Perturbed WZW models and N=(2,2) supersymmetric sigma models with complex structure

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Abstract

We have perturbed N=(2,2) supersymmetric sigma models and WZW models on Lie groups by adding a term containing complex structure to their actions. Then, by using non-coordinate basis, we have shown that the conditions (from the algebraic point of view) for the preservation of the N=(2,2) supersymmetry impose that the complex structure must be invariant. Also, we have shown that the perturbed WZW model with this term is an integrable model.

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1 Introduction

Supersymmetric sigma models are of particular interest, for their intimate connection to complex geometry of target manifold [1, 2] and for their role in effective low-energy actions for supergravity scalars. The N=(2,2) extended supersymmetry in sigma model from the geometrical point of view is equivalent to the existence of bi-Hermitian structure on the target manifold where the complex structures are covariantly constant with respect to torsionful affine connections [1] (see also [3] and references therein). We know that the algebraic structures related to bi-Hermitian relations of the N=(2,2) supersymmetric WZW models are the Manin triples [4, 5]. Furthermore, recently the algebraic structure associated with the bi-Hermitian geometry of the N=(2,2) supersymmetric sigma models on Lie groups has been found in [6]. In [7], we have studied the perturbed N=(2,2) supersymmetric sigma models on Lie groups and have obtained conditions under which the N=(2,2) supersymmetry is preserved. Here in that direction, we perturb N=(2,2) supersymmetric sigma models on Lie groups and also WZW models by adding similar terms containing complex structure to their actions. We show that for the preservation of the N=(2,2) supersymmetry we must have the invariant complex structure. Also, we show that the perturbed WZW model with similar term is an integrable model. The paper is organized as follows.

In section 2 we first review the N=(2,2) supersymmetric sigma models, then by using the method mentioned in [7] we show that the perturbed N=(2,2) supersymmetric sigma model on Lie group has N=(2,2) supersymmetry when the tensor structure in the perturbed term is an invariant complex structure. In section 3, we present an example by use of the Heisenberg Lie group. Some concluding remarks are addressed in section 4.

2 N=(2,2) supersymmetric sigma model on Lie groups perturbed with complex structure

We know that the N=(2,2) supersymmetric sigma models [1] on the manifold M can be shown by the following N=1 supersymmetric sigma models action

\[ S = \int d^2 \sigma d^2 \theta d\Phi^\mu D_- D_+ \Phi^\mu (G_{\mu \nu}(\Phi) + B_{\mu \nu}(\Phi)), \]

where \( \Phi^\mu \) are N=1 superfields with bosonic parts as coordinates of the manifold M such that the bosonic parts of \( G_{\mu \nu} \) and \( B_{\mu \nu} \) are respectively the metric and the antisymmetric tensors on M. This action is manifestly invariant under supersymmetry transformations

\[ \delta^1(\epsilon) \Phi^\mu = i(\epsilon^+ Q_+ + \epsilon^- Q_-) \Phi^\mu, \]

furthermore it has additional non manifest supersymmetry of the form

\[ \delta^2(\epsilon) \Phi^\mu = \epsilon^+ D_+ \Phi^\mu J^\mu_{+ \nu}(\Phi) + \epsilon^- D_- \Phi^\mu J^\mu_{- \nu}(\Phi), \]

where in the above relations \( Q_\pm \) and \( D_\pm \) are supersymmetry generators and superderivatives respectively, and \( J^\mu_{\pm \nu} \in TM \otimes T^* M \) are complex structures. Invariance of the action [1] under the transformations [3] imposes the fact that \( J_\pm \) must be bi-Hermitian complex structure such that their covariant derivations with respect to torsionful affine connections \( \Omega^\pm_{\mu \nu} = \Gamma^\mu_{\nu \rho} \pm G^\mu_{\nu \rho} H_{\sigma \mu \rho} \)

\[ \Omega^\pm_{\mu \nu} = \frac{1}{2} R^A_{\mu \nu} R^B_{\alpha \beta} R C_{\alpha \beta \gamma} H_{\gamma \delta \epsilon}, \]

where \( G_{AB} \) is symmetric ad-invariant non-degenerate bilinear form (ad-invariant metric) and \( H_{ABC} \) is anti-symmetric tensor on Lie algebra g; furthermore \( L^A_{\mu} (R^A_{\mu \nu}) \) and \( L^A_{\mu} (R^A_{\mu \nu}) \) are left (right) invariant\footnote{Note that \( H \) is the torsion three form \( H_{\mu \nu \rho} = \frac{1}{2} (B_{\mu \nu, \sigma} + B_{\rho, \mu \nu} + B_{\sigma \mu \nu}). \)
\footnote{Note that the indices \( A, B, \ldots \) show the Lie algebra indices and Greek indices \( \mu, \nu, \ldots \) show the Lie group manifold indices.}.}

\[ J_{- \nu}(\Phi) = R^A_{\mu} J^A_{- \nu} R^B_{\mu}, \]

\[ J_{+ \nu}(\Phi) = R^A_{\mu} J^A_{+ \nu} R^B_{\mu}, \]

1 Note that \( H \) is the torsion three form \( H_{\mu \nu \rho} = \frac{1}{2} (B_{\mu \nu, \sigma} + B_{\rho, \mu \nu} + B_{\sigma \mu \nu}). \)
2 Note that the indices \( A, B, \ldots \) show the Lie algebra indices and Greek indices \( \mu, \nu, \ldots \) show the Lie group manifold indices.
and their inverses on the Lie group \(G\) respectively and \(J^A_B\) is an endomorphism of the Lie algebra \(g\): \(J : g \rightarrow g\). Then, the conditions of the N=(2,2) supersymmetric sigma model can be rewritten in the following algebraic form \[6\]

\[J^2 = -I,\]  
\[(7)\]

\[\chi_A + J^t \chi_A J^t + J^B_A \chi_B J^t - J^B_A J^t \chi_B = 0,\]  
\[(8)\]

\[J^t G J = G,\]  
\[(9)\]

\[H_A = J^t(H_B J^B_A) + J^t H_A J + (H_B J^B_A) J,\]  
\[(10)\]

\[J^t(H_A + \chi_A G) = (J^t(H_A + \chi_A G))^t,\]  
\[(11)\]

where \((\chi_A)_B^C = -f^C_{AB}\) is the adjoint representation such that \(f_{AB}^C\) is the structure constant of the Lie algebra \(g\) and we have the matrix form \((H_A)_{BC} = H_{ABC}\), furthermore we have

\[\chi_A G)^t = -\chi_A G.\]  
\[(12)\]

These relations show that N=(2,2) supersymmetric sigma models on the Lie groups from the geometric point of view correspond to the bi-Hermitian structures on the Lie groups \[1\] or equivalently the algebraic bi-Hermitian structures \((J,G,H)\) on the Lie algebras \[6\]. For N=2 supersymmetric WZW models on the Lie group \(G\) we have \(H_{ABC} = f_{ABC}\). In this case, relations \[8\] and \[9\] show that we have the Lie bialgebra structures on \(g\) \[5\]; and relation \[10\] reduces to \[8\], and \[11\] is automatically satisfied i.e. Lie bialgebra structure is a special case of algebraic bi-Hermitian structure \((J,G,H)\) with \(H_{ABC} = f_{ABC}\) \[6\].

In \[7\], we have considered the general cases such that the perturbed N=(2,2) supersymmetric sigma models on Lie groups preserve N=(2,2) supersymmetry. Here we assume that the action \[1\] (as sigma models on Lie groups or WZW model) has N=(2,2) supersymmetry, and as a special example is perturbed with the following term

\[S' = \int d^2 \sigma d^2 \theta D_\Phi \Phi^\nu D_\phi G^\mu_\nu \Phi^\mu_\nu G^\nu_\phi \Phi^\nu_\phi J^\lambda_\nu,\]  
\[(13)\]

where the tensor \(J^\lambda_\nu(\Phi)\) is the complex structure that appeared in the second supersymmetry transformations \[3\] of the action \[1\]. Now together with \[1\] we have

\[S'' = \int d^2 \sigma d^2 \theta D_\Phi \Phi^\nu D_\phi G^\mu_\nu \Phi^\mu_\nu (G''_{\mu\nu}(\Phi) + B''_{\mu\nu}(\Phi)),\]  
\[(14)\]

such that

\[G''_{\mu\nu} = G_{\mu\nu} + \frac{1}{2}(G_{\mu\lambda} J^\lambda_\nu + G_{\nu\lambda} J^\lambda_\mu),\]  
\[(15)\]

\[B''_{\mu\nu} = B_{\mu\nu} + \frac{1}{2} (G_{\mu\lambda} J^\lambda_\nu - G_{\nu\lambda} J^\lambda_\mu),\]  
\[(16)\]

with the inverse metric

\[G''_{\mu\nu} = a G_{\mu\nu} + b(G_{\mu\lambda} J^\nu_\lambda + G_{\nu\lambda} J^\mu_\lambda).\]  
\[(17)\]

Such that using the condition \(G'' G''^{-1} = 1\) we have

\[a \delta^{\mu}_\nu + \left(\frac{a}{2} + b\right)(J + A)^{\mu}_\nu + \frac{b}{2} (J + A)(J + A)^{\mu}_\nu = \delta^{\mu}_\nu\]  
\[(18)\]

where \(A = G^{-1} J^t G\). Therefore, we have different cases as follows:

\[\text{case 1)} \quad J + A = 0 \quad \text{or} \quad G^{-1} J^t G = -J,\]  
\[(19)\]
3 Perturbed WZW model with complex structure as an integrable model

which is satisfying for \( a = 1 \) and all values of \( b \),

\[
\text{case 2)} \quad J + A = I \quad \text{or} \quad G^{-1}J^tG = I - J, \tag{20}
\]

which is satisfying for \( a + b = \frac{2}{3} \),

\[
\text{case 3)} \quad J + A = -I \quad \text{or} \quad G^{-1}J^tG = -(I + J), \tag{21}
\]

which is satisfying for \( a - b = 2 \),

\[
\text{case 4)} \quad (J + A)^2 = I \quad \text{or} \quad (J + G^{-1}J^tG)^2 = I, \tag{22}
\]

which is satisfying for \( a = \frac{4}{3} \) and \( b = -\frac{2}{3} \),

\[
\text{case 5)} \quad (J + A)^2 = -I \quad \text{or} \quad (J + G^{-1}J^tG)^2 = -I, \tag{23}
\]

which is satisfying for \( a = \frac{4}{3} \) and \( b = -\frac{2}{3} \). In the above cases, \( I \) is identity matrix. Here, we choose the case 1) where \( a = 1 \) and when \( J^2 = -1 \) then the Hermitian condition \([9]\) is satisfying such that the coefficient of the \( b \) term in \( \text{(17)} \) is automatically zero. Namely, \( G'' = G \) such that \( S'' \) and \( S \) are different in \( B \)-field terms; i.e. \( S \) is perturbed with complex structure \([9]\). For this selection we have the \( g \) case (final case) in \([7]\) where the algebraic bi-Hermitian structure \( (J, G, H) \) is perturbed with \((0, 0, H')\). Now, for having \( S'' \) as a \( \mathbb{N}=(2,2) \) supersymmetric sigma model, we must have the following relation (from relation \( (34) \) of \([7]\))

\[
H'_a J = (H'_a J)^t, \tag{24}
\]

such that

\[
H'_{ABC} = \frac{1}{2} (J^D A f_{DCB} + J^D B f_{DAC} - J^D C f_{DAB}). \tag{25}
\]

Now by substitution \( \text{(25)} \) in \( \text{(21)} \) and using \( \text{(7)-(9)} \) and \( \text{(12)} \) we obtain.

\[
\forall X, Y \in g \quad J[X, Y] = [X, JY], \tag{26}
\]

where this relation with condition \( J^2 = -1 \) shows that \( J \) is an invariant complex structure \([8]\). In this way, the \( \mathbb{N}=(2,2) \) supersymmetry is preserved in the action \( \text{(14)} \) when the tensor \( J^x \) in \( \text{(13)} \) is an invariant complex structure \([8]\). Note that in \([8]\) it is shown that any Lie algebra with invariant complex structure compatible with invariant metric admits an \( \mathbb{N}=2 \) structure. Therefore, in this way, we can perturb the algebraic bi-Hermitian structure by the complex structure of that Lie algebra such that the bi-Hermitian structure is preserved.

3 Perturbed WZW model with complex structure as an integrable model

We know that the WZW model based on Lie group \( G \) takes the following standard form \([9]\)

\[
S_{WZW}(g) = \frac{k}{4\pi} \int_{\Sigma} d\xi^+ \wedge d\xi^- < g^{-1} \partial_+ g, \ g^{-1} \partial_- g >
\]

\[
+ \frac{k}{24\pi} \int_{B} \left< g^{-1} dg, \ [g^{-1}dg, \ g^{-1}dg] \right>, \tag{27}
\]

where the worldsheet \( \Sigma \) is boundary of 3-dimensional manifold \( B (\partial B = \Sigma) \), and \( g^{-1}dg \) with \( g \in G \) is the left invariant one-form on Lie group \( G \) so that

\[
g^{-1}dg = (g^{-1} \partial_\mu g)^A X_A \partial_\alpha x^\mu d\xi^\alpha = L_\mu^A X_A \partial_\alpha x^\mu d\xi^\alpha, \tag{28}
\]

with \( \{X_A\} \) as basis for the Lie algebra \( g \) of the Lie group \( G \) and \( x^\mu \) are Lie group parameters. The WZW action \( \text{(24)} \) then can be rewritten in the following form

\[
S_{WZW}(g) = \frac{k}{4\pi} \int_{\Sigma} d^2\xi \ L^A_{\mu \nu} \ G_{AB} L^B_{\nu \gamma} \partial_+ x^\mu \partial_- x^\nu + \frac{k}{24\pi} \int_{B} d^3\xi^{\alpha \beta \gamma} \ L^A_{\mu \nu} G_{AD} L^B_{\nu \gamma} f^D_{BC} L^C_{\alpha \beta} \partial_\alpha x^\mu \partial_\beta x^\nu \partial_\gamma x^\lambda, \tag{29}
\]
where $G_{AB} = \langle X_A, X_B \rangle$ is non-degenerate ad-invariant metric on $\mathfrak{g}$. Now, we assume that the action (29) (as sigma model on Lie group) is perturbed with the following term

$$S' = k' \int d\xi^+ d\xi^- (g^{-1} \partial_+ g)^A G_{AD} J^D B (g^{-1} \partial_- g)^B,$$

such that $J^D B$ is an endomorphism of $\mathfrak{g}$ i.e $J : \mathfrak{g} \to \mathfrak{g}$. Now, using the vielbein formalism (1) and (3) we have

$$S' = k' \int d\xi^+ d\xi^- L^A G_{AD} J^D B \nu \partial_+ x^\mu \partial_- x^\nu = k' \int d\xi^+ d\xi^- G_{\mu \lambda} J^\nu \partial_+ x^\mu \partial_- x^\nu.$$

Then the general action of the WZW model where perturbed with the term (30) can be rewritten as follows

$$S'' = S + S' = \int d\xi^+ d\xi^- (G_{AB} + B_{AB} + k' G_{AD} J^D B) \nu \partial_+ x^\mu \partial_- x^\nu,$$

such that $S''$ as sigma model have the following invertible metric and anti-symmetric tensor (by choosing $k = 4\pi$

$$G''_{\mu \nu} = L^A G_{AB} \nu^{\nu} B + \frac{k'}{2} (L^A G_{AD} J^D B \nu^{\nu} + L^A G_{AD} J^D B L_{\mu}^{\nu}),$$

$$B''_{\mu \nu} = L^A B_{AB} \nu^{\nu} B + \frac{k'}{2} (L^A G_{AD} J^D B \nu^{\nu} B - L^A G_{AD} J^D B L_{\mu}^{\nu}),$$

with the inverse metric

$$G''_{\mu \nu} = a L^A G^{AB} \nu^{\nu} B + b (L^A G^{AD} J^D B \nu^{\nu} + L^A G^{AD} J^D B L_{\mu}^{\nu}).$$

Similar to the previous section using $G'' G''^{-1} = 1$ and Hermitian condition $J^i G'' J = G''$ we will arrive at $a = 1$ such that the coefficient of the $b$ term must be zero in $G''_{\mu \nu}$. Now, we will prove that the model (32) is an integrable model. For this purpose, we use the formalism presented in [10]. In this direction, we should calculate $H_{\mu \nu}^\lambda$, $\Gamma_{\mu \nu}^\lambda$ and $\Omega_{\mu \nu}^\lambda$; so after some calculation we have

$$\Gamma_{\mu \nu}^\lambda = \frac{1}{2} (-f_{ABC} + 2(\partial_\rho L^A) L^B B \rho^C) L^\lambda A L^B \nu^C,$$

$$H''_{\mu \nu}^\lambda = \frac{1}{2} (H'_{ABC} + k' J^D B f_{A} C - k' J^D C f_{A} B) L^\lambda A L^B \nu^C,$$

$$\Omega_{\mu \nu}^\lambda = \Gamma_{\mu \nu}^\lambda - H''_{\mu \nu}^\lambda = \frac{1}{2} (-H'_{ABC} - f_{A} BC - k' J^D B f_{A} CD + k' J^D C f_{A} CB + k' J^D C f_{A} BD + 2(\partial_\rho L^A) L^B B \rho^C) L^\lambda A L^B \nu^C,$$

where in this calculation we use

$$H_{\mu \nu \lambda} = \frac{1}{2} (\partial_\rho B_{\nu} + \partial_\sigma B_{\lambda} + \partial_\mu B_{\nu \lambda}) = \frac{1}{2} L^A \nu^{\nu} \rho^C H_{ABC},$$

such that for the WZW model we have

$$H_{ABC} = f_{ABC}.$$

In this way, we must obtain a Lax pair as follows [10]

$$[\partial_+ + \alpha_\mu(x) \partial_+ x^\mu] \psi = 0,$$

$$[\partial_- + \beta_\mu(x) \partial_- x^\mu] \psi = 0,$$

where, compatibility condition (the zero curvature representation) of this linear system yields the equation of motion of the system (32), such that $\psi$ is arbitrary function of the fields $x^\mu$; and the matrices $\alpha_\mu(x)$ and $\beta_\mu(x)$ satisfy the following relations [10]

$$\beta_\mu - \alpha_\mu = \mu_\mu,$$

$$\partial_\rho \beta_\nu - \partial_\nu \alpha_\mu + [\alpha_\mu, \beta_\nu] = \Omega_{\mu \nu}^\lambda \mu_\lambda,$$
where equation 42 can then be rewritten as
\[ F_{\mu\nu} = -(\nabla_\mu \mu - \Omega^\lambda_{\mu\nu} \mu_\lambda), \]  
so that the field strength \( F_{\mu\nu} \) and covariant derivative are written as follow
\[ F_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu + [\alpha_\mu, \alpha_\nu], \quad \nabla_\mu X = \partial_\mu X + [\alpha_\mu, X]. \]  
Now for our model 52 we choose
\[ \alpha_\mu = c C^B A L_\mu B X_A, \quad \mu_\mu = d D^A B L_\mu B X_A, \]  
where \( c, d \) are constant and \( C^B A, D^A B \) are constant matrices. By assuming that \( G_{AB} \) and \( J^A B \) are independent of the coordinate of the Lie group \( G \); after some calculation we see that for satisfying relation 48 we must have the following relation
\[ C^A B = J^A B + \delta^A B, \quad D^A B = J^A B + \delta^A B, \]  
with
\[ c = \frac{k'}{k' + 2}, \quad d = \frac{2}{k' + 2}, \]  
such that \( J^A B \) must satisfy the (7)-(9), i.e. it must be an algebraic Hermitian complex structure. In this way, we show that the WZW model 29 which is perturbed with (30), is integrable when \( J \) is a Hermitian complex structure on the Lie algebra \( g \). Furthermore, the equations of motion for this integrable model can be rewritten as the following Lax pair
\[ [\partial_+ + \frac{k'}{k' + 2} (J^A B + \delta^A B) L_\mu B X_A \partial_+ x^\mu] \psi = 0, \]  
\[ [\partial_- + (J^A B + \delta^A B) L_\mu B X_A \partial_- x^\mu] \psi = 0. \]  
But the presence of a spectral parameter in the Lax pair is of crucial importance in extracting conserved quantities 11,12. For this purpose, we will prove that \( k' \) parameter in 52 plays the role of the spectral parameter. In order to show this, we must obtain a Lax pair as follows 11
\[ [\partial_+ + y \hat{\alpha}_\mu (x) \partial_+ x^\mu] \psi = 0, \]  
\[ [\partial_- + z(y) \hat{\beta}_\mu (x) \partial_- x^\mu] \psi = 0, \]  
where \( y \) is a multiplicative spectral parameter and \( z(y) \) is a function of \( y \). The matrices \( \hat{\alpha}_\mu, \hat{\beta}_\mu \) are independent of \( y \). The equivalence of the compatibility condition of the linear system and the equations of motion of the non-linear sigma model results as following equations
\[ A_0 (\lambda), \quad A_1 (\lambda) \]  
In the Lax formulation of classical integrability of a two dimensional field theory (in general) 11 and a two dimensional sigma model (as a special case) 13, the equations of motion for the system can be written as a consistency condition (the zero curvature condition) for the following linear system
\[ [\partial_0 + A_0 (\lambda)] \psi = 0, \]  
\[ [\partial_1 + A_1 (\lambda)] \psi = 0, \]  
where \( \partial_0 = \frac{\partial}{\partial \sigma} \), \( \partial_1 = \frac{\partial}{\partial \tau} \) and \( A_0, A_1 \) are matrices which depend on the fields \( x^\mu \) and the arbitrary spectral parameter \( \lambda \), such that the zero curvature condition is as follows
\[ \{\partial_0 A_1 - \partial_1 A_0 + [A_0, A_1]\} \psi = 0. \]  
For such systems the monodromy matrix has the following form
\[ T(\lambda, \tau) = P \exp(- \int_0^{2\pi} A_1 (\lambda, \sigma, \tau) d\sigma), \]  
where \( P \) represent the path-ordered exponential. Since, traces of powers of the monodromy matrix are independent of time (for the proof of this statement, the assumption of the periodicity condition \( A_0 (\lambda, 0, \tau) = A_0 (\lambda, 2\pi, \tau) \) is necessary 12). The conserved quantities can be obtained from the following relation 12
\[ H^m (\lambda) = Tr [T^m (\lambda, \tau)]. \]  
These quantities are independent of time \( \tau \) and in involution with respect to the Poisson brackets: \( \{H^m (\lambda_1), H^m (\lambda_2)\} = 0 \) (one can find their proofs in 11,12). In the above, we use the light cone coordinates \( \xi^\pm = \frac{\xi \pm \eta}{2} \) instead of \( (\tau, \sigma) \).
In the following forms

Using (52) and (61), the monodromy matrix for model (32) (by assuming that

Therefor, we have the monodromy matrix for WZW model as follows

where equation (56) in the terms of matrices \( \hat{\alpha}_\mu, \hat{\beta}_\nu \) is rewritten as follows

\[ z(\partial_\mu \hat{\beta}_\nu - \Omega^\lambda_{\mu\nu} \hat{\beta}_\lambda) - y(\partial_\nu \hat{\alpha}_\mu - \Omega^\lambda_{\mu\nu} \hat{\alpha}_\lambda) + yz[\hat{\alpha}_\mu, \hat{\beta}_\nu] = 0. \]  

Now, for our model (32) we choose

\[ \hat{\alpha}_\mu = (J^A_B + \delta^A_B)L^B \mu X_A, \quad \hat{\beta}_\nu = (J^A_B + \delta^A_B)L^B \nu X_A, \]  

Using (57), by assuming that \( G_{AB} \) and \( J^A_B \) are independent of the coordinate of the Lie group G, after some calculation we see that for satisfying relation (57) we must have the following relations

\[ y = \frac{k'}{k' + 2}, \quad z = 1. \]  

By substituting the above results in the equations of (64), we will obtain the previous Lax pair (18). Therefore, \( y = \frac{k'}{k' + 2} \) is a spectral parameter where by this parameter the model (32) is an integrable sigma model. In order to clarify this integrability, we will read the general form of conserved quantities for this model as follow. First, we express the equation of motion (45) in the terms of \( (\tau, \sigma) \) coordinates, then we will have Lax pair \( (A_0, A_1) \) in the following forms

\[ A_0(y) = (J^A_B + \delta^A_B)L^B \mu X_A(y + 1 \int \partial_0 x^\mu + \frac{y - 1}{2} \partial_1 x^\mu), \]

\[ A_1(y) = (J^A_B + \delta^A_B)L^B \nu X_A(y - 1 \int \partial_0 x^\mu + \frac{y + 1}{2} \partial_1 x^\mu). \]  

Using (62) and (61), the monodromy matrix for model (32) (by assuming that \( J^A_B \) is independent of the coordinates \( (\tau, \sigma) \)) has the following form

\[ T(y, \tau) = P \exp \left\{ -(J^A_B + \delta^A_B)X_A(y - 1 \int \partial_0 x^\mu) + \frac{y + 1}{2} \partial_1 x^\mu \right\} \]  

According to the above relation for monodromy matrix, conserved quantities mentioned in (53) are dependent on spectral parameter \( y \). Indeed, the monodromy matrix (62) of our model (32) completely coincides with those of WZW model\([10][16][17]\) with this difference that in our model \( \delta^A_B \) is replaced with \( J^A_B + \delta^A_B \). From expanding \( H^\omega(y) \) in \( y \) we obtain an infinite set of conserved quantities. This suggests that model (32) is an integrable field theory\([12]\).

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4In the light cone coordinates \( \xi^\pm = \pm \frac{1}{2} x^\mu \partial_\pm x^\mu \) the equations of motion for WZW model are written in the following Lax form\([10]\)

\[ \partial_+ + x L^A \partial_\pm X_A|\psi = 0, \]

\[ \partial_- + L^A \partial_\pm X_A|\psi = 0. \]  

and Lax pair \( (A_0, A_1) \) are in the following forms

\[ A_0(x) = L^A \partial_\pm X_A(x + 1 \partial_0 x^\mu + \frac{x - 1}{2} \partial_1 x^\mu), \]

\[ A_1(x) = L^A \partial_\pm X_A(x - 1 \partial_0 x^\mu + \frac{x + 1}{2} \partial_1 x^\mu). \]  

Therefor, we have the monodromy matrix for WZW model as follows

\[ T(x, \tau) = P \exp \left\{ -X_A \left[ \frac{y - 1}{2} \int \partial_0 x^\mu d\sigma + \frac{y + 1}{2} \int \partial_1 x^\mu d\sigma \right] \right\}, \]  

where \( x \) is a spectral parameter.
3.1 An example

In the following, we will give example for WZW model perturbed with complex structure on Heisenberg Lie group $A_{4,8}$. Heisenberg Lie algebra with basis $\{X_1,\ldots,X_4\}$ has the following set of non-trivial commutation relations $[14]$

$$[X_2, X_4] = X_2, \quad [X_3, X_4] = -X_3, \quad [X_2, X_3] = X_1.$$  \hfill (67)

Here, we obtain a non-degenerate ad-invariant metric by using the general solution of $[12]$ as follows

$$G_{AB} = \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & 0 & b \end{pmatrix}, \quad a \in \mathbb{R} - \{0\}, \quad b \in \mathbb{R}. \hfill (68)$$

In order to write (27) and (30) explicitly, we need $g^{-1}\partial_\alpha g$. To this end, we use the following parametrization of the Lie group $G$

$$g = e^{x_1 X_1 + \ldots + x_4 X_4}, \quad \text{where} \quad \{X_i\}_{i=1}^4 \text{are generators and parameters of the Lie group } G,$$

where $X_i$ and $x_i$ are generators and parameters of the Lie group $G$, respectively. Inserting our specific choice of the parametrization $[69]$ then $g^{-1}\partial_\alpha g$ takes the following form $[14]$

$$g^{-1}\partial_\alpha g = (\partial_\alpha x^1) X_1 + (\partial_\alpha x^2)(x^1 X_1 + e^{-x^2} X_2) + (\partial_\alpha x^3)(e^{-x^4} X_3) + (\partial_\alpha x^4) X_4, \quad \text{from which we can read the } L_\mu A_i \text{’s and the terms that are being integrated over in } (29) \text{ as follow $[9]$}$$

$$L_\mu^A \ G_{AB} \ L_\mu^B = [\partial_\mu x^3 \partial_\nu x^4 + \partial_\mu x^4 \partial_\nu x^3 - \partial_\mu x^2 \partial_\nu x^3 - \partial_\mu x^3 \partial_\nu x^2 + x^3 \partial_\mu x^2 \partial_\nu x - x^2 \partial_\mu x^4 \partial_\nu x], \quad \text{(71)}$$

$$e^{\alpha_\mu} L_\alpha \ L_\beta ^A \ G_{AD} \ L_\gamma ^B \ L_{\gamma} ^C = -2 e^{\beta_\mu} \partial_{\gamma}[x^3 \partial_\nu x^4 \partial_\beta x^2 - x^4 \partial_\nu x^3 \partial_\beta x^2 - x^2 \partial_\nu x^4 \partial_\beta x^3], \quad \text{(72)}$$

where in the above relations we choose $\beta = 0$, $\alpha = -1$ in $[68]$, and use the adjoint representation $(\gamma^B)_{jk} = -f_{jk}^l$, and also use the following relation

$$L_\alpha = g^{-1}\partial_\alpha g = (g^{-1}\partial_\alpha g)^A X_A = L_\mu^A X_A \partial_\alpha x^\mu. \quad \text{(73)}$$

On the other hand, using integration by parts, the action $[29]$ is reduced to

$$S_{WZW}(g) = \frac{k}{2\pi} \int d^2 \xi \ (\partial_\mu x^1 \partial_\nu x^2 - \partial_\mu x^2 \partial_\nu x^1 + x^3 \partial_\mu x^2 \partial_\nu x - x^2 \partial_\mu x^4 \partial_\nu x). \quad \text{(74)}$$

Now, for calculation of the perturbed term $[30]$ we first choose complex structure compatible with metric $[68]$ in which $\beta = 0$, $\alpha = -1$; from $[6]$ we have

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{(75)}$$

by this selection, the action $[30]$ takes the following form

$$S' = k' \int d^2 \xi \ (e^{-x^4} \partial_\mu x^1 \partial_\nu x^3 - e^{-x^4} \partial_\mu x^3 \partial_\nu x^1 + x^3 e^{-x^4} \partial_\mu x^2 \partial_\nu x - x^2 e^{-x^4} \partial_\mu x^4 \partial_\nu x). \quad \text{(76)}$$

Finally, perturbed WZW model $[32]$ by choosing $k = 4\pi$ takes the following integrable model

$$S'' = \int d^2 \xi \ [k' e^{-x^4} \partial_\mu x^1 \partial_\nu x^3 + 2 \partial_\mu x^1 \partial_\nu x^3 - (1 - k' x^3 e^{-x^4}) \partial_\mu x^2 \partial_\nu x^3$$

$$- k' e^{-x^4} \partial_\mu x^3 \partial_\nu x^1 - (1 + k' x^3 e^{-x^4}) \partial_\mu x^3 \partial_\nu x^2 + 2 x^3 \partial_\mu x^4 \partial_\nu x^2], \quad \text{(77)}$$

Note that we choose $\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in light cone coordinate.
such that the equations of motion can be rewritten in the following Lax form

\[
\left[ \partial_+ + \frac{\partial'}{k'+2} \right] \left( (\partial_+ x^1 + (x^3 - e^{x^4})\partial_+ x^2)X_1 + (\partial_+ x^1 + (x^3 + e^{x^4})\partial_+ x^2)X_2 \\
+ (e^{-x^4}\partial_+ x^3 - \partial_+ x^4)X_3 + (e^{-x^4}\partial_+ x^3 + \partial_+ x^4)X_4 \right] \psi = 0,
\]

(78)

\[
\left[ \partial_- + \left( (\partial_- x^1 + (x^3 - e^{x^4})\partial_- x^2)X_1 + (\partial_- x^1 + (x^3 + e^{x^4})\partial_- x^2)X_2 \\
+ (e^{-x^4}\partial_- x^3 - \partial_- x^4)X_3 + (e^{-x^4}\partial_- x^3 + \partial_- x^4)X_4 \right) \right] \psi = 0.
\]

(79)

4 Conclusion

We have proved that N=(2,2) supersymmetric sigma models on Lie groups when perturbed with complex structure can preserve N=(2,2) supersymmetry if their Lie algebras have invariant complex structure compatible with ad-invariant metric. Also, we have shown that the zero curvature representation and consistency of integrability condition for WZW models perturbed with this term are equivalent to the vanishing of the Nijenhuis tensor for the complex structure and existence of the compatible metric with this complex structure.

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