Toeplitz Determinants Associated with Ma-Minda Classes of Starlike and Convex Functions

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Abstract
A starlike function \( f \) is characterized by the quantity \( zf'(z)/f(z) \) lying in the right half-plane. This paper deals with sharp bounds for certain Toeplitz determinants whose entries are the coefficients of the functions \( f \) for which the quantity \( zf'(z)/f(z) \) takes values in certain specific subset in the right half-plane. The results obtained include several new special cases and some known results.

Keywords Univalent functions · Starlike functions · Convex functions · Toeplitz determinants · Coefficient bounds

1 Introduction

Let \( \mathbb{D} \) be the open unit disk in \( \mathbb{C} \) and let \( \mathcal{A} \) be the class of all analytic functions \( f : \mathbb{D} \rightarrow \mathbb{C} \) having Taylor series \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). Let \( \mathcal{S} \) be the well known subclass of \( \mathcal{A} \) of univalent (\( \equiv \) one-to-one) functions. A set \( D \) is starlike with respect to 0 if \( t w \in D \) for all \( w \in D \) and for all \( t \) with \( 0 \leq t \leq 1 \); it is convex if \( t w_1 + (1-t)w_2 \in D \) for all \( w_1, w_2 \in D \) and for all \( t \) with \( 0 \leq t \leq 1 \). The subclasses of \( \mathcal{S} \) consisting of functions \( f \) for which \( f(\mathbb{D}) \) is starlike with respect to the origin and convex are denoted, respectively, by \( \mathcal{S}^* \) and \( \mathcal{K} \). These classes were introduced and studied aiming at a proof of the famous coefficient conjecture of Bieberbach that \( |a_n| \leq n \) with equality for the Koebe function \( z/(1-z)^2 \) or its rotations; see the survey article by Ahuja (1986) and several references therein for a history on the problem. The concept of subordination is useful in unifying various subclasses of univalent functions. First, let us denote by \( \Omega \) the class of all analytic functions \( w : \mathbb{D} \rightarrow \mathbb{D} \) with \( w(0) = 0 \). A function in \( \Omega \) is known as a Schwarz function. An analytic function \( f \) is said to be subordinate to the analytic function \( F \), written \( f \prec F \) or \( f(z) \prec F(z) \) if \( f \) is univalent in \( \mathbb{D} \), then the subordination \( f(z) \prec F(z) \) holds if and only if \( f(0) = F(0) \) and \( f(D) \subseteq F(D) \). The class \( \mathcal{P} \) of Carathéodory functions consists of all analytic functions \( p : \mathbb{D} \rightarrow \mathbb{C} \) with \( \text{Re} p(z) > 0 \) for \( z \in \mathbb{D} \). The two classes are closely associated as a function \( p \in \mathcal{P} \) if and only if there is a \( w \in \Omega \) with \( p = (1 + w)/(1 - w) \). These functions are characterized analytically as follows:

\[
\mathcal{S}^* = \left\{ f \in \mathcal{A} : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \right\}
\]

\[
= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \frac{1 + z}{1 - z} \right\}, \text{and}
\]

\[
\mathcal{K} = \left\{ f \in \mathcal{A} : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \right\}
\]

\[
= \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \frac{1 + z}{1 - z} \right\}.
\]

Ma and Minda (1992) gave a unified treatment of distortion, growth and covering theorems for the functions \( f \in \mathcal{S}^* \) and \( f \in \mathcal{K} \) for which either of the quantity \( zf'(z)/f(z) \) or \( 1 + zf''(z)/f'(z) \) is subordinate to a more general subordinate function \( \phi \in \mathcal{P} \). In Ma and Minda (1992), it is
assumed that the function $\varphi$ is starlike and the image of unit disk is symmetric with respect to real axis. However, we do not require these conditions in this paper.

**Definition 1** For an analytic univalent function $\varphi$ with positive real part in $\mathbb{D}$, $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi''(0) \in \mathbb{R}$, the classes $S^*(\varphi)$ and $\mathcal{K}(\varphi)$ are defined by

$$S^*(\varphi) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} < \varphi(z) \right\},$$

and

$$\mathcal{K}(\varphi) := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} < \varphi(z) \right\}.$$

Toeplitz matrices and their determinants play an important role in several branches of mathematics and have many applications. For information on applications of Toeplitz matrices to several areas of pure and applied mathematics, we refer to the survey article by Ye and Lim (2016). We recall that Toeplitz symmetric matrices have constant entries along the diagonal. For the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we associate a determinant $T_q(n)$ defined by

$$T_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}.$$

In 2017, Ali et al. (2018) studied Toeplitz determinants $T_q(n)$ for initial values of $n$ and $q$, where the entries of $T_q(n)$ are the coefficients of the functions that are starlike, convex and close to convex. Motivated by Ali et al. (2018), some researchers in the last three years studied $T_q(n)$ for low values of $n$ and $q$, where entries are the coefficients of functions in several subclasses of analytic functions. Some recent work on coefficient problems includes Cho et al. (xxxx); Cudna et al. (2020); Kowalczyk et al. (2017); Lecko et al. (2020).

In this paper, we obtain sharp estimates for Toeplitz determinants $T_2(2)$ and $T_3(1)$ for functions belonging to the classes $S^*(\varphi)$ and $\mathcal{K}(\varphi)$. The functions $K_\varphi$ and $H_\varphi$ defined by

$$\frac{zK_\varphi(z)}{K_\varphi(z)} = \varphi(iz), \quad K_\varphi(0) = K'_\varphi(0) - 1 = 0$$

(1)

and

$$1 + \frac{zH_\varphi(z)}{H_\varphi(z)} = \varphi(iz), \quad H_\varphi(0) = H'_\varphi(0) - 1 = 0,$$

(2)

respectively, belong to the classes $S^*(\varphi)$ and $\mathcal{K}(\varphi)$. We shall use these functions to demonstrate sharpness in certain cases. For a function $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, it is well-known Grenander and Szegö (1958) that $|c_n| \leq 1$. The main results are proved by using this estimate by associating coefficients of the functions in our classes to the functions in the class $\mathcal{P}$. We shall also use estimates for the Feke-Szegö functional for the two classes $S^*(\varphi)$ and $\mathcal{K}(\varphi)$ from Ali et al. (2007) and Ma and Minda (1992). The symmetry of the image of $\varphi$ was used in Ma and Minda (1992) to ensure that the coefficients of $\varphi$ are real and we have assumed it here for the first two coefficients. In Ma and Minda (1992), the univalence was used in defining the function $p_1$ by

$$p_1(z) = \frac{1 - \varphi^{-1}(p(z))}{1 + \varphi^{-1}(p(z))}.$$

However, this requirement can be dropped by defining $p_1$ by (4).

### 2 Main Results

Theorem 1 and Theorem 2, respectively, give the sharp bound for $T_2(2) = a_2^2 - a_2^2$ for functions $f \in S^*(\varphi)$ and $f \in \mathcal{K}(\varphi)$.

**Theorem 1** If $f \in S^*(\varphi)$ and $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $0 < B_1 \leq |B_2 + B_1^2|$, then the Toeplitz determinant $T_2(2)$ satisfies the sharp bound:

$$|T_2(2)| \leq \frac{1}{4} |B_2 + B_1^2|^2 + B_1^2.$$

**Proof** Since $f \in S^*(\varphi)$, there is a function $w$ in the class $\Omega$ of Schwarz functions satisfying that

$$\frac{zf'(z)}{f(z)} = \varphi(w(z)).$$

(3)

Corresponding to the function $w$, define the function $p_1 : \mathbb{D} \to \mathbb{C}$ by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots$$

(4)

so that

$$w(z) = p_1(z) - 1 = \frac{1}{2} c_1 z + \frac{1}{2} \left(c_2 - \frac{1}{2} c_1^2 \right) z^2 + \cdots.$$  

(5)

Clearly, the function $p_1$ is analytic in $\mathbb{D}$ with $p_1(0) = 1$. Since $w \in \Omega$, it follows that $p_1 \in \mathcal{P}$. Using (5) and the Taylor series of $\varphi$ given by $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get
\[ \varphi(w(z)) = 1 + \frac{1}{2}B_1 c_1 z + \left( \frac{1}{2}B_1 \left( c_2 - \frac{1}{2}c_1^2 \right) + \frac{1}{4}B_2 c_1^2 \right) z^2 + \cdots. \]  

(6)

Since \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \), the Taylor series expansion of the function \( \frac{f(z)}{f'(z)} \) is given by

\[ \frac{f'(z)}{f(z)} = 1 + a_2 z + (-a_2^2 + 2a_3)z^2 + \left( a_2^3 - 3a_2a_3 + 3a_4 \right)z^3 \\
+ (-a_2^4 + 4a_2^2a_3 - 2a_2^2 - 4a_2a_4 + 4a_4)z^4 + \cdots. \]

(7)

Using (3), (6) and (7), the coefficients \( a_2 \) and \( a_3 \) can be expressed as a function of the coefficients \( c_i \) of \( p \in \mathcal{P} \) and \( B_i \) of \( \varphi \) as follows:

\[ a_2 = \frac{1}{2}B_1 c_1 \]  

(8)

and

\[ a_3 = \frac{1}{8}((B_1^2 - B_1 + B_2)c_1^2 + 2B_1c_2). \]  

(9)

The equations (8) and (9) (see Ali et al. (2007) for a general result for \( p \)-valent functions) readily shows that

\[ |a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{2} \left( B_2 + B_1^2 - 2\mu B_1^2 \right), & \text{if } 2B_2 \mu \leq B_2 + B_1^2 - B_1; \\
\frac{1}{2}B_1, & \text{if } B_2 + B_1^2 - B_1 \leq 2B_1 \mu \leq B_2 + B_1^2 + B_1; \\
\frac{1}{2} \left( -B_2 - B_1^2 + 2\mu B_1^2 \right), & \text{if } B_2 + B_1^2 + B_1 \leq 2B_1^2 \mu. 
\end{cases} \]  

(10)

Since \(|c_i| \leq 2\), the equation (8) shows that

\[ |a_2| \leq B_1 \]  

(11)

and, when \( B_1 \leq |B_2 + B_1^2| \), the equation (10) readily yields

\[ |a_3| \leq \frac{1}{2} \left| B_1^2 + B_2 \right|. \]  

(12)

Using these estimates for the second and third coefficients given in (11) and (12), we have

\[ |a_3^2 - a_2^2| \leq |a_3|^2 + |a_2|^2 \leq \frac{1}{4} \left( B_1^2 + B_2 \right)^2 + B_1^2. \]

The result is sharp for the function \( K_\varphi \) given by (1). This function \( K_\varphi \) has the Taylor series given by

\[ K_\varphi(z) = z - iB_1 z^2 - \frac{1}{2} \left( B_1^2 + B_2 \right) z^3 + \cdots. \]

The Taylor series can be obtained by noting that \( K_\varphi \) corresponds to the function \( f \) given by (3) when \( w(z) = iz \). In this case, \( p_1(z) = 1 + 2iz - 2z^2 + \cdots \). With \( c_1 = 2i \) and \( c_2 = -2 \), we get \( a_2 = iB_1 \) and \( a_3 = -(B_1^2 + B_2)/2 \). Clearly, for the function \( K_\varphi \), we have

\[ |a_3^2 - a_2^2| = \frac{1}{4} \left( B_1^2 + B_2 \right)^2 + B_1^2 \]

proving the sharpness.

\[ \square \]

**Theorem 2** If \( f \in K(\varphi) \) and \( \varphi(z) = 1 + B_2 z + B_2 z^2 + \cdots \) with \( 0 < B_2 \leq |B_2 + B_1^2| \), then the Toeplitz determinant \( T_2(2) \) satisfies the sharp bound given by

\[ |T_2(2)| \leq \frac{1}{36} \left( B_1^2 + B_2 \right)^2 + \frac{1}{4}B_2^2. \]

**Proof** Let \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) and \( \varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \). Since \( f \in K(\varphi) \), there is a function \( w \) in the class \( \Omega \) of Schwarz functions such that

\[ 1 + \frac{zf''(z)}{f'(z)} = \varphi(w(z)). \]  

(13)

The Taylor series expansion of the function \( f \) given by

\[ f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \]  

shows that

\[ 1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2 z + (-4a_2^2 + 6a_3)z^2 + \cdots. \]  

(14)

Then using (13), (14) and (6), the coefficients \( a_2 \) and \( a_3 \) can be expressed as a function of the coefficients \( c_i \) of \( p \in \mathcal{P} \) given by

\[ a_2 = \frac{1}{4}B_1 c_1, \text{ and } a_3 \]

Using the well-known estimate \(|c_n| \leq 2\) for the function \( p_1 \) with positive real part, it follows that

\[ |a_2| \leq \frac{B_1}{2}. \]  

(15)

For a function \( f \in K(\varphi) \), Ma and Minda (1992) proved the following inequality

\[ |a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{6} \left( B_2 + \frac{3}{2} B_1^2 + B_1^3 \right), & \text{if } 3B_1^2 \mu \leq 2(B_2 + B_1^2 - B_1); \\
\frac{1}{6}B_1, & \text{if } 2(B_2 + B_1^2 - B_1) \leq 3B_1^2 \mu \leq 2(B_2 + B_1^2 + B_1); \\
\frac{1}{6} \left( -B_2 + \frac{3}{2} B_1^2 - B_1^3 \right), & \text{if } 2(B_2 + B_1^2 + B_1) \leq 3B_1^2 \mu. 
\end{cases} \]  

(16)

Since \( B_1 \leq |B_2 + B_1^2| \), the inequality (16) readily gives

\[ |a_3| \leq \frac{1}{6} \left| B_1^2 + B_2 \right|. \]  

(17)

Using the bound for \( a_2 \) and \( a_3 \) given, respectively, by (15) and (17), we get
\[ |a_3^2 - a_1^2| \leq |a_3|^2 + |a_2|^2 \leq \frac{1}{36} (B_1^2 + B_2^2)^2 + \frac{1}{4} B_1^2. \]

The result is sharp for the function \( H_\phi \) defined in (2). Indeed, for this function \( H_\phi \), we have \( a_2 = B_1i/2 \) and \( a_3 = -(B_1^2 + B_2)/6 \) and hence
\[ |a_3^2 - a_1^2| = \frac{1}{36} (B_1^2 + B_2^2)^2 + \frac{1}{4} B_1^2, \]
proving the sharpness of the result.

Theorems 3 and 4 give the sharp bound for the Toeplitz determinant \( T_3(1) \) for functions, respectively, in the classes \( S^*(\phi) \) and \( K(\phi) \).

**Theorem 3** If \( f \in S^*(\phi) \) and \( \phi(z) = 1 + B_1 z + B_2 z^2 + \cdots \), with \( B_1 > 0 \) and \( B_1 - B_1^2 \leq B_2 \leq \frac{3B_1^2}{2} - B_1 \), then the Toeplitz determinant \( T_3(1) \) satisfies the sharp bound:
\[ |T_3(1)| \leq 1 + 2B_1^2 + \frac{1}{4} (B_2 + B_1^2)(3B_1^2 - B_2). \]

**Proof** Since
\[ T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix} = 1 - 2a_2^2 - a_3(a_3 - 2a_2^2) \]
it follows that
\[ |T_3(1)| \leq 1 + 2|a_2|^2 + |a_3||a_3 - 2a_2^2|. \]
Since \( B_1 \leq B_1^2 + B_2 \), the inequality (12) gives
\[ |a_3| \leq \frac{1}{2} (B_1^2 + B_2). \]
(19)
Since \( B_1 + B_2 \leq 3B_1^2 \), the equation (10) readily yields
\[ |a_3 - 2a_2^2| \leq \frac{1}{2} (3B_1^2 - B_2). \]
(20)
Using these estimates for the second and third coefficients given in (11) and (19), and the bound for \( a_3 - 2a_2^2 \) given by (20) in (18), we obtain
\[ |T_3(1)| \leq 1 + 2B_1^2 + \frac{1}{4} (B_2 + B_1^2)(3B_1^2 - B_2). \]
The result is sharp for the function \( K_\phi \) given by (1). For this function \( K_\phi \), we have \( a_2 = iB_1 \) and \( a_3 = -(B_1^2 + B_2)/2 \) and
\[ 1 - 2a_2^2 - a_3(a_3 - 2a_2^2) = 1 + 2B_1^2 + \frac{1}{4} (B_2 + B_1^2)(3B_1^2 - B_2), \]
proving the sharpness of our result.

Theorem 4 If \( f \in K(\phi) \) and \( \phi(z) = 1 + B_1 z + B_2 z^2 + \cdots \), with \( B_1 > 0 \) and \( B_1 - B_1^2 \leq B_2 \leq 2B_1^2 - B_1 \), then the Toeplitz determinant \( T_3(1) \) satisfies the sharp bound:
\[ |T_3(1)| \leq 1 + \frac{1}{2} B_1^2 + \frac{1}{36} (B_2^2 + B_2)(2B_1^2 - B_2). \]

**Proof** The given conditions on \( B_1 \) and \( B_2 \) are the same as \( B_1 \leq B_1^2 + B_2 \) and, \( B_1 + B_2 \leq 2B_1^2 \). Since \( B_1 \leq B_1^2 + B_2 \), the inequality (16) gives
\[ |a_3| \leq \frac{1}{6} (B_1^2 + B_2). \]
(21)
Since \( B_1 \leq 2B_1^2 - B_2 \), the inequality (16) gives
\[ |a_3 - 2a_2^2| \leq \frac{1}{6} (2B_1^2 - B_2). \]
(22)
Using the bound for \( a_2 \) and \( a_3 \) given by (15) and (21) and the bound for \( a_3 - 2a_2^2 \) given by (22) in (18), we get the desired result.

The result is sharp for the function \( H_\phi \) defined in (2). Indeed, for this function \( H_\phi \), we have \( a_2 = B_1i/2 \) and \( a_3 = -(B_1^2 + B_2)/6 \) and hence
\[ 1 - 2a_2^2 - a_3(a_3 - 2a_2^2) = 1 + \frac{1}{2} B_1^2 + \frac{1}{36} (B_2^2 + B_2)(2B_1^2 - B_2), \]
proving the sharpness of the result.

**Remark 1** The problem of finding the sharp bound for \( T_2(2) \) for functions in the classes \( S^*(\phi) \) and \( K(\phi) \) is open when \( B_2 + B_1^2 \leq B_1 \). Similarly, the determination of sharp bounds for \( T_3(1) \) in other cases is open. It may be interesting to extend the results for other classes, in particular, the classes considered in Aouf and Seoudy (2020a) and Aouf and Seoudy (2020b).

### 3 Some Special Cases

Ma and Minda classes of starlike and convex functions include several well-known classes as special cases which have been studied by several authors (see for example Kargar et al. (2019); Mahzoon (2020)). For some of these subclasses, Theorems 1–4 give the sharp bounds for \( |T_2(2)| \) and \( |T_3(1)| \).

3.1 For \(-1 \leq B < A \leq 1\), \( S^*[A,B] := S^*((1 + A z)/(1 + B z)) \) is the familiar class consisting of Janowski starlike functions and \( K[A,B] := K((1 + A z)/(1 + B z)) \) is the class of Janowski convex functions. These classes were initially introduced and studied by Janowski (1973). The series expansion of \( \phi(z) = (1 + A z)/(1 + B z) \) yields
\[
\varphi(z) := \frac{1 + Az}{1 + Bz} = 1 + (A - B)z + B(B - A)z^2 + B^2(A - B)z^3 + \cdots
\]

which implies \( B_1 = (A - B) \) and \( B_2 = -B(A-B) \). If \(|A - 2B| \geq 1\), then, for \( f \in \mathcal{S}^*[A,B], \)
\[
|T_2(2)| \leq (A - B)^2(4 + A^2 - 4AB + 2B^2)/4 \text{ and for } f \in \mathcal{K}[A,B] \text{ we have } |T_2(2)| \leq (A - B)^2(9 + A^2 - 4AB + 2B^2)/36.
\]

If \( B \leq \min\{(A - 1)/2, (3A - 1)/2\}, \text{ then, for } f \in \mathcal{S}^*[A,B], \text{ we have } |T_2(1)| \leq 1 + 2(A - B)^2 + (3A^2 - 5AB + 2B^2)(A^2 - 3AB + 2B^2)/4.
\]

3.3 Sharma et al. (2016) defined and studied the class of functions defined by \( S_C^* = S^*(\varphi_z(z)) \), where \( \varphi_z(z) = 1 + (4/3)z + (2/3)z^2 \). The geometrical interpretation is that a function \( f \) belongs to the class \( S_C^* \) if \( z^f(z)/f(z) \) lies in the region \( \Omega_c \) bounded by the cardioid i.e., \( \varphi_z(\mathbb{D}) := \{x + iy : (9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0\} \). The convex analogous class of the above mentioned class is \( K_C := K(\varphi_z(z)) \). Its geometrical interpretation is that a function \( f \) belongs to the class \( K_C \) if \( 1 + z^f(z)/f(z) \) lies in the region \( \Omega_c \) bounded by the cardioid i.e., \( \varphi_z(\mathbb{D}) := \{x + iy : (9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0\} \).

3.4 Cho et al. (2019) defined and studied the class \( S_{\text{sin}}^* = S^*(1 + \sin z) \). The convex analogous subclass is defined as \( K_{\text{sin}} := K(1 + \sin z) \). Let, the function \( f \in S_{\text{sin}}^* \). Writing the Taylor series expansion for \( \sin z \), we get
\[
\varphi(z) := 1 + \sin z = 1 + z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots
\]

Thus, \( B_1 = 1 \) and \( B_2 = 0 \) which implies \( |T_2(2)| \leq 5/4 = 1.25 \), proved in Zhang et al. (2019), and \( |T_3(1)| \leq 15/4 = 3.75 \) for \( f \in S_{\text{sin}}^* \). Similarly we can obtain \( |T_2(2)| \leq 5/18 = 0.277778 \) and \( |T_3(1)| \leq 14/9 = 1.555556 \) for \( f \in K_{\text{sin}} \).

3.5 Raina and Sokól (2015) defined the class \( S_e^* = S^*(\varphi_e(z)) \), where \( \varphi_e(z) = z + \sqrt{1 + z^2} \). Its convex subclass \( K_e := K(\varphi_e(z)) \). The classes \( S_e^* \) and \( K_e \) consist of functions for which \( z^f(z)/f(z) \) and \( 1 + z^f(z)/f(z) \) lies in the the leftmoon region \( \Omega_e \) defined by \( \varphi_e(\mathbb{D}) := \{w \in \mathbb{C} : |w^2 - 1| < 2|w|\}. \)
Thus, \( f \in S^p \) implies \( zf''(z)/f(z) < z + \sqrt{1 + z^2} \) and therefore we have
\[
\phi_\xi(z) := z + \sqrt{1 + z^2} = 1 + z + \frac{1}{2} z^2 - \frac{1}{8} z^4 + \cdots.
\]

Therefore, \( B_1 = 1 \) and \( B_2 = 1/2 \) which immediately yields \( |T_2(2)| \leq 25/16 = 1.5625 \) and \( |T_3(1)| \leq 63/16 = 3.9375 \) for \( f \in S^p \). Similarly, \( f \in K_\xi \) implies \( 1 + z f''(z)/f(z) \leq \sqrt{1 + z^2} \) and therefore we have \( |T_2(2)| \leq 5/16 = 0.3125 \) and \( |T_3(1)| \leq 25/16 = 1.5625 \).

3.6 Ronning (1993), motivated by Goodman (1991), introduced and studied the parabolic starlike class \( S_p \) and the uniformly convex class \( UCV \) obtained from Ma-Minda class of starlike and convex functions, respectively, by replacing
\[
\varphi(z) := 1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2
= 1 + \frac{8}{\pi^2} z + \frac{16}{3\pi^2} z^2 + \frac{184}{45\pi^2} z^3 + \cdots.
\]
This yields \( B_1 = 8/\pi^2 \) and \( B_2 = 16/3\pi^2 \) and thus we get
\[
|T_2(2)| \leq \frac{128(72 + 12\pi^2 + 5\pi^4)}{(9\pi^8)} \approx 1.015474 \text{ and } |T_3(1)| \leq 1 + 3072/\pi^8 + 512/3\pi^6 + 1088/(9\pi^8) \approx 2.74232
\]
for \( f \in S_p \). For \( f \in UCV \), we get
\[
|T_2(2)| \leq 16(576 + 96\pi^2 + 85\pi^4)/(81\pi^8) \approx 0.204083.
\]

3.7 Yunus et al. (2018) et al. studied the class \( S^p_{\lim} := S^p(1 + \sqrt{2}z + z^2/2) \) associated with the limacon \((4u^2 + 4v^2 - 8u - 5) + 8(4u^2 + 4v^2 - 12u - 3) = 0\). The class \( K_{\lim} := K(1 + \sqrt{2}z + z^2/2) \). Clearly, in this case \( B_1 = \sqrt{2} \) and \( B_2 = 1/2 \) and therefore, we get \( |T_2(2)| \leq 57/16 = 3.5625 \) and \( |T_3(1)| \leq 135/16 = 8.4375 \) for \( f \in S^p_{\lim} \). For \( f \in K_{\lim} \), \( |T_2(2)| \leq 97/144 = 0.673611 \) and \( |T_3(1)| \leq 323/144 = 2.24306 \\

3.8 Wani and Swaminathan (2021), studied the class of functions defined by \( S^p_{\nu} := S^p(\phi_{\nu}(z)) \) and \( K_{\nu} := K(\phi_{\nu}) \), where the function \( \phi_{\nu}(z) := 1 + z - z^3/3 \) maps the unit disk into the interior of the 2-cusped kidney-shaped nephroid. Clearly, here \( B_1 = 1 \) and \( B_2 = 0 \), thereby yielding \( |T_2(2)| \leq 5/4 = 1.25 \) and \( |T_3(1)| \leq 15/4 = 3.75 \) for \( f \in S^p_{\nu} \). For \( f \in K_{\nu} \), \( |T_2(2)| \leq 5/18 = 0.277778 \) and \( |T_3(1)| \leq 14/9 = 1.55556 \\

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