Superconducting ground state of a doped Mott insulator

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Abstract. A d-wave superconducting ground state for a doped Mott insulator is obtained. It is distinguished from a Gutzwiller-projected Bardeen–Cooper–Schrieffer (BCS) superconductor by an explicit separation of Cooper pairing and resonating valence bond (RVB) pairing. Such a state satisfies the precise sign structure of the \( t-J \) model, just as a BCS state satisfies the Fermi–Dirac statistics. This new class of wavefunctions can be intrinsically characterized and effectively manipulated by electron fractionalization with neutral spinons and ‘backflow’ spinons forming a two-component RVB structure. While the former spinon is bosonic, originating from the superexchange correlation, the latter spinon is found to be fermionic, accompanying the hopping of bosonic holons. The low-lying emergent gauge fields associated with such a specific fractionalization are of mutual Chern–Simons type. Corresponding to this superconducting ground state, three types of elementary excitations are identified. Among them a Bogoliubov nodal quasiparticle is conventional, while the other two are neutral excitations of non-BCS type that play crucial roles in higher-energy/temperature regimes. Their unique experimental implications for the cuprates are briefly discussed.
1. Introduction

An important issue in the study of high-$T_c$ cuprates concerns how superconductivity can arise in a doped Mott insulator [1]. The d-wave pairing symmetry has usually been attributed to the fact that the electrons avoid a strong local Coulomb repulsion. But in a doped Mott insulator the on-site Coulomb repulsion is so strong that the Hilbert space of the electrons is also drastically altered. It is thus no longer sufficient just to focus on the relevant attractive interaction as in the framework of the Bardeen–Cooper–Schrieffer (BCS) theory. Rather a fundamental change in the underlying electronic structure should be taken into account before one can meaningfully address the issue of high-$T_c$ superconductivity.

The simplest straightforward method of incorporating superconductivity with the ‘Mott physics’ is to construct a Gutzwiller-projected BCS superconductor. As first envisaged by Anderson [1] in 1987, this class of state is always insulating at half-filling as all the double-occupancy states get projected out and the original Cooper pairs in the BCS wavefunction become neutralized, known as the spin-singlet resonating valence bond (RVB) pairs which are ‘glued’ by the superexchange coupling (the state will be referred to as the fermionic RVB state in the following). Superconductivity arises only away from half-filling when the RVB pairs start to move in the presence of, say, empty sites in the hole-doped case, which become partially charged Cooper pairs. Here the Cooper and RVB pairings are no longer explicitly distinguishable.

The Gutzwiller-projected d-wave BCS state has been studied ([2] and references therein; for a review, see [3]) intensively as a class of variational wavefunctions for the doped Mott insulator. It is also the basis for developing the so-called slave-boson approach [4], which is an electron fractionalization description with the Gutzwiller projection replaced by emergent gauge fluctuations around the spin-charge-separated saddle points. It predicted [5, 6] the presence of a high-temperature pseudogap phase over a finite doping regime and a superconducting dome at lower temperatures, which are both qualitatively consistent with the later experimental measurements in the cuprates [7, 8].

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However, the important long-range antiferromagnetic (AFM) correlations are notably missing in the Gutzwiller-projected BCS state at half-filling. In fact, it is a spin liquid with low-lying fermionic excitations [1, 5, 6], which is in sharp contrast to a long-range AFM order with bosonic spin-wave excitations governed by the two-dimensional (2D) Heisenberg model. Of course one may argue here that once in the presence of some finite concentration of doped holes, the long-range AFM order or correlations will disappear anyway, and hence the Gutzwiller-projected BCS/spin liquid state could become stabilized eventually as a competitive ground state [1, 4].

But it remains a real challenge to understand how the long-range AFM order/correlations of an antiferromagnet/Mott insulator can be effectively destroyed by the motion of the doped holes, and how the associated energy, albeit only a small fraction in the total superexchange energy [9], gets turned off in order to gain the kinetic energy of the doped holes. The issue at the heart of a doped Mott insulator is the competition between the kinetic and superexchange energies—if the AFM correlations are weakened, like in the Gutzwiller-projected BCS state, it would be much more favorable to the hopping of the doped holes; with the enhanced AFM correlations at low doping, on the other hand, the kinetic energy of the doped holes will get strongly suppressed. The novelty inherent from such incompatibility and competition between the hopping and the superexchange processes is thus expected [10] to be responsible for the unconventional nature of the superconducting transition as well as a complex pseudogap phenomenon over a wide temperature regime above $T_c$ in a doped antiferromagnet/Mott insulator.

Therefore, to properly accommodate such novelty, which may provide a basic understanding of the rich and marvelous pseudogap properties in the cuprates [7, 8], both the RVB and Cooper channels should remain generally distinguished even in the superconducting regime, which implies the necessity for one to go beyond the simple Gutzwiller-projected BCS state approach to adequately address the interplay between the magnetism and superconductivity.

Another important property that the wavefunction should obey is an altered statistical sign rule: although the electrons always obey the Fermi–Dirac statistics, which dictates that two electrons of the same spin cannot stay at the same lattice site, new statistics [11] will emerge in a doped Mott insulator where the no double occupancy constraint further enforces that two electrons of opposite spins cannot occupy the same site. In fact, the $t$–$J$ model at half-filling is totally ‘bosonized’ in the restricted Hilbert space where the usual fermion signs are completely diminished: for example, the ground state only possesses the trivial Marshall signs [9, 12], which can be easily gauged away. The nontrivial signs start to reemerge in the doped case, induced by the hopping of the doped holes, which is precisely described by the so-called phase string effect [13–15] in the $t$–$J$ model. The corresponding sign structure is actually independent of temperature, dimensionality, and is of statistical nature [14, 15], which eventually recovers the full Fermi statistical signs only at high doping in the dilute electron limit. Physically the phase string effect also provides an accurate mathematical description of the so-called ‘unrenormalizable phase shift’ first emphasized by Anderson early on [16]. The latter is a total phase shift added up from all the electrons in the ground state, in response to adding/removing an electron into/from the system, due to the strong on-site Coulomb repulsion. Consequently, the irreparable phase string effect/unrenormalizable phase shift will make the Cooper pairing, associated with the doped holes, intrinsically distinguished from the neutral spin RVB pairing caused by superexchange, again suggesting the necessity to go beyond the Gutzwiller-projected BCS state description.
A superconducting ground state distinct from the Gutzwiller-projected BCS state has been previously constructed by the present author and coauthors [17], incorporating the above-mentioned sign structure. Such a ground state can naturally reduce to an insulating AFM state at half-filling, which well accounts for the long-range AFM order as well as the short-range spin–spin correlations with a highly accurate variational superexchange energy, known as the bosonic RVB state [9, 17]. In contrast to the aforementioned fermionic RVB state, the doped holes are quite unfavorable to hop in the bosonic RVB (neutral spin) background, which only involves the spin pairing between different sublattices. It was then shown [17] that the doped holes will force a fundamental change in the RVB structure in order to gain the kinetic energy at finite doping, characterized by emergent ‘backflow spinons’ accompanying the hole hopping [17]. It is these ‘backflow spinons’ that will be associated with the Cooper pairs instead of the original bosonic RVB pairing. Consequently, both the bosonic RVB and Cooper channels remain explicitly separated in the superconducting state (cf equation (84) in [17]). Such a wavefunction description has demonstrated a rich complexity in the pseudogap regime as resulting from the competition between the RVB and Cooper channels.

In this paper, we show that an important simplification in this approach can be made by realizing that the aforementioned ‘backflow spinons’ are actually fermionic upon a closer re-examination of the sign structure. In the previous formulation, they are described in the bosonic representation [17], which causes unnecessary complications because the extra fermionic statistical signs are mixed with the intrinsic phase string effect (cf section 2.1.2). As a result, we obtain a greatly simplified self-consistent description of both the ground state and excitations for the doped Mott insulator.

The key results are summarized in section 2. The general form of the superconducting ground state is presented in section 2.1, which is distinguished from the Gutzwiller-projected BCS superconductor by a novel separation of Cooper and RVB pairings. It precisely satisfies the altered statistical sign rule of the \( t-J \) model in the restricted Hilbert space, which is of mutual semion type instead of the Fermi–Dirac one. In section 2.2, it is shown that such a new class of wavefunctions can be intrinsically characterized by electron fractionalization, where neutral bosonic spinons and ‘backflow’ fermionic spinons together constitute a two-component RVB structure. The low-lying emergent gauge fields associated with such a specific fractionalization are of mutual Chern–Simons type, whose origin can be directly connected to the precise sign structure of the \( t-J \) model. The fractionalization formalism also makes the manipulation of the ground state significantly simplified as the constituent subsystems are more conventional, governed by the effective Hamiltonians presented in section 2.2. Finally, corresponding to this superconducting ground state, three distinctive elementary excitations are briefly discussed in section 2.3. Among them, a Bogoliubov nodal quasiparticle is conventional, while the other two are non-BCS-like neutral excitations that play dominant roles in higher-energy/temperature regimes, controlling the superconducting phase transition and other exotic properties different from a conventional d-wave BCS superconductor.

In section 3, a microscopic justification of the present approach is presented in detail. It is based on a full bosonization formulation known as the phase string representation [14] of the \( t-J \) model, in which the whole nontrivial sign structure is explicitly captured by a topological (mutual statistical) gauge structure. Then we show that such a formalism under the no double occupancy constraint leads to the introduction of the two-component spinons in order to adequately describe the microscopically distinctive superexchange and hopping processes. Such a new exact formulation provides a precise starting point that naturally results in an
Figure 1. Schematic illustration of the Gutzwiller-projected BCS state given in equation (1). (a) Half-filling: singlet electron (Cooper) pairs reduce to the neutral RVB pairs (blue-colored bonds); (b) hole doping: the RVB pairs become partially charged Cooper pairs as they can hop to the hole sites (e.g. as indicated by yellow arrows).

electron-fractionalized description of the superconducting ground state at a finite doping, and a highly accurate bosonic RVB description of the AFM correlation in the zero doping limit. An effective theory of the elementary excitations is also obtained within the same framework. Finally, the conclusion and perspective are presented in section 4.

2. Key results

We present this section to summarize the key equations/results of this work, which basically addresses the issue of how the ground state of a Heisenberg antiferromagnet/Mott insulator can be turned into a superconducting ground state by doping.

2.1. Ground state ansatz

For comparison, let us start with the Gutzwiller-projected BCS state ansatz, proposed [1] for the \( t-J \) Hamiltonian on a 2D square lattice, given by

\[
|\Psi_{RVB}\rangle = \hat{P}_G |d\text{-BCS}\rangle,
\]

where \( |d\text{-BCS}\rangle \) denotes an ordinary d-wave BCS state and \( \hat{P}_G \) is a Gutzwiller projection operator enforcing the following no double occupancy constraint:

\[
\sum_{\sigma} c_{i\sigma}^\dagger c_{i\sigma} \leq 1.
\]

Because of \( \hat{P}_G \), the Cooper pairing in \( |d\text{-BCS}\rangle \) reduces to the neutralized RVB pairing [1] at half-filling, whereas at finite doping the Cooper and RVB pairings are not explicitly distinguished, as schematically illustrated in figure 1.

By contrast, the superconducting ground state obtained in this work may be formally written as

\[
|\Psi_G\rangle = \Lambda_h \left( \sum_{ij} g_{ij} c_{i\uparrow} c_{j\downarrow} \right)^{n_h \over 2} |\text{RVB}\rangle
\]

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Figure 2. Schematic illustration of the present ground state (3) or (20), which is structurally different from the Gutzwiller projected BCS state shown in figure 1. (a) The bosonic RVB pairs in $|\text{RVB}\rangle$: each pair only involves spin partners at opposite sublattice sites; (b) doped holes are created by annihilating (as indicated by the dashed arrows which also represent the backflow spinons in the fractionalization formulation (20)) the spins at the hole sites, and their RVB partners in $|\text{RVB}\rangle$ automatically become associated with the doped holes; (c) these partner spins associated with the holes can also form RVB pairs, which facilitates the Cooper pairing of the doped holes. Note that the important sign structure of the wavefunction, i.e. $\Lambda_h$ in equation (3), is not directly shown here.

in which $|\text{RVB}\rangle$ denotes a neutral spin background that always remains half-filled as a Mott insulator, whereas the Cooper pairs associated with the doped holes are created by annihilating $N_h$ electrons from $|\text{RVB}\rangle$, which are in singlet, d-wave pairing with an amplitude $g_{i,j}$. Apparently such a superconducting state automatically satisfies the no double occupancy constraint without invoking the Gutzwiller projection $\hat{P}_G$ as in equation (1). In particular, it is distinguished from equation (1) by an explicit separation of the Cooper pairing and RVB pairing at finite doping.

Schematically the RVB pairing in $|\text{RVB}\rangle$ and the Cooper pairing in $|\Psi_G\rangle$ are illustrated in figures 2(a) and (b) as well as 2(c), respectively. One easily sees the distinction between the neutral RVB pairing in figure 2(a) and the Cooper pairing in figures 2(b) and (c). In the latter, a pair of holes are involved, in which each hole will be generally associated with a spin via an RVB amplitude (the blue-colored bond in figure 2(b)), since the hole is created by annihilating a spin whose RVB partner already pre-exists in $|\text{RVB}\rangle$. Note that the holes are mobile here and thus their spin partners are also automatically changing with the hopping. Two spin partners associated with a Cooper pair in figure 2(b) can become RVB-paired again to further gain the superexchange energy, which results in the configuration shown in figure 2(c) and serves the driving force for the Cooper pairing. In this sense, the superexchange/RVB pairing provides the ultimate pairing ‘glue’ for superconductivity.

With the Cooper and neutral RVB channels being explicitly differentiated in equation (3), generally some nontrivial phase shift effect will emerge, as introduced by $\Lambda_h$. Specifically, $\Lambda_h$ is given by

$$\Lambda_h \equiv \sum_{\{l_i\}} (n_{l_1}^h n_{l_2}^h \cdots n_{l_{N_h}}^h) \phi_h(l_1, l_2, \ldots, l_{N_h}) e^{-i(\hat{\Omega}_{l_1} + \hat{\Omega}_{l_2} + \cdots + \hat{\Omega}_{l_{N_h}})}, \tag{4}$$

where $n_{l_i}^h = 1 - \sum_{\sigma} c_{l_i \sigma}^\dagger c_{l_i \sigma} \geq 0$ denotes the hole occupation number at site $l_i$, and $\phi_h$ is a bosonic wavefunction symmetric with regard to the hole coordinates $\{l_i\} = l_1, l_2, \ldots, l_{N_h}$, which is
generally present to ensure gauge invariance of the phase shift fields. Here the phase shifts \( \{ \hat{\Omega}_h \} \), associated with the holes, will directly act on the 'ghost' spin liquid state \(|RVB\rangle\) to monitor the background spin correlations. In other words, \( \Lambda_h \) represents an 'entanglement' between these two channels whose physical implications and mathematical definition are to be shown below.

2.1.1. Superconducting phase coherence. Due to the presence of \( \Lambda_h \), injecting a hole into the ground state \(|\Psi_G(N_h)\rangle\) will generally induce a phase shift by

\[
c_i|\Psi_G(N_h)\rangle \sim e^{i\hat{\Omega}_i} |\Psi_G(N_h+1)\rangle.
\]

Thus the wavefunction overlap between the bare hole state and the true ground state of \( N_h + 1 \) holes crucially depends on \( e^{i\hat{\Omega}_i} \).

Furthermore, by noting that the Cooper pairing amplitude already pre-exists via \( g_{ij} \) in equation (3), the superconducting off-diagonal-long-range-order (ODLRO) is essentially determined by (see below)

\[
\langle c_i \uparrow c_j \downarrow \rangle \propto \langle RVB| e^{i(\hat{\Omega}_i+\hat{\Omega}_j)} |RVB\rangle.
\]

Hence, the superconducting phase coherence and the coherence of a Landau (or more precisely, Bogoliubov) quasiparticle will be simultaneously realized. In other words, a 'normal state' obtained by disordering the phase shift factor \( e^{i\hat{\Omega}_i} \) will be intrinsically a non-Fermi liquid with a vanishing quasiparticle weight.

In the following, we provide a simple proof of equation (6) by taking

\[
\varphi_h = \text{constant}
\]

in \( \Lambda_h \) without loss of generality. Note that equation (3) then reduces to

\[
|\Psi_G\rangle \propto \hat{D}^{N_h} |RVB\rangle
\]

with

\[
\hat{D} \equiv \sum_{ij} g_{ij} \hat{D}_{ij}
\]

and

\[
\hat{D}_{ij} \equiv e^{-i(\hat{\Omega}_i+\hat{\Omega}_j)} c_i \uparrow c_j \downarrow.
\]

As shown in the appendix, one has \( \langle \hat{D} \rangle = O(N_h) \) according to equation (8). Then, so long as \( g_{ij} \langle \hat{D}_{ij} \rangle \) is a short-ranged function of \(|i - j|\), an ODLRO can be identified in \(|\Psi_G\rangle\):

\[
g_{ij} \langle \hat{D}_{ij} \rangle = O(\delta)
\]

with \( \delta \) as the doping concentration (\( \delta \equiv N_h/N \), where \( N \) denotes the total number of lattice sites). Generally speaking, equation (11) represents that the Cooper pairing amplitude is formed (with the pairing symmetry determined by \( g_{ij} \)). The true superconducting ODLRO, \( \langle c_i \uparrow c_j \downarrow \rangle \), is thus indeed determined by the phase coherence condition in equation (6), where \( \hat{\Omega}_i \) sensitively depends on the spin correlation in \(|RVB\rangle\).

2.1.2. Sign structure. The phase shift \( \hat{\Omega}_i \) is quantitatively given by

\[
e^{-i\hat{\Omega}_i} = e^{-\frac{i}{2} (\phi_i - \phi_j)}.
\]
in which
\[ \Phi^s_i \equiv \sum_{l \neq i} \theta_j(l) \left( \sum_{\sigma} \sigma n^b_{l\sigma} \right) \tag{13} \]
and
\[ \Phi^0_i \equiv \sum_{l \neq i} \theta_j(l), \tag{14} \]
where \( \theta_j(l) = \text{Im} \ln(z_i - z_l) \) (\( z_i \) is the complex coordinate of site \( i \)), and \( n^b_{l\sigma} \) denotes the spin occupation number (with index \( \sigma \)) at site \( l \), which always satisfies the single occupancy constraint
\[ \sum_{\sigma} n^b_{l\sigma} = 1 \tag{15} \]
acting on the insulating spin state \(|RVB\rangle\).

Then each spin in \(|RVB\rangle\) will contribute to a \( \pm \pi \) vortex via \( \Phi^s_i/2 \) in equation (12) with itself sitting at the vortex core. Conversely, each doped hole will be perceived by the spins in \(|RVB\rangle\) as introducing a \( \pi \) vortex, also via \( \Phi^s_i/2 \), with the hole sitting at the core. It implies that a doped hole and a neutral spin satisfy a ‘mutual semion statistics’ as the phase shift \( \hat{\Omega}_i \) amounts to giving rise to \( \pm \pi \) when one kind of species continuously circles around the other one once. Note that the single valuedness of equation (12) will be ensured by combining with \( \Phi^0_i/2 \). Thus the total phase shift added up in \( \Lambda_h \) represents a nontrivial entanglement between the doped holes and background spins, which will decide a ‘mutual semion statistics’ sign structure in \(|\Psi_G\rangle\) that is fundamentally different from that of a BCS state satisfying the Fermi–Dirac statistics.

One may examine such a sign structure by a thought experiment in which a hole in \(|\Psi_G\rangle\) goes through a closed loop \( c \). At each step of nearest-neighbor moving of the hole, a singular phase 0 or \( \pi \) is generated via a phase shift \( \hat{\Omega}_i \) in \( \Lambda_h \) depending on \( \uparrow \) or \( \downarrow \) spin that the hole ‘exchanges’ with. Note that \( \Lambda_h \) also produces other phase shift contributed by other spins not ‘exchanged’ with the hole, but their effect disappears after counting the total Berry’s phase acquired by the closed-path motion of the hole. In the end, one finds
\[ |\Psi_G\rangle \rightarrow (-1)^{N^\downarrow_{\downarrow}(c)} |\Psi_G\rangle \tag{16} \]
in which \( N^\downarrow_{\downarrow}(c) \) only counts the total number of \( \downarrow \)-spins that the hole has ‘exchanged’ with along the loop \( c \) on a square lattice.

The same sign factor \( (-1)^{N^\downarrow_{\downarrow}(c)} \) has previously been shown to be the precise sign acquired by the hopping of a doped hole through a closed loop \( c \) in the \( t-J \) model, i.e. the phase string effect [13–15]. This effect is proven to be dynamically irreparable and is thus of statistical nature. So the phase shift \( \hat{\Omega} \) defined in equation (12) is necessarily generated by the motion of a doped hole, representing an emergent new statistics [11] in doped Mott insulators with the Hilbert space restricted by the no double occupancy constraint.

It is noted that at finite doping, the exact topological sign structure identified based on the \( t-J \) model is generally given by [15]
\[ \tau_c = (-1)^{N^\downarrow_{\downarrow}(c)} \times (-1)^{N^\uparrow_{\downarrow}(c)}, \tag{17} \]
which appears, say, in the partition function
\[ Z = \sum_c \tau_c Z(c), \tag{18} \]
where \( Z(c) \geq 0 \) for any closed path \( c \) of the multi-hole/spin configurations at arbitrary temperature. Compared to equation (16), an extra sign factor \((-1)^{N_h(c)}\) appears in equation (17) in which \( N_h(c) \) counts the number of hole–hole exchanges on the path \( c \). It is straightforward to verify that fermionic signs of the doped holes created by \( \sum_{ij} g_{ij} c_i^\dagger c_j^\dagger \) in equation (3) can precisely account for such a sign factor. Consequently, combined with the phase shift in \( \Lambda_{h} \), the nontrivial sign structure of the \( t–J \) model is naturally satisfied by the ground state \( |\Psi_1\rangle \) in equation (3) so long as the neutral spin background \( |RVB\rangle \) does not contribute to additional statistical signs as shown below.

2.1.3. \( |RVB\rangle \) as a spin liquid state. As already mentioned, \( |RVB\rangle \) describes a ‘ghost’ spin state, which remains one spin at each site of a square lattice even in the doped case. But the spin state can evolve from an AFM long-range ordered one to a spin liquid with only short-range AFM correlations as the doping concentration is increased. Generically, \( |RVB\rangle \) can be expressed by

\[
|RVB\rangle = \sum_{\{\sigma_i\}} \Phi_{RVB} (\sigma_1, \sigma_2, \ldots, \sigma_N) c_{1\sigma_1}^\dagger c_{2\sigma_2}^\dagger \cdots c_{N\sigma_N}^\dagger |0\rangle
\]

in the electron \( c \)-operator representation. In nature, it is a bosonic state with the wavefunction \( \Phi_{RVB} (\{\sigma_i\}) \equiv \sum_{\text{partition}} \prod_{ij} (-1)^i W_{ij} \) for each given spin configuration \( \{\sigma_i\} = \sigma_1, \sigma_2, \ldots, \sigma_N \). Here the RVB pairing amplitude \((-1)^i W_{ij}\) connects two antiparallel spins denoted by \( i \) (up spin) and \( j \) (down spin), with the summation running over all possible pairing partitions for the given \( \{\sigma_i\} \). The staggered sign \((-1)^i \) (the Marshall sign) is explicitly separated from \( W_{ij} \) such that the latter remains a smooth function of the distance between even and odd lattice sites of a square lattice at different doping concentrations. Such a \( |RVB\rangle \) is a generalized Liang–Docout–Anderson-type bosonic RVB state, which can naturally recover the correct antiferromagnetism in the zero doping limit [9, 17]. But a long-range RVB pairing in the AFM phase will generally destroy the phase coherence condition in equation (6) because the \( \pm \pi \) vortices carried by the two spinon partners of a long-range RVB pair, according to equation (12), do not compensate each other and result in a phase disordering. Only a short-range RVB pairing can lead to a vortex–antivortex binding in equation (6) and thus the superconducting phase coherence. In other words, \( |RVB\rangle \) has to become a spin liquid in the superconducting phase. How the RVB amplitude \( W_{ij} \) evolves with doping and self-consistently becomes short-range in the superconducting state will be shown below.

2.2. Electron fractionalization

What is the physical implication of the explicit separation of the Cooper and RVB pairings in the ground state (3)? In the following, we show that it actually corresponds to a unique electron fractionalization.

2.2.1. Ground state in electron fractionalization form. The ground state \( |\Psi_G\rangle \) in equation (3) can be reformulated as a direct product state

\[
|\Psi_G\rangle = \hat{P} (|\Phi_h\rangle \otimes |\Phi_a\rangle \otimes |\Phi_h\rangle)
\]

(20)
The coefficients, \( \varphi_h, g_{ij}, \) and \( W_{ij}, \) appearing in the original \( |\Psi_G\rangle \) (cf section 2.1), are incorporated into three subsystem states as follows:

\[
|\Phi_h\rangle \equiv \sum_{\{l\}} \varphi_h(l_1, l_2, \ldots) h^\dagger_{i_1} h^\dagger_{i_2} \cdots |0\rangle_h,
\]

and

\[
|\Phi_a\rangle \equiv \exp\left(\sum_{ij} \tilde{g}_{ij} a^\dagger_{i\uparrow} a^\dagger_{j\downarrow}\right) |0\rangle_a,
\]

as well as

\[
|\Phi_b\rangle \equiv \exp\left(\sum_{ij} W_{ij} b^\dagger_{i\uparrow} b^\dagger_{j\downarrow}\right) |0\rangle_b.
\]

Here the bosonic wavefunction \( \varphi_h \) in \( |\Phi_h\rangle \) defines a ‘holon’ state with a bosonic creation operator \( h^\dagger \) acting on a vacuum \( |0\rangle_h; \) \( |\Phi_b\rangle \) defines a neutral ‘spinon’ state with an RVB pairing amplitude \( W_{ij}, \) where \( b^\dagger_\sigma \) as a bosonic creation operator acts on a vacuum \( |0\rangle_b; \) and \( |\Phi_a\rangle \) defines a ‘backflow spinon’ state with the pairing amplitude \( \tilde{g}_{ij} \equiv (-1)^i g_{ij}, \) where \( a^\dagger_\sigma \) denotes a fermionic creation operator acting on a vacuum \( |0\rangle_a. \)

The projection operator \( \hat{P} \) in equation (20) is defined by

\[
\hat{P} \equiv \hat{P}_h \hat{P}_s,
\]

in which \( \hat{P}_s \) will enforce the single-occupancy constraint equation (15) in the spinon state \( |\Phi_b\rangle \) such that

\[
|RVB\rangle \equiv \hat{P}_s |\Phi_b\rangle
\]

with \( n^b_\alpha = b^\dagger_\alpha b_\alpha; \) and \( \hat{P}_b \) will further enforce

\[
n^a_\alpha = n^h_\alpha n^b_\alpha,
\]

such that each \( a \)-spinon always coincides with a holon as \( \sum_\alpha n^a_\alpha = n^h_\alpha \) according to equations (26) and (15) (here \( n^a_\alpha \equiv a^\dagger_\alpha a_\alpha \) and \( n^h_\alpha \equiv h^\dagger_\alpha h_\alpha \) with \( \bar{\sigma} \equiv -\sigma). \) By applying \( \hat{P} \), the physical Hilbert space is restored in equation (20) as schematically shown in figure 2 in which the \( a \)-spinons are indicated by the dashed arrows at the hole sites, while there is always a \( b \)-spinon indicated by a solid arrow at each lattice site.

Note that the phase shift factor \( e^{-i\Delta \hbar} \) in \( \Lambda_N \) has totally disappeared in the above direct-product expression equation (20), where the electrons break up into fractionalized building blocks: the holon \( h^\dagger \), the spinon \( b^\dagger_\sigma \) and the backflow spinon \( a^\dagger_\sigma \), forming three rather ‘conventional’ sub-states. The fractionalization form of the ground state equation (20) can be straightforwardly obtained by substituting into equation (3) the following decomposition form of the electron annihilation operator:

\[
c_{i\sigma} = \hat{P} \tilde{c}_{i\sigma}
\]

with

\[
\tilde{c}_{i\sigma} = h^\dagger_{i\uparrow} a^\dagger_{i\downarrow} (-\sigma)^i e^{i\Delta \hbar},
\]

which acts on the insulating ‘vacuum’ \( |0\rangle_h \otimes |0\rangle_a \otimes |RVB\rangle. \) On the other hand, in the neutral spin state \( |RVB\rangle, \) the \( c^\dagger \)-operator can be re-expressed in terms of the bosonic spinon operator \( b^\dagger_{i\sigma} \)
to result [17] in equation (25) from equation (19), according to the decomposition equation (48) given in section 3, where it is further demonstrated that the full spin operator can be expressed as

$$S_i = \hat{P}\tilde{S}_i,$$

with

$$\tilde{S}_i = S_i^b + S_i^a,$$

where $S_i^b$ denotes the spin operators for $b$-spinons (defined in equations (52) and (53)) and $S_i^a$ for $a$-spinons (defined in equations (73) and (74)).

2.2.2. Effective Hamiltonian. The electron fractionalization form (equation (20)) of the ground state $|\Psi_G\rangle$ will make the manipulation of the superconducting state more easy than in the original form (equation (3)).

Define

$$|\tilde{\Psi}_G\rangle \equiv |\Phi_h\rangle \otimes |\Phi_b\rangle \otimes |\Phi_a\rangle,$$

such that $|\Psi_G\rangle = \hat{P}|\tilde{\Psi}_G\rangle$. Then $c_{i\sigma}|\Psi_G\rangle = \hat{P}\tilde{c}_{i\sigma}|\tilde{\Psi}_G\rangle$ and $S_i|\Psi_G\rangle = \hat{P}\tilde{S}_i|\tilde{\Psi}_G\rangle$, in which $\tilde{c}_{i\sigma}$ and $\tilde{S}_i$ directly act on the fractionalized states.

Based on the $t$–$J$ model, we find that the direct product state $|\tilde{\Psi}_G\rangle$ in equation (31) can be effectively determined as the ground state of the following effective Hamiltonian:

$$H_{\text{eff}} = H_h + H_s + H_a,$$

which is composed of a holon hopping term

$$H_h = -t_h \sum_{\langle ij \rangle} (e^{i\Delta_{ij}^h}) h_i^\dagger h_j + \text{h.c.},$$

a $b$-spinon pairing term

$$H_s = -J_s \sum_{\langle ij \rangle \sigma} (e^{i\alpha_{ij}^b}) b_{i\sigma}^\dagger b_{j\sigma}^{-1} + \text{h.c.},$$

and an $a$-spinon term

$$H_a = -t_a \sum_{\langle ij \rangle \sigma} e^{-i\phi_0} a_{i\sigma}^\dagger a_{j\sigma} - J_a \sum_{\langle ij \rangle \sigma} \eta_{ij} \Delta_{ij}^a + \text{h.c.} + J \sum_{\langle ij \rangle} \left( S_i^a \cdot S_j^b + S_j^b \cdot S_i^a \right),$$

where $\Delta_{ij}^a \equiv \sum_{\sigma} \sigma a_{i\sigma}^\dagger a_{j\sigma}$ and $\eta_{ij} = +(-)$ for $j = i \pm \hat{x}(\hat{y})$ is a d-wave sign factor. Note that the chemical (Lagrangian multiplier) terms implementing

$$\sum_i h_i^\dagger h_i = \sum_{i\sigma} a_{i\sigma}^\dagger a_{i\sigma} = \delta N$$

and

$$\sum_{i\sigma} b_{i\sigma}^\dagger b_{i\sigma} = N$$

are all omitted in equations (33)–(35) for simplicity. One can always add them back in real calculations.

Based on equation (31), the parameters $t_h \sim t$, $J_s \sim J$, $t_a \sim t$ and $J_a \sim J|\langle \Delta^a \rangle|$ in equations (33)–(35) can be determined as variational parameters minimizing the ground state energy of

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\( \hat{P} | \tilde{\Psi}_G \rangle \) at a given doping concentration, which will involve the projection \( \hat{P} \) and whose detailed magnitudes will not affect the general consequences to be outlined in the next subsection.

In \( H_t \) and \( H_s \), the \( h \)-holons and \( b \)-spinons are generally coupled to the U(1)\( \otimes \)U(1) gauge fields, \( A^h_{ij} \) and \( A^b_{ij} \), respectively, in equations (33) and (34), which are topological (mutual Chern–Simons) fields as their gauge-invariant flux strengths in an arbitrary counter-clockwise closed loop \( c \) are constrained to the numbers of spinon and holon matter fields within the enclosed area \( \Sigma_c \), respectively,

\[
\sum_c A^h_{ij} = \pi \sum_{l \in \Sigma_c} (n^b_{l+} - n^b_{l-}) \tag{38}
\]

and

\[
\sum_c A^b_{ij} = \pi \sum_{l \in \Sigma_c} n^b_{l-}. \tag{39}
\]

The link variables, \( A^h_{ij} \) and \( A^b_{ij} \), can be regarded as mediating the mutual statistics coupling between the charge and spin degrees of freedom, i.e. the ‘mutual semion statistics’ entanglement introduced by the phase shift factor \( e^{-i\hbar \phi} \) in the original ground state equation (3). In addition, the constant link field \( \phi^0_{ij} \) in equation (35) describes a nondynamic \( \pi \) flux per plaquette, which is originating from the \( \Phi^0_{ij} \) term in equation (12).

### 2.2.3. Ground state as a mean-field solution

Then the ground state (31) as a self-consistent mean-field solution of \( H_{\text{eff}} \) in equation (32) can be constructed as follows.

First of all, suppose the holon state \( | \Phi_h \rangle \) governed by \( H_h \) in equation (33) becomes Bose-condensed (cf equation (7)). Such holon condensation will then lead to \( A^h_{ij} \to A^h_{ij} \), with \( A^h_{ij} \) depicting a uniform flux \( \sum_c A^h_{ij} = \pi \delta \) per plaquette, in terms of equation (39).

Then \( H_s \) in equation (34) can be diagonalized, resulting in a mean-field solution \( | \Phi_b \rangle \) given in equation (23), in which \( W_{ij} = 0 \) if both \( i \) and \( j \) belong to the same sublattice and decays exponentially at large spatial separations for opposite sublattice sites \( i \) and \( j \): \( | W_{ij} | \propto e^{-r_{ij}/\xi} \) [17]. Here \( r_{ij} \) is the spatial distance and \( \xi \) is the characteristic pair size determined by the doping concentration: \( \xi = a \sqrt{\frac{1}{\pi \delta}} \) (\( a \) is the lattice constant). Hence, the spin background \( | \Phi_b \rangle \) indeed becomes short-range at finite doping with a finite spin gap \( E_g \propto \delta J \) [10, 17]. Once the \( b \)-spinons are all short-range paired up in \( | \Phi_b \rangle \), the fluctuations of \( A^b_{ij} \) would become negligible for the long-wavelength physics, i.e. \( \sum_c A^b_{ij} \approx 0 \) for a large loop \( c \) as compared to \( \xi \), according to equation (38). Self-consistently, the two subsystems of the holons and \( b \)-spinons are decoupled as depicted by \( | \Phi_h \rangle \otimes | \Phi_b \rangle \), as the ground state of

\[
H_{\text{string}} = H_h + H_s, \tag{40}
\]

which is known as the phase string model [10, 18] or the mutual Chern–Simons gauge theory model [19].

It is interesting to point out that \( | \text{RVB} \rangle = \hat{P}_x | \tilde{\Psi}_G \rangle \) is of the same form as the Liang–Docout–Anderson-type RVB wavefunction at half-filling, which has previously been proposed [9] as a very accurate variational ground-state wavefunction for the Heisenberg model. Indeed, at half-filling, in the absence of holes, \( | \Psi_G \rangle \) simply reduces to \( | \text{RVB} \rangle \), with \( \xi \to \infty \) or \( W_{ij} \) obeying the power law at large spatial separation of \( i,j \): \( | W_{ij} | \propto 1/r_{ij}^3 \) [17]. The ground-state energy and staggered magnetization of the AFM ordering determined numerically based...
on such $|\text{RVB}\rangle$, obtained from $H_s$, are highly accurate as compared to the exact numerical results [17]. This indicates that the bosonic RVB mean-field description in equation (34), which reduces to the Schwinger–Boson mean-field theory at half-filling [20], has accurately captured both short-range and long-wavelength correlations of the Heisenberg model in this limit.

Finally, note that the fermionic backflow $a$-spinons in $H_a$ (equation (35)) are gauge neutral without coupling to the internal mutual Chern–Simons gauge fields. Here, the last scattering term with the $b$-spinons in equation (35) can be safely omitted in determining the ground state $|\Phi_1^a\rangle$, due to the above-mentioned gap $E_g$ opening up in the spin excitation involving $b$-spinons in $|\Phi_b\rangle$. Then the bilinear terms in equation (35) can be straightforwardly diagonalized with a proper gauge choice of $\phi_{ij}^0$, to result in equation (22) with a d-wave amplitude $\tilde{g}_{ij}$. Due to the presence of $\phi_{ij}^0$, contributing to a $\pi$-flux per plaquette, the $a$-spinons will form Fermi pockets at both $(0, 0)$ and antinodal point $(\pi, 0)$, etc, and a staggered current loop is expected to be present at $\Delta_{ij}^a \neq 0$. The physical implications of the $a$-spinon excitations will be further discussed later.

To end this section, let us examine the electron pairing parameter $\Delta_{ij}^{SC}$ based on equation (28). It can be expressed in the present fractionalized state by

$$\Delta_{ij}^{SC} \propto \Delta_{ij}^a \langle e^{i(\Omega_{ij}^a + \Omega_{ij}^b)} \rangle.$$  \hspace{1cm} (41)

Namely, the pairing amplitude and symmetry will be determined by the pairing order parameter of the $a$-spinons, and the superconducting phase coherence is decided by equation (6). The latter is realized as the RVB pairing of the $b$-spinons in $|\Phi_b\rangle$ becomes short-range with a finite $\xi$, such that the $\pi$-vortices and -antivortices attached to them, according to equation (13), are all confined to form the vortex–antivortex pairs. Namely, superconductivity will be protected by a ‘ghost’ spin liquid state. As noted before, the superconducting phase coherence can disappear as $\xi \to \infty$ either in the long-range AFM state near half-filling or in the overdoped regime when the RVB pairing in $|\text{RVB}\rangle$ is diminished by doping.

2.3. Elementary excitations

Once the ground-state ansatz equation (3) or its fractionalization form equation (20) is determined, the corresponding low-lying elementary excitations, which reflect the novel correlations in the ground state, will also naturally manifest.

In fact, the effective Hamiltonian equation (32) determines not only the ground state $|\tilde{\Psi}_G\rangle$ in equation (31), but also some nontrivial excited states. In the following, we first show the existence of two novel elementary excitations that are uniquely governed by $H_{\text{eff}}$. Then we show that a conventional Bogoliubov quasiparticle excitation will also appear as a collective mode that goes beyond $H_{\text{eff}}$, which will remain protected within the basic characteristic energy scale $E_g$ in this non-BCS superconducting state.

2.3.1. Spin-roton excitations. In the ground-state equation (20), one does not see the trace of the electrons directly—such a strongly correlated electron system seems entirely fractionalized, as described by the bosonic RVB-paired spinons, Bose-condensed holons and d-wave paired backflow spinons, which form a direct product (a generalized ‘spin-charge separation’) state.

However, we find that the single $b$-spinons and holons will not be truly present in the low-lying energy spectrum to become real elementary excitations. This is because $b$-spinons and holons are not gauge neutral—they carry the ‘gauge charges’ of the mutual Chern–Simons fields, $A^b$ and $A^a$, while they provide the ‘topological sources’ to generate $A^a$ and $A^b$,
respectively, and thus their excitations, by breaking up the correlated patterns formed in the ground state, will generally invite nonlocal responses from the whole system, which would make such excitations too costly \[21\].

For instance, one can imagine spinon excitations created by breaking up an RVB pair in \(|\Phi_b\rangle\), described by the effective Hamiltonian \(H_b\) in equation (34). However, each unpaired \(b\)-spinon will induce vortex-like superfluid currents via \(A^\ast\) from the condensed holons according to \(H_b\) in equation (33), leading to the so-called spinon–vortex composite object which is logarithmically divergent in energy, as discussed in \[22–24\]. Therefore, these spinons can only exist in the RVB pair condensate in the ground state, where such vortex currents get effectively canceled out due to equation (38), but not as a single excitation at low energy. Namely, single \(b\)-spinon excitations will be ‘confined’ \[19\] in the bulk of the superconductor.

On the other hand, an integer \((S = 0 \text{ and } 1)\) spin excitation involving a bound pair of spinons excited in \(|\Phi_b\rangle\) is still allowed \[24, 25\], in which the effects of vortex and antivortex bound to individual spinons get canceled out in the long distance in \(H_b\) such that its excitation energy becomes finite with a mean-field gap \(E_g\) according to \(H_s\). Such a neutral spin mode carrying either an integer \(S = 0 \text{ or } 1\) quantum number is called a spin-roton \[25\]. The spin-roton excitations will not destroy the phase coherence at finite temperature until \(T_c\), where the spin-rotons disassociate into free spinon-vortices \[22–24\]. It has been shown \[25\] that a simple \(T_c\) formula,

\[
T_c \simeq \frac{E_g}{6k_B},
\]

(42)
can be determined with \(E_g \sim \delta J\) denoting the core energy of the spin-rotons, degenerate for \(S = 0 \text{ and } 1\), which is in excellent agreement with the experiments, with the ‘resonance-like’ modes observed in the Raman \(A_{1g}\) channel and neutron scattering measurements consistently interpreted as the spin-roton excitations with \(S = 0 \text{ and } 1\), respectively \[26\]. It also provides a natural explanation why the two modes are energetically degenerate in the experiment \[26\].

Here the spin-rotons are the most essential elementary excitations of non-BCS-type above the superconducting ground-state equation (3) or (20), which directly controls the superconducting phase coherence equation (6) via a characteristic energy \(E_g\). In the AFM long-range ordered state near half-filling, one has \(E_g \rightarrow 0\) such that \(T_c\) vanishes, and the \(S = 1\) spin-roton excitation will naturally reduce to the gapless spin wave.

2.3.2. Fermionic \(a\)-spinon excitation. In the ground-state equation (20), there are two distinct branches of spinons. The \(S = 1\) spin-roton excitations related to the dynamic correlation function of \(S^b_i\) have been discussed above. According to equation (30), the backflow \(a\)-spinons will contribute to another branch of \(S = 1\) excitations as governed by \(H_a\) in equation (35). Here the single \(a\)-spinons are gauge-neutral and can be excited by breaking up the d-wave pairs in \(|\Phi_a\rangle\) (equation (22)), which is expected to provide a characteristically different spectral contribution below the spin-roton ‘resonance’ modes mentioned above (at \(E > E_g\), such an \(S = 1\) mode composed of the \(a\)-spinons may strongly decay into the spin-roton mode via the last term in equation (35)).

Besides contributing to the spin spectral function, a single \(a\)-spinon can also directly appear in the single-particle channel. According to the decomposition equation (28), a coherent term may emerge as the first term in

\[
\tilde{c}_{i\sigma} = h^\dagger_{i\sigma}(-\sigma)^{\dagger} e^{i\tilde{\Omega}_i} + :h^\dagger_{i\sigma}(-\sigma)^{\dagger} e^{i\tilde{\Omega}_i}:
\]

(43)
with the holon condensation \( \langle h_i^+ \rangle = h_0^* \) and the superconducting phase coherence \( \langle e^{i \hat{\phi}_i} \rangle \neq 0 \). In other words, the \( a \)-spinon excitation may be directly probed by ARPES as a coherent term appearing below \( T_c \) with a weight \( |h_0|^2 \propto \delta \) and disappearing above \( T_c \) when \( \langle e^{i \hat{\phi}_i} \rangle = 0 \). Such a coherent term will be nonetheless distinguished from the true quasiparticle excitation, which will be obtained as a collective mode, i.e. a bound state of the holon and \( a \)-spinon by the singular phase shift \( e^{i \hat{\phi}_i} \) from the second term in equation (43), to be given in the following subsection. Since the low-lying \( a \)-spinon excitation will appear near the antinodal region (i.e. the momentum \((0, \pi)\), etc), while the true quasiparticle, after absorbing the phase shift field \( e^{i \hat{\phi}_i} \), is a nodal quasiparticle around \((\pi/2, \pi/2)\), there is a ‘dichotomy’ between these two kinds of excitations in the single-particle channel, which will be explored in detail elsewhere.

Finally, it is noted that in a strong magnetic field, the \( d \)-wave pairing of the \( a \)-spinons may be first broken down by the Zeeman energy before the spin-rotons (of energy \( \gtrsim E_g \)) in \( |RVB\rangle \) get excited to destroy the phase coherence in equation (6). In this case, the \( a \)-spinons in equation (35) will form coherent Fermi pockets with \( \Delta_{ij}^a = 0 \) such that the superconducting order parameter in equation (41) can also vanish. Then a new normal state characterized by small Fermi pockets of the \( a \)-spinons can be realized by applying a sufficiently strong magnetic field without encountering the phase disordering boundary. Of course, such a \( T = 0 \) transition caused by the Zeeman effect should be compared to another possible route to a normal state via the generation of a new type of magnetic vortices at strong magnetic fields.

2.3.3. Quasiparticle as a collective mode. So far, we have discussed two novel elementary excitations in the superconducting state, i.e. the spin rotons and the \( a \)-spinons, which are apparently non-BCS-like as determined by \( H_{\text{eff}} \) in equation (32). In the following, we point out that the conventional Bogoliubov quasiparticle excitation will reemerge as a collective mode in the full \( t-J \) Hamiltonian, although it is not an eigen solution of \( H_{\text{eff}} \). In other words, a Bogoliubov quasiparticle can be regarded as a bound state of the fractionalized building blocks, which are glued by the residual interaction in the original \( t-J \) model \([21, 27]\). However, such a quasiparticle excitation will be stable only in a sufficiently long-range, low-energy regime, where the superconducting state will still behave like a conventional \( d \)-wave BCS superconductor as other exotic modes are not yet excited.

In order to generally trace a quasiparticle excitation, one may directly create a bare hole (particle) by the electron \( c \)-operator on the ground state \( |\Psi_G\rangle \) and then follow its behavior via the following equation of motion (cf section 3.4 for details):

\[
-i \partial_t c_{k\sigma} \simeq -(\epsilon_k - \mu) c_{k\sigma} - \Delta_{k\sigma} c_{-k-\sigma}^+ + \text{decay term} + \text{scattering term},
\]

which is obtained after a linearization in terms of the mean-field order parameters. Here \( \Delta_k \propto \Delta_k^{\text{SC}} \) is given by equation (94). The decay term is given in equation (95), which corresponds to the process discussed in the above subsection that a doped hole dissolves into an \( a \)-spinon, i.e.

\[
\text{decay term} \sim \langle e^{i \hat{\phi}_i} \rangle h_0^* a_{k\sigma}^+
\]

in the superconducting background of \( h_0^* \neq 0 \) and \( \langle e^{i \hat{\phi}_i} \rangle \neq 0 \). Such a coherent mode will appear in the antinodal region where the Fermi pockets of the \( a \)-spinons locate, and disappear once
the superconducting phase coherence gets lost at \( \langle e^{i \hat{\Omega}_i} \rangle = 0 \). The scattering term reads (cf equation (96))

\[
\text{scattering term} = \sum_q \left( 2t \Gamma_{k+q} - J \Gamma_q \right) \left[ \sigma c_{k+q\sigma} S^b_q \sigma c_{k+q-\sigma} S^b_{k-\sigma} \right]
\]

with \( \Gamma_k \equiv \cos k_x a + \cos k_y a \), which represents a process that the doped hole scatters with the spin-roton excitations created by \( S^b \) above the resonance-like energy \( E_g \).

Therefore, if we only focus on the low-lying excitation below the spin-roton energy \( E_g \) and around the nodal region, the scattering and decay terms in equation (44) can be all neglected. Then equation (44) and its quasiparticle counterpart can be combined to give rise to an elementary excitation of the Bogoliubov quasiparticle type \( \alpha_k^{\dagger \sigma} \propto u_k c_{k\sigma}^{\dagger} + \sigma v_k c_{-k-\sigma} \), which leads to

\[
-i \partial_t \alpha_k^{\dagger \sigma} |\Psi_G\rangle = E_k \alpha_k^{\dagger \sigma} |\Psi_G\rangle
\]

and \( \alpha_k^{\dagger \sigma} |\Psi_G\rangle = 0 \), with \( u_k \), \( v_k \) and the energy spectrum \( E_k = \sqrt{(\epsilon_k - \mu)^2 + (\Delta_k)^2} \) given in section 3.4, not different from a usual d-wave nodal quasiparticle in a BCS framework. Here the chemical potential \( \mu \) is determined by requiring \( \sum_k \epsilon_k^{\dagger \sigma} c_{k\sigma} = (1 - \delta) N \), and one arrives at a very important conclusion that a d-wave Bogoliubov quasiparticle excitation is still well preserved in the present non-BCS-like superconducting state. It is not given by the effective Hamiltonian \( H_{\text{eff}} \), but emerges as a collective mode of the original \( t-J \) model, which is ensured by the superconducting ODLRO (41) protected by a finite minimal spin-roton energy \( E_g \).

3. Microscopic justification

In this section, we justify the ground-state ansatz, equations (3) and (20), as well as the elementary excitations outlined in section 2, based on the \( t-J \) model. We shall start with an exact reformulation of the \( t-J \) model in an all-boson formalism in which the hidden sign structure can be explicitly revealed. Such a precise sign structure will then play an essential role in determining the peculiar structure of the ground state and elementary excitations in terms of the electron fractionalization.

3.1. Phase string representation of the \( t-J \) model

The \( t-J \) Hamiltonian is a minimal model of doped Mott insulators, with the Hilbert space restricted by the no double occupancy constraint (2) in the hole-doped case. It has been rigorously demonstrated that the fermion signs are completely diminished at half-filling due to the ‘Mottness’ enforced by the constraint (2), where the \( t-J \) model reduces to the AFM Heisenberg model. The residual fermion signs only start to re-emerge upon doping, which can be mathematically described by the so-called phase string effect [13–15].

In order to explicitly keep track of such a sign structure, the phase string representation of the \( t-J \) model has been previously introduced [14], in which the electron annihilation operator can be fully bosonized by the following decomposition:

\[
c_{i\sigma} = h_i^{\dagger} b_{i\sigma} e^{i \hat{\phi}_{i\sigma}}
\]

in terms of the bosonic holon creation operator \( h_i^{\dagger} \) and the bosonic spinon annihilation operator \( b_{i\sigma} \), with the phase string effect explicitly embedded in the phase factor

\[
e^{i \hat{\phi}_{i\sigma}} \equiv (-\sigma)^{\dagger} e^{i \hat{\phi}_{i}^{\dagger}[\Phi_{i}^{\dagger} - \Phi_{i}^{\dagger -\sigma} \Phi_{i}^{\dagger}]}.
\]
In $e^{i\hat{\Theta}_i}$, $\Phi_i^s$ and $\Phi_i^b$ are defined in equations (13) and (14), respectively, and $\Phi_i^b$ is given as follows:

$$\Phi_i^b = \sum_{l \neq i} \theta_i(l) n_i^b,$$

which is nonlocally associated with the holon occupation number $n_i^h$ at site $l$. Here $e^{i\hat{\Theta}_i}$ also plays a role in restoring the fermionic statistics of $c_i^\sigma$, in the Hilbert space restricted by the single occupancy constraint

$$\sum_{\sigma} n_i^{b\sigma} + n_i^h = 1.$$

Correspondingly, the spin operators $S_i = S_i^b$ with $S_i^b$ are defined by

$$S_i^{bz} = \frac{1}{2} \sum_\sigma \sigma b_i^\sigma b_i^\sigma$$

and

$$S_i^{bh} = (-1)^i \tau_i^b b_{i+1}^\sigma, \quad S_i^{bh} = (-1)^i \tau_i^b b_{i+1}^\sigma e^{-i\phi_i^0},$$

which involve the holon degree of freedom via $\Phi_i^h$ defined above.

Then the $t$–$J$ Hamiltonian

$$H_{t-J} = H_t + H_J$$

$$\equiv -t \sum_{\langle ij \rangle \sigma} c_i^\dagger c_j^\sigma \chi_{\alpha} + \text{h.c.} + J \sum_{\langle ij \rangle} \left( S_i \cdot S_j - \frac{n_i n_j}{4} \right)$$

under the constraint $n_i \equiv \sum_\sigma c_i^\dagger c_i^\sigma \leq 1$ can be reformulated as [14]

$$H_t = -t \sum_{\langle ij \rangle \sigma} \left( h_i^\dagger h_j e^{iA_i^s} (\tau_i^b b_i^\sigma e^{i\phi_i^0} A_i^b) + \text{h.c.} \right)$$

and

$$H_J = -\frac{J}{2} \sum_{\langle ij \rangle} (\Delta_{ij}^s)^\dagger \Delta_{ij}^s,$$

with the bosonic RVB order operator

$$\Delta_{ij}^s = \sum_\sigma e^{-i\sigma A_i^b} b_{i\sigma} b_{j\sigma}$$

for the NN sites.

Here the unique feature is the presence of three link fields: $A_i^s$, $A_i^b$ and $\phi_i^0$, which capture all the nontrivial signs inherent to the model (without them, the model would simply reduce to a pure bosonic one without any phase frustration). They are defined by

$$A_i^s \equiv \frac{1}{2} \sum_{l \neq i,j} \left[ \theta_i(l) - \theta_j(l) \right] \left( \sum_\sigma \sigma n_i^{b\sigma} \right)$$

$$A_i^b \equiv \frac{1}{2} \sum_{l \neq i,j} \left[ \theta_i(l) - \theta_j(l) \right] n_i^h$$

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\[ \phi_{ij}^0 = \frac{1}{2} \sum_{l \neq i, j} [\theta_i(l) - \theta_j(l)]. \] (59)

The flux strengths of \( A_{ij}^s \) and \( A_{ij}^h \) are given in equations (38) and (39), respectively, which are invariant under the gauge transformations

\[ h_i \rightarrow h_i e^{i \psi_i}, \quad A_{ij}^s \rightarrow A_{ij}^s + (\varphi_i - \varphi_j) \] (60)

and

\[ b_{i\sigma} \rightarrow b_{i\sigma} e^{i \rho_i}, \quad A_{ij}^h \rightarrow A_{ij}^h + (\theta_i - \theta_j). \] (61)

Thus the \( t-J \) Hamiltonian in the phase string representation has an intrinsic U(1) \( \otimes \) U(1) gauge structure and according to equations (60) and (61), the holons and spinons will carry the gauge charges of the gauge fields, \( A_{ij}^s \) and \( A_{ij}^h \), respectively. On the other hand, the strengths of \( A_{ij}^s \) and \( A_{ij}^h \) are generated by the local densities of the spinons and holons, respectively, according to equations (38) and (39). Finally, \( \phi_{ij}^0 \) simply describes a uniform \( \pi \) flux per plaquette on a square lattice without any dynamics.

Therefore, the full sign structure hidden in the \( t-J \) model has been explicitly sorted out and captured by the link variables, \( A_{ij}^s \) and \( A_{ij}^h \), as well as \( \phi_{ij}^0 \), in equations (54) and (55) of the phase string/bosonization representation. At half-filling, \( A_{ij}^h \) vanishes such that there is no nontrivial sign left in \( H_J \), while \( H_I = 0 \) under the constraint (51). So the fermionic signs of the electrons totally disappear here, and the residual intrinsic signs only re-emerge as the holes are doped into the system, which are precisely represented by the aforementioned topological gauge fields, and in this sense the \( t-J \) model becomes an intrinsic gauge model in the phase string representation.

### 3.2. Electron fractionalization

Hence the phase string representation of the \( t-J \) model constitutes a suitable starting point, as it smoothly connects the doping problem with the undoped antiferromagnet—the latter can be well described by the bosonic RVB state without involving any sign problem [9, 17].

However, as pointed out in [17], a bosonic spinon defined in the phase string representation, satisfying the constraint (51), is not strictly charge-neutral: that is, it is involved not only in the ‘neutral’ superexchange process described by \( H_J \) (equation (55)), but also in a ‘backflow’ process accompanying holon (charge) hopping in the \( H_I \) term (equation (54)). As a matter of fact, it is shown [17] that the bosonic RVB pairing is incompatible with the backflow accompanying the holon hopping (see section 3.3), and thus it is important to further distinguish these two different types of processes for spinons in this particular representation.

#### 3.2.1. Two-component spinon description

In order to properly accommodate these two distinct correlations of spins, a neutral spinon Hilbert space has been introduced [17] previously, which satisfies the constraint \( \sum_\sigma n_{i\sigma} = 1 \) instead of equation (51), even in the doped case. Such a ‘ghost’ neutral spin state, denoted by \( |\text{RVB}\rangle \), can form a direct product state with the holon state \( |\Phi_h\rangle \)

\[ |\text{RVB}\rangle \otimes |\Phi_h\rangle. \] (62)
Then the excessive spinons coinciding with the holon sites in equation (62) should be removed as they violate the Mott constraint (51). A physical state constrained by equation (51) may then be realized by

$$|\Psi\rangle = \hat{P}_B|RVB\rangle \otimes |\Phi_h\rangle$$

(63)

with \(\hat{P}_B = \cdots (b_{l,\sigma} n_{l}^{h}) (b_{l+1,\sigma} n_{l+1}^{h}) \cdots\) annihilating the unphysical neutral spinons at the holon sites. Here \(\hat{P}_B\) may be regarded as introducing a new type of spinons, namely, the backflow spinons [17], in the ‘vacuum’ state (62). Below we show how to mathematically accurately incorporate such two-component spinon building blocks into the phase string representation.

Define \(a^{\dagger}\) as the creation operator for a backflow spinon, via a one-to-one mapping

$$\langle b_{l,\sigma} n_{l}^{h} |RVB\rangle \otimes |\Phi_h\rangle \mapsto \hat{P}_B \left[ (a_{l,\sigma}^{\dagger} e^{i\Xi_{l,\sigma}}) |0\rangle_{a} \otimes |RVB\rangle \otimes |\Phi_h\rangle \right],$$

(64)

where \(|0\rangle_{a}\) denotes the vacuum state of the \(a\)-spinons and the projector \(\hat{P}_B\) enforces the constraint in equation (26), namely, the occupation number \(n_{l,\sigma}^{b}\) of an \(a\)-spinon will always be equal to the occupation number of the \(b\)-spinon in \(|RVB\rangle\) with the opposite spin index \(\bar{\sigma} = -\sigma\) at site \(l\), if and only if there is a holon sitting at the same site \(l\) in \(|\Phi_h\rangle\) (schematically it is represented by a dashed arrow in figure 2). Here the phase factor \(e^{i\Xi_{l,\sigma}}\) is introduced to simplify the formulation, whose physical meaning will become clear later.

Corresponding to this exact mapping in the Hilbert space, the \(t–J\) Hamiltonian in the phase string representation can be further transformed as follows.

### 3.2.2. The hopping term

Rewrite the hopping term \(H_t\) in equation (54) as

$$H_t = -t \sum_{(i,j)\sigma} \hat{h}_j (b_{i,\sigma} n_{i}^{h}) e^{i\phi_{i,j}^{\sigma} + i\sigma A_{i,j}^{\dagger}} (n_{j}^{h} b_{j,\sigma}^{\dagger}) \hat{h}_j^{i} e^{iA_{i,j}^{\dagger}} + \text{h.c.}$$

(65)

Further note that a shift \(A_{i,j}^{\dagger} \rightarrow A_{i,j}^{\dagger} - \delta A_{i,j}^{\dagger}\) has to be made when the \(b\)-spinon Hilbert space is changed to a neutral spinon basis satisfying \(\sum_{\sigma} n_{l,\sigma}^{b} = 1\) instead of equation (51). Here \(\delta A_{i,j}^{\dagger} = \frac{1}{2} \sum_{l \neq j} [\theta_{l,\sigma} - \theta_{j,\sigma}] (\sum_{\sigma} n_{l,\sigma} n_{j,\sigma}^{\dagger})\) is obtained based on the definition (57). Then under the mapping (64) the hopping term is correspondingly changed to

$$H_t \mapsto \hat{P} \tilde{H}_t$$

with

$$\tilde{H}_t \equiv -t \sum_{(i,j)\sigma} (\hat{h}_j^{i} \hat{h}_j) (a_{l,\sigma}^{\dagger} a_{j,\sigma} e^{-i\phi_{i,j}^{\sigma}}) + \text{h.c.}$$

(66)

In obtaining equation (66), the extra phases arising in equation (65) have been absorbed by making the following choice:

$$e^{i\Xi_{l,\sigma}} = e^{i\sigma \sum_{l,\sigma} \theta_{l,\sigma} n_{l}^{h} n_{l}^{b}}$$

$$= e^{i\sigma \sum_{l,\sigma} \theta_{l,\sigma} n_{l}^{b}}$$

(67)

utilizing equation (26). It is easy to verify that the phase factor \(e^{i\Xi_{l,\sigma}}\) will serve as a 2D Jordan–Wigner operator to precisely make the \(a\)-spinons defined in equation (64) behave as fermions. Note that in the earlier approach [17], without introducing \(e^{i\Xi_{l,\sigma}}\) to absorb the extra phase in equation (65), the backflow spinons are treated in a boson representation which is not as compact as in equation (66).

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Based on the precise mapping (64), one has

\[ |\Psi\rangle \mapsto \hat{P}_B [ |\Phi_a\rangle \otimes |\text{RVB} \rangle \otimes |\Phi_h\rangle ] \]
\[ = \hat{P} [ |\Phi_a\rangle \otimes |\Phi_b\rangle \otimes |\Phi_h\rangle ] \]
\[ \equiv \hat{P} |\tilde{\Psi}\rangle, \quad (68) \]

where \(|\Phi_a\rangle\) denotes the pure \(a\)-spinon state and \(|\text{RVB}\rangle \equiv \hat{P}_s |\Phi_b\rangle\). Generally, \(|\tilde{\Psi}\rangle\) here should be understood as a state expanded in terms of the direct product bases of the \(a\)-spinon, \(b\)-spinon and \(h\)-holon. Hence, in the hopping term

\[ \hat{H}_t |\Psi\rangle \mapsto \hat{P} \tilde{H}_t |\tilde{\Psi}\rangle, \quad (69) \]

\(\tilde{H}_t\) directly acts on the fractionalized state \(|\tilde{\Psi}\rangle\) defined in equation (68).

Corresponding to the mapping in the Hilbert space as given in equation (68), the electron annihilation operator defined in equation (48) in the phase string representation may be reexpressed by

\[ c_{i\sigma} |\Psi\rangle \mapsto \hat{P} \left[ h_{i\sigma} a_{i\sigma}^\dagger (-\sigma)^i e^{i\tilde{\Omega}_i} \right] |\tilde{\Psi}\rangle, \quad (70) \]

which is obtained using equation (67), resulting in the phase shift field \(\tilde{\Omega}_i = (\Phi_i^s - \Phi_i^b)/2\) defined in equation (12).

Similarly, the spin operators \(S_i = S_i^b\) in the phase string representation (cf equations (52) and (53)) can be rewritten under the constraint equation (51) as \(S_i = S_i^b (1 - n_i^h) = S_i^h - n_i^h S_i^b\). Then under the mapping of equation (68), it can be shown that

\[ n_i^h S_i^b |\Psi\rangle \mapsto - \hat{P} S_i^a |\tilde{\Psi}\rangle, \quad (71) \]

using equations (64) and (67), such that

\[ S_i |\Psi\rangle \mapsto \hat{P} \left[ S_i^b + S_i^a \right] |\tilde{\Psi}\rangle, \quad (72) \]

where the \(a\)-spinon spin operators \(S_i^a\) are defined by

\[ S_i^{az} \equiv \frac{1}{2} \sum_\sigma \sigma a_{i\sigma}^\dagger a_{i\sigma} \quad \text{(73)} \]

and

\[ S_i^{a+} \equiv (-1)^i a_{i\uparrow}^\dagger a_{i\downarrow}, \quad S_i^{a-} \equiv (-1)^i a_{i\downarrow}^\dagger a_{i\uparrow} \quad \text{(74)} \]

(we assume the anticommuting relation, e.g. \(a_{i\uparrow}^\dagger a_{i\downarrow} = -a_{i\downarrow} a_{i\uparrow}\), between the \(a\)-spinons of opposite spins without loss of generality).

3.2.3. The superexchange term. For the superexchange term in equation (55), by introducing the factor \((1 - n_i^b)(1 - n_j^b)\) to explicitly enforce the no double occupancy constraint (51) in the enlarged Hilbert space \(|\tilde{\Psi}\rangle\), one finds the following mapping:

\[ H_J |\Psi\rangle \mapsto - J \sum_{(ij)} \hat{P} \left[ (1 - n_i^b)(1 - n_j^h) (\hat{\Delta}_{ij}^a)^\dagger \hat{\Delta}_{ij}^a \right] |\tilde{\Psi}\rangle \]
\[ = \hat{P} \tilde{H}_J |\tilde{\Psi}\rangle, \quad (75) \]
where

\[ \tilde{H}_J = -\frac{J}{2} \sum_{\langle ij \rangle} (\hat{\Delta}^a_{ij})^\dagger \hat{\Delta}^a_{ij} - \frac{J}{2} \sum_{\langle ij \rangle} n^b_i n^b_j (\hat{\Delta}^a_{ij})^\dagger \hat{\Delta}^a_{ij} + \frac{J}{2} \sum_{\langle ij \rangle} (n^b_i + n^b_j)(\hat{\Delta}^a_{ij})^\dagger \hat{\Delta}^a_{ij}. \]  

(76)

\[ \tilde{H}_J \text{ in equation (76) can be further re-expressed as} \]

\[ \tilde{H}_J = -\frac{J}{2} \sum_{\langle ij \rangle} (\hat{\Delta}^a_{ij})^\dagger \hat{\Delta}^a_{ij} - \frac{J}{2} \sum_{\langle ij \rangle} (\hat{\Delta}^a_{ij})^\dagger \hat{\Delta}^a_{ij} + J \sum_{\langle ij \rangle} (S_i^b \cdot S_j^b + S_i^b \cdot S_j^a) + J N_b, \]  

(77)

in which in obtaining the second term on the right-hand side (rhs) of the first line, the relation

\[ n^b_i n^b_j (\hat{\Delta}^a_{ij})^\dagger \hat{\Delta}^a_{ij} \mapsto (\hat{\Delta}^a_{ij})^\dagger \hat{\Delta}^a_{ij} \]  

(78)

according to equation (64) is used, and in obtaining the second line the equality

\[ \frac{1}{2} (\hat{\Delta}^a_{ij})^\dagger \hat{\Delta}^a_{ij} = -S_i^b \cdot S_j^b + \frac{1}{4} \]  

(79)

as well as equation (71) are utilized.

Therefore, by introducing a new kind of ‘backflow’ $a$-spinon, the original $b$-spinon in the phase string formalism will now always describe a ‘half-filled’ neutral spin state constrained by equation (15). Consequently, two distinct processes, involving spins in the hopping and superexchange terms of equations (54) and (55), can be mathematically depicted separately in terms of two kinds of spinons, as in equations (66) and (77). In this new formalism, the mutual Chern–Simons gauge fields, $A^a_{ij}$ and $A^b_{ij}$, are still defined by equations (57) and (58), but $A^a_{ij}$ is now always acting on a half-filling background. The phase difference in $A^a_{ij}$ is then absorbed into the backflow spinon, and naturally turns the latter into a fermion. In the following, we will see that this kind of fractionalization description will be very important for properly constructing a saddle-point state at low doping.

3.3. Mean-field scheme

The $t$–$J$ model has been first reformulated in the phase string representation in order to accurately track of its peculiar and unique sign structure and then expressed in a specific fractionalization formalism in terms of a neutral bosonic spinon, a backflow fermionic spinon and a bosonic holon, in order to properly distinguish the microscopic hopping and superexchange processes in the restricted Hilbert space. Now one is ready to construct an effective theory/ground state based on the above new formulation.

The Schrödinger equation $H_{t-J} |\Psi\rangle = E |\Psi\rangle$ can be rewritten as

\[ \hat{P} (\tilde{H}_{t-J} - E) |\Psi\rangle = 0 \]  

(80)

by using $H_{t-J} |\Psi\rangle \mapsto \hat{P} \tilde{H}_{t-J} |\Psi\rangle$ and $|\Psi\rangle \mapsto \hat{P} |\Psi\rangle$, where

\[ \tilde{H}_{t-J} \equiv \tilde{H}_t + \tilde{H}_J \]  

(81)

as defined in equations (66) and (77).

Based on equation (80), one may further make the following ansatz that

\[ \tilde{H}_{t-J} |\tilde{\Psi}\rangle = E |\tilde{\Psi}\rangle \]  

(82)

holds for both the ground state and low-lying excitation states. Generally speaking, it is a sufficient but not a necessary condition for $\hat{P} |\tilde{\Psi}\rangle$ determined by equation (82) to be an eigenstate.
we shall show that the Bogoliubov quasiparticle excitation is indeed an exception, which satisfies equation (80) but not equation (82), emerging as a collective mode beyond the latter. But in the following we first focus on the solution of equation (82) and develop a generalized mean-field scheme.

Such a mean-field theory will be underpinned by a gauge-invariant bosonic RVB order parameter

$$\Delta_{ij}^\alpha = \langle \hat{\Delta}_{ij}^\alpha \rangle \neq 0,$$  \hspace{1cm} (83)

which was first introduced in [10, 18]. Then $\tilde{H}_J$ in equation (77) can be linearized in terms of the ‘mean-field’ order parameter $\Delta_{ij}^\alpha$, giving rise to an effective Hamiltonian $H_\alpha$ in equation (34) with the order parameter $(\Delta_{ij}^\alpha)_{NN}$ taken as s-wave-like: $(\Delta_{ij}^\alpha)_{NN} = \Delta^s$ and $J_s = J\Delta^s/2$. Self-consistently one always finds the mean field

$$\langle b_\sigma^\dagger b_\sigma e^{i\sigma A_{ij}^\alpha} \rangle_{NN} = 0$$  \hspace{1cm} (84)

such that $(\Delta_{ij}^\alpha)_{NN}$ is the unique order parameter for the bosonic RVB state [10, 18]. Equation (84) implies that the bosonic RVB pairing is indeed incompatible with the hopping, in contrast to the fermionic RVB case, which has been the basis for introducing the backflow spinon to facilitate the hopping process in the first place [17]. It is noted that $H_\alpha$ remains invariant when $(\Delta_{ij}^\alpha)_{NN}$ is changed from s-wave to d-wave-like, via a simple gauge transformation: $b_\sigma \rightarrow (-1)^i b_\sigma$ so long as equation (84) holds.

The kinetic energy of the $a$-spinons will arise from the hopping term $\tilde{H}_t$ in equation (66), where a natural gauge-invariant decoupling gives rise to a pure holon term $H_h$ in equation (33) and the $a$-spinon hopping term in $H_a$ (i.e. the first term on the rhs of equation (35)). The rest of the terms in $H_a$, including the pairing term for the $a$-spinons, all come from $\tilde{H}_J$ in equation (77). The pairing term in equation (35) is obtained by a conventional mean-field decoupling:

$$(\hat{\Delta}_{ij}^\alpha)^\dagger \hat{\Delta}_{ij}^\alpha \rightarrow \sum_\sigma (\Delta_{ij}^\alpha)^\dagger \sigma a_{i\sigma}^\dagger a_{j\sigma}^\vphantom{\dagger} + \text{h.c.} + \cdots$$  \hspace{1cm} (85)

(a linearized $e^{-i\phi_{ij}} a_{i\sigma}^\dagger a_{j\sigma}$ term is incorporated into the first term in equation (35)), with a d-wave order parameter $\Delta_{ij}^\alpha \equiv \langle \Phi_a | \hat{\Delta}_{ij}^\alpha | \Phi_a \rangle = \eta_{ij} |\Delta_{ij}^\alpha|$. Therefore, a ‘pseudogap’ state characterized by the bosonic RVB pairing of equation (83), which persists from half-filling to a finite doping, will be described by an effective Hamiltonian $H_{\text{eff}} = H_s + H_h + H_a$ as given in equation (32) (section 2.2).

It is important to point out that $H_{\text{eff}}$ obtained based on equation (82) is not strictly a conventional mean-field Hamiltonian. For instance, at the mean-field approximation, one would have $t_h = t \sum_{ij} \langle a_{i\sigma}^\dagger a_{j\sigma} e^{-i\phi_{ij}} \rangle \equiv t K^a$ and $t_a = t \langle h_j^\dagger h_j e^{iK^a} \rangle \equiv t H_0$, but under the projection $\hat{P}$, one will have $t_h \sim t_a \sim t$ at low doping. In principle, these parameters together with $J_s$ and $J_a$, appearing in equations (33)–(35), are not determined by a standard self-consistent mean-field procedure. Instead, they will be considered as the variational parameters, to be determined by minimizing the ground state energy of $\hat{P} |\hat{\Psi}_G\rangle$ in equation (20) with regard to the exact $H_{t\rightarrow J}$. Such a variational scheme will decide the magnitudes and doping dependences of these parameters, but their detailed values are not crucial to the qualitative physical consequences discussed in the present work.

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3.4. Quasiparticle as a collective excitation

The above mean-field treatment of $\tilde{H}_{t-J}$ leads to an effective Hamiltonian $H_{eff}$ in equation (32), which determines the fractionalized superconducting ground state (31). Based on $H_{eff}$, two types of unconventional elementary excitations have been identified, i.e. the spin-rotons and the fermionic $a$-spinons, which have been discussed in sections 2.3.1 and 2.3.2, respectively.

A quasiparticle mode, which carries both charge and spin-1/2 quantum numbers, can be created by the electron operator defined in equation (70). In the superconducting phase, with the holon condensation $\langle h_i^+ \rangle \neq 0$ and the phase coherence $\langle e^{i\Omega_i} \rangle \neq 0$, an $a$-spinon excitation may directly appear in the single-particle spectral function according to equation (70). But such a decomposition structure is only a part of the low-energy feature around the antinodal region as discussed in sections 2.3.2 and 2.3.3.

In the following, we demonstrate that a more conventional Bogoliubov quasiparticle will also emerge as an independent low-lying excitation in the superconducting state, in addition to the above non-BCS-type excitations. Being coherent around the nodal region, such a quasiparticle can be regarded as a bound state forming from the elementary building blocks of a holon, an $a$-spinon and a nonlocal phase shift based on equation (70). Such a ‘collective’ mode goes beyond the description of $H_{eff}$ in equation (32), and will be correctly described based on the original full $t$–$J$ Hamiltonian. The hopping term in the latter will provide the necessary binding force for a stable nodal quasiparticle, which has been shown previously [21, 27] in the phase string representation by using the equation-of-motion method. Below we give a similar proof based on the present decomposition given in equation (70).

A quasiparticle excitation state may be generally constructed by

$$\langle k\sigma \rangle_{qp} \equiv a_{k\sigma}^\dagger \Psi_G,$$

(86)

where the creation operator $a_{k\sigma}^\dagger$ is a linear combination of $c_{-k-\sigma}$ and $c_{k\sigma}^\dagger$. The quasiparticle spectrum is given by

$$E_k = \langle k\sigma | H_{t-J} | k\sigma \rangle_{qp} - E_G$$

$$= \langle \Psi_G | a_{k\sigma} \left[ H_{t-J}, a_{k\sigma}^\dagger \right] | \Psi_G \rangle$$

(87)

using $a_{k\sigma} | \Psi_G \rangle = 0$ (assuming $| \Psi_G \rangle$ being normalized).

For the $t$–$J$ model, one generally has [27]

$$[H_t, c_{i\sigma}] = \frac{J}{4} c_{i\sigma} \sum_{j=NN(i)} \left( 1 - n_j^\sigma \right) - \frac{J}{2} \sum_{j=NN(i)} \left( c_{i\sigma} S_j^+ + c_{i-\sigma} S_j^- \right).$$

(88)

and

$$[H_J, c_{i\sigma}] = \frac{J}{4} c_{i\sigma} \sum_{j=NN(i)} \left( 1 - n_j^\sigma \right) - \frac{J}{2} \sum_{j=NN(i)} \left( c_{i\sigma} S_j^+ + c_{i-\sigma} S_j^- \right).$$

(89)

By acting them on the ground state in equation (20) and using the d-wave order parameters: $\Delta_{ij}^{SC} = \langle c_i \sigma j \rangle$ and $K = \sum_{\sigma} \langle c_i \sigma c_{-i} \sigma \rangle$ to linearize the rhs of the equations, one finds

$$[H_{t-J}, c_{i\sigma}] | \Psi_G \rangle \simeq \left( t_{eff} \sum_{j=NN(i)} c_{j\sigma} + \mu c_{i\sigma} \right) | \Psi_G \rangle - J \sum_{j=NN(i)} \Delta_{ij}^{SC} c_{j-\sigma}^\dagger | \Psi_G \rangle$$

+ decay term + scattering term

(90)

where $t_{eff} = t(1 + \delta)/2 + J K/4$ and $\mu$ is the chemical potential.
If one can neglect the higher-order terms on the rhs of equation (90), including a ‘decay term’ and a ‘scattering term’ to be given below, then equation (90) reduces to a linear equation in the $c$-operators, which can be diagonalized via the following Bogoliubov transformation:

$$ \alpha_{k\sigma}^\dagger \propto u_k c_{k\sigma}^\dagger + \sigma v_k c_{-k,-\sigma}. $$

(91)

where $u_k^2 + v_k^2 = 1$, with $u_k^2 = 1 + (\epsilon_k - \mu)/E_k$, $v_k^2 = 1 - (\epsilon_k - \mu)/E_k$ and $2u_kv_k = \Delta_k/E_k$. It leads to

$$ \begin{bmatrix} \Delta_{l-j} & \alpha_{k\sigma}^\dagger \end{bmatrix} |\Psi_G\rangle = E_k \alpha_{k\sigma}^\dagger |\Psi_G\rangle $$

(92)

with a BCS-like spectrum

$$ E_k = \sqrt{(\epsilon_k - \mu)^2 + (\Delta_k)^2}. $$

(93)

Here $\epsilon_k = -2t_{\text{eff}}(\cos k_x + \cos k_y)$ and

$$ \Delta_k \equiv 2J \sum_q (\cos q_x + \cos q_y) \Delta^{SC}_{k+q}, $$

(94)

which can be easily shown to be d-wave if $\Delta^{SC}_k$ is a d-wave superconducting order parameter [27]. (Note that $\Delta_k$ is scaled by an additional factor $|h_0|^2 \propto \delta$ in [21, 27], which is actually an artifact in linearizing [27] the equation of motion.)

By noting that the chemical potential $\mu$ is given by requiring $\langle \sum_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma}\rangle = (1 - \delta)N$, one finds that the quasiparticle state $\alpha_{k\sigma}^\dagger |\Psi_G\rangle$ behaves similarly to a Bogoliubov quasiparticle in a d-wave BCS state, where it is built on a normal state with a large Fermi surface satisfying the Luttinger theorem as decided by $\epsilon_k = \mu$. As a matter of fact, in the above equation-of-motion calculation, if, say, the next-nearest-neighbor hopping $t'$ is added to $H_{l-j}$, a corresponding change in the band structure of $\epsilon_k$ will take place to give rise to a modified Fermi surface just like in ordinary band theory. By contrast, $t'$ can be effectively renormalized to zero in $H_h$ and $H_n$ due to the phase string effect, as to be shown elsewhere, such that it will not directly affect the spin-roton and $a$-spinon excitation spectra. This further illustrates the distinction between a fermionic $a$-spinon and a true Bogoliubov quasiparticle.

Hence a Bogoliubov quasiparticle is always a coherent excitation at a sufficiently low energy as protected by the ODLRO (6). As pointed out above, this Bogoliubov quasiparticle can be viewed as a bound state in the fractionalized ground state, whose wave packet can eventually break down at a higher energy in the superconducting state where the phase coherence (6) is still maintained. To properly understand such stability, one needs to further inspect the higher-order terms in equation (90).

The ‘decay term’ in equation (90) has the following leading contribution:

$$ \text{decay term} \simeq \left\langle e^{i\hat{\alpha}} \right\rangle h_0^* \left[ O a_{i\sigma}^\dagger (-\sigma)^j + \sum_{j=\text{NN}(i)} (P_{ij} a_{j\sigma}^\dagger - Q_{ij} a_{j\sigma})(-\sigma)^j \right] |\Psi_G\rangle $$

(95)

after a linearization with using the order parameters: $H_0$, $K^a$, $\Delta^a_{ij}$ and $h_0^*$, as well as $\langle e^{i\hat{\alpha}} \rangle$. Here the coefficients are given by $O = 3tK^a - J H_0$, $P_{ij} = \frac{1}{2}t H_0 + \frac{3}{8} J K^a \sigma$ and $Q_{ij} = \frac{3}{4} J \Delta^a_{ij}$. Such a decay term represents the fractionalization tendency of a quasiparticle into an $a$-spinon, which is already indicated by the decomposition form (70). It corresponds to a coherent term so long as the phase coherence $\langle e^{i\hat{\alpha}} \rangle$ is maintained, in which the $a$-spinon appears at the antinodal region, and thus in the momentum space is distinguished from the Bogoliubov quasiparticle in the nodal
region. Such a coherent term disappears above \( T_c \), even though the \( a \)-spinon can remain coherent at \( T > T_c \).

Furthermore, the scattering term reads

\[
\text{scattering term} = \sum_{j=NN(i)} \left[ t \left( c_{j \sigma} \sigma S_i^{b\sigma} + c_{j \rightarrow \sigma} S_i^{b\rightarrow \sigma} \right) - \frac{J}{2} \left( c_{i \sigma} \sigma S_j^{b\sigma} + c_{i \rightarrow \sigma} S_j^{b\rightarrow \sigma} \right) \right] |\Psi_G\rangle, \tag{96}
\]

which involves the scattering between the quasiparticle and the spin-roton excitations composed of the \( b \)-spinons. By noting that a spin-roton excitation has a finite ‘resonance energy’ \( E_g \) in the superconducting state (cf section 2.3.1), the scattering in equation (96) may be safely neglected if one only focuses on the nodal quasiparticles below this characteristic energy scale. On the other hand, at an energy scale higher than \( E_g \), a strong scattering between a Bogoliubov quasiparticle and the background neutral spin excitations is expected to dominate the single-particle spectral function in an appropriate momentum region, which will be important to understand the ARPES data, but is beyond the scope of this paper.

4. Conclusion and perspective

The present mechanism for superconductivity resembles, by nature, what has been proposed by Anderson [1, 28] that the spin RVB pairing is turned into the Cooper pairing upon doping. But the basic structure of the superconducting ground state, presented in equation (3), is distinct from the original proposal [1] of the Gutzwiller-projected BCS state in equation (1) in that the neutral RVB and Cooper channels remain clearly differentiated throughout the underdoped superconducting regime, as illustrated by figures 1 and 2, respectively. Such a ground state has intrinsically embedded three essential types of correlations: namely, the AFM correlations in \(|\text{RVB}\rangle\); the Cooper pairing of the doped holes; and mutual influence/competition between the two channels via a mutual statistical phase \( \Lambda_{h} \).

Consequently, the superconductivity arises as a self-organization from these correlations, rather than by default as in a Gutzwiller-projected BCS state at finite doping. Without holes, the long-range AFM order will always win in \(|\text{RVB}\rangle\), which provides a highly accurate description of the ground state of the Heisenberg Hamiltonian. But a sufficient concentration of doped holes will eventually turn the antiferromagnetism in \(|\text{RVB}\rangle\) into a true spin liquid state with short-range AFM correlations. By doing so, the Cooper paired holes can gain phase coherence to realize a high-\( T_c \) superconductivity.

The key underlying such a peculiar structure in the ground state can be attributed to the singular phase shift introduced by the doped holes, which is incorporated in equation (3) by \( \Lambda_{h} \). Such a ‘phase string effect’ induced by the hole hopping is irreparable, representing the exact sign structure of the \( t-J \) model in the no double occupancy constrained Hilbert space, which appears as mutual statistical signs drastically different from the Fermi–Dirac statistical signs. Incorporating this new emergent statistics [11] is thus essential in order to correctly understand the non-Fermi liquid nature of doped Mott insulators.

Correspondingly, there are three distinctive types of elementary excitations. Firstly, the short-range RVB correlations in \(|\text{RVB}\rangle\) are reflected by a finite energy gap \( E_g \) opened up in a neutral spin excitation, called a spin-roton, which only reduces to the gapless spin wave in an AFM state with a long-range RVB pairing, e.g. at half-filling. Such a characteristic \( E_g \) will protect the superconducting phase coherence from the phase disordering effect, caused by \( \Lambda_{h} \), that closely monitors the neutral spin correlations in \(|\text{RVB}\rangle\). In particular, it determines [25] a
unique $T_c$ formula (42). Secondly, the fermionic excitations related to breaking up the Cooper pairs in equation (3) are called backflow spinons ($a$-spinons) as they carry well-defined spins and therefore also contribute to spin excitations. Such backflow spinon excitations will constitute a lower branch of spin excitations, below $E_g$, as the size of the Cooper pairing is usually larger than that of the RVB pairing. Thirdly, the $a$-spinon is distinct from a conventional Bogoliubov particle by a singular phase shift factor $e^{i\hat{\Omega}_1}$. Although the phase coherence $\langle e^{i\hat{\Omega}_1} \rangle \neq 0$ in the superconducting phase can make the $a$-spinon show up in the single-particle channel around the antinodal regime, the local singular fluctuations of $e^{i\hat{\Omega}_1}$ will, in addition, induce a new collective mode as a bound state of the fractionalized particles around the nodal regime. It is nothing but a conventional Bogoliubov nodal quasiparticle.

Some additional unconventional properties can be inferred in such a d-wave superconductor. In the spin channel, two branches of spin excitations near $(\pi, \pi)$, separated by a resonance-like energy scale $E_g$, are responsible for the spin rotons (upper branch) and $a$-spinons (lower branch), respectively. In particular, strong magnetic fields should affect the lower branch first via the Zeeman effect, which can eventually destroy the pairing of the $a$-spinons and thus the superconducting order parameter, leading to small Fermi pockets formed by the fermionic $a$-spinons; on the other hand, in the single-particle channel, a dichotomy of Bogoliubov quasiparticle and the $a$-spinon coherent peaks can appear simultaneously in the superconducting phase in the nodal and antinodal regions, respectively. But the latter should disappear in the single-electron spectral function once the phase of the superconducting order parameter is thermally disordered by spin excitations in $|RVB\rangle$, where each excited spinon in $|RVB\rangle$ automatically induces a current vortex via $e^{i\hat{\Omega}_1}$, forming a spinon-vortex and proliferating in a quantum vortex liquid state [24]; furthermore, in the presence of weak magnetic fields, the Cooper pairing of doped holes provides a small phase stiffness $\rho_s(0) \propto \delta$ at $T = 0$. But $\rho_s$ will be thermally reduced by the Bogoliubov nodal quasiparticles which couple to the external electromagnetic fields with a full electric charge, leading to $\rho_s(T) = \rho_s(0) - aT$ with $a \sim O(1)$. Obviously all of these will have strong experimental implications. But we shall further make a detailed comparison with the cuprate superconductors elsewhere.

Finally, we remark that the present low-energy effective theory described by $H_{\text{eff}}$ in equation (32) resembles the so-called two-fluid models, which have been phenomenologically proposed for the cuprate superconductors ([29] and references therein; [30]) and iron-based superconductors [31] based on different theoretical considerations. The main similarity to these approaches lies in the fact that there exist both an itinerant BCS-type component (i.e. $|\Phi_a\rangle$ in the present case) and a localized spin liquid component (i.e. $|RVB\rangle$ here). The main distinction lies in how the local spin liquid component is mathematically characterized: the full description of it in the present approach is actually given by $|RVB\rangle \otimes |\Phi_h\rangle$, which is described by a mutual Chern–Simons gauge model (equation (40)) that respects spin rotation and time-reversal symmetries [19]. With the doping effect self-consistently incorporated, such a local spin component can naturally evolve from an AFM ordering state to a spin liquid state and exhibit multilevel pseudogap behavior in the underdoped regime [10].

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Appendix. A hidden off-diagonal-long-range-order in ground state (8)

The ground state (3) reduces to equation (8) under the holon condensation condition (7). If the total number of holons is not fixed, the ground state (8) may be further expressed in a compact form

$$|\Psi_G\rangle = e^{\hat{D}}|\text{RVB}\rangle.$$  \hspace{1cm} (A.1)

Since the operator $\hat{D}$ will introduce a pair of holes, satisfying

$$[\hat{N}_h, \hat{D}] = 2\hat{D},$$  \hspace{1cm} (A.2)

where $\hat{N}_h = \sum_i n^h_i$, by using $\hat{N}_h(\hat{D})^n|\text{RVB}\rangle = 2n(\hat{D})^n|\text{RVB}\rangle$, one finds $\hat{N}_h|\Psi_G\rangle = 2\hat{D}|\Psi_G\rangle$ such that

$$\langle \hat{D} \rangle \equiv \frac{\langle \Psi_G | \hat{D} | \Psi_G \rangle}{\langle \Psi_G | \Psi_G \rangle} = \frac{\langle \hat{N}_h \rangle}{2} = \frac{\delta N}{2}$$  \hspace{1cm} (A.3)

at a finite doping concentration $\delta$. The relation clearly indicates that the ground state (8) possesses a new ODLRO as given in equation (11) for a finite spatial separation of $i,j$.

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