LEFT-SYMMETRIC STRUCTURES ON COMPLEX SIMPLE LIE SUPERALGEBRAS

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Abstract. A well-known fact is that there does not exist any compatible left-symmetric structures on a finite-dimensional complex semisimple Lie algebra (see [5]). This result is not valid in semisimple Lie superalgebra case. In this paper, we study the compatible Left-symmetric superalgebra (LSSA for short) structures on complex simple Lie superalgebras. We prove that there is not any compatible LSSA structure on a finite-dimensional complex simple Lie superalgebra except for the classical simple Lie superalgebra $A(m,n)(m \neq n)$ and Cartan simple Lie superalgebra $W(n)(n \geq 3)$. We also classify all compatible LSSAs with a right-identity on $A(0,1)$.

1. Introduction

Let $k$ be a field. A nonassociative algebra $(A, \cdot)$ over $k$ is called a left-symmetric algebra if $(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot z) \cdot x - y \cdot (x \cdot z)$ for all $x, y, z \in A$. A superalgebra $(A = A_0 \oplus A_1, \cdot)$ is called a left-symmetric superalgebra (LSSA for short) if the associator $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is supersymmetric in $x, y$, i.e., $(x, y, z) = (-1)^{|x||y|}(y, x, z)$; or equivalently,

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (-1)^{|x||y|}(y \cdot z) \cdot x - y \cdot (x \cdot z)$$

for all $x, y, z \in A$. We say that an LSSA $N$ is Novikov if it satisfies an additional condition $(z \cdot x) \cdot y = (-1)^{|x||y|}(z \cdot x) \cdot y$ for all $x, y, z \in N$.

To calculate the cohomologies of associative algebras and prove the commutativity of cohomologies space, a kind of graded left-symmetric algebras were first introduced by Gerstenhaber [7] in 1963. Recently, LSSAs, the super-version of left-symmetric algebras, also appeared in several fields of mathematics and mathematical physics as a natural algebraic structure. In [15], a linear basis of the free LSSA was constructed. In 2008, Kong and Bai classified all compatible LSSA on the super-Virasoro algebras satisfying some restricted conditions ([10]). A close relation between LSSAs and the graded classical Yang-Baxter equation in Lie superalgebras was obtained in [16]. As a subclass of LSSAs, Novikov superalgebras were studied extensively in [17, 18, 11, 1, 19, 20, 21]. Because LSSAs are Lie-admissible superalgebras, a fundamental problem asks whether there exist any compatible LSSAs on a give Lie superalgebra. The non-super version of this problem is of importance in geometry. Actually, if $G$ is a connected and simply connected Lie group over the field of real numbers whose Lie algebra is $\mathfrak{g}$, then there is a left-invariant flat and torsion free connection, that is, an affine structure on $G$ if and only if $\mathfrak{g}$ has a compatible left-symmetric algebra ([13, 14]). There are a lot of papers addressing the compatible left-symmetric algebras on a given Lie algebra (see [8, 12, 3, 6, 2]). In particular, an important result given by Chu [5] asserts that there do not exist any compatible left-symmetric algebra on a complex finite-dimensional semisimple Lie algebra.

Date: February 26, 2013.

2010 Mathematics Subject Classification. 17B60, 17B20.

Key words and phrases. Simple Lie superalgebra; left-symmetric superalgebra.
In view of pure mathematics, one can consider the question that asks whether there exist a compatible LSSA structure on a given complex finite-dimensional semisimple Lie superalgebra. We can construct a compatible LSSA structure on the Cartan complex simple Lie superalgebra $W(n)(n \geq 3)$ as follows: Let $\wedge(n)$ denote the Grassmann superalgebra with the generators $\xi_1, \cdots, \xi_n$, whose $\mathbb{Z}_2$-gradation is given by $|\xi_i| = 1$ for $i = 1, \cdots, n$. We write $W(n)$ for Der $\wedge(n)$ and note that it is a Cartan simple Lie superalgebra. Recall that every derivation $d \in W(n)$ can be written in the form

$$d = \sum_i u_i \frac{\partial}{\partial \xi_i}, \ u_i \in \wedge(n),$$

where $\frac{\partial}{\partial \xi_i}$ is the derivation defined by $\frac{\partial}{\partial \xi_i}(\xi_j) = \delta_{ij}$ (see [9]). We define the following product on $W(n)$:

$$u \frac{\partial}{\partial \xi_i} \circ v \frac{\partial}{\partial \xi_j} = u \frac{\partial v}{\partial \xi_i} \frac{\partial}{\partial \xi_j}, \ u, v \in \wedge(n).$$

It is not difficult to check that the superalgebra $(W(n), \circ)$ is an LSSA. This example indicates that the left-symmetric structures on a semisimple Lie superalgebra could be more complicated than the semisimple Lie algebra case.

In this paper, our attention is focused on the compatible LSSA structures on complex simple Lie superalgebras. In section 2, we prove that there is not any compatible LSSA structure on a finite-dimensional complex simple Lie superalgebra except for the classical simple Lie superalgebra $A(m, n)(m \neq n)$ and Cartan simple Lie superalgebra $W(n)(n \geq 3)$. Section 3 deals with the compatible LSSAs on $A(0, 1)$; we construct a family of LSSAs $B_{2,\alpha}$ and two exceptional LSSA $B_1, \overline{B_{2,-1}}$, and prove that given a compatible LSSA $B$ on $A(0, 1)$, if $B$ has a right-identity then it is isomorphic to one of $B_1, \overline{B_{2,-1}}, B_{2,\alpha}$.

Throughout this paper, all (super)algebras are assumed to be finite-dimensional and over the field of complex numbers.

2. **Left-symmetric Superalgebras on Simple Lie Superalgebras**

Let us recall the result due to Kac ([9]): all finite-dimensional simple Lie superalgebras that are not Lie algebras consist of the following classical and Cartan Lie superalgebras

- classical: $A(m, n), n > m \geq 0; A(n, n), n \geq 1; B(m, n), m \geq 0, n \geq 1; C(n), n \geq 3$;
  $D(m, n), m \geq 2, n \geq 1; D(2, 1; \alpha), \alpha \neq 0, -1; F(4); G(3); P(n), n \geq 2$;
  $Q(n), n \geq 2$;
- Cartan: $W(n), n \geq 3; S(n), n \geq 4; \overline{S}(n), n \geq 4, n$ even; $H(n), n \geq 5$.

Note that $D(2, 1; \alpha)$ and $D(2, 1; \beta)$ are isomorphic if and only if $\alpha$ and $\beta$ lie in the same orbit of the group $V$ of order 6 generated by $\alpha \mapsto -1 - \alpha, \alpha \mapsto 1/\alpha$.

**Lemma 2.1.** Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a Lie superalgebra. If $\mathcal{G}_0$ does not have a compatible left-symmetric algebra, then there does not exist a compatible LSSA on $\mathcal{G}$.

**Proof.** If $A$ is a compatible LSSA on the Lie superalgebra $\mathcal{G}$, then $A_0$ is a compatible left-symmetric algebra on $\mathcal{G}_0$, which is a contradiction. \qed

**Lemma 2.2 ([5]).** There does not exist any compatible left-symmetric algebra on a finite-dimensional semisimple Lie algebra over a field of characteristic zero.
Lemma 2.3 ([3]). Let \( g = \alpha \oplus \tilde{\gamma} \) be a finite-dimensional reductive Lie algebra over an algebraically closed field of characteristic zero such that \( \tilde{\gamma} \) is a one-dimensional center and \( \alpha \) is a simple ideal. Then \( g \) has a compatible left-symmetric algebra if and only if \( \alpha \) is of type \( A_n \).

Theorem 2.4. There does not exist any compatible LSSA on a finite-dimensional simple Lie superalgebra except for \( A(m, n) (m \neq n) \) and \( W(n) \).

Proof. Let \( s \) be the even part of the classical Lie superalgebras

\[ A(n, n), B(m, n), D(m, n), D(2, 1; \alpha), F(4), G(3), P(n), Q(n). \]

Then \( s = A_n \oplus A_n, B_m \oplus C_n, D_m \oplus C_n, A_1 \oplus A_1, B_3 \oplus A_1, G_2 \oplus A_1, A_n, A_n \) respectively. Since \( s \) is a semisimple Lie algebra, it does not have a compatible left-symmetric algebra by Lemma 2.2. It follows from Lemma 2.1 that the above simple Lie superalgebras do not have any compatible LSSAs. On the other hand, the even part of \( C(n) \) is isomorphic to the reductive Lie algebra \( C_{n-1} \oplus C \), then \( C(n)_0 \) does not have any compatible left-symmetric algebra for \( n \geq 3 \) by Lemma 2.3. Thus \( C(n)(n \geq 3) \) does not have a compatible LSSA structure, either.

For the \( \mathbb{Z} \)-graded Lie superalgebras \( S(n), \tilde{S}(n), H(n) \) of Cartan type, their \( \mathbb{Z} \) degree zero parts \( S(n)_0 \cong sl(n), \tilde{S}(n)_0 \cong sl(n) \) and \( H(n)_0 \cong so(n) \) are simple Lie algebras and do not have any compatible left-symmetric algebras by Lemma 2.2. Hence there do not exist any compatible LSSAs on \( S(n), \tilde{S}(n) \) or \( H(n) \).

3. Left-Symmetric Superalgebras on \( A(0, 1) \)

In this section, we deals with the left-symmetric superalgebra structures on \( A(0, 1) \). We need some preliminaries on compatible left-symmetric algebra structures on \( gl(2) \).

Lemma 3.1 ([2]). Let \( A \) be a compatible left-symmetric algebra on a reductive Lie algebra \( g \), then \( A \) has a unique right-identity.

Let \( x = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), y = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), h = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), z = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \) be a basis of the Lie algebra \( gl(2) \). Using Lemma 3.1, Burde classified all compatible left-symmetric algebras on \( gl(2) \).

Theorem 3.2 ([3]). Let \( A \) be a compatible left-symmetric algebra on \( gl(2) \). Then it is isomorphic to \( A_1, A_2, \alpha \) or \( A_3 \) defined by the left multiplication operators \( L(x), L(y), L(h), L(z) \) as follows:

\[
\begin{align*}
(1) & \quad \left( \begin{array}{ccc}
0 & 1/2 & -1/2 \\
0 & 1/2 & 0 \\
0 & 1/2 & 0
\end{array} \right), \\
(2) & \quad \left( \begin{array}{ccc}
0 & 0 & -1/2 \\
0 & 1/2 & 0 \\
0 & 1/2 & 0
\end{array} \right), \\
(3) & \quad \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 1/2 & 0
\end{array} \right),
\end{align*}
\]

where \( \alpha \in \mathbb{C} \). Two left-symmetric algebras \( A_2, \alpha \) and \( A_3, \alpha \) are isomorphic if and only if \( \alpha^2 = \tilde{\alpha}^2 \). They are associative if and only if \( \alpha = 0 \).

Lemma 3.3 ([4]). Let \( A \) be an LSSA. If the sub-adjacent Lie superalgebra \( G_A \) is simple, then \( A \) is simple as an LSSA.
We denote by $e_{ij}$ the $3 \times 3$ matrix having 1 in the $(i, j)$ position and 0 elsewhere. Let

\[(3.1) \quad x_1 = e_{23}, x_2 = e_{32}, x_3 = e_{22} - e_{33}, x_4 = 2e_{11} + e_{22} + e_{33};\]

\[(3.2) \quad y_1 = e_{12}, y_2 = e_{13}, y_3 = e_{21}, y_4 = e_{31}\]

be a basis of the simple Lie superalgebra $A(0, 1)$. Then the products are given as follows:

$[x_1, x_2] = x_3, [x_3, x_1] = 2x_1, [x_3, x_2] = -2x_2, [x_1, x_4] = [x_2, x_4] = [x_3, x_4] = 0,$
$[x_1, y_1] = -y_2, [x_1, y_2] = [x_1, y_3] = 0, [x_1, y_4] = y_3,$
$[x_2, y_1] = 0, [x_2, y_2] = -y_1, [x_2, y_3] = y_4, [x_2, y_4] = 0,$
$[x_3, y_1] = -y_1, [x_3, y_2] = y_2, [x_3, y_3] = y_3, [x_3, y_4] = -y_4,$
$[x_4, y_1] = y_1, [x_4, y_2] = y_2, [x_4, y_3] = -y_3, [x_4, y_4] = -y_4,$
$[y_1, y_3] = \frac{1}{2}(x_3 + x_4), [y_1, y_4] = [x_2, y_3] = x_1, [y_2, y_4] = \frac{1}{2}(x_4 - x_3),$
$[y_1, y_1] = [y_1, y_2] = [y_2, y_2] = [y_3, y_3] = [y_3, y_4] = [y_4, y_4] = 0.$

Now we consider the compatible LSSAs with a right identity on $A(0, 1)$. We would like to point out that a left-symmetric algebra has a right identity if and only if the sub-adjacent Lie algebra admits an étale affine representation which leaves a point fixed (see [14]).

**Theorem 3.4.** Let $B$ be a compatible LSSA on $A(0, 1)$. Assume that $B$ has a right identity, then $B$ is isomorphism to $B_1, B_{2, \alpha}$ or $B_{2, -1}$ defined by the left multiplication operators $L(x_i), L(y_j), i, j = 1, 2, 3, 4,$ as follows:

\begin{align*}
(1) & \\
& L(x_1) = \begin{pmatrix} 0 & 1/2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & L(x_2) = \begin{pmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 1 \\ -1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & -1/2 \end{pmatrix}, \\
& L(x_3) = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & L(x_4) = \begin{pmatrix} 1 & -1/2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & -1/2 & 0 & 1 \end{pmatrix}, \\
& L(y_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & L(y_2) = \begin{pmatrix} 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
& L(y_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & L(y_4) = \begin{pmatrix} 0 & 0 & 0 & -1/4 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}. \\
\end{align*}

\begin{align*}
(2) & \\
& L(x_1) = \begin{pmatrix} 0 & 0 & -1 \beta \\ 0 & 0 & 0 \\ 0 & \beta/2 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, & L(x_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 \gamma \\ -\gamma/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}. \\
\end{align*}
Proof. Case (1): Let $B = L(\alpha, \gamma)$. By Theorem 3.2, $B$ is isomorphic to $A_1$, $A_2$, or $A_3$.

Case (1): $B \cong A_1$. Since $L$ is an even linear map, the matrices $L(x_i), L(y_j) (1 \leq i, j \leq 4)$ are of the form

\[
L(x_1) = \begin{pmatrix}
0 & 1/2 & -1/2 & 0 \\
0 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & 0 \\
2 & 0 & 0 & 0
\end{pmatrix},
L(x_2) = \begin{pmatrix}
1/2 & 1/2 & 0 & 0 \\
0 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

where $\beta = 1 + \alpha, \gamma = 1 - \alpha, \alpha \in \mathbb{C}$ and $x_i, y_j, i, j = 1, 2, 3, 4$, are defined by (3.1) and (3.2) respectively. They are simple and not associative.

Proof. Let $B$ be a compatible LSSA on $A(0, 1)$. Then $B_0$ is a compatible left-symmetric algebra on $A(0, 1) \cong \mathfrak{gl}(2)$. By Theorem 3.2, $B_0$ is isomorphic to $A_1, A_2, \alpha$ or $A_3$.
\[ L(y_3) = \begin{pmatrix} 0 & (g_{ij}) \\ (p_{ij}) & 0 \end{pmatrix}, \quad L(y_4) = \begin{pmatrix} 0 & (h_{ij}) \\ (q_{ij}) & 0 \end{pmatrix}, \]

where \((a_{ij}), (b_{ij}), (c_{ij}), (d_{ij}), (e_{ij}), (f_{ij}), (g_{ij}), (h_{ij}), (m_{ij}), (n_{ij}), (p_{ij}), (q_{ij})\) are 4 × 4 matrices. Since

\[ [x_i, y_j] = x_i y_j - y_j x_i, \]

for all \(i, j = 1, 2, 3, 4\), the matrices \((m_{ij}), (n_{ij}), (p_{ij})\) and \((q_{ij})\) can be determined by \((a_{ij}), (b_{ij}), (c_{ij})\) and \((d_{ij})\). Namely,

\[
(m_{ij}) = \begin{pmatrix} a_{11} & b_{11} & c_{11} + 1 & d_{11} - 1 \\ a_{21} + 1 & b_{21} & c_{21} & d_{21} \\ a_{31} & b_{31} & c_{31} & d_{31} \\ a_{41} & b_{41} & c_{41} & d_{41} \end{pmatrix}, \quad (n_{ij}) = \begin{pmatrix} a_{12} & b_{12} + 1 & c_{12} & d_{12} \\ a_{22} & b_{22} & c_{22} - 1 & d_{22} - 1 \\ a_{32} & b_{32} & c_{32} & d_{32} \\ a_{42} & b_{42} & c_{42} & d_{42} \end{pmatrix},
\]

\[
(p_{ij}) = \begin{pmatrix} a_{13} & b_{13} & c_{13} & d_{13} \\ a_{23} & b_{23} & c_{23} & d_{23} \\ a_{33} & b_{33} & c_{33} - 1 & d_{33} + 1 \\ a_{43} & b_{43} - 1 & c_{43} & d_{43} \end{pmatrix}, \quad (q_{ij}) = \begin{pmatrix} a_{14} & b_{14} & c_{14} & d_{14} \\ a_{24} & b_{24} & c_{24} & d_{24} \\ a_{34} - 1 & b_{34} & c_{34} & d_{34} \\ a_{44} & b_{44} & c_{44} + 1 & d_{44} + 1 \end{pmatrix}.
\]

Since \([y_i, y_j] = y_i y_j + y_j y_i\) and \([y_i, y_i] = 0\) for \(i, j = 1, 2, 3, 4\), we have

\[
(e_{ij}) = \begin{pmatrix} 0 & e_{12} & e_{13} & e_{14} \\ 0 & e_{22} & e_{23} & e_{24} \\ 0 & b_{32} & e_{33} & e_{34} \\ 0 & b_{42} & e_{43} & e_{44} \end{pmatrix}, \quad (f_{ij}) = \begin{pmatrix} -e_{12} & 0 & f_{13} & f_{14} \\ -e_{22} & 0 & f_{23} & f_{24} \\ -e_{32} & 0 & f_{33} & f_{34} \\ -e_{42} & 0 & f_{43} & f_{44} \end{pmatrix},
\]

\[
(g_{ij}) = \begin{pmatrix} -e_{13} & -f_{13} + 1 & 0 & g_{14} \\ -e_{23} & -f_{23} & 0 & g_{24} \\ -e_{33} + 1/2 & -f_{33} & 0 & g_{34} \\ -e_{43} + 1/2 & -f_{43} & 0 & g_{44} \end{pmatrix}, \quad (h_{ij}) = \begin{pmatrix} -e_{14} & -f_{14} & -g_{14} & 0 \\ -e_{24} + 1 & -f_{24} & -g_{24} & 0 \\ -e_{34} & -f_{34} - 1/2 & -g_{34} & 0 \\ -e_{44} & -f_{44} + 1/2 & -g_{44} & 0 \end{pmatrix}.
\]

Note that \(B\) has a right identity \(e\), that is, \(x e = x\) for all \(x \in B\). It is easy to see that \(e\) must be an even element. Suppose that \(e = k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4\). From

\[ x_3 = x_3 e = k_1 x_1 - k_2 x_2 + k_3 (x_1 + x_4) + k_4 (-x_1 + x_3), \]

it follows that \(k_1 = 1, k_2 = 0, k_3 = 0, k_4 = 1\), that is, \(e = x_1 + x_4\). By Case (i) of Theorem 3.2, we see that \(e = x_1 + x_4\) is the right identity of \(B_0\). On the other hand, by the assumption that \(e = x_1 + x_4\) is the right identity of \(B\), we have \(y_i (x_1 + x_4) = y_i, i = 1, 2, 3, 4\). Then \((d_{ij})\) can be determined by \((a_{ij})\), that is,

\[
(d_{ij}) = \begin{pmatrix} 2 - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ -1 - a_{21} & 2 - a_{22} & -a_{23} & -a_{24} \\ -a_{31} & -a_{32} & 2 - a_{33} & -a_{34} \\ -a_{41} & -a_{42} & -a_{43} & -a_{44} \end{pmatrix}.
\]

Notice that \((B, L)\) is a representation of Lie superalgebra \(A(0, 1)\), we have \([L(x_1), L(x_4)] = 0\) and thus \((a_{ij})(d_{ij}) - (d_{ij})(a_{ij}) = 0\). A direct computation implies that

\[
(a_{12} = a_{13} = a_{14} = a_{23} = a_{24} = a_{31} = a_{32} = a_{41} = a_{42} = a_{43} = 0), \quad a_{11} = a_{22}, a_{33} = a_{44}.
\]

Since \([L(x_3), L(x_1)] = 2L(x_1)\), it follows from (3.3) that

\[
a_{11} = a_{33} = c_{12} a_{21} = c_{43} a_{34} = 0,
\]

\[
a_{21} (2 + c_{11} - c_{22}) = a_{34} (2 + c_{44} - c_{33}) = 0.
\]
Note that $[L(x_3), L(x_4)] = 0$. Combining (3.3) with (3.4), we have

\begin{equation}
(3.6) \quad c_{13} = c_{14} = c_{23} = c_{24} = c_{31} = c_{32} = c_{41} = c_{42} = 0,
\end{equation}

\begin{equation}
(3.7) \quad (1 + a_{21})(c_{11} - c_{22}) = (1 - a_{34})(c_{33} - c_{44}) = 0.
\end{equation}

By the analogous computations, $[L(x_2), L(x_4)] = 0$, $[L(x_1), L(x_2)] = L(x_3)$, $[L(x_3), L(x_2)] = -2L(x_2)$ and (3.3), (3.4) and (3.6) give rise to the following conditions:

\[ b_{13} = b_{14} = b_{23} = b_{24} = b_{31} = b_{32} = b_{41} = b_{42} = 0, \]
\[ (1 + a_{21})(b_{11} - b_{22}) = (1 + a_{21})b_{12} = 0, c_{34} = a_{34}(b_{44} - b_{33}), \]
\[ c_{12} = c_{43} = 0, c_{22} = -c_{11} = a_{21}b_{12}, c_{33} = -c_{44} = a_{34}b_{43}, \]
\[ c_{21} = a_{21}(b_{11} - b_{22}), (1 - a_{34})(b_{33} - b_{44}) = b_{43}(1 - a_{34}) = 0, \]
\[ b_{11} = -b_{22} = \frac{1}{2}b_{12}c_{21}, b_{33} = -b_{44} = -\frac{1}{2}c_{34}b_{43}, \]
\[ b_{21}(2 + c_{22} - c_{11}) = c_{21}(b_{22} - b_{11}), b_{34}(2 + c_{33} - c_{44}) = c_{34}(b_{33} - b_{44}). \]

It follows from (3.5) and (3.7) that $a_{21} = 0$ or $-1$, $a_{34} = 0$ or 1. Hence there exist four possible cases.

Case (a): $a_{ij} = b_{ij} = c_{ij} = d_{ij} = 0$ for all $i, j = 1, 2, 3, 4$;

Case (b): $a_{21} = -1, b_{12} = -1, c_{11} = -1, c_{22} = 1, b_{11} = a, b_{22} = -a, b_{21} = a^2, c_{21} = -2a, a \in \mathbb{C}$ and other $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are zero;

Case (c): $a_{34} = 1, b_{23} = 1, c_{33} = 1, c_{44} = -1, b_{33} = a, b_{44} = -a, b_{34} = -a^2, c_{34} = -2a, a \in \mathbb{C}$ and other $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are zero;

Case (d): $a_{21} = -1, a_{34} = 1, b_{12} = -1, b_{43} = 1, c_{11} = -1, c_{22} = 1, c_{33} = 1, c_{44} = -1, b_{11} = a, b_{22} = -a, b_{21} = a^2, c_{21} = -2a, b_{33} = b, b_{44} = -b, b_{34} = -b^2, c_{34} = -2b, a, b \in \mathbb{C}$ and other $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are zero.

Since $[L(x_1), L(y_2)] = 0$, we have

\[ a_{21}(b_{12} + 1) + \frac{1}{2}c_{11} = 0. \]

Then the above Cases (b) and (d) do not exist. By $[L(x_1), L(y_3)] = 0$, it follows that

\[ a_{34}(b_{43} - 1) - \frac{1}{2}c_{33} = 0. \]

Then the above Case (c) does not exist. Moreover, by $[L(y_i), L(y_j)] = 0, i = 1, 2, 3, 4, [L(y_1), L(y_2)] = 0$ and Case (a), we have a unique solution

\[ e_{14} = e_{44} = -e_{34} = \frac{1}{4}, f_{14} = -\frac{1}{2} \text{ and other } e_{ij}, f_{ij}, g_{ij} \text{ are zero.} \]

Hence we obtain the superalgebra $B_1$ given by Case (i). We observe that the supercommutator of $B_1$ gives Lie superalgebra $A(0, 1)$ and $(B_1, L)$ is a representation of $A(0, 1)$. Therefore, $B_1$ is a compatible LSSA on $A(0, 1)$.

Case (2): $B_0 \cong A_{2, \alpha}$. Suppose that $e = k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4$ is a right identity. By

\[ x_1 = x_1 e = k_2(\frac{1 + \alpha}{2} x_3 + \frac{1}{2} x_4) - k_3 x_1 + k_4(1 + \alpha)x_1 \]

and

\[ x_2 = x_2 e = k_1(-\frac{1 - \alpha}{2} x_3 + \frac{1}{2} x_4) + k_3 x_2 + k_4(1 - \alpha)x_2, \]
we have $k_1 = k_2 = 0, k_3 = \alpha, k_4 = 1$, that is, $e = \alpha x_3 + x_4$. By the analogous computations, we obtain the superalgebra $B_{2,\alpha}$ with a right identity $e = \alpha x_3 + x_4$. When $\alpha = -1$, we obtain an exceptional superalgebra $\tilde{B}_{2,-1}$ with a right identity $e = -x_3 + x_4$.

Case (3): $B_0 \cong A_3$. It follows from the straightforward calculations that there is not a compatible LSSA on $A(0, 1)$ having a right identity in this case.

By Lemma 3.3, it is obvious that $B_1, B_{2,\alpha}$ and $\tilde{B}_{2,-1}$ are simple. By Theorem 3.2, the even parts of $B_1, B_{2,\alpha}(\alpha \neq 0)$ and $\tilde{B}_{2,-1}$ are not associative, so these LSSAs are not associative. Note that $B_{2,0}$ is not associative since $(x_3 x_3) y_1 - x_3 (x_3 y_1) = 2y_1 \neq 0$. This completes the proof. \hfill \Box

**Lemma 3.5.** $B_{2,\alpha_1} \cong B_{2,\alpha_2}$ if and only if $\alpha_1 = \alpha_2$.

**Proof.** Assume that $B_{2,\alpha_1}$ and $B_{2,\alpha_2}$ are isomorphic. Then their even parts are isomorphic. By Theorem 3.2, we have $\alpha_1 = \pm \alpha_2$. If $B_{2,\alpha}$ and $B_{2,-\alpha}(\alpha \neq 0)$ are isomorphic, then there exists an isomorphism $\varphi : B_{2,\alpha} \rightarrow B_{2,-\alpha}$. Let $P = (p_{ij})$ be the matrix of $\varphi$. Since $\varphi$ is an even linear map, $P$ is of the form

$$
\begin{bmatrix}
  p_{11} & p_{12} & p_{13} & p_{14} \\
  p_{21} & p_{22} & p_{23} & p_{24} \\
  p_{31} & p_{32} & p_{33} & p_{34} \\
  p_{41} & p_{42} & p_{43} & p_{44}
\end{bmatrix}
$$

As $\varphi$ is an isomorphism, the determinant of $P$ cannot be zero. Since $\varphi(x_1 x_3) = \varphi(x_1)\varphi(x_3)$ and $\varphi(x_3 x_1) = \varphi(x_3)\varphi(x_1)$, we have

$$
\begin{align*}
- p_{11} p_{33} + (1 - \alpha) p_{11} p_{34} + p_{31} p_{13} + (1 - \alpha) p_{13} p_{41} &= -p_{11}, \\
p_{21} p_{33} + (1 + \alpha) p_{21} p_{43} - p_{31} p_{23} + (1 + \alpha) p_{41} p_{23} &= -p_{21}, \\
\frac{1}{2} p_{11} p_{23} + \frac{1}{2} p_{21} p_{34} + p_{31} p_{23} + (1 + \alpha^2) p_{31} p_{41} + \alpha p_{31} p_{43} + \alpha p_{41} p_{33} + (1 + \alpha^2) p_{41} p_{43} &= -p_{41}
\end{align*}
$$

(3.8)

and

$$
\begin{align*}
p_{11} p_{33} + (1 - \alpha) p_{11} p_{43} - p_{31} p_{13} + (1 - \alpha) p_{13} p_{41} &= p_{11}, \\
p_{21} p_{33} + (1 + \alpha) p_{21} p_{43} + p_{31} p_{23} + (1 + \alpha) p_{41} p_{23} &= p_{21}, \\
\frac{1}{2} p_{11} p_{23} + \frac{1}{2} p_{21} p_{34} + p_{31} p_{23} + (1 - \alpha^2) p_{31} p_{41} - \alpha p_{31} p_{43} + \alpha p_{31} p_{43} + (1 + \alpha^2) p_{41} p_{43} &= p_{41}
\end{align*}
$$

(3.9)

Combining (3.8) with (3.9), we obtain

$$
\begin{align*}
p_{41} &= 0, (1 - \alpha) p_{11} p_{43} = 0, p_{11} p_{33} - p_{13} p_{31} = p_{11}, \\
(1 + \alpha) p_{21} p_{43} &= 0, p_{31} p_{23} - p_{21} p_{33} = p_{21}, 2p_{31} = p_{21} p_{13} - p_{11} p_{23}, \\
- \alpha p_{11} p_{23} - \alpha p_{13} p_{21} - 2\alpha p_{33} p_{31} + 2(1 - \alpha^2) p_{41} p_{31} &= 0, \\
\frac{1}{2} p_{11} p_{23} + \frac{1}{2} p_{21} p_{13} + p_{31} p_{33} + \alpha p_{31} p_{43} &= 0.
\end{align*}
$$

(3.10)
With analogous arguments for \( \varphi(x, x_j) = \varphi(x_i)\varphi(x_j), 1 \leq i, j \leq 4 \). If \( \alpha \neq 0 \), then

\[
p_{21} = k, p_{12} = \frac{1}{k}, p_{33} = -1, p_{44} = 1,
\]

and other \( p_{ij}(1 \leq i, j \leq 4) \) are zero, where \( 0 \neq k \in \mathbb{C} \). Similarly,

\[
p_{56} = kp_{65} \neq 0, p_{87} = kp_{78} \neq 0, \quad \text{and other } p_{ij}(5 \leq i, j \leq 8) \text{ are zero},
\]

where \( 0 \neq k \in \mathbb{C} \). Observe that \( \varphi(y_1y_3) = \varphi(y_1)\varphi(y_3), \varphi(y_2y_3) = \varphi(y_2)\varphi(x_3) \) and \( \varphi(y_1y_4) = \varphi(y_1)\varphi(y_4) \) give rise to the following conditions on the variables of \( P \):

\[
(3.11) \quad kp_{65}p_{78} = 1,
\]

\[
(3.12) \quad kp_{65}p_{78} = \frac{1 + \alpha}{2},
\]

\[
(3.13) \quad kp_{65}p_{78}(1 - \alpha) = 2.
\]

Substituting (3.11) into (3.12) and (3.13) respectively, we get \( 1 + \alpha = 2 \) and \( 1 - \alpha = 2 \), which is a contradiction. Thus \( B_{2, \alpha} \) and \( B_{2, -\alpha} \) are not isomorphic. \( \square \)

Finally we need to prove

**Theorem 3.6.** \( B_1, B_{2, \alpha} \) and \( \bar{B}_{2, -1} \) are pairwise nonisomorphic.

**Proof.** Firstly, \( B_1 \) is not isomorphic to \( B_{2, \alpha} \) or \( \bar{B}_{2, -1} \), because their even parts are not isomorphic. According to Lemma 3.5, the members of \( B_{2, \alpha} \) are pairwise nonisomorphic. By Theorem 3.2, we have that the even parts of \( \bar{B}_{2, -1} \) and \( B_{2, \alpha} \) for \( \alpha \neq \pm 1 \) are not isomorphic. Hence they are not isomorphic.

Assume that \( B_{2, -1} \) and \( \bar{B}_{2, -1} \) are isomorphic. Let \( \eta : B_{2, -1} \rightarrow \bar{B}_{2, -1} \) be an isomorphism and \( Q = (q_{ij}) \) be the matrix of \( \eta \). Since \( \eta \) is an even linear map, we have that \( Q \) is of the form

\[
\left[
\begin{array}{ccc}
q_{11} & q_{12} & q_{13} \\
q_{21} & q_{22} & q_{23} \\
q_{31} & q_{32} & q_{33} \\
\end{array}
\right]
\left[
\begin{array}{ccc}
q_{14} & q_{15} & q_{16} \\
q_{24} & q_{25} & q_{26} \\
q_{34} & q_{35} & q_{36} \\
\end{array}
\right]
= 0
\]

By \( \eta(x_i, x_j) = \eta(x_i)\eta(x_j), 1 \leq i, j \leq 4 \) and noting that the determinant of \( Q \) cannot be zero, we obtain that

\[
q_{11} = k, q_{22} = \frac{1}{k}, q_{33} = 1, q_{44} = 1, k \neq 0, k \in \mathbb{C} \text{ and other } q_{ij}, 1 \leq i, j \leq 4, \text{ are zero.}
\]

We proceed with analogous computations for \( \eta(x_1y_j) = \eta(x_1)\eta(y_j)(j = 1, 2, 3, 4) \). Thus

\[
q_{55} = q_{56} = q_{57} = q_{58} = 0,
\]

which is a contradiction to the assumption that \( \eta \) is an isomorphism. Thus \( B_{2, -1} \) and \( \bar{B}_{2, -1} \) are not isomorphic. Similarly, \( B_{2, 1} \) is not isomorphic to \( \bar{B}_{2, -1} \). This completes the proof. \( \square \)

We close this paper with the following question.

**Question 3.7.** We would like to point out the problem that asks whether the Lie superalgebras \( A(m, n)(0 \leq m < n) \) except for \( A(0, 1) \) have the compatible LSSAs and how to determine them remains open.

**Acknowledgments**

The author would like to thank Professor Jean-Louis Loday and Yucai Su for their discussion and useful suggestions. This work was supported by NNSF of China (11226051) and the Fundamental Research Funds for the Central Universities (11QNJJ001).
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