Bertini for Macaulay2

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Abstract

Numerical algebraic geometry is the field of computational mathematics concerning the numerical solution of polynomial systems of equations. BERTINI, a popular software package for computational applications of this field, includes implementations of a variety of algorithms based on polynomial homotopy continuation. The MACAULAY2 package Bertini provides an interface to BERTINI, making it possible to access the core run modes of BERTINI in MACAULAY2. With these run modes, users can find approximate solutions to zero-dimensional systems and positive-dimensional systems, test numerically whether a point lies on a variety, sample numerically from a variety, and perform parameter homotopy runs.

1 Numerical algebraic geometry

Numerical algebraic geometry (numerical AG) refers to a set of methods for finding and manipulating the solution sets of systems of polynomial equations. Said differently, given $f : \mathbb{C}^N \to \mathbb{C}^n$, numerical algebraic geometry provides facilities for computing numerical approximations to isolated solutions of $V(f) = \{z \in \mathbb{C}^N | f(z) = 0\}$, as well as numerical approximations to generic points on positive-dimensional components. The book [7] provides a good introduction to the field, while the newer book [2] provides a simpler introduction as well as a complete manual for the software package BERTINI [1].

BERTINI is a free, open source software package for computations in numerical algebraic geometry. The purpose of this article is to present a MACAULAY2 package Bertini that provides an interface to BERTINI. This package uses basic datatypes and service routines for computations in numerical AG provided by the package NAGtypes. It also fits

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the framework of the *NumericalAlgebraicGeometry* package [6], a native MACAULAY2 implementation of a collection of numerical AG algorithms: most of the core functions of *NumericalAlgebraicGeometry* have an option of using BERTINI instead of the native solver.

In the remainder of this section, we very briefly describe a few fundamental concepts of the field. In the subsequent sections, we describe the various run modes of BERTINI that have been implemented in this interface. We conclude with Section 5, which describes how to use BERTINI within *NumericalAlgebraicGeometry*.

1.1 Finding isolated solutions

The core computational engine within BERTINI is *homotopy continuation*. This is a three-stage process for finding a superset of all isolated solutions in $V(f)$. Given a polynomial system $f(z)$, the three steps are as follows:

1. Choose an easily-solved polynomial system $g(z)$ that reflects the structure of $f(z)$, and solve it. Call this set of solutions $S$.

2. Form the homotopy
   \[ H(z, t) = (1 - t)f(z) + \gamma tg(z), \]
   with $\gamma \in \mathbb{C}$ a random complex number. Notice that $H(z, 1) = \gamma g(z)$, the solutions of which are known, and $H(z, 0) = f(z)$, for which we seek the solutions.

3. There is a real curve extending from each solution $z \in S$. Use predictor-corrector methods, adaptive precision, and endgames to track along all of these paths as $t$ goes from 1 to 0.

Assuming $g(z)$ is constructed in one of several canonical ways [7], there is a probability one guarantee that this procedure will result in a superset of all isolated solutions of $f(z) = 0$.

There are many variations on this general technique, and there are many minor issues to consider when implementing this method. However, due to space limitations, we leave the reader to explore the references for more information on this powerful method.

1.2 Finding irreducible components

Given an irreducible algebraic set $X$ of dimension $k$, it is well known that $X$ will intersect almost any linear space of codimension $k$ in a finite set of points. In fact, there is a Zariski open subset of the set of all linear spaces of codimension $k$ for which intersection with $X$ yields some fixed number of points, called the *degree* of $X$, $\text{deg} X$.

This fundamental fact underlies the computation of positive-dimensional irreducible components in numerical algebraic geometry. Suppose algebraic set $Z$ decomposes into irreducible components $Z_{i,j}$,

\[ Z = \bigcup_{i=0}^{\dim Z} \bigcup_{j \in \Lambda_i} Z_{i,j}, \]

where $i$ is the dimension of $Z_{i,j}$ and $j$ is just the index of component $Z_{i,j}$ in dimension $i$, stored in finite indexing set $\Lambda_i$.

In numerical algebraic geometry, the representation $W$ of an algebraic set $Z$ consists of a representation $W_{i,j}$ for each irreducible component $Z_{i,j}$ of $Z$. In particular, *witness set* $W_{i,j}$
is a triple \((f, L_i, \tilde{W}_{i,j})\), consisting of polynomial system \(f\), linear functions \(L_i\) corresponding to a linear space of codimension \(i\), and witness point set \(\tilde{W}_{i,j} = Z_{i,j} \cap V(L_i)\).

There are a variety of ways to compute \(W\), many of which are described in detail in \cite{2}. Most of these methods can be accessed through the package Bertini by using optional inputs to specify the desired algorithm.

## 2 Solving zero-dimensional systems

In the following sections we outline and give examples of the different BERTINI run modes implemented in the interface package Bertini.

### 2.1 Finding solutions of zero-dimensional systems

The method \texttt{bertiniZeroDimSolve} calls BERTINI to solve a polynomial system and returns solutions as a list of \texttt{Points} using the data types from \texttt{NAGtypes}. Diagnostic information, such as the residuals and the condition number, are stored with the coordinates of the solution and can be viewed using \texttt{peek}.

```plaintext
i1 : R=CC[x,y];
i2 : f = {x^2+y^2-1,(x-1)^2+y^2-1};
i3 : solutions=bertiniZeroDimSolve(f)
o3 = {{.5, .866025}, {.5, -.866025}}
i4 : peek solutions_0
o4 = Point{ConditionNumber => 88.2015 }
    Coordinates => {.5, .866025}
    CycleNumber => 1
    FunctionResidual => 3.66205e-15
    LastT => .000390625
    MaximumPrecision => 52
    NewtonResidual => 4.27908e-15
    SolutionNumber => 3
```

Users can specify to use regeneration, an equation-by-equation solving method, by setting the option \texttt{USEREGENERATION} to 1.

```plaintext
i5 : solutions=bertiniZeroDimSolve(f, USEREGENERATION=>1);
```

In common applications, one would like to classify solutions, e.g., separate real solutions from non-real solutions, and, thus, recomputing solutions to a higher accuracy becomes important. The method \texttt{bertiniRefineSols} calls the sharpening module of BERTINI and sharpens a list of solutions to a desired number of digits using Newton’s method.

```plaintext
i6 : refinedSols=bertiniRefineSols(f, solutions, 20);
    (coordinates refinedSols_0)_1
o6 = .86602540378443859659+3.5796948761134507351e-83*ii
```
2.2 Parameter homotopies

Many fields, such as statistics, physics, biochemistry, and engineering, have applications that require solving a large number of systems from a parameterized family of polynomial systems. In such situations, computational time can be decreased by using parameter homotopies. For an example illustrating how parameter homotopies can be used in statistics see [5].

The method \texttt{bertiniParameterHomotopy} calls \textsc{Bertini} to run both stages of a parameter homotopy. First, \textsc{Bertini} assigns a random complex number to each specified parameter and solves the resulting system, then, after this initial phase, \textsc{Bertini} computes solutions for every given choice of parameters using a number of paths equal to the exact root count.

\begin{verbatim}
R=CC[a,b,c][x,y];
f={a*x^2+b*y^2-c, y};
bertiniParameterHomotopy(f,{a,b,c},{1,1,1},{2,3,4})
\end{verbatim}

3 Solving positive-dimensional systems

Given a positive-dimensional system \(f\), the method \texttt{bertiniPosDimSolve} calls \textsc{Bertini} to compute a numerical irreducible decomposition. This decomposition is assigned the type \texttt{NumericalVariety} in \textsc{Macaulay2}. In the default settings, \textsc{Bertini} uses a classical cascade homotopy to find witness supersets in each dimension, removes extra points using a membership test or local dimension test, deflates singular witness points, then factors using a combination of monodromy and a linear trace test.

\begin{verbatim}
R = CC[x,y,z];
f = {(y^2+x^2+z^2-1)*x, (y^2+x^2+z^2-1)*y};
NV = bertiniPosDimSolve f
\end{verbatim}

\texttt{bertiniComponentMemberTest} can be used to test numerically whether a set of points \(p\) lie on the variety \(V\). For every point in \(p\), \texttt{bertiniComponentMemberTest} returns the components to which that point belongs. As for sampling, \texttt{bertiniSample} will sample from a witness set \(W\). These methods call the membership testing and sampling options in \textsc{Bertini} respectively.

\begin{verbatim}
p={{0,0,0}};
bartiniComponentMemberTest (NV, p)
\end{verbatim}
4 Solving homogeneous systems

The package \textit{Bertini} includes functionality to solve a homogenous system that defines a projective variety. In \texttt{Bertini}, the numerical computations are performed on a generic affine chart to compute representatives of projective points. To solve homogeneous equations, set the option \texttt{ISPROJECTIVE} to 1. If the user inputs a square system of $n$ homogeneous equations in $n+1$ unknowns, then the method \texttt{bertiniZeroDimSolve} outputs a list of points in projective space.

If $f$ is a positive-dimensional homogeneous system of equations, then the method \texttt{bertiniPosDimSolve} calls \texttt{Bertini} to compute a numerical irreducible decomposition of the projective variety defined by $f$.

5 Using \textit{Bertini} from \textit{NumericalAlgebraicGeometry}

The \textit{Bertini} package depends on the \textit{NAGtypes} package, a collection of basic datatypes and service routines common to all \texttt{MACAULAY2} packages for numerical AG; e.g., an interface package \texttt{[4]} to another polynomial homotopy continuation solver, \texttt{PHCpack} \texttt{[8]}, also has this dependence. The dependencies between the mentioned packages are depicted on the diagram below: dashed arrows stand for a dependency that is optional and is engaged if an executable for the corresponding software is installed.
While independent from the NumericalAlgebraicGeometry package, our interface provides a valuable option for this package: the user can set BERTINI as a default solver for homotopy continuation tasks.

i23 : needsPackage "NumericalAlgebraicGeometry";
i24 : setDefault(Software=>BERTINI)

An alternative way is to specify the Software option in a particular command:

i25 : CC[x,y]; system = {x^2+y^2-1,x-y};
i26 : sols = solveSystem(system, Software=>M2engine)
o26 = {{-.707107, -.707107}, {.707107, .707107}}
i27 : refsols = refine(system, sols, Bits=>99, Software=>BERTINI);
i28 : first coordinates first refsols
o28 = -.707106781186547524400844362105-2.1376400413426211493428955768e-140*ii
o28 : CC (of precision 100)

The unified framework for various implementations of numerical AG algorithms should be particularly convenient to a MACAULAY2 user doing numerical computations with tools from many packages.

References

[1] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Bertini: Software for numerical algebraic geometry. Available at \url{http://www.nd.edu/~sommese/bertini}.

[2] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Numerically solving polynomial systems with Bertini. To be published by SIAM, 2013.

[3] D.R. Grayson and M.E. Stillman. Macaulay2, a software system for research in algebraic geometry, available at \url{www.math.uiuc.edu/Macaulay2/}.

[4] E. Gross, S. Petrović, and J. Verschelde. Interfacing with PHCpack. J. Softw. Algebra Geom. 5, 20–25, 2013.

[5] J. Hauenstein, J. Rodriguez, and B. Sturmfels. Maximum likelihood for matrices with rank constraints, preprint, arXiv:1210.0198.

[6] A. Leykin. Numerical algebraic geometry. J. Softw. Algebra Geom. 3, 5–10, 2011.
[7] A.J. Sommese and C.W. Wampler. *The numerical solution to systems of polynomials arising in engineering and science*. World Scientific, Singapore, 2005.

[8] J. Verschelde. *Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation* ACM Trans. Math. Softw. 25(2):251–276, 1999. Software available at http://www.math.uic.edu/~jan.