INVERSION FORMULA AND RANGE CONDITIONS FOR A VECTOR MULTI-INTERVAL FINITE HILBERT TRANSFORM IN $L^2$

ALEXANDER KATSEVICH$^1$, MARCO BERTOLA$^{2,3}$ AND ALEXANDER TOVBIS$^1$

Abstract. Given $n$ disjoint intervals $I_j$, on $\mathbb{R}$ together with $n$ functions $\psi_j \in L^2(I_j)$, $j = 1, \ldots, n$, and an $n \times n$ matrix $\Theta$, the problem is to find an $L^2$ solution $\vec{\varphi} = \text{Col}(\varphi_1, \ldots, \varphi_n)$ to the linear system $\chi \Theta H \vec{\varphi} = \vec{\psi}$, where $H = \text{diag}(H_1, \ldots, H_n)$ is a matrix of finite Hilbert transforms and $\chi = \text{diag}(\chi_1, \ldots, \chi_n)$ is a matrix of the corresponding characteristic functions on $I_j$, and $\vec{\psi} = \text{Col}(\psi_1, \ldots, \psi_n)$. Since we can interpret $\chi \Theta H \vec{\varphi}$ as a generalized vector multi-interval finite Hilbert transform, we call the formula for the solution as “the inversion formula” and the necessary and sufficient conditions for the existence of a solution as the “range conditions”. In this paper we derive the explicit inversion formula and the range conditions in two specific cases: a) the matrix $\Theta$ is symmetric and positive definite, and; b) all the entries of $\Theta$ are equal to one. We also prove the uniqueness of solution, that is, that our transform is injective. When the matrix $\Theta$ is positive definite, the inversion formula is given in terms of the solution of the associated matrix Riemann-Hilbert Problem. We also discuss other cases of the matrix $\Theta$.

1. Introduction

We start by reminding the reader of the well known inversion formula and range condition for a finite Hilbert transform $H$ in $L^2$, see, for example, [10]. Let $I = [\alpha, \beta] \subset \mathbb{R}$. The finite Hilbert transform $H : L^2(I) \to L^2(I)$ is defined by

$$(1.1) \quad H[f](z) := \frac{1}{\pi} \int_I \frac{f(\zeta)}{\zeta - z} d\zeta.$$ 

Here and everywhere below, whenever $z$ belongs to the interval of integration, the integral is understood in the Cauchy Principal Value sense. In all other cases, this is just an ordinary integral. The following facts about $H$ are known:

1. The operator $H : L^2(I) \to L^2(I)$ is injective and its range is a proper dense subspace of $L^2(I)$ so that $H$ is not a Fredholm operator;
2. Let $f \in L^2(I)$. Then $f$ is in the range of $H$ if and only if there exists a unique constant $\kappa \in \mathbb{C}$ such that

$$(1.2) \quad -\frac{1}{\pi R(z)} \int_I \frac{R(\zeta) f(\zeta)}{\zeta - z} d\zeta - \frac{\kappa}{R(z)} \in L^2(I),$$

where $R(z) = \sqrt{(z - \beta)(z - \alpha)}$ and $R(z) \sim z$ as $z \to \infty$;
3. If $f$ is in the range of $H$ then

$$(1.3) \quad H^{-1}[f](z) = -\frac{1}{\pi R(z)} \int_I \frac{R(\zeta) f(\zeta)}{\zeta - z} d\zeta - \frac{\kappa}{R(z)},$$

1 Department of Mathematics, University of Central Florida, P.O. Box 161364, 4000 Central Florida Blvd, Orlando, FL 32816-1364, USA.
2 Department of Mathematics and Statistics, Concordia University 1455 de Maisonneuve W., Montréal, Québec, Canada H3G 1M8.
3 SISSA, International School for Advanced Studies, via Bonomea 265, Trieste, Italy.
where the constant $x$ is the same as in (1.2).

These results can be generalized in several directions: for example, one can consider functional spaces $L^p(I)$, $p > 1$, see [10], or more general singular integral transforms, like, for example, the cosh-transform ([1]). In the current paper, we extend the inversion formula (1.3) and the range condition (1.2) to the case of the vector multi-interval finite Hilbert transform $\mathcal{H}$. Here is the general setting of the problem. Given:

- $n \in \mathbb{N}$ disjoint intervals $I_j = [\alpha_j, \beta_j]$ on $\mathbb{R}$, $-\infty < \alpha_1 < \beta_1 < \alpha_2 < \ldots < \alpha_n < \beta_n < \infty$;
- $n$ real-valued functions $\psi_j \in L^2(I_j)$, which we represent as $\tilde{\psi} = \text{Col}(\psi_1, \ldots, \psi_n)$;
- an $n \times n$ real matrix $\Theta = (\theta_{jk})$,

find the range conditions, the inversion formula and study the uniqueness of solution for the following system of singular integral equations

\begin{equation}
I = \bigcup_{j=1}^n I_j, \quad \mathcal{H} = \text{diag}(\mathcal{H}_1, \ldots, \mathcal{H}_n), \quad \chi = \text{diag}(\chi_1, \ldots, \chi_n).
\end{equation}

On $I = \bigcup_{j=1}^n I_j$, where $\tilde{\varphi} = \text{Col}(\varphi_1, \ldots, \varphi_n)$, $\varphi_j \in L^2(I_j)$, and

\begin{equation}
\chi \Theta \mathcal{H} \tilde{\varphi} = \tilde{\psi},
\end{equation}

where $\mathcal{H}_j : L^2(I_j) \mapsto L^2(I)$ denotes the finite Hilbert transform (FHT), which integrates on $I_j$ and evaluates on $I$. Also, $\chi_j$ denotes the characteristic function of $I_j$, $j = 1, 2, \ldots, n$. Whenever appropriate, $\chi_j$ can also be viewed as restriction operators. To the best of our knowledge, this is the first paper that analyzes the equation (1.4).

The problem mentioned above appears naturally in some applications most notably in the study of discrete $\beta$-ensembles, see, for example, [3], where $\Theta$ is considered as the “interaction” matrix, and also in the study of lozenge tilings of polygonal regions. Solution of these problems depends significantly on the particular matrix $\Theta$ in (1.4). Most of the results obtained in this paper results apply to the case where $\Theta$ is a positive definite symmetric matrix, but we will also consider some other situations, for example, the case of all $\theta_{jk} = 1$ (uniform interactions), where $\theta_{jk}$ is the $j$, $k$th entry of $\Theta$. The analysis of the uniform interaction matrix case can be found in Section 6.

The main case of a positive definite symmetric matrix $\Theta$ in (1.4) is considered in Sections 2-4. The main steps of our approach are as follows: we first reduce the operator $\chi \Theta \mathcal{H}$ from (1.4) to $\text{Id} - \tilde{K}$, where $\tilde{K}$ is a Hilbert-Schmidt operator, and then use the approach of [4] to find the resolvent of $\text{Id} - \tilde{K}$ in terms of the solution $\Gamma(z)$ of the corresponding matrix Riemann-Hilbert Problem (RHP). This problem is determined by the intervals $I$ (geometry) and by the matrix $\Theta$. Of course, along the way, we have to prove the invertibility of $\text{Id} - \tilde{K}$.

The plan of the paper is the following. In Section 2 we obtain the $L^2$ solution of (1.4) (the inversion formula) provided that such a solution exists, see Theorem 2.14. This solution is expressed in terms of the matrix $\Gamma(z)$ and the inverse Hilbert transform $\hat{\varphi}(z)$ of the “modified” right-hand side $\hat{\psi}(z)$ of (1.4). We also state necessary conditions for solvability of (1.4), see Lemmas 2.3, 2.4. The condition in Lemma 2.3 is expressed in terms of the right-hand side of (1.4), but the condition in Lemma 2.4 is expressed in terms of the solution $\tilde{\varphi}$ of (1.4). In Section 3 we prove that conditions of Lemmas 2.3, 2.4 are also sufficient, see Theorem 3.1, i.e., they form the range conditions for (1.4). We also express the condition in Lemma 2.4 in terms of $\Gamma(z)$ and $\hat{\varphi}(z)$, see (3.11) and discuss its dependence on the geometry of the intervals, that is, on the points $a_j, b_j$. The invertibility of the operator $\text{Id} - \tilde{K}$ and the uniqueness of the solution of (1.4) are proven in Section 4.

In Section 5 we gradually relax the requirements of positive definiteness and symmetry of $\Theta$. Replacing positive definiteness with a much weaker requirement that $\Theta_j = \text{diag} \Theta$ is invertible, we keep all the above mentioned main results provided that the operator $\text{Id} - \tilde{K}$ is invertible.
Next, removing the symmetry requirement of $\Theta$, we end up with a more complicated expression for the second range condition in Theorem 3.1. Finally, we briefly discuss relaxing the requirement that $\Theta_d$ is invertible, which leads to certain analyticity requirements on some components of $\vec{\psi}$. Moreover, some components $\varphi_j$ can be found as jumps over the corresponding intervals $I_j$ of the analytic continuations of the corresponding $\psi_k$ and their finite Hilbert transforms.

The inversion formula and the range condition for (1.4) with the uniform interaction matrix (i.e., $\theta_{jk} = 1$ for all $j,k$) can be found in Section 6, see Theorem 6.2. The method in this section involves diagonalization of a multi interval FHT, which is based on a special change of variables followed by the application of the Fourier transform. This result is closely related to the classical work done in the 50-s and 60-s, most notably [14]. See also [7, 8, 15, 9, 11, 12] for related earlier work. In these papers the authors use complex-analytic methods to diagonalize and study spectral properties of certain singular operators related to the Hilbert transform. Our approach, while less general, allows to obtain a simple inversion formula and the range condition for the multi interval FHT very quickly and using completely elementary means. These two results, the inversion formula and the range condition, appear to be new.

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2. Solution to (1.4) for positive definite symmetric $\Theta$

In this section, we will use the following result of [10] for a finite interval $I = [\alpha, \beta]$.

**Proposition 2.1.** If $f \in L^2(I)$ and $\frac{f(z)}{R(z)} \in L^1(I)$ then the range condition (1.2) can be written as follows

$$\int_I \frac{f(\zeta)}{R(\zeta)} d\zeta = 0. $$

If $f$ is in the range, the inversion formula (1.3) can be replaced by

$$H^{-1}[f](z) = -\frac{R(z)}{\pi} \int_{I} \frac{f(\zeta)}{R(\zeta)(\zeta - z)} d\zeta. $$

Using (2.2), we immediately obtain that

$$H^{-1}[1](z) = 0, \ z \in I, \text{ and } H^{-1}[1](z) = -i, \ z \in \mathbb{C} \setminus I, $$

where by $H^{-1}[1](z), z \notin I$, we understand the expression (2.2) evaluated at $z$.

Let us split

$$\Theta = \Theta_d + \Theta_o,$$

where

$$\Theta_d := \text{diag}\Theta, \ \Theta_o := \Theta - \Theta_d.$$

For the time being we assume that $\Theta_d$ is invertible.

Assume that a solution $\vec{\varphi} \in L^2(I)$ to (1.4) exists. Then, according to Theorem 4.3, proven below, the solution $\vec{\varphi}$ is unique and equation (1.4) can be written as

$$\chi \Theta_d \mathcal{H} \vec{\varphi} + \chi \Theta_o \mathcal{H} \vec{\varphi} = \vec{\psi}.$$

**Remark 2.2.** A function $\varphi(z) \in L^2(I)$ can be uniquely identified with a vector-function $\vec{\varphi} = \text{Col}(\varphi_1, \ldots, \varphi_n)$, where $\varphi_j(z) = \chi_j(z)\varphi(z)$. With a mild abuse of notations, we will consider $\varphi(z)$ and $\vec{\varphi}$ as the same object in this paper, and use these notations intermittently, as is convenient in a given context.
Let $\mathcal{R} \subset L^2(I)$ denote the range of the operator $\chi \mathcal{H}$. Note that $\vec{\psi} \in \mathcal{R}$ if and only if $\chi \Theta_o \mathcal{H} \vec{\psi} \in \mathcal{R}$ given that the first term in (2.6) is automatically in the range. However, since the $m$-th row of $\Theta_o \mathcal{H} \vec{\psi}$ is analytic on $I_m$, there exists, according to the range condition (2.1), a constant vector $\vec{c} \in \mathbb{R}^n$, such that

(2.7) \[ \chi \Theta_o \mathcal{H} \vec{\psi} - \vec{c} \in \mathcal{R}. \]

Therefore, there must be a constant vector $\vec{c} \in \mathbb{R}^n$, such that $\vec{\psi} - \vec{c} \in \mathcal{R}$. The uniqueness of such $\vec{c}$ follows from the fact that any nonzero constant vector $\vec{c}$ cannot belong to the range, $\vec{c} \notin \mathcal{R}$. Thus, we have proved the following statement.

**Lemma 2.3.** If equation (1.4) is solvable, then there exists a unique constant vector $\vec{c} \in \mathbb{R}^n$ such that $\vec{\psi} - \vec{c} \in \mathcal{R}$.

Given $\vec{\psi} \in L^2(I)$, we denote by $\vec{c}[\vec{\psi}]$ the vector $\vec{c}$ from Lemma 2.3. The existence of $\vec{c}[\vec{\psi}]$ is the first necessary condition for the solvability of (1.4) in $L^2(I)$. The second necessary condition, given in Lemma 2.4 below, follows from the above mentioned arguments.

**Lemma 2.4.** If $\vec{\varphi} \in L^2(I)$ is the solution of (1.4) then

(2.8) \[ \chi \Theta_o \mathcal{H} \vec{\varphi} - \vec{c}[\vec{\psi}] \in \mathcal{R}. \]

Rewriting equation (1.4) as follows

(2.9) \[ \chi \Theta_d \mathcal{H} \vec{\varphi} + \left( \chi \Theta_o \mathcal{H} \vec{\psi} - \vec{c}[\vec{\psi}] \right) = \vec{\nu} - \vec{c}[\vec{\psi}], \]

we obtain

(2.10) \[ \vec{\varphi} + \mathcal{H}^{-1} \left( \Theta_d \Theta_o \mathcal{H} \vec{\varphi} - \Theta_d \vec{c}[\vec{\psi}] \right) = \vec{\nu}, \quad \vec{\nu} := \mathcal{H}^{-1} \Theta_d^{-1} \left( \vec{\psi} - \vec{c}[\vec{\psi}] \right), \]

where $\mathcal{H}^{-1} := \text{diag}(\mathcal{H}_1^{-1}, \ldots, \mathcal{H}_n^{-1})$. Here $\mathcal{H}_j^{-1}$ is given by (1.3) with $I = I_j$.

**Remark 2.5.** Note that in view of our definition of $\mathcal{H}_j$, here and everywhere below, the operator $\mathcal{H}_j^{-1}$ is the inverse not to $\mathcal{H}_j$, but to $\chi_j \mathcal{H}_j$, and the latter coincides with the usual finite Hilbert transform on $I_j$. Note also that whenever $\mathcal{H}_j^{-1}$ acts on a function defined on all of $I$, it is applied only to its restriction on $I_j$.

According to the first formula in (2.3), equation (2.10) becomes

(2.11) \[ \vec{\varphi} + \mathcal{H}^{-1} \Theta_d^{-1} \Theta_o \mathcal{H} \vec{\varphi} = \vec{\nu}, \]

or, in operator form,

(2.12) \[ (\text{Id} - \tilde{K}) \vec{\varphi} = \vec{\nu}, \quad \text{where} \quad \tilde{K} = \begin{bmatrix} 0 & -\frac{\theta_{11}}{\theta_{22}} \mathcal{H}_1^{-1} \mathcal{H}_2 & \cdots & -\frac{\theta_{1n}}{\theta_{22}} \mathcal{H}_1^{-1} \mathcal{H}_n \\ -\frac{\theta_{22}}{\theta_{22}} \mathcal{H}_2^{-1} \mathcal{H}_1 & 0 & \cdots & -\frac{\theta_{22}}{\theta_{22}} \mathcal{H}_2^{-1} \mathcal{H}_n \\ \vdots & \ddots & \ddots & \vdots \\ -\frac{\theta_{22}}{\theta_{22}} \mathcal{H}_n^{-1} \mathcal{H}_1 & -\frac{\theta_{22}}{\theta_{22}} \mathcal{H}_n^{-1} \mathcal{H}_2 & \cdots & 0 \end{bmatrix}. \]

Let $R_j(z) = \sqrt{(z - \alpha_j)(z - \beta_j)}$, where $R_j(z) \sim z$ as $z \to \infty$. We need the following statements.

**Lemma 2.6.** For any $\phi \in L^2(I_k)$ and $j \neq k$ we have

(2.13) \[ - \mathcal{H}_j^{-1} \mathcal{H}_k \phi = \frac{R_j(z)}{\pi} \int_{I_k} \frac{\phi(x)dx}{(x-z)R_j(x)}, \]

where $z \in I_j$, and $\mathcal{H}_j^{-1}$ can be represented in the alternative form (2.2) (with $I$ replaced by $I_k$).
Proof. According to (2.2),
\begin{equation}
- \mathcal{H}_j^{-1} \mathcal{H}_k \phi = \frac{R_j(z)}{\pi^2} \int_{I_j} \frac{dy}{y-z} R_j^{(1)}(y) \int_{I_k} \frac{\phi(x)dx}{(x-y)} = \frac{R_j(z)}{\pi^2} \int_{I_k} \frac{\phi(x)dx}{(x-z)(y-z)} R_j^{(1)}(y).
\end{equation}

Using the identity \( \frac{1}{(x-y)(y-z)} = \frac{1}{x-z} \left[ \frac{1}{x-y} + \frac{1}{y-z} \right] \) and (2.3) we calculate
\begin{equation}
\mathcal{H}_j^{-1} \mathcal{H}_k \phi = - \frac{R_j(z)}{\pi i} \int_{I_k} \frac{\phi(x)dx}{(x-z) R_j^{(1)}(x)}.
\end{equation}

According to Lemma 2.6, the operator \( \hat{K} \) can be written as
\begin{equation}
\hat{K} \varphi(z) = -i \chi \left[ \begin{array}{c}
\theta_{12} R_{12} \hat{H}_1 \left[ \frac{\varphi_1}{R_1} \right] \\
\theta_{22} R_{12} \hat{H}_2 \left[ \frac{\varphi_2}{R_1} \right] \\
\vdots \\
\theta_{n2} R_{12} \hat{H}_n \left[ \frac{\varphi_n}{R_1} \right]
\end{array} \right] = \int_I K(z,x) \varphi(x) dx,
\end{equation}

where the kernel \( K \) of the integral operator \( \hat{K} : L^2(I) \rightarrow L^2(I) \) is given by
\begin{equation}
K(z,x) = 2 \sum_{j,k} \chi_j(z) \chi_k(x) = \frac{2}{\pi i (x-z)}.
\end{equation}

Here the prime notation in the summation symbol in (2.17) means that \( j, k = 1, \ldots, n, k \neq j \). It is clear that \( \hat{K} \) is a Hilbert-Schmidt operator since \( \int_I |K(z,x)|^2 dz dx < \infty \).

The following lemma requires that \( \Theta \) is a positive-definiteness symmetric matrix.

Lemma 2.7. If \( \Theta \) is positive definite then \( \lambda = 1 \) is not in the spectrum of \( \hat{K} \).

Proof. If \( \lambda = 1 \) is in the spectrum of \( \hat{K} \), then \( \lambda = 1 \) must be an eigenvalue of \( \hat{K} \), so that there is a nontrivial \( f \in L^2(I) \) satisfying \( \hat{K} f = f \). Applying the operator \( \Theta d \mathcal{H} \) to both parts of \( \hat{K} f = f \), we obtain equation \( \chi \Theta \mathcal{H} f = \bar{c} \) with \( \bar{c} = \bar{c}(\varphi) \) satisfying (2.8). However, according to Theorem 4.3, the latter equation has only trivial solution in \( L^2(I) \). The argument is completed.

The solution to the equation (2.12) is given by
\begin{equation}
\tilde{\varphi} = \left( \text{Id} - \hat{R} \right)^{-1} \tilde{\nu} = \left( \text{Id} + \hat{R}(1) \right) \tilde{\nu},
\end{equation}

where \( \hat{R}(\lambda) \) denotes the resolvent operator for \( \hat{K} \). The resolvent \( \hat{R} \) is defined by the equation
\begin{equation}
(\text{Id} + \hat{R}) \left( \text{Id} - \frac{1}{\lambda} \hat{R} \right) = \text{Id}.
\end{equation}

According to Lemma 2.7, \( \hat{R}(\lambda) \) is analytic at \( \lambda = 1 \).

In order to construct the resolvent kernel \( \mathbf{R}(z, x, \lambda) \) of \( \hat{R} \) we use the approach of [4], see also [2]. First observe that
\begin{equation}
K(z, x) = \frac{\tilde{f}(z) \bar{g}(x)}{2 \pi i (z-x)} \text{ and } \tilde{f}(z) \bar{g}(z) \equiv 0, \ z \in I,
\end{equation}
where
\[ \tilde{f}(z) := -2\text{Col}(R_{1+}(z)\chi_1(z), \ldots, R_{n+}(z)\chi_n(z)), \]
and \( \bar{g}' \) denotes the transposition of \( \bar{g} \). Integral kernels \( K \) of the form (2.20) are called \textit{integrable}. Equations (2.21) in the vector form become
\[ \tilde{f}(z) = -2\chi(z)R(z)e, \quad \bar{g}(z) = R^{-1}(z)\Theta^{-1}_d\Theta_0\chi(z)e, \]
where
\[ e = \text{Col}(1,1,\ldots,1), \quad R(z) = \text{diag}(R_1(z), \ldots, R_n(z)). \]

Consider the following matrix Riemann-Hilbert Problem (RHP).

\textbf{Riemann-Hilbert Problem 2.8.} Find an \( n \times n \) matrix-function \( \Gamma = \Gamma(z; \lambda), \quad \lambda \in \mathbb{C} \setminus \{0\}, \) which is: a) analytic in \( \mathbb{C} \setminus I \); b) \( \Gamma(\infty; \lambda) = 1 \); c) admits non-tangential boundary values from the upper/lower half-planes that belong to \( L^2_{loc}(I) \); and; d) satisfies the following jump condition on \( I \)
\[ \Gamma_+(z; \lambda) = \Gamma_-(z; \lambda) \left( 1 - \frac{1}{\lambda} \tilde{f}(z)\bar{g}'(z) \right) = \Gamma_-(z; \lambda)\text{V}(z; \lambda). \]

For convenience, we will frequently omit the dependence on \( \lambda \) in the notations.

\textbf{Remark 2.9.} Using standard arguments, one can show that the solution to the RHP 2.8, if it exists, is unique.

In more explicit terms, we note that the jump matrix in (2.24) is given by:
\[ V(z; \lambda) = \begin{bmatrix}
1 & 2\frac{\theta_{11}R_{1+}(z)}{\theta_{11}R_1(z)}\chi_1(z) & \cdots & 2\frac{\theta_{1n}R_{1+}(z)}{\theta_{1n}R_1(z)}\chi_1(z) \\
2\frac{\theta_{21}R_{1+}(z)}{\theta_{21}R_1(z)}\chi_2(z) & 1 & \cdots & 2\frac{\theta_{2n}R_{1+}(z)}{\theta_{2n}R_1(z)}\chi_2(z) \\
\vdots & \vdots & \ddots & \vdots \\
2\frac{\theta_{n1}R_{1+}(z)}{\theta_{n1}R_1(z)}\chi_n(z) & 2\frac{\theta_{n2}R_{1+}(z)}{\theta_{n2}R_1(z)}\chi_n(z) & \cdots & 1
\end{bmatrix}. \]

\textbf{Remark 2.10.} Let \( \Gamma_{j}(z) \) denote the \( j \) th column of \( \Gamma(z) \) and \( \Delta_k(\tilde{f}) \) denote the jump of the vector-function \( \tilde{f} = \tilde{f}(z) \) over \( I_k \), \( j, k = 1, \ldots, n \). It follows from (2.24), (2.25) that
\[ \Delta_j\Gamma_j = 0 \quad \text{and} \quad \Delta_k\Gamma_j = 2\frac{\theta_{kj}R_{k+}}{\theta_{jj}R_j}\Gamma_k, \quad k \neq j. \]

The first equation of (2.26) together with the requirement c) from the RHP 2.8 imply that \( \Gamma_j \) is analytic in \( \mathbb{C} \setminus \cup_{m \neq j}I_m \) for any \( j = 1, \ldots, n \). The second equation of (2.26) yields
\[ \Gamma_j(z) = \frac{1}{\pi i} \sum_{k \neq j} \int_{I_k} \frac{\theta_{kj}R_{k+}(\zeta)\Gamma_k(\zeta)d\zeta}{\theta_{jj}R_j(\zeta)(\zeta - z)}, \]
which implies that \( \Gamma_{j\pm}(z) \) are analytic in the interior of \( I_k \) and are bounded at its endpoints \( \alpha_k, \beta_k \). Thus, we showed that \( \Gamma_{j\pm}(z) \) are bounded at all the endpoints and analytic in the interior of each \( I_j \).

The utility of the RHP 2.8 is demonstrated by the following lemma.
Lemma 2.11. If \( \lambda \) is such that the solution \( \Gamma(z; \lambda) \) of the RHP 2.8 exists, then the kernel \( R \) of the resolvent \( \hat{R} \) defined by (2.19) is given by

\[
R(z, x; \lambda) = \frac{g^\dagger(x)\Gamma^{-1}(x; \lambda)\Gamma(z; \lambda)f(z)}{2\pi i(\lambda - x)}.
\]

The proof of this lemma can be found, for example, in [2], Lemma 3.16.

Remark 2.12. Note that due to the second equation in (2.20): a) the integral kernel \( R(z, x; \lambda) \) given by (2.28) is non-singular; and b) \( \Gamma_+(z)f(z) = \Gamma_-(z)f(z) \) and \( g^\dagger(z)\Gamma^{-1}_+(z) = g^\dagger(z)\Gamma^{-1}_-(z) \) on \( I \). Thus, it does not matter whether we use \( \Gamma_+ \) or \( \Gamma_- \) in the equation (2.28). Indeed, according to (2.24),

\[
\Gamma^{-1}_+(z; \lambda) = \left(1 + \frac{1}{\lambda}f(z)g^\dagger(z)\right)\Gamma^{-1}_-(z; \lambda) \quad \text{on} \ I,
\]

(2.29)

\[
g^\dagger(z)\Gamma^{-1}_+(z; \lambda) = g^\dagger(z)\left(1 + \frac{1}{\lambda}f(z)g^\dagger(z)\right)\Gamma^{-1}_-(z; \lambda) = g^\dagger(z)\Gamma^{-1}_-(z; \lambda)
\]

on \( I \). Similarly, we can show that \( \Gamma(z)f(z) \) has no jump on \( I \).

As we have seen, the existence of \( \Gamma(z; \lambda) \), \( \lambda \in \mathbb{C} \setminus \{0\} \), implies that \( \lambda \) is a regular (non-spectral) point of \( \hat{K} \). In fact, the converse is also true, as is shown by the following lemma.

Lemma 2.13. \( \lambda \in \mathbb{C} \setminus \{0\} \) is a regular point of the operator \( \hat{K} \) with an integrable kernel if and only if the solution \( \Gamma(z; \lambda) \) to the RHP 2.8 exists.

Proof. In view of Lemma 2.11 it is sufficient to prove the existence of \( \Gamma(z; \lambda) \) for any regular (non-spectral) point \( \lambda \) of \( \hat{K} \). Suppose the operator \( \text{Id} - \hat{K} \) with the kernel given by (2.20), (2.21) is invertible. Write the inverse as \( \text{Id} + \hat{R}(\lambda) \). Define

\[
\hat{F}(z; \lambda) := \left(\text{Id} - \frac{\hat{K}}{\lambda}\right)^{-1}\hat{f}(z) = (\text{Id} + \hat{R}(\lambda))\hat{f}(z)
\]

and define the matrix

\[
\Gamma(z; \lambda) := 1 - \int_I \frac{\hat{F}(w; \lambda)\hat{g}(w)dw}{2\pi i(\lambda - w)}.
\]

From (2.31) and the Plemelj-Sokhotski theorem,

\[
\Gamma_+(z; \lambda) - \Gamma_-(z; \lambda) = -\frac{1}{\lambda}\hat{F}(z; \lambda)\hat{g}(z), \quad z \in I,
\]

which implies

\[
\Gamma_+(z; \lambda)\hat{f}(z) = \Gamma_-(z; \lambda)\hat{f}(z) - \frac{1}{\lambda}\hat{F}(z; \lambda)\hat{g}(z)\hat{f}(z) = \Gamma_-(z; \lambda)\hat{f}(z), \quad z \in I.
\]

Thus, \( \Gamma(z; \lambda)\hat{f}(z) \) has no jump across \( I \). We used (2.20) in (2.33). On the other hand, from (2.20), (2.30), and the definition (2.31), we also have

\[
\Gamma_\pm(z; \lambda)\hat{f}(z) = \hat{f}(z) - \int_I \frac{\hat{F}(w; \lambda)\hat{g}(w)\hat{f}(z)dw}{2\pi i(\lambda - w)} = \hat{f} + \frac{1}{\lambda}\hat{K}\hat{F}
\]

\[
= \hat{f} - \left(\text{Id} - \frac{\hat{K}}{\lambda}\right)\hat{F} + \hat{F} = \hat{f} - \left(\text{Id} - \frac{\hat{K}}{\lambda}\right)(\text{Id} + \hat{R}(\lambda))\hat{f} + \hat{F} = \hat{F}(z; \lambda).
\]
Therefore (2.32) becomes
\begin{equation}
(2.35) \quad \Gamma_+(z; \lambda) - \Gamma_-(z; \lambda) = -\frac{1}{\lambda} \Gamma_-(z; \lambda) \bar{f}(z) \bar{g}'(z) \quad \Leftrightarrow \quad \Gamma_+(z; \lambda) = \Gamma_-(z; \lambda) \left(1 - \frac{1}{\lambda} \bar{f}(z) \bar{g}'(z)\right),
\end{equation}
so we recover the jump condition (2.24) for \(\Gamma(z; \lambda)\). The remaining requirements of the RHP 2.8 follow from the definitions (2.30) and (2.31).

We are now ready to formulate the inversion formula.

**Theorem 2.14.** If \(\Theta\) is a positive definite symmetric matrix and if the solution \(\varphi \in L^2(I)\) to (1.4) exists, then
\begin{equation}
(2.36) \quad \varphi(z) = \nu(z) + \int_I \frac{\bar{g}'(x)\Gamma^{-1}(x)\Gamma(z)\bar{f}(x)\nu(x)dx}{2\pi i(z-x)},
\end{equation}
where \(\Gamma(z) = \Gamma(z; 1)\) solves the RHP 2.8, and \(\nu\) and \(\bar{f}, \bar{g}\) are defined by (2.10) and (2.21), respectively.

**Proof.** According to Lemma 2.7, \(\lambda = 1\) is a regular point of the operator \(\hat{K}\). Therefore, according to Lemmas 2.11 and 2.13, \(\Gamma(z)\) exists, and the resolvent kernel is given by (2.28). Thus, the solution of (2.11) is given by (2.36).

**Remark 2.15.** The equation (2.36) can be written component-wise as
\begin{equation}
(2.37) \quad \varphi_m(z) = \nu_m(z) + \frac{R_m(z)}{\pi i} \sum_{k \neq m} \int_{I_k} \sum_{j \neq k} \frac{\theta_{jk} \Gamma^{-1}(x) \Gamma_{jm}(z) \nu_j(x)dx}{\theta_{jj} R_j(x)(x-z)}, \quad z \in I_m, \ m = 1, 2, \ldots, n.
\end{equation}

3. **Necessary and Sufficient Conditions for the Existence of the \(L^2(I)\) Solution to the Equation (1.4)**

In Section 2 we obtained two conditions: a) the existence of the constant (but depending on \(\psi\)) vector \(\bar{c}[\bar{\psi}]\), and b) the requirement (2.8). These conditions are necessary for the existence of an \(L^2(I)\) solution to (1.4). In this section we first express the condition b) in terms of \(\bar{\nu} = \mathcal{H}^{-1} \Theta_d^{-1} \left(\bar{\psi} - \bar{c}[\bar{\psi}]\right)\), where \(\bar{\psi}\) is the original data, see (2.10), and then show that the conditions a) and b) are also sufficient for the existence of the \(L^2(I)\) solution to (1.4).

According to (2.1), the equation (2.8) can be written as follows
\begin{equation}
(3.1) \quad -\frac{1}{\pi i} \int_I R^{-1}(x) \Theta \bar{\varphi} dx = \bar{c}[\bar{\psi}],
\end{equation}
or, component-wise,
\begin{equation}
(3.2) \quad -\frac{1}{\pi i} \int_{I_m} \frac{dz}{R_{m+}(z)} \sum_{k \neq m} \theta_{mk} \int_{I_k} \frac{\varphi_k(x)dx}{x-z} = c_m[\bar{\psi}],
\end{equation}
where \(c_m[\bar{\psi}]\) denotes the \(m\)-th component of \(\bar{c}[\bar{\psi}]\). Changing the order of integration and using the second equation in (2.3), we rewrite (3.2) as follows
\begin{equation}
(3.3) \quad \frac{1}{\pi} \sum_{k \neq m} \theta_{mk} \int_{I_k} \frac{\varphi_k(y)dy}{R_m(y)} = c_m[\bar{\psi}], \ m = 1, 2, \ldots, n.
\end{equation}
where $J_j$, $j = 1, 2$, denote the corresponding terms in (3.4). Recall that for $y \in I_k$, according to (2.21), we have $\Gamma(y)\check{f}(y) = -2\Gamma_k(y)R_{k+}(y)$, where $\Gamma_k$ denotes the $k$-th column of the matrix $\Gamma$. Then, changing the order of integration in $J_2$ (cf. Remark 2.12), we obtain

$$J_2 = \frac{1}{\pi} \int_I \nu(x)g^\dagger(x)\Gamma^{-1}(x) \left[ \sum_{k \neq m} \theta_{mk} \int_{I_k} \frac{\Gamma(y)\check{f}(y)dy}{\pi i(y-x)R_m(y)} \right] dx \quad (3.5)$$

$$= -\frac{1}{\pi} \int_I \nu(x)g^\dagger(x)\Gamma^{-1}(x) \left[ \sum_{k \neq m} \theta_{mk} \int_{I_k} \frac{\Gamma_k(y)R_{k+}(y)dy}{\pi i(y-x)R_m(y)} \right] dx.$$

Let $\hat{\gamma}$ be a large negatively oriented circle containing $I$. Pick a point $x \in I$. Then, using Remark 2.10,

$$-\hat{e}_m = \frac{1}{2\pi i} \int_{\hat{\gamma}} \frac{\Gamma_m(\zeta)d\zeta}{\zeta - x} = \frac{1}{2\pi i} \sum_{k \neq m} \int_{I_k} \frac{\Delta_k \Gamma_m(\zeta)d\zeta}{\zeta - x} - \frac{1}{2} \sum_{j=1}^n [\Gamma_m+(x) + \Gamma_m-(x)] \chi_j(x) \quad (3.6)$$

$$= \frac{1}{\pi i} \sum_{k \neq m} \int_{I_k} \frac{\theta_{km}R_{k+}(\zeta)\Gamma_k-(\zeta)d\zeta}{\theta_{mm}R_m(\zeta)(\zeta - x)} - \frac{1}{2} \sum_{j=1}^n [\Gamma_m+(x) + \Gamma_m-(x)] \chi_j(x).$$

Using the symmetry of $\Theta$, we obtain

$$-\hat{e}_m = \frac{1}{\pi i} \sum_{k \neq m} \int_{I_k} \frac{\theta_{km}R_{k+}(\zeta)\Gamma_k-(\zeta)d\zeta}{\theta_{mm}R_m(\zeta)(\zeta - x)} = \theta_{mm} \left[ \hat{e}_m - \frac{1}{2} \sum_{j=1}^n [\Gamma_m+(x) + \Gamma_m-(x)] \chi_j(x) \right]. \quad (3.7)$$

Substituting (3.7) into (3.5) we obtain

$$J_2 = \frac{\theta_{mm}}{\pi} \int_I \nu(x)g^\dagger(x)\Gamma^{-1}(x) \left[ \hat{e}_m - \frac{1}{2} \sum_{j=1}^n [\Gamma_m+(x) + \Gamma_m-(x)] \chi_j(x) \right] dx, \quad (3.8)$$

where, according to Remark 2.12, $g^\dagger(x)\Gamma^{-1}(x)$ does not have a jump on $I$. Note that $\Gamma^{-1}(x)\Gamma_m(x) = \hat{e}_m$, where $\hat{e}_m$ is the $m$-th standard basis vector and, according to (2.21), (2.22),

$$g^\dagger(x)\chi_j(x)\hat{e}_m = e_j^T \Theta_0 \Theta_d^{-1} R^{-1}(x)e_m \chi_j(x) = \begin{cases} \frac{\theta_{mm}}{\theta_{mm}R_m(x)} \chi_j(x), & \text{if } j \neq m, \\ 0, & \text{if } j = m. \end{cases} \quad (3.9)$$

Thus, according to (2.22) and (3.9),

$$J_2 = \frac{\theta_{mm}}{\pi} \left[ \int_I \nu(x)g^\dagger(x)\Gamma^{-1}(x)m dx - \sum_{k \neq m} \theta_{km} \int_{I_k} \frac{\nu_k(x)dx}{\theta_{mm}R_m(x)} \right]$$

$$= \frac{\theta_{mm}}{\pi} \sum_{k \neq m} \int_{I_k} [\Theta_0 \Theta_d^{-1} R^{-1}(x)\Gamma^{-1}(x)]_{km} \nu_k(x)dx - J_1, \quad (3.10)$$
where $[,]_{km}$ denotes the $k,m$-th entry of the matrix. Substituting (3.10) into (3.4), we obtain the second necessary condition (3.11) in the form

$$c_m[\vec{v}] = \frac{\theta_{mm}}{\pi} \sum_{k \neq m} \int_{I_k} [\Theta_d^{-1}R^{-1}(x)\Gamma^{-1}(x)]_{km} \nu_k(x) dx, \quad m = 1, \ldots, n,$$

where, according to Remark 2.12 and (3.10), the right hand side of (3.11) is independent of the choice of $\Gamma^{-1}$ or $\Gamma^{-1}$.

**Theorem 3.1.** The system (1.4) with a positive-definite symmetric matrix $\Theta$ has a solution $\varphi \in L^2(I)$ if and only if the following two conditions are satisfied:

1. There exists a constant vector $\vec{c}[\vec{v}] \in \mathbb{R}^n$ such that $\vec{v} - \vec{c}[\vec{v}] \in \mathcal{R}$;
2. All the components $c_m[\vec{v}]$ of the vector $\vec{c}[\vec{v}]$ satisfy equations (3.11), where

$$\vec{v} = \text{Col}(\nu_1, \ldots, \nu_n) = \mathcal{H}^{-1}\Theta_d^{-1} \left( \vec{v} - \vec{c}[\vec{v}] \right).$$

The solution $\varphi \in L^2(I)$, if exists, is unique and is given by (2.36).

**Proof.** The necessary part follows from Lemmas 2.3, 2.4. Now assume that the vector $\vec{c}[\vec{v}]$ exists. To prove the sufficient part, notice that equations (2.9) and (2.11) are equivalent if and only if the condition (2.8) holds. Then, if the condition (2.8) holds, it is sufficient to show that (2.11) has a solution $\varphi \in L^2(I)$. Existence of such solution follows from the fact that $\lambda = 1$ is not an eigenvalue of $K$. That completes our argument. \qed

In the particular case $n = 2$ we have

$$c_1[\vec{v}] = \frac{\theta_{21}}{\pi} \int_{I_2} \frac{\Gamma_2(x)\nu_2(x)dx}{R_1(x)}, \quad c_2[\vec{v}] = \frac{\theta_{12}}{\pi} \int_{I_1} \frac{\Gamma_1(x)\nu_1(x)dx}{R_2(x)}.$$

**Corollary 3.2.** In the particular case when $\vec{v} \in \mathcal{R}$ the equations (3.11) from Theorem 3.1 become

$$\sum_{k \neq m} \int_{I_k} [\Theta_d^{-1}\Theta_o\Theta_d^{-1}R^{-1}\Gamma^{-1}]_{km} \mathcal{H}^{-1}[\psi_k] dx = 0$$

for all $m = 1, \ldots, n$.

**Remark 3.3.** In the case $R^{-1}\vec{v} \in L^1(I)$, according to (2.2) and (2.3), the equations (3.11) from Theorem 3.1 become

$$i \int_{I_m} \frac{\psi_m}{R_{m^+}} dx + \theta_{mm} \sum_{k \neq m} \int_{I_k} [R\Theta_d^{-1}\Theta_o\Theta_d^{-1}R^{-1}\Gamma^{-1}]_{km} \mathcal{H}[\psi_k] \frac{dx}{R_k} = 0$$

for all $m = 1, \ldots, n$.

The range condition (3.11) can be viewed as a null-space of some unbounded linear functional in $L^2(I)$ with an everywhere dense domain. It contains three main components: the linear map $\vec{v} \rightarrow c_m[\vec{v}]$, $n$ linear maps $\psi_j \rightarrow \nu_j$, $j = 1, \ldots, n$, and the sum of integrals with the weights $[,]_{km}$. The first map is clearly unbounded. As we show in Lemma 3.5, the weights $[,]_{km}$ in (3.11) are functions analytic in $x$ at least continuous in all $\alpha_j, \beta_j$ when $x \in I_k$. This means that the $n$-dimensional part does not create any additional complications, and the dependence of the range condition on the endpoints becomes essentially the same as in the one-interval case. The latter is outside the scope of this paper.

**Lemma 3.4.** The solution $\Gamma(z)$ to the RHP 2.8 locally analytically depend on $\alpha_j, \beta_j$, $j = 1, \ldots, n$ for any $z \in \hat{C}$ except the endpoints $\alpha_j, \beta_j$, $j = 1, \ldots, n$, where it may have the square root $\sqrt{z-\alpha}$ type singularities. The same statement is true for $\Gamma^{-1}(z)$.
Proof. To show that $\Gamma(z)$ depends analytically on the endpoints, we define "local solutions" near the intervals. For each of the interval $I_\ell$, define

$$P_\ell(z) := 1 + 2\mathfrak{e}_\ell \int_{I_\ell} \frac{[\theta_{11} R_\ell^\pm(w) \theta_{12} R_\ell^\pm(w) \cdots \theta_{nn} R_\ell^\pm(w)]}{(w - z)2\pi i} \, dw,$$

where the 0 in the integrand is in the $\ell$-th position. By the Plemelj–Sokhotski formula if follows that

$$P_{\ell+}(z) - P_{\ell-}(z) = 2\mathfrak{e}_\ell \left[ \begin{array}{c} \theta_{11} R_\ell^+(z) \\ \theta_{12} R_\ell^+(z) \\\\ \vdots \\ \theta_{nn} R_\ell^+(z) \end{array} \right] = \mathfrak{e}_\ell \left[ \begin{array}{c} \theta_{11} R_\ell^+(z) \\ \theta_{12} R_\ell^+(z) \\\\ \vdots \\ \theta_{nn} R_\ell^+(z) \end{array} \right] P_{\ell-}(z)$$

Of course $P_\ell$ fails to solve the jump condition (2.24) on the remaining intervals $I_k, \ k \neq \ell$. Note also that det $P_\ell(z) \equiv 1$ and hence local solutions are analytically invertible.

Let now $D_\ell$, $\ell = 1, \ldots, n$ be mutually disjoint open disks (or regions) such that $I_\ell \subset D_\ell$. We also observe that the local solution $P_\ell(z)$ depends analytically on the endpoints $\alpha_j, \beta_j, j = 1, \ldots, n$, for any $z \in D_\ell$, except at the endpoints $z = \alpha_\ell$ and $z = \beta_\ell$ of $I_\ell$, where it may have the square root $\sqrt{z - \alpha}$ type singularities. Define

$$Q(z) := \begin{cases} \Gamma(z) & z \notin \bigcup_{\ell} D_\ell \\ \Gamma(z) P_\ell^{-1}(z) & z \in D_\ell, \quad \ell = 1, \ldots, n. \end{cases}$$

This new matrix $Q(z)$ has only jumps on the boundaries $\partial D_\ell$ and it satisfies the RHP

$$(3.20) \quad Q_+(z) = Q_-(z) P_\ell(z), \quad z \in \partial D_\ell \quad \text{and} \quad Q(\infty) = 1,$$

where the orientation of the boundary is clockwise. If we now move slightly the endpoints (while keeping the disks $D_\ell$ fixed), the jump-matrices of the RHP (3.20) change analytically; by the analytic Fredholm theorem, so does the solution $Q(z)$. This, in turn implies the statement of the lemma for $\Gamma(z)$. Since det $\Gamma(z) \equiv 1$, the same statement is true for $\Gamma^{-1}(z)$.

Alternatively, we can prove the analyticity of $Q(z)$ by differentiating both sides of (3.20) in $\partial = \mathbb{R}$, where $\alpha$ denotes one of the endpoints. Then the differentiated RHP (3.20) becomes

$$(3.21) \quad \partial Q_+(z) = \partial Q_-(z) P_\ell(z) + Q_-(z) \partial P_\ell(z), \quad z \in \partial D_\ell \quad \text{and} \quad Q(\infty) = 0,$$

which is satisfied by

$$\partial Q = \sum_{\ell=1}^n C_{\partial D_\ell} Q_\ell \partial P_\ell Q_\ell^{-1} = \sum_{\ell=1}^n C_{\partial D_\ell} (\Gamma \partial P_\ell \Gamma^{-1}) Q,$$

where $C_\gamma$ denotes the Cauchy operator over the oriented contour $\gamma$ and we used the fact that $P_\ell^{-1} \partial P_\ell = \partial P_\ell$. (Note that $\Gamma \partial P_\ell \Gamma^{-1}$ is smooth (analytic) on $\partial D_\ell$ for any $\ell$). Thus, $Q(z)$ is analytic in the endpoints for all $z \in \mathbb{C}$. \hfill \Box

Lemma 3.5. Let us fix some index $m \leq n$. The weights $[\Theta_\alpha \Theta_d^{-1} R_1^{-1}(x) \Gamma_1^{-1}(x)]_{km}$ from (3.11) are analytic in $z$ on $I_k$ and continuous in all the points $\alpha_j, \beta_j, j = 1, \ldots, n$ for all $z \in I_k$.

Proof. Note that the $k$th row of the matrix $\Theta_\alpha \Theta_d^{-1} R_1^{-1}$ contains all the radicals $R_j$ in the denominator except for $R_k$, since $\text{diag} \Theta_\alpha = 0$. Then, according to Remark 2.10, the functions $[\Theta_\alpha \Theta_d^{-1} R_1^{-1}(x) \Gamma_1^{-1}(x)]_{km}$ are analytic on the interior of $I_k$. But since these functions coincide on $I_k$, see the comment under (3.11), we conclude that $[\Theta_\alpha \Theta_d^{-1} R_1^{-1}(x) \Gamma_1^{-1}(x)]_{km}$ is analytic in $z$ in some neighborhood of $I_k$. The use of Lemma 3.4 completes the proof. \hfill \Box
4. INVERTIBILITY OF THE OPERATOR $\text{Id} - \hat{K}$ AND UNIQUENESS OF SOLUTION TO (1.4)

In this section we consider the equation
\begin{equation}
\chi \Theta \mathcal{H} \tilde{f} = \tilde{c},
\end{equation}
where $\Theta$ is a symmetric positive definite matrix, and $\tilde{c}$ is a constant vector that may depend on $\tilde{f}$. Theorem 4.3 below shows that the equation (4.1) with any $\tilde{c}$ does not have a nontrivial solution $\tilde{f} \in L^2(I)$. Thus, as it was mentioned in the proof of Lemma 2.7, $\lambda = 1$ is not an eigenvalue of the operator $\text{Id} - \hat{K}$, and the operator $\text{Id} - \hat{K}$ is invertible. Moreover, choosing in (4.1) $\tilde{c} = 0$, we show the uniqueness of an $L^2$ solution to (1.4), provided that such a solution exists.

**Remark 4.1.** Here and in the rest of this section, without any loss of generality, we can assume $\text{diag} \Theta = 1$. Indeed, denote $D = \text{diag} \Theta$. Then, because of symmetry and positive definiteness, all the diagonal entries $D$ are positive. Then $\Theta = D^{1/2} \Theta D^{1/2}$, where $\Theta$ is positive definite, symmetric and $\text{diag} \Theta = 1$. Changing variables $\tilde{f} \mapsto D^{1/2} \tilde{f}$, $\tilde{c} \mapsto D^{-1/2} \tilde{c}$ we obtain (4.1) with the matrix $\Theta$ replaced by $\Theta$.

According to Remark 4.1, we have $\Theta_o = \Theta - 1$ for the off-diagonal part of $\Theta$.

**Lemma 4.2.** If $\tilde{f} \in L^2(I)$ and $\tilde{f}$ solves (4.1), then
\begin{equation}
\tilde{f}(z) = R(z)\tilde{g}(z),
\end{equation}
where each component $g_j(z)$ of $\tilde{g}(z)$ is analytic in $\hat{C} \setminus (\bigcup_{k \neq j} I_k)$.

**Proof.** Let $f_j(z), \ j = 1, 2, \ldots, n$, denote the entries of $\tilde{f}(z)$. According to (4.1), $\chi \Theta_o \mathcal{H} \tilde{f} - \tilde{c}$ must be in the range of $\chi \mathcal{H}$. So, we can write the $j$-th component
\begin{equation}
f_j = -\sum_{k \neq j} \theta_{jk} \mathcal{H}_j^{-1} \mathcal{H}_k f_k + \mathcal{H}_j^{-1} c_j
\end{equation}
on $I_j$, $j = 1, 2, \ldots, n$. We now use (2.3) and Lemma 2.6 to get
\begin{equation}
f_j(z) = \frac{R_j(z)}{\pi i} \sum_{k \neq j} \theta_{jk} \int_{I_k} \frac{f_k(x)dx}{(x - z)R_j(x)} = -i R_j(z) \sum_{k \neq j} \theta_{jk} \mathcal{H}_k \left[ \frac{f_k}{R_j} \right](z) = R_j(z) g_j(z),
\end{equation}
where $z \in I_j$ and $R_j(z)$ is taken on the positive side (the upper shore) of $I_j$. It follows from (4.4) that $g_j$ is analytic in $\hat{C} \setminus (\bigcup_{k \neq j} I_k)$. \hfill $\Box$

Pick any $s, 1/2 < s < 1$, and consider the Hilbert space norm
\begin{equation}
\|f\|_s^2 := \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi, \ f \in C_0^\infty(\mathbb{R}),
\end{equation}
where $\hat{f}$ is the Fourier transform of $f$:
\begin{equation}
\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{ix\xi} dx.
\end{equation}
Let $C_0^\infty(I)$ be the space of smooth functions which vanish at the endpoints of each $I_j$. Define the Hilbert space $\mathcal{H}_s(I)$ as the closure of $C_0^\infty(I)$ in the following norm
\begin{equation}
\|\|f\|_s := \left( \sum_j \|f_j\|_s^2 \right)^{1/2}.
\end{equation}
Here and in what follows, given a function $f \in \hat{H}_s(I)$, $f_j$ denotes the restriction of $f$ onto the interval $I_j$ and extended by zero outside $I_j$. Likewise, a collection of $f_j$ (with the appropriate properties) determines a function $f \in \hat{H}_s(I)$.

Note two facts. (i) Any function $f$, whose pieces $f_j$ satisfy (4.4), belongs to $\hat{H}_s(I)$. This follows from the fact that due to the $\sqrt{\pi x}$-type singularity, the Fourier transform of $f$ satisfies $\tilde{f}(\xi) = O(\xi^{-3/2}), |\xi| \to \infty$; (ii) Any $f \in \hat{H}_s(I)$ is continuous (as is known, $H_s(\mathbb{R}) \subset C(\mathbb{R})$, see [7], Corollary 7.9.4).

Consider the system of equations (4.1), which we write as

$$(4.8) \quad \frac{1}{\pi} \sum_k \theta_{jk} \int_{I_k} \frac{f_k(x)}{x-y} \, dx = c_j, \quad y \in I_j, \quad 1 \leq j \leq n,$$

where $f \in \hat{H}_s(I)$. Observe that (1) the multi-interval Hilbert transform $\mathcal{H} : \hat{H}_s(I) \to H_s(\mathbb{R})$ is bounded. This follows from the fact that each finite Hilbert transform in (4.8) is just the multiplication with $-\text{sgn}(\xi)$ in the Fourier domain; and (2) $H_s(\mathbb{R}) \subset C(\mathbb{R})$. Thus the above equation makes sense pointwise.

**Theorem 4.3.** If $f \in \hat{H}_s(I)$ solves the system (4.8) with a positive definite symmetric matrix $\Theta = (\theta_{jk})$ for some constants $c_j$, then all $c_j = 0$ and $f \equiv 0$.

**Proof.** Pick any $\phi \in C_0^\infty(I)$, multiply the $j$-th equation by $\phi'_j$, integrate over $I_j$, and add the results. Clearly, we get

$$(4.9) \quad \sum_j \int_{I_j} \phi'_j(y) \left\{ \frac{1}{\pi} \sum_k \theta_{jk} \int_{I_k} \frac{f_k(x)}{x-y} \, dx \right\} \, dy = 0.$$

In view of (4.9), we define the following bilinear form:

$$(4.10) \quad J(f, g) := \sum_j \int_{I_j} g'_j(y) \left\{ \frac{1}{\pi} \sum_k \theta_{jk} \int_{I_k} \frac{f_k(x)}{x-y} \, dx \right\} \, dy$$

$$= \frac{1}{\pi} \sum_{j,k} \theta_{jk} \int_{I_j} \int_{I_k} g'_j(y) f'_k(x) \ln \left( \frac{1}{|x-y|} \right) \, dx \, dy, \quad f, g \in C_0^\infty(I).$$

Recall that $f_j$’s are the pieces that make up $f$ (and the same for $g$). With some abuse of notation, we write $J(f) := J(f, f)$. First, $J(f, g) = J(g, f)$. Also, it is easy to see that $J$ is continuous, positive definite, and strictly convex on $\hat{H}_s(I)$. Indeed, extending $f_k$ and $g_j$ by zero outside $I_k$ and $I_j$, respectively, and using that the Fourier transform preserves dot products, we get from (4.10)

$$(4.11) \quad J(f, g) = \frac{1}{2\pi} \sum_{j,k} \theta_{jk} \int_{\mathbb{R}} \left[ -\text{sgn}(\xi) \tilde{f}_k(\xi) \tilde{g}_j(\xi) \right] \, d\xi$$

$$= \frac{1}{2\pi} \sum_{j,k} \theta_{jk} \int_{\mathbb{R}} |\xi| \tilde{f}_k(\xi) \tilde{g}_j(\xi) \, d\xi.$$
Since \( s > 1/2 \), from (4.5) and (4.7) we have
\[
|J(f, g)| \leq \frac{1}{2\pi} \sum_{j,k} |\theta_{jk}| \int_{\mathbb{R}} \sqrt{|\xi|} |f_k(\xi)| \cdot \sqrt{|\xi|} |g_j(\xi)| d\xi \\
\leq \frac{1}{2\pi} \sum_{j,k} |\theta_{jk}| \left( \int_{\mathbb{R}} |\xi||f_k(\xi)|^2 d\xi \right)^{1/2} \left( \int_{\mathbb{R}} |\xi||g_j(\xi)|^2 d\xi \right)^{1/2} \\
\leq c \sum_{j,k} \|f_k\|_s \|g_j\|_s \leq cn \|f\|_s \|g\|_s
\]
for some \( c > 0 \). Thus, \( J \) is continuous and extends to all of \( \mathcal{H}_s(I) \times \mathcal{H}_s(I) \).

Considering \( f \) as a vector, \( \tilde{f}(\xi) = (f_1(\xi), \ldots, f_n(\xi)) \), we have
\[
J(f) = \frac{1}{2\pi} \int_{\mathbb{R}} \xi |(\Theta \tilde{f}(\xi), \tilde{f}(\xi))| d\xi.
\]
Here \((\cdot, \cdot)\) denotes the usual dot product in \( \mathbb{C}^n \). The remaining assertions, convexity and positive definiteness, are now obvious because \( \Theta \) is positive-definite. In particular, (4.13) proves that \( J(f + tg) \) is a parabola with respect to \( t \) for any \( g \in \mathcal{H}_s(I) \), \( g \neq 0 \).

Consider the variation \( J(t\phi) \), \( f, \phi \in \mathcal{H}_s(I) \). We have (cf. (4.10)):
\[
\frac{d}{dt} J(f + t\phi)_{|t=0} = 2J(f, \phi) = 2 \sum_j \int_{I_j} \phi_j'(y) \left\{ \frac{1}{\pi} \sum_k \theta_{jk} \int_{I_k} \frac{f_k(x)}{x-y} dx \right\} dy.
\]
Suppose now that \( f \) satisfies (4.8) and, consequently, \( f \) satisfies (4.9) for all \( \phi \in C^\infty_0(I) \). Then the right-hand side of (4.14) equals zero, and \( f \) is a critical point of \( J(f) \). By convexity, the only critical point is \( f \equiv 0 \). Hence, all \( c_j \) are zero, and any solution \( f \in \mathcal{H}_s(I) \) of (4.8) is necessarily trivial. \( \square \)

5. OTHER CASES FOR THE MATRIX \( \Theta \)

We first observe that in the previous sections the positive-definiteness of matrix \( \Theta \) was used only in two places: (i) to show that \( \Theta_d \) is invertible, and (ii) to show that \( \lambda = 1 \) is not an eigenvalue of the operator \( \mathcal{K} \). Thus, we obtain the following corollary.

**Corollary 5.1.** If matrix \( \Theta \) is symmetric with invertible diagonal part \( \Theta_d \), then Theorem 2.14 (inversion formula) and Theorem 3.1 (range conditions) are still valid provided that \( \lambda = 1 \) is not an eigenvalue of the operator \( \mathcal{K} \) defined by (2.12).

Now, the symmetry of \( \Theta \) is used only in the calculation of the range condition (3.11), namely in the transition from (3.6) to (3.7). Thus, we can state the following corollary.

**Corollary 5.2.** If \( \Theta_d = \text{diag}\Theta \) is invertible and if \( \lambda = 1 \) is not an eigenvalue of the operator \( \mathcal{K} \) defined by (2.12), then Theorem 2.14 and Theorem 3.1 are still valid provided that the range condition (3.11) in Theorem 3.1 is replaced with (3.4), where \( \tilde{f}, \tilde{g} \) are defined by (2.21).

Next we consider the remaining case of a non-invertible matrix \( \Theta_d \). We start with an example of \( n = 2 \). In this example we assume \( \theta_{11} = 0 \), but exclude the trivial case of \( \Theta = 0 \). The first equation of (1.4) (on \( I_1 \)) becomes \( \mathcal{H}_2 \varphi_2 = \frac{\partial \varphi_2}{\partial y} \), provided that \( \theta_{12} \neq 0 \). Since \( \mathcal{H}_2 \varphi_2 \) is analytic in \( \mathbb{C} \setminus I_2 \), we conclude that \( \psi_1 \) can be analytically continued in \( \mathbb{C} \setminus I_2 \), \( \psi_1(\infty) = 0 \), and the jump \( \Delta_2 \psi_1 \) of \( \psi_1 \) over \( I_2 \) must be in \( L^2(I_2) \). These are the range conditions for \( \psi_1 \). Moreover, according to the Plemelj-Sokhotski formula, \( \varphi_2 = \frac{\Delta_2 \psi_1}{2i \theta_{12}} \).
The second equation (on $I_2$) now has the form
\begin{equation}
\theta_{21}H_1\varphi_1 = \psi_2 - \frac{\theta_{22}}{2i\theta_{12}}H_2(\Delta_2\psi_1) =: \tilde{\psi}_2.
\end{equation}
Assuming $\theta_{21} \neq 0$, we can use the previous argument to obtain the range condition for the adjusted right-hand side $\psi_2$: the function $\psi_2$ can be analytically continued in $\tilde{C}\setminus I_1$, it attains zero at infinity and also has an $L^2(I_1)$ jump on $I_1$. In this case $\varphi_1 = \frac{1}{2i\theta_{21}}\Delta_2 \tilde{\psi}_2$.

If $\theta_{21} = 0$, then $\varphi_1$ can be any $L^2(I_1)$ function, and $\psi_2 = \frac{\theta_{22}}{2i\theta_{12}}H_2(\Delta_2\psi_1)$ on $I_2$ becomes the range condition for $\psi_2$.

Consider the remaining case of $\theta_{12} = 0$. Now the first equation (on $I_1$) becomes trivial. Then $\psi_1 \equiv 0$ is the range condition for $\psi_1$. The second equation (on $I_2$) becomes
\begin{equation}
\theta_{22}H_2\varphi_2 = \psi_2 - \theta_{21}H_1\varphi_1.
\end{equation}
If $\theta_{22} = 0$, then $\theta_{21} \neq 0$, and we can repeat our previous arguments to obtain $\varphi_1 = \frac{\Delta_1\psi_1}{2i\theta_{21}}$, which leads to the range conditions for $\psi_2$: it can be analytically continued in $\tilde{C}\setminus I_1$, $\psi_2(\infty) = 0$ and the jump $\Delta_1\psi_2$ of $\psi_2$ over $I_1$ must be in $L^2(I_2)$.

Otherwise, if $\theta_{22} \neq 0$, the right hand side of (5.2) must be in the range of $H_2$ restricted to $I_2$. So, the range conditions for $\psi_2$ are: (i) there exists $c \in \mathbb{R}$ such that $\psi_2 - c$ is in the range of the finite Hilbert transform $\chi_2H_2 : L^2(I_2) \to L^2(I_2)$, and (ii) $c = \frac{\theta_{22}}{2i\theta_{12}} \int_{I_2} \overline{R_2^{-1}}H_1\varphi_1dz$ for some $\varphi_1 \in L^2(I_1)$. The latter condition can always be satisfied since we are free to choose any $\varphi_1 \in L^2(I_1)$; thus the range condition for $\psi_2$ consists only of the condition (i). If it is satisfied, then we solve (5.2) for $\varphi_2$ and, thus, obtain $\varphi$. Of course, this solution is not unique, as the condition (ii) does not determine $\varphi_1$ uniquely. That concludes the case of a $2 \times 2$ matrix $\Theta$ with a non-invertible diagonal.

Based on the example of a $2 \times 2$ matrix $\Theta$ with a non-invertible diagonal we can briefly outline the general $n \times n$ case. We say that a row $k$ is connected with a row $j$ if there is a set of distinct indices $j, m_1, \ldots, m_p, k$ such that $\theta_{jm_1}\theta_{m_1m_2}\cdots\theta_{m_pk} \neq 0$. A row $k$ is called degenerate if either $\theta_{kk} = 0$ or if row $k$ is connected with some row $j$ such that $\theta_{jj} = 0$. A matrix $\Theta$ with non-invertible $\Theta_d$ is called irreducible if all the rows $1, \ldots, n$ are degenerate; otherwise, $\Theta$ is called reducible. In particular, in the above $2 \times 2$ example with $\theta_{11} = 0$, the matrix $\Theta$ is irreducible if either $\theta_{12} \neq 0$ or $\theta_{22} = 0$; otherwise $\Theta$ is reducible.

In the case of an irreducible matrix $\Theta$, all the components $\varphi_j$ of the solution vector $\varphi$ can be found as jumps over $I_j$ of the analytic continuations of the corresponding adjusted right hand sides, see the example of a $2 \times 2$ matrix $\Theta$ described above. The function $\tilde{\psi}_2$ in (5.1) is an example of an adjusted right-hand side. The range conditions over $I_j$ in this case consist of: (i) certain analytic properties of the adjusted $\tilde{\psi}_j$, and (ii) if some $\varphi_j$ can be represented as a jump of different analytic continuations, then all such analytic continuations should have the same jumps over $I_j$.

In the case when the matrix $\Theta$ is reducible, we can bring it to a block-triangular form by interchanging rows and interchanging columns. Such moves do not change the degeneracy of any particular row. Let us move all the degenerate rows and the corresponding columns of $\Theta$ to the positions $1, \ldots, m$, where $m < n$. Then we obtain a lower triangular block matrix
\begin{equation}
\tilde{\Theta} = \begin{bmatrix}
A & 0 \\
B & C
\end{bmatrix}
\end{equation}
where the $m \times m$ matrix $A$ is irreducible, and the $(n-m) \times (n-m)$ matrix $C$ has an invertible diag$C$. Indeed, consider a column $\Theta_j$ of $\Theta$ with $j > m$. If any $\theta_{kj} \neq 0$ with $k \leq m$, then the row $j$ is degenerate, which contradicts the assumption. Finding $\varphi_1, \ldots, \varphi_m$ by solving the first $m$ equations with the corresponding irreducible matrix $A$, we reduce the size of the original problem from $n \times n$ to $(n-m) \times (n-m)$. As it was mentioned above, the matrix $C$ of the reduced system
has invertible \( \text{diag} C \), so it satisfies the requirements of Corollary 5.2. Note that the right hand side of the reduced system depends on the already obtained \( \varphi_1, \ldots, \varphi_m \).

6. The case of uniform interaction matrix \( \Theta \)

In the particular case when all entries \( \theta_{jk} = 1 \), the matrix \( \Theta = (\theta_{jk}) \) is not positive definite, so that the invertibility of the operator \( \text{Id} - \hat{K} \), given by (2.12), cannot be guaranteed using the methods of the previous sections. However, this problem can be reduced to the inversion of the multi-interval finite Hilbert transform:

\[
(\mathcal{H}f)(z) = g(z), \quad z \in I, \quad f, g \in L^2(I),
\]

where

\[
\mathcal{H} : L^2(I) \to L^2(I), \quad (\mathcal{H}f)(z) = \frac{1}{\pi} \int_I \frac{f(x)}{x - z} \, dx.
\]

**Remark 6.1.** In this section the operator \( \mathcal{H} \) is defined by (6.2) instead of (1.5), as is the case in the rest of the paper. We also use here the notations \( f \) and \( g \) instead of \( \varphi \) and \( \psi \), respectively.

Define

\[
\beta_{od}(z) = \prod_{j=1}^{n} (z - \alpha_j), \quad \beta_{ev}(z) = \prod_{j=1}^{n} (z - \beta_j), \quad \beta(z) = \beta_{ev}(z)/\beta_{od}(z),
\]

\[
\phi(z) = \Re \ln \beta(z).
\]

Note that

\[
\arg \beta(z) = \pi, \quad \ln \beta(z) = \phi(z) + i\pi, \quad z \in I.
\]

Using (6.4), we get from (6.3):

\[
\phi'(x) = \frac{Q(x)}{\beta_{od}(x)\beta_{ev}(x)}, \quad Q(x) := \beta_{ev}'(x)\beta_{od}(x) - \beta_{ev}(x)\beta_{od}'(x).
\]

It is known that \( Q(x) \) is positive and bounded away from zero on \( I \) (cf. [6]) and, therefore, \( \phi(x) \) is monotonic and, thus, invertible on each interval \( I_j \). Moreover, it is straightforward to see that the range of \( \phi(x) \) on each interval \( I_j \) is \( \mathbb{R} \). Now we calculate using (6.4) again:

\[
2 \sinh \left( \frac{\phi(x) - \phi(z)}{2} \right) = \frac{(x - z) \sum_{i,j=1}^{n} B_{ij} z^{i-1} x^{j-1}}{\sqrt{\prod_{j=1}^{n} (x - \alpha_j)(x - \beta_j)(z - \alpha_j)(z - \beta_j)}},
\]

where \( B := B(\beta_{ev}, \beta_{od}) = (B_{ij}) \) is the Bézout matrix of the polynomials \( \beta_{ev}(z), \beta_{od}(z) \). Note that \( \prod_{j=1}^{n} (x - \alpha_j)(x - \beta_j)(z - \alpha_j)(z - \beta_j) > 0 \) for any \( z, x \) in the interior of \( I \), and the square root in (6.6) is computed according to the rule

\[
\sqrt{\prod_{j=1}^{n} (x - \alpha_j)(x - \beta_j)(z - \alpha_j)(z - \beta_j)} = -\text{sgn} \beta_{od}(x)\text{sgn} \beta_{od}(z) \prod_{j=1}^{n} |(x - \alpha_j)(x - \beta_j)(z - \alpha_j)(z - \beta_j)|^{\frac{1}{2}}.
\]

Since \( B \) is symmetric,

\[
B = \Omega \text{diag}(\rho_1, \ldots, \rho_n) \Omega,
\]

for an orthogonal matrix \( \Omega \). Then

\[
\sum_{i,j=1}^{n} B_{ij} z^{i-1} x^{j-1} = z_n \Omega \text{diag}(\rho_1, \ldots, \rho_n) \Omega \vec{x} = z_n \Omega \text{diag}(\rho_1, \ldots, \rho_n) \Omega \vec{x} = \sum_{j=1}^{n} \rho_j P_j(z) \vec{p}_j(x),
\]

\[
\sum_{i,j=1}^{n} B_{ij} z^{i-1} x^{j-1} = z_n \Omega \text{diag}(\rho_1, \ldots, \rho_n) \Omega \vec{x} = z_n \Omega \text{diag}(\rho_1, \ldots, \rho_n) \Omega \vec{x} = \sum_{j=1}^{n} \rho_j P_j(z) \vec{p}_j(x),
\]
where \( z_n^+ = (1, z, \ldots, z^{n-1}) \), \( z_n^- = (1, x, \ldots, x^{n-1}) \) and \( (P_1(z), \ldots, P_n(z)) = z_n^+ \Omega t \).

Introduce the isometry of the two spaces
\[
T : L^2(I) \rightarrow L^2_n(\mathbb{R}),
\]
where \( L^2_n(\mathbb{R}) \) is the direct sum of \( n \) copies of \( L^2(\mathbb{R}) \), and \( \oplus_{n=1}^n L^2(\mathbb{R}) \). Here we set \( \|\hat{f}\|^2 = \|\hat{f}_1\|^2 + \cdots + \|\hat{f}_n\|^2 \), where \( \hat{f} = (\hat{f}_1, \ldots, \hat{f}_n) \in L^2_n(\mathbb{R}) \) and \( \|\hat{f}_n\| \) is the conventional \( L^2(I_n) \) norm. Also, in (6.10), \( \phi_k^{-1} \) is the inverse of \( \phi(x) \) on the \( k \)-th interval \( (\alpha_k, \beta_k) \). Changing variables in the definition of \( H \) gives
\[
(THT^{-1}\hat{f})_m(s) = \frac{\text{sgn}(\beta_0(t_m))}{\pi} \sqrt{2} \sum_{k=1}^{n} \int_{\mathbb{R}} \frac{\text{sgn}(\beta_0(x_k))\hat{f}_k(t)}{\sqrt{\phi'(|x_k|)}} dg(t - z_m) dt,
\]
(6.11)
\[
x_k : = \phi_k^{-1}(2t), z_m : = \phi_m^{-1}(2s).
\]
Combining (6.11), (6.5), (6.6), and (6.9) we find
\[
\frac{\text{sgn}(\beta_0(x_k)\beta_0(t_m))}{\sqrt{\phi'(|x_k|)\phi'(|z_m|)}} = \frac{-1}{2\sinh(t - s)} \sum_{j=1}^{n} \frac{\rho_j P_j(x_k)P_j(z_m)}{\sqrt{Q(x_k)Q(z_m)}}
\]
Define the matrix function
\[
\mathcal{M} : = \{M_{jk}(t)\}, \quad M_{jk}(t) := P_j(x_k) \sqrt{\frac{P_j}{Q(x_k)}}, \quad x_k : = \phi_k^{-1}(2t).
\]
It is shown in [6] that \( \{M_{jk}(t)\} \) is an orthogonal matrix for all \( t \in \mathbb{R} \). Substituting (6.12) and (6.13) into (6.11) gives
\[
(THT^{-1}\hat{f})_m(s) = \sum_{j=1}^{n} \int_{\mathbb{R}} \frac{M_j(t)\hat{f}_j(t)}{\pi \sinh(s - t)} dt.
\]
In compact form, (6.14) can be written as follows
\[
THT^{-1}\hat{f} = \mathcal{M}^T K \mathcal{M} \hat{f},
\]
where \( K \) is the operator of component-wise convolution with \( (\pi \sinh(t))^{-1} \).

Let \( \mathcal{F} : L^2_n(\mathbb{R}) \rightarrow L^2_n(\mathbb{R}) \) denote the map consisting of \( n \) component-wise one-dimensional Fourier transforms. Using the integral 2.5.46.2 in [13]:
\[
\int_0^\infty \frac{\sin(\lambda x)}{\sinh x} dx = \frac{\pi}{2} \tanh(\pi \lambda/2),
\]
we get
\[
K = \mathcal{F}^{-1}(i \tanh(\pi \lambda/2)\text{Id}) \mathcal{F},
\]
where \( \lambda \) is the spectral (Fourier) variable. Therefore, (6.14) gives
\[
H \hat{f} = (\mathcal{F} \mathcal{M} T)^{-1}(i \tanh(\pi \lambda/2)\text{Id})(\mathcal{F} \mathcal{M} T) \hat{f}.
\]
Recalling that the operators \( \mathcal{F} \), \( \mathcal{M} \), and \( T \) are all isometries, we immediately obtain \( n \) conditions for the right-hand side of (6.1) to be in the range of the multi-interval finite Hilbert transform and the inversion formula.
Theorem 6.2. A function $g \in L^2(I)$ is in the range of the multi-interval finite Hilbert transform $\mathcal{H}$ given by (6.2) if and only if
\begin{equation}
\frac{1}{\lambda} (\mathcal{FMT}g)_m \in L^2_{\text{loc}}, \quad m = 1, 2, \ldots, n,
\end{equation}
in a neighborhood of $\lambda = 0$. If $g \in L^2(I)$ is in the range of $\mathcal{H}$, then the inversion formula is given by
\begin{equation}
\mathcal{H}^{-1}g = (\mathcal{FMT})^{-1} \left( \frac{1}{i \tanh(\lambda \pi / 2)} \text{Id} \right) (\mathcal{FMT})g.
\end{equation}

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