LOCALIZATIONS, COLOCALIZATIONS AND NON ADDITIVE $\ast$-OBJECTS

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Abstract. Given a pair of adjoint functors between two arbitrary categories it induces mutually inverse equivalences between the full subcategories of the initial ones, consisting of objects for which the arrows of adjunction are isomorphisms. We investigate some cases in which these subcategories may be better characterized. One application is the construction of cellular approximations. Other is the definition and the characterization of (weak) $\ast$-objects in the non additive case.

Introduction

In mathematics the concept of localization has a long history. The origin of the concept is the study of some properties of maps around a point of a topological space. In the algebraic sense, the localization provides a method to invert some morphisms in a category. Making abstraction of some technical set theoretic problems, given a class of morphisms $\Sigma$ in a category $\mathcal{A}$, there is a category $\mathcal{A}[\Sigma^{-1}]$ and a functor $\mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$ universal with the property that it sends any morphism in $\Sigma$ to an isomorphism. This functor will be called a localization, if it has a right adjoint, which will be frequently fully faithful. Dually this functor is called a colocalization provided that it has a left adjoint.

One of the starting point of this paper is the observation that the consequences of the duality between localization and colocalization were not exhausted. For example the concept, borrowed from topology, of cellular approximation in arbitrary category is a particular case of a colocalization, fact remarked for example in [4]. Some results concerning the cellular approximation may be deduced in a formal, categorical way by stressing this duality. On the other hand the same formal techniques are useful in the study of so called $\ast$-modules, defined as in [3].

Now let us present the organization and the main results of the paper. In the first section we set the notations, we define the main notions used throughout of the paper and we record some easy properties concerning these notions.

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In Section 2 are stated the formal results, on which it is based the rest of the paper. There are three main results here: First Theorem 2.4 where are given necessary and sufficient conditions for a pair of adjoint functors to induce an equivalence between the full subcategories consisting of colocal respectively local objects with respect to these functors (for the definition of a (co)local object see Section 1). Second and third Proposition 2.10 and Theorem 2.11 which represent the formal characterization of a non additive (weak) *-object.

Section 3 contains a non additive version of a theorem of Menini and Orsatti in [7]. Consider an object (or a set of objects) \( A \) of a category \( A \), and the category of all contravariant functors \([E^{op},Set]\) where \( E \) is the full subcategory of \( A \) containing the object(s) \( A \), situation which is less general but more comprehensive that the hypotheses of Section 3. Under appropriate assumptions, mutually inverse equivalences between two full subcategories of \( A \) and \([E^{op},Set]\) are represented by \( A \) in the sense that they are realized by restrictions of the representable functor \( H_A = A(A,-) \) and of its left adjoint (see Theorem 3.2).

Provided that \( A \) is a cocomplete, well copowered, balanced category with epimorphic images, and \( A \) is a set of objects of \( A \), it is shown in Section 4 that the inclusion of the subcategory of \( H_A \)-colocal objects has a right adjoint (see Theorem 4.2). Consequently fixing an object \( A \) in such a category, every object \( X \) will have an \( A \)-cellular approximation.

In Section 5 we define and characterize the notions of a (weak) *-act over a monoid, in Proposition 5.2 and Theorem 5.3, providing in this way a translation of the notion of (weak) *-module in these new settings. It is interesting to note that our approach may be continued by developing a theory analogous with so called tilting theory for modules. The Morita theory for the category of acts over monoids is a consequence of our results.

1. Notations and preliminaries

All subcategories which we consider are full and closed under isomorphisms, so if we speak about a class of objects in a category we understand also the respective subcategory. For a category \( A \) we denote by \( A^{-} \) the category of all morphisms in \( A \). We denote by \( A(-,-) \) the bifunctor assigning to any two objects of \( A \) the set of all morphisms between them.

Consider a functor \( H : A \rightarrow B \). The (essential) image of \( H \) is the subcategory \( \text{Im}H \) of \( B \) consisting of all objects \( Y \in B \) satisfying \( Y \cong H(X) \) for some \( X \in A \). In contrast we shall denote by \( \text{im} \alpha \) the categorical notion of image of a morphism \( \alpha \in A^{-} \). A morphism \( \alpha \in A^{-} \) is called an \( H \)-equivalence, provided that \( H(\alpha) \) is an isomorphism. We denote by \( \text{Eq}(H) \) the subcategory of \( A^{-} \) consisting of all \( H \)-equivalences. An object \( X \in A \) is called \( H \)-local (\( H \)-colocal) if, for any \( H \)-equivalence \( \epsilon \), the induced map \( \epsilon_* = A(\epsilon, X) \) (respectively, \( \epsilon^* = A(X, \epsilon) \)) is bijective, that means it is an isomorphism in the category \( Set \) of all sets. We denote by \( C^H \) and \( C_H \) the full
subcategories of \( \mathcal{A} \) consisting of all \( H \)-local, respectively \( H \)-colocal objects. For objects \( X', X \in \mathcal{A} \), we say that \( X' \) is a retract of \( X \) if there are maps \( \alpha : X' \to X \) and \( \beta : X \to X' \) in \( \mathcal{A} \) such that \( \beta \alpha = 1_{X'} \). We record without proof the following properties relative to the above considered notions:

**Lemma 1.1.** The following hold:

a) \( \text{Eq}(H) \) is closed under retracts in \( \mathcal{A}^{-} \).

b) \( \text{Eq}(H) \) satisfies the ‘two out of three’ property, namely if \( \alpha, \beta \in \mathcal{A}^{-} \) are composable morphisms, then if two of the morphisms \( \alpha, \beta, \beta \alpha \) are \( H \)-equivalences, then so is the third.

c) The subcategory \( \mathcal{C}^{\mathcal{H}} \) (respectively \( \mathcal{C}^{\mathcal{H}} \)) is closed under limits (respectively colimits) and retracts in \( \mathcal{A} \).

Moreover if every object of \( \mathcal{A} \) has a left (right) approximation with an \( H \)-local (colocal) object, in a sense becoming precise in the hypothesis of the Lemma below, then we are in the situation of a localization (colocalization) functor, as it may be seen from:

**Lemma 1.2.** If for every \( X \in \mathcal{A} \) there is an \( H \)-equivalence \( X \to X^{H} \) with \( X^{H} \in \mathcal{C}^{\mathcal{H}} \) (respectively, \( X_{H} \to X \) with \( X_{H} \in \mathcal{C}^{\mathcal{H}} \)), then the assignment \( X \mapsto X^{H} \) (respectively, \( X \mapsto X_{H} \)) is functorial and defines a left (right) adjoint of the inclusion functor \( \mathcal{C}^{\mathcal{H}} \to \mathcal{A} \) (\( \mathcal{C}^{\mathcal{H}} \to \mathcal{A} \)). Moreover the left (right) adjoint of the inclusion functor sends every map \( \alpha \in \text{Eq}(H) \) into an isomorphism and it is universal relative to this property.

**Proof.** Straightforward. (The first statement was also noticed in [5, 1.6]).

In the sequel we consider a pair of adjoint functors \( H : \mathcal{A} \to \mathcal{B} \) at the right and \( T : \mathcal{B} \to \mathcal{A} \) at the left, where \( \mathcal{A} \) and \( \mathcal{B} \) are arbitrary categories. We shall symbolize this situation by \( T \dashv H \). Consider also the arrows of adjunction

\[
\delta : T \circ H \to 1_{\mathcal{A}} \quad \text{and} \quad \eta : 1_{\mathcal{B}} \to H \circ T.
\]

Note that, for all \( X \in \mathcal{A} \) and all \( Y \in \mathcal{B} \) we obtain the commutative diagrams in \( \mathcal{B} \) and \( \mathcal{A} \) respectively:

\[
\begin{align*}
H(X) \xrightarrow{\eta_H(X)} & (H \circ T \circ H)(X) & T(Y) \xrightarrow{T(\eta_Y)} & (T \circ H \circ T)(Y) \\
\downarrow^{1_{H(X)}} & & \downarrow^{1_{T(Y)}} & \\
H(X) & & T(Y) & \\
\end{align*}
\]

showing that \( H(X) \) and \( T(Y) \) are retracts of \( (H \circ T \circ H)(X) \), respectively \( (T \circ H \circ T)(Y) \). Corresponding to the adjoint pair considered above, we define the following full subcategories of \( \mathcal{A} \) and \( \mathcal{B} \):

\[
\mathcal{S}_H = \{ X \in \mathcal{A} \mid \sigma_X : (T \circ H)(X) \to X \text{ is an isomorphism} \},
\]

and respectively

\[
\mathcal{S}^T = \{ Y \in \mathcal{B} \mid \eta_Y : Y \to (H \circ T)(Y) \text{ is an isomorphism} \}.
\]
The objects in $S_H$ and $S^T$ are called $\delta$-reflexive, respectively $\eta$-reflexive. Note that $H$ and $T$ restrict to mutually inverse equivalences of categories between $S_H$ and $S^T$ and these subcategories are the largest of $\mathcal{A}$ and $\mathcal{B}$ respectively, enjoying this property.

2. An equivalence induced by adjoint functors

In this section we fix a pair of adjoint functors $T \dashv H$ between two arbitrary categories $\mathcal{A}$ and $\mathcal{B}$, as in Section 1.

**Lemma 2.1.** The following inclusions hold:

a) $S_H \subseteq \text{Im } T \subseteq C_H \subseteq \mathcal{A}$.

b) $S^T \subseteq \text{Im } H \subseteq C^T \subseteq \mathcal{B}$.

**Proof.** a) The first inclusion is obvious. For the second inclusion observe that for all $\epsilon \in \text{Eq}(H)$ and all $Y \in \mathcal{B}$ the isomorphism in $\text{Set}^- 
\epsilon^* = \mathcal{A}(T(Y), \epsilon) \cong \mathcal{B}(Y, H(\epsilon))$
shows that $\epsilon^*$ is bijective. The inclusions from b) follow by duality. □

**Lemma 2.2.** Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$ such that the inclusion functor $I : \mathcal{C} \to \mathcal{A}$ has a right adjoint $R : \mathcal{A} \to \mathcal{C}$ and the arrow of the adjunction $\mu_X : (I \circ R)(X) \to X$ is an $H$-equivalence for all $X \in \mathcal{A}$. Then $\mu_X$ is an isomorphism for all $X \in C_H$, and consequently $C_H \subseteq \mathcal{C}$.

**Proof.** Let $X \in C_H$. Since $\mu_X \in \text{Eq}(H)$, we deduce that the induced map

$$
\mu_X^* : \mathcal{A}(X, (I \circ R)(X)) \to \mathcal{A}(X, X)
$$

is bijective, consequently there is a morphism $\mu'_X : X \to (I \circ R)(X)$ such that $\mu_X \mu'_X = 1_X$. Since $R \circ I \cong 1_\mathcal{C}$ naturally, and $\mu$ is also natural, we obtain a commutative diagram

$$
\begin{array}{ccc}
(I \circ R)(X) & \overset{(I \circ R)(\mu'_X)}{\longrightarrow} & (I \circ R \circ I \circ R)(X) \\
\mu_X & \Downarrow & \mu_{(I \circ R)(X)} \\
X & \underset{\mu'_X}{\longmapsto} & (I \circ R)(X)
\end{array}
$$

showing that $\mu'_X \mu_X = 1_{(I \circ R)(X)}$, hence $\mu_X$ is an isomorphism. □

**Lemma 2.3.** If $S^T = \text{Im } H$ then $S_H = C_H$. Dually if $S_H = \text{Im } T$ then $S^T = C^T$.

**Proof.** Consider an arbitrary object $X \in \mathcal{A}$. By hypothesis $H(X) \in S^T$, so $\eta_{H(X)}$ is an isomorphism. Together with diagrams [1], this implies $H(\delta_X)$ is an isomorphism. Thus $\delta_X : (T \circ H)(X) \to X$ is an $H$-equivalence, and we know $(T \circ H)(X) \in \text{Im } T \subseteq C_H$. As we learned from Lemma 1.2 this means that the assignment $X \mapsto (T \circ H)(X)$ defines a right adjoint of the inclusion of $C_H$ in $\mathcal{A}$. If in addition $X \in C_H$, then $\delta_X$ is an isomorphism by
Lemma 2.2 proving the inclusion $\mathcal{C}_H \subseteq \mathcal{S}_H$. Since the converse inclusion is always true, the conclusion holds.

Remark 2.4. From the proof of Lemma 2.3 we can see that the condition $\mathcal{S}^T = \text{Im} \, H$ implies that the functor $A \to \mathcal{C}_H$, $X \to (T \circ H)(X)$ is the right adjoint of the inclusion functor of $\mathcal{S}_H = \mathcal{C}_H$ into $A$. Dually if $\mathcal{S}_H = \text{Im} \, T$, then the functor $B \to \mathcal{C}^T$, $Y \to (H \circ T)(Y)$ is the left adjoint of the inclusion functor of $\mathcal{S}^T = \mathcal{C}^T$ into $B$.

Theorem 2.5. The following are equivalent:

(i) $\mathcal{S}^T = \mathcal{C}^T$.
(ii) $\mathcal{S}^T = \text{Im} \, H$.
(iii) $\mathcal{S}_H = \mathcal{C}_H$.
(iv) $\mathcal{S}_H = \text{Im} \, T$.
(v) The functors $H$ and $T$ induce mutually inverse equivalence of categories between $\mathcal{C}_H$ and $\mathcal{C}^T$.

Proof. The equivalence of the conditions (i)–(iv) follows by Lemmas 2.1 and 2.3. Finally the equivalent conditions (i)–(ii) are also equivalent to (v), because $\mathcal{S}_H$ and $\mathcal{S}^T$ are the largest subcategories of $A$ and $B$ for which $H$ and $T$ restrict to mutually inverse equivalences.

Corollary 2.6. The adjoint functors $T \dashv H$ induce mutually inverse equivalences $\mathcal{C}_H \rightleftharpoons B$ if and only if $T$ is fully faithful. Dually the adjoint pair induces equivalences $A \rightleftharpoons \mathcal{C}^T$ if and only if $H$ is fully faithful.

Proof. The functor $T$ is fully faithful exactly if the unit of the adjunction $\eta : 1_B \to (H \circ T)$ is an isomorphism, or equivalently, $\mathcal{S}^T = B$. Now, Theorem 2.5 applies.

Theorem 2.5 and Corollary 2.6 generalize [1, Theorem 1.6 and Corollary 1.7], where the work is done in the setting of abelian categories, and the proof stresses the abelian structure. These results may be also compared with [9, Theorem 1.18], where the framework is also that of abelian categories.

We consider next other two subcategories of $A$ and $B$ respectively:

$\mathcal{G}_H = \{X \in A \mid \delta_X \colon (T \circ H)(X) \to X \text{ is an epimorphism}\}$,

$\mathcal{G}^T = \{Y \in B \mid \eta_Y \colon Y \to (H \circ T)(Y) \text{ is a monomorphism}\}$.

The dual character of all considerations in the present Section continues to hold for $\mathcal{G}_H$ and $\mathcal{G}^T$.

Lemma 2.7. The following statements hold:

a) The subcategory $\mathcal{G}_H$ (respectively $\mathcal{G}^T$) is closed under quotient objects (subobjects).

b) $\text{Im} \, T \subseteq \mathcal{G}_H$ (respectively $\text{Im} \, H \subseteq \mathcal{G}^T$).

Proof. a) Let $\alpha : X' \to X$ be an epimorphism in $A$ with $X' \in \mathcal{G}_H$. Since $\delta$ is natural, we obtain the equality $\alpha \delta_{X'} = \delta_X(T \circ H)(\alpha)$, showing that $\delta_X$ is an epimorphism together with $\alpha \delta_{X'}$. 

b) From the diagrams (1), we see that $\delta_{T(Y)}$ is right invertible, so it is an epimorphism for any $Y \in B$. Thus $\text{Im} \ T \subseteq G_H$. □

The subcategory $G_H$ of $A$ is more interesting in the case when $A$ has epimorphic images, what means that it has images and the factorization of a morphism through its image is a composition of an epimorphism followed by a monomorphism (for example, $A$ has epimorphic images, provided that it has equalizators and images, by [8, Chapter 1, Proposition 10.1]). Suppose also that $A$ is balanced, that is every morphism which is both epimorphism and monomorphism is an isomorphism. Thus every factorization of a morphism as a composition of an epimorphism followed by a monomorphism is a factorization through image, by [8, Chapter 1, Proposition 10.2]. With these hypotheses it is not hard to see that the factorization of a morphism through its image is functorial, that means the assignment $\alpha \mapsto \text{im} \alpha$ defines a functor $A^- \to A$.

**Proposition 2.8.** If $A$ is a balanced category with epimorphic images, then the functor $A \to G_H$, $X \mapsto \text{im} \delta_X$ is a right adjoint of the inclusion functor $G_H \to A$.

**Proof.** By hypothesis $\text{im} \delta_X$ is a quotient of $(H \circ T)(Y)$ and $H(T(Y)) \in G_H$, so the functor $A \to G_H$, $X \mapsto \text{im} \delta_X$ is well defined, by Lemma 2.7. Let now $\alpha : X' \to X$ in $A^-$, where $X' \in G_H$ and $X \in A$. Since $\delta_{X'}$ is an epimorphism, it follows

$$\text{im} \alpha = \text{im} (\alpha \delta_{X'}) = \text{im} (\delta_X (T \circ H)(\alpha)) \subseteq \text{im} \delta_X,$$

so $\alpha$ factors through $\text{im} \delta_X$. This means that the map

$$A(X', \text{im} \delta_X) \to A(X', X)$$

is surjective. But it is also injective since the functor $A(X', -)$ preserves monomorphisms, and the conclusion follows. □

**Corollary 2.9.** If $A$ is a balanced category with epimorphic images, then the morphism $\text{im} \delta_X \to X$ is an $H$-equivalence and $C_H \subseteq G_H$.

**Proof.** The second statement of the conclusion follows from the first one by using Proposition 2.8 and Lemma 2.2. But $H$ carries the monomorphism $\text{im} \delta_X \to X$ into a monomorphism in $B$, because $H$ is a right adjoint. Moreover, since $H(\delta_X)$ is right invertible, the same is true for the morphism $H(\text{im} \delta_X) \to H(X)$, as we may see from the commutative diagram

$$
\begin{array}{ccc}
(H \circ T \circ H)(X) & \xrightarrow{H(\delta_X)} & H(X) \\
\downarrow & & \downarrow \\
H(\text{im} \delta_X) & & \\
\end{array}
$$

□
Proposition 2.10. Suppose both \( A \) and \( B \) are balanced categories with epimorphic images. The following are equivalent:

(i) The pair of adjoint functors \( T \dashv H \) induces mutually inverse equivalences \( C_H \rightleftarrows G^T \).

(ii) \( \eta_Y : Y \to (H \circ T)(Y) \) is an epimorphism for all \( Y \in B \).

Proof. (i) \( \Rightarrow \) (ii). Denote \( Y' = \text{im} \eta_Y \). Then the unit \( \eta_Y \) of adjunction factors as \( Y \to Y' \to (H \circ T)(Y) \), where the epimorphism \( Y \to Y' \) is a \( T \)-equivalence by the dual of Corollary 2.9 and \( Y' \to (H \circ T)(Y) \) a monomorphism. Since \((H \circ T)(Y) \in \text{Im} H \subseteq G^T \) and \( G^T \) is closed under subobjects, we deduce \( Y' \in G^T \). Now (i) implies that \( \eta_{Y'} \) is an isomorphism, so the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{id}} & Y' \\
\downarrow{\eta_Y} & & \downarrow{\eta_{Y'}} \\
(H \circ T)(Y) & \xrightarrow{\text{id}} & (H \circ T)(Y')
\end{array}
\]

proves (ii).

(ii) \( \Rightarrow \) (i). Condition (ii) implies that \( T(\eta_Y) \) is an epimorphism, for every \( Y \in B \), since \( T \) preserves epimorphisms. But it is also left invertible by diagrams 4. Thus it is invertible, with the inverse \( \delta_{T(Y)} \). We have just shown that \( S_H = \text{Im} T \), hence Theorem 2.5 tells us that \( C_H \) and \( G^T \) are equivalent via \( H \) and \( T \). Finally, since \( B \) is balanced, clearly \( G^T = S^T = C^T \). \( \square \)

Combining Proposition 2.10 and its dual we obtain:

Theorem 2.11. Suppose both \( A \) and \( B \) are balanced categories with epimorphic images. The following are equivalent:

(i) The pair of adjoint functors \( T \dashv H \) induces mutually inverse equivalences \( G_H \rightleftarrows G^T \).

(ii) \( \delta_X : (T \circ H)(X) \to X \) is a monomorphism for all \( X \in A \) and \( \eta_Y : Y \to (H \circ T)(Y) \) is an epimorphism for all \( Y \in B \).

Remark that [3, Proposition 2.2.4 and Theorem 2.3.8] provide characterizations of (weak) \( * \)-modules which are analogous to Proposition 2.10 and Theorem 2.11 above. These results will be used in Section 5 for defining the corresponding notions in a non additive situation.

3. Representable equivalences

Overall in this section \( A \) is a cocomplete category and \( E \) is small category. Denote by \([E^{\text{op}}, \text{Set}]\) the category of all contravariant functors from \( E \) into \( \text{Set} \). Then we view \( E \) as a subcategory of \([E^{\text{op}}, \text{Set}]\), via the Yoneda embedding \( E \to [E^{\text{op}}, \text{Set}], e \mapsto \mathcal{E}(\cdot, e) \). For simplicity, we shall write \([Y', Y]\) for \([E^{\text{op}}, \text{Set}](Y', Y)\), where \( Y', Y \in [E^{\text{op}}, \text{Set}] \). For every \( Y \in [E^{\text{op}}, \text{Set}] \) denote by \( \mathcal{E} \down Y \) the comma category whose objects are of the form \((e, y)\) with \( e \in E \) and \( y \in Y(e) \) and whose morphisms are

\[
(\mathcal{E} \down Y)((e', y'), (e, y)) = \{ \alpha \in E(e', e) \mid Y(\alpha)(y') = y \}.
\]
The projection functor \( \mathcal{E} \downarrow Y \to \mathcal{E} \) is given by \((e, y) \mapsto e\) and \(\alpha \mapsto \alpha\) for all \((e, y) \in \mathcal{E} \downarrow Y\) and all \(\alpha \in (\mathcal{E} \downarrow Y)((e', y'), (e, y))\). Observe then that the subcategory \(\mathcal{E}\) is dense in \([\mathcal{E}^{\text{op}}, \text{Set}]\), what means, for every \(Y \in [\mathcal{E}^{\text{op}}, \text{Set}]\) it holds
\[
Y \cong \text{colim}((\mathcal{E} \downarrow Y) \to \mathcal{E} \to [\mathcal{E}^{\text{op}}, \text{Set}]) = \text{colim}_{(e,y) \in \mathcal{E} \downarrow Y} \mathcal{E}(-, e),
\]
where the last notation is a shorthand for the previous colimit.

For a functor \(A : \mathcal{E} \to \mathcal{A}\), consider the left Kan extension of \(A\) along the Yoneda embedding:
\[
T_A : [\mathcal{E}^{\text{op}}, \text{Set}] \to \mathcal{A}, T_A(Y) = \text{colim}_{(e,y) \in \mathcal{E} \downarrow Y} A(e),
\]
which may be characterized as the unique, up to a natural isomorphism, colimit preserving functor \([\mathcal{E}^{\text{op}}, \text{Set}] \to \mathcal{A}\), mapping \(\mathcal{E}(-, e)\) into \(A(e)\) for all \(e \in \mathcal{E}\). The functor \(T_A\) has a right adjoint, namely the functor
\[
H_A : \mathcal{A} \to [\mathcal{E}^{\text{op}}, \text{Set}], H_A(X) = A(A(-), X).
\]
In order to use the results of Section \(\text{I}\), we remaind the notations made there, namely let \(\delta : T_A \circ H_A \to 1_A\) and \(\eta : 1_B \to H_A \circ T_A\) be the arrows of adjunction. For simplicity we shall replace in the next considerations the subscript \(H_A\) and the superscript \(T_A\) with \(A\). So objects in \(C, C^A, G_A\) and \(\mathcal{G}^A\) will be called \(A\)-colocal, \(A\)-local, \(A\)-generated, respectively \(A\)-cogenerated.

We consider overall in this section two subcategories \(C \subseteq A\) and \(C' \subseteq [\mathcal{E}^{\text{op}}, \text{Set}]\), such that \(C\) is closed under taking colimits and retracts and \(C'\) is closed under taking limits and retracts.

**Lemma 3.1.** Let \(H : C \rightleftarrows C' : T\) be a pair of adjoint functors \(T \dashv H\), and denote \(A : \mathcal{E} \to \mathcal{A}\) the functor given by \(A(e) = T(\mathcal{E}(-, e))\). If \(\mathcal{E}(-, e) \in \mathcal{C'}\) for all \(e \in \mathcal{E}\) then \(H\) is naturally isomorphic to the restriction of \(H_A\) and \(T\) is naturally isomorphic to the restriction of \(T_A\), \(\text{Im} H_A \subseteq C'\) and \(\text{Im} T_A \subseteq C\). If moreover \(T\) is fully faithful, then arrow \(\delta_X : (T_A \circ H_A)(X) \to X\) is an \(H_A\)-equivalence for all \(X \in \mathcal{A}\).

**Proof.** Using Yoneda lemma, we have the natural isomorphisms for every \(X \in C\), and every \(e \in \mathcal{E}\):
\[
H(X)(e) \cong [\mathcal{E}(-, e), H(X)] \cong A(T(\mathcal{E}(-, e)), X) = A(A(e), X) = H_A(X)(e),
\]
thus \(H(X) \cong H_A(X)\) naturally. Since \(A(e) = T(\mathcal{E}(-, e)) \in C\) for all \(e \in \mathcal{E}\), the closure of \(C\) under colimits and the formula
\[
T_A(Y) = \text{colim}_{(e,y) \in \mathcal{E} \downarrow Y} A(e),
\]
valid for all \(Y \in [\mathcal{E}^{\text{op}}, \text{Set}]\), show that \(\text{Im} T_A \subseteq C\). Therefore the assignment \(Y \mapsto T_A(Y)\) defines a functor \(C' \to C\), which is naturally isomorphic to \(T\), as right adjoints of \(H\).

Further \(\text{Im} T_A \subseteq C\) implies \(\text{Im} H_A \subseteq C'\), since for all \(X \in \mathcal{A}\), we have
\[
(H_A \circ T_A \circ H_A)(X) \cong H((T_A \circ H_A)(X)) \in C',
\]
$H_A(X)$ is a retract of $(H_A \circ T_A \circ H_A)(X)$ and $C'$ is closed under retracts.

Now, the fully faithfulness of $T$ is equivalent to the fact that $H \circ T \cong 1_{C'}$ naturally, thus $\text{Im} H_A \subseteq C'$ implies

$$(H_A \circ T_A \circ H_A)(X) \cong (H \circ T)(H_A(X)) \cong H_A(X),$$

what means that $\delta_X$ is an $H_A$-equivalence.

\begin{proof}
The conclusions concerning $H$ and $T$ follow by Lemma \ref{3.1}. For the rest, we have for all $X \in C$:

$$X \cong (T \circ H)(X) \cong (T_A \circ H_A)(X) \in \text{Im} T_A \subseteq C_A$$

as we have seen in Lemma \ref{2.1}. Thus $C \subseteq C_A$, and dually $C' \subseteq C^A$.

The functor $A \to C$ given by $X \mapsto (T_A \circ H_A)(X)$ is a right adjoint of the inclusion functor $C \to A$. Indeed, for all $X' \in C$ and all $X \in A$, we obtain $X' \cong (T \circ H)(X') \cong (T_A \circ H_A)(X')$ and $H_A(X) \cong (H_A \circ T_A \circ H_A)(X)$ since the counit $\delta_X$ of adjunction is an $H_A$-equivalence, as we observed in Lemma \ref{3.1}. Now the natural isomorphisms

$$A(X', (T_A \circ H_A)(X)) \cong A((T_A \circ H_A)(X'), (T_A \circ H_A)(X))$$

$$\cong [H_A(X'), (H_A \circ T_A \circ H_A)(X)] \cong [H_A(X'), H_A(X)]$$

$$\cong A((T_A \circ H_A)(X'), X) \cong A(X', X)$$

prove our claim. Using again the fact that $\delta_X : (T_A \circ H_A)(X) \to X$ is an $H_A$-equivalence, Lemma \ref{2.2} tells us that $C_A \subseteq C$. The functor $[\mathcal{E}^{\text{op}}, \text{Set}] \to C'$ given by $Y \mapsto (H_A \circ T_A)(Y)$ is also well defined. In a dual manner we show that it is a left adjoint of the inclusion $C' \to [\mathcal{E}^{\text{op}}, \text{Set}]$, and follows $C^A \subseteq C'$.

The equivalences $H : C \cong C' : T$ are called represented by $A : \mathcal{E} \to A$ provided that $H \cong H_A$ and $T \cong T_A$ as in the Theorem \ref{3.2}.

In the work \cite{MeniniOrsatti} of Menini and Orsatti (see also \cite{Menini}), it is given an additive version of Theorem \ref{3.2}. There, our category $\mathcal{E}$ is preadditive with a single object (that means it is a ring), $A$ is an object in $\mathcal{A}$ with endomorphism ring $\mathcal{E}$ (therefore $A : \mathcal{E} \to A$ is a fully faithful functor), and $[\mathcal{E}^{\text{op}}, \text{Set}]$ is replaced with $\text{Mod}(\mathcal{E})$.

### 4. The existence of cellular covers

In this Section consider as in the previous one a cocomplete category $\mathcal{A}$, a functor $A : \mathcal{E} \to \mathcal{A}$, where $\mathcal{E}$ is a small category and construct its left Kan extension $T_A : [\mathcal{E}^{\text{op}}, \text{Set}] \to \mathcal{A}$ along the Yoneda embedding $\mathcal{E} \to [\mathcal{E}^{\text{op}}, \text{Set}]$ which has the right adjoint $H_A : \mathcal{A} \to [\mathcal{E}^{\text{op}}, \text{Set}]$. In addition suppose that $A$ is fully faithful. Note that, this additional assumption means that the
category $\mathcal{E}$ may be identified with a (small) subcategory of $\mathcal{A}$ and $A$ with the inclusion functor. For example, if $\mathcal{E}$ has a single object, then $A$ may be identified with an object of $\mathcal{A}$.

**Lemma 4.1.** If $\mathcal{A}$ is a cocomplete, balanced category with epimorphic images and $A : \mathcal{E} \to \mathcal{A}$ is fully faithful, then it holds:

a) $A(e) \in S_A$ for all $e \in \mathcal{E}$.

b) An object $X \in \mathcal{A}$ is $A$-generated exactly if there is an epimorphism $A' \to X$ with $A'$ a coproduct of objects of the form $A(e)$ with $e \in \mathcal{E}$.

**Proof.** a) Since $A$ is fully faithful, we have the natural isomorphisms:

\[(T_A \circ H_A)(A(e)) = T_A(A(A(-), A(e))) \cong T_A(\mathcal{E}(-, e)) \cong A(e),\]

for every $e \in \mathcal{E}$.

b) Let $A' = \coprod A(e_i) \in \mathcal{A}$ be a coproduct of objects of the form $A(e)$. By the result in a) we deduce

\[A' = \coprod A(e_i) \cong \coprod (T_A \circ H_A)(A(e_i)) \cong T_A(\coprod H_A(A(e_i))) \in \text{Im} T_A,\]

so $A' \in G_A$, since $\text{Im} T_A \subseteq G_A$, inclusion established in Lemma 2.7. If $X \in \mathcal{A}$ such that there is an epimorphism $A' \to X$, then $X \in G_A$, again by Lemma 2.7. Conversely, for every $X \in \mathcal{A}$, the object $H_A(X)$ of $[\mathcal{E}^{\text{op}}, \text{Set}]$ may be written as

\[H_A(X) \cong \colim_{(e,x) \in \mathcal{E}^1H_A(X)} \mathcal{E}(-, e) = \colim_{(e,x) \in \mathcal{A}^1X} \mathcal{E}(-, e),\]

where the comma category $A \downarrow X$ has as objects pairs of the form $(e, x)$ with $e \in \mathcal{E}$ and $x \in A(A(e), X)$. Thus

\[(T_A \circ H_A)(X) \cong \colim_{(e,x) \in \mathcal{A}^1X} T_A(\mathcal{E}(-, e)) \cong \colim_{(e,x) \in \mathcal{A}^1X} A(e),\]

so there is an epimorphism from $\coprod_{(e,x) \in \mathcal{A}^1X} A(e)$ to $(T_A \circ H_A)(X)$. Further the morphism $\delta_X : (T_A \circ H_A)(X) \to X$ is an epimorphism too, for $X \in G_A$. Composing them we obtain the desired epimorphism. 

Recall that a category $\mathcal{A}$ is called well (co)powered if for every object the class of subobjects (respectively quotient objects) is actually a set.

**Theorem 4.2.** If $\mathcal{A}$ is a cocomplete, well copowered, balanced category with epimorphic images, and $A : \mathcal{E} \to \mathcal{A}$ is a fully faithful functor, then the inclusion functor $C_A \to \mathcal{A}$ has a right adjoint, or equivalently, every object in $\mathcal{A}$ has a left $C_A$-approximation.

**Proof.** Combining Lemma 4.1 and Corollary 2.9, we deduce that every object in $C$ is a quotient object of a direct sum of objects of the form $A(e)$, so $\{A(e) \mid e \in \mathcal{E}\} \subseteq C_A$ is a generating set for $C_A$, by [8, Chapter II, Proposition 15.2]. The closure of $C_A$ under colimits implies that the inclusion functor $C_A \to \mathcal{A}$ preserves colimits, and the category $C_A$ inherits from $\mathcal{A}$ the property to be well copowered. Thus the conclusion follows by Freyd’s Special Adjoint Functor Theorem (see [8, Chapter V, Corollary 3.2]).
If $\mathcal{E}$ has a single object and $A$ is a fully faithful functor (i.e. $A$ is an object of $\mathcal{A}$), then $A$-colocal objects are sometimes called $A$-cellular, and an $H_A$-equivalence is called then simply an $A$-equivalence. Our Theorem 4.2 shows that, under reasonable hypotheses (that means $A$ is a cocomplete, well copowered, balanced category with epimorphic images), every object $X$ has an $A$-cellular approximation, what means an $A$-equivalence $C \to X$ with $C$ being $A$-cellular. Hence it is generalized in this way [4, Section 2.1], where is constructed an $A$-cellular approximation for every group.

The same proof that given [4, Lemma 2.6] for the case of the category of groups works for the following consequence of the existence of an $A$-cellular approximation for every object $X \in A$:

**Corollary 4.3.** Let $A : \mathcal{E} \to \mathcal{A}$ be a fully faithful functor, where $\mathcal{A}$ is a cocomplete, well copowered, balanced category with epimorphic images and $\mathcal{E}$ is small category. The following are equivalent for a morphism $\alpha : C \to X$ in $\mathcal{A}$:

(i) $\alpha$ is a left $C_A$-approximation of $X$.

(ii) $\alpha$ is an $H_A$-equivalence and it is initial among all $H_A$-equivalences ending in $X$.

(iii) $C \in C_A$ and $\alpha$ is terminal among all morphisms from an $A$-colocal object to $X$.

Consequently we may use the following more of less tautological formulas for determining the left $C_A$-approximation of an object (see [5, Sections 7.1 and 7.2]):

**Corollary 4.4.** Let $A : \mathcal{E} \to \mathcal{A}$ be a fully faithful functor, where $\mathcal{A}$ is a cocomplete, well copowered, balanced category with epimorphic images and $\mathcal{E}$ is small category. If $\alpha : C \to X$ is the left $C_A$-approximation $X$, then it holds:

a) $C = \lim_{X' \to X} X'$, where $X' \to X$ runs over all $H_A$-equivalences.

b) $C = \colim_{X' \to X} X'$, where $X'$ runs over all $A$-colocal objects.

5. **Acts over Monoids**

We see a monoid $M$ as a category with one object whose endomorphism set is $M$. Thus we consider the category $[M^{op}, \text{Set}]$ of all contravariant functors from this category to the category of sets, and we call it the category of (right) acts over $M$, or simply $M$-acts. Clearly an $M$-act is a set $X$ together with an action $X \times M \to X$, $(x, m) \mapsto xm$ such that $(xm)m' = x(mm')$ and $x1 = x$ for all $x \in X$ and all $m, m' \in M$. Left acts are covariant functors $M \to \text{Set}$, that is sets $X$ together with an action $M \times X \to X$, satisfying the corresponding axioms. For the general theory of acts over monoids and undefined notions concerning this subject we refer to [6]. We should mention here that in contrast with [6] we allow the empty act to be an object in our category of acts, for the sake of (co)completeness. Note
that the category of $M$-acts is balanced and has epimorphic images, by [6, Proposition 1.6.15 and Theorem 1.4.21].

Fix a monoid $M$ and an object $A \in [M^{\text{op}}, \text{Set}]$. In order to use the results of the preceding Sections, we identify $A$ with a fully faithful functor $E \to [M^{\text{op}}, \text{Set}]$ where $E$ is the endomorphism monoid of $A$. Thus $A$ is canonically an $E - M$-biact (see [6, Definition 1.4.24]), so we obtain two functors

$$H_A : [M^{\text{op}}, \text{Set}] \to [E^{\text{op}}, \text{Set}], \quad H_A(X) = [A, X]$$

and

$$T_A : [E^{\text{op}}, \text{Set}] \to [M^{\text{op}}, \text{Set}], \quad T_A(Y) = Y \otimes_E A$$

the second one being the left adjoint of the first (see [6, Definition 2.5.1 and Proposition 2.5.19]). Clearly these functors agree with the functors defined at the beginning of the Section 3.

We say that $A$ is a (weak) $\ast$-act if the above adjoint pair induces mutually inverse equivalences $H_A : G_A \rightleftarrows G^A : T_A$ (respective $H_A : C_A \rightleftarrows G^A : T_A$). Note that our definitions for subcategories $G_A$ and $G^A$ agree with the characterizations of all $A$-generated respectively $A^\ast$-cogenerated modules given in [2, Lemma 2.1.2]. As we may see from Proposition 2.10, our subcategory $C_A$ seems to be the non–additive counterpart of the subcategory of all $A$-presented modules (compare with [2, Proposition 2.2.4]).

In what follows, we need more definitions relative to an $M$-act $A$. First $A$ is called decomposable if there exists two non empty subacts $B, C \subseteq A$ such that $A = B \cup C$ and $B \cap C = \emptyset$ (see [6, Definition 1.5.7]). In this case $A = B \sqcup C$, since coproducts in the category of acts is the disjoint union, by [6, Proposition 2.1.8]. If $A$ is not decomposable, then it is called indecomposable. Second, $A$ is say to be weak self–projective provided that $(H_A \circ T_A)(g)$ is an epimorphism whenever $g : U \to Y$ is an epimorphism in $[E^{\text{op}}, \text{Set}]$ with $U \in S^A$. More explicitly, if $g : U \to Y$ is an epimorphism in $[E^{\text{op}}, \text{Set}]$, then $T_A(g)$ is an epimorphism in $[M^{\text{op}}, \text{Set}]$ and our definition requires that $A$ is projective relative to such epimorphisms for which $U \in S^A$. Third $A$ is called (self–)small provided that the functor $H_A$ preserves coproducts (of copies of $A$).

**Lemma 5.1.** With the notations above, the following are equivalent:

(i) $A$ is small.

(ii) $A$ is self–small.

(iii) $E^{(I)}$ is $\eta$-reflexive for any set $I$, where $E^{(I)}$ denotes the coproduct indexed over $I$ of copies of $E$.

(iv) $E \sqcup E$ is $\eta$-reflexive.

(v) $A$ is indecomposable.

**Proof.** (i)$\Rightarrow$(ii) is obvious.
(ii)⇒(iii). If \( H_A \) commutes with coproducts of copies of \( A \) then

\[
E^{(I)} = \bigoplus_I [A, A] \cong \left[ A, \bigoplus_I A \right] \cong \left[ A, \bigoplus_I (E \otimes E A) \right] \\
\cong \left[ A, \left( E^{(I)} \right) \otimes_E A \right] \cong (H_A \circ T_A) \left( E^{(I)} \right).
\]

(iii)⇒(iv) is obvious.

(iii)⇒(iv). If \( A \) is decomposable, that is \( A = B \sqcup C \) with \( B \neq \emptyset \) and \( C \neq \emptyset \), then let \( i_B : B \to A \) and \( i_C : C \to A \) the canonical injections of this coproduct. Denote also by \( j_1, j_2 : A \to A \sqcup A \) the corresponding canonical injections. The homomorphisms of \( M \)-acts \( j_1 i_B : B \to A \sqcup A \) and \( j_2 i_C : C \to A \sqcup A \) induce a unique homomorphism \( f : A = B \sqcup C \to A \sqcup A \).

Obviously \( f \in (H_A \circ T_A)(E \sqcup E) \) but \( f \notin [A, A] \sqcup [A, A] = E \sqcup E \).

(iv)⇒(i) is [6, Lemma 1.5.37].

\[ \square \]

Proposition 5.2. The following statements hold:

a) If \( A \) is a weak \( * \)-act then \( A \) is weak self–projective.

b) If \( A \) is weak self–projective and indecomposable, then \( A \) is a weak \( * \)-act.

Proof. a) Let \( A \) be a weak \( * \)-act and let \( g : U \to Y \) be an epimorphism in \([E^{op}, Set]\) with \( U \in S^A \). We know by Proposition 2.10 that \( \eta_Y \) is epic, and by the naturalness of \( \eta \) that \((H_A \circ T_A)(g)\eta_U = \eta_Y g \). Since \( \eta_U \) is an isomorphism and \( \eta_Y g \) is an epimorphism we deduce that \((H_A \circ T_A)(g)\) is an epimorphism too.

b) As we have already noticed \( H_A \) preserves coproducts, provided that \( A \) is indecomposable. Thus \( S^A \) is closed under arbitrary coproducts in the category of \( E \)-acts. For a fixed \( Y \in [E^{op}, Set] \) there is an epimorphism \( g : E^{(I)} \to Y \). \( H \) is \( \eta \)-reflexive the same is also true for \( E^{(I)}(g) \). But \((H_A \circ T_A)(g)\) is an epimorphism, since \( A \) is weak self–projective. From the equality \((H_A \circ T_A)(g)\eta_{E^{(I)}} = \eta_Y g \) follows that \( \eta_Y \) is an epimorphism too.

The conclusion follows by Proposition 2.10.

\[ \square \]

Theorem 5.3. The following statements hold:

a) If \( A \) is a \( * \)-act then \( A \) is weak self–projective and \( C_A = \mathcal{G}_A \).

b) If \( A \) is indecomposable, weak self–projective and \( C_A = \mathcal{G}_A \), then \( A \) is a \( * \)-act.

Proof. The both implications follow at once from Proposition 5.2.

\[ \square \]

Remark 5.4. Propositions 2.10 and 5.2 and Theorems 2.11 and 5.3 provide a non additive version of [2, Proposition 2.2.4] respectively [2, Theorem 2.3.8].

In contrast with the case of modules, where the functors are additive, for acts it is not clear that a weak star object must me indecomposable (the non additive version of self–smallness as we may seen from Lemma 5.1).
main obstacle for deducing this implication in the new setting comes from the fact that non additive functors do not have to preserves finite coproducts.

Using the characterization of so called tilting modules given in [2, Theorem 2.4.5], we may define a tilting $M$-act to be a $*$-act $A$ such that the injective envelope of $M$ belongs to $\mathcal{G}_A$. Note that injective envelopes exist in $[\text{Set}^\text{op}, \text{Set}]$ by [6, Corollary 3.1.23]. As a subject for a future research we may ask ourselves which from the many beautiful results which are known for tilting modules do have correspondents for acts.

Our next aim is to infer from our results the Morita–type characterization of an equivalence between categories of acts (see [6, Section 5.3]). In order to perform it we need a couple of lemmas.

**Lemma 5.5.** If the $M$-act $A$ is a generator in $[\text{Set}^\text{op}, \text{Set}]$ then $C_A = \mathcal{G}_A = [\text{Set}^\text{op}, \text{Set}]$.

**Proof.** For a generator $A$ of $[\text{Set}^\text{op}, \text{Set}]$ the equality $\mathcal{G}_A = [\text{Set}^\text{op}, \text{Set}]$ follows by Lemma 4.1. Moreover $M$ is a retract of $A$ by [6, Theorem 2.3.16], therefore $M \in C_A$, since $C_A$ is closed under retracts. Thus a morphism $\epsilon : U \to V$ in $[\text{Set}^\text{op}, \text{Set}]$ is an $A$-equivalence if and only if it is an isomorphism, therefore $C_A = [\text{Set}^\text{op}, \text{Set}]$. $\square$

Recall that the left $E$-act $A$ is said to be pull back flat if the functor $T_A = (- \otimes_E A)$ commutes with pull backs (see [6, Definition 3.9.1]).

**Lemma 5.6.** If the right $M$-act $A$ is indecomposable, weak self projective and the left $E$-act $A$ is pull back flat, then $\mathcal{G}^A = [E^\text{op}, \text{Set}]$.

**Proof.** First observe that $A$ is a weak $*$-act by Proposition 5.2. Hence $\mathcal{G}^A = C^A = S^A$, and this subcategory has to be closed under subacts and limits. Moreover $E(I)$ is $\eta$-reflexive for any set $I$ according to Lemma 5.1. For a fixed $Y \in [E^\text{op}, \text{Set}]$ there is an epimorphism $g : E(I) \to Y$. Take the kernel pair of $g$, that is construct the pull back

$\begin{array}{ccc}
K & \xrightarrow{k_1} & E(I) \\
\downarrow{k_2} & & \downarrow{g} \\
E(I) & \xrightarrow{g} & Y
\end{array}$

The functors $T_A$ and $H_A$ preserve pull backs, the first one by hypothesis and the second one automatically. Moreover $K$ is a subact of $E(I) \times E(I)$ and the closure properties of $S^A$ imply $K \cong (T_A \circ H_A)(K)$. Applying the functor $H_A \circ T_A$ to the above diagram and having in the mind the previous
observations we obtain a pull back diagram

\[
\begin{array}{c}
K \\ k_2
\end{array}
\xrightarrow{k_1} E^{(I)} = (H_A \circ T_A) \left( E^{(I)} \right) \\
\downarrow \downarrow \downarrow \\
E^{(I)} = (H_A \circ T_A) \left( E^{(I)} \right) \xrightarrow{(H_A \circ T_A)(g)} (H_A \circ T_A) \left( Y \right)
\end{array}
\]

Note that \((H_A \circ T_A)(g)\) is an epimorphism by hypothesis. Then we know by [6, Theorem 2.2.44] that both \(g\) and \((H_A \circ T_A)(g)\) are coequalizers for the pair \((k_1, k_2)\). Thus we deduce \(Y \cong (H_A \circ T_A)(Y)\) canonically, so \(Y \in S^A\). Thus \(G^A = S^A = [E^{op}, Set]\). □

Now we are in position to prove the desired Morita–type result:

**Theorem 5.7.** Let \(M\) and \(E\) be two monoids. Then the categories \([M^{op}, Set]\) and \([E^{op}, Set]\) are equivalent via the mutually inverse equivalence functors \(H\) and \(T\) if and only if there is a cyclic, projective generator \(A\) of \([M^{op}, Set]\) such that \(E\) is the endomorphism monoid of \(A\), case in which \(H = H_A\) and \(T = T_A\).

**Proof.** First note that a projective act is indecomposable if and only if it is cyclic in virtue of [6, Propositions 1.5.8 and 3.17.7]. If \(H : [M^{op}, Set] \xrightarrow{\cong} [E^{op}, Set] : T\) are mutually inverse equivalences, then \(H \cong H_A\) and \(T \cong T_A\), where \(A = T(E)\) according to Theorem 3.2. Moreover the endomorphism monoid of \(A\) is \(E\), and \(A\) has to be projective, indecomposable and generator together with \(E\).

Conversely if \(A\) is indecomposable and projective in \([M^{op}, Set]\) then it is a weak \(*\)-act by Proposition 5.2. Since \(A\) is in addition a generator, Lemma 5.5 tells us that \(A\) is a \(*\)-act and \(C_A = G_A = [M^{op}, Set]\) and Theorem 5.3 implies that \(A\) is a \(*\)-act. Finally the left \(E\)-act \(A\) is projective by [6, Corollary 3.18.17], so it is strongly flat by [6, Proposition 3.15.5], that means \(T_A\) commutes both with pull backs and equalizers. Thus \(G^A = [E^{op}, Set]\), according to Lemma 5.6. □

**References**

[1] F. Castaño Iglesias, J. Gómez–Torrecillas, R. Wisbauer, Adjoint functors and equivalences, *Bull. sci. math.* 127 (2003), 379–395.

[2] R. Colpi, K. Fuller, *Equivalence and Duality for Module Categories*, Cambridge University Press, Cambridge, 2004.

[3] R. Colpi, K. Fuller, Tilting objects in abelian categories and quasitilted rings, *Trans. Amer. Math. Soc.*, 359 (2007), 741–765.

[4] E. Dror Farjoun, R. Göbel, Y. Segev, Cellular cover of groups, *J. Pure Appl. Algebra*, 208 (2007), 61–76.

[5] W. Dwyer, Localizations in *Axiomatic, Enriched and Motivic Homotopy Theory*, Proceedings of the NATO ASI (ed. J.P.C. Greenlees), Kluwer, 2004.

[6] M. Kilp, U. Knauer, A. Mikhalev, *Monoids, Acts and Categories* De Gruyter Expositions in Mathematics, 29, Walter de Gruyter, Berlin, New–York, 2000.
[7] C. Menini, A. Orsatti, Representable equivalences between categories of modules and applications, *Rend. Sem. Math. Univ. Padova*, 82 (1989), 203–231.
[8] B. Mitchell, *Theory of Categories*, Academic Press, New York and London, 1965.
[9] C. Modoi, Equivalences induced by adjoint functors, *Comm. Alg.* 31(5) (2003), 2327–2355.
[10] N. Popescu, L. Popescu, *Theory of Categories*, Editura Academiei, București and Sijthoff & Noordhoff International Publishers, 1979.

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