On étale hypercohomology of henselian regular local rings with values in $p$-adic étale Tate twists

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Abstract
Let $R$ be the henselization of a local ring of a semistable family over the spectrum of a discrete valuation ring of mixed characteristic $(0, p)$ and $k$ the residue field of $R$. In this paper, we prove an isomorphism of étale hypercohomology groups $H^{n+1}_{ét}(R, \mathbb{T}_r(n)) \simeq H^1_{ét}(k, W_rΩ^n_{\log})$ for any integers $n \geq 0$ and $r > 0$ where $\mathbb{T}_r(n)$ is the $p$-adic Tate twist and $W_rΩ^n_{\log}$ is the logarithmic Hodge-Witt sheaf. As an application, we prove the local-global principle for Galois cohomology groups over function fields of curves over an excellent henselian discrete valuation ring of mixed characteristic.

1 Introduction

The main objective of this paper is to study a certain étale hypercohomology in the mixed characteristic cases. In order to understand a scheme $X$ over the spectrum of a discrete valuation ring, it is effective to observe the special fiber of $X$. So we start in a positive characteristic situation.

Let $k$ be a field of positive characteristic, $X$ a normal crossing variety over $k$, i.e. a pure-dimensional scheme of finite type over Spec($k$) which is everywhere étale locally isomorphic to $\text{Spec}(k[T_0, \ldots, T_d]/(T_0T_1\cdots T_r))$ for some integer $r$ with $0 \leq r \leq d = \dim X$ (cf. [23, p.707]).

Let us denote the set of points on $X$ of codimension $i$ by $X^{(i)}$. For a point $x \in X$, let $i_x$ be the canonical map $x \hookrightarrow X$. Then we have three kinds of logarithmic Hodge-Witt sheaves: $W_rΩ^n_{X,\log}$ ([15]).$

\lambda^g_{X,r} := \text{Im} \left( d \log : (\mathbb{G}_{m,X})^{(n)} \to \bigoplus_{x \in X^{(0)}} i_x W_rΩ^n_{x,\log} \right) \\
and \\
\nu^g_{X,r} := \text{Ker} \left( \partial : \bigoplus_{x \in X^{(0)}} i_x W_rΩ^n_{x,\log} \to \bigoplus_{x \in X^{(1)}} i_x W_rΩ^{n-1}_{x,\log} \right)
which agree with the logarithmic Hodge-Witt sheaves $W_r\Omega_X^{r, \log}$ \(\text{[16]}\) in the case where $X$ is a smooth variety (cf.\[16\] p.528, Théorème 2.4.2, \[12\]). Moreover, we have inclusions of étale sheaves

$$\lambda^n_{X,r} \subset W_r\omega_X^n \subset v^n_{X,r}$$

\(\text{[22] p.736, Proposition 4.2.1}\) and these inclusions are not equalities in general (cf. \[23\] p.737, Remark 4.2.3). In \[2\] we prove the following:

**Theorem 1.1.** (Gersten resolution, Theorem \[2\]) Let $A$ be a local ring of a normal crossing variety over a field of positive characteristic $p > 0$. Then the sequence

$$0 \to H_1^\et(A, v^n_r) \to \bigoplus_{x \in \Spec(A)^{(0)}} H_1^\et(A_{\et}, v^n_r) \to \bigoplus_{x \in \Spec(A)^{(1)}} H_2^\et(A_{\et}, v^n_r) \to \cdots$$

is exact for any integers $s$ and $r > 0$. Here $H_1^\et(A_{\et}, v^n_r) = \lim_{j\to \infty} H_1^\et(A_{\et}, v^n_r) / U_{\et}$ for a non-negative integer $t$ where $U$ runs through open subschemes of $\Spec(A)$ such that $x \in U$.

**Theorem 1.2.** (Rigidity, Theorem \[2\]) Let $X$ be a normal crossing variety over a field of characteristic $p > 0$. Let $A$ be a local ring of a normal crossing variety $X$ of $x$ in $X$ and $\lambda$ the residue field of $A$. Let $n$ be a non negative integer and $r$ a positive integer. Then the homomorphism

$$H_1^\et(A, \lambda^n_{X,r}) \to H_1^\et(k, \lambda^n_{X,r})$$

is an isomorphism.

In the case where $X$ is a smooth variety over a field of characteristic $p > 0$, Theorem \[12\] is \[22\] p.55, Theorem 5.3. The proof of Theorem \[12\] is reduced to \[22\] p.55, Theorem 5.3 by using the contravariant functoriality of the sheaves $\lambda^n_{X,r}$ for normal crossing varieties $X$ (\[23\] p.734, Corollary 3.5.3). If we replace $\lambda^n_{X,r}$ with $v^n_{X,r}$ in the homomorphism \[12\], the homomorphism \[12\] is not an isomorphism unless $A$ is smooth over a field of positive characteristic (see Remark \[2\] below).

Next, we consider mixed characteristic cases. Let $B$ be a discrete valuation ring of mixed characteristic $(0, p)$ with the quotient field $K$. Let $\mathcal{X}$ be a *semistable family* over $\Spec(B)$, i.e. a regular scheme of pure dimension which is flat of finite type over $\Spec(B)$, $\mathcal{X} \otimes_B K$ is smooth over $\Spec(K)$, and the special fiber $Y$ of $\mathcal{X}$ is a reduced divisor with normal crossings on $\mathcal{X}$.

Let $j$ and $t$ be as follows:

$$\mathcal{X}_K \xrightarrow{j} \mathcal{X} \xleftarrow{t} Y.$$ 

Then there is an exact sequence of sheaves on $\mathcal{X}_{\et}$

$$R^n j_* \mu_{p^r}^{\otimes n} \to \bigoplus_{y \in \eta^{(0)}} i_{y*} W_r^{n-1} \Omega_{y_{\et}}^{n, \log} \to \bigoplus_{y \in \eta^{(1)}} i_{y*} W_r^{n-2} \Omega_{y_{\et}}^{n, \log}$$

where $\mu_{p^r}$ is the sheaf of $p^r$-th roots of unity and each arrow arises from the boundary maps of Galois cohomology groups (cf.\[24\] pp.522–523, Lemma 1.3.1.(1)). Hence we have the morphism

$$R^n j_* \mu_{p^r}^{\otimes n} \to t_* v_{y_{\et}}^{n-1}$$

\(\text{(2)}\)
by the definition of $v_{Y,r}^{n-1}$. In this situation, the $p$-adic Tate twist $\mathcal{I}_r(n)$ is defined by K.Sato as follows:

**Definition 1.3.** ($p$-adic Tate twist $\mathcal{I}_r(n)$, cf. [24, p. 537, Definition 4.2.4]) Let the notations be the same as above. For $n = 0$,

$$\mathcal{I}_r(0)_X := \mathbb{Z}/p^r\mathbb{Z}.$$  

For $n \geq 1$, $\mathcal{I}_r(n)$ be defined as a complex which is fitted into the following distinguished triangle

$$\tau_{\leq n} R j_* \mu_{p^n} \otimes \sigma_X \to \mathcal{I}_r(n) \to \mathcal{I}_r(n+1) \to \tau_{\leq n} R j_* \mu_{p^n} \otimes \sigma_X,$$

where the morphism $\sigma_X$ is induced by the morphism (2).

The main objective of this paper is to study étale hypercohomology groups with values in $\mathcal{I}_r(n)$. By Theorem 3.3 and Theorem 3.4 which are results about the $p$-adic vanishing cycle $\iota^* \nu_{\eta, r}$ due to Bloch-Kato/Hyodo and K.Sato, we have a finite filtration of $H^n_{\text{ét}}(\mathcal{I}_r(n))$ which relates to the logarithmic Hodge-Witt sheaves $\lambda_{Y, r}$ and the modified differential modules (see Remark 3.5 below). Then we are able to prove the following by using Theorem 1.2 and an isomorphism (3):

**Theorem 1.4. (Rigidity, Theorem 3.7)** Let $\mathcal{X}$ be the same as above, $R$ the henselization of a local ring $\mathcal{O}_{\mathcal{X}, x}$ of $x$ in $\mathcal{X}$ and $k$ the residue field of $R$. Let $n$ be a non negative integer and $r$ be a positive integer. Then we have an isomorphism

$$H^{n+1}_{\text{ét}}(R, \mathcal{I}_r(n)) \simeq H^{1}_{\text{ét}}(k, \lambda_{Y, r}).$$

Theorem 1.4 has several applications. Let us denote Bloch’s cycle complex for the Zariski topology ([2], [18]) by $\mathbb{Z}(n)$. For a positive integer $m$, $\mathbb{Z}(n) \otimes \mathbb{Z}/m$ denotes by $\mathbb{Z}/m(n)$. Thus, $\mathbb{Z}(n)_{\text{ét}}$ (resp. $\mathbb{Z}/m(n)_{\text{ét}}$) denotes the étale sheafification of $\mathbb{Z}(n)$ (resp. $\mathbb{Z}/m(n)$).

Then, the first application of Theorem 1.4 is as follows:

**Proposition 1.5. (Proposition 4.5)** Let $R$ be the same as in Theorem 1.4. Then we have

$$H^i_{\text{Zar}}(R, \mathcal{I}_r(n)) = 0$$

for $i = n + 1$.

By Proposition 1.5 and [25, p. 209, Remark 7.2], we have an isomorphism

$$\tau_{\leq n+1}(\mathbb{Z}/p^n(n)) \simeq \mathcal{I}_r(n)$$

in $D^b(\mathcal{X}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ which is the derived category of bounded complexes of étale $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on $X$. In [24, p. 524, Conjecture 1.4.1 (1)], it is conjectured that the truncation in the isomorphism (3) is unnecessary. If $\mathcal{X}$ is smooth over Spec($B$), then this conjecture holds true by [9, p. 786, Corollary 4.4].

Then, by using the isomorphism (3), the second application of Theorem 1.4 is expressed as follows:
THEOREM 1.6. (Gersten resolution, Theorem 4.8 and 4) Let \( \mathcal{X} \) be a semistable family over the spectrum of a discrete valuation ring of mixed characteristic \((0, p)\) and \( R = \mathcal{O}_{\mathcal{X}, x} \) the henselian local ring of \( x \) in \( \mathcal{X} \). Suppose that \( \dim(R) = 2 \). Then the sequence

\[
0 \to H^{n+1}_{\text{ét}}(R, \mathbb{Z}/m(n)) \to H^{n+1}_{\text{ét}}(k(R), \mathbb{Z}/m(n)) \to \bigoplus_{p \in \text{Spec}(R)} H^{n+2}_{\text{ét}}((R_p)_{\text{ét}}, \mathbb{Z}/m(n))
\]

is exact for any integer \( n \geq 0 \) and \( m = p' \). Here \( k(R) \) is the fraction field of \( R \).

Let \( B \) be a regular local ring of dimension at most 1. Let \( l \) be a positive integer which is invertible in \( B \) and \( \mu_l \) the étale sheaf of \( l \)-th roots of unity. Then we have an isomorphism

\[
\tau_{\leq n+1}(\mathbb{Z}/l(n)) \simeq \mu_l^\infty
\]

in \( D^b(\mathcal{X}_{\text{ét}}, \mathbb{Z}/l\mathbb{Z}) \) (see Remark 4.7 below). If \( \mathcal{X} \) is smooth over \( \text{Spec}(B) \), then the truncation in (5) is unnecessary by \([11, \text{Theorem 1.5}]\) and \([9, \text{p.78, Corollary 4.4}]\). In the case where \( R \) is equi-characteristic and \((m, \text{char}(R)) = 1 \), the exactness of the sequence (4) is a part of the Bloch-Ogus Theorem (3) by (5).

Moreover, we are able to prove the following local-global principle as an application of Theorem 1.6 and Theorem 1.4:

THEOREM 1.7. (Corollary 4.9 and Theorem 4.11) Let \( B \) be an excellent henselian discrete valuation ring of mixed characteristic \((0, p)\). Let \( X \) be a regular scheme over \( \text{Spec}(B) \) and \( Y \) the special fiber of \( X \). Let \( \mathcal{O}_{X, p} \) be the henselization of a local ring \( \mathcal{O}_{X, p} \) of \( p \in X \), \( \kappa(p) \) the residue field of \( p \in X \) and \( k(X) \) (resp. \( k(\mathcal{O}_{X, p}) \)) the ring of rational functions on \( X \) (resp. \( \text{Spec}(\mathcal{O}_{X, p}) \)). Suppose that \( \dim(X) = 2 \).

Then the natural map

\[
H^{n+1}_{\text{ét}}(k(X), \mu_p^\infty) \to \bigoplus_{p \in \text{X}^{(1)} \setminus Y^{(0)}} H^{n+1}_{\text{ét}}(\kappa(p), \mu_p^\infty) \oplus \bigoplus_{p \in Y^{(0)}} H^{n+1}_{\text{ét}}(k(\mathcal{O}_{X, p}), \mu_p^\infty)
\]

is injective for \( n \geq 1 \) in the following cases:

(i) \( X \) is a scheme \( \mathcal{X} \) which is a proper and semistable family over \( \text{Spec}(B) \).

(ii) \( X = \mathcal{O}_{X, p}^\times \). Here \( \mathcal{X} \) is the same as in (i) and \( p \in \mathcal{X}^{(2)} \).

This implies that the local-global map

\[
H^{n+1}_{\text{ét}}(k(X), \mu_m^\infty) \to \prod_{p \in \text{X}^{(1)}} H^{n+1}_{\text{ét}}(k(\mathcal{O}_{X, p}), \mu_m^\infty)
\]

is injective for \( n \geq 1 \) and \( m = p' \).

Suppose that \( B \) is equi-characteristic and \((m, \text{char}(B)) = 1 \). In the case (i), Colliot-Thélène raised a question whether the local-global map (6) is injective (6). In [13 p.245, Theorem 3.3.6], Harbater–Hartmann–Krashen provided an affirmative answer to this question in this case.
In the case where $B$ is mixed characteristic $(0, p)$, due to Hu’s method (cf. [14, The proof of Theorem 2.5]). [14, Remark 2.6 (2)], Theorem 1.7 (i) follows from Theorem 1.6 (or Theorem 1.7 (ii)) without assuming $(m, p) = 1$. If $(m, p) = 1$, then the exactness of the sequence (4) has been proved in [21, p.34, Theorem 1]. Theorem 1.7 (i) is an extension of [22, pp.62–63, Theorem 6.3].

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Notations

For a scheme $X$, $X^{(i)}$ denotes the set of points on $X$ of codimension $i$. Moreover, $k(X)$ denotes the ring of rational functions on $X$ and $\kappa(x)$ denotes the residue field of $x \in X$. If $X = \text{Spec}(A)$ for a ring $A$, $k(A)$ denotes $k(\text{Spec}(A))$. For a scheme $X$, $X_{\text{Zar}}, X_{\text{Nis}}$ and $X_{\text{et}}$ denote the category of étale schemes over $X$ equipped with the Zariski, Nisnevich and étale topology, respectively. For a scheme $X$, $D(X)$ denotes the derived category of complexes of étale sheaves of abelian groups on $X$, and $D_+(X) \subset D(X)$ denotes the full subcategory of complexes that are bounded below.

2 Logarithmic Hodge-Witt sheaves

In this section, we study the logarithmic Hodge-Witt sheaves $\nu^n_{X,r}$ and $\lambda^n_{X,r}$ ([23]) for a normal crossing variety $X$.

2.1 $\nu^n_{X,r}$

Let $X$ be a normal crossing variety over a field of characteristic $p > 0$. Let $n$ be a non negative integer and $r$ a positive integer. We show a relation between two Zariski sheaves on $X_{\text{Zar}}$

$$\nu^n_{X,r} := \text{Ker} \left( \partial : \bigoplus_{x \in X^{(0)}} i_{xs} W_r \Omega^n_{x, \log} \rightarrow \bigoplus_{x \in X^{(1)}} i_{xs} W_r \Omega^{n-1}_{x, \log} \right) \tag{7}$$

(which is the image of the étale sheaf $\nu^n_{X,r}$ in [23, p.527, 2.2] under the forgetful functor) and $\mathbb{Z}/p^r(n)$, where $\mathbb{Z}(n)$ is the Bloch’s cycle complex for the Zariski topology ([2], [18]) and $\mathbb{Z}/p^r(n) = \mathbb{Z}(n) \otimes \mathbb{Z}/p^r$.

**Proposition 2.1.** Let $X$ be a normal crossing variety over a field. Then we have

$$\mathcal{H}^i(\mathbb{Z}(n)) = 0$$

for $i \geq n + 1$. 

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Proof. It suffices to show the statement in the case where $X$ is the spectrum of a local ring of a normal crossing variety. Moreover, the stalk of the sheaf $\mathcal{H}^i(\mathbb{Z}(n))$ at a point $x \in X$ is equal to $H^n_{\text{Zar}}(\mathcal{O}_{X,x}, (j_x)^*\mathcal{O}_X)$ by the spectral sequence

$$E_2^{ij} = H^i_{\text{Zar}}(\mathcal{O}_{X,x}, \mathcal{H}^j(\mathbb{Z}(n))) \Rightarrow E^{i+j} = H^{i+j}_{\text{Zar}}(\mathcal{O}_{X,x}, \mathcal{O}_X)$$

and the definition of the Zariski cohomology. Here $j_x : \text{Spec}(\mathcal{O}_{X,x}) \to X$ is the natural map. Hence it suffices to show

$$H^n_{\text{Zar}}(\text{Spec}(A), \mathbb{Z}(n)) = 0$$

for $i \geq n + 1$ in the case where $A$ is a local ring of a normal crossing variety. We prove the equation (8) by induction on $\#(\text{Spec}(A))$.

In the case where $\#(\text{Spec}(A)) = 1$, the equation (8) follows from [9, p.786, Corollary 4.4].

Assume that the equation (8) holds in the case where $\#(\text{Spec}(A)) \leq r$. Let

$$\text{Spec}(A_1), \cdots, \text{Spec}(A_{r+1})$$

be the irreducible components of $\text{Spec}(A)$. Then elements of the set (9) and

$$\text{Spec}(B) = \cap_{j=1}^{r} \text{Spec}(A_j) = \left( \cup_{j=1}^{r} \text{Spec}(A_j) \right) \cap \text{Spec}(A_{r+1})$$

are smooth varieties by the definition of normal crossing variety. Let

$$H^n_{\text{Zar}}(\text{Spec}(B), \mathbb{Z}(n)) = 0$$

for $i \geq n + 1$ by the inductive hypothesis. Since we have

$$H^{n+1}_{\text{Zar}}(\text{Spec}(A_{r+1}), \mathbb{Z}(n)) = 0$$

by [9, p.786, Corollary 4.4], the homomorphism

$$H^n_{\text{Zar}}(\text{Spec}(A_{r+1}) \setminus \text{Spec}(B), \mathbb{Z}(n)) \to H^{n+1}_{\text{Zar}}(\text{Spec}(B), \mathbb{Z}(n-1))$$

is surjective by the localization theorem ( [2, p.277, Theorem (3.1)], [3, p.537, Corollary (0.2)])). Moreover we have

$$\text{Spec}(A) \setminus \text{Spec}(B) = \left( \cup_{j=1}^{r} \text{Spec}(A_j) \setminus \text{Spec}(B) \right) \bigoplus (\text{Spec}(A_{r+1}) \setminus \text{Spec}(B))$$

and a commutative diagram

$$H^n_{\text{Zar}}(\text{Spec}(A_{r+1}) \setminus \text{Spec}(B), \mathbb{Z}(n)) \xrightarrow{\alpha} H^{n+1}_{\text{Zar}}(\text{Spec}(B), \mathbb{Z}(n-1))$$

$$\xrightarrow{\beta} H^n_{\text{Zar}}(\text{Spec}(A) \setminus \text{Spec}(B), \mathbb{Z}(n)) \xrightarrow{\gamma} H^{n+1}_{\text{Zar}}(\text{Spec}(B), \mathbb{Z}(n-1)).$$
So the homomorphism
\[ H^r_{Zar}(\text{Spec}(A) \setminus \text{Spec}(B), \mathbb{Z}(n)) \to H^{r-1}_{Zar}(\text{Spec}(B), \mathbb{Z}(n-1)) \] (11)
is also surjective. Since we have the equation (10) and \( A_j \) (\( 1 \leq j \leq r+1 \)) are local rings of a smooth variety, we have
\[ H^i_{Zar}(\bigcup_{j=1}^{r} \text{Spec}(A_j) \setminus \text{Spec}(B), \mathbb{Z}(n)) = H^i_{Zar}(\text{Spec}(A_{r+1}) \setminus \text{Spec}(B), \mathbb{Z}(n)) = 0 \]
for \( i \geq n+1 \) by the inductive hypothesis and the localization theorem. Hence we have
\[ H^i_{Zar}(\text{Spec}(A) \setminus \text{Spec}(B), \mathbb{Z}(n)) = 0 \] (12)
for \( i \geq n+1 \). By using the localization theorem, the sequence
\[ H^i_{Zar}(\text{Spec}(A) \setminus \text{Spec}(B), \mathbb{Z}(n)) \to H^{i-1}_{Zar}(\text{Spec}(B), \mathbb{Z}(n-1)) \to H^{i+1}_{Zar}(\text{Spec}(A) \setminus \text{Spec}(B), \mathbb{Z}(n)) \]
is exact for any integer \( i \). Hence the equation (8) follows from the equations (10), (12) and the surjectivity of the homomorphism (11). This completes the proof.

**Proposition 2.2.** Let \( X \) be a normal crossing variety over a field.

1. If \( \iota : Z \to X \) is a closed subscheme of codimension \( c \), then the canonical map
   \[ \tau_{\leq n+2}(\mathbb{Z}(n-c)_{\text{Zar}}[-2c]) \to \tau_{\leq n+2}R^{\text{int}}\mathbb{Z}(n)_{\text{Zar}} \]
is a quasi-isomorphism, where \( \mathbb{Z}(n)_{\text{Zar}} \) is the étale sheafification of \( \mathbb{Z}(n) \).

2. Let \( \varepsilon : X_{\text{Zar}} \to X_{\text{Zar}} \) be the change of site map. Then the canonical map induces a quasi-isomorphism
   \[ \mathbb{Z}(n)_{\text{Zar}} \to \tau_{\leq n+1}R\varepsilon_*\mathbb{Z}(n)_{\text{Zar}}. \]

**Proof.** By the same argument as in the proof of [22 Proposition 2.1], Proposition 2.2 follows from Proposition 2.2.2. Moreover, it suffices to show Proposition 2.2.2. Furthermore, it suffices to show Proposition 2.2.2 in the case where \( X \) is the spectrum of a local ring \( A \) of a normal crossing variety.

We prove Proposition 2.2.2 by induction on \#(Spec(\( A^{(0)} \))). Suppose that \#(Spec(\( A^{(0)} \))) = 1. Then we have a quasi-isomorphism
\[ \mathbb{Q}/\mathbb{Z}(n)_{\text{Zar}} \to \tau_{\leq n}R\varepsilon_*\mathbb{Q}/\mathbb{Z}(n)_{\text{Zar}} \]
by [10 Theorem 8.5], [11 Theorem 1.6] and [26]. Moreover we have a quasi-isomorphism
\[ \mathbb{Q}(n)_{\text{Zar}} \to R\varepsilon_*\mathbb{Q}(n)_{\text{Zar}} \]
by [9, p.781, Proposition 3.6]. So Proposition 2.2.2 holds in the case where \#(Spec(\( A^{(0)} \))) = 1 by the five lemma. See also [9, p.774, Theorem 1.2.2].
Assume that Proposition \[2.2\] holds in the case where \(\#(\text{Spec}(A)^{(0)}) \leq r\). Suppose that \(\#(\text{Spec}(A)^{(0)}) = r + 1\) and \(\text{Spec}(A_k) \quad (1 \leq k \leq r + 1)\) are the irreducible components of \(\text{Spec}(A)\).

Let \(\text{Spec}(B) = \text{Spec}(A) \cap \text{Spec}(A_{r+1})\) and \(t : \text{Spec}(B) \to \text{Spec}(A)\) be the closed immersion. Let \(j : U \to \text{Spec}(A)\) be the open complement of \(t\). Then we have

\[
U = \left( \bigcup_{k=1}^{r} \text{Spec}(A_k) \setminus \text{Spec}(B) \right) \bigoplus \left( \text{Spec}(A_{r+1}) \setminus \text{Spec}(B) \right).
\]

Moreover, \(\bigcup_{k=1}^{r} \text{Spec}(A_k) \setminus \text{Spec}(B)\) and \(\text{Spec}(A_{r+1}) \setminus \text{Spec}(B)\) are normal crossing varieties,

\[
\# \left( \left( \bigcup_{k=1}^{r} \text{Spec}(A_k) \setminus \text{Spec}(B) \right)^{(0)} \right) = r \quad \text{and} \quad \# \left( \left( \text{Spec}(A_{r+1}) \setminus \text{Spec}(B) \right)^{(0)} \right) = 1.
\]

Hence Proposition \[2.2\] holds for \(U\) by the assumption. Then Proposition \[2.2\] holds for \(t : \text{Spec}(B) \to \text{Spec}(A)\) by Proposition \[2.1\] and the same argument as in the proof of [22, Proposition 2.1]. Moreover, \(\#(\text{Spec}(B)^{(0)}) = r\) and Proposition \[2.2\] holds for \(\text{Spec}(B)\) by the assumption. Therefore Proposition \[2.2\] follows by Proposition \[2.1\] and the same argument as in the proof of [9, Theorem 1.2.2]. This completes the proof. \(\square\)

**Proposition 2.3.** Let \(X\) be a normal crossing variety over a field of characteristic \(p > 0\). Then we have an isomorphism in \(D^b(X_{zar}, \mathbb{Z}/p^r\mathbb{Z})\)

\[
\mathbb{Z}/p^r(n) \simeq \nu_{n} \mathbb{Z}/p^r\mathbb{Z},
\]

where \(D^b(X_{zar}, \mathbb{Z}/p^r\mathbb{Z})\) is the derived category of bounded complexes of Zariski \(\mathbb{Z}/p^r\mathbb{Z}\)-sheaves on \(X\).

**Proof.** It suffices to show the statement in the case where \(X\) is the spectrum of a local ring of a normal crossing variety. We prove the statement by induction on \(\dim X\). In the case where \(\dim X = 0\), the statement follows from [10, p.491, Theorem 8.3].

Assume that the statement holds in the case where \(\dim X \leq m\). Suppose that \(\dim X = m + 1\). Let \(t : Y \to X\) be a closed immersion of codimension 1, \(Y\) regular and \(j : U \to X\) the complement of \(t\). Let \(Y\) be the spectrum of a regular local ring \(A'\). Then \(A'\) is a local ring of a regular ring of finite type over a field of positive characteristic. By Quillen’s method (cf. [20, §7, The proof of Theorem 5.11]),

\[
A' = \lim_{\to} A'_{i}
\]

(13)

where \(A'_{i}\) is a local ring of a smooth algebra over \(\mathbb{F}_p\) and the maps \(A'_{i} \to A'_{i}\) are flat. So Proposition \[2.3\] holds for \(Y\) by [10, p.491, Theorem 8.3]. Since \(\dim U = m\) and the sequence

\[
\cdots \to t_* \mathbb{Z}/p^r(n-1)[n-2] \to \mathbb{Z}/p^r(n)[n] \to Rj_* \mathbb{Z}/p^r(n)[n] \to \cdots
\]

\(8\)
is a distinguished triangle by [2] p.277, Theorem (3.1), we have a quasi-isomorphism
\[ \tau_{\leq n}(\mathbb{Z}/p^r(n)) \simeq \varphi_{X,r}^n[-n] \]
by (7). Moreover, we have
\[ \mathcal{H}^s(\mathbb{Z}/p^r(n)) = 0 \]
for \( s > n \) by Proposition [2,1]. This completes the proof. □

**Corollary 2.4.** Let \( A \) be a local ring at a point of a normal crossing variety over a field of positive characteristic \( p > 0 \). Then the sequence

\[
\begin{align*}
0 \to H^0(A, \nu^n) & \to \bigoplus_{x \in \text{Spec}(A)^{(0)}} H^0(\kappa(x), \nu^n) \to \bigoplus_{x \in \text{Spec}(A)^{(1)}} H^0(\kappa(x), \nu^{n-1}) \\
& \to \bigoplus_{x \in \text{Spec}(A)^{(2)}} H^0(\kappa(x), \nu^{n-2}) \to \cdots \to \bigoplus_{x \in \text{Spec}(A)^{(j)}} H^0(\kappa(x), \nu^{n-j}) \to \cdots
\end{align*}
\]

is exact for any integer \( n \).

**Proof.** We consider the spectral sequence

\[
E_1^{s,t} = \bigoplus_{x \in \text{Spec}(A)^{(s)}} H^{2n-s+t}_{\text{Zar}}(\kappa(x), \mathbb{Z}/p^r(n-s)) \Rightarrow E_n^{s,t} = H^{2n+s+t}_{\text{Zar}}(A, \mathbb{Z}/p^r(n))
\]

(cf. [9, p.782]). In [9] the spectral sequence (14) is shown in the smooth case. But the construction of the spectral sequence (14) can be carried out without assuming smoothness. By Proposition [2,3] we have \( E_1^{s,t} = 0 \) for \( t \neq -n \) and \( E_1^{s,t} = 0 \) for \( s + t \neq -n \). Hence the statement follows from the spectral sequence (14). □

**Remark 2.5.** If \( A \) is the strict henselization of a local ring of a normal crossing variety over a field of positive characteristic, Corollary 2.4 directly follows from [23, p.716, Corollary 2.2.5 (1)].

**Proposition 2.6.** Let \( A \) be an equidimensional catenary local ring of characteristic \( p > 0 \) and \( F \) be an étale \( p \)-torsion sheaf.

Then we have

\[ H^{s+j}_{A_{\text{ét}}}(F) = 0 \]

for \( r \geq 2 \) and \( x \in \text{Spec}(A)^{(j)} \).

**Proof.** Let \( A_x \) be the local ring of \( A \) at \( x \in \text{Spec}(A) \). Then \( \text{codim}(\{x\}, \text{Spec}(A)) = \dim(A_x) \) and we have

\[
H^r_x(A_{\text{ét}}, F) = \lim_{U \subseteq \text{Spec}(A) \text{ open subscheme containing } x} H^r(U_{\text{ét}}, F) = \lim_{U \subseteq \text{Spec}(A) \text{ open subscheme containing } x} H^r((A_x)_{\text{ét}}, F)
\]

for any integer \( s \geq 0 \) by [19, pp.88–89, III, Lemma 1.16] where \( U \) runs through open subschemes of \( \text{Spec}(A) \) such that \( x \in U \). So it suffices to show the statement in the case where \( x \in \text{Spec}(A) \) is the closed point of \( \text{Spec}(A) \). We prove the statement by induction on \( \dim(A) \).
In the case where \(\dim(A) = 0\), the statement is true because \(A\) is a ring of characteristic \(p > 0\) and \(p\)-cohomological dimension \(\text{cd}_p(A) \leq 1\).

Assume that the statement is true for \(\dim(A_x) \leq j\). Then we prove the statement in the case where \(\dim(A) = j + 1\). In order to prove the statement, we use the spectral sequence
\[
E_1^{s,t} = \bigoplus_{x \in \text{Spec}(A)^{(i)}} H_x^{s+t}(A_{\text{ét}}, F) \Rightarrow E^{s+t} = H^{s+t}_{\text{ét}}(A) (15)
\]
(cf. [7] Part 1, §1). Since \(\dim(A) = j + 1\), we have
\[
E_1^{s,t} = 0
\]
for \(s > j + 1\). So we have
\[
E_2^{s,t} = 0 (16)
\]
for \(s > j + 1\).

Moreover, we have (16) for \(s \leq j\) and \(t \geq 2\) by the assumption. Hence we have
\[
E_2^{j+1,r} = E_{\infty}^{j+1,r} (17)
\]
for \(r \geq 2\). Since \(\text{cd}_p(A) \leq 1\) by [1, Exposé X, Théorème 5.1], we have
\[
E_{\infty}^{j+1,r} = E^{j+r+1} = 0 (18)
\]
for \(r \geq 2\) by the spectral sequence (15). Hence the sequence
\[
\bigoplus_{x \in \text{Spec}(A)^{(j)}} H_x^{j+r}(A_{\text{ét}}, F) \rightarrow \bigoplus_{x \in \text{Spec}(A)^{(j+1)}} H_x^{j+r+1}(A_{\text{ét}}, F) \rightarrow 0
\]
is exact for \(r \geq 2\) by (17) and (15). Therefore we have
\[
H_x^{j+r+1}(A_{\text{ét}}, F) = 0
\]
for \(r \geq 2\) by the assumption. This completes the proof. \(\square\)

**Theorem 2.7.** Let \(A\) be a local ring of a normal crossing variety over a field of positive characteristic \(p > 0\). Then the sequence
\[
0 \rightarrow H^{r}_{\text{ét}}(A, F) \rightarrow \bigoplus_{x \in \text{Spec}(A)^{(0)}} H^{r}_{x}(A_{\text{ét}}, F) \rightarrow \bigoplus_{x \in \text{Spec}(A)^{(1)}} H^{r+1}_{x}(A_{\text{ét}}, F) \rightarrow \cdots
\]
is exact for any integers \(r\) and \(r > 0\).

**Proof.** In the case where \(t < 0\), the statement follows from [23] p.718, Theorem 2.4.2]. In the case where \(t = 0\), the statement follows from Corollary 2.4 and [23] p.718, Theorem 2.4.2]. In the case where \(t \geq 2\), the statement follows from Proposition 2.6 and [1] Exposé X, Théorème 5.1]. So it suffices to show the statement in the case where \(t = 1\). In order to prove the statement, we use the spectral sequence (15) for \(F = F_{r}^{\nu}\). By Corollary 2.4 and [23] p.718, Theorem 2.4.2], we have
\[
E_1^{s,t} = 0
\]
(19)
for $s > 0$ and $t \leq 0$. By Proposition 2.6, we have the equation (19) for $t \geq 2$. Moreover, we have $E^{s+t} = 0$ for $s + t \geq 2$ by [1, Expos´e X, Th´eor`eme 5.1]. Hence we have isomorphisms

$$E_2^{s,t} \simeq \begin{cases} E^1 & (s = 0) \\ 0 & (s > 0) \end{cases}$$

by the spectral sequence (15) for $F = \nu^n$. This completes the proof.

2.2 $\lambda^n_{X,r}$

Let $n$ be a non negative integer and $r$ a positive integer. For a normal crossing variety $X$ over a field of positive characteristic, we define a kind of generalized logarithmic Hodge-Witt sheaves on $X$ by

$$\lambda^n_{X,r} := \text{Im} \left( d \log : (\mathbb{G}_{m,X})^\otimes n \to \bigoplus_{x \in X^{(0)}} i_x \ast \omega^n_{x,\log} \right)$$

(cf. [23, p.726, Definition 3.1.1]). Then we show the following:

**Theorem 2.8.** Let $A$ be the henselization of a local ring of a normal crossing variety over a field of characteristic $p > 0$ and $k$ the residue field of $A$. Then the homomorphism

$$H^1_{\text{ét}}(A, \lambda^n_{A,r}) \rightarrow H^1_{\text{ét}}(k, \lambda^n_{k,r})$$

(20)

is an isomorphism.

**Proof.** Let $C$ be a finitely generated $k$-algebra. Then there exists a surjective $k$-algebra homomorphism

$$k[T_1, \cdots, T_N] \rightarrow C$$

for some integer $N$. Hence $A$ is embedded into an equi-characteristic henselian regular local ring $B$. Let $t : \text{Spec}(A) \rightarrow \text{Spec}(B)$ be the closed immersion. Then the natural pull-back map

$$t^* : \mathbb{G}_{m,B}^\otimes n \rightarrow t_\ast \left( \mathbb{G}_{m,A}^\otimes n \right)$$

(21)

is surjective. Moreover the homomorphism (21) induces the pull-back map

$$t^* : \lambda^n_{B,r} \rightarrow t_\ast \lambda^n_{A,r}$$

(22)

(cf. [23, p.732, Theorem 3.5.1]) and the homomorphism (22) is also surjective. Since $B$ has $p$-cohomological dimension at most 1, we have

$$H^2_{\text{ét}}(B, \text{Ker}(t^*))) = 0.$$
which is induced by the pull-back map \( (22) \) is surjective. Moreover a composition of the homomorphism \((23)\) and the homomorphism

\[
H^1_{\text{ét}}(A, \lambda^n_{A,r}) \to H^1_{\text{ét}}(k, \lambda^n_{k,r})
\]

which is induced by the pull-back map coincides with the homomorphism

\[
H^1_{\text{ét}}(B, \lambda^n_{B,r}) \to H^1_{\text{ét}}(k, \lambda^n_{k,r})
\]

which is induced by the pull-back map by \([23, p.734, Corollary 3.5.3]\). Then the homomorphism \((25)\) is an isomorphism by \([22, Theorem 5.3]\). Therefore the homomorphism \((23)\) is an isomorphism and the homomorphism \((24)\) is also an isomorphism. This completes the proof.

\[\Box\]

Remark 2.9. If we replace \(\lambda^n_{A,r}\) with \(\nu^n_{A,r}\) in the homomorphism \((20)\), the homomorphism \((20)\) is not an isomorphism in general. In fact,

\[
\lambda^n_{A,r} = \mathbb{Z}/p^r
\]

and

\[
\nu^n_{A,r} = \mathbb{Z}/p^r \oplus \cdots \oplus \mathbb{Z}/p^r \quad \text{#(Spec}(A)^{(0)}) \text{ times}
\]

unless \(A\) is smooth over a field of positive characteristic.

3 \quad \text{\textit{\(p\)}}\text{-adic Tate twist} \(\Xi_r(n)\)

Let \(B\) be a discrete valuation ring of mixed characteristic \((0, p)\) with the quotient field \(K\).

Let \(\mathcal{X}\) be a semistable family over \(\text{Spec}(B)\), i.e. a regular scheme of pure dimension which is flat of finite type over \(\text{Spec}(B)\), \(\mathcal{X}_K = \mathcal{X} \otimes K\) is smooth over \(\text{Spec}(K)\), and the special fiber \(Y\) of \(\mathcal{X}\) is a reduced divisor with normal crossings on \(\mathcal{X}\).

Let \(j\) and \(t\) be as follows:

\[
\mathcal{X}_K \xrightarrow{j} \mathcal{X} \xleftarrow{t} Y.
\]

Let \(n\) be a non negative integer and \(r\) a positive integer. In this section, we study the \(p\)-adic Tate twist \(\Xi_r(n)\) (cf.\([24, p.537, Definition 4.2.4]\)). In order to study it, the \(p\)-adic vanishing cycle

\[
M^r := t^* R^j_! \mu_p^{\otimes n}
\]

plays important roles where \(\mu_p\) is the sheaf of \(p^r\)-th roots of unity.

Remark 3.1. In \([24]\), \(\Xi_r(n)\) is defined in the case where the residue field of \(B\) is perfect. But the results in \([24]\) hold true without this assumption as explained in \([25, p.187, Remark 3.7]\).
3.1 Review on the structure of the \( p \)-adic vanishing cycle

Let the notations be the same as above. We define the étale sheaf \( \mathcal{M}^{M}_{n, X_{K}/Y} \) on \( Y \) as
\[
\mathcal{M}^{M}_{n, X_{K}/Y} := (t^{*} j_{e} \mathcal{O}_{X_{K}}^{\infty})^{\otimes n} / J,
\]
where \( J \) denotes the subsheaf which is generated by local sections of the form
\[
x_{1} \otimes \cdots \otimes x_{n} (x_{i} \in t^{*} j_{e} \mathcal{O}_{X_{K}}^{\infty}) \text{ with } x_{i} + x_{j} = 0 \text{ or } 1
\]
for some \( 1 \leq i < j \leq n \). By \([4, (1.2)]\), there is a natural map
\[
\mathcal{M}^{M}_{n, X_{K}/Y} \twoheadrightarrow M_{r}^{\alpha}
\]
and we define the filtrations \( U^{\ast} \) and \( V^{\ast} \) on the \( p \)-adic vanishing cycle \( M_{r}^{\alpha} \) by using this map \((26)\), as follows:

**Definition 3.2.** (cf.\([15, p.546, (1.4)], [24, pp.530–531, Definition 3.3.2]\))

(1) Let \( \pi \) be a prime element of \( B \). Let \( U^{0}_{X_{K}} \) be the full sheaf \( t^{*} j_{e} \mathcal{O}_{X_{K}}^{\infty} \). For \( q \geq 1 \), let \( U^{q}_{X_{K}} \) be the étale subsheaf of \( t^{*} j_{e} \mathcal{O}_{X_{K}}^{\infty} \) which is generated by local sections of the form \( 1 + \pi^{q} \cdot a \) with \( a \in t^{*} \mathcal{O}_{X}^{\infty} \). We define the subsheaf \( U^{q, \mathcal{M}^{M}_{n, X_{K}/Y}} \) as the part which is generated by \( U^{q}_{X_{K}} \otimes (t^{*} j_{e} \mathcal{O}_{X_{K}}^{\infty})^{\otimes(n-1)} \).

(2) We define the subsheaf \( U^{q} M_{r}^{\alpha} (q \geq 0) \) of \( M_{r}^{\alpha} \) as the image of \( U^{q, \mathcal{M}^{M}_{n, X_{K}/Y}} \) under the map \((26)\). We define the subsheaf \( V^{q} M_{r}^{\alpha} (q \geq 0) \) of \( M_{r}^{\alpha} \) as the part which is generated by \( U^{q+1} M_{r}^{\alpha} \) and the image of \( U^{q} M_{r}^{\alpha} \) under the map \((26)\).

Then we give a brief review of the structure of \( M_{r}^{\alpha} \). Let \( \omega_{Y}^{\alpha} \) be the modified differential modules which is defined in \([15, p.546, (1.5)] \) (See also \([24, p.531]\)). For \( q \geq 0 \),
\[
\text{gr}_{U/V}^{q} M_{r}^{\alpha} := U^{q} M_{r}^{\alpha} / V^{q} M_{r}^{\alpha} \quad \text{and} \quad \text{gr}_{V/U}^{q} M_{r}^{\alpha} := V^{q} M_{r}^{\alpha} / U^{q+1} M_{r}^{\alpha}
\]
are expressed by using subsheaves of the modified differential modules \( \omega_{Y}^{\alpha} \) as follows:

**Theorem 3.3.** (Bloch-Kato \([4, pp.112–113, Corollary (1.4.1)] \)/Hyodo \([15, p.548, (1.7) \) Corollary], Sato \([25, pp.184–185, Theorem 3.3] \))

Let the notations be the same as above. Then

(1) The map \((26)\) is surjective, that is, the subsheaf \( U^{0} M_{r}^{\alpha} \) is the full sheaf \( M_{r}^{\alpha} \) for any \( n \geq 0 \) and \( r > 0 \).

(2) Let \( e \) be the absolute ramification index of \( K \), and let \( r = 1 \). Then for \( q \) with \( 1 \leq q < e' := pe/(p-1) \), there are isomorphisms
\[
\text{gr}_{U/V}^{q} M_{r}^{\alpha} \cong \begin{cases} \omega_{Y}^{e-1} / \mathcal{B}_{Y}^{e-1} ((p, q) = 1), & \\
\omega_{Y}^{e-1} / \mathcal{B}_{Y}^{e-1} (p|q), & \end{cases}
\]
\[
\text{gr}_{V/U}^{q} M_{r}^{\alpha} \cong \omega_{Y}^{e-2} / \mathcal{B}_{Y}^{e-2},
\]
where \( \omega_{Y}^{e} = W_{1} \omega_{Y}^{e} \) and \( \mathcal{B}_{Y}^{e} \) (resp. \( \mathcal{B}_{Y}^{e} \)) denotes the image of \( d : \omega_{Y}^{e-1} \rightarrow \omega_{Y}^{e} \) (resp. the kernel of \( d : \omega_{Y}^{e} \rightarrow \omega_{Y}^{e+1} \)).
(3) We have
\[ U^q M^n_1 = V^q M^n_1 = 0 \]
for any \( q \geq e' \).

We define the étale subsheaf \( FM^n_1 \) as the part which is generated by \( U^1 M^n_1 \) and the image of \( (t^* \mathcal{O}_X)_{\otimes n} \) under the map \( \sigma_{t^*} \).

**THEOREM 3.4.** (Sato [24, p.533, Theorem 3.4.2], [25, p.186, Theorem 3.4]) Let \( X \) be the same as in Theorem 3.3. Then there exists a short exact sequence of sheaves on \( Y \)
\[ 0 \to FM^n_1 \to M^n_1 \to \sigma_{t^*} n \to 0, \]
where \( \sigma_{t^*} \) is induced by the boundary map of Galois cohomology groups (cf. [24, p.530, (3.2.5)]). Furthermore there is an isomorphism
\[ FM^n_1 / U^1 M^n_1 \xrightarrow{\sim} \lambda^n_{1,1} \]
sending a symbol
\[ \{x_1, x_2, \ldots, x_n\} \quad (x_i \in t^* \mathcal{O}_X) \]
to
\[ d \log (\bar{x}_1 \otimes \bar{x}_2 \otimes \cdots \otimes \bar{x}_n). \]
Here for a section \( x \in t^* \mathcal{O}_X, \bar{x} \) denotes its residue class in \( \mathcal{O}_Y \).

**REMARK 3.5.** By Definition 1.3 and Theorem 3.4, we have an isomorphism
\[ H^n(t^* \Sigma_1(n)) \cong FM^n_1 \]
for a positive integer \( r \). Hence there is a finite filtration of subsheaves of \( H^n(t^* \Sigma_1(n)) \)
\[ H^n(t^* \Sigma_1(n)) \supseteq U^1 M^n_1 \supseteq \cdots \supseteq U^q M^n_1 \supseteq V^q M^n_1 \supseteq U^{q+1} M^n_1 \supseteq \cdots \]
where we have isomorphisms
\[ H^n(t^* \Sigma_1(n)) / U^1 M^n_1 \cong \lambda^n_{1,1}, \quad \quad (27) \]
\[ U^q M^n_1 / V^q M^n_1 \cong \begin{cases} \omega^n_{q-1} / \mathcal{O}^{q-1} & (p, q) = 1, \\ \omega^n_{q-1} / \mathcal{O}^{p-1} & (p | q) \end{cases} \]
for \( 1 \leq q < e' \), where \( e' = pe / (p - 1) \). For \( 1 \leq q < e' \) and
\[ V^q M^n_1 / U^{q+1} M^n_1 \xrightarrow{\sim} \omega^n_{q-2} / \mathcal{O}^{q-2} \]
for \( 1 \leq q < e' \) and
\[ U^q M^n_1 = V^q M^n_1 = 0 \]
for any \( q \geq e' \) by Theorem 3.3 and Theorem 3.4. Here \( e \) is the absolute ramification index of \( K \).
3.2 Rigidity

The following lemma is useful to compute étale cohomology groups with values in $U^1M^n_1$.

**Lemma 3.6.** Let $A$ be the henselization of a local ring of a normal crossing variety over a field of characteristic $p > 0$. Then

$$H^i_{\text{ét}}(A, \mathcal{B}^n_A) = H^i_{\text{ét}}(A, \mathcal{Z}^n_A) = 0$$

for $i \geq 1$. Here $\mathcal{B}^n_A$ (resp. $\mathcal{Z}^n_A$) is the image of $d : \omega^{n-1}_A \to \omega^n_A$ (resp. the kernel of $d : \omega^n_A \to \omega^{n+1}_A$).

**Proof.** $\mathcal{B}^n_A$ and $\mathcal{Z}^n_A$ are locally free $\mathcal{O}_{\text{Spec}(A)}$-modules after twisting with Frobenius. Therefore the statement follows from [19, p.114, III, Remark 3.8] and [19, p.103, III, Lemma 2.15].

Then we are able to prove the following main result of this paper by applying Theorem 2.8.

**Theorem 3.7.** Let $B$ be a discrete valuation ring of mixed characteristic $(0, p)$, $R$ the henselization of a local ring of a semistable family over $\text{Spec}(B)$ and $k$ the residue field of $R$. Then we have an isomorphism

$$H^{n+1}_{\text{ét}}(R, \mathcal{T}_r(n)) \cong H^1_{\text{ét}}(k, \lambda^n_{k,r})$$

where $\mathcal{T}_r(n)$ is the $p$-adic Tate twist.

**Proof.** Let $\iota : \text{Spec}(A) \hookrightarrow \text{Spec}(R)$ be the special fiber. Then we have an isomorphism

$$H^{n+1}_{\text{ét}}(R, \mathcal{T}_r(n)) \cong H^{n+1}_{\text{ét}}(A, t^*\mathcal{T}_r(n))$$

by [19, p.777, The proof of Proposition 2.2.b)]. Since $A$ has $p$-cohomological dimension at most 1, we have

$$H^s_{\text{ét}}(A, \mathcal{H}^t(t^*\mathcal{T}_r(n))) = 0$$

for $s \geq 2$ and any positive integer $t$. Moreover, we have

$$\mathcal{H}^t(t^*\mathcal{T}_r(n)) = 0$$

for $t \geq n+1$ by the definition of $\mathcal{T}_r(n)$. So we have an isomorphism

$$H^{n+1}_{\text{ét}}(A, t^*\mathcal{T}_r(n)) \cong H^1_{\text{ét}}(A, \mathcal{H}^n(t^*\mathcal{T}_r(n)))$$

by the spectral sequence

$$E^2_2 = H^s_{\text{ét}}(A, \mathcal{H}^t(t^*\mathcal{T}_r(n))) \Rightarrow E^{s+t}_{\text{ét}} = H^{s+t}_{\text{ét}}(A, t^*\mathcal{T}_r(n)).$$

By (28) and (29), we have an isomorphism

$$H^{n+1}_{\text{ét}}(R, \mathcal{T}_r(n)) \cong H^1_{\text{ét}}(A, \mathcal{H}^n(t^*\mathcal{T}_r(n)))$$ (30)
for any positive integer $r$ and so it suffices to show that we have an isomorphism
\[ H^r_{\text{ét}}(A, \mathcal{H}^n(t^*\Sigma_r(n))) \simeq H^r_{\text{ét}}(A, \lambda^n_{A,r}). \]  
(31)

Since we have
\[ H^i_{\text{ét}}(A, U^1M^r) = 0 \]
for $i \geq 1$ by Theorem 3.3 and Lemma 3.6, we have an isomorphism
\[ H^1_{\text{ét}}(A, \mathcal{H}^n(t^*T_r^1(n))) \simeq H^1_{\text{ét}}(A, \lambda^n_{A,1}) \]
by (27). Therefore we have an isomorphism
\[ H^{n+1}_{\text{ét}}(A, t^*\Sigma_1(n)) \simeq H^1_{\text{ét}}(A, \lambda^n_{A,1}) \]
by (29). Since
\[ \mathcal{H}^t(\Sigma_r(n)) = 0 \]
for $t > n$, we have
\[ H^{n+2}_{\text{ét}}(A, t^*\Sigma_r(n)) = 0 \]
by the similar argument as in the proof of the isomorphism (29). Hence the top row in a commutative diagram
\[
\begin{array}{cccccc}
H^{n+1}_{\text{ét}}(A, t^*\Sigma_1(n)) & \rightarrow & H^{n+1}_{\text{ét}}(A, t^*\Sigma_{r+1}(n)) & \rightarrow & H^{n+1}_{\text{ét}}(A, t^*\Sigma_1(n)) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^1_{\text{ét}}(A, \lambda^n_{A,r}) & \rightarrow & H^1_{\text{ét}}(A, \lambda^n_{A,r+1}) & \rightarrow & H^1_{\text{ét}}(A, \lambda^n_{A,1})
\end{array}
\]  
(32)
is exact. Moreover the bottom row in the commutative diagram (32) is exact by Theorem 2.8. If the homomorphism (31) is an isomorphism in the case where $r \leq j$, then the homomorphism (31) is an isomorphism in the case where $r = j + 1$ by applying the snake lemma to the commutative diagram (32). Therefore the statement follows by induction on $r$. \( \square \)

### 3.3 Gersten-type conjecture

**Proposition 3.8.** Let $A$ be a discrete valuation ring of mixed characteristic $(0, p)$ and $X$ a semistable family over $\text{Spec}(A)$. Let $R$ be the local ring at a point of $X$ and $Z$ the closed fiber of $\text{Spec}(R)$. Then we have a spectral sequence
\[ E^{s,t}_1 = \bigoplus_{s \in \text{Spec}(R)^{(i)} \cap Z} H^{s+t}_Z(R_{\text{ét}}, \Sigma_r(n)) \Rightarrow E^{s,t}_2 = H^{s+t}_Z(R_{\text{ét}}, \Sigma_r(n)) \]
for any integer $n$.

**Proof.** For a bounded below complex $F^\bullet$ of étale sheaves on $\text{Spec}(R)$, we have a spectral sequence
\[ \bigoplus_{s \in \text{Spec}(R)^{(i)}} H^s_{\text{ét}}(R_{\text{ét}}, F^\bullet) \Rightarrow E^{s+t} = H^{s+t}(R_{\text{ét}}, F^\bullet) \]  
(33)
Let $i: Z \to \text{Spec}(R)$ be the inclusion of the closed fiber of $\text{Spec}(R)$ and $F^\bullet = i_* i^! T^r$. Then we have
$$H^j_x(R, F^\bullet) = \begin{cases} H^j_x(R, \mathcal{F}_r(n)) & \text{if } x \in Z \\ 0 & \text{if } x \notin Z \end{cases}$$
for any integer $j > 0$. Hence the statement follows from the spectral sequence (33) and (34). \hfill \Box

**Lemma 3.9.** Let $n$ and $N$ be positive integers. Let $i$ be an integer where $i = 0$ or $i = 1$. Consider a spectral sequence $E^s_{i,t} \Rightarrow E^{s+t}$ which satisfies the following conditions:

(a) If $s < 0$ or $s > N$, then $E^s_{i,t} = 0$;

(b) If $s + t = n$ or $s + t = n + 1$, then $E^s_{i,t} = 0$ for $s \neq i$.

Then we have
$$E^n = E^{i,n-i}_2.$$

**Proof.** By the definition of the spectral sequence, we have a filtration
$$E^n = E^n_0 \supset E^n_1 \supset \cdots \supset E^n_N \supset 0$$
which satisfies
$$E^n_k / E^n_{k+1} \simeq E^{k,n-k}_\infty$$
for any integer $k$. So we have
$$E^n_{i+1} = 0 \text{ and } E^n_i = E^n$$
by the conditions. Moreover, we have
$$E^{i,n-i}_2 = E^{i,n-i}_\infty$$
by the conditions. Hence the statement follows. \hfill \Box

**Theorem 3.10.** Let $A$ be a discrete valuation ring of mixed characteristic $(0, p)$, $X$ a semistable family over $\text{Spec}(A)$ and $R = \mathcal{O}_{X,x}$ the local ring at a point $x$ of $X$. Let $Z$ be the closed fiber of $\text{Spec}(R)$. Then the sequence
$$0 \to H^{n+2}_Z(R_{\text{ét}}, \mathcal{F}_r(n)) \to \bigoplus_{x \in Z^{(0)}} H^{n+2}_x(R_{\text{ét}}, \mathcal{F}_r(n)) \to \bigoplus_{x \in Z^{(1)}} H^{n+3}_x(R_{\text{ét}}, \mathcal{F}_r(n))$$
is exact for any integer $r > 0$.

**Proof.** By [24] p.540, Theorem 4.4.7 and Corollary 2.4 the sequence
$$\bigoplus_{x \in Z^{(0)}} H^i_x(R_{\text{ét}}, \mathcal{F}_r(n)) \to \bigoplus_{x \in Z^{(1)}} H^{i+1}_x(R_{\text{ét}}, \mathcal{F}_r(n)) \to \cdots$$
is exact for $s \leq n + 1$. So the statement follows from Proposition 3.8 and Lemma 3.9. \hfill \Box
Remark 3.11. Let $A$ be a discrete valuation ring of mixed characteristic $(0, p)$ and $R$ a local ring of a smooth scheme over $\text{Spec}(A)$. Let $r$ be a positive integer. Then the sequence
\[
\bigoplus_{x \in \text{Spec}(R)^{(0)}} H^s_{\text{ét}}(R_{\text{ét}}, \mathbb{Z}/p^r(n)) \to \bigoplus_{x \in \text{Spec}(R)^{(1)}} H^{s+1}_{\text{ét}}(R_{\text{ét}}, \mathbb{Z}/p^r(n)) \to \cdots
\]
is exact for $s \leq n$ by [9, p.774, Theorem 1.2] and [26]. Hence we can also give an another proof of [22, Theorem 1.1] by Lemma 3.9 and the spectral sequence (33).

4 Applications

Throughout this section, $n$ is a non negative integer and $r$ is a positive integer.

Lemma 4.1. Let $R$ be a discrete valuation ring of mixed characteristic $(0, p)$. Then the homomorphism
\[
H^{n+1}_{\text{ét}}(R, \mathbb{Z}_p(n)) \to H^{n+1}_{\text{ét}}(k(R), \mu_p^{\otimes n})
\]
(35)
is injective.

Proof. Let $m$ be the residue field of $R$. We have a commutative diagram
\[
\begin{array}{ccc}
K^M_n(k(R))/p' & \to & K^M_{n-1}(k(m))/p' \\
\downarrow & & \downarrow \\
H^n_{\text{ét}}(k(R), \mu_p^{\otimes n}) & \to & H_0^0(\kappa(m), \nu^{n-1}_{(\kappa(m), r)})
\end{array}
\]
where $K^M_n(F)$ is the $n$-th Milnor $K$-group for a field $F$. Then the top horizontal arrow is induced by the boundary map (cf. [24, p.529, §3.1]) and so is surjective. Moreover the right vertical map is an isomorphism by [4, p.113, Theorem (2.1)]. Hence the homomorphism
\[
H^n_{\text{ét}}(k(R), \mu_p^{\otimes n}) \to H_0^0(\kappa(m), \nu^{n-1}_{(\kappa(m), r)})
\]
is surjective and the homomorphism
\[
H^{n+1}_{\text{ét}}(R, \mathbb{Z}_p(n)) \to H^{n+1}_{\text{ét}}(R, \tau \leq n R j_* \mu_p^{\otimes n})
\]
is injective where $j : \text{Spec}(k(R)) \to \text{Spec}(R)$ is the generic fiber. Since we have a distinguished triangle
\[
\cdots \to \tau_{\leq n} R j_* \mu_p^{\otimes n} \to \tau_{\leq n+1} R j_* \mu_p^{\otimes n} \to R^{n+1} j_* \mu_p^{\otimes n}[-(n+1)] \to \cdots
\]
and
\[
H^{n+1}_{\text{ét}}(R, \tau_{\leq n} R j_* \mu_p^{\otimes n}) = H^{n+1}_{\text{ét}}(k(R), \mu_p^{\otimes n}),
\]
the homomorphism
\[
H^{n+1}_{\text{ét}}(R, \tau_{\leq n} R j_* \mu_p^{\otimes n}) \to H^{n+1}_{\text{ét}}(k(R), \mu_p^{\otimes n})
\]
is injective. Therefore the statement follows. 

\[\square\]
**Proposition 4.2.** Let $X$ be a semistable family over the spectrum of a discrete valuation ring of mixed characteristic $(0, p)$ and $R = O_{X,x}$ be the henselian local ring of $x$ in $X$. Then the homomorphism (35) is injective.

**Proof.** By [7, Corollary 2.1.3], it suffices to prove the statement in the case where $x$ is in the closed fiber of $X$. Let $\mathfrak{m}$ be the maximal ideal of $R$. Let $a \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $p \in (a) = \mathfrak{p}$. By the assumption of $X$, there exists such a prime ideal $\mathfrak{p} = (a)$. Then $R/\mathfrak{p}$ is a regular local ring of characteristic $p > 0$ and we have a commutative diagram

$$
\begin{array}{ccc}
H^n_{\text{ét}}(R; \mathcal{I}, (n)) & \rightarrow & H^n_{\text{ét}}(R_p; \mathcal{I}, (n)) \\
\downarrow & & \downarrow \\
H^n_{\text{ét}}(R/\mathfrak{p}; \lambda^{n}_{R/\mathfrak{p}, r}) & \rightarrow & H^n_{\text{ét}}(\kappa(\mathfrak{p}); \lambda^{n}_{\kappa(\mathfrak{p}), r})
\end{array}
$$

where the left vertical arrow in (36) is injective by Theorem 3.7. Since $R/\mathfrak{p}$ is a regular local ring, the bottom horizontal arrow in (36) is also injective by Theorem 2.7. Hence the top horizontal arrow in (36) is injective. Therefore the statement follows from Lemma 4.1.

**Lemma 4.3.** Let $R$ be a local ring of a semistable family over a discrete valuation ring of mixed characteristic $(0, p)$. Suppose that $\dim(R) = 2$. Let $\mathfrak{p}$ be a prime ideal such that $\text{ht}(\mathfrak{p}) = 1$, $R/\mathfrak{p}$ is regular and $(\mathfrak{p}) \subset \mathfrak{p}$. Then the homomorphism

$$H^n_{R/\mathfrak{p}}(R; \mathcal{I}, (n)) \rightarrow H^n_{R/\mathfrak{p}}(R_p; \mathcal{I}, (n))$$

is injective.

**Proof.** Let $\mathfrak{m}$ be the maximal ideal of $R$. Since the sequence

$$H^n_{\mathfrak{p}}(R_p; \mathcal{I}, (n)) \rightarrow H^n_{R/\mathfrak{m}}(R; \mathcal{I}, (n)) \rightarrow H^n_{R/\mathfrak{p}}(R; \mathcal{I}, (n)) \rightarrow H^n_{R/\mathfrak{p}}(R_p; \mathcal{I}, (n))$$

is exact by [19, p.92, III, Remark 1.26], it suffices to show that the homomorphism

$$H^n_{\mathfrak{p}}(R_p; \mathcal{I}, (n)) \rightarrow H^n_{R/\mathfrak{m}}(R; \mathcal{I}, (n))$$

is surjective. The homomorphism (37) coincides with $\pm$ the boundary map

$$H^n_{\mathfrak{p}}(k(R/p), \nu^{n-1}) \rightarrow H^n_{\mathfrak{p}}(R/m, \nu^{n-2})$$

by [24, p.547, Theorem 6.1.1], [24, p.540, Theorem 4.4.7] and [25, p.187, Remark 3.7]. Let $K^M_n(F)$ be the $n$-th Milnor $K$-group for a field $F$. Then the homomorphism (38) coincides with $\pm$ the tame symbol

$$K^M_{n-1}(k(R/p))/\mathfrak{p} \rightarrow K^M_n(R/m)/\mathfrak{p}$$

of Milnor $K$-groups (modulo $\mathfrak{p}$) by the definition of the boundary map (cf. [17, p.150, (1.3) (ii)]) and [4, p.113, Theorem (2.1)]. So the homomorphism (37) is surjective. This completes the proof.

$\square$
LEMMA 4.4. Let $B$ be a discrete valuation ring of mixed characteristic $(0, p)$. Let $\mathcal{X}$ be a semistable family over $\text{Spec}(B)$. Then we have an isomorphism in $\mathcal{D}^b(\mathcal{X}_{\text{ét}}, \mathbb{Z}/p'^\infty \mathbb{Z})$

$$\tau_{\leq n}(\mathbb{Z}/p'^\infty(n)_{\text{Nis}}) \simeq \tau_{\leq n}(R\alpha_*\mathbb{Z}/p'^\infty(n)_{\text{ét}}),$$

where $\alpha : \mathcal{X}_{\text{ét}} \to \mathcal{X}_{\text{Nis}}$ is the change of site map and $\mathcal{D}^b(\mathcal{X}_{\text{ét}}, \mathbb{Z}/p'^\infty \mathbb{Z})$ is the derived category of bounded complexes of étale $\mathbb{Z}/p'^\infty \mathbb{Z}$-sheaves on $\mathcal{X}$.

**Proof.** Let $\epsilon : \mathcal{X}_{\text{ét}} \to \mathcal{X}_{\text{Zar}}$ and $\beta : \mathcal{X}_{\text{Nis}} \to \mathcal{X}_{\text{Zar}}$ be the change of site map. Let $j : U \to \mathcal{X}$ be the generic fiber and $i : Z \to \mathcal{X}$ be the closed fiber. Then the Beilinson-Lichtenbaum conjecture holds for $U$ by [11, Theorem 1.1] and the Bloch-Kato conjecture ([26]). Moreover, the Beilinson-Lichtenbaum conjecture holds for $Z$ by Proposition 2.2.2. Therefore we have a quasi-isomorphism

$$\tau_{\leq n}(\mathbb{Z}/p'^\infty(n)_{\text{Zar}}) \simeq \tau_{\leq n}(R\epsilon_*\mathbb{Z}/p'^\infty(n)_{\text{ét}})$$

by [22, p.33, Proposition 2.1] and the same argument as in the proof of [9, Theorem 1.2.2]. Moreover we have quasi-isomorphisms

$$\beta^*R\beta_* \simeq \text{id} \quad \text{and} \quad R\epsilon_* \simeq R\beta_*R\alpha_*.$$

Since $\beta^*$ is exact and $\beta^*(\mathbb{Z}(n)_{\text{Zar}}) \simeq \mathbb{Z}(n)_{\text{Nis}}$, the statement follows from (39).

PROPOSITION 4.5. Let $B$ be a discrete valuation ring of mixed characteristic $(0, p)$ and $R$ the henselization of a local ring of a semistable family over $\text{Spec}(B)$. Then we have

$$H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/p'^\infty(n)) = 0$$

for any integer $r > 0$ where $\mathbb{Z}(n)$ is Bloch’s cycle complex for the Zariski topology ([2], [18]) and $\mathbb{Z}/p'^\infty(n) = \mathbb{Z}(n) \otimes \mathbb{Z}/p'^\infty$.

**Proof.** Let $Z$ be the closed fiber of $\text{Spec}(R)$ and $U$ the generic fiber of $\text{Spec}(R)$. By Proposition 4.2 the homomorphism

$$H^{n+1}_{\text{ét}}(R, \mathbb{Z}/p'^\infty(n)) \to H^{n+1}_{\text{ét}}(U, \mathbb{Z}/p'^\infty)$$

is injective and so the homomorphism

$$H^n_{\text{ét}}(U, \mathbb{Z}/p'^\infty) \to H^{n+1}_{\text{ét}}(Z, \mathbb{Z}/p'^\infty[-(n + 1)])$$

is surjective by [24, p.540, Theorem 4.4.7]. So the homomorphism

$$H^r_{\text{Zar}}(U, \mathbb{Z}/p'^\infty(n)) \to H^{r-1}_{\text{Zar}}(Z, \mathbb{Z}/p'^\infty(n - 1))$$

is surjective by [9, p.774, Theorem 1.2.2], [26] and Proposition 2.3. Hence the homomorphism

$$H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/p'^\infty(n)) \to H^{n+1}_{\text{Zar}}(U, \mathbb{Z}/p'^\infty(n))$$

(40)
is injective by the localization theorem ([9, p.780, Corollary 3.3a]). Moreover, we have the commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
H^{n+1}_{\text{ét}}(U, T_r(n)) \\
\oplus \bigoplus_{x \in \mathbb{Z}^{(0)}} H^{n+1}_{\text{ét}}(R_x, T_r(n))
\end{array}
\end{array}
\quad \xrightarrow{\cong}
\quad
\begin{array}{c}
\begin{array}{c}
H^{n+1}_{\text{ét}}(U, \mu_{p^n}) \\
\oplus \bigoplus_{x \in \mathbb{Z}^{(0)}} H^{n+1}_{\text{ét}}(k(R), \mu_{p^n})
\end{array}
\end{array}
\quad \xrightarrow{\cong}
\quad
\begin{array}{c}
\begin{array}{c}
H^{n+2}_{\text{Zar}}(R, T_r(n)) \\
\oplus \bigoplus_{x \in \mathbb{Z}^{(0)}} H^{n+2}_{\text{Zar}}(k(R), T_r(n))
\end{array}
\end{array}
\]

(41)

where the sequences are exact. Then the left map in the diagram (41) is injective by Proposition 4.2 and the right map in the diagram (41) is injective by Theorem 3.10. So the middle map in the diagram (41) is injective and the homomorphism

\[
H^{n+1}_{\text{ét}}(U, Z_p(n)) \to H^{n+1}_{\text{ét}}(k(R), Z_p(n))
\]

is injective by [9, p.774, Theorem 1.2.4] and [26]. Moreover, the homomorphism

\[
H^{n+1}_{\text{Zar}}(U, Z_p(n)) \to H^{n+1}_{\text{Zar}}(U, p'(n))
\]

is injective by [11, Theorem 1.6] and [26]. Hence the homomorphism

\[
H^{n+1}_{\text{Zar}}(U, Z_p(n)) \to H^{n+1}_{\text{Zar}}(k(R), p'(n))
\]

is injective. By the definition of \(Z_p(n)\) and the same argument in the proof of Proposition 2.1 we have

\[
H^{n+1}_{\text{Zar}}(k(R), Z_p(n)) = H^0_{\text{Zar}}(k(R), \mathcal{H}^{n+1}(Z_p(n))) = 0
\]

and so we have

\[
H^{n+1}_{\text{Zar}}(U, Z_p(n)) = 0. \tag{42}
\]

Therefore the statement follows from the equation (42) and the injectivity of the homomorphism (40).

The following means that [24, p.524, Conjecture 1.4.1 (1)] holds in the case where \(\mathcal{X}\) is a semistable family over the spectrum of a Dedekind domain and \(\dim(\mathcal{X}) = 2\).

**Corollary 4.6.** Let \(\mathcal{X}\) be a semistable family over the spectrum of a discrete valuation ring of mixed characteristic \((0, p)\). Suppose that \(\dim(\mathcal{X}) = 2\). Then we have an isomorphism in \(D^b(\mathcal{X}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})\)

\[
\mathbb{Z}/p'(n)_{\text{ét}} \simeq T_r(n),
\]

where \(\mathbb{Z}/p'(n)_{\text{ét}}\) is the étale sheafification of \(\mathbb{Z}/p'(n)\) and \(D^b(\mathcal{X}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})\) is the derived category of bounded complexes of étale \(\mathbb{Z}/p^n\mathbb{Z}\)-sheaves on \(\mathcal{X}\).

**Proof.** We have a quasi-isomorphism

\[
\tau_{\leq n}(\mathbb{Z}/p'(n)_{\text{ét}}) \simeq \tau_{\leq n}(T_r(n))
\]
by [25] p.209, Remark 7.2. By the definition of $\mathcal{T}_r(n)$, we have

$$\mathcal{H}^i(\mathcal{T}_r(n)) = 0$$

for $i \geq n + 1$. Let $\alpha : \mathcal{X}_{\acute{e}t} \to \mathcal{X}_{\text{Nis}}$ be the canonical map of sites. Then the sheafification $\alpha^*$ is exact and

$$\alpha^* \mathbb{Z}/p^r(n)_{\text{Nis}} = \mathbb{Z}/p^r(n)_{\acute{e}t}.$$ 

Hence it suffices to show that

$$\mathcal{H}^i(\mathbb{Z}/p^r(n)) = 0$$

for $i \geq n + 1$. Since we have

$$\Gamma(\text{Spec}(\mathcal{O}_{\mathcal{X},x}), \mathcal{H}^{n+1}(\mathbb{Z}/p^r(n)_{\text{Nis}})) \simeq H^{n+1}(\mathbb{Z}/p^r(n)_{\text{Zar}}(\text{Spec}(\mathcal{O}_{\mathcal{X},x})))$$

$$\simeq H^{n+1}_{\text{Zar}}(\mathcal{O}_{\mathcal{X},x}, \mathbb{Z}/p^r(n))$$

by [9] p.779, Theorem 3.2 b], it suffices to show that

$$H^i_{\text{Zar}}(\mathcal{O}_{\mathcal{X},x}, \mathbb{Z}/p^r(n)) = 0$$

for $i \geq n + 1$. Here $\mathcal{O}_{\mathcal{X},x}$ is the henselian local ring at a point $x$ of $\mathcal{X}$.

Put $R = \mathcal{O}_{\mathcal{X},x}$. Let $Z \hookrightarrow \text{Spec}(R)$ be a closed immersion of codimension 1 with $U = \text{Spec}(R) \setminus Z$. Suppose that $Z$ is regular and $\text{char}(Z) = p > 0$. By the assumptions of $\mathcal{X}$, we are able to choose such a $Z$. Then we have

$$H^i_{\text{Zar}}(Z, \mathbb{Z}/p^r(n-1)) = 0$$

for $i > n$ by [9] p.786, Corollary 4.4. Since $\text{dim}(U) \leq 1$, we have

$$H^i(\mathbb{Z}(n)_{\text{Zar}}(U)) = 0$$

for $i > n + 1$ by the definition of $\mathbb{Z}(n)_{\text{Zar}}$. Moreover we have

$$H^i_{\text{Zar}}(U, \mathbb{Z}(n)) = H^i(\mathbb{Z}(n)_{\text{Zar}}(U)) = 0$$

for $i > n + 1$ by [9] p.779, Theorem 3.2 b]. Hence we have

$$H^i_{\text{Zar}}(U, \mathbb{Z}/p^r(n)) = 0$$

for $i > n + 1$. Therefore we have

$$H^i_{\text{Zar}}(R, \mathbb{Z}/p^r(n)) = 0$$

for $i > n + 1$ by the localization theorem ([9] p.779, Theorem 3.2 a]). Moreover we have

$$H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/p^r(n)) = 0$$

by Proposition 4.5. This completes the proof.
Remark 4.7. Let \( \mathcal{X} \) be a regular scheme which is finite type over the spectrum of a Dedekind domain. Let \( R \) be the henselization of a local ring \( \mathcal{O}_{\mathcal{X},x} \) and \( l \) a positive integer which is invertible in \( R \). Then we can prove

\[
H_{\text{ét}}^{n+1}(R, \mathbb{Z}/l(n)) = H_{\text{Nil}}^{n+1}(R, \mathbb{Z}/l(n)) = 0
\]

by the similar argument as in Proposition 4.5. Hence we have

\[
\mathcal{H}^{n+1}(\mathbb{Z}/l(n)_{\text{ét}}) = 0
\]

by [9, p.779, Theorem 3.2.b)]. Let \( \iota : \mathcal{Z} \hookrightarrow \text{Spec}(R) \) be a closed immersion of regular subschemes of pure codimension 1. Then we have an isomorphism

\[
\pi_* \mu_l^{\otimes n} \simeq \mu_l^{\otimes (n-1)}[-2]
\]

in \( D^b(\mathbb{Z}_{\text{ét}}, \mathbb{Z}/l\mathbb{Z}) \) by the absolute purity theorem ([8]). Here \( D^b(\mathbb{Z}_{\text{ét}}, \mathbb{Z}/l\mathbb{Z}) \) is the derived category of bounded complexes of étale \( \mathbb{Z}/l\mathbb{Z} \)-sheaves on \( \mathcal{Z} \). On the other hand, we have an isomorphism

\[
\tau_{\leq n+1}(R\text{Spec}(R)) \simeq \tau_{\leq n+1}(\mathbb{Z}/l(n-1)_{\text{ét}})[2]
\]

in \( D^b(\mathbb{Z}_{\text{ét}}, \mathbb{Z}/l\mathbb{Z}) \) by [22, p.33, Proposition 2.1] and [26]. Hence we have an isomorphism

\[
\tau_{\leq n+1}(\mathbb{Z}/l(n)_{\text{ét}}) \simeq \mu_l^{\otimes n}
\]

in \( D^b(\text{Spec}(R)_{\text{ét}}, \mathbb{Z}/l\mathbb{Z}) \) by induction on \( \dim(R) \). Therefore we have an isomorphism

\[
\tau_{\leq n+1}(\mathbb{Z}/l(n)_{\text{ét}}) \simeq \mu_l^{\otimes n}
\]

in \( D^b(\mathcal{X}_{\text{ét}}, \mathbb{Z}/l\mathbb{Z}) \).

Theorem 4.8. Let \( \mathcal{X} \) be a semistable family over the spectrum of a discrete valuation ring of mixed characteristic \((0, p)\) and \( R = \mathcal{O}_{\mathcal{X}, x} \) the henselian local ring of \( x \) in \( \mathcal{X} \). Suppose that \( \dim(R) = 2 \). Then the sequence

\[
0 \to H_{\text{ét}}^{n+1}(R, \mathcal{X}_r(n)) \to H_{\text{ét}}^{n+1}(k(R), \mu_p^{\otimes n}) \xrightarrow{\epsilon} \bigoplus_{p \in \text{Spec}(R)_{(1)}} H_p^{n+2}((R_p)_{\text{ét}}, \mathcal{X}_r(n))
\]

is exact for any integers \( n \geq 0 \) and \( r > 0 \).

Proof. Let \( m \) be the maximal ideal of \( R \). Let \( a \in m \setminus m^2 \) and \( p \in (a) \). By the assumption of \( \mathcal{X} \), there exists such an element \( a \in m \setminus m^2 \). Let us denote \( q = (a) \) and \( U = \text{Spec}(R) \setminus \text{Spec}(R/q) \). Put

\[
H_{R/q}^{n+2}(R_{\text{ét}}, \mathcal{X}_r(n))' = \text{Im} \left( H_{\text{ét}}^{n+1}(U, \mathcal{X}_r(n)) \to H_{R/q}^{n+2}(R_{\text{ét}}, \mathcal{X}_r(n)) \right)
\]

and

\[
H_{\text{ét}}^{n+1}(k(R), \mu_p^{\otimes n})' = \text{Ker} \left( H_{\text{ét}}^{n+1}(k(R), \mu_p^{\otimes n}) \to \bigoplus_{p \in U_{(1)}} H_p^{n+2}((R_p)_{\text{ét}}, \mathcal{X}_r(n)) \right).
\]
Then we have a commutative diagram
\[
\begin{array}{c}
0 \\
\oplus H^\ell_{\text{et}}(\bar{R}_p, \mathcal{I}_r(n)) \\
\oplus H^{n+1}_{\text{et}}(R, \mathcal{I}_r(n)) \rightarrow \bigoplus_{p \in \text{Spec}(R)^{(1)}} H^\ell_{\text{et}}(\kappa(p), \mu_{p'^n}) \\
H^{n+2}_{\text{et}}((R_\text{et})_r, \mathcal{I}_r(n)) \\
0
\end{array}
\] (43)

where the rows are exact. Then the middle vertical arrow is surjective by [9, p.778, Lemma 2.4]. Moreover the right vertical arrow is injective by Lemma 4.3. Hence the surjectivity of the left vertical arrow follows from the snake lemma. Moreover the left vertical arrow is injective by Proposition 4.2. This completes the proof.

Corollary 4.9. Let $R$ be the same as in Theorem 4.8 and $Y$ the special fiber of $\text{Spec}(R)$. Then the natural map
\[
H^{n+1}_{\text{et}}(k(R), \mu_{p'^n}) \rightarrow \bigoplus_{p \in \text{Spec}(R)^{(1)} \setminus Y^{(0)}} H^\ell_{\text{et}}(\kappa(p), \mu_{p'^n}) \oplus \bigoplus_{p \in Y^{(0)}} H^{n+1}_{\text{et}}(k(\bar{R}_p), \mu_{p'^n})
\]
is injective for any integers $n \geq 0$ and $r > 0$. Here $\bar{R}_p$ is the henselization of a local ring $R_p$.

Proof. Put
\[
\text{RH}^{n+1}(R) := \bigoplus_{p \in \text{Spec}(R)^{(1)} \setminus Y^{(0)}} H^\ell_{\text{et}}(\kappa(p), \mu_{p'^n}) \oplus \bigoplus_{p \in Y^{(0)}} H^{n+1}_{\text{et}}(k(\bar{R}_p), \mu_{p'^n}).
\]

Then we have a commutative diagram
\[
\begin{array}{c}
0 \\
\oplus H^\ell_{\text{et}}(\bar{R}_p, \mathcal{I}_r(n)) \\
\oplus H^{n+1}_{\text{et}}(R, \mathcal{I}_r(n)) \rightarrow \bigoplus_{p \in \text{Spec}(R)^{(1)}} H^\ell_{\text{et}}(\kappa(p), \mu_{p'^n}) \\
H^{n+2}_{\text{et}}((R_\text{et})_r, \mathcal{I}_r(n)) \\
0
\end{array}
\]

where the rows are exact by Theorem 4.8 and the absolute purity theorem (cf. [19, p.241, VI, Theorem 5.1]). Hence it suffices to show that the homomorphism
\[
H^{n+1}_{\text{et}}(R, \mathcal{I}_r(n)) \rightarrow \bigoplus_{p \in Y^{(0)}} H^{n+1}_{\text{et}}(\bar{R}_p, \mathcal{I}_r(n))
\]
is injective. By the assumption of $R$, there exists an element $p \in Y^{(0)}$ such that $R/p$ is regular. Moreover, a diagram
\[
\begin{array}{c}
H^{n+1}_{\text{et}}(R, \mathcal{I}_r(n)) \\
H^{n+1}_{\text{et}}(R/p, \mathcal{I}_r(n)) \\
H^1_{\text{et}}(\kappa(p), \lambda_{p^n}) \\
H^1_{\text{et}}(\kappa(p), \lambda_{p^n})
\end{array}
\]
is commutative. Hence the homomorphism
\[ H_{\text{ét}}^{n+1}(R, \mathcal{I}_r(n)) \to H_{\text{ét}}^{n+1}(\mathcal{K}_p, \mathcal{I}_r(n)) \]
is injective by Theorem 3.7 and Theorem 2.7. This completes the proof.

**Lemma 4.10.** Let \( v_i (1 \leq i \leq m) \) be a finite collection of independent discrete valuations on a field \( K \) of characteristic 0. Let \( K_i \) be the henselization of \( K \) at \( v_i \) and \( \kappa(v_i) \) the residue field of \( v_i \) for each \( i \). Suppose that \( \text{char}(\kappa(v_i)) = p > 0 \) for all \( i \). Then the natural map
\[ H_{\text{ét}}^n(K, \mu_p^{\otimes n}) \to \bigoplus_i H_{\text{ét}}^n(K_i, \mu_p^{\otimes n}) \]
is surjective for integers \( n \geq 1 \) and \( r > 0 \).

**Proof.** The statement follows from \([4, \text{p.131, Theorem (5.12)}]\) and \([22, \text{pp.61–62, Lemma 6.2}]\).

**Theorem 4.11.** Let \( X \) be a proper and semistable family over an excellent henselian discrete valuation ring of mixed characteristic \((0, p)\) and \( Y \) the special fiber of \( X \). Let \( r \) be any positive integer. Suppose that \( \dim(X) = 2 \). Then the map
\[ H_{\text{ét}}^{n+1}(k(X), \mu_p^{\otimes n}) \to \bigoplus_{p \in \mathfrak{X}(1) \setminus (0)} H_{\text{ét}}^n(k(p), \mu_p^{\otimes (n-1)}) \oplus \bigoplus_{p \in Y(0)} H_{\text{ét}}^{n+1}(k(\mathfrak{O}_{\mathfrak{X}, p}), \mu_p^{\otimes n}) \]
is injective for \( n \geq 1 \). Here \( \mathfrak{O}_{\mathfrak{X}, p} \) is the henselization of a local ring \( \mathfrak{O}_{\mathfrak{X}, p} \). This implies that the local-global map
\[ H_{\text{ét}}^{n+1}(k(X), \mu_p^{\otimes n}) \to \prod_{p \in \mathfrak{X}(1)} H_{\text{ét}}^{n+1}(k(\mathfrak{O}_{\mathfrak{X}, p}), \mu_p^{\otimes n}) \]
is injective for \( n \geq 1 \).

**Proof.** The proof of the statement is the same as \([14, \text{Theorem 2.5}]\). The statement follows from Corollary 4.9 and Lemma 4.10. We review the proof of \([14, \text{Theorem 2.5}]\) for convenience.

Let
\[
\begin{align*}
X_1 & \overset{j_2}{\leftarrow} X_3 \\
\downarrow j_1 & \quad \downarrow j_1' \\
X_0 & \overset{j_2}{\leftarrow} X_2
\end{align*}
\]
be Cartesian. If \( j_1 \) and \( j_2 \) are étale and a complex \( \mathcal{U} \in D_+(X_1) \), then we have an isomorphism
\[ j_2^* R j_1_*(\mathcal{U}) \cong R j'_1 j_2^*(\mathcal{U}) \] (44)
in \( D_+(X_2) \) by \([19, \text{p.88, III, Theorem 1.15}]\).
Write $\Lambda = \mu_{pr}^{\otimes n}$ and $U := \mathcal{X} \setminus Y$. The natural inclusion $j : U \hookrightarrow \mathcal{X}$ is the complement of the closed immersion $i : Y \to \mathcal{X}$. For each $x \in U$, $\{\bar{x}\}$ denotes its closure in $\mathcal{X}$. Then we have an isomorphism

$$H^s_x(U_{\text{\acute{e}t}}, \Lambda) \cong H^s_x(\mathcal{X}_{\text{\acute{e}t}}, Rj_*\Lambda)$$

for any integer $s$ by (44). Thus the sequence

$$\bigoplus_{x \in U^{(1)}} H^{n+1}_x(U_{\text{\acute{e}t}}, \Lambda) \to H^{n+1}_{\text{\acute{e}t}}(U, \Lambda) \to H^{n+1}_{\text{\acute{e}t}}(k(U), \Lambda) \to \bigoplus_{x \in U^{(1)}} H^{n+2}_x(U_{\text{\acute{e}t}}, \Lambda)$$

corresponds to the sequence

$$\bigoplus_{x \in U^{(1)}} H^{n+1}_x(\mathcal{X}_{\text{\acute{e}t}}, Rj_*\Lambda) \to H^{n+1}_{\text{\acute{e}t}}(\mathcal{X}, Rj_*\Lambda) \to H^{n+1}_{\text{\acute{e}t}}(k(\mathcal{X}), \Lambda) \to \bigoplus_{x \in U^{(1)}} H^{n+2}_x(\mathcal{X}_{\text{\acute{e}t}}, Rj_*\Lambda).$$

Put $\mathcal{F} = t^*Rj_*\Lambda$.

Consider the fiber product

$$\begin{array}{ccc}
Y & \overset{j_3}{\hookrightarrow} & Y \cap \mathcal{X} \setminus \{\bar{x}\} \\
\downarrow & & \downarrow \iota_1 \\
\mathcal{X} & \overset{j_3}{\hookrightarrow} & \mathcal{X} \setminus \{\bar{x}\}
\end{array}$$

where $j_3 : \mathcal{X} \setminus \{\bar{x}\} \to \mathcal{X}$ is the open immersion of $\mathcal{X}$. Then we have an isomorphism

$$(j_3)^*t_*\mathcal{F} \cong (\iota_1)_s((j^*_3)^*\mathcal{F})$$

by the proper base change theorem ([19, pp.223–224, VI, Corollary 2.3]) (or by [19, p.69, II, Theorem 3.2] and [19, p.71, II, Corollary 3.5 (a)]). So we have isomorphisms

$$H^s_{\text{\acute{e}t}}(\mathcal{X}, R(j_3)_s(j_3)^*t_*\mathcal{F}) \cong H^s_{\text{\acute{e}t}}(Y, R(j^*_3)_s(j^*_3)^*\mathcal{F})$$

and

$$H^s_{\{x\}}(\mathcal{X}_{\text{\acute{e}t}}, t_*\mathcal{F}) \cong H^s_{\{x\}}(Y_{\text{\acute{e}t}}, \mathcal{F})$$

for any integer $s$. Hence the unit of the adjunction $id \to t_*t^*$ induces a commutative diagram

\[
\begin{array}{ccccccc}
\bigoplus_{x \in U^{(1)}} H^{n+1}_{\{x\}}(\mathcal{X}_{\text{\acute{e}t}}, Rj_*\Lambda) & \to & H^{n+1}_a(X, Rj_*\Lambda) & \to & H^{n+1}_a(k(\mathcal{X}), \Lambda) & \to & \bigoplus_{x \in U^{(1)}} H^{n+2}_{\{x\}}(\mathcal{X}_{\text{\acute{e}t}}, Rj_*\Lambda) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \in Y^{(1)}} H^{n+1}_a(Y_{\text{\acute{e}t}}, \mathcal{F}) & \to & H^{n+1}_a(Y, \mathcal{F}) & \to & H^{n+1}_a(k(\eta), \mathcal{F}) & \to & \bigoplus_{x \in Y^{(1)}} H^{n+2}_a(Y_{\text{\acute{e}t}}, \mathcal{F})
\end{array}
\]
where the rows are exact. Let $\mathcal{O}_{X, \eta}$ be the henselization of the local ring $\mathcal{O}_{X, \eta}$ of $\eta \in Y^{(0)}$ and $K(\eta)$ its fraction field. Let $j : \text{Spec}(K(\eta)) \to \text{Spec}(\mathcal{O}_{X, \eta})$ be the complement of the closed immersion $i : \text{Spec}(\kappa(\eta)) \to \text{Spec}(\mathcal{O}_{X, \eta})$. Then we have an isomorphism

$$H_{\text{ét}}^{n+1}(\kappa(\eta), \mathcal{F}) \simeq H_{\text{ét}}^{n+1}(\kappa(\eta), i^*Rj_*\Lambda)$$

by (44). So we have isomorphisms

$$H_{\text{ét}}^{n+1}(\kappa(\eta), \mathcal{F}) \simeq H_{\text{ét}}^{n+1}(\mathcal{O}_{X, \eta}, Rj_*\Lambda) \simeq H_{\text{ét}}^{n+1}(K(\eta), \Lambda)$$

by [9, p.777, The proof of Proposition 2.2.b]]. Moreover, we have an isomorphism

$$H_{\text{ét}}^{n-2}(\kappa(x), \Lambda(-1)) \simeq H_{\text{ét}}^n(U_{\text{ét}}, \Lambda)$$

for any integer $s$ by the absolute purity theorem ([19, p.241, VI, Theorem 5.1]). Hence it suffices to show that the homomorphism

$$\gamma : \bigoplus_{x \in U^{(1)}} H_{\text{ét}}^{n-1}(\kappa(x), \Lambda(-1)) \to \bigoplus_{y \in Y^{(1)}} H_{\text{ét}}^{n+1}(Y_{\text{ét}}, \mathcal{F})$$

is injective. By the proper base change theorem, the middle map in the diagram (45) is an isomorphism. So it suffices to show that the homomorphism

$$\gamma : \bigoplus_{x \in U^{(1)}} H_{\text{ét}}^{n-1}(\kappa(x), \Lambda(-1)) \to \bigoplus_{x \in U^{(1)}} H_{\text{ét}}^{n+1}(\kappa(x), \Lambda(-1))$$

is surjective.

Let $x \in U^{(1)}$. Since $\{\tilde{x}\} \to X \to \text{Spec}(A)$ is proper and quasi-finite, so is finite by [19, p.6, I, Corollary 1.10]. Hence $\{\tilde{x}\}$ is the spectrum of the henselian local domain $\mathcal{O}_{X, y}/p_x$ by [19] pp.32–33, I, Theorem 4.2 and [19] p.34, I, Corollary 4.3]. Here $p_x$ is the prime ideal of $\mathcal{O}_{X, y}$ which corresponds to $x \in U^{(1)}$. Therefore $\{\tilde{x}\}$ meets $Y$ at one and only one point and

$$U^{(1)} = \bigcup_{y \in Y^{(1)}} (\text{Spec}(\mathcal{O}_{X, y}) \times_X U)^{(1)}.$$
and \( p \cdot \mathcal{O}_{X,y} \) is a prime ideal of \( \mathcal{O}_{X,y} \). So we have a bijection

\[
P_y := (\text{Spec}(\mathcal{O}_{X,y}) \times_X U)^{(1)} \to (\text{Spec}(\mathcal{O}_{X,y}) \times_X U)^{(1)}.
\]

Hence the map \( \gamma \) decomposes into a direct sum

\[
\gamma = \bigoplus_{y \in Y^{(1)}} \left( \gamma_y : \bigoplus_{x \in P_y} H^{n-1}_{\text{ét}}(\kappa(x), \Lambda(-1)) \to H^{n+1}_{\text{ét}}(\eta_x, \mathcal{F}) \right)
\]

and it suffices to show that the map \( \gamma_y \) is surjective.

Put

\[
V_y = \text{Spec}(\mathcal{O}_{X,y})^{(1)} \setminus (\text{Spec}(\mathcal{O}_{X,y}) \times_X U)^{(1)}.
\]

Let \( (\mathcal{O}_{X,y})_{\eta} \) be the localization of the henselian local ring \( \mathcal{O}_{X,y} \) at \( \eta \in V_y \) and \( K(\eta) \) the fraction field of the henselization of \( (\mathcal{O}_{X,y})_{\eta} \). Since

\[
\mathcal{O}_{Y,y} = \mathcal{O}_{X,y} \otimes \mathcal{O}_{X,y} \mathcal{O}_{Y,y}
\]

by [19, p.37, I, Examples 4.10 (c)], we have a commutative diagram with exact rows

\[
\begin{array}{cccc}
H^n_\text{ét}(K(\eta), \Lambda) & \rightarrow & \bigoplus_{x \in P_y} H^{n-1}_{\text{ét}}(\kappa(x), \Lambda(-1)) & \rightarrow & H^{n+1}_{\text{ét}}(\mathcal{O}_{X,y} \times_X U, \Lambda) \\
\oplus_{\eta \in V_y} H_\text{ét}(K(\eta), \Lambda) & \rightarrow & H^{n+1}(\eta_x, \mathcal{F}) & \rightarrow & H^{n+1}_{\text{ét}}(\mathcal{O}_{X,y}, \mathcal{F}) & \rightarrow & \bigoplus_{\eta \in V_y} H^{n+1}_{\text{ét}}(K(\eta), \Lambda)
\end{array}
\]

(46)

where \( K(\eta) \) is the fraction field of the henselian local ring \( \mathcal{O}_{X,y} \) and

\[
H^{n+1}_{\text{ét}}(K(\eta), \Lambda)' = \text{Ker} \left( H^{n+1}_{\text{ét}}(K(\eta), \Lambda) \rightarrow \bigoplus_{x \in P_y} H^{n}_{\text{ét}}(\kappa(x), \Lambda(-1)) \right).
\]

In the diagram (46), the left vertical map is surjective by Lemma [4.10] the map \( \varphi \) is an isomorphism by the proper base change theorem and the right vertical map is injective by Corollary [4.9]. Therefore the map \( \gamma_y \) is surjective by chasing the diagram. This completes the proof.

Remark 4.12. In the proof of Theorem [4.11] we use the Bloch-Kato conjecture ([26]). But it suffices to use [4] p.131, Theorem (5.12)] which is a special case of the Bloch-Kato conjecture (cf. The proof of Lemma [4.10]).

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