Big Data Quantum Support Vector Clustering

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Clustering is a complex process in finding the relevant hidden patterns in unlabeled datasets, broadly known as unsupervised learning. Support vector clustering algorithm [1] is a well-known clustering algorithm based on support vector machines and Gaussian kernels. In this paper, we have investigated the support vector clustering algorithm in quantum paradigm. We have developed a quantum algorithm which is based on quantum support vector machine and the quantum kernel (Gaussian kernel and polynomial kernel) formulation. The investigation exhibits approximately exponential speed up in the quantum version with respect to the classical counterpart. The classical support vector clustering algorithm converges in \( O(M^2N) \) run times, and the quantum version converges in \( O(\log MN) \) run time, whereas the quantum Gaussian kernel runs in time \( O(\epsilon^{-1}(1+\epsilon)\log N) \), and the quantum d-level polynomial kernel runs in time \( O(d\epsilon^{-1}\log N) \). Clustering identification phase with adjacency matrix exhibits run time complexity of \( O(\sqrt{M^3\log M}) \) in quantum paradigm whereas the run time complexity of classical implementation is of \( O(M^2) \). Here \( M \) is the input objects, \( N \) is the dimension of the object feature and \( \epsilon \) is error. The overall runtime complexity of the proposed quantum clustering algorithm shows exponential speedup as compared to the classical implementation.

KEYWORDS
Quantum Algorithm, Clustering, QRAM

1 INTRODUCTION
Clustering is a popular task in machine learning, where a set of input objects are grouped together based on some similarities. This becomes a more tedious task on big datasets. We have many well-defined clustering algorithms which work very efficiently in their defined domain of interests. K-means [2] [3] is one of the most popular clustering algorithms but has the drawback of defining the number of clusters in advance. Although with the upgraded version, K-means++ [4], this limitation has been handled to some extent. Many supervised clustering approaches [5] [6] have been proposed, where we take the advantages of classification algorithms for clustering. One of the popular classical approaches is to use one-class SVM and extend it to clustering, known as the support vector clustering [1]. Support vector clustering is based on one-class SVM and Gaussian kernel. In support vector clustering, a kernel is used that helps in contour’s formation with input data in a higher dimension. These contour’s boundaries are defined by support vectors and consist of a set of some specific input objects. Each contour boundary is considered as a cluster. Once the contour boundaries are identified, we can separate the clusters with the help of an adjacency matrix. In this paper, we have investigated the support vector clustering in quantum paradigm. In the proposed approach, we have used a quantum support vector machine and quantum Gaussian kernel (and as well as the quantum polynomial kernel) to design the support vector clustering algorithm. Our analysis shows that the proposed quantum version of support vector clustering is around exponential faster than the classical counterpart.

2 SUPPORT VECTOR MACHINE
In this section, we have briefly discussed support vector machine in classical and quantum paradigm.

2.1 Classical Least Square SVM
Support vector machine with kernel trick is a very popular classification techniques, where the input objects are mapped into a higher dimensional input space. It then constructs an optimal separating hyperplane which can be used for classification. A good separation is achieved by the hyperplane which has the largest expanse to the nearest training input objects of any class. In [7], a least square support vector machine (LS-SVM) was
suggested, which is based on the least square method for functional estimation [8]. In LS-SVM, instead of quadratic programming, we solve a linear system of equations to find the solutions. The problem is formulated by use of equality constraints rather than inequality constraints.

With a given training object set of $M$ data objects $\{y_i, x_i\}_{i=1}^M$, where $x_i \in \mathbb{R}^N$ denotes the $i^{th}$ input and $y_i \in \mathbb{R}$ the $i^{th}$ output. The aim of SVM is to construct a classifier of the form:

$$y(x) = sign \left\{ \sum_{i=1}^M \alpha_i y_i K(x, x_i) + b \right\}$$  \hspace{1cm} (1)

Where, $\alpha_i$ are support values, $b$ is a real constant and $K(x, x_i)$ is any kernel function. Let us suppose

$$w^T(x_i) + b \geq +1, \quad \text{if } y_i = +1$$
$$w^T(x_i) + b \leq -1, \quad \text{if } y_i = -1$$

which can be written in the following equivalent formulation

$$y_i [w^T(x_i) + b \geq 1], \quad i = 1, ..., M,$$

and the classifier is obtained as the solution to the following optimization problem:

$$\min_{w, b, e} J_{LS} (w, b, e) = \frac{1}{2} w^T w + \gamma \frac{1}{2} \sum_{i=1}^M e_i^2$$  \hspace{1cm} (2)

which is subject to the following equality constraints

$$y_i [w^T(x_i) + b] = 1 - e_i, \quad i = 1, ..., M$$  \hspace{1cm} (3)

and $\gamma \frac{1}{2} \sum_{i=1}^M e_i^2$ is a penalty term. The Lagrangian function is then defined as

$$L(w, b, e; \alpha) = J_{LS} - \sum_{i=1}^M \alpha_i [y_i [w^T(x_i) + b] - 1 + e_i]$$  \hspace{1cm} (4)

and the conditions for optimality can be written as linear systems [9-10] by taking partial derivatives of the Lagrangian function and eliminating the variables $e_i$ and $w$:

$$\begin{pmatrix} 0 \\
1 \\
K + \gamma^{-1}I \end{pmatrix} \begin{pmatrix} b \\
b^T \\
\alpha \end{pmatrix} = \begin{pmatrix} 0 \\
y \end{pmatrix}.$$

$$\Rightarrow F \begin{pmatrix} b \\
\alpha \end{pmatrix} = \begin{pmatrix} 0 \\
y \end{pmatrix}.$$  \hspace{1cm} (5)

where, $K_{ij} = K(x_i, x_j)$ is the kernel matrix, $I$ is the identity matrix, $\tilde{y} = (y_1, ..., y_M)^T$, $\tilde{1} = (1, ..., 1)^T$, $\tilde{\alpha} = (\alpha_1, ..., \alpha_M)^T$. The support vector machine parameters are then determined by

$$(b, \tilde{\alpha}^T)^T = F^{-1}(0, \tilde{y})^T$$  \hspace{1cm} (6)

Now, an unknown input object $\tilde{x}$ can be classified by the following equation:

$$f(\tilde{x}) = sign(\sum_{j=1}^M \alpha_j y_j K(\tilde{x}_j, \tilde{x}) + b)$$  \hspace{1cm} (7)

2.2 Quantum Least Square Support Vector Machine
The quantum version of least square support vector machine has been discussed in [11] which exhibits an exponential speed up as compared to the classical least square support vector machine during training as well as classification. In the same notion of classical support vector machine, we formulate \( \hat{P}(|b, \tilde{a}\rangle) = |\tilde{y}\rangle \) to determine the quantum support vector machine parameters. The classification of an unknown quantum state |\( x \rangle \) is determined by the success probability \( P \) of swap test between |\( b, \tilde{a}\rangle \) and |\( x \rangle \). If the success probability (\( P \)) < 1/2, then |\( x \rangle \) is classified as +1, otherwise −1.

A kernel matrix plays a crucial role in the dual formulation of equation (4), so the dot product calculation. In the quantum algorithm, the dot product is calculated quantum mechanically [12]. To calculate a dot product of two training instances, first, we need to generate two quantum states |\( \psi \rangle \) and |\( \varphi \rangle \) with an ancilla variable. Then estimate the sum of the squared norms of the two training instances. Now compare the two training instances and perform a projective measurement on the ancilla alone.

Considering the linear kernel \( K_{linear} = x_i^T x_j = \frac{(|x_i|^2 + |x_j|^2) - |x_i - x_j|^2}{2} \), we calculate the dot product for it. We construct the quantum state |\( \psi \rangle = \frac{1}{\sqrt{2}} ([0]|x_i\rangle + [1]|x_j\rangle) \) with the help of QRAM [13], and estimate |\( \varphi \rangle = \frac{1}{\sqrt{|x_i|^2 + |x_j|^2}} (|x_i|0 - |x_j|1) \). Now we evolve the quantum state \( \frac{1}{\sqrt{2}} ([0] - [1]) \otimes |0\rangle \) with the Hamiltonian \( H = (|x_i|0\rangle\langle 0| + |x_j|1\rangle\langle 1|) \otimes \sigma_x \). This results the following state

\[
\frac{1}{\sqrt{2}} (\cos(|x_i|t)|0\rangle - \cos(|x_j|t)|1\rangle) \otimes |0\rangle - i\frac{1}{\sqrt{2}} (\sin(|x_i|t)|0\rangle - \sin(|x_j|t)|1\rangle) \otimes |1\rangle
\]

Measuring the ancilla bit with appropriate \( t \), the complexity of constructing |\( \varphi \rangle \) with accuracy \( \epsilon \) and \( (|x_i|^2 + |x_j|^2) \) is \( O(\epsilon^{-1}) \). The swap test on the ancilla alone with |\( \psi \rangle \) and |\( \varphi \rangle \) is then done. Thus, complexity of evaluating a single dot product \( x_i^T x_j \) with QRAM is \( O(\epsilon^{-1}logN) \).

The speedup gain in quantum implementation can also be achieved during training too. The speedup gain is possible because of quantum implementation of matrix inversion algorithm [14] and non-sparse density matrices [15] and simulating sparse Hamiltonians [16]. To solve \( \hat{P}(|b, \tilde{a}\rangle) = |\tilde{y}\rangle \), we formulate to find the matrix exponential of \( \hat{F} \). The \( \hat{F} \) can be written as \( \hat{F} = (J + K + \gamma^{-1}I) / trF \). Where, \( J = \begin{pmatrix} 0 & 1^T \\ 1 & 0 \end{pmatrix} \) is a star graph and \( K \) is the kernel matrix. We obtain the following exponential

\[
e^{-i\hat{F}\Delta t} = e^{-i\Delta t/|trF|} e^{-i\Delta t/tF} e^{-iK\Delta t/|trF|} + O(\Delta t^2). \tag{8}
\]

by using Lie product formula.

With multiple copies of density matrix \( \rho \), it is possible to implement \( e^{-i\alpha} \) [12], and as \( \hat{K} \) is non-pare Hermitian matrix, computing the matrix inverse \( \hat{K}^{-1} \) can be done. The run time of the exponentiation is then, \( O(logN) \). We evaluate \( e^{-i\hat{K}\Delta t} \) as

\[
e^{-i\hat{L}_K\Delta t}(\rho) \approx \rho - i\Delta t[K, \rho] + O(\Delta t^2). \tag{9}
\]

Where, \( L_K = [K, \rho] \).

Using (9), we obtain the eigenvectors and eigenvalues by doing quantum phase estimation. Also, the |\( b, \tilde{a}\rangle \) is obtained by inverting eigenvalues and expressing \( y \) in the eigenvectors. So the overall run time training complexity is \( O(logMN) \).

The discussion is extended by using a polynomial kernel for handling non-linear datasets. Suppose, \( K(x_i, x_j) = \varphi(x_i) \varphi(x_j) \) is an \( d^{th} \) order polynomial kernel. The support vector machine classification can be performed in higher dimensional space. In this case, each vector is mapped into \( d \)-times
tensor product $|\varphi(x_i)\rangle \equiv |x_i\rangle \otimes ... \otimes |x_i\rangle$. The run time complexity of this quantum polynomial kernel trick is $O(d\log N/e)$. Apart from the quantum polynomial kernel, quantum Gaussian kernel [17], which is our recent work, has also been discussed from computational complexity point of view. The run time complexity of normalized quantum Gaussian kernel trick is $O[\varepsilon^{-1}(1 + e)\log M] \sim O[\varepsilon^{-1}\log M]$ [17].

3 CLASSICAL SUPPORT VECTOR CLUSTERING

We are formulating the classical support vector clustering using least square support vector machine as in the original implementation [1]. This implementation uses a classical Gaussian kernel which transforms the input objects into the higher dimensional space. Gaussian kernel helps in forming tight contours in the higher dimensions. The contours are fenced by the support vectors and these contours are represented as clusters. We modified the equation (7) as follows to make it feasible as per the clustering notion.

$$f(x) = \text{sign} \left( \sum_{\tilde{x}_j \in \text{support vectors}} \alpha_i K(\tilde{x}_i, \tilde{x}) + b \right) \quad (10)$$

The objective is to address a one-class SVM for the clustering implementation that finds a minimal region $R$, which encloses the data objects. The one-class classification can be easily constructed by putting same labels to all the input objects in a binary classification. A $+ve$ value outcome of $f(x)$ implies that $x$ falls within the dense subspace of a specific contour, whereas the $-ve$ value outcome of the decision function (10) imply a sparsely populated region. The objects for which $f(x) < 0$ are known as bounded support vectors, and support vectors are those objects which fall on the contour line. Inside a contour, we have the clustered objects for that specific contour.

The one-class support vector machine is then can be easily extended to a clustering implementation by computing an adjacency matrix $A$ for the objects. In this formulation, $A_{ij} = 1$ if $\tilde{x}_i$ and $\tilde{x}_j$ are enclosed within the same contour and 0 otherwise. Equation (10) helps in determining whether $\tilde{x}_i$ and $\tilde{x}_j$ lie within the same contour or not. In this case equation (10) determines the decision for all the objects in the line that connects $\tilde{x}_i$ and $\tilde{x}_j$, and if the equation (10) results in all $+ve$ values then this concludes the surety that all these objects are within the same contour and bounded by the support vectors $\tilde{x}_i$ and $\tilde{x}_j$.

4 QUANTUM SUPPORT VECTOR CLUSTERING

The quantum support vector clustering implementation has two phases (similar to the case in the classical counterpart). In the first phase, we formulate the cluster boundaries with one-class quantum least square support vector machine and, subsequently, in the second phase, we identify the number of clusters and clustered objects.

4.1 Cluster Boundaries

Quantum support vector machine (SVM) [11] [18] is the classification algorithm which works based on the postulates of quantum mechanics. In our approach, we formulate a quantum support vector machine description of the data set, which is used as the basis for our clustering algorithm. In our formulation, we require a single class quantum support vector machine, so we articulate a single class quantum support vector machine from a binary quantum SVM, which is a straightforward process with a little trick. We just labeled all the input objects with the same label.

Let us begin by representing a classical input object vector for clustering as a $N$ dimensional complex vector in classical sense. We represent this classical data set over $log_2 N$ qubits onto quantum states in quantum RAM (QRAM) [19-25] with $O(\log_2 N)$ steps. Once the data set is mapped to quantum states, the quantum clustering algorithm can further process this data. The QRAM allows us to access the data in quantum parallel and perform memory access in coherent quantum superposition [26]. Suppose $ADR$ is an address register and
it contains a superposition of the addresses $\sum_j \psi_j |j\rangle_{ADR}$. The QRAM will return a superposition of the data in a data register $DR$ which is correlated with the address register, i.e.,

$$\sum_j \psi_j |j\rangle_{ADR} \rightarrow \sum_j \psi_j |j\rangle_{ADR} |D_j\rangle_{DR}$$

where, $D_j$ is the $j^{th}$ memory cell content. In QRAM, it takes only $O(\log MN)$ operations to access the data and it uses $O(MN)$ resources, where $M$ is the number of cluster inputs vector. The next step is to prepare the quantum kernel (with Polynomial or Gaussian kernel) matrix with inner product evaluation $[11][17]$. Kernel matrix is a very important playing factor for least square formulation and objects contour formations in the higher dimension. We have to compute the inverse of the normalized kernel matrix and then carried out the exponentiation of $R^{-1}$, $R$ is the normalized kernel matrix. The computation of a kernel matrix inversion in quantum space has been discussed in $[27]$. 

As discussed in the earlier section, in the quantum setting for binary quantum SVM, a quantum state $|b, a\rangle$ is generated, which helps in classifying an unseen quantum state $|\bar{x}\rangle$. The classifier’s prediction outcome is determined by the success probability $P$ of a swap test between $|b, a\rangle$ and $|\bar{x}\rangle$. Therefore, we solve $F(|b, a\rangle) = |\bar{y}\rangle$, where $F = \frac{P}{1-P}$ with $|F| \leq 1$. The hyper-plan has been described by the quantum inversion algorithm $[28]$. Here $\alpha$ is the distance from the optimal margin, and $b$ is the non-zero offset. If $P < \frac{1}{2}$, the predicted output for $|\bar{x}\rangle$ is considered as $+1$, and otherwise $-1$.

Whether $|\bar{x}_i\rangle$ and $|\bar{x}_j\rangle$ lie within the same contour can be determined by classifying all the objects on the line that connects $|\bar{x}_i\rangle$ and $|\bar{x}_j\rangle$. The $+1$ classification for all the objects guarantees that $|\bar{x}_i\rangle$ and $|\bar{x}_j\rangle$ are in the same contour.

4.2 Clustering Identification

After formulating the quantum one-class support vector machine, we assign the clustering boundaries. Single class quantum SVM can be easily extended to a clustering scheme by computing an adjacency matrix $A$ for the given cluster data, where

$$A_{ij} = \begin{cases} 1, & \text{if } |x_i\rangle \text{ and } |x_j\rangle \text{ are enclosed within the same contour} \\ 0, & \text{otherwise} \end{cases}$$

By formulating $\tilde{F}(|b, a\rangle) = |\tilde{y}\rangle$, and classifying all the points in the line that connects $|x_i\rangle$ and $|x_j\rangle$, we can determine whether $|x_i\rangle$ and $|x_j\rangle$ lies within the same contour. All the objects within a specific contour is considered as separately clustered. So, technically, the number of contours means the number of clusters. In the graph induced by the matrix $A_{ij}$, one can detect the connected components. The number of clusters is determined by the number of graphs induced by the connected components in $A_{ij}$. Let us discuss about finding the number of connected components in $A_{ij}$. For this purposes we have performed a depth-first-search (DFS) in quantum way $[29]$. Here, $A_{ij}$ is an undirected graph matrix. The following algorithm determines the number of clusters and the objects in the clusters:

**clusterFinding(A):**

1. Initialize
   (a) All the vertices $|\bar{x}_i\rangle$ in $A$ as not marked,
   (b) A variable $clusterCount = 1$,
   (c) A one-dimensional dynamic array $storeClusterObjects$
   (d) and a multidimensional dynamic array $clusteredObjects$
2. Loop the following for every vertex $|\bar{x}_i\rangle$ in $A$
   (a) If $|\bar{x}_i\rangle$ is not marked, then call $quantumDepthFirstSearch(|\bar{x}_i\rangle)$
   (b) $clusterCount = clusterCount + 1$
3. Return $clusterCount$, $clusteredObjects$

$quantumDepthFirstSearch(x_i)$:
1. Mark $|\tilde{x}_i\rangle$
2. $storeClusterObjects = |\tilde{x}_i\rangle$
3. Search an adjacency of $|\tilde{x}_i\rangle$, say $|\tilde{x}_j\rangle$, that has not yet been visited using quantum search algorithm
4. Loop the following for every adjacency $|\tilde{x}_j\rangle$ of $|\tilde{x}_i\rangle$.
   (a) If $|\tilde{x}_j\rangle$ is not marked, then call $quantumDepthFirstSearch(|\tilde{x}_j\rangle)$
5. Return $storeClusterObjects$

The algorithm $clusterFinding$ returns the number of clusters, $clusterCount$, and the cluster objects associated with each cluster, $clusteredObjects$. The algorithm $quantumDepthFirstSearch$ is the quantum version of depth-first-search (DFS).

The contours in the cluster data space are governed by two parameters, $\sigma$ and $\gamma$, where $\sigma$ is the scale parameter of quantum Gaussian kernel and $\gamma$ is the soft margin constant. The shape/number of the boundaries in the data space varies with the changes in $\sigma$. Increasing the value of $\sigma$ may result in increasing the number of clusters in the data space. The parameter $\gamma$ controls the number of outliers in the data space.

5 COMPUTATIONAL COMPLEXITY ANALYSIS

First, we analyze the complexity of the classical implementation with least square SVM. The complexity discussion of the whole approach is discussed separately for finding the clustering boundaries and the clustering identification phases. We start the discussion with the clustering boundary phase, where in case of the least square SVM, quadratic programming is circumvented and the parameters are evaluated from the solution of a linear equation system. The authors in [30] have shown that the algorithm converges after approximately $O(M^2)$ kernel evaluations. The complexity of the complete algorithm is $O(M^2N)$, which is a polynomial complexity. In the clustering identification phase, the complexity of finding the number of clusters in adjacency matrix $A$ with depth-first-search contributes $O(M^2)$.

Now let us discuss the complexity of the quantum version, which is our proposed approach. We initially discussed the complexity of finding cluster boundaries. For $N$ dimensional $M$ cluster data points, the least square quantum SVM classification training takes an $O(\log NM)$ with a quantum linear kernel. When implemented with the quantum Gaussian kernel, $O(e^{-1}(1+e)\log N)$ run time complexity of the kernel implementation is under consideration, and with the quantum $d$-level polynomial the consideration of the run time complexity is $O(de^{-1}\log N)$.

The runtime complexity of the clustering identification phase considers two running time contributions. During the quantum search [30], the search will fail once [31], which includes a runtime complexity of $O(\sqrt{M^2\log M})$. Now, we consider the run time complexity of the successful search [32] of the element, which is also $O(\sqrt{M^2\log M})$. Therefore the runtime complexity of the clustering identification phase is $O(\sqrt{M^3\log M})$.

We can see that in both phases, finding clustering boundaries and cluster identification, we achieved exponential and quadratic speed up gains respectively. Therefore, the overall runtime analysis concludes that the proposed quantum version implementation is exponentially faster than the classical implementation.

6 CONCLUSIONS
In our investigation, we have found that the SVM clustering implementation in quantum paradigm exhibited exponential speed up as compared to the classical implementation. We have analyzed the quantum implementation of the SVM clustering with both quantum Gaussian kernel as well as with quantum polynomial kernel, and have concluded that both implementations have shown exponential improvements in the runtime complexity as compared to the classical implementation with the classical Gaussian kernel. Gaussian kernel helps in developing better contours in higher dimensions both in classical implementation [33] as well as in the quantum version. Training of the one-class SVM is also exponentially faster in the quantum version as compared to the classical counterpart. During the clustering identification phase, we have used a quantum version of depth-first search, which shows quadratic speed up as compared to the classical implementation of DFS. DFS is used to identify the number of clusters and clustered objects from the adjacency matrix. We have achieved quantum advantages for speed up gains in multiple stages (one class SVM formulation, kernel formulation, and during depth-first search) in the proposed clustering algorithm. This accomplishes that the overall runtime complexity of the proposed quantum clustering algorithm is exponentially faster than the classical implementation.

REFERENCES

[1] Asa Ben-Hur, David Horn, Hava T. Siegelmann. “Support Vector Clustering”. Journal of Machine Research 2 (2001) 125-137.
[2] Seth Lloyd, Masoud Mohseni, Patrick Rebentrost. "Quantum algorithms for supervised and unsupervised machine learning", arXiv:1307.0411.
[3] MacKay, David (2003), "Chapter 20. An Example Inference Task: Clustering" (PDF). Information Theory, Inference and Learning Algorithms. Cambridge University Press. pp. 284–292. ISBN 0-521-64298-1, MR 2012999.
[4] Ostrofsky, R., Rabani, Y., Schulman, L. J. and Swamy, C. (2006). "The Effectiveness of Lloyd-Type Methods for the k-Means Problem". Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06). IEEE, pp. 165–174.
[5] AM Bensaid, LO Hall, JC Bezdek, LP Clarke. "Partially supervised clustering for image segmentation". Pattern recognition, 1996 - Elsevier. doi.org/10.1016/0031-3203(95)00120-4
[6] Arit Kumar Bishwas, Ashish Mani, Vasile Palade. Book chapter "Quantum Sequence Clustering". Hybrid Intelligent Techniques for Pattern Analysis and Understanding. CRC Press USA, 2017
[7] Suykens J.A.K., Vandewalle J., “Least squares support vector machine classifiers,” Neural Pro- cessing Letters, vol.9,No.3, 1999.
[8] Saunders C., Gammerman A., Vovk V., “Ridge Regression Learning Algorithm in Dual Vari- ables,” Proc. of the 15th Int. Conf. on Machine Learning, ICML-98, Madison-Wisconsin, 1998.
[9] Fletcher R., Pratical methods of optimization, Chichester and NewYork: John Wdey and Sons, 1987.
[10] Fletcher R., Johnson T., “On the stability of null- space methods for KKT systems,” SIAM J. Ma- trix Anal. Appl., Vol.18, No.4, 938-958, 1997.
[11] P. Rebentrost, M. Mohseni and S. Lloyd, arXiv:1307.0471v3.
[12] S. Lloyd, M. Mohseni, and P. Rebentrost, arXiv:1307.0411 (2013).
[13] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. 100,160501 (2008).
[14] S. Lloyd, M. Mohseni, and P. Rebentrost, arXiv:1307.0401 (2013).
[15] D. Berry, G. Ahokas, R. Cleve, and B. Sanders, Comm. Math. Phys. 270, 359 (2007).
[16] A. Harrow, A. Hassidim, and S. Lloyd, Phys. Rev. Lett. 103, 150502 (2009).
[17] Arit Kumar Bishwas, Ashish Mani, Vasile Palade. arxiv.org/pdf/1711.01464
[18] Arit Kumar Bishwas, Ashish Mani and Vasile Palade, Conference Proceedings to IEEE 2nd International Conference on Contemporary Computing and Informatics (IC3I) (2016)
[19] V. Giovannetti, S. Lloyd, L. Maccone, Phys.Rev.Lett. 100, 160501 (2008); arXiv: 0708.1879.
[20] V. Giovannetti, S. Lloyd, L. Maccone, Phys.Rev.A 78, 052310 (2008); arXiv: 0807.4994.
[21] F. De Martini, V. Giovannetti, S. Lloyd, L. Maccone, E. Nagali, L. Sansonii, F. Sciarrino, Phys. Rev. A 80, 010302(R) (2009); arXiv: 0902.0222.
[22] I. Chiorescu, N. Groll, S. Bertaina, T. Mori, S. Miyashita, Phys. Rev. B 82, 024413 (2010).
[23] D.I. Schuster, A. P. Sears, E. Ginossar, L. DiCarlo, L. Frunzio, J. J. L. Morton, H. Wu, G. A. D. Briggs, B. B. Buckley, D. D. Awschalom, R. J. Schoelkopf, Phys. Rev. Lett. 105, 140501 (2010).
[24] Y. Kubo, F. R. Ong, P. Bertet, D. Vion, V. Jacques, D. Zheng, A. Drau, J.-F. Roch, A. Auffeves, F. Jelezko, J. Wrachtrup, M. F. Barthe, P. Bergonzo, D. Esteve, Phys. Rev. Lett. 105, 140502 (2010).
[25] H. Wu, R.E. George, J.H. Wesenberg, K. Miler, D.I. Schuster, R.J. Schoelkopf, K.M. Itoh, A. Ardavan, J.J.L. Morton, G.A.D. Briggs, Phys. Rev. Lett. 105, 140503 (2010).
[26] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[27] S. Lloyd, M. Mohseni, and P. Rebentrost, Nat. Phys. 10, 631 (2014)
[28] A. W. Harrow, A. Hassidim, and S. Lloyd, Phys. Rev. Lett. 103, 150502 (2009).
[29] Bartholomew Furrow. "A panoply of quantum algorithms". Journal Quantum Information & Computation
[30] L. Grover. A fast quantum mechanical algorithm for database search. Proceedings of 28th Annual ACM Symposium on Theory of Computing (STOC), pages 212-219, 1996.

[31] M. Boyer, G. Brassard, P. Høyer and A. Tapp. Tight bounds on quantum searching. Fortschritte Der Physik, 46(4-5), pages 493-505, 1998.

[32] H. Buhrman, R. Cleve, R. de Wolf and C. Zalka. Bounds for Small-Error and Zero-Error Quantum Algorithms. 40th IEEE Symposium on Foundations of Computer Science (FOCS), pages 358-368, 1999. Also cs/9904019.

[33] D.M.J. Tax and R.P.W. Duin. Support vector domain description. Pattern Recognition Letters, 20:1991-1999, 1999.