THE QUOTIENT OF NORMAL RANDOM VARIABLES AND APPLICATION TO ASSET PRICE FAT TAILS

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Abstract. The quotient of random variables with normal distributions is examined and proven to have power law decay, with density $f(x) \approx f_0 x^{-2}$, with the coefficient depending on the means and variances of the numerator and denominator and their correlation. We also obtain the conditional probability densities for each of the four quadrants given by the signs of the numerator and denominator for arbitrary correlation $\rho \in [-1, 1)$. For $\rho = -1$ we obtain a particularly simple closed form solution for all $x \in \mathbb{R}$. The results are applied to a basic issue in economics and finance, namely the density of relative price changes. Classical finance stipulates a normal distribution of relative price changes, though empirical studies suggest a power law at the tail end. By considering the supply and demand in a basic price change model, we prove that the relative price change has density that decays with an $x^{-2}$ power law. Various parameter limits are established.

1. Introduction

A long-standing puzzle in economics and finance has been the "fat tails" phenomenon in relative asset price change and other observations that refer to the empirically observed power-law decay rather than the expected classical exponential. In practical terms, this means unusual events are less rare than expected, resulting in broad implications as discussed below. Given the quite general application of the Central Limit Theorem, it is natural to expect that the frequency of the relative price change as a function of the relative price change would result in a normal, or Gaussian, distribution with exponential decay, i.e., the density has the same tail as $f(x) = (2\pi \sigma^2)^{-1/2} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$ where $\mu$ is the mean and $\sigma^2$ is the variance.

The assumption of a normal distribution for relative price changes dates back to Bachelier’s thesis [2] in 1900, and was further popularized in the Black-Scholes work on options pricing [4]. Champagnat et. al. [9] note several reasons for the saliency of utilizing log-normal price changes. First, they can be "simply interpreted and estimated. Second, closed-form expression exists for several options. Third, they could be embedded in a continuous time process, as the geometric Brownian motion, which models the evolution of the stock over the time. Indeed, many theories, for example the Capital Asset Pricing Model (CAPM) for portfolio management, take their roots in the Gaussian world."

Received by the editors September 28, 2017.

Key words and phrases. Quotient of Normals, Heavy Tails, Fat Tails, Supply/Demand, Tail of Distribution, Leptokurtosis, Asset Price Change, Returns on Stocks.

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One particular application involves "Value-at-Risk" which addresses questions such as: Can we be sure that the investment will retain at least, say, 75% of its current value within a five year time period with a 95% confidence? This methodology is often at the heart of risk analysis of an investment portfolio. The Gaussian assumption facilitates calculations; however, there have been numerous studies that indicate the risk is understated as a result [9], [30], [37]. An early study by Fama [13] provided empirical evidence that there were ten times as many observations of relative price changes than would be expected at four standard deviations from the classical theories stipulating normal distributions. Mandelbrot and Hudson [24] express their perspective in a section heading: "Markets are very, very risky – more risky than the standard theories imagine." In a more generalized context, Taleb [33] has long asserted that unusual events occur far more often than one would expect from the normal distribution. Another aspect of work in this area has involved modeling [23], [24]. The empirical observations of fat tails have also been noted in high frequency trading [10].

Thus, the question of the tail of the distribution of relative price changes is important in several key areas of finance and economics. As a practical matter it would appear that one can investigate empirically, and implement the conclusions without reference to a particular model. While data seems to be abundant, the problem is that obtaining a large amount of it often entails using data from older time periods that may well be irrelevant. Hence, the theoretical examination of the origins of the tail of the distribution becomes crucial. Various explanations have been offered for the observation of fat tails. These differ from our approach in that they stipulate an exogenous influence that alters the natural normal decay. For example, it has been argued that large institutional investors placing trades in less liquid markets tend to cause large spikes [14]. Theoretical models, e.g., [24], using random walk have been used to explain fat tails. We refer to the book by Kemp [21] for a review of the literature. While these may be important factors that lead to fat tails, we demonstrate below that fat tails also arise from the endogenous price formation process.

In the classical approach to finance the basic starting point is the stochastic differential equation for the relative price change, $P^{-1}dP = d\log P$ as a function of time, $t$, and $\omega \in \Omega$ (with $\Omega$ as the sample space):

$$d\log P = \mu dt + \sigma dW.$$ (1.1)

Here $W$ is the standard Brownian motion, so $\Delta W := W(t) - W(t - \Delta t) \sim \mathcal{N}(0, \Delta t)$, i.e., $W$ is normal with the variance as $\Delta t$ and mean 0, and has independent increments. The stochastic differential equation above is shorthand for the integral form (suppressing $\omega$ in notation)

$$\log P(t_2) - \log P(t_1) = \int_{t_1}^{t_2} \mu dt + \int_{t_1}^{t_2} \sigma dW$$ (1.2)

where $\mu$ and $\sigma$ can be constant, functions of $t$, or random variables. For $\mu, \sigma$ constant, and $\Delta t := t_2 - t_1$, one can write

$$\Delta \log P := \log P(t_2) - \log P(t_1) = \mu \Delta t + \sigma \Delta W.$$ (1.3)

With the assumption that $\sigma$ is nearly constant over time, classical finance clearly stipulates that the relative temporal changes in asset prices should be normal. The basic equation (1.1) is obtained from the idea that all information about the asset
is incorporated into the price, and that random world events alter the value on each time interval. Further, the assumption of the existence of a vast arbitrage capital means that the changes in the valuation are immediately reflected in the price, notwithstanding any bias or mistake on the part of the less knowledgeable investors. Consistent with the Central Limit Theorem, it is assumed that the events that alter the asset valuation are normally distributed. But the important and tacit second assumption is that relative price changes inherit this property.

While (1.1) is the basis for a large majority of papers on asset prices and related issues in finance, it is difficult to generalize in some directions. Indeed, with the assumption of infinite arbitrage already built into the model, what can one subtract from infinity in order to obtain the randomness that arises from the finiteness of trader assets and order flow? What is needed then is an approach that takes into account more of the microstructure of trading, i.e., the supply and demand of the asset submitted to the market clearinghouse (see e.g., [5], [27], [29] and references therein).

In this paper we examine the temporal evolution in percentage price changes by modeling price change that utilizes a fundamental approach of supply/demand economics analysis. We show that if one assumes supply and demand are normally distributed, the mathematics of the quotient of normals suffices to yield fat tails. In particular, in the limit of large deviations from the mean, one obtains the result \( f(x) \approx f_0 x^{-2} \) for the density for large \( x \), where the constant \( f_0 \) depends on the means, variances and correlations of supply and demand.

Thus modeling of the relative price change in terms of finite supply and demand lead naturally to the basic statistical problem of the distribution of the quotient of two normal random variables. While there is a long history of the problem, surveyed below, most results concern the mid-range of the distribution, rather than the tail. We obtain a number of exact representations and rigorous bounds on the density conditioned upon the signs of the numerator and denominator, as well as the overall density for all correlations \( \rho \) such that \(|\rho| < 1\). For \( \rho = -1 \) we obtain a particularly simple exact expression for the density for all \( x \in \mathbb{R} \).

### 2. The Model and the Quotient of Normals

We focus on modeling price change in economics; the issues are similar in other disciplines in which quotients arise. Classical economics stipulates that prices remain constant in time when supply and demand are equal [35], [20], thereby defining equilibrium. When demand exceeds supply, prices rise to restore equilibrium, and analogously in the other direction. There is theoretical [5], experimental [7] and empirical [6] evidence that asset price change can be modeled as basic goods, with the supply and demand depending on a number of factors such as the cash/asset ratios.

Let \( S(t; \omega) \) and \( D(t; \omega) \) be the supply and demand, respectively. The most general expression for relative price change (see [5] and references, and [17] p. 165 for motivation for the linear equation below) can be written as

\[
(2.1) \quad \tau \frac{1}{P} \frac{dP}{dt} = g\left( \frac{D - S}{S} \right),
\]

where \( \tau \) is the time scale, \( P(t; \omega) \) is the unit price of the asset, and \( g \) is a differentiable function such that \( g(0) = 0 \), reflecting the assumption that prices do...
not change when demand equals supply, and \( g'(0) = c > 0 \), for some positive constant \( c \), which stipulates that prices rise when demand exceeds supply. If trading is very active, prices will react to small changes in supply/demand imbalances, and the function \( g \) can be assumed to be linear. Also, the constants \( \tau \) and \( c \) can be incorporated into a dimensionless time, yielding our basic starting point:

\[
\frac{1}{P(t; \omega)} \frac{d}{dt} P(t; \omega) = \frac{D(t; \omega)}{S(t; \omega)} - 1.
\]

Rather than considering the randomness directly in terms of prices, as in the classical approach, we assume that supply and demand are random variables. In a liquid market (i.e., frequently traded), the randomly flowing orders for supply and demand can be approximated by normal distributions (as discussed below). However, supply and demand are not likely to be independent, as a random event that increases buying is likely to diminish selling. In general, we can assume that \( D \) and \( S \) are bivariate normal random variables. This leads to the question of estimating the tail of a distribution of (2.2) above, i.e., the quotient of bivariate normal random variables. Our particular interest is for negatively correlated \( D \) and \( S \) but the analysis below will be for the full range of correlations, as similar issues arise in other problems, e.g., physical, biological, in which there is a quotient or random variables.

The supply and demand on a given interval consists of buy and sell orders submitted to the market with some random distribution. The orders will be influenced by news which will arrive from many independent sources, so that the Central Limit Theorem will apply. That is, under a broad set of conditions, the arrival of random orders, and thus, supply and demand, will be approximated well by the normal distribution. However, price formation evolves through a process that is almost deterministic. In other words, if one had a large sample of the same supply/demand graphs, the resulting relative price change would exhibit only a small variance as market makers and short term traders can readily see the very short term market direction. This is a basic consequence of economic game theory (see [35], [29] and references therein). In more practical terms, if we consider the stock of a major company that trades with high volume, there will be a large number of market makers whose only business is to profit from any deviations from the "correct" price, given the order flow. Given a particular supply/demand graph, a significant deviation can only arise from, not one, but many of these market makers erring in the same direction. In summary, the randomness is inherently in the supply and demand curves. For fixed supply/demand functions, the price evolution is nearly deterministic.

The empirical assumption that relative price changes are normal (often stated as price change is log-normal) has been tested, with results that indicate significant deviations from normality (see e.g., [13], [9], [36]). While there are numerous studies on the distribution of trading prices, empirical studies testing the normality of buy/sell orders for major stocks and commodities would be instrumental in understanding the problem of fat-tails in relative price change.

We assume conditions on the orders that are compatible with the Central Limit Theorem, so that, for \( n \) large, the supply and demand are governed by a bivariate normal distribution. Let \( X_1 := D \), \( X_2 := S \) and

\[
R := \frac{X_1}{X_2} - 1.
\]
We assume that \( \vec{X} := (X_1, X_2)^T \) constitutes a bivariate normal distribution having density, for \( \vec{s} \in \mathbb{R}^2 \),

\[
(2.4) \quad f(\vec{s}; \vec{\mu}, \Sigma) := (2\pi)^{-1} |\Sigma|^{-1/2} e^{-\frac{1}{2}(\vec{s} - \vec{\mu})^T \Sigma^{-1} (\vec{s} - \vec{\mu})},
\]

where \( \vec{\mu} := (\mu_1, \mu_2) \) and \( \Sigma \) is the covariance matrix,

\[
(2.5) \quad \Sigma := \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}
\]

with \( \sigma_{12} := E[X_1X_2] \) and \( \sigma_i^2 = E[X_i^2] \) for \( i = 1, 2 \). One has the basic bound \(|\sigma_{12}| < \sigma_1\sigma_2\). Upon defining \( \rho := \sigma_{12} / (\sigma_1\sigma_2) \) one has the result that \( \Sigma^{-1} \) exists if and only if \(|\rho| < 1 \). Note that \( \vec{X} \) is said to have a singular bivariate normal distribution if there exists real numbers \( \mu_a, \mu_b, \sigma_a, \sigma_b \) such that \( \vec{X} \) and \( (\sigma_a Y + \mu_a, \sigma_b Y + \mu_b)^T \) are identically distributed, with \( Y \sim N(0, 1) \). This is equivalent to \(|\Sigma| = 0 \). We will consider the \(|\rho| = 1 \) case later and assume for now that \( (2.5) \) is nonsingular.

An excellent source for relations on multivariate normal distributions is Tong [31].

We let \( f_{X_1/X_2}(x) \) be the density, \( F_{X_1/X_2}(x) \), the (cumulative) distribution function for the variable \( R \). While the random variables \( X_1 \) and \( X_2 \) can take on any values in \( \mathbb{R} \), the primary interest is in positive values which we consider below, with similar results for the remaining quadrants of \( \mathbb{R}^2 \) presented in Appendix B.

The issue of the quotient of normal variables has been studied in a number of contexts, as it arises in a number of biological and physical problems including constructing the genome mapping of plants, imaging ventilation with inert fluorinated gases, and various neurological applications [28]. A classical problem is to estimate parameters (e.g., mean and variance) of the ratio of two populations. An early result by Geary [16] concerned the ratio of two independent normals with zero means. Hinkley [18, 19] obtained a result in the limit as the variance of the denominator approached zero. Kueth [22] and Marsaglia [26] developed complex expressions for the density of the ratio of two independent normals with strictly positive means and variances.

Diaz-Frances and Rubio [11] obtained an important result by proving a theorem that establishes bounds on the difference between the true distribution and the proposed approximation. They summarize a number of results and earlier works, noting that the ratio of independent normals with positive means has no finite moments, and that the shape of the distribution of the quotient "can be bimodal, asymmetric, symmetric, and even close to a normal distribution, depending largely on the values of the coefficient of variation of the denominator."

Much of the recent focus has been on approximating the ratio of the means by a normal distribution; see Diaz-Frances and Rubio [11] and Diaz-Frances and Sprott [12] for a discussion and references, and also Watson [32], Palomino et al. [28], Schneeweiss [30], and Chamberlin and Sprott [8]. For the most part the results involve the mid-range, where a normal approximation is possible, rather than the tail.

The main focus of our paper will be on the tail of the distribution. Although the terminology is not yet standard, one can broadly define "heavy tails" or leptokurtosis as distributions with falloff less rapid than the normal, while "fat tails" consist of power-laws.

While there has been some evidence that the ratio of independent normals, under some conditions on the parameters, will be fat-tailed, there has not
been a comprehensive understanding and analysis of the behavior of the tales of the
distribution. In Section 4 we prove that the ratio of two normals, \( R := \frac{X_1}{X_2} \) with
arbitrary means, variances and correlations \( \rho \in [-1, 1] \) has the fat-tail property,
with the density, \( f_{X_1/X_2}(x) \) approaching zero as \( x^2 \). We also prove bounds on the
coefficient for large \(|x|\), providing a rigorous description of the tail of the quotient
of normals under very general conditions.

3. Calculation of the Probability Density of \( X_1/X_2 \), Conditioned on
Positive \( X_1 \) and \( X_2 \).

We assume \( \mu_1, \mu_2, \sigma_1 \) and \( \sigma_2 \) are all strictly positive. Define

\[
\phi(z) := (2\pi)^{-1/2} e^{-z^2/2}, \quad \Phi(z) := \int_{-\infty}^{z} (2\pi)^{-1/2} e^{-u^2/2} du.
\]

We note that the following probabilities are identical for \( x > 0 \) (and zero for \( x \leq 0 \))

\[
P\left\{ X_1, X_2 > 0 \text{ and } \frac{X_1}{X_2} \leq x \right\} = P\left\{ X_1, X_2 > 0 \text{ and } X_1 \leq x X_2 \right\}
\]

\[
= \int_{0}^{\infty} ds_2 \int_{0}^{xs_2} ds_1 f(s_1, s_2, \mu_1, \mu_2, \Sigma)
\]

\[
= \int_{0}^{\infty} ds_2 \int_{0}^{xs_2} ds_1 \frac{1}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \phi\left( \frac{s_2 - \mu_2}{\sigma_2} \right) \phi\left( \frac{s_1 - \left( \mu_1 + \frac{\rho \sigma_1}{\sigma_2} (s_2 - \mu_2) \right)}{\sigma_1 \sqrt{1 - \rho^2}} \right)
\]

Various expressions for the density of bivariate and multivariate normals, such as
the one above, can be found in [31].

In order to obtain the conditional density we use the identity

\[
\frac{d}{dx} \int_{0}^{xs_2} ds_1 g(s_1, s_2) = s_2 g(xs_2, s_2)
\]

together with an implication of the Dominated Convergence Theorem to write

\[
\frac{d}{dx} \left[\right. \left\{ X_1, X_2 > 0 \text{ and } \frac{X_1}{X_2} \leq x \right\}\right]
\]

\[
= \int_{0}^{\infty} s_2 ds_2 \frac{1}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \phi\left( \frac{s_2 - \mu_2}{\sigma_2} \right) \phi\left( \frac{xs_2 - \left( \mu_1 + \frac{\rho \sigma_1}{\sigma_2} (s_2 - \mu_2) \right)}{\sigma_1 \sqrt{1 - \rho^2}} \right)
\]

\[
= \frac{(2\pi)^{-1}}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \int_{0}^{\infty} s_2 ds_2 e^{-\frac{1}{2} \left( \frac{s_2 - \mu_2}{\sigma_2} \right)^2} - \frac{1}{2} \left( \frac{xs_2 - \left( \mu_1 + \frac{\rho \sigma_1}{\sigma_2} (s_2 - \mu_2) \right)}{\sigma_1 \sqrt{1 - \rho^2}} \right)^2.
\]

Defining the quantities

\[
Q(s) := \left( \frac{s - \mu_2}{\sigma_2} \right)^2 + \left( \frac{xs - \left( \mu_1 + \frac{\rho \sigma_1}{\sigma_2} (s - \mu_2) \right)}{\sigma_1 \sqrt{1 - \rho^2}} \right)^2 = As^2 + 2Bs + C
\]
\[ A := \sigma_2^{-2} + \sigma_1^{-2} (1 - \rho^2)^{-1} \left( x - \rho \frac{\sigma_1}{\sigma_2} \right)^2, \]
\[ B := -\sigma_2^{-2} \mu_2 + \sigma_1^{-2} (1 - \rho^2)^{-1} \left( x - \rho \frac{\sigma_1}{\sigma_2} \right) \left( \mu_2 \rho \frac{\sigma_1}{\sigma_2} - \mu_1 \right), \]
\[ C := \frac{\mu_2^2}{\sigma_2^2} + (1 - \rho^2)^{-1} \left( \frac{\mu_2}{\sigma_2} - \frac{\mu_1}{\sigma_1} \right)^2. \]

(3.6)

where

\[ Q(s) := A \left( s + \frac{B}{A} \right)^2 - \frac{B^2}{A} + C \]

we can express this equation as

\[ \frac{d}{dx} \mathbb{P} \left\{ \ldots \right\} = \frac{(2\pi)^{-1}}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \int_0^\infty sds e^{-\frac{1}{2}Q(s)} \]

(3.7)

\[ = \frac{(2\pi)^{-1}}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} e^{\left( \frac{\mu_2}{\sigma_2} - c \right)} \int_0^\infty sde^{-\frac{1}{2}A(s+\frac{B}{A})^2} ds. \]

(3.8)

Let \( z := \frac{1}{\sqrt{2}} A^{1/2} (s + \frac{B}{A}) \) so \( dz = \frac{1}{\sqrt{2}} A^{1/2} ds \) and \( s = \frac{2}{A^{1/2}} z - \frac{B}{A} \) to transform the integral and obtain

\[ \int_0^\infty sde^{-\frac{1}{2}A(s+\frac{B}{A})^2} ds = \frac{1}{A} e^{-\frac{B^2}{A^2}} - \sqrt{\frac{\pi}{2}} \frac{B}{A^{3/2}} \text{erfc} \left( \frac{B}{\sqrt{2}A^{1/2}} \right) \]

(3.9)

\[ \text{erf}c(z) := 1 - \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2} du. \]

We have then the identity

\[ \frac{d}{dx} \mathbb{P} \left\{ X_1, X_2 > 0 \text{ and } \frac{X_1}{X_2} \leq x \right\} = \frac{(2\pi)^{-1}}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \int_0^\infty sds e^{-\frac{1}{2}Q(s)} \]

(3.10)

\[ = \frac{(2\pi)^{-1}}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} e^{\frac{\mu_2^2}{\sigma_2^2}} \left\{ \frac{1}{A} e^{-\frac{B^2}{A^2}} - \sqrt{\frac{\pi}{2}} \frac{B}{A^{3/2}} \text{erfc} \left( \frac{B}{\sqrt{2}A^{1/2}} \right) \right\} \]

Hence, the density of \( X_1/X_2 \) conditioned on \( X_1, X_2 > 0 \) is given by the basic relation (see [3])

\[ f_{X_1/X_2} (x \mid Q_1) \]

(3.11)

\[ = \frac{(2\pi)^{-1}}{\mathbb{P}(Q_1) \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} e^{\frac{\mu_2^2}{\sigma_2^2}} \left\{ \frac{1}{A} e^{-\frac{B^2}{A^2}} - \sqrt{\frac{\pi}{2}} \frac{B}{A^{3/2}} \text{erfc} \left( \frac{B}{\sqrt{2}A^{1/2}} \right) \right\} \]

where \( Q_1 \) is the set \( X_1 > 0, X_2 > 0 \). The calculation of \( \mathbb{P}(Q_1) \) is carried out in Appendix A.

We can use \( \omega := B/(2A)^{1/2} \) to write the following exact expression for the conditional density:

\[ f_{X_1/X_2} (x \mid Q_1) = \frac{(2\pi)^{-1}}{\mathbb{P}(Q_1) \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \frac{1}{A} e^{-\frac{\omega^2}{2}} h(\omega), \]

(3.12)

\[ h(\omega) := e^{\omega^2} \left\{ e^{-\omega^2} - \sqrt{\pi} \omega \text{erfc}(\omega) \right\} = 1 - \sqrt{\pi} \omega e^{\omega^2} \text{erfc}(\omega). \]
4. Analysis of the logarithm of the conditional density

In order to extract the behavior of the conditional density for large $|x|$ we analyze its logarithm.

**Theorem 4.1.** For $x \geq x_0 := 2 \sigma_1^{2} \sigma_2$ one has the bounds

\[
\frac{\log f_{X_1/X_2} (x \mid Q_1)}{\log x} = -\frac{C}{2} + \log \left( \frac{(2\pi)^{-1}}{\mathbb{P}(Q_1)} \right) + \log h(\omega) - \log \left[ \sigma_1 \sigma_2 \sqrt{1 - \rho^2} A \right] \log x
\]

\[
= -\frac{C}{2} + \log \left( \frac{(2\pi)^{-1}}{\mathbb{P}(Q_1)} \right) - \frac{\log \left( \frac{\sigma_1^{2}}{\sigma_2^{2}} \frac{1}{\sqrt{1 - \rho^2}} \right)}{\log x} + \frac{\log \left( x \frac{\sigma_1}{\sigma_2} \right)}{\log x} \left( 2 + 2 \frac{\sigma_2 \rho}{\log x} + R \left( x ; \frac{\sigma_1}{\sigma_2} \right) \right).
\]

(4.1)

where \( \left| R \left( x ; \frac{\alpha_1}{\alpha_2} \right) \right| \leq \frac{(\frac{\alpha_1}{\alpha_2})^2}{\log x_0} \).

**Proof.** Re-grouping the constants, and taking the logarithm of \( f_{X_1/X_2} (x \mid Q_1) \), we write

\[
\log f_{X_1/X_2} (x \mid Q_1) = -\frac{C}{2} + \log \left( \frac{(2\pi)^{-1}}{\mathbb{P}(Q_1)} \right) + \log h(\omega) - \log \left[ \sigma_1 \sigma_2 \sqrt{1 - \rho^2} A \right].
\]

(4.2)

The decay exponent will be determined by the large $x$ limit of this quantity divided by $\log x$. We first analyze the last term in the expression above:

\[
\sigma_1 \sigma_2 \sqrt{1 - \rho^2} A = \frac{\sigma_2}{\sigma_1} \left( x - \frac{\sigma_1}{\sigma_2} \rho \right)^2 \left[ 1 + \frac{(\frac{\alpha_1}{\alpha_2})^2 (1 - \rho^2)}{(x - \frac{\alpha_1}{\alpha_2} \rho)^2} \right]
\]

(4.3)

Taking the logarithm and dividing by $\log x$, we have

\[
\frac{\log \left[ \sigma_1 \sigma_2 \sqrt{1 - \rho^2} A \right]}{\log x} = \frac{\log \left( \frac{\sigma_2}{\sigma_1} \right)}{\log x} + 2 \frac{\log \left( x - \frac{\sigma_1}{\sigma_2} \rho \right)}{\log x} + \frac{\log \left[ 1 + \frac{(\frac{\alpha_1}{\alpha_2})^2 (1 - \rho^2)}{(x - \frac{\alpha_1}{\alpha_2} \rho)^2} \right]}{\log x}
\]

(4.4)

We examine the last two terms for $x \geq 2x_0$

\[
2 \frac{\log \left( x - \frac{\sigma_1}{\sigma_2} \rho \right)}{\log x} = 2 + 2 \frac{\log \left( 1 - \frac{\sigma_1}{\sigma_2} \rho \right)}{\log x}
\]

(4.5)

\[
\left| 2 \frac{\log \left( x - \frac{\sigma_1}{\sigma_2} \rho \right)}{\log x} - \left( 2 + 2 \frac{\sigma_2 \rho}{\log x} \right) \right| \leq \frac{1}{2} \frac{(\frac{\sigma_1}{\sigma_2} \rho)^2}{x_0 \log x_0} \leq \frac{1}{2} R \left( x_0 ; \frac{\sigma_1}{\sigma_2} \right)
\]
The last logarithm in (4.3) can be bounded (for \( x \geq 2x_0 \)) as

\[
\log \left| 1 + \frac{(\frac{\sigma_1}{\sigma_2})^2 (1 - \rho^2)}{\log x} \right| \leq \frac{2}{2} \log \left( x_0; \frac{\sigma_1}{\sigma_2} \right) \leq \frac{1}{2} R \left( x_0; \frac{\sigma_1}{\sigma_2} \right)
\]

Thus one has

\[
\left| -\log \left[ \frac{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}}{\log x} \right] - \left[ -2 + 2 \frac{\sigma_1 \rho}{\log x} + \frac{\log \left( \frac{\sigma_2}{\sigma_1} \sqrt{1 - \rho^2} \right)}{\log x} \right] \right| \leq R \left( x_0; \frac{\sigma_1}{\sigma_2} \right),
\]

concluding the proof. \( \square \)

In order to extract the relevant part of \( \log \omega \) we write

\[
\omega_0 := \frac{1}{21/2} \left( 1 - \rho^2 \right)^{-1/2} \left( \frac{\mu_2}{\sigma_2} - \frac{\mu_1}{\sigma_1} \right) - \frac{C}{2} = \frac{1}{2} \frac{\mu_2^2}{\sigma_2^2} - \omega_0^2,
\]

and use this to extract the \( x \)-dependent part of the conditional density.

**Theorem 4.2.** (a) For \( x > x_0 := \max \left\{ 2 \frac{M_1}{\sigma_2}, 1 \right\} \) one has the bounds

\[
\log f_{X_1/X_2} (x \mid Q_1) = -C \frac{x}{\log x} + \log \frac{(2\pi)^{-1} \chi_{1}^{-1}(Q_1)}{\log x} + \log h(\omega) - \log \frac{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}}{\log x} + R_4 (x_0)
\]

where \( R_4 \) satisfies \( |R_4 (x_0)| \leq \frac{C_1}{x_0 \log x_0} \) and more specifically

\[
|R_4 (x_0)| \leq R_1 \left( x_0; \frac{\sigma_1}{\sigma_2} \right) + R_2 \left( x_0; \frac{\sigma_1}{\sigma_2} \right) + R_3 \left( x_0; \frac{b_1}{a_1}, a_2 \right)
\]

where \( R_1 \left( x; \frac{\sigma_1}{\sigma_2} \right) := 2x^2 \frac{\sigma_1}{x \log x} \quad R_2 \left( x; \frac{\sigma_1}{\sigma_2} \right) := 2 \left( \frac{\sigma_1}{\sigma_2} \right)^2 \) and

\[
R_3 \left( x; \frac{b_1}{a_1}, a_2 \right) := |\omega_0| \left\{ \frac{b_1}{a_1} + \frac{1}{2a_2^2} + \frac{1}{8a_2^4} \right\} \frac{1}{x_0 \log x_0} \leq \frac{C_1}{x_0 \log x_0}
\]

where \( a_1, a_2, b_1 \) and \( \zeta \) are defined below.

(b) In the limit one has

\[
\lim_{x \to \infty} \log f_{X_1/X_2} (x \mid Q_1) = -2,
\]

and the density can be expressed in the form

\[
f_{X_1/X_2} (x \mid Q_1) \simeq f_0 x^{-2}
\]
\[ f_0 := \frac{(\sigma_1/\sigma_2) \sqrt{1 - \rho^2}}{2\pi^{3/2}} (Q_1) e^{-\frac{\omega_0^2}{2\sigma_2^2}} \left( e^{-\omega_0^2} - \sqrt{\pi \omega_0} \erfc(\omega_0) \right). \]

**Proof.** The main issue is to examine the two terms, \( \log [\sigma_1\sigma_2 \sqrt{1 - \rho^2} A] \) and \( \log h(\omega) \) and extract the part that is significant after division by \( \log x \).

(i) One has

\[
\log [\sigma_1\sigma_2 \sqrt{1 - \rho^2} A] = \log \left( \frac{\sigma_2}{\sigma_1} \sqrt{1 - \rho^2} \right) + 2 \log \left( x - \frac{\sigma_1}{\sigma_2} \rho \right) + \log \left( 1 + \frac{\left( \frac{\sigma_1}{\sigma_2} \right)^2 (1 - \rho^2)}{x - \frac{\sigma_1}{\sigma_2} \rho} \right). 
\]

(4.14)

Upon dividing by \( \log x \) the second of these terms is

\[
\frac{2 \log \left( x - \frac{\sigma_1}{\sigma_2} \rho \right)}{\log x} = 2 + 2 \frac{\log \left( 1 - \frac{\sigma_1}{\sigma_2} \rho \right)}{\log x},
\]

(4.15)

and the latter term is bounded by

\[
2 \left| \frac{\log \left( 1 - \frac{\sigma_1}{\sigma_2} \rho \right)}{\log x} \right| \leq 2 \frac{\sigma_1}{\sigma_2} \frac{|\rho|}{x \log x} \leq R_1 \left( x_0; \sigma_1, \sigma_2 \right).
\]

(4.16)

The last of the terms in (4.14) can be bounded as

\[
0 \leq \log \left( 1 + \frac{\left( \frac{\sigma_1}{\sigma_2} \right)^2 (1 - \rho^2)}{x - \frac{\sigma_1}{\sigma_2} \rho} \right) \leq \frac{1}{2} \frac{\left( \frac{\sigma_1}{\sigma_2} \right)^2 (1 - \rho^2)}{(x - \frac{\sigma_1}{\sigma_2} \rho)^2}
\]

(4.17)

\[
\leq \frac{2 \left( \frac{\sigma_1}{\sigma_2} \right)^2}{x^2}.
\]

Hence, one has the bounds

\[
\frac{\log \left( 1 + \frac{\left( \frac{\sigma_1}{\sigma_2} \right)^2 (1 - \rho^2)}{x - \frac{\sigma_1}{\sigma_2} \rho} \right)}{\log x} \leq R_2 \left( x_0; \sigma_1, \sigma_2 \right).
\]

(4.18)

\[
\left| \frac{-\log [\sigma_1\sigma_2 \sqrt{1 - \rho^2} A]}{\log x} + 2 \right| \leq R_1 \left( x_0; \sigma_1, \sigma_2 \right) + R_2 \left( x_0; \sigma_1, \sigma_2 \right).
\]

(4.19)
(ii) Next, we examine $\omega - \omega_0$ as we focus on $\log h(\omega)$. The term $h(\omega)$ has $x$ dependence. Note that $\omega$ can be written as

$$\omega = \frac{B}{2^{1/2}A^{1/2}} = \frac{-\sigma_2^{-2} \mu_2 + \sigma_1^{-2} (1 - \rho^2)^{-1} \left(x - \rho \frac{\sigma_1}{\sigma_2}\right) \left(\mu_2 \rho \frac{\sigma_1}{\sigma_2} - \mu_1\right)}{\left(\sigma_2^{-2} + \sigma_1^{-2} (1 - \rho^2)^{-1} \left(x - \rho \frac{\sigma_1}{\sigma_2}\right)^2\right)^{1/2}}$$

(4.20)

$$= \frac{1}{2^{1/2}} \left(\frac{a_1 + \frac{b_1}{a_1 x - \rho \sigma_1 / \sigma_2}}{a_2^2 + \frac{1}{(x - \rho \sigma_1 / \sigma_2)^2}}\right)^{1/2}$$

and $\omega_0 = 2^{-1/2}a_1/a_2$ with the definitions

$$a_1 := \left(\frac{\mu_2}{\sigma_2} - \frac{\mu_1}{\sigma_1}\right) \frac{\sigma_2}{\sigma_1} (1 - \rho^2)^{-1}, \quad b_1 := -\frac{\mu_2}{\sigma_2}$$

(4.21)

Using $P := \frac{b_1}{a_1 x - \rho \sigma_1 / \sigma_2}$ and $Q^2 := \left[a_2^2 (x - \rho \sigma_1 / \sigma_2)^2\right]^{-1}$ we write

$$|\omega - \omega_0| = |\omega_0| \frac{1 + P - (1 + Q^2)^{1/2}}{(1 + Q^2)^{1/2}}$$

Basic Taylor series estimates then imply

(4.22) $$\left|1 + P - (1 + Q^2)^{1/2}\right| \leq |P| + \frac{1}{2} Q^2 + \frac{1}{8} Q^4.$$

Upon using $x \geq 2x_0$ so that $x - \rho \sigma_1 / \sigma_2 \geq x_0$, we have the bounds

(4.23) $$|\omega - \omega_0| \leq |\omega_0| \left\{ \left|\frac{b_1}{a_1}\right| \frac{1}{x_0} + \frac{1}{2a_2^2} \frac{1}{x_0^2} + \frac{1}{8a_2^4} \right\}.$$

Thus, for large $x$ we can thus write, for $x \geq 2x_0$

(4.24) $$|\omega - \omega_0| \leq \frac{C_1}{x_0}$$

where $C_1$ depends on $(1 - \rho^2), \sigma_1 / \sigma_2$ and $|\rho \frac{\mu_2}{\sigma_2} - \frac{\mu_1}{\sigma_1}|$.

This yields the bound

(4.25) $$\left|\log h(\omega) - \log h(\omega_0)\right| \leq |\omega - \omega_0| \ |h'(\zeta)|$$

where $\zeta := \inf \{\omega, \omega_0\}$. Hence, we have

(4.26) $$\left|\frac{\log h(\omega)}{\log x} - \frac{\log h(\omega_0)}{\log x}\right| \leq R_3(x_0; \omega_0, |b_1/a_1|, a_2)$$

where

(4.27) $$R_3(x_0; \omega_0, |b_1/a_1|, a_2) \leq \left\{ \left|\frac{b_1}{a_1}\right| + \frac{1}{2a_2^2} + \frac{1}{8a_2^4}\right\} \frac{1}{x_0}.$$

That is, one has the bound, with $C_2$ depending on $\omega_0, |b_1/a_1|, a_2$ and $|h'(\zeta)|$,

(4.28) $$\left|\frac{\log h(\omega)}{\log x} - \frac{\log h(\omega_0)}{\log x}\right| \leq \frac{C_2}{x_0 \log x_0} |h'(\zeta)|.$$
The proof is concluded by observing that the $h'$ term can be bounded using the following classical estimates for the error function:

\[
\omega + \sqrt{\omega^2 + 2} < e^{\omega^2} \int_{\omega}^{\infty} e^{-u^2} du \leq \frac{1}{\omega + \sqrt{\omega^2 + 4/\pi}}
\]

By basic error function series expansions, one can bound $h(\omega)$ by

\[
1 - \frac{2}{1 + \sqrt{1 + \frac{2}{\omega^2}}} \leq h(\omega) \leq 1 - \frac{2}{1 + \sqrt{1 + \frac{4}{\pi\omega^2}}}
\]

For $\omega \geq 0$, one has the additional bounds (see [1])

\[
2\omega - \frac{2(1 + 2\omega^2)}{\omega + \sqrt{\omega^2 + 4/\pi}} \leq h'(\omega) \leq 2\omega - \frac{2(1 + 2\omega^2)}{\omega + \sqrt{\omega^2 + 2}}.
\]

\[\blacksquare\]

We will consider both of the regions $-\omega_0 >> 1$ and $\omega_0 >> 1$ separately after considering the special case of $\rho := -1$.

4.1. The Singular Case $\rho := -1$. Recall that our calculations have been under the assumption of a nonsingular covariance matrix $\Sigma$, i.e. where $|\rho| < 1$. Another interesting case is when $\rho = -1$, which corresponds to two anticorrelated random variables. This is useful in modeling asset pricing as an increase in demand will often be accompanied by a commensurate decrease in supply and vice versa. To this end, let $Y \sim \mathcal{N}(0, 1)$ be a standard normal random variable and suppose demand and supply then take the form, for positive $\mu_1, \mu_2, \sigma_1, \sigma_2$,

\[
D = X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2) = \mu_1 + \sigma_1 Y
\]

\[
S = X_2 \sim \mathcal{N}(\mu, \sigma^2) = \mu_1 - \sigma_2 Y.
\]

Thus, we will have the covariance matrix

\[
\Sigma = \begin{pmatrix}
\sigma_1^2 & -\sigma_1 \sigma_2 \\
-\sigma_1 \sigma_2 & \sigma_2^2
\end{pmatrix}.
\]

Then the price equation becomes

\[
\frac{1}{P} \frac{dP}{dt} = \frac{D}{S} - 1 = \frac{\mu_1 + \sigma_1 Y}{\mu_1 - \sigma_2 Y} - 1 =: R - 1
\]

Under near-equilibrium conditions, the values $\mu_1/\mu_2$ and $\sigma_1/\sigma_2$ will be near unity. Nevertheless, we prove the following more general theorem.

**Theorem 4.3.** Let $\mu_1, \mu_2, \sigma_1, \sigma_2$ be strictly positive, $X_1, X_2$ be bivariate normals with correlation $-1$. Then $R := X_1/X_2$ has density

\[
f_R(x) = \frac{\mu_1 \sigma_2 + \mu_2 \sigma_1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{\mu_1 - \mu_2}{\sigma_2 + \sigma_1}\right)^2}
\]

where $f_R(-\sigma_1/\sigma_2) := 0$. 

Proof. Let $Y \sim N(0, 1)$ be the standard normal. With $R = (\mu_1 + \sigma_1 Y) / (\mu_2 - \sigma_2 Y)$ we use the Theorem of Total Probability to express the distribution function of $R$ as

$$
P \{ R \leq x \} = P \left\{ \frac{\mu_1 + \sigma_1 Y}{\mu_2 - \sigma_2 Y} \leq x, \mu_2 - \sigma_2 Y > 0 \right\} + P \left\{ \frac{\mu_1 + \sigma_1 Y}{\mu_2 - \sigma_2 Y} \leq x, \mu_2 - \sigma_2 Y < 0 \right\}
$$

where we neglect sets of measure zero. Abbreviating the two probabilities by $P_1$ and $P_2$ respectively, we note the two cases determined by the sign of $\sigma_2 x + \sigma_1$. (i) For $\sigma_2 x + \sigma_1 > 0$ we have

$$
P_1 = P \left\{ Y \leq \frac{\mu_2 x - \mu_1}{\sigma_2 x + \sigma_1} \text{ and } Y < \frac{\mu_2}{\sigma_2} \right\} = \int_{-\infty}^{\frac{\mu_2 x - \mu_1}{\sigma_2 x + \sigma_1}} f_Y (s) \, ds,
$$

$$
P_2 = P \left\{ Y \geq \frac{\mu_2 x - \mu_1}{\sigma_2 x + \sigma_1} \text{ and } Y > \frac{\mu_2}{\sigma_2} \right\} = \int_{\frac{\mu_2}{\sigma_2}}^{\infty} f_Y (s) \, ds.
$$

Differentiating the sum with respect to $x$, we note that the latter vanishes, and we obtain, for $\sigma_2 x + \sigma_1 > 0$, the density

$$
f_R (x) = f_Y \left( \frac{\mu_2 x - \mu_1}{\sigma_2 x + \sigma_1} \right) \frac{d}{dx} \left( \frac{\mu_2 x - \mu_1}{\sigma_2 x + \sigma_1} \right) = \frac{e^{-\frac{1}{2} \left( \frac{\mu_2 x - \mu_1}{\sigma_2 x + \sigma_1} \right)^2}}{\sqrt{2\pi}} \left( \frac{\mu_1 \sigma_2 + \mu_2 \sigma_1}{(\sigma_2 x + \sigma_1)^2} \right).
$$

(ii) Next, for $\sigma_2 x + \sigma_1 < 0$ we use the same notation to write

$$
P_1 = \int_{\frac{\mu_2 x - \mu_1}{\sigma_2 x + \sigma_1}}^{\infty} f_Y (s) \, ds,
$$

$$
P_2 = P \left\{ Y < \frac{\mu_2 x - \mu_1}{\sigma_2 x + \sigma_1} \text{ and } Y > \frac{\mu_2}{\sigma_2} \right\} = 0.
$$

so that differentiation with respect to $x$ yields the same result as above, proving the theorem. \qed

Remark 4.4. In the special case $\mu_1 = \mu_2 = \mu$ and $\sigma_1 = \sigma_2 = \sigma$ the density has the form

$$
f_R (x) = \sqrt{\frac{2}{\pi}} \mu e^{-\frac{1}{\sigma^2 (x + 1)^2}}
$$

where $f_R (-1) := 0$.

4.2. The Cauchy Limit. We consider the limit in which $\rho := 0$ and $\mu_1, \mu_2 \to 0+$ with $\sigma_1, \sigma_2$ fixed at 1, $\omega = 0$, $-C/2 = -\frac{\rho A}{\pi} = 0$, $\log h (\omega_0) = 0$. Since $\rho := 0$, the random variables $X_1$ and $X_2$ are independent, and together with $\mu_1 = \mu_2 = 0$, one has

$$
P \{ Q_1 \} = 1/4, \quad \sigma_1 \sigma_2 \sqrt{1 - \rho^2} A = 1 + x^2.
$$

Thus, the expression simplifies to

$$
\log f_{X_1/X_2} (x \mid Q_1) = \log \frac{2}{\pi (1 + x^2)}.
$$
Hence, this is a simple proof of the known result that the quotient of two independent standard normals (i.e., mean 0 and variance 1) yields the Cauchy density

\[ f_{\text{Cauchy}}(x \mid x > 0) = \frac{2}{\pi (1 + x^2)}. \]

Thus, the limit \( \mu_1, \mu_2 \to 0+ \) with \( \sigma_1, \sigma_2 \) set at 1 and \( \rho \) at 0, recovers the classical limit of the Cauchy density, and one obtains

\[
\lim_{x \to \infty} \frac{\log f_{X_1/X_2}(x \mid Q_1)}{\log x} = \lim_{x \to \infty} \frac{\log f_{\text{Cauchy}}(x \mid Q_1)}{\log x} = -2.
\]

4.3. The limit of constant denominator. We consider for positive \( X_1 \) and \( X_2 \) the ratio \( X_1/X_2 \) in the limit in which the denominator approaches a constant, i.e., \( \mu_2 > 0 \) and \( \sigma_2 \to 0 \). Note that when \( \mu_1 \) and \( \mu_2 \) are positive and \( \mu_1/\sigma_1 \) and \( \mu_2/\sigma_2 \) are large, there is a very small difference between the density and conditional density.

The basic tool we use is an asymptotic expression for the integral for \( b > 0 \) and \( a >> 1 \),

\[
I(a, b) := \int_0^\infty e^{-a(s-b)^2} g(s) \, ds \approx \int_{-\infty}^\infty e^{-a(s-b)^2} g(s) \, ds \approx \int_{-\infty}^\infty e^{-a(s-b)^2} g(b) \, ds.
\]

In other words, the integral will be negligible when \( s \) is far from \( b \), since \( a >> 1 \). Let \( z := a^{1/2} (s - b) \) and \( dz = a^{1/2} \, ds \), so we can write

\[
I(a, b) \approx g(b) \int_{-\infty}^\infty e^{-z^2} a^{-1/2} \, dz = \left( \frac{\pi}{a} \right)^{1/2} g(b).
\]

More precisely, for a function \( g \) such that \( |g| \leq 1 \) and \( |g''| \leq M \), one can use Taylor series bounds to obtain

\[
\left| I(a, b) - \left( \frac{\pi}{a} \right)^{1/2} g(b) \right| \leq \frac{\sqrt{\pi}}{4} Ma^{-3/2} + \frac{1}{a^{1/2} b} e^{-ab}.
\]

We start from the early expression (3.3), namely,

\[
f_{X_1/X_2}(x \mid Q_1) := \frac{d}{dx} \mathbb{P} \{ X_1, X_2 > 0 \text{ and } \frac{X_1}{X_2} \leq x \}
\]

\[
= \frac{(2\pi)^{-1}}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \int_0^\infty s_2 ds_2 e^{-\frac{1}{2} \left( \frac{z_2 - \mu_2}{\sigma_2 \sqrt{1 - \rho^2}} \right)^2} e^{-\frac{1}{2} \left( \frac{z_1 - \mu_1 + \rho \sqrt{1 - \rho^2} (z_2 - \mu_2)}{\sigma_1 \sqrt{1 - \rho^2}} \right)^2},
\]

and apply the asymptotic result above with \( g \) as the integrand above with \( a := (2\sigma_2^2)^{-1} \) and \( b := \mu_2 \). This yields (with \( \rho := 0 \), with \( E_1(a, b) \) the error term,

\[
\int_0^\infty s_2 ds_2 e^{-\frac{1}{2} \left( \frac{z_2 - \mu_2}{\sigma_2} \right)^2} e^{-\frac{1}{2} \left( \frac{z_1 - \mu_1 + \rho \sqrt{1 - \rho^2} (z_2 - \mu_2)}{\sigma_1 \sqrt{1 - \rho^2}} \right)^2}
\]

\[
= \mu_2 e^{-\frac{1}{2} \left( \frac{z_2 - \mu_1}{\sigma_1} \right)^2} \left( 2\pi \sigma_2^2 \right)^{1/2} + E_1(a, b)
\]
so that substitution into the integral yields
\[
 f_{X_1/X_2} (x \mid Q_1) = \frac{(2\pi)^{-1}}{\sigma_1 \sigma_2} \mu_2 \exp\left(\frac{-\left(\frac{x - \mu_1}{\sigma_1}\right)^2}{2} \right) \left(2\pi \sigma_2^2\right)^{1/2} + E_1 \left((2\sigma_2^2)^{-1}, \mu_2\right).
\]  
(4.44)

The error term is bounded by
\[
 \left| E_1 \left((2\sigma_2^2)^{-1}, \mu_2\right) \right| \leq \frac{1}{2\pi\sigma_1 \sigma_2} \left(\frac{\sqrt{\pi} 3/2}{4} M \sigma_2^3/2 + \pi^{-1/2} \frac{2\sigma_2^2}{\mu_2} e^{-\left(2\sigma_2^2\right)^{3/2}/\mu_2}\right)
\]  
(4.45)

and vanishes as \(\sigma_2^{3/2} \to 0\).

Recall that the density for a \(N(\mu_1, \sigma_1^2)\) random variable is
\[
 f_{\mu_1, \sigma_1^2} (x) = \frac{1}{(2\pi)^{1/2} \sigma_1} \exp\left(-\frac{(x - \mu_1)^2}{2\sigma_1^2}\right).
\]  
(4.46)

Given random variables \(X_1 \sim N(\mu_1, \sigma_1^2)\) and \(X_2 \sim N(\mu_2, \sigma_2^2)\), in the limit as \(\sigma_2 \to 0\), the denominator is simply division by \(\mu_2\) (since \(X_2 = \mu_2\) at that stage).

Thus, we have that
\[
 \frac{1}{\mu_2} X_1 \sim N\left(\frac{\mu_1}{\mu_2}, \left(\frac{1}{\mu_2}\right)^2 \sigma_1^2\right)
\]  
(4.47)

so the density of \(X_1/\mu_2\) is given by
\[
 f_{X_1/\mu_2} (x) = \frac{1}{(2\pi)^{1/2} \sigma_1} \exp\left(-\frac{(\mu_2 x - \mu_1)^2}{2\sigma_1^2}\right).
\]  
(4.48)

Hence, we see that \(f_{X_1/\mu_2} (x) = f_{X_1/X_2} (x \mid Q_1)\) in the limit as \(\sigma_2 \to 0\) with \(\rho := 0\), as expected.

4.4. The limits \(-\omega_0 >> 1\) and \(\omega_0 >> 1\). Recalling the definition of \(\omega_0\) and \(C\), we have
\[
 C = \frac{\mu_2^2}{\sigma_2^2} + (1 - \rho^2)^{-1} \left(\frac{\mu_2}{\sigma_2} - \frac{\mu_1}{\sigma_1}\right)^2, \quad \omega_0 := \frac{1}{21/2} (1 - \rho^2)^{-1/2} \left(\frac{\rho \mu_2}{\sigma_2} - \frac{\mu_1}{\sigma_1}\right)
\]  
(4.49)

Rewriting (4.2) we have the exact expression
\[
 \log f_{X_1/X_2} (x \mid Q_1) = -\frac{C}{2} + \log \left[\frac{(2\pi)^{-1}}{\mathcal{P}(Q_1)}\right] + \log h (\omega) - \log \left[\sigma_1 \sigma_2 \sqrt{1 - \rho^2} A\right].
\]  
(4.50)

(i) We consider the region \(-\omega_0 >> 1\) which, for \(\sigma_2, \mu_1 > 0, \mu_2 > 0\) and \(|\rho| < 1\) fixed, means \(\sigma_1\) is small. Using the standard bounds \(\[1\]\) for the error function we write, for \(\omega < 0\),
\[
 \log \left[2\sqrt{\pi} (-\omega_0) e^{-\omega_0^2}\right] \leq \log h (\omega_0) \leq \log \left[2\sqrt{\pi} (-\omega_0) e^{-\omega_0^2} + \frac{1}{2\omega_0^2}\right].
\]  
(4.51)
Approximating $\omega$ with $\omega_0$ and utilizing the bounds in the proofs of the theorems above, as well as the $-C/2$ identity we write,

$$
\log f_{X_1/X_2}(x \mid Q_1) \doteq -2 \log x + 2\rho \frac{\sigma_1}{\sigma_2} - \log (1 - \rho^2)^{-1/2} + \log \left( \frac{\sqrt{\pi}}{\mathbb{E}(Q_1)} \right)
$$

(4.52)

$$
- \frac{1}{2} \frac{\mu_2^2}{\sigma_2^2} - \log \left( \frac{\sigma_2}{\sigma_1} \right) + \log (-\omega_0).
$$

Note that the difference between the two sides vanishes as $x \to \infty$. All but the last two terms are clearly bounded for small $\sigma_1$. The last two are

$$
- \log \left( \frac{\sigma_2}{\sigma_1} \right) + \log (-\omega_0) = \log \left( \frac{\sigma_1}{\sigma_2} (-\omega_0) \right)
$$

(4.53)

$$
= \log \left\{ \left[ 2 \left( 1 - \rho^2 \right) \right]^{-1/2} \left( \frac{\mu_1}{\sigma_2} - \frac{\mu_2 \sigma_1}{\sigma_2^2} \right) \right\},
$$

so that the sum of these two are also bounded for small $\sigma_1$. One can then write

$$
\log f_{X_1/X_2}(x \mid Q_1) \doteq -2 \log x + 2\rho \frac{\sigma_1}{\sigma_2} + \log \left( \frac{\sqrt{2}}{\pi} \frac{1}{\mathbb{E}(Q_1)} \right) + \log \left( \frac{\mu_1}{\sigma_2} - \frac{\mu_2 \sigma_1}{\sigma_2^2} \right)
$$

and one has, provided $\sigma_1$ approaches zero more rapidly than $(\log x)^{-1}$, the limit,

$$
\lim_{\sigma_1 \to 0} \frac{\log f_{X_1/X_2}(x \mid Q_1)}{\log x} = -2.
$$

(ii) Next, we consider the $\omega_0 >> 1$ region, which, for $\sigma_1$, $\mu_1 > 0$, $\mu_2 > 0$ and $|\rho| < 1$ fixed, implies $\sigma_2$ is small. This is the limit in which the denominator becomes deterministic, as discussed above. We now attain this limit from the $\log f_{X_1/X_2}(x \mid Q_1)$ expression. From the identities of $C$ and $\omega_0$ above, we have

$$
\frac{C}{2} = \left( \frac{\mu_2}{\sigma_2} \right)^2 - \left( \frac{1}{2(1 - \rho^2)} \right) \left\{ \frac{\rho^2 \left( \frac{\mu_2}{\sigma_2} \right)^2}{\sigma_1^2 \sigma_2} - \frac{2 \rho \mu_1 \mu_2}{\sigma_1 \sigma_2^2} + \frac{\mu_1^2}{\sigma_1^2} \right\}
$$

(4.57)

$$
= \frac{1}{2} \frac{\mu_2^2}{\sigma_2^2} + \frac{\rho \mu_1 \mu_2}{\sigma_1 \sigma_2} - \frac{1}{2(1 - \rho^2)} \frac{\mu_1^2}{\sigma_1^2} \rho
$$

With this expression and $\log h(\omega_0) \doteq 0$ the identity (4.2) yields

$$
\log f_{X_1/X_2}(x \mid Q_1) \doteq -2 \log x + 2\rho \frac{\sigma_1}{\sigma_2} + \log \left( \frac{(2\pi)^{-1}}{\mathbb{E}(Q_1)} \right) + \log \left( \frac{\sigma_2}{\sigma_1 \sqrt{1 - \rho^2}} \right) + \log h(\omega_0) - \log \left( \frac{\sigma_2}{\sigma_1 \sqrt{1 - \rho^2}} \right)
$$

(4.58)

$$
- \frac{1}{2} \frac{\mu_2^2}{\sigma_2^2} + \frac{\rho \mu_1 \mu_2}{\sigma_1 \sigma_2} - \frac{1}{2(1 - \rho^2)} \frac{\mu_1^2}{\sigma_1^2} \rho
$$

As $\sigma_2 \to 0$ the right hand side of this expression diverges as $\sigma_2^{-2}$. This means that the exponent of the decay rate of $X_1/X_2$ diverges to $-\infty$. This is consistent with the previous result and the expectation that when $\sigma_2 \to 0$ the denominator approaches a constant, so that the only randomness is in the Gaussian numerator.
5. Appendix A: Calculation of $\mathbb{P}(Q_1), \ldots, \mathbb{P}(Q_4)$

We label the quadrants $Q_1, \ldots, Q_4$ in the usual way. We define the multivariable distribution function $F(x_1, x_2)$ in terms of the density as

$$F(x_1, x_2) := \int_{-\infty}^{0} ds_1 \int_{-\infty}^{0} ds_2 f(s_1, s_2).$$

We would like to evaluate the probability of $(X_1, X_2)$ being in the first quadrant, i.e., in $Q_1$ which is

$$\mathbb{P}(Q_1) = \int_{0}^{\infty} ds_1 \int_{0}^{\infty} ds_2 f(s_1, s_2).$$

A basic decomposition of the $\mathbb{R}^2$ integrals yields

$$\mathbb{P}(Q_1) = 1 - F(0, \infty) - F(\infty, 0) + F(0, 0).$$

We now evaluate the right hand side. Recall that $\rho \in (-1, 1)$. Let $\delta_\rho := \text{sgn} \rho$ and

$$a_i := (x_i - \mu_i) / \sigma_i$$

so $a_i = \infty$ if $x_i = \infty$ and $a_i = -\mu_i / \sigma_i$ if $x_i = 0$.

We have, using using (2.2.3) in [31] and $\Phi(\infty) = 1$, the identities

$$F(x_1, x_2) = \int_{-\infty}^{\infty} \Phi\left(\frac{\sqrt{|\rho|}z + \alpha_1}{\sqrt{1 - |\rho|}}\right) \Phi\left(\frac{\delta_\rho \sqrt{|\rho|}z + \alpha_2}{\sqrt{1 - |\rho|}}\right) \phi(z) dz,$$

$$F(0, 0) = \int_{-\infty}^{\infty} \Phi\left(\frac{\sqrt{|\rho|}z - \mu_1 / \sigma_1}{\sqrt{1 - |\rho|}}\right) \Phi\left(\frac{\delta_\rho \sqrt{|\rho|}z - \mu_2 / \sigma_2}{\sqrt{1 - |\rho|}}\right) \phi(z) dz,$$

$$F(0, \infty) = \int_{-\infty}^{\infty} \Phi\left(\frac{\sqrt{|\rho|}z - \mu_1 / \sigma_1}{\sqrt{1 - |\rho|}}\right) \phi(z) dz,$$

$$F(\infty, 0) = \int_{-\infty}^{\infty} \Phi\left(\frac{\delta_\rho \sqrt{|\rho|}z - \mu_2 / \sigma_2}{\sqrt{1 - |\rho|}}\right) \phi(z) dz.$$

Combining these with the above expression for $\mathbb{P}(Q_1)$ we have the calculation in closed form.

Note that one can also modify the expression (2.2.3) in [31] to calculate this in another way by deriving the analogous relation for $\tilde{F}(x_1, x_2) := \int_{x_1}^{\infty} ds_1 \int_{x_2}^{\infty} ds_2 f(s_1, s_2)$ so $\mathbb{P}\{X_1 \geq 0, X_2 \geq 0\} = \tilde{F}(0, 0)$.

The calculations for $\mathbb{P}(Q_2), \mathbb{P}(Q_3), \mathbb{P}(Q_4)$ are similar.

Defining $H_T := \{X_2 > 0\}$ and $H_B := \{X_2 < 0\}$ we note

$$\mathbb{P}(H_B) = F(\infty, 0), \quad \mathbb{P}(H_T) = 1 - F(\infty, 0).$$
6. Appendix B: Other Quadrants

We have been considering $X_1/X_2$ when both $X_1$ and $X_2$ are positive. The remaining possibilities are calculated below. We divide $(X_1, X_2)$ space into the usual quadrants $Q_1, Q_2, Q_3$ and $Q_4$ and also let $H_T$ and $H_B$ denote the half-spaces consisting of $X_2 > 0$ and $X_2 < 0$ respectively. Note that sets such as $\{X_2 = 0\}$ will be of measure zero in terms of the multivariate density, which is written in terms of the exponential $\phi$ defined earlier as

$$f(s_1, s_2) = \frac{1}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \phi \left( \frac{s_2 - \mu_2}{\sigma_2} \right) \phi \left( \frac{s_1 - \left( \mu_1 + \rho \frac{\sigma_1}{\sigma_2} (s_2 - \mu_2) \right)}{\sigma_1 \sqrt{1 - \rho^2}} \right).$$

6.1. Obtaining the density without conditioning. (i) Upon using the half-spaces we can write

$$P \{X_1/X_2 \leq x \text{ and } X_2 > 0\} = \int_0^\infty ds_2 \int_{-\infty}^{x s_2} ds_1 f(s_1, s_2)$$

$$P \{X_1/X_2 \leq x \text{ and } X_2 < 0\} = \int_{-\infty}^0 ds_2 \int_{x s_2}^\infty ds_1 f(s_1, s_2).$$

Using the theorem of total probability and ignoring sets of measure zero we write

$$P \{X_1/X_2 \leq x \} = P \{X_1/X_2 \leq x \text{ and } X_2 > 0\} + P \{X_1/X_2 \leq x \text{ and } X_2 < 0\}.$$

Differentiating this expression, we obtain the density (without conditioning)

$$f_{X_1/X_2}(x) = \partial_x P \{X_1/X_2 \leq x\} = \int_0^\infty f(x s_2, s_2) s_2 ds_2 - \int_{-\infty}^0 s_2 f(x s_2, s_2) ds_2.$$

The first integral has already been calculated. Using the symmetry of $\phi$ we can rewrite the second integral as

$$-\int_{-\infty}^0 s_2 f(x s_2, s_2) = \int_{0}^\infty \frac{1}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \phi \left( \frac{s_2 - (-\mu_2)}{\sigma_2} \right) \phi \left( \frac{x s_2 - \left( -\mu_1 + \rho \frac{\sigma_1}{\sigma_2} (s_2 - (-\mu_2)) \right)}{\sigma_1 \sqrt{1 - \rho^2}} \right).$$

We see that this is the same integral as the first with $(\mu_1, \mu_2)$ replaced by $(-\mu_1, -\mu_2)$, and thus has similar properties.

Thus we can express the (non-conditioned) density as

$$f_{X_1/X_2}(x) = \partial_x P \{X_1/X_2 \leq x\} =$$

$$\int_0^\infty s_2 ds_2 \frac{1}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \phi \left( \frac{s_2 - \mu_2}{\sigma_2} \right) \phi \left( \frac{x s_2 - \left( \mu_1 + \rho \frac{\sigma_1}{\sigma_2} (s_2 - \mu_2) \right)}{\sigma_1 \sqrt{1 - \rho^2}} \right)$$

$$+ \frac{1}{\sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \int_0^\infty \phi \left( \frac{z_2 - (-\mu_2)}{\sigma_2} \right) \phi \left( \frac{x z_2 - \left( -\mu_1 + \rho \frac{\sigma_1}{\sigma_2} (z_2 - (-\mu_2)) \right)}{\sigma_1 \sqrt{1 - \rho^2}} \right).$$
Hence, this can be analyzed asymptotically in the same manner as the conditional probability with similar results.

(ii) We can compute the other conditional probabilities in addition to \( f_{X_1/X_2} (x \mid Q_1) \).

For \( X_1 < 0, \ X_2 < 0 \) we have

\[
\begin{align*}
\mathbb{P} \{ X_1/X_2 \leq x \text{ and } X_1 < 0, \ X_2 < 0 \} \\
= \mathbb{P} \{ X_1 \geq x X_2 \text{ and } X_1 < 0, \ X_2 < 0 \} \\
= \begin{cases} 
\int_{-\infty}^{0} ds_2 \int_{s_2}^{0} ds_1 f(s_1, s_2) & \text{if } x > 0 \\
0 & \text{if } x \leq 0
\end{cases}
\end{align*}
\]

Hence, we can differentiate with respect to \( x \) and use the definition of conditional density to obtain, for \( x \geq 0 \),

\[
f_{X_1/X_2} (x \mid Q_3) = \partial_x \mathbb{P} (X_1/X_2 \leq x \mid Q_3)
= [\mathbb{P} (Q_3)]^{-1} \partial_x \int_{-\infty}^{0} ds_2 \int_{s_2}^{0} ds_1 f(s_1, s_2)
\]

\[
(6.7)
\]

and \( f_{X_1/X_2} (x \mid Q_3) = 0 \) if \( x < 0 \).

Similarly, we write

\[
f_{X_1/X_2} (x \mid Q_2) = \begin{cases} 
[\mathbb{P} (Q_2)]^{-1} \int_{0}^{\infty} f(x s_2, s_2) ds_2 & \text{if } x < 0 \\
0 & \text{if } x \geq 0
\end{cases}
\]

\[
(6.8)
\]

and

\[
f_{X_1/X_2} (x \mid Q_4) = \begin{cases} 
- [\mathbb{P} (Q_4)]^{-1} \int_{0}^{\infty} f(x s_2, s_2) ds_2 & \text{if } x < 0 \\
0 & \text{if } x \geq 0
\end{cases}
\]

\[
(6.9)
\]

7. Appendix C: Relations between conditional densities.

For any \( x \in \mathbb{R} \) we have

\[
\mathbb{P} \{ X_1/X_2 \leq x, \ X_2 > 0 \} = \mathbb{P} \{ X_1/X_2 \leq x, \ X_2 > 0, \ X_1 > 0 \}
+ \mathbb{P} \{ X_1/X_2 \leq x, \ X_2 > 0, \ X_1 < 0 \}.
\]

(7.1)

Let \( H_R := \{ X_1 > 0, \ X_2 \in \mathbb{R} \} \) and \( H_L := \{ X_1 < 0, \ X_2 \in \mathbb{R} \} \).

For \( x > 0 \) the probability that \( X_1/X_2 \leq x \) is 1 if \( X_2 > 0 \) and \( X_1 < 0 \), so differentiating the term above yields

\[
\partial_x \mathbb{P} \{ X_1/X_2 \leq x, \ X_2 > 0 \} = \partial_x \mathbb{P} \{ X_1/X_2 \leq x, \ X_2 > 0, \ X_1 > 0 \}.
\]

(7.2)

Rewriting each side using conditional probability, we have,

\[
\partial_x \mathbb{P} \{ X_1/X_2 \leq x \mid X_2 > 0 \} \mathbb{P} \{ X_2 > 0 \}
= \partial_x \mathbb{P} \{ X_1/X_2 \leq x \mid X_2 > 0, \ X_1 > 0 \} \mathbb{P} (Q_1)
\]

(7.3)

sand we have, in terms of conditional probability,

\[
\partial_x \mathbb{P} \{ X_1/X_2 \leq x \mid X_2 > 0 \} \mathbb{P} (H_T)
= \partial_x \mathbb{P} \{ X_1/X_2 \leq x \mid X_2 > 0, \ X_1 > 0 \} \mathbb{P} (Q_1).
\]

(7.4)
The differentiated terms are just conditional densities, and we write

\[ f_{X_1/X_2}(x \mid H_T) \mathbb{P}(H_T) = f_{X_1/X_2}(x \mid Q_1) \mathbb{P}(Q_1). \tag{7.5} \]

For \( x < 0 \) the probability that \( X_1/X_2 \leq x \) is 0 if \( X_1 > 0 \) and \( X_2 > 0 \). Thus we can write, from the first expression,

\[ \partial_x \mathbb{P} \{ X_1/X_2 \leq x, X_2 > 0 \} = \partial_x \mathbb{P} \{ X_1/X_2 \leq x, X_2 > 0, X_1 < 0 \}. \tag{7.6} \]

Using the same procedure as above, we write

\[ \partial_x \mathbb{P} \{ X_1/X_2 \leq x \mid X_2 > 0 \} = \partial_x \mathbb{P} \{ X_1/X_2 \leq x \mid X_2 > 0, X_1 < 0 \} \mathbb{P}(Q_2). \tag{7.7} \]

We can then write the expression in terms of conditional density as

\[ f_{X_1/X_2}(x \mid H_R) \mathbb{P}(H_R) = f_{X_1/X_2}(x \mid Q_2) \mathbb{P}(Q_2). \tag{7.8} \]

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