Fractional Dehn twists and modular invariants

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Abstract In this paper, we establish a relationship between fractional Dehn twist coefficients of Riemann surface automorphisms and modular invariants of holomorphic families of algebraic curves. Specially, we give a characterization of pseudo-periodic maps with nontrivial fractional Dehn twist coefficients. We also obtain some uniform lower bounds of non-zero fractional Dehn twist coefficients.

Keywords fractional Dehn twists, pseudo-periodic maps, modular invariants, families of curves

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1 Introduction

The mapping class group is important in low-dimensional topology, and Dehn twists are the generators of this group. Recently, many authors (see, e.g., [4–8,15,16]) studied fractional Dehn twist coefficients in various aspects of 3-manifold. The study of fractional Dehn twist coefficients dates at least from Gabai and Oertel [2].

Let \( \Sigma_g \) be a closed connected Riemann surface of genus \( g \geq 2 \). The mapping class group \( \text{Mod}(\Sigma_g) \) of \( \Sigma_g \) is the group of isotopy classes of orientation preserving homeomorphism of \( \Sigma_g \). The Nielsen-Thurston classification theorem says that any mapping class \( \phi \in \text{Mod}(\Sigma_g) \) is either periodic, pseudo-Anosov or reducible. The homeomorphism \( \phi \) is reducible if there exist finite simple closed curves \( \mathcal{C} = \{ \gamma_1, \ldots, \gamma_r \} \) on \( \Sigma_g \) such that the restriction of \( \phi \) on \( \Sigma_g \setminus \mathcal{C} \) is either periodic or pseudo-Anosov. If \( \phi \in \text{Mod}(\Sigma_g) \) is periodic or \( \phi \) is reducible and the restriction is periodic, then \( \phi \) is said to be pseudo-periodic. We may assume that \( \mathcal{C} \) satisfies the following additional conditions: (i) \( \gamma_i \) does not bound a disk on \( \Sigma_g \), and (ii) \( \gamma_i \) is not parallel to \( \gamma_j \) if \( i \neq j \) (see [13, Lemma 1.1]). Such \( \mathcal{C} \) is called an admissible system of cut curves.

Given a pseudo-periodic map \( \phi \), a sufficiently high power \( \phi^m \) preserves each cut curve \( \gamma_1, \ldots, \gamma_r \). Denote by \( T_{\gamma_i} \) the (right-hand) Dehn twist of \( F \) along \( \gamma_i \). Then there is a factorization of \( \phi \) into a commutative product

\[ \phi^m = T_{\gamma_1}^{k_1} \cdots T_{\gamma_r}^{k_r}. \]

The fractional Dehn twist coefficient of \( \phi \) along \( \gamma_i \), defined to be \( c(\phi, \gamma_i) = k_i/m \). This definition does not depend on the choice of \( m \), and indeed, \( c(\phi^n, \gamma_i) = n \cdot c(\phi, \gamma_i) \) (see [12, Subsection 2.2.2]).

If \( \phi \in \text{Mod}(\Sigma_g) \) is a pseudo-periodic map of negative twist, i.e., \( c(\phi, \gamma) < 0 \) for each \( \gamma \in \mathcal{C} \), then there exists a local family \( f_{\phi} : S \to \Delta \) whose monodromy homeomorphism around its central fiber is equal (up
to isotopy and conjugation) to \( \phi \). Here, a family of complex projective curves of genus \( g \) is a surjective holomorphic morphism \( f : X \to Y \) whose general fiber is a smooth curve of genus \( g \), where \( X \) is a complex smooth projective surface, and \( Y \) is a complex smooth projective curve of genus \( b \). Moreover, if \( Y = \Delta \) is the unit disk of the complex plane, and only the fiber \( F = f^{-1}(0) \) over the origin is singular, then we call \( f \) a local family. Denote by \( \tilde{F} \) the minimal normal crossing model of \( F \), i.e., \( \tilde{F} \) is normal crossing without redundant \((-1)\text{-}\)curves, and \( \tilde{F}_{\text{red}} \) has at worst ordinary double points as its singularities.

If \( f \) is a family of curves, and \( F_1, \ldots, F_s \) are all the singular fibers of \( f \), then there is a local family \( f_{F_k} \) with the central fiber \( F_k \) for each \( k \). Denote by \( \phi_{F_k} \) the monodromy homeomorphism of \( f_{F_k} \). Then \( \phi_{F_k} \) is a pseudo-periodic map of negative twist (see [13, Theorem 7.1]). We denote by \( \mathcal{G}_{F_k} \) the admissible system of cut curves of \( \phi_{F_k} \).

Let \( \mathcal{M}_g \) be the moduli space of smooth curves of genus \( g \) over the field of complex numbers \( \mathbb{C} \), and \( \Delta_0, \Delta_1, \ldots, \Delta_{[g/2]} \) be the boundary divisors of the Deligne-Mumford compactification \( \overline{\mathcal{M}}_g \). The family \( f \) induces a holomorphic map from \( \mathcal{Y} \) to the moduli space \( \overline{\mathcal{M}}_g \):

\[
J : \mathcal{Y} \to \overline{\mathcal{M}}_g.
\]

For each \( \mathbb{Q} \)-divisor class \( \eta \) of the moduli space \( \overline{\mathcal{M}}_g \), we can define an invariant \( \eta(f) = \deg J^*\eta \) which satisfies the base change property, i.e., if \( f : \tilde{X} \to \tilde{Y} \) is the pullback fibration of \( f \) under a base change \( \pi : \tilde{Y} \to Y \) of degree \( d \), then \( \eta(f) = d \cdot \eta(f) \) (see [17]).

Let \( \delta \) be the \( \mathbb{Q} \)-divisor class corresponding to \( \Delta_i \), and

\[
\delta = \delta_0 + \delta_1 + \cdots + \delta_{[g/2]};
\]

The corresponding modular invariants of \( f \) are \( \delta_i(f) \) and \( \delta(f) \). Hence, \( \delta_i(f) = 0 \) if and only if the image \( J(Y) \) of \( f \) does not intersect with \( \Delta_i \). In particular, if \( f : X \to Y \) is isotrivial, then \( \delta(f) = 0 \).

Modular invariants are basic in the study of fibrations of algebraic surfaces and moduli spaces of algebraic curves (see [9,10,14,17]). In arithmetic algebraic geometry, modular invariants are some heights of algebraic curves, and can be used to give uniformity properties of curves (see [11]).

The first result of this paper is a relationship between fractional Dehn twist coefficients and modular invariants. Precisely, we have the following theorem.

**Theorem 1.1.** Let \( \phi \in \text{Mod}(\Sigma_g) \) be a pseudo-periodic map of negative twist. Then

\[
\delta(f_\phi) = \sum_{\gamma \in \mathcal{G}} |c(\phi, \gamma)|.
\]

In particular, if \( f : X \to Y \) is a family of curves, then

\[
\delta(f) = \sum_{k=1}^s \sum_{\gamma \in \mathcal{G}_{F_k}} |c(\phi_{F_k}, \gamma)|.
\]

We say that a singularity \( p \) in a semistable curve \( F \) is of type \( i \) if its partial normalization at \( p \) consists of two connected components of arithmetic genera \( i \) and \( g - i \), where \( g \geq 0 \), and is connected for \( i = 0 \). The general point of \( \Delta_0 \) is an irreducible curve with a node \( p \) of \( \alpha \)-type, i.e., an ordinary double point, and hence the node \( p \) is of type \( 0 \). The general point of \( \Delta_i \) corresponds to a semistable curve with only one node of type \( i \) (\( i \geq 1 \)) as the following Figure 1.

\[\begin{array}{c}
\text{genus } i \\
\downarrow \\
p \\
\downarrow \\
\text{genus } g - i
\end{array}\]

**Figure 1** Node of type \( i \) (\( i \geq 1 \)}
Topologically, if \( \gamma \) is a non-separated cut curve, then \( \gamma \) is said to be of type 0; if \( \gamma \) is separated, and the least genus of the two connected components is \( i \geq 1 \), then \( \gamma \) is said to be of type \( i \). Set \( \mathcal{C}_i = \{ \gamma \in \mathcal{C} : \gamma \) is of type \( i \} \), \( i \geq 0 \).

**Theorem 1.2.** Let \( \phi \in \text{Mod}(\Sigma_g) \) be a pseudo-periodic map of negative twist. Then

\[ \delta_i(f_\phi) = \sum_{\gamma \in \mathcal{C}_i} |c(\phi, \gamma)|. \tag{1.4} \]

In particular, if \( f : X \to Y \) is a family of curves, then

\[ \delta_i(f) = \sum_{k=1}^\infty \sum_{\gamma \in \mathcal{C}_{k,i}} |c(\phi_{F_k}, \gamma)|. \tag{1.5} \]

As a corollary, we can give a characterization of the pseudo-periodic map of negative twist with the nontrivial fractional Dehn twist coefficient. See [12, Corollary 1.4] for another characterization in the aspect of topology of 3-manifold.

**Corollary 1.3.** Let \( \phi \) be a pseudo-periodic map of negative twist. Then all the fractional Dehn twist coefficients of \( \phi \) vanish if and only if the image \( J(Y) \) of \( f_\phi \) by the moduli map \( J \) is contained in \( \mathcal{M}_g \). Moreover, the fractional Dehn twist coefficients \( c(\phi, \gamma) \) vanish for all the cut curves \( \gamma \) of type \( i \) if and only if \( J(Y) \) does not intersect with the boundary divisor \( \Delta_i \) of \( \mathcal{M}_g \).

Let \( \phi \) be a pseudo-periodic map of negative twist, and \( F \) be the central fiber of \( f_\phi \). We will show that each cut curve \( \gamma \) corresponds to a unique principal chain \( \mathcal{C}_\phi \) of \( F \), and there is a relation between the data of \( F \) with that of \( \mathcal{C} \) in Section 4. Furthermore, we get a formula of \( c(\phi, \gamma) \) as follows.

**Theorem 1.4** (See Theorem 4.5). It holds that

\[ |c(\phi, \gamma)| = \frac{H(\mathcal{C}_\phi)}{m_\gamma}. \]

For the bounds of fractional Dehn twist coefficients, there are many results in different contexts (see [4, Theorem 1], \([7\), Section 7\] and \([8\), Theorem 2.16]). In this paper, we will give some uniform lower bounds of fractional Dehn twist coefficients as follows.

**Theorem 1.5.** Let \( \phi \in \text{Mod}(\Sigma_g) \) be a pseudo-periodic map of negative twist. For each cut curve \( \gamma \) with \( c(\phi, \gamma) \neq 0 \), we have

\[ |c(\phi, \gamma)| \geq \begin{cases} \frac{1}{4g(g+1)^2}, & \text{if } g \text{ is even}, \\ \frac{1}{4g^2(g+2)}, & \text{if } g \text{ is odd}. \end{cases} \tag{1.6} \]

Furthermore, if \( \gamma \) is a cut curve of type \( i \) with \( c(\phi, \gamma) \neq 0 \), then

\[ |c(\phi, \gamma)| \geq \begin{cases} \frac{1}{4g^3}, & \text{if } i = 0, \\ \frac{1}{g(4i+2)(1(g-i)+2)}, & \text{if } i \geq 1. \end{cases} \tag{1.7} \]

The modular invariants of families of curves can be generalized to holomorphic foliations and even to differential equations\(^1\). It is natural to raise the following problem.

**Problem 1.6.** For homeomorphisms which are not pseudo-periodic map of negative twist, is there a relation between fractional Dehn twist coefficients with generalized modular invariants?

**Notations.** If \( m \) and \( n \) are two integers, we set \( \sigma(m, n) \) to be the remainder of the division of \( n \) by \( m \), i.e., \( \sigma(m, n) \) is the integer satisfying \( 0 \leq \sigma(m, n) < n \) and \( \sigma(m, n) \equiv m \) (mod \( n \)).

Let \( a = (a_1, a_2, \ldots, a_r) \) be an ordered sequence of integers, and \( d(a) = \gcd(a_1, \ldots, a_r) \).

It holds that \( \lambda_1.a = \frac{a}{\gcd(a)}, \lambda_2.a = \frac{a}{\gcd(a)}, \sigma_1.a = \sigma(\frac{a}{\gcd(a)}, \lambda_1.a) \) and \( \sigma_2.a = \sigma(\frac{a}{\gcd(a)}, \lambda_2.a) \).

\(^1\) Tan S L. On the Poincaré-Painlevé problem and Chern numbers of a holomorphic foliation. Preprint
2 Modular invariants of families

By blowing up singularities of fibers of \( f : X \to Y \), we can obtain a family \( \tilde{f} : \tilde{X} \to Y \) satisfying that each singular fiber of \( \tilde{f} \) is minimal normal crossing. Let \( \pi : \tilde{Y} \to Y \) be a base change of degree \( d \). The pullback fibration \( f \) of \( \tilde{f} \) under \( \pi \) is defined as the relative minimal model of the desingularization of \( \tilde{X} \times Y \to \tilde{Y} \).

See the following diagram for this construction:

\[
\begin{array}{cccccccccc}
\tilde{X} & \xleftarrow{\rho} & X_2 & \xrightarrow{\rho_2} & X_1 & \xrightarrow{\rho_1} & X \times_Y \tilde{Y} & \xrightarrow{\Pi'} & \tilde{X} \\
\tilde{f} & \downarrow & f_2 & \downarrow & f_1 & \downarrow & \downarrow & f & \downarrow \\
\tilde{Y} & = & \tilde{Y} & = & \tilde{Y} & = & \tilde{Y} & \xrightarrow{\pi} & Y.
\end{array}
\]

Here, \( \rho_1 \) is the normalization, \( \rho_2 \) is the minimal desingularization of \( X_1 \), and \( \rho : X_2 \to \tilde{X} \) is the contraction of the vertical \((-1)\)-curves. We call \( \pi \) a stabilizing base change if \( \tilde{f} \) is semistable.

Now we consider the above construction locally. Let \( F' \) be a fiber of \( f \) over \( p \in Y \). Assume that \( \pi \) is totally ramified over \( p \), i.e., \( \pi^{-1}(p) \) contains only one point \( \tilde{p} \). In this case, \( \pi \) is defined by \( z = w^d \) locally, where \( z \) (resp. \( w \)) is the local parameter of \( Y \) (resp. \( \tilde{Y} \)). Denote by \( \tilde{F} \) the fiber of \( \tilde{f} \) over \( \tilde{p} \in \tilde{Y} \). If \( \tilde{F} \) is semistable, then \( \tilde{F} \) is called the \( d \)-th semistable model of \( F' \).

Set \( \Pi_2 = \Pi' \circ \rho_1 \circ \rho_2 : X_2 \to \tilde{X} \) and \( \Pi = \Pi_2 \circ \tilde{\rho}^{-1} : \tilde{X} \to \tilde{X} \). Then \( \Pi \) is a well-defined rational morphism. For any irreducible smooth component \( C \) of \( \tilde{F} \), we can define the induced morphism \( \Pi_C : \tilde{C} \to C \) by the unique extension, where \( C \) is the image. On the other hand, if \( C \) is an irreducible component of \( \tilde{F} \), we always use \( C \) to denote an irreducible element of \( \Pi^{-1}(C) \). Here, the element is chosen arbitrarily.

Let \( F \) be a singular fiber of \( f \), and \( \tilde{F} \) be its \( d \)-th semistable model. Denote by \( \delta_i(\tilde{F}) \) the number of nodes of type \( i \) in \( \tilde{F} \). Then we define

\[
\delta_i(\tilde{F}) := \frac{\delta_i(\tilde{F})}{d}, \quad i = 0, 1, \ldots, \lfloor g/2 \rfloor,
\]

which is independent of the choice of the semistable model \( \tilde{F} \) of \( F \). Let \( F_1, \ldots, F_s \) be all the singular fibers of \( f \). If we choose a stabilizing base change totally ramified over \( f(F_1), \ldots, f(F_s) \), then the modular invariant \( \delta_i(\tilde{F}) \) of \( \tilde{f} \) is \( \delta_i(\tilde{F}) = \delta_i(F_1) + \cdots + \delta_i(F_s) \) for each \( i \) by [3, Proposition 3.92]. So we have

\[
\delta_i(\tilde{F}) = \delta_i(F_1) + \cdots + \delta_i(F_s), \quad i = 0, 1, \ldots, \lfloor g/2 \rfloor.
\]

An irreducible component \( C \) of \( \tilde{F} \) is said to be principal if either \( C \) is not smooth rational or \( C \) meets other components of \( \tilde{F}_{\text{red}} \) at no less than 3 points.

**Definition 2.1.** Let \( \tilde{F} \) be the minimal normal crossing model of a singular fiber \( F \).

1. An H-J chain \( C \) of \( \tilde{F} \) is the following subgraph of the dual graph \( G(\tilde{F}) \) of \( \tilde{F} \):

\[
\begin{array}{ccccccc}
a_r & a_{r-1} & a_{r-2} & a_1 & n(C) = a_0 \\
\Gamma_r & \Gamma_{r-1} & \Gamma_{r-2} & \Gamma_1 & C
\end{array}
\]

where \( C \) is a principal component of \( \tilde{F} \), \( n(C) = \text{mult}_C(\tilde{F}) \), \( a_j = \text{mult}_{\Gamma_j}(\tilde{F}) \), \( \Gamma_j \cong \mathbb{P}^1 \) is not principal for each \( 1 \leq j \leq r \), and \( \Gamma_r \) meets no other component of \( \tilde{F} \).

2. If \( C_1 \) and \( C_2 \) are two principal components of \( \tilde{F} \) \((C_1 \text{ and } C_2 \text{ may be the same})\). Let \( C \) be the following subgraph of \( G(\tilde{F}) \):

\[
\begin{array}{ccccccc}
n(C_1) = a_0 & a_1 & a_2 & a_r & n(C_2) = a_{r+1} \\
C_1 & \Gamma_1 & \Gamma_2 & \Gamma_r & C_2
\end{array}
\]

where \( n(C_i) = \text{mult}_{C_i}(\tilde{F}) \), \( a_j = \text{mult}_{\Gamma_j}(\tilde{F}) \), and \( \Gamma_j \cong \mathbb{P}^1 \) is not principal. Then we call \( C \) a principal chain between \( C_1 \) and \( C_2 \) with the multiplicity sequence \( a(C) = (a_0, a_1, \ldots, a_{r+1}) \). If there is no confusion, we set \( C = (C_1, C_2) \).
We denote by \( PC(\bar{F}) \) the set of all the principal chains of \( \bar{F} \). If all the nodes of \( \Pi^{-1}(p) \) are of type \( i \) for any node \( p \) of \( C \), then we call \( C \) a principal chain of type \( i \). Denote by \( PC_i(\bar{F}) \) the set of all the principal chains of \( \bar{F} \) of type \( i \).

**Example 2.2.** The following is a dual graph of a singular fiber of genus 2. There are exactly two principal components \( C_1 \) and \( C_2 \). The subgraph between \( C_1 \) and \( C_2 \) is a principal chain, and chains between \( C_i \) and \( \Gamma_j \) \( ((i,j) = (1,1), (1,2), (2,3), (2,4)) \) are all the H-J chains. It holds that

\[
\begin{align*}
3 & \quad \Gamma_1 \quad 6 \quad 5 \quad 4 \quad 2 \\
4 & \quad C_1 \\
6 & \quad C_2 \\
5 & \quad \Gamma_2 \\
4 & \quad \Gamma_3 \\
1 & \quad \Gamma_4.
\end{align*}
\]

3 Fractional Dehn twists

Let \( \Sigma \) be a connected real 2-dimensional manifold with or without boundary. When we emphasize its complex structure, we call \( \Sigma \) a Riemann surface.

Let \( \phi: \Sigma \to \Sigma \) be a pseudo-periodic map. For each cut curve \( \gamma \) in the admissible system \( \mathcal{C} \), there exists a minimal integer \( \alpha \) such that \( \phi^\alpha(\gamma) = \gamma \), i.e., \( \phi^\alpha(\gamma) = \gamma \) as a set and \( \phi^\alpha \) preserving the orientation of \( \gamma \). The curve \( \gamma \) is said to be amphidrome if \( \alpha \) is even and \( \phi^{\alpha/2}(\gamma) = -\gamma \) (where \( -\gamma \) denote the same \( \gamma \) with the opposite directions assigned) and non-amphidrome otherwise. There exists a minimal integer \( L \) such that \( \phi^L \) restricted to an annulus of \( \gamma \) is isotopic to a Dehn twist of \( e \) times \( (e \in \mathbb{Z}) \). The rational number \( ey/L \) is called the screw number of \( \phi \) about \( \gamma \), and is denoted by \( s(\gamma) \). We may always assume that \( s(\gamma) \neq 0 \) for each \( \gamma \in \mathcal{C} \) (see [13, p.5]).

Let \( \phi: \Sigma \to \Sigma \) be a periodic homeomorphism of order \( n \geq 2 \), and \( p \) be a point on \( \Sigma \). There is a positive integer \( m_p \) such that the points \( p, \phi(p), \ldots, \phi^{m_p-1}(p) \) are mutually distinct and \( \phi^m(p) = p \). If \( m_p = n \), we call the point \( p \) a simple point of \( \phi \), while if \( m_p < n \), we call \( p \) a multiple point of \( \phi \).

Let \( \gamma \) be a cut curve in \( \mathcal{C} \), and \( m = m_{\gamma} \) be the smallest positive integer such that \( \phi^m(\gamma) = \gamma \). The restriction of \( \phi^m \) to \( \gamma \) is a periodic map of order, say, \( \lambda \geq 1 \). Let \( q \) be any point on \( \gamma \), and suppose that the images of \( q \) under the iteration of \( \phi^m \) are ordered \( (q, \phi^{m\sigma}(q), \phi^{2m\sigma}(q), \ldots, \phi^{(\lambda-1)m\sigma}(q)) \) viewed in the direction of \( \gamma \), where \( \sigma \) is an integer with \( 0 \leq \sigma \leq \lambda - 1 \), \( \text{gcd}(\sigma, \lambda) = 1 \), and \( \sigma = 0 \) if and only if \( \lambda = 1 \). Then the action of \( \phi^m \) on \( \gamma \) is the rotation of angle \( 2\pi \delta/\lambda \) with a suitable parametrization of \( \gamma \) as an oriented circle. The triple \( (m, \lambda, \sigma) \) is called the valency of \( \gamma \) with respect to \( \phi \).

We define the valency of a boundary curve (i.e., a connected component of the boundary \( \partial \Sigma \)) as its valency with respect to \( \phi \), assuming that it has the orientation induced by the surface \( \Sigma \). The valency of a multiple point \( p \) is defined to be the valency of the boundary curve \( \partial D_p \), oriented from the outside of a disk neighborhood \( D_p \) of \( p \).

Suppose that \( \pi: \Sigma \to \Sigma' \) is the \( n \)-fold cyclic covering induced by \( \phi \), where \( \Sigma' \) is the quotient surface of \( \Sigma \) with respect to \( \phi \). Let \( B_\phi = \{q_1, \ldots, q_s\} \subseteq \Sigma' \) be the set of branch points. If \( \tilde{q}_i \) is a point of the pre-image of \( q_i \), and let the valency of \( \tilde{q}_i \) be \( (m_i, \lambda_i, \sigma_i) \). Then we know that \( m_i \) is the number of points in the pre-image of \( q_i \) and \( \lambda_i = n/m_i \). Since the valencies of points in the pre-image of \( q_i \) are the same, we can define the valency of \( q_i \) to be the valency of \( \tilde{q}_i \).

4 Correspondence between chains and cut curves

Let \( f: S \to \Delta \) be a local family of Riemann surfaces of genus \( g \), \( \bar{F} \) be a semistable model of the singular fiber \( F \) of \( f \), and \( \phi_f: \Sigma_g \to \Sigma_3 \) be the monodromy homeomorphism of \( f \).

**Lemma 4.1.** Let \( \mathcal{C} \) be the admissible system of cut curves of \( \phi_f \). Then there is a 1-1 correspondence between \( \mathcal{C} \) and \( PC(\bar{F}) \) of \( \bar{F} \).
Moreover, there is a 1-1 correspondence between the cut curves $\mathcal{C}_i$ of type $i$ and the principal chains $PC_i(\bar{F})$ of type $i$ of $\bar{F}$ for each $i = 0, 1, \ldots, [g/2]$.

**Proof.** By shrinking each curve $\gamma$ of $\mathcal{C}$ to a point, say $p_\gamma$, we naturally associate a stable Riemann surface. The topological structure of the stable Riemann surface coincides with the moduli point $F^s$ of $f : S \to \Delta$ (see [1, Lemma 4.2]), where $F^s$ is the stable model of $F$. Note that $\bar{F}$ has no H-J chains. The stable curve $F^s$ is obtained by contracting $(-2)$-curves in $\bar{F}$, and hence every principal chain of $\bar{F}$ is contracted to a point in $F^s$. Let $\tilde{C}_\gamma \in PC(\bar{F})$ be the principal chain contracted to the point $p_\gamma$ in $F^s$. Then $\gamma \mapsto \tilde{C}_\gamma$ gives a 1-1 correspondence between the cut curves $\mathcal{C}$ and the principal chains $PC(\bar{F})$ of $\bar{F}$.

Since the stable Riemann surface and $F^s$ are the same, the above correspondence preserves the type, by the definition of the type of principal chains of $\bar{F}$ and that of the type of cut curves.

In the following, for each $\gamma \in \mathcal{C}$, we denote by $\tilde{C}_\gamma \in PC(\bar{F})$ the corresponding principal chain (see Lemma 4.1).

**Theorem 4.2.** There is a correspondence between $PC(\bar{F})$ and the admissible system $\mathcal{C}$ of cut curves of $\phi_f$ satisfying the following:

1. each $\mathcal{C} \in PC(\bar{F})$ corresponds to a cyclic orbit $\gamma_1, \ldots, \gamma_m$ in $\mathcal{C}$ under the permutation caused by $\phi_f$, where
   \[ \Pi^{-1}(\mathcal{C}) := \{ \tilde{C} \in PC(\bar{F}) : \Pi(\tilde{C}) = \mathcal{C} \} = \{ \tilde{C}_{\gamma_1}, \tilde{C}_{\gamma_2}, \ldots, \tilde{C}_{\gamma_m} \} ; \]  
   \[ (4.1) \]

2. each $\gamma \in \mathcal{C}$ corresponds to a unique principal chain $C_\gamma \in PC(\bar{F})$ with $C_\gamma = \Pi(\tilde{C}_\gamma)$.

Moreover, the correspondence preserves the type of cut curves and of principal chains. Precisely, we have

1. (i) each $\mathcal{C} \in PC_i(\bar{F})$ corresponds to a cyclic orbit $\gamma_1, \ldots, \gamma_m$ in $\mathcal{C}$; and

2. (ii) each $\gamma \in \mathcal{C}$ corresponds to a unique principal chain $C_\gamma \in PC_i(\bar{F})$, where $C_\gamma = \Pi(\tilde{C}_\gamma)$.

**Proof.** For every principal chain $\tilde{C} \in PC(\bar{F})$ of $\bar{F}$, $\Pi(\tilde{C}) \in PC(\bar{F})$ is a principal chain of $\bar{F}$. For each $\mathcal{C} \in PC(\bar{F})$, its pre-image $\Pi^{-1}(\mathcal{C})$ consists of principal chains in $PC(\bar{F})$. The pre-image

\[ \Pi^{-1}(\mathcal{C}) = \{ \tilde{C} \in PC(\bar{F}) : \Pi(\tilde{C}) = \mathcal{C} \} \]

is a cyclic orbit induced by the base change $\pi : \bar{Y} \to Y$, by the construction of semistable reduction (see Section 2). By the definitions of the type of principal chains of $\bar{F}$ and $\tilde{F}$, the correspondence preserves the type. Then the result is directly from Lemma 4.1. □

Let $\gamma \in \mathcal{C}$ be a cut curve, and $\gamma_1 = \gamma, \gamma_2, \ldots, \gamma_m$ be the cyclic orbit of $\gamma$ under the permutation caused by $\phi_f$. Then we call $m_\gamma$ the multiplicity of $\gamma$, which is the length of the orbit. Therefore, by Theorem 4.2,

\[ m_\gamma = \# \Pi^{-1}(C_\gamma) \]  
\[ (4.2) \]

It is easy to see that $C_\gamma = \Pi(\tilde{C}_{\gamma_1}) = \cdots = \Pi(\tilde{C}_{\gamma_m})$. Let $C \in PC(\bar{F})$ be a principal chain in Definition 2.1. We define

\[ H(C) := \sum_{i=0}^{r} \frac{\gcd(\gamma_i, \gamma_{i+1})^2}{\gamma_i \gamma_{i+1}}. \]
\[ (4.3) \]

For the pseudo-periodic map $\phi_f$, we may assume that (i) there exists a system of disjoint annular neighborhoods $\{ A_\gamma \}_{i=1}^r$ of the admissible system of cut curves subordinate to $\phi$ such that $\phi(\mathcal{A}) = \mathcal{A}$, where $\mathcal{A} = \bigcup_{i=1}^r A_\gamma$, and (ii) $\phi\big|_\mathcal{B} : \mathcal{B} \to \mathcal{B}$ is a periodic map, where $\mathcal{B} = \Sigma_{\mathcal{A}} - \text{Int}(\mathcal{A})$ (see [13, Theorem 2.1]).

**Theorem 4.3.** Let $\gamma \in \mathcal{C}$ be a cut curve of $\phi_f$, and the multiplicity sequence of $C_\gamma = (C_1, C_2) \in PC(\bar{F})$ be $a_\gamma = (a_0, a_1, \ldots, a_{r+1}).$

1. If $\gamma$ is non-amphidrome, then the valencies of the two boundaries of the annulus $A_\gamma$ are

\[ (d(a_\gamma), \lambda_{1,a_\gamma}, \sigma_{1,a_\gamma}), \quad (d(a_\gamma), \lambda_{2,a_\gamma}, \sigma_{2,a_\gamma}). \]

Moreover, $m_\gamma = d(a_\gamma)$ and $|s(\gamma)| = H(C_\gamma).$
(2) If $\gamma$ is amphidrome, then we may assume that $\text{mult}_{\mathcal{C}_e}(F) = d(a_\gamma)$ and the valencies of the two boundaries of the annulus $A_\gamma$ are the same, which is $(d(a_\gamma), \lambda_1 a_\gamma, \sigma_1 a_\gamma)$. Moreover, $m_\gamma = \frac{1}{2}d(a_\gamma)$ and $|s(\gamma)| = 2H(\mathcal{C}_e)$.

**Proof.** **Step 1.** We describe the construction of the numerical topological space $F_{\text{top}}$ from the pseudo-periodic map $\phi: \Sigma_g \to \Sigma_g$ according to [13]. Here, a numerical topological space is a topological space attached a positive integer to each irreducible component, and the attached positive integer is the multiplicity of the irreducible component.

Now we take a disjoint union $\mathcal{D}$ of invariant disk neighborhoods of all the multiple points in $\mathcal{B}$, and let $\mathcal{B}'$ denote $\mathcal{B} - \text{Int}(\mathcal{D})$. We further classify the annuli in $\mathcal{A}$ into $\mathcal{A}_p$ and $\mathcal{A}_m$ according to their character of being amphidrome or non-amphidrome.

It is proved that (see [13, Theorem 7.4]): There exist numerical topological spaces $Ch(\mathcal{B}')$, $Ch(\mathcal{D})$, $Ch(\mathcal{A}_m)$ and $Ch(\mathcal{A}_p)$ such that (see [13, p.92])

$$F = Ch(\mathcal{B}') \cup Ch(\mathcal{D}) \cup Ch(\mathcal{A}_m) \cup Ch(\mathcal{A}_p)$$

as numerical topological spaces, and the mentioned spaces are defined as follows:

$Ch(\mathcal{B}')$: The action of $\phi|_{\mathcal{B}'}$ is free, and the well-defined quotient space $\mathcal{B}'/(\phi|_{\mathcal{B}'})$ is the topological space of the numerical surface $Ch(\mathcal{B}')$. The attached multiplicity of each connected component $P$ of $\mathcal{B}'/(\phi|_{\mathcal{B}'})$ is the number of the sheets over $P$.

$Ch(\mathcal{D})$: Let $p$ be a multiple point of $\phi$, and $p_1 = p, p_2, \ldots, p_m$ be the cyclic orbit of $p$ under the permutation caused by $\phi$. Let $P_p$ consist of the components of $\mathcal{D}$ containing $p_i$ for some $1 \leq i \leq m_p$. Then there is a numerical surface $Ch(P_p)$ shown in Figure 2, and attached the multiplicities $m_p b_0, m_p b_1, \ldots, m_p b_l$ to the components $D_0, S_1, \ldots, S_l$, where $D_0$ is a disk and $S_i$ are spheres. Here, the sequence of integers $b_0 > b_1 > \cdots > b_l = 1$ satisfies $b_j b_{j+1} \equiv 0 \pmod{b_j}$ ($j = 0, 1, \ldots, l - 1$), and $(m_p, b_0, b_1)$ is the valency of $p$ (see [13, Lemma 3.2]). The space $Ch(\mathcal{D})$ is the union of $Ch(P_p)$, where $P_p$ runs over all the multiple points.

$Ch(\mathcal{A}_m)$: Let $\gamma$ be a non-amphidrome cut curve, and $\gamma_1 = \gamma, \gamma_2, \ldots, \gamma_{m_\gamma}$ be the cyclic orbit of $\gamma$ under the permutation caused by $\phi$. Let $\mathcal{A}_i = \bigcup_{i=1}^{m_i} A_{\gamma_i}$ be the components of $\mathcal{A}$ containing $\gamma_i$. The space $Ch(\mathcal{A}_i)$ is shown in Figure 3, and attached the multiplicities $m_\gamma b_0, m_\gamma b_1, \ldots, m_\gamma b_l$ to the components $D_0, S_1, \ldots, S_{l-1}, D_l$, where $D_i$ ($i = 0, l$) are disks and $S_i$ are spheres. Here, the sequence of integers $b_0, b_1, \ldots, b_l$ satisfies $b_{i-1} + b_{i+1} \equiv 0 \pmod{b_i}$, $b_{i-1} + b_{i+1} \geq 2b_i$ ($i = 1, 2, \ldots, l - 1$),

$$\sum_{i=0}^{l-1} \frac{1}{b_i b_{i+1}} = |s|,$$

**Figure 2** The numerical space $Ch(\mathcal{D}_p)$

**Figure 3** The numerical space $Ch(\mathcal{A}_i)$ ($\gamma$ non-amphidrome)
and \((m_\gamma, b_0, \sigma(b_1, b_0))\) and \((m_\gamma, b_1, \sigma(b_{l-1}, b_l))\) are the valencies of the two boundaries of \(A_\gamma\) (see [13, Lemma 3.3]). The space \(Ch(A_{\gamma})\) is the union of \(Ch(A_{\gamma})\), where \(\gamma\) runs over all the non-amphidrome cut curves.

In this case, we also denote \(Ch(A_{\gamma})\) by \(Ch(A_{\gamma})\)' for simplicity.

\(Ch(A_{sp})\): Let \(\gamma\) be an amphidrome cut curve, and \(\gamma_1 = \gamma, \gamma_2, \ldots, \gamma_m\), be the cyclic orbit of \(\gamma\) under the permutation caused by \(\phi\). Let \(\gamma_i\) be the components of \(A\) containing \(\gamma_i\). The space \(Ch(A_{\gamma})\) is shown in Figure 4, and attached the multiplicities \(2m_\gamma b_0, 2m_\gamma b_1, \ldots, 2m_\gamma b_l, m_\gamma, m_\gamma\) to the components \(D_0, S_1, \ldots, S_l, S_1', S_2'\). Here, \(D_0\) is a disk, and \(S_i\) and \(S_i'\) are spheres. Here, the sequence of integers \(b_0, b_1, \ldots, b_{l-1}, b_l = 1\) satisfies \(b_{i-1} + b_{i+1} \equiv 0 \pmod{b_i}\) \((i = 1, 2, \ldots, l - 1)\),

\[
\sum_{i=0}^{l-1} \frac{1}{b_i b_{i+1}} = \frac{1}{2} |s|, \tag{4.6}
\]

and \((2m_\gamma b_0, \sigma(b_1, b_0))\) is the valency of the boundary of \(A_\gamma\) (see [13, Lemma 3.4]). The space \(Ch(A_{sp})\) is the union of \(Ch(A_{sp})\), where \(\gamma\) runs over all the amphidrome cut curves.

In this case, we denote by \(Ch(A_{\gamma})\)' the subspace of \(Ch(A_{\gamma})\) consisting of \(D_0, S_1, \ldots, S_l\) with their multiplicities.

**Step 2.** Since \(E\) is the admissible system of cut curves, each connected component \(P\) in \(Ch(E')\) is a principal component of \(E\), and each disk \(D_i\) \((i = 0, l)\) in \(Ch(E) \cup Ch(A_{\gamma}) \cup Ch(A_{sp})\) is part of some connected component \(P\) in \(Ch(E')\). Hence, by Definition 2.1, \(Ch(A_{\gamma})\)' is the same with the principal chain \(C_{\gamma} \in PC(F)\) as a numerical topological space for each cut curve \(\gamma\).

If \(\gamma\) is amphidrome, then \(C_{\gamma} = \langle C_1, C_2 \rangle \in PC(F)\) is the dual graph of Figure 4, where we may assume that \(D_0\) is part of \(C_1\) and \(S_1 = C_2\). Hence, \(d(C_{\gamma}) = \text{mult} C_{\gamma}(F)\).

Other results are directly from the construction of \(Ch(A_{\gamma})\) and \(Ch(A_{sp})\) above.

**Remark 4.4.** Let \(C_{H}\) be an H-J chain with the multiplicity \((a_0, a_1, \ldots, a_r)\), where \(a_r | a_0\). From the proof above, either \(C_{H}\) is part of \(Ch(A_{sp})\) or there exists a multiple point \(p\) such that \(C_{H}\) is the same with \(Ch(D_p)\) as a numerical topological space. In the latter case, the valency of the multiple point \(p\) is

\[
\left( a_r, \frac{a_0}{a_r}, \sigma \left( \frac{a_1}{a_r}, \frac{a_0}{a_r} \right) \right).
\]

**Theorem 4.5.** Let \(\phi\) be a pseudo-periodic map of negative twist, \(\gamma \in \mathcal{E}\), and \(C_{\gamma} \in PC(F)\) be the corresponding principal chain of the fiber \(\tilde{F}\) of \(f_\phi\). Then

\[
|c(\phi, \gamma)| = \frac{H(C_{\gamma})}{m_{\gamma}} = \begin{cases} 
\frac{|s|}{m_{\gamma}} & \text{if } \gamma \text{ is non-amphidrome}, \\
\frac{|s|}{2m_{\gamma}} & \text{if } \gamma \text{ is amphidrome}.
\end{cases}
\]

**Proof.** By using the above notations, \(\gamma_1 = \gamma, \gamma_2, \ldots, \gamma_m\) is the cyclic orbit of \(\gamma\), \(A_{\gamma_i}\) are disjoint annuli, and \(A_{\gamma} = \bigcup_{i=1}^{m_{\gamma}} A_{\gamma_i}\). Hence, \(\phi|_{A_{\gamma}} : A_{\gamma} \to A_{\gamma}\) is a homeomorphism, and the restriction of \(\phi\) to the boundary \(\partial A_{\gamma} = \bigcup_{i=1}^{m_{\gamma}} \partial A_{\gamma_i}\) is periodic. We know that \(m_{\gamma}\) is the smallest positive integer such that

\[(i) \; \phi^{m_{\gamma}}(A_{\gamma_i}) = A_{\gamma_i}.\]
If $\gamma$ is amphidrome, then $\alpha_\gamma = 2m_\gamma$ is the smallest positive integer satisfying (i) and (ii) $\phi^{\alpha_\gamma}$ does not interchanged the boundary components. If $\gamma$ is non-amphidrome, then $\alpha_\gamma = m_\gamma$ is the smallest positive integer satisfying (i) and (ii) (see [13, Theorems 2.3 and 2.4]).

Let $l_\gamma$ be a non-zero integer such that $\phi^{l_\gamma} |_{\partial A_\gamma}$ is the identity. Then $l_\gamma$ is a multiple of $\alpha_\gamma$, and $\phi^{|\gamma| : A_i \to A_i}$ is the result of $e_\gamma$ full Dehn twists, where $e_\gamma$ is an integer. Then (see [13, p.22])

$$s(A_i) := e_\gamma\alpha_\gamma/l_\gamma = s(\gamma), \quad i = 1, 2, \ldots, m_\gamma.$$ 

Thus

$$|e(\phi, \gamma)| = |e_\gamma| = \frac{|s_\gamma|}{\alpha_\gamma} = \begin{cases} \frac{|s_\gamma|}{m_\gamma}, & \text{if } \gamma \text{ is non-amphidrome}, \\ \frac{|s_\gamma|}{2m_\gamma}, & \text{if } \gamma \text{ is amphidrome}. \end{cases}$$

Hence by Theorem 4.3, $|e(\phi, \gamma)| = H(\mathcal{C}_\gamma)/m_\gamma$. □

5 Proof of the theorems in Section 1

Proof of Theorem 1.1. From [11, Lemma 3.1], we have

$$\delta(F) = \sum_{i=0}^{\lfloor g/2 \rfloor} \delta_i(F) = \sum_{\mathcal{C} \in PC(\mathcal{F})} H(\mathcal{C}).$$

By (4.2), Theorem 4.5 and the correspondences in Section 4, we have

$$\delta(F) = \sum_{\mathcal{C} \in PC(\mathcal{F})} H(\mathcal{C}) = \sum_{\mathcal{C} \in PC(\mathcal{F}), \mathcal{C} \in \Pi^{-1}(\mathcal{C}_\gamma)} H(\mathcal{C}_\gamma) \sum_{\mathcal{C} \in PC(\mathcal{F})} \frac{H(\mathcal{C}_\gamma)}{m_\gamma} = \sum_{\mathcal{C} \in PC(\mathcal{F})} \sum_{\mathcal{C} \in \Pi^{-1}(\mathcal{C}_\gamma)} |e(\phi, \gamma)| = \sum_{\gamma \in \mathcal{E}} |e(\phi, \gamma)|.$$

The proof is completed. □

Proof of Theorem 1.2. From [11, Lemma 3.1], we have

$$\delta_i(F) = \sum_{\mathcal{C} \in PC_i(\mathcal{F})} H(\mathcal{C}).$$

Then the proof is similar to that of Theorem 1.1, by Lemma 4.1. □

Proof of Theorem 1.5. By the definition of the admissible system $\mathcal{E} = \{\gamma_i\}_{i=1}^r$ of cut curves, we know that $g \geq \#\mathcal{E} = r$. Furthermore, for each $\gamma \in \mathcal{E}$, the cyclic orbit $\gamma_1, \ldots, \gamma_m_\gamma$ is a subset of $\mathcal{E}$, and then $g \geq m_\gamma$.

If $\gamma$ is a cut curve of type 0, thus by the proof of [11, Theorem 1.4] and Theorem 4.5, we have

$$|e(\phi, \gamma)| = \frac{H(\mathcal{C}_\gamma)}{m_\gamma} \geq \frac{1}{4g^3}.$$ 

Similarly, if $\gamma$ is a cut curve of type $i \geq 1$, then

$$|e(\phi, \gamma)| = \frac{H(\mathcal{C}_\gamma)}{m_\gamma} \geq \frac{1}{g(4i + 2)(4(g - i) + 2)}.$$ 

Hence for any $\gamma \in \mathcal{E}$, we have

$$|e(\phi, \gamma)| \geq \min \left\{ \frac{1}{4g}, \min \left\{ \frac{1}{g(4i + 2)(4(g - i) + 2)} : i = 1, 2, \ldots, \lfloor g/2 \rfloor \right\} \right\}$$
\[
\frac{1}{g(\lfloor g/2 \rfloor + 2)(4(g - \lfloor g/2 \rfloor) + 2)} \\
\begin{cases} 
\frac{1}{4g(g + 1)^2}, & \text{if } g \text{ is even,} \\
\frac{1}{4g^2(g + 2)}, & \text{if } g \text{ is odd.}
\end{cases}
\]

The proof is completed.

The above uniform lower bounds of fractional Dehn twist coefficients for each \( g \geq 2 \) are not strict. It is interesting to obtain sharp lower bounds for these coefficients.

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**References**

1. Ashikaga T, Ishizaka M. Classification of degenerations of curves of genus three via Matsumoto-Montesinos’ theorem. Tohoku Math J (2), 2002, 54: 195–226
2. Gabai D, Oertel U. Essential laminations in 3-manifolds. Ann of Math (2), 1989, 130: 41–73
3. Harris J, Morrison I. Moduli of Curves. Graduate Texts in Mathematics, vol. 187. New York: Springer-Verlag, 1998
4. Hedden M, Mark T. Floer homology and fractional Dehn twists. Adv Math, 2018, 324: 1–39
5. Honda K, Kazez W, Matic G. Right-veering diffeomorphisms of compact surfaces with boundary. Invent Math, 2007, 169: 427–449
6. Honda K, Kazez W, Matic G. Right-veering diffeomorphisms of compact surfaces with boundary II. Geom Topol, 2008, 12: 2057–2094
7. Ito T, Kawamuro K. Essential open book foliation and fractional Dehn twist coefficient. Geom Dedicata, 2017, 187: 17–67
8. Kazez W, Roberts R. Fractional Dehn twists in knot theory and contact topology. Geom Dedicata, 2013, 13: 3603–3637
9. Liu X L. Modular invariants and singularity indices of hyperelliptic fibrations. Chin Ann Math Ser B, 2016, 37: 875–890
10. Liu X L, Tan S L. Families of hyperelliptic curves with maximal slopes. Sci China Math, 2013, 56: 1743–1750
11. Liu X L, Tan S L. Uniform bound for the effective Bogomolov conjecture. C R Math Acad Sci Paris, 2017, 355: 205–210
12. Liu Y. A characterization of virtually embedded subsurfaces in 3-manifolds. Trans Amer Math Soc, 2017, 369: 1237–1264
13. Matsumoto Y, Montesinos-Amilibia J M. Pseudo-Periodic Maps and Degeneration of Riemann Surfaces. Lecture Notes in Mathematics, vol. 2030. New York: Springer-Verlag, 2011
14. Norbury P. Stable reduction and topological invariants of complex polynomials. In: Real and Complex Singularities. Hackensack: World Scientific, 2007, 299–322
15. Roberts R. Taut foliations in punctured surface bundles, I. Proc Lond Math Soc (3), 2001, 82: 747–768
16. Roberts R. Taut foliations in punctured surface bundles, II. Proc Lond Math Soc (3), 2002, 83: 443–471
17. Tan S L. Chern numbers of a singular fiber, modular invariants and isotrivial families of curves. Acta Math Vietnam, 2010, 35: 159–172