A Novel Nonlinear Leader-Follower Opinion Dynamics Model With Asynchronous Trust/Distrust Evolution

Lei Shi, Yuhua Cheng, Senior Member, IEEE, Jinliang Shao, Xiaofan Wang, Senior Member, IEEE, and Yan Xu

Abstract—Trust and distrust are common in the opinion interactions among agents in social networks. It is believed in social psychology that opinion difference is usually an important factor affecting the trust/distrust level between neighboring agents. With that in mind, this paper proposes a nonlinear opinion dynamics model with asynchronous evolution of trust/distrust level based on opinion difference, in which the trust/distrust level between neighboring agents is portrayed as a nonlinear weight function of their opinion difference, and the asynchronous setting implies that each agent interacts with the neighbors to update the trust/distrust level and opinion at the times determined by its own will. The influence of an opinion leader with a firm stand on the formation of followers’ opinions is considered. Based on infinite products of nonnegative matrices, a comprehensive theoretical analysis for the opinion dynamics is performed. Numerical simulations based on two well-known networks called the “12 Angry Men” network and the Karate Club network in social psychology are provided to verify the correctness of the theoretical results.

Index Terms—Opinion dynamics; nonlinear model; opinion leaders; asynchronous interaction; trust and distrust.

I. INTRODUCTION

In a social network, the social agents which represent the social actors, e.g., countries, parties, social individuals, will have different opinions when they are confronted with various social issues, and the opinions may change according to the influence of other individuals. Over the past few decades, interest in the opinion formation mechanism of agents in social networks has grown dramatically. In this regard, the DeGroot model [1] is a classic reference, in which each agent utilizes the opinions information from the neighbors to form its opinion on various social issues. It has been shown conclusively in [1] that when the network structure has sufficient connectivity, the social interaction in the DeGroot model tends to facilitate the agents’ opinions to merge into a common opinion, namely, to reach opinion consensus in the whole social network. Based on the DeGroot model, researchers have conducted extensive researches on the mechanisms of opinion formation and constructed some improved models, such as the Friedkin-Johnsen model [2]–[8] and the Hegselmann-Krause model [7]–[9]. Under the existing mechanisms of opinion formation, different dynamics phenomena may emerge in the entire network, such as opinion consensus and opinion polarization, etc.

In many real situations of social networks, it is found that certain agents, called opinion leaders, have more power on influencing the opinions, decisions and actions of most other agents (known as followers) because of their expertise or positions [10]. For instance, in the famous Karate Club network [11], the entire club eventually was split into two new organizations due to conflict between the club supervisor and the club coach. Opinion differences among leaders led to polarization of congressional parties in congress [12]. The Twitter users with higher connectivity and problem participation had a significant impact on the opinions of ordinary Twitter users in the Twitter activism network [13]. Compared to the case in the absence of opinion leaders, it was shown in [14] that opinions tend to spread faster in the presence of opinion leaders. The emergence of these real social networks undoubtedly indicates that the study of the influence of opinion leaders on the formation of other agents’ opinions in social networks has potential application value. To date, many studies have focused on the dynamics of opinion consensus in the presence of an opinion leader, see [15]–[18] for examples. Specifically, following findings from experimental social psychology, the reference [15] considered the opinion dynamics of the DeGroot model via leadership with state and time-dependent characteristics. The opinion dynamics of the Hegselmann-Krause model with an opinion leader was examined in [16], [17]. Dong et al. [18] analyzed the opinion consensus of administrations in the management field.

The aforementioned opinion dynamics models mostly considered the positive interaction relationship between agents, which is also stated as trust, cooperation, or friendliness equivalently. However, as reported in [19], the negative interaction relationship between agents, such as distrust, competition, or antagonism, is ubiquitous in real social networks. The positive and negative interaction relationships are usually represented...
by the edges with positive and negative signs in the signed digraph, respectively. For consistency, in this paper, the positive and negative interaction relationships are collectively referred to as “trust” and “distrust” respectively, and thus the signed network is also called the “trust-distrust network”. In the pioneering study 20, Altanini proposed a simple and instructive dynamics model of trust-distrust networks based on Laplacian feedback designs, and analyzed the dynamics of networks with and without structural balance. Since then, the Altanini model on the trust-distrust network has gained widespread attention (see, e.g., 21, 22).

It is noteworthy that the trust/distrust level between neighboring agents (usually represented by the weight of the edge connecting them) is assumed to be independent of their own opinions in the Altani model. However, it can be observed in daily life that long-term opinion difference will lead two good friends who trust each other to drift away; in turn, holding similar opinions on certain events will weaken the gulf between two agents who distrust each other. In fact, it has been pointed out in “Social psychology” 23 that opinion difference is an important factor affecting the intimacy between social agents in real interaction scenarios, and people usually tend to be more trusting to the agents who have similar opinions with themselves and more distrustful towards the agents who hold different opinions with themselves.

An American classic film called “12 Angry Men” in 1957 is a living example to show the opinion evolution of the agents under the influence of an opinion leader, in which twelve jurors were invited to decide whether a teenager was guilty of killing his father. At the beginning, only juror 8 firmly believed that the boy was acquitted, and the remaining jurors considered the boy guilty. After rounds of discussions, juror 8 finally succeeded in persuading other eleven jurors to agree with him. Moreover, through careful observation of the details of the film, it is found that when juror 8 presented the evidence in support of his own opinion, other jurors showed varying attitudes (i.e., trust or distrust) to juror 8, and the trust/distrust level was affected mainly by their opinion difference with juror 8. Although juror 3, for instance, had the strongest initial distrust level to juror 8, his initial judgment was gradually shaken with jury 8’s statements of sufficient evidences of the boy’s innocence. With the deepening process of the opinion interactions, juror 3 gradually reduced the distrust to juror 8, and finally admitted that the existing evidence is not enough to make the guilty verdict.

Inspired by the above discussions, the main purpose of this paper is to model the relationship between opinion difference and trust/distrust level of neighboring agents according to the real interaction scenario, and analyze the influence of opinion leaders on the formation of followers’ opinions. The main contributions of this paper can be summarized as follows.

1) A nonlinear opinion dynamics model with asynchronous evolution of trust/distrust level based on opinion difference is proposed, where the trust/distrust level between neighboring agents is portrayed as a weight function of their opinion difference, and the asynchronous setting implies that each agent interacts with the neighbors to update the trust/distrust level and opinion at the times determined by its own will. It is noteworthy that the asynchronous evolution mechanism of trust/distrust level is proposed for the first time in this paper, and it has not been taken into account in the existing opinion dynamics models so far.

II) Through the construction of signed digraphs involving asynchronous interaction information, the analysis of dynamics evolution for the nonlinear model is equivalent to the convergence analysis of infinite products of nonnegative matrices. By developing some new approaches based on nonnegative matrix, we solve the convergence problem and establish the algebraic condition which depends on the network structure and weight functions for achieving consensus in presence of a single opinion leader. Further, we explore the dynamics on the trust-distrust network with multiple opinion leaders. Finally, it is shown that the derived dynamics results can be applied to explain the opinion evolution phenomena of real-world networks in social psychology, such as the “12 Angry Men” network and the Karate Club network.

The rest of this paper is arranged as follows. Section II introduces some basic notations and preliminaries. The problem statement is shown in Section III. Section IV presents the main results of this paper. Section V validates the correctness of the theoretical results by some simulation examples. The paper is concluded in Section VI.

II. Preliminaries

A. Notations

Some notations for a real matrix \( S = [s_{ij}]_{n \times n} \) are introduced as follows. \( |S| = [|s_{ij}]_{n \times n} \) stands for a matrix in which each element \( s_{ij} \) is the absolute value of \( s_{ij} \). The \( i \)-th row’s sum of matrix \( S \) is represented by \( |S|_i = \sum_{j=1}^n s_{ij} \). The infinite norm of matrix \( S \) is expressed as \( |||S|||_\infty = \max \{ |S|_i \mid i = 1, 2, \ldots, n \} \). The real matrix \( S \) is nonnegative if \( s_{ij} \geq 0, \quad i, j = 1, 2, \ldots, n \). Let \( S(i : j, p : q) \) be a block matrix consisting of the \( i \)-th row to the \( j \)-th row and the \( p \)-th column to the \( q \)-th column of matrix \( S \), where \( i \leq j, p \leq q \). Denote the left products of matrices \( S_i, i = 1, 2, \ldots, q \) by \( \prod_{i=1}^q S_i = S_q \cdots S_2 S_1 \). 0 is a matrix with all elements being 0. \( |\mathcal{X}| \) represents the number of elements in the set \( \mathcal{X} \).

Definition 1: 24 A real matrix \( S = [s_{ij}]_{n \times n} \) is general row-stochastic if \( |S|_i = 1, \quad i = 1, 2, \ldots, n \).

Lemma 1: Let \( S_1, S_2, \ldots, S_q \) be \( n \times n \) nonnegative matrices and \( \max \{ |S_i|_t \mid t = 1, 2, \ldots, q, \quad i = 1, 2, \ldots, n \} = g \), then \( \| \prod_{i=1}^q S_i \|_\infty \leq g^q \).

Proof: For any \( i \in \{ 1, 2, \ldots, n \} \), we have:

\[
|S_i|_t = \sum_{j_1} |S_{ij_1} A_{j_1} S_{j_1-1} \cdots S_2 S_1| \\
= \sum_{j_1} |S_{ij_1} A_{j_1} S_{j_1-1} \cdots S_2 S_1| \\
= \sum_{j_1} |S_{ij_1} A_{j_1} S_{j_1-1} \cdots S_2 S_1| \\
\leq g^q.
\]

According to the definition of infinite norm of matrix, one has \( \| \prod_{i=1}^q S_i \|_\infty \leq g^q \).
Similar to Lemma 1 we present the corresponding result for the general row-stochastic matrix below.

**Lemma 2:** Let $S_1, S_2, \ldots, S_q$ be general row-stochastic matrices, then $\prod_{i=1}^{q} S_i$ is a general row-stochastic matrix.

### B. Interaction network

Consider a social network with $n$ agents indexed in the set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, where $v_n$ represents the opinion leader and $v_1, v_2, \ldots, v_{n-1}$ stand for the followers. The opinion leader is an agent with a firm stand and its opinion is not affected by the opinions of other agents. The opinions of followers are influenced by those of their neighbors.

The structure of the social network is represented as a signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ represent the sets of nodes and edges, respectively. There is a directed edge $(v_j, v_i)$ in $\mathcal{E}$ if and only if agent $v_i$ takes agent $v_j$ as an in-neighbor and thus agent $v_j$’s opinion is influencing agent $v_i$’s. Denote the signed adjacency matrix by $\mathcal{A} = (a_{ij})$, with the elements being 0, 1 or $-1$, where $a_{ij} \neq 0$ if and only if $(v_j, v_i) \in \mathcal{E}$, otherwise $a_{ij} = 0$. In particular, $a_{ij} = 1$ and $a_{ij} = -1$ represent that agent $v_i$ trust and distrust agent $v_j$, respectively.

The sets of trusted in-neighbors and distrustful in-neighbors of agent $v_i$ are denoted by $N^+_i = \{(v_j, v_i) \in \mathcal{E}, a_{ij} = 1\}$ and $N^-_i = \{(v_j, v_i) \in \mathcal{E}, a_{ij} = -1\}$, respectively. The set of in-neighbors of agent $v_i$ is $N_i = N^+_i \cup N^-_i$. For convenience, the agent’s neighbors refer specifically to its in-neighbors in the rest of this paper. A directed path from node $v_{i_1}$ to node $v_{i_s}$ in a digraph $\mathcal{G}$ is denoted by $\mathcal{P}_{v_{i_1} \rightarrow v_{i_s}} = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{s-1}}, v_{i_s}\}$, where the nodes $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$ are distinct and $(v_{i_{s-1}}, v_{i_s}) \in \mathcal{E}$ for $s = 2, 3, \ldots, z$. The directed distance from $v_{i_1}$ to $v_{i_s}$ is the number of edges in the shortest path from $v_{i_1}$ to $v_{i_s}$.

### C. Asynchronous interaction

Most of the existing literature is dedicated to analyzing opinion dynamics of social networks under the synchronous interaction, in which all agents interact with their neighbors at the same time. However, in real-world social networks, the synchronous interaction is generally not easy to implement due to some objective factors and the subjective will. This consideration shifts our attention to a more general asynchronous interaction scenario in this paper, where each agent interacts with the neighbors to update the trust/distrust level and opinion at the times determined by its own will. In other words, each agent can interact with its neighbors at arbitrary times in the asynchronous interaction scenario. Without loss of generality, we assume that the set of each agent $v_i$’s opinion interaction times is

$$\{s^i_k\} = \{s^i_0, s^i_1, \ldots, s^i_k, \ldots\},$$

in which $s^i_0, s^i_1, \ldots, s^i_k \in \{0, 1, 2, \ldots, t\}$ and $0 = s^i_0 < s^i_1 < \cdots < s^i_k < \cdots$. It is further assumed that $\{s^i_k\}$ satisfies the following condition:

$$s^i_{k+1} - s^i_k \leq h,$$

where $h$ is a positive integer. An example for three agents’ interaction times in the asynchronous interaction scenario is shown in Fig. 1.

**Remark 1:** Condition (1) is given to guarantee that there is no such an agent, who no longer communicates with its neighbors from a certain time. Equivalently, in each finite time interval with length $h$, where $h$ is a constant that can be arbitrarily large, each agent interacts with its neighbors at least once. This is necessary in the asynchronous interaction scenario. Without Condition (1), there may be an agent unable to interact with its neighbors at all times, which obviously cannot guarantee the realization of opinion consensus or opinion polarization. In addition, it is worth noting that the asynchronous interaction considered in this paper includes the synchronous interaction as a special case of $h = 1$. Therefore, the asynchronous dynamics results obtained in this paper can be easily extended to the synchronous interaction scenario.

### III. Model formulation

In a social network, the trust/distrust level between neighbouring agents is usually affected by their opinion difference. For example, it can be observed in the daily interaction that two closely related agents will gradually alienate due to long-term opinion difference, and the similarity of opinions will gradually eliminate the gulf between two distrusting agents.

Take into account such an interaction scenario, we set the trust/distrust level of agent $v_i$ to agent $v_j$ as a function $f_{ij}$ of their opinion difference $|x_j(t) - x_i(t)|$, i.e.,

$$f_{ij}(t) = \begin{cases} f^{ij}_+(|x_j(t) - x_i(t)|), & \text{if } a_{ij} = 1, \\ f^{ij}_-(|x_j(t) - x_i(t)|), & \text{if } a_{ij} = -1, \end{cases}$$

where the weight functions $f^{ij}_+$ and $f^{ij}_-$ are used to quantify the trust and distrust levels of agent $v_i$ to agent $v_j$, respectively. Suppose the weight functions satisfy the following assumptions.

**Assumption 1:** For each agent $v_i$, the functions $f^{ij}_+$, $j = 1, 2, \ldots, n$ are positive, decreasing and bounded, in particular, $\alpha \leq f^{ij}_+(y_2) \leq f^{ij}_+(y_1) \leq \beta$, $j \neq n$ and $\alpha \leq f^{ij}_-(y_2) \leq f^{ij}_-(y_1) \leq \beta$ as $0 \leq y_1 < y_2 < +\infty$.

**Assumption 2:** For each agent $v_i$, the functions $f^{ij}_+$, $j = 1, 2, \ldots, n$ are positive, increasing and bounded, in particular, $\lambda \leq f^{ij}_+(y_1) \leq f^{ij}_+(y_2) \leq \gamma$, $j \neq n$ and $\lambda \leq f^{ij}_-(y_1) \leq f^{ij}_-(y_2) \leq \gamma$ as $0 < y_1 < y_2 < +\infty$.

**Remark 2:** For each edge $(v_j, v_i)$, there is a corresponding weight function $f_{ij}$ to quantify the trust/distrust level of agent $v_i$ to agent $v_j$, which reflects the difference of opinion interaction between different neighboring agents in the social network. In addition, the leader which plays a leading role in the evolution of the entire social network is usually an agent...
An interaction network $\mathcal{G}$ and the corresponding newly constructed signed digraphs $\mathcal{G}(t)$, $t = 0, 1, \ldots, 6$ under the asynchronous interaction setting shown in Fig. 1.

### IV. MAIN RESULTS

In this section, we analyze the opinion dynamics in presence of single and multiple opinion leaders under the asynchronous evolution mechanism of trust/distrust level based on opinion difference, respectively.

Based on the construction of digraph $\mathcal{G}(t)$, we rearrange model (3) as

$$x_i[t+1] = \theta_i(t)x_i[t] + (1-\theta_i(t)) \sum_{v_j \in \mathcal{N}(t)} p_{ij}(t),$$  

(5)

where

$$p_{ij}(t) = \frac{a_{ij}(t)f_{ij}(t)}{\sum_{v_s \in \mathcal{N}(t)} a_{is}(t)f_{is}(t)}.$$  

(6)

Let $x[t] = [x_1[t], x_2[t], \ldots, x_n[t]]^T$, then (5) can be expressed as the following state-space form

$$x[t+1] = \Gamma(t)x[t],$$  

(7)

where $\Gamma(t) = \Theta(t) + (I_n - \Theta(t)) P(t)$ with $P(t) = [p_{ij}(t)]$ and $\Theta(t) = \text{diag} \{\theta_1(t), \ldots, \theta_n(t)\}$. Observing the construction of $\Gamma(t)$, we know that $\Gamma(t)$ is a general row-stochastic matrix with both positive and negative elements.

**Remark 3:** The classical linear DeGroot and Altafini models can also be transformed into the state-space form $x[t+1] = Hx[t]$. In the DeGroot model, $H$ is a nonnegative row-stochastic matrix. In the Altafini model, although $H$ is a matrix containing both positive and negative elements, $|H|$ is a row-stochastic matrix. The Friedkin-Johnsen model can be written as the form $x[t+1] = \Delta x[0] + (I - \Delta) Q x[t]$, where $\Delta$ and $I - \Delta$ are nonnegative diagonal matrices, and $Q$ is a row-stochastic matrix. In Eq. (7), $\Gamma(t)$ is a general row-stochastic matrix in which each element is a nonlinear function of the opinion difference between corresponding neighbouring agents.

Before proceeding, the following notations are introduced,

$$|\mathcal{N}_{\text{max}}^-| = \max\{|\mathcal{N}_i^-| \mid i = 1, 2, \ldots, n - 1\},$$

$$|\mathcal{N}_{\text{min}}^+| = \min\{|\mathcal{N}_i^+| \mid i = 1, 2, \ldots, n - 1\},$$

$$|\mathcal{N}_{\text{min}}^-| = \min\{|\mathcal{N}_i^-| \mid i = 1, 2, \ldots, n - 1\},$$

$$|\mathcal{N}_{\text{max}}^+| = \max\{|\mathcal{N}_i^+| \mid i = 1, 2, \ldots, n - 1\}.$$  

Next, we show that the DeGroot-style interaction rule (4) can effectively guarantee the realization of opinion consensus on the trust-distrust network in presence of a single opinion leader.

**Theorem 1:** Consider model (3) with a single opinion leader. Suppose that the followers’ self-confidence levels satisfy $\theta_i \neq 0$, $i = 1, 2, \ldots, n - 1$. If the interaction network $\mathcal{G}$ contains a spanning tree $\mathcal{T}$ with the root being the leader and all edges representing trust relationships, and the following condition holds

$$p^h - \sigma p^h < 1,$$  

(8)

where $p$ is the longest distance from the leader to the followers on the spanning tree $\mathcal{T}$ and

$$l = 1 + \frac{2(1 - \theta_{\text{min}})|\mathcal{N}_{\text{max}}| \max\{\tau, \mathcal{K}\}}{|\mathcal{N}_{\text{min}}^+| \min\{\alpha, \beta\} - |\mathcal{N}_{\text{max}}^-| \max\{\tau, \mathcal{K}\}},$$

$$\sigma = \min\left\{\theta_{\text{min}}, \frac{1}{|\mathcal{N}_{\text{max}}^-| \min\{\alpha, \beta\} - |\mathcal{N}_{\text{min}}^+| \max\{\tau, \mathcal{K}\}}\right\},$$

using (5), (6) and the inequality $1 - x \geq e^{-x}$, with $x = \sum_{j \neq i} p_{ij}$, we have

$$x_i[t+1] = \theta_i(t)x_i[t] + (1-\theta_i(t)) \sum_{v_j \in \mathcal{N}(t)} p_{ij}(t) \geq e^{-\sum_{j \neq i} p_{ij}} x_i[t] = e^{-\|px\|} x_i[t],$$

and

$$x_i[t+1] = \theta_i(t)x_i[t] + (1-\theta_i(t)) \sum_{v_j \in \mathcal{N}(t)} p_{ij}(t) \leq \theta_i(t)x_i[t] + (1-\theta_i(t)) \max\{\tau, \mathcal{K}\} x_i[t] = \theta_i(t)x_i[t] + (1-\theta_i(t)) \max\{\tau, \mathcal{K}\} x_i[t].$$

Defining

$$x_i[t+1] = \theta_i(t)x_i[t] + (1-\theta_i(t)) \sum_{v_j \in \mathcal{N}(t)} p_{ij}(t),$$

(5)

we can get

$$x_i[t+1] = \theta_i(t)x_i[t] + (1-\theta_i(t)) \sum_{v_j \in \mathcal{N}(t)} p_{ij}(t),$$

(6)

where

$$p_{ij}(t) = \frac{a_{ij}(t)f_{ij}(t)}{\sum_{v_s \in \mathcal{N}(t)} a_{is}(t)f_{is}(t)}.$$
in which \( |\mathcal{A}_{\min}^+| \min\{\alpha, \beta\} > |\mathcal{A}_{\max}^-| \max\{7, \pi\}, \theta_{\max} = \max\{\theta_1, \ldots, \theta_{n-1}\}\) and \( \theta_{\min} = \max\{\theta_1, \ldots, \theta_{n-1}\}\), then opinion consensus is achieved, that is, \( \lim_{t \to \infty} x_i[t] = x_i[0], i = 1, 2, \ldots, n - 1\).

**Proof:** According to the division of the opinion leader and the followers, \( \Gamma(t) \) has a block-matrix form as follows

\[
\Gamma[t] = \begin{bmatrix} \Phi(t) & w(t) \\ 0 & 1 \end{bmatrix},
\]

where \( \Phi(t) = \Theta^*(t) + (I_{n-1} - \Theta^*(t))P(t)(1 : n - 1, 1 : n - 1) \) and \( w(t) = (I_{n-1} - \Theta^*(t))P(t)(1 : n - 1, n : n)\). Note that \( \Gamma(t) \) is a general row-stochastic matrix. By Lemma 2, \( \prod_{t=0}^{\infty} \Gamma(t) \) is a general row-stochastic matrix.

To analyze the dynamics of model \( \mathcal{A} \), the following equation needs to be proved first

\[
\prod_{t=0}^{\infty} \Phi(t) = 0. \tag{9}
\]

According to a known property of infinite norm, i.e., \( \|AB\|_{\infty} \leq \|A\|\|B\|_{\infty} \), we have \( \| \prod_{t=0}^{\infty} \Phi(t) \|_{\infty} \leq \| \prod_{t=0}^{\infty} |\Phi(t)| \|_{\infty} \). And hence, it is sufficient to prove

\[
\left\| \prod_{t=0}^{\infty} |\Phi(t)| \right\|_{\infty} = 0 \tag{10}
\]

for ensuring that Eq. (9) holds. Under the condition \( \theta_i \neq 0, i = 1, 2, \ldots, n - 1 \), the diagonal elements of nonnegative matrix \( |\Phi(t)| \) satisfy \( |\Phi(t)|_{ii} = \theta_i > \sigma, i = 1, 2, \ldots, n - 1 \). Some non-diagonal elements of \( |\Phi(t)| \) satisfy:

\[
|\Phi(t)|_{ij} = (1 - \theta_i(t))|pi_j(t)| > \sigma, \text{ if } (j, i) \in \mathcal{E}(t), a_{ij} = 1.
\]

Further the row sums of \( |\Phi(t)| \) satisfy:

\[
\Lambda_i[|\Phi(t)|] \leq 1 - (1 - \theta_i(t))|pi_i(t)| = 1 - \sigma < 1, \text{ if } (i, n) \in \mathcal{E}(t), a_{in} = 1,
\]

\[
\Lambda_i[|\Phi(t)|] \leq 1, \text{ otherwise}.
\]

In order to get Eq. (10), we first make an isometric division of the time axis, i.e., \( [0, ph), [ph, 2ph), [2ph, 3ph), \ldots \). Then, we prove \( \| \prod_{t=\gamma ph}^{\gamma ph+ph-1} \Phi(t) \|_{\infty} < 1 \) associated with each time interval \( [\gamma ph, \gamma ph + ph - 1], \gamma \in \mathbb{N}. \) Since \( \mathcal{G} \) contains a spanning tree \( \mathcal{T} \) with the root being the leader and all edges representing trust relationships, there exists a directed path \( \mathcal{P}_{v_n \to v_{s_z}} = (v_n, v_{s_1}, v_{s_2}) \cdots (v_{s_{z-1}}, v_{s_z}) \) for each follower \( v_{s_z} \), where \( a_{s_{z-1}z} = a_{s_{z-2}z} = \cdots = a_{s_1z} = 1 \).

According to (10), one knows that each agent updates its opinion at least once in arbitrary time interval with length \( h \). Thus, in the interval \( [\gamma ph, \gamma ph + h), \) there exists \( \gamma ph + l_1 \) such that \( (v_n, v_{s_z}) \in \mathcal{E}^{\gamma ph + l_1} \), which leads to

\[
\Lambda_{s_z}[|\Phi(\gamma ph + l_1)|] \leq l - \sigma. \tag{11}
\]

By the result in Lemma 1, we have

\[
\Lambda_{s_z}[\prod_{t=\gamma ph}^{\gamma ph+l_1} |\Phi(t)|] \leq l^{s_z}; j = 1, 2, \ldots, n - 1. \tag{12}
\]

Combining (11) and (12), it can be derived that

\[
\Lambda_{s_z}[\prod_{t=\gamma ph}^{\gamma ph+l_1} |\Phi(t)|] = \sum_j \prod_{t=\gamma ph}^{\gamma ph+l_1} |\Phi(t)|_{s_zj} \Lambda_{s_z}[\prod_{t=\gamma ph}^{\gamma ph+l_1} |\Phi(t)|] \leq \prod_j \sum_{t=\gamma ph}^{\gamma ph+l_1} |\Phi(t)|_{s_zj} \leq (l - \sigma)^{s_z} \tag{13}
\]

Noting the fact \( |\Phi(t)|_{s_z s_z} \geq \sigma \) for any \( t \in \mathbb{N}, \) holds

\[
\left\| \prod_{t=\gamma ph}^{\gamma ph+l-1} \Phi(t) \right\|_{s_z s_z} \geq \sigma^{\gamma ph + h - l} \tag{14}
\]

According to (13) and (14), the following result is obtained

\[
\Lambda_{s_z}[\prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)|] = \sum_j \prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)|_{s_zj} \Lambda_{s_z}[\prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)|] + \prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)|_{s_z s_z} \Lambda_{s_z}[\prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)|] \leq l^{h - \sigma^h} \tag{15}
\]

Consider the intervals \( [\gamma ph + yh - h, \gamma ph + yh), y = 2, 3, \ldots, z \). In each interval, there exists \( \gamma ph + yh - h + l_y \in [\gamma ph + yh - h, \gamma ph + yh) \) such that \( (s_{y-1}, s_y) \in \mathcal{E}^{\gamma ph + yh - h + l_y} \), and then we have \( |\Phi(\gamma ph + yh - h + l_y)|_{s_y s_{y-1}} \geq \sigma \).

It follows that

\[
\left\| \prod_{t=\gamma ph+yh-h}^{\gamma ph+yh} |\Phi(t)| \right\|_{s_y s_{y-1}} \geq \sigma^{\gamma ph + h - l} \tag{16}
\]

Equivalently, \( |\Phi(t)|_{s_z s_z} \geq \sigma^{\gamma ph + z - h} \), which combines the fact \( \left\| \prod_{t=\gamma ph+zh}^{\gamma ph+zh+2h} |\Phi(t)| \right\|_{s_z s_z} \geq \sigma^{\gamma ph + z - h} \) ensures that

\[
\left\| \prod_{t=\gamma ph+zh}^{\gamma ph+zh+2h} |\Phi(t)| \right\|_{s_z s_z} \geq \sigma^{\gamma ph + z - h} \tag{17}
\]

By (15) and (17), we get that

\[
\Lambda_{s_z}[\prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)|] = \sum_j \prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)|_{s_zj} \Lambda_{s_z}[\prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)|],
\]

\[
\leq \prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)|_{s_z s_z} \Lambda_{s_z}[\prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)|] \leq l^{h - \sigma^h} \tag{18}
\]

for any \( s_z = 1, 2, \ldots, n \). By condition (8), we have

\[
\left\| \prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)| \right\|_{s_z s_z} \leq l^{h - \sigma^h} < 1.
\]

This immediately yields that

\[
\left\| \prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)| \right\|_{\infty} \leq l^{h - \sigma^h} < 1.
\]

\[
\left\| \prod_{t=\gamma ph}^{\gamma ph+l-1} |\Phi(t)| \right\|_{\infty} \leq l^{h - \sigma^h} < 1.
\]
We discuss the cases of \( w^*(t) = \Phi(t) \cdots \Phi(i+1)w(i) \). Since all row sums of \( \prod_{t=0}^{\infty} \Gamma(t) \) are equal to 1, one has \( \lim_{t \to \infty} w^*(t) = 1_{n \times 1} \). This means that \( \lim_{t \to \infty} x_i[t] = x_i[0], i = 1, 2, \ldots, n - 1 \).

As can be seen from Theorem 1, the self-confidence levels of all agents are assumed to be non-zero to achieve opinion consensus in the asynchronous interaction scenario. In fact, opinion consensus can still be achieved when there is no constraint on the self-confidence levels of the agents in the synchronous interaction scenario, as shown in the following corollary.

**Corollary 1:** Let each agent update the opinion by the synchronous model \( x_i[t + 1] = \theta_i x_i[t] + (1 - \theta_i) u_i[t] \), where \( u_i[t] \) is given in (3). Consider the case with a single opinion leader. If the interaction network \( G \) contains a spanning tree with the root being the leader and all edges representing trust relationships, and the following condition holds

\[
P^h - \sigma^h < 1, \quad (20)
\]



where \( \sigma = \frac{(1 - \theta_{\text{max}}) \min\{\alpha_{ij}, \beta_{ij}\}}{\alpha_{\text{max}} \min\{\alpha_{ij}, \beta_{ij}\} - \alpha_{\text{min}} \max\{\alpha_{ij}, \beta_{ij}\}} \), then opinion consensus is achieved.

**Proof:** We discuss the cases of \( \theta_i \neq 0 \) and \( \theta_i = 0 \), respectively. Since the synchronous interaction is a special case of the asynchronous interaction. Consequently, synchronous opinion consensus can be achieved when \( \theta_i \neq 0 \) according to Theorem 1. When \( \theta_i = 0 \), the synchronous model is written as

\[
x[t + 1] = R(t)x[t] = \begin{bmatrix} R_1(t) & R_2(t) \\ 0 & 1 \end{bmatrix} x[t],
\]

where \( R(t) = [r_{ij}(t)] \) is a general row-stochastic matrix in which all diagonal elements are equal to zeros and \( r_{ij}(t) = a_{ij} f_{ij}(t) / \sum_{k \in K(t)} a_{ik} f_{ik}(t) \) for \( i \neq j \). \( R_1(t) \) is a nonnegative matrix. Using a method similar to the analysis of nonnegative matrix \( \Phi(t) \) in Theorem 1, we can deduce that \( \prod_{t=0}^{\infty} R_1(t) = 0 \) when \( G \) contains a spanning tree rooted at the leader. Equivalently, \( \prod_{t=0}^{\infty} R_1(t) = 0 \). It follows that

\[
\prod_{t=0}^{\infty} R(t) = \begin{bmatrix} 0 & 1_{n \times n-1} \\ 0 & 1 \end{bmatrix},
\]

which leads to \( \lim_{t \to \infty} x_i[t] = x_i[0], i = 1, 2, \ldots, n - 1 \), namely, opinion consensus is achieved.

Inspired by real-world networks such as the well-known “12 Angry Men” network in social psychology [23], we study the consensus dynamics for the nonlinear model (3) on the trust-distrust network in presence of a single opinion leader in Theorem 1. As a matter of fact, more common are social networks with multiple opinion leaders, such as the Karate Club network [11] and the Party network [12], etc. With that in mind, we below explore the opinion dynamics on the trust-distrust network with multiple opinion leaders under the asynchronous evolution mechanism of trust/distrust level based on opinion difference. Without loss of generality, we assume that the set of followers and the set of leaders are \( \mathcal{F} = \{v_1, \ldots, v_m\} \) and \( \mathcal{L} = \{v_{m+1}, \ldots, v_n\} \), respectively.

**Theorem 2:** Consider model (3) with multiple opinion leaders. Under the condition in (8), if for each follower \( v_i \in \mathcal{F} \), there exists a directed path \( \mathcal{D}_{v_i} \), with all edges representing trust relationships, where \( v_j \in \mathcal{L} \), then the followers’ final opinions are the linear combinations of the leaders’ opinions, i.e.,

\[
\lim_{t \to \infty} x_i[t] = \sum_{j=1}^{n-m} c_{ij} x_{m+j}[0], \quad i = 1, 2, \ldots, m, \quad (21)
\]

where \( \sum_{j=1}^{n-m} c_{ij} = 1 \).

**Proof:** Let \( x_F[t] = [x_1[t], \ldots, x_m[t]]^T \) and \( x_L[t] = [x_{m+1}[t], \ldots, x_n[t]]^T \). Then, Eq. (7) is expressed as

\[
\begin{bmatrix} x_F[t+1] \\ x_L[t+1] \end{bmatrix} = \Gamma(t) \begin{bmatrix} x_F[t] \\ x_L[t] \end{bmatrix},
\]

where \( \Gamma(t) = [\Gamma_{ij}(t)] \) is a nonnegative matrix. Then using the analysis method similar to the nonnegative matrix \( \Phi(t) \) in Theorem 1, we have \( \lim_{t \to \infty} \Gamma_1(t) = 0 \). Equivalently, \( \lim_{t \to \infty} \Gamma(t) = 0 \). When \( t \to \infty \), the final form of Eq. (21) can be written as

\[
\lim_{t \to \infty} [x_F[t] \quad x_L[t]] = \begin{bmatrix} 0 & \lim_{t \to \infty} \Gamma^*(t) \\ 0 & I_{m \times m} \end{bmatrix} [x_F[0] \quad x_L[0]],
\]

where \( \Gamma^*(t) = \sum_{i=0}^{\infty} \Gamma_1(i + 1) \Gamma_2(i) \). It follows that \( \lim_{t \to \infty} x_F[t] = \lim_{t \to \infty} \Gamma^*(t) x_F[0] \), namely, \( \lim_{t \to \infty} x_i[t] = \sum_{j=1}^{n-m} c_{ij} x_{m+j}[0] \) for any \( i = 1, 2, \ldots, m \), where \( c_{ij} = \lim_{t \to \infty} [\Gamma^*(t)]_{ij} \). As known from Theorem 1, \( \lim_{t \to \infty} \Gamma(t) \) is a general row-stochastic matrix. Equivalently, \( \lim_{t \to \infty} \Gamma^*(t) \) is a general row-stochastic matrix. Therefore, we have \( \sum_{j=1}^{n-m} c_{ij} = 1 \) in which each combination coefficient \( c_{ij} \) is a real number determined by the weight function \( f_{ij} \) and the opinion difference \( |x_i[t] - x_j[t]| \).

When considering the dynamics behavior of the linear DeGroot model in presence of multiple leaders, which is the so-called containment control of multi-agent systems, it has been derived that the followers’ states eventually converge into a convex hull composed of the leaders’ states if the interaction graph contains only positive signed edges, and the final states of the followers are independent of their own initial states (e.g., see [26]). But in Eq. (21) of Theorem 2, there may exist combination coefficients satisfying \( c_{ij} < 0 \) because of the influence of negative signed edges, that is, the followers’ final opinions may be outside the convex hull of the leaders’ opinions. In addition, the followers’ final opinions, described as \( \lim_{t \to \infty} x_i[t] = \sum_{j=1}^{n-m} c_{ij} x_{m+j}[0], i = 1, 2, \ldots, m \), are dependent on their initial opinions since \( c_{ij} \) is determined by the weight function and the opinion difference between neighboring agents. In Section V, the efficiency of Theorem 2 is illustrated by a simulation example on the Karate Club network.

**V. Simulation results**

**Example 1:** In the film “12 Angry Men”, 12 jurors were invited to decide whether a boy was guilty or not. For this
Because each juror actively communicates with other jurors about whether the boy is guilty, we assume that each juror can be influenced by the opinions of his neighbors at any time, that is, the parameter $h$ is set as 1 in the asynchronous interaction. Suppose that the self-confidence level of juror 8 is $\theta_8 = 1$, while the self-confidence levels of the remaining jurors are set to $\theta_i = 0.5$, $i \neq 8$. In Fig. 3(a), red solid lines and black dotted lines represent trust and distrust relationships, respectively. In addition, the weights of red solid lines are quantified by a positive, decreasing, bounded function $f^{+}_{ij}(|x_i[t] - x_j[t]|) = 1 - 0.1|x_i[t] - x_j[t]|$, and the weights of black dotted lines are quantified by a positive, increasing, bounded function $f^{-}_{ij}(|x_i[t] - x_j[t]|) = 0.02|x_i[t] - x_j[t]| + 0.06$, where $|x_i[t] - x_j[t]|$ represents the opinions difference between neighboring jurors. With the mentioned settings, the condition [8] in Theorem 1 holds.

According to the plot of the film “12 Angry Men”, we can observe that juror 3 shows the strongest distrust level to juror 8 at the beginning. However, juror 3’s distrust level to juror 8 gradually decreased as juror 8 stated the evidences round after round, which can be seen from Fig. 4(a). In addition, juror 9 appeared hesitant in convicting the boy for lack of evidence at the initial moment, so he showed trust in juror 8 but with little level when juror 8 stated sufficient evidences of the boy’s innocence at the outset. Along with the opinion interaction, the trust level of juror 9 to juror 8 gradually increased, which can be observed from Fig. 4(b). Finally, all the jurors who thought the boy was guilty at the initial moment finally turned their opinions into “no guilty”. This phenomenon is shown through Fig. 3(b). Moreover, we can also clearly see from Fig. 3(b) that jurors 2, 5, 6, 9, 11 change their opinions faster than jurors 1, 3, 4, 7, 10, 12, which is exactly the same as the details of the film.

Example 2: The Karate Club network proposed in [11] is a classic data set in the field of social network analysis. Considering the dynamics phenomenon in karate club, where the club is split into two small groups each with its core because of the dispute between the supervisor (node 1) and the coach (node 34), the following simulation example is employed to verify the theoretical results.

Consider the interaction structure of Karate Club network shown in Fig. 5(a). The initial opinions of the supervisor and the coach are set to $x_1 = 2$ and $x_{34} = -2$, respectively, and other members’ initial opinions are randomly generated in the interval $[-8, 8]$, in particular, $x_5 = 7$ and $x_{33} = -6$. We set the weight functions $f^{+}_{ij}$ and $f^{-}_{ij}$ to be the same as the “12 Angry
Men” network. According to Eq. (21) in Theorem 2 we know that the final opinions of all members are the linear combinations of the initial opinions of the supervisor and the coach, i.e., \( x_i[t] = c_{i1}x_1[0] + c_{i2}x_34[0] \), where \( c_{i1} + c_{i2} = 1 \). Through calculation, we get \( |c_{i1}| |x_1[0]| - |x_i[t]| = |c_{i2}| |x_{34}[0]| - |x_i[t]| \). This means that the member \( i \)'s opinion is closer to the supervisor’s opinion if \( |c_{i1}| > |c_{i2}| \) (the corresponding node and opinion trajectory are marked as blue in Fig. 5(a) and Fig. 5(b)), while its opinion is closer to that of the coach if \( |c_{i2}| > |c_{i1}| \) (which are marked as yellow in Figs. 5(a) and 5(b), accordingly). For example, it can be calculated that the combination coefficients associated with the members 5 and 33 are \( c_{51} = 0.8374 \), \( c_{52} = 0.1626 \) and \( c_{33,1} = -0.1152 \), \( c_{33,2} = 1.1152 \), respectively. From \( |c_{51}| > |c_{52}| \) and \( |c_{33,1}| < |c_{33,2}| \), we can say that the member 5 tends to support the supervisor, while the member 33 trusts the coach more in this numerical example. As a result, we can also distinguish the two communities of the entire club network through the result obtained in Theorem 2 which is described in details in Fig. 5(b).

VI. CONCLUSION

In this paper, a nonlinear opinion dynamics model with asynchronous evolution of trust/distrust level based on opinion difference has been proposed. The influence of opinion leaders on the formation of followers’ opinions in trust-distrust social networks has been studied. The properties of nonnegative matrix, as well as the construction of signed digraphs, have been used to derive comprehensive theoretical results for opinion dynamics. Our results complement the existing results in the literature regarding trust-distrust social networks. Moreover, the numerical simulations of opinions formation of the members in the “12 Angry Men” network and the Karate Club network have been provided to verify the correctness of our theoretical results.

ACKNOWLEDGMENT

The authors gratefully acknowledge the suggestions and comments by the associate editor and anonymous reviewers.

REFERENCES

[1] M. H. DeGroot, “Reaching a consensus,” J. Am. Stat. Assoc., vol. 69, no. 345, pp. 118–121, 1974.
[2] N. E. Friedkin and E. C. Johnsen, “Social influence networks and opinion change,” Advances in Group Processes, vol. 16, no. 1, pp. 1–29, 1999.
[3] J. Ghaderi and R. Srikant, “Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate,” Automatica, vol. 50, no. 12, pp. 3209–3215, 2014.
[4] P. Jia, A. Mirtabatabaei, N. E. Friedkin, and F. Bullo, “Opinion dynamics and the evolution of social power in influence networks,” SIAM Rev., vol. 57, no. 3, pp. 367–397, 2015.
[5] S. E. Parsegov, A. V. Proskurnikov, R. Tempo, and N. E. Friedkin, “Novel multidimensional models of opinion dynamics in social networks,” IEEE Trans. Autom. Control, vol. 62, no. 5, pp. 2270–2285, 2017.
[6] N. E. Friedkin, “The problem of social control and coordination of complex systems in sociology: A look at the community cleavage problem,” IEEE Contr. Syst. Mag., vol. 35, no. 3, pp. 40–51, 2015.
[7] R. Hegselmann and U. Krause, “Opinion dynamics and bounded confidence models, analysis, and simulation,” Journal of Artificial Societies and Social Simulation, vol. 5, no. 3, 2002.
[8] J. Lorenz, “Continuous opinion dynamics under bounded confidence: a survey,” Int. J. of Modern Phys. C, vol. 18, no. 12, pp. 1819–1838, 2007.
[9] W. Blondel, J. Hendrickx, and J. Tsitsiklis, “On Krause’s multilagent consensus model with state-dependent connectivity,” IEEE Trans. Autom. Control, vol. 54, no. 11, pp. 2586–2597, 2009.
[10] C. H. Roch, “The dual roots of opinion leadership,” The Journal of Politics, vol. 67, no. 1, pp. 110–131, 2005.
[11] W. W. Zachary, “An information flow model for conflict and fission in small groups,” J. Anthropol. Res., vol. 33, no. 4, pp. 452–473, 1977.
[12] E. Heberlig, M. Hetherington, and E. Larson, “The price of leadership: Campaign money and the polarization of congressional parties,” The Journal of Politics, vol. 68, no. 4, pp. 992–1005, 2006.
[13] W. W. Xu, Y. Sang, S. Blasiola, B. Stacy, and W. P. Han, “Predicting opinion leaders in Twitter activism networks: The case of the Wisconsin recall election,” Am. Behav. Sci., vol. 58, no. 10, pp. 1278–1293, 2014.
[14] S. Liu, C. Jiang, Z. Lin, Y. Ding, R. Duan, and Z. Xu, “Identifying effective influencers based on trust for electronic word-of-mouth marketing: A domain-aware approach,” Inf. Sci., vol. 306, pp. 34–52, 2015.
[15] F. Dietrich, S. Martin, and M. Jungers, “Control via leadership of opinion dynamics with state and time-dependent interactions,” IEEE Trans. Autom. Control, vol. 63, no. 4, pp. 1200–1207, 2017.
[16] A. Wongkaew, M. Caponigro, and A. Borzi, “On the control through leadership of the Hegselmann-Krause opinion formation model,” Math. Mod. Meth. Appl. S., vol. 25, no. 3, pp. 565–585, 2015.
[17] Y. Zhao, L. Zhang, M. Tang, and G. Kou, “Bounded confidence opinion dynamics with opinion leaders and environmental noises,” Computers & Operations Research, vol. 74, pp. 205–213, 2016.
[18] Y. Dyng, Z. Ding, L. Martín, and F. Herrera, “Managing consensus based on leadership in opinion dynamics,” Inf. Sci., vol. 397-398, pp. 187–205, 2017.
[19] D. Easley and J. Kleinberg, Networks, Crowds and Markets. Reasoning about a Highly Connected World. Cambridge: Cambridge Univ. Press, 2010.
[20] C. Alahfi, “Consensus problems on networks with antagonistic interactions,” IEEE Trans. Autom. Control, vol. 58, no. 4, pp. 935–946, 2013.
[21] W. Xia, M. Cao, and K. Johansson, “Structural balance and opinion separation in trust-mistrust social networks,” IEEE Trans. Control Netw. Syst., vol. 3, no. 1, pp. 46–56, 2016.
[22] A. Proskurnikov, A. Matveev, and M. Cao, “Opinion dynamics in social networks with hostile camps: Consensus vs. polarization,” IEEE Trans. Autom. Control, vol. 61, no. 6, pp. 1524–1536, 2016.
[23] D. G. Myers, “Social psychology (7th ed.),” Boston, MA: McGraw-Hill.
[24] Y. Chen, J. Lv, X. Yu, and Z. Lin, “Consensus of discrete-time second-order multiagent systems based on infinite products of general stochastic matrices,” SIAM J Control Optim., vol. 51, no. 4, pp. 3274–3301, 2013.
[25] F. Bullo, Lectures on Network Systems, CreateSpace, 1 edition, 2018. With contributions by J. Cortés, F. Dörfler, and S. Martínez. URL: http://motion.me.ucsb.edu/book-lns.
[26] Z. Kan, J. Klotz, E.L. Pasiliao, W. E. Dixon, “containment control for a directed social network with state-dependent connectivity,” Automatica, vol. 56, pp. 86–92, 2015.