Hilbert schemes and symmetric products: 
a dictionary

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Abstract. Given a closed complex manifold $X$ of even dimension, we develop a systematic (vertex) algebraic approach to study the rational orbifold cohomology rings $H^*_{orb}(X^n/S_n)$ of the symmetric products. We present constructions and establish results on the rings $H^*_{orb}(X^n/S_n)$ including two sets of ring generators, universality and stability, as well as connections with vertex operators and $W$ algebras. These are independent of but parallel to the main results on the cohomology rings of the Hilbert schemes of points on surfaces as developed in our earlier works joint with W.-P. Li. We introduce a deformation of the orbifold cup product and explain how it is reflected in terms of modification of vertex operators in the symmetric product case. As a corollary, we obtain a new proof of the isomorphism between the rational cohomology ring of Hilbert schemes $X^{[n]}$ and the ring $H^*_{orb}(X^n/S_n)$ (after some modification of signs), when $X$ is a projective surface with a numerically trivial canonical class; we show that no sign modification is needed if both cohomology rings use $\mathbb{C}$-coefficients.

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1. Introduction

One of the recent surprises is the deep connections among geometry of Helbert schemes $X^{[n]}$ of points on a projective surface $X$, symmetric groups (or more generally wreath products), infinite dimensional Lie algebras, and vertex algebras. The construction in [Na1, Gro] of Heisenberg algebra which acts on the direct sum...
\[
\mathcal{H}_X = \bigoplus_{n=0}^{\infty} H^*(X^{[n]}) \]

of rational cohomology groups of Hilbert schemes made it possible to study the geometry of the Hilbert schemes from a new algebraic viewpoint. Lehç's work \cite{Lehn} initiated the deep interaction between Heisenberg algebra and the ring structure on \( H^*(X^{[n]}) \). The potential of this approach is made manifest in \cite{LQW1, LQW2} where two different sets of ring generators on \( H^*(X^{[n]}) \) were found for an arbitrary \( X \). In \cite{LS2}, a construction of the ring \( H^*(X^{[n]}) \) is made in terms of symmetric group, for \( X \) with numerically trivial canonical class. In \cite{LQW3}, the universality and stability of Hilbert schemes were established, which concern about the relations among the cohomology rings \( H^*(X^{[n]}) \) when \( X \) or \( n \) varies. In \cite{LQW4}, a \( W \) algebra was constructed geometrically acting on \( \mathcal{H}_X \). This is an analogue of the \( W_{1+\infty} \) algebra (cf. e.g. \cite{FKRW, Kac}) and it contains the Heisenberg algebra and Virasoro algebra as subalgebras. The construction of the \( W \) algebra is based on an explicit vertex operator formula for the so-called Chern character operator which plays an important role in the development.

It is well known that the Hilbert scheme \( X^{[n]} \) is a crepant resolution of singularities of the symmetric product \( X^n/S_n \). As inspired from orbifold string theory \cite{DHVW, VW}, the geometry of Hilbert schemes should be “equivalent” to the \( S_n \)-equivariant geometry of \( X^n \). As stated in the footnote 3 of \cite{Gro}, the direct sum \( \bigoplus_{n=0}^{\infty} K_{S_n}(X^n) \otimes \mathbb{C} \) of equivariant \( K \)-groups can be identified with a Fock space of a Heisenberg algebra, cf. \cite{Seg, Wal}. In particular, the size of the cohomology group \( H^*(X^{[n]}) \) coincides with that of the equivariant \( K \)-group \( K_{S_n}(X^n) \) or the orbifold cohomology group of \( X^n/S_n \). (Equivariant \( K \)-groups are related to orbifold cohomology groups by a decomposition theorem due to \cite{BC} and independently \cite{Kuh}.) However the “equivalence” on the level of ring structures is more subtle.

In \cite{CR}, Chen and Ruan introduced an orbifold cohomology \textit{ring} for any orbifold. When \( X \) is a projective surface with a numerically trivial canonical class, Ruan \cite{Ru1} conjectured that the orbifold cohomology ring of the symmetric product \( X^n/S_n \) is isomorphic to the cohomology ring of Hilbert scheme \( X^{[n]} \). This has been established with some sign modification in \cite{LS2, FG}. More precisely, a certain graded Frobenius algebra \( A^{[n]} \) was constructed in Lehç-Sorger \cite{LS2} based on a graded Frobenius algebra \( A \). The cohomology ring \( H^*(X^{[n]}) \) is then shown to be isomorphic to \( A^{[n]} \) for \( A = H^*(X) \), which is subsequently identified in \cite{FG} with the rational orbifold cohomology ring \( H^*_{orb}(X^n/S_n) \) with some modifications of signs in the orbifold cup product, also see \cite{Uri}.

The proof of the ring isomorphism in \cite{LS2} is very ingenious however quite indirect. It made use of earlier results on Hilbert schemes as well as an observation in \cite{FW} on the relation between Goulden’s operator on the symmetric groups and Lehç’s operator in Hilbert schemes. In particular, much remains to be understood about the finer structures of the ring \( H^*_{orb}(X^n/S_n) \) on its own for a general complex manifold \( X \) or about any direct connections between \( H^*_{orb}(X^n/S_n) \) and vertex algebras.

The goal of the present paper is to develop systematically a (vertex) algebraic approach to study the ring \( H^*_{orb}(X^n/S_n) \) for a closed complex manifold \( X \) of even dimension in a self-contained manner which is \textit{independent} of (but parallel to) the development on Hilbert schemes. By examining closely the arguments in \cite{LQW1-4}, we observe that the results obtained therein follow in an axiomatic way from several key constructions and statements obtained in \cite{Na2, Lehn, LQW1}, no
matter how difficult or how long the proofs could be. (This does not mean that we could formulate and prove these results in Hilbert schemes in any easier way.) Thus we formulate several axioms to formalize the setup in order to make it applicable in different situations. We obtain the corresponding key constructions and prove the corresponding statements in \( H^*_{orb}(X^n/S_n) \). These enable us to apply the axioms to obtain the counterparts in the setup of symmetric products of the main results in [LQW1-4] for Hilbert schemes.

Let us explain in more detail. We first note that a Heisenberg algebra acting on \( F_X = \oplus_{k=0}^{\infty} H^*_{orb}(X^n/S_n) \) is available by easily reformulating the equivariant K-group construction in [Seg, Wal]. We then introduce the cohomology classes \( O^k(\alpha, n) \in H^*_{orb}(X^n/S_n) \) using the \( k \)-th power sum of Jucys-Murphy elements in the symmetric groups [Juc, Mur]. We define the operator \( \mathcal{D}^k(\alpha) \in \text{End}(F_X) \) to be the orbifold cup product with \( O^k(\alpha, n) \) in \( H^*_{orb}(X^n/S_n) \) for each \( n \). (As we shall see, the operators \( \mathcal{D}^k(\alpha) \) turn out to be the counterpart of the Chern character operators in [LQW1].) The idea of relating Jucys-Murphy elements to vertex operators has been used in [LT] in the study of class functions of symmetric groups. In particular, when \( X \) is a point, the operator \( \mathcal{D}^1 \) reduces to the Goulden’s operator \( \mathcal{G} \) which admits a vertex operator interpretation [FW].

By studying the interaction of the operators \( \mathcal{D}^k(\alpha) \) and Heisenberg operators, we are able to verify all the axioms. Therefore, as formal consequences of the axiomatization, we obtain the counterparts in the symmetric product setup of all the main results for the cohomology rings of Hilbert schemes in [LQW1-4]. Namely, we prove that \( O^k(\alpha, n) \), as \( 0 \leq k < n \) and \( \alpha \) runs over a linear basis of \( H^*(X) \), form a set of ring generators of \( H^*_{orb}(X^n/S_n) \). We also show that there is another set of ring generators in terms of Heisenberg algebra generators. We establish the stability of the ring \( H^*_{orb}(X^n/S_n) \), which tells us in what sense the orbifold cup product on \( H^*_{orb}(X^n/S_n) \) is independent of \( n \). We further obtain a description of the operators \( \mathcal{D}^k(\alpha) \) as the zero mode of a certain explicit vertex operator when \( X \) has a positive dimension. (Such a description has been given in [LT] when \( X \) is a point. There is however a remarkable difference between these two cases.) The components of these vertex operators generate a \( W \) algebra acting on \( F_X \). The description of the operator \( \mathcal{D}^k(\alpha) \) as the zero mode of a certain explicit vertex operator provides us a new way to construct a sequence of Frobenius algebra \( \mathcal{F}^n \) starting from a Frobenius algebra \( A \), cf. Remark [LT]. (Compare with the different construction in [LS2] of the Frobenius algebra \( A^{[n]} \)).

For a global quotient \( M/G \), we introduce a deformed orbifold cup product on \( H^*_{orb}(M/G) \) (actually first on \( H^*(M, G) \)) depending on a rational (or complex, if we consider the orbifold cohomology group with \( \mathbb{C} \)-coefficient) parameter \( t \). This reduces to the original construction in [CR] for \( t = 1 \), and to the construction of [FG] (also cf. [LS2]) for \( t = -1 \). In the case of symmetric products, we explain how the parameter \( t \) is reflected in terms of some modifications on Heisenberg algebra and vertex operators. By comparing our results on the symmetric products with results on Hilbert schemes in [LQW1], we obtain a new proof of the ring isomorphism between \( H^*(X^{[n]}) \) and \( H^*_{orb}(X^n/S_n) \) (with the sign modification) for \( X \) with a numerically trivial canonical class.

It turns out that the \( t \)-family of deformed ring structures on \( H^*_{orb}(M/G, \mathbb{C}) \) with \( \mathbb{C} \)-coefficient are isomorphic for all nonzero \( t \). As a consequence, using \( \mathbb{C} \) instead of \( \mathbb{Q} \) as coefficients for (orbifold) cohomology groups, there exists a ring
isomorphism between cohomology ring of \( X^{[n]} \) and the original orbifold cohomology ring of \( X^n/S_n \) for \( X \) with a numerically trivial canonical class. This supports the original conjecture of Ruan \([Ru1]\) on the ring isomorphism of hyperkahler resolutions, if we insist on using \( \mathbb{C} \) rather than \( \mathbb{Q} \) as the cohomology coefficients.

As observed in \([Wa1]\), for a given complex manifold \( X \) with a finite group \( \Gamma \) action, the product \( X^n \) affords a natural action of the wreath product \( \Gamma_n \) (which is a finite group given by the semidirect product \( \Gamma^n \rtimes S_n \)). Further, the quotient \( X^n/\Gamma_n \) can be identified with the symmetric product of the orbifold \( X/\Gamma \). The results on the orbifold cohomology ring of symmetric products in this paper will be generalized elsewhere to the symmetric products of a general orbifold.

We find it amazing to have such a wonderful dictionary between results in Hilbert schemes and in symmetric products. To some extent, the results on symmetric products are simpler since the canonical class does not play a role here (besides, \( X \) needs not to be a surface). We present a partial dictionary in a table near the end of the paper. It is also instructive to compare with another dictionary table in \([Wa2]\) between Hilbert schemes and wreath products.

Our results in this paper may shed light on the understanding of the difference between the rings \( H^*(X^{[n]}) \) and \( H^*_{\text{orb}}(X^n/S_n) \) (with/out the sign changes) even when the projective surface has a nontrivial canonical class. A special case of a conjecture made in \([Ru2]\) says that there exists a ring isomorphism between \( H^*_{\text{orb}}(X^n/S_n) \) and \( H^*(X^{[n]}) \) with a quantum corrected product. A partial verification has been made by the Gromov-Witten 1-point function computation for \( X^{[n]} \) in \([LQ]\). Our axiomatization provides a possible strategy for checking Ruan’s conjectural ring isomorphism. Namely, we can use the quantum corrected product to replace the usual cup product to introduce operators analogous to the Chern character operators, and then try to understand their interaction with the usual Heisenberg algebra, and then compare with the results on the ring \( H^*_{\text{orb}}(X^n/S_n) \) obtained in this paper.

The paper is organized as follows. In Sect. \( \text{\ref{sec:2}} \), we review the key constructions and statements in Hilbert schemes which are responsible for the further results. We put an emphasis on the axiomatic nature of these results. We also obtain a variation of Lehn’s theorem in relating the Chern class of certain tautological bundles on \( X^{[n]} \) and Heisenberg generators, and point out an interesting corollary. In Sect. \( \text{\ref{sec:3}} \), we present the corresponding key constructions and prove the corresponding statements in symmetric products. The reader should compare the constructions in Sect. \( \text{\ref{sec:2}} \) and in Sect. \( \text{\ref{sec:3}} \). In Sect. \( \text{\ref{sec:4}} \), we formulate formal consequences in symmetric products of the results in the previous section, which are the counterparts of earlier results on Hilbert schemes. In Sect. \( \text{\ref{sec:5}} \), we explain how a deformation of the orbifold cup product \( H^*_{\text{orb}}(X^n/S_n) \) is reflected in terms of modified Heisenberg algebra and vertex operators. As a corollary, we obtain a new proof of the modified Ruan’s conjecture on the ring isomorphism between \( H^*(X^{[n]}) \) and \( H^*_{\text{orb}}(X^n/S_n) \) when \( X \) has a numerically trivial canonical class. Further, we show that Ruan’s original conjecture holds if we use \( \mathbb{C} \) as cohomology coefficient. In Sect. \( \text{\ref{sec:6}} \), we list several open questions for further research.

**Convention.** All the (orbifold) cohomology groups/rings are assumed to have \( \mathbb{Q} \)-coefficients unless otherwise specified.

### 2. The cohomology ring of Hilbert schemes
2.1. Hilbert schemes of points on surfaces. Let \( X \) be a smooth projective complex surface with the canonical class \( K \) and the Euler class \( e \), and \( X^{[n]} \) be the Hilbert scheme of points in \( X \). We define a bilinear form

\[
(\alpha, \beta) = \int_X \alpha \beta, \quad \alpha, \beta \in H^*(X).
\]

An element in \( X^{[n]} \) is represented by a length-\( n \) 0-dimensional closed subscheme \( \xi \) of \( X \). For \( \xi \in X^{[n]} \), let \( I_\xi \) be the corresponding sheaf of ideals. It is well known that \( X^{[n]} \) is smooth. Sending an element in \( X^{[n]} \) to its support in the symmetric product \( X^n/S_n \), we obtain the Hilbert-Chow morphism \( \pi_n : X^{[n]} \to X^n/S_n \), which is a resolution of singularities. Define the universal codimension-2 subscheme:

\[
\mathcal{Z}_n = \{ (\xi, x) \subset X^{[n]} \times X \mid x \in \text{Supp}(\xi) \} \subset X^{[n]} \times X.
\]

Denote by \( p_1 \) and \( p_2 \) the projections of \( X^{[n]} \times X \) to \( X^{[n]} \) and \( X \) respectively. Let

\[
\mathcal{H}_X = \bigoplus_{n=0}^{\infty} H^*(X^{[n]})
\]

be the direct sum of total cohomology groups (with \( \mathbb{Q} \)-coefficient) of the Hilbert schemes \( X^{[n]} \).

2.2. The Heisenberg Algebra. Nakajima and Grojnowski \cite{Na1, Gro} constructed geometrically a Heisenberg algebra which acts irreducibly on \( \mathcal{H}_X \) with generators \( a_n(\alpha), n \in \mathbb{Z}, \alpha \in H^*(X) \). Below we recall the construction of Nakajima \cite{Na2}.

For \( m \geq 0 \) and \( n > 0 \), let \( Q^{[m,n]} = \emptyset \) and define \( Q^{[m+n,m]} \) to be the closed subset:

\[
\{(\xi, x, \eta) \in X^{[m+n]} \times X \times X^{[m]} \mid \xi \supset \eta \text{ and } \text{Supp}(I_\eta/I_\xi) = \{x\}\}.
\]

Let \( n \geq 0 \). The linear operator \( a_{-n}(\alpha) \in \text{End}(\mathcal{H}_X) \) with \( \alpha \in H^*(X) \) is defined by

\[
a_{-n}(\alpha)(a) = \tilde{p}_1\ast(Q^{[m+n,m]} \cdot \tilde{\rho}^*\alpha \cdot \tilde{p}_2a)
\]

for \( a \in H^*(X^{[m]}) \), where \( \tilde{p}_1, \tilde{\rho}, \tilde{p}_2 \) are the projections of \( X^{[m+n]} \times X \times X^{[m]} \) to \( X^{[m+n]} \times X \times X^{[m]} \) respectively. Define \( a_n(\alpha) \in \text{End}(\mathcal{H}_X) \) to be \((-1)^n\) times the operator obtained from the definition of \( a_{-n}(\alpha) \) by switching the roles of \( \tilde{p}_1 \) and \( \tilde{p}_2 \).

We often refer to \( a_{-n}(\alpha) \) (resp. \( a_n(\alpha) \)) as the creation (resp. annihilation) operator.

**Theorem 2.1.** The operators \( a_n(\alpha) \in \text{End}(\mathcal{H}_X) \) \((n \in \mathbb{Z}, \alpha \in H^*(X))\) generate a Heisenberg (super)algebra with commutation relations given by

\[
[a_m(\alpha), a_n(\beta)] = -m\delta_{m-n}(\alpha, \beta) \cdot \text{Id}_{\mathcal{H}_X}
\]

where \( n, m \in \mathbb{Z}, \alpha, \beta \in H^*(X) \). Furthermore, \( \mathcal{H}_X \) is an irreducible representation of the Heisenberg algebra with the vacuum vector \( |0\rangle = 1 \in H^*(pt) \cong \mathbb{C} \).

The commutator above is understood in the super sense according to the parity of cohomology classes \( \alpha, \beta \) involved.

We define the following cohomology class in \( H^*(X^{[n]}) \) \cite{LQW}:

\[
B_i(\gamma, n) = \frac{1}{(n-i-1)!} \cdot a_{-i-1}(\gamma)a_{-1}(1_X)^{n-i-1}|0\rangle.
\]
We define the normally ordered product: $\phi G$ where $\Delta$ is the conformal weight of the field $\phi$.

For $n \geq 0$ and a homogeneous class $\gamma \in H^*(X)$, let $|\gamma| = s$ if $\gamma \in H^s(X)$, and let $G_i(\gamma, n)$ be the homogeneous component in $H^{\lfloor |\gamma|+2i \rfloor}(X[n])$ of

$$G(\gamma, n) = p_{1*}(\text{ch}(O_{Z_n}) \cdot p_2^* \text{td}(X) \cdot p_2^* \gamma) \in H^*(X[n])$$

where $\text{ch}(O_{Z_n})$ denotes the Chern character of the structure sheaf $O_{Z_n}$ and $\text{td}(X)$ denotes the Todd class. Here and below we omit the Poincaré duality used to switch a homology class to a cohomology class and vice versa. We extend the definition of $G_i(\gamma, n)$ to an arbitrary class $\gamma \in H^*(X)$ by linearity. It turns out to be more convenient to introduce a normalized class

$$G^k(\gamma, n) := k! \cdot G_k(\gamma, n).$$

It was proved in [LQW1] that the cohomology ring of $X[n]$ is generated by the classes $G^i(\gamma, n)$ where $0 \leq i < n$ and $i$ runs over a linear basis of $H^*(X)$.

The Chern character operator $\mathfrak{S}^k(\gamma) \in \text{End}(H_X)$ is defined to be the operator acting on the component $H^*(X[n])$ by the cup product with $G^i(\gamma, n)$ for every $n \geq 0$. We introduce a formal variable $h$ (here and in other places later on) and let $\mathfrak{S}_h(\gamma) = \sum_{i \geq 0} h^i \cdot \mathfrak{S}^i(\gamma)$. A convenient way is to regard $h$ as having ‘cohomology degree’ $-2$ so $\mathfrak{S}_h(\gamma)$ becomes homogeneous of degree $|\gamma|$.

Let $\mathfrak{d} = \mathfrak{S}^1(1_X)$ where $1_X$ is the fundamental cohomology class of $X$. The operator $\mathfrak{d}$ was first introduced in [Lehn] and plays an important role in the theory. For a linear operator $\mathfrak{f} \in \text{End}(H_X)$, define its derivative $\mathfrak{f}'$ by $\mathfrak{f}' = [\mathfrak{d}, \mathfrak{f}]$. The higher derivative $\mathfrak{f}^{(k)}$ is defined inductively by $\mathfrak{f}^{(k)} = [\mathfrak{d}, \mathfrak{f}^{(k-1)}]$.

### 2.4. Vertex operators.

We define the normally ordered product: $a_{m_1}a_{m_2}$ to be $a_m a_m$ when $m_1 \leq m_2$ and $a_m a_m$ when $m_1 > m_2$. We denote

$$\tau_k : H^*(X) \to H^*(X^k) \cong H^*(X)^{\otimes k},$$

where $k \geq 1$, is the linear map induced by the diagonal embedding $\tau_k : X \to X^k$, and $a_{m_1} \cdots a_{m_k}(\tau_k(\alpha))$ denotes $\sum_j a_{m_1}(\alpha_{j,1}) \cdots a_{m_k}(\alpha_{j,k})$ if we write $\tau_k(\alpha) = \sum_j \alpha_{j,1} \otimes \cdots \otimes \alpha_{j,k}$ via the Künneth decomposition of $H^*(X^k)$. We will simply write $\tau_k$ for $\tau_k$ when there is no confusion.

Our convention of vertex operators or fields is to write them in a form

$$\phi(z) = \sum_n \phi_n z^{-n-\Delta}$$

where $\Delta$ is the conformal weight of the field $\phi(z)$. Define the derivative field

$$\partial \phi(z) = \sum_n (-n - \Delta) \phi_n z^{-n-\Delta-1}.$$

We define the normally ordered product: $\phi_1(z) \cdots \phi_k(z)$ as usual (cf. e.g. [Kac]).

For $\alpha \in H^*(X)$, we define a vertex operator $a(\alpha)(z)$ by putting

$$a(\alpha)(z) = \sum_{n \in \mathbb{Z}} a_n(\alpha) z^{-n-1}.$$

The field $a(z)^p : (\tau_s \alpha)$ is defined to be $\sum_j a(\alpha_{i,1})(z) a(\alpha_{i,2})(z) \cdots a(\alpha_{i,p})(z)$ if we write $\tau_s \alpha = \sum_{i=1}^s \alpha_{i,1} \otimes \alpha_{i,2} \otimes \cdots \otimes \alpha_{i,p} \in H^*(X)^{\otimes p}$. We rewrite $a(z)^p : (\tau_s \alpha)$.
componentwise as
\[ a(z)^p : (\tau_\alpha) = \sum_m : a^p : (\tau_\alpha) \ z^{-m-p}, \]
where \( : a^p : (\tau_\alpha) \in \text{End}(H_X) \) is the coefficient of \( z^{-m-p} \) (i.e. the \( m \)-th Fourier component of the field \( : a(z)^p : (\tau_\alpha) \)), and maps \( H^*(X^{[n]}) \) to \( H^*(X^{[n+m]}) \). Similarly, for \( r \geq 1 \), we can define the field \( (\partial^r a(z)) a(z)^p : (\tau_\alpha) \), and define the operator \( (\partial^r a) a^{p-1} : (\tau_\alpha) \) as the coefficient of \( z^{-m-r-p} \) in \( (\partial^r a(z)) a(z)^p : (\tau_\alpha) \).

**2.5. Interactions between Heisenberg algebra and \( \Theta(\alpha) \).** The following theorem is a variation of Lemma 5.8, [LQW1], which generalizes Theorem 4.2, [Lehn]. Clearly the two identities in the following theorem are equivalent.

**Theorem 2.2.** Let \( \gamma, \alpha \in H^*(X) \). Then we have
\[
[\Theta_h(\gamma), a_{-1}(\alpha)] = \exp(h \cdot \text{ad}\,(a_{-1}(\gamma))) \\
[\Theta^k(\gamma), a_{-1}(\alpha)] = a_{-1}^k(\gamma) \alpha, \quad k \geq 0.
\]

**Theorem 2.3.** For \( \alpha \in H^*(X) \), we have
\[
(2.1) \quad \Theta^1(\alpha) = -\frac{1}{6} : a^3 :_0 (\tau_\alpha) - \sum_{n>0} \frac{n-1}{2} : a_n a_{-n} : (\tau_n(K\alpha)).
\]
In particular, for a surface \( X \) with numerically trivial canonical class, we have
\[
(2.2) \quad a = -\frac{1}{6} : a^3 :_0 (\tau_1 1_X).
\]

**Proof.** Observe that both sides of (2.1) annihilates the vacuum vector \( \langle 0 \rangle \). To prove (2.1), it suffices to show that the commutators of both sides of (2.1) with the operators \( a_n(\beta), n \in \mathbb{Z}, \beta \in H^*(X) \), coincide. It was shown in [LQW1] that (i.e. the transfer property)
\[
[\Theta^1(\alpha), a_n(\beta)] = [\Theta^1(1_X), a_n(\alpha\beta)] = a'_n(\alpha\beta),
\]
while \( a'_n(\alpha\beta) \) was computed in [Lehn], Theorem 3.10. We can easily compute the commutator of the right hand side of (2.1) with \( a_n(\beta) \) by using Lemma 3.1 in [LQW3]. These two commutators coincide. \( \square \)

**2.6. Axiomatization.** We claim that Theorem 2.1, Theorem 2.2, and Theorem 2.3 encode all the information about the ring structure of the cohomology ring \( H^*(X^{[n]}) \) for every \( n \). In fact, if we examine closely the proofs of all the main results in [LQW1-4], we see that these three statements (or sometimes some weaker form of Theorem 2.3) were used effectively, together with numerous standard properties concerning \( \tau_n \), the Heisenberg generators etc. The results therein include the theorems on cohomology ring generators in [LQW1], [LQW2], the universality and stability theorems in [LQW3], and the connection with \( \mathcal{W} \) algebras in [LQW4].

This observation leads to a strategy which allows us to treat similar cases in an axiomatic manner. Assume that (A1) there exists a sequence of (finite-dimensional) graded Frobenius algebras \( A^{[n]} \) \((n \geq 0)\) such that \( A = A^{[1]} \). (A2) the direct sum \( \oplus_n A^{[n]} \) affords the structure of a Fock space of a Heisenberg algebra. (A3) There exists a sequence of elements \( G_k(\alpha, n) \in A^{[n]} \) depending on \( \alpha \in A \) (linearly) and a non-negative integer \( k \), which can be used to define operators \( \Theta^k(\alpha) \). The operators \( \Theta^k(\alpha) \), \( \Theta^1(1_X) \) and the Heisenberg generators satisfy the relations as in Theorem 2.2 and (2.3). (Note here that in general there is no counterpart of the
$K$-term in Theorem 2.3, so we have put a stricter axiom here. Variations of this are allowed in different setup.)

The axioms (A1-A3) suffice us to use the same approach as in [LQW1-4] to obtain similar theorems. (For results in [LQW3, LQW4] the fact $e^2 = 0$ is used. This requires a separate treatment when $A \cong \mathbb{C}$ which corresponds to the cohomology ring of $X = pt$.) In fact, many formulas in the Hilbert scheme side are to be simplified by discarding all the terms which involve in a manifest way with the canonical class $K$.

We will follow this axiomatic route when we treat the orbifold cohomology ring of the symmetric products in the sections below.

2.7. Tautological bundles and Heisenberg generators. Given a line bundle $L$ over $X$, we obtain a rank $n$ vector bundle $L[n] = (p_1|z_n) \ast (p_2^*L|z_n)$ over $X^n$, where we recall that $p_1$ and $p_2$ are the projections from $X^n \times X$ to $X^n$ and $X$ respectively, and $Z_n$ is the universal subscheme of $X^n \times X$. We introduce the generating function for Chern classes:

$$c_h(L[n]) = \sum_{i \geq 0} c_i(L[n]) h^i.$$  

The following theorem is a variation of Theorem 4.6 in [Lehn]. We can prove it in a way parallel to the Variant 2 of the proof of Lehn’s theorem, once we make sure to put a suitable power of $h$ in the right places.

**Theorem 2.4.** Let $L$ be a line bundle on $X$. Then

$$\sum_{n=0}^\infty c_h(L[n])z^n = \exp \left( \sum_{r \geq 1} \frac{(-h)^{r-1}}{r} a_{-r}(c_h(L)) z^r \right) \cdot |0\rangle.$$  

When setting $h = 1$, we recover Theorem 4.6, [Lehn]:

$$\sum_{n=0}^\infty c(L[n])z^n = \exp \left( \sum_{r \geq 1} \frac{(-1)^{r-1}}{r} a_{-r}(c(L)) z^r \right) \cdot |0\rangle.$$  

Besides making various gradings involved more transparent, one advantage of our Theorem 2.4 to Theorem 4.6 in [Lehn] is that we may set the value of $h$ to be a number other than 1. Denote by $L[n]^\vee$ the dual bundle of $L[n]$. The Chern classes of a vector bundle and its dual are related to each other: $c_k(L[n]^\vee) = (-1)^k c_k(L[n])$, $k \geq 0$. Setting $h = -1$ in Eq. (2.3), we arrive at the following new observation.

**Corollary 2.5.** Let $L$ be a line bundle on $X$. Then

$$\sum_{n=0}^\infty c(L[n]^\vee)z^n = \exp \left( \sum_{r \geq 1} \frac{1}{r} a_{-r}(c(L^\vee)) z^r \right) \cdot |0\rangle.$$  

3. The orbifold cohomology ring of symmetric products I

3.1. Generalities on orbifold cohomology rings. Let $M$ be a complex manifold of complex dimension $d$ with a finite group $G$ action. Following [BBM, BC, Kuhn], we introduce the space

$$M \circ G = \{(g, x) \in G \times M \mid gx = x\} = \bigsqcup_{g \in G} M^g,$$
and $G$ acts on $M \circ G$ naturally by $h.(g,x) = (hgh^{-1}, hx)$. As a vector space, we define $H^*(M, G)$ to be the cohomology group of $M \circ G$ with rational coefficient (cf. [FC]), or equivalently,

$$H^*(M, G) = \bigoplus_{g \in G} H^*(M^g).$$

The space $H^*(M, G)$ has a natural induced $G$ action, which is denoted by $\text{ad} \ h : H^*(M^g) \to H^*(M^{gh^{-1}})$. As a vector space, the orbit cohomology group $H^*_{\text{orb}}(M/G)$ is the $G$-invariant part of $H^*(M, G)$, which is isomorphic to

$$\bigoplus_{[g] \in G_*} H^*(M^g/Z(g))$$

where $G_*$ denotes the set of conjugacy classes of $G$ and $Z(g) = Z_G(g)$ denotes the centralizer of $g$ in $G$.

For $g \in G$ and $x \in M^g$, write the eigenvalues of the action of $g$ on the complex tangent space $TM_x$ to be $\mu_k = e^{2\pi i r_k}$, where $0 \leq r_k < 1$. The degree shift number (or age) is the rational number $F^g_x = \sum_{k=1}^d r_k$, cf. [Zas]. It depends only on the connected component $Z$ which contains $x$, so we can denote it by $F^g_z$. Then associated to a cohomology class in $H^*(Z)$, we assign the corresponding element in $H^*(M, G)$ (and thus in $H^*_{\text{orb}}(M/G)$) the degree $r + 2F^g_z$.

A ring structure on $H^*_{\text{orb}}(M/G)$ was introduced by Chen and Ruan [CR]. This was subsequently clarified in [FC] by introducing a ring structure on $H^*(M, G)$ first and then passing to $H^*_{\text{orb}}(M/G)$ by restriction. We shall use $\circ$ to denote this product. The ring structure on $H^*(M, G)$ is degree-preserving, and has the property: $\alpha \circ \beta$ lies in $H^*(M^g)$ for $\alpha \in H^*(M^g)$ and $\beta \in H^*(M^h)$.

For $1 \in G$, $H^*(M^1/Z(1)) \cong H^*(M/G)$, and thus we can regard $a \in H^*(M/G)$ to be $a \in H^*_{\text{orb}}(M/G)$ by this isomorphism. Also given $a = \sum_{g \in G} a_g g$ in $\mathbb{Q}[G]$ (resp. $\mathbb{Q}[G]^G$), we may regard $a$ as an element in $H^*(M, G)$ (resp. $H^*_{\text{orb}}(M/G)$) whose component in each $H^*(M^g)$ is $a_g \cdot 1_{M^g} \in H^0(M^g)$.

If $K$ is a subgroup of $G$, then we can define the restriction map from $H^*(M, G)$ to $H^*(M, K)$ by projection to the component $\oplus_{g \in K} H^*(M^g)$ which, when restricted to the $G$-invariant part, induces naturally a degree-preserving linear map $\text{Res}^G_K : H^*_{\text{orb}}(M/G) \to H^*_{\text{orb}}(M/K)$. We define the induction map

$$\text{Ind}^G_K : H^*(M, K) \to H^*_{\text{orb}}(M/G)$$

by sending $\alpha \in H^*(M^h)$, where $h \in K$, to

$$\text{Ind}^G_K(\alpha) = \frac{1}{|K|} \sum_{g \in G} \text{ad} g(\alpha).$$

Note that $\text{Ind}^G_K(\alpha)$ is clearly $G$-invariant. When restricted to the invariant part, we obtain a degree-preserving linear map $\text{Ind}^G_K : H^*_{\text{orb}}(M/K) \to H^*_{\text{orb}}(M/G)$. We often write the restriction and induction maps as $\text{Res}_K, \text{Res}$ and $\text{Ind}^G, \text{Ind}$, when the groups involved are clear from the context. In particular, when $M$ is a point, $H^*_{\text{orb}}(pt/G)$ reduces to the Grothendieck ring $R_{\mathbb{Q}}(G)$ of $G$, and we recover the induction and restriction functors in the theory of finite groups.
3.2. The Heisenberg algebra. Let $X$ be a closed complex manifold of complex dimension $d$. Our main objects are the orbifold cohomology ring $H^*_{orb}(X^n/S_n)$, and the non-commutative ring $H^*(X^n,S_n)$. We denote

$$\mathcal{F}_X = \bigoplus_{n=0}^{\infty} H^*_{orb}(X^n/S_n).$$

We introduce a linear map $\omega_n : H^*(X) \to H^*_{orb}(X^n/S_n)$ as follows: given $\alpha \in H^r(X)$, we denote by $\omega_n(\alpha) \in H^{r+d(n-1)}_{orb}(X^n/S_n)$ the element associated to $n\alpha$ by the isomorphism $H^*((X^n)^{\sigma_n}) \cong H^*(X)$ for any permutation $\sigma_n$ in the conjugacy class $[n] \in (S_n)_\circ$, which consists of the $n$-cycles. We also define $\text{ch}_n : H^*_{orb}(X^n/S_n) \to H^*(X)$ as the composition of the isomorphism $H^*((X^n)^{\sigma_n}) \cong H^*(X)$ with the projection from $H^*_{orb}(X^n/S_n)$ to $H^*((X^n)^{\sigma_n})$.

Let $\alpha \in H^*(X)$. For $n > 0$, we define the creation operator $p_{-n}(\alpha) \in \text{End}(\mathcal{F}_X)$ given by the composition $(k \geq 0)$:

$$H^*_{orb}(X^k/S_k) \xrightarrow{\omega_n(\alpha) \otimes} H^*_{orb}(X^n/S_n) \bigotimes H^*_{orb}(X^k/S_k) \xrightarrow{\cong} H^*_{orb}(X^{n+k}/(S_n \times S_k)) \xrightarrow{\text{Ind}} H^*_{orb}(X^{n+k}/S_{n+k}),$$

and the annihilation operator $p_n(\alpha) \in \text{End}(\mathcal{F}_X)$ given by the composition $(k \geq 0)$:

$$H^*_{orb}(X^{n+k}/S_{n+k}) \xrightarrow{\text{Res}} H^*_{orb}(X^{n+k}/(S_n \times S_k)) \xrightarrow{\cong} H^*_{orb}(X^n/S_n) \bigotimes H^*_{orb}(X^k/S_k) \xrightarrow{\text{ch}_n} H^*(X) \bigotimes H^*_{orb}(X^k/S_k) \xrightarrow{(\alpha,\cdot)} H^*_{orb}(X^k/S_k).$$

We also set $p_0(\alpha) = 0$.

**Theorem 3.1.** The operators $p_n(\alpha) \in \text{End}(\mathcal{F}_X)$ $(n \in \mathbb{Z}, \alpha \in H^*(X))$ generate a Heisenberg (super)algebra with commutation relations given by

$$[p_m(\alpha), p_n(\beta)] = m\delta_{m,-n}(\alpha,\beta) \cdot \text{Id}_{\mathcal{F}_X}$$

where $n,m \in \mathbb{Z}$, $\alpha,\beta \in H^*(X)$. Furthermore, $\mathcal{F}_X$ is an irreducible representation of the Heisenberg algebra with the vacuum vector $|0\rangle = 1 \in H^*(pt) \cong \mathbb{C}$.

This theorem can be proved in the same way as an analogous theorem formulated by using the equivariant $K$-group $K_{S_n}(X^n) \otimes \mathbb{C}$. This analogous theorem was established in [Seg] (see [Wal], Theorem 4 and its proof for detail). In general, equivariant $K$-groups are related to orbifold cohomology groups by a decomposition theorem [Kuhn, BC]. Note that there is a (fundamental!) sign difference in the two commutators of Theorems 2.1 and 3.1.

In particular, for a given $y \in H^i_{orb}(X^{n-1}/S_{n-1})$, by the definition of $p_{-1}(\alpha)$ (where $\alpha \in H^{[\alpha]}(X)$) and the induction map, we can write that

$$p_{-1}(\alpha)(y) = \frac{1}{(n-1)!} \sum_{g \in S_n} \text{ad}_g(\alpha \otimes y) = \frac{(-1)^{|\alpha||y|}}{(n-1)!} \sum_{g \in S_n} \text{ad}_g(y \otimes \alpha).$$
For $0 \leq i < n$, we introduce the following cohomology class in $H^*(X^n)$:

$$P_i(\gamma, n) = \frac{1}{(n-i-1)!} \cdot p_{-i-1}(\gamma) p_{-1}(1_{X})^{n-i-1} |0).$$

3.3. Jucys-Murphy elements. For a permutation $\sigma \in S_n$ of cycle type given by a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of length $\ell = \ell(\lambda)$, we denote $d(\sigma) = d(\lambda) = n - l(\lambda)$. Let $\text{ch} : \bigoplus_{n=0}^{\infty} R(S_n) \to \Lambda$ be the Frobenius characteristic map from the direct sum of (complex) class functions on the symmetric group $S_n$ to the ring $\Lambda$ of symmetric functions in infinitely many variables, cf. [Mac]. Denote by $\eta_n$ and $\varepsilon_n$ the trivial and alternating characters of $S_n$. Then $\text{ch}$ sends $\eta_n$ and $\varepsilon_n$ to the $n$-th complete and elementary symmetric functions in $\Lambda$ respectively. We denote by $p_r$ the $r$-th power sum symmetric function.

Recall [Juc, Mur] that the Jucys-Murphy elements $\xi_{j, n}$ of the symmetric group $S_n$ are defined to be the sums of transpositions:

$$\xi_{j, n} = \sum_{i<j} (i, j), \quad j = 1, \ldots, n.$$

When it is clear from the text, we may simply write $\xi_{j, n}$ as $\xi_j$. Denote by $\Xi_n$ the set $\{\xi_1, \ldots, \xi_n\}$. According to Jucys, the $k$-th elementary symmetric function $e_k(\Xi_n)$ of $\Xi_n = \{\xi_1, \ldots, \xi_n\}$ is equal to the sum of all permutations in $S_n$ having exactly $(n-k)$ cycles. Therefore, we obtain

$$\varepsilon_n = \sum_{\sigma \in S_n} (-1)^{d(\sigma)} \sigma = \sum_{k=0}^{n} (-1)^{k} e_k(\Xi_n) = \prod_{i=1}^{n} (1 - \xi_i).$$

Denote by $\varepsilon_n(h) = \prod_{i=1}^{n} (1 - h\xi_i)$, where $h$ is a formal parameter here and below. We have

$$\sum_{n=0}^{\infty} \text{ch}(\varepsilon_n(h)) z^n = \exp \left( \sum_{r \geq 1} (-h)^{r-1} \frac{p_r}{r} z^r \right).$$

Noting that $\varepsilon_n(1)$ (resp. $\varepsilon_n(-1)$) coincides with the alternating character $\varepsilon_n$ (resp. the trivial character $\eta_n$), we obtain two classical identities involving $\eta_n, \varepsilon_n$, and $p_r$ by setting $h = \pm 1$ in (3.2).

3.4. The cohomology classes $\eta_n(\gamma)$ and $O^k(\alpha, n)$. In the rest of this paper, we will assume that $X$ is a closed complex manifold of even complex dimension $d$. Given $\gamma \in H^*(X)$, we denote

$$\gamma^{(i)} = 1^\otimes i-1 \otimes \gamma \otimes 1^\otimes n-i \in H^*(X^n),$$

and regard it to be a cohomology class in $H^*(X^n, S_n)$ associated to the identity conjugacy class. We define $\xi_i(\gamma) := \xi_i + \gamma^{(i)} \in H^*(X^n, S_n)$. We sometimes write $\xi_i(\gamma)$ as $\xi_{i, n}(\gamma)$ to specify its dependence on $n$ when necessary.

**Definition 3.2.** Given $\gamma \in H^*(X)$, we define $\eta_n(\gamma)$ (resp. $\varepsilon_n(\gamma)$) to be the cohomology class in $H^*_{\text{orb}}(X^n, S_n)$ whose component associated to an element $\sigma$ in the conjugacy class of partition $\lambda$ of $n$ is given by $\gamma^{\otimes (\ell(\lambda))} \in H^*((X^n)^{\sigma}) \cong H^*\otimes (\ell(\lambda))$ (resp. by $(-1)^{d(\lambda)} \gamma^{\otimes (\ell(\lambda))}$). We further define an operator $\eta(\gamma)$ (resp. $\varepsilon(\gamma)$) in $\text{End}(F_X)$ by letting it act on $H^*_{\text{orb}}(X^n, S_n)$ by the orbifold product with $\eta_n(\gamma)$ (resp. $\varepsilon_n(\gamma)$) for every $n$. 
This definition is motivated by its counterpart in terms of equivariant K-groups \( \text{Seg} \). We can show that
\[
\sum_{n=0}^{\infty} \eta_n(\gamma)z^n = \exp \left( \sum_{r \geq 1} \frac{1}{r} p_{-r}(\gamma) z^r \right) \cdot |0|.
\]

Compare with [Wal], Proposition 4.

**Proposition 3.3.** Given \( \gamma \in H^*(X) \), the orbifold cup product of the \( n \) elements \( \xi_i(\gamma) \) \( (i = 1, \ldots, n) \) in \( H^*(X^n, S_n) \) lies in \( H^*_{\text{orb}}(X^n/S_n) \), and furthermore the following identity holds:
\[
\eta_n(\gamma) = \prod_{i=1}^{n} \xi_i(\gamma) = \xi_1(\gamma) \circ \xi_2(\gamma) \circ \ldots \circ \xi_n(\gamma).
\]

**Proof.** It suffices to prove (3.3), since the first claim follows from (3.3) and the fact that \( \eta_n(\gamma) \) is \( S_n \)-invariant.

A typical monomial on the right-hand side of (3.3) is of the form
\[
(\xi_{i_1} \cdot \cdot \cdot \xi_{i_k}) \circ (\gamma^{(j_1)} \cdot \cdot \cdot \gamma^{(j_{n-k})})
\]
where \( i_1 < \ldots < i_k \), \( j_1 < \ldots < j_{n-k} \), and \( \{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \). Here we have used the observation that \( \xi_{i_1} \circ \cdot \circ \xi_{i_k} \) is just the usual multiplication of permutations \( \xi_{i_1} \cdots \xi_{i_k} \) and \( \gamma^{(j_1)} \circ \cdot \circ \gamma^{(j_{n-k})} \) is just the ordinary cup product \( \gamma^{(j_1)} \cdot \cdot \cdot \gamma^{(j_{n-k})} \) in \( H^*(X^n) \cong H^*(X)^{\otimes n} \). Note that every cycle of each permutation \( \sigma \) appearing in \( \xi_{i_1} \cdots \xi_{i_k} \) has exactly one number which does not belong to \( i_1, \ldots, i_k \), and in addition, \( \ell(\sigma) = n - k \) and \( d(\sigma) = k \). Using the definition of the orbifold cup product, we see that the product \( \sigma \circ \gamma^{(j_1)} \cdot \cdot \cdot \gamma^{(j_{n-k})} \) does not involve the obstruction bundles (or the group defects are trivial in the terminology of Lehn-Sorger) and equals \( \gamma^{\otimes \ell(\sigma)} \in H^\otimes(\sigma) \cong H^*(X^n)^\sigma \). This proves (3.3).

If we denote \( \varepsilon_n(\gamma, h) = \prod_{i=1}^{n} (\gamma^{(i)} - h \xi_i) \), we have
\[
\sum_{n=0}^{\infty} \varepsilon_n(\gamma, h)z^n = \exp \left( \sum_{r \geq 1} (-h)^{r-1} \frac{p_{-r}(\gamma)}{r} z^r \right) \cdot \sum_{r \geq 1} \frac{1}{r} \cdot \frac{(\gamma)^k}{k!} z^k.
\]

Regarding \( \xi_i = \xi_i(0) \in H^*(X^n, S_n) \), we denote \( \xi_i^{\circ k} = \xi_i \circ \cdot \circ \xi_i \in H^*(X^n, S_n) \), and define \( e^{-\xi_i} = \sum_{k \geq 0} \frac{1}{k!} (-\xi_i)^{\circ k} \in H^*(X^n, S_n) \).

**Definition 3.4.** For homogeneous \( \alpha \in H^{[\alpha]}(X) \), we define the class \( O^k(\alpha, n) \in H^*_{\text{orb}}(X^n/S_n) \) to be
\[
O^k(\alpha, n) = \sum_{i=1}^{n} (-\xi_i)^{\circ k} \circ \alpha^{(i)} \in H^*_{\text{orb}}(X^n/S_n),
\]
and extends linearly to all \( \alpha \in H^*(X) \). We put \( O(\alpha, n) = \sum_{k \geq 0} \frac{1}{k!} O^k(\alpha, n) = \sum_{i=1}^{n} e^{-\xi_i} \circ \alpha^{(i)} \), and put \( O_h(\alpha, n) = \sum_{k \geq 0} \frac{n^k}{k!} O^k(\alpha, n) \). We further define the operator \( \Omega^k(\alpha) \in \text{End}(\mathcal{F}_X) \) (resp. \( \Omega(\alpha) \), or \( \Omega_h(\alpha) \)) to be the orbifold cup product with \( O^k(\alpha, n) \) (resp. \( O(\alpha, n) \), or \( O_h(\alpha, n) \)) in \( H^*_{\text{orb}}(X^n/S_n) \) for every \( n \geq 0 \).
Remark 3.5. We can see that $O^k(\alpha, n) \in H^*(X^n, S_n)$ is $S_n$-invariant (and thus lies in $H^*_{\text{orb}}(X^n/S_n)$) as follows. For $\gamma \in H^*(X)$, note that $e_j(\xi_1(\gamma), \ldots, \xi_n(\gamma))$ lies in $H^*_{\text{orb}}(X^n/S_n)$, where $e_j(\xi_1(\gamma), \ldots, \xi_n(\gamma))$ $(1 \leq j \leq n)$ is the $j$-th elementary symmetric function in $\xi_i(\gamma)$'s. So $H^*_{\text{orb}}(X^n/S_n)$ contains all symmetric functions in $\xi_i(\gamma)$'s. In particular, $O(e^{-\gamma}, n) = \sum_j (e^{-\xi_i} \circ (e^{-\gamma})^j) = \sum_i e^{-\xi_i(\gamma)} \in H^*_{\text{orb}}(X^n/S_n)$. Letting $\gamma$ vary, we see that $O(\alpha, n)$ and similarly $O^k(\alpha, n)$ lie in $H^*_{\text{orb}}(X^n/S_n)$.

In particular, $D^1(1x) \in \text{End}(\mathcal{F}_X)$ is the generalized Goulden’s operator \textbf{FW}, which will be denoted by $b$. The reason for our convention of putting the sign in front of $\xi_i$ is to make the comparison with the Hilbert scheme side easier. Also, the power sums of Jucys-Murphy elements have been studied in \textbf{LA} which corresponds to our case when $X$ is a point.

Note that the generalized Goulden’s operator $b$ is defined to be the orbifold cup product with $O^1(1x, n)$ in $H^*_{\text{orb}}(X^n/S_n)$. Given an operator $f \in \text{End}(\mathcal{F}_X)$, we denote by $f' = [b, f]$, and $f^{(k+1)} = (f')^k$. We have the following.

**Theorem 3.6.** We have $b = -\frac{1}{b} : p^3 \circ (\tau, 1x)$.

**Remark 3.7.** This proposition is a counterpart of \textbf{2.2}. The proof is essentially the same as the proof in the case when $X$ is a point \textbf{FW} (also cf. \textbf{LS2}). For example, if we look at the proof of Proposition 4.4, \textbf{LS2}, the $\Delta$ and $e$ there should be replaced by our $\tau$ (which equals $-\Delta$) and $-e$ respectively, since we are using the orbifold cup product of \textbf{CR}. Also compare Proposition 5.7 and Remark 5.8 below.

### 3.5. Interactions between Heisenberg algebra and $D^k(\gamma)$

**Theorem 3.8.** Let $\gamma, \alpha \in H^*(X)$. Then we have

\[
[D_h(\gamma), p_{-1}(\alpha)] = \exp(h \cdot \text{ad } b)(p_{-1}(\alpha))
\]

\[
[D^k(\gamma), p_{-1}(\alpha)] = p^{(k)}_{-1}(\alpha), \quad k \geq 0.
\]

**Proof.** For simplicity of signs, we assume that the cohomology classes $\gamma, \alpha$ are of even degree. It suffices to prove the second identity.

Recall that $p_{-1}(\alpha)(y) = \frac{1}{(\alpha, -1)} \sum_{g \in S_n} \text{ad}_g (y \otimes \alpha)$, for $y \in H^*_{\text{orb}}(X^{n-1}/S_{n-1})$. Regarding $S_{n-1}$ as the subgroup $S_{n-1} \times 1$ of $S_n$, we introduce an injective ring homomorphism

\[ i : H^*(X^{n-1}, S_{n-1}) \to H^*(X^n, S_n) \]

by sending $\alpha_{\sigma}$ to $\alpha_{\sigma} \otimes 1_X$, where $\sigma \in S_{n-1}$. Thus

\[
(n-1)! [D^k(\gamma), p_{-1}(\alpha)](y) = (n-1)! [D^k(\gamma) \cdot p_{-1}(\alpha)(y) - p_{-1}(\alpha) \cdot D^k(\gamma)(y)]
\]

\[
= O^k(\gamma, n) \circ \sum_{g \in S_n} \text{ad}_g (y \otimes \alpha) - \sum_{g \in S_n} \text{ad}_g ((O^k(\gamma, n-1) \circ y) \otimes \alpha)
\]

\[
= \sum_{g} \text{ad}_g ((O^k(\gamma, n) - i(O^k(\gamma, n-1))) \circ (y \otimes \alpha))
\]
where we used the fact that $O^k(\gamma, n)$ is $S_n$-invariant. By definition, we have $O^k(\gamma, n) - \iota(O^k(\gamma, n - 1)) = (-\xi_{n,n})^{ok} \circ \gamma^{(n)}$. Thus, we obtain

$$(n-1)! \left[ D^k(\gamma), \mathfrak{p}_{-1}(\alpha) \right](y) = \sum_g \text{ad} g \left[ (-\xi_{n,n})^{ok} \circ \gamma^{(n)} \circ (y \otimes \alpha) \right]$$

$$= \sum_g \text{ad} g \left[ (-\xi_{n,n})^{ok} \circ (y \otimes \gamma \alpha) \right].$$

It remains to prove that

$$(3.5) \sum_{g \in S_n} \text{ad} g \left[ (-\xi_{n,n})^{ok} \circ (y \otimes \gamma \alpha) \right] = (n-1)! p^{(k)}_{-1}(\gamma \alpha)(y).$$

We will prove this by induction. It is clearly true for $k = 0$. Note that $O^1(1_X, n) - \iota(O^1(1_X, n - 1)) = -\xi_{n,n}$. Under the assumption that the formula (3.5) is true for $k$, we have

$\sum_g \text{ad} g \left[ (-\xi_{n,n})^{o(k+1)} \circ (y \otimes \gamma \alpha) \right]$

$$= \sum_g \text{ad} g \left[ (O^1(1_X, n) - \iota(O^1(1_X, n - 1))) \circ (-\xi_{n,n})^{ok} \circ (y \otimes \gamma \alpha) \right]$$

$$= O^1(1_X, n) \circ \sum_g \text{ad} g \left[ (-\xi_{n,n})^{ok} \circ (y \otimes \gamma \alpha) \right]$$

$$- \sum_g \text{ad} g \left[ \iota(O^1(1_X, n - 1)) \circ (-\xi_{n,n})^{ok} \circ (y \otimes \gamma \alpha) \right],$$

since $O^1(\gamma, n)$ is $S_n$-invariant. By using the induction assumption twice, we get

$$\sum_g \text{ad} g \left[ (-\xi_{n,n})^{o(k+1)} \circ (y \otimes \gamma \alpha) \right]$$

$$= (n-1)! O^1(1_X, n) \circ p^{(k)}_{-1}(\gamma \alpha)(y)$$

$$- \sum_g \text{ad} g \left[ (-\xi_{n,n})^{ok} \circ ((O^1(1_X, n - 1) \circ y) \otimes \gamma \alpha) \right]$$

$$= (n-1)! [b \cdot p^{(k)}_{-1}(\gamma \alpha)(y) - p^{(k)}_{-1}(\gamma \alpha)(O^1(1_X, n - 1) \circ y)]$$

$$= (n-1)! p^{(k+1)}_{-1}(\gamma \alpha)(y).$$

So by induction, we have established (3.5) and thus the theorem.

We also have a theorem concerning the operator $\eta(\gamma)$. It generalizes Proposition 4.6 in [LS2] (which corresponds to our special case when $\gamma = 1_X$ and the assumption there that $X$ is a surface is unnecessary).

**Theorem 3.9.** Let $\gamma, \alpha \in H^*(X)$ and we further assume that $\gamma$ can be written as a sum of classes of even degree. Then we have

$$\eta(\gamma) \cdot \mathfrak{p}_{-1}(\alpha) = \mathfrak{p}_{-1}(\gamma \alpha) \cdot \eta(\gamma) - \mathfrak{p}_{-1}^{\pm}(\alpha) \cdot \eta(\gamma),$$

$$\varepsilon(\gamma) \cdot \mathfrak{p}_{-1}(\alpha) = \mathfrak{p}_{-1}(\gamma \alpha) \cdot \varepsilon(\gamma) + \mathfrak{p}_{-1}^{\pm}(\alpha) \cdot \varepsilon(\gamma).$$

**Proof.** The proof of the second formula is similar, so we will prove the first one only. For simplicity of signs in the proof, we assume that the cohomology class $\alpha$ is of even degree.
By definition, $\eta_n(\gamma) = \prod_{i=1}^{n}(\xi_{i:n} + \gamma^{(i)})$. It follows that
\[\eta_n(\gamma) - \eta_{n-1}(\gamma) \otimes \gamma = \xi_{n:n} \circ \iota(\eta_{n-1}(\gamma)).\]
Given $y \in H^*_{\text{orb}}(X^{n-1}/S_{n-1})$, we have
\[(n - 1)! \left[ \eta(\gamma) \cdot p_{-1}(\alpha)(y) - p_{-1}(\gamma_{\alpha}) \cdot \eta(\gamma)(y) \right] = \eta_n(\gamma) \circ \sum_{g \in S_n} \text{adg}(y \otimes \alpha) - \sum_{g \in S_n} \text{adg}[(\eta_{n-1}(\gamma) \circ y) \otimes \gamma_{\alpha}]\]
\[= \sum_{g} \text{adg}[(\eta_{n-1}(\gamma) \circ \gamma) \circ \eta_{n-1}(\gamma) \otimes \gamma_{\alpha}]\]
\[= \sum_{g} \text{adg}[\xi_{n:n} \circ \iota(\eta_{n-1}(\gamma)) \circ \eta_{n-1}(\gamma) \otimes \gamma_{\alpha}]\]
\[= -O^1(1_X, n) \circ \sum_{g} \text{adg}[(\eta_{n-1}(\gamma) \circ y) \otimes \gamma_{\alpha} + \sum_{g} \text{adg}[(O^1(1_X, n - 1) \circ \eta_{n-1}(\gamma) \circ y) \otimes \gamma_{\alpha}]\]
\[= (n - 1)! \left[ -b \cdot p_{-1}(\alpha) \cdot \eta(\gamma)(y) + p_{-1}(\alpha) \cdot b \cdot \eta(\gamma)(y) \right] = -(n - 1)! \left[ \eta_{n-1}(\gamma) \cdot \eta(\gamma)(y) \right].\]
This finishes the proof. \(\square\)

4. The orbifold cohomology ring of symmetric products II

We see from Theorem 3.1, Theorem 3.6 and Theorem 3.8 that the orbifold cohomology rings $H^*_{\text{orb}}(X^n/S_n)$ satisfy the axioms in Subsect. 2.6. Therefore, we can follow the approaches of [LQW1-4] to establish the results in the following subsections. The terms involving the canonical class $K$ of $X$ in various formulas in [LQW1-4] will disappear because there is no $K$-term in Theorem 3.6. We remark that the fact $c^2 = 0$ was used in [LQW3, LQW4] (where $X$ is a surface). Thus, when dealing the orbifold cohomology ring $H^*_{\text{orb}}(X^n/S_n)$ in this section, we sometimes need to treat separately and carefully the case when $X$ is a point (i.e. when $c^2 \neq 0$).

4.1. The ring generators for $H^*_{\text{orb}}(X^n/S_n)$.

**Theorem 4.1.**

(i) Given a closed complex manifold $X$ of even dimension, the orbifold cohomology ring $H^*_{\text{orb}}(X^n/S_n)$ is generated by the cohomology classes $O^i(\alpha, n)$, where $0 \leq i < n$ and $\alpha$ runs over a fixed linear basis of $H^*(X)$;

(ii) The ring $H^*_{\text{orb}}(X^n/S_n)$ is generated by the classes $P_i(\alpha, n)$, where $0 \leq i < n$ and $\alpha$ runs over a fixed linear basis of $H^*(X)$.

**Remark 4.2.** Part (i) is the counterpart of Theorem 5.30 in [LQW1], while part (ii) is the counterpart of Theorem 3.23 in [LQW2].
4.2. The universality of the ring $H^*_{orb}(X^n/S_n)$.

**Theorem 4.3.** Let $X$ be a closed complex manifold of even dimension. The orbifold cohomology ring $H^*_{orb}(X^n/S_n)$ is determined uniquely by the ring $H^*(X)$.

We refer to this theorem, Proposition 4.4 below as the universality of the ring $H^*_{orb}(X^n/S_n)$. The theorem follows from the more quantitative descriptions of the orbifold cup product of ring generators of $H^*_{orb}(X^n/S_n)$ in Proposition 4.4 and Proposition 4.6. It also follows from combining the results of [LS2] and [FG].

Let $s \geq 1$, and let $\alpha_1, \ldots, \alpha_s \in H^*(X)$ be homogeneous cohomology classes. For a partition $\pi = \{\pi_1, \ldots, \pi_j\}$ of $\{1, \ldots, s\}$, we fix the orders of the elements listed in each subset $\pi_i$ ($1 \leq i \leq j$) once and for all, and define $\ell(\pi) = j$, $\alpha_{\pi_i} = \prod_{m \in \pi_i} \alpha_m$, and the sign $\text{sign}(\alpha, \pi)$ by the relation

$$\prod_{i=1}^{j} \alpha_{\pi_i} = \text{sign}(\alpha, \pi) \cdot \prod_{i=1}^{s} \alpha_i.$$

The choice of the orders for the elements listed in each of the subsets $\pi_i$, $1 \leq i \leq \ell(\pi)$ will affect the sign $\text{sign}(\alpha, \pi)$, but will not affect the long expression in Proposition 4.4 below. We denote by $1_{-k} = \frac{p - 1(1)^k}{k!}$ if $k \geq 0$ and $1_{-k} = 0$ if $k < 0$.

**Proposition 4.4.** (Universality) Let $X$ be a closed complex manifold of even dimension $d > 0$. Let $n, s \geq 1$, $k_1, \ldots, k_s \geq 0$, and let $\alpha_1, \ldots, \alpha_s \in H^*(X)$ be homogeneous. Then, the orbifold product $O^{k_1}(\alpha_1, n) \circ \cdots \circ O^{k_s}(\alpha_s, n)$ in $H^*_{orb}(X^n/S_n)$ is a finite linear combination of expressions of the form

$$\text{sign}(\alpha, \pi) \cdot 1 - \left( n - \sum_{i=1}^{\ell(\pi)} \sum_{j=1}^{m_i - r_i} n_{i,j} \prod_{j=1}^{\ell(\pi)} \prod_{i=1}^{l(\pi)} (\tau_{(m_i - r_i)}(\epsilon_i \alpha_{\pi_i})) \cdot 0 \right)$$

where $\pi$ runs over all partitions of $\{1, \ldots, s\}$, $\epsilon_i \in \{1_X, e\}$, $r_i = |\epsilon_i|/d \leq m_i \leq 2 + \sum_{j \in \pi_i} k_j$, $0 < n_{i,1} \leq \ldots \leq n_{i,m_i - r_i}$, $\sum_{j=1}^{m_i - r_i} n_{i,j} \leq \sum_{j \in \pi_i} (k_j + 1)$ for every $i$, and

$$\sum_{i=1}^{\ell(\pi)} \left( m_i - 2 + \sum_{j=1}^{m_i - r_i} n_{i,j} \right) = \sum_{i=1}^{s} k_i.$$

Moreover, all the coefficients in this linear combination are independent of the manifold $X$, the cohomology classes $\alpha_1, \ldots, \alpha_s$, and the integer $n$.

**Remark 4.5.** This proposition is the counterpart of Proposition 5.1, [LQW3].

For the case $d = 0$ (i.e., $X$ is a point), we adopt the simplified notations $p_m$ and $O^k(n)$ for $p_m(1_X)$ and $O^k(1_X, n)$ respectively. We have the following analog of Proposition 4.4.
where \( \pi \) runs over all partitions of \( \{1, \ldots, s\} \), \( m_i, r_i \in \mathbb{Z}_+ \) such that \( 2r_i \leq m_i \leq 2 + \sum_{j \in \pi} k_j, \) \( 0 < n_i,1 \leq \cdots \leq n_i, m_i - 2r_i \), \( \sum_{j \in \pi} n_{i,j} \leq \sum_{j \in \pi} (k_j + 1) \) for every \( i \), and

\[
\ell(\pi) \prod_{i=1}^{\ell(\pi)} \prod_{j=1}^{m_i - 2r_i} p_{-n_{i,j}} \cdot [0]
\]

Moreover, all the coefficients in this linear combination are independent of \( n \).

4.3. The stability of the ring \( H^*_{orb}(X^n/S_n) \).

Theorem 4.7. Let \( X \) be a closed complex manifold of even dimension \( d \). Let \( s \geq 1 \) and \( k_i \geq 1 \) for \( 1 \leq i \leq s \). Fix \( n_{i,j} \geq 1 \) and \( \alpha_{i,j} \in H^*(X) \) for \( 1 \leq j \leq k_i \), and fix \( n \) with \( n \geq \sum_{j=1}^{k_i} n_{i,j} \) for all \( 1 \leq i \leq s \). Then the orbifold cup product

\[
\prod_{i=1}^{s} \left( 1 - (n - \sum_{j=1}^{k_i} n_{i,j}) \prod_{j=1}^{m_i - 2r_i} p_{-n_{i,j}} (\alpha_{i,j}) \cdot [0] \right)
\]

in \( H^*_{orb}(X^n/S_n) \) is equal to a finite linear combination of monomials of the form

\[
1 - (n - \sum_{a=1}^{N} m_a) \prod_{a=1}^{N} p_{-m_a} (\gamma_a) \cdot [0]
\]

where \( \sum_{a=1}^{N} m_a \leq \sum_{i=1}^{s} \sum_{j=1}^{k_i} n_{i,j} \), and \( \gamma_1, \ldots, \gamma_N \) depend only on \( e, \alpha_{i,j}, 1 \leq i \leq s, 1 \leq j \leq k_i \). Moreover, the coefficients in this linear combination are independent of \( \alpha_{i,j} \) and \( n \); they are also independent of \( X \) provided \( d > 0 \).

Remark 4.8. This theorem is the counterpart of Theorem 6.1 in [LQW3].

4.4. The stable ring \( \mathfrak{H}_X \). Given a finite set \( S \) which is a disjoint union of subsets \( S_0 \) and \( S_1 \), we denote by \( \mathcal{P}(S) \) the set of partition-valued functions \( \rho = (\rho(c))_{c \in S} \) on \( S \) such that for every \( c \in S_1 \), the partition \( \rho(c) \) is required to be strict in the sense that \( \rho(c) = (1^{m_1(c)}, 2^{m_2(c)}, \ldots) \) with \( m_r(c) = 0 \) or \( 1 \) for all \( r \geq 1 \).

Now let us take a linear basis \( S = S_0 \cup S_1 \) of \( H^*(X) \) such that \( 1_X \in S_0, S_0 \subset H^{\text{even}}(X) \) and \( S_1 \subset H^{\text{odd}}(X) \). If we write \( \rho = (\rho(c))_{c \in S} \) and \( \rho(c) = (r^{m_r(c)})_{r \geq 1} = (1^{m_1(c)}2^{m_2(c)}\ldots) \), then we introduce the following notations:

\[
\ell(\rho) = \sum_{c \in S} \ell(\rho(c)) = \sum_{c \in S, r \geq 1} m_r(c),
\]

\[
\|\rho\| = \sum_{c \in S} \|\rho(c)\| = \sum_{c \in S, r \geq 1} r \cdot m_r(c),
\]

\[\mathcal{P}_n(S) = \{\rho \in \mathcal{P}(S) \mid \|\rho\| = n\}\]
Given \( \rho = (\rho(c))_{c \in S} = (\rho^m(c))_{c \in S, r \geq 1} \in \mathcal{P}(S) \) and \( n \geq 0 \), we define
\[
p_{-\rho(c)}(c) = \prod_{r \geq 1} p_{-r}(c)^{m_r(c)} = p_{-1}(c)^{m_1(c)} p_{-2}(c)^{m_2(c)} \ldots
\]
\[
p_{\rho}(n) = 1_{-(n-\|\rho\|)} \prod_{c \in S} p_{\rho(c)}(c) \cdot [0] \in H^n_{\text{orb}}(X^n/S_n)
\]
where we fix the order of the elements \( c \in S_1 \) appearing in \( \prod_{c \in S} \) once and for all. It is understood that \( p_{\rho}(n) = 0 \) for \( 0 \leq n < \|\rho\| \).

As \( \rho \) runs over all partition-valued functions on \( S \) with \( \|\rho\| \leq n \), the corresponding \( p_{\rho}(n) \) linearly span \( H^n_{\text{orb}}(X^n/S_n) \), as a corollary to Theorem 4.7 (for \( s = 2 \)), we can write the orbifold cup product in the ring \( H^n_{\text{orb}}(X^n/S_n) \) as
\[
p_{\rho}(n) \circ p_{\sigma}(n) = \sum_{\nu} d_{\rho\sigma}^\nu p_{\nu}(n),
\]
where \( \|\nu\| \leq \|\rho\| + \|\sigma\| \), and the structure coefficients \( d_{\rho\sigma}^\nu \) are independent of \( n \). Even though the cohomology classes \( p_{\nu}(n) \) with \( \|\nu\| \leq n \) in \( H^n_{\text{orb}}(X^n/S_n) \) are not linearly independent, we can show that (cf. Lemma 7.1, [LQW3]) the structure constants \( d_{\rho\sigma}^\nu \) in the formula (4.1) are uniquely determined from the fact that they are independent of \( n \).

**Definition 4.9.** The *stable ring* associated to a closed complex manifold \( X \), denoted by \( \mathfrak{R}_X \), is defined to be the ring with a linear basis formed by the symbols \( p_\rho, \rho \in \mathcal{P}(S) \) and with the multiplication defined by
\[
p_\rho \cdot p_\sigma = \sum_{\nu} d_{\rho\sigma}^\nu p_\nu,
\]
where the structure constants \( d_{\rho\sigma}^\nu \) are from the relations (4.1).

Note that the stable ring does not depend on the choice of a linear basis \( S \) of \( H^*(X) \) containing \( 1_\mathfrak{X} \) since the operator \( p_\rho(\alpha) \) depends on the cohomology class \( \alpha \in H^*(X) \) linearly. Clearly the stable ring \( \mathfrak{R}_X \) itself is super-commutative and associative. The ring \( \mathfrak{R}_X \) captures all the information of the orbifold cohomology rings \( H^*_\text{orb}(X^n/S_n) \) for all \( n \), as we easily recover the relations (4.1) from the ring \( \mathfrak{R}_X \). We summarize these observations into the following.

**Theorem 4.10.** (Stability) For a closed complex manifold \( X \) of even dimension, the cohomology rings \( H^*_\text{orb}(X^n/S_n) \), \( n \geq 1 \) give rise to the stable ring \( \mathfrak{R}_X \) which completely encodes the cohomology ring structure of \( H^*_\text{orb}(X^n/S_n) \) for each \( n \). The stable ring \( \mathfrak{R}_X \) depends only on the cohomology ring \( H^*(X) \).

**Remark 4.11.** This theorem is the counterpart of Theorem 7.1 [LQW3]. The stable ring here is the counterpart of the Hilbert ring introduced in Definition 7.1 of [LQW3]. The stability of the convolution of symmetric groups (which corresponds to our special case when \( X \) is a point) was due to Kerov and Olshanski (cf. [LT]).

When \( \ell(\rho) = 1 \), that is, when the partition \( \rho(c) \) is a one-part partition \( (r) \) for some element \( c \in S \) and is empty for all the other elements in \( S \), we will simply write \( p_\rho = p_{r,c} \). Just as in [LQW3], we can show that the stable ring \( \mathfrak{R}_X \) is isomorphic to the tensor product \( P \otimes E \), where \( P \) is the polynomial algebra generated by \( p_{r,c}, c \in S_0, r \geq 1 \) and \( E \) is the exterior algebra generated by \( p_{r,c}, c \in S_1, r \geq 1 \).
4.5. The \( \mathcal{W} \) algebras. In this subsection, we assume that \( X \) is a closed complex manifold of even dimension \( d > 0 \). Results on this section are counterparts of Sect. 5 of [LQW4]. However, some signs have been modified due to the sign difference between the two Heisenberg algebra commutators in the setups of Hilbert schemes and symmetric products, cf. Theorems 2.1 and 3.1. The modification is done by carefully tracing the procedures in [LQW4].

Let \( \alpha \in H^*(X) \), and \( \lambda = (\cdots (-2)^{m_1} (-1)^{m_2} 1^{m_3} 2^{m_2} \cdots) \) be a generalized partition of the integer \( n = \sum_i i m_i \) whose part \( i \in \mathbb{Z} \) has multiplicity \( m_i \). Define \( \ell(\lambda) = \sum_i m_i \), \( |\lambda| = \sum_i i m_i = n \), \( s(\lambda) = \sum_i i^2 m_i \), \( \lambda' = \prod_i m_i! \), and

\[
p_\lambda(\tau_\lambda \alpha) = \left( \prod_i (p_i)^{m_i} \right) (\tau_{\ell(\lambda)} \alpha).
\]

Let \( -\lambda \) be the generalized partition whose multiplicity of \( i \in \mathbb{Z} \) is \( m_{-i} \). A generalized partition becomes a partition in the usual sense if the multiplicity \( m_i = 0 \) for every \( i < 0 \).

For \( p \geq 0, n \in \mathbb{Z} \) and \( \alpha \in H^*(X) \), define \( \mathcal{J}^p_n(\alpha) \in \text{End}(\mathcal{F}_X) \) to be

\[
p! \left( \sum_{\ell(\lambda) = p + 1, |\lambda| = n} \frac{1}{\lambda} p_\lambda(\tau_\lambda \alpha) + \sum_{\ell(\lambda) = p - 1, |\lambda| = n} \frac{s(\lambda) + n^2 - 2}{24 \lambda!} p_\lambda(\tau_\lambda (e\alpha)) \right)
\]

where the \( \lambda \)'s are generalized partitions. Note that \( \mathcal{J}^0_n(\alpha) = p_n(\alpha) \). We define \( \mathcal{W}_X \) to be the linear span of the identity operator \( \text{Id}_{\mathcal{F}_X} \) and the operators \( \mathcal{J}^p_n(\alpha) \) in \( \text{End}(\mathcal{F}_X) \), where \( p \geq 0, n \in \mathbb{Z} \) and \( \alpha \in H^*(X) \).

The following theorem describes the operator \( \mathcal{D}^k(\alpha) \) in terms of the Heisenberg generators explicitly. It is a counterpart of Theorem 4.6 in [LQW4].

**Theorem 4.12.** Let \( k \geq 0, \) and \( \alpha \in H^*(X) \). Then, \( \mathcal{D}^k(\alpha) = \frac{(-1)^k}{k+1} \mathcal{J}^{k+1}_0(\alpha) \).

In terms of vertex operators, the operator \( \mathcal{J}^p_m(\alpha) \) can be rewritten as:

\[
\frac{1}{(p+1)} : p^{p+1} :_m (\tau_m \alpha) + \frac{1}{24} p(m^2 - 3m - 2p) : p^{p-1} :_m (\tau_m (e\alpha))
\]

\[+ \frac{p(p-1)}{24} : (\partial^2 p) p^{p-2} :_m (\tau_m (e\alpha)).
\]

(4.2)

If we want the coefficients above to be independent of \( m \), we can further rewrite

\[
\mathcal{J}^p_m(\alpha) = \frac{1}{(p+1)} : p^{p+1} :_m (\tau_m \alpha) + \frac{p}{24} (\partial^2 p : p^{p-1} :_m (\tau_m (e\alpha))
\]

\[+ \frac{(p+1)p}{12} : (\partial : p^{p-1} :_m (\tau_m (e\alpha))
\]

\[+ \frac{p(p^2 - p - 2)}{24} : p^{p-1} :_m (\tau_m (e\alpha))
\]

\[+ \frac{p(p-1)}{24} : (\partial^2 p) p^{p-2} :_m (\tau_m (e\alpha)).
\]

The operators \( \mathcal{D}^p(\alpha), p_n(\alpha), \) and \( \mathcal{J}^p_n(\alpha) \) are related in the following way.

**Proposition 4.13.** Given \( p \geq 0, \alpha, \beta \in H^*(X) \), we have

\[
[\mathcal{D}^p(\alpha), p_n(\beta)] = -n \cdot \mathcal{J}^p_n(\alpha \beta).
\]
We introduce an integer $\Omega_{m,n}^{p,q}$ for $m,n,p,q \in \mathbb{Z}$ as follows:

$$
\Omega_{m,n}^{p,q} = mp^3n^2 + 3mp^2n^2q - p^2nq + p^2q^3 - 3mp^2n^2 + pq
+ 3m^2pq - 3mp^2q - m^3q^2p - mpq + m^3pq
+ mpq^2 + 2mp^2 - 3m^2pq^2 - 2m^2nq + 3m^2nq^2 - m^2nq^3.
$$

**Theorem 4.14.** Let $X$ be a closed complex manifold of even dimension $d > 0$. The vector space $\mathcal{W}_X$ is closed under the Lie bracket. More explicitly, for $m,n \in \mathbb{Z}$, and $\alpha, \beta \in H^+(X)$, we have

$$
[\mathcal{J}^m_\alpha, \mathcal{J}^n_\beta] = (qm - pn) \cdot \mathcal{J}^{p+q-1}_m(\alpha \beta) + \frac{\Omega_{m,n}^{p,q}}{12} \cdot \mathcal{J}^{p+q-3}_m(e \alpha \beta)
$$

where $(p,q) \in \mathbb{Z}^2_+$ except for the unordered pairs $(0,0), (1,0), (2,0)$ and $(1,1)$. In addition, for these four exceptional cases, we have

$$
\begin{align*}
[\mathcal{J}^0_m(\alpha), \mathcal{J}^0_n(\beta)] &= m\delta_{m,-n} \int_X (\alpha \beta) \cdot \text{Id}_{F_X}, \\
[\mathcal{J}^1_m(\alpha), \mathcal{J}^0_n(\beta)] &= -n \cdot \mathcal{J}^0_{m+n}(\alpha \beta), \\
[\mathcal{J}^2_m(\alpha), \mathcal{J}^0_n(\beta)] &= -2n \cdot \mathcal{J}^1_{m+n}(\alpha \beta) + \frac{m^3 - m}{6} \delta_{m,-n} \int_X (e \alpha \beta) \cdot \text{Id}_{F_X}, \\
[\mathcal{J}^1_m(\alpha), \mathcal{J}^1_n(\beta)] &= (m - n) \cdot \mathcal{J}^1_{m+n}(\alpha \beta) + \frac{m^3 - m}{12} \delta_{m,-n} \int_X (e \alpha \beta) \cdot \text{Id}_{F_X}.
\end{align*}
$$

**Remark 4.15.** This $\mathcal{W}$ algebra should be viewed as a generalization of the $\mathcal{W}_n$ algebra (cf. e.g. [FKRW, Kac]). The assumption $d > 0$ above ensures that $e^2 = 0$. The case when $d = 0$ (i.e. $X$ is a point) has been treated in [LT].

**Remark 4.16.** Our understanding of $\mathcal{D}^k(\alpha)$ in terms of vertex operators also allows us to recast the study of the ring structure problems from a different perspective starting from vertex algebras. Given an integral lattice $A$ (i.e. a free abelian group with a non-degenerate bilinear form $A \times A \to \mathbb{Z}$), the Fock space $\mathfrak{F}_A$ of a Heisenberg algebra associated to $A$ affords a natural $\mathbb{Z}_+$-grading: $\mathfrak{F}_A = \oplus_{n=0}^\infty \mathfrak{F}^n_A$, and in addition $\mathfrak{F}_A$ carries a natural vertex algebra structure [Bor]. There are numerous operators in $\text{End}(\mathfrak{F}_A)$ arising from the vertex algebra constructions.

Let us assume that $A$ has an additional structure of a graded Frobenius algebra compatible with the given bilinear form on $A$. We may ask if there is any reasonable graded commutative ring structure on $\mathfrak{F}^n_A$ for each $n$ which comes from the vertex algebra structure on $\mathfrak{F}_A$. The answer to this question is affirmative. We may introduce operators $\mathcal{D}^k(\alpha)$ acting on $\mathfrak{F}^n_A$ for each $n$ to be the zero-modes of the vertex operators given in Theorem 4.12 and 4.13 above. The operators $\mathcal{D}^k(\alpha)$ commute with each other by Theorem 4.11. Next, we define elements $O^k(\alpha, n)$ in $\mathfrak{F}^n_A$ by applying the operator $\mathcal{D}^k(\alpha)$ to $\frac{1}{m} \mathfrak{p}_-(1_A)^n |0\rangle \in F^n_A$, where $\mathfrak{p}_-(1_A)$ is a Heisenberg generator. Then, we define the product in $\mathfrak{F}_A$ (which is commutative) by letting $O^k(\alpha, n) \circ O^k(\beta, n) = \mathcal{D}^k(\alpha) \circ \mathcal{D}^k(\beta) \cdot \frac{1}{m} \mathfrak{p}_-(1_A)^n |0\rangle$. By Theorem 4.13 we see that $\mathfrak{F}_A$ is generated as a ring by the elements $O^k(\alpha, n)$'s. In this way, we define a ring structure on $\mathfrak{F}^n_A$ for each $n$ with a set of ring generators.
5. The deformed orbifold cohomology ring of symmetric products

5.1. A deformed orbifold cohomology ring. Given a complex manifold \( M \) with a finite group \( G \) action, we denote by \( H^*(M,G;\mathbb{C}) \) and \( H^*_{\text{orb}}(M/G;\mathbb{C}) \) respectively the counterparts of \( H^*(M,G) \) and \( H^*_{\text{orb}}(M/G) \) with \( \mathbb{C} \)-coefficients.

**Definition 5.1.** Let \( M \) be a complex manifold with a finite group \( G \) action. Let \( t \) be a nonzero complex parameter. We define a product structure, denoted by \( \circ_t \), on \( H^*(M,G;\mathbb{C}) \):

\[
\alpha_g \circ_t \beta_h = t^\epsilon(g,h) \alpha \circ \beta,
\]

where \( g, h \in G \), \( \alpha_g \in H^*(M^g) \), \( \beta_h \in H^*(M^h) \), and \( \epsilon(g,h) = (F^g + F^h - F^{gh})/2 \). For the sake of simplicity, we assume here that \((F^g + F^h - F^{gh})/2\) is an integer for every \( g, h \in G \).

**Remark 5.2.** In the above definition, for brevity, we have omitted the dependence of shift numbers on the connected components. If \( \epsilon(g,h) \) is a rational number for some \( g, h \), we make sense of \( t^\epsilon(g,h) \) by fixing a suitable root of \( t \). This definition is a simple generalization of the signed orbifold product (i.e. our \( t = -1 \) case) introduced in [FG], which in turn was motivated by Lehn and Sorger [LS2] who introduced the sign in the symmetric product setup. The new product \( \circ_t \) remains to be associative thanks to the identity \( \epsilon(g,h) + \epsilon(gh,k) = \epsilon(g,hk) + \epsilon(h,k) \). It induces a graded commutative product structure on \( H^*_{\text{orb}}(M/G;\mathbb{C}) \), the \( G \)-invariant part of \( H^*(M,G;\mathbb{C}) \) (which uses the easy identity \( \epsilon(g,h) = \epsilon(h^{-1}gh) \)). Compare with [FG], Theorem 1.29 and its proof.

**Remark 5.3.** In the above definition, if \( t^\epsilon(g,h) \) are all rational for a given \( t \) and every \( g, h \in G \) (e.g. when \( t \) is rational), it makes sense to talk about the ring product \( \circ_t \) on \( H^*(M,G) \) and \( H^*_{\text{orb}}(M/G) \) with rational coefficients.

**Proposition 5.4.** The family of ring structures \( \circ_t \) on \( H^*(M,G;\mathbb{C}) \) (and resp. on \( H^*_{\text{orb}}(M/G;\mathbb{C}) \)) with \( \mathbb{C} \)-coefficient are isomorphic for all nonzero \( t \in \mathbb{C} \).

**Proof.** For \( t \neq 0 \), we define the linear map from \( H^*(M,G;\mathbb{C}) \) with product \( \circ_t \) to \( H^*(M,G;\mathbb{C}) \) with the (original) product \( \circ = \circ_1 \):

\[
\zeta_t : (H^*(M,G;\mathbb{C}), \circ_t) \to (H^*(M,G;\mathbb{C}), \circ)
\]

by sending \( t^{-F^g/2} \alpha_g \) to \( \alpha_g \), for \( \alpha_g \in H^*(M^g) \). The ring isomorphism follows from the definition of the product \( \circ_t \).

**Remark 5.5.** The deformed product \( \circ_t \), where \( t \in \mathbb{Q} \), on \( H^*(M,G) \) in general can be non-isomorphic for different \( t \), and so this is an interesting deformation. For example, for symmetric product \( X^n/S_n \) associated with a complex surface \( X \), the number \((F^g + F^h - F^{gh})/2\) is always an integer for every \( g, h \in S_n \), but \( F^{gh}/2 \) often not. So in general \( t^{-F^g/2} \) may not be a rational number even when \( t \) is, and thus the isomorphism given in Proposition 5.4 is not valid over \( \mathbb{Q} \).
5.2. The symmetric product case. In this subsection, we have two options. Either we assume \( t \) is chosen such that a cubic root \( t^{1/3} \) of \( t \) is rational, then all the (orbifold) cohomology groups involved use \( \mathbb{Q} \)-coefficients. On the other hand, if we choose to use \( \mathbb{C} \)-coefficients for the cohomology groups, then \( t \) can be any complex number.

Now let \( X \) be a closed complex manifold of even dimension \( d \). Let us fix a cubic root \( t^{1/3} \) of \( t \). We introduce the modified Heisenberg operators: 
\[
\big( t^d \big)^m \partial_m, -n = \partial_m \big( t^{d/3} \big)^n \partial_n \quad \text{if} \quad n \leq 0, \quad \text{and} \quad \big( t^d \big)^m \partial_m \big( t^{-d/6} \big)^n \partial_n \quad \text{if} \quad n > 0.
\]
Then the Heisenberg algebra commutation relations in Theorem 3.1 becomes
\[
\big[ t^d \big]^m \partial_m, \big( t^{d/3} \big)^n \partial_n \big( t^{-d/6} \big) = t^{d/6} \bar{m} \delta_{m, -n}(\alpha, \beta) \cdot \text{Id}_{\mathcal{F}_X}.
\]
We introduce the modified vertex operator: 
\[
\big( t^d \big)^m \partial_m(z) = \sum_{n \in \mathbb{Z}} \big( t^{d/3} \big)^n \partial_n(\alpha) z^{n+1}.
\]
We modify the definition of the operators \( \mathcal{D}(\alpha), \mathcal{D}(\delta), \) and \( \mathcal{D}_h(\alpha) \) by using the product \( \alpha \) instead of \( \alpha \), and denote by the resulting operators by \( \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha), \big( t^d \big)^m \partial_m \big( t^{-d/6} \big)^n \partial_n \mathcal{D}(\delta) \), and \( \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}_h(\alpha) \). We denote by \( \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha) = \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\delta) = \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}_h(\alpha) \). We denote by 
\[
\big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\delta) = \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}_h(\alpha).
\]
We denote by \( \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha) = \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\delta) = \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}_h(\alpha) \). We denote by 
The same argument as earlier leads to the following theorem.

**Theorem 5.6.** Let \( \gamma, \alpha \in H^* (X) \). Then we have
\[
\big[ \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}_h(\gamma), \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha) \big] = \exp(\hbar \cdot \text{adj}(t^d(\gamma) \alpha)) \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha), \quad \hbar \geq 0.
\]

**Theorem 5.7.** We have 
\[
\big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n (\tau_1) = -\frac{1}{6} \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha).
\]

**Remark 5.8.** An outline of a proof of Theorem 5.7 goes as follows. For notational simplicity, we suppress the dependence on cohomology classes of \( \partial_m \) and \( \partial_n(\alpha) \) below. Given \( \sigma \in S_{n} \), it is well known that the degree shift number \( F^\sigma = \frac{d}{2} \cdot d(\sigma) \). Let us look at the product of a transposition \( (a, b) \) with \( \alpha \in H^* (X)^\sigma \). If \( a, b \) do not lie in the same cycle of \( \sigma \), then \( (a, b) \sigma \) is obtained from \( \sigma \) by combining the two cycles in \( \sigma \) containing \( a, b \) respectively. It follows that \( \epsilon((a, b), \sigma) = 0 \). If \( a, b \) lie in a same cycle of \( \sigma \), then \( (a, b) \sigma \) is obtained from splitting the cycle of \( \sigma \) containing \( a, b \) into two cycles, and thus \( \epsilon((a, b), \sigma) = \frac{d}{2} \). See the proof of Theorem 2, [FW] for some illustration by concrete examples. This explains the factor \( t^{d/2} \) below (compare with Remark 5.7):
\[
\big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha) = -\frac{1}{6} \sum_{m, n \geq 0} \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha).
\]
On the other hand, one observes that
\[
-\frac{1}{6} \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha) = -\frac{1}{2} \sum_{m, n \geq 0} \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha).
\]
which coincide with \( \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha) \) by using the definition of \( \mathcal{D}(\alpha) \) and the above formula for \( \big( t^d \big)^m \partial_m \big( t^{d/3} \big)^n \partial_n \mathcal{D}(\alpha) \).

**Remark 5.9.** Formula (5.1), Theorem 5.6 and Theorem 5.7 indicate that the counterparts of Theorem 3.1, Theorem 3.6, and Theorem 3.8 hold if we use the product \( \partial_i(\alpha) \) on \( H^* (X)/\mathbb{Z}_n \) instead of \( \alpha \). Therefore the results established in section 4 also carry over for the product \( \partial_i(\alpha) \) with appropriate modifications.
Remark 5.10. Starting from a graded Frobenius algebra $A$, we can obtain a family of Frobenius algebra structures on $\mathfrak{F}_n^A$ depending on $t$ by using the modified Heisenberg algebra etc. When setting $t = -1$, the algebra $\mathfrak{F}_n^A$ should be isomorphic to $A^{[n]}$ given in \cite{LS2}.

5.3. A cohomology ring isomorphism. Let $X$ be a projective surface. We have seen that both $F_X = \oplus_n H^*_\text{orb}(X^n/S_n)$ and $H_X = \oplus_n H^*(X^n)$ are Fock spaces of the same size. By sending $p_{-n_1}(\alpha_1) \cdots p_{-n_k}(\alpha_k)(0)$ to $a_{-n_1}(\alpha_1) \cdots a_{-n_k}(\alpha_k)(0)$, where $n_1, \ldots, n_k > 0$, $\alpha_1, \ldots, \alpha_k \in H^*(X)$, we have defined a linear isomorphism $\Theta : F_X \rightarrow H_X$, which induces a linear isomorphism $\Theta_n : H^*_\text{orb}(X^n/S_n) \rightarrow H^*(X^n)$ for each $n$.

**Theorem 5.11.** Let $X$ be a projective surface with numerically trivial canonical class. The linear map $\Theta_n : H^*_\text{orb}(X^n/S_n) \rightarrow H^*(X^n)$ is a ring isomorphism, if we use the product $\circ$ in $H^*_\text{orb}(X^n/S_n)$.

**Proof.** Noting that $d = 2$ and $t = -1$, we can take $t^d/6 = -1$. Thus, $tp_n = p_n$ if $n \leq 0$, and $tp_n = -p_n$ if $n > 0$ (we keep using $t$ instead of $-1$ here and below for notational convenience.) The Heisenberg algebra commutators for the $t^i p_n$ in (5.1) and for the $a_n$ in Theorem 2.1 exactly match. Comparing (2.1) and (5.1), we see that $\Theta$ sends $G^1(1_X, n) = c_1(O[n])$ (where $O$ denotes the trivial line bundle over $X$) to $O^1(1_X, n) = -\sum_{i=1}^n \xi_i$. Note that the operator $\delta$ and $t^i b$ are defined in terms of $G^1(1_X, n)$ and $O^1(1_X, n)$ respectively. By Theorem 2.1 and Theorem 5.5, the operator $\delta$ matches exactly with $t^i b$. Then it follows from comparing Theorems 2.1 and Proposition 5.6 that the operator $\Theta^k(\gamma)$ coincide with $\Omega^k(\gamma)$. If we recall the definitions of $\Theta^k(\gamma)$ and $\Omega^k(\gamma)$, the theorem follows now from Theorem 4.1 (i) and its Hilbert scheme counterpart Theorem 1.2 in \cite{LQW1}.

**Remark 5.12.** This ring isomorphism has been earlier established in a different way by combining the results in \cite{LS2, FG} (also cf. \cite{Ur}.)

Modifying $\Theta$, we introduce a linear isomorphism $\tilde{\Theta} : F_X \rightarrow H_X$ by sending $\sqrt{-1} \sum_{a=1}^k p_{-n_1}(\alpha_1) \cdots p_{-n_k}(\alpha_k)(0)$ to $a_{-n_1}(\alpha_1) \cdots a_{-n_k}(\alpha_k)(0)$. This induces a linear isomorphism $\tilde{\Theta}_n : H^*_\text{orb}(X^n/S_n; \mathbb{C}) \rightarrow H^*(X^n; \mathbb{C})$ for each $n$. Note that both $\Theta_1$ and $\tilde{\Theta}_1$ are simply the identity map on the cohomology group of the surface $X$.

**Theorem 5.13.** Let $X$ be a projective surface with numerically trivial canonical class. The linear map $\tilde{\Theta}_n : H^*_\text{orb}(X^n/S_n; \mathbb{C}) \rightarrow H^*(X^n; \mathbb{C})$ is a ring isomorphism from the cohomology ring of Hilbert scheme with $\mathbb{C}$-coefficient to the standard orbifold cohomology ring of the symmetric product with $\mathbb{C}$-coefficient.

**Proof.** Note that $p_{-n}(\alpha)(0)$ corresponds to an $n$-cycle whose shift number is $(n - 1)$ and a permutation associated to $p_{-n_1}(\alpha_1) \cdots p_{-n_k}(\alpha_k)(0)$ has shift number $\sum_{a=1}^k n_a - k$. Thus, the map $\tilde{\Theta}_n$ is the composition of the ring isomorphism $\Theta_n$ with the ring isomorphism $\zeta_t$ for $t = -1$ defined in the proof of Proposition 5.4.

The above theorem supports the original conjecture of Ruan \cite{Ru} if the cohomology coefficient is $\mathbb{C}$ rather than $\mathbb{Q}$. Of course, the surface example at the end of Section 2 of \cite{FG} is no longer a counterexample over $\mathbb{C}$, since all symmetric bilinear form over $\mathbb{C}$ can be diagonalizable. Our results refresh the hope that...
Ruan’s Conjecture may be valid for any hyperkahler resolution, once we insist on the cohomology coefficient being $\mathbb{C}$.

6. Open questions

In this section, we list some open problems for further research.

Question 1. Understand the cohomology ring structure of the Hilbert scheme $X^{[n]}$ when $X$ is a quasi-projective surface. While certain degeneracies occur in connection between the Chern character operators and vertex operators (cf. [Lehn], Sect. 4.4, and [LS1] for the affine plane case), we expect that most of the geometric statements, such as those on ring generators, universality and stability, should remain valid in the quasi-projective case.

Question 1’. Understand the orbifold cohomology ring structure of the symmetric products $X^n/S_n$ for a non-closed complex manifold $X$.

Question 2. Use the axiomatization in Sect. 2.6 to check Ruan’s conjecture on the isomorphism between the (signed) orbifold cohomology ring of the symmetric product $X^n/S_n$ and the quantum corrected cohomology ring of the Hilbert scheme $X^{[n]}$, when $X$ is an arbitrary (quasi-)projective surface.

Question 3. Is there a family of ring structures on the rational cohomology group of the Hilbert scheme $X^{[n]}$ depending on a rational parameter $t$, such that when $t = -1$ it is the standard one and that it becomes isomorphic to the deformed orbifold cohomology ring $(H^*_{orb}(X^n/S_n), \omega_t)$ when $X$ has a numerically trivial canonical class? We may ask similar questions for crepant resolutions of orbifolds.

Question 4. Why is the theory of vertex algebras so effective in the study of the geometry of Hilbert schemes and symmetric products? On the other hand, when the canonical class $K$ of the surface $X$ is not numerically trivial, $K$ becomes an obstruction in connection between Hilbert schemes and vertex algebras. How is this related to the quantum corrections on Hilbert schemes as proposed by Ruan?

Question 5. The appearance of $W$ algebras indicates connections to completely integrable systems. How to see this in the framework of Hilbert schemes and symmetric products?

Question 6. How to understand the orbifold cohomology ring of the symmetric products $X^n/S_n$ for $X$ of odd complex dimension, or even for a more general manifold $X$?

We end this paper with a table comparing the pictures of Hilbert schemes and symmetric products (see above). The reader may compare with another table in [Wa2] which relate the pictures of Hilbert schemes and wreath products.

References

[BBM] P. Baum, J. Brylinski and R. MacPherson, Cohomologie équivariante délocalisée, C.R. Acad. Sci. Paris 300 (1985), 605–608.
[BC] P. Baum and A. Connes, Chern character for discrete groups, In: Y. Matsumoto et al (eds.), A Fête of Topology, Academic Press, 1988.
[Bor] R. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Natl. Acad. Sci. USA 83 (1986), 3068–3071.
[CR] W. Chen and Y. Ruan, A new cohomology theory for orbifold, math.AG/0004129.
### Table 1. A DICTIONARY

| Hilbert Scheme $X^{[n]}$ | Symmetric Product $X^n/S_n$ |
|--------------------------|----------------------------|
| $\mathcal{H}_X = \oplus_n H^*(X^{[n]})$ | $\mathcal{F}_X = \oplus_n H^*_{\text{orb}}(X^n/S_n)$ |
| cup product               | (signed) orbifold cup product |
| Heisenberg generator $a_n(\alpha)$ | Heisenberg generator $p_n(\alpha)$ |
| total Chern class $c(L^{[n]})$ | $\varepsilon_n(c(L))$ |
| $c(L^{[n]}^*)$            | $\eta_n(c(L^*))$ |
| Chern roots of $L^{[n]}$  | $c_1(L)^{(i)} - \xi_i$ |
| class $G^k(\alpha, n)$   | class $O^k(\alpha, n)$ |
| Lehnm’s operator $\mathfrak{d}$ | generalized Goulden’s operator $\mathfrak{b}$ |

[BH] L. Dixon, J.A. Harvey, C. Vafa, and E. Witten, *Strings on orbifolds*, Nuclear Phys. B 261 (1985), 678–686.

[FG] B. Fantechi, L. Göttsche, *Orbifold cohomology for global quotients*, math.AG/0104207.

[FKRW] E. Frenkel, V. Kac, A. Radul, and W. Wang, $W_{1+\infty}$ and $W(gl_N)$ with central charge $N$, Commun. Math. Phys. 170 (1995), 337–357.

[FW] I. Frenkel and W. Wang, *Virasoro algebra and wreath product convolution*, J. Alg. 242 (2001), 656–671.

[Gou] I. Goulden, *A differential operator for symmetric functions and the combinatorics of multiplying transpositions*, Trans. Amer. Math. Soc. 344 (1994), 421–440.

[Gro] I. Grojnowski, *Instantons and affine algebras I: the Hilbert scheme and vertex operators*, Math. Res. Lett. 3 (1996), 275–291.

[Juc] A. Jucys, *Symmetric polynomials and the center of the symmetric group rings*, Rep. Math. Phys. 5 (1974), 107–112.

[Kac] V. Kac, *Vertex Algebras for Beginners*, Second Edition, University Lecture Series 10, AMS, Providence, Rhode Island, 1998.

[Kuhn] N. Kuhn, *Character rings in algebraic topology*, In: Advances in Homotopy, London Math. Soc. Lect. Notes Series 139 (1989), 111–126.

[LT] A. Lascoux and J.-Y. Thibon, *Vertex operators and the class algebras of symmetric groups*, Preprint, math.CO/0102041.

[Lehn] M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. 136 (1999), 157–207.

[LS1] M. Lehn and C. Sorger, *Symmetric groups and the cup product on the cohomology of Hilbert schemes*, Duke Math. J. (to appear), math.AG/0009131.

[LS2] ——, *The cup product of the Hilbert scheme for K3 surfaces*, math.AG/0012166.

[LQ] W.-P. Li and Z. Qin, *On 1-point Gromov-Witten invariants of the Hilbert schemes of points on surfaces*, Preprint.

[LQW1] W.-P. Li, Z. Qin, and W. Wang, *Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces*, Math. Ann. (to appear), math.AG/0009132.

[LQW2] ——, *Generators for the cohomology ring of Hilbert schemes of points on surfaces*, Intern. Math. Res. Notices No. 20 (2001) 1057–1074, math.AG/0009167.

[LQW3] ——, *Universality and stability of cohomology rings of Hilbert schemes of points on surfaces*, Preprint, math.AG/0107132.

[LQW4] ——, *Hilbert schemes and W algebras*, Intern. Math. Res. Notices (to appear), math.AG/0111047.

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd Ed., Clarendon Press, Oxford, 1995.

[Mur] G. Murphy, *A new construction of Young’s seminormal representation of the symmetric group*, J. Alg. 69 (1981), 287–291.

[Na1] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. Math. 145 (1997), 379–388.

[Na2] ——, *Lectures on Hilbert schemes of points on surfaces*, Univ. Lect. Ser. 18, Amer. Math. Soc. (1999).

[Rui] Y. Ruan, *Stringy geometry and topology of orbifolds*, math.AG/0011144.
[Ru2] ——, Cohomology ring of crepant resolutions of orbifolds, math.AG/0108195.
[Seg] G. Segal, Equivariant K-theory and symmetric products, Preprint, 1996.
[Uri] B. Uribe, Orbifold Cohomology of the Symmetric Product, math.AT/0109127.
[VW] C. Vafa and E. Witten, A strong coupling test of S-duality, Nucl. Phys. B 431 (1994), 3–77.
[Wa1] W. Wang, Equivariant K-theory, wreath products, and Heisenberg algebra, Duke Math. J. 103 (2000), 1–23.
[Wa2] ——, Algebraic structures behind Hilbert schemes and wreath products, Contemp. Math. (to appear), math.QA/0011103.
[Zas] E. Zaslow, Topological orbifold models and quantum cohomology rings, Commun. Math. Phys. 156 (1993), 301–331.