SIMPLIFIED NUMERICAL FORM OF UNIVERSAL FINITE TYPE INVARIANT OF GAUSS WORDS

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Abstract. In the present paper, we study the finite type invariants of Gauss words. In the Polyak algebra techniques, we reduce the determination of the group structure to transformation of a matrix into its Smith normal form and we give the simplified form of a universal finite type invariant by means of the isomorphism of this transformation. The advantage of this process is that we can implement it as a computer program. We obtain the universal finite type invariant of degree 4, 5, and 6 explicitly. Moreover, as an application, we give the complete classification of Gauss words of rank 4 and the partial classification of Gauss words of rank 5 where the distinction of only one pair remains.

1. Introduction

One of concerns in knot theory is the classification of knots, which is mainly studied by invariants of knots. A finite type invariant is one of the most important classes of knot invariants. V. A. Vassiliev introduced finite type invariants to study topology of the space of all knots [14]. Finite type invariants are also known as Vasiliev knot invariants. Later, the definition of finite type invariants was simplified by J. Birman and X. Lin in [2].

A knot is the image of a smooth embedding of $S^1$ into $\mathbb{R}^3$. We also express a knot as a knot diagram in $\mathbb{R}^2$, which is a smooth immersion of $S^1$ into $\mathbb{R}^2$ with transversal double points such that the two paths at each double point are assigned to be an over path and an under path respectively. Such a double point is called a crossing. Reading off labels of crossings and crossing information starting from a fixed base point, we can interpret knots as words with additional data. For such words, considering equivalence relations that are analogy of Reidemeister moves, which are called homotopy moves, we can investigate knots combinatorially.

Generalized notions of knots have been introduced. L. Kauffman introduced the theory of virtual knots by using combinatorially extended knot diagrams which are called virtual knot diagrams [7]. A virtual knot diagram is a planar graph of valency four endowed with the following structure: each vertex either has an overcrossing and undercrossing (in other words, real crossing) or is marked by a virtual crossing. Then, he define virtual knots as the quotient of a set of virtual knot diagrams with respect to an equivalence relation generated by the virtual Reidemeister moves. V. Turaev extended the theory of virtual knots and virtual links from the aspect of Gauss codes to nanowords and nanophrases in his papers [11, 13]. Nanowords are generalizations of knots and some other knot theoretical objects. Gauss words are defined as the simplest version of nanowords and homotopy classes of Gauss words are equivalent to free knots with a base point in [9, 10].

In [5], M. Goussarov, M. Polyak and O. Viro extended the notion of the finite type invariants to virtual knots and applied the theory of finite type invariants of classical knots to virtual knots. They constructed an abelian group, called Polyak
algebra, and the explicit form of universal finite type invariants that is written as count of subdiagrams for a diagram. Furthermore, in [4], A. Gibson and N. Ito defined the finite type invariants of nanophrases in a similar way to Goussarov, Polyak and Viro’s, and they investigated the finite type invariants of nanophrases of lower degree. They also obtained the finite type invariant of Gauss words of degree 4.

Gauss words are sequences of letters so that all letters appear exactly twice or not in it. Because Gauss words are words that have no additional information, only configuration of letters determines homotopy classes of Gauss words. V. Turaev conjectured that all Gauss words are homotopic to the empty word [12], however, Gibson [3] and Manturov [9] proved independently the existence of a Gauss word not homotopic to the empty word. In [4], the same result was also proven by means of a finite type invariant. It is known that there exist infinite homotopy classes of such Gauss words. Because nanowords are Gauss words with additional data, we regard Gauss words as invariants of knots and of their generalizations.

An argument of Polyak algebra is useful for studies using computer programs. For virtual knots, computing the group defined from Polyak algebra, D. Bar-Natan et al. conjectured relations between dimensions of spaces of finite type invariants and dimensions of spaces of weight systems [1]. Note that their computation was carried out when Polyak algebra is defined over some field while Polyak algebra in our situation is defined over \( \mathbb{Z} \).

The universal finite type invariant of Gauss words can be written as a map counting isomorphic subwords of a word. In this paper, we present a simplified form of the universal finite type invariant, which is a map between words and a direct sum of cyclic groups. To obtain the form of the invariant, we determine the structure of the abelian group \( H_n \), which is defined by truncating words of rank more than \( n \) in Polyak algebra. We can reduce this determination of the group structure to transformation of a matrix into its Smith normal form, which is a popular technique in computer algebra. Then, we obtain a direct sum of cyclic groups that is isomorphic to \( H_n \). Also, the transformation matrix in this process gives the isomorphism between the group \( H_n \) and the direct sum of cyclic groups. Composing the isomorphism with the original universal finite type invariant, we have the simplified form.

Our computer program to carry out this process determined the group structure for rank less than or equal to 7. In addition, we constructed the universal invariants of degree 4, 5, and 6. As an application of these invariants, we show the partial classification of Gauss words of rank less than or equal to 5.

This paper is organized as follows. In Section 2, we review the definitions and notations on Gauss words and finite type invariants. We recall Ito and Gibson’s results in [4]. In Section 3, we present an algorithm to determine the group structure and give our computational results for truncated Polyak algebra. In Section 4, we transform the universal finite type invariants into the simplified numerical form and present explicitly the finite type invariant of the degree 4 and 5. In section 5, we apply our main result to the classification of Gauss words of at most 5 letters.

2. Gauss words and finite type invariants

In this section, we recall some definitions and facts on Gauss words and finite type invariants of them. The definitions and notations in this section are those of a restricted version of [4]. We can also refer to [11, 12, 13] for basic definitions.

A word of length \( n \) is a sequence of \( n \) letters. If a word is length 0 we call it an empty word. In this paper we consider Gauss words, which are sequences of letters so that all letters appear exactly twice or not in it. The rank of a Gauss word is
the number of distinct letters appearing in it. Clearly, the rank of a Gauss word is the half of length of it. Two Gauss words \( w_1 \) and \( w_2 \) are isomorphic if there is a bijection between the sets of letters appearing in \( w_1 \) and \( w_2 \) so that a word created by mapping all letters of \( w_1 \) coincides with \( w_2 \).

We define homotopy moves for Gauss words as the following.

\[
H_1: xAy \leftrightarrow xy \\
H_2: xAyBzA \leftrightarrow xyz \\
H_3: xAyBzA \leftrightarrow xByCAzCBt
\]

Here, \( x, y, z, \) and \( t \) are arbitrary words that can be empty words. Homotopy is the equivalence relation generated by isomorphisms and homotopy moves of the three types.

Let \( P \) be the set of homotopy classes of Gauss words and \( \mathbb{Z}P \) be the free abelian group generated by the elements of \( P \). We define a semi-letter \( \dot{A} \) for a Gauss word of the form \( xAyAz \), where \( x, y, \) and \( z \) are arbitrary words that can be empty words.

A word including \( \dot{A} \) in \( \mathbb{Z}P \) is defined by

\[
x\dot{A}y\dot{A}z = xAyAz - xyz.
\]

Semi-letters define a class of homotopy invariants as follows. Let \( v \) be a homotopy invariant for Gauss words taking values in an abelian group. We extend linearly a homotopy invariant to \( \mathbb{Z}P \). A homotopy invariant \( v \) is a finite type invariant if there exists an integer \( n \) so that \( v(w) = 0 \) for any Gauss word \( w \) including more than \( n \) semi-letters. We call such a least integer \( n \) the degree of the finite type invariant \( v \). A finite type invariant \( v: \mathbb{Z}P \to G \) of degree \( n \) is a universal invariant of degree \( n \) if for any finite type invariant \( v' \) of degree \( n: \mathbb{Z}P \to H \), there exists a homomorphism \( f \) so that the following diagram is commutative.

\[
\begin{array}{ccc}
\mathbb{Z}P & \xrightarrow{v} & G \\
\downarrow{v'} & & \downarrow{f} \\
& & \mathbb{Z}I
\end{array}
\]

Goussarov et al. described a universal invariant of virtual knots as an angle bracket formula [5]. Gibson and Ito [4] extended the formula to nanophrases. Let \( w_1 \) and \( w_2 \) be words. The word \( w_1 \) is a subword of the word \( w_2 \) if we can generate \( w_1 \) by deleting some letters from \( w_2 \). Then we write \( w_1 \prec w_2 \). For any word \( w, w' \), itself and the empty word are subwords of \( w \). We define an angle bracket \( \langle w_1, w_2 \rangle \) to be the number of subwords of \( w_2 \) isomorphic to \( w_1 \). We extend linearly the angle bracket to \( \mathbb{Z}P \).

Let \( \mathbb{Z}I \) be the free abelian group generated by \( \mathbb{Z} \) isomorphism classes of Gauss words. Then \( \mathbb{Z}P \) coincides with \( \mathbb{Z}I \) modulo homotopy moves \( H_1 \) to \( H_3 \). We consider three other relations on \( \mathbb{Z}I \) than the homotopy moves.

\[
G_1: xAy = 0 \\
G_2: xAyBzA + 2xAyAz = 0 \\
G_3: xAyBzA + xAyACzBt + xAyACzCt + xAyCzBt + xAyACzCt + xAyCzBt
\]

Here, \( x, y, z, \) and \( t \) are arbitrary words so that each term of word in the above relations is a Gauss word. Let \( G \) be the group given by \( \mathbb{Z}I \) modulo these three types of relations, which is called Polyak algebra. Note that Polyak algebra has the structure of an algebra defined by concatenation of words, however, we do not use it in this paper. To define a group \( G_n \), we introduce another relation \( G_4 \) for a positive integer \( n \):

\[
G_4: \text{If the rank of a word } w \text{ is greater than } n \text{ then } w = 0.
\]
Note that $G_4$ depends on the rank $n$. Let $G_n$ be the group given by $\mathbb{Z} \mathcal{I}$ modulo the relations $G_1$, $G_2$, $G_3$, and $G_4$. We can decompose $G_n$ into the direct sum of two groups: one is the group generated by the empty word and the other is the group generated by the other generators. Because the empty word does not appear in the relations $G_1$ to $G_4$, it generates $\mathbb{Z}$. Therefore, we have

$$G_n = \mathbb{Z} \oplus H_n,$$

where $H_n$ be the group generated by Gauss words except the empty word whose rank is less than or equal to $n$.

We define a map $\theta : \mathbb{Z} \mathcal{I} \to \mathbb{Z} \mathcal{I}$ by

$$\theta(p) = \sum_{q \triangleright p} q.$$

We also define an additive map $O_n : \mathbb{Z} \mathcal{I} \to \mathbb{Z} \mathcal{I}$ by $O_n(p) = p$ for a word $p$ of rank less than or equal to $n$ and $O_n(p) = 0$ for a word $p$ of rank greater than $n$. Then, the map $\theta$ induces an isomorphism $\hat{\theta}$ from $\mathbb{Z} \mathcal{P}$ onto $G$ (see Proposition 5.7 in [4]). Also, the map $O_n$ induces a map from $G$ to $G_n$. We define the map $\tilde{\Gamma}_n : \mathbb{Z} \mathcal{P} \to G_n$ as the composite of $\theta$ and $O_n$:

Let $P_n$ be the set of homotopy classes of Gauss words whose rank is $n$ or less. We also have

$$\tilde{\Gamma}_n(p) = \sum_{q \triangleright p} O_n(p) = \sum_{q \in P_n} \langle q, p \rangle q.$$

The following result is crucial.

**Proposition 2.1** (Gibson and Ito [4 Proposition 5.9]). The map $\tilde{\Gamma}_n$ is a universal invariant of degree $n$.

We define $\Gamma_n : \mathbb{Z} \mathcal{P} \to H_n$ as the composite of $\tilde{\Gamma}_n$ and the natural map from $G_n$ to $H_n$. Because for $\sum a_i w_i \in \mathbb{Z} \mathcal{P}$ where $a_i$ are integers

$$\tilde{\Gamma}_n \left( \sum a_i w_i \right) = \left( \sum a_i, \Gamma_n \left( \sum a_i w_i \right) \right),$$

to analyze $\tilde{\Gamma}_n$ is equivalent to analyzing $\Gamma_n$. Later we give a numerical form of $\Gamma_n$.

The structures of $G_n$ for smaller $n$ have been determined.

**Proposition 2.2** (Gibson and Ito [4]). For Gauss words, $G_0$, $G_1$, $G_2$, and $G_3$ are isomorphic to $\mathbb{Z}$ and $G_4$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

We determine the structures of $G_n$ for $n = 5, 6, 7$ in the next section.

3. **Computational determination of the group $H_n$**

It is well-known that every finitely generated abelian group is (noncanonically) isomorphic to the direct sum of cyclic groups, and when we are given generators and relations on the group, we can construct an isomorphism in terms of matrix theory (See [8] for example). We rephrase it in a suitable form for our situation.
Proposition 3.1. Let $M$ be a free abelian group with basis $w_1, \ldots, w_s$. Let $N$ be the subgroup generated by $\left\{ \sum_{i=1}^s a_{ji} w_i \mid j = 1, \ldots, t \right\}$ for $a_{ji} \in \mathbb{Z}$ and $j = 1, \ldots, t$, which corresponds to relations of $M$ so that $\sum_{i=1}^s a_{ji} w_i = 0$. Then, there are a nonzero integer $p$ and positive integers $d_1, \ldots, d_l$ with $l = s - p$ such that $M/N$ is isomorphic to

$$Z^p \oplus \bigoplus_{i=1}^l (\mathbb{Z}/d_i \mathbb{Z}),$$

and such that if $l$ is greater than 1 then $d_{i-1}$ divides $d_i$ for $i = 2, \ldots, l$. These integers $d_1, \ldots, d_l$ and $p$ are uniquely determined.

Moreover, if $A$ denotes the $s \times t$ matrix $(a_{ij})$ and $S$ denotes the $s \times t$ matrix

$$
\begin{pmatrix}
  d_1 & 0 & \cdots & 0 \\
  d_2 & \ddots & \ddots & 0 \\
  0 & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 & d_l
\end{pmatrix}
$$

then we can transform $A$ into $S$ by row and column operations, where row (respectively column) operations mean either interchanging two rows (respectively columns), adding a multiple of a row (respectively column) to another, or multiplying a row (respectively column) by $-1$. In other words, there are a $\mathbb{Z}$-invertible $s \times s$ matrix $U$ and a $\mathbb{Z}$-invertible $t \times t$ matrix $V$ so that $S = UAV$. The matrix $S$ is called the Smith normal form of $A$ and $d_i$ are called its elementary divisors.

Because the group $H_n$ is finitely generated, it is isomorphic to a group of the form (8). The second type relations G2 restrict possibilities of the structure of $H_n$ further.

Proposition 3.2. The group $H_n$ is isomorphic to a group of the form

$$\bigoplus_{i=1}^{n-3} (\mathbb{Z}/2^i \mathbb{Z})^{p_i},$$

for some nonnegative integers $p_i$.

In particular, for an arbitrary Gauss word $w$ of rank $m$ in $H_n$, we have

$$2^{n-m+1} w = 0.$$

Proof. From second type relation G2, for an arbitrary Gauss word $w$ of rank $m$, there exists a Gauss word $w'$ of rank $m + 1$ so that $w' + 2w = 0$. If $m = n$ then the word $w'$ has rank $n + 1$ and $w' = 0$ in $H_n$. Therefore, we have $2w = 0$. Easy induction proves the second assertion for $m < n$. Obviously, under the relation (11), $H_n$ must be of the form as in (10). $\square$

To determine the structure of $H_n$ is equivalent to computing the Smith normal form of the matrix obtained from the generators and the relations. To be more precise, we carried out the following procedures and determined $H_n$, for $n = 5, \ldots, 7$

1. Generate all Gauss words whose rank is less than or equal to $n + 1$ excluding the empty word.
2. Remove words of the form $xAAy$ and words of rank $n + 1$ from the set of Gauss words generated at the first step.
3. Keep all Gauss words generated at the second step as a set of generators for $H_n$. 

(11)
(4) Generate all relations for the set of Gauss words generated at the first step; to be more precise, generate second type relations from Gauss words of the form $xAByBAz$ and generate third type relations from Gauss words of the form $xAByACzBCt$.

(5) Remove words of the form $xAAy$ and words of rank $n+1$ from the set of relations generated at the fourth step.

(6) Define a matrix from generators obtained at the third step and relations obtained at the fifth step.

(7) Compute the Smith normal form of the matrix by using row and column operations.

Smith normal form computation has two main difficulties. One is coefficient growth: the absolute value of a coefficient becomes larger and larger as the transformation proceeds. Such a large coefficient raises an overflow error on computation. Fortunately, Proposition 3.2 allows us to carry out the transformation on $\mathbb{Z}/(2^n-1)\mathbb{Z}$.

|   | number of generators | number of second type relations | number of third type relations | number of total unique relations | 
|---|---------------------|---------------------------------|-------------------------------|---------------------------------|
| $H_4$ | 42                  | 161                             | 62                            | 97                              |
| $H_5$ | 371                 | 1806                            | 672                           | 998                             |
| $H_6$ | 4026                | 23736                           | 8652                          | 12287                           |
| $H_7$ | 51870               | 358644                          | 128926                        | 176591                          |
| $H_8$ | 773185              | 6129164                         | 2181235                       | 2900594                         |

Table 1. Numbers of generators and relations for $H_n$. To obtain a group isomorphic to $H_n$, we need to transform a matrix into Smith normal form. The row size of the matrix is the number of generators and the column size is the number of total unique relations. Note that numbers of relations in the second and third columns count duplicated relations.

The other difficulty is “fill-in” on transformation of a sparse matrix, which occurs also on Gaussian elimination. The matrix obtained from our relations and generators is extremely sparse because the numbers of relations and generators are very large (Table 1) while the relations have at most only 8 terms. The sparsity of a matrix is getting lost gradually during the operations of the matrix. This is quite a difficult problem and has been studied as an elimination game of a chordal graph on graph theory [6]. In our case, because about 50% of relations have only one or two terms, we can delay fill-in by eliminating these relations at the beginning.

We consider the matrix size as a rough estimate of computation amount. The number of isomorphism classes of rank $n$ Gauss words is $(2n-1) \cdot (2n-3) \cdots 3 \cdot 1$. The number of generators excluding words of the form $xAAy$ is also multiplied by about $2n+1$ as the rank $n$ increases to $n+1$. The growth of the number of corresponding relations is of the same order as the number of generators. Then, the number of the nonzero matrix entries increases roughly at an order of $4n^2$.

Table 1 presents the actual number of generators and the number of relations in our computation.

The growth of computation for the rank prevented us from determining the group $H_8$; in fact, our computer program to obtain $H_7$ took more than one week and it probably takes hundreds of days to obtain $H_8$. To conclude this section, we give our computational result for $H_5$, $H_6$, and $H_7$ as a proposition.
Proposition 3.3. We have the following.

(12) \( G_0 \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^6 \oplus \mathbb{Z}/4\mathbb{Z} \)
(13) \( G_6 \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{32} \oplus (\mathbb{Z}/4\mathbb{Z})^6 \oplus \mathbb{Z}/8\mathbb{Z} \)
(14) \( G_7 \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{188} \oplus (\mathbb{Z}/4\mathbb{Z})^{32} \oplus (\mathbb{Z}/8\mathbb{Z})^6 \oplus \mathbb{Z}/16\mathbb{Z} \)

4. Simplified form of the universal invariant

Let \( Q_n = \{ w_1, \ldots, w_s \} \) be the finite set of isomorphism classes of Gauss words of rank \( n \) or less excluding the empty word and let an integer \( t \) be the number of relations. Then, \( \{ w_1, \ldots, w_s \} \) is the basis of \( \mathbb{Z}Q_n \) and \( \mathbb{Z}Q_n \) is isomorphic to \( \mathbb{Z}^s \).

Defining a matrix \( A \) from the generators of \( \mathbb{Z}Q_n \) and the relations, we transformed the matrix \( A \) into the Smith normal form \( S \) by using row operations \( U \) and column operations \( V \) in the last section. The matrix \( U \) induces an isomorphism between \( \mathbb{Z}^s/\text{Im } A \) and \( \mathbb{Z}^s/\text{Im } S \). From the definitions of \( H_n \) and \( A \), it holds that \( H_n \) is isomorphic to \( \mathbb{Z}^s/\text{Im } A \).

\[
\begin{array}{ccc}
\mathbb{Z}Q_n & \xrightarrow{\sim} & H_n \\
\xrightarrow{A} & \xrightarrow{\sim} & \xrightarrow{U} \\
\mathbb{Z}^s & \xrightarrow{V} & \mathbb{Z}^s/\text{Im } S \\
\end{array}
\]

Because the map \( \Gamma_n : \mathbb{Z}P \to H_n \) is a universal invariant of degree \( n \), so is the composite of \( \Gamma_n \) and the isomorphism \( H_n \cong \mathbb{Z}^s/\text{Im } S \). To exhibit this universal invariant, we need to determine the image of each element of \( Q_n \). Because the map

\[
U \circ \Gamma_n(p) = U \left( \sum_{i=1}^{s} (w_i, p)w_i \right) = \sum_{i=1}^{s} (w_i, p)U(w_i)
\]

gives the universal invariant from \( \mathbb{Z}P \) to \( \mathbb{Z}^s/\text{Im } S \), it is sufficient to calculate \( U(w_i) \) in order to obtain the explicit form.

Suppose that the Smith normal form \( S = UAV \) has the form \( [1] \), i.e., the elementary divisors of the matrix \( A \) are \( d_1, \ldots, d_l \). It follows from Proposition 3.2 that we have \( l = s \) and \( d_j = 2^{q_j} \) for some nonnegative integer \( q_j \) and \( j = 1, \ldots, s \). If \( d_j = 1 \) for all \( j \), then \( U(w_i) = 0 \) in \( \mathbb{Z}^s/\text{Im } S \) for all \( i \) and hence the invariant \( U \circ \Gamma_n \) is trivial. This situation occurs in the case of \( \Gamma_n \) for \( n = 0, \ldots, 3 \). We consider the case when \( d_{k-1} = 1 \) and \( d_k \neq 1 \). Under the identification \( \mathbb{Z}Q_n \cong \mathbb{Z}^s \), \( w_i \) is assumed to map to the \( i \)-th unit vector \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \), where \( v^T \) denotes the transpose of a vector \( v \). Let \( v_i \) be the \((s-k+1)\)-dimensional vector consisting of the last \( s-k+1 \) entries of \( Ue_i \) i.e., \( v_i = (x_{k+1}, \ldots, x_s)^T \). From the diagram \( [15] \), we see that \( U(w_i) \neq 0 \) in \( \mathbb{Z}^s/\text{Im } S \) if and only if \( v_i \neq 0 \) in \( \oplus_{j=k}^{s} \mathbb{Z}/d_j \mathbb{Z} \). Therefore, by adding subscript \( n \) to \( U \), \( s \), \( w_i \), and \( v_i \) in order to point out dependency on the degree \( n \) of the finite type invariant, we obtain the simplified numerical form \( \Gamma_n \) of the universal invariant \( \Gamma_n \):

\[
\Gamma_n(p) = U_n \circ \Gamma_n(p) = \sum_{i=1}^{s} (w_{n,i}, p)v_{n,i}.
\]

Note that the map \( \Gamma_n \) is a map from \( \mathbb{Z}P \) to \( \oplus_{j=k}^{s} \mathbb{Z}/d_j \mathbb{Z} \).
We obtained the universal finite type invariant for \( n = 4, 5 \) from our computation in this way. All words of rank 4 or less except the following words \( w_1, \ldots, w_6 \) map to 0 in \( \mathbb{Z}^s/\text{Im } S \) via \( U_4 \):

\[
\begin{align*}
  w_1 &= ABACDCBD, \quad w_2 = ABCACDBD, \\
  w_3 &= ABCADBDc, \quad w_4 = ABCBDACD, \\
  w_5 &= ABCDBDAC, \quad w_6 = ABCDCADB. 
\end{align*}
\]

Then, we obtain the explicit form of \( \bar{\Gamma}_4 \), which is the same as the invariant of Proposition 8.2 in [3].

**Proposition 4.1** (Gibson and Ito [3]). We define \( w_1, \ldots, w_6 \) by (18). The map \( \bar{\Gamma}_4 : \mathbb{Z} P \to \mathbb{Z}/2\mathbb{Z} \) defined by

\[
\bar{\Gamma}_4(p) = \left( \sum_{i=1}^{6} (w_i, p) \right) \mod 2
\]

is the universal finite type invariant of degree 4.

For \( n = 5 \) Table 2 shows pairs of words and nonzero vectors in \( (\mathbb{Z}/2\mathbb{Z})^6 \oplus \mathbb{Z}/4\mathbb{Z} \). This is enough to obtain \( \bar{\Gamma}_5 \).

**Proposition 4.2.** Let \( W_5 \) be the set of words appearing in Table 2 and \( v(w) \) be the corresponding vector in \( (\mathbb{Z}/2\mathbb{Z})^6 \oplus \mathbb{Z}/4\mathbb{Z} \) for \( w \) in \( W_5 \). The map \( \bar{\Gamma}_5 : \mathbb{Z} P \to (\mathbb{Z}/2\mathbb{Z})^6 \oplus \mathbb{Z}/4\mathbb{Z} \) defined by

\[
\bar{\Gamma}_5(p) = \sum_{w \in W_5} (w_i, p)v(w)
\]

is the universal finite type invariant of degree 5.

**Remark 4.3.** The expression of \( \Gamma_5 \) is not unique. Indeed, the values \( v(w) \) depends on the choice of an isomorphism \( H_5 \cong (\mathbb{Z}/2\mathbb{Z})^6 \oplus \mathbb{Z}/4\mathbb{Z} \).

5. Homotopy classification of Gauss words of rank 4 and 5

As an application of the simplified form of the universal invariant, we classify completely Gauss words of rank 4. We also apply the universal invariant of degree 6 to Gauss words of rank 5 and we obtain the classification with only one unclassified pair of two Gauss words. We recall moves derived from \( H_1, H_2, \) and \( H_3 \).

**Lemma 5.1** (Turaev [12]). The following pairs of words are in same homotopy class of words:

\[
\begin{align*}
  H_4: \ & xAByABz \leftrightarrow xyz, \\
  H_5: \ & xAByCAzBCt \leftrightarrow xBAyACzCBt, \\
  H_6: \ & xAByCAzCBt \leftrightarrow xBAyACzBCt, \\
  H_7: \ & xAByACzCBt \leftrightarrow xBAyCAzBCt, 
\end{align*}
\]

where \( x, y, z, \) and \( t \) are arbitrary words that can be empty words.

Here, we give the classification of Gauss words of rank 4 or less by using homotopy moves \( H_1 \) to \( H_7 \) and \( \bar{\Gamma}_5 \).

**Theorem 5.2.** Gauss words of rank 4 or less are classified into the following four homotopy classes:

\[
\begin{align*}
  & \{ ABACDCBD, ABCBDACD, ABCDCADB \}, \\
  & \{ ABCACDBD, ABCADBDC, ABCDBDAC \}, \\
  & \{ ABACBDCD \}, \text{ and} \\
  & \text{the set of other words}. 
\end{align*}
\]
Table 2. Words of rank 5 mapping to nonzero elements in \((\mathbb{Z}/2\mathbb{Z})^6 \oplus \mathbb{Z}/4\mathbb{Z}\) via the isomorphism \(U_5\) and their values. In Proposition 3.2 we let \(W_5\) be the set of words in the right column and \(v(w)\) be the vector in the left column corresponding to \(w\) in \(W_5\).
Proof. By using the moves from H1 to H7, we can easily show that words of rank at most 4 except for the seven words exhibited in the list of the statement are homotopic to the empty word. For the first two sets in the list, the elements in each set are homotopic to each other via the moves H3, H5, H6, or H7. Applying the universal finite type invariant $\bar{\Gamma}_5$, we have

\begin{align*}
\bar{\Gamma}_5(ABACDCBD) &= (0, 0, 0, 0, 0, 1), \\
\bar{\Gamma}_5(ABCACDBD) &= (0, 1, 1, 1, 1, 0, 3), \\
\bar{\Gamma}_5(ABACBDCD) &= (0, 1, 1, 1, 1, 0, 0), \\
\bar{\Gamma}_5(\emptyset) &= (0, 0, 0, 0, 0, 0, 0).
\end{align*}

Therefore, we see that these four sets are distinct homotopy classes. □

The universal invariant $\bar{\Gamma}_6$ obtained from our computation gives the following theorem, which shows a partial classification of Gauss words of rank 5 or less. We omit the specific description of $\bar{\Gamma}_6$ because it consists of 2545 correspondences between words and integer vectors.

**Theorem 5.3.** The universal finite type invariant $\bar{\Gamma}_6$ classifies Gauss words of rank 5 or less as shown in Table 3. This classification is complete except for the distinction between the two sets of words; the classification of the two sets $\{w_1, w_2\}$ and $\{w_3, w_4\}$ remains unknown, where

\begin{align*}
w_1 &= ABCDEBEACED, & w_2 &= ABCDECAEBD, \\
w_3 &= ABCADBECDE, & w_4 &= ABCACDEDBE,
\end{align*}

and $w_1$ is homotopic to $w_2$ and $w_3$ is homotopic to $w_4$.

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**References**

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Table 3. Partial classification of Gauss words of rank 5 or less under the universal invariant $\bar{\Gamma}_6$. Words excluded from the table are homotopic to the empty word or words of rank at most 4. The invariant $\bar{\Gamma}_6$ distinguishes words in a row from words in another row. The words in each row except the first row are homotopic to each other. The classification of the two sets in the first row remains unknown.
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