Thermodynamics of a class of large quantum systems

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Abstract. C∗-dynamical approaches to the second law of thermodynamics are reviewed. Then, the Clausius equality and quasistatic processes are reinvestigated within this context.

1. Introduction
An understanding of a relation between reversible dynamics and irreversible phenomena is one of the fundamental problems in statistical mechanics. Recent progress in the research of mesoscopic systems brings a new aspect into this problem. In these systems, coherence (a quantum dynamical aspect) may be observed in a dissipative transport (an irreversible phenomenon) and the two aspects should be discussed simultaneously. Usually, a mesoscopic system couples with much larger environments and the interaction is not weak. As a result, the system cannot be clearly distinguished from the environments. Therefore, it is natural to deal with a mesoscopic system plus its environments as an infinitely extended system.

Statistical mechanics of infinitely extended systems has been developed so far [1, 2, 3] and, recently, their nonequilibrium properties are studied intensively. Those include analytical studies on nonequilibrium steady states of harmonic crystals [4, 5], a one-dimensional gas [6], unharmonic chains [7], an isotropic XY-chain [8], systems with asymptotic abelianness [9], a one-dimensional quantum conductor [10], an interacting fermion-spin system [11], fermionic junction systems [12], a quasi-spin model of superconductors [13] and a bosonic junction system with the Bose-Einstein condensate [14] (see also reviews [15, 16]). In this article, we review some works on the second law of thermodynamics in infinitely extended quantum systems and reinvestigate the Clausius equality and quasi-static processes within this context.

2. C∗-dynamical systems [3]
One of promising approaches dealing with infinitely extended quantum systems is a C∗-algebraic approach [3]. It starts from a set \( \mathcal{F} \) of finite observables (a C∗-algebra), which is a complete linear space with a norm \( \| \cdot \| \), where a product \( AB \) and antilinear involution \( *: A \to A^* \) (\( \forall A, B \in \mathcal{F} \)) are defined and whose norm satisfies \( \| AB \| \leq \| A \| \| B \| \) and the C∗-property: \( \| A^* A \| = \| A \|^2 \).

An autonomous time evolution is described by a strongly continuous one-parameter group of *-automorphisms \( \tau_t \) (\( t \in \mathbb{R} \)), namely, \( \tau_t \) is a linear map satisfying \( \tau_t(AB) = \tau_t(A)\tau_t(B), \tau_t(A^*) = (\tau_t(A))^* \), \( \tau_0 = I \) (I: the identity map), \( \tau_t\tau_s = \tau_{t+s} \) and \( \lim_{t \to 0} \| \tau_t(A) - A \| = 0 \) (\( \forall A \in \mathcal{F} \)). Then, according to the theory of semigroups, there exists a densely defined generator \( \delta \) of \( \tau_t \):

\[
\lim_{t \to 0} \left\| \delta(A) - \frac{1}{t} \{ \tau_t(A) - A \} \right\| = 0, \quad (\forall A \in D(\delta))
\]
where $D(\delta)$ is the domain of $\delta$. For finite systems, the generator $\delta$ is the commutator with the Hamiltonian $H$: $\delta(A) = i[H, A]$ and $\tau_t$ is a map $\tau_t(A) = e^{iHt}Ae^{-iHt}$. As we shall see later, nonautonomous time evolution can be described by a one-parameter family of $*$-automorphisms.

States are introduced by listing expectation values. Namely, each state is identified with a linear map $\omega$ from $A \in \mathcal{F}$ to an expectation value $\omega(A)$. The positivity condition $\omega(A^*A) \geq 0$ and normalization condition $\omega(1) = 1$ ($1 \in \mathcal{F}$ is the identity) are required. From the practical point of view, such a specification is natural since states are determined through measurements.

Canonical states are necessary for the investigation of thermal properties. As easily seen, the grand canonical average $\langle \cdots \rangle_{gc}$ with temperature $\beta^{-1}$ and chemical potential $\mu$ over a finite dimensional algebra of observables is uniquely determined by the Kubo-Martin-Schwinger (KMS) condition: $\langle AB \rangle_{gc} = \langle A\sigma_{i\beta}(B) \rangle_{gc}$ where $\sigma_s(A) = e^{(H-\mu N)s}Ae^{-(H-\mu N)s}$ is a ‘time evolution’ generated by the ‘Hamiltonian’ $H - \mu N$ with $H$ the Hamiltonian and $N$ the total number of particles. This argument can be generalized to the $C^*$-algebraic approach because the ‘time evolution’ can be defined as a strongly continuous one-parameter group. One then introduce an analog $\omega$ of a canonical state, called a $\sigma$-KMS state at inverse temperature $\beta$, as the one satisfying $\omega(A\sigma_{i\beta}(B)) = \omega(BA)$ ($\forall A, B \in F_{\text{KMS}}$) where $\sigma_s$ is a strongly continuous one-parameter group of $*$-automorphisms and $F_{\text{KMS}}$ is a norm dense $*$-subalgebra of $\mathcal{F}$.

A local equilibrium state can be represented as a KMS state. For example, let us consider a system consisting of $M$ independent infinitely extended systems. Let $\mathcal{F}_j$ be an algebra of dynamical variables belonging to the $j$th subsystem ($j = 1 \cdots M$), where a time evolution group $\tau_t^{(j)} : \mathcal{F}_j \rightarrow \mathcal{F}_j$ is defined. Now we introduce a strongly continuous group of $*$-automorphisms on $\mathcal{F}_{\text{MR}} \equiv \mathcal{F}_1 \otimes \mathcal{F}_2 \cdots \otimes \mathcal{F}_M$ by

$$\sigma_s^{loc} = \tau_{\beta_1 s}^{(1)} \otimes \tau_{\beta_2 s}^{(2)} \cdots \otimes \tau_{\beta_M s}^{(M)}$$

where $\beta_j$ is the inverse temperature of the $j$th subsystem. Then, the $\sigma^{loc}$-KMS state $\omega_{loc}$ at $\beta = 1$ is a local equilibrium state, which, in the finite dimensional case, corresponds to the density matrix $\rho_{loc} \sim \exp\{-\sum_j \beta_j H_j\}$ ($H_j$ is the Hamiltonian of the $j$th subsystem).

### 3. $L^1$-asymptotic abelian property

In thermodynamics, environments are assumed to stay in equilibrium under arbitrary processes and their details are considered to be unimportant. Hence, thermodynamic environments would be well-modelled by systems with appropriate ergodicity. As one of such an example, we consider systems satisfying the $L^1$-asymptotic abelian property.

The time evolution $\tau_t$ is said to satisfy the $L^1(\mathcal{G})$-asymptotic abelian property if there exists a norm dense $*$-subalgebra $\mathcal{G}$ such that

$$\int_{-\infty}^{+\infty} dt \| [A, \tau_t(B)] \| = \int_{-\infty}^{+\infty} dt \| A\tau_t(B) - \tau_t(B)A \| < +\infty$$

holds for $A \in \mathcal{G}$ consisting of even number of fermion operators and any $B \in \mathcal{G}$ [3]. If both $A$ and $B$ consist of odd number of fermion operators, the commutator $[A, \tau_t(B)]$ should be replaced by the anticommutator $[A, \tau_t(B)]_+ = A\tau_t(B) + \tau_t(B)A$. This property implies rapid decay of correlations and is satisfied by free fermions in $\mathbb{R}^d (d \geq 1)$ (Example 5.4.9 of [3]). Note that, if a system admits bound states, it does not satisfy the $L^1$-asymptotic abelian condition as there exist bounded constants of motion, i.e., observables $C$ satisfying $\tau_t(C) = C$.

For any $V = V^* \in \mathcal{G}$, let us consider the perturbed time evolution $\tau^V_t$ which is defined by

$$\frac{d\tau^V_t(A)}{dt} = \tau^V_t \left( \delta(A) + i[V, A] \right), \quad \tau^V_0(A) = A, \quad (A \in D(\delta))$$

with $\delta$ the generator of $\tau_t$. Then, from the $L^1$-asymptotic abelian property, one has
(a) Existence of Møller morphisms $\gamma_{\pm}$: The limits $\gamma_{\pm}(A) = \lim_{t \to \pm \infty} \tau_{\pm}^{V_{t}}(A)$ exist in norm for $\forall A \in \mathcal{F}$ and $\forall V = V^* \in \mathcal{G}$, and they define automorphisms (cf. Prop. 5.4.10 of [3]).

(b) Return-to-equilibrium: Let $\omega^V$ be a $\tau^V$-KMS state at $\beta$, then $\omega^V \circ \gamma_{\pm}$ is $\tau$-KMS and $\lim_{t \to \pm \infty} \omega^V(\tau_t(A)) = \omega^V(\gamma_{\pm}(A))$ (cf. Prop. 5.4.10 of [3]).

(c) Strong mixing\(^1\): Any factor state\(^2\) $\omega$ satisfies $\lim_{t \to \pm \infty} |\omega(\tau_t(B)C) - \omega(AC)\omega(\tau_t(B))| = 0$ for all $A, B, C \in \mathcal{F}$ (cf. Example 4.3.24 of [3]).

4. Passivity

Within the context of $C^*$-dynamical systems, Pusz and Woronowicz [17] studied the response of KMS states to cyclic perturbations. They have considered a $C^*$-algebra $\mathcal{F}$ where a strongly continuous group $\tau_t$ of automorphisms is given. Let $\delta$ be the generator of $\tau_t$ and the function $P_t = P^*_t \in \mathcal{F}$ of $t$ be twice continuously differentiable in norm such that $P_t = 0$ ($t \leq 0$ or $t \geq T$), then the differential equation

$$\frac{d}{dt}\tau_t^P(A) = \tau_t^P(\delta(A) + i[P_t, A]) \quad \tau_0^P(A) = A \quad (A \in D(\delta))$$

defines a time evolution $\tau_t^P$. Pusz and Woronowicz [17] have shown that the work performed on a $\tau$-KMS state $\omega$ at $\beta(\neq 0)$ is non-negative:

$$L^P(\omega) \equiv \int_0^T dt \omega\left(\tau_t^P\left(\frac{dP_t}{dt}\right)\right) = -\frac{i}{\beta} S(\omega|\omega_T) \geq 0$$

where $\tau_t^P \in \mathcal{F}$ is the unique unitary element satisfying $\tau_t^P(A) = \Gamma_t^P\tau_t(A)\Gamma_t^{P^*}$ and $S(\omega|\omega_T)$ stands for the relative entropy between the initial equilibrium state $\omega$ and the state $\omega_T$ at time $t = T$. The relative entropy $S(\omega|\omega')$ between two states $\omega$ and $\omega'$ was introduced by Araki [3, 20, 21, 22, 23] as a $C^*$-generalization of $S(\rho|\rho') = \text{Tr}(\rho - \rho')/2$ two density matrices $\rho$ and $\rho'$. A finite-dimensional version of (4) was also given by Lenard [18] (See also [19]). If $-i\omega(U^*\delta(U)) \geq 0$ holds for a class of unitary elements $U$, $\omega$ is called passive [17].

Pusz and Woronowicz [17] have mentioned that Carnot’s inequality follows from (4). Here we show that Clausius inequality can also be derived. Let us consider a system of $M$ independent subsystems, then it is described by an algebra: $\mathcal{F}_{\text{MR}} = \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M$. Suppose that the system is in a local equilibrium state $\omega_{\text{loc}}$ discussed in Sec. 2, namely $\omega_{\text{loc}}$ is the $\sigma_{\text{loc}}$-KMS state at $\beta = 1$ with respect to $\sigma_{\text{loc}}$ given by (1). Then, consider an engine which takes deterministic internal states and which can exchange heat with individual reservoirs and perform work to external systems. A cyclic operation of this engine can be described by a set of dynamical perturbations $P_j(t) \in \mathcal{F}_j$ ($j = 1, \ldots, M$) to individual reservoirs, where $P_j(t) \equiv 0$ ($t < 0$ or $t > T$). Now we consider the perturbed ‘evolution’ $\sigma_{P}$ defined by $\frac{d}{dt}\sigma_{P}(A) = \sigma_{P}(\delta_{\text{loc}}(A) + i \sum_{j=1}^{M} \beta_j [P_j(\beta_j s), A])$, $\sigma_{P}(A)|_{s=0} = A$, $\forall A \in D(\delta_{\text{loc}})$ where $\delta_{\text{loc}}$ is the generator of $\sigma_{\text{loc}}$ and $P_j$ is an abbreviation of $\mathbf{1} \otimes \cdots \otimes P_j \otimes \mathbf{1}$. It is easy to see that $\sigma_{P} = \tau_{\beta_{1}s}^{(1)} \otimes \cdots \otimes \tau_{\beta_{M}s}^{(M)}$ where $\tau_{\beta_{j}s}^{(j)}$ is the solution of $\frac{d}{dt}\tau_t^{(j)}(A) = \tau_t^{(j)}(\delta(A) + i[P_j(t), A])$, $\tau_0^{(j)}(A) = A$ with $\delta_j$ the generator of $\tau_t^{(j)}$. Then, (4) gives

$$0 \leq \int_{0}^{T} ds \omega_{\text{loc}}\left(\sigma_{P}^{e}(\sum_j \beta_j \frac{dP_j(\beta_j s)}{ds})\right) = \sum_j \beta_j \int_{0}^{T} ds \omega_{\text{loc}}\left(\tau_t^{(j)}(\delta(A) + i[P_j(t), A])\right) = -\int_{0}^{T} ds \omega_{\text{loc}}\left(\tau_t^{(j)}(\delta(A) + i[P_j(t), A])\right) - \sum_j \beta_j Q_j$$

\(^1\) This property holds under a weaker asymptotic abelian condition in the norm sense: $\lim_{t \to \pm \infty} ||A, \tau_t(B)|| = 0$. For the sake of simpler arguments, here we restrict ourselves to the $L^1$-asymptotic abelian case.

\(^2\) Given a state $\omega$ over a $C^*$-algebra $\mathcal{F}$, it can be represented as a subalgebra $\pi_\omega(\mathcal{F})$ of the algebra $\mathcal{B}$ of all bounded operators on some Hilbert space (GNS representation). Let $\pi_\omega(\mathcal{F})' = \{ a \in \mathcal{B} : [a, b] = 0, \forall b \in \pi_\omega(\mathcal{F}) \}$ and $\pi_\omega(\mathcal{F})'' = \{ c \in \mathcal{B} : [c, a] = 0, \forall a \in \pi_\omega(\mathcal{F}) \}$, then $\omega$ is called a factor state iff $\pi_\omega(\mathcal{F})' \cap \pi_\omega(\mathcal{F})'' = \mathbf{1}$, where $\mathbf{1}$ is the set of complex numbers and $\mathbf{1}$ is the unit of $\mathcal{F}$. A KMS state representing pure phase is factor.
where $\bar{T} = \max_j T/\beta_j$ and $Q_j \equiv -\int_0^T dt \, \omega_{\text{loc}} \left( \frac{dP_j(t)}{dt} \right) = -\int_0^T dt \, \omega_{\text{loc}} \left( \tau_{t} \left( \frac{dP_j(t)}{dt} \right) \right)$ with $\tau_{t} = \tau_{t}^{(1)} \otimes \cdots \otimes \tau_{t}^{(M)}$. Note that $Q_j$ is the energy taken from the $j$th reservoir, or the heat taken from the $j$th reservoir. Namely, the Clausius inequality is derived. Note that such an engine may be realized by the classical ideal gas since fluctuations are negligible, but not by mesoscopic systems because their states are easily disturbed by the reservoirs.

5. Relative entropy change

Ojima, Hasegawa and Ichiyanagi [24] investigated an infinitely extended system driven by an interaction $V(t) = -\sum_j A_j X_j(t)$ where $X_j(t)$ and $A_j$ are, respectively, c-number forces and the conjugate system variables. They proved that the relative entropy between the initial state $\omega_{t_0}$ and the state $\omega_t$ at time $t$ is related to the thermodynamic entropy change: $S(\omega_{t_0} | \omega_t) = \beta \int_{t_0}^t ds \sum_j \omega_s (\delta(A_j)) X_j(s)$ where $\delta$ is the generator of the unperturbed time evolution $\tau$, the initial state $\omega_{t_0}$ is assumed to be $\tau$-KMS state at temperature $\beta$, and $\omega_s$ stands for the state at time $s$. Based on the view that the microscopic and macroscopic time scales are well separated, they compared the initial and final time average of $\frac{d}{dt} S(\omega_{t_0} | \omega_t)$ with the macroscopic entropy production and proved its positivity for almost periodic forces $X_j(t)$. A similar situation was studied by Fröhlich, Merkli, Schwarz, Ueltschi [25], but they identify the entropy with the true-state average of the logarithm of a reference density operator.

The relative-entropy formula was generalized by Ojima [26] to a composite system obeying an autonomous time evolution and he showed that the relative entropy production of a steady state, if it exists, is non-negative. Jakšić and Pillet [27, 11, 15] studied the relative entropy of a finite system coupled with several infinite reservoirs where both energy and mass transports are possible [16]: Let $F_j$ be an algebra of dynamical variables belonging to the $j$th reservoir, where a time evolution group $\tau_{t}^{(j)} : F_j \rightarrow F_j$ and a gauge transformation group $\alpha_{s}^{(j)} : F_j \rightarrow F_j$ are defined $(j = 1 \cdots M)$. Let $F_0$ be the corresponding algebra for the finite system. We, then, introduce an initial state $\omega$ as the $\tau$-KMS state at $\beta = 1$ with respect to a strongly continuous group of $*$-automorphisms on $F = F_0 \otimes F_1 \otimes F_2 \cdots \otimes F_M = F_0 \otimes F_M$ defined by

$$
\sigma_s = \sigma_s^{(0)} \otimes \tau_{t}^{(1)} \otimes \cdots \otimes \tau_{t}^{(M)}\alpha_{s}^{(1)} \otimes \cdots \otimes \alpha_{s}^{(M)}
$$

where $\beta_j$ and $\mu_j$ are, respectively, the inverse temperature and chemical potential of the $j$th reservoir and $\sigma_s^{(0)}(A) = \rho_s^{-\delta s} \rho_s^s (A \in F_0)$ with $\rho_s$ the density matrix of the initial system state. Suppose that the time evolution group $\tau_{t}^{(j)}$ is generated by $\delta(V)(A) = \sum_{j=1}^{M} \delta_j(A) + i[V,A]$, where $\delta_j$ is the generator of $\tau_{t}^{(j)}$ and $V = V^* \in \mathcal{F}$ is a gauge-invariant interaction. Then, the relative entropy $S(\omega | \omega_t)$ between the initial state $\omega$ and the state $\omega_t \equiv \omega \circ \tau_{t}^{V}$ at time $t$ is related to the heat flow $J_j^t \equiv -\delta_j(V) + \mu_j g_j(V)$ to the $j$th reservoir as

$$
S(\omega | \omega_t) = \sum_{j=1}^{M} \beta_j \int_{0}^{t} \omega_s (J_j^s) ds - \omega_t (\ln \rho_s \otimes 1_R) + \omega_t (\ln \rho_s \otimes 1_R)
$$

It is a group of automorphisms and, if the $j$th subsystem is finite, reduces to the map: $A \rightarrow e^{(N_j s)_{A} e^{-\delta (N_j s)}}$ where $N_j$ is the total number of particles. It is assumed to commute with the unperturbed evolution: $\tau_{t}^{(j)} \alpha_{s}^{(j)} = \alpha_{s}^{(j)} \tau_{t}^{(j)}$. 

\[^{3}\text{It is a group of automorphisms and, if the } j \text{th subsystem is finite, reduces to the map: } A \rightarrow e^{(N_j s)_{A} e^{-\delta (N_j s)}} \text{ where } N_j \text{ is the total number of particles. It is assumed to commute with the unperturbed evolution: } \tau_{t}^{(j)} \alpha_{s}^{(j)} = \alpha_{s}^{(j)} \tau_{t}^{(j)}.\]
where $g_j$ is the generator of the transformation $\alpha^{(j)}_s$ and $1_R$ is the unit of $\mathcal{F}_{MR}$. The meaning of $J^\eta_j$ is clear in the finite dimensional case. Then, in terms of a local Hamiltonian $H_j$ and a number operator $N_j$, one has $\delta_j(A) = i[H_j, A]$ and $g_j(A) = i[N_j, A]$. Let $H = \sum_{j=1}^{N} H_j + V$, then

$$J^\eta_j = -i[H_j, V] + \mu \delta_j[i[N_j, V] = i[H, H_j] - \mu_j i[H, N_j] = \frac{d}{dt} \tau^V_i (H_j) - \mu_j \frac{d}{dt} \tau^V (N_j)|_{t=0}$$

which, indeed, represents non-systematic energy flow, or heat flow, to the $j$th reservoir.

If the evolution $\tau^V_i$ is $L^1(\mathcal{G})$-asymptotic abelian and $V \in \mathcal{G}$, $\lim_{t \to \infty} \omega_t(A) = \omega_+(A)$ exists for any $A \in \mathcal{F}$ and defines a nonequilibrium steady state (NESS) $\omega_+ [9]$. Then, the NESS value of the entropy production is non-negative [26, 27]: $\lim_{t \to \infty} \frac{d}{dt} S(\omega_t|\omega_i) = \sum_{j=1}^{M} \beta_j \omega_+(J^\eta_j) \geq 0$.

Although this observation is consistent with thermodynamics, one cannot regard $\frac{d}{dt} S(\omega_t|\omega_i)$ as the entropy production of the whole (isolated) system as it may become negative [24].

6. Quasistatic process and Clausius equality

The Clausius inequality derived from passivity is not satisfactory as it cannot be applied to mesoscopic systems. The relative entropy production can be identified with the thermodynamic entropy production but for limited cases. And there might be an arbitrariness in the introduction of a reference state. Then, a useful insight into a full understanding of the relation between dynamics and thermodynamics would be obtained from the reinvestigation of a textbook issue, namely, the response of a finite system interacting with a single infinite reservoir.

Let $\mathcal{F}_S$ be a finite-dimensional algebra of system dynamical variables and let $\mathcal{F}_R$ be that of reservoir dynamical variables, where a time evolution group $\tau^R_t : \mathcal{F}_R \to \mathcal{F}_R$ and a gauge transformation group $\alpha^R_s : \mathcal{F}_R \to \mathcal{F}_R$ are defined. These groups are naturally extended to the algebra $\mathcal{F} = \mathcal{F}_S \otimes \mathcal{F}_R$ as $\delta S \otimes \tau^R_t$ and $\alpha^S_s \otimes \alpha^R_t$ with $\delta S$ the identity map on $\mathcal{F}_S$, and their generators are denoted, respectively, as $\delta_R$ and $g_R$. The system and the reservoir are coupled by a gauge invariant interaction $\lambda V$: $V = V^* \in D(\delta_R) \cap D(g_R) \subset \mathcal{F}$, $\lambda \in \mathbb{R}$, $g_R(V) + i[N \otimes 1_R, V] = 0$ where $N$ is the number of particles in the system and $1_R$ is the unit of $\mathcal{F}_R$. We are interested in the response of the whole system under a time-dependent perturbation: $W(t) \otimes 1_R$ where $W(t) \in \mathcal{F}_S$ is twice continuously differentiable in norm, $[N, W(t)] = 0$ and $W(t) = W_0$ for $t \leq 0$.

Initially, the whole system is prepared to be an equilibrium state $\omega$ of the inverse temperature $\beta$ and the chemical potential $\mu$, which is a KMS state at $\beta$ with respect to $\delta_s$ defined by

$$\frac{d\delta_s(A)}{ds} = \delta_s(\delta_R(A) - \mu g_R(A) + i[W_0 \otimes 1_R + \lambda V - \mu N \otimes 1_R, A]) , \quad \delta_s(A)|_{s=0} = A ,$$

where $A \in D(\delta_R) \cap D(g_R)$. The time evolution $\tau^W_t$ is given by the solution of

$$\frac{d\tau^W_t(A)}{dt} = \tau^W_t(\delta_R(A) + i[W(t) \otimes 1_R + \lambda V, A]) , \quad \tau^W_t(A)|_{t=0} = A , \quad (A \in D(\delta_R)) ,$$

and the state at time $t$ by $\omega_t \equiv \omega \circ \tau^W_t$. Now, we define the system-energy increase $Z_T$ induced by the mass flow, the work $W_T$ done on the system and the heat $Q_T$ absorbed by the system during the time interval $T$ as follows:

$$
\begin{align*}
Z_T &= \mu \{\omega_T(N \otimes 1_R) - \omega(N \otimes 1_R)\} \\
W_T &= \int_0^T dt \omega_t \left( \frac{d}{dt} W(t) \otimes 1_R \right) \\
Q_T &= \{\omega_T(W(T) \otimes 1_R + \lambda V) - \omega(W_0 \otimes 1_R + \lambda V)\} - W_T - Z_T
\end{align*}
$$

(a) Stepwise perturbation: Suppose $W(t) = W_f (t \geq t_0)$, where $0 < t_0 \ll T$. Then, if the initial state $\omega$ is the unique $\hat{\sigma}$-KMS and the evolution $\hat{\tau}^{(1)}_t$ given by

$$\frac{d\hat{\tau}^{(1)}_t(A)}{dt} = \hat{\tau}^{(1)}_t(\delta_R(A) + i[W_f \otimes 1_R + \lambda V, A]) , \quad \hat{\tau}^{(1)}_t(A)|_{t=0} = A , \quad (A \in D(\delta_R)) ,$$


is $L^1(\mathcal{G})$-asymptotic abelian and $W_f - W_0 \in \mathcal{G}$, one has

$$\lim_{\lambda \to 0} \lim_{T \to +\infty} \beta Q_T = S(\rho_f) - S(\rho_i) - S(\rho_f|\rho_0)$$

where $S(\rho_\alpha) = -\text{Tr}\{\rho_\alpha \ln \rho_\alpha\}$ ($\alpha = i, f$) is the von Neumann entropy of the density matrix $\rho_\alpha$, $S(\rho_f|\rho_0) = \text{Tr}\{\rho_0 (\ln \rho_0 - \ln \rho_f)\} \geq 0$ is the relative entropy between density matrices $\rho_f$ and $\rho_0$, and $\rho_i = e^{-\beta(W_0-\mu N)}/\text{Tr} e^{-\beta(W_0-\mu N)}$, $\rho_f = e^{-\beta(W_f-\mu N)}/\text{Tr} e^{-\beta(W_f-\mu N)}$, $\rho_0 = u_t^* \rho_0 u_t$.

The unitary element $u_t \in \mathcal{F}_S$ is the solution of $\frac{d}{dt} u_t = \dot{u}_t W(t)$, $u_{|t=0} = 1_S$.

(b) Staircase perturbation: Suppose $T = \sum_{j=1}^{N} T_j$ and the interaction $W(t)$ has a staircase form: $W(t) = W_0 + (j - 1 + \varphi(t - \tilde{T}_j - 1))(W_f - W_0)/N$ for $\tilde{T}_{j-1} \leq t \leq \tilde{T}_j$ ($\tilde{T}_0 = \sum_{k=1}^{j} T_k$, $\tilde{T}_{j} \equiv 0$), where $\varphi(t)$ is a twice continuously differentiable real-valued function with $\varphi(0) = 0$, $\varphi(t) = 1$ ($\forall t \geq t_0$). Let us denote $W_j = W_0 + j(W_f - W_0)/N$ and define groups $\tilde{\sigma}_t^{(j)}$ and $\tilde{\tau}_t^{(j)}$ by

$$\frac{d\tilde{\sigma}_t^{(j)}(A)}{ds} = \tilde{\sigma}_t^{(j)}(\delta_R(A) - \mu g R(A) + i[W_f \otimes 1_R + \lambda V - \mu N \otimes 1_R, A]), \quad \tilde{\sigma}_t^{(j)}(A)_{|s=0} = A$$

$$\frac{d\tilde{\tau}_t^{(j)}(B)}{dt} = \tilde{\tau}_t^{(j)}(\delta_R(B) + i[W_f \otimes 1_R + \lambda V, B])$$

where $A \in D(\delta_R) \cap D(g_R)$, $B \in D(\delta_R)$. Then, if every $\tilde{\sigma}_t^{(j)}$-KMS state $\omega_j$ at inverse temperature $\beta$ is unique, every evolution $\tilde{\tau}_t^{(j)}$ is $L^1(\mathcal{G})$-asymptotically abelian and $W_f - W_0 \in \mathcal{G}$, one has

$$\lim_{V \to 0} \lim_{\Gamma_{\lambda \to 0}} \lim_{T \to +\infty} \beta Q_T = S(\rho_f) - S(\rho_i) + O\left(\frac{1}{N}\right)$$

**NB:** Because of the return-to-equilibrium property, the final state (and every intermediate state in case (b)) of the whole system is an equilibrium state. The limit of $\lambda \to 0$ implies that the coupling between the system and the reservoir is negligibly small. Thus, the processes treated in (a) and (b) precisely correspond to those in the classical thermodynamics [30]. The Clausius inequality follows from (a): $\lim_{\lambda \to 0} \lim_{T \to +\infty} \beta Q_T \leq S(\rho_f) - S(\rho_i)$ because of the positivity of the relative entropy. On the other hand, (b) implies that, if the whole system changes very slowly ($N \gg 1$) so that the whole system is in equilibrium at every instant, the Clausius equality holds. Thus, the thermodynamic entropy is given by the von Neumann entropy, as expected, and the process described in (b) is nothing but a quasistatic process. Note that, as $\lambda V$ is responsible for the equilibration, the weak coupling limit is given by the von Neumann entropy, after the long term limits.

**Proof of (a):** Let $\Gamma_t$ and $\Gamma_t^{(1)}$ be solutions of $\frac{d}{dt} \Gamma_t = i\Gamma_t \mathcal{L}_S \otimes \tau_t^R(W(t) \otimes 1_R + \lambda V)$, $\Gamma_t^{(1)}_{|t=0} = 1$ and $\frac{d}{dt} \Gamma_t^{(1)} = i\Gamma_t^{(1)} \mathcal{L}_S \otimes \tau_t^R(W_f \otimes 1_R + \lambda V)$, $\Gamma_t^{(1)}_{|t=0} = 1$, respectively, where $1 \equiv 1_S \otimes 1_R$. Then,

$$Q_T = -i\omega((\delta_R - \mu g R)(\Gamma_{t_0}))\Gamma_t^{*}$$

$$= -i\omega((\delta_R - \mu g R)(\Gamma_{t_0}))\Gamma_t^{*} + \omega\left(\Gamma_{t_0}(\Gamma_{t_0}^{(1)*}\tilde{\tau}_t^{(1)}((W_f - \mu N) \otimes 1_R + \lambda V)\Gamma_{t_0}^{(1)*})\Gamma_{t_0}^{*}\right)$$

$$\Rightarrow \lim_{T \to +\infty} \omega\left(\Gamma_{t_0}((W_f - \mu N) \otimes 1_R + \lambda V)\Gamma_{t_0}^{*}\right) = \lim_{T \to +\infty} \omega\left(\tilde{\tau}_t^{(1)}((W_f - \mu N) \otimes 1_R + \lambda V)\right)$$

Because the return-to-equilibrium property follows from the $L^1(\mathcal{G})$-asymptotic abelianness of $\tilde{\tau}_t^{(1)}$ and $W_f - W_0 \in \mathcal{G}$, the right-hand side agrees with $\omega\left(\tilde{\tau}_t^{(1)}((W_f - \mu N) \otimes 1_R + \lambda V)\right)$, where
respectively, where τ
\[\tau\]
\|ω\| ≤ \|u(t_0(W_f - μN)u^*_{t_0} \otimes 1_R)\| \leq K_1\|1_R - u(t_0 \otimes t_R)\| \leq |λ|t_0K_1\|V\| ,
\|ω((\delta_R - μgR)(Γ_{t_0}))\| ≤ \|\|\delta_R - μgR\|Γ_{t_0}\|\| \leq |λ|t_0 \sup_{0 \leq s \leq t_0} \|\|\delta_R - μgR\|Γ_{s,t_0} \otimes τ^R_{t_0}(V)\|\| ,
with K_1 = 2|W_f - μN|\. Thus, because of \max\{|ω(Γ_{t_0}S_{t_0} \otimes τ^R_{t_0}(λV)Γ_{t_0}^*)|, |ω(λV)|\} ≤ |λ||V|,
\lim_{T \to -∞} Q_T = 1_{ω}(W_f - μN) \otimes 1_R - ω(u(t_0(W_f - μN)u_{t_0}^* \otimes 1_R) + O(λ)
Remind that ω and ω_1 are λV-perturbed KMS states of the product states ρ_i \otimes ω_R and ρ_j \otimes ω_R, respectively, where ω_R is the reservoir grand canonical state (a KMS state at β with respect to τ^R_{t_0(α)}\. Then, as shown in [29], the stability of KMS states (Theorem 5.4.4 of [3]) leads to
\lim_{λ \to -∞} \lim_{T \to -∞} βQ_T = \beta Tr(ρ_f(W_f - μN)) - β Trρ_{t_0}(u(t_0(W_f - μN)u_{t_0}^* = S(ρ_f) − S(ρ_i) - S(ρ_j)\|ρ_{t_0}0\|
Outine of the proof of (b): The formula (8) is still valid. And the equation for Γ_{t} gives
\frac{d}{dt}\tilde{\tau}^{(j)}_t = i\frac{\tilde{\tau}^{(j)}_t}{(W_{j-1} + ϕ(t)ΔW) \otimes 1_R + λV}\Delta W = (W_f - W_0)/N with \tilde{\tau}_t = 1. Because of the L^1-asymptotic abelianess of the evolutions \tilde{\tau}_t, one finds that, when T_j \to +∞, \tilde{\tau}_T(A) \to \omega_j(A), ω_s(U\tilde{\tau}_j^{(j)}(A)U^*) \to \omega_j(A)\. Thus,
\lim_{λ \to -∞} \lim_{T_1,...,T_N \to -∞} Q_T = -i \lim_{λ \to -∞} \lim_{T_1,...,T_N \to -∞} \sum_{j=1}^{N} ω((\delta_R - μgR)(Γ_{T_j})^\tau_{j} - (\delta_R - μgR)(Γ_{T_{j-1}})\partial T_{j-1})
= \lim_{λ \to -∞} \sum_{j=1}^{N} \{ω_j((W_j - μN) \otimes 1_R + λV) - ω_{j-1}(Γ_{T_j}((W_j - μN) \otimes 1_R + λT_{t_0}(V))\tilde{\tau}^{(j)*})\}
= S(ρ_f) - S(ρ_i) - \sum_{j=1}^{N} S(ρ_j)\tilde{ρ}_{j-1}
where ρ_j = e^{-β(W_j - μN)/Tr}e^{-β(W_j - μN)}, \tilde{ρ}_{j-1} = u^{(j)*}_{t_0} ρ_{j-1} u^{(j)}_{t_0} and u^{(j)}_t is the solution of u^{(j)}_0 = 1_S, \frac{d}{dt}u^{(j)}_t = i u^{(j)}_t (W_{j-1} + ϕ(t)ΔW)\. As S(ρ_j)\tilde{ρ}_{j-1} = O(1/N^2)\, one obtains the desired result.

7. Conclusion
After having reviewed some C* dynamical approaches to the second law of thermodynamics, we have investigated ‘thermodynamic’ processes where the initial and final states of the whole system are equilibrium states and the system-reservoir coupling is negligibly small. The Clausius inequality is shown to hold for a finite system where the entropy production is given by the relative entropy between the initial and process-dependent intermediate states. For a very slowly varying process where the whole system is in equilibrium at every instant (a dynamical counter part of a quasistatic process), the Clausius equality holds. Namely the standard thermodynamic arguments can be fully reproduced in a C* dynamical system. This observation seems to confirm that the entropy production is non-negative only for a certain class of processes and to indicate the absence of clearcut distinction between processes with non-negative entropy production and those without it. For a full description of statistical aspects of mesoscopic systems, investigation of fluctuations such as done by Sewell [31] is inevitable and will be discussed elsewhere.
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