EXPLICIT CONSTANTS IN AVERAGES INVOLVING THE MULTIPLICATIVE ORDER

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Abstract. Let $a > 1$. Denote by $\ell_a(p)$ the multiplicative order of $a$ modulo $p$. We look for an estimate of sum of $\ell_a(p)$ over primes $p \leq x$ on average. When we average over $a \leq N$, we observe a statistic of $\text{CLi}(x)$.

P. J. Stephens [S, Theorem 1] proved this statistic for $N > \exp(c_1 \sqrt{\log x})$ for some positive constant $c_1$. Upon this result, the value of $c_1$ is at least $12e^9$ in [S], we reduce this value of $c_1$ to $3.42$ by a different method. In fact, [S, Theorem 1, 3] hold with $N > \exp(3.42 \sqrt{\log x})$, and [S, Theorem 2, 4] hold with $N > \exp(4.8365 \sqrt{\log x})$. Also, we improve the range of $y$, from $y \geq \exp((2 + \epsilon) \sqrt{\log x \log \log x})$ in [LP, Theorem 1], to $y > \exp(3.42 \sqrt{\log x})$.

1. Introduction

We use $p$, $q$ to denote prime numbers, and use $c_i$ to denote an absolute positive constants. Let $a \geq 1$ be an integer. Denote by $\ell_a(p)$ the multiplicative order of $a$ modulo prime $p$. Artin’s Conjecture on Primitive Roots (AC) states that for non-square non-unit $a$, $a$ is a primitive root modulo infinitely many primes $p$. Thus, $\ell_a(p) = p - 1$ for infinitely many primes $p$. Assuming the Generalized Riemann Hypothesis (GRH) for Dedekind zeta functions for Kummer extensions, C. Hooley [H] showed that $\ell_a(p) = p - 1$ for positive proportion of primes $p \leq x$. Thus, we expect that $\ell_a(p)$ is large, and close to $p - 1$ for large number of primes $p$. On average, we expect that $\ell_a(p)/(p - 1)$ behaves like a constant. P. J. Stephens (see [S, Theorem 1]) showed that if $N > \exp(c_1 \sqrt{\log x})$ then for any positive constant $A$,

$$N^{-1} \sum_{a \leq N} \sum_{p \leq x} \frac{\ell_a(p)}{p - 1} = \text{CLi}(x) + O \left( \frac{x}{\log^A x} \right),$$

where $C$ is Stephens’ constant:

$$C = \prod_p \left( 1 - \frac{p}{p^3 - 1} \right).$$

Although the value of the positive constant $c_1$ is not explicitly given in [S], but we see that $c_1$ is at least $12e^9$ (see [S Lemma 7]). The optimal value of $c_1$ along Stephens’ method is any positive number greater than $2\sqrt{2}e \approx 7.6885$. We replace $7.6885$ by $3.42$ by a different method.

Theorem 1.1. If $N > \exp(3.42 \sqrt{\log x})$, then for any positive constant $D$,

$$N^{-1} \sum_{a \leq N} \sum_{p \leq x} \frac{\ell_a(p)}{p - 1} = \text{CLi}(x) + O \left( \frac{x}{\log^D x} \right).$$

Similarly, we give an explicit constant $c_1$ in [S, Theorem 2].

Theorem 1.2. If $N > \exp(4.8365 \sqrt{\log x})$, then for any positive constant $E$,

$$N^{-1} \sum_{a \leq N} \left( \sum_{p < x} \frac{\ell_a(p)}{p - 1} - \text{CLi}(x) \right)^2 \ll \frac{x^2}{\log^E x}.\quad (2)$$

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Subsequently, we obtain an improvement of average results on
\[ P_a(x) = \{ p \leq x | a \text{ is a primitive root modulo } p \}. \]
(see \[ \text{S0} \] Theorem 1, 2]:)
If \( N > \exp(3.42 \sqrt{\log x}) \), then for any positive constant \( D \),
\begin{equation}
N^{-1} \sum_{a \leq N} P_a(x) = A \pi(x) + O \left( \frac{x}{\log^D x} \right)
\end{equation}
where \( A = \prod_p \left( 1 - \frac{1}{p^{\lambda(n) - 1}} \right) \) is Artin’s constant. Also, the normal order result:
If \( N > \exp(4.8365 \sqrt{\log x}) \), then for any positive constant \( E \),
\begin{equation}
N^{-1} \sum_{a \leq N} (P_a(x) - A \pi(x))^2 \ll \frac{x^2}{\log^E x}.
\end{equation}

Stephens also proved that the average number of prime divisors of \( a^n - b \) also asymptotic to \( C \text{Li}(x) \) in \[ \text{S} \] Theorem 3], and proved normal order result in \[ \text{S}, \text{Theorem 4} \]. The number \( N \) is rather large compared to those of Theorem 1, 2. \((N > x (\log x)^{c_2} \text{ in } \[ \text{S}, \text{Theorem 3} \], and \( N > x^2 (\log x)^{c_2} \text{ in } \[ \text{S}, \text{Theorem 4} \) respectively.) He mentioned that these could probably be improved by using the large sieve inequality as in Theorem 1, 2. However, he did not carry out the improvement in \[ \text{S} \]. Here, we state the improvement and prove them.

**Theorem 1.3.** If \( N > \exp(3.42 \sqrt{\log x}) \), then for any positive constant \( D \),
\begin{equation}
N^{-2} \sum_{a \leq N} \sum_{b \leq N} \sum_{\text{for some } n} \frac{p \leq x}{p | a^n - b} 1 = C \text{Li}(x) + O \left( \frac{x}{\log^D x} \right).
\end{equation}

**Theorem 1.4.** If \( N > \exp(4.8365 \sqrt{\log x}) \), then for any positive constant \( E \),
\begin{equation}
N^{-2} \sum_{a \leq N} \sum_{b \leq N} \left( \sum_{\text{for some } n} \frac{p \leq x}{p | a^n - b} 1 - C \text{Li}(x) \right)^2 \ll \frac{x^2}{\log^E x}.
\end{equation}

In \[ \text{C} \], Carmichael’s lambda function \( \lambda(n) \) is defined by the exponent of the group \((\mathbb{Z}/n\mathbb{Z})^*\). We say that \( a \) is a \( \lambda \)-primitive root modulo \( n \) if the order \( \ell_a(n) \) of \( a \) modulo \( n \) is exactly \( \lambda(n) \). Following the definitions and notations in \[ \text{LP} \],
\[ R(n) = \#\{ a \in (\mathbb{Z}/n\mathbb{Z})^* : a \text{ is a } \lambda \text{-primitive root modulo } n \}, \]
\[ N_a(x) = \#\{ n \leq x : a \text{ is a } \lambda \text{-primitive root modulo } n \}. \]
S. Li \[ \text{L} \] proved that for \( y \geq \exp((\log x)^{3/4}) \),
\begin{equation} \frac{1}{y} \sum_{a \leq y} N_a(x) \sim \sum_{n \leq x} \frac{R(n)}{n}. \end{equation}
This was further improved by S. Li and C. Pomerance \[ \text{LP} \], where they obtained (7) with range of \( y \):
\begin{equation} y \geq \exp((2 + \epsilon)\sqrt{\log x \log \log x}). \end{equation}
We prove (7) with a wider range of \( y \):

**Theorem 1.5.** If \( y > \exp(3.42 \sqrt{\log x}) \), then there exists a positive constant \( c_2 \) such that
\begin{equation} \frac{1}{y} \sum_{a \leq y} N_a(x) = \sum_{n \leq x} \frac{R(n)}{n} + O \left( x \exp(-c_2 \sqrt{\log x}) \right). \end{equation}
2. Proof of Theorems

2.1. Proof of Theorem 1.1 and 1.2. We begin with the following elementary lemma (see [B]):

**Lemma 2.1.** Let \( r \geq 1 \) and define \( \tau_r(a) \) to be the number of ways to write \( a \) as an ordered product of \( r \) positive integers. If \( N \geq 1 \), then we have

\[
\sum_{a \leq N} \tau_r(a) \leq \frac{1}{(r-1)!} N (\log N + 1)^{r-1}.
\]

The proof is by induction.

**Corollary 2.1.** Let \( c > 0 \). If \( N \geq 1 \) and \( r - 1 \leq c \log N \), then

\[
\sum_{a \leq N} \tau_r(a) \leq \frac{(1 + c)^{r-1}}{(r-1)!} N \log^{r-1} N.
\]

We define \( \tau'_r(a) \) to be the number of ways of writing \( a \) as ordered product of \( r \) positive integers, each of which does not exceed \( N \).

**Lemma 2.2.** We have

\[
\sum_{a \leq N} (\tau'_r(a))^2 \leq \left( \sum_{a \leq N} \tau_r(a) \right)^r.
\]

**Proof.**

\[
\sum_{a \leq N} (\tau'_r(a))^2 = \sum_{a_1, \ldots, a_r \leq N} \tau'_r(a_1) \cdots a_r \leq \sum_{a_1, \ldots, a_r \leq N} \tau'_r(a_1) \cdots \tau'_r(a_r)
= \left( \sum_{a \leq N} \tau'_r(a) \right)^r = \left( \sum_{a \leq N} \tau_r(a) \right)^r.
\]

\[\square\]

**Corollary 2.2.** Let \( c > 0 \). If \( N \geq 1 \) and \( r - 1 \leq c \log N \), then

\[
\sum_{a \leq N} (\tau'_r(a))^2 \leq \left( \frac{(1 + c)^{r-1}}{(r-1)!} N \log^{r-1} N \right)^r.
\]

We follow the notations in [S], and give a critical upper estimate of the following quantities:

**Lemma 2.3.** Let

\[
S_4 = \sum_{p \leq x} \sum_{\chi \pmod{p}} \frac{1}{\text{ord} \chi} \left| \sum_{a \leq N} \chi(a) \right|
\]

and

\[
S_{10} = \sum_{p \leq x} \sum_{q \leq x} \sum_{\chi \pmod{pq}} \frac{1}{\text{ord} \chi} \left| \sum_{a \leq N} \chi(a) \right|.
\]

The sum \( \sum^* \) denotes the sum over non-principal primitive characters. Then by H"older inequality and the large sieve inequality,

\[
S_4 \ll x^{1-\frac{1}{2r}} \left( x^\frac{1}{2r} + N^{\frac{1}{2r}} \right) \left( \sum_{a \leq N} (\tau'_r(a))^2 \right)^{\frac{1}{2r}}.
\]
and

\[ S_{10} \ll x^{1 + \frac{1}{2r}} (x^2 + N^\frac{1}{r}) \left( \sum_{n \leq N^r} (\tau_n(a))^2 \right)^{\frac{1}{r}}. \]

**Corollary 2.3.** By taking \( N^{r-1} < x^2 \leq N^r \), we have for \( r - 1 \leq c \log N \),

\[ S_4 \ll xN^{\frac{1}{2} + \frac{1}{2r}} \left( \frac{(1 + c)^{r-1}}{(r - 1)!} N \log^{r-1} N \right)^{\frac{1}{2}} = xN^{\frac{1}{2} + \frac{1}{2r}} \left( \frac{(1 + c)^{r-1}}{(r - 1)!} \log^{r-1} N \right)^{\frac{1}{2}}. \]

By taking \( N^{r-1} < x^4 \leq N^r \), we have for \( r - 1 \leq c \log N \),

\[ S_{10} \ll x^2 N^{\frac{1}{2} + \frac{1}{2r}} \left( \frac{(1 + c)^{r-1}}{(r - 1)!} N \log^{r-1} N \right)^{\frac{1}{2}} = x^2 N^{\frac{1}{2} + \frac{1}{2r}} \left( \frac{(1 + c)^{r-1}}{(r - 1)!} \log^{r-1} N \right)^{\frac{1}{2}}. \]

By \([S0]\), we may assume that \( N = \exp(K \sqrt{\log x}) \). Then \( N^{\frac{1}{2r}} = \exp(O(K^2)) = (\log x)^{O(1)} \) and we have the following:

\[ \log N = K \sqrt{\log x}, \]

\[ \log \log N = \log K + \frac{1}{2} \log \log x, \]

\[ r - 1 < \frac{2 \log x}{\log N} = \frac{2}{K} \sqrt{\log x} \leq r. \]

By Stirling’s formula, we have

\[ S_4 \ll x(\log x)^{O(1)} N \exp \left( -\frac{1}{4} \log N + \frac{1}{2}(r - 1) \log(1 + c) - \frac{1}{2} \log(r - 1)! + \frac{r-1}{2} \log \log N \right) \]

\[ \ll xN \exp \left( \sqrt{\log x} \left( -\frac{K}{4} + \frac{1}{K} \log(1 + c) - \frac{1}{K} \log 2 + \frac{1}{K} + \frac{2 \log K}{K} + o(1) \right) \right). \]

From \( r - 1 \leq c \log N \), we have \( \frac{2}{K} \leq cK \). Taking \( c = \frac{2}{K^2} \), we have

\[ S_4 \ll xN \exp \left( \sqrt{\log x} \left( -\frac{K}{4} + \frac{1}{K} \log \left(1 + \frac{2}{K^2}\right) - \frac{1}{K} \log 2 + \frac{1}{K} + \frac{2 \log K}{K} + o(1) \right) \right). \]

If \( K \geq 3.42 \), then we see that

\[ f_1(K) = -\frac{K}{4} + \frac{1}{K} \log \left(1 + \frac{2}{K^2}\right) - \frac{1}{K} \log 2 + \frac{1}{K} + \frac{2 \log K}{K} < 0. \]

We finally obtain that

**Lemma 2.4.** If \( N > \exp(3.42 \sqrt{\log x}) \), then there is a positive constant \( c_2 \) such that

\[ S_4 \ll xN \exp \left( -c_2 \sqrt{\log x} \right). \]

Now, we deal with \( S_{10} \). Let \( N = \exp(K \sqrt{\log x}) \). By \([S0]\), we may assume that \( K \leq 6 \sqrt{\log \log x} \). Then \( N^{\frac{1}{2r}} = \exp(O(K^2)) = (\log x)^{O(1)} \) and we have the following:

\[ \log N = K \sqrt{\log x}, \]

\[ \log \log N = \log K + \frac{1}{2} \log \log x, \]

\[ r - 1 < \frac{4 \log x}{\log N} = \frac{4}{K} \sqrt{\log x} \leq r. \]
By Stirling’s formula, we have
\[
S_{10} \ll x^2 \log x \left( \frac{1}{4} \log N + \frac{1}{2} \log(1 + c) - \frac{1}{2} \log(1 + c) + \frac{r - 1}{2} \log \log N \right)
\]
\[
\ll x^2 N \exp \left( \frac{K}{4} + \frac{2}{K} \log \left( 1 + \frac{4K}{K^2} \right) - \frac{2}{K} \log 4 + \frac{2}{K} + \frac{4 \log K}{K} + o(1) \right).
\]
Theorem 1.1. As in [S, Theorem 1], we define a character sum
\[
\tilde{S}_{10} = \sum_{p \leq x \atop p \neq q} \sum_{\chi \mod p} \sum_{a \leq N} \frac{1}{\ord(\chi)} \sum_{a \leq N} \chi(a).
\]
where \(\Sigma^*\) is over non-principal characters. Then
\[
\tilde{S}_{10} \ll x^2 N \exp \left( -c_3 \sqrt{\log x} \right).
\]
For the estimate for \(\tilde{S}_{10}\), note that
\[
\tilde{S}_{10} \ll S_{10} + xS_4.
\]

Lemma 2.5. If \(N > \exp(4.8365 \sqrt{\log x})\), then there is a positive constant \(c_3\) such that
\[
S_{10} \ll x^2 N \exp \left( -c_3 \sqrt{\log x} \right).
\]
Denote by \(\tilde{S}_{10}\) the following character sum:
\[
\tilde{S}_{10} = \sum_{p \leq x \atop p \neq q} \sum_{\chi \mod p} \sum_{a \leq N} \frac{1}{\ord(\chi)} \sum_{a \leq N} \chi(a).
\]
where \(\Sigma^*\) is over non-principal characters. Then
\[
\tilde{S}_{10} \ll x^2 N \exp \left( -c_3 \sqrt{\log x} \right).
\]
Proof of Theorem 1.1. As in [S, Theorem 1], we define a character sum \(c_r(\chi)\) where \(\chi\) is a Dirichlet character modulo \(p\). For \(r \mid p - 1\), define
\[
c_r(\chi) = \frac{1}{p - 1} \sum_{a \leq p} \chi(a).
\]
Then we have
\[
S_3 = N^{-1} \sum_{a \leq p} \sum_{\ell_a(p) = \frac{p - 1}{r}} \sum_{|w| \leq x} \frac{1}{\ell_a(p) - \frac{p - 1}{r}} w^{-1} = N^{-1} \sum_{a \leq p} \sum_{|w| \leq x} \sum_{\chi \mod p} \sum_{a \leq N} c_w(\chi) \chi(a)
\]
\[
= N^{-1} \sum_{p \leq x \atop p \mid a} \sum_{|w| \leq x} \sum_{\chi \mod p} c_w(\chi) \chi(a)
\]
\[
= N^{-1} \sum_{p \leq x \atop p \mid a} \sum_{|w| \leq x} \frac{\phi(p - 1)}{w(p - 1)} \left( N + O(1) + O \left( \frac{N}{p} \right) \right) + O \left( N^{-1} \sum_{p \leq x \atop p \mid a} \sum_{|w| \leq x} \sum_{\chi \mod p} c_w(\chi) \chi(a) \right)
\]
\[
= \sum_{p \leq x \atop p \mid a} \frac{\phi(p - 1)}{w(p - 1)} + \frac{x}{N \log x} + O(\log \log x) + O \left( \frac{S_4 \log x \log \log x}{N} \right)
\]
\[
= C \text{Li}(x) + O \left( \frac{x}{\log^3 x} \right) + O \left( x \exp \left( -c_4 \sqrt{\log x} \right) \right)
\]
by Lemma 2.4 and [S, Lemma 12]. This completes the proof, since the second error term is dominated by the first.
Proof of Theorem 1.2. As in [S, Theorem 2], let

\[ S_5 = N^{-1} \sum_{\alpha \leq N} \left( \sum_{p \leq x} \frac{\ell_\alpha(p)}{p-1} - C \text{Li}(x) \right)^2, \]

then

\[ S_5 = N^{-1} \sum_{p \leq x} \sum_{q \leq x} \sum_{a \leq N} \frac{\ell_\alpha(p)\ell_\alpha(q)}{(p-1)(q-1)} - C^2 \text{Li}^2(x) + O \left( \frac{x^2}{\log^E x} \right). \]

Let

\[ S_6 = N^{-1} \sum_{p \leq x} \sum_{q \leq x} \sum_{a \leq N} \frac{\ell_\alpha(p)\ell_\alpha(q)}{(p-1)(q-1)}. \]

Stephens decomposed $S_6$ into three parts:

\[
S_6 = N^{-1} \sum_{p \leq x} \sum_{q \leq x} \sum_{a \leq N} \frac{1}{w} \sum_{\chi_1 \pmod{p}} c_w(\chi_1) \sum_{\chi_2 \pmod{q}} c_t(\chi_2) \sum_{a \leq N} \chi_1 \chi_2(a)
\]

\[ = S_7 + 2S_8 + S_9 \]

where $S_7$, $S_8$, $S_9$ are the contributions to $S_6$ when both $\chi_1, \chi_2$ are principal, when one of $\chi_1, \chi_2$ is principal, and when neither $\chi_1, \chi_2$ is principal, respectively. Then

\[ S_7 = C^2 \text{Li}^2(x) + O \left( \frac{x^2}{\log^E x} \right) + O \left( \frac{x^2}{N \log^2 x} \right), \]

\[ S_8 \ll \frac{x^2}{\log^E x}, \]

and

\[ S_9 \ll \frac{S_{10} \log^2 x \log \log x}{N}. \]

Then we have by Lemma 2.5,

\[ S_5 \ll C^2 \text{Li}^2(x) - C^2 \text{Li}^2(x) + O \left( \frac{x^2}{\log^E x} \right) + O \left( \frac{x^2}{N \log^2 x} \right) + O \left( x^2 \exp \left( -c_5 \sqrt{\log x} \right) \right). \]

Since the last two error terms are dominated by $\frac{x^2}{\log^E x}$, this completes the proof of Theorem 1.2. \qed

2.2. Proof of Theorem 1.3 and 1.4.
Proof of Theorem 1.3. As in [2, Theorem 3], we write the sum as

\[
N^{-2} \sum_{b \leq N} \sum_{p \leq x} \sum_{c_{w}(\chi) \leq N} \sum_{\chi(\alpha)} \frac{\phi(p-1)}{p-1} \left( N + O(1) + \left( \frac{N}{p} \right) \right) + O \left( \frac{\tau_2(p-1)}{\log x} \right)
\]

To treat the last error term, let \( c_2 \) be the positive constant in Lemma 2.4. Choose any positive constant \( c_6 \) smaller than \( c_2 \), and split the sum into two parts:

\[
N^{-1} \sum_{p \leq x} \sum_{\chi(\alpha)} \frac{1}{\text{ord} \chi} \sum_{a \leq N} \chi(a) = \Sigma_1 + \Sigma_2,
\]

where \( \Sigma_1 \) is the sum over \( p \)'s with \( \tau_2(p-1) < \exp(c_6 \sqrt{\log x}) \), and \( \Sigma_2 \) is the sum over remaining \( p \)'s. Then by Lemma 2.4,

\[
\Sigma_1 \ll N^{-1} \exp(c_6 \sqrt{\log x}) x N \exp(-c_2 \sqrt{\log x}) \ll x \exp(-c_7 \sqrt{\log x}).
\]

By an elementary estimate \( \sum_{n \leq x} \tau_2(n)^3 \ll x \log^7 x \),

\[
\Sigma_2 \ll N^{-1} \sum_{p \leq x} \tau_2(p-1)^2 N \ll \sum_{p \leq x} \frac{\tau_2(p-1)^3}{\exp(c_6 \sqrt{\log x})} \ll x \exp(-c_7 \sqrt{\log x}).
\]

Thus, we obtain

\[
N^{-2} \sum_{b \leq N} \sum_{p \leq x} \sum_{\chi(\alpha)} \frac{\phi(p-1)}{p-1} + O \left( x \exp(-c_7 \sqrt{\log x}) \right).
\]
Now, we treat the first sum on the right side. Again, by writing it with character sums,

\[ N^{-1} \sum_{b \leq N} \sum_{p \leq x \mid p-1} \sum_{\ell_b(p) \mid \frac{p-1}{w}} \phi \left( \frac{p-1}{w} \right) p-1 = N^{-1} \sum_{p \leq x \mid p-1} \sum_{\ell_b(p) \mid \frac{p-1}{w}} \phi \left( \frac{p-1}{w} \right) p-1 \sum_{b \leq N} \sum_{\ell_b(p) \mid \frac{p-1}{w}} 1 \]

\[ = N^{-1} \sum_{p \leq x \mid p-1} \sum_{\ell_b(p) \mid \frac{p-1}{w}} \phi \left( \frac{p-1}{w} \right) p-1 \sum_{\chi \pmod{p}} c_{tw}(\chi) \sum_{b \leq N} \chi(b) \]

\[ = N^{-1} \sum_{p \leq x \mid p-1} \sum_{\ell_b(p) \mid \frac{p-1}{w}} \phi \left( \frac{p-1}{w} \right) p-1 \phi \left( \frac{p-1}{tw} \right) \left( N + O(1) + O \left( \frac{N}{p} \right) \right) + E \]

\[ = \sum_{p \leq x \mid p-1} \sum_{\ell_b(p) \mid \frac{p-1}{w}} \phi \left( \frac{p-1}{w} \right) p-1 \sum_{\chi \pmod{p}} c_{tw}(\chi) \sum_{b \leq N} \chi(b) \]

where

\[ (41) \]

We split the sum as before,

\[ E = N^{-1} \sum_{p \leq x \mid p-1} \sum_{\ell_b(p) \mid \frac{p-1}{w}} \phi \left( \frac{p-1}{w} \right) p-1 \sum_{\chi \pmod{p}} c_{tw}(\chi) \sum_{b \leq N} \chi(b) \]

\[ \leq \sum_{p \leq x \mid p-1} \sum_{\ell_b(p) \mid \frac{p-1}{w}} \phi \left( \frac{p-1}{w} \right) p-1 \sum_{\chi \pmod{p}} \frac{1}{\ord \chi} \sum_{b \leq N} \chi(b) \]

\[ = N^{-1} \sum_{p \leq x \mid p-1} \sum_{\ell_b(p) \mid \frac{p-1}{w}} \phi \left( \frac{p-1}{w} \right) p-1 \sum_{\chi \pmod{p}} \frac{1}{\ord \chi} \sum_{b \leq N} \chi(b) \]

\[ = \Sigma_3 + \Sigma_4, \]

where \( \Sigma_3 \) is over \( p \)'s with \( \tau_3(p-1) < \exp(c_6 \sqrt{\log x}) \), and \( \Sigma_4 \) is over remaining \( p \)'s. By the same argument, we have

\[ (42) \quad \Sigma_3 + \Sigma_4 \ll x \exp(-c_7 \sqrt{\log x}). \]

Therefore,

\[ (43) \quad N^{-2} \sum_{b \leq N} \sum_{p \leq x \mid p-1} \phi \left( \frac{p-1}{w} \right) p-1 \sum_{\chi \pmod{p}} \frac{1}{\ord \chi} \sum_{b \leq N} \chi(b) \]

\[ = \sum_{p \leq x \mid p-1} \sum_{\ell_b(p) \mid \frac{p-1}{w}} \phi \left( \frac{p-1}{w} \right) p-1 \phi \left( \frac{p-1}{tw} \right) + O \left( x \exp(-c_7 \sqrt{\log x}) \right). \]

By the elementary identity \( \sum_{d \mid n} \phi(d) = n \), we have

\[ (44) \quad N^{-2} \sum_{b \leq N} \sum_{p \leq x \mid p-1} \phi \left( \frac{p-1}{w} \right) p-1 \sum_{\chi \pmod{p}} \frac{1}{\ord \chi} \sum_{b \leq N} \chi(b) \]

\[ = \sum_{p \leq x \mid p-1} \sum_{\ell_b(p) \mid \frac{p-1}{w}} \phi \left( \frac{p-1}{w} \right) + O \left( x \exp(-c_7 \sqrt{\log x}) \right). \]

Then Theorem 1.3 follows by [S, Lemma 12]. \( \square \)
Proof of Theorem 1.4. Using the same argument as for Theorem 1.2 we deduce that

\[ N^{-2} \sum_{a \leq N} \sum_{b \leq N} \left( \sum_{p \leq x, \text{for some } n} 1 - CLi(x) \right)^2 = N^{-2} \sum_{a \leq N} \sum_{b \leq N} \sum_{q \leq x} \sum_{q \neq p} \sum_{q \mid a^n - b} \sum_{\ell_b(p) \mid \ell_a(q)} 1 - C^2 Li^2(x) + O \left( \frac{x^2 \log^E x}{\log x} \right) \]

\[ = N^{-2} \sum_{p \leq x} \sum_{q \leq x} \sum_{q \neq p} \sum_{q \mid a^n - b} \sum_{\ell_b(p) \mid \ell_a(q)} 1 - C^2 Li^2(x) + O \left( \frac{x^2 \log^E x}{\log x} \right) \]

\[ = N^{-2} \sum_{p \leq x} \sum_{q \leq x} \sum_{q \neq p} \sum_{q \mid a^n - b} \sum_{a \leq N} \sum_{b \leq N} \chi_1 \chi_3(a) \chi_2 \chi_4(b) \]

\[ - C^2 Li^2(x) + O \left( \frac{x^2 \log^E x}{\log x} \right). \]

We prove that if any one of \( \chi_i, i = 1, 2, 3, 4 \) is non-principal, then they contribute \( O(x^2 / \log^E x) \). In this case, either \( \chi_1 \chi_3 \) or \( \chi_2 \chi_4 \) is non-principal. Suppose that \( \chi_1 \chi_3 \) is non-principal. Then, the contribution is

\[ \ll N^{-1} (\log \log x)^4 \sum_{p \leq x} \sum_{q \leq x} \sum_{u \mid p-1} \sum_{u \mid q-1} \chi_1 \chi_2 \chi_3 \chi_4 \sum_{\ell_b(p) \mid \ell_a(q)} \frac{1}{\text{ord}(\chi_1) \text{ord}(\chi_2) \text{ord}(\chi_3) \text{ord}(\chi_4)} \sum_{a \leq N} \chi_1 \chi_3(a) \]

\[ \ll N^{-1} (\log \log x)^4 \sum_{p \leq x} \sum_{q \leq x} \sum_{u \mid p-1} \sum_{u \mid q-1} \chi_1 \chi_2 \chi_3 \chi_4 \sum_{\ell_b(p) \mid \ell_a(q)} \frac{1}{\text{ord}(\chi_1) \text{ord}(\chi_3)} \cdot \frac{1}{\text{ord}(\chi_2) \text{ord}(\chi_4)} \sum_{a \leq N} \chi_1 \chi_3(a) \]

\[ \ll N^{-1} (\log \log x)^4 \sum_{p \leq x} \sum_{q \leq x} \sum_{u \mid p-1} \sum_{u \mid q-1} \chi_1 \chi_2 \chi_3 \chi_4 \sum_{\ell_b(p) \mid \ell_a(q)} \frac{1}{\text{ord}(\chi_1) \text{ord}(\chi_3)} \sum_{a \leq N} \chi_1 \chi_3(a) \]

\[ = E_1 + E_2, \]

where \( E_1 \) is the sum over \( p, q \)'s with \( \tau_3(p-1) \tau_3(q-1) \tau_2(p-1) \tau_2(q-1) < \exp(c_8 \sqrt{\log x}) \), and \( E_2 \) is the sum over remaining \( p, q \)'s. Here, we let \( c_8 \) be a positive number smaller than \( c_3 \) in Lemma 2.5. By Lemma 2.5,

\[ E_1 \ll x^2 \exp(-c_9 \sqrt{\log x}). \]

For \( E_2 \), we have

\[ E_2 \ll \sum_{p \leq x} \sum_{q \leq x} \tau_3(p-1) \tau_3(q-1) \tau_2(p-1)^2 \tau_2(q-1)^2 \tau_3(p-1) \tau_3(q-1) \tau_2(p-1) \tau_2(q-1) \exp(c_8 \sqrt{\log x}) \]

\[ \ll \sum_{p \leq x} \tau_3(p-1)^2 \tau_2(p-1)^3 \tau_3(q-1)^2 \tau_2(q-1)^3 \exp(-c_8 \sqrt{\log x}) \]

\[ \ll x^2 \exp(-c_9 \sqrt{\log x}). \]

The case when \( \chi_2 \chi_4 \) is non-principal, is treated similarly and it also contributes \( \ll x^2 \exp(-c_9 \sqrt{\log x}). \)
Now, we find that the main contribution is when every character $\chi_1, \ldots, \chi_4$ is principal. In fact,

$$N^{-2} \sum_{p \leq x} \sum_{q \leq x} \sum_{q \nmid p} \sum_{|w| = q-1} \phi \left( \frac{p-1}{w} \right) \phi \left( \frac{p-1}{w} \right) \phi \left( \frac{q-1}{u} \right) \phi \left( \frac{q-1}{u} \right) \left( N + O(1) + O \left( \frac{N}{p} \right) + O \left( \frac{N}{q} \right) \right)^2$$

$$= N^{-2} \sum_{p \leq x} \sum_{q \leq x} \sum_{q \nmid p} \sum_{|w| = q-1} \phi \left( \frac{p-1}{w} \right) \phi \left( \frac{q-1}{u} \right) \left( N^2 + O(N) + O \left( \frac{N^2}{p} \right) + O \left( \frac{N^2}{q} \right) \right)$$

$$= \sum_{p \leq x} \sum_{q \leq x} \sum_{q \nmid p} \sum_{|w| = q-1} \phi \left( \frac{p-1}{w} \right) \phi \left( \frac{q-1}{u} \right) \left( N \log \log x + O \left( \frac{x^2}{N \log^2 x} \right) \right) + O \left( \frac{x^2}{\log^2 x} \right) \right)$$

Therefore,

$$N^{-2} \sum_{a \leq N} \sum_{b \leq N} \left( \sum_{p \leq x} \frac{1}{\phi(n)} \left( 1 - C\text{Li}(x) \right) \right)^2 = C^2 \text{Li}^2(x) - C^2 \text{Li}^2(x) + O \left( \frac{x^2}{\log^2 x} \right).$$

This completes the proof of Theorem 1.4. \square

2.3. Proof of Theorem 1.5. Following the definitions in [LP],

$$\Delta_q(n) = \# \{ \text{cyclic factors } C_{q^r} \text{ in } (\mathbb{Z}/n\mathbb{Z})^\ast : q^r || \lambda(n) \},$$

then

$$R(n) = \phi(n) \prod_{q || \lambda(n)} \left( 1 - \frac{1}{q^{\Delta_q(n)}} \right).$$

Let $\text{rad}(m)$ denote the largest square-free divisor of $m$. Let

$$E(n) = \{ a \in (\mathbb{Z}/n\mathbb{Z})^\ast : a^{\frac{\lambda(n)}{\text{rad}(n)}} \equiv 1 \pmod{n} \},$$

and we say that $\chi$ is elementary character if $\chi$ is trivial on $E(n)$. For each square free $h|\phi(n)$, let $\rho_n(h)$ be the number of elementary characters mod $n$ or order $h$. Then

$$\rho_n(h) = \prod_{q || h} \left( q^{\Delta_q(n)} - 1 \right).$$

For a character $\chi \mod n$, let

$$c(\chi) = \frac{1}{\phi(n)} \sum_{b} \chi(b),$$

where the sum is over $\lambda$-primitive roots in $[1, n]$. Then

$$|c(\chi)| \leq \bar{c}(\chi),$$

where

$$\bar{c}(\chi) = \begin{cases} \frac{1}{\rho_n(\text{ord}(\chi))}, & \text{if } \chi \text{ is elementary,} \\ 0, & \text{otherwise.} \end{cases}$$

For the proof of above, see [LP Proposition 2].
Let \( X(n) \) be the set of non-principal elementary characters mod \( n \). In [L], it is shown that

\[
\sum_{a \leq y} N_a(x) = y \sum_{n \leq x} \frac{R(n)}{n} + B(x, y) + O(x \log x),
\]

where

\[
B(x, y) = \sum_{n \leq x} \sum_{\chi \in X(n)} c(\chi) \sum_{a \leq y} \chi(a).
\]

Following the proof in [LP],

\[
|B(x, y)| \leq \sum_{d \leq x} |\mu(d)| S_d,
\]

where

\[
S_d \ll \frac{xy}{d^2} \exp \left( 3 \sqrt{\frac{\log x}{\log \log x}} \right).
\]

We use this when \( d \) is large.

Let \( \chi_{0,n} \) be the principal character modulo \( n \). For positive integer \( k \) and reals \( w, z \), define

\[
F(k, z) = \sum_{\rad(m) \mid k} \frac{1}{m} \sum_{\chi \mod k}^* \bar{c}(\chi \chi_{0,km}) \left| \sum_{a \leq z} \chi(a) \right|,
\]

\[
T(w, z) = \sum_{k \leq w} F(k, z),
\]

\[
S(w, z) = w \sum_{k \leq w} \frac{1}{k} F(k, z) = T(w, z) + w \int_1^w \frac{1}{u^2} T(u, z) \, du,
\]

and

\[
S_d \leq S \left( \frac{x}{d}, \frac{y}{d} \right).
\]

We want to estimate \( S_d \) using the estimate of \( T(w, z) \). Because of the integral in \( S(w, z) \), we need an estimate of \( T(u, z) \) for \( 1 \leq u \leq w \). First, assume that \( w \leq z^{\frac{3}{2}} \). By applying Pólya-Vinogradov inequality, Li and Pomerance showed in [LP, Lemma 5] that

\[
T(w, z) \ll w^{\frac{3}{4}} \exp \left( 3 \sqrt{\frac{\log w}{\log \log w}} \right) \ll wz^{\frac{3}{4}} \exp \left( 3 \sqrt{\frac{\log w}{\log \log w}} \right).
\]

Suppose now that \( w > z^{\frac{3}{2}} \). By Hölder inequality with \( r = \left\lceil \frac{2w}{\log w} \rightceil \),

\[
T(w, z)^{2r} \leq A^{2r-1} B,
\]

where

\[
A = \sum_{k \leq w} \frac{1}{m} \sum_{\rad(m) \mid k}^* \bar{c}(\chi \chi_{0,km}) \frac{2r}{2r},
\]

and

\[
B = \sum_{k \leq w} \frac{1}{m} \sum_{\chi \mod k}^* \left| \sum_{a \leq z} \chi(a) \right| \sum_{k \leq w} \frac{k}{\phi(k)} \sum_{\chi \mod k}^* \left| \sum_{a \leq z} \chi(a) \right|^{2r}.
\]

Then by \( 0 \leq \bar{c}(\chi \chi_{0,km}) \leq 1 \),

\[
A \ll w \exp \left( 3 \sqrt{\frac{\log w}{\log \log w}} \right).
\]
By large sieve inequality,
\[(56) \quad B \ll (w^2 + z^r) \sum_{a \leq z^r} (\tau'_r(a))^2\]

If \(w > z^{\frac{3}{2}}\), then \(w \frac{1}{w} > z \frac{3}{w}\). By the method in Lemma 2.4, we have

**Lemma 2.6.** If \(z > \exp(4.18\sqrt{\log w})\), then
\[(57) \quad T(w, z) \ll wz^\frac{11}{16} \exp \left( \sqrt{\log w} \left( f_1(4.18) + \frac{4.18}{4} + \epsilon \right) \right)\]

**Lemma 2.7.** If \(\exp(3.419906\sqrt{\log w}) < z \leq \exp(16\sqrt{\log w})\), then
\[(58) \quad T(w, z) \ll wz^\frac{3}{16} \exp \left( \sqrt{\log w} \left( f_1(3.419906) + \frac{3.419906}{4} + \epsilon \right) \right)\]

Therefore, by \(S(w, z) = T(w, z) + w \int_1^w T(u, z)du\),

**Lemma 2.8.** If \(z > \exp(4.18\sqrt{\log w})\), then
\[(59) \quad S(w, z) \ll wz^\frac{13}{16} \exp \left( \sqrt{\log w} \left( f_1(4.18) + \frac{4.18}{4} + 2\epsilon \right) \right) + w \log w \cdot z^\frac{7}{w}\]

**Lemma 2.9.** If \(\exp(3.419906\sqrt{\log w}) < z \leq \exp(16\sqrt{\log w})\), then
\[(60) \quad S(w, z) \ll wz^\frac{3}{16} \exp \left( \sqrt{\log w} \left( f_1(3.419906) + \frac{3.419906}{4} + 2\epsilon \right) \right) + w \log w \cdot z^\frac{7}{w}\]

Suppose that \(y > \exp(4.2\sqrt{\log x})\). If \(\frac{y}{d} > \exp \left( 4.18 \sqrt{\log \frac{x}{d}} \right)\), then by Lemma 2.8,
\[
S \left( \frac{x}{d}, \frac{y}{d} \right) \ll \frac{x}{d} \left( \frac{y}{d} \right)^{\frac{11}{16}} \exp \left( \sqrt{\log \frac{x}{d}} \left( f_1(4.18) + \frac{4.18}{4} + 2\epsilon \right) \right) \ll \frac{xy}{d^\frac{11}{16}} \exp \left( \sqrt{\log \frac{x}{d}} \left( f_1(4.18) + \frac{4.18}{4} + 2\epsilon \right) \right)
\]
\[
\ll \frac{xy}{d^\frac{11}{16}} \exp \left( \sqrt{\log \frac{x}{d}} \left( -\frac{3}{16} \cdot 4.2 + f_1(4.18) + \frac{4.18}{4} + 2\epsilon \right) \right)
\]

Since \(-\frac{3}{16} \cdot 4.2 + f_1(4.18) + \frac{4.18}{4} < 0\), the contribution of these \(d\)'s is
\[
\ll xy \exp \left( -c_{11} \sqrt{\log x} \right)
\]

If \(\frac{y}{d} \leq \exp \left( 4.18 \sqrt{\log \frac{x}{d}} \right)\), then clearly \(d \geq \exp \left( 0.02 \sqrt{\log x} \right)\). Then by (50), the contribution of these \(d\)'s is
\[
\ll xy \exp \left( -c_{12} \sqrt{\log x} \right).
\]

Therefore, we obtain the following weaker version of Theorem 1.5:

**Theorem 2.1.** If \(y > \exp(4.2\sqrt{\log x})\), then
\[(61) \quad \frac{1}{y} \sum_{a \leq y} N_a(x) = \sum_{n \leq x} \frac{R(n)}{n} + O \left( x \exp \left( -c_{13} \sqrt{\log x} \right) \right) \]

Now, assume that \(\exp(3.42\sqrt{\log x}) < y \leq \exp(16\sqrt{\log x})\), then for large \(x\) and for \(\frac{y}{d} \geq \exp \left( 3.419907 \sqrt{\log x} \right)\),
\[
\exp \left( 16 \sqrt{\log \frac{x}{d}} \right) \geq \exp \left( \frac{16 \sqrt{\log x}}{d} \right) \geq \frac{y}{d} \geq \exp \left( 3.419907 \sqrt{\log x} \right) > \exp \left( 3.419906 \sqrt{\log \frac{x}{d}} \right).
\]

Thus by Lemma 2.9,
\[(62) \quad S \left( \frac{x}{d}, \frac{y}{d} \right) \ll \frac{x}{d} \left( \frac{y}{d} \right)^\frac{3}{4} \exp \left( \sqrt{\log \frac{x}{d}} \left( f_1(3.419906) + \frac{3.419906}{4} + 2\epsilon \right) \right) + \frac{x}{d} \log x \cdot \left( \frac{y}{d} \right)^\frac{7}{8} \].
We sum this for \( y \geq \exp \left( 3.419907 \sqrt{\log x} \right) \). By \(-\frac{3.419907}{4} + f_1(3.419906) + \frac{3.419906}{4} < 0\), the contribution is
\[
\ll xy \exp \left( \sqrt{\log x} \left( -\frac{3.419907}{4} + f_1(3.419906) + \frac{3.419906}{4} + 2\epsilon \right) \right) + \frac{xy \log x}{y^\frac{1}{8}}
\ll xy \exp \left( -c_{14} \sqrt{\log x} \right).
\]

If \( y < \exp \left( 3.419907 \sqrt{\log x} \right) \), then \( d > \exp \left( 0.000093 \sqrt{\log x} \right) \). By (50), the contribution of these \( d \)'s is
\[
\ll xy \exp \left( -c_{15} \sqrt{\log x} \right).
\]

This completes the proof of Theorem 1.5.

3. SOME NUMERICAL DATA AND CALCULUS REMARKS

The numerical values 3.419906, 3.419907, and 3.42 are positive numbers greater than the unique solution to the equation \( f_1(K) = 0 \). The values of \( f_1(K) \) for these numbers are negative. Thus, 3.42 in Theorem 1.1, 1.3, 1.5 can be replaced by any positive number greater than the unique solution to \( f_1(K) = 0 \), which is numerically 3.41990570065660 \cdots \). Similarly, 4.2 can be replaced by any positive number greater than the unique solution to \( -\frac{3}{16} K + f_1(K) + \frac{K}{4} = 0 \), which is numerically 4.17980309602625 \cdots \). Also, 4.8365 in Theorem 1.2, 1.4 can be replaced by any positive number greater than 4.83647702390563 \cdots \). The function \( f_1(K) + \frac{K}{4} \) which can be simplified as \( \frac{1}{K} \left( \log \left( \frac{K^2}{2} + 1 \right) + 1 \right) \), is a decreasing function for \( K > 0 \) and it converges to 0 as \( K \to \infty \). The author used Wolfram Alpha for numerical calculations.

4. FURTHER DEVELOPMENTS

The author thinks that the corresponding normal order result for Theorem 1.5 could probably be done such as: If \( y > \exp(4.8365 \sqrt{\log x}) \), then
\[
(63) \quad \frac{1}{y} \sum_{a \leq y} \left( N_a(x) - \sum_{n \leq x} \frac{R(n)}{n} \right)^2 \ll x^2 \exp \left( -c_{16} \sqrt{\log x} \right).
\]

The author also thinks that, in Theorem 1.1, 1.2, 1.3 and 1.4, if we replace \( CLi(x) \) by \( \sum_{p \leq x} \sum_{w \mid p-1} \phi(w) \frac{w^{-1}}{w(w-1)} \), then \( (\log x)^D \), \( (\log x)^E \) in the error terms may be replaced by \( \exp \left( c_{17} \sqrt{\log x} \right) \).

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