The energy-momentum tensor of perturbed tachyon matter

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Abstract

We add an initial nonhomogeneous perturbation to an otherwise homogeneous condensing tachyon background and compute its space time energy-momentum tensor from worldsheet string theory. We show that in the far future the energy-momentum tensor corresponds to nonhomogeneous pressureless tachyon matter.
**Introduction** The condensation of the tachyonic mode (or modes) present in an unstable brane system in string theory was found to lead to pressureless tachyon matter [1–3]. This process has generated wide interest in cosmology. Initially, it was speculated that tachyon matter could be a novel form of dark matter, but since then tachyon condensation has been applied to a variety of different scenarios, too numerous to attempt to give a fair list of references. In this short paper we will focus on one aspect. From very early on, there has been interest in understanding the effect of fluctuations to the condensing tachyon field, and their cosmological implications. This was first studied in the simple effective tachyon field theory model in [4]. It is important – but also more challenging – to go beyond the simple toy model and to understand how initial fluctuations could be taken into account and analyzed in string theory. A first elementary question could be to see what kind of an imprint small fluctuations leave on the energy-momentum tensor of the resulting tachyon matter. Recent new tricks, applying random matrix theory to worldsheet calculations in a condensing tachyon background, have made it possible to study the problem analytically in this paper. The answer should be of interest, e.g., to cosmological models where gravity waves are generated by tachyon inhomogeneities [5].

We focus on bosonic open string theory. In the presence of a background open string tachyon field, the worldsheet action is

\[ S = S_0 + S_T = \frac{1}{2\pi} \int_{\text{disk}} (-\partial X^0 \bar{\partial} X^0 + \partial X^I \bar{\partial} X^I) + \lambda \oint dt \, T(X(t)) . \]  

(1)

Suppose that \( T(X) \) is divided into a homogeneous rolling background (taken to be the simple exponential profile) and small nonhomogeneous perturbations,

\[ T(X(t)) = \lambda e^{X^0(t)} + \delta T(X^0, \vec{X}) . \]  

(2)

As is well known, the possible field configurations must satisfy the beta function equations which can be interpreted as the equations of motion of the effective field theory. Let us consider what this means for the perturbations \( \delta T \), and turn off the homogeneous profile for a moment.

The perturbations should be marginal deformations, i.e., in the expansion

\[ \delta T(X^0, \vec{X}) = \int d\vec{k} \, \delta \tilde{T}_\vec{k} \int dt \, e^{-i\omega_k X^0(t) + i\vec{k} \cdot \vec{X}(t)} , \]  

(3)

the frequency \( \omega_k \) is not independent but must satisfy the tachyon on-shell condition

\[ \omega_k^2 - \vec{k}^2 = -1 . \]  

(4)

However, since the deformation is in the exponent inside the worldsheet path integral, for a sizable deviation the deformation must continue to be marginal even with respect to an already deformed theory. This means that the deforming operator must
be exactly marginal (or “mutually self-local”, see [6]); for the tachyon operator this
condition can be satisfied only if \( \omega = \pm i \) (leading to the homogeneous deformation)
or \( \omega = \pm i/\sqrt{2}, \lvert \vec{k} \rvert = 1/\sqrt{2} \). The latter condition means that a nonhomogeneous large
deformation can at most be a superposition of a left- and a right-moving monochromatic tachyonic mode. A reason for this requirement is that in the Taylor series expansion of the tachyonic deformation one encounters higher powers of tachyon op-
erators, and the operator products must be regular. If, on the other hand, we restrict
ourselves to a small deformation, and then consider only the leading term in the expansion,

\[
Z_{\text{def}} = \int \mathcal{D}X^\mu e^{-S_0} \int \delta T
\]

\[
\approx \int \mathcal{D}X^\mu e^{-S_0} \left( 1 - \int dk \delta \hat{T}_{\vec{k}} \int dt e^{-i\omega_k X^0(t) + i\vec{k} \cdot \vec{X}(t)} \right),
\]

(5)

it is sufficient to consider marginal deformations and we can consider generic superpo-
sitions of different momenta \( \vec{k} \) as long as the on-shell condition (4) is satisfied. (Here \( \vec{k} = \vec{k}_{||} \) denotes the momentum in directions parallel to the decaying brane.)

The homogeneous rolling tachyon background \( e^{X^0} \) corresponds to an exactly marginal
deformation

\[
\delta S_{\text{roll}} = \lambda \int dt e^{X^0(t)}
\]

(6)

and leads to the formation of pressureless tachyon matter. This was first found in [1]
by calculating the spacetime energy-momentum tensor (for a slightly different tachyon
profile) in the boundary state formalism. We will base our analysis on the approach
of [7] (see also [8, 9]), which studied the exponential profile and calculated the energy-
momentum tensor by a different approach. It is defined as a functional derivative of
the spacetime effective action with respect to the metric,

\[
T^{\mu\nu}(x^\mu) = \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{spacetime}}}{\delta g_{\mu\nu}} \bigg|_{g_{\mu\nu} = \eta_{\mu\nu}}.
\]

(7)

On the other hand the action \( S_{\text{spacetime}} \) is given by the worldsheet disk partition function,

\[
S_{\text{spacetime}} = Z_{\text{disk}} = \int \mathcal{D}X^\mu e^{-S_0[g] - \delta S_{\text{roll}}}.
\]

(8)

with a general space time metric in the worldsheet action,

\[
S_0[g] = \frac{1}{2\pi} \int d^2z g_{\mu\nu} \partial X^\mu \overline{\partial X^\nu}.
\]

(9)

The energy-momentum tensor turns out to be

\[
T^{\mu\nu} = K \left( Z'_{\text{disk}}(x^0) \eta^{\mu\nu} + A^{\mu\nu}(x^0) \right),
\]

(10)
where \( Z'_{\text{disk}}(x^0) \) is the disk partition function (the prime indicates that the zero mode \( x^0 \) is left unintegrated) and \( A^{\mu\nu} \) is the one-point function

\[
A^{\mu\nu}(x^0) = 2 \left< : \partial X^\mu(0) \bar{\partial} X^\nu(0) : e^{-\delta S_{\text{roll}}} \right> \tag{11}
\]

in the rolling tachyon background \( \text{(6)} \). The result for the energy-momentum tensor is

\[
T_{00} = -T_p \quad T_{ij}(x^0) = \delta_{ij} T_p (1 + 2\pi \lambda e^{x^0})^{-1}, \tag{12}
\]

with a constant energy density and with pressure components decaying exponentially to zero at late times.

**Perturbed energy-momentum tensor** We will next calculate the spacetime energy-momentum tensor in the presence of the initial perturbation \( \text{(3)}, \text{(5)} \) in the rolling tachyon background \( \text{(6)} \). It becomes

\[
T_{\mu\nu} = T_{\mu\nu}^{(0)}(x^0) + \Delta T_{\mu\nu}(x), \tag{13}
\]

where \( T_{\mu\nu}^{(0)}(x^0) \) is the unperturbed result \( \text{(12)} \) and \( \Delta T_{\mu\nu}(x) \) is the perturbation which we want to calculate. It is given by

\[
\Delta T_{\mu\nu}(x) = K \int d\vec{k} \delta T_{\vec{k}} (\Delta Z'_{\text{disk}}(x) \eta^{\mu\nu} + \Delta A^{\mu\nu}(x)), \tag{14}
\]

where the perturbation to the disk partition function \( \Delta Z'_{\text{disk}} \) and the perturbation \( \Delta A \) involves

\[
\Delta Z'_{\text{disk}}(x) = \left< e^{\xi X^0(\tau)} e^{-\delta S_{\text{roll}}} \right> \left< e^{i\vec{k} \cdot \vec{X}} \right>', \tag{15}
\]

\[
\Delta A^{\mu\nu}(x) = 2 \left< : \partial X^\mu(0) \bar{\partial} X^\nu(0) : e^{\xi X^0(\tau) + i\vec{k} \cdot \vec{X}(\tau)} e^{-\delta S_{\text{roll}}} \right> \tag{16}
\]

where we introduced \( \xi = -i\omega \). The \( \Delta Z'_{\text{disk}}(x) \) could be calculated from the recent result (see Eq. (21) of [11])

\[
\mathcal{A}_1(\xi) = \int dx^0 e^{\xi x^0} \Delta Z'_{\text{disk}}(x) \bigg|_{k=0} = (2\pi \lambda)^{-1} \eta(\xi) G(1 + \xi)^2 G(2 - \xi) G(2\xi + 1) \tag{17}
\]

by undoing the zero mode integral. However, we will analyze both terms of \( \text{(14)} \) simultaneously. We perform a Taylor expansion of the boundary deformation. With the help of results from [11] we obtain

\[
\Delta T_{\mu\nu}(x) = K \int d\vec{k} \delta \bar{T}_{\vec{k}} e^{\xi x^0 + i\vec{k} \cdot \vec{x}} \sum_{N=0}^\infty (-z)^N \left< I_\xi(N) \eta^{\mu\nu} + \Delta A_k^{\mu\nu}(N) \right>, \tag{18}
\]

\footnote{Because of normal ordering, \( \left< e^{i\vec{k} \cdot \vec{X}} \right>' = e^{i\vec{k} \cdot \vec{x}} \).}
where

\[
I_\xi(N) = \frac{1}{N!} \int \frac{d\tau}{2\pi} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \left\langle e^{\xi X^0} \prod_{i=1}^{N} e^{X^0(t_i)} \right\rangle'
\]

\[
= \frac{1}{N!} \int \frac{d\tau}{2\pi} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \prod_{1 \leq i < j \leq N} |e^{it_i} - e^{it_j}|^2 \prod_{i=1}^{N} |e^{it_i} - e^{it_j}|^2
\]

(20)

\[
= \prod_{j=1}^{N} \frac{\Gamma(j) \Gamma(j + 2\xi)}{\Gamma(j + \xi)^2}.
\]

(21)

The more challenging task is to calculate the coefficients \(\Delta A^{\mu\nu}(N)\) and then resum the series. We start the analysis from a generating function

\[
C(\chi^{(i)}; z^{(i)}, \tilde{z}^{(i)}) = \left\langle e^{\chi^{(i)} X^\mu(z^{(i)}, z^{(i)})} e^{\chi^{(2)} X^\nu(z^{(2)}, \tilde{z}^{(2)})} : e^{\xi X^0(0) + i\tilde{k} \cdot X(0)} \prod_{i=1}^{N} e^{X^0(t_i)} : \right\rangle'
\]

\[
= |1 - z^{(1)} \tilde{z}^{(2)}|^{\chi^{(1)} X^\mu} \prod_{j=1,2} \left| 1 - z^{(j)} \tilde{z}^{(j)} \right|^{\chi^{(j)} X^\mu} \prod_{j} \left| 1 - \chi^{(j)} \right|^2 \chi^{(j)} X^\mu
\]

\[
\times \prod_{i,j} \left| e^{it_i} - e^{it_j} \right|^2 \prod_{i} \left| 1 - e^{it_i} \right|^2 \prod_{i<j} \left| e^{it_i} - e^{it_j} \right|^2,
\]

(22)

where we used rotational symmetry to fix \(\tau = 0\) and denoted \(\xi^\mu = (\xi, -i\tilde{k})\). It is then straightforward to calculate the term

\[
\Delta A^{\mu\nu}_k(N) = \frac{2}{N!} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \frac{\partial^4 C}{\partial z^{(1)} \partial \tilde{z}^{(2)} \partial \chi^{(1)} \partial \chi^{(2)}} \Bigg|_{z^{(j)} = \tilde{z}^{(j)} = \chi^{(j)} = 0}.
\]

(23)

We find,

\[
\Delta A^{00}_k(N) = \frac{2}{N!} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \left( \left| \chi + \sum_i e^{it_i} \right|^2 - \frac{1}{2} \right) \prod_i \left| 1 - e^{it_i} \right|^2 \prod_{i<j} \left| e^{it_i} - e^{it_j} \right|^2
\]

\[
= \frac{N^2 + 2\xi N - \xi^2 + 2\xi^4}{(\xi + N)^2} I_\xi(N)
\]

(24)

\[
\Delta A^{0j}_k(N) = \frac{2\xi}{N!} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \left( \chi + \sum_i e^{it_i} \right) \prod_i \left| 1 - e^{it_i} \right|^2 \prod_{i<j} \left| e^{it_i} - e^{it_j} \right|^2
\]

\[
= -\frac{2ikj \xi^2}{\xi + N} I_\xi(N)
\]

(25)

\[
\Delta A^{ij}_k(N) = \frac{2}{N!} \left( \xi \delta^{ij} + \frac{\delta^{ij}}{2} \right) \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \prod_i \left| 1 - e^{it_i} \right|^2 \prod_{i<j} \left| e^{it_i} - e^{it_j} \right|^2
\]

\[
= (-2k^i k^j + \delta^{ij}) I_\xi(N),
\]

(26)
where the integrals can be calculated by using Szegő polynomials [10].

In total, the series coefficients in the expansion of $\Delta T^{\mu\nu}$ are the following:

\[ \Delta T^{00}(N) = \frac{2\xi^2(\xi^2 - 1)}{(\xi + N)^2} I_\xi(N) \] (27)
\[ \Delta T^{0j}(N) = -\frac{2ik\xi^2}{N + \xi} I_\xi(N) \] (28)
\[ \Delta T^{ij}(N) = (2k^i k^j + 2\delta^{ij}) I_\xi(N). \] (29)

The only $N$-dependent term in $\Delta T^{ij}(N)$ is $I_\xi(N)$. Comparing with (17) we see that the pressure components are proportional to $\mathcal{A}_1$. At this stage we can already check that the total energy-momentum current (13) is conserved. Conservation of energy requires

\[ \partial_\mu T^{\mu 0}(x) = 2K \int d\vec{k} \delta \hat{T}_k \xi^2 e^{\xi x^0 + i\vec{k} \cdot \vec{x}} \sum_{N=0}^\infty \frac{\xi^2 - 1 + \vec{k}^2}{\xi + N} I_\xi(N)(-z)^N \] (30)

to vanish, which is indeed the case for $\xi^2 + \vec{k}^2 = 1$. The momentum conservation equation reads

\[ \partial_\mu T^{\mu i}(x) = -2iK \int d\vec{k} \delta \hat{T}_k k^i e^{\xi x^0 + i\vec{k} \cdot \vec{x}} \sum_{N=0}^\infty \left( \xi^2 + \vec{k}^2 - 1 \right) I_\xi(N)(-z)^N, \] (31)

which vanishes similarly.

**Asymptotic behavior** The energy-momentum tensor is still given in the form of a series expansion. However, our main interest is in its asymptotic behavior as $x^0 \rightarrow \pm \infty$. In these cases we can find analytic expressions for the leading terms. We can easily extract the leading behavior of $T^{\mu\nu}$ at past infinity $x^0 \rightarrow -\infty$. The components are given by the $N = 0$ terms

\[ \Delta T^{00}(x) = 2K \int d\vec{k} \delta \hat{T}_k e^{\xi x^0 + i\vec{k} \cdot \vec{x}}(\xi^2 - 1) \left[ 1 + \mathcal{O}\left(e^{x^0}\right) \right] \] (32)
\[ \Delta T^{0j}(x) = -2K i \int d\vec{k} \delta \hat{T}_k k^j e^{\xi x^0 + i\vec{k} \cdot \vec{x}} \xi \left[ 1 + \mathcal{O}\left(e^{x^0}\right) \right] \] (33)
\[ \Delta T^{ij}(x) = 2K \int d\vec{k} \delta \hat{T}_k (-k^i k^j + \delta^{ij}) e^{\xi x^0 + i\vec{k} \cdot \vec{x}} \left[ 1 + \mathcal{O}\left(e^{x^0}\right) \right]. \] (34)

To extract the leading behavior at future infinity $x^0 \rightarrow \infty$, we can use a contour integration trick which is described in Subsection 2.2 in [11]. Defining the analytic continuation of $\Delta T^{\mu\nu}(N)$ to complex values,

\[ \Delta \hat{T}^{\mu\nu}(s) \equiv \Delta T^{\mu\nu}(N \rightarrow -s) \] (35)

\footnote{The expression for $\Delta A_{\vec{k}}^{00}(N)$ cannot be proven directly by using the results in [10], but is found in a generalization of this calculation. We thank H. Schomerus for discussions on this point.}
the asymptotic behavior of $\Delta T^{\mu \nu}(x)$ in (18) is determined by the residues of the first few poles of $z^{-s} \Delta T^{\mu \nu}(s)/\sin(\pi s)$ on the positive real $s$-axis. Following the analysis of [11], the residue contributions at $s = \xi, \xi + 1$ give

$$\Delta T^{00}(x) = 2K \int d\vec{k} \delta \hat{T}_k e^{i\vec{k} \cdot \vec{x}}(\omega_k^2 + 1) \omega_k^2 \left[ x^0 A_1(-i\omega_k) - B(-i\omega_k) \right] + \mathcal{O}(e^{-x^0})$$

$$\Delta T^{0j}(x) = -2iK \int d\vec{k} \delta \hat{T}_k k^j e^{i\vec{k} \cdot \vec{x}} \omega_k^2 A_1(-i\omega_k)$$

$$+ 2iK \int d\vec{k} \delta \hat{T}_k \frac{k^j e^{i\vec{k} \cdot \vec{x}}}{(2\pi \lambda)^{1-\omega_k+1}} \omega_k^2 \left[ C(-i\omega_k) + D(-i\omega_k) \log(2\pi \lambda) \right]$$

$$+ x^0 D(-i\omega_k) \log(2\pi \lambda) + \mathcal{O}(e^{-x^0})$$

$$\Delta T^{ij}(x) = 2K \int d\vec{k} \delta \hat{T}_k \frac{e^{i\vec{k} \cdot \vec{x}}}{(2\pi \lambda)^{1-\omega_k+1}} (-k^i k^j + \delta^{ij})$$

$$\times \left[ C(-i\omega_k) + D(-i\omega_k) \log(2\pi \lambda) + x^0 D(-i\omega_k) \right] e^{-x^0} \left[ 1 + \mathcal{O}(e^{-x^0}) \right].$$

First, we see that perturbations in the tachyon field give a nonhomogeneous contribution to the energy-momentum tensor of tachyon matter. Moreover, the leading contribution to the energy density $\Delta T^{00}$ actually grows linearly as a function of time. This is compensated by the nonzero momentum flow $\Delta T^{0i}$ that guarantees energy conservation. The pressure components $\Delta T^{ij}$ decay exponentially. The decaying terms depend on the coefficients

$$C(\xi) = \frac{\partial}{\partial s} \left[ \frac{\pi G(\xi + 1)^2 \Gamma(\xi - s + 2) G(2\xi - s + 1) G(-s + 1)}{\sin \pi s G(2\xi + 1) G(-s + 2)} \right]_{s=\xi+1}$$

$$D(\xi) = -\frac{A_1(\xi)}{\Gamma(-\xi)},$$

which were extracted from Eq. (30) of [11]. Here $A_1(\xi)$ is the one-point amplitude of Eq. (17), and $G(\xi)$ is the Barnes $G$ function. The nonvanishing terms depend on the $A_1(\xi)$, and on the coefficient

$$B(\xi) = \frac{\partial}{\partial s} \left[ \frac{\pi (2\pi \lambda)^{-s} G(\xi + 1)^2 G(2\xi - s + 1) G(-s + 1)}{\sin \pi s G(2\xi + 1) G(-s + 2)} \right]_{s=\xi}.$$ (38)

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