GALOIS SUBFIELDS OF TAME DIVISION ALGEBRAS

TIMO HANKE, DANNY NEFTIN, AND ADRIAN WADSWORTH

Abstract. We show that a finite-dimensional tame division algebra $D$ over a Henselian field $F$ has a maximal subfield Galois over $F$ if and only if its residue division algebra $\overline{D}$ has a maximal subfield Galois over the residue field $\overline{F}$.

This generalizes the mechanism behind several known noncrossed product constructions to a crossed product criterion for all tame division algebras, and in particular for all division algebras if the residue characteristic is 0. If $\overline{F}$ is a global field, the criterion leads to a description of the location of noncrossed products among tame division algebras, and their discovery in new parts of the Brauer group.

1. Introduction

A division algebra $D$ finite-dimensional over its center $F$ is called a crossed product if it contains a maximal subfield which is Galois over $F$; otherwise it is a noncrossed product. The question of existence of noncrossed products arose in the 1930’s, and was answered affirmatively by Amitsur in 1972 [2]. Subsequently, their existence over more familiar fields $F$ has been studied by many authors, see for example [13, 18, 3, 7, 4].

Recall that for a Henselian valued field $F$, the valuation of $F$ extends uniquely to a valuation of $D$ for every finite-dimensional division algebra $D$ over $F$, see for example [12, §1]. In case $D$ is inertially split, i.e. split by an unramified extension of $F$, it was known long ago that $D$ is a crossed product only if the residue division algebra $\overline{D}$ is a crossed product [12 Thm. 5.15(b)]. This criterion traces back to Saltman [16] who used it to construct new noncrossed products of higher index from ones already known. By a more complete criterion [9], any inertially split $D$ is a crossed product if and only if $\overline{D}$ contains a maximal subfield Galois over the residue field $\overline{F}$. Note that this is a stronger condition than saying $\overline{D}$ is a crossed product since $\overline{F}$ may be a proper subfield of the center of $\overline{D}$. This criterion goes back to Brussel [3], who used such an argument over complete rank 1 valued fields to obtain noncrossed products over fields as elementary as $\mathbb{Q}((t))$. Subsequently, the criterion has led to a description of the “location” of crossed and noncrossed products among all inertially split division algebras over $F$, when $F$ is Henselian with global residue field. [8, 6].

2000 Mathematics Subject Classification. Primary 16S35, Secondary.

Key words and phrases. division algebra, tame Brauer group, noncrossed product, Henselian valuation, Galois maximal subfield, residue division ring.
In this paper we consider the larger class of division algebras $D$ which are \emph{tamely ramified} (or \emph{tame} for short) over their center a Henselian field $F$, i.e. split by a tamely ramified extension of $F$. In particular, these include all division algebras whose degree is prime to the residue characteristic. Our main result, Theorem 1.1, generalizes the criterion mentioned above to tame division algebras:

\begin{theorem}
Let $F$ be a Henselian field, and $D$ a finite-dimensional tamely ramified division algebra with center $F$. Then $D$ has a maximal subfield Galois over $F$ if and only if $\overline{D}$ has a maximal subfield Galois over $\overline{F}$.
\end{theorem}

Theorem 1.1 is useful in determining which tame division algebras are crossed products over all Henselian fields $F$ whose residue field is sufficiently well understood. When $\overline{F}$ has cohomological dimension 1 or is a local field, we deduce that all finite-dimensional tame division algebras with center $F$ are crossed products. When $\overline{F}$ is a global field we describe the location of noncrossed products among tame division algebras, extending [8] and [6], and locating noncrossed products in new parts of the Brauer group, see §4.2.

The main difficulty in proving Theorem 1.1 lies in the “only if” implication. Given a maximal subfield $M$ of $D$, Galois over $F$, neither the residue field $\overline{M}$ itself nor the compositum of $\overline{M}$ with the center of $\overline{D}$ need to be a maximal subfield of $\overline{D}$. Hence in the tame case the construction is significantly different from the inertially split case, where it was enough to consider $\overline{M} \cdot Z(\overline{D})$, cf. [9].

Our construction of the desired maximal subfield of $\overline{D}$ uses the theory of graded division algebras which provides a one-to-one correspondence between tame division algebras over a Henselian field $F$ and graded division algebras over its associated graded field $\text{gr}(F)$ [11]. It associates to $D$ a graded division algebra $\text{gr}(D)$ over $\text{gr}(F)$, and to a maximal subfield $M$ of $D$ Galois over $F$ a maximal graded subfield $\text{gr}(M)$ Galois over $\text{gr}(F)$. Most importantly, it equips $\text{gr}(D)$ with canonical subalgebras which can be entwined with $\text{gr}(M)$ to form a maximal graded subfield $M'$ of $\text{gr}(D)$ with residue field which is maximal in $\overline{D}$ and Galois over $\overline{F}$. This $M'$ lifts to a maximal subfield $M'$ of $D$ Galois over $F$.

The proof of our theorem gives a good illustration of the utility of the graded approach in working with valued division algebras. Many properties of $D$ are faithfully reflected in $\text{gr}(D)$, but $\text{gr}(D)$ has a simpler structure which is often considerably easier to work with, as demonstrated by our use of its canonical subalgebras.

We thank Uzi Vishne, Eric Brussel, and Jack Sonn for helpful discussions. We also thank Kelly McKinnie for her interest and questions which led us to a simplification of our proof. This material is based upon work supported by the National Science Foundation under Award No. DMS-1303990.
2. Graded and valued algebras

We first recall the basic definition and facts concerning graded algebras, based on [11] with the exception of Section 2.3 which is based on [10] and [14]. A more extensive treatment of these facts will appear in [19].

Throughout the section we let $\Gamma$ be a torsion-free abelian group.

2.1. Graded rings and division algebras. A graded ring $D$ with grade group $\Gamma$ (or a $\Gamma$-graded ring) is a ring with a direct sum decomposition $D = \bigoplus_{\gamma \in \Gamma} D_\gamma$, where each $D_\gamma$ is an additive abelian group and $D_{\gamma} \cdot D_{\delta} \subseteq D_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. Set $\Gamma_D = \{ \gamma \in \Gamma | D_\gamma \neq \{0\} \}$ and call the elements in $D_\gamma$, $\gamma \in \Gamma$, homogenous.

A graded homomorphism $\varphi : D \to E$ of $\Gamma$-graded rings is a homomorphism which preserves the grading, i.e. $\varphi(D_\gamma) \subseteq E_\gamma$ for all $\gamma \in \Gamma$.

A graded subring of $D$ is a subring $E \subseteq D$ such that $E = \bigoplus_{\gamma \in \Gamma_D} (D_\gamma \cap E)$. Such a decomposition defines a $\Gamma$-grading on $E$ with $E_\gamma = D_\gamma \cap E$. Note that with $E$ the centralizer $C_D(E)$ is also a graded subring of $D$, hence in particular the center $Z(D)$ is a graded subring of $D$.

Let $D$ be a graded ring with $1 \neq 0$. Then $D$ is said to be a graded division ring if every nonzero element of $D_\gamma$, $\gamma \in \Gamma_D$, is a unit. In this case $D_0$ is a division ring, and multiplication by any nonzero element from $D_\gamma$ induces an isomorphism $D_\gamma \cong D_0$ of $D_0$-module; hence, $D_\gamma$ is a rank 1-module over $D_0$.

Commutative graded division rings are called graded fields. If $D$ is a graded division ring whose center contains a graded field $F$ as a graded subring, then $D$ is called a graded division algebra over $F$. In this case $D_0$ is a division algebra over $F_0$. Moreover, for a graded $F$-subalgebra $E \subseteq D$, $D$ is free as an $E$-module, cf. [11], Paragraph preceding (1.6)], and the dimension $[D : E]$ is defined as the rank of $D$ as an $E$-module. We shall assume throughout that all graded division algebras are finite-dimensional over their centers. The degree of a graded division algebra $D$ is $\text{deg } D := \sqrt{[D : Z(D)]}$.

Given two $\Gamma$-graded algebras $D$ and $E$ over $F$, the tensor product $D \otimes_F E$ is also a $\Gamma$-graded algebra with $(D \otimes_F E)_\gamma$ generated by all $d_\alpha \otimes e_\beta$ where $d_\alpha \in D_\alpha, e_\beta \in E_\beta$, and $\alpha + \beta = \gamma$. The double centralizer theorem is available in the graded setting [11], Proposition 1.5] and implies, using the same argument as in the ungraded setting, that for a graded division algebra $D$ over $F$, $[M : F] \leq \text{deg } D$ for any graded subfield $M$, with equality if and only if $M$ is a maximal graded subfield of $D$.

2.2. Ramification. The following ramification properties of graded division rings are analogues to ramification properties of valued division rings. Let $D$ be a graded division ring, and $E \subseteq D$ a graded division subring. Then one easily obtains the fundamental equality, cf. [11] (1.7), p. 79,

$$[D : E] = [D_0 : E_0] |\Gamma_D : \Gamma_E|.$$  \hfill (2.1)

We say that $D$ is unramified over $E$ if $\Gamma_D = \Gamma_E$ (i.e. $[D : E] = [D_0 : E_0]$); it is totally ramified over $E$ if $[D : E] = |\Gamma_D : \Gamma_E|$ (i.e. $D_0 = E_0$).
Let $A \subseteq D$ be a graded subring that is also a graded division ring and contains $E$. Then $D/E$ is unramified (resp. totally ramified) if and only if $D/A$ and $A/E$ are each unramified (resp. totally ramified).

Let $F$ be a graded subfield of $Z(D)$, so that $D$ is a graded $F$-division algebra. For every $F_0$-subalgebra $A$ of $D_0$ there is a unique graded division $F$-subalgebra $A \subseteq D$ with $A_0 = A$ and $\Gamma_A = \Gamma_D$. This $A$ is generated over $F$ by $A$ and is canonically isomorphic to $A \otimes_{F_0} F$.

Note that the intersection $A \cap B$ of two graded subrings of $D$ is a graded subring with $(A \cap B)_\gamma = A_\gamma \cap B_\gamma$. In the totally ramified case we have:

Lemma 2.1. Let $A, B$ be two graded subrings of $D$ that are also graded division rings. Assume $D$ is totally ramified over $B$. Then,

(i) $B = \oplus_{\gamma \in \Gamma_B} D_\gamma$;

(ii) $\Gamma_{A \cap B} = \Gamma_A \cap \Gamma_B$.

Proof. Clearly, $B \subseteq B' := \oplus_{\gamma \in \Gamma_B} D_\gamma$. For every $\gamma \in \Gamma_B$, since $D_\gamma$ is a rank-1 module over $D_0 = B_0$, one has $B_\gamma = D_\gamma$. Hence, $B' = B$, showing (i).

The inclusion $\Gamma_{A \cap B} \subseteq \Gamma_A \cap \Gamma_B$ is obvious. Conversely, for $\gamma \in \Gamma_A \cap \Gamma_B$, using $B_\gamma = D_\gamma$, we have $(A \cap B)_\gamma = A_\gamma \cap B_\gamma \neq \{0\}$, completing (ii). \qed

2.3. Graded field extensions. Let $L/F$ be a finite extension of $\Gamma$-graded fields. See [10, §2] for proofs of the properties recalled in this paragraph. As $\Gamma$ is torsion-free, $F$ is an integral domain and we can form its field of quotients $q(F)$. Then, $q(L) = L \otimes_F q(F)$, as $[L : q(F)] < \infty$ and $L$ has no zero divisors. Hence, $[L : F] = [q(L) : q(F)]$. Moreover, the ring $F$ is integrally closed, and $L$ is the integral closure of $F$ in $q(L)$. In addition, for any homogeneous $c$ in $L$, the minimal polynomial $m_{q(F),c}$ of $c$ over $q(F)$ has homogeneous coefficients in $F$. In any graded field extension of $F$, every root of $m_{q(F),c}$ is homogeneous of the same degree as $c$, and there are graded field extensions of $F$ over which $m_{q(F),c}$ splits. It follows that every $F$-algebra automorphism of $L$ as ungraded rings actually preserves the grading on $L$. Therefore, the group of graded ring automorphisms $\text{Aut}(L/F)$ is canonically isomorphic to $\text{Aut}(q(L)/q(F))$.

We say that $L$ is tame (or tamely ramified) over $F$ if the field extension $L_0/F_0$ is separable and $\text{char} F_0 \nmid [\Gamma_L : \Gamma_F]$. We say that $L$ is normal over $F$ if for every homogeneous $c \in L$, its minimal polynomial $m_{q(F),c}$ splits over $L$. More restrictively, $L/F$ is Galois if it is Galois as an extension of ungraded commutative rings, or, equivalently, if $F$ is the fixed-ring of $\text{Aut}(L/F)$. These properties of $L/F$ and are equivalent to corresponding properties of their quotient fields, as follows:

| $L/F$ | $q(L)/q(F)$ | Reference |
|-------|-------------|-----------|
| tame  | separable   | [10] Theorem 3.11(a) |
| normal| normal      | [14] Lemma 1.2 |
| Galois| Galois      | [10] Theorem 3.11(b) |
Moreover, if \( L/F \) is Galois the map \( M \mapsto q(M) \) gives a one-to-one correspondence between graded subfields \( M \) with \( F \subseteq M \subseteq L \) and subfields \( M \) with \( q(F) \subseteq M \subseteq q(L) \) which preserves degrees and Galois groups \([10, \text{Proposition 5.1}]\). In addition, by passage to quotient fields, we have: if \( L/F \) is normal and \( M \) is a graded field with \( F \subseteq M \subseteq L \) then:

\[
(2.3) \quad M/F \text{ is normal if and only if } \sigma(M) = M \text{ for every } \sigma \in \text{Aut}(M/F).
\]

Consider commuting graded subfields \( K, L \) of a graded division algebra \( D \) which each contain \( F := Z(D) \). The *compositum* \( K \cdot L \) is the \( F \)-algebra generated by \( K \) and \( L \) in \( D \). Picking \( F \)-bases \( \{ k_i \}_{i \in I} \) and \( \{ \ell_j \}_{j \in J} \) for \( K \) and \( L \), respectively, which consist of homogenous elements, one has

\[
K \cdot L = \sum_{i \in I, j \in J} F_0 k_i \ell_j,
\]

and hence \( K \cdot L \) is a graded subfield of \( D \) (since it is finite-dimensional over \( F \)). We say that \( K \cdot L \) is linearly disjoint over \( E := K \cap L \) if the natural surjection \( K \otimes_F L \to K \cdot L \) is an isomorphism. In particular, \( K \) and \( L \) are linearly disjoint over \( E \) if and only if \( |K \cdot L : E| = |K : E||L : E| \).

**Lemma 2.2.** Let \( D \) be a graded division algebra over \( F \). Let \( K \) and \( L \) be commuting graded subfields of \( D \) containing \( F \), with \( L \) Galois over \( K \cap L \). Then \( K \) and \( L \) are linearly disjoint over \( K \cap L \).

**Proof.** Let \( E := K \cap L \). Then,

\[
q(E) = (K \cap L) \otimes_F q(F) = (K \otimes_F q(F)) \cap (L \otimes_F q(F)) = q(K) \cap q(L).
\]

Moreover, since \( q(K) \cdot q(L) \) is generated over \( q(F) \) by \( K \) and \( L \), we have \( q(K) \cdot q(L) = q(K \cdot L) \). As \( q(L)/q(E) \) is Galois, \( q(K) \) and \( q(L) \) are linearly disjoint over \( q(E) \); hence,

\[
[K \cdot L : K] = [q(K) \cdot q(L) : q(K)] = [q(L) : q(E)] = [L : E].
\]

Therefore, \( K \) and \( L \) are linearly disjoint over \( E \). \( \square \)

We shall also need the following lemma concerning totally ramified extensions:

**Lemma 2.3.** Let \( F \subseteq K \subseteq M \) be graded fields such that \( M/K \) is totally ramified. If \( M/F \) is normal (resp. Galois) then \( K/F \) is normal (resp. Galois).

**Proof.** Since \( M/K \) is totally ramified, by Lemma 2.1 we have \( K = \bigoplus_{\gamma \in \Gamma_M} M_\gamma \). As each \( \sigma \in \text{Gal}(M/F) \) is degree-preserving, this shows that \( \sigma(K) = K \). If \( M \) is normal over \( F \) then by \( (2.3) \), \( K \) is also normal over \( F \). If \( M/F \) is tame then by \( (2.2) \), \( K/F \) is tame. Both statements are proved. \( \square \)

Finally, we note that for any finite graded field extension \( L/F \), if \( c \in L_0 \), then since the roots of its minimal polynomial \( m_{q(F), c} \) all have degree 0, the polynomial has coefficients in \( F_0 \); so, \( m_{q(F), c} = m_{F_0, c} \). Hence,

\[
(2.4) \quad \text{if } L/F \text{ is normal (resp. Galois) then } L_0/F_0 \text{ is normal (resp. Galois).}
\]
2.4. Canonical subalgebras of graded division algebras. The following canonical algebras and their properties were introduced in [11]. Let $D$ be a graded division algebra over its center $F$. Then $D$ has the following canonical subalgebras:

- $U = D_0 \otimes_{F_0} F = \text{the maximal subalgebra of } D \text{ unramified over } F$,
- $Z = Z(D_0) \otimes_{F_0} F = \text{the center of } U$,
- $C = \text{the centralizer } C_D(U)$,
- $E = \text{the centralizer } C_D(Z) = U \otimes_{Z} C$.

Note that $U$ and hence $Z$, $C$, and $E$, were chosen canonically and hence are invariant under conjugation by nonzero elements of $D$, $\gamma \in \Gamma_D$. Moreover, as $D$ is totally ramified over $U$, one has $U_0 = E_0 = D_0$. In particular, as $[D : E] = [Z : F]$ by the graded double centralizer theorem, we have

\[ [D : E] = |\Gamma_D : \Gamma_E| = [Z : F] = [Z_0 : F_0]. \tag{2.5} \]

The center $Z(D_0) = Z_0$ clearly contains $F_0$ but is not necessarily equal to it. In fact, $Z_0/F_0$ is Galois with abelian Galois group which is described as follows: Let $\text{int}(d_\gamma)$ denote the inner automorphism which sends $x \in D$ to $d_\gamma xd_\gamma^{-1}$. Let $\theta_D : \Gamma_D \to \text{Gal}(Z_0/F_0)$ be the homomorphism for which $\theta_D(\gamma)$ is the restriction of $\text{int}(d_\gamma)$ to $Z_0$ for any nonzero $d_\gamma \in D$. Then $\theta_D$ is well defined, surjective, and its kernel is $\Gamma_E$. [11] Proposition 2.3. Hence $\text{Gal}(Z_0/F_0) \cong \Gamma_D/\Gamma_E$. Note also that as $C_0 = C_{D_0}(D_0) = Z_0$, the graded algebra $C$ is totally ramified over $Z$.

We shall need the following properties of maximal graded subfields of $C$:

**Lemma 2.4.** Let $T$ be a maximal graded subfield of $C$. Then:

(i) $T$ is Galois over $F$;

(ii) $\Gamma_T = \Gamma_{C_0(T)}$.

**Proof.** By the graded double centralizer theorem [11] Proposition 1.5] and dimension count,

\[ C_0(T) = C_0(U) \otimes_{Z} C_0(T) = U \otimes_{Z} T. \]

As $\Gamma_U = \Gamma_Z \subseteq \Gamma_T$, we get $\Gamma_{C_0(T)} = \Gamma_T$, proving (ii).
To show (i), we first claim that $T/Z$ is Galois. Since $C$ is totally ramified over $Z$, its graded subfield $T$ is also totally ramified over $Z$. Hence, by Lemma 2.1

$\square$

By [11] Proposition 2.3, we have $\text{char} \ F_0 \nmid |\Gamma_C : \Gamma_Z|$, and hence $\text{char} \ F_0 \nmid |\Gamma_T : \Gamma_Z|$. This shows that $T$ is tame over $Z$. As $T$ is also totally ramified over $Z$, and $\text{char} \ F_0 \nmid |\Gamma_T : \Gamma_Z|$, [10] Proposition 3.3 implies that $q(T)/q(Z)$ is a Kummer extension, hence Galois. Thus, $T/Z$ is Galois (see (2.2)), proving the claim.

As $Z$ is Galois and unramified over $F$, we have $\text{Gal}(Z/F) \cong \text{Gal}(Z_0/F_0)$. Let $\sigma \in \text{Gal}(Z/F)$. Since $\theta_D$ is surjective, there is a unit $d \in D_\gamma$ for some $\gamma \in \Gamma$, such that $\text{int}(d)|_{Z_0} = \sigma|_{Z_0}$, and hence $\text{int}(d)|_Z = \sigma$. Since $d$ lies in one of the components $D_\gamma$, $\text{int}(d)$ preserves $C$. Thus, (2.6) shows that the graded automorphism $\text{int}(d)$ preserves $T$. Since $Z/F$ and $T/Z$ are Galois and since every automorphism in $\text{Gal}(Z/F)$ extends to a graded automorphism of $T$, it follows that $T$ is Galois over $F$, as required.

2.5. Tame division algebras. All division algebras considered in this paper are assumed to be finite-dimensional. Let $F$ be a Henselian field and $D$ a division algebra with center $F$, $\overline{D}$ the residue division algebra of $D$ with respect to the unique extension of the valuation on $F$ to $D$, and let $\Gamma_D$ be the value group (a totally ordered abelian group).

Recall that $D$ is tame (or tamely ramified over $F$) if and only if

$[D:F] = [\overline{D}: \overline{F}]|\Gamma_D : \Gamma_F|,$

$Z(\overline{D})$ is separable over $\overline{F}$, and $\text{char} \ F \nmid |\ker(\theta_D) : \Gamma_F|$, cf. [12] §6. Furthermore, $D$ is said to be inertial over $F$ if it is tame and also unramified, i.e. $\Gamma_D = \Gamma_F$.

The tame Brauer group $\text{TBr}(F)$ is the subgroup of $\text{Br}(F)$ which consists of classes $[D]$ of tame division algebras $D$ with center $F$. Since $D$ is tame if and only if $D$ is split by the maximal tamely ramified field extension of $F$, $\text{TBr}(F)$ is a subgroup of $\text{Br}(F)$. Denote the degree of $D$ by $\text{deg} \ D := \sqrt{|D:F|}$, write $\text{ind}[D]$ for the Schur index $\text{ind}[D] := \text{deg} \ D$, and let $[D]^Z = [D \otimes_F Z] \in \text{Br}(Z)$ for any extension $Z/F$.

We shall need the following lemma from [12] which describes properties that are preserved under tensor products with inertial algebras.

**Lemma 2.5.** Let $I, D$ be central division algebras over $F$. Assume $I$ is inertial and $D$ is tame. Let $D'$ be the division algebra underlying $[I \otimes_F D]$. Then $Z(\overline{D'}) \cong Z(\overline{D})$, $\Gamma_{D'} = \Gamma_D$, $[\overline{D}] = [\overline{I} \otimes_F \overline{D}]$ in $\text{Br}(Z(\overline{D}))$, and the following ratio is preserved:

$\frac{\text{deg} \ D}{\text{deg} \ D'} = \frac{\text{deg} \ D'}{\text{deg} \ D'}$.

**Proof.** All assertions are proved in [12] Corollary 6.8, except for the last, which is derived as follows. Recall that there is a well-defined homomorphism

$\theta_D : \Gamma_D \rightarrow \text{Gal}(Z(\overline{D})/\overline{F})$
Let $v$ be the valuation on $D$. For $\gamma \in \Gamma_D$ take any nonzero $c \in D$ with $v(c) = \gamma$. Then, for any $z \in D$ with $v(z) \geq 0$ and $\overline{z} \in Z(D)$, define $\theta_D(\gamma)(\overline{z}) = czc^{-1}$. By [12, Corollary 6.8], $\theta_D = \theta_D$. Since $D$ is tame, by (2.7)
$$[D : F] = [\overline{D} : \overline{F}] \cdot |\Gamma_D : \Gamma_F| \cdot |\ker \theta_D : \Gamma_F| \cdot |\Im \theta_D|.$$ By [12, Prop. 1.7], $[\overline{D} : \overline{F}] = |\Im \theta_D|$, hence by taking square roots we obtain:
$$(2.9) \quad \deg_D = \deg \overline{D} \cdot |\Im \theta_D| \cdot \sqrt{|\ker \theta_D : \Gamma_F|}.$$ Thus,
$$\frac{\deg_D}{\deg_D} = |\Im \theta_D| \cdot \sqrt{|\ker \theta_D : \Gamma_F|} = |\Im \theta_D| \cdot \sqrt{|\ker \theta_D : \Gamma_F|} = \frac{\deg_D}{\deg_D}. \square$$

2.6. The correspondence. A tame division algebra $D$ with value group $\Gamma$ yields a $\Gamma$-graded division ring $\text{gr}(D)$ with components $\text{gr}(D)_\gamma = D_{\geq \gamma} / D_{> \gamma}$, $\gamma \in \Gamma$, where
$$D_{\geq \gamma} = \{x \in D \mid v(x) \geq \gamma\} \text{ and } D_{> \gamma} = \{x \in D \mid v(x) > \gamma\}.$$ Furthermore, $\text{gr}(D)$ is a graded division algebra over $\text{gr}(F)$ with $\text{gr}(D)_0 = \overline{D}$, and $\Gamma_{\text{gr}(D)} = \Gamma$. Thus, (2.11) and (2.7) together show that
$$(2.10) \quad [\text{gr}(D) : \text{gr}(F)] = [D : F].$$ Also, the tameness of $D$ implies that $Z(\text{gr}(D)) = \text{gr}(F)$ by [11, Proposition 4.3].

The map $D \mapsto \text{gr}(D)$ gives a degree-preserving bijection [11, Theorem 5.1] between tame division algebras with center $F$ (up to isomorphism) and graded division algebras with center $\text{gr}(F)$ (up to isomorphism). By [11, Corollary 5.7], this correspondence is functorial under field extensions $L/F$; hence, $L$ is a maximal subfield of $D$ if and only if $\text{gr}(L)$ is a maximal graded subfield of $\text{gr}(D)$. By [14, Theorem 1.5], if $L/F$ is normal then so is $\text{gr}(L)/\text{gr}(F)$.

On the level of fields, by [10, Theorem 5.2], there is a correspondence between tame graded field extensions of $\text{gr}(F)$ and tame field extensions of $F$, which preserves degrees and Galois groups. In particular, to every tame graded field extension $L$ of $\text{gr}(F)$ there corresponds a unique tamely ramified field extension $L$ of $F$, called the tame lift of $L$ over $F$, such that $\text{gr}(L) \cong L$ as graded fields and $[L : F] = [L : \text{gr}(F)]$. Moreover, $L$ is Galois over $F$ if and only if $L$ is Galois over $\text{gr}(F)$.

---

1This definition slightly differs from the definition of $\theta_D$ in [12, p. 133], where $\theta_D$ is defined on $\Gamma_D/\Gamma_F$. 
3. Maximal subfields of tame graded division algebras

Throughout this section we fix a graded division algebra $D$ with center $F$, and let $Z, C, U$, and $E$ be its canonical subalgebras (introduced in §2.4). We first prove the graded version of Theorem 1.1:

**Theorem 3.1.** A finite-dimensional graded division algebra $D$ has a graded maximal subfield Galois (resp. normal) over $F$ if and only if $D_0$ has a maximal subfield Galois (resp. normal) over $F_0$.

The following Proposition gives the “if” implication of Theorem 3.1.

**Proposition 3.2.** Let $M$ be a maximal subfield of $D_0$, $L = M \otimes_{F_0} F$, and $T$ a maximal graded subfield of $C$. Then $M := L \cdot T$ is a maximal graded subfield of $D$. Moreover, if $M/F_0$ is Galois (resp. normal) then $M/F$ is Galois (resp. normal).

**Proof.** Since $M$ is maximal it contains $Z(D_0)$. By definition of $Z, U, C$, cf. §2.4, we have $Z \subseteq L \subseteq U$, and $L$ and $T$ commute. Hence, $M$ is a graded subfield of $D$. Since $L/Z$ is inertial we have:

$$[L:Z] = [M:Z(D_0)] = \deg D_0 = \deg U.$$

As $L/Z$ is unramified, and $T/Z$ is totally ramified one has $L \cap T = Z$. By Lemma 2.4, $T/Z$ is Galois. Hence, Lemma 2.2 implies

$$[M:Z] = [L:Z] \cdot [T:Z] = \deg U \cdot \deg C = \deg E.$$

This shows that $M$ is a maximal graded subfield of $E$, cf. end of §2.1 hence also a maximal graded subfield of $D$ by (2.5).

Furthermore, if $M = L_0$ is Galois (resp. normal) over $F_0$, then $L$ is Galois (resp. normal) over $F$. As $T$ is Galois over $F$ by Lemma 2.4 we get that $M$ is Galois (resp. normal) over $F$. □

For a maximal graded subfield $M$ of $D$, the field $M_0$ need not be a maximal subfield of $D_0$. We will therefore modify $M$ to enlarge the degree-0 part. We start with the following observation:

**Lemma 3.3.** Let $M$ be a maximal graded subfield of $D$. Then, $M_0$ is a maximal subfield of $D_0$ if and only if $M \supseteq Z$ and $|\Gamma_M : \Gamma_Z| = \deg C$. This holds if $M \cap C$ is a maximal graded subfield of $C$.

**Proof.** If $M_0$ is a maximal subfield of $D_0$ then $M_0 \supseteq Z(D_0) = Z_0$, so $M \supseteq Z$ by definition of $Z$. Hence, we assume $M \supseteq Z$ throughout the proof. Then $M \subseteq C_D(Z) = E$, so $M$ is a maximal graded subfield of $E$. We have

$$[M_0 : Z_0] \cdot |\Gamma_M : \Gamma_Z| = [M : Z] = \deg E = \deg U \cdot \deg C = \deg D_0 \cdot \deg C.$$

Hence, $M_0$ is a maximal subfield of $D_0$ (i.e. $[M_0 : Z_0] = \deg D_0$) if and only if $|\Gamma_M : \Gamma_Z| = \deg C$. 

Suppose now that $\mathcal{M} \cap \mathcal{C}$ is a maximal graded subfield of $\mathcal{C}$. Then, since $\mathcal{M} \subseteq C_E(\mathcal{M} \cap \mathcal{C})$, by Lemma 2.4 we have

$$\Gamma_\mathcal{M} \subseteq \Gamma_{C_E(\mathcal{M} \cap \mathcal{C})} = \Gamma_{\mathcal{M} \cap \mathcal{C}} \subseteq \Gamma_\mathcal{M},$$

hence, as $\mathcal{M} \cap \mathcal{C}$ is totally ramified over $\mathcal{Z}$, $|\Gamma_\mathcal{M} : \Gamma_\mathcal{Z}| = |\mathcal{M} \cap \mathcal{C} : \mathcal{Z}| = \deg \mathcal{C}$. □

The following Proposition gives the “only if” implication of Theorem 3.1, and completes its proof.

**Proposition 3.4.** Let $\mathcal{M}$ be a maximal graded subfield of $\mathcal{D}$ and let $\mathcal{T}$ be a maximal graded subfield of $\mathcal{C}$. Then $\mathcal{M}' := (\mathcal{M} \cap C_D(\mathcal{T})) \cdot \mathcal{T}$ is a maximal graded subfield of $\mathcal{D}$ for which $\mathcal{M}'_0$ is a maximal subfield of $\mathcal{D}_0$.

Furthermore, if $\mathcal{M}$ is Galois (resp. normal) over $\mathcal{F}$ then $\mathcal{M}'$ is Galois (resp. normal) over $\mathcal{F}$ and $\mathcal{M}'_0$ is Galois (resp. normal) over $\mathcal{F}_0$.

The proof relies on the following lemma:

**Lemma 3.5.** Let $\mathcal{A}, \mathcal{B}$ be graded subalgebras of $\mathcal{D}$ containing $\mathcal{F}$ such that $\mathcal{A}$ is a graded field and $\mathcal{B} \subseteq \mathcal{C}$. Define $\mathcal{A}' := \mathcal{A} \cap C_D(\mathcal{B})$. Then, $\mathcal{A}/\mathcal{A}'$ is totally ramified and $[\mathcal{A} : \mathcal{A}'] \leq [\mathcal{B} : \mathcal{A} \cap \mathcal{B}]$.

**Proof.** Since $\mathcal{B} \subseteq \mathcal{C}$, one has $\mathcal{U} = C_D(\mathcal{C}) \subseteq C_D(\mathcal{B})$, hence $C_D(\mathcal{B})_0 = \mathcal{U}_0 = \mathcal{D}_0$. Thus, $\mathcal{D}$ is totally ramified over $C_D(\mathcal{B})$. Hence, (i) $\mathcal{A}/\mathcal{A}'$ is totally ramified (for $\mathcal{A}'_0 = \mathcal{A}_0 \cap C_D(\mathcal{B})_0 = \mathcal{A}_0 \cap \mathcal{D}_0 = \mathcal{A}_0$), and (ii) $\Gamma_A' = \Gamma_A \cap \Gamma_{C_D(\mathcal{B})}$ by Lemma 2.4(ii). Since $\mathcal{A}$ is a graded field, we have $\mathcal{A} \subseteq C_D(\mathcal{A}) \subseteq C_D(\mathcal{A} \cap \mathcal{B})$; also, $C_D(\mathcal{B}) \subseteq C_D(\mathcal{A} \cap \mathcal{B})$. These together yield,

$$[\mathcal{A} : \mathcal{A}'] = |\Gamma_A : \Gamma_A'| = |\Gamma_A : \Gamma_A \cap \Gamma_{C_D(\mathcal{B})}| = |\Gamma_A + \Gamma_{C_D(\mathcal{B})} : \Gamma_{C_D(\mathcal{B})}|$$

$$\leq |\Gamma_{C_D(\mathcal{A} \cap \mathcal{B})} : \Gamma_{C_D(\mathcal{B})}| \leq [C_D(\mathcal{A} \cap \mathcal{B}) : C_D(\mathcal{B})] = [\mathcal{B} : \mathcal{A} \cap \mathcal{B}],$$

with the last equality given by the graded double centralizer theorem. □

**Proof of Proposition 3.4.** By Lemma 2.4, $\mathcal{T}$ is Galois over $\mathcal{F}$. Thus, by Lemma 2.2, $\mathcal{T}$ is linearly disjoint from $\mathcal{M} \cap C_D(\mathcal{T})$ over their intersection $(\mathcal{M} \cap C_D(\mathcal{T})) \cap \mathcal{T} = \mathcal{M} \cap \mathcal{T}$. Hence, by definition of $\mathcal{M}'$, we have $[\mathcal{M}' : \mathcal{M} \cap C_D(\mathcal{T})] = [\mathcal{T} : \mathcal{M} \cap \mathcal{T}]$. Lemma 3.5 applied with $\mathcal{A} = \mathcal{M}$ and $\mathcal{B} = \mathcal{T}$, states that this dimension is greater than or equal to $[\mathcal{M} : \mathcal{M} \cap C_D(\mathcal{T})]$, so $[\mathcal{M}' : \mathcal{T}] \geq [\mathcal{M} : \mathcal{T}]$. Since $\mathcal{M}$ is maximal it follows that $\mathcal{M}'$ is maximal. Moreover, since $\mathcal{M}'$ contains $\mathcal{T}$, $\mathcal{M}'_0$ is a maximal subfield of $\mathcal{D}_0$ by Lemma 3.3.

Assume that $\mathcal{M}$ is Galois (resp. normal) over $\mathcal{F}$. The application of Lemma 3.5 above also showed that $\mathcal{M}$ is totally ramified over $\mathcal{M} \cap C_D(\mathcal{T})$. Hence, by Lemma 2.3, $\mathcal{M} \cap C_D(\mathcal{T})$ is Galois (resp. normal) over $\mathcal{F}$. Since, by Lemma 2.4, $\mathcal{T}$ is Galois over $\mathcal{F}$, we get that $\mathcal{M}'$ is Galois (resp. normal) over $\mathcal{F}$. Hence, $\mathcal{M}'_0$ is Galois (resp. normal) over $\mathcal{F}_0$, by (2.4). □

**Remark 3.6.** (i) For a graded subfield $\mathcal{M}$ of $\mathcal{D}$ which is not necessarily maximal, the proof gives $[\mathcal{M}' : \mathcal{F}] \geq [\mathcal{M} : \mathcal{F}]$, where $\mathcal{M}' := (\mathcal{M} \cap C_D(\mathcal{T})) \cdot \mathcal{T}$. 
The proof shows that Propositions 3.2 and 3.4, and hence also Theorem 3.1 hold more generally when \( F \) is a proper subfield of \( Z(D) \) under the assumption that \( T/F \) is Galois.

Theorem 1.1 follows from the following corollary:

**Corollary 3.7.** Let \( F \) be a Henselian field, and let \( D \) be a tame division algebra with center \( F \). The following are equivalent:

1. \( D \) has a maximal subfield Galois over \( F \).
2. \( \overline{D} \) has a maximal subfield Galois over \( F \).
3. \( D \) has a maximal subfield Galois and tamely ramified over \( F \).

Moreover, the list can be extended by the three conditions \((a'), (b'), (c')\) which are obtained from \((a), (b), (c)\) by replacing ‘Galois’ with ‘normal’.

**Proof of Corollary 3.7.** Trivially, \((a), (b), (c)\) imply \((a'), (b'), (c')\) respectively, and \((c')\) implies \((a')\). Nontrivially, by [16, Lemma 3], \((a')\) implies \((a)\). Since \( D \) is tame, \( Z(D)/F \) is separable. Therefore, as was noted in [5, Prop. 14.2, p. 59], also \((b')\) implies \((b)\). We will show \((b) \Rightarrow (c)\) and \((a') \Rightarrow (b')\), then the proof is completed:

\[
\begin{array}{c}
(b) \rightarrow (c) \rightarrow (a) \\
\downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \\
(b') \quad (c') \rightarrow (a')
\end{array}
\]

\((b) \Rightarrow (c): \) Suppose \( \overline{D} = gr(D)_{0} \) has a maximal subfield \( M \) Galois over \( F = gr(F)_{0} \). By Proposition 3.2, \( gr(D) \) has a maximal graded subfield \( M \) that is Galois over \( Z(gr(D)) \). But \( Z(gr(D)) = gr(F) \), as \( D \) is tame. Let \( M' \) be the tame lift of \( M \) over \( F \) (cf. §2.6), i.e. the unique tame Galois extension of \( F \) with \( gr(M') = M \). By the functoriality mentioned in §2.6 this \( M' \) is a maximal subfield of \( D \), since it splits \( D \) and

\[
[M' : F] = [gr(M') : gr(F)] = deg gr(D) = deg D.
\]

\((a') \Rightarrow (b'): \) Let \( M \) be a maximal subfield of \( D \) that is normal over \( F \). Then \( gr(M)/gr(F) \) is normal (cf. §2.6). As \( D \) is defectless over \( F \), i.e., equality (2.10) holds, \( M \) must also be defectless over \( F \). Thus,

\[
gr(M) : gr(F) = [M : F] = deg D = deg gr(D),
\]

showing that \( gr(M) \) is a maximal graded subfield of \( gr(D) \). By Proposition 3.3, \( gr(D) \) has a maximal graded subfield \( M' \) normal over \( F \) and such that \( M'_{0} \) is a maximal subfield of \( D = gr(D)_{0} \). By (2.4), \( M'_{0}/F \) is normal. \( \square \)

4. Tamely ramified noncrossed products

4.1. Simple residue fields. It is a fundamental question to determine which division algebras over a given field \( F \) are crossed products. As a corollary to
Theorem 1.1 we obtain an answer when $F$ is Henselian and division algebras over the residue field $K := \overline{F}$ are sufficiently well understood. Let $\text{cd} G_K$ denote the cohomological dimension of the absolute Galois group $G_K$ of $K$.

**Corollary 4.1.** Let $F$ be a Henselian field whose residue field $K$ is a local field\(^2\), real closed field, or satisfies $\text{cd} G_K \leq 1$, then every tame central division algebra over $F$ is a crossed product.

**Proof.** By Theorem 1.1 it suffices to show that $\overline{D}$ has a maximal subfield which is Galois over $K$. If $\text{cd} G_K \leq 1$, then $\overline{D} = Z(\overline{D})$ is a field which, since $D$ is tame, is Galois over $K$. If $K$ is a real closed field, then either $K$ or $K(\sqrt{-1})$ is a maximal subfield of $\overline{D}$ which is Galois over $K$. If $K$ is a local field then $Z(\overline{D})$ has extensions of arbitrary degree which are Galois over $K$, simply by composing $Z(\overline{D})$ with unramified extensions of $K$. This gives the desired result since over local fields every field of degree $\text{deg} \overline{D}$ over $Z(\overline{D})$ is a maximal subfield of $\overline{D}$. □

Note that (1) if $K$ is real closed the assertion can be proved directly without using Corollary 4.1; (2) examples of fields $K$ for which $\text{cd} G_K \leq 1$ include finite fields, and by Tsen’s theorem \cite[§19.4]{15}, function fields of curves over algebraically closed fields.

4.2. **Global residue fields.** Let $\Gamma$ be the value group of the Henselian valuation on $F$. We consider next the simplest residue field $K := \overline{F}$ for which noncrossed products exist over $F$, namely when $K$ is a global field \cite{3}.

The tame Brauer group $\text{TBr}(F)$ is described by a generalized Witt theorem \cite[Proposition 3.5]{1} as a direct sum:

$$\text{TBr}(F) \cong \text{Br}(K) \oplus \text{Hom}(G_K, \Delta/\Gamma) \oplus T,$$

where $\Delta$ is the divisible hull of $\Gamma$, and $T$ is a subgroup consisting of classes of some totally ramified division algebras. Moreover, the subgroup of $\text{TBr}(F)$ corresponding to $\text{Br}(K) \oplus \text{Hom}(G_K, \Delta/\Gamma)$ (resp. $\text{Br}(K)$) is the subgroup of classes of inertially split division algebras (resp. inertial division algebras), as described by a generalization of Witt’s theorem, see \cite[Satz 2.3]{17} or \cite[(5.4), Th. 5.6]{12}.

For fixed $\chi \in \text{Hom}(G_K, \Delta/\Gamma)$ and $\eta \in T$, we call the preimage of $\chi + \eta$ under (1.1) the fiber over $\chi + \eta$. Note that the isomorphism (1.1) is not entirely canonical and a different choice will give us a different fiber. However, none of our results depends on this choice. To describe the location of noncrossed products in $\text{TBr}(F)$, we ask for which $\chi$ and $\eta$ the fiber over $\chi + \eta$ contains noncrossed products? This problem was answered in \cite{8} and \cite{6} for the inertially split subgroup, i.e. when $\eta = 0$. In the following we combine Theorem 1.1 with the methods of \cite{8} and \cite{6} to answer this problem for the entire group $\text{TBr}(F)$.

---

\(^2\)We call a field *local* if it is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime $p$.

\(^3\)The decomposition of $\text{TBr}(F)$ is described in \cite{11} on the level of primary components.
To this end, we fix \( \chi \) and \( \eta \) and let \( \mathfrak{C} \subseteq \text{TBr}(F) \) be the fiber over \( \chi + \eta \). For any \( c \in \text{TBr}(F) \) we write \( \mathfrak{r} \) (resp. \( \mathbb{Z}(\mathfrak{r}) \)) for the class (resp. the center) of the residue algebra of the division algebra in \( c \). By Lemma 2.3, \( \mathbb{Z} := \mathbb{Z}_c := \mathbb{Z}(\mathfrak{r}) \) and \( \text{ind} \, c \vdash \text{ind} \mathfrak{r} \) are independent of the choice of \( c \in \mathfrak{C} \).

Note that \( \mathbb{Z}/\mathfrak{K} \) is abelian, as division algebras in \( \mathfrak{C} \) are tame. If \( \mathbb{Z}/\mathfrak{K} \) is cyclic, we say that it is of \emph{infinite height} if for every integer \( m \), \( \mathbb{Z}/\mathfrak{K} \) embeds into a cyclic extension \( L/\mathfrak{K} \) with \( [L : \mathfrak{K}] = m \). We will prove

\textbf{Theorem 4.2.} Let \( \mathbb{Z}/\mathfrak{K} \) be the maximal subextension of \( \mathbb{Z}/\mathfrak{K} \) of order prime to char \( \mathfrak{K} \). Then \( \mathfrak{C} \) consists of crossed products if and only if \( \mathbb{Z}/\mathfrak{K} \) is cyclic of infinite height.

Moreover, we will show that if \( \mathfrak{C} \) contains one noncrossed product it contains infinitely many of them.

Theorem 4.2 is already known if \( \mathfrak{C} \) consists of inertially split division algebras by [8, 6]. Furthermore, for such \( \mathfrak{C} \), [8, 6] prove the existence of index bounds which essentially separate crossed and noncrossed products within the fiber. We do not know if such bounds exist in fibers which are not inertially split. Nevertheless, we prove that unless \( \mathbb{Z}'/\mathfrak{K} \) is cyclic of infinite height there is a number \( m \) (depending only on \( \mathbb{Z}' \)) such that \( \mathfrak{C} \) contains noncrossed products of every index divisible by \( m \), see Remark 4.7.

\section*{4.3. Residue classes, Galois covers, and their local degrees.}

For \( m \in \mathbb{N} \), let \( \mathfrak{C}_m \) be the set of \( c \in \mathfrak{C} \) with \( m \mid \text{ind} \, c \), and \( \overline{\mathfrak{C}}_m \) (resp. \( \overline{\mathfrak{C}} \)) the set of residue classes of \( \mathfrak{C}_m \) (resp. \( \mathfrak{C} \)). Note that by Lemma 2.3, \( \alpha + c = \alpha^\mathfrak{K} + \mathfrak{r} \in \text{Br}(\mathbb{Z}) \) for all \( \alpha \in \text{Br}(\mathfrak{K}) \), \( c \in \text{Br}(F) \), where \( \alpha + c \) is defined via (4.1). Hence, for any \( \beta \in \overline{\mathfrak{C}}_m \),

\begin{equation}
\overline{\mathfrak{C}}_m = \{ \alpha^\mathfrak{K} + \beta : \alpha \in \text{Br}(\mathfrak{K}), \text{ind}(\alpha^\mathfrak{K} + \beta) = m \}.
\end{equation}

By Theorem 1.1 the following conditions are equivalent:

\begin{enumerate}
\item[(A)] \( \mathfrak{C}_m \) consists entirely of crossed products
\item[(\overline{A}_m)] Every class in \( \overline{\mathfrak{C}}_m \) has a splitting field \( L \), Galois over \( \mathfrak{K} \), with \( [L : \mathfrak{K}] = m \).
\end{enumerate}

We call \( L \supseteq \mathfrak{K} \) an \emph{\( m \)-cover} of \( \mathfrak{K}/\mathfrak{K} \) if \( L \) is Galois over \( \mathfrak{K} \) and \( [L : \mathfrak{K}] = m \). The cover \( L \) is cyclic if \( L/\mathfrak{K} \) is cyclic. For a prime \( \mathfrak{p} \) of \( \mathfrak{K} \), let \( K_\mathfrak{p} \) denote the completion at \( \mathfrak{p} \) and let \( [L : \mathfrak{K}]_\mathfrak{p} := [L_{\mathfrak{p}} : \mathfrak{K}_{\mathfrak{p}}] \) for any prime \( \mathfrak{p}' \) of \( L \) dividing \( \mathfrak{p} \).

Let \( \beta \in \text{Br}(\mathbb{Z}) \). Recall that for a prime \( \mathfrak{p} \) of \( \mathbb{Z} \), the index \( \text{ind}_{\mathfrak{p}} \beta \) of \( \beta^\mathfrak{p} \) equals its exponent \( \exp_{\mathfrak{p}} \beta \). By the Albert-Brauer-Hasse-Noether theorem [15, §18.4],

\begin{equation}
\beta \text{ is split by } L \text{ if and only if } \text{ind}_{\mathfrak{p}} \beta \mid [L : \mathfrak{K}]_\mathfrak{p} \text{ for every prime } \mathfrak{p} \text{ of } \mathbb{Z}.
\end{equation}

This allows to translate (\overline{A}_m) into conditions on the local degrees of covers, as follows. Let \( d_\mathfrak{p}(m) := m \) for every finite prime \( \mathfrak{p} \) of \( \mathfrak{K} \), \( d_\mathfrak{p}(m) = \gcd(m, 2) \) for every real prime \( \mathfrak{p} \) of \( \mathfrak{K} \) which is unramified in \( \mathfrak{Z} \), and \( d_\mathfrak{p}(m) := 1 \) otherwise.

Let \( S \) be a finite set of primes of \( \mathfrak{K} \). An \emph{\( m \)-cover} \( L \) of \( \mathfrak{K}/\mathfrak{K} \) has \emph{full local degree in} \( S \) if \( [L : \mathfrak{K}]_\mathfrak{p} = d_\mathfrak{p}(m) \) for all \( \mathfrak{p} \in S \).
Proposition 4.3. Assume $\mathcal{C}_m \neq \emptyset$. There exists a finite set $T$ of primes of $K$ such that $(B_m) \Rightarrow (\overline{A}_m) \Rightarrow (B'_m)$, where $(B_m)$ and $(B'_m)$ are the following conditions:

$(B_m)$ for every $S$, $Z/K$ has an $m$-cover with full local degree in $S$.

$(B'_m)$ for every $S$ disjoint from $T$, $Z/K$ has an $m$-cover with full local degree in $S$.

Proof. $(B_m) \Rightarrow (\overline{A}_m)$: Let $\beta \in \overline{A}_m$, and $S$ the set of primes $p$ of $K$ such that the restriction of $\beta$ to $\mathbb{Z}_p$ is nontrivial for some prime $\mathfrak{p} | p$ of $Z$. Applying $(B_m)$ to $S$, we obtain an $m$-cover $L$ of $Z$ with full local degree in $S$. By (4.3), $L$ splits $\beta$, as required.

$(\overline{A}_m) \Rightarrow (B'_m)$: For every $p | [Z : K]$, let $\mathfrak{p}_1^{(p)}, \mathfrak{p}_2^{(p)}, \ldots$ be any enumeration of the primes of $K$ so that $p^n \mid [Z : K]_{\mathfrak{p}_i^{(p)}}$ implies $p^n \mid [Z : K]_{\mathfrak{p}_i^{(p)}}$ for all $i, n \in \mathbb{N}$. Let $T$ be the set $\{\mathfrak{p}_1^{(p)}, \mathfrak{p}_2^{(p)} \mid p \text{ divides } m\}$.

Let $S$ be disjoint from $T$ and $\beta \in \overline{A}_m$. Define $S'$ to be the subset of primes $p \in S$ for which $\operatorname{ind}_p \beta < d_p(m)$ for all $\mathfrak{p} | p$. Since $S$ is disjoint from $T$, we can apply [6, Lemma 2.5] to obtain a class $\alpha \in \operatorname{Br}(K)$ such that $\operatorname{ind}_p \alpha^Z = m$, and $\operatorname{ind}_p \alpha^Z = d_p(m)$ for all $\mathfrak{p} \in S'$ and $\mathfrak{p} | p$. Furthermore, the proof of [6, Lemma 2.5] gives $\operatorname{ind}_p \alpha^Z = 1$ for all $\mathfrak{p} | p$ with $p \in S \setminus S'$.

Let $\gamma := \alpha^2 + \beta$. Since $\exp_{\mathfrak{p}} \beta < \exp_{\mathfrak{p}} \alpha^Z = d_p(m)$ for all $\mathfrak{p} | p$, $p \in S'$, we have $\operatorname{ind}_{\mathfrak{p}} \gamma = \exp_{\mathfrak{p}} \gamma = d_p(m)$ for all $\mathfrak{p} | p, p \in S'$. For every $p \in S \setminus S'$ there is a prime $\mathfrak{p} | p$, such that $\operatorname{ind}_{\mathfrak{p}} \beta = d_p(m)$, and hence, as $\operatorname{ind}_{\mathfrak{p}} \alpha^Z = 1$, one has $\operatorname{ind}_{\mathfrak{p}} \gamma = d_p(m)$. By enlarging $S$, we may assume that $S'$ contains a finite prime $p$ and hence that $\operatorname{ind} \gamma = \operatorname{ind}_{\mathfrak{p}} \gamma = m$, where $\overline{\mathfrak{p}} | p$.

By applying $(A_m)$ to $\gamma$, we obtain an $m$-cover of $Z/K$ which splits $\gamma$. Thus, by (4.3), $L$ has full local degree in $S$, as required. $\square$

Remark 4.4. Assume $\mathcal{C}_m \neq \emptyset$. The proof reveals that if $(B'_m)$ fails then there are in fact infinitely many noncrossed products in $\mathcal{C}_m$. Indeed if $(B'_m)$ fails for $S$, it fails for every set $S'$ which contains $S$ and is disjoint from $T$. Since $\gamma \in \overline{A}_m$ was constructed with nontrivial completions at all primes of $S'$, there are infinitely many $\gamma \in \overline{A}_m$ for which $(A_m)$ fails. Thus, there are infinitely many noncrossed products in $\mathcal{C}_m$.

4.4. Proof of Theorem 4.2. Set $\ell = \operatorname{char}(K)$. We use the following lemmas:

Lemma 4.5. ([6, Lemma 2.12]) There is a finite set $S_0$ such that every $p^n$-cover $L$ with full local degree in $S_0$ has abelian kernel $A = \operatorname{Gal}(L/Z)$, for which the conjugation action of $\operatorname{Gal}(Z/Z')$ on $A$ is trivial.

Lemma 4.6. Let $p \neq \ell$ be a prime. There exists a finite set $S_0$ disjoint from $T$ such that every $p^n$-cover $L$ of $Z/K$ with full local degree in $S_0$ contains a $p^n$-cover $L'$ of $Z'/K$.

Furthermore, if $L$ has full local degree in a set $S \supseteq S_0$ then so does $L'$. 

Proof. Let $S_0$ be as in Lemma 4.5 and let $B_\ell = \text{Gal}(Z/Z')$. Let $A_\ell = \text{Gal}(L/Z')$ since $|A|$ and $|B_\ell|$ are relatively prime, the group extension

$$1 \rightarrow A \rightarrow A_\ell \rightarrow B_\ell \rightarrow 1$$

is split by the Schur-Zassenhaus theorem. Since $B_\ell$ acts trivially on $A$, $A_\ell = A \oplus \hat{B}_\ell$ with $\hat{B}_\ell \cong B_\ell$. In particular, $A_\ell$ is abelian. Letting $G = \text{Gal}(L/K)$ and $B = \text{Gal}(Z'/K)$ the group extension $1 \rightarrow A_\ell \rightarrow G \rightarrow B \rightarrow 1$ induces an action of $B$ on $A_\ell$. Being a characteristic subgroup of $A_\ell$, $\hat{B}_\ell$ is $B$-invariant, hence normal in $G$. The fixed field $L' \subseteq L$ of $\hat{B}_\ell$ is then a cover of $Z'/K$ with associated group extension

$$1 \rightarrow A_\ell/\hat{B}_\ell \rightarrow G/\hat{B}_\ell \rightarrow B \rightarrow 1.$$ 

Since $A_\ell/\hat{B}_\ell \cong A$ it is a $p^n$-cover.

Full local degree in $S$ is inherited from $L$ since the completions of $L'$ and of $Z$ are linearly disjoint over a completion of $Z'$.

Proof of Theorem 4.2. \hfill \box

Fix $\beta \in \overline{T}$ and let $m = \text{ind} \beta$. By (4.2) we see that $\mathcal{C}_{m'} \neq \emptyset$ for every $m \mid m'$. For a prime $p \neq \ell$, let $p^n$ (resp. $2^s$ if $p = 2$) denote the number of $p$-power (resp. 2-power) roots of unity in $Z$ (resp. in $Z(\sqrt{-1})$).

Assume $Z'/K$ is cyclic of finite height or noncyclic. We first claim that there is a prime $p$ for which $(B'_{p^n})$ fails for all sufficiently large $n$. Assume first that $Z'/K$ is noncyclic and let $p \neq \ell$ be a prime for which the $p$-Sylow subgroup of $\text{Gal}(Z/K)$ is noncyclic. By [6, Proposition 3.3], $(B'_{p^n})$ fails for all $n > 2s_p$ if $p$ is odd and for $n > 2(r_2 + 2)$ if $p = 2$.

Assume next that $Z'/K$ is cyclic of finite height, and fix $m \in \mathbb{N}$ such that $Z'/K$ has no cyclic $m$-cover. We can further assume that $m$ is a prime power. Indeed, writing $m = \prod_p p^{s_p}$, $p$ prime, if there are cyclic $p^{s_p}$-covers for all $p \mid m$, their composite gives a cyclic $m$-cover of $Z'/K$. Let $m = p^{s_p}$ and let $n > n_p + s_p + 1$. By [8, Theorem 6.4] there is a set $S$ disjoint from $T$, such that $Z'/K$ has no $p^n$-cover with full local degree in $S$. By Lemma 4.3 there is a finite set $S_0$ such that $Z/K$ has no $p^n$-cover $L$ with full local degree in $S_0 \cup S$. Hence, $(B'_{p^n})$ fails, proving the claim.

Fix an $n$ for which $(B'_{p^n})$ fails and such that $p^n$ is at least the largest $p$-power dividing $m$. Letting $m' = \text{lcm}(m, p^n)$, by Lemma 4.5 every $m'$-cover with full local degree in a set $S \supseteq S_0$ has an abelian kernel and hence contains a $p^n$-cover with full local degree in $S$. Hence, $(B'_{m'})$ fails. As $\mathcal{C}_{m'} \neq \emptyset$, Proposition 4.3 implies that $(A_{m'})$ fails.

Conversely, assume that $Z'/K$ is cyclic of infinite height. By [8, Theorem 6.3], for every $m$ prime to $\ell$, $Z'/K$ has an $m$-cover with full local degree. Taking composites with $Z$, we get that $(B_m)$ holds for every $m$ prime to $\ell$. By [6, Lemma 2.10], $(B_m)$ holds for all $n$. By taking composites, we see that $(B_m)$ holds for all $m \in \mathbb{N}$. Thus, by Proposition 4.3 $(A_m)$ holds for all $m$. \hfill \box

Footnote 4: If $Z'/K$ is non-exceptional we can choose $n := n_p + s_p + 1$, see [8].
Remark 4.7. Assuming $C$ does not consist of crossed products, the proof reveals that there is $p^{np} \in \mathbb{P}$, $p$ prime, such that $(A_m)$ fails for every $m \in \mathbb{N}$ with $C_m \neq \emptyset$ and $p^{np} | m$. By Remark [4.4] for such $m$, $C_m$ contains infinitely many noncrossed products. If the $p$-Sylow subgroup of $\text{Gal}(\mathbb{Z}/K)$ is noncyclic and $p \neq \ell$, we can choose $n_p = 2s_p + 1$ if $p$ is odd and $n_2 = 2(r_2 + 2) + 1$ if $p = 2$. If $\mathbb{Z}'/K$ is cyclic with no cyclic $p^{b_p}$-cover, we can choose $n_p = k_p + s_p + 2$.

References

[1] E. Aljadeff, J. Sonn, and A. R. Wadsworth. Projective Schur groups of Henselian fields. *J. Pure Appl. Algebra*, 208:833–851, 2007.

[2] S. Amitsur. On central division algebras. *Israel J. Math.*, 12:408–420, 1972.

[3] E. Brussel. Noncrossed products and nonabelian crossed products over $\mathbb{Q}(t)$ and $\mathbb{Q}((t))$. *Amer. J. Math.*, 117:377–393, 1995.

[4] E. Brussel, K. McKinnie, and E. Tengan. Indecomposable and noncrossed product division algebras over function fields of smooth $p$-adic curves. *Adv. Math.*, 226:4316–4337, 2011.

[5] T. Hanke. *A Direct Approach to Noncrossed Product Division Algebras*. Dissertation, Universität Potsdam, 2001; available at arXiv:1109:1911.

[6] T. Hanke, D. Neftin, and J. Sonn. Noncrossed product bounds over Henselian fields. Submitted, 2013; available at arXiv:1209.0238.

[7] T. Hanke. A twisted laurent series ring that is a noncrossed product. *Israel J. Math.*, 150:199–203, 2005.

[8] T. Hanke and J. Sonn. The location of noncrossed products in the Brauer group of Laurent series fields over global fields. *Math. Ann.*, 350:313–337, 2010.

[9] T. Hanke. Galois subfields of inertially split division algebras. *J. Algebra*, 346:147–151, 2011.

[10] Y.-S. Hwang and A. R. Wadsworth. Algebraic extensions of graded and valued fields. *Communications in Algebra*, 27(2):821–840, 1999.

[11] Y.-S. Hwang and A. R. Wadsworth. Correspondences between valued division algebras and graded division algebras. *J. Algebra*, 220(1):73–114, 1999.

[12] B. Jacob and A. R. Wadsworth. Division algebras over Henselian fields. *J. Algebra*, 128:126–179, 1990.

[13] B. Jacob and A. R. Wadsworth. A new construction of noncrossed product algebras. *Trans. Amer. Math. Soc.*, 293:693–721, 1986.

[14] K. Mounirh and A. R. Wadsworth. Subfields of nondegenerate tame semiremified division algebras. *Communications in Algebra*, 39(2):462–485, 2011.

[15] R. Pierce. *Associative Algebras*. Springer-Verlag, New York, 1982.

[16] D. Saltman. Noncrossed products of small exponent. *Proc. Amer. Math. Soc.*, 68:165–168, 1978.

[17] W. Scharlau. Über die Brauer-Gruppe eines Hensel-Körpers. *Abh. Math. Sem. Univ. Hamburg*, 33:243–249, 1969.

[18] J.-P. Tignol. Cyclic and elementary abelian subfields of Malcev-Neumann division algebras. *J. Pure Appl. Algebra*, 42:199–220, 1986.

[19] J.-P. Tignol and A. R. Wadsworth. *Value Functions on Simple Algebras, and Associated Graded Algebras*, book in preparation.
Lehrstuhl D für Mathematik, RWTH Aachen Templergraben 64, D-52062 Aachen, Germany
E-mail address: hanke@math.rwth-aachen.de

Department of Mathematics, University of Michigan, Ann Arbor, 530 Church St., Ann Arbor, MI 48109-1043 USA
E-mail address: neftin@umich.edu

Department of Mathematics, University of California, San Diego, 9500 Gilman Drive, La Jolla, California 92093-0112 USA
E-mail address: arwadsworth@ucsd.edu