Stability analysis of piecewise defined nonlinear ambiguous system by direct Lyapunov method

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Abstract. In this paper the piecewise defined nonlinear ambiguous system is considered. The stability of such objects is investigated using separately excited DC generator with two excitation windings with relay control system taking into account magnetic hysteresis phenomenon. The Lyapunov function for this system is designed by direct Lyapunov method and its generalization through Dini derivatives that allows approving the stability of generator with nonlineairties.

1. Introduction
Stability analysis of nonlinear control system is the essential part of the control system design. Known stability analysis approaches (frequency criterions, in particular Popov’s criterion, circular criterion and others, describing function method, etc. [1]-[3]) possess significant restrictions and are applicable to the system with single definite nonlinearity of specific class.

The more powerful tool of stability analysis is the second, or direct, Lyapunov method [4] that is based on Lyapunov function properties investigation. Such approach allows defining the structure and domain of synthesized controller parameters under which the control objective is solved. One of the direct method drawbacks is the absence of universal Lyapunov function designing techniques and existence of the requirements to it continuous differentiability. Last constraint, as it is seen in [5]-[7], can be bypassed with implementation of Dini derivatives mathematical tool.

In this paper stability analysis of nonlinear stabilization current system of the separately excited (SE) DC generator with two excitation windings (EW) taking into account the nonlinearity of the magnetic curve (MC) that describes ambiguous defined performance and relay characteristic of the EW control device by generalized direct Lyapunov method is considered.

2. Mathematical description of the SE DC generator
The electrical machine design, in particular SE DC, assumes the ferromagnetic materials utilization. If ferromagnetic subject is affected by nonzero external magnetic field magnetization process will occur that characterized with dependence between magnetic induction of the object and external magnetic field intensity.

Feeding the EW of the SE DC generator that was not magnetic exposed leads to the magnetization process according to main MC. Operating principle of the investigated object presumes output pulse alternating current series generation with exponential decreasing amplitude \( A \), increase \( T_f \) and decline...
$T_c$ front time duration, stabilization peak time duration $T$ and pauses between pulses $t_{\text{pause}}$. As a result of cycled magnetic impaction the magnetic hysteresis phenomenon is observed that represents the lagged magnetic induction of the object from external magnetic field intensity.

The MC parameters are estimated from electromotive force (EMF) $E_d$ of the SEDC generator[8],[9]:

$$f_{MC}(i_{EW}) = \frac{E_s}{w_{MEW} + w_{AEW}} c \cdot n = \frac{P_s}{30 c_n} \frac{E_s}{w_{MEW} + w_{AEW}} \frac{1}{\omega} = \frac{k_{EW} E_s}{(w_{MEW} + w_{AEW})} \frac{1}{\omega}$$

(1)

where $i_{EW}$ – excitation current; $w_{MEW}, w_{AEW}$ – number of turns in main excitation winding (MEW) and additional excitation winding (AEW); $c_e$ – constructive constant; $n$ – rotation frequency of the system; $\omega$ – angular rotation speed of the system.

In the case main MC EMF is defined by the value of experimental idling characteristic (IC) that is approximated by polynomial. Then $E_d$ till the low peak formation is changed conform to hysteresis loop of SEDC generator which mathematical description based on Takacs $T(x)$-model [10] and obtained in [11]. Hence EMF generating process of any pulse pair can be presented as equationsystem:

$$E_s = f_s(i_{EW}) = E_{s\text{max}}$$

(2)

$$\begin{align*}
  f_s(x) &= a_n x^n + a_1 x^1 + a_0 x^0 + a_i x_i, \quad (t < T_j) \land (j > 1); \\
  f_s(x_{\text{max}}, i_{EW}) &= \left(\frac{x + x_i}{x_{\text{max}} + x_i}\right), \quad (t <= ((j - 1)T + T_j)) \land (j > 1); \\
  \tanh(x + a_n) - b, \quad (t > ((j - 1)T + T_j)) \land (t <= (j - 1)T + 0.5T + T_j); \\
  \tanh(x - a_n) + b, \quad (t > (j - 1)T + 0.5T + T_j)) \land (t <= (j - 1)T + T). 
\end{align*}$$

(3)

where:

$$x = \frac{i_{EW}}{i_{EW\text{max}}}; \quad x_{\text{max}} = A; \quad x_{\text{max}} = -A;$$

$$b = \frac{E_{s\text{max}}}{f_s(i_{EW}) \cdot \tanh(x_{\text{max}}) \cdot \text{sign}(x_{\text{max}}) \cdot \text{sign}(x_{\text{max}} a_n)} +$$

$$+ (1 - \frac{x_{\text{max}} - x_i}{x_{\text{max}} - x_i}) \cdot \text{sign}(x_{\text{max}}) \cdot \frac{f_s(i_{EW})}{E_{s\text{max}}} \cdot \tanh(x_{\text{max}}) \cdot \text{sign}(x_{\text{max}} a_n);$$

(4)

where $i_{EW\text{max}}, E_{d\text{max}}$ - rated values of the excitation current and EMF accordingly; $x$ - normed value of the excitation current; $x_{\text{max}}$ - initial peak value of the hysteresis loop that takes either positive value of current pulse pair (descending hysteresis loop) or negative value (ascending hysteresis loop); $x_{\text{max}}$ - final value of the hysteresis loop; $a_n$ - the displacement coefficient along abscissa axis that equals 0.5 in this case (defines the coercive force); $b$ - the displacement coefficient along ordinate axis (defines the remanence); $x_i$ - normed crossing value of the descending hysteresis loop and abscissa axis for the current pulse pair; $j$ - number of the current pulse pair.

As a power source both of MEW and AEW the voltage source inverter pulse width converter realizing the relay control law is used. The MEW control system is carried out according to principle of the negative feedback generator amateur current that is compared with reference signal. Then depending on error sign arriving to input of the regulator with relay characteristic the MEW voltage is formed:

$$u_{MEW} = f_e(i_{\text{ref}} - i_n) = f_e(e) = u_{MEW\text{max}} \begin{cases} +1, e > 0; \\ -1, e < 0. \end{cases}$$

(4)

where $u_{MEW\text{max}}$ – rated value of MEW voltage.
AEW relay control system based on principle negative feedback of the derivative amateur current and, similarly to (4), the AEW voltage is defined either positive rated or negative AEW rated voltage value $u_{AEW}^{max}$.

If it is assumed that DC generator is rotated with stabilized speed $\omega$ then the electromagnetic processes in SE DC generator with two EW with relay control system taking into account magnetic hysteresis phenomenon through excitation $i_EW$ and amateur $i_a$ currents can be described by the following differential equations:

\[
\begin{align*}
\frac{di_{EW}}{dt} &= \frac{1}{(L_{MEW} + L_{AEW})} \left[ u_{MEW}^{max} f(e) + u_{AEW}^{max} - \frac{1}{\omega} \left( r_{MEW} + r_{AEW} \right) i_EW - \frac{(w_{MEW} + w_{AEW}) k_{EW}}{\omega} f_i(i_EW) \right], \\
\frac{di_a}{dt} &= \frac{1}{(L_a + L_f)} \left[ f_1'(i_{EW}) - (r_a + R_f) i_a \right]
\end{align*}
\]

where

\[
f_1 = f_1'(i_{EW}) = E_{s\max} \begin{cases} \alpha_1 + 3\alpha_2 x_2^2 + 5\alpha_3 x_4^4 + 7\alpha_4 x_6^6 + 9\alpha_5 x_8^8, & (t < T_f) \land (i = 1); \\
\frac{f_{ic}(x_{max} l_{EW}^{max}) (\frac{1}{(x_{max} + x_2)} + \frac{x_2}{(x_{max} + x_2)})}{(x_{max} + x_2)}, & (t <= ((i-1)T_f + T_f) \land (i > 1)); \\
\text{sech}^2(x_i + a_i), & (t > ((i-1)T_f + T_f)) \land (t <= ((i-1)T_f + 0.5T_f + T_f)); \\
\text{sech}^2(x_i - a_i), & (t > ((i-1)T_f + 0.5T_f + T_f)) \land (t <= ((i-1)T_f + T_f)). \end{cases}
\]

where $L_{MEW}$ - inductance and active resistance of the MEW accordingly; $L_{AEW}$ - inductance and active resistance of the AEW accordingly; $L_a$, $r_a$ - inductance and active resistance of the amateur winding accordingly; $L_f$, $f_f$ - inductance and active resistance of the load accordingly.

The transients of (5) are obtained in Matlab/Simulink and graphs of the amateur current (a) and current error (b) are presented in Figure 1.

![Figure 1. Output current pulse series in SE DC generator.](image-url)
3. Direct Lyapunov method

This method, as it is shown in [4], [12]-[14], is universal for stability analysis of the nonlinear systems and presumes usage the Lyapunov function definition.

Consider the following system:

\[
\frac{dx}{dt} = f(t, x)
\]

with domain of \( f \) of the form \( \Gamma_i = \{x, t : t \in I, \|x\| < H, H = \text{const} > 0 \text{with } H = +\infty \} \) at that \( f \) is continuous by \( t \) and \( x \) and continuously differentiable by \( x \). Besides that it is introduced additional restriction for \( f \):

\[
f(t, 0) = 0
\]

i.e. the system (7) assumes a trivial solution.

**Definition 1.** The real scalar function of vector argument and time:

\[
V(t, x) : \Gamma_i \rightarrow R^1
\]

defined in some domain

\[
\Gamma_i = \{x, t : t \in I, \|x\| < H \}
\]

such as \( \Gamma_i \subseteq \Gamma_i^H \), is called Lyapunov function for the system (7) in domain (10) that satisfies the following conditions:

1) \( V(t, x) \) is real scalar function: \( V(t, x) \) is continuous by \( t \) and \( x \); \( V(t, x) \geq 0 \) at any \( t \in \Gamma_i^b \).

**Definition 2.** Consider continuous by \( t \) and \( x \) Lyapunov function \( V(t, x) \), then the equation:

\[
\frac{dV(t, x)}{dt} = \frac{\partial V(t, x)}{\partial t} + \sum_{j=1}^{n} \frac{\partial V(t, x)}{\partial x_j} f_j(t, x) = \frac{\partial V(t, x)}{\partial t} + [\text{grad}_x V(t, x)]^T f(t, x)
\]

said to be derivative by \( t \) of Lyapunov function \( V(t, x) \) calculated according to the system (7).

**Theorem 1.** Consider continuous by \( t \) and \( x \) Lyapunov function \( V(t, x) \) (9) in \( \Gamma_i^b \) (10), such as in this domain:

1) \( V(t, x) \geq W(x) > 0 \), where \( W(x) \) – some independent from time Lyapunov function; \( V(t, 0) = W(0) = 0 \forall t \in I \), i.e. \( V(t, x) \) is positive definite Lyapunov function in \( \Gamma_i^b \) (10);

2) complete derivative of Lyapunov function (11) of (7) \( \frac{dV(t, x)}{dt} \leq 0 \), i.e. \( \frac{dV(t, x)}{dt} \) is negative definite Lyapunov function in \( \Gamma_i^b \) (10).

Then the trial solution \( x = 0 \) (9) of the system (7) is stable by Lyapunov at \( t \rightarrow +\infty \).

To define the Lyapunov function for the system (5) error function should be calculated at any time. Therefore \( f(e) \) is approximated and substituted by trigonometrical function of the error of (5) namely hyperbolical tangent \( \tanh(k \cdot e) \), where \( k \) – an infinite large gain.

Due to the state variables of (5) are piecewise defined through (3) then the condition of continuous differentiable of (9) is disrupted. As it is seen, for example, in [1] and [2], in such case the generalization of direct Lyapunov method is permissible with utilization Dini derivatives that will be pointed in the next section.
4. Dini derivatives

**Definition 3.** Let, conform to [1], \( f : G \rightarrow R \) be a function, where \( G \subset R \), \( h \in G \). Then the following states are acceptable:

1) Value \( D^+ f(h) = \lim_{x \to h^+} \frac{f(x) - f(h)}{x - h} \) is right-hand upper Dini derivative of \( f \) at the point \( h \);

2) Value \( D^- f(h) = \lim_{x \to h^-} \frac{f(x) - f(h)}{x - h} \) is right-hand lower Dini derivative of \( f \) at the point \( h \);

3) Value \( D^- f(h) = \lim_{x \to h^-} \frac{f(x) - f(h)}{x - h} \) is left-hand upper Dini derivative of \( f \) at the point \( h \);

4) Value \( D^- f(h) = \lim_{x \to h^-} \frac{f(x) - f(h)}{x - h} \) is left-hand lower Dini derivative of \( f \) at the point \( h \).

**Theorem 2.** The significant condition of continuous of the function at the range is a fixed sign of one of the Dini derivatives \( D^+ f(x) \) at this range. Meanwhile:

1) If \( D^+ f(x) > 0 \), \( x \in (a, b) \) then \( f \) is strictly increasing at \((a, b)\);

2) If \( D^+ f(x) \geq 0 \), \( x \in (a, b) \) then \( f \) is increasing at \((a, b)\);

3) If \( D^+ f(x) < 0 \), \( x \in (a, b) \) then \( f \) is strictly decreasing at \((a, b)\);

4) If \( D^+ f(x) \leq 0 \), \( x \in (a, b) \) then \( f \) is decreasing at \((a, b)\).

Therefore to proof the continuous differentiability of the system (5) it is necessary to show existence the fixed sign Dini derivative at the points where the derivative amateur current sign is changed, i.e. at the final time of the each MC part, which graphs are depicted in Figure 2.

![Figure 2. MC of SE DC generator.](image)

The examples of Dini derivatives calculation are carried out for first current pulse pair according to (2) and Figure 2.

**Table 1.** Dini derivatives for current pulse pair of SE DC generator.

| Dini derivatives at h point | \( h = i_{EW,\text{max}1} \) | \( h = -i_{EW,\text{max}1} \) | \( h = x_1 \) | \( h = i_{EW,\text{max}2} \) |
|-----------------------------|-----------------|-----------------|-----------------|-----------------|
| \( D^+ f_e (h) \)           | 0               | \(-k_{21}\)     | \( k_{31}\)     | \(-k_{41}\)     |
| \( D^- f_e (h) \)           | 0               | \( k_{22}\)     | \( k_{32}\)     | \(-k_{42}\)     |
\[D \cdot f_s(h) \quad k_{i3} \quad 0 \quad -k_{s3} \quad -k_{d3}
\]

\[D \cdot f_s(h) \quad k_{i4} \quad 0 \quad k_{s4} \quad k_{d4}
\]

It is necessary to note that the obtained results (see Table 1) can be extended to following pulse pairs with accuracy to decrement coefficient of its amplitudes. In this regard it can be made a conclusion that at any time of the sign change of the derivative amateur current there are at least two Dini derivatives that satisfies the theorem 2. Hence, the electromagnetic processes in SE DC generator are continuous differentiable.

5. Lyapunov function for SE DC generator with relay control system and hysteresis loop

Here it will be shown that Lyapunov function derived from [15], where it was made assumption that magnetization in generator occurred conform to main MC, is also acceptable to hysteresis loop case.

To conveniently using of the system (5) there are introduced following substitutions:

\[x_1 = i_l \; ; \; x_2 = i_a ;\]

\[a_{i1} = \frac{r_{em} + r_{cad}}{(L_{em} + L_{cad})} \cdot c_{i1} = \left(\frac{w_{em} + w_{cad}}{\omega_s (L_{em} + L_{cad})}\right) \cdot c_{i2} = \frac{1}{(L_{em} + L_{cad})} \cdot a_{i2} = \frac{r_a + R_t}{(L_a + L_r)} \cdot c_{i2} = \frac{1}{(L_a + L_r)} \cdot c_{i2}.
\]

The reference signal and its time derivative can be represented by Fourier series form:

\[f_i(t) = x_{ref}(t) = s_0 + C_h \cos(G^T h(t)) + S_h \sin(G^T h(t))
\]

\[f_{ao}(t) = \frac{dx_{ref}(t)}{dt} = s_{ao} + C_{ao} \cos(G^T h(t)) + S_{ao} \sin(G^T h(t))
\]

where

\[T_{pulse} = 2(T_f + T_s + T_p + T_{pause}); s_0 = \frac{1}{T_a} \int_0^{T_{pulse}} f_i dt, s_{d0} = \frac{1}{T_a} \int_0^{T_{pulse}} f_{ao} dt;
\]

\[G_h = [g_1 \; g_2 \; \ldots \; g_{h-1} \; g_h], g_i = i \cdot \frac{2\pi}{T_{pulse}}, i = 1, h;
\]

\[C_h = [c_1 \; c_2 \; \ldots \; c_{h-1} \; c_h], c_i = -\frac{2}{T_{pulse}} \int_0^{T_{pulse}} f_i \cos(g_i) dt, i = 1, h;
\]

\[S_h = [s_1 \; s_2 \; \ldots \; s_{h-1} \; s_h], s_i = \frac{2}{T_{pulse}} \int_0^{T_{pulse}} f_i \sin(g_i) dt, i = 1, h;
\]

\[C_{ao} = [c_{d1} \; c_{d2} \; \ldots \; c_{d(h-1)} \; c_{dh}], c_{di} = \frac{2}{T_{pulse}} \int_0^{T_{pulse}} f_{ao} \cos(g_i) dt, i = 1, h;
\]

\[S_{ao} = [s_{d1} \; s_{d2} \; \ldots \; s_{d(h-1)} \; s_{dh}], s_{di} = \frac{2}{T_{pulse}} \int_0^{T_{pulse}} f_{ao} \sin(g_i) dt, i = 1, h;
\]

where \(T_{pulse}\) – period of the signal; \(h\) – number of harmonics at Fourier series expansion.

Then the system (5) relatively \(e\) and \(x_1\) takes the view:

\[\frac{dx_1}{dt} = -a_{i1} x_1 - c_{i1} f_s(x_1) + c_{i2} u_{MEW_{max}} \tanh(ke) + c_{i2} u_{AEW_{max}} \tanh(f_{ao}(t));
\]

\[\frac{de}{dt} = -c_{21} f_s(x_1) - a_{22} e + f_{ao}(t),
\]

(15)
where \( f_{11}(t) = f_{1}(t) + a_{22} f_{id}(t) \).

Consider the independent from time Lyapunov function of the system (15):

\[
V_{g}(x_{1}, e, t) = \ln(\cosh(-a_{11} x_{1} - c_{11} f_{g}'(x_{1}) + c_{12} u_{AEW_{\max}} \tanh(ke) + u_{AEW_{\max}} c_{12} \tanh(f_{id}(t)))) - 0.5 c_{12} u_{AEW_{\max}} \tanh^{2}(ke) \tag{16}
\]

Hence according to (11) derivative of (14) can be estimated:

\[
dV_{g}(t, x)/dt = \tanh(-a_{11} x_{1} - c_{11} f_{g}'(x_{1}) + c_{12} u_{AEW_{\max}} \tanh(ke) + u_{AEW_{\max}} c_{12} \tanh(f_{id}(t))) \cdot
\]

\[
- u_{AEW_{\max}} c_{12} \tanh^{2}(f_{id}(t)) + (-a_{11} x_{1} - c_{11} f_{g}'(x_{1}) + c_{12} u_{AEW_{\max}} \tanh(ke) +
\]

\[
+ u_{AEW_{\max}} c_{12} \tanh(f_{id}(t))) \tanh(-a_{11} x_{1} - c_{11} f_{g}'(x_{1}) + c_{12} u_{AEW_{\max}} \tanh(ke) +
\]

\[
+ u_{AEW_{\max}} c_{12} \tanh(f_{id}(t)) \tanh(-a_{11} x_{1} - c_{11} f_{g}'(x_{1}) + c_{12} u_{AEW_{\max}} \tanh(ke) + u_{AEW_{\max}} c_{12} \tanh(f_{id}(t)))) \cdot
\]

\[
- c_{12} u_{AEW_{\max}} \tanh^{2}(ke) k - c_{12} u_{AEW_{\max}} \tanh(ke) \tanh^{2}(ke) k \]

where

\[
f_{2} = f_{g}''(x_{1}) = E_{\max} \begin{cases} 6 a_{i} x_{i} + 20 a_{i} x_{i}^{2} + 42 a_{i} x_{i}^{3} + 72 a_{i} x_{i}^{4}, & (t < T_{f}) \land (i = 1); \\
0, & (t <= ((i - 1)T + T_{f}) \land (i > 1)); \\
2 \tanh^{2}(x_{i} + a_{i}) \tanh(x_{i} + a_{i}), & (t > ((i - 1)T + T_{f})) \land (t <= (i - 1)T + 0.5T + T_{f}); \\
2 \tanh^{2}(x_{i} - a_{i}) \tanh(x_{i} + a_{i}), & (t > (i - 1)T + 0.5T + T_{f}) \land (t <= (i - 1)T + T). 
\end{cases} \tag{18}
\]

**Figure 3.** Lyapunov function \( V_{g}(x_{1}, e, t) \).

To prove that Lyapunov function \( V_{g}(x_{1}, e, t) \) (16) is positive definite and continuous by \( x_{1}, e, t \) it is presented its graph in Figure 3.

To prove that derivative of Lyapunov function \( dV_{g}(x_{1}, e, t)/dt \) (17) is negative definite and continuous by \( x_{1}, e, t \) it is depicted its graph in Figure 4.
As it can be seen from Figures 3 and 4 Lyapunov function \( V_g(x_1,e,t) \) (16) is positive definite and its derivative \( dV_g(x_1,e,t)/dt \) (17) – negative definite. These satisfy the theorem 1 requirements that lead to stability of the system (15) according to direct Lyapunov function method.

6. Conclusions
In this paper the stability analysis of piecewise defined nonlinear system with implementation generalized by Dini derivatives second Lyapunov method is carried out. The investigated object presents SE DC generator with two EW with relay control system taking into account magnetic hysteresis phenomenon that is described by ambiguous nonlinear characteristic. It is proofed that for both magnetization process according to main MC curve and conform to hysteresis loop the positive definite Lyapunov function and negative definite Lyapunov function derivative exists and has with accuracy to nonlinearity of SE DC generator MC curve the similar form that points to the stability of the considered system.

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