THE SHIMURA–W ALDSPURGER CORRESPONDENCE FOR $\text{Mp}_{2n}$

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ABSTRACT. We generalize the Shimura–Waldspurger correspondence, which describes the tempered part of the automorphic discrete spectrum of the metaplectic group $\text{Mp}_{2n}$, to the metaplectic group $\text{Mp}_{2n}$ of higher rank.

1. Introduction

In a seminal 1973 paper [67], Shimura revolutionized the study of half integral weight modular forms by establishing a lifting

$$\{\text{Hecke eigenforms of weight } k + \frac{1}{2} \text{ and level } \Gamma_0(4)\} \to \{\text{Hecke eigenforms of weight } 2k \text{ and level } \text{SL}_2(\mathbb{Z})\}.$$  

He proved this by using Weil’s converse theorem and a Rankin–Selberg integral for the standard $L$-function of a half integral weight modular form. Subsequently, Niwa [59] and Shintani [68] explicitly constructed the Shimura lifting and its inverse by using theta series lifting. Then, in two influential papers [77] [78], Waldspurger studied this construction in the framework of automorphic representations of the metaplectic group $\text{Mp}_2$, which is a nonlinear two-fold cover of $\text{SL}_2 = \text{Sp}_2$. Namely, he described the automorphic discrete spectrum of $\text{Mp}_2$ precisely in terms of that of $\text{PGL}_2 = \text{SO}_3$ via the global theta lifts between $\text{Mp}_2$ and (inner forms of) $\text{SO}_3$. Subordinate to this global result is the local Shimura correspondence, which is a classification of irreducible genuine representations of $\text{Mp}_2$ in terms of that of $\text{SO}_3$ and was also established by Waldspurger. For an expository account of Waldspurger’s result taking advantage of 30 years of hindsight and machinery, the reader can consult [15].

The goal of the present paper (and its sequel) is to establish a similar description of the automorphic discrete spectrum of $\text{Mp}_{2n}$, which is a nonlinear two-fold cover of $\text{Sp}_{2n}$, in terms of that of $\text{SO}_{2n+1}$. As in the case of $\text{Mp}_2$, it is natural to attempt to use the global theta lifts between $\text{Mp}_{2n}$ and (inner forms of) $\text{SO}_{2n+1}$ to relate these automorphic discrete spectra. However, we encounter a difficulty. For any irreducible cuspidal automorphic representation $\pi$ of $\text{Mp}_{2n}$, there is an obstruction to the nonvanishing of its global theta lift to $\text{SO}_{2n+1}$ given by the vanishing of the central $L$-value $L(\frac{1}{2}, \pi)$. Thus, if we would follow Waldspurger’s approach, then we would need the nonvanishing of the central $L$-value $L(\frac{1}{2}, \pi, \chi)$ twisted by some quadratic Hecke character $\chi$. The existence of such $\chi$ is supplied by Waldspurger [78] in the case of $\text{Mp}_2$ (or equivalently $\text{PGL}_2$) as a consequence of the nonvanishing of a global theta lift, and its new proof and an extension to the case of $\text{GL}_2$ are given by Friedberg–Hoffstein [11]. However, in the higher rank case, this seems to be a very difficult question in analytic number theory. The main novelty of this paper is to overcome this inherent difficulty when $L(\frac{1}{2}, \pi) = 0$.

1.1. Near equivalence classes. We now describe our results in more detail. Let $F$ be a number field and $A$ the adèlle ring of $F$. We denote by $\text{Sp}_{2n}$ the symplectic group of rank $n$ over $F$ and by $\text{Mp}_{2n}(A)$
the metaplectic two-fold cover of $\text{Sp}_{2n}(\mathbb{A})$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \text{Mp}_{2n}(\mathbb{A}) \longrightarrow \text{Sp}_{2n}(\mathbb{A}) \longrightarrow 1.$$ 

Let $L^2(\text{Mp}_{2n})$ be the subspace of $L^2(\text{Sp}_{2n}(F) \backslash \text{Mp}_{2n}(\mathbb{A}))$ on which $\{\pm 1\}$ acts as the nontrivial character, where we regard $\text{Sp}_{2n}(F)$ as a subgroup of $\text{Mp}_{2n}(\mathbb{A})$ via the canonical splitting. Then one of the basic problems is to understand the spectral decomposition of the unitary representation $L^2(\text{Mp}_{2n})$ of $\text{Mp}_{2n}(\mathbb{A})$ and our goal is to establish a description of its discrete spectrum

$$L^2_{\text{disc}}(\text{Mp}_{2n})$$

in the style of Arthur’s conjecture formulated in [16, Conjecture 25.1], [14, §5.6].

We first describe the decomposition of $L^2_{\text{disc}}(\text{Mp}_{2n})$ into near equivalence classes (which are coarser than equivalence classes) of representations. Here we say that two irreducible representations $\pi = \bigotimes_v \pi_v$ and $\pi' = \bigotimes_v \pi'_v$ of $\text{Mp}_{2n}(\mathbb{A})$ are nearly equivalent if $\pi_v$ and $\pi'_v$ are equivalent for almost all places $v$ of $F$. This decomposition will be expressed in terms of $A$-parameters defined as follows. Consider a formal (unordered) finite direct sum

$$(1.1) \quad \psi = \bigoplus_i \phi_i \boxtimes S_{d_i},$$

where

- $\phi_i$ is an irreducible cuspidal automorphic representation of $\text{GL}_{n_i}(\mathbb{A})$;
- $S_{d_i}$ is the unique $d_i$-dimensional irreducible representation of $\text{SL}_2(\mathbb{C})$.

We call $\psi$ an elliptic $A$-parameter for $\text{Mp}_{2n}$ if

- $\sum_i n_i d_i = 2n$;
- if $d_i$ is odd, then $\phi_i$ is symplectic, i.e. the exterior square $L$-function $L(s, \phi_i, \wedge^2)$ has a pole at $s = 1$;
- if $d_i$ is even, then $\phi_i$ is orthogonal, i.e. the symmetric square $L$-function $L(s, \phi_i, \text{Sym}^2)$ has a pole at $s = 1$;
- if $(\phi_i, d_i) = (\phi_j, d_j)$, then $i = j$.

If further $d_i = 1$ for all $i$, we say that $\psi$ is tempered. For each place $v$ of $F$, let $\psi_v = \bigoplus_i \phi_{i,v} \boxtimes S_{d_i}$ be the localization of $\psi$ at $v$. Here we regard $\phi_{i,v}$ as an $n_i$-dimensional representation of $\text{L}_{F_v}$ via the local Langlands correspondence [38, 25, 26, 64], where

$$L_{F_v} = \begin{cases} 
\text{the Weil group of } F_v & \text{if } v \text{ is archimedean;} \\
\text{the Weil–Deligne group of } F_v & \text{if } v \text{ is nonarchimedean.}
\end{cases}$$

Note that $\psi_v$ gives rise to an $A$-parameter $\psi_v : L_{F_v} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{Sp}_{2n}(\mathbb{C})$. We associate to it an $L$-parameter $\phi_{\psi_v} : L_{F_v} \rightarrow \text{Sp}_{2n}(\mathbb{C})$ by

$$\phi_{\psi_v}(w) = \psi_v \left( w, \left( \frac{|w|^{1/2}}{|w|^{-1/2}} \right) \right).$$

Our first result is:

**Theorem 1.1.** Fix a nontrivial additive character $\Psi$ of $F \backslash \mathbb{A}$. Then there exists a decomposition

$$L^2_{\text{disc}}(\text{Mp}_{2n}) = \bigoplus_{\psi} L^2(\psi(\text{Mp}_{2n})).$$
where the direct sum runs over elliptic $A$-parameters $\psi$ for $\text{Mp}_{2n}$ and $L^2_\psi(\text{Mp}_{2n})$ is a full near equivalence class of irreducible representations $\pi$ in $L^2_{\text{disc}}(\text{Mp}_{2n})$ such that the $L$-parameter of $\pi_v$ (relative to $\Psi_v$; see Remark 5.3 below) is $\phi_{\psi_v}$ for almost all places $v$ of $F$.

Thus, to achieve our goal, it remains to describe the decomposition of $L^2_\psi(\text{Mp}_{2n})$ into equivalence classes of representations. In this paper, we carry this out when $\psi$ is tempered.

Remark 1.2. It immediately follows from Theorem 1.1 that $\text{Mp}_{2n}$ has no embedded eigenvalues, i.e. any family of eigenvalues of unramified Hecke algebras on the automorphic discrete spectrum of $\text{Mp}_{2n}$ is distinct from that on the automorphic continuous spectrum of $\text{Mp}_{2n}$. This is an analog of Arthur’s result [6, Theorem 5], [7], which he needed to establish in the course of his proof of the classification of automorphic representations of orthogonal and symplectic groups. However, in our case, we first establish the classification (with the help of theta lifts) and then deduce from it the absence of embedded eigenvalues.

1.2. Local Shimura correspondence. As in Waldspurger’s result [77, 78], our result will be expressed in terms of the local Shimura correspondence. Fix a place $v$ of $F$ and assume for simplicity that $v$ is nonarchimedean. For the moment, we omit the superscript $v$ from the notation, so that $F$ is a nonarchimedean local field of characteristic zero. Let $\text{Irr} \text{Mp}_{2n}$ be the set of equivalence classes of irreducible genuine representations of the metaplectic group $\text{Mp}_{2n}$ over $F$. Then the local Shimura correspondence is a classification of $\text{Irr} \text{Mp}_{2n}$ in terms of $\text{Irr} \text{SO}(V)$, where $V$ is a $(2n+1)$-dimensional quadratic space over $F$.

To be precise, recall that there are precisely two such quadratic spaces with trivial discriminant (up to isometry). We denote them by $V^+$ and $V^-$ so that $\text{SO}(V^\pm)$ is split over $F$. In [20], the first-named author and Savin showed that for any nontrivial additive character $\Psi$ of $F$, there exists a bijection (relative to $\Psi$)

$$
\theta : \text{Irr} \text{Mp}_{2n} \longleftrightarrow \text{Irr} \text{SO}(V^+) \sqcup \text{Irr} \text{SO}(V^-)
$$

satisfying natural properties:

- $\theta$ is compatible with the Langlands classification (modulo tempered representations);
- $\theta$ is compatible with the theory of $R$-groups (modulo square-integrable representations);
- $\theta$ preserves the square-integrability of representations;
- $\theta$ preserves the genericity of tempered representations;
- $\theta$ preserves various invariants such as $L$ and $\epsilon$-factors, Plancherel measures, and formal degrees [17].

Note that an analogous result in the archimedean case was proved by Adams–Barbasch [2, 3] almost 20 years ago.

On the other hand, the local Langlands correspondence, established by Arthur [7] for $\text{SO}(V^+)$ and by Mœglin–Renard [57] for $\text{SO}(V^-)$, gives a partition

$$
\text{Irr} \text{SO}(V^+) \sqcup \text{Irr} \text{SO}(V^-) = \bigsqcup_{\psi} \Pi_{\psi}(\text{SO}(V^\pm)),
$$

where the disjoint union runs over equivalence classes of $L$-parameters $\psi : L_F \to \text{Sp}_{2n}(\mathbb{C})$ and $\Pi_{\psi}(\text{SO}(V^\pm))$ is the associated Vogan $L$-packet equipped with a bijection

$$
\Pi_{\psi}(\text{SO}(V^\pm)) \longleftrightarrow \hat{S}_{\psi},
$$

where $S_{\psi}$ is the component group of the centralizer of $\psi$ in $\text{Sp}_{2n}(\mathbb{C})$ and $\hat{S}_{\psi}$ is the group of continuous characters of $S_{\psi}$. Composing this with the local Shimura correspondence, we obtain a local Langlands
correspondence for \( \text{Mp}_{2n} \) (relative to \( \Psi \)):

\[
\text{Irr} \, \text{Mp}_{2n} = \bigsqcup_{\psi} \Pi_{\psi}(\text{Mp}_{2n})
\]

with

\[
\Pi_{\psi}(\text{Mp}_{2n}) \leftrightarrow \hat{S}_{\psi},
\]

which inherits various properties of the local Langlands correspondence for \( \text{SO}(V^\pm) \). We emphasize that the \( L \)-packets \( \Pi_{\psi}(\text{Mp}_{2n}) \) satisfy endoscopic character relations (see \([13,3]\) below).

1.3. Multiplicity formula for \( \text{Mp}_{2n} \). Suppose again that \( F \) is a number field. We now describe the multiplicity of any representation of \( \text{Mp}_{2n}(A) \) in \( L^2_{\psi}(\text{Mp}_{2n}) \) when \( \psi \) is tempered, i.e. \( \psi \) is a multiplicity-free sum

\[
\psi = \bigoplus_{i} \phi_i
\]

of irreducible symplectic cuspidal automorphic representations \( \phi_i \) of \( \text{GL}_{n_i}(A) \). We formally define the centralizer of \( \psi \) as a free \( \mathbb{Z}/2\mathbb{Z} \)-module

\[
S_{\psi} = \bigoplus_{i} (\mathbb{Z}/2\mathbb{Z}) a_i
\]

with a basis \( \{ a_i \} \), where \( a_i \) corresponds to \( \phi_i \). For any place \( v \) of \( F \), this gives rise to a local \( L \)-parameter \( \psi_v : L_{F_v} \to \text{Sp}_{2n}(\mathbb{C}) \) together with a canonical map \( S_{\psi} \to S_{\psi,v} \). Thus, we obtain a compact group \( S_{\psi,A} = \prod_v S_{\psi,v} \) equipped with the diagonal map \( \Delta : S_{\psi} \to S_{\psi,A} \). For any \( \eta = \bigotimes_v \eta_v \in \hat{S}_{\psi,A} \), we may form an irreducible genuine representation

\[
\pi_{\eta} = \bigotimes_v \pi_{\eta_v}
\]

of \( \text{Mp}_{2n}(A) \), where \( \pi_{\eta_v} \in \Pi_{\psi_v}(\text{Mp}_{2n}) \) is the representation of \( \text{Mp}_{2n}(F_v) \) associated to \( \eta_v \in \hat{S}_{\psi_v} \). (Due to the possible failure of the Ramanujan conjecture for general linear groups, we cannot say that \( \pi_{\eta} \) is tempered.) Finally, we define a quadratic character \( \epsilon_{\psi} \) of \( S_{\psi} \) by setting

\[
\epsilon_{\psi}(a_i) = \epsilon(\frac{1}{2}, \phi_i),
\]

where \( \epsilon(\frac{1}{2}, \phi_i) \in \{ \pm 1 \} \) is the root number of \( \phi_i \).

Our second result (under the hypothesis that Arthur’s result \([7]\) extends to the case of nonsplit odd special orthogonal groups; see \([3,1]\) and \([6,2]\) below for more details) is:

**Theorem 1.3.** Let \( \psi \) be an elliptic tempered \( A \)-parameter for \( \text{Mp}_{2n} \). Then we have

\[
L^2_{\psi}(\text{Mp}_{2n}) \cong \bigoplus_{\eta \in \hat{S}_{\psi,A}} m_{\eta} \pi_{\eta},
\]

where

\[
m_{\eta} = \begin{cases} 
1 & \text{if } \Delta^* \eta = \epsilon_{\psi}; \\
0 & \text{otherwise}. 
\end{cases}
\]

**Remark 1.4.** The description of the automorphic discrete spectrum of \( \text{Mp}_{2n} \) is formally similar to that of \( \text{SO}_{2n+1} \), except that the condition \( \Delta^* \eta = \epsilon_{\psi} \) is replaced by \( \Delta^* \eta = 1 \) in the case of \( \text{SO}_{2n+1} \).

As an immediate consequence of Theorems \([1,1]\) and \([1,3]\) we obtain the following generalization of Waldspurger’s result \([77, \text{p. 131}]\).

**Corollary 1.5.** The tempered part of \( L^2_{\text{disc}}(\text{Mp}_{2n}) \) (which is defined as \( \bigoplus_{\psi} L^2_{\psi}(\text{Mp}_{2n}) \), where the direct sum runs over elliptic tempered \( A \)-parameters \( \psi \) for \( \text{Mp}_{2n} \)) is multiplicity-free.
1.4. Idea of the proof. The main ingredient in the proof of Theorems 1.1 and 1.3 is Arthur’s book [7]. It is natural to attempt to transport Arthur’s result for the odd special orthogonal group $SO_{2n+1}$ to the metaplectic group $Mp_{2n}$ by using global theta lifts. However, as explained above, the difficulty arises when the central $L$-value vanishes.

To circumvent this difficulty, we consider the theta lift between $Mp_{2n}$ and $SO_{2r+1}$ with $r \gg 2n$, i.e. the one in the stable range. More precisely, let $\pi$ be an irreducible summand of $L^2_{disc}(Mp_{2n})$. If $\pi$ is cuspidal, then by the Rallis inner product formula, the global theta lift $\theta(\pi)$ to $SO_{2r+1}(\mathbb{A})$ is always nonzero and square-integrable. Even if $\pi$ is not necessarily cuspidal (so that the Rallis inner product formula is not available), J. S. Li [42] has developed a somewhat unconventional method for lifting $\pi$ to an irreducible summand $\theta(\pi)$ of $L^2_{disc}(SO_{2r+1})$. Then Arthur’s result attaches an elliptic $A$-parameter $\psi'$ to $\theta(\pi)$, which turns out to be of the form

$$\psi' = \psi \otimes S_{2r-2n}$$

for some elliptic $A$-parameter $\psi$ for $Mp_{2n}$ (see Proposition 3.1). We now define the $A$-parameter of $\pi$ as $\psi$, which proves Theorem 1.1.

To prove Theorem 1.3, we apply Arthur’s result to the near equivalence class $L^2_{\psi'}(SO_{2r+1})$ and transport its local-global structure to $L^2_{\psi}(Mp_{2n})$. For this, we need the following multiplicity preservation: if

$$L^2_{\psi}(Mp_{2n}) \cong \bigoplus_\pi m_\pi \pi,$$

then

$$L^2_{\psi'}(SO_{2r+1}) \cong \bigoplus_\pi m_\pi \theta(\pi).$$

Since the above result of J. S. Li amounts to the theta lift from $Mp_{2n}$ to $SO_{2r+1}$, we need the theta lift from $SO_{2r+1}$ to $Mp_{2n}$ in the opposite direction. In fact, J. S. Li [42] has also developed a method which allows us to lift an irreducible summand $\sigma$ of $L^2_{\psi'}(SO_{2r+1})$ (which is no longer cuspidal so that the conventional method does not work) to an irreducible subrepresentation $\theta(\sigma)$ of the space of automorphic forms on $Mp_{2n}(\mathbb{A})$, which is realized as a Fourier–Jacobi coefficient of $\sigma$ (as in the automorphic descent; see [23, 24]). However, we do not know a priori that $\theta(\sigma)$ occurs in $L^2_{disc}(Mp_{2n})$. The key innovation in this paper is to show that $\theta(\sigma)$ is cuspidal if $\psi$ is tempered (see Proposition 4.4). From this, we can deduce the multiplicity preservation and hence obtain a multiplicity formula for $L^2_{\psi'}(Mp_{2n})$ when $\psi$ is tempered (see Proposition 4.4).

However, there is still an issue: we do not know a priori that the local structure of $L^2_{\psi}(Mp_{2n})$ transported from $L^2_{\psi'}(SO_{2r+1})$ agrees with the one defined via the local Shimura correspondence. In other words, we have to describe the local theta lift from $SO_{2r+1}$ to $Mp_{2n}$ in terms of the local Shimura correspondence (see Proposition 6.1). This is the most difficult part in the proof of Theorem 1.3 and will be proved as follows.

- We consider the theta lift of representations in the local $A$-packet $\Pi_{\psi'}(SO_{2r+1})$ to $Mp_{2n}$, where $\psi'$ is a local $A$-parameter of the form

$$\psi' = \psi \otimes S_{2r-2n}$$

for some local $L$-parameter $\psi$ for $Mp_{2n}$. For our global applications, we may assume that $\psi$ is almost tempered. Then we can reduce the general case to the case of good $L$-parameters for smaller metaplectic groups, where we say that an $L$-parameter $\psi$ is good if any irreducible summand of $\psi$ is symplectic. This will be achieved by using irreducibility of some induced representations (see Lemma 5.5), which is due to Mœglin [49, 50, 51, 52] in the nonarchimedean case and to Mœglin–Renard [54] in the complex case, and which will be proved in Appendix 11 below in the real case.
• If \( \psi \) is good, then we appeal to a global argument. As in our previous paper [13], we can find a global elliptic tempered \( \mathcal{A} \)-parameter \( \Psi \) such that \( \Psi_{v_0} = \psi \) for some \( v_0 \) and \( \Psi_v \) is non-good for all \( v \neq v_0 \), and then apply Arthur’s multiplicity formula (viewed as a product formula) to extract information at \( v_0 \) from the knowledge at all \( v \neq v_0 \). Strictly speaking, we need to impose more conditions on \( \Psi \) and the most crucial one is the nonvanishing of the central \( L \)-value \( L(\frac{1}{2}, \Psi) \), which makes the argument more complicated than that in [13]. This will be achieved by a refinement of the globalization of Sakellaridis–Venkatesh [62] (see Proposition 6.5).

1.5. **Endoscopy for \( \text{Mp}_2n \).** Finally, we remark that Wen-Wei Li [43, 44, 45] has developed the theory of endoscopy for \( \text{Mp}_2n \) and has stabilized the elliptic part of the trace formula for \( \text{Mp}_2n \), which should yield a definition of local \( L \)-packets for \( \text{Mp}_2n \) satisfying endoscopic character relations. In this paper, the local \( L \)-packets for \( \text{Mp}_2n \) are defined via the local Shimura correspondence and we do not know a priori that they satisfy the endoscopic character relations. However, this was established by Adams [1] and Renard [61] in the real case. Moreover, using the main result of this paper as a key input, Caihua Luo [46], a student of the first-named author, has recently proved the endoscopic character relations in the nonarchimedean case.

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**Notation.** If \( F \) is a number field and \( G \) is a reductive algebraic group defined over \( F \), we denote by \( \mathcal{A}(G) \) the space of automorphic forms on \( G(\mathbb{A}) \), where \( \mathbb{A} \) is the adèle ring of \( F \). If \( G = \text{Mp}_2n \), we understand that \( \mathcal{A}(G) \) consists only of genuine functions. We denote by \( \mathcal{A}_{\text{disc}}(G) \) and \( \mathcal{A}_{\text{cusp}}(G) \) the subspaces of square-integrable automorphic forms and cusp forms on \( G(\mathbb{A}) \), respectively.

If \( F \) is a local field and \( G \) is a reductive algebraic group defined over \( F \), we denote by \( \text{Irr} G \) the set of equivalence classes of irreducible smooth representations of \( G \), where we identify \( G \) with its group of \( F \)-valued points \( G(F) \). If \( G = \text{Mp}_2n \), we understand that \( \text{Irr} G \) consists only of genuine representations.

For any irreducible representation \( \pi \), we denote by \( \pi^\vee \) its contragredient representation. For any abelian locally compact group \( S \), we denote by \( \hat{S} \) the group of continuous characters of \( S \). For any positive integer \( d \), we denote by \( S_d \) the unique \( d \)-dimensional irreducible representation of \( \text{SL}_2(\mathbb{C}) \).

2. **Some results of J. S. Li**

In this section, we recall some results of J. S. Li [41, 42] on theta lifts and unitary representations of low rank which will play a crucial role in this paper.

2.1. **Metaplectic and orthogonal groups.** Let \( F \) be either a number field or a local field of characteristic zero. Let \( \text{Sp}_{2n} \) be the symplectic group of rank \( n \) over \( F \). If \( F \) is local, we denote by \( \text{Mp}_{2n} \) the metaplectic two-fold cover of \( \text{Sp}_{2n} \):

\[
1 \longrightarrow \{\pm 1\} \longrightarrow \text{Mp}_{2n} \longrightarrow \text{Sp}_{2n} \longrightarrow 1.
\]

This cover does not split unless \( F = \mathbb{C} \), in which case we have \( \text{Mp}_{2n} \cong \text{Sp}_{2n} \times \{\pm 1\} \). If \( F \) is global, we denote by \( \text{Mp}_{2n}(\mathbb{A}) \) the metaplectic two-fold cover of \( \text{Sp}_{2n}(\mathbb{A}) \):

\[
1 \longrightarrow \{\pm 1\} \longrightarrow \text{Mp}_{2n}(\mathbb{A}) \longrightarrow \text{Sp}_{2n}(\mathbb{A}) \longrightarrow 1.
\]

This cover splits over \( \text{Sp}_{2n}(F) \) canonically.
Let $V$ be a quadratic space over $F$, i.e., a finite-dimensional vector space over $F$ equipped with a nondegenerate symmetric bilinear form $b : V \times V \to F$. If $F$ is local, we denote by $\varepsilon(V) \in \{\pm 1\}$ the Hasse–Witt invariant of $(V, b)$. We assume that $V$ is odd-dimensional:
\[
\dim V = 2r + 1.
\]
We denote by $O(V)$ the orthogonal group of $(V, b)$ and by $SO(V) = O(V)^0$ the special orthogonal group of $(V, b)$. Note that
\[
O(V) = SO(V) \times \{\pm 1\}.
\]
If $V$ is the split space with trivial discriminant, i.e., the orthogonal direct sum $\mathbb{H}^r \oplus F$ with $\mathbb{H}$ being the hyperbolic plane and $F$ equipped with a bilinear form $b(x, y) = 2xy$, we write
\[
SO(V) = SO_{2r+1}.
\]
Then $SO_{2r+1}$ is split over $F$.

2.2. **Theta lifts.** Suppose first that $F$ is local and fix a nontrivial additive character $\Psi$ of $F$. Let $\omega$ be the Weil representation of $\text{Mp}_{2n} \times O(V)$ with respect to $\Psi$. For any irreducible genuine representation $\pi$ of $\text{Mp}_{2n}$, the maximal $\pi$-isotypic quotient of $\omega$ is of the form
\[
\pi \boxtimes \Theta(\pi)
\]
for some representation $\Theta(\pi)$ of $O(V)$. Then, by the Howe duality [29, 79, 22], $\Theta(\pi)$ has a unique irreducible (if nonzero) quotient $\theta(\pi)$. We regard $\theta(\pi)$ as a representation of $SO(V)$ by restriction. By (2.1), $\theta(\pi)$ remains irreducible if nonzero.

Similarly, for any irreducible representation $\tilde{\sigma}$ of $O(V)$, we define a representation $\Theta(\tilde{\sigma})$ of $\text{Mp}_{2n}$ with its unique irreducible (if nonzero) quotient $\theta(\tilde{\sigma})$. We now assume that $r \geq n$. Let $\sigma$ be an irreducible representation of $SO(V)$. By the conservation relation [70], there exists at most one extension $\tilde{\sigma}$ of $\sigma$ to $O(V)$ such that $\theta(\tilde{\sigma})$ is nonzero, in which case we write $\theta(\tilde{\sigma}) = \theta(\sigma)$. (We interpret $\theta(\sigma)$ as zero if such $\tilde{\sigma}$ does not exist.)

Suppose next that $F$ is global and fix a nontrivial additive character $\Psi$ of $F \setminus \mathbb{A}$. Let $\pi = \bigotimes_v \pi_v$ be an abstract irreducible genuine representation of $\text{Mp}_{2n}(\mathbb{A})$. Assume that the theta lift $\theta(\pi_v)$ of $\pi_v$ to $SO(V)(F_v)$ is nonzero for all places $v$ of $F$. Then $\theta(\pi_v)$ is irreducible for all $v$ and is unramified for almost all $v$. Hence we may define an abstract irreducible representation
\[
\theta^{\text{abs}}(\pi) = \bigotimes_v \theta(\pi_v)
\]
of $SO(V)(\mathbb{A})$. We call $\theta^{\text{abs}}(\pi)$ the abstract theta lift of $\pi$ to $SO(V)(\mathbb{A})$. On the other hand, if $\pi$ is an irreducible genuine cuspidal automorphic representation of $\text{Mp}_{2n}(\mathbb{A})$, then we may define its global theta lift $\Theta^{\text{aut}}(\pi)$ as the subspace of $\mathcal{A}(SO(V))$ spanned by all automorphic forms of the form
\[
\theta(\phi, f)(h) = \int_{\text{Sp}_{2n}(F) \setminus \text{Mp}_{2n}(\mathbb{A})} \theta(\phi)(g, h) \overline{f(g)} \, dg
\]
for $\phi \in \omega$ and $f \in \pi$. Here $\omega$ is the Weil representation of $\text{Mp}_{2n}(\mathbb{A}) \times O(V)(\mathbb{A})$ with respect to $\Psi$ and $\theta : \omega \to \mathcal{A}(\text{Mp}_{2n} \times O(V))$ is the theta distribution. If $\Theta^{\text{aut}}(\pi)$ is nonzero and contained in $\mathcal{A}_{\text{disc}}(SO(V))$, then
\[
\Theta^{\text{aut}}(\pi) \cong \theta^{\text{abs}}(\pi)
\]
by [36] Corollary 7.1.3. In particular, $\Theta^{\text{aut}}(\pi)$ is irreducible.
2.3. Unitary representations of low rank. The notion of rank for unitary representations was first introduced by Howe [28] in the case of symplectic groups and was extended by J. S. Li [41] to the case of classical groups. Following [41, §4], we say that an irreducible unitary representation of $SO_{2r+1}$ is of low rank if its rank (which is necessarily even) is less than $r - 1$. Such representations are obtained by theta lifts as follows.

Assume that $2n < r - 1$. In particular, the reductive dual pair $(Mp_{2n}, SO_{2r+1})$ is in the stable range (see [40, Definition 5.1]). If $F$ is local, then for any irreducible genuine representation $\pi$ of $Mp_{2n}$, its theta lift $\theta(\pi)$ to $SO_{2r+1}$ is nonzero. Moreover, if $\pi$ is unitary, then so is $\theta(\pi)$ by [40]. In [41], J. S. Li showed that this theta lift provides a bijection

\[
\begin{align*}
\{\text{irreducible genuine unitary representations of } Mp_{2n}\} & \times \{\text{quadratic characters of } F^\times\} \\
\downarrow & \\
\{\text{irreducible unitary representations of } SO_{2r+1} \text{ of rank } 2n\}
\end{align*}
\]

which sends a pair $(\pi, \chi)$ in the first set to a representation $\theta(\pi) \otimes (\chi \circ \nu)$ of $SO_{2r+1}$, where $\nu : SO_{2r+1} \to F^\times/(F^\times)^2$ is the spinor norm.

This result has a global analog. Let $F$ be a number field and $\sigma = \bigotimes_v \sigma_v$ an irreducible unitary representation of $SO_{2r+1}(\mathbb{A})$ which occurs as a subrepresentation of $A(SO_{2r+1})$. Then, by [27, 42 Lemma 3.2], the following are equivalent:

- $\sigma$ is of rank $2n$;
- $\sigma_v$ is of rank $2n$ for all $v$;
- $\sigma_v$ is of rank $2n$ for some $v$.

Suppose that $\sigma$ satisfies the above equivalent conditions. Then, for any $v$, there exist a unique irreducible genuine unitary representation $\pi_v$ of $Mp_{2n}(F_v)$ and a unique quadratic character $\chi_v$ of $F_v^\times$ such that

\[
\sigma_v \cong \theta(\pi_v) \otimes (\chi_v \circ \nu).
\]

By [42] Proposition 5.7], $\chi_v$ is unramified for almost all $v$ and the abstract character $\chi = \bigotimes_v \chi_v$ of $\mathbb{A}^\times$ is in fact automorphic. This implies that $\theta(\pi_v)$ and hence $\pi_v$ are unramified for almost all $v$. Hence we may define an abstract representation $\pi = \bigotimes_v \pi_v$ of $Mp_{2n}(\mathbb{A})$, so that

\[
\sigma \cong \theta^{\text{abs}}(\pi) \otimes (\chi \circ \nu).
\]

2.4. Some inequalities. Finally, we recall a result of J. S. Li [42] which allows us to lift square-integrable (but not necessarily cuspidal) automorphic representations of $Mp_{2n}(\mathbb{A})$ to $SO_{2r+1}(\mathbb{A})$. For any irreducible genuine representation $\pi$ of $Mp_{2n}(\mathbb{A})$, we define its multiplicities $m(\pi)$ and $m_{\text{disc}}(\pi)$ by

\[
m(\pi) = \dim \text{Hom}_{Mp_{2n}(\mathbb{A})}(\pi, A(Mp_{2n})),
\]

\[
m_{\text{disc}}(\pi) = \dim \text{Hom}_{Mp_{2n}(\mathbb{A})}(\pi, A_{\text{disc}}(Mp_{2n})).
\]

Obiously, $m_{\text{disc}}(\pi) \leq m(\pi)$. Likewise, if $\sigma$ is an irreducible representation of $SO_{2r+1}(\mathbb{A})$, we have its multiplicities $m(\sigma)$ and $m_{\text{disc}}(\sigma)$.

**Theorem 2.1** (J. S. Li [42]). Assume that $2n < r - 1$. Let $\pi$ be an irreducible genuine unitary representation of $Mp_{2n}(\mathbb{A})$ and $\theta^{\text{abs}}(\pi)$ its abstract theta lift to $SO_{2r+1}(\mathbb{A})$. Then we have

\[
m_{\text{disc}}(\pi) \leq m_{\text{disc}}(\theta^{\text{abs}}(\pi)) \leq m(\theta^{\text{abs}}(\pi)) \leq m(\pi).
\]
3. Near equivalence classes and $A$-parameters

In this section, we attach an $A$-parameter to each near equivalence class in $L^2_{\text{disc}}(\text{Mp}_{2n})$.

3.1. The automorphic discrete spectrum of $SO(V)$. We first describe the automorphic discrete spectrum

$$L^2_{\text{disc}}(SO(V)) = L^2_{\text{disc}}(SO(V)(F) \backslash SO(V)(A)),$$

where $V$ is a $(2r + 1)$-dimensional quadratic space over a number field $F$. If $SO(V)$ is split over $F$, then it follows from Arthur’s book [7] that

$$L^2_{\text{disc}}(SO(V)) = \bigoplus_{\psi} L^2_{\psi}(SO(V)),$$

where the direct sum runs over elliptic $A$-parameters $\psi$ for $SO(V)$ (or equivalently those for $\text{Mp}_{2n}$) and $L^2_{\psi}(SO(V))$ is a full near equivalence class of irreducible representations $\sigma$ in $L^2_{\text{disc}}(SO(V))$ such that the $L$-parameter of $\sigma_v$ is $\phi_{\psi_v}$ for almost all places $v$ of $F$. Even if $SO(V)$ is not necessarily split over $F$, this decomposition is expected to hold.

3.2. Attaching $A$-parameters. Let $C$ be a near equivalence class in $L^2_{\text{disc}}(\text{Mp}_{2n})$. Then $C$ gives rise to a collection of $L$-parameters

$$\phi_v : L_{\text{Fv}} \rightarrow \text{Sp}_{2n}(\mathbb{C})$$

for almost all $v$ such that for any irreducible summand $\pi$ of $C$, the $L$-parameter of $\pi_v$ (relative to $\Psi_v$) is $\phi_v$ for almost all $v$. Here $\Psi$ is the fixed nontrivial additive character of $F\backslash A$. Then we have:

**Proposition 3.1.** There exists a unique elliptic $A$-parameter $\psi$ for $\text{Mp}_{2n}$ such that $\phi_{\psi_v} = \phi_v$ for almost all $v$.

**Proof.** To prove the existence of $\psi$, we fix an integer $r > 2n + 1$ and consider the abstract theta lift from $\text{Mp}_{2n}(A)$ to $\text{SO}_{2r+1}(A)$. Choose any irreducible summand $\pi$ of $C$. Since $m_{\text{disc}}(\pi) \geq 1$, we deduce from Theorem 2.11 that $m_{\text{disc}}(\theta_{\text{abs}}(\pi)) \geq 1$, i.e. $\theta_{\text{abs}}(\pi)$ occurs in $L^2_{\text{disc}}(\text{SO}_{2r+1})$. Hence, as explained in §3.1 above, Arthur’s result [7] attaches an elliptic $A$-parameter $\psi'$ to $\theta_{\text{abs}}(\pi)$.

We show that $\psi'$ contains $S_{2r-2n}$ as a direct summand. Consider the partial standard $L$-function $L^S(s, \theta_{\text{abs}}(\pi))$ of $\theta_{\text{abs}}(\pi)$, where $S$ is a sufficiently large finite set of places of $F$. If we write $\psi' = \bigoplus_i \phi_i \boxtimes S_{d_i}$ as in (1.1), then

$$L^S(s, \theta_{\text{abs}}(\pi)) = \prod_i \prod_{j=1}^{d_i} L^S(s + \frac{d_i + 1}{2} - j, \phi_i).$$

Note that $L^S(s, \phi_i)$ is holomorphic for $\text{Re} s > 1$ for all $i$, and it has a pole at $s = 1$ if and only if $\phi_i$ is the trivial representation of $\text{GL}_1(A)$. On the other hand, by the local theta correspondence for unramified representations, the $L$-parameter of $\theta(\pi_v)$ is

$$\phi_v \oplus \bigoplus_{j=1}^{2r-2n} \left( \bigoplus_i \right) \zeta^{2r-2n-j}(s + r - n + \frac{1}{2} - j).$$

for almost all $v$. Hence

$$L^S(s, \theta_{\text{abs}}(\pi)) = L^S(s, \pi) \prod_{j=1}^{2r-2n} \zeta^S(s + r - n + \frac{1}{2} - j).$$
Since \( L^S(s, \pi) \) is holomorphic for \( \text{Re} s > n + 1 \) (see e.g. [31] Theorem 9.1), it follows from (3.3) that 
\[ L^S(s, \theta^\text{abs}(\pi)) \] 
is holomorphic for \( \text{Re} s > r - n + \frac{1}{2} \) but has a pole at \( s = r - n + \frac{1}{2} \). This and (3.1) imply that \( \psi' \) contains \( S_{2t} \) as a direct summand for some \( t \geq r - n \). If \( t \) is the largest such integer, then 
\[ L^S(s, \theta^\text{abs}(\pi)) \] 
has a pole at \( s = t + \frac{1}{2} \). This forces \( t = r - n \).

Thus, we may write
\[ \psi' = \psi \oplus S_{2r-2n} \]
for some elliptic \( A \)-parameter \( \psi \). This and (3.2) imply that \( \phi_{\psi_v} = \phi_v \) for almost all \( v \). Moreover, by the strong multiplicity one theorem [31] Theorem 4.4, \( \psi \) is uniquely determined by this condition. \( \square \)

We now denote by \( L^2_\psi(\text{Mp}_2n) \) the near equivalence class \( C \), where \( \psi \) is the \( A \)-parameter attached to \( C \) by Proposition 3.1. Then we have a decomposition
\[ L^2_{\text{disc}}(\text{Mp}_2n) = \bigoplus_{\psi} L^2_\psi(\text{Mp}_2n), \]
where the direct sum runs over elliptic \( A \)-parameters \( \psi \) for \( \text{Mp}_2n \). Note that \( L^2_\psi(\text{Mp}_2n) \) is possibly zero for some \( \psi \). This completes the proof of Theorem 1.1.

**Remark 3.2.** Suppose that
\[ L^2_\psi(\text{Mp}_2n) \cong \bigoplus_{\pi} m_{\pi} \pi, \]
where the direct sum runs over irreducible genuine representations \( \pi \) of \( \text{Mp}_2n(\mathbb{A}) \) and \( m_{\pi} \) is the multiplicity of \( \pi \) in \( L^2_\psi(\text{Mp}_2n) \). If \( r > 2n + 1 \) and \( \psi' = \psi \oplus S_{2r-2n} \), then it follows from Theorem 2.1 and the Howe duality that
\[ L^2_{\psi'}(\text{SO}_{2r+1}) \cong \bigoplus_{\pi} m_{\pi} \theta^\text{abs}(\pi). \]

In Corollary 4.2 below, we show that the above embedding is an isomorphism if \( \psi \) is tempered.

**Remark 3.3.** By Arthur’s multiplicity formula [7], Theorem 1.5.2, once we know that local \( A \)-packets are multiplicity-free, we have
\[ m_{\text{disc}}(\sigma) \leq 1 \]
for all irreducible representations \( \sigma \) of \( \text{SO}_{2r+1}(\mathbb{A}) \). On the other hand, the multiplicity-freeness of local \( A \)-packets was proved by Mœglin [49, 50, 51, 52] in the nonarchimedean case, and by Mœglin [53] and Mœglin–Renard [54] in the complex case, but is not fully known in the real case, though there has been progress by Arancibia–Mœglin–Renard [5] and Mœglin–Renard [55, 56]. Hence, by Theorem 2.1 \( L^2_{\text{disc}}(\text{Mp}_2n) \) is multiplicity-free at least when \( F \) is totally imaginary.

4. **The case of elliptic tempered \( A \)-parameters**

In this section, we study the structure of \( L^2_\psi(\text{Mp}_2n) \) for elliptic tempered \( A \)-parameters \( \psi \).

4.1. **A key equality.** For any irreducible genuine representation \( \pi \) of \( \text{Mp}_2n(\mathbb{A}) \), we define the multiplicity \( m_{\text{cusp}}(\pi) \) by
\[ m_{\text{cusp}}(\pi) = \dim \text{Hom}_{\text{Mp}_2n}(\mathbb{A})(\pi, A_{\text{cusp}}(\text{Mp}_2n)). \]
Obviously, \( m_{\text{cusp}}(\pi) \leq m_{\text{disc}}(\pi) \leq m(\pi) \).

**Proposition 4.1.** Let \( \psi \) be an elliptic tempered \( A \)-parameter for \( \text{Mp}_2n \). Let \( \pi \) be an irreducible genuine representation of \( \text{Mp}_2n(\mathbb{A}) \) such that the \( L \)-parameter of \( \pi_v \) (relative to \( \Psi_v \)) is \( \psi_v \) for almost all \( v \). Then we have
\[ m_{\text{cusp}}(\pi) = m_{\text{disc}}(\pi) = m(\pi). \]
Proof. It suffices to show that for any realization $V \subset A(Mp_{2n})$ of $\pi$, we have $V \subset A_{\text{cusp}}(Mp_{2n})$. Suppose on the contrary that $V \not\subset A_{\text{cusp}}(Mp_{2n})$ for some such $V$. Then the image $\mathcal{V}_P$ of $V$ under the constant term map

$$A(Mp_{2n}) \rightarrow A_{P}(Mp_{2n})$$

is nonzero for some proper parabolic subgroup $P$ of $Sp_{2n}$. Here $A_{P}(Mp_{2n})$ is the space of genuine automorphic forms on $N(\mathbb{A})M(F)\backslash Mp_{2n}(\mathbb{A})$, where $M$ and $N$ are a Levi component and the unipotent radical of $P$, respectively. Assume that $P$ is minimal with this property, so that $V_P$ is contained in the space of cusp forms in $A_{P}(Mp_{2n})$. Then, as explained in [37, p. 205], $\pi$ is a subrepresentation of

$$\text{Ind}_{P(\mathbb{A})}^{\tilde{M}(\mathbb{A})}(\rho)$$

for some irreducible cuspidal automorphic representation $\rho$ of $\tilde{M}(\mathbb{A})$, where $\tilde{P}(\mathbb{A})$ and $\tilde{M}(\mathbb{A})$ are the preimages of $P(\mathbb{A})$ and $M(\mathbb{A})$ in $Mp_{2n}(\mathbb{A})$, respectively. If $M \sim \prod_i \text{GL}_{k_i} \times \text{Sp}_{2n_0}$ with $\sum_i k_i + n_0 = n$, then $\rho$ is of the form

$$\rho \cong \left( \bigotimes_i \tilde{\tau}_i \right) \otimes \pi_0$$

for some irreducible cuspidal automorphic representations $\tilde{\tau}_i$ and $\pi_0$ of $\text{GL}_{k_i}(\mathbb{A})$ and $Mp_{2n_0}(\mathbb{A})$, respectively. Here, as in [17, §2.6], we define the twist $\tilde{\tau}_i = \tau_i \otimes \chi_\Psi$ of $\tau_i$ by the genuine quartic automorphic character $\chi_\Psi$ of the two-fold cover of $\text{GL}_{k_i}(\mathbb{A})$. By Proposition 3.1, $\pi_0$ has a weak lift $\tau_0$ to $GL_{2n_0}(\mathbb{A})$. Hence $\pi$ has a weak lift to $GL_{2n}(\mathbb{A})$ of the form

$$\left( \bigoplus_i (\tau_i \boxplus \tilde{\tau}_i) \right) \boxplus \tau_0,$$

where $\boxplus$ denotes the isobaric sum.

On the other hand, if we write $\psi = \bigoplus_i \phi_i$ for some pairwise distinct irreducible symplectic cuspidal automorphic representations $\phi_i$ of $\text{GL}_{n_i}(\mathbb{A})$, then $\pi$ has a weak lift to $GL_{2n}(\mathbb{A})$ of the form

$$\bigoplus_i \phi_i.$$  

By the strong multiplicity one theorem [31, Theorem 4.4], the two expressions (4.1) and (4.2) must agree. However, $\tau_i$ in (4.1) either is non-self-dual or is self-dual but occurs with multiplicity at least 2, whereas $\phi_i$ in (4.2) is self-dual and occurs with multiplicity 1. This is a contradiction. Hence we have $V \subset A_{\text{cusp}}(Mp_{2n})$ as required. $\square$

4.2. The multiplicity preservation. As a consequence of Proposition 4.1 we deduce:

Corollary 4.2. Let $\psi$ be an elliptic tempered $A$-parameter for $Mp_{2n}$. Suppose that

$$L^2_\psi(Mp_{2n}) \cong \bigoplus_{\pi} m_\pi \pi.$$  

If $r > 2n + 1$ and $\psi' = \psi \oplus S_{2r-2n}$, then

$$L^2_{\psi'}(SO_{2r+1}) \cong \bigoplus_{\pi} m_\pi \theta^{\text{abs}}(\pi).$$

Proof. For any irreducible genuine unitary representation $\pi$ of $Mp_{2n}(\mathbb{A})$, we have

$$m_{\text{disc}}(\pi) = m_{\text{disc}}(\theta^{\text{abs}}(\pi))$$
by Theorem 2.1 and Proposition 4.1. In view of Remark 3.2 it remains to show that for any irreducible summand $\sigma$ of $L^2_\psi(\text{SO}_{2r+1})$, there exists an irreducible summand $\pi$ of $L^2_\psi(\text{Mp}_{2n})$ such that $\sigma \cong \theta^{\text{abs}}(\pi)$. Since the $L$-parameter of $\sigma_v$ is

$$\phi_{\psi_v'} = \psi_v \oplus \left( \bigoplus_{j=1}^{2r-2n} | \cdot |^{-n+\frac{r}{2}-j} \right)$$

for almost all $v$, it follows from the local theta correspondence for unramified representations that

$$\sigma_v \cong \theta(\pi_{\psi_v})$$

for almost all $v$, where $\pi_{\psi_v}$ is the irreducible genuine unramified representation of $\text{Mp}_{2n}(F_v)$ whose $L$-parameter (relative to $\Psi_v$) is $\psi_v$. Note that such $\pi_{\psi_v}$ is unitary (see [71] and Remark 5.3 below). As explained in [22], such $\sigma_v$ is of rank $2n$, and hence so is $\sigma$. Hence there exist a unique irreducible genuine unitary representation $\pi$ of $\text{Mp}_{2n}(\mathbb{A})$ and a unique quadratic automorphic character $\chi$ of $\mathbb{A}^\times$ such that

$$\sigma \cong \theta^{\text{abs}}(\pi) \otimes (\chi \circ \nu).$$

Then we have

$$\sigma_v \cong \theta(\pi_v) \otimes (\chi_v \circ \nu)$$

for all $v$. Recalling the bijection (2.2), we deduce from this and (4.4) that $\pi_v \cong \pi_{\psi_v}$ and $\chi_v$ is trivial for almost all $v$. Since $\chi$ is automorphic, it must be trivial, so that $\sigma \cong \theta^{\text{abs}}(\pi)$. Hence, by (4.3), we have $m_{\text{disc}}(\pi) = m_{\text{disc}}(\sigma) > 0$.

4.3. Multiplicity formula for $\text{SO}_{2r+1}$. Corollary 4.2 allows us to infer a local-global structure of $L^2_\psi(\text{Mp}_{2n})$ with a multiplicity formula from Arthur’s result [7], which we now recall. Let $\psi'$ be an elliptic $A$-parameter for $\text{SO}_{2r+1}$ and write

$$\psi' = \bigoplus_i \phi_i \boxtimes S_{d_i}$$

as in (1.1). Let

$$S_{\psi'} = \bigoplus_i (\mathbb{Z}/2\mathbb{Z})a_i'$$

be a free $\mathbb{Z}/2\mathbb{Z}$-module with a basis $\{a_i'\}$, where $a_i'$ corresponds to $\phi_i \boxtimes S_{d_i}$, and put

$$\tilde{S}_{\psi'} = S_{\psi'}/\Delta(\mathbb{Z}/2\mathbb{Z}).$$

For each place $v$ of $F$, we regard the localization $\psi'_v$ of $\psi'$ at $v$ as a local $A$-parameter $\psi'_v : L_{F_v} \times \text{SL}_2(\mathbb{C}) \to \text{Sp}_{2r}(\mathbb{C})$. Let $S_{\psi'_v}$ be the component group of the centralizer of $\psi'_v$ in $\text{Sp}_{2r}(\mathbb{C})$ and put $\tilde{S}_{\psi'_v} = S_{\psi'_v}/\langle z_{\psi'_v} \rangle$, where $z_{\psi'_v}$ is the image of $-1 \in \text{Sp}_{2r}(\mathbb{C})$ in $S_{\psi'_v}$. Then we have a canonical map $\tilde{S}_{\psi'} \to \tilde{S}_{\psi'_v}$. Thus, we obtain a compact group $\tilde{S}_{\psi', A} = \prod_v \tilde{S}_{\psi'_v}$ equipped with the diagonal map $\Delta : \tilde{S}_{\psi'} \to \tilde{S}_{\psi', A}$.

To each local $A$-parameter $\psi'_v$, Arthur [7] assigned a finite set of (possibly zero, possibly reducible) semisimple representations of $\text{SO}_{2r+1}(F_v)$ of finite length:

$$\Pi_{\psi'_v}(\text{SO}_{2r+1}) = \{ \sigma_{\eta'_v} | \eta'_v \in \tilde{S}_{\psi'_v} \}.$$ 

If $\psi'$ is tempered, then $\Pi_{\psi'_v}(\text{SO}_{2r+1})$ is simply the local $L$-packet associated to $\psi'_v$ (regarded as a local $L$-parameter) by the local Langlands correspondence. In particular, $\sigma_{\eta'_v}$ is nonzero and irreducible for any $\eta'_v$. However, if $\psi'$ is nontempered, then Arthur’s result [7] does not provide explicit knowledge of the representations $\sigma_{\eta'_v}$. Fortunately, in the nonarchimedean case, Mœglin’s results [49, 50, 51, 52] provide an alternative explicit construction of $\Pi_{\psi'_v}(\text{SO}_{2r+1})$ and rather precise knowledge of the properties of $\sigma_{\eta'_v}$ such as nonvanishing, multiplicity-freeness, and irreducibility. These results will be reviewed in [5] below.
For any \( \eta' = \bigotimes_v \eta'_v \in \hat{S}_{\psi', \mathbb{A}} \), we may form a semisimple representation \( \sigma_{\eta'} = \bigotimes_v \sigma_{\eta'_v} \) of \( \text{SO}_{2r+1}(\mathbb{A}) \). Let \( \epsilon_{\psi'} \) be the quadratic character of \( \hat{S}_{\psi'} \) defined by [7 (1.5.6)]. Then Arthur’s multiplicity formula [7, Theorem 1.5.2] asserts that

\[
L^2_{\psi'}(\text{SO}_{2r+1}) \cong \bigoplus_{\eta' \in \hat{S}_{\psi', \mathbb{A}}} m_{\eta'} \sigma_{\eta'},
\]

where

\[
m_{\eta'} = \begin{cases} 
1 & \text{if } \Delta^* \eta' = \epsilon_{\psi'}; \\
0 & \text{otherwise}.
\end{cases}
\]

4.4. **Structure of** \( \text{L}^2_{\psi'}(\text{Mp}_{2n}) \). Finally, with the help of Corollary 4.3, we can transfer the structure of \( \text{L}^2_{\psi'}(\text{SO}_{2r+1}) \) to \( \text{L}^2_{\psi'}(\text{Mp}_{2n}) \) for any elliptic tempered \( \mathbb{A} \)-parameter \( \psi \) for \( \text{Mp}_{2n} \), where \( \psi' = \psi \oplus S_{2r-2n} \) with \( r > 2n + 1 \). If we write \( \psi = \bigoplus_i \phi_i \) for some pairwise distinct irreducible symplectic cuspidal automorphic representations \( \phi_i \) of \( \text{GL}_{n_i}(\mathbb{A}) \), then \( \hat{S}_{\psi} \) and \( \hat{S}_{\psi'} \) are of the form

\[
\hat{S}_{\psi} = \bigoplus_i (\mathbb{Z}/2\mathbb{Z})a_i, \quad \hat{S}_{\psi'} = \hat{S}_{\psi} \oplus (\mathbb{Z}/2\mathbb{Z})a'_0,
\]

where \( a_i \) and \( a'_0 \) correspond to \( \phi_i \) and \( S_{2r-2n} \), respectively. In particular, the natural map

\[
i : \hat{S}_{\psi} \hookrightarrow \hat{S}_{\psi'} \rightarrow \hat{S}_{\psi'}
\]

is an isomorphism. Put \( \epsilon_{\psi'} = \iota^* \epsilon_{\psi'} \).

**Lemma 4.3.** We have

\[
\epsilon_{\psi}(a_i) = \epsilon(\frac{1}{2}, \phi_i).
\]

Moreover, if the central \( L \)-value \( L(\frac{1}{2}, \psi) \) does not vanish, then \( \epsilon_{\psi} \) is trivial.

**Proof.** The first assertion follows from the definition of \( \epsilon_{\psi'} \) and the fact that

\[
\text{Ad} \circ \psi' = (\text{Ad} \circ \psi) \oplus (\psi \boxtimes S_{2r-2n}) \oplus S_3 \oplus S_7 \oplus \cdots \oplus S_{4r-4n-1}.
\]

If \( L(\frac{1}{2}, \psi) \neq 0 \), then since \( L(\frac{1}{2}, \psi) = \prod_i L(\frac{1}{2}, \phi_i) \), we have \( L(\frac{1}{2}, \phi_i) \neq 0 \) and hence \( \epsilon(\frac{1}{2}, \phi_i) = 1 \) for all \( i \).

Thus, the second assertion follows from the first one.

Similarly, for any place \( v \) of \( F \), the natural map

\[
i_v : S_{\psi_v} \hookrightarrow S_{\psi'_v} \rightarrow \hat{S}_{\psi'_v}
\]

is an isomorphism. For any \( \eta'_v \in \hat{S}_{\psi'_v} \), we write

\[
\sigma_{\eta'_v} = \bigoplus_i m_{\eta'_v,i} \sigma_{\eta'_v,i}
\]

for some positive integers \( m_{\eta'_v,i} \) and some pairwise distinct irreducible representations \( \sigma_{\eta'_v,i} \) of \( \text{SO}_{2r+1}(F_v) \). Put \( \eta'_v = \iota_v^* \eta'_v \) and

\[
\pi_{\eta'_v} = \bigoplus_i m_{\eta'_v,i} \theta(\sigma_{\eta'_v,i}),
\]

where \( \theta(\sigma_{\eta'_v,i}) \) is the theta lift to \( \text{Mp}_{2n}(F_v) \). Thus, for any \( \eta = \bigotimes_v \eta_v \in \hat{S}_{\psi, \mathbb{A}} \), we may form a semisimple genuine representation \( \pi_\eta = \bigotimes_v \pi_{\eta_v} \) of \( \text{Mp}_{2n}(\mathbb{A}) \).
Proposition 4.4. Let \( \psi \) be an elliptic tempered \( A \)-parameter for \( \text{Mp}_{2n} \). Then we have
\[
L^2_{\psi}(\text{Mp}_{2n}) \cong \bigoplus_{\eta \in \hat{S}_{\psi, A}} m_{\eta} \tilde{\pi}_{\eta},
\]
where
\[
m_{\eta} = \begin{cases} 
1 & \text{if } \Delta^* \eta = \epsilon_{\psi}; \\
0 & \text{otherwise}. 
\end{cases}
\]

Proof. Suppose that
\[
L^2_{\psi}(\text{SO}_{2r+1}) \cong \bigoplus_{\sigma} m_{\sigma} \sigma.
\]
Then, by Corollary 4.2 and the Howe duality, we have
\[
L^2_{\psi}(\text{Mp}_{2n}) \cong \bigoplus_{\sigma} m_{\sigma} \theta_{\text{abs}}(\sigma).
\]
This and the multiplicity formula (4.5) imply the assertion. \( \square \)

Hence, to complete the proof of Theorem 1.3, it remains to describe \( \tilde{\pi}_{\eta} \) in terms of the local Shimura correspondence. This will be established in Proposition 6.1 below.

Remark 4.5. In the above argument, we have fixed an integer \( r > 2n + 1 \) and do not know a priori that \( \tilde{\pi}_{\eta} \) is independent of the choice of \( r \). This seems not immediate and will be deduced from the description of \( \tilde{\pi}_{\eta} \) in terms of the local Shimura correspondence. While this is a purely local problem, we will address it by a global argument in \( \S 6 \) below.

5. Local \( L \) and \( A \)-packets

In this section, we review the representation theory of metaplectic and orthogonal groups over local fields. In particular, we state irreducibility of some induced representations which will play an important role in an inductive argument in \( \S 6 \) below.

5.1. \( L \)-parameters. Let \( F \) be a local field of characteristic zero and put
\[
L_F = \begin{cases} 
\text{the Weil group of } F & \text{if } F \text{ is archimedean}; \\
\text{the Weil–Deligne group of } F & \text{if } F \text{ is nonarchimedean}. 
\end{cases}
\]
Then the local Langlands correspondence \([28, 25, 26, 64]\) provides a bijection
\[
\text{Irr GL}_n \leftrightarrow \{ \text{n-dimensional representations of } L_F \}. 
\]
Let \( \phi \) be an \( n \)-dimensional representation of \( L_F \). We may regard \( \phi \) as an \( L \)-parameter \( \phi : L_F \to \text{GL}_n(\mathbb{C}) \). We say that:

- \( \phi \) is symplectic if there exists a nondegenerate antisymmetric bilinear form \( b : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \) such that \( b(\phi(w)x, \phi(w)y) = b(x, y) \) for all \( w \in L_F \) and \( x, y \in \mathbb{C}^n \);
- \( \phi \) is tempered if the image of the Weil group of \( F \) under \( \phi \) is relatively compact in \( \text{GL}_n(\mathbb{C}) \).

If \( \phi \) is irreducible and symplectic, then it is tempered. Let \( \tau \) be the irreducible representation of \( \text{GL}_n \) associated to \( \phi \). Then we have:

- \( \tau \) is essentially square-integrable if and only if \( \phi \) is irreducible;
- \( \tau \) is tempered if and only if \( \phi \) is tempered.
Let \( \psi \) be a \( 2n \)-dimensional symplectic representation of \( L_F \). We may regard \( \psi \) as an \( L \)-parameter \( \psi : L_F \to \text{Sp}_{2n}(\mathbb{C}) \). We decompose \( \psi \) as

\[
\psi = \bigoplus_i m_i \phi_i
\]

for some positive integers \( m_i \) and some pairwise distinct irreducible representations \( \phi_i \) of \( L_F \). We say that:

- \( \psi \) is good if \( \phi_i \) is symplectic for all \( i \);
- \( \psi \) is tempered if \( \phi_i \) is tempered for all \( i \);
- \( \psi \) is almost tempered if \( \phi_i \cdot |^{-s_i} \) is tempered for some \( s_i \in \mathbb{R} \) with \( |s_i| < \frac{1}{2} \) for all \( i \).

If \( \psi \) is good, then it is tempered. Also, any localization of a global elliptic tempered \( A \)-parameter for \( \text{Mp}_{2n} \) is almost tempered.

For any \( 2n \)-dimensional symplectic representation \( \psi \) of \( L_F \), we may write

\[
\psi = \phi \oplus \phi^\vee \oplus \psi_0,
\]

where

- \( \phi \) is a \( k \)-dimensional representation of \( L_F \) whose irreducible summands are all non-symplectic;
- \( \psi_0 \) is a \( 2n_0 \)-dimensional good symplectic representation of \( L_F \);
- \( k + n_0 = n \).

More explicitly, if \( \psi \) is of the form (5.1), then \( \psi_0 \) is given by

\[
\psi_0 = \bigoplus_{i \in I_0} m_i \phi_i,
\]

where \( I_0 = \{ i \mid \phi_i \text{ is symplectic} \} \). Let \( S_\psi \) be the component group of the centralizer of \( \psi \) in \( \text{Sp}_{2n}(\mathbb{C}) \) and \( z_\psi \) the image of \( -1 \in \text{Sp}_{2n}(\mathbb{C}) \) in \( S_\psi \). Then \( S_\psi \) is a free \( \mathbb{Z}/2\mathbb{Z} \)-module of the form

\[
S_\psi = \bigoplus_{a \in I_0} (\mathbb{Z}/2\mathbb{Z}) a_i,
\]

where \( a_i \) corresponds to \( \phi_i \), and \( z_\psi = (m_i a_i) \). In particular, we have a natural identification \( S_{\psi_0} = S_\psi \).

### 5.2. Representation theory of \( \text{SO}(V) \)

The local Langlands correspondence [38, 65, 66, 7, 57] provides a partition

\[
\text{Irr} \text{SO}(V) = \bigsqcup_i \Pi_\psi(\text{SO}(V))
\]

and bijections

\[
\bigsqcup_i \Pi_\psi(\text{SO}(V)) \leftrightarrow S_\psi,
\]

where the first disjoint union runs over equivalence classes of \( 2n \)-dimensional symplectic representations \( \psi \) of \( L_F \) and the second disjoint union runs over isometry classes of \( (2n + 1) \)-dimensional quadratic spaces \( V \) over \( F \) with trivial discriminant. For any \( \sigma \in \Pi_\psi(\text{SO}(V)) \), we have:

- \( \sigma \) is square-integrable if and only if \( \psi \) is good and multiplicity-free;
- \( \sigma \) is tempered if and only if \( \psi \) is tempered.
We write $\sigma = \sigma_\eta$ if $\sigma$ corresponds to $\eta \in \hat{S}_\psi$ under the bijection \((5.3)\). Let $\hat{S}_{\psi, V} \subset \hat{S}_\psi$ be the subset of all $\eta$ such that $\sigma_\eta$ is a representation of $\SO(V)$. Then we have

$$\hat{S}_{\psi, V} \subset \{ \eta \in \hat{S}_\psi \mid \eta(z_\psi) = \varepsilon(V) \},$$

with equality when $F$ is nonarchimedean or $F = \mathbb{C}$.

Write $\psi = \phi \oplus \phi^\vee \oplus \psi_0$ as in \((5.2)\). We have $\Pi_\psi(\SO(V)) = \emptyset$ unless the $F$-rank of $\SO(V)$ is greater than or equal to $k$, in which case there exists a $(2n_0 + 1)$-dimensional quadratic space $V_0$ over $F$ with trivial discriminant such that $V = \mathbb{H}^k \oplus V_0$. Let $Q$ be a parabolic subgroup of $\SO(V)$ with Levi component $\GL_k \times \SO(V_0)$. Let $\tau$ be the irreducible representation of $\GL_k$ associated to $\phi$. Then, by the inductive property of the local Langlands correspondence, for any $\eta \in \hat{S}_{\psi, V}$, $\sigma_\eta$ is equal to an irreducible subquotient of

$$\Ind_Q^{\SO(V)}(\tau \otimes \sigma_{\eta_0}),$$

where we write $\eta$ as $\eta_0$ if we regard it as a character of $S_{\psi_0}$ via the identification $S_{\psi_0} = S_\psi$.

**Lemma 5.1.** If $\psi$ is almost tempered, then $\Ind_Q^{\SO(V)}(\tau \otimes \sigma_0)$ is irreducible for any $\sigma_0 \in \Pi_{\psi_0}(\SO(V_0))$.

**Proof.** If $\psi$ is tempered, then by definition, $\Pi_\psi(\SO(V))$ consists of all irreducible summands of $\Ind_Q^{\SO(V)}(\tau \otimes \sigma_0)$ for all $\sigma_0 \in \Pi_{\psi_0}(\SO(V_0))$. Hence the irreducibility of $\Ind_Q^{\SO(V)}(\tau \otimes \sigma_0)$ is a consequence of the bijectivity of \((5.3)\) and the fact that $S_{\psi_0} = S_\psi$.

Thus, if $\psi$ is almost tempered, then by induction in stages, it remains to show that $\Ind_Q^{\SO(V)}(\tau \otimes \sigma_0)$ is irreducible if

- $\tau$ is an irreducible representation of $\GL_k$ whose $L$-parameter is of the form $\bigoplus_i \phi_i \cdot |^{s_i}$ for some tempered representations $\phi_i$ of $L_F$ and some $s_i \in \mathbb{R}$ with $0 < |s_i| < \frac{1}{2}$;
- $\sigma_0$ is an irreducible tempered representation of $\SO(V_0)$.

If $F$ is archimedean, then this follows from a result of Speh–Vogan \([69]\) (see also \([72]\) Chapter 8). If $F$ is nonarchimedean, then this follows from a result of Mœglin–Waldspurger \([58]\) \(\S 2.14\) and a conjecture of Gross–Prasad and Rallis \([18]\) Appendix B].

**5.3. Representation theory of $\Mp_{2n}$**. Fix a nontrivial additive character $\Psi$ of $F$. The local Shimura correspondence \([2, 3, 20]\) asserts that the theta lift (relative to $\Psi$) induces a bijection

$$\theta : \text{Irr } \Mp_{2n} \leftrightarrow \bigsqcup_V \text{Irr } \SO(V)$$

satisfying natural properties, where the disjoint union runs over isometry classes of $(2n + 1)$-dimensional quadratic spaces $V$ over $F$ with trivial discriminant. Composing this with the local Langlands correspondence for $\SO(V)$, we obtain a partition

$$\text{Irr } \Mp_{2n} = \bigsqcup_\psi \Pi_\psi(\Mp_{2n})$$

and bijections

$$\Pi_\psi(\Mp_{2n}) \leftrightarrow \hat{S}_\psi,$$

where the disjoint union runs over equivalence classes of $2n$-dimensional symplectic representations $\psi$ of $L_F$. Since $\theta$ preserves the square-integrability and the temperedness of representations, for any $\pi \in \Pi_\psi(\Mp_{2n})$, we have:
• \(\pi\) is square-integrable if and only if \(\psi\) is good and multiplicity-free;
• \(\pi\) is tempered if and only if \(\psi\) is tempered.

We write \(\pi = \pi_\eta\) if \(\pi\) corresponds to \(\eta \in \hat{S}_\psi\) under the bijection (5.4).

Write \(\psi = \phi \oplus \phi^\vee \oplus \psi_0\) as in (5.2). Let \(P\) be a parabolic subgroup of \(Sp_{2n}\) with Levi component \(GL_k \times Sp_{2n_0}\) and \(\tilde{P}\) the preimage of \(P\) in \(Mp_{2n}\). Let \(\tau\) be the irreducible representation of \(GL_k\) associated to \(\phi\) and \(\tilde{\tau} = \tau \otimes \chi\psi\) its twist by the genuine quartic character \(\chi\psi\) of the two-fold cover of \(GL_k\) given in [17, §2.6]. Then, by the inductive property of the local Shimura correspondence, for any \(\eta \in \hat{S}_\psi\), \(\pi_\eta\) is equal to an irreducible subquotient of

\[
\text{Ind}_{\tilde{P}}^{Mp_{2n}}(\tilde{\tau} \otimes \pi_0),
\]

where we write \(\eta\) as \(\eta_0\) if we regard it as a character of \(S_{\psi_0}\) via the identification \(S_{\psi_0} = S_\psi\).

**Lemma 5.2.** If \(\psi\) is almost tempered, then \(\text{Ind}_{\tilde{P}}^{Mp_{2n}}(\tilde{\tau} \otimes \pi_0)\) is irreducible for any \(\pi_0 \in \Pi_{\psi_0}(Mp_{2n_0})\).

**Proof.** If \(\psi\) is tempered, then by the inductive property of the local Shimura correspondence, \(\Pi_{\psi}(Mp_{2n})\) consists of all irreducible summands of \(\text{Ind}_{\tilde{P}}^{Mp_{2n}}(\tilde{\tau} \otimes \pi_0)\) for all \(\pi_0 \in \Pi_{\psi_0}(Mp_{2n_0})\). Hence the irreducibility of \(\text{Ind}_{\tilde{P}}^{Mp_{2n}}(\tilde{\tau} \otimes \pi_0)\) is a consequence of the bijectivity of (5.4) and the fact that \(S_{\psi_0} = S_\psi\).

Thus, if \(\psi\) is almost tempered, then by induction in stages, it remains to show that \(\text{Ind}_{\tilde{P}}^{Mp_{2n}}(\tilde{\tau} \otimes \pi_0)\) is irreducible if

- \(\tau\) is an irreducible representation of \(GL_k\) whose \(L\)-parameter is of the form \(\bigoplus_i |\phi_i|^{s_i}\) for some tempered representations \(\phi_i\) of \(L_F\) and some \(s_i \in \mathbb{R}\) with \(0 < |s_i| < \frac{1}{2}\);
- \(\pi_0\) is an irreducible tempered representation of \(Mp_{2n_0}\).

If \(F = \mathbb{C}\), then this follows from a result of Speh–Vogan [69] (see also [72, Chapter 8]). If \(F = \mathbb{R}\), then the argument in [69] should also work, but they only consider real reductive \textit{linear} Lie groups. For the sake of completeness, we give a proof in Appendix A below. Suppose that \(F\) is nonarchimedean. Then this follows from a result of Atobe [8, Theorem 3.13], but we include the proof for the convenience of the reader. We may assume that \(\pi = \text{Ind}_{\tilde{P}}^{Mp_{2n}}(\tilde{\tau} \otimes \pi_0)\) is a standard module. Let \(\pi'\) be the unique irreducible quotient of \(\pi\). By the local Shimura correspondence, the theta lift \(\sigma_0 = \theta(\pi_0)\) to \(SO(V_0)\) is nonzero and tempered for a unique \((2n_0 + 1)\)-dimensional quadratic space \(V_0\) over \(F\) with trivial discriminant. Put \(\sigma = \text{Ind}_Q^{SO(V)}(\tau \otimes \sigma_0)\), where \(V = \mathbb{H}^k \oplus V_0\) and \(Q\) is the standard parabolic subgroup of \(SO(V)\) with Levi component \(GL_k \times SO(V_0)\). As shown in the proof of Lemma 5.1, \(\sigma\) is irreducible. Then, by [20], we have

\[
\theta(\pi') = \sigma.
\]

On the other hand, let \(\pi''\) be an irreducible subrepresentation of \(\pi\). As in the proof of [20, Theorem 8.1], it follows from Kudla’s filtration (see e.g. [20, Proposition 7.3]) and [20, Lemma 7.4], which continues to hold for \(s > -\frac{1}{2}\), that there exists a surjection

\[
\text{Ind}_Q^{SO(V)}(\tau^\vee \otimes \Theta(\pi_0)) \longrightarrow \Theta(\pi'').
\]

Since \(\Theta(\pi_0) = \theta(\pi_0)\) by [20] and any irreducible representation of an odd special orthogonal group is self-dual, the left-hand side is

\[
\text{Ind}_Q^{SO(V)}(\tau^\vee \otimes \sigma_0) \cong \text{Ind}_Q^{SO(V)}(\tau \otimes \sigma_0)^\vee \cong \sigma.
\]

Hence

\[
\theta(\pi'') = \sigma.
\]
By (5.5), (5.6), and the Howe duality, we have \( \pi' \cong \pi'' \). However, since \( \pi' \) occurs in \( \pi \) with multiplicity one (see e.g. [9] Remark 4.5), and \( \pi' \) and \( \pi'' \) are a quotient and a subrepresentation of \( \pi \), respectively, we have \( \pi = \pi' = \pi'' \). In particular, \( \pi \) is irreducible.

\[ \text{Remark 5.3} \] Suppose that \( F \) is nonarchimedean of odd residual characteristic and \( \Psi \) is of order zero. Then we may regard \( \pi \) by the local theta correspondence for unramified representations, the \( \text{Ind}_{B}^{\text{Mp}}(\chi_{\Psi} \cdot |s_{1} \otimes \cdots \otimes \chi_{\Psi} \cdot |^{s_{n}}) \),

where \( B \) is a Borel subgroup of \( \text{Sp}_{2n} \), \( \tilde{B} \) is the preimage of \( B \) in \( \text{Mp}_{2n} \), and \( s_{1}, \ldots, s_{n} \in \mathbb{C} \). In this case, if \( \pi \) is irreducible, \( \pi \) is the unique unramified subquotient of a principal series representation

\[ \text{Ind}_{B}^{\text{Mp}}(\chi_{\Psi} \cdot |s_{1} \otimes \cdots \otimes \chi_{\Psi} \cdot |^{s_{n}}) \]

where \( B \) is a Borel subgroup of \( \text{Sp}_{2n} \), \( \tilde{B} \) is the preimage of \( B \) in \( \text{Mp}_{2n} \), and \( s_{1}, \ldots, s_{n} \in \mathbb{C} \). In this case, by the local theta correspondence for unramified representations, the \( L \)-parameter of \( \pi \) (relative to \( \Psi \)) is

\[ \bigoplus_{i=1}^{n} (| \cdot |^{s_{i}} \otimes | \cdot |^{-s_{i}}) \]

Also, as shown in the proof of Lemma 5.2, the representation (5.7) is irreducible if \( |\text{Re } s_{i}| < \frac{1}{2} \) for all \( i \). Hence, by [39] Lemma 3.3, the representation (5.7) is unitary if either \( s_{i} \in \sqrt{-1} \mathbb{R} \) or \( s_{i} \in \mathbb{R} \) with \( |s_{i}| < \frac{1}{2} \) for all \( i \).

5.4. Some A-packets for \( \text{SO}_{2r+1} \). Let \( \psi' \) be a \( 2r \)-dimensional symplectic representation of \( L_{F} \times \text{SL}_{2}(\mathbb{C}) \). We may regard \( \psi' \) as an \( A \)-parameter \( \psi' : L_{F} \times \text{SL}_{2}(\mathbb{C}) \rightarrow \text{Sp}_{2r}(\mathbb{C}) \) and associate to it an \( L \)-parameter \( \phi_{\psi'} : L_{F} \rightarrow \text{Sp}_{2r}(\mathbb{C}) \) by

\[ \phi_{\psi'}(w) = \psi' \left( w, \left( |w|^{\frac{1}{2}}, |w|^{-\frac{1}{2}} \right) \right) \]

Let \( S_{\psi'} \) be the component group of the centralizer of \( \psi' \) in \( \text{Sp}_{2r}(\mathbb{C}) \) and put \( \hat{S}_{\psi'} = S_{\psi'}/(z_{\psi'}) \), where \( z_{\psi'} \) is the image of \(-1 \in \text{Sp}_{2r}(\mathbb{C}) \) in \( S_{\psi'} \). Then Arthur [7] assigned to \( \psi' \) an A-packet:

\[ \Pi_{\psi'}(\text{SO}_{2r+1}) = \{ \sigma_{\eta'} | \eta' \in \hat{S}_{\psi'} \}, \]

where \( \sigma_{\eta'} \) is a (possibly zero, possibly reducible) semisimple representation of \( \text{SO}_{2r+1} \) of finite length.

From now on, we only consider an \( A \)-parameter \( \psi' \) of the form

\[ \psi' = \psi \oplus S_{2r-2n} \]

for some \( 2n \)-dimensional symplectic representation \( \psi \) of \( L_{F} \) with \( 2n < r - 1 \). Then we have:

**Proposition 5.4.** Assume that \( \psi \) is almost tempered. Then \( \sigma_{\eta'} \) is nonzero and irreducible for any \( \eta' \in \hat{S}_{\psi'} \). Moreover, \( \Pi_{\psi'}(\text{SO}_{2r+1}) \) is multiplicity-free, i.e. the \( \sigma_{\eta'} \)'s are pairwise distinct.

This proposition is largely due to Mœglin [49, 50, 51, 52] when \( F \) is nonarchimedean, to Mœglin [53] and Mœglin–Renard [54] when \( F = C \), and to Arancibia–Mœglin–Renard [5] and Mœglin–Renard [55, 56] when \( F = \mathbb{R} \) and \( \psi \) is good. The reader can also consult an efficient and concise exposition of Mœglin’s results by B. Xu [80]. In the next section, we will give a proof of Proposition 5.4 based on theta lifts. For this, we need the following irreducibility of some induced representations.

Write \( \psi = \phi \oplus \phi' \oplus \psi_{0} \) as in (5.2). Put \( \psi_{0}' = \psi_{0} \oplus S_{2r-2n} \), so that \( \psi' = \phi \oplus \phi' \oplus \psi_{0}' \). Let \( Q' \) be a parabolic subgroup of \( \text{SO}_{2r+1} \) with Levi component \( \text{GL}_{k} \times \text{SO}_{2r-2k+1} \). Let \( \tau \) be the irreducible representation of
GL_k associated to \( \phi \). Then, by the definition and the inductive property of \( A \)-packets (see [7]), for any \( \eta' \in \hat{S}_{\psi'} \), \( \sigma_{\eta'} \) is equal to the semisimplification of

\[
\text{Ind}^{SO_{2r+1}}_{Q'}(\tau \otimes \sigma'_{\eta'_0}),
\]

where we write \( \eta' \) as \( \eta'_{0} \) if we regard it as a character of \( \bar{S}_{\psi'} \) via the identification \( \bar{S}_{\psi'} = \bar{S}_{\psi} \).

**Lemma 5.5.** Assume that \( \psi \) is almost tempered. Then \( \text{Ind}^{SO_{2r+1}}_{Q'}(\tau \otimes \sigma'_{\eta'_0}) \) is irreducible for any irreducible subrepresentation \( \sigma'_{0} \) of any representation in the \( A \)-packet \( \Pi_{\psi'_0}(SO_{2r-2k+1}) \).

**Proof.** The assertion was proved in a more general context by Mœglin [51, §3.2], [52, Proposition 5.1] when \( F \) is nonarchimedean, and by Mœglin–Renard [54, §6] when \( F = \mathbb{C} \). In Appendix B below, we will give a proof based on the Kazhdan–Lusztig algorithm when \( F = \mathbb{R} \), noting that \( \sigma'_{0} \) has half integral infinitesimal character by [10, Lemme 3.4]. \( \square \)

6. Comparison of local theta lifts

As explained in [4.4] to finish the proof of Theorem 1.3, it remains to describe the local theta lift from \( SO_{2r+1} \) to \( \text{Mp}_{2n} \) with \( r > 2n + 1 \) in terms of the local Shimura correspondence. Namely, we need to compare the local theta correspondences for the following reductive dual pairs:

- \((\text{Mp}_{2n}, \text{SO}_{2n+1})\) in the equal rank case (and its inner forms);
- \((\text{Mp}_{2n}, \text{SO}_{2r+1})\) in the stable range.

To distinguish them, we keep using \( \theta \) to denote the theta correspondence for the former but change it to \( \vartheta \) for the latter.

Let \( F \) be a local field of characteristic zero and fix a nontrivial additive character \( \Psi \) of \( F \). Let \( \psi \) be a \( 2n \)-dimensional symplectic representation of \( L_F \) and put \( \psi' = \psi \oplus S_{2r-2n} \) with \( r > 2n + 1 \). Then we have a natural isomorphism

\[
\iota : S_{\psi} \longrightarrow S_{\psi'} \longrightarrow \bar{S}_{\psi'}.
\]

We now state the main result of this section.

**Proposition 6.1.** Assume that \( \psi \) is almost tempered. Then, for any \( \eta' \in \hat{S}_{\psi'} \), we have

\[
\vartheta(\sigma_{\eta'}) = \pi_{\eta},
\]

where \( \eta = \iota^* \eta' \).

The rest of this section is devoted to the proof of Propositions 5.4 and 6.1.

6.1. Reduction to the case of good \( L \)-parameters. We proceed by induction on \( n \).

**Lemma 6.2.** Propositions 5.4 and 6.1 hold for \( n = 0 \).

**Proof.** If \( n = 0 \), then \( \psi' = S_{2r} \) and \( \Pi_{\psi'}(SO_{2r+1}) = \{ \sigma_1 \} \), where \( \sigma_1 \) is the trivial representation of \( SO_{2r+1} \). Moreover, the theta lift \( \vartheta(\sigma_1) \) to \( \text{Mp}_0 \) is the genuine 1-dimensional representation of \( \text{Mp}_0 = \{ \pm 1 \} \), which is also the theta lift of the trivial representation of \( \text{SO}_1 = \{ 1 \} \). This completes the proof. \( \square \)

From now on, we assume that \( n > 0 \).
Lemma 6.3. Assume that Propositions 5.4 and 6.1 hold for all $2n_0$-dimensional good symplectic representations of $L_F$ for all $n_0 < n$. Then they also hold for all $2n$-dimensional almost tempered non-good symplectic representations of $L_F$.

Proof. Let $\psi$ be a $2n$-dimensional almost tempered non-good symplectic representation of $L_F$. Write $\psi = \phi \oplus \phi' \oplus \psi_0$ as in (5.2). In particular, $\psi_0$ is a $2n_0$-dimensional good symplectic representation of $L_F$ with $n_0 < n$. Put $\psi'_0 = \psi_0 \oplus S_{2r-2n}$, so that $\psi' = \phi \oplus \phi' \oplus \psi'_0$. Let $\eta' \in \mathcal{S}_{\psi'}$ and put $\eta = \iota^* \eta'$. We write $\eta'$ (resp. $\eta$) if we regard it as a character of $\mathcal{S}_{\psi'}$ (resp. $\mathcal{S}_{\psi_0}$) via the natural identification.

Then, by definition, $\sigma_{\eta'}$ is the semisimplification of $\text{Ind}^{\text{SO}_{2r+1}}_{\text{SO}_{2r-2k+1}} (\tau \otimes \sigma_{\eta'_0})$, where $Q'$ is the standard parabolic subgroup of $\text{SO}_{2r+1}$ with Levi component $\text{GL}_k \times \text{SO}_{2r-2k+1}$ and $\tau$ is the irreducible representation of $\text{GL}_k$ associated to $\phi$. Since $\sigma_{\eta'_0}$ is nonzero and irreducible by assumption, so is $\sigma_{\eta'}$ by Lemma 5.5. Moreover, the theta lift $\vartheta(\sigma_{\eta'})$ to $\text{Mp}_{2n_0}$ is $\pi_{\eta_0}$ by assumption. Hence it follows from the induction principle [35], [2, Corollary 3.21], [3, Theorem 8.4], which easily extends to the case at hand, that there exists a nonzero equivariant map

$$\omega \longrightarrow \text{Ind}_{\text{P}}^{\text{Mp}_{2n}} (\tilde{\tau} \otimes \pi_{\eta_0}) \otimes \text{Ind}^{\text{SO}_{2r+1}}_{\text{SO}_{2r-2k+1}} (\tau \otimes \sigma_{\eta'_0}),$$

where $P$ is the standard parabolic subgroup of $\text{Sp}_{2n}$ with Levi component $\text{GL}_k \times \text{Sp}_{2n_0}$. Since $\text{Ind}_{\text{P}}^{\text{Mp}_{2n}} (\tilde{\tau} \otimes \pi_{\eta_0})$ is irreducible by Lemma 5.2, this implies that the theta lift $\vartheta(\sigma_{\eta'})$ to $\text{Mp}_{2n}$ is $\pi_\eta$. Thus, since $\Pi_\psi(\text{Mp}_{2n})$ is multiplicity-free, so is $\Pi_\psi(\text{SO}_{2r+1})$. This completes the proof. $\square$

We may now assume that Propositions 5.4 and 6.1 hold for all $2n$-dimensional almost tempered non-good symplectic representations of $L_F$. It remains to show that they also hold for all $2n$-dimensional good symplectic representations of $L_F$. In particular, we have finished the proof for $F = \mathbb{C}$ since any irreducible representation of $L_{\mathbb{C}}$ is 1-dimensional and hence non-symplectic.

Later, we also need the following description of the theta lift from $\text{SO}_{2r+1}$ to $\text{Mp}_{2n}$.

Lemma 6.4. Assume that $F$ is nonarchimedean or $F = \mathbb{C}$, and that $\psi$ is almost tempered. Let $\phi_{\psi'}$ be the $L$-parameter associated to $\psi'$ by (5.8) and $\sigma'$ the irreducible representation in the $L$-packet $\Pi_{\phi_{\psi'}}(\text{SO}_{2r+1})$ associated to the trivial character of $S_{\phi_{\psi'}}$. Then we have

$$\vartheta(\sigma') = \pi_1,$$

where $\pi_1$ is the irreducible representation in the $L$-packet $\Pi_\psi(\text{Mp}_{2n})$ associated to the trivial character of $S_{\psi}$.

Proof. If $\psi$ is non-good, then it follows from [7, Proposition 7.4.1] and Proposition 5.4 that $\sigma'$ is the representation in the $A$-packet $\Pi_{\psi'}(\text{SO}_{2r+1})$ associated to the trivial character of $S_{\psi'}$. Hence, by Proposition 6.1, we have $\vartheta(\sigma') = \pi_1$.

We may now assume that $\psi$ is good (and hence tempered), so that $F$ is nonarchimedean. By definition, $\sigma'$ is the unique irreducible quotient of the standard module

$$\text{Ind}^{\text{SO}_{2r+1}}_{Q'} (| \cdot |^{r-n-\frac{1}{2}} \otimes | \cdot |^{r-n-\frac{3}{2}} \otimes \cdots \otimes | \cdot |^{\frac{3}{2} \otimes \sigma}),$$

where $Q'$ is the standard parabolic subgroup of $\text{SO}_{2r+1}$ with Levi component $(\text{GL}_1)^{r-n} \times \text{SO}_{2n_1}$ and $\sigma$ is the irreducible tempered representation in the $L$-packet $\Pi_\psi(\text{SO}_{2n_1})$ associated to the trivial character of $S_{\psi}$. Since the theta lift of $\sigma$ to $\text{Mp}_{2n}$ is $\pi_1$, we have $\vartheta(\sigma') = \pi_1$ by [21, Proposition 3.2]. $\square$
6.2. Multiplicity formula and globalization. To finish the proof of Propositions 5.4 and 6.1, we appeal to a global argument. One of the global ingredients is Arthur’s multiplicity formula (4.5) and the following variant.

Let \( F \) be a number field and \( A \) the ad\'ele ring of \( F \). Let \( \mathbb{V} \) be a \((2n+1)\)-dimensional quadratic space over \( F \) with trivial discriminant. Recall the (expected) decomposition of \( L^2_d(\text{SO}(\mathbb{V})) \) into near equivalence classes described in \( \S 3.1 \). We only consider the near equivalence class \( L^2_\Psi(\text{SO}(\mathbb{V})) \) associated to an elliptic tempered \( A \)-parameter \( \Psi \) for \( \text{SO}(\mathbb{V}) \). As in \( \S 1.3 \), we formally define the centralizer \( S_\Psi \) equipped with a canonical map \( S_\Psi \to S_\Psi \) for each \( v \) and the diagonal map \( \Delta : S_\Psi \to S_\Psi \). For any \( \eta = \otimes_v \eta_v \in \hat{S}_\Psi \) such that \( \eta_v \in \hat{S}_{\Psi_v} \) for all \( v \), we may form an irreducible representation \( \Sigma_\eta \) of \( \text{SO}(\mathbb{V})(A) \), where \( \Sigma_\eta \) is the representation of \( \text{SO}(\mathbb{V})(F_v) \) associated to \( \eta_v \) by the local Langlands correspondence described in \( \S 5.2 \). Then Arthur’s multiplicity formula (which has not been established if \( \text{SO}(\mathbb{V}) \) is nonsplit but will be assumed in this paper) asserts that

\[
L^2_\Psi(\text{SO}(\mathbb{V})) \cong \bigoplus_\eta m_\eta \Sigma_\eta,
\]

where the direct sum runs over continuous characters \( \eta \) of \( S_\Psi \) such that \( \eta_v \in \hat{S}_{\Psi_v} \) for all \( v \) and

\[
m_\eta = \begin{cases} 1 & \text{if } \Delta^* \eta = 1; \\ 0 & \text{otherwise}. \end{cases}
\]

In particular, \( L^2_\Psi(\text{SO}(\mathbb{V})) \) is multiplicity-free and \( m_{\text{disc}}(\Sigma_\eta) = m_\eta \).

We will use Arthur’s multiplicity formula

- to globalize a local representation to a global automorphic representation;
- to extract local information from a global product formula.

To make the argument work, we first globalize a local \( L \)-parameter to a global \( A \)-parameter suitably. For this, we need the following refinement of the globalization of Sakellaridis–Venkatesh [62, Theorem 16.3.2], [30, Corollary A.8], which will be proved in Appendix C below.

**Proposition 6.5.** Let \( S \) and \( S' \) be nonempty finite sets of places of \( F \) such that

- \( S \) contains no archimedean places;
- \( S' \) contains all archimedean places and some nonarchimedean place;
- \( S \cap S' = \emptyset \).

For each \( v \in S \), let \( \phi_v \) be a \( 2n \)-dimensional irreducible symplectic representation of \( L_{\mathbb{R}} \). Then there exists an irreducible symplectic cuspidal automorphic representation \( \Phi \) of \( \text{GL}_{2n}(A) \) such that

- \( \Phi_v = \phi_v \) for all \( v \in S \);
- \( \Phi_v \) is a sum of \( 1 \)-dimensional representations for all \( v \notin S \cup S' \);
- \( L \left( \frac{1}{2}, \Phi \right) \neq 0 \).

We also need the following globalization (noting that any irreducible symplectic representation of \( L_{\mathbb{R}} \) is \( 2 \)-dimensional), which can be easily deduced from a result of Waldspurger [78, Théorème 4].

**Proposition 6.6.** Assume that \( F \) is totally real. Let \( S \) and \( S' \) be nonempty finite sets of places of \( F \) such that

- \( S \) contains all archimedean places;
- \( S' \) contains no archimedean places and some nonarchimedean place;
• $S \cap S' = \emptyset$.

For each $v \in S$, let $\phi_v$ be a 2-dimensional irreducible symplectic representation of $L_{\mathbb{F}_v}$. Then there exists an irreducible symplectic cuspidal automorphic representation $\Phi$ of $GL_2(\mathbb{A})$ such that

• $\Phi_v = \phi_v$ for all $v \in S$;
• $\Phi_v$ is a sum of 1-dimensional representations for all $v \notin S \cup S'$;
• $L(\frac{1}{2}, \Phi) \neq 0$.

Later, we will use the following consequence of these globalizations.

**Corollary 6.7.** Let $F$ be a nonarchimedean local field of characteristic zero or $F = \mathbb{R}$. Let $\psi$ be a 2$n$-dimensional good symplectic representation of $L_F$.

(i) Let $\mathbb{F}$ be a totally imaginary number field (resp. a real quadratic field) if $F$ is nonarchimedean (resp. if $F = \mathbb{R}$) with distinct places $v_0, v_1$ of $\mathbb{F}$ such that $F_{v_0} = F_{v_1} = F$. Then there exists an elliptic tempered $A$-parameter $\Psi$ for $Mp_{2n}$ such that

• $\Psi_{v_0} = \Psi_{v_1} = \psi$;
• the natural maps $S_\psi \to S_{\psi_{v_0}}$ and $S_\psi \to S_{\psi_{v_1}}$ agree;
• $L(\frac{1}{2}, \Psi) \neq 0$.

(ii) Assume that $\psi$ is reducible. Let $\mathbb{F}$ be a totally imaginary number field (resp. $\mathbb{F} = \mathbb{Q}$) if $F$ is nonarchimedean (resp. if $F = \mathbb{R}$) with a place $v_0$ of $\mathbb{F}$ such that $F_{v_0} = F$. Then there exists an elliptic tempered $A$-parameter $\Psi$ for $Mp_{2n}$ such that

• $\Psi_{v_0} = \psi$;
• $\Psi_v$ is non-good for all $v \neq v_0$;
• the natural map $S_\psi \to S_{\psi_{v_0}}$ is surjective;
• the natural map $S_\psi \to \prod_{v \neq v_0} S_{\psi_v}$ is injective;
• $L(\frac{1}{2}, \Psi) \neq 0$.

**Proof.** Put $S_0 = \{v_0, v_1\}$ in case (i); $S_0 = \{v_0\}$ in case (ii). We may write $\psi = \bigoplus_i \phi_i$ for some (not necessarily distinct) $2n_i$-dimensional irreducible symplectic representations $\phi_i$ of $L_F$. For each $i$, choose a nonempty finite set $S_i$ of nonarchimedean places of $\mathbb{F}$ such that

• $|S_i| \geq 2$ for all $i$;
• $S_0 \cap S_i = \emptyset$ for all $i$;
• $S_i \cap S_j = \emptyset$ for all $i \neq j$.

Since we can always find a $2n_i$-dimensional irreducible symplectic representation of $L_{\mathbb{F}_v}$ for all nonarchimedean $v$ (see e.g. [63 §2]) and any irreducible representation of $L_C$ is 1-dimensional, it follows from Propositions 6.3, 6.4 and 6.6 that there exist irreducible symplectic cuspidal automorphic representations $\Phi_i$ of $GL_{2n_i}(\mathbb{A})$ such that

• $\Phi_{i,v} = \phi_i$ for all $v \in S_0$;
• $\Phi_{i,v_i}$ is irreducible for some $v_i \in S_i$;
• $\Phi_{i,v}$ is a sum of 1-dimensional representations for all $v \notin S_0 \cup S_i$;
• $L(\frac{1}{2}, \Phi_i) \neq 0$.

In particular, the $\Phi_i$'s are pairwise distinct.

We show that $\Psi = \bigoplus_i \Phi_i$ satisfies the required conditions. In case (i), this is easy. In case (ii), since $\psi$ is reducible, $\Psi_v$ contains a 1-dimensional irreducible summand (which is non-symplectic) and hence is non-good for all $v \neq v_0$. Also, since $\Phi_{i,v_i}$ is irreducible but $\Phi_{j,v_i}$ is a sum of 1-dimensional representations
for all $j \neq i$, the natural map $S_\psi \to \prod_i S_{\psi_i}$ is bijective, so that the natural map $S_\psi \to \prod_{v \neq v_0} S_{\psi_v}$ is injective. The remaining conditions can be easily verified.

6.3. The case of good $L$-parameters. We now prove Propositions 5.4 and 6.1 for any $2n$-dimensional good symplectic representation $\psi$ of $L_F$ when $F$ is nonarchimedean or $F = \mathbb{R}$. We give a somewhat roundabout argument below, though we can streamline it by using results of Mœglin [49, 50, 51, 52] and Mœglin–Renard [55, 56], which were not fully available when this paper was first written.

Put $\psi' = \psi \oplus S_{2r-2n}$ with $r > 2n + 1$. For any irreducible representation $\sigma'$ of $SO_{2r+1}$ and any character $\eta'$ of $\hat{S}_{\psi'}$, we define the multiplicity $m(\sigma', \eta')$ by

$$m(\sigma', \eta') = \dim \text{Hom}_{SO_{2r+1}}(\sigma', \sigma_{\eta'}) ,$$

where $\sigma_{\eta'}$ is the representation in the $A$-packet $\Pi_{\psi'}(SO_{2r+1})$ associated to $\eta'$.

**Lemma 6.8.** Let $\sigma'$ be an irreducible representation of $SO_{2r+1}$. Then, for any $\eta' \in \hat{S}_{\psi'}$, we have

$$m(\sigma', \eta') \leq 1,$$

with equality for at most one $\eta'$. Namely, $\Pi_{\psi'}(SO_{2r+1})$ is multiplicity-free.

**Proof.** The assertion was proved in a more general context by Mœglin [51] when $F$ is nonarchimedean and by Mœglin–Renard [56] when $F = \mathbb{R}$. Here we give a proof based on theta lifts.

We may assume that $m(\sigma', \eta') > 0$ for some $\eta'$. Let $\mathbb{F}, v_0, v_1$, and $\Psi$ be as given in Corollary 6.7 (i), so that $\Psi_{v_0} = \Psi_{v_1} = \psi$. Put $\Psi' = \psi \oplus S_{2r-2n}$. Since $L(\frac{1}{2}, \Psi) \neq 0$, it follows from Lemma 4.3 that the quadratic character $\epsilon_{\Psi'}$ of $\hat{S}_{\psi'}$ is trivial. We define an abstract irreducible representation $\Sigma' = \bigotimes_v \Sigma'_v$ of $SO_{2r+1}(\mathbb{A})$ by setting

- $\Sigma'_{v_0} = \Sigma'_{v_1} = \sigma'$;
- $\Sigma'_v$ to be the irreducible representation in the $L$-packet $\Pi_{\Phi_{\psi_v}}(SO_{2r+1})$ associated to the trivial character of $S_{\psi_v}$ if $v \neq v_0, v_1$.

By [7] Proposition 7.4.1], $\Sigma'_v$ is an irreducible summand of the representation in the $A$-packet $\Pi_{\Phi_{\psi_v}}(SO_{2r+1})$ associated to the trivial character of $\hat{S}_{\psi_v}$ if $v \neq v_0, v_1$. Since the pullback of $\eta' \otimes \eta'$ under the natural map

$$\hat{S}_{\psi'} \longrightarrow \hat{S}_{\psi_{v_0}} \times \hat{S}_{\psi_{v_1}} = \hat{S}_{\psi} \times \hat{S}_{\psi'}$$

is trivial for any $\eta' \in \hat{S}_{\psi'}$, we have an embedding

$$\left( \bigoplus_{\eta' \in \hat{S}_{\psi'}} (\sigma_{\eta'} \otimes \sigma_{\eta'}) \right) \otimes \left( \bigotimes_{v \neq v_0, v_1} \Sigma'_v \right) \hookrightarrow L^2_{\psi'}(SO_{2r+1})$$

by the multiplicity formula (4.5). In particular,

$$m_{\text{disc}}(\Sigma') \geq \sum_{\eta' \in \hat{S}_{\psi'}} m(\sigma', \eta') \geq 0.$$

We now consider theta lifts. Fix a nontrivial additive character $\Psi_{\mathbb{A}}$ of $\mathbb{F}\backslash \mathbb{A}$ such that $\Psi_{\mathbb{A}, v_0}$ and $\Psi_{\mathbb{A}, v_1}$ belong to the $(\mathbb{F}^\times)^2$-orbit of the fixed nontrivial additive character $\Psi$ of $F$. Let $\Pi = \overline{\sigma_{\text{abs}}(\Sigma')}$ be the abstract theta lift to $Mp_{2n}(\mathbb{A})$ relative to $\Psi_{\mathbb{A}}$. Since $\Psi$ is tempered, it follows from Proposition 4.1 and Corollary 4.2 that $\Pi$ is an irreducible summand of $L^2_{\psi}(Mp_{2n})$ and that

$$m_{\text{cusp}}(\Pi) = m_{\text{disc}}(\Pi) = m_{\text{disc}}(\Sigma') > 0.$$
For any realization \( \mathcal{V} \subset \mathcal{A}_{\text{cusp}}(Mp_{2n}) \) of \( \Pi \), let \( \Theta^\text{aut}(\mathcal{V}) \) be the global theta lift to \( \text{SO}(\mathcal{V})(\mathbb{A}) \) relative to \( \Psi_{\mathbb{A}} \), where \( \mathcal{V} \) is a \((2n+1)\)-dimensional quadratic space over \( \mathbb{F} \) with trivial discriminant. Since \( \Psi \) is tempered, we deduce from the tower property and the local theta correspondence for unramified representations that \( \Theta^\text{aut}_\mathcal{V}(\mathcal{V}) \) is cuspidal (but possibly zero). Moreover, we have
\[
L^S(\frac{1}{2}, \Pi) = L^S(\frac{1}{2}, \Psi) \neq 0,
\]
where \( S \) is a sufficiently large finite set of places of \( \mathbb{F} \) and \( L^S(s, \Pi) \) is the partial standard \( L \)-function of \( \Pi \) relative to \( \Psi_{\mathbb{A}} \). Hence, by the Rallis inner product formula \([36, 19, 81]\) and the local Shimura correspondence \([2, 3, 20]\), there exists a unique \( \mathcal{V} \) such that \( \Theta^\text{aut}_\mathcal{V}(\mathcal{V}) \) is nonzero for any realization \( \mathcal{V} \) of \( \Pi \). For this unique \( \mathcal{V} \), let \( \Sigma = \theta^{\text{abs}}(\Pi) \) be the abstract theta lift to \( \text{SO}(\mathcal{V})(\mathbb{A}) \) relative to \( \Psi_{\mathbb{A}} \). Then, by the multiplicity preservation \([13, \text{ Proposition } 2.6]\), we have
\[
m_{\text{disc}}(\Sigma) \geq m_{\text{cusp}}(\Sigma) = m_{\text{cusp}}(\Pi).
\]

Also, it follows form the local theta correspondence for unramified representations that \( \Sigma \) is an irreducible summand of \( L^2_{\Psi}(\text{SO}(\mathcal{V})) \). Since \( \Psi \) is tempered, the multiplicity formula \((6.1)\) implies that
\[
m_{\text{disc}}(\Sigma) = 1.
\]
Thus, combining these (in)equalities, we obtain
\[
1 \geq \sum_{\eta' \in \hat{S}} m(\sigma', \eta')^2 > 0.
\]
This completes the proof. \( \square \)

Let \( \text{JH}(\Pi_{\psi'}(\text{SO}_{2r+1})) \) be the set of equivalence classes of irreducible representations \( \sigma' \) of \( \text{SO}_{2r+1} \) such that \( m(\sigma', \eta') > 0 \) for some \( \eta' \).

**Lemma 6.9.** The theta lift induces an injection
\[
\vartheta : \text{JH}(\Pi_{\psi'}(\text{SO}_{2r+1})) \hookrightarrow \Pi_{\psi}(\text{Mp}_{2n}).
\]

**Proof.** We retain the notation of the proof of Lemma 6.8. In particular, for any \( \sigma' \in \text{JH}(\Pi_{\psi'}(\text{SO}_{2r+1})) \), we have an irreducible summand \( \Sigma' \) of \( L^2_{\psi'}(\text{SO}_{2r+1}) \) such that
\[
\begin{align*}
\bullet & \quad \Sigma'_{\vartheta_0} = \sigma'; \\
\bullet & \quad \Pi = \theta^{\text{abs}}(\Sigma') \text{ is an irreducible summand of } L^2_{\psi}(\text{Mp}_{2n}); \\
\bullet & \quad \Sigma = \theta^{\text{abs}}(\Pi) \text{ is an irreducible summand of } L^2_{\psi}(\text{SO}(\mathcal{V})).
\end{align*}
\]
Since the L-parameter of \( \Sigma'_{\vartheta_0} = \theta(\Pi_{\vartheta_0}) \) is \( \Psi_{\vartheta_0} = \psi \), we have
\[
\vartheta(\sigma') = \vartheta(\Sigma'_{\vartheta_0}) = \Pi_{\vartheta_0} \in \Pi_{\psi}(\text{Mp}_{2n}).
\]
This and the Howe duality imply the assertion. \( \square \)

We now show that the map \( \vartheta \) in Lemma 6.9 is in fact surjective.

**Lemma 6.10.** The theta lift induces a bijection
\[
\vartheta : \text{JH}(\Pi_{\psi'}(\text{SO}_{2r+1})) \leftrightarrow \Pi_{\psi}(\text{Mp}_{2n}).
\]

**Proof.** Let \( \eta' \in \hat{S}_{\psi'} \) and put \( \eta = \imath^* \eta' \). Let \( \Sigma_\eta \) be the irreducible representation in the \( L \)-packet \( \Pi_{\psi}(\text{SO}(V)) \) associated to \( \eta \), where \( V \) is the \((2n+1)\)-dimensional quadratic space over \( \mathbb{F} \) with trivial discriminant such that \( \eta \in \hat{S}_{\psi,V} \). Let \( \mathcal{F}, \vartheta_0, v_1 \), and \( \Psi \) be as given in Corollary 6.7, \( (i) \), so that \( \Psi_{\vartheta_0} = \psi_{v_1} = \psi \). Then there exists a unique \((2n+1)\)-dimensional quadratic space \( \mathcal{V} \) over \( \mathbb{F} \) with trivial discriminant such that
\[
\begin{align*}
\end{align*}
\]
Let $\Sigma = \bigotimes_v \Sigma_v$ of $\text{SO}(V)(\mathbb{A})$ by setting

- $\Sigma_v = \Sigma_v = \Sigma_v$;
- $\Sigma_v$ to be the irreducible representation in the $L$-packet $\Pi_{\phi_v}(\text{SO}(V))$ associated to the trivial character of $S_{\phi_v}$ if $v \neq v_0, v_1$.

Then, by the multiplicity formula (6.1, $\Sigma$ is an irreducible summand of $L_2^\varphi(\text{SO}(V))$. In fact, since $\Psi$ is tempered, it follows from the argument in the proof of Proposition 1.1 that $\Sigma$ is cuspidal.

Let $\Pi = \theta^{\text{abs}}(\Sigma)$ be the abstract theta lift to $\text{Mp}_{2n}(\mathbb{A})$ (relative to $\Psi_\mathbb{A}$ as in the proof of Lemma 6.8), so that

$$\Pi_{v_0} = \theta(\Sigma_{v_0}) = \theta(\Sigma_{v_1}) = \pi_{\eta} \in \Pi_\psi(\text{Mp}_{2n}).$$

Since $L(\frac{1}{2}, \Psi) \neq 0$, we can deduce from the argument in the proof of Lemma 6.8 that the global theta lift $\Theta^{\text{aut}}(\Sigma)$ to $\text{Mp}_{2n}(\mathbb{A})$ is nonzero and cuspidal, so that $\Theta^{\text{aut}}(\Sigma) \cong \Pi$ is an irreducible summand of $L_2^\varphi(\text{Mp}_{2n})$. Hence, by Corollary 4.2, the abstract theta lift $\Sigma' = \psi^{\text{abs}}(\Pi)$ to $\text{SO}_{2r+1}(\mathbb{A})$ is an irreducible summand of $L_2^\varphi(\text{SO}_{2r+1})$, where $\Psi' = \Psi \oplus S_{2r-2n}$. Since $\Pi'_{v_0} = \psi'$, this implies that

$$\Sigma'_{v_0} \in \text{JH}(\Pi_{\psi'}(\text{SO}_{2r+1})).$$

On the other hand, we have

$$\psi(\Sigma'_{v_0}) = \Pi_{v_0} = \pi_{\eta}.$$

Hence the map in Lemma 6.9 is surjective.

Finally, we show that Propositions 5.4 and 6.1 hold for $\psi$. We consider the irreducible case and the reducible case separately.

**Lemma 6.11.** Assume that $\psi$ is irreducible. Then Propositions 5.4 and 6.1 hold for $\psi$.

**Proof.** Let $\sigma'_1$ be the representation in the $A$-packet $\Pi_{\phi_1'}(\text{SO}_{2r+1})$ associated to the trivial character of $\tilde{S}_{\psi'}$. Then, by [11, Proposition 7.4.1], $\sigma'_1$ contains the irreducible representation $\sigma'$ in the $L$-packet $\Pi_{\phi_1'}(\text{SO}_{2r+1})$ associated to the trivial character of $S_{\phi_1'}$. Since

$$\# \text{JH}(\Pi_{\phi_1'}(\text{SO}_{2r+1})) = 2$$

by the irreducibility of $\psi$ and Lemma 6.10, we may write $\text{JH}(\Pi_{\phi_1'}(\text{SO}_{2r+1})) = \{\sigma', \sigma''\}$ for some irreducible representation $\sigma''$ of $\text{SO}_{2r+1}$.

We show that $\sigma'_1$ is irreducible. Suppose on the contrary that $\sigma'_1$ is reducible. By Lemma 6.8, we have $\sigma'_1 = \sigma' \oplus \sigma''$. Let $F, v_0, v_1$, and $\Psi$ be as given in Corollary 6.7 [1], so that $S_{v_0} = S_{v_1} = \psi$. Put $\Psi' = \Psi \oplus S_{2r-2n}$. Since $L(\frac{1}{2}, \Psi) \neq 0$, it follows from Lemma 4.3 that the quadratic character $e_{\psi'}$ of $\tilde{S}_{\psi'}$ is trivial. We define an abstract irreducible representation $\Sigma' = \bigotimes_v \Sigma'_v$ of $\text{SO}_{2r+1}(\mathbb{A})$ by setting

- $\Sigma'_{v_0} = \sigma'$;
- $\Sigma'_{v_1} = \sigma''$;
- $\Sigma'_v$ to be the irreducible representation in the $L$-packet $\Pi_{\phi_1'}(\text{SO}_{2r+1})$ associated to the trivial character of $S_{\phi_1'}$ if $v \neq v_0, v_1$.

Since $\sigma'_1 = \sigma' \oplus \sigma''$, it follows from the multiplicity formula (4.5) that $\Sigma'$ is an irreducible summand of $L_2^\varphi(\text{SO}_{2r+1})$. Let $\Pi = \psi^{\text{abs}}(\Sigma')$ be the abstract theta lift to $\text{Mp}_{2n}(\mathbb{A})$ (relative to $\Psi_\mathbb{A}$ as in the proof of Lemma 6.8). Then, as shown in the proof of Lemma 6.8, there exists a unique $(2n + 1)$-dimensional
quadratic space $V$ over $F$ with trivial discriminant such that the abstract theta lift $\theta^\text{abs}_V(\Pi)$ to $SO(V)(A)$ is nonzero. On the other hand, if $v \neq v_0, v_1$ (so that $v$ is not real), then by Lemma 6.3, $\Pi_v = \vartheta(\Sigma'_v)$ is the irreducible representation in the $L$-packet $\Pi_{\psi_v}(Mp_{2n})$ associated to the trivial character of $S_{\psi_v}$, so that 
\[ \varepsilon(V_v) = 1. \]

Also, by Lemma 6.10, we have 
\[ \{ \Pi_{v_0}, \Pi_{v_1} \} = \{ \vartheta(\Sigma'_{v_0}), \vartheta(\Sigma'_{v_1}) \} = \{ \vartheta(\sigma'), \vartheta(\sigma'') \} = \Pi_\psi(Mp_{2n}), \]
so that 
\[ \varepsilon(V_{v_0}) \cdot \varepsilon(V_{v_1}) = -1. \]

This contradicts the fact that $\prod_v \varepsilon(V_v) = 1$. Hence $\sigma'_1$ is irreducible and $\sigma'_1 = \sigma'$.

By Lemma 6.8 and 6.10, it remains to show that $\vartheta(\sigma')$ is associated to the trivial character of $S_\psi$. If $F$ is nonarchimedean, then this follows from Lemma 6.4. Suppose that $F = \mathbb{R}$ (so that $n = 1$) and that $\vartheta(\sigma')$ is associated to the nontrivial character of $S_\psi$. From the above argument with the following modifications:

- $F = \mathbb{Q}$;
- $\psi$ is an elliptic tempered $A$-parameter for $Mp_2$ such that $\Psi_\infty = \psi$ and $L(1/2, \psi) \neq 0$ (see Proposition 6.3);
- $\Sigma'_\infty = \sigma'$;
- $\Sigma'_v$ is associated to the trivial character of $S_{\phi_\psi}$ if $v$ is nonarchimedean,

we can deduce that there exists a 3-dimensional quadratic space $V$ over $F$ with trivial discriminant such that $\varepsilon(V_v) = 1$ for all nonarchimedean $v$ but $\varepsilon(V_\infty) = -1$. This is a contradiction and completes the proof.

**Lemma 6.12.** Assume that $\psi$ is reducible. Then Propositions 5.4 and 6.1 hold for $\psi$.

**Proof.** Let $\eta' \in \hat{S}_\psi$ and put $\eta = \psi' \eta'$. Let $F, v_0$, and $\psi$ be as given in Corollary 6.7 (iii), so that $\Psi_{v_0} = \psi$. Since the natural map $\left( \prod_{v \neq v_0} S_{\psi_v} \right) \rightarrow \hat{S}_\psi$ is surjective, there exists a continuous character $\bigotimes_v \xi_v$ of $S_{\psi_\mathbb{A}}$ such that 
\begin{align*}
(6.2) & \quad \xi_{v_0} = \eta, \\
(6.3) & \quad \left( \bigotimes_v \xi_v \right) \circ \Delta = 1.
\end{align*}

Then there exists a unique $(2n + 1)$-dimensional quadratic space $V$ over $F$ with trivial discriminant such that $\xi_v \in \hat{S}_{\psi_v}V_v$ for all $v$. We define an abstract irreducible representation $\Sigma = \bigotimes_v \Sigma_v$ of $SO(V)(A)$ by setting 
\[ \Sigma_v = \Sigma_{\xi_v} \in \Pi_{\psi_v}(SO(V)) \]
for all $v$. Then, by the multiplicity formula (6.1), $\Sigma$ is an irreducible summand of $L^2_\psi(SO(V))$.

Let $\Pi = \theta^\text{abs}(\Sigma)$ be the abstract theta lift to $Mp_{2n}(A)$ (relative to a fixed nontrivial additive character $\Psi_\mathbb{A}$ of $F\backslash \mathbb{A}$ such that $\Psi_{A,v_0}$ belongs to the $(F^\times)^2$-orbit of $\Psi$), so that 
\[ \Pi_v = \theta(\Sigma_v) = \theta(\Sigma_{\xi_v}) = \pi_{\xi_v} \in \Pi_{\psi_v}(Mp_{2n}) \]
for all $v$. Then, as shown in the proof of Lemma 6.10, the abstract theta lift $\Sigma' = \theta^\text{abs}(\Pi)$ to $SO_{2r+1}(A)$ is an irreducible summand of $L^2_{\psi'}(SO_{2r+1})$, where $\psi' = \psi \oplus S_{2r-2n}$. This implies that for any $v$, $\Sigma'_v$ is an irreducible summand of $\sigma_{\xi'_v} \in \Pi_{\psi'_v}(SO_{2r+1})$ for some $\xi'_v \in \hat{S}_{\psi'_v}$. By Lemma 6.8 such $\xi'_v$ is unique. In
fact, if \( v \neq v_0 \) (so that \( v \) is not real), then since \( \Psi_v \) is non-good, we can apply Propositions 5.4 and 6.1 to obtain \( \Sigma'_v = \sigma \xi_v \) with

(6.4) \[ \iota_v^* \xi'_v = \xi_v, \]

where \( \iota_v : \mathcal{S}_{\Psi_v} \to \mathcal{S}_{\Psi'_v} \) is the natural isomorphism. On the other hand, since \( L(\frac{1}{2}, \Psi) \neq 0 \), it follows from Lemma 4.3 that the quadratic character \( \epsilon_{\Psi'_v} \) of \( \mathcal{S}_{\Psi'_v} \) is trivial. Hence, by the multiplicity formula (4.5), we must have

(6.5) \[ \left( \bigotimes_{v} \xi'_v \right) \circ \Delta = 1. \]

Thus, since the natural map \( \hat{\mathcal{S}}_{\Psi_{v_0}} \to \hat{\mathcal{S}}_{\Psi} \) is injective, we deduce from (6.3), (6.4), (6.5) that

\[ \iota_{v_0}^* \xi_{v_0} = \xi_{v_0}. \]

Hence we have \( \iota_{v_0}^* \xi_{v_0} = \eta \) by (6.2), so that

(6.6) \[ \xi_{v_0} = \eta' \]

by the definition of \( \eta \). In particular, \( \sigma_{\eta'} \in \Pi_{\Psi}(SO_{2r+1}) \) is nonzero for any \( \eta' \in \hat{\mathcal{S}}_{\Psi} \). By Lemmas 6.8 and 6.10, this implies that \( \sigma_{\eta'} \) is irreducible for any \( \eta' \). Moreover, by (6.6), we have

\[ \vartheta(\sigma_{\eta'}) = \vartheta(\Sigma'_{v_0}) = \Pi_{v_0} = \pi_{\eta}. \]

This completes the proof.

This completes the proof of Propositions 5.4 and 6.1 and hence of Theorem 1.3.

\[ \square \]

Appendix A. Irreducibility of Some Standard Modules of \( \text{Mp}_{2n}(\mathbb{R}) \)

In this appendix, we finish the proof of Lemma 5.2 in the real case (see Proposition A.3 below). The argument is standard, but we include it for the convenience of the reader.

A.1. Notation. Let \( G = \text{Mp}_{2n}(\mathbb{R}) \) be the metaplectic two-fold cover of \( \text{Sp}_{2n}(\mathbb{R}) \), which we realize as \( \text{Sp}_{2n}(\mathbb{R}) = \left\{ g \in \text{GL}_{2n}(\mathbb{R}) \mid g \begin{pmatrix} -1 & \mathbf{1}_n \\ \mathbf{1}_n \\ \end{pmatrix} t g = \begin{pmatrix} -1 & \mathbf{1}_n \\ \mathbf{1}_n \\ \end{pmatrix} \right\} \).

We define a maximal compact subgroup \( K \) of \( G \) as the preimage in \( G \) of \( \left\{ g \in \text{Sp}_{2n}(\mathbb{R}) \mid t g = g \right\} \).

Let \( \theta \) be the Cartan involution of \( G \) corresponding to \( K \). Let \( \mathfrak{g}_0 = \text{Lie} G \) be the Lie algebra of \( G \) and \( \mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \) its complexification; analogous notation is used for other groups.

For any non-negative integers \( k, l, m \) such that \( k + 2l + m = n \), we define a \( \theta \)-stable Cartan subalgebra \( h^{k,l,m}_0 \) of \( \mathfrak{g}_0 \) as follows. For \( a = (a_1, \ldots, a_k) \in \mathbb{R}^k \), put

\[ h^{k,0,0}(a) = \begin{pmatrix} \mathbf{a} \\ -\mathbf{a} \end{pmatrix} \in \mathfrak{sp}_{2k}(\mathbb{R}), \]

where \( \mathbf{a} = \text{diag}(a_1, \ldots, a_k) \). For \( z = (z_1, \ldots, z_l) \in \mathbb{C}^l \) with \( z_i = x_i + \sqrt{-1}y_i \), put

\[ h^{0,l,0}(z) = \begin{pmatrix} x & y \\ -y & -x \end{pmatrix} \in \mathfrak{sp}_{2l}(\mathbb{R}), \]
where \( x = \text{diag}(x_1, \ldots, x_l) \) and \( y = \text{diag}(y_1, \ldots, y_l) \). For \( \vartheta = (\vartheta_1, \ldots, \vartheta_m) \in \mathbb{R}^m \), put

\[
h^{0,0,m}(\vartheta) = \begin{pmatrix} -\vartheta \\ \vartheta \end{pmatrix} \in \mathfrak{sp}_m(\mathbb{R}),
\]

where \( \vartheta = \text{diag}(\vartheta_1, \ldots, \vartheta_m) \). Let \( h^{k,l,m}(a, z, \vartheta) \) be the image of

\[
(h^{k,0,0}(a), h^{0,l,0}(z), h^{0,0,m}(\vartheta))
\]

under the natural embedding

\[
\mathfrak{sp}_k(\mathbb{R}) \oplus \mathfrak{sp}_l(\mathbb{R}) \oplus \mathfrak{sp}_m(\mathbb{R}) \hookrightarrow \mathfrak{sp}_n(\mathbb{R}).
\]

Then we set

\[
\mathfrak{h}_0^{k,l,m} = \{ h^{k,l,m}(a, z, \vartheta) \mid a \in \mathbb{R}^k, z \in \mathbb{C}^l, \vartheta \in \mathbb{R}^m \}.
\]

These \( \mathfrak{h}_0^{k,l,m} \) with \( k + 2l + m = n \) form a set of representatives for the \( G \)-conjugacy classes of Cartan subalgebras of \( \mathfrak{g}_0 \).

Fix such \( k, l, m \) and write \( \mathfrak{h}_0 = \mathfrak{h}_0^{k,l,m} \). We define a basis \( e_1, \ldots, e_n \) of \( \mathfrak{h}^* \) by

\[
e_i(h) = a_i \quad (1 \leq i \leq k);
\]

\[
e_{k+2i-1}(h) = x_i + \sqrt{-1}y_i \quad (1 \leq i \leq l);
\]

\[
e_{k+2i}(h) = x_i - \sqrt{-1}y_i \quad (1 \leq i \leq l);
\]

\[
e_{k+2l+i}(h) = \sqrt{-1}d_i \quad (1 \leq i \leq m)
\]

for \( h = h^{k,l,m}(a, z, \vartheta) \). Note that

\[
\theta(e_i) = -e_i \quad (1 \leq i \leq k);
\]

\[
\theta(e_{k+2i-1}) = -e_{k+2i} \quad (1 \leq i \leq l);
\]

\[
\theta(e_{k+2l+i}) = e_{k+2l+i} \quad (1 \leq i \leq m).
\]

Using the above basis, we identify \( \mathfrak{h}^* \) with \( \mathbb{C}^n \). Let \( \langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C} \) be the standard bilinear form:

\[
\langle \lambda, \mu \rangle = \lambda_1 \mu_1 + \cdots + \lambda_n \mu_n
\]

for \( \lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n) \in \mathfrak{h}^* \cong \mathbb{C}^n \). We denote by \( \Delta \) the set of roots of \( \mathfrak{h} \) in \( \mathfrak{g} \):

\[
\Delta = \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ \pm 2e_i \mid 1 \leq i \leq n \}.
\]

For any subspace \( \mathfrak{f} \) of \( \mathfrak{g} \) stable under the adjoint action of \( \mathfrak{h} \), we denote by \( \Delta(\mathfrak{f}) \) the set of roots of \( \mathfrak{h} \) in \( \mathfrak{f} \) and put

\[
\rho(\mathfrak{f}) = \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{f})} \alpha.
\]

### A.2. Discrete series

As in [3, §3], genuine (limits of) discrete series representations of \( G \) are classified as follows. Suppose that \( \mathfrak{h}_0 = \mathfrak{h}_0^{0,0,n} \), so that \( \mathfrak{h}_0 = \mathfrak{t}_0 \). Let \( \Delta_c \) be the set of compact roots and take a positive system

\[
\Delta_c^+ = \{ e_i - e_j \mid 1 \leq i < j \leq n \}.
\]

Then a genuine discrete series representation of \( G \) is parametrized by its Harish-Chandra parameter \( \lambda \in \sqrt{-1} \mathfrak{t}_0^* \) of the form

\[
\lambda = (a_1, \ldots, a_r, -b_1, \ldots, -b_s),
\]

where

- \( a_i, b_j \in \mathbb{Z} + \frac{1}{2} \);
- \( a_1 > \cdots > a_r > 0 \) and \( 0 < b_1 < \cdots < b_s \).
• $a_i \neq b_j$ for all $i, j$.

More generally, a genuine limit of discrete series representation of $G$ is parametrized by a pair $(\lambda, \Psi)$ consisting of $\lambda \in \sqrt{-1}h_0^*$ of the form

$$
\lambda = (a_1, \ldots, a_1, \ldots, a_r, \ldots, a_r, -a_r, \ldots, -a_r, \ldots, -a_1, \ldots, -a_1),
$$

where

• $a_i \in \mathbb{Z} + \frac{1}{2}$;
• $a_1 > \cdots > a_r > 0$;
• $m_i, n_j \geq 0$;
• $m_i + n_i > 0$ and $|m_i - n_i| \leq 1$ for all $i$,

and a positive system $\Psi$ of $\Delta$ such that

• $\Delta_c^+ \subset \Psi$;
• $\langle \alpha, \lambda \rangle \geq 0$ for all $\alpha \in \Psi$;
• if $\alpha$ is a simple root in $\Psi$ such that $\langle \alpha, \lambda \rangle = 0$, then $\alpha$ is noncompact (see the condition (F-1) in [76]).

Note that, given such $\lambda$, there are precisely $2^t$ positive systems $\Psi$ satisfying the above conditions, where $t$ is the number of indices $i$ such that $m_i = n_i > 0$.

**Remark A.1.** The $L$-parameter of the representation associated to $(\lambda, \Psi)$ is

$$
\bigoplus_{i=1}^r (m_i + n_i)D_{a_i},
$$

where for $a \in \frac{1}{2}\mathbb{Z}$, we denote by $D_a$ the 2-dimensional representation of $L_\mathbb{R}$ induced from the character $z \mapsto (z/\bar{z})^a$ of $L_\mathbb{C} = \mathbb{C}^\times$. Note that

• $D_{-a} = D_a$;
• $D_a$ is irreducible if and only if $a \neq 0$;
• $D_a$ is symplectic if and only if $a \in \mathbb{Z} + \frac{1}{2}$.

In particular, the above $L$-parameter is good and the associated $L$-packet consists of $2^r$ representations.

**A.3. Standard modules.** We will use Vogan’s version of the Langlands classification for real reductive Lie groups in Harish-Chandra’s class [76]. Suppose again that $h_0 = h_0^{k,l,m}$ is arbitrary. Let $H$ be the centralizer of $h_0$ in $G$. Then $H$ is the preimage in $G$ of a Cartan subgroup of $\text{Sp}_{2n}(\mathbb{R})$ isomorphic to

$$(\mathbb{R}^\times)^k \times (\mathbb{C}^\times)^l \times (S^1)^m.$$  

Let $t_0$ and $a_0$ be the $+1$ and $-1$ eigenspaces of $\theta$ in $h_0$, respectively. Put $T = H \cap K$ and $A = \exp(a_0)$, so that

$$H = T \times A.$$  

Let $M$ be the centralizer of $a_0$ in $G$. Then $M$ is the preimage in $G$ of a Levi subgroup of $\text{Sp}_{2n}(\mathbb{R})$ isomorphic to

$$\text{GL}_1(\mathbb{R})^k \times \text{GL}_2(\mathbb{R})^l \times \text{Sp}_{2m}(\mathbb{R}).$$  

For the inducing data of a standard module, we take an irreducible representation of $M$ as follows. Let $\tilde{\text{GL}}_d(\mathbb{R})$ be the two-fold cover of $\text{GL}_d(\mathbb{R})$ given in [17, §2.5]. Let $\chi_\psi$ be the genuine quartic character of
$\widetilde{\text{GL}}_1(\mathbb{R})$ given in [17 §2.6] relative to a fixed nontrivial additive character $\psi$ of $\mathbb{R}$. For $1 \leq i \leq k$, let $\chi_i$ be a character of $\text{GL}_1(\mathbb{R})$ of the form

$$\chi_i = \text{sgn}^{\delta_i} \otimes \chi_\psi \otimes | \cdot |^{\nu_i}$$

for some $\delta_i \in \{ 0, 1 \}$ and some $\nu_i \in \mathbb{C}$. For $1 \leq i \leq l$, let $\tau_i$ be an irreducible representation of $\widetilde{\text{GL}}_2(\mathbb{R})$ of the form

$$\tau_i = D_{\kappa_i} \otimes (\chi_\psi \circ \widetilde{\text{det}}) \otimes \text{det} |^{\nu'_i}$$

for some $\kappa_i \in \frac{1}{2} \mathbb{Z}$ and some $\nu'_i \in \mathbb{C}$, where $D_{\kappa_i}$ is the relative (limit of) discrete series representation of $\text{GL}_2(\mathbb{R})$ of weight $2|\kappa_i| + 1$ with central character trivial on $\mathbb{R}_+^\times$ and $\widetilde{\text{det}}$ is the natural lift of the determinant map given in [17 §2.6]:

$$\begin{array}{ccc}
\widetilde{\text{GL}}_2(\mathbb{R}) & \xrightarrow{\text{det}} & \widetilde{\text{GL}}_1(\mathbb{R}) \\
\downarrow & & \downarrow \\
\text{GL}_2(\mathbb{R}) & \xrightarrow{\text{det}} & \text{GL}_1(\mathbb{R})
\end{array}$$

Note that $\tau_i$ does not depend on the choice of $\psi$ since $D_{\kappa_i} \otimes (\text{sgn} \circ \text{det}) \simeq D_{\kappa_i}$. Let $\pi'$ be a genuine (limit of) discrete series representation of $\text{Mp}_{2m}(\mathbb{R})$ associated to $(\lambda', \Psi')$ as in [A.2]. Then

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi'$$

descends to an irreducible representation of $M$.

Put

$$\gamma = (\lambda, \nu) \in \mathfrak{h}^* = \mathfrak{t}^* \oplus \mathfrak{a}^*,$$

where

$$\lambda = (0, \ldots, 0, \kappa_1, -\kappa_1, \ldots, \kappa_l, -\kappa_l, \lambda'_1, \ldots, \lambda'_m),$$

$$\nu = (\nu_1, \ldots, \nu_k, \nu'_1, \ldots, \nu'_1, 0, \ldots, 0).$$

Assume that the condition (F-2) in [76] (which is explicated in [3, Lemma 4.3]) holds:

(i) if $\nu_i = \pm \nu_j$, then $\delta_i = \delta_j$;
(ii) if $\nu'_i = 0$, then $\kappa_i \in \mathbb{Z}$.

Choose a parabolic subgroup $P = MN$ of $G$ with Levi component $M$ and unipotent radical $N$ such that

$$\text{Re} \langle \alpha, \nu \rangle \geq 0$$

for all roots $\alpha$ of $\mathfrak{a}$ in $\mathfrak{n}$. Then, by [76 Proposition 2.6], the normalized parabolic induction

$$\text{Ind}^G_P(\pi)$$

has a unique irreducible quotient $J^G_P(\pi)$. Note that $J^G_P(\pi)$ is tempered if and only if $\text{Re} \nu_i = \text{Re} \nu'_j = 0$ for all $i, j$, in which case $\text{Ind}^G_P(\pi)$ is irreducible. Moreover, every irreducible genuine representation of $G$ arises in this way (see [76 Theorem 2.9]).

Remark A.2. The $L$-parameter of $J^G_P(\pi)$ is

$$\phi \oplus \phi' \oplus \phi',$$
where \( \phi \) is given by

\[
\phi = \left( \bigoplus_{i=1}^{k} \text{sgn} \delta_i \mid {}^i \nu_i \right) \oplus \left( \bigoplus_{j=1}^{l} \mathcal{D}_{\kappa_j} \mid {}^j \nu_j \right)
\]

and \( \phi' \) is the \( L \)-parameter of \( \pi' \) (see Remark A.1). Note that any irreducible summand of \( \phi \) is non-symplectic by the condition (F-2) in [76] (see the above conditions (i), (ii)).

Finally, for any real root \( \alpha \in \Delta \), we consider the following “parity condition”:

- if \( \alpha = \pm(e_i - e_j) \) with \( 1 \leq i < j \leq k \), then either
  - \( \delta_i = \delta_j \) and \( \nu_i - \nu_j \in 2\mathbb{Z} + 1 \); or
  - \( \delta_i \neq \delta_j \) and \( \nu_i - \nu_j \in 2\mathbb{Z} \);
- if \( \alpha = \pm(e_i + e_j) \) with \( 1 \leq i < j \leq k \), then either
  - \( \delta_i = \delta_j \) and \( \nu_i + \nu_j \in 2\mathbb{Z} + 1 \); or
  - \( \delta_i \neq \delta_j \) and \( \nu_i + \nu_j \in 2\mathbb{Z} \);
- if \( \alpha = \pm 2e_i \) with \( 1 \leq i \leq k \), then \( \nu_i \in \mathbb{Z} + \frac{1}{2} \);
- if \( \alpha = \pm(e_{k+2i-1} + e_{k+2i}) \) with \( 1 \leq i \leq l \), then either
  - \( \kappa_i \in \mathbb{Z} \) and \( \nu'_i \in \mathbb{Z} + \frac{1}{2} \); or
  - \( \kappa_i \in \mathbb{Z} + \frac{1}{2} \) and \( \nu'_i \in \mathbb{Z} \).

With the above notation, we now state the main result of this appendix.

**Proposition A.3.** Assume that there exists no root \( \alpha \in \Delta \) such that either

(i) \( \alpha \) is complex and satisfies \( 2\langle \alpha, \gamma \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z} \), \( \langle \alpha, \gamma \rangle > 0 \), and \( \langle \theta \alpha, \gamma \rangle < 0 \); or

(ii) \( \alpha \) is real and satisfies the parity condition.

Then Ind\(^G\)\(_P\)\( (\pi) \) is irreducible. In particular, if \( |\text{Re} \nu_i|, |\text{Re} \nu'_j| < \frac{1}{2} \) for all \( i, j \), then Ind\(^G\)\(_P\)\( (\pi) \) is irreducible.

A.4. Proof of Proposition A.3. We first express the standard module Ind\(^G\)\(_P\)\( (\pi) \) as cohomological induction from a principal series representation. By [34] XI.8 combined with [34] Lemma 11.202, we may write

\[
\pi = (u\mathcal{R}^{m,M \cap K}_{b,T})^{\text{dim} \varpi} (\zeta \otimes \chi_{\rho(\varpi)}),
\]

where

- \( (u\mathcal{R}^{m,M \cap K}_{b,T})^{i} \) is the functor defined by [34] (11.71d);
- \( b = \mathfrak{h} \oplus \mathfrak{v} \) is a \( \theta \)-stable Borel subalgebra of \( \mathfrak{m} \) with Levi component \( \mathfrak{h} \) and nilpotent radical \( \mathfrak{v} \) such that
  \[
  \langle \alpha, \lambda \rangle \geq 0
  \]
  for all \( \alpha \in \Delta(\mathfrak{v}) \);
- \( \zeta \) is the character of \( H \) given by
  \[
  \zeta = \chi_1 \otimes \cdots \otimes \chi_k \otimes \xi_1 \otimes \cdots \otimes \xi_l \otimes \eta_1 \otimes \cdots \otimes \eta_m,
  \]
  where
  - \( \xi_i \) is the character of \( \mathbb{C}^\times \times \{ \pm 1 \} \) given by \( \xi_i(z, \epsilon) = \epsilon \cdot (z/\bar{z})^{\eta_i} \cdot (\bar{z}z)^{\nu_i} \);
  - \( \eta_i \) is the genuine character of the nonsplit two-fold cover of \( S^1 \) whose square descends to the character \( z \mapsto z^{2\nu_i} \) of \( S^1 \);
- \( \chi_{\rho(\varpi)} \) is the character of \( H \) such that
  - \( \chi_{\rho(\varpi)} \) factors through the image of \( H \) in \( \text{Sp}_{2n}(\mathbb{R}) \);
  - \( \chi_{\rho(\varpi)} \) is trivial on \( (\mathbb{R}^\times)^k \);
Hence, noting that for all \( \alpha \) we are reduced to the following irreducibility:

\[
\text{(A.2)} \quad |\alpha| \geq 0
\]

for all \( \alpha \in \Delta(u) \). Then, by [34, Theorem 11.225], we have

\[
\text{Ind}_P^G(\pi) \cong L_{\dim \mathfrak{w} \mathfrak{t}}(\text{Ind}_{P \cap L}^L(\xi \otimes \chi_{\rho(u)}^{-1})),
\]

where

- \( L_{\mathfrak{t}} \) is the functor defined by [34, (5.3a)];
- \( P \cap L = H(N \cap L) \) is a Borel subgroup of \( L \) with Levi component \( H \) and unipotent radical \( N \cap L \);
- \( \chi_{\rho(u)} \) is the character of \( H \) such that
  - \( \chi_{\rho(u)} \) factors through the image of \( H \) in \( \text{Sp}_{2n}(\mathbb{R}) \);
  - \( \chi_{\rho(u)} \) is trivial on \( (\mathbb{R}^\times)^k \);
  - the differential of \( \chi_{\rho(u)} \) is \( \rho(u) \) (which is analytically integral).

Assume for a moment that

\[
\text{(A.2)} \quad |\text{Re} \langle \alpha, \nu \rangle| \leq \langle \alpha, \lambda \rangle
\]

for all \( \alpha \in \Delta(u) \). Then, by [34, Corollary 11.227], \( \text{Ind}_P^G(\pi) \) is irreducible if \( \text{Ind}_{P \cap L}^L(\xi \otimes \chi_{\rho(u)}^{-1}) \) is irreducible. Hence, noting that

\[
\chi_{\rho(u)} = \chi_1' \otimes \cdots \otimes \chi_k' \otimes \zeta_1' \otimes \cdots \otimes \zeta_i' \otimes \eta_1' \otimes \cdots \otimes \eta_m',
\]

where

- \( \chi_i' \) is the trivial character of \( \mathbb{R}^\times \);
- \( \zeta_i' \) is a character of \( \mathbb{C}^\times \) of the form \( \zeta_i'(z) = (z/\bar{z})^{a_i} \) for some \( a_i \in \mathbb{Z} \);
- \( \eta_i' \) is a character of \( S^1 \) of the form \( \eta_i'(z) = z^{b_i} \) for some \( b_i \in \mathbb{Z} \),

we are reduced to the following irreducibility.

- The principal series representation of \( M_{p2k}(\mathbb{R}) \) induced from \( \chi_1 \otimes \cdots \otimes \chi_k \) is irreducible. Indeed, as in [72, Theorem 4.2.25], this can be deduced from the following:
  - the principal series representation of \( GL_d(\mathbb{R}) \) induced from any unitary character is irreducible (see e.g. [18]);
  - the principal series representation of \( M_{p2d}(\mathbb{R}) \) induced from any generic unitary character is irreducible (see the proof of Lemma [5, 2]);
  - for \( \epsilon_1, \epsilon_2 \in \{0, 1\} \) and \( s_1, s_2 \in \mathbb{C} \), the principal series representation of \( GL_2(\mathbb{R}) \) induced from \( \text{sgn}^{\epsilon_1} \cdot |\cdot |^{s_1} \otimes \text{sgn}^{\epsilon_2} \cdot |\cdot |^{s_2} \) is irreducible if and only if either
    * \( \epsilon_1 = \epsilon_2 \) and \( s_1 - s_2 \notin 2\mathbb{Z} + 1 \); or
    * \( \epsilon_1 \neq \epsilon_2 \) and \( s_1 - s_2 \notin 2\mathbb{Z} \setminus \{0\} \);
  - for \( s \in \{0, 1\} \) and \( s \in \mathbb{C} \), the principal series representation of \( M_{p2}(\mathbb{R}) \) induced from \( \text{sgn}^s \chi_{\psi} \cdot |\cdot |^s \) is irreducible if and only if \( s \notin \mathbb{Z} + \frac{1}{2} \).
- For \( \kappa \in \frac{1}{2} \mathbb{Z} \) and \( s \in \mathbb{C} \), the principal series representation of \( U(1, 1) \) induced from the character \( z \mapsto (z/\bar{z})^k \cdot (z\bar{z})^s \) of \( \mathbb{C}^\times \) is irreducible if and only if either
  - \( \kappa \in \mathbb{Z} \) and \( s \notin \mathbb{Z} + \frac{1}{2} \); or
  - \( \kappa \in \mathbb{Z} + \frac{1}{2} \) and \( s \notin \mathbb{Z} \).
Thus, in view of the condition (1) in Proposition [A.3] we have shown that \(\text{Ind}_{\Pi}^G(\pi)\) is irreducible under the assumption (A.2).

We now consider the general case. We reduce it to the case where \(\gamma\) as in (A.1) satisfies the condition (A.2) by using the translation functor. Fix a positive system \(\Delta^+(l)\) of \(\Delta(l)\) such that

\[
\Re \langle \alpha, \gamma \rangle \geq 0
\]

for all \(\alpha \in \Delta^+(l)\). Then \(\Delta^+ = \Delta^+(l) \cup \Delta(u)\) is a positive system of \(\Delta\). We denote by \(\Delta(\gamma)\) the set of integral roots defined by \(\gamma\):

\[
\Delta(\gamma) = \left\{ \alpha \in \Delta \mid 2\frac{\langle \alpha, \gamma \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \right\}.
\]

Put \(\Delta^+(\gamma) = \Delta(\gamma) \cap \Delta^+\). Then we have

\[
\langle \alpha, \gamma \rangle \geq 0
\]

for all \(\alpha \in \Delta^+(\gamma)\). Indeed, if \(\langle \alpha, \gamma \rangle < 0\) for some \(\alpha \in \Delta(\gamma) \cap \Delta(u)\), then since \(\langle \alpha, \lambda \rangle \geq 0\), we have \(\langle \alpha, \nu \rangle < 0\) and hence

\[
\langle \theta \alpha, \gamma \rangle = \langle \alpha, \lambda \rangle - \langle \alpha, \nu \rangle > 0.
\]

Namely, \(-\alpha\) satisfies the condition (1) in Proposition [A.3] which contradicts the assumption. Let \(\mu \in h^*\) be an integral weight, i.e. \(\mu = (\mu_1, \ldots, \mu_n)\) with \(\mu_i \in \mathbb{Z}\). Then we have \(\Delta(\gamma + \mu) = \Delta(\gamma)\). Recall that the translation functor \(\psi_{\gamma+\mu}\) for \(G\) is defined by

\[
\psi_{\gamma+\mu}(X) = P_{\gamma}(P_{\gamma+\mu}(X) \otimes F_{-\mu})
\]

for any \((g, K)\)-module \(X\) of finite length, where \(P_{\gamma}\) is the projection to the \(\gamma\)-primary component and \(F_{-\mu}\) is the (non-genuine) finite-dimensional irreducible \((g, K)\)-module with extreme weight \(-\mu\). The translation functor for \(M\) is defined similarly and is also denoted by \(\psi_{\gamma+\mu}\) (see [33, §XIV.12]). We now take \(\mu\) of the form \(\mu = (t\rho(u), \mu')\) for some positive integer \(t\) and some integral weight \(\mu' \in a^*\) such that

- \(\langle \alpha, \mu' \rangle > 0\) for all \(\alpha \in \Delta^+(l)\);
- \(|\Re \langle \alpha, \nu + \mu' \rangle| < \langle \alpha, \lambda + t\rho(u) \rangle\) for all \(\alpha \in \Delta(u)\).

Then we have:

- \(\gamma + \mu\) is regular;
- \(\gamma + \mu\) satisfies (A.2);
- \(\Delta^+(\gamma) = \{ \alpha \in \Delta(\gamma) \mid \langle \alpha, \gamma + \mu \rangle > 0 \}\).

Moreover, if \(\tilde{\pi}\) is the irreducible representation of \(M\) associated to

\[
\tilde{\delta}_i \equiv \delta_i + \mu_i \mod 2, \quad \tilde{\nu}_i = \nu_i + \mu_i,
\]

\[
\tilde{\kappa}_i = \kappa_i + \frac{1}{2}(\mu_{k+2i-1} - \mu_{k+2i}), \quad \tilde{\nu}'_i = \nu_i' + \frac{1}{2}(\mu_{k+2i-1} + \mu_{k+2i}),
\]

\[
\tilde{\lambda}'_i = \lambda'_i + \mu_{k+2i+1}, \quad \tilde{\Psi}' = \Psi',
\]

then we have shown that \(\text{Ind}_{\Pi}^G(\tilde{\pi})\) is irreducible. On the other hand, by [34, Theorem 7.237], we have

\[
\psi_{\gamma+\mu}(\tilde{\pi}) = \pi.
\]

Hence it follows from the argument in the proof of [33, Theorem 14.67] combined with [72, Lemma 7.2.18] that

\[
\psi_{\gamma+\mu}(\text{Ind}_{\Pi}^G(\tilde{\pi})) = \text{Ind}_{\Pi}^G(\psi_{\gamma+\mu}(\tilde{\pi})) = \text{Ind}_{\Pi}^G(\pi).
\]

From this and [34, Theorem 7.229] (which asserts that under the integral dominance condition, the translation functor sends an irreducible \((g, K)\)-module to either an irreducible \((g, K)\)-module or zero), we deduce that \(\text{Ind}_{\Pi}^G(\pi)\) is irreducible. This completes the proof.
Appendix B. Irreducibility of some non-standard modules of \( SO_{2n+1}(\mathbb{R}) \)

In this appendix, we finish the proof of Lemma 5.5 in the real case (see Proposition B.4 below). The argument is the same as that in [47, §4], but we include it for the convenience of the reader.

B.1. Notation. Let \( G \) be a real reductive linear Lie group with abelian Cartan subgroups. Let \( g_0 = \text{Lie} G \) be the Lie algebra of \( G \) and fix a Cartan involution \( \theta \) of \( g_0 \). We denote by \( K \) the maximal compact subgroup of \( G \) associated to \( \theta \). Then we have a Cartan decomposition \( g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \), where \( \mathfrak{k}_0 = \text{Lie} K \) and \( \mathfrak{p}_0 \) are the +1 and −1 eigenspaces of \( \theta \) in \( g_0 \), respectively. Fix a nondegenerate invariant symmetric bilinear form

\[
\langle \cdot, \cdot \rangle : g_0 \times g_0 \rightarrow \mathbb{R}
\]

such that

- \( \langle \cdot, \cdot \rangle \) is preserved by \( \theta \);
- \( \langle \cdot, \cdot \rangle \) is negative definite on \( \mathfrak{k}_0 \) and positive definite on \( \mathfrak{p}_0 \).

Let \( g = g_0 \otimes_{\mathbb{R}} \mathbb{C} \) be the complexification of \( g_0 \) and \( Z(g) \) the center of the universal enveloping algebra of \( g \). Let \( \text{Ad}(g) \) be the identity component of the automorphism group of \( g \).

Let \( H \) be a \( \theta \)-stable Cartan subgroup of \( G \). Let \( h_0 = \text{Lie} H \) be the corresponding Cartan subalgebra of \( g_0 \) (so that \( H \) is the centralizer of \( h_0 \) in \( G \)) and \( h = h_0 \otimes_{\mathbb{R}} \mathbb{C} \) the complexification of \( h_0 \). Let \( \langle \cdot, \cdot \rangle : h^* \times h^* \rightarrow \mathbb{C} \) be the bilinear form induced by (B.1). We denote by \( \Delta(g, h) \) the set of roots of \( h \) in \( g \). Let \( W(g, h) = W(\Delta(g, h)) \) be the associated Weyl group and put \( W(G, H) = N(G, H)/H \), where \( N(G, H) \) is the normalizer of \( H \) in \( G \). Then we may regard \( W(G, H) \) as a subgroup of \( W(g, h) \). For any regular element \( \gamma \in h^* \), we denote by \( \Delta(\gamma) \) the set of integral roots defined by \( \gamma \):

\[
\Delta(\gamma) = \left\{ \alpha \in \Delta(g, h) \mid 2\langle \alpha, \gamma \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z} \right\}.
\]

Then \( \Delta(\gamma) \) is a root system. Let \( W(\gamma) = W(\Delta(\gamma)) \) be the associated Weyl group. We may define a positive system \( \Delta^+(\gamma) \) of \( \Delta(\gamma) \) by

\[
\Delta^+(\gamma) = \{ \alpha \in \Delta(\gamma) \mid \langle \alpha, \gamma \rangle > 0 \}.
\]

Let \( \Pi(\gamma) \) be the set of simple roots in \( \Delta^+(\gamma) \). We define a homomorphism \( \chi_\gamma : Z(g) \rightarrow \mathbb{C} \) as the composition of the Harish-Chandra isomorphism \( Z(g) \cong S(h)^W(g, h) \) with evaluation at \( \gamma \).

Fix a \( \theta \)-stable maximally split Cartan subgroup \( H^s \) of \( G \) and write \( \Delta = \Delta(g, h^s) \). Fix a regular element \( \xi \in (h^s)^* \). For any \( \gamma \in h^* \) such that \( \chi_\gamma = \chi_\xi \), there exists an isomorphism \( i_\gamma : (h^s)^* \rightarrow h^* \) such that

- \( i_\gamma(\xi) = \gamma \);
- \( i_\gamma \) is induced by some element \( g \in \text{Ad}(g) \).

Since \( \xi \) is regular, \( i_\gamma \) does not depend on the choice of \( g \). We define an automorphism \( \theta_\gamma \) of \( (h^s)^* \) by

\[
\theta_\gamma = i_\gamma^{-1} \circ \theta \circ i_\gamma,
\]

which depends only on the \( K \)-conjugacy class of \( \gamma \). For \( \alpha \in \Delta(\xi) \) and \( w \in W(\xi) \), put

\[
\alpha_\gamma = i_\gamma(\alpha) \in \Delta(\gamma),
\]

\[
w_\gamma = i_\gamma(w) \in W(\gamma).
\]

Let \( \Lambda = \Lambda^G \) be the subgroup of \( \hat{H}^s \) (where \( \hat{H}^s \) is the group of continuous characters of \( H^s \)) consisting of weights of finite-dimensional representations of \( G \). For any \( \lambda \in \Lambda \), we denote by \( \hat{\lambda} \in (h^s)^* \) the differential of \( \lambda \). Then the homomorphism \( \lambda \mapsto \hat{\lambda} \) splits over the root lattice \( \mathbb{Z}\Delta \) canonically (see [72, Lemma 0.4.5]).
For any $\xi \in (\mathfrak{h}^*)^*$, we denote by $\xi + \lambda$ the set of formal symbols $\xi + \lambda$ with $\lambda \in \Lambda$. Note that $W(\xi)$ acts on $\xi + \Lambda$ (see [72, Definition 7.2.21]).

We denote by $R(\mathfrak{g}, K)$ the Grothendieck group of the category of $(\mathfrak{g}, K)$-modules of finite length. For any $(\mathfrak{g}, K)$-module $X$ of finite length, we denote by $[X]$ the image of $X$ in $R(\mathfrak{g}, K)$.

**B.2. Regular characters.** Following [76, Definition 2.2], we call a triple $\gamma = (H, \Gamma, \bar{\gamma})$ a regular character for $G$ if

- $H$ is a $\theta$-stable Cartan subgroup of $G$;
- $\Gamma$ is a continuous character of $H$;
- $\bar{\gamma} \in \mathfrak{h}^*$ is an element such that
  - if $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is an imaginary root, then $\langle \alpha, \bar{\gamma} \rangle$ is a nonzero real number;
  - the differential of $\Gamma$ is $\bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi)$, where $\Psi$ is the positive system of imaginary roots such that $\langle \alpha, \bar{\gamma} \rangle > 0$ for all $\alpha \in \Psi$, $\rho(\Psi)$ is half the sum of the roots in $\Psi$, and $\rho_c(\Psi)$ is half the sum of the compact roots in $\Psi$.

If further $\bar{\gamma}$ is regular, we define the length $\ell(\gamma) = \ell^G(\gamma)$ of $\gamma$ by

$$\ell(\gamma) = \frac{1}{2} \left| \{ \alpha \in \Delta^+(\mathfrak{g}) \mid \theta \alpha \notin \Delta^+(\bar{\gamma}) \} \right| + \frac{1}{2} \dim a_0 \in \frac{1}{2} \mathbb{Z},$$

where $a_0$ is the $-1$ eigenspace of $\theta$ in $\mathfrak{h}_0$.

To any regular character $\gamma = (H, \Gamma, \bar{\gamma})$ for $G$ such that $\bar{\gamma}$ is regular, we may associate a $(\mathfrak{g}, K)$-module $X(\gamma) = X^G(\gamma)$ of finite length with infinitesimal character $\bar{\gamma}$ as follows (see [76, Definition 2.3]). Let $M$ be the centralizer of $a_0$ in $G$. Then there exists a unique relative discrete series $(\mathfrak{m}, M \cap K)$-module $X^M(\gamma)$ such that

- $X^M(\gamma)$ has infinitesimal character $\bar{\gamma}$;
- $X^M(\gamma)$ has a lowest $(M \cap K)$-type of highest weight $\Gamma|_{H \cap K}$.

Choose a parabolic subgroup $P = MN$ of $G$ with Levi component $M$ and unipotent radical $N$ such that $\Re \langle \alpha, \bar{\gamma} \rangle \leq 0$ for all roots $\alpha$ of $\mathfrak{a}$ in $\mathfrak{n}$. Then $X(\gamma)$ is given by

$$X(\gamma) = \text{Ind}_P^G(X^M(\gamma)).$$

We recall some properties of $X(\gamma)$.

- $[X(\gamma)]$ depends only on the $K$-conjugacy class of $\gamma$.
- $X(\gamma)$ has a unique irreducible $(\mathfrak{g}, K)$-submodule $X(\gamma)$.
- $\bar{X}(\gamma)$ depends only on the $K$-conjugacy class of $\gamma$.
- For any irreducible $(\mathfrak{g}, K)$-module $X$ with regular infinitesimal character, we have $X \cong \bar{X}(\gamma)$ for some $\gamma$.

For any $\theta$-stable Cartan subgroup $H$ of $G$ and any regular element $\xi \in (\mathfrak{h}^*)^*$, we denote by $R^G(H, \xi)$ the set of regular characters $\gamma = (H, \Gamma, \bar{\gamma})$ for $G$ such that $\chi_{\bar{\gamma}} = \chi_\xi$. Put

$$R^G(\xi) = \bigcup_H R^G(H, \xi).$$
where the union runs over \( \theta \)-stable Cartan subgroups \( H \) of \( G \). Later, we also need the following notion.

**Definition B.1.** We say that \( H \) is \( \xi \)-integral if \( \mathcal{R}^G(H, \xi) \neq \emptyset \).

### B.3. Coherent families

In this subsection, we recall some properties of coherent families.

Fix a regular element \( \xi \in (h^*)^* \). Following \cite{72}, Definition 7.2.5, we call a map

\[
\Theta : \xi + \Lambda \longrightarrow \mathcal{R}(g, K)
\]

a coherent family on \( \xi + \Lambda \) if

- \( \Theta(\xi + \lambda) \) has infinitesimal character \( \xi + \bar{\lambda} \);
- for any finite-dimensional representations \( F \) of \( G \), we have
  \[
  \Theta(\xi + \lambda) \otimes F = \sum_{\mu \in \Delta(F)} \Theta(\xi + \lambda + \mu),
  \]
  where \( \Delta(F) \) is the multiset of weights of \( H^s \) in \( F \) (counted with multiplicity).

Then the following properties hold.

- For any coherent family \( \Theta \) on \( \xi + \Lambda \) and any \( \lambda \in \Lambda \) such that \( \xi + \bar{\lambda} \) is dominant for \( \Delta^+(\xi) \) (but possibly singular), we have
  \[
  \Theta(\xi + \lambda) = \psi_{\xi}^{\xi + \lambda}(\Theta(\xi))
  \]
  by \cite{72} Proposition 7.2.22, where \( \psi_{\xi}^{\xi + \lambda} \) is the translation functor (see \cite{72} Definition 4.5.7).

- For any \((g, K)\)-module \( X \) of finite length with infinitesimal character \( \xi \), there exists a unique coherent family \( \Theta_X \) on \( \xi + \Lambda \) such that
  \[
  \Theta_X(\xi) = [X]
  \]
  by \cite{72} Theorem 7.2.7 and Corollary 7.2.27.

We denote by \( \mathcal{C}(\xi + \Lambda) \) the free \( \mathbb{Z} \)-module of coherent families on \( \xi + \Lambda \). Then we may define a representation \( W(\xi) \) on \( \mathcal{C}(\xi + \Lambda) \) by

\[
(w\Theta)(\xi + \lambda) = \Theta(w^{-1}(\xi + \lambda))
\]
for \( w \in W(\xi) \) and \( \Theta \in \mathcal{C}(\xi + \Lambda) \), which we call the coherent continuation representation (see \cite{72} Definition 7.2.28).

For any \( \gamma \in \mathcal{R}^G(\xi) \), we define coherent families \( \Theta_\gamma = \Theta_G^\gamma \) and \( \bar{\Theta}_\gamma = \bar{\Theta}_G^\gamma \) on \( \xi + \Lambda \) by

\[
\Theta_\gamma = \Theta_{X(\gamma)}, \quad \bar{\Theta}_\gamma = \bar{\Theta}_{X(\gamma)}.
\]

Put

\[
\text{Std}(G, \xi) = \{ \Theta_\gamma | \gamma \in \mathcal{R}^G(\xi) \}, \quad \text{Irr}(G, \xi) = \{ \bar{\Theta}_\gamma | \gamma \in \mathcal{R}^G(\xi) \}.
\]

Then both \( \text{Std}(G, \xi) \) and \( \text{Irr}(G, \xi) \) are bases of \( \mathcal{C}(\xi + \Lambda) \), so that we may define a bijection \( \Theta \mapsto \bar{\Theta} \) from \( \text{Std}(G, \xi) \) to \( \text{Irr}(G, \xi) \) by \( \Theta_\gamma \mapsto \bar{\Theta}_\gamma \) for \( \gamma \in \mathcal{R}^G(\xi) \). Moreover, we may write

\[
\bar{\Theta}_\gamma = \sum_{\Theta \in \text{Std}(G, \xi)} M(\Theta, \bar{\Theta}_\gamma) \Theta
\]
for some \( M(\Theta, \bar{\Theta}_\gamma) \in \mathbb{Z} \).

Let \( P \) be a parabolic subgroup of \( G \) with Levi component \( M \) such that \( H^s \subset M \). In particular, \( M \) is \( \theta \)-stable and \( \Lambda^G \subset \Lambda^M \). Also, the parabolic induction functor \( \text{Ind}_M^G \) induces a homomorphism

\[
\text{Ind}_M^G : \mathcal{R}(m, M \cap K) \longrightarrow \mathcal{R}(g, K),
\]
which depends only on $M$. For any coherent family $\Theta^M$ on $\xi + \Lambda^G$, we may define a coherent family $\text{Ind}_M^G(\Theta^M)$ on $\xi + \Lambda^G$ by

$$\text{Ind}_M^G(\Theta^M)(\xi + \lambda) = \text{Ind}_M^G(\Theta^M(\xi + \lambda))$$

for $\lambda \in \Lambda^G$ (see [69, Lemma 5.8]). Then we have

$$\text{Ind}_M^G(\Theta^M) = \Theta^G$$

for $\gamma \in R^M(\xi)$, noting that $R^M(\xi) \subset R^G(\xi)$.

B.4. The Kazhdan–Lusztig algorithm. In this subsection, we recall the Kazhdan–Lusztig algorithm for real reductive Lie groups, which determines the coefficients $M(\Theta, \Theta_\gamma)$ in (B.3).

Fix a regular element $\xi \in (\mathfrak{h}^*)^*$. Recall the cross action of $W(\xi)$ on $R^G(\xi)$:

$$w \times \gamma = (H, w^{-1}_\gamma \times \Gamma, w^{-1}_\gamma \gamma)$$

for $w \in W(\xi)$ and $\gamma = (H, \Gamma, \bar{\gamma}) \in R^G(\xi)$, where $w_\gamma$ is as in (B.3) and $w^{-1}_\gamma \times \Gamma$ is the cross product given in [72] Definition 8.3.1. This descends to an action of $W(\xi)$ on $\text{Std}(G, \xi)$ such that $w \times \Theta_\gamma = \Theta_{w \gamma}$ for $w \in W(\xi)$ and $\gamma \in R^G(\xi)$.

Let $\alpha \in \Pi(\xi)$ and $\gamma = (H, \Gamma, \bar{\gamma}) \in R^G(\xi)$. If the root $\alpha_\gamma$ as in (B.2) either is noncompact imaginary, or is real and satisfies the parity condition [72] Definition 8.3.11, then we have the Cayley transform of $\Theta_{\gamma}$ through $\Theta$ (which is a subset of $\text{Std}(G, \xi)$). We recall some details in turn.

- Suppose first that $\alpha_\gamma$ is noncompact imaginary. Following [72] Definition 8.3.4, we say that $\alpha_\gamma$ is type I (resp. type II) if the reflection in $W(\mathfrak{g}, \mathfrak{h})$ with respect to $\alpha_\gamma$ does not belong to (resp. belongs to) $W(G, H)$. Let $c^\alpha(\gamma)$ be the Cayley transform of $\gamma$ through $\alpha$, i.e. $c^\alpha(\gamma)$ is the subset of $R^G(\xi)$ given in [72] Definition 8.3.6] of the form

$$c^\alpha(\gamma) = \{\gamma^\alpha\}, \quad \gamma^\alpha = (H^\alpha, \Gamma^\alpha, \bar{\gamma}^\alpha)$$

if $\alpha_\gamma$ is type I, and

$$c^\alpha(\gamma) = \{\gamma^\alpha, \gamma^\alpha_\pm\}, \quad \gamma^\alpha_\pm = (H^\alpha, \Gamma^\alpha_\pm, \bar{\gamma}^\alpha)$$

if $\alpha_\gamma$ is type II, where $H^\alpha$ is the $\theta$-stable Cartan subgroup of $G$ given in [72] Definition 8.3.4. Then the subset

$$c^\alpha(\Theta_{\gamma}) = \{\Theta_{\gamma'} | \gamma' \in c^\alpha(\gamma)\}$$

of $\text{Std}(G, \xi)$ depends only on the $K$-conjugacy class of $\gamma$.

- Suppose next that $\alpha_\gamma$ is real and satisfies the parity condition [72] Definition 8.3.11. Following [72] Definition 8.3.8], we say that $\alpha_\gamma$ is type I (resp. type II) if $\alpha_\gamma : H \cap K \rightarrow \{\pm 1\}$ is not surjective (resp. is surjective). Let $c_{\alpha}(\gamma)$ be the Cayley transform of $\gamma$ through $\alpha$, i.e. $c_{\alpha}(\gamma)$ is the subset of $R^G(\xi)$ given in [72] Definitions 8.3.14 and 8.3.16] of the form

$$c_{\alpha}(\gamma) = \{\gamma^\pm_{\alpha}, \gamma^-_{\alpha}\}, \quad \gamma^\pm_{\alpha} = (H^\alpha, \Gamma^\pm_{\alpha}, \bar{\gamma}^\pm_{\alpha})$$

if $\alpha_\gamma$ is type I, and

$$c_{\alpha}(\gamma) = \{\gamma^\alpha\}, \quad \gamma^\alpha = (H^\alpha, \Gamma^\alpha, \bar{\gamma}^\alpha)$$

if $\alpha_\gamma$ is type II, where $H^\alpha$ is the $\theta$-stable Cartan subgroup of $G$ given in [72] Definition 8.3.8]. Then the subset

$$c_{\alpha}(\Theta_{\gamma}) = \{\Theta_{\gamma'} | \gamma' \in c_{\alpha}(\gamma)\}$$

of $\text{Std}(G, \xi)$ depends only on the $K$-conjugacy class of $\gamma$. 

THE SHIMURA–WALDSPURGER CORRESPONDENCE FOR MP_{2n}
Let $\mathcal{H}(W(\xi))$ be the Hecke algebra of $W(\xi)$ over $\mathbb{Z}[q]$, where $q$ is an indeterminate. Note that the specialization at $q = 1$ gives a surjection $\mathcal{H}(W(\xi)) \to \mathbb{Z}[W(\xi)]$. Then, by \[73\] Definition 5.2 (see also \[73\] Definition 12.3 and Proposition 12.5), there exists an action of $\mathcal{H}(W(\xi))$ on
\[ C(\xi + \Lambda)_q = C(\xi + \Lambda) \otimes_{\mathbb{Z}} \mathbb{Z}[q] \]
determined by the cross action and the Cayley transforms. Moreover, by \[73\] Lemma 14.5, the specializat\on of $C(\xi + \Lambda)_q$ at $q = 1$ is isomorphic to the coherent continuation representation tensored with the sign representation of $W(\xi)$. More explicitly, this isomorphism is induced by the surjection $\epsilon : C(\xi + \Lambda)_q \to C(\xi + \Lambda)$ given by
\[ \epsilon(q^i \Theta_\gamma) = (-1)^{\ell^I(\gamma)} \Theta_\gamma \]
for $i \geq 0$ and $\gamma \in R^G(\xi)$, where the integral length $\ell^I(\gamma)$ of $\gamma$ is given by
\[ \ell^I(\gamma) = \ell(\gamma) - c_0(G) \]
for some choice of $c_0(G) \in \frac{1}{2} \mathbb{Z}$ such that $\ell(\gamma) \in \mathbb{Z}$ for all $\gamma \in R^G(\xi)$ (see \[73\] Definition 12.1)).

Finally, we recall the Kazhdan–Lusztig algorithm for real reductive Lie groups.

**Theorem B.2** (\[74\], \[4\] Theorem 16.22]). For any $\gamma, \delta \in R^G(\xi)$, we have
\[ M(\Theta_\gamma, \Theta_\delta) = (-1)^{\ell^I(\gamma) - \ell^I(\delta)} P_{\gamma,\delta}(1), \]
where $M(\Theta_\gamma, \Theta_\delta)$ is the integer defined by \[B.5\] and $P_{\gamma,\delta}(q)$ is the Kazhdan–Lusztig–Vogan polynomial defined in terms of the $\mathcal{H}(W(\xi))$-module $C(\xi + \Lambda)_q$. In particular, $M(\Theta_\gamma, \Theta_\delta)$ can be computed by an algorithm which depends only on the $\mathcal{H}(W(\xi))$-module structure on $C(\xi + \Lambda)_q$.

**B.5. Comparison of Hecke algebra module structures.** Let $G_1$ and $G_2$ be two real reductive linear Lie groups with abelian Cartan subgroups. For $i = 1, 2$, fix a Cartan involution $\theta_i$ of $(g_i)_0 = \text{Lie} G_i$ and let $K_i$ be the maximal compact subgroup of $G_i$ associated to $\theta_i$. Fix a $\theta_i$-stable maximally split Cartan subgroup $H^*_i$ of $G_i$ and a regular element $\xi_i \in (h^*_i)^*$.

We now assume that the following conditions hold.

(i) There exists an isomorphism
\[ H^*_1 \cong H^*_2. \]

(ii) Let $f : H^*_1 \to H^*_2$ be the isomorphism induced by the isomorphism in \(i\). Then we have
\[ f(\Lambda^{G_1}) \subset \Lambda^{G_2}. \]

(iii) Let $f : (h^*_1)^* \to (h^*_2)^*$ be the isomorphism induced by the isomorphism in \(i\) and put $\xi_2 = f(\xi_1)$. Then $\xi_2$ is regular.

(iv) The isomorphism in \(iii\) induces an isomorphism
\[ f : \Delta(\xi_1) \to \Delta(\xi_2) \]
of root systems. This induces an isomorphism
\[ f : W(\xi_1) \to W(\xi_2) \]
of the associated Weyl groups.

(v) There exists a bijection
\[ \varphi : \text{Std}(G_1, \xi_1) \to \text{Std}(G_2, \xi_2). \]
(vi) Let $\gamma_i \in \mathcal{R}^{G_i}(\xi_i)$ be such that $\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}$. Then we have

$$f \circ \theta_{\gamma_1} = \theta_{\gamma_2} \circ f.$$ 

This implies that

$$\ell^{G_1}(\gamma_1) = \ell^{G_2}(\gamma_2),$$

and that for any $\alpha \in \Delta(\xi_1)$, $\alpha_{\gamma_1}$ is imaginary (resp. real, resp. complex) if and only if $f(\alpha)_{\gamma_2}$ is imaginary (resp. real, resp. complex).

(vii) Let $\gamma_1 \in \mathcal{R}^{G_1}(\xi_i)$ and $\alpha \in \Delta(\xi_1)$ be such that $\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}$ and such that $\alpha_{\gamma_1}$ is imaginary (and hence so is $f(\alpha)_{\gamma_2}$). Then $\alpha_{\gamma_1}$ is noncompact if and only if $f(\alpha)_{\gamma_2}$ is noncompact, in which case $\alpha_{\gamma_1}$ is type I (resp. type II) if and only if $f(\alpha)_{\gamma_2}$ is type I (resp. type II).

(viii) Let $\gamma_1 \in \mathcal{R}^{G_1}(\xi_i)$ and $\alpha \in \Delta(\xi_1)$ be such that $\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}$ and such that $\alpha_{\gamma_1}$ is real (and hence so is $f(\alpha)_{\gamma_2}$). Then $\alpha_{\gamma_1}$ satisfies the parity condition if and only if $f(\alpha)_{\gamma_2}$ satisfies the parity condition, in which case $\alpha_{\gamma_1}$ is type I (resp. type II) if and only if $f(\alpha)_{\gamma_2}$ is type I (resp. type II).

(ix) The bijection in (v) is compatible with the cross action: for $w \in W(\xi_1)$ and $\gamma \in \mathcal{R}^{G_1}(\xi_1)$, we have

$$\varphi(w \times \Theta_{\gamma}) = f(w) \times \varphi(\Theta_{\gamma}).$$

(x) The bijection in (v) is compatible with the Cayley transforms: for $\alpha \in \Pi(\xi_1)$ and $\gamma \in \mathcal{R}^{G_1}(\xi_1)$, we have

$$\varphi(c^\alpha(\Theta_{\gamma})) = c^{f(\alpha)}(\varphi(\Theta_{\gamma}))$$

if $\alpha_{\gamma}$ is noncompact imaginary, and

$$\varphi(c_\alpha(\Theta_{\gamma})) = c_{f(\alpha)}(\varphi(\Theta_{\gamma}))$$

if $\alpha_{\gamma}$ is real and satisfies the parity condition.

The bijection in (v) induces isomorphisms

$$\varphi : \mathcal{C}(\xi_1 + \Lambda^{G_1}) \rightarrow \mathcal{C}(\xi_2 + \Lambda^{G_2})$$

and

$$\varphi_q : \mathcal{C}(\xi_1 + \Lambda^{G_1})_q \rightarrow \mathcal{C}(\xi_2 + \Lambda^{G_2})_q$$

of $\mathbb{Z}$-modules and $\mathbb{Z}[q]$-modules, respectively. By the definition of the $\mathcal{H}(W(\xi_i))$-module structure on $\mathcal{C}(\xi_i + \Lambda^{G_i})_q$, the above conditions imply that $\varphi_q$ is equivariant under the action of $\mathcal{H}(W(\xi_1)) \cong \mathcal{H}(W(\xi_2))$.

From this and the commutative diagram

$$\begin{array}{ccc}
\mathcal{C}(\xi_1 + \Lambda^{G_1})_q & \xrightarrow{\varphi_q} & \mathcal{C}(\xi_2 + \Lambda^{G_2})_q \\
\downarrow & & \downarrow \\
\mathcal{C}(\xi_1 + \Lambda^{G_1}) & \xrightarrow{\varphi} & \mathcal{C}(\xi_2 + \Lambda^{G_2})
\end{array}$$

induced by the specialization at $q = 1$ defined by (B.6) (with a suitable choice of $c_0(G_1)$ in the definition of the integral length; see (B.7)), we can deduce that $\varphi$ is an isomorphism of the coherent continuation representations of $W(\xi_1) \cong W(\xi_2)$. Moreover, by Theorem [B.2] we have

$$M(\varphi(\Theta_{\gamma}), \varphi(\Theta_{\delta})) = M(\Theta_{\gamma}, \Theta_{\delta})$$

for all $\gamma, \delta \in \mathcal{R}^{G_1}(\xi_1)$ and hence

$$\varphi(\Theta) = \varphi(\Theta)$$

for all $\Theta \in \mathrm{Std}(G_1, \xi_1)$. In particular, $\varphi$ induces a bijection from $\mathrm{Irr}(G_1, \xi_1)$ to $\mathrm{Irr}(G_2, \xi_2)$. 
Lemma B.3. For $i = 1, 2$, let $\Xi_i \in \mathcal{C}(\xi_i + \Lambda^G_i)$ and $\lambda_i \in \Lambda^G_i$ be such that $\varphi(\Xi_1) = \Xi_2$ and $f(\lambda_1) = \lambda_2$. Assume that there exists an irreducible $(g_2, K_2)$-module $X_2$ such that

$$\Xi_2(\xi_2 + \lambda_2) = [X_2].$$

Then there exists an irreducible $(g_1, K_1)$-module $X_1$ such that

$$\Xi_1(\xi_1 + \lambda_1) = [X_1].$$

Proof. The assertion was proved by Matumoto [47, Lemma 4.1.3] when Cartan subgroups of $G_i$ are all connected, but the argument works in the general case. We include the proof for the convenience of the reader.

Choose $w_1 \in W(\xi_1)$ such that $w_1(\xi_1 + \bar{\lambda}_1)$ is dominant for $\Delta^+(\xi_1)$ and write

$$w_1\Xi_1 = \sum_{\Theta \in \text{Irr}(G_1, \xi_1)} a_\Theta \Theta$$

for some $a_\Theta \in \mathbb{Z}$. Put $w_2 = f(w_1) \in W(\xi_2)$, so that $\varphi(w_1\Xi_1) = w_2\Xi_2$. Then we have

$$\sum_{\Theta \in \text{Irr}(G_1, \xi_1)} a_\Theta \varphi(\Theta)(w_2(\xi_2 + \lambda_2)) = \varphi(w_1\Xi_1)(w_2(\xi_2 + \lambda_2))$$

$$= (w_2\Xi_2)(w_2(\xi_2 + \lambda_2))$$

$$= \Xi_2(\xi_2 + \lambda_2)$$

$$= [X_2].$$

On the other hand, since $w_2(\xi_2 + \bar{\lambda}_2)$ is dominant for $\Delta^+(\xi_2)$, we deduce from [B.41 and 75, Theorem 7.6] (see also [69, Theorem 6.18]) that for any $\bar{\Upsilon} \in \text{Irr}(G_2, \xi_2)$, $\bar{\Upsilon}(w_2(\xi_2 + \lambda_2))$ is either $[X]$ for some irreducible $(g_2, K_2)$-module $X$ or zero, and that there exists a unique $\bar{\Upsilon}_0 \in \text{Irr}(G_2, \xi_2)$ such that

$$\bar{\Upsilon}_0(w_2(\xi_2 + \lambda_2)) = [X_2].$$

Hence, noting that $\varphi(\bar{\Theta}) \in \text{Irr}(G_2, \xi_2)$ for $\bar{\Theta} \in \text{Irr}(G_1, \xi_1)$, we have

$$a_{\bar{\Theta}_0} = 1$$

for $\bar{\Theta}_0 = \varphi^{-1}(\bar{\Upsilon}_0)$, and either $a_{\bar{\Theta}_0} = 0$ or $\varphi(\bar{\Theta})(w_2(\xi_2 + \lambda_2)) = 0$ for $\bar{\Theta} \neq \bar{\Theta}_0$. Moreover, recalling the definition of $\tau$-invariants (see [75, Definition 5.3]), we can also deduce from [B.41 and 75, Theorem 7.6] that

$$\varphi(\bar{\Theta})(w_2(\xi_2 + \lambda_2)) = 0 \iff \bar{\Theta}(w_1(\xi_1 + \lambda_1)) = 0$$

for all $\bar{\Theta} \in \text{Irr}(G_1, \xi_1)$. Thus, we obtain

$$\Xi_1(\xi_1 + \lambda_1) = (w_1\Xi_1)(w_1(\xi_1 + \lambda_1))$$

$$= \sum_{\Theta \in \text{Irr}(G_1, \xi_1)} a_\Theta \bar{\Theta}(w_1(\xi_1 + \lambda_1))$$

$$= \bar{\Theta}_0(w_1(\xi_1 + \lambda_1))$$

$$= [X_1]$$

for some irreducible $(g_1, K_1)$-module $X_1$. □
B.6. Some non-standard modules of $\text{SO}_{2n+1}(\mathbb{R})$. Let $G = \text{SO}_{2n+1}(\mathbb{R})$ be the split odd special orthogonal group, which we realize as

$$\text{SO}_{2n+1}(\mathbb{R}) = \{ g \in \text{SL}_{2n+1}(\mathbb{R}) \mid tg\begin{pmatrix} 1_{n+1} & 0 \\ 0 & -1_n \end{pmatrix}g = \begin{pmatrix} 1_{n+1} & 0 \\ 0 & -1_n \end{pmatrix} \}.$$ 

We define a Cartan involution $\theta$ of $G$ by

$$\theta(g) = t\, g^{-1}.$$ 

Let $K$ be the maximal compact subgroup of $G$ associated to $\theta$. We define a bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{R}$ as in \((B.1)\) by

$$\langle X, Y \rangle = \frac{1}{d} \text{tr}(XY).$$

For any non-negative integers $k, l, m$ such that $k + 2l + m = n$, we define a $\theta$-stable Cartan subalgebra $\mathfrak{h}_0^{k,l,m}$ of $\mathfrak{g}_0$ as follows. For $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, put

$$h^{k,0,0}(a) = \begin{pmatrix} a \\ \mathbf{0} \end{pmatrix} \in \mathfrak{so}_{2k}(\mathbb{R}),$$

where $\mathbf{a} = \text{diag}(a_1, \ldots, a_k)$. For $z = (z_1, \ldots, z_l) \in \mathbb{C}^l$ with $z_i = x_i + \sqrt{-1}y_i$, put

$$h^{0,l,0}(z) = \begin{pmatrix} y & x \\ -x & y \end{pmatrix} \in \mathfrak{so}_{4l}(\mathbb{R}),$$

where $\mathbf{x} = \text{diag}(x_1, \ldots, x_l)$ and $\mathbf{y} = \text{diag}(y_1, \ldots, y_l)$. For $\vartheta = (\vartheta_1, \ldots, \vartheta_m) \in \mathbb{R}^m$, put

$$h^{0,0,m}(\vartheta) = \text{diag}(\vartheta_1, \ldots, \vartheta_m, 0, -\vartheta_{m+1}, \ldots, -\vartheta_m) \in \mathfrak{so}_{2m+1}(\mathbb{R}),$$

where

$$\vartheta_i = \begin{pmatrix} 0 & 0 \\ -\vartheta_i & 0 \end{pmatrix}$$

and $m_1 = \lceil \frac{m+1}{2} \rceil$. Let $h^{k,l,m}(a, z, \vartheta)$ be the image of

$$(h^{k,0,0}(a), h^{0,l,0}(z), h^{0,0,m}(\vartheta))$$

under the natural embedding

$$\mathfrak{so}_{2k}(\mathbb{R}) \oplus \mathfrak{so}_{4l}(\mathbb{R}) \oplus \mathfrak{so}_{2m+1}(\mathbb{R}) \hookrightarrow \mathfrak{so}_{2n+1}(\mathbb{R}).$$

Then we set

$$\mathfrak{h}_0^{k,l,m} = \{ h^{k,l,m}(a, z, \vartheta) \mid a \in \mathbb{R}^k, z \in \mathbb{C}^l, \vartheta \in \mathbb{R}^m \}.$$ 

These $\mathfrak{h}_0^{k,l,m}$ with $k + 2l + m = n$ form a set of representatives for the $G$-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$. Let $H^{k,l,m}$ be the centralizer of $\mathfrak{h}_0^{k,l,m}$ in $G$. Then $H^{k,l,m}$ is a $\theta$-stable Cartan subgroup of $G$ isomorphic to

$$\mathbb{R}^k \times (\mathbb{C}^l)^{\frac{n-1}{2}} \times (S^1)^m.$$ 

Note that $W(\mathfrak{g}, h^{k,l,m}) \cong W(B_n)$ and

$$(B.8) \quad W(G, H^{k,l,m}) \cong W(B_k) \times (\mathfrak{g}_l \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^l) \times W(B_{m_1}) \times W(B_{m_2}),$$

where $\mathfrak{g}_d$ is the symmetric group of degree $d$, $W(B_d) = \mathfrak{g}_d \times (\mathbb{Z}/2\mathbb{Z})^d$ is the Weyl group of type $B_d$, $m_1 = \lceil \frac{m+1}{2} \rceil$, and $m_2 = \lceil \frac{m}{2} \rceil$ (see e.g. \[73\] Proposition 4.16).

Fix non-negative integers $k, l, m$ such that $k + 2l + m = n$ and write $\mathfrak{h}_0 = \mathfrak{h}_0^{k,l,m}$. Let $M$ be the centralizer of $a_0$ in $G$, where $a_0$ is the $-1$ eigenspace of $\theta$ in $\mathfrak{h}_0$. Then $M$ is a Levi subgroup of $G$ isomorphic to

$$\text{GL}_1(\mathbb{R})^k \times \text{GL}_2(\mathbb{R})^l \times \text{SO}_{2m+1}(\mathbb{R}).$$
We consider an irreducible representation $\pi$ of $M$ of the form
\[
\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi',
\]
where
- $\chi_i$ is a character of $\text{GL}_1(\mathbb{R})$ of the form
  \[
  \chi_i = \text{sgn}^{\delta_i} \otimes | \cdot |^{\nu_i}
  \]
  for some $\delta_i \in \{0, 1\}$ and some $\nu_i \in \mathbb{C}$;
- $\tau_i$ is an irreducible representation of $\text{GL}_2(\mathbb{R})$ of the form
  \[
  \tau_i = D_{\kappa_i} \otimes | \det |^{\nu'_i}
  \]
  for some $\kappa_i \in \frac{1}{2} \mathbb{Z}$ and some $\nu'_i \in \mathbb{C}$, where $D_{\kappa_i}$ is the relative (limit of) discrete series representation of $\text{GL}_2(\mathbb{R})$ of weight $2|\kappa_i| + 1$ with central character trivial on $\mathbb{R}_+^\times$;
- $\pi'$ is an irreducible representation of $\text{SO}_{2m+1}(\mathbb{R})$ with infinitesimal character
  \[
  \lambda' = (\lambda'_1, \ldots, \lambda'_m) \in (\mathfrak{h}^{m,0,0})^* \cong \mathbb{C}^m
  \]
  (with the identification given in B.7 below).

Choose a parabolic subgroup $P$ of $G$ with Levi component $M$.

We now state the main result of this appendix.

**Proposition B.4.** Assume that
- if $\nu_i = \pm \nu_j$, then $\delta_i = \delta_j$;
- if $\nu'_i = 0$, then $\kappa_i \in \mathbb{Z}$;
- $|\text{Re} \nu_i|, |\text{Re} \nu'_i| < \frac{1}{2}$ for all $i, j$;
- $\lambda'_i \in \mathbb{Z} + \frac{1}{2}$ for all $i$.

Then the normalized parabolic induction $\text{Ind}_P^G(\pi)$ is irreducible.

**B.7. Proof of Proposition B.4.** Put
\[
G_1 = \text{SO}_{2n+1}(\mathbb{R}), \quad G_2 = \text{SO}_{2(n-m)+1}(\mathbb{R}) \times \text{SO}_{2m+1}(\mathbb{R}).
\]

We define embeddings $\iota : \text{SO}_{2(n-m)+1}(\mathbb{R}) \hookrightarrow G_1$ and $\iota' : \text{SO}_{2m+1}(\mathbb{R}) \hookrightarrow G_1$ by
\[
\iota \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} 1_m,
\]
\[
\iota' \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} 1_{n-m},
\]
where $a, b, c, d, a', b', c', d' \in M_{n,m,n-m}(\mathbb{R}), \quad b \in M_{n-m,n-m+1}(\mathbb{R}), \quad c \in M_{n-m+1,n-m}(\mathbb{R}), \quad d \in M_{n-m+1,n-m+1}(\mathbb{R}), \quad a' \in M_{m+1,m+1}(\mathbb{R}), \quad b' \in M_{m+1,m}(\mathbb{R}), \quad c' \in M_{m,m+1}(\mathbb{R}), \quad d' \in M_{m,m}(\mathbb{R})$.

For $i = 1, 2$, let $\theta_i$ be the Cartan involution of $G_i$ as in B.6 and $K_i$ the maximal compact subgroup of $G_i$ associated to $\theta_i$. We take a $\theta_i$-stable maximally split Cartan subgroup $H_i^s$ of $G_i$ given by
\[
H_1^s = H^{n,0,0}, \quad H_2^s = H^{n-m,0,0} \times H^{m,0,0}.
\]

Then we have an isomorphism $H_2^s \to H_1^s$ given by
\[
(h, h') \mapsto \iota(h)\iota'(h').
\]
This induces an isomorphism \( f : \hat{H}_1 \rightarrow \hat{H}_2 \).

**Lemma B.5.** We have \( f(\Lambda^{G_1}) \subset \Lambda^{G_2} \).

**Proof.** Let \( \mu \in \Lambda^{G_1} \), so that \( \mu \) occurs in some finite-dimensional representation \( F \) of \( G_1 \). Then \( f(\mu) \) occurs in the representation \( \iota^* F \otimes (\iota')^* F \) of \( G_2 \). Hence \( f(\mu) \in \Lambda^{G_2} \). \( \Box \)

Also, the isomorphism \((B.9)\) induces an isomorphism

\[
(f, h_{\xi}^*) \rightarrow (h_{\xi}^*).
\]

We define a basis \( e_1^*, \ldots, e_n^* \) of \((h_{\xi}^*)^* = (h_{n,0,0})^*\) by

\[
e_i^*(h_{n,0,0}(a)) = a_i.
\]

Fix a regular element \( \xi_1 = (x_1, \ldots, x_n) \in (h_{\xi}^*)^* \subset \mathbb{C}^n \) (with the identification using the above basis) such that

\[
x_i \notin \mathbb{Z} + \frac{1}{2} \quad (1 \leq i \leq n - m);
\]

\[
x_i \in \mathbb{Z} + \frac{1}{2} \quad (n - m < i \leq n).
\]

Put \( \xi_2 = f(\xi_1) \). Since \( f(\Delta(g_1, h_{\xi}^*)) \supset \Delta(g_2, h_{\xi}^*)), \xi_2 \) is regular.

**Lemma B.6.** The isomorphism \((B.10)\) induces an isomorphism \( f : \Delta(\xi_1) \rightarrow \Delta(\xi_2) \) of root systems.

**Proof.** Since

\[
\Delta(g_1, h_{\xi}^*) \cap f^{-1}(\Delta(g_2, h_{\xi}^*))) = \{ \pm e_i^* \pm e_j^* | 1 \leq i \leq n - m < j \leq n \},
\]

it follows from \((B.11)\) that

\[
2\langle \langle \alpha, \xi_1 \rangle \rangle \notin \mathbb{Z}
\]

for all \( \alpha \in \Delta(g_1, h_{\xi}^*) \cap f^{-1}(\Delta(g_2, h_{\xi}^*)) \). This implies the assertion. \( \Box \)

Recall that

\[
H^{k', l', m'} \quad (k' + 2l' + m' = n), \quad H^{p, q, r} \times H^{p', q', r'} \quad (p + 2q + r = n - m, p' + 2q' + r' = m)
\]

form a set of representatives for the \( K_i \)-conjugacy classes of \( \theta_i \)-stable Cartan subgroups of \( G_i \) for \( i = 1, 2 \), respectively.

**Lemma B.7.** (i) If the \( \theta_1 \)-stable Cartan subgroup \( H^{k', l', m'} \) of \( G_1 \) is \( \xi_1 \)-integral, then \( m' \leq m \).

(ii) If the \( \theta_2 \)-stable Cartan subgroup \( H^{p, q, r} \times H^{p', q', r'} \) of \( G_2 \) is \( \xi_2 \)-integral, then \( r = 0 \).

**Proof.** We only prove \((i)\); the proof of \((ii)\) is similar. Put \( H_1 = H^{k', l', m'} \) and \( h_1 = h^{k', l', m'} \). We define a basis \( e_1, \ldots, e_n \) of \( h_1^* \) by

\[
e_i(h) = a_i \quad (1 \leq i \leq k');
\]

\[
e_{k'+2l-1}(h) = x_i + \sqrt{-1} y_i \quad (1 \leq i \leq l');
\]

\[
e_{k'+2l}(h) = -x_i + \sqrt{-1} y_i \quad (1 \leq i \leq l');
\]

\[
e_{k'+2r+1}(h) = \sqrt{-1} \vartheta_i \quad (1 \leq i \leq m').
\]
for $h = h^{k',l',m'}(a,z,\vartheta)$. Note that
\[ \theta(e_i) = -e_i \quad (1 \leq i \leq k'); \]
\[ \theta(e_{k'+2i-1}) = e_{k'+2i} \quad (1 \leq i \leq l'); \]
\[ \theta(e_{k'+2l'+i}) = e_{k'+2l'+i} \quad (1 \leq i \leq m'). \]
Then there exists a unique isomorphism $j : (\mathfrak{h}_1^*)^* \to \mathfrak{h}_1^*$ such that
\begin{itemize}
  \item $j(e_i^*) = e_i$ for all $i$;
  \item $j$ is induced by some element in $\text{Ad}(\mathfrak{g}_1)$.
\end{itemize}
Let $\gamma_1 = (H_1, \Gamma_1, \bar{\gamma}_1) \in R_{G_1}(\xi_1)$. Then $j$ is $W(\mathfrak{g}_1, \mathfrak{h}_1)$-conjugate to $i_{\gamma_1}$. Under the identification $\mathfrak{h}_1^* \cong \mathbb{C}^n$ using the above basis, we write
\[ \bar{\gamma}_1 = (u_1, \ldots, u_n), \quad \rho(\Psi) - 2\rho_c(\Psi) = (v_1, \ldots, v_n), \]
where $\Psi$ is the positive system of imaginary roots as in $[B.2]$. Then we have $v_i \in \mathbb{Z} + \frac{1}{2}$ for all $k' < i \leq n$. Since $\bar{\gamma}_1 + \rho(\Psi) - 2\rho_c(\Psi)$ is the differential of a character of $H_1 \cong (\mathbb{R}^x)^{k'} \times (\mathbb{C}^x)^{l'} \times (S^1)^{m'}$, we must have
\[ u_{k'+2i-1} + v_{k'+2i-1} + u_{k'+2i} + v_{k'+2i} \in \mathbb{Z} \quad (1 \leq i \leq l'); \]
\[ u_{k'+2l'+i} + v_{k'+2l'+i} \in \mathbb{Z} \quad (1 \leq i \leq m'), \]
so that
\[ u_{k'+2i-1} + u_{k'+2i} \in \mathbb{Z} \quad (1 \leq i \leq l'); \]
\[ u_{k'+2l'+i} \in \mathbb{Z} + \frac{1}{2} \quad (1 \leq i \leq m'). \]
Hence, noting that $j(\xi_1)$ is $W(\mathfrak{g}_1, \mathfrak{h}_1)$-conjugate to $i_{\gamma_1}(\xi_1) = \bar{\gamma}_1$, we deduce from $[B.11]$ that $m' \leq m$. \hfill $\square$

We now define a map
\[ \varphi' : \text{Std}(G_2, \xi_2) \longrightarrow \text{Std}(G_1, \xi_1) \]
as follows. Let $\gamma_2 = (H_2, \Gamma_2, \bar{\gamma}_2) \in R_{G_2}(\xi_2)$. Replacing $\gamma_2$ by a $K_2$-conjugate if necessary, we may assume that
\[ H_2 = H^{p,q,r} \times H^{p',q',r'} \]
with $p + 2q + r = n - m$ and $p' + 2q' + r' = m$. By Lemma $[B.7]$, we have $r = 0$. Put
\[ H_1 = \{ (h,h') \mid h \in H^{p,q,0}, h' \in H^{p',q',r'} \}. \]
Then $H_1$ is a $\theta_1$-stable Cartan subgroup of $G_1$ and is $K_1$-conjugate to $H^{p+p',q+q',r'}$. Moreover, we have an isomorphism $H_2 \to H_1$ given by $(h,h') \mapsto (h)\zeta(h')$. This induces isomorphisms $\phi : \bar{H}_1 \to \bar{H}_2$ and $\phi : \mathfrak{h}_1^* \to \mathfrak{h}_2^*$, which in turn induces an embedding
\[ W(\mathfrak{g}_2, \mathfrak{h}_2) \longrightarrow W(\mathfrak{g}_1, \mathfrak{h}_1). \]
We identify $W(\mathfrak{g}_2, \mathfrak{h}_2)$ with its image in $W(\mathfrak{g}_1, \mathfrak{h}_1)$.

**Lemma B.8.** We have
\[ W(G_2, H_2) = W(\mathfrak{g}_2, \mathfrak{h}_2) \cap W(G_1, H_1). \]

**Proof.** The assertion follows from $[B.8]$. \hfill $\square$

Put $\gamma_1 = (H_1, \Gamma_1, \bar{\gamma}_1)$, where
\[ \Gamma_1 = \phi^{-1}(\Gamma_2), \quad \bar{\gamma}_1 = \phi^{-1}(\bar{\gamma}_2). \]
Replacing $\gamma$ $K$ $\phi$ $p$ where $\gamma$ have $K$. Hence we may define $\varphi'$ by

$$\varphi'(\Theta_{\gamma_2}) = \Theta_{\gamma_1}.$$  

We also define a map $\varphi : \text{Std}(G_1, \xi_1) \rightarrow \text{Std}(G_2, \xi_2)$ as follows. Let $\gamma_1 = (H_1, \Gamma_1, \bar{\gamma}_1) \in \mathcal{R}^{G_1}(\xi_1)$. Replacing $\gamma_1$ by a $K_1$-conjugate if necessary, we may assume that

$$H_1 = H^{k', l', m'}$$

with $k' + 2l' + m' = n$. Write $\bar{\gamma}_1 = (u_1, \ldots, u_n)$ as in the proof of Lemma B.7 and put

$$p' = \{1 \leq i \leq k' \mid u_i \in \mathbb{Z} + \frac{1}{2}\}, \quad q' = \frac{1}{2} \{1 \leq i \leq 2l' \mid u_{k' + i} \in \mathbb{Z} + \frac{1}{2}\},$$

$$r' = \{1 \leq i \leq m' \mid u_{k' + 2l' + i} \in \mathbb{Z} + \frac{1}{2}\}.$$

Then it follows from the proof of Lemma B.7 that

$$q' \in \mathbb{Z}, \quad r' = m', \quad p' + 2q' + r' = m.$$  

Put

$$H_2 = H^{p, q, 0} \times H^{l', q', r'},$$

where $p = k' - p'$ and $q = l' - q'$. Then $H_2$ is a $\theta_2$-stable Cartan subgroup of $G_2$. Replacing $\gamma_1$ by a $K_1$-conjugate again, we may now assume that

$$H_1 = \{i(h)t'(h') \mid h \in H^{p, q, 0}, h' \in H^{l', q', r'}\}.$$  

Let $\phi : \hat{H}_1 \rightarrow \hat{H}_2$ and $\phi : h_1^* \rightarrow h_2^*$ be the isomorphisms induced by the isomorphism $H_2 \rightarrow H_1$ given by $(h, h') \mapsto i(h)t'(h')$. Put $\gamma_2 = (H_2, \Gamma_2, \bar{\gamma}_2)$, where

$$\Gamma_2 = \phi(\Gamma_1), \quad \bar{\gamma}_2 = \phi(\bar{\gamma}_1).$$  

Replacing $\gamma_1$ by a $W(G_1, H_1)$-conjugate if necessary, we may further assume that $\chi_{\gamma_2} = \chi_{\xi_2}$. Then we have $\gamma_2 \in \mathcal{R}^{G_2}(\xi_2)$, and by Lemma B.8 the $K_2$-conjugacy class of $\gamma_2$ is uniquely determined by the $K_1$-conjugacy class of $\gamma_1$. Hence we may define $\varphi$ by

$$\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}.$$  

By construction, we have:

**Lemma B.9.** The two maps $\varphi$ and $\varphi'$ are inverses of each other. Moreover, the conditions (4), ..., (8) in [B.5] hold.

Finally, as in [B.6] we define a Levi subgroup $M_i$ of $G_i$ with respect to the $\theta_i$-stable Cartan subgroup $H^{k_i, l_i, m_i}$ of $G_i$ for $i = 1, 2$, respectively. Then we have $H_1^* \subset M_i$ and

$$M_i \cong \text{GL}_1(\mathbb{R})^k \times \text{GL}_2(\mathbb{R})^l \times \text{SO}_{2m+1}(\mathbb{R}).$$

Since $M_2 = M_3 \times \text{SO}_{2m+1}(\mathbb{R})$ for some Levi subgroup $M_3 \cong \text{GL}_1(\mathbb{R})^k \times \text{GL}_2(\mathbb{R})^l$ of $\text{SO}_{2(n-m)+1}(\mathbb{R})$, we may identify $M_2$ with $M_1$ via the isomorphism $M_2 \rightarrow M_1$ given by $(h, h') \mapsto i(h)t'(h')$. Let $P_i$ be a parabolic subgroup of $G_i$ with Levi component $M_i$. Note that $P_2 = P_3 \times \text{SO}_{2m+1}(\mathbb{R})$ for some parabolic subgroup $P_3$ of $\text{SO}_{2(n-m)+1}(\mathbb{R})$ with Levi component $M_3$. Recall the irreducible representation

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi'$$
of $M_1$ as in [B.3]. Put
\[
\xi_1 = (\nu_1, \ldots, \nu_k, \kappa_1 + \nu'_1, \kappa_1 - \nu'_1, \ldots, \kappa_i + \nu'_i, \kappa_i - \nu'_i, \lambda_1, \ldots, \lambda_m) \in (\frak{h}_1^\dagger)^* \cong \mathbb{C}^n,
\]
so that $\text{Ind}_{P_1}^{G_1}(\pi)$ has infinitesimal character $\xi'_1$. Fix a positive system $\Delta^+$ of $\Delta(\mathfrak{g}_1, \mathfrak{h}_1^\dagger)$ such that
\[
\text{Re} \langle \alpha, \xi'_1 \rangle \geq 0
\]
for all $\alpha \in \Delta^+$ and let $\rho(\Delta^+)$ be half the sum of the roots in $\Delta^+$. Choose a sufficiently large positive integer $t$ such that
\[
\xi_1 = \xi'_1 + 2t\rho(\Delta^+)
\]
is regular. Then we have $\Delta^+(\xi_1) = \Delta(\xi_1) \cap \Delta^+$, and by the assumption on $\pi$, $\xi_1$ satisfies (B.11). By construction, we have
\[
\varphi(\Theta_{\gamma_1}^{G_1}) = \Theta_{\gamma_2}^{G_2}
\]
for all $\gamma \in \mathcal{R}(\xi_1) = \mathcal{R}(\xi_2)$. Since $\Theta_{\gamma_1}^{G_1} = \text{Ind}_{M_1}^{G_1}(\Theta_{\gamma_1}^{M_1})$ and $\text{Ind}_{M_1}^{G_1}$ is additive, we have
\[
\varphi(\text{Ind}_{M_1}^{G_1}(\Theta)) = \text{Ind}_{M_2}^{G_2}(\Theta)
\]
for all $\Theta \in \text{Irr}(M_1, \xi_1) = \text{Irr}(M_2, \xi_2)$. On the other hand, by (B.4) and [75, Theorem 7.6], there exists $\Theta \in \text{Irr}(M_1, \xi_1)$ such that
\[
\bar{\Theta}(\xi_1 + \lambda_1) = [\pi],
\]
where $\lambda_1 \in \Lambda^{G_1}$ with $\lambda_1 = -2t\rho(\Delta^+)$. Put $\Xi_i = \text{Ind}_{M_i}^{G_i}(\Theta)$ and $\lambda_2 = f(\lambda_1)$, so that
\[
\Xi_i(\xi_i + \lambda_i) = [\text{Ind}_{P_i}^{G_i}(\pi)].
\]
Then, applying Lemma [B.3] to $\Xi_i$ and $\lambda_i$, we can reduce the irreducibility of $\text{Ind}_{P_1}^{G_1}(\pi)$ to that of
\[
\text{Ind}_{P_2}^{G_2}(\pi) = \text{Ind}_{P_3}^{SO(2n-m)+1(\mathbb{R})}(\chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l) \otimes \pi'.
\]
Since $\text{Ind}_{P_2}^{G_2}(\pi)$ is a standard module (with a suitable choice of $P_2$), its irreducibility follows from a result of Speh–Vogan [69] (see also [72, Chapter 8]) and the assumption on $\pi$. This completes the proof.

Appendix C. Globalization

In this appendix, we prove a refinement of the globalization of Sakellaridis–Venkatesh (Proposition 5.5).

Let $F$ be a number field and $\mathbb{A}$ the adele ring of $F$. Let $S_\infty$ be the set of archimedean places of $F$. Fix a nonempty finite set $S$ of nonarchimedean places of $F$ and a nonarchimedean place $v_0$ of $F$ such that $v_0 \notin S$. Then Proposition 5.5 is an immediate consequence of the following:

Proposition C.1. For each $v \in S \cup \{v_0\}$, let $\tau_v$ be an irreducible square-integrable representation of $\text{GL}_2(F_v)$ such that $L(s, \tau_v, \wedge^2)$ has a pole at $s = 0$. Assume that $\tau_{v_0}$ is supercuspidal. Then there exists an irreducible cuspidal automorphic representation $T$ of $\text{GL}_2(\mathbb{A})$ such that

- $T_v = \tau_v$ for all $v \in S \cup \{v_0\}$;
- $T_v$ is a principal series representation for all $v \notin S_\infty \cup S \cup \{v_0\}$;
- $L(s, T, \wedge^2)$ has a pole at $s = 1$;
- $L(\frac{1}{2}, T) \neq 0$.

To prove this proposition, we globalize a generic representation of $\text{Mp}_{2n}(F_v)$ to a globally generic automorphic representation of $\text{Mp}_{2n}(\mathbb{A})$. More precisely, let $N$ be the standard maximal unipotent subgroup of $\text{Sp}_{2n}$. We regard $N(\mathbb{A})$ as a subgroup of $\text{Mp}_{2n}(\mathbb{A})$ via the canonical splitting. Fix a nontrivial additive character $\Psi$ of $F \backslash \mathbb{A}$. As in [16, §12], $\Psi$ gives rise to a generic character of $N(F) \backslash N(\mathbb{A})$, which we denote again by $\Psi$. 


Proposition C.2. For each \( v \in S \cup \{v_0\} \), let \( \pi_v \) be an irreducible \( \Psi_v \)-generic square-integrable representation of \( \text{Mp}_{2n}(F_v) \). Assume that \( \pi_{v_0} \) is supercuspidal, and that if \( n = 1 \), then \( \pi_{v_0} \) is not the odd Weil representation relative to \( \Psi_{v_0} \). Then there exists an irreducible globally \( \Psi \)-generic cuspidal automorphic representation \( \Pi \) of \( \text{Mp}_{2n}(\mathbb{A}) \) such that

- \( \Pi_v = \pi_v \) for all \( v \in S \cup \{v_0\} \);
- \( \Pi_v \) is a principal series representation for all \( v \notin S_\infty \cup S \cup \{v_0\} \).

Proposition C.1 can be easily deduced from Proposition C.2 and [30] Proposition 4.3. We include the proof for the convenience of the reader.

Proof of Proposition C.1. For each \( v \in S \cup \{v_0\} \), let \( \pi_v \) be the descent of \( \tau_v \) to \( \text{Mp}_{2n}(F_v) \) relative to \( \Psi_v \) (see [23], [24], [30] Theorem 3.1). Then \( \pi_v \) satisfies the conditions in Proposition C.2. Let \( \Pi \) be as given in Proposition C.2. Let \( \Sigma = \Theta^{\text{aut}}(\Pi) \) be the global theta lift to \( \text{SO}_{2n+1}(\mathbb{A}) \) relative to \( \Psi \). Since \( \theta(\pi_{v_0}) \) is supercuspidal by [32] Theorem 2.2, \( \Sigma \) is cuspidal. By [12], \( \Sigma \) is nonzero and globally generic. Hence \( \Sigma \) is irreducible, and by the Rallis inner product formula [36, [19, [81], we have

\[
L\left(\frac{1}{2}, \Pi\right) \neq 0.
\]

Moreover, it follows from the local Shimura correspondence [20] that

- \( \Sigma_v \) is square-integrable for all \( v \in S \cup \{v_0\} \);
- \( \Sigma_v \) is a principal series representation for all \( v \notin S_\infty \cup S \cup \{v_0\} \).

We now take \( \mathcal{T} \) to be the functorial lift of \( \Sigma \) to \( \text{GL}_{2n}(\mathbb{A}) \). Then \( L(s, \mathcal{T}, \wedge^2) \) has a pole at \( s = 1 \), and we have

\[
L\left(\frac{1}{2}, \mathcal{T}\right) = L\left(\frac{1}{2}, \Sigma\right) = L\left(\frac{1}{2}, \Pi\right) \neq 0.
\]

Also, since \( \mathcal{T}_v \) is the functorial lift of \( \Sigma_v \), it is a principal series representation for all \( v \notin S_\infty \cup S \cup \{v_0\} \). Finally, by [30] Proposition 4.3, we have

\[
\mathcal{T}_v = \tau_v
\]

for all \( v \in S \cup \{v_0\} \). In particular, \( \mathcal{T}_{v_0} \) is supercuspidal and hence \( \mathcal{T} \) is cuspidal.

It remains to prove Proposition C.2, which is a refinement of of [62] Theorem 16.3.2, [30] Corollary A.8. We need to modify their argument to control the localization \( \Pi_v \) at nonarchimedean \( v \) outside \( S \cup \{v_0\} \).

Proof of Proposition C.2. We first introduce some notation. By abuse of notation, we write

\[
G_v = \text{Mp}_{2n}(F_v), \quad G_S = \text{Mp}_{2n}(F_S), \quad G(\mathbb{A}) = \text{Mp}_{2n}(\mathbb{A})
\]

where \( F_S = \prod_{v \in S} F_v \). If \( F_v \) is nonarchimedean of odd residual characteristic, we regard \( K_v = \text{Sp}_{2n}(O_v) \) as a subgroup of \( G_v \) via the standard splitting, where \( O_v \) is the integer ring of \( F_v \). Also, we regard \( G(F) = \text{Sp}_{2n}(F) \) as a subgroup of \( G(\mathbb{A}) \) via the canonical splitting. Put

\[
N_S = N(F_S), \quad \Psi_S = \bigotimes_{v \in S} \Psi_v, \quad \pi_S = \bigotimes_{v \in S} \pi_v.
\]

Let \( C^\infty_c(G_v) \) be the space of genuine smooth functions on \( G_v \) with compact support. Let \( C^\infty_c(N_v \backslash G_v, \Psi_v) \) be the space of genuine smooth functions \( f \) on \( G_v \) such that

- \( \text{supp} f \) is compact modulo \( N_v \);
- \( f(xg) = \Psi_v(x)f(g) \) for all \( x \in N_v \) and \( g \in G_v \).
Then we have a map $\mathcal{P}_v : C_c^\infty(G_v) \to C_c^\infty(N_v\backslash G_v, \Psi_v)$ defined by
\[
(\mathcal{P}_v \phi)(g) = \int_{N_v} \phi(xg)\overline{\Psi_v(x)} \, dx.
\]
Let $C_c^\infty(N_S\backslash G_S, \Psi_S)$ be defined similarly. For any automorphic form $\phi$ on $G(\mathbb{A})$, we define its Whittaker–Fourier coefficient $\mathcal{W}_\phi$ by
\[
\mathcal{W}_\phi(g) = \int_{N(F)\backslash N(\mathbb{A})} \phi(xg)\overline{\Psi(x)} \, dx.
\]
Let $L^2_{\text{cusp}}(G)$ be the genuine cuspidal spectrum of $Mp_{2n}$. We define $L^2_{\text{cusp}, \Psi}(G)$ as the orthogonal complement in $L^2_{\text{cusp}}(G)$ of the closure of the space of genuine cusp forms $\phi$ on $G(\mathbb{A})$ such that $\mathcal{W}_\phi = 0$. Then, by the uniqueness of Whittaker models, $L^2_{\text{cusp}, \Psi}(G)$ is multiplicity-free. Fix a finite set $S_0$ of nonarchimedean places of $F$ such that
\begin{itemize}
  \item $S \cap S_0 = \emptyset$;
  \item $v_0 \notin S_0$;
  \item $F_v$ is of odd residual characteristic and $\Psi_v$ is of order zero if $v \notin S_\infty \cup S \cup \{v_0\} \cup S_0$.
\end{itemize}
Let $\{\Pi_i\}$ be the set of irreducible summands of $L^2_{\text{cusp}, \Psi}(G)$ such that
\begin{itemize}
  \item $\Pi_{i,v_0} = \pi_{v_0}$;
  \item $\Pi_{i,v}$ is a principal series representation for all $v \in S_0$;
  \item $\Pi_{i,v}$ is unramified (i.e. $\Pi_{i,v}$ has a nonzero $K_v$-fixed vector) for all $v \notin S_\infty \cup S \cup \{v_0\} \cup S_0$.
\end{itemize}
As in [62, §16.4], we show that $\pi_S$ is weakly contained in the Hilbert space direct sum $\bigoplus \Pi_i$ (regarded as a representation of $G_S$). Since $\pi_S$ belongs to the support of the Plancherel measure for $L^2(N_S\backslash G_S, \Psi_S)$, it suffices to prove the following: for any compact subset $\Omega \subset G_S$ and any $f_S \in C_c^\infty(N_S\backslash G_S, \Psi_S)$, the function
\[
y \mapsto \langle R_yf_S, f_S \rangle, \quad y \in \Omega
\]
is the restriction to $\Omega$ of a convex combination of diagonal matrix coefficients of all $\Pi_i$. Here $\langle \cdot , \cdot \rangle$ is a hermitian inner product and $R_y$ is the right translation by $y$. For each $v \notin S_\infty \cup S$, we choose $f_v = \mathcal{P}_v \phi_v \in C_c^\infty(N_v\backslash G_v, \Psi_v)$, where $\phi_v \in C_c^\infty(G_v)$ is given as follows:
\begin{itemize}
  \item if $v = v_0$, then $\phi_{v_0}$ is a matrix coefficient of $\pi_{v_0}$;
  \item if $v \in S_0$, then $\phi_v$ belongs to the wave packet associated to a Bernstein component consisting only of irreducible principal series representations (e.g. one containing a sufficiently ramified principal series representation);
  \item if $v \notin S_\infty \cup S \cup \{v_0\} \cup S_0$, then
    \[
    \phi_v(g) = \begin{cases}
    \epsilon & \text{if } g \in \epsilon \cdot K_v \text{ with } \epsilon \in \{-1,1\}, \\
    0 & \text{otherwise}.
    \end{cases}
    \]
\end{itemize}
By [69, Lemma 4.4], we may further assume that $\langle f_v, f_v \rangle = 1$. For each $v \in S_\infty$, we choose $f_v \in C_c^\infty(N_v\backslash G_v, \Psi_v)$ later. Put
\[
f = f_S \otimes \left( \bigotimes_{v \notin S} f_v \right).
\]
We define a Poincaré series $P_f$ on $G(\mathbb{A})$ by
\[
P_f(g) = \sum_{\gamma \in N(F)\backslash G(F)} f(\gamma g),
\]
where the sum converges absolutely. Then we have
\[
\langle P_f, P_f \rangle \leq \int_{N(F) \backslash G(\mathbb{A})} |f(g)P_f(g)| \, dg
\leq \int_{N(\mathbb{A}) \backslash G(\mathbb{A})} |f(g)| \int_{N(F) \backslash N(\mathbb{A})} |P_f(xg)| \, dx \, dg < \infty,
\]
so that \( P_f \) is square-integrable over \( G(F) \backslash G(\mathbb{A}) \). Since \( \pi_{v_0} \) is supercuspidal, \( P_f \) is cuspidal by the condition on \( f_{v_0} \). For any genuine cusp form \( \phi \) on \( G(\mathbb{A}) \) such that \( W_{\phi} = 0 \), we have
\[
\langle P_f, \phi \rangle = \int_{N(F) \backslash G(\mathbb{A})} f(g)\overline{\phi(g)} \, dg
= \int_{N(\mathbb{A}) \backslash G(\mathbb{A})} f(g)\overline{W_{\phi}(g)} \, dg = 0.
\]
Hence \( P_f \in L^2_{\text{cusp},\Psi}(G) \). Moreover, by the conditions on \( f_v \), we have \( P_f \in \bigoplus_i \Pi_i \). Thus, as in [62 (16.8)], it remains to prove the following: there exists \( f_v \in C_c^\infty(N_v \backslash G_v, \Psi_v) \) for \( v \in S_\infty \) such that
\[
\langle R_y f_S, f_S \rangle = \langle R_y P_f, P_f \rangle
\]
for all \( y \in \Omega \). This was proved in [62 §16.4], but we include the proof for the convenience of the reader. We choose \( f_v \in C_c^\infty(N_v \backslash G_v, \Psi_v) \) for \( v \in S_\infty \) such that \( \langle f_v, f_v \rangle = 1 \) and \( \text{supp } f_v \) is sufficiently small, so that there exists a compact subset \( U \subset G(\mathbb{A}) \) satisfying the following conditions:
- \( \Omega \subseteq U \);
- \( \text{supp } R_y f \subset N(\mathbb{A}) \cdot U \) for all \( y \in \Omega \);
- \( N(\mathbb{A}) \subset N(F) \cdot U \);
- \( G(F) \cap U_4 \cdot N(\mathbb{A}) = N(F) \), where \( U_4 = U \cdot U \cdot U \cdot U \).

For any \( y \in \Omega \), we have
\[
\langle R_y P_f, P_f \rangle = \int_{N(F) \backslash G(\mathbb{A})} f(gy)\overline{P_f(g)} \, dg
= \int_{N(F) \backslash G(\mathbb{A})} \sum_{\gamma \in N(F) \backslash G(F)} f(gy)\overline{f(\gamma g)} \, dg.
\]
Let \( \gamma \in G(F) \). If \( f(gy)\overline{f(\gamma g)} \neq 0 \) for some \( g \in G(\mathbb{A}) \), then \( gy \in N(\mathbb{A}) \cdot U \) and \( \gamma g \in N(\mathbb{A}) \cdot U \subset N(F) \cdot U \cdot U \). Replacing \( \gamma \) by an element in \( N(F) \cdot U \) if necessary, we may assume that \( \gamma g \in U \cdot U \). Then we have \( \gamma = \gamma g \cdot y \cdot (gy)^{-1} \in U_4 \cdot N(\mathbb{A}) \), so that \( \gamma \in N(F) \). Hence
\[
\langle R_y P_f, P_f \rangle = \int_{N(F) \backslash G(\mathbb{A})} f(gy)\overline{f(g)} \, dg = \langle R_y f, f \rangle = \langle R_y f_S, f_S \rangle.
\]
Thus, we have shown that \( \pi_S \) is weakly contained in \( \bigoplus_i \Pi_i \). As shown in the proof of Proposition 4.1, the global theta lift \( \Theta^{\text{aut}}(\Pi_i) \) to \( \text{SO}_{2n+1}(\mathbb{A}) \) is cuspidal. Hence the assertion follows from the analog of [30 Lemma A.2] for \( \text{Mp}_{2n} \) combined with [30 Proposition A.7], noting that \( \Pi_{i,v} \) is an irreducible principal series representation as in Remark 5.3 for all \( v \notin S_\infty \cup S \cup \{v_0\} \cup S_0 \).
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