N=1 SUSY Conformal Block Recursive Relations

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Abstract
We present explicit recursive relations for the four-point superconformal block functions that are essentially particular contributions of the given conformal class to the four-point correlation function. The approach is based on the analytic properties of the superconformal blocks as functions of the conformal dimensions and the central charge of the superconformal algebra. The results are compared with the explicit analytic expressions obtained for special parameter values corresponding to the truncated operator product expansion. These recursive relations are an efficient tool for numerically studying the four-point correlation function in Super Conformal Field Theory in the framework of the bootstrap approach, similar to that in the case of the purely conformal symmetry.

1 Introduction
Recent progress in Liouville field theory \cite{1,2,3} and two-dimensional quantum Liouville gravity \cite{4,5,6} implies new applications in string theory. A very important step in this direction is the supersymmetric extension of the methods used in the bosonic case. This is especially interesting because there is not yet a supersymmetric generalization of the matrix model technique; the super-Liouville approach introduced by Polyakov \cite{7} therefore remains the only promising approach. One of the open problems here is to construct the complete set of explicit correlation functions in the minimal supergravity (minimal superstring theory). As in the bosonic conformal field theory, the main method here is based on solving the conformal bootstrap equations. The conformal block functions \cite{8} play an important role in this program. Unfortunately, closed analytic expressions for these functions could be found only for some special values of the conformal dimensions of the fields under consideration. We here present recursive relations for the four-point conformal block functions in the Neveu–Schwarz sector of the N=1 SUSY conformal field theory analogous to those found in \cite{9} in the bosonic case. Successive iterations of these relations converge for sufficiently small $x$ and allow efficiently calculating series expansions for the correlation functions.

The paper is arranged as follows. In Sec. 2, we very briefly recall some necessary facts about the N=1 SUSY conformal field theory (see \cite{10,11} for more details). In Sec. 3, we introduce the superconformal block functions and describe their analytic properties. In Sec. 4, we describe
the singularities of the superconformal blocks as functions of the field dimensions. In Sec. 5, we discuss the asymptotic behavior of the conformal blocks as functions of the central charge of the superconformal algebra $c$ and their singularities in $c$. We then introduce the recursive relations for the superconformal block functions. In Sec. 6, we discuss a special degenerate case resulting in the ordinary differential equations for the superconformal block functions under consideration and verify the recursive relations for the corresponding parameter choices. We present our conclusions in Sec. 7.

2 $\mathcal{N}=1$ superconformal field theory

The symmetry in the superconformal field theory is generated by the holomorphic and antiholomorphic components of the supercurrent $S$ and the stress tensor $T$. In terms of the Laurent components of $S$ and $T$, the algebra takes the conventional form of the Neveu–Schwarz–Ramond (NSR) algebra

$$[L_n; L_m] = (n - m)L_{n+m} + \frac{c}{8}(n^3 - n)\delta_{n,-m},$$

$$\{G_r; G_s\} = 2L_{r+s} + \frac{1}{2}c \left( r^2 - \frac{1}{4} \right) \delta_{r,-s},$$

$$[L_n; G_r] = \left( \frac{1}{2}n - r \right) G_{n+r},$$

where

$$r, s \in \mathbb{Z} + \frac{1}{2} \quad \text{for NS sector},$$

$$r, s \in \mathbb{Z} \quad \text{for R sector}.$$

In a Liouville-like manner, we write the central charge

$$c = 1 + 2Q^2, \quad \text{where } Q = b + \frac{1}{b}. \quad (2)$$

Local fields form the highest-weight representations of the NSR algebra. Each representation $[\Phi_\Delta]$ consists of a primary field with the conformal dimension $\Delta$ and all its superconformal descendants. The primary NS superfields\(^4\) are

$$\Phi_\Delta(z) = \Phi_\Delta(x) + \theta \Psi_\Delta(x) \quad \text{with } \Psi_\Delta(z) = G_{-\frac{1}{2}} \Phi_\Delta(z), \quad (3)$$

where $x$ and $\theta$ are the holomorphic coordinates of the (2+2)-dimensional superspace ($\theta$ is the anticommuting, “odd” coordinate). We also introduce the convenient parameterization

$$\Delta(\lambda) = \frac{Q^2}{8} - \frac{\lambda^2}{2}. \quad (4)$$

The field $\Phi_{mn}$ with $m$ and $n$ being either both even or both odd positive integers corresponds to the “degenerate” primary field in the NS sector with the conformal dimension $\Delta = \Delta(\lambda_{mn}),$

$$\lambda_{mn} = \frac{mb^{-1} + nb}{2}. \quad (5)$$

\(^4\)For simplicity, we consider only the holomorphic part.
The general form of the descendant operator in the conformal class \([\Phi_\Delta]\) is

\[L_{\vec{k}} |\Delta\rangle = L_{-k_1} \cdots L_{-k_n} G_{-r_1} \cdots G_{-r_m} \Phi_\Delta,\]  

(6)

where \(\vec{k}\) denotes \(\{k_i, r_j\}\), which is an ordered set of positive integers and half-integers correspondingly. The relation \(\sum_i k_i + \sum_j r_j = N\) fixes the particular level in the Verma module corresponding to the superconformal family \([\Phi_\Delta]\). As usual, the Ward identities restrict the possible coordinate dependence of the correlation functions. In particular, the two-point functions are completely determined:

\[\langle \Phi_1(z_1) \Phi_2(z_2) \rangle \sim \frac{\delta_{\Delta_1, \Delta_2}}{|z_{12}|^{\Delta_1}},\]  

(7)

where \(z_{ik} = x_i - x_k + \theta_i \theta_k\). The three-point function depends on an arbitrary function of the “odd” superprojective invariant of three points,

\[\theta_{123} = \frac{z_{23} \theta_1 + z_{31} \theta_2 + z_{12} \theta_3 - \theta_1 \theta_2 \theta_3}{(z_{12} z_{13} z_{23})^{1/2}},\]  

(8)

and is presented as

\[\langle \Phi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle = \frac{C_{123} + |\theta_{123}|^2 \tilde{C}_{123}}{|z_{12}|^2 |\Delta_1 + 2 \Delta_2 - \Delta_3| |z_{13}|^2 |\Delta_1 + \Delta_3 - \Delta_2| |z_{23}|^2 |\Delta_2 + \Delta_3 - \Delta_1|}.\]  

(9)

The “degenerate” primary field \(\Phi_{mn}\) has a singular vector at the level \(N = mn/2\) [12]. It is convenient to introduce a “singular-vector creation operator” \(D_{m,n}\) such that the singular vector appears when \(D_{m,n}\) is applied to \(\Phi_{mn}\). We fix the normalization by taking the coefficient of the leading term to be unity,

\[D_{mn} = G_{mn}^* - \frac{1}{2} + \ldots\]  

The first nontrivial null vector in the NS sector occurs on the level \(N = \frac{3}{2}\):

\[D_{13} \Phi_{13} = \left( L_{-1} G_{-\frac{3}{2}} + b^2 G_{-\frac{3}{2}} \right) \Phi_{13} = 0.\]  

(10)

3 Four-point correlation function and conformal blocks

We consider the four-point correlation function of the primary superfields in the NS sector of the \(N=1\) SUSY conformal theory. For the four-point correlation function, there are three independent superprojective invariants, one “even” and two “odd” (see, e.g., [11]). Using the superconformal invariance (similarly to the consideration in [5]), we can write

\[\langle \Phi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \Phi_4(z_4) \rangle = |z_{34}|^{2 \gamma_{34}} |z_{13}|^{2 \gamma_{13}} |z_{23}|^{2 \gamma_{23}} |z_{12}|^{2 \gamma_{12}} g(z, \bar{z}, \tau_1, \bar{\tau}_1, \tau_2, \bar{\tau}_2),\]  

(11)

where

\[\gamma_{34} = -2 \Delta_4,\]

\[\gamma_{13} = -\Delta_1 - \Delta_4 + \Delta_4 + \Delta_2,\]

\[\gamma_{23} = -\Delta_2 - \Delta_3 + \Delta_4 + \Delta_1,\]

\[\gamma_{12} = -\Delta_4 - \Delta_1 - \Delta_2 + \Delta_3,\]  

(12)
and we choose three independent invariants:

\[ z = \frac{z_{41} z_{23}}{z_{43} z_{21}}, \]
\[ \tau_1 = \theta_{213}, \]
\[ \tau_2 = -[z(z - 1)]^{1/2} \theta_{214}. \] (13)

Taking the superprojective invariance into account, we can, for example, fix

\[ \theta_1 = 0, \quad \theta_2 = 0, \quad \theta_3 = R \eta, \]
\[ x_1 = 0, \quad x_2 = 1, \quad x_3 = R, \quad x_4 = x, \] (14)

where \( R \to \infty \). The function \( g \) is related to the correlation function of the boson components of supermultiplet (3):

\[ \langle \Phi_1(0) \Phi_2(1) \Phi_3(\infty) \Phi_4(z) \rangle = g(z, \bar{z}, 0, 0, 0, 0) = g_0(z, \bar{z}). \] (15)

In what follows, we restrict ourself to considering superconformal blocks contributing to the correlation function \( g_0 \). Generalizing to the other components is straightforward.

Similarly to the case of the purely conformal symmetry [8], we can write the \( s \)-channel expansion for the correlation function \( g_0 \):

\[ \langle \Phi_1(x) \Phi_2(0) \Phi_3(1) \Phi_4(\infty) \rangle = \sum_{\Delta, \Delta_i} \left[ C_{12}^{\Delta} C_{34}^{\Delta} F_0(\Delta, \Delta_i, x) F_0(\Delta, \Delta_i, \bar{x}) \right. \\
\left. + \tilde{C}_{12}^{\Delta} \tilde{C}_{34}^{\Delta} F_1(\Delta, \Delta_i, x) F_1(\Delta, \Delta_i, \bar{x}) \right]. \] (16)

The superconformal blocks \( F_0 \) and \( F_1 \) are defined similarly to the bosonic case (see, e.g., [14] for details):

\[ F_0(\Delta, \Delta_i, c, x) = x^{\Delta - \Delta_1 - \Delta_2} \sum_{N \geq 0} x^N N_{12} \langle N \mid N \rangle_{34}, \] (17)

\[ F_1(\Delta, \Delta_i, c, x) = x^{\Delta - \Delta_1 - \Delta_2} \sum_{N > 0} x^N N_{12} \langle N \mid N \rangle_{34}, \] (18)

where the vector \( |N\rangle \) is the \( N \)th-level descendent contribution of the intermediate state with the conformal dimension \( \Delta \) appearing in the operator product expansion (OPE) \( \Phi(x) \Phi(0) \):

\[ [\Phi_1(x) \Phi_2(0)]_\Delta = x^{\Delta - \Delta_1 - \Delta_2} \sum_{N=0}^{\infty} x^N |N\rangle_{12}, \] (19)

where \( |N\rangle_{12} = Q_{12}(N, \Delta)|\Delta\rangle \), \( Q_{12}(N, \Delta) = \sum \beta_{12}^{(k)} (\bar{k}) L_k^{(N)} \), and \( L_k^\Delta \) is defined in [10] (we assume summation over all \( N \)th-level descendents). The vectors \( |N\rangle_{12} \) depend not only on the conformal dimension \( \Delta \) and central charge \( c \) but also on the dimensions of the fields \( \Phi_1 \) and \( \Phi_2 \), and the operator \( Q \) is hence supplied with the appropriate subscript. The vectors \( |\bar{N}\rangle_{12} = \bar{Q}(N, \Delta)|\Delta\rangle \) arising in the OPE \( \Psi_1(x) \Phi_2(0) \),

\[ [\Psi_1(x) \Phi_2(0)]_\Delta = x^{\Delta - \Delta_1 - \Delta_2 - \frac{1}{2}} \sum_{N=0}^{\infty} x^N |\bar{N}\rangle_{12}, \] (20)
are completely determined by the superconformal symmetry. Namely, the superconformal constraints lead to relations for the vectors of the chain that grows from the vacuum vector $|\Delta\rangle$,

\[
\begin{align*}
G_k |N\rangle_{12} &= |N - k\rangle_{12}, \\
G_k |N\rangle_{12} &= [\Delta + 2k\Delta_1 - \Delta_2 + N - k]|N - k\rangle_{12},
\end{align*}
\] (21)

for $k > 0$. From definition (17), (18) and from the properties of Eqs. (21), we can deduce that starting from the level $mn/2$, the functions $F_0$ and $F_1$ have simple poles for $\Delta = \Delta_{mn}(c)$. Similarly, this relation shows that as functions of the central charge $c$, the conformal blocks have one simple pole for each pair of positive integers $m$ and $n$ ($n > 1$) at $c = c_{mn}(\Delta)$, where

\[
c_{mn} = 5 + 2(T_{mn} + T_{mn}^{-1}),
\]

\[
T_{mn} = \frac{1 - 4\Delta - mn + \sqrt{[(mn - 1) + 4\Delta]^2 - (m^2 - 1)(n^2 - 1)}}{n^2 - 1}.
\]

The residues of the functions $F_0$ and $F_1$ at these poles are proportional to the conformal block functions corresponding to the invariant subclass that appears on the $mn/2$ level with the highest vector being a new primary field with the conformal dimension $\Delta_{mn} = \Delta + mn/2$ (see an analogous consideration in [9] and also [14]). The singular part is briefly discussed in the next section.

4 Singular structure of the chain vectors

We will consider the singular structure of the chain vectors in more detail in a subsequent publication. Here, we present the main results concerning the singularities of the chain vectors introduced in (21). For $\Delta \to \Delta_{mn}$,

\[
|N = \frac{mn}{2}\rangle \rightarrow \frac{X_{mn}}{\Delta - \Delta_{mn}}D_{mn}|\Delta\rangle,
\]

\[
|N = \frac{mn}{2}\rangle \rightarrow \frac{\bar{X}_{mn}}{\Delta - \Delta_{mn}}D_{mn}|\Delta\rangle.
\]

To recover the dependence of the coefficient functions $X_{mn}$ and $\bar{X}_{mn}$ on the external dimensions, we observe that for $\Delta = \Delta_{mn}(c)$, the chain vectors should still be well defined if certain “fusion” relations [8] between $\Delta$ and $\Delta_i$ are satisfied. Investigating the “fusion” rules based on analyzing the structure functions [10] leads to the expressions

\[
X_{mn} = \begin{cases} 
2 - \frac{mn}{2} \frac{P_{mn}(\lambda_1 + \lambda_2)P_{mn}(\lambda_1 - \lambda_2)}{\rho_{mn}'}, & m, n \text{ even,} \\
2 - \frac{mn}{2} \frac{P_{mn}(\lambda_1 + \lambda_2)P_{mn}(\lambda_1 - \lambda_2)}{\rho_{mn}'}, & m, n \text{ odd}
\end{cases}
\]

\[
\bar{X}_{mn} = \begin{cases} 
2 - \frac{mn}{2} \frac{\bar{P}_{mn}(\lambda_1 + \lambda_2)\bar{P}_{mn}(\lambda_1 - \lambda_2)}{\rho_{mn}'}, & m, n \text{ even,} \\
2 - \frac{mn}{2} \frac{\bar{P}_{mn}(\lambda_1 + \lambda_2)\bar{P}_{mn}(\lambda_1 - \lambda_2)}{\rho_{mn}'}, & m, n \text{ odd.}
\end{cases}
\]

5
Here,
\[ P_{mn} = \prod_{(r,s) \in [m,n]} (\lambda - \lambda_{rs}), \tag{28} \]
where
\[ [m,n] = \{1 - m : 4 : m - 3, 1 - n : 4 : n - 3\} \cup \{3 - m : 4 : m - 1, 3 - n : 4 : n - 1\} \tag{29} \]
for \(m\) and \(n\) both even and
\[ [m,n] = \{3 - m : 4 : m - 3, 1 - n : 4 : n - 1\} \cup \{1 - m : 4 : m - 1, 3 - n : 4 : n - 3\} \tag{30} \]
for \(m\) and \(n\) both odd;
\[ \hat{P}_{mn} = \prod_{(r,s) \in [m,n]} (\lambda - \lambda_{rs}), \tag{31} \]
where
\[ [m,n] = \{1 - m : 4 : m - 3, 3 - n : 4 : n - 1\} \cup \{3 - m : 4 : m - 1, 1 - n : 4 : n - 3\} \tag{32} \]
for \(m\) and \(n\) both even and
\[ [m,n] = \{3 - m : 4 : m - 3, 3 - n : 4 : n - 3\} \cup \{1 - m : 4 : m - 1, 1 - n : 4 : n - 1\} \tag{33} \]
for \(m\) and \(n\) both odd. A simple analysis of (21) shows that the factor \(r_{mn}'\) depends only on \(c\) and \(\Delta\). It will be clarified in the subsequent publication that this coefficient is related to the norm of the corresponding singular vector,
\[ \|D_{mn}(\Delta)\|^2 = r_{mn}'(\Delta - \Delta_{mn}), \tag{34} \]
as \(\Delta \to \Delta_{mn}\). Taking the results in [13] into account, we write
\[ r_{mn}' = \frac{r_{mn}}{\lambda_{mn}}, \tag{35} \]
where
\[ r_{mn} = 2^{1-mn} \prod_{(k,l) \in [m,n]} (kb^{-1} + lb) \tag{36} \]
and
\[ [m,n] = \{1 - m : 2 : m - 1, 1 - n : 2 : n - 1\} \cup \{2 - m : 2 : m, 2 - n : 2 : n\} \setminus (0,0). \tag{37} \]
The expressions of the form \(a : d : b\) (“from \(a\) to \(b\) step \(d\)”) in the above formulas denote sets of numbers \(a, a + d, a + 2d, \ldots, b\). The symbol \(\{A, B\}\) denotes the set of pairs \((k,l)\) with \(k\) and \(l\) independently ranging the sets \(A\) and \(B\), and \(\{A_1, B_1\} \cup \{A_2, M_2\}\) is the standard union of two sets. Finally, \(\ldots \setminus (0,0)\) means that the pair \((0,0)\) is excluded.

In the same way, the chain vectors for \(N > mn/2\) also have simple poles at \(\Delta = \Delta_{mn}\), and hence
\[
|N \geq \frac{mn}{2} | \rightarrow \begin{cases} 
\frac{X_{mn}}{\Delta - \Delta_{mn}} Q \left(N - \frac{mn}{2}, \Delta_{mn} + \frac{mn}{2}\right) D_{mn}(\Delta), & N - \frac{mn}{2} \text{ integer}, \\
\frac{X_{mn}}{\Delta - \Delta_{mn}} Q \left(N - \frac{mn}{2}, \Delta_{mn} + \frac{mn}{2}\right) D_{mn}(\Delta), & N - \frac{mn}{2} \text{ half-integer}, 
\end{cases} \tag{38}
\]
\[
|N \geq \frac{mn}{2} | \rightarrow \begin{cases} 
\frac{X_{mn}}{\Delta - \Delta_{mn}} \hat{Q} \left(N - \frac{mn}{2}, \Delta_{mn} + \frac{mn}{2}\right) D_{mn}(\Delta), & N - \frac{mn}{2} \text{ integer}, \\
\frac{X_{mn}}{\Delta - \Delta_{mn}} \hat{Q} \left(N - \frac{mn}{2}, \Delta_{mn} + \frac{mn}{2}\right) D_{mn}(\Delta), & N - \frac{mn}{2} \text{ half-integer}. 
\end{cases} \tag{39}
\]
5 Recursive relations

The consideration in the preceding section leads to relations for the conformal blocks as functions of the internal conformal dimension $\Delta$:

\[
F_0(\Delta \to \Delta_{mn}, \Delta_i, c, x) = \begin{cases} 
\frac{R_{mn}}{\Delta - \Delta_{mn}} F_0(\Delta_{m,-n}, \Delta_i, c, x), & m, n \text{ even}, \\
\frac{\bar{R}_{mn}}{\Delta - \Delta_{mn}} F_1(\Delta_{m,-n}, \Delta_i, c, x), & m, n \text{ odd}, 
\end{cases} 
\]

(40)

\[
F_1(\Delta \to \Delta_{mn}, \Delta_i, c, x) = \begin{cases} 
\frac{\bar{R}_{mn}}{\Delta - \Delta_{mn}} F_0(\Delta_{m,-n}, \Delta_i, c, x), & m, n \text{ even}, \\
\frac{R_{mn}}{\Delta - \Delta_{mn}} F_1(\Delta_{m,-n}, \Delta_i, c, x), & m, n \text{ odd}, 
\end{cases} 
\]

where

\[
R_{mn} = X_{mn}^{(12)} X_{mn}^{(34)} r_{mn} \quad \text{and} \quad \bar{R}_{mn} = \bar{X}_{mn}^{(12)} \bar{X}_{mn}^{(34)} r_{mn}. 
\]

(41)

Hence, the residues at the poles of the conformal blocks as functions of the central charge $c$ are also completely determined:

\[
F_0(\Delta, \Delta_i, c \to c_{mn}, x) = \begin{cases} 
\frac{R_{mn}'}{c - c_{mn}} F_0\left(\Delta + \frac{mn}{2}, \Delta_i, c_{mn}, x\right), & m, n \text{ even}, \\
\frac{\bar{R}_{mn}'}{c - c_{mn}} F_1\left(\Delta + \frac{mn}{2}, \Delta_i, c_{mn}, x\right), & m, n \text{ odd}, 
\end{cases} 
\]

(42)

\[
F_1(\Delta, \Delta_i, c \to c_{mn}, x) = \begin{cases} 
\frac{\bar{R}_{mn}'}{c - c_{mn}} F_0\left(\Delta + \frac{mn}{2}, \Delta_i, c_{mn}, x\right), & m, n \text{ even}, \\
\frac{R_{mn}'}{c - c_{mn}} F_1\left(\Delta + \frac{mn}{2}, \Delta_i, c_{mn}, x\right), & m, n \text{ odd}, 
\end{cases} 
\]

where $c_{mn}$ is defined in (22) and (23) and the coefficients $R_{mn}$ and $R_{mn}'$ are essentially the same and differ only in the change of variable

\[
R_{mn}' = R_{mn}(c_{mn}) \cdot \left(\frac{\partial \Delta_{mn}}{\partial c}\right)^{-1}, \quad \frac{\partial \Delta_{mn}}{\partial c} = \frac{1}{16} \frac{(1 - n^2)T_{mn} - (1 - m^2)T_{mn}^{-1}}{T_{mn} - T_{mn}^{-1}}. 
\]

(43)

(44)

For $c = \infty$, relations (21) are simplified and can be solved explicitly. This leads to expressions for the asymptotic values of $F_0$ and $F_1$ in terms of hypergeometric functions:

\[
F_0(c \to \infty) = f_0(\Delta, \Delta_i, x) = x^{\Delta - \Delta_1 - \Delta_2 - 1/2} F_1(\Delta + \Delta_1 - \Delta_2, \Delta + \Delta_3 - \Delta_4, 2\Delta, x), \\
F_1(c \to \infty) = f_1(\Delta, \Delta_i, x) = \frac{1}{2\Delta} x^{\Delta - \Delta_1 - \Delta_2 + 1/2} F_1(\Delta + \Delta_1 - \Delta_2 + 1/2, \Delta + \Delta_3 - \Delta_4 + 1/2, 2\Delta + 1, x). 
\]

(45)
It is therefore clear that we can write the following relations for the conformal blocks $F_0$ and $F_1$:

\[
F_0(\Delta, \Delta_i, c, x) = f_0(\Delta, \Delta_i, x) + \sum_{\{m,n\} \text{ even}} R'_{mn} \frac{c}{c - c_{mn}} F_0(\Delta_{m,-n}, \Delta_i, c_{mn}, x) + \sum_{\{m,n\} \text{ odd}, n > 1} \bar{R}'_{mn} \frac{c}{c - c_{mn}} F_1(\Delta_{m,-n}, \Delta_i, c_{mn}, x),
\]

\[
F_1(\Delta, \Delta_i, c, x) = f_1(\Delta, \Delta_i, x) + \sum_{\{m,n\} \text{ even}} R'_{mn} \frac{c}{c - c_{mn}} F_1(\Delta_{m,-n}, \Delta_i, c_{mn}, x) + \sum_{\{m,n\} \text{ odd}, n > 1} \bar{R}'_{mn} \frac{c}{c - c_{mn}} F_0(\Delta_{m,-n}, \Delta_i, c_{mn}, x).
\]

(46)

We can expand $F_0$ and $F_1$ in $x$ by iterating Eqs. (46) and (47):

\[
F_0(\Delta, \Delta_i, c, x) = x^{\Delta - \Delta_1 - \Delta_2} \sum_{k=0}^{\infty} F_0^{(k)} x^k,
\]

\[
F_1(\Delta, \Delta_i, c, x) = x^{\Delta - \Delta_1 - \Delta_2 + 1/2} \sum_{k=0}^{\infty} F_1^{(k)} x^k.
\]

(48)

For brevity, the dependence on the external dimensions, which is always the same, is omitted below. Using recursive relations (46) and (47), we easily find the first few terms of the series expansions:

\[
F_0^{(0)} = f_0^{(0)}(\Delta),
\]

\[
F_0^{(1)} = f_0^{(1)}(\Delta),
\]

\[
F_0^{(2)} = f_0^{(2)}(\Delta) + \frac{R'_{13}}{c - c_{13}} f_1^{(0)} \left(\Delta + \frac{3}{2}\right) + \frac{R'_{22}}{c - c_{22}} f_0^{(0)}(\Delta + 2),
\]

\[
F_0^{(0)} = f_1^{(0)}(\Delta),
\]

\[
F_0^{(1)} = f_1^{(1)}(\Delta) + \frac{R'_{13}}{c - c_{13}} f_0^{(0)} \left(\Delta + \frac{3}{2}\right),
\]

\[
F_0^{(2)} = f_1^{(2)}(\Delta) + \frac{R'_{13}}{c - c_{13}} f_1^{(0)} \left(\Delta + \frac{3}{2}\right) + \frac{R'_{22}}{c - c_{22}} f_1^{(0)}(\Delta + 2) + \frac{R'_{15}}{c - c_{15}} f_0^{(0)} \left(\Delta + \frac{5}{2}\right).
\]

(49)

Using Eqs. (41) and also the expressions obtained in the preceding section, we can write the first coefficients explicitly:

\[
F_0^{(0)} = 1,
\]

\[
F_0^{(1)} = (\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)(2\Delta)^{-1},
\]

(50)

(51)
\[ F^{(2)}_0 = (\Delta + \Delta_1 - \Delta_2)(1 + \Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)(1 + \Delta + \Delta_3 - \Delta_4) \]
\[ \times (4\Delta(1 + 2\Delta))^{-1} \]
\[ + (\Delta^2 - 3(\Delta_1 - \Delta_2)^2 + 2\Delta(\Delta_1 + \Delta_2)) \]
\[ \times (\Delta^2 - 3(\Delta_3 - \Delta_4)^2 + 2\Delta(\Delta_3 + \Delta_4))(2\Delta(3 + 2\Delta)(-3 + 3c + 16\Delta))^{-1} \]
\[ + (\Delta_1 - 2(\Delta_1 - \Delta_2)^2 + \Delta_2 + \Delta(-1 + 2\Delta_1 + 2\Delta_2)) \]
\[ \times (\Delta_3 - 2(\Delta_3 - \Delta_4)^2 + \Delta_4 + \Delta(-1 + 2\Delta_3 + 2\Delta_4)) \]
\[ \times ((c + 2(-3 + c)\Delta + 4\Delta^2)(3 + 4\Delta(2 + \Delta)))^{-1}, \quad (52) \]
\[ F^{(0)}_1 = (2\Delta)^{-1}, \quad (53) \]
\[ F^{(1)}_1 = (1 + 2\Delta + 2\Delta_1 - 2\Delta_2)(1 + 2\Delta + 2\Delta_3 - 2\Delta_4)(8\Delta(1 + 2\Delta))^{-1} \]
\[ + 4(\Delta_1 - \Delta_2)(\Delta_3 - \Delta_4)((c + 2(-3 + c)\Delta + 4\Delta^2)\Delta(1 + 2\Delta))^{-1}, \quad (54) \]
\[ F^{(2)}_1 = 128^{-1}(1 + 2\Delta + 2\Delta_1 - 2\Delta_2)(3 + 2\Delta + 2\Delta_3 - 2\Delta_2)(1 + 2\Delta + 2\Delta_3 - 2\Delta_4) \]
\[ \times (3 + 2\Delta_1 + 2\Delta_3 - 2\Delta_4)(\Delta(1 + 3\Delta + 2\Delta^2))^{-1} \]
\[ + (3 + 2\Delta + 2\Delta_1 - 2\Delta_2)(\Delta_1 - \Delta_2)(3 + 2\Delta + 2\Delta_3 - 2\Delta_4)(\Delta_3 - \Delta_4) \]
\[ \times ((c + 2(-3 + c)\Delta + 4\Delta^2)(3 + 4\Delta(2 + \Delta)))^{-1} \]
\[ + (1 + 4\Delta_1 + 4\Delta_2 + 2(-6(\Delta_1 - \Delta_2)^2 + \Delta(-1 + 2\Delta_1 + 2\Delta_2))) \]
\[ \times (\Delta_1, \Delta_2 \rightarrow \Delta_3, \Delta_4)(64\Delta(1 + \Delta)(2 + \Delta)(5 - 11\Delta + 2\Delta^2 + 3c(1 + \Delta)))^{-1} \]
\[ + (3(-1 + 4\Delta_1 + 4\Delta_2) + 4(\Delta^2 - 3(\Delta_1 - \Delta_2)^2 + \Delta(1 + 2\Delta_1 + 2\Delta_2))) \]
\[ \times (\Delta_1, \Delta_2 \rightarrow \Delta_3, \Delta_4)(64\Delta(2 + \Delta)(3 + 2\Delta)(-3 + 3c + 16\Delta))^{-1}. \quad (55) \]

6 Differential equations corresponding to the null vector (1,3)

Because singular vector [10] vanishes, the differential equation

\[
\left\{-b^{-2}\partial_4 D_4 + \sum_{i=1}^{3} \left\{ \frac{2\Delta_i}{z_{4i}}\theta_{4i} + \frac{1}{z_{4i}}\left[ 2\theta_{4i}\partial_i - D_i \right] \right\} \right\} \cdot \left\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4) \right\rangle = 0 \quad (56)
\]

holds, where

\[ D = \partial_\theta - \theta\partial_\tau. \quad (57) \]

With [11], this equation reduces to the differential equation for \( g(z, \bar{z}, \tau_1, \bar{\tau}_1, \tau_2, \bar{\tau}_2) \)

\[
\left[ -b^{-2}\partial_4 D_4 - b^{-2}\frac{\gamma_{34}}{z_{34}}(\theta_{43}\partial_4 - D_4) + \sum_{i=1}^{3} \left\{ \frac{2\Delta_i}{z_{4i}}\theta_{4i} + \frac{1}{z_{4i}}\left( 2\theta_{4i}\partial_i - D_i \right) \right\} \right]
\]
\[ + \frac{\gamma_{12}(\theta_{41} + \theta_{42})}{z_{41}z_{42}} + \frac{\gamma_{13}(\theta_{41} + \theta_{43})}{z_{41}z_{43}} + \frac{\gamma_{23}(\theta_{42} + \theta_{43})}{z_{42}z_{43}} + \frac{\gamma_{34}\theta_{43}}{z_{43}^2} \right\} g = 0, \quad (58)
\]

In accordance with [15], we present \( g = g_0(z) + g_1(z)\tau_1 + g_2(z)\tau_2 + g_3(z)\tau_1\tau_2 \) (also keeping in mind the antiholomorphic dependence). Equation [58] splits into two independent systems of
ordinary differential equations for $g_i$. For the purpose of this paper, we explicitly write the system for $g_0$ and $g_3$:

\[-b^{-2}zg''_0 + \frac{3z - 2}{z - 1}g'_0 + b^{-2}g'_3 + \left[\frac{\gamma_{13}}{z} + \frac{\gamma_{23}}{z - 1}\right]g_0 + \frac{1 - 2z}{z(z - 1)}g_3 = 0,\]  
\[-b^{-2}g''_0 + \frac{1 - 3z}{z(z - 1)}g'_0 + \left[\frac{2\Delta_1}{z^2} + \frac{2\Delta_2}{z - 1} + \frac{2\gamma_{12}}{z(z - 1)}\right]g_0 + \frac{1}{z(z - 1)}g_3 = 0.\]  

(59)  

(60)

Three independent solutions for $g_0$ with a diagonal monodromy near $x = 0$ are just the $s$-channel conformal blocks [17,18] with the special parameter choices $\Delta_1 = \Delta_{13}$, $\Delta_2 = \Delta(\lambda_1)$, $\Delta_3 = \Delta(\lambda_2)$, $\Delta_4 = \Delta(\lambda_3)$, and $\Delta = \Delta^{(\pm)} = \Delta(\lambda_1 \pm b)$ or $\Delta = \Delta^{(0)} = \Delta(\lambda_1)$,

\[g_0^{(\pm)} = x^{\Delta - \Delta_1 - \Delta_2} \sum_{n=0}^{\infty} A_n^{(\pm)} x^n = F_0(\Delta^{(\pm)}, \Delta_i, c, x),\]  
\[g_0^{(0)} = x^{\Delta - \Delta_1 - \Delta_2 + 1/2} \sum_{n=0}^{\infty} A_n^{(0)} x^n = F_1(\Delta^{(0)}, \Delta_i, c, x),\]  

(61)  

(62)

corresponding to the overall normalization $A_0^{(\pm)} = 1, A_0^{(0)} = 1/(2\Delta(\lambda_1))$. The first terms in the series expansion can be easily found by substituting these expansions in differential equations (59) and (60) and solving the recursive relations for the coefficients order by order (we did this using Mathematica),

\[A_0^{(+)} = 1,\]  
\[A_1^{(+)} = (1 + 2b^2 - 3b^4 - 8b^2\lambda_1 - 4b^2\lambda_2^2 - 4b^2\lambda_3^2 + 4b^2\lambda_3^2)(4(-1 + b^2 + 2b\lambda_1))^{-1},\]  
\[A_2^{(+)} = (21 + 4b^2 - 11b^4 - 84b^6 - 19b^8 - 62b\lambda_1 - 152b^3\lambda_1 + 380b^5\lambda_1 - 184b^7\lambda_1 + 18b\lambda_1 + 8b^2\lambda_1^2 + 432b^5\lambda_1^2 + 96b^8\lambda_1^2 + 112b^3\lambda_1^3 - 416b^5\lambda_1^3 + 176b^7\lambda_1^3 - 112b^9\lambda_1^3 + 128b^6\lambda_1^3 + 32b^3\lambda_1^3 + 72b^3\lambda_1^3 + 176b^4\lambda_1^3 - 40b^6\lambda_1^3 + 4b\lambda_1^3 - 288b^5\lambda_1^3 + 64b^3\lambda_1^3 - 8b^2\lambda_1^3 - 224b^5\lambda_1^3 + 128b^6\lambda_1^3 + 64b^5\lambda_1^3 - 48b^4\lambda_1^3 + 32b^2\lambda_1^3 + 24b^2\lambda_1^3 - 144b^4\lambda_1^3 + 56b^6\lambda_1^3 - 112b^3\lambda_1^3 + 288b^5\lambda_1^3 - 8b^7\lambda_1^3 - 16b^4\lambda_1^3 - 4b^2\lambda_1^3 - 64b^5\lambda_1^3 + 64b^4\lambda_1^3 - 64b^3\lambda_1^3 - 32b^2\lambda_1^3)\]  
\[\times (64(-1 + b\lambda_1)(-3 + b^2 + 2b\lambda_1)(-1 + b^2 + 2b\lambda_1))^{-1},\]  
\[A_0^{(0)} = 4b^2((1 + b^2)(1 + b^2) + (3 + b^2 + 2b\lambda_1))^{-1},\]  
\[A_1^{(0)} = (-2b^4(9 + 6b^2 + b^4 - 4b^2\lambda_1^2 - 8\lambda_2^2 - 4b^2\lambda_3^2 + 8\lambda_3^2 + 4b^2\lambda_3^2))\]  
\[\times ((1 + b^2 + 2b\lambda_1)(3 + b^2 + 2b\lambda_1)(1 + b^2 + 2b\lambda_1))^{-1},\]  
\[A_2^{(0)} = (b^2(-72 - 54b^4 - 32b^6 + 48b^6 + 62b^6 + 14b^{10} + b^{12} + 32b^2\lambda_1^2 + 288b^4\lambda_1^2 - 8b^6\lambda_1^2 - 8b^6\lambda_1^2 - 8b^{10}\lambda_1^2 - 32b^6\lambda_1^2 + 16b^8\lambda_1^2 + 544b^2\lambda_2^2 + 128b^4\lambda_2^2 + 200b^6\lambda_2^2 - 80b^8\lambda_2^2 - 8b^{10}\lambda_2^2 - 128b^4\lambda_2^2 + 32b^8\lambda_2^2 + 128b^8\lambda_2^2 + 96b^6\lambda_2^2 + 16b^8\lambda_2^2 - 256b^8\lambda_2^2 + 208b^8\lambda_2^2 + 544b^2\lambda_3^2 + 48b^6\lambda_3^2 + 8b^{10}\lambda_3^2 - 64b^8\lambda_3^2 + 32b^8\lambda_3^2 + 32b^8\lambda_3^2 + 128b^6\lambda_3^2 + 96b^6\lambda_3^2 + 16b^8\lambda_3^2 + 256b^8\lambda_3^2 - 192b^6\lambda_3^2 - 32b^8\lambda_3^2 + 128b^6\lambda_3^2 + 96b^6\lambda_3^2 + 16b^8\lambda_3^2))\]  
\[\times (2(1 + b^2 - 2b\lambda_1)(3 + b^2 + 2b\lambda_1)(5 + b^2 - 2b\lambda_1))^{-1},\]  
\[\times (1 + b^2 + 2b\lambda_1)(3 + b^2 + 2b\lambda_1)(5 + b^2 + 2b\lambda_1))^{-1}.\]  

(63)  

(64)  

(65)  

(66)  

(67)  

(68)
These expressions coincide with the expressions for the corresponding coefficients (50) and (53) calculated for the same parameter choices and thus confirm recursive relations (46) and (47).

7 Conclusion

Recursive relations (46) and (47) for the superconformal block functions together with the supporting verification in Sec. 6 is our main result in this paper. These relations allow efficiently evaluating the superconformal blocks for any parameter values. They are appropriate for numerically calculating the four-point correlation functions of the primary fields in the NS sector in general and of the degenerate primary fields in particular. We note that for the four-point conformal blocks (and also superconformal blocks), there exists [9, 15] the so-called $q$-representation of the recursive relations, which is much more favorable because it converges in the whole complex plane with three punctures.

Representations like (16) can also be written for higher multipoint correlation functions. They involve the multipoint conformal blocks, which are much more complicated than the four-point blocks. We hope that the consideration in this paper will help to approach the extremely interesting question of generalizing the recursive relations to higher multipoint conformal blocks and in particular the question of the corresponding generalized $q$-representations.

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9 Note Added

After this paper was sent to arXiv I have learned that the recursive relations for the superconformal block functions apparently equivalent to (46) and (47) were proposed simultaneously in [16]. I am grateful to L. Hadasz for bringing the paper [16] to my attention.

References

[1] A. Zamolodchikov and Al. Zamolodchikov. Structure constants and conformal bootstrap in Liouville field theory. *Nucl. Phys.* B477 (1996) 577–605 (hep-th/9506136)

[2] H. Dorn and H. Otto. Two and three point functions in Liouville theory. *Nucl. Phys.* B429 (1994) 375–388 (hep-th/9403141)

[3] J. Teschner. Liouville theory revisited. *Class. Quant. Grav.* 18 (2001) R153–R222 (hep-th/0104158)

[4] Al. Zamolodchikov. Higher equations of motion in Liouville field theory. *Int. J. Mod. Phys.* A19S2 (2004) 510–523 (hep-th/0312279)
[5] Al. Zamolodchikov. Three-point function in Minimal Liouville Gravity. *Theor. Math. Phys.* **142** (2005) 183–196 (hep-th/0505063)

[6] A. Belavin and Al. Zamolodchikov. Integrals over moduli spaces, ground ring, and four-point function in minimal Liouville gravity. *Theor. Math. Phys.* **147** (2006) 729–754

[7] A. Polyakov. Quantum geometry of fermionic strings. *Phys. Lett.* **B103** (1981) 211–213

[8] A. Belavin, A. Polyakov, and A. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nucl. Phys. B241* (1984) 333–380

[9] Al. Zamolodchikov. Conformal symmetry in two dimensions: An explicit recurrence formula for the conformal partial wave amplitude. *Commun. Math. Phys.* **96** (1984) 419–422

[10] A. Zamolodchikov and R. Pogosian. Operator algebra in two-dimensional superconformal field theory. *Sov. J. Nucl. Phys.* **47** (1988) 929–936

[11] L. Alvarez-Gaume and P. Zaugg. Structure constants in the $N=1$ superoperator algebra. *Annals Phys.* **215** (1992) 171–230 (hep-th/9109050)

[12] V. G. Kac. *Infinite-dimensional Lie algebras.* Prog. Math., Vol. 44, Birkhäuser, Boston, 1984.

[13] A. Belavin and Al. Zamolodchikov. Higher equations of motion in $N=1$ SUSY Liouville field theory. *JETP Lett.* **84** (2006) 496–502 (hep-th/0610316)

[14] A. B. Zamolodchikov and Al. B. Zamolodchikov. Conformal field theory and 2-D critical phenomena. Part III. Conformal bootstrap and degenerate representations of conformal algebra. ITEP-90-31 (1990)

[15] Al. Zamolodchikov. *Teor. Mat. Fiz.* **73** (1987) 103

[16] L. Hadasz, Z. Jaskolski, P. Suchanek. Recursion representation of the Neveu-Schwarz superconformal block. (hep-th/0611266)