A Local Logarithmic Conformal Field Theory

Matthias R. Gaberdiel\textsuperscript{a} and Horst G. Kausch\textsuperscript{b}

\textsuperscript{a} Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, U.K.

\textsuperscript{b} Department of Mathematical Sciences, University of Durham, South Road, Durham DH1 3LE, U.K.

13 July 1998

Abstract

The local logarithmic conformal field theory corresponding to the triplet algebra at $c = -2$ is constructed. The constraints of locality and duality are explored in detail, and a consistent set of amplitudes is found. The spectrum of the corresponding theory is determined, and it is found to be modular invariant. This provides the first construction of a non-chiral rational logarithmic conformal field theory, establishing that such models can indeed define bona fide conformal field theories.

PACS: 11.25.Hf
Keywords: Conformal field theory

1 Introduction

Recently, a new class of chiral conformal field theories whose correlation functions have logarithmic branch cuts has attracted some attention. These models are believed to be important for the description of certain statistical models, in particular in the theory of (multi)critical polymers \cite{1,2}, two-dimensional turbulence \cite{3,4}, and the quantum Hall effect \cite{5}. There have also been suggestions that some of the so-called logarithmic operators (which appear in these theories) might correspond to normalisable zero modes for string backgrounds \cite{6}.

\textsuperscript{*}Email: M.R.Gaberdiel@damtp.cam.ac.uk

\textsuperscript{†}Email: H.G.Kausch@dur.ac.uk
By now, quite a number of such models have been analysed. They include the WZNW model on the supergroup $GL(1, 1)$ \([10]\), the $c = -2$ model \([11,12]\), gravitationally dressed conformal field theories \([13]\) and some critical disordered models \([14–17]\). Singular vectors of some Virasoro models have been constructed in \([18]\), correlation functions have been calculated in \([19,20]\), and more structural properties of logarithmic conformal field theories have been analysed in \([21]\).

The new class of “logarithmic” chiral conformal field theories is also of interest for more conceptual considerations of conformal field theory. In particular, there exist logarithmic models which behave in many respects like ordinary (non-logarithmic) chiral conformal field theories, and it is not yet clear in which way these models differ structurally from conventional theories. For example, there exists a series of “quasirational logarithmic” Virasoro models \([12]\), and a series of “rational logarithmic” models, the simplest of which is the triplet algebra at $c = -2$ \([22]\). Here quasirational means that a countable set of representations of the chiral algebra closes under fusion (with finite fusion rules), and rational that the same holds for a certain finite set of representations, including all (finitely many) irreducible representations. For these rational models Zhu’s algebra \([23]\) is finite dimensional, and it should be possible to read off all properties of the whole chiral theory from the vacuum representation. In particular, one should be able to decide whether a rational meromorphic conformal field theory (i.e. a chiral algebra, for which Zhu’s algebra is finite-dimensional) leads to a logarithmic theory or not, without actually constructing all the amplitudes. As yet little progress has been made in this direction, although it is believed that unitary (rational) meromorphic conformal field theories always lead to non-logarithmic theories.

The only rational logarithmic model which has been studied in detail so far, the aforementioned triplet algebra at $c = -2$, possesses another oddity (apart from the appearance of indecomposable reducible representations which lead to logarithmic correlation functions), and it is quite possible that this is true in more generality \([24]\): although the theory possesses a finite fusion algebra, the matrices corresponding to the reducible representations cannot be diagonalised, and a straightforward application of Verlinde’s formula does not make sense. This is mirrored by the fact that the modular transformation properties of some of the characters cannot be described by constant matrices as they depend on the modular parameter $\tau$. This might suggest that these logarithmic rational theories only make sense as chiral theories, and that they do not correspond to modular invariant (non-chiral) conformal field theories. It is the purpose of this paper to demonstrate that, at least for the case of the triplet algebra at $c = -2$, this is not the case. The resulting theory is in every aspect a standard (non-chiral) conformal field theory but for the property that it does not factorise into standard chiral theories. Among other things, this demonstrates that a non-chiral conformal field theory has significantly more structure than the two chiral theories it is built from.

In the process of constructing this non-chiral theory, we shall meet a number of novel difficulties. For example, in order to satisfy the locality constraint, the non-chiral representation corresponding to two indecomposable chiral representations is not simply the ordinary tensor product of the two chiral representations, but only a certain quotient thereof. Furthermore, in order to obtain a theory with finite multiplicities, it will be necessary to
identify states from different (non-chiral) representations (corresponding to different indecomposable chiral representations); this will imply that the two different indecomposable (non-chiral) representations combine to form a single representation with two fundamental vectors, from which the representation is generated by the action of the chiral algebras. Both of these features could have been predicted from the requirement to obtain a modular invariant theory, but it is gratifying to see that they can also be understood as arising in this way.

Our general strategy is motivated by the observation that the fundamental objects of a (chiral) conformal field theory are the correlation functions. In particular, as has been demonstrated in [25], all data of a chiral conformal field theory can be recovered from the complete set of amplitudes, and it is clear that the same holds for non-chiral conformal field theories. In order to construct our theory, we therefore have to determine all amplitudes and show that they satisfy the relevant properties. In fact, it is not actually necessary to determine all amplitudes of the theory, but it suffices to construct the two-, three- and four-point functions of the fundamental fields (the fields that correspond to the fundamental states of the different representations). Indeed, given the two- and three-point functions of the fundamental fields, all other amplitudes can be derived from these, and the consistency conditions of all amplitudes can be reduced to those being obeyed by the four-point functions. This reduces the problem of constructing the theory to a finite computation which can be done in principle.

Unfortunately, for the theory in question, some of the four-point functions involving two or more indecomposable representations are very complicated, and it is not feasible to calculate them explicitly. However, since we can determine all two- and three-point functions, all amplitudes of the theory are (in principle) determined, and the question is only that of showing the consistency of the resulting theory. We can then make use of the observation that the three-point functions (and therefore the operator product expansions) of the fundamental fields agree with those of a certain free field realisation of the theory. Because of the above arguments, this implies that all amplitudes of the two theories coincide, and since the free field theory is consistent thus establishes the consistency of our theory.

In the course of the construction we shall also explain how the spectrum of the theory can be read off from the amplitudes. For the theory in question, there exist only finitely many sectors, and each sector appears with multiplicity one. We can then determine the partition function of the resulting theory, and it turns out that it is indeed modular invariant.

The paper is organised as follows. In sections 2 and 3 we describe in some detail the general strategy of our approach. In section 4 the relevant amplitudes (including the three-point functions of the fundamental fields) are explicitly constructed. We then explain the free field realisation in terms of symplectic fermions in section 5, and demonstrate that the OPEs coincide. Finally in section 6, we discuss the modular invariance of the resulting theory. We have also included a number of appendices, where some of the more technical details of our calculations can be found.
2 Correlation functions

Let us first explain how the correlation functions of the theory can be determined. Let $S(w)$ be a meromorphic field of conformal weight $h$, and denote by $S_n$ the corresponding modes 

$$S(w) = \sum_n w^{n-h} S_n.$$ 

The action of the modes on a product of two fields is described by the two different comultiplication formulae of [26]; on $N$ fields, using successively these comultiplication products, there are $N$ different formulae which are given as 

$$\Delta^i(S_n) = S_n^{(i)} + \sum_{j \neq i} \sum_{k=-h+1}^{\infty} \left(\frac{n+h-1}{k+h-1}\right) (z_j - z_i)^{n-k} S_k^{(j)},$$

where $i \in \{1, \ldots, N\}$ and $S_k^{(j)}$ is the mode $S_k$ acting on the $j$-th factor in the tensor product. (These formulae can be obtained from the ones of [27] by the adjoint action of the translation operator.) The fusion product of these $N$ fields is defined as the quotient space of the direct tensor product by the relations of the form

$$e^{z_i L - 1} \Delta^i(S_n) e^{-z_i L - 1} = e^{z_j L - 1} \Delta^j(S_n) e^{-z_j L - 1}.$$ 

The correlation function $\langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle$ of a given set of $N$ fields is non-trivial if the vacuum representation is contained in the multiple fusion product of the corresponding representations. This is equivalent to the property that there exists a (non-trivial) linear functional $\Phi$ on the fusion product $(V_1 \otimes \cdots \otimes V_N)_f$ (where $V_j$ is the representation corresponding to $\phi_j$ and the subscript $f$ denotes that this is the quotient space of the direct tensor product) which is induced by the vacuum state (at infinity), and which therefore satisfies

$$\Phi \circ \Delta^i(S_n) = 0, \quad \forall i, n < h.$$ (1)

Furthermore, if there exist non-trivial null-vectors in the vacuum representation, these will give rise to additional constraints. The dimension of the space of solutions for the linear functional $\Phi$ satisfying these conditions is precisely the multiplicity with which the vacuum representation appears in the multiple fusion product of the representations. The results of our analysis of fusion [22] determine therefore the number of different solutions.

To find the different correlation functions explicitly, we shall use the following method. First we observe that any functional $\Phi$, satisfying the above constraints, is already uniquely determined by its value on $V_1^0 \otimes V_2^0 \otimes V_3^0 \otimes \cdots \otimes V_N^0$, where $V_j^0$ denotes the highest weight space of a representation $V$, i.e. the space of states which is not in the image of the action of the negative modes, and $V_N^0$ denotes the special subspace [28]. Indeed, we can use the relations (1) to remove modes $S_k$ with $k < -h$ from the states $\phi_i$. This allows us to determine the value of the functional for arbitrary vectors in terms of those on the $N$-fold tensor product of the special subspaces $V_1^s \otimes \cdots \otimes V_N^s$ [28].

We can then use the $2h - 2$ relations coming from

$$\Phi \circ e^{z_i L - 1} \Delta^i(S_n) e^{-z_i L - 1} = 0,$$
where $-h < n < h$ (in which case the formula is actually independent of $i$), which can be written as $\Phi \circ \Delta^0(S_n) = 0$ with

$$\Delta^0(S_n) = \sum_j \sum_{k=-h+1}^n \left( \frac{n-h-1}{k+h-1} \right) z_j^{n-k} S_{(j)}^k. \quad (2)$$

In particular, we can use these formulae to remove the negative modes of $S$ on two of the states $\phi_1$ and $\phi_2$, say. At each stage in the reduction process, when commuting annihilation modes through creation modes we might again produce modes $S_k$ with $k \leq -h$ but the sum of the conformal weights of the $n$ states decreases at each step and, therefore, we can reduce any correlation function to a functional on $\mathcal{V}_1^0 \otimes \mathcal{V}_2^0 \otimes \mathcal{V}_3^0 \otimes \cdots \otimes \mathcal{V}_N^0$. (This argument is analogous to the argument given in [28] for the case of three representations.)

This argument implies that the number of different solutions is bounded by the dimension of the space $\mathcal{V}_1^0 \otimes \mathcal{V}_2^0 \otimes \mathcal{V}_3^0 \otimes \cdots \otimes \mathcal{V}_N^0$. In general, the dimension of this space need not agree with the multiplicity of the vacuum representation in the $N$-fold fusion product, as there might exist the analogue of the spurious subspace of Nahm [28]. Once we have found all relations, we can determine the dependence of the correlation functions on $z_i$ by solving the system of first order differential equations which are obtained by identifying $\frac{d}{dz_i} \leftrightarrow L_{-1}^{(i)}$.

In practice, since we are mainly interested in the amplitudes of the fundamental fields, we shall follow a slightly different approach. We first solve the first order differential equations arising from the comultiplication [2], and this determines the two- and three-point functions up to a constant. The four-point functions depend then on an arbitrary function of the cross-ratio, and by using the defining null vector for one of the fundamental fields we obtain a higher order differential equation for this function which we then solve.

Non-chiral amplitudes are obtained by considering suitable linear combinations of products of chiral amplitudes. These amplitudes are required to be local, and this will constrain the way in which the various chiral correlation functions can be combined. For example, for a two-point function, locality requires that

$$\langle \phi_1(e^{2\pi i z}, e^{-2\pi i \bar{z}}) \phi_2(0,0) \rangle = \langle \phi_1(z, \bar{z}) \phi_2(0,0) \rangle$$.

On the other hand, because of (2) and the property that $L_0$ can be identified with the scaling generator of the Möbius group, the two-point function has to satisfy

$$z \partial_z \langle \phi_1(z, \bar{z}) \phi_2(0,0) \rangle + \langle L_0 \phi_1(z, \bar{z}) \phi_2(0,0) \rangle + \langle \phi_1(z, \bar{z}) L_0 \phi_2(0,0) \rangle = 0,$$

and the analogous relation for the barred coordinate. We can integrate this differential equation along a circle around the origin, and find

$$\langle \phi_1(e^{-2\pi i z}, e^{2\pi i \bar{z}}) \phi_2(0,0) \rangle = e^{2\pi i (h_1 - h_1 + h_2 - h_2)} \langle e^{2\pi i S} \phi_1(z, \bar{z}) e^{2\pi i S} \phi_2(0,0) \rangle,$$

where $(h_j, \bar{h}_j)$ are the left and right conformal weights of the states $\phi_j$ and $S = L_0^{(n)} - \bar{L}_0^{(n)}$ is the nilpotent part of $L_0 - \bar{L}_0$. The conditions for the two-point function to be local are thus

$$h_1 - \bar{h}_1 + h_2 - \bar{h}_2 \in \mathbb{Z}, \quad \langle S^n \phi_1(z, \bar{z}) S^m \phi_2(0,0) \rangle = 0 \quad \forall n, m \in \mathbb{Z}_{\geq 0}, \ m + n > 0.$$
This has to hold for any combination of \( \phi_1 \) and \( \phi_2 \). Since every \( N \)-point function involving \( \phi_1 \), say, can be expanded in terms of such two-point functions (by defining \( \phi_2 \) to be a suitable contour integral of the product of the remaining \( N - 1 \) fields), it follows that we have to have

\[
h - \bar{h} \in \mathbb{Z}, \quad S\phi = 0,
\]

where \((h, \bar{h})\) are the conformal weights of any non-chiral field \( \phi \).

The triplet algebra whose local conformal field theory we want to construct has four indecomposable chiral representations which close under fusion \([22]\). There are two irreducible representations, \( V_{-1/8} \), generated from an \( su(2) \)-singlet state \( \mu \) of weight \( h = -1/8 \), and \( V_{3/8} \), generated from an \( su(2) \)-doublet \( \nu^\alpha \) of weight \( h = 3/8 \). The corresponding non-chiral irreducible representations are the “diagonal” tensor products

\[
V_{-1/8,-1/8} = V_{-1/8} \otimes \bar{V}_{-1/8}, \quad V_{3/8,3/8} = V_{3/8} \otimes \bar{V}_{3/8}
\]

with cyclic states \( \mu = \mu \otimes \bar{\mu} \) and \( \nu^{\alpha\bar{\alpha}} = \nu^\alpha \otimes \bar{\nu}^{\bar{\alpha}} \), respectively. In this case the above constraint is manifestly satisfied as \( S \equiv 0 \) on \( V_{-1/8,-1/8} \) and \( V_{3/8,3/8} \).

The other two representations are reducible (but indecomposable) and are characterised by the diagrams

Here, each vertex represents a representation of the chiral algebra, and an arrow \( A \rightarrow B \) indicates that the representation \( B \) is in the image of \( A \) under the action of the chiral algebra. In the bottom row, the representations have conformal weight \( h = 0 \), and in the top row \( h = 1 \). The representation \( \mathcal{R}_0 \) is generated from a cyclic vector \( \omega \) of \( h = 0 \), which is a singlet under \( su(2) \). It forms a Jordan block for \( L_0 \) with \( \Omega \), and the defining relations are \([22]\)

\[
L_0^j \omega = \Omega, \quad L_0 \Omega = 0, \\
W_0^a \omega = 0, \quad W_0^a \Omega = 0.
\]

The four states \( L_{-1} \omega \) and \( W_{-1}^a \omega \), collectively denoted by \( X_{-1}^j \omega \), form two doublets under \( su(2) \).

The representation \( \mathcal{R}_1 \) is generated from a doublet \( \phi^\pm \) of weight \( h = 1 \). It has two ground states \( \xi^\pm \) at \( h = 0 \) and another doublet \( \psi^\pm \) at \( h = 1 \) forming an \( L_0 \) Jordan block.
with $\phi^\pm$. The defining relations are [22]

$$
\begin{align*}
L_1\phi^\alpha &= -\xi^\alpha, & W_1^a\phi^\alpha &= t_{\beta}^{\alpha\alpha} \xi^\beta, \\
L_0\phi^\alpha &= \phi^\alpha + \psi^\alpha, & W_0^a\phi^\alpha &= 2t_{\beta}^{\alpha\alpha} \phi^\beta, \\
L_0\xi^\alpha &= 0, & W_0^a\xi^\alpha &= 0, \\
L_{-1}\xi^\alpha &= \psi^\alpha, & W_{-1}^a\xi^\alpha &= t_{\beta}^{\alpha\alpha} \psi^\beta, \\
L_0\psi^\alpha &= \psi^\alpha, & W_0^a\psi^\alpha &= 2t_{\beta}^{\alpha\alpha} \psi^\beta.
\end{align*}
$$

The two states $\xi^\alpha$ form two singlets under $su(2)$ and are thus represented by a pair of vertices in the above diagram.

Suppose now that we want to construct a non-chiral local field corresponding to the tensor product of the chiral states $\psi \otimes \bar{\psi} \in \mathcal{R}_0 \otimes \bar{\mathcal{R}}_0$. In order for this field to be local, the action of $S$ must vanish on $\psi \otimes \bar{\psi}$. In general, however, $S(\psi \otimes \bar{\psi})$ does not vanish in $\mathcal{R}_0 \otimes \bar{\mathcal{R}}_0$, and this must mean that it is only possible to associate local fields to a certain quotient space of $\mathcal{R}_0 \otimes \bar{\mathcal{R}}_0$.

To determine this quotient space, we observe that $S$ commutes with the action of both chiral algebras. This implies that every state which is in the image space of $S$, is in the subrepresentation generated from $S(\omega \otimes \bar{\omega})$, where $\omega \otimes \bar{\omega}$ is the cyclic state for the representation $\mathcal{R}_0 \otimes \bar{\mathcal{R}}_0$ with respect to the action of both chiral algebras. It is therefore clear that the space by which we have to quotient has to contain at least the subspace $\mathcal{N}_{00}$ which is the subrepresentation generated from $S(\omega \otimes \bar{\omega}) = \Omega \otimes \bar{\omega} - \omega \otimes \Omega$. For $\mathcal{R}_1 \otimes \bar{\mathcal{R}}_1$, the situation is analogous, and $\mathcal{N}_{11}$ is generated from $\phi^\alpha \otimes \bar{\psi}^\alpha - \psi^\alpha \otimes \bar{\phi}^\alpha$. The maximal (non-chiral) representations which can correspond to local fields are thus of the form

$$
\begin{align*}
\mathcal{R}_{00} &= (\mathcal{R}_0 \otimes \bar{\mathcal{R}}_0) / \mathcal{N}_{00}, & \mathcal{R}_{11} &= (\mathcal{R}_1 \otimes \bar{\mathcal{R}}_1) / \mathcal{N}_{11};
\end{align*}
$$

their structure can be summarised schematically as

Here $\omega$ is the equivalence class of states in $\mathcal{R}_{00}$ which contains as a representative $(\omega \otimes \bar{\omega})$, and we have $\Omega = L_0\omega = \bar{L}_0\omega$ (as $S\omega = 0$). Likewise, $\phi^{\alpha\bar{\alpha}}$ is the equivalence class in $\mathcal{R}_{11}$.
with representative \((\phi^\alpha \otimes \bar{\phi}^\alpha)\). We shall always use the convention that non-chiral states are denoted by bold symbols.

This solves the locality constraints for two-point functions. For higher point amplitudes, locality will require that we combine only suitable combinations of chiral correlation functions. This will eliminate already a large number of possibilities, but in order to constrain the amplitudes further, we have to turn to different considerations relating to the spectrum of the non-chiral theory.

3 The spectrum of the theory and the operator product expansion

The considerations of the previous section only determine the (chiral) correlation functions and the (non-chiral) amplitudes up to some integration constants. In terms of the chiral theory alone, there is no way of obtaining further restrictions on these constants, but for the non-chiral theory there are further constraints which come about as follows. The local fields of the non-chiral theory are well-defined operators on the whole space of non-chiral states, and are in one-to-one correspondence with these states. (Indeed, given a complete set of amplitudes, we can reconstruct a dense subspace of states as the natural quotient space\(^1\) of the free vector space of formal products of fields, and for this dense subspace there exists a one-to-one correspondence between states and fields. The actual space of states is then defined as the weak completion, using the natural weak topology induced by the amplitudes \([25]\).) This construction is by the way not possible for a chiral theory as the chiral amplitudes carry additional labels (specifying the different “fusion channels”).

It therefore makes sense to ask how large the space of states is, and in particular, what the multiplicities of the various subsectors are. This will depend crucially on the various normalisation constants of the amplitudes, and for a generic choice, these multiplicities will all be infinite. It is therefore an interesting question whether there exists a special solution for the amplitudes for which the multiplicities are finite. This is, in particular, necessary for a finite partition function, and only in this case can we expect to obtain a modular invariant theory.

The simplest situation (which is the one which shall be relevant in the following) arises if all sectors of the theory appear with multiplicity one, so that every field is uniquely characterised by its representation properties with respect to the chiral algebra (and there are no additional labels referring to the different copies of the given representation). Using the action of the meromorphic fields, every amplitude can be rewritten in terms of those that only involve the fundamental fields, i.e. the fields whose corresponding states are the fundamental states. Furthermore, it is sufficient to know the two- and three-point functions of the fundamental fields, as these determine already the operator product expansions (OPEs), from which higher amplitudes can be constructed by the so-called gluing process.

Conversely, given a set of two- and three-point functions of the fundamental fields, we can use these to construct higher amplitudes. In general, there are different ways

\(^1\)We quotient by all states whose amplitudes vanish identically.
of obtaining a given \( n \)-point function, and a priori, it is not clear whether the different functions so constructed agree. A sufficient (and necessary) condition for them to agree is that they do so in the case of the four-point functions, in which case the condition is usually referred to as duality. (This is sufficient, as every decomposition of the sphere into “pairs of pants” can be related to any other one by a succession of “simple moves”, where each simple move only involves four fields.) In order to construct a theory with trivial multiplicities, it is therefore sufficient to determine the two-, three- and four-point functions of the fundamental fields, and show that they are local and satisfy the duality relations; this is what we shall do in the following.

In order to determine the \( n \)-point functions involving the reducible representations from those of the irreducible ones, it is useful to determine the operator product expansions (OPEs) of the fundamental fields explicitly. These (non-chiral) OPEs are effectively built from the two chiral OPEs corresponding to the chiral representations, and their information in turn is encoded in the chiral fusion rules. For example, the (chiral) fusion product of \( \mu \) with itself contains only the representation \( R_0 \), which, at lowest level, has a two-dimensional Jordan block for \( L_0 \). Since \( L_0 \) can be identified with the scale generator of the Möbius transformations it follows that the chiral OPE for \( \mu(x) \mu \) has a power series expansion of the form (compare [20])

\[
\mu(x)\mu = x^{\frac{1}{4}} \sum_{n=0}^{\infty} x^n (X_n + Y_n \ln x),
\]

where \( X_n, Y_n \) are states in \( R_0 \) of weight \( n \). The states for \( n > 1 \) are in the representation generated by those at \( n = 0 \), and so by acting with positive modes on them, using the comultiplication relations, we can find recursive relations for \( X_n, Y_n \) with \( n > 0 \) in terms of \( X_0 \) and \( Y_0 \). It is therefore sufficient to determine only the lowest level terms.

Without loss of generality, we can identify \( X_0 = \omega \) with a cyclic vector of \( R_0 \). Then, the chiral OPE is of the form

\[
x^{-\frac{1}{4}} \mu(x)\mu = \omega + \Omega \ln x + \frac{1}{2} L_{-1} \omega x + \left( \frac{13}{16} L_{-1}^2 \omega - \frac{3}{8} L_{-2} \omega + \frac{19}{8} L_{-2} \Omega - \frac{3}{8} L_{-2} \Omega \ln x \right) x^2 + \cdots,
\]

where \( \Omega = L_0 \omega \) and thus \( L_0 \Omega = 0 \).

In a similar way we can determine the other chiral OPEs. The only slight complication arises due to the \( su(2) \) tensorial structure; for a definition of the various \( su(2) \) tensors see appendix D.

\[
x^{\frac{1}{4}} \mu(x)\nu^\alpha = \xi^\alpha + \left( \frac{1}{2} \phi^\alpha + \frac{1}{2} \psi^\alpha \ln x \right) x + \left( \frac{1}{8} L_{-1} \phi^\alpha - \frac{3}{16} L_{-1} \psi^\alpha + \frac{1}{4} L_{-2} \xi^\alpha + \frac{1}{8} L_{-1} \psi^\alpha \ln x \right) x^2 + \cdots,
\]
\[ x^\frac{1}{2} \nu^\alpha(x) \nu^\beta = d^\alpha\beta \left( \omega' + \Omega \ln x + \frac{1}{2} L_{-1} \omega' x + \left( \frac{13}{16} L_{-1}^2 \omega' - \frac{3}{8} L_{-2} \omega' + \frac{19}{8} L_{-2} \Omega - \frac{3}{8} L_{-2} \ln x \right) x^2 + \cdots \right) + t^\alpha\beta \left( 4 W^a_\omega x + W^a \omega^2 x^2 + \cdots \right). \]

The vector \( \omega' \) is a cyclic vector of the representation \( R_0 \), and \( \Omega = L_0 \omega' \). Here we have included a prime for \( \omega \) as the cyclic vector of the representation \( R_0 \) is not uniquely fixed. Apart from the usual freedom of rescaling \( \omega \), we can always add to a given cyclic vector \( \omega \) a multiple of \( \Omega = L_0 \omega' \). (This is a special feature of not completely reducible representations — if the fusion product is completely reducible, because of Schur’s lemma, the only remaining freedom is a constant for each irreducible component.)

The (non-chiral) OPEs are obtained by multiplying the chiral OPEs. However, if the corresponding theory is indeed local, only the states in \( R_{j\bar{j}} \) can occur in the OPE. To first order in \( x \) we therefore find (the expressions to second order in \( x \) and \( \bar{x} \) are listed in appendix A)

\[ |x|^{-\frac{1}{2}} \mu(x) \mu = \omega + \ln |x|^2 \Omega + \cdots, \]
\[ |x|^{\frac{1}{2}} \mu(x) \nu^\alpha = \xi^{\alpha\bar{\alpha}} + \frac{1}{2} \rho^{\alpha\bar{\alpha}} x + \frac{1}{2} \rho^{\alpha\bar{\alpha}} \bar{x} + \frac{1}{4} \left( \phi^{\alpha\bar{\alpha}} + \ln |x|^2 \psi^{\alpha\bar{\alpha}} \right) |x|^2 + \cdots, \]
\[ |x|^{\frac{1}{2}} \nu^\alpha(x) \nu^\beta = -\frac{1}{4} d^\alpha\beta d^{\alpha\bar{\beta}} \left( \omega' + (\ln |x|^2 + 4) \Omega' + \cdots \right). \]

In the last OPE we have made use of the freedom to redefine \( \omega' \) and \( \Omega' \) for later convenience. We can regard these OPEs as defining the states \( \omega, \Omega \) and \( \omega', \Omega' \), and it is therefore not clear whether \( \omega = \omega' \) and \( \Omega = \Omega' \). Indeed, we do not yet know whether all amplitudes involving \( \omega \) agree with those involving \( \omega' \), and similarly for \( \Omega \) and \( \Omega' \).

4 Amplitudes

In this section we shall construct the relevant amplitudes explicitly. As a first step we fix the normalisation of the two-point functions of the irreducible fundamental fields

\[ \langle \mu(z_1) \mu(z_2) \rangle = D |z_{12}|^{1/2}, \]
\[ \langle \nu^{\alpha\bar{\alpha}}(z_1) \nu^{\beta\bar{\beta}}(z_2) \rangle = -\frac{1}{4} D d^{\alpha\beta} d^{\alpha\bar{\beta}} |z_{12}|^{-3/2}, \]

where \( z_{12} = z_1 - z_2 \). We have also included a factor of \(-1/4\) for later convenience. We can then use the OPEs (3, 5) to deduce the one-point functions of the reducible representations

\[ \langle \Omega \rangle = \langle \Omega' \rangle = \langle \xi^{\alpha\bar{\alpha}} \rangle = 0, \quad \langle \omega \rangle = D, \quad \langle \omega' \rangle = D. \]
Next, we calculate the chiral four-point functions involving four irreducible fields, following the strategy of section 2. We then form the most general linear combination of two such four-point functions which give rise to local amplitudes

\[
\langle \mu(z_1) \mu(z_2) \mu(z_3) \mu(z_4) \rangle = -\pi C_0 \left| \frac{z_{12}z_{34}z_{14}z_{23}}{z_{13}z_{24}} \right|^{\frac{1}{2}} \left[ K(x)\bar{K}(\bar{x}) + \bar{K}(x)K(\bar{x}) \right],
\]

\[
\langle \mu(z_1) \mu(z_2) \mu(z_3) \nu^{\alpha\dot{\alpha}}(z_4) \rangle = -B_0^{\alpha\dot{\alpha}} \left| \frac{z_{12}z_{13}z_{23}}{z_{14}z_{24}z_{34}} \right|^{\frac{1}{2}},
\]

\[
\langle \mu(z_1) \mu(z_2) \nu^{\alpha\dot{\alpha}}(z_3) \nu^{\beta\dot{\beta}}(z_4) \rangle = \frac{\pi}{4} C_1 d^{\alpha\beta} d^{\alpha\dot{\beta}} \left| \frac{z_{12}z_{13}z_{24}}{z_{14}z_{23}z_{34}} \right|^{\frac{1}{2}} \left[ E(x)\bar{D}_2(x) + \bar{D}_2(x)E(x) \right],
\]

\[
\langle \nu^{\alpha\dot{\alpha}}(z_1) \nu^{\beta\dot{\beta}}(z_2) \nu^{\gamma\dot{\gamma}}(z_3) \nu^{\delta\dot{\delta}}(z_4) \rangle = \frac{\pi}{16} C_2 \left| \frac{z_{13}z_{24}}{z_{12}z_{34}z_{14}z_{23}} \right|^{\frac{1}{2}} \times \left( \delta^{\alpha\dot{\alpha}} \delta^{\beta\dot{\beta}} \delta^{\gamma\dot{\gamma}} \delta^{\delta\dot{\delta}} - d^{\alpha\dot{\alpha}} d^{\beta\dot{\beta}} \delta^{\gamma\dot{\gamma}} \delta^{\delta\dot{\delta}} \right),
\]

The functions \( K, E, D_2, F_1, F_2 \) and their \((\cdot)\) counterparts are related to complete elliptic integrals (see appendix C). \( C_0, C_1, C_2, B_0^{\alpha\dot{\alpha}} \) and \( B_1^{\alpha\dot{\alpha}} \) are (at this stage) arbitrary constants. The factors of \( \pi \) were introduced for later convenience. The cross-ratio \( x \) is defined by \( x = (z_{12}z_{34})/(z_{13}z_{24}) \), where as always from now on \( z_{ij} = z_i - z_j \). The above expressions are well-defined for \( x \) near the origin. Using the formulae of appendix C, we can analytically continue the amplitudes to different values for \( x \) which can then be written in terms of amplitudes where the fields are in a different order (and the corresponding cross-ratio is again small); the relevant expressions are given in appendix E.

Next, we use the OPEs (3 - 5) to determine the non-chiral three-point functions of two irreducible representations and one reducible representation. In general, there are different limits of a given four-point amplitude that should give rise to the same three-point amplitude, and the duality relations require that all these three-point functions are indeed the same. For the situation in question, this follows manifestly from the expressions above and the ones of appendix E.
Let us consider first the states of weight $(0, 0)$ in the reducible representation.

\[
\langle \mu(z_1)\mu(z_2)\omega(z_3) \rangle = -C_0|z_{12}|^\frac{1}{2} \left( 8 \ln 2 + \ln \left| \frac{z_{13}z_{23}}{z_{12}} \right|^2 \right),
\]

\[
\langle \mu(z_1)\mu(z_2)\omega'(z_3) \rangle = -C_1|z_{12}|^\frac{1}{2} \left( 8 \ln 2 + \ln \left| \frac{z_{13}z_{23}}{z_{12}} \right|^2 \right),
\]

\[
\langle \mu(z_1)\mu(z_2)\Omega(z_3) \rangle = C_0|z_{12}|^\frac{1}{2},
\]

\[
\langle \mu(z_1)\mu(z_2)\Omega'(z_3) \rangle = C_1|z_{12}|^\frac{1}{2},
\]

\[
\langle \mu(z_1)\mu(z_2)\xi^{\gamma\tilde{\gamma}}(z_3) \rangle = -B_0^{\gamma\tilde{\gamma}}|z_{12}|^\frac{1}{2},
\]

\[
\langle \nu^{\alpha\tilde{\alpha}}(z_1)\nu^{\beta\tilde{\beta}}(z_2)\omega(z_3) \rangle = \frac{1}{4}C_1a^{\alpha\beta}d^{\alpha\beta}|z_{12}|^{-\frac{3}{2}} \left( 8 \ln 2 - 4 + \ln \left| \frac{z_{13}z_{23}}{z_{12}} \right|^2 \right),
\]

\[
\langle \nu^{\alpha\tilde{\alpha}}(z_1)\nu^{\beta\tilde{\beta}}(z_2)\omega'(z_3) \rangle = \frac{1}{4}C_2a^{\alpha\beta}d^{\alpha\beta}|z_{12}|^{-\frac{3}{2}} \left( 8 \ln 2 - 4 + \ln \left| \frac{z_{13}z_{23}}{z_{12}} \right|^2 \right),
\]

\[
\langle \nu^{\alpha\tilde{\alpha}}(z_1)\nu^{\beta\tilde{\beta}}(z_2)\Omega(z_3) \rangle = -\frac{1}{4}C_1a^{\alpha\beta}d^{\alpha\beta}|z_{12}|^{-\frac{3}{2}},
\]

\[
\langle \nu^{\alpha\tilde{\alpha}}(z_1)\nu^{\beta\tilde{\beta}}(z_2)\Omega'(z_3) \rangle = -\frac{1}{4}C_2a^{\alpha\beta}d^{\alpha\beta}|z_{12}|^{-\frac{3}{2}},
\]

\[
\langle \nu^{\alpha\tilde{\alpha}}(z_1)\nu^{\beta\tilde{\beta}}(z_2)\xi^{\gamma\tilde{\gamma}}(z_3) \rangle = \frac{1}{4}B_1^{\gamma\tilde{\gamma}}a^{\alpha\beta}d^{\alpha\beta}|z_{12}|^{-\frac{3}{2}}.
\]

The fields $\omega'$ and $\Omega'$ have the same functional behaviour as $\omega$ and $\Omega$, respectively, and only the relative normalisation of the three-point functions differs. This was to be expected since they correspond to isomorphic representations. However, from the point of view of the non-chiral theory, the state $\omega$ (and likewise for $\Omega$, etc.) will appear with non-trivial multiplicity unless

\[
\frac{C_1}{C_0} = \frac{C_2}{C_1} = \frac{B_0^{\alpha\tilde{\alpha}}}{B_0^{\alpha\tilde{\alpha}}} \equiv 1,
\]

in which case $\omega' = \omega, \Omega' = \Omega$. Since we want to construct a theory with trivial multiplicities, we therefore choose

\[
C_2 = C_1 = C_0, \quad B_0^{\alpha\tilde{\alpha}} = B_1^{\alpha\tilde{\alpha}} = \Theta^{\alpha\tilde{\alpha}}C_0.
\]

Apart from the identification of $\mathcal{R}'_{0,0}$ with $\mathcal{R}_{0,0}$, the results also imply

\[
\xi^{\alpha\tilde{\alpha}} = -\Theta^{\alpha\tilde{\alpha}}\Omega.
\]
At level \((1, 1)\) of the reducible representation we find

\[
\langle \mu(z_1) \mu(z_2) X_{-1}^2 X_{-1}^j \omega(z_3) \rangle = 0,
\]

\[
\langle \mu(z_1) \mu(z_2) \phi^{\gamma \gamma}(z_3) \rangle = 0,
\]

\[
\langle \mu(z_1) \nu^{\beta \beta}(z_2) X_{-1}^j X_{-1}^j \omega(z_3) \rangle = -\frac{1}{4} b^\alpha_\beta b^\beta_\gamma C_0|z_{12}|^2 |z_{13}|^{-1} |z_{23}|^{-3},
\]

\[
\langle \mu(z_1) \nu^{\beta \beta}(z_2) \phi^{\gamma \gamma}(z_3) \rangle = -\frac{1}{4} d^\beta \gamma d^\beta \gamma C_0|z_{12}|^2 |z_{13}|^{-1} |z_{23}|^{-3},
\]

\[
\langle \nu^{\alpha \alpha}(z_1) \nu^{\beta \beta}(z_2) X_{-1}^j X_{-1}^j \omega(z_3) \rangle = 0,
\]

\[
\langle \nu^{\alpha \alpha}(z_1) \nu^{\beta \beta}(z_2) \phi^{\gamma \gamma}(z_3) \rangle = 0,
\]

where

\[
b^\alpha_\beta = (\delta^\alpha_\beta, -t^\alpha_\beta), \quad \text{for} \ X^j = (L, W^a).
\]

The three-point functions with \(\omega'\) are the same as for \(\omega\). Furthermore, we note that we can identify (at least on the level of the above amplitudes)

\[
X_{-1}^j X_{-1}^j \omega = b^\gamma_\beta \gamma \beta \Theta^\gamma \phi^{\gamma \gamma},
\]

where \(b^i_\beta = b^i_\beta d_{i\gamma}\). The other states at level \((1, 1)\) are \(\phi^{\gamma \gamma}\) with three-point functions

\[
\langle \mu(z_1) \mu(z_2) \phi^{\gamma \gamma}(z_3) \rangle = -\frac{1}{4} \Theta^{\gamma \gamma} C_0 |z_{12}|^2 \left| \frac{z_{13} + z_{23}}{z_{13} z_{23}} \right|^2,
\]

\[
\langle \mu(z_1) \nu^{\beta \beta}(z_2) \phi^{\gamma \gamma}(z_3) \rangle = \frac{1}{4} d^\beta \gamma d^\gamma \gamma C_0 |z_{12}|^2 |z_{13}|^{-1} |z_{23}|^{-3} \times
\]

\[
\left( 8 \ln 2 + \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 + \frac{2 z_{13} + z_{23}}{z_{12}} + \frac{2 z_{13} + z_{23}}{z_{12}} \right),
\]

\[
\langle \nu^{\alpha \alpha}(z_1) \nu^{\beta \beta}(z_2) \phi^{\gamma \gamma}(z_3) \rangle = \frac{1}{4} C_0 |z_{12}|^{-\frac{3}{2}} |z_{13} z_{23}|^{-1} \times
\]

\[
\left( \frac{1}{4} d^\alpha \beta d^\beta \gamma \Theta^{\gamma \gamma} |z_{13} + z_{23}|^2 + d^\alpha \beta h^\gamma_\eta \Theta^{\gamma \gamma} |z_{13} + z_{23}||z_{12}|
\]

\[
+ h^\alpha \beta \gamma d^\alpha \beta \Theta^{\gamma \gamma} |z_{12} + z_{23}|^2 + 4 h^\alpha \beta \gamma h^\beta \gamma \Theta^{\gamma \gamma} |z_{12}|^2 \right).
\]

Our choice of the normalisation constants \(b\) so far guarantees that the irreducible sub-representations of \(R_{00}\) and \(R_{11}\) (which are at level \((0, 0)\) and \((1, 1)\)) only have trivial multiplicity in the non-chiral theory. Let us analyse now how many states appear at level \((1, 0)\). Using the OPEs \(\{3 - 5\}\) together with the four-point functions of the irreducible representations we find

\[
\langle \mu(z_1) \mu(z_2) L_{-1} \omega(z_3) \rangle = C_0 |z_{12}|^2 \left| \frac{z_{13} + z_{23}}{z_{13} z_{23}} \right|^2,
\]

\[
\langle \mu(z_1) \mu(z_2) W^a_{-1} \omega(z_3) \rangle = 0,
\]

\[
\langle \mu(z_1) \mu(z_2) \rho^{\gamma \gamma}(z_3) \rangle = \frac{1}{2} \Theta^{\gamma \gamma} C_0 |z_{12}|^\frac{1}{2} \left| \frac{z_{13} + z_{23}}{z_{13} z_{23}} \right|^2,
\]

13
\[
\langle W_{-1}^a \mu(z_1) \mu(z_2) L_{-1} \omega(z_3) \rangle = 0 ,
\]
\[
\langle W_{-2}^a \mu(z_1) \mu(z_2) W_{-1}^b \omega(z_3) \rangle = \frac{1}{2} g^{\alpha \beta} C_0 |z_{12}|^4 z_{13} z_{23} z_{12}^{-1} ,
\]
\[
\langle W_{-2}^a \mu(z_1) \mu(z_2) \rho^{\gamma \delta}(z_3) \rangle = t^\gamma_\eta \Theta^{\eta \gamma} C_0 |z_{12}|^4 z_{13} z_{23} z_{12}^{-1} ,
\]
\[
\langle \mu(z_1) \nu^{\alpha \delta}(z_2) X^i_{-1} \omega(z_3) \rangle = -\frac{1}{2} b^\beta \Theta^\alpha_{\beta \gamma} C_0 |z_{12}|^4 z_{13} z_{23} z_{12}^{-1} ,
\]
\[
\langle \mu(z_1) \nu^{\alpha \delta}(z_2) \rho^{\gamma \delta}(z_3) \rangle = -\frac{1}{2} d\xi d^\alpha \Theta^{\eta \gamma} C_0 |z_{12}|^4 z_{13} z_{23} z_{12}^{-1} ,
\]
\[
\langle \nu^{\alpha \delta}(z_1) \nu^{\beta \delta}(z_2) L_{-1} \omega(z_3) \rangle = -\frac{1}{4} d^\alpha d^\beta \Theta^{\eta \gamma} C_0 |z_{12}|^4 z_{13} z_{23} z_{12}^{-1} ,
\]
\[
\langle \nu^{\alpha \delta}(z_1) \nu^{\beta \delta}(z_2) W_{-1}^\gamma \omega(z_3) \rangle = -\frac{1}{2} d\xi d^\beta \Theta^{\eta \gamma} C_0 |z_{12}|^4 z_{13} z_{23} z_{12}^{-1} ,
\]
\[
\langle \nu^{\alpha \delta}(z_1) \nu^{\beta \delta}(z_2) \rho^{\gamma \delta}(z_3) \rangle = -\frac{1}{8} d\xi d^\beta \Theta^{\eta \gamma} C_0 |z_{12}|^4 z_{13} z_{23} z_{12}^{-1} -\frac{1}{2} h^\alpha_{\beta \gamma} d^\beta \Theta^{\eta \gamma} C_0 |z_{12}|^4 z_{13} z_{23} z_{12}^{-1} .
\]

We observe that the states \( X_{-1}^j \omega \) and \( \rho^{\gamma \delta} \) are linearly dependent provided that
\[
det \Theta = 1 .
\]

As we shall show below this constraint is also required by the consistency of the amplitude \( \langle \mu \mu \omega \omega \rangle \). We therefore have
\[
\rho^{\gamma \delta} = \frac{1}{2} \Theta^{\gamma \delta} L_{-1} \omega + 2 t^\gamma_\alpha \Theta^\alpha_{\gamma \delta} W_{-1}^\alpha \omega
\]
and
\[
L_{-1} \omega = \tilde{\Theta}_{\alpha \alpha} \rho^{\alpha \delta} , \quad W_{-1}^\alpha \omega = t^\alpha_\beta \tilde{\Theta}_{\alpha \beta} \rho^{\beta \alpha} ,
\]
where \( t^\gamma_\alpha = g_{\alpha \beta} b^\gamma_{\beta} \) and \( \tilde{\Theta} \) is the inverse of \( \Theta \),
\[
\tilde{\Theta}_{\alpha \alpha} = \frac{1}{\det \Theta} \begin{pmatrix} \Theta^{--} & -\Theta^{+-} \\ -\Theta^{-+} & \Theta^{++} \end{pmatrix} , \quad \Theta^{\alpha \delta} \tilde{\Theta}_{\alpha \delta} = \delta^\gamma_\gamma , \quad \tilde{\Theta}_{\alpha \alpha} \Theta^{\alpha \beta} = \delta^\beta_\beta .
\]

In the same way we can express \( \tilde{\rho}^{\gamma \delta} \) in terms of \( \tilde{X}_{-1}^j \omega \) and vice versa. This implies that the two reducible representations combine to form one representations \( \mathcal{R} \) whose structure is summarised in the following diagram:
The representation $\mathcal{R}$ does not have one cyclic state, but instead is generated from a state $\omega$ of weight $(0,0)$ and the four states $\phi^{\alpha\bar{\alpha}}$ of weight $(1,1)$. The non-trivial defining relations are

\begin{align*}
L_0 \omega &= \Omega, & W_0^a \omega &= 0, \\
L_0 \Omega &= 0, & W_0^a \Omega &= 0, \\
L_{-1} \omega &= \tilde{\Theta}^{\alpha\alpha} \rho^{\bar{\alpha}\bar{\alpha}}, & W_{-1}^a \omega &= i^{\alpha\bar{\alpha}} \tilde{\Theta}^{\alpha\alpha} \rho^{\bar{\alpha}\bar{\alpha}}, \\
L_0 \rho^{\alpha\bar{\alpha}} &= \rho^{\alpha\bar{\alpha}}, & W_0^a \rho^{\bar{\alpha}\bar{\alpha}} &= 2t_{\beta}^{\alpha\bar{\alpha}} \rho^{\beta\bar{\beta}}, \\
L_1 \rho^{\alpha\bar{\alpha}} &= \Theta^{\alpha\alpha} \Omega, & W_1^a \rho^{\bar{\alpha}\bar{\alpha}} &= i^{\alpha\bar{\alpha}} \Theta^{\alpha\alpha} \Omega, \\
L_{-1} \rho^{\alpha\bar{\alpha}} &= \psi^{\alpha\bar{\alpha}}, & W_{-1}^a \rho^{\bar{\alpha}\bar{\alpha}} &= i^{\alpha\bar{\alpha}} \psi^{\bar{\alpha}\bar{\alpha}}, \\
L_0 \phi^{\alpha\bar{\alpha}} &= \phi^{\alpha\bar{\alpha}} + \psi^{\alpha\bar{\alpha}}, & W_0^a \phi^{\bar{\alpha}\bar{\alpha}} &= 2t_{\beta}^{\alpha\bar{\alpha}} \phi^{\beta\bar{\beta}}, \\
L_1 \phi^{\alpha\bar{\alpha}} &= \psi^{\alpha\bar{\alpha}}, & W_1^a \phi^{\bar{\alpha}\bar{\alpha}} &= 2t_{\beta}^{\alpha\bar{\alpha}} \psi^{\beta\bar{\beta}}, \\
L_{-1} \phi^{\alpha\bar{\alpha}} &= -\rho^{\alpha\bar{\alpha}}, & W_{-1}^a \phi^{\bar{\alpha}\bar{\alpha}} &= -i^{\alpha\bar{\alpha}} \rho^{\bar{\alpha}\bar{\alpha}},
\end{align*}


\begin{align*}
\langle \omega \omega \rangle &= -C_0 \left( 8 \ln 2 + 2 \ln |z_{12}|^2 \right), \\
\langle \omega \Omega \rangle &= C_0, \\
\langle \Omega \Omega \rangle &= 0, \\
\langle \omega \rho^{\beta\bar{\beta}} \rangle &= \Theta^{\beta\bar{\beta}} C_0 z_{12}^{-1}, \\
\langle \omega \rho^{\bar{\beta}\beta} \rangle &= \Theta^{\beta\bar{\beta}} C_0 z_{12}^{-1}, \\
\langle \omega \phi^{\beta\bar{\beta}} \rangle &= -\Theta^{\beta\bar{\beta}} C_0 |z_{12}|^{-2}, \\
\langle \rho^{\alpha\bar{\alpha}} \rho^{\beta\bar{\beta}} \rangle &= -\delta^{\alpha\beta} \delta^{\bar{\alpha}\bar{\beta}} C_0 z_{12}^{-2}, \\
\langle \rho^{\alpha\bar{\alpha}} \rho^{\bar{\beta}\beta} \rangle &= 0, \\
\langle \rho^{\alpha\bar{\alpha}} \phi^{\beta\bar{\beta}} \rangle &= \delta^{\alpha\beta} \delta^{\bar{\alpha}\bar{\beta}} C_0 z_{12}^{-2} z_{12}^{-1},
\end{align*}

together with their anti-chiral counterparts.

Given the above three-point functions we can now apply the OPEs (3 – 5) again, and deduce the two-point functions for the reducible representations $\mathcal{R}$. We find

\begin{align*}
\langle \omega \omega \rangle &= -C_0 \left( 8 \ln 2 + 2 \ln |z_{12}|^2 \right), \\
\langle \omega \Omega \rangle &= C_0, \\
\langle \Omega \Omega \rangle &= 0, \\
\langle \omega \rho^{\beta\bar{\beta}} \rangle &= \Theta^{\beta\bar{\beta}} C_0 z_{12}^{-1}, \\
\langle \omega \rho^{\bar{\beta}\beta} \rangle &= \Theta^{\beta\bar{\beta}} C_0 z_{12}^{-1}, \\
\langle \omega \phi^{\beta\bar{\beta}} \rangle &= -\Theta^{\beta\bar{\beta}} C_0 |z_{12}|^{-2}, \\
\langle \rho^{\alpha\bar{\alpha}} \rho^{\beta\bar{\beta}} \rangle &= -\delta^{\alpha\beta} \delta^{\bar{\alpha}\bar{\beta}} C_0 z_{12}^{-2}, \\
\langle \rho^{\alpha\bar{\alpha}} \rho^{\bar{\beta}\beta} \rangle &= 0, \\
\langle \rho^{\alpha\bar{\alpha}} \phi^{\beta\bar{\beta}} \rangle &= \delta^{\alpha\beta} \delta^{\bar{\alpha}\bar{\beta}} C_0 z_{12}^{-2} z_{12}^{-1},
\end{align*}
\[ \langle \rho^\alpha \psi^\beta \rangle = 0, \]
\[ \langle \phi^\alpha \phi^\beta \rangle = d^\alpha \beta C_0 |z_{12}|^{-4} (8 \ln 2 + 2 \ln |z_{12}|^2), \]
\[ \langle \phi^\alpha \psi^\beta \rangle = -d^\alpha \beta d^\alpha \beta C_0 |z_{12}|^{-4}, \]
\[ \langle \psi^\alpha \psi^\beta \rangle = 0. \]

We can also determine further OPEs from the three-point functions involving one reducible and two irreducible fields. For example, it follows from the amplitudes \( \langle \mu \mu \omega \rangle \) and \( \langle \mu \nu^{\alpha \beta} \omega \rangle \) that the OPE of \( \mu \) with \( \omega \) takes the form

\[ \frac{D}{C_0} \mu(x) \omega = -(8 \ln 2 + 2 \ln |x|^2)\mu + 4 \Theta^{\alpha \beta} d_{\alpha \beta} \nu^{\alpha \beta} |x| + \cdots. \]

Other OPEs can be determined similarly and can be found in appendix A.

Next, we consider the four-point functions of two irreducible and two reducible representations. The simplest case is the amplitude \( \langle \mu \mu \omega \omega \rangle \) which, after imposing the locality constraints, takes the form

\[ |z_{12}|^{-\frac{3}{2}} \langle \mu \mu \omega \omega \rangle = A_2 + A_1 \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 + A_1 \ln \left| \frac{z_{14} z_{24}}{z_{12}} \right|^2 \]
\[ + A_0 \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \ln \left| \frac{z_{14} z_{24}}{z_{12}} \right|^2 - A_0 \left( \ln \frac{\sqrt{z_{13} z_{23}} - \sqrt{z_{14} z_{24}}}{\sqrt{z_{13} z_{23}} + \sqrt{z_{14} z_{24}}} \right)^2, \]

where \( A_i \) are at this stage arbitrary constants. In order to determine these we impose the duality relations, i.e. we use the different OPEs to relate this function to three-point functions which we have already determined. For example, using the above OPE of \( \mu \) with \( \omega \) we find to lowest order in \( x \)

\[ \langle \mu(z_1) \mu(z_2 + x) \omega(z_2) \omega(z_3) \rangle \sim -(8 \ln 2 + 2 \ln |x|^2) \frac{C_0}{D} \langle \mu(z_1) \mu(z_2) \omega(z_3) \rangle \]
\[ + 4 \frac{C_0}{D} \Theta^{\alpha \beta} d_{\alpha \beta} \langle \mu(z_1) \nu^{\beta} (z_2) \omega(z_3) \rangle |x|. \]

This is to be compared with the same limit of the above four-point function

\[ |z_{12}|^{-\frac{3}{2}} \langle \mu(z_1) \mu(z_2 + x) \omega(z_2) \omega(z_3) \rangle \sim A_2 + A_1 \left( \ln |x|^2 \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \right) \]
\[ + A_0 \left( \ln |x|^2 \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 - 8 \frac{z_{13}}{z_{12} z_{23}} |x| \right), \]

and we thus find that

\[ A_i = (8 \ln 2)^i \frac{C_0^2}{D} \quad \text{and} \quad \Theta^{\alpha \beta} d_{\alpha \beta} \Theta^{\alpha \beta} = 2. \]

Since \( \Theta^{\alpha \beta} d_{\alpha \beta} \Theta^{\alpha \beta} = 2 \det(\Theta) \) the second constraint requires \( \det(\Theta) = 1 \) which is what we required previously (8).
Similarly, we can determine the other three- and four-point functions of the fundamental fields, and check their consistency with duality; some of these amplitudes are explicitly given in appendix B. Unfortunately, the expressions become rather complicated, and it is not feasible to determine all of the amplitudes involving arbitrary combinations of the fundamental fields explicitly. However, we have determined all the relevant three-point functions (see the above formulae and appendix B), and this specifies the theory uniquely. The question reduces then to whether the theory so specified is indeed consistent; this will be answered in the affirmative by relating the theory (with this choice of normalisation constants) to a free field theory that is consistent \[30\]. This free field theory will be described in the following section.

5 Symplectic fermions

In this section we shall show that the conformal field theory that we have discussed so far is the bosonic sector of a model of free “symplectic” fermions. Here we shall only summarise the essential features of this fermion model; further details may be found in \[30\] and \[3\].

The chiral algebra of the symplectic fermion model is generated by a two-component fermion field $\chi^\alpha$ of conformal weight one whose anti-commutator is given by

$$\{\chi^\alpha_m, \chi^\beta_n\} = md^{\alpha\beta}\delta_{m+n},$$

where $d^{\alpha\beta}$ is the same anti-symmetric tensor as before. This algebra has a unique irreducible highest weight representation generated from a highest weight state $\Omega$ satisfying $\chi^\alpha_m\Omega = 0$ for $m \geq 0$. We may call this module the vacuum representation. It contains the triplet $W$-algebra since

$$L_{-2}\Omega = \frac{1}{2}d_{\alpha\beta}\chi_{-1}^\alpha\chi_{-1}^\beta\Omega,$$

$$W_{-3}^a\Omega = t_{a\beta}\chi_{-2}^\alpha\chi_{-1}^\beta\Omega$$

satisfy the triplet algebra \[3\,31\].

Let us consider the maximal generalised highest weight representation of this chiral algebra that contains the vacuum representation. It is freely generated by the negative modes $\chi^\alpha_m, m < 0$ from a four dimensional space of ground states. This space is spanned by two bosonic states $\Omega$ and $\omega$, and two fermionic states, $\theta^\alpha$, and the action of the zero-modes $\chi^\alpha_0$ is given as

$$\chi^\alpha_0\omega = -\theta^\alpha,$$

$$\chi^\alpha_0\theta^\beta = d^{\alpha\beta}\Omega,$$

$$L_0\omega = \Omega.$$

Imposing the locality constraints as in section \[3\], the corresponding non-chiral representation is freely generated by the negative modes $\chi^\alpha_m, \bar{\chi}^\bar{\alpha}_m, m < 0$ from the ground space representation

$$\chi^\alpha_0\omega = -\theta^\alpha,$$

$$\chi^\alpha_0\theta^\beta = d^{\alpha\beta}\Omega,$$

$$\chi^\alpha_0\bar{\theta}^\bar{\alpha} = \Theta^{\alpha\bar{\alpha}}\Omega,$$

$$\chi^\alpha_0\bar{\theta}^\bar{\alpha} = -\Theta^{\alpha\bar{\alpha}}\Omega.$$
The space of ground states contains two bosonic states, $\Omega$ and $\omega$, and two fermionic states since the four fermionic states $\theta^\alpha$ and $\bar{\theta}^{\bar{\alpha}}$ are related as
\[
\theta^\alpha = \Theta^\alpha_{\bar{\alpha}} d_{\bar{\alpha}\bar{\beta}} \bar{\theta}^{\bar{\beta}}, \quad \bar{\theta}^{\bar{\alpha}} = -\Theta^{\alpha\bar{\alpha}} d_{\alpha\beta} \theta^\beta.
\]
One can show (see [34] for further details) that the bosonic sector of this representation is isomorphic to the representation $\mathcal{R}$. For example, the higher level states of $\mathcal{R}$ can be expressed as fermionic descendents as
\[
\rho^\alpha_{\bar{\alpha}} = \chi^\alpha_{\bar{\scriptscriptstyle 1}} \bar{\theta}^{\bar{\alpha}}, \quad \bar{\rho}^{\bar{\alpha}}_{\alpha} = -\chi_{\bar{\scriptscriptstyle 1}}^\alpha \theta^\alpha,
\psi^\alpha_{\bar{\alpha}} = \chi_{\bar{\scriptscriptstyle 1}}^\alpha \bar{\chi}^{\bar{\alpha}}, \quad \bar{\psi}^{\bar{\alpha}}_{\alpha} = \chi_{\bar{\scriptscriptstyle 1}}^\alpha \bar{\chi}^{\bar{\alpha}}.
\]

The other two representations of the triplet model, the irreducible representations $\mathcal{V}_{-1/8, -1/8}$ and $\mathcal{V}_{3/8, 3/8}$, also have an interpretation in terms of the symplectic fermion theory: they correspond to the bosonic sector of the (unique) $\mathbb{Z}_2$-twisted representation. In this sector, the fermions are half-integrally moded, but all bosonic operators (including the triplet algebra generators that are bilinear in the fermions) are still integrally moded. The ground state of the twisted sector, $\mu$, has conformal weight $h = \bar{h} = -1/8$ and satisfies
\[
\chi^\alpha_{\bar{\scriptscriptstyle 1}} \mu = \bar{\chi}_{\bar{\scriptscriptstyle 1}}^\alpha \mu = 0, \quad \text{for } r > 0,
\]
while
\[
\nu^{\alpha\bar{\alpha}} = \chi^\alpha_{\bar{\scriptscriptstyle 1}} \bar{\chi}^{\bar{\alpha}}_{\bar{\scriptscriptstyle 1}} \mu.
\]

With respect to the symplectic fermions, $\omega$ and $\mu$ are cyclic states, and all amplitudes can be reduced to those involving $\mu$ and the four ground states of the vacuum representation. This can be done using the (fermionic) comultiplication formula and its twisted analogue [32]. The amplitudes involving the ground states can then be determined by solving differential equations. We have determined some of these amplitudes, and we have checked that they reproduce all the three-point functions of the fundamental fields of our logarithmic theory. This implies that all amplitudes of the two theories agree, and since the fermion theory is consistent, thus establishes the consistency of our logarithmic theory. We should stress that the agreement between the two set of amplitudes only holds if we make the specific choices for the normalisation constants (6) and (8).

6 Discussion and Conclusion

Let us first make three comments about the theory we have just constructed.

\[\text{We have also checked that some of the four-point amplitudes agree, for example } \langle \mu \bar{\mu} \omega \bar{\omega} \rangle, \langle \mu \nu \omega \bar{\omega} \rangle, \langle \nu \nu \omega \bar{\omega} \rangle \text{ and } \langle \omega \omega \omega \bar{\omega} \rangle.\]
6.1 Normalisation constants

In order to give a unified treatment, let us classify the amplitudes according to their grade, where each irreducible representations contributes one, and each reducible representation contributes two to the overall grade. For example, the four-point functions of four irreducible representations are at grade four, and so are all the two- and three-point functions that can be derived from them by the use of the OPE. The grade of the amplitude $\langle \mu \mu \omega \omega \rangle$ is then for example six, etc.

As we have seen in section 4, all amplitudes of grade two are proportional to $D$ while all amplitudes of grade four are proportional to $C_0$. It is therefore natural to introduce the parameters $\Lambda$ and $O$ by

$$D = \Lambda^2 O, \quad C_0 = \Lambda^4 O,$$

so that every amplitude at grade $g$ is proportional to $\Lambda^g O$. Here, $\Lambda$ corresponds to the freedom to rescale the field $\mu$, and since we have fixed the normalisations of $\nu^{\alpha\bar{\alpha}}$, $\omega$ and $\phi^{\alpha\bar{\alpha}}$ relative to that of $\mu$ (see (3–5)), to the freedom to rescale all fields with the appropriate power; this leads to the term $\Lambda^g$ in the normalisation of the amplitude. The second parameter, $O$, can be identified with the normalisation of the amplitude functional. In ordinary local conformal field theories, the parameter $O$ is fixed by the condition that the amplitudes satisfy the cluster property, i.e. that to leading order every $n + m$-point amplitude is the product of an $n$- and an $m$-point amplitude. This condition is essentially equivalent to the uniqueness of the vacuum, i.e. to the property that the theory has only one state with vanishing conformal weight. In our case this condition is not satisfied as there are two states of conformal weight 0 in the theory, $\omega$ and $\Omega$. As a consequence, the cluster property does not hold for any choice of $O$, and hence we cannot fix $O$ in this way.

Finally, the last (free) parameter corresponds to the matrix $\Theta^{\alpha\bar{\alpha}}$ which describes the coupling between the left and right $SU(2)$. By performing a global chiral (or anti-chiral) $SU(2)$ transformation we can always bring this matrix into standard form,

$$\Theta^{\alpha\bar{\alpha}} = d^{\alpha\bar{\alpha}}.$$

6.2 Scale invariance

It is clear that the logarithms introduce a length scale into the description and that therefore manifest scale invariance is broken. However, the amplitudes are in fact invariant under the dilatations $z_i \mapsto e^{\lambda} z_i$ if the corresponding transformation on the states (apart from the usual factor of $\exp(\lambda (h + \bar{h}))$) is given as

$$\omega \mapsto \omega + 2\lambda \Omega, \quad \phi^{\alpha\bar{\alpha}} \mapsto \phi^{\alpha\bar{\alpha}} + 2\lambda \psi^{\alpha\bar{\alpha}}.$$

This is a direct consequence of the fact that $L_0$ can be identified with the scale operator, and that $L_0$ does not act diagonally on $\omega$ and $\phi$. It is also easy to check the scale invariance of the amplitudes explicitly.
6.3 Partition function

We can also read off immediately the partition function of the resulting theory. First of all, the characters of the non-chiral irreducible representations are simply the product of a left and right chiral character

\[ \chi_{V_{-1/8}}(\tau) = \chi_{V_{-1/8}}(\tau) \bar{\chi}_{V_{-1/8}}(\bar{\tau}) = |\eta(\tau)^{-1}\theta_{0,2}(\tau)|^2, \]

\[ \chi_{V_{3/8}}(\tau) = \chi_{V_{3/8}}(\tau) \bar{\chi}_{V_{3/8}}(\bar{\tau}) = |\eta(\tau)^{-1}\theta_{2,2}(\tau)|^2. \]

To determine the character of the reducible representation \( R \), let us recall that each vertex in the diagrammatical representation of \( R \) corresponds to an irreducible representation of the left and right triplet algebra. Putting the different contributions together we find

\[ \chi_{\mathcal{R}}(\tau) = 2\chi_{V_0}(\tau)\bar{\chi}_{V_0}(\bar{\tau}) + 2\chi_{V_1}(\tau)\bar{\chi}_{V_1}(\bar{\tau}) + 2\chi_{V_0}(\tau)\bar{\chi}_{V_1}(\bar{\tau}) + 2\chi_{V_1}(\tau)\bar{\chi}_{V_0}(\bar{\tau}) \]

\[ = 2|\chi_{V_0}(\tau) + \chi_{V_1}(\tau)|^2 \]

\[ = 2|\eta(\tau)^{-1}\theta_{1,2}(\tau)|^2, \]

where \( V_0 \) and \( V_1 \) denote the irreducible representations with conformal weights 0 and 1, respectively \([22]\). The partition function of the full theory is thus

\[ Z = \chi_{V_{-1/8},-1/8}(\tau) + \chi_{V_{3/8,3/8}}(\tau) + \chi_{\mathcal{R}}(\tau) = |\eta(\tau)|^{-2} \sum_{k=0}^{3} |\theta_{k,2}(\tau)|^2, \]

and this is indeed modular invariant. Actually, it is the same partition function as that of a free boson compactified on a circle of radius \( \sqrt{2} \) \([33]\). However, in our case the partition function is not simply the sum of products of left- and right- chiral representations of the chiral algebra as the non-chiral representation \( \mathcal{R} \) is not the tensor product of a left- and right-chiral representation, but only a quotient thereof. As we have explained before, this follows directly from locality.

6.4 Conclusions

We have constructed a consistent local logarithmic theory, the first such theory to be understood in detail. Schematically speaking, its (non-chiral) fusion rules are described by

\[ \mu \otimes \mu = \omega \]

\[ \mu \otimes \nu = \omega \]

\[ \nu \otimes \nu = \omega \]

\[ \mu \otimes \omega = \mu \oplus \nu \]

\[ \nu \otimes \omega = \mu \oplus \nu \]

\[ \omega \otimes \omega = 2\omega. \]

We have constructed the two-, three- and some of the four-point amplitudes of the fundamental fields, and have shown that they satisfy the locality and duality constraints.
have also shown that this theory is equivalent to a the bosonic subtheory of a free fermionic
term, thereby establishing that it is indeed consistent. We have determined the partition
function of the theory, and shown that it is indeed modular invariant. Apart from the
appearance of logarithms in some of the correlation functions, this theory defines a bona
fide local conformal field theory.

Acknowledgements: We would like to thank Wolfgang Eholzer, Michael Flohr, Peter
Goddard, Gérard Watts for useful discussions.

M.R.G. is grateful to Jesus College, Cambridge, for a Research Fellowship. This work
has also been supported in part by PPARC.

Appendix

A Operator product expansions

This appendix lists the relevant operator product expansions. Terms up to order $O(x, \bar{x})$
for states in $\mathcal{R}$ and up to order $O(x^{1/2}, \bar{x}^{1/2})$ for states in $\mathcal{V}_{-1/8, -1/8}$ and $\mathcal{V}_{3/8, 3/8}$ are given.
We have included the arbitrary constant $\Lambda$ whose significance is explained in section 3.

\[
|x|^{-\frac{1}{2}} \mu(x) \mu = (\omega + \ln |x|^2 \Omega) + \frac{1}{2} L_{-1} \omega x + \frac{1}{2} \bar{L}_{-1} \omega \bar{x} + \frac{1}{4} L_{-1} \bar{L}_{-1} \omega |x|^2 + \cdots,
\]

\[
|x|^3 \nu^\alpha(x) \nu^\beta = -\theta^\alpha \theta^\beta \omega + \frac{1}{2} \rho^\alpha \rho^\beta x + \frac{1}{2} \rho^\alpha \bar{x} + \frac{1}{4} \beta^\alpha \beta^\beta \omega |x|^2 + \cdots,
\]

\[
|x|^3 \nu^\alpha(x) \nu^\beta = \frac{1}{4} d^{\alpha \beta} d_{\gamma \delta} \phi^{\gamma \delta} + \frac{1}{4} \theta^{\alpha \beta} \phi^{\gamma \delta} + \frac{3}{4} \beta^{\alpha \beta} \phi^{\gamma \delta} |x|^2 + \cdots,
\]

\[
\Lambda^{-2} \mu(x) \omega = -(8 \ln 2 + \ln |x|^2) \mu + 4 \theta_{\alpha \beta} \nu^{\alpha \beta} |x| + \cdots,
\]

\[
\Lambda^{-2} |x| \nu(x) \omega = -\theta_{\alpha \beta} \mu - (8 \ln 2 - 4 + \ln |x|^2) \nu^{\alpha \beta} |x| + \cdots,
\]

\[
\Lambda^{-2} |x|^2 \mu(x) \phi^{\alpha \beta} = \frac{1}{4} \theta^{\alpha \beta} \mu - (8 \ln 2 - 2 + \ln |x|^2) \nu^{\alpha \beta} |x| + \cdots,
\]

\[
\Lambda^{-2} |x|^3 \nu^{\alpha \beta}(x) \phi^{\gamma \delta} = \frac{1}{4} d^{\alpha \beta} d_{\gamma \delta} (8 \ln 2 + \ln |x|^2) \mu + \frac{3}{4} \theta^{\gamma \delta} \phi^{\gamma \delta} |x| + \cdots,
\]

\[
\Lambda^{-2} |x|^3 \nu^{\alpha \beta}(x) \phi^{\gamma \delta} = \frac{1}{4} d^{\alpha \beta} d_{\gamma \delta} (8 \ln 2 + \ln |x|^2) \mu + \frac{3}{4} \theta^{\gamma \delta} \phi^{\gamma \delta} |x| + \cdots,
\]
\[ \Lambda^{-2} \omega(x) \omega = - (4 \ln 2 + \ln |x|^2) \left[ 2 \omega + (4 \ln 2 + \ln |x|^2) \Omega \right. \\
\left. + \tilde{\Theta}_{\bar{a}a} \rho^{\bar{a}a} x + \tilde{\Theta}_{\bar{a}a} \tilde{\rho}^{\bar{a}a} \bar{x} \right] + \tilde{\Theta}_{\bar{a}a} \phi^{\bar{a}a} |x|^2 + \ldots, \]

\[ \Lambda^{-2} |x|^2 \phi(x)^{\alpha\bar{a}} = - \Theta^{\alpha\bar{a}} \omega \\
- (\Theta^{\alpha\bar{a}} L_{-1} \omega - (4 \ln 2 + \ln |x|^2) \rho^{\alpha\bar{a}}) x \\
- (\Theta^{\alpha\bar{a}} \tilde{L}_{-1} \omega - (4 \ln 2 + \ln |x|^2) \tilde{\rho}^{\alpha\bar{a}}) \bar{x} \\
- (4 \ln 2 + \ln |x|^2)(4 \ln 2 - 2 + \ln |x|^2) \psi^{\alpha\bar{a}} \\
+ 2(4 \ln 2 - 1 + \ln |x|^2) \phi^{\alpha\bar{a}} + \Theta^{\alpha\bar{a}} \tilde{\Theta}_{\bar{a}a} \phi^{a\bar{a}}) |x|^2 + \ldots, \]

\[ \Lambda^{-2} |x|^4 \phi(x)^{\alpha\bar{a}} \phi^{\beta\bar{b}} = \Theta^{\alpha\bar{a}} \Theta^{\beta\bar{b}} \Omega \\
+ d^{\beta\bar{b}} d^{\alpha\bar{a}} \left[ 2(4 \ln 2 + 1 + \ln |x|^2) \omega \\
+ (4 \ln 2 + \ln |x|^2)(4 \ln 2 + 2 + \ln |x|^2) \Omega \right] \\
+ 2 \left( 4 \ln 2 + \ln |x|^2 \right) \Theta^{\gamma\bar{b}} d^{\alpha\bar{a}} d_{\gamma\bar{b}} \rho^{\alpha\bar{a}} + \Theta^{\gamma\bar{b}} d^{\alpha\bar{a}} d_{\gamma\bar{b}} \tilde{\rho}^{\alpha\bar{a}} \bar{x} \\
+ 2 \left( 4 \ln 2 + \ln |x|^2 \right) \Theta^{\gamma\bar{b}} d^{\alpha\bar{a}} d_{\gamma\bar{b}} \rho^{\alpha\bar{a}} + \Theta^{\gamma\bar{b}} d^{\alpha\bar{a}} d_{\gamma\bar{b}} \tilde{\rho}^{\alpha\bar{a}} \bar{x} \\
- 2 \left[ d^{\alpha\bar{a}} d^{\beta\bar{b}} \tilde{\Theta}_{\bar{a}a} \left( 2 \phi^{a\bar{a}} + (4 \ln 2 + \ln |x|^2) \psi^{a\bar{a}} \right) \\
+ \Theta^{\alpha\bar{a}} \psi^{\beta\bar{b}} - (4 \ln 2 + \ln |x|^2) \Theta^{\beta\bar{b}} \psi^{a\bar{a}} \right] |x|^2 + \ldots. \]

The operator product of \( \Omega \) with any field \( S \) is simply given by \( \Omega(x) S = \Lambda^2 S \) to all orders. For \( \Lambda^2 = 1 \), \( \Omega \) can be thought of as the unit operator, except that its one-point function vanishes, \( \langle \Omega \rangle = 0 \).

## B Amplitudes

In this appendix we list some of the higher grade amplitudes of the theory that we have checked to be consistent with the OPEs and the lower grade amplitudes determined before. For simplicity we have set \( \Lambda = \mathcal{O} = 1 \). These parameters can be restored by multiplying an amplitude of grade \( g \) by a factor of \( \Lambda^g \).

The fundamental 3-point amplitudes of reducible representations are

\[ \langle \omega \omega \omega \rangle = 48(\ln 2)^2 + 8 \ln 2 \ln |z_{12} z_{13} z_{23}|^2 \\
+ 2 \left( \ln |z_{12}|^2 \ln |z_{13}|^2 + \ln |z_{12}|^2 \ln |z_{23}|^2 + \ln |z_{13}|^2 \ln |z_{23}|^2 \right) + \left( \ln |z_{12}|^2 + \ln |z_{13}|^2 + \ln |z_{23}|^2 \right) \], 22
\[ \langle \phi^{\alpha\beta} \omega \rangle = \Theta^{\alpha\alpha} \left[ \frac{1}{|z_{12}|^2} \left( 4 \ln 2 + \ln \left| \frac{z_{13}^2 z_{23}}{z_{12}} \right|^2 \right) \right. \\
+ \frac{1}{|z_{13}|^2} \left( 4 \ln 2 + \ln \left| \frac{z_{12} z_{23}}{z_{13}} \right|^2 \right) \\
\left. + \frac{z_{23}}{|z_{12} z_{13}|^2} \left( 4 \ln 2 + \ln \left| \frac{z_{12} z_{13}}{z_{23}} \right|^2 \right) \right], \\
\langle \phi^{\alpha\beta} \phi^{\gamma\delta} \omega \rangle = \Theta^{\alpha\alpha} \Theta^{\beta\beta} \left[ \frac{1}{|z_{12}|^2} + \frac{1}{|z_{13} z_{23}|^2} \right. \\
+ d^{\alpha\beta} d^{\gamma\delta} \left[ \left( 4 \ln 2 - 2 + \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \right) \right. \\
\times \left( 4 \ln 2 + \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 + \frac{z_{12}^2}{z_{13} z_{23}} + \frac{z_{13}^2}{z_{12} z_{23}} \right) \\
- \left( 4 \ln 2 + 2 \ln |z_{13}|^2 + \frac{z_{12}}{z_{23}} + \frac{z_{13}}{z_{23}} \right) \right. \\
\times \left( 4 \ln 2 + 2 \ln |z_{23}|^2 - \frac{z_{12}}{z_{13}} - \frac{z_{13}}{z_{13}} \right) \left. \right], \\
\langle \phi^{\alpha\beta} \phi^{\gamma\delta} \phi^{\epsilon\kappa} \rangle = - \frac{T_1}{|z_{12} z_{13} z_{23}|^2} \\
\times \left[ \left( \frac{z_{13} + z_{23}}{z_{13} z_{23}} \right)^2 \left( 4 \ln 2 + 1 + \ln \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \right) \right. \\
\left. + \frac{z_{12} - z_{23}}{z_{12} z_{23}} \right]^2 \left( 4 \ln 2 + 1 + \ln \left| \frac{z_{12} z_{23}}{z_{13}} \right|^2 \right) \right. \\
\left. + \frac{z_{12} + z_{13}}{z_{12} z_{13}} \right]^2 \left( 4 \ln 2 + 1 + \ln \left| \frac{z_{12} z_{13}}{z_{23}} \right|^2 \right) \right. \\
\left. \frac{1}{2} \left( |z_{12}|^2 + |z_{13}|^2 + |z_{23}|^2 \right) \right] \\
\left. \times \left[ T_2 (z_{13} z_{23} - z_{23} z_{13}) + T_3 (|z_{12}|^2 + |z_{13}|^2 + |z_{23}|^2) \right], \right. \\
\left. \right. \\
\right. \\
\\text{where} \\
T_1 = \Theta^{\alpha\alpha} d^{\beta\gamma} d^{\gamma\delta} z_{13} z_{13} - \Theta^{\alpha\beta} d^{\beta\gamma} d^{\alpha\gamma} z_{13} z_{23} - \Theta^{\beta\gamma} d^{\alpha\gamma} d^{\beta\gamma} z_{23} z_{13} + \Theta^{\beta\gamma} d^{\alpha\gamma} d^{\beta\gamma} z_{23} z_{23}, \\
T_2 = \Theta^{\alpha\alpha} d^{\beta\gamma} d^{\gamma\delta} + \Theta^{\beta\beta} d^{\alpha\gamma} d^{\delta\gamma} + \Theta^{\gamma\gamma} d^{\alpha\delta} d^{\beta\delta}, \\
T_3 = \Theta^{\alpha\beta} d^{\beta\gamma} d^{\alpha\gamma} - \Theta^{\beta\alpha} d^{\alpha\gamma} d^{\beta\gamma}. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
Using the tensor identities in appendix D, it is not difficult to see that \( T_1 \) and \( T_2 \) are completely symmetric under the exchange of any two of the three fields, whereas \( T_3 \) is completely anti-symmetric. Furthermore, it is easy to check that \( (z_{13} z_{23} - z_{23} z_{13}) \) is completely anti-symmetric, and it thus follows that the above amplitude is completely symmetric.
The four-point amplitudes of grade four were already given for a certain domain in the cross-ratio $x$ in section 1. Other regions of $x$ are described by different orderings of the fields, where the corresponding cross-ratio is again small.

\[
\langle \mu(z_1) \nu^{\alpha\dot{\alpha}}(z_2) \mu(z_3) \nu^{\beta\dot{\beta}}(z_4) \rangle = -\frac{\pi}{4} d^{\alpha\dot{\beta}} d^{\beta\dot{\beta}} \left| \frac{z_{13}^2}{z_{12} z_{24}^2 z_{23} z_{34}} \right|^{\frac{1}{2}} \times \left[ D_1(x) \bar{D}_1(\bar{x}) + \bar{D}_1(x) D_1(\bar{x}) \right],
\]

\[
\langle \mu(z_1) \nu^{\alpha\dot{\alpha}}(z_2) \nu^{\beta\dot{\beta}}(z_3) \mu(z_4) \rangle = \frac{\pi}{4} d^{\alpha\dot{\beta}} d^{\beta\dot{\beta}} \left| \frac{z_{13} z_{14}^2 z_{24}^2}{z_{12} z_{23}^2 z_{34}} \right|^{\frac{1}{2}} \left[ D_2(x) \bar{E}(\bar{x}) + \bar{E}(x) D_2(\bar{x}) \right].
\]

It is easy to check (remembering that we have set $C_1 = 1$) that the above four-point amplitudes give rise to the same three-point amplitudes as the ones determined in section 1.

Some four-point amplitudes of grade six are

\[
|z_{12}|^{-\frac{1}{2}} \langle \mu \nu \mu \nu \rangle = \left( 8 \ln 2 + \ln \left| \frac{z_{13}^2 z_{23}^2}{z_{12}} \right| \right) \left( 8 \ln 2 + \ln \left| \frac{z_{14}^2 z_{24}^2}{z_{12}} \right| \right) - H(x, \bar{x})^2,
\]

\[
|z_{12}|^{\frac{1}{2}} \langle \mu \nu^{\alpha\dot{\alpha}} \nu \omega \rangle = \Theta^{\alpha\dot{\alpha}} \left[ \frac{z_{13}}{z_{23}} \left( 8 \ln 2 + \ln \left| \frac{z_{14}^2 z_{24}^2}{z_{12}} \right| \right) + \frac{z_{14}}{z_{24}} \left( 8 \ln 2 + \ln \left| \frac{z_{13}^2 z_{23}^2}{z_{12}} \right| \right) \right. 
\]

\[
+ \left. \left( \sqrt{\frac{z_{13} z_{14}}{z_{23} z_{24}}} + \sqrt{\frac{z_{13} z_{14}}{z_{24} z_{23}}} \right) H(x, \bar{x}) \right],
\]

\[
|z_{12}|^{\frac{3}{2}} \langle \nu^{\alpha\dot{\alpha}} \nu^{\beta\dot{\beta}} \nu \omega \rangle = \Theta^{\alpha\dot{\alpha}} \Theta^{\beta\dot{\beta}} \left[ \sqrt{\frac{z_{13}^2 z_{24}^2}{z_{14} z_{23}}} - \sqrt{\frac{z_{14}^2 z_{23}^2}{z_{13} z_{24}}} \right. 
\]

\[
- \frac{1}{4} d^{\alpha\dot{\beta}} d^{\beta\dot{\beta}} \left[ \left( 8 \ln 2 - 4 + \ln \left| \frac{z_{13}^2 z_{23}^2}{z_{12}} \right| \right) \right. 
\]

\[
\times \left. \left( 8 \ln 2 - 4 + \ln \left| \frac{z_{14} z_{24}^2}{z_{12}} \right| \right) \right] 
\]

\[
- \left( H(x, \bar{x}) + 2 \sqrt{\frac{z_{13} z_{24}^2}{z_{14} z_{23}}} + 2 \sqrt{\frac{z_{13} z_{23}^2}{z_{14} z_{24}}} \right) 
\]

\[
\times \left( H(x, \bar{x}) + 2 \sqrt{\frac{z_{14} z_{23}^2}{z_{13} z_{24}}} + 2 \sqrt{\frac{z_{14} z_{24}^2}{z_{13} z_{23}}} \right),
\]

where

\[
H(x, \bar{x}) = \ln \left| \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} \right|^2 = \ln \left| \sqrt{\frac{z_{13} z_{24}^2}{z_{14} z_{23}}} - \sqrt{\frac{z_{14} z_{23}^2}{z_{13} z_{24}}} \right|^2.
\]
The fundamental four-point amplitude of grade eight is
\[
\langle \omega \omega \omega \omega \rangle = 256(\ln 2)^3 + 32(\ln 2)^2(\omega-\omega) + 8 \ln 2 \left[ (\omega-\omega-\omega) - (\omega=\omega) \right] \\
+ \left[ 2(\omega-\omega-\omega-\omega) - 2(\omega=\omega-\omega) - 2(\nabla) \right],
\]
where
\[
(\omega-\omega) = \sum_{ij} \ln |z_{ij}|^2,
\]
\[
(\omega-\omega-\omega) = \sum_{ijk} \ln |z_{ij}|^2 \ln |z_{jk}|^2,
\]
\[
(\omega-\omega-\omega-\omega) = \sum_{ijkl} \ln |z_{ij}|^2 \ln |z_{jk}|^2 \ln |z_{kl}|^2,
\]
\[
(\omega=\omega) = \sum_{ij} (\ln |z_{ij}|^2)^2,
\]
\[
(\omega=\omega-\omega) = \sum_{ijkl} (\ln |z_{ij}|^2)^2 \ln |z_{kl}|^2,
\]
\[
(\nabla) = \sum_{ijkl} \ln |z_{ij}|^2 \ln |z_{jk}|^2 \ln |z_{kl}|^2.
\]
The sums are over pairwise distinct labelled graphs (i.e. graphs with vertices); labelled graphs that differ by a graph symmetry are only counted once.

### C Elliptic Integrals

The four-point functions can be expressed in terms of complete elliptic integrals, \( K, \tilde{K}, E, \tilde{E} \). These are related to the hypergeometric series near 0 or 1,
\[
K(x) = 2F_1(1/2, 1/2; 1; x), \quad E(x) = 2F_1(-1/2, 1/2; 1; x).
\]
\[
\tilde{K}(x) = K(1-x), \quad \tilde{E}(x) = E(1-x).
\]
The other functions appearing in four-point functions are
\[
D_1(x) = E(x) - (1-x)K(x) = 2x(1-x)K'(x),
\]
\[
D_2(x) = -E(x) + K(x) = -2xE'(x),
\]
\[
\tilde{D}_1(x) = \tilde{E}(x) - x\tilde{K}(x),
\]
\[
\tilde{D}_2(x) = -\tilde{E}(x) + \tilde{K}(x),
\]
\[
F_1(x) = 2xE(x) - x(2-x)K(x),
\]
\[
F_2(x) = (2-x)E(x) - \frac{1}{2}(2-2x+x^2)K(x),
\]
\[
\tilde{F}_1(x) = 2x\tilde{E}(x) - x^2\tilde{K}(x),
\]
\[
\tilde{F}_2(x) = (2-x)\tilde{E}(x) - \frac{1}{2}(2-x^2)\tilde{K}(x).
\]
The functions with a \( \tilde{\cdot} \) have a logarithmic branch cut at zero, which can be seen from the analytic continuation for \( |\arg x| < \pi \) (c.f. Erdélyi et al. [34]),

\[
K(1 - x) = -\frac{1}{\pi} \ln(x/16)K(x) + 2M(x),
\]
\[
E(1 - x) = -\frac{1}{\pi} \ln(x/16)D_2(x) - 2N(x),
\]

where

\[
M(x) = \sum_{n=1}^{\infty} \frac{(1/2)_n^2}{n!} h_n x^n,
\]
\[
N(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1/2)_n (1/2)_n}{n!(n+1)!} (h_n + h_{n-1}) x^n,
\]

and

\[
h_n = \psi(1) - \psi(n+1) - \psi(1/2) + \psi(n+1/2).
\]

Furthermore, for analytic continuation to infinity one has

\[
x^{-1/2} K(1/x) = K(x) + i\tilde{K}(x), \quad x^{-1/2} \tilde{K}(1/x) = \tilde{K}(x),
\]
\[
x^{1/2} E(1/x) = D_1(x) + i\tilde{D}_1(x), \quad x^{1/2} \tilde{E}(1/x) = \tilde{E}(x).
\]

**D su(2) tensors.**

We choose a Cartan-Weyl basis for \( su(2) \) so that the non-vanishing structure constants are

\[
f_{\pm}^{0} = \pm 1, \quad f_{0}^{\mp} = \pm 2,
\]

and the metric is given by

\[
g_{00} = 1, \quad g_{\pm}^{\mp} = 2.
\]

The spin 1/2 representation is given by the matrices \( t_{a}^{\alpha}_{\beta} \) whose non-vanishing entries are

\[
t_{\pm}^{0} = \pm \frac{1}{2}, \quad t_{\pm}^{\mp} = 1.
\]

The anti-symmetric tensor \( d^{\alpha\beta} \) and its inverse \( d_{\alpha\beta} \), normalised to \( d^{\pm\mp} = d_{\mp\pm} = \pm 1 \), can be used to raise and lower indices in the spin 1/2 representation. The other tensors appearing in correlation functions are

\[
t_{\alpha\beta}^{a} : \quad t_{\alpha\beta}^{a} = t_{\alpha}^{\gamma} d_{\beta}^{\gamma},
\]
\[
t_{\pm}^{0} = \frac{1}{2}, \quad t_{\pm}^{\mp} = \pm 1,
\]
\[
t_{\alpha\beta}^{a} : \quad t_{\alpha\beta}^{a} = t_{\alpha}^{\gamma} d_{\beta}^{\gamma},
\]
\[
t_{\pm}^{0} = -\frac{1}{2}, \quad t_{\pm}^{\mp} = \pm 1.
\]
\[ h_{\eta}^{\alpha\beta\gamma} : \quad h_{\eta}^{\alpha\beta\gamma} = 2t^{\alpha\alpha\beta}g_{ab}t_{\eta}^{b\gamma} = -\frac{1}{2} (d^{\alpha\gamma}\delta_{\eta}^{\beta} + d^{\beta\gamma}\delta_{\eta}^{\alpha}) , \]
\[ h_{\pm\mp\mp}^{\pm\pm\mp} = \mp 1 , \quad h_{\pm\mp}^{\mp\pm\mp} = \mp \frac{1}{2} , \]
\[ h_{\mp\mp\mp}^{\pm\pm\mp} = -1 , \quad h_{\mp\mp\mp}^{\mp\pm\mp} = \frac{1}{2} . \]

In addition the following tensor relations hold:
\[ d^{\alpha\beta}d^{\gamma\delta} - d^{\alpha\gamma}d^{\beta\delta} + d^{\alpha\delta}d^{\beta\gamma} = 0 , \]
\[ \Theta^{\alpha\bar{\alpha}} \Theta^{\beta\bar{\beta}} - \Theta^{\alpha\bar{\beta}} \Theta^{\beta\bar{\alpha}} - d^{\alpha\beta}d^{\bar{\alpha}\bar{\beta}} = 0 , \]
\[ \Theta^{\alpha\bar{\alpha}} d^{\beta\gamma} + \Theta^{\beta\bar{\beta}} d^{\gamma\alpha} + \Theta^{\gamma\bar{\alpha}} d^{\alpha\beta} = 0 , \]
\[ \Theta^{\alpha\bar{\alpha}} d^{\bar{\beta}\bar{\gamma}} + \Theta^{\beta\bar{\beta}} d^{\bar{\gamma}\alpha} + \Theta^{\gamma\bar{\alpha}} d^{\alpha\bar{\beta}} = 0 . \]

References

[1] H. Saleur, Polymers and percolation in two dimensions and twisted \( N = 2 \) supersymmetry, Nucl. Phys. B382 (1992) 486, [hep-th/9111007].

[2] M. A. Flohr, On modular invariant partition functions of conformal field theories with logarithmic operators, Int. J. Mod. Phys. A11 (1996) 4147, [hep-th/9509166].

[3] H. G. Kausch, Curiosities at \( c = -2 \), preprint DAMTP 95-52, [hep-th/9510149].

[4] M. J. Martins, B. Nienhuis and R. Rietman, An intersecting loop model as a solvable super spin chain, [cond-math/9709051].

[5] M. R. Rahimi Tabar and S. Rouhani, The Alfven effect and conformal field theory, Nuovo Cimento 112B (1997) 1079, [hep-th/9507166].

[6] M. Flohr, Two-dimensional turbulence: yet another conformal field theory solution, Nucl. Phys. B482 (1996) 567, [hep-th/9606130].

[7] M. R. Rahimi Tabar and S. Rouhani, A logarithmic conformal field theory solution for two dimensional magnetohydrodynamics in presence of the Alfven effect, Europhys. Lett. 37 (1997) 447, [hep-th/9606143].

[8] V. Gurarie, M. Flohr and C. Nayak, The Haldane-Rezayi quantum hall state and conformal field theory, Nucl. Phys. B498 (1997) 513, [cond-math/9701212].

[9] I. I. Kogan and N. E. Mavromatos, World-Sheet Logarithmic Operators and Target Space Symmetries in String Theory, Phys. Lett. B375 (1996) 111, [hep-th/9512210].

[10] L. Rozansky and H. Saleur, Quantum field theory for the multi-variable Alexander-Conway polynomial, Nucl. Phys. B376 (1992) 461, [hep-th/9203069].
[11] V. Gurarie, Logarithmic operators in conformal field theory, Nucl. Phys. B410 (1993) 535, hep-th/9303160.

[12] M. R. Gaberdiel and H. G. Kausch, Indecomposable Fusion Products, Nucl. Phys. B477 (1996) 293, hep-th/9604026.

[13] A. Bilal and I. I. Kogan, On gravitational dressing of 2D field theories in chiral gauge, Nucl. Phys. B449 (1995) 569, hep-th/9503209.

[14] J.-S. Caux, I. I. Kogan and A. M. Tsvelik, Logarithmic Operators and Hidden Continuous Symmetry in Critical Disordered Models, Nucl. Phys. B466 (1996) 444, hep-th/9511134.

[15] J.-S. Caux, I. I. Kogan, A. Lewis and A. M. Tsvelik, Logarithmic operators and dynamical extension of the symmetry group in the bosonic SU(2)_0 and SUSY SU(2)_2 WZNW models, Nucl. Phys. B489 (1997) 469, hep-th/9606138.

[16] Z. Maassarani and D. Serban, Non-unitary conformal field theory and logarithmic operators for disordered systems, Nucl. Phys. B489 (1997) 603, hep-th/9605062.

[17] J.-S. Caux, N. Taniguchi and A. M. Tsvelik, Disordered Dirac fermions: Multifractality termination and logarithmic conformal field theories, preprint, cond-mat/9801055, to appear in Nucl. Phys. B.

[18] M. Flohr, Singular vectors in logarithmic conformal field theories, Nucl. Phys. B514 (1998) 523, hep-th/9707090.

[19] A. Shafiekhani and M. R. Rahimi Tabar, Logarithmic operators in conformal field theory and the \( W_{\infty} \)-algebra, Int. J. Mod. Phys. A12 (1997) 3723, hep-th/9604007.

[20] M. R. Rahimi Tabar, A. Aghamohammadi and M. Khorrami, The logarithmic conformal field theories, Nucl. Phys. B497 (1997) 555, hep-th/9610168.

[21] F. Rohsiepe, On reducible but indecomposable representations of the Virasoro algebra, preprint BONN-TH-96-17, hep-th/9611160.

[22] M. R. Gaberdiel and H. G. Kausch, A rational logarithmic conformal field theory, Phys. Lett. B 386 (1996) 131, hep-th/9606050.

[23] Y. Zhu, Vertex operator algebras, elliptic functions and modular forms, J. Amer. Math. Soc. 9 (1996) 237.

[24] M. Flohr, On Fusion Rules in Logarithmic Conformal Field Theories, Int. J. Mod. Phys. A12 (1997) 1943, hep-th/9605151.

[25] M. R. Gaberdiel and P. Goddard, Axiomatic conformal field theory, hep-th/9810019.

[26] M. R. Gaberdiel, Fusion in conformal field theory as the tensor product of the symmetry algebra, Int. J. Mod. Phys. A9 (1994) 4619, hep-th/9307183.
[27] M. R. Gaberdiel, Fusion rules of chiral algebras, Nucl. Phys. B417 (1994) 130, hep-th/9309105.

[28] W. Nahm, Quasirational fusion products, Int. J. Mod. Phys. B8 (1994) 3693, hep-th/9402039.

[29] G. Moore and N. Seiberg, Classical and quantum conformal field theory, Comm. Math. Phys. 123 (1989) 177.

[30] H. G. Kausch, Symplectic fermions, Durham University preprint DTP-98/39, in preparation.

[31] H. G. Kausch, Extended conformal algebras generated by a multiplet of primary fields, Phys. Lett. B259 (1991) 448.

[32] M. R. Gaberdiel, Fusion of twisted representations, Int. J. Mod. Phys. A12 (1997) 5183, hep-th/9607036.

[33] P. Ginsparg, Curiosities at $c = 1$, Nucl. Phys. B295 (1988) 153.

[34] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions, McGraw-Hill, (1953).