Measurement Dependence Inducing Latent Causal Models

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Abstract

We consider the task of causal structure learning over measurement dependence inducing latent (MeDIL) causal models. We show that this task can be framed in terms of the graph theoretical problem of finding edge clique covers, resulting in a simple algorithm for returning minimal MeDIL causal models (minMCMs). This algorithm is non-parametric, requiring no assumptions about linearity or Gaussianity. Furthermore, despite rather weak assumptions about the class of MeDIL causal models, we show that minimality in minMCMs implies three rather specific and interesting properties: first, minMCMs provide lower bounds on (i) the number of latent causal variables and (ii) the number of functional causal relations that are required to model a complex system at any level of granularity; second, a minMCM contains no causal links between the latent variables; and third, in contrast to factor analysis, a minMCM may require more latent than measurement variables.

1 Introduction

Despite the many theoretical and practical difficulties, establishing and understanding causal relationships remains one of the fundamental goals of scientific research. Consequently, many different approaches have been developed, with applications spanning a diverse range of fields, e.g., from epidemiology to econometrics to neuroimaging [12, 15, 16, 22, 30].

The Rubin causal model or RCM [11, 19, 20], employing the potential outcomes framework, draws heavily from one of the great successes of these many different causal approaches, the ‘gold standard’ of clinical trials, randomized controlled experiments. The Rubin causal model was primarily developed for these applications by focusing on how to quantify the strength of the (potentially indirect) causal effects of treatments or interventions.

Other approaches have focused on different aspects of causation, such as learning the causal structure among a set of variables. Roughly speaking, causal structure learning (CSL) typically focuses on identifying which variables (possibly including unobserved, latent variables) are directly causally related and how these direct causal relations form a structure over which indirect causal relations exist. Instead of the RCM, CSL employs the structural equation model (SEM) and especially its representation as a directed acyclic graph (DAG) [16, 17, 26, 30].

Advocates of the SEM and DAG argue that their models are general enough to subsume the RCM [10, 16, 23]. Nevertheless, we think it is important to emphasize the different contexts in which these models were developed and the different problems to which they are typically applied. This paper is an example of how analyzing such differences can lead to interesting theoretical insights as well as fruitful new algorithms for scientific application.

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More specifically, the basic approach in CSL (namely, the IC and PC algorithms [28, 32]) is to select a set of variables with the assumption that there exist no latent common causes of any of the selected variables (of course there have been many extensions and alternatives that relax this assumption, but the point here is to give a basic overview). These selected variables are then considered to be nodes in a DAG and their causal structure is learned, possibly even from purely observational data. Advocates of the RCM criticize this somewhat indiscriminate representation of variables, arguing that failing to specify treatments and effects beforehand can lead to non-meaningful causal statements. For instance, the causal statement IQ causes income, which can be expressed in a DAG, would be rejected in the potential outcomes framework, because a treatment that randomly assigns IQ to individual units, i.e., subjects, would be infeasible and arguably even nonsensical.

One way to obviate this criticism of CSL is to focus on measurement models [25]. These models explicitly distinguish between latent causal concepts and their measurements. This separation means that even though treatments are not specified beforehand, there is ambiguity left about what the latent variables and interventions thereon could be, while measurements are explicitly specified (e.g., there can be a latent structure relating the measurement variables “IQ test score” and “reported earned-income”). However, this separation comes at a cost. Inference over latent models is an ill-posed problem, i.e., there exists an infinitely large set of latent models that are consistent with a given set of observations [29]. In recent years, research on measurement models has focused on the introduction of additional assumptions that render this inference problem well-posed [13, 14, 25, 27]. However, we are interested in exploring a different approach. Instead of making strong assumptions about unmeasured variables, we accept the ambiguity of measurement models and study which causal properties characterize the latent variables regardless of which treatments and interventions they represent.

In Section 2 we define measurement dependence inducing latent (MeDIL) causal models to be the class of latent measurement models in which measurement variables can only be effects (and not causes—contrary to the usual definition measurement models explored by others), making no further assumptions about linearity or parametrizations of the distributions. We then introduce the notions of observational consistency and minimality, allowing us to, for a given (estimated) distribution of measurement variables, construct a minimal MeDIL causal model (minMCM). Then, in Section 3 by framing minMCMs as edge clique covers (ECCs) of the undirected dependency graph over measurement variables, we note how two notions of minimality emerge. Subsequently, despite our nonrestrictive assumptions and notion of minimality in minMCMs, we are able to prove (i) that a minMCM lower bounds the number of latent variables or the number of functional causal relations (depending on which notion of minimality is used), (ii) that the latent variables of the minMCM are all pairwise independent, and (iii) that (somewhat surprisingly) the minMCM can have more latent causes than measured variables. Then, in Section 4 we describe an algorithm for learning minMCMs from only unconditional (in)dependencies.

Our contribution in this paper is theoretical: by re-examining some of the foundational issues of causal discovery methods and questioning some of the common assumptions upon which these methods rely, we are able to articulate unexplored connections to problems in graph theory and produce a novel algorithm. Instead of attempting to estimate causal relations between latent variables, we employ CSL to shed light on the minimal complexity of the causal structure underlying observed data, regardless of its latent causal representation. In its abstract goal, our work is related to much of the work on latent measurement models and that of [24] in particular, as well as factor analysis [3]. However, we use weaker assumptions concerning linearity and non-Gaussianity, and a more specific notion of measurement variables, resulting in a significantly different approach, with connections to causal feature learning [4] and causal consistency [1, 18].

2 Minimal MeDIL causal models

We begin with a formal definition of measurement dependence inducing latent (MeDIL) causal models, before discussing the notion of observational consistency and its implications about minimality in such models.
We use functional causal models (FCMs) to describe causal relations in complex systems.

**Definition 1** (Functional Causal Model). A *functional causal model* is a triple \( M = (V, F, \epsilon) \), where

- \( V \) is the set of (endogenous) random variables,
- \( F \) is a set of functions defining each endogenous variable as a function of its direct causes (i.e., parents or \( \text{pa}(V_i) \)) and its corresponding exogenous random variable, so that for each \( V_i \in V \), we have \( V_i := f_i(\text{pa}(V_i), \epsilon_i) \). Furthermore, \( F \) is constrained such that no \( V_i \) is a direct cause of itself or any of its causes, removing the possibility of causal cycles.
- \( \epsilon \) defines a joint probability distribution over the exogenous (or noise) variables, with a corresponding \( \epsilon_i \) for each \( V_i \in V \), and with \( \epsilon_i \) being independent with \( \epsilon_j \) for each \( \epsilon_i, \epsilon_j \in \epsilon \).

The use of FCMs for causal modeling is criticized by advocates of the Rubin-Neyman framework, because endogenous variables in a FCM are used to represent causally effective concepts, interventions on these concepts, and measurements thereof \[11\]. One way to accommodate this criticism is by separating the set of endogenous variables into causally effective variables and their measurements, leading to the idea of MeDIL causal models:

**Definition 2** (Measurement Dependence Inducing Latent Causal Model (MCM)). A graphical MCM is a DAG, given by the triple \( G = (L, M, E) \). \( L \) and \( M \) are disjoint sets of vertices, while \( E \) is a set of directed edges between these vertices, subject to the following constraints:

1. all vertices in \( M \) have in-degree of at least 1 and out-degree of 0
2. all vertices in \( L \) have out-degree of at least 1
3. \( E \) contains no cycles

There are no further constraints as to the variety of distributions and functional causal relations that MCMs can represent, i.e., they are non-parametric and their arrows can represent arbitrary functional relations between variables. The formal constraints 1. and 2. in Definition 2 are to ensure that MCMs are applicable to settings in which we can explicitly separate into disjoint sets the measured effect variables \( M \) whose probabilistic dependencies must therefore be mediated by latent causes \( L \).

However, this separation comes at a cost. While causal structure learning over FCMs is a well-studied problem \[16, 30\], the explicit separation of cause and effect and the corresponding latent structure in MCMs introduces its own difficulties for inference. Namely, many latent models are consistent with a given probability distribution over observed effects, making the task of inferring a single latent model ill-posed. This is usually addressed by introducing strong assumptions, e.g., linearity. We, however, want to study what we can learn from CSL over MCMs without such assumptions. In order to help explain this consistency of different latent models and illustrate our strategy for restricting the problem so that inference is well-posed, consider the following definition and example.

**Definition 3** (Observational Consistency). A MCM is *observationally consistent* with a probability distribution over measurement variables if it is capable of inducing the pairwise dependencies (which can estimated from samples) of that distribution. Furthermore, two MCMs are observationally consistent with each other if there exists a distribution of measurements that they are both capable of inducing. This can be seen as a weakening of the notion of observational equivalence for use in our correspondingly weakened (by the inclusion of latents) notion of MCMs.

**Example 4** (Observational Consistency). Suppose we have data consisting of peoples’ answers to a questionnaire with four questions designed to measure depression and stress. We assume that

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1 We prefer this terminology to Structural Equation Model (SEM) because the latter is often used in econometrics to include assumptions of linear functions and Gaussian noise, while the former emphasizes our use of a non-parametric form that (arguably) subsumes RCMs \[10, 16\].

2 *observational or Markov equivalence* \[16\, pp. 16–20\] means two DAGs have the same skeletons and colliders, while observational consistency means that two MCMs have the same undirected dependency graphs over measurement variables (e.g., Figure 1).
answer to one question cannot cause the answer to another and therefore that the observed answers as well as any observed association between answers are the result of latent causes, such as depression or stress. Define random variables $M = \{M_1, M_2, M_3, M_4\}$ corresponding to answers to the four questions, and let them have only the following two pairwise independencies:

$$M_1 \perp \perp M_4 \quad \text{and} \quad M_2 \perp \perp M_3$$

The pairwise dependency structure between variables in $M$ is shown in Figure 1(a) and three observationally consistent MCMs are shown in 1(b), 1(c), 1(d). As this example demonstrates, multiple latent models can give rise to the same set of observed dependences.

![Figure 1](image.png)

Figure 1: (a) undirected dependency graph over $M$—notice two missing edges corresponding to independencies; (b) minimal MCM over $M$; (c) non-minimal MCM observationally consistent with $M$; (d) MCM corresponding to ICA or FA

One way of addressing this problem is by assuming linear dependencies and then selecting an appropriate subset of observed variables whose covariance matrices can be used to identify a unique number and structure of latent variables. [24, 25] We take a different approach, by making no such linearity assumptions and instead employing Ockham’s razor to pick a minimal MCM ($\text{minMCM}$) (e.g., Figure 1(b)).

**Definition 5** (Minimal MeDIL causal model ($\text{minMCM}$)). A $\text{minMCM}$ for a set of measurement variables $M$ is any least expressive (i.e., minimal) MCM that is observationally consistent with $M$.

As [17] note, a latent causal model’s expressive power can be measured by the (in)dependencies it induces over the measured variables, with more dependencies corresponding to more expressive power. Additionally, note that in our case minimality implies the Causal Faithfulness and Causal Markov assumptions:

1. in addition to being observationally consistent with its set of measurements, a $\text{minMCM}$ must induce the measurements without violating faithfulness (more dependencies can be induced by an unfaithful model than a faithful one, e.g., Figure 1(d))

2. considering arbitrary subsets of the latents, $Z \subseteq L$, there are as few dependencies of the form $M_i \notin \mathcal{L} | Z$ as (faithfully) possible, i.e., such dependencies only exist in an $\text{minMCM}$ if implied by $M$ without conditioning on (subsets of) the latents (e.g., unlike Figure 1(c)).

Thus, instead of using the (linear) covariance matrix itself as in [23], finding a $\text{minMCM}$ only requires independencies, which can be estimated from non-linear measures of dependence [3]. Estimating a set of independencies from the data before attempting CSL is also necessary for other methods, however our method needs only to consider unconditional independencies (unlike PC and IC, which require conditional independencies [28, 52]) between measurement variables, greatly reducing the number of independencies that must be estimated. The sufficiency of only considering unconditional independencies for our method follows from Proposition 6.

**Proposition 6.** In a MCM, the set of unconditional (in)dependencies over measurement variables fully determines the set of conditional (in)dependencies over measurement variables.

**Proof.** The Causal Markov and Causal Faithfulness assumptions (CMA and CFA, respectively) imply that two variables are probabilistically independent if and only if they are $d$-separated. Recall
Figure 2: (a) $D(M)$—undirected dependency graph of $M = \{M_1, \ldots, M_8\}$; (b) MCM, where each $C_i$ corresponds to a maximal clique in $D(M)$—dashed red edges/vertices are redundant for vertex-minimality while blue dotted edges/vertices are redundant for edge-minimality; (c) vertex-minimal $\min MCM$ of $D(M)$; (d) edge-minimal $\min MCM$ of $D(M)$

from Definition 2 that all dependence relations (and therefore, by the CMA and CFA, d-connections) between measurement variables are mediated by latent variables. Hence, all measurement variables have out-degree 0, and so any measurement variable in a path between two other measurement variables must be a collider and any dependent measurement variables must share at least one latent parent. This means that the set of unconditional (in)dependencies over measurement variables fully determines the set of conditional (in)dependencies as follows: for all $M_i, M_j, M_k \in M$,

- $M_i \perp \!\!\!\!\!\perp M_j \implies M_i \not\perp \!\!\!\!\!\perp M_j | M_k$
- $M_i \perp \!\!\!\!\!\perp M_j \implies \begin{cases} M_i \perp \!\!\!\!\!\perp M_j | M_k, & \text{if } M_i \perp \!\!\!\!\!\perp M_k \text{ or } M_j \perp \!\!\!\!\!\perp M_k \\ M_i \not\perp \!\!\!\!\!\perp M_j | M_k, & \text{otherwise} \end{cases}$
As we will see in Section 4, even though estimating conditional independencies is not required for our method, doing so nevertheless can help determine whether any of the assumptions have been violated.

3 Minimal MeDIL causal models as edge clique coverings

We can now present our main insight:

**Proposition 7.** The problem of finding a minMCM for a set of measurement variables can be framed as the graph theoretical problem of finding a minimum edge clique covering (ECC) [6–8, 21] over the corresponding undirected dependency graph of the measurement variables.

**Proof.** For a given set of measurement variables $M$, denote the undirected dependency graph as $D(M)$, e.g., Figure 1(a), where an edge represents dependence and the lack of an edge independence. Proposition 5 tells us that $D(M)$, though it only encodes unconditional (in)dependencies, contains all necessary information for characterizing observationally consistent MCMs. Consider the MCM $G = \langle L, M, E \rangle$ constructed from a set of cliques $C$ comprising a minimum ECC over $D(M)$ using the following procedure: (i) posit a latent $L_C \in L$ iff $C \in C$ and (ii) posit a directed edge $E \in E$ from the latent $L_C$ to the measurement variable $M$ iff $M \in C$. In other words, $G$ is a MCM with measurement variables $M$, one latent for each clique in the minimum ECC over $D(M)$, and an edge from each latent to exactly the measurement variables in the corresponding clique.

Note that $G$ is not only observationally consistent with $D(M)$ but also captures its independencies and is thus faithful, satisfying criterion 1. of Definition 5. Furthermore, the construction of $G$ from a minimum ECC ensures that latents are only posited when necessitated by the dependencies between measurements, satisfying criterion 2. of Definition 5. Thus, $G$ is an minMCM for $D(M)$.

A minimum ECC can be minimal in two related but distinct ways: the original and more well-studied approach seeks the smallest number of cliques needed to cover all edges (this is equivalent to the intersection number [7]), while another justifiable approach is to seek an ECC requiring the fewest assignments of vertices to cliques. The corresponding interpretation for minMCMs is vertex-minimal (fewer cliques imply fewer latents imply fewer total vertices) and edge-minimal (fewer assignments of measurement vertices to cliques implies fewer directed edges from latent to measurement vertices), resulting in Proposition 8. There are some undirected dependency graphs for which the vertex-minimal and edge-minimal minMCMs are identical, such as figures 1 and 3, but this identity does not hold generally [6] (see Figure 2). In either approach to minimality, the resulting minMCM induces the same set of dependencies over measurement variables and thus has the same expressive power (w.r.t. the measurement variables). We thus see no straightforwardly principled way of picking one approach over the other, and so we present both in hopes that practitioners will use whichever one (or both) they judge most sensible/interesting for their particular application.

Regardless of which notion of minimality is used, minMCMs have some interesting properties. First, they lower bound (i) the number of causal concepts or (ii) the number of functional causal relations that are required to model measurements of a complex system at any level of granularity (Proposition 8). Second, minMCMs contain no causal links between the latent variables (Proposition 9). Finally, in contrast to factor analysis, a minMCM may require more latent than measurement variables (Proposition 10).

**Proposition 8.** For a given set of unconditional pairwise dependencies among measurement variables $M$, a minMCM gives a lower bound on the number of latent variables or edges (depending on the measure of minimality is used) required in any (faithful and observationally consistent) MCM.

**Proof.** This is a direct consequence of the construction of minMCMs from either the clique-minimum or assignment-minimum ECC of $D(M)$, as described in Proposition 7.

**Proposition 9.** In a minMCM, each latent variable is $d$-separated from every other latent variable.

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4 A minimum ECC over an undirected graph is a collection of cliques that exactly covers its edges, where an edge $E = (V_i, V_j)$ is covered by clique $C$ iff $V_i, V_j \in C$. 
Proof. Intuitively, this is a result of the definition of a minMCM being minimal in the sense of least expressive (and thus having as few latents or edges): if two latent variables are d-connected, then the dependencies among measurement variables that they induce could also instead be induced by a single latent variable (which also results in fewer edges). A minMCM has no redundant latent variables or edges and therefore no d-connected latent variables. For example, note in the MCMs in figures 1(b) and 1(c) induce the same d-separations over the measurement variables, but that 1(b) with its d-separated latents has the fewer latents and fewer total edges. More formally, this also follows directly from procedure for constructing an minMCM in Proposition 7 and Algorithm 1.

![Diagram](image)

Figure 3: (a) example D(M) for which the minMCM (b) has 6 measurement variables and 7 latent variables

Proposition 10. There exist minMCMs containing more latent than measurement variables.

Proof. This follows from the graph theoretical characterization of minMCMs: there are at least as many latent variables as the intersection number of D(M), which in a graph with n vertices is (non-trivially) upper bounded by $\frac{n^2}{4}$ [7]. A simple example can be found when D(M) is as in Figure 3, resulting in n = 6 nodes and an intersection number of $i = 7$.

4 An minMCM-finding algorithm and its complexity

The procedure in the proof of Proposition 7 for constructing a minMCM from an undirected dependency graph leads directly to the following algorithm.

Algorithm 1: constructing a minimal MeDIL causal model (minMCM)

**Input**: undirected dependency graph, D(M), over the measurement variables M

**Output**: vertex-minimal or assignment-minimal MCM $G$ over M

1. initialize edgeless graph with a vertex for each $M \in M$;
2. find a clique-minimum or assignment-minimum edge clique cover of D(M), using the algorithm in Fig. 3 of [8] or the algorithm FIND-AM of [6], respectively;
3. for each clique $C$ in the cover do
   4. add vertex $L$ with edges directed to each $M \in C$;
5. end

Notice that Line 2 in Algorithm 1 is to find a minimum ECC of D(M). Nearly all of the computational complexity of Algorithm 1 comes from this step, which is known to be an NP-hard problem, and so the choice of an efficient ECC-finding algorithm and implementation is especially important.

In case a clique-minimum ECC (and therefore vertex-minimum minMCM) is preferred, [8] provides an exact algorithm. The exact algorithm finds an ECC in $O(f(2^k) + n^4)$ time, where $k$ is the number of cliques in the ECC and $n$ is the number of vertices in the undirected graph, and is thus fixed-parameter tractable. Furthermore, [5] gives a lower bound on the complexity of the clique-minimum ECC problem and argues that the algorithm is probably optimal. [8] also provide a free/libre implementation of their algorithm, showing that it runs in a reasonable amount of time (less than 10 minutes) up to $k=100$ [6, 8].

In case an assignment-minimum ECC (and therefore edge-minimum minMCM) is preferred, [6] provides an exact algorithm. Though they do not offer an analyses of its complexity, it is essentially a
backtracking algorithm based on [2]'s maximal clique finding algorithm, which has time complexity of \( O(3^{3n/3}) \), and so this assignment-minimum ECC finding algorithm has an even larger complexity. Though [6] do not provide a free/libre implementation of their algorithm, they claim that it runs in a reasonable amount of time (also within 10 minutes for up to 100 cliques).

5 Discussion

Reiterating the contribution of this paper: we are motivated by (i) an interest in exploring what can be learned about the causal structure of latent variables while relaxing some typical assumptions (such as linearity or Gaussianity), as well as (ii) the RCM and potential outcomes framework emphasis and explicit representation of the theoretical distinctiveness of latent causes (treatments) and their measured effects. By re-examining foundational issues of causal discovery methods, we are able to draw on some untapped advances in graph theory to produce a novel, principled method for causal structure learning over latent variables. Instead of estimating functional causal relations between (latent) variables, we provide insights into the representation of latent causes via their measured effects, resulting in a method that is non-parametric and requires no assumptions about linearity or Gaussianity.

Having in the preceding sections presented our minMCM finding algorithm and its supporting theory, we now briefly discuss important considerations for its application as well as promising directions for future work.

Notice that the input to Algorithm 1 is an undirected dependency graph, while in practice one does not have direct knowledge of the (in)dependencies themselves but only samples of the measurement variables. It is therefore necessary to first estimate the independencies before applying this algorithm. For other causal inference methods, one common and rather easy way of doing this is to use the covariance as a test statistic in permutation-based hypothesis testing. However, covariance is only a measure of linear dependence and has the property that “\( X \perp \!\!\!\perp Y \implies \text{cov}(X, Y) = 0 \)” but not the converse. Thus, covariance is lacking as a statistic for hypothesis testing of independence, especially if one is interested in also detecting non-linear dependencies. Fortunately, better-suited test statistics can be found in, e.g., the Herbert-Schmidt Independence Criterion [9] or the distance covariance [31], which in addition to measuring non-linear dependence also have the property that “\( X \perp \!\!\!\perp Y \iff \text{dist.cov}(X, Y) = 0 \)”.

Of course, after estimating the independencies, one must actually apply Algorithm 1. Note that Line 2 requires finding either a clique-minimal or assignment-minimal ECC [6, 8]. Though algorithms exist for both, the free/libre source code of the former is outdated, unmaintained, and not easily compilable on any up-to-date operating system, while no free/libre source code is provided for the latter. To remedy this, we will provide a free/libre Python implementation of Algorithm 1, both ECC finding algorithms, as well as software for performing permutation-based hypothesis testing using the distance covariance.

Though applications of Algorithm 1 are beyond the scope of this paper, we note the following potential pitfall. It is a constraint-based algorithm, relying on estimated independencies. Thus, errors in the inference of minMCMs come not from Algorithm 1 itself but rather from the estimation of independencies that it (along with many other causal inference methods) requires as input. In this regard, a single incorrectly estimated independence can in the (unlikely) worst case result in incorrectly doubling or halving the number of estimated latents or edges. In any case, as mentioned at the end of Section 2, further estimates of conditional independencies can help corroborate or refute the estimated unconditional independencies. More detailed examination is needed to make this more theoretically precise as well as to determine how much of a problem this is likely to pose for real data.

One final caveat for interpreting minMCMs is that, for complex graphs, there can be multiple minimum ECCs (for both types of minimality), each with the same minimum number of cliques or assignments. Thus, while using a minMCM to reason about the minimum number of edges or latents

3The code and documentation will be available at the project homepage, https://medil.causal.dev
4This would be when the inclusion/exclusion of a single edge in an \( n \geq 3 \) vertex undirected dependency graph makes the difference between the graph having \( 2(n-2) \) maximal cliques that are all edges and \( n-2 \) maximal cliques that are all triangles.
is always valid, stronger conclusions may require that the graph $D(M)$ admits only one minMCM (which is simple enough to test) or that further assumptions or background knowledge are used to justify one minMCM over another observationally consistent one. To this end, the MCM corresponding to maximal cliques (e.g., Figure 2(b)) may be especially interesting, because it contains all observationally consistent MCMs (including the minMCMs in 2(c) and 2(d)).

Some obvious next steps for future work include finding minMCMs from real data and comparing them to the results of other CSL methods as well as factor analysis. An interesting direction for future work would be (i) extending the algorithm to make better use of the results of interventions, and (ii) investigating and creating score-based analogs to this method, allowing for the inclusion of priors about independence structure (e.g., clustering coefficients, or the (in)dependence of certain groups of measurements). This could potentially lead to the development of a causal, non-parametric generalization of factor analysis. Another interesting direction for possible future work is in relating minMCMs—which offer a minimal or most-macrosopic characterization of the set of observationally consistent latent models—to work on causal feature learning [4] and causally consistent transformations between different causal representations [18] as well as various other ways of abstracting causal models [1].

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