Quantization of the Classical Maxwell-Nordström Fields

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Abstract  The classical electromagnetic and gravitomagnetic fields in the vacuum, in (3 + 2) dimensions, described by the Maxwell-Nordström equations, are quantized. These equations are rederived from the field tensor which follows from a five-dimensional form of the Dirac equation. The electromagnetic field depends on the customary time $t$, and the hypothetical gravitomagnetic field depends on the second time variable $u$. The total field energy is identified with the component $T_{44}$ of the five-dimensional energy-stress tensor of the electromagnetic and gravitomagnetic fields. In the ground state, the electromagnetic field and the gravitomagnetic field energies cancel out. The quanta of the gravitomagnetic field have spin 1.

1. Introduction

The Maxwell-Nordström equations [10] refer to pseudo-orthogonal five-dimensional space. They were rederived [3, 4], from the iterated form of a five-dimensional Dirac equation [5, 6, 7]. In the original Nordström theory [10], the fifth coordinate is a real spatial coordinate. When imaginary character is ascribed to the fifth coordinate, the Maxwell-Nordström equations describe electromagnetic phenomena together with hypothetical gravitomagnetic phenomena [4]. In that theory we are dealing with electromagnetic field and with gravitomagnetic field. The electromagnetic waves are periodic in the customary time variable $t$, while the gravitomagnetic waves are periodic in the second time variable $u$ [4].

The quantization of the classical Maxwell-Nordström fields can be accomplished in the customary way of quantizing the classical electromagnetic field, provided that the expressions for the total field energy and field momentum of the electromagnetic and gravitomagnetic fields are known, and that it is accepted that the respective two types of waves are periodic in two different times. The field momentum of these two fields was already determined in [4]. The total field energy can conveniently be determined by applying Sommerfeld’s method [13] in the calculation of the stress-energy tensor in the Minkowski space.

We will determine the stress-energy tensor in the two flat spaces $E(3,2)$ and $E(4,1)$. These are connected with the groups $SO(3,2)$ and $SO(4,1)$, under which the respective Maxwell-Nordström equations are covariant [3, 4]. The
total field energy will be identified with the respective $T_{44}$ components of the stress-energy tensor for those two cases. The result of calculation of the stress-energy tensor plays a pivotal role for the outcome of the quantization procedure, concerning the resultant ground-state energy of the electromagnetic and gravitomagnetic fields. It comes out that the signs of the two energy terms which appear in the $T_{44}$ component of the stress-energy tensor, the electromagnetic energy and the hypothetical gravitomagnetic energy, can be the same or opposite. These two terms have the same sign in $(4+1)$ dimensions, when the fifth coordinate is real, and have opposite signs in $(3+2)$ dimensions, when the fifth coordinate is imaginary. In the latter case the ground-state energy of the quantized total field vanishes.

2. The Five-dimensional Stress-Energy Tensor

For the method of determining the stress-energy tensor in $(3+2)$ and $(4+1)$ dimensions, we refer to Sommerfeld [13]. All definitions of the relevant quantities and the respective equations can be specified in $(3+2)$ or $(4+1)$ dimensions. The formal difference between the two cases is that the fifth coordinate and the fifth component of the five-potential are respectively imaginary or real. We will perform the calculations in the case of $(3+2)$ dimensions, and next will indicate in which way are the final results modified when $(4+1)$-dimensional space-time is considered.

In $(3 + 2)$-dimensional pseudo-orthogonal space the coordinates of a point are: $x_1, x_2, x_3, x_4 = ict$, and $x_5 = icu$, where $c$ denotes the speed of light in the vacuum, and $u$ denotes the second time coordinate. The form of $x_5$ implies that the speed of the gravitomagnetic waves is assumed to be equal to the speed of light $c$ [4]. If another speed $c' = \text{const} \ c$ were assumed in the expression for $x_5$, a constant factor would appear in the formulas, which is irrelevant for the quantization of the fields, and for the cancellation of the ground-state energies of electromagnetic and gravitomagnetic fields.

We define the five-potential, [4],

$$ \vec{A} = (A_1, A_2, A_3, A_4, A_5) = \left( A_x, A_y, A_z, \frac{i}{c} \phi, \frac{im}{c^2} \chi \right) $$

(1)

which consists of the three real components $A_x, A_y, A_z$, which are referred to the Cartesian system of coordinates $(x_1, x_2, x_3)$, and of two imaginary components, which are proportional to a scalar electric potential $\phi$ and a scalar gravitational potential $\chi$. Under rotations in $(3+2)$ dimensions, the components of the five-potential $\vec{A}$ transform like the components of the five-vector $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$.

The respective components of the five-dimensional field tensor have the form [4],
\[ \partial_j A_k - \partial_k A_j = B_i, \quad i, j, k = 1, 2, 3 \]  
\[ \partial_j A_4 - \partial_4 A_j = -\frac{i}{c} E_j, \quad j = 1, 2, 3 \]  
\[ \partial_j A_5 - \partial_5 A_j = -\frac{i}{c} G_j, \quad j = 1, 2, 3 \]  
\[ \partial_4 A_5 - \partial_5 A_4 = -\frac{1}{c} G_0 \]  

where \( B_i \) and \( E_j \) are the components of the magnetic induction and of the electric field, respectively, while \( G_j \) is proportional to the hypothetical gravitational field \( G'_j \), \([4]\), namely

\[ \vec{G} = \frac{m}{e} \vec{G}' \]  

where \( m \) denotes the rest mass and \( e \) the electric charge of an electron, and \( G_0 \) is proportional to the Brans-Dicke scalar field in the Kaluza-Klein theory \([11]\) which here is denoted by \( G'_0 \), namely \([4]\),

\[ G_0 = \frac{m}{e} G'_0 \]  

To avoid writing the factor \( m/e \) in the following formulas, we will use the symbols \( \vec{G} \) and \( G_0 \).

The field six-vectors in the Minkowski space \([13]\) are replaced in \((3 + 2)\) dimensions by the field ten-vectors. The field ten-vector is defined by \([4]\),

\[ F = (c \vec{B}, -i \vec{E}, -i \vec{G}, -G_0) \]  

In turn, the excitation ten-vector is defined by

\[ f = \sqrt{\varepsilon_0 \mu_0} F = (\vec{H}, -ic \vec{D}, -i\varepsilon_0 c \vec{G}, -\varepsilon_0 c G_0) \]  

where \( \vec{B} \) denotes the magnetic induction, \( \vec{E} \) - the electric field, \( \vec{G} = \frac{m}{e} \vec{G}' \), with \( \vec{G}' \) denoting the gravitational field, and \( G_0 = \frac{m}{e} G'_0 \), with \( G'_0 \) being the counterpart of the Brans-Dicke scalar; \( \vec{H} \) denotes the magnetic field intensity, \( \vec{D} \) is the electric displacement, \( \varepsilon_0 \) denotes the electric permeability, and \( \mu_0 \) the magnetic susceptibility of the vacuum.

These two ten-vectors have the following matrix form,

\[ F = \begin{bmatrix} 0 & cB_z & -cB_y & -iE_x & -iG_x \\ -cB_z & 0 & cB_x & -iE_y & -iG_y \\ cB_y & -cB_x & 0 & -iE_z & -iG_z \\ iE_x & iE_y & iE_z & 0 & -G_0 \\ iG_x & iG_y & iG_z & G_0 & 0 \end{bmatrix} \]  

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and

\[
f = \begin{bmatrix}
0 & H_z & -H_y & -icD_x & -i\varepsilon_0cG_x \\
-H_z & 0 & H_x & -icD_y & -i\varepsilon_0cG_y \\
H_y & -H_x & 0 & -icD_z & -i\varepsilon_0cG_z \\
icD_x & icD_y & icD_z & 0 & -\varepsilon_0cG_0 \\
ic\varepsilon_0cG_x & ic\varepsilon_0cG_y & ic\varepsilon_0cG_z & \varepsilon_0cG_0 & 0 
\end{bmatrix}
\]  

(11)

In the Minkowski subspace, the ten-vectors in Eqs. (8) and (9), and the respective matrices \(F\) and \(f\) in Eqs. (10) and (11), reduce to those given in [13].

We now define the Lagrangian density \(\Lambda\) in the form

\[
\Lambda = \frac{1}{2c} f \cdot F = \frac{1}{2} (\vec{H} \cdot \vec{B} - \vec{D} \cdot \vec{E}) - \frac{1}{2} \varepsilon_0 (\vec{G}^2 - \vec{G}_0^2)  
\]  

(12)

where the dot (\(\cdot\)) denotes the scalar product of the two ten-vectors, \(F\) and \(f\) defined in Eqs. (8) and (9). This Lagrangian density is invariant under the transformations of the \(SO(3, 2)\) group, owing to the covariance under that group of the ten-vectors appearing in the scalar product in Eq. (12).

We now observe that to obtain from the above expressions the respective expressions when the fifth coordinate \(x_5\) is real, as it is in the original Nordström theory [10, 3], we have to omit the factor \(i\) by the fifth component of the five-potential in Eq. (1), and in Eqs. (8) through (12), and to replace \(i\vec{G}\) or \(iG_j\) by \(\vec{G}\) or \(iG_j\), respectively, and \(G_0\) by \(iG_0\). In the Minkowski space, the Lagrangian in Eq. (12) reduces to that given in [13].

Extending on five dimensions the calculations in [13], we define the stress-energy tensor \(T\) with the components,

\[
T_{nm} = -\frac{1}{c} \sum_{r=1}^{5} F_{nr} f_{mr} + \delta_{nm}\Lambda  
\]  

(13)

where \(F_{nr}\) and \(f_{mr}\) are the elements of the matrices in Eqs. (10) and (11), respectively. The tensor properties of this quantity follow from the behaviour of the ten-vectors \(f\) and \(F\) under the \((3+2)\)-dimensional rotations. The tensor \(T\) is symmetric since the ten-vector \(f\) is proportional to the ten-vector \(F\).

From Eqs. (12) and (13), we calculate the components of the stress-energy tensor \(T\) in the form,

\[
T_{11} = H_x B_x + D_x E_x - U - \frac{1}{2} \varepsilon_0 (\vec{G}^2 - 2\vec{G}_x^2 - \vec{G}_0^2)  
\]

\[
T_{12} = H_x B_y + D_y E_x + \varepsilon_0 G_x G_y  
\]

\[
T_{13} = H_x B_z + D_z E_x + \varepsilon_0 G_x G_z  
\]

\[
T_{14} = -\frac{i}{c} (\vec{E} \times \vec{H})_x - i\varepsilon_0 G_x G_0  
\]
\[ T_{15} = -\frac{i}{c}(\vec{G} \times \vec{H})_x - iD_x G_0 \]

\[ T_{22} = H_y B_y + D_y E_y - U - \frac{1}{2}\varepsilon_0(G^2 - 2G^2_y - G_0^2) \]

\[ T_{23} = H_y B_z + D_y E_y + \varepsilon_0 G_y G_z \]

\[ T_{24} = -\frac{i}{c}(\vec{E} \times \vec{H})_y - \varepsilon_0 G_y G_0 \]

\[ T_{25} = -\frac{i}{c}(\vec{G} \times \vec{H})_y - iD_y G_0 \]

\[ T_{33} = H_z B_z + D_z E_z - U - \frac{1}{2}\varepsilon_0(G^2 - 2G^2_z - G_0^2) \]

\[ T_{34} = -\frac{i}{c}(\vec{E} \times \vec{H})_z - \varepsilon_0 G_z G_0 \]

\[ T_{35} = -\frac{i}{c}(\vec{G} \times \vec{H})_z - iD_z G_0 \]

\[ T_{44} = U - \frac{1}{2}\varepsilon_0(G^2 + G_0^2) \]

\[ T_{55} = \frac{1}{2}(\vec{H} \cdot \vec{B} - \vec{D} \cdot \vec{E}) - \frac{1}{2}\varepsilon_0(G_0^2 - G^2) \] (14)

where \( U \) denotes the electromagnetic field energy,

\[ U = \frac{1}{2}(\vec{H} \cdot \vec{B} + \vec{D} \cdot \vec{E}) \] (15)

If the speed of the gravitomagnetic waves were assumed to be \( ac \), where \( a = \text{const} \), the factor \( a^2 \) would appear in front of \( G^2 \) and \( G_0^2 \) in these formulas.

We observe that the trace of the stress-energy tensor,

\[ \sum_{r=1}^{5} T_{rr} = \frac{1}{2}(\vec{H} \cdot \vec{B} - \vec{D} \cdot \vec{E}) - \frac{1}{2}\varepsilon_0(G_0^2 - G^2) = \Lambda \] (16)

is invariant under the \((3 + 2)\)-dimensional rotations.

The form of the \( T_{44} \) component of the stress-energy tensor is of special interest, since it determines the total field energy. In Eqs. (14), the densities of the electromagnetic field energy and of the gravitomagnetic field energy appearing in \( T_{44} \) have opposite signs. Consequently, it will appear that the ground-state energies of the two fields cancel out.

For a real fifth coordinate, owing to the above indicated alterations in the formulas, connected with the passage from \((3 + 2)\) to \((4 + 1)\) dimensions, namely, \( i\vec{G} \to \vec{G} \) and \( G_0 \to iG_0 \), (the passage from imaginary to real \( x_5 \)-coordinate), the two energy densities acquire the same sign. Consequently, the ground-state energies of the two fields will not cancel out.
3. The Quantization of Maxwell-Nordström Fields

We consider the stress-energy tensor in (3 + 2) dimensions in Eq. (14). Its $T_{44}$ component is identified with the total energy density $W$ of the two fields,

$$ W = \frac{1}{2} (\vec{H} \cdot \vec{B} + \vec{D} \cdot \vec{E}) - \frac{1}{2} \varepsilon_0 (G^2 + G'_0) $$

(17)

The density of field momentum connected with the electromagnetic and gravitomagnetic fields is determined by the Pointing vector [4],

$$ \vec{P} = \frac{1}{c^2} (\vec{E} \times \vec{H} + \vec{G} \times \vec{H}) $$

(18)

We assume, as it was done in [4], that the vector potential $\vec{A}(\vec{r}, t, u)$ is equal to the sum of two terms, one depending on time $t$ and the other on time $u$,

$$ \vec{A}(\vec{r}, t, u) = \vec{A}_1(\vec{r}, t) + \vec{A}_2(\vec{r}, u) $$

(19)

In the vacuum we can assume that

$$ A_4 = 0, \quad G'_0 = \text{const} $$

(20)

The last condition has a counterpart in fixing of the scalar field which appears in the Kaluza-Klein theory, i.e. the Brans-Dicke field, in order to obtain Einstein equations of general relativity and Maxwell equations (see [11]). We notice that the conditions $A_4 = 0$ and $G'_0 = \text{const}$, imply the loss of covariance of the five-potential $\vec{A}$ in Eq. (1) under the group $SO(3, 2)$.

Assuming that the electric field in Eq. (3) depends on time $t$ and the gravitational field in Eq. (4) depends on time $u$, we obtain [4] from those equations and from Eq. (19) the expressions

$$ \vec{E} = - \frac{\partial \vec{A}_1}{\partial t}, \quad \text{and} \quad \vec{G} = - \frac{\partial \vec{A}_2}{\partial u}, $$

(21)

With $\vec{H} = \mu_0^{-1} \vec{B}$, where $\mu_0$ denotes the magnetic susceptibility of the vacuum, with $\vec{D} = \varepsilon_0 \vec{E}$ and $\vec{B} = \text{curl} \vec{A}$, we obtain from Eq.(17) the expression for the total energy density $W$ in the form

$$ W = \frac{1}{2} \left[ \frac{1}{\mu_0} \text{(curl} \vec{A})^2 + \varepsilon_0 \left( \frac{\partial \vec{A}_1}{\partial t} \right)^2 - \left( \frac{\partial \vec{A}_2}{\partial u} \right)^2 \right], $$

(22)

and from Eq. (18) the field momentum density $\vec{P}$ in the form

$$ \vec{P} = - \frac{1}{c^2} \left( \frac{\partial \vec{A}_1}{\partial t} + \frac{\partial \vec{A}_2}{\partial u} \right) \times (\text{curl} \vec{A}) $$

(23)
The total field energy and momentum is obtained from these expressions by integrating over space.

We expand the vector potential in two series of which the first is connected with the part of the vector potential depending on the time $t$, and the second with the part of the vector potential depending on the time $u$,

$$
\vec{A} = \vec{A}_1(t) + \vec{A}_2(u) = \sqrt{\frac{\mu_0 c^2}{V}} \left\{ \sum_{\vec{k}, \mu} \vec{u}_{k, \mu} \left( a_{k, \mu} e^{i(\vec{k} \cdot \vec{r} - \omega t)} + a^*_{k, \mu} e^{-i(\vec{k} \cdot \vec{r} - \omega t)} \right) + \sum_{\vec{q}, \nu} \vec{v}_{q, \nu} \left( b_{q, \nu} e^{i(\vec{q} \cdot \vec{r} - \omega u)} + b^*_{q, \nu} e^{-i(\vec{q} \cdot \vec{r} - \omega u)} \right) \right\} = \vec{A}^* (24)
$$

where $\vec{k}$ and $\vec{q}$ are the wave vectors of the electromagnetic and gravitomagnetic fields, respectively, where $a_{k, \mu}$ and $b_{q, \nu}$ are linearly independent, complex amplitudes and where $*$ denotes the conjugate complex quantity. According to Eq. (21), the unit vectors $\vec{u}_{k, \mu}$ and $\vec{v}_{q, \nu}$, determine the directions of the fields $\vec{E}$ and $\vec{G}$, respectively. The indices $\mu, \nu$ denote the states of transverse polarization. The condition $\text{div} \vec{A} = 0$, implies that

$$
\vec{u}_{k, \mu} \cdot \vec{k} = 0, \quad \mu = 1, 2; \quad \vec{v}_{q, \nu} \cdot \vec{q} = 0, \quad \nu = 1, 2
$$

$$
\vec{u}_{k_1} \cdot \vec{u}_{k_2} = 0, \quad \vec{v}_{q_1} \cdot \vec{v}_{q_2} = 0 (25)
$$

The conditions in Eq. (25) yield for each term under the two sums the expressions which in turn lead for each term to the conditions

$$
\omega = kc, \quad \omega' = qc (26)
$$

We now will perform the consecutive quantization steps in the customary way [1, 2]. In the energy density expression in Eq. (22), we firstly consider the term $\text{curl} \vec{A}$, which is proportional to the following expression,

$$
\text{curl} \vec{A} \sim i \left[ \vec{c}_x (q_y u_z - q_z u_y) - \vec{c}_y (q_x u_z - q_z u_x) + \vec{c}_z (q_x u_y - q_y u_x) \right] (27)
$$

We also have

$$
\vec{q} \times \vec{u}_{q, \lambda} = \vec{c}_x (q_y u_z - q_z u_y) - \vec{c}_y (q_x u_z - q_z u_x) + \vec{c}_z (q_x u_y - q_y u_x) (28)
$$

Taking these two expressions into account we obtain
\[
\text{curl } \vec{A} = \sqrt{\frac{\mu_0 c^2}{V}} \left\{ \sum_{\vec{k},\mu} \left[ i(\vec{k} \times \vec{u}_{k,\mu}^e) a_{k,\mu} e^{i(\vec{k}' \cdot \vec{r} - \omega t)} - i(\vec{k} \times \vec{u}_{k,\mu}^e) a_{k,\mu}^* e^{-i(\vec{k}' \cdot \vec{r} - \omega t)} \right] + \sum_{\vec{q},\nu} \left[ i(\vec{q} \times \vec{v}_{q,\nu}^e) b_{q,\nu} e^{i(\vec{q}' \cdot \vec{r} - \omega' t)} - i(\vec{q} \times \vec{v}_{q,\nu}^e) b_{q,\nu}^* e^{-i(\vec{q}' \cdot \vec{r} - \omega' t)} \right] \right\}
\]

(29)

From \((\text{curl } \vec{A})^2\), appearing in the energy expression, we therefore obtain the expressions containing the following terms:

\[-(\vec{k} \times \vec{u}_{k,\mu}) \cdot (\vec{k}' \times \vec{u}_{k',\mu'}) = (\vec{k}' \cdot \vec{k})(\vec{u}_{k'} \cdot \vec{u}_k) - (\vec{u}_{k'} \cdot \vec{u}_k)(\vec{k}' \cdot \vec{u}_k) \quad (30)\]

\[-(\vec{k} \times \vec{u}_{k,\mu}) \cdot (\vec{q}' \times \vec{v}_{q',\nu'}) = (\vec{k}' \cdot \vec{q}'')(\vec{u}_{k\mu} \cdot \vec{v}_{q',\nu'}) - (\vec{u}_{k\mu} \cdot \vec{q}'')(\vec{k}' \cdot \vec{v}_{q',\nu'}) \quad (31)\]

\[-(\vec{q} \times \vec{v}_{q\nu}) \cdot (\vec{k}' \times \vec{u}_{k',\mu'}) = (\vec{q}' \cdot \vec{k}'')(\vec{v}_{q\nu} \cdot \vec{u}_{k',\mu'}) - (\vec{v}_{q\nu} \cdot \vec{k}'')(\vec{q}' \cdot \vec{u}_{k',\mu'}) \quad (32)\]

\[-(\vec{q} \times \vec{v}_{q\nu}) \cdot (\vec{q}' \times \vec{v}_{q',\nu'}) = (\vec{q}' \cdot \vec{q}'')(\vec{v}_{q\nu} \cdot \vec{v}_{q',\nu'}) - (\vec{v}_{q\nu} \cdot \vec{q}'')(\vec{q}' \cdot \vec{v}_{q',\nu'}) \quad (33)\]

It was shown in [4] that plane gravitomagnetic waves are transverse. For quantization to be possible, the terms in Eqs. (31) and (32) have to disappear. We consider the two situations: (1) The electromagnetic wave and the gravitomagnetic wave move in the same direction, and the vector \(\vec{E}\) of the electromagnetic wave is perpendicular to the vector \(\vec{G}\) of the gravitomagnetic wave. This means that

\[\vec{u}_{k} \cdot \vec{v}_{q} = \vec{u}_{k} \cdot \vec{v}_{q'} = 0 \quad (34)\]

(2) These two waves move in mutually perpendicular directions, and the field vectors \(\vec{E}\) and \(\vec{G}\) are parallel. In both cases the terms in Eqs. (31) and (32) disappear.

We next substitute the expansion of the vector potential in Eq. (18) into the terms \(\partial \vec{A}_2 / \partial t\) and \(\partial \vec{A}_2 / \partial u\) in the energy expression in Eq. (22). We then obtain for the electromagnetic terms the conditions,

\[-\frac{\omega^2}{c^2} - (\vec{k}' \cdot \vec{k}) = 0 \quad \text{when} \quad \vec{k}' = -\vec{k} \quad (35)\]

\[-\frac{2\omega^2}{c^2} \delta_{\mu\nu} \quad \text{when} \quad \vec{k}' = \vec{k} \]
and for the gravitomagnetic field the conditions,

\[ \frac{\omega'^2}{c^2} - (\vec{q}' \cdot \vec{q}) = 2 \frac{\omega'^2}{c^2} \delta_{\nu'}^\nu \quad \text{when} \quad \vec{q}' = -\vec{q} \]

\[ \frac{\omega'^2}{c^2} - (\vec{q}' \cdot \vec{q}) = 0 \quad \text{when} \quad \vec{q}' = \vec{q} \]

Consequently, we obtain the total field energy in the form,

\[ W = \sum_{\vec{k}, \mu} \frac{\omega^2}{c^2} (a_{\vec{k} \mu}^* a_{\vec{k} \mu} + a_{\vec{k} \mu}^* a_{\vec{k} \mu}^*) - \sum_{\vec{q}, \nu} \frac{\omega'^2}{c^2} (b_{\vec{q} \nu}^* b_{\vec{q} \nu} + b_{\vec{q} \nu} b_{\vec{q} \nu}) \]

where the first term represents electromagnetic energy and the second term represents gravitomagnetic energy. These two terms have opposite signs. For a real fifth coordinate, both terms would have positive sign. This is a direct consequence of the expression for the component \( T_{44} \) of the stress-energy tensor in Eq. (14).

The total field momentum is calculated in an analogous way and we obtain,

\[ \vec{P} = \sum_{\vec{k}, \mu} \omega \vec{k} (a_{\vec{k} \mu}^* a_{\vec{k} \mu} + a_{\vec{k} \mu}^* a_{\vec{k} \mu}^*) + \sum_{\vec{q}, \nu} \omega' \vec{q} (b_{\vec{q} \nu}^* b_{\vec{q} \nu} + b_{\vec{q} \nu} b_{\vec{q} \nu}) \]

The contributions to the total field momentum from both fields have the same sign.

We now will quantize the classical electromagnetic and gravitomagnetic fields by replacing the amplitudes \( a_{\vec{k} \mu} \) and \( b_{\vec{q} \nu} \) and their complex conjugates by the respective operators and their Hermitian conjugates, in the form,

\[ a_{\vec{k} \mu} \rightarrow C_{\vec{k}} \alpha_{\vec{k} \mu} \quad \text{and} \quad a_{\vec{k} \mu}^* \rightarrow C_{\vec{k}}^* \alpha_{\vec{k} \mu}^* \]

\[ b_{\vec{q} \nu} \rightarrow D_{\vec{q}} \beta_{\vec{q} \nu} \quad \text{and} \quad b_{\vec{q} \nu}^* \rightarrow D_{\vec{q}}^* \beta_{\vec{q} \nu}^* \]

with real normalization factors \( C_{\vec{k}} \) and \( D_{\vec{q}} \). We then obtain the operators of total field energy \( \mathcal{W} \) and total field momentum \( \mathcal{P} \) in the form,

\[ \mathcal{W} = \sum_{\vec{k}, \mu} \frac{\omega^2}{c^2} C_{\vec{k}}^2 (\alpha_{\vec{k} \mu}^* \alpha_{\vec{k} \mu} + \alpha_{\vec{k} \mu}^* \alpha_{\vec{k} \mu}^*) - \sum_{\vec{q}, \nu} \frac{\omega'^2}{c^2} D_{\vec{q}}^2 (\beta_{\vec{q} \nu}^* \beta_{\vec{q} \nu} + \beta_{\vec{q} \nu} \beta_{\vec{q} \nu}) \]

and

\[ \mathcal{P} = \sum_{\vec{k}, \mu} \omega \vec{k} C_{\vec{k}}^2 (\alpha_{\vec{k} \mu}^* \alpha_{\vec{k} \mu} + \alpha_{\vec{k} \mu}^* \alpha_{\vec{k} \mu}^*) + \sum_{\vec{q}, \nu} \omega' \vec{q} D_{\vec{q}}^2 (\beta_{\vec{q} \nu}^* \beta_{\vec{q} \nu} + \beta_{\vec{q} \nu} \beta_{\vec{q} \nu}) \]
We assume the validity of the commutation relations,
\[ \alpha_{k\mu} \alpha_{k\mu}^\dagger - \alpha_{k\mu}^\dagger \alpha_{k\mu} = \delta_{k\bar{k}} \delta_{\mu\mu'} \]
\[ \beta_{q\nu} \beta_{q\nu}^\dagger - \beta_{q\nu}^\dagger \beta_{q\nu} = \delta_{q\bar{q}} \delta_{\nu\nu'} \] (43)
and that all other combinations of these operators commute. We also write,
\[ C_{\vec{k}} = \sqrt{\frac{\hbar}{2\omega_{\vec{k}}}} \quad \text{and} \quad D_{\vec{q}} = \sqrt{\frac{\hbar}{2\omega_{\vec{q}'}}} \] (44)
and the operators \( W \) and \( P \) then take the form,
\[ W = \frac{1}{2} \sum_{\vec{k},\mu} \hbar \omega_{\vec{k}} (\alpha_{\vec{k}\mu} \alpha_{\vec{k}\mu}^\dagger + \alpha_{\vec{k}\mu}^\dagger \alpha_{\vec{k}\mu}) - \frac{1}{2} \sum_{\vec{q},\nu} \hbar \omega_{\vec{q}'} (\beta_{\vec{q}\nu} \beta_{\vec{q}\nu}^\dagger + \beta_{\vec{q}\nu}^\dagger \beta_{\vec{q}\nu}) \] (45)
and
\[ P = \sum_{\vec{k},\mu} \hbar \vec{k} (\alpha_{\vec{k}\mu} \alpha_{\vec{k}\mu}^\dagger + \alpha_{\vec{k}\mu}^\dagger \alpha_{\vec{k}\mu}) + \sum_{\vec{q},\nu} \hbar \vec{q} (\beta_{\vec{q}\nu} \beta_{\vec{q}\nu}^\dagger + \beta_{\vec{q}\nu}^\dagger \beta_{\vec{q}\nu}) \] (46)
with the eigenvalues
\[ W = \sum_{\vec{k},\mu} \hbar \omega (N_{\vec{k},\mu} + \frac{1}{2}) - \sum_{\vec{q},\nu} \hbar \omega' (N_{\vec{q},\nu} + \frac{1}{2}) \] (47)
and
\[ P = \sum_{\vec{k},\mu} \hbar \vec{k} (N_{\vec{k},\mu} + \frac{1}{2}) + \sum_{\vec{q},\nu} \hbar \vec{q} (N_{\vec{q},\nu} + \frac{1}{2}) \] (48)
We are dealing with quanta of electromagnetic field and with quanta of the hypothetical gravitomagnetic field.

From Eq. (47) it follows that the ground state energy of the system of harmonic oscillators representing electromagnetic and gravitomagnetic field is equal to zero.

4. The Spin of the Quanta of a Gravitomagnetic Field

To determine the spin of a quantum of a gravitomagnetic wave we follow the argument in [12], adapted for the present case. In the vacuum we have, [4],
\[ \text{div} \vec{B}(\vec{r}, u) = 0, \]
\[ \text{div} \vec{G}(\vec{r}, u) = 0, \]
\[ \text{curl} G(\vec{r}, u) = -\frac{\partial \vec{B}(\vec{r}, u)}{\partial u} \] (49)
We define the vector
\[ \vec{F} = \frac{1}{\sqrt{2}}(\vec{G} + ic\vec{B}) \] (50)

From Eqs. (49) and (50), we then find that
\[ \text{div } \vec{F} = 0, \]
\[ i\partial_u \vec{F} = c \text{curl } \vec{F} \] (51)

With \( \varepsilon_{lmn} \) denoting the Levi-Civita tensor, we rewrite the last equation in the form
\[ i\partial_u F_l = c\varepsilon_{lmn}\partial_m F_n \] (52)

Introducing the angular momentum matrices,
\[ s_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad s_2 = i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad s_3 = i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \] (53)

with \( \partial_m = \frac{i}{\hbar}p_m \), we can rewrite Eq.(52) in the form
\[ -c(s_m)_{ln} p_m F_n = i\hbar \partial_u F_l \] (54)

or in the form
\[ -c(s \cdot \vec{p})_{ln} F_n = i\hbar \partial_u F_l \] (55)

which can be interpreted as the Schrödinger equation for the quantum of the gravitomagnetic field, with the Hamiltonian,
\[ -c(s \cdot \vec{p}) = \mathcal{H} \] (56)

while \( \vec{F} \) in Eq.(50) is the wave function.

The condition \( \text{div } \vec{F} = 0 \) can be rewritten in the form
\[ \vec{p} \cdot \vec{F} = 0 \] (57)

We now verify that the Schrödinger equation in Eq.(55) has a plane wave solution
\[ \vec{F}(\vec{r}, u) = \vec{f} e^{i\vec{q} \cdot \vec{r} - \omega' u} \] (58)
where $\vec{f}_q$ is represented by the column matrix

$$\vec{f}_q = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$ (59)

Introducing into Eq.(55) the matrices $s_l$ in Eq. (53) we obtain the equations

$$\omega' f_x + icq_z f_y - icq_y f_z = 0$$
$$-icq_z f_x + \omega' f_y + icq_y f_z = 0$$
$$icq_y f_x - icq_x f_y + \omega' f_z = 0$$ (60)

which have nonzero solutions when

$$(\omega')^2 = c^2 q^2$$ (61)

There follows the conclusion that we are dealing with the spin-one quantum, of the gravitomagnetic field.

5. Conclusions

The notion of a second time variable is not alien to physics. A brief review of papers in which the second time variable is considered was given in [4]. In a Kaluza-Klein theory with two times, the Schwarzschild type solution of the five-dimensional Einstein equations in the vacuum was determined in [8, 9]. The two independent parameters of that solution are related with mass and electric charge, respectively. That solution exhibits a Schwarzschild radius, whose magnitude is predominantly determined by the electric charge. It was shown that the perihelie motion of a test particle in four-dimensional relativity has a counterpart in five dimensions in the perinucleic motion of a negatively-charged test particle. With the quantization conditions of the older quantum theory included into the five-dimensional geometry, the perinucleic motion of an electron leads to the fine structure of line spectra which is analogous to that determined by Sommerfeld’s formula for hydrogen-like atoms.

The Maxwell-Nordström equations were rederived in [3, 4] from the field tensor implied by the iterated form of a five-dimensional Dirac equation [5, 6, 7]. From the Maxwell-Nordström equations with two times there follows the existence of hypothetical gravitomagnetic phenomena, together with the electromagnetic phenomena. In particular, plane gravitomagnetic waves with a transverse polarization are implied by those equations [4].

In this paper the quantization of the classical electromagnetic and gravitomagnetic fields has lead to the conclusion that in $(3 + 2)$ dimensions, the
ground-state energies of electromagnetic and gravitomagnetic fields in the vacuum cancel out. It has been shown that the quanta of the hypothetical gravitomagnetic field have spin 1.

The idea that a remedy for the infinite energy of the ground state of the quantized electromagnetic field may be sought in finding another energy which will cancel that infinity out, was expressed by Wesson [14]. It seems that we are dealing with such a case.

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