Phase Transition in Gauge Theories and the Planck Scale Physics.

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Abstract

The present paper is based on the modified part of the review “Random Dynamics and Multiple Point Model” by L.V.Laperashvili, H.B.Nielsen, D.A.Ryzhikh and N.Stillits, in preparation for publication in Russian, which contains the results of our joint activity with H.B.Nielsen concerning the investigations of phase transitions in gauge theories. In this review we have presented the main ideas of the Nielsen’s Random Dynamics (RD) and his achievements (with co-authors) in the Anti-Grand Unification Theory (AGUT) and Multiple Point Model (MPM). We have considered also the theory of Scale Relativity (SR) by L.Nottale, which has a lot in common with RD: both theories lead to the discreteness of our space-time, giving rise to the new description of physics at very small distances. In this paper we have demonstrated the possibility of $[SU(5)]^3$ SUSY unification with superparticles of masses $M \approx 10^{18.3}$ GeV and calculated its critical point – critical value of the inverse finestructure constant – at $\alpha_{5,\text{crit}}^{-1} = \alpha_{5}(\mu_{P}) \approx 34.0$ (close to $\alpha_{GUT}^{-1} \approx 34.4$) with a hope that such an unified theory approaches the (multi)critical point at the Planck scale.

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This review and new investigations are dedicated to the 60th jubilee of the outstanding physicist of Denmark Holger Bech Nielsen, professor of the Niels Bohr Institute, whose achievements are well-known in the World physics.

1. Introduction: Random Dynamics

The goal of physics always has been to describe an enormous phenomena existing in Nature by a few number of rules called as "the laws of physics". Trying to look insight the Nature and considering the physical processes at small and very small distances, physicists have made attempts to explain the well–known laws of physics as a consequence of the more fundamental laws of Nature. The contemporary physics (high energy physics, nuclear physics, solid state physics, astrophysics, cosmology) essentially is based on the quantum theory and theory of general relativity. But on the present level of our knowledge we strongly suspect that these theories are not fundamental, they are a consequence of the more fundamental laws of physics. On this way, Random Dynamics (RD) was suggested and developed in Refs. [2–12] as a theory of physical processes proceeding at small distances of order of the Planck length \( \lambda_P = M_{Pl}^{-1} \):

\[
M_{Pl} = 1.22 \cdot 10^{19} \text{ GeV.}
\]

RD tries to derive the laws of physics known today by the use of almost no assumptions.

The modern physics of electroweak and strong interactions is described by the Standard Model (SM) unifying the Glashow–Salam–Weinberg electroweak theory and QCD, theory of strong interactions (see [13]).

The gauge group of SM is:

\[
SMG = SU(3)_c \times SU(2)_L \times U(1)_Y,
\]

which describes the present elementary particle physics up to the scale \( \sim 100 \text{ GeV.} \)

SM contains 19 degrees of freedom: three independent gauge coupling constants \( g_i \) (i=1,2,3 correspond to the groups U(1), SU(2), SU(3)); three masses of leptons, six quark masses, the Higgs boson mass, four quark mixing angles and two topological angles \( \theta_{SU(2)} \) and \( \theta_{QCD} \) from the topological terms in SU(2) and SU(3), respectively. A large part of these degrees of freedom are related to the coupling of the Higgs sector to itself and to the matter sector. But at this stage of our knowledge we have already had questions which have no answers at present: Why has SM the Lie group \( S(U(2) \times U(3)) \)? Why have we three generations of the "fundamental" particles (at least, within the energies of today experiment)? Do more generations exist? And more fundamental question: Why quantum physics at small distances?

These questions lead to the conclusion that SM is not a TOE (Theory of Everything). It looks like that SM is only a low-energy limit of a more fundamental theory. The efforts to construct such a fundamental theory have led to Grand Unified Theories (GUT), especially supersymmetric GUT (SUSY GUT) which had an aim to give an unified selfconsistent description of electroweak and strong interactions by one simple group of symmetry SU(5), SU(6),
SO(10), or $E_8$ (see reviews [14], [15]). But at present time the experiment does not indicate any manifestation of these theories.

The next step to search the fundamental theory was a set of theories describing the extended objects: string, superstring and M–theories. They came into existence due to the necessity of unification of electroweak and strong interactions with gravity [16–19].

RD is an alternative of these theories. The above mentioned theories are based on some fixed axioms. If namely one of these axioms is changed a little bit, the theory also is changed and often becomes not even consistent. In RD we have the very opposite case. RD is based on the very general assumptions which take place at fundamental scale.

Wondering what fundamental laws of physics lead to the description of the low–energy SM phenomena, observed by today experiment, we can consider two possibilities:

1. A simple and fine theory underlies in physics: at very small (Planck scale) distances in our continuous space–time, there exists a theory with a high symmetry, for example, such an extension of superstrings as M–theory [20], which unifies all known theories of superstrings (I, IIA, IIB, SO(32), $E_8 \times E_8$, etc.) with 11–dimensional supergravity.

2. The fundamental laws of Nature are so complicated that it is preferable to think that at very small distances there are no laws at all: our space–time is discrete and the physical processes are described randomly. Then the fundamental law of Nature is one, which is randomly chosen from a large set of sufficiently complicated theories, and this one leads to the laws observed in the low–energy limit by our experiment.

The item 2 is a base of RD theory.

2. Theory of Scale Relativity

The theory of scale relativity [21] is also related with item 2 of Introduction and has a lot in common with RD. This is a theory representing a new approach to understanding a quantum mechanics and various problems of scales in today’s physics. As it was mentioned in Introduction, the contemporary physics is based on two main theories: the theory of relativity (special and general, which include classical physics) and quantum mechanics (developed into quantum field theories). Both theories are very effective and precise in their predictions, but they are founded on completely different grounds, even contradictory in appearance, and are described by quite different mathematical apparatus. General relativity is based on the fundamental physical principles, such as principles of general covariance and equivalence. The mathematical description of this theory follows from these principles. On the contrary, quantum mechanics is an axiomatic theory (at least at present time), and its mathematical apparatus is not understood on a more fundamental level yet. Such a situation shows an apparent contradiction in physics: two opposite theories, classical and quantum, cohabit there. In particular, gravity up to now has not selfconsistent description in terms of the quantum field theory. These and other signs indicate that physics is still in infancy: present theory is not able to describe the two “tails” of the physical world – very small and very large time and length scales.

It is clear now that the observed properties of the quantum world cannot be reproduced by Riemannian geometry. The space–time description cannot be based on particular fields, but only on the universal properties of the matter. That is, we have a necessity to introduce new concepts in physics.
The author of theory of scale relativity L.Nottale [21], his collaborators [22] and B.G.Sidharth [23] assumed that the previous geometrical description of the quantum properties of microphysics is impossible. They suggested to consider the non–differentiable space–time and build the microphysical world on the concept of the fractal space–time. Moreover, this non–differentiability implies an explicit dependence of space–time on scale. The principle of relativity can be applied not only to motion, but also to scale transformations, because the resolution of measurements must be taken into account in definition of the coordinate system. The resolution of the measurement apparatus plays in quantum physics a completely new role with respect to the classical, since the result of measurements depends on it, as a consequence of Heisenberg’s relations. Any set of physical data has its sense only when it is accompanied by the measurement errors or uncertainties. More generally, the resolution characterizes the system under consideration. Complete information about the measurement of position and time can be obtained when not only space–time coordinate (t,x,y,z), but also resolutions (Δt, Δx, Δy, Δz) are given. As a result, we expect space–time to be described by a metric element based on generalized, explicitly scale–dependent, metric potentials:

\[ g_{\mu\nu} = g_{\mu\nu}(t, x, y, z; \Delta t, \Delta x, \Delta y, \Delta z). \] (3)

2.1. The concept of fractals

It was shown in Refs. [22] that a continuous but non–differentiable space–time is necessarily fractal. Here a word fractal [24] means an object or space showing structures at all scales, or on a wide range of scales.

Let us consider a fractal ”curve” in the \( \mathbb{R}^2 \) or \( \mathbb{C} \) plane. It can be obtained from an initial curve \( \mathbf{F}_1 \) made up of \( p \) segments of equal length \( 1/q \) which connect the origin of the coordinate system XY to the point \([0,1]\) (see Fig.1). If \( \omega_{j+1} \) is the polar angle of the \( j \)th segment and \( Z_j = X_j + iY_j = (1/q)^{\sum_{k=0}^{j-1} e^{i\omega_k}} \) is the complex coordinate of a breaking point \( P_j \), then we have the following relations:

\[ \sum_{k=0}^{p-1} e^{i\omega_k} = q, \quad Z_{j+1} - Z_j = \frac{1}{q} e^{i\omega_j}. \] (4)

A curve \( \mathbf{F}_2 \) can be built by substituting each segment of \( \mathbf{F}_1 \) by \( \mathbf{F}_2 \) itself, scaled at its length \( 1/q \), as it is shown in Fig.2. Substitution of each segment of \( \mathbf{F}_n \) by \( q^{-n}\mathbf{F}_1 \) giving \( \mathbf{F}_{n+1} \) leads to the fractal curve \( \mathbf{F} \) which is a result of an infinite sequence of these steps.

The curve \( \mathbf{F} \) can be parametrized by a real number \( x \in [0,1] \) developed in the counting base \( p \) in the form:

\[ x = 0.x_1x_2... = \sum_{k=1}^{\infty} x_k p^{-k}. \] (5)

In this case we have so called ”the Peano curve” [24].

The fractal is completely defined when the complex coordinate \( Z(x) \) of the point on \( \mathbf{F} \) parametrized by \( x \) is known. According to the above building process of \( \mathbf{F} \), it is easy to obtain \( Z(x) \):

\[ Z(x) = Z_{x_1} + q^{-1} e^{i\omega_{x_1}} [Z_{x_2} + q^{-1} e^{i\omega_{x_2}} [Z_{x_3} + ...]]. \] (6)
and finally:

\[ Z(x) = q \sum_{k=1}^{\infty} Z_{x_k} e^{i \sum_{l=1}^{k-1} \omega_{xl}} q^{-k}. \] (7)

It is well-known [24] that a fractal dimension lies between 1 and 2, and a fractal curve within a plane is intermediate between a line and a surface. Indeed, while it may be built by adding segments, it may be also be obtained by deleting surfaces [24].

The fractal dimension is given by

\[ D = \log p/\log q. \]

The length and surface of the \( F_n \) are:

\[ L \bigg/ L_0 = (p/q)^n = q^{n(D-1)}, \]

\[ S \bigg/ S_0 = (p/q^2)^n = q^{n(D-2)}. \] (8)

We see that however small is the difference of parameters \( x(2) - x(1) \) for two points \( M_2 \) and \( M_1 \) on the fractal, the distance in the plane \( |Z(x_2) - Z(x_1)| \) vanishes, then the fractal length goes to infinity (but surface vanishes). From this ”paradox” we conclude that the description of the fractal curve \( F \) by the real coordinate \( x \) is insufficient and another formalism is needed. But we don’t consider here the ”non–standard analysis” of fractals, referring to the book [24].

2.2. Standard fractal scale-invariant laws

Let us consider a non–differentiable coordinate system. The basic theorem of the theory of scale relativity [22] implies that the length of the fractal curve \( L \) is an explicit function of the resolution interval \( \epsilon \): \( L = L(\epsilon) \). At the first step it was assumed that this function obeys the simplest possible (first order) scale differential equation:

\[ \frac{d \ln L}{d \ln (\lambda/\epsilon)} = \delta, \] (9)

where \( \delta \) is a constant. The solution of Eq.(9) is a fractal power–law dependence:

\[ L = L_0(\lambda/\epsilon)^\delta, \] (10)

here \( \delta \) is the scale dimension: \( \delta = D - D_T \), where \( D \) and \( D_T \) are the fractal and topological dimensions, respectively.

The group of scale transformations has the Galilean structure: it transforms in a scale transformation \( \epsilon \to \epsilon' \) as

\[ \ln L(\epsilon') = \ln L(\epsilon) + \delta(\epsilon) \ln \frac{\epsilon}{\epsilon'}, \text{ with } \delta(\epsilon') = \delta(\epsilon). \] (11)

From Eq.(11) the structure of the Galileo group is confirmed by the law of composition of dilations \( \epsilon \to \epsilon' \to \epsilon'' \), which leads to the relation \( \ln \rho'' = \ln \rho + \ln \rho' \), with \( \rho = \epsilon'/\epsilon, \rho' = \epsilon''/\epsilon' \) and \( \rho'' = \epsilon''/\epsilon. \)
2.3. Breaking of scale symmetry

One can assume also the existence of a first order differential equation for the function $L(\epsilon)$ which is similar to the renormalization group equation (RGE):

$$\frac{dL}{d\ln \epsilon} = \beta(L).$$

(12)

Here a scale variable $\epsilon$ is a resolution.

The function $\beta(L)$ is a priori unknown, but it is possible to consider its Taylor expansion giving a perturbative series:

$$\frac{dL}{d\ln \epsilon} = a + bL + ...$$

(13)

The solution of this equation, with $\lambda$ as a constant of integration, has the following form:

$$L = L_0[1 + (\frac{\lambda}{\epsilon})^\delta],$$

(14)

where $\delta = -b$.

From Eq.(14) we see for $\delta > 0$ a small-scale fractal behavior of $L$ which is broken at larger scales, but for $\delta < 0$ this fractal behavior takes place for large scales and is broken at smaller scales.

2.4. Special scale relativity

We know that the Galileo group of motion is a degeneration of the more general Lorentz group. Assuming that the same is true for scale laws, we can use a principle of scale relativity and consider instead of the Galilean law of composition of dilations:

$$\ln(\frac{\epsilon'}{\lambda}) = \ln \rho + \ln(\frac{\epsilon}{\lambda})$$

(15)

the more general Lorentzian law:

$$\ln(\frac{\epsilon'}{\lambda}) = \frac{\ln \rho + \ln(\epsilon/\lambda)}{1 + \ln \rho \ln (\epsilon/\lambda)/\ln^2(\lambda_P/\lambda)}.$$  

(16)

The scale dimension $\delta$ becomes itself a variable (see [22]):

$$\delta(\epsilon) = \frac{1}{\sqrt{1 - \ln^2(\epsilon/\lambda)/\ln^2(\lambda_P/\lambda)},}$$

(17)

where $\lambda$ is the fractal–nonfractal transition scale.

2.5. Fundamental scale

In the law given by Eq.(17) there exists a minimal scale of space–time resolution $\epsilon_{min} = \lambda_P$, which is invariant under dilations and contractions. It plays the same role for scales as the velocity of light for motion in the Einstein’s special theory of relativity. This invariant length scale is the minimal, i.e. fundamental scale of Nature. It is natural to identify it with the Planck scale, $\lambda_P = (\hbar G/c^3)^{1/2}$. The Planck length and time scales thus appear as natural units of length and time intervals.

If the fundamental scale exists in Nature, then our (3+1)–dimensional space is discrete on the fundamental level. This hypothesis is a base of RD theory by H.B.Nielsen [2]. It is an initial point of view, but not an approximation.
3. Lattice Theories

The lattice model of gauge theories is the most convenient formalism for the realization of RD ideas. In the simplest case we can imagine our space–time as a regular hypercubic (3+1)–lattice with the parameter \( a \) equal to the fundamental scale:

\[
a = \lambda_P = 1/M_{Pl} \sim 10^{-33} \text{sm.}
\]  

(18)

3.1. Mathematical structure of lattice gauge theories

A lattice contains sites, links and plaquettes. Link variables defined on the edges of the lattice are fundamental variables of the lattice theory. These variables are simultaneously the elements of gauge group \( G \), describing a symmetry of the corresponding lattice gauge theory:

\[
U(x \rightarrow y) \in G.
\]  

(19)

It is easy to understand the sense of this variable turning to the differential geometry of the continuum space–time in which our gauge fields \( \hat{A}_\mu(x) \) exist, where the quantity

\[
\hat{A}_\mu(x) = gA^\mu_\mu(x)t^j
\]  

(20)

contains the generator \( t^j \) of the group \( G \). For \( G = SU(3) \) we have \( t^j = \lambda^j/2 \), where \( \lambda^j \) are the well–known Gell–Mann matrices.

For \( G = U(1) \):

\[
\hat{A}_\mu(x) = gA_\mu(x).
\]  

(21)

Such a space geometrically is equivalent to curvilinear space and an operator, which compares fields at different points, is an operator of the parallel transport between the points \( x, y \):

\[
U(x, y) = Pe^i \int_{C_{xy}} \hat{A}_\mu(x) dx^\mu,
\]  

(22)

where \( P \) is the path ordering operator and \( C_{xy} \) is a curve from point \( x \) till point \( y \). Moreover, the operator:

\[
W = Tr(Pe^i \int_{C_{xy}} \hat{A}_\mu(x) dx^\mu)
\]  

(23)

is the well–known Wilson loop. In the case of scalar field \( \phi(x) \), interacting with gauge field \( A_\mu \), we have an additional gauge invariant observable:

\[
\phi^+(y)[Pe^i \int_{C_{xy}} \hat{A}_\mu(x) dx^\mu] \phi(x).
\]  

(24)

The link variable \( 19 \) is a lattice version of Eq.(22):

\[
U(x \rightarrow y) = e^{i\Theta_\mu(n)} \equiv U_\mu(n).
\]  

(25)

This link variable connects the point \( n \) and the point \( n + a_\mu \), where the index \( \mu \) indicates the direction of a link in the hypercubic lattice with parameter \( a \). Considering the infinitesimal increment of the operator \( 22 \) in the continuum limit, we have:

\[
\Theta_\mu(n) = a \hat{A}_\mu(x).
\]  

(26)

Plaquette variables are not independent because they are products of link variables:

\[
U_p \equiv U(\square) \equiv U(\Box)U(\Box)U(\Box)U(\Box).
\]  

(27)
3.2. Lattice actions

The lattice action $S[\mathcal{U}]$ is invariant under the gauge transformations on a lattice:

$$\mathcal{U}(x \to y) \to \Lambda(x)\mathcal{U}(x \to y)\Lambda^{-1}(y),$$

where $\Lambda(x) \in G$. The simplest action $S[\mathcal{U}]$ is given by the following expression:

$$S[\mathcal{U}] = \sum_q \frac{\beta_q}{\dim q} \sum_p \text{ReTr} \mathcal{U}_p^{(q)}.$$  \hfill (28)

Here $q$ is the index of representation of the group $G$, $\dim q$ is the dimension of this representation, and $\beta_q = 1/g_q^2$, where $g_q$ is the coupling constant of gauge fields corresponding to the representation $q$.

The path integral

$$Z = \int D\mathcal{U}(\to) e^{-S[\mathcal{U}]}$$  \hfill (29)

which is an analogue of the partition function, describes the lattice gauge theory in the Euclidean four-dimensional space. It is necessary to construct the lattice field theory such that for $a \to 0$, i.e. in the continuum limit, it leads to a regularized smooth gauge theory of fields $A^i_\mu(x)$ (here $j$ is the symmetry subscript). In the opposite case, a passage to the continuum limit is not unique [25].

Let us consider the simplest case of the group $G = U(1)$, using the only representation of this group in Eq.(28):

$$S[\mathcal{U}_p] = \beta \sum_p \text{Re}\mathcal{U}_p.$$  \hfill (30)

Here the quantity $\mathcal{U}_p$ is given by Eq.(27) in which the link variables $\mathcal{U}(\to)$ are complex numbers with their moduli equal to unity, i.e.,

$$\mathcal{U}(x \to y) = \{z | z \in \mathbb{C}, |z| = 1\}. \hfill (31)$$

In the lattice model, the Lorentz gauge condition has the form

$$\prod_{x \to y} \mathcal{U}(x \to y) = 1.$$  \hfill (32)

Introducing the notation

$$z = e^{i\Theta}, \hfill (33)$$

we can write:

$$\mathcal{U}_p = e^{i\Theta_p}. \hfill (34)$$

The variables $\mathcal{U}_p$ satisfy the identity:

$$\prod_{\square \in \text{lattice cube}} \mathcal{U}(\square) = I.$$  \hfill (35)
which is called the Bianchi identity. In Eq. (35) the product is taken over all plaquettes belonging to the cell (cube) of the hypercubic lattice.

From Eqs. (30) and (34), the simplest lattice \( U(1) \) action has the form:

\[
S[U_p] = \beta \sum_p \cos \Theta_p.
\] (36)

For the compact lattice QED: \( \beta = 1/e_0^2 \), where \( e_0 \) is the bare electric charge.

The lattice \( SU(N) \) gauge theories were first introduced by K.Wilson [26] for studying the problem of confinement. He suggested the following simplest action:

\[
S = -\frac{\beta}{N} \sum_p \text{Re}(Tr U_p),
\] (37)

where the sum runs over all plaquettes of a hypercubic lattice and \( U_p \equiv U(\square) \) belongs to the fundamental representation of \( SU(N) \).

Monte Carlo simulations of these simple Wilson lattice theories in four dimensions showed a (or an almost) second–order deconfining phase transition for \( U(1) \) [27], [28], a crossover behavior for \( SU(2) \) and \( SU(3) \) [29], [30], and a first–order phase transition for \( SU(N) \) with \( N \geq 4 \) [31]. Bhanot and Creutz [32], [33] have generalized the simple Wilson theory by introducing two parameters in the \( SU(N) \) action:

\[
S = \sum_p \left[ -\frac{\beta_f}{N} \text{Re}(Tr U_p) - \frac{\beta_A}{N^2 - 1} \text{Re}(Tr_A U_p) \right],
\] (38)

where \( \beta_f \), \( Tr \) and \( \beta_A \), \( Tr_A \) are respectively the lattice constants and traces in the fundamental and adjoint representations of \( SU(N) \). The phase diagrams obtained for the generalized lattice \( SU(2) \) and \( SU(3) \) theories [38] by Monte Carlo methods in Refs. [32], [33] (see also [34]) are shown in Fig.3(a,b). They indicated the existence of a triple point which is a boundary point of three first–order phase transitions: the "Coulomb–like" and confining \( SU(N)/Z_N \) phases meet together at this point. From the triple point emanate three phase border lines which separate the corresponding phases. The \( Z_N \) phase transition is a "discreteness" transition, occurring when lattice plaquettes jump from the identity to nearby elements in the group. The \( SU(N)/Z_N \) phase transition is due to a condensation of monopoles (a consequence of the non-trivial \( \Pi_1 \) of the group).

Monte Carlo simulations of the \( U(1) \) gauge theory described by the two-parameter lattice action [35], [36]:

\[
S = \sum_p \left[ \beta_{\text{lat}} \cos \Theta_p + \gamma_{\text{lat}} \cos 2\Theta_p \right], \quad \text{where} \quad U_p = e^{i\Theta_p},
\] (39)

also indicate the existence of a triple point on the corresponding phase diagram: "Coulomb–like", totally confining and \( Z_2 \) confining phases come together at this triple point (see Fig.4).

Lattice theories are given in reviews [25]. The next efforts of the lattice simulations for the \( SU(N) \) gauge theories are presented in the review [37].

3.3. Lattice artifact monopoles

Lattice monopoles are responsible for the confinement in lattice gauge theories what is confirmed by many numerical and theoretical investigations (see reviews [38] and papers [39]).
In the compact lattice gauge theory the monopoles are not physical objects: they are lattice artifacts driven to infinite mass in the continuum limit. Weak coupling ("Coulomb") phase terminates because of the appearance for non-trivial topological configurations which are able to change the vacuum. These topological excitations are closed monopole loops (or universe lines of the monopole-antimonopole pairs). When these monopole loops are long and numerous, they are responsible for the confinement. But when they are dilute and small, the Coulomb ("free photon") phase appears. Banks et al. [40] have shown that in the Villain form of the $U(1)$ lattice gauge theory [41] it is easy to exhibit explicitly the contribution of the topological excitations. The Villain lattice action is:

$$S_V = \left(\frac{\beta}{2}\right) \sum_p (\Theta_p - 2\pi k)^2, \quad k \in \mathbb{Z}. \quad (40)$$

In such a model the partition function $Z$ may be written in a factorized form:

$$Z = Z_C Z_M, \quad (41)$$

where $Z_C$ is a part describing the photons:

$$Z_C \sim \int_{-\infty}^{\infty} d\Theta \exp{\left(-\frac{\beta}{2} \sum_{\square} \Theta^2(\square)\right)}, \quad (42)$$

and $Z_M$ is the partition function of a gas of the monopoles [42], [43]:

$$Z_M \sim \sum_{m \in \mathbb{Z}} \exp{-2\pi^2 \beta \sum_{x,y} m(x)v(x-y)m(y)}. \quad (43)$$

In Eq. (43) $v(x-y)$ is a lattice version of $1/r$-potential and $m(x)$ is the charge of the monopole sitting in an elementary cube $c$ of the dual lattice, which can be simply expressed in terms of the integer variables $n_p$:

$$m = \sum_{p \in \partial c} n_p, \quad (44)$$

where $n_p$ is a number of Dirac strings passing through the plaquettes of the cube $c$.

The Gaussian part $Z_C$ provides the usual Coulomb potential, while the monopole part $Z_M$ leads, at large separations, to a linearly confining potential.

It is more complicated to exhibit the contribution of monopoles even in the $U(1)$ lattice gauge theory described by the simple Wilson action (36). Let us consider the Wilson loop as a rectangle of length $T$ in the 1-direction (time) and width $R$ in the 2-direction (spacelike distance), then we can extract the potential $V(R)$ between two static charges of opposite signs:

$$V(R) = -\lim_{T \to \infty} \frac{1}{T} \log <W>, \quad (45)$$

and obtain:

$$V(R) = -\frac{\alpha(\beta)}{R} \quad \text{in the "Coulomb" phase,} \quad (46)$$

$$V(R) = \sigma R - \frac{\alpha(\beta)}{R} + O(\frac{1}{R^3}) + \text{const} \quad \text{in the confinement phase.} \quad (47)$$
3.4. The behavior of electric fine structure constant $\alpha$ near the phase transition point. "Freezing" of $\alpha$

The lattice investigators were not able to obtain the lattice triple point values of $\alpha_{c,\text{crit}}$ by Monte Carlo simulations method. Only the critical value of the effective electric fine structure constant $\alpha$ was obtained in Ref. [36] in the compact QED described by the Wilson and Villain actions (36) and (40), respectively:

$$\alpha_{\text{crit}}^{\text{lat}} \approx 0.20 \pm 0.015 \quad \text{and} \quad \tilde{\alpha}_{\text{crit}}^{\text{lat}} \approx 1.25 \pm 0.10 \quad \text{at} \quad \beta_T \equiv \beta_{\text{crit}} \approx 1.011.$$  (48)

Here:

$$\alpha = \frac{e^2}{4\pi} \quad \text{and} \quad \tilde{\alpha} = \frac{g^2}{4\pi},$$  (49)

are the electric and magnetic fine structure constants, containing the electric charge $e$ and magnetic charge $g$.

The result of Ref. [36] for the behavior of $\alpha(\beta)$ in the vicinity of the phase transition point $\beta_T$ is shown in Fig.5(a) for the Wilson and Villain lattice actions. Fig.5(b) demonstrates the comparison of the function $\alpha(\beta)$ obtained by Monte Carlo method for the Wilson lattice action and by theoretical calculation of the same quantity. The theoretical (dashed) curve was calculated by so-called "Parisi improvement formula" [44]:

$$\alpha(\beta) = \left[4\pi\beta W_p\right]^{-1}.$$  (50)

Here $W_p = \langle \cos \Theta_p \rangle$ is a mean value of the plaquette energy. The corresponding values of $W_p$ are taken from Ref. [35].

The theoretical value of $\alpha_{\text{crit}}$ is less than the "experimental" (Monte Carlo) value (48):

$$\alpha_{\text{crit}}(\text{in lattice theory}) \approx 0.12.$$  (51)

This discrepancy between the theoretical and "experimental" results is described by monopole contributions: the fine structure constant is renormalized by an amount proportional to the susceptibility of the monopole gas [42]:

$$K = \frac{\alpha_{\text{crit}}(\text{Monte Carlo})}{\alpha_{\text{crit}}(\text{lattice theory})} \approx \frac{0.20}{0.12} \approx 1.66.$$  (52)

Such an enhancement of the critical fine structure constant is due to vacuum monopole loops [43].

According to Fig.5(c):

$$\alpha_{\text{crit},\text{theor.}}^{-1} \approx 8.$$  (53)

This result does not coincide with the lattice result (48), which gives the following value:

$$\alpha_{\text{crit},\text{lat.}}^{-1} \approx 5.$$  (54)

The deviation of theoretical calculations of $\alpha(\beta)$ from the lattice ones, which is shown in Fig.5(b,c), has the following explanation: "Parisi improvement formula" (50) is valid in
Coulomb phase where the mass of artifact monopoles is infinitely large and photon is massless. But in the vicinity of the phase transition (critical) point the monopole mass $m \to 0$ and photon acquires the non-zero mass $m_0 \neq 0$ on the confinement side. This phenomenon leads to the "freezing" of $\alpha$: the effective electric fine structure constant is almost unchanged in the confinement phase and approaches to its maximal value $\alpha = \alpha_{\text{max}}$. The authors of Ref. [45] predicted that in the confinement phase, where we have the formation of strings, the fine structure constant $\alpha$ cannot be infinitely large, but has the maximal value:

$$\alpha_{\text{max}} \approx \frac{\pi}{12} \approx 0.26,$$

(55)
due to the Casimir effect for strings. The authors of Ref. [46] developed this viewpoint in spinor QED: the vacuum polarization induced by thin "strings"–vortices of magnetic flux leads to the suggestion of an analogue of the "spaghetti vacuum" [47] as a possible mechanism for avoiding the divergences of perturbative QED. According to Ref. [46], the non–perturbative sector of QED arrests the growth of the effective $\alpha$ to infinity and confirms the existence of $\alpha_{\text{max}}$. This phenomenon was called "the freezing of $\alpha". We see that Fig.5(a) demonstrates the tendency to freezing of $\alpha$ in the vicinity of the phase transition point $\beta = \beta_T$ for the compact QED.

The analogous phenomenon of the "freezing" of $\alpha_s$ was considered in QCD in Refs. [48].

4. Higgs Monopole Model and Phase Transition in the Regularized U(1) Gauge Theory

The simplest effective dynamics describing the confinement mechanism in the pure gauge lattice U(1) theory is the dual Abelian Higgs model of scalar monopoles [49] (see also Refs. [38] and [39]).

In the previous papers [50]–[53] the calculations of the U(1) phase transition (critical) coupling constant were connected with the existence of artifact monopoles in the lattice gauge theory and also in the Wilson loop action model [53].

In Ref. [53] we (L.V.L. and H.B.Nielsen) have put forward the speculations of Refs. [50]–[52] suggesting that the modifications of the form of the lattice action might not change too much the phase transition value of the effective continuum coupling constant. The purpose was to investigate this approximate stability of the critical coupling with respect to a somewhat new regularization being used instead of the lattice, rather than just modifying the lattice in various ways. In [53] the Wilson loop action was considered in the approximation of circular loops of radii $R \geq a$. It was shown that the phase transition coupling constant is indeed approximately independent of the regularization method: $\alpha_{\text{crit}} \approx 0.204$, in correspondence with the Monte Carlo simulation result on lattice: $\alpha_{\text{crit}} \approx 0.20 \pm 0.015$ (see Eq.(48) and [36]).

But in Refs. [54]–[57], instead of using the lattice or Wilson loop cut–off, we have considered the Higgs Monopole Model (HMM) approximating the lattice artifact monopoles as fundamental pointlike particles described by the Higgs scalar field. Considering the renormalization group improvement of the effective Coleman–Weinberg potential [58], [59], written in Ref. [56] for the dual sector of scalar electrodynamics in the two–loop approximation, we have calculated the U(1) critical values of the magnetic fine structure constant $\tilde{\alpha}_{\text{crit}} = g_{\text{crit}}^2/4\pi \approx 1.20$ and electric fine structure constant $\alpha_{\text{crit}} = \pi/g_{\text{crit}}^2 \approx 0.208$ (by the Dirac relation). These values coincide with the lattice result [18]. The next subsections follow the review [57] of the HMM calculations of the U(1) critical couplings.
4.1. The Coleman-Weinberg effective potential in HMM

As it was mentioned above, the dual Abelian Higgs model of scalar monopoles (shortly HMM) describes the dynamics of confinement in lattice theories. This model, first suggested in Refs. [49], considers the following Lagrangian:

\[
L = -\frac{1}{4g^2} F^2_{\mu\nu} (B) + \frac{1}{2} |(\partial_\mu - i B_\mu) \Phi|^2 - U(\Phi), \quad \text{where} \quad U(\Phi) = \frac{1}{2} \mu^2 |\Phi|^2 + \frac{\lambda}{4} |\Phi|^4
\]  

(56)

is the Higgs potential of scalar monopoles with magnetic charge \( g \), and \( B_\mu \) is the dual gauge (photon) field interacting with the scalar monopole field \( \Phi \). In this model \( \lambda \) is the self-interaction constant of scalar fields, and the mass parameter \( \mu^2 \) is negative. In Eq.(56) the complex scalar field \( \Phi \) contains the Higgs (\( \phi \)) and Goldstone (\( \chi \)) boson fields:

\[
\Phi = \phi + i\chi.
\]  

(57)

The effective potential in the Higgs Scalar ElectroDynamics (HSED) was first calculated by Coleman and Weinberg [58] in the one–loop approximation. The general method of its calculation is given in the review [59]. Using this method, we can construct the effective potential for HMM. In this case the total field system of the gauge (\( B_\mu \)) and magnetically charged (\( \Phi \)) fields is described by the partition function which has the following form in Euclidean space:

\[
Z = \int [DB][D\Phi][D\Phi^+] e^{-S},
\]  

(58)

where the action \( S = \int d^4x L(x) + S_{gf} \) contains the Lagrangian (56) written in the Euclidean space and gauge fixing action \( S_{gf} \).

Let us consider now a shift:

\[
\Phi(x) = \Phi_b + \hat{\Phi}(x),
\]  

(59)

with \( \Phi_b \) as a background field, and calculate the following expression for the partition function in the one-loop approximation:

\[
Z = \int [DB][D\Phi][D\Phi^+] \exp\{-S(B, \Phi_b) - \int d^4x [\frac{\delta S(\Phi)}{\delta \Phi(x)}]_{\Phi=\Phi_b} \hat{\Phi}(x) + h.c.]\}
\[
= \exp\{-F(\Phi_b, g^2, \mu^2, \lambda)\}.
\]  

(60)

Using the representation (57), we obtain the effective potential:

\[
V_{eff} = F(\phi_b, g^2, \mu^2, \lambda)
\]  

(61)

given by the function \( F \) of Eq.(60) for the real constant background field \( \Phi_b = \phi_b = \text{const.} \). In this case the one–loop effective potential for monopoles coincides with the expression of the effective potential calculated by the authors of Ref. [58] for scalar electrodynamics and extended to the massive theory (see review [59]):

\[
V_{eff}(\phi_b^2) = \frac{\mu^2}{2} \phi_b^2 + \frac{\lambda}{4} \phi_b^4 + \frac{1}{64\pi^2} [3g^4 \phi_b^4 \log \frac{\phi_b^2}{M^2} + ...
\]
\[(\mu^2 + 3\lambda b^2)^2 \log \frac{\mu^2 + 3\lambda b^2}{M^2} + (\mu^2 + \lambda b^2)^2 \log \frac{\mu^2 + \lambda b^2}{M^2}\] + C, \quad (62)

where \(M\) is the cut–off scale and \(C\) is a constant not depending on \(b^2\).

The effective potential (61) has several minima. Their position depends on \(g^2, \mu^2\) and \(\lambda\). If the first local minimum occurs at \(b^2 = 0\) and \(V_{\text{eff}}(0) = 0\), it corresponds to the so-called "symmetrical phase", which is the Coulomb-like phase in our description. Then it is easy to determine the constant \(C\) in Eq.(62):

\[C = -\frac{\mu^4}{16\pi^2} \log \frac{\mu}{M}, \quad (63)\]

and we have the effective potential for HMM described by the following expression:

\[V_{\text{eff}}(b^2) = \frac{\mu^2}{2} b^2 + \frac{\lambda}{4} b^4 + \frac{\mu^4}{64\pi^2} \log \left(\frac{\mu^2 + 3\lambda b^2}{M^2} + \frac{\mu^2 + \lambda b^2}{M^2}\right). \quad (64)\]

Here \(\lambda_{\text{run}}\) is the running self–interaction constant given by the expression standing in front of \(b^4\) in Eq.(62):

\[\lambda_{\text{run}}(b^2) = \lambda + \frac{1}{16\pi^2} [3g^4 \log \frac{b^2}{M^2} + 9\lambda^2 \log \frac{\mu^2 + 3\lambda b^2}{M^2} + \lambda^2 \log \frac{\mu^2 + \lambda b^2}{M^2}]. \quad (65)\]

The running squared mass of the Higgs scalar monopoles also follows from Eq.(62):

\[\mu^2_{\text{run}}(b^2) = \mu^2 + \frac{\lambda \mu^2}{16\pi^2} [3 \log \frac{\mu^2 + 3\lambda b^2}{M^2} + \log \frac{\mu^2 + \lambda b^2}{M^2}]. \quad (66)\]

As it was shown in Ref. [58], the effective potential can be improved by consideration of the renormalization group equation (RGE).

### 4.2. Renormalization group equations in HMM

The RGE for the effective potential means that the potential cannot depend on a change in the arbitrary parameter – renormalization scale \(M\):

\[\frac{dV_{\text{eff}}}{dM} = 0. \quad (67)\]

The effects of changing it are absorbed into changes in the coupling constants, masses and fields, giving so–called running quantities.

Considering the RG improvement of the effective potential [58], [59] and choosing the evolution variable as

\[t = \log(\phi^2/M^2), \quad (68)\]

we have the following RGE for the improved \(V_{\text{eff}}(\phi^2)\) with \(\phi^2 \equiv b^2\) [60]:

\[(M^2 \frac{\partial}{\partial M^2} + \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_g \frac{\partial}{\partial g^2} + \beta_{\mu^2} \mu^2 \frac{\partial}{\partial \mu^2} - \gamma \phi^2 \frac{\partial}{\partial \phi^2})V_{\text{eff}}(\phi^2) = 0, \quad (69)\]
where $\gamma$ is the anomalous dimension and $\beta_{(\mu^2)}$, $\beta_\lambda$ and $\beta_g$ are the RG $\beta$–functions for mass, scalar and gauge couplings, respectively. RGE (69) leads to the following form of the improved effective potential \([58]\):

$$V_{\text{eff}} = \frac{1}{2}\mu_{\text{run}}^2(t)G^2(t)\phi^2 + \frac{1}{4}\lambda_{\text{run}}(t)G^4(t)\phi^4.$$  

(70)

In our case:

$$G(t) = \exp\left[-\frac{1}{2}\int_0^t dt' \gamma(g_{\text{run}}(t'), \lambda_{\text{run}}(t'))\right].$$  

(71)

A set of ordinary differential equations (RGE) corresponds to Eq.(69):

$$\frac{d\lambda_{\text{run}}}{dt} = \beta_\lambda(g_{\text{run}}(t), \lambda_{\text{run}}(t)),$$

(72)

$$\frac{d\mu_{\text{run}}^2}{dt} = \mu_{\text{run}}^2(t)\beta_{(\mu^2)}(g_{\text{run}}(t), \lambda_{\text{run}}(t)),$$

(73)

$$\frac{dg_{\text{run}}^2}{dt} = \beta_g(g_{\text{run}}(t), \lambda_{\text{run}}(t)).$$

(74)

So far as the mathematical structure of HMM is equivalent to HSED, we can use all results of the scalar electrodynamics in our calculations, replacing the electric charge $e$ and photon field $A_\mu$ by magnetic charge $g$ and dual gauge field $B_\mu$.

Let us write now the one–loop potential (64) as

$$V_{\text{eff}} = V_0 + V_1,$$

(75)

where

$$V_0 = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4, \quad V_1 = \frac{1}{64\pi^2}[3g^4\phi^4 \log \frac{\phi^2}{M^2} + (\mu^2 + 3\lambda\phi^2)^2 \log \frac{\mu^2 + 3\lambda\phi^2}{M^2}$$

$$+ (\mu^2 + \lambda\phi^2)^2 \log \frac{\mu^2 + \lambda\phi^2}{M^2} - 2\mu^4 \log \frac{\mu^2}{M^2}]$$

(76)

We can plug this $V_{\text{eff}}$ into RGE (69) and obtain the following equation (see \([59]\)):

$$(\beta_\lambda \frac{\partial}{\partial \lambda} + \beta_{(\mu^2)} \frac{\partial}{\partial \mu^2} - \gamma \phi^2 \frac{\partial}{\partial \phi^2})V_0 = -M^2 \frac{\partial V_1}{\partial M^2}.$$  

(77)

Equating $\phi^2$ and $\phi^4$ coefficients, we obtain the expressions of $\beta_\lambda$ and $\beta_{(\mu^2)}$ in the one–loop approximation:

$$\beta_\lambda^{(1)} = 2\gamma \lambda_{\text{run}} + \frac{5\lambda_{\text{run}}^2}{8\pi^2} + \frac{3g_{\text{run}}^4}{16\pi^2},$$

(78)

$$\beta_{(\mu^2)}^{(1)} = \gamma + \frac{\lambda_{\text{run}}}{4\pi^2}.$$  

(79)
The one–loop result for $\gamma$ is given in Ref. [58] for scalar field with electric charge $e$, but it is easy to rewrite this $\gamma$–expression for monopoles with charge $g = g_{\text{run}}$:

$$\gamma^{(1)} = -\frac{3g^2_{\text{run}}}{16\pi^2}. \quad (80)$$

Finally we have:

$$\frac{d\lambda_{\text{run}}}{dt} \approx \beta^{(1)}_{\lambda} = \frac{1}{16\pi^2} (3g^4_{\text{run}} + 10\lambda^2_{\text{run}} - 6\lambda_{\text{run}}g^2_{\text{run}}), \quad (81)$$

$$\frac{d\mu^2_{\text{run}}}{dt} \approx \beta^{(1)}_{(\mu^2)} = \frac{\mu^2_{\text{run}}}{16\pi^2} (4\lambda_{\text{run}} - 3g^2_{\text{run}}), \quad (82)$$

The expression of $\beta_g$–function in the one–loop approximation also is given by the results of Ref. [58]:

$$\frac{dg^2_{\text{run}}}{dt} \approx \beta^{(1)}_g = \frac{g^4_{\text{run}}}{48\pi^2}. \quad (83)$$

The RG $\beta$–functions for different renormalizable gauge theories with semisimple group have been calculated in the two–loop approximation [61]–[66] and even beyond [67]. But in this paper we made use the results of Refs. [61] and [64] for calculation of $\beta$–functions and anomalous dimension in the two–loop approximation, applied to the HMM with scalar monopole fields. The higher approximations essentially depend on the renormalization scheme [67]. Thus, on the level of two–loop approximation we have for all $\beta$–functions:

$$\beta = \beta^{(1)} + \beta^{(2)}, \quad (84)$$

where

$$\beta^{(2)}_{\lambda} = \frac{1}{(16\pi^2)^2} (-25\lambda^3 + \frac{15}{2}g^2\lambda^2 - \frac{229}{12}g^4\lambda - \frac{59}{6}g^6), \quad (85)$$

and

$$\beta^{(2)}_{(\mu^2)} = \frac{1}{(16\pi^2)^2} (\frac{31}{12}g^4 + 3\lambda^2). \quad (86)$$

The gauge coupling $\beta^{(2)}_g$–function is given by Ref. [61]:

$$\beta^{(2)}_g = \frac{g^6}{(16\pi^2)^2}. \quad (87)$$

Anomalous dimension follows from calculations made in Ref. [64]:

$$\gamma^{(2)} = \frac{1}{(16\pi^2)^2} \frac{31}{12}g^4. \quad (88)$$

In Eqs. (84)–(88) and below, for simplicity, we have used the following notations: $\lambda \equiv \lambda_{\text{run}}$, $g \equiv g_{\text{run}}$ and $\mu \equiv \mu_{\text{run}}$. 

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4.3. The phase diagram in HMM

Now we want to apply the effective potential calculation as a technique for the getting phase diagram information for the condensation of monopoles in HMM. As it was mentioned in the subsection 4.1, the effective potential \( V_{\text{eff}} \) can have several minima. Their positions depend on \( g^2, \mu^2 \) and \( \lambda \). If the first local minimum occurs at \( \phi = 0 \) and \( V_{\text{eff}}(0) = 0 \), it corresponds to the Coulomb–like phase. In the case when the effective potential has the second local minimum at \( \phi = \phi_{\text{min}} \neq 0 \) with \( V_{\text{eff}}^{\text{min}}(\phi_{\text{min}}^2) < 0 \), we have the confinement phase. The phase transition between the Coulomb–like and confinement phases is given by the condition when the first local minimum at \( \phi = 0 \) is degenerate with the second minimum at \( \phi = \phi_0 \). These degenerate minima are shown in Fig. 6 by the curve 1. They correspond to the different vacua arising in this model. And the dashed curve 2 describes the appearance of two minima corresponding to the confinement phases (see details in the next Section).

The conditions of the existence of degenerate vacua are given by the following equations:

\[
V_{\text{eff}}(0) = V_{\text{eff}}(\phi_0^2) = 0, \quad (89)
\]

\[
\frac{\partial V_{\text{eff}}}{\partial \phi} \bigg|_{\phi=0} = \frac{\partial V_{\text{eff}}}{\partial \phi} \bigg|_{\phi=\phi_0} = 0, \quad \text{or} \quad V'_{\text{eff}}(\phi_0^2) \equiv \frac{\partial V_{\text{eff}}}{\partial \phi^2} \bigg|_{\phi=\phi_0} = 0, \quad (90)
\]

and inequalities

\[
\frac{\partial^2 V_{\text{eff}}}{\partial \phi^2} \bigg|_{\phi=0} > 0, \quad \frac{\partial^2 V_{\text{eff}}}{\partial \phi^2} \bigg|_{\phi=\phi_0} > 0. \quad (91)
\]

The first equation (89) applied to Eq.(70) gives:

\[
\mu_{\text{run}}^2 = -\frac{1}{2} \lambda_{\text{run}}(t_0) \phi_0^2 G^2(t_0), \quad \text{where} \quad t_0 = \log(\phi_0^2/M^2). \quad (92)
\]

Calculating the first derivative of \( V_{\text{eff}} \) given by Eq.(90), we obtain the following expression:

\[
V'_{\text{eff}}(\phi^2) = \frac{V_{\text{eff}}(\phi^2)}{\phi^2} (1 + 2 \frac{d \log G}{dt}) + \frac{1}{2} \frac{d \mu_{\text{run}}^2}{dt} G^2(t)
\]

\[
+ \frac{1}{4} \left( \lambda_{\text{run}}(t) + \frac{d \lambda_{\text{run}}}{dt} + 2 \lambda_{\text{run}} \frac{d \log G}{dt} \right) G^4(t) \phi^2. \quad (93)
\]

From Eq.(71), we have:

\[
\frac{d \log G}{dt} = -\frac{1}{2} \gamma. \quad (94)
\]

It is easy to find the joint solution of equations

\[
V_{\text{eff}}(\phi_0^2) = V'_{\text{eff}}(\phi_0^2) = 0. \quad (95)
\]

Using RGE (72), (73) and Eqs. (72)–(94), we obtain:

\[
V'_{\text{eff}}(\phi_0^2) = \frac{1}{4}(-\lambda_{\text{run}} \beta(\mu^2) + \lambda_{\text{run}} + \beta \lambda - \gamma \lambda_{\text{run}}) G^4(t_0) \phi_0^2 = 0, \quad (96)
\]
\[ \beta_\lambda + \lambda_{\text{run}} (1 - \gamma - \beta_{(\mu^2)}) = 0. \]  

(97)

Substituting in Eq. (97) the functions \( \beta^{(1)}_\lambda \), \( \beta^{(1)}_{(\mu^2)} \) and \( \gamma^{(1)} \) given by Eqs. (78)–(80), we obtain in the one–loop approximation the following equation for the phase transition border:

\[ g^4_{PT} = -2\lambda_{\text{run}} \left( \frac{8\pi^2}{3} + \lambda_{\text{run}} \right). \]  

(98)

The curve (98) is represented on the phase diagram \( (\lambda_{\text{run}}; g^2_{\text{run}}) \) of Fig. 7 by the curve ”1” which describes the border between the Coulomb–like phase with \( V_{\text{eff}} \geq 0 \) and the confinement one with \( V_{\text{eff}} \leq 0 \). This border corresponds to the one–loop approximation.

Using Eqs. (80)-(88), we are able to construct the phase transition border in the two–loop approximation. Substituting these equations into Eq. (97), we obtain the following phase transition border curve equation in the two–loop approximation:

\[ 3y^2 - 16\pi^2 + 6x^2 + \frac{1}{16\pi^2} (28x^3 + \frac{15}{2}x^2y + \frac{97}{4}xy^2 - \frac{59}{6}y^3) = 0, \]  

(99)

where \( x = -\lambda_{PT} \) and \( y = g^2_{PT} \) are the phase transition values of \( -\lambda_{\text{run}} \) and \( g^2_{\text{run}} \). Choosing the physical branch corresponding to \( g^2 \geq 0 \) and \( g^2 \to 0 \), when \( \lambda \to 0 \), we have received curve 2 on the phase diagram \( (\lambda_{\text{run}}; g^2_{\text{run}}) \) shown in Fig. 7. This curve corresponds to the two–loop approximation and can be compared with the curve 1 of Fig. 7, which describes the same phase transition border calculated in the one–loop approximation. It is easy to see that the accuracy of the one–loop approximation is not excellent and can commit errors of order 30%.

According to the phase diagram drawn in Fig. 7, the confinement phase begins at \( g^2 = g^2_{\text{max}} \) and exists under the phase transition border line in the region \( g^2 \leq g^2_{\text{max}} \), where \( e^2 \) is large: \( e^2 \geq (2\pi/g_{\text{max}})^2 \), due to the Dirac relation (see below). Therefore, we have:

\[ g^2_{\text{crit}} = g^2_{\text{max}} \approx 18.61 \quad \text{— in the one–loop approximation,} \]

\[ g^2_{\text{crit}} = g^2_{\text{max}} \approx 15.11 \quad \text{— in the two–loop approximation.} \]  

(100)

Comparing these results, we obtain the accuracy of deviation between them of order 20%.

The results (100) give:

\[ \tilde{\alpha}_{\text{crit}} = \frac{g^2_{\text{crit}}}{4\pi} \approx 1.48, \quad \text{— in the one–loop approximation,} \]  

(101)

\[ \tilde{\alpha}_{\text{crit}} = \frac{g^2_{\text{crit}}}{4\pi} \approx 1.20, \quad \text{— in the two–loop approximation.} \]  

(102)

Using the Dirac relation for elementary charges:

\[ eg = 2\pi, \quad \text{or} \quad \alpha \tilde{\alpha} = \frac{1}{4}, \]  

(103)

we get the following values for the critical electric fine structure constant:

\[ \alpha_{\text{crit}} = \frac{1}{4\tilde{\alpha}_{\text{crit}}} \approx 0.17 \quad \text{— in the one–loop approximation,} \]  

(104)
\[ \alpha_{\text{crit}} = \frac{1}{4\alpha_{\text{crit}}} \approx 0.208 \quad \text{in the two–loop approximation.} \quad (105) \]

The last result coincides with the lattice values \([48]\) obtained for the compact QED by Monte Carlo method \([36]\).

Writing Eq.(74) with \(\beta_g\) function given by Eqs.(83), (84), and (87), we have the following RGE for the monopole charge in the two–loop approximation:

\[ \frac{dg_{\text{run}}^2}{dt} \approx \frac{g_{\text{run}}^4}{48\pi^2} + \frac{g_{\text{run}}^6}{(16\pi^2)^2}, \quad (106) \]

or

\[ \frac{d \log \tilde{\alpha}}{dt} \approx \frac{\tilde{\alpha}}{12\pi} (1 + 3 \frac{\tilde{\alpha}}{4\pi}). \quad (107) \]

The values \([100]\) for \(g_{\text{crit}}^2 = g_{\text{max}1,2}^2\) indicate that the contribution of two loops described by the second term of Eq.(106), or Eq.(107), is about 0.3, confirming the validity of perturbation theory.

In general, we are able to estimate the validity of two–loop approximation for all \(\beta\)–functions and \(\gamma\), calculating the corresponding ratios of two–loop contributions to one–loop contributions at the maxima of curves 1 and 2:

\[
\begin{align*}
\lambda_{\text{crit}} &= \lambda_{\text{run}}^{\text{max}1} \approx -13.16 \\
g_{\text{crit}}^2 &= g_{\text{max}1}^2 \approx 18.61 \\
\gamma^{(2)} \approx -0.0080 \\
\beta_{\mu^2}^{(2)} \approx -0.0826 \\
\beta_{\lambda}^{(2)} \approx 0.1564 \\
\beta_g^{(2)} \approx 0.3536
\end{align*}
\]

Here we see that all ratios are sufficiently small, i.e. all two–loop contributions are small in comparison with one–loop contributions, confirming the validity of perturbation theory in the 2–loop approximation, considered in this model. The accuracy of deviation is worse (~30%) for \(\beta_g\)–function. But it is necessary to emphasize that calculating the border curves 1 and 2 of Fig.7, we have not used RGE \([87]\) for monopole charge: \(\beta_g\)–function is absent in Eq.(97).

Therefore, the calculation of \(g_{\text{crit}}^2\) according to Eq.(99) does not depend on the approximation of \(\beta_g\) function. The above–mentioned \(\beta_g\)–function appears only in the second order derivative of \(V_{\text{eff}}\) which is related with the monopole mass \(m\) (see the next Section).

Eqs.(18) and (103) give the result \([54]\):

\[ \alpha_{\text{crit}}^{-1} \approx 5, \quad (109) \]

which is important for the phase transition at the Planck scale predicted by the Multiple Point Model (MPM) (see below).
5. Approximate Universality of the Critical Coupling Constants

The review of all existing results for $\alpha_{\text{crit}}$ and $\tilde{\alpha}_{\text{crit}}$ gives:

1) $\alpha_{\text{lat}}^{\text{crit}} \approx 0.20 \pm 0.015$,
   $\tilde{\alpha}_{\text{lat}}^{\text{crit}} \approx 1.25 \pm 0.10$ (110)

   – in the Compact QED with the Wilson lattice action [36];

2) $\alpha_{\text{lat}}^{\text{crit}} \approx 0.204$,
   $\tilde{\alpha}_{\text{lat}}^{\text{crit}} \approx 1.25$ (111)

   – in the model with the Wilson loop action [53];

3) $\alpha_{\text{crit}} \approx 0.1836$,
   $\tilde{\alpha}_{\text{crit}} \approx 1.36$ (112)

   – in the Compact QED with the Villain lattice action [68];

4) $\alpha_{\text{crit}} = \alpha(A) \approx 0.208$,
   $\tilde{\alpha}_{\text{crit}} = \tilde{\alpha}(A) \approx 1.20$ (113)

   – in the HMM ([56], [57] and the present paper).

It is necessary to emphasize that the functions $\alpha(\beta)$ in Fig.5(a), describing the behavior of the effective electric fine structure constant in the vicinity of the phase transition point, are different for the Wilson and Villain lattice actions in the U(1) lattice gauge theory, but the critical values of $\alpha$ coincide for both theories [36].

Hereby we see an additional arguments for our previously hoped (in Refs. [51] and [53]) "approximate universality" of the first order phase transition couplings: the fine structure constant is at the critical point approximately the same one independent of various parameters of the different (lattice, etc.) regularization.

The most significant conclusion for MPM, predicting the values of gauge couplings being so as to arrange just the multiple critical point where all phases existing in the theory meet, is possibly that our calculations suggest the validity of an approximate universality of the critical couplings, in spite of the fact that we are concerned with the first order phase transitions. We have shown that one can crudely calculate the phase transition couplings without using any specific lattice, rather only approximating the lattice artifact monopoles as fundamental (pointlike) particles condensing. The details of the lattice – hypercubic or random, with multiplaquette terms or without them, etc., – also the details of the regularization – lattice or Wilson loops, lattice or HMM – do not matter for the value of the phase transition coupling so much. Critical couplings depend only on groups with any regularization. Such an approximate universality is, of course, absolutely needed if there is any sense in relating lattice phase transition couplings to the experimental couplings found in Nature. Otherwise, such a comparison would only make sense if we could guess the true lattice in the right model, what sounds too ambitious.
6. Triple Point

In this section we demonstrate the existence of the triple point on the phase diagram of HMM.

Considering the second derivative of the effective potential:

$$V''_{\text{eff}}(\phi^2) \equiv \frac{\partial^2 V_{\text{eff}}}{\partial (\phi^2)^2},$$  \hspace{1cm} (114)

we can calculate it for the RG improved effective potential (70):

$$V''_{\text{eff}}(\phi^2) = \frac{V'_{\text{eff}}(\phi^2)}{\phi^2} + \left( -\frac{1}{2} \mu_{\text{run}}^2 + \frac{1}{2} d^2 \mu_{\text{run}}^2 \frac{d \lambda_{\text{run}}}{dt} + 2 \frac{d \mu_{\text{run}}^2}{dt} \frac{d \log G}{dt} \right) + \mu_{\text{run}}^2 \frac{d^2 \log G}{dt^2} + 2 \mu_{\text{run}}^2 \left( \frac{d \log G}{dt} \right)^2 G^2 \phi^2 + \left( \frac{1}{2} d \lambda_{\text{run}} + \frac{1}{4} d^2 \lambda_{\text{run}} + 2 \frac{d \lambda_{\text{run}}}{dt} \frac{d \log G}{dt} \right) G^4(t).$$ \hspace{1cm} (115)

Let us consider now the case when this second derivative changes its sign giving a maximum of $V_{\text{eff}}$ instead of the minimum at $\phi^2 = \phi_0^2$. Such a possibility is shown in Fig.6 by the dashed curve ”2”. Now the two additional minima at $\phi^2 = \phi_1^2$ and $\phi^2 = \phi_2^2$ appear in our theory. They correspond to the two different confinement phases for the confinement of electrically charged particles if they exist in the system. When these two minima are degenerate, we have the following requirements:

$$V_{\text{eff}}(\phi_1^2) = V_{\text{eff}}(\phi_2^2) < 0 \quad \text{and} \quad V'_{\text{eff}}(\phi_1^2) = V'_{\text{eff}}(\phi_2^2) = 0,$$ \hspace{1cm} (116)

which describe the border between the confinement phases ”conf.1” and ”conf.2” presented in Fig.8. This border is given as a curve ”3” at the phase diagram ($\lambda_{\text{run}}; g_{\text{run}}^4$) drawn in Fig.8. The curve ”3” meets the curve ”1” at the triple point A. According to the illustration shown in Fig.6, it is obvious that this triple point A is given by the following requirements:

$$V_{\text{eff}}(\phi_0^2) = V'_{\text{eff}}(\phi_0^2) = V''_{\text{eff}}(\phi_0^2) = 0.$$ \hspace{1cm} (117)

In contrast to the requirements:

$$V_{\text{eff}}(\phi_0^2) = V'_{\text{eff}}(\phi_0^2) = 0,$$ \hspace{1cm} (118)

giving the curve ”1”, let us consider now the joint solution of the following equations:

$$V_{\text{eff}}(\phi_0^2) = V''_{\text{eff}}(\phi_0^2) = 0.$$ \hspace{1cm} (119)

For simplicity, we have considered the one–loop approximation. Using Eqs.(115), (92) and (81)-(83), it is easy to obtain the solution of Eq.(119) in the one–loop approximation:

$$\mathcal{F}(\lambda_{\text{run}}, g_{\text{run}}^2) = 0,$$ \hspace{1cm} (120)

where

$$\mathcal{F}(\lambda_{\text{run}}, g_{\text{run}}^2) = 5g_{\text{run}}^6 + 24\pi^2 g_{\text{run}}^4 + 12\lambda_{\text{run}} g_{\text{run}}^4 - 9\lambda_{\text{run}}^2 g_{\text{run}}^2.$$
The dashed curve "2" of Fig.8 represents the solution of Eq. (120) which is equivalent to Eqs. (119). The curve "2" is going very close to the maximum of the curve "1". Assuming that the position of the triple point A coincides with this maximum, let us consider the border between the phase "conf.1", having the first minimum at nonzero $\phi_1$ with $V_{\text{eff}}''(\phi_1) = c_1 < 0$, and the phase "conf.2", which reveals two minima with the second minimum being the deeper one and having $V_{\text{eff}}''(\phi_2^2) = c_2 < 0$. This border (described by the curve "3" of Fig.8) was calculated in the vicinity of the triple point A by means of Eqs. (116) with $\phi_1$ and $\phi_2$ represented as $\phi_{1,2} = \phi_0 \pm \epsilon$ with $\epsilon << \phi_0$. The result of such calculations gives the following expression for the curve "3":

$$g_{PT}^4 = \frac{5}{2}(5\lambda_{\text{run}} + 8\pi^2)\lambda_{\text{run}} + 8\pi^4.$$ (122)

The curve "3" meets the curve "1" at the triple point A.

The piece of the curve "1" to the left of the point A describes the border between the Coulomb–like phase and the phase "conf.1". In the vicinity of the triple point A the second derivative $V_{\text{eff}}''(\phi_0^2)$ changes its sign leading to the existence of the maximum at $\phi^2 = \phi_0^2$, in correspondence with the dashed curve "2" of Fig.6. By this reason, the curve "1" of Fig.8 does not already describe a phase transition border up to the next point B when the curve "2" again intersects the curve "1" at $\lambda(B) \approx -12.24$. This intersection (again giving $V_{\text{eff}}''(\phi_0^2) > 0$) occurs surprisingly quickly.

The right piece of the curve "1" to the right of the point B separates the Coulomb–like phase and the phase "conf.2". But between the points A and B the phase transition border is going slightly above the curve "1". This deviation is very small and cannot be distinguished on Fig.8.

It is necessary to note that only $V_{\text{eff}}''(\phi^2)$ contains the derivative $dg_{\text{run}}^2/dt$. The joint solution of equations (117) leads to the joint solution of Eqs. (98) and (120). This solution was obtained numerically and gave the following triple point values of $\lambda_{\text{run}}$ and $g_{\text{run}}^2$:

$$\lambda(A) \approx -13.4073, \quad g_{(A)}^2 \approx 18.6070.$$ (123)

The solution (123) demonstrates that the triple point A exists in the very neighborhood of the maximum of the curve (128). The position of this maximum is given by the following analytical expressions, together with their approximate values:

$$\lambda(A) \approx -\frac{4\pi^2}{3} \approx -13.2, \quad g^2(A) = g_{\text{crit}}^2|_{\lambda_{\text{run}} = \lambda(A)} \approx \frac{4\sqrt{2}}{3} \pi^2 \approx 18.6.$$ (124)

Finally, we can conclude that the phase diagram shown in Fig.8 gives such a description: there exist three phases in the dual sector of the Higgs scalar electrodynamics – the Coulomb–like phase and confinement phases "conf.1" and "conf.2".

The border "1", which is described by the curve (98), separates the Coulomb–like phase (with $V_{\text{eff}} \geq 0$) and confinement phases (with $V_{\text{eff}}^\text{min}(\phi_0^2) < 0$). The curve "1" corresponds to the joint solution of the equations $V_{\text{eff}}(\phi_0^2) = V_{\text{eff}}'(\phi_0^2) = 0$. 

\[+36\lambda_{\text{run}}^3 + 80\pi^2\lambda_{\text{run}}^2 + 64\pi^4\lambda_{\text{run}}.\] (121)
The dashed curve "2" represents the solution of the equations $V_{\text{eff}}(\phi_0^2) = V''_{\text{eff}}(\phi_0^2) = 0$.

The phase border "3" of Fig.8 separates two confinement phases. The following requirements take place for this border:

$V_{\text{eff}}(\phi_{1,2}^2) < 0, \quad V_{\text{eff}}(\phi_1^2) = V_{\text{eff}}(\phi_2^2), \quad V'_{\text{eff}}(\phi_1^2) = V'_{\text{eff}}(\phi_2^2) = 0,$

$V''_{\text{eff}}(\phi_1^2) > 0, \quad V''_{\text{eff}}(\phi_2^2) > 0.$

(126)

The triple point A is a boundary point of all three phase transitions shown in the phase diagram of Fig.8. For $g^2 < g_{(A)}^2$ the field system, described by our model, exists in the confinement phase, where all electric charges have to be confined.

Taking into account that monopole mass $m$ is given by the following expression:

$$\frac{d^2V_{\text{eff}}}{d\phi^2}\bigg|_{\phi=\phi_0} = m^2,$$

we see that monopoles acquire zero mass in the vicinity of the triple point A:

$$V''_{\text{eff}}(\phi_{0A}^2) = \frac{1}{4\phi_{0A}^2} \frac{d^2V_{\text{eff}}}{d\phi^2}\bigg|_{\phi=\phi_{0A}} = \frac{m_{(A)}^2}{4\phi_{0A}^2} = 0.$$

(128)

This result is in agreement with the result of compact QED [68]: $m^2 \to 0$ in the vicinity of the critical point.

7. "ANO–Strings", or the Vortex Description of the Confinement Phases

As it was shown in the previous subsection, two regions between the curves "1", "3" and "3", "1", given by the phase diagram of Fig.8, correspond to the existence of the two confinement phases, different in the sense that the phase "conf.1" is produced by the second minimum, but the phase "conf.2" corresponds to the third minimum of the effective potential. It is obvious that in our case both phases have nonzero monopole condensate in the minima of the effective potential, when $V_{\text{eff}}^{\text{min}}(\phi_{1,2} \neq 0) < 0$. By this reason, the Abrikosov–Nielsen–Olesen (ANO) electric vortices (see Refs. [69], [70]) may exist in these both phases, which are equivalent in the sense of the "string" formation. If electric charges are present in a model (they are absent in HMM), then these charges are placed at the ends of the vortices—"strings" and therefore are confined. But only closed "strings" exist in the confinement phases of HMM. The properties of the ANO—"strings" in the U(1) gauge theory were investigated in Ref. [55].

As in Refs. [69], [70], in the London’s limit ($\lambda \to \infty$) the dual Abelian Higgs model, described by the Lagrangian (56), gives the formation of monopole condensate with amplitude $\phi_0$, which repels and suppresses the electromagnetic field $F_{\mu\nu}$ almost everywhere, except the region around the vortex lines. In this limit, we have the following London equation:

$$\text{rot } j^{(m)} = \delta^{-2} \vec{E},$$

(129)

where $j^{(m)}$ is the microscopic current of monopoles, $\vec{E}$ is the electric field strength and $\delta$ is the penetration depth. It is clear that $\delta^{-1}$ is the photon mass $m_\gamma$, generated by the Higgs
mechanism. The closed equation for $\mathbf{E}$ follows from the Maxwell equations and Eq. (129) just in the London’s limit.

In our case $\delta$ is defined by the following relation:

$$\delta^{-2} \equiv m_V^2 = g^2 \phi_0^2.$$  

(130)

On the other hand, the field $\phi$ has its own correlation length $\xi$, connected to the mass of the field $\phi$ (”the Higgs mass”):

$$\xi = m_S^{-1}, \quad m_S^2 = \lambda \phi_0^2.$$  

(131)

The London’s limit for our ”dual superconductor of the second kind” corresponds to the following relations:

$$\delta >> \xi, \quad m_V << m_S, \quad g << \lambda,$$  

(132)

and ”the string tension” – the vortex energy per unit length (see Ref. 69) – for the minimal electric vortex flux $2\pi$, is:

$$\sigma = \frac{2\pi}{g^2 \delta^2} \ln \frac{\delta}{\xi} = 2\pi \phi_0^2 \ln \frac{m_S}{m_V}, \quad \text{where} \quad \delta/\xi = \frac{m_S}{m_V} >> 1.$$  

(133)

We see that in the London’s limit ANO–theory implies the mass generation of the photons, $m_V = 1/\delta$, which is much less than the Higgs mass $m_S = 1/\xi$.

Let us wonder now in the question whether our ”strings” are thin or not. The vortex may be considered as thin, if the distance between the electric charges sitting at its ends, i.e. the string length $L$, is much larger than the penetration length $\delta$:

$$L >> \delta >> \xi.$$  

(134)

It is obvious that only rotating ”strings” can exist as stable states. In the framework of classical calculations, it is not difficult to obtain the mass $M$ and angular momentum $J$ of the rotating ”string”:

$$J = \frac{1}{2\pi \sigma} M^2, \quad M = \frac{\pi}{2} \sigma L.$$  

(135)

The following relation follows from Eqs. (135):

$$L = 2\sqrt{\frac{2J}{\pi \sigma}},$$  

(136)

or

$$L = \frac{2g \delta}{\pi} \sqrt{\ln \frac{m_S}{m_V}}.$$  

(137)

For $J = 1$ we have:

$$\frac{L}{\delta} = \frac{2g}{\pi} \sqrt{\ln \frac{m_S}{m_V}}.$$  

(138)
what means that for $m_S > m_V$ the length of this "string" is small and does not obey the requirement $L/\delta >> 1$. It is easy to see from Eq.(137) that in the London’s limit the "strings" are very thin ($L/\delta >> 1$) only for the enormously large angular momenta $J >> 1$.

The phase diagram of Fig.8 shows the existence of the confinement phase for $\alpha \geq \alpha(A)$. This means that the formation of (closed) vortices begins at the triple point $\alpha = \alpha(A)$: for $\alpha > \alpha(A)$ we have a nonzero $\phi_0$ giving rise to existence of vortices.

In Section 3 we have shown that the lattice investigations lead to the freezing of the electric fine structure constant at the value $\alpha = \alpha_{max}$ and mentioned that the authors of Ref. [45] predicted: $\alpha_{max} = \frac{\pi}{12} \approx 0.26$.

Let us estimate now the region of values of the magnetic charge $g$ in the confinement phase considered in this paper:

$$g_{min} \leq g \leq g_{max},$$
$$g_{max} = g(A) \approx \sqrt{15.1} \approx 3.9,$$
$$g_{min} = \sqrt{\frac{\pi}{\alpha_{max}}} \approx 3.5.$$ (139)

Then for $m_S = 10m_V$, say, we have from Eq.(138) the following estimation of the "string" length for $J = 1$:

$$1.5 \leq \frac{L}{\delta} \leq 1.8.$$ (140)

We see that in the $U(1)$ gauge theory the low-lying states of "strings" correspond to the short and thick vortices.

In general, the way of receiving of the Nambu–Goto strings from the dual Abelian Higgs model of scalar monopoles was demonstrated in Ref. [71].

8. Phase Transition Couplings in the Regularized SU(N) Gauge Theories

It was shown in a number of investigations (see for example [38], [39] and references there), that the confinement in the SU(N) lattice gauge theories effectively comes to the same U(1) formalism. The reason is the Abelian dominance in their monopole vacuum: monopoles of the Yang–Mills theory are the solutions of the U(1)–subgroups, arbitrary embedded into the SU(N) group. After a partial gauge fixing (Abelian projection by ‘t Hooft [72]) SU(N) gauge theory is reduced to an Abelian $U(1)^{N-1}$ theory with $N - 1$ different types of Abelian monopoles. Choosing the Abelian gauge for dual gluons, it is possible to describe the confinement in the lattice SU(N) gauge theories by the analogous dual Abelian Higgs model of scalar monopoles.

8.1. The "abelization" of monopole vacuum in the non-Abelian theories

A lattice imitates the non–perturbative vacuum of zero temperature SU(2) and SU(3) gluodynamics as a condensate of monopoles which emerge as leading non–perturbative fluctuations of non–Abelian SU(N) gauge theories in the gauge of Abelian projections by G.’t Hooft [72].

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It is possible to find such a gauge, in which monopole degrees of freedom, hidden in the given field configuration, become explicit.

Let us consider the SU(N) gluodynamics. For any composite operator \( X \in \) the adjoint representation of SU(N) group (\( X \) may be \( (F_{\mu\nu})_{ij} \), where \( i, j = 1, 2, \ldots N \) we can find such a gauge:

\[
X \rightarrow X' = VXXV^{-1},
\]

where the unitary matrix \( V \) transforms \( X \) to diagonal \( X' \):

\[
X \rightarrow X' = VXXV^{-1} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N).
\]

We can choose the ordering of \( \lambda_i \):

\[
\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N.
\]

The matrix \( X' \) belongs to the Cartain, or Maximal Abelian subgroup of the SU(N) group:

\[
U(1)^{N-1} \in SU(N).
\]

Let us consider the field \( A_\mu \) in the diagonal gauge:

\[
\bar{A}_\mu = V(A_\mu + \frac{i}{g} \partial_\mu)V^{-1}.
\]

This field transforms according to the subgroup \( U(1)^{N-1} \): its diagonal elements

\[
(a_\mu)_i \equiv (\bar{A}_\mu)_{ii}
\]

transform as Abelian gauge fields (photons):

\[
(a_\mu)_i \rightarrow (a'_\mu)_i = (a_\mu)_i + \frac{1}{g} \partial_\mu \alpha_i,
\]

but its non–diagonal elements

\[
(c_\mu)_{ij} \equiv (\bar{A}_\mu)_{ij} \quad \text{with} \quad i \neq j
\]

transform as charged fields:

\[
(c'_\mu)_{ij} = \exp[i(\alpha_i - \alpha_j)](c_\mu)_{ij},
\]

where \( i, j = 1, \ldots, N \).

According to G.’t Hooft [72], if some \( \lambda_i \) coincide, then singularities having properties of monopoles appear in the ”Abelian” part of the non–Abelian gauge fields. Indeed, let us consider the strength tensor of the ”Abelian gluons”:

\[
(f_{\mu\nu})_i = \partial_\mu(a_\nu)_i - \partial_\nu(a_\mu)_i
\]

\[
= VF_{\mu\nu}V^{-1} + ig[V(A_\mu + \frac{i}{g} \partial_\mu)V^{-1}, V(A_\nu + \frac{i}{g} \partial_\nu)V^{-1}].
\]

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The monopole current is:

\[(K_\mu)_i = \frac{1}{8\pi} \varepsilon_{\mu\nu\rho\sigma} \partial_\nu (f_{\rho\sigma})_i,\]  

(149)

and it is conserved:

\[\partial_\mu (K_\mu)_i = 0.\]  

(150)

The initial \(F_{\mu\nu}\) had not singularities. Therefore, all singularities can come from the commutator, which is contained in Eq. (148).

The magnetic charge \(m_i(\Omega)\) in 3d–volume \(\Omega\) is:

\[m_i(\Omega) = \int_{\Omega} d^3\sigma (K_\mu)_i = \frac{1}{8\pi} \int_{\partial\Omega} d^2\sigma (f_{\mu\nu})_i.\]  

(151)

If \(\lambda_1 = \lambda_2\) (coincide) at the point \(x^{(1)}\) in 3d–volume \(\Omega\), then we have a singularity on the curve in 4d–space, which is a world–line of the magnetic monopole, and \(x = x^{(1)}\) is a singular point of gauge transformed fields \(\bar{A}_\mu\) and \((a_\mu)_i\).

As it was shown by ’t Hooft:

\[(f_{\mu\nu})_i \sim O(|x - x^{(1)}|^{-2})\]  

(152)

only in the vicinity of \(x^{(1)}\), where it behaves as a magnetic field of the point–like monopole.

Finally, we have the following conclusions:

1) The initial potentials \(A_\mu\) and strength tensor \(F_{\mu\nu}\) had not singularities.
2) At large distances \((f_{\mu\nu})_i\) doesn’t have a behaviour

\[(f_{\mu\nu})_i \sim O(|x - x^{(1)}|^{-2})\]

and we have monopoles only near \(x = x^{(1)}\).
3) Fields \(\bar{A}_\mu\) and \((a_\mu)_i\) are not classical solutions: they are a result of the quantum fluctuations.
4) Any distribution of fields in the vacuum can undergo the Abelian projection.

We have seen that in the SU(N) gauge theories quantum fluctuations (non–perturbative effects) reveal an Abelian vacuum monopoles and suppress the non–diagonal components of the strength tensor \((F_{\mu\nu})_{ij}\). As it is shown below, this phenomenon gives very important consequences for the Planck scale physics.

Using the idea of ”abelization” of monopole vacuum in the SU(N) lattice gauge theories, we have developed in Ref. [56] a method of theoretical estimation of the SU(N) critical couplings.

8.2. Monopoles strength group dependence

Lattice non–Abelian gauge theories also have lattice artifact monopoles. We suppose that only those lattice artifact monopoles are important for the phase transition calculations which have the smallest monopole charges. Let us consider the lattice gauge theory with the gauge group \(SU(N)/Z_N\) as our main example. That is to say, we consider the adjoint representation action and do not distinguish link variables forming the same one multiplied by any element of the center of the group. The group \(SU(N)/Z_N\) is not simply connected and has the first homotopic group \(\Pi_1(SU(N)/Z_N)\) equal to \(Z_N\). The lattice artifact monopole with the smallest magnetic
charge may be described as a three–cube (or rather a chain of three–cubes describing the time track) from which radiates magnetic field corresponding to the $U(1)$ subgroup of gauge group $SU(N)/\mathbb{Z}_N$ with the shortest length insight of this group, but still homotopically non-trivial. In fact, this $U(1)$ subgroup is obtained by the exponentiating generator:

$$\frac{\lambda_8}{2} = \frac{1}{\sqrt{2N(N-1)}} \begin{pmatrix} N - 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}.$$  \hspace{1cm} (153)

This specific form is one gauge choice; any similarity transformation of this generator would describe physically the same monopole. If one has somehow already chosen the gauge monopoles with different but similarity transformation related generators, they would be physically different. Thus, after gauge choice, there are monopoles corresponding to different directions of the Lie algebra generators in the form $U^{\frac{\lambda_8}{2}} U^+.$

Now, when we want to apply the effective potential calculation as a technique for the getting phase diagram information for the condensation of the lattice artifact monopoles in the non-abelian lattice gauge theory, we have to correct the abelian case calculation for the fact that after gauge choice we have a lot of different monopoles. If a couple of monopoles happens to have their generators just in the same directions in the Lie algebra, they will interact with each other as Abelian monopoles (in first approximation). In general, the interaction of two monopoles by exchange of a photon will be modified by the following factor:

$$\frac{Tr(U_1^{\frac{\lambda_8}{2}} U_1^+ U_2^{\frac{\lambda_8}{2}} U_2^+)}{Tr(\frac{\lambda_8}{2})^2}.$$ \hspace{1cm} (154)

We shall assume that we can correct these values of monopole orientations in the Lie algebra in a statistical way. That is to say, we want to determine an effective coupling constant $\tilde{g}_{eff}$ describing the monopole charge as if there is only one Lie algebra orientationwise type of monopole. It should be estimated statistically in terms of the monopole charge $\tilde{g}_{genuine}$ valid to describe the interaction between monopoles with generators oriented along the same $U(1)$ subgroup. A very crude intuitive estimate of the relation between these two monopole charge concepts $\tilde{g}_{genuine}$ and $\tilde{g}_{eff}$ consists in playing that the generators are randomly oriented in the whole $N^2 - 1$ dimensional Lie algebra. When even the sign of the Lie algebra generator associated with the monopole is random – as we assumed in this crude argument – the interaction between two monopoles with just one photon exchanged averages out to zero. Therefore, we can get a non-zero result only in the case of exchange by two photons or more. That is, however, good enough for our effective potential calculation since only $\tilde{g}^4$ (but not the second power) occurs in the Coleman – Weinberg effective potential in the one–loop approximation (see \cite{58, 59}). Taking into account this fact that we can average imagining monopoles with generators along a basis vector in the Lie algebra, the chance of interaction by double photon exchange between two different monopoles is just $\frac{1}{N^2 - 1}$, because there are $N^2 - 1$ basis vectors in the basis of the Lie algebra. Thus, this crude approximation gives:

$$\tilde{g}_{eff}^4 = \frac{1}{N^2 - 1} \tilde{g}_{genuine}^4. \hspace{1cm} (155)$$

Note that considering the two photons exchange which is forced by our statistical description, we must concern the forth power of the monopole charge $\tilde{g}.$
The relation (155) was not derived correctly, but its validity can be confirmed if we use a more correct statistical argument. The problem with our crude estimate is that the generators making monopole charge to be minimal must go along the shortest type of U(1) subgroups with non-trivial homotopy.

**Correct averaging**

The $\lambda_8$–like generators $U^{\lambda_8^* \lambda_8^*} U^+$ maybe written as

$$U^{\lambda_8^* \lambda_8^*} U^+ = -\sqrt{\frac{1}{2N(N-1)}} 1 + \sqrt{\frac{N}{2(N-1)}} P,$$  (156)

where $P$ is a projection metrics into one–dimensional state in the $N$ representation. It is easy to see that averaging according to the Haar measure distribution of $U$, we get the average of $P$ projection on “quark” states with a distribution corresponding to the rotationally invariant one on the unit sphere in the $N$–dimensional $N$–Hilbert space.

If we denote the Hilbert vector describing the state on which $P$ shall project as

$$P = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix},$$  (157)

then the probability distribution on the unit sphere becomes:

$$P\left( \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \right) \prod_{i=1}^{N} d\psi_i \propto \delta(\sum_{i=1}^{N} |\psi_i|^2 - 1) \prod_{i=1}^{N} d(|\psi_i|^2).$$  (158)

Since, of course, we must have $|\psi_i|^2 \geq 0$ for all $i = 1, 2, ..., N$, the $\delta$–function is easily seen to select a flat distribution on a $(N - 1)$–dimensional equilateral simplex. The average of the two photon exchange interaction given by the correction factor (154) squared (numerically):

$$\frac{Tr(U_1^{\lambda_8^* \lambda_8^*} U_1^+ U_2^{\lambda_8^* \lambda_8^*} U_2^+)^2}{Tr((\frac{\lambda_8^*}{2})^2)^2}$$  (159)

can obviously be replaced by the expression where we take as random only one of the “random” $\lambda_8$–like generators, while the other one is just taken as $\frac{\lambda_8^*}{2}$, i.e. we can take say $U_2 = 1$ without changing the average.

Considering the two photon exchange diagram, we can write the correction factor (obtained by the averaging) for the fourth power of magnetic charge:

$$\frac{\tilde{g}^{\lambda_8^* \lambda_8^*}^{\lambda_8^* \lambda_8^*}}{\tilde{g}_{\text{genuine}}^4} = \text{average}\left\{ \frac{Tr(U_1^{\lambda_8^* \lambda_8^*} U_1^+ U_1^{\lambda_8^* \lambda_8^*} U_1^+)^2}{Tr((\frac{\lambda_8^*}{2})^2)^2} \right\}.$$  (160)
Substituting the expression (156) in Eq.(161), we have:

\[
\frac{\bar{g}^4_{\text{eff}}}{\bar{g}^4_{\text{genuine}}} = \text{average}\left\{ \frac{\text{Tr}\left(\frac{\lambda_8}{2}(-\sqrt{\frac{1}{2N(N-1)}} + \sqrt{\frac{N}{2(N-1)}}P\right)^2}{\text{Tr}\left((\frac{\lambda_8}{2})^2\right)} \right\}. 
\]

(161)

Since \(\frac{\lambda_8}{2}\) is traceless, we obtain using the projection (157):

\[
\text{Tr}\left(\frac{\lambda_8}{2}P\sqrt{\frac{N}{2(N-1)}}\right) = -\frac{1}{2}(N-1) + \frac{N}{2(N-1)}|\psi_1|^2. 
\]

(162)

The value of the square \(|\psi_1|^2\) over the simplex is proportional to one of the heights in this simplex. It is obvious from the geometry of a simplex that the distribution of \(|\psi_1|^2\) is

\[
dP = (N-1)(1 - |\psi_1|^2)^{(N-2)}d(|\psi_1|^2), 
\]

where, of course, \(0 \leq |\psi_1|^2 \leq 1\) only is allowed. In Eq.(163) \(P\) is a probability. By definition:

\[
\text{average}\{f(|\psi_1|^2)\} = (N-1) \int_0^1 f(|\psi_1|^2)(1 - |\psi_1|^2)^{(N-2)}d(|\psi_1|^2). 
\]

(164)

Then

\[
\frac{\bar{g}^4_{\text{eff}}}{\bar{g}^4_{\text{genuine}}} = \frac{N^2}{(N-1)} \int_0^1 \left( \frac{1}{N} - |\psi_1|^2 \right)^2 (1 - |\psi_1|^2)^{(N-2)}d(|\psi_1|^2) 
\]

(165)

\[
= \frac{N^2}{N-1} \int_0^1 (1 - y - \frac{1}{N})^2 y^{N-2}dy 
\]

(166)

\[
= \frac{1}{N^2 - 1} 
\]

(167)

and we have confirmed our crude estimation (158).

**Relative normalization of couplings**

Now we are interested in how \(\bar{g}^2_{\text{genuine}}\) is related to \(\alpha_N = \frac{g_N^2}{4\pi}\).

We would get the simple Dirac relation:

\[
g(1) \cdot \bar{g}_{\text{genuine}} = 2\pi, 
\]

(168)

if \(g(1) \equiv g_{U(1)\text{-subgroup}}\) is the coupling for the U(1)–subgroup of SU(N) normalized in such a way that the charge quantum \(g(1)\) corresponds to a covariant derivative \(\partial_{\mu} - g(1)A_{\mu}^{U(1)}\).

Now we shall follow the convention – usually used to define \(\alpha_N = \frac{g_N^2}{4\pi}\) – that the covariant derivative for the \(N\)–plet representation is:

\[
D_{\mu} = \partial_{\mu} - g_N \frac{\lambda^a}{2} A_{\mu}^a 
\]

(169)

with

\[
\text{Tr}\left(\frac{\lambda^a}{2} \frac{\lambda^b}{2}\right) = \frac{1}{2} \delta^{ab}, 
\]

(170)
and the kinetic term for the gauge field is

\[ L = -\frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu}, \]  

(171)

where

\[ F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_N f^{abc} A^b_\mu A^c_\nu. \]  

(172)

Especially if we want to choose a basis for our generalized Gell–Mann matrices so that one basic vector is our “\( \lambda^8 \)”, then for \( A^8_\mu \) we have the covariant derivation \( \partial_\mu - g_N \frac{\lambda^8}{2} A^8_\mu \). If this covariant derivative is written in terms of the \( U(1) \)–subgroup, corresponding to monopoles with the Dirac relation (168), then the covariant derivative has a form \( \partial_\mu - g(1) A^8_\mu \cdot M \). Here \( M \) has the property that \( \exp(2\pi M) \) corresponds to the elements of the group \( SU(N)/Z_N \) going all around and back to the unit element. Of course, \( M = \frac{g_8}{g(1)} \cdot \frac{\lambda^8}{2} \) and the ratio \( g_N/g(1) \) must be such one that \( \exp(2\pi \frac{g_N}{g(1)} \frac{\lambda^8}{2}) \) shall represent – after first return – the unit element of the group \( SU(N)/Z_N \). Now this unit element really means the coset consisting of the center elements \( \exp(i2\pi k/N) \in SU(N), (k \in \mathbb{Z}) \), and the requirement of the normalization of \( g(1) \) ensuring the Dirac relation (168) is:

\[ \exp(i2\pi \frac{g_N}{g(1)} \frac{\lambda^8}{2}) = \exp(i\frac{2\pi}{N})1. \]  

(173)

This requirement is satisfied if the eigenvalues of \( \frac{g_N}{g(1)} \frac{\lambda^8}{2} \) are modulo 1 equal to \(-\frac{1}{N} \), i.e. formally we might write:

\[ \frac{g_N}{g(1)} \frac{\lambda^8}{2} = -\frac{1}{N} \quad (\text{mod } 1). \]  

(174)

According to (153), we have:

\[ \frac{g_N}{g(1)} \cdot \frac{1}{\sqrt{2N(N-1)}} \begin{pmatrix} N-1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{pmatrix} = -\frac{1}{N} \quad (\text{mod } 1), \]  

(175)

what implies:

\[ \frac{g_N}{g(1)} = \sqrt{\frac{2(N-1)}{N}}, \]  

(176)

or

\[ \frac{g_N^2}{g(1)^2} = \frac{2(N-1)}{N}. \]  

(177)
8.3. The relation between U(1) and SU(N) critical couplings

Collecting the relations (177), (168) and (155), we get:

\[ \alpha_{U(1)}^{-1} = N_{\text{crit}} = \frac{4\pi}{g_{N}^{2}(1)} = \frac{N}{2(N-1)} \cdot \frac{4\pi}{g_{U(1)}^{2}} = \frac{N}{2(N-1)} \cdot \frac{\tilde{g}_{\text{genuine}}^{2}}{\pi} \]

\[ = \frac{N}{2(N-1)} \sqrt{N^{2} - 1} \cdot \frac{\tilde{g}_{\text{eff}}^{2}}{\pi} = \frac{N}{2(N-1)} \sqrt{N^{2} - 1} \cdot \frac{4\pi}{g_{U(1)}^{2}} = \frac{N}{2} \sqrt{\frac{N+1}{N-1}} \alpha_{U(1)}^{-1}, \tag{178} \]

where

\[ g_{U(1)} \tilde{g}_{\text{eff}} = 2\pi \tag{179} \]

and \( \alpha_{U(1)} = g_{U(1)}^{2}/4\pi \).

The meaning of this result is that provided that we have \( \tilde{g}_{\text{eff}} \) the same for \( SU(N)/Z_{N} \) and U(1) gauge theories the couplings are related according to Eq. (178).

We have a use for this relation when we want to calculate the phase transition couplings considering the scalar monopole field responsible for the phase transition in the gauge groups \( SU(N)/Z_{N} \). Having in mind the ”Abelian” dominance in the SU(N) monopole vacuum, we must think that \( \tilde{g}_{\text{eff}} \text{crit} \) coincides with \( g_{\text{crit}} \) of the U(1) gauge theory. Of course, here we have an approximation taking into account only monopoles interaction and ignoring the relatively small selfinteractions of the Yang–Mills fields. In this approximation we obtain the same phase transition (triple point, or critical) \( \tilde{g}_{\text{eff}} \text{crit} \)–coupling which is equal to \( g_{\text{crit}} \) of U(1) whatever the gauge group SU(N) might be. Thus we conclude that for the various groups U(1) and \( SU(N)/Z_{N} \), according to Eq. (178), we have the following relation between the phase transition couplings:

\[ \alpha_{U(1), \text{crit}}^{-1} = \frac{N}{2} \sqrt{\frac{N+1}{N-1}} \alpha_{U(1), \text{crit}}^{-1} \tag{180} \]

Using the relation (180), we obtain:

\[ \alpha_{U(1), \text{crit}}^{-1} : \alpha_{2, \text{crit}}^{-1} : \alpha_{3, \text{crit}}^{-1} = 1 : \sqrt{3} : 3/\sqrt{2} = 1 : 1.73 : 2.12. \tag{181} \]

These relations will be used below for the explanation of predictions of MPM.

9. Multiple Point Model and Critical Values of the U(1) and SU(N) Fine Structure Constants

Investigating the phase transition in the dual Higgs monopole model, we have pursued two objects. From one side, we had an aim to explain the lattice results. But we had also another aim.

According to MPM, at the Planck scale there exists a multiple critical point, which is a boundary point of the phase transitions in U(1), SU(2) and SU(3) sectors of the fundamental regularized gauge theory G. It is natural to assume that the objects responsible for these transitions are the physically existing Higgs scalar monopoles, which have to be introduced into theory as fundamental fields. Our calculations indicate that the corresponding critical couplings coincide with the lattice ones, confirming the idea of Ref. [51].

The results of the present paper are very encouraging for the Anti–Grand Unification Theory (AGUT), which always is used in conjunction with the Multiple Point Model (MPM).
9.1. **G-theory, or Anti–grand unification theory (AGUT)**

Most efforts to explain the Standard Model (SM) describing well all experimental results known today are devoted to Grand Unification Theories (GUTs). The supersymmetric extension of the SM consists of taking the SM and adding the corresponding supersymmetric partners [73]. The Minimal Supersymmetric Standard Model (MSSM) shows [74] the possibility of the existence of the grand unification point at

$$\mu_{\text{GUT}} \sim 10^{16} \text{ GeV}.$$  

Unfortunately, at present time experiment does not indicate any manifestation of the supersymmetry. In this connection, the Anti–Grand Unification Theory (AGUT) was developed in Refs. [2], [6] - [10], [50] - [57] and [75] - [78] as a realistic alternative to SUSY GUTs. According to this theory, supersymmetry does not come into the existence up to the Planck energy scale [1]. The Standard Model (SM) is based on the group SMG described by Eq.(2). AGUT suggests that at the scale $$\mu_G \sim \mu_{\text{Pl}} = M_{\text{Pl}}$$ there exists the more fundamental group $$G$$ containing $$N_{\text{gen}}$$ copies of the Standard Model Group SMG:

$$G = \text{SMG}_1 \times \text{SMG}_2 \times \ldots \times \text{SMG}_{N_{\text{gen}}} \equiv (\text{SMG})^{N_{\text{gen}}}, \quad (182)$$

where $$N_{\text{gen}}$$ designates the number of quark and lepton generations.

If $$N_{\text{gen}} = 3$$ (as AGUT predicts [10]), then the fundamental gauge group G is:

$$G = (\text{SMG})^3 = \text{SMG}_{1\text{st gen.}} \times \text{SMG}_{2\text{nd gen.}} \times \text{SMG}_{3\text{rd gen.}}, \quad (183)$$

or the generalized one:

$$G_f = (\text{SMG})^3 \times U(1)_f, \quad (184)$$

which was suggested by the fitting of fermion masses of the SM (see Refs. [75]).

Recently a new generalization of AGUT was suggested in Refs. [77]:

$$G_{\text{ext}} = (\text{SMG} \times U(1)_{B-L})^3, \quad (185)$$

which takes into account the see–saw mechanism with right-handed neutrinos, also gives the reasonable fitting of the SM fermion masses and describes all neutrino experiments known today.

By reasons considered in the last Section of this review, we prefer not to use the terminology ”Anti-grand unification theory, i.e. AGUT”, but call the theory with the group of symmetry $$G$$, or $$G_f$$, or $$G_{\text{ext}}$$, given by Eqs.(182)-(185), as ”G–theory”, because, as it will be shown below, we have a possibility of the Grand Unification near the Planck scale using just this theory.

The group $$G_f$$ contains the following gauge fields: 3 × 8 = 24 gluons, 3 × 3 = 9 W-bosons and 3 × 1 + 1 = 4 Abelian gauge bosons. The group $$G_{\text{ext}}$$ contains: 3 × 8 = 24 gluons, 3 × 3 = 9 W-bosons and 3 × 1 + 3 × 1 = 6 Abelian gauge bosons.

At first sight, this $$(\text{SMG})^3 \times U(1)_f$$ group with its 37 generators seems to be just one among many possible SM gauge group extensions. However, it is not such an arbitrary choice. There are at least reasonable requirements (postulates) on the gauge group G (or $$G_f$$, or $$G_{\text{ext}}$$) which have uniquely to specify this group. It should obey the following postulates (the first two are also valid for SU(5) GUT):
1. G or $G_f$ should only contain transformations, transforming the known 45 Weyl fermions (= 3 generations of 15 Weyl particles each) – counted as left handed, say – into each other unitarily, so that G (or $G_f$) must be a subgroup of U(45): $G \subseteq U(45)$.

2. No anomalies, neither gauge nor mixed. AGUT assumes that only straightforward anomaly cancellation takes place and forbids the Green-Schwarz type anomaly cancellation \[79\].

3. AGUT should NOT UNIFY the irreducible representations under the SM gauge group, called here SMG (see Eq.(2)).

4. G is the maximal group satisfying the above-mentioned postulates.

There are five Higgs fields named $\phi_{WS}$, S, W, T, $\xi$ in AGUT extended by Froggatt and Nielsen \[75\] with the group of symmetry $G_f$ given by Eq.(184). These fields break AGUT to the SM what means that their vacuum expectation values (VEV) are active. The field $\phi_{WS}$ corresponds to the Weinberg–Salam theory, $<S> = 1$, so that we have only three free parameters – three VEVs $<W>$, $<T>$ and $<\xi>$ to fit the experiment in the framework of this model. The authors of Refs. \[75\] used them with aim to find the best fit to conventional experimental data for all fermion masses and mixing angles in the SM (see Table I).

The result is encouraging. The fit is given by the $\chi^2$ function (called here $\bar{\chi}^2$). The lowest value of $\bar{\chi}^2(\approx 1.87)$ gives the following VEVs:

$$
<S> = 1; \quad <W> = 0.179; \quad <T> = 0.071; \quad <\xi> = 0.099.
$$

The extended AGUT by Nielsen and Takanishi \[77\], having the group of symmetry $G_{ext}$ (see Eq.(185)), was suggested with aim to explain the neutrino oscillations. Introducing the right-handed neutrino in the model, the authors replaced the assumption 1 and considered U(48) group instead of U(45), so that $G_{ext}$ is a subgroup of U(48): $G_{ext} \subseteq U(48)$. This group ends up having 7 Higgs fields falling into 4 classes according to the order of magnitude of the expectation values:

1) The smallest VEV Higgs field plays role of the SM Weinberg–Salam Higgs field $\phi_{WS}$ having the weak scale value $<\phi_{WS}> = 246 \text{ GeV}/\sqrt{2}$.

2) The next smallest VEV Higgs field breaks all families $U(1)_{(B-L)}$ group, which is broken at the see–saw scale. This VEV is $<\phi_{(B-L)}> \approx 10^{12} \text{ GeV}$. Such a field is absent in the "old" extended AGUT.

3) The next 4 Higgs fields are W, T, $\xi$ and $\chi$, which have VEVs of the order of a factor 10 to 50 under the Planck unit. That means that if intermediate propagators have scales given by the Planck scale, as it is assumed in AGUT in general, then they will give rise to suppression factors of the order 1/10 each time they are needed to cause a transition. The field $\chi$ is absent in the "old" $G_f$-AGUT. It was introduced in Refs. \[77\] for the purpose of the study of neutrinos.

4) The last one, with VEV of the same order as the Planck scale, is the Higgs field S. It had VEV $<S> = 1$ in the "old" extended AGUT by Froggatt and Nielsen (with $G_f$ group of symmetry), but this VEV is not equal to unity in the "new" extended AGUT. Therefore there is a possibility to observe phenomenological consequences of the field S in the Nielsen–Takanishi model.

Typical fit to the masses and mixing angles for the SM leptons and quarks in the framework of the $G_{ext}$-AGUT is given in Table II.
In contrast to the "old" extended AGUT by Froggatt–Nielsen (called here as $G_f$–theory),
the new results of $G_{ext}$–theory by Nielsen–Takanishi are more encouraging.

We conclude that the $G$–theory, in general, is successful in describing of the SM experiment.

### 9.2. Multiple Point Principle

AGUT approach is used in conjunction with the Multiple Point Principle proposed by D.L.Bennett and H.B.Nielsen [51]. According to this principle, Nature seeks a special point – the Multiple Critical Point (MCP) – which is a point on the phase diagram of the fundamental regularized gauge theory $G$ (or $G_f$, or $G_{ext}$), where the vacua of all fields existing in Nature are degenerate having the same vacuum energy density. Such a phase diagram has axes given by all coupling constants considered in theory. Then all (or just many) numbers of phases meet at the MCP.

MPM assumes the existence of MCP at the Planck scale, insofar as gravity may be "critical" at the Planck scale.

The philosophy of MPM leads to the necessity to investigate the phase transition in different gauge theories. A lattice model of gauge theories is the most convenient formalism also for the realization of the MPM ideas. As it was mentioned above, in the simplest case we can imagine our space–time as a regular hypercubic (3+1)–lattice with the parameter $a$ equal to the fundamental (Planck) scale: $a = \lambda_P = 1/M_{Pl}$. In general, the lattice results are very encouraging for MPM.

### 9.3. AGUT-MPM prediction of the Planck scale values of the U(1), SU(2) and SU(3) fine structure constants

The usual definition of the SM coupling constants:

$$\alpha_1 = \frac{5}{3} \frac{\alpha}{\cos^2 \theta_{MS}}, \quad \alpha_2 = \frac{\alpha}{\sin^2 \theta_{MS}}, \quad \alpha_3 \equiv \alpha_s = \frac{g_s^2}{4\pi}, \quad (187)$$

where $\alpha$ and $\alpha_s$ are the electromagnetic and SU(3) fine structure constants, respectively, is given in the Modified minimal subtraction scheme ($\overline{MS}$). Here $\theta_{MS}$ is the Weinberg weak angle in $\overline{MS}$ scheme. Using RGE with experimentally established parameters, it is possible to extrapolate the experimental values of three inverse running constants $\alpha_i^{-1}(\mu)$ (here $\mu$ is an energy scale and $i=1,2,3$ correspond to U(1), SU(2) and SU(3) groups of the SM) from the Electroweak scale to the Planck scale. The precision of the LEP data allows to make this extrapolation with small errors (see [74]). Assuming that these RGEs for $\alpha_i^{-1}(\mu)$ contain only the contributions of the SM particles up to $\mu \approx M_{Pl}$ and doing the extrapolation with one Higgs doublet under the assumption of a "desert", the following results for the inverses $\alpha_Y^{-1}$ (here $\alpha_Y \equiv \frac{3}{5} \alpha_1$) were obtained in Ref. [51] (compare with [74]):

$$\alpha_Y^{-1}(\mu_{Pl}) \approx 55.5; \quad \alpha_2^{-1}(\mu_{Pl}) \approx 49.5; \quad \alpha_3^{-1}(\mu_{Pl}) \approx 54.0. \quad (188)$$

The extrapolation of $\alpha_Y^{-1}(\mu)$ up to the point $\mu = \mu_{Pl}$ is shown in Fig.9.

According to AGUT, at some point $\mu = \mu_G < \mu_{Pl}$ (but near $\mu_{Pl}$) the fundamental group $G$ (or $G_f$, or $G_{ext}$) undergoes spontaneous breakdown to its diagonal subgroup:

$$G \rightarrow G_{\text{diag.subgr.}} = \{g, g, g || g \in SMG\}, \quad (189)$$
which is identified with the usual (low–energy) group SMG. The point \( \mu_G \approx 10^{18} \text{ GeV} \) also is shown in Fig.9, together with a region of G–theory, where AGUT works.

The AGUT prediction of the values of \( \alpha_i(\mu) \) at \( \mu = \mu_{Pl} \) is based on the MPM assumption about the existence of the phase transition boundary point MCP at the Planck scale, and gives these values in terms of the corresponding critical couplings \( \alpha_{i,crit} \):\[ \alpha_i(\mu_{Pl}) = \frac{\alpha_{i,crit}}{N_{gen}} = \frac{\alpha_{i,crit}}{3} \quad \text{for} \quad i = 2, 3 \quad \text{(also for} \quad i > 3), \quad (190) \]

and
\[ \alpha_1(\mu_{Pl}) = \frac{\alpha_{1,crit}}{\frac{1}{2}N_{gen}(N_{gen} + 1)} = \frac{\alpha_{1,crit}}{6} \quad \text{for} \quad U(1). \quad (191) \]

There exists a simple explanation of the relations (190) and (191). As it was mentioned above, the group G breaks down at \( \mu = \mu_G \). It should be said that at the very high energies \( \mu_G \leq \mu \leq \mu_{Pl} \) (see Fig.9) each generation has its own gluons, own W’s, etc. The breaking makes only linear combination of a certain color combination of gluons which exists in the SM below \( \mu = \mu_G \) and down to the low energies. We can say that the phenomenological gluon has a strength that is \( \sqrt{3} \) times smaller, if as we effectively assume that three AGUT SU(3) couplings are equal to each other. Then we have the following formula connecting the fine structure constants of G–theory (e.g. AGUT) and low energy surviving diagonal subgroup \( G_{\text{diag.subg.}} \subseteq (SMG)^3 \) given by Eq.(189):

\[ \alpha^{-1}_{\text{diag},i} = \alpha^{-1}_{1\text{st gen.},i} + \alpha^{-1}_{2\text{nd gen.},i} + \alpha^{-1}_{3\text{rd gen.},i}. \quad (192) \]

Here \( i = U(1), SU(2), SU(3) \), and \( i=3 \) means that we talk about the gluon couplings. For non–Abelian theories we immediately obtain Eq.(190) from Eq.(192) at the critical point (MCP).

In contrast to non-Abelian theories, in which the gauge invariance forbids the mixed (in generations) terms in the Lagrangian of G–theory, the U(1)–sector of AGUT contains such mixed terms:

\[ \frac{1}{g^2} \sum_{p,q} F_{\mu\nu}^p F_q^{\mu\nu} = \frac{1}{g^2_{11}} F_{\mu\nu,1} F_{1}^{\mu\nu} + \frac{1}{g^2_{12}} F_{\mu\nu,1} F_{2}^{\mu\nu} + \ldots + \frac{1}{g^2_{23}} F_{\mu\nu,2} F_{3}^{\mu\nu} + \frac{1}{g^2_{33}} F_{\mu\nu,3} F_{3}^{\mu\nu}, \quad (193) \]

where \( p, q = 1, 2, 3 \) are the indices of three generations of the AGUT group \((SMG)^3\). The last equation explains the difference between the expressions (190) and (191).

It was assumed in Ref. [51] that the MCP values \( \alpha_{i,crit} \) in Eqs.(190) and (191) coincide with the triple point values of the effective fine structure constants given by the generalized lattice SU(3)–, SU(2)– and U(1)–gauge theories described by Eqs.(38) and (39). Also it was used a natural assumption that the effective \( \alpha_{crit} \) does not change its value (at least too much) along the whole borderline ”3” of Fig.4 for the phase transition ”Coulomb–confinement” in the U(1) lattice gauge theory with the generalized (two parameters) lattice Wilson action (39).

Now let us consider \( \alpha_y^{-1} \approx \alpha^{-1} \) at the point \( \mu = \mu_G \approx 10^{18} \text{ GeV} \) shown in Fig.9. If the point \( \mu = \mu_G \) is very close to the Planck scale \( \mu = \mu_{Pl} \), then according to Eqs.(188) and (191), we have:

\[ \alpha_{1\text{st gen.}} \approx \alpha_{2\text{nd gen.}} \approx \alpha_{3\text{rd gen.}} \approx \frac{\alpha_{1\text{st gen.}}(\mu_G)}{6} \approx 9, \quad (194) \]
what is almost equal to the value (53):

\[ \alpha_{\text{crit.,theor}} \approx 8 \]

obtained by ”Parisi improvement method” (see Fig.5(c)). This means that in the U(1) sector of AGUT we have \( \alpha \) near the critical point. Therefore, we can expect the existence of MCP at the Planck scale.

As it was mentioned above, the lattice investigators were not able to obtain the lattice triple point values of \( \alpha_{i,\text{crit}} \) (\( i = 1, 2, 3 \) correspond to U(1), SU(2) and SU(3) groups) by Monte Carlo simulation methods. These values were calculated theoretically by Bennett and Nielsen in Ref. [51]. Using the lattice triple point values of \( (\beta_A; \beta_f) \) and \( (\beta_{\text{lat}}, \gamma_{\text{lat}}) \) (see Fig.3(a,b) and Fig.4), they have obtained \( \alpha_{i,\text{crit}} \) by the ”Parisi improvement method”:

\[ \alpha_{\gamma,\text{crit}}^{-1} \approx 9.2 \pm 1, \quad \alpha_{\alpha,\text{crit}}^{-1} \approx 16.5 \pm 1, \quad \alpha_{\beta,\text{crit}}^{-1} \approx 18.9 \pm 1. \] (195)

Assuming the existence of MCP at \( \mu = \mu_{Pl} \) and substituting the last results in Eqs.(190) and (191), we have the following prediction of AGUT [51]:

\[ \alpha_{\gamma}^{-1}(\mu_{Pl}) \approx 55 \pm 6; \quad \alpha_{\alpha}^{-1}(\mu_{Pl}) \approx 49.5 \pm 3; \quad \alpha_{\beta}^{-1}(\mu_{Pl}) \approx 57.0 \pm 3. \] (196)

These results coincide with the results [188] obtained by the extrapolation of experimental data to the Planck scale in the framework of the pure SM (without any new particles) [51], [74].

Using the relation (180), we obtained the result (181), which in our case gives the following relations:

\[ \alpha_{\gamma,\text{crit}}^{-1} : \alpha_{\alpha,\text{crit}}^{-1} : \alpha_{\beta,\text{crit}}^{-1} = 1 : \sqrt{3} : \frac{3}{\sqrt{2}} = 1 : 1.73 : 2.12. \] (197)

Let us compare now these relations with the MPM prediction.

For \( \alpha_{\gamma,\text{crit}}^{-1} \approx 9.2 \) given by the first equation of (195), we have:

\[ \alpha_{\gamma,\text{crit}}^{-1} : \alpha_{\alpha,\text{crit}}^{-1} : \alpha_{\beta,\text{crit}}^{-1} = 9.2 : 15.9 : 19.5. \] (198)

In the framework of errors the last result coincides with the AGUT–MPM prediction (195). Of course, it is necessary to take into account an approximate description of confinement dynamics in the SU(N) gauge theories, which was used in our investigations.

10. The possibility of the Grand Unification Near the Planck Scale

We can see new consequences of the extension of G–theory, if G–group is broken down to its diagonal subgroup \( G_{\text{diag}} \), i.e. SM, not at \( \mu_G \sim 10^{18} \) GeV, but at \( \mu_G \sim 10^{15} \) or \( 10^{16} \) GeV. In this connection, it is very attractive to consider the gravitational interaction.

10.1. ”Gravitational finestructure constant” evolution

The gravitational interaction between two particles of equal masses M is given by the usual classical Newtonian potential:

\[ V_g = -G \frac{M^2}{r} = - \left( \frac{M}{M_{Pl}} \right)^2 \frac{1}{r} = - \frac{\alpha_g(M)}{r}, \] (199)

37
which always can be imagined as a tree–level approximation of quantum gravity.

Then the quantity:

$$\alpha_g = \left( \frac{\mu}{\mu_{Pl}} \right)^2$$

(see also Refs. [22, 30, 81]) plays a role of the running "gravitational finestructure constant" and the evolution of its inverse quantity is presented in Fig.10 together with the evolutions of $\alpha_{1,2,3}^{-1}(\mu)$ (here we have returned to the consideration of $\alpha_1$ instead of $\alpha_Y$).

Then we see the intersection of $\alpha_g^{-1}(\mu)$ with $\alpha_1^{-1}(\mu)$ in the region of $G$–theory at the point:

$$(x_0, \alpha_1^{-1}(\mu)),$$

where

$$x_0 \approx 18.3, \quad \alpha_1^{-1} \approx 34.4,$$

and $x = \log_{10}\mu$.

10.2. The consequences of the breakdown of $G$-theory at $\mu_G \sim 10^{15}$ or $10^{16}$ GeV

Let us assume now that the group of symmetry $G$ undergoes the breakdown to its diagonal subgroup not at $\mu_G \sim 10^{18}$ GeV, but at $\mu_G \sim 10^{15}$ GeV, i.e. before the intersection of $\alpha_2^{-1}(\mu)$ and $\alpha_3^{-1}(\mu)$ at $\mu \sim 10^{16}$ GeV. Why is it important?

As a consequence of behavior of the function $\alpha_1^{-1}(\beta)$ near the phase transition point, shown in Fig.5c, we have to expect the change of the evolution of $\alpha_1^{-1}(\mu)$ in the region $\mu > \mu_G$ shown in Fig.9 by dashed lines. Instead of these dashed lines, we must see the decreasing of $\alpha_1^{-1}(\mu)$, when they approach MCP, if this MCP really exists at the Planck scale.

According to Fig.5c, it seems that in the very vicinity of the phase transition point (i.e. also near the MCP at $\mu = \mu_{Pl}$), we cannot describe the behavior of $\alpha_1^{-1}(\mu)$ by the one–loop approximation RGE.

It is well known, that the one–loop approximation RGEs for $\alpha_1^{-1}(\mu)$ (see for example [13]) can be described in our case by the following expression:

$$\alpha_1^{-1}(\mu) = \alpha_1^{-1}(\mu_{Pl}) + \frac{b_1}{4\pi} \log(\frac{\mu^2}{\mu_{Pl}^2}),$$

(202)

where $b_1$ are given by the following values:

$$b_1 = (b_1, b_2, b_3) =$$

$$(-\frac{4N_{gen}}{3} - \frac{1}{10}N_S, \frac{22}{3}N_V - \frac{4N_{gen}}{3} - \frac{1}{6}N_S, 11N_V - \frac{4N_{gen}}{3}).$$

(203)

The integers $N_{gen}$, $N_S$, $N_V$ are respectively the numbers of generations, Higgs bosons and different vector gauge fields of given "colors”.

For the SM we have:

$$N_{gen} = 3, \quad N_S = N_V = 1,$$

(204)
and the corresponding slopes (203) describe the evolutions of $\alpha^{-1}_i(\mu)$ up to $\mu = \mu_G$ presented in Fig.10.

But in the region $\mu_G \leq \mu \leq \mu_{Pl}$, when $G$–theory works, we have $N_V = 3$ (here we didn’t take into account the additional Higgs fields which can change the number $N_S$), and the one–loop approximation slopes are almost 3 times larger than the same ones for the SM. In this case, it is difficult to understand that such evolutions give the MCP values of $\alpha^{-1}_i(\mu_{Pl})$, which are shown in Fig.11. These values were obtained by the following way:

$$
\begin{align*}
\alpha^{-1}_{1, \text{crit}} &\equiv \alpha^{-1}_1(\mu_{Pl}) \approx 6 \cdot \frac{2}{3} \alpha^{-1}_{U(1), \text{crit}} \approx 13, \\
\alpha^{-1}_{2, \text{crit}} &\equiv \alpha^{-1}_2(\mu_{Pl}) \approx 3 \cdot \sqrt{3} \alpha^{-1}_{U(1), \text{crit}} \approx 19, \\
\alpha^{-1}_{3, \text{crit}} &\equiv \alpha^{-1}_3(\mu_{Pl}) \approx 3 \cdot \frac{3}{\sqrt{2}} \alpha^{-1}_{U(1), \text{crit}} \approx 24,
\end{align*}
$$

(205)

where we have used the relation (180) with

$$
\alpha_{U(1), \text{crit}} = \frac{\alpha_{\text{crit}}}{\cos^2 \theta_{\text{MS}}} \approx 0.77 \alpha_{\text{crit}},
$$

(206)

taking into account our HMM result (105): $\alpha_{\text{crit}} \approx 0.208$, which coincides with the lattice result (18) and gives:

$$
\alpha^{-1}_{U(1), \text{crit}} \approx 3.7.
$$

(207)

In the case when $G$–group undergoes the breakdown to the SM not at $\mu_G \sim 10^{18}$ GeV, but at $\mu_G \sim 10^{15}$ GeV, the artifact monopoles of non-Abelian SU(2) and SU(3) sectors of $G$–theory begin to act more essentially.

According to the group dependence relation (180) (but now we expect that it is very approximate), there exists, for example, the following estimation at $\mu_G \sim 10^{15}$ GeV:

$$
\alpha^{-1}_{U(1)}(\mu_G) \sim 7 \quad \text{for} \quad \text{SU}(3)_{1\text{st gen.}}, \text{etc.}
$$

(208)

which is closer to MCP than the previous value $\alpha^{-1}_Y \sim 9$, obtained for the AGUT breakdown at $\mu_G \sim 10^{18}$ GeV.

It is natural to assume that $\beta$–functions of SU(2) and SU(3) sectors of $G$–theory change their one–loop approximation behavior in the region $\mu > 10^{16}$ GeV and $\alpha^{-1}_{2,3}(\mu)$ begin to decrease, approaching the phase transition (multiple critical) point at $\mu = \mu_{Pl}$. This means that the asymptotic freedom of non–Abelian theories becomes weaker near the Planck scale, what can be explained by the influence of artifact monopoles condensation. It looks as if these $\beta$–functions have singularity at the phase transition point and, for example, can be approximated by the following expression:

$$
\frac{d\alpha^{-1}}{dt} = \frac{\beta(\alpha)}{\alpha} \approx A(1 - \frac{\alpha}{\alpha_{\text{crit}}})^{-\nu} \quad \text{near the phase transition point}.
$$

(209)

This possibility is shown in Fig.11 for $\nu \approx 1$ and $\nu \approx 2.4$.

Here it is worth-while to comment that such a tendency was revealed in the vicinity of the confinement region by the forth–loop approximation of $\beta$–function in QCD (see Ref. [82].
10.3. Does the $[\text{SU}(5)]^3$ SUSY unification exist near the Planck scale?

Approaching the MCP in the region of $G$–theory ($\mu_G \leq \mu \leq \mu_{Pl}$), $\alpha^{-1}_3(\mu)$ shows the necessity of intersection of $\alpha^{-1}_2(\mu)$ with $\alpha^{-1}_3(\mu)$ at some point of this region if $\mu_G \sim 10^{15}$ or $10^{16}$ GeV (see Fig.11). If this intersection takes place at the point $(\alpha^{-1}_0, \alpha^{-1}_G)$ given by Eq.(201), then we have the unification of all gauge interactions (including the gravity) at the point:

$$(x_{\text{GUT}}; \alpha^{-1}_{\text{GUT}}) \approx (18.3; 34.4), \quad (210)$$

where $x = \log_{10} \mu$(GeV). Here we assume the existence of $[\text{SU}(5)]^3$ SUSY unification with superparticles of masses

$$M \approx 10^{18.3} \text{GeV}. \quad (211)$$

The scale $\mu_{\text{GUT}} = M$, given by Eq.(211), can be considered as a SUSY breaking scale.

Figures 12(a,b) demonstrate such a possibility of unification. We have investigated the solutions of joint intersections of $\alpha^{-1}_g(\mu)$ and all $\alpha^{-1}_i(\mu)$ at different $x_{\text{GUT}}$ with different $\nu$ in Eq.(209). These solutions exist from $\nu \approx 0.5$ to $\nu \approx 2.5$. The possibility of poles ($\nu = 1, 2$) was obtained for QCD beta–function near the confinement region (see [83]).

The unification theory with $[\text{SU}(5)]^3$–symmetry was suggested first by S.Rajpoot [84]. It is essential that in this theory the critical point, obtained by means of Eqs.(190), (180) and (207), is given by the following value:

$$\alpha^{-1}_{5, \text{crit}} \approx 3 \cdot 5 \sqrt{\frac{3}{2}} \alpha^{-1}_{U(1), \text{crit}} \approx 34.0. \quad (212)$$

The point (212) is shown in Figures 12(a,b) for the cases:

1. $\nu \approx 1$, $\alpha^{-1}_{\text{GUT}} \approx 34.4$, $x_{\text{GUT}} \approx 18.3$ and $\mu_G \approx 10^{16}$ shown in Fig.12(a);
2. $\nu \approx 2.4$, $\alpha^{-1}_{\text{GUT}} \approx 34.4$, $x_{\text{GUT}} \approx 18.3$ and $\mu_G \approx 10^{15}$ shown in Fig.12(b).

We see that the point (212) is very close to the unification point $\alpha^{-1}_{\text{GUT}} \approx 34.4$, given by Eq.(210). This means that the unified theory, suggested here as the $[\text{SU}(5)]^3$ SUSY unification, approaches the confinement phase at the Planck scale. But the confinement of all SM particles is impossible in our world, because then they have to be confined also at low energies what is not observed in the nature.

It is worth–while to mention that using the Zwanziger formalism for the Abelian gauge theory with electric and magnetic charges (see Refs. 85–87 and 55), the possibility of unification of all gauge interactions at the Planck scale was considered in Ref. 78 in the case when unconfined monopoles come to the existence near the Planck scale. They can appear only in G–theory 88, because RGEs for monopoles strongly forbid their deconfinement in the SM up to the Planck scale. But it is not obvious that they really exist in G–theory. This problem needs more careful investigations, because our today knowledge about monopoles is still very poor.

The unified theory, suggested in this Section, essentially differs in its origin from that one, which was considered in Ref. 88, because it does not assume the existence of deconfining monopoles up to the Planck scale, but assumes the influence of artifact monopoles near the phase transition (critical) point.

Considering the predictions of this unified theory for the low–energy physics and cosmology, maybe in future we shall be able to answer the question: ”Does the unification of $[\text{SU}(5)]^3$ SUSY type really exist near the Planck scale?”
In conclusion, it is necessary to comment that in the SR theory [21–23] all evolutions investigated in this section have to be considered only as functions of the variable $\mu = 1/r$, where $r$ is a distance. The evolutions are different if $\mu$ is the energy scale. The explanation is given in Refs. [21–23], and we refer to them.
11. Conclusions

1. In this review we have presented the main ideas of the Nielsen’s Random Dynamics (RD).
2. We have considered the theory of Scale Relativity (SR) by L.Nottale, which has something in common with RD and predicts the existence of the fundamental (minimally possible) length in Nature equal to the Planck length.
3. We have discussed that RD and SR lead to the discreteness of our space-time, giving rise to the new description of physics at very small distances.
4. We have reviewed the lattice gauge theories.
5. We have used the approximation of lattice artifact monopoles as fundamental pointlike particles described by the Higgs scalar fields and considered the dual Abelian Higgs model of scalar monopoles, or shortly the Higgs Monopole Model (HMM), as a simplest effective dynamics describing the confinement mechanism in the pure gauge lattice theories.
6. Using the Coleman–Weinberg idea of the RG improvement of the effective potential, we have considered this potential with $\beta$–functions calculated in the two–loop approximation.
7. The phase transition between the Coulomb–like and confinement phases has been investigated in the U(1) gauge theory by the method developed in the Multiple Point Model (MPM), where degenerate vacua are considered.
8. We have presented the calculation of the phase transition values of $\alpha = e^2/4\pi$ and $\tilde{\alpha} = g^2/4\pi$. Comparing the result $\alpha_{\text{crit}} \approx 0.17$ and $\tilde{\alpha}_{\text{crit}} \approx 1.48$, obtained in the one–loop approximation, with the result $\alpha_{\text{crit}} \approx 0.208$ and $\tilde{\alpha}_{\text{crit}} \approx 1.20$, obtained in the two–loop approximation, we have shown the coincidence of the critical values of electric and magnetic fine structure constants calculated in the two–loop approximation of HMM with the lattice result $[36]: \alpha_{\text{crit}} \approx 0.20 \pm 0.015$ and $\tilde{\alpha}_{\text{crit}} \approx 1.25 \pm 0.10$.
9. Comparing the one–loop and two–loop contributions to beta–functions, we have demonstrated the validity of HMM perturbation theory in solution of the phase transition problem in the U(1) gauge theory.
10. Investigating the phase transition in HMM, we have pursued two objects: the first aim was to explain the lattice results, but the second one was to confirm the MPM prediction, according to which at the Planck scale there exists a Multiple Critical Point (MCP).
11. We have given reviews of the Anti-Grand Unification Theory (AGUT, or G–theory) and MPM.
12. We have compared the predictions of AGUT and MPM for the Planck scale value of $\alpha_{\text{Y}}^{-1}(\mu)$ with its lattice and HMM results. This comparison gives arguments in favour of the existence of MCP at the Planck scale.
13. We have argued ”an approximate universality” of the critical coupling constants which is very important for AGUT and MPM.
14. We have used the ’t Hooft idea $[72]$ about the Abelian dominance in the monopole vacuum of non–Abelian theories: monopoles of the Yang–Mills theories are the solutions of the U(1)–subgroups, arbitrary embedded into the SU(N) group.
15. Choosing the Abelian gauge and taking into account that the direction in the Lie algebra of monopole fields are gauge independent, we have found an average over these directions and obtained the group dependence relation between the phase transition fine structure constants for the groups $U(1)$ and $SU(N)/\mathbb{Z}_N$:

$$\alpha_{N,\text{crit}}^{-1} = \frac{N}{2} \sqrt{\frac{N + 1}{N - 1}} \alpha_{U(1),\text{crit}}^{-1}.$$
16. The significant conclusion for MPM was that in the case of the AGUT breakdown at \( \mu_G \sim 10^{18} \) GeV, using the group dependence of critical couplings and AGUT prediction of the values of \( \alpha_i(\mu) \) at the Planck scale given in terms of the corresponding critical couplings \( \alpha_{i,crit} \), we have obtained the following relations:

\[
\alpha^{-1}_{Y,crit} : \alpha^{-1}_{2,crit} : \alpha^{-1}_{3,crit} = 1 : \sqrt{3} : 3/\sqrt{2} = 1 : 1.73 : 2.12.
\]

We have shown that for \( \alpha^{-1}_{Y,crit} \approx 9.2 \) the last equation gives the following result:

\[
\alpha^{-1}_{Y,crit} : \alpha^{-1}_{2,crit} : \alpha^{-1}_{3,crit} = 9.2 : 15.9 : 19.5,
\]

which confirms the Bennett–Nielsen AGUT–MPM prediction:

\[
\alpha^{-1}_{Y,crit} \approx 9.2 \pm 1, \quad \alpha^{-1}_{2,crit} \approx 16.5 \pm 1, \quad \alpha^{-1}_{3,crit} \approx 18.9 \pm 1.
\]

17. We have considered the gravitational interaction between two particles of equal masses \( M \), given by the Newtonian potential, and presented the evolution of the quantity:

\[
\alpha_g = \left( \frac{\mu}{\mu_{Pl}} \right)^2,
\]

which plays a role of the running "gravitational fine-structure constant".

18. We have shown that the intersection of \( \alpha^{-1}_g(\mu) \) with \( \alpha^{-1}_1(\mu) \) occurs in the region of G–theory at the point \( (x_0, \alpha^{-1}_{0}) \) with the following values:

\[
\alpha^{-1}_0 \approx 34.4, \quad x_0 \approx 18.3,
\]

where \( x = \log_{10} \mu(\text{GeV}) \).

19. Using the lattice indications of the decreasing of \( \alpha^{-1}_i(\mu) \), when they approach the phase transition points, we have argued that near the MCP at \( \mu = \mu_{Pl} \) the behavior of \( \alpha^{-1}_i(\mu) \) cannot be described by the one–loop approximation for RGE.

20. We have calculated the MCP values \( \alpha^{-1}_i(\mu_{Pl}) \):

\[
\alpha^{-1}_{1, crit} \approx 13, \quad \alpha^{-1}_{2, crit} \approx 19, \quad \alpha^{-1}_{3, crit} \approx 24,
\]

using the group dependence relation for critical couplings and our HMM result:

\[
\alpha_{crit} \approx 0.208.
\]

21. We have considered a new possibility of the breakdown of G–theory at \( \mu_G \sim 10^{15} \) or \( 10^{16} \) GeV and a possible role of monopoles at high energies: in this case the abelian artifact monopoles of the non-Abelian SU(2) and SU(3) sectors of AGUT begin to act more essentially and the evolutions of \( \alpha^{-1}_{2,3}(\mu) \) begin to decrease approaching the MCP, what leads to the necessity of intersection of \( \alpha^{-1}_2(\mu) \) with \( \alpha^{-1}_3(\mu) \) at some point in the region \( \mu_G \leq \mu \leq \mu_{Pl} \).

22. We have discussed that at present time there exists a possibility of the intersection between \( \alpha^{-1}_2(\mu) \) and \( \alpha^{-1}_3(\mu) \) at the point \( x_{GUT} \approx 18.3 \) with different \( \alpha^{-1}_{GUT} \). In particular,

\[
(x_{GUT}; \alpha^{-1}_{GUT}) \approx (18.3; 34.4)
\]

can be the point of unification of all gauge interactions (including the gravity).
23. It is natural to assume the existence of the $[\text{SU}(5)]^3$ SUSY unification having super-particles of masses

$$M \approx 10^{18.3} \text{GeV},$$

considered as a SUSY breaking scale.

24. We have demonstrated the possibility of $[\text{SU}(5)]^3$ SUSY unification and the existence of its critical point at

$$\alpha_{5,\text{crit}}^{-1} = \alpha_{5}^{-1}(\mu_{Pl}) \approx 34.0$$

what means that such an unified theory approaches the confinement phase at the Planck scale, but does not reach it.

25. We have demonstrated that the Higgs scalar monopoles imitating the artifact monopoles of our discrete space–time play an essential role at very high (Planck scale) energies.
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Fig. 1.



Fig. 2.
Fig. 3(a,b) Phase diagrams for the SU(2) and SU(3) lattice gauge theories with the generalized Wilson lattice action.

The result of Monte–Carlo simulations:
(a) – for SU(2) gauge theory,
(b) – for SU(3) gauge theory.
Fig. 4 The phase diagram for U(1) when the two-parameter lattice action is used. This type of action makes it possible to provoke the confinement $Z_2$ (or $Z_3$) alone. The diagram shows the existence of a triple (critical) point. From this triple point emanate three phase borders: the phase border ”1” separates the totally confining phase from the phase where only the discrete subgroup $Z_2$ is confined; the phase border ”2” separates the latter phase from the totally Coulomb–like phase; and the phase border ”3” separates the totally confining and totally Coulomb–like phases.
Fig. 5 (a, b) (a) The renormalized electric fine structure constant plotted versus $\beta/\beta_T$ for the Villain action (circles) and the Wilson action (crosses). The points are obtained by the Monte-Carlo simulations method for the compact QED;
(b) The behavior of the effective electric fine structure constant in the vicinity of the phase transition point obtained with the lattice Wilson action. The dashed curve corresponds to the theoretical calculations by the ”Parisi improvement method”.

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Fig. 5c The behavior of the inverse effective electric fine structure constant in the vicinity of the phase transition point plotted versus $\beta/\beta_T$ for the simple Wilson lattice action. The dashed curve corresponds to the theoretical calculations by the "Parisi improvement method".
Fig. 6 The effective potential $V_{\text{eff}}$: the curve 1 corresponds to the "Coulomb"-"confinement" phase transition; curve 2 describes the existence of two minima corresponding to the confinement phases.
Fig. 7 The one-loop (curve 1) and two-loop (curve 2) approximation phase diagram in the dual Abelian Higgs model of scalar monopoles.
Fig. 8 The phase diagram ($\lambda_{\text{run}}; g^4 \equiv g^4_{\text{run}}$), corresponding to the Higgs monopole model in the one-loop approximation, shows the existence of a triple point A ($\lambda_{(A)} \approx -13.4; g^2_{(A)} \approx 18.6$). This triple point is a boundary point of three phase transitions: the ”Coulomb-like” phase and two confinement phases (”conf.1” and ”conf.2”) meet together at the triple point A. The dashed curve ”2” shows the requirement: $V_{\text{eff}}(\phi_0^2) = V''_{\text{eff}}(\phi_0^2) = 0$. Monopole condensation leads to the confinement of the electric charges: ANO electric vortices (with electric charges at their ends, or closed) are created in the confinement phases ”conf.1” and ”conf.2”.
Fig. 9 The evolution of three inverse running constants $\alpha^{-1}_i(\mu)$, where $i=1,2,3$ correspond to U(1), SU(2) and SU(3) groups of the SM. The extrapolation of their experimental values from the Electroweak scale to the Planck scale was obtained by using the renormalization group equations with one Higgs doublet under the assumption of a ”desert”. The precision of the LEP data allows to make this extrapolation with small errors. AGUT works in the region $\mu_G \leq \mu \leq \mu_{Pl}$. 
Fig. 10 The intersection of the inverse "gravitational fine-structure constant" $\alpha_g^{-1}(\mu)$ with $\alpha_1^{-1}(\mu)$ occurs at the point $(x_0, \alpha_0^{-1})$:

$$\alpha_0^{-1} \approx 34.4,$$

and

$$x_0 \approx 18.3,$$

where

$$x = \log_{10} \mu(GeV).$$
Fig. 11 The case of the breakdown of G-theory at $\mu_G$. The evolution of $\alpha_{1,2,3}^{-1}(\mu)$ decreases near the Planck scale and approaches to the Multiple Critical Point at $\mu = \mu_{Pl}$. It is shown the possibility of unification of all gauge interactions, including gravity, at $\alpha_{GUT}^{-1} \approx 34.4$ and $x_{GUT} \approx 18.3$, where $x = \log_{10} \mu (GeV)$:

a) $\mu_G \approx 10^{16} (GeV)$, $\nu \approx 1$;

b) $\mu_G \approx 10^{15} (GeV)$, $\nu \approx 2.4$.  

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Fig. 12 The possibility of the \([\text{SU}(5)]^3\) SUSY unification of all gauge interactions, including gravity. A dashed line corresponds to the evolution of \(\alpha_5^{-1}(\mu)\) approaching to the critical point at \(\alpha_5^{-1}(\mu_{Pl}) \approx 34.0:\)

\[\begin{align*}
  a) & \quad \mu_G \approx 10^{16}(GeV), \quad \nu \approx 1; \\
  b) & \quad \mu_G \approx 10^{15}(GeV), \quad \nu \approx 2.4.
\end{align*}\]
Table I. Best fit to conventional experimental data. All masses are running masses at 1 GeV except the top quark mass $M_t$ which is the pole mass.

|       | Fitted       | Experimental |
|-------|--------------|--------------|
| $m_u$ | 3.6 MeV     | 4 MeV        |
| $m_d$ | 7.0 MeV     | 9 MeV        |
| $m_e$ | 0.87 MeV    | 0.5 MeV      |
| $m_c$ | 1.02 GeV    | 1.4 GeV      |
| $m_s$ | 400 MeV     | 200 MeV      |
| $m_{\mu}$ | 88 MeV | 105 MeV      |
| $M_t$ | 192 GeV     | 180 GeV      |
| $m_b$ | 8.3 GeV     | 6.3 GeV      |
| $m_{\tau}$ | 1.27 GeV | 1.78 GeV    |
| $V_{us}$ | 0.18    | 0.22         |
| $V_{cb}$ | 0.018  | 0.041        |
| $V_{ub}$ | 0.0039 | 0.0035       |

Table II. Best fit to conventional experimental data in the "new" AGUT.

|       | Fitted       | Experimental |
|-------|--------------|--------------|
| $m_u$ | 3.1 MeV     | 4 MeV        |
| $m_d$ | 6.6 MeV     | 9 MeV        |
| $m_e$ | 0.76 MeV    | 0.5 MeV      |
| $m_c$ | 1.29 GeV    | 1.4 GeV      |
| $m_s$ | 390 MeV     | 200 MeV      |
| $m_{\mu}$ | 85 MeV | 105 MeV      |
| $M_t$ | 179 GeV     | 180 GeV      |
| $m_b$ | 7.8 GeV     | 6.3 GeV      |
| $m_{\tau}$ | 1.29 GeV | 1.78 GeV    |
| $V_{us}$ | 0.21    | 0.22         |
| $V_{cb}$ | 0.023  | 0.041        |
| $V_{ub}$ | 0.0050 | 0.0035       |

All masses are running masses at 1 GeV except the top quark mass $M_t$ which is the pole mass. The lowest value of $\chi^2$ is $\approx 1.46$. 
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