PRESERVING OLD (\([\omega]^{|\aleph_0|}, \supseteq\)) IS PROPER

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Abstract. We give some sufficient and necessary conditions on a forcing notion \(Q\) for preserving the forcing notion (\([\omega]^{|\aleph_0|}, \supseteq\)) is proper. They cover many reasonable forcing notions.

ANOTATED CONTENT

§0 Introduction

[I.e. Definition 0.1, we define the problem and some variants.]

§1 Properness of \(\mathbb{P}_{\mathcal{A}[V]}\) and CH

[Under CH, if non-meagerness of \((<2)^V\) is preserved then \(\mathbb{P}_{\mathcal{A}_r[V]}\) is proper, (1.1). If \(V\) fail CH, then usually \(\mathbb{P}_{\mathcal{A}_r[V]}\) is not proper after a forcing adding a new real and satisfying a relative of being proper, e.g. satisfies c.c.c. or is any true creature forcing.]

§2 General sufficient conditions

[If \(V\) satisfies CH and \(Q\) is c.c.c. then \(\models Q \Rightarrow \mathbb{P}_{\mathcal{A}[V]}\) is proper", in (2.1). In (2.3) we replace \(\mathcal{A}^V\) by a forcing notion \(\mathbb{Q}\) adding no \(\omega\)-sequence, \(Q\) is c.c.c. even in \(V^\mathbb{Q}\). Instead "\(Q\) satisfies the c.c.c." it suffices to demand \(Q\). Lastly, (2.6) prove some proper forcing does not preserve.]
0. Introduction

Gitman proved that $\text{Pr}_1(Q, P(\omega)|V)$ (see definition below, $\mathbb{P}_{P(\omega)} = \mathbb{P}_{A_1}[V]$ is the forcing notion $\{A \in V : A \subseteq \omega, |A| = \aleph_0, A \subseteq^* B\}$ where of course $A \supseteq^* B$ means $B \subseteq^* A$ when $Q$ is adding Cohen (even $2^{\aleph_0}$); but no other examples were known even Sacks forcing. Also for e.g. $V \models \text{"V = L"}$, we did not know a forcing making it not proper.

We investigate the question “$\text{Pr}_1(Q, R)$”, the proper forcing $Q$ preserves that the (old) $R$ is proper for various $R$'s.

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Let us state the problem and relatives.

Definition 0.1. 1) Let $\text{Pr}_1(Q, P)$ means: $Q, P$ are forcing notions, $Q$ is proper and $\models_Q \text{"P, i.e. } P^V \text{ is a proper forcing"}$.
2) For $A \subseteq P(\omega)$ let $P_A = A \cap [\omega]^\omega \cup \{\omega\}$ ordered by inverse almost inclusion.
3) Let $A_1 = A_1[v] = ([\omega]^{\aleph_0})^V$.

Observation 0.2. A necessary condition for $\text{Pr}_1(Q, P)$ is: $Q$ is proper and

\[ (*)_1 \text{ if } \chi \text{ large enough, } N < (H(\chi), \in) \text{ is countable, } Q, P \in n, q_1 \in Q \text{ is } (N, Q) \text{-generic and } r_1 \in N \cap P \text{ then we can find } (q_2, r_2) \text{ such that:} \]

\[ \circ (a) \quad q_1 \leq Q q_2 \]
\[ (b) \quad r_1 \leq R r_2 \]
\[ (c) \quad q_2 \models \text{"} r_2 \text{ is } (N[G_0], P) \text{-generic"}. \]

Definition 0.3. 1) We define $\text{Pr}^-(Q, P) = \text{Pr}_2(Q, P)$ as the necessary condition from [0,2
2) Let $\text{Pr}_3(Q, P)$ mean that $Q, P$ are forcing notions and for some $\lambda$ and stationary $S \subseteq |\lambda|^{\aleph_0}$ from $V$ we have $\models_Q \text{"P is } S\text{-proper"}$.
3) $\text{Pr}_4(Q, P)$ similarly but $S \subseteq V^{Q^2}$, still $S \subseteq ([\lambda]^{\aleph_0})^V$.
4) $\text{Pr}_5(Q, P)$ is the statement (A) of [0,3,4] below.

Claim 0.4. 1) $\text{Pr}^-(Q, P)$ means that for $\lambda$ large enough, letting $S = ([\lambda]^{\aleph_0})^V$, we have $\models_Q \text{"P is } S\text{-proper"}$.
2) $\text{Pr}_1(Q_1, P) \Rightarrow \text{Pr}_2(Q, P) \Rightarrow \text{Pr}_3(Q, P)$.
3) Also $\text{Pr}_3(Q, P) \Rightarrow \text{Pr}_4(Q, P) \Rightarrow \text{Pr}_5(Q, P)$.
4) If $Q, P$ are forcing notions, $\chi$ large enough, then $(A) \iff (B)$ where

\[ (A) \text{ for some countable } N \prec (H(\chi), \in) \text{ and for some } q \in Q, p \in P \text{ we have} \]
\[ (a) \quad q \text{ is } (N, Q) \text{-generic} \]
\[ (b) \quad q \models \text{"} p \text{ is } (N[G_0], P) \text{-generic"} \]
\[ (B) \text{ for some } q_* \in Q, p_* \in P \text{ we have } \text{Pr}(Q_{\geq q_*}, P_{\geq p_*}). \]

Proof. Easy.
1. Properness of $\mathbb{P}_{A,\{V\}}$ and CH

Claim 1.1. 1) Assume $V_0 \models \text{CH}$, $V_1 \supseteq V_0$, e.g. $V_1 = V_0^Q$ and let $A = A_+\{V_0\}$. Then $V_1 \models \text{"P}_A$ is proper", i.e. $\text{Pr}_1(Q,\mathbb{P}_A)$ when $V_1 \models \text{"if } \omega_1^V \text{ is not collapsed then } \langle \omega \rangle_V^0 \text{ is non-meagre".}$

Proof. If $V_1 \models \langle \omega \rangle_V^0 \text{ is countable" then recalling } V_0 \models \text{CH clearly } V_1 \models \langle A \rangle$ is countable" so we know $\mathbb{P}_A$ is proper in $V_1$. So from now on we assume $\omega_1^V$ is not collapsed.

Secondly in $V_0$, there is a dense $A \subseteq A$ downward dense in it, which under $\subseteq^+$ is downward a tree isomorphic to $T = \omega_1^{\omega_1}$ (the tree of length $\omega_1$). In $V_0$ there is a sequence $\check{T} = \langle T_\alpha : \alpha < \omega_1 \rangle$ which is $\subseteq^+$-increasing continuous with union $T$ and each $T_\alpha$ countable. Also there is $\bar{C} = \{ C_\delta : \delta < \omega_1 \text{ limit} \} \in V_0$ such that $C_\delta \subseteq \delta = \text{sup}(C_\delta)$, $\text{otp}(C_\delta) = \omega$. Let $T_\delta' = T_\delta \restriction \{ \eta \in T_\delta : \ell g(\eta) \in C_\delta \} \in V_0$.

In $V_1$ let $N = (\mathcal{H}(\chi),\in)$ be countable such that $\check{T} \in N$ and let $\omega = \omega_1 \cap N$ clearly $\check{T} \cap N = T_\delta$. We have to prove the statements

\[(*)_0 \text{ "for every } p \in P_A \cap N \text{ there is } q \in P_A \text{ above } p \text{ which is } (N,\mathbb{P}_A)-\text{generic".}\]

As $V_0 \models \text{CH}$ and the density of $T$ this is equivalent to

\[(*)_1 \text{ for every } \nu \in T \cap N = T_\delta \text{ there is } \eta \in \check{T} \text{ which is } (N,T)-\text{generic and } \nu \leq_T \eta.\]

In $V_0$ we let $\check{S} = (S_\delta : \delta < \omega_1 \text{ a limit ordinal})$ where $S_\delta = \{ \check{\nu} : \check{\nu} = \langle \nu_n : n < \omega \rangle \} \subseteq \check{T}$ is $\subseteq^+$-increasing continuous, moreover $\ell g(\nu) = \text{the } n\text{-th member of } C_\delta$. As $\langle \forall \nu \in T_\delta \exists \rho \in T_\delta' \rangle \models \langle \forall \nu < \check{T} \rho \in T_\delta' \rangle$, clearly $(*)_1$ is equivalent to

\[(*)_2 \text{ for every } \nu \in T_\delta' \text{ there is } \check{\nu} \in S_\delta \text{ such that } \nu \in \text{Rang}(\check{\nu}) \text{ and } \check{\nu} \text{ induce a subset of } T_\delta \text{ generic over } N \text{ (i.e. } \langle \forall A | A \in N \rangle \models (N,\mathbb{P}_A)\text{-generic over } N \text{.)} \]

Now a sufficient condition for $(*)_2$ is

\[(*)_3 \text{ as a set of } \omega \text{-branches of the tree } T_\delta', \text{ is non-meagre.}\]

But in $V_0, T_\delta$ and $\omega^\omega$ are isomorphic and $S_\delta$ is the set of all $\omega$-branches of $T_\delta'$, so by an assumption $(*)_3$ holds so we are done. \[1.1.\]

Discussion 1.2. However, there can be $A \subseteq \mathcal{P}(\omega)$ such that $(A, \subseteq^*)$ is a variation of Souslin tree.

Claim 1.3. 1) We have $\text{Pr}_1(Q,\mathbb{P}_{A,\{V\}})$ when:

(a) $\aleph_1^{\mathbb{P}_A} = \aleph_1$
(b) $\models Q | \lambda = \aleph_1$ where $\lambda = (2^{\aleph_0})^V$
(c) moreover in $V^Q$ letting $\langle u_i : i < \aleph_1 \rangle$ be a $\subseteq^+$-increasing continuous sequence of countable subsets of $\lambda$ with union $\lambda$, the set $\{ i : u_i \in V \}$ contains a club of $\omega_1$
(d) forcing with $Q$ preserves $\langle \omega \rangle_V^0 \text{ is non-meagre".}$

\[1\]this is trivial as $V_0 \models \text{CH}$, always there is a dense tree with $\check{\eta}$ levels by the celebrated theorem of Balcar-Pelant-Simon.
2) Assume the forcing notion $Q$ satisfies (a) + (d), $Pr(A, \mathbb{V})$ as witnessed by $S$ and $Q$ is proper.

Then the forcing notion $Q \ast \text{Levy}(\mathbb{V})$ preserves $\mathbb{V}$ is proper when: if $Q$ forces (c) hold.

Proof. Like Theorem 1.4.

\begin{theorem}
We have $\mathbb{V}$ is not proper when:

(a) $\mathbb{V} \models 2\aleph_0 \geq \aleph_2$

(b) $\lambda = \aleph_2$ or just $\lambda$ is regular, $\aleph_2 \leq \lambda \leq 2^{\aleph_0}$ and $\alpha < \lambda \Rightarrow \text{cf}(\alpha, \subseteq) < \lambda$

(c) $\text{h} < \lambda$

(d) the forcing notion $Q$ adds at least one real and is $\lambda$-newly proper, see Definition 1.5 below.
\end{theorem}

Before proving 1.4

Definition 1.5. For $\ell \in \{1, 2\}$ and $\lambda > \kappa$ we say that a forcing notion $Q$ is $(\lambda, \kappa)$-newly proper (omitting $\kappa_\ell$ means $\kappa = \aleph_0$ and we define newly $(\lambda, < \chi)$-proper similarly) when: if $N = \langle (N, \nu) : \nu \in \omega \rangle$ satisfies $\Diamond$ below and $Q \in N_{\infty}$, $p \in Q \cap N_{\infty}$ then we can find $q, \eta$ such that $\exists_\ell$ below holds where:

\begin{itemize}
  \item[\Diamond] for some cardinal $\chi > \lambda$
    \begin{itemize}
      \item[(a)] $N_\eta \prec \langle \mathcal{H}(\chi), \in, <_\chi \rangle$ is countable
      \item[(b)] if $\nu \prec \eta$ then $N_\nu \prec N_\eta$
      \item[(c)] $N_\eta \cap N_{\infty} = N_\eta \cap N_{\infty}$ if $\kappa = \aleph_0$ and $N_\eta \cap N_{\infty} = N_\eta \cap N_{\infty}$ generally where $N_\eta \cap N_{\infty} = \{ \nu \in N_\eta : |\nu| \leq \kappa \}$
      \item[(d)] $\nu_\eta \in N_\eta \cap \{ \nu_\eta : m < \ell g(\eta) \}$ hence $\nu_\eta \not\in \cup(N_\nu : \neg(\nu \subseteq \eta) \land \nu \in \omega \rangle \lambda$\}
      \item[(e)] $\nu_\eta \in \ell g(\eta)$ and $\ell < \ell g(\eta) \Rightarrow \nu_\eta \subseteq \nu_\eta$
    \end{itemize}
\end{itemize}

For a proper forcing notion adding a new real it is quite easy to be $\aleph_1$-newly proper; e.g.

Claim 1.6. Assuming $2^{\aleph_0} \geq \lambda = \text{cf}(\lambda) > \aleph_1$, sufficient conditions for $\mathbb{V}$ is $\lambda$-newly proper are:

\begin{itemize}
  \item[(a)] $Q$ is c.c.c. and add a new real
  \item[(b)] $Q$ is Sacks forcing
  \item[(c)] $Q$ is a tree-like creature forcing in the sense of Roslanowski-Shelah $\text{RoSh:470}$.
\end{itemize}

Proof. Easy; for clause (a) we use $q = p$; in $\mathbb{V}$ in the definition. For clauses (b),(c) we use fusion but in the next step use members of $N_\eta \cap Q$ for $\nu \in \aleph_0$ we get as many distinct $\eta$’s as we can.
Proof. Proof of [1.3] Let $\chi$ be large enough and for transparency, $x \in H(\chi)$.

By Rubin-Shelah [RuSh:117] in $V$ there are sequences $\langle N_\eta : \eta \in \omega^\omega \rangle; \langle \nu_\eta : \eta \in \omega^\omega \rangle$ such that:

- $\Box_1$ (a) $N_\eta \prec (H(\chi), \in)$
- (b) $\forall \cdot x \in N_\eta$
- (c) $N_\eta$ is countable
- (d) $N_{\eta_1} \cap N_{\eta_2} = N_{\eta_1 \cap \eta_2}$
- (e) $\nu_\eta \in \ell_g(n)(\omega^\omega)$
- (f) $\nu_\eta \in N_\eta$
- (g) if $\eta_1 \in \omega^\omega(\lambda)$ and $\neg(\eta \leq \eta_1)$ then $\nu_\eta \notin N_{\eta_1}$
- (h) $\nu_\eta(\ell_1) = \nu_{\eta_2}(\ell_2) \Rightarrow \ell_1 = \ell_2 \land \eta_1(\ell_1 + 1) = \eta_2(\ell_2 + 1)$

Now for each $\eta \in \omega^\omega$ let $N_\eta = \cup\{N_\eta \cap k : k < \omega\}$; we can add:

- (i) $\ell_g(\eta) = n + 1$ then $\nu_\eta(n) > \sup(N_{\eta_1 \cap \lambda})$ and even $\nu_\eta(n) > \sup\{N_\rho \cap \lambda : \rho \in \omega^\omega(\eta_1(n))\}$
- (j) if $\eta \in \omega^\omega$ is increasing, then $\sup(N_\eta \cap \lambda) = \sup(\text{Rang}(\eta))$.

Why is this sufficient? By Balcar-Pelant-Simon [BPS80] there is $T \subseteq [\omega]^{\aleph_0}$ such that

- $\Box_2$ (a) $(T, * \supseteq)$ is a tree with $h$ levels ($h$ is a cardinal invariant, a regular cardinal $\in [\aleph_1, 2^{\aleph_0}]$, with a root and each node has $2^{\aleph_0}$ many immediate successors, i.e. $T$ has splitting to $2^{\aleph_0}$)
- (b) $T$ is dense in $([\omega]^{\aleph_0}, * \supseteq)$, i.e. $\text{P}_{P(\omega)}V = \text{P}_{\text{A}^0_1(V)}$
- recalling [1.12].

Choose $\tilde{h}$ such that

- $\Box_3 \tilde{h} = \langle h_p : p \in T \rangle$ satisfies $h_p$ is one to one from $\text{succ}_T(p)$ onto $2^{\aleph_0} \setminus \{h_{p_1}(p_2) : p_1 <_T p_2 \leq_T p \land p_2 \in \text{succ}_T(p_1)\}$.

So without loss of generality

- $\Box_4 T \in N_{\omega^\omega}$ and $\tilde{h} \in N_{\omega^\omega}$.

As $Q$ is newly $\lambda$-newly proper there are $\eta, \eta$ as in $\Box_1$ of Definition [1.3]. Let $G \subseteq Q$ be generic over $V$ such that $q \in G$, let $\eta = \eta[G]$ and $M_2 := N_{\eta[G]} := \cup\{N_\eta[n][G] : n < \omega\}$, so $M \prec (H(\chi))^V[G], H(\chi)^V, e)$ is countable, pedantically $\langle |M|, H(\chi)^V \cap |M|, e \rangle$ is countable, and $M \prec \langle H(\chi)^V[G], H(\chi)^V, e \rangle$; is proper, hence some $p_\ast \in \text{P}_{\text{A}^0_1(V)}$ is $(M_2, \text{P}_{\text{A}^0_1(V)})$-generic. But $T$ is dense in $\text{P}_{\text{A}^0_1(V)}$ so without loss of generality $p_\ast \in T$ and $p_\ast$ is $(M_2, \tilde{T})$-generic.

Clearly $h \in N_{\omega^\omega}$ or we may demand this, so without loss of generality $\eta \in \omega^\omega \lambda \Rightarrow N_\eta \cap h = N_{\omega^\omega} \cap h$. For any $\alpha < \lambda$ let

$$I_\alpha = \{p \in T : \text{ for some } p_0 \in T \text{ we have } p \in \text{succ}(p_0) \text{ and } h_{p_0}(p) = \alpha\}$$

and letting $T_\alpha$ be the $\alpha$-th level of $T$.
\[ \mathcal{I}_\alpha^+ = \{ p \in \mathbb{P}_{\mathcal{A}_\alpha[V]} : p \text{ is above some member of } \mathcal{T}_\alpha \}. \]

Now clearly (in \( V \) and in \( V[G] \)):

1. \( \mathcal{I}_\alpha \) is a pre-dense subset of \( \mathcal{T} \) (and of \( \mathbb{P}_{\mathcal{A}_\alpha[V]} \))
2. \( \mathcal{I}_\alpha^+ \) is dense open decreasing with \( \alpha \)
3. if \( p \in \mathbb{P}_{\mathcal{A}_\alpha[V]} \) then for every large enough \( \alpha < \lambda, p \notin \mathcal{I}_\alpha^+ \).

Also if \( \alpha \in \lambda \cap N_\eta[G] \) then \( \mathcal{I}_\alpha \in N_\eta[G] \) and the set \( \{ p \in \mathcal{T} \cap N_\eta[G] : p \leq_T p_\alpha \} \) is not empty, let \( p_\alpha^* \) be in it and let its level in \( \mathcal{T} \) be \( \gamma_\alpha^* \).

Also by the choice of \( \bar{h} \) (and genericity) clearly

4. \( \text{Rang}(h_\alpha) \) is equal to \( u := (2^{\aleph_0}) \cap N_\eta[G] \).

Lastly,

5. \( h_\alpha \in V \).

[Why? As its domain, \( N_{<\alpha} \cap h \) belongs to \( V \) and \( h_\alpha(\gamma) \) is defined from \( (\mathcal{T}, \gamma, p_\alpha) \in V \) and \( \mathcal{T} \) is a tree.]

\[ \text{Claim 1.7. We have } \neg \text{Pr}_1(Q, \mathbb{P}_{\mathcal{A}_\alpha[V]}) \text{ when } \]

- \( 2^{\aleph_0} \geq \lambda \) if \( \text{cf}(\lambda) > \kappa = h \)
- \( \alpha < \lambda \Rightarrow \text{cf}([\alpha]^{\kappa \cup} \subseteq \kappa) \leq \lambda \)
- \( Q \) is \( (\lambda, \kappa) \)-newly proper.

\[ \text{Proof. Similar to 1.4.} \]

\[ \text{Conclusion 1.8. If } h < 2^{\aleph_0} \text{ and } Q \text{ is a } (h^+, h)\text{-newly proper then } \neg \text{Pr}_1(Q, \mathbb{P}_{\mathcal{A}_\alpha[V]}). \]
2. General sufficient conditions

Claim 2.1. Assume CH, i.e. \( V \models CH \).

If \( Q \) is c.c.c. then \( \text{Pr}_2(Q, P_{A,V}) \).

Remark 2.2. 1) This works replacing \( P_{A,V} \) by any \( \aleph_1 \)-complete \( P \) and strengthening the conclusions to \( \text{Pr}_1 \), see 2.3.

2) See Definition 0.3(1).

Proof. Let \( P = P_{A,V} \). The point is

\[(\ast) \text{ if } r \in P \text{ and } \models Q \text{ "} \mathcal{I} \text{ is a dense open subset of } P \text{" then there is } r' \text{ such that}

\[(a) \ r \leq_P r'
\]

\[(b) \ |_Q \text{ "} r' \in \mathcal{I} \subseteq P \text{".}
\]

Why (\ast) holds? We try (all in \( V \)) to choose \((r_\alpha, q_\alpha)\) by induction on \( \alpha < \omega_1 \) such that

\[(\Diamond) \ (a) \ r_0 = r
\]

\[(b) \ r_\alpha \in R \text{ is } \leq_P \text{-increasing}
\]

\[(c) \ q_\alpha \in Q
\]

\[(d) \ q_\alpha, q_\beta \text{ are incompatible in } Q \text{ for } \beta < \alpha
\]

\[(e) \ q_\alpha \models Q \text{ "} r_\alpha+1 \in \mathcal{I} \text{".}
\]

We cannot succeed because \( Q \models \text{ c.c.c.} \).

For \( \alpha = 0 \) no problem as only clause (a) is relevant.

For \( \alpha \) limit - easy as \( P \) is \( \aleph_1 \)-complete (and the only relevant clause is (b)).

For \( \alpha = \beta + 1 \), we first ask:

Question: Is \( \langle q_\gamma : \gamma < \beta \rangle \) a maximal antichain of \( Q \)?

If yes, then \( r_\beta \) is as required: if \( G_Q \subseteq Q \) is generic over \( V \) then for some \( \gamma < \beta \), \( q_\gamma \in G_Q \) hence \( r_\gamma+1 \in \mathcal{I}[G_Q] \) but \( \mathcal{I}[G_Q] \) is a dense subset of \( P \) and is open and \( r_\gamma+1 \leq_P r_\beta \) so \( r_\beta \in I[G_Q] \).

If no, let \( q_\beta \in Q \) be incompatible with \( q_\gamma \) for every \( \gamma < \beta \). Recalling \( |_Q \text{ "} \mathcal{I} \text{ is dense and open} \text{"} \) the set \( X_\beta = \{ r \in P : \text{ for some } q, q_\beta \leq Q q \text{ and } q \models \text{ "} r \in \mathcal{I} \text{"} \} \) is a dense subset of \( P \) hence there is a member of \( X_\beta \) above \( r_\beta \), let \( r_\alpha \) be such member. By \( r_\alpha \in X_\beta \), there is \( q, q_\beta \leq q \) such that \( q \models r_\alpha \in \mathcal{I} \). But we could have chosen \( q_\beta \) as such \( q \), contradiction, hence (\ast) indeed holds and this is clearly enough. \( \square \)

We can weaken the demand on the second forcing (here \( P_{A,V} \)) and strengthen the conclusion to \( \text{Pr}(Q, P_{A,V}) \).

Claim 2.3. If (A) then (B) where:

\[(A) \ (a) \ P, Q \text{ are forcing notions}
\]

\[(b) \ Q \text{ is c.c.c. moreover } \models R \text{ "} Q \text{ is c.c.c.} \text{"}
\]

\[(c) \ \text{forcing with } P \text{ and no new } \omega \text{-sequences from } \lambda
\]

\[(d) \ Q \text{ has cardinality } \leq \lambda
\]

\[(B) \ (a) \text{ if } P \text{ is proper in } V \text{ then } \text{Pr}_2(Q, P)
\]

\(2\) if you assume \( Q, R \) are proper, \( \lambda = \aleph_0 \) the proof may be easier to read.
(b) for every $Q$-name $\mathcal{I}$ of a dense open subset of $\mathbb{R}$, the set 
$\mathcal{J} = \{ r \in \mathbb{P} : \models_{Q} "r \in \mathcal{I}" \}$ is dense and open.

Proof. Let $\langle q_\varepsilon : \varepsilon < \kappa := |Q| \rangle$ list $Q$.
For every $r \in \mathbb{P}$ we define a sequence $\eta_r$ of ordinals $< \kappa$ as follows:

$\odot_1 \eta_r(\alpha)$ is the minimal ordinal $\varepsilon < \kappa$ such that
(a) $q_\varepsilon \models "r \in \mathcal{I}"$
(b) if $\beta < \alpha$ then $q_\varepsilon, q_{\eta_r(\beta)}$ are incompatible in $Q$.

Now

$\odot_2 (a) \ \eta_r$ is well defined
(b) $\ell g(\eta_r) < \omega_1$.

[Why? As $Q \models$ c.c.c.]

Note

$\odot_3$ if $r_1 \leq_p r_2$ then either $\eta_{r_1} \leq \eta_{r_2}$ or for some $\alpha < \ell g(\eta_{r_1})$ we have

$\eta_{r_1} |\alpha = \eta_{r_2} |\alpha$

$\eta_{r_1}(\alpha) > \eta_{r_2}(\alpha)$.

[Why? Think about the definition.]
For $s \in \mathbb{P}$ let $\eta_s^* be \cap \{ \eta_{s_1} : s \leq_p s_1 \}$, i.e. the longest common initial segment of 
$\{ \eta_{s_1} : s \leq_p s_1 \}$. Clearly $s_1 \leq_R s_2 \Rightarrow \eta_{s_1}^* \leq \eta_{s_2}^*$. So

$\odot_4 \ \eta^* = \cup \{ \eta_s^* : s \in G_P \}$ is an $\mathbb{P}$-name of a sequence of pairwise incompatible 
members of $Q$

but forcing with $\mathbb{P}$ preserve "$Q \models$ c.c.c." so $\ell g(\eta^*)$ is countable in $V[G_P]$. But 
forcing by $\mathbb{P}$ adds no new $\omega$-sequences to $\kappa = |Q|$ (and $Q$ is infinite) and $V[G_R]$ has 
the same $\aleph_1$ as $V$ and

$\odot_5 \ \eta^*$ is a sequence of countable length of ordinals $< \kappa$ so is old, hence

$\odot_6$ the following set is dense open in $\mathbb{P}$

$\mathcal{J} = \{ r \in \mathbb{P} : r$ forces $(\models_{P}) that $\eta^*_r = \eta_r^* \text{ for some } \eta_r^* \in V \}$

$\odot_7$ if $r \in \mathcal{J}$ then $\langle q_{\eta_r^*(\varepsilon)} : \varepsilon < \ell g(\eta^*_r) \rangle$ is a maximal antichain of $Q$.

[Why? As in the proof of 2.1]
Fix $r_*, \in \mathcal{J} \subseteq \mathbb{P}$ and $\alpha < \ell g(\eta_{r_*})$ let

$(*)_1 \ \mathcal{J}_{r_*, \alpha} = \{ r \in \mathbb{R} : r_* \leq_p r$ and $q_{\eta_{r_*}(\alpha)} \text{ forces (for } \models_Q \text{) that } r \in \mathcal{I} \}$.

[Why? Assume $\mathbb{P} \models "r_* \leq_r 1"$ so $r_1 \models_P "\eta^*(\alpha) = \eta^*_{r_*}(\alpha)"$ hence for some $r_2$ we have $\mathbb{P} \models "r_1 \leq_r r_2"$ and $\eta^*(\alpha + 1) \leq \eta^*_{r_2}$, so by clause (a) of $\odot_1$ we have $q_{\eta^*_{r_2}(\varepsilon)} \models Q "r_2 \in \mathcal{I}"$ hence $r_2 \in \mathcal{J}_{r_*, \alpha}$ as required.]

So
In \( \alpha \) 2, \( J_{r_\ast, \alpha} \) is a dense open subset of \( P_{\geq r_\ast} \) (i.e. above \( r_\ast \)).

As forcing with \( P \) add no new \( \omega \)-sequence
\[
(\ast)_3 J_{r_\ast}^+ := \cap \{ J_{r_\ast, \alpha} : \alpha < \ell g(\eta^*_r) \} \text{ is dense open in } R \text{ above } r_\ast.
\]

Remark 2.4. Should be similar.

Claim 2.4. In 2.1, 2.3 we can replace “c.c.c.” by strongly proper.

Proof. We use the proof of \([Sh:f, Ch.17, Sec.2]\) and references. We repeat in short.

Proof. 5) Even (A) of 0.4(3) fail, i.e.
\[
\neg \Pr_5(Q, P_{\mathcal{A}[\mathcal{V}]})
\]

We use a finite iteration so let \( P_0 \) be the trivial forcing notion, \( P_{k+1} = P_k \ast Q_k \) for \( k \leq 3 \) and the \( P_k \)-name \( Q_k \) is defined below.

Step A: \( Q_0 = \text{Levy}(\aleph_1, 2^{\aleph_0}) \) so \( \vdash_{Q_0} \text{“CH”} \).

Step B: \( Q_1 \) is Cohen forcing.

Step C: In \( V^{P_1}, Q_2 \) in the Levy collapse \( 2^{\aleph_0} \) to \( \aleph_1 \), i.e. \( Q_1 = \text{Levy}(\aleph_1, \aleph_2)^{V[Q_0]} \).

Step D: Let \( T = (\omega_1, \omega_1)^{V[P_1]} \) a tree, so \( \lim_{\omega_1}(T)^{V[P_1]} = \lim_{\omega_1}(T)^{V[P_2]} = \lim_{\omega_1}(T)^{V[P_3]} \)

\[
(\ast)_1 \text{ in } V^{P_1}, T \text{ is isomorphic to a dense subset of } P_{\mathcal{A}[P_1]}.\]

So in \( V^{P_3} \) there is a list \( \langle \eta^*_\varepsilon : \varepsilon < \omega_1 \rangle \) of \( \lim_{\omega_1}(T)^{V[P_1]} \). Let \( \langle \eta^*_\varepsilon \rangle_{\varepsilon < \omega_2} \) be pairwise disjoint end segments.

Step E: In \( V^{P_3} \) there is \( Q_3 \), a c.c.c. forcing notion specializing \( T \) in the sense of \([Sh:74]\), i.e. \( h_s : T \rightarrow \omega, h \) is increasing in \( T \) except on the end segment \( \eta^*_\varepsilon | [\gamma_\varepsilon, \omega_1) \), i.e. \( \rho < \tau \land h_s(\rho) = h(\tau) \Rightarrow (\exists \varepsilon)[\rho, \rho \in \{ \eta^*_\varepsilon | [\gamma_\varepsilon, \omega_1) \}]

\]

\( \therefore \) after forcing with \( P_4 = Q_0 \ast Q_1 \ast Q_2 \ast Q_3 \), i.e. in \( V^{P_4} \) the forcing notion \( P_{\mathcal{A}[V]} \) is not proper, in fact it collapses \( \aleph_1 \).

Why? Recall \( (\ast)_1 \) and note
\[
(\ast)_2 \mathcal{I}_n := \{ \rho \in T : (\forall \nu)(\rho \leq \tau \land h_s(\nu) \neq n) \} \text{ is dense open in } T \text{ (in } V^{\aleph_0 \ast Q_1})\]
and trivially
\( \bigcap_n I_n = \emptyset \); in fact if \( G \subseteq T \) is generic, then

(A) \( G \) is a branch of \( T \) of order type \( \omega_1 \) let its name be \( (\rho_\gamma : \gamma < \omega_1) \)

(B) letting \( \gamma_n = \text{Min}\{\gamma < \omega_2 : \rho_\gamma \in I_n\} \) we have \( \models T \langle \{\gamma_n : n < \omega\} \) is unbounded in \( \omega_1 \rangle \).

\[ \square \]

2.6 References

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