SPHERICAL AFFINE CONES IN EXCEPTIONAL CASES
AND RELATED BRANCHING RULES

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Abstract. Given a complex simply connected simple algebraic group $G$ of exceptional type and a maximal parabolic subgroup $P \subset G$, we classify all triples $(G, P, H)$ such that $H \subset G$ is a maximal reductive subgroup acting spherically on $G/P$. In addition we derive branching rules for $\text{res}_P^G(V_{k\omega_i}), k \in \mathbb{N}$, where $\omega_i$ is the fundamental weight associated to $P$.

This is the first of two parts of a project to classify all such triples and corresponding branching rules for all simply connected simple algebraic groups.

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1. Introduction

Given a reductive algebraic group $G$, a reductive subgroup $H$ and some irreducible $G$-module $V$, then $V$ is also a $H$-module in a natural way. An obvious problem is to find branching rules that describe the decomposition of the $H$-module $V$ into irreducible components.

We will deal with this problem in the situation where $G$ is a complex simply connected simple algebraic group of exceptional type. The subgroup structure of these groups has been studied in great detail and we want to consider maximal reductive subgroups of $G$. The maximal closed connected subgroups are listed in Theorem 1 of [Sei91]. These groups are either semisimple or parabolic. So the maximal reductive subgroups are easily obtained by adding the Levi factors of the maximal parabolic groups which are maximal reductive in $G$ to the list of maximal semisimple subgroups. The modules $V$ that we consider are those having as highest weights a multiple of a fundamental weight.
We will approach this problem by working with spherical varieties. We consider the flag variety $G/P$ where $P$ is a maximal parabolic subgroup of $G$. Of special interest to us are the flag varieties of that form, that are $H$-spherical, i.e. they contain an open orbit for a Borel subgroup of $H$. The property of being spherical can also be described in a representation-theoretic way. Namely a normal affine $G$-variety is spherical if and only if its coordinate ring is a multiplicity-free $G$-module \[ VK78 \]. Let $\tilde{Y}$ denote the affine cone over $G/P$. Then the flag variety is $H$-spherical if and only if all restrictions of the homogeneous components of the coordinate ring of $\tilde{Y}$ to $H$ are multiplicity-free. These homogeneous components are exactly the irreducible submodules of the coordinate ring $C[\tilde{Y}]$ and they are of shape $V_{k\omega_i}^\ast$. In the case of sphericity we can derive branching rules for these modules.

So the content of this paper is twofold. We classify the spherical $H$-varieties $G/P$ and furthermore we derive branching rules for the simple $G$-submodules of the coordinate ring of the affine cones in the spherical cases. The results are summarized in Table 1. A flag variety $G/P$ is $H$-spherical if and only if the branching rules for the corresponding modules $V$ are given in the table.

### 2. Notation

We work over the field of complex numbers throughout the article. $G$ always denotes a simply connected simple algebraic group of exceptional type. Within $G$ we choose a Borel subgroup $B$, a maximal torus $T$ and thereby define a set $\{\alpha_1, \ldots, \alpha_r\}$ of simple roots which are labeled according to Bourbaki-notation. The system of roots of $G$ is denoted by $\Phi$, the system of positive roots of $G$ is denoted by $\Phi^+$ and $(a_1, \ldots, a_r)$ stands for the root $\sum_{i=1}^r a_i \alpha_i$. Further $X_\alpha$ denotes a non-trivial element of the root space associated to $\alpha$. Let $\Lambda^+$ be the set of dominant weights related to $B$ and $T$.

The irreducible $G$-module of highest weight $\lambda \in \Lambda^+$ is denoted by $V_{\lambda}$. The fundamental weights of $G$ are $\omega_1, \ldots, \omega_r$ and $\omega_1^\ast, \ldots, \omega_r^\ast$ are the fundamental weights such that $(V_{\omega_i})^\ast = V_{\omega_i^\ast}$, where $(V_{\omega_i})^\ast$ is the dual of $V_{\omega_i}$. If we write $k\omega_i$, then $k \in \mathbb{N}$.

Let $H$ denote a reductive subgroup of $G$ with root system $\Phi_H$ and analogous to $G$ we use the notation $(b_1, \ldots, b_s)_H := \sum_{i=1}^s b_i \beta_i$ where $\{\beta_1, \ldots, \beta_s\}$ is a set of simple roots of $\Phi_H$ given by the Borel subgroup $B_H = B \cap H$. The fundamental weights of $H$ are denoted by $\lambda_1, \ldots, \lambda_s$, if $H$ is semisimple. When $H$ is a Levi subgroup, $\lambda_1, \ldots, \lambda_s$ denote the fundamental weights of the semisimple part of $H$.

Lastly $\mathfrak{b}$ denotes the Lie algebra of $B_L$, $\mathfrak{u}$ the Lie algebra of $U_L$ the unipotent radical of $B_L$ and $\mathfrak{h}$ the Lie algebra of the maximal torus $T$ of $B_L$.

### 3. Main results and outline of proof

We will now summarize the results and give an outline of the proof. In this paper we will derive the branching rules stated in the following table. Further we show that if $\text{res}^G_H(V_{k\omega_i})$ is given in the table, then $G/P_{\omega_i^\ast}$ is a spherical $H$-variety. Conversely, if a maximal reductive subgroup $H \subset G$ does not appear in the table, then the varieties $G/P_{\omega_i}$ are not $H$-spherical.
Note that for the subgroups $D_5 \times \mathbb{C}^* \subset E_6$ and $E_6 \times \mathbb{C}^* \subset E_7$ the weight of the $\mathbb{C}^*$-action depends on the embedding of $\mathbb{C}^*$. The embedding that we chose is given in the corresponding sections.

**Table 1**

| $G$   | $H$   | $\omega$ | $\text{res}_G^H(V_\omega)$                                                                 |
|-------|-------|----------|------------------------------------------------------------------------------------------|
| $G_2$ | $A_2$ | $k\omega_1$ | \begin{align*} V_{a_1\lambda_1+a_2\lambda_2} \quad & \text{if } a_1+a_2 \leq k \\ V_{(a_1+a_3)\lambda_1+(a_2+a_3)\lambda_2} \quad & \text{if } a_1+a_2+a_3 = k \end{align*} |
|       |       | $k\omega_2$ | \begin{align*} V_{a_1\lambda_1+a_2\lambda_2} \quad & \text{if } a_1 \geq k \\ V_{(a_1+a_3)\lambda_1+(a_2+a_3)\lambda_2} \quad & \text{if } a_1+a_2+a_3 = k \end{align*} |
| $F_4$ | $B_4$ | $k\omega_1$ | $V_{a_1\lambda_1+a_2\lambda_4}$                                                                |
|       |       | $k\omega_2$ | $V_{(a_1+a_2)\lambda_1+(a_3+a_4)\lambda_2+(a_1+a_5)\lambda_3+(a_2+a_4)\lambda_4}$                                      |
|       |       | $k\omega_3$ | $V_{(a_1+a_5)\lambda_1+a_2\lambda_4+a_3\lambda_3+(a_4+a_5)\lambda_4}$                                      |
|       |       | $k\omega_4$ | $V_{a_1\lambda_2+a_2\lambda_4}$                                                                |
| $E_6$ | $A_5 \times A_1$ | $k\omega_1$ | $V_{a_1\lambda_1+a_2\lambda_4+a_3\lambda_5} \otimes V_{a_5\lambda_6}$                                      |
|       |       | $k\omega_6$ | $V_{a_1\lambda_1+a_2\lambda_4+a_3\lambda_5} \otimes V_{a_5\lambda_6}$                                      |
| $F_4$ |       | $k\omega_1$ | $V_{a_1\lambda_4}$                                                                |
|       |       | $k\omega_2$ | $V_{a_1\lambda_1+a_2\lambda_4}$                                                                |
|       |       | $k\omega_3$ | $V_{a_1\lambda_1+a_2\lambda_4} \otimes V_{a_5\lambda_6}$                                      |
|       |       | $k\omega_5$ | $V_{a_1\lambda_1+a_2\lambda_4} \otimes V_{a_5\lambda_6}$                                      |
|       |       | $k\omega_6$ | $V_{a_1\lambda_4}$                                                                |
| $C_4$ |       | $k\omega_1$ | $V_{a_1\lambda_2+a_2\lambda_4}$                                                                |
|       |       | $k\omega_6$ | $V_{a_1\lambda_2+a_2\lambda_4}$                                                                |
| $D_5 \times \mathbb{C}^*$ | $k\omega_1$ | $a_1 \geq k$ | $V_{a_1\lambda_1+a_2\lambda_4} \otimes V_{-a_1+a_2+4a_3}$                                      |
|       | $k\omega_2$ | $a_1 \geq k$ | $V_{a_1\lambda_1+a_2\lambda_4+a_3\lambda_5} \otimes V_{-a_1+a_2+3a_3}$                                      |
|       | $k\omega_3$ | $a_1 \geq k$ | $V_{a_1\lambda_1+a_2\lambda_4+a_3\lambda_5+(a_4+a_5)\lambda_4+a_5\lambda_5} \otimes V_{2a_1+4a_2+2a_3+3a_4-a_5-a_6}$                                      |
|       | $k\omega_7$ | $a_1 \geq k$ | $V_{a_1\lambda_1+a_2\lambda_4+a_3\lambda_5+(a_4+a_5)\lambda_4+a_5\lambda_5} \otimes V_{2a_1+4a_2+2a_3+3a_4-a_5-a_6}$                                      |
| $E_7$ | $A_7$ | $k\omega_7$ | $V_{a_2\lambda_2+a_3\lambda_4+a_4\lambda_6}$                                      |
| $E_6 \times \mathbb{C}^*$ | $k\omega_1$ | $a_1 \geq k$ | $V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6} \otimes V_{2a_1-a_3}$                                      |
|       | $k\omega_2$ | $a_1 \geq k$ | $V_{a_1\lambda_1+(a_2+a_6)\lambda_2+a_3\lambda_4+a_4\lambda_4+a_5\lambda_5+a_6\lambda_6} \otimes V_{-a_1+3a_2+a_3-a_5-2a_6}$                                      |
Table 1

| $G$   | $H$   | $\omega$ | $\text{res}^G_H(V_{\omega})$ |
|-------|-------|---------|-----------------------------|
| $k\omega_7$ | $\bigoplus_{a_1+a_2=a_3+a_4=k} V_{a_1\lambda_1+a_2\lambda_6} \otimes V_{-a_1+a_2+3a_3-3a_4}$ |
| $D_6 \times A_1$ | $k\omega_7$ | $\bigoplus_{a_1+2a_2+a_3=6} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6} \otimes V_{a_2\lambda_7}$ |

To obtain the previous table we shall adapt the proof of Proposition 4.4 in [FL10] by Feigin and Littelmann. But first we will introduce some additional notation.

Let $P_1 \supset B$ denote the maximal parabolic subgroup of $G$ associated to the fundamental weight $\omega_i$. We shall consider the natural action of $H$ on the projective varieties $Y = G/P_i$. The affine cone over $Y$ is denoted by $\hat{Y}$ and the stabilizer of $\mathcal{T} \in G/P_i$ is denoted by $H_\mathcal{T}$. The group $H_\mathcal{T}$ is a parabolic subgroup of $H$. Its opposite parabolic subgroup in $H$ is denoted by $Q$. Furthermore let $Q^u$ be its unipotent radical and let $L$ be the Levi-subgroup $H_\mathcal{T} \cap Q$ with Borel subgroup $B_L$ defined by the simple roots of $H$ that appear in $L$. If we consider the orbit $O = H.\mathcal{T} \simeq H/H_\mathcal{T}$ with normal bundle $N$ having fiber $N$ at $\mathcal{T}$ then $N$ has the structure of an $L$-module since $L \subset H_{\mathcal{T}}$.

If no confusion can arise we will write $P$ instead of $P_i$ from now on.

The proof is divided into two parts. First we will determine in which cases $Y$ is a spherical $H$-variety. This part of the proof is conducted in four steps.

**Step 1:** We apply the Brion-Luna-Vust Local Structure Theorem [BLV86] to get the following proposition.

**Proposition 1:** There exists a locally closed affine subvariety $Z \subset Y$ such that $\mathcal{T} \in Z$, $Z$ is stable under the action of $L$, $Q^u.Z$ is open in $Y$ and the canonical map $Q^u \times Z \to Q^u.Z$ is an isomorphism of varieties.

**Proof:** Note that since the Borel subgroup $B_H$ is a subgroup of $P$, it is contained in the stabilizer $H_\mathcal{T}$ of $\mathcal{T} \in Y$. Thus $H_\mathcal{T}$ is a parabolic subgroup of $H$.

Now we can apply the Local Structure Theorem to this situation and obtain the proposition. $\square$

**Step 2:** We have the following proposition.

**Proposition 2:** The variety $Y$ is $H$-spherical if and only if $Z$ is a spherical $L$-variety.

**Proof:** Assume $Z$ is spherical, i.e. a Borel subgroup of $L$ has a dense orbit in $Z$. Let $B_L$ be the Borel subgroup $B_H \cap L \subset L$ and let $B_L^-$ be the opposite Borel subgroup. Then $B_L^- = Q^uB_L^-$ is a Borel subgroup of $H$. Let $z \in Z$ be an element such that $B_L^-z$ is dense in $Z$. Since $Q^u.Z$ is dense in $Y$, so is $B_H^-z = Q^u(B_L^-z)$. Hence $Y$ is a spherical $H$-variety.

If on the other hand $Y$ is $H$-spherical, then $B_H^-y = Q^u(B_L^-.y)$ is open in $Y$ for some $y \in Y$. Since $Q^u.Z$ is open in $Y$ we can assume that $y \in Z$. Now if $Q^u(B_L^-y)$ is dense in $Y$ it follows that $B_L^-y$ is dense in $Z$. $\square$
Step 3: Now $N$ is isomorphic to the tangent space $T_T Z$ and thanks to Luna’s Slice Theorem $Y$ is $H$-spherical if and only if $N$ is $L$-spherical.

Step 4: It remains to compute $N$ and to check in which cases it is a spherical $L$-module. Note that we have

$$N \cong (\text{Lie } G/\text{Lie } P_i)/(\text{Lie } H/\text{Lie } H_L).$$

So if $\Phi_H \subset \Phi$, then we can describe $N$ as the root spaces that occur in $T_T Y = \text{Lie } G/\text{Lie } P_i$ but not in $T_T (H/H_L)$. These are all the root spaces $\mathbb{C}X_\alpha$ such that $\alpha$ is negative and $\mathbb{C}X_\alpha \not\subset \text{Lie } P_i$ as well as $\mathbb{C}X_\alpha \not\subset \text{Lie } H$.

Remark. There is an algorithm by F. Knop [Kno97, Thm. 3.3] to check whether a given $L$-module is spherical. But in order for this paper to be self-contained we compute an explicit $X \in N$ such that $B_L.X$ is a dense orbit in $N$ in the spherical cases.

The second part is to compute the restrictions of the $G$-modules $V_{k\omega_i^*}$ to $H$. It is well-known that

$$\mathbb{C}[\hat{Y}] = \bigoplus_{k \geq 0} V_{k\omega_i^*}$$

where $V_{k\omega_i^*}$ corresponds to the homogeneous functions of degree $k$ on $\hat{Y}$. In order to derive branching rules for $V_{k\omega_i^*}$ we need to determine the $U_H$-invariants of $V_{k\omega_i^*}$.

Because $\hat{Y}$ is a spherical $(H \times \mathbb{C}^*)$-variety and because $U_H = U_{H \times \mathbb{C}^*}$, we know from Lemma 1 in [Lit94] that the ring $\mathbb{C}[\hat{Y}]^{U_H}$ is a polynomial ring with some set of generators $f_j$ of degree $d_j$, $1 \leq j \leq s$, where $s$ is the number of generators. Thus we have the following branching rules in this situation.

**Theorem 3:** Let $\eta_j$ denote the weight of $f_j$ with respect to $H$ and suppose $G/P_i$ is a spherical $H$-variety. Then we get

$$\text{res}^G_H(V_{k\omega_i^*}) = \bigoplus_{a_1 \eta_1 + \ldots + a_s \eta_s = k} V_{a_1 \omega_1 + \ldots + a_s \omega_s}.$$

We need to compute the number of generators, i.e. the dimension of $\mathbb{C}[\hat{Y}]^{U_H}$.

**Proposition 4:** We have

$$\dim \mathbb{C}[\hat{Y}]^{U_H} = \dim N - \dim(\text{generic } U_L\text{-orbit}) + 1.$$ 

**Proof:** We know that $\dim \mathbb{C}[\hat{Y}]^{U_H} = \text{trdeg } \mathbb{C}(\hat{Y})^{U_H}$ and by a theorem of Rosenthal we know that $\text{trdeg } \mathbb{C}(\hat{Y})^{U_H} = \dim \hat{Y} - \dim(\text{generic } U_H\text{-orbit})$ (paragraph II.4.3.E in [Kra84, p. 143]).

So the proposition is an immediate corollary of the following lemma. □

**Lemma 5:** Let $Y$, $N$, $U_L$ and $U_H$ be defined as above. Let $O_1$ be a generic $U_H$-orbit in $Y$ and $O_2$ be a generic $U_L$-orbit in $N$. Then

$$\dim Y - \dim O_1 = \dim N - \dim O_2.$$ 

**Proof:** Let $O \subset Y$ be the open subset of $X$ such that $\dim U_H.x$ is maximal for all $x \in O$ (i.e. $U_H.x$ is an generic orbit). We have $O \cap Q^a.Z \neq \emptyset$, because $Q^a.Z$ is open and dense in $Y$. 

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Let \( x = qz \) be an element in \( O \cap Q^u.Z \). We know that \( U_H = U_L.Q^u = Q^u.U_L \). So we have \( U_H.x = U_H.(qz) = U_L Q^u(qz) = U_L.Q^u.z = U_H.z \) and we can assume that \( U_H.x \) is a generic \( U_H \)-orbit in \( Y \) with \( x \in Z \).

Suppose \( y \) is an element of the stabilizer \((U_H)_x\) of \( x \). Then we have \( y = q.u \) for some \( q \in Q^u, u \in U_L \). So it follows from the Local Structure Theorem that \( q = \text{id} \) and \( ux = x \). Thus we get \((U_H)_x = (U_L)_x\).

With \( \dim Y = \dim Z + \dim Q^u \) (Local Structure Theorem) we get

\[
\dim Y - \dim U_H.x = \dim Q^u + \dim Z - \dim U_H.z \\
= \dim Z - (\dim U_H.x - \dim Q^u) \\
= \dim Z - (\dim U_H - \dim(U_H)_x - \dim Q^u) \\
= \dim Z - (\dim U_H - \dim Q^u - \dim(U_L)_x) \\
= \dim Z - (\dim U_L - \dim(U_L)_x) \\
= \dim Z - \dim U_L.x.
\]

\( \square \)

4. The maximal reductive subgroups of the exceptional groups

We want to list all maximal reductive subgroups of the exceptional algebraic groups. G. Seitz listed all maximal closed connected subgroups in arbitrary characteristics. We recall his results for the case that the ground field is \( \mathbb{C} \) ([Sei91], Thm. 1).

**Theorem 6:** Let \( G \) be a simple algebraic group of exceptional type and let \( X \) be maximal among the proper closed connected subgroups of \( G \). Then either \( X \) contains a maximal torus of \( G \) or \( X \) is semisimple and the pair \((G, X)\) is given below. Moreover, maximal subgroups of each type exist and are unique up to conjugacy in \( \text{Aut}(G) \).

| \( G \) | \( X \) simple | \( X \) not simple |
|---|---|---|
| \( G_2 \) | \( A_1 \) |       |
| \( F_4 \) | \( A_1 \) | \( A_1 \times G_2 \) |
| \( E_6 \) | \( A_2, G_2, F_4, C_4 \) | \( A_2 \times G_2 \) |
| \( E_7 \) | \( A_1, A_2 \) | \( A_1 \times A_1, A_1 \times G_2, A_1 \times F_4, G_2 \times C_3 \) |
| \( E_8 \) | \( A_1, B_2 \) | \( A_1 \times A_2, G_2 \times F_4 \) |

Since the maximal subgroups that do not contain a maximal torus are semisimple they are also maximal reductive subgroups of \( G \).

It remains to identify the maximal reductive subgroups that are contained in a maximal subgroup of maximal rank. These groups fall in two categories. Some are the maximal parabolic subgroups of \( G \) and the others are so called subsystem subgroups. There is an algorithm (cf. paragraph no. 17 of [Dyn57] or [BdS49]) that determines these subgroups: Start with the Dynkin diagram of \( G \) and adjoin the smallest root \( \delta \) to obtain the extended Dynkin diagram. By removing a node from the extended diagram you arrive at the Dynkin diagram of a subgroup of \( G \). By Theorem 5.5 and the subsequent remark in
these groups are maximal. Since they are semisimple they are also maximal reductive.

To complete the list we need to consider the maximal parabolic subgroups of $G$. Any reductive subgroup of a parabolic can be assumed to be a subgroup of its Levi factor by Theorem 1 in [LS96]. By considering the Dynkin diagrams it is transparent that the Levi subgroups need not be maximal reductive but can be subgroups of a subsystem subgroup. A simple case by case check shows that there are only two Levi groups, that are maximal reductive.

Summarizing this we have the following maximal reductive subgroups containing a maximal torus.

| $G$ | subsystem subgroups | Levi subgroups |
|-----|---------------------|----------------|
| $G_2$ | $A_2, A_1 \times A_1$ |                |
| $F_4$ | $A_1 \times C_3, A_2 \times A_2, A_3 \times A_1, B_4$ | $D_5 \times \mathbb{C}^*$ |
| $E_6$ | $A_5 \times A_1, A_2 \times A_2 \times A_2$ | $D_5 \times \mathbb{C}^*$ |
| $E_7$ | $D_6 \times A_1, A_5 \times A_2, A_3 \times A_3 \times A_1, A_7$ | $E_6 \times \mathbb{C}^*$ |
| $E_8$ | $A_1 \times E_7, A_2 \times E_6, A_3 \times D_5, A_4 \times A_4$ | $A_5 \times A_2 \times A_1, A_7 \times A_1, D_8, A_8$ |

5. THE EXCEPTIONAL GROUP OF TYPE $G_2$

We will now consider the simply connected simple algebraic group $G$ of type $G_2$. The long roots of its root system form a subsystem of type $A_2$ and we will consider the subsystem subgroup $H$ obtained in this way. The simple roots of $H$ are given by

$$(1,0)_{A_2} = (3,1) \text{ and } (0,1)_{A_2} = (0,1).$$

Using the same methods as before we can prove:

**Theorem 7:** The varieties $G/P_1$ and $G/P_2$ are $H$-spherical.

**Proof:**

Case $G/P_1$: We compute

$L = \langle T, U_{\pm(0,1)} \rangle$.

and

$N = \mathbb{C}X_{-(1,0)_{G_2}} \oplus \mathbb{C}X_{-(1,1)_{G_2}} \oplus \mathbb{C}X_{-(2,1)_{G_2}}$.

If we define $X := X_{-(1,1)} + X_{-(2,1)}$ we have $[b,X] = N$, which shows that $N$ is $L$-spherical. It follows that $G/P_1$ is a spherical $H$-variety.

Case $G/P_2$: In this case we can compute that $L = T$ and

$N = \mathbb{C}X_{-(1,1)} \oplus \mathbb{C}X_{-(2,1)}$.

The module $N$ consists of two linearly independent root spaces and since $T$ is 2-dimensional $N$ is obviously $L$-spherical. That implies that $G/P_2$ is a spherical $H$-variety. □
**Theorem 8:** Let $G$ be of type $G_2$ and $H$ of type $A_2$. Then we have the following branching rules:

1. $\text{res}_H^G(V_{\omega_1}) = \bigoplus_{a_1 + a_2 \leq k} V_{a_1 \lambda_1 + a_2 \lambda_2}$,
2. $\text{res}_H^G(V_{\omega_2}) = \bigoplus_{a_1 + a_2 + a_3 = k} V_{(a_1 + a_3) \lambda_1 + (a_2 + a_3) \lambda_2}$.

**Remark.** In $G_2$ the fundamental weights are self-dual.

**Proof:**

1. We use “LiE” to compute the restriction of $V_{\omega_1}$ and get

$$\text{res}_H^G(V_{\omega_1}) = \mathbb{C} \oplus V_{\lambda_1} \oplus V_{\lambda_2}.$$  

Let $f_0, f_1, f_2$ be highest weight vectors of these representations. We need to show that $\mathbb{C}[\hat{Y}]^{U_H}$ is generated by these elements, i.e. we need to show that the dimension of $\mathbb{C}[\hat{Y}]^{U_H}$ is 3.

By considering $X_{-1,0} \in N$ we immediately see that the $U_L$-orbit of this element is of codimension 2. Thus $\dim \mathbb{C}[\hat{Y}]^{U_H} = 3$ and since we have already found three algebraically independent elements the branching rules follow immediately.

2. We use “LiE” to compute

$$\text{res}_H^G(V_{\omega_2}) = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_1 + \lambda_2}.$$  

Let $f_1, f_2, f_3$ be highest weight vectors of these modules. We know that $U_L$ is the maximal torus in this case and so the unipotent radical is just the identity. A generic orbit in $N$ is of dimension 0. And since $N$ is 2-dimensional, its codimension is 2. That means a generic $U_H$-orbit has codimension 3 in $\hat{Y}$ and that is also the dimension of $\mathbb{C}[\hat{Y}]^{U_H}$. We have already found three linearly independent elements which form a generating set. The branching rules follow immediately. \(\square\)

**Proposition 9:** The varieties $G/P_i$ are not spherical $H$-varieties if $H$ is any other maximal reductive subgroup of $G_2$.

**Proof:** We have the following maximal reductive subgroups besides $A_2$: $A_1 \times A_1$ and $A_1$. If we compute the dimensions of a Borel subgroup in each case and the dimensions of $G/P_i$ we obtain:

|      | $G/P_1$ | $G/P_2$ |
|------|---------|---------|
| dim  | 5       | 5       |

and

| $H$  | $A_1 \times A_1$ | $A_1$ |
|------|------------------|-------|
| dim $B_H$ | 4  | 2    |

So $\dim B_H < \dim G/P_i$, $i = 1, 2$ for these subgroups. \(\square\)
6. The Exceptional Group of Type $F_4$

In this section let $G$ be the group of type $F_4$.
Let $H$ be the subgroup of type $B_4$ in $G$. This is a subsystem subgroup so from the Dynkin-diagram of $F_4$ we pass on to the extended Dynkin-diagram by adding the smallest root $\delta$ to the system of simple roots.

![Dynkin Diagram]

By removing the simple root $\alpha_4$ we obtain a root-subsystem of type $B_4$ and thus we find the corresponding subgroup $H \subset G$.

Explicitly we can choose the roots

$$(1, 0, 0, 0)_{B_4} = (0, 1, 2, 2), \quad (0, 1, 0, 0)_{B_4} = (1, 0, 0, 0),$$

$$(0, 0, 1, 0)_{B_4} = (0, 1, 0, 0), \quad (0, 0, 0, 1)_{B_4} = (0, 0, 1, 0),$$

which form a set of simple roots of a root subsystem of type $B_4$ in $F_4$.

We have the following theorem:

**Theorem 10:** The varieties $G/P_i$, $i = 1, \ldots, 4$, are spherical $H$-varieties.

**Proof:** We need to check that $N$ is a spherical $L$-module in each case.

**Case $G/P_1$:** In this case we have

$$L = \langle T, U_{\pm(0,1,2,2)}, U_{\pm(0,1,0,0)}, U_{\pm(0,0,1,0)}, U_{\pm(0,1,1,0)}, U_{\pm(0,1,2,0)} \rangle$$

and

$$N = \mathbb{C}X_{-(1,2,3,1)} \oplus \mathbb{C}X_{-(1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1)}.$$  

The Borel subgroup $B_L$ of $L$ obviously contains the maximal torus $T$ of $G$. Since $N$ consists of four root spaces with linearly independent roots and $T$ is 4-dimensional we know that there is a dense $B_L$-orbit in $N$. Hence $N$ is $L$-spherical and that implies that $G/P_1$ is $H$-spherical.

**Case $G/P_2$:** Here we have

$$L = \langle T, U_{\pm(0,0,0,0)}, U_{\pm(0,0,1,0)} \rangle.$$  

We compute $N$ in the same way as in the previous case and get

$$N = \mathbb{C}X_{-(0,1,1,1)} \oplus \mathbb{C}X_{-(0,1,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \mathbb{C}X_{-(1,2,2,1)} \oplus \mathbb{C}X_{-(1,2,3,1)}.$$  

We check the sphericity on the level of Lie algebras. Consider the element

$$X := X_{-(1,2,1,2)} + X_{-(0,1,2,1)} + X_{-(1,1,1,1)} + X_{-(1,2,3,1)}$$

in $N$. Then $[\mathfrak{b}, X] = N$. That means that $N$ is a spherical $L$-variety and therefore $G/P_2$ is a spherical $H$-variety.

**Case $G/P_3$:** We get

$$N = \mathbb{C}X_{-(0,0,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1)} \oplus \mathbb{C}X_{-(0,1,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \mathbb{C}X_{-(1,2,2,1)} \oplus \mathbb{C}X_{-(1,2,3,1)}.$$
If we consider

\[ X := X_{-(1,2,3,1)} + X_{-(1,2,2,1)} + X_{-(1,1,1,1)} + X_{-(0,1,2,1)} \in N \]

we have that \([b, X] = N\), i.e. \(N\) is a spherical \(L\)-variety and that means that \(G/P_3\) is a spherical \(H\)-variety.

**Case** \(G/P_4\): In this case we have

\[
L = \langle T, U_{\pm(1,0,0,0)}, U_{\pm(0,1,0,0)}, U_{\pm(0,0,1,0)}; \\
U_{\pm(1,1,0,0)}, U_{\pm(1,1,1,0)}, U_{\pm(1,1,1,1)}; \\
U_{\pm(0,1,2,0)}, U_{\pm(1,2,0)}, U_{\pm(1,2,2,0)} \rangle
\]

and

\[
N = \mathbb{C}X_{-(0,0,0,1)} \oplus \mathbb{C}X_{-(0,0,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1)} \oplus \\
\mathbb{C}X_{-(1,2,1,1)} \oplus \mathbb{C}X_{-(1,1,1,1)}
\]

The module \(N\) has the following structure.

\[
\begin{array}{c}
X_{-(1,2,3,1)} \overset{(0,0,1,0)}{\longrightarrow} X_{-(1,2,2,1)} \overset{(0,1,0,0)}{\longrightarrow} X_{-(1,1,2,1)} \overset{(1,0,0,0)}{\longrightarrow} X_{-(1,1,1,1)} \\
\cdot \cdot \cdot \\
X_{-(0,0,0,1)} \overset{(0,0,1,0)}{\longrightarrow} X_{-(0,0,1,1)} \overset{(0,1,0,0)}{\longrightarrow} X_{-(0,1,2,1)} \overset{(1,0,0,0)}{\longrightarrow} X_{-(1,1,1,1)}
\end{array}
\]

We have \(L = \mathbb{C}^* \times SO_7\) and \(N\) is an irreducible \(L\)-module of dimension 8. There exists only one such module which is the Spin\(_7\)-module. That \(N\) is a spherical \(L\)-module was proven by Victor Kac [Kac 1977, Thm. 3, p. 208]. It follows that \(G/P_4\) is a spherical \(H\)-module. \(\square\)

The spherical cases imply the following branching rules.

**Theorem 11:** Let \(G\) be of type \(F_4\) and \(H\) of type \(B_4\). Then we have the following branching rules:

1. \(\text{res}_H^G(V_{k\omega_1}) = \bigoplus_{a_1 + a_2 = k} V_{a_1 \lambda_2 + a_2 \lambda_4}\)
2. \(\text{res}_H^G(V_{k\omega_2}) = \bigoplus_{a_1 + a_3 = k} V_{(a_1 + a_2) \lambda_1 + (a_3 + a_4) \lambda_2 + (a_1 + a_3) \lambda_3 + (a_2 + a_4) \lambda_4}\)
3. \(\text{res}_H^G(V_{k\omega_3}) = \bigoplus_{a_1 + a_5 = k} V_{(a_1 + a_2) \lambda_1 + a_2 \lambda_2 + a_4 \lambda_3 + (a_4 + a_5) \lambda_4}\)
4. \(\text{res}_H^G(V_{k\omega_4}) = \bigoplus_{a_1 + a_2 \leq k} V_{a_1 \lambda_2 + a_2 \lambda_4}\)

**Remark.** In \(F_4\) the fundamental weights are self-dual.
Proof:

i): Standard computations yield

\[ \text{res}^G_H(V_{\omega_1}) = V_{\lambda_2} \oplus V_{\lambda_4}. \]

Let now \( f_1, f_2 \in V_{\omega_1} \) be highest weight vectors of \( V_{\lambda_2} \) and \( V_{\lambda_4} \) respectively. We will show that \( \mathbb{C}[\hat{Y}]^{U_H} \) is generated by these degree 1 elements. We know that \( \mathbb{C}[\hat{Y}]^{U_H} \) is a polynomial ring. The grading and weights of \( f_1 \) and \( f_2 \) imply that they are algebraically independent. To rule out the possibility that there are generators of degree two or higher we need to show that the Krull dimension of \( \mathbb{C}[\hat{Y}]^{U_H} \) is 2.

Thus we need to find a generic \( U_L \)-orbit in \( N \) and compute its codimension. Since we have found 2 algebraically independent elements in \( \mathbb{C}[\hat{Y}]^{U_H} \), we already know that the codimension must be at least 2.

Consider the Lie algebra \( I \) of \( L \). From above we know that the Lie algebra \( u \) of \( U_L \), is

\[ u = C X_{(0,1,2,2)} \oplus C X_{(0,1,0,0)} \oplus C X_{(0,0,1,0)} \oplus C X_{(1,1,1,1)} \oplus C X_{(0,1,2,0)}. \]

Define \( X := X_{(1,2,3,1)} \in N \). Then

\[
\begin{align*}
[X_{(0,1,2,2)}, X] &= 0, \\
[X_{(0,0,1,0)}, X] &= X_{(1,2,2,1)}, \\
[X_{(0,0,1,0)}, X] &= X_{(1,1,1,1)}, \\
[X_{(0,1,2,0)}, X] &= X_{(1,1,1,1)},
\end{align*}
\]

which shows that the orbit of \( X \) is of dimension 3. Thus a generic orbit has dimension at least 3 with codimension at most 1. By Proposition 4 we know that the codimension must be at least 2.

ii): In this case we need to find generators of \( \mathbb{C}[\hat{Y}]^{U_H} \). One can use the software “LiE” to compute

\[ \text{res}^G_H(V_{\omega_2}) = V_{\lambda_1 + \lambda_3} \oplus V_{\lambda_2 + \lambda_4} \oplus V_{\lambda_2 + \lambda_4} \oplus V_{\lambda_3}. \]

Let \( f_1, \ldots, f_5 \) be highest weight vectors of these irreducible modules.

Consider \( X := X_{(1,1,2,1)} + X_{(1,2,3,1)} \in N \) and let \( u \) be the Lie-algebra of \( U_L \) the unipotent radical of \( L \). The stabilizer of this element is just 0, which means that the dimension of a generic \( U_L \)-orbit is 2 with codimension 4. This implies that the codimension of a generic \( U_H \)-orbit in \( \hat{Y} \) is 5. Thus \( \mathbb{C}[\hat{Y}]^{U_H} \) is generated by its degree 1 elements and the assertion follows.

iii): We need to find generators of \( \mathbb{C}[\hat{Y}]^{U_H} \). One can use “LiE” to compute

\[ \text{res}^G_H(V_{\omega_3}) = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus V_{\lambda_1 + \lambda_4}. \]

Let \( f_1, \ldots, f_5 \) be highest weight vectors of these irreducible modules.

Consider \( X := X_{(1,1,1,1)} + X_{(1,2,2,1)} \in N \) and take an element \( u \in u \) with \( u = aX_{(1,0,0,0)} + bX_{(0,1,0,0)} + cX_{(1,1,0,0)}. \) Then

\[
[u, X] = 0
\]

\[
\Rightarrow aX_{(1,1,1,1)} + bX_{(1,1,2,1)} + c(X_{(0,1,2,1)} + X_{(0,0,1,1)})
\]

\[
\Rightarrow a = b = c = 0 \Rightarrow u = 0
\]

and hence a generic \( U_L \)-orbit has dimension 3 with codimension 4. That means that \( \mathbb{C}[\hat{Y}]^{U_H} \) is of dimension 5 and generated by the elements \( f_i \).
In this case we need to find generators of $\mathbb{C}[\hat{Y}]^{U_H}$. We use “LiE” to compute
\[
\text{res}^G_H(V_{\omega_4}) = \mathbb{C} \oplus V_{\lambda_1} \oplus V_{\lambda_4}.
\]
Let $f_1, \ldots, f_3$ be highest weight vectors of these irreducible modules.
Consider $X := X_{(1,2,3,1)}$. We know that for
\[
X_{(1,0,0,0)}, X_{(0,1,0,0)}, X_{(1,1,0,0)} \in \mathfrak{u}
\]
we have
\[
[X_{(1,0,0,0)}, X] = [X_{(0,1,0,0)}, X] = [X_{(1,1,0,0)}, X] = 0
\]
and thus the generic stabilizer is at most of dimension 3. The generic orbit is at least of dimension 6 and thus its codimension is at most 2. This means that a generic $U_H$-orbit in $\hat{Y}$ is of dimension less or equal to 3.

Since we have found 3 algebraically independent elements the dimension of $\mathbb{C}[\hat{Y}]^{U_H}$ is exactly 3 and this finishes the proof. □

**Proposition 12:** The varieties $G/P_i$ are not spherical $H$-varieties if $H$ is any other maximal reductive subgroup of $F_4$.

**Proof:** We have the following maximal reductive subgroups besides $B_4$: $A_1 \times C_3$, $A_2 \times A_2$, $A_3 \times A_1$, $A_1 \times G_2$ and $A_1$. If we compute the dimensions of a Borel subgroup in each case and the dimensions of $G/P_i$ we obtain:

| $G/P_i$   | $G/P_2$ | $G/P_3$ | $G/P_4$ |
|----------|---------|---------|---------|
| dim      | 15      | 20      | 20      | 15      |
| $H$      | $A_1 \times C_3$ | $A_2 \times A_2$ | $A_3 \times A_1$ | $A_1 \times G_2$ | $A_1$ |
| dim $B_H$| 14      | 10      | 11      | 10      | 2      |

So we have $\dim B_H < \dim G/P_i$ for $i = 1, \ldots, 4$ in each case. □

7. **The Exceptional Group of Type $E_6$**

We will now turn to the group of type $E_6$. First we calculate the dimensions of the Borel subgroups of the maximal reductive subgroups as well as the dimensions of $G/P_i$ for $i = 1, \ldots, 6$.

| $H$      | $A_5 \times A_1$ | $A_2 \times A_2 \times A_2$ | $D_5 \times \mathbb{C}^*$ | $A_2 \times G_2$ | $G_2$ | $A_2$ | $F_4$ | $C_4$ |
|----------|-------------------|-------------------------------|--------------------------|------------------|-------|-------|-------|-------|
| dim $B_H$| 22                | 15                            | 26                       | 13               | 8     | 5     | 28    | 20    |
and

| dim |
|-----|
| 16 |
| 21 |
| 25 |
| 29 |
| 25 |
| 16 | G/P_1 G/P_2 G/P_3 G/P_4 G/P_5 G/P_6 |

Thus we get the following proposition.

**Proposition 13:** Let $G$ be the simply connected simple algebraic group of type $E_6$ and let $H$ be a maximal reductive subgroup of type $A_2 \times A_2 \times A_2$, $A_2 \times G_2$, $G_2$ or $A_2$.

Then $G/P_i$ is not $H$-spherical for $i = 1, \ldots, 6$.

**Proof:** In these cases we have $\dim B_H < \dim G/P_i$ for $i = 1, \ldots, 6$. \hfill $\square$

Now we will consider the remaining groups and first we start with the subsystem subgroup of type $A_5 \times A_1$.

**Theorem 14:** Let $G$ be the simply connected simple algebraic group of type $E_6$ and let $H$ be the maximal reductive subgroup of type $A_5 \times A_1$. Then $G/P_i$ and $G/P_6$ are spherical $H$-varieties. The varieties $G/P_2, \ldots, G/P_5$ are not $H$-spherical.

**Proof:** The dimension of a Borel subgroup of a group of type $A_5 \times A_1$ is 22. Since we have $\dim G/P_3 = 25$, $\dim G/P_4 = 29$, $\dim G/P_5 = 25$ these varieties cannot be spherical.

We know that $\omega_3^6 = \omega_2$ in type $E_6$. Now if $G/P_2$ was a spherical $H$-variety, $\text{res}^G_H(V_{\omega_2})$ would be multiplicity-free for all $k \in \mathbb{N}$ by what has been said above. But with “LiE” we compute

$$\text{res}^G_H(V_{\omega_2}) = \ldots \oplus 2(V_{2\lambda_3} \otimes V_{3\lambda_6}) \oplus \ldots$$

which means that there are multiplicities in this case.

To prove that $G/P_3$ and $G/P_6$ are spherical $H$-varieties we proceed as in the cases above. We will show how $H$ is embedded in $G$. For doing so we consider the extended Dynkin-diagram of type $E_6$ again by adding the smallest root $\delta$ to the simple roots. Now omitting the root $\alpha_2$ we obtain the embedding of $A_5 \times A_1$ in $E_6$.

Explicitly we get the following set of simple roots:

$(1,0,0,0,0,0)_{A_5 \times A_1} = (1,0,0,0,0,0) \quad (0,1,0,0,0,0)_{A_5 \times A_1} = (0,0,1,0,0,0)$

$(0,0,1,0,0,0)_{A_5 \times A_1} = (0,0,0,1,0,0) \quad (0,0,0,1,0,0)_{A_5 \times A_1} = (0,0,0,0,1,0)$

$(0,0,0,0,1,0)_{A_5 \times A_1} = (0,0,0,0,0,1) \quad (0,0,0,0,1,0)_{A_5 \times A_1} = (1,2,2,3,2,1)$

**Case $G/P_1$:** We compute

$$L = \langle T, U_{\pm(0,0,1,0,0,0)}, U_{\pm(0,0,0,1,0,0)}, U_{\pm(0,0,0,0,1,0)}, U_{\pm(0,0,0,0,0,1)} \rangle$$

$$U_{\pm(0,0,1,1,0,0)}, U_{\pm(0,0,0,1,1,0)}, U_{\pm(0,0,0,0,1,1)},$$

$$U_{\pm(0,0,1,1,1,0)}, U_{\pm(0,0,0,1,1,1)}, U_{\pm(0,0,1,1,1,1)} \rangle$$
and

\[ N = CX_{-(1,1,1,1,0)} \oplus CX_{-(1,1,1,1,1)} \oplus CX_{-(1,1,1,2,1,0)} \oplus CX_{-(1,1,1,1,1,1)} \oplus CX_{-(1,1,2,2,1,0)} \oplus CX_{-(1,1,1,1,1,1)} \oplus CX_{-(1,1,2,2,1,1)} \oplus CX_{-(1,1,2,2,2,1,0)} \oplus CX_{-(1,1,2,2,2,1,1)} \oplus CX_{-(1,1,2,3,2,1)}. \]

Now let \( X := X_{-(1,1,2,3,2,1)} + X_{-(1,1,1,1,1,1)} \). We have

\[ [h, X] = \langle X_{-(1,1,2,3,2,1)}, X_{-(1,1,1,1,1,1)} \rangle, \]

since the roots are linearly independent. Next we compute

\[
\begin{align*}
[X_{(0,0,0,1,0,0)}, X] &= X_{-(1,1,2,2,2,2,1)} \quad [X_{(0,0,0,1,1,0)}, X] = X_{-(1,1,1,2,2,1,1)} \\
[X_{(0,0,1,1,0,0)}, X] &= X_{-(1,1,1,1,2,2,1)} \quad [X_{(0,0,1,1,1,0)}, X] = X_{-(1,1,1,2,1,1,1)} \\
[X_{(0,0,0,1,1,1)}, X] &= X_{-(1,1,2,2,1,0)} \quad [X_{(0,0,1,1,1,1)}, X] = X_{-(1,1,1,2,1,0)} \\
[X_{(0,0,0,0,0,1)}, X] &= X_{-(1,1,1,1,1,0)} \quad [X_{(0,0,0,0,1,1)}, X] = X_{-(1,1,1,1,0,0)}
\end{align*}
\]

and these computations show that we have ten linearly independent vectors in \([b, X] = [b, X] = N \Rightarrow N\) is a spherical \(H\)-variety. Hence \(G/P_1\) is a spherical \(H\)-variety.

Case \(G/P_6\): The \(H\)-sphericity of \(G/P_6\) is an immediate corollary of the following theorem which states that \(C[Y]\) is multiplicity free. \(\square\)

**Theorem 15:** Let \(G\) be the simply connected simple algebraic group of type \(E_6\) and let \(H \subset G\) be the maximal reductive subgroup of type \(A_5 \times A_1\).

Then we have the following branching rules:

\[i) \quad \text{res}^G_H(V_{k\lambda}) = \bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_2+a_2\lambda_4+a_3\lambda_5} \otimes V_{a_3\lambda_6}, \]

\[ii) \quad \text{res}^G_H(V_{k\lambda}) = \bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_5+a_3\lambda_4} \otimes V_{a_1\lambda_6}. \]

**Remark.** In \(E_6\) we have \(\omega_1^* = \omega_6, \omega_2^* = \omega_2, \omega_3^* = \omega_5, \) and \(\omega_4^* = \omega_4\).

**Proof:** \(\text{i})\) With “LiE” we compute

\[ \text{res}^G_H(V_{k\omega_1}) = (V_{\lambda_4} \otimes \mathbb{C}) \oplus (V_{\lambda_1} \otimes V_{\lambda_6}), \]

\[ \text{res}^G_H(V_{k\omega_6}) = (V_{2\lambda_4} \otimes \mathbb{C}) \oplus (V_{\lambda_1+\lambda_4} \otimes V_{\lambda_6}) \oplus (V_{\lambda_1} \otimes V_{2\lambda_6}) \oplus (V_{\lambda_2} \otimes \mathbb{C}). \]

There are at least two generators of degree 1 and of weights \((\lambda_4, 0)\) and \((\lambda_1, \lambda_6)\) and one generator of degree 2 and of weight \((\lambda_2, 0)\) for \(C[Y]^{U/H}\) with \(Y = G/P_1\). In the proof of the previous theorem we have found an element \(X \in N\) with a \(U_L\)-orbit of codimension 2. So it follows that \(\dim C[Y]^{U/H} = 3\) and the branching rules follow immediately.

\(\text{i)}\) Theses branching rules follow directly from \(\text{ii})\) by noting that \(\omega_1 = \omega_6^*, \lambda_1 = \lambda_5, \lambda_2 = \lambda_4\) and \(\lambda_6 = \lambda_6. \quad \square\)

**Theorem 16:** Let \(G\) be the simply connected simple algebraic group of type \(E_6\) and let \(H\) be the maximal reductive subgroup of type \(F_4\). Then \(G/P_i, i \neq 4, \) are spherical \(H\)-varieties. The variety \(G/P_4\) is not \(H\)-spherical.

**Proof:** If we have the Dynkin diagrams
of $E_6$ and $F_4$, then we have an embedding of the simple Lie-algebra $F_4$ in $E_6$ by choosing the following root vectors

$$X_x := X_{(0,1,0,0,0,0)},\quad X_z := \frac{1}{\sqrt{2}}(X_{(0,0,1,0,0,0)} + X_{(0,0,0,0,1,0)})$$

$$X_y := X_{(0,0,0,1,0,0)},\quad X_u := \frac{1}{\sqrt{2}}(X_{(1,0,0,0,0,0)} + X_{(0,0,0,0,0,1)})$$

([Dyn57], p. 258, Table 24] with different numbering of the Dynkin diagrams). Now we consider the associated algebraic subgroup of $E_6$.

**Case $G/P_1$:** We compute

$$N = \mathbb{C}X_{-(1,1,1,2,2,1)}.$$

So $N$ is obviously $L$-spherical and thus $G/P_1$ is $H$-spherical.

**Case $G/P_2$:** The $H$-sphericity of $Y = G/P_6$ is an immediate corollary of the following theorem which states that $\mathbb{C}[\hat{Y}]$ is multiplicity free.

**Case $G/P_2$:** In this case we get

$$N = \mathbb{C}X_{-(0,1,0,1,1,0)} \oplus \mathbb{C}X_{-(0,1,0,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)},$$

If we define $X := X_{-(1,1,1,2,2,1)}$ then we have:

$$[X_{(0,0,0,1)}F_4, X] = X_{-(1,1,1,2,2,1)},\quad [X_{(0,0,1,1)}F_4, X] = X_{-(0,1,1,2,2,1)},$$

$$[X_{(0,1,1,1)}F_4, X] = X_{-(0,1,1,1,1)},\quad [X_{(0,1,2,1)}F_4, X] = X_{-(0,1,1,1,1)},$$

$$[X_{(0,1,2,2)}F_4, X] = X_{-(0,1,0,1,1,1)}.$$

With $[b, X] = \mathbb{C}X$ we get $[b, X] = N$ and it follows that $N$ is a spherical $L$-module.

**Case $G/P_3$:** In this case we get

$$N = \mathbb{C}X_{-(0,0,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1)},$$

Set $X := X_{-(1,1,1,2,2,1)} + X_{-(0,1,1,2,1,1)}$. Then we have

$$[b, X] = \mathbb{C}X_{-(1,1,1,2,2,1)} \oplus \mathbb{C}X_{-(0,1,1,2,1,1)}.$$

since the roots of the root vectors defining $X$ are linearly independent. Furthermore we have

$$[X_{(0,0,0,1)}F_4, X] = X_{-(0,1,1,2,2,1)},\quad [X_{(1,0,0,0)}F_4, X] = X_{-(0,1,1,1,1,1)},$$

$$[X_{(1,1,0,0)}F_4, X] = X_{-(0,0,1,1,1,1)}.$$

So $[b, X] = N \Rightarrow N$ is a spherical $L$-module and this implies that $G/P_3$ is $H$-spherical.

**Case $G/P_5$:** The $H$-sphericity of $Y = G/P_5$ is an immediate corollary of the following theorem which states that $\mathbb{C}[\hat{Y}]$ is multiplicity free. □

We can derive branching rules in the cases where $G/P_i$ is a spherical $H$-variety.
Theorem 17: Let $G$ be the simple simply connected algebraic group of type $E_6$ and $H$ be the subgroup of type $F_4$. Then we have the branching rules:

\begin{align*}
i) \quad & \text{res}_H^G(V_{k\omega_1}) = \bigoplus_{a_1 \leq k} V_{a_1\lambda_4}, \\
ii) \quad & \text{res}_H^G(V_{k\omega_2}) = \bigoplus_{a_1+a_2=k} V_{a_1\lambda_1+a_2\lambda_4}, \\
iii) \quad & \text{res}_H^G(V_{k\omega_3}) = \bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_3+a_3\lambda_4}, \\
v) \quad & \text{res}_H^G(V_{k\omega_4}) = \bigoplus_{a_1 \leq k} V_{a_1\lambda_4}.
\end{align*}

Proof: $v)$ In this case we work with $Y = G/P_1$. With “LiE” we compute

$$\text{res}_H^G(V_{\omega_1}) = C \oplus V_{\lambda_4}.$$ 

Since $N$ is 1-dimensional in this case, each $U_L$-orbit is 0-dimensional with codimension 1. So $\dim C[\tilde{Y}][U_H] = 2$ and $\dim C[\tilde{Y}][U_H]$ is generated by its degree-1-elements. The branching rules follow.

i) These branching rules follow directly from $v)$ by noting that $\omega_1 = \omega_6^*$ and $\lambda_4^* = \lambda_4$.

ii) In this case we work with $Y = G/P_2$. With “LiE” we compute

$$\text{res}_H^G(V_{\omega_2}) = V_{\lambda_1} \oplus V_{\lambda_4},$$

so there are two generators of degree 1. The module $N$ is of dimension 6 and we have seen that $X_{-(1,1,2,2,1)} \in N$ is an element such that $U_L \cdot X$ is of dimension 5. So $\dim C[\tilde{Y}][U_H] \leq 2$ and hence $C[\tilde{Y}][U_H]$ is generated by its degree-1-elements. The branching rules follow immediately.

iv) In this case we work with $G/P_3$. With “LiE” we compute

$$\text{res}_H^G(V_{\omega_3}) = V_{\lambda_1} \oplus V_{\lambda_3} \oplus V_{\lambda_4},$$

so again there are 3 generators of degree 1. The module $N$ is of dimension 5 and $X_{-(1,1,2,2,1),e_6} + X_{-(0,1,1,2,1),e_6}$ is an element of $N$ with a 3-dimensional $U_L$-orbit (cf. proof of previous theorem). So $\dim C[\tilde{Y}][U_H] \leq 3$. It follows that $C[\tilde{Y}][U_H]$ is generated by its degree-1-elements and so the branching rules follow.

iii) These branching rules follow directly from $v)$ by noting that $\omega_3 = \omega_5^*$ and $\lambda_3^* = \lambda_3$. \hfill \Box

Theorem 18: Let $G$ be the simply connected simple algebraic group of type $E_6$ and let $H$ be the maximal reductive subgroup of type $F_4$. Then $G/P_1$ and $G/P_6$ are spherical $H$-varieties. The varieties $G/P_2, \ldots, G/P_5$ are not $H$-spherical.

Proof: That $G/P_2, \ldots, G/P_5$ are not $H$-spherical follows by dimension reasons.

For the other two cases we consider the Dynkin diagrams.
We compute
\[ x := \frac{1}{\sqrt{2}}(X(0,1,1,1,0,0) + X(0,0,0,0,0,0)), \quad y := \frac{1}{\sqrt{2}}(X(1,0,0,0,0,0) + X(0,0,0,0,0,1)) \]
\[ z := \frac{1}{\sqrt{2}}(X(0,0,1,0,0,0) + X_{-(0,0,0,0,1,0)}), \quad u := X(0,0,0,1,0,0) \]
This implies that
\[ b \Rightarrow A \]
Further we get
\[ \text{Case } G/P_1: \text{ We compute} \]
\[ N = C X_{-(1,1,1,1,1)} \oplus C X_{-(1,1,2,1,1,1)} \oplus C X_{-(1,1,2,2,1,1)}, \]
We define \( X := X_{-(1,1,2,3,2,1)} + X_{-(1,1,1,1,1)} \). Then we have
\[ [b, X] = C X_{-(1,1,2,3,2,1)} \oplus C X_{-(1,1,1,1,1)} \]
Further we get
\[ [X(0,0,0,1), C_4], X] = X_{-(1,1,2,2,2,1)}, \quad [X(0,0,1,1), C_4], X] = X_{-(1,1,2,2,1,1)}, \]
This implies that \( [b, X] \) contains five linearly independent vectors of \( N \) \( \Rightarrow [b, X] = N \). Hence \( N \) is \( L \)-spherical.

Case \( G/P_0 \): The \( H \)-sphericity of \( Y = G/P_0 \) is an immediate corollary of the following theorem which states that \( \mathbb{C}[Y] \) is multiplicity free.

From the spherical cases we can derive the following branching rules:

**Theorem 19:** Let \( G \) be the simply connected simple algebraic group of type \( E_6 \) and \( H \) be the subgroup of type \( C_4 \).

Then we have the following branching rules:

1. \( \text{res}_{H}^{G}(V_{\lambda_1}) = \bigoplus_{a_1+2a_2+2a_3=k} V_{a_1 \lambda_2 + a_2 \lambda_4} \)
2. \( \text{res}_{H}^{G}(V_{\lambda_2}) = \bigoplus_{a_1+2a_2+2a_3=k} V_{a_1 \lambda_2 + a_2 \lambda_4} \)

*Proof:* 

i) Here we are in the case \( Y = G/P_1 \). With “LiE” we compute
\[ \text{res}_{H}^{G}(V_{\lambda_2}) = V_{\lambda_2} \text{ and } \text{res}_{H}^{G}(V_{\lambda_3}) = \mathbb{C} \oplus V_{2 \lambda_2} \oplus V_{\lambda_4} \]
So there is one generator of degree 1 and two of degree 2 in \( \mathbb{C}[\hat{Y}]^{U_H} \). From the calculations in the proof of the previous theorem we know that \( X_{-(1,1,2,3,2,1)} + X_{-(1,1,1,1,1,1)} \) is an element of \( N \) whose \( U_L \)-orbit is of codimension 2. Hence \( \dim \mathbb{C}[\hat{Y}]^{U_H} \leq 3 \). But since we have already found three generators we know that \( \dim \mathbb{C}[\hat{Y}]^{U_H} = 3 \). The branching rules follow immediately.
Theses branching rules follow directly from ii) by noting that $\omega_1 = \omega_6^*$, $\lambda_2^* = \lambda_2$ and $\lambda_4^* = \lambda_4$. \hfill $\square$

Next we will consider the Levi subgroup $H$ of $G$ that is obtained by omitting the simple root $\alpha_1$. From the Dynkin diagram of $E_6$ we see that $H$ is the group $D_5 \times \mathbb{C}^*$. 

Theorem 20: Let $G$ be the simply connected simple algebraic group of type $E_6$ and let $H$ be the Levi subgroup $D_5 \times \mathbb{C}^*$. Then $G/P_i$ is a spherical $H$-variety for $i \neq 4$. The variety $G/P_4$ is not $H$-spherical.

Proof: This is proven in [Lit94]. \hfill $\square$

Theorem 21: Let $G$ be the simply connected simple algebraic groups of type $E_6$ let $H \subset G$ be the Levi subgroup $D_5 \times \mathbb{C}^*$. Then we have the following branching rules.

\begin{align*}
  i) \quad \text{res}_H^G(V_{k\omega_1}) &= \bigoplus_{a_1 + a_2 + a_3 = k} V_{a_1 \lambda_1 + a_2 \lambda_4} \otimes V_{-a_1 + a_2 + 4a_3}, \\
  ii) \quad \text{res}_H^G(V_{k\omega_2}) &= \bigoplus_{a_1 + a_2 + a_3 + a_4 = k} V_{a_1 \lambda_2 + a_2 \lambda_4 + a_3 \lambda_5} \otimes V_{-3a_2 + 3a_3}, \\
  iii) \quad \text{res}_H^G(V_{k\omega_3}) &= \bigoplus_{a_1 + \ldots + a_6 = k} V_{(a_1 + a_6)\lambda_1 + a_2 \lambda_2 + a_3 \lambda_3 + a_4 + a_6} \lambda_4 + a_5 \lambda_5 \otimes V_{2a_1 - 4a_2 + 2a_3 + 5a_4 - a_5 - 3a_6}, \\
  iv) \quad \text{res}_H^G(V_{k\omega_4}) &= \bigoplus_{a_1 + \ldots + a_6 = k} V_{(a_1 + a_6)\lambda_1 + a_2 \lambda_2 + a_3 \lambda_3 + a_4 + a_6 + a_1 + (a_5 + a_6)\lambda_5} \otimes V_{-2a_1 + 4a_2 - 2a_3 + a_4 - 5a_5 + a_6}, \\
  v) \quad \text{res}_H^G(V_{k\omega_5}) &= \bigoplus_{a_1 + a_2 + a_3 = k} V_{a_1 \lambda_1 + a_2 \lambda_5} \otimes V_{2a_1 - a_2 - 4a_3}.
\end{align*}

Proof: From paragraph 1.4 in [Lit94] we get the following branching rules.

\begin{align*}
  i) \quad \text{res}_H^G(V_{k\omega_1}) &= \bigoplus_{a_1 + a_2 + a_3 = k} V_{(a_1 - a_2)\omega_1 + a_2 \omega_3 + a_1 \omega_6}, \\
  ii) \quad \text{res}_H^G(V_{k\omega_2}) &= \bigoplus_{a_1 + a_2 + a_3 + a_4 = k} V_{-(a_1 + 2a_2)\omega_1 + a_2 \omega_2 + a_2 \omega_3 + a_1 \omega_5}, \\
  iii) \quad \text{res}_H^G(V_{k\omega_3}) &= \bigoplus_{a_1 + \ldots + a_6 = k} V_{-(a_2 + a_3 + a_5 + 2a_6)\omega_1 + a_5 \omega_2 + (a_4 + a_6) \omega_3 + a_3 \omega_4 + a_2 \omega_5 + (a_1 + a_6) \omega_6}, \\
  iv) \quad \text{res}_H^G(V_{k\omega_4}) &= \bigoplus_{a_1 + \ldots + a_6 = k} V_{-(a_1 + 2a_3 + a_4 + 2a_5 + a_6)\omega_1 + (a_5 + a_6) \omega_2 + a_4 \omega_3 + a_3 \omega_4 + a_2 \omega_5 + (a_1 + a_6) \omega_6}, \\
  v) \quad \text{res}_H^G(V_{k\omega_5}) &= \bigoplus_{a_1 + a_2 + a_3 = k} V_{-(a_2 + a_3)\omega_1 + a_2 \omega_2 + a_1 \omega_6}.
\end{align*}

We would like to write these highest weights in terms of the fundamental weights of $D_5$ and $\mathbb{C}^*$. We have $\omega_6 = \lambda_1$, $\omega_5 = \lambda_2$, $\omega_4 = \lambda_3$, $\omega_3 = \lambda_4$ and $\omega_2 = \lambda_5$, where $\lambda_i$ are the fundamental weights of $D_5$ and we fix the coweight $3\omega_1^\vee = 4a_1^\vee + 3a_2^\vee + 5a_3^\vee + 6a_4^\vee + 4a_5^\vee + 2a_6^\vee$ which determines the highest weights for $\mathbb{C}^*$. Thus we get the branching rules in the theorem. \hfill $\square$
8. The Exceptional Group of Type $E_7$

Let $G$ be of type $E_7$ with the following Dynkin-Diagram.

For this group there are only a few cases of sphericity as we will see. As we did in the last section we start by calculating the dimensions of the Borel subgroups of the maximal reductive subgroups as well as the dimensions of $G/P_i$ for $i = 1, \ldots, 7$.

We have

| $G/P_i$   | 33 | 42 | 47 | 53 | 50 | 42 | 27 |
|-----------|----|----|----|----|----|----|----|
| dim       |    |    |    |    |    |    |    |

For the Borel subgroups $B_H$ we have:

| $H$ | $A_7$ | $E_6 \times C^*$ | $A_3 \times A_3 \times A_1$ | $A_5 \times A_2$ | $D_6 \times A_1$ | $A_1 \times A_1$ |
|-----|-------|------------------|-----------------------------|-----------------|----------------|----------------|
| dim $B_H$ | 35 | 43 | 20 | 25 | 38 | 4 |

| $H$ | $A_1 \times G_2$ | $G_2 \times C_3$ | $A_1 \times F_4$ | $A_1$ | $A_2$ |
|-----|------------------|------------------|-----------------|-------|-------|
| dim $B_H$ | 10 | 20 | 30 | 2 | 5 |

So we can rule out a lot of cases by dimension comparison.

**Proposition 22:** Let $G$ be the simply connected simple algebraic group of type $E_7$. If $H$ is a maximal reductive subgroup of type $A_3 \times A_3 \times A_1$, $A_5 \times A_2$, $A_1 \times A_1$, $A_1 \times G_2$, $G_2 \times C_3$, $A_1$ or $A_2$, then $G/P_i$ is not a spherical $H$-variety for $i = 1, \ldots, 7$. □

**Proof:** In these cases we have dim $B_H < $ dim $G/P_i$ for $i = 1, \ldots, 7$. □

Now we turn to the remaining subgroups and start with the subgroup of type $A_7$. This is a subsystem subgroup so we add the smallest root $\delta$ to the simple roots and consider the extended Dynkin diagram.

By omitting the simple root $\alpha_2$ we obtain the embedding of the root system $A_7$ into $E_7$. Explicitly we get

\[ (1,0,0,0,0,0,0)_{A_7} = (1,0,0,0,0,0,0), (0,1,0,0,0,0,0)_{A_7} = (0,0,1,0,0,0,0), \]
\[ (0,0,1,0,0,0,0)_{A_7} = (0,0,0,1,0,0,0), (0,0,0,1,0,0,0)_{A_7} = (0,0,0,0,1,0,0), \]
\[ (0,0,0,0,1,0,0)_{A_7} = (0,0,0,0,0,1,0), (0,0,0,0,0,1,0)_{A_7} = (0,0,0,0,0,0,1), \]
\[ (0,0,0,0,0,0,1)_{A_7} = (1,2,2,3,2,1,0). \]

Now we consider the corresponding subsystem subgroup $H$.

**Theorem 23:** Let $G$ be the simply connected simple algebraic group of type $E_7$ and $H$ the maximal reductive subgroup of type $A_7$. Then $G/P_7$ is a spherical $H$-variety whereas $G/P_i$ is not $H$-spherical for $i \neq 7$. 
Proof: By dimension comparison $G/P_i$ can only be spherical for $i = 1$ or $i = 7$. We know that for $E_7$ we have $\omega_i^* = \omega_i$. And with LiE we compute

$$\text{res}_{G_i}^G(V_{4\omega_i}) = \ldots \oplus 2V_{\lambda_i} \oplus \ldots .$$

This shows that we have multiplicities in this case and $G/P_i$ is not a spherical $H$-variety.

For $G/P_7$ we use the same methods as above. We compute

$$N = \C X_{-(0,1,0,1,1,1)} \oplus \C X_{-(0,1,1,1,1,1)} \oplus \C X_{-(1,1,1,1,1,1)} \oplus \C X_{-(1,1,2,1,1,1)} \oplus \C X_{-(1,1,2,2,1,1)} \oplus \C X_{-(1,1,2,2,2,1)} \oplus \C X_{-(1,2,2,2,2,1)} \oplus \C X_{-(1,2,3,3,2,1)} .$$

Define $X := X_{-(1,1,2,3,3,2,1)} + X_{-(1,1,2,2,1,1)} + X_{-(0,1,0,1,1,1)}$. The roots of the root-vectors in $X$ are linearly independent. Thus we get that

$$[\mathfrak{b}, X] := \langle X_{-(1,1,2,3,3,2,1)}_{E_7}, X_{-(1,1,2,2,1,1)}_{E_7}, X_{-(0,1,1,1,1,1)}_{E_7} \rangle$$

and further

$$[X_{(0,0,1,0,0,0,0)}, X] = X_{-(1,1,2,2,1,1,1)}, \quad [X_{(0,0,0,0,1,0,0)}, X] = X_{-(1,1,2,3,2,1,1)} ,$$

$$[X_{(1,0,1,0,0,0,0)}, X] = X_{-(0,1,1,1,1,1,1)}, \quad [X_{(0,0,1,1,0,0,0)}, X] = X_{-(1,1,1,1,1,1)},$$

$$[X_{(0,0,0,1,1,0,0)}, X] = X_{-(1,1,2,2,2,1,1)}, \quad [X_{(0,0,0,0,0,1,1)}, X] = X_{-(1,1,2,3,2,1,1)},$$

$$[X_{(0,1,0,1,0,0,0)}, X] = X_{-(0,1,1,1,1,1,1)}, \quad [X_{(0,0,1,1,1,0,0)}, X] = X_{-(1,1,2,2,2,1,1)} ,$$

$$[X_{(0,0,1,1,0,1,0)}, X] = X_{-(1,1,2,2,2,1,1)}, \quad [X_{(0,0,1,1,1,0), X}] = X_{-(0,1,1,2,2,2,1)},$$

$$[X_{(0,0,1,1,0,1), X}] = X_{-(1,1,2,2,2,1,1)}, \quad [X_{(0,0,1,1,1,1,0)}, X] = X_{-(0,1,1,2,2,2,1)} .$$

This shows that $\dim[\mathfrak{b}, X] = 15 = \dim N \Rightarrow [\mathfrak{b}, X] = N \Rightarrow N$ is a spherical $L$-module. And thus $G/P_7$ is a spherical $H$-variety. \hfill $\square$

Since $G/P_7$ is a spherical $H$-variety we can derive branching rules for $V_{\omega_7}^* = V_{\omega_7}$.

**Theorem 24:** Let $G$ be the simply connected simple algebraic group of type $E_7$ and $H$ the maximal reductive subgroup of type $A_7$. Then

$$\text{res}_{H}^G(V_{\omega_7}) = \bigoplus_{2a_1 + a_2 + 2a_3 + a_4 = k} V_{a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4} .$$

Proof: With “LiE” we compute

$$\text{res}_{H}^G(V_{\omega_7}) = V_{\lambda_2} \oplus V_{\lambda_6} .$$

So there are two generators of degree 1 of weight $\lambda_2$ and $\lambda_6$. Further we have

$$\text{res}_{H}^G(V_{2\omega_7}) = \mathbb{C} \oplus V_{2\lambda_2} \oplus V_{2\lambda_6} \oplus V_{\lambda_2 + \lambda_6} \oplus V_{\lambda_4} ,$$

which shows that there are 2 generators of degree 2 which are of weight 0 and $\lambda_1$. This shows that $\dim \mathbb{C}[Y]^U_H \geq 4$.

In the proof of the previous theorem we have found an $X \in N$ such that $U_L \cdot X$ is of codimension 3. It follows that $\dim \mathbb{C}[Y]^U_H = 4$ and we have found four generators. The branching rules follow immediately. \hfill $\square$
Next we will consider the Levi subgroup $E_6 \times \mathbb{C}^\ast$, which is obtained by omitting the simple root $\alpha_7$ in the Dynkin-diagram.

**Theorem 25:** Let $G$ be the simply connected simple algebraic group of type $E_7$ and $H \subset G$ the Levi subgroup of type $E_6 \times \mathbb{C}^\ast$. Then $G/P_i$ and $G/P_7$ are spherical $H$-varieties whereas $G/P_i$, $i = 2, \ldots, 6$ are not spherical $H$-varieties.

**Proof:** This was proven in [Lit94].

We get the following branching rules from the spherical cases.

**Theorem 26:** Let $G$ be the simply connected simple algebraic group of type $E_7$ and $H$ the Levi subgroup of type $E_6 \times \mathbb{C}^\ast$. Then we have the following branching rules.

- i) $\text{res}_H^G(V_{k\omega_1}) = \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6} \otimes V_{2a_1-2a_3}$,
- ii) $\text{res}_H^G(V_{k\omega_2}) = \bigoplus_{a_1+a_2+a_3+2a_4+a_5+a_6+a_7=k} V_{a_1\lambda_1+(a_2+a_7)\lambda_2+a_3\lambda_3+a_4\lambda_4+a_5\lambda_5+a_6\lambda_6 \otimes V_{-a_1+3a_3-2a_6}}$,
- iii) $\text{res}_H^G(V_{k\omega_7}) = \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_1+a_2\lambda_6} \otimes V_{-a_1+a_2+3a_3-3a_4}$.

**Proof:** From paragraph 1.4 in [Lit94] we get the following branching rules.

- i) $\text{res}_H^G(V_{k\omega_1}) = \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\omega_1+a_2\omega_2+a_3\omega_6-(a_2+2a_3)\omega_7}$,
- ii) $\text{res}_H^G(V_{k\omega_2}) = \bigoplus_{a_1+a_2+a_3+2a_4+a_5+a_6+a_7=k} V_{a_1\omega_1+(a_2+a_7)\omega_2+a_3\omega_3+a_4\omega_4+a_5\omega_5+a_6\omega_6+(a_1+a_3+2a_4+2a_5+2a_6+a_7)\omega_7}$,
- iii) $\text{res}_H^G(V_{k\omega_7}) = \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\omega_1+a_2\omega_6+(a_3-a_1-a_2-a_4)\omega_7}$.

We have $\omega_i = \lambda_i$ for $i = 1, \ldots, 6$ and we fix the coweight $2\omega_7^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 6\alpha_4^\vee + 5\alpha_5^\vee + 4\alpha_6^\vee + 3\alpha_7^\vee$ which determines the highest weights for $\mathbb{C}^\ast$. Thus we get the branching rules in the theorem.

Now we will turn to the subgroup of $E_7$ of type $D_6 \times A_1$. We will consider the extended Dynkin-diagram of $E_7$ again by adding the smallest root $\delta$ to the simple roots.

![Diagram]

If we omit the simple root $\alpha_6$ we have a sub-diagram of type $D_6 \times A_1$ and consider the the corresponding subsystem subgroup. Explicitly we can choose the following simple roots:

- $(1,0,0,0,0,0,0)_H = (0,1,1,2,2,2,1)$, $(0,1,0,0,0,0,0,0)_H = (1,0,0,0,0,0,0,0)$,
- $(0,0,1,0,0,0,0,0)_H = (0,0,1,0,0,0,0,0)$, $(0,0,0,1,0,0,0,0)_H = (0,0,0,1,0,0,0,0)$,
- $(0,0,0,0,1,0,0,0)_H = (0,1,0,0,0,0,0,0)$, $(0,0,0,0,0,1,0,0)_H = (0,0,0,0,1,0,0,0)$,
- $(0,0,0,0,0,0,1)_H = (0,0,0,0,0,0,1)$.
**Theorem 27:** Let $G$ be the simply connected simple algebraic group of type $E_7$. If $H$ is the subgroup of type $D_6 \times A_1$ then $G/P_i$ is a spherical $H$-variety and $G/P_i$ is not a spherical $H$-variety for $i = 1, \ldots, 6$.

**Proof:** Dimension comparison shows that $G/P_3, \ldots, G/P_6$ are not $H$-spherical. For $G/P_1$ we can compute the restriction of $V_{k\omega_i}$ (note that $\omega_i^* = \omega_i$ for $E_7$) with LiE and get

$$\text{res}_H^G(V_{k\omega_i}) = \ldots \oplus 2(V_{2\lambda_6} \otimes V_{2\lambda_7}) \oplus \ldots .$$

Thus there are multiplicities in this case and we know that the $H$-variety $G/P_1$ is not $H$-spherical.

**Case $G/P_2$:** We compute

$$N = \mathbb{C}X_{-(0,0,0,0,1,1,1)} \oplus \mathbb{C}X_{-(0,0,0,0,1,1,1)} \oplus \mathbb{C}X_{-(0,0,0,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,0,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,2,1,1)} \oplus \mathbb{C}X_{-(1,2,3,2,1,1)} .$$

Now define $X := X_{-(1,2,2,3,2,1,1)} + X_{-(1,0,1,1,1,1,1)}$. The roots of these two root vectors are linearly independent and we have

$$[\mathfrak{h}, X] = \langle X_{-(1,2,2,3,2,1,1)}, X_{-(1,0,1,1,1,1,1)} \rangle$$

Further we have

$$[X_{(1,0,0,0,0,0,0)}, X] = X_{-(0,1,1,1,1,1,1)}, \quad [X_{(0,1,0,0,0,0,0)}, X] = X_{-(1,1,2,2,1,1,1)} ,$$

$$[X_{(1,0,1,1,0,0,0)}, X] = X_{-(0,0,0,0,1,1,1)}, \quad [X_{(0,1,1,0,1,0,0)}, X] = X_{-(0,1,1,2,1,1,1)} ,$$

$$[X_{(0,1,0,1,1,0,0)}, X] = X_{-(0,0,0,0,1,1,1)}, \quad [X_{(0,1,1,1,1,0,0)}, X] = X_{-(0,1,2,2,1,1,1)} ,$$

$$[X_{(1,0,1,1,1,0,0)}, X] = X_{-(0,0,0,0,1,1,1)}, \quad [X_{(1,0,1,1,2,1,1)}, X] = X_{-(1,1,1,1,1,1,1)} ,$$

$$[X_{(1,1,1,1,1,0,0)}, X] = X_{-(0,1,1,2,1,1,1)}, \quad [X_{(1,1,1,1,2,1,0)}, X] = X_{-(1,1,1,1,1,1,1)} ,$$

$$[X_{(1,1,1,2,1,1,1)}, X] = X_{-(0,1,1,1,1,1,1)}, \quad [X_{(1,1,2,2,1,1,0)}, X] = X_{-(0,1,0,1,1,1,1)} .$$

So we have $\dim [\mathfrak{h}, X] = 16 = \dim N$. This implies that $N$ is a spherical $L$-module and thus $G/P_2$ is a spherical $H$-variety.

From the sphericity of $G/P_i$ we can derive branching rules for $V_{k\omega_i^*} = V_{k\omega_i}$.

**Theorem 28:** Let $G$ be the simply connected simple algebraic group of type $E_7$ and let $H$ be a maximal reductive subgroup of type $D_6 \times A_1$. Then

$$\text{res}_H^G(V_{k\omega_i}) = \bigoplus_{a_1 + 2a_2 + a_3 = k} V_{a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_6} \otimes V_{a_1\lambda_7} .$$

**Proof:** With “LiE” we compute

$$\text{res}_H^G(V_{\omega_i}) = (V_{\lambda_1} \otimes V_{\lambda_7}) \oplus (V_{\lambda_6} \otimes \mathbb{C}) .$$
So there are two generators of degree 1 with weights \((\lambda_1, \lambda_7)\) and \((\lambda_6, 0)\). Further we have
\[
\text{res}_H^G(V_{2\omega_7}) = (V_{2\lambda_1} \otimes V_{2\lambda_7}) \oplus (V_{\lambda_1 + \lambda_6} \otimes V_{\lambda_7}) \oplus (V_{2\lambda_6} \otimes \mathbb{C}) \oplus (V_{\lambda_2} \otimes \mathbb{C}).
\]
Thus there is a further generator of degree 2 and weight \(\lambda_2\) and we know that \(\dim \mathbb{C}[\hat{Y}]^{U_H} \geq 3\).

In the proof of the previous theorem we have seen that there is an \(X \in N\) such that \(\dim U_H X\) is of codimension 2. It follows that \(\dim \mathbb{C}[\hat{Y}]^{U_H} = 3\). The branching rules follow.

The last maximal reductive subgroup of \(G\) where a sphericity of \(G/P_i\) can occur is the group \(H\) of type \(A_1 \times F_4\). From the table with dimensions of \(G/P_i\) we know that only \(G/P_7\) can be a spherical \(H\)-variety. But with LiE we compute
\[
\text{res}_H^G(V_{4\omega_7}) = \ldots \oplus 2(V_{4\lambda_1} \otimes V_{\lambda_8}) \oplus \ldots
\]
and thus there are multiplicities in this case. We have shown:

**Theorem 29:** Let \(G\) be the simply connected simple group of type \(E_7\) and \(H\) the maximal subgroup of type \(A_1 \times F_4\).

Then \(G/P_i\) (\(i = 1, \ldots, 7\)) is not a spherical variety.

\[
\begin{array}{c|cccccccc}
H & E_7 \times A_1 & E_6 \times A_2 & A_3 \times D_5 & A_4 \times A_4 & A_5 \times A_2 \times A_1 \\
\text{dim } B_H & 72 & 47 & 34 & 28 & 27 \\
H & A_7 \times A_1 & D_8 & A_8 & G_2 \times F_4 & A_2 \times A_1 & C_2 & A_1 \\
\text{dim } B_H & 37 & 72 & 44 & 36 & 6 & 6 & 2 \\
\end{array}
\]

The dimensions of the varieties \(G/P_i\) (\(i = 1, \ldots, 8\)) are:

\[
\begin{array}{cccccccc}
G/P_1 & G/P_2 & G/P_3 & G/P_4 & G/P_5 & G/P_6 & G/P_7 & G/P_8 \\
\text{dim} & 78 & 92 & 98 & 106 & 104 & 97 & 83 & 57 \\
\end{array}
\]

By dimension comparison there are only two possibilities of sphericity. If we take the maximal reductive subgroup \(H_1\) of type \(E_7 \times A_1\) or the maximal reductive subgroup \(H_2\) of type \(D_8\), then the variety \(G/P_8\) can be spherical for \(H_1\) or \(H_2\). But we can compute the following restrictions by using LiE
\[
\text{res}_{H_1}^G(V_{8\omega_8}) = \ldots \oplus 2(V_{1\lambda_1 + 2\lambda_7} \otimes V_{2\lambda_8}) \oplus \ldots,
\]
\[
\text{res}_{H_2}^G(V_{4\omega_8}) = \ldots \oplus 2V_{\lambda_8} \oplus \ldots,
\]
which show that there are multiplicities in these cases. So there are no spherical cases for \(G\). We have shown:

**Theorem 30:** Let \(G\) be the simply connected simple algebraic groups of type \(E_8\). Let \(H\) be one of its maximal reductive subgroups.

Then \(G/P_i\) (\(i = 1, \ldots, 8\)) is not a spherical variety.
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