Minimal two-spheres with constant curvature in the quaternionic projective space

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Abstract: In this paper we completely classify the homogeneous two-spheres, especially, the minimal homogeneous ones in the quaternionic projective space $\mathbb{HP}^n$. According to our classification, more minimal constant curved two-spheres in $\mathbb{HP}^n$ are obtained than Ohnita conjectured in [13].

Keywords: minimal two-sphere; Gauss curvature; quaternionic projective space.

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1 Introduction

The study of minimal constant curved two-spheres in symmetric spaces has a long history. E.Calabi [5] and do Carmo and Wallach [6] proved that any isometric minimal immersion from $S^3_K$ into $S^n(1)$ is congruent to a linearly full one in $S^{2m}(1)$ with $K = 2/m(m+1)$ for some positive integer $m$. S. Bando, Y. Ohnita [11] and J. Bolton, G.R. Jensen, M. Rigoli, L. M. Woodward [3] proved that the linearly full minimal $S^3_K$ in $\mathbb{CP}^m$ must be one of the Veronese surfaces with $K = 4/(n+2j(n−j))$ for some integer $0 ≤ j ≤ n$. These perfect works lead to the study of minimal two-spheres with constant curvature in symmetric spaces. For example, Zh.Q.Li and Zh.H.Yu [11] classified the minimal constant curved two-spheres in the complex Grassmann manifold $G(2,4)$, X.X.Jiao and C.K.Peng [12] classified the holomorphic ones in $G(2,5)$. L.Delisle, V.Hussin and W.J.Zakrzewski proved [2] [8] these two results again from the viewpoint of the

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Grassmannian sigma-model, and they also proposed conjectures w.r.t. the holomorphic constant curved two-spheres in complex Grassmannians. Recently, C.K.Peng and X.W.Xu completely classified the homogenous minimal two-spheres in the complex Grassmann manifold $G(2,n)$, and C.K.Peng, J.Wang and X.W.Xu completely classified the homogenous minimal ones in the complex hyperquadric $Q_n$.

Naturally, one can consider the corresponding problem when the target space is quaternionic projective space $\mathbb{HP}^n$. We call a minimal immersion $f$ from $S^2$ into $\mathbb{HP}^n$ is proper means that $f(S^2)$ is not contained in any totally geodesic submanifold of $\mathbb{HP}^n$. Without loss of generality, we only consider the proper ones in this paper. Base on the works of homogeneous harmonic maps into projective space, Y.Ohnita proposed the following conjecture:

**Conjecture 1.1.** Let $f$ be a proper minimal constant curved immersion from $S^2$ maps into projective space $\mathbb{HP}^n$. Without loss of generality, we only consider the proper ones in this paper. Base on the works of homogeneous harmonic maps into projective space, Y.Ohnita proposed the following conjecture:

**Theorem 1.2.** Let $f : S^2 \rightarrow \mathbb{HP}^n$ be a proper homogeneous minimal immersion. Then, in terms of homogeneous coordinates, $f$ is congruent to one of the following:

1. $f_\lambda : [a,b] \mapsto [\Phi_{\lambda,m+1}], \lambda \in \{3,5,\ldots,2n+1\}$, and the Gauss curvature $K = 8/[(2n+2)^2 - (\lambda^2 + 1)]$;
2. $f_1 : [a,b] \mapsto [\Phi_{1,m+1}]$ and the Gauss curvature $K = 4/[n(n+2)]$;
3. $f_{\lambda,m,t} : [a,b] \mapsto [\cos t\Phi_{\lambda,m}, \sin t\Phi_{\lambda,m}]$ for some positive weight $\lambda \in \{3,5,\ldots,n\}$, $t \in (0,\pi/2)$, $2m = n + 1$, and the Gauss curvature $K = 8/[n(n+1)^2 - (\lambda^2 + 1)]$;
4. $f_{m_1,m_2} : [a,b] \mapsto [\sqrt{m_1/(m_1+m_2)} \Phi_{m_1,m_1}, \sqrt{m_2/(m_1+m_2)} \Phi_{m_2,m_2}]$ for some positive $m_1 \leq m_2$ so that $m_1 + m_2 = n + 1$ is an even, and the Gauss curvature $K = 4/(m_1^2 + m_2^2)$;
5. $f_{m_1,m_2} : [a,b] \mapsto [\sqrt{m_1/(m_1+m_2)} \Phi_{m_1,m_1}, \sqrt{m_2/(m_1+m_2)} \Phi_{m_2,m_2}]$ for some positive $m_1 < m_2$ so that $m_1 + m_2 = n + 1$ is an odd, and the Gauss curvature $K = 4/(m_1^2 + m_2^2 - 1)$, where $\Phi_{\lambda,m}$ is defined in the end of Section 2.1.

All these minimal two-spheres obtained are not congruent to each other, up to a rigidity of $\mathbb{HP}^n$. So, by comparing Conjecture 1.1 with Theorem 1.2, we obtain three families of minimal two-spheres more than Y.Ohnita conjectured. Particularly, there is a family of minimal two-spheres in $\mathbb{HP}^n$ which depend on a parameter $t \in (0,\pi/2)$. This phenomenon appeared in Theorem 1.2 is quite different from the case of complex projective space (see [13]), due to the non-commutativity of quaternion.

In this paper we first show that there exists a quaternionic representation $\rho$ of $SU(2)$ associated to each homogeneous immersion $f$ from $S^2$ into $\mathbb{HP}^n$ so that $f(S^2)$ is a $\rho(SU(2))$-orbit in $\mathbb{HP}^n$. Then, in order to characterize the geometry of such orbit, we find the "best" base point...
in it. That is, we prove that such an orbit must contain a point \( P_0 \) spanned by vectors belonging to a weight space of the quaternionic representation \( \rho \). To study the minimal orbits, we obtain a minimality criterion by using the method of moving frame which is inspired from S.S.Chern and J.G.Wolfson’s work [4]. By using this criterion, up to a rigidity of \( \text{HP}^n \), we show that the representation \( \rho \) is a direct sum of two different irreducible representations at the most. Thus, we completely classify the homogenous and minimal homogeneous two-spheres in \( \text{HP}^n \), see Theorem 3.3, Theorem 4.4 respectively.

2 Preliminaries

2.1 The quaternionic representation of \( SU(2) \)

For completeness, we review some facts on the unitary and quaternionic representations of the special unitary group \( SU(2) \).

The special unitary group \( SU(2) \) is defined by

\[
SU(2) = \left\{ g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1, \ a, b \in \mathbb{C} \right\}.
\]

Its Lie algebra \( su(2) \) is real spanned by

\[
\mathbf{e}_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]

where \( i^2 = -1 \). The Maurer-Cartan forms of \( SU(2) \) is determined by

\[
\Theta := dgg^{-1} = \begin{pmatrix} i\omega & \varphi \\ -\bar{\varphi} & -i\omega \end{pmatrix},
\]

where \( \omega, \varphi \) are real and complex one-forms respectively. The Maurer-Cartan equation is given by

\[
d\Theta = \Theta \wedge \Theta,
\]

which implies

\[
d(i\omega) = -\varphi \wedge \bar{\varphi}, \quad d\varphi = (2i\omega) \wedge \varphi.
\]

Let \( T = \{ \text{diag} \{ e^{it}, e^{-it} \} \mid t \in \mathbb{R} \} \) be a maximal torus subgroup of \( SU(2) \). It is known that the set \( SU(2)/T = \{ [g] = Tg \mid g \in SU(2) \} \) is the complex projective space \( \mathbb{CP}^1 \), which is diffeomorphic to \( S^2 \). The canonical metric \( \varphi \bar{\varphi} \) on \( \mathbb{CP}^1 \) has constant curvature 4 by (2.2).

Let \( V_n \) be the \((n+1)\)-dimensional complex vector space of all complex homogeneous polynomials of degree \( n \) w.r.t. the two complex variables \( z_0 \) and \( z_1 \). We define a Hermitian inner product \((, )\) on \( V_n \) by

\[
(f, g) := \sum_{k=0}^{n} k!(n-k)! a_k \bar{b}_k,
\]
for \( f = \sum_{k=0}^{n} a_k z_0^{k-n} k, \ g = \sum_{k=0}^{n} b_k z_0^{k-n} k \in V_n \). So, we know \( \{ v_{k,n} = z_0^{k-n} k / \sqrt{k!(n-k)!} \mid 0 \leq k \leq n \} \) is a unitary basis for \( V_n \). A unitary representation \( \rho_n \) of \( SU(2) \) on \( V_n \) is defined by
\[
\rho_n(g)f(z_0, z_1) := f((z_0, z_1)g^{-1}) = f(\bar{a}z_0 + (\bar{b} - az_1)
\]
for \( g \in SU(2) \) and \( f \in V_n \). Under the basis \( \{ v_{k,n} : 0 \leq k \leq n \} \), we get a matrix representation \( \rho_n : SU(2) \to U(n+1), g \mapsto \rho_n(g) \), and \( \rho_n(g) \) is described by \( v_{k,n} \rho_n(g) := \sum_{k=0}^{n} \Lambda_{l,k}(g)v_{l,n} \), where
\[
\Lambda_{l,k}(g) = \frac{\Gamma(n-l)!}{k!(n-k)!} \sum_{p+q=l} \binom{k}{p} \binom{n-k}{q} (-1)^{q-k} a^p b^{k-p}
\]
which satisfies \( \Lambda_{l,k}(g) = (-1)^{k+l} \Lambda_{n-l,n-k}(g) \). The action of \( su(2) \) on \( V_n \) is as follows:
\[
v_{k,n} d \rho_n(\varepsilon) := \frac{d}{dt} (\rho_n(\exp t \varepsilon)(v_{k,n})) |_{t=0}, \tag{2.3}
\]
for \( 0 \leq k \leq n \) and any element \( \varepsilon \in su(2) \). In particular, when \( \varepsilon \) takes \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \), we have
\[
v_{k,n} d \rho_n(\varepsilon_1) = (n-2k) i v_{k,n}, \tag{2.4}
v_{k,n} d \rho_n(\varepsilon_2) = -a_{k-1,n} v_{k-1,n} + a_{k,n} v_{k+1,n}, \tag{2.5}
v_{k,n} d \rho_n(\varepsilon_3) = -a_{k-1,n} i v_{k-1,n} - a_{k,n} i v_{k+1,n}, \tag{2.6}
\]
for \( 0 \leq k \leq n \) and \( a_{k,n} = \sqrt{(k+1)(n-k)} \). Set
\[
\lambda_k := n-2k, \ \ v_{\lambda_k,n} := v_{k,n}, \ \ a_{\lambda_k,n} = a_{k,n} = \sqrt{(n+1)^2 - (\lambda_k - 1)^2}/2.
\]
An element in \( \Delta_n = \{ \lambda_0, \ldots, \lambda_n \} \) is called a weight of \( \rho_n \), and \( \lambda_0 \) is called the highest weight. The representation space \( V_n \) has an orthonormal decomposition w.r.t. the Hermitian inner product \( (, ) \), i.e., \( V_n = V_{\lambda_0} \oplus \cdots \oplus V_{\lambda_n} \), where \( V_{\lambda_k} = \text{Span}_C \{ v_{\lambda_k,n} \} \) is called the weight space w.r.t. the weight \( \lambda_k \).

Suppose \( n = 2m-1 \in N^+ \) is an odd number. Define \( u_{\lambda_k,n}, u_{-\lambda_k,n} \) as follows
\[
u_{\lambda_k,n} := v_{\lambda_k,n}, \ \ u_{-\lambda_k,n} := (-1)^k i v_{-\lambda_k,n}, \tag{2.7}
\]
where \( 0 \leq k \leq m-1 \). Under the basis \( \{ u_{\lambda_k,n}, u_{-\lambda_k,n}, \ldots, u_{-\lambda_0,n}, u_{-\lambda_m,n}, \ldots, u_{-\lambda_{m-1},n} \} \), for every \( g \in SU(2) \), one can check that \( \rho_n(g) \) satisfies
\[
J \rho_n(g) = \rho_n(g) J, \ \text{for} \ J = \left( \begin{array}{cc} 0 & -I_m \\ I_m & 0 \end{array} \right).
\tag{2.8}
\]
We identify \( V_{2m-1} \) with \( H^m \) by
\[
v = \sum_{k=0}^{m-1} (a_k u_{\lambda_k,n} + b_k u_{-\lambda_k,n}) \mapsto (a_0 + b_0 j, \ldots, a_{m-1} + b_{m-1} j), \tag{2.9}
\]

where \( j \in \mathbf{H} \) and \( j^2 = -1 \). From this identification, it is convenient to define

\[
j u_{\lambda_k,n} := u_{-\lambda_k,n}.
\]  

(2.10)

Thus, an element in \( V_{2m-1} \) can be written as \( v = \sum_{k=0}^{m-1} (a_k + b_k j) u_{\lambda_k,n} \). When \( V_{2m-1} \) is viewed as a quaternionic linear space, the property (2.8) ensures us obtain a quaternionic representation of \( SU(2) \) denoted by \( \rho_m \). In terms of matrix, we have

\[
\rho_m : SU(2) \rightarrow Sp(m), \quad g \mapsto \left( \Xi_{kl}(g) := \Lambda_k l(g) + (-1)^{k+l+1} i\Lambda_2 m_{k-l}(g) j \right),
\]

(2.11)

where \( 0 \leq k, l \leq m - 1 \) and \( Sp(m) = \{ A \in M(m; \mathbf{H}) : AA^* = I_m \} \) is the unitary symplectic group.

It’s well known that \( \{ (V_n, \rho_n) : n = 0, 1, 2, \cdots \} \) are all inequivalent unitary representations of \( SU(2) \). By the Theorem 6.3 in [2], we know that \( \{ (V_{2m-1}, \rho_m) : m = 1, 2, \cdots \} \) are all inequivalent proper unitary quaternionic representations of \( SU(2) \). Since any quaternionic representation \( (V, \rho) \) is completely reducible, for a proper one, we can write

\[
V = \bigoplus_{\alpha=1}^{s} V_{2m_\alpha-1}, \quad \rho = \bigoplus_{\alpha=1}^{s} \rho_{m_\alpha}.
\]

(2.12)

In terms of weights, under the natural inner product, we also have the orthogonal decomposition

\[
V = \bigoplus_{\lambda > 0} U_{\lambda} = \bigoplus_{\lambda > 0} \left( \bigoplus_{\alpha} U_{\lambda, \alpha} \right), \quad U_{\lambda, \alpha} = V_{\lambda, 2m_\alpha-1} \oplus V_{-\lambda, 2m_\alpha-1}.
\]

(2.13)

Here, the "orthogonal decomposition" is related to the inner product \( (\cdot, \cdot) \) defined by

\[
(z, w) := \sum_{k=0}^{n} p_k \bar{q}_k,
\]

for \( z = \sum_{k=0}^{n} p_k u_{\lambda_k,n}, \quad w = \sum_{k=0}^{n} q_k u_{\lambda_k,n} \). Notice that \( d\rho \) is a real representation of \( su(2) \), we extend naturally \( d\rho \) to be a complex representation of \( \mathfrak{sl}(2; \mathbb{C}) \), which is still denoted by \( d\rho \). Set

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

which forms a basis of \( \mathfrak{sl}(2; \mathbb{C}) \). In the sequel, we denote \( d\rho(\sigma_1), d\rho(\sigma_2), d\rho(\sigma_3) \in \text{End}(V) \) by \( H, A, B \) respectively. Alternatively, they are determined by

\[
H = -i d\rho(\varepsilon_1), \quad A = \frac{d\rho(\varepsilon_2) - i d\rho(\varepsilon_3)}{2}, \quad B = \frac{d\rho(\varepsilon_2) + i d\rho(\varepsilon_3)}{2},
\]

(2.14)

which satisfy

\[
[H, A] = 2A, \quad [H, B] = -2B, \quad [A, B] = H.
\]

(2.15)
Throughout this subsection we agree on the following ranges of indices:

\[ 0 \leq k \leq m_\alpha - 1, \quad 0 \leq k \leq m_\alpha - 2, \]

It follows from (2.4)–(2.7), (2.10) and (2.14), we have

\[ u_{\lambda k, n_\alpha} H = \lambda k u_{\lambda k, n_\alpha}, \quad (j u_{\lambda k, n_\alpha}) H = -\lambda k (j u_{\lambda k, n_\alpha}), \quad 0 \leq k \leq m_\alpha - 1, \]

\[ u_{\lambda k, n_\alpha} A = -a_{\lambda k, n_\alpha} u_{\lambda k-2, n_\alpha}, \quad 0 \leq k \leq m_\alpha - 2, \]

\[ u_{\lambda m_{\alpha-1}, n_\alpha} A = -(-1)^{m_\alpha} \lambda a_{\lambda m_{\alpha-1}, n_\alpha} (j u_{\lambda m_{\alpha-1}, n_\alpha}), \]

The pull-back of Maurer-Cartan forms of \( Aut(V) \) is

\[ dp \rho^{-1} = (i\omega) H + \phi A - \phi B. \]  

It follows that

\[ zd\rho = ((i\omega)(zH) + \phi(zA) - \phi(zB)) \rho. \]  

Throughout this paper we will agree on the following conventions:

- \( n_\alpha = 2m_\alpha - 1, \) for \( m_\alpha \in \mathbb{N}^+; \)
- \( \Delta_\alpha := \{n_\alpha, n_\alpha - 2, \ldots, -(n_\alpha - 2), -n_\alpha\}; \)
- \( a_{\lambda, \alpha} := a_{\lambda, n_\alpha}, \) and \( a_{\lambda, \alpha} = 0 \) if \( \lambda \notin \Delta_\alpha; \)
- \( u_{\lambda, \alpha} := u_{\lambda, n_\alpha}, V_{\lambda, \alpha} = \text{Span}_\mathbb{C}\{u_{\lambda, \alpha}\}, \) and \( V_{\lambda, \alpha} = \{0\} \) if \( \lambda \notin \Delta_\alpha; \)
- \( \phi_{\lambda, m} := u_{\lambda, n_\alpha} \rho_m(g), \) \( g \in SU(2), \) for \( \lambda > 0. \)

### 2.2 Geometry of surfaces in quaternionic projective space \( 
\mathbb{H}^n \)

Throughout this subsection we agree on the following ranges of indices:

\[ 0 \leq A, B, \ldots, n, \quad 1 \leq \alpha, \beta, \ldots, n. \]

The standard inner product \( (\cdot, \cdot) \) of \( \mathbb{H}^{n+1} \) is defined by

\[ (z, w) = \sum A z_A w_A = \sum_A z_A w_A, \]

for \( z = (z_0, z_1, \ldots, z_n), \) \( w = (w_0, w_1, \ldots, w_n) \in \mathbb{H}^{n+1}. \) A unitary frame of \( \mathbb{H}^{n+1} \) is an ordered set of \( n + 1 \) linearly independent vectors \( Z_0, Z_1, \ldots, Z_n \) satisfying

\[ (Z_A, Z_B) = \delta_{AB}. \]

Taking the exterior derivative of \( Z_A, \) we have

\[ dZ_A = \sum_B \omega_{A B} Z_B, \]  

\[ \sum = \sum. \]
where \( \omega_{ab} \) are \( \mathbb{H} \)-valued one forms. We identify the space of unitary frames with the unitary symplectic group \( Sp(n+1) \). Then, \( \omega_{AB} \) are the Maurer-Cartan forms of \( Sp(n+1) \), which satisfy the Maurer-Cartan equations:

\[
d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}.
\]

(2.20)

The quaternionic projective space \( HP^n \) is the set of all one-dimensional subspaces in \( \mathbb{H}^{n+1} \). An element of \( HP^n \) can be defined by the unitary vector \( Z_0 \), up to a factor of absolute 1. Its orthogonal vectors \( Z_\alpha \) are defined up to a transformation of \( Sp(n) \). So, \( HP^n \) has a \( Sp(1) \times Sp(n) \)-structure. Therefore, the form

\[
ds^2 = \sum_\alpha \omega_0 \bar{\omega}_\alpha
\]

(2.21)

defines a Riemannian metric on \( HP^n \).

Let \( M \) be an oriented Riemannian surface and let \( f : M \rightarrow HP^n \) be a smooth immersion. The induced metric on \( M \) by

\[
ds^2_M = \bar{\phi} \phi,
\]

where \( \phi \) is complex-valued one-form. \( \phi \) is defined up to a complex factor of absolute value 1. The first structure equation of \( ds^2_M \) is given by

\[
d\phi = \eta \wedge \phi,
\]

(2.22)

where the one-form \( \eta \) is purely imaginary. The Gauss curvature \( K \) is described by

\[
d\eta = -\frac{K}{2} \phi \wedge \bar{\phi}.
\]

(2.23)

Locally, we have

\[
f^* \omega_{\bar{a}b} = \phi a_\alpha + \bar{\phi} b_\alpha,
\]

(2.24)

and using (2.20), (2.22) and (2.23), we obtain

\[
\phi \wedge Da_\alpha + \bar{\phi} \wedge Db_\alpha = 0,
\]

(2.25)

where

\[
Da_\alpha = da_\alpha + \eta a_\alpha - \theta_{0\bar{0}} a_\alpha - \theta'_{0\bar{0}} b_\alpha + a_\beta \omega_{\beta \alpha},
\]

(2.26)

\[
Db_\alpha = db_\alpha - \eta b_\alpha - \theta_{0\bar{0}} b_\alpha - \theta'_{0\bar{0}} a_\alpha + b_\beta \omega_{\beta \alpha},
\]

(2.27)

and

\[
\theta_{0\bar{0}} = \frac{\omega_{0\bar{0}} - i \omega_{\bar{0}0} \bar{i}}{2}, \quad \theta'_{0\bar{0}} = \frac{\omega_{0\bar{0}} + i \omega_{\bar{0}0} \bar{i}}{2}.
\]

(2.28)

Because \( HP^n \) is non-Kahlerian, the definition of \( Da_\alpha \) (resp. \( Db_\alpha \)) involves \( b_\alpha \) (resp. \( a_\alpha \)). By (2.25), we can set

\[
Da_\alpha = \phi p_\alpha + \phi q_\alpha, \quad Db_\alpha = \phi q_\alpha + \phi r_\alpha.
\]

(2.29)

Similar to (4), \( f \) is minimal if and only if

\[
q_\alpha = 0, \quad \text{mod } \phi \quad \text{or} \quad r_\alpha = 0, \quad \text{mod } \phi.
\]

(2.30)
Set the local $H^{n+1}$-valued functions $X, Y$ as

$$X = \sum_{\alpha} a_{\alpha} Z_{\alpha}, \quad Y = \sum_{\alpha} b_{\alpha} Z_{\alpha},$$

(2.31)

and define the covariant derivative of $X$ and $Y$ by

$$DX = dX + \eta X - \theta_{00} X - \theta'_{00} Y,$$

(2.32)

$$DY = dY - \eta Y - \theta_{00} X - \theta'_{00} Y.$$  

(2.33)

We give a criterion to measure the minimality of $f$ in terms of $DX$ or $DY$, that is

**Proposition 2.1.** The smooth immersion $f$ is minimal if and only if one of the following holds:

(a) $DX \equiv 0$, mod $Z_0$, $\varphi$;

(b) $DY \equiv 0$, mod $Z_0$, $\varphi$.

**Proof.** It follows from the facts

$$DX = \sum_{\alpha} Da_{\alpha} Z_{\alpha} + \sum_{\beta} a_{\beta} \omega_{\beta0} Z_0, \quad DY = \sum_{\alpha} Db_{\alpha} Z_{\alpha} + \sum_{\beta} b_{\beta} \omega_{\beta0} Z_0.$$  

This completes the proof. 

\[\square\]

### 3 Homogeneous two-spheres in $HP^n$

In this section we first determine the relations between the homogeneous two-spheres and the two-dimensional $\rho(SU(2))$-orbits in $HP^n$, where $\rho$ is a quaternionic representation of $SU(2)$. Then, all the two-dimensional $\rho(SU(2))$-orbits in $HP^n$ are determined.

**Theorem 3.1.** Let $f : S^2 \rightarrow HP^n$ be a homogeneous immersion, then there exists a quaternionic representation $\rho$ of $SU(2)$ such that $f(S^2)$ is a two-dimensional $\rho(SU(2))$-orbit in $HP^n$.

**Proof.** It is similar to the proof of Theorem 3.2 in [9] or Theorem 3.3 in [15], we omit the details.

\[\square\]

In general, a $\rho(SU(2))$-orbit in $HP^n$ is a principle orbit, i.e., it is a three-dimensional orbit. For the two-dimensional $\rho(SU(2))$-orbits, we have

**Lemma 3.2.** Let $M$ be a $\rho(SU(2))$-orbit in $HP^n$, then $M$ is two-dimensional if and only if there exists a point $P_0 \in M$ s.t. $P_0$ is invariant under the $\rho(T)$-action.

**Proof.** Let $P$ be a point in $M$. Since $\text{dim} M = 2$, the isotropy group $G$ at $P$ is a 1-dimensional subgroup of $SU(2)$. Therefore, there exists a nonzero element $\epsilon \in \mathfrak{su}(2)$ s.t. $T_{P_0} G = \text{Span}_{\mathbb{R}} \{ \epsilon \}$. Notice that $\epsilon \in \mathfrak{su}(2)$, there exist a fixed $g \in SU(2)$ s.t. $g^{-1} \epsilon g = c \epsilon$ for some nonzero real number $c$. It follows that the isotropy group at $P_0 = P \rho(g) \in M$ contains the maximal torus subgroup $T$, i.e., $P_0$ is invariant under $\rho(T)$-action. The sufficiency follows from the fact that there is no two-dimensional subgroup in $SU(2)$.

\[\square\]
**Remark.** By taking derivative, the condition "$P_0$ is invariant under $\rho(T)$" is equivalent to

$$ H(P_0) \subseteq P_0, \tag{3.1} $$

where $P_0$ is viewed as a linear subspace and $H$ is defined in (2.14).

Let $\rho$ be a quaternionic representation of $SU(2)$. Then $\rho(SU(2))$ acts on $\text{HP}^n$ by

$$ SU(2) \times \text{HP}^n \longrightarrow \text{HP}^n, \quad (g, P) \mapsto P \rho(g). \tag{3.2} $$

Therefore, for a two-dimensional $\rho(SU(2))$-orbit $M$ in $\text{HP}^n$, we can obtain an immersion $f$ from $S^2$ into $\text{HP}^n$ as follows

$$ f : S^2 \simeq SU(2)/T \longrightarrow \text{HP}^n, \quad [g] \mapsto P_0 \rho(g), \tag{3.3} $$

where $P_0$ is a fixed point in $M$. According to Theorem 3.1, we only need to study the two-dimensional $\rho(SU(2))$-orbits in $\text{HP}^n$ if we study the homogeneous immersions from $S^2$ into $\text{HP}^n$.

Let $\rho = \rho_{m_1} \oplus \cdots \oplus \rho_{m_s}$ be a quaternionic representation of $SU(2)$. Let $M$ be a linearly full two-dimensional $\rho(SU(2))$-orbit in $\text{HP}^n$. Suppose $P_0 = [z]$ is a point obtained in Lemma 3.2.

According to the decomposition (2.13), we can write

$$ z = \sum_{\lambda > 0} \sum_{\alpha} (c_{\lambda, \alpha} + d_{\lambda, \alpha} j) u_{\lambda, \alpha}, \quad \text{for } c_{\lambda, \alpha}, d_{\lambda, \alpha} \in \mathbb{C}. \tag{3.4} $$

Without loss of generality, we further assume $d_{\lambda_0, \alpha_0} = 0$ for some $\lambda_0$ and $\alpha_0$. If not, we use $(\bar{c}_{\lambda_0, \alpha_0} - d_{\lambda_0, \alpha_0} j) z$ instead of $z$. The condition (3.1) reduces to

$$ zH = pz, \tag{3.5} $$

for some fixed $p = p_1 + p_2 j \in \mathbb{H}$. In terms of coefficients, (3.4) is equivalent to

$$ \lambda c_{\lambda, \alpha} = p_1 c_{\lambda, \alpha} - p_2 d_{\lambda, \alpha}, \quad -\lambda d_{\lambda, \alpha} = p_1 d_{\lambda, \alpha} + p_2 \bar{c}_{\lambda, \alpha}, \tag{3.6} $$

for all $\lambda, \alpha$. Notice that $d_{\lambda_0, \alpha_0} = 0$, it follows from (3.5) we have $p_1 = \lambda_0$, $p_2 = 0$. This implies

$$ \lambda = p_1 = \lambda_0, \quad d_{\lambda, \alpha} = 0, \quad \text{for all } \lambda, \alpha, \tag{3.7} $$

by the linearly full assumption. Thus, we have

$$ z = \sum_{\alpha} c_{\alpha} u_{\lambda, \alpha}, \quad c_{\alpha} \in \mathbb{C}, \tag{3.8} $$

for a positive weight $\lambda$.

In summary, we completely classify all homogeneous two-spheres in $\text{HP}^n$. That is

**Theorem 3.3.** Let $\rho = \rho_{m_1} \oplus \cdots \oplus \rho_{m_s}$, $m_1 + \cdots + m_s = n + 1$, be a quaternionic representation of $SU(2)$. If $M$ is a two-dimensional $\rho(SU(2))$-orbit in $\text{HP}^n$, then there exists a point $[z] \in M$ and a positive weight $\lambda$ s.t.

$$ z = \sum_{\alpha} c_{\alpha} u_{\lambda, \alpha}, \quad c_{\alpha} \in \mathbb{C}. $$
4 Minimal homogeneous two-spheres in $\text{HP}^n$

Let $\rho = \rho_{m_1} \oplus \cdots \oplus \rho_{m_s}$, $m_1 + \cdots + m_s = n + 1$ be a quaternionic representation of $\text{SU}(2)$, and let $M$ be a two-dimensional $\rho(\text{SU}(2))$-orbit in $\text{HP}^n$. Suppose $[z] \in M$ is the point we get in Theorem 3.3 i.e., $z = \sum_{\alpha} c_{\lambda, \alpha} u_{\lambda, \alpha}$, $c_{\alpha} \in \mathbb{C}$, for some positive weight $\lambda$. We further assume $|z|^2 = 1$. Set

$$Z_0 = z \rho.$$ (4.1)

Taking the exterior derivative of (4.1) and using (2.16), (2.17), we have

$$dZ_0 = (\lambda(\iota \omega)z + \varphi z A - \varphi z B) \rho.$$ (4.2)

We define

$$dZ_0 \equiv \varphi X + \bar{\varphi} Y, \mod Z_0.$$ (4.3)

Notice that $z \in U_{\lambda}$, $z A \in U_{\lambda - 2}$ for $\lambda > 1$, $z A \in U_{\lambda}$ for $\lambda = 1$, $z B \in U_{\lambda + 2}$, we have

$$X = z A \rho - \ell z \rho, \quad Y = -z B \rho,$$ (4.4)

where $\ell = (zA, z)$. It’s clear that $\ell = 0$ when $\lambda > 1$, and $\ell$ has the possibility to be nonzero when $\lambda = 1$. The fact $(X, Y)$ = 0 means that the orbit $M$ is conformal in $\text{HP}^n$, and the induced metric is

$$K = \frac{4}{|X|^2 + |Y|^2} \rho \bar{\rho}.$$ (4.5)

Notice that $|X|^2 + |Y|^2$ is a constant and the metric $\rho \bar{\rho}$ has the curvature 4, we know that the induced metric (4.5) has constant curvature

$$K = \frac{4}{|X|^2 + |Y|^2}.$$ (4.6)

Now we give a simple expression of the criteria (a) in Proposition 2.1 for later use. Notice that $\eta = 2 i \omega$ by (2.2), and substituting (4.2) into (2.28), we have $\theta_{\ell 0} = i \omega$, $\theta'_{0 0} = \varphi \ell$. From the definition (2.32), we get

$$DX = \varphi \left( z A^2 - (\ell z) A - (\ell z) B \right) \rho - \bar{\varphi} \left( z A B - (\ell z) B \right) \rho.$$ (4.7)

Thus, according to Proposition 2.1 $M$ is minimal in $\text{HP}^n$ if and only if

$$z A B - (\ell z) B = p z$$ (4.8)

for fixed $p = p_1 + p_2 j \in \mathbb{H}$. We will study the geometry of minimal orbits in following two cases: $\lambda > 1$ and $\lambda = 1$.

**Case I.** Suppose that $\lambda > 1$, and hence $\ell = 0$. From (2.16), by comparing the coefficients, (4.7) is equivalent to

$$a_{\lambda, \alpha}^2 = p, \quad \text{for all } \alpha.$$ (4.9)

This implies that $n_1 = \cdots = n_s = n_0 = 2m_0 - 1$. So, the associated quaternionic representation $\rho$ takes the form $\rho = \rho_{m_0} \oplus \cdots \oplus \rho_{m_0}$, and the corresponding minimal immersion is given by

$$[a, b] \mapsto [c_1 \phi_{\lambda, m_0}, \ldots, c_s \phi_{\lambda, m_0}],$$ (4.10)

where $c_{\alpha} \in \mathbb{C}$ are nonzero.
Proposition 4.1. Up to an isometry of $\mathbb{H}^n$, the immersion \((4.10)\) is congruent to
\[
f_{\lambda,m_0} : [a, b] \mapsto [\cos t \phi_{\lambda,m_0}, \sin t \phi_{\lambda,m_0}, 0, \ldots, 0],
\]
where $t \in [0, \pi/2)$.

Remark. If the parameter $t = 0$ and $f$ is proper, we obtain the immersion \[
f_{\lambda} : S^2 \to \mathbb{H}^n, \quad [a, b] \mapsto [\phi_{\lambda,n+1}^0],\]
which firstly appeared in [13].

Proof. Set $e := (c_1, \ldots, c_s)$. Notice that the choice of $e$ is unique up to a factor (multiplied on the left) of absolute $1$, without loss of generality, we can find a matrix $T \in SO(s)$ s.t. $cT = (\cos t, \sin t, 0, \ldots, 0)$. Then the proposition holds by choosing the isometry $T \otimes I_m$ of $\mathbb{H}^n$. □

Next, we calculate the Gauss curvature of the immersion \((4.11)\). By \((2.16)\) and \((4.4)\), we obtain
\[
K = \frac{4}{a_{\lambda,m_0}^2 + a_{\lambda+2,m_0}^2} = \frac{8}{(n_0 + 1)^2 - (\lambda^2 + 1)}.
\]
So by \((4.6)\), the Gauss curvature is
\[
K = \frac{4}{a_{\lambda,m_0}^2 + a_{\lambda+2,m_0}^2}, \quad (4.12)
\]

Case II. Suppose that $\lambda = 1$. Since $z = \sum_{\alpha=1}^s e_{\alpha}u_{1,\alpha}$, $e_{\alpha} \in \mathbb{C}$, we have
\[
zA = \sum_{\alpha=1}^s \left( (-1)^{m_{\alpha}+1} i c_{\alpha} a_{1,\alpha} \right) u_{1,\alpha}
\]
by \((2.16)\), and then,
\[
(zA, z) = \left( \sum_{\alpha=1}^s (-1)^{m_{\alpha}+1} a_{1,\alpha} c_{\alpha}^2 \right) j.
\]
From \((4.13)\), we find \( (e^{\pm \tau}z)A, e^{\pm \tau}z \) = \( e^{\pm \tau} (zA, z) \). So, we can further assume $\ell = (zA, z) = (\ell' \bar{\ell}) j$, where $\ell' \in \mathbb{R}$. In terms of components, \((4.3)\) is equivalent to
\[
a_{1,\alpha}^2 c_{\alpha} + (-1)^{m_{\alpha}} \ell' a_{1,\alpha} \bar{c_{\alpha}} = \bar{p}_1 c_{\alpha}, \quad \text{for all } \alpha.
\]
Splitting $c_{\alpha}$, $p_1$ into real and imaginary parts, i.e., $c_{\alpha} = c_{\alpha}' + c_{\alpha}'' i$, $p_1 = p_1' + p_1'' i$. Then, \((4.14)\) are equivalent to
\[
(a_{1,\alpha}^2 + (-1)^{m_{\alpha}} \ell' a_{1,\alpha} - p_1') c_{\alpha} + p_1'' c_{\alpha} = 0,
\]
\[
-p_1'' c_{\alpha}' + (a_{1,\alpha}^2 + (-1)^{m_{\alpha}} \ell' a_{1,\alpha} - p_1') c_{\alpha}'' = 0,
\]
for all $\alpha$. Notice that $c_{\alpha} \neq 0$, we know that the determinant of the coefficient matrix of the system $\((4.15)\)$ and $\((4.16)\)$ is equal to zero, which follows that $a_{1,\alpha}^2$ satisfy the real quadratic equation
\[
x^2 - ((\ell')^2 + 2p_1') x + |p_1'|^2 = 0.
\]
This means that there are two different $m_\alpha$ at most. So, the associated representation $\rho$ can be written as

$$\rho = \rho_{m_1} \oplus \cdots \oplus \rho_{m_1},$$  

(4.17)
or

$$\rho = \rho_{m_1} \oplus \cdots \oplus \rho_{m_1} \oplus \rho_{m_2} \oplus \cdots \oplus \rho_{m_2}, \; m_1 < m_2.$$  

(4.18)

**Proposition 4.2.** Up to an isometry of $\text{HP}^n$, the minimal immersion (4.19) is congruent to

$$f_1 : [a,b] \mapsto [\phi_{1,m_1}, 0, \ldots, 0]$$  

(4.20)
or

$$f_{m_1, m_1} : [a,b] \mapsto \frac{\sqrt{7}}{2} [\phi_{1,m_1}, 1, \phi_{1,m_1}, 0, \ldots, 0].$$  

(4.21)

**Proof.** For this subcase, we have $\ell' = (-1)^{m_1+1} a_{1,n_1} \sum c_\alpha^2$. Then the minimality condition (4.14) can be written as

$$(a_{1,n_1}^2 - p_1)c_\alpha = a_{1,n_1}^2 (\sum c_\alpha^2)\overline{c_\alpha}$$  

(4.22)

for all $\alpha$, where $\sum c_\alpha \overline{c_\alpha} = 1$. Multiplying by $c_\alpha$ on both sides of (4.22) and summating for all $\alpha$, we obtain

$$(a_{1,n_1}^2 - p_1)\sum c_\alpha^2 = a_{1,n_1}^2 (\sum c_\alpha^2),$$

which implies $p_1 = 0$ or $\sum c_\alpha^2 = 0$.

If $p_1 = 0$, then from (4.22), we get $c_\alpha = (\sum c_\alpha^2)\overline{c_\alpha}$. Since we assume $\ell'$ is real, then $\sum c_\alpha^2$ is also real. Therefore, we must have $\sum c_\alpha^2 = 1$ and $c_\alpha$ are real for all $\alpha$. Now we can find a matrix $T \in \text{SO}(s)$ s.t. $(c_1, \cdots, c_s) T = (1, 0, \ldots, 0)$. Then the immersion (4.19) is congruent to (4.20) by choosing the isometry $T \otimes I_{m_1}$ of $\text{HP}^n$.

If $\sum c_\alpha^2 = 0$, i.e. $\ell' = 0$, then the choosing of $z$ is still allowed a $U(1)$-transformation. Similar to the proof of Proposition 4.1, the immersion (4.19) is congruent to

$$[a,b] \mapsto [\cos t \phi_{1,m_1}, i \sin t \phi_{1,m_1}, 0, \ldots, 0],$$

where $t \in [0, \pi)$. It follows from $\sum c_\alpha^2 = 0$ that $t = \pi/4$. Thus we complete the proof.

Next, we calculate the Gauss curvatures of the immersions (4.20) and (4.21). For the immersion (4.20), by (4.16), (4.44) and straightforward computation, we obtain $|X|^2 + |Y|^2 = |zB|^2 = a_{1,n_1}^2$. So, by (4.6), the Gauss curvature is

$$K = \frac{4}{a_{1,n_1}^2} = \frac{16}{(n_1 + 1)^2 - 4}.$$  

(4.23)
For the immersion (4.21), by \( \ell' = 0 \), we have \( |X|^2 + |Y|^2 = |zA|^2 + |zB|^2 = a_{1,n_1}^2 + a_{3,n_1}^2 \). Then the Gauss curvature is
\[
K = \frac{4}{a_{1,n_1}^2 + a_{3,n_1}^2} = \frac{8}{(n_1 + 1)^2 - 2}.
\]

(II.II) If the associate representation \( \rho \) takes form of (4.18), we split
\[
z = \sum_{\alpha = 0}^{s_1} c_{\alpha} u_{1,n_1} + \sum_{\beta = s_1+1}^{s} c_{\beta} u_{1,n_2},
\]
where \( c_{\alpha}, c_{\beta} \in \mathbb{C} \) and \( \sum_{\alpha} c_{\alpha} \bar{c}_{\alpha} + \sum_{\beta} c_{\beta} \bar{c}_{\beta} = 1 \). Then the corresponding immersion is given by
\[
[a, b] \mapsto [c_1 \phi_{1,m_1}, \ldots, c_{s_1} \phi_{1,m_1}, c_{s_1+1} \phi_{1,m_2}, \ldots, c_s \phi_{1,m_2}].
\]

Proposition 4.3. Up to an isometry of \( \mathbb{H}^n \), the minimal immersion (4.25) is congruent to
\[
f_{m_1,m_2}: [a, b] \mapsto \left[ \sqrt{\frac{m_1}{m_1+m_2}} \phi_{1,m_1}, \sqrt{\frac{m_2}{m_1+m_2}} \phi_{1,m_2}, 0, \ldots, 0 \right]
\]
when \( m_1 + m_2 \) is an even, or
\[
f'_{m_1,m_2}: [a, b] \mapsto \left[ \sqrt{\frac{m_1}{m_1+m_2}} \phi_{1,m_1}, \sqrt{\frac{m_2}{m_1+m_2}} \phi_{1,m_2}, 0, \ldots, 0 \right]
\]
when \( m_1 + m_2 \) is an odd, respectively, where \( m_1 < m_2 \).

Proof. For this subcase, the minimality condition (4.14) can be written as
\[
(a_{1,n_1}^2 - p_1)c_{\alpha} = (-1)^{m_1+1} \ell' a_{1,n_1} c_{\alpha},
\]
\[
(a_{1,n_2}^2 - p_1)c_{\beta} = (-1)^{m_2+1} \ell' a_{1,n_2} c_{\beta},
\]
for all \( \alpha, \beta \), where
\[
\ell' = (-1)^{m_1+1} a_{1,n_1} \sum_{\alpha} c_{\alpha} \bar{c}_{\alpha} + (-1)^{m_2+1} a_{1,n_2} \sum_{\beta} c_{\beta} \bar{c}_{\beta}.
\]
We first claim that \( \ell' \neq 0 \). Otherwise, if \( \ell' = 0 \), then from (4.28) and (4.29), we get \( p_1 = a_{1,n_1}^2 = a_{1,n_2}^2 \), i.e. \( m_1 = m_2 \), which leads to a contradiction. Multiplying by \( \bar{c}_{\alpha} \) and \( \bar{c}_{\beta} \) on both sides of (4.28) and (4.29), respectively, and summatting for all \( \alpha \) and \( \beta \), we obtain
\[
(a_{1,n_1}^2 - p_1) \sum_{\alpha} c_{\alpha} \bar{c}_{\alpha} = (-1)^{m_1+1} \ell' a_{1,n_1} \sum_{\alpha} c_{\alpha} \bar{c}_{\alpha},
\]
\[
(a_{1,n_2}^2 - p_1) \sum_{\beta} c_{\beta} \bar{c}_{\beta} = (-1)^{m_2+1} \ell' a_{1,n_2} \sum_{\beta} c_{\beta} \bar{c}_{\beta}.
\]
By using the assumption \( \ell' \) is real and adding (4.31) and (4.32), we have
\[
p_1 = a_{1,n_1}^2 \sum_{\alpha} c_{\alpha} \bar{c}_{\alpha} + a_{1,n_2}^2 \sum_{\beta} c_{\beta} \bar{c}_{\beta} - (\ell')^2;
\]

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so \( p_1 \) is also real. Therefore, it follows from (4.28) and (4.29) that \( c_\alpha = \pm \bar{c}_\alpha \) for all \( \alpha \) and 
\( c_\beta = \pm \bar{c}_\beta \) for all \( \beta \). Moreover, we have
\[
a_{1,n_1}^2 - p_1 = \pm (-1)^{m_1 + 1} \ell' a_{1,n_1}, \quad a_{1,n_2}^2 - p_1 = \pm (-1)^{m_2 + 1} \ell' a_{1,n_2}.
\]
Without lose of generality, we first consider \( c_\alpha = \bar{c}_\alpha \) for all \( \alpha \) and \( c_\beta = \bar{c}_\beta \) for all \( \beta \), i.e. \( c_\alpha, c_\beta \) are all real. Then \( \sum c_\alpha^2 + \sum c_\beta^2 = 1 \) and
\[
a_{1,n_1}^2 - p_1 = (-1)^{m_1 + 1} \ell' a_{1,n_1}, \quad a_{1,n_2}^2 - p_1 = (-1)^{m_2 + 1} \ell' a_{1,n_2},
\]
which gives
\[
a_{1,n_1}^2 - a_{1,n_2}^2 = \ell' ((-1)^{m_1 + 1} a_{1,n_1} - (-1)^{m_2 + 1} a_{1,n_2}),
\]
so
\[
\ell' = (-1)^{m_1 + 1} a_{1,n_1} + (-1)^{m_2 + 1} a_{1,n_2}.
\] (4.33)
Combining (4.30) and (4.33), we obtain
\[
\sum c_\beta^2 = (-1)^{m_1 + m_2 + 1} a_{1,n_2} a_{1,n_1} \sum c_\alpha^2 > 0,
\]
which follows that \( m_1 + m_2 \) is an odd number and
\[
\sum c_\alpha^2 = \frac{a_{1,n_1}}{a_{1,n_1} + a_{1,n_2}} = \frac{m_1}{m_1 + m_2}, \quad \sum c_\beta^2 = \frac{a_{1,n_2}}{a_{1,n_1} + a_{1,n_2}} = \frac{m_2}{m_1 + m_2}.
\] (4.34)
Therefore, by similar argument as the proof of Proposition 4.1, the immersion (4.28) is congruent to (4.27).

Next, we only need to consider \( c_\alpha = \bar{c}_\alpha \) for all \( \alpha \) and \( c_\beta = -\bar{c}_\beta \) for all \( \beta \), i.e. \( c_\alpha \) are all real and \( c_\beta \) are all purely imaginary. Then \( \sum c_\alpha^2 - \sum c_\beta^2 = 1 \) and
\[
a_{1,n_1}^2 - p_1 = (-1)^{m_1 + 1} \ell' a_{1,n_1}, \quad a_{1,n_2}^2 - p_1 = (-1)^{m_2 + 1} \ell' a_{1,n_2},
\]
which gives
\[
a_{1,n_1}^2 - a_{1,n_2}^2 = \ell' ((-1)^{m_1 + 1} a_{1,n_1} + (-1)^{m_2 + 1} a_{1,n_2}),
\]
so
\[
\ell' = (-1)^{m_1 + 1} a_{1,n_1} - (-1)^{m_2 + 1} a_{1,n_2}.
\] (4.35)
Combining (4.30) and (4.35), we obtain
\[
\sum c_\beta^2 = (-1)^{m_1 + m_2 + 1} a_{1,n_2} a_{1,n_1} \sum c_\alpha^2 < 0,
\]
which follows that \( m_1 + m_2 \) is an even number and
\[
\sum c_\alpha^2 = \frac{a_{1,n_1}}{a_{1,n_1} + a_{1,n_2}} = \frac{m_1}{m_1 + m_2}, \quad \sum c_\beta^2 = -\frac{a_{1,n_2}}{a_{1,n_1} + a_{1,n_2}} = -\frac{m_2}{m_1 + m_2}.
\] (4.36)
Therefore, by similar argument as the proof of Proposition 4.1, the immersion (4.25) is congruent to (4.26). Thus we complete the proof.

Now, we calculate the Gauss curvatures of the immersions (4.26) and (4.27). For the immersion (4.26), by (2.16), (4.4), (4.34) and straightforward computation, we obtain

$$|X|^2 = |zA - \ell z|^2 = a_1^2 \sum_{\alpha} c_{\alpha}^2 + a_2^2 \sum_{\beta} c_{\beta}^2$$

$$= m_1^2 \frac{m_1}{m_1 + m_2} + m_2^2 \frac{m_2}{m_1 + m_2} = m_1 m_2,$$

and

$$|Y|^2 = |zB|^2 = a_3^2 \sum_{\alpha} c_{\alpha}^2 + a_4^2 \sum_{\beta} c_{\beta}^2$$

$$= (m_1 - 1)^2 \frac{m_1}{m_1 + m_2} + (m_2 - 1)^2 \frac{m_2}{m_1 + m_2} = m_1^2 - m_1 m_2 + m_2^2 - 1.$$ 

So by (4.6), the Gauss curvature is

$$K = \frac{4}{m_1^2 + m_2^2 - 1}. \quad (4.37)$$

For the immersion (4.26), we can obtain the expression of the Gauss curvature by similar calculation, which is also given by (4.37).

In summary, we completely classify the minimal homogeneous 2-spheres in quaternionic projective space $\mathbb{HP}^n$. That is

**Theorem 4.4.** Let $f : S^2 \to \mathbb{HP}^n$ be a linearly full homogeneous minimal immersion. Then, in terms of homogeneous coordinates, $f$ is congruent to one of the following:

1. $f_\lambda : [a, b] \mapsto [\phi_{\lambda, n+1}], \lambda \in \{3, 5, \ldots, 2n + 1\}$, and the Gauss curvature $K = 8/[(2n + 2)^2 - (\lambda^2 + 1)];$

2. $f_1 : [a, b] \mapsto [\phi_{1, n+1}]$ and the Gauss curvature $K = 4/[n(n + 2)];$

3. $f_{\lambda, m, t} : [a, b] \mapsto [\cos t\phi_{\lambda, m}, \sin t\phi_{\lambda, m}]$ for some positive weight $\lambda \in \{3, 5, \ldots, n\}$, $t \in (0, \pi/2)$, $2m = n + 1$, and the Gauss curvature $K = 8/[n + 1)^2 - (\lambda^2 + 1)];$

4. $f_{m_1, m_2} : [a, b] \mapsto [\sqrt{m_1/(m_1 + m_2)} \phi_{1, m_1}, \sqrt{m_2/(m_1 + m_2)} \phi_{1, m_2}]$ for some positive $m_1 \leq m_2$ so that $m_1 + m_2 = n + 1$ is an even, and the Gauss curvature $K = 4/(m_1^2 + m_2^2 - 1);$

5. $f_{m_1, m_2} : [a, b] \mapsto [\sqrt{m_1/(m_1 + m_2)} \phi_{1, m_1}, \sqrt{m_2/(m_1 + m_2)} \phi_{1, m_2}]$ for some positive $m_1 < m_2$ so that $m_1 + m_2 = n + 1$ is an odd, and the Gauss curvature $K = 4/(m_1^2 + m_2^2 - 1).$

**Proof.** From the Proposition 4.1, 4.2 and 4.3 we know that $f$ is congruent to $f_\lambda$, or $f_1$, or $f_{\lambda, m, t}$, or $f_{m_1, m_2}$, or $f_{m_1, m_2}$. The expressions for the Gauss curvature $K$ follows from (4.12), (4.23), (4.24) and (4.37), respectively. □

We should point out the congruence of $f_{\lambda, m, t}$ w.r.t. the parameter $t$, that is

**Proposition 4.5.** $f_{\lambda, m, t_1}$ is not congruent to $f_{\lambda, m, t_2}$ for any $t_1, t_2 \in (0, \pi/2)$, $t_1 < t_2.$
Proof. Suppose that $f_{\lambda,m,t_1}$ is congruent to $f_{\lambda,m,t_2}$ for $t_1,t_2 \in (0,\pi/2)$, then there exists fixed $p \in Sp(1), P \in Sp(2m)$ such that
\[
p(\cos t_1, i \sin t_1) \rho_m \oplus \rho_m(g) = (\cos t_2, i \sin t_2) \rho_m \oplus \rho_m(g) P
\]
for all $g \in SU(2)$. Set $p = p' + p'' j$ and $P = (p'_{AB}) + (p''_{AB}) j$. Since $\{\Lambda_{AB} \mid 0 \leq A,B \leq 2m-1\}$ are linearly independent in $L^2(SU(2))$ by Peter-Weyl’s theorem, we obtain
\[
p'' = p''_{AB} = p'_{s1} = p'_{m+s+1} = p'_{m+s+1} = 0, s \neq l,
\]
and
\[
(p' + \bar{p'}) \cos t_1 = 2 \cos t_2 p'_{11},
(p' - \bar{p'}) \cos t_1 = 2i \sin t_2 p'_{m+11},
(p' - \bar{p'}) \sin t_1 = -2i \cos t_2 p'_{m+1},
(p' + \bar{p'}) \sin t_1 = 2 \sin t_2 p'_{m+1} + 1,
\]
because by comparing the coefficients of $\Lambda_{AB}$. The identities (4.40) and (4.41) imply that $t_1 = t_2$. \[\Box\]

Base on the study of homogeneous harmonic maps into projective space, Ohnita [13] conjectured that all proper minimal constant curved minimal two-spheres in $\mathbb{H}P^n$ are \{f_{\lambda} : \lambda = 1,3,\ldots,2n+1\}, up to a rigidity of $\mathbb{H}P^n$. Notice that $\mathbb{H}P^n$ is a rank one compact symmetric space, together with Theorem 4.4, we propose the following conjecture, still called Ohnita’s conjecture:

**Conjecture 4.6.** Let $f$ be a proper minimal constant curved immersion from $S^2$ into $\mathbb{H}P^n$, then $f$ is congruent to one of $f_{\lambda}, f_{\lambda,m,t}, f_{m1}, f_{m2}, f'_{m1}, f'_{m2}$, where $\lambda \in \{1,3,\ldots,2n+1\}$.

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