A GELFAND–NAIMARK TYPE THEOREM

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Abstract. Let $X$ be a completely regular space. For a non-vanishing self-adjoint Banach subalgebra $H$ of $C_B(X)$ which has local units we construct the spectrum $\text{sp}(H)$ of $H$ as an open subspace of the Stone–Čech compactification of $X$ which contains $X$ as a dense subspace. The construction of $\text{sp}(H)$ is simple. This enables us to study certain properties of $\text{sp}(H)$, among them are various compactness and connectedness properties. In particular, we find necessary and sufficient conditions in terms of either $H$ or $X$ under which $\text{sp}(H)$ is connected, locally connected and pseudocompact, strongly zero-dimensional, basically disconnected, extremally disconnected, or an $F$-space.

1. Introduction

Throughout this article by a space we mean a topological space. We adopt the definitions of [6], in particular, completely regular spaces as well as compact spaces (and therefore locally compact spaces) are assumed to be Hausdorff. The field of scalars is assumed to be the complex field $\mathbb{C}$, though all results hold true (with the same proof) in the real setting.

Let $X$ be a space. We denote by $C(X)$ the set of all continuous scalar valued mappings on $X$ and we denote by $C_B(X)$ the set of all bounded elements of $C(X)$. The set $C_B(X)$ is a Banach algebra with pointwise addition and multiplication and supremum norm. For any $f$ in $C(X)$, the zero-set of $f$ is defined to be $f^{-1}(0)$ and is denote by $z(f)$, the cozero-set of $f$ is defined to be $X \setminus z(f)$ and is denote by $\text{coz}(f)$, and the support of $f$ is defined to be $\text{cl}_X \text{coz}(f)$ and is denoted by $\text{supp}(f)$. The set of all zero-sets of $X$ is denote by $z(X)$ and the set of all cozero-sets of $X$ is denote by $\text{coz}(X)$. We denote by $C_0(X)$ the set of all $f$ in $C(X)$ which vanish at infinity (that is, $|f|^{-1}(\varepsilon, \infty)$ is compact for any positive $\varepsilon$). Also, we denote by $C_{00}(X)$ the set of all $f$ in $C(X)$ whose support is compact.

In [10] the second author has obtained a commutative Gelfand–Naimark type theorem which shows that for a locally separable metrizable space $X$ the set $C_s(X)$ of all continuous bounded complex valued mappings whose support is separable, where $C_s(X)$ is provided with the supremum norm, is a Banach algebra which is isometrically isomorphic to $C_0(Y)$ for some locally compact space $Y$. The space $Y$ (which is unique up to homeomorphism) has been constructed explicitly as a subspace of the Stone–Čech compactification of $X$. Furthermore, the space $Y$ is countably compact, and if $X$ is non-separable, is also non-normal. In addition $C_0(Y) = C_{00}(Y)$. The theorems in [10] have motivated a series of subsequent

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results. (See [11], [12] and [14]. See also [1] for a relevant result.) This in particular includes our result in this paper in which, when $X$ is a completely regular space, for a non-vanishing self-adjoint Banach subalgebra of $C_B(X)$ which has local units we find a locally compact space $Y$ such that $H$ is isometrically isomorphic to $C_0(Y)$. This result also follows from the celebrated commutative Gelfand–Naimark theorem. Here, we construct $Y$ explicitly as an open subspace of the Stone–Čech compactification of $X$. The space $Y$ contains $X$ as a dense subspace. Furthermore, it is unique up to homeomorphism and therefore coincides with the spectrum of $H$. The simple construction of $Y$ enables us to study some of its properties, among them are various compactness and connectedness properties. In particular, we find necessary and sufficient conditions in terms of either $H$ or $X$ under which $\text{sp}(H)$ is connected, locally connected and pseudocompact, strongly zero-dimensional, basically disconnected, extremally disconnected, or an $F$-space.

Throughout this article we will make critical use of the theory of the Stone–Čech compactification. We review some of the basic properties of the Stone–Čech compactification in the following and refer the reader to the texts [6] and [7] for further possible reading.

**The Stone–Čech compactification.** Let $X$ be a completely regular space. A compactification of $X$ is a compact space which contains $X$ as a dense subspace. The Stone–Čech compactification of $X$, denoted by $\beta X$, is the (unique) compactification of $X$ which is characterized among all compactifications of $X$ by the property that every mapping in $C_B(X)$ is extendable to a (unique) mapping in $C(\beta X)$. For a mapping $f$ in $C_B(X)$ we denote this continuous extension by $f_\beta$. We make use of the following properties.

- The space $X$ is locally compact if and only if $X$ is open in $\beta X$.
- For any $X \subseteq T \subseteq \beta X$ we have $\beta T = \beta X$.
- For any open subspace $V$ of $\beta X$ we have $\text{cl}_{\beta X} V = \text{cl}_{\beta X} (V \cap X)$.
- The closure in $\beta X$ of every open and closed subspace of $X$ is open and closed in $\beta X$.
- For any two open and closed subspaces $U$ and $V$ of $X$ we have $\text{cl}_{\beta X} (U \cap V) = \text{cl}_{\beta X} U \cap \text{cl}_{\beta X} V$.

In particular, disjoint open and closed subspaces of $X$ have disjoint closures in $\beta X$.

2. **The Representation Theorem**

In this section we prove our representation theorem. This, for a completely regular space $X$, provides a way to represent a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which has local units as $C_0(Y)$ for some (unique) locally compact space $Y$.

We now proceed with a few definitions and lemmas.

For a completely regular space $X$ and a subset $H$ of $C_B(X)$ the following subspace $\lambda H X$ of $\beta X$ has been defined in [11] (originally in [9] and [13]) and plays a crucial role in our study.
Definition 2.1. Let $X$ be a completely regular space. For a subset $H$ of $C_B(X)$ let

$$\lambda_H X = \bigcup_{f \in H} \text{int}_{\beta X} \text{cl}_{\beta X} \text{coz}(f).$$

Observe that $\lambda_H X$ is open in (the compact space) $\beta X$ and is therefore locally compact.

Recall that for a space $X$, a subset $H$ of $C_B(X)$ is said to

- be self-adjoint if $H$ contains the complex conjugate $\overline{f}$ of any element $f$ in $H$ (where $\overline{f}(x) = \overline{f(x)}$ for any $x$ in $X$).
- be non-vanishing if for any $x$ in $X$ there is some $f$ in $H$ such that $f(x) \neq 0$.
- separate points of $X$ if for any distinct elements $x$ and $y$ in $X$ there is some $f$ in $H$ such that $f(x) \neq f(y)$.
- have local units if for any closed subspace $A$ in $X$ and any neighborhood $U$ of $A$ in $X$ contained in $\text{supp}(h)$ for some $h$ in $H$, there is some $f$ in $H$ such that $f|_A = 1$ and $f|_{X \setminus U} = 0$.

Lemma 2.2. Let $X$ be a completely regular space and let $H$ be a subset of $C_B(X)$. Let $f$ be in $H$ and let $f_\beta : \beta X \to \mathbb{C}$ be the continuous extension of $f$. Then

$$\text{coz}(f_\beta) \subseteq \lambda_H X.$$  

Proof. Note that $\text{coz}(f_\beta) \subseteq \text{cl}_{\beta X} \text{coz}(f_\beta)$ and $\text{cl}_{\beta X} \text{coz}(f_\beta) = \text{cl}_{\beta X} (X \cap \text{coz}(f_\beta))$, as $\text{coz}(f_\beta)$ is open in $\beta X$ (and $X$ is dense in $\beta X$). Also, $X \cap \text{coz}(f_\beta) = \text{coz}(f)$, as $f_\beta$ extends $f$. Thus $\text{coz}(f_\beta) \subseteq \text{int}_{\beta X} \text{cl}_{\beta X} \text{coz}(f)$. But $\text{int}_{\beta X} \text{cl}_{\beta X} \text{coz}(f) \subseteq \lambda_H X$ by the definition of $\lambda_H X$. \qed

Lemma 2.3. Let $X$ be a completely regular space and let $H$ be a non-vanishing subset of $C_B(X)$. Then

$$X \subseteq \lambda_H X.$$  

Proof. Let $x$ be in $X$. There is some $f$ in $H$ such that $f(x) \neq 0$. By Lemma 2.2 we have $\text{coz}(f_\beta) \subseteq \lambda_H X$. But then $x$ is in $\text{coz}(f_\beta)$, as $f_\beta$ extends $f$. \qed

Definition 2.4. Let $X$ be a completely regular space and let $H$ be a non-vanishing subset of $C_B(X)$. For any $f$ in $H$ denote

$$f_H = f_\beta |_{\lambda_H X}.$$  

Note that $X \subseteq \lambda_H X$ by Lemma 2.3 thus, $f_H$ extends $f$.

Lemma 2.5. Let $X$ be a completely regular space and let $H$ be a non-vanishing subset of $C_B(X)$. Then, $f_H$ is in $C_0(\lambda_H X)$ for any $f$ in $H$.

Proof. Let $f$ be in $H$. Let $\epsilon > 0$. Then $|f_\beta|^{-1}((\epsilon, \infty)) \subseteq \lambda_H X$, as $|f_\beta|^{-1}((\epsilon, \infty)) \subseteq \text{coz}(f_\beta)$ and $\text{coz}(f_\beta) \subseteq \lambda_H X$ by Lemma 2.2. Thus

$$|f_H|^{-1}((\epsilon, \infty)) = \lambda_H X \cap |f_\beta|^{-1}((\epsilon, \infty)) = |f_\beta|^{-1}((\epsilon, \infty))$$

is closed in $\beta X$ and is therefore compact. \qed

Lemma 2.6. Let $X$ be a completely regular space and let $H$ be a self-adjoint subalgebra of $C_B(X)$. Let $A$ be a compact subspace of $\lambda_H X$. Then

$$A \subseteq \text{cl}_{\beta X} \text{coz}(f)$$

for some $f$ in $H$. 

Proof. Suppose that $A$ is compact. Using the definition of $\lambda_X$, we have

$$A \subseteq \text{int}_{\beta_X} \text{cl}_{\beta_X} \text{coz}(h_1) \cup \cdots \cup \text{int}_{\beta_X} \text{cl}_{\beta_X} \text{coz}(h_n)$$

for some $h_1, \ldots, h_n$ in $H$. In particular,

$$A \subseteq \text{cl}_{\beta_X} \text{coz}(h_1) \cup \cdots \cup \text{cl}_{\beta_X} \text{coz}(h_n) = \text{cl}_{\beta_X} \text{coz}(h)$$

where

$$h = |h_1|^2 + \cdots + |h_n|^2 = h_1\overline{h_1} + \cdots + h_n\overline{h_n}.$$  

Then $h$ is in $H$, as $H$ is self-adjoint. \hfill \square

The following lemma is a corollary of the Stone–Weierstrass theorem (see Theorem 8.1 and Corollary 8.3 of [5], Chapter V) and will be used in the proof of our next theorem.

**Lemma 2.7.** Let $X$ be a locally compact space and let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_0(X)$ which separates points of $X$. Then

$$H = C_0(X).$$

The following version of the Banach–Stone theorem (see Theorem 7.1 of [4]) will be used in the proof of our theorem. It states that for a locally compact $X$ the topology of $X$ determines and is determined by the algebraic structure of $C_0(X)$. (It turns out that for a locally compact space $X$ even the ring structure of $C_0(X)$ suffices to determine the topology $X$; see [2].)

**Lemma 2.8.** Let $S$ and $T$ be locally compact spaces. Then the normed algebras $C_0(S)$ and $C_0(T)$ are isometrically isomorphic if and only if the spaces $S$ and $T$ are homeomorphic.

We are now at a point to prove our main result.

**Theorem 2.9.** Let $X$ be a completely regular space. Let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which has local units. Then $H$ is isometrically isomorphic to $C_0(Y)$ for some unique (up to homeomorphism) locally compact space $Y$, namely, for $Y = \lambda_X$. Moreover, the following are equivalent:

1. $H$ is unital.
2. $H$ contains $1$.
3. $Y$ is compact.
4. $Y = \beta X$.

**Proof.** Let

$$\psi : H \rightarrow C_0(\lambda_X)$$

be defined by $\psi(f) = f_H$ for any $f$ in $H$. The definition makes sense by Lemma 2.5. We show that $\psi$ is an isometric isomorphism. It is clear that $\psi$ is injective and that $\psi$ is a homomorphism. (To see this, let $f$ and $g$ be in $H$. Then $(f + g)_H = f_H + g_H$, as $(f + g)_H$ and $f_H + g_H$ are continuous mappings on $\lambda_X$ which are both identical to $f + g$ on $X$, which is contained and is therefore dense in $\lambda_X$ by Lemma 2.4. Similarly, $(fg)_H = f_Hg_H$. Also, if $f_H$ equals to $g_H$ then the restrictions of $f_H$ and $g_H$ to $X$ are also equal.) It therefore suffices to show that $\psi$ is surjective and preserves norms.
To show that \( \psi \) is an isometry, let \( f \) be in \( H \). By continuity of \( f \) we have
\[
|f_H|_H(\lambda H X) = |f_H|(\overline{cl}_\beta X) \subseteq \overline{|f_H|(X)} = \overline{|f|(X)} \subseteq [0, \|f\|],
\]
where the bar denotes the closure in \( \mathbb{R} \). This implies that \( \|f_H\| \leq \|f\| \). That \( \|f\| \leq \|f_H\| \) is clear, as \( f_H \) extends \( f \). This shows that \( \psi \) is an isometry.

Next, we show that \( \psi \) is surjective, that is
\[
\psi(H) = \mathcal{C}_0(\lambda H X).
\]
Note that \( \psi(H) \) is a subalgebra of \( \mathcal{C}_0(\lambda H X) \) which is also closed in \( \mathcal{C}_0(\lambda H X) \), as it is an isometric image of the complete normed space \( H \). By Lemma 2.7 it therefore suffices to verify that \( \psi(H) \) satisfies the following conditions:
1. \( \psi(H) \) does not vanish on \( \lambda H X \);
2. \( \psi(H) \) separates points of \( \lambda H X \);
3. \( \psi(H) \) is a self-adjoint subalgebra of \( \mathcal{C}_0(\lambda H X) \).

To show (2), suppose that \( x \) and \( y \) are distinct elements of \( \lambda H X \). Let \( U \) be an open neighborhood of \( x \) in \( \beta X \) such that \( cl_\beta U \) does not contain \( y \) and
\[
(2.1) \quad cl_\beta U \subseteq \lambda H X.
\]
Let \( U' \) be an open neighborhood of \( x \) in \( \beta X \) such that \( cl_\beta U' \subseteq U \). Let
\[
A = cl_X(X \cap U').
\]
Then \( cl_\beta A \subseteq U \), and thus \( X \cap U \) is a neighborhood of \( A \) in \( X \). By (2.1), and since \( cl_\beta U \) is compact, it follows from Lemma 2.6 that \( cl_\beta U \subseteq cl_\beta \text{coz}(h) \) for some \( h \) in \( H \). In particular \( U \subseteq cl_\beta \text{coz}(h) \), which, intersecting with \( X \), gives \( X \cap U \subseteq cl_X \text{coz}(h) = \text{supp}(h) \). That is, \( X \cap U \) is a neighborhood in \( X \) of the closed subspace \( A \) of \( X \) which is contained in \( \text{supp}(h) \). By our assumption, there is some \( f \) in \( H \) such that
\[
f \mid_A = 1 \quad \text{and} \quad f \mid_{X \setminus U} = 0.
\]
Note that \( cl_\beta U' = cl_\beta (X \cap U') \). We have
\[
f_H(x) = f_\beta(x) \in f_\beta(cl_\beta U') \subseteq f_\beta(cl_\beta A) \subseteq f_\beta(cl_\beta X \cap U') \subseteq f_\beta(X \setminus U) \subseteq f(X \setminus U) \subseteq [0,1],
\]
where the bar denotes the closure in \( \mathbb{C} \) and \( f_\beta : \beta X \to \mathbb{C} \) denotes the continuous extension of \( f \). Observe that
\[
cl_\beta (X \cap U') \cup cl_\beta (X \setminus U) = \beta X.
\]
Therefore, by the choice of \( U \), we have
\[
f_H(y) = f_\beta(y) \in f_\beta(\beta X \setminus cl_\beta X U') \subseteq f_\beta(cl_\beta X (X \setminus U)) \subseteq f_\beta(X \setminus U) = f(\beta X \setminus U) \subseteq [0,1].
\]
This implies that \( f_H(x) \neq f_H(y) \). Thus \( \psi(H) \) separates points of \( \lambda H X \). This shows (2).

Note that the above argument also proves (1), as for any \( x \) in \( \lambda H X \) one can choose some \( y \) in \( \lambda H X \) different from \( x \) (which is always possible provided that \( X \) is not a singleton!) and find an element \( f_H \) in \( \psi(H) \) for some \( f \) in \( H \) which does not vanish at \( x \).

To prove (3), observe that for any \( f \) in \( H \), since \( H \) is self-adjoint, \( \overline{f} \) is in \( H \), and therefore \( \overline{f}_H \) is in \( \psi(H) \). But \( \overline{f}_H \) and \( \overline{f}_H \) are identical (as they are continuous mappings on \( \lambda H X \) which agree on the dense subspace \( X \) of \( \lambda H X \)).

This shows that \( \psi \) is an isometric isomorphism, and therefore, \( H \) and \( \mathcal{C}_0(Y) \) are isometrically isomorphic, where \( Y = \lambda H X \). The fact that \( Y \) is unique (up to homeomorphism) is immediate and follows from Lemma 2.8.
We now verify the final assertion of the theorem. Let $H$ be unital with the unit element $u$. By our assumption, since $H$ is non-vanishing, for every $x$ in $X$ there exists some $f_x$ in $H$ such that $f_x(x) \neq 0$. But $u(x)f_x(x) = f_x(x)$ which yields $u(x) = 1$. That is $u = 1$ and thus $H$ contains $1$. Clearly, if $H$ contains $1$ then $Y = \beta X$, by the definition of $\lambda_H X$. Also, if $Y$ is compact, since it contains $X$, it contains its closure in $\beta X$. Therefore $Y = \beta X$. Finally, if $Y = \beta X$, then $H$ is unital, as it is isometrically isomorphic to $C_0(Y)$ and the latter is so, since $C_0(Y) = C_B(Y)$.

Remark 2.10. The closed subalgebra $H$ of $C_B(X)$ in the statement of Theorem 2.9 can be thought of as being a $C^*$-algebra with the standard operation $*: H \to H$ of complex conjugation (which maps $f$ to $\overline{f}$ for every $f$ in $H$). In this case, as is pointed out in the proof of Theorem 2.9 we have $\psi(*) = \psi(f)$ for every $f$ in $H$, and thus, the mapping $\psi$ is an isometric $*$-isomorphism. In particular, $H$ and $C_0(Y)$ are then isometrically $*$-isomorphic.

In the following we give examples of spaces $X$ and non-vanishing self-adjoint closed subalgebra $H$ of $C_B(X)$ which have local units (thus satisfying the requirements in Theorem 2.1).

Let $\mathcal{P}$ be a topological property. Then
- $\mathcal{P}$ is closed hereditary if any closed subspace of a space which has $\mathcal{P}$ also has $\mathcal{P}$.
- $\mathcal{P}$ is preserved under countable closed unions if any space which is a countable union of its closed subspaces each having $\mathcal{P}$ also has $\mathcal{P}$.

A space $X$ is called locally-$\mathcal{P}$ if every point of $X$ has a neighborhood in $X$ which has $\mathcal{P}$.

Example 2.11. Let $\mathcal{P}$ and $\mathcal{Q}$ be topological properties such that
- $\mathcal{P}$ and $\mathcal{Q}$ are closed hereditary.
- A space with both $\mathcal{P}$ and $\mathcal{Q}$ is Lindelöf.
- $\mathcal{P}$ is preserved under countable closed unions.
- A space with $\mathcal{Q}$ having a dense subspace with $\mathcal{P}$ has $\mathcal{P}$.

Let $X$ be a normal locally-$\mathcal{P}$ space with $\mathcal{Q}$. Let

$$C_0^\mathcal{P}(X) = \{ f \in C_B(X) : |f|^{-1}(1/n, \infty) \text{ has } \mathcal{P} \text{ for each } n \}.$$ 

We check that $C_0^\mathcal{P}(X)$ is a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which has local units. The fact that $C_0^\mathcal{P}(X)$ is a non-vanishing closed subalgebra of $C_B(X)$ is proved in Theorem 3.3.9 of [14]. It is also clear that $C_0^\mathcal{P}(X)$ contains $f$ if it contains $f$. That is, $C_0^\mathcal{P}(X)$ is self-adjoint. To show that $C_0^\mathcal{P}(X)$ has local units let $A$ be a closed subspace of $X$ and $U$ be a neighborhood of $A$ in $X$ which is contained in $\text{supp}(h)$ for some $h$ in $H$. Observe that

$$C_0^\mathcal{P}(X) = \{ f \in C_B(X) : \text{supp}(f) \text{ has } \mathcal{P} \}$$

by Theorem 3.3.10 of [14]. Since $X$ is normal, there is some $f$ in $C_B(X)$ such that $f|_A = 1$ and $f|_{X \setminus U} = 0$. Then $f$ is in $C_0^\mathcal{P}(X)$, as $\text{supp}(f)$ has $\mathcal{P}$, since $\text{supp}(f)$ is contained in $\text{supp}(h)$ as a closed subspace, the latter has $\mathcal{P}$ and $\mathcal{P}$ is closed hereditary.

Specific examples of topological properties $\mathcal{P}$ and $\mathcal{Q}$ which satisfy the above requirements are given in Example 3.3.13 of [14]. This includes the case when $\mathcal{P}$ is the Lindelöf property and $\mathcal{Q}$ is paracompactness (and particularly, the case when
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is either the Lindelöf property, second countability or separability and \( \mathcal{D} \) is metrizability).

Let \( X \) be a completely regular space. In Theorem 2.9 for a non-vanishing self-adjoint closed subalgebra \( H \) of \( C_B(X) \) which has local units we have shown that \( H \) is isometrically isomorphic to \( C_0(\lambda_H X) \). It also follows from the commutative Gelfand–Naimark theorem that \( H \) is isometrically isomorphic to \( C_0(Y) \) in which \( Y \) is the spectrum (or the maximal ideal space) of \( H \) which has the Gelfand (or the Zariski) topology. The uniqueness part of Theorem 2.9 now implies that \( Y = \lambda_H X \). We state this fact formally as a theorem.

**Notation 2.12.** Let \( X \) be a space. We denote the spectrum of a closed subalgebra \( H \) of \( C_B(X) \) (considered as a \( C^* \)-subalgebra of \( C_B(X) \) under the standard operation of complex conjugation) by \( \text{sp}(H) \).

**Theorem 2.13.** Let \( X \) be a completely regular space. Let \( H \) be a non-vanishing self-adjoint closed subalgebra of \( C_B(X) \) which has local units. Then
\[
\text{sp}(H) = \lambda_H X.
\]

3. Compactness and connectedness properties of spectrum

In this section we use the representation given in Theorem 2.9 to study various properties of the spectrum.

**Lemma 3.1.** Let \( X \) be a normal space. Let \( H \) be a non-vanishing self-adjoint closed subalgebra of \( C_B(X) \) which has local units. Let
\[
\psi : H \rightarrow C_0(\lambda_H X)
\]
be defined by \( \psi(f) = f_H \) for any \( f \) in \( H \). Then
\[
\psi^{-1}(C_0(\lambda_H X)) = \{ f \in H : \text{supp}(f) \subseteq \text{int}_X \text{supp}(h) \text{ for some } h \in H \}.
\]

**Proof.** Let \( f \) be in \( \psi^{-1}(C_0(\lambda_H X)) \). Then \( f_H \) is in \( C_0(\lambda_H X) \), that is, \( \text{supp}(f_H) \) is a compact subspace of \( \lambda_H X \). Let \( U \) be an open subspace of \( \beta X \) such that
\[
\text{supp}(f_H) \subseteq U \subseteq \text{cl}_{\beta X} U \subseteq \lambda_H X.
\]
Since \( \text{cl}_{\beta X} U \) is compact, by Lemma 2.6 we have
\[
\text{cl}_{\beta X} U \subseteq \text{cl}_{\beta X} \text{coz}(h)
\]
for some \( h \) in \( H \). Clearly, \( \text{coz}(f) \subseteq \text{coz}(f_H) \), as \( f_H \) extends \( f \). In particular \( \text{coz}(f) \subseteq \text{supp}(f_H) \), and thus since the latter is closed in \( \beta X \) (as it is compact) \( \text{cl}_{\beta X} \text{coz}(f) \subseteq \text{supp}(f_H) \). Now, using the above relations, we have
\[
\text{cl}_{\beta X} \text{coz}(f) \subseteq U \subseteq \text{cl}_{\beta X} \text{coz}(h),
\]
which if we intersect each side with \( X \) it yields
\[
\text{supp}(f) = \text{cl}_X \text{coz}(f) \subseteq X \cap U \subseteq \text{cl}_X \text{coz}(h) = \text{supp}(h).
\]
Therefore \( \text{supp}(f) \subseteq \text{int}_X \text{supp}(h) \).

For the converse, let \( f \) be in \( H \) such that \( \text{supp}(f) \subseteq \text{int}_X \text{supp}(h) \) for some \( h \) in \( H \). Since \( X \) is normal, by the Urysohn lemma, there is a continuous mapping \( g : X \rightarrow [0,1] \) such that
\[
g|_{\text{supp}(f)} = 1 \quad \text{and} \quad g|_{X \setminus \text{int}_X \text{supp}(h)} = 0.
\]
Let $g_β : βX → [0, 1]$ be the continuous extension of $g$. Note that
\[ g_β^{-1}((1/2, 1]) ⊆ cl_{βX}g_β^{-1}((1/2, 1]) = cl_{βX}(X ∩ g_β^{-1}((1/2, 1])) = cl_{βX}g_β^{-1}((1/2, 1]) \]
and
\[ cl_{βX}g_β^{-1}((1/2, 1]) ⊆ cl_{βX}supp(h) = cl_{βX}coz(h), \]
using the definition of $g$. Therefore
\[ g_β^{-1}((1/2, 1]) ⊆ int_{βX}cl_{βX}coz(h). \]
Also
\[ cl_{βX}coz(f) ⊆ g_β^{-1}(1) \]
by the definition of $g$ and $int_{βX}cl_{βX}coz(h) ⊆ λ_H X$ by the definition of $λ_H X$. Thus $cl_{βX}coz(f) ⊆ λ_H X$. It now follows that
\[ supp(f_H) = cl_{λ_H X}coz(f_H) = cl_{λ_H X}(X ∩ coz(f_H)) = cl_{λ_H X}(coz(f)) = λ_H X ∩ cl_{βX}coz(f) = cl_{βX}coz(f) \]
is compact. (Note that $X ⊆ λ_H X$ by Lemma 2.13.) Therefore $f_H$ is in $C_{00}(λ_H X)$. That is, $f$ is in $ψ^{-1}(C_{00}(λ_H X))$. □

Let $X$ be a locally compact space. It is known that $C_0(X) = C_{00}(X)$ if and only if every $σ$-compact subspace of $X$ is contained in a compact subspace of $X$. (See Problem 7G.2 of [7].) In particular, $C_0(X) = C_{00}(X)$ implies that $X$ is countably compact. Every countably compact paracompact space is necessarily compact. (See Theorem 5.1.20 of [6].) Therefore, if $X$ is non-compact, then $C_0(X) = C_{00}(X)$ implies that $X$ is non-paracompact (and thus, in particular, non-metrizable and non-Lindelöf). (Every Lindelöf space, in particular every compact space, and every metrizable space is paracompact; see Theorems 5.1.1–5.1.3 of [6].)

**Theorem 3.2.** Let $X$ be a normal space. Let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_β(X)$ which has local units. The following are equivalent:

1. For any $f$ in $H$ there is some $h$ in $H$ with $supp(f) ⊆ int_Xsupp(h)$.
2. Every $σ$-compact subspace of $sp(H)$ is contained in a compact subspace of $sp(H)$; in particular, $sp(H)$ is countably compact.
3. $C_0(sp(H)) = C_0(sp(H))$.

**Proof.** Conditions (2) and (3) are equivalent by the observation made preceding the statement of the theorem. (Note that $sp(H) = λ_H X$ by Theorem 2.13.)

Let the mapping $ψ : H → C_0(λ_H X)$ be defined by $ψ(f) = f_H$ for any $f$ in $H$. Then $ψ$ is an isometric isomorphism by (the proof of) Theorem 2.7. Note that (1) is equivalent to $H = ψ^{-1}(C_{00}(λ_H X))$ by Lemma 3.1. But this holds if and only if $ψ(H) = C_0(λ_H X)$ (and since $ψ$ is surjective) if and only if $C_0(λ_H X) = C_{00}(λ_H X)$. This is equivalent to (2). □

In the above theorem we considered compactness properties (such as countable compactness) of the spectrum. In the next few theorems we consider connectedness properties of the spectrum. (Compare Theorems 3.2.2 and 3.3.3 with Theorems 3.2.10 and 3.2.11 of [14], respectively.)

We first consider the case of usual connectedness. We will use the well known fact that the Stone–Čech compactification $βX$ of a completely regular space $X$ is
connected if and only if $X$ is connected. (See Problem 6L of [7].) Recall that an algebra is said to be indecomposable if it has no idempotent except 0 (and 1, if it is unital).

**Theorem 3.3.** Let $X$ be a completely regular space. Let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which has local units. The following are equivalent:

1. $X$ is connected.
2. $\text{sp}(H)$ is connected.
3. $H$ is indecomposable.

**Proof.** Note that $\lambda_H X (= \text{sp}(H))$ by Theorem 2.13 contains $X$ by Lemma 2.3, and is contained in $\beta X$. Therefore $\beta(\lambda_H X) = \beta X$. Thus, $X$ is connected if and only if $\beta X$ is connected if and only if $\beta(\lambda_H X)$ is connected. This shows the equivalence of (1) and (2).

(1) implies (3). Suppose that $H$ is not indecomposable. Then $H$ has an idempotent $f$ other than 0 and 1. Clearly, $f$ is $\{0,1\}$-valued. In particular, $f^{-1}(0)$ and $f^{-1}(1)$ form a separation for $X$, and $X$ is therefore disconnected.

(3) implies (1). Suppose that $X$ is disconnected. Let $U$ and $V$ be a separation for $X$. Then $U$ and $V$ are disjoint closed subspaces of $X$ and therefore by our assumption there is some $f$ in $H$ such that $f|_U = 0$ and $f|_V = 1$. That is, $f$ is the characteristic function $\chi_V$ on $X$. Clearly, $f$ is idempotent and $f$ is neither 0 nor 1. That is $H$ is not indecomposable. □

We do not know how Theorem 3.3 can be formulated in the context of local connectedness. For possible future reference we record this below as an open question.

**Question 3.4.** Let $X$ be a completely regular space. For a non-vanishing self-adjoint closed subalgebra $H$ of $C_B(X)$ find necessary and sufficient conditions (in terms of either $X$ or $H$) for $\text{sp}(H)$ to be locally connected.

As we have just pointed out, we do not know the local connectedness version of Theorem 3.3; however, we can obtain some results in the presence of a compactness property called pseudocompactness. Recall that a space $X$ is called pseudocompact if there is no unbounded continuous scalar valued mapping on $X$.

In the next theorem we need to use the following lemma which is due to Henriksen and Isbell in [8] (that (1) implies (2) is actually due to Banaschewski in [3]).

**Lemma 3.5.** Let $X$ be a completely regular space. The following are equivalent:

1. $X$ is locally connected and pseudocompact.
2. $\beta X$ is locally connected.

**Theorem 3.6.** Let $X$ be a completely regular space. Let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which has local units. The following are equivalent:

1. $\text{sp}(H)$ is locally connected and pseudocompact.
2. $X$ is locally connected and pseudocompact.

**Proof.** Note that $\beta(\lambda_H X) = \beta X$, as $X \subseteq \lambda_H X \subseteq \beta X$ by Lemma 2.3. It now follows from Lemma 3.5 that $X$ is locally connected and pseudocompact if and only if $\beta X = \beta(\lambda_H X)$ is locally connected and pseudocompact if and only if $\lambda_H X (= \text{sp}(H))$ by Theorem 2.13 is locally connected and pseudocompact. □
In the next few results we study various (dis)connectedness properties of the spectrum. These properties are total disconnectedness, zero-dimensionality, strong zero-dimensionality, basic disconnectedness, extreme disconnectedness, and being an $F$-space. There is some disagreement on definition of some of these properties, so we define them below to avoid confusion. Recall that two subspaces $A$ and $B$ of a space $X$ are called completely separated if there is a continuous mapping $f : X \to [0, 1]$ such that $f \mid_A = 1$ and $f \mid_B = 0$.

Let $X$ be a completely regular space. The space $X$ is called

1. **totally disconnected** if $X$ does not contain any connected subspace of cardinality larger than one, or, equivalently, if every component of $X$ is a singleton.

2. **zero-dimensional** if the set of all open and closed subspaces of $X$ forms an open base for $X$.

3. **strongly zero-dimensional** if every two completely separated subspaces of $X$ are separated by two disjoint open and closed subspaces.

4. **basically disconnected** if the closure of every cozero-set of $X$ is open.

5. **extremally disconnected** if the closure of every open subspace of $X$ is open.

6. **an $F$-space** if any two disjoint cozero-sets in $X$ are completely separated.

See Section 6.2 of [6] for definitions of (1)–(3) and (5), and see Problems 1H and 14N of [7] for definitions of (4) and (6), respectively. It is known that (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (6). (See Theorem 6.2.1 of [6] for (2) $\Rightarrow$ (1), and Theorem 6.2.6 of [6] for (3) $\Rightarrow$ (2). To see (4) $\Rightarrow$ (3), let $X$ be a basically disconnected space. Then $\beta X$ is basically disconnected by Problem 6M.1 of [7]. But every basically disconnected space is zero-dimensional by Problem 4K.8 of [7]. Therefore $\beta X$ is zero-dimensional, and thus is strongly zero-dimensional by Theorem 6.2.7 of [6], as it is compact. But $\beta X$ is strongly zero-dimensional if and only if $X$ is strongly zero-dimensional by Theorem 6.2.12 of [6]. The implication (5) $\Rightarrow$ (4) is clear. The implication (4) $\Rightarrow$ (6) follows from Problem 14N.4 of [7].)

**Lemma 3.7.** Let $X$ be a completely regular space. Let $G$ and $H$ be non-vanishing self-adjoint closed subalgebras of $C_B(X)$ which have local units. Then $G = H$ if $\lambda_G X = \lambda_H X$.

**Proof.** Suppose that $\lambda_G X = \lambda_H X$. Let

$$\phi : G \to C_0(\lambda_G X) \quad \text{and} \quad \psi : H \to C_0(\lambda_H X)$$

where $\phi(g) = g \mid_X$ and $\psi(h) = h \mid_X$ for any $g$ in $G$ and $h$ in $H$. Then $\phi$ and $\psi$ are isomorphisms by (the proof of) Theorem 2.3, where

$$\phi^{-1}(g) = g \mid_X \quad \text{and} \quad \psi^{-1}(h) = h \mid_X$$

for any $g$ in $C_0(\lambda_G X)$ and $h$ in $C_0(\lambda_H X)$. The mapping

$$\psi^{-1} \phi : G \to C_0(\lambda_G X) = C_0(\lambda_H X) \to H$$

is such that

$$\psi^{-1}\phi(g) = \psi^{-1}(g \mid_X) = g \mid_X = g$$

for any $g$ in $G$. Thus, in particular $G \subseteq H$. Similarly $H \subseteq G$. Therefore $G = H$. □

Let $\mathcal{P}$ be either strong zero-dimensionality, basic disconnectedness, extreme disconnectedness, or being an $F$-space. Let $X$ be a completely regular space. It is known that $X$ has $\mathcal{P}$ if and only if $\beta X$ has $\mathcal{P}$. (See Theorem 6.2.12 of [6].)
for strong zero-dimensionality, Problem 6M.1 of \cite{7} for basic disconnectedness and extreme disconnectedness and Theorem 14.25 of \cite{7} for being an \textit{F}-space.)

**Theorem 3.8.** Let $X$ be a completely regular space. Let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which has local units. Let $\mathcal{P}$ be either strong zero-dimensionality, basic disconnectedness, extreme disconnectedness, or being an $F$-space. The following are equivalent:

1. $\text{sp}(H)$ has $\mathcal{P}$.
2. $X$ has $\mathcal{P}$.

**Proof.** Note that $\lambda_H X$ has $\mathcal{P}$ if and only if $\beta(\lambda_H X)$ has $\mathcal{P}$. But $\beta(\lambda_H X) = \beta X$, as $X \subseteq \lambda_H X \subseteq \beta X$ by Lemma 2.3. Therefore $\text{sp}(H)$ ($= \lambda_H X$ by Theorem 2.13) has $\mathcal{P}$ if and only if $\beta X$ has $\mathcal{P}$ if and only if $X$ has $\mathcal{P}$. \hfill $\Box$

**Theorem 3.9.** Let $X$ be a completely regular space. Let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which has local units. Let $\text{sp}(H)$ be either strong zero-dimensionality, basically disconnected, or extremally disconnected. Then $H = \{ f \in H : f \text{ is an idempotent} \}$.

Here the bar denotes the closure in $C_B(X)$.

**Proof.** Note that $\lambda_H X$ ($= \text{sp}(H)$ by Theorem 2.13) is strongly zero-dimensional, as basic disconnectedness and extreme disconnectedness are stronger than strong zero-dimensionality. Let $G = \overline{I}$ where $I$ is the subalgebra of $H$ generated by the set of all its idempotents. We verify that $G$ is a non-vanishing self-adjoint closed subalgebra of $C_B(X)$ which has local units such that $\lambda_G X = \lambda_H X$. Corollary 3.7 will then imply that $G = H$.

To show that $G$ is non-vanishing, let $x$ be in $X$. Then $x$ is in $\lambda_H X$. Since $\beta X$ is zero-dimensional, there is an open and closed neighborhood $U$ of $x$ in $\beta X$ such that $U \subseteq \lambda_H X$. By Lemma 2.6 then $U \subseteq \text{cl}_{\beta X} \text{coz}(h)$ for some $h$ in $H$. In particular $X \cap U \subseteq \text{cl}_{\beta X} \text{coz}(h) = \text{supp}(h)$. Note that $X \cap U$ is open and closed in $X$. Therefore, by our assumption, there is an $f$ in $H$ such that $f \mid_{X \cap U} = 1$ and $f \mid_{X \setminus U} = 0$. Observe that $f$ is in $G$, as it is idempotent, and $f(x) \neq 0$.

We now show that $G$ is self-adjoint. Note that elements of $I$ are sums of the form

$$\sum_{k=1}^{n} \alpha_k f_{j_1}^k \cdots f_{j_k}^k$$

where $\alpha_j$'s are scalars and $f_j^k$'s are idempotent elements of $H$. It is now clear that for any elements $f$ in $I$ its conjugate $\overline{f}$ is also in $I$. Let $g$ be in $G$. Then $f_n \to g$ for some sequence $f_1, f_2, \ldots$ in $I$. But $\overline{f_n} \to \overline{g}$ and $\overline{f_1}, \overline{f_2}, \ldots$ are in $I$. Therefore $\overline{g}$ is in $G$.

Next, we show that $G$ has local units. Suppose that $A$ is a closed subspaces of $X$ and $U$ is a neighborhood of $A$ in $X$ which is contained in $\text{supp}(g)$ for some $g$ in $G$. Note that $g$ is also in $H$. Thus, by our assumption, there is some $f$ in $H$ such that $f \mid_A = 1$ and $f \mid_{X \setminus U} = 0$. Now, since $X$ is strongly zero-dimensional, there is an open and closed subspace $V$ of $X$ such that $A \subseteq V \subseteq U$. Again, $V$ is a closed subspace of $X$ which is also a neighborhood of itself in $X$ and is contained in $\text{supp}(g)$. Therefore, by our assumption, there is some $h$ in $H$ such that $h \mid_V = 1$ and $h \mid_{X \setminus V} = 0$. Observe that $h$ is an idempotent in $H$ and is therefore in $G$. Also, $h \mid_A = 1$ and $h \mid_{X \setminus U} = 0$, as $A \subseteq V$ and $X \setminus U \subseteq X \setminus V$. 


Finally, we show that $\lambda_G X = \lambda_H X$. It is clear that $\lambda_G X \subseteq \lambda_H X$, as $G \subseteq H$. To check the reverse inclusion, let $t$ be in $\lambda_H X$. By an argument similar to the one we used to check that $G$ is non-vanishing we can find an open and closed neighborhood $U$ of $t$ in $\beta X$ such that $f = \chi_{(X \cap U)}$ is in $H$. Observe that $cl_{\beta X}(X \cap U) = cl_{\beta X} U = U$ is open and closed in $\beta X$ and

$$U = int_{\beta X} cl_{\beta X} U = int_{\beta X} cl_{\beta X} (X \cap U) = int_{\beta X} cl_{\beta X} coz(f) \subseteq \lambda_G X.$$ 

Thus $t$ is in $\lambda_G X$. Therefore $\lambda_H X \subseteq \lambda_G X$. □

It is known that in the class of locally compact paracompact spaces, total disconnectedness, zero-dimensionality and strong zero-dimensionality are all equivalent. (See Theorem 6.2.10 of [6].)

**Theorem 3.10.** Let $X$ be a completely regular space. Let $H$ be a non-vanishing self-adjoint closed subalgebra of $C_b(X)$ which has local units. Let $sp(H)$ be paracompact (in particular, metrizable or Lindelöf). The following are equivalent:

1. $sp(H)$ is totally disconnected.
2. $sp(H)$ is zero-dimensional.
3. $sp(H)$ is strongly zero-dimensional.
4. $X$ is strongly zero-dimensional.

**Proof.** Note that $sp(H)$ ( = $\lambda_H X$ by Theorem 2.13) is locally compact (by its definition). Thus (1)–(3) are equivalent if $sp(H)$ is also paracompact. The equivalence of (3) and (4) follows from Theorem 3.8. □

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