Concentration-compactness principle of singular Trudinger-Moser inequality involving $N$-Finsler–Laplacian operator

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Abstract: In this paper, suppose $F: \mathbb{R}^N \rightarrow [0, +\infty)$ be a convex function of class $C^2(\mathbb{R}^N \backslash \{0\})$ which is even and positively homogeneous of degree 1. We establish the Lions type concentration-compactness principle of singular Trudinger-Moser Inequalities involving $N$-Finsler–Laplacian operator. Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a smooth bounded domain. \{u_n\} $\subset W^{1,N}_0(\Omega)$ be a sequence such that anisotropic Dirichlet norm $\int_\Omega F^N(\nabla u_n)dx = 1$, $u_n \rightharpoonup u \neq 0$ weakly in $W^{1,N}_0(\Omega)$. Then for any $0 < p < p_N(u) := (1 - \int_\Omega F^N(\nabla u)dx)^{-\frac{N}{N-1}}$, we have
\[ \int_\Omega e^{\lambda_N(1-\frac{p}{p_N(u)})\frac{\|u_n\|^{N}}{F\alpha(x)^{\beta}}}dx < +\infty, \]
where $0 \leq \beta < N$, $\lambda_N = N \frac{\kappa_N}{\kappa^\perp_N}$ and $\kappa_N$ is the volume of a unit Wulff ball. This conclusion fails if $p \geq p_N(u)$. Furthermore, we also obtain the corresponding concentration-compactness principle in the entire Euclidean space $\mathbb{R}^N$.

Keywords: $N$-Finsler–Laplacian; Singular Trudinger-Moser inequality; Anisotropic Dirichlet norm; Concentration-compactness principle

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1 Introduction and main results

This paper is concerned with concentration-compactness-principle of singular Trudinger-Moser inequality involving $N$-Finsler-Laplacian operator. In order to give
our motivation, let’s recall some known results. Suppose \( \Omega \subset \mathbb{R}^N (N \geq 2) \) be a bounded smooth domain. \( W^{1,N}_0(\Omega) \hookrightarrow L^q(\Omega) \) for \( 1 \leq q < \infty \), but the embedding \( W^{1,N}_0(\Omega) \not\hookrightarrow L^\infty(\Omega) \), one can see the counterexample by taking \( u(x) = (-\ln |\ln |x||)_+ \) as \( \Omega \) is the unit ball. It was proposed independently by Yudovich [3], Pohozaev [4], Peetre [5] and Trudinger [6] that \( W^{1,N}_0(\Omega) \) is embedded in the Orlicz space \( L^{\varphi_\alpha}(\Omega) \) determined by the Young function \( \varphi_\alpha(t) = e^{\alpha|t|^{\frac{N}{N-1}}} - 1 \) for some positive number \( \alpha \). Moser [7] sharpened the results of Trudinger [6] and established the following inequality

\[
\sup_{u \in W^{1,N}_0(\Omega), \|\nabla u\|_{N} \leq 1} \int_{\Omega} e^{\alpha|u|^{\frac{N}{N-1}}} \, dx < +\infty, \forall \alpha \leq \alpha_N, \tag{1.1}
\]

where \( \alpha_N := \frac{N^{1/(N-1)}}{\omega_{N-1}} \) and \( \omega_{N-1} \) is the surface measure of the unit sphere in \( \mathbb{R}^N \). Moreover, the supremum in (1.1) is \( +\infty \) if \( \alpha > \alpha_N \). Inequality (1.1) is now referred as Trudinger-Moser inequality and plays an important role in geometric analysis and partial differential equations (see [8]). Using a rearrangement argument and a change of variables, Adimurthi-Sandeep [1] generalized the Trudinger-Moser inequality to a singular version as follows:

\[
\sup_{u \in W^{1,N}_0(\Omega), \int_{\Omega} |\nabla u|^{N-1} \, dx \leq 1} \int_{\Omega} e^{\alpha|u|^{\frac{N}{N-1}}} |x|^{\beta} \, dx < +\infty, \tag{1.2}
\]

where \( 0 \leq \beta < N, 0 < \alpha \leq \alpha_N(1-\frac{\beta}{N}) \), \( \alpha_N = \frac{N^{1/(N-1)}}{\omega_{N-1}} \) and \( \omega_{N-1} \) is the surface measure of the unit sphere in \( \mathbb{R}^N \). Moreover, this inequality is sharp, i.e., when \( \alpha > \alpha_N(1-\frac{\beta}{N}) \), the supremum is infinity. Trudinger-Moser inequalities for unbounded domains were proposed by D. M. Cao [9] in dimension two and J. M. do Ó [10], Adachi-Tanaka [11] in high dimension. Ruf [12] (for the case \( N = 2 \)), Li and Ruf [13] (for the general case \( N \geq 2 \)) obtained the Trudinger-Moser inequality in the critical case by replacing the Dirichlet norm with the standard Sobolev norm in \( W^{1,N}(\mathbb{R}^N) \). Obviously, if \( \beta = 0 \), then (1.2) reduces to the famous Trudinger-Moser inequality. Subsequently, the inequality (1.2) was extended to the entire Euclidean space \( \mathbb{R}^N \) by Adimurthi-Yang [2].

An important result is concentration-compactness principle with Trudinger-Moser inequality due to P. L Lions [26]. More precisely, let \( \{u_n\} \subset W^{1,N}_0(\Omega) \) be a se-
quence such that $\|\nabla u_n\|_N = 1$, $u_n \rightarrow u \neq 0$ weakly in $W^{1,N}_0(\Omega)$. Then for any $0 < p < p_N := (1 - \int_\Omega |\nabla u|^N)^{-\frac{1}{N-1}}$, it holds
\[
\int_\Omega e^{\alpha_N p |u_n|^N} \frac{\nabla u_n}{|\nabla u_n|^N} \, dx < +\infty.
\] (1.3)

Roughly speaking, the concentration-compactness principle tells us that, if a sequence $\{u_n\} \subset W^{1,N}_0(\Omega)$ converges weakly to some function $u \in W^{1,N}_0(\Omega)$, and does not concentrate at one point in $\Omega$, then an inequality like (1.3) holds along the sequence $\{u_n\}$, with a constant larger than $N\omega_{N-1}^{1/(N-1)}$, depending on $\|\nabla u\|_N$. In [26], the author only proved the case of $0 < p < p_N := (1 - \int_\Omega |\nabla u^*|^N)^{-\frac{1}{N-1}}$, we know $p_N^* \leq p_N$ by Poláč-Szegő inequality $\int_\Omega |\nabla u^*|^N \, dx \leq \int_\Omega |\nabla u|^N \, dx$, here $u^*$ is the radially decreasing symmetry of $u$. We should pay attention to the recent work in [27] by Černý et al. The authors present a new proof of this relevant principle for $0 < p < p_N$. Moreover, this approach allows one to treat functions with unrestricted boundary values in bounded domains. Concentration-compactness principle is a powerful tool in proving existence of extremal functions and existence of solutions to boundary value problems. It has been extended to the singular version in [14]. Their results can be stated as follows: let $\{u_n\} \subset W^{1,N}_0(\Omega)$ be a sequence such that $\|\nabla u_n\|_N = 1$, $u_n \rightarrow u \neq 0$ weakly in $W^{1,N}_0(\Omega)$, $\nabla u_n \rightarrow \nabla u$ a.e. in $\Omega$. Then for any $0 \leq \beta < N$ and $0 < p < (1 - \|\nabla u\|_N^{N})^{-\frac{1}{N-1}}$, there holds
\[
\int_\Omega e^{\alpha_N (1 - \frac{d}{N}) |u_n|^N} \frac{\nabla u_n}{|\nabla u_n|^N} \, dx < +\infty.
\] (1.4)

More concentration-compactness principle on unbounded domain and the Heisenberg group, we refer the reader to [21, 22, 23].

Another interesting research is that Trudinger-Moser inequality has been generalized to the case of anisotropic norm. In this paper, denote that $F \in C^2(\mathbb{R}^N \setminus 0)$ is a positive, convex and homogeneous function, $F_{\xi_i} = \frac{\partial F}{\partial \xi_i}$ and its polar $F^a(x)$ represents a Finsler metric on $\mathbb{R}^N$. We will replace the isotropic Dirichlet norm $\|u\|_{W^{1,N}_0(\Omega)} = (\int_\Omega |\nabla u|^N \, dx)^{\frac{1}{N}}$ by the anisotropic Dirichlet norm $(\int_\Omega F^N(\nabla u) \, dx)^{\frac{1}{N}}$ in $W^{1,N}_0(\Omega)$. In [17], Wang and Xia proved the following result:

**Theorem A.** Suppose $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a smooth bounded domain. Let $u \in W^{1,N}_0(\Omega)$ and $(\int_\Omega F^N(\nabla u) \, dx) \leq 1$. Then there exists a constant $C(N)$, such
that
\[ \int_{\Omega} e^{\lambda u^{N-1}} \, dx \leq C(N)|\Omega|, \] (1.5)
where \( 0 < \lambda \leq \lambda_N = N^{N-1} \frac{1}{\kappa_N} \) and \( \kappa_N = \{|x \in \mathbb{R}^N : F^o(x) \leq 1\} \). \( \lambda_N \) is sharp in the sense that if \( \lambda > \lambda_N \) then there exists a sequence \((u_n)\) such that \( \int_{\Omega} e^{\lambda u^{N-1}} \, dx \) diverges. In [28], the authors obtained the existence of extremal functions for the sharp geometric inequality (1.5).

For the minimization problem of \( \int_{\Omega} F^N(\nabla u) \, dx \), we know that its Euler equation contains an operator of the form
\[ Q_N u := \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (F^{N-1}(\nabla u) F_{\xi_i}(\nabla u)), \]
which is called \( N \)-Finsler-Laplacian operator. When \( N = 2 \) and \( F(\xi) = |\xi| \), \( Q_2 \) is just the ordinary Laplacian. The operator \( Q_N \) is closely related to a smooth, convex hypersurface in \( \mathbb{R}^N \). It has been studied in some literatures, see [19] and the references therein. We denote \( \kappa_N = \{|x \in \mathbb{R}^N : F^o(x) \leq 1\} \) is the volume of a unit Wulff ball. Recently, by using a convex symmetrization approach proposed in [19], which is the extension of Schwarz symmetrization in [30], X. Zhu [18] derived the following results.

**Theorem 1.1.** (see [18]) Let \( \Omega \subset \mathbb{R}^N (N \geq 2) \) be a smooth bounded domain. Then
\[ \sup_{u \in W^{1,N}_0(\Omega), \int_{\Omega} F^N(\nabla u) \, dx \leq 1} \int_{\Omega} e^{\lambda u^{N-1}} \frac{dx}{F^o(x)^{1/N}} < +\infty, \] (1.6)
and
\[ \sup_{u \in W^{1,N}(\mathbb{R}^N), \int_{\mathbb{R}^N} F^N(\nabla u) \, dx + \tau |u|^N \leq 1} \int_{\mathbb{R}^N} \Phi(\lambda |u|^N) \frac{dx}{F^o(x)^{1/N}} < \infty, \] (1.7)
where \( 0 \leq \beta < N, \tau > 0, \Phi(s) := e^{s} - \sum_{k=0}^{N-2} \frac{s^k}{k!}, 0 < \lambda \leq \lambda_N(1 - \beta \frac{1}{N}), \lambda_N = N^{N-1} \frac{1}{\kappa_N} \) and \( \kappa_N \) is the volume of a unit Wulff ball. Moreover, the above inequalities are sharp, i.e., when \( \lambda > \lambda_N(1 - \beta \frac{1}{N}) \), the supremum is infinity.

In this paper, we will establish the Lions type concentration-compactness principle of singular Trudinger-Moser Inequalities under the anisotropic norm.

**Theorem 1.2** Let \( \Omega \subset \mathbb{R}^N (N \geq 2) \) be a smooth bounded domain. \( \{u_n\} \subset W^{1,N}_0(\Omega) \)
be a sequence such that \( \int_{\Omega} F^N(\nabla u_n) dx = 1 \), \( u_n \rightharpoonup u \neq 0 \) weakly in \( W^{1,N}_0(\Omega) \). Then for any
\[
0 < p < p_N(u) := (1 - \int_{\Omega} F^N(\nabla u) dx)^{-\frac{1}{N-1}},
\]
we have
\[
\int_{\Omega} e^{\lambda_N(1 - \frac{\beta}{N})p|u_n|^{\frac{N}{N-1}}} dx < +\infty
\]
where \( 0 \leq \beta < N \), \( \lambda_N = \frac{N}{N-1} \kappa_N^{-\frac{1}{N-1}} \) and \( \kappa_N \) is the volume of a unit Wulff ball. Moreover, this conclusion fails if \( p \geq p_N(u) \).

**Theorem 1.3** Suppose \( \{u_n\} \subset W^{1,N}(\mathbb{R}^N) \) be a sequence such that \( \int_{\mathbb{R}^N}(F^N(\nabla u_n) + |u_n|^N) dx = 1 \), \( u_n \rightharpoonup u \neq 0 \) weakly in \( W^{1,N}(\mathbb{R}^N) \). Then for any
\[
0 < p < \bar{p}_N(u) := (1 - \int_{\mathbb{R}^N}(F^N(\nabla u) + |u|^N) dx)^{-\frac{1}{N-1}},
\]
we have
\[
\int_{\mathbb{R}^N} \Phi(\lambda_N(1 - \frac{\beta}{N})p|u_n|^{\frac{N}{N-1}}) dx < +\infty
\]
where \( 0 \leq \beta < N \), \( \Phi(s) := e^s - \sum_{k=0}^{N-2} \frac{s^k}{k!} \), \( \lambda_N = \frac{N}{N-1} \kappa_N^{-\frac{1}{N-1}} \) and \( \kappa_N \) is the volume of a unit Wulff ball. Moreover, this conclusion fails if \( p \geq \bar{p}_N(u) \).

This paper is organized as follows: In Section 2, we give some preliminaries. In Section 3, we establish the Lions type concentration-compactness principle of singular Trudinger-Moser Inequality under the anisotropic Dirichlet norm. In Section 4, we obtain the corresponding concentration-compactness principle in the entire Euclidean space \( \mathbb{R}^N \).

## 2 preliminaries

In this section, we will give some preliminaries for our use later.

Let \( F : \mathbb{R}^N \to [0, +\infty) \) be a convex function of class \( C^2(\mathbb{R}^N \setminus \{0\}) \) which is even and positively homogeneous of degree 1, so that
\[
F(t\xi) = |t|F(\xi) \quad \text{for any} \quad t \in \mathbb{R}, \ \xi \in \mathbb{R}^N.
\]
We also assume that \( F(\xi) > 0 \) for any \( \xi \neq 0 \) and \( Hess(F^2) \) is positive definite in \( \mathbb{R}^N \setminus \{0\} \). A typical example is \( F(\xi) = (\sum_i |\xi_i|^q)^{\frac{1}{q}} \) for \( q \in [1, \infty) \).
Let $F^o$ be the support function of $K := \{ x \in \mathbb{R}^N : F(x) \leq 1 \}$, which is defined by

$$F^o(x) := \sup_{\xi \in K} \langle x, \xi \rangle,$$

so $F^o : \mathbb{R}^N \to [0, +\infty)$ is also a convex, homogeneous function of class $C^2(\mathbb{R}^N \setminus \{0\})$. From [19], $F^o$ is dual to $F$ in the sense that

$$F^o(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)}, \quad F(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F^o(\xi)}.$$

Consider the map $\phi : S^{N-1} \to \mathbb{R}^N$, $\phi(\xi) = F^o(\xi)$. Its image $\phi(S^{N-1})$ is smooth, convex hypersurface in $\mathbb{R}^N$, which is called the Wulff shape (or equilibrium crystal shape) of $F$. Then $\phi(S^{N-1}) = \{ x \in \mathbb{R}^N | F^o(x) = 1 \}$ (see [16], Proposition 2.1).

We also give some simple properties of the function $F$, which follows directly from the assumption on $F$, also see [17, 25].

**Lemma 2.1.** There hold

(i) $|F(x) - F(y)| \leq F(x + y) \leq F(x) + F(y)$;

(ii) $\frac{1}{C} \leq |\nabla F(x)| \leq C$ and $\frac{1}{C} \leq |\nabla F^o(x)| \leq C$ for some $C > 0$ and any $x \neq 0$;

(iii) $\langle x, \nabla F(x) \rangle = F(x)$, $\langle x, \nabla F^o(x) \rangle = F^o(x)$ for any $x \neq 0$.

**Remark 2.2.** Since $\text{Hess}(F^2)$ is positive definite in $\mathbb{R}^N \setminus \{0\}$. Then by Xie and Gong [29], $\text{Hess}(F^N)$ is also positive definite in $\mathbb{R}^N \setminus \{0\}$. Moreover, for a bounded smooth domain $\Omega \subset \mathbb{R}^N (N \geq 2)$, we know that $Q_2$ is a uniformly elliptic operator in any compact subsets of $\Omega \setminus \{x | \nabla u(x) = 0\}$, see [16].

We will use the convex symmetrization which is defined in [19]. The convex symmetrization generalizes the Schwarz symmetrization (see [30]). Let us consider a measured function $u$ on $\Omega \subset \mathbb{R}^N$, one dimensional decreasing rearrangement of $u$ is

$$u^\sharp(t) = \sup\{s \geq 0 : | \{ x \in \Omega : |u(x)| > s \} | > t \} \quad \text{for} \quad t \in \mathbb{R}. \quad (2.2)$$

The convex symmetrization of $u$ with respect to $F$ is defined as

$$u^*(x) = u^\sharp(\kappa_N F^o(x)^N) \quad \text{for} \quad x \in \Omega^* \quad. \quad (2.3)$$

Here $\kappa_N F^o(x)^N$ is just the Lebesgue measure of a homothetic Wulff ball with radius $F^o(x)$ and $\Omega^*$ is the homothetic Wulff ball centered at the origin having the same
measure as $\Omega$. In [19], the authors proved a Pólya-Szegö principle and a comparison result for solutions of the Dirichlet problem for elliptic equations for the convex symmetrization, which generalizes the classical results for Schwarz symmetrization due to Talenti [30].

**Lemma 2.3.** (see [19]) If $u \in W^{1,p}_0(\Omega)$ for $p \geq 1$. Then $u^* \in W^{1,p}_0(\Omega^*)$ and

$$\int_{\Omega} F^p(\nabla u) dx \geq \int_{\Omega^*} F^p(\nabla u^*) dx.$$  

Next, we denote $D_u(\mu) = \{ x \in \Omega : |u(x)| \geq \mu \}$. It is easily derived

$$u^*(x) = \sup \{ \mu : F^o(x) \leq r, \kappa_N r^N = |D_u(\mu)| \}. \quad (2.4)$$

We claim: for any $p \in [0, N)$, it holds

$$\left( \frac{1}{(F^o)^p} \right) \ast (x) \leq \frac{1}{F^o(x)^p}. \quad (2.5)$$

In fact, by (2.4), we have

$$\left( \frac{1}{(F^o)^p} \right) \ast (x) = \sup \{ \mu : F^o(x) \leq r, \kappa_N r^N = |D(F^o)^{-p}(\mu)| \}. \quad (2.6)$$

According to our notation, we have

$$D_{(F^o)^{-p}}(\mu) = \{ x \in \Omega : (F^o(x))^{-p} \geq \mu \}$$

$$= \{ x \in \Omega : F^o(x) \leq \frac{1}{\mu^{1/p}} \}$$

$$\subset \{ x \in \mathbb{R}^N : F^o(x) \leq \frac{1}{\mu^{1/p}} \} \quad (2.7)$$

Thus $|D_{(F^o)^{-p}}(\mu)| \leq \kappa_N \frac{1}{\mu^{N/p}}$, combing with (2.6), we have $\kappa_N r^N \leq \kappa_N \frac{1}{\mu^{N/p}}$, so $\mu \leq \frac{1}{rp}$. Therefore

$$\left( \frac{1}{(F^o)^p} \right) \ast (x) = \sup \{ \mu : F^o(x) \leq r, \kappa_N r^N = |D_{(F^o)^{-p}}(\mu)| \}$$

$$\leq \inf \{ \frac{1}{r^p} : F^o(x) \leq r \} = \frac{1}{(F^o(x))^p} \quad (2.8)$$

Our claim is proved.

Now suppose $h$ and $\varphi$ be real-valued functions defined for $x \in \Omega$ with $h$ integrable over $\Omega$. Let $\varphi$ be measurable over $\Omega$ and satisfy the condition $-\infty < \varphi_0 \leq \varphi(x) \leq \varphi_1 < \infty$, set $D_{\varphi}(t) = \{ x \in \Omega : \varphi(x) \geq t \}$. Then Lemma 2.3 in [20] implies

$$\int_{\Omega} h\varphi dx = \varphi_0 \int_{\Omega} h dx + \int_{\varphi_0}^{\varphi_1} dt \int_{D_{\varphi}(t)} h dx. \quad (2.9)$$
Lemma 2.4. Assume that \( f : [\varphi_0, \varphi_1] \rightarrow \mathbb{R}^+ \) is an increasing function. Then we have
\[
\int_{\Omega} hf(\varphi)dx \leq \int_{\Omega^*} h^* f(\varphi^*)dx. \tag{2.10}
\]

Proof. On one hand,
\[
\int_{\Omega} hf(\varphi)dx = f(\varphi_0) \int_{\Omega} hdx + \int_{\frac{f(\varphi_1)}{f(\varphi_0)}} dt \int_{\{x \in \Omega : f(\varphi) \geq t\}} hdx = f(\varphi_0) \int_{\Omega} hdx + \int_{\frac{f(\varphi_1)}{f(\varphi_0)}} dt \int_{\{x \in \Omega : \varphi \geq f^{-1}(t)\}} hdx. \tag{2.11}
\]

On the other hand, since inf \( \varphi = \inf \varphi^* \) and sup \( \varphi = \sup \varphi^* \),
\[
\int_{\Omega^*} h^* f(\varphi^*)dx = f(\varphi_0) \int_{\Omega^*} h^* dx + \int_{\frac{f(\varphi_1)}{f(\varphi_0)}} dt \int_{\{x \in \Omega^* : f(\varphi^*) \geq t\}} h^* dx = f(\varphi_0) \int_{\Omega^*} h^* dx + \int_{\frac{f(\varphi_1)}{f(\varphi_0)}} dt \int_{\{x \in \Omega^* : \varphi^* \geq f^{-1}(t)\}} h^* dx. \tag{2.12}
\]

Notice that \( \int_{\Omega} hdx = \int_{\Omega^*} h^* dx \) and Lemma 2.2 in [20] implies
\[
\int_{\{x \in \Omega : \varphi \geq f^{-1}(t)\}} hdx \leq \int_{\{x \in \Omega^* : \varphi^* \geq f^{-1}(t)\}} h^* dx. \tag{2.13}
\]
The assertion now follows immediately.

3 Lions type concentration-compactness principle in bounded domain

In this section, we will prove Lions type concentration-compactness principle of singular Trudinger-Moser Inequalities under the anisotropic Dirichlet norm, which can be referred to [22] and Lemma 2.3 in [28]. This is the extension of Concentration-Compactness Principle due to P. L. Lions [26].

Proof of Theorem 1.2. From the weak semicontinuity of the norm in \( W^{1,N}_0(\Omega) \), we have
\[
\int_{\Omega} F^N(\nabla u)dx \leq \lim inf_{n \to \infty} \int_{\Omega} F^N(\nabla u_n)dx = 1.
\]

Firstly, let \( 0 < \int_{\Omega} F^N(\nabla u)dx < 1 \), we give the proof by contradiction. Assume that there exists some \( p_1 < p_N(u) \) and a subsequence of \( \{u_n\} \) (still denote \( u_n \)) such that
\[
\sup_n \int_{\Omega} e^{\lambda N(1-\frac{1}{p_1})p_1|u_n|^\frac{N}{N-1}} dx = +\infty. \tag{3.1}
\]
Set $\Omega^p = \{ x \in \Omega : |u_n(x)| \geq L \}$, where $L$ is a positive constant. Let $v_n = |u_n| - L$, for any $\epsilon > 0$ and some positive constant $C$, by Young inequality $a^\beta b^{\alpha} \leq \epsilon a + \epsilon^{-\frac{\alpha}{\beta}} b$, $\frac{1}{q} + \frac{1}{q} = 1$ we have

$$
|u_n|^{\frac{N}{N-1}} \leq |v_n|^{\frac{N}{N-1}} + C|v_n|^{\frac{N}{N-1}}L + L^{\frac{N}{N-1}}
$$

$$
= |v_n|^{\frac{N}{N-1}} + C(|v_n|^{\frac{N}{N-1}})^{\frac{q}{q-1}} (L^{\frac{N}{N-1}})^{\frac{q-1}{q}} + L^{\frac{N}{N-1}}
$$

$$
\leq |v_n|^{\frac{N}{N-1}} + C \cdot \left( \frac{\epsilon}{C} |v_n|^{\frac{N}{N-1}} + \left( \frac{\epsilon}{C} \right)^{\frac{1}{q-1}} L^{\frac{N}{N-1}} \right) + L^{\frac{N}{N-1}}
$$

$$
=: (1 + \epsilon) v_n^{\frac{N}{N-1}} + C_L^{\frac{N}{N-1}}. \quad (3.2)
$$

Since $0 \leq \beta < N$, we have

$$
\int_{\Omega} e^{\lambda N(1-\frac{\beta}{N})p_1 u_n|^{\frac{N}{N-1}}} F^\beta(x) dx = \int_{\Omega \setminus \Omega_0} e^{\lambda N(1-\frac{\beta}{N})p_1 u_n|^{\frac{N}{N-1}}} F^\beta(x) dx + \int_{\Omega \setminus \Omega_0} e^{\lambda N(1-\frac{\beta}{N})p_1 u_n|^{\frac{N}{N-1}}} F^\beta(x) dx
$$

$$
\leq \int_{\Omega \setminus \Omega_0} e^{\lambda N(1-\frac{\beta}{N})p_1 u_n|^{\frac{N}{N-1}}} F^\beta(x) dx + e^{\lambda N(1-\frac{\beta}{N})p_1 L^{\frac{N}{N-1}}} \int_{\Omega \setminus \Omega_0} \frac{1}{F^\beta(x)} dx
$$

$$
\leq \int_{\Omega \setminus \Omega_0} e^{\lambda N(1-\frac{\beta}{N})p_1 u_n|^{\frac{N}{N-1}}} F^\beta(x) dx + C(L, N, \beta), \quad (3.3)
$$

and then

$$
\sup_n \int_{\Omega_L^p} e^{\lambda N(1-\frac{\beta}{N})p_1 u_n|^{\frac{N}{N-1}}} F^\beta(x) dx = +\infty. \quad (3.4)
$$

From (3.2), we have

$$
\int_{\Omega_L^p} e^{\lambda N(1-\frac{\beta}{N})p_1 u_n|^{\frac{N}{N-1}}} F^\beta(x) dx \leq e^{\lambda N(1-\frac{\beta}{N})p_1 C_L^{\frac{N}{N-1}}} \int_{\Omega_L^p} \frac{e^{(1+\epsilon)\lambda N(1-\frac{\beta}{N})p_1 u_n|^{\frac{N}{N-1}}}}{F^\beta(x)} dx. \quad (3.5)
$$

Thus

$$
\sup_n \int_{\Omega_L^p} e^{\lambda N(1-\frac{\beta}{N})p_1 u_n|^{\frac{N}{N-1}}} F^\beta(x) dx = \sup_n \int_{\Omega_L^p} e^{\lambda N(1-\frac{\beta}{N})p_1 u_n|^{\frac{N}{N-1}}} F^\beta(x) dx = +\infty, \quad (3.6)
$$

where $p_1 = (1 + \epsilon)p_1 < p_N(u)$. Now, we define

$$
T^L(u) = \min\{L, |u|\} \text{sign}(u) \quad \text{and} \quad T_L(u) = u - T^L(u)
$$

From the assumption $0 < \int_{\Omega} F^N(\nabla u) dx < 1$, we choose $L$ large enough such that

$$
1 - \int_{\Omega} F^N(\nabla u) dx > \left( \frac{p_1}{p_N(u)} \right)^{N-1}. \quad (3.7)
$$
Since $T_L(u_n)$ is bounded in $W^{1,N}_0(\Omega)$, hence, up to a subsequence, $T_L(u_n) \rightharpoonup T_L(u)$ in $W^{1,N}_0(\Omega)$ and $T_L(u_n) \to T_L(u)$ a.e. in $\Omega$. Combining (3.6) and (1.6), up to a subsequence, we have

$$
\limsup_{n \to \infty} \int_{\Omega_L^n} F^N(\nabla v_n) dx = \limsup_{n \to \infty} \int_{\Omega_L^n} F^N(\nabla T_L(u_n)) dx \geq 1,
$$
which implies

$$
\int_{\Omega_L^n} F^N(\nabla v_n) dx = \int_{\Omega} F^N(\nabla T_L(u_n)) dx \geq \left( \frac{1}{p_1} \right)^{N-1} + o_n(1). \tag{3.8}
$$
Thus,

$$
\left( \frac{1}{p_1} \right)^{N-1} + \int_{\Omega} F^N(\nabla T_L(u_n)) dx + o_n(1)
\leq \int_{\Omega} F^N(\nabla T_L(u_n)) dx + \int_{\Omega_\Omega^n} F^N(\nabla u_n) dx
= \int_{\Omega_\Omega^n} F^N(\nabla u_n) dx + \int_{\Omega_\Omega^n} F^N(\nabla u_n) dx = 1.
$$
The above inequality, the weak lower semicontinuity of norm, and (3.7) yield

$$
\bar{p}_1 \geq \frac{1}{(1 - \liminf_{n \to \infty} \int_{\Omega} F^N(\nabla T_L(u_n)) dx)^{1 \over N-1}}
\geq \frac{1}{(1 - \int_{\Omega} F^N(\nabla T_L(u)) dx)^{1 \over N-1}}
\geq \frac{\bar{p}_1}{p_N(u)} \frac{1}{(1 - \int_{\Omega} F^N(\nabla u) dx)^{1 \over N-1}} = \bar{p}_1,
$$
which is a contradiction. Secondly, let $\int_{\Omega} F^N(\nabla u) dx = 1$, we can repeat the process of first case and get

$$
\sup_n \int_{\Omega_L^n} \frac{e^{\lambda N(1 - \beta N \bar{p}_n \beta \nabla v_n)} F^N(\nabla v_n) dx}{F^N(\nabla v_n)} dx = +\infty,
$$
where $\bar{p}_1 = (1 + \epsilon)p_1$. Then we have

$$
\limsup_{n \to \infty} \int_{\Omega_L^n} F^N(\nabla v_n) dx = \limsup_{n \to \infty} \int_{\Omega} F^N(\nabla T_L(u_n)) dx \geq \left( \frac{1}{p_1} \right)^{N-1},
$$
thus,

$$
\int_{\Omega} F^N(\nabla T_L(u)) dx \leq \liminf_{n \to \infty} \int_{\Omega} F^N(\nabla T_L(u_n)) dx
= 1 - \limsup_{n \to \infty} \int_{\Omega} F^N(\nabla T_L(u_n)) dx
\leq 1 - \left( \frac{1}{\bar{p}_1} \right)^{N-1}. \tag{3.9}
$$
On the other hand, since \( \int_{\Omega} F^N(\nabla u)dx = 1 \), we can choose \( L > 0 \) in such a way that
\[
\int_{\Omega} F^N(\nabla T_L(u))dx > 1 - \frac{1}{2} \left( \frac{1}{p_1} \right)^{N-1},
\]
which is contradiction, and the proof is finished in second case.

Next, we prove the sharpness of \( p_N(u) \). It suffices to construct a sequence \( \{u_n\} \subset W^{1,N}_0(\Omega) \) and a function \( u \in W^{1,N}_0(\Omega) \) such that
\[
\int_{\Omega} F^N(\nabla u_n)dx = 1, \quad u_n \rightharpoonup u \neq 0 \quad \text{in} \quad W^{1,N}_0(\Omega),
\]
\[
\left( \int_{\Omega} F^N(\nabla u)dx \right)^{\frac{1}{N}} = \delta < 1 \quad \text{and} \quad \int_{\Omega} \frac{e^{\lambda_N(1-\frac{\delta}{N})(1-\delta^N)\frac{N}{N-1} |u_n|^N}}{F^o(x)^{\beta}}dx \to +\infty.
\]

For \( n \in \mathbb{N} \), let \( r > 0 \), we define
\[
\omega_n(x) = \begin{cases} \frac{1}{N} \kappa_N^N n^{N-1}, & 0 \leq F^o(x) \leq re^{-\frac{N}{N}}, \\ \kappa_N^N \log(r/F^o(x))n^{-\frac{1}{N}}, & re^{-\frac{N}{N}} \leq F^o(x) \leq r, \\ 0, & F^o(x) \geq r. \end{cases}
\]
A straightforward calculation yields
\[
\int_{\Omega} F^N(\nabla \omega_n)dx = 1, \quad \omega_n \rightharpoonup 0 \quad \text{in} \quad W^{1,N}_0(\Omega).
\]
Set \( R = 3r \), define
\[
u(x) = \begin{cases} A, & 0 \leq F^o(x) \leq \frac{2}{3}R, \\ 3A - 3A F^o(x), & \frac{2}{3}R \leq F^o(x) \leq R, \\ 0, & F^o(x) \geq R, \end{cases}
\]
where \( A \) is a positive constant to be chosen in such a way that \( (\int_{\Omega} F^N(\nabla u)dx)^{\frac{1}{N}} = \delta < 1 \). Denote \( \mathcal{W}(R) = \{ x \in \mathbb{R}^N : F^o(x) \leq R \} \) be a Wulff ball centered at the origin. Let \( u_n = u + (1 - \delta^N)\frac{1}{N} \omega_n \), since \( \nabla u \) and \( \nabla \omega_n \) have disjoint supports, we have
\[
\int_{\Omega} F^N(\nabla u_n)dx = \int_{\mathcal{W}(R)} F^N(\nabla u)dx + (1 - \delta^N) \int_{\mathcal{W}(R)} F^N(\nabla \omega_n)dx = 1
\]
and $u_n \rightarrow u$ in $W^{1,N}_0(\Omega)$. Thus

$$\int_\Omega e^{\lambda N (1 - \frac{\beta}{N}) (1 - \delta N)} \frac{F_0(x)^\beta}{F_0(x)} \left| u_n \right|^N dx \geq \int_{W(re^{-\frac{\mu}{\lambda}})} e^{\lambda N (1 - \frac{\beta}{N}) (1 - \delta N)} \frac{F_0(x)^\beta}{F_0(x)} \left| u_n \right|^N dx$$

$$= \int_{W(re^{-\frac{\mu}{\lambda}})} e^{\lambda N (1 - \frac{\beta}{N}) (1 - \delta N)} \frac{F_0(x)^\beta}{F_0(x)} \left| u_n \right|^N dx$$

$$= e^{\frac{\lambda N}{N - \beta} \kappa_N \frac{1}{\kappa_N} (1 - \frac{\beta}{N}) (C + \frac{N - 1}{N - \beta})} \int_{W(re^{-\frac{\mu}{\lambda}})} \frac{1}{F_0(x)^\beta} dx$$

$$\geq e^{C_1 + \frac{(1 - \frac{\beta}{N}) N - 1}{N - \beta}} e^{\frac{N - 1}{N - \beta} \kappa_N} e^{\frac{N - 1}{N - \beta} \kappa_N} e^{(1 - \frac{\beta}{N}) n} \rightarrow +\infty (n \rightarrow +\infty)$$

where $0 \leq \beta < N$ and $C, C_1, C_2$ are positive constants.

\[\square\]

4 Lions type concentration-compactness principle in $\mathbb{R}^N$

As the similar procedure in Theorem 1.2, we can immediately get Theorem 1.3.

Proof of Theorem 1.3. Since

$$\int_{\mathbb{R}^N} (F^N(\nabla u) + |u|^N) dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} (F^N(\nabla u_n) + |u_n|^N) dx = 1.$$ 

We discuss it in two cases.

Case 1: Let $0 < \int_{\mathbb{R}^N} (F^N(\nabla u) + |u|^N) dx < 1$, we give the proof by contradiction. Assume that there exists some $p_1 < \bar{p}_N(u)$ and a subsequence of $\{u_n\}$ (still denote $u_n$) such that

$$\sup_n \int_{\mathbb{R}^N} \Phi(\lambda N (1 - \frac{\beta}{N}) p_1 |u_n|^\frac{N}{N - 1}) dx = +\infty. \quad (4.1)$$

Set $\Omega_L^u = \{ x \in \mathbb{R}^N : |u_n(x)| \geq L \}$, where $L$ is a positive constant. Since $0 \leq \beta < N$,
we have
\[ \int_{\mathbb{R}^N} \Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |u_n|^{\frac{N}{N-1}}) F^o(x)^N \, dx = \int_{\Omega^*_L} \Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |u_n|^{\frac{N}{N-1}}) F^o(x)^N \, dx + \int_{\mathbb{R}^N \setminus \Omega^*_L} \Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |u_n|^{\frac{N}{N-1}}) F^o(x)^N \, dx \]
\[ \leq \int_{\Omega^*_L} \Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |u_n|^{\frac{N}{N-1}}) F^o(x)^N \, dx + C \int_{\mathbb{R}^N \setminus \Omega^*_L} |u_n|^{\frac{N}{N-1}} F^o(x)^N \, dx \]
\[ \leq \int_{\Omega^*_L} \Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |u_n|^{\frac{N}{N-1}}) F^o(x)^N \, dx + C \int_{F^o(x) \leq 1} \frac{1}{F^o(x)^\beta} \, dx + C \int_{F^o(x) > 1} |u_n|^N \, dx \]
\[ \leq \int_{\Omega^*_L} \Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |u_n|^{\frac{N}{N-1}}) F^o(x)^N \, dx + C(p_1, L, N, \beta), \quad (4.2) \]

and then
\[ \sup_n \int_{\Omega^*_L} \Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |u_n|^{\frac{N}{N-1}}) F^o(x)^N \, dx = +\infty. \quad (4.3) \]

Let \( v_n = u_n - L \), for any \( \epsilon > 0 \), we have
\[ |u_n|^{\frac{N}{N-1}} \leq (1 + \epsilon) v_n^{\frac{N}{N-1}} + C_o L^{\frac{N}{N-1}}. \quad (4.4) \]

Notice that
\[ \int_{\Omega^*_L} \Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |u_n|^{\frac{N}{N-1}}) F^o(x)^N \, dx \leq e^{\lambda_N (1 - \frac{\beta}{N}) p_1 C_o L^{\frac{N}{N-1}}} \int_{\Omega^*_L} e^{(1+\epsilon)\lambda_N (1 - \frac{\beta}{N}) p_1 |v_n|^{\frac{N}{N-1}}} F^o(x)^N \, dx. \quad (4.5) \]

Thus
\[ \sup_n \int_{\Omega^*_L} e^{\lambda_N (1 - \frac{\beta}{N}) p_1 |v_n|^{\frac{N}{N-1}}} F^o(x)^N \, dx = \sup_n \int_{\Omega^*_L} e^{\lambda_N (1 - \frac{\beta}{N}) p_1 |v_n|^{\frac{N}{N-1}}} F^o(x)^N \, dx = +\infty, \quad (4.6) \]

where \( \overline{p}_1 = (1 + \epsilon) p_1 < \overline{p}_N(u) \). Now, we define
\[ T^L(u) = \min\{L, \|u\| \} \text{sign}(u) \quad \text{and} \quad T_L(u) = u - T^L(u) \]

and choose \( L \) so large that
\[ \frac{1 - \int_{\mathbb{R}^N} (F^N(\nabla u) + \|u\|^N) \, dx}{1 - \int_{\mathbb{R}^N} (F^N(\nabla T^L(u)) + \|T^L(u)\|^N) \, dx} > \left( \frac{\overline{p}_1}{\overline{p}_N(u)} \right)^{N-1}. \quad (4.7) \]

Since \( T^L(u_n) \) is bounded in \( W^{1,N}(\mathbb{R}^N) \), hence, up to a subsequence, \( T^L(u_n) \rightharpoonup T^L(u) \) in \( W^{1,N}(\mathbb{R}^N) \) and \( T^L(u_n) \rightarrow T^L(u) \) a. e. in \( \mathbb{R}^N \). Combing (4.6) and (4.7), up to a
subsequence, we have
\[
\limsup_{n \to \infty} \int_{\Omega} F^N \left( \frac{N-1}{p_1^N} \nabla v_n \right) dx \geq 1.
\]
Thus
\[
\int_{\Omega} F^N(\nabla v_n) dx = \int \Omega F^N(\nabla T_L(u_n)) dx \geq \left( \frac{1}{p_1} \right)^{N-1} + o_n(1). \tag{4.8}
\]
Then we have
\[
\left( \frac{1}{p_1} \right)^{N-1} + \int_{\mathbb{R}^N} F^N(\nabla T_L(u_n)) dx + \int_{\mathbb{R}^N} |T_L(u_n)|^N dx + o_n(1)
\leq \left( \frac{1}{p_1} \right)^{N-1} + \int_{\mathbb{R}^N} F^N(\nabla T_L(u_n)) dx + \int_{\mathbb{R}^N} |u_n|^N dx + o_n(1)
\leq \int_{\mathbb{R}^N} F^N(\nabla T_L(u_n)) dx + \int_{\mathbb{R}^N} F^N(\nabla T_L(u_n)) dx + \int_{\mathbb{R}^N} |u_n|^N dx
= \int_{\Omega} F^N(\nabla T_L(u_n)) dx + \int_{\mathbb{R}^N \setminus \Omega} F^N(\nabla u_n) dx + \int_{\mathbb{R}^N} |u_n|^N dx = 1.
\]
From (4.7), it holds
\[
\bar{p}_1 \geq \frac{1}{(1 - \liminf_{n \to \infty} \int_{\mathbb{R}^N} (F^N(\nabla T_L(u_n)) + |T_L(u_n)|^N) dx)^{-\frac{1}{N-1}}}
\geq \frac{1}{(1 - \int_{\mathbb{R}^N} (F^N(\nabla T_L(u)) + |T_L(u)|^N) dx)^{-\frac{1}{N-1}}}
> \frac{1}{\bar{p}_N(u) (1 - \int_{\mathbb{R}^N} (F^N(\nabla u) + |u|^N) dx)^{-\frac{1}{N-1}}} = \bar{p}_1,
\]
which is a contradiction. The proof is finished in the first case.

Case 2: Let \( \int_{\mathbb{R}^N} (F^N(\nabla u) + |u|^N) dx = 1 \). Since \( u_n \rightharpoonup u \) and \( W^{1,N}(\mathbb{R}^N) \) is a uniformly convex Banach space, we know that \( u_n \to u \) in \( W^{1,N}(\mathbb{R}^N) \). Thus, by Proposition 1 in [16], there exists some \( v \in W^{1,N}(\mathbb{R}^N) \), such that up to a subsequence, \( |u_n(x)| \leq v(x) \) a.e. in \( W^{1,N}(\mathbb{R}^N) \). Denote
\[
D = \{ x \in \mathbb{R}^N : \int_{\mathbb{R}^N} F^N(\nabla v) dx \leq 1, v(x) > 1 \},
\]
we have
\[
\int_{\mathbb{R}^N \setminus D} \Phi(\lambda_N(1 - \frac{\beta}{N}p_1|v|^\frac{N}{\beta})) \frac{|v|^\frac{N}{\beta}}{F_0(\beta)} dx \leq C(p_1, N, \beta).
\]
Indeed,
\[ \int_{\mathbb{R}^N \setminus D} \frac{\Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |v|^N)}{F^\alpha(x)^3} \, dx \]
\[ \leq \int_{\{v(x) > 1\}} \frac{1}{F^\alpha(x)^3} \sum_{k=N-1}^{\infty} \frac{[\lambda_N (1 - \frac{\beta}{N}) p_1]^k |u|^k}{k!} \, dx \]
\[ \leq \int_{\{v(x) > 1\}} \frac{1}{F^\alpha(x)^3} \sum_{k=N-1}^{\infty} \frac{[\lambda_N (1 - \frac{\beta}{N}) p_1]^k |u|^N}{k!} \, dx \]
\[ \leq \int_{\{F^\alpha(x) \geq 1\}} \sum_{k=N-1}^{\infty} \frac{[\lambda_N (1 - \frac{\beta}{N}) p_1]^k |u|^N}{k!} \, dx + \int_{\{F^\alpha(x) < 1\}} \sum_{k=N-1}^{\infty} \frac{[\lambda_N (1 - \frac{\beta}{N}) p_1]^k |u|^N}{k!} \, dx \]
\[ \leq C(p_1, N, \beta) \]

From Lemma 2.4 and (2.5), we have
\[ \int_{\mathbb{R}^N} \frac{\Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |u_n|^N)}{F^\alpha(x)^3} \, dx \]
\[ \leq \int_{\mathbb{R}^N} \frac{\Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |v|^N)}{F^\alpha(x)^3} \, dx \]
\[ \leq \int_{\mathbb{R}^N \setminus D} \frac{\Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |v|^N)}{F^\alpha(x)^3} \, dx + \int_{D} \frac{\Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |v|^N)}{F^\alpha(x)^3} \, dx \]
\[ \leq C(p_1, N, \beta) + \int_{\mathcal{W}(R)} \left( \frac{1}{F^\alpha(x)^3} \right) \Phi(\lambda_N (1 - \frac{\beta}{N}) p_1 |v|^N) \, dx \]
\[ \leq C(p_1, N, \beta) + \int_{\mathcal{W}(R)} e^{\lambda_N (1 - \frac{\beta}{N} p_1 |v|^N)} \frac{|u_n|^N}{F^\alpha(x)^3} \, dx \] (4.9)

where \( \mathcal{W}(R) = \{ x \in \mathbb{R}^N : F^\alpha(x) \leq R \} \) be a Wulff ball and \( |\mathcal{W}(R)| = |D| \). We know that \( \int_{\mathcal{W}(R)} F^N(\nabla u^*) \, dx \leq \int_{\mathcal{W}(R)} F^N(\nabla u) \, dx \) by Lemma 2.3. Hence, the result follows from (1.6).

Next, we prove the sharpness of \( \bar{p}_N(u) \). It suffices to construct a sequence \( \{u_n\} \subset W^{1,N}(\mathbb{R}^N) \) and a function \( u \in W^{1,N}(\mathbb{R}^N) \) such that
\[ \int_{\mathbb{R}^N} (F^N(\nabla u_n) + |u_n|^N) \, dx = 1, \quad u_n \rightharpoonup u \neq 0 \quad \text{in} \quad W^{1,N}(\mathbb{R}^N), \]
\[ \left( \int_{\mathbb{R}^N} (F^N(\nabla u_n) + |u_n|^N) \, dx \right)^{\frac{1}{N}} = \delta < 1 \]
and
\[ \int_{\mathbb{R}^N} \frac{\Phi(\lambda_N (1 - \frac{\beta}{N}) \bar{p}_N(u) |u_n|^N)}{F^\alpha(x)^3} \, dx \to +\infty. \]
For $n \in \mathbb{N}$, let $r > 0$, we define
\[
\omega_n(x) = \begin{cases} 
\frac{1}{N} \kappa_N \log \left( \frac{N}{N-n} \right), & 0 \leq F^o(x) \leq re^{-\frac{n}{N}} , \\
\frac{1}{N} \kappa_N \log \left( \frac{F^o(x)}{N} \right), & re^{-\frac{n}{N}} \leq F^o(x) \leq r, \\
0, & F^o(x) \geq r.
\end{cases}
\]
A straightforward calculation yields
\[
w_n \to 0 \quad \text{in} \quad W^{1,N}_0(\Omega), \quad \int_{\mathbb{R}^N} F^N(\nabla \omega_n) dx = 1, \quad \int_{\mathbb{R}^N} |\omega_n|^N dx \to 0.
\]
Set $R = 3r$, define
\[
u(x) = \begin{cases} 
A, & 0 \leq F^o(x) \leq \frac{2}{3}R, \\
3A - \frac{3A}{R} F^o(x), & \frac{2}{3}R \leq F^o(x) \leq R, \\
0, & F^o(x) \geq R,
\end{cases}
\]
where $A$ is a positive constant to be chosen in such a way that \( \left( \int_{\mathbb{R}^N} (F^N(\nabla u) + |u|^N) dx \right)^{\frac{1}{N}} = \delta < 1 \). Denote $W(R) = \{ x \in \mathbb{R}^N : F^o(x) \leq R \}$ be a Wulff ball centered at the origin. Set $v_n = u + (1 - \delta^N) \frac{1}{N} \omega_n$, we have
\[
\int_{\mathbb{R}^N} F^N(\nabla v_n) dx = \int_{W(R)} F^N(\nabla u) dx + (1 - \delta^N) \int_{W(R)} F^N(\nabla w_n) dx = \int_{W(R)} F^N(\nabla u) dx + (1 - \delta^N).
\]
Moreover, we have
\[
\int_{\mathbb{R}^N} |v_n|^N dx = \int_{\mathbb{R}^N} |u + (1 - \delta^N) \frac{1}{N} \omega_n|^N dx = \int_{\mathbb{R}^N} |u|^N dx + r_n,
\]
where $r_n = O(n^{-\frac{1}{N}})$ as $n \to +\infty$. Thus we have $\int_{\mathbb{R}^N} (F^N(\nabla u) + |v|^N) dx = 1 + r_n$. Let $u_n = \frac{v_n}{(1+r_n)^{\frac{1}{N}}}$, it holds
\[
\int_{\mathbb{R}^N} (F^N(\nabla u_n) + |u_n|^N) dx = 1, \quad u_n \to u \quad \text{in} \quad W^{1,N}(\mathbb{R}^N).\]
Then
\[
\int_{\mathbb{R}^N} \frac{\Phi(\lambda_N (1 - \frac{\beta}{N}) p_N(u) |u_n|^\frac{N}{N-1})}{F^\alpha(x)^\beta} dx \\
\geq \int_{W(re^{-\frac{N}{4}})} \frac{e^{\lambda_N (1 - \frac{\beta}{N}) (1 - \frac{\delta}{N}) - \frac{\beta}{N} N |A+1-\frac{\beta}{N} N^\beta |^\frac{N}{N-1}}}{F^\alpha(x)^\beta} dx + C(u)
\]
\[
= \int_{W(re^{-\frac{N}{4}})} \frac{e^{\lambda_N (1 - \frac{\beta}{N}) (1 + r_n) - \frac{\beta}{N} N |C+\omega_n|^\frac{N}{N-1}}}{F^\alpha(x)^\beta} dx + C(u)
\]
\[
= e^{N \frac{\lambda_N}{\beta N} \frac{1}{N-1} (1 - \frac{\beta}{N}) (1 + r_n) - \frac{\beta}{N} N |C+\omega_n|^\frac{N}{N-1}} \int_{W(re^{-\frac{N}{4}})} \frac{1}{F^\alpha(x)^\beta} dx + C(u)
\]
\[
\geq e^{C_1 + (1 + r_n) \frac{1}{N} ((1 - \frac{\beta}{N}) n)^\frac{N-1}{N-1} - \frac{\beta}{N} N e^{-\frac{\beta}{N} n}} \int_{W(re^{-\frac{N}{4}})} \frac{1}{F^\alpha(x)^\beta} dx + C(u)
\]
\[
\geq C_2 e^{C_1 + (1 + r_n) \frac{1}{N} ((1 - \frac{\beta}{N}) n)^\frac{N-1}{N-1} - \frac{\beta}{N} N e^{-\frac{\beta}{N} n}} + C(u) \rightarrow +\infty (n \rightarrow +\infty)
\]
where $0 \leq \beta < N$ and $C, C_1, C_2$ are positive constants.

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