Faber-Krahn inequality for Robin problem involving p-Laplacian

Qiuyi Dai
Department of Mathematics, Hunan Normal University
Changsha Hunan 410081, P.R.China
E-mail: daiqiuyi@yahoo.com.cn

Yuxia Fu
Department of Applied Mathematics, Hunan University
Changsha Hunan 410082, P.R.China
E-mail: fyuxia@yahoo.com.cn

Abstract
The eigenvalue problem for the p-Laplace operator with Robin boundary condition is considered in this paper. A Faber-Krahn type inequality is proved. More precisely, it is shown that amongst all the domains of fixed volume, the ball has the smallest first eigenvalue.

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Preface
The result of this paper was announced at first on a conference held at the Wuhan Institute of Physics and Mathematics in May 2007. The final version of this paper was finished in September 2008 when the first author worked as a research fellow at The Australian National University. As soon as we completed our paper, we sent a copy of our preprint to D.Daners (see item 6 in the reference of [3]) since our result is related to a previous paper [5] of him. Five months later, D.Bucur and D.Daners give an alternative proof of our result in February 2009. Though the paper has been published (see [3]), their proof depends completely on Proposition 2.2, Corollary 2.3 and Proposition 2.7 of this paper which, to our knowledge, can not been found in other

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1 Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be an open bounded smooth domain, we consider the following eigenvalue problem

\[
\begin{cases}
-\text{div}(\vert \nabla u \vert^{p-2} \nabla u) = \lambda \vert u \vert^{p-2}u & \text{in } \Omega, \\
\vert \nabla u \vert^{p-2} \frac{\partial u}{\partial \nu} + \beta \vert u \vert^{p-2}u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $1 < p < +\infty$, $\nu$ is the outward unit normal of $\partial \Omega$ and $\beta$ is a non-negative constant.

The p-Laplacian $\text{div}(\vert \nabla u \vert^{p-2} \nabla u)$ arises in many applications such as non-Newtonian fluids, quasi-regular and quasi-conformal mapping theory and Finsler geometry etc. An important special case of the p-Laplacian is the well known Laplacian $\Delta u = \text{div}(\nabla u)$ which corresponds to $p = 2$. Problem (1.1) is called Dirichlet when $\beta = +\infty$, Neumann when $\beta = 0$, and Robin when $0 < \beta < +\infty$.

The main purpose of this paper is to prove a Faber-Krahn type inequality for the Robin problem of the p-Laplacian. This inequality says that amongst all the domains of fixed volume, the ball has the smallest first eigenvalue. The study of this kinds of inequalities can be traced back to 1877 [20]. Let $B$ denote a ball in $\mathbb{R}^N$, and $\lambda^D_1(\Omega)$ denote the first eigenvalue of the following eigenvalue problem

\[
\begin{cases}
-\Delta \psi = \lambda \psi & x \in \Omega, \\
\psi = 0 & x \in \partial \Omega.
\end{cases}
\]

Rayleigh [20] conjectured that

\[
\lambda^D_1(\Omega) \geq \lambda^D_1(B) \quad \text{for } \Omega \subset \mathbb{R}^N \quad \text{with } |\Omega| = |B|,
\]

and the equality hold if and only if $\Omega = B$. This conjecture was proved independently by Faber [8] and Krahn [16, 17] in the 1920’s by making use of Schwartz symmetrization. Since then, the inequality (1.3) was known as Faber-Krahn inequality. In 1999, a proof of Faber-Krahn type inequality for the Dirichlet problem of the p-Laplacian was given by T. Bhattacherya [1]. Recently, Faber-krahn type inequality was generalized to Robin problem of the Laplacian by M.H.Bossel [2] for dimension $N = 2$, and by D.Daners [5] for dimension $N \geq 3$ but left the equality case open. A little bit later, D.Daners and J.Kennedy complete the proof of equality case in [6]. Note that the generalization of the Faber-Krahn inequality from Dirichlet problem to Robin problem is not trivial as, unlike in the Dirichlet problem, the first eigenvalue of Robin problem is not monotone as the domain expands (see [11]). For more information of the Faber-Krahn type inequality on manifold , we refer to [13].
Since the level surface of the first eigenfunction of Robin problem intersects with the boundary \( \partial \Omega \), the Schwartz symmetrization of the first eigenfunction generally does not decrease its Dirichlet integral and hence the Schwartz symmetrization method does not apply to the proof of Faber-Krahn inequality for Robin problem. Therefore, new approach must be employed in the proof of the Faber-Krahn inequality for Robin problem. The two crucial tools used by D.Daners \[5\] to prove the Faber-Krahn inequality for Robin problem of the Laplacian are the Bessel functions and a new formula for the first eigenvalue by making use of level sets of the corresponding eigenfunction. To prove the Faber-Krahn type inequality for Robin problem of the \( p \)-Laplacian with \( p \neq 2 \), we mainly face two difficulties. One is the lack of Bessel functions and the other is the degeneracy of the operator. The tools we use to overcome these difficulties are some new abstract propositions of the first eigenfunction and some approximation procedure.

The main results of this paper can be stated as the following

**Theorem 1.1.** Let \( 1 < p < +\infty \) and \( \lambda_1(\Omega) \) be the first eigenvalue of problem (1.1) with \( 0 < \beta < +\infty \). If \( B \) is an open ball such that \( |B| = |\Omega| \), then \( \lambda_1(B) \leq \lambda_1(\Omega) \).

**Remark.** Theorem 1.1 is proved under the assumption that \( \Omega \) is smooth. However, by an approximation method similar to that used in \[5\], we can prove that Theorem 1.1 is still true for the domains of Lipschitz type.

**Theorem 1.2.** Let \( 1 < p < +\infty \) and \( B \) be a ball satisfying \( |B| = |\Omega| \). If \( \lambda_1(\Omega) = \lambda_1(B) \), then, up to a translation, we have \( \Omega = B \).

We also point out here that a symmetry result due to Gidas, Ni and Nirenberg \[10\] plays a crucial role in the proof of Theorem 1.2 when \( p = 2 \) (see D.Daners and J.Kennedy \[6\]). However, this kind of result is not available for \( p \)-Laplace equation when \( p > 2 \) and \( p \neq N \) (see however \[15\] for the case \( p = N \)). Fortunately, we can prove a symmetry result needed in the proof of Theorem 1.2 in the special case of eigenvalue problem, though we can not prove more general symmetry result as in \[10\].

The contents of the rest of this paper are as follows: §2 The First Eigenvalue and Eigenfunction. §3 Level Sets Formula of \( \lambda_1(\Omega) \). §4 The lower bound of \( \lambda_1(\Omega) \). §5 Proof of Theorem 1.1. §6 Proof of Theorem 1.2.

### 2 The First Eigenvalue and Eigenfunction

In this section, we give definition and some properties of the first eigenvalue and its corresponding eigenfunction of problem (1.1). We focus on the case \( 0 < \beta < +\infty \), since the case \( \beta = +\infty \) has been resolved and the case \( \beta = 0 \) is trivial.

Let \( \mathcal{K} = \{ u \in W^{1,p}(\Omega); ||u||_{L^p(\Omega)} = 1 \} \). Let

\[
\lambda_1(\Omega) = \inf \{ \int_{\Omega} |\nabla u|^p + \beta \int_{\partial\Omega} |u|^p; \ u \in \mathcal{K} \}
\]

be the first eigenvalue and \( \psi \) the corresponding eigenfunction of problem (1.1).
Proposition 2.1. Let $\lambda_1(\Omega)$ be defined as in (2.1). Then $\lambda_1(\Omega) > 0$ can be achieved by some positive function $\psi$.

Proof. Define functional $\Phi(u)$ on $\mathcal{K}$ by

$$
\Phi(u) = \int_{\Omega} |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p dx.
$$

It is obvious that $\Phi(u)$ is a convex functional. By Theorem 1.3 in Chapter 5 of [12], $\Phi(u)$ is weakly lower semi-continuous on $\mathcal{K}$. Let $\{u_j\}_{j=1}^\infty$ be a minimum sequence of $\lambda_1$ on $\mathcal{K}$, that is, $\int_{\Omega} |u_j|^p dx = 1$ and

$$
\int_{\Omega} |\nabla u_j|^p dx + \beta \int_{\partial\Omega} |u_j|^p dx \to \lambda_1(\Omega), \text{ as } j \to +\infty.
$$

Since $\{u_j\}$ is bounded in $W^{1,p}(\Omega)$ and the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, there exists $u \in W^{1,p}(\Omega)$ such that

$$
u_j \to u \text{ weakly in } W^{1,p}(\Omega),
$$

$$u_j \to u \text{ strongly in } L^p(\Omega).
$$

Hence, by the weakly lower semi-continuity of $\Phi(u)$, we have

$$
\Phi(u) \leq \lim_{j \to +\infty} \Phi(u_j) = \lambda_1(\Omega).
$$

On the other hand, we have $\lambda_1(\Omega) \leq \Phi(u)$ due to $u \in \mathcal{K}$ and the definition of $\lambda_1(\Omega)$. Thus $\lambda_1(\Omega) = \Phi(u)$. Let $\psi = |u|$ , it is easy to check that $\lambda_1(\Omega) = \Phi(\psi)$. Moreover, $\psi$ is positive in $\bar{\Omega}$ by the strong maximum principle (see Lemma 2.6 below) . Thus, we complete the proof of proposition 2.1.

Proposition 2.2. Let $\lambda_1(\Omega)$ be the first eigenvalue of problem (1.1). Then $\lambda_1(\Omega)$ is simple in the sense that if $\psi_1 > 0$ and $\psi_2 > 0$ are two eigenfunctions corresponding to $\lambda_1(\Omega)$, then $\psi_2 = C\psi_1$ and $C$ is a constant.

Proof. Suppose that $\psi_1$ and $\psi_2$ are two eigenfunctions corresponding to $\lambda_1(\Omega)$, and $\psi_1, \psi_2 > 0$. Then $\psi_i, i = 1, 2$ satisfy

\begin{equation}
(2.2)
\begin{cases}
-\text{div}(|\nabla \psi_i|^{p-2}\nabla \psi_i) = \lambda |\psi_i|^{p-2} \psi_i & \text{in } \Omega, \\
|\nabla \psi_i|^{p-2} \frac{\partial \psi_i}{\partial n} + \beta |\psi_i|^{p-2} \psi_i = 0 & \text{on } \partial\Omega.
\end{cases}
\end{equation}

Let

$$
\eta_1 = \psi_1 - \psi_2, \quad \eta_2 = \psi_2 - \psi_1
$$

Multiplying equation (2.2) by $\eta_i$ ($i = 1, 2$) and integrating by parts, we obtain

$$
\int_{\Omega} |\nabla \psi_i|^{p-2}\nabla \psi_i \cdot \nabla \eta_i + \beta \int_{\partial\Omega} \psi_i^{p-1} \eta_i - \lambda_1 \int_{\Omega} \psi_i^{p-1} \eta_i = 0, \quad (i = 1, 2).
$$
It follows
\begin{equation}
\int_{\Omega} (1 + (p - 1)(\frac{\psi_2}{\psi_1})^p)|\nabla \psi_1|^p + \int_{\Omega} (1 + (p - 1)(\frac{\psi_1}{\psi_2})^p)|\nabla \psi_2|^p
- \int_{\Omega} (p(\frac{\psi_2}{\psi_1})^{p-1}|\nabla \psi_1|^{p-2} + p(\frac{\psi_1}{\psi_2})^{p-1}|\nabla \psi_2|^{p-2})\nabla \psi_1 \cdot \nabla \psi_2 = 0.
\end{equation}

Noticing that $\nabla (\ln \psi_1) = \frac{\nabla \psi_1}{\psi_1}$, (2.3) can be rewritten as
\begin{equation}
\int_{\Omega} (\psi_1^p + (p - 1)\psi_2^p)|\nabla \ln \psi_1|^p + (\psi_2^p + (p - 1)\psi_1^p)|\nabla \ln \psi_2|^p
= p\int_{\Omega} (\psi_2^p)|\nabla \ln \psi_1|^{p-2} + \psi_1^p|\nabla \ln \psi_2|^{p-2})\nabla \ln \psi_1 \cdot \nabla \ln \psi_2
\end{equation}

Hence
\begin{equation}
\int_{\Omega} (\psi_1^p - \psi_2^p)(|\nabla \ln \psi_1|^p - |\nabla \ln \psi_2|^p)
= p\int_{\Omega} \psi_2^p|\nabla \ln \psi_1|^{p-2}(\nabla \ln \psi_1) \cdot (\nabla \ln \psi_2 - \nabla \ln \psi_1)
- p\int_{\Omega} \psi_1^p|\nabla \ln \psi_2|^{p-2}(\nabla \ln \psi_2) \cdot (\nabla \ln \psi_2 - \nabla \ln \psi_1).
\end{equation}

Observing that (see [18])
\begin{equation}
|\xi_2|^p - |\xi_1|^p \geq p|\xi_1|^{p-2}\xi_1 \cdot (\xi_2 - \xi_1) + C(p)\frac{|\xi_2 - \xi_1|^p}{2^p - 1}, \forall \xi_1, \xi_2 \in \mathbb{R}^n,
\end{equation}
we obtain
\begin{equation}
|\nabla \ln \psi_1|^p - |\nabla \ln \psi_2|^p
\geq p|\nabla \ln \psi_2|^{p-2}(\nabla \ln \psi_2) \cdot (\nabla \ln \psi_1 - \nabla \ln \psi_2) + C_1(p)\frac{|\nabla \ln \psi_1 - \nabla \ln \psi_2|^p}{2^p - 1},
\end{equation}
and
\begin{equation}
|\nabla \ln \psi_2|^p - |\nabla \ln \psi_1|^p
\geq p|\nabla \ln \psi_1|^{p-2}(\nabla \ln \psi_1) \cdot (\nabla \ln \psi_2 - \nabla \ln \psi_1) + C_2(p)\frac{|\nabla \ln \psi_2 - \nabla \ln \psi_1|^p}{2^p - 1},
\end{equation}
where $C(p), C_1(p)$ and $C_2(p)$ are positive constants depend only on $p$.

From (2.5), (2.7) and (2.8), we deduce
\[-\frac{C_1(p) + C_2(p)}{2^p - 1}\int_{\Omega} (\frac{1}{\psi_2^p} + \frac{1}{\psi_1^p})|\nabla \ln \psi_1 - \nabla \ln \psi_2|^p \geq 0.
\]
This implies that $\nabla (\ln \psi_1 - \ln \psi_2) = 0$, namely, $\psi_2 = C\psi_1$. This completes the proof of Proposition 2.2.

**Corollary 2.3.** If $\Omega = B(0)$ is a ball, then the first eigenfunction $\psi$ of problem (1.1) is radially symmetry, that is, $\psi(x) = \psi(r)$ with $r = |x|$.

**Proof.** The conclusion of Corollary 2.3 comes immediately from the simplicity of $\lambda_1(\Omega)$ and the rotational invariance of problem (1.1).

To state our next proposition of the first eigenfunction, we need the following two lemmas which were proved in the appendix of [21].
Lemma 2.4. (Weak comparison principle) Let $\Omega \in \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial \Omega$. Let $u_1, u_2 \in W^{1,p}(\Omega)$ satisfy

$$-\text{div}(|\nabla u_1|^{p-2}\nabla u_1) \leq -\text{div}(|\nabla u_2|^{p-2}\nabla u_2)$$

in weak sense. Then $u_1 \leq u_2$ on $\partial \Omega$ implies $u_1 \leq u_2$ in $\Omega$.

Lemma 2.5. (Hopf’s lemma) Let $\Omega \in \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial \Omega$. Let $u \in C^1(\Omega)$ satisfy

$$(2.9) \begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) \geq 0 & x \in \Omega, \\ u > 0 & x \in \Omega. \end{cases}$$

If $u = 0$ at $x_0 \in \partial \Omega$, then $\frac{\partial u}{\partial \nu}(x_0) < 0$, where $\nu$ denotes the unit outward vector normal to $\partial \Omega$.

We also need the following strong maximum principle which is a special case of Theorem 1.1 in [19].

Lemma 2.6. (Strong maximum principle) If $u \in C^1(\Omega)$ satisfies the following inequalities in weak sense

$$(2.10) \begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) \geq 0 & x \in \Omega, \\ u \geq 0 & x \in \Omega. \end{cases}$$

Then, $u(x_0) = 0$ for some $x_0 \in \Omega$ implies $u(x) \equiv 0$ in $\Omega$.

Proposition 2.7. Let $B_R(0)$ be a ball in $\mathbb{R}^N$ with radius $R$ and center $0$. If $\psi(x) = \psi(r)$ denotes the first eigenfunction of problem (1.1) on $B_R(0)$, then $\psi'(r) < 0$ for any $0 < r \leq R$.

**Proof:** For any fixed $r_0 \in (0, R)$, we have

$$\begin{cases} -\text{div}(|\nabla \phi(r_0)|^{p-2}\nabla \phi(r_0)) \leq -\text{div}(|\nabla \phi(x)|^{p-2}\nabla \phi(x)) & x \in B_{r_0}(0) \\ \phi(x) = \phi(r_0) & x \in \partial B_{r_0}(0). \end{cases}$$

Hence, by lemma 2.4, we have

$$\phi(x) \geq \phi(r_0) \quad x \in B_{r_0}(0).$$

Since $\phi(x)$ is not a constant, it follows from lemma 2.6 that

$$\phi(x) > \phi(r_0) \quad x \in B_{r_0}(0).$$

Let $w(x) = \phi(x) - \phi(r_0) = \phi(r) - \phi(r_0)$. Then, $w(x)$ satisfies

$$\begin{cases} -\text{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \lambda_1 \phi(x) > 0 & x \in B_{r_0}(0) \\ w(x) > 0 & x \in B_{r_0}(0) \\ w(x) = 0 & x \in \partial B_{r_0}(0). \end{cases}$$
Consequently, lemma 2.5 implies that $\phi'(r_0) < 0$. Noting that $r_0$ is arbitrary, the conclusion of proposition 2.7 then follows.

We conclude this section with the following proposition which is essential for the proof of Theorem 1.1.

**Proposition 2.8.** Let $B_R(0)$ be a ball in $\mathbb{R}^N$ with radius $R$ and center 0. Let $\psi(x) = \psi(r)$ denote the first eigenfunction of problem (1.1) on $B_R(0)$. If $g(r) = |\psi'(r)|/\psi(r)$, then $g'(r) > 0$ for $0 < r < R$, and $g(r) \leq \beta^{1/(p-1)}$ for any $r \in [0, R]$.

**Proof:** It follows from Proposition 2.7 and the standard regularity theory of elliptic equations that $\psi \in C^\infty(B_R(0) \setminus \{0\})$. Consequently, $0 < g \in C^\infty(0, R)$. Now, we compute

$$g' = \left(\frac{\psi'}{\psi}\right)' = -\frac{\psi''}{\psi} + g^2, \quad (2.11)$$

$$g'' = -\frac{\psi'''}{\psi} + 3gg' - g^3. \quad (2.12)$$

From the equation satisfied by $\psi$, we have

$$-(p-1)|\psi'|^{p-2}\psi'' - \frac{N-1}{r}|\psi'|^{p-2}\psi' = \lambda_1|\psi|^{p-1}. \quad (2.13)$$

It follows that

$$-(p-1)|\psi'|^{p-2}\psi'' - \frac{N-1}{r}\psi' = \lambda_1\frac{\psi}{g^{p-2}}. \quad (2.14)$$

Differentiating the above equation, we obtain

$$-(p-1)|\psi'|^{p-2}\psi''' - \frac{N-1}{r}|\psi'|^{p-2}\psi' - \frac{N-1}{r^2}g = \frac{\lambda_1}{g^{p-3}} - \frac{(p-2)\lambda_1g'}{g^{p-1}}. \quad (2.15)$$

Since $-\frac{\psi''}{\psi} = g' - g^2$, it follows from (2.14) that

$$\lambda_1 = -(p-1)g^{p-2}\psi' + \frac{N-1}{r}g^{p-1} = (p-1)g^{p-2}g' - (p-1)g^p + \frac{N-1}{r}g^{p-1}. \quad (2.16)$$

Hence

$$-\frac{\lambda_1}{g^{p-3}} = -(p-1)gg' + (p-1)g^3 - \frac{N-1}{r}g^2,$$

and

$$-\frac{\psi''}{\psi} = g^3 + \frac{(N-1)g}{(p-1)r^2} - fg',$$

where $f = g + \frac{N-1}{(p-1)r} + \frac{(p-2)\lambda_1}{(p-1)g^{p-1}}$. Substituting this equation into (2.12), we infer that for any $r \in (0, R)$

$$g''(r) + [f(r) - 3g(r)]g'(r) = \frac{(N-1)g(r)}{(p-1)r^2} > 0.$$
we claim that \( g' \neq 0 \) in \((0, R)\). For if there exists \( r_0 \in (0, R) \) such that \( g'(r_0) = 0 \), then \( g''(r_0) > 0 \). Hence \( r_0 \) is a minimum point of \( g \). Since \( g \geq 0 \) and \( g(0) = 0 \), it follows from the continuity of \( g \) that \( g(r_0) = 0 \). This contradicts with the fact that \( g > 0 \) in \((0, R)\). Consequently, \( g' \) has definite sign in \((0, R)\). This implies immediately that \( g' > 0 \) in \((0, R)\). For if \( g' \leq 0 \) in \((0, R)\), then we have \( g(r) \leq g(0) = 0 \) in \((0, R)\), a contradiction. Finally, the a priori estimate \( g(r) \leq \beta_1 p^{-1} \) for any \( r \in [0, R] \) follows from the facts that \( g' > 0 \) and \( g(R) = \beta_1 p^{-1} \). This completes the proof of Proposition 2.8.

3 Level Sets Formula of \( \lambda_1(\Omega) \)

For an open set \( U \subset \Omega \), we define the interior and exterior boundary of \( U \) respectively by
\[
\partial I U = \partial U \cap \Omega, \quad \partial E U = \partial U \cap \partial \Omega.
\]
Then \( \partial U = \partial I U \cup \partial E U \) is a disjoint union. For any \( \varphi \in C(\Omega) \) and \( \varphi(x) \geq 0 \), we define a functional \( H_{\Omega}(U, \varphi) \) by
\[
H_{\Omega}(U, \varphi) = \frac{1}{|U|} \left( \int_{\partial I U} \varphi \, d\sigma + \int_{\partial E U} \beta \, d\sigma - (p - 1) \int_U \varphi^{\frac{p}{p-1}} \, dx \right),
\]
where \( \sigma \) is the \((N - 1)\)-dimensional Hausdorff measure defined on \( \partial U \) and \(|U|\) is the Lebesgue measure of \( U \). Since \( \varphi \) is continuous on \( \partial I U \) and \( U \), all integrals in \( H_{\Omega}(U, \varphi) \) are well defined. In the following, we reformulate \( \lambda_1(\Omega) \) by \( H_{\Omega}(U, \varphi) \). To this end, we always denote by \( \psi \) the first eigenfunction of \((1.1)\) and sometimes denote \( \psi \) by \( \psi_{\Omega} \) when we want to emphasize the dependence of \( \psi \) on the domain \( \Omega \). Furthermore, we choose \( \psi \) so that \( \psi > 0 \) and \( \|\psi\|_{L^\infty(\Omega)} = 1 \). By regularity results of DiBenedetto \[7\] and Tolksdorf \[23, 24\], we know that \( \psi \) belongs to \( C^{1, \alpha}(\Omega) \) for some \( 0 < \alpha < 1 \). Let
\[
m = \min_{x \in \Omega} \psi(x).
\]
Then, by Hopf’s boundary point Lemma we have \( m > 0 \). For any \( t \in (m, 1) \), we denote by \( U_t \) the level set of \( \psi \), that is
\[
U_t = \{ x \in \Omega; \psi(x) > t \},
\]
then \( U_t \) is open and the interior boundary of \( U_t \) is the level surface
\[
S_t = \partial I U_t = \{ x \in \Omega; \psi(x) = t \}.
\]
Hence, \( S_t = \emptyset \), if \( t \notin (m, 1) \). Bearing all notations \( \psi, U_t \) and \( S_t \) in mind, we prove

**Proposition 3.1.** Let \( \lambda_1(\Omega) \) be the first eigenvalue of problem \((1.1)\), \( \psi \) be the corresponding eigenfunction, and \( H_{\Omega}(U, \varphi) \) be defined as in \((3.1)\). Then
\[
\lambda_1(\Omega) = H_{\Omega}(U_t, \frac{\nabla \psi^{p-1}}{\psi^{p-1}}) \quad \text{for} \quad t \in (m, 1).
\]
As in [5], second partial derivatives of eigenfunction will be involved in the proof of Proposition 3.1. However, it is well known that, in general, the best possible regularity results of Problem (1.1) is $C^{1,\alpha}$. Hence, to prove Proposition 3.1, we consider the following regularized problem

\[
\begin{aligned}
-\text{div} (|\varepsilon u \varepsilon|^2 \frac{\partial^2}{\partial x^2} u \varepsilon) &= \lambda_\varepsilon |u \varepsilon|^{p-2} u \varepsilon - \varepsilon (|\varepsilon u \varepsilon|^2 + |\nabla u \varepsilon|^2) \frac{\partial^2}{\partial x^2} u \varepsilon, \quad x \in \Omega, \\
(\varepsilon u \varepsilon^2 + |\nabla u \varepsilon|^2) \frac{\partial^2}{\partial \nu} u \varepsilon + \beta u \varepsilon^{p-1} &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]

where $1 < p < +\infty$, $\nu$ is the outward unit normal of $\partial \Omega$.

Define $\lambda_\varepsilon^1(\Omega)$ by

$$
\lambda_\varepsilon^1(\Omega) = \inf \left\{ \int_\Omega (\varepsilon u \varepsilon^2 + |\nabla u \varepsilon|^2)^\frac{p}{2} + \beta \int_{\partial \Omega} u \varepsilon^p; \ u \varepsilon \in K \right\}.
$$

Then, we have

**Lemma 3.2.** For any $\varepsilon > 0$, $\lambda_\varepsilon^1(\Omega)$ is attained by a positive function $\psi \varepsilon \in K$. Moreover, up to a subsequence, we have $\lim_{\varepsilon \to 0} \lambda_\varepsilon^1(\Omega) = \lambda_1(\Omega)$, and

$$
\lim_{\varepsilon \to 0} \psi \varepsilon = \hat{\psi} \quad \text{in} \quad C^1(\overline{\Omega})
$$

where $\lambda_1(\Omega)$ is the first eigenvalue of problem (1.1) and $\hat{\psi}$ is a corresponding eigenfunction with $||\hat{\psi}||_{L^p(\Omega)} = 1$.

**Proof.** The conclusion that $\lambda_\varepsilon^1(\Omega)$ is attained by a positive function $\psi \varepsilon$ can be proved in the same way as that of Proposition 2.1. To prove the second part of Lemma 3.2, We first note that $\psi \varepsilon$ is bounded in $W^{1,p}(\Omega)$ when $\varepsilon$ is small enough. Hence, up to a subsequence, we may assume that

$$
\psi \varepsilon \rightharpoonup \tilde{\psi} \quad \text{weakly in} \quad W^{1,p}(\Omega), \quad \text{as} \ \varepsilon \to 0.
$$

Since $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, we also have $\tilde{\psi} \in K$. Because convex functional is weakly lower semi-continuous, we have

\[
\int_\Omega |\nabla \tilde{\psi}|^p + \beta \int_{\partial \Omega} |\tilde{\psi}|^p \leq \liminf_{\varepsilon \to 0} \left( \int_\Omega |\nabla \psi \varepsilon|^p + \beta \int_{\partial \Omega} |\psi \varepsilon|^p \right)
\]

On the other hand, if $\tilde{\psi}$ is the first eigenfunction of problem (1.1) with $||\tilde{\psi}||_{L^p(\Omega)} = 1$, then by the definition of $\lambda_1(\Omega)$ and $\lambda_\varepsilon^1(\Omega)$, we have

\[
\lambda_1(\Omega) = \int_\Omega |\nabla \tilde{\psi}|^p + \beta \int_{\partial \Omega} |\tilde{\psi}|^p \leq \int_\Omega |\nabla \psi \varepsilon|^p + \beta \int_{\partial \Omega} |\psi \varepsilon|^p
\]

\[
= \lambda_\varepsilon(\Omega) \leq \int_\Omega (\varepsilon \tilde{\psi}^2 + |\nabla \tilde{\psi}|^2)^\frac{p}{2} + \beta \int_{\partial \Omega} \tilde{\psi}^p
\]

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Let $\varepsilon \to 0$ on the both side of the above inequality, we obtain

\begin{equation}
\lim_{\varepsilon \to 0} \lambda_\varepsilon(\Omega) = \lim_{\varepsilon \to 0} \int_{\Omega} |\nabla \psi_\varepsilon|^p + \beta \int_{\Omega} |\psi_\varepsilon|^p = \int_{\Omega} |\nabla \hat{\psi}|^p + \beta \int_{\Omega} |\hat{\psi}|^p = \lambda_1(\Omega).
\end{equation}
(3.6)

From (3.4) and (3.6), we infer that

\begin{align*}
\int_{\Omega} |\nabla \hat{\psi}|^p + \beta \int_{\partial \Omega} |\hat{\psi}|^p &\leq \int_{\Omega} |\nabla \hat{\psi}|^p + \beta \int_{\partial \Omega} |\hat{\psi}|^p = \lambda_1(\Omega).
\end{align*}

Hence, $\hat{\psi}$ is a minimizer of $\lambda_1(\Omega)$. This implies that $\hat{\psi} = \hat{\psi}$ due to the simplicity of $\lambda_1(\Omega)$ and $||\hat{\psi}||_{L^p(\Omega)} = ||\hat{\psi}||_{L^p(\Omega)} = 1$. Consequently, $\psi_\varepsilon \to \hat{\psi}$ weakly in $W^{1,p}(\Omega)$ as $\varepsilon \to 0$. Finally, by the regularity theory of Tolksdorf [23, 24] and DiBenedetto [7], we know that for $\varepsilon \in (0,1)$, there exists $\alpha \in (0,1)$ and a positive constant $C$ independent of $\varepsilon$ such that $||\psi_\varepsilon||_{C^{1,\alpha}(\overline{\Omega})} \leq C$. Hence, up to a subsequence, $\psi_\varepsilon$ converges to $\hat{\psi}$ in $C^1(\overline{\Omega})$. This completes the proof of Lemma 3.2.

**Proof of Proposition 3.1.** For any fixed $t \in (m, 1)$, let $\nu$ denote the outward unit vector normal to $\partial U_t$. If we denote by $\psi_\varepsilon$ the solution of Problem (3.3) obtained in Lemma 3.2, then by the standard regularity theory of elliptic equations we know that $\psi_\varepsilon \in C^\infty(\Omega)$. Hence, by divergence Theorem, we have

\begin{equation}
- \int_{\partial U_t} \frac{(\varepsilon \psi_\varepsilon + |\nabla \psi_\varepsilon|^2)^{p-2}}{\psi_\varepsilon^p} \frac{\partial \psi_\varepsilon}{\partial \nu} d\sigma \\
= - \int_{U_t} \text{div} \left( \frac{(\varepsilon \psi_\varepsilon + |\nabla \psi_\varepsilon|^2)^{p-2}}{\psi_\varepsilon^p} \nabla \psi_\varepsilon \right) dx \\
= - \int_{U_t} \text{div} \left( \frac{(\varepsilon \psi_\varepsilon + |\nabla \psi_\varepsilon|^2)^{p-2}}{\psi_\varepsilon^p} \nabla \psi_\varepsilon \right) dx + (p-1) \int_{U_t} \frac{(\varepsilon \psi_\varepsilon + |\nabla \psi_\varepsilon|^2)^{p-2}}{\psi_\varepsilon^p} \frac{|\nabla \psi_\varepsilon|^2}{|\psi_\varepsilon|^2} dx \\
= \lambda_1^\varepsilon(\Omega)|U_t| - \varepsilon \int_{U_t} \frac{(\varepsilon \psi_\varepsilon + |\nabla \psi_\varepsilon|^2)^{p-2}}{\psi_\varepsilon^p} \frac{|\nabla \psi_\varepsilon|^2}{|\psi_\varepsilon|^2} dx + (p-1) \int_{U_t} \frac{(\varepsilon \psi_\varepsilon + |\nabla \psi_\varepsilon|^2)^{p-2}}{\psi_\varepsilon^p} \frac{|\nabla \psi_\varepsilon|^2}{|\psi_\varepsilon|^2} dx.
\end{equation}
(3.7)

Passing to the limit in (3.7) as $\varepsilon \to 0$, we obtain

\begin{equation}
- \int_{\partial U_t} \frac{|\nabla \psi|^{p-2}}{\psi^{p-1}} \frac{\partial \psi}{\partial \nu} d\sigma = \lambda_1(\Omega)|U_t| + (p-1) \int_{U_t} \frac{|\nabla \psi|^p}{\psi^p} dx.
\end{equation}
(3.8)

Since $\lambda_1(\Omega)$ is simple, we have

\begin{equation}
- \int_{\partial U_t} \frac{|\nabla \psi|^{p-2}}{\psi^{p-1}} \frac{\partial \psi}{\partial \nu} d\sigma = \lambda_1(\Omega)|U_t| + (p-1) \int_{U_t} \frac{|\nabla \psi|^p}{\psi^p} dx.
\end{equation}
(3.9)

By the boundary condition, we have

\begin{equation}
\beta = - \frac{|\nabla \psi|^{p-2}}{\psi^{p-1}} \frac{\partial \psi}{\partial \nu}, \quad x \in \partial E U_t,
\end{equation}
where $E U_t \subset \partial \Omega$. Noticing further that $|\nabla \psi| = -\frac{\partial \psi}{\partial \nu}$ on $S_t$, we obtain from the definitions of $S_t$ and $\partial E U_t$ that

\begin{equation}
- \int_{\partial U_t} \frac{|\nabla \psi|^{p-2}}{\psi^{p-1}} \frac{\partial \psi}{\partial \nu} d\sigma = \int_{S_t} \frac{|\nabla \psi|^{p-2}}{\psi^{p-1}} |\nabla \psi| d\sigma + \int_{\partial E U_t} \beta \ d\sigma.
\end{equation}
(3.10)

Now, the conclusion of Proposition 3.1 follows from (3.9) and (3.10).
4 The lower bound of $\lambda_1(\Omega)$

In this section, we give a lower bound of $\lambda_1(\Omega)$. Let

$$M_\beta = \{ \varphi(x) \in C(\Omega); \varphi(x) \geq 0, \lim_{x \to z} \varphi(x) \leq \beta, \forall z \in \partial \Omega \}. \tag{4.1}$$

Keep in use the same notations $\psi$, $U_t$ and $S_t$ as in the previous section. Since $\psi(x) \in C^1(\Omega)$, it is easy to see that $((\nabla \psi)^{p-1}) \in M_\beta$ if and only if $\psi$ is a constant on $\partial \Omega$. In fact, if $\psi$ is a constant on $\partial \Omega$ then $\frac{\partial \psi}{\partial \nu} = -|\nabla \psi|$ on $\partial \Omega$. Hence

$$\frac{|\nabla \psi|^{p-1}}{\psi^{p-1}} = -\frac{|\nabla \psi|^{p-2} \partial \psi}{\psi^{p-1}} = \beta \text{ on } \partial \Omega.$$ 

This implies that $((\nabla \psi)^{p-1}) \in M_\beta$.

On the other hand, if $((\nabla \psi)^{p-1}) \in M_\beta$, then

$$\frac{|\nabla \psi|^{p-1}}{\psi^{p-1}} \leq \beta = -\frac{|\nabla \psi|^{p-2} \partial \psi}{\psi^{p-1}} \leq \frac{|\nabla \psi|^{p-1}}{\psi^{p-1}} \forall \, x \in \partial \Omega.$$ 

Hence $\frac{\partial \psi}{\partial \nu} = -|\nabla \psi|$ for all $x \in \partial \Omega$, which implies that $\psi$ is a constant on $\partial \Omega$.

The main results of this section can be stated as

**Theorem 4.1.** For every $\varphi \in M_\beta$, there exists a set $I \subset [0, 1]$ with positive measure such that

$$\lambda_1(\Omega) \geq H_\Omega(U_t, \varphi) \text{ for all } t \in I. \tag{4.2}$$

**Theorem 4.2.** Let $\varphi(x) \in M_\beta$, and $\psi$ be the first eigenfunction of problem (1.1). If $\varphi \neq \frac{|\nabla \psi|^{p-1}}{\psi^{p-1}}$, then there exists a set $I \subset [m, 1]$ with positive measure such that

$$H_\Omega(U_t, \varphi) < \lambda_1(\Omega) \text{ for all } t \in I.$$ 

To prove Theorems, we prove some lemmas first. For any given $\varphi$ and $\varphi \geq 0$, let

$$\omega(x) := \varphi(x) - \frac{|\nabla \psi|^{p-1}}{\psi^{p-1}}, \quad x \in \Omega. \tag{4.3}$$

Then we have

**Lemma 4.3.** For any $\varphi \in M_\beta$, let $\omega$ be defined as (4.3). Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\omega(x) \leq \varepsilon$ for all $x \in \Omega$ with $\text{dist}(x, \partial \Omega) < \delta$.

**Proof.** Since $\frac{|\nabla \psi(x)|^{p-1}}{\psi(x)^{p-1}}$ is continuous on the compact set $\overline{\Omega}$, we have that for any fixed $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$\left| \frac{|\nabla \psi(x)|^{p-1}}{\psi(x)^{p-1}} - \frac{|\nabla \psi(z)|^{p-1}}{\psi(z)^{p-1}} \right| < \frac{\varepsilon}{2} \quad \text{for any } x, z \in \overline{\Omega}, \ |x - z| < \delta_0. \tag{4.4}$$
For any fixed \( z \in \partial \Omega \) and the above fixed \( \varepsilon \), by the assumption that \( \lim_{x \to z} \varphi \leq \beta \), we can choose \( r_z > 0 \) such that

\[
\sup_{x \in B(z, r_z) \cap \Omega} \varphi(x) \leq \beta + \frac{\varepsilon}{2},
\]

that is

\[
(4.5) \quad \varphi(x) - \beta \leq \frac{\varepsilon}{2}, \quad \text{for all } x \in B(z, r_z) \cap \Omega,
\]

where \( B(z, r_z) \) denotes the ball with radius \( r_z \) and center \( z \). Since the set \( \{B(z, r_z), z \in \partial \Omega\} \) of balls form an open cover of the compact set \( \partial \Omega \), we can select a finite sub-cover \( \{B(z_i, r_i)\}_{i=1}^{n} \) with \( r_i = r_{z_i} \). Let \( \delta \leq \min\{r_1, r_2, \ldots, r_n, \delta_0\} \) be so small that \( x \in \bigcup_{i=1}^{n} B(z_i, r_i) \) whenever \( x \in \Omega \) satisfying \( \text{dist}(x, \partial \Omega) < \delta \). Then, for any \( x \in \Omega \) with \( \text{dist}(x, \partial \Omega) < \delta \), there exists \( i_0 \in \{1, 2, \ldots, n\} \) such that \( x \in B(z_{i_0}, r_{i_0}) \). By the boundary condition, we have

\[
(4.6) \quad \beta = -\frac{|\nabla \psi|^{p-2} \frac{\partial \psi}{\partial \nu}(z_{i_0})}{\psi^{p-1}} \leq \frac{|\nabla \psi|^{p-1}}{\psi^{p-1}}(z_{i_0}).
\]

It follows from (4.4), (4.5) and (4.6) that for any \( x \in \Omega \) with \( \text{dist}(x, \partial \Omega) < \delta \), we have

\[
\omega(x) = \varphi(x) - \frac{|\nabla \psi(x)|^{p-1}}{\psi^{p-1}} = \varphi(x) - \beta + \beta - \frac{|\nabla \psi(x)|^{p-1}}{\psi^{p-1}} - \frac{|\nabla \psi(z_{i_0})|^{p-1}}{\psi^{p-1}} - \frac{|\nabla \psi(x)|^{p-1}}{\psi^{p-1}}
\]

\[
\leq \frac{\varepsilon}{2} + \frac{|\nabla \psi(z_{i_0})|^{p-1}}{\psi^{p-1}} - \frac{|\nabla \psi(x)|^{p-1}}{\psi^{p-1}}
\]

\[
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

This is just the desired conclusion of Lemma 4.3.

**Lemma 4.4.** Suppose that \( \varphi \in C(\Omega) \) is non-negative such that \( \varphi \in L^1(U) \) for every open set \( U \subset \Omega \). Let \( \omega \) be defined as (4.3). Set

\[
F(t) := \int_{t}^{1} \frac{1}{\tau} \int_{S_{\tau}} \omega d\sigma d\tau, \quad \text{for } t \in (m, 1).
\]

Then \( F \) is absolutely continuous on \( (\varepsilon, 1) \) for all \( \varepsilon \in (0, 1) \) and

\[
\frac{d}{dt} F(t) = -\frac{1}{t} \int_{S_t} \omega d\sigma,
\]

for almost all \( t \in (0, 1) \).

**Proof.** Fix \( \varepsilon \in (0, 1) \). By the assumption \( \varphi \in C(\Omega) \cap L^1(U_{\varepsilon}) \) and the co-area formula, we have

\[
\int_{\varepsilon}^{1} \frac{1}{\tau} \int_{S_{\tau}} \varphi d\sigma d\tau = \int_{U_{\varepsilon}} \frac{\varphi}{\psi} |\nabla \psi| \, dx < \infty
\]

and

\[
\int_{\varepsilon}^{1} \frac{1}{\tau} \int_{S_{\tau}} \frac{|\nabla \psi|^{p-1}}{\psi^{p-1}} d\sigma d\tau = \int_{U_{\varepsilon}} \frac{|\nabla \psi|^{p}}{\psi^{p}} \, dx < \infty.
\]
Let
\[ f(\tau) := \frac{1}{\tau} \int_{S_\tau} \omega \, d\sigma, \]
Then \( f(\tau) \in L^1((\varepsilon, 1)) \), thus \( F(t) = \int_1^t f(\tau) \, d\tau \) is absolutely continuous on \((\varepsilon, 1)\) and differentiable almost everywhere. Moreover
\[ F'(t) = -f(t) = -\frac{1}{t} \int_{S_t} \omega \, d\sigma. \]
This completes the proof of Lemma 4.4.

To state our next Lemma, we recall more regularity results of the first eigenfunction \( \psi \). By the boundary condition and the Hopf’s boundary point Lemma, we know that \( \psi(x) > 0 \) for any \( x \in \partial \Omega \). Consequently \( |\nabla \psi|(x) > 0 \) for any \( x \in \partial \Omega \). Since \( \partial \Omega \) is compact and \( |\nabla \psi| \in C(\overline{\Omega}) \), it is easy to prove that there exists positive number \( \alpha > 0 \) and a neighborhood \( N \) of \( \partial \Omega \) in \( \Omega \) such that \( |\nabla \psi|(x) \geq \alpha > 0 \) for any \( x \in N \). This implies that \( p\)-Laplace is uniformly elliptic in \( N \). Hence, by the interior regularity theorem of elliptic equations, we know that \( \psi \in C^\infty(N) \). If we let \( m = \min \{ \psi(x); x \in \overline{\Omega} \} \) and \( K = \{ x \in \Omega; \psi(x) = m \} \), then by strong maximum principle we know that \( K \subset \partial \Omega \). Noticing furthermore that \( K \) is compact, there exists \( t_0 \in (m, 1) \) small enough such that
\[ S_t \subset N \quad \text{for any} \quad t \leq t_0. \]

An argument similar to that used by Daners in [5] implies the following lemma since all computations in [5] are local.

**Lemma 4.5.** Let \( U_t \) and \( S_t \) be defined as in section 3. Then \( U_t \) is a Lipschitz domain, moreover, there exist \( t_1 \in (m, t_0) \) and a constant \( C > 0 \) independent of \( t \) such that \( \sigma(S_t) \leq C \sigma(\partial \Omega) \) for all \( t \in (m, t_1) \).

**Proof of Theorem 4.1.** We give a proof by contradiction. Suppose that there exists \( \varphi \in \mathcal{M}_\beta \) such that
\[ \lambda_1(\Omega) < H_\Omega(U_t, \varphi) \quad \text{for almost all} \quad t \in (m, 1). \]
Let \( \omega \) be defined as \([4.3]\) and \( F(t) \) be defined as in Lemma 4.4, that is
\[ \omega(x) := \varphi(x) - \frac{|\nabla \psi|^{p-1}}{\psi^{p-1}}, \quad x \in \Omega. \]
and
\[ F(t) := \int_t^1 \frac{1}{\tau} \int_{S_\tau} \omega \sigma d\tau, \quad \text{for all} \quad t \in (m, 1). \]
Then by \([4.3]\), the definition of \( H_\Omega(U_t, \varphi) \) and Proposition 3.1, we have
\[ \int_{S_t} \omega \, d\sigma - (p-1) \int_{U_t} (|\nabla \psi|^p - \frac{|\nabla \psi|^p}{\psi^p}) \, dx = |U_t|[H_\Omega(U_t, \varphi) - \lambda_1(\Omega)] > 0. \]
By Taylor’s expansion, there holds
\[
\varphi^{p-1} - \frac{\left|\nabla \varphi\right|^{p}}{p^{p-1}} = \left(\frac{\left|\nabla \varphi\right|^{p-1}}{p^{p-1}} + \omega\right)^{p-1} - \frac{\left|\nabla \varphi\right|^{p}}{p^{p-1}} = \frac{p}{p-1}\left(\frac{\left|\nabla \varphi\right|^{p-1}}{p^{p-1}} + \omega\right)^{\frac{1}{p-1}} + \frac{1}{2}\frac{1}{p-1}\xi^{\frac{2}{p-1}}\omega^{2},
\]
where \(\xi\) is a nonnegative function with value between \(\varphi\) and \(\frac{\left|\nabla \varphi\right|^{p-1}}{p^{p-1}}\).

From (4.10), (4.11), the co-area formula and the definition of \(H\), we obtain
\[
\int_{S_{t}} \omega d\sigma > p\int_{U_{t}} \left|\frac{\nabla \varphi}{\varphi}\right| \omega dx = p\int_{t}^{1} \int_{S_{t}} \frac{1}{\tau} \omega d\sigma d\tau = pF(t)
\]
for almost all \(t \in (m, 1)\).

It follows from Lemma 4.4 and the above inequality that
\[
\frac{d}{dt}(t^{p}F(t)) = -t^{p}f(t) + pt^{p-1}F(t) = t^{p-1}(\int_{S_{t}} \omega d\sigma + pF(t)) < 0
\]
for almost all \(t \in (m, 1)\).

Hence, the function \(t^{p}F(t)\) is strictly decreasing on \((m, 1)\). Since \(F(1) = 0\) and \(F(t)\) is continuous on \((m, 1)\), there exists \(\eta > 0\) and \(t_{2} \in (m, 1)\) such that \(F(t) > \eta\) for \(t \in (m, t_{2}]\). On the other hand, by Lemma 4.5, there exists \(t_{3} \in (m, t_{2}]\) and a constant \(C > 0\) such that \(\sigma(S_{t}) \leq C\sigma(\partial \Omega)\) for \(t \in (m, t_{3})\). Set
\[
\varepsilon_{0} = \frac{\eta}{C\sigma(\partial \Omega)}.
\]
For this fixed \(\varepsilon_{0}\), it follows from Lemma 4.3 that there exists \(\delta_{0} > 0\) such that \(\omega(x) \leq \varepsilon_{0}\) for any \(x \in \Omega\) with \(\text{dist}(x, \partial \Omega) < \delta_{0}\). Noticing that \(\psi\) attains its strict minimum on \(\partial \Omega\), we can choose \(0 < t_{4} < t_{3}\) so small that \(\text{dist}(x, \partial \Omega) < \delta_{0}\) for any \(x \in S_{t}\) and \(t \in (m, t_{4})\). Hence, for any \(t \in (m, t_{4})\), there holds
\[
 pn\eta \leq pF(t) < \int_{S_{t}} \omega d\sigma \leq \varepsilon_{0}\sigma(S_{t}) \leq \varepsilon_{0}C\sigma(\partial \Omega) \leq \eta
\]
which is a contradiction. Thus we complete the proof of Theorem 4.1.

**Proof of Theorem 4.2.** We give a proof by contradiction. Assume that \(\varphi \neq \frac{\left|\nabla \varphi\right|^{p-1}}{p^{p-1}}\) and that
\[
H_{2}\left(U_{t}, \varphi\right) \geq \lambda_{1}(\Omega), \quad \text{for almost all } t \in (m, 1).
\]

Similar to the proof of Theorem 4.1, by the definition of \(H_{2}\left(U_{t}, \varphi\right)\) and Proposition 3.1, we have
\[
\int_{S_{t}} \omega d\sigma \geq pF(t) + \frac{1}{2} \frac{p}{p-1} \int_{U_{t}} \xi^{\frac{2}{p-1}}\omega^{2}dx \quad \text{for almost all } t \in (m, 1),
\]
and
\[
\frac{d}{dt}(t^{p}F(t)) \leq \frac{1}{2} \frac{p}{p-1} \int_{U_{t}} \xi^{\frac{2}{p-1}}\omega^{2}dx \leq 0, \quad \text{for almost all } t \in (m, 1),
\]
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where $\xi$ is a nonnegative function with value between $\varphi$ and $\frac{|\nabla \psi|^{p-1}}{\psi^{p-1}}$. Hence $t^p F(t)$ is nonincreasing in $(m, 1)$. Since $\omega(x) \in C(\Omega)$, $\omega(x) \neq 0$ and $\bigcup_{t \in (m, 1)} U_t = \Omega$, there exists $t_0 \in (m, 1)$ such that

$$ \int_{U_{t_0}} \frac{2-p}{p-1} \omega^2 dx > 0 \tag{4.14} $$

Moreover, if $t_1, t_2 \in (m, 1)$ satisfy $t_1 < t_2$, then we have $U_{t_2} \subset U_{t_1}$. Hence, the map

$$ t \mapsto \int_{U_t} \frac{2-p}{p-1} \omega^2 dx $$

is non-increasing in $(m, 1)$ and $\int_{U_1} \frac{2-p}{p-1} \omega^2 dx = 0$ due to $U_1 = \emptyset$.

Let

$$ t^* = \sup \{ t \in (m, 1), \int_{U_t} \frac{2-p}{p-1} \omega^2 dx > 0 \}. $$

From (4.14), we know that $t^* \in (m, 1]$ and thus $t^p F(t)$ is strictly decreasing on $(m, t^*)$ and non-increasing on $[t^*, 1]$, similar to the proof of Theorem 4.1, there exists $t_3 \in (m, t^*)$ such that for any $t \in (m, t_3)$,

$$ p\eta < p F(t) < \int_{S_t} \omega \, d\sigma \leq \varepsilon \sigma(S_t) \leq \eta, $$

which is a contradiction. Hence, we complete the proof of Theorem 4.2.

5 Proof of Theorem 1.1

This section devotes to prove Theorem 1.1. To this end, we denote by $\lambda_1(\Omega)$ the first eigenvalue of problem (1.1) on the domain $\Omega$ and $\psi_\Omega$ denotes its corresponding eigenfunction. Furthermore, $B = B_R(0)$ denotes the ball with radius $R$ and center $0$ such that $|B| = |\Omega|$. Let $U_t$ be the level set and $S_t$ be the level surface of $\psi_\Omega$ at level $t$ defined in section 2, and $B_{r(t)}(0)$ be the ball with radius $r(t)$ and center $0$ such that $|B_{r(t)}(0)| = |U_t|$. Define

$$ \Phi_B(x) = \frac{|\nabla \psi_B(x)|^{p-1}}{\psi_B^{p-1}(x)} \quad \text{for} \quad x \in B_R(0). $$

By Corollary 2.3, $\Phi_B$ is radially symmetry. So, we only need to consider the radial function

$$ G(r) = \Phi_B(|x|) = \frac{|\psi_B'(r)|^{p-1}}{\psi_B^{p-1}(r)} = g^{p-1}(r) \quad \text{for} \quad r \in (0, R) $$

where $g(r)$ is the function defined in Proposition 2.8. Then by Proposition 2.8, we know that $G(r)$ is strictly increasing in $(0, R)$. Consequently, $G(r) \leq G(R) = \beta$ for any $r \in [0, R]$. we construct our test function as the following.
For any \( t \in (m, 1) \) and \( x \in S_t \), we set
\[
\Phi(x) = G(r(t)).
\]

It is obvious that \( \Phi \) is well defined since \( \Omega \) is a disjoint union of \( S_t, t \in (m, 1] \). Moreover, \( \Phi \in \mathcal{M}_\beta(\Omega) \) due to \( \Phi \) is continuous and \( \Phi(x) \leq \beta \) for all \( x \in \overline{\Omega} \). It is also not too difficult to see that
\[
\int_{U_t} |\Phi|^{\frac{p}{p-1}} \, dx = \int_{B_{r(t)}} \Phi^\frac{p}{p-1} \, dx.
\]

Since by the construction the level sets of \( \Phi \) and \( \Phi_B \) have the same measure. Now, we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Since \( \Phi \in \mathcal{M}_\beta \), we conclude from Theorem 4.1 that there exist a set \( I \subset (m, 1) \) with positive measure such that
\[
\lambda_1(\Omega) \geq H\Omega(U_t, \Phi) \quad \text{for all } t \in I.
\]

Noticing that \( \sigma(\partial B_{r_t}) \leq \sigma(\partial U_t) \) for all \( t \in (m, 1] \), and \( \Phi(x) = G(r(t)) \leq \beta \) when \( x \in S_t \), we have
\[
\int_{\partial B_{r(t)}} \Phi_B(x) \, d\sigma = G(r(t))\sigma(\partial B_{r(t)}) \leq G(r(t))\sigma(\partial U_t)
\]
\[
= G(r(t))(\int_{S_t} d\sigma + \int_{\partial E U_t} d\sigma)
\]
\[
\leq \int_{S_t} \Phi \, d\sigma + \int_{\partial E U_t} \beta \, d\sigma.
\]

Hence, from (5.1), (5.3) and the definitions of \( H_B(B_{r(t)}(0), \Phi_B) \) and \( H\Omega(U_t, \Phi) \), we have
\[
(5.4) \quad H_B(B_{r(t)}(0), \Phi_B) \leq H\Omega(U_t, \Phi) \quad \forall t \in (m, 1).
\]

Since, by Proposition 3.1, we have \( \lambda_1(B) = H_B(B_{r(t)}(0), \Phi_B) \) for any \( t \in (m, 1) \), it follows from (5.2) and (5.4) that
\[
\lambda_1(\Omega) \geq \lambda_1(B).
\]

This completes the proof of Theorem 1.1.

**6 Proof of Theorem 1.2**

This section devotes to prove Theorem 1.2. To this end, we keep in use of all notations in section 5, and prove some lemmas first.

**Lemma 6.1.** Suppose that \( \Omega \) satisfies that \( \lambda_1(\Omega) = \lambda_1(B) \) with \( |\Omega| = |B| \). Then
\[
\Phi = \frac{|
abla \psi_{\Omega}|^{p-1}}{\psi_{\Omega}^{p-1}} \quad \text{and} \quad H\Omega(U_t, \Phi) = \lambda_1(B),
\]

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for almost all \( t \in (m, 1) \).

**Proof.** If \( \lambda_1(\Omega) = \lambda_1(B) \), then by Proposition 3.1 and (5.4), we have \( \lambda_1(\Omega) = \lambda_1(B) = H_B(B(r(t)), G) \leq H_{\Omega}(U_t, \Phi) \) for almost all \( t \in (m, 1) \). Hence by Theorem 4.2, \( \Phi = \frac{\nabla \psi_\Omega}{\psi_\Omega} \). Again, by Proposition 3.1, we obtain \( H_{\Omega}(U_t, \Phi) = \lambda_1(\Omega) = \lambda_1(B) \), for almost all \( t \in (m, 1) \).

**Lemma 6.2.** Let \( \psi_\Omega \) be the eigenfunction corresponding to the first eigenvalue \( \lambda_1(\Omega) \), and \( U_t \) be the level set of \( \psi_\Omega \). Then \( H_{\Omega}(U_t, \Phi) = \lambda_1(B) \) if and only if \( U_t \) is a ball and \( \sigma(\partial E U_t) = 0 \).

**Proof.** It follows from Proposition 3.1 that \( \lambda_1(B) = H_B(B(r(t)), G) \) for all \( t \in (m, 1) \). By the construction of \( G \) and \( \Phi \), we know that the level sets of \( G \) and \( \Phi \) have the same measure. Hence

\[
\int_{U_t} |\Phi|^{\frac{p-1}{p}} \, dx = \int_{B_{r(t)}} |G|^{\frac{p-1}{p}} \, dx, \quad \text{for all } t \in (m, 1).
\]

Using the definitions of \( H_{\Omega}(U, \varphi) \) and \( \Phi \), we have

\[
H_{\Omega}(U_t, \Phi) = \frac{1}{|U_t|} \int_{\partial U_t} \Phi \, d\sigma + \int_{\partial E U_t} \beta d\sigma - (p-1) \int_{U_t} \Phi^{\frac{p}{p-1}} \, dx,
\]

\[
= \frac{1}{|B_{r(t)}|} [G(r(t))\sigma(S_t) + \beta \sigma(\partial E U_t) - (p-1) \int_{B_{r(t)}} G^{\frac{p}{p-1}} \, dx].
\]

If \( U_t \) is a ball and \( \sigma(\partial E U_t) = 0 \), then \( \sigma(S_t) = \sigma(\partial B_{r(t)}) \) and

\[
H_{\Omega}(U_t, \Phi) = \frac{1}{|B_{r(t)}|} [G(r(t))\sigma(\partial B_{r(t)}) - (p-1) \int_{B_{r(t)}} G^{\frac{p}{p-1}} \, dx]
\]

\[
= H_B(B_{r(t)}, G) = \lambda_1(B).
\]

Conversely, if \( H_{\Omega}(U_t, \Phi) = \lambda_1(B) \), then for this \( t \),

\[
G(r(t))\sigma(S_t) + \beta \sigma(\partial E U_t) = G(r(t))\sigma(\partial B_{r(t)}).
\]

Noticing that \( S_t = \partial U_t - \partial E U_t \), we have

\[
\sigma(\partial E U_t)(\beta - G(r(t))) = G(r(t))(\sigma(\partial B_{r(t)}) - \sigma(U_t)).
\]

This is only possible when \( \sigma(\partial E U_t) = 0 \) and \( \sigma(\partial B_{r(t)}) = \sigma(U_t) \), since \( 0 < G(r(t)) < \beta \) for all \( t \in (m, 1) \) and \( |B_{r(t)}| = |U_t| \) implies \( \sigma(\partial B_{r(t)}) \leq \sigma(U_t) \). But we know that the ball is the unique minimizer of the isoperimetric inequality. Hence, \( U_t = B_{r(t)} + z \) for some \( z \in \mathbb{R}^N \). This completes the proof of Lemma 6.2.

**Lemma 6.3.** Assume that \( u(x) \geq 0 \) satisfies that \( -\text{div}(|\nabla u|^{p-2} \nabla u) = \lambda u^{p-1} \) in \( \Omega \) for some \( \lambda > 0 \). Suppose further that for some \( t > 0 \) the level set \( \{ x \in \Omega, u(x) > t \} = B_{r(t)}(x_0) \) is a ball with radius \( r(t) \) and center \( x_0 \). If \( u \in C(B_{r(t)}(x_0)) \) and \( \sigma(\partial E B_{r(t)}(x_0)) = 0 \), then \( u \) is radially symmetric with respect to \( x_0 \) in \( B_{r(t)}(x_0) \).

This lemma is crucial to the proof of Theorem 1.2. In the case \( p = 2 \), the conclusion of the Lemma 6.3 is a famous result due to Gidas, Ni and Nirenberg [10] (see also
Corollary 3.4 in [9]. In the case $1 < p < 2$, the conclusion of Lemma 6.3 was given in [4]. In the case $p = N$, the conclusion of Lemma 6.3 was proved in [15]. However, the conclusion of Lemma 6.3 for the case $p > 2$ and $p \neq N$ is not available so far. Here, we give a proof of Lemma 6.3 for all $p \in (1, +\infty)$.

Proof of Lemma 6.3. By the assumption, we know that for the same $t > 0$ in the above Lemma, $u(x)$ is a solution of the following Dirichlet problem

$$
(-\div (|\nabla v|^{p-2} \nabla v)) = \lambda v^{p-1} \quad \text{in } B_{r(t)}(x_0),
$$

$$
v \geq 0 \quad \text{in } B_{r(t)}(x_0),
$$

$$
v = t \quad \text{on } \partial B_{r(t)}(x_0).
$$

By Lemma 2.4 and Lemma 2.5, we know that any solution of problem (6.3) is strictly positive in $\Omega$. Since problem (6.3) is invariant under rotation, we can prove Lemma 6.3 by proving that uniqueness theorem is valid for (6.3). To this end, we denote $B_{r(t)}(x_0)$ by $\Omega$ for simplicity, and suppose that $v_1 > 0$ and $v_2 > 0$ are two solutions of problem (6.3). Then $v_i(x)$ $(i = 1, 2)$ satisfy

$$
(-\div (|\nabla v_i|^{p-2} \nabla v_i)) = \lambda |v_i|^{p-2} v_i \quad \text{in } \Omega,
$$

$$
v_i = t \quad \text{on } \partial \Omega.
$$

Let

$$
\eta_1 = v_1 - v_2 v_1^{1-p} = \frac{v_1^p - v_2^p}{v_1^{p-1}}, \quad \eta_2 = v_2 - v_1 v_2^{1-p} = \frac{v_2^p - v_2^p}{v_2^{p-1}}.
$$

It is obvious that $\eta_i = 0$ $(i = 1, 2)$ on $\partial \Omega$. Multiplying equation (6.3) by $\eta_i$ $(i = 1, 2)$ and integrating by parts, we obtain

$$
\int_\Omega |\nabla v_i|^{p-2} \nabla v_i \cdot \nabla \eta_i - \lambda \int_\Omega v_i^{p-1} \eta_i = 0, \quad (i = 1, 2).
$$

By a similar argument to that used in the proof of Proposition 2.2, we infer that $\nabla (\ln v_1 - \ln v_2) = 0$, namely, $v_2 = Cv_1$ for some constant $C$. Since $v_1(x) = v_2(x) = t$ for $x \in \partial \Omega$, we obtain that $C = 1$ and $v_1(x) \equiv v_2(x)$ on $\overline{\Omega}$. Hence, the solution of problem (6.3) is unique, and hence, the symmetry result of Lemma 6.3 follows.

Proof of Theorem 1.2. Let $\Omega$ satisfy $\lambda_1(\Omega) = \lambda_1(B)$ and $|\Omega| = |B|$, $U_t$ be the level set of eigenfunction $\psi_\Omega$ correspond to $\lambda_1(\Omega)$. Then by Lemma 6.1, $H_{\Omega}(U_t, \Phi) = \lambda_1(B)$ for almost all $t \in (m, 1)$, and so, $U_t$ is a ball for any $t \in (m, 1)$ and $\sigma(\partial E U_t) = 0$ by Lemma 6.2. At this stage, Lemma 6.3 implies that $\psi_\Omega$ is radially symmetry inside $U_t$, and all interior level sets $U_{\tau}$ for $\tau \in (t, 1)$ are concentric balls. In particular, for all $t \in (m, 1]$, the level sets $U_t$ are concentric balls. Therefore, $\Omega = \bigcup_{t \in (m, 1]} U_t$ is a ball.

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