From axion quality and naturalness problems to a high-quality $\mathbb{Z}_{4N}$ QCD relaxion

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We highlight general issues associated with quality and naturalness problems in theories of light QCD-axions, axion-like particles, and relaxions. We show the presence of Planck-suppressed operators generically lead to scalar coupling of axions with the SM. We present a new class of $\mathbb{Z}_{4N}$ QCD relaxion models that can address both the QCD relaxion CP problem as well as its quality problem. This new class of models also leads to interesting experimental signatures, which can be searched for at the precision frontier.

I. INTRODUCTION

The Standard Model of particle physics (SM) is an extremely successful yet incomplete description of nature. It cannot account for the observed neutrino masses and mixings, the matter anti-matter asymmetry, and the origin of Dark Matter (DM). Even within the framework of the SM, we have the Higgs-hierarchy and the Strong CP problems. On top of that, the effect of gravity is expected to be significant at the Planck scale despite the lack of knowledge about its quantum nature. In particular, quantum gravity is expected to violate global symmetries in the UV, implying the existence of symmetry-breaking operators suppressed by powers of the Planck mass $M_{Pl} = 2.4 \times 10^{18}$ GeV in the framework of effective field theory (EFT). For an axion field $\Phi$ with a global Peccei-Quinn (PQ) symmetry \[1\], one for instance expects, among others, operators of the form

$$\mathcal{L} \supset \frac{1}{2} \left( \frac{c_N \Phi^N + \text{h.c.}}{M_{Pl}} \right) \mathcal{O}, \quad (I.1)$$

where $N$ is an integer, $c_N$ is a dimensionless EFT parameter, and $\mathcal{O}$ is any dimension-four operator consistent with the unbroken gauge symmetries. Expanding $\Phi = f e^{i \phi/f}$, this Lagrangian generates a shift-symmetric potential of the form

$$V_\Delta = |c_N| \Delta^N \cos \left( \frac{N \phi}{f} + \beta \right) \mathcal{O}, \quad (I.2)$$

where $\Delta \equiv f/M_{Pl}$ and $\beta = \text{arg}(c_N)$ is an arbitrary phase which is generically $O(1)$. Note that, if CP is not broken by gravity then $\beta = 0$. The dimension $N$ of the PQ-breaking operator in Eqs. (I.1-I.2) is dictated by the unbroken gauge symmetries of the underlying theory.

The leading contribution to PQ-breaking arises from a constant operator multiplied by $M_{Pl}^4$ to match the dimension. This definition fixes $N > 4$ so that these operators are suppressed in the limit $M_{Pl} \to \infty$ (see e.g. \[2\], \[3\]). This implies a contribution $V_\Delta = |c_N| \Delta^N M_{Pl}^4 \cos (N \phi/f + \beta)$ to the scalar field theory. If this field is identified with the QCD axion \[4\], \[5\], then the coefficient $|c_N| \Delta^N$ in Eq. (I.2) cannot be too large or else it will spoil the solution to the strong CP problem; this is the so-called axion quality problem \[10\], \[12\], and it can be solved by either (a) fine-tuning, e.g. taking $|c_N| \ll 1$, (b) taking $f$ very small (which is constrained by measurements of axion couplings to matter), or (c) forbidding operators of dimension $N$ up to some large value, for example by imposing some unbroken gauge symmetry (e.g. $\mathbb{Z}_N$).

We describe the constraints on these operators in greater detail in the subsections below.

II. AXION PHENOMENOLOGY

A. Axion-like particles and naturalness

A general axion-like particle (ALP) which is not coupled to QCD does not exhibit a quality problem related to the vacuum structure (see next section), and therefore it might seem that the presence of Planck-suppressed operators would be harmless. However, these same operators can induce large contributions to the ALP mass, leading to a fine-tuning problem.

Planck-suppressed operators can also generate ALP couplings to the SM scalar operators, which we discuss in Section II.C. In the absence of any CP violation, ALPs interact with the SM scalar operators quadratically at the leading order, whereas if gravity does not respect CP, i.e. for $\beta \neq 0$ in Eq. (I.2), these interactions are generated at linear order.

An ALP is defined by its mass $m$ and coupling with the SM pseudoscalar operators. These couplings are associated with an energy scale, which we will identify with $f$. To analyze the effect of Planck-suppressed operators, we consider an ALP potential of

$$V_{ALP}(\phi) = -m^2 f^2 \cos \left( \frac{\phi}{f} \right), \quad (II.1)$$
which defines the ALP mass $m$. However, the second derivative of the potential induced by Planck-suppressed operators in Eq. (1.2) is

$$V''(\phi) = |c_N| \Delta N^{-2} M_{P1}^2 N^2 \cos \left( \frac{N \phi}{f} + \beta \right). \quad (II.2)$$

Therefore, at leading order in $\phi/f \ll 1$, we have a bare contribution to the mass $m^2$ and a correction of order

$$\delta m^2 \approx |c_N| \cos \beta \Delta N^{-2} N^2 M_{P1}^2. \quad (II.3)$$

Such corrections satisfy $\delta m^2 \ll m^2$ only if

$$\left| \frac{|c_N| \cos \beta \Delta N^{-2} N^2 M_{P1}^2}{m^2} \right| \ll 1. \quad (II.4)$$

Assuming $c_N \sim \cos \beta \sim 1$, one can translate the inequality (II.4) into an upper bound on $f$ as a function of $N$ in order to have negligible fine-tuning of the ALP mass. We illustrate these limits for ALP masses $m = 1, 10^{-7}, 10^{-14}$ eV using the red, blue, and green dotted lines (respectively) in Figure 1.

B. QCD axion quality and naturalness

QCD axions [4–9] exhibit a quality problem when the contribution of Planck-suppressed operators contribute significantly to a shift in the low-energy vacuum of the potential [10–12]. At low energy, QCD axions have a potential of the form [13–14]

$$V_a(\phi) = -\Lambda_{QCD}^3 (m_u + m_d) \times \sqrt{1 - \frac{2z}{(1 + z)^2} \left[ 1 - \cos \left( \frac{\phi}{f} + \hat{\theta} \right) \right]}, \quad (II.5)$$

where $z = m_u/m_d \approx 0.47$ [15] is the ratio of up and down quark masses, $\Lambda_{QCD} = (\langle q\bar{q} \rangle)^{1/3}$ is the QCD scale defined by the quark condensate, and $\hat{\theta}$ is the effective CP violating angle. At leading order in $z \ll 1$ (and ignoring an irrelevant constant), we have

$$V_a(\phi) \approx -\Lambda_{QCD}^4 \cos \left( \frac{\phi}{f} + \hat{\theta} \right), \quad (II.6)$$

where for simplicity we define $\Lambda_a = (\Lambda_{QCD}^3 m_u)^{1/4} \approx 84$ MeV.

In the presence of the leading Planck-suppressed operator, one can find the minimum of the QCD-axion potential as

$$0 = V'(\langle \phi \rangle) = |c_N| \Delta N N M_{P1} \sin(N\epsilon + \beta') + \Lambda_a^4 \sin \epsilon \approx |c_N| \Delta N N M_{P1} \sin \beta' + \Lambda_a^4 \epsilon, \quad (II.7)$$

where $\epsilon \equiv \langle \phi \rangle/f - \theta$ and $\beta' \equiv \beta - N\theta$ which is generically $O(1)$. Non-observation of the neutron electric dipole moment (EDM) implies that $|\epsilon| \lesssim 10^{-10}$ (see e.g. [16–17]), so in the last step we have expanded in small $\epsilon$, $Ne \ll 1$.

In order to not spoil the QCD axion solution to the strong CP problem, one must require

$$|\epsilon| = \left| \frac{|c_N| \sin \beta' \Delta N N M_{P1}^2}{\Lambda_a^4} \right| \lesssim 10^{-10}. \quad (II.8)$$

At leading order in $N$ this gives (for $c_N \simeq \sin \beta' \simeq 1$)

$$N \gtrsim \log \left( \frac{(10^{-10} \Lambda_a/M_{P1})^4}{19 - \log (f/10^{10} \text{GeV})} \right) = \frac{201}{\log (\Delta)} \quad (II.9)$$

see also [18]. So for PQ quality to be preserved, one needs to forbid operators with $N \lesssim 10$ (13) for $f = 10^{10}$ (10$^{12}$) GeV. A simple way to do this is with a gauged $Z_N$ symmetry (see Section II C).

The inequality of (II.8) is illustrated by the black solid line in Figure 1. Comparing the QCD case to an ALP where $m^2 f^2 \approx \Lambda_a^4$, we observe a natural suppression of $10^{-10} N$ in the ALP naturalness condition in Eq. (II.4), relative to Eq. (II.8). Further, ALPs can populate a wider space of values for $m$ and $f$, allowing for more freedom in parameter inputs. Still, it is intriguing that the requirement of natural ALP mass given in (II.4) is nearly as restrictive as the quality problem for QCD axions.

As we point out above, generically scalar fields acquire large mass corrections from Planck-suppressed operators. Therefore in principle there is another constraint on the quality of the QCD axion, arising from fine-tuning of the axion mass, though this is always weaker than the constraint above (this was also pointed out in [18]). Finally, note that in principle one could satisfy Eq. (II.8) even at small $N$ by tuning the EFT coefficient $|c_N| \ll 1$ or the phase parameter $|\beta'| = |\beta - N\theta| \ll 1$. However, this quickly leads to a fine-tuning as bad as (or worse than) the original strong CP problem.

C. High-quality, natural $Z_N$ QCD Axion

It was shown in [19] that an extended sector with $N$ copies of the SM, related by a $Z_N$ symmetry, can lead to a QCD-like axion of mass much smaller than that of canonical QCD axion, due to additional suppression by $\sim z^N$ in the effective QCD scale. This idea was further investigated in [3] and shown to simultaneously admit a viable ultralight axion DM candidate [20]. If this $Z_N$ symmetry is gauged, it can protect the theory from Planck-suppressed operators in Eq. (I.1).

Let us consider $N$ copies of the SM which are related to each other by a $Z_N$ symmetry which is non-linearly realized by the axion field $\phi$, as

$$Z_N : \text{SM}_k \rightarrow \text{SM}_{k+1 (\text{mod } N)} \quad (II.10)$$

$$\phi \rightarrow \phi + \frac{2\pi k}{N}, \quad (II.11)$$
This is apparent in the effective potential of the theory QCD case because the effective QCD scale is shifted.

As shown in Eq. (II.5), the axion will receive contributions from all the sectors; the combined potential can be written as

\[ \mathcal{V}_\text{tot} (\phi) = \sum_{k=0}^{N-1} V (\phi + 2\pi k N) , \]  

(II.13)

where, the axion potential in each sector is

\[ V(x) = -\Lambda^3_{\text{QCD}} m_a \sqrt{1 + z^2 + 2z \cos x} , \]

as shown in Eq. (II.5).

At low energies, this theory differs from the generic QCD case because the effective QCD scale is shifted. This is apparent in the effective potential of the theory \[ N \] (see Eq. (2.30)):

\[ V_N(\phi) \simeq -\sqrt{1 - z^2} \frac{\sin \beta}{\pi N} z^{N-1} \Lambda^3_{\text{QCD}} m_a \cos \left( \frac{N \phi}{f} + \tilde{\theta} \right) , \]  

(II.14)

The requirement \( V'(\phi) = V'_N(\phi) + V_N(\phi) = 0 \) implies

\[ |c| = \left| \frac{|c_N| \sin \beta \Delta^N M^4_{11}}{\Lambda^3} \right| \kappa \lesssim 10^{-10} , \]

(II.15)

where \( \kappa \equiv z^{N-1} \sqrt{(1 - z^2)/(\pi N)} \).

The \( Z_N \) axion case of Refs. \[ 3, 19, 20 \] is illustrated by the black dashed line in Figure 1. The symmetry provides a mechanism for suppressing operators up to some large \( N \) relative to the vanilla QCD case; however, at any given \( N \), the inequality (II.15) has a natural enhancement of order \( 1/\kappa \gg 1 \) relative to the minimal QCD axion (c.f. Eq. (II.8)).

D. Challenges associated with the QCD relaxion idea

The relaxion framework, proposed in \[ 21 \] provides a new insight on the hierarchy problem, which does not require TeV-scale new physics, but rather implies a non-trivial cosmological evolution of the Higgs mass. The original relaxion model was based on the QCD axion model \[ 21 \].

However, as the back-reaction and the rolling potential are sequestered, the relaxion stopping point corresponds to sizeable phase, and generically cannot be set to zero. It was noticed in the original paper \[ 21 \] as well. Furthermore, as was shown in \[ 26 \], and further derived below for the QCD-relaxion model, the peculiar nature of the relaxion dynamics implies that the relaxion stops at a highly non-generic point in the field space. At this point, the mass is parametrically suppressed, and the phase is predicted to be very close to \( \pi/2 \), a mechanism dubbed the relaxed relaxion. In \[ 27 \], a solution was proposed to this problem; however, it required non-classical evolution of the relaxion and thus, led to further problems associated with the measure problem \[ 28 \].

In addition to that, a successful relaxation of the Higgs mass requires large hierarchy between the scales of the rolling potential and the back-reaction potential \[ 29 \] and thus, the relaxion setup rely on a carefully designed potential derived from the clockwork mechanism \[ 30, 33 \], which is based on a \( U(1) \) global symmetry. The resulting construction suffers from a fairly severe quality problem, unless the relaxion is rather heavy \[ 34 \]. In Section III, we propose a new construction that addresses both of the above challenges.

E. Axion/ALP couplings from unknown Planck physics

As mentioned previously, the Planck-suppressed PQ-breaking operators in Eq. (I.2) give rise to SM couplings. This is, as we discuss below, due to the fact that the additional terms may be misaligned in phase relative to the terms induced by the IR QCD instantons. In the presence of CP violation, the resulting couplings can be linear

\[ 3 \] Note that there could also be portal couplings between sectors, though we postpone discussion of this to Section IIIIB

\[ 2 \] See \[ 22 \] this for a possible generalisation of the back-reaction potential, and \[ 23, 24 \] for non-inflationary relaxation mechanism.
in the field, whereas if CP is conserved the leading couplings are quadratic. In addition to that, the QCD axion always induces a scalar interaction with the nucleons at the quadratic order of the axion field \( \phi \) [35].

For low-energy phenomenology, we consider ALP/axion interaction with the electrons, photons, or gluons; the Lagrangian of such interactions can be written as

\[
\mathcal{L} \supset \frac{\phi}{M_{\text{Pl}}} \left[ d_{m_e}^{(1)} m_e \bar{e} e + \frac{d_\alpha}{4} F^2 + \frac{d_\beta}{2g} G^2 \right] + \frac{\phi^2}{2 M_{\text{Pl}}^2} \left[ d_{m_e}^{(2)} m_e \bar{e} e + \frac{d_{\alpha}^2}{4} F^2 + \frac{d_{\beta}^2}{2g} G^2 \right].
\]

(II.16)

where, \( e \) is the electron field, \( F^2 = F^{\mu\nu} F_{\mu\nu} \), \( G^2 = \frac{1}{4} \text{Tr}(G^a \mu \nu G^a \mu \nu) \), \( F_{\mu\nu} \) (\( G_{\mu\nu} \)) is the electromagnetic (QCD) field strength. Also, \( g \) is the QCD gauge coupling and \( \beta (g) \) is the beta function. Such couplings can be searched for via the equivalence principle violations and/or fifth forces experiments [36–42], or oscillation of fundamental constants (for a review, see for example [43]; for proposals, see [20,44,53]; and for experiments providing bounds on oscillations see [54,55]). Note that, one can also consider ALP/axion interaction with \( m_q q \bar{q} \), where \( q = u, d \) denotes the light quarks; see e.g. [64] for bounds on such couplings.

To see how the above interactions are generated from Eq. (I.2), one can expand the cosine part up to quadratic order to find

\[
\cos \left( \frac{N \phi}{j} + \beta \right) = \cos \beta - \sin \beta \frac{N \phi}{j} \cos \beta - \frac{\cos \beta}{2} \left( \frac{N \phi}{j} \right)^2 + \cdots.
\]

(II.17)

Comparing Eqs. (I.2) and (II.17), we can easily identify

\[
d_X^{(1)} = |e_N| N \sin \beta \Delta^{N-1}, \quad d_X^{(2)} = |e_N|^2 N^2 \cos \beta \Delta^{N-2},
\]

(II.18)

for \( X = m_e, \alpha, g \), which we will refer to as the quality couplings of the theory (due to their possible connection with the quality problem). As discussed before, if gravity respects CP, then \( \beta = 0 \) and thus, there is no linear scalar coupling between ALP and SM. However, the quadratic interactions are present both for the CP-violating and CP-conserving cases.

Experimental searches for equivalence principle violations and fifth forces [37–41] have led to stringent constraints on light scalars with couplings \( d_X \) as above. In particular, for the linear gluon coupling \( d_{g}^{(1)} \lesssim 10^{-3} \) \((10^{-6})\) for all particle masses \( m \lesssim 10^{-6} \) \((10^{-14})\) eV (see [41,64] and refs. therein), for the linear electron coupling \( d_{e}^{(1)} \lesssim 1 \) \((10^{-2})\) for \( m \lesssim 10^{-6} \) \((10^{-14})\) eV, and for the linear photon coupling \( d_{\gamma}^{(1)} \lesssim 1 \) \((10^{-1})\) for \( m \lesssim 10^{-6} \) \((10^{-14})\) eV (see [32] and refs. therein). Constraints on the quadratic couplings are weaker, but as we shall see below, still relevant.

One can also search for these couplings through direct detection of oscillation of fundamental constants from the oscillation of the bosonic DM field [44,66]. This variation is characterized at leading order by

\[
\frac{\delta X}{X_0} \approx \frac{d_X^{(j)}}{j M_{\text{Pl}}^2} \sim \frac{d_X^{(j)}}{j} \left( \frac{2 \rho_{\text{DM}}}{m M_{\text{Pl}}} \right)^j,
\]

(II.19)

where \( \rho_{\text{DM}} \) is the density of DM in the vicinity of the experiment, and \( j = 1 \) \((2)\) for linear \((\text{quadratic})\) coupling to \( \phi \). The typical value for the local density is \( \rho_{\text{local}} = 0.4 \) GeV/cm\(^3\), though it can be larger if the field becomes bound to the Earth or Sun [67,68]. Substituting Eq. (II.18), we can write the above equations in a compact form

\[
\frac{\delta X(\phi)}{X_0} \approx \frac{N^j \Delta^{N-j}}{j} \left( \frac{2 \rho_{\text{DM}}}{m M_{\text{Pl}}} \right)^j
\]

(II.20)

for \( X = m_e, \alpha, g \), where we have taken \(|e_N| \approx \sin \beta \approx \cos \beta \approx 1\). For comparison, present experimental sensitivity to \( \delta m_e/m_e \) is at the level of \( 10^{-16} \) for microwave clocks, but somewhat higher for molecular clocks with some prospect to improve to \( 10^{-21} \) in the coming years; for the \( \alpha \)-coupling, current optical clock searches can achieve \( 10^{-18} \), and a nuclear clock could potentially reach \( 10^{-23} \) (see [15] and references therein). See [64] for a discussion about the precision probes related to the gluons and quarks couplings.

For QCD axions, owing to the suppression required to resolve the quality problem, direct searches for quality couplings is challenging. The linear coupling \((j = 1)\) term in Eq. (II.20) gives

\[
\left( \frac{\delta X(\phi)}{X_0} \right)_{\text{QCD}} \sim 10^{-98} \left( \frac{10^{-13} \text{eV}}{m} \right) \sqrt{\frac{\rho_{\text{DM}}}{\rho_{\text{local}}}} \left( \frac{\rho_{\text{DM}}}{\rho_{\text{local}}} \right)^{j-1}.
\]

(II.21)

for \( f = 10^{10} \) GeV \((N = 10)\), and even smaller for \( f = 10^{12} \) GeV \((N = 13)\) and/or for quadratic couplings \((j = 2)\).

The scale of these couplings is exceedingly small, even for ALPs. For quadratic couplings \((j = 2)\), there is a simple expression for the coupling of Eq. (II.18) such that it satisfies the condition \( \delta m \ll m \) of Eq. (II.1):

\[
d_X^{(2)} \ll \frac{m^2}{M_{\text{Pl}}} = 10^{-56} \left( \frac{m}{\text{eV}} \right)^2,
\]

(II.22)

which is far out of reach of experimental searches for the foreseeable future. For linear couplings \((j = 1)\), the condition is more complicated but can be written as

\[
d_X^{(1)} \ll \frac{m^2 \tan \beta}{M_{\text{Pl}}^2} N,
\]

(II.23)
which is suppressed by an additional factor of $\Delta/N \ll 1$. Therefore, natural couplings are out of reach for now.

Relative to the case of QCD axions, where additional fine-tuning of the phase parameter $\beta'$ spoiled the solution of the strong CP problem (see Section II.B), for ALPs the problem is naturalness of the mass. Therefore it is more compelling to ask what level of fine-tuning might be required to produce an ALP with the desired properties. Rather than $\beta' \ll 1$, here we may require $\beta' - \pi/2 \ll 1$ so that $\cos \beta' \approx 1$. Expanding in this limit, Eq. (II.4) is equivalent to

$$|c_N| \times |\beta' - \pi/2| \ll \frac{m^2 f^2}{M_{Pl}^4} \frac{M_N^2}{N^2 f N},$$

(II.24)

i.e. one either tunes $|c_N| \ll 1$ or $|\beta' - \pi/2| \ll 1$ or both.

The level of tuning of an ALP theory with a given $N$ and $m$ is given in Fig. 2 where “tuning” is defined by the right-hand-side (RHS) of Eq. (II.24). We see that there is a trade-off between the level of tuning in the model (which prefers larger $N$ and smaller $f$) and the possibility of direct detection (which prefers smaller $N$ and larger $f$).

It matters how the tuning of parameters is accomplished. If $|c_N| \ll 1$, then all quality couplings $d_X^{(j)}$ are also strongly suppressed (see Eq. (II.18)). If $|\beta' - \pi/2| \ll 1$, then the quadratic couplings become suppressed whereas the linear couplings remain of order $d_X^{(1)} \approx N \Delta N^{-1}$. Finally, one might imagine a UV model with a bare mass term $m_3^2 < 0$ and a fine cancellation $\delta m^2 - |m_3^2| \equiv m^2$ where $m$ is the ultralight mass one searches for in experiment (this is analogous to Higgs fine-tuning); in this case neither linear nor quadratic couplings are necessarily suppressed by the tuning of the theory.

The quality couplings to Planck-suppressed operators $\mathcal{O}_{SM} = F^{\mu\nu} F_{\mu\nu}$ for $j = 1$ (linear coupling to SM) are shown in the left panel of Figure 3 and $j = 2$ (quadratic couplings to SM) are shown in the right panel. The region already ruled out by EP tests is given in grey, and the natural region of coupling space is highlighted in blue. The horizontal lines correspond to Eq. (II.18) for the labelled values of $N$ and $f$, assuming $c_N \sim \beta' \sim O(1)$. We observe that even in the case of a high-density solar halo or Earth halo, a future nuclear clock with precision at the level of $\delta\alpha/\alpha \sim 10^{-23}$ (blue dashed line) will still not be sufficient to probe these Planck-suppressed couplings.

III. HIGH-QUALITY QCD RELAXION

We combine elements of $Z_N$ QCD axion model with the relaxion, in a way that can ameliorate the challenges described in Section II.B. The relaxation of the axion field will preserve the QCD axion solution to the strong CP problem, giving rise to a low-mass $Z_N$ QCD axion which also relaxes the electroweak (EW) scale via the relaxion mechanism.

We again consider $N$ copies of the SM related by a $Z_N$ symmetry, with an effective potential given in Eq. (II.14). We will use the fact that the QCD axion potential depends on the Higgs vev through the quark masses, and thus, it can be used as a trigger for the relaxation of the Higgs mass [21, 69]. Note that for our purpose, we will only be interested in the shape of the potential and its dependence on Higgs vev.

Starting from a high-energy cut-off $\Lambda$, the EW scale is set by the dynamics of a axion-like field, usually known as a relaxion. The relaxion-Higgs potential can be written as

$$V(\phi, H) = (\Lambda^2 - g_3^2) |H|^2 + \lambda |H|^4 + V_{\text{roll}}(\phi) + V_{\text{br}}(\phi, \langle H \rangle),$$

(III.1)

where $V_{\text{roll}} = -g_3^2 \Lambda^2 \phi$ (a dimensionless constant) and $V_{\text{br}}$ is called the “back-reaction” potential as this back-reacts to the motion of the relaxion and is only active when $\langle H \rangle \neq 0$. In our case, we will take $V_{\text{br}} = V_N(\phi)$ in Eq. (II.14). For $N = 1$, the back-reaction potential depends linearly on the Higgs vev through $m_u = y_u \langle H \rangle$, with $y_u$ the Yukawa coupling of the up quark, in contrast to the quadratic case discussed in [23]. Note that with this definition, the SM Higgs vacuum expectation value (vev) would be $\langle H \rangle = v \approx 174$ GeV. See Appendix A for general details about the relaxion mechanism and constraints.

3 Analogous estimates for other SM operators, e.g. in Eq. (II.16), are straightforward. Since neither the couplings Eq. (II.18) nor the tuning constraint (II.4) depend on the SM operator, our estimations of the magnitude of the coupling strength is unchanged in such cases.
In order to successfully solve the strong CP problem, we need to find at least one sector in which effective theta angle would be a multiple of 4 in this case. To do that, we will require \( \epsilon_b, \gamma \equiv v/v' \), with

\[
\delta_0 \equiv \frac{\epsilon_b}{\Lambda} + N^2 \kappa \theta_0 \gamma \pm \delta - O(\delta^2),
\]

where we define

\[
\epsilon_b^2 \equiv \frac{\Lambda^2_{\text{QCD}} y_u}{v'^2} = \frac{\Lambda^2}{v'^4} \epsilon_b^3, \quad \gamma \equiv v/v'.
\]

We are assuming that the Higgs mass is relaxed starting from some cut-off \( \Lambda \) to the value \( v' > v \) in the \( k = 0 \) sector, and to \( v \) in the SM sector. This amounts to a fine-tuning of order \( \gamma^2 = (v/v')^2 \).

The relaxation stopping point would be close to \( \pi/2 \) in the \( k = 0 \) sector which dominates the relaxation potential. However in the \( k \)th sector the stopping point would be shifted by \( 2\pi k/N \) as per the structure of the potential as seen in Eq. (II.13). So, if we identify the SM at the \( k_{\text{SM}} = (3N/4)-1 \) sector which is shifted from the dominating sector by \( 2\pi k_{\text{SM}}/N = 3\pi/2 \), then in our SM the effective theta angle would be \( \theta_0 + 3\pi/2 \sim \delta_0 \). We reiterate that selecting the SM out of \( N \) sectors as the one with minimum at \( 3\pi/2 \) amounts to tuning of \( 1/N \). This also implies the constraint that \( N \) be a multiple of 4 in this case.

In this setup, the back-reaction potential can be written as

\[
V_{\text{br}}(\phi) \sim -\Lambda_{\text{QCD}}^3 y_u' v' \cos \left( \frac{\phi}{f} \right) (1 - \epsilon_b \gamma)
\]

\[
-\Lambda_{\text{QCD}}^3 y_u v \kappa \cos \left( \frac{N\phi}{f} \right), \quad (\text{III.3})
\]

where we ignore \( \tilde{\theta} \) for the purposes of this section. If \( \epsilon_b \gamma \ll 1 \), we can treat the term proportional to \( \cos(N\phi/f) \) as a perturbation and the relaxation stopping point can be written as \( \theta_0 = \pi/2 - \delta_0 \), with

\[
\delta_0 \simeq \frac{\epsilon_b^2}{8} + N^2 \kappa \theta_0 \gamma \pm \delta - O(\delta^2), \quad (\text{III.4})
\]

Note that \( 0 \leq \{\epsilon_b, \gamma\} \leq 1 \), and the \( Z_N \) symmetry is restored for \( \epsilon_b, \gamma \to 1 \).}

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4 We denote all quantities in the \( k = 0 \) sector with a \((')\).
model, i.e. $N \equiv N/4$. In this sense, the underlying symmetry of this theory is $Z_{4N}$, with $N \equiv N/4$.

In order to solve the strong CP problem successfully one requires $\delta_0 \lesssim \theta_{\text{CP}} = \theta_{10}10^{-10}$, the limit on the CP violating phase from neutron EDM experiments; at present, $\theta_{10} = 2.7$ but is expected to improve in the future [22]. To avoid any additional tuning, one would expect each term in $\delta_0$, defined in Eq. (III.4), to separately less than $\theta_{\text{CP}}$. This implies additional conditions, namely

$$e^2 \lesssim \theta_{\text{CP}} \Rightarrow \frac{\gamma^3}{\epsilon_b} \lesssim 1800 \theta_{10}, \quad (\text{III.7})$$

$$\delta \lesssim \theta_{\text{CP}} \Rightarrow \frac{\gamma}{\epsilon_b} \lesssim 6 \left( \frac{\Lambda}{10^6 \text{ GeV}} \right)^2 \theta_{10}, \quad (\text{III.8})$$

$$\epsilon_b \gamma \lesssim \frac{\theta_{\text{CP}}}{N^2 \kappa} = \frac{\theta_{10} 10^{-10}}{N^2 \pi 10^{-1}} \left( \frac{\pi N}{1 - z^2} \right)^{-1}. \quad (\text{III.9})$$

where we have taken $\Lambda_0/v \simeq 5 \times 10^{-4}$.

Note that, in addition to the QCD and relaxion parameters, we have two additional free parameters $\epsilon_b$ and $\gamma$, constrained by three inequalities. The cut-off of the Higgs mass $\Lambda$ is also constrained by the consistency of the effective theory as $f \gtrsim \Lambda \gtrsim 4 \pi v'$. Other constraints from the success of the relaxion mechanism are described in Appendix A; the upshot is that a successful relaxation of the EW scale requires the additional condition

$$\epsilon_b^{5/2} \sqrt{\gamma} \lesssim 24 \pi^2 \left( \frac{10^4 M_P^4}{\Lambda_{11}^4 v^4} \right). \quad (\text{III.10})$$

Finally, as mentioned above, we must ensure that the values of $\epsilon_b$ and $\gamma$ are consistent with the change of the QCD scale as the Higgs vev changes $\theta$, i.e. $\Lambda_{\text{QCD}}^N \gtrsim \Lambda_{\text{QCD}}(\gamma)$, which roughly translates to the constraint $\Lambda_{\text{QCD}}(\gamma) \gtrsim 4 \pi v'$. Intriguingly, the parameter space consistent with the QCD axion solution of the strong CP problem (above) as well as the constraints from a successful relaxion mechanism (see Appendix A) is exceedingly predictive; c.f. top-left panel of Figure 4. This leads to a prediction that, for $N = 8$, the relaxed axion mass is

$$\epsilon_b \lesssim \gamma. \quad (\text{III.11})$$

We illustrate the available model space for a few benchmark parameter inputs in Figure 4. Note that the $\epsilon^2$ inequality (III.7), does not appear, as this constraint is always much weaker than the others.

After the rolling stops, the mass of relaxion will be relaxed to a value modified from the naive expectation; it can be written as

$$m_{\phi}^2 = \frac{\Lambda_{\text{QCD}}^3 y_u y_y'}{f_a^2} \times \delta$$

$$= (m_{\phi}^2)_{\text{QCD}} \frac{\delta}{\epsilon_b \gamma}. \quad (\text{III.12})$$

where $(m_{\phi}^2)_{\text{QCD}} \equiv \Lambda_{\text{QCD}}^4/f^2$. In particular, it is suppressed by $\delta \ll 1$ but enhanced by $(\epsilon_b \gamma)^{-1} \gg 1$ relative to the naive QCD expectation. Note that one can express $\delta$ in terms of theory parameters as

$$\delta \simeq 4 \times 10^{-11} \left( \frac{10^6 \text{ GeV}}{\Lambda} \right) \sqrt{\frac{\gamma}{\epsilon_b}}. \quad (\text{III.13})$$

So, finally the (rel)-axion mass can be written as,

$$m_{\phi}^2 \approx 1.3 \times 10^{-7} \left( \frac{10^{-7}}{\epsilon_b^\gamma} \right)^{1/2} \left( \frac{10^6 \text{ GeV}}{\Lambda} \right). \quad (\text{III.14})$$

In Figure 4, the pink dashed lines denote the parameter space satisfying $m_{\phi} \simeq (m_{\phi})_{\text{QCD}}$.

The minimal working example of our model has $N = 4$. However, the present-day constraint (95% C.L.) on the CP-violating phase $\theta_{\text{CP},0}$ $\simeq 2.7 \times 10^{-10}$ [24] implies that $N = 4$ is marginally excluded (the minimum value for $N = 4$ in our model is $\theta_{10} \simeq 3$). At this current experimental margin, there is a narrow range of parameters that satisfy all constraints for $N = 8$; the predicted set of parameters is very near to

$$N = 8, \quad \theta_{\text{CP}} \simeq 2.7 \times 10^{-10}, \quad \epsilon_b \simeq 4 \times 10^{-6}, \quad \Lambda \simeq 3 \times 10^6 \text{ GeV}, \quad \gamma \simeq 10^{-3}. \quad (\text{III.15})$$

There is a mild dependence on $N$ through $\kappa$ in the constraint of Eq. (III.9); when $N > 8$, this constraint becomes weaker and therefore a larger parameter space becomes available (see Figure 4). The parameter space for $N = 8$ can be fully explored provided a future constraint at the level of $\theta_{10} \lesssim 2.4$.

We can determine the limits to our parameter space analytically by determining the intersection points of the constraints (III.8–III.11) as functions of $\Lambda$, $\theta_{\text{CP}}$, and $N$. At the intersection of (III.8–III.10), we have

$$\theta_{\text{CP}, \text{int}} = \left( \frac{32 \pi^2 N^2 \kappa}{3} \right)^{1/8} \left( \frac{\Lambda_{\text{int}}}{M_{\text{Pl}}} \right), \quad (\text{III.17})$$

which implies $\theta_{10, \text{int}}(N) = 3.35 \exp(-0.09N)^{3/16}$, which is effectively the minimum $\theta_{10}$ value possible for a given $N$. Stronger experimental constraints on $\theta_{10}$ can therefore push our model to larger $N$. The ratio of symmetry-breaking parameters at the intersection point is

$$\left( \frac{\epsilon_b}{\gamma} \right)_{\text{int}} = \left( \frac{3}{2^6 \pi^{18} N_{18} \kappa^9} \right)^{1/16} \left( \frac{M_{\text{Pl}}^3 \Lambda_{\text{int}}}{v^2} \right)^{1/4}, \quad (\text{III.18})$$

The requirement of up and down quark masses are lighter than the corresponding QCD scale in the $k = 0$ sector, leads to a constraint on $\epsilon_b$ as a function of $\gamma$ which depends on the ratio of $y_u/y_d$. However, this condition can be satisfied for the whole range of $\gamma$ our model admits, for at least one value of the $y_u$ ratio and, thus, is not shown in Fig. 4.
FIG. 4: Parameter space for the $Z_N$ QCD relaxion using as parameter inputs (a) $\theta_{10} = 2.7, \gamma = 10^{-3}$, (b) $\theta_{10} = 2.7, \gamma = 10^{-1}$, and (c) $\theta_{10} = 0.9, \gamma = 5 \times 10^{-3}$. The two symmetry-breaking parameters $\epsilon_b$ and $\gamma$ are defined in Eq. (III.2). The constraints shown in green, red, blue, and purple are given in Eqs. (III.8-III.11) (respectively). The dashed line illustrates where the relaxion mass is equal to $(m_\phi)_\text{QCD} \equiv \Lambda_4^4/f^2$; above the line, $m_\phi < (m_\phi)_\text{QCD}$.

which approaches unity at $N \to 23$. Recall that the constraint (III.11) requires $\epsilon_b \lesssim \gamma$. Therefore any $N \geq 24$ will give the same prediction for $\theta_{10} = 0.7$. However, as can be seen from Fig. 4(b), to reach $\gamma \sim 10^{-1}$, we need $N \geq 28$. As a result, the parameter space of our model is fully bounded by $4 \leq N \leq 28$ and $0.7 \lesssim \theta_{10} \lesssim 3$. This range of CP-violating phase will be probed within the next five years by neutron EDM experiments [73]. It is also worth noting that in this model $10^{-3} \lesssim \gamma \lesssim 10^{-1}$; the cut-off scale in this model is also bounded, between $10^5 \text{GeV} \lesssim \Lambda \lesssim 10^7 \text{GeV}$.

At the intersection of (III.8-III.10), the mass of the relaxion can also easily be calculated:

$$\left( \frac{m_{\phi}^2}{(m_{\phi})_{\text{QCD}}} \right)_{\text{int}} = N^2 k = 0.56 N^{3/2} \sqrt{1 - z^2 z^{-N-1}}.$$  

(III.19)

Owing to the narrowness of our parameter space, this estimate roughly holds across the full range of model parameters.

A. Quality of the $Z_{AN}$ QCD relaxion

As before, we combine the low-energy axion potential with that induced by Planck-suppressed operators in Eq. (I.2) to see whether the latter will spoil the solution to the strong CP problem. The combined potential is

$$V(\phi) = |c_N| \Delta^N M_{Pl}^4 \cos \left( \frac{N \phi}{f} + \delta' \right) - m_{\phi}^2 f^2 \cos \left( \frac{\phi}{f} \right) = |c_N| \Delta^N M_{Pl}^4 \cos \left( \frac{N \phi}{f} + \delta' \right) - \frac{\Delta^4 a}{\epsilon_b \gamma} \cos \left( \frac{\phi}{f} \right).$$  

(III.20)

The first derivative is

$$0 = V'(\langle \phi \rangle) \approx |c_N| \Delta^N N M_{Pl}^4 \sin \delta' + \frac{\delta \Delta^4 a}{\epsilon_b \gamma} \epsilon, \quad (III.21)$$

which implies the constraint

$$|\epsilon| = \left| \frac{|c_N| \sin \delta'}{\Delta^N N M_{Pl}^4} \frac{\epsilon_b \gamma}{\delta} \right| \approx 10^{-10}. \quad (III.22)$$

For the case of $N = 8$, the leading Planck-suppressed operator is proportional to $\Phi^8/M_{Pl}^4$ + h.c., i.e. Eq. (III.22) with $N = 8$. For the parameters given in Eq. (III.15), the constraint gives $f_{\text{max}} \sim 10^8 \text{ GeV}$, very near the magnitude of the black lines in Figure 4. As $N$ grows, Eq. (III.22) approaches the QCD case in Eq. (II.8), since the additional factor $\epsilon_b \gamma/\delta$ is bounded (roughly) by $1 - 10^8$ over the available parameter space.
of the model \( N \geq 8 \), but the inequality depends on even-higher powers of \( f^N \). Therefore within a factor of \( \mathcal{O}(\text{few}) \) in \( f_{\text{max}} \), the quality of the \( Z_N \) QCD relaxion is equivalent to that of \( Z_N \) QCD (Section II.C) at any given \( N \).

**B. Direct searches for a \( Z_{4N} \) QCD relaxion**

Here we outline the phenomenological implication of our QCD relaxion. In our model, the axion has a CP-violating phase of \( \delta_0 \). Like the usual relaxion models, due to the relaxion-Higgs mixing angle

\[
\sin \theta_{h\phi} \simeq \frac{\Lambda^4}{v^3} \theta_0 , \tag{III.23}
\]

the QCD-axion also has scalar interaction with the SM. See [26, 75, 76] for a detailed discussion of relaxion phenomenology.

The QCD axion also induces a scalar interaction with the SM fields in the presence of a CP-violating phase of the form of

\[
\mathcal{L} \supset -g_{\phi NN} \phi \bar{N} N , \tag{III.24}
\]

through the pion-nucleon sigma term as noted in [77–80]. Using \( \partial \ln m_N / \partial \ln m_N \simeq 0.06 \) [35, 81], and \( m_u m_d / (m_u + m_d)^2 \simeq 0.22 \), we obtain

\[
g_{\phi NN} \simeq 1.3 \times 10^{-2} \frac{m_N}{f} \delta_0 . \tag{III.25}
\]

The predictive range of \( 0.7 \lesssim (\delta_0 / 10^{-10}) \lesssim 3 \), limits the strength of the scalar interaction of the QCD-axion to the SM as

\[
\frac{9 \times 10^{-24}}{f_{11}} \lesssim g_{\phi NN} \lesssim \frac{4 \times 10^{-23}}{f_{11}} , \tag{III.26}
\]

where, \( f_{11} = f / (10^{11} \text{GeV}) \) and we have used \( m_N \sim 1 \text{GeV} \). The strongest bound on \( g_{\phi NN} \) comes from the experiments looking for the existence of fifth force and/or violation of equivalence principle (EP) [37–41]. The bound from EP violation searches, for the axion mass around \( 10^{-6} \text{eV} \), is \( g_{\phi NN} \lesssim 10^{-21} \), which becomes stronger as we go to the lower masses. Note that, in our model, the mass of the QCD relaxion is slightly lighter than the QCD axion. Thus, for a given \( f \), one should be careful about analysing the EP bounds.

The QCD axion also has pseudoscalar interaction with the SM fermions as, \( \mathcal{L} \supset -g^a_{\phi} \bar{\psi} i\gamma^5 \psi \) with \( g^a_{\phi} = C_{\phi} m_{\psi} / f \). The coefficient \( C_{\phi} \) depends on QCD axion models [4–9, 14]. Many experimental efforts are concentrated on probing QCD axion through its pseudoscalar interaction with the SM (see e.g. [82] and Refs. therein). In our model, the product of the scalar and the pseudoscalar coupling of the QCD relaxion to the nucleon can be written as

\[
g^a_{\phi} g_{\phi NN} = 1.3 \times 10^{-2} \frac{C_N m^2_N}{f^2} \delta_0 , \tag{III.27}
\]

where \( C_N \) is some model dependent coefficient of the nucleons arising from the pseudoscalar interaction of the axion to protons and/or neutrons [13]. In our model, the strength of the axion-nucleon scalar interaction is bounded and using Eq. (III.26) one can more specifically limit the product of axion-proton pseudoscalar and axion-nucleon scalar coupling as,

\[
\frac{4 \times 10^{-35}}{f_{11}^2} \lesssim |g^a_{\phi} g_{\phi NN}| \lesssim \frac{2 \times 10^{-34}}{f_{11}^2} . \tag{III.28}
\]

Note that, in the above estimate we use the axial coupling strength of proton \( C_p = -0.47 \) which is obtained in the KSVZ QCD axion model. Another QCD axion model such as DFSZ may provide a different value of \( C_p \) [14]. Note that, the above parameter range will be probed by the ARIADNE experiment whose projected reach is \( |g^a_{\phi} g_{\phi NN}| \lesssim 10^{-36} \) for \( f \sim 10^{11} \text{GeV} \) [83, 84].

The QUAx experiment is also looking for similar scalar-pseudoscalar interaction, using the pseudoscalar electron coupling \( g^a_{\phi} \) rather than \( g_{\phi NN} \). They provide the current constraint on \( |g^a_{\phi} g_{\phi NN}| \lesssim 5.7 \times 10^{-32} \) in the mass range of \( 10^{-5} \gtrsim m_{\phi} / \text{eV} \gtrsim 6 \times 10^{-13} \) by updating their previous result by \( \mathcal{O}(10^2) \) [85, 86]. We estimate the the range of \( |g^a_{\phi} g_{\phi NN}| \) in our model as

\[
\frac{10^{-38}}{f_{11}^2} \lesssim |g^a_{\phi} g_{\phi NN}| \lesssim \frac{6 \times 10^{-38}}{f_{11}^2} , \tag{III.29}
\]

where we use \( C_{\phi} = 1/3 \); this parameter is model-dependent, and this value is on the larger side of model-parameter possibilities [87]. Although our predicted range is beyond the current experimental reach, our model presents an opportunity for scalar and pseudoscalar searches to work together to confirm (or refute) the existence of such axions in a complementary way.

**IV. DISCUSSION**

In this work we analysed how Planck-suppressed (quality) operators affect the low-energy dynamics of theories involving QCD axions or axion like particles (ALPs). For the QCD axion, the quality operators lead to the well-known QCD axion quality problem, whereas for ALPs, they may lead to a fairly severe fine-tuning problem. Quality operators also induce scalar interaction between the Standard Model (SM) fields and the QCD axion/ALPs. In the absence of CP violation, we obtain SM-ALP scalar interaction in quadratic order of the ALP field, whereas if CP is broken by gravity, ALP-SM scalar interactions are generated even at linear order. These interactions can be probed by various precision experiments. The strength of the scalar and pseudoscalar interactions are closely related, and therefore these search strategies can complement one another.

We also provide a framework for addressing both the Higgs hierarchy and the strong CP problems together. We invoke a relaxation mechanism where the Higgs mass
is scanned during inflation and the QCD axion plays the role of the relaxion. We show that a $\mathbb{Z}_{4N}$-symmetric back-reaction potential which is broken explicitly by a small parameter can address both of these problems simultaneously. We show that one of the sector has effective CP violating phase $\theta_0 \lesssim \mathcal{O}(10^{-10})$ and identify this sector as our SM. This amounts to a linear tuning of $\mathcal{O}(N)$. Our model cannot fully ameliorate the hierarchy problem, as it leaves a little hierarchy to address. The mass of QCD relaxion obtained in our model is lighter than that of the canonical QCD axion.

In our model, we predict a narrow range of CP-violating phase of $0.7 \lesssim \theta_0/10^{-10} \lesssim 3$. This range of CP-violating phase will be probed within the next five years by neutron electric dipole moment experiments [73]. Due to the underlying $\mathbb{Z}_{4N}$ symmetry which can be gauged, this model exhibits better protection against quality operators than the vanilla QCD axion/relaxion models. Also, a highly constrained relaxation scenario suggests that the parameter space of our model is fully bounded by $4 \leq N \leq 28$. Due to the predicted narrow range of CP-violating phase, our model can also be used as target of experiments like ARIADNE [83,84] and/or QUAX [85,86] which are sensitive to the product of scalar and CP-violating phase, our model can also be used as target of experiments like ARIADNE [85,86], which are sensitive to the product of scalar and pseudoscalar interaction of the QCD axion to the SM. In this section we discuss the relaxation of the Higgs mass parameter. For the case of QCD relaxion, the back-reaction potential depends linearly on the Higgs vev as opposed to the quadratic case discussed in [26]. A generic back-reaction potential which depends linearly on the Higgs vev can be written as

$$V_{br} = -\Lambda_b^3 \langle H \rangle \cos(\phi/f), \quad (A.1)$$

where $\Lambda_b$ is the back-reaction scale. Following the notation of the main text Eq. (III.1), the total relaxation potential can be written as

$$V(\phi, H) = (\Lambda^2 - gA\phi)|H|^2 + \lambda|H|^4 - gA^3 \phi - \Lambda_b^3 \langle H \rangle \cos(\phi/f). \quad (A.2)$$

Below we set the Higgs quartic coupling $\lambda = 1$ for notational convenience. We are interested in understanding the evolution of the relaxation close to the EW scale (v) Higgs mass. In that case, the minimum of the potential can be found by solving two equations: $\partial V(\phi, H)/\partial \phi = 0$ and $\partial V(\phi, H)/\partial \phi = 0$. If $|\partial^2 V/\partial^2 H| \gg |\partial^2 V/\partial^2 \phi|$, then one can set the Higgs at its instantaneous minimum by solving $\partial V(\phi, H)/\partial H = 0$. Using perturbation theory around the EW vacuum, one finds the relaxation-dependent Higgs vev as

$$v^2(\theta_a) = v^2 \left( -\frac{\Lambda^2}{v^2} + \frac{gAf \theta_a}{v^2} + \frac{\Lambda_b^3}{2v^2} \cos \theta_a \right) + \mathcal{O}\left(\frac{\Lambda_b^6}{v^6}\right),$$

where we write $\theta_a = \phi/f$. The perturbative expansion of the Higgs vev is valid as long as

$$\Lambda_b^3 \ll \Lambda^3.$$

From Eq. (I.6) one can see that the above condition is easily satisfied for QCD axion. Setting the Higgs to its relaxation dependent vev, we obtain the effective potential of the relaxion as

$$V_{eff}(\theta_a) = -gA^3 f \theta_a - (v^2(\theta_a))^2 - \frac{\Lambda_b^3}{2} \cos \theta_a. \quad (A.4)$$

Thus, the relaxion encounters the first minimum when $V_{eff}(\theta_a) = 0$ and we find,

$$-gA^3 f - 2v^2(\theta)\Delta v^2(\theta) + \frac{\Lambda_b^3}{2} \sin \theta_a = 0. \quad (A.5)$$

By setting the EW scale as $gA^3 f \approx \Lambda_b^3 v$, and defining a small parameter $\delta^2 = \Lambda_b^3/(v^2 A^3) \ll 1$, we find that the Higgs vev changes only incrementally as

$$\frac{\Delta v^2}{v^2} = \frac{v^2(\theta_a + 2\pi) - v^2(\theta_a)}{v^2} = \pi gAf \quad \Rightarrow \quad \pi \delta^2 = \pi \delta^2. \quad (A.6)$$

Following the calculation of [26], by realizing $\theta_a \rightarrow 2\pi m + \theta_a$ where $m \in \mathbb{Z}$ and $\theta_a \in [0, 2\pi]$ and then properly adjusting $m$ we find,

$$v^2_m(\theta_a) = v^2 \left( 1 + m\pi \delta^2 + \frac{1}{2} \delta^2 \theta_a + \frac{\Lambda_b^3}{4v^2} \cos \theta_a \right). \quad (A.7)$$

As this work was finalized, [88] appeared, which also address issues related to the QCD relaxion by changing the relaxion evolution during inflation. We also note the recent work [59], which discusses the relationship of Planck-suppressed operators and fifth forces.

Appendix A: Review of the relaxion mechanism

In this section we discuss the relaxation of the Higgs mass parameter. For the case of QCD relaxion, the back-reaction potential depends linearly on the Higgs vev as opposed to the quadratic case discussed in [26]. A generic back-reaction potential which depends linearly on the Higgs vev can be written as

$$V_{br} = -\Lambda_b^3 \langle H \rangle \cos(\phi/f), \quad (A.1)$$

where $\Lambda_b$ is the back-reaction scale. Following the notation of the main text Eq. (III.1), the total relaxation potential can be written as

$$V(\phi, H) = (\Lambda^2 - gA\phi)|H|^2 + \lambda|H|^4 - gA^3 \phi - \Lambda_b^3 \langle H \rangle \cos(\phi/f). \quad (A.2)$$

Below we set the Higgs quartic coupling $\lambda = 1$ for notational convenience. We are interested in understanding the evolution of the relaxation close to the EW scale (v) Higgs mass. In that case, the minimum of the potential can be found by solving two equations: $\partial V(\phi, H)/\partial \phi = 0$ and $\partial V(\phi, H)/\partial \phi = 0$. If $|\partial^2 V/\partial^2 H| \gg |\partial^2 V/\partial^2 \phi|$, then one can set the Higgs at its instantaneous minimum by solving $\partial V(\phi, H)/\partial H = 0$. Using perturbation theory around the EW vacuum, one finds the relaxation-dependent Higgs vev as

$$v^2(\theta_a) = v^2 \left( -\frac{\Lambda^2}{v^2} + \frac{gAf \theta_a}{v^2} + \frac{\Lambda_b^3}{2v^2} \cos \theta_a \right) + \mathcal{O}\left(\frac{\Lambda_b^6}{v^6}\right),$$

where we write $\theta_a = \phi/f$. The perturbative expansion of the Higgs vev is valid as long as

$$\Lambda_b^3 \ll \Lambda^3.$$

From Eq. (I.6) one can see that the above condition is easily satisfied for QCD axion. Setting the Higgs to its relaxation dependent vev, we obtain the effective potential of the relaxion as

$$V_{eff}(\theta_a) = -gA^3 f \theta_a - (v^2(\theta_a))^2 - \frac{\Lambda_b^3}{2} \cos \theta_a. \quad (A.4)$$

Thus, the relaxion encounters the first minimum when $V_{eff}(\theta_a) = 0$ and we find,

$$-gA^3 f - 2v^2(\theta)\Delta v^2(\theta) + \frac{\Lambda_b^3}{2} \sin \theta_a = 0. \quad (A.5)$$

By setting the EW scale as $gA^3 f \approx \Lambda_b^3 v$, and defining a small parameter $\delta^2 = \Lambda_b^3/(v^2 A^3) \ll 1$, we find that the Higgs vev changes only incrementally as

$$\frac{\Delta v^2}{v^2} = \frac{v^2(\theta_a + 2\pi) - v^2(\theta_a)}{v^2} = \pi gAf \quad \Rightarrow \quad \pi \delta^2 = \pi \delta^2. \quad (A.6)$$

Following the calculation of [26], by realizing $\theta_a \rightarrow 2\pi m + \theta_a$ where $m \in \mathbb{Z}$ and $\theta_a \in [0, 2\pi]$ and then properly adjusting $m$ we find,

$$v^2_m(\theta_a) = v^2 \left( 1 + m\pi \delta^2 + \frac{1}{2} \delta^2 \theta_a + \frac{\Lambda_b^3}{4v^2} \cos \theta_a \right). \quad (A.7)$$

NOTE ADDED

As this work was finalized, [88] appeared, which also address issues related to the QCD relaxion by changing the relaxion evolution during inflation. We also note the recent work [59], which discusses the relationship of Planck-suppressed operators and fifth forces.

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From Eq. \[(A.5)\] we get,
\[
\frac{\sin \theta_a}{2} \left(1 + \frac{v^2}{\sin^2(\theta_a)}\right) = \frac{v^2}{\sin^2(\theta_a)} + v^2 \Lambda^2 .
\] (A.8)

Note that, the effective potential written before and the above equation is valid only when the Higgs vev is close to \(v\). By expanding \(v_n(\theta_a)\) close to \(v\) we find, the above equation admits a solution when,
\[
\sin \theta_a = 1 - \frac{m \pi}{2} + \frac{\nu^2}{4} - \frac{\delta^2 \theta_a - \frac{\Lambda^3_b}{8 \nu^3} \cos \theta_a + v^2 \frac{\Lambda^2}{2}}. \quad (A.9)
\]

It is easy to see that, the above equation has two solutions close to \(\theta_a \sim \pi/2\). As, the Higgs vev, only increases incrementally with a small parameter \(\delta\), we find the relaxation stopping point as
\[
\theta_a - \pi/2 \equiv \delta_0 = \frac{\Lambda^3_b}{8 \nu^3} + \frac{\delta^2}{4} \equiv \alpha \delta , \quad (A.10)
\]
where \(\alpha\) is some \(O(1)\) number. The mass of relaxation at the first minimum can be written as
\[
m^2 = \frac{\Lambda^3_b \nu}{f^2} \times \delta , \quad (A.11)
\]

significantly reduced by the small parameter \(\delta\) compared to the naive expected value.

**Constraints:** For a successful relaxation of the Higgs mass we require the following conditions:
\[
f \gtrsim \Lambda \gtrsim \Lambda_{\text{min}} = 4 \pi v. \quad (A.12)
\]

Here we are considering scanning of the Higgs mass during inflation. We require a separate inflaton sector dominates the energy of the universe during inflation and the classical evolution of the relaxion dominates over quantum spreading during inflation. These two requirements lead (respectively) to the constraints
\[
3H_1^2 M_{\phi 1}^2 \gtrsim \Lambda^4 \text{ and } (\Delta \phi)_{c1} = \frac{g \Lambda^3}{3 H_1^2} \gtrsim \frac{H_1}{2 \pi} , \quad (A.13)
\]
where \(H_1\) is the Hubble scale during inflation.

We also want the relaxion to be cosmologically stable in the first minima. This leads to the following constraint:
\[
\frac{8 \pi^2}{3} (g \Lambda^3 f) \delta^3 \gtrsim H_1^4 . \quad (A.14)
\]

In the case of a QCD relaxion, the back-reaction potential depends on the temperature and thus, it is only significant when \(H_1 < \Lambda_{\text{QCD}}\). In this section we only consider inflationary based-relaxation of the Higgs mass with a back-reaction potential which depends linearly on Higgs vev.

Now let us consider the back-reaction potential of our interest as given in Eq. \[(3.3)\],
\[
V_{\text{br}}(\phi) \approx -\Lambda^3_{\text{QCD}} y_u v' \cos(\theta_a) (1 - \epsilon_b \gamma) - \Lambda^3_{\text{QCD}} y_u v \kappa \cos(N \theta_a) . \quad (A.15)
\]

Note that, as discussed in the relaxation is happening at the \(k = 0\) sector where all the quantities are denoted by \(\gamma'\). We see that in \(V_{\text{br}}\) the coefficient of \(\cos \theta_a\) term is responsible for relaxion whereas the coefficient of \(\cos(N \theta_a)\) term has contribution independent of the relaxing Higgs. To use the result of previous discussion, we can make the following replacements
\[
v \rightarrow v', \quad \Lambda^3_b \rightarrow \Lambda^3_{\text{QCD}} y_u' (1 - \epsilon_b \gamma) , \quad (A.16)
\]
\[
V_{\text{eff}}(\theta_a) \rightarrow V_{\text{eff}}(\theta_a) - \Lambda^3_{\text{QCD}} y_u' v \kappa \cos(N \theta_a) .
\]

In the limit, \(\Lambda^3_{\text{QCD}} y_u v \kappa \ll \Lambda^3_{\text{QCD}} y_u' v' (1 - \epsilon_b \gamma)\), using the above substitution, one obtains the relaxation stopping point as \(\theta_a - \pi/2 = \delta_0\) where
\[
\delta_0 = \frac{y_u' \Lambda^3_{\text{QCD}}}{8 \nu^3} + \frac{N^2 \epsilon_b \gamma \Lambda^3_{\text{QCD}} v}{y_u' \Lambda^3_{\text{QCD}} v'} + \alpha \delta + O(\delta^3). \quad (A.17)
\]

In the above equation we also use \((1 - \epsilon_b \gamma) \simeq 1\). In the main text, for all the purposes we set \(\alpha = 1\). With the definition of \(\epsilon_b = \Lambda^3_{\text{QCD}} y_u/\Lambda^3_{\text{QCD}} y_u'\) and \(\gamma = v/v'\), we get back Eq. \[(3.10)\].

Using the substitution \[(A.16)\], we obtain the expression for relaxion mass
\[
m^2 = \frac{y_u' \Lambda^3_{\text{QCD}} v'}{f^2} \times \delta = \frac{y_u \Lambda^3_{\text{QCD}} v}{f^2} \times \frac{\delta}{\epsilon_b \gamma} . \quad (A.18)
\]

All the constraints discussed before translate to this case with proper substitution given in Eq. \[(A.16)\]. The additional constraint in this scenario comes from the fact that, as we consider both QCD and QCD' potential are temperature-dependent, we need
\[
H_1 < \Lambda_{\text{QCD}}, \quad \Lambda_{\text{QCD}}' . \quad (A.19)
\]

Written explicitly, Eq. \[(A.12)\], becomes
\[
f \gtrsim \Lambda \gtrsim 4 \pi v'. \quad (A.20)
\]

The form of Eq. \[(A.13)\] and \[(A.14)\] do not change. However now one needs to replace
\[
g \Lambda^3 f \approx y_u' (\Lambda^3_{\text{QCD}})^3 v' = \frac{\Lambda^4}{\epsilon_b \gamma} . \quad (A.21)
\]

Recall we define \(\Lambda_{\kappa} = (\Lambda^3_{\text{QCD}} m_u)^{1/4} = (\Lambda^3_{\text{QCD}} y_u v)^{1/4}\) in the main text. Also, with the prime notation,
\[
\delta^2 = \frac{y_u' (\Lambda^3_{\text{QCD}})^3}{v \Lambda^2} = \frac{\Lambda^4}{v^2 \Lambda^2} \frac{\gamma}{\epsilon_b} . \quad (A.22)
\]

In our parameter estimation, the constraints arising from a separate inflaton sector which dominates the energy of the universe during inflation Eq. \[(A.13)\] (left side), as well as stability of the first minimum Eq. \[(A.14)\], were the most important. To estimate this constraint the blue lines in Figure 3, we used Eq. \[(A.21)\] to fix \(g \Lambda^3 f\), and Eq. \[(A.13)\] (left side) to fix \(H_1\); substituting both into Eq. \[(A.14)\] and solving for \(\epsilon_b\) recovers Eq. \[(3.10)\].
It is straightforward to see that Eq. (A.13) (right side) and Eq. (A.19) are trivially satisfied. Observe from Eq. (A.14) that $H_1^2 \lesssim 8\pi A_1^2 \beta^3/(3e_5 \gamma)$; this is at most $H_1 \sim \text{keV}$ for the largest $\delta$ values we achieve, which are $\mathcal{O}(10^{-10})$, and even if $e_5 \gamma \rightarrow 1$. Then Eq. (A.13) (right side) implies $H_1^2 \lesssim 2\pi A_1^2/(3e_5 \gamma f)$, which is satisfied even if $f \rightarrow M_{Pl}$. Thus, in our case the requirement of classical evolution of the relaxion dominates over quantum spreading during inflation, provides a weaker constraint than the one provide by the cosmological stability of the relaxion.
