Towards a Gauge Invariant Scattering Theory of Cylindrical Gravitational Waves

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Chapter 1

Introduction

One of the most important and unresolved questions of canonical gravity is the so called “problem of time”. In quantum theory time is not regarded as an observable in the usual sense, since it is not represented by an operator. Time is rather treated as a parameter which, as in classical mechanics, describes the evolution of a system. In the time-dependent Schrödinger equation, for example, time occurs only as parameter and not as operator. In the conventional Copenhagen interpretation the notion of a measurement made at a particular time is a fundamental ingredient. An observable is something whose value can be measured at a fixed time.

In general relativity the role of time is very different. The four dimensional spacetime, i.e. a four dimensional manifold with a Lorentzian metric and an appropriate topology, can be foliated in many different ways as a one-parameter family of spacelike hypersurfaces. Each of these parameters can be viewed as a possible definition of time. There are many such foliations and there is no way to select a particular one to consider it as ‘natural’. Hence, in general relativity, time is not a fixed quantity, as it is in quantum theory. Time is dependent on the chosen foliation of the manifold and is closely linked to a fixed choice of the metric. Such a definition of time does not give any hint how it should be measured, and is so a non-physical quantity.

The gravity group of the institute of theoretical physics deals with asymptotically flat gravitational models. This kind of systems seems to remove the “problem of time” in the asymptotic, since the parameter of time can be chosen in a unique manner. The dynamics of the system can be determined, and one can attempt to study theories like the scattering one in a quantum theoretical way. The hope is to understand the structure of the theory of such models in order to be able to state something in the direction of a quantum gravity.

In this diploma thesis such an asymptotically flat model is investigated. The model I have chosen in this paper is pure gravity - no matter fields are present - with cylindrical symmetry. The symmetries of the cylindrical gravitational wave help to solve the Einstein equations and thus, to know the metric which describes spacetime of the treated toy model. The cylindrically symmetric gravitational wave, also called Einstein-Rosen wave, was discovered by A. Einstein and N. Rosen [1]. To give an idea what gravitational waves are we cite [2]: Just as one identifies as water waves small ripples rolling across the ocean,
so one gives the name gravitational waves to small ripples rolling across spacetime. In the seventies of the last century this field theory was again picked up by K. Kuchař [3]. The model was solved in the canonical ADM formalism and afterwards quantized. Twenty years later, C.G. Torre [4] found a complete set of observables of the cylindrically symmetric gravitational wave.

This work is mainly based on the contributions mentioned above. In chapter two the general cylindrical symmetric metric is studied and mathematical simplifications of it are discussed in detail. Then the Einstein-Rosen metric is introduced and the corresponding Einstein equations are derived. In the third chapter, we determine the solution of the Einstein equations and check the boundary conditions the solving function is supposed to meet. In the following chapter the ADM formalism is presented, firstly for the general spacetime then for the cylindrically symmetric one and finally for the Einstein-Rosen wave, which is the interesting case for our consideration. The fifth chapter is devoted to the derivation of the dynamical variables by solving the Hamilton equation. Moreover it is shown how in Torre’s work the set of observables for the cylindrical gravitational waves is obtained. In the last chapter the condition is investigated, which phase space functions have to fulfill in order to be declared as observables. Then the Poisson algebra of the observables is studied and quantum theoretical conclusions to these quantities are stated. In the appendix a detailed derivation of the Hankel transformation is noted, as well as the orthonormality relation of the Bessel functions of the first kind. This relation plays an important role in the mathematical part of this diploma thesis. Lengthy computations and verifications have also been displaced into the appendix in order to preserve the overlook while reading.

Conventions and notations:
Spacetime has the signature $S = 2$, so the signs of the diagonal elements of the Lorentzian metric are $\left(\mathbf{-}, +, +, +\right)$. We use natural units that means $\hbar = c = 16\pi G = 1$. In order to distinguish clearly the coordinate $R$ from the Ricci scalar $R$, we write throughout the coordinate in the italic style and the curvature scalar in the Roman style. The Greek indices run over $0, 1, 2, 3$ while the Latin indices only over $1, 2, 3$, corresponding to the three space coordinates. Derivatives in general are denoted as follows: $\frac{\partial}{\partial T} f(T, R) \equiv f_T, \frac{\partial^2}{\partial T^2} f(T, R) \equiv f_{TT}$. In particular the derivative with respect to the time $t$ is denoted with a dot, the one with respect to the radius $r$ with a prime. All these conventions hold unless otherwise specified.
Chapter 2

Structure of the Cylindrical Spacetime

2.1 Line Element with Cylindrical Symmetry

A method to quantize a gravitational system is to foliate the four dimensional spacetime in a (3+1)-dimensional general relativity and subsequently to reduce it to an equivalent (1+1)-dimensional model, which is coupled to a massless scalar field. An example therefore is the cylindrically symmetric Einstein-Rosen wave. In this section we are going to study the line element for the mentioned kind of gravitational waves.

We start with considering the line element, which defines cylindrical spacetime, in its most general form. Afterwards we will deduce from the general form, the line element in the Einstein-Rosen coordinates through conformal deformations and coordinate transformations. First of all, we take for convenience the cylindrical coordinate system to describe points in spacetime, namely the set of variables \((t, r, ϕ, z)\) with \(t ∈ (−∞, ∞), \ r ∈ [0, ∞), \ ϕ ∈ [0, 2π)\) and \(z ∈ (−∞, ∞)\). The most general line element with cylindrical symmetry reads:

\[
ds^2 = g_{αβ} x^α x^β = g_{00} \ dt^2 + 2 \ g_{01} \ dt \ dr + g_{11} \ dr^2 + g_{22} \ dϕ^2 + g_{33} \ dz^2, \tag{2.1}
\]

with obviously

\[
x^0 = t, \quad x^1 = r, \quad x^2 = ϕ, \quad x^3 = z. \tag{2.2}
\]

To get a cylindrically symmetric line element the coefficients of the metric \(g_{αβ}\) has to depend only on the coordinates \(t\) and \(r\). As we are going to see, only in that case the line element is invariant with respect to changes of the coordinates \(ϕ\) and \(z\). So the metric is a function of the two coordinates \(t\) and \(r\), and of the form

\[
g_{αβ}(t, r) = \begin{pmatrix}
g_{00}(t, r) & g_{01}(t, r) & 0 & 0 \\
g_{10}(t, r) & g_{11}(t, r) & 0 & 0 \\
0 & 0 & g_{22}(t, r) & 0 \\
0 & 0 & 0 & g_{33}(t, r)
\end{pmatrix}, \tag{2.3}
\]
where, due to the symmetry property of both indices of the metric tensor \( g_{\alpha\beta} \), \( g_{01}(t, r) = g_{10}(t, r) \). After these settings, we now derive the symmetries of spacetime by analyzing the line element (2.1). For this inspection we define the z-axis as the symmetry axis for the following four symmetries:

1. Rotations around the symmetry axis \( z \): \( \varphi \rightarrow \varphi' = \varphi + \Delta \varphi \).
2. Translations in direction of the \( z \)-axis: \( z \rightarrow z' = z + \Delta z \).
3. Reflections in all surfaces containing the symmetry axis: \( \varphi \rightarrow \varphi' = -\varphi \).
4. Reflections in all surfaces perpendicular to the \( z \)-axis: \( z \rightarrow z' = -z \).

If we substitute in the line element \( z \) with \( z' \) and \( \varphi \) with \( \varphi' \) the line element will not change. This is the reason why we have pretended \( g_{\alpha\beta} \) to depend only on \( t \) and \( r \), otherwise the line element would change because of the substitutions. Seeing that, \( ds^2 \) is invariant under the four transformations noted above, just the four symmetries. The line element is symmetric to the following two dimensional transformation group:

\[
\begin{align*}
\varphi & \rightarrow \varphi' = \varphi + \Delta \varphi \quad & \text{(rotation around } z \text{-axis)} \\
 z & \rightarrow z' = z + \Delta z \quad & \text{(translation in direction of the } z \text{-axis)}
\end{align*}
\]

All transformations of such a group are called isometries. Each of these isometries is generated by the related Killing vector field, for the cylindrical coordinates \((t, r, \varphi, z)\) the two orthogonal vectors

\[
\xi^\alpha_{(\varphi)} = (0, 0, 1, 0), \quad \xi^\alpha_{(z)} = (0, 0, 0, 1),
\]

which are determined by the Killing equation

\[
g_{\alpha\beta,\nu} \xi^\nu_{\alpha} + g_{\nu\beta} \xi^\nu_{\alpha} + g_{\alpha\nu} \xi^\nu_{\beta} = 0 \tag{2.7}
\]

or equivalently

\[
\partial_\kappa g_{\rho\sigma} = 0. \tag{2.8}
\]

The two isometries and consequently both Killing fields implicate conservation laws of the dynamic. Indeed the cylindrical gravitational fields are characterized by the existence of a two-parameter Abelian group of motions \( \mathbb{G}_2 \) with both, to each other orthogonal, hyper-surface-orthogonal spacelike Killing vectors \( \xi^\nu_{(\varphi)} \), \( \xi^\nu_{(z)} \).

### 2.2 Simplification of the Cylindrically Symmetric Metric

In order to obtain solutions of the Einstein equations written for the general cylindrically symmetric metric (2.3), we have to simplify it without losing physically relevant
2.2 Simplification of the Cylindrically Symmetric Metric

properties. The method, which in our case is going to be applied, consists in deforming conformally a part of the four dimensional cylindrical metric (2.3), and subsequently to find a coordinate transformation, which leads into an inertial system IS. Also we will see that the mentioned coordinate transformation brings the metric, dependent on the new coordinates, into diagonal form.

2.2.1 Conformal Deformation

In the beginning, we would like to start with defining the conformal deformation in space time as follows:

\[ g_{\alpha\beta} = \Omega^{-2} g'_{\alpha\beta}. \]  

(2.9)

The function \( \Omega = \Omega(t, r) \) is a real and positive scalar function. We consider now a conformal deformation in two dimensions. In this case the metric \( g_{\lambda\kappa}, \) with \( \lambda, \kappa = 0, 1 \) labeling \( (t, r), \) corresponds to a regular \( 2 \times 2 \) matrix. From this it follows that

\[ g_{\lambda\kappa} g^{\lambda\kappa} = \delta^\lambda_\lambda \delta^\kappa_\kappa = \sum_0^1 \delta^\lambda_\lambda = 2, \]

(2.10)

where in section 2.2 \( \lambda, \kappa, \rho, \sigma, \mu \) and \( \nu \) run over 0 and 1. Then we are going to show that there is always a conform deforming factor such that the metric \( g_{\lambda\kappa} \) can be expressed through a flat metric \( g'_{\lambda\kappa}. \)

**Theorem:** Every two dimensional metric \( g_{\lambda\kappa} \) is conformally flat.

**Proof:** We begin with affiliating the transformation law between the Ricci scalar of a conformally deformed metric, \( R[g'_{\lambda\kappa}], \) and the curvature scalar of the original one, \( R[g_{\lambda\kappa}]. \) From the definition of the conformal deformation (2.9) it follows:

\[ R'_{\lambda\kappa} = \Omega^2 g_{\lambda\kappa}. \]

(2.11)

The Ricci scalar \( R \) is defined by

\[ R = g^{\lambda\kappa} R_{\lambda\kappa} = g^{\lambda\kappa} R^\rho_{\lambda\rho\kappa}, \]

(2.12)

where \( R_{\lambda\kappa} \) is the Ricci tensor and \( R^\rho_{\lambda\rho\kappa} \) the contracted Riemann tensor. The transformed curvature scalar reads:

\[ R'_{\lambda\kappa} = g'^{\lambda\kappa} R'_{\lambda\kappa} = g'^{\lambda\kappa} R'^\rho_{\lambda\rho\kappa}. \]

(2.13)

If we put in (2.13) instead of \( g'_{\lambda\kappa} \) consequently \( \Omega^2 g_{\lambda\kappa} \) and write down the transformed Ricci scalar \( R' \), we get

\[ R' = \Omega^2 g_{\lambda\kappa} \left[ \partial_\rho \Gamma^\rho_{\lambda\kappa} - \partial_\kappa \Gamma^\rho_{\lambda\rho} + \Gamma^\rho_{\nu\rho} \Gamma^{\nu}_{\lambda\kappa} - \Gamma^\rho_{\nu\kappa} \Gamma^{\nu}_{\lambda\rho} \right]. \]

(2.14)

Evidently, the Christoffel symbol

\[ \Gamma^\rho_{\lambda\kappa} = \frac{1}{2} g^{\rho\sigma} [g_{\sigma\kappa, \lambda} + g_{\sigma\lambda, \kappa} - g_{\lambda\kappa, \sigma}] \]

(2.15)
has to be transformed. Its transformation law is given by utilizing (2.11), as done for the
curvature scalar (2.12).

\[
\Gamma'_{\lambda\kappa}^{\rho} = \frac{1}{2} \Omega^2 g^{\rho\sigma} \left[ \left( \Omega^{-2} g_{\sigma\kappa} \right) \Lambda_{,\lambda} + \left( \Omega^{-2} g_{\sigma\lambda} \right) \Lambda_{,\kappa} - \left( \Omega^{-2} g_{\lambda\kappa} \right) \Lambda_{,\sigma} \right].
\]  

(2.16)

If one performs the partial derivatives and the relation

\[
\ln \Omega_{,\rho} = (\ln \Omega)_{,\rho} = \Omega_{,\rho} \Omega
\]

is employed, then the transformed Christoffel symbol is obtained:

\[
\Gamma'_{\lambda\kappa}^{\rho} = \Gamma_{\lambda\kappa}^{\rho} - \ln \Omega_{,\lambda} \delta_{\kappa}^{\rho} - \ln \Omega_{,\kappa} \delta_{\lambda}^{\rho} + \ln \Omega_{,\sigma} g_{\lambda\kappa} g^{\rho\sigma}.
\]  

(2.18)

This intermediate result is introduced in (2.14) and the transformation law for the Ricci
scalar is given:

\[
R'[g'_{\lambda\kappa}] = \Omega^2 R[g_{\lambda\kappa}] + \Omega^2 \left( 2 \partial_{\kappa} \partial^{\kappa} \ln \Omega + 2 \Gamma^\kappa_{\kappa} \ln \Omega_{,\rho} \right).
\]  

(2.19)

To this end we have used the fact that the covariant partial derivative of a scalar field
(scalar function) is a covariant vector field.

\[
V_{\rho} \equiv \Phi_{,\rho} = \partial_{\rho} \Phi = \ln \Omega_{,\rho}.
\]  

(2.20)

From this it follows the repeatedly applied equation

\[
g^{\lambda\kappa} \partial_{\kappa} V_{\lambda} = g^{\lambda\kappa} \left( \partial_{\kappa} V_{\lambda} \right)
\]

\[
= g^{\lambda\kappa} \partial_{\kappa} \left( V^{\mu} g_{\mu\lambda} \right)
\]

\[
= \delta^{\kappa}_{\mu} \partial_{\kappa} V^{\mu} + V^{\mu} g^{\lambda\kappa} g_{\mu\lambda,\kappa}
\]

\[
= \partial_{\kappa} V^{\kappa} + V^{\mu} g^{\lambda\kappa} g_{\mu\lambda,\kappa}.
\]  

(2.21)

Further more the subsequent expression is relevant also:

\[
2 V^{\rho} \Gamma^\lambda_{\rho\lambda} = g^{\lambda\kappa} g_{\lambda\kappa,\rho} V^{\rho}.
\]  

(2.22)

In case that \( \Omega \) is a scalar field, the right hand side of the equation (2.19) can be noted
more compactly. So, we apply the (two dimensional) Laplacian operator \( \triangle \) onto a scalar
field in the two dimensional spacetime.

\[
\triangle \phi = \nabla_{\kappa} \left( g^{\lambda\kappa} \nabla_{\lambda} \phi \right)
\]

\[
= \nabla_{\kappa} V^{\kappa}
\]

\[
= \partial_{\kappa} V^{\kappa} + \Gamma^\kappa_{\rho\kappa} V^{\rho}
\]

\[
= \partial_{\kappa} \partial^{\kappa} \Phi + \Gamma^\kappa_{\rho\kappa} \partial^{\rho} \Phi,
\]  

(2.23)

whereby we know that the covariant derivative of a scalar field is equal to the covariant
partial derivative of the same. From (2.20) we see that \( \Phi = \ln \Omega \), and by putting it into
(2.19), we achieve that the transformation law for the Ricci scalar reduces to the following compact formula:

\[ R'[g'_{\lambda\kappa}] = \Omega^2 R[g_{\lambda\kappa}] + 2 \Omega^2 \nabla \ln \Omega. \]  

(2.24)

Now we set the two dimensional conformally deformed metric \( g'_{\lambda\kappa} \) to be flat. In this case the transformed Ricci scalar \( R'[g'_{\lambda\kappa}] \) vanishes. From the transformation law (2.24) it follows then:

\[ \nabla \ln \Omega + \frac{1}{2} R[g_{\lambda\kappa}] = 0 \]  

(2.25)

This equation is the known generalized inhomogeneous Klein-Gordon differential equation for the massless scalar field \( \Phi = \ln \Omega \). This kind of differential equation can be worked out by performing a change of variables to the double null coordinates:

\[ u = t + r, \quad v = t - r. \]  

(2.26)

In [3] it is shown that by the substitution to the double null coordinates the differential equation (2.25) possesses for all regular \( \Omega > 0 \) a solution. So it exists throughout a scalar function \( \Omega \) such that the Klein-Gordon equation is fulfilled. Thus, there is a conformal deforming scalar function \( \Omega \) in such a way that the metric \( g_{\lambda\kappa} \) is expressed by a flat metric \( g'_{\lambda\kappa} \).

### 2.2.2 Coordinate Transformation into the Inertial System

In the previous paragraph it has been showed that the \((t, r)\)-part of the cylindrically symmetric line element is conformally flat. By virtue of this property it can be expressed by a two dimensional flat metric.

\[ g_{\lambda\kappa} \, dx^\lambda \, dx^\kappa = \Omega^{-2} g'_{\lambda\kappa} \, dx^\lambda \, dx^\kappa, \]  

(2.27)

\[ g'_{00} = \Omega^{-2} g_{00}, \quad g'_{01} = \Omega^{-2} g_{01}, \quad g'_{11} = \Omega^{-2} g_{11}. \]  

(2.28)

The scalar function is chosen such that \( R'[g'_{\lambda\kappa}] = 0 \) holds, in order the metric \( g'_{\lambda\kappa} \) to be flat. But by then, the four dimensional metric (2.3) has not been simplified really, since the number of its non-vanishing components did not decrease. Such a simplification would be an advantage to find solutions to the vacuum Einstein equations, as the number of vanishing components would rise. At this place it is again important to remind that the physical model has not to be trivialized by the possible elimination of some components of the Einstein equations. The issue is only to obviate the mathematical impediments by a suitable choice of coordinates.

The theorem about the conformal flatness of two dimensional metrics just allow us to find a global coordinate transformation to an inertial system, as the conformally flat metric can be described by a flat metric \( g'_{\lambda\kappa} \). Now the importance of the proved theorem appears evidently on the score of the suited simplification of the Einstein equations by reducing the number of non-vanishing components of the four dimensional metric (2.3).
the coordinates \( \bar{t} \) and \( \bar{r} \) of the inertial system the conformally deformed two dimensional \((t, r)\)-part of (2.3), namely \( g'_{\lambda\kappa} \), just transmute to the Minkowskian standard metric:

\[
g'_{\lambda\kappa}(t, r) = \begin{pmatrix}
g_{00} & g_{01} \\
g_{01} & g_{11}
\end{pmatrix}
\]

\[
\text{coordinate transformation } \varphi \quad \eta_{\lambda\kappa}(\bar{t}, \bar{r}) = \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}.
\]

(2.29)

Through the conformal deformation \( \Omega \) and the coordinate transformation \( \varphi \) we get from the conformally flat metric \( g_{\lambda\kappa}(t, r) \) a diagonal matrix, namely

\[
g_{\lambda\kappa}(t, r) \quad \Omega, \varphi \quad \bar{g}_{\lambda\kappa}(\bar{t}, \bar{r}) = \bar{\Omega}^2(\bar{t}, \bar{r}) \eta_{\lambda\kappa}(\bar{t}, \bar{r}) = \begin{pmatrix}
-\bar{\Omega}^2 & 0 \\
0 & \bar{\Omega}^2
\end{pmatrix}.
\]

(2.30)

The conformal deformation and the coordinate transformation simplify the exclusively on \( t \) and \( r \) depending four dimensional metric (2.3) significantly, as it is brought into diagonal form without changing anything on the physical aspect. The four dimensional metric reads newly:

\[
\bar{g}_{\alpha\beta}(\bar{t}, \bar{r}) = \begin{pmatrix}
-\bar{\Omega}^2 & 0 & 0 & 0 \\
0 & \bar{\Omega}^2 & 0 & 0 \\
0 & 0 & g_{22}(\bar{t}, \bar{r}) & 0 \\
0 & 0 & 0 & g_{33}(\bar{t}, \bar{r})
\end{pmatrix},
\]

(2.31)

with the knowledge that \( g_{22}(\bar{t}, \bar{r}) \) and \( g_{33}(\bar{t}, \bar{r}) \) are invariant with respect to the coordinate transformation. This fact is shown to hold in equation (2.41) if we are allowed to anticipate \( g_{22}(\bar{t}, \bar{r}) = R^2 e^{-\psi} \) and \( g_{33}(\bar{t}, \bar{r}) = e^{\psi} \) in case of the Einstein-Rosen spacetime.

At last we would like to add an illustration for a better intuitively understanding of the coordinate transformation into the inertial system. The starting point is a flat geometry, which is represented by the so-called flat metric \( g'_{\lambda\kappa} \) in (2.29). However we probably would not assert on the basis of the treated metric that a two dimensional flat geometry is described, since for this case we rather would expect the Minkowskian standard metric. Nevertheless, we get zero by calculating the Ricci scalar of \( g'_{\lambda\kappa} \); then we say the metric is flat. How should we understand intuitively this apparent paradox? To this end, we consider a flat disk at rest in an inertial system. With respect to the coordinates \((\bar{t}, \bar{r})\) of an inertial system, an observer A measures the length \( C = 2\pi \) for the circumference of the disk with the unit radius. Now we make the assumption that an other observer B sizes the disk with respect to a coordinate system rotating around the symmetry axis, which intersects the middle point of the disk. We emphasize that the rotating coordinate system is accelerated. The observer B also measures the radius to be unit, but a circumference \( C' \) smaller than \( 2\pi \). This is an effect due to the Lorentz contraction. So the observer B, confident in his ratios, would not assert the measured surface to be a disk, but rather a paraboloid. Thus, he would confirm a curved geometry and consequently a two dimensional metric, which is different from the Minkowskian standard metric, presumably with four non-vanishing components. However the Ricci scalar of his metric is zero implicating a flat geometry.

In a certain sense, he has applied for his gaging the “wrong” coordinate system, namely an accelerated one, which intuitively speaking, distorts the geometry. So it is recommendable
to undertake a coordinate transformation to the inertial system to see the geometry the “right” way. That the Ricci scalar of a flat metric vanishes with respect to all coordinate systems is not surprisingly, since a scalar is independent on coordinates. That is the reason why also observer B has to get a vanishing Ricci scalar for his metric.

By means of this Gedankenexperiment, we see that the physical model is not influenced at all by a coordinate transformation. In fact the properties of the two dimensional geometry are not changed, it remains flat with respect to all coordinate systems.

2.3 Einstein-Rosen Metric

After the introductory sections, we shall consider more concretely the real subject of our study, namely the Einstein-Rosen gravitational waves. The utilized metric for this model is the Einstein-Rosen metric, namely

$$g_{\alpha \beta}^{ER}(T, R) = \begin{pmatrix} -e^{\Gamma - \psi} & 0 & 0 & 0 \\ 0 & e^{\Gamma - \psi} & 0 & 0 \\ 0 & 0 & R^2 e^{-\psi} & 0 \\ 0 & 0 & 0 & e^\psi \end{pmatrix}.$$ (2.32)

The functions $\Gamma$ and $\psi$ are dependent on both Einstein-Rosen coordinates $T$ and $R$ exclusively. The complete coordinate system of course is eked by the other two spatial coordinates $\varphi$ and $z$. As we notice, the Einstein-Rosen metric is cylindrically symmetric, since it is a special case of the general cylindrically symmetric metric (2.3). The metric (2.32) depends on the time coordinate $T \in [0, \infty)$ and on the spatial coordinate $R \in [0, \infty)$ only. Just this requirement was the second requisite in section 2.1 a metric is supposed to possess in order to be cylindrically symmetric. Thus, the Einstein-Rosen metric inherits all properties of cylindrical metrics and which are listed in section 1.1.

In the following it is showed, how one gets from the general cylindrically symmetric metric (2.3) the Einstein-Rosen metric, which is the important special case for our purpose. If we set in the general metric (2.3)

$$g_{22}(t, r) = R^2 e^{-\psi}, \quad g_{33}(t, r) = e^\psi,$$ (2.33)

we remark that only the two dimensional $(t, r)$-part

$$g_{\lambda \kappa}(t, r) = \begin{pmatrix} g_{00}(t, r) & g_{01}(t, r) \\ g_{01}(t, r) & g_{11}(t, r) \end{pmatrix}$$ (2.34)

has to be brought into the diagonal form

$$g_{\lambda \kappa}^{ER}(T, R) = \begin{pmatrix} -e^{\Gamma - \psi} & 0 \\ 0 & e^{\Gamma - \psi} \end{pmatrix} = e^{\Gamma - \psi} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$ (2.35)

\footnote{Compare with equation (2) in section II of [3]}
where \( \lambda \) and \( \kappa \) run again over 0 and 1. In the original coordinates \((t, r, \varphi, z)\) the functions \( R \) and \( \psi \) are dependent on \( t \) and \( r \). In Kuchař’s paper [3] both functions are defined by the Killing vectors (2.6):

\[
R(t, r) \doteq \sqrt{\xi^\alpha_{\varphi} \xi_{\alpha(\varphi)} \xi^\beta_{(z)} \xi_{\beta(z)}}
\]

(2.36)

\[
\psi(t, r) \doteq \ln(\xi^\alpha_{(z)} \xi_{\alpha(z)}).
\]

(2.37)

If the right hand side of both equations (2.36) and (2.37) are calculated by means of both Killing vectors and the metric

\[
g_{\lambda\kappa}(t, r) = \begin{pmatrix}
g_{00} & g_{01} & 0 & 0 \\
g_{01} & g_{11} & 0 & 0 \\
0 & 0 & R^2 e^{-\psi} & 0 \\
0 & 0 & 0 & e^\psi
\end{pmatrix},
\]

(2.38)

then the functions are got. Now we bring \( g_{\lambda\kappa}(t, r) \) into diagonal form. The metric is two dimensional and dependent only on \( t \) and \( r \). After we have learned in section 2.2 how this kind of metric can be simplified, the following plan appears to be suited:

1. We find a scalar function \( \Omega \) such that the metric \( g_{\lambda\kappa} \) can be expressed by a flat metric \( g'_{\lambda\kappa} \). To this end the Klein-Gordon differential equation for the massless scalar field \( \Omega \) (2.25) must be solved.

2. By a general coordinate transformation \( \varphi \), a coordinate system \((\bar{t}, \bar{r})\) of an inertial system IS is found, wherein the flat metric \( g'_{\lambda\kappa}(\bar{t}, \bar{r}) \) assumes the Minkowskian standard form.

The result of this procedure is the metric which is conformally flat in the \((t, r)\)-part. This metric is noted in the second section of Kuchař’s paper [3],

\[
\bar{g}_{\alpha\beta}(\bar{t}, \bar{r}) = \begin{pmatrix}
-e^{\bar{\gamma}-\psi} & 0 & 0 & 0 \\
0 & e^{\bar{\gamma}-\psi} & 0 & 0 \\
0 & 0 & R^2 e^{-\psi} & 0 \\
0 & 0 & 0 & e^\psi
\end{pmatrix},
\]

(2.39)

for the case that the solution of the Klein-Gordon differential equation (2.25) is the subsequently conform deforming factor:

\[
\Omega^2 = e^{\bar{\gamma}-\psi},
\]

(2.40)

\[
\bar{\gamma} = \bar{\gamma}(\bar{t}, \bar{r}), \quad \psi = \bar{\psi}(\bar{t}, \bar{r}) = \psi(t, r).
\]

(2.41)

The functions \( R \) and \( \psi \) do not change because of the coordinate transformation \( \varphi \), since they behave like scalars\(^2\), as one can take from the equations (2.36) and (2.37).

\(^2\)Now the provision we made for the metric (2.31) is justifiable in virtue of the behavior of \( R \) and \( \psi \) with respect to coordinate transformations.
We shortly focus on the Einstein-Rosen coordinate \( R \). To develop a more concrete vision, as also to convince us that the Einstein-Rosen coordinate is a kind of radius in the cylindrical space time, we add a short illustration. We concentrate on a two-dimensional cylindrical surface \( t = \text{const.}, r = \text{const.} \) around the axis of symmetry. Let be the height of the cylindrical surface \( \Delta z = 1 \). Thus the area of the described surface is \( 2\pi R \), the same as the area for the surface of a cylinder with height one and radius \( R \) in Euclidean space. So we can identify \( R \) in (2.31) with a kind of radius in the cylindrical spacetime. To make sure that this quantity is spacelike, a special regard should be given to the vector \( R_{,\nu}(t, r) \), which can be spacelike \( (R_{,\nu} R_{,\nu} > 0) \), timelike \( (R_{,\nu} R_{,\nu} < 0) \) or lightlike \( (R_{,\nu} R_{,\nu} = 0) \). In order to deal with the model of cylindrically symmetric gravitational waves, we are forced to choose the vector \( R_{,\nu}(t, r) \) spacelike everywhere in spacetime. Of course, it would be quite senseless to speak of a timelike or a lightlike radius vector, at least if the coordinate \( R \) is supposed to form with the timelike Einstein-Rosen coordinate \( T \) an orthogonal coordinate system.

### 2.3.1 Conformal Coordinate Transformation

Comparing the metric \( \bar{g}_{\alpha \beta} (\bar{t}, \bar{r}) \) with the Einstein-Rosen metric \( g^{ER}_{\alpha \beta}(T, R) \), we notice that a further simplification of the first metric is possible. By a coordinate transformation \( \vartheta \) into the coordinate system \( (T, R, \varphi, z) \), the function \( R(\bar{t}, \bar{r}) \) can be converted into a coordinate, namely one of the Einstein-Rosen coordinates. Looking at both metrics above, we remark that besides a further conformal deformation in the \((t, r)\)-part just a change of coordinates transports the one metric in the other (please notice \( \lambda, \kappa = 0, 1 \)). Then the functions \( T(\bar{t}, \bar{r}) \) and \( R(\bar{t}, \bar{r}) \) adopt the role of coordinates \( T \) and \( R \) in the orthogonal Einstein-Rosen coordinate system \( (T, R, \varphi, z) \). While in these coordinates the function \( \psi(t, r) \) will not change as it is a scalar, the function \( \gamma(\bar{t}, \bar{r}) \), on the contrary, will change due to the coordinate transformation, according to the relation,

\[
\gamma(\bar{t}, \bar{r}) \rightarrow \Gamma(T, R) = \gamma - \ln \left( R'^{2} - T'^{2} \right),
\]

where the primes are in this case derivatives with respect to \( \bar{r} \). Since the change of coordinates influences the time and the radius, we require the following connection between the \((t, r)\)-parts of the considered metrics:

\[
-d\bar{t}^{2} + d\bar{r}^{2} = \bar{\Omega}^{-2} (-dT^{2} + dR^{2}).
\]

Thereby it is not about a general coordinate transformation, but about a so-called conformal coordinate transformation. From this requirement we are going to get an additional condition on the coordinates \( T \) and \( R \). We compute the total differential of the functions \( T(\bar{t}, \bar{r}) \) and \( R(\bar{t}, \bar{r}) \) and set them in the right hand side of equation (2.44) after multiplying with \( \bar{\Omega}^{2} \). So we get:

\[
\bar{\Omega}^{2} (-d\bar{t}^{2} + d\bar{r}^{2}) = \quad -(T_{,\bar{t}}^{2} - R_{,\bar{t}}^{2}) d\bar{t} + (R_{,\bar{r}}^{2} - T_{,\bar{r}}^{2}) d\bar{r}^{2} - T_{,\bar{t}} R_{,\bar{r}} d\bar{t} d\bar{r}. \]
The equation (2.45) holds only if one of the following two settings is applied:

\[ T, \bar{t} = R, \bar{r}, \quad T, \bar{r} = R, \bar{t} \]  \hspace{1cm} (2.46)

\[ T, \bar{t} = -R, \bar{r}, \quad T, \bar{r} = -R, \bar{t} \]  \hspace{1cm} (2.47)

We have assumed that the order of the partial derivatives is commutable, what for coordinate transformations surely apply. Derivating partially the first equation in (2.46) by \( \bar{t} \) and the second in (2.46) by \( \bar{r} \), and then subtracting the second result from the first, we get

\[ \Box T(\bar{t}, \bar{r}) \doteq T,_{\bar{t}\bar{t}} - T,_{\bar{r}\bar{r}} = 0. \]  \hspace{1cm} (2.48)

Derivating partially the first equation in (2.46) by \( \bar{r} \) and the second in (2.46) by \( \bar{t} \), and then subtracting the first result from the second, we get

\[ \Box R(\bar{t}, \bar{r}) \doteq R,_{\bar{t}\bar{t}} - R,_{\bar{r}\bar{r}} = 0. \]  \hspace{1cm} (2.49)

So, the Einstein-Rosen coordinates \( T \) and \( R \) not only build by construction an orthogonal system and are timelike respectively spacelike, but also they have to be harmonically with respect to the coordinates \( \bar{t} \) and \( \bar{r} \). Due to the invariance of \( R \) regarding coordinate transformations, they are harmonically with respect to the original coordinates \( t \) and \( r \), too. Of course the coordinate \( T \) is also a scalar and therefore invariant to an arbitrary change of coordinates, as the quantity follows by integrating the equations (2.46) or (2.47).

In order to verify whether the function \( R(\bar{t}, \bar{r}) \) is harmonic, we write the Einstein equations for the metric (2.39):

\[ \tilde{G}_{\alpha\beta} = \bar{R}_{\alpha\beta} - \frac{1}{2} \bar{R} \bar{g}_{\alpha\beta} = 0 \]  \hspace{1cm} (2.50)

By performing the Ricci tensor \( \bar{R}_{\alpha\beta} \) and the Ricci scalar \( \bar{R} \) for the metric \( \bar{g}_{\alpha\beta} \), we obtain the following, for our consideration relevant, components:

\[ \tilde{G}_{22} = R(\bar{\gamma}_{,\bar{t}\bar{t}} - \bar{\gamma}_{,\bar{r}\bar{r}}) + \frac{1}{2} R(\psi^2 - \psi^2_{,\bar{r}}) = 0 \]  \hspace{1cm} (2.51)

\[ \tilde{G}_{33} = R(\bar{\gamma}_{,\bar{t}\bar{t}} - \bar{\gamma}_{,\bar{r}\bar{r}}) + \frac{1}{2} R(\psi^2 - \psi^2_{,\bar{r}}) + 2(R,_{\bar{t}\bar{t}} - R,_{\bar{r}\bar{r}}) = 0 \]  \hspace{1cm} (2.52)

Equation (2.49) just appears by subtracting \( \tilde{G}_{33} \) from \( \tilde{G}_{22} \).

\[ R,_{\bar{t}\bar{t}} - R,_{\bar{r}\bar{r}} = 0. \]  \hspace{1cm} (2.53)

We assume the derivative of the Einstein-Rosen coordinate \( R \) to be by construction a spacelike vector \( (R,_{\nu}, R,_{\nu'} > 0) \) and the coordinate itself to be perpendicularly on the time coordinate \( T \). Then \( R \) is fit to be Einstein coordinate, since it is showed it to be harmonic with respect as well to the variables \( (\bar{t}, \bar{r}) \) as to \( (t, r) \). At last we would like to note both Einstein-Rosen coordinates \( T \) and \( R \). To this end we solve both D’Alembert equations (2.48) and (2.49) for one time dimension and one space dimension. This sort of differential

\[ ^3 \text{In the subsequent section the way how the tensor and the scalar are performed is explained in a more detailed way.} \]
equation can be worked out by writing them in double null coordinates, i.e. the following substitution of variables is suggested:

\[
\bar{u} = \bar{t} + \bar{r} \tag{2.54}
\]
\[
\bar{v} = \bar{t} - \bar{r}. \tag{2.55}
\]

If we note the differential operators in terms of the new coordinates, then we get for the time and radius derivatives

\[
\frac{\partial}{\partial \bar{t}} = \frac{\partial \bar{u}}{\partial \bar{t}} \frac{\partial}{\partial \bar{u}} + \frac{\partial \bar{v}}{\partial \bar{t}} \frac{\partial}{\partial \bar{v}} = \frac{\partial}{\partial \bar{u}} + \frac{\partial}{\partial \bar{v}}, \tag{2.56}
\]
\[
\frac{\partial}{\partial \bar{r}} = \frac{\partial \bar{u}}{\partial \bar{r}} \frac{\partial}{\partial \bar{u}} + \frac{\partial \bar{v}}{\partial \bar{r}} \frac{\partial}{\partial \bar{v}} = \frac{\partial}{\partial \bar{u}} - \frac{\partial}{\partial \bar{v}}, \tag{2.57}
\]

and putting them in (2.48) and (2.49), they yield the D’Alembert differential equations in the double null coordinates:

\[
\frac{\partial^2}{\partial \bar{u} \partial \bar{v}} T(\bar{u}, \bar{v}) = 0 \tag{2.58}
\]
\[
\frac{\partial^2}{\partial \bar{u} \partial \bar{v}} R(\bar{u}, \bar{v}) = 0. \tag{2.59}
\]

So, the general solution of the differential equation (2.59) can be found. For the derivative with respect to \( \bar{v} \) the following is compelled to be set:

\[
\frac{\partial}{\partial \bar{v}} R(\bar{u}, \bar{v}) = f(\bar{v}). \tag{2.60}
\]

Only in case that the derivative with respect to \( \bar{v} \) yields a function depending on \( \bar{v} \) exclusively, the following derivative with respect to \( \bar{u} \) vanishes. In that case the D’Alembert equation (2.59) is fulfilled. The indefinite integration over \( \bar{v} \) of equation (2.60) yields the general form of the solution for the \( R \)-coordinate:

\[
\int d\bar{v} \frac{\partial}{\partial \bar{v}} R(\bar{u}, \bar{v}) = \int d\bar{v} f(\bar{v})
\]
\[
\rightarrow R(\bar{u}, \bar{v}) = F(\bar{v}) + G(\bar{u}). \tag{2.61}
\]

Analogously the general solution for the time coordinate \( T \) reads:

\[
\int d\bar{v} \frac{\partial}{\partial \bar{v}} R(\bar{u}, \bar{v}) = \int d\bar{v} h(\bar{v})
\]
\[
\rightarrow T(\bar{u}, \bar{v}) = H(\bar{v}) + K(\bar{u}). \tag{2.62}
\]
As we know, both Einstein-Rosen coordinates are linked to each other by the relations (2.46) or (2.47). From the first identity in (2.46) the connection in double null variables follows:

\[ T,\bar{r} = G,\bar{u} + F,\bar{v} = K,\bar{u} + H,\bar{v}. \]  

(2.63)

From the second identity in (2.46) we get a similar expression performing in the same way

\[ T,\bar{t} = G,\bar{u} - F,\bar{v} = K,\bar{u} + H,\bar{v}. \]  

(2.64)

Adding both identities written in the double null coordinates we obtain

\[ \frac{\partial G}{\partial \bar{u}} = \frac{\partial K}{\partial \bar{u}}, \]

\[ \rightarrow G(\bar{u}) = K(\bar{u}), \]  

(2.65)

and by putting this result in (2.63) or (2.64) trivially

\[ \frac{\partial F}{\partial \bar{v}} = \frac{\partial H}{\partial \bar{v}}, \]

\[ \rightarrow F(\bar{v}) = -H(\bar{v}). \]  

(2.66)

We remind on the fact that in (2.65) and (2.66) functions depending on \( \bar{u} \) respectively \( \bar{v} \) are not allowed to occur due to the indefinite integrations. Otherwise the condition

\[ \frac{\partial}{\partial \bar{u}} \frac{\partial}{\partial \bar{v}} R(\bar{u}, \bar{v}) = \frac{\partial}{\partial \bar{u}} f(\bar{v}) = 0, \]

\[ f(\bar{v}) = \frac{\partial}{\partial \bar{v}} F(\bar{v}), \]  

(2.67)

(2.68)

would be violated. Replacing the double null variables by the original coordinates \( \bar{t} \) and \( \bar{r} \) the Einstein-Rosen coordinates read:

\[ T(\bar{t}, \bar{r}) = G(\bar{t} + \bar{r}) - F(\bar{t} - \bar{r}), \]

\[ R(\bar{t}, \bar{r}) = G(\bar{t} + \bar{r}) + F(\bar{t} - \bar{r}). \]  

(2.69)

(2.70)

As we have seen, the conditions (2.46) and (2.47), which arise from the crucial requirement the transformation between \( (\bar{t}, \bar{r}) \) and \( (T,R) \) to be conformal, is authoritative. If we construct the coordinate \( R(\bar{t}, \bar{r}) \) then the coordinate \( T(\bar{t}, \bar{r}) \) emerge directly from the relations (2.46) and (2.47). It has been showed that the function \( R(\bar{t}, \bar{r}) \) is actually harmonic by solving the Einstein equations written for the metric (2.31). Rather due to the relations (2.46) and (2.47) it is also proved that the time coordinate \( T(\bar{t}, \bar{r}) \), which corresponds to the Einstein-Rosen coordinate \( R(\bar{t}, \bar{r}) \), is also harmonic. Therewith the differential equation (2.48) is fulfilled, too.
2.4 **Einstein Equations in the Einstein-Rosen Coordinates**

The model of the cylindrically symmetric gravitational waves does not include the presence of matter fields. Thus, we consider a general relativity without matter fields, that means a vacuum spacetime. So, we are interested in the vacuum Einstein equations

\[ G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 0. \]  

(2.71)

In this case the energy-momentum tensor \( T_{\alpha\beta} \) is zero and the Einstein equations \( G_{\alpha\beta} - \kappa T_{\alpha\beta} = 0 \) reduce to the vacuum equations (2.71).

We now would like to calculate the vacuum equations (2.71) in the Einstein-Rosen coordinates system, since, as we are going to see, they assume a very convenient form therein. So, the metric \( g_{\alpha\beta} \) is obviously the Einstein-Rosen metric (2.32). In virtue of the cylindrical symmetry, only six components of the Einstein tensor (2.71) do not vanish, namely:

\[
\begin{align*}
G_{00} &= R_{00} - \frac{1}{2} R g_{00} \quad & G_{01} &= R_{01} \\
G_{10} &= R_{10} \quad & G_{11} &= R_{11} - \frac{1}{2} R g_{11} \\
G_{22} &= R_{22} - \frac{1}{2} R g_{22} \quad & G_{33} &= R_{33} - \frac{1}{2} R g_{33}.
\end{align*}
\]  

(2.72)

(2.73)

(2.74)

Because of the symmetry of the Ricci tensor \( R_{\alpha\beta} \) in its indices, the components \( G_{01} \) and \( G_{10} \) are identical, and so one of them can be neglected, for instance \( G_{10} \). Now only five components are relevant for the further computation of the Einstein field equations. In the apposite appendix C the various non vanishing components of the Ricci tensor

\[
R_{\alpha\beta} = R_{\alpha\mu\beta}^\mu = \partial_\mu \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\mu_{\alpha\mu} + \Gamma^\mu_{\rho\mu} \Gamma^\rho_{\alpha\beta} - \Gamma^\mu_{\rho\beta} \Gamma^\rho_{\alpha\mu}
\]  

(2.75)

already have been performed, as well as the Ricci scalar

\[
R = g^{\alpha\beta} R_{\alpha\beta}.
\]  

(2.76)

Then the five Einstein equations read:

\[
\begin{align*}
G_{00} &= \frac{1}{2} \left( \psi_T^2 + \psi_R^2 \right) - \frac{1}{R} \Gamma_{,R} = 0 \\
G_{01} &= \psi_T \psi_R - \frac{1}{R} \Gamma_{,T} = 0 \\
G_{11} &= \frac{1}{2} \left( \psi_T^2 + \psi_R^2 \right) - \frac{1}{R} \Gamma_{,R} = 0 \\
G_{22} &= -\Gamma_{,RR} - \Gamma_{,TT} - \frac{1}{2} \left( \psi_R^2 \psi_T^2 \psi_T^2 - \psi_T^2 \psi_R^2 \right) = 0 \\
G_{33} &= 2 \left( \psi_T \psi_{TT} - \psi_T \psi_{,RR} - \frac{1}{R} \psi_T \psi_R \right) + G_{22} = 0.
\end{align*}
\]  

(2.77)
We recommend the reader to distinguish the Roman R, the Ricci scalar, from the italic \( R \), one of the Einstein-Rosen coordinates.

Straight away one remarks that the component \( G_{00} \) is identical to \( G_{11} \). Obviously only one of them is indispensable to obtain the complete set of the field equations, say \( G_{00} \). Just this equation yields one of the differential equations determining \( \psi \) and \( \Gamma \), namely \( (2.79) \). Then of course we set in \( G_{33} \) the component \( G_{22} \) equal zero, so to get the Bessel differential equation \( (2.78) \). From the component \( G_{01} \) the last Einstein equation \( (2.80) \) is read out. The result is a compact set of the Einstein field equations:

\[
\begin{align*}
\psi_{,TT} - \psi_{,RR} - \frac{1}{R} \psi_{,R} &= 0 \quad (2.78) \\
\Gamma_{,R} &= \frac{1}{2} R \left( \psi_{,R}^2 + \psi_{,T}^2 \right) \quad (2.79) \\
\Gamma_{,T} &= R \psi_{,T} \psi_{,R} \quad (2.80)
\end{align*}
\]

This set of equations has a familiar structure:

- Equation \( (2.78) \) looks exactly like the usual wave equation for the cylindrically symmetric massless scalar field \( \psi(T, R) \) propagating on a Minkowskian spacetime background.
- The succeeding equation \( (2.79) \) describes the energy density of the massless scalar field \( \psi \) in the cylindrical coordinates.
- At last, through equation \( (2.80) \) the radial energy current density is determined.

As explained in the beginning of this chapter, the foliated \((3+1)\)-dimensional general relativity has been reduced to a \((1+1)\)-dimensional model by the cylindrical symmetry. Further on, the model really is coupled to a massless scalar field, since such a field solves the Einstein equations written for the simplified cylindrical metric, namely the Einstein-Rosen metric.

Last but not least we wish to notice that the set of the Einstein equations, reduced by the symmetry property of the Einstein-Rosen metric, can be deduced in a easier and faster manner. The whole calculation can be slashed to the task of complying

\( R_{\alpha\beta} = 0 \).

The proof that the equations \( (2.71) \) are equivalent to the vanishing Ricci tensor is very short. The main step is to multiply \( (2.71) \) from the left with the Einstein-Rosen metric \( g_{\alpha\beta}^{ER} \). One gets \( G = -R \). This result is inserted in \( (2.71) \) wherefrom the following is given:

\[
- G_{\alpha\beta} = R_{\alpha\beta}. \quad (2.82)
\]

In our case of a vacuum spacetime the assertion \( R_{\alpha\beta} = 0 \) follows trivially, as \( G_{\alpha\beta} = 0 \).

Again the performed Ricci tensor for the Einstein-Rosen metric in Appendix C is taken for eliciting the set of the Einstein equations. From \( R_{22} = 0 \) as also from \( R_{33} = 0 \) we
get the wave equation for the massless scalar field (2.78), both equations are equivalent. Adding $R_{00}$ and $R_{11}$ together and setting the sum equal to zero, (2.79) is obtained. Finally, $R_{01} = R_{10} = 0$ yields the energy current density (2.80) of the scalar field. The main simplification in computing the Einstein equations with the second method is of course the fact that no Ricci scalar is needed.
Chapter 3

Einstein-Rosen Wave

3.1 Solution of the Wave Equation

In the present chapter we calculate the parameters $\psi(T, R)$ and $\Gamma(T, R)$, which occur in the Einstein-Rosen metric and so determine the Einstein-Rosen wave. The cylindrically symmetric wave equation (2.78) is solved by a band of solutions. The parameter $\psi(T, R)$ is a superposition of all allowed solutions. To keep track of, we note the fundamental wave equation again:

$$\psi_{TT} - \psi_{RR} - R^{-1} \psi_R = 0$$  \hspace{1cm} (3.1)

The equation is obviously linear. The general solution of this differential equation is the cylindrically symmetric massless scalar field $\psi(T, R)$ propagating on a Minkowskian background. The function $\psi(T, R)$ is the sum over all modes. A mode is an oscillation with a particular frequency $\omega = k = p, (c = \hbar = 1)$, $k$ the wave number and $p$ the momentum of the mode. As the frequency is a continuous parameter ($k \in \mathbb{R}_0^+$) the sum corresponds to an integration over all modes, i.e. over all $k \in \mathbb{R}_0^+$, or equivalently over all momenta, $p \in \mathbb{R}_0^+$. Thus, the ansatz is the well-known Fourier decomposition by frequencies. The method also is known under the name “separation of variables”. Thus we arrange the following ansatz:

$$\psi_k(T, R) = \varphi_k(T) \chi_k(R)$$  \hspace{1cm} (3.2)

This ansatz is put in the differential equation (3.1) and thereupon we divide by the product $\varphi_k(T) \chi_k(R)$, in order to obtain

$$\frac{\varphi_{k,TT}}{\varphi_k} = \frac{X_{k,RR}}{\chi_k} + \frac{1}{R} \frac{X_{k,R}}{\chi_k}.$$  \hspace{1cm} (3.3)

The ansatz (3.2) presumes the functions $\varphi_k$, $\chi_k$ to depend only on $T$ and $R$ respectively. This fact constrains the right and the left side of the equation (3.3) to be equal to a constant $k^2$ with $k \in \mathbb{R}_0^+$. Otherwise the treated differential expression does not hold anymore. Then
we set
\[
\frac{\varphi_{k,TT}}{\varphi_k} = -k^2 \quad \text{(3.4)}
\]
\[
\frac{\chi_{k,RR}}{\chi_k} + \frac{1}{R} \frac{\chi_{k,R}}{\chi_k} = -k^2, \quad k \in \mathbb{R}_0^+. \quad \text{(3.5)}
\]

The first relation is the ordinary free harmonic oscillator differential equation with oscillation frequency \(k\). This equation is solved by a linear combination of both solutions
\[
\varphi_k^{(1)}(T) = A(k) e^{ikT} \quad \text{and} \quad \varphi_k^{(2)}(T) = B(k) e^{-ikT}. \quad \text{(3.6)}
\]
So we find for the general solution of the time part of (3.1) the announced overlay:
\[
\varphi_k(T) = A(k) e^{ikT} + B(k) e^{-ikT}. \quad \text{(3.7)}
\]

The functions \(A(k)\) and \(B(k)\) are in relationship with each other. As we know, gravitational waves, like electromagnetic waves, are real valued waves. From this it follows that the solution of the differential equation (3.1) has to be real. The function \(\psi(T;R)\) is real if and only if the following holds:
\[
B(k) = A^*(k), \quad \text{(3.8)}
\]
whereas \(A^*(k)\) is the complex conjugate of \(A(k)\). Thereupon the solution to the time part of the wave equation reads:
\[
\varphi_k(T) = A(k) e^{ikT} + A^*(k) e^{-ikT}. \quad \text{(3.9)}
\]

Subsequently we consecrate ourselves to the spatial part (3.5) of the wave equation. Rewriting the equation into the form
\[
R^2 \chi_{k,RR} + R \chi_{k,R} + k^2 R^2 \chi_k = 0 \quad \text{(3.10)}
\]
we notice the evident similarity to the Bessel differential equation
\[
z^2 \frac{d^2}{dz^2} w(z) + z \frac{d}{dz} w(z) + (z^2 - n^2) w(z) = 0. \quad \text{(3.11)}
\]
In addition we set in (3.11) \(n = 0\) and in (3.10) \(s \doteq kR, \quad s \in \mathbb{R}\). The spatial differential equation is indeed the Bessel differential equation with index \(n = 0\).
\[
s^2 \frac{d^2}{dR^2} \chi_k + s \frac{d}{dR} \chi_k + s^2 \chi_k = 0, \quad \text{(3.12)}
\]
with \(s = s(k; R)\). The Bessel function of first order solve the equations with \(n = 0\). These are (see \[5\]): \(J_0(kR), Y_0(kR), H_0^{(+)}(kR), H_0^{(-)}(kR)\), where \(H_0^{(\pm)}(kR) = J_0(kR) \pm iY_0(kR)\). For the solution \(\psi(T,R)\) the following boundary condition is given [3]:
\[
\lim_{R \to 0} R \psi_{;T} \psi_{;R} = 0 \quad \text{(3.13)}
\]
Since the Weber function $Y_0(kR)$ contains the logarithm function depending just on $R$, all expressions involving the considered function are invalid solutions, as the limes in (3.13) does not exist. For this reason the cylindrically symmetric Bessel function $J_0(kR)$ is the last possibility to solve the spatial differential equation. For the further considerations we choose the representation for the Bessel function $J_0(kR)$ given in appendix A, namely (A.19), but with $z = kR$ and $n = 0$. Setting $\sin \varphi = - \cos \vartheta$ with $\vartheta = \varphi + \frac{\pi}{2}$ and applying what is proved in (A.12), we get for $J_0(kR)$:

$$J_0(kR) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, e^{ikR\cos \varphi} \tag{3.14}$$

Inserting $J_0(kR)$ in (3.13) we obtain

$$\lim_{R \to 0} R \varphi_{k,T}(T) J_0(kR) \frac{\partial}{\partial R} J_0(kR) = 0. \tag{3.15}$$

Thus, the Bessel function $J_0(kR)$ solves the spatial differential equation and the resulting wave function $\psi_k(T,R)$ the boundary condition too.

$$\psi_k(T,R) = J_0(kR) \left[ A(k) e^{ikT} + A^*(k) e^{-ikT} \right]. \tag{3.16}$$

The above function is the solution for a specific mode, namely the one with frequency $\omega = k$. To obtain the general solution for the wave equation (3.1), we have to sum together all modes, thus to integrate over $k$, accordingly to the Fourier theorem. Therewith the general solution of the wave equation for the cylindrically symmetric massless scalar field propagating on a Minkowskian spacetime is:

$$\psi(T,R) = \int_0^\infty dk \, J_0(kR) \left[ A(k) e^{ikT} + A^*(k) e^{-ikT} \right]. \tag{3.17}$$

If we compare the solution (3.17) with the Hankel transformation (A.22) in appendix A for the case $n = 0$, we remark a close similitude. In fact the function (3.17) can be written as the Hankel transformation of $\frac{1}{k} \varphi_k(T)$, that is to say

$$\psi(T,R) = \int_0^\infty dk \, \frac{1}{k} J_0(kR) \left[ \frac{A(k)}{k} e^{ikT} + \frac{A^*(k)}{k} e^{-ikT} \right]. \tag{3.18}$$

Because of the close relation of $\psi(T,R)$ to the Hankel transformation, we define newly the expression $\frac{1}{k} \varphi_k(T)$.

We set

$$\tilde{\psi}(k,T) \doteq \frac{1}{k} \varphi_k(T) \tag{3.19}$$

with the Hankel coefficients

$$a(k) \doteq \frac{A(k)}{k}, \quad a^*(k) \doteq \frac{A^*(k)}{k}. \tag{3.20}$$
Therewith the scalar field can be written as the Hankel transformation of $\tilde{\psi}(k, T)$:

$$
\psi(T, R) = \int_0^\infty dk \; k J_0(kR) \tilde{\psi}(k, T)
= \int_0^\infty dk \; k J_0(kR) \left[ a(k) e^{ikT} + a^*(k) e^{-ikT} \right].
$$

(3.21)

In order to be sure that the Hankel transformation of $\psi(T, R)$ is a well defined function, it is essential to give a look to the property the Hankel coefficients $a(k)$ and $a^*(k)$ have to fulfill. As we can learn from appendix A, the two dimensional Fourier transformation with cylindrical symmetry is just the Hankel transformation - we can imagine that the Hankel transformation of $\psi(T, R)$ exists only if $\tilde{\psi}(k, T)$, and in detail $a(k) = \frac{A(k)}{k}$ and $a^*(k) = \frac{A^*(k)}{k}$, is integrable. In addition to this requirement, we would like that the function we get through the transformation, namely $\psi(T, R)$, is again integrable over the radius $R$. This would assure the possibility to transform back $\psi(T, R)$ to the functions $a(k)$ and $a^*(k)$. This means mathematically expressed:

$$
\mathcal{H} : \psi(R; T) \rightarrow \tilde{\psi}(k; T) \quad \text{(Hankel transformation)}
$$

(3.22)

$$
\mathcal{H}^{-1} : \tilde{\psi}(T; k) \rightarrow \psi(R; T) \quad \text{(Inverse transformation)}
$$

(3.23)

and thus the identity

$$
\mathcal{H}^{-1} \left[ \mathcal{H} \left[ \psi(R; T) \right] \right] = \mathcal{H}^{-1} \left[ \tilde{\psi}(T; k) \right] = \psi(T, R).
$$

(3.24)

In case of the Fourier transformation there is a function space $\mathcal{S}$ (Schwartz space), which guarantees the existence of the inverse map of a Fourier transformed function, if the treated function is element of $\mathcal{S}$. This is ensured by the following theorem:

**Fourier inversion theorem:**

The Fourier transform is a bicontinuous bijection from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$. Its inverse map is the inverse Fourier transform, i.e. $(\hat{f})^- = f = (\hat{f})^\ast$. The Fourier transform and the inverse map are denoted by $\hat{f}$ and $\check{f}$ respectively [6].

Since the Hankel transformation is nothing but the two dimensional Fourier transformation with cylindrical symmetry it is assumed that an analogous theorem holds, which presupposes the Hankel coefficients to be element of the Schwartz space in order to assure the identity (3.24). So, if it is set $A(k)$ and $A^*(k)$ to be element of the Schwartz space, then the functions $a(k)$ and $a^*(k)$ are too, since the functions in $\mathcal{S}$ fall off faster than every power of their argument increasing to infinity. So we managed to assure the integrability of the wave function $\psi(T, R)$ and, of course, of its Hankel transformed function $\tilde{\psi}(k, T)$ also.

### 3.2 Finiteness of the Energy $\Gamma(T, R)$

The integrability property of the wave function $\psi(T, R)$ is a vital condition for the function $\Gamma(T, R)$. As it is shown in the appendix B, it turns out that the energy of the scalar field,
3.2 Finiteness of the Energy $\Gamma(T, R)$

contained in a disk $\Delta z = 1$, is given (up to a factor $2\pi$) by this same function

$$
\Gamma(T, R) = \int_0^R dR' \Gamma_{R, R'}(T, R').
$$

(3.25)

It is now important to prove that the energy remains finite for the radius $R$ at infinity$^1$.

$$
\Gamma(T, \infty) = \frac{1}{2} \int_0^\infty dR \frac{R}{R'} \left( \psi_{T, R}^2 + \psi_{R, R}^2 \right).
$$

(3.26)

Obviously, the properties of $\psi(T, R)$ play a main role for the finiteness of the energy. We first start with treating a single mode of $\psi(T, R)$ with the specific frequency $k$.

$$
\psi_k(T, R) = J_0(kR) \left[ A(k) e^{ikT} + A^*(k) e^{-ikT} \right].
$$

(3.27)

As we will integrate over $R$, we concentrate for the time being only on the $R$-depending part

$$
\psi_k(R) = J_0(kR).
$$

(3.28)

The plain wave is not normalizable as is generally known$^2$.

$$
||\psi_k(R)||^2 = \int_0^\infty dR R J_0(kR) J_0(k'R) = \frac{1}{k'} \delta(k - k').
$$

(3.29)

In such cases a wave packet is introduced, which, for instance, could be formed by a Gauss profile $A(k)$. This function is an element of the Schwartz space, and so falls off faster than every power of the argument $k$.

$$
\bar{\psi}(R) = \int_0^\infty dk A(k) J_0(kR).
$$

(3.30)

Now the wave is normalizable, since the following integral is smaller than infinity due to the wave packet, which vanishes at infinity.

$$
||\bar{\psi}(R)||^2 = \int_0^\infty dk |A(k)|^2 < \infty.
$$

(3.31)

Comparing (3.30) with (3.27) or directly with (3.17) we remark promptly that the wave function (3.17) is furnished already with its wave packet, and is therefore normalizable:

$$
||\psi(T, R)||^2 = \int_0^\infty dR R \psi^2(T, R) = \int_0^\infty dk k \left[ a(k) e^{ikT} + a^*(k) e^{-ikT} \right]^2.
$$

(3.32)

1. The next equation is given by the expression (2.79) for the energy density of the massless scalar field $\psi(T, R)$.

2. For eq. (3.29): The orthonormality relation is presented in appendix A. Replace $x$ with $R$ in equation (A.26).
So we can verify the finiteness of the energy $\Gamma(T, R)$ if we integrate over the whole space at the fixed time $T$. For the derivatives of the wave function with respect to $T$ and $R$ we get

$$\psi_{,T} = i \int_0^\infty dk \, k^2 J_0(kR) \varphi_{(-)}(k; T)$$

(3.34)

$$\psi_{,R} = - \int_0^\infty dk \, k^2 J_1(kR) \varphi_{(+)}(k; T),$$

(3.35)

where we define

$$\varphi^{(\mp)} \equiv a(k) \, e^{ikT} \mp a^*(k) \, e^{-ikT}.$$ 

(3.36)

If we square both derivatives and integrate them over the radius $R$, we obtain:

$$\int_0^\infty dR \, R \, \psi_{,T}^2(T, R) = - \int_0^\infty dk \, k^3 \varphi_{(-)}^2(k; T)$$

(3.37)

$$\int_0^\infty dR \, R \, \psi_{,R}^2(T, R) = \int_0^\infty dk \, k^3 \varphi_{(+)}^2(k; T),$$

(3.38)

where we have used the orthonormality relation for the Bessel function

$$\int_0^\infty dR \, R \, J_n(kR) \, J_n(k'R) = \frac{1}{k'} \delta(k - k'), \quad \forall n \in \mathbb{Z}.$$ 

(3.39)

Both integrals are put now in (3.26), so to get the finite quantity

$$\Gamma(T, \infty) = 2 \int_0^\infty dk \, k^3 \, a(k) \, a^*(k)$$

$$= 2 \int_0^\infty dk \, k^3 \, |a(k)|^2$$

$$= 2 \int_0^\infty dk \, k \, |\Lambda(k)|^2, \quad \Lambda(k), a(k) \in \mathcal{S}.$$ 

(3.40)

QED

We conclude the present chapter affirming that the elicited wave function $\psi(T, R)$ is well-defined, since for the two cases, $R \to 0$ and $R \to \infty$, the function remains bounded providing at the same time its normalizibility and a finite energy of the field.

---

3See (A.26), for the case $x = R$. 
Chapter 4

ADM Formalism

In the previous chapters we have presented how the Einstein equations can be solved if the symmetries of the metric are known. In our case it is about the Einstein-Rosen spacetime and so the Einstein equations have been solved for the Einstein-Rosen metric (2.32). The solution to the differential equations of second order is then, as we have seen, the massless scalar field $\psi$. In general the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0 \quad (4.1)$$

are obtained by varying the following action with respect to the metric $g_{\alpha\beta}$, and discarding the resulting divergences$^1$:

$$S[g_{\alpha\beta}(x)] = \frac{1}{16\pi G} \int d^4x \mathcal{L} = \frac{1}{16\pi G} \int \sqrt{g(x)} R(x) d^4x, \quad (4.2)$$

where $\mathcal{L}$ stands for the Lagrangian density and $g(x)$ for the determinant of the metric $g_{\alpha\beta}$ at the spacetime point $x$. In the Einstein equations above, $R_{\alpha\beta}$ denotes the Ricci tensor, $R$ the Ricci scalar and finally $G$ the universal gravitational constant. In the early sixties of the last century R. Arnowitt, S. Deser and C.W. Misner, short ADM, developed the so called canonical formalism for general relativity [8]. A reason for constructing such a formalism to describe dynamic in general relativity is that an action containing canonical conjugate variables is a possible basis to develop a quantum theory of gravity by canonical methods.

In order to introduce canonical variables, a frame is needed, where they can be related on. To this end, spacetime, in the ADM formalism, is split into three spatial dimensions and one time dimension, in literature shortly noted by (3+1). One can now attempt to imagine spacetime as a composite of hypersurfaces (3-space surfaces), which are fixed at arbitrary points in time. So a hypersurface $\mathcal{G}_{t_0}$ could be fixed at the time $t = t_0$, just as the hypersurfaces $\mathcal{G}_{t_k}$ at an arbitrary time $t = t_k$. Here we would like to consider two specific hypersurfaces $\mathcal{G}_{t_0}$ and $\mathcal{G}_{t_1}$ to different times $t_0 < t_1$. In the following we try to give an idea how points in the first hypersurface $\mathcal{G}_{t_0}$ are connected with the corresponding points

$^1$See [7].
in the later hypersurface $G_t$. In the ADM formalism two functions are brought in, which measure the translation in spacetime. The lapse of the proper time between the 'lower' $G_{t_0}$ and the 'upper' $G_t$ hypersurface is measured by the so called lapse function $N(t, x_1, x_2, x_3)$. On the other hand the shift function $N_i(t, x_1, x_2, x_3)$ measures the spatial shift, i.e. this function specifies the position in the 'upper' hypersurface where to the point $x^i$ of the 'lower' hypersurface have to be placed. The infinitesimal translation in spacetime is given by the interval $ds^2$ between the point $x^\alpha = (t, x^i)$ and the point $x^{\beta} + dx^{\beta} = (t + dt, x^i + dx^i)$ in the 'upper' hypersurface, namely by the relation

$$ds^2 = g_{ik}(dx^i + N^i dt)(dx^k + N^k dt) - (N dt)^2 = g_{\alpha\beta}dx^\alpha dx^\beta.$$  \hspace{1cm} (4.3)

Thus, the 4-metric has been constructed out of the 3-metric $g_{ik}$, the lapse function and the shift function:

$$
\begin{pmatrix}
g_{00} & g_{0k} \\
g_{0i} & g_{ik}
\end{pmatrix} = 
\begin{pmatrix}
N_i N^i - N^2 & N_k \\
N_i & g_{ik}
\end{pmatrix}.
\hspace{1cm} (4.4)
$$

If total divergences are discarded from the Lagrangian density of the action $\mathcal{L}$, then the functional $S$ can be brought into the form

$$S = \int dt \int d^3x \ g^{1/2}(K_{ik}K^{ik} - K^2 + R).$$

\hspace{1cm} (4.5)

Here, $R$ is again the Ricci scalar and $g^{1/2}$ the square root of the determinant of the 3-metric $g_{ik}$. $K^2$ is the squared trace of the tensor $K_{ik}$: $K = K_i^i = g^{ik}K_{ik}$. The new second order tensor $K_{ik}$ is the extrinsic curvature of the spacelike hypersurface. The extrinsic curvature is calculated out of the 3-metric, the lapse and the shift function, according to the formula

$$K_{ik} = \frac{1}{2N}(N_i|k + N_k|i - g_{ik,0}),$$

\hspace{1cm} (4.6)

where the stroke means the covariant derivative in 3-space. As an illustration to the extrinsic curvature, we cite [2]:

*The extrinsic curvature measures the fractional shrinkage and deformation of a figure lying in the spacelike hypersurface $\Sigma$ that takes place when each point is carried forward a unit interval of proper time “normal” to the hypersurface out into the enveloping spacetime. No enveloping spacetime? No extrinsic curvature!*

Varying the action $(4.5)$ with respect to $g_{ik,0}$ we obtain the momentum $\pi^{ik}$, which is canonical conjugate to the 3-metric $g_{ik}$:

$$\pi^{ik} = \frac{\delta \mathcal{L}}{\delta g_{ik,0}} = -g^{1/2}(K^{ik} - K g^{ik}),$$

\hspace{1cm} (4.7)

where the Lagrange density $\mathcal{L}$ is the integrand of the action $(4.5)$ and $16\pi G = 1$ is set. Therewith it is scheduled what, in the ADM formalism, the generalized coordinates and the belonging canonical momentum are, namely the 3-metric $g_{ik}$ and the momentum $\pi^{ik}$. Just the introduction of the momentum

$$\pi^{ik} = \frac{\delta \mathcal{L}}{\delta g_{ik,0}}$$

\hspace{1cm} (4.8)
makes it possible to transform the second order partial differential equations of the Einstein
equations to an equivalent set of first order differential equations. This set will then depend
on the dynamic variables $g_{ik}$ and $\pi^{ik}$. The action functional is then converted into the
Hamiltonian form, which is more convenient for the developing of a quantum theory of
gravity by canonical methods, since the Hamiltonian enters the action.

$$S = \int dt \int d^3x \left( \pi^{ik} g_{ik,0} - N\mathcal{H} - N_i\mathcal{H}' \right),$$

(4.9)

with $S = S[g_{ik}, \pi^{ik}, N, N_i]$. The lapse $N$ and the shift function $N_i$ in (4.9) play the role
of Lagrange multipliers, whereby the function $\mathcal{H}$ is the so called super-Hamiltonian
and the function $\mathcal{H}'$ the supermomentum. These quantities are expressed by the canonical
conjugate variables $g_{ik}$ and $\pi^{ik}$,

$$\mathcal{H} = g^{-1/2} \left( \pi^{ik} \pi_{ik} - \frac{1}{2} \pi^2 \right) - g^{1/2} R,$$

(4.10)

$$\mathcal{H}'^i = -2\pi^{ik}_{;k} - g^{ij} \left( 2g_{j,k,l} - g_{jk,l} \right) \pi^{jk}.$$

(4.11)

Analogously to $K$ in (4.5), $\pi$ denotes the trace of $\pi^{ik}$ and so $\pi^2$ is nothing but the square
of the trace of $\pi^{ik}$: $\pi = \pi^i_i = g_{ik} \pi^{ik}$. The commas in the right hand side of the equation
of the supermomentum represent the common partial derivative with respect to the space
coordinates $k, l = 1, 2, 3$. Now, if the action (4.9) is varied with respect to the Lagrange
multipliers $N$ and $N_i$, then the constraints of the system are obtained:

$$\mathcal{H} = 0,$$

(4.12)

$$\mathcal{H}'^i = 0.$$

(4.13)

Varying whereas with respect to the dynamical variables $g_{ik}$ and $\pi^{ik}$, the set of canonical
equations is yield, which is well known from the Hamilton formalism and which, apart from
the constraints, replace the Einstein equations in the form (4.1).

$$g_{ik,0} = \frac{\delta H}{\delta \pi^{ik}}, \quad \pi^{ik}_{;0} = \frac{\delta H}{\delta g_{ik}},$$

(4.14)

with the Hamiltonian

$$H = \int d^3x \left( N\mathcal{H} + N_i\mathcal{H}'^i \right)$$

(4.15)

We can see the relation (4.15) by recalling the definition of the Hamilton function:

$$H = p^i \dot{q}_i - L,$$

(4.16)

where $p^i$ are the canonical conjugate momenta to the generalized coordinates $q_i$. In our
case we have then,

$$L = \int d^3x \left( \pi^{ik} g_{ik,0} - N\mathcal{H} - N_i\mathcal{H}'^i \right),$$

(4.17)

$$p^i \dot{q}_i = \int d^3x \pi^{ik} g_{ik,0};$$

(4.18)

which reproduce equation (4.15) if inserted in (4.16).
4.1 Reduced ADM Action for the Cylindrically Symmetric Spacetime

We dedicate this section to the ADM action for the cylindrically symmetric spacetime. Due to the symmetries of the model the 4-metric simplifies drastically, as it is shown in chapter one, see (2.2.1), (2.2.2). The first simplification is that the cylindrically symmetric metric depends only on the coordinates $t$ and $r$, since it is invariant with respect to changes in the coordinates $\varphi$ and $z$. Secondly, the metric acquires diagonal form by specific conformal transformations \(^2\), a fact which simplifies the task of solving the Einstein equations. These properties of course have influence on all quantities of the ADM formalism. We start with the lapse and the shift functions, which are related through equation (4.4) to the 4-metric, in our case the cylindrically symmetric one. The only component of the cylindrically symmetric metric, which in one hand is non vanishing and in the other hand is in relation with the lapse and the shift functions, is (2.31)

$$g_{00} = -\Omega^2(t, r), \quad (4.19)$$

where in the following we use the coordinates $t$ and $r$ in the conformal transformed metric instead of $\tilde{t}$ and $\tilde{r}$. This component is only dependent on the coordinate $t$ and $r$, such that also the Lagrangian multipliers $N(t, r)$ and $N^i(t, r)$ of the ADM action in the Hamiltonian form only depend on these coordinates. Further on two components of the shift function drop out due to the symmetries, namely the ones which measures the changes in the $\varphi$ and $z$ coordinates. So, only the lapse function $N(t, r)$ and one shift function $N^1(t, r)$, namely that one which measures translations in the radius direction, are left. Also the 3-metric depends only on the time and radius coordinates and as part of the conformal transformed metric, it is in diagonal form, see (2.31). Since the canonical conjugate momenta are related to the 3-metric by the equation

$$\pi^{ik} = \frac{\delta L}{\delta g_{ik,0}}, \quad (4.20)$$

one can imagine that the momenta inherit the properties of the 3-metric. In [3] the symmetry reduction of the momenta is presented at length, so that we note only the results. The non vanishing components of the momentum tensor $\pi^{ik}$ are the following, with 1, 2, 3 labeling $r, \varphi, z$:

$$\pi^{11}(t, r), \quad \pi^{22}(t, r), \quad \pi^{33}(t, r). \quad (4.21)$$

As expected, there remain only the components left, which are canonical conjugate to the corresponding non vanishing components of the 3-metric,

$$g_{11}(t, r), \quad g_{22}(t, r), \quad g_{33}(t, r). \quad (4.22)$$

If in the following discussion we take the metric (2.39) suggested in [3] as an example for a cylindrically symmetric spacetime, we see that the metric is given by the quantities $\gamma$, $R$

\(^2\)See (2.30), (2.31).
and $\psi$. The equations of motion, i.e the Einstein equations in the ADM form will finally be differential equations of those variables. So it is throughout correct to look at these quantities as generalized coordinates and to note the canonical conjugate momenta with respect to those generalized coordinates

$$
\pi^{11}(t, r) = \pi_R e^{-\gamma}, \quad \pi^{22}(t, r) = \frac{1}{2} R^{-1} \pi_R e^{\psi}, \quad \pi^{33}(t, r) = (\pi_R + \frac{1}{2} R \pi_R + \pi_\psi) e^{\psi},
$$

(4.23)

where our calculation yields for the component $\pi^{22}$ a factor $R^{-1}$ instead of the factor $R$ noted in the paper [3]. Thus, it has been achieved to bring the expression $\pi^{ik} g_{ik, 0}$ of the action (4.9) in the following canonical form:

$$
\pi^{ik} g_{ik, 0} = \pi_R \dot{\gamma} + \pi_R \dot{R} + \pi_\psi \dot{\psi},
$$

(4.24)

where this form holds exclusively for the cylindrically symmetric spacetime and especially for the metric (2.39) depending on $\gamma$, $R$ and $\psi$. The dots in equation (4.24) denote the partial derivative $\partial_t$. If (4.23) is solved for the momenta conjugate to the generalized coordinates $\gamma$, $R$ and $\psi$ then we get

$$
\pi_\gamma = e^{\psi - \gamma} \pi^{11},
\pi_R = 2 R e^{-\psi} \pi^{22},
\pi_\psi = \pi^{33} e^\psi - \pi^{22} R^2 e^\psi - \pi^{11} e^{\gamma - \psi}.
$$

(4.25-4.27)

Now it is straight forward to calculate the super-Hamiltonian and the supermomentum. To this end one inserts in (4.10) for $\pi^{ik}$ the three non vanishing momenta $\pi^{11}$, $\pi^{22}$ and $\pi^{33}$ and for $g_{ik}$ the non vanishing components of the 3-metric,

$$
g_{11} = e^{\gamma - \psi}, \quad g_{22} = R^2 e^{-\psi}, \quad g_{33} = e^\psi.
$$

(4.28)

The same is inserted in (4.11) for the computation of the supermomenta. To simplify the reduced ADM action anymore, the super-Hamiltonian and the lapse function as also the supermomentum and the shift function are rescaled respectively with the factors $e^{1/2(\gamma - \psi)}$ and $e^{\psi - \gamma}$. The rescaled constraints read

$$
\tilde{\mathcal{H}} = -\pi_\gamma \pi_R + \frac{1}{2} R^{-1} \pi_\psi^2 + 2 R'' - \gamma' R' + \frac{1}{2} R \psi'^2,
\tilde{\mathcal{H}}_1 = -2 \pi'_\gamma + \gamma' \pi_\gamma + R' \pi_R + \psi' \pi_\gamma,
\tilde{\mathcal{H}}_2 = \tilde{\mathcal{H}}_3 = 0,
$$

(4.29-4.31)

where the primes stand for the derivative with respect to the radius coordinate $r$. It is not surprising to see only one supermomentum, which does not vanish. We should take into account, there is only one non vanishing shift function, namely $N_1(t, r)$. The variation of the ADM action in Hamiltonian form with respect to the rescaled Lagrange multipliers yields only the following two constraints,

$$
\tilde{\mathcal{H}} = 0,
\tilde{\mathcal{H}}_1 = 0.
$$

(4.32-4.33)
Inserting (4.24) and the rescaled constraints in the expression (4.9) for the ADM action, we get finally the symmetric reduced ADM action for the case of the cylindrically symmetric spacetime described by the metric (2.39).

\[ S = 2\pi \int_{-\infty}^{\infty} dt \int_{0}^{\infty} dr \left( \pi_\gamma \dot{\gamma} + \pi_R \dot{R} + \pi_\psi \dot{\psi} - \tilde{N} H - \tilde{N}^1 \tilde{H}^1 \right) \]  

(4.34)

4.2 Reduced ADM Action for the Einstein-Rosen Wave

After bringing the general cylindrically symmetric metric (2.3) into diagonal form by a conformal and a coordinate transformation into the inertial system, a further simplification of the diagonal metric was performed. By another conformal transformation of the coordinates\(^\text{3}\) into the Einstein-Rosen coordinates were introduced. Exactly this step has to be repeated, i.e. to introduce the Einstein-Rosen coordinates in the canonical formalism such that the structure of the super-Hamilton and the supermomentum get more simplified. The Einstein-Rosen radius \(R\) is already present in the canonical formalism, so that it remains only to enter the Einstein-Rosen time \(T\). To this end the function \(T\) is defined by the extrinsic curvature\(^\text{4}\) and is converted to a canonical coordinate by a canonical transformation. The canonical coordinate \(T\) is then identified with the momentum \(\pi_\gamma\), according to

\[ \pi_\gamma = -T', \]  

(4.35)

where the prime denotes the derivative with respect to \(r\). The canonical transformation also allows to identify \(-\gamma'\) in (4.29) and (4.30) with the canonically conjugate momentum of the Einstein-Rosen time,

\[ \pi_T = -\gamma'. \]  

(4.36)

In section 2.3.1 we have seen that the introduction of the Einstein-Rosen coordinates changed \(\gamma\) to the ‘energy’ function \(\Gamma\) of the massless scalar field following the relation (2.43). Through this transformation also the momentum \(\pi_T\) changes according to (4.36). The result is the introduction of a new momentum \(\Pi_T\), which forces the implementation of a new momentum canonically conjugate to \(R\), namely \(\Pi_R\). The relations between \(\pi_T\) and \(\Pi_T\) and between \(\pi_R\) and \(\Pi_R\) are taken from [3]:

\[ \Pi_T = \pi_T + \left[ \ln(R'^2 + T'^2) \right]' \]  

(4.37)

\[ \Pi_R = \pi_R + \left[ \ln \left( \frac{R' + T'}{R' - T'} \right) \right]' \]  

(4.38)

So the new set of the canonical variables for the reduced action is: \((T, \Pi_T)\), \((R, \Pi_R)\), \((\psi, \pi_\psi)\). The canonical momentum \(\pi_\psi\) is unaffected by the canonical transformation, since

\(^3\)See section 2.3 and especially 2.3.1.

\(^4\)See section VII in [3].
\(\psi\) is untouched by it. Analogously \(\psi\) did not change by the conformal transformation (2.44).

Introducing the new canonical variables \((T, \Pi_T), (R, \Pi_R)\) and \((\psi, \pi_\psi)\) into the rescaled super-Hamilton (4.29) and the supermomentum (4.30), we obtain:

\[
\tilde{\mathcal{H}} = R'\Pi_T + T'\Pi_R + \frac{1}{2}(R^{-1}\pi_\psi^2 + R\psi'^2),
\]
\[
\tilde{\mathcal{H}}_1 = T'\Pi_T + R'\Pi_R + \psi'\pi_\psi.
\]

Of course the constraints equations (4.32) and (4.33) remain unaffected by the reparametrization, such that also for the reduced action in the Einstein-Rosen coordinates the constraints equation read

\[
\tilde{\mathcal{H}} = 0,
\]
\[
\tilde{\mathcal{H}}_1 = 0.
\]

The new canonical variables bring then the reduced ADM action in the subsequent simplified form, whereby \(S = S[T, \Pi_T, R, \Pi_R, \psi, \pi_\psi]\):

\[
S = 2\pi \int_{-\infty}^{\infty} dt \int_0^\infty dr (\Pi_T \dot{T} + \Pi_R \dot{R} + \pi_\psi \dot{\psi} - \tilde{N}\tilde{\mathcal{H}} - \tilde{N}^1\tilde{\mathcal{H}}_1).
\]

By varying the action (4.43) with respect to \(\tilde{N}\) and \(\tilde{N}^1\) the constraint equations (4.41) and (4.42) are derived.
Chapter 5

Observables of the Einstein-Rosen Wave

In this chapter we dedicate us to the main topic of this work. It is presented how a set of phase space functions can be derived from the dynamical variables, namely the scalar field $\psi$ and the canonically conjugate momentum $\pi_\psi$. The phase space functions turn out to be observables of the Einstein-Rosen waves. The condition the phase space functions have to fulfill in order to be considered as observables will be discussed in chapter six.

Before we can look after the phase space functions, it is necessary to express the canonically conjugate momentum $\pi_\psi$ as a functional of the scalar field $\psi$ (3.17).

5.1 Canonical Momentum $\pi_\psi$

The canonically conjugate momentum $\pi_\psi$ is derived following the method presented in Torre’s work [4]. In the canonical formalism the dynamic of a field is described by the Hamilton equation

$$\dot{\psi} = \{\psi, H\},$$

where the dot denotes the time derivative $\partial_t$. The scalar field $\psi$ is well known, being the solution to the wave equation (2.78). The solution, we have found in chapter two, is

$$\psi(T, R) = \int_0^\infty dk J_0(kR) \left[ A(k) e^{ikT} + A^*(k) e^{-ikT} \right].$$

(5.2)

The Hamilton function

$$H = \int_0^\infty dr (\tilde{N}\tilde{H} + \tilde{N}^1\tilde{H}_1)$$

(5.3)

is given by the integral over the linear combination of the super-Hamiltonian $\tilde{H}$ and the supermomentum $\tilde{H}_1$, which were obtained for the reduced action of the Einstein-Rosen spacetime.

$$\tilde{H} = R'\Pi_T + T'\Pi_R + \frac{1}{2}(R^{-1}\pi_\psi^2 + R\psi'^2),$$

(5.4)

$$\tilde{H}_1 = T'\Pi_T + R'\Pi_R + \psi'\pi_\psi.$$
The coordinates \( T(r) \) and \( R(r) \) are part of an embedding \( X^\alpha : \Sigma \rightarrow \mathbb{R}^4 \), \( \Sigma \) being the hypersurface at \( t = \text{const.} \), which maps the hypersurface \( \Sigma \) into the flat spacetime. The embeddings \( X^\alpha(r) = (T(r), R(r), \varphi, z) \) are arbitrary cylindrically symmetric slices on the interior of the spacetime and approach the hypersurface \( T = 0 \) for \( r \rightarrow \infty \), where \( T \) is the Minkowskian time. The coordinates \( T(r) \) and \( R(r) \) do not depend on \( \varphi \) and \( z \) due to the cylindrically symmetric model.

If the Hamilton equation (5.1) is solved then the canonically conjugate momentum \( \pi_\psi \) is obtained as a functional of the scalar field \( \psi \). In the following prearrangements are presented, which allow to solve the Hamilton equation in a facilitated manner. It is then possible to write the time derivative, denoted by a dot in (5.1), by a functional derivative of the embedding variables \( X^\alpha(r) = (T(r), R(r), \varphi, z) \). The definition of the unit vector on a given hypersurface is defined by

\[
X^\alpha_a n_\alpha = 0, \quad g^{\alpha\beta} n_\alpha n_\beta = -1,
\]

where the index \( a \) runs over \((1, 2, 3)\) labeling \((R, \varphi, z)\). The metric \( g_{\alpha\beta} \) is the Minkowskian metric written for the cylindrically symmetric coordinates. The diagonal components are \((-1, 1, R^2, 1)\), while all other components vanish. The matrix \( X^\alpha_a \) reads

\[
X^\alpha_a = \begin{pmatrix}
T' & 0 & 0 \\
R' & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
\]

From the condition (5.6) the two possible normal covectors on the hypersurface \( \Sigma \) are obtained:

\[
n^{(1)}_\alpha = (-R', T', 0, 0), \quad n^{(2)}_\alpha = (R', -T', 0, 0).
\]

For the further calculation we choose without restricting generality the first one, which points out of the hypersurface in the negative time direction. The normal covector is normalized on \(-1\) by the second condition (5.7).

\[
n^\alpha = g^{\alpha\beta} n_\beta = (R', T', 0, 0),
\]

\[
A n^\alpha n_\alpha = A (-R'^2 + T'^2) = -1,
\]

\[
\rightarrow \quad A = \frac{1}{R'^2 - T'^2}.
\]

For convenience we assign the normalizing factor \( A \) to the covector, so to get in the further calculation a more compact form for the momentum \( \pi_\psi(r) \),

\[
n_\alpha = \frac{1}{R'^2 - T'^2} (-R', T', 0, 0), \quad n^\alpha = (R', T', 0, 0).
\]
5.1 Canonical Momentum $\pi_\psi$

Hence, the induced metric on the hypersurface is given by the embedding variables,

$$g_{ab} = X^\alpha_{,a} g_{\alpha\beta} X^\beta_{,b}. \quad (5.14)$$

The components in the diagonal of the matrix $g_{ab}$ read

$$(R'^2 - T'^2, R^2, 1), \quad (5.15)$$

all other components are zero. The projection operator $X^a_\alpha$, which takes a spacetime vector into a vector in the hypersurface, is expressed by the induced metric,

$$X^a_\alpha = g_{\alpha\beta} X^\beta_{,b} g^{-1}_{ba} = \begin{pmatrix} \frac{-T'}{R'^2 - T'^2} & 0 & 0 \\ \frac{R'}{R'^2 - T'^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.16)$$

If the constraint functions (5.4) and (5.5) are taken together to a covector then it is possible to express the Hamilton functions as follows:

$$H = \int_0^\infty dr \tilde{N}^\alpha \tilde{\mathcal{H}}_\alpha, \quad (5.17)$$

where $\alpha = 0, 1, 2, 3$. Of course $\tilde{\mathcal{H}}_2 = 0 = \tilde{\mathcal{H}}_3$ due to the symmetries, as we have learned in section (4.1). In this notation the constraint equations read: $\mathcal{H}_\alpha = 0$. Then also the Einstein-Rosen momenta $\Pi_T$ and $\Pi_R$ in (5.4) and (5.5) respectively canonically conjugate to the embeddings $T(r)$ and $R(r)$ are taken together to a covector,

$$\Pi_\alpha = (\Pi_T, \Pi_R, 0, 0). \quad (5.18)$$

Thus the Hamilton function in (5.17) is newly expressed by,

$$\tilde{\mathcal{H}}_\alpha = \Pi_\alpha - g^{-1/2} n_\alpha h + X^a_\alpha h_a, \quad (5.19)$$

$$h = \frac{1}{2} \left( \pi_\psi^2 + R^2 \psi'^2 \right), \quad (5.20)$$

$$h_a = (\pi_\psi \psi', 0, 0). \quad (5.21)$$

The constraint function in this form allows to solve the Hamilton equation in an easier way.

$$\dot{\psi}(t, r) = \left\{ \psi(t, r), \int_0^\infty dr' \tilde{N}^\alpha (r') \tilde{\mathcal{H}}_\alpha (r') \right\}. \quad (5.22)$$

In order to compute the Poisson bracket, it is necessary to choose a lapse and a shift function. In our case the fixing of the $\tilde{N}^\alpha$ is not needed, as the introduction of embeddings selects a precise folium (an instant of time) i.e. $t = const.$, what just corresponds to the choice of $\tilde{N}^\alpha$. In fact, only the $r$-dependence of the quantities will be noted, for example
\( \psi(r) \) instead of \( \psi(T(t, r), R(t, r)) \). The choice of the embeddings permits to replace the time derivative by the functional derivative.

\[
\frac{d}{dt} = \int_0^\infty dr \left( \frac{d}{dt} X^\alpha(r) \right) \frac{\delta}{\delta X^\alpha(r)} = \int_0^\infty dr \tilde{N}^\alpha(r) \frac{\delta}{\delta \tilde{N}^\alpha(r)}.
\]

(5.23)

where it has been utilized: \( \frac{d}{dt} X^\alpha(t, r) = \tilde{N}^\alpha \). If the operator (5.23) is applied on the time derivative of the wave function, one gets

\[
\dot{\psi}(t, r) = \int_0^\infty dr' \tilde{N}^\alpha(r') \frac{\delta \psi(r)}{\delta X^\alpha(r')} = \int_0^\infty dr' \tilde{N}^\alpha(r') \left\{ \psi(r), \tilde{H}_\alpha(r') \right\}.
\]

(5.24)

The right hand side is obtained by simplifying the same side of the Hamilton equation (5.22), considering that Poisson brackets are linear in both arguments, so

\[
\{ \psi, H \} = \left\{ \psi(r), \int_0^\infty dr' \tilde{N}^\alpha(r') \tilde{H}_\alpha(r') \right\} = \int_0^\infty dr' \tilde{N}^\alpha(r') \{ \psi(r), \tilde{H}_\alpha(r') \}.
\]

(5.25)

Comparing the integrands in (5.24), the following has to hold obviously.

\[
\frac{\delta \psi(r)}{\delta X^\alpha(r')} = \{ \psi(r), \tilde{H}_\alpha(r') \}.
\]

(5.26)

If in the above equation the Poisson bracket is solved then the conditional equation for the canonical momentum \( \pi_\psi(r) \) comes out.

\[
\{ \psi(r), \tilde{H}_\alpha(r') \} = \int_0^\infty dr'' \left( \frac{\delta \psi(r)}{\delta \tilde{H}_\alpha(r')} \frac{\delta \tilde{H}_\alpha(r')}{\delta \pi_\psi(r'')} - \frac{\delta \psi(r)}{\delta \pi_\psi(r'')} \frac{\delta \tilde{H}_\alpha(r')}{\delta \psi(r'')} \right)
\]

\[
= \int_0^\infty dr'' \delta(r - r'') \delta(r - r') [-g^{-1/2} n_\alpha \pi_\psi(r') + X^1_\alpha \psi'(r')]
\]

\[
= -g^{-1/2} n_\alpha \pi_\psi(r') \delta(r - r') + X^1_\alpha \psi'(r') \delta(r - r').
\]

(5.27)

The canonical equation \( \dot{\psi} = \{ \psi, H \} \) has been reduced to

\[
\frac{\delta \psi(r)}{\delta X^\alpha(r')} = \delta(r - r') \left[ -g^{-1/2} n_\alpha \pi_\psi(r') + X^1_\alpha \psi'(r') \right]
\]

(5.28)

To get the momentum, we set this form of the canonical equation in (5.24), remembering \( g^{-1/2} = R^{-1} \), \( g \) is the determinant of the Minkowskian metric in cylindrical coordinates.

\[
\int_0^\infty dr' \tilde{N}^\alpha(r') \frac{\delta \psi(r)}{\delta X^\alpha(r')} = \int_0^\infty dr' \tilde{N}^\alpha(r') \delta(r - r') \left[ X^1_\alpha \psi'(r') - R^{-1} n_\alpha \pi_\psi(r') \right]
\]

\[
\int_0^\infty dr' \tilde{N}^\alpha(r') \frac{\partial \psi(r)}{\partial X^\alpha(r')} \delta(r - r') = \tilde{N}^\alpha(r) \left[ X^1_\alpha \psi'(r) - R^{-1} n_\alpha \pi_\psi(r) \right]
\]

\[
n^\alpha(r) \frac{\partial \psi(r)}{\partial X^\alpha(r')} = n^\alpha(r) X^1_\alpha \psi'(r) - R^{-1} n^\alpha n_\alpha \pi_\psi(r).
\]

(5.29)
Since the smear function $\tilde{N}_\alpha(r)$ can be arbitrarily chosen, it is allowed to substitute it with the unit vector $n^\alpha$. As $n^\alpha n_\alpha = -1$ and $n^\alpha X_\alpha^1 = 0$, the formula for the momentum finally follows from (5.29):

$$\pi_\psi(r) = R(r) \left[ R_{,r} \psi_{,T}(T(r), R(r)) + T_{,r} \psi_{,R}(T(r), R(r)) \right].$$

(5.30)

The canonical momentum $\pi_\psi$ is then given, as suited, by a functional of the massless scalar field $\psi$. Inserting the derivatives with respect to the embeddings the momentum is then written as a functional of the mode expansion $A$ and $A^*$ contained in the field $\psi$.

### 5.2 Cauchy Data

The cylindrically symmetric wave equation (2.78) as also equivalently the Hamilton equation (5.1) define the initial value problem for the scalar field $\psi$. If initial data are given, also known as Cauchy data, the dynamics of the field $\psi(T, R)$ is uniquely described by the differential equation (2.78). If the problem is solved in the canonical formalism then an initial value of the field $\psi(T, R)$ and of the momentum $\pi_\psi(T, R)$ must be given to uniquely fix the dynamics of the field by the conditional equation of the field and momentum. The field $\psi(T, R)$ is known as solution to the wave equation (2.78), see for instance (5.2). The momentum $\pi_\psi(T, R)$ is obtained by inserting (3.34) and (3.35) in equation (5.30). Then the momentum $\pi_\psi(T, R)$ reads:

$$\pi_\psi(T, R) = iR \int_0^\infty dk \, k J_0(kR) \left[ A(k) e^{ikT} - A^*(k) e^{-ikT} \right] R_{,r}
- R \int_0^\infty dk \, k J_1(kR) \left[ A(k) e^{ikT} + A^*(k) e^{-ikT} \right] T_{,r}.$$  

(5.31)

Now we affiliate the Cauchy data $\psi_0(r)$ and $\pi_\psi_0(r)$, which are calculated for $T(r) = 0$ and $R(r) = r$:

$$\psi_0(r) \equiv \psi(T, R)|_{T=0, R=r} = \int_0^\infty dk \, J_0(kr) \left[ A(k) + A^*(k) \right],$$  

(5.32)

$$\pi_\psi_0(r) \equiv \pi_\psi(T, R)|_{T=0, R=r} = ir \int_0^\infty dk \, J_0(kr) \left[ A(k) - A^*(k) \right].$$  

(5.33)

In order to enter the initial data $\psi_0(r)$ and $\pi_\psi_0(r)$ in the general solution $\psi(T, R)$ and $\pi_\psi(T, R)$, $\psi_0(r)$ is adapted to $\pi_\psi_0(r)$. Subsequently the sum of both is Hankel transformed.

$$ikr\psi_0(r) = ir \int_0^\infty dk \, J_0(kr) \left[ A(k) + A^*(k) \right]$$

$$\pi_\psi_0(r) + ikr\psi_0(r) = 2ir \int_0^\infty dk \, J_0(kr) A(k)$$

$$2iA(k) = \int_0^\infty dr \, J_0(kr) \left[ \pi_\psi_0(r) + ikr\psi_0(r) \right].$$  

(5.34)
Hence, the expansion coefficients $A(k)$ and $A^*(k)$ are given as functionals of the Cauchy data $\psi_0(r)$ and $\pi_{\psi_0}(r)$ for $T(r) = 0$ and $R(r) = r$.

$$A(k) = \frac{1}{2} \int_0^\infty dr \ J_0(kr) \left[ kr \psi_0(r) - i \pi_{\psi_0}(r) \right],$$

(5.35)

$$A^*(k) = \frac{1}{2} \int_0^\infty dr \ J_0(kr) \left[ kr \psi_0(r) + i \pi_{\psi_0}(r) \right].$$

(5.36)

By inserting the mode expansions (5.35) and (5.36) in the general solution (5.2) and in the momentum (5.31), we managed to express the canonical quantities $\psi(T, R)$ and $\pi_\psi(T, R)$ as quantities of the Cauchy data $\psi_0(r)$ and $\pi_{\psi_0}(r)$.

$$\psi[A(k), A^*(k); T(r), R(r)] = \int_0^{\infty} dk \ J_0(kR) \left[ \frac{1}{2} \int_0^{\infty} dr \ J_0(kr) \left( kr \psi_0(r) - i \pi_{\psi_0}(r) \right) e^{ikT} \right.\left. + \frac{1}{2} \int_0^{\infty} dr \ J_0(kr) \left( kr \psi_0(r) + i \pi_{\psi_0}(r) \right) e^{-ikT} \right],$$

(5.37)

$$\pi_\psi[A(k), A^*(k); T(r), R(r)] = iR \int_0^{\infty} dk \ J_0(kR) \left[ \frac{1}{2} \int_0^{\infty} dr \ J_0(kr) \left( kr \psi_0(r) - i \pi_{\psi_0}(r) \right) e^{ikT} \right.\left. - \frac{1}{2} \int_0^{\infty} dr \ J_0(kr) \left( kr \psi_0(r) + i \pi_{\psi_0}(r) \right) e^{-ikT} \right] R_{rr} - R \int_0^{\infty} dk \ J_1(kR) \left[ \frac{1}{2} \int_0^{\infty} dr \ J_0(kr) \left( kr \psi_0(r) - i \pi_{\psi_0}(r) \right) e^{ikT} \right.\left. + \frac{1}{2} \int_0^{\infty} dr \ J_0(kr) \left( kr \psi_0(r) + i \pi_{\psi_0}(r) \right) e^{-ikT} \right] T_{rr}.$$

(5.38)

Hence, the dynamics is uniquely given by the initial data $\psi_0(r)$ and $\pi_{\psi_0}(r)$ and so the initial value problem has been solved completely.

### 5.3 Observables $A(k)$ and $A^*(k)$

In the final part of the present chapter we would like to deduce the phase space functions $A(k)$ and $A^*(k)$, which turn out to be observables of the cylindrically symmetric gravitational waves. By deriving the field $\psi$ and its conjugate momentum $\pi_\psi$ as functional of the Cauchy data (5.32) and (5.33), we also calculated the phase space functions $A(k)$ and $A^*(k)$ for a specific hypersurface, namely for $T(r) = 0$ and $R(r) = r$. Of course, our goal is a generalization of these mode expansions, that is to say for the phase space functions of an arbitrary hypersurface. To this end, we perform some Hankel transformations of the canonical variables $\psi$ and $\pi_\psi$, whose sum finally will lead to the desired set of observables.
5.3 Observables $A(k)$ and $A^*(k)$

In the beginning we start to transform the momentum while multiplying it with a phase,

$$
i \int_0^\infty \! dr \ e^{ikT} J_0(kR) \pi_\psi(r) = - e^{ikT} [A(k) e^{ikT} - A^*(k) e^{-ikT}]$$

$$- i \int_0^\infty \! dr \ R \int_0^\infty \! dk \ J_0(kR) J_1(kR)$$

$$\times [A(k) e^{ikT} + A^*(k) e^{-ikT}] T_{\pi \psi} e^{ikT}. \quad (5.39)$$

The field is treated in a similar way, in order to obtain the integral term in the right hand side of the equation above.

$$i \int_0^\infty \! dr \ e^{ikT} k J_1(kR) T_{\psi}(r) = i \int_0^\infty \! dr \ R \int_0^\infty \! dk \ k J_0(kR) J_1(kR)$$

$$\times [A(k) e^{ikT} + A^*(k) e^{-ikT}] T_{\psi} e^{ikT}. \quad (5.40)$$

Further on $\psi(r)$ is transformed once more in order to get something related to the first part of the transformation (5.39).

$$\int_0^\infty \! dr \ e^{ikT} k R, r J_0(kR) \psi(r) = e^{ikT} [A(k) e^{ikT} + A^*(k) e^{-ikT}]. \quad (5.41)$$

If the sum over the three transformations is taken, we easily obtain the observables.

$$2A^*(k) = \int_0^\infty \! dr \ e^{ikT} \{ \psi(r) \ k R \left[ i T_{\psi} J_1(kR) + R_{\psi} J_0(kR) \right] + i J_0(kR) \pi_\psi \}. \quad (5.42)$$

Thus, the observables on an arbitrary hypersurface are:

$$A(k) = \frac{1}{2} \int_0^\infty \! dr \ e^{-ikT} \{ \psi(r) \ k R \left[ R_{\psi} J_0(kR) - i T_{\psi} J_1(kR) \right] - i J_0(kR) \pi_\psi \}, \quad (5.43)$$

$$A^*(k) = \frac{1}{2} \int_0^\infty \! dr \ e^{ikT} \{ \psi(r) \ k R \left[ R_{\psi} J_0(kR) + i T_{\psi} J_1(kR) \right] + i J_0(kR) \pi_\psi \}. \quad (5.44)$$

These phase space functions yield just the mode expansions (5.35) and (5.36) if $T(r) = 0$ and $R(r) = r$ are set in the observables above. So (5.43) and (5.44) are really generalizations of the phase space functions (5.35) and (5.36).
Chapter 6

Algebra of the Observables

For the developing of a quantum theory of the cylindrically symmetric gravitational waves, observables of the system are needed, which enter the theory as operators. So it is of crucial importance to investigate the gauge invariant property of the phase space functions proposed to be observables of the model. Then, if the derived phase space functions $A$ and $A^*$ on an arbitrary hypersurface are suggested to be observables of the treated model, gauge invariance of these functions has to be assured. In the first part of this chapter we present the condition a phase space function has to fulfill in order to be considered as an observable. Then we will show that the phase space functions $A$ and $A^*$ are observables. In the second part we will calculate the Poisson brackets of the observables in order to derive their algebra.

6.1 Gauge Invariant Phase Space Function

A classical observable is a phase space function, which is gauge invariant on the constraint surface [9]. First class constraints $\gamma_a$ of a system are quantities, which are the generators of infinitesimal gauge transformations. A phase space function, which remains constant on the orbit in the constraint surface is gauge invariant. This is equivalent to the vanishing Poisson brackets of the phase space function $F$ and the first class constraints $\gamma_a$,

$$\{F, \gamma_a\} \approx 0.$$  \hspace{1cm} (6.1)

The bracket is calculated on the whole phase space and afterwards its value is taken on the constraint surface. The weak equality symbol $\approx$ is to be understood in this way. In the following section an example for this kind of calculation is given. If (6.1) is fulfilled then we are allowed to consider the phase space function $F$ as an observable of the system.

6.1.1 Gauge Invariance of $A(k)$ and $A^*(k)$

In order to find out whether the mode coefficients $A(k)$ and $A^*(k)$ of the cylindrically symmetric scalar wave are gauge invariant, we have to assure that the Poisson bracket
of these functionals with the first class constraints vanishes on the constraint surface, according to the formula (6.1). The first class constraints are the super-Hamiltonian (5.4) and the supermomentum (5.5). Thus, \( A(k) \), and therewith also \( A^*(k) \), are observables if the following formula holds:

\[
\{ A(k), \tilde{\mathcal{H}}_\alpha(r) \} \approx 0.
\] (6.2)

The constraint submanifold \( \Gamma \) is the surface where \( \tilde{\mathcal{H}}_\alpha = 0 \). Then, saying the Poisson bracket (6.2) has to vanish on the constraint surface means that one evaluates firstly the bracket and secondly, if still necessary, \( \tilde{\mathcal{H}}_\alpha = 0 \) is set, so to investigate whether the bracket really vanishes. The relation (6.2) can also be rewritten equivalently as,

\[
\{ A(k), \tilde{\mathcal{H}}_\alpha(r) \}|_{\Gamma; \tilde{\mathcal{H}}_\alpha=0} = 0.
\] (6.3)

As shown in [4] the Poisson bracket (6.2) vanish, such that the phase space function are observables. The calculation is performed for a scalar field propagating in curved spacetime. The cylindrically symmetric scalar field on a Minkowskian background is then a special case of the presented computation.

### 6.2 Poisson Brackets

In this section we are going to calculate the Poisson brackets of the phase space functions \( A(k) \) and \( A^*(k) \). The motivation therefore is to elaborate a Poisson algebra of these observables. To this end we introduce the functions \( Q(r) \) and \( P(r) \) written in [4], where just for consistency we retain our normalization.

\[
Q(r) = \int_0^\infty dk \ J_0(kr) \left[ A(k) + A^*(k) \right],
\] (6.4)

\[
P(r) = i r \int_0^\infty dk \ J_0(kr) \left[ A(k) - A^*(k) \right],
\] (6.5)

where \( Q(r) = \psi_0(r) \) (5.32) and \( P(r) = \pi_\psi(r) \) (5.33). As we know from section 5.2, the functions \( Q(r) \) and \( P(r) \) are related to \( A(k) \) and \( A^*(k) \) by a Hankel transformation, which will be undone in the next step so to describe the phase space functions as functionals of \( Q(r) \) and \( P(r) \).

\[
A(k) = \frac{1}{2} \int_0^\infty dr \ J_0(kr) \left[ kr \ Q(r) - i \ P(r) \right],
\] (6.6)

\[
A^*(k) = \frac{1}{2} \int_0^\infty dr \ J_0(kr) \left[ kr \ Q(r) + i \ P(r) \right].
\] (6.7)

#### 6.2.1 Algebra of \( A(k) \) and \( A^*(k) \)

It is straight forward to compute the Poisson brackets for the observables \( A(k) \) and \( A^*(k) \).

\[
\{ A(k), A^*(k') \} = \int_0^\infty dr \left[ \frac{\delta A(k) \ \delta A^*(k')}{\delta Q(r) \ \delta P(r)} - \frac{\delta A(k) \ \delta A^*(k')}{\delta P(r) \ \delta Q(r)} \right],
\] (6.8)
Inserting the functional derivatives in (6.8), we obtain the coefficients $A(k)$ and $A^*(k)$ to be observables, which are canonical conjugate to each other.

\[
\{A(k), A^*(k')\} = \int_0^\infty dr \left[ -\frac{i}{4} k r J_0(kr) J_0(k' r) + \frac{i}{4} k' r J_0(k' r) J_0(kr) \right],
\]

\[
= -\frac{i}{4} k \frac{1}{k'} \delta(k - k') + \frac{i}{4} k' \frac{1}{k} \delta(k - k'),
\]

\[
= \frac{i}{2} \delta(k - k').
\]

\[
\{A^*(k), A^*(k')\} = \int_0^\infty dr \left[ \frac{i}{4} k' r J_0(kr) J_0(k' r) - \frac{i}{4} k r J_0(k r) J_0(k' r) \right],
\]

\[
= \frac{i}{4} k' \frac{1}{k} \delta(k - k') - \frac{i}{4} k \frac{1}{k'} \delta(k - k'),
\]

\[
= 0.
\]

Hence, the well-defined algebra for the phase space functions $A(k)$ and $A^*(k)$ of the cylindrical gravitational wave has been elicited:

\[
\{A(k), A^*(k')\} = \frac{i}{2} \delta(k - k'),
\]

\[
\{A(k), A(k')\} = 0,
\]

\[
\{A^*(k), A^*(k')\} = 0.
\]

### 6.2.2 Algebra of $Q(r)$ and $P(r)$

Analogously, the Poisson brackets of $Q(r)$ and $P(r)$ can be calculated, so to obtain the Poisson algebra of these observables.

\[
\{Q(r), P(r')\} = \int_0^\infty dk \left[ \frac{\delta Q(r) \delta P(r')}{\delta A(k) \delta A^*(k)} - \frac{\delta Q(r) \delta P(r')}{\delta A^*(k) \delta A(k)} \right],
\]

\[
\frac{\delta Q(r)}{\delta A(k)} = J_0(kr),
\]

\[
\frac{\delta Q(r)}{\delta A^*(k)} = J_0(kr),
\]

\[
\frac{\delta P(r')}{\delta A(k)} = i k r' J_0(kr'),
\]

\[
\frac{\delta P(r')}{\delta A^*(k)} = -i k r' J_0(kr').
\]
From this it follows immediately the Poisson algebra for $Q(r)$ and $P(r)$:

$$\{Q(r), P(r')\} = -2i \delta(r - r'), \quad \{Q(r), Q(r')\} = 0, \quad \{P(r), P(r')\} = 0. \quad (6.18)$$

It is not surprising to get the same Poisson algebra for $A(k)$ and $A^∗(k)$ as well as for $Q(r)$ and $P(r)$ until normalization and sign. The functions $Q(r)$ and $P(r)$ are, from a mathematical point of view, just expressions for the Hankel transformed real and imaginary parts of the mode coefficients $A(k)$ and $A^∗(k)$.

### 6.3 Quantum Theoretical Property

The Poisson algebra (6.14) for the observables $A(k)$ and $A^∗(k)$ has an interesting feature if investigated from a quantum theoretical point of view. As known the Poisson brackets multiplied by the complex $i$ correspond to the commutators of quantum theory. Multiplying the Poisson brackets in (6.14) with the complex $i$, the commutation relations for the creator and the annihilator operators of the scalar field - see for instance [10] - are obtained up to a normalizing factor $\sqrt{2}$. Hence, the observables $A(k)$ and $A^∗(k)$ can just be identified with the creator and annihilator respectively.

$$A \rightarrow A, \quad A^∗ \rightarrow A^\dagger. \quad (6.19)$$

Viewing now the cylindrically symmetric scalar field as a wave function in quantum field theory and applying the operators (6.19) and (6.19) on the wave function, the Fock space of the system can be constructed, which spans the Hilbert space of the quantum states of the cylindrically symmetric scalar field.
Chapter 7

Conclusion

By a conformal deformation and a coordinate transformation into the inertial system the general cylindrically symmetric metric is brought into diagonal form. The introduction of the Einstein-Rosen coordinates allows a further simplification of the metric. The symmetries and the special form of the metric reduce the Einstein equations to a set of three differential equations. The master equation is solved by the cylindrically symmetric massless scalar field propagating on a Minkowskian background.

The introduction of the reduced ADM action for the Einstein-Rosen wave, and therewith of the super-Hamilton and supermomentum gives the opportunity to derive the momentum, which is canonically conjugate to the scalar field, by solving the Hamilton equation of it. Combining the scalar field with its canonically conjugate momentum and performing several Hankel transformations it is possible to isolate the mode expansions contained in the function of the scalar field. Just these phase space functions turn out to be gauge invariant and hence observables of the model. This is shown by investigating the Poisson bracket of the phase space function and the first class constraints. As it is required for observables the Poisson bracket vanishes.

In order to guarantee the phase space functions to be canonical conjugate to each other the Poisson brackets of them are computed. The result assure the expected property and allow the correspondence of the observables to the annihilator and creator operators in the formulation of the quantum theory for the cylindrically symmetric scalar field on a Minkowskian background. Acting on the scalar field, now considered as a wave function, the Fock space can be developed, spanning therewith the Hilbert space of the quantum states of the cylindrically symmetric scalar field.
Appendix A

Orthonormality Relation for the Bessel Function of the First Kind

A.1 One Dimensional Fourier Transformation in Cartesian Coordinates

In the beginning we shall study the following integral:

\[ \int_{-\infty}^{\infty} dk \ e^{-ikx} e^{ikx'} = \int_{-\infty}^{\infty} dk \ e^{i(x-x')k}. \] (A.1)

Unfortunately this integral is divergent! In order to obviate this trouble, a suitable regularisation is carried out. To this end a new measure is defined, which contains a proper damping, whereas \( \epsilon > 0 \).

\[ [dk] \doteq (\theta(-k) \ e^{\epsilon k} + \theta(k) \ e^{-\epsilon k}) \ dk. \]

Therewith the integral exists and can be calculated.

\[ \int_{-\infty}^{\infty} [dk] e^{-i(x-x')k} = \int_{-\infty}^{0} dk \ e^{\epsilon k} e^{i(x-x')k} + \int_{0}^{\infty} dk \ e^{-\epsilon k} e^{i(x-x')k} \]

\[ = \frac{1}{\epsilon - i(x-x')} - \frac{1}{\epsilon + i(x-x')}, \] with \( u \doteq x - x' \)

\[ = \frac{2\epsilon}{\epsilon^2 + u^2}. \]

The next step is to perform the limes \( \epsilon \to 0 \) to achieve the value of the initial integral \( (A.1)^1 \)

\[ \int_{-\infty}^{\infty} dk \ e^{i(x-x')k} = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} [dk] e^{i(x-x')k} \]

\[ = \lim_{\epsilon \to 0} \frac{2\epsilon}{\epsilon^2 + u^2} = 2\pi \delta(x-x') \] (A.2)

---

\(^1\)In [11] we find: \( \lim_{\epsilon \to 0} \frac{1}{\epsilon^2 + \pi^2} = \delta(x) \), for \( \epsilon > 0 \).
If the distribution \((A.2)\) is applied on a test function \(f \in \mathcal{S}\), so obviously the result is not normalized. To adjust this non-conformance it is sufficient to assess the integral \((A.1)\) with \(\frac{1}{2\pi}\). Now we are allowed to undertake the identity map of the Fourier transformation in one dimension. Let’s define the transformation and its inverse respectively as follows:

\[
\begin{align*}
 f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ikx} \tilde{f}(k) \\
 \tilde{f}(k) &= \int_{-\infty}^{\infty} dx' \, e^{-ikx'} f(x')
\end{align*}
\]

The identity map reads

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \, e^{ikx} \int_{-\infty}^{\infty} dk \, e^{-ikx'} f(x'), \quad (A.3)
\]

whereby the integration order has been changed, since the integral over \(k\) is regularizable, as shown above. So we obtain

\[
\begin{align*}
 f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dk \, e^{i(x-x')k} f(x') \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \, 2\pi \delta(x-x') \, f(x') = f(x). \quad (A.4)
\end{align*}
\]

Finally, the identity map \((A.3)\) is normalized on one by the factor \(N_{d=1} = \frac{1}{2\pi}\).

### A.2 Two Dimensional Fourier Transformation in Cartesian Coordinates

In order to understand better the switch from the transformation in Cartesian coordinates to the one in cylindrical coordinates, first we shall write down the relations in the first mentioned coordinates.

We take two vectors from the two dimensional vector space \(\mathbb{R}^2\), namely \(\vec{x} \doteq (x_1, x_2)\) and \(\vec{k} \doteq (k_1, k_2)\). The Fourier transformation in \(d = 2\) and its inverse are defined respectively as follows:

\[
\begin{align*}
 f(\vec{x}) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d^2 k \, e^{i\vec{k}\vec{x}} \tilde{f}(\vec{k}) \quad (A.5) \\
 \tilde{f}(\vec{k}) &= \int_{-\infty}^{\infty} d^2 x' \, e^{-i\vec{k}\vec{x}'} \tilde{f}(\vec{x}'). \quad (A.6)
\end{align*}
\]

\(^2\mathcal{S}\) is the Schwartz space.
Applying the inverse on the transformation we get as is well known the identity map:

\[
\begin{align*}
  f(\vec{x}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d^2k \ e^{i\vec{k}\cdot\vec{x}} \frac{1}{2\pi} \int_{-\infty}^{\infty} d^2x' \ e^{-i\vec{k}\cdot\vec{x}'} \ \tilde{f}(\vec{x}') \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d^2x' \ \int_{-\infty}^{\infty} d^2k \ e^{i(\vec{x}-\vec{x}')\cdot\vec{k}} \ \tilde{f}(\vec{x}') \\ &= \int_{-\infty}^{\infty} dx'_1 \int_{-\infty}^{\infty} dx'_2 \ \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_1 \ e^{i(x_1-x'_1)k_1} \\
  &\quad \quad \quad \quad \times \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_2 \ e^{i(x_2-x'_2)k_2} \ \tilde{f}(x'_1, x'_2) \\
  &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d^2x' \ (2\pi)^2 \delta(\vec{x}-\vec{x}') \tilde{f}(\vec{x}) \\
  &= f(\vec{x}).
\end{align*}
\] (A.7)

In the separated representation of (A.8) we observe twice a one dimensional integration over \(k_1\) resp. \(k_2\) of the form (A.1). Belonging to (A.2) each of them are equal to \(2\pi \delta(x_i-x'_i), i = 1, 2\). Consequently, we understand the reason why in (A.5) the normalizing factor in two dimensions \(N_{d=2} = \frac{1}{(2\pi)^2}\) appears, analogously to the Fourier-transformation in one dimension.

### A.3 Hankel Transformation

We start now with noting the two components of both vectors \(\vec{x} \doteq (x_1, x_2)\) and \(\vec{k} \doteq (k_1, k_2)\) in cylindrical coordinates. That is

\[
x_1 = x \cos \alpha, \quad x_2 = x \sin \alpha \quad k_1 = k \cos \beta, \quad k_2 = k \sin \beta.
\]

For further need, we prepare a third vector from \(\mathbb{R}^2\) in the same coordinates: \(\vec{x}' \doteq (x'_1, x'_2)\) with \(x'_1 = x' \cos \alpha', x'_2 = x' \sin \alpha'\). Thus, we get for the following products:

\[
\vec{k}\vec{x} = kx \cos(\alpha - \beta), \quad \vec{k}\vec{x}' = kx' \cos(\alpha' - \beta).
\]

Please note that the scalars \(k, x\) and \(x'\) \(\in \mathbb{R}\) are the absolute values of the homonymic vectors. Furthermore the determinants of the Jacobi matrix for the integral over \(x'\) and \(k\) give the measures \(dx'\) times \(x'\) and \(dk\) times \(k\). A function \(g(\vec{x}) = g(x_1, x_2)\) and its Fourier transformed function \(\tilde{g}(\vec{x}') = \tilde{g}(x'_1, x'_2)\) can be noted in polar coordinates as follows:

\[
\begin{align*}
  g(\vec{x}) &= h(x, \alpha) = f(x)e^{i\alpha} \\
  \tilde{g}(\vec{x}') &= \tilde{h}(x', \alpha') = \tilde{f}(x')e^{i\alpha'}.
\end{align*}
\] (A.9) (A.10)

\[\]Belonging to (A.2) the integral (A.1) is regularizable. Thus, the integration order is arbitrary.
If we write down the Fourier transformation identity map in cylindrical coordinates, then it reads:

\[
\begin{align*}
\int_{0}^{\infty} dk \int_{-\pi}^{\pi} d\beta \ e^{ikx \cos(\beta - \alpha)} \int_{0}^{\infty} dx' \int_{-\pi}^{\pi} d\alpha' \ e^{-ikx' \cos(\alpha' - \beta)} \ e^{i\alpha' \tilde{f}(x')}
\end{align*}
\]

We treat now the last equation in order to obtain finally the identity (A.7) containing the Bessel function \(J_n(kx)\). For this purpose we prove that an integral of the form

\[
\int_{0}^{2\pi} d\varphi \ e^{ikx \cos(\varphi - \varphi_0)}
\]

is equal to

\[
\int_{0}^{2\pi} d\vartheta \ e^{ikx \cos \vartheta}.
\]

This is possible due to the symmetry property of the trigonometric functions if a whole period is taken. That is gladly our case.

\[
\int_{0}^{2\pi} d\varphi \ e^{ikx \cos(\varphi - \varphi_0)} = \int_{0}^{\varphi_0} d\varphi \ e^{ikx \cos(\varphi - \varphi_0)} + \int_{\varphi_0}^{2\pi} d\varphi \ e^{ikx \cos(\varphi - \varphi_0)} = \int_{\varphi_0}^{2\pi} d\varphi \ e^{ikx \cos(\varphi - \varphi_0)} + \int_{2\pi}^{\varphi_0} d\varphi \ e^{ikx \cos(\varphi - \varphi_0)} = \int_{\varphi_0}^{2\pi} d\varphi \ e^{ikx \cos(\varphi - \varphi_0)} \]

whereas \(\vartheta = \varphi - \varphi_0\). QED

Thereafter we return to the crucial equation (A.11). We perform a first substitution of variables, namely

\[
\beta - \alpha = \frac{\pi}{2} + \beta'.
\]

The transformation map (A.11) changes to

\[
\begin{align*}
f(x) e^{i\alpha} &= \frac{1}{(2\pi)^2} \int_{0}^{\infty} dx' \int_{0}^{\infty} dk \int_{-\frac{3\pi}{2} - \alpha}^{\frac{\pi}{2} - \alpha} d\beta' e^{-ikx \sin \beta'} \times \int_{-\pi}^{\pi} d\alpha' e^{-ikx' \cos(\alpha' - \beta - \frac{\pi}{2})} e^{i\alpha' \tilde{f}(x')}
\end{align*}
\]

In doing so we have employed the properties of the cosine function

\[
\begin{align*}
\cos(\varphi) &= \cos(-\varphi) & (A.15) \\
\cos\left(\frac{\pi}{2} + \varphi\right) &= -\sin(\varphi). & (A.16)
\end{align*}
\]
In virtue of the periodicity of the trigonometric functions, the interval \([-\frac{3}{2}\pi - \alpha, \frac{3}{2} - \alpha]\) can be traced back to the original interval \([-\pi, \pi]\), cp. (A.12). Then a second substitution is undertaken, namely
\[
\alpha' - \alpha - \beta' = \vartheta, \tag{A.17}
\]
whereby one gets for the transformation newly:
\[
f(x) e^{ina} = e^{ina} \int_{0}^{\infty} dx' x' \int_{0}^{\infty} dk k \frac{1}{(2\pi)} \int_{-\pi}^{\pi} d\beta' e^{-ikx \sin \beta' + in\beta'}
\times \frac{1}{(2\pi)} \int_{-\pi}^{\pi} d\vartheta e^{-ikx' \sin \vartheta + in\vartheta} \tilde{f}(x'). \tag{A.18}
\]
Also for this step the periodicity of the trigonometric functions has been of use, since one integrates at each case over a complete period. Further on the identity \(\cos(\vartheta - \frac{\pi}{2}) = \cos(\frac{\pi}{2} - \vartheta) = \sin \vartheta\) has been utilized, too. The integral representation of the Bessel function \(J_n(z)\) is given in [12] by
\[
J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi e^{i(n\varphi - z \sin \varphi)}. \tag{A.19}
\]
By setting \(\vartheta = \varphi + \pi\) and making use of \(\cos(\varphi + \pi) = -\cos \varphi\) and \(\sin(\varphi + \pi) = -\sin \varphi\), it can be shown that the Bessel function \(J_n(z)\) is a real function, i.e. \(J_n(z) = J_n^*(z)\). Comparing both integrals over \(\beta'\) and \(\vartheta\) in (A.18) with the integral representation of the Bessel function \(J_n(z)\), we remark that the integrals in (A.18) are just the Bessel functions. After latter cognitions the sums over \(\beta'\) and \(\vartheta\) can be substituted with the Bessel function \(J_n(kx)\) and \(J_n(kx')\), replacing \(z\) with \(kx\) respectively with \(kx'\). Thus, we get:
\[
f(x) = \int_{0}^{\infty} dx' x' \int_{0}^{\infty} dk k J_n(kx) J_n(kx') \tilde{f}(x') = f(x) \tag{A.20}
\]
If we are interested in the Fourier transformation on the dependence of the vector’s \(\vec{x}\) absolute value only, then the function \(f\) in (A.18) is no more subject to the angle \(\alpha\). Recapitulating we put down:
\[
g(|\vec{x}|) = f(x) = \int_{0}^{\infty} dx' x' \int_{0}^{\infty} dk k J_n(kx) J_n(kx') \tilde{f}(x') = f(x). \tag{A.21}
\]
Analogously to the identity map in Cartesian coordinate (A.7), the Fourier transformation and its inverse, described in the new variables, can be obtained comparing just the form in (A.7) and the gained form in cylindrical coordinates in (A.20). The so-called Hankel transformations read:
\[
f(x) = \int_{0}^{\infty} dk k J_n(kx) \tilde{f}(k)
\tilde{f}(k) = \int_{0}^{\infty} dx' x' J_n(kx') f(x'). \tag{A.22}
\]
\(^4\)The integral succession can be switched arbitrary, as the regularisation for the one dimensional case in section A.1 guarantees the existence of the integrals.
These equations are called Hankel transformation. They are the analogous expressions to the two dimensional Fourier transformation with cylindrical symmetry. The normalization factor for this transformation is also given in the last two equations, namely $N_{\text{cyl.coord.}} = 1$. So we deduce that the Bessel function defined in [12] already are normalized. Observing properly (A.11), we remark that the initial factor $\frac{1}{(2\pi)^2}$ has been absorbed in both Bessel functions $J_n(kx), J_n(kx')$ belonging to the definition in [12].

As a main result of this appendix we want to write down explicitly the normalization factor for the orthonormality relation of the Bessel function of the first kind $J_n(kx)$. We can take the relation directly from the expression (A.20):

$$f(x) = \int_0^\infty dx' x' \int_0^\infty dk k J_n(kx)J_n(kx') f(x') = f(x)$$  \hspace{1cm} (A.23)

The orthonormality relation is just the integral over $k$, which obviously has to be equal to the Dirac $\delta$-function multiplied with the normalizing factor.

$$\int_0^\infty dk k J_n(kx)J_n(kx') = \frac{1}{x'} \delta(x - x'),$$  \hspace{1cm} (A.24)

with the normalization factor

$$N_{x'} \doteq \frac{1}{x'}.$$  \hspace{1cm} (A.25)

For the Hankel transform $\tilde{f}(k)$ there is of course the analogous orthonormality relation:

$$\int_0^\infty dx x J_n(kx)J_n(k'x) = \frac{1}{k'} \delta(k - k'),$$  \hspace{1cm} (A.26)

with the corresponding normalization factor

$$N_{k'} \doteq \frac{1}{k'}.$$  \hspace{1cm} (A.27)
Appendix B

The Energy $\Gamma(T, R)$

The metric (2.32) is determined among other things also by the function $\Gamma(T, R)$. This function is given indirectly by the gradient formed by the derivatives $\Gamma_T(T, R)$ (2.80) and $\Gamma_R(T, R)$ (2.79), which describe the energy density of the massless scalar field $\psi(T, R)$ on a Minkowsikian spacetime background and respectively the radial energy current density.

$$\vec{\nabla}\Gamma(T, R) = \begin{pmatrix} \Gamma_T \\ \Gamma_R \end{pmatrix}$$ (B.1)

In order to get the function $\Gamma(T, R)$ we have to integrate the gradient field (B.1). Starting from the point $\Gamma(0, 0)$ we would like to reach the arbitrary point $\Gamma(T, R)$ by choosing an arbitrary path. However, at this place we make use of a fundamental property of gradient fields. This sort of fields are conservative. This property implicates that all path from the starting point $\Gamma(0, 0)$ to the arbitrary point $\Gamma(T, R)$ are equivalent. So we are free to choose a particular path without restricting generality.

$$\Gamma(T, R) = \Gamma(0, 0) + \int_0^T dT' \Gamma_T(T', 0) + \int_0^R dR' \Gamma_R(T, R').$$ (B.2)

We prove now that

$$\Gamma(T, R) = \int_0^R dR' \Gamma_R(T, R'),$$ (B.3)

if we deal in the following with a pure radiative field without sources on the axis of symmetry ($z$-axis).

**Proof**: If we have no sources on the axis of symmetry, the spatial geometry must be locally Euclidean on the mentioned axis. So the circumference of a small circle $R=$const., $T=$const., $z=$const. is the $2\pi$ multiple of its proper radius. For $R=$const., $T=$const., $z=$const. the Einstein-Rosen line element

$$g^{ER}_{\alpha\beta}(T, R) x^\alpha x^\beta = e^{\Gamma-\psi} (-dT^2 + dR^2) + R^2 e^{-\psi} d\varphi^2 + e^{\psi} dz^2$$ (B.4)
reduces to the formula
\[ ds^2 = R^2 e^{-\psi} d\varphi^2 \]  
\[ \Rightarrow ds^2 = R e^{-\frac{1}{2} \psi} d\varphi. \]  
(B.5)  
(B.6)

As we have Euclidean space near the axis of symmetry, we can set the radius \( r = R e^{-\frac{1}{2} \psi} \), with \( r \) the radius in Euclidean geometry. The Euclidean circumference turns out to be
\[ C = 2\pi r = 2\pi R e^{-\frac{1}{2} \psi}, \]  
(B.7)

for a fixed value of \( R \). In case that \( R \) is variable, i.e. it is no more the radius of the small circle around the \( z \)-axis, then the radius \( r \) is given by
\[ r' = \int_0^R dR' e^{\frac{1}{2} (\Gamma - \psi)}, \]  
(B.8)

for constant \( \varphi \). In this case the circumference \( C' \) is different:
\[ C' = 2\pi r' = 2\pi \int_0^R dR' e^{\frac{1}{2} (\Gamma - \psi)}. \]  
(B.9)

Since the geometry near the \( z \)-axis (\( R \to 0 \)) is Euclidean, because there is no source, both circumferences has to be equal to each other.
\[ \lim_{R \to 0} \frac{C}{C'} = 1. \]  
(B.10)

And therefrom
\[ \lim_{R \to 0} \frac{R e^{-\frac{1}{2} \psi}}{\int_0^R dR' e^{\frac{1}{2} (\Gamma - \psi)}} = 1. \]  
(B.11)

With the help of the method by “de l’Hopital”, we get for the lines
\[ \lim_{R \to 0} \frac{1 + \frac{1}{2} R \frac{\partial}{\partial R} \psi(T, R)}{e^{\frac{1}{2} \Gamma(T, R)}} = 1 \Rightarrow \frac{1}{e^{\frac{1}{2} \Gamma(T, 0)}} = 1. \]  
(B.12)

Obviously the last equation holds only if
\[ \Gamma(T, 0) = 0, \quad \forall T \in \mathbb{R}. \]  
(B.13)

And especially it follows of course
\[ \Gamma(0, 0) = 0. \]  
(B.14)

The derivative of (B.13) with respect to the time \( T \) is clearly zero, too. Then the assertion has been proved:
\[ \Gamma(T, R) = \int_0^R dR' \Gamma_{,R'}(T, R'). \]  
(B.15)

QED

Finally, we see that, since \( \Gamma_{,R} \) is the energy density, the function \( \Gamma(T, R) \) describes the energy of the massless scalar field \( \psi \) (up to a factor \( 2\pi \)) contained in a disk with radius \( R \) and a thickness of \( \Delta z = 1 \) at the time \( T \).
B.1 Computation of $\Gamma(T, R)$

We evaluate now the energy $\Gamma(T, R)$ explicitly. Therefore we integrate the left side of (B.15) with respect to the radius $R$.

$$\Gamma(T, R) = \int_0^R dR' \Gamma_{T, R'}(T, R') = \frac{1}{2} \int_0^R dR' R' (\psi_T^2 + \psi_{R'}^2), \quad (B.16)$$

with

$$\psi(T, R) = \int_0^\infty dk \, J_0(kR) \left[ a(k) e^{ikT} + a^*(k) e^{-ikT} \right] \quad (B.17)$$

The dependence $\Gamma(T, R)$ on $R$ is due to the Bessel function in $\psi(T, R)$. Indeed only the Bessel function is subject to the radius $R$. So the integration over $R$ concerns this function only. In [5] there is the formula 11.3.29 for integrating this kind of Bessel function product. The results are:

$$\int_0^R dR' R' J_0(kR') J_0(k'R') = \frac{R}{k^2 - k'^2} \left[ k J_1(kR') J_1(k'R) - k' J_0(kR) J_1(k'R) \right]$$

$$\int_0^R dR' R' J_1(kR') J_1(k'R') = \frac{R}{k^2 - k'^2} \left[ k J_2(kR') J_1(k'R) - k' J_1(kR) J_2(k'R) \right] \quad (B.18)$$

Therewith the integration over $R$ has been performed, and the resulting large expression can be cast in a compact form, if the following is set:

$$a_1(k) \doteq a(k) \quad a_1(k') \doteq a(k')$$

$$a_2(k) \doteq a^*(k) \quad a_2(k') \doteq a^*(k') \quad (B.19)$$

The integration over $R$ produces eight addends. It is advantageous to notice that four of them are nothing but the complex conjugate of the other four. Together with the definitions in (B.19), this property is helpful to cast the function $\Gamma(T, R)$ in a very short form:

$$\Gamma(T, R) = \frac{1}{2} \int_0^\infty dk \int_0^\infty dk' \left[ K_{11} a_1(k) a_1(k') + K_{12}^* a_2(k) a_2(k') \right. + K_{12} a_1(k) a_2(k') + K_{11}^* a_2(k) a_1(k') \right], \quad (B.20)$$

where

$$K_{11}^* = K_{22} \quad K_{12}^* = K_{21} \quad (B.21)$$
with

\[ K_{11} = K_{11}[k, k'; T, R] = k k' \frac{R}{k^2 - k'^2} e^{-(k+k')T} \left[ k J_2(kR) J_1(k'R) - k' J_1(kR) J_2(k'R) \right. \]
\[ \left. - k J_1(kR) J_0(k'R) + k' J_0(kR) J_1(k'R) \right]; \]

\[ K_{12} = K_{12}[k, k'; T, R] = k k' \frac{R}{k^2 - k'^2} e^{-(k-k')T} \left[ k J_1(kR) J_0(k'R) - k' J_0(kR) J_1(k'R) \right. \]
\[ \left. + k J_2(kR) J_1(k'R) - k' J_1(kR) J_2(k'R) \right]. \]
Appendix C

Ricci Tensor and Ricci Scalar of the Einstein-Rosen Metric

In this appendix we present the calculation of the Ricci tensor and the Ricci scalar of the Einstein-Rosen metric. The computation has been performed with the help of the software MAPLE 6, in particular with a specific package, “with(tensor)” wherein the two tools “RICCI” (for Ricci tensor) and “RS” (for Ricci scalar) are contained. In the following two sections there are the results of both computations. The motivation for this task are shown in the section 1.4.

The Einstein-Rosen metric \(\text{(2.32)}\), whereof the Ricci tensor and the Ricci scalar have been elicited reads:

\[
g_{\alpha\beta} = \begin{pmatrix}
  -e^{\Gamma(T,R)-\psi(T,R)} & 0 & 0 & 0 & 0 \\
  0 & e^{\Gamma(T,R)-\psi(T,R)} & 0 & 0 & 0 \\
  0 & 0 & R^2e^{-\psi} & 0 & 0 \\
  0 & 0 & 0 & e^\psi & 0 \\
  0 & 0 & 0 & 0 & e^\psi \\
\end{pmatrix}
\]

C.1 Ricci Tensor

By virtue of the symmetry of the Einstein-Rosen metric the Ricci tensor has the following five non-vanishing components:

\[
R_{00} = -\frac{1}{2R} \left[ R \Gamma_{,RR} - R \psi_{,RR} - R \Gamma_{,TT} + R \psi_{,TT} + \Gamma_{,R} - \psi_{,R} - R \psi_{,T}^2 \right] \quad \text{(C.1)}
\]

\[
R_{01} = -\frac{1}{2} \left[ \frac{\Gamma_{,T}}{R} - \psi_{,T} \psi_{,R} \right] \quad \text{(C.2)}
\]

\[
R_{10} = -\frac{1}{2} \left[ \frac{\Gamma_{,T}}{R} - \psi_{,T} \psi_{,R} \right] \quad \text{(C.3)}
\]

\[
R_{11} = -\frac{1}{2R} \left[ R \psi_{,RR} - R \Gamma_{,RR} + R \Gamma_{,TT} - R \psi_{,TT} + \psi_{,R} + \Gamma_{,R} - R \psi_{,R}^2 \right] \quad \text{(C.4)}
\]
\[ R_{22} = -\frac{1}{2}R e^{-\Gamma} [R \psi, \psi_R + \psi, R - R \psi, \psi_{TT}] \quad (C.5) \]
\[ R_{33} = \frac{1}{2R} e^{2\psi - \Gamma} [R \psi, \psi_R - R \psi, \psi_{TT} + \psi, R]. \quad (C.6) \]

We remark that the two components \( R_{01} \) and \( R_{10} \) are equal to each other. This is rather what we expected, since the Ricci tensor is symmetric in the two indices.

### C.2 Ricci Scalar

For the Ricci scalar we obtained the following expression:

\[ R = -\frac{1}{2R} e^{-\Gamma + \psi} \left[ 2R \Gamma, \psi_{TT} - 2R \Gamma, \psi_{RR} + 2R \psi, \psi_{RR} - 2R \psi, \psi_{TT} + 2 \psi, \psi_R + R \psi^2, \psi - R \psi^2, \psi_R \right]. \quad (C.7) \]
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