ENHANCED GAUGE SYMMETRY 
IN 
M-THEORY 

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ABSTRACT 
In this article we examine some points in the moduli space of M-Theory at which there arise enhanced gauge symmetries. In particular, we examine the “trivial” points of enhanced gauge symmetry in the moduli space of M-Theory on $S^1 \times S^1 / \mathbb{Z}_2$ as well as the points of enhanced gauge symmetry in the moduli space of M-Theory on $K3$ and those in the moduli space of M-Theory on $T^5 / \mathbb{Z}_2 \times S^1$. Also, we employ the above enhanced gauge symmetries to derive the existence of some points of enhanced gauge symmetry in the moduli space of the Type IIA string theory.
1. Introduction

In the past year and a half much has changed in the field of string theory. Early results relating the two Type II string theories [8] and the two Heterotic string theories [1] have been combined with newer results relating the Type II string theory and the Heterotic string theory [9][12] as well as the Type I string theory and the Heterotic string theory [16] to obtain a single “String Theory.” Also, there has been very much progress in interpreting some, if not all, properties of String Theory in terms of M-Theory [2][3][6][7][14][16][18]. In addition recent progress has been made in interpreting the properties of String Theory and M-Theory in terms of F-Theory [11][12][13]. In this article, as I am “morally” opposed to breaking Lorentz symmetry, we will not consider F-Theory, but we will examine some points of enhanced gauge symmetry which arise in the moduli space of M-Theory compactified on various manifolds. In addition, we will be examining the implications that these points of enhanced gauge symmetry have for String Theory. In doing so we will derive the existence of some standard points of enhanced gauge symmetry in the Type IIA moduli space. So, let us start by giving an overview of the various relations which we will explore in this paper.

First and foremost, we should note a standard relation [17] which we will employ numerous times in this paper. If a Heterotic string is compactified on a circle $S^1$, then it has a relatively simple moduli space which is simply $[0, \infty]$, corresponding to the radius of the $S^1$ upon which the Heterotic string is compactified. At generic points in this moduli space, the Heterotic theory possess a $U(1) \times U(1)$ gauge symmetry corresponding to the two one-forms introduced by the compactification of the ten-metric and ten-two-form. However, at a special radius $R = \sqrt{\alpha'}$, the so-called “self-dual” radius, the $U(1) \times U(1)$ gauge symmetry is enhanced to an $SU(2) \times SU(2)$ gauge symmetry. We will combine this standard relation with the more complicated dualities between String Theory and M-Theory as well as dualities among the various string theories to examine some points of enhanced gauge symmetry which arise in the moduli space of M-Theory as well as points of enhanced gauge symmetry which arise in the Type IIA moduli space.

We will first examine some relatively standard points of enhanced gauge symmetry in the moduli space of M-Theory. We will then move onto less standard points of enhanced gauge symmetry. We will first, as a “warm-up,” take a brief look at the points of enhanced gauge symmetry in the moduli space of M-Theory on $S^1 \times S^1/\mathbb{Z}_2$. The existence of these

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1 Not, of course, including internal radii.
points of enhanced gauge symmetry follows almost directly from the points of enhanced
gauge symmetry in the moduli space of the Heterotic string on $S^1$. Witten and Horava [16] found that M-Theory on $S^1/Z_2$ is equivalent to the $E_8 \times E_8$ Heterotic string theory. Hence, as direct result, M-Theory on $S^1 \times S^1/Z_2$ is equivalent to the Heterotic string theory on $S^1$. Thus, M-Theory on $S^1 \times S^1/Z_2$, as a result of the points of enhanced gauge symmetry present in the moduli space of the Heterotic string theory on $S^1$, possess an enhanced gauge symmetry at the self-dual radius of the $S^1$ factor in $S^1 \times S^1/Z_2$.

Next we will examine the points of enhanced gauge symmetry present in the moduli space of M-Theory compactified on $K3$. Witten [14] has shown that M-Theory on $K3$ is equivalent to the Heterotic string theory on $T^3$. Now, as we previously mentioned, the Heterotic string theory on $S^1$ possess an enhanced gauge symmetry at a particular value of the $S^1$ radius. Similarly, the Heterotic string theory on $T^3$ possess various points of enhanced gauge symmetry corresponding to the various radii of the three-torus $T^3$ becoming self-dual. Hence, M-Theory on $K3$ should also possess various points of enhanced
gauge symmetry in its moduli space corresponding to varying the parameters which define the $K3$ compactification. In addition, we will examine what the points of enhanced gauge symmetry in the moduli space of M-Theory on $K3$ imply about enhanced gauge symmetries of the Type IIA theory on $K3$.

Finally, we will examine the points of enhanced gauge symmetry present in the moduli space of M-Theory compactified on $T^5/Z_2 \times S^1$. As shown by Dasgupta, Mukhi, and Witten [3][18], M-Theory on $T^5/Z_2 \times S^1$ is equivalent to the Type IIB string theory on $K3 \times S^1$. Furthermore, Seiberg, Dine, and Huet [8] long ago showed that the Type IIB string theory on $K3 \times S^1$ is equivalent to the Type IIA string theory on $K3 \times S^1$, where a T-Duality transformation is performed on the final $S^1$ factor. Also, by way of six-dimensional string-string duality [3][14], the Type IIA string theory on $K3 \times S^1$ is equivalent to the Heterotic string theory on $T^5$. Hence, chaining together the above relations, M-Theory on $T^5/Z_2 \times S^1$ is equivalent to the Heterotic string theory on $T^5$. Thus, as varying the radii of the $T^5$ on the Heterotic side leads to enhanced gauge symmetries, the above equivalence dictates that varying the parameters defining the $T^5/Z_2 \times S^1$compactification leads to enhanced gauge symmetries at points in the moduli space of M-Theory on $T^5/Z_2 \times S^1$. We will then find that the enhanced gauge symmetries at points in the moduli space of M-Theory on $T^5/Z_2 \times S^1$ imply the existence of enhanced gauge symmetries involving the Type IIA string theory on $T^5/Z_2$. With all this in mind let us start by looking at the “easy” case of M-Theory on $S^1 \times S^1/Z_2$. 

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2. Enhanced Gauge Symmetries: M-Theory on $S^1 \times S^1 / \mathbb{Z}_2$

In this section, as a “warm-up” exercise, we will derive the points of enhanced gauge symmetry present in the moduli space of M-Theory on the manifold $S^1 \times S^1 / \mathbb{Z}_2$. To do so, however, we will need to make use of the relation, derived by Witten and Horava [16], between M-Theory on $S^1 / \mathbb{Z}_2$ and the $E_8 \times E_8$ Heterotic string theory. So, as a first step in our derivation, we will take a quick look at their result.

2.1. M-Theory on $S^1 / \mathbb{Z}_2 \sim E_8 \times E_8$ Heterotic String Theory

In this subsection we will quickly review the equivalence, established by Witten and Horava [16], between M-Theory on $S^1 / \mathbb{Z}_2$ and the $E_8 \times E_8$ Heterotic string theory. So, let us start by considering the fact that the low-energy limit of M-Theory is eleven-dimensional supergravity [16]. Hence, the low-energy limit of M-Theory employs a set of gamma matrices $\Gamma^I$ which, as odd dimensions support only a single chirality, satisfy the following relation

$$\Gamma^1 \Gamma^2 \cdots \Gamma^{11} = 1. \quad (2.1)$$

Now, consider placing M-Theory on the manifold $X_{10} \times S^1$, where $X_{10}$ is an arbitrary ten-manifold, then modding the theory by a $\mathbb{Z}_2$ which acts by a sign change on the eleventh-coordinate $x_{11} \rightarrow -x_{11}$. In this manner we obtain M-Theory on $X_{10} \times S^1 / \mathbb{Z}_2$. As the low-energy limit of M-Theory is eleven-dimensional supergravity, M-Theory’s low-energy limit is invariant with respect to a supersymmetry generated by an arbitrary constant Majorana spinor $\epsilon$. Dividing by the above $\mathbb{Z}_2$ rids us of half this supersymmetry. If we choose our signs properly, we can require that the supersymmetries generated by $\epsilon$ satisfy

$$\Gamma^{11} \epsilon = \epsilon. \quad (2.2)$$

This, along with (2.1), implies that $\epsilon$ satisfies

$$\Gamma^1 \Gamma^2 \cdots \Gamma^{10} \epsilon = \epsilon. \quad (2.3)$$

Hence, the supersymmetries not projected out by this $\mathbb{Z}_2$ action are chiral in a ten-dimensional sense.

Thus, the low-energy limit of M-Theory on $X_{10} \times S^1 / \mathbb{Z}_2$ reduces to a ten-dimensional theory with one chiral supersymmetry. If M-Theory on $X_{10} \times S^1 / \mathbb{Z}_2$ is actually equivalent to a ten-dimensional string theory, then this discovery limits our choices to the Type I
string theory or either of the two Heterotic string theories. We will now show that if M-Theory on the manifold $X_{10} \times S^1 / \mathbb{Z}_2$ is equivalent to any string theory, then it is the $E_8 \times E_8$ Heterotic string theory on $X_{10}$.

We will do so by considering gravitational anomalies. As shown long ago by Witten and Alvarez-Gaume [19], the only theories in which there may exist local gravitational anomalies are those theories with spacetime dimension of the form $4n + 2$, for some integer $n$. Hence, there exist no local gravitational anomalies in eleven-dimensions, as 11 cannot be written in the form $4n + 2$ for some integer $n$. (One should also note that Witten [20] long ago established that global gravitational anomalies may only exist in theories with spacetime dimension of the form $8n + 1$ or $8n$; hence, there are no global gravitational anomalies in eleven-dimensions.) So, with all these “negative” results in relation to eleven-dimensional gravitational anomalies, we seem to be out of luck. But, this is actually not the case. The fact that the manifold $X_{10} \times S^1 / \mathbb{Z}_2$ has a boundary actually comes to the rescue.

Consider the eleven-dimensional Rarita-Schwinger field and also think of our compactification on $X_{10} \times S^1 / \mathbb{Z}_2$ as a compactification on $X_{10} \times S^1$ mod the $\mathbb{Z}_2$ action $x_{11} \to -x_{11}$. Hence, the Rarita-Schwinger field may be decomposed into its Fourier modes along the $S^1$ direction. The non-zero Fourier modes along this direction have mass of order $1/R_{11}$, where $R_{11}$ is the radius of the $S^1$; hence, they can not contribute to the anomaly. Only the zero mode may contribute as it is generically massless. From a ten-dimensional point-of-view, however, this massless mode leads to a chiral ten-dimensional Rarita-Schwinger field. As 10 is of the form $4n + 2$ this ten-dimensional chiral Rarita-Schwinger field can lead to an anomaly. The computation of this anomaly will allow us to connect M-Theory and the $E_8 \times E_8$ Heterotic string theory; so, let us proceed with its computation.

Consider an arbitrary diffeomorphism of the eleven-dimensional manifold $X_{10} \times S^1 / \mathbb{Z}_2$ generated by a vector $v^I$ such that $\delta x^I = \epsilon^I v^I$. The change in the effective action under this diffeomorphism is of the following form,

$$
\delta \Gamma = i\epsilon^I \int_{X_{10} \times S^1 / \mathbb{Z}_2} d^{11}x \sqrt{g} v^I W_I, \tag{2.4}
$$

where $g$ is the eleven-dimensional metric and $W_I$ is the anomalous variation in the effective Lagrangian, computable from local data [16]. Now, as we mentioned earlier, there are no local gravitational anomalies in eleven-dimensions. Hence, the contribution to the anomaly from a non-boundary point is zero and the above integrand only has a non-zero support
on the boundary of $X_{10} \times S^1/Z_2$. Let us notate the two boundary components of the manifold $X_{10} \times S^1/Z_2$ as $A'$ and $A''$. Hence, the anomaly takes the form

$$\delta \Gamma = i\epsilon' \int_{A'} d^{10}x \sqrt{g'}v^I W_I' + i\epsilon' \int_{A''} d^{10}x \sqrt{g''}v^I W_I'', \quad (2.5)$$

where $g'$ and $g''$ are the restriction of $g$ to $A'$ and $A''$ respectively and $W_I'$ and $W_I''$ are constructed from data at $A'$ and $A''$ respectively.

If we take the metric on $X_{10} \times S^1/Z_2$ to be a standard metric on $S^1/Z_2$ and an arbitrary metric on $X_{10}$, then we can see that the above anomaly is simply the standard gravitational anomaly in ten dimensions. Hence, each boundary component of $X_{10} \times S^1/Z_2$ contributes one-half of the standard ten-dimensional chiral gravitational anomaly. Thus, as the anomaly is not zero, there must be extra massless modes we do not yet know about living on the boundary of $X_{10} \times S^1/Z_2$. Also, as the boundary is ten-dimensional, these massless boundary modes will have to be vector multiplets as the ten dimensional vector multiplet is the only ten-dimensional supermultiplet in which all particles have spin less than 1 [16]. So, one can practically guess at this point, though a thorough justification requires a bit more work, that M-Theory on $X_{10} \times S^1/Z_2$ is related to the $E_8 \times E_8$ Heterotic string theory on $X_{10}$. As our above comments suggest, each boundary component of $X_{10} \times S^1/Z_2$ contributes one-half the standard ten-dimensional anomaly, one would guess that one $E_8$ propagates $A'$ and the second $E_8$ propagates on $A''$. If we would have considered either of the $N = 1$ $SO(32)$ string theories, then we would have been forced to place the entire $SO(32)$ multiplet on $A'$ or $A''$ violating the above symmetry. Hence, as sketched above, M-Theory on $X_{10} \times S^1/Z_2$ is equivalent to the $E_8 \times E_8$ Heterotic string theory on $X_{10}$. Furthermore, aping arguments put forth by Witten in [14], we find that the $S^1/Z_2$ radius $R_{11}$ as measured in the eleven-dimensional metric and the ten-dimensional $E_8 \times E_8$ Heterotic string theory coupling constant $\lambda_{10,H}$ are related in the following manner

$$R_{11} = \lambda_{10,H}^{2/3}. \quad (2.6)$$

In addition, again aping the arguments put forth by Witten in [14], we find that the ten-dimensional metric $G_{10,H}$ on $X_{10}$ given by the $E_8 \times E_8$ Heterotic string theory is related to the ten-dimensional metric $g_{10,M}$ on $X_{10}$ given by M-Theory in the following manner

$$g_{10,M} = R_{11}^{-1}G_{10,H}. \quad (2.7)$$

Now, let us employ the equivalence sketched above along with (2.6) and (2.7) to derive some information about the points of enhanced gauge symmetry in the moduli space of M-Theory on $S^1 \times S^1/Z_2$. 

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2.2. M-Theory on $S^1 \times S^1 / \mathbb{Z}_2$ : Enhanced Gauge Symmetries

In this subsection we will take a look at the points of enhanced gauge symmetry in the moduli space of M-Theory on the manifold $S^1 \times S^1 / \mathbb{Z}_2$. As a first step in this direction, we should note that M-Theory on $S^1 \times S^1 / \mathbb{Z}_2$, as sketched in the previous subsection, is equivalent to the Heterotic string theory on $S^1$. Furthermore, as mentioned in the introduction, the Heterotic string theory on a $S^1$ possess a point of enhanced gauge symmetry in its moduli space. This point occurs at the so-called self-dual radius of the $S^1$ on which the Heterotic string theory is compactified. Taking $\alpha' = 1$, this self-dual radius is given by $R_{10,H}^2 = 1$, where $R_{10,H}$ is the radius of the $S^1$ as measured in the $E_8 \times E_8$ Heterotic string theory metric. Now, let us interpret this in terms of M-Theory.

As mentioned in the previous subsection, the M-Theory metric $g_{10,M}$ on the ten-manifold $X_{10}$ and the $E_8 \times E_8$ Heterotic string theory metric $G_{10,H}$ on the same manifold $X_{10}$ are related, $g_{10,M} = R_{11}^{-1} G_{10,H}$. As a result, distances measured in the Heterotic string theory metric are scaled as compared to distances measured in the M-Theory metric. In particular, any distance $D_H$ in the Heterotic string theory metric is related to a the same distance $D_M$ as measured in the M-Theory metric by $D_H R_{11}^{-1/2} = D_M$. So, in particular, we may employ this relation to interpret the self-dual radius of the Heterotic string theory in a M-Theory context.

The self-dual radius in the Heterotic string theory occurs at $R_{10,H}^2 = 1$. If we multiply both sides of this equation by $R_{11}^{-1}$ and employ the above scaling between M-Theory and Heterotic theory distances, then we find that the points of enhanced gauge symmetry in the M-Theory moduli space occur on the curve

$$R_{10}^2 R_{11} = 1,$$  \hspace{1cm} (2.8)

where $R_{10}$ is the $S^1$ radius as measured in the M-Theory metric and $R_{11}$ is the $S^1 / \mathbb{Z}_2$ radius as measured in the M-Theory metric. A graphical representation of this result is shown in Figure 1.
As we have indicated in Figure 1, some regions in the moduli space of M-Theory on $S^1 \times S^1 / \mathbb{Z}_2$ lend themselves to a “natural” interpretation in terms of various weakly coupled string theories. Let us next take a moment to comment on the various limits proposed in Figure 1.

Let us first consider the limit in which both $R_{10}$ and $R_{11}$ are taken to infinity in the M-Theory compactification on $S^1 \times S^1 / \mathbb{Z}_2$. Obviously, as $R_{10}$ corresponds to the $S^1$ radius and $R_{11}$ to the $S^1 / \mathbb{Z}_2$ radius, the limit in which both $R_{10}$ and $R_{11}$ go to infinity corresponds to M-Theory on $R / \mathbb{Z}_2$. Now, let us comment on the Type IIA string theory limit.

As Figure 1 indicates, the Type IIA string theory limit corresponds to the limit in which $R_{11}$ goes to infinity and $R_{10}$ to zero. Let us try and understand this a bit better. As was shown by Witten [14] M-Theory compactified on an $S^1$ is equivalent to the Type IIA string theory. This equivalence is similar to the Heterotic/M-Theory equivalence we sketched above and the details of its derivation are presented elsewhere [2]; so, we will not review them here. As we are considering M-Theory on $S^1 \times S^1 / \mathbb{Z}_2$, Witten’s equivalence between M-Theory on $S^1$ and the Type IIA string theory [14] implies that M-Theory on $S^1 \times S^1 / \mathbb{Z}_2$ is equivalent to the Type IIA string theory on $S^1 / \mathbb{Z}_2$. Now, as shown by Witten [14], the ten-dimensional coupling constant of the Type IIA string theory $\lambda_{10, IIA}$ and the radius $R_{10}$ of the $S^1$ as measured in the M-Theory metric are related as follows $R_{10} = \lambda_{10, IIA}^{2/3}$. Hence, the weak coupling limit of the Type IIA string theory occurs when
we take \( R_{10} \) to zero, as indicated in Figure 1. Also, as indicated in Figure 1, the limit in which \( R_{11} \) goes to infinity and \( R_{10} \) to zero corresponds to the Type IIA string theory on \( S^1_r/\mathbb{Z}_2 \). The \( S^1_r/\mathbb{Z}_2 \) factor follows from the Type IIA string theory interpretation of the limit \( R_{11} \to \infty \). As \( R_{11} \) is measured in the M-Theory metric, we can not automatically assume that taking the limit \( R_{11} \to \infty \), and thus \( S^1/\mathbb{Z}_2 \to \mathbb{R}/\mathbb{Z}_2 \) in the M-Theory picture, corresponds to a similar story in the Type IIA string theory picture. We must translate this limit to a limit taken in the Type IIA string theory metric.

This translation between the two limits, and hence the two metrics, is analogous to the metric translation between the ten-dimensional Heterotic string theory metric and the ten-dimensional M-Theory metric put forth at the end of last subsection. In fact the ten-dimensional Type IIA metric \( G_{10,IIA} \) and the ten-dimensional M-Theory metric \( g_{10,M} \) are related by 

\[
g_{10,M} = R_{10}^{-1} G_{10,IIA}.
\]

Hence, we find that the \( S^1/\mathbb{Z}_2 \) radius as measured in the M-Theory metric, becomes \( R_{11}/2 \) when measured in the Type IIA metric. Hence, in the limit \( R_{11} \to \infty \) and \( R_{10} \to 0 \) the radius of the \( S^1/\mathbb{Z}_2 \) factor as measured in the Type IIA metric can take on any value depending upon how one goes to infinity. In particular, consider taking a variable \( q \) to infinity \( q \to \infty \), then defining the limits of \( R_{10} \) and \( R_{11} \) by their relation to \( q \) such that \( R_{11} = q \) and \( R_{10} = r^2 q^{-2} \). Doing so and taking the limit as \( q \to \infty \) one finds that the \( S^1/\mathbb{Z}_2 \) radius as measured in the Type IIA metric is simply \( R_{11}/2 \). So, “generically” we have found that in the limit \( R_{10} \to 0 \) and \( R_{11} \to \infty \) the radius of \( S^1/\mathbb{Z}_2 \) as measured in the Type IIA string theory metric is finite. However, we have also shown that the ten-dimensional Type IIA string theory coupling constant \( \lambda_{10,IIA} \) goes to zero in the same limit. Hence, at a “generic,” finite compactification radius of \( S^1/\mathbb{Z}_2 \), the Type IIA string theory is a “good” description of the limit \( R_{11} \to \infty \) and \( R_{10} \to 0 \) as it is weakly coupled at its compactification radius, justifying its inclusion in Figure 1. Next let us consider the limit in which \( R_{11} \to 0 \) and \( R_{10} \to \infty \).

As we sketched in the previous subsection, M-Theory on the manifold \( X_{10} \times S^1/\mathbb{Z}_2 \) is equivalent to an \( E_8 \times E_8 \) Heterotic string theory on \( X_{10} \). So, in particular, M-Theory on \( S^1 \times S^1/\mathbb{Z}_2 \) is equivalent to the Heterotic string theory on \( S^1 \), where, in accord with (2.3), the ten-dimensional coupling constant \( \lambda_{10,H} \) of the Heterotic string theory is related to \( R_{11} \) by \( \lambda_{10,H} = R_{11}^{3/2} \). So, as we are considering the limit in which \( R_{11} \to 0 \), this limit corresponds to the weakly coupled Heterotic theory on \( S^1 \), i.e. \( \lambda_{10,H} \to 0 \). Furthermore, as we are also considering the limit in which \( R_{10} \to \infty \), as measured by in M-Theory metric, the \( S^1 \) factor of \( S^1 \times S^1/\mathbb{Z}_2 \) in this M-Theory limit becomes \( \mathbb{R} \). However, this
limit is taken in the M-Theory metric and we must interpret this limit in the metric of the Heterotic string theory.

We can do this by simply employing the relation between the ten-dimensional metric of M-Theory and the ten-dimensional metric of the Heterotic string theory presented in equation (2.7). This relation implies that the $S^1$ radius $R_{10,H}$ as measured in the Heterotic theory is

$$R_{10,H} = R_{11}^{1/2} R_{10}. \quad (2.7)$$

So, in the limit $R_{10} \to \infty$ and $R_{11} \to 0$ the radius of the $S^1$ factor in the M-Theory compactification on $S^1 \times S^1 / \mathbb{Z}_2$ as measured in the Heterotic string theory metric does not actually go to infinity, but can be taken to any value depending upon the manner in which the limit is taken. For instance, consider taking a variable $q$ to infinity $q \to \infty$ and then defining $R_{11}$ by $R_{11} = r'^2 q^{-2}$ and $R_{10}$ by $R_{10} = q$. In this case the $S^1$ radius as measured in the Heterotic string theory metric is $R_{10,H} = R_{11}^{1/2} R_{10} = r'$. So, as $r'$ is arbitrary the $S^1$ radius as measured in the Heterotic string theory metric can take on any value in this limit. However, as we previously proved, the ten-dimensional Heterotic string theory coupling constant $\lambda_{10,H}$ in this limit goes to zero. So, at “generic” finite values of $r'$ the Heterotic string theory is weakly coupled at its compactification radius and thus a good description of the limit in which $R_{11} \to 0$ and $R_{10} \to \infty$.

Finally, let us look at the limit in which $R_{10} \to 0$ and $R_{11} \to 0$. In this limit, as the diagram suggests, the theory can be interpreted in terms of a Type I string theory or an $SO(32)$ Heterotic string theory. Let us see why this is the case. As shown by Witten and Horava [16], the Type I string theory and the $SO(32)$ Heterotic string theory are “dual.” Furthermore, as established by Narin and clarified by Ginsparg [5], the $SO(32)$ and $E_8 \times E_8$ Heterotic string theories are equivalent when compactified on an $S^1$. So, employing these two relations one may write the coupling constant and $S^1$ radius in the limit $R_{10} \to 0$ and $R_{11} \to 0$ in terms of the variables of an $SO(32)$ Heterotic string theory and/or a Type I string theory. As the details of this derivation were presented elsewhere [16], we will not repeat them here, but we will simply quote the results.2 The ten-dimensional Type I coupling constant $\lambda_{10,I}$ and the radius $R_{10,I}$ of the $S^1$ as measured in the Type I metric are

$$\lambda_{10,I} = \frac{R_{11}}{R_{10}}, \quad R_{10,I} = \frac{1}{R_{11} R_{10}^{1/2}}. \quad (2.9)$$

2 Note, the $R_{10}$ we are employing is the $R_{11}$ of [16], and the $R_{11}$ of [10] is our $R_{10}$.
Furthermore, we may also express the $SO(32)$ Heterotic coupling constant $\lambda_{10,H'}$ and the radius $R_{10,H'}$ of the $S^1$ as measured in the $SO(32)$ Heterotic metric in a similar manner,

$$\lambda_{10,H'} = \frac{R_{10}}{R_{11}},$$
$$R_{10,H'} = \frac{1}{R_{10} R_{11}^{1/2}}. \quad (2.10)$$

Hence, in the limit $R_{10} \rightarrow 0$ and $R_{11} \rightarrow 0$ one can see that the couplings $\lambda_{10,I}$ and $\lambda_{10,H'}$ can be made to go to finite values. Also, in this same limit one can see that the radii $R_{10,I}$ and $R_{10,H'}$ both go to infinity. Thus, “generically,” the $SO(32)$ Heterotic and Type I string theories will be weakly coupled at their compactification radii when $R_{10} \rightarrow 0$ and $R_{11} \rightarrow 0$. However, one should note that $\lambda_{10,I}$ and $\lambda_{10,H'}$ are inverses of one another; thus, at any given point “near” the origin in Figure 1 one theory will a “better” description than the other. However, along the line $R_{10} = R_{11}$ both the theories have coupling order one and are equally “bad” descriptions for $R_{10,I}$ and $R_{10,H'}$ order one.

This leads us to an interesting region in the moduli space of M-Theory on $S^1 \times S^1/\mathbb{Z}_2$. Notated by a “cloud” in Figure 1, it is the region surrounding the point where the line $R_{11} = R_{10}$ meets the curve $R_{10}^2 R_{11} = 1$. This is an interesting region in that we have predicted an enhanced gauge symmetry should occur in this region along $R_{10}^2 R_{11} = 1$, but it seems we have no perturbative “handle” on the physics occurring in this region. According to our above comments, both the Type I and $SO(32)$ Heterotic theory have coupling constants and compactification radii of order $1$ in this region. Similarly, tracing our results for the Type IIA and $E_8 \times E_8$ Heterotic string theories we find that both of these theories have coupling constants and compactification radii of order $1$ in this region. Hence, nothing seems to be weakly coupled at its compactification radius in this region, but yet, strangely enough, we can predict the existence of an enhanced gauge symmetry there. It would be very interesting to try and further understand this region of enhanced gauge symmetry from a perturbative view-point. Perhaps F-Theory on $T^2 \times S^1/\mathbb{Z}_2$ may come to the rescue?

3. Enhanced Gauge Symmetries : M-Theory on $K3$

In this section we will examine the enhanced gauge symmetries present in the moduli space of M-Theory on the four-manifold $K3$. However, to do so we will have need of a relation, derived by Witten [14], between M-Theory on $K3$ and the Heterotic string theory on $T^3$. So, we will first take a quick look at this relation then later employ it to examine the points of enhanced gauge symmetry in the moduli space of M-Theory on $K3$. 

3.1. M-Theory on K3 $\sim$ Heterotic String Theory on $T^3$.

In this subsection we will briefly review the equivalence, established by Witten $[14]$, between M-Theory on $K^3$ and the Heterotic string theory on $T^3$. The easiest manner in which to derive this relation, and the one which we shall use, employs string-string duality in six-dimensions.

In six-dimensions there is a string-string duality $[9][14]$ which states that the Heterotic string on $T^4$ is equivalent to the Type IIA string on the manifold $K^3$. From a slightly different view-point we can consider this as an equivalence between the Type IIA string theory on $K^3$ and the Heterotic string theory on $T^3 \times S^1$. Furthermore, we may view the Heterotic string on $T^3 \times S^1$ as a Heterotic theory on $T^3$ compactified on an $S^1$. From this final point-of-view we can consider the seven-dimensional Heterotic coupling constant $\lambda^7_{H}$ and the $S^1$ radius $R^7_{H}$ of $T^3 \times S^1$ as measured in the metric of the Heterotic string theory on $T^3$. These two quantities are related in a simple manner to the six-dimensional coupling constant $\lambda^6_{H}$ of the Heterotic string theory on $T^3 \times S^1$,

$$\frac{1}{\lambda^6_{H}} = \frac{R^7_{H}}{\lambda^7_{H}}. \quad (3.1)$$

Now, by way of six-dimensional string-string duality, we may further relate $\lambda^6_{H}$, and thus $\lambda^7_{H}$, to the Type IIA string theory coupling constant in six-dimensions $\lambda^6_{IIA}$. According to standard string-string duality in six-dimensions $[2][14]$, $\lambda^6_{H}$ and $\lambda^6_{IIA}$ are inverses of one another. Hence, we have

$$\lambda^6_{IIA} = \frac{1}{\lambda^6_{H}} = \frac{R^{1/2}_{7,H}}{\lambda^7_{H}}. \quad (3.2)$$

In addition, standard string-string duality $[2][14]$ allows us to relate the Type IIA metric in six-dimensions $G^6_{IIA}$ to the Heterotic metric $G^6_{H}$ in six-dimensions. They are related by,

$$G^6_{H} = \lambda^2_{6,H} G^6_{IIA} = \frac{\lambda^2_{7,H} R^7_{H}}{G^6_{IIA}}, \quad (3.3)$$

where the second equality follows from (B.1).

Now, let us consider taking the limit in which the $S^1$ factor of the Heterotic compactification on $T^3 \times S^1$ goes to $\mathbb{R}$, i.e. the limit in which $R^7_{H} \to \infty$. In this limit we will end up with the Heterotic string theory on $T^3$ and not $T^3 \times S^1$; hence, this limit should, if our
conjectured equivalence, M-Theory on $K3 \sim$ Heterotic theory on $T^3$, holds, be equivalent to M-Theory on $K3$. Let us show this is the case.

In the limit $R_{7,H} \rightarrow \infty$ the $S^1$ of the Heterotic compactification on $T^3 \times S^1$ is becoming "large." Hence, the states of the Heterotic string which wind about the $S^1$ are becoming very massive with mass of order $R_{7,H}$. Also, the states of the Heterotic string which have a non-zero momentum in the $S^1$ direction are becoming very light with mass of order $1/R_{7,H}$. In addition, as the three-torus $T^3$ is not varying at all, the states with momentum and/or windings about $T^3$ are unaffected in this limit. As the Type IIA string theory on $K3$ is equivalent to the Heterotic string theory on $T^3 \times S^1$ [9][14], the limit $R_{7,H} \rightarrow \infty$ also may be interpreted as a particular limiting case of the Type IIA string theory on $K3$. Let us consider what this particular limit is.

As the Heterotic string theory possess a T-Duality symmetry which exchanges the various radii of the $T^4 = T^3 \times S^1$ compactification, there is not a unique map from radii of the Heterotic compactification to the parameters defining the Type IIA compactification. However, one can choose a map, and this map will be equivalent to a class of other maps by way of a T-Duality transformation. So, with this in mind, let us choose a map from the radii of the Heterotic string theory to the parameters defining the compactification of the Type IIA string theory on $K3$.

As we will show below, we may make a map from the moduli space of the Heterotic string theory on $T^3 \times S^1$ to the moduli space of the Type IIA string theory on $K3$ which maps the $S^1$ radius to the volume of the $K3$ factor. In particular, if we denote the $K3$ volume as measured in the ten-dimensional Type IIA string theory metric as $V_{10,IIA}$, then the $S^1$ radius $R_{7,H}$ is related to $V_{10,IIA}$ in the following manner,

$$V_{10,IIA} = R_{7,H}^2. \quad (3.4)$$

Now, let us prove this relation is true.

Consider the Type IIA string theory in ten-dimensions. The low-energy effective action for the Type IIA string theory is Type IIA supergravity in ten-dimensions [17]. The bosonic fields present in Type IIA supergravity in ten-dimensions include a metric $G_{10,IIA}$, a dilaton $\phi$, a one-form $A$, and a three-form $A_3$. The bosonic portion of the low-energy effective action for these fields is schematically [14],

$$\int d^{10}x \sqrt{G_{10,IIA}} \left(e^{-\phi} R + |dA|^2 + |dA_3|^2 + \cdots \right). \quad (3.5)$$
However, when we compactify this theory on $K3$ we get a set of various fields. In particular $A$ yields a six-dimensional one-form $a$ and no zero-forms, $H^1(K3) = 0$. Also, $A_3$ yields a six-dimensional three-form $a_3$ as well as 22 six-dimensional one-forms $C^I$, $H^2(K3)$ is 22 dimensional. These all lead to a six-dimensional action with the general form,

$$\int d^6x \sqrt{G_{6,IIA}} \left( \frac{1}{\lambda_6^{2,IIA}} R + V_{10,IIA} |da|^2 + V_{10,IIA} |da_3|^2 + |dC^I|^2 + \cdots \right). \quad (3.6)$$

Now, in six-dimensions a three-form “vector” potential is dual to a one-form vector potential. Hence, in the above action we may exchange the three-form $a_3$ for a one-form $\tilde{a}$ where $V_{10,IIA} da_3 = \ast \tilde{d}a$. Doing so we find,

$$\int d^6x \sqrt{G_{6,IIA}} \left( \frac{1}{\lambda_6^{2,IIA}} R + V_{10,IIA} |da|^2 + V_{10,IIA}^{-1} |\tilde{d}a|^2 + |dC^I|^2 + \cdots \right). \quad (3.7)$$

Now, to get a physical “feel” for what the above action “means” one should consider the fact that the canonical action for a one-form $A$ in six-dimensions with effective charge $e$ is,

$$\int d^6x \frac{1}{4e^2} |dA|^2. \quad (3.8)$$

Hence, in the action (3.7) we have a one-form $a$ with effective charge order $V_{10,IIA}^{-1/2}$. Also, we have a one-form $\tilde{a}$ with effective charge order $V_{10,IIA}^{1/2}$ and 22 one-forms $C^I$ with effective charge order 1.

Now, as was found in [2,14], the mass of RR charged particles with charge $e$ and coupling $\lambda_6,IIA$ is of order $e/\lambda_6,IIA$. Hence, if we let $\lambda_6,IIA$ be constant and take $V_{10,IIA} \to \infty$, then we see that the RR particles charged with respect to the one-form $a$ have a mass which goes to zero in this limit and RR particles charged with respect to $\tilde{a}$ have a mass which goes to infinity in this limit. This is analogous to behavior we found for the Heterotic string on $T^3 \times S^1$ in the limit $R_{7,H} \to \infty$, and the matching of these two behaviors will be the basis for our map from the moduli space of the Heterotic string theory to the moduli space of the Type IIA string theory.

In particular, if we look at the mass of lightest RR particles in the $V_{10,IIA} \to \infty$ limit in the Type IIA string theory on $K3$, then we find they are charged with respect to the one-form $a$, and hence, we see that their mass is given by

$$M_{6,IIA} = \frac{1}{V_{10,IIA}^{1/2} \lambda_6,IIA}. \quad (3.9)$$
Similarly, if we consider the Heterotic string theory on $T^3 \times S^1$ in the limit $R_{7,H} \to \infty$, then we can see that the lightest states are those which have a non-zero momentum about $S^1$. Their mass is of order,

$$M_{6,H} = \frac{1}{R_{7,H}}$$

(3.10)

So, if we wish to identify these two classes of particles with vanishing mass, then we should identify $M_{6,IIA}$ and $M_{6,H}$. However, before doing so we should note that the six-dimensional Type IIA metric is scaled relative to the six-dimensional Heterotic metric in accord with (3.3). Hence, we must also scale the masses before equating them. Doing so, we find,

$$M_{6,H} = \lambda_{6,H}^{-1} M_{6,IIA} = \lambda_{6,IIA} M_{6,IIA},$$

(3.11)

where we have employed (3.2) to write the second equality. Hence, (3.9), (3.10), and (3.11) together imply

$$V_{10,IIA} = R_{7,H}^2,$$

(3.12)

as was promised in (3.4).

Now, as we have the relation (3.12) between $R_{7,H}^2$ and $V_{10,IIA}$, we can relate the Heterotic string theory limit $R_{7,H} \to \infty$ to a limit in the Type IIA string theory. The limit in which the Heterotic string theory on $T^3 \times S^1$ has the radius $R_{7,H}$ of the $S^1$ factor go to infinity corresponds, by way of (3.12), to the volume of the $K3$ factor in the Type IIA compactification going to infinity. Now, let us relate this to M-Theory.

As proven by Witten [14], M-Theory on $X_{10} \times S^1$ is equivalent to the Type IIA string theory on $X_{10}$. So, in particular, Type IIA string theory on $K3$ is equivalent to M-Theory on $K3 \times S^1$, and thus by way of string-string duality [2] M-Theory on $K3 \times S^1$ is equivalent to the Heterotic string theory on $T^3 \times S^1$. Witten also proved [14] the M-Theory metric $g_{10,M}$ on $X_{10}$ and the Type IIA metric $G_{10,IIA}$ on $X_{10}$ are related by,

$$g_{10,M} = \lambda_{10,IIA}^{-2/3} G_{10,IIA}.$$

(3.13)

So, as the M-Theory metric is a scaled version of the Type IIA metric, we can see that if the $K3$ volume goes to infinity as measured in the Type IIA metric, then the $K3$ volume as measured in the M-Theory metric need not go to infinity. To understand this let us look at the relation between the Type IIA metric and the M-Theory metric in a bit more detail.
Consider the fact that the Type IIA coupling constant in ten-dimensions is related to the Type IIA coupling constant $\lambda_{6,IIA}$ of the Type IIA string theory on $K3$ by the standard “compactification relation” obtained by integrating over $K3$,

$$\frac{V_{10,IIA}}{\lambda^2_{10,IIA}} = \frac{1}{\lambda^2_{6,IIA}}. \quad (3.14)$$

In turn, as a result of (3.2) and (3.12) we have,

$$\lambda_{10,IIA} = \frac{R^{3/2}}{\lambda^{2/3}}_{7,H}. \quad (3.15)$$

Now, as we know that the Type IIA metric and the M-Theory metric are related as in (3.13), we see that the volume of $K3$ in the M-Theory metric $V_{10,M}$ is actually finite in the limit $R_{7,H} \to \infty$. We have,

$$V_{10,M} = \lambda^{-4/3}_{10,IIA} V_{10,IIA} = \lambda^{4/3}_{7,H} R^{-2}_{7,H} V_{10,IIA} = \lambda^{4/3}_{7,H}, \quad (3.16)$$

where the first equality follows from (3.13), the second from (3.15), and the final from (3.12).

Furthermore, Witten [14] also found that in the equivalence between M-Theory on $X_{10} \times S^1$ and the Type IIA string theory, the $S^1$ radius $R_{11}$ as measured in the M-Theory metric is related to the Type IIA coupling constant $\lambda_{10,IIA}$ in the following manner,

$$R_{11} = \frac{\lambda^{2/3}}{\lambda_{10,IIA}}. \quad (3.17)$$

Employing (3.15) we have,

$$R_{11} = \frac{R_{7,H}}{\lambda^{2/3}}_{7,H}. \quad (3.18)$$

Hence, in the limit $R_{7,H} \to \infty$ for finite $\lambda_{7,H}$, the radius $R_{11}$ of the $S^1$ in the M-Theory compactification on $K3 \times S^1$ goes to infinity.

In summary, we have found that, by way of string-string duality and the Type IIA/M-Theory equivalence, M-Theory on $K3 \times S^1$ is equivalent to the Heterotic string theory on $T^3 \times S^1$. Furthermore, in the limit in which the $S^1$ radius of the Heterotic compactification goes to infinity, the above equivalence dictates that the $S^1$ radius of the M-Theory compactification also goes to infinity. Thus, this limit establishes the equivalence between the Heterotic string theory on $T^3$ and M-Theory on $K3$, as promised at the beginning of this subsection.
3.2. M-Theory on K3: Enhanced Gauge Symmetries

In this subsection we will examine the points of enhanced gauge symmetry which occur in the moduli space of M-Theory on K3. As we established in the previous subsection, M-Theory on K3 is equivalent to the Heterotic string theory on $T^3$. Hence, as the Heterotic string theory on $T^3$ possess various points of enhanced gauge symmetry in its moduli space, M-Theory on K3 must also possess various points of enhanced gauge symmetry in its moduli space. So, let us now examine these points of enhanced gauge symmetry.

Consider the Heterotic string theory compactified on $T^3$. Generically, the three-torus $T^3$ is specified by six parameters, three radii and three “angles.” As the “angles” are not central to our concerns here, we will ignore them by assuming that the three-torus $T^3$ upon which the Heterotic string theory is compactified is a “right” torus and thus specified by giving three radii. Let us notate these three radii, as measured in the ten-dimensional metric of an un-compactified Heterotic string theory, by $R_{i,H}$, where $i \in \{8, 9, 10\}$. Now, with this limitation imposed upon the Heterotic string theory moduli space, let us consider the construction of a map from the points of enhanced gauge symmetry in the Heterotic string theory moduli space to the corresponding points of enhanced gauge symmetry in the M-Theory moduli space. We will start this process by ascertaining which states of M-Theory on K3 may give rise to the gauge particles with which we are concerned.

M-Theory [18] is a theory with two-branes and five-branes. However, as is well-known from standard quantum field theory, gauge fields are zero-branes. Hence, to obtain an enhanced gauge symmetry in M-Theory on K3 we must exhibit a zero-brane gauge field which results from the two-branes and five-branes of M-Theory. The manner in which to do this was first found by Stromminger [10]. One can take a $p$-brane, say, and wrap it around a $q$-cycle of the compact manifold in question, which is K3 in our case. This produces a $(p-q)$-brane in the non-compactified portion of the theory. So, for instance, in our case we are starting with a two-brane and a five-brane in eleven-dimensions. We can wrap the two-brane or five-brane about any cycle in the homology ring of K3. However, as our goal is to obtain zero-branes in the resultant seven-dimensional theory, we should look for two-cycles and/or five-cycles. But, K3 is four-dimensional; hence, it possess no five-cycles. Our only hope then, is to look for two-cycles. In fact, K3 has 22 two-cycles [2]. So, there are plenty of two-cycles around which we can wrap the M-Theory two-brane to obtain a zero-brane in seven-dimensions. Next, let us consider how this gives us or does not give us the massless gauge particles for which we are looking.
Consider M-Theory compactified on $K3$. As shown above, the “wrapped” two-branes give rise to zero-branes in the resultant seven-dimensional theory. The zero-branes in the resultant seven-dimensional theory will have mass of order the area of the two-cycle about which they are wrapped. In particular $[15]$, the mass of these zero-branes is the tension of the two-brane times the area of the two-cycle. Let us notate this by,

$$M_{i,M} = T_2 A_{i,M}$$

(3.19)

where $A_{i,M}$ is the area of the $i$-th two-cycle as measured in the M-Theory metric, $T_2$ is the tension of the two-brane, and $M_{i,M}$ is the mass of the resultant zero-brane as measured in the M-Theory metric. Hence, as the area of a given two-cycle goes to zero we obtain a massless zero-brane in the seven-dimensional world. It is such massless zero-branes which we will identify with the “enhanced” gauge fields of the Heterotic theory on $T^3$. But, to do so we will need a few more details about both M-Theory on $K3$ and the Heterotic theory on $T^3$.

In the Heterotic string theory on $T^3$ for a given $S^1$ of radius $R_{i,H}$, there are states which wind about this $S^1$ and have mass order $R_{i,H}$. Similarly, there are also non-winding states which have a non-zero momentum along this $S^1$ and have mass of order $1/R_{i,H}$, and there are states which do not wind about the $S^1$ and have no momentum along $S^1$. However, by way of our general discussion above, the non-winding states are one-branes in seven dimensions, and the winding states are zero-branes in seven-dimensions. So, if we were to try and identify one of these three types of states with the wrapped two-brane states of M-Theory, we would try and identify the winding states with the wrapped two-brane states. Furthermore, this identification agrees with now standard results of string theory $[17]$. In the Heterotic string theory on $T^3$, the states which are becoming massless at the points of enhanced gauge symmetry are Heterotic strings which wind and have momentum along the $S^1$ which is becoming “critical.” Hence, as we wish to identify the wrapped two-branes of M-Theory with the gauge particles of the Heterotic string theory’s enhanced gauge symmetry, we will identify the wrapped M-Theory two-branes with the Heterotic one-branes which are wrapped and have momentum along an $S^1$.

As we have established on a conceptual level that we are identifying the wrapped two-branes of M-Theory with the wrapped string states of the Heterotic theory, let us now employ this information to derive a relation between the coordinates on the moduli space of M-Theory and the coordinates on the moduli space of the Heterotic string theory. To
do so, we will need to look at the mass relation for the Heterotic one-branes which wrap about and have a non-zero momentum along a given $S^1$. In particular, consider an $S^1$ with radius $R_{i,H}$, then the mass of such states is given by \[ M_{i,H} = \frac{1}{R_{i,H}} + R_{i,H} - 2. \] (3.20)

Now, to identify the coordinates parameterizing the moduli space of M-Theory on $K3$ and those parameterizing the moduli space of the Heterotic string theory on $T^3$ we must identify $M_{i,M}$ and $M_{i,H}$. However, we should be careful in doing so as the masses are measured in different seven-dimensional metrics. Hence, to write an equality expressing the relation of $M_{i,M}$ to $M_{i,H}$ we must find a relation between the M-Theory seven-metric $g_{7,M}$ and the Heterotic seven-metric $G_{7,H}$. However, as our line of inquiry does not require an equality, we will simply satisfy ourselves with a proportionality. We have,

\[ T_2 A_{i,M} \sim \left( \frac{1}{R_{i,H}} + R_{i,H} - 2 \right). \] (3.21)

This is the map for which we have been looking, the map from the moduli space of M-Theory on $K3$ to the moduli space of the Heterotic string theory on $T^3$. Of course one should note, as was the case in our identification of $V_{10,IIA}$ and $R_{6,H}^2$, this identification does not take into account the action of the T-Duality group on the moduli of the Heterotic theory, and hence would be changed greatly if one acted on the moduli with such a transformation.

Now, at this point the observant reader may feel a bit cheated. The $K3$ upon which M-Theory is compactified contains 22 two-cycles. Hence, there exist 22 ways of obtaining an enhanced gauge symmetry when a two-cycle collapses. This in-turn leads to 22 different $M_{i,M}$’s. However, on the Heterotic side we only had 3 $M_{i,H}$’s corresponding to the three different radii of $T^3$. So, it seems that the M-Theory side contains may more ways of obtaining an enhanced gauge symmetry than the Heterotic string theory side does. This is, however, is not really the case.

The Heterotic string theory, in addition to the three radii explicitly in $T^3$, contains 16 internal radii. So, in total we now have 19 radii, but $19 \neq 22$. However, any one of the three radii of $T^3$ give two enhanced gauge symmetries, i.e. as one of the radii $R_{i,H}$ of $T^3$ becomes self-dual, there exist two $U(1)$’s which become two $SU(2)$’s. Hence, they are double counted. Thus, on the Heterotic side we now have $16+3+3 = 22$ ways of obtaining an enhanced gauge symmetry. This matches exactly with the 22 two-cycles of $K3$ that
may collapse and give an enhanced gauge symmetry on the M-Theory side. So, we were not really cheating in our derivation, just holding our cards a little close to our chest.

Next, let us see if we can learn anything new about String Theory from the existence of these points of enhanced gauge symmetry in the moduli space of M-Theory.

3.3. Surprises?

In this subsection we will quickly examine some implications that the points of enhanced gauge symmetry in the moduli space of M-Theory on $K3$ have upon String Theory. In particular, we will examine the points of enhanced gauge symmetry present in moduli space of the Type IIA string theory on $K3$ which can be derived from the points of enhanced gauge symmetry present in the moduli space of M-Theory on $K3$.

Consider M-Theory on $K3$, as we saw above, when various two-cycles in $K3$ go to zero area there arise various points of enhanced gauge symmetry in the moduli space of M-Theory on $K3$. Furthermore, as Witten proved [14], M-Theory on $S^1$ is equivalent to the Type IIA string theory. Hence, M-Theory on $K3 \times S^1$ is equivalent to Type IIA string theory on $K3$. Furthermore, as there exist points of enhanced gauge symmetry in the moduli space of M-Theory on $K3$, there also exist points of enhanced gauge symmetry in the moduli space of M-Theory on $K3 \times S^1$. In particular, some of them arise in the same manner we discussed earlier, a two-brane wraps about a collapsing two-cycle. So, due the equivalence of M-Theory on $K3 \times S^1$ and the Type IIA string theory on $K3$, one should expect points of enhanced gauge symmetry to arise in the moduli space of the Type IIA string theory on $K3$. In fact these symmetries do arise, as has been known [15]. Type IIA string theory on $K3$ is equivalent to the Heterotic string theory on $T^4$ [9]. Hence, as there exist points of enhanced gauge symmetry in the moduli space of the Heterotic string theory on $T^4$ there should also exist corresponding points in the moduli space of the Type IIA string theory on $K3$.

The means by which some of the points of enhanced gauge symmetry arise was established by Witten [15]. Witten found that some of the points of enhanced gauge symmetry in the moduli space of the Type IIA string theory on $K3$ arise from two-branes of the Type IIA string theory wrapping about collapsing two-cycles of $K3$, the exact explanation we put forward in the context of M-Theory. Furthermore, by way of the equivalence of M-Theory on $S^1$ to the Type IIA string theory and the vanishing of $H_1(K3)$, we see that the behaviors are exactly the same. Hence, we have found that the behaviors of the Type IIA string theory on $K3$ and M-Theory on $K3$ “dove-tail” very nicely. Both obtain points
of enhanced gauge symmetry in their respective moduli spaces from the wrapping of two-branes about collapsing two-cycles and, by way of the M-Theory/Type IIA equivalence \[14\], we see that these are indeed one and the same behavior.

4. Enhanced Gauge Symmetry in M-Theory : M-Theory on $T^5/\mathbb{Z}_2 \times S^1$

In this section we will examine the enhanced gauge symmetries which appear in the moduli space of M-Theory on $T^5/\mathbb{Z}_2 \times S^1$. To do so, however, we will make use of the equivalence between M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ and the Heterotic string theory on $T^5$. So, we will, in the next subsection, derive this result.

4.1. M-Theory on $T^5/\mathbb{Z}_2 \times S^1 \sim$ Heterotic String Theory on $T^5$.

In this subsection we will prove that M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ is equivalent to the Heterotic string theory on $T^5$. M-Theory on $S^1$, as was proven by Witten \[14\], is equivalent to the Type IIA string theory. Hence, M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ is equivalent to the Type IIA string theory on $T^5/\mathbb{Z}_2$. By, definition \[16\], the Type I' string theory on $T^5$ is equivalent to the Type IIA string theory on $T^5/\mathbb{Z}_2$. Furthermore, the Type I' string theory on $T^5$ is equivalent, by way of T-Duality, to the Type I string theory on $T^5$. Also, the Type I theory on $T^5$ is equivalent by way of Type I/Heterotic “duality” \[16\] to the Heterotic string theory on $T^5$ \[17\].

So, if we trace through all these relations, we find that M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ is equivalent to the Heterotic string theory on $T^5$. Another way of looking at this is by way of the equivalence of M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ with the Type IIB string theory on $K3 \times S^1$. This equivalence was established by Dasgupta, Mukhi, and Witten \[3\][18]. By way of T-Duality \[8\] the Type IIB string theory on $K3 \times S^1$ is equivalent to the Type IIA string theory on $K3 \times S^1$. Furthermore, by way of string-string duality the Type IIA string theory on $K3 \times S^1$ is equivalent to the Heterotic string theory on $T^5$. So, chaining together these results one again finds that M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ is equivalent to the Heterotic string theory on $T^5$. Now, let us employ this to examine the points of enhanced gauge symmetry in the moduli space of M-Theory on $T^5/\mathbb{Z}_2 \times S^1$.  

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4.2. M-Theory on $T^5/\mathbb{Z}_2 \times S^1$: Enhanced Gauge Symmetries

Now, as we have established that M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ is equivalent to the Heterotic string theory on $T^5$, let us use this information to examine the points of enhanced gauge symmetry in the moduli space of M-Theory on $T^5/\mathbb{Z}_2 \times S^1$.

As we mentioned in the previous section, the Heterotic string theory on $T^5/\mathbb{Z}_2 \times S^1$, in the previous section we dealt with $T^3$ but the remarks apply also to $T^5$, possess points of enhanced gauge symmetry in its moduli space. These points occur in the moduli space when a given radii of $T^5$ becomes self-dual or one of the sixteen internal radii becomes self-dual. Hence, by way of the equivalence of M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ with the Heterotic string theory on $T^5$, M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ should also possess points of enhanced gauge symmetry in its moduli space. To examine these points let us “step-wise” compactify M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ by compactifying first on $T^5/\mathbb{Z}_2$, then compactifying the resultant theory on $S^1$.

As the low-energy limit of M-Theory in eleven-dimensions is eleven-dimensional supergravity \[16\], M-Theory possess a three-form $A_3$ in its spectrum. This three-form \[1\] gives rise to the two-brane and, through duality, the five-brane of M-Theory in eleven-dimensions. Now, when we compactify M-Theory on $T^5/\mathbb{Z}_2$ the $A_3 \wedge A_3 \wedge F_4$ interaction \[1\] of eleven-dimensional supergravity implies that $A_3$ is odd under the action of $\mathbb{Z}_2$. Hence, $A_3$ does not yield any vectors or three-forms when M-Theory is compactified upon $T^5/\mathbb{Z}_2$. The compactification of $A_3$ yields five two-forms and ten-scalars\[3\]. These two-forms \[1\] give rise to one-branes in six-dimensions. Now, upon compactifying the resultant six-dimensional theory on a further $S^1$, these one-branes can wrap about the $S^1$. Hence, as $1 - 1 = 0$ these one-branes yield zero-branes in five-dimensions. These zero-branes can yield some of the points of enhanced gauge symmetry for which we are looking.

As we mentioned in the previous section, the mass of a zero-brane which arises from a $p$-brane wrapping about a $p$-cycle is the tension of the $p$-brane, $T_p$ say, times the area of the $p$-cycle, $A_p$ say. So, if we notate the tension of a one-brane which arises from compactifying M-Theory on $T^5/\mathbb{Z}_2$ as $T_1$, then the mass of the five-dimensional zero-brane resultant from wrapping this one-brane about the $S^1$ of radius $R_{6,M}$ is of order $T_1 R_{6,M}$. Thus, we can easily obtain massless zero-branes in five-dimensions in the limit $R_{6,H} \rightarrow 0$. These are some of the gauge particles for which we are looking.

\[3\] One may easily derive this by looking at the cohomology ring of $T^5$. 

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However, one may wonder if there are any other means through which enhanced gauge symmetries may arise in the compactification of M-Theory on $T^5/\mathbb{Z}_2 \times S^1$. A quick look at the Heterotic side will tell us that indeed this is the case. As the Heterotic theory on $T^5$ is equivalent to M-Theory on $T^5/\mathbb{Z}_2 \times S^1$, we can check to see if we have explored all of the enhanced gauge symmetries by counting the various ones present in the Heterotic compactification on $T^5$. In the Heterotic theory, we have sixteen internal radii to adjust as well as five others corresponding to the $T^5$ upon which the Heterotic string is compactified. As we found five-two forms and thus five one-branes in the compactification of M-Theory on $T^5/\mathbb{Z}_2$, these can map to the five different enhanced gauge symmetries present in the Heterotic theory which arise as a result of varying the $T^5$ radii. However, as the Heterotic string theory also possess sixteen internal radii we seem at a bit of a loss. However, we can rectify this by examining the “twisted” sector of M-Theory on $T^5/\mathbb{Z}_2$.

To do so we will have to take a step back from the task at hand and examine for a moment the origin of twisted sector states in the compactification of M-Theory on $T^5/\mathbb{Z}_2$. This particular subject was considered in detail by Dasgupta, Mukhi, and Witten. We will simply touch upon the relevant portions of their work; if the reader is interested in a more detailed account, they should consult the primary sources referenced above.

In the work of Dasgupta, Mukhi, and Witten, it was proven that M-Theory on $T^5/\mathbb{Z}_2$ is equivalent to the Type IIB string theory on $K3$. So, in examining the twisted sector of M-Theory on $T^5/\mathbb{Z}_2$ we are free to examine either the Type IIB string theory on $K3$ or M-Theory on $T^5/\mathbb{Z}_2$. We will, for the moment, study the Type IIB string theory on $K3$. Type IIB string theory on $K3$ yields a chiral $N = 4$ theory. The supergravity multiplet of this chiral $N = 4$ theory contains a graviton, five self-dual two-forms, and various gravitinos. This gravitational multiplet is, however, anomalous in six-dimensions and hence requires the addition of various other multiplets to cancel the anomaly. The choice of such multiplets, thankfully, is rather easy. There is actually only one possible type of matter multiplet in six-dimensional $N = 4$ chiral supergravity, a tensor multiplet. This tensor multiplet contains an anti-self-dual two-form, five scalars, and various chiral spinors. Cancellation of the anomaly due to the gravitational multiplet requires that the six-dimensional theory possess twenty one such tensor multiplets.

Now, looking at our previous results, M-Theory on $T^5/\mathbb{Z}_2$ possess five two-forms. The two-forms may be split into self-dual and anti-self-dual portions. Hence, we obtain five anti-self-dual two-forms and five self-dual two-forms. Including the other bosonic and fermionic modes in the compactification of M-Theory on $T^5/\mathbb{Z}_2$ one finds that the five
self-dual two-forms are part of the gravitational multiplet and the five anti-self-dual two-forms are part of five tensor multiplets. Hence, from the untwisted sector one obtains just the gravitational multiplet and five tensor multiplets. However, this spectrum, as it does not contain the requisite twenty one tensor multiplets, is anomalous. This is the origin of the “twisted” states. One must add to the theory extra states not present in the standard compactification of M-Theory on $T^5/Z_2$ to yield a non-anomalous theory. The question now is, exactly how does one add such states to the theory?

This was answered by Witten \[18\]. Witten found that there are an infinite variety of ways to add these states to the theory, but each of these many ways must satisfy a set of constraints which we will presently review.

Consider the three-form $A_3$ in eleven-dimensional M-Theory. As this three-form \[1\] is a gauge field, objects may possess an electric or magnetic charge with respect to $A_3$. Witten found that one could take the fixed points of $T^5/Z_2$ as magnetic sources of $A_3$ charge. This was, however, subject to the constraints that the sum of all magnetic charges on $T^5/Z_2$ must be zero and no fixed point may have charge less than $-1/2$. Furthermore, the manifold $T^5/Z_2$ must also possess the $21 - 5 = 16$ tensor multiplets which are needed for anomaly cancellation.

Now, let us consider an example of such a configuration which satisfies all of the above constraints. One could consider placing a magnetic charge $-1/2$ at all $2^5 = 32$ of $T^5/Z_2$’s fixed points and also placing at 16 of the fixed points 16 different five-branes. A five-brane is actually a magnetic source of $A_3$ charge \[1\] \[18\] with magnetic charge 1. The five-brane also supports a chiral $N = 4$ tensor multiplet on its world-volume. Hence, with this configuration we have satisfied all of the above requirements. The sum of the magnetic charges is $32(-1/2) + 16(1) = 0$, no fixed point has charge less than $-1/2$, and we have the sixteen extra tensor multiplets supported by the sixteen five-branes. In another more general configuration one could consider placing a magnetic charge $-1/2$ at each of the 32 fixed points and locating the sixteen five-branes at generic points on $T^5/Z_2$. This again satisfies all the above constraints, and hence, it is a configuration which the theory may take.

Now, let us consider what this implies about M-Theory on $T^5/Z_2 \times S^1$. As we mentioned earlier, we have five enhanced gauge symmetries arising from the five one-branes of M-Theory on $T^5/Z_2$ wrapping about the collapsing one-cycle $S^1$. We need sixteen more ways to obtain an enhanced gauge symmetry so as to match up with the Heterotic theory
on $T^5$. The fact that we also require sixteen five-branes on $T^5/Z_2$ is, as we will see, more than just a coincidence.

As we stated previously, each one of the five-branes supports a chiral $N = 4$ tensor multiplet. This tensor multiplet consists of an anti-self-dual two-form along with five scalars and various chiral fermions. The physical interpretation of these scalars can be ascertained by considering the standard bosonic string. The bosonic world-sheet theory generically has twenty-six scalars, however, one can rid the theory of two, the dimension of the world-sheet, of these twenty-six scalars as a result of the action of the world-sheet diffeomorphism group. In a similar light, consider the world-volume of the five-brane before gauge fixing. As M-Theory is an eleven-dimensional theory the world-volume theory would generically have eleven scalars corresponding to the eleven coordinates. However, one could rid the theory of six of these scalars by employing the diffeomorphism group of the world-volume theory. This would leave the world-volume theory with five scalars, the exact number we found earlier by other means. These five scalars, which are part of the tensor multiplet, thus correspond to physical displacements of the five-brane in the ambient eleven-dimensional world.

Let us consider a configuration of five-branes and $A_3$ charges in which the the sixteen five-branes reside at sixteen of the thirty-two fixed points of $T^5/Z_2$ and in addition there is a magnetic charge of $-1/2$ at each of the thirty-two fixed points. This configuration, as we found earlier, satisfies all the relevant constraints. Now, as we have a physical interpretation for the five scalars on the five-brane world-volume, we can interpret them in this configuration. For a given five-brane the values of its five scalars give its position on the five manifold $T^5/Z_2$. In this configuration the scalars of the five-branes all take on values which land the five-branes at the fixed points of $T^5/Z_2$. However, one could also consider a case in which the values of the five scalars for any given five-brane vary a bit taking the five-brane away from its fixed point. As this new situation also satisfies all the relevant constraints, it is also physical. So, letting the world-volume scalars vary, the generic configuration consists of a set of sixteen five-branes at generic locations on $T^5/Z_2$ determined by the values of the various world-volume scalars. Now, let us consider compactifying this generic configuration on $S^1$.

A five-brane is a six-dimensional object; $T^5/Z_2$ is a five-dimensional object. In the generic configuration of five-branes which we are considering, any given five-brane and $T^5/Z_2$ are transverse. Hence, when we compactify on $S^1$ all sixteen of the five-branes must wrap around the $S^1$ just as a result of dimension counting. Now, as the five-brane
world-volume possess a chiral $N = 4$ tensor multiplet which supports, among other fields, an anti-self-dual tensor, the five-brane world-volume also supports a one-brane. This one-brane, upon compactifying on $S^1$, has states which wrap about this $S^1$ and states which do not wrap about this $S^1$. The states which wrap about this $S^1$ appear in the resultant five-dimensional world as zero-branes and hence, if rendered massless, the gauge fields for which we are looking. So, again, we are now in a familiar situation. A $p$-brane wrapping about a $p$-cycle. As we found earlier, the mass of such a state is the tension of the $p$-brane, $T_p$ say, times the area of the $p$-cycle, $A_p$ say. In the case at hand, it is the tension of the one-brane on the five-brane world-volume times the length of the $S^1$ upon which we are compactifying. Hence, as the $S^1$ collapses or the one-brane tension vanishes we obtain an enhanced gauge symmetry in the same manner we found in earlier examples. However, now our counting works correctly.

As we previously mentioned, M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ is equivalent to the Heterotic string theory on $T^5$. Varying the radii of the $T^5$ as well as the sixteen internal radii of the Heterotic string theory one can see that there are twenty one different ways to obtain an enhanced gauge symmetry in the Heterotic string on $T^5$. Previously, we were unable to find all these different enhanced gauge symmetries in the M-Theory compactification on $T^5/\mathbb{Z}_2 \times S^1$; now, with our above comments on the various five-branes in the theory, we are able to do so. As we found above, the untwisted sector of M-Theory on $T^5/\mathbb{Z}_2$ possess five two-forms. These five two-forms give rise to five one-branes. Upon compactifying further on an $S^1$ these five one-branes can wrap about this $S^1$ and give rise to five different ways of obtaining an enhanced gauge symmetry as the $S^1$ collapses. These five different ways of obtaining an enhanced gauge symmetry correspond to the five different radii of the $T^5$ factor in the Heterotic string theory compactification becoming self-dual. Now, on the Heterotic side we have $21 - 5 = 16$ other ways in which to obtain an enhanced gauge symmetry. Originally, on the M-Theory side we did not know the origin of these sixteen extra enhanced gauge symmetries, now we do. Generically, the sixteen five branes in the M-Theory compactification on $T^5/\mathbb{Z}_2$ are transverse to $T^5/\mathbb{Z}_2$. So, upon further compactification on $S^1$ they wrap about $S^1$. Each of these five-branes also supports a tensor multiplet which contains a two-form. Thus, each five-brane has a one-brane living on its world-volume. Upon wrapping these sixteen five-branes on $S^1$, each of these sixteen one-branes may also wrap about this $S^1$. Counting dimensions, these sixteen wrapped one-branes give rise to sixteen zero-forms in five-dimensions, and as the $S^1$ collapses or the one-brane tension vanishes, these sixteen one-branes give rise to sixteen “enhanced”
gauge fields. This mechanism thus provides the sixteen missing ways in which to obtain an enhanced gauge symmetry in the M-Theory picture. Hence, we now have an understanding of all twenty one ways in which to obtain an enhanced gauge symmetry in the M-Theory compactification on $T^5/\mathbb{Z}_2 \times S^1$.

4.3. Surprises?

In this subsection we will briefly comment on the enhanced gauge symmetries present in the moduli space of the Type IIA string theory on $T^5/\mathbb{Z}_2$. As M-Theory on $S^1$ is equivalent to the Type IIA string theory \cite{14}, M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ is equivalent to the Type IIA string theory on $T^5/\mathbb{Z}_2$. Hence, the enhanced gauge symmetries in the moduli space of M-Theory on $T^5/\mathbb{Z}_2 \times S^1$ demand the existence of enhanced gauge symmetries in the moduli space of the Type IIA string theory on $T^5/\mathbb{Z}_2$.

Now, these enhanced gauge symmetries in the moduli space of the Type IIA string theory on $T^5/\mathbb{Z}_2$ actually come as no surprise. Type IIA string theory on $T^5/\mathbb{Z}_2$ is by definition equivalent to Type I' string theory on $T^5$. Type I' string theory on $T^5$ is T-Dual to Type I string theory on $T^5$ \cite{16}. Also, Type I string theory on $T^5$ is equivalent to the Heterotic string theory on $T^5$ by way of the Heterotic/Type I duality \cite{16}. Hence, as the Heterotic string theory on $T^5$ exhibits points of enhanced gauge symmetry in its moduli space, so the Type IIA string theory on $T^5/\mathbb{Z}_2$ should also exhibit points of enhanced gauge symmetry in its moduli space. Notice that we did not rely upon M-Theory to reach this conclusion. So, our derivation of the existence of these points of enhanced gauge symmetry from a M-Theory point-of-view acts as a very strong check on the various relations between M-Theory and String Theory and the various enhanced gauge symmetries of both.

5. Conclusion

In this article we have examined various points of enhanced gauge symmetry in the moduli space of M-Theory on $S^1 \times S^1/\mathbb{Z}_2$, M-Theory on $K3$, and M-Theory on $T^5/\mathbb{Z}_2 \times S^1$. In each case we found an interesting interplay of M-Theory and String Theory symmetries. But, our inquiry left open a few interesting questions which deserve further study. It would be nice to understand the points of enhanced gauge symmetry in the strong coupling region of Figure 1 from a perturbative point-of-view. Also, it would be nice to explore in a bit more detail the interplay of the one-branes on the world-volume of the five-brane and their interaction with spacetime fields. A bit of this interplay between the one-branes on the
five-brane world-sheet and the spacetime fields is given by our above comments. Also, some of this interplay is suggested by the comments made by Witten in [18]. However, it would be interesting to fully understand these one-branes and how they affect spacetime physics.
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