( (\bar{\lambda}_{i,h}, \bar{u}_{i,h}) \) has the property that \( \bar{\mu}_{i,h} = 1/\bar{\lambda}_{i,h} \) is closest to \( \mu = 1/\lambda \). The corresponding spectral projector \( E_{i,h} : V \mapsto \text{span}\{\bar{u}_{i,h}\} \) is defined as follows

\[ a(E_{i,h} w, \bar{u}_{i,h}) = a(w, \bar{u}_{i,h}), \quad \text{for} \quad w \in V. \]

Then the following error estimate holds

\[ \|u - E_{i,h} u\|_a \leq \sqrt{1 + \frac{\bar{\mu}_{i,h}}{\delta_{i,h}^2} \eta_{i,h}^2(V_h)\|(I - P_h)u\|_a}, \]  

(2.15)
where \( \eta_h(V_h) \) is defined in (2.10) and \( \delta_{\lambda,h} \) is defined as follows
\[
\delta_{\lambda,h} := \min_{j \neq i} |\bar{\mu}_{j,h} - \mu| = \min_{j \neq i} \left| \frac{1}{\lambda_{j,h}} - \frac{1}{\lambda_i} \right|. \tag{2.16}
\]
Furthermore, the eigenvector approximation \( \bar{u}_{i,h} \) has following error estimate in \( \| \cdot \|_b \)-norm
\[
\| u - E_{i,h}u \|_b \leq \left( 1 + \frac{\bar{\mu}_{1,h}}{\delta_{\lambda,h}} \right) \eta_h(V_h) \| u - E_{i,h}u \|_a. \tag{2.17}
\]

For simplicity of notation, we assume that the eigenvalue gap \( \delta_{\lambda,h} \) has a uniform lower bound which is denoted by \( \delta_\lambda \) (which can be seen as the “true” separation of the eigenvalue \( \lambda \) from others) in the following parts of this paper. This assumption is reasonable when the mesh size is small enough. We refer to [17, Theorem 4.6] and Lemma 2.3 in this paper for details of the dependence of error estimates on the eigenvalue gap. Then we have the following simple version of the error estimates based on Lemma 2.3.

**Corollary 2.1.** Under the conditions of Lemma 2.3, the following error estimates hold
\[
\| u - E_{i,h}u \|_a \leq \sqrt{1 + \frac{1}{\lambda_{1,h}} \eta_h^2(V_h)} \| (I - \mathcal{P}_h)u \|_a, \tag{2.18}
\]
\[
\| u - E_{i,h}u \|_b \leq \left( 1 + \frac{1}{\lambda_{1,h}} \right) \eta_h(V_h) \| u - E_{i,h}u \|_a. \tag{2.19}
\]

In the following part of this section, we consider the error estimates for the first \( k \) eigenpair approximations associated with \( \lambda_1 \leq \cdots \leq \lambda_k \).

**Theorem 2.1.** Let us define the spectral projection \( E_{k,h} : V \rightarrow \text{span}\{\bar{u}_{1,h}, \cdots, \bar{u}_{k,h}\} \) as follows
\[
a(E_{k,h}w, \bar{u}_{i,h}) = a(w, \bar{u}_{i,h}), \quad i = 1, \cdots, k \text{ for } w \in V. \tag{2.20}
\]
Then the associated exact eigenfunctions \( u_1, \cdots, u_k \) of eigenvalue problem (2.3) have the following error estimates
\[
\| u_i - E_{k,h}u_i \|_a \leq \sqrt{1 + \frac{1}{\lambda_{k+1,h}} \eta_h^2(V_h)} \| (I - \mathcal{P}_h)u_i \|_a, \quad 1 \leq i \leq k, \tag{2.21}
\]
where
\[
\delta_{k,i,h} = \min_{k < j \leq N_h} \left| \frac{1}{\lambda_{j,h}} - \frac{1}{\lambda_i} \right|. \tag{2.22}
\]
Furthermore, these \( k \) exact eigenvectors have following error estimate in \( \| \cdot \|_b \)-norm
\[
\| u_i - E_{k,h}u_i \|_b \leq \left( 1 + \frac{\mu_{k+1}}{\delta_{k,i,h}} \right) \eta_h(V_h) \| u_i - E_{k,h}u_i \|_a, \quad 1 \leq i \leq k. \tag{2.23}
\]
**Proof.** Similarly to the duality argument in the finite element method, the following inequality holds
\[
\| (I - \mathcal{P}_h)u_i \|_b = \sup_{\| g \|_b = 1} b((I - \mathcal{P}_h)u_i, g) = \sup_{\| g \|_b = 1} a((I - \mathcal{P}_h)u_i, Tg) = \sup_{\| g \|_b = 1} a((I - \mathcal{P}_h)u_i, (I - \mathcal{P}_h)Tg) \leq \eta_h(V_h) \| (I - \mathcal{P}_h)u_i \|_a. \tag{2.24}
\]
Since \((I - E_{k,h}) \mathcal{P}_h u_i \in V_h\) and \((I - E_{k,h}) \mathcal{P}_h u_i \in \text{span}\{\bar{u}_{k+1,h}, \ldots, \bar{u}_{N_h,h}\}\), the following orthogonal expansion holds
\[(I - E_{k,h}) \mathcal{P}_h u_i = \sum_{j=k+1}^{N_h} \alpha_j \bar{u}_{j,h}, \quad (2.25)\]
where \(\alpha_j = a(\mathcal{P}_h u_i, \bar{u}_{j,h})\). From Lemma 2.2, we have
\[
\alpha_j = a(\mathcal{P}_h u_i, \bar{u}_{j,h}) = \bar{\lambda}_{j,h} b(\mathcal{P}_h u_i, \bar{u}_{j,h}) = \frac{\bar{\lambda}_{j,h}}{\lambda_h - \lambda} b(u_i - \mathcal{P}_h u_i, \bar{u}_{j,h}) \leq \frac{1}{\mu - \bar{\mu}_{j,h}} b(u_i - \mathcal{P}_h u_i, \bar{u}_{j,h}). \quad (2.26)
\]
From the orthogonal property of eigenvectors \(\bar{u}_{1,h}, \ldots, \bar{u}_{N_h,h}\), the following equalities hold
\[
1 = a(\bar{u}_{j,h}, \bar{u}_{j,h}) = \bar{\lambda}_{j,h} b(\bar{u}_{j,h}, \bar{u}_{j,h}) = \bar{\lambda}_{j,h} \| \bar{u}_{j,h} \|^2_b,
\]
which leads to the following property
\[
\| \bar{u}_{j,h} \|^2_b = \frac{1}{\bar{\lambda}_{j,h}} = \bar{\mu}_{j,h}. \quad (2.27)
\]
From (2.12) and definitions of eigenvectors \(\bar{u}_{1,h}, \ldots, \bar{u}_{N_h,h}\), we have following equalities
\[
a(\bar{u}_{j,h}, \bar{u}_{k,h}) = \delta_{j,k}, \quad b\left( \frac{\bar{u}_{j,h}}{\| \bar{u}_{j,h} \|_b}, \frac{\bar{u}_{k,h}}{\| \bar{u}_{k,h} \|_b} \right) = \delta_{j,k}, \quad 1 \leq j, k \leq N_h. \quad (2.28)
\]
Then from (2.25), (2.26), (2.27) and (2.28), we have following estimates
\[
\| (I - E_{k,h}) \mathcal{P}_h u_i \|_a^2 = \left\| \sum_{j=k+1}^{N_h} \alpha_j \bar{u}_{j,h} \right\|_a^2 = \sum_{j=k+1}^{N_h} \alpha_j^2 = \frac{1}{\mu - \bar{\mu}_{j,h}} b(u_i - \mathcal{P}_h u_i, \bar{u}_{j,h})^2 \leq \frac{1}{\| \bar{u}_{j,h} \|_b} \sum_{j=k+1}^{N_h} \| \bar{u}_{j,h} \|^2_b \leq \frac{\bar{\mu}_{j+1,h}}{\delta_{k,i,h}} \sum_{j=k+1}^{N_h} \| \bar{u}_{j,h} \|^2_b \leq \frac{\bar{\mu}_{j+1,h}}{\delta_{k,i,h}} \| \bar{u}_{j,h} \|^2_b \leq \frac{\bar{\mu}_{j+1,h}}{\delta_{k,i,h}} \| u_i - \mathcal{P}_h u_i \|^2_b. \quad (2.29)
\]
where the last inequality holds since \(\frac{\bar{u}_{1,h}}{\| \bar{u}_{1,h} \|_b}, \ldots, \frac{\bar{u}_{N_h,h}}{\| \bar{u}_{N_h,h} \|_b}\) are the normal-orthogonal basis for the space \(V_h\) in the sense of the inner product \(b(\cdot, \cdot)\).

Combining (2.8) and (2.29) leads to the following inequality
\[
\| (I - E_{k,h}) \mathcal{P}_h u_i \|_a^2 \leq \frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}} \eta_a(V_h)^2 \| (I - \mathcal{P}_h) u_i \|_a^2 \leq \frac{\bar{\mu}_{k+1,h}}{\delta_{k,i,h}} \eta_a(V_h)^2 \| (I - \mathcal{P}_h) u_i \|_a^2. \quad (2.30)
\]

From (2.30) and the orthogonal property \(a(u_i - \mathcal{P}_h u_i, (I - E_{k,h}) \mathcal{P}_h u_i) = 0\), we have following error estimate
\[
\| u_i - E_{k,h} u_i \|_a^2 = \| u_i - \mathcal{P}_h u_i \|_a^2 + \| (I - E_{k,h}) \mathcal{P}_h u_i \|_a^2 \]

This is the desired result (2.33).

Similarly, from (2.8), (2.25), (2.26), (2.27) and (2.28), we have the following estimates

\[\|(I - E_k,h)P_hu_i\|_b^2 = \left\| \sum_{j=k+1}^{N_h} \alpha_j \bar{u}_{j,h} \right\|_b^2 = \sum_{j=k+1}^{N_h} \alpha_j^2 \|\bar{u}_{j,h}\|_b^2\]

\[= \frac{1}{\mu_i - \bar{\mu}_{j,h}} \left( \frac{1}{\mu_i} \right)^2 b \left( u_i - \bar{\mathcal{P}_h}u_i, \bar{u}_{j,h} \right) \|\bar{u}_{j,h}\|_b^2 \leq \frac{1}{\delta_{k,i,h}} \sum_{j=k+1}^{N_h} \|\bar{u}_{j,h}\|_b^2 \left( u_i - \bar{\mathcal{P}_h}u_i, \bar{u}_{j,h} \right)^2 \]

\[\leq \frac{1}{\delta_{k,i,h}} \sum_{j=k+1}^{N_h} \bar{\mu}_{j,h} \left( u_i - \bar{\mathcal{P}_h}u_i, \bar{u}_{j,h} \right)^2 \leq \frac{\mu_k + 1}{\delta_{k,i,h}} \|u_i - \bar{\mathcal{P}_h}u_i\|_b^2 \leq \frac{\mu_k + 1}{\delta_{k,i,h}} \|u_i - \bar{\mathcal{P}_h}u_i\|_b^2,\]

which leads to the inequality

\[\|(I - E_k,h)P_hu_i\|_b \leq \frac{\mu_k + 1}{\delta_{k,i,h}} \|u_i - \bar{\mathcal{P}_h}u_i\|_b.\]  

(2.31)

From (2.24), (2.31) and the triangle inequality, we have the following error estimates for the eigenvector approximations in the \(\|\cdot\|_b\)-norm

\[\|u_i - \bar{E}_k,h u_i\|_b \leq \|u_i - \bar{P}_h u_i\|_b + \|(I - E_k,h)\mathcal{P}_h u_i\|_b \]

\[\leq \left(1 + \frac{\mu_k + 1}{\delta_{k,i,h}}\right) \|(I - \mathcal{P}_h) u_i\|_b \leq \left(1 + \frac{\mu_k + 1}{\delta_{k,i,h}}\right) \eta_a(V_h) \|(I - \mathcal{P}_h) u_i\|_a\]

\[\leq \left(1 + \frac{\mu_k + 1}{\delta_{k,i,h}}\right) \eta_a(V_h) \|u_i - \bar{E}_k,h u_i\|_a.\]

This is the second desired result (2.33) and the proof is complete.

Similarly, we assume that the eigenvalue gap \(\delta_{k,i}\) has a uniform lower bound which is denoted by \(\bar{\delta}_{k,i}\) (which can be seen as the “true” separation of the eigenvalue \(\lambda_i\) from the unwanted eigenvalues) in the following parts of this paper. This assumption is reasonable when the mesh size is small enough. Then we have the following simple version of the error estimates based on Theorem 2.1.

**Corollary 2.2.** Under the conditions of Theorem 2.1, the following error estimates hold

\[\|u_i - \bar{E}_k,h u_i\|_a \leq \left(1 + \frac{1}{\lambda_{k+1}} - \eta_a(V_h)\right) \|(I - \mathcal{P}_h) u_i\|_a, \quad 1 \leq i \leq k,\]  

(2.32)

\[\|u_i - \bar{E}_k,h u_i\|_b \leq \eta_a(V_h) \|u_i - \bar{E}_k,h u_i\|_a, \quad 1 \leq i \leq k,\]  

(2.33)

where \(\eta_a(V_h)\) is defined as follows

\[\eta_a(V_h) = \left(1 + \frac{\mu_k + 1}{\bar{\delta}_{k,i}}\right) \eta_a(V_h).\]

**Remark 2.1.** When \(1 \leq i \leq k\) in (2.18), it is easy to find that the estimate (2.32) is less than (2.18) since we have the following inequalities

\[\frac{1}{\lambda_{k+1}} \leq \frac{1}{\lambda_i}, \quad \frac{1}{\bar{\delta}_{k,i}} \leq \frac{1}{\bar{\delta}_i}.\]

From Lemma 2.3, Theorem 2.1 and their proofs, we can extend the error estimates in this section to the case that the subspace is \(V_{H,h}\) and the space \(V\) is replaced by \(V_h\). This understanding will be used to deduce the error estimates for the augmented subspace methods in the following section.
3 Augmented subspace method and its error estimates

In this section, we first present the augmented subspace method for solving the eigenvalue problem (2.5). This method contains solving auxiliary linear boundary value problem in the finer finite element space $V_h$ and the eigenvalue problem on the augmented subspace $V_{H,h}$ which is built by the coarse finite element space $V_H$ and a finite element function in the finer finite element space $V_h$. Then, the new convergence analysis is given for this augmented subspace method. We will find the new convergence result is sharper than the existed results in [16, 20, 21, 22, 23].

In order to define the augmented subspace method, we first generate a coarse mesh $T_H$ with the mesh size $H$ and the coarse linear finite element space $V_H$ is defined on the mesh $T_H$. For simplicity, in this paper, we assume the coarse space $V_H$ is a subspace of the finer element space $V_h$ which is defined on the finer mesh $T_h$.

For some given eigenfunction approximations $u_1^{(\ell)}, \ldots, u_k^{(\ell)}$ which are approximations for the first $k$ eigenfunctions $\bar{u}_{1,h}, \ldots, \bar{u}_{k,h}$ of (2.5), we can do the following augmented subspace iteration step which is defined by Algorithm 1 to improve the accuracy of $u_1^{(\ell)}, \ldots, u_k^{(\ell)}$.

Algorithm 1: Augmented subspace method for the first $k$ eigenpairs

1. If $\ell = 1$, we define $\bar{u}_{1,h}^{(\ell)} = u_1^{(\ell)}$, $i = 1, \ldots, k$, and the augmented subspace $V_{H,h} = V_H + \text{span}\{\bar{u}_{1,h}^{(\ell)}, \ldots, \bar{u}_{k,h}^{(\ell)}\}$. Then solve the following eigenvalue problem: Find $(\lambda_{1,h}^{(\ell)}, u_{1,h}^{(\ell)}) \in \mathcal{R} \times V_{H,h}$ such that $a(u_{1,h}^{(\ell)}, u_{1,h}^{(\ell)}) = 1$ and

   \[ a(u_{1,h}^{(\ell)}, v_{H,h}) = \lambda_{1,h}^{(\ell)} b(u_{1,h}^{(\ell)}, v_{H,h}), \quad \forall v_{H,h} \in V_{H,h}, \quad i = 1, \ldots, k. \]  

2. Solve the following linear boundary value problems: Find $\bar{u}_{i,h}^{(\ell+1)} \in V_h$ such that

   \[ a(\bar{u}_{i,h}^{(\ell+1)}, v_h) = \lambda_{i,h}^{(\ell)} b(u_{i,h}^{(\ell)}, v_h), \quad \forall v_h \in V_h, \quad i = 1, \ldots, k. \]  

3. Define the augmented subspace $V_{H,h} = V_H + \text{span}\{\bar{u}_{1,h}^{(\ell+1)}, \ldots, \bar{u}_{k,h}^{(\ell+1)}\}$ and solve the following eigenvalue problem: Find $(\lambda_{1,h}^{(\ell+1)}, u_{1,h}^{(\ell+1)}) \in \mathcal{R} \times V_{H,h}$ such that $a(u_{1,h}^{(\ell+1)}, u_{1,h}^{(\ell+1)}) = 1$ and

   \[ a(u_{1,h}^{(\ell+1)}, v_{H,h}) = \lambda_{1,h}^{(\ell+1)} b(u_{1,h}^{(\ell+1)}, v_{H,h}), \quad \forall v_{H,h} \in V_{H,h}, \quad i = 1, \ldots, k. \]  

Solve (3.3) to obtain $(\lambda_{1,h}^{(\ell+1)}, u_{1,h}^{(\ell+1)}), \ldots, (\lambda_{k,h}^{(\ell+1)}, u_{k,h}^{(\ell+1)})$.

4. Set $\ell = \ell + 1$ and go to Step 2 for the next iteration until convergence.

Theorem 3.1. Let us define the spectral projection $E_{k,h}^{(\ell+1)} : V \to \text{span}\{\bar{u}_{1,h}^{(\ell+1)}, \ldots, \bar{u}_{k,h}^{(\ell+1)}\}$ for any integer $\ell \geq 0$ as follows

   \[ a(E_{k,h}^{(\ell+1)} w, u_{i,h}^{(\ell+1)}) = a(w, u_{i,h}^{(\ell+1)}), \quad i = 1, \ldots, k \quad \text{for } w \in V. \]  

There exist exact eigenfunction $\bar{u}_{1,h}, \ldots, \bar{u}_{k,h}$ of (2.5) such that the resultant eigenfunction approximations $u_1^{(\ell+1)}, \ldots, u_k^{(\ell+1)}$ have the following error estimate

   \[ \|\bar{u}_{i,h} - E_{k,h}^{(\ell+1)} u_{i,h}\| \leq \lambda_{i,h} \left( \frac{\gamma_0^2(V_H)}{\lambda_{k+1}(\delta_{k,i}^{(\ell+1)})^2} \left( 1 + \frac{\mu_{k+1}}{\delta_{k,i}^{(\ell+1)}} \right) \|u_{i,h} - E_{k,h}^{(\ell)} u_{i,h}\| \right)^{1/2}. \]
Furthermore, the following $\| \cdot \|_a$-norm error estimate hold

$$
\| \bar{u}_{i,h} - E_k^{(\ell+1)} \bar{u}_{i,h} \|_a \leq \left( 1 + \frac{\mu_{k+1}}{\delta_{k,i}^{(1)}} \right) \eta_a(V_H) \| \bar{u}_{i,h} - E_k^{(\ell+1)} \bar{u}_{i,h} \|_a. \tag{3.6}
$$

Proof. First, let us consider the error estimate for the initial approximations $u_{1,h}^{(1)}, \ldots, u_{k,h}^{(1)}$. From Corollary 2.2, there exist exact eigenvectors $\bar{u}_{1,h}, \ldots, \bar{u}_{k,h}$ such that the following error estimates for the eigenvector approximations $u_{1,h}^{(1)}, \ldots, u_{k,h}^{(1)}$ hold for $i = 1, \ldots, k$

$$
\| \bar{u}_{i,h} - E_k^{(1)} \bar{u}_{i,h} \|_a \leq \sqrt{\frac{\eta_a^2(V_H)}{\lambda_{k+1}(\delta_{k,i}^{(1)})^2} (1 - \mathcal{P}_{H,h}) \| \bar{u}_{i,h} \|_a}
$$

and

$$
\| \bar{u}_{i,h} - E_k^{(1)} \bar{u}_{i,h} \|_a \leq \frac{\lambda_{k+1}(\delta_{k,i}^{(1)})^2}{1 + \eta_a^2(V_H)} \| (I - \mathcal{P}_{H,h}) \bar{u}_{i,h} \|_a,
$$

where we have used the inequality $\eta_a(V_H) \leq \eta_a(V_H)$ since $V_H \subset V_H$.

Then the result (3.6) holds for $\ell = 1$. Here the induction method is adopted to prove that (3.5) and (3.6) hold for any $\ell \geq 1$. For this aim, we assume the estimates (3.5) and (3.6) holds for $\ell - 1$. Then let us prove that they also hold for $\ell$ based on this assumption.

From Algorithm 1, it is easy to know that $u_{1,h}^{(\ell)}, \ldots, u_{k,h}^{(\ell)}$ is the orthogonal basis for the space $\text{span}\{u_{1,h}^{(\ell)}, \ldots, u_{k,h}^{(\ell)}\}$. We define the $b(\cdot, \cdot)$-orthogonal projection operator $\pi_{k,h}^{(\ell)}$ to the space $\text{span}\{u_{1,h}^{(\ell)}, \ldots, u_{k,h}^{(\ell)}\}$. Then there exist $k$ real numbers $q_1, \ldots, q_k \in \mathbb{R}$ such that $\pi_{k,h}^{(\ell)} \bar{u}_{i,h}$ has following expansion

$$
\pi_{k,h}^{(\ell)} \bar{u}_{i,h} = \sum_{j=1}^k q_j u_{j,h}^{(\ell)}.	ag{3.8}
$$

From the orthogonal property of the projection operator $\mathcal{P}_{H,h}$, (3.33), (3.2), (3.7), (3.8) and induction assumption, the following inequalities hold

$$
\| \bar{u}_{i,h} - \mathcal{P}_{H,h} \bar{u}_{i,h} \|_a^2 = a(\bar{u}_{i,h} - \mathcal{P}_{H,h} \bar{u}_{i,h}, \bar{u}_{i,h} - \mathcal{P}_{H,h} \bar{u}_{i,h})
$$

$$
= a\left( \bar{u}_{i,h} - \sum_{j=1}^k \tilde{\lambda}_{j,h} \frac{q_j}{\lambda_{j,h}^{(\ell)}} u_{j,h}^{(\ell)}, \bar{u}_{i,h} - \mathcal{P}_{H,h} \bar{u}_{i,h} \right)
$$

$$
= \tilde{\lambda}_{i,h} b\left( \bar{u}_{i,h} - \sum_{j=1}^k \frac{q_j}{\lambda_{j,h}^{(\ell)}} u_{j,h}^{(\ell)}, \bar{u}_{i,h} - \mathcal{P}_{H,h} \bar{u}_{i,h} \right)
$$

$$
\leq \tilde{\lambda}_{i,h} \| \bar{u}_{i,h} - \pi_{k,h}^{(\ell)} \bar{u}_{i,h} \|_a \| \bar{u}_{i,h} - \mathcal{P}_{H,h} \bar{u}_{i,h} \|_b
$$

$$
\leq \tilde{\lambda}_{i,h} \| \bar{u}_{i,h} - E_k^{(\ell)} \bar{u}_{i,h} \|_a \| \bar{u}_{i,h} - \mathcal{P}_{H,h} \bar{u}_{i,h} \|_b
$$

$$
\leq \tilde{\lambda}_{i,h} \eta_a(V_H) \| \bar{u}_{i,h} - E_k^{(\ell)} \bar{u}_{i,h} \|_a \eta_a(V_H) \| \bar{u}_{i,h} - \mathcal{P}_{H,h} \bar{u}_{i,h} \|_a.	ag{3.9}
$$
Similarly to the proof of Theorem 2.1, we have the following estimate
\[ \|\tilde{u}_{i,h} - P_{H,h}\tilde{u}_{i,h}\|_a \leq \tilde{\lambda}_{i,h}\eta_a(V_H)\eta_a(V_H)\|\tilde{u}_{i,h} - E_{k,h}^{(\ell)}\bar{u}_{i,h}\|_a. \]  
(3.10)

Combining Corollary 2.2 and (3.10) leads to the following estimate
\[ \|\tilde{u}_{i,h} - E_{k,h}^{(\ell+1)}\bar{u}_{i,h}\|_a \leq \tilde{\lambda}_{i,h}\sqrt{1 + \frac{\eta_a^2(V_H)}{\lambda_{k+1}(\delta_{k,i})}\eta_a(V_H)\|\tilde{u}_{i,h} - E_{k,h}^{(\ell)}\bar{u}_{i,h}\|_a}. \]  
(3.11)

Similarly to the proof of Theorem 2.1, we have the following \( \| \cdot \|_b \) error estimate
\[ \|\tilde{u}_{i,h} - E_{k,h}^{(\ell+1)}\bar{u}_{i,h}\|_b \leq \left( 1 + \frac{\mu_{k+1}}{\delta_{k,i}^{(\ell+1)}} \right) \eta_a(V_H)\|\tilde{u}_{i,h} - E_{k,h}^{(\ell+1)}\bar{u}_{i,h}\|_a. \]  
(3.12)

From (3.11) and (3.12), we know that the estimates (3.5) and (3.6) holds for the integer \( \ell \). Then the proof is complete.

**Remark 3.1.** From the convergence result (3.5) in Theorem 3.1, in order to accelerate the convergence rate, we should decrease the term \( \eta_a(V_H) \) which depends on the coarse space \( V_H \). Then enlarging the subspace \( V_H \) can accelerate the convergence.

**Remark 3.2.** In this paper, we are only concerned with the error estimates for the eigenvector approximation since the error estimates for the eigenvalue approximation can be easily deduced from the following error expansion
\[
0 \leq \tilde{\lambda}_i - \bar{\lambda}_{i,h} = \frac{(A(\bar{u}_{i,h} - \psi), \bar{u}_{i,h} - \psi)}{\|\psi\|^2} - \tilde{\lambda}_{i,h} \frac{(\bar{u}_{i,h} - \psi, \bar{u}_{i,h} - \psi)}{\|\psi\|^2} \leq \frac{\|\bar{u}_{i,h} - \psi\|^2}{\|\psi\|^2},
\]
where \( \psi \) is the eigenvector approximation for the exact eigenvector \( \bar{u}_{i,h} \) and
\[
\tilde{\lambda}_i = \frac{(A\psi, \psi)}{\|\psi\|^2}.
\]

It is obvious that the parallel computing method can be used for Step 2 of Algorithm 1 since each linear equation can be solved independently. Furthermore, the augmented subspace method can be used to design a complete parallel scheme for eigenvalue problems. For this aim, we give another version of the augmented subspace method for only one (may be not the smallest one) eigenpair. The corresponding numerical method is defined by Algorithm 2. This idea has already been proposed and analyzed in [23]. But, we will give a sharper error estimate for this type of method.

In this section, we assume the given eigenpair approximation \( (\tilde{\lambda}_{i,h}, \bar{u}_{i,h}) \) in \( \mathcal{R} \times V_h \) with different superscript is closet to an exact eigenpair \( (\tilde{\lambda}_i, \bar{u}_i) \) of (2.5). Based on these settings, we can give the following convergence result for the augmented subspace method defined by Algorithm 2.

**Theorem 3.2.** For \( \ell \geq 1 \), according to the eigenpair approximation \( (\tilde{\lambda}_{i,h}, \bar{u}_{i,h}) \) in \( \mathcal{R} \times V_h \), we define the spectral projectors \( E_{k,h}^{(\ell)} : V \mapsto \text{span}\{u_{k,h}^{(\ell)}\} \) as follows
\[
a(E_{k,h}^{(\ell)}w, u_{k,h}^{(\ell)}) = a(w, u_{k,h}^{(\ell)}), \quad \text{for } w \in V.
\]

Then the eigenpair approximation \( (\tilde{\lambda}_{i,h}, \bar{u}_{i,h}) \) in \( \mathcal{R} \times V_h \) produced by Algorithm 2 satisfies the following error estimates
\[
\|\bar{u}_i - E_{k,h}^{(\ell+1)}\bar{u}_i\|_a \leq \tilde{\lambda}_i \sqrt{1 + \frac{\eta_a^2(V_H)}{\lambda_i}\eta_a(V_H)\|\bar{u}_i - E_{k,h}^{(\ell)}\bar{u}_i\|_a}, \]  
(3.16)

\[
\|\bar{u}_i - E_{k,h}^{(\ell+1)}\bar{u}_i\|_b \leq \left( 1 + \frac{\eta_a(V_H)}{\lambda_i} \right) \eta_a(V_H)\|\bar{u}_i - E_{k,h}^{(\ell+1)}\bar{u}_i\|_a. \]  
(3.17)
Algorithm 2: Augmented subspace method for one eigenpair

1. If $\ell = 1$, we define $\tilde{u}_h^{(\ell)} = u_h^{(\ell)}$ and the augmented subspace $V_{H,h} = V_H + \text{span}\{\tilde{u}_h^{(\ell)}\}$. Then solve the following eigenvalue problem: Find $(\lambda_h^{(\ell)}, u_h^{(\ell)}) \in \mathcal{R} \times V_{H,h}$ such that $a(u_h^{(\ell)}, u_h^{(\ell)}) = 1$ and
   \[ a(u_h^{(\ell)}, v_{H,h}) = \lambda_h^{(\ell)} b(u_h^{(\ell)}, v_{H,h}), \quad \forall v_{H,h} \in V_{H,h}. \]  
   (3.13)

2. Solve the following linear boundary value problem: Find $\tilde{u}_h^{(\ell+1)} \in V_h$ such that
   \[ a(\tilde{u}_h^{(\ell+1)}, v_h) = \lambda_h^{(\ell)} b(\tilde{u}_h^{(\ell)}, v_h), \quad \forall v_h \in V_h. \]  
   (3.14)

3. Define the augmented subspace $V_{H,h} = V_H + \text{span}\{\tilde{u}_h^{(\ell+1)}\}$ and solve the following eigenvalue problem: Find $(\lambda_h^{(\ell+1)}, u_h^{(\ell+1)}) \in \mathcal{R} \times V_{H,h}$ such that $a(u_h^{(\ell+1)}, u_h^{(\ell+1)}) = 1$ and
   \[ a(u_h^{(\ell+1)}, v_{H,h}) = \lambda_h^{(\ell+1)} b(u_h^{(\ell+1)}, v_{H,h}), \quad \forall v_{H,h} \in V_{H,h}. \]  
   (3.15)

Solve (3.15) and the output $(\lambda_h^{(\ell+1)}, u_h^{(\ell+1)})$ is chosen such that $u_h^{(\ell+1)}$ has the largest component in $\text{span}\{\tilde{u}_h^{(\ell+1)}\}$ among all eigenfunctions of (3.15).

4. Set $\ell = \ell + 1$ and go to Step 2 for the next iteration until convergence.

Proof. First, let us consider the error estimate for the initial approximations $u_h^{(1)}$. From Corollary 2.1, there exist exact eigenfunction $\bar{u}_h$ of (2.5) such that the following error estimates hold for the eigenvector approximation $u_h^{(1)}$

\[ \|\bar{u}_h - E_h^{(1)} \bar{u}_h\|_a \leq \sqrt{1 + \frac{\eta_2(V_h)}{\lambda_1 \delta_h^2}} \| (I - \mathcal{P}_{H,h}) \bar{u}_h\|_a \]

\[ \leq \sqrt{1 + \frac{\eta_2(V_h)}{\lambda_1 \delta_h^2}} \| (I - \mathcal{P}_{H,h}) \bar{u}_h\|_a, \]

and

\[ \|\bar{u}_h - E_h^{(1)} \bar{u}_h\|_b \leq \left( 1 + \frac{1}{\lambda_1 \delta_h} \right) \eta_a(V_{H,h}) \|\bar{u}_h - E_h^{(1)} \bar{u}_h\|_a \]

\[ \leq \left( 1 + \frac{1}{\lambda_1 \delta_h} \right) \eta_a(V_H) \|\bar{u}_h - E_h^{(1)} \bar{u}_h\|_a, \]  
   (3.18)

where we have used the inequality $\eta_a(V_{H,h}) \leq \eta_a(V_H)$ since $V_H \subset V_{H,h}$.

Then the result (3.17) holds for $\ell = 1$. Here the induction method is adopted to prove that (3.16) and (3.17) hold for any $\ell \geq 1$. For this aim, we assume the estimates (3.16) and (3.17) holds for $\ell - 1$. Then let us prove that they also hold for $\ell$ based on this assumption.

We define the $b(\cdot, \cdot)$-orthogonal projection operator $\pi_h^{(\ell)}$ to the space $\text{span}\{u_h^{(\ell)}\}$. Then there exists a real number $q \in \mathcal{R}$ such that $\pi_h^{(\ell)} \bar{u}_h = qu_h^{(\ell)}$. Then from the orthogonal property of the projection operator $\mathcal{P}_{H,h}$, (2.14), (3.7), (3.14) and the induction assumption, the following inequalities hold

\[ \|\bar{u}_h - \mathcal{P}_{H,h} \bar{u}_h\|_a^2 = a(\bar{u}_h - \mathcal{P}_{H,h} \bar{u}_h, \bar{u}_h - \mathcal{P}_{H,h} \bar{u}_h) \]
Combining Lemma 2.3, Corollary 2.1 and (3.20), we have the following estimate

\[
\| \hat{u}_h - P_{H,h} \hat{u}_h \| \leq \lambda_h \left( \frac{1}{\lambda_h^{\ell+1}} a_h^{(\ell+1)} \right) \| \hat{u}_h - P_{H,h} \hat{u}_h \|
\]

where we also used the inequality \( \eta_a(V_{H,h}) \leq \eta_a(V_H) \) since \( V_H \subset V_{H,h} \).

From (3.19), we have the following estimate

\[
\| \hat{u}_h - P_{H,h} \hat{u}_h \| \leq \lambda_h \left( 1 + \frac{1}{\lambda_1 \delta_\lambda} \right) \eta_a^2(V_H) \| \hat{u}_h - E_h^{(\ell)} \hat{u}_h \|.
\]  

(3.20)

Combining Lemma 2.3, Corollary 2.1 and (3.20), we have the following estimate

\[
\| \hat{u}_h - E_h^{(\ell+1)} \hat{u}_h \| \leq \lambda_h \left( 1 + \frac{\eta_a^2(V_H)}{\lambda_1 \delta_\lambda} \right) \eta_a^{\ell+1}(V_H) \| \hat{u}_h - u_h^{(\ell)} \|.
\]

(3.21)

Similarly to the proof of Lemma 2.3, the following \( \| \cdot \|_a \)-error estimate hold

\[
\| \hat{u}_h - E_h^{(\ell+1)} \hat{u}_h \| \leq \left( 1 + \frac{1}{\lambda_1 \delta_\lambda} \right) \eta_a(V_H) \| \hat{u}_h - E_h^{(\ell+1)} \hat{u}_h \|.
\]

(3.22)

From (3.21) and (3.22), we know that the estimates (3.16) and (3.17) also holds for \( \ell \). Then the proof is complete.

\[\square\]

**Corollary 3.1.** Under the conditions of Theorem 3.2, the eigenfunction approximation \( u_h^{(\ell+1)} \) has following error estimates

\[
\| \hat{u}_h - E_h^{(\ell+1)} \hat{u}_h \|_a \leq (\gamma(\lambda_h))^{\ell} \| \hat{u}_h - E_h^{(1)} \hat{u}_h \|_a,
\]

(3.23)

\[
\| \hat{u}_h - E_h^{(\ell+1)} \hat{u}_h \|_b \leq \left( 1 + \frac{1}{\lambda_1 \delta_\lambda} \right) \eta_a(V_H) \| \hat{u}_h - E_h^{(\ell+1)} \hat{u}_h \|_a.
\]

(3.24)

where

\[
\gamma(\lambda_h) = \lambda_h \left( 1 + \frac{\eta_a^2(V_H)}{\lambda_1 \delta_\lambda} \right) \eta_a^\ell(V_H).
\]

(3.25)

The error estimate for the eigenvalue approximations \( \lambda_h^{(\ell)} \) can be deduced from Theorem 3.2 and Remark 3.2.

4 The application to second order elliptic eigenvalue problem

In this section, we will show the applications of augmented subspace methods to the second order elliptic eigenvalue problem. These numerical schemes can improve the efficiency for solving the
By solving the eigenvalue problem directly on the fine space where the computing domain is set to be the unit square \( \Omega = (0,1) \times (0,1) \). The exact finite element eigenfunction \( \bar{u}_h \) is obtained by solving the eigenvalue problem directly on the fine space \( V_h \).

Here, we are concerned with the second order elliptic eigenvalue problem, i.e., in (2.3), the bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are defined as follows

\[
a(u, v) = \int_{\Omega} \nabla u \cdot A \nabla v d\Omega, \quad b(u, v) = \int_{\Omega} \rho uv d\Omega,
\]

where \( \Omega \subset \mathbb{R}^d \) \((d=2,3)\) is a bounded domain, \( A \in \left(W^{1,\infty}(\Omega)\right)^{d \times d} \) a uniformly positive definite matrix on \( \Omega \) and \( \rho \in W^{0,\infty}(\Omega) \) is a uniformly positive function on \( \Omega \). We pose homogeneous Dirichlet boundary condition to the problem and it means here \( V = H_0^1(\Omega) \) and \( W = L^2(\Omega) \) (cf. [1]). In order to use the finite element discretization method, we employ the meshes defined in section 2.

Here the augmented subspace methods defined by Algorithms 1 and 2 are applied to the second order elliptic eigenvalue problem. The main ingredient is to discuss the way to construct the coarse space \( V_H \) based on the fine space \( V_h \). There have two obvious ways to produce the coarse space \( V_H \). In the first way, the coarse space \( V_H \) and fine space \( V_h \) are defined on the same mesh denoted by \( T_H \) in this section. But the degree of the fine space \( T_h \) is higher than that of the coarse space \( V_H \). This means the coarse space \( V_H \) is chosen as the linear finite element space. The second way to produce the coarse space is based on the two-grid idea from [24]. In this way, the coarse space \( V_H \) is defined on the coarse grid \( T_H \) but the fine space \( V_h \) is defined on the finer grid \( T_h \). In these two ways, the coarse space \( V_H \) are both chosen as the linear finite element space on the mesh \( T_H \), we have the following estimate for the quantity \( \eta_a(V_H) \) (cf. [6, 9])

\[
\eta_a(V_H) \leq CH,
\]

where the constant depends on the matrix \( A \), scalar \( \rho \) and the shape of the mesh \( T_H \).

Based on Theorems 3.1 and 3.2, the convergence result can be concluded with the following inequalities

\[
\|\bar{u}_{i,h} - E_{k,h}^{(\ell+1)} u_{i,h}\|_a \leq C(CH)^{2\ell} \|\bar{u}_{i,h} - E_{k,h}^{(1)} u_{i,h}\|_a, \quad (4.2)
\]

\[
\|\bar{u}_{i,h} - E_{k,h}^{(\ell+1)} \bar{u}_{i,h}\|_b \leq CH \|\bar{u}_{i,h} - E_{k,h}^{(\ell+1)} \bar{u}_{i,h}\|_a, \quad (4.3)
\]

and

\[
\|\bar{u}_h - E_{h}^{(\ell+1)} \bar{u}_h\|_a \leq (CH)^{2\ell} \|\bar{u}_h - E_{h}^{(1)} \bar{u}_h\|_a, \quad (4.4)
\]

\[
\|\bar{u}_h - E_{h}^{(\ell+1)} \bar{u}_h\|_b \leq CH \|\bar{u}_h - E_{h}^{(\ell+1)} \bar{u}_h\|_a. \quad (4.5)
\]

The aim of this section is to check these convergence results by some numerical examples. In these numerical experiments, Algorithms 1 and 2 are implemented for solving the following standard Laplace eigenvalue problem: Find \( (\lambda, u) \in \mathcal{R} \times H_0^1(\Omega) \) such that

\[
\begin{align*}
-\Delta u &= \lambda u, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial\Omega, \\
\|u\|_1^2 &= 1,
\end{align*}
\]

where the computing domain is set to be the unit square \( \Omega = (0,1) \times (0,1) \).

In all numerical testes, the initial eigenfunction approximation is produced by solving the eigenvalue problem (4.6) on the coarse space \( V_H \). The exact finite element eigenfunction \( \bar{u}_h \) is obtained by solving the eigenvalue problem directly on the fine space \( V_h \).
4.1 Augmented subspace by low order finite element space

In the first subsection, we check the convergence results (4.2)-(4.5) for the fine space is chosen as the high order finite element space. In these tests, the initial eigenfunction approximation is produced by solving the eigenvalue problems on the coarse space $V_H$. Then we do the iteration steps by the augmented subspace method defined by Algorithms 1 and 2.

In the first way, the spaces $V_H$ and $V_h$ are defined on the same mesh $T_H$ but with different order of finite element methods. Here, $V_H$ is chosen as the linear finite element space and the fine mesh $V_h$ is 4-th order finite element space defined on the mesh $T_H$.

In order to validate the convergence results stated in (4.2)-(4.5), we check the numerical errors corresponding to the linear finite element space $V_H$ with different sizes $H$. The aim here is to check the dependence of the convergence rate on the mesh size $H$. The coarse mesh $T_H$ is set to be the regular type of uniform mesh. Figure 1 shows the corresponding convergence behaviors for the first eigenfunction by Algorithm 1 (or Algorithm 2) with the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8$, $\sqrt{2}/16$, $\sqrt{2}/32$ and $\sqrt{2}/64$. We can find the corresponding convergence rate are 0.044633, 0.012493, 0.0032218 and 0.00081231. These results show that the augmented subspace method defined by Algorithms 1 and 2 should have second order convergence which validates the results (4.2)-(4.5).

Here, we also check the performance of Algorithm 1 for computing the smallest 4 eigenpairs. Figure 2 shows the corresponding convergence behaviors for the smallest 4 eigenfunctions by Algorithm 1 with the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8$, $\sqrt{2}/16$, $\sqrt{2}/32$ and $\sqrt{2}/64$. We can find the corresponding convergence rate are 0.35452, 0.12177, 0.032864 and 0.007999. Furthermore, from Figures 1 and 2, we can find the convergence rate for the 4-th eigenfunction is slower than that for the 1-st eigenfunction which is consistent with
Theorem 1.

Errors of augmented subspace iteration

\begin{align*}
\text{Errors of augmented subspace iteration} \\
\text{Errors of augmented subspace iteration} \\
\text{Errors of augmented subspace iteration} \\
\text{Errors of augmented subspace iteration}
\end{align*}

Figure 2: The convergence behaviors for the smallest 4 eigenfunction by Algorithm 1 with the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8$, $\sqrt{2}/16$, $\sqrt{2}/32$ and $\sqrt{2}/64$. The corresponding convergence rates are $0.35452$, $0.12177$, $0.032864$ and $0.007999$.

The next task is to check the performance of Algorithm 2 for computing the only 4-th eigenpair. Figure 3 shows the corresponding convergence behaviors for the only 4-th eigenfunctions by Algorithm 2 with the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8$, $\sqrt{2}/16$, $\sqrt{2}/32$ and $\sqrt{2}/64$. The corresponding convergence rate shown in Figure 3 are $0.35918$, $0.12588$, $0.035169$ and $0.009017$. These results show that the augmented subspace method defined by Algorithm 2 has second order convergence which validate the results (4.4)-(4.5).

4.2 Augmented subspace by the finite element space on the coarse mesh

In the second subsection, $V_h$ is chosen as the linear finite element space defined on the finer mesh $T_h$. For this aim, we start from the coarse mesh $T_H$ to produce the finer mesh by the regular refinement. In the numerical tests here, we set the size $h = 1/256$ for the finer mesh $T_h$. Here, $V_H$ is chosen as the linear finite element space defined on the coarse mesh $T_H$. The initial eigenfunction approximation is also produced by solving the eigenvalue problems on the coarse space $V_H$. Then we do the iteration steps by the augmented subspace method defined by Algorithms 1 and 2.

In order to validate the convergence results stated in (4.2)-(4.5), we also check the numerical errors corresponding to the linear finite element space $V_H$ with different sizes $H$. The aim is to check the dependence of the convergence rate on the mesh size $H$. Here, the coarse mesh $T_H$ is also set to be the regular type of uniform mesh.

Figure 4 shows the convergence behaviors for the first eigenfunction by the augmented subspace methods corresponding to the coarse mesh sizes $H = \sqrt{2}/4$, $\sqrt{2}/8$, $\sqrt{2}/16$ and $\sqrt{2}/32$. The corresponding convergence rates are $0.13142$, $0.048523$, $0.013652$ and $0.0035056$. These results
show that the augmented subspace method defined by Algorithms 1 and 2 should have second order convergence which also validates the results (4.2)-(4.3).

Then, we check the performance of Algorithm 1 for computing the smallest 4 eigenpairs. Figure 5 shows the corresponding convergence behaviors for the smallest 4 eigenfunctions by Algorithm 1 with the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8$, $\sqrt{2}/16$, $\sqrt{2}/32$ and $\sqrt{2}/64$. We can find that the corresponding convergence rate are 0.31838, 0.09979, 0.026024 and 0.006825. Furthermore, from Figures 4 and 5, we can find the convergence rate for the 4-th eigenfunction is slower than that for the 1-st eigenfunction which is consistent with Theorem 1.

The final task is to check the performance of Algorithm 2 for computing the only 4-th eigenpair. Figure 6 shows the corresponding convergence behaviors for the only 4-th eigenfunctions by Algorithm 2 with the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8$, $\sqrt{2}/16$, $\sqrt{2}/32$ and $\sqrt{2}/64$. The corresponding convergence rate shown in Figure 6 are 0.33687, 0.11207, 0.030571 and 0.0077354. These results show that the augmented subspace method defined by Algorithm 2 has second order convergence which validates the results (4.4)-(4.5).

5 Concluding remarks

In this paper, some enhanced error estimates for the augmented subspace method are deduced for solving eigenvalue problems. We have derived higher order convergence rates than existing results. Based on these new results, we can also produce the corresponding sharper error estimates for the multigrid and multilevel methods which are designed based on the augmented subspace methods and the sequence of grids.
Figure 4: The convergence behaviors for the first eigenfunction by Algorithm 1 corresponding to the coarse mesh size $H = \sqrt{2}/4$, $\sqrt{2}/8$, $\sqrt{2}/16$ and $\sqrt{2}/32$. The corresponding convergence rates are 0.13142, 0.048523, 0.013652 and 0.0035056.

Figure 5: The convergence behaviors for the smallest 4 eigenfunction by Algorithm 1 with the coarse space being the linear finite element space on the mesh with size $H = \sqrt{2}/8$, $\sqrt{2}/16$, $\sqrt{2}/32$ and $\sqrt{2}/64$. The corresponding convergence rates are 0.31838, 0.09979, 0.026024 and 0.0068251.
Figure 6: The convergence behaviors for the only 4-th eigenfunction by Algorithm 2 with the coarse space being the linear finite element space on the mesh with size \( H = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32 \) and \( \sqrt{2}/64 \). The corresponding convergence rates are 0.33687, 0.11207, 0.030571 and 0.0077354.

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