The maximum number of cliques in graphs without long cycles

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Abstract

The Erdős–Gallai Theorem states that for \( k \geq 3 \) every graph on \( n \) vertices with more than \( \frac{1}{2}(k-1)(n-1) \) edges contains a cycle of length at least \( k \). Kopylov proved a strengthening of this result for 2-connected graphs with extremal examples \( H_{n,k,t} \) and \( H_{n,k,2} \). In this note, we generalize the result of Kopylov to bound the number of \( s \)-cliques in a graph with circumference less than \( k \). Furthermore, we show that the same extremal examples that maximize the number of edges also maximize the number of cliques of any fixed size. Finally, we obtain the extremal number of \( s \)-cliques in a graph with no path on \( k \)-vertices.

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1 Introduction

In [4], Erdős and Gallai determined \( ex(n, P_k) \), the maximum number of edges in an \( n \)-vertex graph that does not contain a copy of the path on \( k \) vertices, \( P_k \). This result was a corollary of the following theorem:

**Theorem 1.1** (Erdős and Gallai [4]). *Let \( G \) be an \( n \)-vertex graph with more than \( \frac{1}{2}(k-1)(n-1) \) edges, \( k \geq 3 \). Then \( G \) contains a cycle of length at least \( k \).*

To obtain the result for paths, suppose \( G \) is an \( n \)-vertex graph with no copy of \( P_k \). Add a new vertex \( v \) adjacent to all vertices in \( G \), and let this new graph be \( G' \). Then \( G' \) is an \( n+1 \)-vertex graph with no cycle of length \( k+1 \) or longer, and so \( e(G) + n = e(G') \leq \frac{1}{2}kn \) edges.

**Corollary 1.2** (Erdős and Gallai [4]). *Let \( G \) be an \( n \)-vertex graph with more than \( \frac{1}{2}(k-2)n \) edges, \( k \geq 2 \). Then \( G \) contains a copy of \( P_k \).*

Both results are sharp with the following extremal examples: for Theorem 1.1 when \( k-2 \) divides \( n-1 \), take any connected \( n \)-vertex graph whose blocks (maximal connected subgraphs with no cut vertices) are cliques of order \( k-1 \). For Corollary 1.2 when \( k-1 \) divides \( n-1 \), take the \( n \)-vertex graph whose connected components are cliques of order \( k-1 \).

There have been several alternate proofs and sharpenings of the Erdős-Gallai theorem including results by Woodall [15], Lewin [13], Faudree and Schelp [5, 6], and Kopylov [12] – see [8] for further details.

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The strongest version was that of Kopylov who improved the Erdős–Gallai bound for 2-connected graphs. To state the theorem, we first introduce the family of extremal graphs.

Fix \( k \geq 4, n \geq k, \frac{k}{2} > a \geq 1 \). Define the \( n \)-vertex graph \( H_{n,k,a} \) as follows. The vertex set of \( H_{n,k,a} \) is partitioned into three sets \( A, B, C \) such that \( |A| = a, |B| = n - k + a \) and \( |C| = k - 2a \) and the edge set of \( H_{n,k,a} \) consists of all edges between \( A \) and \( B \) together with all edges in \( A \cup C \).

Note that when \( a \geq 2 \), \( H_{n,k,a} \) is 2-connected, has no cycle of length \( k \) or longer, and 
\[
e(H_{n,k,a}) = \binom{k-a}{2} + (n - k + a)a.
\]

**Definition.** Let \( f_s(n, k, a) := \binom{k-a}{s} + (n - k + a)\binom{a}{s-1} \), where \( f_2(n, k, a) = e(H_{n,k,a}) \).

By considering the second derivative, one can check that \( f_s(n, k, a) \) is convex in \( a \) in the domain \([1, \lceil(k-1)/2\rceil]\), thus it attains its maximum at one of the endpoints \( a = 1 \) or \( a = \lceil(k-1)/2\rceil \).

**Theorem 1.3** (Kopylov [12]). Let \( n \geq k \geq 5 \) and let \( t = \lfloor k-1/2 \rfloor \). If \( G \) is a 2-connected \( n \)-vertex graph with
\[
e(G) \geq \max\{f_2(n, k, 2), f_2(n, k, t)\},
\]
then either \( G \) has a cycle of length at least \( k \), or \( G = H_{n,k,2} \), or \( G = H_{n,k,t} \).

It is straightforward to check that any 2-connected graph that is not a triangle has a cycle of length 4 or greater, and so the theorem covers all nontrivial choices of \( k \). This theorem also implies Theorem 1.1 by applying induction to each block of the graph.

We consider a generalized Turán-type problem. Fix graphs \( T \) and \( H \), and define the function \( ex(n, T, H) \) to be the maximum number of (unlabeled) copies of \( T \) in an \( H \)-free graph on \( n \) vertices. When \( T = K_2 \), we have the usual extremal number \( ex(n, T, H) = ex(n, H) \).

There are many notable papers studying the \( ex(n, T, H) \) function for different combinations of \( T \) and \( H \). Erdős [3] proved that for \( s \leq r \), among all \( n \)-vertex graphs that forbid \( K_{r+1} \), the Turán
graph (i.e., the balanced complete r-partite graph) maximizes the number of copies of \(K_s\). Hatami, Hladký, Král’, Norine, and Razborov \([11]\) and independently Grzesik \([9]\) proved \(ex(n, C_5, K_3) = (n/5)^5\) whenever \(n\) is divisible by 5 using the method of flag algebras. On the other hand, Bollobás and Győr [2] proved \((1+o(1)) \frac{1}{3\sqrt{3}} n^{3/2} \leq ex(n, K_3, C_5) \leq (1+o(1)) \frac{5}{4} n^{3/2}\), and later Győri and Li \([10]\) proved an upper bound for \(ex(n, K_3, C_{2k+1})\) in terms of \(ex(n, C_{2k})\). This bound was improved by Füredi and Özkahya \([7]\) and then later improved again by Alon and Shikhelman \([1]\). In the same paper, Alon and Shikhelman proved \(ex(n, K_s, K_r, t) = \Theta(n^{s-1}/r)\) for certain values of \(r, s,\) and \(t\), among other results.

Furthermore, such generalized Turán-type results for graphs can be instrumental for proving related extremal results in hypergraphs. For example, Füredi and Özkahya \([7]\) used their upper bounds for the number of triangles in graphs without cycles of fixed lengths to give an upper bound for the number of hyperedges in 3-uniform hypergraphs without Berge-cycles of a fixed length.

In this note, we give an upper bound for the number of \(s\)-cliques in a graph without cycles of length \(k\) or greater (i.e., circumference less than \(k\)). We also obtain \(ex(n, K_s, P_k)\).

**Definition.** For \(s \geq 2\), let \(N_s(G)\) denote the number of unlabeled copies of \(K_s\) in \(G\), e.g., \(N_2(G) = e(G)\).

Our main result is a generalization of Kopylov’s result, Theorem \([1.3]\). In particular, we show that the same extremal examples that maximize the number of edges among \(n\)-vertex 2-connected graphs with circumference less than \(k\) also maximize the number of cliques of any size. Our main results are the following:

**Theorem 1.4.** Let \(n \geq k \geq 5\) and let \(t = \lfloor \frac{k-1}{2} \rfloor\). If \(G\) is a 2-connected \(n\)-vertex graph with circumference less than \(k\), then

\[
N_s(G) \leq \max\{f_s(n, k, 2), f_s(n, k, t)\}.
\]

Again, this theorem is sharp with the same extremal examples \(H_{n,k,2}\) and \(H_{n,k,t}\).

This theorem implies the cliques version of Theorem \([1.1]\)

**Corollary 1.5.** Let \(n \geq k \geq 4\). If \(G\) is an \(n\)-vertex graph with circumference less than \(k\), then

\[
N_s(G) \leq \frac{n-1}{k-2} \binom{k-1}{s}.
\]

Unlike the edges case, Theorem \([1.4]\) unfortunately does not easily imply \(ex(n, K_s, P_k)\). However, a Kopylov-style argument very similar to the proof of Theorem \([1.4]\) gives the result for paths.

**Theorem 1.6.** Let \(n \geq k \geq 4\) and let \(G\) be an \(n\)-vertex connected graph with no path on \(k\) vertices. Let \(t = \lfloor (k-2)/2 \rfloor\). Then \(N_s(G) \leq \max\{f_s(n, k-1, 1), f_s(n, k-1, t)\} \).

We have sharpness examples \(H_{n,k-1,1}\) and \(H_{n,k-1,t}\). Finally, using induction on the number of components gives the following result:

**Corollary 1.7.** \(ex(n, K_s, P_k) = \frac{n}{k-1} \binom{k-1}{s} \).

And the same extremal examples as for Corollary \([1.2]\) apply.
The proofs for Corollary 1.5, Theorem 1.6 and Theorem 1.7 are given in Section 3 of this paper. We first prove Theorem 1.4.

2 Proof of Theorem 1.4

Let \( G \) be an edge-maximal counterexample. Then \( G \) is \( k \)-closed, i.e., adding any additional edge to \( G \) creates a cycle of length at least \( k \). In particular, for any nonadjacent vertices \( x \) and \( y \) of \( G \), there exists a path of at least \( k - 1 \) edges between \( x \) and \( y \). We will use the following lemma:

Lemma 2.1 (Kopylov [12]). Let \( G \) be a 2-connected \( n \)-vertex graph with a path \( P \) of \( m \) edges with endpoints \( x \) and \( y \). For \( v \in V(G) \), let \( d_P(v) = |N(v) \cap V(P)| \). Then \( G \) contains a cycle of length at least \( \min\{m + 1, d_P(x) + d_P(y)\} \).

Our first goal is to show that \( G \) contains a large “core”, i.e., a subgraph with large minimum degree. For this, we use the notion of disintegration.

Definition: For a natural number \( \alpha \) and a graph \( G \), the \( \alpha \)-disintegration of a graph \( G \) is the process of iteratively removing from \( G \) the vertices with degree at most \( \alpha \) until the resulting graph has minimum degree at least \( \alpha + 1 \) or is empty. This resulting subgraph \( H = H(G, \alpha) \) will be called the \((\alpha + 1)\)-core of \( G \). It is well known that \( H(G, \alpha) \) is unique and does not depend on the order of vertex deletion (for instance, see [14]).

Let \( H(G, t) \) denote the \((t + 1)\)-core of \( G \), i.e., the resulting graph of applying \( t \)-disintegration to \( G \). We claim that \( H(G, t) \) is nonempty.

Suppose \( H(G, t) \) is empty. In the disintegration process, every time a vertex of degree at most \( t \) is removed, we delete at most \((\frac{t}{s-1})\) copies of \( K_s \). For the last \( \ell \leq t \) vertices, we remove at most \((\frac{\ell-1}{s-1})\) copies of \( K_s \) with each deletion. Thus

\[
N_s(G) \leq (n-t)\binom{t}{s-1} + (t-1)\binom{s-1}{s-1} + \cdots + \binom{0}{s-1}
\]

\[
= (n-t)\binom{t}{s-1} + \binom{t}{s-1} + \cdots + \binom{t-1}{s-1}
\]

\[
= (n-(t+1))\binom{t}{s-1} + \binom{t+1}{s-1}
\]

\[
\leq f_s(n,k,t),
\]

a contradiction.

Therefore \( H(G, t) \) is nonempty. Next we show that

\( H(G, t) \) is a complete graph.

If there exists a nonedge of \( H(G, t) \), then in \( G \), there is a path of length at least \( k - 1 \) edges with these vertices as its endpoints. Among all nonadjacent pairs of vertices in \( H(G, t) \), choose \( x, y \) such that there is a longest path \( P \) in \( G \) with endpoints \( x \) and \( y \). By maximality of \( P \), all neighbors
of \(x\) in \(H(G, t)\) lie in \(P\): if \(x\) has a neighbor \(x' \in H(G, t) - P\), then either \(x'y \in E(G)\) and \(x'P\) is a cycle of length at least \(k\), or \(x'y \notin E(G)\) and so \(x'P\) is a longer path. Similar for \(y\). Hence, by Lemma 2.1, \(G\) has a cycle of length at least \(\min\{k, d_P(x) + d_P(y)\} = \min\{k, 2(t + 1)\} = k\), a contradiction.

Now let \(r = |V(H(G, t))|\). Each vertex in \(H(G, t)\) has degree at least \(t + 1\), so \(r \geq t + 2\). Also, if \(r \geq k - 1\), as \(G\) is 2-connected and \(H(G, t)\) is a clique, we can extend a path on \(r\) vertices of \(H(G, t)\) to a cycle of length at least \(r + 1 \geq k\), a contradiction. Therefore \(t + 2 \leq r \leq k - 2\). In particular, \(2 \leq k - r \leq t\). Apply \((k-r)\)-disintegration to \(G\), and let \(H(G, k - r)\) be the resulting graph. Then \(H(G, t) \subseteq H(G, k - r)\).

If \(H(G, t) = H(G, k - r)\), then

\[
N_s(G) \leq \binom{r}{s} + (n - r)\binom{k - r}{s - 1} = f_s(n, k, k - r) \leq \max\{f_s(n, k, 2), f_s(n, k, t)\}
\]

by the convexity of \(f_s\). Therefore, \(H(G, t)\) is a proper subgraph of \(H(G, k - r)\), and there must be a nonedge between a vertex in \(H(G, t)\) and a vertex in \(H(G, k - r)\). Among all such pairs, choose \(x \in H(G, t)\) and \(y \in H(G, k - r)\) to have a longest path \(P\) between them. As before, \(P\) contains at least \(k - 1\) edges, and each neighbor of \(x\) in \(H(G, t)\) and each neighbor of \(y\) in \(H(G, k - r)\) lie in \(P\). Then \(G\) contains a cycle of length at least \(\min\{k, (r - 1) + (k - r + 1)\} = k\), a contradiction.  

\section{Proof of Corollary 1.5, Theorem 1.6, and Corollary 1.7}

\textbf{Proof of Corollary 1.5} Define \(g_s(n, k) = \frac{n - 1}{k - 2}\binom{k - 1}{s}\) and \(t = \lfloor\frac{k - 1}{2}\rfloor\). One can check that when \(n \geq k\),

\[
g_s(n, k) \geq \max\{f_s(n, k, t), f_s(n, k, 2)\}.
\]

Fix a graph \(G\) on \(n\) vertices with circumference less than \(k\). If \(G\) is disconnected, simply apply induction to each component of \(G\) to obtain the desired result. Therefore we may assume \(G\) is connected. We induct on the number of blocks of \(G\). First suppose \(k \geq 5\). If \(G\) is a block, i.e., 2-connected, then either \(n \leq k - 1\), and so \(N_s(G) \leq \binom{|V(G)|}{s} \leq g_s(n, k)\), or \(n \geq k\), and so by Theorem 1.4 \(N_s(G) \leq \max\{f_s(n, k, t), f_s(n, k, 2)\} \leq g_s(n, k)\).

Otherwise, consider the \textit{block-cut tree} of \(G\)—the tree whose vertices correspond to blocks of \(G\) such that two vertices in the tree are adjacent if and only if the corresponding blocks in \(G\) share a vertex. Let \(B_1\) be a block in \(G\) corresponding to a leaf-vertex in the block-cut tree such that \(B_1\) and its complement are connected by the cut vertex \(v\). Set \(B_2 = G - B_1 + \{v\}\). Apply the induction hypothesis to \(B_1\) and \(B_2\) to obtain

\[
N_s(G) = N_s(B_1) + N_s(B_2) \leq g_s(|B_1|, k) + g_s(n - |B_1| + 1, k) = \frac{|B_1| - 1}{k - 2}\binom{k - 1}{s} + \frac{(n - |B_1| + 1) - 1}{k - 2}\binom{k - 1}{s} = g_s(n, k).
\]

If \(k = 4\), then either \(G\) is a forest or \(G\) has circumference 3. In the second case, each block of \(G\) is
either a triangle or an edge. Thus \( N_s(G) \leq g_s(n,k) \) in both cases. \qed

The proof of Theorem 1.6 follows the same steps as the proof of Theorem 1.4. As some details here will be omitted to prevent repetition, it is advised that the reader first reads the proof of Theorem 1.4.

**Proof of Theorem 1.6.** Suppose for contradiction that \( N_s(G) > \max\{f_s(n,k-1,1), f_s(n,k-1,t)\} \) where \( t = \lceil (k-2)/2 \rceil \). Let \( G_0 \) be the graph obtained by adding a dominating vertex \( v_0 \) adjacent to all of \( V(G) \). Then \( G_0 \) is 2-connected, has \( n+1 \) vertices, and contains no cycle of length \( k+1 \) or greater. Let \( G' \) be the \( k+1 \)-closure of \( G_0 \) (i.e., add edges to \( G_0 \) until any additional edge creates a cycle of length at least \( k+1 \)). Denote by \( N'_s(G') \) the number of \( K_s \)'s in \( G' \) that do not contain \( v_0 \). Thus \( N'_s(G') \geq N'_s(G_0) = N_s(G) \). Apply \((t+1)\)-disintegration to \( G' \), where if necessary, we delete \( v_0 \) last. Let \( H(G', t+1) \) be the resulting graph of the disintegration. If \( H(G', t+1) \) is empty, then at the time of deletion each vertex has at most \( t \) neighbors that are not \( v_0 \). Hence

\[
N'_s(G') \leq (n-(t+1)) \left( \frac{t}{s-1} \right) + \left( \frac{t+1}{s} \right) \leq f_s(n,k-1,t),
\]

a contradiction.

The same argument as in the proof of Theorem 1.4 also shows that \( H(G', t+1) \) is a complete graph, otherwise there would be a cycle of length at least \( 2(t+2) \geq (k-1) + 2 \) in \( G' \). Note that \( v_0 \) must be contained in \( H(G', t+1) \) as it is adjacent to all vertices in \( G' \). Set \( |V(H(G', t+1))| = r \) where \( t+3 \leq r \leq k-1 \) (and so \( k - r \geq 1 \)). In particular, \((k+1) - r \leq t+1 \). Apply \((k+1-r)\)-disintegration to \( G' \). If \( H(G', t+1) \neq H(G', k+1-r) \), then again we can find a cycle of length at least \((r-1) + k+2 - r = k+1 \). Otherwise, suppose \( H(G', t+1) = H(G', k+1-r) \). In \( H(G', t+1) \), the number of \( s \)-cliques that do not include \( v_0 \) is \( \left( \frac{r-1}{s} \right) \), and in \( V(G) - V(H(G', k+1-r)) \), every vertex had at most \( k-r \) neighbors that were not \( v_0 \) at the time of its deletion. We have

\[
N'_s(G') \leq \left( \frac{r-1}{s} \right) + (n+1-r) \left( \frac{k-r}{s-1} \right) = f_s(n,k-1,k-r) \leq \max\{f_s(n,k-1,1), f_s(n,k-1,t)\},
\]

a contradiction. \qed

**Proof of Corollary 1.7.** Define \( h_s(n,k) = \frac{n}{k-1} \left( \frac{k-1}{s} \right) \), and note that when \( n \geq k \),

\[
h_s(n,k) \geq \max\{f_s(n,k-1,t), f_s(n,k-1,1)\}.
\]

We induct on the number of components in \( G \). First suppose \( k \geq 4 \). If \( G \) is connected, then either \( n \leq k-1 \), in which case \( N_s(G) \leq \left( \frac{|V(G)|}{s} \right) \leq h_s(n,k) \), or \( n \geq k \) and \( N_s(G) \leq \max\{f_s(n,k-1,1), f_s(n,k-1,t)\} \leq h_s(n,k) \). Otherwise if \( G \) is not connected, let \( C_1 \) be a component of \( G \). Then \( N_s(G) = N_s(C_1) + N_s(G - C_1) \leq h_s(|C_1|,k) + h_s(n-|C_1|,k) = h_s(n,k) \).

If \( k = 3 \) (the cases \( k \leq 2 \) are not interesting), then the longest path in \( G \) has two vertices. It follows that \( G \) is the union of a matching and isolated vertices. Therefore \( N_s(G) \leq h_s(n,k) \). \qed
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References

[1] N. Alon and C. Shikhelman, Many $T$ copies in $H$-free graphs, J. Combin. Theory Ser. B. 121 (2016), 146–172.

[2] B. Bollobás and E. Győri, Pentagons vs. triangles, Discrete Math. 308 (2008), 4332–4336.

[3] P. Erdős, On the number of complete subgraphs contained in certain graphs, Magyar Tud. Akad. Mat. Kut. Int. Közl, 7 (1962), 459-474.

[4] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar. 10 (1959), 337–356.

[5] R. J. Faudree and R. H. Schelp, Ramsey type results, Infinite and Finite Sets, Colloq. Math. J. Bolyai 10, (ed. A. Hajnal et al.), North-Holland, Amsterdam, 1975, pp. 657–665.

[6] R. J. Faudree and R. H. Schelp, Path Ramsey numbers in multicolorings, J. Combin. Theory Ser. B. 19 (1975), 150–160.

[7] Z. Füredi and L. Özkahya, On 3-uniform hypergraphs without a cycle of a given length, Discrete Applied Mathematics 216, Part 3, (2017), 582-588.

[8] Z. Füredi and M. Simonovits, The history of degenerate (bipartite) extremal graph problems, Bolyai Math. Studies 25 pp. 169–264, Erdős Centennial (L. Lovász, I. Ruzsa, and V. T. Sós, Eds.) Springer, 2013. Also see: arXiv:1306.5167.

[9] A. Grzesik, On the maximum number of five-cycles in a triangle-free graph, J. Combin. Theory Ser. B. 102.5 (2012), 1061-1066.

[10] E. Győri and H. Li, The maximum number of triangles in $C_{2k+1}$-free graphs, Combinatorics, Probability and Computing 21(1-2), (2012), 187-191.

[11] H. Hatami, J. Hladký, D. Král’, S. Norine, and A. Razborov, On the number of pentagons in triangle-free graphs, J. Combin. Theory Ser. A. 120 (2013) no. 3, 722–732.

[12] G. N. Kopylov, Maximal paths and cycles in a graph, Dokl. Akad. Nauk SSSR 234 (1977), 19–21. (English translation: Soviet Math. Dokl. 18 (1977), no. 3, 593–596.)

[13] M. Lewin, On maximal circuits in directed graphs, J. Combin. Theory Ser. B. 18 (1975), 175–179.

[14] B. Pittel, J. Spencer, and N. Wormald, Sudden emergence of a giant $k$-core in a random graph, J. Combin. Theory Ser. B. 67 (1996), 111151.

[15] D. R. Woodall, Maximal circuits of graphs I, Acta Math. Acad. Sci. Hungar. 28 (1976), 77–80.