Haldane–Wu statistics and Rogers dilogarithm

Andrei G. Bytsko

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Abstract

The Haldane–Wu exclusion statistics is considered from the generalized extensive statistics point of view and certain related mathematical aspects are investigated. A series representation for the corresponding generating function is proven. Equivalence of two formulae for the central charge, derived for the Haldane-Wu statistics via the thermodynamic Bethe ansatz, is established. As a corollary, a series representation with a free parameter for the Rogers dilogarithm is found. It is shown that the generating function, the entropy, and the central charge for the Gentile statistics majorize those for the Haldane–Wu statistics (under appropriate choice of parameters). From this, some dilogarithm inequality is derived.

1 Introduction

Consider (1+1)-dimensional system of relativistic particles on an interval of length $L$. If the particle interaction is described by a factorizable scattering matrix then the boundary condition for the wave function of a particle has the form

$$\exp(iLm_k \sinh \theta_k) \prod_{l \neq k}^{N} S_{kl}(\theta_k - \theta_l) = \varsigma_k, \quad k = 1, \ldots, N,$$

where $\theta_k$ and $m_k$ are the rapidity and the mass of the particle, $S_{kl}(\theta)$ is the two–particle scattering matrix, and $N$ is the total number of particles. The phases $\varsigma_k$ can be different for different particles (their exact values are not relevant for our purposes). For simplicity we consider the case when all particles belong to the same species and have mass $m$.

Analysis of the multiparticle system in the thermodynamic limit ($L \to \infty$, but the density $N/L$ remains finite) is based on the thermodynamic Bethe ansatz. Apart from the system it uses the thermodynamic equilibrium condition, i.e., the condition of minimum of the free energy $F$ ($F = E - TS$, where $T$ — the temperature, $E$ — the total energy, $S$ — the entropy of the system). Thus, the initial data for the thermodynamic Bethe ansatz are the two–particle scattering matrix $S(\theta)$, the spectrum of particle masses, and the statistics which governs filling in states in the momentum space. The latter, so–called exclusion statistics, determines the exact form of the entropy of the system.

For one-dimensional systems, the exclusion statistics is not necessarily of fermion or boson type but can depend nontrivially on the number of particles already present in a given state. For instance, a generalized extensive statistics is defined by a choice of generating function $f(t)$ such that

$$\left( f(t) \right)^N = \sum_{n \geq 0} W(N, n) t^n,$$
where \( W(N, n) \) – the number of possible ways for \( n \) identical particles to occupy \( N \) states. It is natural to impose the condition \( f(0) = 1 \) that implies that the vacuum is realized with the probability one independently on the size of a system.

The thermodynamic Bethe ansatz allows one to obtain certain information about the ultra-violet (i.e., higher temperature) limit of the system under consideration. In particular, it allows one to find the effective central charge for the corresponding conformal model. For instance, in the case of a generalized extensive statistics, the effective central charge is given by the following formula \[2\]

\[
c = \frac{6}{\pi^2} \left[ \int_0^{x_0} \frac{dt}{t} \ln f(t) - \frac{1}{2} \ln x_0 \ln f(x_0) \right]. \tag{3}
\]

Here \( x_0 \) is the positive root of the equation

\[
\ln x_0 + \Phi \ln f(x_0) = 0, \tag{4}
\]

which is unique if \( f(t) \) is monotonically increasing and \( \Phi \geq 0 \). From the physical point of view, \( \Phi \) is related to the asymptotics of the scattering matrix, \( 2\pi i \Phi = \ln S(-\infty) - \ln S(\infty) \), but we will treat \( \Phi \) just as a free non-negative parameter.

2 Haldane–Wu statistics

The Haldane–Wu statistics \[3, 4\] is one of the most studied cases of an exotic statistics (see, e.g., \[4, 5, 6, 7, 8, 9\]). It has applications, for instance, in the quantum Hall effect theory. For this statistics, the number of possible ways for \( n \) identical particles to occupy \( N \) states is given by

\[
W_g(N, n) = \frac{(N + (1 - g)n + g - 1)!}{n!(N - gn + g - 1)!}, \tag{5}
\]

where \( 0 \leq g \leq 1 \). The Haldane–Wu statistics interpolates between fermions \((g = 1)\) and bosons \((g = 0)\).

The Haldane–Wu statistics is asymptotically extensive in the following sense. For a generalized extensive statistics \[3\], the entropy density is defined as

\[
s(\mu) = \lim_{N \to \infty} \frac{1}{N} \ln W(N, \mu N). \tag{6}
\]

One can show that (see, e.g., \[4\])

\[
s(\mu) = \ln f(x) - \mu \ln x, \tag{7}
\]

where \( x \equiv x(\mu) \) is the positive root of the equation (the prime denotes a derivative)

\[
x f'(x) = \mu f(x). \tag{8}
\]

It follows then that

\[
f(x(\mu)) \equiv f(\mu) = \exp\{s(\mu) - \mu \partial_{\mu} s(\mu)\}. \tag{9}
\]

In the case of the Haldane–Wu statistics, application of the Stirling formula to \[3\] yields

\[
s_g(\mu) = (1 + \mu(1 - g)) \ln(1 + \mu(1 - g)) - \mu \ln \mu - (1 - g\mu) \ln(1 - g\mu). \tag{10}
\]

Now, comparison with \[3\] shows that

\[
f_g(\mu) = \frac{1 + (1 - g)\mu}{1 - g\mu}. \tag{11}
\]
and, therefore, equation (8) acquires the form
\[(gf(t) + 1 - g) t f'(t) = f^2(t) - f(t) . \] (12)
Whence, determining the integration constant from the condition \(f(0) = 1\), we obtain
\[f(t) - 1 = t (f(t))^{1-g} . \] (13)
If \(f^{1-g}\) on the r.h.s. is understood as \(\exp[(1 - g) \ln f]\), where \(\Im(\ln f) = 0\) for \(f > 0\), then for \(0 \leq g \leq 1\) equation (13) has unique positive solution. Equations (11) and (13) are well-known in the context of exotic exclusion statistics \([4, 5, 6]\).

Notice that the solution to (13) satisfies a duality relation:
\[f_1(t) - g(t) = t (f(t))^{1-g} . \] (14)
Furthermore, it follows from (12) that \(t f'/f - 1 > 0\), that is \(f(t)\) is a monotonically increasing function. From (13) we infer (with the help of the \(g > 1\) counterpart of (48)) also that
\[f(t) < 1 . \] (15)
For non-negative \(t\). Actually, the r.h.s. of (15) gives the asymptotics of \(f(t)\) for large \(t\).

Using equation (13), we can compute derivatives of \(f(t)\) at \(t = 0\) in a recursive way:
\[f(t) = \frac{\partial}{\partial t} (f_1 - g(t))|_{t=0} . \] (16)
First few values allow us to conjecture that \(f(t)\) is given by the following Taylor series
\[f(t) = 1 + t + \sum_{n=2}^{\infty} \left( \prod_{k=2}^{n} (1 - \frac{gn}{k}) \right) t^n . \] (17)
This series for \(f(t)\) was suggested in \([8]\); some combinatorial arguments were given for it in \([8]\) (for positive integer values of \(g\)). Furthermore, it was also suggested in \([10, 5, 8]\) that logarithm of \(f(t)\) is given by the series
\[\ln f(t) = t + \sum_{n=2}^{\infty} \left( \frac{1}{n} \prod_{k=1}^{n-1} (1 - \frac{gn}{k}) \right) t^n . \] (18)
We will prove the following statement.

**Proposition 1** The series (17) and (18) are absolutely convergent for
\[|t| < \ln t_0 = -g \ln g - (1 - g) \ln(1 - g) . \] (19)
On this interval, the series (17) and (18) are, respectively, the positive solution of equation (13) and its logarithm. Moreover, for an integer \(m\) we have on the same interval
\[(f(t))^m = 1 + mt + \sum_{n=2}^{\infty} \left( m \prod_{k=2}^{n} \left( 1 + \frac{m-1-gn}{k} \right) \right) t^n . \] (20)

**Proof.** Let \(f_n\) and \(w_n\), \(n = 0, 1, 2, \ldots\) denote, respectively, the coefficients of \(t^n\) in the series (17) and (18) (so that \(w_0 = 0\) and \(f_0 = f_1 = w_1 = 1\)). Notice that they can be written in terms of the gamma–function:
\[f_n = \frac{\Gamma(1 + (1-g)n)}{n! \Gamma(2-gn)} = -\frac{\sin \pi gn}{\pi n!} \Gamma(1 + (1-g)n) \Gamma(gn - 1) , \] (21)
\[w_n = \frac{\Gamma((1-g)n)}{n! \Gamma(1-gn)} = \frac{\sin \pi gn}{\pi n!} \Gamma((1-g)n) \Gamma(gn) . \] (22)
Let us denote \( \tilde{f}_n = f_n / \sin \pi g n \) and \( \tilde{w}_n = w_n / \sin \pi g n \). Applying the Stirling formula (for large \( z \) and \( \delta \ll z \)) in the form \( \ln \Gamma(z + \delta) - \ln \Gamma(z) = \delta \ln z + o(1) \), we find

\[
\lim_{n \to \infty} \ln \left| \frac{\tilde{f}_{n+1}}{\tilde{f}_n} \right| = \lim_{n \to \infty} \ln \left| \frac{\tilde{w}_{n+1}}{\tilde{w}_n} \right| = g \ln g + (1 - g) \ln(1 - g). \quad (23)
\]

Thus, the series \( \sum_{n \geq 1} \tilde{f}_n t^n \) and \( \sum_{n \geq 1} \tilde{w}_n t^n \) and, hence, the series (17)–(18) converge absolutely on the interval (13).

In order to prove the second assertion of the proposition we observe that equation (9), being multiplied by \( f_g m^{-2} \), acquires the form

\[
m = 1 : \quad (1 - g)t (\ln f_g) = f_g - 1 - gt f_g', \quad (24)
m \neq 0, 1 : \quad f_g m - g m t (f_g m) = f_g m - 1 + (1 - g) m t (f_g m - 1)'. \quad (25)
\]

Similarly, for the function \( h_g(t) = f_g(t) - 1 \) equation (12) yields

\[
m \neq 0, 1 : \quad h_g m - g m t (h_g m)' = -h_g m - 1 + \frac{1}{m} t (h_g m)' . \quad (26)
\]

From (24)–(26) we derive relations between the Taylor coefficients

\[
w_n = \frac{1 - gn}{(1 - g)n} f_n, \quad n = 1, 2, \ldots, \quad (27)
\]

\[
f_g[n] = \frac{m(m - 1 + (1 - g)n)}{(m - 1)(m - gn)} f_g[n-1], \quad n = 0, 1, \ldots, \quad (28)
\]

\[
h_g[n] = \frac{m(n + 1 - m)}{(m - 1)(m - gn)} h_g[n-1], \quad n \geq m = 2, 3, \ldots, \quad (29)
\]

Here \( f_g[n] \) and \( h_g[n] \) are, respectively, Taylor coefficients of the series \( (f_g(t))^m = \sum_{n \geq 0} f_g[n] t^n \) and \( (h_g(t))^m = \sum_{n \geq m} h_g[n] t^n \). Solving equations (28)–(29), we find

\[
f_g[n] = n f_n \frac{\Gamma(2 - gn) \Gamma(m + (1 - g)n)}{\Gamma(1 + (1 - g)n) \Gamma(m + 1 - gn)}, \quad (30)
\]

\[
h_g[n] = m h_g[n] \frac{(n - 1)! \Gamma(2 - gn)}{(n - m)! \Gamma(m + 1 - gn)}. \quad (31)
\]

Substituting \( m = n \) into (21) and taking into account that \( h_g[n] = f_n \) and \( h_g[0] = 1 \) for all \( n \geq 1 \), we obtain exactly formula (21) for the coefficients of the series (17). The assertion that the series (18) is logarithm of the series (17) follows now from the relation (27).

Finally, combining (30) with (24), we find the formula

\[
f_g[n] = m \frac{\Gamma(m + (1 - g)n)}{n! \Gamma(m + 1 - gn)}, \quad n = 1, 2, \ldots \quad (32)
\]

that yields the series expansion (19). Analysis of absolute convergence of this series on interval (13) is done in the same way as for series (17) and (18). Although we have considered only positive values of \( m \), an easily verified relation \( (-1)^n f_g[1-g,n] = f_g[-n,n] \) together with the duality relation (14) show that (20) holds for negative \( m \) as well.

Let us remark that, if we assume validity of (30) for \( m = 1 - g \), then we can use relation \( f_{n+1} = f_g[1-g] \) (that follows from (13)) to obtain a recurrence relation. Solution of this relation coincides with (21). This indicates that formula (20) holds also for non-integer \( m \). Another evidence for this is that series (18) and (20) are consistent in the sense that \( \lim_{m \to 0} (f_g^n - 1)/m = \ln f_g \).
3 Central charge for Haldane–Wu statistics

Strictly speaking, formula (5) for counting of states in the Haldane–Wu statistics needs additional conventions for finite \( n \) and \( N \). It however is sufficient for constructing the corresponding thermodynamic Bethe ansatz along the same lines as in the case of the ordinary statistics. This approach does not use explicit form of \( f_g \) and leads to the following expression for the effective central charge

\[
\mathcal{c}_g = \frac{6}{\pi^2} L(y_0),
\]

(33)

where \( y_0 \) is the positive root of the equation

\[
\ln y_0 = (\Phi + g) \ln(1 - y_0).
\]

(34)

The r.h.s. of (33) contains the Rogers dilogarithm that is defined as

\[
L(x) = -\frac{1}{2} \int_0^x dt \left( \frac{\ln(1 - t)}{t} + \frac{\ln t}{1 - t} \right) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2} \ln x \ln(1 - x).
\]

(35)

On the other hand, since the Haldane–Wu statistics is asymptotically extensive, the corresponding effective central charge \( \mathcal{c}_g \) should also be given by the general formula (3) if we substitute \( f = f_g \). Thus, we have two expressions, rather different at the first site, for the effective central charge in the Haldane–Wu statistics. Since the thermodynamic Bethe ansatz derivation of the formula for an effective central charge involves a nontrivial limit and uses some additional assumptions, it appears to be instructive to provide a direct proof of equivalence of the two expressions for \( \mathcal{c}_g \).

**Proposition 2** Let \( 0 \leq g \leq 1, \Phi \geq 0, \) and \( f_g(t) \) be the positive solution of equation (13). Then the following equality holds

\[
\int_0^{x_0} \frac{dt}{t} \ln f_g(t) - \frac{1}{2} \ln x_0 \ln f_g(x_0) = L \left( 1 - \frac{1}{f_g(x_0)} \right) = L(y_0),
\]

(36)

where \( y_0 \) is the positive root of equation (34), and \( x_0 \) is the positive root of equation

\[
\ln x_0 + \Phi \ln f_g(x_0) = 0.
\]

(37)

**Proof.** Notice that, since \( f_g(t) \) increases monotonically, equation (34) has unique positive solution \( x_0 \). Furthermore, \( x_0 \leq 1 \) because \( f_g(0) = 1 \).

Consider the function \( y(t) = 1 - 1/f_g(t) \). It allows us to rewrite equation (13) as \( t = y(1-y)^{-g} \). Therefore

\[
\int_0^{x_0} \frac{dt}{t} \ln f_g(t) - \frac{1}{2} \ln t \ln f_g(t) = -\int d(\ln y - g \ln(1 - y)) \ln(1 - y) \]

(38)

\[
+ \frac{1}{2} \ln y \ln(1 - y) \ln(1 - y) = -\int \frac{dy}{y} \ln(1 - y) + \frac{1}{2} \ln y \ln(1 - y).
\]

Comparison of the last expression with the definition (35) yields the first equality in (36). Further, employing equations (13) and (34), we obtain

\[
\ln y(x_0) = \ln f_g(x_0) - 1 - \ln f_g(x_0) = \ln x_0 - g \ln f_g(x_0)
\]

\[
= -(\Phi + g) \ln f_g(x_0) = (\Phi + g) \ln(1 - y(x_0)).
\]

(39)

Since (34) has unique positive solution for \( (\Phi + g) \geq 0 \), we conclude that \( y(x_0) = y_0 \). \( \square \)

Let us now formulate a mathematical corollary of Propositions (1) and (2).
Proposition 3 Let \(0 < g < 1\) and \(\Phi \geq 0\) and let \(y_0\) be the positive root of (14). Then
\[
\sum_{n=1}^{\infty} \sin \pi gn \frac{\Gamma((1-g)n) \Gamma(gn)}{\pi n!} \left( y_0(1-y_0)^{-g} \right)^n + \frac{\Phi}{2}(\ln(1-y_0))^2 = L(y_0) \tag{40}
\]
if \(t = y_0(1-y_0)^{-g}\) satisfies condition (19).

Proof. Indeed, by Proposition 1 we can substitute the series (17) into the integral on the l.h.s. of (18) and carry out term–wise integration. The resulting series converges to the value of the integral if the condition of absolute convergence (19) is satisfied. The quantity \(x_0\) entering the l.h.s. of (18) is the solution to equations (13) and (37) which are equivalent, after the change of variables \(y_0 = 1 - 1/f_g(x_0)\), to equation (34) and the relation \(y_0 = x_0(1-y_0)^g\).

An interesting feature of identity (40) is that, although its l.h.s. involves \(g\) and \(\Phi\) in essentially different ways, its r.h.s. depends only on the value of \(\nu \equiv (g + \Phi)\). Thus, for a fixed \(y_0\), identity (40) provides a representation for dilogarithm \(L(y_0)\) as a series with a free parameter. As an example, consider three special cases, namely, \(\nu = 2, 1, \frac{1}{2}\). For these values we have, respectively, \(y_0 = 1 - \rho, \frac{1}{2}, \rho\), where \(\rho = (\sqrt{5} - 1)/2\). It is known (see, e.g., (12)) that these are the only algebraic points on the interval \((0, 1)\), where \(\frac{\pi}{2}L(y_0)\) takes rational values (which are \(\frac{5}{2}, \frac{1}{2}, \frac{1}{3}\), respectively). Thus, keeping \(g\) as a free parameter, we obtain for the special values of \(\nu\) the following identities
\[
\sum_{n=1}^{\infty} \sin \pi gn \frac{\Gamma((1-g)n) \Gamma(gn)}{\pi n!} \rho^{(2-g)n} + \frac{2-\rho}{2}(\ln \rho)^2 = \frac{\pi^2}{15}, \tag{41}
\]
\[
\sum_{n=1}^{\infty} \sin \pi gn \frac{\Gamma((1-g)n) \Gamma(gn)}{\pi n!} 2^{(g-1)n} + \frac{1-g}{2}(\ln 2)^2 = \frac{\pi^2}{12}, \tag{42}
\]
\[
\sum_{n=1}^{\infty} \sin \pi gn \frac{\Gamma((1-g)n) \Gamma(gn)}{\pi n!} \rho^{(1-2g)n} + (1-2g)(\ln \rho)^2 = \frac{\pi^2}{10}. \tag{43}
\]
Here \(0 < g < 1\) in (41)–(42), whereas the upper bound for \(g\) in (13) is determined from the convergence condition (19) (approximately, \(g < 0.88\)).

4 Gentile statistics

Another interesting case of extensive statistics (which appeared already in (11) and is sometimes called the Gentile statistics) arises when we chose in (2) the following generating function
\[
F_G(t) = 1 + t + t^2 + \ldots + t^G, \tag{44}
\]
which also interpolates between fermions \((G = 1)\) and bosons \((G = \infty)\). For this statistics, the general formula (3) for the effective central charge acquires the form (2)
\[
\tilde{c}_G = \frac{6}{\pi^2} \left[ L(x_0) - \frac{1}{G+1}L(x_0^G+1) \right], \tag{45}
\]
where \(x_0\) is the positive root of equation (3) for \(f(t) = F_G(t)\).

The maximal value of \(\mu\) for which equation (3) has positive root is interpreted (because \(\mu = n/N\) in formula (3)) as the maximal occupation number for a single state. It is easy to see that this number is \(\mu_{\text{max}} = G\) for the Gentile statistics and \(\mu_{\text{max}} = 1/g\) for the Haldane–Wu statistics. In both cases the entropy density \(s(\mu)\) is a concave function such that \(s(0) = s(\mu_{\text{max}}) = 0\). Therefore, it is natural to compare properties of the Gentile statistics with parameter \(G\) and the Haldane–Wu statistics with parameter \(g = 1/G\). It was conjectured in (2) that the former statistics majorizes the latter. Here we will prove the following statement.
Proposition 4 Let $1 < G < \infty$ and $g = 1/G$. Then the Gentile statistics majorizes the Haldane–Wu statistics in the sense that
\[ F_G(t) > f_G(t) \]  
for $t > 0$.

Proof. To prove this assertion, it is again useful to use the function $y(t) = 1 - 1/f_G(t)$; equation (13) then acquires the form $t = y(1 - y)^{-g}$. Hence
\[ (1 - t) \left( F_G(t) - f_G(t) \right) = 1 - t^{1+\frac{1}{g}} + (t - 1) f_G(t) = y(1 - y)^{-g-1} \phi_g(y), \]  
where $\phi_g(y) = 1 - y^{\frac{1}{g}} - (1 - y)^g$. Let us show that $\phi_g(y) > 0$ for $0 < t < 1$, i.e., for $0 < y < y_0$, where $y_0$ is the positive root of equation $y_0 = (1 - y_0)^g$. The inequality
\[ (1 - y)^g < 1 - gy, \]  
that holds for $0 < g < 1$ and $0 < y < 1$, leads to the estimate $\phi_g(y) > gy - y^{\frac{1}{g}}$. Consequently, $\phi_g(y) > 0$ for $0 < y \leq \tilde{y}$, where $\tilde{y}$ is the positive root of equation $g\tilde{y} = \tilde{y}^{\frac{1}{g}}$. For $y > \tilde{y}$ we find
\[ \phi_g'(y) = g(1 - y)^{g-1} - \frac{1}{g} y^{\frac{1}{g}-1} < (1 - y)^{g-1} - 1 - \frac{1}{g} (1 - y(1 + g)), \]  
where we again used inequality (48). On the other hand, for $y \leq y_0$, it follows from (48) that $y < y_0 < \frac{1}{1-g}$. Therefore $\phi_g'(y) < 0$ on the interval $\tilde{y} < y < y_0$. And since $\phi_g(y_0) = 1 - y_0^\frac{1}{g} - (1 - y_0)^g = 1 - \frac{1}{g} y_0 - y_0 = 0$, we conclude that $\phi_g(y) > 0$ on this interval as well. Thus, the r.h.s. of (47) is positive for $0 < t < 1$. Using that $\phi_g(y) = \phi_g'(1-y)$ (notice that inequality (48) reverses for $g > 1$), we can analogously show that the r.h.s. of (47) is negative for $t > 1$. Finally, for $t = 1$ we have (also cf. (13))
\[ f_G(1) = \frac{1}{1 - y_0} < 1 + \frac{1}{g} = F_G(1), \]  
which completes the proof. \qed

Let us remark that, as seen from the proof, $G$ in (46) does not have to be an integer if we write $F_G(t)$ as $(1 - t^{G+1})/(1 - t)$. Actually, doing so, we can consider also the case $0 < G < 1$. In this case inequality (46) reverses as can be shown by a proper modification of the above proof. However, for a physical interpretation, the case of non-integer $G$ is less natural.

Proposition 4 can be used to establish inequalities between physical quantities related to the statistics in question. For example, we will prove the following.

Proposition 5 Let $\tilde{s}_G(\mu)$ and $\tilde{c}_G$ be the entropy density and the effective central charge for the Gentile statistics, and let $s_G(\mu)$ and $c_G$ be the entropy density and the effective central charge for the Haldane–Wu statistics. Then, for $1 < G < \infty$ and $g = 1/G$, the following inequalities hold
\[ \tilde{s}_G(\mu) > s_G(\mu), \]  
\[ \tilde{c}_G > c_G, \]  
where $0 < \mu < G$ in (54).

Proof. For fixed value of $\mu$, equations (7) and (8) define the entropy density as a functional of the generating function, $s = s[f]$T. Taking a small variation of the function $f$ (which involves also variation of $x$ via (8)), we obtain
\[ \delta s[f] = \delta (\ln f - \mu \ln x) = \frac{\delta f}{f} + \frac{f'}{f} \delta x - \frac{\mu}{x} \delta x = \frac{\delta f}{f}, \]  
(53)
where the last equality took into account equation (8).

Analogously, for fixed value of $\Phi$, equations (3) and (4) define the functional $c[f]$. For a small variation of the function $f$ we find

$$
\delta\left(\frac{\pi^2}{6} [c[f]]\right) = \frac{1}{2} \left(\ln f(x_0) \delta x - \frac{\ln x_0}{f(x_0)} \delta f - \ln f'(x_0) f(x_0) \delta x\right) + \int_0^{x_0} \frac{dt}{t} \delta f(t) = \int_0^{x_0} \frac{dt}{t} \delta f(t),
$$

where we used equation (8) and its consequence, $f(x_0) \delta x + \Phi x_0 (\delta f + f'(x_0) \delta x) = 0$.

Now consider $\psi_a(t) = aF_2^1(t) + (1 - a)f_2(t)$ for $a \in [0, 1]$. This function is positive for all $t$ and moreover, due to Proposition 4, $\delta \psi_a(t) = \delta \alpha (F_2^1(t) - f_2(t)) > 0$ if $\delta \alpha > 0$ and $t > 0$. This, together with (53)–(54), implies that $s[\psi_a]$ and $c[\psi_a]$ are monotonically growing functions of $a$ and hence relations (11)–(12) follow.

Relation (52) gives us an inequality involving the Rogers dilogarithm at specific arguments. Let us formulate it explicitly.

**Proposition 6** Let $\Phi \geq 0$ and $0 \leq g \leq 1$. Let $x_0$ and $y_0$ be, respectively, the positive roots of equations

$$
\ln x_0 = \Phi \ln(1 - x_0) - \Phi \ln\left(1 - x_0^{1 + \frac{1}{g}}\right),
$$

$$
\ln y_0 = (\Phi + g) \ln(1 - y_0).
$$

Then

$$
L(x_0) - \frac{g}{1 + g} L\left(x_0^{1 + \frac{1}{g}}\right) \geq L(y_0)
$$

(57)

and the equality takes place if and only if $g = 0$ or $g = 1$.

**Proof.** For $g = 0^+$ we have $x_0^\frac{1}{g} = 0$. Then equations (55)–(56) yield $y_0 = x_0$ and (57) is obviously an equality. For $g = 1$ equations (53)–(54) yield $y_0 = \frac{x_0}{1 + x_0}$. Then (57) becomes an equality due to the Abel identity $L(t^2) = 2L(t) - 2L(\frac{t}{1+t})$ that holds for any $t$ on the interval $[0, 1]$.

For $0 < g < 1$ the inequality in (57) follows from relation (24) in Proposition 4, equations (63)–(64), formula (47), and equation (3) for $f(t) = (1 - t^{1 + \frac{1}{g}})/(1 - t)$.

In the simplest case, $\Phi = 0$, we have $x_0 = 1$ and Proposition 4 reduces to the estimate

$$
L(y) \leq \frac{1}{1 + g} \frac{\pi^2}{6} \quad \text{for} \quad y = (1 - g)^g \quad \text{and} \quad 0 < g < 1.
$$

(58)

The case $\Phi = 1$ can be interpreted as related to the $A_2$ affine Toda model [2] and to the Calogero–Sutherland model model with the coupling constant $\lambda = g [9]$.

**Remark.** After this manuscript had been written the author was informed that multivariable analogues of formulae (53) and (54) were obtained in [13] and [14] by means of the multivariable Lagrange inversion theorem.

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Institut für theoretische Physik, Freie Universität Berlin
Arnimallee 14, 14195 Berlin, Germany
and
Steklov Institute for Mathematics
Fontanka 27, 191011 St.Petersburg, Russia