Newtonian gravitational multipoles as group-invariant solutions

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Abstract
A family of vector fields that are the infinitesimal generators of determined one-parameter groups of transformations are constructed. It is shown that these vector fields represent symmetries of the system of differential equations interrelated by the axially symmetric Laplace equation and a certain supplementary equation. Group-invariant solutions of this system of equations are obtained by means of two alternative methods, and it is proved that these solutions turn out to be the family of axisymmetric potentials related to specific gravitational multipoles. The existence of these symmetries provides us with a generalization of the fact that the Newtonian monopole is defined by the solution of the Laplace equation with spherical symmetry, and it allows us to extract from all solutions of this equation those with the prescribed Newtonian multipole moments.

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1. Introduction

In Newtonian gravity the gravitational field of a bounded mass distribution is given as a solution of the Poisson equation. As is known, that solution can be expanded in a power series of the inverse radial coordinate in the neighbourhood of infinity, in such a way that some suitable quantities, referred to as multipole moments (MM), characterize the specific solution. These (MM) quantities are related to the structure of the mass distribution by means of integrals defined over the source, and hence the solution provides a well-defined and physically meaningful description of Newtonian gravity.

When addressing the description of gravity in the vacuum, we look for solutions of the Laplace equation, and the above-mentioned multipole expansion allows us to identify the arbitrary constants of the general solution in terms of the multipole moments (MM) of the...
source. Thus, the general well-behaved solution of the Laplace equation at infinity (or far enough from the source) is a series expansion in powers of $1/r$ the terms of which represent the different contributions of each MM that provide both the physical characteristics of the source and a description of the exterior gravitational field.

Let us stress two relevant features of this description of the gravitational solutions: first, because of the linearity of the Laplace equation that series can be seen as the sum of exact solutions, and hence the partial sums of that series become a new solution of the equation. We refer to these partial sums as the multipole solutions of Newtonian gravity.

Secondly, we wish to emphasize that the definition of MM gives them a well-known physical meaning in terms of certain integrals defined over the mass distribution.

These characteristics allow us to consider the multipole expansion series as a perturbative approach to the interpretation of the gravitational field by the following two criteria: the description of the field is performed with respect to a point sufficiently far away from the source that higher powers of the inverse radial coordinate can be neglected; alternatively, we truncate the multipole expansion series at a suitable multipole solution because the mass distribution of the source provides infinitesimal higher-order contributions.

Therefore, MM play a fundamental role in the construction of solutions of Newtonian gravity since multipole solutions can be understood as corrections in the description of the gravitational field of the first term of the multipole expansion (the monopole solution). This first term of the succession of partial sums represents the gravitational field generated by a bounded spherical source. The spherical feature characterizes that specific solution univocally and, in fact, if the spherical symmetry condition is imposed on the solutions of the Laplace equation, then a unique solution of the series is obtained: the gravitational monopole. One way to introduce that constraint is by requiring the solution to be invariant under the action of the symmetry group $SO(3)$. In spherical coordinates $\{r, \theta, \phi\}$, the infinitesimal generators of that group are

\begin{align}
J_3 &= \frac{\partial}{\partial \phi}, \\
J_2 &= -\cos \phi \frac{\partial}{\partial \theta} + \sin \phi \cot \theta \frac{\partial}{\partial \phi}, \\
J_1 &= \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \cot \theta \frac{\partial}{\partial \phi}. 
\end{align}

The invariance condition over a gravitational potential $\phi$ leads to the equations $\frac{\partial \phi}{\partial \phi} = 0, \frac{\partial \phi}{\partial \theta} = 0$, and therefore the Laplace equation reduces to the following equation:

\begin{align}
\frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = 0, \\
or, equivalently,
2\phi_r + r\phi_{rr} = 0,
\end{align}

whose unique solution (different from the trivial constant solution) is simply the gravitational monopole $\phi = -\frac{a}{r}$ (with the imposed condition $a > 0$, from the demand of positive mass).

This is the point of departure in the line of enquiry pursued here: is there any kind of symmetry that describes gravitational multipole solutions? Might it be possible to find some differential equations such as (3) that restrict the solutions of the Laplace equation to those with the prescribed multipole characteristic? The spherical symmetry, in the way described above, becomes a universal symmetry in the sense that it does not depend on the equation we wish to solve, namely, the Laplace equation. This is not our aim for the symmetries we
are seeking, but the claim is the existence of some group of transformations that will be a
symmetry of the Laplace equation, which allows us to choose a specific kind of solution from
the general equation.

In what follows we shall see that it is possible to construct a family of vector fields that are
the infinitesimal generators of a one-parameter group of transformations. Furthermore, this
group turns out to be a symmetry of a system of equations consisting of the Laplace equation
and another equation, henceforth called supplementary equation.

Thus, by means of the symmetry, the resolution of the Laplace equation, when we are
looking for multipole solutions, simplifies the system of equations to be solved, and the
problem proves to be easier, since we only need to integrate the supplementary equation
(which is a very simple ODE in the angular variable alone), choosing those given by the
correspondent invariance condition for the integration functions.

This work is organized as follows. In section 2, along a first subsection we offer a brief
review of gravitational multipole solutions in Newtonian gravity, and we recall what multipole
moments are and their physical meaning. In order to avoid too many complicated formulae in
the main text, we have used an appendix to explain details of these contents.

In a second subsection we find some differential equations that are fulfilled by the partial
sums of the series that describe gravitational multipole solutions. These equations represent
a family of additional conditions to the axially symmetric Laplace equation, acting as a
restriction to its general solution.

In section 3, we construct a family of vector fields that play the role of infinitesimal
generators of some one-parameter group of transformations. It is proved that these groups
are symmetries of a system of equations consisting of the Laplace equation and a specific
supplementary equation. In a first step, we show the so-called monopole–dipole symmetry
and then we generalize the symmetry to the $2^N$-pole-order moment case, where the notion of
generalized symmetries must be introduced.

Section 4 is devoted to calculating the groups of transformations for these vector fields that
generate the symmetries. Finally, in section 5 we show that the calculation of group-invariant
solutions of the axially symmetric Laplace equation leads to the gravitational multipole
solutions of Newtonian gravity.

A conclusion section contains some comments about the aims achieved and possible
future generalizations.

Finally, two appendices are included: appendix A contains information about classical
multipole moments, and appendix B is devoted to complete the calculation of the functions
involved in the construction of the symmetries.

2. Multipole solutions in Newtonian gravity

2.1. The multipole moments and gravitational multipole solution

In Newtonian gravity we need to introduce completely symmetric and trace-free tensor fields
to define the so-called classical multipole moments. As can be seen in the appendix A, for
the case of axial symmetry only one component of each tensor is independent, and the exterior
gravitational field generated by an axially symmetric source of mass density $\rho(z, \theta)$ is given
by the following potential:

$$\phi = -G \sum_{n=0}^{\infty} \frac{M_n}{r^{n+1}} P_n(\cos \theta),$$  \hspace{1cm} (4)
where \((r, \theta)\) are the radial and polar coordinates of the exterior point with respect to any origin on the axes of symmetry; \(P_n\) are Legendre polynomials and the constants \(M_n\) are the multipole moments \((A.10)\) of the source.

Just as the Poisson equation applies for the description of gravity generated by a mass distribution, so does the Laplace equation for the vacuum case. Thus, if we consider both cases in our configuration space \(M\), the problem to solve could be divided into different domains \((M = \Omega_\rho \cup \Omega_\nu)\) for each region as follows:

\[
\Delta \phi = \begin{cases} 
\rho(\vec{x}), & \vec{x} \in \Omega_\rho, \\
0, & \vec{x} \in \Omega_\nu,
\end{cases}
\]

\((5)\)

\(\Omega_\rho, \Omega_\nu\) being the domains of the region containing all the mass distributions and vacuum regions, respectively. In the \(\Omega_\nu\) domain, the general solution with axial symmetry and good asymptotic behaviour (i.e. decaying to zero at infinity or far from the source)

\[
\Delta \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) = 0
\]

is given by the series

\[
\phi = \sum_{n=0}^{\infty} a_n \frac{n+1}{r^{n+1}} P_n(\cos \theta),
\]

\((7)\)

where \(a_n\) are arbitrary constants without any physical meaning. Nevertheless, since we must demand continuity of the function \(\phi\) at the boundary of regions \(\Omega_\rho\) and \(\Omega_\nu\), then these constants \(a_n\) are (apart from the constant \(G\)) identical to the multipole moments \(M_n\) and acquire the meaning of those quantities.

### 2.2. Some properties of multipole solutions

We shall now show some properties of the series \((4)\). The derivative of \(\phi\) with respect to the variable \(r\) proves to be (let us take \(G = 1\))

\[
\phi_r = \sum_{n=0}^{\infty} (n+1) \frac{M_n}{n+1} P_n(y) = -\frac{\phi}{r} + \frac{1}{r} \sum_{n=0}^{\infty} n \lambda_n P_n(y), \quad \lambda_n \equiv \frac{M_n}{n+1}, \quad y \equiv \cos \theta,
\]

\((8)\)

and hence we have the following relation:

\[
r \phi_r + \phi = R_1
\]

\((9)\)

\[
R_1 = \sum_{n=1}^{\infty} n \lambda_n P_n(y).
\]

\((10)\)

The above expression can be read as a linear first-order differential equation for the potential \(\phi\); it is clear that since \(R_1\), the right part of equation \((9)\), does not depend on \(\lambda_0\), the first term of the series \((4)\) is a solution of the corresponding homogeneous differential equation.

The derivative of \(\phi\) with respect to the variable \(y\) is

\[
\phi_y = -\sum_{n=0}^{\infty} \lambda_n \partial_y P_n(y) = -\lambda_1 - \sum_{n=2}^{\infty} \lambda_n \partial_y P_n(y),
\]

\((11)\)

and hence the following relation holds:

\[
r \phi_y + \phi + y \phi_y = \sum_{n=2}^{\infty} n \lambda_n P_n(y) - \sum_{n=2}^{\infty} y \lambda_n \partial_y P_n(y) = R_2,
\]

\((12)\)
by means of the property of the Legendre polynomials
\[ n P_n(y) + P'_{n-1}(y) = y P'_n(y). \]
Again by considering expression (12) as a linear first-order differential equation for the potential \( \phi \), we realize that since \( R_2 \) (13) does not depend on \( \lambda_1 \), the partial sum of order two (i.e., the sum of the first two terms) of the series (4) is a solution of the corresponding homogeneous differential equation (12).

Moreover, since the first term of \( R_2 \) (13) is \( -\lambda_2 \), we can add to the first member of equation (12) the second derivative (multiplied by a suitable factor) of \( \phi \) with respect to the variable \( y \) (see equation (14)) in order to obtain a new differential equation for \( \phi \) with an independent term of higher order on \( \lambda_n \). By means of a recursive calculation, and considering successive derivatives of \( \phi \) with respect to the variable \( y \),

\[
\frac{\partial^i \phi}{\partial y^i} = -\sum_{n=2}^{\infty} \lambda_n \frac{\partial^i P_n(y)}{\partial y^i} = -\lambda_i i! L_{i,i} - \sum_{n=i+1}^{\infty} \lambda_n \sum_{k=i}^{n} \frac{k!}{(k-i)!} y^{k-i} L_{n,k},
\]

where \( L_{n,k} \) are the coefficients of the Legendre polynomial of degree \( n \) for the power \( k \) of its variable \( y \), we can obtain the following differential equation for any value of \( N \geq 2 \):

\[
r\phi_r + \phi + y \phi_y + \sum_{n=2}^{N} h_n(y) \partial^n \phi = R_{N+1},
\]

where \( R_{N+1} \) is a series on \( \lambda_n \) of order \( 3 \), \( \lambda_{N+1} \), and \( h_n(y) \) are unknown polynomials of \( y \) that we need to introduce to obtain an independent term \( R_{N+1} \) of that order. In fact, starting from (10) and (13), \( R_k \) (with \( k \geq 1 \)) are series on \( \lambda_n \) which can be written as follows:

\[
R_k = \sum_{n=k}^{\infty} \lambda_n C_{k,n}(y),
\]

that is to say, it can be seen that, for all \( k \geq 1 \), \( C_{k,0}(y) = 0 \) with \( n < k \), in particular, by using (14) and (16) (note that \( L_{i,i} = (2i-1)!!/i! \)) one can quickly conclude that if one chooses \( R_k \) recursively in the following way

\[
R_{k+1} = R_k + \frac{C_{k,k}(y)}{(2k-1)!!} \partial^k \phi,
\]

then, the \( \lambda_k \) term of each \( R_{k+1} \) will vanish for all \( k \geq 1 \), and therefore from equation (15) the functions \( h_n(y) \) are given by

\[
h_n(y) = \frac{C_{n,n}(y)}{n! L_{n,n}} = \frac{C_{n,n}(y)}{(2n-1)!!},
\]

In appendix B we develop the recurrence relation (17) in order to obtain the degree and coefficients of the polynomials \( h_n(y) \), which is needed for the construction of the symmetries and the proof of theorems in following sections.

In conclusion, we have that a solution of the corresponding homogeneous equation (15) for any \( N \) is the partial sum of order \( N + 1 \) of the series (4). Therefore, the homogeneous part of equation (15) can be considered as a supplementary condition that must satisfy a solution of the Laplace equation in order to be the gravitational multipole solution of order \( N \), or in other words, can be understood as a condition to truncate the series (4).

\[^3\] The order of the series must be understood as the smallest value of the index \( n \) for a non-null term \( \lambda_n C_{k,n} \) of the series. Thus, the sum (16) starts at the value of that order. (Note that \( C_{1,0}(y) = 0 \) and \( C_{2,1}(y) = 0 \) since \( C_{1,n}(y) = n P_n(y) \) and \( C_{2,n}(y) = -\delta_2 P_{n-1}(y) \)).
3. Multipole symmetries

Let
\[ \mathbf{v} = \xi(x, u) \frac{\partial}{\partial r} + \tau(x, u) \frac{\partial}{\partial y} + \sigma(x, u) \frac{\partial}{\partial u} \]  
(19)
be a vector field on an open subset \( M \subset X \times U \), where \( X = \mathbb{R}^2 \), with coordinates \( x = (r, y) \), is the space representing the independent variables, and \( U = \mathbb{R} \), with coordinate \( u \), represents the dependent variable.

Let \( \Delta_{\nu}(x, u^{(n)}) = 0 \) be a system of \( \nu \) differential equations defined over \( M \). As is known [9], if \( G \) is a local group of transformations acting on \( M \) and
\[ \mathrm{pr}^{(n)}\mathbf{v}[\Delta_{\nu}(x, u^{(n)})] = 0, \]
(20)
whenever \( \Delta_{\nu}(x, u^{(n)}) = 0 \), for every infinitesimal generator \( \mathbf{v} \) of \( G \), then \( G \) is a symmetry group of the system, where the vector field \( \mathrm{pr}^{(n)}\mathbf{v} \) is the \( n \)th prolongation of \( \mathbf{v} \) defined on the corresponding jet space \( M^{(n)} \subset X \times U^{(n)} \), (whose coordinates represent the independent variables, the dependent variable and the derivatives of the dependent variable up to order \( n \))
\[ \mathrm{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum J \sigma_J(x, u^{(n)}) \frac{\partial}{\partial u_J}, \]
(21)
where the summation being over all multi-indices \( J = (j_1, \ldots, j_k) \), with \( 1 \leq j_k \leq 2, 1 \leq k \leq n \), and the notation \( u_J = \frac{\partial u}{\partial x_J} \) is used.

A scalar function \( \Psi(x, u) = 0 \) on \( M \) is an invariant of the vector field \( \mathbf{v} \) if the following equation is fulfilled:
\[ \xi(x, u) \frac{\partial \Psi}{\partial r} + \tau(x, u) \frac{\partial \Psi}{\partial y} + \sigma(x, u) \frac{\partial \Psi}{\partial u} = 0, \]
(22)
and taking into account that \( D_r(\Psi) = 0 = \frac{\partial \Psi}{\partial u} + \frac{\partial \Psi}{\partial u} \) and \( D_y(\Psi) = 0 = \frac{\partial \Psi}{\partial u} \), (where \( D \) denotes total derivative) then equation (22) can be written in terms of the function \( u(x) \) as follows:
\[ \xi(x, u) \frac{\partial u}{\partial r} + \tau(x, u) \frac{\partial u}{\partial y} - \sigma(x, u) = 0. \]
(23)

3.1. The monopole–dipole symmetry

Let
\[ \mathbf{v} = r \frac{\partial}{\partial r} + y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u} \]  
(24)
be a vector field considered as the infinitesimal generator of some group \( G_{\text{MD}} \). Looking for invariants \( \Psi(r, y, u) \) of this vector field (i.e: \( r\Psi_r + y\Psi_y - u\Psi_u = 0 \)) we end up with the following condition for the variable \( u \):
\[ ru_r + yu_y + u = 0, \]
(25)
which is merely the homogeneous equation (12) satisfied by the gravitational potential describing the monopole–dipole solution of the Laplace equation.

We shall now check whether the one-parameter group generated by the infinitesimal generator \( \mathbf{v} \) (24) is a symmetry group of the Laplace equation. In order to do so, we first calculate the second prolongation (21) of \( \mathbf{v} \) by means of formula (see formula (2.39) in [9] for details)
\[ \sigma_J(x, u^{(2)}) = D_f(\sigma(x, u) - ru_r - yu_y) + ru_{r_r} + yu_{y_y}. \]
(26)
where \( u_{J,x} \equiv \frac{\partial u_{J}}{\partial x} \), and we have
\[
\text{pr}^{(2)} v = v - 2u_{r} \frac{\partial}{\partial u_{r}} - 2u_{y} \frac{\partial}{\partial u_{y}} - 3u_{rr} \frac{\partial}{\partial u_{rr}} - 3u_{yy} \frac{\partial}{\partial u_{yy}} - 3u_{ry} \frac{\partial}{\partial u_{ry}}. \tag{27}
\]
If this is applied to the Laplace equation (with axial symmetry)
\[
\triangle u \equiv r^2 u_{rr} + 2ru_{r} - 2u_{yy} + (1 - y^2)u_{yy} = 0, \tag{28}
\]
we obtain
\[
\text{pr}^{(2)} v [\triangle u] = -[\triangle u] - 2u_{yy}. \tag{29}
\]
Therefore, we can state the following.

**Theorem 1.** The system of equations \( \triangle u(x, u^{(n)}) = 0 \) given by
\[
\begin{align*}
\triangle_1 (x, u^{(n)}) &\equiv \triangle u = 0 \\
\triangle_2 (x, u^{(n)}) &\equiv u_{yy} = 0
\end{align*} \tag{30}
\]
where \( \triangle_1 \) being the Laplace equation (28) and \( \triangle_2 \), the so-called supplementary equation, admits a symmetry group whose infinitesimal generator is \( v \).

**Proof.** Since the second prolongation of \( v \) acting on the supplementary equation is \( \text{pr}^{(2)} v [\triangle_2] = -3u_{yy} \), and according to (29) we have \( \text{pr}^{(2)} v [\triangle_1] = 0 = \text{pr}^{(2)} v [\triangle_2] \) whenever \( \triangle u(x, u^{(n)}) = 0 \), we conclude with the proof. \( \Box \)

### 3.2. \( 2^N \)-pole-order symmetry

The homogeneous differential equation from expression (15) is the equation that must be fulfilled by the gravitational multipole solution up to \( 2^N \)-pole order. In analogy to the previous case, we now introduce a new vector field and claim that the condition satisfied by an invariant function of that vector field on \( M \) reproduces the homogeneous equation (15). This kind of vector field is as follows:
\[
v = r \frac{\partial}{\partial r} + y \frac{\partial}{\partial y} - \left[ u + \sum_{n=2}^{N} h_n(y) \frac{\partial^n u}{\partial u} \right] \frac{\partial}{\partial u}. \tag{31}
\]
As can be seen immediately in the above expression, the coefficient function of the \( \frac{\partial}{\partial u} \) derivative at \( v \) depends not only on \( x \) and \( u \) but also on the derivatives of \( u \). This leads to a significant generalization of the notion of symmetry group, obtained by relaxing the geometric assumption that as long as the coefficient function of the vector field depends on \( x \) and \( u \) such dependence will generate a (local) one-parameter group of transformation acting on the underlying space \( M \). Noether was the first to recognize that it is possible to significantly extend the application of symmetry group methods by including derivatives of the dependent variables in the infinitesimal generators of the transformations\(^4\).

Henceforth, we use the term *generalized* vector field to refer to that kind of vector field (31) and we shall denote by \( \sigma [u] = \sigma [x, u^{(n)}] \) any smooth differential function depending on \( x \), \( u \) and derivatives of \( u \) up to order \( n \) defined for \( (x, u^{(n)}) \in M^{(n)} \subset X \times U^{(n)} \).

Given a generalized vector field such as (31), its infinite prolongation is the formally infinite sum
\[
\text{pr}_{v} = r \frac{\partial}{\partial r} + y \frac{\partial}{\partial y} + \sum_{j} \sigma^{(j)} \frac{\partial}{\partial u_{j}}, \tag{32}
\]
\(^4\) A complete discussion of the curious history of generalized symmetries can be found in [9].
where $\sigma^r$ is given by (26). As is known, a generalized vector field $v$ is a generalized infinitesimal symmetry [9] of a system of $v$ differential equations $\Delta_v(x, u^{(n)}) = 0$ if and only if $\text{prv}[\Delta_v] = 0$ for every smooth solution $u = f(x)$, in direct analogy with the infinitesimal symmetry criterion given previously.

The generalized vector field $v$ (31) has the prolongation

$$\text{prv} = v + \sigma^r \frac{\partial}{\partial u_r} + \sigma^y \frac{\partial}{\partial u_y} + \sigma^{rr} \frac{\partial}{\partial u_{rr}} + \sigma^{yy} \frac{\partial}{\partial u_{yy}} + \cdots,$$

(33)

where the only relevant components for our case are (note that in the following expressions the prime sign is used to denote derivation of functions $h_n(y)$ with respect to variable $y$)

$$\sigma^r = -2u_r - \sum_{n=2}^{N} h_n(y) \partial^n_y u_r,$$  

(34)

$$\sigma^y = -2u_y - \sum_{n=2}^{N} \left( h_n(y) \partial^{n+1}_y u + h'_n(y) \partial^n_y u \right),$$  

(35)

$$\sigma^{rr} = -3u_{rr} - \sum_{n=2}^{N} h_n(y) \partial^n_y u_{rr},$$  

(36)

$$\sigma^{yy} = -3u_{yy} - \sum_{n=2}^{N} \left( h_n(y) \partial^{n+2}_y u + 2h'_n(y) \partial^{n+1}_y u + h''_n(y) \partial^n_y u \right).$$  

(37)

By applying this prolongation to the Laplace equation (28), we obtain

$$\text{prv}[\Delta_1] = -\Delta_1[u] - 2u_{yy} - \sum_{n=2}^{N} h_n(y) \left( 2r \partial^n_y u_r - 2y \partial^{n+1}_y u + r^2 \partial^n_y u_{rr} + (1 - y^2) \partial^{n+2}_y u \right) - \sum_{n=2}^{N} h'_n(y) \left( -2y \partial^n_y u + (1 - y^2) 2 \partial^{n+1}_y u \right) + h''_n(y) \left( 1 - y^2 \right) \partial^n_y u.$$  

(38)

It is straightforward to see that

$$\text{prv}[\Delta_1] = -\Delta_1[u] - 2u_{yy} - \sum_{n=2}^{N} h_n(y) \left( D^n_y[\Delta_1[u]] + n(n+1) \partial^n_y u + 2ny \partial^{n+1}_y u \right) - \sum_{n=2}^{N} h'_n(y) \left( -2y \partial^n_y u + (1 - y^2) 2 \partial^{n+1}_y u \right) + h''_n(y) \left( 1 - y^2 \right) \partial^n_y u,$$  

(39)

and whenever $\Delta_1[u] = 0$ (i.e., over all smooth solutions $u = f(x)$ of the Laplace equation), we have

$$\text{prv}[\Delta_1[u]] = -2u_{yy} - \sum_{n=2}^{N} \left[ \Omega_n(y) \partial^n_y u + \Pi_n(y) \partial^{n+1}_y u \right].$$  

(40)

where

$$\Omega_n(y) \equiv n(n+1)h_n(y) - 2yh'_n(y) + (1 - y^2)h''_n(y),$$  

(41)

$$\Pi_n(y) \equiv 2nh_n(y) + 2(1 - y^2)h'_n(y).$$  

(42)
From the above expressions (40), and considering expressions (B.3), (B.4), for the functions $h_n(y)$, it can be seen that

$$\Omega_k(y) + \Pi_{k-1}(y) = 0, \quad 3 \leq k \leq N. \quad (43)$$

That is to say, it can be proved that the functions $h_n(y)$ defined by (B.3) satisfy (for $k \geq 3$):

$$k(k+1)h_k(y) - 2y h'_k(y) + (1 - y^2)h''_k(y) + 2(1 - y^2)h'_{k-1}(y) = 0. \quad (44)$$

And thus, since $\Omega_2(y) = -2$, for every vector field $v$ (31) we have calculated from (40)–(43) that

$$\text{pr}_v[\Delta_1[u]] = -\Pi_N \frac{\partial^{N+1}u}{\partial y^{N+1}}, \quad N \geq 2, \quad (45)$$

and therefore, since $\Pi_N \equiv 2N y h_N(y) + 2(1 - y^2)h'_N(y) \neq 0$, we have that $\text{pr}_v[\Delta_1] = 0$ if and only if

$$\frac{\partial^{N+1}u}{\partial y^{N+1}} = 0, \quad (46)$$

an equation that we shall call *supplementary* equation and denote by $\Delta_2[u]$. With all these results we can state the following.

**Theorem 2.** The generalized vector field $v$ (31) is a generalized infinitesimal symmetry of the system of equations $\Delta_\nu(x, u^{(n)}) = 0$ given by

$$\begin{cases} 
\Delta_1(x, u^{(n)}) & \equiv \Delta u = 0 \\
\Delta_2(x, u^{(n)}) & \equiv \frac{\partial^{N+1}u}{\partial y^{N+1}} = 0.
\end{cases} \quad (47)$$

**Proof.** We shall show that for every smooth solution $u = f(x)$ of the system we have $\text{pr}_v[\Delta_1] = 0$.

From equation (46), it is obvious that $\text{pr}_v[\Delta_1[u]] = 0$. Now, with respect to the supplementary equation we have that

$$\text{pr}_v[\Delta_2[u]] = \sigma^{\overline{N+1}}_{\overline{N}} y \cdots y,$$  

(48)

where $\sigma^{\overline{N+1}}_{\overline{N}} y \cdots y$ is the coefficient of the prolongation of $v$ defined as follows:

$$\sigma^{\overline{N+1}}_{\overline{N}} y \cdots y \equiv D_y^{\overline{N+1}} \left( -u - \sum_{n=2}^{N} h_n(y) \partial^n_y u - r u_r - y u_y \right) + r \frac{\partial^{N+1}u}{\partial y^{N+1}} u_r + y \frac{\partial^{N+2}u}{\partial y^{N+2}} u. \quad (49)$$

Accordingly, by developing the total derivative we have

$$\sigma^{\overline{N+1}}_{\overline{N}} y \cdots y = -(N+2) \frac{\partial^{N+1}u}{\partial y^{N+1}} u - \sum_{k=0}^{N+1} \binom{N+1}{k} \sum_{n=2}^{N} \partial_y^{N+1-k} h_n(y) \partial_{y^{N+k}} u. \quad (50)$$

The first term at the right-hand side of (50) is proportional to the left part of equation $\Delta_2^{(N)}[u]$ and therefore, it will be zero for all solution $u$ of the system (47). Moreover, we can see that all derivatives of $u$ with respect to $y$ appearing in the sums of (50) are higher than $N + 3$, because, as we already know, the functions $h_n(y)$ are polynomials of degree $n - 2$ with respect to $y$, and hence all derivatives $\partial_y^{N+1-k} h_n(y)$ vanish for $N + 1 - k > n - 2$, or equivalently $n + k < N + 3$.\]
For these reasons, the derivatives of \( u \) with respect to \( y \) appearing in (50) can be written as total derivatives of \( \Delta_2^{(N)}[u] \), and therefore, we can finally write the prolongation acting on \( \Delta_2^{(N)}[u] \) as follows:

\[
\text{prv}\left[ \Delta_2^{(N)}[u] \right] = -(N + 2)[\Delta_2^{(N)}[u]] - \sum_{k=0}^{N+1} \binom{N+1}{k} \Delta_y^{N+1-k} h_n(y) D_y^{n+k-N-1}[\Delta_2^{(N)}[u]].
\]

(51)

With this argument, it is clear that (51) is zero whenever \( \Delta_2^{(N)} u = 0 \), and so we conclude the proof.

4. The group of transformations

Given a vector field \( v \), the parametrized maximal integral curve \( \Upsilon(\epsilon, x) \) passing through \( x \) in \( M \) is called the flow generated by \( v \). As is known, the flow is exactly the same as a local group action on the manifold \( M \) or the so-called one-parameter group of transformation.

Therefore, the vector field \( v \) is called the infinitesimal generator of the action, since by Taylor’s theorem in local coordinates we have that

\[
\Upsilon(\epsilon, x) \equiv \exp(\epsilon v)(x) = x + \epsilon \xi(x) + O(\epsilon^2),
\]

(52)

where \( \xi = (\xi^1, \ldots, \xi^n) \) are the coefficients of \( v \), and the notation referred to as exponentiation of the vector field is used. The orbits of the one-parameter group action are the maximal integral curves of the vector field, and there is a one-to-one correspondence between local one-parameter groups of transformations and their infinitesimal generators.

4.1. Monopole–dipole group \( G_{MD} \)

The computation of the flow generated by the vector field \( v \) (19), which is the infinitesimal generator of the monopole–dipole symmetry, is done by solving the following system of ordinary differential equations:

\[
\begin{align*}
    r &= \frac{d}{d\epsilon}(e^\epsilon r) \bigg|_{\epsilon=0}, \\
y &= \frac{d}{d\epsilon}(e^\epsilon y) \bigg|_{\epsilon=0}, \\
-u &= \frac{d}{d\epsilon}(e^{-\epsilon} u) \bigg|_{\epsilon=0},
\end{align*}
\]

(53)

which leads to

\[
\exp(\epsilon v)(x, u) = (e^\epsilon r, e^\epsilon y, e^{-\epsilon} u).
\]

(54)

Thus, the one-parameter group \( G_{MD} \) generated by \( v \) of (19) is given by the following transformation:

\[
(\tilde{x}, \tilde{u}) = (e^\epsilon r, e^\epsilon y, e^{-\epsilon} u),
\]

(55)

i.e., this symmetry group is a kind of scaling symmetry on \( M \subseteq X \times U \).

Since the group \( G_{MD} \) was proved to be a symmetry group, if \( u = f(x) \) is a solution of the system of differential equations (30), so are the functions:

\[
\tilde{u} = e^{-\epsilon} u = e^{-\epsilon} f(r, y) = e^{-\epsilon} f(e^{-\epsilon} \tilde{r}, e^{-\epsilon} \tilde{y}).
\]

(56)
4.2. \(2^N\)-pole-order groups \(G^{(N)}_{M^n}\)

With respect to the generalized vector field \(v\) (31), some comments should be made about its related group of transformations. First, its one-parameter group can no longer act geometrically on the underlying domain \(M\) because the coefficients of \(v\) depend on derivatives of \(u\), which are also being transformed. Nor can we define a prolonged group action on any finite jet space \(M^{(m)}\), since the coefficients of \(pr^{(m)}v\) will depend on still higher order derivatives of \(u\) than appear in \(M^{(m)}\). As is known, the best way to resolve this is to define an action of the group on a space of smooth functions as follows [9]:

\[
\left[ \exp(\epsilon v)\right] f(x) \equiv u(x, \epsilon),
\]

(57)

where \(u(x, \epsilon)\) is the solution (provided it exists) to the Cauchy problem of the system of evolution equations

\[
\begin{align*}
\frac{\partial u}{\partial \epsilon} &= Q(x, u^{(m)}), \\
u(x, 0) &= f(x).
\end{align*}
\]

(58)

\(Q(x, u^{(m)}) \equiv Q[u]\) being the so-called characteristic of the evolutionary vector field \(v_Q\) defined as follows [8, 9]:

\[
v_Q \equiv Q[u] \frac{\partial}{\partial u},
\]

(59)

and associated with any generalized vector field (see formula (5.7) in [9] for details). In our case, the characteristic of the evolutionary vector field associated with (31) is

\[
Q[u] = -u - \sum_{n=2}^{N} h_n(y) D^n u + r u_x - y u_y.
\]

(60)

As is known [9], if \(P[u]\) is any differential function, and \(u(x, \epsilon)\) a smooth solution to (58), then the prolongation of the evolutionary vector field determines the infinitesimal change in \(P\) under the one-parameter group generated by \(v_Q\). If we assume convergence of the entire Taylor series in \(\epsilon\) of the group action, we obtain the Lie series:

\[
P[\exp(\epsilon v_Q)f] = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} (pr v_Q)^n P[f],
\]

(61)

where the power \(n\) at the prolongation denotes the successive application of \(pr v_Qn\) times. In particular, if \(P[u] = u\) then (61) provides the formal series solution to the evolutionary system (58):

\[
u(x, \epsilon) = f(x) + \epsilon pr v_Q(f) + \frac{1}{2} \epsilon^2 (pr v_Q)^2(f) + O(\epsilon^3),
\]

(62)

where, by means of the formula (see (5.6) in [9]) \(pr v_Q = \sum_J D_J Q[u] \frac{\partial}{\partial u^J}\), we have that

\[
(pr v_Q)^2(f) \equiv pr v_Q(Q)\bigg|_f = \left(-Q - \sum_{n=2}^{N} h_n(y) D^n Q - r D_x Q - y D_y Q\right)\bigg|_f.
\]

(63)

This formal series (62) is no longer of practical use, since we are forced to assume that the solution to the Cauchy problem (58) is uniquely determined provided the initial data \(f(x)\)

\(^5\) That means, from the definition of the group action (57) we have that \(P[\exp(\epsilon v_Q)f] = P[f] + \epsilon pr v_Q(P)f + O(\epsilon^2)\), since \(\frac{d}{d\epsilon} P[u] = pr v_Q(P)\).
is chosen in some appropriate space of functions. Accordingly, the resulting flow will be on the above function space. Verification of this hypothesis involves overcoming a very difficult problem regarding the existence and uniqueness of solutions. And moreover, (62) does not give an explicit expression of the complete transformation group but the infinitesimal one, and, as was mentioned above, one has to assume its convergence.

Nevertheless, we have presented here these results because, as we shall see in the following section, Lie series solution (62), (63), provides us with an argument to introduce a procedure for calculating the \( G_{M}^{(N)} \) group-invariant solutions by considering those solutions of the system of differential equations that make the characteristic \( Q \) zero.

An alternative method for calculating the transformation group is now approached, as follows [10]. Let us consider the infinite prolongation (32) of the generalized vector field \( v \), and let us write down the infinite system of ordinary differential equations

\[
\frac{dr}{d\epsilon} = r, \quad \frac{dy}{d\epsilon} = y, \quad \frac{du}{d\epsilon} = \sigma^J.
\]  

From this system of equations, we can define the flow of \( v \) on the infinite jet space to be the solution of the above system with given initial values \( (x, u^{(\infty)}) = (x_0, u_0^0) \), \( u_0^0 \equiv f(r, y) \) being a solution of the system of equations (47):

\[
\exp[\epsilon pr]v (x, u^{(\infty)}) = (x(\epsilon), u^{(\infty)}(\epsilon)).
\]  

In our case, from equation (64) we must handle the following system of coupled differential equations (we only write the actually relevant equations for our purpose, which is no more than obtaining the flow on the manifold \( M \)):

\[
\frac{dr}{d\epsilon} = r, \quad \frac{dy}{d\epsilon} = y, \quad \frac{du}{d\epsilon} = -u - \sum_{n=2}^{N} h_n(y)u^{(n)}_y \quad \text{etc.}
\]  

where the notation \( u^{(k)}_y \equiv \frac{\partial^k}{\partial y^k} u \) has been used. The first two equations in (66) lead to the already known transformation of the independent variables \( (r = e^{\epsilon} r_0, y = e^{\epsilon} y_0) \), whereas the following ones (67), (68) turn out to be decoupled, as we will see now, when we take into account some considerations about the variable \( u \) and the functions \( h_n(y) \). In order to see this, let us consider the above equations (68) when the total derivative \( D^k_y \) has been developed:

\[
\frac{du}{d\epsilon} = -(k + 1)u^{(k)}_y - D^k_y \left[ \sum_{n=2}^{N} h_n(y)u^{(n)}_y \right].
\]  

Let us consider now the derivatives appearing in the right-hand side of equation (69). On the one hand, the derivatives of the functions \( h_n(y) \) (which have been denoted by \( h_n^{(j)}(y) \)) vanish for \( j > n - 2 \) since the degree with respect to \( y \) of these functions is \( n - 2 \). And on the other hand, the variable \( u \), which is a function of the independent variables \( r \) and \( y \), is a solution of the system of equations (47) and therefore, \( \partial^k u = 0, \forall k > N \). Thus, the order of the derivatives of \( u \) appearing in (69) fulfills the condition \( n + k - j \leq N \). With these considerations, the
non-vanishing terms of the sum of index \( j \) in (69) correspond to the following range of the index \( j \): \( n + k - N \leq j \leq n - 2 \); and hence, that equation can be written as follows:

\[
\frac{du_y}{d\epsilon} = -(k + 1)u_y^{(k)} - \sum_{n=2}^{N} \sum_{j=n+k-N}^{n-2} \binom{k}{j} h^{(j)}(y)u_y^{(n+j-k)}.
\]  

(70)

Moreover, for \( k = N \) and \( k = N - 1 \) in (70) the lower value of the index \( j \) is higher than the upper value and, therefore, the only term in the right-hand side of the corresponding differential equations of (70) is the first one, whereas for higher values of \( k \), it is straightforward to see that the corresponding differential equations can be written as follows:

\[
\frac{du_y^{(N)}}{d\epsilon} = -(N + 1)u_y^{(N)},
\]

\[
\frac{du_y^{(N-1)}}{d\epsilon} = -Nu_y^{(N-1)},
\]

\[
\frac{du_y^{(N-a)}}{d\epsilon} = -(N - a + 1)u_y^{(N-a)} - \sum_{n=2}^{N} \sum_{j=n-a}^{a-2} \binom{N-a}{j} h^{(j)}(y)u_y^{(n+j-a-k)}
\]

\[
= -(N - a + 1)u_y^{(N-a)} - \sum_{n=2}^{N} \sum_{j=0}^{a-2} \binom{N-a}{n-a+j} h^{(n-a+j)}(y)u_y^{(N-j)},
\]

(71)

where \( a \) takes values from 2 to \( N - 1 \).

These decoupled equations can be integrated successively with the given initial values previously mentioned \( (x, u^{(\infty)}) = (x^0, u^0, u_y^0) \), and after that we can solve, with these solutions, equation (67) which gives us the group transformation of the variable \( u \).

As a matter of illustration, let us consider, for example, the case \( N = 2 \); i.e., let us try to solve the above equations (67), (71) in order to obtain the action of the group of transformation \( G_{M^2} \), which represents a symmetry of the system of equations (47). The resulting equations for this case are as follows:

\[
\frac{du}{d\epsilon} = -u + \frac{1}{3}u_{yy},
\]  

(72)

\[
\frac{du}{d\epsilon} = -2u_y,
\]  

(73)

\[
\frac{du_{yy}}{d\epsilon} = -3u_{yy},
\]  

(74)

and therefore the solution of (74), which is unique derivative involved in (72), is given by

\[
u_{yy}(x, \epsilon) = e^{-3\epsilon}u_{yy}^0,
\]  

(75)

with the initial condition \( u_{yy}(x, 0) = u_{yy}^0 \). And the solution of equation (72) is

\[
u(x, \epsilon) = e^{-\epsilon}\left[u^0 + \frac{1}{5}u_{yy}^0(1 - e^{-2\epsilon})\right],
\]

(76)

where \( u(x, \epsilon = 0) = u^0 \) has been considered as the initial condition.

Thus, the one-parameter group generated by \( v (31) \) in \( M \) for the case \( N = 2 \) is given by

\[(\tilde{x}, \tilde{u}) = \left[e^{\epsilon r}, e^{\epsilon y}, e^{-\epsilon}\left(u^0 + \frac{1}{5}u_{yy}^0(1 - e^{-2\epsilon})\right)\right].
\]

(77)
5. Group-invariant solutions

A solution \( u = f(x) \) of a system of differential equations is said to be \( G \)-invariant if it remains unchanged by all group transformations in \( G \), meaning that for each element of the group both the function \( f \) and that transformed by the action of this element agree with their common domains of definition. If \( G \) is a symmetry group of a system of differential equations \( \Delta \), then we can find all the \( G \)-invariant solutions to \( \Delta \) by solving a reduced system of equations, which will involve fewer independent variables than the original system.

5.1. \( G_{MD} \)-invariant solutions

According to (55) global invariants of the group \( G_{MD} \) are \( \kappa = y/r \) and \( \mu = uy \), and hence a group-invariant solution takes the form

\[
u = \frac{1}{y} f(\kappa).
\]

Solving for the derivatives of \( u \) with respect to \( r \) and \( y \) in terms of those of \( f \) with respect to \( \kappa \) and so on, and substituting those expressions in the Laplace equation, leads to the following linear, constant coefficient differential equation:

\[
f'' \kappa^2 - 2f' \kappa + 2f = 0,
\]

whose general solution is \( f = a\kappa^2 + b\kappa \). Therefore, the invariant solution of the Laplace equation by the action of group \( G_{MD} \) is simply the gravitational monopole–dipole solution

\[
u(x) = a \frac{y}{r^2} + b \frac{1}{r}.
\]

An alternative way to obtain group-invariant solutions of the system of differential equations (47) is as follows. First, we solve the supplementary equation, which is a PDE but which involves only one coordinate, its general solution being \( u(r, y) = A(r)y + B(r) \). Now, we make this solution invariant under the action of \( G_{MD} \) (55); i.e., the transformed function \( \tilde{u}(\tilde{r}, \tilde{y}) \) (56) functionally becomes the same with \( u(\tilde{r}, \tilde{y}) \), and therefore we may conclude that the only solutions for \( A(r) \) and \( B(r) \) are

\[
A(r) = \frac{a}{r^2}, \quad B(r) = \frac{b}{r},
\]

whose transformation by the action of the group proves to be

\[
\tilde{u}(\tilde{r}, \tilde{y}) = e^{-\epsilon} u(r, y) = e^{-\epsilon} \left( \frac{a}{r^2} e^{-2\epsilon} \tilde{y} + \frac{b}{r} e^{-\epsilon} \right) = a \frac{\tilde{y}}{r^2} + b \frac{1}{r}.
\]

As seen, this method is equivalent to the previous one and proves to be easier to develop because we do not need either to search for the characteristics of the vector field or to solve the resulting differential equation (79) after substituting the global invariants and their derivatives in the Laplace equation.

5.2. \( G_{MN}^{(N)} \)-invariant solutions

We present now two equivalent methods to obtain the group-invariant solutions for the case of the generalized vector field \( v \) (31): firstly, by using the explicit expression for the group of transformations obtained by integrating (67), (71) for any value of \( N \); and secondly, by using expressions (62)–(63) which outline the infinitesimal flow by means of a formal Lie series in terms of the characteristic of the evolutionary vector field.
(A) For the first method we proceed in analogy with the alternative way seen in the previous case of monopole–dipole symmetry. Let us illustrate the method by considering, as an example, the case $N = 2$: we solve the supplementary equation of the system of equations (47) i.e., $u \equiv f(r, y) = A(r) + B(r)y + C(r)y^2$ and require this solution to be invariant under the action of the group $G^{(2)}(M_2)$ (77).

Since group $G^{(2)}(M_2)$ was proved to be a symmetry group, if $u^0 = f(x)$ is a solution of the system of differential equations (47), so are the functions:

$$u = e^{-e}u^0 = e^{-e}\left[f(r, y) + \frac{1}{6}f_{yy}(r, y)(1 - e^{-2e})\right].$$

Thus, the transformed function $\tilde{u}(\tilde{r}, \tilde{y})$ will be invariant under the action of the group if the following equations are fulfilled:

$$A(\tilde{r}) = e^{-e}\left[A(\tilde{r} e^{-e}) + \frac{1}{3}(1 - e^{-2e})C(\tilde{r} e^{-e})\right],$$
$$B(\tilde{r}) = e^{-2e}B(\tilde{r} e^{-e}),$$
$$C(\tilde{r}) = e^{-3e}C(\tilde{r} e^{-e}),$$

which have the following unique kind of solutions:

$$A(r) = \frac{a}{r} - \frac{1}{3}c r^3, \quad B(r) = \frac{b}{r^2}, \quad C(r) = \frac{c}{r^3}.$$  (87)

The resulting invariant solution turns out to be

$$u(r, y) \equiv f(r, y) \equiv \left(\frac{a}{r} - \frac{1}{3}c r^3\right) + y\left(\frac{b}{r^2}\right) + \left(\frac{c}{r^3}\right)y^2 = \frac{a}{r} + \frac{b}{r^2}P_1(y) + \frac{2c}{3r^3}P_2(y),$$

the gravitational multipole solution possessing monopole, dipole and quadrupole moments.

(B) The second method to obtain group-invariant solutions has to be with the action of the group defined in (57). When the formal Lie series (62), (63) was previously outlined, we mentioned that the complete action of the group is not needed to obtain group-invariant solutions, because an alternative procedure arises from those expressions. The argument is supported by the following.

Proposition. The functions $u = f(x)$, solutions of the system of differential equations $\triangle \nu(x, u^{(n)}) = 0$ given by (47), are group-invariant solutions by the action of $G^{(N)}(M_2)$ (the symmetry group whose infinitesimal generator is given by the generalized vector field (31)) if and only if the characteristic of the associated evolutionary vector field $\nu_Q$ is zero over those functions.

Proof. Since the component of the evolutionary vector field is simply the characteristic $Q[u]$, if a solution $u = f(x)$ is group invariant this obviously means that $\nu_Q(u) = 0$, and therefore $Q(x, u^{(n)}) = 0$. Conversely, from the previous expression (62–63) we see that the prolongation of the evolutionary vector and all its successive prolongations $(pr\nu_Q)^n$ depend on $Q$ and its total derivatives, and hence the action of the group (62) on every solution $u = f(x) = u(x, 0)$ that makes $Q|u = 0$ leads to the equality $u(x, \epsilon) = u(x, 0)$, i.e. the solution is group invariant because the group transforms the solution into itself. \qed

By applying this proposition, we can obtain group-invariant solution of the system of differential equations $\triangle \nu(x, u^{(n)}) = 0$ (47) as follows. We first solve $\triangle \nu^{(N)}$ and demand that
the characteristic of this solution be zero. In fact, that condition, $Q|_u = 0$, is exactly the corresponding homogeneous equation (15):

$$Q|_u = 0 = -u - r \partial_r u - y \partial_y u - \sum_{n=2}^{N} h_n(y) \partial^u_y u.$$  

(89)

The general solution of supplementary equation $\Delta^{(N)}_2$ from (47) is

$$u(r, y) = \sum_{k=0}^{N} F_k(r)y^k.$$  

(90)

If we force that solution to make the characteristic zero, we find that

$$\sum_{k=0}^{N} \left( F_k y^k + F'_k y^k r + k F_k y^k \right) + \sum_{n=2}^{N} h_n(y) \partial^u_y u = 0.$$  

(91)

It is clear that

$$\partial^u_y u = \sum_{k=n}^{N} F_k \frac{k!}{(k-n)!} y^{k-n},$$  

(92)

and therefore the second sum in the expression (91) turns out to be

$$\sum_{n=2}^{N} h_n(y) \partial^u_y u = \sum_{n=2}^{N} \sum_{l=0}^{n-2} H_{nl} y^l \sum_{p=0}^{N-n} \frac{F_{p+n}(r)(p+n)!}{p!} y^p,$$  

(93)

where the following notation has been used for the functions $h_n(y)$ (B.3):

$$h_n(y) \equiv \sum_{i=0}^{n-2} H_{ni} y^i.$$  

(94)

If we rearrange the sums in powers of the variable $y$ we have

$$\sum_{n=2}^{N} h_n(y) \partial^u_y u = \sum_{k=0}^{N-2} y^k \left[ \sum_{j=0}^{k} \sum_{p=k+2}^{N} F_p(r) H_{p-k+j} \frac{p!}{(k-j)!} \right].$$  

(95)

The equation for the characteristic (91) is now

$$\sum_{k=0}^{N} y^k \left[ (k+1) F_k + F'_k r \right] + \sum_{k=0}^{N-2} y^k \left[ \sum_{j=0}^{k} \sum_{p=k+2}^{N} F_p(r) H_{p-k+j} \frac{p!}{(k-j)!} \right] = 0,$$  

(96)

and so for each power in the variable $y$, we have the following equations:

$$(k + 1) F_k + F'_k r = 0, \quad k = N, N - 1,$$

$$k = 0, \ldots, N - 2,$$  

(97)

whose solutions provide us with the corresponding functions for the potential $u$ (90) as follows:

$$F_N(r) = \frac{c_N}{r^{N+1}},$$

$$F_{N-1}(r) = \frac{c_{N-1}}{r^N},$$

$$F_k(r) = \frac{c_k}{r^{k+1}} - \frac{1}{r^{k+1}} \sum_{j=0}^{k} \frac{1}{(k-j)!} \sum_{p=k+2}^{N} H_{p-k+j} \int F_p(r) r^k dr,$$  

(98)

$$k = 0, \ldots, N - 2.$$
Let us finish this section with an example. We look for the $G^{(3)}_{M}$ group-invariant solutions of the Laplace equation. From the above expressions (98) we have that

\[ F_3(r) = \frac{c_3}{r^4}, \quad F_2(r) = \frac{c_2}{r^5}, \quad F_1(r) = \frac{c_1}{r^2} - \frac{3}{5} \frac{c_3}{r^4}, \]  

which is the general solution of the equation $2F_1(r) + F'_1(r)r - \frac{6}{5} \frac{c_3}{r^5} = 0$, and for $k = 0$ in (98), we have

\[ F_0(r) = \frac{c_0}{r} - \frac{2}{3} \frac{c_2}{r^3}, \]

which is the general solution of the equation $F_0(r) + F'_0(r)r - \frac{2}{3} \frac{c_2}{r^3} = 0$.

Finally, the invariant solution $u(r, y)$ is

\[ u(r, y) = \left( \frac{c_0}{r} - \frac{1}{3} \frac{c_2}{r^3} \right) + y \left( \frac{c_1}{r^2} - \frac{3}{5} \frac{c_3}{r^4} \right) + y^2 \left( \frac{c_2}{r^3} \right) + y^3 \left( \frac{c_3}{r^4} \right) \]

\[ = \frac{c_0}{r} + \frac{c_1}{r^2} + 2 \frac{c_2}{r^3} P_2(y) + \frac{2}{3} \frac{c_3}{r^4} P_3(y), \]

i.e., the partial sum of order four of series (4) representing the gravitational solution constructed with the mass, the dipole, the quadrupole and the octupole multipole moments.

6. Conclusion

The gravitational multipole solutions of Newtonian gravity are well known, and of course it is not the aim of this work to discover them but to introduce new insights into the mathematical description of the solutions of the gravitational potentials with a prescribed number of multipole moments. For the case of Newtonian gravity, it is known that these solutions are given by the partial sums of the series (4). In this work, we have shown that these solutions can be viewed as group-invariant solutions of certain one-parameter groups of transformations $G_{MD}$ (monopole–dipole case) and $G^{(N)}_{M}$ (general case).

The result obtained is a kind of generalization of the fact that the Newtonian monopole is defined by the solution with spherical symmetry of the Laplace equation. The existence of some symmetries of the Laplace equation is proved, which allows us to restrict the solutions of that equation to those with the prescribed MM. In order to do so, it is necessary to append a so-called supplementary equation to the Laplace equation in order to provide a system that will admit the defined group as a symmetry.

The infinitesimal generators of the group as well as the action of the group have been constructed for each symmetry (in the general case we calculate specifically the action of the group $G^{(2)}_{M}$ to illustrate the procedure). And finally it is proved that the group-invariant solutions are exactly the gravitational multipole solutions.

The supplementary equation arises as a constraint imposed on the Laplace equation in order to maintain the symmetry of the system of both equations, but this restriction itself does not limit the solution of the Laplace equation to those desired gravitational multipole solutions if, in addition, one does not demand an asymptotically well-behaved condition at infinity. For example, for the case $N = 2$, the group $G^{(2)}_{M}$ is a symmetry of the system of equations (47).
whose general solution is

\[ u(r, y) = a_2 - \frac{a_1}{r} + \frac{b_2 y + \frac{2c_1 P_2(y)}{3r^3} + rb_1 y + \frac{2}{3} r^2 c_1 P_2(y)}{r^2} + \frac{2c_2 P_2(y)}{3r^3} + \frac{rb_1 y + \frac{2}{3} r^2 c_1 P_2(y)}{r^2}, \tag{103} \]

where the last two terms must be rejected for asymptotic reasons.

Alternatively, both methods proposed to obtain the group-invariant solutions for the general case (subsections 5.2A and 5.2B) lead univocally to the specific gravitational multipole solutions. In particular, first method provides a smart and mathematically standard procedure, with the additional advantage that we are dealing with an algebraic condition that is easier to solve than a differential one\(^6\). Of course we already know the solution of the axially symmetric Laplace equation and the interpretation of the truncated series from (4), but the theoretical result given by the existence of some extra symmetries of the Laplace equation is that it provides us with a procedure to obtain the gravitational multipole solutions with prescribed multipole moments without solving that differential equation.

A clearly more relevant feature of these results has to be stressed in the sense that they serve as a trial for a future generalization to general relativity (GR). Apart from the fact that the nonlinearity of GR gravitation does introduce calculation problems, the static- and axisymmetric-vacuum metrics are described by two metric functions; one of them (which provides the other one by means of a quadrature) is actually a solution of the Laplace equation. Whereas multipole moments in Newtonian gravity provide physical meaning to the coefficients arising from the general solution series (7), the GR scenario requires complicated computations to obtain the relation between those coefficients and relativistic multipole moments (RMM) [1, 2, 4].

Some authors have devoted much effort in researching techniques aimed at obtaining solutions of stationary axisymmetric-vacuum solutions of Einstein field equations with prescribed multipole moments. In [3, 4] the monopole–quadrupole solution was obtained and [5] addresses the MJQ approximate solution. In [4], the authors obtained algebraic conditions \(a_n = a_n(M_n)\) relating the coefficients of the series (7) to the RMM, and a metric function that is a solution of the Laplace equation is given explicitly. In [6, 7], the authors developed a very interesting and useful method for generating the coefficients \(a_n\) needed to construct the gravitational multipole solution with prescribed RMM. Since the existence and uniqueness of that kind of solution in GR can be proved [7], it would be an exceptional goal to establish the existence of some kind of ‘symmetry’ that would generalize Birkhoff’s theorem for a solution with a given set of RMM. The uniqueness theorems of partial differential equations are, in general, very difficult to solve and a topic of research in Mathematical Physics, and they would not strictly be the aim of future works. However, the proposal outlined here consists of considering the algebraic relations \(a_n = a_n(M_n)\) from a different mathematical point of view, hopefully searching for the existence of some kind of ‘symmetry’ rather than a boundary condition problem from which those relations arose.

Since it is the Laplace equation that we wish to solve, the problem could be oriented towards finding the appropriate system of coordinates or some basis of functions where the

\[ u(r, y) = A(r) + B(r)y + C(r)y^2, \]

\[ 2rA'(r) + r^2A''(r) + 2C(r) = 0, \]

\[ 2rB'(r) + r^2B''(r) - 2B(r) = 0, \]

\[ 2rC'(r) + r^2C''(r) - 6C(r) = 0. \]
already known gravitational multipole solutions, as in the case of Newtonian gravity, could be considered as the only solutions satisfying suitable symmetry conditions.

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Appendix A

Let us consider the Newtonian gravitational potential of a mass distribution with density \( \rho(\vec{z}) \), given by the following solution of the Poisson equation:

\[
\Phi(\vec{x}) = -G \int_{V} \frac{1}{R} \rho(\vec{z}) \, d^3\vec{z},
\]

where \( G \) is the gravitational constant; the integral is extended to the volume of the source; \( \vec{z} \) is the vector that gives the position of a generic point inside the source; and \( R \) is the distance between that point and any exterior point \( P \) defined by its position vector \( \vec{x} \). Let us now make an expansion of this potential in a power series of the inverse of the distance from the origin to the point \( \vec{P} \) \((r \equiv |\vec{x}|)\) by means of a Taylor expansion\(^7\) of the term \( \frac{1}{R} \) around the origin of coordinates, where \( R \equiv \sqrt{(x^i - z^i)(x_i - z_i)} \). The result is the following:

\[
\Phi(\vec{x}) = -\frac{GM}{r} - G \sum_{l=1}^{\infty} \frac{1}{l! r^{l+1}} Q^{i_1...i_l}_{n_1...n_l}, \quad n^i \equiv \frac{x^i}{r},
\]

where \( M \) represents the total mass of the source, i.e.,

\[
M \equiv \int_{V} \rho(\vec{z}) \, d^3\vec{z},
\]

and the quantities \( Q^{i_1...i_l}_{n_1...n_l} \) are completely symmetric and trace-free tensor fields defined as follows:

\[
Q^{i_1...i_l}_{n_1...n_l} \equiv (2l + 1)! T \int_{V} z^{i_1} ... z^{i_l} \rho(\vec{z}) \, d^3\vec{z},
\]

where the symbol \( T \) denotes the subtraction of the traces.

By using spherical harmonic functions it is possible to give a better representation of the multipole moments from the above-mentioned expansion of the function \( \frac{1}{R} \) if it is written in terms of the Legendre polynomials as follows:

\[
\frac{1}{R} \equiv \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} z^l P_l(\cos \beta),
\]

where \( z \equiv |\vec{z}| \) and \( \beta \) is the angle defined between the position vector of the exterior point \( \vec{x} \) and the position vector of the interior point of integration \( \vec{z} \).

\(^7\) See [11] for details.
Let us write the Legendre polynomials in terms of the spherical harmonics

$$P_l(\cos \beta) = \frac{4\pi}{2l + 1} \sum_{m=-l}^{m=l} Y_l^m(\theta, \phi) Y_l^m(\hat{\theta}, \hat{\phi}), \tag{A.6}$$

where \((\theta, \phi)\) and \((\hat{\theta}, \hat{\phi})\) are the angular spherical coordinates of the position vectors \(\vec{x}\) and \(\vec{z}\), respectively. Thus, the gravitational potential \(\Phi(\vec{x})\) generated by a mass distribution outside the source is given by

$$\Phi(\vec{x}) = -\frac{GM}{r} - 4\pi G \sum_{l=1}^{\infty} \frac{1}{2l + 1} \sum_{m=-l}^{m=l} D_l^m Y_l^m(\theta, \phi), \tag{A.7}$$

with \(D_l^m\) being some quantities defined as follows:

$$D_l^m = \int_V z^l Y_l^m(\hat{\theta}, \hat{\phi}) \rho(\vec{z}) d^3\vec{z}. \tag{A.8}$$

For each value of \(l\), there exist \(2l + 1\) complex quantities \(D_l^m\). Nevertheless, since \(Y_{-m} = (-1)^m Y_{m}^*\), only \(2l + 1\) real quantities are independent.

From expressions (A.7) and (A.2) it is clear that the completely symmetric and trace-free tensors \(Q^{i_1 \ldots i_l}\) are equivalent to the \(2l + 1\) real independent quantities \(^8 D_l^m\), i.e.,

$$\frac{1}{l!} Q^{i_1 \ldots i_l} n_{i_1} \ldots n_{i_l} = \frac{4\pi}{2l + 1} \sum_{m=-l}^{m=l} D_l^m Y_l^m(\theta, \phi). \tag{A.9}$$

Finally, if we consider axial symmetry the number of quantities needed to define the multipole moments decreases. The only spherical harmonic that provides a non-vanishing integral in the expressions of \(D_l^m\) is the corresponding \(Y_{0}^l\), since the solution does not depend on the azimuthal angle. Therefore, only one quantity remains to define the multipole moment of order \(l\):

$$M_l = 2\pi \int z^{l+2} \rho(z, \hat{\theta}) P_l(\cos \hat{\theta}) \sin \hat{\theta} d\hat{\theta} dz, \tag{A.10}$$

with \(\rho(z, \hat{\theta})\) being the density of the axially symmetric distribution; the integral is extended to the volume of the source, and therefore \(z\) represents the radius of the integration point and \(\hat{\theta}\) the corresponding polar angle.

**Appendix B**

The functions \(h_n(y)\) of equation (15) have been defined by (18) in terms of some quantities \(C_{k,n}(y)\) from (16). By using formula (14) in the recurrence relation (17) we immediately get

$$C_{k+1,n}(y) = C_{k,n}(y) - \frac{C_{k,k}(y)}{(2k - 1)!!} \partial^k y P_n(y), \quad n > k. \tag{B.1}$$

This recurrence relation is solved by just plugging it into itself as many times as needed, but nevertheless it is not necessary to solve all the quantities of these double-index functions \(C_{k,n}\), since we are only looking for the quantities \(C_{n,n}\). As has been said, by developing the recurrence relation (B.1) (with increasing value of \(k\)) starting from \(C_{2,n}\) we easily see that

$$C_{k,n} = C_{2,n} - \sum_{j=2}^{k-1} \frac{C_{j,j}(y)}{(2j - 1)!!} \partial^j y P_n(y), \quad n \geq k. \tag{B.2}$$

\(^8\) A completely symmetric tensor \(T_{i_1 \ldots i_r}\) of rank \(r\) in a manifold of dimension 3 only has \((r + 2)(r + 1)/2\) independent components. In addition, the trace-free condition, i.e., \(T_{i_1 \ldots i_r} g^{i_1 i_r} = 0\), imposes \(r(r - 1)/2\) constraints, and therefore only \(2r + 1\) components of a completely symmetric and trace-free tensor in dimension three are independent.
Finally, by taking $k = n$ and considering (18) and $C_{2,n} = -\partial_y P_{n-1}(y)$ we get the following relation to calculate the functions $h_n(y)$:

$$h_n(y) = -\frac{1}{(2n-1)!!} \left[ \partial_y P_{n-1} + \sum_{k=2}^{n-1} h_k(y) \partial_k^2 P_n(y) \right], \quad n \geq 3, \quad h_2(y) = -\frac{1}{3}.$$

From these relations (B.3) we can see that the functions $h_n(y)$ are polynomials in the variable $y$ of degree $n - 2$:

$h_2(y) = -\frac{1}{3!!},$

$h_3(y) = \frac{2}{5!!} y,$

$h_4(y) = -\frac{1}{7!!}(1 + 4y^2),$

$h_5(y) = \frac{2}{9!!}(3y + 4y^3),$

$h_6(y) = -\frac{2}{11!!}(1 + 12y^2 + 8y^4).$

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