Path Dependent McKean-Vlasov SDEs with Hölder Continuous Diffusion*

Xing Huang a), Xucheng Wang a)

a)Center for Applied Mathematics, Tianjin University, Tianjin 300072, China
xinghuang@tju.edu.cn, wxc3018233022@163.com

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Abstract

In this paper, the well-posedness for one-dimensional path dependent McKean-Vlasov SDEs with $\alpha(\alpha \geq \frac{1}{2})$-Hölder continuous diffusion is investigated. Moreover, the associated quantitative propagation of chaos in the sense of Wasserstein distance, total variation distance as well as relative entropy is studied.

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1 Introduction

Distribution dependent SDEs can be used to characterize the nonlinear Fokker-Planck-Kolmogorov equations. They are also called McKean-Vlasov SDEs due to the pioneer work in [18]. On the other hand, McKean-Vlasov SDE can be viewed as the limit equation of a single particle in the mean field interacting particle system, which is related to the propagation of chaos [24], so it is also called mean field SDE. Recently, there are plentiful results on McKean-Vlasov SDEs. With respect to the well-posedness, one can refer to [1, 4, 5, 12, 13, 19, 23, 25] and references therein, see also [14] for the path dependent case with singular drifts. In [4, 5, 12, 13, 23], the diffusion is assumed to be uniformly elliptic. For the propagation of chaos, see [2, 3, 6, 8, 9, 11, 15, 17, 24, 27]. One can also refer to [10, 22, 26] for the long time behavior of mean field interacting particle system and McKean-Vlasov SDEs.

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The aim of this paper is to investigate the well-posedness and propagation of chaos of one-dimensional path dependent McKean-Vlasov SDEs with $\alpha(\alpha \geq \frac{1}{2})$-Hölder continuous diffusion. With respect to the well-posedness, we do not assume that the diffusion is elliptic.

Throughout the paper, fix a constant $r > 0$. Let $C = C([-r, 0]; \mathbb{R})$. For any $f \in C([-r, \infty); \mathbb{R})$, $t \geq 0$, define $f_t \in C$ as $f_t(s) = f(t + s), s \in [-r, 0]$, which is called the segment process. Let $\mathcal{P}(\mathbb{R})$ be the set of all probability measures in $\mathbb{R}$ equipped with the weak topology. Define $\mathcal{P}_1(\mathbb{R}) = \{\mu \in \mathcal{P}(\mathbb{R}) : \mu(|\cdot|) < \infty\}$.

It is well known that $\mathcal{P}_1(\mathbb{R})$ is a Polish space under the Wasserstein distance

$$\mathbb{W}_1(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left( \int_{\mathbb{R} \times \mathbb{R}} |x - y| \pi(dx, dy) \right), \ \mu, \nu \in \mathcal{P}_1(\mathbb{R}),$$

where $\mathcal{C}(\mu, \nu)$ is the set of all couplings of $\mu$ and $\nu$. By the adjoint formula, it holds

$$\mathbb{W}_1(\mu, \nu) = \sup_{\|f\|_{\text{lip}} \leq 1} |\mu(f) - \nu(f)|,$$

where

$$\|f\|_{\text{lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Recall that for two probability measures $\mu, \nu$ on some measurable space $(E, \mathcal{E})$, the entropy and total variation distance are defined as follows:

$$\text{Ent}(\nu|\mu) := \begin{cases} \int_E (\log \frac{d\nu}{d\mu})d\nu, & \text{if } \nu \text{ is absolutely continuous with respect to } \mu, \\ \infty, & \text{otherwise}, \end{cases}$$

and

$$\|\mu - \nu\|_{\text{var}} := \sup_{|f| \leq 1} |\mu(f) - \nu(f)|.$$

By Pinsker’s inequality (see [20]),

$$(1.1) \quad \|\mu - \nu\|_{\text{var}}^2 \leq 2\text{Ent}(\nu|\mu), \ \mu, \nu \in \mathcal{P}(E),$$

here $\mathcal{P}(E)$ denotes all probability measures on $(E, \mathcal{E})$.

Let $W = (W(t))_{t \geq 0}$ be a one-dimensional standard Brownian motion on a complete filtration probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Consider the following one-dimensional path dependent McKean-Vlasov SDE:

$$(1.2) \quad dX(t) = b(t, X(t), \mathcal{L}_X(t))dt + B(t, X_t, \mathcal{L}_X(t))dt + \sigma(t, X(t))dW(t),$$

where $b : [0, \infty) \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$, $B : [0, \infty) \times \mathcal{C} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$, $\sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are measurable and $X_0$ is an $\mathcal{F}_0$-measurable $\mathcal{C}$-valued random variable. Define the uniform norm $\|\xi\|_{\infty} := \sup_{s \in [-r, 0]} |\xi(s)|, \xi \in \mathcal{C}$.
Definition 1.1. A continuous process \((X(t))_{t \geq -r}\) on \(\mathbb{R}\) is called a strong solution of (1.2) in \(\mathcal{P}_1(\mathbb{R})\), if for any \(t \geq 0\), \(X(t)\) is \(\mathcal{F}_t\)-measurable, \(\mathbb{E}\|X_t\|_{\infty} < \infty\), and \(\mathbb{P}\)-a.s.

\[
X(t) = X(0) + \int_0^t (b(s, X(s), \mathcal{L}_{X(s)}) + B(s, X_s, \mathcal{L}_{X(s)}))ds + \int_0^t \sigma(s, X(s))dW(s), \quad t \geq 0.
\]

Throughout the paper, we fix \(T > 0\), and consider the solution on \([0, T]\).

The remainder of this paper is organized as follows: In Section 2, the strong well-posedness of path dependent classical SDEs is addressed by Yamada-Watanabe’s approximation; In Section 3, the well-posedness and quantitative propagation of chaos for path dependent McKean-Vlasov SDEs are investigated.

2 Multi-dimensional Path Dependent Classical SDEs with Hölder Continuous Diffusion

In this section, we study the well-posedness of multi-dimensional path dependent classical SDEs with Hölder continuous diffusion. The main tool is the Yamada-Watanabe approximation, see [16]. Before going on, we state a time nonhomogeneous version of [21, Theorem 2.3]. More precisely, consider path dependent SDE on \(\mathbb{R}^m\):

\[
(2.1) \quad dX(t) = f(t, X_t)dt + g(t, X_t)d\tilde{W}(t), \quad X_0 = \xi \in \mathcal{C}^m,
\]

here \(f : [0, T] \times \mathcal{C}^m \to \mathbb{R}^m\), \(g : [0, T] \times \mathcal{C}^m \to \mathbb{R}^m \otimes \mathbb{R}^d\) and \(\tilde{W}(t)\) is a \(d\)-dimensional standard Brownian motion on a complete filtration probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})\).

Proposition 2.1. Assume that \(f\) and \(g\) are bounded on bounded sets. Suppose that for any \(t \in [0, T]\), \(f(t, \cdot)\) and \(g(t, \cdot)\) are continuous in \((\mathcal{C}^m, \|\cdot\|_{\infty})\) and there exists a constant \(K \in \mathbb{R}\) such that

\[
2\langle f(t, \xi) - f(t, \eta), \xi(0) - \eta(0) \rangle + \|g(t, \xi) - g(t, \eta)\|_{HS}^2 \leq K\|\xi - \eta\|_{\infty}^2, \quad t \in [0, T], \xi, \eta \in \mathcal{C}^m.
\]

Then (2.1) has a unique non-explosive strong solution on \([0, T]\).

Since the proof of Proposition 2.1 is completely the same with that of [21, Theorem 2.3], we omit it here.

The following stochastic Gronwall lemma comes from [21, Lemma 5.2], which is crucial in the proof of the main result of this section.

Lemma 2.2. Let \(Z\) be a continuous adapted non-negative stochastic process which satisfies the inequality

\[
Z(t) \leq K \int_0^t \sup_{u \in [0, s]} Z(u)ds + M(t) + C, \quad t \geq 0,
\]

where \(C \geq 0\), \(K > 0\) and \(M\) is a continuous local martingale with \(M(0) = 0\). Then for any \(p \in (0, 1)\), there exist finite constants \(c_1(p), c_2(p)\) (not depending on \(K, C, T\) and \(M\)) such that

\[
\mathbb{E}(\sup_{s \in [0, t]} |Z(s)|^p) \leq C^p c_1(p) e^{c_2(p)Kt}, \quad t \geq 0.
\]
For $x \in \mathbb{R}^d$, we denote $x_i$ as the $i$-th component of $x$, that is $x = (x_1, x_2, \cdots, x_d)$. Consider

$$
(2.2) \quad dX(t) = F(t, X(t))dt + H(t, X(t))dt + G(t, X(t))dW(t), \quad X_0 = \xi \in \mathcal{C}^d,
$$

where $F = (F_1, F_2, \cdots, F_d) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $H = (H_1, H_2, \cdots, H_d) : [0, T] \times \mathcal{C}^d \to \mathbb{R}^d$, $G : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$. We make the following assumption.

(A1) $F$ is locally bounded in $[0, T] \times \mathbb{R}^d$. For any $t \in [0, T]$, $F(t, \cdot)$ is continuous, and there exists a constant $K_1 \in \mathbb{R}$ such that

$$(F_i(t, x) - F_i(t, y))\text{sgn}(x_i - y_i) \leq K_1|x - y|, \quad t \in [0, T], x, y \in \mathbb{R}^d, 1 \leq i \leq d.$$

(A2) There exist $m$ real valued functions $(G_i)_{1 \leq i \leq d}$ on $[0, T] \times \mathbb{R}$ such that

$$G(t, x) = \text{diag}(G_1(t, x_1), G_2(t, x_2), \cdots, G_1(t, x_d)), \quad x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d, t \in [0, T].$$

Moreover, there exist constants $(a_i)_{1 \leq i \leq d} \subset [\frac{1}{2}, 1]$ and $K_2 \geq 0$ such that

$$|G_i(t, z) - G_i(t, \bar{z})| \leq K_2|z - \bar{z}|^{a_i}, \quad |G_i(t, 0)| \leq K_2, \quad z, \bar{z} \in \mathbb{R}, t \in [0, T], 1 \leq i \leq d.$$

(A3) There exists a constant $K_3 \geq 0$ such that

$$|H(t, \xi) - H(t, \eta)| \leq K_3\|\xi - \eta\|_{\infty}, \quad |H(t, 0)| \leq K_3, \quad \xi, \eta \in \mathcal{C}^d, t \in [0, T].$$

Now, we provide the main result in this section.

**Theorem 2.3.** Assume (A1)-(A3). Then for any $\xi \in \mathcal{C}^d$, (2.2) has a unique strong solution $(X^\xi(t))_{t \in [-r, T]}$ with initial value $X_0^\xi = \xi$ and for any $q \in (0, 2]$, there exists a constant $C(T, q) > 0$ such that

$$
(2.3) \quad \mathbb{E} \sup_{t \in [-r, T]} |X^\xi(t)|^q \leq C(T, q)(1 + \|\xi\|_\infty^q).
$$

Moreover, for any $p \in (0, 1)$,

$$
(2.4) \quad \mathbb{E} \sup_{t \in [-r, T]} |X^\xi(t) - X^n(t)|^p \leq C(p, T)\|\xi - \eta\|_\infty^p
$$

for some constant $C(p, T) > 0$.

**Proof.** Step 1. Existence of the strong solution.

For $\varepsilon \in (0, 1)$, note $\int_{\epsilon/e^{\frac{1}{2}}}^\varepsilon \frac{1}{x} dx = 1$, so there exists a continuous function $\psi_\varepsilon : [0, \infty) \to [0, \infty)$ with support $[\varepsilon/e^{\frac{1}{2}}, \varepsilon]$ such that

$$0 \leq \psi_\varepsilon(x) \leq \frac{2\varepsilon}{x}, \quad x \in [\varepsilon/e^{\frac{1}{2}}, \varepsilon], \quad \int_{\varepsilon/e^{\frac{1}{2}}}^\varepsilon \psi_\varepsilon(u) du = 1.$$
Let
\[ V_\varepsilon(x) := \int_0^{|x|} \int_0^y \psi_\varepsilon(z) dz dy, \quad x \in \mathbb{R}. \]
Then \( V_\varepsilon \in C^2 \),
\begin{equation}
|x| - \varepsilon \leq V_\varepsilon(x) \leq |x|, \quad 0 \leq \text{sgn}(x)V_\varepsilon'(x) \leq 1, \quad x \in \mathbb{R}, \tag{2.5}
\end{equation}
and
\begin{equation}
0 \leq V_\varepsilon''(x) \leq \frac{2\varepsilon}{|x|} 1_{[\varepsilon/\alpha^i, \varepsilon]}(|x|), \quad x \in \mathbb{R}. \tag{2.6}
\end{equation}
Let \( \rho \in C_0^\infty(\mathbb{R}) \) with \( \rho \geq 0 \) and \( \int_{\mathbb{R}} \rho(x) dx = 1 \) be supported in \([-1, 1]\). For any \( n \geq 1 \), define
\[ \rho^n(x) = n \rho(nx), \quad x \in \mathbb{R} \]
and
\[ G^n_i(t, \cdot) = G_i(t, \cdot) * \rho^n, \quad 1 \leq i \leq d, \quad t \in [0, T] \]
and
\[ G^n(t, x) = \text{diag}(G^n_1(t, x_1), G^n_2(t, x_2), \ldots, G^n_d(t, x_d)), \quad x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d, \quad t \in [0, T]. \]
(A2) implies that
\begin{equation}
\lim_{n \to \infty} \sup_{t \in [0,T], x \in \mathbb{R}} |G^n_i(t, x) - G_i(t, x)| \leq \int_{\mathbb{R}} |y|^{\alpha_i} \rho^n(y) dy \leq K_2 \int_{\mathbb{R}} |y|^{\alpha_i} \rho(y) dy = K_2 \int_{\mathbb{R}} \frac{1}{n^{\alpha_i}} |x|^{\alpha_i} \rho(x) dx = 0, \quad 1 \leq i \leq d \tag{2.7}
\end{equation}
and for any \( n \geq 1 \), there exists a constant \( K_n \geq 0 \) such that
\begin{equation}
|G^n(t, x) - G^n(t, y)| \leq K_n |x - y|, \quad |G^n(t, 0)| \leq 2\sqrt{d} K_2, \quad t \in [0, T], x, y \in \mathbb{R}^d, \tag{2.8}
\end{equation}
where for the second inequality, it is sufficient to note that
\[ |G^n_i(t, 0)| \leq \int_{\mathbb{R}} |G_i(t, -y)| n \rho(ny) dy \leq K_2 \int_{\mathbb{R}} |y|^{\alpha_i} n \rho(ny) dy + K_2 \leq 2K_2, \quad 1 \leq i \leq d. \]
According to Proposition 2.1, it follows from (A1), (A3) and (2.8) that for any \( n \geq 1 \), the SDE
\begin{equation}
dX^n(t) = F(t, X^n(t)) dt + H(t, X^n(t)) dt + G^n(t, X^n(t)) d\bar{W}(t), \quad X^n_0 = \xi \tag{2.9}
\end{equation}
has a unique non-explosive strong solution. Moreover, in view of the second inequality of (2.8) and

\[(2.10) \quad |G^n_i(t, z) - G^n_i(t, \bar{z})| \leq K_2|z - \bar{z}|^{\alpha_i}, \quad t \in [0, T], \ z, \bar{z} \in \mathbb{R}, \ 1 \leq i \leq d, \ n \geq 1,\]

this together with (A1) and (A3) implies that there exists a constant \(C(T) > 0\) such that

\[(2.11) \quad \sup_{n \geq 1} \mathbb{E} \sup_{t \in [-r, T]} |X^n(t)|^2 \leq C(T)(1 + \|\xi\|_\infty^2).\]

In fact, by (A1), (A3), (2.10) and the second inequality in (2.8), it holds

\[(2.12) \quad \langle F(t, x), x \rangle \leq C(T)(1 + |x|^2), \quad |H(t, \xi)| \leq C(T)(1 + \|\xi\|_\infty),
\quad |G^n(t, x)| \leq C(T)(1 + |x|), \quad x \in \mathbb{R}^d, \ t \in [0, T], \ \xi \in \mathcal{C}^d, \ n \geq 1.\]

For each integer \(N \geq 1\), define \(\tau^N_n = \inf \{t \in [0, T] : |X^n(t)| \geq N\} \) and \(\inf \emptyset = \infty\) by convention. By Itô’s formula, we derive from (2.12) that

\[|X^n(t \wedge \tau^N_n)|^2 \leq \xi(0)^2 + C + C \int_0^{t \wedge \tau^N_n} \sup_{u \in [-r, s]} |X^n(u)|^2 ds \]
\[+ 2 \int_0^{t \wedge \tau^N_n} \langle X^n(s), G^n(s, X^n(s)) d\bar{W}(s) \rangle \]

for some constant \(C > 0\). Applying BDG’s inequality, (2.12) and Grönwall’s inequality, it is standard to derive (2.11). For any \(1 \leq i \leq d\), let \(X^{n,i}\) be the \(i\)-th component of \(X^n\). For any \(m, n \geq 1\), it follows from Itô’s formula that

\[V_\varepsilon(X^{m,i}(t) - X^{n,i}(t)) = \int_0^t V_\varepsilon'(X^{m,i}(s) - X^{n,i}(s))(F_i(s, X^m(s)) - F_i(s, X^n(s))) ds \]
\[+ \int_0^t V_\varepsilon'(X^{m,i}(s) - X^{n,i}(s))(H_i(s, X^m) - H_i(s, X^n)) ds \]
\[+ \frac{1}{2} \int_0^t V_\varepsilon''(X^{m,i}(s) - X^{n,i}(s))(G_i^m(s, X^{m,i}(s)) - G_i^n(s, X^{n,i}(s)))^2 ds \]
\[+ \int_0^t V_\varepsilon'(X^{m,i}(s) - X^{n,i}(s))(G_i^m(s, X^{m,i}(s)) - G_i^n(s, X^{n,i}(s))) d\bar{W}^i(s).\]

This combined with (2.5) implies that for any \(1 \leq i \leq d\),

\[\sum_{i=1}^d |X^{m,i}(t) - X^{n,i}(t)| - d\varepsilon \]
\[\leq \int_0^t \sum_{i=1}^d V_\varepsilon'(X^{m,i}(s) - X^{n,i}(s))(F_i(s, X^m(s)) - F_i(s, X^n(s))) ds \]
\[+ \int_0^t \sum_{i=1}^d V_\varepsilon'(X^{m,i}(s) - X^{n,i}(s))(H_i(s, X^m) - H_i(s, X^n)) ds\]
\[
+ \frac{1}{2} \int_0^t \sum_{i=1}^d \left( V_{\varepsilon}'(X^{m,i}(s) - X^{n,i}(s))(G_i^m(s, X^{m,i}(s)) - G_i^n(s, X^{n,i}(s)))^2 ds
\]
\[
+ \int_0^t \sum_{i=1}^d V_{\varepsilon}'(X^{m,i}(s) - X^{n,i}(s))(G_i^m(s, X^{m,i}(s)) - G_i^n(s, X^{n,i}(s)))dW^i(s)
\]
\[
= J_{1}^{m,n} + J_{2}^{m,n} + J_{3}^{m,n} + J_{4}^{m,n}, \quad t \in [0, T].
\]
By (2.5) and (A1), we conclude that
\[
J_{1}^{m,n} = \int_0^t \sum_{i=1}^d \left[ V_{\varepsilon}'(X^{m,i}(s) - X^{n,i}(s))\operatorname{sgn}(X^{m,i}(s) - X^{n,i}(s))
\times (F_i(s, X^{m}(s)) - F_i(s, X^{n}(s)))\operatorname{sgn}(X^{m,i}(s) - X^{n,i}(s)) \right] ds
\]
\[
\leq K_1^+ d \int_0^t |X^m(s) - X^n(s)| ds, \quad t \in [0, T].
\]
(2.5) and (A3) yield that
\[
J_{2}^{m,n} \leq K_3 d \int_0^t \|X^m - X^n\|_{\infty} ds, \quad t \in [0, T].
\]
Note that (2.6) and (2.10) derive
\[
J_{3}^{m,n} \leq 2K_2^2 \int_0^t \sum_{i=1}^d \varepsilon^{2\alpha_i} ds + 2 \int_0^t \sum_{i=1}^d \varepsilon^{1}(G_i^m(s, X^{n,i}(s)) - G_i^n(s, X^{n,i}(s)))^2 ds
\]
\[
\leq 2tK_2^2 \sum_{i=1}^d \varepsilon^{2\alpha_i} + 2te^2 \sum_{s \in [0, t], z \in \mathbb{R}} \sup_{i=1} \|G_i^m(s, z) - G_i^n(s, z)\|^2
\]
\[
=: H^{m,n,\varepsilon}(t), \quad t \in [0, T].
\]
So, (2.5) and Lemma 2.2 imply that for any \( p \in (0, 1) \), there exist constants \( c_1(p), c_2(p) > 0 \) such that
\[
\mathbb{E} \sup_{t \in [0, T]} |X^m(t) - X^n(t)|^p \leq c_1(p) e^{c_2(p)(K_1^+ + K_3)dT} (H^{m,n,\varepsilon}(T) + d\varepsilon)^p.
\]
Thanks to (2.7), we derive
\[
\limsup_{m,n \to \infty} |H^{m,n,\varepsilon}(T)| \leq 2TK_2^2 \sum_{i=1}^d \varepsilon^{2\alpha_i}.
\]
So, for \( p \in (0, 1) \), it holds
\[
\limsup_{m,n \to \infty} \mathbb{E} \sup_{t \in [0, T]} |X^m(t) - X^n(t)|^p \leq c_1(p) e^{c_2(p)(K_1^+ + K_3)dT} (2TK_2^2 \sum_{i=1}^d \varepsilon^{2\alpha_i} + d\varepsilon)^p.
\]
Letting $\varepsilon \to 0$, we conclude that for any $p \in (0, 1)$,

$$\limsup_{m,n\to\infty} \mathbb{E} \sup_{t\in[0,T]} |X^m(t) - X^n(t)|^p = 0.$$ 

So, there exists a continuous stochastic process $\{\bar{X}(t)\}_{t\in[-r,T]}$ satisfying $\bar{X}_0 = \xi$ and

\begin{equation}
\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T]} |X^n(t) - \bar{X}(t)|^p = 0.
\end{equation}

This yields that there exists a subsequence $\{n_k\}_{k \geq 1}$ such that $\mathbb{P}$-a.s.

\begin{equation}
\lim_{k \to \infty} \sup_{t\in[0,T]} |X^{n_k}(t) - \bar{X}(t)| = 0, \quad \sup_{k \geq 1} \sup_{t\in[-r,T]} (|X^{n_k}(t)| + |\bar{X}(t)|) < \infty.
\end{equation}

Moreover, (2.11) and Fatou’s Lemma imply

$$\mathbb{E} \sup_{t\in[-r,T]} |\bar{X}(t)|^2 \leq C(T) (1 + \|\xi\|_{L^\infty}^2).$$

So, by the local boundedness of $F, H$, the continuity of $F(s,\cdot), H(s,\cdot)$, (2.14) and the dominated convergence theorem, we conclude that $\mathbb{P}$-a.s.

$$\lim_{k \to \infty} \sup_{t\in[0,T]} \left| \int_0^t (F(s, X^{n_k}(s)) + H(s, X^{n_k}(s)))ds - \int_0^t (F(s, \bar{X}(s)) + H(s, \bar{X}(s)))ds \right| = 0.$$ 

Moreover, by Markov’s inequality, BDG’s inequality, (2.7), (2.10) and (2.13), for any $\varepsilon > 0$ and $p \in (0, 1)$, we have

\begin{align*}
\limsup_{k \to \infty} \mathbb{P} \left( \sup_{t\in[0,T]} \left| \int_0^t \left[ G^{n_k}(s, X^{n_k}(s)) - G(s, \bar{X}(s)) \right]d\bar{W}(s) \right| \geq \varepsilon \right) \\
\leq \limsup_{k \to \infty} \frac{1}{\varepsilon^p} \mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t \left[ G^{n_k}(s, X^{n_k}(s)) - G(s, \bar{X}(s)) \right]d\bar{W}(s) \right|^p \\
\leq c(p) \limsup_{k \to \infty} \frac{1}{\varepsilon^p} \mathbb{E} \left( \int_0^T \sum_{i=1}^d \left| G^{n_k}_i(s, X^{n_k,i}(s)) - G_i(s, \bar{X}^i(s)) \right|^2 ds \right)^{\frac{p}{2}} \\
\leq c(p) T \frac{\varepsilon}{p} \limsup_{k \to \infty} \left( \sum_{i=1}^d \sup_{s\in[0,T],x\in\mathbb{R}} \left| G^{n_k}_i(s, x) - G_i(s, x) \right|^2 \right)^{\frac{p}{2}} \\
+ c(p) T^{\frac{p}{2}} K c \frac{\varepsilon}{p} \limsup_{k \to \infty} \mathbb{E} \left( \sup_{s\in[0,T]} \sum_{i=1}^d \left| X^{n_k,i}(s) - \bar{X}^i(s) \right|^{2\alpha_i} \right)^{\frac{p}{2}} = 0.
\end{align*}

Therefore, replacing $n$ by $n_k$ in (2.9) and letting $k \to \infty$, it holds $\mathbb{P}$-a.s.

$$\bar{X}(t) = \int_0^t F(s, \bar{X}(s))ds + \int_0^t H(s, \bar{X}(s))ds + \int_0^t G(s, \bar{X}(s))d\bar{W}(s), \quad t \in [0,T].$$
This means that \( \{ \tilde{X}(t) \}_{t \in [-r,T]} \) is a strong solution to (2.2).

**Step 2. Uniqueness of the strong solution.**

Let \( X^\xi(t) \) be the solution to (2.2) with initial value \( \xi \in \mathcal{C}^d \). By the same argument to derive (2.11), we obtain

\[
\mathbb{E} \sup_{t \in [-r,T]} |X^\xi(t)|^2 \leq C(T)(1 + \|\xi\|_\infty^2).
\]

So, Jensen’s inequality implies (2.3). For any \( 1 \leq i \leq d \), let \( X^\xi,i \) be the \( i \)-th component of \( X^\xi \). Applying Itô’s formula, for any \( 1 \leq i \leq d \), we have

\[
V'_\varepsilon(X^\xi,i(t) - X^\eta,i(t)) = V'_\varepsilon(\xi^i(0) - \eta^i(0))
\]

\[
+ \int_0^t \left. V''(X^\xi,i(s) - X^\eta,i(s)) \right\} \{ F_i(s, X^\xi(s)) - F_i(s, X^\eta(s)) \} ds
\]

\[
+ \int_0^t \left. V''(X^\xi,i(s) - X^\eta,i(s)) \right\} \{ H_i(s, X^\xi(s)) - H_i(s, X^\eta(s)) \} ds
\]

\[
+ \frac{1}{2} \int_0^t \left. V''(X^\xi,i(s) - X^\eta,i(s)) \right\} \{ G_i(s, X^\xi,i(s)) - G_i(s, X^\eta,i(s)) \}^2 ds
\]

By (A1)-(A3), (2.5) and (2.6), it holds

\[
|X^\xi(t) - X^\eta(t)| \leq d\varepsilon + K^2 T \sum_{i=1}^d \varepsilon^{2\alpha_i} + C(T)\|\xi - \eta\|_\infty
\]

\[
+ C \int_0^t \sup_{u \in [0,s]} |X^\xi(u) - X^\eta(u)| ds + M_t, \ t \in [0,T]
\]

for a martingale \( M_t \) and some constants \( C, C(T) > 0 \). Then for any \( p \in (0,1) \), applying Lemma 2.2, we get

\[
\mathbb{E} \sup_{t \in [0,T]} |X^\xi(t) - X^\eta(t)|^p \leq c_1(p)e^{c_2(p)T} \left( d\varepsilon + K^2 T \sum_{i=1}^d \varepsilon^{2\alpha_i} + C(T)\|\xi - \eta\|_\infty \right)^p.
\]

Letting \( \varepsilon \to 0 \), we derive (2.4), which yields the uniqueness of the strong solution of (2.2). \( \square \)

## 3 Path Dependent McKean-Vlasov SDEs with Hölder Continuous Diffusion

Throughout this section, we make the following assumption.

**(H)** Assume that the following conditions hold.
Proof.

\[ b(t, x, \mu) - b(t, y, \nu) \text{sgn}(x - y) \leq K_b(\mathbb{W}_1(\mu, \nu) + |x - y|), \quad t \in [0, T], \]

where \( \text{sgn}(\cdot) \) means the sign function.

(H\(\sigma\)) There exist constants \( K_\sigma \geq 0 \) and \( \alpha \in [\frac{1}{2}, 1] \) such that

\[ |\sigma(t, x) - \sigma(t, y)| \leq K_\sigma |x - y|^\alpha, \quad |\sigma(t, 0)| \leq K_\sigma, \quad x, y \in \mathbb{R}, t \in [0, T] \]

(HB) There exists a constant \( K_B \geq 0 \) and a probability measure \( m \) on \([-r, 0] \) such that for any \( \xi, \eta \in \mathcal{C}, \mu, \nu \in \mathcal{P}_1(\mathbb{R}), t \in [0, T], \)

\[ |B(t, \xi, \mu) - B(t, \eta, \nu)| \leq K_B \{ |\xi - \eta|_{L^1(m)} + \mathbb{W}_1(\mu, \nu) \}, \quad |B(t, 0, \delta_0)| \leq K_B, \]

where \( \delta_0 \) is the Dirac measure at the point 0.

### 3.1 Well-posedness

**Theorem 3.1.** Assume (H). Then for any \( X_0 \in L^1(\Omega \rightarrow (\mathcal{C}, \| \cdot \|_\infty); \mathcal{F}_0, \mathbb{P}), (1.2) \) has a unique strong solution \( (X(t))_{t \in [-r, T]} \) with initial value \( X_0 \) and there exists a constant \( C(T) > 0 \) such that

\[ \mathbb{E} \sup_{t \in [0, T]} \|X_t\|_\infty \leq C(T)(1 + \mathbb{E}\|X_0\|_\infty). \]  

Moreover, for two solutions \( X(t) \) and \( \tilde{X}(t) \),

\[ \sup_{t \in [0, T]} \mathbb{E}|X(t) - \tilde{X}(t)| \leq C(T)\mathbb{E} \left\{ |X(0) - \tilde{X}(0)| + K_B \int_{-r}^0 m([-r, u])|X(u) - \tilde{X}(u)|du \right\}. \]  

**Proof.** For \( \mu \in C([0, T]; \mathcal{P}_1(\mathbb{R})), x \in \mathbb{R} \) and \( \xi \in \mathcal{C} \), let \( b^\mu(t, x) = b(t, x, \mu), B^\mu(t, \xi) = B(t, \xi, \mu_t). \) Consider

\[ dX^\mu(t) = b^\mu(t, X^\mu(t))dt + B^\mu(t, X^\mu(t))dW(t), \quad t \in [0, T]. \]  

By (H) and Theorem 2.3, (3.3) is strongly well-posed and let \( \Phi_t(\mu) = \mathcal{L}X^\mu(t), t \in [0, T], \) where \( (X^\mu(t))_{t \in [-r, T]} \) solves (3.3) with \( X^\mu_0 \in L^1(\Omega \rightarrow (\mathcal{C}, \| \cdot \|_\infty); \mathcal{F}_0, \mathbb{P}). \) In view of (HB),

\[ b(t, x, \mu)\text{sgn}(x) \leq C_0(T)(1 + |x| + \mu(| \cdot |)), \quad t \in [0, T] \]

holds for some constant \( C_0(T) > 0 \). So, by the similar argument to derive (2.11), we get

\[ \mathbb{E}(\sup_{s \in [-r, t]} |X^\mu(s)|^2 |\mathcal{F}_0) \leq C(T)^2 \left( 1 + \|X^\mu_0\|_\infty^2 + \int_0^t \mu_s(| \cdot |)^2ds \right), \quad t \in [0, T], \]  

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which yields

$$\mathbb{E}( \sup_{s \in [-r, t]} |X^\mu_t(s)|) \leq C(T) \left( 1 + \mathbb{E}\|X^\mu_0\|_\infty + \left( \int_0^t \mu_s(|\cdot|)^2 ds \right)^{\frac{1}{2}} \right), \quad t \in [0, T]$$

for some constant $C(T) \geq 0$. By Itô’s formula, it follows that

$$V_\varepsilon(X^\mu(t) - X^\nu(t)) = V_\varepsilon(X^\mu(0) - X^\nu(0)) + \int_0^t V_\varepsilon'(X^\mu(s) - X^\nu(s)) \{b(s, X^\mu(s), \mu_s) - b(s, X^\nu(s), \nu_s)\} ds$$

$$+ \int_0^t V_\varepsilon'(X^\mu(s) - X^\nu(s)) \{B(s, X^\mu, \mu_s) - B(s, X^\nu, \nu_s)\} ds$$

$$(3.7)$$

$$+ \frac{1}{2} \int_0^t V_\varepsilon''(X^\mu(s) - X^\nu(s)) \{\sigma(s, X^\mu(s)) - \sigma(s, X^\nu(s))\}^2 ds$$

$$+ \int_0^t V_\varepsilon'(X^\mu(s) - X^\nu(s)) \{\sigma(s, X^\mu(s)) - \sigma(s, X^\nu(s))\} dW(s)$$

$$=: I_{1,\varepsilon} + I_{2,\varepsilon}(t) + I_{3,\varepsilon}(t) + I_{4,\varepsilon}(t) + I_{5,\varepsilon}(t).$$

Using (2.5), we get

$$I_{1,\varepsilon} \leq |X^\mu(0) - X^\nu(0)|.$$

Moreover, it follows from (2.5) and (HB) that

$$I_{2,\varepsilon}(t) \leq K^+ \int_0^t \{|X^\mu(s) - X^\nu(s)| + \mathbb{W}_1(\mu_s, \nu_s)\} ds, \quad t \in [0, T].$$

By (2.5), (HB) and Fubini’s theorem, we arrive at

$$I_{3,\varepsilon}(t) \leq K_B \int_0^t \{\|X^\mu - X^\nu\|_{L^1(m)} + \mathbb{W}_1(\mu_s, \nu_s)\} ds$$

$$= K_B \int_{-r}^{0} \left( \int_{\theta}^{t+\theta} |X^\mu(u) - X^\nu(u)| du \right) m(d\theta) + K_B \int_0^t \mathbb{W}_1(\mu_s, \nu_s) ds$$

$$\leq K_B \int_{-r}^{0} \left( \int_{0}^{t} |X^\mu(u) - X^\nu(u)| du \right) m(d\theta)$$

$$+ K_B \int_{-r}^{0} \left( \int_{\theta}^{t} |X^\mu(u) - X^\nu(u)| du \right) m(d\theta) + K_B \int_0^t \mathbb{W}_1(\mu_s, \nu_s) ds$$

$$\leq K_B \left( \int_{0}^{t} |X^\mu(u) - X^\nu(u)| du \right)$$

$$+ K_B \int_{-r}^{0} m([-r, u]) \{X^\mu(u) - X^\nu(u)| du + K_B \int_0^t \mathbb{W}_1(\mu_s, \nu_s) ds, \quad t \in [0, T].$$
Furthermore, by \((H\sigma)\), (2.6) and using \(\alpha \in [1/2, 1]\), we deduce
\[
I_{4,\varepsilon}(t) \leq K_{\sigma}^2 T \varepsilon^{2\alpha}, \quad t \in [0, T].
\]

In addition, by (2.5), \((H\sigma)\) and (3.6), we have \(\mathbb{E}I_{5,\varepsilon}(t) = 0\). Taking expectation in (3.7), using (2.5) and letting \(\varepsilon \downarrow 0\), there exists a constant \(C > 0\) such that
\[
\mathbb{E}|X^\mu(t) - X^\nu(t)| \leq \mathbb{E}|X^\mu(0) - X^\nu(0)| + K_B \int_{-\varepsilon}^{0} m([0, r]) \mathbb{E}|X^\mu(u) - X^\nu(u)|du
\]
\[
+ C \int_{0}^{t} \mathbb{E}|X^\mu(s) - X^\nu(s)|ds + (K_0^+ + K_B) \int_{0}^{t} \mathbb{W}_1(\mu_s, \nu_s)ds.
\]

It follows from Grönwall’s inequality that
\[
\mathbb{E}|X^\mu(t) - X^\nu(t)| \leq e^{Ct} \left\{ \mathbb{E}|X^\mu(0) - X^\nu(0)| + K_B \int_{-\varepsilon}^{0} m([0, r]) \mathbb{E}|X^\mu(u) - X^\nu(u)|du \right\}
\]
\[
+ C(T) \int_{0}^{t} \mathbb{W}_1(\mu_s, \nu_s)ds.
\]

So, when \(X^\mu_0 = X^\nu_0\), for \(\lambda = 2C(T)\), we get
\[
\sup_{t \in [0,T]} e^{-\lambda t} \mathbb{W}_1(\Phi_t(\mu), \Phi_t(\nu)) \leq \frac{1}{2} \sup_{t \in [0,T]} e^{-\lambda t} \mathbb{W}_1(\mu_t, \nu_t).
\]

Set
\[
E_\lambda := \left\{ \mu \in C([0, T]; \mathcal{P}(\mathbb{R})) : \mu_0 = \mathcal{L}_{X^\nu(0)} \right\}
\]
equipped with the complete metric
\[
\rho(\mu, \nu) := \sup_{t \in [0,T]} e^{-\lambda t} \mathbb{W}_1(\mu_t, \nu_t), \quad \mu, \nu \in E_\lambda.
\]

Then \(\Phi\) is strictly contractive in \(E_\lambda\). Consequently, the Banach fixed point theorem together with the definition of \(\Phi\) implies that there exists a unique \(\mu \in E_\lambda\) such that
\[
\Phi_t(\mu) = \mu_t = \mathcal{L}_{X^\nu(t)}, \quad t \in [0, T].
\]

Finally, taking \(\mu_t = \mathcal{L}_{X^\nu(t)}\) in (3.6), (3.1) follows from Grönwall’s inequality. Similarly, taking \(\mu_t = \mathcal{L}_{X(t)}, \nu_t = \mathcal{L}_{\tilde{X}(t)}\), \(X^\mu(t) = X(t), X^\nu(t) = \tilde{X}(t)\) in (3.8), (3.2) holds by Grönwall’s inequality.

\[\square\]

### 3.2 Propagation of Chaos

Let \(N \geq 1\) be an integer and \((X^i_0, W^i(t))_{1 \leq i \leq N}\) be i.i.d. copies of \((X_0, W(t))\) with \(\mathcal{F}_0\)-measurable \(\mathcal{C}\)-valued random variable \(X_0\). Consider
\[
dX^i(t) = b(t, X^i(t), \mathcal{L}_{X^i(t)})dt + B(t, X^i_t, \mathcal{L}_{X^i(t)})dt + \sigma(t, X^i(t))dW^i(t), \quad 1 \leq i \leq N.
\]
Let
\[ \hat{\mu}^N_t = \frac{1}{N} \sum_{j=1}^N \delta_{X^j(t)}. \]

Consider the stochastic $N$-interacting particle system:
\[ dX^i(t) = b(t, X^i(t), \hat{\mu}^N_t)dt + B(t, X^i_t, \hat{\mu}^N_t)dt + \sigma(t, X^i(t))dW^i(t), \quad X^i_0 = X^i, \]
where $\hat{\mu}^N_t$ is the empirical distribution corresponding to $X^1(t), \ldots, X^N(t)$, i.e.
\[ \hat{\mu}^N_i := \frac{1}{N} \sum_{j=1}^N \delta_{X^j, i}(t). \]

Applying Theorem 2.3, the well-posedness of the stochastic $N$-interacting particle system (3.10) can be proved in the following lemma.

**Lemma 3.2.** Assume (H) and $X^i_0 \in L^1(\Omega \to (\mathcal{C}, \|\cdot\|_\infty); \mathcal{F}_0, \mathbb{P})$, $1 \leq i \leq N$. Then, for each $N \geq 1$, (3.10) admits a unique strong solution $\{(X^i(t))_{t \in [-r, T]}\}_{1 \leq i \leq N}$ and
\[ \mathbb{E} \sup_{t \in [-r, T]} |X^i(t)| \leq C(T)(1 + \mathbb{E}\|X^i_0\|_\infty), \quad 1 \leq i \leq N \]
holds for some constant $C(T) > 0$.

**Proof.** For $x := (x_1, x_2, \ldots, x_N)^* \in \mathbb{R}^N$, $\xi := (\xi_1, \xi_2, \ldots, \xi_N)^* \in \mathcal{C}^N$, set $\tilde{\mu}^N_x = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ and
\[ \hat{b}(t, x) := (b(t, x_1, \tilde{\mu}^N_x), \ldots, b(t, x_N, \tilde{\mu}^N_x))^*, \quad \hat{B}(t, \xi) := (B(t, \xi_1, \tilde{\mu}^N_{\xi_1}), \ldots, B(t, \xi_N, \tilde{\mu}^N_{\xi_N}))^*, \]
\[ \hat{\sigma}(t, x) := \text{diag}(\sigma(t, x_1), \ldots, \sigma(t, x_N)), \quad \hat{W}(t) := (W^1(t), \ldots, W^N(t))^*, \quad t \in [0, T]. \]

Then it is clear that $(\hat{W}(t))_{t \in [0, T]}$ is an $N$-dimensional Brownian motion and (3.10) can be reformulated as
\[ d\hat{X}(t) = \hat{b}(t, \hat{X}(t))dt + \hat{B}(t, \hat{X}_t)dt + \hat{\sigma}(t, \hat{X}(t))d\hat{W}(t), \quad \hat{X}_0 = (X^1_0, X^2_0, \ldots, X^N_0)^*. \]

Note that
\[ \mathbb{W}_1 \left( \frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \frac{1}{N} \sum_{i=1}^N \delta_{\bar{x}_i} \right) \leq \frac{1}{N} \sum_{i=1}^N |x_i - \bar{x}_i|, \quad x_i, \bar{x}_i \in \mathbb{R}, 1 \leq i \leq N. \]

It is not difficult to see from (Hb), (HB) and (3.13) that $\hat{b}$ is locally bounded in $[0, T] \times \mathbb{R}^N$, for any $t \in [0, T], \hat{b}(t, \cdot)$ is continuous,
\[ (\hat{b}(t, x) - \hat{b}(t, y))\text{sgn}(x_i - y_i) = (b(t, x_i, \tilde{\mu}^N_x) - b(t, y_i, \tilde{\mu}^N_y))\text{sgn}(x_i - y_i) \leq K_b(\|x_i - y_i\| + \mathbb{W}_1(\tilde{\mu}^N_x, \tilde{\mu}^N_y)) \leq K_b^+(\|x_i - y_i\| + \frac{1}{N} \sum_{i=1}^N |x_i - y_i|) = K_b^+(1 + N^{-\frac{1}{2}})\|x - y\|, \quad x, y \in \mathbb{R}^N, 1 \leq i \leq N, \]
and

\[|\hat{B}(t, \xi) - \hat{B}(t, \eta)|^2 \leq \sum_{i=1}^{N} |B(t, \xi_i; \mu_{\xi(i)}^N) - B(t, \eta_i; \mu_{\eta(i)}^N)|^2 \]

\[\leq 2K_B^2 \sum_{i=1}^{N} (||\xi_i - \eta_i||^2 + \mathbb{W}_1(\mu_{\xi(i)}^N, \mu_{\eta(i)}^N)^2) \]

\[\leq 2K_B^2 \sum_{i=1}^{N} (||\xi_i - \eta_i||^2 + |\xi_i(0) - \eta_i(0)|^2) \]

\[\leq 4K_B^2 \sum_{i=1}^{N} ||\xi_i - \eta_i||^2_{\infty}, \xi, \eta \in \mathcal{C}^N. \]

(3.15)

So, (3.14), (3.15) and (Hσ) yield that (A1)-(A3) hold for \(\hat{b}, \hat{B}, \hat{\sigma}, N\) replacing \(F, H, G, d\) respectively. Therefore, according to Theorem 2.3, for each \(N \geq 1\), (3.12) and consequently (3.10) admits a unique strong solution \(\{X_{i,N}(t)\}_{t \in [-r, T]}\) for each \(N \geq 1\).

Finally, by Itô's formula, (3.4), (HB) and (Hσ), there exists a constant \(C > 0\) such that

\[|X_{i,N}(t)|^2 \leq |X_i(0)|^2 + C \int_0^t \left[ 1 + |X^{i,N}(s)|^2 + \frac{1}{N} \sum_{j=1}^{N} |X^{j,N}(s)|^2 \right] ds \]

\[+ C \int_0^t \|X_{i,N}(s)\|_{L^p(m)}^2 ds + \int_0^t 2X_{i,N}(s)\sigma(s, X_{i,N}(s))dW^i(s).\]

Using the same argument to derive (3.5), we arrive at

\[\mathbb{E}(\sup_{t \in [-r, T]} |X_{i,N}(t)|^2 |\mathcal{F}_0) \leq C(T)(1 + \|X_i^0\|_{\infty}^2).\]

This implies (3.11) by Jensen's inequality with respect to conditional expectation. \(\square\)

Finally, we give the quantitative propagation of chaos.

**Theorem 3.3.** Assume that \(\mathbb{E}\|X^i_0\|_{\infty}^p < \infty\) for some \(p > 1\) and \(p \neq 2\). Let \(\mu_t = \mathcal{L}_{X^i(t)}\).

(1) Then there exists a constant \(C(p, T) > 0\) depending only on \(p, T\) such that

\[\sup_{t \in [0, T]} \mathbb{E}|X^i(t) - X_{i,N}(t)| \leq C(p, T)(1 + (\mathbb{E}\|X^i_0\|_{\infty}^p)^{\frac{1}{p}})(N^{-1/2} + N^{-\frac{p-1}{p}}),\]

(3.16)

and consequently,

\[\sup_{t \in [0, T]} \mathbb{E}\mathbb{W}_1(\mu_{\hat{\sigma}(t)}^N, \mu_t) \leq C(p, T)(1 + (\mathbb{E}\|X^i_0\|_{\infty}^p)^{\frac{1}{p}})(N^{-1/2} + N^{-\frac{p-1}{p}}).\]

(3.17)
(2) If in addition, $\sigma^2 \geq \delta$ for some $\delta > 0$ and there exists a constant $K \geq 0$ such that

\[
|b(t, x, \mu) - b(t, x, \nu)| + |B(t, \xi, \mu) - B(t, \xi, \nu)| \\
\leq K(1 \wedge \mathbb{W}_1(\mu, \nu)), \quad \mu, \nu \in \mathcal{P}_1(\mathbb{R}), \ t \in [0, T], \ x \in \mathbb{R}, \ \xi \in \mathcal{C}.
\]

then there exists a constant $C(p, T) > 0$ depending only on $p, T$ such that for any $1 \leq k \leq N$,

\[
\sup_{t \in [0, T]} \left\| \mathcal{L}^{(X^{1,N}(t), X^{2,N}(t), \ldots, X^{k,N}(t))} - \mu_t^{\otimes k} \right\|_{\text{var}}^2 \\
\leq 2 \sup_{t \in [0, T]} \text{Ent} \left( \mu_t^{\otimes k} \right) \left( \mathcal{L}^{(X^{1,N}(t), X^{2,N}(t), \ldots, X^{k,N}(t))} \right) \\
\leq kC(p, T) \left( 1 + \left( \mathbb{E} \|X_0^i\|_p^p \right)^{\frac{1}{p}} \right) \left( N^{-\frac{1}{2}} + N^{-\frac{p-1}{p}} \right),
\]

where $\mu_t^{\otimes k} = \prod_{i=1}^k \mu_t$, the $k$-independent product of $\mu_t$.

**Proof.** Applying Itô’s formula, it holds

\[
\begin{align*}
V_{\varepsilon}(X^i(t) - X^{i,N}(t)) & \\
& = \int_0^t \left[ V_{\varepsilon}^i(X^i(s) - X^{i,N}(s)) \left( b(s, X^i(s), \tilde{\mu}_s^N) - b(s, X^{i,N}(s), \mu_s) \right) + \frac{1}{2} V_{\varepsilon}''(X^i(s) - X^{i,N}(s)) \left( \sigma(s, X^i(s)) - \sigma(s, X^{i,N}(s)) \right)^2 \right] ds \\
& + \frac{1}{2} \int_0^t V_{\varepsilon}''(X^i(s) - X^{i,N}(s)) \left( \sigma(s, X^i(s)) - \sigma(s, X^{i,N}(s)) \right)^2 ds \\
& + \int_0^t V_{\varepsilon}^i(X^i(s) - X^{i,N}(s)) \left( \sigma(s, X^i(s)) - \sigma(s, X^{i,N}(s)) \right) dW^i(s).
\end{align*}
\]

By the same argument to derive (3.8) and adopting the triangle inequality for $\mathbb{W}_1$, we arrive at

\[
\mathbb{E}|X^i(t) - X^{i,N}(t)| \leq C \int_0^t \left\{ \mathbb{E}|X^i(s) - X^{i,N}(s)| + \mathbb{E}\mathbb{W}_1(\mu_s, \tilde{\mu}_s^N) + \mathbb{E}\mathbb{W}_1(\tilde{\mu}_s^N, \mu_s^N) \right\} ds,
\]

where $\tilde{\mu}^N$ was introduced in (3.9). By [7, Theorem 1], there exists a constant $C(p, T) > 0$ such that

\[
\mathbb{E}\mathbb{W}_1(\mu_t, \tilde{\mu}_t^N) \leq C(p, T) \left( 1 + \left( \mathbb{E}\|X_0^i\|_p^p \right)^{\frac{1}{p}} \right) \left( N^{-1/2} + N^{-\frac{p-1}{p}} \right).
\]

As a result, it follows from (3.13) and (3.19) that

\[
\begin{align*}
\mathbb{E}|X^i(t) - X^{i,N}(t)| & \\
& \leq C_1 \int_0^t \left\{ \mathbb{E}|X^i(s) - X^{i,N}(s)| + C(p, T) \left( 1 + \left( \mathbb{E}\|X_0^i\|_p^p \right)^{\frac{1}{p}} \right) \left( N^{-1/2} + N^{-\frac{p-1}{p}} \right) \right\} ds
\end{align*}
\]
for some constant $C_1 > 0$. Consequently, we derive (3.16) by (3.1), (3.11) and Grönwall’s inequality. Finally, note that

$$\mathbb{W}_1(\hat{\mu}_s^N, \mu_s) \leq \mathbb{W}_1(\hat{\mu}_s^N, \mu_s^N) + \mathbb{W}_1(\hat{\mu}_s^N, \mu_s) \leq \frac{1}{N} \sum_{i=1}^{N} |X_i^N(s) - X_i(s)| + \mathbb{W}_1(\hat{\mu}_s^N, \mu_s),$$

which together with (3.16) and (3.19) yields (3.17).

(2) Rewrite (3.10) as

$$dX_i^{i,N}(t) = b(t, X_i^{i,N}(t), \mu_t)dt + B(t, X_i^{i,N}, \mu_t)dt + \sigma(t, X_i^{i,N}(t))d\tilde{W}_i(t), \quad 1 \leq i \leq k$$

with

$$d\tilde{W}_i(t) = \hat{\Gamma}_i(t)dt + dW_i(t), \quad 1 \leq i \leq k$$

and

$$\hat{\Gamma}_i(t) = \frac{1}{2}[b(t, X_i^{i,N}(t), \mu_t) - b(t, X_i^{i,N}(t), \hat{\mu}_t^N) + B(t, X_i^{i,N,N}(t), \mu_t) - B(t, X_i^{i,N}(t), \hat{\mu}_t^N)].$$

It follows from (3.18) and $\sigma^2 \geq \delta$ that there exists a constant $C > 0$ such that

$$|\hat{\Gamma}_i(t)| \leq C(\mathbb{W}_1(\hat{\mu}_t^N, \mu_t) \wedge 1), \quad t \in [0, T], 1 \leq i \leq k.$$

Let

$$R_t^k = \exp \left\{ -\frac{1}{2} \sum_{i=1}^{k} \int_0^t (\hat{\Gamma}_i(s), dW_i(s)) - \frac{1}{2} \sum_{i=1}^{k} \int_0^t |\hat{\Gamma}_i(s)|^2ds \right\}, \quad t \in [0, T].$$

(3.20) and Girsanov’s theorem imply that \( \{R_t^k\}_{t \in [0, T]} \) is a martingale and \((\hat{\Gamma}_i(t))_{1 \leq i \leq k, t \in [0, T]} \) is a $k$-dimensional Brownian motion under \( \mathbb{Q}_T^k = \hat{\mathbb{R}}_T^k \mathbb{P} \) and

$$\mathcal{L}(X_1^{i,N}(t), X_2^{i,N}(t), \cdots, X_k^{i,N}(t)) | \mathbb{Q}_T^k = \mu_t^{\otimes k}, \quad t \in [0, T].$$

This implies that

$$\mu_t^{\otimes k}(f) = \mathbb{E}[R_t^k f(X_1^{i,N}(t), X_2^{i,N}(t), \cdots, X_k^{i,N}(t))]
= \mathbb{E}[R_t^k f(X_1^{i,N}(t), X_2^{i,N}(t), \cdots, X_k^{i,N}(t))], \quad f \in \mathcal{B}(\mathbb{R}^k), t \in [0, T].$$

So, there exists a constant $C > 0$ such that

$$\text{Ent}(\mu_t^{\otimes k} | \mathcal{L}(X_1^{i,N}(t), X_2^{i,N}(t), \cdots, X_k^{i,N}(t)))
= \mathbb{E}(R_t^k \log R_t^k) = \frac{1}{2} \sum_{i=1}^{k} \int_0^t \mathbb{E}^{\mathbb{Q}_T^k} |\hat{\Gamma}_i(s)|^2 ds \leq C^2 k \int_0^t \mathbb{E}^{2 k}_T (\mathbb{W}_1(\hat{\mu}_s^N, \mu_s) \wedge 1)^2 ds, \quad t \in [0, T].$$

This together with Pinsker’s inequality (1.1) yields

$$||\mu_t^{\otimes k} - \mathcal{L}(X_1^{i,N}(t), X_2^{i,N}(t), \cdots, X_k^{i,N}(t))||_\text{var}
\leq 2 \text{Ent}(\mu_t^{\otimes k} | \mathcal{L}(X_1^{i,N}(t), X_2^{i,N}(t), \cdots, X_k^{i,N}(t)))
\leq 2 C^2 k \int_0^t \mathbb{E}^{\mathbb{Q}_T^k}_T (\mathbb{W}_1(\hat{\mu}_s^N, \mu_s)) ds.$$

The proof is finished by (3.21) and (3.19).
Remark 3.4. For quantitative propagation of chaos, one can refer to [11] and references therein for the convolution type distribution dependent SDEs. Since we only assume that the drift is Lipschitz continuous under $L^1$-Wasserstein distance and the estimate in [7, Theorem 1] for the convergence rate of empirical distribution of i.i.d. random variables plays crucial role, the order of the quantitative propagation of chaos may be not optimal.

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