Transmuted of Rayleigh Distribution with Estimation and Application on Noise Signal

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Abstract. This paper deals with transforming one parameter Rayleigh distribution, into transmuted probability distribution through introducing a new parameter (ʎ), since this studied distribution is necessary in representing signal data distribution and failure data model the value of this transmuted parameter |ʎ| ≤ 1, is also estimated as well as the original parameter (ϴ) by methods of moments and maximum likelihood using different sample size (n=25, 50, 75, 100) and comparing the results of estimation by statistical measure (mean square error, MSE).

Keywords: Rayleigh Distribution, Exponentiated Transformation, Transmuted Rayleigh, Moment’s Method, Maximum Likelihood Method, Application of audio signal.

1. Introduction
Many researcher’s work on modifying the given probability distribution, through using either exponentiated to another parameter or using modification to by applying some transformation (like marshall-olkin) which is used in reference Vasile Preda(2011) when the failure rate model of any distribution is not constant, as well as using transmutation to obtain a new family which we apply it in our research to expand Rayleigh one parameter into two parameters one through transmuted. The one parameter Rayleigh distribution may have high precision in data fitting, but sometimes we want to obtain another family to have better representation for data without any effect of original probability models, this done through exponentiated transformation, or through exponentiated to another parameters, may give best fitting and more flexibility of data, so here we extend one parameter Rayleigh through transmuted of this distribution applying quadratic rank transformation, were this used by Show and Buckely (2009), and Aryal and Tosokos (2009 & 2011)[5], and Aryal (2013), so here we continue the work of extension by applying it to a one parameter Rayleigh to obtain a new p.d.f of two transmuted Rayleigh as well as deriving the new p.d.f and the new CDF, also the formula of rth
moments about origin were derived, then we used moments method for estimation, as well as maximum likelihood method, the application of the studied model is done on data which represent audio signal with noise at \((n=100)\).

2. Theoretical Aspect

The p.d.f of one scale parameter Rayleigh distribution is:

\[ f(x; \theta) = \frac{2x}{\theta} \left( e^{-\frac{x^2}{\theta}} \right) \quad x, \theta > 0 \]  

(1)

Where \(x\) is random variable, \(\theta\) is scale parameter.

The cumulative distribution function is:

\[ F_X(x) = \left( 1 - e^{-\frac{x^2}{\theta}} \right) \]  

(2)

Also, the mean and variance are:

\[ E(x, \theta) = \sqrt{\frac{\pi \theta}{4}} \]  

\[ \nu(x) = \theta \left( 1 - \frac{\pi}{4} \right) \]

The random variable \((x)\) with Rayleigh (one parameter \(\theta\)), can be transmuted using quadratic rank transmutation, this help in finding more flexible family distribution and is more benefit in representing signal system, also radical distance. The transmutation is done using:

\[ G(x) = (1 + \lambda)F(x) - \lambda[F(x)]^2 |\lambda| \leq 1 \]  

(3)

Transmuted Rayleigh from by applying equation (3); is obtained and explained in equation (4):

\[ g(x) = (1 + \lambda)f(x) - 2\lambda F(x)f(x) |\lambda| \leq 1 \]

\[ g(x) = (1 + \lambda) \frac{2x}{\theta} \left( e^{-\frac{x^2}{\theta}} \right) - 2\lambda \left( 1 - e^{-\frac{x^2}{\theta}} \right) \left( \frac{2x}{\theta} \right) \left( e^{-\frac{x^2}{\theta}} \right) \]

\[ g(x) = \frac{2x}{\theta} \left( e^{-\frac{x^2}{\theta}} \right) \left[ (1 + \lambda) - 2\lambda \left( 1 - e^{-\frac{x^2}{\theta}} \right) \right] \]

\[ g(x) = \frac{2x}{\theta} \left( e^{-\frac{x^2}{\theta}} \right) \left[ (1 - \lambda) + e^{-\frac{x^2}{\theta}} \right] \]  

(4)

This is the new generated p.d.f of transmuted Rayleigh is:

\[ g(x) = (1 + \lambda)f(x) - 2\lambda F(x)f(x) \]

\[ = f(x)[(1 + \lambda) - 2\lambda F(x)] \]  

(5)
\[ g(x) = \frac{2x}{\theta} \left( e^{\frac{x^2}{\theta}} \right) \left[ (1 + \lambda) - 2\lambda \left( 1 - e^{\frac{x^2}{\theta}} \right) \right] \]

\[ g(x) = \frac{2x}{\theta} \left( e^{\frac{x^2}{\theta}} \right) \left[ (1 - \lambda) + 2\lambda \left( e^{\frac{x^2}{\theta}} \right) \right] \] (6)

The p.d.f is new transmuted Rayleigh with two parameters (\( \theta \) is scale parameter, and \( \lambda \) is transmution parameter).

Now we derive the formula for \( r^{th} \) moments about origin by applying equation (7);

\[ \mu_r' = E(x^r) = \int_0^{\infty} x^r g(x)dx \] (7)

\[ = (1 - \lambda) \int_0^{\infty} x^r \left( e^{\frac{x^2}{\theta}} \right) dx + (2\lambda) \int_0^{\infty} x^r \left( e^{\frac{x^2}{\theta}} \right) \left( \frac{2x}{\theta} \right) \left( e^{\frac{x^2}{\theta}} \right) dx \]

\[ = (1 - \lambda) \int_0^{\infty} x^{r+1} \left( e^{\frac{x^2}{\theta}} \right) dx + \frac{(4\lambda)}{\theta} \int_0^{\infty} x^{r+1} \left( e^{-\frac{x^2}{\theta}} \right) dx \] (8)

\[ \mu_r' = I_1 + I_2 \]

\[ I_1 = (1 - \lambda) \int_0^{\infty} x^{r+1} \left( e^{\frac{x^2}{\theta}} \right) dx \]

Let

\[ Z = \frac{x^2}{\theta} \rightarrow \theta Z = x^2 \rightarrow x = \sqrt{\theta} Z \frac{1}{2} \rightarrow dx = \sqrt{\theta} \frac{1}{2\sqrt{Z}} dZ \]

\[ I_1 = (1 - \lambda) \frac{2}{\theta} \int_0^{\infty} \left( \sqrt{\theta} Z \frac{1}{2} \right)^{r+1} \left( e^{-Z} \right) \frac{1}{2\sqrt{Z}} dZ \]

\[ I_1 = \frac{(1-\lambda)}{\theta} \int_0^{\infty} Z^{\frac{r+2}{2}} (e^{-Z}) dZ = (1 - \lambda) \left( \theta Z \frac{1}{2} \right)^{r+1} \Gamma \left( \frac{r}{2} + 1 \right) \] (9)

\[ I_2 = \frac{(4\lambda)}{\theta} \int_0^{\infty} x^{r+1} \left( e^{-\frac{x^2}{\theta}} \right) dx \]

Let

\[ Z = 2 \frac{x^2}{\theta} \rightarrow \theta Z = x^2 \rightarrow x = \sqrt{\theta} Z \frac{1}{2} \rightarrow dx = \sqrt{\theta} \frac{1}{2\sqrt{Z}} dZ \]

\[ I_2 = \frac{(4\lambda)}{\theta} \int_0^{\infty} \left( \sqrt{\theta} Z \frac{1}{2} \right)^{r+1} \left( e^{-Z} \right) \frac{1}{2\sqrt{Z}} dZ \]

\[ I_2 = \frac{(4\lambda)}{\theta} \left( \sqrt{\theta} \frac{1}{2} \right)^{r+1} \int_0^{\infty} Z^r (e^{-Z}) dZ \]

\[ I_2 = \lambda \left( \frac{\theta}{Z} \right)^{r+1} \Gamma \left( \frac{r}{2} + 1 \right) \] (10)
Then the formula of \( r \)-th moment about origin is;

\[
\mu'_r = (1 - \lambda) \theta^r \Gamma \left( \frac{r}{2} + 1 \right) + \lambda \left( \frac{\theta}{2} \right)^r \Gamma \left( \frac{r}{2} + 1 \right)
\]

Applying \( \mu'_r = \frac{\sum_{i=1}^{n} x_i^2}{n} \) for \( r = 1, 2 \), we obtain moment estimator of \((\theta \text{ and } \lambda)\)

3. Maximum Likelihood Estimator

Let \((x_1,x_2,...,x_n)\) be a random variable from \([g(x)]\) then;

\[
L = \prod_{i=1}^{n} g(x_i) = \frac{2^n}{\theta^n} \prod_{i=1}^{n} x_i e^{-\frac{\sum_{i=1}^{n} x_i^2}{\theta}} \prod_{i=1}^{n} \left( 1 - \lambda + 2\lambda e^{-\frac{x_i^2}{\theta}} \right)
\]

\[
\log L = n \log 2 - n \log \theta + \sum_{i=1}^{n} \log x_i - \frac{\sum_{i=1}^{n} x_i^2}{\theta} + \sum_{i=1}^{n} \log \left( 1 - \lambda + 2\lambda e^{-\frac{x_i^2}{\theta}} \right)
\]

\[
\frac{\partial \log L}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i^2}{\theta^2} + \sum_{i=1}^{n} \frac{2\lambda e^{-\frac{x_i^2}{\theta}} \left( -\frac{x_i^2}{\theta^2} \right)}{\left( 1 - \lambda + 2\lambda e^{-\frac{x_i^2}{\theta}} \right)}
\]

\[
\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} x_i^2 - 2\lambda \sum_{i=1}^{n} x_i e^{-\frac{x_i^2}{\theta}} \left[ \frac{x_i^2}{\theta^2} \left( 1 - \lambda + 2\lambda e^{-\frac{x_i^2}{\theta}} \right) \right]}{n} \quad (11)
\]

Since \((|\lambda| \leq 1)\), we can restricted \((\lambda)\) by this region, we do not estimate it.

4. Simulation

From the CDF of transmuted Rayleigh, we generate random samples of various size \((n = 25,50,75,100)\), using different sets of two parameters \((\lambda, \theta)\), where \((|\lambda| \leq 1)\).

**Table (1):** Prior values of \((\lambda \& \theta)\) given for simulation

| \(\theta\) (scale parameter) | \(\lambda\) (transmuted parameter) |
|-----------------------------|----------------------------------|
| 1.5                         | 0.9                              |
| 3                           | 0.7                              |
| 4.5                         | 0.4                              |

**Table (2):** MLE and MOM Estimates for \((\lambda \& \theta)\) when priors values \(\theta = 1.5, \lambda = 0.9\)

| Estimator | MOM | MLE | Best |
|-----------|-----|-----|------|
|           |     |     |      |
| Sample Size (n) | Estimator (MSE) | \( \hat{\theta} \)   | \( \hat{\lambda} \)   | \( \hat{\theta} \)   | \( \hat{\lambda} \)   | \( \lambda \text{ MOM}, \hat{\theta}_{\text{MLE}} \) | \( \lambda \text{ MLE}, \hat{\theta}_{\text{MOM}} \) |
|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|--------------------------------|--------------------------------|
| 25             | Estimator (MSE) | 1.2203          | 0.9785          | 0.9979          | 0.8997          | (0.0224)                     | (0.00364)                     |
|                |                 |                 |                 |                 |                 | (0.01304)                   | (0.01236)                     |
| 50             | Estimator (MSE) | 1.0114          | 0.96631         | 1.0321          | 0.9946          | (0.01302)                   | (0.0061)                      |
|                |                 |                 |                 |                 |                 | (0.0041)                   | (0.003)                       |
| 75             | Estimator (MSE) | 1.0304          | 0.90662         | 1.0314          | 0.8863          | (0.00738)                   | (0.00182)                     |
|                |                 |                 |                 |                 |                 | (0.00715)                   | (0.00124)                     |
| 100            | Estimator (MSE) | 1.2205          | 0.9936          | 1.0031          | 0.7793          | (0.0094)                    | (0.00126)                     |
|                |                 |                 |                 |                 |                 | (0.00521)                   | (0.00124)                     |

**Table 3:** MLE and MOM Estimates for \( (\lambda & \theta) \) when priors values \( (\theta = 3, \lambda = 0.7) \)

| Sample Size | Estimator (MSE) | MOM \( \hat{\theta} \) | MOM \( \hat{\lambda} \) | MLE \( \hat{\theta} \) | MLE \( \hat{\lambda} \) | Best \( \lambda \text{ MLE}, \hat{\theta}_{\text{MOM}} \) | Best \( \lambda \text{ MOM}, \hat{\theta}_{\text{MLE}} \) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|--------------------------------|--------------------------------|
| 25          | Estimator (MSE) | 3.0645          | 0.9907          | 3.2461          | 0.9981          | (0.0952)                     | (0.0891)                      |
|             |                 |                 |                 |                 |                 | (0.0987)                   | (0.0617)                      |
| 50          | Estimator (MSE) | 3.9475          | 1.0062          | 3.8890          | 1.0041          | (0.3882)                    | (0.0991)                      |
|             |                 |                 |                 |                 |                 | (0.2061)                   | (0.1691)                      |
| 75          | Estimator (MSE) | 3.0062          | 0.8751          | 3.0094          | 0.8082          | (0.0632)                    | (0.0762)                      |
|             |                 |                 |                 |                 |                 | (0.0605)                   | (0.0415)                      |
| 100         | Estimator (MSE) | 3.0061          | 0.7726          | 3.0021          | 0.80061         | (0.0457)                    | (0.0329)                      |
|             |                 |                 |                 |                 |                 | (0.0241)                   | (0.0061)                      |
Table 4: MLE and MOM Estimates for ($\lambda$ & $\theta$) when priors values ($\theta = 4.5, \lambda = 0.4$)

| Sample Size | Estimator (MSE) | MOM | MLE | Best |
|-------------|-----------------|-----|-----|------|
|             | $\hat{\lambda}$ | $\hat{\theta}$ | $\hat{\lambda}$ | $\hat{\theta}$ | $\hat{\lambda}$ MOM, $\hat{\theta}$ MLE |
| 25          | 5.4706          | 0.9901 | 5.3201 | 0.9802 |
|             | (0.4921)        | (0.0781) | (0.4761) | (0.0851) |
| 50          | 4.9906          | 0.8006 | 4.2031 | 0.7064 |
|             | (0.0321)        | (0.12542) | (0.1467) | (0.0332) |
| 75          | 4.6972          | 0.5006 | 4.0031 | 0.4806 |
|             | (0.08211)       | (0.03115) | (0.0445) | (0.0025) |
| 100         | 4.6652          | 0.5003 | 3.9980 | 0.4772 |
|             | (0.07013)       | (0.0224) | (0.0332) | (0.0013) |

Application

We applied Rayleigh probability distribution on audio signal which have some noise, and noise refers to unwanted audio, so Rayleigh probability distribution work on choosing the best audio signal through replacing the best estimates of parameters through of probability function, and also we apply the data from distribution and the original data and explain these opinion in the following figures.

All figures and results are applied in MATLAB environment. Figure 1 shows the original audio signal and original audio signal with noise.

![Figure 1 original and original with noise signal](image)

Now we applied some Rayleigh probability distribution to choose the best set of parameter.
original & X' & S^2 & S & C.V  \\
--- & --- & --- & --- & ---  \\
1.0085 & 0.0064 & 0.0798 & 7.9763  \\

| noise | X'  | S^2 | S     | C.V  |
|-------|------|-----|-------|------|
| \(\lambda = 0.9\) \(\theta = 1.5\) | 0.7134 | 0.0039 | 0.0622 | 6.2188 |
| \(\lambda = 0.7\) \(\theta = 3\) | 0.4292 | 0.0026 | 0.0512 | 5.1157 |
| \(\lambda = 0.4\) \(\theta = 4.5\) | 0.2670 | 0.0013 | 0.0364 | 3.6423 |

**Conclusion**

The best estimator for \((\theta)\) for small size \((n = 25)\) and moderate sample size \((n = 50)\) and large sample \((n = 75)\), are \((\hat{\theta}_{MLE}, \hat{\lambda}_{MLE})\). This is related to the important properties of estimation by maximum likelihood method.

\(\hat{\lambda}_{MOM}\) is best by MOM with percentage \((3/24 * 100)\) while \(\hat{\lambda}_{MLE}\) is best with percentage \((9/24 * 100)\) and \(\hat{\theta}_{MOM}\) is best with percentage \((3/24)\) while \(\hat{\theta}_{MLE}\) is best with percentage \((9/24 * 100\%)\).

In the application of noise audio signal we applied three value of the two parameter \((\lambda & \theta)\) and we see that \((\lambda = 0.4 & \theta = 4.5)\) are the best estimator when \((n = 100)\) since it give the smallest coefficient of variation which is the measure of dispersion.

**Summary**

For original data, and at computed values of \((\lambda = 0.9, \theta = 1.5), (\lambda = 0.7, \theta = 3), (\lambda = 0.4, \theta = 4.5)\) we find that the best estimator that gives smallest C.V (coefficient of variation).

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