Abstract. I describe the advantage of using singular value decomposition as a diagnostic tool for exploring the potential and limitations of seismic data. Using stellar models coupled with the expected errors in seismic and complementary data we can predict the precision in the stellar parameters. This in turn allows us to quantify if and to what extent we can distinguish between various descriptions of the interior physical processes. This method can be applied to a wide range of astrophysical problems, and here I present one such example which shows that the convective core overshoot parameter can be constrained with one identified mode if the pulsating component is in an eclipsing binary system.

1. Introduction

It is of extreme benefit to have a reliable method to explore both the scientific potential and the limitations of a particular astrophysical system. This paper is dedicated to discussing how to use some properties of singular value decomposition (SVD) to do just this. We can use SVD to do such things as predict the precision of global quantities, explore the impact of extending a data set from 3 months to 6 months (e.g. with Kepler), and probe the scientific benefit of complementary observables. If we understand the potential of the system, then we can carefully choose the scientific question that we would like to pursue. As a specific example, we will test if an eclipsing binary system containing a pulsating component has sufficient information to allow us to constrain the convective core overshoot parameter, \( \alpha_{ov} \). This method of using SVD was exploited by Brown et al. (1994), and more recently in the literature, by Miglio & Montalbán (2005), Creevey et al. (2007) and Creevey (2008a). Press et al. (1992) while concentrating on SVD from a numerical point of view, also briefly discusses some of these properties.

2. Observational data and interpretation

Normally in astrophysical problems, one is confronted with comparing a set of observational data with an astrophysical model for scientific interpretation. This model should describe as best as possible the physical or chemical processes observed. The input to this model are a set of parameters of the system, which we shall denote by \( \mathbf{P} \). Often we know what the parameters are, but we do not know their quantities. For example, in a spectroscopic binary system, such parameters are the orbital period, the mass ratio and the inclination of the system with respect to the observer. Given a specific set of input \( \mathbf{P} \), this model...
will produce a set of expected observables, which we denote by \( \mathbf{B} \). In the binary system example, \( \mathbf{B} \) could be the radial velocities of both components. Scientific interpretation often then involves comparing the real observational data, \( \mathbf{O} \), to \( \mathbf{B} \) to infer the values of \( \mathbf{P} \).

If the system is linear, or if it can be described locally as linear, the following equation can determine the true parameters of the system, by calculating (iteratively) the parameter changes \( \delta \mathbf{P} \) to make to an initial guess \( \mathbf{P}_0 \):

\[
\delta \mathbf{P} = \mathbf{VW}^{-1} \mathbf{U} \frac{\mathbf{O} - \mathbf{B}}{\epsilon},
\]

where \( \epsilon \) are the measurement errors. Here, we can see that the parameter changes needed to match the observations come from mainly a product of two terms: the latter term in Eq. 1 is simply the scaled difference between the observations and the expected observables — the discrepancies — and the first term in Eq. 1 is the inverse of the sensitivity matrix (calculated from stellar models), expressed in its SVD form. (This matrix is: \( \frac{\partial \mathbf{B}}{\partial \mathbf{P}} / \epsilon \)). The amount by which we need to change any parameter to successfully match \( \mathbf{O} \) with \( \mathbf{B} \) comes from a product of the discrepancies in the observations, and a set of linear vectors (given by \( \mathbf{U} \) and \( \mathbf{V} \)) describing the relationship between each observable and each parameter. The rest of this paper is not dedicated to solving these equations, but rather to exploring the information contained in \( \mathbf{U} \) and \( \mathbf{V} \) to gain an understanding of the system under study.

### 2.1. Relationship vectors versus the sensitivity matrix

Let’s concentrate on a more concrete example. Imagine that the system under study is a single isolated pulsating star. The model consists of a stellar evolution and structure model, coupled to an adiabatic oscillation code. Here I use the Aarhus Stellar Evolution Code [Christensen-Dalsgaard 1982, 2007a], coupled to the ADIabatic PuLsation code [Christensen-Dalsgaard 2007b]. The \( \mathbf{P} \) are mass \( M \), age \( \tau \), initial mass fractions of the elements \( X \) and \( Z \), and a mixing-length parameter to describe the outer convection zone \( \alpha \). The \( \mathbf{B} \) are the classical observables of effective temperature \( T_{\text{eff}} \), luminosity \( L_* \) and metallicity \( [M/H] \), and the average seismic quantities: the small and large frequency separations, \( \langle \delta \nu \rangle \) and \( \langle \Delta \nu \rangle \). These quantities are the average values of the observed \( \delta \nu_{l,n} = \nu_{l,n} - \nu_{l,n+2} \) and \( \Delta \nu_{l,n} = \nu_{l,n} - \nu_{l,n-1} \) where \( \nu_{l,n} \) are the oscillation modes of degree \( l \) and radial order \( n \). The errors on each of the observations are typical: 200 K, 0.1 \( L_\odot \), 0.1, 1.3 \( \mu \text{Hz} \), 1.3\( \mu \text{Hz} \).

Table 1 shows an example of the sensitivity matrix of the system just described, taking as reference a star with parameters similar to the Sun. Each element is a partial derivative of one observable (specified in the leftmost column) with respect to one parameter (specified in the top row). For example, the partial derivative of \( L_* \) with respect to \( \tau \) is 50.5. Note that these values take \( \epsilon \) into account, and will change if we assume some other values of \( \epsilon \).

If there were a discrepancy in only one of the observables, say, \( \langle \delta \nu \rangle \), it is not so clear by reading this matrix which parameters and by how much each of these will need to change to reconcile \( \mathbf{O} \) with \( \mathbf{B} \), mainly because \( \langle \delta \nu \rangle \) is sensitive to each parameter. However, representing this matrix in its SVD form allows us to do this very simply.
Table 1. Sensitivity matrix: partial derivatives of some stellar observables with respect to the global parameters, divided by the measurement errors

|                  | M  | τ  | X  | Z  | α  |
|------------------|----|----|----|----|----|
| \( T_{\text{eff}} \) | 36.5 | 0.3 | -70.5 | -421.9 | 3.4 |
| \( L_{\star} \)    | 58.5 | 50.5 | -71.1 | -390.0 | 6.2 |
| \( \log(Z/X) \)    | 0.0  | 0.0  | -6.1  | 217.3  | 0.0 |
| \( \langle \Delta \nu \rangle \) | -3180.9 | -58.1 | 3913.82 | 18874.3 | 235.5 |
| \( \langle \delta \nu \rangle \) | -264.0 | -13.9 | 384.5 | 1628.7 | 4.18 |

Figure 1. The decomposition matrices \( U \) and \( V \), describing the relationship between the stellar observables (left panel) and the stellar parameters (right panel) for a single \( 1 \, M_{\odot} \) star.

Figure 1 is a graphical representation of the SVD of the matrix given in Table 1. In the decomposition both \( U \) and \( V \) are orthonormal matrices that span, respectively, the observable and the parameter spaces. Therefore each element is a value that varies between -1 and 1, and in Figure 1 each value is represented by a triangle, whose magnitude and direction is proportional to it.

Instead of thinking of these matrices as two separate matrices, we should interpret them rather as sets of vectors that are related according to their position in the matrix. For example, row 1 in \( U \) (vector 1 or \( U_1 \)) corresponds only to vector 1 in \( V \) and it is related by the first (highest) singular value \( W_1 \).

Now, we can investigate which of the parameters should be changed in order to reconcile the discrepancy in the observable \( \langle \delta \nu \rangle \). Inspecting the column containing this observable on the left panel, we see that the largest component is contained in the 5th row of the matrix \( U \) (\( U_5 \)). This implies that the parameter changes given by \( V_5 \) are those necessary to reconcile the discrepancy in the observation. However, \( V_5 \) (the bottom row on the right panel) has mainly one non-zero value, and this corresponds to \( \tau \). This means that in order to resolve the discrepancy in \( \langle \delta \nu \rangle \), one would need to adjust the value of \( \tau \). The amount by which \( \tau \) needs to be changed is proportional to the discrepancy in the observation.
and inversely proportional to the corresponding singular value, in this case $1/W_5$. $W_5$ is the smallest singular value, so $1/W_5$ is the largest inverse. Now, we can also see that resolving discrepancies in the observables that appear in vectors $U_1, U_2,$ etc. would cause smaller $\delta P$ than those in $U_5$ because $1/W_1, 1/W_2,$ etc. are much smaller than $1/W_5$. This implies that the parameters that appear in the topmost vectors are only allowed to be adjusted by a small amount, while those in the lowest vectors can vary more, i.e. the parameters appearing in $V_1, V_2,$ etc. have tighter constraints, and thus have the smallest uncertainties associated with them. In fact, the uncertainties $\sigma_j$ for each $j$ parameter come in a neat and compact form:

$$\sigma_j^2 = \sum_{k=1}^{N} \frac{V_{jk}^2}{W_{kk}}.$$  

(2)

Lets take as another example the observable $\langle \Delta \nu \rangle$. This observables appears mainly in $U_1$, implying that the parameter adjustments given by $V_1$ will resolve any discrepancy in this observation (if one exists). In this case, we would need to increase $M$ and decrease $X$ by the same amount, while also increasing the value of $Z$ by a larger amount. Again, the actual value by which we need to adjust these parameters is proportional to $1/W_1$ and the discrepancy in $\langle \Delta \nu \rangle$. We can now begin to understand that these matrices show quite directly which observables contribute to determining each parameter.

As a third example, if the uncertainty in the mixing-length parameter $\alpha$ were quite large and it was in our interest to decrease it, given that $\alpha$ appears mainly in $V_3$ and $V_4$, a reduction in the errors on the observables appearing in $U_3$ and $U_4$ would cause the desired decrease in $\sigma(\alpha)$ — these responsible observables are mainly $T_{\text{eff}}$ and $L_*$ (in this particular case).

One must also take some precaution with this interpretation: these results are sensitive to the observational errors that we assume, the set of observables that we take into account, and the range of parameters that we are studying. Here we assume a single star with solar characteristics, but these relationships will change if the star is at a different evolutionary stage, metallicity, and mass — for a given reference set of parameters, these relationships extend to parameter ranges where we can still assume linearity with respect to the reference set.

I showed briefly how we can interpret the decomposition matrices, and how useful the properties of SVD are for understanding the system under study. Not only can we investigate the role that each observable plays in determining the system parameters, but also SVD shows us if an observable has no important role, i.e. if the observable is redundant (in the case of an over-determined system). Being redundant implies that this observable can be used to test specific physical phenomena because they are independent of the stellar model chosen to represent the star.
Figure 2. Parameter adjustments $\delta P$ needed to most adequately fit $O$ when the model used for the inversion is incorrect. The last four parameters are rotational velocity of component A, component B, the distance to the system and the inclination. The values for $v_A$ and $v_B$ fall outside of the figure.

3. Determining the convective core overshoot parameter

Let’s suppose now that the system is a main sequence eclipsing spectroscopic binary system with component masses of roughly 1.8 and 1.1 $M_\odot$ and the 1.8 $M_\odot$ is a $\delta$ Scuti star. If we consider both photometric and spectroscopic light curve observations, then the observables will comprise of things such as the effective temperatures $T_{\text{eff}}A$, and ratio $T_{\text{eff}}B/T_{\text{eff}}A$, inclination $i$, orbital period $\Pi$, radii $R_A$, $R_B$, mass estimates $M_A\sin i$, $M_B\sin i$, and an identified oscillation mode $\nu$ from the $\delta$ Scuti component. An analysis of these observables following the discussion in Sec. 2 shows that all of the parameters of the system (now including two stars that we assume are coeval) are well-constrained (Creevey 2008b). We pose the following question: Can the single identified oscillation frequency be used to learn about the convective core overshoot parameter $\alpha_{ov}$?

In order to set about answering this question, we simulated a set of observations $O$, from a model with $\alpha_{ov} = 0.3$. We then used Eq. 1 iteratively to recover the input set of parameters, while using the correct model in the inversion. The parameters converged to the original set to within a small amount, we denote these by $P_F$. We obtain the uncertainties, $\sigma(P)$ from Eq. 2.

Now we change the inversion model to $\alpha_{ov} = 0.2$, and we try to recover the original observations using again, Eq. 2, while use the fit $P_F$ as the initial guess. Because we are using the incorrect physical model to fit the observations, then we should expect to see some non-zero values of $\delta P$. The question is, is the difference in the observables sufficient to be able to detect? We can argue that if there are $\delta P$ that are larger than the $\sigma(P)$ calculated, then we conclude that there is enough information in $B$ to detect an error in the physical assumptions.

Figure 2 shows $\delta P/\sigma(P)$ for each $P$ after the inversion with the incorrect ($\alpha = 0.2$) model. The set of observables include one identified mode. The
Figure 3. Discrepancies in $O$ when they are fit to the $\alpha_{ov} = 0.2$ model. The best set of $P$ result in large deviations for some observables.

The horizontal dotted lines represent $\pm 1-\sigma(P)$. It is clear that the eclipsing binary observables are capable of providing different $P_F$ solutions based on diverging physical assumptions — implying that there is sufficient information there to learn about $\alpha_{ov}$.

But how can we distinguish between the two fit parameter sets given that both make physical sense? Figure 3 shows the new discrepancies in the observables, where $B$ are calculated from the $\alpha_{ov} = 0.2$ models (our assumption for the inversion) with the fit parameters corresponding to those in Fig. 2. The large discrepancy in three of the observables indicates that our model is incorrect. The identified mode is the observable that provides most evidence that $\alpha_{ov} = 0.2$ is incorrect, its discrepancy is 40 times the allowed amount away from the observed value. Repeating this for various values of $\alpha_{ov}$ allows us to recover correctly the input value of 0.3.

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