Flavour–symmetric type-II Dirac neutrino seesaw mechanism

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We propose a Standard Model extension with underlying $A_4$ flavour symmetry where small Dirac neutrino masses arise from a Type–II seesaw mechanism. The model predicts the “golden” flavour-dependent bottom-tau mass relation, requires an inverted neutrino mass ordering and non-maximal atmospheric mixing angle. Using the latest neutrino oscillation global fit we derive restrictions on the oscillation parameters, such as a correlation between $\delta_{\text{CP}}$ and $m_{\text{lightest}}$. 

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I. INTRODUCTION

Full quark-lepton correspondence within the Standard Model would suggest neutrinos to be Dirac fermions, with the lepton mixing matrix completely analogous to the CKM matrix. However, if the ultimate description of particle physics is a four-dimensional quantum field theory, this situation is unlikely since, on general grounds, one expects neutrinos to be Majorana type. In addition, the associated mechanisms to account for small neutrino mass as a consequence of their charge neutrality, such as the celebrated seesaw mechanism in its various realisations, all lead to Majorana neutrinos.

From the experimental side, however, despite great efforts over several decades, neutrinoless double beta decay has not yet been detected. According to the black-box theorem, such detection would provide the only robust way to establish the Majorana nature of neutrinos. Hence, for the time being, one must keep an open mind and re-examine the foundations of our ideological prejudices against having neutrinos as Dirac fermions.

Till recently, one argument against Dirac neutrinos has been the absence of convincing realizations of the seesaw mechanism for this case. However this argument is fragile, and Dirac seesaw mechanisms have been shown to exist, both within the type-I, as well as type-II realizations, for a brief classification see. Moreover, the existence of right-handed states, required for Dirac masses, may be necessary in order to have a consistent high energy completion, or for realizing a higher symmetry, such as the gauged B-L symmetry present in the conventional SO(10) seesaw scenarios.

Dirac neutrinos also arise in schemes with extra space-time dimensions such as string constructions, where right-handed states are needed to ensure anomaly cancellation through a generalized Green-Schwarz mechanism. Alternatively, Dirac neutrinos arise from five-dimensional anti de Sitter space warped solutions to the hierarchy problem. The truth of the matter is that the nature of neutrinos remains as mysterious as the mechanism providing their small masses, and hence we cannot simply rule it out.

In this letter, we consider the possibility of Dirac neutrinos resulting from a family symmetry construction in which the small neutrino mass arises a la seesaw, thus complementing the idea proposed in. In addition to shedding light upon the pattern of neutrino oscillations, we also require the flavour structure to provide the generalised bottom-tau mass relation,

\[ \frac{m_\tau}{\sqrt{m_e m_\mu}} = \frac{m_b}{\sqrt{m_d m_s}}, \]

proposed in. This successful “golden” mass relation is derived without the need for invoking grand unification in a number of models, all of which lead to particular restrictions on the neutrino oscillation sector. The paper is organized as follows: In the next section we introduce the model, in section we present the predictions and discuss the results, and finally we conclude in.

II. THE MODEL

This model is constructed so that the smallness of the neutrino mass has a dynamical origin, given by the Type-II seesaw mechanism for Dirac neutrinos, illustrated in Figure. In order to explain neutrino oscillation data in this scenario we introduce the flavour symmetry $A_4 \otimes Z_3 \otimes Z_2$. After symmetry breaking there is an Abelian discrete symmetry $Z_3$ which forbids the appearance of Majorana operators at the loop level in the broken phase, thus protecting the Diracness of neutrinos.

¹ For the possibility of having radiative Dirac neutrino mass models see Refs. 

19.
A. Lepton sector

The particle assignments for the lepton sector and scalars are given in Table I. The $SU(2)$ scalar doublets $H^d = (H^d_1, H^d_2, H^d_3)$ and $\phi = (\phi_1, \phi_2, \phi_3)$ transform as triplets under $A_4$, where each component can be written as follows

$$H^d_i = \begin{pmatrix} h_{d_i}^+ \\ i h_{d_i} \end{pmatrix}, \quad \phi_i = \begin{pmatrix} \phi_0^i \\ i \phi_i \end{pmatrix},$$

(2)

| $SU(2)_L \otimes U(1)_Y$ | (2, $-1/2$) | (1, $-1$) | (1, 0) | (2, $1/2$) | (2, $-1/2$) | (1, 0) |
|--------------------------|-------------|-------------|---------|-------------|-------------|---------|
| $A_4$                    | 3           | 3           | 3       | 3           | 3           | 3 or 1L |
| $Z_3$                    | $\omega^2$  | $\omega$    | $\omega$| 1           | 1           | 1       |
| $Z_2$                    | +           | +           | −       | +           | −           | −       |

Table I. Charge assignments for the particles involved in the neutrino mass generation mechanism, where $\omega^3 = 1$.

The vacuum expectation value (vev) alignments of these scalar triplets given by $\langle H^d \rangle = (v_{h_d^1}, v_{h_d^2}, v_{h_d^3})$ and $\langle \phi \rangle = (v_{\phi_1}, v_{\phi_2}, v_{\phi_3})$. The complex scalar $\sigma$ responsible for inducing the small induced vev of $\phi$ could transform either as triplet or singlet under $A_4$. On the other hand all leptons, both left- and right-handed, including the right-handed neutrinos $\nu_R = (\nu_{R_1}, \nu_{R_2}, \nu_{R_3})$, transform as $A_4$ triplets. Given the charges under the cyclic groups $Z_3 \otimes Z_2$, one can easily see that the $Z_3$ remains unbroken after symmetry breaking, because all scalars are blind under this symmetry. Such residual $Z_3$ symmetry forbids the Majorana mass terms $M_R \nu_R \nu_R$ as well as the dimension-5 operators: $LH^d LH^d$, $L\tilde{\phi}L\tilde{\phi}$ and $LH^d L\tilde{\phi}$. This symmetry also forbids higher order operators derived from the product of the previous dimension-5 operators and $(H^d \dagger H^d)^n$, $(\phi \dagger \phi)^n$ and $(H^d \dagger \tilde{\phi})^n$ as well as $\nu_R \nu_R \sigma^n$. Finally, the $Z_2$ charges are assigned as $(-)$ to $\nu_R$, $\sigma$ and $\phi$ and $+$ to the other particles. As a result they act in a complementary way to the $Z_3$, forbidding the unwanted renormalisable Yukawa couplings: $\bar{L} \tilde{\phi} \ell_R$ and $\bar{L} \tilde{H}^d \nu_R$, where $\tilde{\phi} = i \sigma_2 \phi^*$ and $\tilde{H}^d = i \sigma_2 H^d^*$. 

Figure 1. Neutrino mass generation in the Type-II seesaw for Dirac neutrinos [13–15]
In accordance with the previous discussion, the relevant part of Yukawa Lagrangian for the leptons is given as:

\[ \mathcal{L}_Y \supset Y_\ell^i \left[ \bar{L}, H^d \right]_{3i} \ell_R + Y_\nu^i \left[ \bar{L}, \phi \right]_{3i} \nu_R + \text{h.c.}, \]

(3)

where the symbol \([a, b]_3\) stands for the two ways of contracting two triplets of \(A_4\), \(a\) and \(b\), into a triplet, as shown in the Appendix. Before proceeding we summarize our model structure by saying that, compared with the minimal Standard Model case, here one has an extra scalar iso-doublet \(\phi\), the iso-singlet \(\sigma\) and the right-handed neutrino states \(\nu_R\), all triplets under \(A_4\). After symmetry breaking, neutrinos get a small type-II seesaw mass, as a result of the small vacuum expectation value (vev) \(\langle \phi \rangle\) which is induced by the vev of \(\sigma\) as proposed in [13–15].

### III. FLAVOUR PREDICTIONS AND NUMERICAL RESULTS

Two aspects of the flavour problem concern the explanation of mass hierarchies of quark and leptons, as well as to account for the structure of mixing in each of these sectors, so disparate from each other. We will see how our model leads to a successful “golden” mass formula relating quark and lepton masses, despite the absence of grand unification. This is a flavour generalization of bottom-tau unification previously proposed in [12, 24–28]. In addition we will derive the corresponding specific predictions for the lepton mixing matrix describing neutrino oscillations. This arises from the study of the charged and neutral lepton sector. While no predictions are made for the CKM quark mixing matrix, it can be adequately fit in a simple way, see reference [25]. We now spell out the detailed flavour predictions of our model.

#### A. Charged fermions and the generalised bottom-tau mass relation

The complete particle assignment of our model is inspired in the one in [25], and is shown in Table II including both gauge as well as flavour transformation properties.

| \(SU(2)_L \otimes U(1)_Y\) | \(Q\) | \(L\) | \(u_R\) | \(d_R\) | \(\ell_R\) | \(\nu_R\) | \(H^u_d\) | \(H^d\) | \(\phi\) | \(\sigma\) |
|----------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \(A_4\)              | 3     | 3     | 1     | 1     | 3     | 3     | 3     | 3     | 3     | 3/14  |
| \(Z_3\)              | 1     | \(\omega^2\) | 1     | \(\omega\) | 1     | \(\omega\) | 1     | 1     | 1     |       |
| \(Z_2\)              | +     | +     | +     | +     | +     | -     | +     | -     | -     |       |
| \(Z_2^*\)            | +     | -     | +     | +     | +     | +     | -     | +     | +     |       |

Table II. Particle content and quantum numbers for the model.

From Eq. (3) one sees that the charged lepton mass matrix, following [24, 26], can be parametrised as:

\[ m_\ell = \begin{pmatrix} 0 & a_\ell \alpha e^{i\theta_\ell} & b_\ell \\ b_{\ell} \alpha e^{i\theta_\ell} & 0 & e^{i\theta_\ell} \alpha \rho_\ell \\ a_{\ell} e^{i\theta_\ell} & b_{\ell} \rho_\ell & 0 \end{pmatrix}, \]

(4)

where \(a_\ell = v_{h_2} (Y^1_\ell + Y^3_\ell)\) and \(b_\ell = v_{h_2} (Y^2_\ell + Y^4_\ell)\) are real Yukawa couplings, \(\theta_\ell\) is a unremovable complex phase\(^2\) and the \(H^d\) vev alignment is parameterised as \(\langle H^d \rangle = (v_{h_2}, v_{h_2}, v_{h_2}) = v_{h_2} (\rho_\ell, 1, \alpha_\ell)\), with \(\alpha_\ell = v_{h_2} / v_{h_2}\) and \(\rho_\ell = v_{h_2} / v_{h_2}\).

\(^2\) We assume that the non-conservation of CP symmetry comes entirely from the neutrino sector. Thus in our analysis we fixed \(\theta_\ell = 0\).
The bi–unitary invariants of the squared mass matrix $M_\ell^2 = m_\ell m_\ell^\dagger$ are determined as:

\[ \text{Tr} M_\ell^2 = m_1^2 + m_2^2 + m_3^2, \]
\[ \det M_\ell^2 = m_1^2 m_2^2 m_3^2, \]
\[ (\text{Tr} M_\ell^2)^2 - \text{Tr}(M_\ell^2)^2 = 2m_1^2 m_2^2 + 2m_2^2 m_3^2 + 2m_1^2 m_3^2. \]

We work under the assumptions $\rho_\ell \gg \alpha_\ell$, $\rho_\ell \gg 1$, $b_\ell > a_\ell$ and $\rho_\ell \gg \frac{b_\ell}{a_\ell}$, which, at leading order, ensure adequate family mass hierarchy as well as mixing patterns. One can show from Eqs. (3) that:

\[ (b_\ell \rho_\ell)^2 \approx m_3^2, \]
\[ (b_\ell^2 \rho_\ell \alpha_\ell)^2 \approx m_1^2 m_2^2 m_3^2, \]
\[ (a_\ell b_\ell^2)^2 \approx m_2^4. \]

Solving the system in Eqs. (8–10), we can find the approximate expressions:

\[ a_\ell \approx \frac{m_2}{m_3} \sqrt{\frac{m_1 m_2}{\alpha_\ell}}, \quad b_\ell \approx \sqrt{\frac{m_1 m_2}{\alpha_\ell}} \quad \text{and} \quad \frac{\rho_\ell}{\alpha_\ell} \approx \frac{m_3}{\sqrt{m_1 m_2}}. \]

Notice that also the down–type quarks couple to the $H^d$ and hence have the same flavour structure. This implies that the parameters $\rho_\ell$ and $\alpha_\ell$ in Eq. (11) are common to the charged leptons and the down–type quarks, i.e. $\rho_\ell = \rho_d$ and $\alpha_\ell = \alpha_d$. From this we derive the generalised bottom–tau mass relation in Eq. (11):

\[ \frac{m_\tau}{\sqrt{m_\nu m_\mu}} = \frac{m_b}{\sqrt{m_\nu m_\mu}}, \]

in a straightforward way. This generalised down quark–charged lepton mass relation, Eq. (11), follows from our flavour group assignments. Although it has been obtained also in other realizations of $A_4$ family symmetry [23, 25, 26], these are not equivalent.

### B. Fermion masses and mixing

In this section we focus on the lepton mixing matrix, because it is in this sector that our model makes non-trivial predictions. However the CKM matrix describing quark mixing will be adequately described and this provides an input for the lepton mixing matrix. In analogy with the previous subsection, Eq. (3) gives the neutrino mass matrix, which can also be parametrised as:

\[ m_\nu = \begin{pmatrix} 0 & a_\nu a_\nu & b_\nu e^{i\theta_\nu} \\ b_\nu e^{i\theta_\nu} a_\nu & 0 & a_\nu \rho_\nu \\ a_\nu & b_\nu e^{i\theta_\nu} \rho_\nu & 0 \end{pmatrix}, \]

where $a_\nu = v_{\phi_3}(Y_\nu^1 + Y_\nu^2)$ and $b_\nu = v_{\phi_3}(Y_\nu^2 + Y_\nu^4)$ are real Yukawa couplings, $\theta_\nu$ is the complex phase that cannot be rotated away under $SU(2)$ transformations, and characterizes the strentgh of CP violation in the lepton sector. The vev–alignment of $\phi$ can be written as $\langle \phi \rangle = (v_{\phi_1}, v_{\phi_2}, v_{\phi_3}) = v_{\phi_3} (\rho_\nu, 1, \alpha_\nu)$, with $\alpha_\nu = v_{\phi_3}/v_{\phi_2}$ and $\rho_\nu = v_{\phi_3}/v_{\phi_2}$.

From the invariants Eqs. (5) of the squared neutrino mass matrix $M_\nu^2 = m_\nu m_\nu^\dagger$:

\[ \text{Tr}(M_\nu^2) = (a_\nu^2 + b_\nu^2)(1 + \alpha_\nu^2 + \rho_\nu^2), \]
\[ \det(M_\nu^2) = (a_\nu^2 + b_\nu^2 + 2a_\nu^2 b_\nu^2 \cos(3\theta_\nu)) \alpha_\nu^2 \rho_\nu^2, \]
\[ \frac{1}{2} [(\text{Tr} M_\nu^2)^2 - \text{Tr}(M_\nu^2)] = a_\nu^2 b_\nu^2 (1 + \alpha_\nu^2 + \rho_\nu^2) + (a_\nu^4 + b_\nu^4) (\rho_\nu^2 + \alpha_\nu^2(1 + \rho_\nu^2)), \]
we performed a numerical scan over the parameter regions for the solutions of Eqs. (13–15) that reproduce the measured elements of the leptonic mixing matrix $V = U^\dagger_l U_\nu$. To do this we use as inputs the 3$\sigma$ values for the three neutrino mixing angles and the two mass squared differences from the global fit [1]. The $U_\nu$ matrix comes from the bi–unitary transformation for the neutrinos $U^\dagger_\nu m_\nu V_\nu = D_\nu$, where $D = \text{diag} (m_\nu_1, m_\nu_2, m_\nu_3)$, while the $U_l$ comes from the equivalent transformation for the charged leptons, e.g. $U^\dagger_l m_\ell V_\ell = D_\ell$.

We now turn to the CKM matrix describing quark mixing. Although we have no family symmetry prediction for the CKM matrix, we notice that it can be accommodated in the same way as described in [25]. This fixes the value for the $\alpha_d$ parameter which enters also in the leptonic sector. In order to adequately fit the CKM matrix we need $\alpha_d = \alpha_l \approx 1.58$ [25]. Taking into account the numerical values for the charged lepton masses $m_e = 0.511006$ MeV, $m_\mu = 105.656$ MeV and $m_\tau = 1776.96$ MeV and assuming that the complex phase $\theta_\ell = 0$, as in Refs. [24–26], we find that the resulting contribution from the charged lepton sector to the neutrino mixing matrix is fixed and close to be diagonal, see for instance [24].

C. Neutrino oscillation predictions

In order to determine the neutrino oscillation predictions of the model, we have performed a numerical scan in which parameters are varied randomly in the ranges

\[ \alpha_\nu \in [-10, 10], \quad \rho_\nu \in [-10, 10] \quad \text{and} \quad \theta_\nu \in [0, 2\pi]. \]  

Only those choices for which the undisplayed and well-measured oscillation parameters are within 3$\sigma$ of the values obtained in the latest neutrino oscillation global fit of Ref. [1] are kept. This way we have obtained the model–allowed regions in terms of the “interesting” and poorly determined oscillation parameters $\theta_{23}$, $\delta_{CP}$ as well as the lightest neutrino mass eigenvalue. These are displayed in shaded (green) regions in the Figures 2, 3 and 4. In contrast, the unshaded regions are the 90 and 99%CL regions obtained directly from the unconstrained three–neutrino oscillation global fit [1], see Figures 4 and 5 in reference [1].

We find that the model is only compatible with the inverted ordering for the neutrino mass eigenvalues. The consistent parameter regions for the atmospheric mixing angle $\sin \theta_{23}$ vs. the lightest neutrino mass $m_3$ are given in Fig. 2 while the CP violating Dirac phase $\delta_{CP}$ vs. $m_3$ are displayed in Fig. 3.

From the plot given in Fig. 3 one sees that the allowed region for the lightest neutrino mass $m_3$ is within the range $[6.4 \times 10^{-4}$ eV, $2.7 \times 10^{-5}$ eV]. We can see that only masses above $\sim 0.002$ eV allow $\delta_{CP} = 0$, signifying no CP violation, while for lower masses such value is always non–zero. The shaded areas in Fig. 3 are obtained from a numerical scan that filters those parameter choices for which the well-measured undisplayed oscillation parameters lie within 3$\sigma$ of the best fit values obtained in the latest neutrino oscillation global fit in Ref. [1]. These should be compared with the unshaded 90 and 99%CL regions obtained directly in the unconstrained three–neutrino oscillation global fit [1].

IV. CONCLUSIONS

We have proposed a $A_4 \otimes Z_3 \otimes Z_2$ flavour extension of the Standard Model where the small neutrino masses are generated from a type II Dirac see-saw mechanism. The model addresses both aspects of the flavour problem: the explanation of mass hierarchies of quark and leptons, as well as restricting the structure of the lepton mixing matrix. Concerning the first point, our model leads to a successful “golden” mass relation between quark en lepton masses, proposed previously. In addition, the model provides flavour predictions for the lepton mixing matrix relevant in order to account for neutrino oscillations. First of all, inverted neutrino mass ordering and non-maximal atmospheric mixing angle are predicted. While this is at odds with the results of the latest neutrino oscillation global fit in [1], we stress...
that the neither the preference for normal ordering nor the indication for a given octant are currently statistically significant, since the general three-neutrino fit gives four possible closely separated local minima. In any case, taken at face value, the model at hand would suggest a slight preference for the higher octant, since it predicts inverted neutrino mass ordering. We have also found a positive hint for CP violation, $\delta_{CP} \neq 0$, if $m_{\text{lightest}} \lesssim 0.002$ eV, while bigger masses are consistent with CP conserving solutions. Concerning the CKM quark mixing matrix, we also saw that, although no definite predictions are made, the required CKM matrix elements can be adequately described, and they also fix the contribution to the neutrino mixing matrix that comes from the charged lepton sector. Finally, we note that the residual flavour symmetry forbids the Majorana mass terms at any order and provides, by construction, a natural realization of a type II Dirac see-saw mechanism for small neutrino masses.
Figure 4. The allowed regions of the atmospheric mixing angle and \( \delta_{\text{CP}} \) are indicated in shaded (green). They result from a numerical scan keeping only those choices that lie within 3\( \sigma \) of their preferred best fit values Ref. [1]. The unshaded regions are 90 and 99\%CL regions obtained directly in the unconstrained three-neutrino oscillation global fit [1].

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Appendix A: The \( A_4 \) product representation

The non abelian discrete group \( A_4 \), or the group of the even permutation of four elements, has four irreducible representations [29,31]: three singlets \( 1_1, 1_2, \) and \( 1_3 \) and one triplet \( 3 \) and two generators: \( S \) and \( T \) following the relations \( S^2 = T^3 = (ST)^3 = I \). The one-dimensional unitary representations are

\[
\begin{align*}
1_1 & : S = 1, \ T = 1, \\
1_2 & : S = 1, \ T = \omega, \\
1_3 & : S = 1, \ T = \omega^2,
\end{align*}
\]

where \( \omega^3 = 1 \). In the basis where \( S \) is real diagonal,

\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\quad \text{and} \quad
T = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

The product rule for the singlets are

\[
\begin{align*}
1_1 \otimes 1_1 &= 1_2 \otimes 1_3 = 1, \\
1_2 \otimes 1_2 &= 1_3, \\
1_3 \otimes 1_3 &= 1_2,
\end{align*}
\]
and triplet multiplication rules are

\[(ab)_1 = a_1 b_1 + a_2 b_2 + a_3 b_3,\]
\[(ab)_2 = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3,\]
\[(ab)_3 = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3,\]
\[(ab)_4 = (a_2 b_3, a_3 b_1, a_1 b_2),\]
\[(ab)_5 = (a_3 b_2, a_1 b_3, a_2 b_1),\]

where \(a = (a_1, a_2, a_3)\) and \(b = (b_1, b_2, b_3)\).
[21] P. Chen et al., *Warped flavor symmetry predictions for neutrino physics*, JHEP 01 (2016) 007, arXiv:1509.06683 [hep-ph]

[22] A. Aranda et al., *Dirac neutrinos from flavor symmetry*, Phys. Rev. D89 (2014) 3 033001, arXiv:1307.3553 [hep-ph]

[23] S. Morisi and E. Peinado, *An A(4) model for lepton masses and mixings*, Phys. Rev. D80 (2009) 113011, arXiv:0910.4389 [hep-ph]

[24] S. Morisi, E. Peinado, Y. Shimizu and J. W. F. Valle, *Relating quarks and leptons without grand-unification*, Phys. Rev. D84 (2011) 036003, arXiv:1104.1633 [hep-ph]

[25] S. F. King, S. Morisi, E. Peinado and J. W. F. Valle, *Quark-Lepton Mass Relation in a Realistic A_4 Extension of the Standard Model*, Phys. Lett. B724 (2013) 68–72, arXiv:1301.7065 [hep-ph]

[26] S. Morisi et al., *Quark-Lepton Mass Relation and CKM mixing in an A_4 Extension of the Minimal Supersymmetric Standard Model*, Phys. Rev. D88 (2013) 036001, arXiv:1303.4394 [hep-ph]

[27] C. Bonilla, S. Morisi, E. Peinado and J. W. F. Valle, *Relating quarks and leptons with the T_7 flavour group*, Phys. Lett. B742 (2015) 99–106, arXiv:1411.4883 [hep-ph]

[28] B. Carballo-Perez, E. Peinado and S. Ramos-Sanchez, *Δ(54) flavor phenomenology and strings*, JHEP 12 (2016) 131, arXiv:1607.06812 [hep-ph]

[29] E. Ma and G. Rajasekaran, *Softly broken A(4) symmetry for nearly degenerate neutrino masses*, Phys. Rev. D64 (2001) 113012, arXiv:hep-ph/0106291 [hep-ph]

[30] K. S. Babu, E. Ma and J. W. F. Valle, *Underlying A(4) symmetry for the neutrino mass matrix and the quark mixing matrix*, Phys. Lett. B552 (2003) 207–213, arXiv:hep-ph/0206229 [hep-ph]

[31] S. F. King et al., *Neutrino Mass and Mixing: from Theory to Experiment*, New J.Phys. 16 (2014) 045018, arXiv:1402.4271 [hep-ph]