The ERA of FOLE: Foundation

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Abstract. This paper discusses the representation of ontologies in the first-order logical environment FOLE. An ontology defines the primitives with which to model the knowledge resources for a community of discourse. These primitives consist of classes, relationships and properties. An ontology uses formal axioms to constrain the interpretation of these primitives. In short, an ontology specifies a logical theory. This paper continues the discussion of the representation and interpretation of ontologies in the first-order logical environment FOLE. The formalism and semantics of (many-sorted) first-order logic can be developed in both a classification form and an interpretation form. Two papers, the current paper, defining the concept of a structure, and “The ERA of FOLE: Superstructure”, defining the concept of a sound logic, represent the classification form, corresponding to ideas discussed in the “Information Flow Framework”. Two papers, “The FOLE Table”, defining the concept of a relational table, and “The FOLE Database”, defining the concept of a relational database, represent the interpretation form, expanding on material found in the paper “DatabaseSemantics”. Although the classification form follows the entity-relationship-attribute data model of Chen, the interpretation form incorporates the relational data model of Codd. A fifth paper “FOLE Equivalence” proves that the classification form is equivalent to the interpretation form. In general, the FOLE representation uses a conceptual structures approach, that is completely compatible with the theory of institutions, formal concept analysis and information flow.

Keywords: entity, attribute, relationship, schema, universe, structure.
1 Introduction

1.1 Philosophy

A conceptual model for a community represents the information needed by the community — the content, relationships and constraints necessary to describe the community. The content consists of things of significance to the community (entities), and characteristics of those things (attributes). The relationships are associations between those things. The entities are the core concepts that are used for representing the semantics of the community. Entities are described by attributes, which are the various properties, characteristics, modifiers, aspects or features of entities. Hence, the entity-relationship-attribute (ERA) formalism is a ternary representation for knowledge, since it uses three kinds for representation: entities, attributes and relations. In contrast, the first-order logical environment FOLE (Kent [14]) followed the knowledge representation approach of traditional many-sorted first-order logic (MSFOL). Both the original FOLE formalism and the MSFOL formalism are binary representations for knowledge, since they use two kinds for representation: entities and relations.

However, the first-order logical environment FOLE can very naturally represent the ERA data model. The idea is that the original FOLE relation represented a nexus of roles, where the roles were played by the original FOLE entities. In order to represent the ERA data model, we think of the original FOLE relations as the new FOLE entities described by a nexus of features or aspects, where the aspects are represented by the new FOLE attributes, which replace the original FOLE entities. In the FOLE representation of the ERA data model, entities and their attributes are primary notions, whereas relationships are secondary notions that are subsumed by other constructs. Some relations (foreign keys, subtypes, sums, ...) have a special representation in FOLE; whereas, other relations can be resolved into concepts (entities) with a nexus of roles.

Tbl. 1 shows the terminological correspondence between the basic components of (old/new) FOLE and ERA. For example, the original FOLE entity type is renamed the new FOLE attribute type (sort), and this corresponds to the ERA attribute type (data type); and the original FOLE relation instance is renamed the new FOLE entity instance (key), and this corresponds to the ERA entity.

| FOLE (old) | FOLE (new) | ERA       |
|------------|------------|-----------|
| relation   | entity     | ∼ entity  |
| entity     | ∼ attribute| ∼ attribute|

Table 1. FOLE-ERA Correspondence

As commonly observed, an entity is a thing capable of an independent existence that can be uniquely identified. In natural language, an entity corresponds to a noun. A relationship links entities, and corresponds to a verb in natural
language. Entities and relationships can both have attributes. In natural language, a relational attribute corresponds to a role or case. Inclusion and subtype relationships are special kinds of relationships. A data model can be visualized in terms of entities, relationships and attributes. But in general, relationships can be conceptualized by being converted to entities.

Hence, a data model is more simply conceptualized in terms of entities and attributes. When doing so, there is an implied boundary around the conceptualization, which converts an entity’s collection of attributes into a list (possibly infinite in size or arity); a signature is the list of attribute types (sorts) associated with an entity type, whereas a tuple is the list of attribute instances (values) associated with an entity instance (key). Entity types can be mapped to the associated signature, and entity instances (primary keys) identify and can be mapped to the associated tuple (horizontal dimension of Fig. 1). In general, types classify instances. Hence, entity types classify keys and sorts classify values (vertical dimension of Fig. 1). Implicit from the ERA data model is an entity type system and multi-sorted logic, which uses boolean operators and quantification, and is defined in terms of signature-based fibers of formulas (queries) in Kent [16].

In review, the simplest way to handle things is first to distinguish types from instances along the FOLE classification dimension in Fig. 1 and second to view things (either types or instances) as participating in Whitehead’s fundamental prehension relationship (Sowa [21]) along the FOLE hypergraph dimension in Fig. 1, which links a prehending thing called an entity to a prehended thing called an attribute: “an entity has an attribute”. The ERA data model of FOLE uses an inclusive prehension for things; hence, it is a mixed data model; some entities are not attributes, some attributes are not entities, and some are both. For any type in the overlap $E \cap A$, any instance of that type is also in the overlap. Foreign keys are examples of things that are both entities and attributes, things in the overlap.
The author’s “Systems Consequence” paper (Kent [12]) is a very general theory and methodology for specification and inter-operation of systems of information resources. The generality comes from the fact that it is independent of the logical/semantic system (institution) being used. This is a wide-ranging theory, based upon ideas from information flow (Barwise and Seligman [1]), formal concept analysis (Ganter and Wille et al. [5]), the theory of institutions (Goguen et al. [7]), and the lattice of theories notion (Sowa [21]), for the integration of both formal and semantic systems independent of logical environment. In order to better understand the motivations of that paper and to be able more readily to apply its concepts, in the future it will be important to study system consequence in various particular logical/semantic systems. This paper aims to do just that for the logical/semantic system of relational databases. The paper, which was inspired by and which extends a recent set of papers on the theory of relational database systems (Spivak [22],[23]), is linked with work on the Information Flow Framework (IFF [25]) connected with the ontology standards effort (SUO), since relational databases naturally embed into first order logic. We offer both an intuitive and a technical discussion. Corresponding to the notions of primary and foreign keys, relational database semantics takes two forms: a distinguished form where entities are distinguished from relations, and a unified form where relations and entities coincide. The distinguished form corresponds to the theory presented in the paper (Spivak [22]). We extend Spivak’s treatment of tables from the static case of a single entity classification (type specification) to the dynamic case of classifications varying along infomorphisms. Our treatment of relational databases as diagrams of tables differs from Spivak’s sheaf theory of databases. The unified form, a special case of the distinguished form, corresponds to the theory presented in the paper (Spivak [23]). The unified form has a graphical presentation, which corresponds to the sketch theory of databases (Johnson and Rosebrugh [11]) and the resource description framework (RDF). This paper, which is the first step to connect relational databases with system consequence, is concerned with the semantics of relational databases. Later papers will discuss various formalisms of relational databases, such as first order logic (Kent [16]) and relational algebra (Kent [20]).
1.2 Knowledge Representation

Many-sorted (multi-sorted) first-order predicate logic represents a community’s “universe of discourse” as a heterogeneous collection of objects by conceptually scaling the universe according to types. The relational model (Codd [4]) is an approach for the information management of a “community of discourse”, using the semantics and formalism of (many-sorted) first-order predicate logic. The relational model was initially discussed in two papers: “A Relational Model of Data for Large Shared Data Banks” by Codd [4] and “The Entity-Relationship Model – Toward a Unified View of Data” by Chen [2]. The relational model follows many-sorted logic by representing data in terms of many-sorted relations, subsets of the Cartesian product of multiple domains. All data is represented horizontally in terms of tuples, which are grouped vertically into relations. A database organized in terms of the relational model is called a relational database. The relational model provides a method for modeling the data stored in a relational database and for defining queries upon it.

1.3 First Order Logical Environment

**Basics.** The first-order logical environment FOLE is a category-theoretic representation for many-sorted (multi-sorted) first-order predicate logic. The relational model can naturally be represented in FOLE. The FOLE approach to logic, and hence to databases, relies upon two mathematical concepts: (1) lists and (2) classifications. Lists represent database signatures and tuples; classifications represent data-types and logical predicates. FOLE represents the header of a database table as a list of sorts, and represents the body of a database table as a set of tuples classified by the header. The notion of a list is common in category theory. The notion of a classification is described in two books: “Information Flow: The Logic of Distributed Systems” by Barwise and Seligman [1] and “Formal Concept Analysis: Mathematical Foundations” by Ganter and Wille [5].

**Architecture.** A series of papers provides a rigorous mathematical basis for FOLE by defining an architectural semantics for the relational data model, thus providing the foundation for the formalism and semantics of first-order logical/relational database systems. This architecture consists of two hierarchies of two nodes each: the classification hierarchy and the interpretation hierarchy.

- Two papers provide a precise mathematical basis for FOLE classification. The current paper “The ERA of FOLE: Foundation”, develops the notion of a FOLE

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1 Examples include: an academic discipline; a commercial enterprise; library science; the legal profession; etc.
2 Following the original discussion of FOLE (Kent [14]), we use the term mathematical context for the concept of a category, the term passage for the concept of a functor, and the term bridge for the concept of a natural transformation. A context represents some “species of mathematical structure”. A passage is a “natural construction on structures of one species, yielding structures of another species” (Goguen [6]).
structure, following the entity-relationship model of Chen [2]. This provides a basis for the paper “The ERA of FOLE: Superstructure” [16], which develops the notion of a FOLE sound logic.

- Two papers provide a precise mathematical basis for FOLE interpretation. Both of these papers expand on material found in the paper “Database Semantics” [13]. The paper “The FOLE Table” [17], develops the notion of a FOLE table, following the relational model of Codd [4]. This provided a basis for the paper “The FOLE Database” [17], which develops the notion of a FOLE relational database.

The architecture of FOLE is pictured briefly on the right and more completely in Fig. 1 of the preface of the paper [19]. This consists of two hierarchies of two nodes each. The paper “FOLE Equivalence” [19] proves that FOLE sound logics are equivalent to FOLE databases.

In the relational model there are two approaches for database management: the relational algebra, which defines an imperative language, and the relational calculus, which defines a declarative language. The paper “Relational Operations in FOLE” [20] represents relational algebra by expressing the relational operations of database theory in a clear and implementable representation. The relational calculus will be represented in FOLE in a future paper.

1.4 Overview

The first-order logical environment FOLE (Kent [14]) is a framework for defining the semantics and formalism of logic and databases in an integrated and coherent fashion. Institutions in general, and logical environments in particular, give equivalent heterogeneous and homogeneous representations for logical systems. FOLE is an institution, since “satisfaction is invariant under change of notation”. FOLE is a logical environment, since “satisfaction respects structure linkage”. As an institution, the architecture of FOLE consists of languages as indexing components, structures to represent semantic content, specifications to represent formal content, and logics to combine formalism with semantics. FOLE structures are interpreted as relational/logical databases.

In §2 and §3 we show how the ERA data model is represented in FOLE by connecting elements of the ERA data model to components of the FOLE structure concept. §2 discusses the direct lower-level connection between the ERA elements (attributes, entities, relations) and the FOLE components (type domains and entity classifications). §3 discusses the abstract higher-level representation.

3 The theory of classifications and infomorphisms is discussed in the book Information Flow by Barwise and Seligman [1].
of the ERA data model within the FOLE architecture. In addition, we give a rudimentary description of the interpretation of FOLE structures in §3.4. In §4 we connect FOLE to Sowa’s knowledge representation hierarchy (Sowa [21]) and through linearization to the Olog data model (Spivak and Kent [24]).

This paper, which is concerned with the FOLE foundation, is illustrated in Fig. 7 of §3.3 and is centered on the mathematical context of structures. FOLE structures sit at the bottom of the classification form of the FOLE architecture. The FOLE Superstructure (Kent [16]), which is concern with the formalism and semantics of first-order logic, and the FOLE interpretation, which is concerned with database interpretation, are presented in the two papers that follow this one. Two further papers are pending on the integration of federated systems of knowledge: one discusses integration over a fixed type domain and the other discusses integration over a fixed universe.

4 In a direct fashion, we show how the ERA entity notion is represented by the FOLE entity classification, and how the ERA attribute notion is represented by the FOLE type domain (attribute classification). In an indirect fashion, we show how the ERA relation notion is represented in general by the FOLE designation plus mixed data model, and in particular by the sequents in FOLE specifications (discussed further in Kent [16]).

5 Following the original discussion of FOLE (Kent [14]), we use “mathematical context” for the mathematical term “category”, “passage” for the term “functor”, and “bridge” for the term “natural transformation”.

Table 2. Figures and Tables

| Fig. 1 | ERA Data Model in FOLE |
| Fig. 2 | Example |
| Fig. 3 | Structure |
| Fig. 4 | Interpreted Structure |
| Fig. 5 | Structure Morphism |
| Fig. 6 | Interpreted Structure Morphism |
| Fig. 7 | FOLE Foundation |
| Fig. 8 | Analogy |
2 ERA Data Model

2.1 Attributes.

In the ERA data model, attributes are represented by a typed domain consisting of a collection of data types. In FOLE, a typed domain is represented by an attribute classification \( \mathcal{A} = \langle X, Y, \models_{\mathcal{A}} \rangle \) consisting of a set of attribute types (sorts) \( X \), a set of attribute instances (data values) \( Y \) and an attribute classification relation \( \models_{\mathcal{A}} \subseteq X \times Y \). For each sort (attribute type) \( x \in X \), the data domain of that type is the \( \mathcal{A}\text{-extent} \mathcal{A}_x = \text{ext}_{\mathcal{A}}(x) = \{ y \in Y \mid y \models_{\mathcal{A}} x \} \). The passage \( X \xrightarrow{\text{ext}_{\mathcal{A}}} Y \) maps a sort \( x \in X \) to its data domain \( \mathcal{A}\text{-extent} \mathcal{A}_x \subseteq Y \).

An \( X\text{-signature} \) (header) is a sort list \( \langle I, s \rangle \), where \( I \xrightarrow{\cdot} X \) is a map from an indexing set (arity) \( I \) to the set of sorts \( X \). A more visual representation for this signature is \( (\cdots s_i \cdots | i \in I) \). The mathematical context of \( X\text{-signatures} \) is \( \text{List}(X) \). \( \text{List}(Y) \) is a map from an indexing set (arity) \( J \) to the set of data values \( Y \). A more visual representation for this tuple is \( (\cdots t_j \cdots | j \in J) \). The mathematical context of \( Y\text{-tuples} \) is \( \text{List}(Y) \). The attribute list classification \( \text{List}(\mathcal{A}) = \langle \text{List}(X), \text{List}(Y), \models_{\text{List}(\mathcal{A})} \rangle \) has \( X\text{-signatures} \) as types and \( Y\text{-tuples} \) as instances, with classification by common arity and universal \( \mathcal{A}\text{-classification} \): a \( Y\text{-tuple} \( \langle J, t \rangle \) is classified by an \( X\text{-signature} \( \langle I, s \rangle \) when \( J = I \) and \( t_k \models_{\mathcal{A}} s_k \) for all \( k \in J = I \).

2.2 Entities.

We distinguish between an entity instance and an entity type. An entity type is a category of existence; entity types classify entity instances. There might be many instances of an entity type, and an entity instance can be classified by many types. An entity instance (entity, for short) is also called an object. Every entity is uniquely identified by a key. In FOLE, entities and their types are collected together locally in an entity classification \( \mathcal{E} = \langle R, K, \models_{\mathcal{E}} \rangle \) consisting of a set of entity types \( R \), a set of entity instances (keys) \( K \) and an entity classification relation \( \models_{\mathcal{E}} \subseteq R \times K \). In the database interpretation in \( \text{ERA} \) each entity type \( r \in R \) is regarded to be the name for a relation (or table) in the database: for each entity type (relation name) \( r \in R \), the set of primary keys for that type is the \( \mathcal{E}\text{-extent} \mathcal{E}_r = \text{ext}_{\mathcal{E}}(r) = \{ k \in K \mid k \models_{\mathcal{E}} r \} \).

\(^6\) \( \text{List}(X) \) is the comma context \( \text{List}(X) = (\text{Set} \downarrow X) \) of \( X\text{-signatures} \), where an object \( (I, s) \) is an \( X\text{-signature} \) and a morphism \( (I', s') \xrightarrow{h} (I, s) \) is an arity function \( I' \xrightarrow{h} I \) that preserves signatures \( h \cdot s = s' \); visually, \( (\cdots t_j \cdots | j \in J) \) is \( \langle \cdots s_i \cdots | i \in I \rangle \).

\(^7\) The header for a database table is a signature (list of sorts) \( \langle I, s \rangle \in \text{List}(X) \). Pairs \( (i : s) \) from a signature \( (I, s) \) are called attributes (see §3.4). Examples of attributes are ‘(name : Str)’, ‘(age : Natno)’.
2.3 Relations.

Here we discuss how the relational aspect of the ERA data model is handled in FOLE. Some relations are special. One example is subtyping, which specifies that one category of existence is more general than another. This arises when representing the taxonomic aspect of ontologies. Subtyping is handled by the binary sequents $\varphi \vdash \psi$ in FOLE specifications (discussed further in the (Kent 16)).

Some many-to-one relationships can be represented as attributes. But in general, many-to-many relationships are represented in FOLE as entities, whose attributes, each of which plays a thematic role for the relationship, may be other entities.\(^8\)

Consider the example (Fig. 2) of a simple entity-relationship-attribute diagram. Here we have three entities (represented by rectangles), two relationships (represented by diamonds) and numerous attributes (represented by ovals). The works_on relationship is many-to-many, and so we can represent this in FOLE as an entity type Activity with four attributes: entry_date of sort Date, job_descr of sort String, employee of sort Employee, and project of sort Project. Note that attributes employee and project are foreign keys of the Activity entity. Since the works_for relationship is many-to-one without any attributes of its own, we can represent this as an attribute called dept of sort Department. This is a foreign key of the Employee entity.

\(^8\) A sequent $\varphi \vdash \psi$ expresses interpretation widening between formulas.

\(^9\) As an example, the “marriage” binary relation can be represented as a Marriage entity with wife and husband attributes that are themselves Person entities.

\(^{10}\) The Employee type, which plays the employee thematic role for the works_on relationship, is both an entity type and an attribute type (sort); any value in the Employee data domain is a key of the Employee entity and a foreign key of the Activity entity.
Fig. 2. Example
3 FOLE Components

3.1 Schema.

The type aspect of the ERA data model is gathered together into a schema. A schema $S = \langle R, \sigma, X \rangle$ consists of a set of sorts (attribute types) $X$, a set of entity types $R$ and a signature map $R \overset{\sigma}{\rightarrow} \text{List}(X)$. Within the schema $S$, we think of each $r \in R$ as being an entity type that is locally described by the associated $X$-signature $\sigma(r) = \langle I, s \rangle \in \text{List}(X)$. A more visual representation for this signature mapping is $r \overset{\sigma}{\rightarrow} \langle \cdots s_i \cdots | i \in I \rangle$. An ERA-style visualization might be $\text{Person} \overset{\text{name}}{\rightarrow} \text{String}$ or $\text{Employee} \overset{\text{dept}}{\rightarrow} \text{Department}$.

The entity type $r$ in the ERA data model corresponds to the relation symbol $r$ in FOLE/MSFOL. Either representation is a kind of nexus. A schema corresponds to a multi-sorted first-order logical language in the FOLE/MSFOL approach to knowledge representation. In the database interpretation of FOLE (Kent [18]), we think of $r$ as being a relation name with associated header $\sigma(r) = \langle I, s \rangle$.

We formally link schemas with morphisms. A schema morphism $S_2 = \langle R_2, \sigma_2, X_2 \rangle \overset{\langle r, f \rangle}{\rightarrow} \langle R_1, \sigma_1, X_1 \rangle = S_1$ from schema $S_2$ to schema $S_1$ consists of an sort function $f : X_2 \rightarrow X_1$ and an entity type function $r : R_2 \rightarrow R_1$, which preserve signatures by satisfying the condition $r \cdot \sigma_1 = \sigma_2 \cdot \sum f$.

$$\begin{align*}
S_2 & \left\{ \begin{array}{l}
  r_2 \in R_2 \\
  \sigma_2 \downarrow \\
  \langle I, s \rangle = \langle \cdots s_i \cdots | i \in I \rangle \\
  s_i \in X_2 \\
  \sum f
\end{array} \right\}
\quad
\text{\rightarrow}
\quad
\begin{array}{l}
  r(r_2) \in R_1 \\
  \sigma_1 \\
  \langle \cdots f(s_i) \cdots | i \in I \rangle = \sum f(I, s) \\
  f(s_i) \in X_1
\end{array}
\quad
S_1
\end{align*}$$

Let $\text{Sch}$ denote the mathematical context of schemas and their morphisms.

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11 There is an associated arity function $R \overset{\sigma \circ set}{\rightarrow} \text{Set} : r \overset{\sigma}{\rightarrow} \langle I, s \rangle \overset{\text{set}}{\rightarrow} I$.
12 Formulas based on relation symbols can be inductively defined, thus forming extended schemas (Kent [13]). Terms composed of function symbols can be added as constraints between formulas.
3.2 Universe.

The instance aspect of the ERA data model is gathered together into a universe. A universe $\mathcal{U} = (K, \tau, Y)$ consists of a set of values (attribute instances) $Y$, a set of keys (entity instances) $K$ and a tuple map $K \xrightarrow{\tau} \text{List}(Y)$. Within the universe $\mathcal{U}$, we think of each key $k \in K$ as being an identifier or name for an object that is locally described by the associated tuple of values $\tau(k) = (J, t) \in \text{List}(Y)$. A more visual representation for this tuple mapping is $k \xrightarrow{\tau} (\cdots t_j \cdots | j \in J)$. Note that, no typing has been mentioned here and no typing restrictions are required. In a universe by itself, we do not require the data values $t_j$ to be members of any special data-types.

An element of a universe $\mathcal{U} = (K, \tau, Y) \in \text{Univ}$ is a key $(k \xrightarrow{\tau} (J, t))$ with associated list. We can think of such universe elements as object descriptions without attached typing or as tuples untethered from a database table. They develop meaning by being classified by schema elements $(r \xrightarrow{\sigma} (I, s))$ in a structure (§3.3). Hence, a FOLE universe is like the key-value-list store at the heart of Google’s Spanner database (Google [5]):

“Spanner’s data model is not purely relational, in that rows must have names. More precisely, every table is required to have an ordered set of one or more primary-key columns. This requirement is where Spanner still looks like a key-value store: the primary keys form the name for a row, and each table defines a mapping from the primary-key columns to the non-primary-key columns.”

We semantically link universes with morphisms. A universe morphism $\mathcal{U}_2 = (K_2, \tau_2, Y_2) \xleftarrow{(k,g)} (K_1, \tau_1, Y_1) = \mathcal{U}_1$ consists of a value (attribute instance) function $Y_2 \xleftarrow{g} Y_1$ and a key (entity instance) function $K_2 \xleftarrow{k} K_1$, which preserve tuples (instance lists) by satisfying the condition $k \cdot \tau_2 = \tau_1 \cdot \sum g$.

\[
\begin{align*}
\mathcal{U}_2 & \quad \begin{cases}
\quad k(k_1) \in K_2 \\
\quad \tau_2 \downarrow \\
\quad \sum g(J, t) = (\cdots g(t_j) \cdots | j \in J) \\
\quad g(t_j) \in Y_2
\end{cases} & \quad \begin{cases}
\quad k_1 \in K_1 \\
\quad \downarrow \tau_1 \\
\quad \sum g(J, t) = (\cdots t_j \cdots | j \in J) \\
\quad t_j \in Y_1
\end{cases} \\
\mathcal{U}_1 & \quad \end{align*}
\]

Let $\text{Univ}$ denote the mathematical context of universes and their morphisms.

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13 They are somewhat like genes (bits of DNA) without the genomic structure that provides interpretation.

14 When the universe $\mathcal{U}$ is the instance aspect of a FOLE structure $\mathcal{M}$ with typed domain $\mathcal{A}$, in the database interpretation of that structure (§5.3 and Kent [17]), we think of the entity instance $k$ as being a primary key that indexes a row $\tau(k) = (I, t)$ in the table associated with the relation symbol $r \in R$ with associated header $\sigma(r) = (I, s)$. A more visual representation for this tuple mapping is $k \xrightarrow{\tau} (\cdots t_i \cdots | i \in I, t_i \in \mathcal{A}_{s_i})$, where $\mathcal{A}_{s_i}$ is the data-type for sort $s_i \in X$. Here, we do require the data values $t_i$ to be members of the special data-types $\mathcal{A}_{s_i}$. 
3.3 Structure.

The complete ERA data model is incorporated into the notion of a (model-theoretic) structure in the FOLE representation of knowledge.

Structures. A FOLE structure $\mathcal{M} = \langle \mathcal{E}, \sigma, \tau, \mathcal{A} \rangle$ is a hypergraph of classifications (Fig. 3) — a two-dimensional construct with the following components:

- attribute classification: typed domain $\mathcal{A} = \text{attr}(\mathcal{M}) = \langle X, Y, \models \mathcal{A} \rangle$
- entity classification: $\mathcal{E} = \text{ent}(\mathcal{M}) = \langle R, K, \models \mathcal{E} \rangle$
- type hypergraph: schema $\mathcal{S} = \text{sch}(\mathcal{M}) = \langle R, \sigma, X \rangle$
- instance hypergraph: universe $\mathcal{U} = \text{univ}(\mathcal{M}) = \langle K, \tau, Y \rangle$

and a list designation $\langle \sigma, \tau \rangle : \mathcal{E} \models \text{List}(\mathcal{A})$ with signature map $R \xrightarrow{\sigma} \text{List}(X)$ and tuple map $K \xrightarrow{\tau} \text{List}(Y)$, whose defining condition states that: if entity $k \in K$ is of type $r \in R$, then the description tuple $\tau(k) = \langle J, t \rangle$ is the same "size" ($J = I$) as the signature $\sigma(r) = \langle I, s \rangle$ and each data value $t_n$ is of sort $s_n$; or interpretively ($\S$ 3.4 and Kent [17]), in a database table all rows are classified by the table header. A FOLE structure embodies the idea of an ERA data model (compare Fig. 3 with Fig. 1). Each community of discourse that incorporates the ERA data model will have its own local FOLE structure. [15]

The entity and attribute-list classifications $\mathcal{E}$ and $\text{List}(\mathcal{A})$ are equivalent [16] to their extent diagrams $R \xrightarrow{\text{ext}_r} \text{Set}$ and $\text{List}(X) \xrightarrow{\text{tup}_A} \text{Set}$, and the list designation is equivalent to its extent diagram morphism $\text{ext} : \langle \sigma, \tau \rangle : \langle R, \text{ext}_{\mathcal{E}} \rangle \rightarrow \langle \text{List}(X), \text{ext}_{\text{List}(\mathcal{A})} \rangle$ consisting of the signature map $R \xrightarrow{\sigma} \text{List}(X)$

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15 In anticipation of the discussion in § 3.4, we illustrate the associated tabular interpretation [3] on the right side of Fig. 3.

16 Any classification $\mathcal{A} = \langle X, Y, \models \mathcal{A} \rangle$ is equivalent to its extent map $X \xrightarrow{\text{ext}_A} \wp Y$. 

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Fig. 3. Structure
and the bridge $\text{ext}_E \Rightarrow \sigma \circ \text{ext}_{\text{List}(A)}$, whose $r^{th}$-component is the tuple function $K(r) = \text{ext}_E(r) \rightarrow \text{ext}_{\text{List}(A)}(\sigma(r))$ (see the tabular interpretation in §3.1). Hence, any structure $\mathcal{M}$ has the interpretive presentation in Fig. 4 (see the discussion on linearization §4.2).

In the concept of a FOLE structure we have abstracted the (primary) keys from the tuples that they described. The key-embedding construction replaces keys into their tuples.

**Definition 1.** (key embedding) Any FOLE structure $\mathcal{M} = (\mathcal{E}, \sigma, \tau, A)$ with signature map $R \xrightarrow{\sigma} \text{List}(X) : r \mapsto \langle I, s \rangle$ and tuple map $\tau \xrightarrow{\tau} \text{List}(Y) : k \mapsto \langle I, t \rangle$ has a companion key embedding structure $\mathcal{\mathcal{M}} = (\mathcal{E}, \dot{\sigma}, \dot{\tau}, A)$ consisting of entity classification $\mathcal{E}$, parallel sum typed domain $\mathcal{E}+A = (R+X, K+Y, =_{\mathcal{E}+A})$, schema $(R, \dot{\sigma}, R+X)$ with signature map $R \xrightarrow{\dot{\sigma}} \text{List}(R+X) : r \mapsto \langle 1, r \rangle + \langle I, s \rangle$, and universe $(K, \dot{\tau}, K+Y)$ with tuple map $\dot{\tau} \xrightarrow{\dot{\tau}} \text{List}(K+Y) : k \mapsto \langle 1, k \rangle + \langle I, t \rangle$. The signature and tuple maps are injective.

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17 We can think of the entity classification $\mathcal{E} = (R, K, =_{\mathcal{E}})$ as a type domain. For each sort (attribute type) $r \in R$, the data domain of that type is the $\mathcal{E}$-extent $\mathcal{E}_r = \text{ext}_{\mathcal{E}}(r) = \{ k \in K \mid k =_{\mathcal{E}} r \}$. The passage $R \xrightarrow{\text{ext}_{\mathcal{E}}} \varnothing K$ maps a sort $r \in R$ to its data domain ($\mathcal{E}$-extent) $\mathcal{E}_r \subseteq K$. 

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Structure Morphisms. In order to allow communities of discourse to interoperate, we define the notion of a morphism between two structures that respects the ERA data model. A structure morphism $\mathcal{M}_2 \xleftarrow{\langle r,k,f,g \rangle} \mathcal{M}_1$ (Fig. 5) from source structure $\mathcal{M}_2 = (\mathcal{E}_2, \langle \sigma_2, \tau_2 \rangle, \mathcal{A}_2)$ to target structure $\mathcal{M}_1 = (\mathcal{E}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{A}_1)$ is defined in terms of the hypergraph and classification morphisms between the source and target structure components (projections):

- **typed domain morphism**: $\mathcal{A}_2 = \text{attr}(\mathcal{M}_2) \xrightarrow{\langle f,g \rangle} \text{attr}(\mathcal{M}_1) = \mathcal{A}_1$
- **entity infomorphism**: $\mathcal{E}_2 = \text{ent}(\mathcal{M}_2) \xrightarrow{\langle r,k \rangle} \text{ent}(\mathcal{M}_1) = \mathcal{E}_1$
- **schema morphism**: $\mathcal{S}_2 = \text{sch}(\mathcal{M}_2) \xrightarrow{(r,f)} \text{sch}(\mathcal{M}_1) = \mathcal{S}_1$
- **universe morphism**: $\mathcal{U}_2 = \text{univ}(\mathcal{M}_2) \xrightarrow{(k,g)} \text{univ}(\mathcal{M}_1) = \mathcal{U}_1$

satisfying the conditions

- **typed domain morphism**: $g(y_1) \models_{\mathcal{A}_2} x_2 \iff y_1 \models_{\mathcal{A}_1} f(x_2)$
- **entity infomorphism**: $k(k_1) \models_{\mathcal{E}_2} r_2 \iff k_1 \models_{\mathcal{E}_1} r(r_2)$
- **schema morphism**: $r \cdot \sigma_1 = \sigma_2 \cdot \Sigma_f$
- **universe morphism**: $k \cdot \tau_2 = \tau_1 \cdot \Sigma_g$

The designation defining condition states that for any $k_1 \in K_1$ and $r_2 \in R_2$,

$$\left( k(k_1) \models_{\mathcal{E}_2} r_2 \iff k_1 \models_{\mathcal{E}_1} r(r_2) \right) \text{ implies } \left( \tau_2(k(k_1)) = \Sigma_3(\tau_1(k_1)) \models_{\text{List}(A_2)} \sigma_2(r_2) \iff \tau_1(k_1) \models_{\text{List}(A_1)} \Sigma_f(\sigma_2(r_2)) = \sigma_1(r(r_2)) \right).$$
Structure morphisms compose component-wise. Let \textbf{Struc} denote the context of structures and structure morphisms. In the appendix \S\,A we develop \textbf{Struc} as a fibered mathematical context in two orientations: either as the Grothendieck construction of the schema indexed mathematical context of structures \( \text{Sch}^{\text{op}} \overset{\text{struc}}{\to} \text{Cxt} \) or as the Grothendieck construction of the universe indexed mathematical context of structures \( \text{Univ}^{\text{op}} \overset{\text{struc}}{\to} \text{Cxt} \). The schema indexed mathematical context of structures is used to establish the institutional aspect of FOLE.

Any structure morphism \( \mathcal{M}_2 \overset{(r,k,f,1_Y)}{\longrightarrow} \mathcal{M}_1 \) with identity value map \( Y \overset{1_Y}{\rightarrow} Y \) has the interpretive presentation in Fig. 6.

\begin{itemize}
  \item entity type map: \( R_2 \overset{r}{\rightarrow} R_1 \)
  \item sort map: \( X_2 \overset{f}{\leftrightarrow} X_1 \)
  \item key map: \( K_2 \overset{k}{\leftrightarrow} K_1 \)
  \item entity infomorphism: \( r \cdot \text{ext}_{A_1} = \text{ext}_{A_2} \cdot k^{-1} \)
  \item type domain morphism: \( \Sigma_f \cdot \text{ext}_{\text{List}(A_1)} = \text{ext}_{\text{List}(A_2)} \)
  \item schema morphism: \( r \cdot \sigma_1 = \sigma_2 \cdot \Sigma_f \)
  \item universe morphism: \( \varphi r_2 \supseteq k^{-1} \cdot \varphi r_1 \)
\end{itemize}

\begin{center}
\begin{tikzpicture}
  \node (X1) at (0,0) {\text{List}(X_1)};
  \node (X2) at (3,0) {\text{List}(X_2)};
  \node (Y1) at (0,-3) {\text{List}(Y)};
  \node (Y2) at (3,-3) {\text{List}(Y)};
  \node (R1) at (6,0) {R_1};
  \node (R2) at (6,-3) {R_2};

  \draw[->] (X1) to node [above] {$\Sigma_f$} (X2);
  \draw[->] (Y1) to node [below] {$\varphi r_1$} (Y2);
  \draw[->] (R1) to node [left] {$r$} (R2);
  \draw[->] (X2) to node [below] {$\Sigma_f \cdot \text{ext}_{\text{List}(A_2)}$} (Y2);
  \draw[->] (X1) to node [above] {$\text{ext}_{\text{List}(A_1)}$} (Y1);
  \draw[->] (R2) to node [above] {$\sigma_2$} (X2);
  \draw[->] (R1) to node [above] {$\sigma_1$} (X1);
  \draw[->] (Y2) to node [below] {$\varphi r_2$} (X2);
  \draw[->] (Y1) to node [below] {$\varphi r_1$} (X1);

  \end{tikzpicture}
\end{center}

Fig. 6. Interpreted Structure Morphism

---

\(^{18}\) Any infomorphism \( \mathcal{A}_2 \overset{(f,g)}{\leftrightarrow} \mathcal{A}_1 \) has the equivalent condition \( f \cdot \text{ext}_{\mathcal{A}_1} = \text{ext}_{\mathcal{A}_2} \cdot g^{-1} \).
Proof. The following conditions must hold.

Domain integrity specifies that all columns in a relational database

\[ \mathcal{M}_2 = (E_2, \sigma_2, \tau_2, E_2+A_2) \xrightarrow{(r,k,f,g)} (E_1, \sigma_1, \tau_1, E_1+A_1) = \mathcal{M}_1. \]

with the following components:

- **typed domain morphism**
  \[ E_2+A_2 \xrightarrow{(r+f,k+g)} E_1+A_1 \]
- **entity infomorphism**
  \[ E_2 \xrightarrow{(r,k)} E_1 \]
- **schema morphism**
  \[ S_2 = (R_2, \sigma_2, R_2+X_2) \xrightarrow{(r,r+f)} (R_1, \sigma_1, R_1+X_1) = S_1 \]
- **universe morphism**
  \[ U_2 = (K_2, \tau_2, K_2+Y_2) \xleftarrow{(k,k+g)} (K_1, \tau_1, K_1+Y_1) = U_1 \]

Proof. The following conditions must hold.

- **typed domain morphism**
  \[ k(k_1) \mid x \xrightarrow{r_2} \ x_2 \]
  \[ g(y_1) \mid y_1 \xrightarrow{f(x_2)} f(r_1) \]
- **entity infomorphism**
  \[ k(k_1) \mid x \xrightarrow{r_2} \]
  \[ g(y_1) \mid y_1 \xrightarrow{f(r_2)} \]
- **schema morphism**
  \[ r \cdot \sigma_1 = \sigma_2 \cdot \sum_{r+f} \]
- **universe morphism**
  \[ k \cdot \tau_2 = \tau_1 \cdot \sum_{k+g} \]

We use the comparable conditions for the original structure morphism \( \mathcal{M}_2 \xrightarrow{(r,k,f,g)} \mathcal{M}_1 \). The entity infomorphism condition is given. The type domain morphism condition is straightforward. We show the schema morphism condition. The universe morphism condition is similar. The schema morphism condition for the original structure morphism is \( r \cdot \sigma_1 = \sigma_2 \cdot \sum_f \); that is, for any \( r_2 \in R_2 \), if \( \sigma_2(r_2) = \langle I_2, s_2 \rangle \) and \( \sigma_1(r(r_2)) = \langle I_1, s_1 \rangle \), then \( \langle I_1, s_1 \rangle = \sum_f(I_2, s_2) \). Hence, the schema morphism condition for the key-embedding structure morphism holds, since \( \sigma_1(r(r_2)) = \langle 1, r(r_2) \rangle + \langle I_1, s_1 \rangle = \sum_f(I_2, s_2) = \sum_f(\langle I_2, s_2 \rangle) \]

Integrity Constraints. Integrity constraints help preserve the validity and consistency of data. Here we briefly explain how various integrity constraints are represented in the ERA data model of FOLE.

**Entity:** (primary key rule) Entity integrity states that every table must have a primary key and that the column or columns chosen to be the primary key should be unique and not null. In the ERA data model of FOLE, entity integrity asserts that the universe \( U = \langle K, \tau, Y \rangle \) of a structure \( \mathcal{M} \) is well-defined.

**Domain:** Domain integrity specifies that all columns in a relational database must be declared upon a defined domain. In the ERA data model of FOLE, domain integrity asserts that the schema \( S = \langle R, \sigma, X \rangle \) and the list designation \( \langle \sigma, \tau \rangle : E \equiv \text{List}(A) \) of a structure \( \mathcal{M} \) are well-defined.

**Referential:** (foreign key rule) Referential integrity states that the foreign-key value of a source table refers to a primary key value of a target table. In the ERA data model of FOLE, referential integrity asserts that the ERA data model of FOLE is a mixed data model.
Algebra. For simplicity of presentation, this paper and the paper on FOLE superstructure (Kent [16]) use a simplified form of FOLE, in contrast to the full form presented in Kent [14]. In this paper and Kent [16], schemas are used in place of (many-sorted) first-order logical languages. Schemas are simplified logical languages without function symbols. The main practical result is that signature morphisms \( \langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle \) are replaced by term vectors \( \langle I', s' \rangle \xrightarrow{t} \langle I, s \rangle \) in the full version of FOLE. Signature morphisms are simplified term vectors without function symbols. In the full version of FOLE, equations can be defined between parallel pairs of term vectors \( \langle I', s' \rangle \xrightarrow{t,t'} \langle I, s \rangle \), thus allowing the use of equational presentations and their congruences. Also in the full version of FOLE, the tuple map along signature morphisms becomes the algebraic operation along term vectors; hence, formula flow (substitution/quantification) in Kent [16] is lifted from being along signature morphisms to being along term vectors.

Fig. 7. FOLE Foundation

19 The tuple relational calculus is a query language for relational databases. In order to use the tuple calculus in the FOLE, we need to enrich with many-sorted constant declarations and equational presentations. Constant declarations are first-order logical languages with sorted nullary function symbols. A constant \( c \) of sort \( x \) is an \( x \)-sorted nullary function symbol \( x \xrightarrow{c} \langle \emptyset, 0_X \rangle \).

20 Let \( \text{FOLE-ARCH} \) denote Fig. 1 in Kent [14]. \( \text{FOLE-ARCH} \) is the 3-dimensional visualization of the fibered architecture of FOLE. The upper right quadrant of Fig. 7 corresponds to the the 2-D prism below \( \text{Rel} \) in \( \text{FOLE-ARCH} \). As indicated in \( \text{FOLE-ARCH} \), to move from the simple version of the FOLE foundation used here (Fig. 7) to the full version in \( \text{FOLE-ARCH} \), we lift from sort sets to algebraic languages and from typed domains to many-sorted algebras.
3.4 Interpretation.

In the model theory for traditional many-sorted first-order logic, a (possible world, model) structure corresponds to an interpretation of relation symbols (entity types) in terms of relations in a typed domain. The FOLE approach to logic replaces n-tuples with lists, defines quantification/substitution along signature morphisms (Kent [16]) (or term vectors in the full version [14]), and following databases, incorporates identifiers (keys) for data value lists (tuples) (here and in the paper on FOLE tables Kent [17]). The FOLE approach modifies the idea of model-theoretic interpretation as follows. 21

We assume that the traditional many-sorted first-order logic language is represented by the schema $S = \langle R, \sigma, X \rangle$ and that the typed domain is represented by the attribute classification $A = \langle X, Y, \mid \tau = A \rangle$. We further assume that these are components of a structure $M = \langle E, \langle \sigma, \tau \rangle, A \rangle$.

Traditional: In the traditional approach, an entity type $r \in R$ is interpreted as the set of descriptors of entities in the extent of $r$. For X-signature $\sigma(r) = \langle I, s \rangle \in \text{List}(X)$, this is the subset of $Y$-tuples $I_M(r) = \varphi(\text{ext}(r)) \in \varphi\text{tup}_A(I, s) = \varphi\text{ext}_\text{List}(A)(I, s)$, an element of the fiber relational order $\langle \text{Rel}_A(I, s), \subseteq \rangle$. This defines the traditional interpretation function

$$R \xrightarrow{I_M} \text{Rel}(A).$$  

For all $r \in R$, we have the relationships

$$\varphi(\text{ext}(r)) = I_M(r) \quad \text{ext}(r) \subseteq \tau^{-1}(I_M(r)).$$  

Definition 3. The inequality $\text{ext}(r) \subseteq \tau^{-1}(I_M(r))$ says that $\text{ext}(r)$ is not the morphic closure of itself w.r.t. the tuple map $K \xrightarrow{\tau} \text{List}(Y)$. A structure $M$ is called extensive when the right hand expression in (2) is an equality: $\text{ext}(r) = \tau^{-1}(I_M(r))$ for any entity type $r \in R$. 22

Any structure $M$ with an injective tuple map $K \xrightarrow{\tau} \text{List}(Y)$ has an associated extensive structure. An example is the key-embedding structure $\hat{M}$.

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21 Hence, the notions of (1) a many-sorted first-order logic interpretation, (2) a FOLE structure, and (3) an ERA data model are all equivalent; and each can be implemented as a relational database with associated logic.

22 The fibered context $\text{Rel}(A)$ is defined in the paper on FOLE tables (Kent [17]). An object of $\text{Rel}(A)$, called an $A$-relation, is a pair $\langle I, s, R \rangle$ consisting of an indexing $X$-signature $(I, s)$ and a subset of $A$-tuples $R \in \varphi\text{tup}_A(I, s) = \varphi\text{ext}_\text{List}(A)(I, s)$.

23 Philosophical note: In the knowledge resources for a community, the entities are of first importance. The tuples in $\text{List}(Y)$ are descriptors, which may or may not have an identity. An entity consists of an identifier $k \in K$ and its descriptor $\tau(k) \in \varphi(\tau(K))$. Tuples with identity are those in $\varphi(\tau(K)) \subseteq \text{List}(Y)$. Two entities that have the same descriptor are said to be “descriptor-equivalent”.
Tabular: In the database approach, an entity type $r \in R$ is interpreted as a table with the entities (both the keys and their descriptors) being explicit. For $X$-signature $\sigma(r) = (I, s) \in \text{List}(X)$, this is the $(I, s)$-indexed $\mathcal{A}$-table

$$K(r) \xrightarrow{\tau_r} \text{tup}_\mathcal{A}(I, s)$$

$$\tau_r(k) = (I, t), I \xrightarrow{\tau} Y$$

$T_M(r) = \langle K(r), \tau_r \rangle$ (visualized above) consisting of the key set $K(r) = \text{ext}_r(\mathcal{E}) \subseteq \varnothing K$ and the (descriptor) tuple function $K(r) \xrightarrow{\tau_r} \text{tup}_\mathcal{A}(I, s)$, a restriction of the tuple function $K \xrightarrow{\tau} \text{List}(Y)$ for the universe $U = \langle K, \tau, Y \rangle$. The tuple function factors through the traditional interpretation $\tau_r : K(r) \rightarrow I_M(r) \hookrightarrow \text{tup}_\mathcal{A}(I, s)$. This defines the tabular interpretation function

$$R \xrightarrow{T_M} \text{Tbl}(\mathcal{A}).$$

The companion key-embedding structure $\mathcal{M}$ (Def. 1) is most easily understood from its interpretation tables (visualized below).

$$K(r) \xrightarrow{\tau_r} \text{tup}_\mathcal{E}+\mathcal{A}(1+I, r+s)$$

$$\tau_r(k) = (1+I, k+t), 1 \xrightarrow{\tau} K, I \xrightarrow{\tau} Y$$

$\mathcal{M}$ is visualized above.

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24 The fibered context $\text{Tbl}(\mathcal{A})$ is defined in the paper on FOLE tables (Kent [17]). An object of $\text{Tbl}(\mathcal{A})$, which is called an $\mathcal{A}$-table, is a triple $(I, s, K, t)$ consisting of an $X$-signature $(I, s)$, a set of entities (keys) $K$, and a tuple function $K \xrightarrow{\tau} \text{tup}_\mathcal{A}(I, s)$ mapping each entity in $K$ to its descriptor $\mathcal{A}$-tuple.
4 Connections

4.1 Analogy.

In the original paper (Kent [14]) explaining the first-order logical environment FOLE, there was an analogy between the top-level ontological categories discussed in (Sowa [21]) and the components of the first-order logical environment FOLE. We recast that here in terms of the FOLE approach to the ERA data model. The analogy between the Sowa hierarchy and FOLE is illustrated graphically in Fig. 8. The twelve categories displayed in the hierarchy (we do not concern ourselves here with the temporal dimension given by the continuent-occurent distinction) and the primitives from which they are generated are arranged in the matrix of Tbl. 3 with the corresponding FOLE terminology in parentheses.

The physical-abstract distinction, which corresponds to the Heraclitus distinction physis-logos, is represented by the FOLE classification dimension, a connection between FOLE schemas and FOLE universes. The abstract category corresponds to FOLE types, either entities or attributes. The physical category corresponds to FOLE instances, either entities or attributes. The independent-relative-mediating triad is represented by the FOLE hypergraph dimension, a connection between FOLE entities and FOLE attributes. The independent category (firstness) corresponds to FOLE entities that are not attributes. The relative category (secondness) corresponds to FOLE attributes that are not entities. The mediating category (thirdness) corresponds to FOLE list maps between entities and attributes. The triads actuality-prehension-nexus and form-proposition-intention correspond to Whitehead’s categories of existence.

Fig. 8. Analogy

Here we explain this in more detail. As discussed in §3.3 a FOLE structure is a two-dimensional combination of a classification and a hypergraph. The entire hierarchy of the top-level ontological categories is represented by

\[ \text{hierarchy of top-level categories} \]

\[ \text{FOLE structure components} \]
a FOLE (model-theoretic) structure $\mathcal{M} = \langle \mathcal{E}, \langle \sigma, \tau \rangle, \mathcal{A} \rangle$. This is appropriate, since a (model-theoretic) structure represents the knowledge in the local world of a community of discourse. The form-proposition-intention triad is represented by a FOLE schema $\mathcal{S} = \langle R, \sigma, X \rangle$ with signature function $R \rightarrow \text{List}(X)$. The actuality-prehension-nexus triad is represented by a FOLE universe $\mathcal{U} = \langle K, \tau, Y \rangle$ with tuple function $K \rightarrow \text{List}(Y)$. The firstness subgraph of independent(actuality,form) is represented by a FOLE entity classification $\mathcal{E} = \langle R, K, \models \mathcal{E} \rangle$ between entity instances (keys) and entity types (or a classification between keys and logical formula, more generally; see Kent [16]). The secondness subgraph of relative(prehension,proposition) is represented by a FOLE attribute classification $\mathcal{A} = \langle X, Y, \models \mathcal{A} \rangle$ between attribute instances (data values) and attribute types (sorts). The thirdness subgraph of mediating(nexus,intention) is represented by a FOLE list designation $\langle \sigma, \tau \rangle : \mathcal{E} \Rightarrow \text{List}(\mathcal{A})$. A FOLE entity type loosely corresponds to a form = independent $\land$ abstract. A FOLE entity (key) loosely corresponds to an actuality = independent $\land$ physical. A FOLE sort (attribute type), which is not necessarily linked from a FOLE entity, loosely corresponds to a proposition = relative $\land$ abstract. A FOLE data value (attribute instance) loosely corresponds to a prehension $\land$ abstract.

### 4.2 Linearization.

The “ontology log” $\text{Olog}$ formalism (Spivak and Kent [24]) is a category-theoretic model for knowledge representation. As we indicate below, there is a sense in which the FOLE representation subsumes the $\text{Olog}$ representation (and vice-versa). Let $\mathcal{M}$ be a FOLE structure with schema $\mathcal{S} = \langle R, \sigma, X \rangle$, universe $\mathcal{U} = \langle K, \tau, Y \rangle$, type domain $\mathcal{A} = \langle X, Y, \models \mathcal{A} \rangle$, entity classification $\mathcal{E} = \langle R, K, \models \mathcal{E} \rangle$, and list designation $\langle \sigma, \tau \rangle : \mathcal{E} \Rightarrow \text{List}(\mathcal{A})$.

---

25 This section is closely related to the discussion of the sketch and interpretation associated with a unified relational database in § 4 of Kent [13].
By using the extent operator (§ 3.3), the following classifications and sets are informationally equivalent.

| classification | set |
|----------------|-----|
| \( \mathcal{E} \) | \( \prod \text{ext}_\mathcal{E} \) \( \prod \text{ext}_{\text{List}(A)} \) \( \{ (r, k) \mid r \in R, k \in K, r \models \mathcal{E} r \} \) |
| \( \mathcal{A} \) | \( \prod \text{ext}_\mathcal{A} \) \( \{ (x, y) \mid x \in X, y \in Y, r \models \mathcal{A} x \} \) |
| \( \text{List}(A) \) | \( \prod \text{ext}_{\text{List}(A)} \) \( \{ (I, s, t) \mid (I, s) \in \text{List}(X), (I, t) \in \text{List}(Y), (I, t) \models_{\text{List}(A)} (I, s) \} \) |

The designation property defines a function \( \prod \text{ext}_\mathcal{E} \prod \text{ext}_{\text{List}(A)} \) which is the sum of the extent diagram morphism \( \text{ext}_{(\sigma, r)} = \langle \mathcal{E}, \mathcal{A} \rangle \rightarrow \langle \text{List}(X), \text{ext}_{\text{List}(A)} \rangle \) (Fig. 1 of § 3.3). By flattening the resulting lists, for each \( (r, k) \in \prod \text{ext}_\mathcal{E} \), we know that \( (s_i, t_i) \in \prod \text{ext}_\mathcal{A} \) for each index \( i \in \alpha(r) = I \) in the common arity. Hence, the following linearization set \(^{26}\) is equivalent in information to the structure \( \mathcal{M} \):

\[
\text{lin}(\mathcal{M}) = \left\{ (r, k) \mid (I, s) \in \prod \text{ext}_\mathcal{E}, \sigma(r) = \langle I, s \rangle, \tau(k) = \langle I, t \rangle, i \in I \right\}.
\]

The FOLE representation is binary, since it has two kinds of type, sorts (attribute types) and entity types. The \( \text{Olog} \) representation is unary, since it has only one kind of type, the abstract concept. \(^{25}\) However, the FOLE representation can be transformed to the \( \text{Olog} \) representation by the process of linearization. If we restrict FOLE to the unified model, identifying entities with attributes \( \mathcal{E} = \mathcal{A} \), by separating the type-instance information in the linearization set \( \text{lin}(\mathcal{M}) \) via the extent operator along the classification dimension (Fig. 1), we get the basis for the \( \text{Olog} \) data model, consisting of three notions: types, aspects, and facts. Types, which represent things, are depicted by nodes in \( \text{Olog} \); aspects, which represent functional relationships between things, are depicted by edges in \( \text{Olog} \); and facts, which represent assertions, are depicted by path equations in \( \text{Olog} \).

\( \text{Olog} \) types and aspects are covered by the linearization process discussed here. \( \text{Olog} \) facts correspond to the formalism discussed (indirectly) in the FOLE superstructure (Kent \([16]\)) and (more directly) in the full form of FOLE (Kent \([14]\)).

\[\begin{align*}
\text{typ} &\quad \xrightarrow{(r.i)} \quad \text{typ} \\
\text{ext} &\quad \xrightarrow{\sigma} \quad \text{typ} \\
\text{ext} &\quad \xrightarrow{\tau} \quad \text{typ} \\
\text{M} &\quad \xrightarrow{M(i)} \quad \text{M} \\
\text{M} &\quad \xrightarrow{M(s_i)} \quad \text{M} \\
k &\quad \xrightarrow{\text{ext}(s_i)} \quad t_i
\end{align*}\]

\( S \) a schema context

\( M \) an interpretation passage

\( \text{Set} \) the context of sets

\(^{26}\) The linearization set \( \text{lin}(\mathcal{M}) \) loosely corresponds to the following: (1) the table used in the entity-attribute-value (EAV) data model, where data is recorded in three columns: the entity component is a foreign key into an object information table, the attribute component is a foreign key into an attribute definition table, and the value component is the value of the attribute; and (2) the labelled directed graph used in the resource description framework (RDF) data model, where each subject-predicate-object triple is regarded as an edge in the graph.

\(^{27}\) See § 4.4 of \([24]\) for further discussion of binary/unary knowledge representations.
**01og Database Schema S:** By projecting the type components out of the
linearization set \( \text{lin}(M) \), define a graph \( G \) whose node set is \( R = X \) and whose
edge set is \( \prod_{r \in R = X} \alpha(r) \). Picture the graph \( G \) as follows:

\[
G = \left\{ r \xrightarrow{(r,i)} s_i \mid \sigma(r) = \langle I, s_i \rangle, \ i \in I, \ with \ r, s_i \in R = X \right\}.
\]

The schema mathematical context \( S = G^* \) is the path context of graph \( G \).
The graph \( G \) corresponds to a many-sorted unary algebraic language \( \Omega = \langle X, \Omega \rangle \), where \( \Omega \) is essentially the opposite of \( G \): the collection of sets of function (operator) symbols \( \Omega = \{ \Omega_{s_i}(1,r) \mid \text{sort } s_i \in X, \text{unary signature } \langle 1, r \rangle \in \text{List}(X) \} \) with each \( (r,i) \in \Omega_{s_i}(1,r) \) being an \( s_i \)-sorted \( r = (1,r) \)-ary function symbol \( s_i \xrightarrow{(r,i)} r = \langle 1, r \rangle \).

The opposite of the schema context is a subcontext (no cotupling) of the term context \( \text{Term}_\Omega \xrightarrow{\text{inc}} \text{S}^{\text{op}} \). An 01og fact corresponds to a linear \( \Omega \)-equation \( \langle t = \hat{t} \rangle : \langle I, s \rangle \rightarrow \langle 1, s' \rangle \).

**01og Database Instance M:** Using extent, define a passage \( S = G^* \xrightarrow{\mathcal{M}} \text{Set} \)
which maps a node \( r \) in \( S \) to the extent \( M(r) = K(r) = \text{ext}(r) \) and maps an
edge \( r \xrightarrow{(r,i)} s_i \) in \( S \) to the function \( M(r) \xrightarrow{\mathcal{M}(r,i)} \text{tup}(s_i) \). The
functional language \( \mathcal{O} = \langle X, \mathcal{O} \rangle \) mentioned above has an \( \mathcal{O} \)-algebra \( \langle A, \delta \rangle \)
consisting of the \( X \)-sorted collection of extents \( \{ A_x = \text{ext}(x) \subseteq Y \mid x \in X \} \),
where \( \delta \) assigns the \( (1,r) \)-ary \( s_i \)-sorted function \( M(r) = A_r = A^{(1,r)} \xrightarrow{\delta(r,i)} M(r,i) \)
\( A_{s_i} = M(s_i) \) to each function symbol \( s_i \xrightarrow{(r,i)} r = \langle 1, r \rangle \). The interpretation
passage \( \text{T} \xrightarrow{\mathcal{A}} \text{Set} \) inductively define by this \( \mathcal{O} \)-algebra contains the
database instance as a subfunctor \( M = \text{inc}^{\text{op}} \circ \mathcal{A} \). Any equation satisfied by
the 01og database interpretation \( M \) is also satisfied by the interpretation
passage \( \mathcal{A} \).

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28 Define \( \langle X, \Omega \rangle \)-term \( \langle I, s \rangle \xrightarrow{\pi_r} \langle 1, r \rangle = r \) as the cotupling of \( \{ s_i \xrightarrow{(r,i)} r = \langle 1, r \rangle \mid i \in I \} \),
so that \( s_i \xrightarrow{(r,i)} r = \langle 1, r \rangle \) is the composite \( s_i \xrightarrow{\pi_r} 
\langle I, s \rangle \xrightarrow{r} \langle 1, r \rangle \).

29 A projection \( M(r) = \text{ext}(r) \xrightarrow{\tau_r} \text{tup}(I, s) \xrightarrow{\text{tup}(\pi_i)} \text{tup}(1, s_i) \xrightarrow{\text{ext}(s_i)} \mathcal{M}(s_i) \) of
the tabular interpretation in § 35.

30 \( \delta \) assigns the table \( M(r) = \text{ext}(r) \xrightarrow{\delta} \text{tup}(I, s) \xrightarrow{\text{tup}(\pi_i)} \text{tup}(1, s_i) \xrightarrow{\text{ext}(s_i)} \mathcal{M}(s_i) \) (a
tupling) to the cotupling \( \langle I, s \rangle \xrightarrow{\pi_r} \).

31 The interpretation passage satisfies an \( \Omega \)-equation \( \langle t = \hat{t} \rangle : \langle I, s \rangle \rightarrow \langle I', s' \rangle \) when the
operations coincide \( \mathcal{A}(t) = \mathcal{A}(\hat{t}) : \mathcal{A}(I, s) \rightarrow \mathcal{A}(I', s') \) in \( \text{Set} \) (see Kent [14]).
5 Conclusion

The ERA data model describes the conceptual model for a community of discourse, which can be used as the foundation for designing relational databases. The first-order logical environment FOLE provides a rigorous and principled approach to distributed inter-operable first-order information systems, which integrates ontologies and databases into a unified framework. In this paper, we have discussed in detail the representation of elements of the ERA data model with components of the FOLE logical environment, described interpretation basics, and illustrated the connections between FOLE and other data models in knowledge representation. We complete the presentation of the FOLE logical environment by discussing the formalism and semantics of many-sorted logic in the paper Kent [16] and database interpretation in the papers Kent [17] and Kent [18].
A The Fibered Context of Structures.

In order to allow communities of discourse to interoperate, we define the notion of a morphism between two structures that respects the ERA data model. This describes the mathematical context of structures $\text{Struc}$ as a fibered mathematical context in two orientations: the Grothendieck construction of the schema indexed mathematical context $\text{Sch}^{\text{op} \cdot \text{struc}} \to \text{Cxt}$ or the Grothendieck construction of the universe indexed mathematical context $\text{Univ}^{\text{op} \cdot \text{struc}} \to \text{Cxt}$. The schema indexed mathematical context of structures is used in the paper [16] on FOLE superstructure to establish the institutional aspect of the FOLE. Each orientation is developed in three steps: the structure fiber context for a single indexing object, the structure fiber passage along an indexing morphism, the fibered mathematical context $\text{Struc}$ of structures and structure morphisms as the Grothendieck construction for this orientation. This dual development of the mathematical context of structures $\text{Struc}$ (Prop. 2) is based upon and mirrors a dual development of the mathematical context of classifications $\text{Cls}$ (Prop. 1).
A.1 An Exemplar.

The mathematical context of classifications $\text{Cls}$ is a fibered mathematical context in two orientations: the Grothendieck construction of the type set indexed mathematical context $\mathbf{Set}^{\text{op}} \times \text{Cls} \to \text{Cxt}$ or the Grothendieck construction of the instance set indexed mathematical context $\mathbf{Set}^{\text{op}} \times \text{Cls} \eto \mathbf{Cxt}$.

**Proposition 1.** Any infomorphism $C_2 = \langle X_2, Y_2, \models_2 \rangle \xrightarrow{(f,g)} \langle X_1, Y_1, \models_1 \rangle = C_1$ in the mathematical context of classifications $\text{Cls}$, with type function projection $X_2 \xrightarrow{f} X_1$ and instance function projection $Y_2 \xleftarrow{g} Y_1$, has dual factorizations

$\xymatrix@C=40pt@R=30pt{\text{Cls}(X_2) \ar[r] & C_2 \ar[r]_{g}^{(1_{X_2}, g)} & \text{cls}^y_f(C_1) = \langle X_2, Y_1, \models_f \rangle \\
\langle X_2, Y_1, \models_g \rangle = \text{cls}^x_g(C_2) \ar[u]_{f}^{(f, 1_{Y_1})} \ar[r]_{f} \ar[u]_{f} & C_1 \ar[u]_{f}^{(f, 1_{Y_1})} &} \quad \text{where } \text{cls}^y_f(C_1) = f^{-1}(C_1) = \langle X_2, Y_1, \models_f \rangle = \langle X_2, Y_1, \models_g \rangle = g^{-1}(C_2) = \text{cls}^x_g(C_2).

**Proof.** $y_1 \models_g x_2 \iff g(y_1) \models_2 c_2 x_2 \iff y_1 \models_1 c_1, f(x_2) \iff y_1 \models_f x_2$.

The top-right factorization consists of a $\text{Cls}(X_2)$-morphism $C_2 \xrightarrow{g} \text{cls}^y_f(C_1)$ and the $C_1^{\text{th}}$ component $\text{cls}^y_f(C_1) \xrightarrow{f} \text{C}_1$ of a bridge $\text{cls}^y_f \circ \text{inc}_{X_2} \xrightarrow{\chi_f} \text{inc}_{X_1}$. This factors through the $X_2$-classification $\text{cls}^y_f(C_1)$, which is the fiber passage image along the type function $X_2 \xrightarrow{f} X_1$ of the $X_1$-classification $C_1$.

The left-bottom factorization consists of the $C_2^{\text{th}}$ component $C_2 \xrightarrow{g} \text{cls}^x_g(C_2)$ of a bridge $\text{cls}^x_g \circ \text{inc}_{Y_2} \xleftarrow{\chi_g} \text{inc}_{Y_1}$ and a $\text{Cls}(Y_1)$-morphism $\text{cls}^x_g(C_2) \xrightarrow{f} \text{C}_1$. This factors through the $Y_1$-classification $\text{cls}^x_g(C_2)$, which is the fiber passage image along the instance function $Y_2 \xleftarrow{g} Y_1$ of the $Y_2$-classification $C_2$. 
A.2 Schema Orientation.

The fibered mathematical context $\text{Struc}$ of structures and structure morphisms can be developed as the Grothendieck construction of a schema indexed mathematical context. This approach using schema indexing corresponds to the use of type-set indexing in §A.3 and was the approach used in the paper (Kent [14]).

1. For a fixed schema $S$, we define the structure fiber context $\text{Struc}(S)$.
2. For a schema morphism $S_2 \xrightarrow{(r,f)} S_1$, we define the structure fiber passage $\text{Struc}(S_2) \xleftarrow{\text{struc}^{(r,f)}} \text{Struc}(S_1)$.
3. We define the fibered mathematical context $\text{Struc}$ of structures and structure morphisms to be the Grothendieck construction of the schema indexed mathematical context $\text{Sch}^{\text{op} \text{struc}^\leftarrow} \xrightarrow{\text{Cxt}}$.

Fixed Schema. Let $S = \langle R, \sigma, X \rangle \in \text{Sch}$ be a fixed schema. We define the notion of a morphism between two (fixed schema) $S$-structures that respects the ERA data model. In these morphisms, the schema remains fixed, but the attribute and entity instances (data values and keys) are formally linked by maps that respect universe tuple, typed domain extent and entity interpretation. An $S$-structure morphism $\mathcal{M}_2 \xleftarrow{\mathcal{K}_{0}} \mathcal{M}_1$ over a fixed schema $S = \langle R, \sigma, X \rangle$ is a universe morphism $\text{univ}(\mathcal{M}_2) \xleftarrow{(k, g)} \text{univ}(\mathcal{M}_1)$, where $\text{attr}(\mathcal{M}_2) \xrightarrow{(1_x, g)} \text{attr}(\mathcal{M}_1)$ is an infomorphism in $\text{Cls}(X)$ over the sort set $X$, and $\text{ent}(\mathcal{M}_2) \xrightarrow{(1_y, k)} \text{ent}(\mathcal{M}_1)$ is an infomorphism in $\text{Cls}(R)$ over the entity type set $R$. $S$-structure morphisms compose component-wise. Let $\text{Struc}(S)$ denote the fiber context of structures over the fixed schema $S$.

Structure Fiber Passage. We define the indexed context $\text{Sch}^{\text{op} \text{struc}^\leftarrow} \text{Cxt}$. Given a schema $S$, there is a fiber context of structures $\text{Struc}(S)$ with that schema. Given a schema morphism $S_2 \xrightarrow{(r,f)} S_1$, there is a fiber passage of structures $\text{Struc}(S_2) \xleftarrow{\text{struc}^{(r,f)}} \text{Struc}(S_1)$: a structure $\mathcal{M}_1 = \langle E_1, \langle \sigma_1, \tau_1 \rangle, A_1 \rangle \in \text{Struc}(S_1)$ is mapped to a structure $\mathcal{M}_2 = \text{struc}^{(r,f)}(\mathcal{M}_1) = \langle r^{-1}(E_1), \langle \sigma_2, \tau_1 \rangle, f^{-1}(A_1) \rangle \in \text{Struc}(S_2)$. $\mathcal{M}_2$ is called the reduct of $\mathcal{M}_1$ and $\mathcal{M}_1$ is called the expansion of $\mathcal{M}_2$.

Hence, the target type domain is the inverse image of the source type domain $\text{attr}(\mathcal{M}_1) = g^{-1}(\text{attr}(\mathcal{M}_2)) = \text{cls}^{\gamma}_\text{r}(\text{attr}(\mathcal{M}_2))$ and the target entity classification is the inverse image of the source entity classification $\text{ent}(\mathcal{M}_1) = k^{-1}(\text{ent}(\mathcal{M}_2)) = \text{cls}^{\gamma}_\text{e}(\text{ent}(\mathcal{M}_2))$. By combining entity/attribute inverse image classifications, the target structure $\mathcal{M}_1$ is the inverse image $\text{struc}^{\gamma}_{(k,g)}(\mathcal{M}_2) = \langle k^{-1}(\text{ent}(\mathcal{M}_2)), \langle \sigma, \tau_1 \rangle, g^{-1}(\text{attr}(\mathcal{M}_2)) \rangle$ of the source structure $\mathcal{M}_2$ (§A.3). The two are bridged $\mathcal{M}_2 \xleftarrow{(k,g)} \text{struc}^{\gamma}_{(k,g)}(\mathcal{M}_2) = \mathcal{M}_1$ by the structure morphism.
$\mathcal{M}_2$. The two are linked $\text{struc}^\wedge_{(r,f)}(\mathcal{M}_1) \xrightarrow{\tilde{x}} \mathcal{M}_1$ by a structure morphism, which is the $\mathcal{M}_1^\text{th}$ component of a bridge $\text{struc}^\wedge_{(r,f)} \circ \text{inc}_{\mathcal{S}_2} \xrightarrow{\tilde{x}} \text{inc}_{\mathcal{S}_1}$.

\begin{align*}
\text{Struc}(\mathcal{S}_2) & \xrightarrow{\text{struc}^\wedge_{(r,f)}} \text{Struc}(\mathcal{S}_1) \\
\text{inc}_{\mathcal{S}_2} & \xrightarrow{\tilde{x}} \text{inc}_{\mathcal{S}_1}
\end{align*}

Multiple Universes. The Grothendieck construction of the schema indexed mathematical context $\text{Sch}^\text{op} \xrightarrow{\text{struc}^\wedge} \text{Cxt}$ is the fibered mathematical context $\text{Struc}$ of structures and structure morphisms. A structure $\mathcal{M}$ is as described in § 3.3 (Fig. 3). A structure morphism $\mathcal{M}_2 \xrightarrow{(r,k,f,g)} \mathcal{M}_1$ from source structure $\mathcal{M}_2$ to target structure $\mathcal{M}_1$ consists of a schema morphism $\mathcal{S}_2 \xrightarrow{(r,f)} \mathcal{S}_1$ from source schema $\mathcal{S}_2$ to target structure $\mathcal{S}_1$ and a morphism

$\mathcal{M}_2 = (\mathcal{E}_2, \langle \sigma_2, \tau_2 \rangle, \mathcal{A}_2) \xrightarrow{(k,g)} (\mathcal{E}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{A}_1) = \text{struc}^\wedge_{(r,f)}(\mathcal{M}_1)$

in the fiber mathematical context of structures $\text{Struc}(\mathcal{S}_2)$. Hence, a structure morphism satisfies the following conditions.

- **list preservation** $r \cdot \sigma_1 = \sigma_2 \cdot \Sigma_f$
- $k \cdot \tau_2 = \tau_1 \cdot \Sigma_g$
- \text{infomorphisms} $k_1 |\Rightarrow_{\mathcal{E}_1} r(\tau_2) \iff k(k_1) |\Rightarrow_{\mathcal{E}_2} \tau_2$
- $g(y_1) |\Rightarrow_{\mathcal{A}_2} f(x_2) \iff g(\sigma_1) |\Rightarrow_{\mathcal{A}_1} \tau_1$

Thus, a structure morphism $\mathcal{M}_2 \xrightarrow{(r,k,f,g)} \mathcal{M}_1$ (Fig. 3 in § 3.3) from source structure $\mathcal{M}_2 = (\mathcal{E}_2, \langle \sigma_2, \tau_2 \rangle, \mathcal{A}_2)$ to target structure $\mathcal{M}_1 = (\mathcal{E}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{A}_1)$ is defined in terms of the hypergraph and classification morphisms between the source and target structure components (projections):

- **universe morphism** $\mathcal{U}_2 = \text{univ}(\mathcal{M}_2) \xrightarrow{(k,g)} \text{univ}(\mathcal{M}_1) = \mathcal{U}_1$
- **schema morphism** $\mathcal{S}_2 = \text{sch}(\mathcal{M}_2) \xrightarrow{(r,f)} \text{sch}(\mathcal{M}_1) = \mathcal{S}_1$
- **typed domain morphism** $\mathcal{A}_2 = \text{attr}(\mathcal{M}_2) \xrightarrow{(f,g)} \text{attr}(\mathcal{M}_1) = \mathcal{A}_1$
- **entity infomorphism** $\mathcal{E}_2 = \text{ent}(\mathcal{M}_2) \xrightarrow{(r,k)} \text{ent}(\mathcal{M}_1) = \mathcal{E}_1$

Structure morphisms compose component-wise. Let $\text{Struc}$ denote the context of structures and structure morphisms. (Fig. 7 in § 3.3)
A.3 Universe Orientation.

As evident from the type-instance duality in Fig. 8 and Fig. 5 of § 3, the fibered mathematical context \( \text{Struc} \) of structures and structure morphisms can be developed from the dual standpoint — the Grothendieck construction of a universe indexed mathematical context. This approach using universe indexing corresponds to the use of instance-set indexing in § 3.1.

1. For a fixed universe \( \mathcal{U} \), we define the structure fiber context \( \text{Struc}(\mathcal{U}) \).
2. For a universe morphism \( \mathcal{U}_2 \xleftarrow{(k,g)} \mathcal{U}_1 \), we define the structure fiber passage
   \[ \text{Struc}(\mathcal{U}_2) \xrightarrow{\text{struc}_g}\mathcal{U}_2 \xleftarrow{(k,g)} \text{Struc}(\mathcal{U}_1) \].
3. We define the fibered mathematical context \( \text{Struc} \) of structures and structure morphisms to be the Grothendieck construction of the universe indexed mathematical context \( \text{Univ} \xrightarrow{\text{struc}} \text{Cxt} \).

Fixed Universe. Let \( \mathcal{U} = (K, \tau, Y) \in \text{Univ} \) be a fixed universe. We define the notion of a morphism between two (fixed universe) \( \mathcal{U} \)-structures that respects the ERA data model. In these morphisms, the universe remains fixed, but the attribute types (sorts) and entity types are formally linked by maps that respect schema signature, typed domain extent and entity interpretation. A \( \mathcal{U} \)-structure morphism \( \mathcal{M}_2 \xrightarrow{(r,f)} \mathcal{M}_1 \) over a fixed universe \( \mathcal{U} = (K, \tau, Y) \) is a schema morphism \( \text{sch}(\mathcal{M}_2) \xrightarrow{(r,f)} \text{sch}(\mathcal{M}_1) \), where \( \text{attr}(\mathcal{M}_2) \xrightarrow{(f,1)} \text{attr}(\mathcal{M}_1) \) is an infomorphism in \( \text{Cls}(Y) \) over the value set \( Y \), and \( \text{ent}(\mathcal{M}_2) \xrightarrow{(r,1)} \text{ent}(\mathcal{M}_1) \) is an infomorphism in \( \text{Cls}(K) \) over the key set \( K \). Here, \( \mathcal{M}_2 \) is the reduct of \( \mathcal{M}_1 \) and \( \mathcal{M}_1 \) is the expansion of \( \mathcal{M}_2 \). The two are bridged \( \mathcal{M}_2 = \text{struc}_{(r,f)}(\mathcal{M}_1) \xrightarrow{(r,f)} \mathcal{M}_1 \) by the structure morphism. \( \mathcal{U} \)-structure morphisms compose component-wise. Let \( \text{Struc}(\mathcal{U}) \) denote the fiber context of structures over the fixed universe \( \mathcal{U} \).

Structure Fiber Passage. We define the indexed context \( \text{Univ} \xrightarrow{\text{struc}} \text{Cxt} \). Given a universe \( \mathcal{U} \), there is a fiber context of structures \( \text{Struc}(\mathcal{U}) \) with that universe. Given a universe morphism \( \mathcal{U}_2 \xleftarrow{(k,g)} \mathcal{U}_1 \), there is a fiber passage of

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33 Hence, the source type domain is the inverse image of the target type domain \( \text{attr}(\mathcal{M}_2) = f^{-1}(\text{attr}(\mathcal{M}_1)) = \text{cls}(\text{attr}(\mathcal{M}_1)) \) and the source entity classification is the inverse image of the target entity classification \( \text{ent}(\mathcal{M}_2) = r^{-1}(\text{ent}(\mathcal{M}_1)) = \text{cls}(\text{ent}(\mathcal{M}_1)) \). By combining entity/attribute inverse image classifications, the source structure \( \mathcal{M}_2 \) is the inverse image \( \text{struc}_{(r,f)}(\mathcal{M}_1) = (r^{-1}(\text{ent}(\mathcal{M}_1)), (\sigma_2, \tau), f^{-1}(\text{attr}(\mathcal{M}_1))) \) of the target structure \( \mathcal{M}_1 \) (§ 3.2).

34 When this definition is extended to formulas, one gets the notion of an interpretation of first-order logic (extended to the many-sorted case) given in (Barwise and Selman [1]).
structures $\text{Struct}(\mathcal{U}_2) \xrightarrow{\text{struc}^{\mathcal{U}}_{k,g}} \text{Struct}(\mathcal{U}_1)$: a structure $\mathcal{M}_2 = \langle \mathcal{E}_2, \langle \sigma_2, \tau_2 \rangle, A_2 \rangle \in \text{Struct}(\mathcal{U}_2)$ is mapped to a structure $\mathcal{M}_1 = \text{struc}^{\mathcal{U}}_{k,g}(\mathcal{M}_2) = \langle k^{-1}(\mathcal{E}_2), \langle \sigma_2, \tau_1 \rangle, g^{-1}(A_2) \rangle \in \text{Struct}(\mathcal{U}_1)$. The two are linked $\mathcal{M}_2 \xrightarrow{\chi_{x_2}} \text{struc}^{\mathcal{U}}_{k,g}(\mathcal{M}_2)$ by a structure morphism, which is the $\mathcal{M}_2^{\text{th}}$ component of a bridge $\text{inc}_{\mathcal{U}_2} \xrightarrow{\chi_{(k,g)}} \text{struc}^{\mathcal{U}}_{k,g} \circ \text{inc}_{\mathcal{U}_1}$.

Multiple Universes. The Grothendieck construction of the universe indexed mathematical context $\text{Univ}^{\text{op}}_{k,g} \xrightarrow{\text{Struct}} \text{Cxt}$ is the fibered mathematical context $\text{Struct}$ of structures and structure morphisms. A structure $\mathcal{M}$ is as described in §3.3 (Fig. 3). A structure morphism $\mathcal{M}_2 \xleftarrow{(r,k,f,g)} \mathcal{M}_1$ from source structure $\mathcal{M}_2$ to target structure $\mathcal{M}_1$ consists of a universe morphism $\mathcal{U}_2 \xrightarrow{(k,g)} \mathcal{U}_1$ to target structure $\mathcal{U}_1$ and a morphism

\[ \text{struc}^{\mathcal{U}}_{k,g}(\mathcal{M}_2) = \langle k^{-1}(\mathcal{E}_2), \langle \sigma_2, \tau_1 \rangle, g^{-1}(A_2) \rangle \xrightarrow{(r,f)} \langle \mathcal{E}_1, \langle \sigma_1, \tau_1 \rangle, A_1 \rangle = \mathcal{M}_1 \]

in the fiber mathematical context of structures $\text{Struct}(\mathcal{U}_1)$. Hence, a structure morphism satisfies the following conditions.

- list preservation
  \[ r \cdot \sigma_1 = \sigma_2 \cdot \Sigma f \]
  \[ k \cdot \tau_2 = \tau_1 \cdot \Sigma g \]
- infomorphisms
  \[ k_1 \models_{\mathcal{E}_1} r(r_2) \iff k(k_1) \models_{\mathcal{E}_2} r_2 \]
  \[ y_1 \models_{\mathcal{A}_1} f(x_2) \iff g(y_1) \models_{\mathcal{A}_2} x_2 \]

Thus, a structure morphism $\mathcal{M}_2 \xleftarrow{(r,k,f,g)} \mathcal{M}_1$ (Fig. 3 in §3.3) from source structure $\mathcal{M}_2 = \langle \mathcal{E}_2, \langle \sigma_2, \tau_2 \rangle, A_2 \rangle$ to target structure $\mathcal{M}_1 = \langle \mathcal{E}_1, \langle \sigma_1, \tau_1 \rangle, A_1 \rangle$ is defined in terms of the hypergraph and classification morphisms between the source and target structure components (projections):

- universe morphism $\mathcal{U}_2 = \text{univ}(\mathcal{M}_2) \xrightarrow{(k,g)} \text{univ}(\mathcal{M}_1) = \mathcal{U}_1$
- schema morphism $\mathcal{S}_2 = \text{sch}(\mathcal{M}_2) \xrightarrow{(r,f)} \text{sch}(\mathcal{M}_1) = \mathcal{S}_1$
- typed domain morphism $\mathcal{A}_2 = \text{attr}(\mathcal{M}_2) \xrightarrow{(f,g)} \text{attr}(\mathcal{M}_1) = \mathcal{A}_1$
- entity infomorphism $\mathcal{E}_2 = \text{ent}(\mathcal{M}_2) \xrightarrow{(r,k)} \text{ent}(\mathcal{M}_1) = \mathcal{E}_1$
Structure morphisms compose component-wise. Let $\text{Struct}$ denote the context of structures and structure morphisms. (Fig. 7 in § 3.3)

**Proposition 2.** (compare Prop. 1) Any structure morphism $\mathcal{M}_2 \xrightarrow{(r,k,g)} \mathcal{M}_1$, with schema morphism projection $\mathcal{S}_2 \xrightarrow{(r,f)} \mathcal{S}_1$ and universe morphism projection $\mathcal{U}_2 \xleftarrow{(k,g)} \mathcal{U}_1$, has dual factorizations (see the diagram below).

\[
\begin{array}{c}
\text{Struc}(S_2) \\
\mathcal{M}_2 \xrightarrow{(k,g)} \text{struc}^\uparrow_{(r,f)}(\mathcal{M}_1) \\
\mathcal{M}_2 \xleftarrow{(k,g)} \| (r,f) \\
\text{struc}^\gamma_{(k,g)}(\mathcal{M}_2) \xrightarrow{(r,f)} \mathcal{M}_1 \\
\text{Struc}(U_1) \\
\end{array}
\]

The top-right factorization (corresponding to the schema orientation of § A.2) consists of a $\text{Struc}(S_2)$-morphism $\mathcal{M}_2 \xrightarrow{(k,g)} \text{struc}^\uparrow_{(r,f)}(\mathcal{M}_1)$ and the $\mathcal{M}_1^{th}$ component $\text{struc}^\gamma_{(k,g)}(\mathcal{M}_2) \xrightarrow{(r,f)} \mathcal{M}_1$ of a bridge $\text{struc}^\gamma_{(k,g)} \circ \text{inc}_{S_2} \xrightarrow{(r,f)} \mathcal{M}_2 \xrightarrow{(k,g)} \text{inc}_{S_1}$.

The left-bottom factorization (corresponding to the universe orientation of § A.3) consists of the $\mathcal{M}_2^{th}$ component $\mathcal{M}_2 \xrightarrow{(k,g)} \text{struc}^\gamma_{(k,g)}(\mathcal{M}_2)$ of a bridge $\text{inc}_{U_2} \xrightarrow{(k,g)} \mathcal{M}_2 \xrightarrow{(r,f)} \mathcal{M}_1$.

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