ON THE DOUBLE ROMAN BONDAGE NUMBERS OF GRAPHS

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ABSTRACT. For a graph $G = (V, E)$, a double roman dominating function (DRDF) is a function $f : V \rightarrow \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$ for some vertex $v$, then $v$ has at least two neighbors assigned 2 under $f$ or one neighbor $w$ with $f(w) = 3$, and if $f(v) = 1$ then $v$ has at least one neighbor $w$ with $f(w) \geq 2$. The weight of a DRDF $f$ is the sum $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a DRDF on a graph $G$ is the double Roman domination number of $G$ and is denoted by $\gamma_{dR}(G)$. The double roman bondage number of $G$, denoted by $b_{dR}(G)$, is the minimum cardinality among all edge subsets $B \subseteq E(G)$ such that $\gamma_{dR}(G - B) > \gamma_{dR}(G)$. In this paper we study the double roman bondage number in graphs. We determine the double roman bondage number in several families of graphs, and present several bounds for the double roman bondage number. We also study the complexity issue of the double roman bondage number and prove that the decision problem for the double roman bondage number is NP-hard even when restricted to bipartite graphs.

1. Introduction

Throughout this paper all graphs are finite, simple and undirected. We denote the vertex set and the edge set of a graph $G$ by $V = V(G)$ and $E = E(G)$, respectively. Let $G = (V, E)$ be a graph of order $n$. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u | uv \in E\}$, and the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v$ is $\deg_G(v) = |N(v)|$. The maximum (respectively, minimum) degree among the vertices of $G$ is denoted by $\Delta(G)$ (respectively, $\delta(G)$). The open neighborhood of a set $S \subseteq V$ is $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood of $S$ is $N[S] = N(S) \cup S = \cup_{v \in S} N[v]$. A vertex with exactly one neighbor is called a leaf and its unique neighbor is a support vertex. A strong support vertex is a support vertex adjacent to at least two leaves. The distance $d_G(x,y)$ (or briefly $d(x,y)$) between vertices $x$ and $y$ of a graph $G$ is the length of shortest path connecting them. The girth $g(G) = g$ of $G$ is the length of a shortest cycle in $G$, and $g(G) = \infty$ when $G$ is a forest. A $k$-partite graph is a graph which its vertex set can be partitioned into $k$ sets $V_1, V_2, \cdots, V_k$ such that every edge of the graph has an end point in $V_i$ and an end point in $V_j$ for some $1 \leq i \neq j \leq k$. A complete $k$-partite graph is a $k$-partite graph that every vertex of each partite set is adjacent to all vertices of the other partite sets. We denote by $K_{n_1,n_2,\ldots,n_k}$ the complete $k$-partite graph where $|V_i| = n_i$. 

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for $1 \leq i \leq k$. In the case $k = 2$, the $k$-partite and complete $k$-partite graph are called bipartite and complete bipartite graphs. We denote by $P_n, C_n, K_n$ and $\overline{K_n}$, the path, the cycle, the complete graph and the empty graph of order $n$, respectively. For a graph $G$ and a nonempty subset $S \subseteq V(G)$, the vertex-induced subgraph, denoted by $G[S]$, is the subgraph of $G$ with vertex-set $S$ and edges incident to members of $S$. For a subset $S$ of vertices, we refer to $G - S$ as the subgraph of $G$ induced by $V(G) \setminus S$. If $S = \{v\}$, then the subgraph $G - S$ is denoted by $G - v$. For a nonempty subset $X \subseteq E(G)$, we denote by $G - X$ the spanning subgraph of $G$ obtained by deleting the edges of $X$ from $G$. If $X = \{e\}$, then we denote $G - X$ by $G - e$. A planar graph is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints. A connected graph $G$ is called 2-connected, if for every vertex $x \in V(G)$, the graph $G - x$ is connected. The join of two graphs $G$ and $H$, $G \vee H$, is the graph with vertex-set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The graph $K_1 \vee C_{n-1}$ is called a wheel and is denoted by $W_n$.

A set $S \subseteq V$ is called a dominating set if $N[S] = V$. The domination number, $\gamma(G)$ of $G$, is the minimum cardinality of a dominating set in $G$. A dominating set of $G$ of cardinality $\gamma(G)$ is called a $\gamma$-set of $G$ or just a $\gamma(G)$-set. For other graph theory notation and terminology not given here we refer to [10].

Let $f : V \to \{0, 1, 2\}$ be a function having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a neighbor $u \in N(v)$ with $f(u) = 2$. Such a function is called a Roman dominating function. The weight of a Roman dominating function is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function on $G$ is called the Roman domination number of $G$ and is denoted by $\gamma_R(G)$. A Roman dominating function on $G$ of weight $\gamma_R(G)$ is called a $\gamma_R$-function of $G$ or just a $\gamma_R(G)$-function. A Roman dominating function $f$ can be represented as a triple $f = (V_0, V_1, V_2)$ (or $f = (V_0', V_1', V_2')$), where $V_i = \{v : f(v) = i\}$ for $i = 0, 1, 2$. The mathematical concept of Roman domination, defined and discussed by Stewart [15], and ReVelle and Rosing [14], and subsequently developed by Cockayne et al. [6]. Several variants of Roman domination already have been defined and studied. For a graph $G = (V,E)$, a double roman dominating function (DRDF) is a function $f : V \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then $v$ has at least two neighbors assigned 2 under $f$ or one neighbor $w$ with $f(w) = 3$, and if $f(v) = 1$ then $v$ has at least one neighbor $w$ with $f(w) \geq 2$. The weight of a double Roman dominating function $f$ is the sum $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a DRDF is called double Roman domination number of $G$ and is denoted by $\gamma_{dR}(G)$. The concept of double roman domination is defined by Beeler, Haynes and Hedetniem [3], and further studied in, for example [1] [12] [17]. Beeler et al. [5] observed that in a DRDF of minimum weight no vertex needs to be assigned the value 1. In fact for every DRDF $f : V \to \{0, 1, 2, 3\}$, there is a DRDF $f' : V \to \{0, 2, 3\}$ with $w(f') \leq w(f)$. Thus, since $\gamma_{dR}(G)$ is the minimum weight among all double Roman dominating functions on $G$, without loss of generality, we only consider double Roman
dominating functions with no vertex assigned 1. We use the notation \( f = (V^f_0, V^f_2, V^f_3) \) for a DRDF \( f : V \rightarrow \{0, 2, 3\} \), where \( V^f_i = \{v : f(v) = i\} \), for \( i = 0, 2, 3 \).

Bauer, Harary, Nieminen and Suffel [4] introduced the concept of bondage number in graphs. The bondage number \( b(G) \) of a nonempty graph \( G \) is the minimum cardinality among all sets of edges \( E' \subseteq E(G) \) for which \( \gamma(G - E') > \gamma(G) \). This concept has been further studied for various domination variants, see for example, [2, 3, 7, 8, 11, 13, 16].

In this paper we consider the concept of bondage number for the double roman domination number. The double roman bondage number of a graph \( G \), denoted by \( b_{dR}(G) \), is the minimum cardinality among all edge subsets \( B \subseteq E(G) \) such that \( \gamma_{dR}(G - B) > \gamma_{dR}(G) \). The organization of the paper is as follows. In Section 2, we present some preliminary results and determine the double Roman bondage number in some families of graphs. In Section 3, we present various bounds for the double Roman bondage number. In Section 4, we study complexity issue of the double Roman bondage number, and show that the decision problem of the double Roman bondage number is NP-hard even when restricted to bipartite graphs. We make use of the following.

**Theorem 1.1.** [1] Let \( G \) be a connected graph of order \( n \geq 3 \). Then

1. \( \gamma_{dR}(G) = 3 \) if and only if \( \Delta(G) = n - 1 \).
2. \( \gamma_{dR}(G) = 4 \) if and only if \( G = \overline{K}_2 \cup H \), where \( H \) is a graph with \( \Delta(H) \leq |V(H)| - 2 \).
3. \( \gamma_{dR}(G) = 5 \) if and only if \( \Delta(G) = n - 2 \) and \( G \neq \overline{K}_2 \cup H \) for any graph \( H \) of order \( n - 2 \).

**Theorem 1.2.** [1] Let \( n \) be a positive integer. Then

\[
\gamma_{dR}(P_n) = \begin{cases} 
  n, & \text{if } n \equiv 0 \pmod{3}, \\
  n + 1, & \text{if } n \not\equiv 0 \pmod{3}.
\end{cases}
\]

\[
\gamma_{dR}(C_n) = \begin{cases} 
  n + 1, & \text{if } n \equiv 1, 5 \pmod{6}, \\
  n, & \text{if } n \not\equiv 1, 5 \pmod{6}.
\end{cases}
\]

**Lemma 1.3.** [17] Let \( 1 \leq n_1 \leq n_2 \leq \cdots \leq n_r \) be integers. Then

\[
\gamma_{dR}(K_{n_1, n_2, \ldots, n_r}) = \begin{cases} 
  3 & \text{if } n_1 = 1, \\
  4 & \text{if } n_1 = 2, \\
  6 & \text{n}_1 \geq 3.
\end{cases}
\]

**Theorem 1.4.** [18] If \( G \) is a planar graph of girth \( g \leq \infty \), then

\[
|E(G)| \leq \frac{g(n(G) - 2)}{g - 2}.
\]

**Corollary 1.5.** [18] If \( G \) is a planar graph of order \( n(G) \geq 3 \), then

\[
|E(G)| \leq 3n(G) - 6.
\]
Thus we conclude that \[ \text{Lemma 1.7.} \]

Let \( v \) be a vertex of a planar graph \( G \) with \( \deg_G(v) \geq 3 \), and let \( E_v = \{ xy \mid x, y \in N(v), xy \notin E(G) \} \). Then there exists a subset \( S \subseteq E_v \) such that \( H = G + S \) is still a planar graph and \( H[N(v)] \) is 2-connected.

2. Preliminaries and Exact values

In this section we present some preliminary results, and determine the double Roman bondage number in several families of graphs including paths, cycles, complete graphs and complete bipartite graphs. We first determine the double bondage number of paths \( P_n \).

**Theorem 2.1.** For any \( n \geq 1 \), \( b_{dR}(P_n) = 1 \).

**Proof.** Let \( V(P_n) = \{ v_1, v_2 \ldots v_n \} \). If \( n = 3k \), then by Theorem 1.2, \( \gamma_{dR}(P_n - v_2v_3) = \gamma_{dR}(P_2 \cup P_{n-2}) = n + 2 > \gamma_{dR}(P_n) \) and so \( b_{dR}(P_n) = 1 \). If \( n = 3k + 1 \), then by Theorem 1.2, \( \gamma_{dR}(P_n - v_2v_3) = \gamma_{dR}(P_2 \cup P_{n-2}) = 3 + n - 1 = n + 2 \) and so \( b_{dR}(P_n) = 1 \). Now assume that \( n = 3k - 1 \). Then by Theorem 1.2, \( \gamma_{dR}(P_n - v_1v_2) = \gamma_{dR}(P_{n-1}) = 2 + n - 1 + 1 = n + 2 \) and so \( b_{dR}(P_n) = 1 \). \( \square \)

**Theorem 2.2.** For any \( n \geq 3 \) we have:

\[
\gamma_{dR}(C_n) = \begin{cases} 
1, & \text{if } n \equiv 2 \text{ or } 4 \pmod{6}; \\
2, & \text{if } n \not\equiv 2 \text{ or } 4 \pmod{6}.
\end{cases}
\]

**Proof.** Let \( V(C_n) = \{ v_1, v_2 \ldots v_n \} \). Clearly removing any vertex of \( C_n \) leaves a \( P_3 \). If \( n \equiv 2 \) or \( n \equiv 4 \pmod{6} \), then according to Theorem 1.2, \( \gamma_{dR}(C_n) = n \), while \( \gamma_{dR}(P_3) = n + 1 \), and so \( b_{dR}(C_n) = 1 \). Next assume that \( n \not\equiv 2 \) or \( n \not\equiv 4 \pmod{6} \). Then Theorem 1.2 leads to \( \gamma_{dR}(C_n) = \gamma_{dR}(P_n) \). We thus obtain that \( b_{dR}(C_n) \geq 2 \). On the other hand because of \( b_{dR}(C_n) \geq 2 \), for each edge \( e_1 \in E(C_n) \) we have \( \gamma_{dR}(C_n - e_1) = \gamma_{dR}(P_n) \).

By Theorem 2.1 there exists an edge \( e_2 \in E(C_n) \) such that \( \gamma_{dR}(C_n - \{ e_1, e_2 \}) = \gamma_{dR}(P_n - e_2) = \gamma_{dR}(P_n) = \gamma_{dR}(C_n) \), thus \( b_{dR}(C_n) \leq 2 \). Consequently, \( b_{dR}(C_n) = 2 \). \( \square \)

**Corollary 2.3.** If \( G \) is a graph of order \( n \geq 3 \) with exactly \( k \geq 1 \) vertices of degree \( n - 1 \), then \( b_{dR}(G) = \lceil k/2 \rceil \).

**Proof.** Since \( k \geq 1 \), we have \( \gamma_{dR}(G) = 3 \) by Theorem 1.1. Let \( S = \{ v \in G, \deg_G(v) = n-1 \} \). Then \( S \) is a clique of \( G \). Consider a minimum edge cover \( E' \) of \( S \). We know that \( |E'| = \lceil k/2 \rceil \). Now \( G - E' \) has no vertex of degree \( n-1 \), and so \( \gamma_{dR}(G - E') = 3 = \gamma_{dR}(G) \).

Thus we conclude that \( b_{dR}(G) \leq \lceil k/2 \rceil \).

Now suppose that \( b_{dR}(G) = l \), and let \( S \) be an edge set of size \( l \) such that \( \gamma_{dR}(G - S) > \gamma_{dR}(G) \). Note that \( S \) covers at most \( 2l \) vertices of \( G \). If \( 2l < k \), then \( G - S \) has at least one vertex of degree \( n-1 \), and hence \( \gamma_{dR}(G - S) = 3 = \gamma_{dR}(G) \), a contradiction.

Therefore \( 2l \geq k \), and we conclude that \( b_{dR}(G) \geq \lceil k/2 \rceil \). \( \square \)

As a consequence of Corollary 2.3, we have the following.
Corollary 2.4. If \( n \geq 3 \), then \( b_{dR}(K_n) = \lceil n/2 \rceil \), \( b_{dR}(W_n) = 1 \).

We next determine the double bondage number of complete multipartite graphs.

Lemma 2.5. Let \( 1 \leq n_1 \leq n_2 \leq \cdots \leq n_r \) be integers. Then

\[
b_{dR}(K_{n_1,n_2,\ldots,n_r}) = \begin{cases} 
\left\lceil \frac{1}{2} \right\rceil & \text{if } n_1 = \cdots = n_l = 1, n_{l+1} \geq 2 \\
\left\lceil \frac{2}{3} \right\rceil & \text{if } n_1 = \cdots = n_l = 2, n_{l+1} \geq 3 \\
3(r-1) + 1 & \text{if } n_1 = n_2 = \cdots = n_r = 3 \\
\sum_{i=1}^{r-1} n_i & \text{if otherwise.}
\end{cases}
\]

Proof. Let \( G = K_{n_1,n_2,\ldots,n_r} \) and \( V_1, V_2, \ldots, V_r \) be partite sets of \( V(G) \), where \( |V_j| = n_j \) for \( 1 \leq j \leq r \). Also suppose that \( X \subseteq E(G) \) such that \( \gamma_{dR}(G) < \gamma_{dR}(G - X) \). Let \( H \) be the subgraph of \( G \) induced by \( X \) and let \( T = V(H) \).

If \( |V_1| = |V_2| = \cdots = |V_l| = 1 \) and \( |V_{l+1}| \geq 2 \), then \( b_{dR}(K_{n_1,n_2,\ldots,n_r}) = \left\lceil \frac{1}{2} \right\rceil \) by Corollary 2.3. Now suppose that \( |V_1| = |V_2| = \cdots = |V_l| = 2 \) and \( |V_{l+1}| \geq 3 \). If \( T \cap V_i = \emptyset \) for some \( 1 \leq i \leq l \), then \( G - X \cong \overline{K_2} \vee K \) for some graph \( K \) such that \( \Delta(K) \leq |V(K)| - 2 \). Hence \( \gamma_{dR}(G) = \gamma_{dR}(G - X) = 4 \) by Theorem 1.1, which is a contradiction. Hence \( T \cap V_i \neq \emptyset \) for each \( 1 \leq i \leq l \). Then \( |T| \geq l \). Therefore \( |X| \geq \left\lceil \frac{3}{2} \right\rceil \) and we conclude that \( b_{dR}(G) \geq \left\lceil \frac{1}{2} \right\rceil \). For \( 1 \leq i \leq l \), choose \( v_i \in V_i \) and consider \( Y = \{v_1v_2, v_3v_4, \cdots, v_{l-1}v_l\} \) if \( l \) is even and \( Y = \{v_1v_2, v_3v_4, \cdots, v_{l-2}v_{l-1}, v_1v_l\} \) if \( l \) is odd. In both cases \( G - Y \cong \overline{K_2} \vee K \) for each graph \( K \). Hence \( \gamma_{dR}(G - Y) \geq 5 \) by Theorem 1.1, and therefore \( b_{dR}(G) \leq \left\lceil \frac{1}{2} \right\rceil \).

Now suppose that \( n_i \geq 3 \), for any \( 1 \leq i \leq r \). In this case \( \gamma_{dR}(G) = 6 \) by Lemma 1.3. Therefore \( V_i \subseteq T \) for each \( i \), except probably one \( i \), since if \( v_i \in V_i \setminus T \) for \( t \in \{t_1, t_2\} \), then \( \{V_{t_1}, V_{t_2}\} \) dominate all vertices of \( G - X \) and hence \( \gamma_{dR}(G - X) \leq 6 \), which is a contradiction. We consider two cases.

Case 1. For each \( 1 \leq i \leq r \), \( n_i = 3 \).

Suppose that \( \bigcup_{i=1}^{r-1} V_i \subseteq T \). Then \( |V_r \cap T| \geq 2 \), since otherwise \( \gamma_{dR}(G - X) = 6 \). First suppose that \( |V_r \cap T| = 2 \). Assume that \( V_r = \{y, y', y''\} \) and \( y \in V_r \setminus T \). If there exists \( z \in \bigcup_{i=1}^{r-1} V_i \) such that \( z \) is adjacent to \( y' \) and \( y'' \) in \( G - X \), then \( (V \setminus \{z, y\}, \emptyset, \{z, y\}) \) is a DRDF, and hence \( \gamma_{dR}(G - X) \leq 6 \), which is a contradiction. So for any \( z \in \bigcup_{i=1}^{r-1} V_i \), there exists an edge \( zy' \) or \( zy'' \) which belongs to \( X \). On the other hand there exists \( x \in \bigcup_{i=1}^{r-1} V_i \), which is not adjacent to \( y', y'' \) in \( G - X \), since otherwise \( (V \setminus \{y, y', y''\}, \{y, y', y''\}, \emptyset) \) is a DRDF for \( G - X \), which is a contradiction. Hence \( |X| \geq 3(r-1) + 1 \). Now suppose that \( V_r \subseteq T \). Hence \( H \) is a spanning subgraph of \( G \). If for any vertex \( x \in V(G) \) we have \( \deg_H(x) \geq 2 \), then

\[
2|X| = \sum_{x \in V} \deg_H(x) \geq 2(3r)
\]

and so \( |X| \geq 3r > 3(r-1) + 1 \). Suppose that there exists a vertex \( x \in V(G) \), with \( \deg_H(x) = 1 \). Without lose of generality, suppose that \( x \in V_1 = \{x, x', x''\} \) and \( x \) is adjacent to \( y \in V_r \). If \( y \) is adjacent to \( x' \) and \( x'' \) in \( G - X \), then \( (V \setminus \{x, y\}, \emptyset, \{x, y\}) \) is a DRDF for \( G - X \), which is a contradiction. Hence \( yx' \) or \( yx'' \) belongs to \( X \).
Also there are two edges $y't_1, y''t_2 \in X$ for some vertices $t_1, t_2$ of $G$. If there exists a vertex $z \in \bigcup_{i=2}^{r-1} V_i$, such that $z$ is adjacent to all vertices $y, x', x''$ in $G - X$, then $(V \setminus \{z, x\}, \emptyset, \{z, x\})$ is a DRDF for $G - X$, which is impossible. Hence for any $z \in \bigcup_{i=2}^{r-1} V_i$, $X \cap \{zx', zx'', zy\} \neq \emptyset$. Therefore

$$|X| \geq 3(r - 2) + 2 + 2 = 3(r - 1) + 1.$$  

Hence in this case $b_{dr}(G) \geq 3(r - 1) + 1$. On the other hand for $x, x' \in V_1, y \in V_2$ consider the set $X = \{xz; z \in \bigcup_{i=2}^{r-1} V_i, x'y\}$. Clearly $G - X = K_1 \cup K$, and $K \not\cong K_2 \lor L$ for any graph $L$. This means that $\gamma_{dr}(G - X) \geq 2 + 5 = 7$, and hence $b_{dr}(G) \leq 3(r - 1) + 1$. We conclude that $b_{dr}(G) = 3(r - 1) + 1$.

**Case 2.** For at least one $i, n_i \geq 4$. The argument of this case is similar to case 1. \(\square\)

### 3. Bounds for the Double Roman Bondage Number

In this section we present various bounds for the double Roman bondage number. We begin with the following.

**Theorem 3.1.** If $G$ is a graph, and $xyz$ a path of length 2 in $G$, then

1. $b_{dr}(G) \leq \deg_G(x) + \deg_G(y) + \deg_G(z) - 3 - |N(x) \cap N(y)|$.
   If $x$ and $z$ are adjacent, then
2. $b_{dr}(G) \leq \deg_G(x) + \deg_G(y) + \deg_G(z) - 4 - |N(x) \cap N(y)|$.

**Proof.** Let $H$ be the graph obtained from $G$ by removing the edges incident to $x, y$ and $z$ with exception of $yz$ and all edges between $y$ and $N(x) \cap N(y)$. In $H$, the vertex $x$ is isolated, $z$ is leaf, $y$ is adjacent to $z$, and all neighbors of $y$ in $H$, if any, lie in $N_G(x)$.

Let $f = (V_0, V_2, V_3)$ be a $\gamma_{dr}(H)$-function. Then $x \in V_2$ and, without loss of generality, assume that $z \in V_0 \cup V_2$. If $z \in V_0$, then $y \in V_3$ and therefore $(V_0 \cup \{x\}, V_2 \setminus \{x\}, V_3)$ is a DRDF on $G$ of weight less than $f$, and (1) as well as (2) are proved.

Now assume that $z \in V_2$. Obviously $y \not\in V_2$. If $y \in V_2$, then $(V_0 \cup \{z\}, V_2 \setminus \{y, z\}, V_3 \cup \{y\})$ is a DRDF on $H$ of weight less than $f$, that is a contradiction. However, if $y \in V_0$, then $((V_0 \cup \{x, z\}) \setminus \{y\}, V_2 \setminus \{x, z\}, V_3 \cup \{y\})$ is a DRDF of $G$ of weight less than $w(f)$, and again (1) and (2) are proved. \(\square\)

Applying Theorem 3.1 on the path $xyz$ such that one of the vertices $x, y$ or $z$ has minimum degree, we obtain the next result immediately.

**Corollary 3.2.** If $G$ is a connected graph of order $n \geq 3$, then

$$b_{dr}(G) \leq \delta(G) + 2\Delta(G) - 3.$$  

**Corollary 3.3.** If a graph $G$ has a support vertex $v$ of degree at least three such that all of its neighbors except one are leaves, then $b_{dr}(G) \leq 2$. 
It follows that $v$ on the path $v_1v_2$ in the case $\deg_G(v) = k = 3$, we obtain $b_{dR}(G) \leq 2$ immediately. Assume now that $\deg_G(v) = k \geq 4$. Let $f = (V_0, V_2, V_3)$ be a $\gamma_{dR}$-function of $G - vv_1$. It follows that $v_1 \in V_2$ and, without loss of generality, assume that $v \in V_3$. Therefore $(V_0 \cup \{v_1\}, V_2 \setminus \{v_1\}, V_3)$ is a DRDF on $G$ of weight $\gamma_{dR}(G - vv_1) - 2$, and thus $b_{dR}(G) = 1$.

**Corollary 3.4.** For any tree $T$ with at least three vertices, $b_{dR}(T) \leq 2$.

**Proof.** If $T$ has a support vertex $v$ of degree at least three such that all of its neighbors except one is leaf, then $b_{dR}(T) \leq 2$ by Corollary 3.3. So assume that for any support vertex $v$ either $\deg_T(v) = 2$ or $v$ has at least two neighbors which are not leaves. Let $P = v_1, v_2, ..., v_k$ be a longest path of $T$. By the assumption, $\deg_T(v_2) = 2$. If $g$ is a $\gamma_{dR}(T - \{v_1v_2, v_2v_3\})$-function, then $g(v_1) = g(v_2) = 2$. Now if we let $h(v_1) = 0, h(v_2) = 3$ and $h = g$ for other vertices of $T$, then $h$ is a DRDF on $T$ of weight less than $w(g)$ and so $\gamma_{dR}(T - \{v_1v_2, v_2v_3\}) > \gamma_{dR}(T)$. Thus the proof is complete.

**Problem:** Characterize trees $T$ with $b_{dR}(T) = 1$ or $b_{dR}(T) = 2$.

We next improve Theorem 3.1.

**Theorem 3.5.** If $G$ is a connected graph of order $n \geq 2$ and $uv \in E(G)$ then

$$b_{dR}(G) \leq \deg_G(u) + \deg_G(v) - 1 - |N(u) \cap N(v)|.$$  

**Proof.** It is not hard to see that $\gamma_{dR}(G - U) > \gamma_{dR}(G)$, where $U = \{tu|t \in N(u)\} \cup \{sv|s \in N(v)\} \setminus \{(u,v)\}$. Note that $|U| = \deg_G(u) + \deg_G(v) - 1 - |N(u) \cap N(v)|$.

**Corollary 3.6.** If $G$ is a connected graph, then $b_{dR}(G) \leq \Delta(G) + \delta(G) - 1$.

Note that Corollary 3.6 improves Corollary 3.2. By Corollary 1.6 we obtain the following improvement of Corollary 3.6 if the graph is planar.

**Theorem 3.7.** If $G$ is a connected graph and $uvw$ is a path of $G$, then $b_{dR}(G) \leq \deg_G(u) + \deg_G(v) - 1$.

**Proof.** It is not hard to see that $\gamma_{dR}(G - U) > \gamma_{dR}(G)$, where $U = \{tu|t \in N(u)\} \cup \{sv|s \in (N(v) \setminus \{w\})\}$. Since $|U| = \deg_G(u) + \deg_G(v) - 1$, the proof is complete.

**Remark 3.8.** If $G$ is a planar graph and $g \geq 4$, then $\delta(G) \leq 3$. Also $b_{dR}(G) \leq \Delta(G) + \delta(G) - 1 \leq \Delta(G) + 2$. If $G$ is a planar graph and $g \geq 6$, then $\delta(G) \leq 2$. Also $b_{dR}(G) \leq \Delta(G) + \delta(G) - 1 \leq \Delta(G) + 1$.

**Theorem 3.9.** If $G$ is a connected planar graph of order $n \geq 2$ without vertices of degree five, then $b_{dR}(G) \leq 7$.

**Proof.** If $A = \{v \in V(G)|\deg_G(v) \leq 4\} = \{v_1, v_2, ..., v_k\}$, then Corollary 1.6 and the hypothesis imply that $A \neq \emptyset$. Suppose on the contrary that $b_{dR}(G) \geq 8$. In view of Theorem 3.7 and the assumption, we deduce that $d_G(x, y) \geq 3$ for any two distinct
vertices \(x, y \in A\). Using Lemma 1.7, we define \(H_0 = G\) and \(H_i = H_{i-1} + S_i\) for \(1 \leq i \leq k\), where \(S_i\) is a subset of \(E_{v_i} = \{xy| x, y \in N(v_i), xy \notin E(H_{i-1})\}\) such that \(H_{i-1} + S_i\) is still a planar graph and \(H_i[N(v_i)]\) is 2-connected when \(\deg_H(v_i) \geq 3\).

Now let \(x \in A\) and \(y \in N_G(x)\). If \(\deg_G(x) \leq 2\), then it follows from the assumption and Theorem 3.7 that \(\deg_H(y) \geq 7\) and so \(\deg_{H_k}(y) \geq 7\). Assume next that \(\deg_G(x) = 3\). By the assumption and Theorem 3.5, we obtain

\[8 \leq \deg_G(x) + \deg_G(y) - |N_G(x) \cap N_G(y)| - 1 = \deg_G(y) - |N_G(x) \cap N_G(y)| + 2.\]

If \(|N_G(x) \cap N_G(y)| \geq 1\), then we deduce that \(\deg_{H_k}(y) \geq \deg_G(y) \geq 7\). In the remaining case \(N_G(x) \cap N_G(y) = \emptyset\), inequality chain leads to \(\deg_G(y) \geq 6\) and thus \(\deg_{H_k}(y) \geq 8\). Finally assume that \(\deg_G(x) = 4\). By the assumption and Theorem 3.5, we obtain

\[8 \leq b_{dR}(G) \leq \deg_G(x) + \deg_G(y) - |N_G(x) \cap N_G(y)| - 1 = \deg_G(y) - |N_G(x) \cap N_G(y)| + 3.\]

If \(|N_G(x) \cap N_G(y)| \geq 2\), then we deduce that \(\deg_{H_k}(y) \geq \deg_G(y) \geq 7\). If \(|N_G(x) \cap N_G(y)| = 1\), then \(\deg_G(y) \geq 6\) and so \(\deg_{H_k}(y) \geq 7\). In the remaining case \(N_G(x) \cap N_G(y) = \emptyset\), we observe that \(\deg_G(y) \geq 5\) and thus \(\deg_{H_k}(y) \geq 7\).

According to Lemma 1.7, the graph \(H_k\) is planar. However, since \(d_G(x, y) \geq 3\) for any two distinct vertices \(x, y \in A\), we observe that \(H_k - A\) is a planar graph with minimum degree at least 6. This is a contradiction to Corollary 1.6 and the proof is complete.

**Theorem 3.10.** If \(G\) is a connected planar graph of order \(n \geq 2\), then \(b_{dR}(G) \leq 8\).

**Proof.** Let

\[A_3 = \{v \in V(G)| \deg_G(v) \leq 3\},\]

\[A_4 = \{v \in V(G)| \deg_G(v) = 4\},\]

\[A_5 = \{v \in V(G)| \deg_G(v) = 5\}.\]

If \(A_5 = \emptyset\), then Theorem 3.9 implies the desired result. Thus we can assume \(A_5 \neq \emptyset\). Suppose on the contrary that \(b_{dR}(G) \geq 9\). In view of Theorem 3.7 and the assumption, if \(x \in A_3 \cup A_4\) and \(y \in A_3 \cup A_4 \cup A_5\), then \(d_G(x, y) \geq 3\). In addition, if \(x \in A_3\), then \(\deg_G(y) \geq 7\) for all \(y \in N_G(x)\), by theorem 3.5.

Let \(I\) be a maximum independent subset of \(A_5\). Then \(A_5 \subseteq I \cup N(I)\) and \(N(A_4) \cap N(I) = \emptyset\). Now let \(A_4 \cup I = \{v_1, v_2, \ldots, v_k\}\) and \(H = G - A_3\). Applying Lemma 1.7, we define \(H_0 = H\) and \(H_i = H_{i-1} + S_i\) for \(1 \leq i \leq k\), where \(S_i\) is a subset of \(E_{v_i} = \{xy| x, y \in N(v_i), xy \notin E(H_{i-1})\}\) such that \(H_{i-1} + S_i\) is still a planar graph and \(H_i[N(v_i)]\) is 2-connected. We proceed with the following claims.

**Claim 1.** If \(A_4 \neq \emptyset\), then \(\deg_{H_k}(y) \geq 7\) for each vertex \(y \in N_G(A_4)\).

To see this, assume that \(A_4 \neq \emptyset\). Let \(x \in A_4\) and \(y \in N_G(x)\). Then Theorem 3.5 and the assumption imply that

\[9 \leq b_{dR}(G) \leq \deg_G(x) + \deg_G(y) - |N_G(x) \cap N_G(y)| - 1 = \deg_G(y) - |N_G(x) \cap N_G(y)| + 3.\]
If \(|N_G(x) \cap N_G(y)| \geq 1\), then we deduce that \(\deg_{H_k}(y) \geq \deg_G(y) \geq 7\). If \(N_G(x) \cap N_G(y) = \emptyset\), then the inequality chain leads to \(\deg_G(y) \geq 6\) and thus \(\deg_{H_k}(y) \geq 8\).

**Claim 2.** \(\deg_{H_k}(y) \geq 7\) for each vertex \(y \in N_G(I)\).

To see this, let \(x \in I\) and \(y \in N_G(x)\). Then Theorem [3.3] and the assumption imply that
\[
9 \leq b_{dr}(G) \leq \deg_G(x) + \deg_G(y) - |N_G(x) \cap N_G(y)| - 1 = \deg_G(y) - |N_G(x) \cap N_G(y)| + 4.
\]
If \(|N_G(x) \cap N_G(y)| \geq 2\), then we deduce that \(\deg_{H_k}(y) \geq \deg_G(y) \geq 7\). If \(|N_G(x) \cap N_G(y)| = 1\), then \(\deg_G(y) \geq 6\) and so \(\deg_{H_k}(y) \geq 7\). In the remaining case \(N_G(x) \cap N_G(y) = \emptyset\), we observe that \(\deg_G(y) \geq 5\) and thus \(\deg_{H_k}(y) \geq 7\).

Combining Claims 1 and 2, we find that \(G^* = H_k - A_4\) is a planar graph with the following properties. The minimum degree of \(G^*\) is 5, \(I = \{v \in V(G^*)| \deg_{G^*}(v) = 5\}\) is an independent set in \(G^*\) and \(\deg_{G^*}(v) \geq 7\) for each vertex \(v \in N_{G^*}(I) = N_G(I)\). Let \(B\) be the bipartite graph with the bipartition \(I\) and \(N(I)\) and the edge set \(\{uv \in E(G^*)| u \in I, v \in N(I)\}\). Then \(B\) is a bipartite planar graph with exactly \(5|I|\) edges. Applying Theorem [1.4] with girth \(g \geq 4\), we obtain \(5|I| \leq 2|I| + 2|N(I)| - 4\) (note that this bound remains valid if \(g = \infty\), that means that \(B\) is a forest) and therefore \(|N(I)| \geq \frac{3}{2}|I| + 2\). Altogether we find
\[
|E(G^*)| = \frac{1}{2} \sum_{v \in V(G^*)} \deg_{G^*}(v) \\
\quad \geq \frac{1}{2}(5|I| + 7|N(I)| + 6(|V(G^*)| - |I| - |N(I)|)) \\
\quad = 3|V(G^*)| + \frac{1}{2}|N(I)| - \frac{1}{2}|I| \\
\quad \geq 3|V(G^*)| + \frac{1}{4}|I| + |I| > 3|V(G^*)| - 6
\]
This is a contradiction to Corollary [1.5] and the proof is complete. \(\square\)

4. **Complexity of double Roman bondage number**

In this section, we study the NP-hardness of the double Roman bondage number problem. The decision problem of the double Roman bondage number problem is stated as follows.

**Double Roman bondage number problem (DRBN):**

**Instance:** A nonempty graph \(G\) and a positive integer \(k\).

**Question:** Is \(b_{dr}(G) \leq k\)?

We will prove the NP-hardness of the double Roman bondage number problem by transforming from a known NP-complete problem, namely 3-satisfiability problem that is known to be NP-complete [9].
At first we recall some terms for the 3SAT problem. Let \( U \) be a set of Boolean variables. If \( u \) is a variable in \( U \), then \( u \) and \( \overline{u} \) are literals over \( U \). A truth assignment for \( U \) is a mapping \( t: U \rightarrow \{ T, F \} \). We call \( u \) true under \( t \) if \( t(u) = T \), otherwise it is called false under \( t \). The literal \( u \) is true under \( t \) if and only if the variable \( u \) is true under \( t \); the literal \( \overline{u} \) is true if and only if the variable \( u \) is false. A clause over \( U \) is a set of literals over \( U \). A clause represents the disjunction of literals and is satisfied by a truth assignment if and only if at least one of its members is true under that assignment. A collection \( \xi \) of clauses over \( U \) is satisfiable if and only if there exists some truth assignment for \( U \) that simultaneously satisfies all the clauses in \( \xi \). Such a truth assignment is called a satisfying truth assignment for \( \xi \). The 3-SAT is specified as follows.

**3-satisfiability problem (3-SAT):**

**Instance:** A collection \( \xi = \{D_1, D_2, \ldots, D_m\} \) of clauses over a finite set \( U \) of variables such that \( |D_j| = 3 \) for \( j = 1, 2, \ldots, m \).

**Question:** Is there a truth assignment for \( U \) that satisfies all the clauses in \( \xi \)?

**Theorem 4.1.** DRBN is NP-hard even for bipartite graphs.

*Proof. Let \( U = \{u_1, u_2, \ldots, u_n\} \) and \( \xi = \{D_1, D_2, \ldots, D_m\} \) be an arbitrary instance of 3-SAT. We construct a bipartite graph \( G \) and a positive integer \( k \) such that \( \xi \) is satisfiable if and only if \( b_{DR}(G) \leq k \). The graph \( G \) is constructed as follows. For each \( i = 1, 2, \ldots, n \), corresponding to the variable \( u_i \in U \), we associate a graph \( H_i \) with vertex set \( V(H_i) = \{u_i, \overline{u}_i, v_i, v_i', x_i, y_i, z_i\} \) and edge set

\[
E(H_i) = \{u_i z_i, u_i v_i, v_i w_i, \overline{u}_i z_i, \overline{u}_i v_i, v'_i w_i, w_i z_i, y_i v_i, y_i v'_i, y_i z_i, x_i v_i, x_i v'_i\}.
\]

For each \( j = 1, 2, \ldots, m \), corresponding to the clause \( D_j = \{p_j, q_j, r_j\} \in \xi \), associate a single vertex \( c_j \) and add edge set \( E_j = \{c_j p_j, c_j q_j, c_j r_j\}, 1 \leq j \leq m \). Next, add a cycle \( C_8 = l_1 l_2 l_3 l_4 l_5 l_6 l_7 l_8 l_1 \), and join \( l_2 \) and \( l_4 \) to each vertex \( c_j \) with \( 1 \leq j \leq m \). Finally add a new vertex \( l_9 \) and join it to both \( l_1 \) and \( l_5 \), and set \( k = 1 \). Figure 1 shows an example of the graph \( G \) when \( U = \{u_1, u_2, u_3, u_4\} \) and \( \xi = \{D_1, D_2, D_3\} \), where \( D_1 = \{u_1, \overline{u}_2, u_4\}, D_2 = \{u_1, \overline{u}_2, u_4\} \) and \( D_3 = \{u_2, u_3, \overline{u}_4\} \).
It is easy to see that $G$ is a bipartite graph, and the construction can be accomplished in polynomial time. Let $k = 1$. We will prove that $\xi$ is satisfiable if and only if $b_{dR}(G) = 1$. We proceed with a series of claims namely Claim 1, Claim 2, Claim 3 and Claim 4.

**Claim 1.** $\gamma_{dR}(G) \geq 6n + 8$. If equality hold, then for any $\gamma_{dR}$-function $f$ on $G$,

i) $f(H_i) = 6$, for any $i = 1, \ldots, n$,

ii) $|\{u_i, \overline{u}_i\} \cap V_3| \leq 1$, for any $i = 1, \ldots, n$,

iii) $\{u_i, \overline{u}_i\} \cap V_2 = \emptyset$, for any $i = 1, \ldots, n$,

iv) $f(l_1) = f(l_3) = f(l_5) = f(l_7) = 2$ and $\{c_j : j = 1, 2, \ldots, m\} \subseteq V_0$.

**Proof of Claim 1.** Let $f$ be a $\gamma_{dR}$-function of $G$, and $i \in \{1, 2, \ldots, n\}$.

If $f(u_i) \geq 2$ and $f(\overline{u}_i) \geq 2$, then $\sum_{v \in N[u_i]} f(v) \geq 2$ and so $\sum_{v \in V(H_i)} f(v) \geq 6$. If $f(u_i) = 0$ or $f(\overline{u}_i) = 0$, then by considering $H_i - u_i - \overline{u}_i$ and $H_i - u_i$ or $H_i - \overline{u}_i$ and Theorem [1.1] we obtain that $\sum_{v \in V(H_i)} f(v) \geq 6$. If $f(u_i) = 2$ and $f(\overline{u}_i) = 0$ or $f(u_i) = 3$ and $f(\overline{u}_i) = 0$, then similarly we observe that $\sum_{v \in V(H_i)} f(v) \geq 6$. On the other hand $\sum_{i=1}^{\alpha} f(l_i) \geq 8$. Consequently, $\gamma_{dR}(G) \geq 6n + 8$. Assume that $\gamma_{dR}(G) = 6n + 8$. Then $\sum_{i=1}^{\alpha} f(l_i) = 8$ and $f(H_i) = \sum_{v \in V(H_i)} f(v) = 6$ for $i = 1, 2, \ldots, n$. 

Suppose that there is an integer $j \in \{1, 2, \ldots, n\}$ such that $f(u_j) = f(\overline{u_j}) = 3$. Since $\sum_{v \in N[u_j]} f(v) \geq 2$, and $\sum_{v \in N[\overline{u_j}]} f(v) \geq 2$, we find that $f(H_j) > 6$, a contradiction. Thus $|\{u_i, \overline{u_i}\} \cap V_3| \leq 1$, for any $i = 1, \ldots, n$. Parts (iii) and (iv) are proved similarly. Thus the proof of Claim 1 is complete. ♦

**Claim 2.** $\gamma_{dR}(G) = 6n + 8$ if and only if $\xi$ is satisfiable.

**Proof of Claim 2.** Assume that $\gamma_{dR}(G) = 6n + 8$ and let $f$ be a $\gamma_{dR}(G)$-function. By Claim 1, at most one of $f(u_i)$ and $f(\overline{u_i})$ is equal to 3 for each $i = 1, 2, \ldots, n$. Define a mapping $t : U \rightarrow \{T, F\}$ by

$$
t(u_i) = \begin{cases} 
T & \text{if } f(u_i) = 3 \text{ or } f(u_i) \neq 3 \text{ and } f(\overline{u_i}) \neq 3, \\
F & \text{if } f(\overline{u_i}) = 3.
\end{cases}
$$

We now show that $t$ is a satisfying truth assignment for $\xi$. It is sufficient to show that every clause in $\xi$ is satisfied by $t$. To this end, we arbitrarily choose a clause $D_j \in \xi$ with $1 \leq j \leq m$. By Claim 1, $f(c_j) = f(l_2) = f(l_4) = 0$. Thus there exists some $i$ with $1 \leq i \leq n$ such that $f(u_i) = 3$ or $f(\overline{u_i}) = 3$, where $c_j$ is adjacent to $u_i$ or $\overline{u_i}$. Assume that $c_j$ is adjacent to $u_i$, where $f(u_i) = 3$. Since $u_i$ is adjacent to $c_j$ in $G$, the literal $u_i$ is in the clause $D_j$ by the construction of $G$. Since $f(u_i) = 3$, it follows that $t(u_i) = T$ which implies that the clause $D_j$ is satisfied by $t$. Next assume that $c_j$ is adjacent to $\overline{u_i}$, where $f(\overline{u_i}) = 3$. Since $\overline{u_i}$ is adjacent to $c_j$ in $G$, the literal $\overline{u_i}$ is in the clause $D_j$. Since $f(\overline{u_i}) = 3$, it follows that $t(u_i) = F$. Thus, $t$ assigns $\overline{u_i}$ the truth value $T$, that is, $t$ satisfies the clause $D_j$. Since $j$ is chosen arbitrarily, thus $t$ satisfies all the clauses in $\xi$. Consequently, $\xi$ is satisfiable.

Conversely, assume that $\xi$ is satisfiable, and let $t : U \rightarrow \{T, F\}$ be a satisfying truth assignment for $\xi$. Create a function $f$ on $V(G)$ as follows: if $t(u_i) = T$, then let $f(u_i) = f(v') = 3$, and if $t(u_i) = F$, then let $f(\overline{u_i}) = f(v_i) = 3$. Let $f(l_1) = f(l_3) = f(l_5) = f(l_7) = 2$. Clearly, $f(V(G)) = 6n + 8$. Since $t$ is a satisfying truth assignment for $\xi$, at least one of literals in $D_j$ is true under the assignment $t$, for each $j = 1, 2, \ldots, m$. It follows that the corresponding vertex $c_j$ in $G$ is adjacent to at least one vertex $w$ with $f(w) = 3$, since $c_j$ is adjacent to each literal in $D_j$ by the construction of $G$. Thus $f$ is a DRDF of $G$, and so $\gamma_{dR}(G) \leq f(G) = 6n + 8$. Since by Claim 1, $\gamma_{dR}(G) \geq 6n + 8$, we conclude that $\gamma_{dR}(G) = 6n + 8$. Thus the proof of Claim 2 is complete. ♦

**Claim 3.** $\gamma_{dR}(G - e) \leq 6n + 9$ for any $e \in E(G)$.

**Proof of Claim 3.** For any edge $e \in E(G)$, it is sufficient to construct a DRDF $f$ on $G - e$ with weight $4n + 9$. We first assume that $e \in \{l_1l_2, l_1l_8, l_1l_9, l_3l_4, l_6l_7\}$ or $e = c_jl_4$, $c_ju_i$ or $e = c_j\overline{u}_i$, for some $j = 1, 2, \ldots, m$, and $i = 1, 2, \ldots, n$. We define a function $f$ by $f(l_2) = f(l_5) = f(l_8) = 3$, $f(u_i) = f(v'_i) = 3$ for each $i = 1, 2, \ldots, n$, and $f(v) = 0$ otherwise. Then $f$ is a DRDF of $G - e$ with $f(G - e) = 6n + 9$. If $e \in \{l_5l_4, l_5l_6, l_5l_9, l_2l_3, l_6l_8\}$, then we define $f$ by $f(l_1) = f(l_4) = f(l_6) = 3$, $f(u_i) = f(v'_i) = 3$ for each $i = 1, 2, \ldots, n$, and $f(v) = 0$ otherwise. Then $f$ is a DRDF of $G - e$ with $f(G - e) = 6n + 9$. If $e$ is not incident with $u_i$ or $v'_i$ for each $i$, then we
define \( f(l_1) = f(l_4) = f(l_6) = 3 \) and \( f(u_i) = f(v'_i) = 3 \), and \( f(v) = 0 \) otherwise.

If \( e \) is not incident with \( v_i \) or \( v_i' \), then we define \( f \) by \( f(l_1) = f(l_4) = f(l_6) = 3 \), \( f(u_i) = f(v_i) = 3 \), and \( f(v) = 0 \) otherwise. If \( e = u_i v_i \) or \( u_i v_i' \), for some \( i \), then we define \( f \) by \( f(l_1) = f(l_4) = f(l_6) = 3 \), \( f(x_i) = f(z_i) = 3 \) for each \( i = 1, 2, \ldots, n \), and \( f(v) = 0 \) otherwise. Then \( f \) is a DRDF of \( G - e \) with \( f(G - e) = 6n + 9 \) and thus \( \gamma_{dR}(G - e) \leq 6n + 9 \). \( \diamondsuit \)

**Claim 4.** \( \gamma_{dR}(G) = 6n + 8 \) if and only if \( b_{dR}(G) = 1 \).

**Proof of Claim 4.** Assume \( \gamma_{dR}(G) = 6n + 8 \) and consider the edge \( e = l_1 l_2 \).

Suppose \( \gamma_{dR}(G) = \gamma_{dR}(G - e) \). Let \( f' \) be a \( \gamma_{dR} \)-function of \( G - e \). It is clear that \( f' \) is also a \( \gamma_{dR} \)-function on \( G \). By Claim 1, we have \( f'(e_j) = 0 \) for each \( j = 1, 2, \ldots, m \) and \( f(l_2) = f(l_4) = f(l_6) = f(l_8) = f(l_9) = 0 \). But then \( f'(N[l_2]) = 2 \), a contradiction.

Hence, \( \gamma_{dR}(G) < \gamma_{dR}(G - e) \), and so \( b_{dR}(G) = 1 \).

Conversely, assume that \( b_{dR}(G) = 1 \). By Claim 1, we have \( \gamma_{dR}(G) \geq 6n + 8 \). Let \( e' \) be an edge such that \( \gamma_{dR}(G) < \gamma_{dR}(G - e') \). By Claim 3, we have \( \gamma_{dR}(G - e') \leq 6n + 8 \). Thus, \( 6n + 8 \leq \gamma_{dR}(G) < \gamma_{dR}(G - e') \leq 6n + 9 \), which yields \( \gamma_{dR}(G) = 6n + 8 \). \( \diamondsuit \)

By Claims 2 and 4, \( b_{dR}(G) = 1 \) if and only if there is a truth assignment for \( U \) that satisfies all clauses in \( \xi \). Since the construction of the double Roman bondage number instance is straightforward from a 3-satisfiability instance, the size of the double Roman bondage number instance is bounded above by a polynomial function of the size of 3-satisfiability instance. It follows that this is a polynomial reduction and the proof is complete. \( \square \)

**References**

[1] H. Abdollahzadeh Ahangar, M. Chellali and S.M. Sheikholeslami, On the double Roman domination in graphs, *Discrete Appl. Math.* 103 (2017) 245–258.

[2] S. Akbari, M. Khatirinejad and S. Qajar, A note on Roman bondage number of planar graphs, *Graphs Combin.* 29 (2013), 327–331.

[3] A. Bahremandpour, F.-T. Hu, S.M. Sheikholeslami and J.-M. Xu, On the Roman bondage number of a graph, *Discrete Math. Alg. Appl.* 5 (1) (2013), 1350001 (15 pages).

[4] D. Bauer, F. Harary, J. Nieminen and C.L. Suffel, Domination alteration sets in graphs, *Discrete Math.* 47 (1983), 153–161.

[5] R.A. Beeler, T.W. Haynes and S.T. Hedetniemi, Double Roman domination, *Discrete Appl. Math.* 211 (2016), 23–29.

[6] E.J. Cockayne, P.M. Dreyer Jr., S.M. Hedetniemi and S.T. Hedetniemi, On Roman domination in graphs, *Discrete Math.* 278 (2004) 11–22.

[7] J.E. Dunbar, T.W. Haynes, U. Teschner and L. Volkmann, Bondage, insensitivity, and reinforcement, in: T.W. Haynes, S.T. Hedetniemi and P.J. Slater (Eds.), Domination in Graphs: Advanced Topics (Marcel Dekker, New York, 1998), 471–489.

[8] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, The bondage number of a graph, *Discrete Math.* 86 (1990), 47–57.

[9] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, San Francisco, 1979.

[10] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[11] B.L. Hartnell and D.F. Rall, Bounds on the bondage number of a graph, *Discrete Math.* 128 (1994), 173–177.
[12] N. Jafari Rad and H. Rahbani, Some progress on double Roman domination in graphs, Discuss. Math. Graph Theory (to appear).

[13] N. Jafari Rad and L. Volkmann, Roman bondage in graphs, Discuss. Math. Graph Theory 31 (4) (2011), 763–773.

[14] C. S. ReVelle and K. E. Rosing, Defendens imperium Romanum: a classical problem in military strategy, Amer Math. Monthly 107 (2000), 585–594.

[15] I. Stewart, Defend the roman empire!, Sci. Amer. 281 (6) (1999), 136–139.

[16] J.-M. Xu, On bondage numbers of graphs a survey with some comments, International J. Combin. (2013), Article ID 595210, 34 pages, 2013. doi:10.1155/2013/5952.

[17] L. Volkmann, Double Roman domination and domatic numbers of graphs, Comm. Combin. Optim. 3 (2018), 71–77.

[18] D.B. West, Introduction to Graph Theory, 2nd Edition, Prentice-Hall, Inc. (2001).

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