Kurosh rank of intersections of subgroups of free products of orderable groups

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Abstract

We prove that the Kurosh rank of the intersection of two subgroups $H$ and $K$ of a free product of orderable groups is bounded above by the product of the Kurosh ranks of $H$ and $K$.

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1 Introduction

Let $H$ and $K$ be subgroups of a free group $F$, and $\tau(H) = \max\{0, \text{rank}(H) - 1\}$. It was an open problem dating back to the 1950’s to find a bound of the rank of $H \cap K$ in terms of the ranks of $H$ and $K$.

In [8], Hanna Neumann proved the following

$$\tau(H \cap K) \leq 2 \cdot \tau(H) \cdot \tau(K).$$

(1)

The Hanna Neumann Conjecture says the (1) holds replacing the 2 by a 1. Later, in [9], Walter Neumann improved this to

$$\sum_{g \in K \setminus F/H} \tau(H^g \cap K) \leq 2 \cdot \tau(H) \cdot \tau(K),$$

(2)

where $H^g = g^{-1}Hg$. The Strengthened Hanna Neumann Conjecture, introduced by Walter Neumann, says that (2) holds replacing the 2 by a 1.

These two conjectures have received a lot of attention, and recently, Igor Mineyev [6] proved that both conjectures are true. The fact that $F$ is orderable plays a crucial role in Mineyev’s proof.

1.1 Definition. A group is orderable if it admits a total order which is invariant under the right multiplication action. In particular, orderability is inherited by subgroups and implies torsion freeness.

It is natural to consider to what extent this result can be generalised to intersections of subgroups of free products. In this case, it turns out that the appropriate concept to consider is the Kurosh rank. This problem has already been studied in the case of general free products (see [5], for example). It is also worthwhile mentioning that S. Ivanov already established in [4] a version
of the Hanna Neumann bound for intersection of factor free subgroups in a free product of orderable groups.

We recall the well known Kurosh Subgroup Theorem, whose proof can be found in, for example, [2].

1.2 Theorem (Kurosh Subgroup Theorem). Let $G = *_{i \in I} A_i$ be a free product and $H$ a subgroup of $G$. Then

$$
H = *(H \cap A_i^g) * F,
$$

where the $g$ ranges over a set of double coset representatives in $A_i \setminus G/H$ for each $i \in I$ and $F$ is a free group.

In view of the theorem, one would like to define the Kurosh rank of $H$ with respect to the free product $*_{i \in I} A_i$ as the number of non-trivial factors $(H \cap A_i^g)$ in (3) plus the rank of $F$. One has to prove that this number is independent of the double coset representatives. This is done in [5, Lemma 3.4].

However we prefer to give a different definition based on groups acting on trees. Our notation and basic reference for groups acting on trees is [2].

1.3 Definition (Kurosh rank). Let $G$ be a group and $T$ a $G$-tree with trivial edge stabilisers and $H$ a subgroup of $G$.

Let $c(H) \in \mathbb{N} \cup \{\infty\}$ be the number of vertices $vH \in VT/H$ such that $v$ has a non-trivial $H$-stabiliser. This is well defined since it is independent of the choice of the representative of $vH$.

The Kurosh rank of $H$ with respect to $T$ is defined to be

$$
\kappa_T(H) := c(H) + \text{rank}(T/H),
$$

where the $\text{rank}(T/H)$, is the rank of fundamental group of the graph $T/H$ (which is a free group).

The reduced Kurosh rank of $H$ with respect to $T$ is defined to be

$$
\overline{\kappa}_T(H) := \max\{0, \kappa_T(H) - 1\}.
$$

1.4 Remark. We note that the Kurosh rank of a subgroup depends on the free product decomposition and not just the isomorphism type of the subgroup. Therefore, suppose we took the free product of two surface groups, $S_1, S_2$. Then a free subgroup of infinite rank inside $S_1$ would have Kurosh rank 1, whereas a free subgroup of infinite rank meeting no conjugate of either $S_1$ or $S_2$ would have infinite Kurosh rank.

Our main result is the following

1.5 Theorem A. Let $G$ be an orderable group and $T$ a $G$-tree with trivial edge stabilisers. Let $H, K$ be subgroups of $G$. Then

$$
\sum_{g \in K \setminus G/H} \overline{\kappa}_T(H^g \cap K) \leq \overline{\kappa}_T(H) \overline{\kappa}_T(K).
$$

1.6 Remark. The main interest is when all these quantities are finite, but it is also true when they are infinite, in which case we adopt the convention that $0.\infty = 0$. 

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Our proof of Theorem A, is basically the same as the simplified version of Mineyev’s proof due to Warren Dicks [3]. Dicks explained how one could convert Mineyev’s brilliant Hilbert-module proof into a short Bass-Serre-theory proof, which Mineyev did in [7].

We note that the main contribution of this paper is the proof of Theorem 2.7, which says that the reduced Kurosh rank is equal to the number of orbits of edges in the Dicks tree.

To obtain examples of groups and trees for Theorem A, we can consider $G$ to be a graph of groups with orderable vertex groups and trivial edge groups, and $T$ the corresponding Bass-Serre tree. In this case, the group is a free product of orderable groups. By the Kurosh Subgroup Theorem, any subgroup of $G$ is a free product of orderable groups, and hence by the main result of [1], the group $G$ is itself orderable.

In particular, taking $G$ the fundamental group of a graph of groups with trivial vertex and edge groups, and $T$ its Bass-Serre tree, Theorem A becomes the desired inequality of the Strengthened Hanna Neumann conjecture.

### 2 Main Argument

Throughout this section $G$ will be an orderable group and $T$ a $G$-tree with trivial edge stabilisers. $G$ will act on $T$ on the right.

An element $g \in G$ is called elliptic if it fixes a point in $T$ and is called hyperbolic if it does not.

Given a hyperbolic $g \in G$, the axis of $g$, denoted $A_g$, consists of the subtree of points displaced by the minimal amount by $g$ (with respect to the path metric). This is always non-empty and homeomorphic to the real line.

Associated to any non-trivial subgroup (not necessarily finitely generated), $H \leq G$ there is a minimal $H$-invariant subtree $T_H$ of $T$. In general, this will be the union of the axes of hyperbolic elements of $H$ except when every element of $H$ is elliptic. In this case, a result of Serre says that any finite set of elements of $H$ have a common fixed point and therefore, as edge stabilisers are trivial, there will be a unique point for the whole of $H$, in which case $T_H$ will be the fixed vertex for $H$.

#### 2.1 Proposition. Let $H$ be a subgroup of $G$. Let $T'$ be a $H$-subtree of $T$. Then $\kappa_{T'}(H) = \kappa_T(H)$.

In particular, $T_H/H$ is finite (as a graph) if and only if $\kappa_T(H)$ is finite.

**Proof.** This is an easy exercise in Bass-Serre theory, bearing in mind that $T_H \subset T'$ and $T_H$ has no proper $H$-invariant subtree. ☐

#### 2.2 Remark. If, in Theorem A, either $\kappa_T(H)$ or $\kappa_T(K)$ is equal to infinity, then the theorem holds. So without loss of generality we can assume that $\kappa_T(H)$ and $\kappa_T(K)$ are finite. Hence $\kappa_T(H \cup K)$ is also finite, and in view of Proposition 2.1 we can change $G$ by $\langle H \cup K \rangle$, $T$ by $T_{H \cup K}$ and hence we can assume that $T/G$ is finite.

Throughout the rest of the section $T/G$ will be a finite graph.

We fix an order $<$ of $G$, and we use it to construct an order on the edges of $T$. We first put any order on $ET/G$, which is a finite set, and we denote it again by $<$. Then, we order lexicographically $(ET/G) \times G$, and use the natural
bijection of this set with $ET$ to order it. That is $(eG, g) \leq (fG, h)$ if and only if $eG < fG$ or $eG = fG$ and $g \leq h$. This ordering on the edges of $T$ is invariant under the action of $G$. We henceforth fix this ordering.

2.3 Definition (Dick’s Trees). Let $T'$ be a subtree of $T$.

An edge $e$ of $T'$ is a $T'$-bridge if there is a reduced bi-infinite path in $T'$, containing $e$ and in which $e$ is the $<$-largest edge.

For any subgroup $H$ of $G$, we call an edge an $H$-bridge if it is a $T_H$-bridge. Note that $H$ acts freely on the set of $H$-bridges.

Note that if $T_0 \subseteq T_1$ are subtrees of $T$ then any $T_0$-bridge is a $T_1$-bridge. Hence, if $H \leq K$ are subgroups of $G$, then any $H$-bridge is also a $K$-bridge.

An $H$-island is a component of $T_H$ after all the $H$-bridges have been removed. Note that $H$ acts on the set of $H$-islands.

The Dicks $H$-tree, $T_H$, is the $H$-tree whose vertices are the $H$-islands and whose edges are the $H$-bridges. Note that all the edge stabilisers in $T_H$ are trivial.

2.4 Proposition. Let $1 \neq H$ be a subgroup of $G$ with $\kappa_T(H) < \infty$. Suppose that for every $H$-island $I$ in $T_H$, the $H$-stabiliser $H_I$ has $\kappa_T(H_I) = 1$. Then $\pi_T(H)$ is equal to $|ET_H/H|$, the number of orbits of edges in Dicks $H$-tree.

Proof. The action of $H$ on $T_H$ induces a free product decomposition of $H = H_1 \ast \ldots \ast H_k \ast F$, where the $H_i$ are vertex groups and $F$ is a free group, acting freely on $T_H$. Then $\pi_T(H) = k + \text{rank}(F) - 1$.

Similarly, the action of $H$ on $T_H$ also induces a free product decomposition $H = B_1 \ast \ldots \ast B_n \ast K$, where the $B_i$ are the vertex groups, that is, stabilisers of $H$-islands, and $K$ is a group acting freely on $T_H$. Since every vertex stabiliser in $T_H$ has Kurosh rank one, there are exactly $n$ vertices in $T_H/H$.

Recall that the rank of a graph is the rank of the fundamental group and for a finite graph is equal to the number of edges minus the number of vertices plus 1. Then $T_H/H$ is a finite graph of $n$ vertices and whose rank is equal to $\text{rank}(K)$. Therefore, the number of edges of $T_H/H$ is equal to $n + \text{rank}(K) - 1$.

By hypothesis, $\kappa_T(B_i) = 1$ for $i = 1, \ldots, n$. Then, for $i = 1, \ldots, n$, either $B_i$ is a subgroup of a $G$-stabilisers of a vertex, or $B_i$ is an infinite cyclic group acting freely on $T_H$. After rearranging the indexes, we can assume that $B_1, \ldots, B_r$ fix a vertex of $T_H$ and $B_{r+1}, \ldots, B_n$ are infinite cyclic groups acting freely in $T_H$. Thus $B_1 \ast \ldots \ast B_n \ast K$ is a free group of rank $n - r + \text{rank}(K)$.

We claim that $k = r$ and $\langle \langle H_1, \ldots, H_k \rangle \rangle = \langle \langle B_1, \ldots, B_r \rangle \rangle$.

Since $H_i$ fixes a vertex in $T_H$, it stabilises the island containing that vertex. This means that $H_i$ is contained in a $H$-conjugate of $B_{i(j)}$ for some $1 \leq j(i) \leq r$. Since the edge stabilisers in $T_H$ are trivial, $H_i$ can not fix more one vertex. This means that $k \leq r$ and that $\langle \langle H_1, \ldots, H_k \rangle \rangle \leq \langle \langle B_1, \ldots, B_r \rangle \rangle$.

Conversely, note that if $B_j$ fixes a vertex in $T_H$, then $B_j$ is contained in a $H$-conjugate of $H_{i(j)}$ for some $1 \leq i(j) \leq r$. As the Kurosh rank of each $B_j$ is one, there is at most one fixed vertex in each $H$-island, hence $B_j$ can not be contained in conjugates of two different $H_i$. This means that $k \geq r$ and that $\langle \langle H_1, \ldots, H_k \rangle \rangle \geq \langle \langle B_1, \ldots, B_r \rangle \rangle$.

This completes the proof of the claim. Therefore,

$$F \simeq \frac{H}{\langle \langle H_1, \ldots, H_k \rangle \rangle} = \frac{H}{\langle \langle B_1, \ldots, B_r \rangle \rangle} \simeq B_{r+1} \ast \ldots \ast B_n \ast K.$$
Hence the $\pi_T(H) = k + \text{rank}(F) - 1 = r + (n - r) + \text{rank}(K) - 1$. \hfill \Box

Therefore the goal will be to show that for the tree $T_H$, the stabiliser of any vertex has Kurosh rank 1 (to obtain our main result Theorem 2.7). Equivalently, we need to show that stabilisers of $H$-islands have Kurosh rank less than 2 (Proposition 2.5) and are non-trivial (Proposition 2.6).

2.5 Proposition. Suppose that $H \leq G$ and that $\kappa_T(H) \geq 2$. Then there is a $H$-bridge in $T_H$.

Proof. If $H$ fixes a vertex, $T_H$ will simply be this fixed vertex and $\kappa(H) \leq 1$, contradicting the hypothesis.

Now suppose that $T_H$ is a single line. If some $h \in H$ fixes a vertex, then $h^2$ fixes $T_H$ and hence $h^2 = 1$. This is impossible since $H$ is orderable, and hence torsion-free. Thus $H$ acts freely on $T_H$, i.e. $H \cong \mathbb{Z}$, so has Kurosh rank 1, again contradicting the hypothesis.

Hence there exist two hyperbolic elements $g, h \in H$ with distinct axes, $A_g \neq A_h$. These two might intersect non-trivially, but they can only intersect in finitely many edges. If they did intersect in an infinite ray, then the commutator $ghg^{-1}h^{-1}$ would fix an infinite subray and hence an edge. However, the action is free on the edge set, and hence the commutator would be the trivial element, implying that $g$ and $h$ commute and therefore have the same axis, contradicting our choice of $g$ and $h$.

Now choose a vertex $v \in A_g$ and let $p$ denote a path from $vg^{-1}$ to $vg$ with <largest edge $e$. By replacing $g$ with $g^{-1}$ if necessary, we may assume that $e > eg$. It follows that $e > eg > eg^2 > \ldots > eg^n \ldots$. Therefore $e$ is the largest edge in the infinite ray starting with $e$ and continuing in the positive direction of the axis. (Note that the action of an hyperbolic element $g$ on its axis $A_g$ induces an orientation of $A_g$ with respect to which $g$ translates in the positive direction.) Likewise $eg^n$ is the largest edge in the infinite ray starting with $eg^n$.

It follows that every infinite ray $p_\infty$ in $A_g$ starting at any vertex of $A_g$ and going in the positive direction has a <largest edge. Similarly, every infinite ray $r_\infty$ in $A_h$ starting at any vertex of $A_h$ and going in the positive direction has a <largest edge.

Thus there will exist a reduced bi-infinite path of the form $p_\infty^{-1} \cdot q \cdot r_\infty$ where $q$ is a finite path from $A_g$ to $A_h$; in the case where $A_g$ and $A_h$ are disjoint, $q$ is the path from one axis to the other, and in the case where they intersect, $q$ is a subpath of the intersection, possibly a single vertex. In either case, $p_\infty^{-1} \cdot q \cdot r_\infty$ has a <largest edge which is then a $H$-bridge in $T_H$. \hfill \Box

2.6 Proposition. Let $H$ be a non-trivial subgroup of $G$ with $\kappa_T(H) < \infty$ and let $I$ be an $H$-island in $T_H$ with stabiliser $H_I$. Then $H_I$ is non-trivial.

Proof. If there is no $H$-bridge, then $H_I = H$ and, since $H$ is non-trivial, $H_I$ is non-trivial.

Consider the set $\{ e : e$ is a bridge whose initial point is in $I \}$. Note that $T_H/H$ is a finite graph, therefore if the set above is infinite, it must contain edges $e$ and $eh$ for some $1 \neq h \in H$. Clearly, by looking at initial points, $h$ is a non-trivial element preserving $I$ and we would have that $H_I$ would be non-trivial.
Therefore, we may assume that the set above is finite and list the elements in order, $e_1 < e_2 < \ldots < e_s$.

Then $e_1$ is the largest edge in a reduced bi-infinite path in $T_H$. The tail of this path is a ray whose initial vertex is in $I$ and hence the entire ray must remain in $I$ due to the minimality of $e_1$. Therefore, again as $T_H/H$ is finite, there are two distinct edges of $I$ in the same $H$ orbit, and therefore $H_I$ is non-trivial (and note that this is really a contradiction, since it implies that the set above is infinite unless $T_H$ is a single vertex).

We now summarise the three propositions in one Theorem.

2.7 Theorem. Let $G$ be an orderable group and $T$ an $G$-tree with trivial edge stabilisers and finitely many orbits of edges. Let $H$ be a subgroup of $G$. Then $\pi_T(H)$ is equal to $|ET_H/H|$, the number of orbits of edges in Dicks $H$-tree.

Proof. If $H$ is the trivial group, then the theorem holds. So we assume $H$ non-trivial. If $\kappa_T(H) = \infty$, then $T_H/H$ is infinite and from Proposition 2.4 $T_H/H$ is also infinite, and the theorem holds. So we assume $\kappa_T(H) < \infty$. For every $H$-island $I$, by Proposition 2.6 its $H$-stabiliser $H_I$ is non-trivial. By Proposition 2.5 if $\kappa(H_I) \geq 2$ then there would be a $T_H$-bridge, which would imply the existence of an $H$-bridge in $I$, contradicting the definition of $I$. Then $\kappa(H_I) = 1$. The theorem now follows from Proposition 2.4.

Proof of Theorem A. If either $H$ or $K$ is trivial, the theorem trivially holds. Hence, we assume that $H$ and $K$ are non-trivial.

By Remark 2.2 the theorem also holds when the Kurosh rank of $H$ or $K$ is infinite. Therefore, again by Remark 2.2, we can assume that $T/G$, $T_H/H$ and $T_K/K$ are finite.

The inclusions $T_{(H \cap K)} \subseteq T_{H^g} = (T_H)^g$ and $T_{(H \cap K)} \subseteq T_K$ induce a (diagonal) graph map, sending $e(H^g \cap K)$ to $(e^{-1}H, eK)$. Therefore we get a map,

$$\bigcup_{g \in K \setminus G/H} T_{(H \cap K)}/(H^g \cap K) \to T_H/H \times T_K/K$$

which is injective on edges as edge stabilisers are trivial and the union is over distinct double coset representatives.

In turn, this induces a graph map

$$\bigcup_{g \in K \setminus G/H} T_{(H \cap K)}/(H^g \cap K) \to T_H/H \times T_K/K$$

(4)

which is also injective on edges.

By Theorem 2.7 we know that the number of edges in $T_H/H$ is equal to $\pi_T(H)$. And similarly for $K$ and $H^g \cap K$ whenever it is non-trivial.

Therefore, the injectivity of the map (4) on edges, gives us the result.

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