Shooting-projection method for a small object moving under the influence of a force

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Abstract. We consider a small object in 3D moving under the influence of a force that may depend explicitly on time, on the position of the object, and on its velocity. The classical mechanics equations are assumed to hold. If the position of the object is specified at some initial and some final time, obtaining the trajectory of the object requires the solution of a two-point boundary value problem. To solve the problem various numerical techniques can be applied. This paper extends the recently proposed shooting-projection method to 3D. We introduce a Lagrangian from which, applying the principle of least action, the projection trajectory is derived. Analysis of the action reveals the meaning of the projection trajectory. Using the shooting-projection method, the considered two-point boundary value problem is solved for the case of a projectile motion in the presence of air resistance and wind.

1. Introduction

A small object of mass \( m \) travels under the influence of a force \( \mathbf{F} \in \mathbb{R}^3 \). We consider a general case where the force \( \mathbf{F} \) may depend on the time \( t \) explicitly (time-dependent force), on the position \( \mathbf{r} \in \mathbb{R}^3 \) of the object (non-uniform force field), and on the velocity \( \mathbf{v} = \dot{\mathbf{r}} \) of the object (e.g. dissipative force). The equation of motion [1, 2] for the object is

\[
m \ddot{\mathbf{r}} = \mathbf{F}(t, \mathbf{r}, \mathbf{v}).
\]

If the position of the object is specified at some initial time \( t = a \) and some final time \( t = b \), we get a two-point boundary value problem (TPBVP) [3, 4] of the form:

\[
\ddot{\mathbf{r}} = \mathbf{f}(t, \mathbf{r}, \dot{\mathbf{r}}), \ t \in [a, b],
\]

\[
\mathbf{r}(a) = \mathbf{r}_a, \ \mathbf{r}(b) = \mathbf{r}_b,
\]

where \( \mathbf{f} = \mathbf{F}/m \) is the force per unit mass. When \( \mathbf{f} \) is a nonlinear function of the position \( \mathbf{r} \) and/or the velocity \( \mathbf{v} \), the problem is nonlinear. In general, one needs to use a numerical method in order to solve equations (1)-(2). Some of the more important numerical methods for the solution of nonlinear two-point boundary value problems are the shooting-methods [5, 6], the finite difference methods [7-11].
(also known as relaxation methods [12]), the quasi-linearization method [13, 14], and the monotone iterative methods [15-17]. All of the enumerated methods are iterative methods. Brief survey of these methods and other methods and results are presented in [18, 19]. Recently, a new iterative method, called shooting-projection method, has been proposed [20]. Some of the benefits of the method are the following: (i) compared to the other shooting methods and the finite difference methods, it exhibits greater stability in certain situations; (ii) the method is quite easy to implement since it does not require calculation of derivatives nor pre-bracketing of the root; (iii) it requires only one initial guess and the solution of only one initial value problem (IVP) per iteration. In this paper the shooting-projection method is implemented in 3D to describe the motion of an object traveling under the influence of a force (1) when boundary conditions are imposed (2).

2. Shooting-projection method in 3D

Let \( \mathbf{v}_a \) be some arbitrary initial velocity and let \( \mathbf{r}(t; \mathbf{v}_a) \) be a solution to equation (1) satisfying the initial conditions

\[
\mathbf{r}(a; \mathbf{v}_a) = \mathbf{r}_a, \quad \dot{\mathbf{r}}(a; \mathbf{v}_a) = \mathbf{v}_a.
\]  

(3)

The trajectory \( \mathbf{r}(t; \mathbf{v}_a) \) is an IVP solution. It will be called a shooting trajectory. The shooting trajectory satisfies the first boundary condition but typically it does not satisfy the second boundary condition, i.e. \( \mathbf{r}(b; \mathbf{v}_a) \neq \mathbf{r}_b \). If we can find an initial velocity \( \mathbf{v}_a \) such that \( \mathbf{r}(b; \mathbf{v}_a) = \mathbf{r}_b \), then the shooting trajectory \( \mathbf{r}(t; \mathbf{v}_a) \) will be a solution to the TPBVP (1)-(2).

Let \( \mathbf{r}^* = \mathbf{r}(t; \mathbf{v}_a) \) be the shooting trajectory satisfying the initial conditions (3) and let \( \mathbf{r}^* \in \mathbb{R}^3 \) be a twice differentiable function of \( t \) which satisfies both boundary conditions (2):

\[
\mathbf{r}^*(a) = \mathbf{r}_a, \quad \mathbf{r}^*(b) = \mathbf{r}_b.
\]

(4)

The function \( \mathbf{r}^* \) does not need to satisfy the equation of motion (1). To implement the shooting-projection method, we introduce the Lagrangian

\[
\mathcal{L} = \frac{m}{2} (\mathbf{v}^* - \ddot{\mathbf{v}})^2,
\]

(5)

where \( \mathbf{v}^* = \ddot{\mathbf{r}}^* \) and \( \ddot{\mathbf{v}} = \ddot{\mathbf{r}} \). Note that \( \ddot{\mathbf{v}} \) is a known (given) function of time. In the next section we prove that (5) is the Lagrangian of an object of mass \( m \) and position \( \mathbf{r}^* \) travelling between \( (a, \mathbf{r}_a) \) and \( (b, \mathbf{r}_b) \) and experiencing, at any time \( t \in [a, b] \), a force \( \mathbf{f}(t, \mathbf{r}^*(t), \dot{\mathbf{r}}^*(t)) \). This force depends on \( t \) but does not depend on \( \mathbf{r}^* \) or \( \mathbf{v}^* \). To find the motion of the object, i.e. \( \mathbf{r}^*(t), t \in [a, b] \), we require that the function \( \mathbf{r}^* \) minimize the action

\[
S = \int_a^b \mathcal{L} \, dt.
\]

(6)

Please note that minimizing \( S \) is equivalent to minimizing the \( H^1 \)-seminorm of \( \mathbf{r}^* - \ddot{\mathbf{r}} \) as in [20]. The function \( \mathbf{r}^* \) that minimizes (6) will be called a projection trajectory. If a function \( \mathbf{r}^* \) that satisfies both boundary conditions (4) should minimize \( S \), then the Euler-Lagrange equation must hold:

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{v}}^*} \right) - \frac{\partial \mathcal{L}}{\partial \mathbf{r}^*} = 0.
\]

Performing the differentiation yields the differential equation that the projection trajectory \( \mathbf{r}^* \) should satisfy:

\[
\dddot{\mathbf{r}}^* = \ddot{\ddot{\mathbf{r}}}.
\]

(7)

Since the shooting trajectory \( \ddot{\mathbf{r}} \) satisfies the equation of motion (1), we can replace in (7) the acceleration \( \dddot{\mathbf{r}} \) by the force \( \mathbf{f}(t, \ddot{\mathbf{r}}, \dot{\mathbf{r}}) \), and then expand \( \mathbf{f} \) around \( (t, \mathbf{r}^*, \dot{\mathbf{r}}^*) \) to obtain

\[
\dddot{\mathbf{r}}^* = \mathbf{f}(t, \mathbf{r}^*, \dot{\mathbf{r}}^*) + \frac{\partial \mathbf{f}}{\partial t}(t, \mathbf{r}^*, \dot{\mathbf{r}}^*) \cdot (\dddot{\mathbf{r}} - \mathbf{r}^*) + \frac{\partial \mathbf{f}}{\partial \mathbf{r}}(t, \mathbf{r}^*, \dot{\mathbf{r}}^*) \cdot (\dddot{\mathbf{r}} - \mathbf{r}^*) + \cdots
\]

(8)
Equation (8) tells us that if the force field $f$ is uniform, i.e. $\partial f / \partial r = 0$, and independent of $v$, i.e. $\partial f / \partial v = 0$, then the projection trajectory $r^*$ is an exact solution to (1), and hence to the TPBV (1)-(2). If $\partial f / \partial r \neq 0$ and/or $\partial f / \partial v \neq 0$, then $r^*$ is an approximate solution to (1), hence to (1)-(2). In general, the closer $v_a$ to $V_a$, the better the approximation. We introduce the notation $v^*_a = v^*(a)$. It is logical to assume that $v^*_a$ will be, in general, closer to $V_a$ than $v_a$ is. Hence, we can use $v^*_a$ as a new initial velocity to find a new shooting trajectory, and so on. This is the shooting-projection iterative procedure. To find $v^*_a$ as a function of $v_a$, we integrate (7) on $[a, b]$ and then integrate the result on $[a, b]$ to get

$$r^*(b) - r^*(a) - v^*(a)(b - a) = \tilde{f}(b; v_a) - \tilde{f}(a; v_a) - \tilde{v}(a; v_a)(b - a).$$

Taking into account that the projection trajectory $r^*$ satisfies the boundary conditions (4) and that the shooting trajectory $\tilde{f}$ satisfies the initial conditions (3) we finally get

$$v^*_a = v_a - (b - a)^{-1}(\tilde{f}(b; v_a) - r_b).$$

(9)

This is the shooting-projection iteration formula in 3D. Given an initial velocity $v_a$ we can find the shooting trajectory $\tilde{f}(t; v_a)$ that satisfies the initial conditions (3) and the equation of motion (1). Using (9) we obtain a new initial velocity $v^*_a$. From the new initial velocity, we find a new shooting trajectory, and so on until the shooting trajectory satisfies the second boundary condition. Iteration (9) is a fixed point iteration. It converges locally and (at least) linearly to $V_a$ if any matrix norm of the Jacobian (w.r.t $v_a$) of the right hand side of (9) evaluated at $V_a$ is less than one [21]. Alternatively, the convergence of the iteration can be guaranteed by requiring that the right hand side of (9) be a contractive mapping in some domain $D$ around $V_a$ and the starting guess for the initial velocity lies in $D$.

The proposed iterative procedure is very easy to implement since it requires only one initial guess $v_a$ and the solution of only one initial value problem (1), (3) per iteration. Calculation of the derivatives of $f$ with respect to $r$ and $v$ are not necessary as in the shooting by Newton method. Pre-bracketing of the root $V_a$ is not necessary as in the shooting by bisection method. As shown in [20], in certain situations, the method is more stable than the other shooting and finite difference methods. Note that the method is as easy to implement in 3D as in 1D which is not the case with many of the other numerical methods. The convergence of the method is linear, and this is a disadvantage compared to the shooting by Newton method, where the convergence is quadratic. The slower convergence, however, is, to some extent, offset by the much fewer calculations at each iteration step.

3. Interpretation of the projection trajectory

Using the Lagrangian (5), we rewrite the action (6) as follows:

$$S = \int_a^b m \frac{1}{2} (v^* - \tilde{v})^2 dt = \int_a^b m \frac{1}{2} v^2 dt - \int_a^b m v^* \tilde{v} dt + \int_a^b m \tilde{v}^2 dt.$$

The second term on the right-hand side is integrated by part:

$$\int_a^b m v^* \tilde{v} dt = \int_a^b m r^* \tilde{v} dt = m r^* \tilde{v}\bigg|_a^b - \int_a^b m r^* \tilde{v} dt = m r^* \tilde{v}\bigg|_a^b - \int_a^b r^* F(t, \tilde{r}, \tilde{v}) dt.$$

Hence, the action can be written as

$$S = \int_a^b \left( m \frac{1}{2} v^2 - U(r^*, t) \right) dt + m r_a \tilde{v}(a; v_a) - m r_b \tilde{v}(b; v_a),$$

(10)

where

$$U(r^*, t) = -r^* F(t, \tilde{r}(t; v_a), \tilde{v}(t; v_a)) - m \tilde{v}^2(t; v_a).$$

(11)

The last two terms in (10) are constants and do not affect the variation of $S$. The function (11) can be regarded as a time-dependent potential. Note that the last term of (11) is a function of time only and can
be dropped if one wishes [1, 2]. To calculate the force acting on the object, we take the negative gradient of (11):

\[- \frac{\partial U(r^*, t)}{\partial r^*} = F(t, \tilde{r}(t; \tilde{v}_a), \tilde{v}(t; \tilde{v}_a)). \quad (12)\]

Therefore, the projection trajectory \( r^* \) describes the motion of an object traveling from \((a, r_0)\) to \((b, r_b)\) under the influence of the uniform \( \partial F/\partial r^* = 0 \) and velocity-independent \( \partial F/\partial v^* = 0 \) but time-dependent force (12). This force, however, is exactly the same force that acts on the object while traveling from \((a, r_0)\) to \((b, r^*(b; v_0))\) along the shooting trajectory. The difference between the shooting and the projection trajectory comes from the different initial velocities, namely \( v_a \) for the shooting trajectory and \( v_0 \) for the projection trajectory. An object with initial velocity \( v_a \) would travel along the projection trajectory \( r^* \) if it experienced a force (12). In reality, however, the force that the object would experience along \( r^* \) is slightly different, namely \( F(t, r^*, v^*) \). The projection trajectory \( r^* \) is not a solution to the TPBVP (1)-(2), but it is an approximate solution to the problem, as can be seen from equation (8).

4. Application to projectile motion

In this section we apply the shooting-projection method to projectile motion in the presence of air resistance (drag, friction) and wind. The following problem is considered. A spherical object with diameter \( d = 6 \text{ cm} \) and mass \( m = 100 \text{ g} \) is launched from position \( r_0 = (0,0,10)^T \) \( m \), where the third component \((z\text{-component})\) of the radius vector denotes the altitude. What must the initial velocity of the object be so that after \( \tau = 2.3 \text{ s} \) the object hits a target located at \( r_{\text{target}} = (40,0,15)^T \) \( m \). Take into account the air resistance and a wind with velocity \( u(r) = (10,14,0)^T (z/10)^a \text{ m/s} \), where \( z \) is third component of the radius vector \( r \), i.e. the altitude in meters, and \( \alpha \) is the Hellmann exponent [22]. Use a value \( \alpha = 0.27 \).

The equation of motion for the considered problem is

\[ \dot{\tilde{r}} = g - c|\tilde{v} - u(r)|(|\tilde{v} - u(r)|, \quad (13)\]

where \( \tilde{r} = r(t) \) is the radius vector indicating the position of the object at time \( t \), \( \tilde{v} = \dot{r} \) is the velocity of the object relative to the ground, \( g = (0,0,-9.8)^T \text{m/s}^2 \) is the gravitational force per unit mass (gravitational acceleration), while the second term is the drag force [23] per unit mass. The vector \( \tilde{v} - u(r) \) is the velocity of the object relative to the air. The coefficient \( c \) is given by \( c = \rho \pi C_d/2m \), where \( \rho \) is the density of the air, \( A = \pi (d/2)^2 \) is the area of the cross section of the object, \( C_d \) is the dimensionless drag coefficient, which for a sphere is \( C_d = 0.47 \), and \( m \) is the mass of the object. Taking into account the values of \( d \) and \( m \), and \( \rho = 1.225 \text{ kg/m}^3 \), we get \( c = 0.00814 \text{ m}^{-1} \). Note that the magnitude of the drag force is proportional to the square of \( |\tilde{v} - u| \). The expression for the drag force is valid for high values of the Reynolds number (\( \text{Re} > \sim 1000 \)), which holds for the considered case.

Using an initial velocity \( v_0 = (20,0,12)^T \text{ m/s} \) as a starting guess, we apply the shooting projection method. First, we have to find the shooting trajectory \( \tilde{r}(t; v_0) \). It satisfies the equation of motion (13) and the initial conditions

\[ \tilde{r}(0; v_0) = r_0, \quad \dot{\tilde{r}}(0; v_0) = v_0. \quad (14)\]

To find the shooting trajectory, the equation of motion, subject to initial conditions (14), is integrated numerically by the Heun’s method [11] with time-step 0.1s. Then, we calculate the new initial velocity

\[ v_0 = v_0 - \tau^{-1}(\tilde{r}(\tau; v_0) - r_{\text{target}}), \]

and improve the initial condition (14) by replacing \( v_0 \) with \( \tilde{v}_0 \). Then, a new shooting trajectory is found and so on until the second boundary condition is satisfied with certain precision, i.e. until the object hits the target. The first four shooting trajectories are shown in figure 1 and figure 2. In the figures, the target
is denoted by red circle. The first shooting trajectory is the one that is most deviated from the target and each next trajectory gets closer to the target.

Figure 1. Side view (z vs x) of the first four shooting trajectories.

Figure 2. Top view (y vs x) of the first four shooting trajectories.

As can be seen in figure 1, in the first shooting, the object goes horizontally farther away than required and climbs a bit higher. Figure 2 shows that, due to the wind, there is a considerable deviation from the target in the y direction.

The vector connecting the location of the target and the final position of the object is the deviation from the target:

\[ \mathbf{e}(\mathbf{v}_0) = \mathbf{r}(\tau; \mathbf{v}_0) - \mathbf{r}_{\text{target}} \]

The magnitude of the deviation is the distance to the target. Table 1 shows the initial velocity and the corresponding deviation and distance to the target for the first eight shooting trajectories. The results show that the first correction of the velocity reduces the distance to the target about five times. Each next correction reduces the distance between three and four times. Thus, at the eighth shooting, the magnitude of the deviation is less than one millimeter. Other initial guesses for \( \mathbf{v}_0 \) were tried and each time the iterative procedure converged successfully to the same solution.

Table 1. Initial velocity, deviation from the target, and distance to the target for the first eight shooting trajectories.

| \( \mathbf{v}_0 \)   | \( \mathbf{e}(\mathbf{v}_0) \) | \( |\mathbf{e}(\mathbf{v}_0)| \) |
|----------------------|-------------------------------|-------------------------------|
| initial velocity, m/s | deviation from the target, m | distance to the target, m     |
| x-component          | y-component                   | z-component                   |
| x-component          | y-component                   | z-component                   |

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5. Conclusion
This paper extended the shooting-projection method to 3D. The equation of motion of classical mechanics for a small object under the influence of a force was considered. Boundary conditions were imposed and a Lagrangian was introduced that, applying the principle of least action, yielded the differential equation for the projection trajectory. It was shown that the projection trajectory describes a real physical motion of an object that moves as though it was a subject to the same force that acts along the shooting trajectory. Integrating the equation for the projection trajectory yields an iteration formula for improving the initial velocity, i.e. the initial condition. The proposed iterative method was applied successfully to solve a particular boundary value problem for a projectile motion in the presence of air resistance and wind.

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