Spin–orbit coupling induced displacement and hidden spin textures in spin-1 Bose–Einstein condensates

Shu-Wei Song1, Yi-Cai Zhang1, Lin Wen1 and Hanquan Wang2

1 Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, People’s Republic of China
2 School of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan Province 650221, People’s Republic of China

E-mail: ssw@iphy.ac.cn

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Abstract
We analytically and numerically investigate the ground state of spin–orbit coupled spin-1 Bose–Einstein condensates in an external parabolic potential. When the spin–orbit coupling is introduced, spatial displacement exists between the atom densities of components with different magnetic quantum numbers. The analytical calculations show this displacement reaches a maximum when the spin–orbit coupling strength is comparable with that of the trapping potential. As the spin–orbit coupling strength gets larger and larger, the spatial displacement decreases at a rate inversely proportional to the spin–orbit coupling strength. Correspondingly, periphery half-skyrmion textures arise; this displacement can be reflected by the non-uniform magnetic moment in the $z$ direction. With the manipulation of the external trap, the local magnitude of the non-uniform magnetic moment can be increased evidently. This kind of increase of the local magnetic moment is also observed in the square vortex lattice phase of the condensate.

(Some figures may appear in colour only in the online journal)

1. Introduction

Spin–orbit (SO) coupling is an essential mechanism for most spintronics devices and leads to many fundamental phenomena in condensed matter physics and atomic physics. For example, SO coupling gives rise to the quantum spin Hall effect in electronic condensed matter systems [1]. Although the spinor Bose–Einstein condensates (BECs) without SO coupling have been explored extensively [2–16], the investigation of their SO coupled counterparts is now in full swing. Recently, artificial external Abelian or non-Abelian gauge potentials coupled to neutral atoms have been generated by controlling atom–light interaction [17–20], which provides the possibility of spin control in neutral atom systems through laser fields. In the presence of artificial Abelian or non-Abelian gauge potentials, neutral atoms behave like electrons in an electro magnetic field or electrons with SO coupling and abundant phenomena arise.

Thus, SO coupled cold atom systems have attracted intense attention recently [21–40].

For example, the condensate may become self-trapped as a result of the SO coupling and the nonlinearity, resembling the so-called chiral confinement [33]. Dynamical oscillations of the condensate with SO coupling are studied in [34], where an oscillation period, similar to the Zitterbewegung oscillation, was found. Besides, SO coupled BECs with dipole–dipole interactions [36] and a rotating trap [37, 38], have also been studied. In [39], SO coupled atomic spin-2 BEC was investigated and square or triangular density patterns were reported. Both patterns evolve continuously into striped forms with increased asymmetry of the SO coupling.

The SO coupled spinor condensate develops a spontaneous plane wave phase or stripe phase because of the interaction energy [41–43]. In this case, the harmonic length of the external trap is much greater than the wavelength of the stripe, indicating a weak external trap compared to the
SO coupling strength. In two-component condensates, the phase diagram including half-quantum vortex, vortex lattice and stripe states was obtained in [23]. For the case with strong external traps and strong SO coupling, a half-quantum vortex state and skyrmion lattice patterns can arise [24], where the atomic interaction is relatively weak. However, spin-1 BECs in the presence of an external trap with neither a too strong nor too weak SO coupling strength, i.e., the dimensionless SO coupling strength $\kappa \sim 1$, have not been studied enough. The atom–atom interaction needs to be large enough to produce a trapped plane wave or stripe state in an external trap. In the trapped plane wave or stripe state, nontrivial detailed structures exist, as we will see in the following sections.

In this paper, we both analytically and numerically investigate certain characteristics of the spin-1 BECs with a SO coupling strength comparable to the external parabolic potential. Compared with the homogeneous case, we find that the ground state wavefunctions of different components of the condensate dislocate and spatial displacement exists between the atom densities of components with different magnetic quantum numbers. Actually, the calculations show that a smaller displacement would also occur in SO coupled two-component condensates. For the feasibility of experimental observation, we study this phenomena in SO coupled spin-1 condensates. This spatial displacement can be reflected by the non-uniform magnetic moment distribution in the $z$ direction. In particular, we find associated hidden half-skyrmion spin textures in the usual plane wave and stripe phases. The ‘skyrmion’ is named after the nuclear physicist Tony Skyrme, who studied a certain nonlinear field theory for interacting pions; he showed quantized and topologically stable field configurations as solutions of such field theories [44]. A skyrmion is a topological field configuration which defines a nontrivial surjective mapping from real space to an order parameter space with a nontrivial topology [45]. The half-skyrmion was originally hypothesized as the half-instanton in particle physics [46] and later studied in quantum hall systems [47]. It carries half a unit of topological charge; they have been created in the spinor BEC with a constraining magnetic field [48]. Our numerical calculations show that the displacement in the trapped stripe state is more complex. By applying an anisotropic external potential, the trapped stripe state develops a spin texture characterized by aligned half-skyrmions.

The rest of the paper is organized as follows. In section 2, we propose a variational approach to study the displacement of density distributions between different components in the plane wave phase. In section 3, we numerically confirm the analytical results obtained in section 2 and exhibit the hidden spin texture in the plane wave phase. Section 4 focuses on the hidden spin textures in the stripe phase. Finally, the conclusions are summarized in section 5.

2. Theoretical framework

We consider the spin-1 BECs with Rashba SO coupling in the $xy$ plane confined in an external parabolic potential. The model Hamiltonian in the second quantized form is given by

$$H = \hat{H}_0 + \hat{H}_P + \hat{H}_{\text{int}}$$

where

$$\hat{H}_0 = \int \text{d} r \Psi^\dagger \left[ \frac{1}{2M} \left( \hat{\mathbf{p}}^2 + 2\gamma (\hat{\mathbf{p}}_x S_z + \hat{\mathbf{p}}_z S_x) \right) \right] \Psi,$$

$$\hat{H}_P = \int \text{d} r \Psi \tilde{V}(r) \Psi,$$

$$\hat{H}_{\text{int}} = \int \text{d} r \left[ \frac{c_0}{2} \psi_d^\dagger \psi_d \psi_a^\dagger \psi_a + \frac{c_2}{2} \psi_d^\dagger \psi_d \mathbf{F}_{ab} \mathbf{F}_{ab} \right].$$

The spin-independent and spin-dependent interactions are $\gamma$ and $\mathbf{F} = (F_x, F_y, F_z)$, respectively; $\psi_d$ and $\psi_a$ are the spin-1 matrices. The spin-independent and spin-dependent interactions are denoted as $c_0 = 4\pi\hbar^2 (a_0 + 2a_2)/3M$ and $c_2 = 4\pi\hbar^2 (a_2 - a_0)/3$, where $a_0$ and $a_2$ are the s-wave scattering lengths corresponding to the total spin of the two colliding bosons 0 and 2, respectively. The SO coupling strength $\gamma$ is experimentally related to the wavelength of the laser beams and exact experimental setups. As proposed in [50], $\gamma = 2\pi\hbar \sin(\theta/2)/\lambda$, where $\theta$ and $\lambda$ are the angle between two Raman beams and the wavelength of the laser, respectively. In reference [51], the tetrapod setup used to generate the Rashba SO coupled BEC requires $\gamma = \sqrt{2\pi}\hbar/\lambda$.

In order to gain some intuition, we first focus on the homogeneous condensate. Thus, the single-particle spectrum of the Hamiltonian (1) has the following three branches with different helicity in the momentum space (see figure 1):

$$E_0 = \frac{p^2}{2M}, \quad E_{\pm 1} = \frac{p^2 \pm 2\sqrt{2}\gamma |p|}{2M}.$$ (4)

The single-particle ground state is in the negative helicity branch with $|p| = \sqrt{2}\gamma$; the wavefunction is given by

$$u_- = \begin{pmatrix} -\frac{1}{\sqrt{2}} e^{-ip_y} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$ (5)
that all the single-particle eigenstates with the same variational method. The energy spectrum in figure 1 shows precise outcomes of the SO coupled condensate. Hence, we emphasize that the numerical results are more be qualitatively captured using even this crude approximation.

density distributions of different components is sturdy and can to proceed analytically. Precisely speaking, the Gaussian-profile density distribution with \(|\psi_1^*\psi_0 + \psi_0^*\psi_1|\) is plotted in figure 3. From figure 3(a) with \(\kappa = 2.0\), we find that the displacement does not change dramatically with respect to \(\eta\) and chooses values 0.2 ~ 0.35. More calculations show that the displacement function

\[
\begin{align*}
H &= \int \text{d}r \left\{ \sum_{m=0,\pm 1} \left[ -\frac{1}{2} \psi_m^* \nabla^2 \psi_m + \frac{1}{2} (x^2 + y^2) |\psi_m|^2 \right] + \kappa \left[ \psi_1^*(-i\partial_\delta - \partial_\eta) \psi_1 + \psi_1^*(-i\partial_\delta + \partial_\eta) \psi_1 + \psi_0^*(-i\partial_\delta - \partial_\eta) \psi_0 + \psi_0^*(-i\partial_\delta + \partial_\eta) \psi_0 \right] \\
&\quad + \frac{\alpha_0}{2} (|\psi_1|^2 + |\psi_0|^2 + |\psi_{-1}|^2) \\
&\quad + \frac{\alpha_2}{2} (|\psi_1|^2 - |\psi_{-1}|^2)^2 + 2|\psi_1^*\psi_0 + \psi_0^*\psi_{-1}|^2 \right\}
\end{align*}
\]  
(6)

where \(\alpha_0 = \frac{NM}{\sqrt{\hbar \omega_0 c_0}}\), \(\alpha_2 = \frac{NM}{\sqrt{\hbar \omega_0 c_2}}\) (\(N\) is the total atom number) and \(\kappa = \frac{1}{\sqrt{\hbar \omega_0 \tau}}\). The length, time and energy are respectively scaled in units of \(a_\perp = \sqrt{\hbar / M \omega_0}\), \(\tau = 1 / \omega_0\) and \(\hbar \omega_0\).

In order to explore the characteristics of the displacement, we approximate the density distribution by the Gaussian function; such an approximation would make it easier to proceed analytically. Precisely speaking, the Gaussian distribution approximation is crude because of the interaction, so the Thomas–Fermi distribution would be a better approximation. However, the displacement between the density distributions of different components is sturdy and can be qualitatively captured using even this crude approximation. Hence, we emphasize that the numerical results are more precise outcomes of the SO coupled condensate.

To explore the spatial displacement effect, we use the variational method. The energy spectrum in figure 1 shows that all the single-particle eigenstates with the same \(|p|\) are circularly degenerate, regardless of the direction of the plane wave vector [42]. The numerical results show that the plane wave vector chooses the diagonal direction because the square computational domain with zero boundary is used in the numerical scheme. So without generality, we can suppose a plane wave propagating along the \(x\) axis and use the ansatz:

\[
\begin{pmatrix}
\psi_1 \\
\psi_0 \\
\psi_{-1}
\end{pmatrix} = C \exp(i k_0 x) \begin{pmatrix}
-\frac{1}{2} \exp(-k_0^2 \eta (x^2 + (y - \delta)^2)) \\
\frac{1}{2} \exp(-k_0^2 \eta (x^2 + (y - \delta)^2)) \\
\frac{1}{2} \exp(-k_0^2 \eta (x^2 + (y + \delta)^2))
\end{pmatrix},
\]

(7)

where \(\eta\) is a positive dimensionless variational parameter related to the width of the condensate, \(C = 2 \frac{\sqrt{\kappa}}{2\pi b_{\text{qD}}}\) is the normalization coefficient and \(k_0 = \sqrt{\kappa} \tau\) is the dimensionless wave vector. Here we introduce the displacement \(\delta_0\) from the origin in the \(y\) direction. A displacement in the \(x\) direction can also be supposed and this displacement is found to be zero to minimize the total mean-field energy. Substituting the variational wavefunction equation (7) into the mean-field Hamiltonian, the ground state can be obtained by solving the equation set composed of \(\partial H / \partial \delta_0 = 0 \ (m = 0, \pm 1)\).

Without the SO coupling term in the Hamiltonian, the condensate feels comfortable when the distribution centres of the three components occupy the centre of the external parabolic potential. It is not the same situation once the SO coupling is included. In the presence of the SO coupling, the Gaussian-profile density distribution with \(\delta_1 = \delta_0 = \delta_{-1} = 0\) is not the ground state. We can introduce the effective virtual force to determine the deviating direction of the Gaussian-profile density of each component. We define the effective virtual force: \(f_1 = -\partial H_s / \partial \delta_0 \big|_{\delta_0=0} \ (i = 0, \pm 1)\). After substituting the ansatz in equation (7) into the SO part of the total energy, we have

\[
f_1 = \sqrt{2} k_0^2 \eta, \quad f_0 = 0, \quad f_{-1} = -\sqrt{2} k_0^2 \eta.
\]

(8)

Since \(\eta\) and \(\kappa\) are positive parameters, the directions of the dragging forces have been fixed. The component with magnetic quantum number \(m = 1\) feels a dragging force pointing to the left side of the propagating plane wave direction and the component with magnetic quantum number \(m = -1\) feels the same force but in the opposite direction, while no force acts on the components with the magnetic quantum number \(m = 0\). Based on these effective virtual forces, we suppose \(\delta_1 = \delta_{-1} = \delta, \delta_0 = 0\). After substituting the ansatz in equation (7) into the SO part of equation (6), we have

\[
H_s = -\kappa \exp\left(-1/2 \eta k_0^2 \delta^2\right) \sqrt{2} \left( k_0 + \eta k_0^2 \delta \right).
\]

(9)

Starting from \(H_s\), we find that the minimum functional energy requires the condition:

\[
\delta = -1 + \sqrt{1 + 4 \eta} \over 2 \sqrt{2} \kappa > 0.
\]

(10)

In figure 2, the SO part of the total mean-field energy with respect to \(\delta_1 (\delta_1 = \delta)\) and \(\delta_{-1} (\delta_{-1} = -\delta)\) is plotted. The light-red balls, which deviate from the equilibrium position, feel forces as expressed in equation (8). We know that spin currents accompany charge currents in high-mobility two-dimensional electron systems [52]; thus spin accumulation will be induced with the presence of the SO coupling. Starting from the single-particle Hamiltonian in equation (1), we can obtain the force felt by the components of the condensate in the Heisenberg picture by following the calculations in [53]:

\[
f = 2y^2 / (\hbar M^2) (\hat{p} \times \hat{e}_y) F_z.
\]

(11)

Here we should mention that the above effective virtual forces introduced in equation (8) come from small deviations from the ground state (equilibrium position), which is not the same as the true force of the particles during dynamical motions, as expressed in equation (11).

The dependence of \(\delta\) on \(\eta\) and \(\kappa\) is plotted in figure 3. From figure 3(a) with \(\kappa = 2.0\), we find that the displacement does not change dramatically with respect to \(\eta\) and chooses values 0.2 ~ 0.35. More calculations show that the displacement function
of the SO coupling strength is $\delta = 1/\sqrt{2} \kappa$ as $\eta \to 0$ (meaning an unlimited condensate width corresponding to the case of zero external potential). This limit behaviour is not affected by the interactions between particles. Figure 3(b) shows the dependence of $\delta$ on $\kappa$ with $\eta = 0.0096$, which is consistent with the size of the condensate obtained numerically. For the black dotted line, only the SO part of the mean-field energy is minimized. As $\kappa$ increases, i.e., is approaching the strong SO coupling regime, the displacement diminishes at a rate proportional to $1/\kappa$. From figure 3(b), we can see from the black dotted line that the displacement diverges as the SO coupling strength gets weaker and weaker ($\kappa \to 0$), which is nonphysical. This divergence can be cancelled by counting in the external potential, as shown by the solid red line of figure 3(b).

To minimize the mean-field energy, there needs to be a displacement of the density distribution between different components of the condensate, resulting in a polarized magnetic momentum distribution ($\mathcal{F}_z$). It is necessary to notice that the polarized magnetic momentum distribution has nothing to do with the spin-dependent force expressed in equation (11). Through minimizing the mean-field energy, the polarized magnetic momentum distribution in the plane wave phase is the consequence of the cooperation of all the components of the condensate in the presence of the SO coupling and the external trap.

For the stripe phase of the condensate, the matter wave is composed of two coherent wave vector states that propagate face to face along the $x$ axis. We can choose the ansatz:

\[
\begin{pmatrix}
\psi_1 \\
\psi_0 \\
\psi_{-1}
\end{pmatrix}
= C \begin{pmatrix}
-\frac{i}{2} \exp(-k_0^2 \eta (x^2 + (y + \delta)^2)) (\exp(i k_0 x) - \exp(-i k_0 x)) \\
\frac{\sqrt{2}}{k_0^2 \eta} \exp(-k_0^2 \eta (x^2 + y^2)) (\exp(i k_0 x) + \exp(-i k_0 x)) \\
-\frac{i}{2} \exp(-k_0^2 \eta (x^2 + (y - \delta)^2)) (\exp(i k_0 x) - \exp(-i k_0 x))
\end{pmatrix}
\]

where $C = k_0 \sqrt{\eta/\pi}$ is the normalization coefficient. Starting from the SO part of equation (6), a displacement formula similar to equation (10) can be found:

\[
\sqrt{2} k_0^3 \eta \delta \exp(-1/2 k_0^2 \eta \delta^2) = 0.
\]

Equation (13) requires $\delta = 0$, i.e., the displacement which occurs in the plane wave phase does not reside in the stripe phase. However, as we point out in section 4, there is actually a small displacement in the stripe phase due to the atomic interaction. As the external potential becomes more and more anisotropic, this displacement manifests itself more clearly.
3. Numerical results in the plane wave phase and explorations by manipulation of the external potential

In this section we numerically investigate the displacement pointed out in section 2. Corresponding to the mean-field Hamiltonian (6), we obtain the ground state using the imaginary-time evolution method.

As shown in the previous analytical calculations, the displacement $\delta$ is small; it is usually difficult to identify the existence of this kind of displacement by only keeping eyes on the density distribution. In figure 4(a), we show the plane wave phase of the spin-1 condensate. Theoretical explorations show that the plane wave phase can be realized in the $^{87}$Rb condensate because of the spin-exchange interaction coefficient $c_2 < 0$ [42].

Displacements in the anisotropic external potential are more complex for the stripe phase and we need to consider an additional degree of freedom to describe the displacement, as we will introduce in section 4.

Figure 4. (a) The density (upper panels) and phase (bottom panels) distributions of the wavefunction for the $^{87}$Rb condensate in the plane wave phase with $\kappa = 2.0$ and total atom number $N = 1 \times 10^4$. (b) The non-uniform density of the magnetic moment along the $z$ direction, $F_x$, in the plane wave phase. The units of the space coordinates and $F_x$ are $a_\perp = \sqrt{\hbar/M\omega}$ and $\hbar$, respectively.
By resorting to the density of the magnetic moment along the $z$ direction:
\[
\mathcal{F}_z = |\psi_1|^2 - |\psi_{-1}|^2,
\]
we can specify this kind of displacement. If the density distribution of particles in component $\psi_1$ coincides with that in component $\psi_{-1}$, a uniform $\mathcal{F}_z$ would be expected. Otherwise, non-uniform $\mathcal{F}_z$ appear, indicating the displacement of particle density distributions in components $\psi_1$ and $\psi_{-1}$. We show the distribution of $\mathcal{F}_z$ in figure 4(b). From figure 4(b), it is clearly shown that $\mathcal{F}_z$ chooses positive values in the top right corner of the panel, i.e., the density distribution of the $\psi_1$ component has been dragged along the direction $(\hat{x} + \hat{y})/\sqrt{2}$ ($\hat{x}$ and $\hat{y}$ are unit vectors along axes $x$ and $y$, respectively). The phase distribution of the wavefunction in figure 4(a) shows that the plane wave propagates along the direction $(\hat{x} - \hat{y})/\sqrt{2}$. The numerical results agree with the analytical calculations in section 2.

As the small displacement can be reflected by the polarized magnetic moment distribution, possible ways to magnify the local magnetic moment would be useful. We implement an anisotropic external trapping potential in the $xy$ plane and show the response of the local magnetic moment to this anisotropism. In figure 5(a), we show the variation of the peak value of $\mathcal{F}_z$ with respect to the $\omega_z/\omega$, where $\omega_z$ is the trapping frequency in the squeezing direction in the $xy$ plane. As $\omega_z/\omega$ increases, the local magnetic moment gets larger and larger, which is helpful to assure the small displacement in the experiments. For the square lattice structure of the condensate, half-skyrmion lattices would appear; the squeezing of the external potential takes up the role of magnifying the local $z$ component of the magnetic moment, which is similar to that in the plane wave phase (figure 5(b)). We should mention that as $\omega_z/\omega$ gets larger, the condensate crosses from the square lattice phase to the plane wave phase. As the trapping frequency in the squeezing direction gets larger and larger, the spatial density displacement is more and more evident; thus the in situ imaging of the BEC (non-destructive imaging) should be favoured to identify this displacement.

Corresponding to the displacement, half-skyrmions appear in the periphery of the condensate. For the origin of the skyrmion spin texture, it would be better to start with the single-particle energy spectrum. As the SO coupling strength gets larger and larger, the single-particle energy spectrum is inclined to expand into a form similar to the Landau levels. The interaction causes the eigen-functions with the different quantum number $m$ to be occupied; the superposition of all these eigen-functions would result in skyrmion spin texture states as found in [24]. As the interaction gets larger, the half-skyrmions will be expelled to the periphery of the condensate.

4. Hidden spin textures in the stripe phase

Through analytical and numerical calculations, we know that there is spatial displacement between the atom densities of components $\psi_1$ and $\psi_{-1}$ for the plane wave phase. This displacement occurs in the direction perpendicular to the propagating direction of the plane wave. In this section, we focus on the stripe phases. The stripe phase is predicted to exist in the spinor condensate of $^{23}$Na with the spin-exchange interaction $\kappa > 0$ [42].

In the following, we take the parameters $\kappa = 2.0$ indicating that the SO coupling strength is comparable to the strength of the parabolic trap. The density and phase distributions of component $\psi_1$ are plotted in figure 6(a). Figure 6(b) shows the spin texture of the stripe phase. The hidden half-skyrmions can be clearly seen in the periphery of the condensate. The spin texture of a pair of half-skyrmions is shown in the bottom of figure 6(b). By applying external manipulation on the condensate, i.e., by squeezing the condensate in a certain direction, the spin textures can manifest themselves more clearly, which is shown by the panels in the bottom of figure 7. The colour on the arrows represents the local magnetic moment in the $z$ direction (the moduli of the magnetic moment have been normalized to unity). As the colour changes from blue to dull-red, the relative atom number changes between components $\psi_1$ and $\psi_{-1}$. A line of
Figure 6. (a) The density and phase distributions of $\psi_1$ in the stripe phase and (b) the corresponding spin texture, showing the hidden half-skyrmions in the periphery of the $^{23}$Na condensate with SO coupling strength $\kappa = 2.0$ and total atom number $N = 1 \times 10^4$. The spin texture in the bottom is the magnification of the squared part in the top panel.

Figure 7. (a) The top panel shows the phase distribution of the $\psi_1$ component in the $^{23}$Na condensate obtained by squeezing the condensate with $\omega_\perp/\omega = 8$, where $\omega_\perp$ is the trapping frequency in the squeezing direction. The SO coupling strength is $\kappa = 2.0$ and the total atom number is $N = 5 \times 10^4$. (b) The panels in the bottom are the corresponding spin texture. The spin texture on the right is the magnification of the squared area in the left panel. The unit of the space coordinates is $a_\perp = \sqrt{\hbar/M\omega}$. 
paired half-skyrmions can be seen. The panel on the right is the magnification of the squared area in the panel on the left.

From the spin texture in figure 7, there seems to be a small displacement between components $\psi_1$ and $\psi_{-1}$ because of the evident colour contrast in the paired half-skyrmions line.

In order to understand the numerical results in figure 7 qualitatively, we introduce the following ansatz:

$$
\psi_1 = \frac{1}{2} C \exp(-i k_0 x) \exp(-k_0^2 \eta (x^2 + R(y - y_0 + \delta)^2)) + \exp(-i k_0 x) \exp(-k_0^2 \eta (x^2 + R(y - y_0 - \delta)^2)),$$

$$
\psi_0 = \frac{\sqrt{2}}{2} C \exp(-k_0^2 \eta (x^2 + R y^2)) \exp(i k_0 x) + \exp(-i k_0 x),
$$

$$
\psi_{-1} = \frac{1}{2} C \exp(-i k_0 x) \exp(-k_0^2 \eta (x^2 + R(y + y_0 + \delta)^2)) + \exp(-i k_0 x) \exp(-k_0^2 \eta (x^2 + R(y + y_0 - \delta)^2)).
$$

(15)

where, $C = k_0 \sqrt{\eta}$ is the normalization coefficient and $R$ fixes the ratio of the size scales of the condensate in the $x$ and $y$ directions. Repeating the same calculations as in section 2, we can obtain the dependence of the energy on parameters $\delta$ and $y_0$. Here $\delta$ represents the displacement between the two Gaussian packets bearing counter-propagating plane waves within the same component and $y_0$ characterizes the average displacement of packets in $m = 1$ and $m = -1$ components.

From figure 8(a), where the parabolic potential and SO parts of the energy are plotted as a function of $\delta$ and $y_0$, we can see that the energetic minimum is shifted in the negative direction of the $\delta$ axis (marked by the light-red ball), while there is no shift in the $y_0$ axis. By counting in the non-zero $\delta$ in the ansatz (15), the phase distribution of the wavefunction shows the same configuration with that of the numerical result in figure 7. In reference [55], a variational ansatz representing a modified striped state was proposed. The phase distribution in this paper is consistent with the particle current found in [55].

Thus far, the roles of the interaction have not been included in the analytical calculations. In figure 8(b), we plot the total Hamiltonian energy $H$ as a function of $\delta$ and $y_0$. There are two energetic minimums (marked by light-red balls) around the centre. Therefore, the role of the interaction is to drag the energetic minimum off the centre ($\delta = y_0 = 0$). The colour distribution (blue to dull-red) in figure 7(b), which represents the non-uniform magnetic momentum component $f_z$, indicates that a non-zero $y_0$ has been chosen. Thus, we can conclude that displacement occurs between the two Gaussian packets of the two components bearing the co-propagating plane waves, which is similar to that in the case of the trapped plane wave phase. This displacement is reflected by the non-zero $\delta$. Because of the interaction between atoms, a smaller overlapping density distribution between the Gaussian packets bears the non-zero $y_0$ is favourable, indicating that the ground state favours the approximate wavefunction in (15) with non-zero $y_0$.

We finally consider some aspects of the experimental parameters. In recent experiments, $^{87}$Rb atoms are used to realize SO coupled BECs [17–19, 56]. By referring to the experimental setups in [17, 18, 50], the dimensionless SO coupling strength $\kappa = \frac{\eta}{\M} = \frac{2\hbar}{\M} \sin(\theta/2)$. It is crucial to choose values of the wavelength of the laser beam $\lambda$ and the angle $\theta$ between two Raman beams to achieve $\kappa \sim 2$, where the density spatial displacement reaches a maximum according to our analytical calculation. For trapping frequencies $\omega_x = \omega_y = \omega = (2\pi) \times 62$ Hz and $\omega_z = (2\pi) \times 688$ Hz, the angle between two Raman beams is $\theta \sim 21.28^\circ$ with the laser wavelength $\lambda \sim 800$ nm. As proposed in [54], the polarization-dependent phase-contrast imaging can possibly be used to identify non-uniform magnetic momentum distributions accompanying the density spatial displacement in this paper. In addition, the spatial displacement between the atom densities of components remains when the radial trapping frequency varies in a wide experimental parameter span. In figure 9, we show the non-uniform density of the magnetic moment in the $^{87}$Rb condensate with $\omega = (2\pi) \times 1$ Hz, $(2\pi) \times 10$ Hz, $(2\pi) \times 40$ Hz, $(2\pi) \times 80$ Hz and $(2\pi) \times 120$ Hz ($\omega_z = 10\omega$). Thus, the spatial density displacement and the corresponding spin textures exist in a large parameter range of the experimental trapping frequencies. It should be noticed that the minimization of the spin-dependent interaction energy...
with an increasing $c_2 (c_2 > 0)$ tends to erase the local spin magnetic momentum, even though there is nontrivial spin textures in the stripe phase of the condensate. The competition between the SO coupling and the spin-dependent interaction will determine the final spin texture. However, this would not happen in the plane wave phase of the condensate for the spin-dependent interaction parameter $c_2 < 0$.

5. Conclusions

In summary, we have analytically and numerically studied the spatial displacement between the atom densities of components with different magnetic quantum numbers in SO coupled spin-1 BECs with external potential, which results in the hidden spin textures in the trapped plane wave and stripe phases. The polarized magnetic moment distribution in the $z$ direction, $F_z$, can mark the existence of the displacement. Manipulation of the external potential can be utilized to magnify the local $F_z$ in both the plane wave and the square lattice phases. Although $F_z$ in the stripe phase is small, nontrivial spin texture dwells in the condensate. As the squeezing of the external trap gets harder and harder, the hidden half-skyrmions align themselves, forming into a paired half-skyrmion line. With the mean-field energy minimized, the displacement that we obtained is the consequence of the cooperation of all the components of the condensate in the presence of the SO coupling and the external trap. Thus, for a spinor condensate with a SO coupling strength that is comparable to the external trap, the usual plane wave phase or stripe phase are revised, bringing nontrivial detail structures. This paper should help us to understand the cooperative roles of the SO coupling and the external potential in determining the ground state of the spinor condensates.

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Figure 9. The non-uniform distributions of the magnetic moment along the $z$ direction with different radial trapping frequencies $\omega$ ($\omega = \omega_x = \omega_z$) in $^{87}$Rb condensate with a total atom number $N = 1 \times 10^4$ and dimensionless SO coupling strength $\kappa = 2.0$. The unit of the space coordinates is $a_\perp = \sqrt{\hbar/M\omega}$. 

$\omega = (2\pi) \times 100Hz$ $\omega = (2\pi) \times 10Hz$ $\omega = (2\pi) \times 120Hz$ $\omega = (2\pi) \times 80Hz$ $\omega = (2\pi) \times 40Hz$ $\omega = (2\pi) \times 1Hz$
