On three-dimensional quasi-Stäckel Hamiltonians

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Abstract
A three-dimensional integrable generalization of the Stäckel systems is proposed. A classification of such systems is obtained, which results in two families. The first family is the direct sum of the two-dimensional system which is equivalent to the representation of the Schottky–Manakov top in the quasi-Stäckel form and a Stäckel one-dimensional system. The second family is probably a new three-dimensional system. The system of hydrodynamic type, which we get from this family in the usual way, is a three-dimensional generalization of the Gibbons–Tsarev system. A generalization of the quasi-Stäckel systems to the case of any dimension is discussed.

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1. Stäckel Hamiltonians

The Stäckel systems [1] have been considered in many works; see e.g. [2, 3] and references therein, and [4]. Associated with this approach, questions about the related problems on the separation of variables were considered in, for example, the works [5, 6]. The popularity of these systems can be explained by the fact that a number of important finite-dimensional integrable systems can be brought to the Stäckel form by a change of variables. The goal of this article is the study of quasi-Stäckel systems which can be described as integrable deformations of Stäckel systems with the same principal parts.

First, we recall the general construction of the classical Stäckel system with \( n \) degrees of freedom. The corresponding Hamiltonians are defined by a system of linear algebraic equations:

\[
\sum_{k=0}^{n-1} S_{ik}^{-1} H_k = \psi (p_i, q_i) \Rightarrow H_k = \sum_{i=1}^{n} S_{ik} \psi (p_i, q_i), \quad S_{ik}^{-1} = f_k (p_i, q_i).
\] (1)
The matrix $S_{ik}$ is called the Stäckel matrix. Let us consider the simplest inverse Stäckel matrix $S_{ik}^{-1} = q_i^{n-k-1}, i = 1, \ldots, n, k = 0, \ldots, n-1$ (this is the so-called Benenti case [7, 8]) and the function $\psi(p_k, q_k) = s(q_k) p_k^2 + u(q_k)$. This approach is usual for obtaining the representation of the Stäckel system in terms of separated variables. Solving of the system (1) yields a family of Hamiltonians which is commutative with respect to the canonical Darboux–Poisson bracket $\{q_i, p_j\} = \delta_{ij}$:

$$H(\alpha) = \sum_{k=1}^{n} (s(q_k) p_k^2 + u(q_k)) \prod_{j \neq k} \frac{\alpha - q_j}{q_k - q_j}, \quad \{H(\alpha), H(\beta)\} = 0. \quad (2)$$

The coefficients of the generating function $H(\alpha) = \sum_{k=0}^{n-1} H_k \alpha^{n-k-1}$ provide a set of Hamiltonians in involution. Therefore the Stäckel system described by these Hamiltonians is integrable in the Liouville sense.

2. Quasi-Stäckel Hamiltonians in the $n = 3$ case

The quasi-Stäckel systems in the two-dimensional case were introduced in [9]. Analogous systems were considered in [10–12]; see also references therein. We recall that the Hamiltonians of quasi-Stäckel systems are Hamiltonians of Stäckel systems plus magnetic (linear in momenta) terms.

The Hamiltonians $h_i$ of the quasi-Stäckel systems in the Benenti case are defined for arbitrary $n$ as follows:

$$\sum_{k=0}^{n-1} q_i^k h_{n-1-k} = S(q_i) p_i^2 + \sum_{j=1}^{n} z_{ij}(\vec{q}) p_j + u_i(\vec{q}). \quad (3)$$

One can obtain a generating function $h(\alpha)$ analogously to that for the Stäckel case:

$$h(\alpha) = \sum_{k=1}^{n} \left( S(q_k) p_k^2 + \sum_{i=1}^{n} z_{i,k}(\vec{q}) p_i + u_k(\vec{q}) \right) \prod_{j \neq k} \frac{\alpha - q_j}{q_k - q_j}. \quad (4)$$

In order to characterize the integrable cases, we find the Hamiltonians $h_i$ as the coefficients of the generating function $h(\alpha) = \sum_{k=0}^{n-1} h_k \alpha^{n-k-1}$ (4), and compute the commutators and equate them to zero:

$$\{h_i, h_j\} = 0, \quad i, j = 0, \ldots, n-1. \quad (5)$$

Unlike for the Stäckel case, these commutators do not vanish identically, and in order to obtain a full classification one has to determine the functions $S, z_{i,j}, u_i$ from this set of $\frac{n(n-1)}{2}$ equations which are quadratic in the momenta. This is a difficult problem for $n > 3$ and we consider only the case $n = 3$ in this article.

The first step is to solve the equations corresponding to the vanishing of the quadratic in the momenta terms. This can be done by straightforward, although tedious computation and we arrive at the following statement.

Statement 1. If the family (4) is commutative, then the coefficients $z_{ij}$ can be brought to the form

$$z_{ij} = \Delta_{ij} \frac{\sqrt{S(q_i)} \sqrt{S(q_j)}}{q_i - q_j}, \quad z_{ii} = 0 \quad (6)$$

where the $\Delta_{ij}$ are some constants.
Notice that, strictly speaking, the diagonal coefficients may be of the form \( z_{ii} = S(q_i)\frac{\partial^2}{\partial q_i^2} F(q) \), but one can set \( z_{ii} = 0 \) by use of the canonical transformation \( p_i \to p_i - \frac{1}{2} \frac{\partial}{\partial q_i} S(q_i) \).

The analysis of the rest of the equations brings us to the following two (non-Stäckel) cases.

1. The symmetric case: \( \Delta_{i,i} = 0, \quad \Delta_{i\neq j} = -\delta, \quad S(x) = a(x)^2 \) where \( a(x) \) is a quadratic polynomial.

2. The non-symmetric case: \( \Delta_{2,3} = \Delta_{3,2} = -\delta \) and all other \( \Delta_{i,j} = 0 \) (up to change of enumeration), and \( S \) is a polynomial of sixth degree.

These cases describe all possible three-dimensional quasi-Stäckel systems.

We recall that the dynamics of the quasi-Stäckel system is defined by Hamiltonians \( h_i \). Now we consider these two cases in detail.

### 2.1. The symmetric case

In the general symmetric case the final answer is given by the following statement.

**Statement 2.** In the symmetric case 1 the resulting commutative family of Hamiltonians is of the form (4)

\[
h(\alpha) = \sum_{k=1}^3 \left( S(q_k) p_k^2 + \sum_{i=1}^3 z_{i,j}(q) p_i + u_k(q) \right) \prod_{j \neq k} \frac{\alpha - q_j}{q_k - q_j}, \quad [h(\alpha), h(\beta)] = 0
\]  

(7)

where

\[ S(x) = a^2(x), \quad z_{i,j}(q) = -\delta \frac{a(q_i) a(q_j)}{q_i - q_j}, \quad i \neq j, \quad z_{i,i}(q) = 0, \quad a(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0, \]

\[ u_1(q) = u(q_1, q_2, q_3), \quad u_2(q) = u(q_2, q_3, q_1), \quad u_3(q) = u(q_3, q_1, q_2)
\]

and

\[
u(x, y, z) = -\frac{\delta^2}{4} \left( 3S(x) \left( \frac{1}{(x-y)^2} + \frac{1}{(x-z)^2} \right) - S'(x) \left( \frac{1}{x-y} + \frac{1}{x-z} \right) + \frac{1}{5} S''(x) \right).
\]

**Remark.**

1. In fact, \( u(x, y, z) \) is defined up to a quadratic polynomial \( b(x) = b_0 x^2 + b_1 x + b_2 \), but one can exclude this through the shifts \( h_1 \to h_1 - h_i \).

2. In the case \( a_3 = 0 \), that is if \( a(x) \) is a quadratic polynomial, the canonical transformation

\[ p_i \to p_i + \frac{\partial}{\partial q_i} G(q_1, q_2, q_3), \quad G(x, y, z) = \frac{\delta}{2} \log((x-z)(x-y)(y-z))
\]

allows us to exclude the potential \( u \) completely. In this case, the family (7) takes the form

\[
h(\alpha) = \sum_{k=1}^3 \left( a^2(q_k) p_k^2 + a(q_k) \delta \sum_{j \neq k} \frac{a(q_j) p_k - a(q_j) p_i}{q_k - q_i} \right) \prod_{j \neq k} \frac{\alpha - q_j}{q_k - q_j}.
\]

(9)

Using the relation \( h(\alpha) = \sum_{k=0}^2 h_k q^{2-k} \) we obtain Hamiltonians \( h_0, h_1, h_2 \) in involution:

\( \{h_0, h_1\} = 0, \quad \{h_1, h_2\} = 0, \quad \{h_2, h_0\} = 0 \); therefore the system (9) is a Liouville integrable. Let us discuss some applications of this system to the problems of hydrodynamic type. To this end, we introduce three times \( t, \tau, \xi \) which describe the full evolution of the system:

\[ A_t = \{A, h_0\}, \quad A_\tau = \{A, h_1\}, \quad A_\xi = \{A, h_2\}
\]

(10)

and define \( I_1 = q_1 + q_2 + q_3, \quad I_2 = q_1 q_2 + q_2 q_3 + q_3 q_1, \quad I_3 = q_1 q_2 q_3 \).
We introduce two more functions: the hydrodynamic type system
\[
\left\{ \begin{array}{l}
V_k = (I_1 - q_k)q_k \frac{\partial}{\partial q_k} \left( I_1 - q_1 \right) \frac{\partial}{\partial q_1} \end{array} \right. \right\}
\]
that any solution of (10) satisfies the hydrodynamic type system
\[
q_{k,\tau} + (I_1 - q_k)q_{k,\tau} = \frac{\delta}{2} a''(q_k) a(q_k) - a^2(q_k)\delta \prod_{j\neq k} \frac{1}{q_k - q_j}, \quad I_{1,\tau} = -I_{2,\tau}.
\]
Quite similarly, the computation of \( \frac{\partial q_k}{\partial q_k} - I_1 \frac{\partial q_k}{\partial q_k} \) brings us to another hydrodynamic type system:
\[
q_k,\xi \frac{\partial}{\partial q_k} a(q_k)\delta \sum_{(i,j,k) = (1,2,3)} \frac{a(q_k) q_j}{(q_i - q_k)(q_j - q_k)}, \quad I_{1,\xi} = I_{3,\xi}.
\]

The system (11) is a generalization of the celebrated Gibbons–Tsarev system [13]. Some systems analogous to (11), (12) were considered in [11] and recently in [14]. Both systems can be generalized to the case of any dimension.

We do not know, at the moment, the separation of variables for system (9) and whether it is solvable.

But we will show that in the special symmetric case \( a(x) = 1 \) the system is solvable. In the rest of this section, we consider this case in detail and obtain its general solution. To this end, we write the system (3) with \( a(x) = 1 \) in a ‘separated’ form (in the \( \delta = 0 \) case this system is indeed separated and coincides with the St¨ackel one) as follows:
\[
q_k^2 h_0 + q_k h_1 + h_2 - p_k^2 - \delta \sum_{j\neq k} \frac{p_k - p_j}{q_k - q_j} = 0, \quad k = 1, 2, 3.
\]

We solve system (13), linear with respect to \( h_k \), define \( h_k = H_k + V_k \delta, \quad k = 0, 1, 2 \) and obtain
\[
H_0 = \sum_k p_k^2 \prod_{j\neq k} \frac{1}{q_k - q_j}, \quad H_1 = \sum_k p_k^3 (q_k - I_1) \prod_{j\neq k} \frac{1}{q_k - q_j}, \quad H_2 = \sum_k p_k^2 \prod_{j\neq k} \frac{1}{q_k - q_j}, \quad V_0 = 0, \quad V_1 = \sum_k p_k \prod_{j\neq k} \frac{1}{q_k - q_j}, \quad V_2 = \sum_k p_k (2q_k - I_1) \prod_{j\neq k} \frac{1}{q_k - q_j}.
\]

We introduce two more functions:
\[
V_3 = \sum_k p_k \left( I_1^2 - 2I_2 - 2q_k^2 \right) \prod_{j\neq k} \frac{1}{q_k - q_j}, \quad V_4 = \sum_k p_k
\]

to obtain a Poisson algebra with seven generators \( H_0, H_1, H_2, V_1, V_2, V_3, V_4 \):
\[
\left\{ \begin{array}{l}
\{ H_0, H_1 \} = 0, \quad \{ H_1, H_2 \} = 0, \quad \{ H_2, V_1 \} = 0, \quad \{ H_0, V_1 \} = 0, \quad \{ H_0, V_2 \} = 0, \quad \{ H_0, V_3 \} = 0, \quad \{ H_0, V_4 \} = 0, \quad \{ H_1, V_2 \} = 0, \quad \{ V_1, V_3 \} = 0, \quad \{ V_1, V_4 \} = 0, \quad \{ V_2, V_3 \} = 2V_1, \quad \{ V_2, V_4 \} = V_1, \quad \{ V_3, V_4 \} = 2V_2, \quad \{ H_1, V_1 \} = -V_1^2, \quad \{ H_1, V_2 \} = -V_1 V_2, \quad \{ H_1, V_3 \} = -V_2^2, \quad \{ H_1, V_4 \} = 2H_0, \quad \{ H_2, V_1 \} = \{ H_2, V_2 \} = -V_1 V_3, \quad \{ H_2, V_3 \} = -V_2 (V_3 + 2V_4) - 2H_1, \quad \{ H_2, V_4 \} = H_1.
\end{array} \right.
\]

The main reason that the system is solvable is not just the Liouville integrability but also the fact that \( \{ h_0, V_i \} = 0, \quad i = 1, \ldots, 4 \). This allows us to introduce the notation \( V_i = v_i(\tau, \xi) \), and to find the momenta \( p_1, p_2, p_3 \) from the linear system \( V_1 = v_1(\tau, \xi), \quad V_2 = v_2(\tau, \xi), \quad V_4 = v_4(\tau, \xi) \). The variables \( q_i \) are found then by use of only the first Hamiltonian equation:
\[
q_{i,\tau} = \frac{\partial h_i}{\partial p_i}.
\]
after substituting in the momenta obtained. In order to determine $q_i$ it is convenient to consider the dynamical systems

$$\frac{\partial I_j}{\partial t_\beta} = [I_j, H_\beta], \quad j = 1, 2, 3, \quad \beta = 0, 1, 2, \quad t_{\beta=0} = t, \quad t_{\beta=1} = \tau, \quad t_{\beta=2} = \xi. \quad (17)$$

First, we find the $t$-dynamics of $I_1$:

$$I_1 = -2v_1(\tau, \xi) + c_1(\tau, \xi), \quad I_2 = -v_1(\tau, \xi)(-v_1(\tau, \xi)t^2 + c_1(\tau, \xi)t) + v_2(\tau, \xi)t + c_2(\tau, \xi), \quad I_3 = -\frac{1}{\tau^2}c_1(\tau, \xi)^3 - v_1(\tau, \xi)v_2(\tau, \xi)t^2 + v_2(\tau, \xi)c_1(\tau, \xi)t + v_3(\tau, \xi)(t) + c_3(\tau, \xi). \quad (18)$$

Then step by step we restore the dependence on $\tau$ and $\xi$ using the additional identities

$$V_1 = v_3(\tau, \xi), \quad h_1 = \tilde{h}_1,$$

where the $\tilde{h}_i$ are constant values of the Hamiltonians $h_i$.

Finally, we obtain the general solution of the quasi-Stäckel system under consideration as the zeros of the cubic equation

$$(q - q_1)(q - q_2)(q - q_3) = q^3 - t_1q^2 + I_2q - I_3 = 0$$

with the coefficients

$$I_1 = \frac{2(t - t_0) + \lambda r}{\tau_0 - \tau} - \lambda t_1, \quad I_2 = \frac{(t - t_0 - \frac{1}{2}\lambda r)^2}{(\tau_0 - \tau)^2} - \frac{1}{4}\lambda^2(t_0 - \tau)^2 + \lambda^2 t_2(t_0 - \tau) - \lambda^2 t_1^2, \quad I_3 = \lambda t_1 \left(\frac{t - t_0 - \frac{1}{2}\lambda r}{\tau_0 - \tau} + t_1\lambda\right)^2 - \lambda^2(t - t_0)t_2 + \frac{1}{2}\lambda^3 r t_2 - (\lambda^3 t_1 t_2 + \delta)(t_0 - \tau) - \frac{1}{4}\lambda^3(t_1 - 2t_2)(t_0 - \tau)^3$$

where we use the notation

$$f(\lambda \xi) = C_1 + C_2\lambda \xi + C_3 e^{-\lambda \xi} + C_4 e^{\lambda \xi}, \quad t_0 = f(\lambda \xi), \quad t_1 = f'(\lambda \xi), \quad t_2 = f''(\lambda \xi), \quad t_3 = f'''(\lambda \xi), \quad r = \lambda \xi((t_1 - t_3)^2 + t_3^2 - t_2^2) + t_0(t_1 - t_3) + t_2(3t_1 - 2t_3), \quad h_0 = \frac{1}{4}\lambda^2, \quad h_1 = -\frac{1}{4}C_2\lambda^3, \quad h_2 = \lambda^4(C_1C_4 + \frac{1}{4}C_3^2).$$

This solution depends on the full set of integration constants $t_0, C_1, C_2, C_3, C_4, \lambda$.

### 2.2. The non-symmetric case

In this case we obtain the one-dimensional Stäckel system (recall that we can assume that the $H_i$ are constant)

$$x^2 H_0 + x H_1 + H_2 - S(x)p_1^2 = 0.$$ 

The action in this case is of the form $S = \int dx \sqrt{\frac{y H_0 + y H_1 + H_2}{S(x)}}$.

The remaining two-dimensional (quasi-Stäckel) system reads

$$y^2 H_0 + y H_1 + H_2 - S(y) p_2^2 - \delta \frac{S'(y)}{S(y)} = 0, \quad z^2 H_0 + z H_1 + H_2 - S(z) p_3^2 - \delta \frac{S'(z)}{S(z)} = 0,$$

where $S$ is a polynomial of sixth degree [15].
Notice that the Hamiltonians $H_i$ defined by these systems are three-dimensional, that is, they depend on all variables $q_i$, $p_i$, $i = 1, \ldots, 3$. At the moment, only the partial separation of variables is known for this system [16].

In this article we considered a classification of three-dimensional quasi-Stäckel systems. We obtain only two families: the symmetric three-dimensional system and non-symmetric ones. In the case $a(x) = 1$ the symmetric system is solvable and we obtain the general solution of this system. The calculations become very involved for the case $n > 3$ and at the moment one can hardly expect to obtain a full classification even for the $n = 4$ case. On the other hand, system (13) admits a straightforward generalization for the case $n > 3$. A preliminary result is that in the $n = 4$ case we obtain $\{h_i, h_j\} = 0$, $i, j = 0, 1, 2, 3$, except for the case of the relation $\{h_2, h_3\} \neq 0$. This is a surprising fact and we have no Liouville integrability in this case.

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References

[1] Stäckel P 1891 Über die Integration der Hamilton Jacobiischen Differential Gleichung Mittelst Separation der Variabeln (Halle: Habilitationsschrift)
[2] Eisenhart L P 1934 Separable systems of Stäckel Ann. Math. 35 284–305
[3] Błaszak M 2009 Bi-Hamiltonian representation of Stäckel systems Phys. Rev. E 79 056607
[4] Błaszak M and Sergyeyev A 2011 Generalized Stäckel systems Phys. Lett. A 375 2617
[5] Sklyanin E K 1995 Separation of variables—new trends Prog. Theor. Phys. Suppl. 118 35
[6] Kalnins E 1986 Separation of Variables for Riemannian Spaces of Constant Curvature (New York: Wiley)
[7] Benenti S 1997 Intrinsic characterization of the variable separation in the Hamilton–Jacobi equation J. Math. Phys. 38 6578
[8] Benenti S 1992 Inertia tensors and Stäckel systems in the Euclidean spaces Rend. Sem. Mat. Univ. Politec. Torino 50 315
[9] Marikhin V G and Sokolov V V 2005 On quasi-Stäckel Hamiltonians Russ. Math. Surv. 60 981
[10] Dorizzi B, Grammaticos B, Ramani A and Winternitz P 1985 Integrable Hamiltonian systems with velocity-dependent potentials J. Math. Phys. 26 3070
[11] Ferapontov E V and Fordy A P 1997 Non-homogeneous systems of hydrodynamic type, related to quadratic Hamiltonians with electromagnetic term Physica D 108 350
[12] Yehia H M 2007 Atlas of two-dimensional irreversible conservative Lagrangian mechanical systems with a second quadratic integral J. Math. Phys. 48 082902
[13] Gibbons J and Tsarev S P 1996 Reductions of the Benney equations Phys. Lett. A 211 19
[14] Odesskii A V and Sokolov V V 2013 Non-homogeneous systems of hydrodynamic type possessing Lax representations Commun. Math. Phys. 324 47
[15] Marikhin V G and Sokolov V V 2005 Separation of variables on a non-hyperelliptic curve Reg. Chaot. Dyn. 10 59
[16] Marikhin V G and Sokolov V V 2010 Transformation of a pair of commuting Hamiltonians quadratic in momenta to a canonical form and real partial separation of variables for the Clebsch top Reg. Chaot. Dyn. 15 652