Abstract

Motivated by call center practice, we propose a tractable model for GI/GI/n + GI queues in the efficiency-driven (ED) regime. We use a one-dimensional diffusion process to approximate the virtual waiting time process that is scaled in both space and time, with the number of servers and the mean patience time as the respective scaling factors. Using this diffusion model, we obtain the steady-state distributions of virtual waiting time and queue length, which in turn yield simple formulas for performance measures such as the service level and the effective abandonment fraction. These formulas are generally accurate when the mean patience time is several times longer than the mean service time and the patience time distribution does not change rapidly around the mean virtual waiting time. For practical purposes, these formulas outperform existing results that rely on the exponential service time assumption.

To justify the diffusion model, we formulate an asymptotic framework by considering a sequence of queues, in which both the number of servers and the mean patience time go to infinity. We prove that the space-time scaled virtual waiting time process converges in distribution to the one-dimensional diffusion process. A fundamental result for proving the diffusion limit is a functional central limit theorem (FCLT) for the superposition of renewal processes. We prove that the superposition of many independent, identically distributed stationary renewal processes, after being centered and scaled in space and time, converges in distribution to a Brownian motion. As a useful technical tool, this theorem characterizes the service completion process in heavy traffic, allowing us to greatly simplify the many-server analysis when service times follow a general distribution.

1 Introduction

Queues with many parallel servers are building blocks for modeling call center operations; see [Gans et al. (2003)] and [Aksin et al. (2007)] for comprehensive reviews. A large call center faces a great amount of traffic that is stochastic and time-varying. Since the rate of incoming calls changes over time, the system may become overloaded during peak hours. Waiting on a phone line, a customer may hang up before being connected to an agent. This phenomenon, referred to as customer abandonment, is present in almost all call centers and becomes prominent when the system is overloaded. A call center may also be intentionally operated in an overloaded regime. Nowadays, more and more firms outsource their call centers to save costs. In service-oriented call centers,
staffing costs usually dominate the expenses of customer delay and abandonment. As pointed out by Whitt (2006), the rational operational regime for these call centers is the efficiency-driven (ED) regime that emphasizes capacity utilization over the quality of service. In the ED regime, the service capacity is set below the customer arrival rate by a moderate fraction. Because the lost demands of abandoning customers compensate for the excess of customer arrivals over the service capacity, a call center operated in the ED regime can still achieve reasonable service quality. More specifically, the mean customer waiting time is comparable to the mean service time, a moderate fraction of customers abandon the system, and all agents are almost always busy.

Service requirements for an outsourced call center are specified in the service-level agreement (SLA) between the firm and the call center provider. The SLA includes performance objectives such as the average service time, the acceptable abandonment rate, and the acceptable customer delay time. One important service level objective is a specified percentage of customers to be served within a given delay, e.g., “80% of calls should be answered within one minute.” Based on this service level, Baron and Milner (2009) studied SLA design using the M/M/n + M model, and Mandelbaum and Zeltyn (2009) studied call center staffing using the M/M/n + GI model. Despite the wide use of the above models, it was pointed out by Brown et al. (2005) that the exponential service time distribution is not a realistic assumption for call center customers. Therefore, these models may not be able to provide adequate estimates for the performance measures required by the SLA. Call center managers may need a more accurate but still tractable model for performance analysis and staff deployment.

For queues in the ED regime, a fluid model proposed by Whitt (2006) is useful for estimating several performance measures, including the fraction of abandoning customers, the mean queue length, and the mean virtual waiting time. In the M/M/n + GI setting, the accuracy of the fluid model was studied by Bassamboo and Randhawa (2010). They proved that in the steady state, the accuracy gaps of fluid approximations for the mean queue length and the rate of customer abandonment do not increase with the arrival rate. As a deterministic model, however, the fluid model cannot be used to predict the percentage of customers to be served within a time limit. A refined model is thus necessary for estimating such a measure.

The focus of this paper is a diffusion model for many-server queues in the ED regime. Both the service and patience time distributions are assumed to be general. Using this diffusion model, we obtain the steady-state distributions of virtual waiting time and queue length, which in turn yield approximate formulas for performance measures such as the service level mentioned above. These formulas are able to produce accurate estimates, especially when customer patience times are relatively long compared with their service times. Empirical studies suggest that this requirement is realistic for service-oriented call centers. For example, it was reported by Mandelbaum et al. (2001) and Mandelbaum and Zeltyn (2013) that in the call center of an Israeli bank, the mean customer patience time was several times longer than the mean service time. By numerical experiments, we demonstrate that for practical purposes, this diffusion model outperforms existing models that rely on the exponential service time assumption; see Section 4.2.

The general service time assumption is a major challenge in the analysis of many-server queues. In the literature, the studies of many-server queues with a general service time distribution usually involve the analysis of one or several infinite-dimensional processes, which are used for tracking customer age or residual times. The resulting approximate models are also infinite-dimensional,
typically in the form of two-parameter or measure-valued processes; see, e.g., [Whitt (2006), Kang and Ramanan (2010), Kaspi and Ramanan (2011), Zhang (2013), and Kaspi and Ramanan (2013)]. Because these approximate models are either deterministic or too complex to be used for estimating a distribution, explicit formulas have been absent from the literature for the steady-state virtual waiting time and queue length distributions. Estimation of these distributions relies heavily on simulation; see, e.g., Blanchet and Lam (2014). The performance formulas provided in this paper can fill this gap for queues in the ED regime.

We use a one-dimensional Ornstein–Uhlenbeck (OU) process to approximate the virtual waiting time process. We obtain this diffusion model by scaling a many-server queue in both space and time, and then replacing the centered and scaled arrival, service completion, and abandonment processes with mutually independent Brownian motions. Depending on the service time distribution through the first two moments, this model allows us to obtain Gaussian approximations for the steady-state virtual waiting time and queue length distributions. In contrast, the approximate models in the literature are derived by scaling many-server queues in space only. As a result, the general service time distribution needs to be incorporated in the approximate model, leading to an infinite-dimensional Markovian representation. It is well known that when a many-server queue is critically loaded, the system performance depends on the entire service time distribution and differs from that of a queue with one or several servers significantly; see, e.g., [Dai et al. (2010), Mandelbaum and Momčilović (2012), and Kaspi and Ramanan (2013)]. From a macroscopic perspective in both space and time, we demonstrate that in the ED regime, the dynamics of a many-server queue could be as simple as that of a single-server queue, even though the service time distribution is assumed to be general. A one-dimensional diffusion process, depending on the service time distribution only by its first two moments, may suffice to capture the dynamics of the many-server system. (One may also refer to He (2013), an earlier version of this paper, where we proved a common diffusion limit for systems with an exponential patience time distribution, with either a single or many servers in the overloaded regime, using a different method than the one in this paper.) Such a simple model is much more attractive for analysis and control purposes.

The diffusion model and the performance formulas are rooted in the limit theorems presented in Section 3. For queues in the ED regime, [Dai et al. (2010)] proved a multi-dimensional diffusion limit for the GI/Ph/n + M model, and [Huang et al. (2014)] proved a one-dimensional diffusion limit for the GI/M/n + GI model. In addition, [Huang et al. (2014)] applied the obtained diffusion limit to delay announcement in call centers. In this paper, we consider a sequence of GI/GI/n + GI queues indexed by the number of servers n, and assume that the mean patience time goes to infinity as n goes large. In this asymptotic framework, the virtual waiting time process of each queue is scaled in both space and time, with the number of servers and the mean patience time being the respective scaling factors. The joint scaling scheme is essential to obtain a Brownian approximation for the service completion process, eventually leading to a one-dimensional diffusion limit. In the previous studies such as [Dai et al. (2010), Mandelbaum and Momčilović (2012), and Kaspi and Ramanan (2013)], the queueing processes are scaled by the number of servers only. In this case, only when the service time distribution is exponential, will the scaled queueing process converge to a one-dimensional diffusion process. To the best of our knowledge, Theorem 1 in this paper is the first rigorous result that identifies a one-dimensional diffusion limit for many-server queues with a general service time assumption.
The technique of joint scaling in space and time was adopted by Whitt (2003, 2004), Gurvich (2004), and Atar (2012) for many-server queues with an exponential service time distribution. Whitt (2004) considered the M/M/n/r + M model and proved that in the ED regime, the queue length process has a diffusion limit when the product of the number of servers and the mean patience time goes to infinity. A critically loaded regime, known as the nondegenerate slowdown regime, was studied by Whitt (2003), Gurvich (2004), and Atar (2012). In this regime, the diffusion limit for the queue length process is proved to be either a reflected OU process when the patience time distribution is exponential, or a reflected Brownian motion when there is no customer abandonment. Because of the exponential service time assumption, it is certain that those diffusion limits are one-dimensional. The ability of space-time scaling to simplify the analysis of many-server queues is barely manifested by those papers. In this sense, it is the general service time assumption that distinguishes our work from others. By means of space-time scaling, we provide a paradigm for building a tractable model for many-server queues with a general service time distribution.

Proving the diffusion limit requires new fundamental tools. Theorem 3 in this paper is a functional central limit theorem (FCLT) for the superposition of renewal processes. We prove that the superposition of $n$ independent, identically distributed (iid) stationary renewal processes, after being centered and scaled in space and time, converges in distribution to a Brownian motion as $n$ goes large. For a many-server system in heavy traffic, the space-time scaled service completion process is characterized by this theorem, which allows us to bypass the analysis of the queue’s infinite-dimensional state process. To apply this theorem, we consider a sequence of perturbed systems that are asymptotically equivalent to the original queues. We assume that servers in a perturbed system are always busy so that the service completion process is the superposition of $n$ renewal processes. The simplified dynamics of the perturbed system enable us to follow the procedure in Huang et al. (2014), proving the diffusion limit by a continuous mapping approach.

We would summarize the contributions of this paper as follows. First, the explicit formulas obtained from the diffusion model are practical tools for performance estimation and staff deployment in efficiency-driven service systems. In spite of their simple expressions, these formulas are superior to the widely used formulas relying on the exponential service time assumption. Second, we establish limit theorems to justify the diffusion model. By means of space-time scaling, we prove a one-dimensional diffusion limit for many-server queues with a general service time distribution. This joint scaling approach is useful for building tractable models for queues in the ED regime. Third, we prove an FCLT for the superposition of renewal processes. It is a fundamental result that characterizes the service completion process of a many-server queue in heavy traffic. This theorem enables us to bypass the infinite-dimensional analysis in proving the diffusion limit.

The remainder of the paper is organized as follows. The diffusion model and the performance formulas are introduced in Section 2. The limit theorems for the diffusion model are presented in Section 3. We examine and discuss the approximate formulas by numerical experiments in Section 4. Section 5 is dedicated to the proof of the diffusion limit for the virtual waiting time process, and Section 6 presents the proof for the queue length limit. The paper is concluded in Section 7. We leave the proof of the FCLT and the proofs of technical lemmas to the appendix.

Let us close this section with frequently used notation. The space of functions $f : \mathbb{R}_+ \to \mathbb{R}^k$ that are right-continuous on $[0, \infty)$ and have left limits on $(0, \infty)$ is denoted by $\mathbb{D}^k$ (with $\mathbb{D} = \mathbb{D}^1$), which is endowed with the Skorokhod $J_1$ topology (see, e.g., Billingsley (1999)). For $f \in \mathbb{D}$, we use $f(t-)$. 


to denote the left limit of $f$ at $t$ and $\Delta f(t)$ the increment at $t$, i.e., $\Delta f(t) = f(t) - f(t-)$. For $T > 0$, we use $\int_0^T |df(t)|$ to denote the total variation of $f$ over $[0, T]$. For $f' \in \mathbb{D}$ that is nondecreasing and takes values in $\mathbb{R}_+$, $f \circ f'$ is the composed function in $\mathbb{D}$, i.e., $(f \circ f')(t) = f(f'(t))$. We use $e$ for the identity function on $\mathbb{R}_+$ and $\chi$ the constant one function on $\mathbb{R}_+$, i.e., $e(t) = t$ and $\chi(t) = 1$ for $t \geq 0$.

## 2 Diffusion model and performance formulas

Consider a GI/GI/$n$ + GI queue, whose customer arrival process is a renewal process and service times are iid nonnegative random variables. Customers are served by $n$ identical servers. Upon arrival, a customer gets into service if an idle server is available; otherwise, he waits in a buffer with infinite room. Waiting customers are served on the first-come, first-served basis, and the servers are not allowed to idle if there are customers waiting. Each customer has a random patience time. When a customer’s waiting time exceeds his patience time, the customer abandons the system without being served. The patience times are iid nonnegative random variables, and the sequences of interarrival, service, and patience times are mutually independent.

Let $\lambda$ be the customer arrival rate and $\mu$ the service rate of each server. The traffic intensity satisfies $\rho = \lambda/(n\mu) > 1$. When a many-server queue becomes overloaded, all servers will be almost always busy, so the fraction of abandoning customers will be around

$$\alpha = \frac{\rho - 1}{\rho}. \quad (2.1)$$

Assume that both interarrival times and service times have finite variances, with squared coefficients of variation $c^2_a$ and $c^2_s$, respectively. Let $\Theta$ be the distribution function of patience times and $\gamma$ the mean patience time. Assume that $\Theta$ is absolutely continuous with a bounded, strictly positive density function $f_\Theta$.

Suppose that at time $t \geq 0$, a hypothetical customer with infinite patience arrives at the queue. Let $W(t)$ be the virtual waiting time at $t$, i.e., the amount of time this hypothetical customer has to wait before getting into service. As the queue comes into the steady state, the virtual waiting time process fluctuates around a mean level $w$ that can be determined as follows: Because $\Theta(w)$ is the fraction of customers whose patience times are less than $w$, it should be approximately equal to the fraction of abandoning customers. Then, $\Theta(w) = (\rho - 1)/\rho$, which yields

$$w = \Theta^{-1}\left(\frac{\rho - 1}{\rho}\right). \quad (2.2)$$

To represent the fluctuation of the virtual waiting time process around this equilibrium level, we introduce a centered and scaled version of $W$ by

$$\tilde{W}(t) = \sqrt{n}(W(\gamma t) - w).$$

We refer to $\tilde{W}$ as the diffusion-scaled virtual waiting time process. To obtain this process, we scale the virtual waiting time process in both space and time after removing the mean $w$. Besides the commonly used scaling in space by the number of servers, we also change the time scale of the
process by using the **mean patience time** as the scaling factor.

We approximate the diffusion-scaled virtual waiting time process by an OU process, which is given by the following stochastic differential equation

\[
\dot{W}(t) = \dot{W}(0) + \dot{M}(t) - \rho \gamma f_\Theta(w) \int_0^t \dot{W}(u) \, du \quad \text{for } t \geq 0.
\]

Here, \( \dot{M} \) is a driftless Brownian motion with \( \dot{M}(0) = 0 \) and variance

\[
\sigma_m^2 = \frac{c_a^2 + \rho c_s^2 + \rho - 1}{\rho \mu}.
\]

The OU process is a reasonable approximate model because the virtual waiting time process is mean-reverting: As the virtual waiting time fluctuates around \( w \), the instantaneous abandonment rate from the buffer fluctuates accordingly. If the probability density function of patience times does not change much around \( w \), the relative variation in the abandonment rate will be approximately proportional to the relative variation in the virtual waiting time. When the virtual waiting time is either too long or too short, the increased or decreased abandonment rate will pull it back toward the mean level \( w \).

For the diffusion approximation to be accurate, the mean patience time \( \gamma \), serving as the scaling factor in time, should be *relatively long* compared with the mean service time. This requirement can be justified by Theorem 1 where the ED regime is formulated into an asymptotic framework and \( \dot{W} \) is proved to be the limit of a sequence of diffusion-scaled virtual waiting time processes. Although the mean patience time goes to infinity in this asymptotic framework, the diffusion model may still produce accurate performance estimates when it is just several times longer than the mean service time. We will discuss the influence of the mean patience time in Section 4.1.

The OU process is strongly ergodic with a Gaussian steady-state distribution (see, e.g., Karlin and Taylor (1981)). More specifically, the steady-state distribution of \( \dot{W} \) has mean zero and variance

\[
\sigma_w^2 = \frac{c_a^2 + \rho c_s^2 + \rho - 1}{2\rho^2 \mu \gamma f_\Theta(w)}.
\]

Let \( W(\infty) \) be the virtual waiting time in the steady state and \( \dot{W}(\infty) \) be the diffusion-scaled version. Because \( \dot{W} \) is an approximation of \( \dot{W} \), their steady-state distributions are expected to be close, i.e.,

\[
\mathbb{P}[\dot{W}(\infty) > a] \approx 1 - \Phi\left( \frac{a}{\sigma_w} \right) \quad \text{for } a \in \mathbb{R},
\]

where \( \Phi \) is the standard Gaussian distribution function. Consequently, the steady-state virtual waiting time approximately follows a Gaussian distribution with mean \( w \) and variance

\[
\sigma_w^2 = \frac{c_a^2 + \rho c_s^2 + \rho - 1}{2\rho^2 \mu \gamma f_\Theta(w)}.
\]

The percentage of customers to be served within a specified delay is referred to as the *service level* in practice. Let \( \zeta \) be a random variable that has distribution \( \Theta \) and is independent of \( \dot{W}(\infty) \).
The service level within $d \geq 0$ can be approximated by
\[ P[W(\infty) \leq \zeta \wedge d] \approx \int_0^{\infty} \Phi_w(u \wedge d) f_\Theta(u) \, du \] (2.5)
where
\[ \Phi_w(u) = \Phi \left( \frac{\rho(u - w) \sqrt{2n\mu f_\Theta(w)}}{\sqrt{c_a^2 + \rho c_s^2 + \rho - 1}} \right). \]

In many service systems, the fraction of abandoning customers whose actual waiting times exceed a short delay is an important measure of the quality of service and customer satisfaction. We refer to this fraction as the effective abandonment fraction because it excludes those abandoning customers whose effort of waiting is insignificant. For queues in the ED regime, the fraction of abandoning customers out of those whose waiting times exceed $d \geq 0$ can be approximated by
\[ P[\zeta \wedge W(\infty) > d] \approx \int_d^{\infty} (1 - \Phi_w(u)) f_\Theta(u) \, du. \] (2.6)
Note that when $d \geq w$, we cannot use the fluid model by Whitt (2006) to estimate this fraction. This is because the steady-state virtual waiting time is equal to $w$ in the fluid model, by which the estimate of $P[\zeta \wedge W(\infty) > d]$ must be zero for $d \geq w$.

We are also interested in the distribution of the steady-state queue length. For $0 < u < w$, the probability that a customer who arrived $u$ time units ago is still waiting in the buffer is around $1 - \Theta(u)$. The mean queue length (i.e., the mean number of customers in the buffer) can thus be approximated by
\[ q = \int_0^w \lambda(1 - \Theta(u)) \, du. \] (2.7)

Let $X(t)$ be the number of customers in the system at time $t$, which fluctuates around $n + q$ as the queue comes into the steady state. A centered and scaled version of $X$ is defined by
\[ \tilde{X}(t) = \frac{1}{\sqrt{n\gamma}} (X(\gamma t) - n - q). \]

We refer to $\tilde{X}$ as the diffusion-scaled queue length process.

Let $X(\infty)$ be the number of customers in the steady state and $\tilde{X}(\infty)$ be the diffusion-scaled version. Theorem 3 in Section 3 implies that $\tilde{X}(\infty)$ approximately follows a Gaussian distribution with mean zero and variance
\[ \sigma_x^2 = \frac{\mu(c_a^2 + \rho c_s^2 + \rho - 1)}{2\rho^2 \gamma f_\Theta(w)} + \frac{\rho \mu}{\gamma} \int_0^w \Theta(u)(1 - \Theta(u)) \, du + \frac{\rho \mu c_a^2}{\gamma} \int_0^w (1 - \Theta(u))^2 \, du, \]

i.e.,
\[ P[\tilde{X}(\infty) > a] \approx 1 - \Phi \left( \frac{a}{\sigma_x} \right) \] for $a \in \mathbb{R}$. (2.8)

Hence, the steady-state number of customers approximately follows a Gaussian distribution with
mean $n + q$ and variance

$$
\sigma_x^2 = \frac{n\mu(c_a^2 + \rho c_a^2 + \rho - 1)}{2\rho^2 f_{\Theta}(w)} + n\rho\mu \int_0^w \Theta(u) (1 - \Theta(u)) \, du + n\rho\mu c_a^2 \int_0^w (1 - \Theta(u))^2 \, du. \tag{2.9}
$$

\section{Limit theorems}

We present the underlying theorems of the diffusion model in this section. To formulate the ED regime, let us consider a sequence of $\text{G}/\text{GI}/n + \text{GI}$ queues indexed by the number of servers. We do not require the arrival processes to be renewal, but simultaneous arrival of two or more customers is not allowed. In each queue, the number of initial customers, the arrival process, the sequence of service times, and the sequence of patience times are mutually independent. All these queues have the same traffic intensity $\rho > 1$ and the same service time distribution. Since the service rate $\mu$ is identical in all systems, the arrival rate of the $n$th system is $\lambda_n = n\rho\mu$. Assume that the mean patience time goes to infinity as $n$ goes large, i.e.,

$$
\gamma_n \to \infty \quad \text{as } n \to \infty. \tag{3.1}
$$

We do not require any assumption on the increasing rate of $\gamma_n$ towards infinity. Because the patience time distribution changes with $n$, it is necessary to define a normalized patience time distribution for all queues. Let $H$ be the distribution function of a nonnegative random variable with mean one, i.e., $\int_0^\infty u \, dH(u) = 1$. Assume that $H$ is absolutely continuous with a bounded, strictly positive density function $f_H$, i.e., there exists some $\kappa > 0$ such that

$$
0 < f_H(u) < \kappa \quad \text{for } u \geq 0. \tag{3.2}
$$

Using this normalized distribution, we define the distribution function of patience times in the $n$th system by

$$
\Theta_n(u) = H\left(\frac{u}{\gamma_n}\right) \quad \text{for } u \geq 0. \tag{3.3}
$$

Clearly, the mean patience time in the $n$th system is equal to $\gamma_n$. By (2.2), the mean virtual waiting time in the $n$th system is $w_n = \Theta_n^{-1}((\rho - 1)/\rho)$. We may thus define the normalized mean virtual waiting time by

$$
\bar{w} = H^{-1}\left(\frac{\rho - 1}{\rho}\right),
$$

which satisfies $\bar{w} = w_n/\gamma_n$ for all $n \in \mathbb{N}$.

Let $F$ be the distribution function of service times. We impose a mild regularity condition on $F$, which is

$$
\limsup_{u \downarrow 0} \frac{1}{u} (F(u) - F(0)) < \infty, \tag{3.4}
$$

and assume that the third moment of $F$ is finite, i.e.,

$$
\int_0^\infty u^3 \, dF(u) < \infty. \tag{3.5}
$$
Let $F_e$ be the equilibrium distribution of $F$, given by

$$F_e(t) = \mu \int_0^t (1 - F(u)) \, du \quad \text{for } t \geq 0.$$ 

We assign service times to customers according to the following procedure. Let \{\xi_{j,k} : j, k \in \mathbb{N}\} be a double sequence of independent nonnegative random variables. For each $j \in \mathbb{N}$, $\xi_{j,1}$ follows distribution $F_e$ and $\xi_{j,k}$ follows distribution $F$ for $k \geq 2$. In the $n$th system, assume that all $n$ servers are busy at time zero. For $j = 1, \ldots, n$, $\xi_{j,1}$ is assigned to the initial customer served by the $j$th server as the residual service time at time zero. For $k \geq 2$, $\xi_{j,k}$ is the service time of the $k$th customer served by the $j$th server. By this assignment, for all $j, k \in \mathbb{N}$, the $k$th service time by the $j$th server is identical in all systems that have at least $j$ servers.

Let $E_n(t)$ be the number of arrivals in the $n$th system during time interval $(0, t]$. Define the diffusion-scaled arrival process $\tilde{E}_n$ by

$$\tilde{E}_n(t) = \frac{1}{n\gamma_n} (E_n(\gamma_n t) - \lambda_n \gamma_n t).$$

Let $N$ be a renewal process whose interrenewal times have mean one and variance $c^2_a$. If $E_n$ is a renewal process with $E_n(t) = N(\lambda_n t)$, it follows from the FCLT for renewal processes that

$$\tilde{E}_n \Rightarrow \hat{E} \quad \text{as } n \to \infty,$$

where $\hat{E}$ is a driftless Brownian motion with $\hat{E}(0) = 0$ and variance $\rho \mu c^2_a$. To allow for more general arrival processes, we take the convergence in (3.6) as an assumption rather than require each $E_n$ to be a renewal process.

Let $W_n(t)$ be the virtual waiting time at $t$ in the $n$th system, whose diffusion-scaled version is

$$\tilde{W}_n(t) = \sqrt{\frac{n}{\gamma_n}} (W_n(\gamma_n t) - \gamma_n \bar{w}).$$

Assume that there exists a random variable $\hat{W}(0)$ such that

$$\tilde{W}_n(0) \Rightarrow \hat{W}(0) \quad \text{as } n \to \infty.$$ (3.7)

The first theorem states the diffusion limit for the virtual waiting time process in the ED regime.

**Theorem 1.** Assume that the sequence of $G/GI/n + GI$ queues described above has a common traffic intensity $\rho > 1$ and satisfies conditions (3.1)–(3.7). Then,

$$\tilde{W}_n \Rightarrow \hat{W} \quad \text{as } n \to \infty,$$

where $\hat{W}$ is the OU process given by the following stochastic differential equation

$$\hat{W}(t) = \hat{W}(0) + \hat{M}(t) - \rho f_H(\bar{w}) \int_0^t \hat{W}(u) \, du \quad \text{for } t \geq 0.$$ (3.8)

Here, $\hat{M}$ is a driftless Brownian motion with $\hat{M}(0) = 0$ and variance $(c^2_a + \rho c^2_s + \rho - 1)/\rho \mu).$
Remark. Let \( h \) be the hazard rate function of \( H \), i.e., 
\[ h(u) = \frac{f_H(u)}{1 - H(u)} \] 
for \( u \geq 0 \). Because 
\[ H(\bar{w}) = \frac{\rho - 1}{\rho}, \] 
we have \( h(\bar{w}) = \rho f_H(\bar{w}) \). The diffusion limit depends on the interarrival and service time distributions through their first two moments, and depends on the normalized patience time distribution through the hazard rate at \( \bar{w} \). This is because by centering and space-time scaling, the arrival and service completion processes are replaced by Brownian motions in the limit process. Since the virtual waiting time process fluctuates around the equilibrium level, the influence on the scaled abandonment process is mostly dictated by the normalized patience hazard rate at \( \bar{w} \).

The mean queue length in the \( n \)th system can be computed by
\[
q_n = \int_0^{w_n} \lambda_n(1 - \Theta_n(u)) \, du = n \gamma_n \rho \mu \int_0^{\bar{w}} (1 - H(u)) \, du. \tag{3.9}
\]

Let \( X_n(t) \) be the number of customers at time \( t \geq 0 \) in the \( n \)th system. Then, the diffusion-scaled queue length process is given by
\[
\tilde{X}_n(t) = \frac{1}{\sqrt{n \gamma_n}} (X_n(\gamma_n t) - n - q_n). \tag{3.10}
\]

Put
\[
\tilde{W}_n(t) = \frac{1}{\gamma_n} W_n(\gamma_n t), \tag{3.11}
\]
which is the virtual waiting time at \( t \) in the \( n \)th time-scaled system. For a given \( t \geq 0 \), the second theorem concerns the limit of the diffusion-scaled queue length at time \( t + \tilde{W}_n(t) \).

**Theorem 2.** Under the conditions of Theorem 1, for any fixed \( t \geq 0 \),
\[
\tilde{X}_n(t + \tilde{W}_n(t)) \Rightarrow \mu \tilde{W}(t) + \hat{G}(t) \quad \text{as} \quad n \to \infty,
\]
where \( \tilde{W} \) is the OU process defined by \( 3.8 \) and \( \hat{G}(t) \) is a Gaussian random variable with mean zero and variance
\[
\hat{\sigma}^2_g = \rho \mu \int_0^{\bar{w}} H(u)(1 - H(u)) \, du + \rho \mu c_a \int_0^{\bar{w}} (1 - H(u))^2 \, du.
\]

In addition, \( \tilde{W}(t) \) and \( \hat{G}(t) \) are mutually independent.

**Remark.** Let \( t \) go to infinity. Then, \( \mu \tilde{W}(t) + \hat{G}(t) \) converges in distribution to a Gaussian random variable with mean zero and variance
\[
\hat{\sigma}^2_x = \frac{\mu (c_n^2 + \rho c_a^2 + \rho - 1)}{2 \rho^2 f_H(\bar{w})} + \rho \mu \int_0^{\bar{w}} H(u)(1 - H(u)) \, du + \rho \mu c_a \int_0^{\bar{w}} (1 - H(u))^2 \, du.
\]

When \( n \) is large, the distribution of \( \tilde{X}_n(\infty) \) should be close to this Gaussian distribution, which leads to formula \( 2.8 \).

The third theorem plays an essential role in proving the previous two theorems. This theorem is an FCLT for the superposition of time-scaled, stationary renewal processes, which are defined as follows. For \( t \geq 0 \) and \( j \in \mathbb{N} \), let
\[
N_j(t) = \max\{k \geq 0 : \xi_{j,1} + \cdots + \xi_{j,k} \leq t\},
\]
where \( \{\xi_{j,k} : j, k \in \mathbb{N}\} \) is the double sequence of random variables defined earlier. If \( \xi_{j,1} > t \), we take \( N_j(t) = 0 \) by convention. Because \( \xi_{j,1} \) follows distribution \( F_e \) and \( \xi_{j,k} \) follows distribution \( F \) for \( k \geq 2 \), \( \{N_j : j \in \mathbb{N}\} \) is a sequence of iid stationary renewal processes.

**Theorem 3.** Let \( \{N_j : j \in \mathbb{N}\} \) be a sequence of iid stationary renewal processes, i.e., the delay distribution \( F_e \) of each renewal process is the equilibrium distribution of the interrenewal distribution \( F \). Assume that \( F \), having mean \( 1/\mu \) and squared coefficient of variation \( c_s^2 \), satisfies (3.4) and (3.5). Let

\[
B_n(t) = \sum_{j=1}^{n} N_j(t)
\]

(3.12)

and \( \{\gamma_n : n \in \mathbb{N}\} \) be a sequence of positive numbers such that \( \gamma_n \to \infty \) as \( n \to \infty \). Then,

\[
\tilde{B}_n \Rightarrow \hat{B} \text{ as } n \to \infty,
\]

where

\[
\hat{B}_n(t) = \frac{1}{\sqrt{n\gamma_n}} (B_n(\gamma_n t) - n\mu \gamma_n t)
\]

(3.13)

and \( \hat{B} \) is a driftless Brownian motion with \( \hat{B}(0) = 0 \) and variance \( \mu c_s^2 \).

**Remark.** To better understand Theorem 3, let us compare this result with two other FCLTs. By the FCLT for renewal processes, \( \{(N_1(\ell t) - \ell \mu t)/\sqrt{\ell} : t \geq 0\} \) converges in distribution to a Brownian motion as \( \ell \) goes to infinity; see, e.g., Theorem 5.11 in Chen and Yao (2001). Clearly, the increments of this time-scaled renewal process become independent of its history as the scaling factor gets large. Whitt (1985) proved an FCLT for the superposition of stationary renewal processes. It states that \( \{\sum_{j=1}^{n}(N_j(t) - \mu t)/\sqrt{n} : t \geq 0\} \) converges in distribution to a zero-mean Gaussian process that has stationary increments and continuous sample paths. In this FCLT, the superposition process is scaled in space only. The covariance function of each stationary renewal process is retained in the limit Gaussian process, which, in general, is not a Brownian motion; see Theorem 2 in Whitt (1985). In our theorem, each superposition process is scaled in both space and time. Squeezing the time scale erases the dependence of the increments of \( \tilde{B}_n \) to the history. The limit of these space-time scaled superposition processes should thus be a Gaussian process with independent, stationary increments and continuous sample paths, which must be a Brownian motion.

In the ED regime, all servers are nearly always busy, so the service completion process is almost identical to the superposition of many renewal processes. Theorem 3 implies that the space-time scaled service completion process can be approximated by a Brownian motion, which allows us to bypass the analysis of the infinite-dimensional age or residual process in proving a limit process. Hence, by zooming out our perspective in both space and time, we may obtain a one-dimensional diffusion model for many-server queues with a general service time distribution.

**4 Numerical experiments and discussion**

In this section, we examine the diffusion model by numerical experiments. We first study the influence of the mean patience time on the accuracy of approximation, and then use the diffusion model to solve a staffing problem.
Table 1: Performance estimates for the M/GI/100 + M queue with $\mu = 1.0$ and $\rho = 1.2$; simulation results (with 95% confidence intervals) are compared with approximate results (in italics).

| Patience | Abd. fraction | M/D/100 + M | M/E_2/100 + M | M/LN/100 + M |
|----------|---------------|-------------|---------------|--------------|
| $\gamma = 1.0$ | 0.1668 | 0.1851 | 0.005322 | 20.02 | 73.11 |
| 0.1667 | 0.1823 | 0.005000 | 20.00 | 70.00 |
| $\gamma = 5.0$ | 0.1667 | 0.9142 | 0.02639 | 99.99 | 364.1 |
| 0.1667 | 0.9116 | 0.025000 | 100.0 | 350.0 |
| $\gamma = 10$ | 0.1667 | 1.826 | 0.05487 | 200.0 | 749.2 |
| 0.1667 | 1.823 | 0.050000 | 200.0 | 700.0 |

4.1 Influence of the mean patience time

Serving as the respective scaling factors in space and time, the number of servers and the mean patience time will affect how close the queue’s performance is to the diffusion approximation. By Theorem 1, both scaling factors are required to approach infinity in order for the diffusion-scaled virtual waiting time process to converge. A diffusion model can generally produce satisfactory performance approximations when there are at least tens of servers (see, e.g., Garnett et al. (2002), Dai and He (2013), and Huang et al. (2014)). However, it is not immediately clear how large the mean patience time should be in order for the proposed diffusion model to be sufficiently accurate. We would thus evaluate the influence of the mean patience time on the accuracy of approximation.

Assume that the queue has a Poisson arrival process with rate $\lambda = 120$ and 100 servers with service rate $\mu = 1.0$, so the traffic intensity is $\rho = 1.2$. The patience time distribution is exponential with mean $\gamma = 1.0, 5.0, $ or $10$. The service time distribution may be deterministic, Erlang (with
are compared with diffusion approximations (in italics). We need the diffusion model for completing this task. Theorems 1 and 2 imply that produce accurate approximations for mean performance measures in the ED regime.

The fluid model, however, cannot be used for estimating variances because of its deterministic nature. We need the diffusion model for completing this task. Theorems 1 and 2 imply that

Table 2: Tail probabilities for the steady-state virtual waiting time and queue length in the M/GI/100 + M queue with \( \mu = 1.0 \) and \( \rho = 1.2 \); simulation results (with 95% confidence intervals) are compared with diffusion approximations (in italics).

| Patience | \( P[W(\infty) > a] \) | \( P[X(\infty) > a] \) |
|----------|----------------|----------------|
|          | \( a = 0.5 \) | \( a = 1.0 \) | \( a = 2.0 \) | \( a = 0.5 \) | \( a = 1.0 \) | \( a = 2.0 \) |
| \( \gamma = 1.0 \) | 0.2584 | 0.09269 | 0.003869 | 0.2559 | 0.1131 | 0.01140 |
|          | \( \pm 0.00014 \) | \( \pm 0.000078 \) | \( \pm 0.000018 \) | \( \pm 0.00014 \) | \( \pm 0.000089 \) | \( \pm 0.00031 \) |
| \( \gamma = 5.0 \) | 0.2505 | 0.08689 | 0.003138 | 0.2707 | 0.1200 | 0.01120 |
|          | \( \pm 0.0019 \) | \( \pm 0.0016 \) | \( \pm 0.00017 \) | \( \pm 0.0013 \) | \( \pm 0.00031 \) |
| \( \gamma = 10 \) | 0.2539 | 0.09004 | 0.003419 | 0.2840 | 0.1252 | 0.01089 |
|          | \( \pm 0.00046 \) | \( \pm 0.00023 \) | \( \pm 0.000050 \) | \( \pm 0.00029 \) | \( \pm 0.00093 \) |

For service time data from an Israeli call center, we also test such a distribution with deterministic and Erlang distributions are used to represent scenarios where service times have small variability. Brown et al. (2005) reported that a log-normal distribution provides a good fit for service time data from an Israeli call center. We also test such a distribution with \( c_s^2 = 2 \), which yields more variable service times.

The estimates of several performance measures, including the fraction of abandoning customers, the mean and variance of the steady-state virtual waiting time, and the mean and variance of the steady-state queue length, are listed in Table 1. We use (2.1), (2.2), (2.4), (2.7), and (2.9) to compute the respective approximate results. The formulas for the fraction of abandoning customers, the mean virtual waiting time, and the mean queue length are identical to those obtained from the fluid model by Whitt (2006). These fluid approximations agree with the simulation results very well, which is consistent with the conclusion drawn by Whitt (2006): The fluid model is able to produce accurate approximations for mean performance measures in the ED regime.

The fluid model, however, cannot be used for estimating variances because of its deterministic nature. We need the diffusion model for completing this task. Theorems 1 and 2 imply that
diffusion approximations are more accurate when the mean patience time is longer. Comparing the variance results in Table 1 however, we can see that an adequate diffusion approximation may not require the mean patience time to be large. With a mean patience time that is comparable to the mean service time, the approximate variances are satisfactory when service times are deterministic or follow an Erlang distribution. This observation can be explained as follows. Because all servers are almost always busy, the service completion process is close to the superposition of iid renewal processes. As we discussed in the previous section, by squeezing the time scale, the increments of the service completion process will become more and more independent of the history. Then by Theorem 3 we may use a Brownian motion to replace the space-time scaled service departure process to obtain the diffusion model. If the variability in service times is not significant, a moderate scaling factor in time is sufficient for the Brownian approximation to work well. Therefore, with a deterministic or Erlang service time distribution, the approximate variances are close to the simulation results even for $\gamma = 1.0$. A larger scaling factor is necessary if the variability in service times is more considerable. When the service time distribution is log-normal with $c_s^2 = 2.0$, the approximate variances are no longer accurate for $\gamma = 1.0$. In order for the increments of the time-scaled service completion process to be sufficiently independent of the history, the mean patience time should be at least several times longer than the mean service time. We can see that the approximate variances are satisfactory for $\gamma = 5.0$ and 10.

We also compute tail probabilities for the steady-state distributions of diffusion-scaled virtual waiting time and queue length. Approximations by (2.3) and (2.8) are compared with simulation results in Table 2, the observation from which is consistent with what we found from Table 1. With the deterministic or Erlang service time distribution, the approximate distributions are satisfactory when the mean patience time is comparable to or longer than the mean service time; when service times have a larger variance, the mean patience time needs to be at least several times longer than the mean service time, in order for Gaussian approximations to be accurate.

To illustrate how the number of customers converges to a Gaussian random variable in the
Figure 2: The hazard rate of the hyperexponential distribution.

steady state, we examine an M/H$_2$/100 + M queue whose service times follow a hyperexponential distribution with $\mu = 1.0$ and $c^2 = 3.0$. There are two types of customers in the queue, and the service times of either type are iid exponential random variables. The fraction of the first type is 59.16% and its mean service time is 0.1691; the fraction of the second type is 40.84% and its mean service time is 2.203. We are interested in this queue because the exact distribution of the steady-state number of customers can be computed by the matrix-analytic method (see, e.g., Latouche and Ramaswami (1999)). We may also approximate the distribution of $X(\infty)$ by

$$P[X(\infty) = i] \approx \frac{1}{\sigma_x} \phi\left(\frac{i - n - q}{\sigma_x}\right) \text{ for } i = 0, 1, \ldots,$$

where $\phi$ is the standard Gaussian density function and $q$ and $\sigma_x$ are given by (2.7) and (2.9), respectively. We compare these two distributions in Figure 1. Although the Gaussian approximation cannot capture the exact distribution for $\gamma = 1.0$, it becomes a good fit for $\gamma = 5.0$.

Through the above numerical examples, we can tell that both the variability in service times and the ratio of the mean patience time to the mean service time have influence on the accuracy of approximation. We would thus introduce the following quantity

$$R_{svp} = \frac{c_s}{\gamma \mu},$$

or the ratio of the standard deviation of service times to the mean patience time, as an index of accuracy when a queue is approximated by the diffusion model. This quantity is called the service-variability-to-patience ratio (SVPR). (Essentially, the SVPR is an index of how close the space-time scaled service completion process is to a Brownian motion.) By extensive numerical experiments, we observe that the diffusion approximations are generally accurate when $R_{svp} < 0.5$.

4.2 Staffing using the diffusion model

In typical service-oriented call centers, the variability in customer service times is not significant and most customers are relatively patient when they are waiting for service. For example, by analyzing a set of operational data from an Israeli call center, Mandelbaum et al. (2001) reported that the
customer service times had moderate variability, ranging from \( c_s^2 = 1.54 \) to \( c_s^2 = 5.83 \) for different months, and that the mean patience time was about four times longer than the mean service time. Both moderate service variability and long customer patience suggest that the diffusion model is appropriate for modeling call centers in the ED regime. Next, let us study an application of the diffusion model to call center staffing.

Consider an M/LN/\( n + H_2 \) queue, which has a Poisson arrival process, a log-normal service time distribution, and a hyperexponential patience time distribution. There are two types of customers with different abandonment behaviors, and the patience times of either type are iid exponential random variables. We assume that 98% of customers have long patience times with mean 1000 seconds and 2% of customers have short patience times with mean 6.0 seconds. We use such a distribution to represent a typical pattern of abandonment in call centers: While most customers would wait patiently for their service, a small fraction of customers would hang up within seconds if they cannot be served immediately. The hazard rate function of patience times is plotted in Figure 2. We adjusted the parameters in order for the hyperexponential distribution to imitate the patience time distribution in the call center of a large U.S. bank, whose operational data were analyzed by Mandelbaum and Zeltyn (2013). By checking Figure 2 in their paper, one can see that our hazard rate is just a “replica” of the smoothed hazard rate of their patience time data.

In this queue, the service rate is taken to be \( \mu = 1/230 \) per second, which is obtained from the estimated mean service time in the U.S. call center; see Figure 16 in Mandelbaum and Zeltyn (2013). Their paper, however, does not provide information about the variance of service time. We would consider two scenarios where the log-normal service time distribution has \( c_s^2 = 3.0 \) and \( c_s^2 = 5.0 \), respectively.

Let the customer arrival rate be fixed at \( \lambda = 1.0 \) per second. We would like to determine the minimum number of servers that is required for at least 80% of customers to receive service within 120 seconds. The given delay in this example is around one half of the mean service time, which
Figure 4: The effective abandonment fraction (the fraction of abandoning customers out of those whose waiting times exceed 60 seconds) in the M/LN/n + H2 queue with $\lambda = 1.0$, $\mu = 1/230$, and the hyperexponential patience time distribution illustrated in Figure 2, simulation results are compared with the estimates by (2.6) and the estimates by Zeltyn and Mandelbaum (2005).

is a reasonable requirement because customer waiting times should be comparable to service times in order for the queue to be in the ED regime. For any fixed number of servers, the service level within the given delay can be estimated by (2.5). We compare the service level estimates with simulation results in Figure 3 and find good agreement in both scenarios. To reach the service level objective, the estimates from the diffusion model recommend 211 servers for $c_s^2 = 3.0$ and 213 servers for $c_s^2 = 5.0$, while the simulation results recommend 211 and 212 servers, respectively. It can be seen that the estimation error is more apparent when $c_s^2 = 5.0$, which is consistent with the fact that this scenario has a larger SVPR.

An existing method is also considered for the purpose of comparison. Zeltyn and Mandelbaum (2005) modeled a call center as an M/M/n + GI queue and studied performance approximations for this model. Their results are based on the analysis of a Markov chain studied by Baccelli and Hebuterne (1981), and some results are used for call center staffing in Mandelbaum and Zeltyn (2009). The approximate distribution of the steady-state virtual waiting time can be found in (6.12) in their paper. This approximation turns out to be identical to our Gaussian approximation if we take $c_a^2 = c_s^2 = 1.0$ in (2.3), so we may still use (2.5) to estimate the service level by their approximation. The resulting service level estimates are marked with “Z–M” in Figure 3.

The approximate results by Zeltyn and Mandelbaum (2005) are based on the assumption that both interarrival times and service times are exponentially distributed. Their estimates may not be accurate when these distributions are non-exponential. We find a wide discrepancy in Figure 3 between the simulation results and their service level estimates. Because their approximation assumes $c_s^2 = 1.0$, the estimates suggest 208 servers in both scenarios. If this recommendation is adopted, the actual service level will be 74.4% for $c_s^2 = 3.0$ and 70.9% for $c_s^2 = 5.0$; in neither case can the service level objective be fulfilled. When the variability in arrival and service times is relatively large, one tends to underestimate the staffing level using the M/M/n + GI model, in which case the required service level can hardly be achieved.

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The fraction of abandoning customers is also a common concern in call center staffing. This time we would like to determine the minimum number of servers in the $M/\text{LN}/n + H_2$ queue, such that the fraction of abandoning customers is less than 5%. Excluding customers abandoning the system within a short delay, this effective abandonment fraction is used more widely as a measure of customer satisfaction than the total percentage of abandoning customers. Figure 4 compares the respective estimates by (2.6) and by Zeltyn and Mandelbaum (2005) with simulation results. As in the previous example, the estimates by (2.6) agree well with the simulation results, whereas the results by Zeltyn and Mandelbaum (2005) cannot capture the actual fractions. To achieve the above target, the simulation results recommend 205 servers for $c_s^2 = 3.0$ and 206 servers for $c_s^2 = 5.0$, and the estimates by (2.6) suggest 205 and 207 servers, respectively. In contrast, the results by Zeltyn and Mandelbaum (2005) yield 202 servers in both scenarios. If one follows this recommendation, the effective abandonment fraction will be 5.55% for $c_s^2 = 3.0$ and 6.15% for $c_s^2 = 5.0$, failing to reach the performance objective in both cases.

5 Diffusion limit of virtual waiting time

This section is dedicated to the proof of Theorem 1. A sequence of perturbed systems is introduced in Section 5.1. In Section 5.2, we first show that the perturbed systems are asymptotically equivalent to the original queues, then prove the diffusion limit for virtual waiting time processes in the perturbed systems, and finally finish the proof of the theorem by using the asymptotic equivalence. The proof procedure in Section 5.2 partially follows that of Theorem 1 in Huang et al. (2014).

5.1 Perturbed systems

Consider the virtual waiting time process in the $n$th queue. Let $w_{n,k}$ be the offered waiting time of the $k$th customer arriving after time zero, which is the amount of time the $k$th customer would wait until getting into service, provided that his patience is infinite. If we use $\zeta_{n,k}$ to denote the patience time of the $k$th customer, the number of customers who arrived during $(0, t]$ but will eventually abandon the system can be counted by

$$L_n(t) = \sum_{k=1}^{E_n(t)} 1\{\zeta_{n,k} \leq w_{n,k}\}.$$

Suppose that all customers arriving after time $t$ are rejected immediately. Then, the virtual waiting time at $t$ turns out to be the amount of time from $t$ until an idle server appears, i.e.,

$$W_n(t) = \inf\{u \geq 0 : X_n(0) + E_n(t) - L_n(t) - D_n(t + u) < n\},$$

where $D_n(t)$ is the number of service completions during $(0, t]$. Let $a_{n,k}$ be the arrival time of the $k$th customer. Because no two customers arrive at the same time, it follows from Lemma 3.2 in Dai and He (2010) that

$$w_{n,k} = W_n(a_{n,k}-).$$

All servers are almost always busy in the ED regime. For $j = 1, \ldots, n$, because $\{\xi_{j,k} : k \in \mathbb{N}\}$ is the sequence of service times to be finished by the $j$th server, the service completion process from
this server is identical to the renewal process $N_j$ until it begins to idle. Therefore, the departure process $D_n$ is identical to the superposition of $N_1, \ldots, N_n$ until the first idle server appears. Let 

$$
\tau_n = \inf\{t \geq 0 : W_n(t) = 0\}
$$

be the time that the first idle server appears. Then, $\tau_n > 0$ because all servers are busy at time zero. The departure process satisfies

$$
D_n(t) = B_n(t) \quad \text{for } 0 \leq t \leq \tau_n,
$$

where $B_n(t)$ is given by (3.12). As the superposition of $n$ iid stationary renewal processes, $B_n$ is more analytically tractable than $D_n$. The equivalence between these two processes up to time $\tau_n$ allows us to explore a perturbed system that has simplified dynamics. This perturbed system is asymptotically equivalent to the original queue as $n$ goes large.

Because $W_n(t) > 0$ for $0 \leq t < \tau_n$, it follows from (5.1) that with probability one,

$$
X_n(0) + E_n(t) - L_n(t) - D_n(t + W_n(t)) = n - 1 \quad \text{for } 0 \leq t \leq \tau_n.
$$

Because $D_n(t + u) = B_n(t + u)$ for $0 \leq t \leq \tau_n$ and $0 \leq u \leq W_n(t)$, we further have

$$
X_n(0) + E_n(t) - L_n(t) - B_n(t + W_n(t)) = n - 1 \quad \text{for } 0 \leq t \leq \tau_n.
$$

Let us introduce a new process $V_n$ by using a slightly modified dynamical equation

$$
X_n(0) + E_n(t) - R_n(t) - B_n(t + V_n(t)) = n - 1 \quad \text{for } t \geq 0,
$$

where

$$
R_n(t) = \sum_{k=1}^{E_n(t)} 1\{\zeta_{n,k} \leq V_n(t) - (a_{n,k})\}.
$$

By a standard sample path argument, one can show that (5.2) has a unique solution $V_n$. Clearly,

$$
W_n(t) = V_n(t) \quad \text{for } 0 \leq t \leq \tau_n
$$

on each sample path, and $\tau_n$ can thus be defined alternatively by

$$
\tau_n = \inf\{t \geq 0 : V_n(t) = 0\}.
$$

Taking $t = 0$ in (5.2), we obtain $B_n(V_n(0)) = X_n(0) - (n - 1)$, which allows us to write (5.2) into

$$
B_n(t + V_n(t)) = B_n(V_n(0)) + E_n(t) - R_n(t).
$$

We refer to $V_n$ as the virtual waiting time process in the $n$th perturbed system. The dynamical equation (5.5) is identical to equation (9) in Huang et al. (2014), while the latter is derived for their original system.

The perturbed system can be envisioned as a queue where no server is allowed to idle. If a server finds no waiting customers upon a service completion, she begins to serve a customer who
has not arrived yet. In the perturbed system, all servers are always busy and the departure process from each server is a stationary renewal process.

5.2 Asymptotic equivalence

We first prove a fluid limit for the virtual waiting time processes in the perturbed systems. This limit allows us to establish the asymptotic equivalence between the original queues and the perturbed systems, which implies that these two sequences of systems have an identical diffusion limit.

Lemma 1 is a modified version of Theorem 4.1 in Pang et al. (2007), stating the continuity of a map defined by an integral equation. This map will be used extensively in the subsequent proofs.

Lemma 1. For any \( f \in \mathbb{D} \), let \( x \) be a function in \( \mathbb{D} \) such that

\[
  x(t) = f(t) - \int_0^t g(x(u)) \, du,
\]

where \( g : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous with \( g(0) = 0 \). Then for each \( f \in \mathbb{D} \), there is a unique \( x \in \mathbb{D} \) such that (5.6) holds. Let \( \psi : \mathbb{D} \to \mathbb{D} \) be the function that maps \( f \) to \( x \). Then, \( \psi \) is a continuous map when \( \mathbb{D} \) (as both the domain and the range) is endowed with the \( J_1 \) topology.

Write the dynamical equation (5.5) into the fluid-scaled form

\[
  \bar{\mathbf{B}}_n(t + \bar{\mathbf{V}}_n(t)) = \bar{\mathbf{B}}_n(\bar{\mathbf{V}}_n(0)) + \bar{\mathbf{E}}_n(t) - \bar{\mathbf{R}}_n(t),
\]

where

\[
  \begin{align*}
  \bar{\mathbf{E}}_n(t) &= \frac{1}{n\gamma_n} E_n(\gamma_n t), \quad \bar{\mathbf{B}}_n(t) = \frac{1}{n\gamma_n} B_n(\gamma_n t), \quad \bar{\mathbf{R}}_n(t) = \frac{1}{n\gamma_n} R_n(\gamma_n t), \quad \bar{\mathbf{V}}_n(t) = \frac{1}{\gamma_n} V_n(\gamma_n t).
  \end{align*}
\]

By (3.6), the fluid-scaled arrival process satisfies

\[
  \bar{E}_n \Rightarrow \rho \mu e \quad \text{as } n \to \infty.
\]

As a scaled version of the superposition of renewal processes, \( \bar{B}_n \) satisfies a functional strong law of large numbers (FSLLN), as is stated in the proposition below. We leave the proof to the appendix.

Proposition 1. Under the conditions of Theorem 3,

\[
  \bar{B}_n \to \mu e \quad \text{almost surely as } n \to \infty.
\]

The following result states that in these perturbed systems, both the fluid-scaled virtual waiting time processes and their total variations are stochastically bounded.

Lemma 2. Under the conditions of Theorem 7,

\[
  \lim_{a \to \infty} \limsup_{n \to \infty} \mathbb{P}\left[ \sup_{0 \leq t \leq T} (t + \bar{\mathbf{V}}_n(t)) > a \right] = 0
\]

and

\[
  \lim_{a \to \infty} \limsup_{n \to \infty} \mathbb{P}\left[ \int_0^T |d\bar{\mathbf{V}}_n(t)| > a \right] = 0 \quad \text{for } T > 0.
\]
The next lemma establishes an asymptotic relationship in the fluid scaling, allowing us to approximate the abandonment process by an integral of the virtual waiting time process.

**Lemma 3.** Under the conditions of Theorem 1,

\[ \sup_{0 \leq t \leq T} \left| \bar{R}_n(t) - \rho \mu \int_0^t H(\bar{V}_n(u)) \, du \right| \to 0 \quad \text{for } T > 0. \]

Now we can prove the fluid limit for the virtual waiting time processes in the perturbed systems.

**Proposition 2.** Under the conditions of Theorem 1,

\[ \bar{V}_n \Rightarrow \bar{w} \chi \quad \text{as } n \to \infty. \]

**Proof.** By (5.7), we obtain

\[ \bar{V}_n(t) = \bar{V}_n(0) + \bar{I}_n(t) + (\rho - 1)t - \rho \int_0^t H(\bar{V}_n(u)) \, du, \]

where

\[ \bar{I}_n(t) = \frac{1}{\mu} \left( \bar{B}_n(\bar{V}_n(0)) - \mu \bar{V}_n(0) \right) - \frac{1}{\mu} \left( \bar{B}_n(t + \bar{V}_n(t)) - \mu(t + \bar{V}_n(t)) \right) + \frac{1}{\mu} \left( \bar{E}_n(t) - \rho \mu t \right) \]

\[ - \frac{1}{\mu} \left( \bar{R}_n(t) - \rho \mu \int_0^t H(\bar{V}_n(u)) \, du \right). \]

Then, \( \bar{V}_n = \psi(\bar{V}_n(0)\chi + \bar{I}_n + (\rho - 1)e) \), where \( \psi \) is the map defined by (5.6) with \( g(u) = \rho H(u) \) for \( u \in \mathbb{R} \). By (3.2), \( H \) is Lipschitz continuous with \( H(0) = 0 \), so \( \psi \) is a continuous map. Because \( \bar{w}\chi = \psi(\bar{w}\chi + (\rho - 1)e) \), Lemma 7 and the continuous mapping theorem (see, e.g., Theorem 5.2 in Chen and Yao (2001)) will lead to the assertion once we prove \( \bar{V}_n(0) \Rightarrow \bar{w} \) and \( \bar{I}_n \Rightarrow 0 \) as \( n \to \infty \).

The convergence of \( \bar{V}_n(0) \) follows from (3.7) and (5.3), and the convergence of \( \bar{I}_n \) follows from (5.9), Proposition 1, and Lemmas 1–3.

The fluid limit allows us to establish the asymptotic equivalence between the perturbed systems and the original queues. Put \( \bar{\tau}_n = \tau_n/\gamma_n \), which is the instant when the first idle server appears in the \( n \)th time-scaled perturbed system. The next proposition states that \( \bar{\tau}_n \) goes to infinity in probability as \( n \) increases.

**Proposition 3.** Under the conditions of Theorem 1,

\[ \lim_{n \to \infty} \mathbb{P}[\bar{\tau}_n \leq T] = 0 \quad \text{for all } T > 0. \]

**Proof.** By (5.4) and (5.8), \( \bar{\tau}_n = \inf\{t \geq 0 : \bar{V}_n(t) = 0\} \), which yields

\[ \mathbb{P}[\bar{\tau}_n \leq T] = \mathbb{P}\left[ \inf_{0 \leq t \leq T} \bar{V}_n(t) = 0 \right]. \]

Then, the assertion follows from Proposition 2.

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Remark. By (3.11), (5.3), and Propositions 2 and 3 we obtain
\[ \bar{W}_n \Rightarrow \bar{w} \chi \quad \text{as } n \to \infty, \] (5.12)
which is the fluid limit for the virtual waiting time processes in the original queues.

The diffusion-scaled virtual waiting time process in the \( n \)th perturbed system is defined by
\[ \tilde{V}_n(t) = \sqrt{\frac{n}{\gamma n}} (V_n(\gamma_n t) - \gamma_n \bar{w}). \] (5.13)
Then,
\[ \tilde{W}_n(t) = \tilde{V}_n(t) \quad \text{for } 0 \leq t \leq \bar{\tau}_n. \] (5.14)
Proposition 3 implies that \( \tilde{W}_n \) and \( \tilde{V}_n \) are asymptotically equal over any finite time interval. This allows us to consider the diffusion limit of \( \tilde{V}_n \) in order to obtain that of \( \tilde{W}_n \).

For notational convenience, write \( \bar{a}_{n,k} = a_{n,k}/\gamma_n \) and \( \bar{\zeta}_{n,k}/\gamma_n \), which are the scaled arrival and patience times of the \( k \)th customer. Put
\[ \tilde{R}_n(t) = \frac{1}{\sqrt{n \gamma n}} \sum_{k=1}^{E_n(\gamma_n t)} \left( I(\bar{\zeta}_{n,k} \leq V_n(\bar{a}_{n,k} -)) - \Theta_n(V_n(a_{n,k} -)) \right). \]
By (3.13), (5.8), and (5.13), we can write (5.5) into the diffusion-scaled form
\[ \tilde{V}_n(t) = \tilde{V}_n(0) + \tilde{M}_n(t) - \tilde{\Upsilon}_n(t) - \rho f_H(\bar{w}) \int_0^t \tilde{V}_n(u) \, du, \] (5.15)
where
\[ \tilde{M}_n(t) = \frac{1}{\mu} \left( \tilde{B}_n(\tilde{V}_n(0)) - \tilde{B}_n(t + \tilde{V}_n(t)) - \tilde{R}_n(t) \right) + \frac{1}{\rho \mu} \tilde{E}_n(t) \] (5.16)
and
\[ \tilde{\Upsilon}_n(t) = \frac{1}{\mu \sqrt{n \gamma n}} \sum_{k=1}^{E_n(\gamma_n t)} \left( H(V_n(\bar{a}_{n,k} -)) - H(\bar{w}) \right) - \rho f_H(\bar{w}) \int_0^t \tilde{V}_n(u) \, du. \] (5.17)
Equations (5.15)–(5.17) follow equations (16)–(17) in Huang et al. (2014).

To obtain the limit process of \( \tilde{V}_n \), we should first obtain the limit processes of \( \tilde{M}_n \) and \( \tilde{\Upsilon}_n \). The convergence of \( \tilde{M}_n \) can be deduced by the next lemma, which states the joint convergence of the diffusion-scaled arrival, service completion, and abandonment processes in the perturbed systems.

**Lemma 4.** Under the conditions of Theorem 7
\[ (\tilde{E}_n, \tilde{B}_n, \tilde{R}_n) \Rightarrow (\hat{E}, \hat{B}, \hat{R}) \quad \text{as } n \to \infty, \]
where \( \hat{E}, \hat{B}, \) and \( \hat{R} \) are independent driftless Brownian motions with variances \( \rho \mu c^2_a, \mu c^2_s, \) and \( (\rho - 1)\mu/\rho, \) respectively.

The process \( \tilde{\Upsilon}_n \) turns out to be an error term converging to zero.
Lemma 5. Under the conditions of Theorem 1,
\[ \tilde{\Upsilon}_n \Rightarrow 0 \quad \text{as } n \to \infty. \]

Using these convergence results, we can obtain the diffusion limit for the perturbed systems.

Proposition 4. Under the conditions of Theorem 1,
\[ \tilde{V}_n \Rightarrow \hat{W} \quad \text{as } n \to \infty, \]
where \( \hat{W} \) is the OU process given by (3.8).

Proof. By (3.7) and (5.14), \( \tilde{V}_n(0) \Rightarrow \hat{W}(0) \) as \( n \to \infty \). Using Proposition 2 and Lemma 4, we obtain \( \tilde{M}_n \Rightarrow \hat{M} \) as \( n \to \infty \), where \( \hat{M} \) is a driftless Brownian motion with \( \hat{M}(0) = 0 \) and variance \( (c_a^2 + \rho c_b^2 + \rho - 1)/\rho \mu \). Then, by Lemma 5,
\[ \tilde{V}_n(0) \chi + \tilde{M}_n - \tilde{\Upsilon}_n \Rightarrow \hat{W}(0) \chi + \hat{M} \quad \text{as } n \to \infty. \]

Because \( \tilde{V}_n = \psi(\tilde{V}_n(0) \chi + \tilde{M}_n - \tilde{\Upsilon}_n) \) and \( \hat{W} = \psi(\hat{W}(0) \chi + \hat{M}) \), where \( \psi \) is the map given by (5.6) with \( g(u) = \rho f_H(\bar{w})u \) for \( u \in \mathbb{R} \), the assertion follows from Lemma 1 and the continuous mapping theorem. \( \square \)

Let us complete the proof of Theorem 1 by using the asymptotic equivalence.

Proof of Theorem 1. By (5.14),
\[ \mathbb{P}\left[ \sup_{0 \leq t \leq T} |\tilde{W}_n(t) - \tilde{V}_n(t)| > 0 \right] \leq \mathbb{P}[\tilde{\tau}_n \leq T] \quad \text{for } T > 0. \]

Then, Proposition 3 implies that \( \tilde{W}_n - \tilde{V}_n \Rightarrow 0 \) as \( n \to \infty \). The theorem follows from Proposition 4 and the convergence-together theorem (see Theorem 5.4 in Chen and Yao (2001)). \( \square \)

6 Gaussian limit of queue length

We prove Theorem 2 in this section. With a general patience time distribution, it is difficult to prove a diffusion limit for the queue length process in the ED regime. Huang et al. (2014) proved a one-dimensional limit process of queue length process for the GI/M/n + GI model with a general scaling of patience-time distributions; they used the diffusion limit of virtual waiting time to obtain the limit queue length process. In the present study, we consider the marginal distribution of the queue length at a particular time, which also allows us to infer the steady-state queue length distribution.

Let \( Q_n(t) = (X_n(t) - n)^+ \) be the queue length in the \( n \)th system at time \( t \). Because customers are first-come first-served, those who arrived before \( t \) must have either entered service or abandoned the system by \( t + W_n(t) \). In other words, customers who are waiting at \( t + W_n(t) \) must have arrived
during \((t, t + W_n(t)]\) and have not abandoned the system by \(t + W_n(t)\). This observation yields

\[
Q_n(t + W_n(t)) = \sum_{k=\mathcal{E}_n(t)+1}^{E_n(t+\bar{W}_n(t))} 1_{\{\zeta_{n,k} > t+\bar{W}_n(t)-a_{n,k}\}}.
\]

For \(0 \leq t < \tau_n\), because \(Q_n(t + W_n(t)) = X_n(t + W_n(t)) - n\), it follows from (3.10) and (3.11) that

\[
\tilde{X}_n(t + \bar{W}_n(t)) = \frac{1}{\sqrt{n\gamma_n}}(Q_n(\gamma_n t + W_n(\gamma_n t)) - q_n),
\]

where \(q_n\) is given by (3.9). Then,

\[
\tilde{X}_n(t + \bar{W}_n(t)) = \frac{1}{\sqrt{n\gamma_n}} \sum_{k=\mathcal{E}_n(\gamma_n t)+1}^{E_n(\gamma_n t+W_n(\gamma_n t))} 1_{\{\zeta_{n,k} > \gamma_n t+W_n(\gamma_n t)-a_{n,k}\}} - \sqrt{n\gamma_n \rho \mu} \int_0^{\bar{w}} (1 - H(u)) \, du.
\]

By (5.8), we can further decompose the scaled queue length into

\[
\tilde{X}_n(t + \bar{W}_n(t)) = \mu \bar{W}_n(t) + \tilde{G}_n'(t) + \tilde{G}_n''(t) + \tilde{Y}_n(t) + \tilde{Y}_n'(t),
\]

where

\[
\tilde{G}_n'(t) = \sqrt{n\gamma_n} \int_t^{t+\bar{w}} (1 - H(t + \bar{w} - u)) \, d\bar{E}_n(u) - \sqrt{n\gamma_n \rho \mu} \int_t^{t+\bar{w}} (1 - H(t + \bar{w} - u)) \, du,
\]

\[
\tilde{G}_n''(t) = \frac{1}{\sqrt{n\gamma_n}} \sum_{k=\mathcal{E}_n(\gamma_n t)+1}^{E_n(\gamma_n t+W_n(\gamma_n t))} \left( H(t + \bar{W}_n(t) - \bar{a}_{n,k}) - 1_{\{\zeta_{n,k} \leq t+\bar{W}_n(t)-\bar{a}_{n,k}\}} \right),
\]

\[
\tilde{Y}_n(t) = \tilde{E}_n(t + W_n(t)) - \tilde{E}_n(t + \bar{w}) - \sqrt{n\gamma_n} \int_{t+\bar{w}}^{t+\bar{W}_n(t)} H(t + \bar{W}_n(t) - u) \, d\bar{E}_n(u)
\]

\[
+ (\rho - 1) \mu \bar{W}_n(t) - \sqrt{n\gamma_n \rho \mu} \int_0^{\bar{w}} (H(\bar{W}_n(t) - u) - H(\bar{w} - u)) \, du,
\]

\[
\tilde{Y}_n'(t) = \sqrt{n\gamma_n} \int_t^{t+\bar{w}} (H(t + \bar{w} - u) - H(t + \bar{W}_n(t) - u)) \, d\bar{E}_n(u)
\]

\[
- \sqrt{n\gamma_n \rho \mu} \int_t^{t+\bar{w}} (H(t + \bar{w} - u) - H(t + \bar{W}_n(t) - u)) \, du.
\]

To obtain the limit distribution of \(\tilde{X}_n(t + \bar{W}_n(t))\), we need to analyze each term on the right side of (6.1). The joint convergence of the first three terms is given by Lemma 6.

**Lemma 6.** Under the conditions of Theorem 7, for any fixed \(t \geq 0\),

\[
(\bar{W}_n(t), \tilde{G}_n'(t), \tilde{G}_n''(t)) \Rightarrow (\bar{W}(t), \tilde{G}'(t), \tilde{G}''(t)) \quad \text{as } n \to \infty,
\]

as stated in the previous sentences.
where \( \hat{W} \) is the OU process given by (3.8),
\[
\hat{G}'(t) = \int_t^{t+\bar{w}} (1 - H(t + \bar{w} - u)) \, d\hat{E}(u),
\]
and
\[
\hat{G}''(t) = \int_0^\infty \sqrt{\rho \mu H(\bar{w} - u)(1 - H(\bar{w} - u))} \, d\hat{S}(u).
\]
Here, \( \hat{S} \) is a standard Brownian motion that is independent of \( \hat{E}, \hat{B}, \) and \( \{\hat{W}(u) : 0 \leq u \leq t\} \).

The other two random variables \( \tilde{Y}_n(t) \) and \( \tilde{Y}'_n(t) \) are error terms converging to zero.

**Lemma 7.** Under the conditions of Theorem 1, for any fixed \( t \geq 0 \),
\[
(\tilde{Y}_n(t), \tilde{Y}'_n(t)) \Rightarrow (0, 0) \quad \text{as } n \to \infty.
\]

Using the previous two lemmas, we are ready to prove Theorem 2.

**Proof of Theorem 2.** Put \( \tilde{G}_n(t) = \tilde{G}'_n(t) + \tilde{G}''_n(t) + \tilde{Y}_n(t) + \tilde{Y}'_n(t) \). By Lemmas 6 and 7,
\[
(\tilde{W}_n(t), \tilde{G}_n(t)) \Rightarrow (\hat{W}(t), \hat{G}(t)) \quad \text{as } n \to \infty,
\]
from which the convergence of \( \tilde{X}_n(t + \tilde{W}_n(t)) \) follows.

7 Conclusion and future work

A diffusion model was proposed for GI/GI/n + GI queues in the ED regime. We adopted a space-time scaling approach, in which the number of servers and the mean patience time are used as the respective scaling factors in space and time, to obtaining a one-dimensional diffusion approximation for the virtual waiting time process. Using this diffusion model, we derived the steady-state distributions of virtual waiting time and queue length, along with approximate formulas for other performance measures. These approximations are generally accurate when the mean patience time is several times longer than the mean service time and the patience time distribution does not change rapidly around the mean virtual waiting time.

One limitation of the diffusion model is as follows. As we discussed in Section 3, the diffusion limit given by (3.8) depends on the normalized patience time distribution only through the hazard rate at \( \bar{w} \). In consequence, all approximate formulas derived from the diffusion model are dictated by the patience time hazard rate (or equivalently, the patience time probability density) at the mean virtual waiting time. These approximations will generally be accurate, if the patience time hazard rate does not change much around the mean virtual waiting time (as the case in Figure 2). In call centers, this assumption would be valid if waiting customers do not receive real-time information about the queue. However, if delay announcements are made in the call center, customers may decide whether to hang up according to what they hear about the queue. As a result, the patience time hazard rate may change rapidly after an announcement time. In this case, performance approximations that depend on the patience time distribution only through “a single point” may no longer produce satisfactory results. We would thus need a diffusion model that could incorporate...
the patience time hazard rate on a neighborhood of the mean virtual waiting time. Such models were derived for GI/M/\(n\) + GI queues by [Reed and Tezcan (2012)] in the critically loaded regime and by [Huang et al. (2014)] in the ED regime. Furthermore, Huang et al. (2014) applied the results to the analysis of systems with delay announcement. In the future, we would also extend our diffusion model by including the entire patience time distribution. To this end, the current asymptotic framework needs to be modified so that both the space-time scaling and the hazard rate scaling used by [Reed and Tezcan (2012)] and [Huang et al. (2014)] can be combined into the same asymptotic framework.

**Appendix**

We prove Theorem 3 is in Section A.1 and prove all technical lemmas in Sections A.2 and A.3.

### A.1 Proof of the FCLT

Let \( S_{j,k} \) be the \( k \)th partial sum of \( \{\xi_{j,\ell} : \ell \in \mathbb{N}\} \), i.e.,

\[
S_{j,k} = \sum_{\ell=1}^{k} \xi_{j,\ell} \text{ for } j \in \mathbb{N}.
\]

We take \( S_{j,0} = 0 \) by convention. Let us first present the proof of Proposition 1.

**Proof of Proposition 1.** Because \( S_{j,N_j(t)}(t) \leq t \leq S_{j,N_j(t)+1}(t) \) for \( t > 0 \), then

\[
\frac{\sum_{j=1}^{n} S_{j,N_j(\gamma_n t)}}{\sum_{j=1}^{n} N_j(\gamma_n t)} \leq \frac{n \gamma_n t}{\sum_{j=1}^{n} N_j(\gamma_n t)} \leq \frac{\sum_{j=1}^{n} S_{j,N_j(\gamma_n t)+1}(t)}{\sum_{j=1}^{n} N_j(\gamma_n t)}
\]

provided that \( \sum_{j=1}^{n} N_j(\gamma_n t) > 0 \). Note that

\[
\sum_{j=1}^{n} S_{j,N_j(\gamma_n t)+1} = \sum_{j=1}^{n} N_j(\gamma_n t) + \sum_{j=1}^{n} \sum_{k=2}^{N_j(\gamma_n t)+1} \xi_{j,k}.
\]

Because \( \lim_{n \to \infty} N_j(\gamma_n t) = \infty \) almost surely for \( t > 0 \), then

\[
\lim_{n \to \infty} \frac{\sum_{j=1}^{n} \sum_{k=2}^{N_j(\gamma_n t)+1} \xi_{j,k}}{\sum_{j=1}^{n} N_j(\gamma_n t)} = \frac{1}{\mu}
\]

almost surely by the strong law of large numbers. In addition, \( \lim_{n \to \infty} \sum_{j=1}^{n} N_j(\gamma_n t)/n = \infty \) almost surely for \( t > 0 \), which implies that

\[
\lim_{n \to \infty} \frac{\sum_{j=1}^{n} \xi_{j,1}}{\sum_{j=1}^{n} N_j(\gamma_n t)} = 0.
\]

Therefore,

\[
\lim_{n \to \infty} \frac{\sum_{j=1}^{n} S_{j,N_j(\gamma_n t)+1}}{\sum_{j=1}^{n} N_j(\gamma_n t)} = \frac{1}{\mu}.
\]
Lemma 9. Under the conditions of Theorem 3, for all $B < 0$ and Yao (2001), there exists $\tilde{B}$ such that
\[
\tilde{B}_n(t) = 0 \text{ and } \tilde{B}_0(t) = 0 \quad \text{for } t \geq 0.
\]
Then, it follows from Proposition 1 and the random time-change theorem (see Theorem 5.3 in Chen)
\[
\mu \xi_k
\]
continuous function, the assertion follows from Theorem VI.2.15 in Jacod and Shiryaev (2002). □

Lemma 8. Let
\[
\tilde{B}_n(t) = \frac{1}{\sqrt{n \gamma_n}} \sum_{j=1}^{n} N_j(\gamma_n t) \quad \text{for } t \geq 0.
\]
Under the conditions of Theorem 3,
\[
\tilde{B}_n \Rightarrow \tilde{B} \quad \text{as } n \to \infty.
\]
Proof. Let $\{\xi_k : k \in \mathbb{N}\}$ be a sequence of iid random variables following distribution $F$. Put
\[
\tilde{B}_n(\gamma_n t) = \frac{1}{\sqrt{n \gamma_n}} \sum_{j=1}^{[n \gamma_n]} (1 - \mu \xi_j) \quad \text{for } t \geq 0.
\]
Because $\mu \xi_k$ has mean one and variance $c_k^2$, by Donsker’s theorem, $\tilde{B}_n \Rightarrow \tilde{B}'$ as $n \to \infty$, where $\tilde{B}'$ is a driftless Brownian motion with $\tilde{B}'(0) = 0$ and variance $c_2$. By (5.8),
\[
\tilde{B}_n(\tilde{B}_n(t)) = \frac{1}{\sqrt{n \gamma_n}} \sum_{k=1}^{n \gamma_n} (1 - \mu \xi_k) = \frac{1}{\sqrt{n \gamma_n}} \sum_{k=1}^{n} (1 - \mu \xi_k).
\]
Then, it follows from Proposition 1 and the random time-change theorem (see Theorem 5.3 in Chen and Yao (2001)) that $\tilde{B}_n \circ \tilde{B}_n \Rightarrow 1/2 \tilde{B}$ as $n \to \infty$. Because $\tilde{B}_n$ has the same distribution as $\tilde{B}_n \circ \tilde{B}_n$ and $1/2 \tilde{B}$ has the same distribution as $\tilde{B}$, the lemma follows. □

The next lemma is a technical result for proving the convergence of $\tilde{B}_n$.

Lemma 9. Under the conditions of Theorem 3, for all $0 \leq r \leq s \leq t$ and all $n \in \mathbb{N}$, there exists $0 < c < \infty$ such that
\[
\mathbb{E}[(\tilde{B}_n(s) - \tilde{B}_n(r))^2(\tilde{B}_n(t) - \tilde{B}_n(s))^2] \leq c(t - r)^2.
\]
Proof. Let $\tilde{N}_j(u) = N_j(u) - \mu u$ for $u \geq 0$ and $j = 1, \ldots, n$. Because $N_j$ is a stationary renewal process, by inequalities (7) and (8) in Whitt (1985), there exists $c_1 < \infty$ such that
\[
\mathbb{E}[(\tilde{N}_j(s) - \tilde{N}_j(r))^2] \leq c_1(s - r)
\]
and
\[
\mathbb{E}[(\tilde{N}_j(s) - \tilde{N}_j(r))^2(\tilde{N}_j(t) - \tilde{N}_j(s))^2] \leq c_1(t - r)^2
\]
for all $0 \leq r \leq s \leq t$. The regularity condition $[3,4]$ is required for inequality (A.3) to hold. In
addition, it follows from (A.2) and Hölder’s inequality that
\[
E[|\tilde{N}_j(s) - \tilde{N}_j(r)||\tilde{N}_j(t) - \tilde{N}_j(s)||] \leq c_1(s - r)^{1/2}(t - s)^{1/2} \leq c_1(t - r).
\] (A.4)

Because \(N_1, \ldots, N_n\) are iid processes,
\[
E[(\tilde{B}_n(s) - \tilde{B}_n(r))^2(\tilde{B}_n(t) - \tilde{B}_n(s))^2] = \frac{1}{n\gamma_n^2}E[(\tilde{N}_1(\gamma_n s) - \tilde{N}_1(\gamma_n r))^2(\tilde{N}_1(\gamma_n t) - \tilde{N}_1(\gamma_n s))^2]
\]
\[
+ \frac{n - 1}{n\gamma_n^2}E[(\tilde{N}_1(\gamma_n s) - \tilde{N}_1(\gamma_n r))^2]E[(\tilde{N}_1(\gamma_n t) - \tilde{N}_1(\gamma_n s))^2]
\]
\[
+ \frac{2(n - 1)}{n\gamma_n^2}E[(\tilde{N}_1(\gamma_n s) - \tilde{N}_1(\gamma_n r))(\tilde{N}_1(\gamma_n t) - \tilde{N}_1(\gamma_n s))]^2
\]
\[
\leq c_1(t - r)^2 + c_2^2(s - r)(t - s) + 2c_1^2(t - r)^2,
\]
in which the inequality is obtained by (A.2)–(A.4). The lemma follows with \(c = 3c_1^2 + c_1\).

Proof of Theorem 3

For \(j \in \mathbb{N}\), let
\[
U_j(t) = S_{j,N_j(t)+1} - t
\] (A.5)
be the residual time at \(t \geq 0\). Note that \(U_j(0) = \xi_{j,1}\). Because \(N_1, \ldots, N_n\) are iid stationary renewal processes, \(U_1(t), \ldots, U_n(t)\) are iid random variables following distribution \(F_e\) for all \(t \geq 0\), each having mean
\[
m_e = \int_0^\infty t \, dF_e(t) = \frac{1 + c_2^2}{2\mu}
\]
and variance
\[
\sigma_e^2 = \int_0^\infty t^2 \, dF_e(t) - m_e^2 = \frac{\mu}{3} \int_0^\infty t^3 \, dF(t) - m_e^2.
\]
Then, \(\sigma_e^2 < \infty\) by (3.5). Put
\[
\hat{U}_n(t) = \frac{1}{\sqrt{n\gamma_n}} \sum_{j=1}^n (U_j(\gamma_n t) - m_e),
\]
and we have \(E[\hat{U}_n(t)]^2 = \sigma_e^2/\gamma_n \rightarrow 0\) as \(n \rightarrow \infty\). This implies that \(\hat{U}_n(t) \Rightarrow 0\) as \(n \rightarrow \infty\) for \(t \geq 0\).

By Theorem 3.9 in [Billingsley (1999)],
\[
(\hat{U}_n(t_1), \ldots, \hat{U}_n(t_\ell)) \Rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\] (A.6)
for any \(\ell \in \mathbb{N}\) and \(0 \leq t_1 < \cdots < t_\ell\). By (A.5),
\[
U_j(t) = \xi_{j,1} + \sum_{k=2}^{N_j(t)+1} \xi_{j,k} - t = U_j(0) + \sum_{k=2}^{N_j(t)+1} (\xi_{j,k} - \mu^{-1}) + \mu^{-1}N_j(t) - t.
\]

Then, by (3.13) and (A.1), we obtain
\[
\hat{B}_n(t) = -\mu\hat{U}_n(0) + \mu\hat{U}_n(t) + \hat{B}_n'(t).
\] (A.7)
We deduced from (A.6), (A.7), and Lemma 8 that
\[(\bar{B}_n(t_1), \ldots, \bar{B}_n(t_l)) \Rightarrow (\bar{B}(t_1), \ldots, \bar{B}(t_l)) \text{ as } n \to \infty.\]

Finally, it follows from Lemma 9 and Theorem 13.5 in Billingsley (1999) (with condition (13.13) replaced by (13.14)) that \(\bar{B}_n \Rightarrow \bar{B}\) as \(n \to \infty\).

A.2 Proofs of lemmas for Theorem 1

Proof of Lemma 2. By (5.7), \(\bar{B}_n(t + \bar{V}_n(t)) \leq \bar{B}_n(\bar{V}_n(0)) + \bar{E}_n(T)\) for \(0 \leq t \leq T\). Because \(\bar{B}_n\) have nondecreasing sample paths,
\[\mathbb{P}\left[ \sup_{0 \leq t \leq T} (t + \bar{V}_n(t)) > a \right] \leq \mathbb{P}[\bar{B}_n(a) \leq \bar{B}_n(\bar{V}_n(0)) + \bar{E}_n(T)] \text{ for } a > 0.\]

By (3.7), (5.3), and Proposition 1, \(\bar{B}_n(\bar{V}_n(0)) \Rightarrow \mu \bar{w}\) as \(n \to \infty\). Then, (5.10) follows from (5.9).

Because \(e + \bar{V}_n\) has nondecreasing sample paths (see Lemma 3.3 in Dai and He (2010)), we have
\[(\bar{V}_n(t_2) - \bar{V}_n(t_1))^+ \leq (t_2 + \bar{V}_n(t_2)) - (t_1 + \bar{V}_n(t_1)) \text{ and } (\bar{V}_n(t_2) - \bar{V}_n(t_1))^+ \leq t_2 - t_1 \text{ for } 0 \leq t_1 \leq t_2.\]

Hence, the total variation of \(\bar{V}_n\) over \([0, T]\) satisfies
\[\int_0^T |d\bar{V}_n(t)| \leq 2T + \bar{V}_n(T) - \bar{V}_n(0).\]

Then, (5.11) follows from (5.10).

Martingale arguments are extensively involved in subsequent proofs. Let us define the associated filtrations. In the nth system, let \(v_{n,k}\) be the service time of the kth customer arriving after time zero. Note that \(v_{n,k}\) is not identical to \(\xi_{n,k}\), since the latter is the kth service time finished by the nth server. To keep track of the history of the queue, we define a filtration \(\mathcal{F}_{n,i} : i = 0, 1, \ldots\) by
\[\mathcal{F}_{n,i} = \sigma\{a_{n,k+1}, v_{n,k}, \xi_{n,k} : k \leq i\},\]
where \(a_{n,k}\) and \(\xi_{n,k}\) are the arrival and patience times of the kth customer. By Lemma 3.1 in Dai and He (2010), \(V_n(a_{n,k}^-)\) is \(\mathcal{F}_{n,i}\)-measurable for all \(k \leq i + 1\). Modifying the above filtration, we construct a continuous-time filtration \(\mathcal{F}_{n}(t) : t \geq 0\) by
\[\mathcal{F}_{n}(t) = \mathcal{F}_{n,\lfloor n\gamma t \rfloor}.

Proof of Lemma 3. Put
\[\bar{Z}_n(t) = \frac{1}{n\gamma_n} \sum_{k=1}^{\lfloor n\gamma t \rfloor} (1_{\{\xi_{n,k} \leq V_n(a_{n,k}^-)\}} - \Theta_n(V_n(a_{n,k}^-))).\]

Then, \(\{(\bar{Z}_n(t), \mathcal{F}_{n}(t)) : t \geq 0\}\) is a martingale with quadratic variation
\[[Z_n](t) = \frac{1}{n^2 \gamma_n^2} \sum_{k=1}^{\lfloor n\gamma t \rfloor} (1_{\{\xi_{n,k} \leq V_n(a_{n,k}^-)\}} - \Theta_n(V_n(a_{n,k}^-)))^2.\]
(see Lemma 4.2 in [Dai and He (2010)]). Clearly, \( \lim_{n \to \infty} |\bar{Z}_n(t)| = 0 \) for \( t > 0 \). We also have

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |\Delta \bar{Z}_n(t)| = 0
\]

because \( |\Delta \bar{Z}_n(t)| \leq 1/(n\gamma_n) \). Then, it follows from the martingale FCLT (see Theorem 7.1.4 in [Ethier and Kurtz (1986)]) that \( \bar{Z}_n \Rightarrow 0 \) as \( n \to \infty \). By (5.9) and the random time-change theorem,

\[
\bar{Z}_n \circ \bar{E}_n \Rightarrow 0 \quad \text{as} \quad n \to \infty. \tag{A.8}
\]

Put

\[
\Gamma_n(t) = \sum_{k=1}^{E_n(t)} \Theta_n(V_n(a_{n,k-})) \quad \text{and} \quad \bar{\Gamma}_n(t) = \frac{1}{n\gamma_n} \Gamma_n(\gamma_n t).
\]

By (3.3) and (5.8), \( \bar{\Gamma}_n(t) \) can be written into a Riemann–Stieltjes integral

\[
\bar{\Gamma}_n(t) = \int_0^t H(\bar{V}_n(u)) \, d\bar{E}_n(u).
\]

Using integration by parts, we obtain

\[
\bar{\Gamma}_n(t) - \rho \mu \int_0^t H(\bar{V}_n(u)) \, du = (\bar{E}_n(t) - \rho \mu t) H(\bar{V}_n(t)) - \int_0^t (\bar{E}_n(u) - \rho \mu u) \, dH(\bar{V}_n(u)).
\]

By (5.9), the first term on the right side satisfies

\[
\sup_{0 \leq t \leq T} |(\bar{E}_n(t) - \rho \mu t) H(\bar{V}_n(t))| \Rightarrow 0 \quad \text{as} \quad n \to \infty.
\]

For the second term, it follows from (3.2) that

\[
\sup_{0 \leq t \leq T} \left| \int_0^t (\bar{E}_n(u) - \rho \mu u) \, dH(\bar{V}_n(u)) \right| \leq \sup_{0 \leq t \leq T} |\bar{E}_n(t) - \rho \mu t| \int_0^T |dH(\bar{V}_n(t))| \\
\leq \kappa \sup_{0 \leq t \leq T} |\bar{E}_n(t) - \rho \mu t| \int_0^T |d\bar{V}_n(t)|.
\]

By (5.9) and (5.11),

\[
\sup_{0 \leq t \leq T} \left| \int_0^t (\bar{E}_n(u) - \rho \mu u) \, dH(\bar{V}_n(u)) \right| \Rightarrow 0 \quad \text{as} \quad n \to \infty.
\]

It follows that

\[
\sup_{0 \leq t \leq T} \left| \bar{\Gamma}_n(t) - \rho \mu \int_0^t H(\bar{V}_n(u)) \, du \right| \Rightarrow 0 \quad \text{as} \quad n \to \infty. \tag{A.9}
\]

Because \( \bar{R}_n(t) = \bar{Z}_n(\bar{E}_n(t)) + \bar{\Gamma}_n(t) \), the lemma follows from (A.8) and (A.9).
Proof of Lemma 4.2} Put

\[
\tilde{Z}_n(t) = \frac{1}{\sqrt{n\gamma_n}} \sum_{k=1}^{[n\gamma_n t]} \left(1_{\{\tilde{\zeta}_{n,k}\leq \nu_n(\bar{a}_{n,k})\}} - H(\tilde{V}_n(\bar{a}_{n,k})) \right)
\]

and

\[
\tilde{Z}_n'(t) = \frac{1}{\sqrt{n\gamma_n}} \sum_{k=1}^{[n\gamma_n t]} (1_{\{\tilde{\zeta}_{n,k}\leq \bar{w}\}} - H(\bar{w})).
\]

Clearly, \(\mathbb{P}[\tilde{\zeta}_{n,k} \leq \bar{w}] = H(\bar{w}) = (\rho - 1)/\rho\) for \(n, k \in \mathbb{N}\). Then, \(\tilde{Z}_n' \Rightarrow \tilde{Z}\) as \(n \to \infty\) by Donsker’s theorem, where \(\tilde{Z}\) is a driftless Brownian motion with variance \((\rho - 1)/\rho^2\). Because \(\tilde{E}_n, \tilde{B}_n\), and \(\tilde{Z}_n'\) are mutually independent, it follows from (3.6) and Theorem 3 that

\[
(\tilde{E}_n, \tilde{B}_n, \tilde{Z}_n') \Rightarrow (\tilde{E}, \tilde{B}, \tilde{Z}) \quad \text{as } n \to \infty,
\]

where \(\tilde{E}, \tilde{B}, \) and \(\tilde{Z}\) are also mutually independent.

By Lemma 4.2 in [Dai and He (2010)], \(\{(\tilde{Z}_n(t) - \tilde{Z}_n'(t), \mathcal{F}(t)): t \geq 0\}\) is a martingale with

\[
[\tilde{Z}_n - \tilde{Z}_n'](t) = \frac{1}{n\gamma_n} \sum_{k=1}^{[n\gamma_n t]} \left(1_{\{\tilde{\zeta}_{n,k}\leq \nu_n(\bar{a}_{n,k})\}} - 1_{\{\tilde{\zeta}_{n,i}\leq \bar{w}\}} + H(\bar{w}) - H(\tilde{V}_n(\bar{a}_{n,k})) \right)^2.
\]

Because \(\tilde{\zeta}_{n,k}\) is independent of \(\mathcal{F}_{n,k-1}\) and \(\tilde{V}_n(\bar{a}_{n,k})\) is \(\mathcal{F}_{n,k-1}\)-measurable,

\[
\mathbb{E}\left[1_{\{\tilde{\zeta}_{n,k}\leq \nu_n(\bar{a}_{n,k})\}} - 1_{\{\tilde{\zeta}_{n,i}\leq \bar{w}\}} + H(\bar{w}) - H(\tilde{V}_n(\bar{a}_{n,k})) \right] \leq H(\tilde{V}_n(\bar{a}_{n,k})) + H(\bar{w}) - 2H(\tilde{V}_n(\bar{a}_{n,k}) \wedge \bar{w}) - (H(\bar{w}) - H(\tilde{V}_n(\bar{a}_{n,k})))
\]

\[
= |H(\tilde{V}_n(\bar{a}_{n,k})) - H(\bar{w})|.
\]

Hence,

\[
\mathbb{E}[|\tilde{Z}_n - \tilde{Z}_n'|(t)] \leq \frac{1}{n\gamma_n} \sum_{k=1}^{[n\gamma_n t]} \mathbb{E}[|H(\tilde{V}_n(\bar{a}_{n,k})) - H(\bar{w})|] \leq t \cdot \mathbb{E}\left[ \sup_{1 \leq k \leq n\gamma_n t} |H(\tilde{V}_n(\bar{a}_{n,k})) - H(\bar{w})| \right].
\]

For any \(\varepsilon > 0\) and \(t > 0\), it follows from (3.2) that

\[
\mathbb{P}\left[ \sup_{1 \leq k \leq n\gamma_n t} |H(\tilde{V}_n(\bar{a}_{n,k})) - H(\bar{w})| > \varepsilon \right]
\leq \mathbb{P}\left[ \sup_{1 \leq k \leq n\gamma_n t} |\tilde{V}_n(\bar{a}_{n,k}) - \bar{w}| > \frac{\varepsilon}{\kappa} \right]
\leq \mathbb{P}\left[ \sup_{1 \leq k \leq E_n^{\mu^{-1}\gamma_n t}} |\tilde{V}_n(\bar{a}_{n,k}) - \bar{w}| > \frac{\varepsilon}{\kappa} \right] + \mathbb{P}[E_n^{\mu^{-1}\gamma_n t} < n\gamma_n t]
\leq \mathbb{P}\left[ \sup_{1 \leq u \leq \mu^{-1} t} |\tilde{V}_n(u) - \bar{w}| > \frac{\varepsilon}{\kappa} \right] + \mathbb{P}[\tilde{E}_n^{\mu^{-1} t} < t].
\]
Then, we deduce from (5.9) and Proposition 2 that
\[
\lim_{n \to \infty} \mathbb{P}\left[ \sup_{1 \leq k \leq n^{\gamma t}} |H(\bar{V}_n(a_{n,k}^-)) - H(\bar{w})| > \varepsilon \right] = 0.
\]
Because |H(\bar{V}_n(a_{n,k}^-)) - H(\bar{w})| \leq 1, we further obtain
\[
\lim_{n \to \infty} \mathbb{E}\left[ \sup_{1 \leq k \leq n^{\gamma t}} |H(\bar{V}_n(a_{n,k}^-)) - H(\bar{w})| \right] = 0.
\]
Then, \( \mathbb{E}[|\tilde{Z}_n - \tilde{Z}'_n|(t)] \to 0 \) and thus \( [\tilde{Z}_n - \tilde{Z}'_n](t) \to 0 \) as \( n \to \infty \) for all \( t > 0 \). In addition, because \( |\Delta(\tilde{Z}_n - \tilde{Z}'_n)(t)| \leq 2/\sqrt{n^{\gamma t}} \), it follows from the martingale FCLT that \( \tilde{Z}_n - \tilde{Z}'_n \Rightarrow 0 \) as \( n \to \infty \), which, along with (A.10) and the convergence-together theorem, implies that
\[
(\tilde{E}_n, \tilde{B}_n, \tilde{Z}_n) \Rightarrow (\hat{E}, \hat{B}, \hat{Z}) \quad \text{as} \quad n \to \infty.
\]
Because \( \tilde{R}_n = \tilde{Z}_n \circ \tilde{E}_n \), we conclude the proof using (5.9) and the random time-change theorem. ∎

**Proof of Lemma 5** Put
\[
\tilde{Y}'_n(t) = \frac{1}{\mu \sqrt{n^{\gamma t}}} \sum_{k=1}^{E_n(\gamma t)} \left( H(\bar{V}_n(a_{n,k}^-)) - H(\bar{w}) \right) - \sqrt{n^{\gamma t}} \rho \int_0^t (H(\bar{V}_n(u)) - H(\bar{w})) \, du.
\]
By (5.8) and integration by parts,
\[
\tilde{Y}'_n(t) = \frac{\sqrt{n^{\gamma t}}}{\mu} \int_0^t (H(\bar{V}_n(u^-)) - H(\bar{w})) \, d\bar{E}_n(u) - \frac{\sqrt{n^{\gamma t}}}{\mu} \int_0^t (H(\bar{V}_n(u)) - H(\bar{w})) \, du
\]
\[
= \frac{\sqrt{n^{\gamma t}}}{\mu} (H(\bar{V}_n(t)) - H(\bar{w})) (\bar{E}_n(t) - \rho \mu t) - \frac{\sqrt{n^{\gamma t}}}{\mu} \int_0^t (\bar{E}_n(t) - \rho \mu t) \, dH(\bar{V}_n(u))
\]
\[
= \frac{1}{\mu} (H(\bar{V}_n(t)) - H(\bar{w})) \bar{E}_n(t) - \frac{1}{\mu} \int_0^t \bar{E}_n(u^-) \, dH(\bar{V}_n(u)) + \frac{1}{\mu} \sum_{0 < u \leq t} \Delta \bar{E}_n(u) \Delta H(\bar{V}_n(u)).
\]
Consider the three terms on the right side. By (3.6) and Proposition 2
\[
\sup_{0 \leq t \leq T} \left| (H(\bar{V}_n(t)) - H(\bar{w})) \bar{E}_n(t) \right| \to 0 \quad \text{as} \quad n \to \infty.
\]
Using the continuous mapping theorem and Theorem 3.9 in [Billingsley (1999)], we obtain
\[
(\bar{E}_n, H \circ \bar{V}_n) \Rightarrow (\hat{E}, H(\bar{w}) \chi) \quad \text{as} \quad n \to \infty.
\]
By (3.2), \( \int_0^t |dH(\bar{V}_n(u))| \leq \kappa \int_0^t |d\bar{V}_n(u)| \). Then, it follows from (5.11) that
\[
\lim_{a \to \infty} \lim_{n \to \infty} \mathbb{P}\left[ \int_0^T |dH(\bar{V}_n(t))| > a \right] = 0,
\]
which implies that \( \{H \circ \bar{V}_n : n \in \mathbb{N}\} \) is uniformly tight (see Definition 7.4 in [Kurtz and Protter (1996) for the definition of uniform tightness]). Because \( \int_0^t H(\bar{w}) \hat{E}(u^-) \, d\chi(u) = 0 \) for all \( t > 0 \), it
follows from Theorem 7.10 in [Kurtz and Protter (1996)] that

\[
\sup_{0 \leq t \leq T} \left| \int_0^t \tilde{E}_n(u- \, dH(\tilde{V}_n(u)) \right| \Rightarrow 0 \quad \text{as } n \to \infty.
\]

In addition, the third term satisfies

\[
\sup_{0 \leq t \leq T} \left| \sum_{0 < s \leq T} \Delta \tilde{E}_n(t) \Delta H(\tilde{V}_n(s)) \right| \leq \sup_{0 \leq t \leq T} |\Delta \tilde{E}_n(t)| \int_0^T |dH(\tilde{V}_n(t))|.
\]

By (3.6), (A.11), and the fact that \( \hat{E} \) has almost sure continuous sample paths,

\[
\sup_{0 \leq t \leq T} \left| \sum_{0 < s \leq T} \Delta \tilde{E}_n(t) \Delta H(\tilde{V}_n(s)) \right| \Rightarrow 0 \quad \text{as } n \to \infty.
\]

Therefore, \( \tilde{Y}'_n \Rightarrow 0 \) as \( n \to \infty \).

Rewrite (5.15) into

\[
\tilde{V}_n(t) = \tilde{V}_n(0) + \tilde{M}_n(t) - \tilde{Y}'_n(t) - \sqrt{\frac{n}{\gamma_n}} \rho \int_0^t (H(\tilde{V}_n(u)) - H(\bar{w})) \, du.
\]

It follows from (3.2) and (5.13) that

\[
|\tilde{V}_n(t)| \leq |\tilde{V}_n(0) + \tilde{M}_n(t) - \tilde{Y}'_n(t)| + \sqrt{\frac{n}{\gamma_n}} \rho \int_0^t |H(\tilde{V}_n(u)) - H(\bar{w})| \, du
\]

\[
\leq |\tilde{V}_n(0) + \tilde{M}_n(t) - \tilde{Y}'_n(t)| + \kappa \rho \int_0^t |\tilde{V}_n(u)| \, du.
\]

Using Gronwall’s inequality (see Lemma 21.4 in [Kallenberg (2002)]), we obtain

\[
|\tilde{V}_n(t)| \leq |\tilde{V}_n(0) + \tilde{M}_n(t) - \tilde{Y}'_n(t)| \exp(\kappa pt).
\]

By (3.7), (5.14), (5.16), Lemma 4, and the convergence of \( \tilde{Y}'_n \), we have

\[
\lim_{a \to \infty} \limsup_{n \to \infty} \mathbb{P} \left[ \sup_{0 \leq t \leq T} |\tilde{V}_n(0) + \tilde{M}_n(t) - \tilde{Y}'_n(t)| > a \right] = 0,
\]

which implies that \( \{\tilde{V}_n : n \in \mathbb{N}\} \) is stochastically bounded.

Put

\[
\varphi(t) = \begin{cases} 
(t - \bar{w})^{-1}(H(t) - H(\bar{w})) & \text{for } t \neq \bar{w}, \\
f_H(\bar{w}) & \text{for } t = \bar{w}.
\end{cases}
\]

By (5.8) and (5.13),

\[
\sqrt{\frac{n}{\gamma_n}} \int_0^t (H(\tilde{V}_n(u)) - H(\bar{w})) \, du - f_H(\bar{w}) \int_0^t \tilde{V}_n(u) \, du = \int_0^t (\varphi(\tilde{V}_n(u)) - f_H(\bar{w})) \tilde{V}_n(u) \, du.
\]

Because \( \lim_{u \to \bar{w}} \varphi(u) = f_H(\bar{w}) \) and \( \{\tilde{V}_n : n \in \mathbb{N}\} \) is stochastically bounded, it follows from Propo-
sition \( 2 \) that
\[
\sup_{0 \leq t \leq T} \left| \frac{1}{\sqrt{n}} \int_0^t \left( H(V_n(u)) - H(\bar{w}) \right) du - f_H(\bar{w}) \int_0^t \tilde{V}_n(u) du \right| \to 0 \quad \text{as} \quad n \to \infty,
\]
which, along with the convergence of \( \bar{\Sigma}'_n \), leads to the assertion of the lemma.

\( \Box \)

A.3 Proofs of lemmas for Theorem \( 2 \)

Proof of Lemma \( 6 \). Using integration by parts, we obtain
\[
\tilde{G}'_n(t) = \int_t^{t+\bar{w}} \tilde{E}_n(u) dH(t + \bar{w} - u) + \tilde{E}_n(t + \bar{w}) - (1 - H(\bar{w})) \tilde{E}_n(t).
\]
It follows from (3.6) that
\[
\tilde{G}'_n(t) \Rightarrow \int_t^{t+\bar{w}} \hat{E}(u) dH(t + \bar{w} - u) + \hat{E}(t + \bar{w}) - (1 - H(\bar{w})) \hat{E}(t) \quad \text{as} \quad n \to \infty.
\]
Using integration by parts again, we have
\[
\int_t^{t+\bar{w}} \hat{E}(u) dH(t + \bar{w} - u) + \hat{E}(t + \bar{w}) - (1 - H(\bar{w})) \hat{E}(t) = \int_t^{t+\bar{w}} (1 - \hat{E}(t + \bar{w} - u)) d\hat{E}(u).
\]
Then, \( \tilde{G}'_n(t) \Rightarrow \hat{G}'(t) \) as \( n \to \infty \).

Write \( a'_{n,k} = a_{n,k+E_n(\gamma_n t)} \) and \( \zeta'_{n,k} = \zeta_{n,k+E_n(\gamma_n t)} \), which are the arrival and patience times of the \( k \)th customer arriving after \( \gamma_n t \), respectively. Put
\[
\tilde{K}_n(u) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor n \gamma_n u \rfloor} \left( H(\bar{w} - \bar{b}_{n,k}) - 1_{\{\zeta'_{n,k} \leq \bar{w} - \bar{b}_{n,k}\}} \right)
\]
with \( \bar{b}_{n,k} = k/(\rho \mu n \gamma_n) \). Then, \( \tilde{K}_n \) is a martingale with quadratic variation
\[
[\tilde{K}_n](u) = \frac{1}{n \gamma_n} \sum_{k=1}^{\lfloor n \gamma_n u \rfloor} (H(\bar{w} - \bar{b}_{n,k}) - 1_{\{\zeta'_{n,k} \leq \bar{w} - \bar{b}_{n,k}\}})^2.
\]
Because
\[
E \left[ (H(\bar{w} - \bar{b}_{n,k}) - 1_{\{\zeta'_{n,k} \leq \bar{w} - \bar{b}_{n,k}\}})^2 \right] = H(\bar{w} - \bar{b}_{n,k}) (1 - H(\bar{w} - \bar{b}_{n,k})),
\]
by the weak law of large numbers (see, e.g., Theorem 5.14 of Klenke (2014)), we obtain
\[
[\tilde{K}_n](u) \Rightarrow \rho \mu \int_0^{u/\rho \mu} H(\bar{w} - s) (1 - H(\bar{w} - s)) ds.
\]
Since \( |\Delta \tilde{K}_n(u)| \leq 1/\sqrt{n \gamma_n} \) for \( u \geq 0 \), we further have
\[
\lim_{n \to \infty} \sup_{0 \leq u \leq T} |\Delta \tilde{K}_n(u)| = 0 \quad \text{for} \quad T > 0.
\]
Then, it follows from the martingale FCLT that $\tilde{K}_n \Rightarrow \tilde{K}$ as $n \to \infty$, where

$$\tilde{K}(u) = \int_0^{u/\rho u} \sqrt{\rho u H(\bar{w} - s)(1 - H(\bar{w} - s))} \, d\tilde{S}(s).$$

Put

$$\tilde{K}_n'(u) = \frac{1}{\sqrt{n \gamma_n}} \sum_{k=1}^{n \gamma_n u} \left( H(\tilde{W}_n(t) - \bar{a}_{n,k}^t) - 1_{\{\tilde{c}_{n,k} \leq \tilde{W}_n(t) - \bar{a}_{n,k}^t\}} \right).$$

By similar arguments as in the proof of Lemma 4, we can prove that $\tilde{K}_n - \tilde{K}_n' \Rightarrow 0$ as $n \to \infty$.

By (5.15), Lemmas 4 and 5, Proposition 4, and the asymptotic equivalence between $\tilde{V}_n$ and $\tilde{W}_n$, we deduce that $(\tilde{E}_n, \tilde{W}_n) \Rightarrow (\tilde{E}, \tilde{W})$, so that $(\tilde{W}_n, \tilde{G}_n') \Rightarrow (\tilde{W}, \tilde{G}')$ as $n \to \infty$. Because $\tilde{K}_n$ is independent of $(\tilde{W}_n(t), \tilde{G}_n'(t))$, we further obtain

$$(\tilde{W}_n(t) \chi, \tilde{G}_n'(t) \chi, \tilde{K}_n) \Rightarrow (\tilde{W}(t) \chi, \tilde{G}'(t) \chi, \tilde{K}) \quad \text{as} \quad n \to \infty,$$

where $\tilde{W}(t)$, $\tilde{G}(t)$, and $\tilde{K}$ are mutually independent. Because $\tilde{K}_n - \tilde{K}_n' \Rightarrow 0$ as $n \to \infty$ and

$$\tilde{G}_n''(t) = \tilde{K}_n'(\tilde{E}_n(t + \tilde{W}_n(t)) - \tilde{E}_n(t)),$$

the assertion of the lemma follows from (5.9), (5.12), and the random time-change theorem. \qed

**Proof of Lemma 7.** Since $\tilde{E}_n \Rightarrow \tilde{E}$ and $\tilde{E}$ has almost sure continuous sample paths, then by (5.12),

$$\tilde{E}_n(t + \tilde{W}_n(t)) - \tilde{E}_n(t + \bar{w}) \Rightarrow 0 \quad \text{as} \quad n \to \infty.$$

Because

$$\left| \sqrt{n \gamma_n} \int_{t + \bar{W}_n(t)}^{t + \bar{W}_n(t) + u} H(t + \bar{W}_n(t) - u) \, d\tilde{E}_n(u) \right| \leq H(|\bar{W}_n(t) - \bar{w}|) |\bar{W}_n(t) (\tilde{E}_n(t + \bar{W}_n(t)) - \tilde{E}_n(t + \bar{w}))|,$$

it follows from (5.9), (5.12), and Theorem 1 that

$$\sqrt{n \gamma_n} \int_{t + \bar{W}_n(t)}^{t + \bar{W}_n(t) + u} H(t + \bar{W}_n(t) - u) \, d\tilde{E}_n(u) \Rightarrow 0 \quad \text{as} \quad n \to \infty.$$

Write

$$\varphi(u, \delta) = \begin{cases} \delta^{-1}(H(u + \delta) - H(u)) & \text{for} \ \delta \neq 0, \\ f_H(u) & \text{for} \ \delta = 0. \end{cases}$$

Then, $\lim_{\delta \to 0} \varphi(u, \delta) = f_H(u)$ for $u \geq 0$. Using the fact that $H(\bar{w}) = (\rho - 1)/\rho$, we obtain

$$(\rho - 1) \mu \bar{W}_n(t) - \sqrt{n \gamma_n \rho u} \int_0^{\bar{w}} (H(\bar{W}_n(t) - u) - H(\bar{w} - u)) \, du$$

$$= \rho \mu \bar{W}_n(t) \int_0^{\bar{w}} (f_H(\bar{w} - u) - \varphi(\bar{w} - u, \bar{W}_n(t) - \bar{w})) \, du.$$ 

By (3.2), we have $|f_H(\bar{w} - u) - \varphi(\bar{w} - u, \bar{W}_n(t) - \bar{w})| \leq 2\kappa$. It follows from (5.12), Theorem 1, and
the dominated convergence theorem that

\[(\rho - 1)\mu \tilde{W}_n(t) - \sqrt{n}n\mu \int_0^{\tilde{\omega}} (H(\tilde{W}_n(t) - u) - H(\bar{w} - u)) \, du \to 0 \quad \text{as } n \to \infty.\]

Hence, \(\tilde{Y}_n(t) \to 0\) as \(n \to \infty\).

Put \(\tilde{H}_n(u) = H(t + \tilde{w} - u) - H(t + \tilde{W}_n(t) - u)\) for \(u \geq 0\). By (5.12) and Theorem 3.9 in [Billingsley (1999)], we obtain

\[(\tilde{E}_n, \tilde{H}_n) \Rightarrow (\hat{E}, 0) \quad \text{as } n \to \infty.\]

By similar arguments as in the proof of Lemma 5, we can show that \(\{\tilde{H}_n : n \in \mathbb{N}\}\) is uniformly tight. Using integration by parts, we obtain

\[\tilde{Y}_n'(t) = \int_t^{t+\tilde{\omega}} \tilde{E}_n(u) \, d\tilde{H}_n(u) + \tilde{H}_n(t + \tilde{w})\tilde{E}_n(t + \tilde{w}) - \tilde{H}_n(t)\tilde{E}_n(t).\]

Then, it follows from Theorem 7.10 in [Kurtz and Protter (1996)] that

\[\int_t^{t+\tilde{\omega}} \tilde{E}_n(u) \, d\tilde{H}_n(u) \to 0 \quad \text{as } n \to \infty,\]

which, along with (3.6), implies that \(\tilde{Y}_n'(t) \to 0\) as \(n \to \infty\). The joint convergence follows from Theorem 3.9 in [Billingsley (1999)].

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