Notes on rate equations in nonlinear continuum mechanics

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Abstract: The paper gives an introduction to rate equations in nonlinear continuum mechanics which should obey specific transformation rules. Emphasis is placed on the geometrical nature of the operations involved in order to clarify the different concepts. The paper is particularly concerned with common classes of constitutive equations based on corotational stress rates and their proper implementation in time for solving initial boundary value problems. Hypoelastic simple shear is considered as an example application for the derived theory and algorithms.

Keywords: constitutive equation, corotational rate, objectivity, large deformation, stress integration, hypoelastic, simple shear

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1 Introduction

Many problems in physics and engineering science can be formalized as a set of balance equations for the quantity of interest subject to a number of initial and/or boundary conditions. Additional closure relations are often required which connect the primary unknowns with the dependent variables and render the set of equations mathematically well-posed. The most important closure relations in continuum mechanics \cite{29,65,66,122,121,48,42} are employed to determine the state of stress from the state of strain and are referred to as the constitutive equations. Rate constitutive equations describe the rate of change of stress as a function of the strain rate and a set of state variables.

The choice of a reference system to formulate the problem under consideration is a matter of convenience and, from a formal viewpoint, all reference systems are equivalent. There are in fact preferred systems in nonlinear continuum mechanics, particularly the one being fixed in space (Eulerian or spatial description), and the other using fixed coordinates assigned to the particles of the material body in a certain configuration in space (Lagrangian or material description) \cite{65,120}. Lagrangian coordinate lines are convected during the motion of the body, and referring to them leads to the convected description \cite{66,104}. The arbitrary Lagrangian-Eulerian (ALE) formulation is an attempt to generalize the material and spatial viewpoints and to combine their advantages \cite{45,123,14,15,5,7,8,9}. The equivalence of reference systems for all these descriptions requires that each term of the governing equations represents an honest tensor field which transforms according to the transformation between the reference systems — a property referred to as objectivity or, more generally, covariance \cite{29,66}. A covariant formulation of continuum mechanics, in the stricter sense, takes up the geometric point of view that does not rely on a linear Euclidian space \cite{132,31,95}.

As an illustrating example of different reference systems, consider a bar in simple tension which undergoes a rigid rotation. Then in a fixed spatial (i.e. Eulerian) reference system the stress field transforms objectively if its components transform with the matrix of that rigid rotation. In a Lagrangian reference system, on the other hand, the stress components remain unaffected by such rigid motion because it does not stretch material lines. For reasons of consistency it is required that, if the stress transforms objectively under rigid motions, the constitutive equation should transform accordingly. This claim is commonly referred to as material frame indifference \cite{79,83,121} and has been the focus of much controversy during the last decades \cite{18,92,112}.

Further complexity is introduced if time derivatives are involved, as in rate constitutive equations, because both the regarded quantity and the reference system are generally time-dependent. This has led to the definition of countless objective rates of second-order tensors that transform according to the change of the reference system; see \cite{39,67,68,72,87} for early discussions. Many objective rates are particular manifestations of the Lie derivative \cite{66,92,95}, but not all \cite{32,53}. Today the most prominent examples include the Zaremba-Jaumann rate \cite{51,133} and the Green-Naghdi rate \cite{36} falling into the category of so-called objective corotational rates.

This paper\footnote{This paper is a revised and updated version of a preprint shared in 2017 \cite{6}.} gives an introduction to basic notions of rate equations in nonlinear continuum mechanics. It is not intended as a review article and does not provide a comparative study of recent developments in the field. We are particularly concerned with common classes of constitutive equations based on corotational stress rates and their proper implementation in time for solving large deformation mechanical initial boundary value problems.

The structure of the remaining paper is as follows. Section 2 addresses kinematics, stress and balance of momentum as well as fundamentals of constitutive theory. Various rates of second-order tensor fields are reviewed in Section 3, and classes of constitutive equations that employ such rates are summarized in Section 4. Section 5 derives rate forms of virtual power which are implemented in nonlinear finite element methods to solve mechanical initial boundary value problems. In Section 6 we discuss procedures to integrate rate equations of second-order tensors over a finite time interval. We also provide detailed derivations of
two widely-used numerical integration algorithms that retain the property of objectivity on a discrete level. Applications of theory and algorithms are presented in Section 4 using the popular example of hypoelastic simple shear. The paper closes in Section 8 with some concluding remarks. Since we make extensive use of geometrical concepts and notions which have not yet become standard practice in continuum mechanics, they are briefly introduced in Appendix A.

2 Continuum Mechanics

2.1 Motion of a Body

The starting point of any study about objectivity and rate equations in continuum mechanics is the motion of a material body in the ambient space. As a general convention, we use upper case Latin for coordinates, vectors, and tensors of the reference configuration, and objects related to the Lagrangian formulation. Lower case Latin relates to the current configuration, the ambient space, or to the Eulerian formulation.

Definition 2.1. The ambient space, $\mathcal{S}$, is an $m$-dimensional Riemannian manifold with metric $g$, and the reference configuration of the material body is the embedded submanifold $B \subset \mathcal{S}$ with metric $G$ induced by the spatial metric. We assume that both $B$ and $\mathcal{S}$ have the same dimension. Points (or locations) in space are denoted by $x \in \mathcal{S}$, and $X \in B$ are the places of the particles of the body in the reference configuration. For reasons of notational brevity, we refer to $B$ as the body and to $X$ as a particle. Particles carry the properties of the material under consideration.

Definition 2.2. The configuration of $B$ in $\mathcal{S}$ at time $t \in [0, T] \subset \mathbb{R}$ is an embedding

$$\varphi_t : B \to \mathcal{S}$$

$$X \mapsto x = \varphi_t(X),$$

and the set $\mathcal{C} \overset{\text{def}}{=} \{\varphi_t : B \to \mathcal{S}\}$ is called the configuration space. The deformation of the body is the diffeomorphism $B \to \varphi_t(B)$. The motion of $B$ in $\mathcal{S}$ is a family of configurations dependent on time $t \in I \subset \mathbb{R}$, i.e., a curve $c : I \to \mathcal{C}, t \mapsto c(t) = \varphi_t$, and with $\varphi_t(\cdot) \overset{\text{def}}{=} \varphi(\cdot, t)$ at fixed $t$. We assume that this curve is sufficiently smooth. $\varphi_t(B)$ is referred to as the current configuration of the body at time $t$, and $x = \varphi_t(X)$ is the current location of the particle $X$.

Definition 2.3. The differentiable atlas of $\mathcal{S}$ consists of charts $(\mathcal{V}, \sigma)$, where $\mathcal{V}(x) \subset \mathcal{S}$ is a neighborhood of $x \in \mathcal{S}$ and $\sigma(x) = \{x^1, \ldots, x^m\}_x \overset{\text{def}}{=} \{x^1\}_{x} \in \mathbb{R}^m$. The holonomic basis of the tangent space at $x$ is $\{\frac{\partial}{\partial x^i}\}_x \in T_x\mathcal{S}$, $\{dx^i\}_x \in T_x^*\mathcal{S}$ is its dual in the cotangent space, and the metric coefficients on $\mathcal{S}$ are

$$g_{ij}(x) \overset{\text{def}}{=} \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_x = g \left( \frac{\partial}{\partial x^i}(x), \frac{\partial}{\partial x^j}(x) \right)$$

at every $x \in \mathcal{V}$, taken with respect to the local coordinates $\{x^1\}_x$. The torsion-free connection $\nabla$ has coefficients denoted by $\gamma^j_{ik}$.

Definition 2.4. The charts of neighborhoods $U(X) \subset B$ are denoted by $(U, \beta)$, with local coordinate functions $\beta(X) = \{X^j\}_X \in \mathbb{R}^m$. Therefore, $\{\frac{\partial}{\partial x^j}\}_X \in T_XB$ is the holonomic basis $X$, and the dual basis is $\{dx^j\}_X \in T_X^*B$. The metric of the ambient space induces a metric on $B$, with metric coefficients $G_{ij}(X) \overset{\text{def}}{=} \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_X$ at every $X \in U \subset B$.

Definition 2.5. In accordance with Definition 2.6 the localization of the motion his the map

$$\sigma \circ \varphi_t \circ \beta^{-1}|_{\beta(\varphi_t^{-1}(x) \cap U)} ,$$

with $\varphi_t^{-1}(x) \cap U$ assumed non-empty, and $\varphi_t^t(X^j) \overset{\text{def}}{=} (x^i \circ \varphi_t \circ \beta^{-1})(X^j)$ are the spatial coordinates associated with that localization.

Definition 2.6. It is assumed that both $B$ and $\mathcal{S}$ are oriented with the same orientation, and their volume densities be $dV$ and $dv$, respectively. The relative volume change is given by Proposition 2.14, that is,

$$dv \circ \varphi = J \ dV,$$

where $J(X,t)$ is the Jacobian of the motion $\varphi$. 

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**Definition 2.7.** Let \( \varphi_t \) be a continuously differentiable, i.e. \( C^1 \)-motion of \( B \) in \( S \), then

\[
V_t(x) \overset{\text{def}}{=} \frac{\partial \varphi_t}{\partial t}(X) = \frac{\partial \varphi_t}{\partial s}(B) \frac{\partial}{\partial x} \Bigg|_{B(X)} = V_t^s(X) \frac{\partial}{\partial x}(x)
\]

is called the Lagrangian or material velocity field over \( \varphi_t \) at \( X \), where \( x = \varphi_t(X) \), \( V_t(X) \overset{\text{def}}{=} V(X,t) \) for \( t \) being fixed, and, \( V_t : B \rightarrow TS \). Provided that \( \varphi_t \) is also regular, the spatial or Eulerian velocity field of \( \varphi_t \) is defined through

\[
v_t \overset{\text{def}}{=} V_t \circ \varphi_t^{-1} : \varphi_t(B) \rightarrow TS,
\]

so that \( v_t \) is the “instantaneous” velocity at \( x = \varphi_t(B) \subset S \), and \( V(X,t) = v(\varphi(X,t), t) \). By abuse of language, both \( V \) and \( v \) are occasionally called the material velocity in order to distinguish it from other, non-material velocity fields.

**Definition 2.8.** Depending on whether \( x = \varphi(X,t) \in S \) or \( X \in B \) serve as the independent variables describing a physical field, one refers to \( q_t : \varphi_t(B) \rightarrow T^r_s(S) \) as the Eulerian or spatial formulation and to \( Q_t \overset{\text{def}}{=} (q_t \circ \varphi_t) : B \rightarrow T^r_s(S) \) as the Lagrangian or material formulation of that field, respectively.

**Proposition 2.1.** For a regular \( C^1 \)-motion, the Lie derivative of an arbitrary, possibly time-dependent, spatial tensor field \( t_i \in \mathfrak{T}_{ti}(S) \) along the spatial velocity \( v \) can be expressed by

\[
L_v t_i = \varphi_t \upharpoonright \frac{d}{dt}(\varphi_t \downharpoonright t_i).
\]

**Proof.** By Definition A.35 \( L_v(t_i) = \psi_{t,s} \upharpoonright \frac{d}{dt}(\psi_{t,s} \downharpoonright t_i) \), where \( \psi_{t,s} \), with \( s,t \in [t_0,T] \subset \mathbb{R} \), is the time-dependent flow generated by the spatial velocity on \( S \) (Definition A.34). By Definition 2.7 the latter is obtained from

\[
\psi_{t,s} = \varphi_t \circ \varphi_s^{-1} : S \supset \varphi_s(B) \rightarrow \varphi_t(B) \subset S.
\]

The assertion follows by applying the chain rule for pushforward and pullback (Proposition A.35), and noting that \( (\varphi_s \downharpoonright)^{-1} = (\varphi_s^{-1}) \downharpoonright = \varphi_s \upharpoonright \).

**Proposition 2.2.**

\[
\frac{\partial J}{\partial t} = J(\text{tr} \, d) \circ \varphi.
\]

**Proof.** \( \varphi \downharpoonright dV = J dV \) by Definition 2.6 in conjunction with Proposition A.14 so \( J dV \) is a time-dependent volume form on \( B \). Hence, from Propositions A.15 and 2.4

\[
dV \frac{\partial J}{\partial t} = \frac{\partial}{\partial t} (J dV) = \varphi \downharpoonright \mathcal{L}_v dV = \varphi \downharpoonright ((\text{div} \, v) \, dV) = ((\text{div} \, v) \circ \varphi) J dV,
\]

that is, \( \frac{\partial}{\partial t} J = J(\text{div} \, v) \circ \varphi \). Since skew-symmetric tensors have zero trace, \( \text{div} \, v = \text{tr} \, l = \text{tr} \, d \).

**Definition 2.9.** The material time derivative of an arbitrary time-dependent tensor field \( q_t \in \mathfrak{T}^r_s(\varphi(B)) \) is defined through

\[
\dot{q}(x,t) \overset{\text{def}}{=} \frac{\partial q}{\partial t} \bigg|_x (x,t) + (v \cdot \nabla q)(x,t),
\]

where \( \dot{q} \in \mathfrak{T}^r_s(\varphi(B)) \), \( x = \varphi(X,t) \), and the term \( \frac{\partial}{\partial t} q \) is called the local time derivative of \( q \).

### 2.2 Deformation Gradient and Strain

**Definition 2.10.** The deformation gradient at \( X \in B \) is the tangent map over \( \varphi \) at \( X \in B \), that is, \( F(X) \overset{\text{def}}{=} T\varphi(X) : T_X B \rightarrow T_{\varphi(X)} S \) (cf. Definition A.26); the time-dependency has been dropped for notational brevity.
Definition 2.13. By Definitions 2.10, A.28, and A.29.

Proof. (ii)

This circumstance would justify the replacement of mappings are suppressed)

Definition 2.14. Let \( t \in \mathcal{T}_2^e(S) \), \( s \in \mathcal{T}_2^o(S) \), \( T \in \mathcal{T}_2^o(B) \), and \( S \in \mathcal{T}_2^o(B) \), then (compositions with point mappings are suppressed)

\[
\varphi \downarrow t = F^T \cdot t \cdot F \in \mathcal{T}_2^o(B),
\varphi \downarrow s = F^{-1} \cdot s \cdot F^T \in \mathcal{T}_2^o(B),
\varphi \uparrow T = F^{-T} \cdot T \cdot F^{-1} \in \mathcal{T}_2^o(S),
\varphi \uparrow S = F \cdot S \cdot F^T \in \mathcal{T}_2^o(S).
\]

Proof. By Definitions 2.10, A.28 and A.29.

Remark 2.2. The pullback and pushforward operators involve the tangent map \( T\varphi = F \), and not \( \varphi \) itself. This circumstance would justify the replacement of \( \varphi \downarrow \) by the symbol \( F \downarrow \), referred to as the \( F \)-pullback, and \( \varphi \uparrow \) by \( F \uparrow \), called the \( F \)-pushforward.

Definition 2.11. The right Cauchy-Green tensor or deformation tensor is the tensor field defined through \( C \defeq (F^T \circ \varphi) \cdot F \in \mathcal{T}_1^o(B) \).

Definition 2.12. The Green-Lagrange strain or material strain is defined by \( E \defeq \frac{1}{2}(C - I) \), in which \( I \) is the second-order identity tensor on \( B \), with components \( \delta_{ij} \).

Remark 2.3. Note that both \( C \) and \( E \) are proper strain measures on the material body \( B \), and that \( E^b = \frac{1}{2}(C^b - G) \), where \( G = G_{ij} \, dx^i \otimes dx^j \) is the metric on \( B \).

Definition 2.13. The left Cauchy-Green tensor is a spatial or Eulerian strain measure defined through \( b \defeq (F \circ \varphi^{-1}) \cdot F^T \in \mathcal{T}_1^o(S) \). In a local chart of \( S \),

\[
b = G^{ij} g_{jk} F_i^j F_k^s \frac{\partial}{\partial x^i} \otimes dx^i.
\]

The base points have been suppressed. The components of \( F^T \) are given by Proposition A.31.

Definition 2.14. The Euler-Almansi strain or spatial strain is defined by \( e_{EA} \defeq \frac{1}{2}(i - c) \), in which \( c \defeq b^{-1} \) is called the Finger tensor, and \( i \) is the second-order identity tensor on \( S \) with components \( \delta_{ij} \).

Proposition 2.4. Let \( g \in \mathcal{T}_2^e(S) \) be the spatial metric, and \( \varphi \) a regular configuration, then (i) \( C^b = \varphi \downarrow g \), (ii) \( \varphi \downarrow G = c^b \), and (iii) \( \varphi \uparrow E^b = e_{EA}^b \).

Proof. The proofs of (i) and (ii) can be done in local coordinates with the aid of the formulas presented in this section; cf. [4] for details. From this, (iii) becomes

\[
\varphi \uparrow E^b = \frac{1}{2}(\varphi \uparrow C^b - \varphi \uparrow G) = \frac{1}{2}(g - c^b) = e_{EA}^b.
\]

Remark 2.4. It should be emphasized that associated tensors are different objects. For brevity, however, the same name is used for all of them; e.g. all \( C \), \( C^b \), and \( C^\sharp \) denote the right Cauchy-Green tensor. We note that, given a configuration \( \varphi \), the tensor \( C^b \) plays a role of a material metric induced by the spatial metric \( g \), whereas \( c^b \) plays a role of a spatial metric induced by the material metric \( G \). Quite recently, Fiáš [53] and Kolev and Desmorat [55] have brought the significance of \( C^b \) and \( c^b \) in the theory of finite deformations and the definition of objective rates into focus.

Definition 2.15. If \( \varphi : B \to S \) is a regular configuration, then the deformation gradient has a unique right polar decomposition \( F = R \cdot U \), and a unique left polar decomposition \( F = V \cdot R \). The two-point tensor \( R(X) : T_X B \to T_X S \), where \( x = \varphi(X) \), includes the rotatory part of the deformation and is proper orthogonal, that is, \( R^{-1} = R^T \) resp. \( \det R = +1 \). The right stretch tensor \( U(X) : T_X B \to T_X B \) and the left stretch tensor \( V(x) : T_x S \to T_x S \) are symmetric and positive definite for every \( X \in B \) and \( x \in S \), respectively.
Remark 2.5. It will be usually clear from the context whether $V$ denotes the left stretch tensor or the material velocity, respectively, whether $U$ denotes the right stretch tensor or the material displacement.

Recall that pushforward and pullback of a tensor generally do not commute with index raising and lowering (Remark 2.4). This undesirable feature, however, disappears if the alteration of the pushed (pulled) tensor is carried out according to the pushed (pulled) metric tensor, as already mentioned in \cite[p. 70]{66} and \cite[sect. 6.2]{91} and considered in more detail in the following propositions.

Proposition 2.5. Let $\varphi : B \to S$ be a configuration, $t \in T^1_1(S)$ be a mixed second-order spatial tensor, and let the pulled tensor with all indices lowered be obtained through a pulled index lowering operator defined by the pulled spatial metric,

$$(\varphi \downarrow t) \uparrow b \overset{\text{def}}{=} (\varphi \downarrow t) \cdot (\varphi \downarrow g) = (\varphi \downarrow t) \cdot C^b.$$

Then, the following commutative rule holds:

$$(\varphi \downarrow t)\uparrow b = \varphi \downarrow (t^b) \in \mathcal{T}^0_2(B).$$

Proof. Direct application of Proposition 2.3, without indicating point mapping compositions, yields

$$(\varphi \downarrow t)\uparrow b = (\varphi \downarrow t) \cdot (\varphi \downarrow g) = (F^T \cdot t \cdot F^{-T}) \cdot (F^T \cdot g \cdot F) = F^T \cdot (t \cdot g) \cdot F = \varphi \downarrow (t^b).$$

Proposition 2.6. Both $R$-pushforward and $R$-pullback always commute with index raising and lowering, e.g. $R\uparrow (T^b) = (R\uparrow T)^b$.

Proof. By Proposition 2.3 and noting that $R$ is orthogonal, i.e. $R^{-1} = R^T$.

Definition 2.16. The Lagrangian or material logarithmic strain is defined through the spectral decomposition

$$\varepsilon \overset{\text{def}}{=} \ln U = \sum_{\alpha=1}^{m} (\ln \lambda_\alpha) \Psi_{(\alpha)} \otimes \Psi_{(\alpha)} \in \mathcal{T}^1_1(B),$$

where $\lambda_\alpha$ and $\Psi_{(\alpha)}$, with $\alpha \in \{1, \ldots, m\}$, are the eigenvalues and principal axes of the right stretch tensor $U$, respectively. The eigenvalues play the role of principal stretches. The Eulerian or spatial logarithmic strain reads

$$\varepsilon \overset{\text{def}}{=} \ln V = \sum_{\alpha=1}^{m} (\ln \lambda_\alpha) \psi_{(\alpha)} \otimes \psi_{(\alpha)} \in \mathcal{T}^1_1(\varphi(B)),$$

where $\psi_{(\alpha)} = R \cdot \Psi_{(\alpha)}$ are the principal axes of $V$. In the literature, the Eulerian logarithmic strain is often referred to as the Hencky strain.

Definition 2.17. For $m = 3$, the tuples of the three principal axes $\Psi_{(\alpha)}$ and $\psi_{(\alpha)}$, with $\alpha \in \{1, 2, 3\}$, are also called the Lagrangian triad and Eulerian triad, respectively. Given a Cartesian coordinate system, define the matrices $\Psi \overset{\text{def}}{=} (\Psi_1, \Psi_2, \Psi_3)$ and $\psi \overset{\text{def}}{=} (\psi_1, \psi_2, \psi_3)$ as well as the diagonal matrix $A \overset{\text{def}}{=} \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, then the component matrices of the spectral representations of the right stretch tensor and left stretch tensor, respectively, can be written \cite[sect. 6.2]{66}

$$U = \Psi A \Psi^T \quad \text{and} \quad V = \psi A \psi^T.$$
Definition 2.18. Let \( \varphi_t : \mathcal{B} \rightarrow \mathcal{S} \) be a regular \( C^1 \)-motion, then the Lagrangian or material rate of deformation tensor \( D \) is defined by \( 2D(X,t) \overset{\text{def}}{=} \frac{\partial}{\partial t} C(X,t) = 2 \frac{\partial}{\partial t} E(X,t) \). The Eulerian or spatial rate of deformation tensor field \( d \) is defined by \( d_i^k \overset{\text{def}}{=} \varphi_t ^j (D_i^j) \), where \( d_i : \mathcal{S} \rightarrow T \mathcal{S} \otimes T^* \mathcal{S} \) is a spatial tensor field for fixed time \( t \).

Proposition 2.8. \( d^i = L_v (e_{E A}^i) = \frac{1}{2} L_v g \).

Proof. By Definition 2.18 together with Propositions 2.3 and 2.11.

Proposition 2.9. \( d_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i) \) resp. \( d^i = \frac{1}{2}((\nabla v)^T + \nabla v) \).

Proof. By Proposition 2.8 Definition A.31 Propositions A.13 and A.7 and noting that the spatial metric is time-independent.

Definition 2.19. The spatial velocity gradient is defined by

\[
\varepsilon \overset{\text{def}}{=} (\nabla v)^T = \left( \left( \frac{\partial}{\partial t} F \right) \cdot F^{-1} \right) \circ \varphi^{-1} \overset{\text{def}}{=} \dot{F} \cdot F^{-1} .
\]

Moreover, \( \varepsilon \overset{\text{def}}{=} d + \omega \), where \( d = \frac{1}{2}(l + l^T) \) is the spatial rate of deformation tensor (Definition 2.18) and

\[
\omega \overset{\text{def}}{=} \frac{1}{2}(l - l^T) = \frac{1}{2} (\nabla v)^T - \nabla v
\]

is called the vorticity, with \( \omega_t : \mathcal{S} \rightarrow T \mathcal{S} \otimes T^* \mathcal{S} \) for fixed \( t \).

Definition 2.20. The infinitesimal strain is the linear approximation (linearization) to the Green-Lagrange strain about a stress-free and undeformed state in the direction of an infinitesimal displacement \( u \):

\[
\varepsilon_{\text{lin}} \overset{\text{def}}{=} \text{LIN}_u E = \frac{1}{2}((\nabla u)^T + \nabla u) \quad \text{resp.} \quad (\varepsilon_{\text{lin}})_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i) .
\]

2.3 Stress and Balance of Momentum

We are particularly concerned with isothermal mechanical initial boundary value problems that are governed by conservation of mass and balance of linear and angular momentum. This section summarizes some basic relations for which detailed derivations are available in the standard textbooks; e.g. [48, 65, 66, 122]. Notations and definitions of the previous section are used throughout. In addition, let the material body be in its reference configuration at time \( t = 0 \) such that

\[
\varphi_0(\mathcal{B}) = \mathcal{B} \quad \text{and} \quad J(X,0) = 1 .
\]

Moreover, we assume that subsets \( U \subset \mathcal{B} \) of the material body and subsets \( \varphi_t(U) \subset \varphi_t(\mathcal{B}) \subset \mathcal{S} \) embedded in the ambient space have at least piecewise \( C^1 \)-continuous boundaries \( \partial U \) and \( \partial(\varphi_t(U)) = \varphi_t(\partial U) \), respectively. The outward normals to these boundaries are denoted by \( N^* \in \Gamma(T^* \mathcal{B}) \) and \( n^* \in \Gamma(T^* \mathcal{S}) \), respectively.

Definition 2.21. A Cauchy traction vector field is a generally time-dependent vector field on the boundary \( \partial(\varphi_t(\mathcal{B})) \) representing the force per unit area acting on an oriented surface element with outward normal \( n^* \). If the ambient space is the linear Euclidian space, i.e. \( \mathcal{S} = \mathbb{R}^m \), then \( \int_{\partial(\varphi_t(U))} t \, da \) represents the total surface force acting on the body. The Cauchy traction vector at time \( t \) and point \( x \in \partial(\varphi_t(\mathcal{B})) \) is written

\[
t(x,t,n^*(x)) = t_e(x,n^*(x)) \in T_x \mathcal{S} .
\]

Theorem 2.1 (Cauchy’s Stress Theorem). Let the Cauchy traction vector field \( t \) be a continuous function of its arguments, then there exists a unique time-dependent spatial \( (\frac{\partial}{\partial t}) \)-tensor field \( \sigma_t \in \mathcal{S}^0(\mathcal{S}) \) such that

\[
t = \sigma \cdot n^* , \quad \text{resp.} \quad t^i(x,t,n^*(x)) = \sigma^{ij}(x,t) n_j(x) \quad \text{in spatial coordinates} \ x^i ,
\]

that is, \( t \) depends linearly on \( n^* \).

Definition 2.22. The tensor \( \sigma(x,t) = \sigma^{ij}(x,t) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \), as well as its associates with components \( \sigma^i_j, \sigma_i^j \) and \( \sigma_{ij} \), respectively, are referred to as the Cauchy stress.
Definition 2.23. Let $\rho_t : \varphi_1(B) \to \mathbb{R}$ be the spatial mass density of the body, $b_t \in \Gamma(TS)$ the external force per unit mass, and $\sigma_t \in \mathcal{S}_0^1(S)$ the Cauchy stress, with $\rho_t(x) = \rho(x,t)$, $b_t(x) = b(x,t)$, and $\sigma_t(x) = \sigma(x,t)$ at fixed $t$. Moreover, let $p$ satisfy conservation of mass such that $\frac{d}{dt} \int_{\varphi_1(B)} \rho \, dv = 0$, then $\rho$, $b$, and $\sigma$ satisfy spatial balance of linear momentum if

$$\dot{\rho} b = \rho b + \text{div} \sigma,$$

where $\dot{v}_s \in \Gamma(TS)$ is the spatial acceleration field and a superposed dot denotes the material time derivative (Definition 2.24).

Definition 2.24. The first Piola-Kirchhoff stress is the tensor field $P_t : B \to TS \otimes T^*B$ obtained by applying the Piola transformation (Definition 2.26) to the second leg of the Cauchy stress, that is,

$$P_t(X) \equiv J_t(X) \left( (\sigma_t \cdot F_i^{-T}) \circ \varphi_i \right) (X)$$

for every $X \in B$, and $\sigma = \sigma^I$ being understood.

Remark 2.7. In Definition 2.24, the placement of parentheses and the composition with the point map are important: as $\sigma(x,t) \cdot F_i^{-1}(x,t)$ has its values at $(x,t)$, one has to switch the point arguments. In material coordinates $\{X^i\}$, spatial coordinates $\{x^i\}$, and by omitting the point maps and arguments, one has $P^{IJ} = J^{ij}(F_i^{-1})_j^I$. Similar to the deformation gradient, $P_t(X)$ is a two-point tensor at every $X \in B$, having the one “material” leg at $X$, and a “spatial” leg at $x = \varphi(X,t) \in S$.

Proposition 2.10. Since $t = \sigma \cdot n^*$ is the force per unit of deformed area in the current configuration of the body, $T = P \cdot N^*$ resp. $T^* = P^{IJ} N^I$ is the same force measured per unit reference area (or undeformed area), and

$$(t \, dA) \circ \varphi = T \, dA$$

Proof. By Definition 2.24 and Proposition A.17.

Proposition 2.11. Let $\rho_{ref}(X) \equiv \rho(\varphi(X,0),0) J(X,0)$ be the reference mass density at time $t = 0$, $V(X,t)$ be the material velocity, and $B(X,t) \equiv b(\varphi(X,t),t)$, then spatial balance of linear momentum (Definition 2.23) has the equivalent Lagrangian resp. material form

$$\rho_{ref} \frac{\partial V}{\partial t} = \rho_{ref} B + \text{DIV} P.$$

Proof. Conservation of mass requires $\rho(\varphi(X,t),t) J(X,t) = \rho_{ref}(X)$ for all $X \in B$ by Theorem A.2 and Proposition A.14. Moreover, $\text{DIV} P = J(\text{div} \sigma \circ \varphi)$ by the Piola identity (Theorem A.4).

Definition 2.25. The second Piola-Kirchhoff stress $S_t \in \mathcal{S}_0^1(B)$, with $S_t(X) = S(X,t)$ holding $t$ fixed, is the tensor field obtained by pullback of the first leg of $P_t$ that is,

$$S_t \equiv F_i^{-1} \cdot P_t = J_t F_i^{-1} \cdot ((\sigma_t \cdot F_i^{-T}) \circ \varphi_i).$$

In components, $S^{IJ} = J(F^{-1})_i^j (F^{-1})_j^I \sigma^{ij}$.

Definition 2.26. The Kirchhoff stress is defined through $\tau \equiv (J \circ \varphi^{-1}) \sigma$.

Proposition 2.12. $S = \varphi \downarrow \tau$.

Proof. By the previous definitions and Proposition 2.3.

Definition 2.27. Let $R$ be the rotation two-point tensor obtained from polar decomposition of the deformation gradient (cf. Definition 2.15), then the corotated Cauchy stress is defined through $R$-pullback (cf. Remark 2.2) of the Cauchy stress:

$$\bar{\sigma} \equiv R \downarrow \sigma.$$
Proposition 2.13. \( R \)-pullback commutes with index raising and index lowering, yielding
\[
\mathcal{E}^\sharp = R^{-1} \cdot \left( (\sigma^\sharp \cdot R^T) \circ \varphi \right) = \left( (R^T \cdot \sigma^\sharp) \circ \varphi \right) \cdot R \quad \text{and} \quad \mathcal{E}^\flat = \left( (R^T \cdot \sigma^\flat) \circ \varphi \right) \cdot R,
\]
where \( \sigma^\sharp \) is the associated Cauchy stress with all indices raised and \( \sigma^\flat = \sigma_{ij} \, dx^i \otimes dx^j \) is the associated Cauchy stress with all indices lowered (cf. Definition A.1).

**Proof.** By Proposition 2.3 again, and noting that \( R \) is proper orthogonal, i.e. \( R^{-1} = R^T \).

**Theorem 2.2** (Symmetry of Cauchy Stress). Let conservation of mass and balance of linear momentum be satisfied, then balance of angular momentum is satisfied if and only if
\[
\sigma = \sigma^T \quad \text{resp.} \quad \sigma^{ij} = \sigma^{ji},
\]
that is, if the Cauchy stress is symmetric. Symmetric of Cauchy stress is equivalent to symmetry of the second Piola-Kirchhoff stress, i.e. \( S = S^T \).

2.4 Constitutive Theory and Frame Invariance

For isothermal mechanical problems governed by balance of linear momentum (Definition A.20) alone, the motion \( \varphi : B \times [0,T] \to \mathcal{S} \) is generally treated as the primary unknown. The reference mass density, \( \rho_{\text{ref}} \), and the external force per unit mass, \( b \), are usually given. The Jacobian \( J \) is known by the knowledge of \( \varphi \), hence the current density \( \rho \) can be determined from \( \rho = J^{-1} \rho_{\text{ref}} \). The acceleration \( \dot{\nu} \) can likewise be derived from \( \varphi \); equivalently, the \( m \) components of \( \dot{\nu} \) can be determined from the set of \( m \) equations of balance of linear momentum. Therefore, in three dimensions one is left with six unknowns: the independent stress components of \( \sigma = \sigma^T \). To close the set of model equations, these stress components are usually determined from suitable constitutive equations.

Sets of axioms based on rational thermomechanical principles are routinely postulated to constrain and simplify the constitutive equations. These will not be repeated here. Instead we refer to [121] and the key papers and lecture notes [25, 24, 23, 36, 79, 80, 81, 82] particularly concerned with constitutive theory. Additional citations are given in the text.

**Definition 2.28.** A relative motion or change of observer\(^2\) is a time-dependent family of orientation-preserving diffeomorphisms \( \theta_t : \mathcal{S} \to \mathcal{S}' \). A relative rigid motion or change of Euclidian observer requires that \( \theta_t = \theta_t^\text{iso} \) is a spatial isometry preserving the distance of every two points:
\[
g' = \theta_t^\text{iso} \upharpoonright g, \quad \text{that is,} \quad g(u,w) = g'(\theta_t^\text{iso} \upharpoonright u, \theta_t^\text{iso} \upharpoonright w),
\]
where \( u,w \in \Gamma(T\mathcal{S}), \theta_t^\text{iso} \upharpoonright u = (T\theta_t^\text{iso} \cdot u) \circ \theta_t^\text{iso} \), and the tangent map
\[
T_x \theta_t^\text{iso} \overset{\text{def}}{=} Q_t(x) : T_x \mathcal{S} \to T_{\theta_t(x)} \mathcal{S}'
\]
is proper orthogonal at every \( x \in \mathcal{S} \) by Proposition A.9 such that \( Q_t^{-1} = Q_t^T \) and \( \det Q_t = +1 \). In this case \( Q_t \) is called a rotation, with \( Q_t(x) = Q(x,t) \) at fixed \( t \).

For notational brevity the index “\( t \)” will be dropped in what follows. We also refrain from explicitly indicating the dependence of a function on a mapping; e.g. for a scalar field \( f : \mathcal{B} \to \mathbb{R} \), a map \( \varphi : \mathcal{B} \to \mathcal{S} \), and \( x \in \mathcal{S} \), we simply write \( f(x) \) instead of the correct \( (f \circ \varphi^{-1})(x) \).

**Definition 2.29.** A tensor field \( \mathbf{s} \in \mathfrak{T}_p^0(\mathcal{S}) \) on the ambient space is called spatially covariant under the action of a relative motion \( \theta : \mathcal{S} \to \mathcal{S}' \) if it transforms according to pushforward
\[
\mathbf{s}' = \theta \upharpoonright \mathbf{s} \in \mathfrak{T}_p^0(\mathcal{S}') \, .
\]
The field \( \mathbf{s} \) is called objective (or frame-invariant) if the transformation according to pushforward is restricted to relative rigid motions \( \theta = \theta^\text{iso} \).

\(^2\)Both are equivalent provided that the different observers use charts having the same orientation relative to the orientation of the spatial volume density \( \mathbf{d} \).
Definition 2.30. A constitutive operator $H$ is understood as a map between dual material tensor fields. However, it can be equivalently formulated in terms of spatial fields by using the transformation rules outlined in the previous sections. Conceptually, but without loss of generality, the constitutive response is denoted by

$$S = H(C, A) \quad \text{and} \quad \sigma = h(F, g, \alpha),$$

in the material description and spatial description, respectively. Besides $S$, $C$, $\sigma$, $F$, and $g$, which have been defined in the sections above, the probably non-empty sets $A = \{A_1, \ldots, A_k\}$ resp. $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ consist of generally tensor-valued internal state variables (or history variables).

Remark 2.8. From a formal viewpoint, a constitutive operator is a tensor bundle morphism between dual tensor bundles over the same base space $[92]$. Bundle morphisms formalize mappings between tensor fields and guarantee that the domain and co-domain of the constitutive operator are evaluated at the same base point and the same time instant; see $[90, 98]$ for more details on bundles and morphisms.

Remark 2.9. The list of arguments of the constitutive operator in Definition 2.30 is meant conceptually: the stress tensor is a function of a deformation tensor and some state variables. In the material description, the right Cauchy-Green tensor $C$ has been chosen as the deformation tensor. By Propositions 2.3 and 2.4(i) the pushforward of $C$ is a function of the deformation gradient $F$ and the spatial metric $g$, which have thus been chosen as arguments in the spatial description of the constitutive operator. This is similar to the formulation of the free energy in $[103, \text{eq. (3.3)}]$. In contrast to that reference, however, we do not indicate the dependency on the material metric tensor $G$ in the reference configuration, which is needed to form invariants from $C$ and $A$.

Minimal requirements in the formulation of constitutive equations are the following principles.

Principle 2.1 (Objectivity (Frame Invariance)). Tensor fields in descriptions of constitutive behavior must be frame-invariant, i.e., they must transform objectively under a change of Euclidian observer $\theta^\text{iso} : S \to S'$. For spatial tensor fields $s \in \mathfrak{T}^p_q(S)$, the principle is expressed by Definition 2.29:

$$s' = \theta^\text{iso} \upharpoonright s \in \mathfrak{T}^p_q(S').$$

Principle 2.2 (Constitutive Frame Invariance). Any constitutive equation must conform to the principle of constitutive frame invariance (CFI) which requires that material fields, fulfilling the equation formulated by an observer, will also fulfill the equation formulated by another Euclidian observer and vice versa. The principle is expressed by the equivalence of constitutive response

$$S = H(C, A) \iff S' = H'(C', A')$$

for any relative rigid motion resp. change of Euclidian observer $\theta^\text{iso} : S \to S'$.

Remark 2.10. CFI has been introduced by Romano and co-workers $[90, 92, 95]$ in the context of a rigorous geometric constitutive theory. It is intended as a substitute to the classical, but improperly stated principle of material frame-indifference (MFI) $[83, 121]$, which has been introduced by Noll $[79]$ as the “principle of objectivity of material properties”. MFI and the related concepts of indifference with respect to superposed rigid body motions (IRBM), Euclidian frame indifference (EFI), and form-invariance (FI), cf. $[112, 138]$, have been the focus of much controversy over the years, until recently. In contrast to that, CFI employs basic and properly settled geometric notions to account for the fact that distinct observers will formulate distinct constitutive relations involving distinct material tensors.

Definition 2.31. The pushforward of a constitutive operator by a relative motion $\theta : S \to S'$ is defined consistent with Definition 2.29 by the identity

$$\left(\theta \upharpoonright H\right)(\theta \upharpoonright C, \theta \upharpoon-right A) = \theta \upharpoon-right (H(C, A)).$$

Proposition 2.14. (See $[52]$ prop. 9.1) A constitutive equation conforms to the principle of CFI if and only if the constitutive operator is frame invariant, that is,

$$H' = \theta^\text{iso} \upharpoon-right H, \quad \text{or equivalently,} \quad h' = \theta^\text{iso} \upharpoon-right h,$$

for any change of Euclidian observer $\theta^\text{iso} : S \to S'$. 

The Objectivity Principle \[2.1\] and the CFI Principle \[2.2\] require that any constitutive equation must conform to them. In this work we are particularly concerned with spatial rate constitutive equations, according to the following definition.

**Definition 2.32.** A *spatial rate constitutive equation* is understood as a map between a rate of strain and a rate of stress in the spatial description. The spatial rate constitutive equations considered here take the general form

\[
\tilde{s} \overset{\text{def}}{=} h(s, g, \alpha, d) \overset{\text{def}}{=} m(s, g, \alpha) : d,
\]

where \(\tilde{s}\) represents any objective rate of any spatial stress measure \(s\) satisfying \((\tilde{s})' = \theta^{\text{iso}} \circ \tilde{s}\) in accordance with Principle \[2.1\] and \(d\) is the spatial rate of deformation tensor. The metric tensor \(g\) is included since it is needed to form scalar invariants from \(s\), \(d\) and \(\alpha\).

**Remark 2.11.** The particular classes of constitutive equations that fall into the category formalized by Definition 2.32 include hypoelasticity, hypoplasticity, and hypoplasticity. These will be discussed in more detail in Sect. 4.

**Remark 2.12.** The requirement of objectivity (Principle \[2.1\]) alone is too weak to results in a unique definition of stress rate. In fact, infinitely many objective stress rates satisfying \((\tilde{s})' = \theta^{\text{iso}} \circ \tilde{s}\) could be obtained simply by adding, to a particular definition of objective rate, terms that vanish under any rigid motion \(\theta^{\text{iso}} : S \to S'\). Therefore, a large amount of literature is concerned with the discussion and/or development of objective rates \[39, 67, 68, 72, 84, 115, 118, 117, 127, 130\]. The decisive conclusion for a rate, e.g. in a constitutive equation, could not be drawn from its objectivity property alone, but has to consider the intended application of that rate. This has already been realized by Prager \[87\], see also \[72, 39\], who pointed out the desirability of an additional requirement in defining objective stress rates, particularly in the context of elasto-plastic rate constitutive equations. Plastic yield criteria are commonly formulated as functions of scalar invariants of the stress tensor; cf. Sect. 4.2. The elastic response, on the other hand, is prescribed as a function between the rate of deformation and the stress rate. For a composite of both elastic and plastic constituents to make sense, zero change in stress (vanishing stress rate) should not change the value of the yield function. Therefore, the following principle should be added; application of the principle will be exemplified in the following sections.

**Principle 2.3 (Stationary Invariants (Prager’s Requirement)).** Any definition of stress rate \(\tilde{s}\), where \(s\) represents any spatial stress measure, must conform to Prager’s requirement \[53\], according to which vanishing of the stress rate implies stationary invariants of the stress tensor. Let \(I(s(t))\) be a scalar function, then the principle is expressed by the equivalence

\[
\tilde{s} = 0 \quad \Leftrightarrow \quad \frac{d}{dt} I(s(t)) = 0.
\]

**Remark 2.13.** The scalar function \(I(s(t))\) in Principle 2.3 could be, for example, one of the principal invariants (Definition A.20) or the Frobenius norm \(\|s\| = \sqrt{\text{tr}(s^2)}\) (Definition A.17 and Remark A.3) of the stress tensor. Some scalar functions could be obtained through application of the metric tensor. In these cases, Principle 2.3 could be fulfilled if the objective rate of the metric tensor vanishes, \(\tilde{g} = 0\); see also \[89\] sect. 5]. However, other choices for \(I(s(t))\) are possible which do not involve any metric.

**Remark 2.14.** Prager’s requirement, originally proposed for elastic-perfectly plastic materials \[87\], has been extended to elasto-plasticity with kinematic hardening, which contains an additional stress-like tensor whose rate prescribes evolution of the yield function \[130, 20\]. Accordingly, the principle of stationary invariants is replaced by a more general *yielding-stationarity criterion*, which demands that all tensor rates must be of the same kind of objective rates. Beyond continuum mechanics and plasticity theory, Prager’s requirement has also gained attention in general relativity and applied mathematics; see, for example, \[88, 114\].
3 Rates of Tensor Fields

3.1 Fundamentals

In the following sections we inspect the transformation properties of the common tensor fields in spatial rate constitutive equations under the action of any relative motion (resp. change of observer) and under the action of a relative rigid motion (resp. change of Euclidian observer). In particular, a distinction is drawn between spatially covariant rates, objective rates, and corotational rates of second-order tensors.

In accordance with Definition 2.28, and by dropping the index $t$ in what follows, let $\theta : S \to S'$ denote a relative motion, and $\theta = \theta^{\text{iso}}$ if the relative motion is an isometry, i.e. rigid. The tangent map of a relative motion is generally time-dependent and denoted by $F^{\theta} \overset{\text{def}}{=} T\theta$, and the proper orthogonal tangent map of a relative rigid motion is the rotation two-point tensor field denoted by $Q \overset{\text{def}}{=} T\theta^{\text{iso}}$. Here and in the following we assume that both $F^{\theta}$ and $Q$ are continuously differentiable in time.

**Proposition 3.1.** The deformation gradient of a motion $\varphi : B \to S$ is spatially covariant.

**Proof.** By Definition 2.29, Proposition 2.3 and the chain rule,

$$F' = F^{\theta} \cdot F = \theta \upharpoonright F.$$  

Composition with the point mappings $\varphi$ and $\theta$ have been suppressed. $\square$

**Definition 3.1.** The spin of a relative rigid motion is defined through $\Lambda \overset{\text{def}}{=} ˙Q \cdot Q^T$.

**Proposition 3.2.** Orthogonality of $Q$ implies skew-symmetry of the spin, that is, $\Lambda = -\Lambda^T$.

**Proof.** By direct calculation,

$$I = \dot{Q}^T \cdot Q = Q^T \cdot Q + Q^T \cdot \dot{Q} = \Lambda + \Lambda^T = 0.$$  

$\square$

**Proposition 3.3.** Let $l = d + \omega$ be the spatial velocity gradient, being composed of the spatial rate of deformation $d$ and the vorticity $\omega$ (cf. Definition 2.19). Then,

(i) both $l$ and $\omega$ are neither spatially covariant, nor objective, and

(ii) $d$ is objective, but not spatially covariant.

**Proof.** (i) Spatial covariance is a stronger version of objectivity under relative rigid motions, thus it suffices to prove that the velocity gradient is not objective. First, note that $F' = \theta^{\text{iso}} \upharpoonright F = Q \cdot F$ by Proposition 3.1. Moreover, by Propositions 2.3 and 2.6, and Definition 2.19, one has

$$l' = (F') \cdot (F')^{-1} = (Q \cdot F + Q \cdot \dot{F}) \cdot F^{-1} \cdot Q^T = Q \cdot Q^T + Q \cdot l \cdot Q^T = l_{\theta^{\text{iso}}} + \theta^{\text{iso}} \upharpoonright l,$$

which is clearly non-objective, that is, it does not conform to Definition 2.29. Substitution of $l = d + \omega$, with $d$ defined as the symmetric part of $l$, shows that $l_{\theta^{\text{iso}}} = \omega_{\theta^{\text{iso}}}$ and

$$\omega' = Q \cdot \dot{Q}^T + Q \cdot \omega \cdot Q^T = \omega_{\theta^{\text{iso}}} + \theta^{\text{iso}} \upharpoonright \omega,$$

which proofs the first assertion.

(ii) A direct consequence of the proof of (i) is that $d$ is indeed objective under relative rigid motions:

$$d' = Q \cdot d \cdot Q^T = \theta^{\text{iso}} \upharpoonright d.$$  

In the case where $\theta : S \to S'$ is an arbitrary relative motion, the tangent $F_\theta$ is generally not orthogonal. Definition 2.19 and Proposition 3.1 yield

$$l' = (\dot{F'}) \cdot (F')^{-1} = \dot{F}_\theta \cdot F_\theta^{-1} + F_\theta \cdot l \cdot F_\theta^{-1} = l_{\theta} + \theta \upharpoonright l.$$  

Therefore, $d' = \theta \upharpoonright d$ if and only if $l_{\theta} \equiv \omega_{\theta}$, that is, if $d_{\theta} \equiv 0$. $\square$
Proposition 3.4. Cauchy stress $\sigma$ is spatially covariant while its material time derivative $\dot{\sigma}$ is not even objective.

Proof. Transformation of the Cauchy traction vector field $t$ using Cauchy’s stress theorem 2.1 and Proposition 2.1 yields

$$t' = \theta \uparrow t = F_\theta \cdot (\sigma \cdot n^*) = (F_\theta \cdot \sigma \cdot F_\theta^T) \cdot (F_\theta^{-T} \cdot n^*) = (\theta \uparrow \sigma) \cdot (\theta \uparrow n^*) = \sigma' \cdot (n^*)'.$$

For a relative rigid motion $\theta = \theta^{iso}$ this becomes

$$t' = \theta^{iso} \uparrow t = Q \cdot t = Q \cdot (\sigma \cdot n^*) = (Q \cdot \sigma \cdot Q^T) \cdot (Q \cdot n^*) = (\theta^{iso} \uparrow \sigma) \cdot (\theta^{iso} \uparrow n^*) = \sigma' \cdot (n^*)',$$

by using the property $Q \cdot Q^T = Q^T \cdot Q = I$. Therefore, the Cauchy stress is spatially covariant. However, its material time derivative is not even objective because

$$(\sigma') = \overline{Q \cdot \sigma \cdot Q^T} = \dot{Q} \cdot \sigma \cdot Q^T + Q \cdot \dot{\sigma} \cdot Q^T + Q \cdot \sigma \cdot Q^T \neq Q \cdot \dot{\sigma} \cdot Q^T.$$

The Lie derivative (Definition A.35) is a geometric object that has an important property: if a tensor is spatially covariant, then its Lie derivative also is.

Theorem 3.1 (Spatial Covariance of Lie Derivative). Let $\varphi : \mathcal{B} \to \mathcal{S}$ be the motion of a material body $\mathcal{B}$ with spatial velocity $v$, and let $\theta : \mathcal{S} \to \mathcal{S}'$ be a relative motion such that $\varphi' = \theta \circ \varphi$ is the superposed motion of $\mathcal{B}$ with spatial velocity $v'$. Moreover, let $s \in \mathcal{T}_\theta^b(\mathcal{S})$ be a spatially covariant tensor field such that $s' = \theta \uparrow s$, then

$$L_v s' = \theta \uparrow (L_v s).$$

Proof. Let $\varphi'(X^I) = (x^i \circ \varphi)(X^I)$, $\theta^i(\varphi') = (x^i \circ \theta)(\varphi')$, and $(\varphi')^i(X^I) = (x^i \circ \varphi')(X^I)$ be the spatial coordinates $x^i$ on $\mathcal{S}$ arising from the localizations of $\varphi$, $\theta$, and $\varphi' = \theta \circ \varphi$, respectively. Then, by the chain rule and Definition A.35

$$(v')^i = \frac{\partial (\varphi')^i}{\partial t} \circ (\varphi')^{-1} = \frac{\partial (x^i \circ \theta \circ \varphi)}{\partial t} \circ (\varphi')^{-1}$$

$$= \frac{\partial \theta^i}{\partial t} \circ \theta^{-1} + \frac{\partial \theta^i}{\partial \varphi} \left( \frac{\partial \varphi^j}{\partial t} \circ \varphi^{-1} \right) \circ \theta^{-1} = w^i + \frac{\partial \theta^i}{\partial \varphi} v^i \circ \theta^{-1},$$

that is, $v' = w + \theta \uparrow v$, where $w$, with components $w^i$, represents the spatial velocity of $\theta$. The rest of the proof can be done as in [66] pp. 101–102], which is repeated here for completeness. By Proposition A.12

$$L_v s' = \frac{\partial s'}{\partial t} + L_{w'} + \theta \uparrow s' = \frac{\partial s'}{\partial t} + L_w s' + \theta \uparrow (L_v (\theta \uparrow s')).$$

In accordance with Proposition 2.1 the flow associated with $w$ is given by $\theta_t \circ \theta_s^{-1}$, for $s$, $t \in \mathbb{R}$. Definition A.35 and Proposition A.3 then yield

$$L_w s' = \frac{\partial s'}{\partial t} + L_{w'} + \theta \uparrow s' = \frac{d}{dt} (\theta_t \circ \theta_s^{-1}) \uparrow s' \bigg|_{t=s} + \theta \uparrow (L_v s)$$

$$= \frac{d}{dt} \theta_t \downarrow \theta_s^{-1} \downarrow (\theta_t \uparrow (\theta_s \downarrow s')) \bigg|_{t=s} + \theta \uparrow (L_v s) = \frac{d}{dt} \theta_t \uparrow (\theta_s \downarrow s') \bigg|_{t=s} + \theta \uparrow (L_v s)$$

$$= \frac{d}{dt} \theta_s \uparrow s \bigg|_{t=s} + \theta \uparrow (L_v s) = \theta \uparrow \left( \frac{d}{dt} s \bigg|_{t=s} + L_v s \right) = \theta \uparrow (L_v s).$$

Some well-known, spatially covariant, and thus objective stress rates can be directly obtained from the Lie derivative. Note that, with respect to spatial coordinates $x^i$, the components of the Lie derivative of the contravariant Kirchhoff stress $\tau^i \in \mathcal{T}_\theta^b(\mathcal{S})$ are given by

$$(L_v \tau)^{ij} = \dot{\tau}^{ij} - \nabla_k \tau^{kji} - \nabla_k \tau^{ijk} v^j,$$
where the general coordinate formula of Proposition [A.13] has been applied. From this one obtains the coordinate-invariant expression \( L_v(\tau^i) = \dot{\tau}^i - l \cdot \tau^i - \tau^i \cdot l^T \). A similar relation holds for the covariant Kirchhoff stress \( \tau^i \in T^2_0(S) \):
\[
L_v(\tau^b) = \dot{\tau}^b + T^b \cdot \tau^b + \tau^b \cdot l.
\]
Moreover, by recalling that the Kirchhoff stress is defined through \( \tau = J \sigma \), and that the spatial form of Proposition [2.2] is \( J = L_v J = J \text{div} v \), one has
\[
J^{-1} L_v (\tau^i) = L_v (\sigma^i) + \sigma^i \text{div} v = \sigma^i - l \cdot \sigma^i - \sigma^i \cdot l^T + \sigma^i \text{tr} d.
\]

**Definition 3.2.** The rates defined through
\[
\dot{\tau}^{\text{On}} \overset{\text{Def}}{=} L_v (\tau^i), \quad \dot{\tau}^{\text{Ol}} \overset{\text{Def}}{=} L_v (\tau^b), \quad \text{and} \quad \hat{\sigma}^{\text{T}} \overset{\text{Def}}{=} J^{-1} L_v (\tau^i)
\]
are called the upper Oldroyd rate and lower Oldroyd rate of Kirchhoff stress [84] [12], respectively, and the Truesdell rate of Cauchy stress [13] [17].

### 3.2 Corotational Rates

The tensor rates defined previously are members of so-called *objective non-corotational rates*. This category has remarkable properties (see, for example, Remark [3.3] and [72] [20]), but also some drawbacks, as will be discussed next.

**Proposition 3.5.** The Oldroyd rates and the Truesdell rate do not conform to Principle 2.3.

*Proof.* According to Principle 2.3, vanishing of each of the Oldroyd rates and the Truesdell rate of stress must keep the stress invariants stationary. Stationarity of any tensor invariant requires vanishing of the time derivative of the invariant. By Definition [A.20] in conjunction with Definition [A.17] and Remark [A.3] invariants could be formed with the metric tensor. Therefore, it suffices to show that the Oldroyd rates and the Truesdell rate do not vanish for a non-trivial metric. To give an example, consider the stress invariant \( \text{tr} \tau = \tau^i \cdot g^i \). Stationarity of this invariant requires
\[
\frac{d}{dt} (\text{tr} \tau) = L_v (\tau^i \cdot g^i) = L_v (\tau^i) \cdot g^i + \tau^i \cdot L_v (g^i) = 0,
\]
which, upon \( L_v (\tau^i) = \dot{\tau}^{\text{On}} = 0 \), requires \( L_v (g^i) = \dot{g}^{\text{Ol}} = 0 \). By Proposition 2.8, however, \( L_v (g^i) = 2 d^i \) in general. Similarly, the components of the upper Oldroyd rate of the inverse metric \( g^i \), by Definition [3.2] and Proposition [A.13] are
\[
(\dot{g}^{\text{On}})^{ij} = \dot{g}^{ij} = g^{ij} (\nabla_k v^j) - g^{ik} (\nabla_j v^i) = -(\nabla^j v^i + \nabla^i v^j) = -2 d^{ij},
\]
or, in basis-free notation, \( \dot{g}^{\text{On}} = L_v (g^i) = -2 d^i \). Therefore, the upper and lower Oldroyd rates of the metric do not vanish if the flow associated with the velocity \( v \) is not isometry satisfying \( d = 0 \). A similar result is obtained for the Truesdell rate, because these three rates do not commute with index raising and index lowering (Remark [A.11]). Consequently, those stress invariants formed with the metric tensor are not stationary when employing these objective stress rates.

The drawbacks associated with the Oldroyd and Truesdell rates, and other objective rates, are avoided by corotational rates, as discussed next.

**Definition 3.3.** Let \( s \) be a second-order spatial tensor field continuously differentiable in time and let \( A = -A^T \) be a spin tensor, then
\[
\dot{s} \overset{\text{Def}}{=} s - A \cdot s + s \cdot A
\]
is called the corotational rate of \( s \) defined by the spin \( A \).

**Definition 3.4.** Let \( A(x,t) = -A^T(x,t) \) be a given spin tensor for all \( x \in \varphi (B,t) \) and \( t \in [0, T] \), with \( \varphi (B,0) = B \). Consider the following evolution equation
\[
\frac{\partial R}{\partial t} = (A \circ \varphi) \cdot R, \quad \text{with} \quad R|_{t=0} = I,
\]
where \( R(X,t) : T_X B \rightarrow T_{\varphi(X,t)} S \) is a proper orthogonal two-point tensor for fixed \( X \in B \) and each \( t \in [0, T] \), such that \( R^T \cdot R = I_B, R \cdot R^T = I_S, \) and \( \text{det} R = +1 \). Solutions to the problem generate a *one-parameter group of rotations* to which \( R \) belongs, thus \( A \) is called the generator of that group [19] [102].
Remark 3.1. From the previous definition the term corotational can be justified as follows. In a rotating Euclidian frame with spin $A = R \cdot R^\top$ the Cauchy stress is given by $\sigma' = R \ddot{\gamma} \cdot \sigma = R^T \cdot \gamma \cdot R$. Then, the corotational rate $\dot{\sigma}$ represents the rate of change of $\sigma'$ observed in the fixed frame where $\sigma'$ is measured. Clearly,

$$\dot{\sigma} = R \cdot (\sigma') \cdot R^\top,$$

or equivalently,

$$\ddot{\gamma} \cdot \sigma = \frac{\partial (R \ddot{\gamma} \cdot \sigma)}{\partial t}.$$ 

Proposition 3.6. All corotational rates satisfy Prager’s requirement (Principle 2.3). 

Proof. A straightforward proof employing the chain rule and the isotropy property of scalar invariants is provided in [74].

There are infinitely many objective rates and corotational rates. Not every corotational rate is objective, and vice versa. Whether or not a corotational rate is objective depends on its defining spin tensor. This aspect is worth to be considered in more detail in what follows.

Definition 3.5. The Zaremba-Jaumann rate of Cauchy stress [51, 133] is obtained from Definition 3.3 by setting $A \overset{\text{def}}{=} \omega$, where $\omega = \frac{1}{2} (I - T^T) \in \mathfrak{so}(S)$ is the vorticity tensor according to Definition 2.19:

$$\dot{\sigma}_{\text{ZJ}} \overset{\text{def}}{=} \dot{\sigma} - \omega \cdot \dot{\sigma} + \dot{\sigma} \cdot \omega.$$ 

Proposition 3.7. Definition 3.5 identically applies for all associated tensor fields $s \in \mathfrak{so}(S)$, $s^2 = g^k \cdot s \in \mathfrak{so}^2(S)$, and $s^3 = g^4 \cdot s \in \mathfrak{so}^3(S)$ irrespective of index placement. That is, any corotational rate of the metric tensor vanishes.

Proof. We proof this, without loss of generality, for the Zaremba-Jaumann rate. Keeping the property $\nabla (v^\flat) = (\nabla v)^\flat$ in mind, then the components of the Zaremba-Jaumann rate of the inverse metric $g^i$ are

$$(g^{ZJ})^{ij} = \dot{g}^{ij} - \omega^i k g^{kj} - \omega^j k g^{ik} = -\omega^i k g^{kj} + g^k \omega_i^j = -\omega^i j + \omega^i j = 0.$$ 

Definition 3.6. Let $F = R \cdot U$ denote the right polar decomposition of the deformation gradient, with $R$ being proper orthogonal. Similar to the velocity gradient given by the relation $\frac{\partial}{\partial t} F = (l \circ \varphi) \cdot F$, let the spatial rate of relative rotation $\Omega$ be defined through

$$\frac{\partial R}{\partial t} \overset{\text{def}}{=} (\Omega \circ \varphi) \cdot R.$$ 

Choosing the spin $A \overset{\text{def}}{=} \Omega$ in Definition 3.5 then yields the Green-Naghdi rate of Cauchy stress [36]:

$$\dot{\sigma}_{\text{GN}} \overset{\text{def}}{=} \dot{\sigma} - \Omega \cdot \dot{\sigma} + \dot{\sigma} \cdot \Omega.$$ 

Proposition 3.8. Vorticity $\omega = \frac{1}{2} (I - T^T)$ associated with the Zaremba-Jaumann rate and spatial rate of rotation $\Omega = R \cdot R^T$ associated with the Green-Naghdi rate are related by

$$\omega = \Omega + \frac{1}{2} R \cdot (U \cdot U^{-1} - U^{-1} \cdot U) \cdot R^T.$$ 

Proof. By time differentiation of $F = R \cdot U$. 

Remark 3.2. The tensor $\Omega$ is a kind of angular velocity field describing the rate of rotation of the material, whereas $\omega$ describes the rate of rotation of the principal axes of the rate of deformation tensor $d = l - \omega$ [26]. In contrast to $\Omega$, vorticity contains terms due to stretching. Therefore, the Green-Naghdi rate (Definition 3.5) is identical to the material time derivative of the Cauchy stress in the absence of rigid body rotation, while the Zaremba-Jaumann rate (Definition 3.6) is generally not. The Green-Naghdi rate requires knowledge of total material motion resp. material deformation through $R = T \circ \varphi \cdot U^{-1}$, while the Zaremba-Jaumann rate, by virtue of vorticity, is derivable from the instantaneous motion at current time; in fact $l$ is the generator of $F$ through $\frac{\partial}{\partial t} F = l \cdot F$. This renders the Zaremba-Jaumann rate computationally inexpensive and particularly attractive to numerical methods that do not store any past material motion [12, 13, 5, 8].
By using Proposition 3.8 it can be shown that $\omega = \Omega$ resp. $\tilde{\sigma}^{ZJ} = \tilde{\sigma}^{GN}$ if and only if the motion of the material body is a rigid rotation, a pure stretch, or if the current configuration has been chosen as the reference configuration such that $F = R = I$, $U = I$, and $\dot{F} = \dot{R} + I \cdot \dot{U}$; see also [21], pp. 54–55 and 26, 27, 28. The last condition is used in Sect. 6 to compare different time integration algorithms for large deformations based on the Zaremba-Jaumann and Green-Naghdi rates.

**Proposition 3.9.** Let $\theta : S \rightarrow S'$ be a relative motion superposed to the motion $\varphi : B \rightarrow S$ of a material body, then both the Zaremba-Jaumann rate and the Green-Naghdi rate of Cauchy stress

(i) transform objectively if $\theta = \theta^{iso}$ is rigid, but they

(ii) are not spatially covariant.

**Proof.** (i) Recall that $\sigma' = \tilde{\theta}^{iso} \sharp \sigma = Q \cdot \sigma \cdot Q^T$,

$$Q \cdot \sigma \cdot Q^T = (\sigma') - Q \cdot \sigma' - Q \cdot \sigma' \cdot Q^T = (\sigma') - Q \cdot \sigma' + \sigma',$$

and $Q \cdot \omega = \tilde{\Omega} = \sigma' - \dot{Q} \cdot \sigma' + \dot{\sigma}'$, that is, $\tilde{\sigma}^{ZJ}$ is objective. Similarly, one has $\sigma' = \tilde{\omega} = \sigma' - \tilde{\Omega}$, showing that the Green-Naghdi rate $\tilde{\sigma}^{GN}$ is objective, too [26, 52].

(ii) Recall that if $\theta : S \rightarrow S'$ is an arbitrary relative motion with generally non-orthogonal tangent $F_{\theta}$, then $\tilde{\theta} = l_{\theta} + \theta \circ l$ by the proof of Proposition 3.3(ii), where $l_{\theta} = F_{\theta} \cdot F_{\theta}^{-1}$. Moreover,

$$\omega' = \frac{1}{\theta} (l_{\theta} - l_{\theta}^T + \theta \circ l - (\theta \circ l)^T) = \omega_{\theta} + \theta \circ l.$$

Since $\sigma' = \theta \circ \sigma$ by Proposition 3.4, it is easy to show that

$$(\sigma') = \theta_{\ast}(\sigma) + l_{\theta} \cdot \sigma' + \sigma' \cdot l_{\theta}^T,$$

for $\sigma \equiv \sigma^2$ being understood. Now proceed as in the proof of (i), clearly,

$$\theta \circ (\tilde{\sigma}^{ZJ}) = \theta \circ (\sigma - \omega \cdot \sigma + \sigma \cdot \omega) = \theta \circ (\sigma - \omega + \sigma \cdot \omega) = \sigma' \cdot (\theta \circ \omega)$$

$$= \sigma' - l_{\theta} \cdot \sigma' + \sigma' \cdot l_{\theta}^T - \omega \cdot \sigma' - \sigma' \cdot \omega$$

Then it follows immediately that $\theta \circ (\tilde{\sigma}^{ZJ}) = (\tilde{\sigma}^{ZJ})'$ if and only if $l_{\theta} \equiv \omega_{\theta}$, i.e. if $\theta$ is a rigid motion with $d_{\theta} \equiv 0$. To proof (ii) for the Green-Naghdi rate, note that $\tilde{\Omega}' = \tilde{\Omega}' = F_{\theta} \cdot R$, which leads to

$$\tilde{\sigma}^{GN} = \Omega' = (\tilde{R}) \cdot (\tilde{R})^{-1} = (\tilde{F}_{\theta} \cdot R + \tilde{F}_{\theta} \cdot R) \cdot (\tilde{F}_{\theta}^{-1} \cdot F_{\theta}^{-1} = l_{\theta} + \theta \circ \Omega.$$

$\Omega'$ is not skew unless $l_{\theta}$ is skew, that is, unless $F_{\theta}$ is a pure rotation. The rest of the proof is similar to that for $\tilde{\sigma}^{ZJ}$.

**Remark 3.3.** Proposition 3.3(ii) is remarkable because one would never have seen it in classical continuum mechanics in linear Euclidean space. Marsden and Hughes [66, box 6.1] draw a proof from the transformation property of the Lie derivative (Theorem 3.1).

**Proposition 3.10.** (See also [103, p. 222].) Let $\Theta = R \circ \sigma$ be the corotated Cauchy stress, and let $V \circ \eta$ denote the pullback by the left stretch tensor $V$ (cf. Remark 2.2), then the Green-Naghdi rate of Cauchy stress can be obtained from

$$\tilde{\sigma}^{GN} = \frac{\partial \Theta}{\partial t} = L_{(V \circ \eta)} \sigma.$$
expresses logarithmic strain. More generally, assume that \( \phi \) the change of a strain measure. Indeed, it is well known \([122]\) that, under uniaxial extension or compression, symmetric part of the velocity gradient, \( \cdots \) and it has not been known whether or not it is really a rate of

3.5

Remark

3.6

Remark

Remarks 3.4. Objective rates of a spatial second-order tensor fields have been claimed to be Lie derivatives or linear combinations thereof \([66, 103, 102]\). There are, however, objective rates which cannot be written this way \([32, 53]\). We also remark that, although the Green-Naghdi rate is related to some Lie derivative through Proposition 3.10, it is not spatially covariant. The restriction arises from the flow generated by the “stretched” spatial velocity field \( \mathbf{V} \uparrow \mathbf{v} \) being employed.

Remark 3.5. The spatial rate of deformation or stretching \( \mathbf{d} \) is a fundamental kinematic quantity. In quoting Xiao et al. \([127]\), however, it should be noticed that “[...] by now the stretching has been known simply as a symmetric part of the velocity gradient, [...] and it has not been known whether or not it is really a rate of the change of a strain measure.” Indeed, it is well known \([122]\) that, under uniaxial extension or compression,

\[
e = e_0 + \int_0^t \mathbf{d}(\tau) \, d\tau
\]

expresses logarithmic strain. More generally, assume that \( \varphi : \mathbf{B} \to \mathbf{S} \) is a kind of motion of a three-dimensional body \( \mathbf{B} \) in \( \mathbf{S} = \mathbb{R}^3 \) for which the principal axes of stretch are fixed, and let \( \Gamma \), in a Cartesian coordinate system, be the diagonal matrix containing the principal stretches \( \lambda_1, \lambda_2, \lambda_3 \) (the eigenvalues of \( \mathbf{V} \)). Then, according to \([26]\), the rate of deformation is the diagonal matrix

\[
\mathbf{d} = \dot{\mathbf{I}} \Gamma^{-1},
\]

with components \( d_{11} = \dot{\lambda}_1 / \lambda_1 = (\ln \mathbf{V})_{11} = \dot{\varepsilon}_{11}, d_{22} = \dot{\varepsilon}_{22}, \text{ and } d_{33} = \dot{\varepsilon}_{33} \), and \( \mathbf{e} = \ln \mathbf{V} \) being the spatial logarithmic strain or Hencky strain (Definition 2.16). Certain corotational and objective rates of \( \mathbf{e} \) can equal the rate of deformation for certain particular left stretch tensors \( \mathbf{V} \) \([10, 11]\). In their seminal paper Xiao et al. \([127]\) finally prove that for all \( \mathbf{V} \) there is a unique corotational rate, called the *logarithmic rate*, of the Hencky strain \( \mathbf{e} \) which is identical to the rate of deformation:

\[
\mathbf{d} = (\ln \mathbf{V})^{\log} = \dot{\mathbf{e}}^{\log} \overset{\text{def}}{=} \dot{\mathbf{e}} - \mathbf{\Omega}^{\log} \cdot \mathbf{e} + \mathbf{e} \cdot \mathbf{\Omega}^{\log}.
\]

The calculation of the so-called logarithmic spin \( \mathbf{\Omega}^{\log} \), however, is complicated for general cases \([127]\). Another important identity is that the upper Oldroyd rate of the so-called Finger strain \( \mathbf{a} \overset{\text{def}}{=} \frac{1}{2} (\mathbf{b} - \mathbf{I}) \), where \( \mathbf{b} = \mathbf{F} \cdot \mathbf{F}^T \) is the left Cauchy-Green tensor, equals spatial rate of deformation \([20, 42]\):

\[
\dot{\mathbf{a}}^{\log} = L_v(\mathbf{a}^T) = \dot{\mathbf{a}} - \mathbf{I} \cdot \mathbf{a}^T - \mathbf{a} \cdot \mathbf{I}^T = \mathbf{d}^{\log}.
\]

Therefore, \( \mathbf{d} \) is indeed an honest strain rate.

Remark 3.6. The corotational rate of an objective tensor is not necessarily objective. Objective corotational rates are defined by spin tensors \( \mathbf{A} \) that fulfill certain kinematical requirements. Examples of such spin tensors are \( \mathbf{A} = \mathbf{\omega} \) and \( \mathbf{A} = \mathbf{\Omega} \), resulting in the Zaremba-Jaumann rate and the Green-Naghdi rate, respectively. In deriving a general form of spin tensors that define objective corotational rates, the relationship between \( \mathbf{\omega} \) and \( \mathbf{\Omega} \) is of particular importance. Proposition 3.5 provides an expression of this relationship using the \( \mathbf{R} \)-pushforward of a Lagrangian tensor defined by the right stretch tensor and its rate. Other expressions
employ the left Cauchy-Green tensor $b$ and rate of deformation $d$. With respect to the principal axes of the left Cauchy-Green tensor, which coincide with those of the left stretch tensor (see Definitions [2.15 and 2.17], Hill [44, eq. (1.45)], see also [70, eq. (8.14)], has obtained the component expression

$$\omega_{\alpha\beta} - \Omega_{\alpha\beta} = \frac{\lambda_\alpha - \lambda_\beta}{\lambda_\alpha + \lambda_\beta} d_{\alpha\beta}$$

(no summation over repeated indices),

with $\lambda_\alpha$ being the eigenvalues of the left stretch tensor $V$ and $\alpha, \beta \in \{1, \ldots, m\}$. The symbolic form of this expression in $m$-dimensional space reads (cf. [85, eq. (62)] and [128, eq. (44)])

$$\Omega = \omega + \sum_{\alpha \neq \beta} \frac{\lambda_\alpha - \lambda_\beta}{\lambda_\alpha + \lambda_\beta} b_\alpha \cdot d \cdot b_\beta,$$

where $b_\alpha \equiv \prod_{\beta = 1, \beta \neq \alpha}^m \frac{b - \lambda_\beta^2 i}{\lambda_\alpha^2 + \lambda_\beta^2}$ are the *eigenprojections* of $b = \sum_1^m \lambda_\alpha^2 b_\alpha$ [71, sect. 4.5], and $\lambda_\alpha^2$ are the eigenvalues of $b$. Xiao et al. [128] show that this expression for $A = \Omega$ is a particular example of the general form

$$A = \omega + \Upsilon(b, d)$$

of a spin tensor defining an objective corotational rate, where $\Upsilon(b, d)$ is an anti-symmetric tensor-valued isotropic function. Note that the above relationship reveals the basic role of $\omega$. Moreover, if $\tilde{\sigma}^*$ is the objective corotational rate defined by the spin $A$, then [128, theorem 1]

$$\tilde{\sigma}^* = \tilde{\sigma}^{ZJ} + \sigma \cdot \Upsilon(b, d) - \Upsilon(b, d) \cdot \sigma.$$

Finally, we note that Meng and Chen [71] recently derived basis-free relations between the vorticity tensor $\omega$ and particular spin tensors, including the rate of relative rotation $\Omega$, taking the form $\Omega = \omega + z(V) : d$, where $z(V)$ is an isotropic fourth-order tensor-valued function of the left stretch tensor.

We shall conclude our survey of objective rates with the following observation, emphasizing the prominent role of the Zaremba-Jaumann rate of Cauchy stress.

**Proposition 3.11.** Let the motion $\varphi_t : B \to S$ be an orientation-preserving isometry (rigid rotation) for all $t \in [0, T]$, then (i) the discussed objective non-corotational rates ($\tilde{\tau}^{\text{Ou}}, \tilde{\tau}^{\text{Ol}}, \tilde{\tau}^{\text{Tr}}$) and (ii) the objective corotational rate $\tilde{\sigma}^{\text{GN}}$ coincide with the Zaremba-Jaumann rate of Cauchy stress, $\tilde{\sigma}^{ZJ}$.

**Proof.** We provide a full proof here, although some facts have been already shown previously.

Note that any orientation-preserving isometric motion requires $\det T \varphi = \det F = J = +1$ as well as $F^{-1} = F^T$, for all $t \in [0, T]$, by Definition [2.19] and Proposition [2.6]. By this, Kirchhoff stress and Cauchy stress coincide, i.e., $\tau = \sigma$. Moreover, since the deformation gradient $F$ of such motions is proper orthogonal, i.e., $F^{-1} = F^T$ for all $t$, the velocity gradient $l = \dot{F} \cdot F^{-1}$ is skew-symmetric:

$$0 = \dot{F} \cdot F^T = \dot{F} \cdot F^T + F \cdot \dot{F}^T = l + l^T \quad \Leftrightarrow \quad l = -l^T.$$

Therefore, the velocity gradient represents a spin tensor through Proposition [3.2] and its symmetric part $d \equiv 0$ by Definition [2.19]. As a consequence, $l \equiv \omega$, and $L_v g = 0$ by Proposition [2.8] which proofs (i).

Assertion (ii) is readily proven through Proposition [3.8], or by using the first or second equation in the previous Remark [3.8] and noting that $d \equiv 0$. More generally, assertion (ii) holds for all objective corotational rates defined by any spin $A = \omega + \Upsilon(b, d)$, as in Remark [3.8] provided that $\Upsilon(b, 0) = 0$, which is true for all commonly-used spin tensors; cf. [128, sect. 3]. □

## 4 Rate Constitutive Equations

There are basically two main groups of rate-independent constitutive equations (or material models) that are used in computational solid mechanical applications at large deformation. The elements of the first group are typically based on thermodynamical principles postulated at the outset, and they are commonly addressed with the prefix “hyper”: hyperelasticity, hyperelasto-plasticity, and hyperplasticity. The constitutive equations belonging to the second group usually ignore balance of energy and the axiom of entropy production. Many of them are are based on an *ad hoc* extension of existing small-strain constitutive equations to the finite
deformation range. Elements of the second group are called Eulerian or spatial rate constitutive equations and are commonly addressed with the prefix “hypo”: hypoelasticity, hypoplasticity, and hypoplasticity.

The following section gives a general introduction to spatial rate constitutive equations belonging to the second group. In spite of their shortcomings discussed, for example, in [105, 102], we point out that these material models remain widely used in computational continuum mechanics. This is because the same integration algorithms can be employed at both infinitesimal and finite deformations, as will be shown in Sect. 5. Many, if not the majority of finite element codes in solid mechanics employ rate constitutive equations for problems involving small or large inelastic deformations.

In this section we address only rate constitutive equations accounting for finite deformations. Readers who are not familiar with elasticity and classical elasto-plasticity at small strains should consult introductory texts on plasticity theory [21, 102]. We remark, however, that the general formulas presented here carry over to the case of infinitesimal deformations if the objective stress rate and rate of deformation are replaced with the common material time derivatives of stress and infinitesimal strain, respectively:

\[ \dot{\sigma} \Rightarrow \dot{\sigma} \quad \text{and} \quad d \Rightarrow \dot{\varepsilon}_{\text{lin}}. \]

The term material model or just model will be used as a synonym for constitutive equation. Without indicating it further, stress measures are taken with all indices raised, and strain measures with all indices lowered, e.g. \( \sigma_{\text{def}}^{\flat} = \sigma^i_{\flat} \) and \( d_{\text{def}}^{\flat} = d^{\flat}. \) The dependence of a function on a point map, for example, on the motion \( \varphi, \) will be usually clear from the context. Moreover, we do not indicate time-dependence of a function explicitly, hence the argument or index \( t \) will be suppressed.

**Definition 4.1.** It proves convenient to define the following measures of stress and rate of deformation.

(i) The negative mean Cauchy stress and the Cauchy stress deviator

\[ p \overset{\text{def}}{=} -\frac{1}{3} \text{tr} \sigma \quad \text{and} \quad \sigma_{\text{dev}} \overset{\text{def}}{=} \sigma + pg^*, \]

respectively.

(ii) The von Mises stress or equivalent shear stress

\[ q \overset{\text{def}}{=} \sqrt{3J_2} = \sqrt{\frac{2}{3} \| \sigma_{\text{dev}} \|}, \]

in which

\[ J_2 \overset{\text{def}}{=} \frac{1}{2} \text{tr}(\sigma_{\text{dev}}^2) = \frac{1}{2} (I_1(\sigma))^2 - I_2(\sigma) \]

\[ = \frac{1}{2} \sum_{i,j} s_{ij} s_{ij} = \frac{1}{2} (s_{11}^2 + s_{22}^2 + s_{33}^2 + 2s_{12}^2 + 2s_{23}^2 + 2s_{13}^2) = \frac{1}{2} (s_1^2 + s_2^2 + s_3^2) \]

\[ = \frac{1}{6} ((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2) \]

is the negative second principal invariant of the Cauchy stress deviator. Here \( I_1(\sigma) \) and \( I_2(\sigma) \) denote the first and second the principal invariants of the Cauchy stress, respectively. Moreover, \( s_{ij}, \) with \( i,j \in \{1, \ldots, 3\} \) are the components of \( \sigma_{\text{dev}} \) and \( \sigma_1, \sigma_2, \sigma_3 \) the principal stresses in three-dimensional Euclidian space.

(iii) The equivalent shear strain rate and the volumetric strain rate

\[ d_{\text{iso}} \overset{\text{def}}{=} \sqrt{\frac{2}{3} \text{tr}(d_{\text{dev}}^2)} \quad \text{and} \quad d_{\text{vol}} \overset{\text{def}}{=} \text{tr} d = q_{k_k}, \]

respectively, with \( d_{\text{dev}} \overset{\text{def}}{=} d - \frac{1}{3} (\text{tr} d) g. \)

### 4.1 Hypoelasticity

The use of spatial rate constitutive equations to characterize the mechanical behavior of materials is very attractive, especially from a numerical viewpoint. In addition, there is only a limited number of materials, e.g. rubber, whose elastic response resp. stress state can be derived as a whole, either from a finite strain measure (say \( C \)), or a free energy function. Truesdell [118] points out:
While the last few years have brought physical confirmation to the [hyperelastic; note from the author] finite strain theory for rubber, there remain many physical materials which are linearly elastic under small enough strain but which in large strain behave in a fashion the finite strain theory is not intended to represent.

This observation led to the development of hypoelastic rate constitutive equations [117, 118, 121].

**Definition 4.2.** The general *hypoelastic constitutive equation* is defined through

$$\dot{\sigma}^* \overset{\text{def}}{=} h(\sigma, g, d) = a(\sigma, g) : d \quad \text{(linearity in } d),$$

where $\dot{\sigma}^*$ can be any objective rate of Cauchy stress, and $a(\sigma, g)$ is a spatial fourth-order tensor-valued function. To achieve the equivalence $h(\sigma, g, d) = a(\sigma, g) : d$, the function $h$ is required to be continuously differentiable in a neighborhood of $d = 0$, so that $h$ is linear in $d$; note that $a(\sigma, g) = Dh(\sigma, g, 0)$, and that $h(\sigma, g, 0) = 0$, i.e. zero rate of deformation produces zero objective stress rate. If only rate-independent response should be modeled, then $h$ must be *positively homogeneous* of first degree in $d$, i.e. $h(\sigma, g, ad) = a h(\sigma, g, d)$ for all $a > 0$.

**Definition 4.3.** A material is *hypoelastic of grade* $n$, if $a(\sigma, g)$ is a polynomial of degree $n$ in the components of $\sigma$ [117, 121]. For $n = 0$, representing *hyperelasticity of grade zero*, the tensor $a(g)$ is independent of $\sigma$. The simplest *ad hoc* choice compatible with this idea is the constant isotropic elasticity tensor

$$a^{ijkl} = K g^{ij} g^{kl} + 2G (g^{ik} g^{jl} + g^{il} g^{jk} - \frac{1}{3} g^{ij} g^{kl}).$$

Here $g^{ij}$ are the components of the inverse metric, $K = \lambda + \frac{2}{3} \mu$ is the *bulk modulus* or *modulus of compression*, $G = \mu$ is the *shear modulus*, and $\lambda, \mu$ are the Lamé constants. The considered grade-zero hypoelastic rate constitutive equation takes the equivalent forms

$$\dot{\sigma}^* \overset{\text{def}}{=} K (\text{tr } d) g^2 + 2G d_{\text{dev}}^4 \quad \text{resp.} \quad (\dot{\sigma}^*)^{ij} \overset{\text{def}}{=} K d_k^{ij} g^{ij} + 2G d_{\text{dev}}^{ij}$$

and

$$\dot{\sigma}^* \overset{\text{def}}{=} \lambda (\text{tr } d) g^2 + 2\mu d^2.$$

**Remark 4.1.** Within the hypoelasticity framework the stress is not necessarily path-independent such that hypoelastic constitutive equations generally produce non-zero dissipation in a closed cycle [102]. Therefore, a hypoelastic model is not necessarily integrable towards an elastic model [119, 2]. Bernstein [17] proposed conditions to proof if a certain hypoelastic model represents an elastic or even hyperelastic material, i.e. elastic in the sense of Cauchy and Green, respectively. If a certain hypoelastic model is elastic, additional conditions must hold so that the model represents a hyperelastic material. Simo and Pister [105] show that any grade-zero hypoelastic constitutive equation with constant isotropic tensor according to Definition 4.3 cannot represent an elastic material. Instead, the components of $a$ must be nontrivial functions of the Jacobian $J$ of the motion, and must also reduce to the linear elastic case for $J = 1$ [105]. For some applications, it is indeed desirable that the hypoelastic model for large deformations be the rate form of some constitutive equation for hyperelasticity. In this context, the particular case of Hooke-like isotropic hyperelastic material has been considered by Korobeynikov [59].

**Remark 4.2.** Xiao et al. [127, 126] proved that the grade-zero hypoelastic constitutive equation $\dot{\sigma}^* \overset{\text{def}}{=} K (\text{tr } d) g^4 + 2G d_{\text{dev}}^4$, is exactly integrable to define an isotropic elastic constitutive equation in the sense of Cauchy if and only if the stress rate $\dot{\sigma}^*$ on the left hand side is the so-called *logarithmic stress rate*

$$\dot{\sigma}^\log \overset{\text{def}}{=} \dot{\sigma} - \Omega^\log \cdot \sigma + \sigma \cdot \Omega^\log.$$

The resulting finite strain constitutive equation is (see also Remark 3.5)

$$\sigma = K (\text{tr } e) g^4 + 2G e_{\text{dev}}^4,$$

where $e \overset{\text{def}}{=} \ln V$ is the spatial logarithmic strain. Furthermore, Xiao et al. [129] show that if $\sigma$ is replaced with the Kirchhoff stress $\tau = J \sigma$, then the integrable-exactly hypoelastic constitutive rate equation $\dot{\tau}^\log \overset{\text{def}}{=} \lambda (\text{tr } d) g^4 + 2\mu d^4$ defines the isotropic hyperelastic (i.e. Green-elastic) relation

$$\tau = \lambda (\ln(\det V)) g^4 + 2\mu e^4.$$
In order to circumvent the integrability issue of hypoelasticity, Neff et al. [22] recently considered only those fourth-order tangent stiffness tensors \( a(\sigma, g) \) that are induced by a given invertible Cauchy-elastic constitutive model. They showed that for this elastic model there is a relation between the Zaremba-Jaumann rate (as well as for the Green-Naghdi rate and logarithmic rate) of Cauchy stress and a constitutive requirement involving logarithmic strain [2], yielding a representation for the induced stiffness tensors \( a(\sigma, g) \). This could be achieved by making use of a novel corotational stability postulate which, in our notation, reads

\[
\dot{\sigma}^{ZJ} : d > 0, \quad \text{for all } d \neq 0.
\]

**Remark 4.3.** The stress rate \( \dot{\sigma}^{*} \) in the general hypoelastic constitutive equation (Definition 4.2),

\[
\dot{\sigma}^{*} \overset{\text{def}}{=} a^{*} : d
\]

could be any objective stress rate of Cauchy stress; the constitutive equation could also be stated in terms of Kirchhoff stress \( \tau \). In quoting Truesdell and Noll [21, p. 404], we note that “[…] any advantage claimed for one such rate over another is pure illusion.” Indeed, any objective stress rate could be chosen provided that the right hand side of the constitutive equation is properly adjusted. Then, for different choices of objective rate the tangent moduli \( a^{*} \) will differ for the same material, as indicated by the superscript. If, for example, \( \dot{\sigma}^{ZJ} \overset{\text{def}}{=} a^{ZJ} : d \) represents the grade-zero hypoelastic constitutive equation of Definition 4.3 in terms of the Zaremba-Jaumann rate, then the constant isotropic elasticity tensor \( a^{ZJ} \) possesses major symmetries. If this constitutive equation is stated in terms of a different stress rate, say, Truesdell rate of Cauchy stress, then by Definitions 5.2 and 5.3

\[
\sigma^{\text{Tr}} = \dot{\sigma} - l \cdot \sigma - \sigma \cdot I^T + \sigma \cdot \text{tr} d = \dot{\sigma} - (d + \omega) \cdot \sigma - \sigma \cdot (d + \omega^T) + \sigma \cdot \text{tr} d
\]

\[
= \dot{\sigma}^{ZJ} - (d \cdot \sigma + \sigma \cdot d) + \sigma \cdot \text{tr} d = (a^{ZJ} - a + \sigma \otimes g^\ast) : d
\]

\[
\overset{\text{def}}{=} a^{\text{Tr}} : d,
\]

where \( a^\ast \overset{\text{def}}{=} d \cdot \sigma + \sigma \cdot d \) is also symmetric, but \( \sigma \otimes g^\ast \), and hence \( a^{\text{Tr}} \), are not. Therefore, changes in stress rate require consistent adjustment of the material tangent tensor to represent the same material behavior, but symmetry properties might get lost. These observations are of particular importance in finite element implementations of constitutive equations. For comprehensive discussions and more relations between material tangent tensors in terms of different stress rates we refer to [13, sect. 5.4 and box 5.1] and [85].

**Remark 4.4.** Although Truesdell and Noll [121, p. 405] point out that hypoelasticity of grade zero “[…] is not invariant under change of invariant stress rate” several article are concerned with the following question [22, 134]: which objective stress rate should be applied to hypoelasticity of grade zero with constant isotropic elasticity tensor according to Definition 4.3? That question arises after Dienes [26] and others show that for hypoelasticity of grade zero the choice of the Zaremba-Jaumann rate of the Cauchy stress would lead to oscillating stress response in simple shear, which is indeed unacceptable (cf. Sect. 7). Nowadays, researchers agree that the question as posed is meaningless because the claim for a constant isotropic elasticity tensor under large deformations is yet unacceptable [105]. However, for arbitrary rate constitutive equations the question remains: how to choose the stress rate? According to Atluri [3] and Nemat-Nasser [77], it is not the Zaremba-Jaumann rate that generates the spurious stresses, but the constitutive rate equation relating the Zaremba-Jaumann rate of the response functions to their dependent variables. In particular, Atluri [3, p. 145] points out that

[…] all stress-rates are essentially equivalent when the constitutive equation is properly posed [i.e. if the terms by which the rates differ are incorporated into the constitutive equation; note from the author].

Maybe this and other issues associated with rate constitutive equations could be considered obsolete if stated properly within the context of recent developments, e.g., [95, 53].

**4.2 Hypoelasto-Plasticity**

Elasto-plastic constitutive equations in finite element codes for large deformation solid mechanical applications are mostly based on an \( \text{ad hoc} \) extension of classical small-strain elasto-plasticity to the finite deformation range [102]. The presumed “elastic” part is described by a hypoelastic model, hence the term hypoelasto-plasticity has been coined for that class of constitutive equations. In classical plasticity theory, plastic flow is understood as an irreversible process characterized in terms of the past material history.
Definition 4.4. The past material history up to current time \( t \in \mathbb{R} \) is defined as a map
\[
[-\infty, t] \ni \tau \mapsto \{\sigma(x, \tau), \alpha(x, \tau)\},
\]
where \( \alpha(x, \tau) \overset{\text{def}}{=} \{\alpha_1(x, \tau), \ldots, \alpha_k(x, \tau)\} \) is a set of (possibly tensor-valued) internal state variables, often referred to as the hardening parameters or plastic variables.

Definition 4.5. Let \( \mathcal{T}_2^{\text{sym}} \) be the set of symmetric (\( \mathbb{R}^2 \))-tensor fields, and let the internal plastic variables \( \alpha \overset{\text{def}}{=} \{\alpha_1, \ldots, \alpha_k\} \) belong to the set identified through \( \mathcal{H} \overset{\text{def}}{=} \{\alpha \mid \alpha \in \mathcal{H}\} \). A state of an elasto-plastic material is the pair \((\sigma, \alpha) \in \mathcal{T}_2^{\text{sym}} \times \mathcal{H}\). The ad hoc extension of classical small-strain elasto-plasticity to the finite deformation range then consists of the following elements \([102]\):

(i) Additive decomposition. The spatial rate of deformation tensor is additively decomposed into elastic and plastic parts:
\[
d = d^e + d^p, \quad \text{in components,} \quad d_{ij} \overset{\text{def}}{=} d_{ij}^e + d_{ij}^p.
\]

(ii) Stress response. A hypoelastic rate constitutive equation of the form
\[
\dot{\sigma}^* = a(\sigma, g) : (d - d^e)
\]
characterizes the “elastic” response, where \( \dot{\sigma}^* \) represents any objective stress rate.

(iii) Elastic domain and yield condition. A differentiable function \( f : \mathcal{T}_2^{\text{sym}} \times \mathcal{H} \to \mathbb{R} \) is called the yield condition, and
\[
\mathcal{A}_\sigma \overset{\text{def}}{=} \{(\sigma, \alpha) \in \mathcal{T}_2^{\text{sym}} \times \mathcal{H} \mid f(\sigma, g, \alpha) \leq 0\}
\]
is the set of admissible states in stress space; the explicit dependency on the metric \( g \) is necessary in order to define invariants of \( \sigma \) and \( \alpha \). An admissible state \((\sigma, \alpha) \in \mathcal{A}_\sigma \) satisfying \( f(\sigma, g, \alpha) < 0 \) is said to belong to the elastic domain or to be an elastic state, and for \( f(\sigma, g, \alpha) = 0 \) the state is an elasto-plastic state lying on the yield surface. States with \( f > 0 \) are not admissible.

(iv) Flow rule and hardening law. The evolution equations for \( d^p \) and \( \alpha \) are called the flow rule and hardening law, respectively:
\[
d^p \overset{\text{def}}{=} \lambda m(\sigma, g, \alpha) \quad \text{and} \quad \dot{\alpha}^* \overset{\text{def}}{=} -\lambda h(\sigma, g, \alpha).
\]
Here \( m \) and \( h \) are prescribed functions, and \( \lambda \geq 0 \) is called the consistency parameter or plastic multiplier. The flow rule is called associated if \( m = D_\sigma f \), and non-associated if \( m \) is obtained from a plastic potential \( g \neq f \) as \( m = D_\sigma g \). Within the isotropic hardening laws \( \alpha \) usually represents the current radius of the yield surface, whereas \( \alpha \) represents the center of the yield surface (back stress) in kinematic hardening laws.

(v) Loading/unloading and consistency conditions. It is assumed that \( \lambda \geq 0 \) satisfies the loading/unloading conditions
\[
\lambda \geq 0, \quad f(\sigma, g, \alpha) \leq 0, \quad \text{and} \quad \lambda f(\sigma, g, \alpha) = 0,
\]
as well as the consistency condition
\[
\lambda \dot{f}(\sigma, g, \alpha) = 0.
\]

Example 4.1 (Von Mises Plasticity). Consider a well-known hypoelasto-plastic rate constitutive equation which is commonly referred to as \( J_2 \)-plasticity with isotropic hardening or von Mises plasticity in computational solid mechanics \([14, 49, 102]\). This model is applicable to metals and other materials because it includes the von Mises yield condition
\[
f(\sigma, g, \sigma^\nu) \overset{\text{def}}{=} q(\sigma, g) - \sigma^\nu,
\]
where \( q \) is the von Mises stress (Definition \([43]\), and \( \sigma^\nu \) is the current yield stress given by the linear hardening rule
\[
\sigma^\nu(\epsilon^p) \overset{\text{def}}{=} \sigma^{\nu_0} + E^p \epsilon^p.
\]
The initial yield stress \( \sigma^{\nu_0} \) and the plastic modulus \( E^p \) are material constants in addition to the elastic constants \( E \) and \( \nu \) (or \( K \) and \( G \)), and the equivalent plastic strain \( \epsilon^p \) is understood as a function of the plastic rate of deformation tensor \( d^p \). Including the linear hardening rule produces bilinear elasto-plastic response with isotropic hardening mechanism. Bilinear in this context means that a one-dimensional bar in
simple tension behaves elastic with Young’s modulus \( E \) until reaching the initial yield stress. Then plastic flow occurs and the material hardens according to the linear hardening rule. The *elasto-plastic tangent modulus* is given by the constant
\[
E^t \overset{\text{def}}{=} \frac{E E^p}{E + E^p}.
\]
Let the hypoelastic response be characterized by
\[
\dot{\sigma}^{Z1} = a(g) : (d - d^p),
\]
where \( a(g) \) is the constant isotropic elasticity tensor (Definition 4.3). Plastic flow is assumed to be associated, that is,
\[
d^p \overset{\text{def}}{=} d^p_{\text{dev}} = \lambda \frac{\partial f}{\partial \sigma} = \lambda \frac{3}{2q} \sigma^{\text{dev}} = \lambda \sqrt{\frac{3}{2}} n^\circ,
\]
where \( n \overset{\text{def}}{=} \sigma_{\text{dev}}/\|\sigma_{\text{dev}}\| \), with \( \text{tr} n = 0 \). Therefore, plastic straining is purely deviatoric, and the *hardening law*, representing the evolution of the radius of the von Mises yield surface, is given by
\[
\dot{\sigma}^Y = E^p \dot{\varepsilon}^p = E^p \sqrt{\frac{2}{3}} \text{tr}((d^p)^2) = \lambda E^p \quad \text{resp.} \quad \dot{\varepsilon}^p = \lambda.
\]
After substitution into the consistency condition during plastic loading, the plastic multiplier is obtained as
\[
\lambda = \frac{2G}{3G + E^p} \sqrt{\frac{3}{2}} n : d_{\text{dev}},
\]
which completes the model. Some algebraic manipulation finally results in the hypoelasto-plastic spatial rate constitutive equation
\[
\dot{\sigma}^{Z1} = \mathbf{a}^{\text{sp}}(\sigma, g, \varepsilon^p) : d,
\]
in which the *elasto-plastic material tangent tensor* is given by
\[
\mathbf{a}^{\text{sp}}(\sigma, g, \varepsilon^p) \overset{\text{def}}{=} \frac{6G^2}{3G + E^p} n \otimes n
\]
at plastic loading, and by \( \mathbf{a}^{\text{sp}}(\sigma, g, \varepsilon^p) = a(g) \) at elastic loading and unloading, and neutral loading, respectively. The distinction of these types of loading is done with the aid of the yield condition and hardening rule, that is, the dependency of the function \( \mathbf{a}^{\text{sp}} \) on \( \varepsilon^p \) is implicit.

### 4.3 Hypoplasticity

The notion of hypoplasticity, which is entirely different from that of hypoelasto-plasticity, has been introduced by Kolymbas [54], but the ideas behind are much older. Starting in the 1970’s [38, 54], the development of hypoplastic rate constitutive equations has a clear focus on granular materials and applications in soil mechanics [12, 30, 37, 55, 78, 124].

**Definition 4.6.** The general *hypoplastic constitutive equation* for isotropic materials takes the form
\[
\dot{\sigma}^\star = h(\sigma, g, \alpha, d),
\]
where \( \dot{\sigma}^\star \) represents any objective stress rate and \( \alpha(x, t) \overset{\text{def}}{=} \{ \alpha_1(x, t), \ldots, \alpha_k(x, t) \} \) is a set of (possibly tensor-valued) internal state variables.

Hypoplasticity can be understood as a generalization of hypoelasticity. In contrast to hypoelasticity, the hypoplastic response function \( h \) is generally *nonlinear* in \( d \) in order to describe dissipative behavior. Hypoplastic constitutive modeling basically means to fit the almost arbitrary tensor-valued response function \( h \) to experimental data. That makes it to an *deductive* design approach, whereas elasto-plastic constitutive modeling is *inductive*. A basic requirement is that the desired function be as simple as possible. In the simplest hypoplastic model the objective stress rate \( \dot{\sigma}^\star \) is regarded a nonlinear function of \( g \) and \( d \) only.
Example 4.2. Consider the case where $\mathbf{\sigma}^* \overset{\text{def}}{=} \mathbf{h}(\mathbf{\sigma}, \mathbf{g}, \mathbf{d})$. If rate-independent material should be described, then $\mathbf{h}$ is required to be positively homogeneous of first degree in $\mathbf{d}$, so that for every $a > 0$, $\mathbf{h}(\mathbf{\sigma}, \mathbf{g}, a\mathbf{d}) = a\mathbf{h}(\mathbf{\sigma}, \mathbf{g}, \mathbf{d})$. In this case, however, 
\[
\mathbf{h}(\mathbf{\sigma}, \mathbf{g}, \mathbf{d}) = \frac{\partial \mathbf{h}(\mathbf{\sigma}, \mathbf{g}, \mathbf{d})}{\partial \mathbf{d}} : \mathbf{d} = \mathbf{m}(\mathbf{\sigma}, \mathbf{g}, \mathbf{d}) : \mathbf{d}
\]
by Euler’s theorem on homogeneous functions, where $\mathbf{m}$ is a spatial fourth-order tensor-valued function that explicitly depends on $\mathbf{d}$; a proof can be done by differentiating both sides of $f(a\mathbf{d}) = a f(\mathbf{d})$ with respect to $a$ and then applying the chain rule. The values of $\mathbf{m}$ are referred to as material tangent tensors, as for other classes of rate constitutive equations. Due to constitutive frame invariance (Principle \[2.2\]) the response function $\mathbf{h}$ must be isotropic in all variables. It then follows that $\mathbf{\sigma}^* \overset{\text{def}}{=} \mathbf{h}(\mathbf{\sigma}, \mathbf{g}, \mathbf{d})$ has the representation (see [121, eq. (13.7)] and [56, eq. (15)])
\[
\mathbf{\sigma}^* = \psi_0 \mathbf{g}^2 + \psi_1 \mathbf{\sigma} + \psi_2 \mathbf{d} + \psi_3 \mathbf{\sigma}^2 + \psi_4 \mathbf{d}^2 + \psi_5 (\mathbf{\sigma} \cdot \mathbf{d} + \mathbf{d} \cdot \mathbf{\sigma}) \\
+ \psi_6 (\mathbf{\sigma}^2 \cdot \mathbf{d} + \mathbf{d} \cdot \mathbf{\sigma}^2) + \psi_7 (\mathbf{\sigma} \cdot \mathbf{d}^2 + \mathbf{d}^2 \cdot \mathbf{\sigma}) + \psi_8 (\mathbf{\sigma}^2 \cdot \mathbf{d}^2 + \mathbf{d}^2 \cdot \mathbf{\sigma}^2),
\]
where $\psi_0, \ldots, \psi_8$ are polynomials of the ten basic invariants $\text{tr} \mathbf{\sigma}, \text{tr}(\mathbf{\sigma}^2), \text{tr}(\mathbf{\sigma}^3), \text{tr} \mathbf{d}, \text{tr}(\mathbf{d}^2), \text{tr}(\mathbf{d}^3), \text{tr}(\mathbf{\sigma} \cdot \mathbf{d}), \text{tr}(\mathbf{\sigma}^2 \cdot \mathbf{d}), \text{tr}(\mathbf{\sigma}^2 \cdot \mathbf{d}^2)$. Note that in the above equation the symbol $^\text{def}$ denoting index raising has been omitted for the $\mathbf{d}$-terms.

5 Rate Forms of Virtual Power

5.1 Initial Boundary Value Problem and Principle of Virtual Power

In this and subsequent sections, we do not explicitly indicate compositions with point mappings and often suppress function arguments in order to ease notation.

Stated loosely, an initial boundary value problems (IBVP) is a set of differential equations together with a set of initial conditions and boundary conditions that describe the problem under consideration. A mechanical IBVP can be stated precisely, as follows, by making use of the definitions and relations in Section 2.3; see also [2]

Definition 5.1. A mechanical IBVP in the updated Lagrangian (UL) formulation is the problem of finding the spatial velocity $\mathbf{v}$, the spatial mass density $\rho$, the Cauchy stress $\mathbf{\sigma}$ and material state variables $\mathbf{\alpha}$ on $\varphi_t(\mathcal{B})$ for every $t \in [0, T]$ provided that for a reference mass density $\rho_{\text{ref}}$ and a body force per unit mass $\mathbf{b}$ given,

(i) conservation of mass $\dot{\rho} + \rho \text{div} \mathbf{v} = 0$,

(ii) balance of linear momentum $\dot{\mathbf{v}} = \rho \mathbf{b} + \text{div} \mathbf{\sigma}$, and

(iii) balance of angular momentum $\mathbf{\sigma} = \mathbf{\sigma}^T$ hold,

(iv) the stress $\mathbf{\sigma}$ and state variables $\mathbf{\alpha}$ are obtained through constitutive equations,

(v) for the boundary $\partial(\varphi_t(\mathcal{B})) \overset{\text{def}}{=} \partial_t(\varphi_t(\mathcal{B})) \cup \partial_r(\varphi_t(\mathcal{B})) \cup \partial_c(\varphi_t(\mathcal{B}))$ of the body in its current configuration with unit outward normals $\mathbf{n}^*$ there are prescribed

(a) $\mathbf{v}_t = \tilde{\mathbf{v}}_t$ on $\partial_r(\varphi_t(\mathcal{B}))$ (velocity boundary conditions),

(b) $\mathbf{\sigma}_t \cdot \mathbf{n}^* = \mathbf{t}$ on $\partial_c(\varphi_t(\mathcal{B}))$ (traction boundary conditions),

(c) contact constraints on $\partial_c(\varphi_t(\mathcal{B}))$ (contact boundary conditions), and

(vi) $\mathbf{v}_t, \mathbf{\sigma}_t$, and $\mathbf{\alpha}_t$ are given at $t = 0$ (initial conditions).

Remark 5.1. The UL formulation of the IBVP in Definition 5.1 refers to the current configuration of the body $\varphi_t(\mathcal{B})$ as the reference configuration. The UL formulation is one commonly used for large deformation problems in solid mechanics [10] [13]. It is different from the Total Lagrangian (TL) formulation referring to the initial configuration at $t = 0$, and fundamentally different from the Eulerian formulation used in fluid dynamics. The latter refers to a spatially fixed domain through which convective terms enter the balance equations.
Remark 5.2. Note that balance of linear momentum in the form of Definition 5.1(ii) implies conservation of mass [55]. Conservation of mass (i) can thus be solved independently, i.e., it serves merely as an evolution equation for the mass density. Moreover, balance of angular momentum (iii) boiled down to the symmetry of the Cauchy stress, a condition which can be incorporated into (ii). Therefore, balance of linear momentum (ii) is the only independent balance equation of the mechanical IBVP in the form of Definition 5.1 that needs to be solved.

Closed-form analytical solutions for mechanical IBVP (Definition 5.1) are available only for a few simple cases. Most problems need to be approximated and solved numerically. The popular finite element method is based on a weak (or variational) formulation of IBVP. The weak form of the mechanical IBVP, by Remark 5.2, mainly consists of the the weak form of the balance of linear momentum, which is equivalent to the principle of virtual power. In deriving the principle of virtual power in the updated Lagrangian formulation, some additional terminology is required.

Definition 5.2. Let the space of admissible velocities on \( \varphi_t(B) \) be
\[
W \equiv \{ v_t : \varphi_t(B) \to TS \mid v_t = \tilde{v}_t \text{ on } \partial_t(\varphi_t(B)) \},
\]
where \( \partial_t(\varphi_t(B)) \) is the part of the boundary of the current configuration with prescribed velocities (Definition 5.1(v)). The space of admissible spatial variations (or virtual velocities) is then defined through
\[
V_t \equiv \{ \delta v_t : \varphi_t(B) \to TS \mid \delta v_t = 0 \text{ on } \partial_t(\varphi_t(B)) \},
\]
which contains all vector fields vanishing on the boundary \( \partial_t(\varphi_t(B)) \). It is emphasized that \( \delta v_t \) is time-independent, i.e., the index \( t \) serves as a label, and not as a parameter.

Proposition 5.1. Let the balance of linear momentum (Definition 5.1(ii)) be satisfied at every \( x \in \varphi_t(B) \) and \( t \in [0,T] \), and let \( \delta v_t \in V_t \) be an admissible variation. Then,
\[
\int_{\varphi_t(B)} (\text{div} \, \sigma + \rho b - \rho \dot{\bar{v}}) \cdot \delta v_t \, dv = 0.
\]

Proof. This transformation employs the fundamental lemma of the calculus of variations and is a standard exercise in textbooks about finite element methods; e.g. [13].

Definition 5.3. The integral form in Proposition 5.1 is called the weak or variational form of balance of linear momentum, or principle of virtual power, and the differential form in Definition 5.1(ii) the strong form.

Definition 5.4. The virtual velocity gradient is defined through \( \delta \dot{v}_t \equiv (\nabla (\delta v_t))^T \equiv \delta \dot{F}_t \cdot F_t^{-1} \). Accordingly, the virtual rate of deformation tensor and the virtual vorticity tensor are
\[
\delta d_t \equiv \frac{1}{2}(\delta \dot{v}_t + \delta \dot{v}_t^T) \quad \text{and} \quad \delta \omega_t \equiv \frac{1}{2}(\delta \dot{v}_t - \delta \dot{v}_t^T),
\]
respectively.

Proposition 5.2. Assume that there are no contact constraints (\( \partial_t(\varphi_t(B)) = \emptyset \)) and balance of angular momentum \( \sigma = \sigma^T \) holds, then the principle of virtual power (Proposition 5.1) is equivalent to
\[
P(\varphi_t; \delta v_t) \equiv \int_{\varphi_t(B)} \sigma : \delta d_t \, dv + \int_{\varphi_t(B)} \rho (\dot{\bar{v}} - b) \cdot \delta v_t \, dv - \int_{\partial_t(\varphi_t(B))} \dot{t} \cdot \delta v_t \, da = 0.
\]

Proof. Let \( \delta \dot{v}_t \) be continuously differentiable, then by the product rule (Proposition A.10) and noting that \( \sigma \) is symmetric, the term \( (\text{div} \, \sigma) \cdot \delta v_t \) in Proposition 5.1 becomes
\[
(\text{div} \, \sigma) \cdot \delta v_t = \text{div} (\sigma \cdot \delta v_t) - \sigma : \nabla (\delta v_t) = \text{div} (\sigma \cdot \delta v_t) - \sigma : \delta d_t.
\]
Substitution into the integral in Proposition 5.1 and application of the divergence theorem [13] then gives
\[
\int_{\varphi_t(B)} \sigma : \delta d_t \, dv + \int_{\varphi_t(B)} \rho (\dot{\bar{v}} - b) \cdot \delta v_t \, dv - \int_{\partial_t(\varphi_t(B))} n \cdot \sigma \cdot \delta v_t \, da = 0
\]
after some rearrangement. Note \( \delta v_t \) vanishes on \( \partial_t\varphi_t(B) = \partial_\bar{t}\varphi_t(B) \cap \partial_r\varphi_t(B) \) by definition (contact constraints are omitted), and \( \delta v_t \) is arbitrary on \( \partial_t\varphi_t(B) \) where \( \sigma \cdot n = \bar{t} \) is prescribed. Therefore, the assertion follows. \( \square \)
The next concluding definition is due to [66] and [102].

**Definition 5.5.** The variational or weak form of the IBVP in UL description (Definition 5.4) without contact constraints is the problem of finding the velocity field \( \mathbf{v} \), the stress field \( \sigma \), the mass density field \( \rho \), and the internal state variables \( \alpha \) such that conservation of mass holds, and

\[
P(\phi_t; \delta \mathbf{v}_t) = 0, \quad \text{for all } \delta \mathbf{v}_t \in V_t,
\]

subject to prescribed boundary and initial conditions.

**Remark 5.3.** In the above weak forms of balance of linear momentum, the index raising and index lowering operations, which make the equations well-posed, are hidden. It is understood that the involved quantities are compatible with respect to tensor contraction, that is, the spatial metric \( g \) is included in the weak balance of momentum to perform index raising and lowering. However, the metric is not necessary to state weak balance of momentum. As the virtual velocity \( \delta \mathbf{v}_t \) is a vector field, one recognizes from the last term in Propositions 5.2 that forces are in fact 1-forms and not vector fields [101], [66]. According to [101], a more general form of the weak balance of momentum in Propositions 5.2 would be the force functional

\[
F(\delta \mathbf{v}_t) \defeq \int_{\varphi(B)} \beta(\delta \mathbf{v}_t) - \int_{\partial \varphi(B)} \zeta(\delta \mathbf{v}_t).
\]

Here \( \beta(\delta \mathbf{v}_t) \) and \( \zeta(\delta \mathbf{v}_t) \) are the virtual power densities of the body force and surface force, respectively, which can be defined on general manifolds without a metric. In [101], the virtual velocity \( \delta \mathbf{v}_t \) and the force functional \( F(\delta \mathbf{v}_t) \) are regarded as the primitive objects, from which the surface and body force forms can be derived.

### 5.2 Derivation of Rate Formulations

Solution of the weak initial boundary value problem (Definition 5.5) by the finite element method is generally based on an incremental procedure [10], [13]: direct solution is possible only for a limited number of linear problems. If solution is advanced implicitly in time (implicit FEM), the governing equations are usually linearized and then solved iteratively using variants of Newton’s method. However, instead of a consistent linearization of the weak balance of momentum (virtual power), finite element codes often employ a quasi-linearized form derived from the rate of virtual power [101], [66], [102]. The use of the rate formulation will retain the continuum material tangent of the rate formulation in the resulting quasi-linearized equations. From a numerical viewpoint, this puts unnecessary constraints on the step size, and destroys the quadratic rate of asymptotic convergence of Newton’s method in implicit integration methods. A critical assessment is given by [74] in section 7.2.3.

In determining an expression for the rate form of virtual power, \( \dot{P}(\phi_t; \delta \mathbf{v}_t) \), it is assumed that the loads are deformation-independent, that is, they are dead loads depending only on the reference configuration. For example, dead loads are inertia forces, but not pressure loads on a deforming structure. The practical advantage of deformation-independent loads is that their contribution to the principle of virtual power in the current configuration of a body can be evaluated using the reference configuration. Consequently, the \( \dot{P} \)-terms involving these quantities have zero value, resulting in

\[
\dot{P}(\phi_t; \delta \mathbf{v}_t) = \frac{d}{dt} \int_{\varphi(B)} \sigma : \delta \mathbf{d} \mathbf{v}.
\]

**Proposition 5.3.** Let the external loads be deformation-independent, then the rate of virtual power (Propositions 5.2) reads

\[
\dot{P}(\phi_t; \delta \mathbf{v}_t) = \int_{\varphi(B)} \left( \dot{\sigma}_t^T - 2 \delta \mathbf{d} \cdot \sigma_t + \sigma_t \operatorname{tr} \mathbf{d} \right) : \delta \mathbf{d} + \sigma_t : \dot{\mathbf{l}}_t \cdot \delta \mathbf{l}_t \mathbf{d} \mathbf{v}.
\]

**Proof.** The index \( t \) is omitted for the proof. The rate form with respect to the configuration \( \varphi(B) \) will be derived by pulling it back to the reference configuration \( B \), performing the time derivation, and then pushing forward the rate form to the current configuration. This is similar to the approach in [74] to derive \( \Delta P \).
Let \( \sigma \defeq \sigma^j \) and \( \delta \mathbf{d} \defeq \delta \mathbf{d}^j \), then by changing variables (Theorem A.2), using Proposition 2.3 and noting that \( \dot{F} = \nabla \mathbf{v} \cdot \dot{F} \),

\[
\int_{\varphi(B)} \sigma : \delta \mathbf{d} \, d\mathbf{v} = \int_{\varphi(B)} \varphi \downarrow (\sigma : \delta \mathbf{d}) \, d\mathbf{V} = \int_{\varphi(B)} (\varphi \downarrow \sigma) : \dot{\varphi} \downarrow (\delta \mathbf{d}) \, d\mathbf{V} = \int_{\varphi(B)} (\dot{\sigma} \cdot F^{-1} - \delta \mathbf{d} \cdot F^{-1}) : J \, d\mathbf{V} = \int_{\varphi(B)} (\dot{\sigma} - \delta \mathbf{d} \cdot F^{-1}) : \delta \mathbf{F} \, d\mathbf{V}.
\]

The last identity is due to the symmetry of \( \sigma \), that is, \( \sigma : \delta \mathbf{F} \cdot F^{-1} = \sigma : \delta \mathbf{d} \). Differentiation of \( \int_{\varphi(B)} \sigma : \delta \mathbf{d} \, d\mathbf{v} \) in time then yields

\[
\dot{P} = \int_{\varphi(B)} ((\dot{\sigma} - \delta \mathbf{d} \cdot F^{-1}) : \delta \mathbf{F}) \, d\mathbf{V}.
\]

Recall that the rate of \( \delta \mathbf{F} \) contributes zero to \( \dot{P} \), because \( \delta \mathbf{v} \) is time-independent by Definition 5.2 and defined for the configuration \( \varphi(B) \) at fixed time only. The term \( \dot{F}^{-1} = \frac{\partial F^{-1}}{\partial t} \) can be expanded by differentiating \( F^{-1} \cdot \dot{F} = I \) in time:

\[
0 = \frac{\partial F^{-1}}{\partial t} \cdot F + F^{-1} \cdot \frac{\partial F}{\partial t} \quad \Leftrightarrow \quad \frac{\partial F^{-1}}{\partial t} = -F^{-1} \cdot \frac{\partial F}{\partial t} \cdot F^{-1},
\]

so \( \dot{F}^{-1} = -F^{-1} \cdot \dot{F} \cdot F^{-1} \). Moreover, \( \dot{J} = J \, \text{div} \mathbf{v} = J \, \text{tr} \mathbf{d} \) by Proposition 2.2 and

\[
(F^{-1} \cdot \dot{\sigma}) : \delta \mathbf{F} = (F^{-1} \cdot \dot{\sigma}) : \delta \mathbf{d} \mathbf{F} = \dot{\sigma} : (F^{-1} \cdot \delta \mathbf{d} \mathbf{F}) = \dot{\sigma} : \delta \mathbf{t} = \dot{\sigma} : \delta \mathbf{t}.
\]

By changing variables again (note that the integral term in parentheses is a scalar, hence \( \varphi \uparrow (\cdot) = (\cdot) \circ \varphi^{-1} \)), it then follows that

\[
\dot{P} = \int_{\varphi(B)} (\dot{\sigma} - \delta \mathbf{d} \cdot F^{-1}) : \delta \mathbf{F} \, d\mathbf{v} = \int_{\varphi(B)} (\dot{\sigma} - \dot{\mathbf{F}} \cdot F^{-1} \cdot \mathbf{d} + \text{tr} \mathbf{d} \cdot (F^{-1} \cdot \delta \mathbf{d})) : \delta \mathbf{F} \, d\mathbf{v}.
\]

The objective Zaremba-Jaumann stress rate has been defined through \( \dot{\sigma}^{ZJ} \defeq \dot{\sigma} - \mathbf{w} : \mathbf{d} + \mathbf{w} \cdot \mathbf{d} \cdot \mathbf{d} \), so that the previous becomes

\[
\dot{P} = \int_{\varphi(B)} (\dot{\sigma}^{ZJ} - 2 \mathbf{d} \cdot \mathbf{d} + \text{tr} \mathbf{d} \cdot \mathbf{d}) : \delta \mathbf{F} \, d\mathbf{v}.
\]

Using various identities, the terms with \( \delta \mathbf{t} \) condense to

\[
(\mathbf{w} : \mathbf{d} - \mathbf{d} \cdot \mathbf{w} + \text{tr} \mathbf{d} \cdot \mathbf{w}) : \delta \mathbf{t} = \mathbf{w} : \mathbf{d} : \delta \mathbf{t} - \mathbf{w} : \mathbf{t} : \delta \mathbf{t} + \text{tr} \mathbf{d} \cdot \mathbf{w} : \delta \mathbf{t}
\]

\[
= \mathbf{w} : \delta \mathbf{t} + \mathbf{t} \cdot \mathbf{w} : \delta \mathbf{t} + \text{tr} \mathbf{d} \cdot \mathbf{w} : \delta \mathbf{t}
\]

\[
= (\mathbf{I} - \text{tr} \mathbf{d}) \cdot \mathbf{d} : \delta \mathbf{t} + \text{tr} \mathbf{d} \cdot \mathbf{w} : \delta \mathbf{t}
\]

\[
= \mathbf{d} : \delta \mathbf{t} + \mathbf{d} : \mathbf{t} \cdot \mathbf{w}.
\]

so finally,

\[
\dot{P} = \int_{\varphi(B)} ((\dot{\sigma}^{ZJ} - 2 \mathbf{d} \cdot \mathbf{d} + \text{tr} \mathbf{d} \cdot \mathbf{d}) : \delta \mathbf{F} + \mathbf{d} : \mathbf{t} \cdot \mathbf{w}) \, d\mathbf{v}.
\]

The various stress rates introduced in Section 3 motivate alternative expressions for the rate \( \dot{P} \).
Proposition 5.4. Let $\tau \overset{\text{def}}{=} \tau^*$ be the Kirchhoff stress, then Proposition 5.3 is equivalent to

$$\dot{P}(\varphi_t; \delta \nu_t) = \int_B \left( \dot{\tau}^0 + l_t \cdot \tau_t \right) : \delta \nu_t \, dV, \quad (i)$$

$$\dot{P}(\varphi_t; \delta \nu_t) = \int_B \left( \dot{\tau}^{ZJ} : \delta \nu_t - \frac{1}{2} \tau_t : \delta \left( 2d_t \cdot d_t - l_t^T \cdot l_t \right) \right) \, dV, \quad (ii)$$

and

$$\dot{P}(\varphi_t; \delta \sigma_t) = \int_{\varphi_t(B)} \left( \dot{\sigma}^{\text{TN}} : \delta \nu_t + \sigma_t : \delta \left( \frac{1}{2} l_t^T \cdot l_t \right) \right) \, dV. \quad (iii)$$

Proof. In proving the assertions, the index $t$ is again omitted.

(i) With $\sigma = \sigma^t$ and $\tau = \tau^t$ presumed, Section 3 proves that the stress rates $L_\sigma \tau = \dot{\tau}^0$, $\dot{\sigma}^{ZJ}$ and $\dot{\sigma}^{\text{TN}}$ are related by

$$J^{-1} L_\sigma \tau = \dot{\sigma}^{\text{TN}} = \dot{\sigma} - l \cdot \sigma - \sigma \cdot l^T + \sigma \text{div} \nu = \dot{\sigma} - (d + \omega) \cdot \sigma - \sigma \cdot (d + \omega)^T + \sigma \text{tr} d$$

$$= \dot{\sigma} - 2d \cdot \sigma - \omega \cdot \sigma + \sigma \cdot \omega + \sigma \text{tr} d$$

$$= \dot{\sigma}^{ZJ} - 2d \cdot \sigma + \sigma \text{tr} d,$$

thus $\dot{P}$ in Proposition 5.3 equals

$$\dot{P} = \int_B \left( J^{-1} L_\sigma \tau : \delta d_t + \sigma : l^T \cdot \delta l \right) J \, dV.$$

Since $\sigma = J^{-1} \tau$ and $\sigma : l^T \cdot \delta l = l \cdot \sigma : \delta l = J^{-1} l \cdot \tau : \delta l$, assertion (i) follows.

(ii) The identity $\dot{\sigma}^{ZJ} = J^{-1} \dot{\tau}^{ZJ} - \sigma \text{tr} d$ results from

$$\dot{\tau}^{ZJ} = \dot{\tau} - \omega \cdot \tau + \tau \cdot \omega = J \dot{\sigma} + J \dot{\sigma} - J \omega \cdot \sigma + J \sigma \cdot \omega = J (\text{tr} d) \sigma + J \dot{\sigma}^{ZJ}.$$

Moreover, by symmetry of $d$ and $\sigma$,

$$2d \cdot \sigma : \delta d - \sigma : l^T \cdot \delta l = \sigma : \left( 2d \cdot \delta d - l^T \cdot \delta l \right) = \frac{1}{2} \sigma : \delta \left( 2d \cdot d - l^T \cdot l \right).$$

Substitution into Proposition 5.3 and rearrangement then yields

$$\dot{P} = \int_B \left( \dot{\sigma}^{ZJ} : \delta d - \frac{1}{2} \sigma : \delta \left( 2d \cdot d - l^T \cdot l \right) \right) \, dV$$

$$= \int_B \left( J^{-1} \dot{\tau}^{ZJ} : \delta d - (\text{tr} d) \sigma : \delta d - \frac{1}{2} \sigma : \delta \left( 2d \cdot d - l^T \cdot l \right) \right) \, dV$$

$$= \int_B \left( \dot{\tau}^{ZJ} : \delta d - \frac{1}{2} \tau : \delta \left( 2d \cdot d - l^T \cdot l \right) \right) \, dV,$$

as desired.

(iii) From the proof of (i) above, $\dot{\sigma}^{\text{TN}} = \dot{\sigma}^{ZJ} - 2d \sigma + \sigma \text{tr} d$, and $\sigma : l^T \cdot \delta l = \frac{1}{2} \sigma : (l^T \cdot \delta l + \delta l^T \cdot l) = \sigma : \delta \left( \frac{1}{2} l^T \cdot l \right)$ due to symmetry of $\sigma$. Then the assertion (iii) is proved to hold by substitution into Proposition 5.3.

Remark 5.4. The form of $\dot{P}$ in Proposition 5.3 is connected with [74], while [5.2i] was first derived by Hill [33], see also [102, eq. (7.2.21)]. The form [5.2ii] of $\dot{P}$ is in complete agreement with eq. (5) in [69] by setting $J = 1$; note this is an instantaneous condition in the UL formulation of initial boundary value problems, in which reference configuration coincides with the current configuration at time $t$. The form [5.2iii], however, does not correspond to the UL formulations found in [131] [11] [10], although they look similar. The latter authors derived the linearized virtual work from an incremental form, as Nagtegaal [72], and not based on a rate form.

Remark 5.5. It is straightforward to set up the rate form $\dot{P}$ of the weak balance of momentum in terms of the Green-Naghdi stress rate, $\dot{\sigma}^{\text{GN}} \overset{\text{def}}{=} \sigma - \Omega \cdot \sigma + \sigma \cdot \Omega$, by substituting the relation specified in the Remark 5.2 between $\Omega$ and the vorticity tensor $\omega$ related to the Zaremba-Jaumann rate, $\dot{\sigma}^{ZJ} = \sigma - \omega \cdot \sigma + \sigma \cdot \omega$.  

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Remark 5.6. In Proposition 5.3, the term that stems from the change of volume, \((\text{tr } d_i) \sigma_i : \delta d_i\), will lead to a non-symmetric stiffness matrix in implementations of the finite element method, even if the material tangent tensor \(m\) defined through the constitutive rate equation \(\dot{\sigma} = m : d\) is itself symmetric. Therefore, this term is often neglected by assuming that the volume changes will be negligible for large strain conditions [14] [22].

6 Objective Time Integration

6.1 Fundamentals and Geometrical Setup

Determination of the motion \(\varphi : B \times [0, T] \to S\) from balance of linear momentum (Definition 2.22) requires the total Cauchy stress \(\sigma\) at every time \(t \in [0, T]\). If a given constitutive equation calculates only a rate of stress but not total stress, then the latter represents the solution of an initial value problem. A formal description of this situation is given below. For simplicity, we consider only rate constitutive equations that determine an objective corotational rate of Cauchy stress according to Definition 3.3. Recall that this term is often neglected by assuming that the volume changes will be negligible for large strain conditions [50] [22].

Definition 6.1. The considered class of corotational rate constitutive equations takes the general form

\[
\dot{\sigma}^\star \overset{\text{def}}{=} h(\sigma, \alpha, d) \overset{\text{def}}{=} m(\sigma, \alpha) : d,
\]

where \(\dot{\sigma}^\star\) is any objective corotational rate of Cauchy stress, \(\alpha\) is a set of state variables in addition to stress, \(d\) is the rate of deformation, and \(m\) is the material tangent tensor. By defining

\[
\tilde{h}(\sigma, \alpha, d, \Lambda) \overset{\text{def}}{=} h(\sigma, \alpha, d) + \Lambda \cdot \sigma - \sigma \cdot \Lambda,
\]

where \(\Lambda = -A^T\) is the spin tensor associated with \(\dot{\sigma}^\star\), the constitutive equation takes the equivalent form \(\dot{\sigma} = \tilde{h}(\sigma, \alpha, d, \Lambda)\). Moreover, each element in the set \(\alpha\) is assumed to have evolution equations similar to those of stress, that is, \(\dot{\alpha} = k(\sigma, \alpha, d, \Lambda)\).

Definition 6.2. An incremental decomposition of time is a disjoint union

\[
[0, T] \overset{\text{def}}{=} \bigcup_{n=0}^{N-1} [t_n, t_{n+1}],
\]

motivating a sequence \((t_0 = 0, t_1 = t_0 + \Delta t_1, \ldots, t_{n+1} = t_n + \Delta t_{n+1}, \ldots, t_N = T)\) of discrete time steps \(t_{n+1} = t_n + \Delta t_{n+1}\) with time increment \(\Delta t_{n+1}\). For simplicity, we assume that the time increment is constant, that is, \(\Delta t_{n+1} \equiv \Delta t\) such that \(t_{n+1} = (n+1) \Delta t\) for \(t_0 = 0\). Let \(\varphi_n(B)\) and \(\varphi_{n+1}(B)\) be configurations of the material body \(B\) at time \(t_n\) and \(t_{n+1}\), respectively, then the incremental decomposition of stress is accordingly defined by

\[
\sigma_{n+1} \overset{\text{def}}{=} \sigma_n + \Delta \sigma,
\]

in which

\[
\sigma_n : \varphi_n(B) \to T^*_0(S), \quad \sigma_n \overset{\text{def}}{=} \sigma(t_n), \quad \text{and} \quad \Delta \sigma = \int_{t_n}^{t_{n+1}} \dot{\sigma}(t) \, dt.
\]

Similar holds for the state variables.

Definition 6.3. A time integration of the rate constitutive equation in Definition 6.1 determines the stress and material state, \((\sigma, \alpha)\), at time \(t = t_{n+1}\) by considering the differential equations

\[
\dot{\sigma}(t) = \tilde{h}(t, \sigma(t), \alpha(t), d(t), \Lambda(t)) \quad \text{and} \quad \dot{\alpha}(t) = k(t, \sigma(t), \alpha(t), d(t), \Lambda(t))
\]

subject to the initial condition \(\{\sigma, \alpha\}_{t=t_n} = \{\sigma_n, \alpha_n\}\). The time integration is called incrementally objective [20] if the stress is exactly updated (i.e. without the generation of spurious stresses) for rigid motions \(\varphi : B \to S\) over the incremental time interval \([t_n, t_{n+1}]\), that is, if

\[
\sigma_{n+1} = \varphi^\dagger \sigma_n = Q \cdot \sigma_n \cdot Q^T,
\]

where \(Q \overset{\text{def}}{=} T \varphi\) is proper orthogonal. The same is required for tensor-valued state variables, if any.
Since the rate constitutive equations are generally non-linear functions of their arguments, the time integration must be carried out numerically by employing suitable time integration methods; also called stress integration methods in the present context. The choice of the stress integration method plays a crucial role in numerical simulation of solid mechanical problems because it affects the stability of the solution process and the accuracy of the results. Most stress integration methods are customized for small-strain elasto-plastic resp. hypoelasto-plastic constitutive rate equations that include yield conditions. Early works include [62, 63, 75, 99, 106, 107, 125], and a comprehensive treatise is that of Simo and Hughes [102].

The time integration is usually split into two different phases: the objective update, which becomes necessary only for finite deformation problems, and the actual integration of the stress rate. The initial value problem associated with stress integration can be solved either by explicit schemes or implicit schemes. Explicit stress integration methods are formulations using known quantities at the beginning of the time step, like the forward Euler scheme. The procedure is straightforward, and the resulting equations are almost identical to the analytical setup. However, the simplicity of the implementation fronts the stability constraint and the forward Euler scheme. The procedure is straightforward, and the resulting equations are almost identical to the analytical setup. However, the simplicity of the implementation fronts the stability constraint and error accumulation during calculation, since generally no yield condition is enforced. Accuracy can be increased by partitioning the time increment into a number of substeps, and to perform automatic error control [107, 108].

Implicit stress integration methods are based on quantities taken with respect to the end of the time step, like the backward Euler scheme. Operator-split procedures are preferred for elasto-plastic models to solve the coupled system of nonlinear equations. From a geometric standpoint, the implicit stress update with operator-split projects an elastically estimated trial state onto the yield surface. The plastic multiplier serves as the projection magnitude; the plastic multiplier is zero in case of elastic loading, unloading, and neutral loading. The yield condition is naturally enforced at the end of the time increment. Therefore, at the same increment size, implicit algorithms can be more accurate than explicit algorithms. The numerical implementation is, however, more complicated because generally the plastic multiplier has to be obtained from the yield condition by an iterative procedure. This also generates computational overburden. In spite of this, implicit stress integration methods became standard in small-strain elasto-plasticity and hypoelasto-plasticity.

**Remark 6.1.** Consider the initial value problem defined through

\[ \dot{y}(t) = f(t, y(t)) \]

subject to the initial condition \((t_n, y_n)\). A solution to that problem is a function \(y\) that solves the differential equation for all \(t \in [t_n, t_{n+1}]\) and satisfies \(y(t_n) = y_n\). Different approaches are available to obtain an approximate solution. The explicit *forward Euler method*, for example, uses a first-order approximation to the time derivative:

\[ \dot{y}(t) \approx \frac{y(t_n + \Delta t) - y(t_n)}{\Delta t}. \]

Setting \(y(t_n) \equiv y_n\) and noting that \(\dot{y}(t_n) = f(t_n, y(t_n))\) by definition, then

\[ y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n). \]

In contrast to explicit methods, the implicit integration methods use quantities defined at the end of the time increment. Since these are generally unknown, they have to be estimated and subsequently corrected by an iteration. For example, the *backward Euler method* uses the approximation

\[ \dot{y}(t) \approx \frac{y(t_{n+1}) - y(t_{n+1} - \Delta t)}{\Delta t}, \]

by which

\[ y_{n+1} = y_n + \Delta t \cdot f(t_{n+1}, y_{n+1}). \]

Combination of both methods yields the *generalized midpoint rule*

\[ y_{n+1} = y_n + \Delta t \cdot f_{n+\theta}, \]

where

\[ f_{n+\theta} \equiv \theta f(t_{n+1}, y_{n+1}) + (1 - \theta) f(t_n, y_n), \quad \text{mit} \quad \theta \in [0, 1]. \]

The *Crank-Nicolson method* is obtained by setting \(\theta = \frac{1}{2}\).
The rotational terms of the stress rate (Definition 6.1) present at finite deformations render the integration of rate constitutive equations expensive compared to the infinitesimal case. Subsequent to the work of Hughes and Winget [50], who have introduced the notion of incremental objectivity formalized in Definition 6.3, several authors have developed or improved incrementally objective algorithms, e.g. [31, 49, 52, 59, 63, 102, 104, 61]. A comparative analysis is provided in [60]. One basic methodology in formulating objective integration methods utilizes a corotated or rotation-neutralized representation. Within this approach, the basic quantities and evolution equations are locally transformed to a rotating coordinate system that remains unaffected by relative rigid motions; the local coordinate system “corotates” with the relative rotation of the body. Then, the constitutive equation is integrated in the corotated representation by using the algorithms outlined above, and is finally rotated back to the current spatial configuration at time \( t_n \). The main advantage of this class of algorithms is that the integration of the rate constitutive equation can be carried out by the same methods at both infinitesimal and finite deformations.

The remainder of this section is largely based on [102, ch. 8] and [49]. It introduces the integration of rate constitutive equations for finite deformation problems. The main concern is the numerical method designed to achieve the objectives, we use the following.

To achieve the objectives, we use the following.

Definition 6.4. Let time be incrementally decomposed according to Definition 6.2 such that \( t_{n+1} = t_n + \Delta t \in [0, T] \), with \( t_0 = 0 \) and \( \Delta t > 0 \). Moreover, let

\[
\varphi(B, t_n) = \varphi_n(B) \triangleq \{ x_n = \varphi_n(X) | X \in B \} \subset S = \mathbb{R}^m
\]

be a given configuration of \( B \) at time \( t_n \in [0, T] \), and

\[
u \triangleq \varphi_n(B) \rightarrow TS
\]

a given incremental spatial displacement field imposed on \( \varphi_n(B) \) which is constant over the time interval \([t_n, t_{n+1}]\). The incremental material displacement field is then defined through \( U = u \circ \varphi_n \).

Definition 6.5. The spatial finite strain tensor \( e \) at time \( t \in [0, T] \) is defined conceptually through

\[
e(x, t) \triangleq \frac{\partial \varphi_n(X)}{\partial x} (x, 0) + \int_0^t \frac{\partial \varphi_n(X)}{\partial \tau} (x, \tau) d\tau,
\]

where \( e(x, 0) \) is given. The overall accuracy of the stress integration method is then affected by the approximate evaluation of the finite strain increment \( \Delta e \triangleq \int_{t_n}^{t_{n+1}} \frac{\partial \varphi_n(X)}{\partial \tau} (x, \tau) d\tau \) constant in \([t_n, t_{n+1}]\).

According to Simo and Hughes [102, p. 279], the aim in developing objective integration algorithms

\[
\text{...} \quad \text{is to find algorithmic approximations for spatial rate-like objects, in terms of the incremental displacements \( u(x_n) \) and the time increment \( \Delta t \), which exactly preserve proper invariance under superposed rigid body motions \( \text{...} \).}
\]

To achieve the objectives, we use the following.

Definition 6.6. A one-parameter family of configurations is the linear interpolation between \( \varphi_n \) and \( \varphi_{n+1} \) defined through

\[
\varphi_{n+\theta} \triangleq \theta \varphi_{n+1} + (1 - \theta) \varphi_n, \quad \text{with} \quad \theta \in [0, 1].
\]

Conceptually, the intermediate configuration \( \varphi_{n+\theta} \) is related to an intermediate time \( t_{n+\theta} = \theta t_{n+1} + (1 - \theta) t_n = t_n + \theta \Delta t \). The configuration \( \varphi_{n+1} \) in the Euclidean ambient space \( \mathbb{R}^m \) can be determined by adding the incremental displacements to the configuration at time \( t_n \), that is, \( \varphi_{n+1}(X) = \varphi_n(X) + U(X) \in TX \mathbb{R}^m \) for all \( X \in B \). Accordingly, the deformation gradient of \( \varphi_{n+\theta} \) is given by the relationship

\[
F_{n+\theta} = T \varphi_{n+\theta} = \theta F_{n+1} + (1 - \theta) F_n, \quad \text{with} \quad \theta \in [0, 1].
\]
The relative incremental deformation gradient of the configuration \( \varphi_{n+\theta}(B) \) with respect to the configuration \( \varphi_n(B) \) is then defined through

\[
f_{n+\theta} \overset{\text{def}}{=} F_{n+\theta} \cdot F_n^{-1}, \quad \text{with } \theta \in [0, 1].
\]

**Definition 6.7.** The relative incremental displacement gradient is the tensor field \( \nabla_{n+\theta}u \in \mathfrak{T}_1^1(S) \) which has the local representative

\[
(\nabla_{n+\theta}u)(x_{n+\theta}) \overset{\text{def}}{=} \left( \frac{\partial u^i(x_{n+\theta})}{\partial x_j} + u^i(x_{n+\theta})\gamma_{j\,k}^i \right) dx_{n+\theta}^k \otimes \frac{\partial}{\partial x_{n+\theta}^i}.
\]

at \( x_{n+\theta} = \varphi_{n+\theta}(X) \), where \( u^i(x_{n+\theta}) \) are the components of the incremental displacements referred to the configuration \( \varphi_{n+\theta}(B) \). The spatial connection coefficients \( \gamma_{j\,k}^i \) are understood to be taken with respect to \( x_{n+\theta} \). In a spatial Cartesian coordinate system \( \{z^b\} \), recall that \( \nabla_{n+\theta}u \) can likewise be expressed by

\[
(\nabla_{n+\theta}u)(x_{n+\theta}) \equiv \frac{\partial u(x_{n+\theta})}{\partial x_{n+\theta}^d} = \frac{\partial u^b(x_{n+\theta})}{\partial z_{n+\theta}^d} e_d \otimes e_b,
\]

where \( z_{n+\theta}^d = z^d(x_{n+\theta}) \).

**Definition 6.8.** The relative right Cauchy-Green tensor \( C_{n+1} \) and the relative Green-Lagrange strain \( E_{n+1} \) of the configuration \( \varphi_{n+1}(B) \) with respect to the configuration \( \varphi_n(B) \) are defined by

\[
C_{n+1} \overset{\text{def}}{=} f_{n+1}^T \cdot f_{n+1} \quad \text{and} \quad 2E_{n+1} \overset{\text{def}}{=} C_{n+1} - I \in \mathfrak{T}_1^1(\varphi_n(B)),
\]

respectively; composition with the map \( \varphi \) has been dropped. \( I \) is the unit tensor on \( \varphi_n(B) \). On the other hand, define the relative left Cauchy-Green tensor \( b_{n+1} \) and the relative \( \text{Euler-Almansi strain} \) \( e_{EA_{n+1}} \) of the configuration \( \varphi_{n+1}(B) \) relative to \( \varphi_n(B) \) through

\[
b_{n+1} \overset{\text{def}}{=} f_{n+1} \cdot f_{n+1}^T \quad \text{and} \quad 2e_{EA_{n+1}} \overset{\text{def}}{=} i - b_{n+1}^{-1} \in \mathfrak{T}_1^1(S),
\]

respectively. Here \( i \) is the unit tensor on \( S \).

By these definitions and the basic relationships of Section 2, the next results are obtained; see [102] secs. 8.1 and 8.3 for full proofs with respect to Cartesian frames.

**Proposition 6.1.** Let \( [t_n, t_{n+1}] \) be an incremental time interval, \( t_{n+1} = t_n + \Delta t \), and let \( \theta \in [0, 1] \). Then,

(i) the relative incremental deformation and displacement gradients are equivalent to

\[
f_{n+\theta} = I + \theta(\nabla_n u)^T \quad \text{and} \quad (\nabla_{n+\theta}u)^T = (\nabla_n u)^T \cdot f_{n+\theta}^{-1},
\]

respectively, where \( \nabla_n u(x_n) \equiv \partial u(x_n) / \partial x_n \) in Cartesian coordinates,

(ii) objective approximations to the spatial rate of deformation in \( [t_n, t_{n+1}] \) are

\[
d_{n+\theta} = \frac{1}{\Delta t} f_{n+\theta}^{-T} \cdot E_{n+1}^\phi \cdot f_{n+\theta}^{-1},
\]

\[
d_{n+\theta} = \frac{1}{\Delta t} f_{n+\theta}^{-T} \cdot f_{n+1}^T \cdot (e_{EA_{n+1}})^b \cdot f_{n+1} \cdot f_{n+\theta}, \quad \text{and}
\]

\[
d_{n+\theta} = \frac{1}{2\Delta t} \left( (\nabla_{n+\theta}u)^T + \nabla_{n+\theta}u + (1 - 2\theta)(\nabla_{n+\theta}u)^T \cdot \nabla_{n+\theta}u \right),
\]

(iii) an algorithmic approximations to the vorticity is

\[
\omega_{n+\theta} = \frac{1}{2\Delta t} \left( (\nabla_{n+\theta}u)^T - \nabla_{n+\theta}u \right).
\]

**Proposition 6.2.** Let \( s \in \mathfrak{T}_3^2(S) \) be a covariant second-order spatial tensor field in \( [t_n, t_{n+1}] \), then (objective) algorithmic approximations to its Lie derivative are provided through

\[
L_s f_{n+\theta} = \frac{1}{\Delta t} f_{n+\theta}^{-T} \cdot f_{n+1}^T \cdot s_{n+1} \cdot f_{n+\theta}. 
\]

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\textbf{Proof.} This follows from the second equation in Proposition 6.1(ii) by similarity to Proposition 2.1 and using Propositions 2.2 and 2.3.

\textbf{Remark 6.2.} Propositions 6.1 and 6.2 are remarkable in the sense that, by Definition 6.1, the approximations to the discrete rate of deformation, the discrete vorticity, and the discrete Lie derivative in \([t_n, t_{n+1}]\) can be obtained with the aid of either total or incremental deformation gradients. These discrete spatial variables thus do not depend on the choice of a reference configuration, which is consistent to the continuous theory.

\textbf{Definition 6.9.} The \textit{algorithmic finite strain increment} or \textit{incremental finite strain tensor} is defined by

\[
\Delta \varepsilon_{n+\theta} \overset{\text{def}}{=} d_{n+\theta} \Delta t,
\]

where \(d_{n+\theta}\) is according to Proposition 6.1(ii). Hence, \(\Delta \varepsilon_{n+\theta}\) is likewise incrementally objective.

\textbf{Remark 6.3.} For the choice \(\theta = 0\), \(\Delta \varepsilon_{n+\theta}\) coincides with the relative Green-Lagrange strain \(E_{n+1}\). In case of \(\theta = 1\), \(\Delta \varepsilon_{n+\theta}\) is identical to the relative Euler-Almansi strain \(e_{n+1}^{\text{EA}}\). This follows directly from Definition 6.8 and Proposition 6.1(ii). Hughes [49] has shown that Definition 6.9 is a first-order approximation \([\text{eq. (41)}]\) and \([89, \text{eq. (36)}]\). Hughes [50] has also shown that Definition 6.9 is a first-order approximation for the choice \(\theta = 1\), referred to as the \textit{midpoint strain increment}, the approximation is linear in \(u\) (cf. Proposition 6.1(ii)). Therefore, the midpoint strain increment is the most attractive expression from the viewpoint of implementation.

\section{6.2 Algorithm of Hughes and Winget}

The algorithm of Hughes and Winget [50] is probably the most widely used objective stress integration method in nonlinear finite element programs. It considers a class of constitutive rate equations of the form

\[
\dot{\sigma}^{ZJ} = m(\sigma, \alpha) : d
\]

or, equivalently,

\[
\dot{\sigma} \overset{\text{def}}{=} m(\sigma, \alpha) : d + \omega \cdot \sigma - \sigma \cdot \omega,
\]

where \(\dot{\sigma}^{ZJ}\) is the Zaremba-Jaumann rate of Cauchy stress (Definition 3.3). Recall from Sect. 3.2 that \(\dot{\sigma}^{ZJ}\) is a corotational rate of \(\sigma\) defined by the spin \(\omega = -\omega^T\). The spin generates a one-parameter group of proper orthogonal transformations by solving

\[
\dot{\mathbf{R}} = \omega \cdot \mathbf{R}, \quad \text{subject to} \quad \mathbf{R}_{|t=0} = I,
\]

where \(\mathbf{R}\) is a proper orthogonal two-point tensor (cf. Definition 3.3).

Based on the observations summarized in Remark 6.3, Hughes and Winget [50] employ time-centered (\(\theta = \frac{1}{2}\)) approximations of \(d\) and \(\omega\). This leads to a \textit{midpoint strain increment} and \textit{midpoint spin increment} as follows.

\textbf{Proposition 6.3.} Let \(\Delta \varepsilon_{n+1/2} \overset{\text{def}}{=} d_{n+1/2} \Delta t\) and \(\Delta \mathbf{R}_{n+1/2} \overset{\text{def}}{=} \omega_{n+1/2} \Delta t\), then

\[
\Delta \varepsilon_{n+1/2} = \frac{1}{2}(\nabla (\nabla + 1/2 u)^T + \nabla (\nabla + 1/2 u)) \quad \text{and} \quad \Delta \mathbf{R}_{n+1/2} \overset{\text{def}}{=} \frac{1}{2}(\nabla (\nabla + 1/2 u)^T - \nabla (\nabla + 1/2 u))
\]

respectively, where

\[
(\nabla (\nabla + 1/2 u)^T = 2(f_{n+1} - I)(f_{n+1} + I)^{-1}.
\]

\textbf{Proof.} By straightforward application of Proposition 6.1. Similar expressions have been derived in \[86\] eq. (41) and \[30\] eq. (36).

The proof of the next statement is one of the main concerns of [50].

\textbf{Proposition 6.4.} The \textit{generalized midpoint rule} (Remark 6.1), with \(\theta = \frac{1}{2}\), is used to approximately integrate \(\dot{\mathbf{R}} = \omega \cdot \mathbf{R}\) subject to \(\mathbf{R}_{|t=0} = I\), resulting in the proper orthogonal relative rotation

\[
\Delta \mathbf{R} = \mathbf{R}_{n+1} \cdot \mathbf{R}_{n+1}^T = (I - \frac{1}{2} \Delta \mathbf{R}_{n+1/2})^{-1}(I + \frac{1}{2} \Delta \mathbf{R}_{n+1/2}).
\]

\textbf{Definition 6.10.} The \textit{objective integration algorithm of Hughes and Winget [50]} can be summarized as

\[
\sigma_{n+1} \overset{\text{def}}{=} \sigma_{n+1} + \Delta \sigma,
\]

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where

\[ \Delta \sigma \overset{\text{def}}{=} h(\sigma'_{n+1}, \alpha'_{n+1}, \Delta \tilde{e}_{n+1/2}) \], \quad \sigma'_{n+1} \overset{\text{def}}{=} \Delta R \overset{\cdot}{\longrightarrow} \sigma_n = \Delta \sigma \cdot \Delta R^T, \quad \alpha'_{n+1} \overset{\text{def}}{=} \Delta R \overset{\cdot}{\longrightarrow} \alpha_n, \]

\( \Delta R \) is calculated according to Proposition 6.4 and \( \Delta R \overset{\cdot}{\longrightarrow} \) denotes the associated pushforward. The update, properly adjusted, has to be applied to any tensor-valued material state variable. The complete procedure is in Alg. 1 and incremental objectivity has been proven in [50-49].

**Algorithm 1:** Objective integration of rate equations according to Hughes and Winget [50].

**Input:** geometry \( x_n \), incremental displacements \( u \), stress \( \sigma_n \), and state variables \( \alpha_n \)

**Output:** \( \sigma_{n+1}, \alpha_{n+1}, \) and material tangent tensor \( m \)

1. compute \( f_{n+1} = I + (\nabla_n u)^T \) and \( (\nabla_{n+1/2} u)^T = 2(f_{n+1} - I)(f_{n+1} + I)^{-1} \);
2. obtain midpoint strain increment \( \Delta \tilde{e}_{n+1/2} \) and spin increment \( \Delta \tilde{e}_{n+1/2} \) (Prop. 6.3);
3. compute relative rotation \( \Delta R = (I - \frac{1}{2}\Delta \tilde{e}_{n+1/2})^{-1}(I + \frac{1}{2}\Delta \tilde{e}_{n+1/2}) \);
4. rotate stress and state variables by \( \Delta R \), resulting in \( \sigma'_{n+1} \) and \( \alpha'_{n+1} \), respectively;
5. integrate constitutive equation as for infinitesimal deformations: \( \sigma_{n+1} = \sigma'_{n+1} + \Delta \sigma \), where \( \Delta \sigma = h(\sigma'_{n+1}, \alpha'_{n+1}, \Delta \tilde{e}_{n+1/2}) \);
6. compute material tangent tensor \( m \) if necessary;

**Remark 6.4.** The stress update can be interpreted as follows. The full amount of relative rotation \( \Delta R \) over the time increment \( [t_n, t_{n+1}] \) is applied instantaneously to the stress at time \( t_n, \sigma_n, \) in order to account for rigid body motion. The rotated stress \( \sigma'_{n+1} \), more precisely, the \( \Delta R \)-pushforward of \( \sigma_n \), the rotated state variables \( \alpha'_{n+1} \), etc., are then passed to the procedure that integrates the rate constitutive equation without any rotational terms by the methods outlined in Section 6.1. It is emphasized that no choice of such an integration procedure, e.g. explicit or implicit, is defined by Hughes and Winget’s algorithm. However, in case where the material tangent tensor, \( m(\sigma, \alpha) \), is an isotropic function of its arguments and explicit stress integration is employed, the stress increment can be obtained in closed-form from

\[ \Delta \sigma = h(\sigma'_{n+1}, \alpha'_{n+1}, \Delta \tilde{e}_{n+1/2}) = m(\sigma'_{n+1}, \alpha'_{n+1}) \cdot \Delta \tilde{e}_{n+1/2}. \]

**Remark 6.5.** Alg. 1 places a restriction to the magnitude of \( |\Delta \tilde{e}_{n+1/2}| = |\omega_{n+1/2}|\Delta t \). From the approximation \( \frac{1}{2}||\Delta \tilde{e}_{n+1/2}|| \approx |\omega_{n+1/2}|\Delta t \), it follows that \( \Delta R \) according to Proposition 6.4 is defined only for \( ||\frac{1}{2}\Delta \tilde{e}_{n+1/2}|| < 180^\circ \).

### 6.3 Modified Algorithm

The Hughes-Winget algorithm (Alg. 1) applies the full rotation before the stress is updated. This could deteriorate accuracy if the time step and incremental rotation are considerably large, like in finite element methods that advance the solution implicitly in time [11, sec. 15.1]. The restriction to the magnitude of \( \Delta \tilde{e}_{n+1/2} \) when the rotational update uses \( \Delta R \) according to Proposition 6.4 see Remark 6.5 induces another difficulty. A modified algorithm proposed in [11, sec. 15.1] and refined in [16] performs the rotational updates and stress update in three steps so that the stress update is centered in time at \( t_{n+1/2} \). Beyond that, the restriction with the Hughes-Winget rotation (Prop. 6.4) could be removed if one drops the approximation \( \tan ||\frac{1}{2}\Delta \tilde{e}_{n+1/2}|| \approx ||\frac{1}{2}||\Delta \tilde{e}_{n+1/2}|| \).

As in Section 6.2 let us consider again constitutive rate equations in terms of the corotational Zaremba-Jaumann rate of Cauchy stress, \( \tilde{\sigma}_{n+1} = m(\sigma, \alpha) \cdot d \). The skew-symmetric spin tensor defining the corotational rate is the vorticity \( \omega \) (Definition 2.19), which generates a one-parameter group of orthogonal transformations through an evolution equation \( \mathcal{R} = \omega \cdot \mathcal{R} \) in accordance with Definition 6.4. The exponential map transforms skew-symmetric matrices into orthogonal matrices and provides an appropriate extension to the Hughes-Winget rotation, as shown in detail in [102] sect. 8.3.2:.

**Proposition 6.5.** The appropriate extension to the midpoint rule in Proposition 6.4, used to integrate \( \mathcal{R} = \omega \cdot \mathcal{R} \) subject to \( \mathcal{R}|_{t=0} = I \), results in

\[ \Delta \mathcal{R} = \mathcal{R}_{n+1} \cdot \mathcal{R}_n^T = \exp(\Delta \tilde{e}_{n+1/2}), \]

where \( \Delta \tilde{e}_{n+1/2} \) is the midpoint spin increment according to Proposition 6.3.
Definition 6.11. The half-step relative rotation associated with the midpoint configuration \( \varphi_{n+1/2} = \frac{1}{2}(\varphi_{n+1} + \varphi_n) \) is defined through \( \Delta \mathcal{R}_{1/2} = \exp(\frac{1}{2} \Delta \mathbf{r}_{n+1/2}) \)

Proposition 6.6. \( \mathcal{R}_{n+1} = \exp\left(\frac{1}{2} \Delta \mathbf{r}_{n+1/2} + \frac{1}{2} \Delta \mathbf{r}_{n+1/2}\right) \cdot \mathcal{R}_n = \Delta \mathcal{R}_{1/2} \cdot \mathcal{R}_{n+1/2} \).

\[ \text{Proof.} \] By Proposition 6.5 and Definition 6.6 see also [102 sect. 8.3.2]. \( \square \)

The complete integration procedure, which could be considered as a modification of the Hughes-Winget algorithm (Alg. 1), is summarized in Alg. 2.

Algorithm 2: Objective integration of rate equations (modified algorithm).

Input: geometry \( x_n \), incremental displacements \( u \), stress \( \sigma_n \), and state variables \( \alpha_n \)

Output: \( \sigma_{n+1} \), \( \alpha_{n+1} \), and material tangent tensor \( m \)

1. obtain midpoint strain increment \( \Delta \mathbf{e}_{n+1/2} \) and spin increment \( \Delta \mathbf{r}_{n+1/2} \) (Prop. 6.3);
2. compute half-step relative rotation \( \Delta \mathcal{R}_{1/2} = \exp\left(\frac{1}{2} \Delta \mathbf{r}_{n+1/2}\right) \);
3. rotate stress and state variables by \( \Delta \mathcal{R}_{1/2} \), resulting in \( \sigma_{n+1/2} \) and \( \alpha_{n+1/2} \), respectively;
4. integrate constitutive equation as for infinitesimal deformations: \( \Delta \sigma = h(\sigma_{n+1/2}, \alpha_{n+1/2}, \Delta \mathbf{e}_{n+1/2}) \);
5. compute material tangent tensor \( m \) if necessary;
6. rotate stress and state variables again by \( \Delta \mathcal{R}_{1/2} \), resulting in \( \sigma_{n+1} \) and \( \alpha_{n+1} \), respectively;

Remark 6.6. A suitable parametrization of the exponential map \( \exp(\Delta \mathbf{r}) \), or \( \exp\left(\frac{1}{2} \Delta \mathbf{r}\right) \) offering a straightforward implementation is in terms of quaternion parameters. See [102 Box 8.3] for details.

Remark 6.7. It should be noted that Proposition 6.5 places no restriction on the magnitude of \( \Delta \mathbf{r} \), in contrast to the classical approximation introduced by Hughes and Winget [50] (Proposition 6.4), see Remark 6.5.

Remark 6.8. The extension of the midpoint rule employing the exponential map (Proposition 6.5) is not restricted to the vorticity tensor \( \mathbf{w} \) but applies to any spin tensor generating a group of rotations according to Definition 3.4. For example, \( \mathbf{w} \) could be replaced by \( \mathbf{\Omega} = \mathbf{R} \cdot \mathbf{R}^T \), where \( \mathbf{R} \) is the rotation tensor resulting from the polar decomposition of the deformation gradient.

Remark 6.9. Computation of the components of \( \Delta \mathbf{e}_{n+1/2} \) and \( \Delta \mathbf{r}_{n+1/2} \) in the context of finite element methods is almost straightforward. For example, the midpoint strain increment can be obtained from \( \Delta \mathbf{e}_{n+1/2} = B_{n+1/2} \cdot \mathbf{u} \), where \( B_{n+1/2} = B(x_{n+1/2}) \) is the element strain operator matrix evaluated at the midpoint configuration and \( \mathbf{u} \) is the element incremental displacement vector. The computation of \( \Delta \mathbf{r}_{n+1/2} \) is similar. Note that under plane strain and axisymmetric conditions in the, say \( xy \)-plane, \( \Delta \mathbf{r}_{n+1/2} \) has a single independent component. It can be calculated from \( \Delta \mathbf{r}_{xy} = \mathbf{q}_{n+1/2} \cdot \mathbf{u} \), where \( \mathbf{q}_{n+1/2} = q(x_{n+1/2}) \) is an operator that delivers the component of the skew-symmetrized gradient of the element nodal incremental displacement vector \( \mathbf{u} \). Both \( \mathbf{B} \) and \( \mathbf{q} \) depend on the actual element type and order of interpolation.

6.4 Algorithms Using a Corotated Configuration

The class of algorithms discussed in the following are ideally suited for corotational rate constitutive equations (Definition 6.1). These algorithms go back at least to Nagtegaal and Veldpaus [74] and Hughes [49]. Recall from Section 5.2 that any corotational rate of a spatial second-order tensor involves a spin \( \Lambda = \mathbf{R}^T \). The spin generates a one-parameter group of rotations associated with the initial value problem \( \mathbf{R} = \Lambda \cdot \mathbf{R} \) subject to \( \mathbf{R}_{t=0} = \mathbf{I} \), see Definition 5.3, where \( \mathbf{R} \) is proper orthogonal, i.e. a rotation. The crucial observation that leads to the considered class of algorithms can then be stated as follows.

Proposition 6.7. Let \( \dot{\mathbf{\sigma}}^* = \dot{\mathbf{\sigma}} - \Lambda \cdot \mathbf{\sigma} + \mathbf{\sigma} \cdot \Lambda \) be any corotational rate of Cauchy stress defined by the spin tensor \( \Lambda = \mathbf{R} \cdot \mathbf{R}^T \), then

\[
\dot{\mathbf{\sigma}}^* = h(\mathbf{\sigma}, \alpha, \mathbf{d}) \quad \text{and} \quad \mathbf{R} \frac{\partial}{\partial t} (\mathbf{R} \downarrow \mathbf{\sigma}) = \mathbf{R} \uparrow (\mathbf{R} \downarrow h(\mathbf{\sigma} \downarrow \alpha, \mathbf{R} \downarrow \mathbf{\sigma}, \mathbf{R} \downarrow \mathbf{d}))
\]

are equivalent rate constitutive equations.
where $A$ is then defined by corotated configuration.

Pushforward by $R$ on both sides then yields
\[
(R \downarrow \circ R \downarrow)(h(\sigma, \alpha, d)) = R \downarrow (R \downarrow h(R \downarrow \sigma, R \downarrow \alpha, R \downarrow d))
\]
as desired.

The proposition formalizes how to replace a corotational rate by the usual time derivative. Consequently, a corotational rate constitutive equation can be integrated by transforming all variables to the corotating $R$-system, performing the update of the stress and state variables, and then rotating the updated stress tensor back to the current configuration.

**Definition 6.12.** Let $\dot{\sigma}^* = m(\sigma, \alpha) : d$ be a corotational rate constitutive equation for the Cauchy stress defined by the spin $A = R \cdot R^T$ (see also Sect. 6.2). A general class of objective algorithms based on a corotated configuration is then defined by
\[
\sigma_{n+1} \overset{\text{def}}{=} R_{n+1} : (S_n + \Delta S_{n+\theta}) \cdot R^T_{n+1},
\]
where $\theta \in [0,1]$,
\[
S_n = R_n \downarrow \sigma_n = R^T_n \cdot \sigma_n \cdot R_n, \quad \Delta S_{n+\theta} = h_{n+\theta}(S_{n+\theta}, A_{n+\theta}, \Delta E_{n+\theta}),
\]
and $R \overset{\text{def}}{=} R \downarrow \alpha$. The stress increment $\Delta S_{n+\theta}$ is evaluated at some rotation-neutralized intermediate configuration specified by the actual integration algorithm using $\theta \in [0,1]$ and an associated rotation $R_{n+\theta}$. This evaluation is denoted conceptually, but without loss of generality, by the response function $h_{n+\theta}$ representing an explicit or implicit stress integration method (cf. Sect. 6.1). The tensor $\Delta E_{n+\theta}$ is called the algorithmic corotated finite strain increment, and $\Delta E_{n+\theta}$ (Definition 6.9) is regarded as given. The algorithm is incrementally objective provided that $R_n$, $R_{n+\theta}$, and $R_{n+1}$ are properly determined:

Case (i): $A = \Omega$, $R = R$. The rate constitutive equation is formulated in terms of the Green-Naghdi stress rate, that is, $\dot{\sigma}^* = \dot{\sigma}^{GN}$. Recall that $R$ is the rotation tensor resulting from the polar decomposition of the deformation gradient, and $\Omega = R \cdot R^T$.

Case (ii): $A = \omega$, $R \neq R$. The rate constitutive equation is formulated in terms of the Zaremba-Jaumann stress rate, that is, $\dot{\sigma}^* = \dot{\sigma}^{ZJ}$, requiring proper integration of $R = \omega \cdot R$ (cf. Sect. 6.2).

**Remark 6.10.** The above algorithm employs the generalized midpoint rule (Remark 6.1), that is, $S_{n+1} = S_n + \Delta S_{n+\theta}$, to emphasize that the general procedure is not affected by the choice of $\theta \in [0,1]$. If, for example, an explicit stress integration procedure ($\theta = 0$) is applied to the rate constitutive equation in the $R$-system, then the stress increment can be calculated in closed-form:
\[
\Delta S_n = (R \downarrow m)(S_n, A_n) : \Delta E_n.
\]

**Remark 6.11.** The most obvious procedure to determine $R_{n+1}$ and $R_{n+\theta}$ in case (i) of Definition 6.12 is the polar decomposition of the total deformation gradients $F_{n+1}$ and $F_{n+\theta}$, respectively. Alternative procedures that circumvent polar decomposition have been proposed by Flanagan and Taylor [134] and Simo and Hughes [102, sec. 8.3.2]. An algorithmic approximations to the vorticity tensor $\omega$ in case of the Zaremba-Jaumann rate (case (ii) in Definition 6.12) is provided by Proposition 6.11(iii). The rotation group can be approximately integrated, for example, by using the general algorithm of Simo and Hughes [102, sec. 8.3.2], or by the particular procedure of Hughes and Winget [59]; see also algorithm of Hughes [19] below.

The widely-used algorithm of Hughes [19] can be obtained from the general objective integration algorithm in Definition 6.12 by making particular approximations to the orthogonal group of rotations and by using time-centering, i.e. $\theta = \frac{1}{2}$, in the calculation of the algorithmic finite strain increment. Time-centering is employed in accordance with the incrementally objective algorithm developed by Hughes and Winget [59]; see Sect. 8.2. Hughes [49] originally uses the example of von Mises plasticity (cf. Example 4.1) and carried out implicit time integration in the corotated configuration. However, Definition 6.12 generally places no restrictions on the actual integration procedure (explicit, implicit, or generalized midpoint rule).
**Definition 6.13.** The corotated midpoint strain increment is defined by

\[
\Delta \mathbf{e}_{n+1/2}^\text{def} = \mathbf{R}_n^T \cdot \Delta \tilde{\mathbf{e}}_{n+1/2} \cdot \mathbf{R}_{n+1/2},
\]

where \(\Delta \tilde{\mathbf{e}}_{n+1/2} \equiv \frac{1}{2}((\nabla n_{+1/2} u) + \nabla n_{+1/2} u)^T \) is the second-order accurate midpoint strain increment (cf. Proposition 6.5 and Remark 6.7).

**Remark 6.12.** Since \(\Delta \tilde{\mathbf{e}}_{n+1/2} = d_{n+1/2} \Delta t\) by Definition 6.11 an algorithmic approximation to the corotated rate of deformation tensor is thus given by

\[
\mathbf{D}_{n+1/2} = \mathbf{R}_n^T \cdot d_{n+1/2} \cdot \mathbf{R}_{n+1/2}.
\]

Summation of \(\Delta \mathbf{e}_{n+1/2} = \mathbf{D}_{n+1/2} \Delta t\) over a time interval \([t_0, t_{n+1}]\) gives an excellent approximation to the Lagrangian logarithmic strain \(49\). Therefore, an algorithmic approximation to the Eulerian logarithmic strain (Definition 2.16) can be obtained by applying Proposition 2.7(iii), leading to

\[
\ln V_{n+1} = \mathbf{R}_{n+1} \cdot \left( \mathbf{e}_0 + \sum_n \Delta \mathbf{e}_{n+1/2} \right) \cdot \mathbf{R}_{n+1}^T,
\]

where \(\mathbf{e}_0\) is given.

The rotations \(\mathbf{R}_{n+1}\) and \(\mathbf{R}_{n+1/2}\) need to be determined in order to complete the algorithm of Definition 6.12. In case of \(\mathbf{R} = \mathbf{R}\), or equivalently, \(\hat{\sigma}^* = \hat{\sigma}^\text{GN}\), Hughes [49] suggests polar decomposition of the total deformation gradients \(\mathbf{F}_{n+1} = \mathbf{f}_{n+1} \cdot \mathbf{F}_n\) and \(\mathbf{F}_{n+1/2} = \frac{1}{2}(\mathbf{F}_{n+1} + \mathbf{F}_n)\) in order to determine \(\mathbf{R}_{n+1}\) and \(\mathbf{R}_{n+1/2}\), respectively. In case of \(\hat{\sigma}^* = \hat{\sigma}^\text{ZJ}\), where \(\mathbf{R} \neq \mathbf{R}\), the rotation and half-step rotation are defined through

\[
\mathbf{R}_{n+1} = \Delta \mathbf{R} \cdot \mathbf{R}_n \quad \text{and} \quad \mathbf{R}_{n+1/2} = \Delta \mathbf{R}^{1/2} \cdot \mathbf{R}_n,
\]

where \(\Delta \mathbf{R}\) is the time-centered approximation to the incremental rotation according to Hughes and Winget [50]; see Proposition 6.3. For computation of the proper orthogonal square root \(\Delta \mathbf{R}^{1/2}\) the reader is referred to the literature, e.g., [35, 47, 49, 116]. The complete integration procedure using the midpoint strain increment is summarized in Alg. 3.

**Algorithm 3:** Objective integration of rate equations according to Hughes [49].

**Input:** geometry \(x_n\), incremental displacements \(u\), stress \(\sigma_n\), state variables \(\alpha_n\), and rotation \(\mathbf{R}_n\).

**Output:** \(\sigma_{n+1}, \alpha_{n+1}\), and material tangent tensor \(m\).

1. compute \(\mathbf{f}_{n+1} = \mathbf{I} + (\nabla n u)^T\) and \((\nabla n_{+1/2} u)^T = 2(\mathbf{f}_{n+1} - \mathbf{I})(\mathbf{f}_{n+1} + \mathbf{I})^{-1};\)
2. obtain midpoint strain increment \(\Delta \mathbf{e}_{n+1/2}\) and rotation increment \(\Delta \tilde{\mathbf{f}}_{n+1/2}\) (Prop. 6.3);
3. switch corotational rate \(\hat{\sigma}^*\) do
   4. case Green-Naghdi rate \(\hat{\sigma}^\text{GN}\) (\(\mathbf{R} = \mathbf{R}\)) do
      5. compute \(\mathbf{f}_{n+1} = \mathbf{f}_{n+1} \cdot \mathbf{F}_n\) and \(\mathbf{F}_{n+1/2} = \frac{1}{2}(\mathbf{F}_{n+1} + \mathbf{F}_n);\)
      6. perform polar decomposition to obtain \(\mathbf{R}_{n+1}\) and \(\mathbf{R}_{n+1/2};\)
   7. case Zaremba-Jaumann rate \(\hat{\sigma}^\text{ZJ}\) (\(\mathbf{R} \neq \mathbf{R}\)) do
      8. compute \(\Delta \mathbf{R} = (\mathbf{I} - \frac{1}{2} \Delta \tilde{\mathbf{f}}_{n+1/2})^{-1}(\mathbf{I} + \frac{1}{2} \Delta \tilde{\mathbf{f}}_{n+1/2})\) and \(\Delta \mathbf{R}^{1/2};\)
      9. update \(\mathbf{R}_{n+1} = \Delta \mathbf{R} \cdot \mathbf{R}_n\) and \(\mathbf{R}_{n+1/2} = \Delta \mathbf{R}^{1/2} \cdot \mathbf{R}_n;\)
10. corotate midpoint strain increment: \(\Delta \mathbf{e}_{n+1/2} = \mathbf{R}_{n+1/2} \cdot \Delta \tilde{\mathbf{e}}_{n+1/2} \cdot \mathbf{R}_{n+1/2};\)
11. corotated state and stress variables by \(\mathbf{R}_{n+1}\), resulting in \(\mathbf{e}_n\) and \(\mathbf{a}_n\), respectively;
12. integrate constitutive equation as for infinitesimal deformations: \(\mathbf{e}_{n+1} = \mathbf{e}_n + \Delta \mathbf{e},\) where \(\Delta \mathbf{e} = h(\mathbf{e}_n, \mathbf{a}_n, \mathbf{m}, \Delta \mathbf{e}_{n+1/2});\)
13. compute material tangent tensor if necessary;
14. back-rotate updated stress to the current configuration: \(\sigma_{n+1} = \mathbf{R}_{n+1} \cdot \mathbf{e}_{n+1} \cdot \mathbf{R}_{n+1}^T;\)
15. back-correct updated state variables and material tangent tensor to the current configuration;

**Remark 6.13.** According to the basic Definition 6.8 the deformation gradient of the motion is updated by \(\mathbf{F}_{n+1} = \mathbf{f}_{n+1} \cdot \mathbf{F}_n\), where \(\mathbf{F}_n = \mathbf{R}_n \cdot \mathbf{U}_n\), and \(\mathbf{R}_n\) is proper orthogonal. Now, suppose that the current configuration at time \(t_n\) is taken as the reference configuration, i.e. \(\mathbf{B} = \varphi_n(\mathbf{B})\), and no data is available of configurations prior to \(t_n\) such that \(\mathbf{F}_n = \mathbf{R}_n = \mathbf{R}_n = \mathbf{I}\) and \(\mathbf{U}_n = \mathbf{I}\). Then, by Remark 6.2 one has...
Cauchy-Green tensors are given by $\omega \in B$. Remark 6.4. If $\Delta R$, in case of $A = \omega$, is determined from Proposition 6.4 then the same restriction is placed to the magnitude of $\|\omega_{n+1/2}\|\Delta t$ as in Alg. 1; see Remark 6.5.

Remark 6.5. The algorithms based on a corotated description go beyond the one of Hughes and Winget [50] outlined in Sect. 6.2. The difference is that the algorithm of Hughes and Winget [50] rotates the stress and state variables to the current configuration before passing it to the constitutive equation, whereas the algorithms using a corotated configuration use rotation-neutralized variables for calculation of the stress rate. However, all these algorithms satisfy the requirement of incremental objectivity provided that the rotation tensors are properly determined.

Remark 6.6. The main advantage of the investigated algorithms of [50, 49] is that the integration of the constitutive rate equation can be carried out by the same methods at both infinitesimal and finite deformations. That is, the objective algorithms comply with the usual small-strain algorithms if deformations are infinitesimal. From a computational viewpoint this is very attractive, because the same material model subroutine can be employed for both cases without changes. The “rotational” part of the stress update, then, is done outside the subroutine. The algorithms, however, rely heavily on the use of corotational rate constitutive equations. If the desired constitutive rate equation is based on a non-corotational rate, like the Truesdell and Oldroyd rates (Definition 3.2), then additional terms need to be handled.

7 Application: Hypoelastic Simple Shear

Hypoelastic simple shear is an excellent problem to analyze fundamental relations in nonlinear continuum mechanics and to test implementations of objective time integration algorithms for rate equations. This is because material deformations due to simple shear include both finite strains and finite rotations; it is in fact a compound action of pure shear and pure rotation. Several papers are concerned with analytical solutions of simple shear, mostly in connection with a discussion of objective stress rates for constitutive rate equations [22, 20, 27, 51, 52, 53]. They also serve as references for the numerical solution. Notations and definitions of the previous sections are used throughout.

7.1 Analytical Solution

Definition 7.1. Let $B \subset S = \mathbb{R}^3$ be the initial configuration of a material body in the Euclidian space, $X \in B$ the initial location of a material particle, and $\varphi : B \times [0, T] \to S$ the motion of the body. Let $\{Z^1, Z^2, Z^3\}_X \overset{\text{def}}{=} \{Z^A\}_X$ and $\{z^1, z^2, z^3\}_3 \overset{\text{def}}{=} \{z^a\}_3$ respectively denote the coordinate tuples of $X$ and $x = \varphi(X, t) \in S$ with respect to an ortho-normalized frame in $S = \mathbb{R}^3$. Simple shear then prescribes a planar parallel motion of the form

$$z^a \overset{\text{def}}{=} \varphi^a(Z^A, t)$$

where the $\varphi^a, a \in \{1, 2, 3\}$, are respectively defined through

$$\varphi^1 \overset{\text{def}}{=} Z^1 + k(t)Z^2, \quad \varphi^2 \overset{\text{def}}{=} Z^2, \quad \varphi^3 \overset{\text{def}}{=} Z^3,$$

and $k(t) \in \mathbb{R}$ with initial condition $k(0) = 0$. The problem statement is depicted in Fig. 1.

Through the definition of an ortho-normalized frame of reference, every second-order tensor can be represented by a $(3 \times 3)$-matrix of its components with respect to that frame. In particular, the deformation gradient takes the form

$$F = \left( \frac{\partial \varphi^a}{\partial Z^A} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I + \nabla u,$$

where $J = \det F = 1$ (zero volume change) and $\nabla u$ is the displacement gradient. The right and left Cauchy-Green tensors are given by

$$C = U^T U = \begin{pmatrix} 1 & k & 0 \\ k & 1 + k^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = V V^T = \begin{pmatrix} 1 + k^2 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

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\[
\beta = k/2 \quad \kappa/2 \quad Z_2, \ z_2
\]

\[
\Psi_1, \Psi_2, \Psi_3
\]

\[
U = RU
\]

undeformed

deformed

Figure 1: Simple shear and schematic diagram of associated right polar decomposition \( F = RU \).

respectively. Here \( U \) denotes the right stretch tensor and \( V \) is the left stretch tensor, which can be obtained from the right and left polar decompositions \( F = RU \) and \( F = VR \), respectively. \( R \) is the proper orthogonal rotation, which is often referred to in the literature as the material rotation.

To solve for \( R \) and \( U \), let \( \Psi_1, \Psi_2, \Psi_3 \) be the ortho-normalized eigenvectors, and \( \lambda_1, \lambda_2, \lambda_3 \) the eigenvalues of \( C \). The eigenvalues are all real-valued and positive, because \( C \) is symmetric and positive definite. Define the matrices

\[
A^2 \defeq \begin{pmatrix}
\lambda_1^2 & 0 & 0 \\
0 & \lambda_2^2 & 0 \\
0 & 0 & \lambda_3^2
\end{pmatrix} \quad \text{and} \quad \Psi \defeq (\Psi_1, \Psi_2, \Psi_3),
\]

so that \( A^2 = \Psi^T C \Psi \) is the principal axis transformation of \( C \) in \( \{Z^A\} \). From this, one obtains the component matrix \( U = \Psi A \Psi^T \) of the right stretch, and finally, \( R = FU^{-1} \); concerning the subtleties with this matrix notation, the reader is referred to Remark 2.7.

The right polar decomposition \( F = RU \) describes a stretch \( U \) of the material body in the direction of the principal axes \( \Psi_\alpha \), followed by a rotation \( R \) (Fig. 1). Within the left polar decomposition \( F = VR \), the body is first rotated, and then stretched by \( V \) in the direction of the rotated principal axes \( \psi_\alpha = R \Psi_\alpha \). Since

\[
V = RU R^T = (R \Psi) A (\Psi R)^T = \psi A \psi^T,
\]

where \( \psi \defeq (\psi_1, \psi_2, \psi_3) \), the stretches \( U \) and \( V \) have the same eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \), called the principal stretches; hence \( b \) has the same eigenvalues as \( C \). Like before, the three principal stretches are real-valued and positive. Having the principal stretches, one is able to determine the Lagrangian logarithmic strain \( \varepsilon = \ln U \) and Eulerian logarithmic strain \( \varepsilon = \ln V \) (Definition 2.16), which play an important role in nonlinear continuum mechanics. Recall from Proposition 2.7(iii) that both are related by

\[
\varepsilon = R^T e R
\]

using matrix notation.

The particular eigenvalue problem \( C \Psi_\alpha = \lambda^2 \Psi_\alpha \) associated with simple shear results in the characteristic polynomial

\[
0 = \det \begin{pmatrix}
1 - \lambda^2 & k & 0 \\
k & 1 + k^2 - \lambda^2 & 0 \\
0 & 0 & 1 - \lambda^2
\end{pmatrix}
= (1 - \lambda^2)(1 + k^2 - \lambda^2)(1 - \lambda^2) - (1 - \lambda^2)k^2.
\]
It immediately follows $\sqrt{\lambda^2} = \lambda_3 = 1$, that is, the eigenvector $\psi_3$ is equal to the basis vector in $Z^3$-direction. In the remaining two dimensions, the characteristic polynomial reduces to $(1 - \lambda^2)(1 + k^2 - \lambda^2) - k^2 = 0$ resp. $\lambda^2 + \lambda^2 - 2 = 2 + k^2$. Hence, the other two eigenvalues can be obtained from

$$(\lambda^2)_{1/2} = \frac{2 + k^2}{2} \pm \sqrt{\left(\frac{2 + k^2}{2}\right)^2 - 1},$$

and they are related by $\lambda_2 = \lambda_1^{-1}$. This yields

$$\lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} \sqrt{1 - \sin \beta} \sqrt{2} \\ \sqrt{1 + \sin \beta} \sqrt{2} \\ \sqrt{1 - \sin \beta} \sqrt{2} \end{pmatrix},$$

and finally

$$U = \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ \sin \beta & \frac{1 + \sin^2 \beta}{\cos \beta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ -\sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

in which $\beta(t)$ has been defined through $k(t) = 2 \tan \beta(t)$.

The spatial velocity gradient $l = d + \omega$ is readily available from

$$l = \dot{F} F^{-1} = \begin{pmatrix} 0 & \dot{k} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that the spatial rate of deformation and the vorticity take the form

$$d = \begin{pmatrix} 0 & \dot{k}/2 & 0 \\ \dot{k}/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 0 & \dot{k}/2 & 0 \\ -\dot{k}/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively. Moreover, spatial rate of rotation is given by

$$\Omega = \dot{R} R^{-1} = \begin{pmatrix} 0 & \dot{\beta} & 0 \\ -\dot{\beta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
Non-dimensional stress $\sigma_{12}/G$ [-]

Strain $k/2$ [-]

Figure 2: Comparison of the shear stress in hypoelastic simple shear using the Zaremba-Jaumann and Green-Naghdi stress rates.

In both cases, $\sigma^{33} = 0$ holds. The functions for the shear stress component $\sigma^{12} = \sigma^{21}$ are plotted in Fig. 2. It can be seen that shear stress increases monotonically when using the Green-Naghdi stress rate. However, the Zaremba-Jaumann stress rate results in unphysical harmonic oscillation of the stress when applied to hypoelasticity of grade zero.

7.2 Numerical Solution

Let $k = 0.4$ ($\beta = 0.1974$) and $G = 5000 \text{kN m}^{-2}$, then the formulas above yield

$$
A = \begin{pmatrix}
1.2198 & 0 & 0 \\
0 & 0.8198 & 0 \\
0 & 0 & 1
\end{pmatrix},
\Psi = \begin{pmatrix}
0.633399 & -0.77334 & 0 \\
0.77334 & 0.633399 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

$$
U = \begin{pmatrix}
0.98058 & 0.19612 & 0 \\
0.19612 & 1.05903 & 0 \\
0 & 0 & 1
\end{pmatrix},
R = \begin{pmatrix}
0.98058 & 0.19612 & 0 \\
-0.19612 & 0.98058 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

$$
\varepsilon = \begin{pmatrix}
-0.03897 & 0.19483 & 0 \\
0.19483 & 0.03897 & 0 \\
0 & 0 & 0
\end{pmatrix},
\varepsilon = \begin{pmatrix}
0.03897 & 0.19483 & 0 \\
0.19483 & -0.03897 & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

Note that $\text{tr} \varepsilon = \text{tr} \varepsilon = 0$, that is, logarithmic strain is consistent with isochoric response in simple shear. Moreover, using the Zaremba-Jaumann stress rate results in

$$
\sigma^{11} = -\sigma^{22} = 394.7 \text{kN m}^{-2} \quad \text{and} \quad \sigma^{12} = 1947.1 \text{kN m}^{-2},
$$

and for the Green-Naghdi stress rate,

$$
\sigma^{11} = -\sigma^{22} = 387.2 \text{kN m}^{-2} \quad \text{and} \quad \sigma^{12} = 1948.9 \text{kN m}^{-2}.
$$

While both stress rates are approximately equal at $k = 0.4$, they significantly differ at $k = 1.0$ (cf. Fig. 2). For the Zaremba-Jaumann stress rate,

$$
\sigma^{11} = -\sigma^{22} = 2298.5 \text{kN m}^{-2} \quad \text{and} \quad \sigma^{12} = 4207.4 \text{kN m}^{-2},
$$

and for the Green-Naghdi stress rate,

$$
\sigma^{11} = -\sigma^{22} = 2079.5 \text{kN m}^{-2} \quad \text{and} \quad \sigma^{12} = 4348.9 \text{kN m}^{-2}.
$$

Two series of numerical simulations have been carried out using implementations of the objective integration algorithm of Hughes [19] (Alg. 3) outlined in Sect. 6.4. One series employed the Zaremba-Jaumann stress

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Relative error \[%\] and the other the Green-Naghdi stress rate. Here we used the fact that, by Remark 6.13 in conjunction with Remark 3.2, the Green-Naghdi rate reduces to the Zaremba-Jaumann rate if the current configuration is taken as the reference configuration. Clearly, for the calculations employing the Zaremba-Jaumann rate the total deformation gradient in each calculational cycle was set equal to the incremental deformation gradient (Definition 6.6).

In each calculation the maximum shear strain applied was $k = 1.0 \ (\beta = \pi/4)$, but the number of substeps to reach the maximum was continuously increased respectively the size of the applied strain increments was continuously decreased. Fig. 3 shows that the relative error between the numerically calculated stress and the exact solutions presented above is reduced with increasing number of substeps.

8 Conclusions

We have presented basic notions of rate equations in nonlinear continuum mechanics by placing emphasis on the geometrical background. The application of these notions to second-order tensors has led to a clear distinction between the properties their rates may possess under different transformations: objective, covariant, and corotational. Objectivity in constitutive theory has been formalized by the basic principle of constitutive frame invariance, which is intended as a substitute to the classical principle of material frame-indifference. We have then discussed classes of objective and corotational rate constitutive equations as well as rate forms of virtual power which form a basis for numerical methods solving large deformation initial boundary value problems. Concerning the numerical integration of rate constitutive equations in time, the focus has been on classical formulations using the Green-Naghdi and Zaremba-Jaumann corotational stress rates as well as on incrementally objective integration algorithms employed by several finite element codes. Finally, simple shear of hypoelastic material at finite deformations has been considered as an example application of both the fundamental relations and the numerical algorithms. The analytical and numerical results presented may also be used for the verification of future developments.

The focus of this paper has been on well-established aspects of nonlinear continuum mechanics and finite element methods, but some recent developments have also been referenced. It can be concluded that rate constitutive equations and their integration in time are still active research areas, even for the simplest hypoelastic case. One reason is the fact that the implications of the geometrical approach set forth by Marsden and Hughes [66] and continued by others, e.g., [92, 95, 53], are yet not fully understood. They indicate, however, that the classical, nowadays standard formulations and numerical algorithms need to be revised.

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A Differential Geometry

This appendix summarizes some basic notions of differential geometry essential for the main text. Differential geometry has been found to be the most natural way in formulating continuum mechanics, and the choice of objective rates in finite elastoplasticity: General results on the uniqueness of the logarithmic rate. Proceedings of the Royal Society of London. Series A, 456:1865–1882, 2000.

A.1 Manifolds

A topological space $M$ referred to as a Hausdorff space that carries the information of relations or interconnections between the points. A topological space is familiar with linear algebra and calculus in linear spaces.

Definition A.2. A manifold $M$ is called a metric space over all possible values of a coordinate index variable whenever it appears twice, and as both a subscript and Einstein summation convention is forced in the present paper. By this convention, the sum is taken including the homeomorphism $\phi: \mathcal{M} \rightarrow \mathcal{N}$, where $n = \dim(M)$. The tuple $(x^i)_X$ is called the coordinates of $X$ in the chart $(U, \beta)$. An atlas of $\mathcal{M} \equiv \bigcup_{i \in \mathcal{I}} U_i$ is a collection $\mathfrak{A}(\mathcal{M}) \equiv \{ (U_i, \beta_i) \}_{i \in \mathcal{I}}$ of a finite number of charts that covers $\mathcal{M}$.

Definition A.3. Let $U(X) \subset M$ be an open neighborhood of the point $X \in M$, then the pair $(U, \beta)$ including the homeomorphism

$$\beta : \mathcal{M} \supset U \rightarrow X \subset \mathbb{R}^n$$

$$X \mapsto \beta(X) = \{ x^1, x^2, \ldots, x^n \}_X \equiv \{ x^i \}_X, \exists \beta^{-1},$$

is called a chart or local coordinate system on $\mathcal{M}$, where $n = \dim(M)$. The tuple $(x^i)_X$ is called the coordinates of $X$ in the chart $(U, \beta)$. An atlas of $\mathcal{M} \equiv \bigcup_{i \in \mathcal{I}} U_i$ is a collection $\mathfrak{A}(\mathcal{M}) \equiv \{ (U_i, \beta_i) \}_{i \in \mathcal{I}}$ of a finite number of charts that covers $\mathcal{M}$.

Definition A.4. A chart transition or change of coordinates is a composition

$$\beta' \circ \beta^{-1} : \beta' (U \cap U') \rightarrow \beta (U \cap U'),$$
in which \((\mathcal{U}, \beta), (\mathcal{U}', \beta')\) are charts on \(\mathcal{M}\), and \(\mathcal{U} \cap \mathcal{U}' \neq \emptyset\). An atlas is called \textit{differentiable}, if for every two charts the chart transition is differentiable.

**Definition A.5.** A \textit{differentiable manifold} is a Hausdorff space with differentiable atlas.

**Definition A.6.** Let \(\varphi : \mathcal{M} \to \mathcal{N}\) be continuous, \(\mathcal{U}(X) \subset \mathcal{M}\) and \(\mathcal{V}(x) \subset \mathcal{N}\) neighborhoods of \(X \in \mathcal{M}\) and \(x \in \mathcal{N}\), respectively, and let \((\mathcal{U}, \beta), (\mathcal{V}, \sigma)\) be charts. Then for non-empty \(\varphi^{-1}(\mathcal{V}) \cap \mathcal{U}\), then the \textit{localization} of \(\varphi\),

\[
\sigma \circ \varphi \circ \beta^{-1}|_{\varphi^{-1}(\mathcal{V}) \cap \mathcal{U}} : \beta^{-1}(\mathcal{V}) \cap \mathcal{U} \to \sigma(\mathcal{V} \cap \varphi(\mathcal{U})),
\]

describes the chart transition concerning \(\varphi\) with respect to \(\beta\) and \(\sigma\). The map \(\varphi\) is called \textit{differentiable at} \(X \in \varphi^{-1}(\mathcal{V}) \cap \mathcal{U}\), if its localization is differentiable at \(\beta(X)\). A bijective differentiable map \(\varphi\) is referred to as a \textit{diffeomorphism}, if both \(\varphi\) and \(\varphi^{-1}\) are continuous differentiable.

**Remark A.1.** In this paper we simply assume that every chart transition is a diffeomorphism. If \(x^i\) are the coordinate functions of \((\mathcal{V}, \sigma)\) and \(X^j\) are those of \((\mathcal{U}, \beta)\), then it would be convenient to define

\[
\varphi^i \overset{\text{def}}{=} x^i \circ \varphi \circ \beta^{-1} \quad \text{resp.} \quad \varphi'(X^j) \overset{\text{def}}{=} (x^i \circ \varphi \circ \beta^{-1})(X^j).
\]

**Definition A.7.** A map \(\varphi : \mathcal{M} \to \mathcal{N}\) is called an \textit{embedding}, if \(\varphi(\mathcal{M}) \subset \mathcal{N}\) is a submanifold in \(\mathcal{N}\) and \(\mathcal{M} \to \varphi(\mathcal{M})\) is a diffeomorphism.

**Definition A.8.** Let \(\mathcal{M}\) be an \(n\)-dimensional differentiable manifold, \(\mathcal{U} \subset \mathcal{M}\) a subset, and \((\mathcal{U}, \beta)\) a chart with coordinate functions \(\beta(X) = \{x^i\}_{X}\) for every \(X \in \mathcal{U}\). The \textit{tangent space} \(T_X \mathcal{M}\) at \(X\) is a local vector space spanned by the vectors of the \textit{holonomic basis} \(\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\} \overset{\text{def}}{=} \{\frac{\partial}{\partial x^i}\}_X\). Conceptually,

\[
T_X \mathcal{M} \overset{\text{def}}{=} \{X\} \times \mathcal{V}_n.
\]

The disjoint union \(T \mathcal{M} \overset{\text{def}}{=} \bigcup_{X \in \mathcal{M}} T_X \mathcal{M}\) of all tangent spaces at all points of the manifold is called the \textit{tangent bundle} of \(\mathcal{M}\). An element \((X, w) \in T \mathcal{M}\), called a \textit{tangent vector}, will often be denoted by \(w_X\), or just \(w\) if the base point \(X\) is clear from the context.

**Proposition A.1.** For the previous situation, the coordinate differentials \(\{dx^1, \ldots, dx^n\}_X \overset{\text{def}}{=} \{dx^i\}_X\) form a dual basis at \(X\).

**Proof.** By \(dx^i(X) \cdot \frac{\partial}{\partial x^j}(X) = \frac{\partial x^i}{\partial x^j}(X) = \delta^i_j\), where \(\delta^i_j\) is the Kronecker delta.

**Definition A.9.** The co-vector space dual to the tangent space \(T_X \mathcal{M}\) is called the \textit{cotangent space} \(T^*_X \mathcal{M} \overset{\text{def}}{=} \{X\} \times \mathcal{V}_n^*\), and elements of \(T^*_X \mathcal{M}\) are called \textit{differential 1-forms}, or just \textit{1-forms}. The union \(T^* \mathcal{M} \overset{\text{def}}{=} \bigcup_{X \in \mathcal{M}} T^*_X \mathcal{M}\) is referred to as the \textit{cotangent bundle} of \(\mathcal{M}\).

### A.2 Tensors and Tensor Fields

**Definition A.10.** A \(\overset{\text{\(p\)}}{\omega}\)-\textit{tensor} \(T(X)\) at point \(X\) of a differentiable manifold \(\mathcal{M}\) is a multilinear mapping

\[
T(X) : T_X^\mathcal{M} \times \ldots \times T_X^\mathcal{M} \times T_X^\mathcal{M} \times \ldots \times T_X^\mathcal{M} \to \mathbb{R}.
\]

The space of all \(\overset{\text{\(p\)}}{\omega}\)-tensors at all points \(X \in \mathcal{M}\) is denoted by \(T^\mathcal{M}\). If \(\mathcal{N}\) is another differentiable manifold, a \(\overset{\text{\(p\)}}{\omega} \overset{\text{\(q\)}}{\gamma}\)-\textit{two-point tensor over a map} \(\varphi : \mathcal{M} \to \mathcal{N}\) is a multilinear mapping

\[
T(X) : T_{\varphi(X)}^\mathcal{N} \times \ldots \times T_{\varphi(X)}^\mathcal{N} \times T_{\varphi(X')}^\mathcal{N} \times \ldots \times T_{\varphi(X')}^\mathcal{N} \to \mathbb{R}.
\]
Definition A.11. Let \((U, \beta)\), where \(U \subset M\), be a local chart on \(M\) such that \(\{ \frac{\partial}{\partial x^i} \}\) \(\in T_X M\) is a local basis at \(X \in U\), and \(\{dx^i\} \in T^*_X M\) is its dual. The components of a \((p,q)\)-tensor in the chart \((U, \beta)\) are then defined through
\[
T^{i_1\ldots i_p}_{j_1\ldots j_q} \overset{\text{def}}{=} T \left( dx^{i_1}, \ldots, dx^{i_p}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_q}} \right).
\]
Based on index placements, \(T\) is said to be contravariant of order \(p\) and covariant of order \(q\).

Proposition A.2. Under a chart transition with Jacobian matrix \(\frac{\partial x'}{\partial x}\) and its inverse \(\frac{\partial x}{\partial x'}\) the components of a \((p,q)\)-tensor transform according to the rule
\[
T'^{i_1'\ldots i_p'}_{j_1'\ldots j_q'} = \frac{\partial x'}{\partial x} T \frac{\partial x}{\partial x'} \left( dx'^{i_1}, \ldots, dx'^{i_p}, \frac{\partial}{\partial x'^{j_1}}, \ldots, \frac{\partial}{\partial x'^{j_q}} \right) = T^{i_1\ldots i_p}_{j_1\ldots j_q}.
\]
Proof. The assertion follows by multilinearity of a tensor and the transformation properties of \(dx^i\) and \(\frac{\partial}{\partial x^i}\).

Remark A.2. From the definition of a tensor it should be clear that every 1-form \(\alpha = a_i dx^i\) is a \((1,0)\)-tensor, and every vector \(v = v^i \frac{\partial}{\partial x^i}\) is a \((0,1)\)-tensor since \(\alpha(v) = a_i v^i \in \mathbb{R}\).

Definition A.12. The operation \(\alpha' \cdot v \overset{\text{def}}{=} \alpha \cdot v\) is called the (single) contraction of the tensors \(\alpha\) and \(v\). In a local chart \(X\) with coordinate functions \(\{x^i\}_X\), one has
\[
\alpha' \cdot v = (a_i dx^i) \cdot \left( v^j \frac{\partial}{\partial x^j} \right) = a_i v^j \left( dx^i \cdot \frac{\partial}{\partial x^j} \right) = a_i v^j s^j = a_i v^i.
\]

Dependence on the point \(X\) being understood. In general, the contraction two tensors \(T\) and \(S\) in the \(i\)-th covariant slot of \(T\) and the \(j\)-th contravariant slot of \(S\) is defined as if the covariant slot is a 1-form and the contravariant slot is a vector. If the slots are not specified, and \(T^a_{bcd}\) and \(S_{ijkl}\) are the components of \(T\) and \(S\), respectively, then the (single) contraction \(T \cdot S\) simply means \(T^a_{bcd} S_{ijkl}\) in components. The double contraction condenses the last two slots of \(T\) and \(S\):

\[
T : S, \quad \text{in components,} \quad T^a_{bcd} S_{ijkl}.
\]
Moreover, the contraction of a \((1,1)\)-tensor \(T\) is called its trace, written \(\text{tr}T = T^i_i\).

Definition A.13. Let \(T \in T^p_q (M)\) and \(S \in T^r_s (M)\) at a point \(X \in M\), then the tensor product \(T \otimes S\) is the \((p+r,q+s)\)-tensor defined by
\[
(T \otimes S) (a^1, \ldots, a^p, v_1, \ldots, v_q, b^1, \ldots, b^r, w_1, \ldots, w_s)
\overset{\text{def}}{=} T (a^1, \ldots, a^p, v_1, \ldots, v_q) S (b^1, \ldots, b^r, w_1, \ldots, w_s),
\]
where \(v_1, \ldots, v_q, w_1, \ldots, w_s \in T_X M\) and \(a^1, \ldots, a^p, b^1, \ldots, b^r \in T^*_X M\).

Proposition A.3. A \((p,q)\)-tensor \(T\) has the local representative
\[
T (X) = T^{i_1\ldots i_p}_{j_1\ldots j_q} (X) \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_q}.
\]
and \(T^p_q (M) \overset{\text{def}}{=} T^{p\text{--fold}}_q M \otimes \ldots \otimes T^{q\text{--fold}}_p M \otimes T^* M \otimes \ldots \otimes T^* M\).

Proof. By Definitions A.11 and A.13.

Definition A.14. A Riemannian manifold is the pair \((M,g)\), where \(M\) is a differentiable manifold and \(g\) is a metric. If \(U \subset M\) is a subset and \((U, \beta)\) a chart with coordinate functions \(\beta(X) = \{x^i\}_X\) for every \(X \in U\), then the metric can be locally represented by
\[
g (X) \overset{\text{def}}{=} g_{ij} (X) dx^i \otimes dx^j, \quad \text{where} \quad g_{ij} (X) \overset{\text{def}}{=} \left( \frac{\partial}{\partial x^i} \cdot \frac{\partial}{\partial x^j} \right)_X \geq 0
\]
are the metric coefficients at every point \(X \in M\) and \(\langle \cdot, \cdot \rangle\) is the inner product associated with the metric \(g\) on \(M\).
Definition A.15. Contracting the metric tensor with the inverse metric \( g^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \), where \( g_{ik} g^{kj} = \delta^j_i \), gives the second-order identity tensor on \( \mathcal{M} \),

\[
I_M \overset{\text{def}}{=} g \cdot g^{-1} = \delta^j_i \, dx^i \otimes \frac{\partial}{\partial x^j} \, = \, dx^i \otimes \frac{\partial}{\partial x^i}.
\]

The fourth-order symmetric identity tensor or symmetrizer \( I_M \), with components

\[
1_{ij}^{kl} \overset{\text{def}}{=} \frac{1}{2} \left( \delta^k_i \delta^l_j + \delta^k_j \delta^l_i \right),
\]
yields the symmetric part \( \text{Sym}(T) \overset{\text{def}}{=} 1_M : T \) of a second-order tensor \( T \in T^1_1(\mathcal{M}) \).

Definition A.16. Let \( S \in T^q_p(\mathcal{M}) \), then \( S^\flat \in T^0_{p+q}(\mathcal{M}) \) is the associated tensor with all indices lowered, and \( S^\sharp \in T^p_{p+q}(\mathcal{M}) \) is the associated tensor with all indices raised. Here \( ^\flat \) is called the index lowering operator, and \( ^\sharp \) is the index raising operator.

Definition A.17. The Frobenius norm of \( T \in T^1_1(\mathcal{M}) \) is defined through \( \|T\| \overset{\text{def}}{=} \sqrt{\text{tr}(T^2)} \), where \( T^\sharp \overset{\text{def}}{=} T^\phi \cdot T^\flat \in T^1_1(\mathcal{M}) \) is the squared tensor \( T \).

Remark A.3. Tensor indices can be raised by the inverse metric coefficients, and lowered by the metric coefficients. For example, let \( T = T_i^j \frac{\partial}{\partial x^i} \otimes dx^j \in T^1_1(\mathcal{M}) \) in a given chart, then the associated tensors are

\[
T^b = g \cdot T = g_{ik} T^k_j \, dx^i \otimes dx^j \quad \text{and} \quad T^d = T \cdot g^{-1} = T^i_k g^{kj} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^k}.
\]

Remark A.4. Note that \( g \overset{\text{def}}{=} g^\flat = (I_M)^\flat \) and \( g^{-1} \overset{\text{def}}{=} g^\sharp = (I_M)^\sharp \). Moreover, the trace of \( T \in T^1_1(\mathcal{M}) \) can be written \( \text{tr}T = T^\flat : g \).

Definition A.18. Let \( \mathcal{M}, \mathcal{N} \) be Riemannian manifolds and \( T(X) : T^1_1(\mathcal{M}) \to T^1_1(\mathcal{N}) \) a general two-point tensor over a diffeomorphism \( \varphi : \mathcal{M} \to \mathcal{N} \). Moreover, let \( U \in T^1_1(\mathcal{M}) \) and \( v \in T^1_1(\mathcal{N}) \) be vectors on \( \mathcal{M} \) and \( \mathcal{N} \), respectively, with \( X \in \mathcal{M} \) and \( x = \varphi(X) \in \mathcal{N} \), then the transpose of \( T \) is the linear map \( T^T(x) : T^1_1(\mathcal{N}) \to T^1_1(\mathcal{M}) \) defined through

\[
\langle v, T(U) \rangle_x \overset{\text{def}}{=} \langle T^T(v), U \rangle_X.
\]

For \( \varphi = \text{Id} \) resp. \( \mathcal{N} = \mathcal{M} \) the transpose of an ordinary (one-point) tensor is obtained.

Proposition A.4. The components of \( T^T \) are given by

\[
(T^T)^i_j(x) = g_{ij}(x) T^j_l(\varphi^{-1}(x)) G^{IJ}(\varphi^{-1}(x))
\]

with respect to local bases \( \{ \frac{\partial}{\partial x^i} \} \in T^1_1(\mathcal{M}) \) and \( \{ \frac{\partial}{\partial x^j} \} \in T^1_1(\mathcal{N}) \), where \( g_{ij}(x) \) are the metric coefficients on \( \mathcal{N} \) and \( G^{IJ}(X) \) are the inverse metric coefficients on \( \mathcal{M} \).

Proof. By the definitions of the transpose, metric and inverse metric; see [66, 4] for details. \( \square \)

Definition A.19. With \( T, U, v \) be as before, the operations

\[
T^{-1} \cdot T(U) = U \quad \text{and} \quad T^{-T} \cdot T^T(v) = v
\]

involve the inverse \( T^{-1}(X) \) and the inverse transpose \( T^{-T}(x) \). Moreover, a two-point tensor \( T(X) : T^1_1(\mathcal{M}) \to T^1_1(\mathcal{N}) \) is called orthogonal provided that \( T^T \cdot T = I_M \) and \( T \cdot T^T = I_N \). If \( T \) is orthogonal and the determinant \( \det T = +1 \), then \( T \) is called proper orthogonal.

Definition A.20. On a three-dimensional Riemannian manifold, let \( \lambda_k \in \mathbb{R}, k \in \{1, 2, 3\} \), denote the eigenvalues of a second-order tensor \( T \). Given a \( 3 \times 3 \)-matrix representation of \( T \), the Cayley-Hamilton theorem states that this matrix satisfies its own characteristic polynomial

\[
\det(T - \lambda_k I) = \lambda_k^3 - I_1(T) \lambda_k^2 + I_2(T) \lambda_k - I_3(T) = 0.
\]

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The coefficients \( I_1(T), I_2(T), I_3(T) \) are called the \textit{principal invariants} of the tensor, with

\[
I_1(T) \overset{\text{def}}{=} \text{tr} T = \lambda_1 + \lambda_2 + \lambda_3 ,
\]
\[
I_2(T) \overset{\text{def}}{=} \text{det} T \text{tr}(T^{-1}) = \frac{1}{2}((\text{tr} T)^2 - \text{tr}(T^2)) ,
\]
\[
I_3(T) \overset{\text{def}}{=} \text{det} T = \lambda_1\lambda_2\lambda_3 .
\]

In what follows, \( M \) and \( N \) are differentiable manifolds, \( \varphi : M \to N \) is a diffeomorphism, \( (U, \beta) \) and \( (V, \sigma) \) are charts of \( U \subset M \) and \( V \subset N \), respectively, and \( \varphi^i(X^I) \overset{\text{def}}{=} (x^i \circ \varphi \circ \beta^{-1})(X^I) \) are the coordinates \( x^i \) on \( N \) arising from the coordinates \( X^I \) on \( U \) via localization of \( \varphi \).

**Definition A.21.** The tangent bundle of \( M \) has been denoted \( TM \). In a more rigorous definition, it is the triple \( (T \mathcal{M}, \tau \mathcal{M}, \mathcal{M}) \) including the projection \( \tau \mathcal{M} : T \mathcal{M} \to \mathcal{M} \). At \( X \in \mathcal{M} \), with \( \dim(\mathcal{M}) = n \), the tangent space \( \tau^{-1} \mathcal{M}(X) = T_X \mathcal{M} = \{X \} \times \mathcal{V}_u \) is called \textit{fibre over} \( X \), and \( \mathcal{V}_u \) is the fibre space. If \( (TN, \tau_N, \mathcal{N}) \) is another tangent bundle, then a continuous map \( \varphi : M \to N \) induces the bundle \((\varphi^*TN, \tau_{\mathcal{N}}^N, \mathcal{M})\) with \( \tau_{\mathcal{N}}^N : \varphi^*TN \to \mathcal{M} \). The restriction of \( \varphi^*TN \) to \( x = \varphi(X) \in \mathcal{N} \) is the tangent space \( T_{\varphi(X)}\mathcal{N} \).

**Definition A.22.** A \textit{vector field} \( \mathbf{v} \) on \( M \) is identified with the \textit{tangent bundle section}

\[
\nu : M \to TM ,
\]

with \( \tau_M(\mathbf{v}(X)) = X, \forall X \in M \). A \textit{1-form field} \( \kappa : \mathcal{M}_M \to \mathcal{M}^{*}\mathcal{M} \) is a section of the cotangent bundle: \( a^* : M \to \mathcal{T}\mathcal{M} \). The sets of all sections of \( T\mathcal{M} \) and \( \mathcal{T}\mathcal{M} \) are denoted by \( \Gamma(T\mathcal{M}) \) and \( \Gamma(\mathcal{T}\mathcal{M}) \), respectively. As well, if some manifold \( \mathcal{N} \) has the tangent bundle \( T\mathcal{N} \), \( \mathbf{u} \in \Gamma(T\mathcal{N}) \) is a vector field, and \( \varphi : M \to \mathcal{N} \) is continuous, then the related \textit{vector field over} \( \varphi \) is the \textit{induced section} \( \nu \circ \varphi : M \to T\mathcal{N} \) defined through \( (\nu \circ \varphi)(X) \overset{\text{def}}{=} \nu(\varphi(X)) \).

**Definition A.23.** The \textit{local basis sections} of \( T\mathcal{M} \) restricted to \( U \),

\[
\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\} : U \to T\mathcal{M}|_U
\]
define a local basis for all \( X \in \mathcal{M} \), and \( \left\{ dx^1, \ldots, dx^n \right\} : U \to \mathcal{T}\mathcal{M}|_U \) are their duals. Hence, for every fibre \( \tau^{-1}_\mathcal{M}(X) \) at \( X \in U \), \( \nu(X) = \nu(X) \frac{\partial}{\partial x^i}(X) \) and \( a^*(X) = a_i(X) dx^i(X) \), respectively. For the fields to be continuously differentiable, the mappings \( x^i \to v^i(x^i) \) and \( x^i \to a_j(x^i) \) on \( \beta(U) \subset \mathbb{R}^n \) are required to be continuously differentiable.

**Remark A.5.** One may construct tensor fields of any order by fibrewise tensor-multiplication of vector and 1-form fields. For example, if \( \mathbf{w} \in \Gamma(T\mathcal{M}) \) and \( \mathbf{b}^* \in \Gamma(\mathcal{T}\mathcal{M}) \), a \( (1^1) \)-tensor field \( T \) would be

\[
T \overset{\text{def}}{=} (\mathbf{w} \otimes \mathbf{b}^*) \in \Gamma(T\mathcal{M} \otimes \mathcal{T}\mathcal{M}) ,
\]

and \( (\mathbf{w} \otimes \mathbf{b}^*)(X) \overset{\text{def}}{=} \mathbf{w}(X) \otimes \mathbf{b}^*(X) \). Thus \( T \) is a section of the \( (1^1) \)-tensor bundle \( T_1^1(\mathcal{M}) = T\mathcal{M} \otimes \mathcal{T}\mathcal{M} \to \mathcal{M} \). Two-point tensor fields over maps \( \varphi : M \to \mathcal{N} \) are defined analogously by taking into account the sections induced by \( \varphi \).

**Definition A.24.** The \( (p^q) \)-tensor bundle of \( M \) is denoted by \( T^p_q(M), \tau_M, \mathcal{M} \), or just \( T^p_q(M) \), and the set of all sections of it is denoted by \( \mathcal{T}^p_q(\mathcal{M}) \overset{\text{def}}{=} \Gamma(T^p_q(\mathcal{M})) \).

**Definition A.25.** The \textit{index lowering} and \textit{index raising} operators for tensors carry over to tensor fields by defining the so-called \textit{musical isomorphisms} \( ^{\flat} : T\mathcal{M} \to \mathcal{T}\mathcal{M} \) and \( ^{\sharp} : \mathcal{T}\mathcal{M} \to T\mathcal{M} \), respectively.

### A.3 Pushforward and Pullback

**Definition A.26.** The \textit{tangent map} and \textit{cotangent map} over \( \varphi \) are defined through

\[
T\varphi : T\mathcal{M} \to T\mathcal{N} , \quad \frac{\partial}{\partial X^I} \mapsto \frac{\partial}{\partial X^i} \frac{\partial}{\partial x^i}
\]
\[
\mathcal{T}\varphi : T^*\mathcal{N} \to T^*\mathcal{M} , \quad dx^i \mapsto \frac{\partial}{\partial X^i} dx^i ,
\]

respectively.
Definition A.30. An diffeomorphism \( \varphi : M \to N \) between Riemannian manifolds \((M, G)\) and \((N, g)\) is called an \textit{isometry} if
\[
g = \varphi \uparrow G, \quad \text{or equivalently,} \quad G(U, V) = g(\varphi \uparrow U, \varphi \uparrow V),
\]
for \( U, V \in \Gamma(TM) \).
Proposition A.6. The tangent map of an isometry \( \varphi: M \to N \) is orthogonal, and proper orthogonal, with \( \det(T\varphi) = +1 \), if \( \varphi \) is also orientation-preserving.

Proof. Let \( G \) be the metric on \( M \), and \( g \) the metric on \( N \). Then, by Definition A.27 and the definition of an isometry,

\[
(U, V)_X = (T\varphi \cdot U, T\varphi \cdot V)|_{x=\varphi(x)}
\]

for every \( X \in M \). On the other hand, Definition A.18 of the transpose yields

\[
(T\varphi)^T \cdot (T\varphi \cdot U, T\varphi \cdot V)_X = \langle (T\varphi)^T \cdot (T\varphi \cdot U), V \rangle_X
\]

Comparison of both equations shows that

\[
(T\varphi)^T \cdot T\varphi = I_M,
\]

that is, \( (T\varphi)^{-1} = (T\varphi)^T \) at every \( X \in M \).

Proofing the second assertion requires the notion of orientation, which is briefly introduced below. \( \square \)

A.4 Tensor Analysis

Definition A.31. Let \( v, w \in \Gamma(TN) \) be vector fields on a Riemannian manifold \( N \), and \( v \) continuously differentiable. In a chart \((V, \sigma)\) on \( N \) with coordinates \( x^I \), the covariant derivative of \( v \) is the proper \((1)\)-tensor field defined through

\[
\nabla v(x) = \nabla_j v^i(x) \, dx^j \otimes \frac{\partial}{\partial x^i} \equiv \left( \frac{\partial v^i}{\partial x^j} + v^k \gamma^i_{k \, j} \right)(x) \, dx^j \otimes \frac{\partial}{\partial x^i},
\]

and the covariant derivative of \( v \) along \( w \) is the proper vector field defined through

\[
\nabla_w v(x) \equiv w(x) \cdot \nabla v(x) \equiv \left( \frac{\partial v^i}{\partial x^j} w^j + v^k w^j \gamma^i_{k \, j} \right)(x) \frac{\partial}{\partial x^i}.
\]

If \( a^i \in \Gamma(T^*N) \) is continuously differentiable, then \( \nabla a^i \in \mathfrak{T}^0_2(N) \), defined by

\[
\nabla a^i(x) \equiv \left( \frac{\partial a_i}{\partial x^j} - a_k \gamma^i_{k \, j} \right)(x) \, dx^i \otimes dx^j,
\]

is a proper \((0)\)-tensor field. The term “proper” is meant in the sense that the components of the covariant derivative transform under chart transitions according to the tensorial transformation rule (Proposition A.2). In particular, under a chart transition such that \( x^I \mapsto x'^I \), the connection coefficients transform according to

\[
\gamma^i_{k \, j} \rightarrow \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^j}{\partial x^l} \gamma^l_{k \, i} + \frac{\partial x'^j}{\partial x^l} \frac{\partial^2 x^m}{\partial x^k \partial x'^l},
\]

with \( i, j, k, l, m, m' \in \{1, \ldots, n_{\text{dim}}\} \).

Definition A.32. The operator \( \nabla: \Gamma(TN) \times \Gamma(TN) \rightarrow \Gamma(TN) \) introduced in Definition A.31 is referred to as the connection on \( N \), and \( \gamma^i_{k \, j} \) are the connection coefficients. A connection \( \nabla \) is called torsion-free if \( \gamma^i_{k \, j} = \gamma^i_{j \, k} \). In case of a Riemannian manifold the connection coefficients are called Christoffel symbols of the second kind.

Theorem A.1. Let \( N \) be a Riemannian manifold, and \( g_{ij} \) and \( g^{ij} \) be the coefficients of the metric and inverse metric, respectively, then there is a unique and torsion-free connection whose coefficients are given by

\[
\gamma^i_{k \, j} \equiv \frac{1}{2} g^{kl} \left( \frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^l} - \frac{\partial g_{kl}}{\partial x^j} \right).
\]

Proof. Detailed derivations can be found, for example, in [19, 66, 110]. \( \square \)

Proposition A.7. Let \( N \) be a Riemannian manifold with connection \( \nabla \) and metric \( g \), then

\[
\nabla g = 0.
\]
Proposition A.8. On a Riemannian manifold $\mathcal{N}$, $\nabla(u^s) = (\nabla u)^s$ for any $u \in \Gamma(T\mathcal{N})$.

**Proof.** This follows from a straightforward calculation by using Proposition A.7.

**Definition A.33.** The *divergence* of a tensor field $t \in \mathcal{T}_0^p(\mathcal{N})$ is defined as the contraction of its covariant derivative $\nabla t$ on the last contravariant leg. For examples, if $t$ is a $(3\!\!1)$-tensor field, then

$$(\text{div } t)^i_{jk} \equiv \nabla_k t^{ijk}.$$

**Proposition A.9.** Let $\mathcal{N}$ be a Riemannian manifold with metric coefficients $g_{ij}$ in a positively oriented chart and $v \in \Gamma(T\mathcal{N})$ a vector field, then

$$\text{div } v = \nabla_i v^i = \frac{1}{\sqrt{|\det g_{kl}|}} \frac{\partial}{\partial x^i} \left( \sqrt{|\det g_{kl}|} v^i \right).$$

**Proof.** By Definition A.31 and Proposition A.33, see [3] for details.

Weak or variational formulations of balance of momentum (principle of virtual power) often employ the product rule for the divergence, which will be derived here for a particular case.

**Proposition A.10.** Let $v \in \Gamma(T\mathcal{N})$ and $T \in \mathcal{T}_1^1(\mathcal{N})$, then the following product rule holds:

$$\text{div}(T \cdot v) = \text{div } T \cdot v + T : \nabla v.$$

**Proof.** By choosing a local chart on $\mathcal{N}$ with coordinates $x^i$ and making use of the properties of the Kronecker delta, Definitions A.31 and A.33 yield

$$\text{div}(T \cdot v) = \nabla_i (T^i_j v^j) = \nabla_i (T^i_j v^j) v^j \delta^i_1 + T^k_j (\nabla_i v^j) \delta^i_k \delta^j_1$$

$$= (\nabla_i T^i_j) v^j \frac{\partial}{\partial x^i} + T^k_j (\nabla_i v^j) \frac{\partial}{\partial x^k} \otimes dx^j : (dx^i \otimes \frac{\partial}{\partial x^i})$$

$$= (\nabla_i T^i_j) dx^j \cdot (v^j \frac{\partial}{\partial x^i}) + (T^k_j \frac{\partial}{\partial x^k} \otimes dx^j) : (\nabla_i v^j dx^i \otimes \frac{\partial}{\partial x^i})$$

$$= \text{div } T \cdot v + T : \nabla v.$$

**Definition A.34.** The evolution in time of a differentiable manifold $\mathcal{N}$ is described by a mapping $\psi_{t,s} : \mathcal{N} \rightarrow \mathcal{N}$, where $t, s$ are points in a time interval $\mathcal{I} \subset \mathbb{R}$. The mapping $\psi_{t,s}$ is called a *time-dependent flow* on $\mathcal{N}$ provided that

$$\psi_{t,s} \circ \psi_{s,r} = \psi_{t,r} \quad \text{and} \quad \psi_{t,t} = \text{id}_\mathcal{N}.$$

**Remark A.10.** If $X_s = \psi_{t,s}(X_s) \in \mathcal{N}$ is the starting point at starting time $t = s$, then $X = c(t) = \psi_{t,s}(X_s) = \psi(X_s, s, t)$ is the point at $t = t$, for $s, t$ fixed. Hence, $c : \mathcal{I} \rightarrow \mathcal{N}$ is a curve on $\mathcal{N}$ with the initial condition $c(s) = X_s$. The flow $\psi_{t,s}$ is closely connected to a time-dependent vector field $u : \mathcal{N} \times \mathcal{I} \rightarrow TN$ through $u(\psi_{t,s}(X_s), t) = \dot{c}(t)$, with $c(s) = X_t$. Conversely, $c(t)$ is the unique integral curve of $u$ starting at $X_s$ at time $t = s$; thus $u$ generates the flow, so $\psi_{t,s}$ needs not to be given explicitly.
Figure 4: Lie derivative of a time-independent vector field $\mathbf{v}$ along a time-dependent vector field $\mathbf{u}$; reprint from [4, fig. 3.5]

**Definition A.35.** The Lie derivative of a time-dependent tensor field $T_t \in \mathfrak{T}^p_q(N)$ along a time-dependent vector field $u_t \in \Gamma(TN)$ is defined by

$$L_u T \overset{\text{def}}{=} \lim_{\Delta t \to 0} \frac{\psi_{t,s} \downarrow T_t - T_s}{\Delta t} = \frac{d}{dt} \psi_{t,s} \downarrow T_t \bigg|_{t=s} ,$$

where $\Delta t \overset{\text{def}}{=} t - s$, and $\psi_{t,s} \downarrow$ denotes the pullback concerning the flow $\psi_{t,s}$ associated with $u_t$. Therefore, the Lie derivative approximately answers the question how a tensor field $T$ changes under some flow. The so-called autonomous Lie derivative is obtained by holding $t$ fixed in $T_t$, that is,

$$\mathcal{L}_u T \overset{\text{def}}{=} \lim_{\Delta t \to 0} \frac{\psi_{t,s}^* T_s - \psi_{s,s}^* T_s}{\Delta t} = \frac{d}{dt} \psi_{t,s}^* T_s \bigg|_{t=s} = L_u T - \frac{\partial T}{\partial t} .$$

If $T$ is time-independent, $L_u T \equiv \mathcal{L}_u T$. Fig. 4 illustrates the concept.

**Proposition A.11.** The Lie derivative of the tensor field $T_t$ along the vector field $u_t$ is obtained by pulling back the tensor field according to the flow associated with $u_t$ at some starting time, performing the common time derivative, and then pushing forward the result using the inverse of the pullback. Clearly,

$$L_u T \overset{\text{def}}{=} \psi_{t,s} \uparrow \frac{d}{dt}(\psi_{t,s} \downarrow T_t) .$$

**Proof.** We refer to [4] and [1, sect. 5.4] for a detailed discussion.

**Proposition A.12.** Let $\varphi : M \to N$ be a diffeomorphism, $u_t, v_t \in \Gamma(TN)$, and $T \in \mathfrak{T}^p_q(N)$, then

$$\mathcal{L}_{u_t+v} T = \mathcal{L}_u T + \mathcal{L}_v T \quad \text{and} \quad \varphi \downarrow (\mathcal{L}_u T) = \mathcal{L}_{(\varphi \circ u)}(\varphi \downarrow T) .$$

**Proof.** See, for example, [6].

**Proposition A.13.** Let $N$ be a Riemannian manifold, and $u_t \in \Gamma(TN)$ be a time-dependent vector field. In a chart $(V, \sigma)$ on $N$ with coordinates $x^i$, the components of the Lie derivative of a time-dependent tensor field $T_t \in \mathfrak{T}^i_j(M)$ along $u_t$ are computed from

$$(L_u T)^i_j = \frac{\partial T^i_j}{\partial t} + u^k \frac{\partial T^i_j}{\partial x^k} - T^k_j \frac{\partial u^i}{\partial x^k} + T^i_k \frac{\partial u^k}{\partial x^j} .$$

If $N$ has a torsion-free connection $\nabla$, then

$$(L_u T)^i_j = \frac{\partial T^i_j}{\partial t} + u^k \nabla_k T^i_j - \nabla_k T^j_k u^i + \nabla_j T^i_k u^k .$$

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Proof. We refer again to [66] for detailed proof.

Remark A.11. As pushforward and pullback do not commute with index raising and lowering, the Lie derivative also does not commute with these operations in general, that is, for example, $L_u(T^\sigma) \neq (L_uT)^\sigma$.

From now on, we let $\mathcal{M}$ and $\mathcal{N}$ be Riemannian manifolds with some orientation, $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ be an orientation-preserving diffeomorphism, $(\mathcal{U}, \beta)$ be a positively oriented chart of $\mathcal{U} \subset \mathcal{M}$ with respect to the orientation of $\mathcal{M}$, and $(\mathcal{V}, \sigma)$ be a positively oriented chart of $\mathcal{V} \subset \mathcal{N}$, with non-empty $\varphi^{-1}(\mathcal{V}) \cap \mathcal{U} \subset \mathcal{M}$. Furthermore, let $\varphi'(X^I) \equiv (x^i \circ \varphi \circ \beta^{-1})(X^I)$ be the coordinates $x^i$ on $\mathcal{N}$ arising from the coordinates $X^I$ on $\mathcal{U}$ via localization of $\varphi$.

Remark A.12. A precise definition of orientation requires exterior calculus, which is probably one of the most exotic fields of modern differential geometry, at least from an engineer’s point of view. The interested reader is referred to the literature suggested at the beginning of this appendix.

Definition A.36. Let the tuple $(w_1, \ldots, w_n)$ of vector fields $w_1, \ldots, w_n \in \Gamma(T\mathcal{N})$ be positively oriented with respect to the orientation of $\mathcal{N}$, then the volume density $dw$ on $\mathcal{N}$ is defined through

$$dw(w_1, \ldots, w_n) \equiv \sqrt{\det(w_1, w_2)} \cdots \sqrt{\det(w_1, w_n)},$$

where $\det(w_1, w_2)$ is the determinant of the matrix $(W_{ij})$ whose elements are given by the inner products $W_{ij} \equiv \langle w_i, w_j \rangle$.

Remark A.13. Note that $dw(e_1, \ldots, e_n) = 1$ for a positively oriented ortho-normalized basis $\{e_1, \ldots, e_n\}$ in $T\mathcal{N}$. In ordinary $\mathbb{R}^3$, the volume of the parallelepiped spanned by the three vectors $w_1, w_2, w_3 \in \mathbb{R}^3$ is given by $V(w_1, w_2, w_3) \equiv \sqrt{\det(w_1, w_2)}$ provided that $w_1, w_2$, and $w_3$ are positively oriented.

Proposition A.14. Let $dV$ and $dw$ be the volume densities on $\mathcal{M}$ and $\mathcal{N}$, respectively, then

$$\varphi^* dw = dw \circ \varphi = J_\varphi dV,$$

where

$$J_\varphi(X) = \det \left( \frac{\partial \varphi^i}{\partial X^j} \right) \frac{\sqrt{\det g_{ij}(\varphi(X))}}{\sqrt{\det g^{ij}(X)}}$$

using local coordinates.

Proof. The proof is most easily obtained using local representatives of $dV$ and $dw$; cf. [466].

Definition A.37. The proper scalar field $J_\varphi : \mathcal{M} \rightarrow \mathbb{R}$ introduced by Proposition A.14 is called the Jacobian of $\varphi$ or relative volume change with respect to $dV$ and $dw$. Since $\varphi$ was assumed orientation-preserving, $J_\varphi > 0$.

Proposition A.15. (See [466] for a proof.) $L_u dw = (\text{div } u) dw$.

Theorem A.2 (Change of Variables). Let $f : \varphi(\mathcal{M}) \rightarrow \mathbb{R}$, then

$$\int_{\varphi(\mathcal{M})} f dw = \int_{\mathcal{M}} \varphi^* (f dw) = \int_{\mathcal{M}} (f \circ \varphi) J_\varphi dV.$$

Proof. This is well-known from the analysis of real functions. The last expression is a consequence of Proposition A.14.

The following relations, including the divergence theorem, play a fundamental role in both differential geometry and continuum mechanics. A full derivation is beyond the scope of this paper and can be found elsewhere, e.g. [464][66].

Definition A.38. Let $\mathcal{N}$ be an oriented $n$-dimensional manifold with compatible oriented boundary $\partial \mathcal{N}$ such that the normals to $\partial \mathcal{N}$, $n^i \in \Gamma(T^*\mathcal{N})$, point outwards. The area density $da \equiv dw|_{\partial \mathcal{N}}$ is the volume density on $(n-1)$-dimensional $\partial \mathcal{N}$ induced by the volume density $dw$ on $\mathcal{N}$. Conceptually, we write $dw = n^i \wedge da$ to emphasize that $dw$ and $da$ are linked by the outward normals.
Theorem A.3 (Divergence Theorem). Let $w \in \Gamma(TN)$ be a vector field, then for the situation defined above,

$$\int_{\mathcal{N}} (\text{div } w) \, dv = \int_{\partial \mathcal{N}} w \cdot n^* \, da.$$ 

Proposition A.16. Let $\mathcal{M}$ be oriented and $\varphi : \mathcal{M} \to \mathcal{N}$ an orientation preserving diffeomorphism with tangent $F = T\varphi$. Let $dA$ and $da$ be the volume forms on $\partial \mathcal{M}$ and $\partial (\varphi(\mathcal{M}))$, respectively. Then $da = dA \circ \varphi^{-1}$ if and only if the outward normals on $\partial (\varphi(\mathcal{M}))$ and $\partial \mathcal{M}$ are related by

$$n^* = \varphi_* (J\varphi N^*) = (J\varphi \circ \varphi^{-1}) F^{-T}. (N^* \circ \varphi^{-1}),$$

where $n^* \in \Gamma(T^*\mathcal{N})$, $N^* \in \Gamma(T^*\mathcal{M})$, and $J\varphi$ is the Jacobian of $\varphi$.

Definition A.39. Let $\mathcal{M}$, $\varphi$, etc., be as before, then the Piola transform of a spatial vector field $y \in \Gamma(TN)$ is the vector field on $\mathcal{M}$ given by

$$Y \overset{\text{def}}{=} J\varphi \varphi^{-1} y = J\varphi F^{-1} \cdot (y \circ \varphi) \in \Gamma(TM).$$

Theorem A.4 (Piola Identity). If $Y$ is the Piola transform of $y$, then the divergence operators $\text{DIV}$ on $\mathcal{M}$ and $\text{div}$ on $\mathcal{N}$ are related by

$$\text{DIV } Y = (\text{div } y \circ \varphi) J\varphi.$$

Proposition A.17.

$$\int_{\partial \varphi(\mathcal{M})} y \cdot n^* \, da = \int_{\partial \mathcal{M}} Y \cdot N^* \, dA \quad \text{resp.} \quad (y \cdot n^* \, da) \circ \varphi = Y \cdot N^* \, dA.$$ 

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