Abstract

A defender dispatches patrollers to circumambulate a perimeter to guard against potential attacks. The defender decides on the time points to dispatch patrollers and each patroller’s direction and speed, as long as the long-run rate patrollers are dispatched is capped by some constant. An attack at any point on the perimeter requires the same fixed amount of time, during which it will be detected by each passing patroller independently with the same probability. The defender wants to maximize the probability of detecting an attack before it completes, while the attacker wants to minimize it. We study two scenarios, depending on whether the patrollers are undercover or wear a uniform. Conventional wisdom would suggest that the attacker gains advantage if he can see the patrollers going by so as to learn about patrol pattern and time his attack, but we show that the optimal detection probability remains the same in both scenarios.

Keywords: Search and surveillance; Games/group decisions; Probability.

1 Introduction

Consider a two-person zero-sum game played by a defender and an attacker. The defender is tasked to secure the perimeter of an area of interest by dispatching patrollers from a base located on the perimeter. Each patroller departs from the base, circumambulates the perimeter either clockwise or counterclockwise at possibly variable speed, and returns to the base.
base to complete a full circle. The defender decides on the patrol schedule—the time points at which patrollers depart from the base, the moving direction of each patroller, and each patroller’s speed—as long as the long-run rate at which patrollers depart from the base is capped by a constant $\lambda > 0$. For example, the defender can dispatch patrollers according to a Poisson process at rate $\lambda$, or at fixed intervals of $1/\lambda$ time units, and ask each patroller to go in the same clockwise direction at the same constant speed.

The attacker seeks to carry out an attack on the perimeter—such as penetrating the perimeter or planting a hidden bomb. An attack takes a fixed amount of time $t > 0$, during which the attack will be detected by each passing patroller independently with probability $p \in (0, 1]$. The attacker can choose when to arrive at the perimeter, which point on the perimeter to carry out an attack, and how long to wait at the attack point before initiating the attack. The attacker’s objective is to minimize the probability of getting detected by any patroller during his attack, while the defender’s objective is the opposite.

We consider two scenarios. In the first scenario, the patrollers are invisible to the attacker. Examples include high-altitude unmanned aerial vehicles and undercover patrol agents. In the second scenario, the attacker can hide and see the patrollers going by so as to learn about the patrol pattern and time his attack. Examples include ground robots, low-altitude drones, and soldiers in uniform. Conventional wisdom would suggest that the attacker can gain advantage by learning about the patrol pattern in real time and use that knowledge to decide when to initiate an attack. Somewhat surprisingly, we show that the defender can achieve the same detection probability in both scenarios.

Patrol problems arise in many real-life situations. Police officers patrol cities and highways; security guards patrol museums and airport lobbies; soldiers patrol military bases and borders. Earlier works focus on allocating police patrol resources among different areas to maximize the overall performance (Chaiken and Dormont, 1978; Chelst, 1978; Larson, 1972; Olson and Wright, 1975; Szechtman et al., 2008; Portugal and Rocha, 2013). Besides police patrol in urban areas, there are specialized patrol models for rural areas (Birge and Pollock, 1989) and on highways (Lee et al., 1979; Taylor et al., 1985). These earlier works assume that the frequencies of crimes at different locations remain constant and are known to the patrol force.

In recent years, there has been a growing interest in taking a game-theoretic approach to study patrol problems. Alpern et al. (2011) study a patrolling game played on a graph, where
Patrollers that are visible to the attacker when near are often considered in the context of robot patrol. Basilico et al. (2009, 2012) study a graph patrolling game in which the intruder sees the patroller and uses that knowledge to decide when and where to initiate an attack. Agmon et al. (2008a,b) place multiple robots on a perimeter and randomize their movements so that the intruder cannot predict the robots’ locations with certainty. Zoroa et al. (2012) also study perimeter patrol, but in their model, the intruder can move along the perimeter during an attack while the patroller can choose any point on the perimeter to inspect in each time period. The vast majority of earlier patrol models discretize time and space for mathematical tractability. In this paper, we treat time and space as continuum to improve model realism.

2 Main Results

To facilitate discussion, we first define and solve a patrol game in which each patroller is required to circumambulate the perimeter in the same clockwise direction at the same constant speed, and can be seen when going by a hiding attacker. We later explain why the value of the game remains the same if we relax these assumptions.

Consider a patrol game between a defender and an attacker. The defender dispatches patrollers from a base located on a perimeter subject to a long-run rate \( \lambda > 0 \). Each patroller circumambulates the perimeter in the same direction at the same constant speed before returning to the base. For this reason, it is inconsequential which point on the perimeter the attacker chooses to attack, because the interarrival times between two consecutive patrollers at any point on the perimeter is dictated by the difference between their departure times from the base. Upon arrival at an arbitrary point on the perimeter, the attacker hides and waits there—for as long as he wants—so as to decide when to initiate an attack. While hiding,
the attacker can observe each patroller going by in real time, but cannot foretell the arrival times of future patrollers. An attack requires \( t > 0 \) time units to complete, during which it will be detected by each passing patroller independently with probability \( p \in (0, 1] \). The defender wants to maximize the detection probability while the attacker wants to minimize it. Denote this game by \( \Gamma(\lambda, t, p) \), and write \( V(\lambda, t, p) \) for the value of the game. We begin with a lemma that is key to our results.

**Lemma 1** Consider a nonnegative integer valued random variable \( N \) whose expected value is equal to some constant \( c \). For any \( p \in (0, 1) \), we want to choose the distribution of \( N \) to minimize \( E[(1 - p)^N] \). If \( c \) is an integer, then it is optimal to let \( P\{N = c\} = 1 \). If \( c \) is not an integer, then it is optimal to require \( N \) to take on only the two integers surrounding \( c \); in other words, it is optimal to let

\[
\begin{align*}
  P\{N = \lfloor c \rfloor\} &= \lceil c \rceil - c; \\
  P\{N = \lceil c \rceil\} &= c - \lfloor c \rfloor.
\end{align*}
\]

**Proof.** This result can be established via the concept of convex ordering between two probability distributions. Using Equation (2.A.7) in Shaked and Shanthikumar (1994), it is straightforward to check that the distribution given in the lemma is the smallest in the sense of convex ordering—the one with least variability. Because \( \phi(x) = (1 - p)^x \) is a convex function, the distribution in the lemma minimizes \( E[\phi(N)] \), which concludes the proof. A different proof based on algebra can be found in Theorem 2 in Lin et al. (2014).

We next solve the patrol game \( \Gamma(\lambda, t, p) \) by showing that each player has a strategy to guarantee the value of the game.

**Theorem 2** Consider the patrol game \( \Gamma(\lambda, t, p) \), and write \( m = \lfloor \lambda t \rfloor \) and \( r = \lambda t - \lfloor \lambda t \rfloor \).

The value of the game is given by

\[
V(\lambda, t, p) = \begin{cases} 
1 - (1 - p)^m, & \text{if } r = 0, \\
1 - r(1 - p)^{m+1} - (1 - r)(1 - p)^m, & \text{if } r > 0.
\end{cases}
\]  

(1)

An equivalent and more concise—albeit less expressive—formula is

\[
V(\lambda, t, p) = 1 - (\lambda t - \lfloor \lambda t \rfloor)(1 - p)^{\lceil \lambda t \rceil} - (1 - \lambda t + \lfloor \lambda t \rfloor)(1 - p)^{\lfloor \lambda t \rfloor}.
\]
Proof. We first show that the detection probability in (1) is an upper bound for the value of the game. Consider an attack strategy with which the attacker initiates the attack when the patrol process reaches equilibrium. For example, the attacker can initiate an attack at time $T$ for some very large $T$. Write $N$ for the number of patrollers who come by during the attack time $t$. Because the long-run rate of the patrol process is $\lambda$ and the attacker initiates the attack when the patrol process reaches equilibrium, it follows that $E[N] = \lambda t$.

The probability this attack will be detected is

$$1 - E[(1 - p)^N],$$

since each of the $N$ patrollers will overlook the attack independently with probability $1 - p$. From Lemma 1, to maximize this detection probability, it is optimal to require $N$ to take on only the two integers surrounding $\lambda t$, or just $\lambda t$ if it happens to be an integer. Such a choice yields the detection probability in (1). In other words, the proposed attack strategy guarantees the detection probability to be no more than (1).

We next give a patrol strategy that achieves (1) to conclude the proof. If $\lambda t$ is an integer so $r = 0$, dispatch patrollers at fixed intervals of $1/\lambda$ time units. The attacker can hide and wait, but regardless of when he initiates the attack, the attack interval will encounter exactly $m$ patrollers, so the defender guarantees detection probability in (1) if $r = 0$.

If $\lambda t$ is not an integer so $r > 0$, define $\Delta \equiv t/(m + 1)$. Consider the patrol strategy that dispatches a blue patroller at time points $jt + k\Delta$ for $j = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots, m$; and a red patroller at time points $jt$ with probability $r$, for $j = 0, 1, 2, \ldots$. An example with $\lambda t = 3.2$ is shown in Figure 1. Regardless of when the attack begins, any attack interval will encounter exactly $m$ blue patrollers, and 1 red patroller with probability $r$. In other words, each attack will encounter $m + 1$ patrollers with probability $r$, or $m$ patrollers with probability $1 - r$, so the defender guarantees detection probability in (1) if $r > 0$.

Because each player has a strategy to guarantee detection probability in (1), it is the value of the game, and the proof is completed. \qed

One may conjecture that dispatching patrollers according to a Poisson process would work well, because the attacker cannot learn about the future of the Poisson process from its past. The downside of dispatching patrollers according to a Poisson process, however, is that some attacks would encounter many patrollers while some encounter very few or even
none at all, which overall results in a lower detection probability than that in [1]—according to Lemma [1]. If the attack time is relatively short or the patrol resource is very limited, so that $\lambda t < 1$, then according to Theorem [2] we have that $m = 0$ and $r = \lambda t$. It is optimal for the defender to dispatch one patroller with probability $r = \lambda t$ after each $t$ time units, so that every attack—regardless of when it starts—will encounter one patroller with probability $\lambda t$ or no patroller with probability $1 - \lambda t$. The time between two consecutive patrollers is distributed as a geometric random variable with parameter $\lambda t$ multiplied by $t$. In the limit as $t \to 0$, this interarrival time converges to the exponential distribution with rate $\lambda$, so the optimal patrol process converges to the Poisson process with rate $\lambda$.

An interesting and important question is whether the defender can increase the detection probability by making the patrollers invisible to the attacker, or conversely, how much advantage the attacker gains by observing the patrollers going by to time his attack. Somewhat surprisingly, the value of the game remains the same, because each player’s strategy presented in the proof of Theorem [2] is feasible in both scenarios. That is, the attacker’s optimal strategy in $\Gamma(\lambda, t, p)$ does not require him to observe the patrol pattern to time his attack even if he is capable of doing so. Hence, we have shown the following.

**Corollary 3** Consider the patrol game $\Gamma(\lambda, t, p)$ but the patrollers are invisible to the attacker. Each player’s optimal strategy for the game $\Gamma(\lambda, t, p)$ is still optimal, and the value of the game remains the same as in [1].

In the case where the attacker cannot see the patrollers, the defender has a simpler patrol strategy that is also optimal: Dispatch the first patroller at a random time according to the
uniform distribution over $[0, 1/\lambda]$, and subsequent patrollers at fixed intervals of $1/\lambda$ time units. Regardless of when the attack begins, the time elapsed between the start of the attack and the arrival time of the first patroller follows a uniform distribution over $[0, 1/\lambda]$, so the detection probability is still given by (1). This simpler patrol strategy, however, is not optimal if the attacker can see the patrollers, for the attacker can initiate the attack immediately after a patroller goes by to obtain detection probability $1 - (1 - p)^m$ even if $r > 0$.

Can the defender increase the detection probability if some patrollers circumambulate the perimeter clockwise and some others counterclockwise, or if they are allowed to move at different speeds or even change speed during a trip? The answer is no. As long as each patroller is not allowed to turn around during a patrol trip, each patroller goes by each point on the perimeter exactly once during a trip. Hence, the long-run patrol rate observed at any point on the perimeter is still capped by $\lambda$, so the same analysis in Theorem 2 applies. In other words, the optimal patrol strategy requires each patroller to move in the same direction at the same constant speed. The choice of the patroller’s (same constant) speed is inconsequential, because the time between consecutive patrollers passing by any point on the perimeter is dictated by the time between their departures from the base. Hence, we have proved the following.

**Corollary 4** Consider the patrol game $\Gamma(\lambda, t, p)$ but allow the defender to dispatch each patroller in either clockwise or counterclockwise direction, and allow each patroller to vary her own speed circumambulating the perimeter. Each player’s optimal strategy for the game $\Gamma(\lambda, t, p)$ is still optimal, and the value of the game remains the same as in (1).

3 Concluding Remarks

While we present the problem in the context of perimeter patrol, the model applies to any patrol problem in which the patrollers depart from a base, follow the same route, and return to the same base. For example, police cars may patrol the same route in a city, leaving and returning to the same police station.

If each passing patroller can detect an attacker in hiding with some probability, then it only makes the attacker’s hide-and-wait strategy worse, so our results still hold. If the
attacker is allowed to wait, withdraw, and try again at a later time, the results still hold. The optimal patrol strategy neutralizes any tactics the attacker may have and produces the same detection probability regardless of when the attack begins.

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