MONODROMY WEIGHT FILTRATION IS INDEPENDENT OF $l$

Tomohide Terasoma

§1 Weight monodromy filtration for nilpotent monodromy

In this section, we recall several fundamental properties of monodromy weight filtration for nilpotent monodromy and prove some approximation theory. Let $K_0$ a local field of characteristic $p$ with a finite residue field $k = F_q$. The integer ring of $K_0$ is denoted by $R_0$. Let $F$ be an algebraic closure of $F_q$ and $R = R_0 \otimes_{F_q} F$. The fraction field of $R$ is denoted by $K$ and we use the notations

\[
\text{Spec}(R_0) = S_0, \text{Spec}(K_0) = \eta_0, \text{Spec}(k_0) = s_0,
\]
\[
\text{Spec}(R) = S, \text{Spec}(K) = \eta, \text{Spec}(k) = s.
\]

The algebraic closure of $K$ is denoted by $\bar{K}$ and $\text{Spec}(\bar{K})$ is denoted by $\bar{\eta}$. Let $X_0$ be a scheme over $S_0$ whose generic geometric fiber $X_\eta = X_0 \times_{S_0} \eta$ is smooth. Denote by $X$ the fiber product $X_0 \times_F F$. Then by a theorem of Grothendieck, the action of the Galois group $I = \text{Gal}(\bar{K}/K)$ on $H^i(X_\bar{\eta}, Q_l)$ is quasi unipotent. Let $J$ be an open compact subgroup of $I$ whose action on $H^i(X_\bar{\eta}, Q_l)$ is unipotent. The action of $J$ factors through the maximal tame quotient $J^t$ of $J$. Let $U$ a topological generator of $J^t \simeq \mathbb{Z}'(1)$ and $N$ be the logarithm of the action of $U$ on $H^i(X_\bar{\eta}, Q_l)$. In [D], Deligne introduced an increasing filtration $W^+$ with the following properties:

1. $N(W_k H^i(X_\bar{\eta}, Q_l)) \subset W_{k-2} H^i(X_\bar{\eta}, Q_l)$
2. The induced map $N : Gr^W_k (H^i(X_\bar{\eta}, Q_l)) \to Gr^W_{k-2} (H^i(X_\bar{\eta}, Q_l))$ induces an isomorphism;

\[
N^k : Gr^W_k (H^i(X_\bar{\eta}, Q_l)) \xrightarrow{\sim} Gr^W_{k-2} (H^i(X_\bar{\eta}, Q_l)).
\]

We define the primitive part $P_k (H^i(X_\bar{\eta}, Q_l))$ of $Gr^W_k (H^i(X_\bar{\eta}, Q_l)) (k \geq 0)$ by the kernel of $N^{k+1}$. Let $G$ be an open compact subgroup of $\text{Gal}(\bar{K}/K_0)$ such that $J = G \cap \text{Gal}(K_0/K)$ acts $H^i(X_\bar{\eta}, Q_l)$ nilpotently. Let $L_0$ be the corresponding extension of $K_0$ and $F_{q'}$ be the residue field. Then we have the exact sequence

\[
1 \to J \to G \to \text{Gal}(F/F_{q'}) \to 1.
\]

Since the action of $J$ on the the associated graded vector space $Gr^W_k (H^i(X_\bar{\eta}, Q_l))$ is trivial, $\text{Gal}(F/F_{q'})$ acts on this vector space.

Before studying the action of $\text{Gal}(F/F_{q'})$ on $Gr^W_k (H^i(X_\bar{\eta}, Q_l))$, we try to approximate the local situation by the global situation. Let $g_0 : Y_0 \to C_0$ be a projective generic geometrically smooth morphism of relative dimension $n$, where $C_0$ is a projective curve over $F_{q'}$. Let $p$ be an $F_{q'}$ valued point of $C_0$.

We denote by $O_p$ and $K_p$ by the completion of the structure sheaf of $C_0$ at the point $p$ and its quotient field respectively. Let $\bar{K}_p$ be the algebraic closure of $K_p$, and $\xi = \text{Spec}(\bar{K}_p)$. We denote by $Y_\xi$ the base extension of $Y_0$ to $\xi$.
Lemma 1.1. Let \( f_0 : \mathcal{X}_0 \rightarrow S_0 \) be as above and \( l_1, \ldots, l_s \) be finite set of primes different from \( p \). There exist a projective smooth curve \( C_0 \) over \( \mathbb{F}_q \), and a projective geometric generically smooth morphism \( Y_0 \rightarrow C_0 \) of relative dimension \( n \) with the following property:

1. There exists an isomorphism

\[
H^i(Y_\xi, \mathbb{Q}_{l_i}) \simeq H^i(X_\eta, \mathbb{Q}_{l_i})
\]

for all \( i = 1, \ldots, s \).

2. There exists the isomorphism

\[
\begin{array}{ccc}
\text{Gal}(\bar{K}_0/K_0)/P_{K_0,i} & \longrightarrow & \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) \\
\simeq & & \\
\text{Gal}(\bar{K}_p/K_p)/P_{K_p,i} & \longrightarrow & \text{Gal}(\bar{F}_q/\mathbb{F}_q),
\end{array}
\]

which is compatible with the isomorphism of (1.1), where \( P_{K_0,i} \) and \( P_{K_p,i} \) is the kernel of

\[
\text{Gal}(\bar{K}_0/K) \rightarrow \text{Aut}(H^i(X_\eta, \mathbb{Q}_{l_i}))
\]

and

\[
\text{Gal}(\bar{K}_p/K_p \otimes_{\mathbb{F}_q} \mathbb{F}) \rightarrow \text{Aut}(H^i(Y_\xi, \mathbb{Q}_{l_i}))
\]

respectively.

Proof. Let us write \( R_0 = \mathbb{F}_q[[t]] \). By fixing a projective model of \( \mathcal{X}_0 \) on \( S_0 \), we can take a finitely generated sub ring \( \mathbb{F}_q[t, x_1, \ldots, x_m] \) of \( \mathbb{F}_q[[t]] \) which contains all the coefficients of \( \mathcal{X}_0 \). The element corresponding to \( x_i \) is denoted by \( \xi_i(t) \in \mathbb{F}_q[[t]] \). Let \( M_0 = \text{Spec}(\mathbb{F}_q[t, x_1, \ldots, x_m]) \). By the definition of \( M_0 \) there exists a projective variety \( \mathcal{X}_{M_0} \rightarrow M_0 \) of relative dimension \( n \) such that the base change of \( \mathcal{X}_{M_0} \) by the map \( S_0 \rightarrow M_0 = \mathcal{X}_0 \). Let \( M = M_0 \otimes_{\mathbb{F}_q} \mathbb{F}, \mathcal{X}_M = \mathcal{X}_{M_0} \otimes_{\mathbb{F}_q} \mathbb{F}, \) and \( f_M : \mathcal{X}_M \rightarrow M \).

Since it is generically smooth, there exists an open set \( U_0 \) of \( M_0 \) such that \( R^i(f_M)_* \) is a smooth sheaf on \( U = U_0 \otimes_{\mathbb{F}_q} \mathbb{F} \). The complement of \( U_0 \) in \( M_0 \) is denoted by \( D_0 \). By taking a finite covering of \( M_0 \) and alterations [J], Theorem 7.3, we have a proper dominant morphism \( \phi : N_0 \rightarrow M_0 \) and an action of a group \( G \) on \( N_0 \) over \( M_0 \) with the following property:

1. Let \( Q(N_0) \) and \( Q(M_0) \) be the rational function field of \( N_0 \) and \( M_0 \). The field extension \( Q(N_0)^G \) of \( Q(M_0) \) is purely inseparable.
2. The variety \( N_0 \) is smooth and \( E_0 = \phi^{-1}(D_0) \) is a \( G \)-strict normal crossing divisor.
3. Let \( \mathbb{F}_{q'} \) be the constant field of \( N_0 \), \( N = N_0 \otimes_{\mathbb{F}_{q'}} \mathbb{F}, E = E_0 \otimes_{\mathbb{F}_{q'}} \mathbb{F}, \) \( f_N : \mathcal{X}_N \rightarrow N \) be the base change of \( f_M \) by the morphism \( N \rightarrow M \). Let \( \delta \) be a geometric point of \( N - E \). Then the image of the geometric monodromy action of \( \pi_1(N - E, \delta) \) on the geometric fiber of \( R^i(f_N)_* \mathbb{Q}_{l_i} \) at \( \delta \) is a pro-\( l_i \) group.

Since the field \( Q(N_0) \) is a finite extension of \( Q(M_0) \), by choosing an embedding of \( Q(N_0) \) into the algebraic closure of \( K_0 \), the composite field \( K_0 = K_0 \otimes \mathbb{Q}(N_0) \) is finite over \( \mathbb{Q}(N_0) \).
a finite extension of $K_0$. Let $T_0$ be the trait corresponding to the extension $L_0$. Then, by using valuative criterion of properness, we have the following diagram.

$$
\begin{array}{ccc}
T_0 & \longrightarrow & N_0 \\
\downarrow & & \downarrow \\
S_0 & \longrightarrow & M_0
\end{array}
$$

(1.2)

Let $t_0$ be the image of the closed point of $T_0$ in $N_0$. Let $G_0$ be the subgroup of $G$ which preserves the image of $T_0$. By blowing up $t_0$, we get the similar diagram as (1.2). Repeating the process, we may assume that the image of the trait $T_0$ intersects only one component $E_{0,i}$ of $E_0$ transversally. Let $m$ be the maximal ideal corresponding to $t_0$ and $\hat{N}_{0,m}$ be the completion of $N_0$ at $m$ and $\hat{N}_m = \hat{N}_{0,m} \times_{F_q F} F$. Since the image of $T_0$ meets $E_{0,i}$ transversally, the natural homomorphism

$$
\pi_1(T - \{t\})^{(l_i)} \cong \mathbb{Z}_{l_i}(1) \rightarrow \pi_1(\hat{N}_m - E_i)^{(l_i)} \cong \mathbb{Z}_{l_i}(1)
$$

is an isomorphism, where $T = T_0 \times_{F_q F} F$ and $E_i = E_{0,i} \times_{F_q F} F$. Here we denote by $\pi_1(T - \{t\})^{(l)}$ the maximal pro-$l$ quotient of the fundamental group of $T - \{t\}$. Now we take a curve $D_0$ in $N_0$ passing through $t_0$ meeting $E_{0,i}$ transversally and equivariant under the action of $G_0$. Let $Z_0 = (f_{N_0})^{-1}(C_0)$ and take a quotient $Y_0 = Z_0/G_0$ and $C_0 = D_0/G_0$ of $Z_0$ and $D_0$ under the action of $G_0$. Let $p_0$ be the image of $t_0$ under the quotient map. Then the induced morphisms $Y_0 \rightarrow C_0$ and $p_0$ satisfies the required properties.

As an application of this approximation lemma, we have the following Deligne’s theorem for varieties on local fields.

**Lemma 1.2 (see also [D]).** The action of $\text{Gal}(F/F_q)$ on $\text{Gr}^W_1(H^i(X_\eta, \mathbb{Q}_l))$ is of pure weight $i + k$.

**Definition.** The filtration $W$ introduced here is called the weight monodromy filtration of $H^i(X_\eta, \mathbb{Q}_l)$.

### §2 Independence of $l$ in the Global Situation

Let $C_0$ be a proper smooth curve over $F_q$, $f_0 : X_0 \rightarrow C_0$ be a projective flat relative $n$-dimensional morphism with generic geometrically smooth fiber. Let $X = X_0 \otimes_{F_q F} F$, $C = C_0 \otimes_{F_q F} F$, and $K_0$ and $K$ be the function field of $C_0$ and $C$ respectively. Let $s_0$ be a $F_q$-valued point of $C_0$ and $s$ be the spectrum of the algebraic closure of the residue field of $s_0$. For a rational function $g \in K$ and a character $\chi : \mu_d(F) \rightarrow \mathbb{Q}_l^\times$, the associate Kummer sheaf on $C$ is denoted by $\text{Kum}(g, \chi)$. The twist of $\chi$ by an element of Galois group $\sigma \in \text{Gal}(F_q(\mu_d)/F_q)$ is denoted by $\chi^\sigma$. Let $\mathcal{F}_0$ be an étale sheaf on $C_0$ with a finite geometric monodromy. The restriction of $\mathcal{F}_0$ to $C$ is denoted by $\mathcal{F}$.

**Lemma 2.1.** There exists an element $g \in K_0$, a positive integer $d$, and an injective character $\chi : \mu_d \rightarrow \mathbb{Q}_l^\times$ such that

1. The support $\text{Supp}(g)$ of the divisor $(g)$ in $C$ is non-empty and the finite set $\text{Supp}(f) \cup \{s_0\}$ contains all the points in $C$ where the morphism $f$ or the sheaf $\mathcal{F}$ are not smooth.
(2) \( f(s_0) = 1, \)
(3) the order of the class \( g \) in \( K^\times/K^d \) is \( d \), and
(4) for all \( x \in \text{Supp}(g) \) and \( \sigma \in \text{Gal}(F_{q}/F_q) \), the semi-simplification of the local monodromy group on \( R^if_*\mathbb{Q}_l \otimes \text{Kum}(g, \chi^\sigma) \otimes F \) at \( x \) has no fixed part.

**Proof.** Fix one closed point \( x_1 \) in \( C_0 \) different from \( s_0 \). By Riemann-Roch theorem, we can choose a rational function \( g \) on \( C_0 \)

1. whose order at \( x_1 \) is 1,
2. if \( f \) or \( F \) are not smooth at a point \( x \) different from \( s_0 \), then \( \text{ord}_x(g) \neq 0 \), and
3. regular at \( s_0 \) and \( g(s_0) = 1 \).

Choose a sufficiently big \( d \) and \( \chi \) such that \( \text{Kum}(g, \chi) \) has the property (4). Then we get the required \( g, d, \) and \( \chi \).

**Remark 2.2.** The condition (1) and (4) of Lemma 2.1 implies

\[
H^0_c(U, R^if_*\mathbb{Q}_l \otimes \text{Kum}(g, \chi^\sigma) \otimes F) = H^2_c(U, R^if_*\mathbb{Q}_l \otimes \text{Kum}(g, \chi^\sigma) \otimes F) = 0
\]

for all \( \sigma \in \text{Gal}(F_{q'/F_q}) \).

Now we introduce a covering \( \tilde{C}_0, \tilde{X}_0 \) of \( C_0 \) and \( X_0 \). Let \( F_{q'} = F_q(\mu_d) \) and \( C_1 = C_0 \otimes_{F_q} F_{q'} \). By Kummer theory, the \( d \)-root of \( g \) defines a finite cyclic covering \( \tilde{C}_0 \) of \( C_1 \). By the composite \( \tilde{C}_0 \to F_{q'} \to F_q \), \( \tilde{C}_0 \) is considered as a curve on \( F_q \). Note that it is not always geometrically connected. Let \( \pi : \tilde{C}_0 \to C_0 \) be the natural projection. Then we have

\[
G = \text{Aut}(\tilde{C}_0/C_0) \simeq \mu_d \rtimes \text{Gal}(F_{q'/F_q}).
\]

The induced representation \( \text{Ind}_{\mu_d}^G(\chi) \) is denoted by Ind and the group ring \( \mathbb{Q}_l[G] \) is denoted by \( A \). The Kummer sheaf \( K_0 \) on \( C_0 \) is defined by

\[
K_0 = \pi_*\mathbb{Q}_l \otimes_A \text{Ind}.
\]

Then it is easy to see that

\[
K_0 \mid_C \simeq \oplus_{\sigma \in \text{Gal}(F_{q'/F_q})} \text{Kum}(g, \chi^\sigma).
\]

Let \( \tilde{X}_0 = X_0 \times_{C_0} \tilde{C}_0 \) and \( \tilde{X} = \tilde{X}_0 \otimes_{F_q} F \). The projection \( \tilde{X}_0 \to C_0 \) and \( \tilde{X} \to C \) is denoted by \( \tilde{f}_0 \) and \( \tilde{f} \) respectively. Then the group \( G \) acts on \( \tilde{X}_0 \) and \( \tilde{X} \). Let \( W_0 = C_0 - \{s_0\}, U_0 = W_0 - \text{Supp}(g) \) and \( W = W_0 \otimes_{F_q} F, U = U_0 \otimes_{F_q} F \). The natural inclusions \( U \to W, W \to C \) and \( U \to C \) are denoted by \( j_1, j_2 \) and \( j_3 \) respectively.

**Proposition 2.3.** The action of the Frobenius \( \text{Frob}_{F_q} \) on

\[
(2.1) \quad H^1(C, (j_2)_*(j_1)!((R^if_*\mathbb{Q}_l \otimes_A \text{Ind}) \otimes F))
\]

is pure of weight \( i + 1 \).

**Proof.** Let \( \tilde{X}^0 = X_0 \times_{F_{q'}} F \) be the connected component of \( \tilde{X} \), and \( \tilde{f}^0 : \tilde{X}^0 \to C \) be the natural projection. Then the \( \text{Gal}(F/F_q) \)-module (2.1) is isomorphic to the induced representation of the \( \text{Gal}(F/F_q') \)-module

\[
(2.2) \quad H^1(C, (j_2)_*(j_1)!((R^if_*\mathbb{Q}_l \otimes_{\mathbb{Q}_l[\mu_d]} \mathbb{Q}_l(\chi)) \otimes F))
\]

\[
\otimes H^1(C, j_3^!(j_1,R^if_*\mathbb{Q}_l \otimes \text{Kum}(g, \chi) \otimes F))
\]

or
Proposition 2.4. The characteristic polynomial of $Q_{\tilde{R}}$ for zeta function of the sheaf $\zeta$ is independent of $\tau$. By Remark 2.2 and Lefschetz trace formula, we have the following equality

$$(j_1)! (R^i f^* \mathbb{Q}_l \otimes \text{Kum}(g, \chi) \otimes \mathcal{F}) \simeq (j_1)_* (R^i f_* \mathbb{Q}_l \otimes \text{Kum}(g, \chi) \otimes \mathcal{F}).$$

Therefore we have

$$(j_2)_* (j_1)! (R^i f^* \mathbb{Q}_l \otimes \text{Kum}(g, \chi) \otimes \mathcal{F}) \simeq (j_3)_* (R^i f_* \mathbb{Q}_l \otimes \text{Kum}(g, \chi) \otimes \mathcal{F}).$$

Since $f$ is projective smooth on $U$, $R^i f^* \mathbb{Q}_l \otimes \text{Kum}(g, \chi) \otimes \mathcal{F}$ is punctually pure of weight $i$. Therefore by the theorem of purity in [D], Theorem 3.2.3, we get the proposition.

Now we fix an identification $\mathbb{Q}_l \cong \mathbb{Q}_{l'}$ for primes $l$ and $l'$ different from $p$. Let $D_0 \to C_0$ be a Galois covering with the finite Galois group $G$ and $\tau$ a $\mathbb{Q}_l$-valued representation of $G$. Let $F_0(\tau)$ be the $l$-adic sheaf associated to the representation $\tau$. The restriction of $\mathcal{F}$ to $C$ is denoted by $\mathcal{F}$. We compare elements of $\mathbb{Q}_l$ and $\mathbb{Q}_{l'}$ via the isomorphism $\iota$.

**Proposition 2.4.** The characteristic polynomial of $\text{Frob}_{\mathbb{F}_q}$ on

$$H^1_c(U, R^i \tilde{f}_* \mathbb{Q}_l \otimes A \text{Ind} \otimes \mathcal{F}(\tau))$$

is independent of $l$.

**Proof.** By Remark 2.2 and Lefschetz trace formula, we have the following equality for zeta function of the sheaf $R^i (\tilde{f}_0)_* \mathbb{Q}_l \otimes A \text{Ind} \otimes F_0(\tau)$ on $U_0$.

$$\det(1 - t \text{Frob}_{\mathbb{F}_q} | H^1_c(U, R^i \tilde{f}_* \mathbb{Q}_l \otimes A \text{Ind} \otimes \mathcal{F}(\tau)))$$

$$= \prod_{x \in [U_0]} \det(1 - t \text{Frob}_x | (R^i \tilde{f}_* \mathbb{Q}_l \otimes A \text{Ind} \otimes \mathcal{F}(\tau))^{-1})$$

$$= \prod_{x \in [U_0]} \det(1 - t \text{Frob}_x | H^i((\tilde{f})^{-1}(\bar{x}), \mathbb{Q}_l) \otimes A \text{Ind} \otimes \mathcal{F}(\tau))^{-1}$$

Since $\tilde{f}$ is projective smooth at $x \in U$, the right hand side is independent of $l$ by the classical Weil conjecture and relation between Zeta function and the Frobenius action and the action of $G$ on the $\mathbb{F}$ rational points of $\tilde{f}^{-1}(x)$. Therefore we get the proposition.

§3 Proof of the main theorem

Let $K_0$ be a local field of characteristic $p > 0$ with a residue field $\mathbb{F}_q$. Let $X_\eta$ be a projective geometrically smooth variety of dimension $n$ over $K_0$. The variety $X_\eta$ is called globalizable if there exist projective varieties $C_0, X_0$ over $\mathbb{F}_q$ of dimension one and $n + 1$, a morphism $g : X_0 \to C_0$, a point $p \in C_0(\mathbb{F}_q)$ and an isomorphism of local field between $K_0$ and the quotient field $K_p$ of the completion of the structure sheaf of $C_0$ at $p$ such that the base change of $X_0$ by the morphism $\text{Spec}(K_0) \to \text{Spec}(K_p) \to C_0$ is isomorphic to $X_\eta$. Let $\tau$ be a representation of $\pi_1(C_0)$ with a finite image and $\mathbb{Q}_l(\tau)$ be the representation space of $\tau$. 

**Proposition 3.1.** The characteristic polynomial of the Frobenius action \( \text{Frob}_{\kappa(s_0)} \) on the space \( (H^i(X_\eta, Q_l) \otimes Q_l(\tau))^\text{Gal}(K_{\ell}/K_{\ell}) \) is independent of \( l \).

**Proof.** To show the independence of \( l \), it is enough to compare two primes \( l_1 \) and \( l_2 \). By using Lemma 1.1, we may assume that \( X_\eta \) is globalizable. We use the same notation for \( X_0, C_0, p_0 \) and so on. First we choose a rational function \( g \) on \( C_0 \), an integer \( d \), and a character \( \chi : \mu_d(F) \to Q_l^\times \) satisfying the properties in Lemma 2.1. We consider varieties \( \bar{C}_0, \bar{C}, \bar{X}_0, \bar{X} \) and so on as in §2. Now we consider the exact sequence of sheaves on \( C \):

\[
0 \to (j_3)_!(R^i\bar{f}_*Q_l \otimes A \text{Ind}) \otimes \mathcal{F}(\tau) \to (j_2)_!( (j_1)_!(R^i\bar{f}_*Q_l \otimes A \text{Ind}) \otimes \mathcal{F}(\tau)) \to i_s^*( (j_3)_!(R^i\bar{f}_*Q_l \otimes A \text{Ind} \otimes \mathcal{F}(\tau))) \to 0,
\]

where \( i_s : \{s\} \to C \) is the natural inclusion. Taking the cohomology, we have the long exact sequence:

\[
0 \to H^0(Spec(K), H^i(\bar{X}_\eta, Q_l) \otimes_A \text{Ind} \otimes \bar{Q}_l(\tau)) \to H^1_c(U, R^i\bar{f}_*Q_l \otimes_A \text{Ind} \otimes \mathcal{F}(\tau)) \to H^1(C, (j_2)_!(R^i\bar{f}_*Q_l \otimes A \text{Ind} \otimes \mathcal{F}(\tau))) \to 0.
\]

The weight of

\[
H^0(Spec(K), H^i(\bar{X}_\eta, Q_l) \otimes_A \text{Ind} \otimes \bar{Q}_l(\tau))
\]

is less than or equal to \( i \), and that of \( H^1(C, (j_2)_!(R^i\bar{f}_*Q_l \otimes A \text{Ind}) \otimes \mathcal{F}(\tau)) \) is purely \( i+1 \). Therefore the characteristic polynomial of the Frobenius action on (3.1) is independent of \( l \). By the property Lemma 2.1 (2) of the choice of \( g \), (3.1) is isomorphic to

\[
H^0(Spec(K), H^i(\bar{X}_\eta Q_l) \otimes Q_l(\tau)) \otimes r,
\]

where \( r = [F_q(\mu_d) : F_q] \), and we get the theorem.

By using Poincare duality and local duality, we have the following corollary.

**Corollary 3.2.** The characteristic polynomial of the Frobenius action on \( H^1(Spec(K), H^i(\bar{X}_\eta, Q_l) \otimes Q_l(\tau)) \) is independent of \( l \).

Now we recall some properties of the weight monodromy filtration \( \text{W} \). Let \( G_0 \) and \( G \) be the absolute Galois groups of \( K_0 \) and \( K \) respectively. There exists an open compact subgroup \( M^0 \) which acts on \( H^i(\bar{X}_\eta, Q_l) \) unipotently and the corresponding quotient of \( M^0 \) is denoted by \( M^0 \). It is isomorphic to \( Z_l(1) \). Then the image \( M \) of \( G \) in \( \text{Aut} H^i(\bar{X}_\eta, Q_l) \) contains \( M^0 \) and we may assume that \( M^0 \) is a normal subgroup of \( M \) by changing \( M^0 \) sufficiently small. Then the quotient group \( M/M^0 \) acts on \( Z_l(1) \) by the conjugation and the corresponding character is denoted by \( \beta \). Then the relation of \( N : G^W_{k} \to G^W_{k-2} \) and \( g \in M/M^0 \) is given by \( Ng = \beta(g)^{-1}gN \). Therefore, by the universal property of the filtration \( \text{W} \), the action of the group \( M/M^0 \) preserves the filtration \( \text{W} \) and as a consequence \( M/M^0 \) acts on the associate graded module \( G^W_{k} = G^W_{k}(H^i(\bar{X}_\eta, Q_l)) \). Again by the relation of \( N \) and \( g \), the group \( M/M^0 \) acts on the primitive part \( \text{P} \). The corresponding representation of
$P_k$ is denoted by $\alpha_k$. Then as representations of $M/M^0$, $Gr^W_k, H^i(X_{\tilde{\eta}}, Q_l)^{M^0}$, and the coinvariant $H^i(X_{\tilde{\eta}}, Q_l)_{M^0}$ under the action of $M^0$ are isomorphic to

$$Gr^W_k = \bigoplus_{m \geq 0} \alpha_{k+2m} \otimes \beta^m,$$

$$H^i(X_{\tilde{\eta}}, Q_l)^{M^0} = \bigoplus_{m \geq 0} \alpha_m \otimes \alpha^m,$$

$$H^i(X_{\tilde{\eta}}, Q_l)_{M^0} = \bigoplus_{m \geq 0} \beta_m.$$

It is easy to see that all the intersection of the kernel of $\alpha$ and $\alpha_k$ corresponds to the maximal nilpotent subgroup for the action of $M$ on $\text{Aut}(H^1(X_{\tilde{\eta}}, Q_l))$. Changing notation, this group is denoted by $M^0$. Let $\tilde{M}^0$ be the inverse image of $M^0$ under the natural map $G \to M$. Let $N_0$ be an open compact subgroup of $G_0$ where the intersection $N_0 \cap G$ is contained in $\tilde{M}_0$. The corresponding extension of $K_0$ is denoted by $L_0$. The residue field of $L_0$ is denoted by $F_{q'}$.

**Theorem 3.3.**

1. The representation $\alpha_k$ and $\beta$ of $G$ is independent of $l$. Especially, $\tilde{M}^0$ does not depend on $l$ and the dimension of $P_k$ is independent of $l$.

2. The characteristic polynomial of the Frobenius action $Frob_{F_{q'}}$ on $P_k$ is independent of $l$.

**Proof of (1).** Let $\tau$ be a finite dimensional irreducible representation with finite image $H$. Then there exists a Galois covering $\pi : D \to C$ and a lifting $q$ of $p$ such that the quotient field $K_{D,q}$ of the completion of the structure sheaf $\mathcal{O}_D$ of $D$ at $q$ corresponds to the quotient $H$ of $G$. Then the curve $C$, $D$ and all the morphism in $\text{Gal}(D/C)$ is defined over a finite extension $F_{q'}$ of $F_q$. The model of $C$ and $D$ defined on $F_{q'}$ is denoted by $C_1$ and $C_2$ respectively. Let $C_1'$ be the covering between $D \to C$ corresponding to the stabilizer $\text{Stab}_q(G)$ of $q$. Since $H \simeq \text{Stab}_q(G) \simeq \text{Gal}(D_1/C_1')$, we can consider a sheaf $\mathcal{F}'_1(\tau)$ on $C_1'$. We apply Proposition 3.1 to $\mathcal{F}_1(\tau)$ and $C_1'$. Then the characteristic polynomial of the Frobenius action on $(Q_l(\tau) \otimes H^i(X_{\tilde{\eta}}, Q_l))^G$ is independent of $l$. Let $\tilde{M}^0_l$ be the maximal unipotent subgroup for the representation $H^i(X_{\tilde{\eta}}, Q_l)$. Then there exists a finite normal subgroup $\tilde{M}^1_l$ of $\tilde{M}^0_l$ such that the restriction of $\tau$ to $\tilde{M}^1_l$ is trivial. Then the characteristic polynomial of Frobenius action on

$$(Q_l(\tau) \otimes H^i(X_{\tilde{\eta}}, Q_l))^G = ((Q_l(\tau) \otimes H^i(X_{\tilde{\eta}}, Q_l))(\tilde{M}^1_l)^{\tilde{M}^0_l}$$

is independent of $l$. The dimension of weight $k$-part of the above equality is nothing but the multiplicity of $\tau^*$ in $\alpha_k \otimes \beta^k$. Since $\tau$ is the arbitrary irreducible representation of $G$ with finite image, $\alpha_k \otimes \beta^k$ is independent of $l$. Similarly, the characteristic polynomial of Frobenius action on

$$H^1(\text{Spec}(K), H^i(X_{\tilde{\eta}}, Q_l) \otimes Q_l(\tau))$$

is independent of $l$. The dimension of weight $k$-part of the above equality is nothing but the multiplicity of $\tau^*$ in $\alpha_k \otimes \beta^k$. Since $\tau$ is the arbitrary irreducible representation of $G$ with finite image, $\alpha_k \otimes \beta^k$ is independent of $l$.
is independent of \( l \) and we have the \( l \)-independence of \( \alpha_k \) \((k \geq 0)\). This proves (1) of Theorem 3.3. Therefore by the characterization of \( \tilde{M}_l^0 \) gives as above, it is independent of \( l \).

Let \( D \) be a covering of \( C \) and \( q \in D \) a lifting of \( p \) such that the field \( K_{D,q} \) defined as above corresponds to the sub group \( M^0 \) of \( G \). Then \( D, C \) are defined over some finite extension \( F_{q'} \) of \( F_q \) and the model over \( F_{q'} \) are denoted by \( D_1 \) and \( C_1 \) respectively. We may assume \( q \in D_1(F_{q'}) \). We apply Proposition 3.1 to \( q \in D_1 \) and the trivial representation of \( \pi_1(D_1) \). Then the characteristic polynomial of Frobenius action on \( H^1(\bar{X}_{\eta}, \bar{Q}_l)\tilde{M}_l^0 \) is independent of \( l \). Considering weight of \( P_k \), the characteristic polynomial of Frobenius on \( P_k \) is independent of \( l \).

**Remark 3.4.**

(1) By the independence of \( l \) for \( P_k \) implies the independence for the Swan conductor.

(2) Since \( \beta \) is independent of \( l \), it is order 2 character by Chevotalev density theorem.

**References**

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