On the Obfuscation Complexity of Planar Graphs

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Abstract

Being motivated by John Tantalo’s Planarity Game, we consider straight line plane drawings of a planar graph \( G \) with edge crossings and wonder how obfuscated such drawings can be. We define \( \text{obf}(G) \), the obfuscation complexity of \( G \), to be the maximum number of edge crossings in a drawing of \( G \). Relating \( \text{obf}(G) \) to the distribution of vertex degrees in \( G \), we show an efficient way of constructing a drawing of \( G \) with at least \( \text{obf}(G)/3 \) edge crossings. We prove bounds \( (\delta(G)^2/24 - o(1))n^2 \leq \text{obf}(G) < 3n^2 \) for an \( n \)-vertex planar graph \( G \) with minimum vertex degree \( \delta(G) \geq 2 \).

The shift complexity of \( G \), denoted by \( \text{shift}(G) \), is the minimum number of vertex shifts sufficient to eliminate all edge crossings in an arbitrarily obfuscated drawing of \( G \) (after shifting a vertex, all incident edges are supposed to be redrawn correspondingly). If \( \delta(G) \geq 3 \), then \( \text{shift}(G) \) is linear in the number of vertices due to the known fact that the matching number of \( G \) is linear. However, in the case \( \delta(G) \geq 2 \) we notice that \( \text{shift}(G) \) can be linear even if the matching number is bounded. As for computational complexity, we show that, given a drawing \( D \) of a planar graph, it is NP-hard to find an optimum sequence of shifts making \( D \) crossing-free.

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1 Introduction

This note is inspired by John Tantalo’s Planarity Game [10] (another implementation is available at [13]). An instance of the game is a straight line drawing of a planar graph with many edge crossings. In a move the player is able to shift one vertex of the graph to a new position; the incident edges will be redrawn correspondingly. The objective is to achieve a crossing-free drawing in a possibly smaller number of moves.

Let us fix some relevant terminology. By a drawing we will always mean a straight line plane drawing of a graph where no vertex is an inner point of any edge. An edge crossing in a drawing $D$ is a pair of edges having a common inner point. The number of edge crossings in $D$ will be denoted by $obf(D)$. We define the obfuscation complexity of a graph $G$ to be the maximum $obf(D)$ over all drawings $D$ of $G$. This graph parameter will be denoted by $obf(G)$.

Given a drawing $D$ of a planar graph $G$, let $shift(D)$ denote the minimum number of vertex shifts making $D$ crossing-free. The shift complexity of $G$, denoted by $shift(G)$, is the maximum $shift(D)$ over all drawings of $G$.

Our aim is a combinatorial and a complexity-theoretic analysis of the Planarity Game from the standpoint of a game designer. The latter should definitely have a library of planar graphs $G$ with large $shift(G)$. Generation of planar graphs with large $obf(G)$ is also of interest. Though large obfuscation complexity does not imply large shift complexity (see discussion in Section 4.4), the designer can at least expect that a large $obf(D)$ will be a psychological obstacle for a player to play optimally on $D$.

A result of direct relevance to the topic is obtained by Pach and Tardos [8]. Somewhat surprisingly, they prove that even cycles have large shift complexity, namely, $n - O((n \log n)^{2/3}) \leq shift(C_n) \leq n - \lfloor \sqrt{n} \rfloor$.

We first address the obfuscation complexity. In Section 2 we relate this parameter of a graph to the distribution of its vertex degrees. This gives us an efficient way of constructing a drawing $D$ of a given graph $G$ so that $obf(D) \geq obf(G)/3$. As another consequence, we prove that $obf(G) \geq (\delta(G)^2/24 - o(1))n^2$ for an $n$-vertex planar graph with minimum vertex degree $\delta(G) \geq 2$. On the other hand, we prove an upper bound $obf(G) < 3n^2$. In Section 3 we discuss the relationship between the shift complexity of a planar graph and its matching number. We also show that the shift complexity of a drawing is NP-hard to compute. Section 4 contains concluding remarks and questions.
**Related work.** Investigation of the parameter \( \text{shift}(G) \) is well motivated from a graph drawing perspective. Several results were obtained in this area independently of our work and appeared in [3, 9, 2] soon after the present note was submitted to the journal. The Planarity Game is also mentioned in [3, 9] as a source of motivation.

Goaos et al. [3] independently prove that computing \( \text{shift}(D) \) for a given drawing \( D \) is an NP-hard problem, the same result as stated in our Theorem 8. They use a different reduction, allowing them to show that \( \text{shift}(D) \) is even hard to approximate. Our reduction has another advantage: It shows that it is NP-hard to untangle even drawings of as simple graphs as matchings.

Spillner and Wolff [9] and Bose et al. [2] obtain general upper bounds for \( \text{shift}(G) \), which quantitatively improve the classical Wagner-Fáry-Stein theorem (cf. Theorem 4 in Section 3). The stronger of their bounds [2] claims that \( \text{shift}(G) \leq n - \sqrt{n/9} \) for any planar \( G \). Even better bounds are established for trees [3] and outerplanar graphs [9]. The series of papers [3, 9, 2] gives also lower bounds on the variant of \( \text{shift}(G) \) for a broader notion of a “bad drawing”.

**Notation.** We reserve \( n \) and \( m \) for, respectively, the number of vertices and the number of edges in a graph under consideration. We use the standard notation \( K_n, K_{s,t}, \) and \( C_n \) for, respectively, complete graphs, complete bipartite graphs, and cycles. The vertex set of a graph \( G \) will be denoted by \( V(G) \). By \( kG \) we mean the disjoint union of \( k \) copies of \( G \). The number of edges emanating from a vertex \( v \) is called the degree of \( v \) and denoted by \( \deg v \). The minimum degree of a graph \( G \) is defined by \( \delta(G) = \min_{v \in V(G)} \deg v \). A set of pairwise non-adjacent vertices (resp., edges) is called an independent set (resp., a matching). The maximum cardinality of an independent set (resp., a matching) in a graph \( G \) is denoted by \( \alpha(G) \) (resp., \( \nu(G) \)) and called the independence number (resp., the matching number) of \( G \). A graph is \( k \)-connected if it stays connected after removal of any \( k - 1 \) vertices.

## 2 Estimation of the obfuscation complexity

Note that \( \text{obf}(G) \) is well defined for an arbitrary, not necessary planar graph \( G \). As a warm-up, consider a few examples.

\[ \text{obf}(K_n) = \binom{n}{3} \]. Indeed, let \( D \) be a drawing of \( K_n \). \( \text{obf}(D) \) is computable as follows. We start with the initial value 0 and, tracing through all
pairs \{e, e'\} of non-adjacent edges, increase it by 1 once e and e' cross. Consider the set $S$ of 4 endpoints of e and e'. In fact, $S$ corresponds to exactly 3 pairs of edges. If the convex hull of $S$ is a triangle, then none of these three pairs is crossing. If it is a quadrangle, then 1 of the three pairs is crossing and 2 are not. It follows that $obf(D)$ does not exceed the number of all possible $S$. This upper bound is attained if every $S$ has a quadrangular hull, for instance, if the vertices of $D$ lie on a circle.

$obf(K_{s,t}) = \binom{s}{2}\binom{t}{2}$. The upper bound is provable by the same argument as above, where a 4-point set $S$ has 2 points in the $s$-point part of $V(D)$ and 2 points in the $t$-point part. Such an $S$ corresponds to 2 pairs of non-adjacent edges, at most 1 of which is crossing. This upper bound is attained if we put the two vertex parts of $K_{s,t}$ on two parallel lines.

$obf(C_n) = n(n - 3)/2$ if $n$ is odd. The value of $n(n - 3)/2$ is attained by the $n$-pointed star drawing of $C_n$. This is the maximum by a simple observation: $n(n - 3)/2$ is the total number of pairs of non-adjacent edges in $C_n$.

Let us state the upper bound argument we just used for the odd cycles in a general form. Given a graph $G$ with $m$ edges, let

$$\epsilon(G) = \binom{m}{2} - \sum_{v \in V(G)} \binom{\deg v}{2}.$$  

Note that $\epsilon(G) = \frac{1}{2}(m(m + 1) - \sum_v \deg^2 v)$, where the latter term is closely related to the variance of the vertex degrees. Since $\epsilon(G)$ is equal to the number of pairs of non-adjacent edges in $G$, we have $obf(G) \leq \epsilon(G)$. Notice also a lower bound in terms of $\epsilon(G)$.

**Theorem 1.** $\epsilon(G)/3 \leq obf(G) \leq \epsilon(G)$. Moreover, a drawing $D$ of $G$ with $obf(D) \geq \epsilon(G)/3$ is efficiently constructible.

**Proof.** Fix an arbitrary $n$-point set $V$ on a circle. We use the probabilistic method to prove that there is a drawing $D$ with $V(D) = V$ having at least $\epsilon(G)/3$ edge crossings. Let $D$ be a random straight line embedding of $G$ with $V(D) = V$, which is determined by a random map of $V(G)$ onto $V$. For each pair $e, e'$ of non-adjacent vertices of $G$, we define a random variable $X_{e,e'}$ by
$X_{e,e'} = 1$ if $e$ and $e'$ cross in $D$ and $X_{e,e'} = 0$ otherwise. Let $S$ be a 4-point subset of $V$. Under the condition that the set of endpoints of $e$ and $e'$ in $D$ is $S$, these edges cross one another in $D$ with probability 1/3. It follows that $X_{e,e'} = 1$ with probability 1/3. Note that $obf(D) = \sum_{\{e,e'\}} X_{e,e'}$. By linearity of the expectation, we have $E[obf(D)] = \sum_{\{e,e'\}} E[X_{e,e'}] = \frac{1}{3} \epsilon(G)$ and hence $obf(D) \geq \frac{1}{3} \epsilon(G)$ for at least one instance $D$ of $D$. Such a $D$ is efficiently constructible by standard derandomization techniques, namely, by the method of conditional expectations, see, e.g., [1, Chapter 15].

As a consequence of Theorem 1, we have $obf(G) = \Theta(n^2)$ for a planar $G$ whenever $\delta(G) \geq 2$ (the latter condition excludes the cases like $obf(K_{1,s}) = 0$). Indeed, $\epsilon(G) < \frac{9}{2} n^2$ because $m < 3n$ for any planar graph. This bound is sharp in the sense that $\epsilon(G) \geq \frac{9}{2} n^2 - O(n)$ for maximal planar graphs of bounded vertex degree. A sharp lower bound for $\epsilon(G)$ is stated below.

**Theorem 2.** $\epsilon(G) \geq \left( \frac{\delta(G)^2}{8} - o(1) \right)n^2$ for a planar graph $G$ with $\delta(G) \geq 2$. The constant $\delta(G)^2/8$ cannot be better here.

**Proof.** Let $A_k(G) = \{ v \in V(G) : \deg v < k \}$ and denote

$$a_k(G) = |A_k(G)| \quad \text{and} \quad s_k(G) = \sum_{v \in V(G) \setminus A_k(G)} \deg v.$$  

West and Will [12] prove that, if $k \geq 12$, then for every planar $G$ on $n \geq \frac{3}{2}k-1$ vertices we have

$$a_k(G) \geq \frac{(k-8)n + 16}{k-6}$$

and

$$s_k(G) < 2n - 16 + \frac{12(n-8)}{k-6}.$$  

We begin with the bound

$$\epsilon(G) > \frac{1}{2} \left( m^2 - \sum_{v \in V(G)} \deg^2 v \right).$$

Set $\delta = \delta(G)$. Let $\sigma = s_k(G)/n$ (to simplify the notation, we do not indicate the dependence of $\sigma$ on $k$). Suppose that $k$ is large enough, namely, $k \geq 14$. 

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Note that $0 \leq \sigma < 2 + 12/(k - 6)$. We now estimate $m$ from below and $\sum_v \deg^2 v$ from above.

$$m = \frac{1}{2} \sum_v \deg v = \frac{1}{2} \left( \sum_{v \in A_k(G)} \deg v + \sum_{v \notin A_k(G)} \deg v \right) \geq \frac{1}{2} \left( \delta(G) a_k(G) + s_k(G) \right) > \frac{1}{2} \left( \frac{\delta(k - 8)}{k - 6} + \sigma \right) n.$$

Furthermore,

$$\sum_v \deg^2 v = \sum_{v \in A_k(G)} \deg^2 v + \sum_{v \notin A_k(G)} \deg^2 v < (k - 1)^2 n + f(\sigma) n^2,$$

where

$$f(\sigma) = \begin{cases} 
2 + (\sigma - 2)^2 & \text{if } 2 \leq \sigma < 2 + 12/(k - 6), \\
1 + (\sigma - 1)^2 & \text{if } 1 \leq \sigma < 2, \\
\sigma^2 & \text{if } 0 \leq \sigma < 1.
\end{cases}$$

Thus,

$$\epsilon(G) > g(\sigma) n^2 - \frac{(k - 1)^2}{2} n, \text{ where } g(\sigma) = \frac{1}{2} \left( \frac{1}{4} \left( \frac{\delta(k - 8)}{k - 6} + \sigma \right)^2 - f(\sigma) \right).$$

A routine calculation shows that

$$\min \left\{ g(\sigma) : 0 \leq \sigma < 2 + \frac{12}{k - 6} \right\} = g(0) = \frac{\delta^2}{8} \left( \frac{k - 8}{k - 6} \right)^2.$$

We conclude that

$$\epsilon(G) > \frac{\delta^2}{8} \left( \frac{k - 8}{k - 6} \right)^2 n^2 - \frac{(k - 1)^2}{2} n > \left( \frac{\delta^2}{8} - \frac{\delta^2}{2(k - 6)} - \frac{(k - 1)^2}{2 n} \right) n^2$$

whenever $k \geq 14$ and $n \geq \frac{3}{2} k - 1$. Recall that $\delta(G) \leq 5$ for any planar $G$. If we make $k$ a function of $n$ that grows to the infinity slower than $\sqrt{n}$, then the factor in front of $n^2$ becomes $\delta^2/8 - o(1)$ and we arrive at the claimed bound.

The optimality of the constant $\delta^2/8$ is ensured by regular planar graphs (i.e., cycles and cubic, quartic, and quintic planar graphs).
As was already mentioned, for planar graphs we have $obf(G) \leq \epsilon(G) < \frac{9}{7} n^2$, where the bound for $\epsilon(G)$ cannot be improved. However, for $obf(G)$ we can do somewhat better.

**Theorem 3.** $obf(G) < 3 n^2$ for a planar graph $G$ on $n$ vertices.

**Proof.** Note that, if $K$ is a subgraph of $H$, then $obf(K) \leq obf(H)$. It therefore suffices to prove the theorem for the case that $G$ is a maximal planar graph, that is, a triangulation. Let $E$ be a (crossing-free, not necessary straight line) plane embedding of $G$. Denote the number of triangular faces in $E$ by $t$ and note that $3t = 2m$. Based only on facial triangles, let us estimate from below the number of non-crossing edge pairs in an arbitrary straight line drawing $D$ of $G$. Let $P$ denote the set of all pairs of adjacent edges occurring in facial triangles. Here we have $|P| = 3t$ edge pairs which are non-crossing in $D$. Furthermore, for each pair of edge-disjoint facial triangles $\{T, T'\}$ we take into account pairs of non-crossing edges $\{e, e'\}$ with $e$ from $T$ and $e'$ from $T'$. Since at most $3t/2$ pairs of facial triangles can share an edge, there are at least $\left(\frac{t}{2}\right) - \frac{3t}{2}$ such $\{T, T'\}$. We split this amount into two parts. Let $A$ consist of vertex-disjoint $\{T, T'\}$ and $B$ consist of $\{T, T'\}$ sharing one vertex. As easily seen, every $\{T, T'\}$ in $A$ gives us at least 3 edge pairs $\{e, e'\}$ which are non-crossing in $D$. Every $\{T, T'\}$ in $B$ contributes at least 2 pairs of non-adjacent edges and exactly 4 pairs of adjacent edges. However, 2 of the latter 4 edge pairs can participate in $P$. We conclude that in $D$ there are at least $|P| + (3|A| + 4|B|)/4$ non-crossing edge pairs. The factor of 1/4 in the latter term is needed because an edge pair $\{e, e'\}$ can be contributed by 4 triangle pairs $\{T, T'\}$. Thus,

$$obf(D) \leq \left(\frac{m}{2}\right) - 3t - \frac{3}{4} \left(\frac{t}{2} - \frac{3t}{2}\right) < \frac{1}{2} m^2 - \frac{3}{8} t^2 = \frac{1}{3} m^2.$$

Since $m < 3n$ as a simple consequence of Euler’s formula, we have $obf(D) < 3n^2$. As $D$ is arbitrary, the bound for $obf(G)$ follows.

### 3 Estimation of the shift complexity

A basic fact about $shift(G)$ is that this number is well defined.
Theorem 4 (Wagner, Fáry, Stein (see, e.g., [6])). Every planar graph $G$ has a straight line plane drawing. In other words, shift($G$) $\leq n - 3$ if $n \geq 3$.

If we seek for lower bounds, the following example is instructive despite its simplicity: shift($mK_2$) $= m - 1$. It immediately follows that

$$\text{shift}(G) \geq \nu(G) - 1.$$ 

Theorem 5. Let $G$ be a connected planar graph on $n$ vertices.

1. If $\delta(G) \geq 3$ (in particular, if $G$ is 3-connected) and $n \geq 10$, then
   \[ \text{shift}(G) \geq \frac{n - 1}{3}. \]

2. If $G$ is 4-connected, then $\text{shift}(G) \geq \frac{n - 3}{2}$.

3. There is an infinite family of connected planar graphs $G$ with $\delta(G) = 2$ and $\text{shift}(G) \leq 2$.

Proof. Item 1 follows from the fact that, under the stated conditions on $G$, we have $\nu(G) \geq (n + 2)/3$ (Nishizeki-Baybars [5]). Item 2 is true because every 4-connected planar $G$ is Hamiltonian (Tutte [11]) and hence $\nu(G) \geq (n-1)/2$ in this case. Item 3 is due to the bound $\text{shift}(K_{2,s}) \leq 2$. The latter follows from the elementary fact of plane geometry stated in Lemma 6 below.

Lemma 6. For any finite set of points $Z$ there are two points $x$ and $y$ such that the segments with one endpoint in $\{x, y\}$ and the other in $Z$ do not cross each other and have no inner points in $Z$.

Proof. Let $L$ denote the set of all lines going through at least two points in $Z$. Fix the direction “upward” not in parallel to any line in $L$. Pick up $x$ above every line in $L$ and $y$ below every line in $L$.

The next question we address is this: How close is relationship between shift($G$) and $\nu(G)$? By Theorem 5, if $\delta(G) \geq 3$ then both graph parameters are linear. However, if $\delta(G) \leq 2$, the existence of a large matching is not the only cause of large shift complexity.

Theorem 7. There is a planar graph $G_s$ on $3s + 3$ vertices with $\delta(G_s) = 2$ such that $\nu(G_s) = 3$ and $\text{shift}(G_s) \geq 2s - 6$. 

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Proof. A suitable $G_s$ can be obtained as follows: take the multigraph which is triangle with multiplicity of every edge $s$ and make it graph by inserting a new vertex in each of the $3s$ edges (see Fig. 1). Using Lemma 6, it is not hard to show that $\text{shift}(G_s) \leq 2s + 3$. We now construct a drawing $D_s$ of $G_s$ with $\text{shift}(D_s) \geq 2s - 6$. Put vertices $z_1, \ldots, z_{3s}$ in this order in a line and the remaining vertices $c_0, c_1, c_2$ somewhere else in the plane. Connect $z_i$ with $c_j$ iff $j \neq i \mod 3$. Therewith $D_s$ is specified. Denote the fragment of $D_s$ induced on $\{z_1, z_2, z_4, z_5, c_0, c_1, c_2\}$ by $F$. It is not hard to see that $F$ cannot be disentangled by moving only $c_0, c_1, c_2$. In fact, if in place of $z_1, z_2, z_4, z_5$ we take any quadruple $z_i, z_j, z_k, z_l$ with $i < j < k < l, i \equiv k \mod 3,$ and $j \equiv l \mod 3$, this will give us a fragment completely similar to $F$. To destroy all such fragments, we need to move at least two vertices in every triple $z_{3h+1}, z_{3h+2}, z_{3h+3} (0 \leq h < s)$ with possible exception for at most 3 of them. Therefore, making $2(s - 3)$ shifts is unavoidable. 

Finally, we prove a complexity result.

**Theorem 8.** Computing the shift complexity of a given drawing is an NP-hard problem.

**Proof.** In fact, this hardness result is true even for drawings of graphs $mK_2$. Given such a drawing $D$, consider its intersection graph $S_D$ whose vertices are the edges of $D$ with $e$ and $e'$ adjacent in $S_D$ iff they cross one another in $D$. Since computing the independence number of intersection graphs of segments in the plane is known to be NP-hard (Kratochvíl-Nešetřil [4]), it
suffices for us to express $\alpha(S_D)$ as a simple function of $shift(D)$. Fix an optimal way of untangling $D$ and denote the set of edges whose position was not changed by $E$. Clearly, $E$ is an independent set in $S_D$ and hence $shift(D) \geq m - |E| \geq m - \alpha(S_D)$. On the other hand, $shift(D) \leq m - \alpha(S_D)$. Indeed, fix an independent set $I$ in $S_D$ of the maximum size $\alpha(S_D)$. Then $D$ can be untangled this way: we leave the edges in $I$ unchanged and shrink each edge not in $I$ by shifting one endpoint sufficiently close to the other endpoint. Thus, $\alpha(S_D) = m - shift(D)$, as desired.

4 Concluding remarks and problems

1. By Theorem 1 we have $\frac{1}{3}\epsilon(G) \leq obf(G) \leq \epsilon(G)$. The upper bound cannot be improved in general as $obf(C_n) = \epsilon(C_n)$ for odd $n$. Can one improve the factor of $\frac{1}{3}$ in the lower bound?

2. By Theorems 1, 2, and 3 we have $(\delta(G))^2/24 - o(1)n^2 \leq obf(G) \leq 3n^2$ where $\delta(G) \geq 2$ is necessary for the lower bound. Optimize the factors in the left and the right hand sides.

3. As follows from the proof of Theorem 1 there is an $n$-point set $V$ (in fact, this can be an arbitrary set on the border of a convex body) with the following property: Every graph $G$ of order $n$ has a drawing $D$ with $V(D) = V$ such that $obf(D) \geq \frac{1}{3}obf(G)$. Can this uniformity result be strengthened? Is there an $n$-point set $V$ on which one can attain $obf(D) = obf(G)$ for all $n$-vertex $G$?

4. The following remarks show that the obfuscation and the shift complexity of a drawing have, in general, rather independent behavior.

Maximum $obf(D)$ does not imply maximum $shift(D)$. Consider $3K_{1,s}$, the union of 3 disjoint copies of the $s$-star. It is not hard to imagine how a drawing attaining $obf(3K_{1,s}) = 3s^2$ should look (where every two non-adjacent edges cross) and it becomes clear that such a drawing can be untangled just by 2 shifts. However, $shift(3K_{1,s}) \geq s$ is provable similarly to Theorem 7 (an upper bound $shift(3K_{1,s}) \leq s + 2$ follows from Lemma 6).

Maximum $shift(D)$ does not imply maximum $obf(D)$. The simplest example is given by a drawing of the disjoint union of $K_2$ and $K_{1,2}$ with only one edge crossing.
Large \( \text{obf}(D) \) does not imply large \( \text{shift}(D) \). This can be shown by drawings of \( \text{obf}(K_{2,s}) \). Indeed, we know that \( \text{obf}(K_{2,s}) = \binom{s}{2} \) from Section 2 and \( \text{shift}(K_{2,s}) \leq 2 \) from Section 3 (the latter bound is exact if \( s \geq 4 \)).

Large \( \text{shift}(D) \) does not imply large \( \text{obf}(D) \). Pach and Tardos \[8\] Fig. 2 show a drawing \( D \) of the cycle \( C_n \) with linear \( \text{shift}(D) \) and \( \text{obf}(D) = 1 \).

5. In spite of the observation we just made that large \( \text{obf}(D) \) does not imply large \( \text{shift}(D) \), in some interesting cases it does. Pach and Solymosi \[7\] prove that every system \( S \) of \( m \) segments in the plane with \( \Omega(m^2) \) crossings has two disjoint subsystems \( S_1 \) and \( S_2 \) with both \( |S_1| = \Omega(m) \) and \( |S_2| = \Omega(m) \) such that every segment in \( S_1 \) crosses all segments in \( S_2 \). As \( \text{shift}(S) \geq \min\{|S_1|,|S_2|\} \), this result has an interesting consequence: If \( D \) is a drawing of \( mK_2 \) with \( \text{obf}(D) = \Omega(m^2) \), then \( \text{shift}(D) = \Omega(m) \).

6. Theorem 8 shows that computing \( \text{shift}(D) \) for a drawing \( D \) of a graph \( G \) can be hard even in the cases when computing \( \text{shift}(G) \) is easy. Is \( \text{shift}(G) \) hard to compute in general? Theorem 1 shows that \( \text{obf}(G) \) is polynomial-time approximable within a factor of 3. Is exact computation of \( \text{obf}(G) \) NP-hard (Amin Coja-Oghlan)?

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