RELATIVE DONALDSON-THOMAS THEORY
FOR CALABI-YAU 4-FOLDS

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Abstract. Given a complex 4-fold $X$ with an (Calabi-Yau 3-fold) anti-canonical divisor $Y$, we study relative Donaldson-Thomas invariants for this pair, which are elements in the Donaldson-Thomas cohomologies of $Y$. We also discuss gluing formulas which relate relative invariants and $DT_3$ invariants for Calabi-Yau 4-folds.

1. Introduction

Donaldson-Thomas invariants were proposed by Donaldson and Thomas [18], and defined in Thomas’s thesis [53]. They count stable sheaves on Calabi-Yau 3-folds, which are related to many other interesting subjects, including Gopakumar-Vafa conjecture on BPS numbers in string theory [23], [25], [32] and MNOP conjecture [42], [43], [44], [49] relating $DT_3$ invariants to Gromov-Witten invariants. The generalization of $DT_3$ invariants to count strictly semi-stable sheaves is due to Joyce and Song [31] using Behrend’s result [4].

Kontsevich and Soibelman proposed generalized as well as motivic $DT$ theory for Calabi-Yau 3-categories [33], which was later studied by Behrend-Bryan-Szendroi [5] for Hilbert schemes of points. The wall-crossing formula [33], [31] is an important structure for Bridgeland’s stability condition [10] and Pandharipande-Thomas invariants [50], [54].

As a categorification of Donaldson-Thomas invariants, Brav, Bussi, Dupont, Joyce and Szendroi [8] and Kiem and Li [32] recently defined a cohomology theory for Calabi-Yau 3-folds whose Euler characteristic is the $DT_3$ invariant. The point is that moduli spaces of simple sheaves on Calabi-Yau 3-folds are critical points of holomorphic functions locally [9], [31], and we could consider perverse sheaves of vanishing cycles of these functions. They glued these local perverse sheaves and defined its hypercohomology as $DT_3$ cohomology. In general, such a gluing requires a square root of the determinant line bundle of the moduli space [26], [33], [48].

As an extension of Donaldson-Thomas invariants to Calabi-Yau 4-folds, Borisov and Joyce [7] and the authors [12], [13] developed $DT_4$ invariants (or ‘holomorphic Donaldson invariants’) which count stable sheaves on Calabi-Yau 4-folds. It is desirable to construct a TQFT type structure for these $DT_4$ and $DT_3$ theories. The purpose of this paper is to make some initial steps in this direction. We remark that Joyce also has a program of establishing TQFT structures on Calabi-Yau 3 and 4-folds [30] using Pantev-Tøen-Vaquié-Vezzosi’s shifted symplectic structures on derived schemes [51].

Our set-up is a smooth Calabi-Yau 3-fold $Y = s^{-1}(0)$ as an anti-canonical divisor of a complex projective 4-fold $X$, where $s \in \Gamma(X, K_X^{-1})$. Then $1/s$ is nowhere vanishing inside $X \setminus Y$ which gives a trivialization of its canonical bundle. Thus $X \setminus Y$ is an open Calabi-Yau 4-fold which has a compactification $X$ by adding a compact Calabi-Yau 3-fold.

We consider any Gieseker moduli space $\mathfrak{M}_X$ of semi-stable sheaves which consists of slope-stable bundles only, and assume there exists a restriction morphism (see Theorem 3.1 for its existence) $r: \mathfrak{M}_X \to \mathfrak{M}_Y$ to a Gieseker moduli space of stable sheaves on $Y$. The deformation-obstruction theory associated to $r$ is described as follows: for any stable bundle $E \in \mathfrak{M}_X$, we have an exact sequence

$$0 \to H^1(X, \text{End}_0 E \otimes K_X) \to H^1(X, \text{End}_0 E) \to H^1(Y, \text{End}_0 E|_Y) \to H^2(X, \text{End}_0 E \otimes K_X) \to H^2(X, \text{End}_0 E) \to H^2(Y, \text{End}_0 E|_Y) \to H^3(X, \text{End}_0 E \otimes K_X) \to H^3(X, \text{End}_0 E) \to 0.$$

Note that the transpose of the above sequence with respect to Serre duality pairings on $X$ and $Y$ is itself [18]. This is the key property for the definition of relative $DT_4$ virtual cycles, which we define for the following three good cases.
Case I. [Rigid case] If every $E \in \mathcal{M}_X$ satisfies $H^1(Y, \text{End}_0E|_Y) = 0$, then the above long exact sequence breaks into canonical isomorphisms

$$H^1(X, \text{End}_0E) \cong H^1(X, \text{End}_0E)^*, \quad H^2(X, \text{End}_0E) \cong H^2(X, \text{End}_0E)^*.$$ 

Similar to the case of Calabi-Yau 4-folds [7], $\mathcal{M}_X$ will have a $(-2)$-shifted symplectic structure in the sense of PTVV [51]. By Borisov-Joyce [7], there exists a virtual cycle, which we define to be the relative $DT_4$ virtual cycle.

Case II. [Surjective case] If $r : \mathcal{M}_X \to \mathcal{M}_Y$ is a surjective map between smooth moduli spaces (throughout this paper, unless specified otherwise, smooth moduli spaces mean their Kuranishi maps are zero), we obtain a canonical isomorphism

$$H^2(X, \text{End}_0E) \cong H^2(X, \text{End}_0E)^*,$$

which endows the obstruction bundle $\text{Ob}_{2\mathcal{M}_X}$ with a non-degenerate quadratic form. Then the relative $DT_4$ virtual cycle is defined to be the Euler class of the self-dual subbundle of $\text{Ob}_{2\mathcal{M}_X}$ as in Definition 5.12 [13].

Case III. [Injective case] If $r : \mathcal{M}_X \to \mathcal{M}_Y$ is an injective map between smooth moduli spaces, we obtain an exact sequence

$$0 \to H^1(X, \text{End}_0E) \to H^1(Y, \text{End}_0E|_Y) \to H^2(X, \text{End}_0E \otimes K_X)$$

$$\to H^2(X, \text{End}_0E) \to H^2(Y, \text{End}_0E|_Y) \to H^3(X, \text{End}_0E \otimes K_X) \to 0.$$

This determines a surjective map

$$s : \text{Ob}_{2\mathcal{M}_X} \to N^*_{\mathcal{M}_X/\mathcal{M}_Y}$$

from the obstruction bundle of $\mathcal{M}_X$ to the conormal bundle of $\mathcal{M}_X$ inside $\mathcal{M}_Y$, and a non-degenerate quadratic form on the reduced bundle $\text{Ob}_{2\mathcal{M}_X} \cong \ker(s)$). As in Case II, we define the relative $DT_4$ virtual cycle $[\mathcal{M}_X^{\text{rel}}]_{\text{vir}} \in H_*(\mathcal{M}_X, \mathbb{Z})$ to be the Euler class of the self-dual subbundle of $\text{Ob}_{2\mathcal{M}_X}$. Note that when $\mathcal{M}_X$ is smooth, $r$ is injective and a neighbourhood of $r(\mathcal{M}_X) \subseteq \mathcal{M}_Y$ is smooth, $[\mathcal{M}_X^{\text{rel}}]_{\text{vir}}$ can also be defined in a similar way. It is easy to check these definitions of relative $DT_4$ virtual cycles in Cases I-III are all compatible. We compute examples for relative $DT_4$ virtual cycles in Proposition 3.10.

To sum up, we define the notion of admissibility for Gieseker moduli spaces.

**Definition 1.1.** Let $Y$ be a smooth anti-canonical divisor of a projective 4-fold $X$, and $\mathcal{M}_X$ be a Gieseker moduli space of semi-stable sheaves. $\mathcal{M}_X$ is admissible with respect to $(X, Y)$ if

(i) $\mathcal{M}_X$ consists of slope-stable bundles only, and

(ii) there exists a restriction morphism $r : \mathcal{M}_X \to \mathcal{M}_Y$ to a Gieseker moduli space of stable sheaves on $Y$ [4].

**Theorem/Definition 1.2.** If $\mathcal{M}_X$ is admissible with respect to $(X, Y)$, then the relative $DT_4$ virtual cycle exists, i.e.

$$[\mathcal{M}_X^{\text{rel}}]_{\text{vir}} \in H_*(\mathcal{M}_X, \mathbb{Z})$$

provided that any one of the following conditions holds,

(1) $r(\mathcal{M}_X)$ is rigid, i.e. $H^1(Y, \text{End}_0E|_Y) = 0$ for any $E \in \mathcal{M}_X$; or

(2) $r$ is surjective between smooth moduli spaces; or

(3) $r$ is injective between smooth moduli spaces (at least when restricted to a neighbourhood of $r(\mathcal{M}_X)$ in $\mathcal{M}_Y$).

Furthermore, $[\mathcal{M}_X^{\text{rel}}]_{\text{vir}}$ will be defined over integer if $r : \mathcal{M}_X \to \mathcal{M}_Y$ is orientable in relative $DT_4$ theory [4].

In general, the virtual dimension of $[\mathcal{M}_X^{\text{rel}}]_{\text{vir}}$ is not zero, and we introduce the $\mu$-map to cut it down and define the relative $DT_4$ invariant. The relative $DT_4$ invariant is a map

$$v(\mathcal{M}_X) : \text{Sym}^* (H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, \ldots]) \to H^*(\mathcal{M}_Y, \mathcal{P}_{\mathcal{M}_Y}^*),$$

where $\mathcal{P}_{\mathcal{M}_Y}^*$ is the perverse sheaf constructed by Brav-Bussi-Dupont-Joyce-Szendroi [8] and Kiem-Li [32]. In Cases I-III, if $\mathcal{M}_Y$ is smooth, the perverse sheaf $\mathcal{P}_{\mathcal{M}_Y}^*$ is the $C$-constant sheaf (up to some degree shift) and the relative $DT_4$ invariant $v(\mathcal{M}_X)$ is defined by pairing the relative $DT_4$ virtual cycle, $\mu$-map and pull-back classes from $H^*(\mathcal{M}_Y, \mathbb{C})$.

So far, we only work with holomorphic bundles (i.e. $\mathcal{M}_Y$ consists of bundles only). To extend to other coherent sheaves, say ideal sheaves of subschemes, one difficulty is that we will not have

\footnote{Theorem 3.1 ensures we many examples}

\footnote{See Theorem 4.7 for the existence of orientations in relative $DT_4$ theory}
restriction maps in the usual sense as subschemes would sit inside the divisor \(Y \subseteq X\). To handle this issue, we introduce Li-Wu’s good degenerations of Hilbert schemes [39]. Li-Wu’s idea is to blow up the divisor once subschemes sit inside \(Y\) (like neck-stretching in Donaldson theory) and we are then reduced to consider subschemes which are ‘transversal’ to \(Y\).

Working with their good degenerations, we study extensions of the above Cases I-III to ideal sheaf cases. In particular, we study the obstruction theory of moduli spaces of relative ideal sheaves (Lemma 5.3) and discuss a gluing formula (Theorem 5.12) based on certain conjectures. We also compute examples on relative \(DT_4\) virtual cycles (Example 4.1 – Example 4.3) based on Definition 5.4 which include

**Example 1.3.** (Generic quintic in \(P^4\), Example 4.4)

We take \(X = P^4\) which contains a generic quintic 3-fold \(Y = Q\) as its anti-canonical divisor, and consider the primitive curve class \([H] \in H^2(X, \mathbb{Z})\). Ideal sheaves of curves representing this class have Chern character \(c = (1, 0, 0, -PD([H]), \frac{1}{2})\) and we denote their moduli space by \(I_2(X, [H]) \cong \text{Gr}(2, 5)\). The generic quintic \(Q\) contains 2875 rigid degree 1 rational curves and \(I_2(X, [H])\) contains a finite subset \(S\) with 2875 points. \(I_2(X, [H])\) has a well-defined restriction morphism to \(\text{Hilb}^5(Q)\). To extend the morphism across those 2875 points, we introduce Li-Wu’s good degenerations to torsion sheaves and verify Conjectures 5.8, 5.9 in [39].

We adapt Li-Wu’s good degenerations to torsion sheaves and verify Conjectures 5.8, 5.9 in the following case.

**Example 1.5.** (Relative \(DT_4/DT_3\), Example 4.5)

Let \(X = Y_1 \times P^1\) which contains \(Y = Y_1 \times \{0\} \sqcup (Y_1 \times \infty)\) as an anti-canonical divisor, where \(Y_1\) is a compact Calabi-Yau 3-fold. We denote \(\mathcal{M}_c(Y_1)\) to be a Gieseker moduli space of torsion-free semi-stable sheaves on \(Y_1\) with Chern character \(c \in H^{even}(Y_1, Q)\) (we assume there is no strictly semi-stable sheaf), and denote \(\mathcal{M}_c(X)\) to be the moduli space of sheaves on \(X\) which are push-forward of stable sheaves in \(\mathcal{M}_c(Y_1 \times t)\) for some \(t = (\mathcal{M}_c(Y_1) \times P^1)\). To have a well-defined restriction map, we introduce \(X[1]_0 = \Delta_- \cup X \cup \Delta_1\), where \(\Delta_{\pm 1} \cong Y_1 \times \mathbb{P}^1\), and consider the relative moduli space \(\mathcal{M}_c(X[1]_0)\) with

\[
\mathcal{M}_c(X[1]_0) \cong \mathcal{M}_c(Y_1) \times \mathbb{P}^1.
\]

Conjectures 5.8, 5.9 hold and the relative \(DT_4\) virtual cycle satisfies

\[
[\mathcal{M}_c^{rel}(X[1]_0)]^{vir} = DT_3(\mathcal{M}_c(Y_1)) \cdot [P^1] \in H_2(\mathcal{M}_c(X[1]_0), \mathbb{Z}),
\]

where \(DT_3(\mathcal{M}_c(Y_1))\) is the Donaldson-Thomas invariant defined by Thomas [39].

**Theorem 1.6.** For \((X = Y_1 \times P^1, Y = Y_1 \times \{0, \infty\})\), where \(Y_1\) is a compact Calabi-Yau 3-fold, we have a restriction map

\[
\mathcal{M}_c(X[1]_0) \cong \mathcal{M}_c(Y_1) \times \mathbb{P}^1 \rightarrow pt,
\]

from the moduli space of relative torsion sheaves coming from push-forward of stable sheaves in \(\mathcal{M}_c(Y_1 \times t)\), where \(\mathcal{M}_c(Y_1)\) is a Gieseker moduli space of torsion-free semi-stable sheaves on \(Y_1\) consisting of no strictly semi-stable sheaf.

The relative \(DT_4\) virtual cycle exists and satisfies

\[
[\mathcal{M}_c^{rel}(X[1]_0)]^{vir} = DT_3(\mathcal{M}_c(Y_1)) \cdot [P^1] \in H_2(\mathcal{M}_c(X[1]_0), \mathbb{Z}),
\]

as its anti-canonical divisor,
where \( DT_3(\mathcal{M}_c(Y)) \) is the \( DT_3 \) invariant of \( Y \) (CY$_3$) with respect to Chern character \( c \).

Finally, we give a coherent description of the orientability issues involved in Cases I-III (Proposition 6.1) and prove the existence of orientations for projective 4-folds with torsion-free homologies.

**Theorem 1.7.** (Theorem 6.4)

Let \( Y \) be a smooth anti-canonical divisor of a projective 4-fold \( X \), and \( \mathcal{M}_X \) be a moduli space of stable bundles on \( X \) with fixed Chern classes.

Suppose \( Tor(H_*(X,\mathbb{Z})) = 0 \) and there exists a well-defined restriction morphism \( r : \mathcal{M}_X \to \mathcal{M}_Y \) to a Gieseker moduli space of stable sheaves on \( Y \).

Then the morphism \( r : \mathcal{M}_X \to \mathcal{M}_Y \) is orientable in relative \( DT_4 \) theory, i.e. there exists a square root \( L_X^\frac{1}{2} \) of \( L_Y \) (\( L_X^\frac{1}{2} \otimes L_Y^{\frac{1}{2}} \cong L_Y \)) such that

\[
c_1(L_X) = r^*c_1(L_Y^\frac{1}{2}),
\]

where \( L_X \) (resp. \( L_Y \)) is the determinant line bundle of \( \mathcal{M}_X \) (resp. \( \mathcal{M}_Y \)).

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2. Review of basic facts in DT theory

2.1. Some basic facts in \( DT_4 \) theory. We start with a complex projective Calabi-Yau 4-fold \( (X,\mathcal{O}_X(1)) \) (\( Hol(X) = SU(4) \)) with a Ricci-flat Kähler metric \( g \) [58], a Kähler form \( \omega \), a holomorphic four-form \( \Omega \), and a topological bundle with a Hermitian metric \( (E,h) \). We define

\[
*_* = (\Omega,\rho) \circ * : \Omega^{0,2}(X,EndE) \to \Omega^{0,2}(X,EndE),
\]

with \( *_* = 1 \) and it splits the corresponding harmonic subspace into (anti-)self-dual parts.

The \( DT_4 \) equations are defined to be

\[
\begin{aligned}
F_{0,2}^0 &= 0 \\
F \wedge \omega^3 &= 0,
\end{aligned}
\]

where the first equation is \( F_{0,2}^0 + *_*F_{0,2}^0 = 0 \) and we assume \( c_1(E) = 0 \) for simplicity in the moment map equation \( F \wedge \omega^3 = 0 \).

We denote \( \mathcal{M}^{DT_4}(X,g,[\omega],c,h) \) or simply \( \mathcal{M}^{DT_4}_c \) to be the space of gauge equivalence classes of solutions to the \( DT_4 \) equations on \( E \) with \( ch(E) = c \).

We take \( \mathcal{M}^{bdl}_c \) to be the moduli space of slope-stable holomorphic bundles with fixed Chern character \( c \). By the Donaldson-Uhlenbeck-Yau’s theorem [55], we can identify it with the moduli space of gauge equivalence classes of solutions to the holomorphic HYM equations

\[
\begin{aligned}
F_{0,2}^{bdl} &= 0 \\
F \wedge \omega^3 &= 0.
\end{aligned}
\]

By Lemma 4.1 [13], if \( ch_3(E) \in H^{2,2}(X,\mathbb{C}) \), then \( F_{0,2}^{bdl} = 0 \Rightarrow F_{0,2}^0 = 0 \). In particular, if \( \mathcal{M}^{bdl}_c \neq \emptyset \), then \( \mathcal{M}^{DT_4}_c \cong \mathcal{M}^{bdl}_c \) as sets.

The comparison of analytic structures is given by

**Theorem 2.1.** (Theorem 1.1 [13]) We assume \( \mathcal{M}^{bdl}_c \neq \emptyset \) and fix \( d_A \in \mathcal{M}^{DT_4}_c \), then

1. there exists a Kuranishi map \( \tilde{k} \) of \( \mathcal{M}^{bdl}_c \) at \( d_A \) (the (0,1) part of \( d_A \)) such that \( \tilde{k}_+ \) is a Kuranishi map of \( \mathcal{M}^{DT_4}_c \) at \( d_A \), where

\[
\tilde{k}_+ = \pi_+(\tilde{k}) : H^{0,1}(X,EndE) \xrightarrow{\tilde{k}} H^{0,2}(X,EndE) \xrightarrow{\pi_+} H^{0,2}(X,EndE)
\]

and \( \pi_+ \) is projection to the self-dual forms;

2. the closed imbedding between analytic spaces possibly with non-reduced structures \( \mathcal{M}^{bdl}_c \hookrightarrow \mathcal{M}^{DT_4}_c \) is also a homeomorphism between topological spaces.
Remark 2.2. By Proposition 10.10 [13], the map \(\tilde{\kappa}\) satisfies \(Q_{\text{Serre}}(\tilde{\kappa}, \tilde{\kappa}) \geq 0\), where \(Q_{\text{Serre}}\) is the Serre duality pairing on \(H^{0,2}(X, \text{End}E)\).

To define Donaldson type invariants using \(\mathcal{M}_c^{DT_4}\), we need to give it a good compactification (such that it carries a deformation invariant fundamental class). For this purpose, we introduce \(\mathcal{M}_c(X, \mathcal{O}_X(1))\) or simply \(\mathcal{M}_c\) to be the Gieseker moduli space of \(\mathcal{O}_X(1)\)-stable sheaves on \(X\) with given Chern character \(c\). Motivated by Theorem 2.4, we make the following definition.

Definition 2.3. ([13]) We call a \(C^\infty\)-scheme, \(\overline{\mathcal{M}}_c^{DT_4}\), generalized \(DT_4\) moduli space if there exists a homeomorphism

\[ \mathcal{M}_c \to \overline{\mathcal{M}}_c^{DT_4} \]

such that at each closed point of \(\mathcal{M}_c\), say \(F\), \(\overline{\mathcal{M}}_c^{DT_4}\) is locally isomorphic to \(\kappa_+^{-1}(0)\), where

\[ \kappa_+ = \pi_+ \circ \kappa : \text{Ext}^1(F, F) \to \text{Ext}^2(F, F), \]

\(\kappa\) is a Kuranishi map at \(F\) and \(\text{Ext}^2(F, F)\) is a half dimensional real subspace of \(\text{Ext}^2(F, F)\) on which the Serre duality quadratic form is real and positive definite.

Remark 2.4.
1. The existence of generalized \(DT_4\) moduli spaces is proved by Borisov-Joyce [7]. The authors proved their existence as real analytic spaces in certain cases and defined the corresponding virtual fundamental classes [12, 13].
2. For fixed data \((X, \mathcal{O}_X(1), c)\), generalized \(DT_4\) moduli spaces may not be unique. However, they all carry the same virtual fundamental class.

The proof of Borisov-Joyce’s gluing result is divided into two parts. Firstly, they use good local models of \(\mathcal{M}_c\), i.e. local ‘Darboux charts’ in the sense of Brav, Busi and Joyce [9]. Then they chose the half dimensional real subspace \(\text{Ext}^2(F, F)\) appropriately and use partition of unity and homotopical algebra to glue \(\kappa_+\). We state our interested version of the local ‘Darboux charts’ and give a proof using gauge theory.

Theorem 2.5. (Brav-Bussi-Joyce [9], Corollary 5.20, see also Theorem 10.7 [13])

Let \(\mathcal{M}_c\) be a Gieseker moduli space of stable sheaves on a compact Calabi-Yau 4-fold \(X\). Then for any closed point \(F \in \mathcal{M}_c\), there exists an analytic neighborhood \(U_F \subseteq \mathcal{M}_c\), a holomorphic map near the origin

\[ \kappa : \text{Ext}^1(F, F) \to \text{Ext}^2(F, F) \]

such that \(Q_{\text{Serre}}(\kappa, \kappa) = 0\) and \(\kappa^{-1}(0) \cong U_F\) as complex analytic spaces possibly with non-reduced structures, where \(Q_{\text{Serre}}\) is the Serre duality pairing on \(\text{Ext}^2(F, F)\).

Proof. (See the Proof of Theorem 10.7 [13]) The point is that we can use Seidel-Thomas twists [31, 51] to transfer the problem to a problem on moduli spaces of holomorphic bundles and then notice \(\chi_2(E) \wedge \Omega = 0\) for holomorphic bundle \(E\), where \(\Omega\) is the holomorphic top form. \(\square\)

To make sense of virtual fundamental classes of generalized \(DT_4\) moduli spaces as homology classes in \(\mathcal{M}_c\)’s, one would in general rely on Joyce’s D-manifolds theory [29] or Fukaya-Oh-Ohta-Ono’s theory of Kuranishi spaces [21] or Hofer’s polyfolds theory [24, 57]. Assuming this part, which is claimed by Borisov-Joyce, we have

Theorem 2.6. (Borisov-Joyce [1, 30])

Let \(X\) be a complex projective Calabi-Yau 4-fold, and \(\mathcal{M}_c\) be a Gieseker moduli space of stable sheaves which is compact (it is true if the degree and rank of sheaves are coprime). Then there exists a generalized \(DT_4\) moduli space \(\overline{\mathcal{M}}_c^{DT_4}\). If we further assume \(\overline{\mathcal{M}}_c^{DT_4}\) is orientable in the sense of D-manifold, then the virtual fundamental class of \(\overline{\mathcal{M}}_c^{DT_4}\) exists and is a well-defined homology class, i.e.

\[ [\overline{\mathcal{M}}_c^{DT_4}]^{\text{vir}} \in H_4(\mathcal{M}_c, \mathbb{Z}), \]

which coincides with earlier definitions of \(DT_4\) virtual cycles (Definition 5.3, 5.12, 5.14 [13]).

We can furthermore define the \(DT_4\) invariant by pairing the above cycle with \(\mu\)-map as in Definition 5.15 [13]. With appropriate choice of orientations, \(DT_4\) invariants are invariant under deformations of complex structures of \(X\).

Because of the Serre duality for \(\text{Ext}^*(F, F)\), the existence of an orientation on a generalized \(DT_4\) moduli space \(\overline{\mathcal{M}}_c^{DT_4}\) (in the sense of D-manifold) is equivalent to \(c_1(L_X) = 0 \in H^2(\mathcal{M}_c, \mathbb{Z})\) for the determinant line bundle \(L_X\) with \(L_X|_F = (\Lambda^{\text{top}} \text{Ext}^\text{even}(F, F))^{-1} \otimes \Lambda^{\text{top}} \text{Ext}^\text{odd}(F, F)\).

In Corollary 10.17 of [13], the authors proved \(c_1(L_X) = 0\) for Calabi-Yau 4-folds \(X\)’s satisfying \(H^{\text{odd}}(X, \mathbb{Z}) = 0\). More generally, we have
This motivates later definitions of relative $DT_4$ invariants.

2.2. Some basic facts in $DT_3$ theory. Moduli spaces of simple sheaves on CY$_3$’s are locally critical points of holomorphic functions [9], [31] and we can consider the perverse sheaves of vanishing cycles of these functions. The expected cohomology which categorifies $DT_3$ invariant is defined by first gluing these local perverse sheaves and then taking its hypercohomology.

Theorem 2.8. (Brav-Bussi-Dupont-Joyce-Szendroi [8], Kiem-Li [32]) Let $Y$ be a Calabi-Yau 3-fold over $\mathbb{C}$, and $M$ a classical moduli scheme of simple coherent sheaves or simple complexes of coherent sheaves on $Y$, with natural (symmetric) obstruction theory $\phi : E^* \to \mathcal{L}^*$. Suppose we are given a square root of $\det(E^*)$, then there exists a perverse sheaf $\mathcal{P}_M$ uniquely up to canonical isomorphism such that the Euler characteristic of its hypercohomology is the Donaldson-Thomas invariant [53].

2.3. An overview of TQFT type structures in $DT_3$ theories. In this subsection, we give an overview of TQFT structures in gauge theories on Calabi-Yau 3-folds and 4-folds.

We take a smooth (Calabi-Yau) 3-fold $Y$ in a complex projective 4-fold $X^+$ as its anti-canonical divisor, and consider a moduli space $\mathcal{M}_{X^+}$ of stable bundles with fixed Chern classes on $X^+$ which has a well-defined restriction morphism $r : \mathcal{M}_{X^+} \to \mathcal{M}_Y$ to a moduli space of stable sheaves on $Y$. This would determine a class $v(\mathcal{M}_{X^+}) \in H^*(\mathcal{P}_Y^*)$.

Given another complex projective 4-fold $X^-$ which contains $Y$ as its anti-canonical divisor, we form a singular space $X_0 = X^+ \cup_Y X^-$. When $X_0$ admits a smooth deformation $X_t$, $X_t$ will be a family of CY$_4$’s provided that the normal bundle of $Y$ in $X^\pm$ is trivial. As the perverse sheaf in Theorem 2.8 is self-dual under the Verdier duality, i.e. $\mathcal{P}_{\mathcal{M}_Y} \cong \mathcal{P}^*_{\mathcal{M}_Y}$, we can define $(v(\mathcal{M}_{X^+}), v(\mathcal{M}_{X^-}))$ using the Verdier duality on $H^*(\mathcal{M}_Y, \mathcal{P}_{\mathcal{M}_Y}^*)$ as long as $\mathcal{M}_Y$ is compact.

Ignoring the stability issue and contributions from general coherent sheaves, we ask the following question which can be regarded as a complexification of Chern-Simons-Donaldson-Floer TQFT structure for 3 and 4-manifolds (see Atiyah [1]).

Question 2.9. What is the relation between $DT_4$ invariants and relative $DT_4$ invariants, namely, comparing $DT_4(\mathcal{M}_{X})$ with $\langle v(\mathcal{M}_{X^+}), v(\mathcal{M}_{X^-}) \rangle$?

3. Relative $DT_4$ invariants for holomorphic bundles

3.1. Definitions of relative $DT_4$ invariants. In this section, we restrict to some good cases and define rigorously the relative $DT_4$ invariant mentioned before, i.e. $v(\mathcal{M}_{X^+}) \in H^*(\mathcal{P}_Y^*)$.

To have a well-defined restriction map, in this section, we assume all Gieseker moduli spaces of semi-stable sheaves on 4-folds consist of slope-stable bundles only. This will serve as a model for the later study of relative $DT_4$ invariants for ideal sheaves.

Let $Y$ be a smooth anti-canonical divisor of a smooth projective 4-fold $X$, $\mathcal{M}_X$ be a Gieseker moduli space which is admissible with respect to $(X, Y)$ (see Definition 1.1), and $r : \mathcal{M}_X \to \mathcal{M}_Y$ be the restriction morphism to a Gieseker moduli space on $Y$. We recall the following criterion which ensures that we have such morphism $r$ in many cases.

Theorem 3.1. (Flenner [20]) Let $(X, \mathcal{O}_X(1))$ be a complex n-dimensional normal projective variety with $\mathcal{O}_X(1)$ very ample.
We take $F$ to be a $\mathcal{O}_X(1)$-slope semi-stable torsion-free sheaf of rank $r$. $d$ and $1 \leq c \leq n - 1$ are integers such that
\[
\left(\frac{n + d}{d}\right) - cd - 1/d > \deg(\mathcal{O}_X(1)) \cdot \max\left(\frac{r^2 - 1}{4}, 1\right).
\]

Then for a generic complete intersection $Y = H_1 \cap \cdots \cap H_c$ with $H_i \in |\mathcal{O}_X(d)|$, $F|_Y \cong F \otimes \mathcal{O}_Y$ is $\mathcal{O}_X(1)|_Y$-slope semi-stable on $Y$.

**Remark 3.2.** For $X = \mathbb{P}^4$, $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^4}(1)$ is very ample. We take $c = 1$ and $d = 5$, then any rank $r \leq 5$ semi-stable sheaf on $\mathbb{P}^4$ remains semi-stable when restricted to a generic quintic 3-fold inside.

The deformation-obstruction theory associated to the restriction morphism $r$ is described by the following exact sequence.

**Lemma 3.3.** We take a stable bundle $E \in \mathcal{M}_X(x)$, and assume $Y$ is connected, then we have a long exact sequence,
\[
0 \rightarrow H^1(X, EndE \otimes K_X) \rightarrow H^1(X, EndE) \rightarrow H^1(Y, EndE|_Y) \rightarrow \]
\[
\rightarrow H^2(X, EndE \otimes K_X) \rightarrow H^2(X, END) \rightarrow H^2(Y, END|_Y) \rightarrow \]
\[
\rightarrow H^3(X, EndE \otimes K_X) \rightarrow H^3(X, END) \rightarrow 0.
\]

**Proof.** We tensor $0 \rightarrow \mathcal{O}_X(-Y) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$ with $EndE$ and take its cohomology. \hfill \Box

We note that the transpose of the above sequence with respect to Serre duality pairing on $X$ and $Y$ remains the same. This will be the key property used in the following definitions of relative DT invariants.

**Case I: when $\mathcal{M}_Y$ is of expected dim.** If we assume $H^1(Y, EndE|_Y) = 0$ for any $E \in \mathcal{M}_X$, then $H^2(Y, EndE|_Y) = 0$ and $\mathcal{M}_Y$ contains components of finite points, labeled by $E_1, \ldots, E_m$ which come from restrictions of bundles on $X$. We denote $\mathcal{M}_{X,E}$ to be components of $\mathcal{M}_X$ such that $r(\mathcal{M}_{X,E}) = E_i$. By Lemma 3.3, we have canonical isomorphisms
\[
H^3(X, EndE)^* \cong H^1(X, EndE), \quad H^2(X, EndE)^* \cong H^2(X, END).
\]

In fact, we apply Theorem 2.13 of [11] and take the induced shifted symplectic structure on the stable loci as in [7], $\mathcal{M}_{X,E}$ has a $(-2)$-shifted symplectic structure in the sense of [7]. Analogous to Theorem 2.6, we can define the relative $DT_4$ virtual cycle $[\mathcal{M}^{rel}_{X,E}]|_{vir} \in H_n(\mathcal{M}_{X,E}, \mathbb{Z}_n)$ with $n = 1 - \chi(X, END)$. The cycle will be defined over integer if the associated D-manifold of $\mathcal{M}_{X,E}$ is orientable.

The virtual dimension is not zero in general, we introduce a $\mu$-map as in Definition 5.15 [13].

**Definition 3.4.** We denote the universal sheaf of $\mathcal{M}_X$ by $\mathfrak{F} \rightarrow \mathcal{M}_X \times X$.

The $\mu$-map is
\[
\mu : H_*(X) \otimes \mathbb{Z}[x_1, x_2, \ldots] \rightarrow H^*(\mathcal{M}_X),
\]
\[
\mu(\gamma, P) = P(c_1(\mathfrak{F}), c_2(\mathfrak{F}), \ldots)/\gamma.
\]

Pairing virtual cycles with $\mu$-maps defines the relative invariants.

**Definition 3.5.** Let $\mathcal{M}_X$ be a Gieseker moduli space of semi-stable sheaves which is admissible with respect to $(X, Y)$ (see Definition 1.1), and $r : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ be the restriction morphism.

We assume $H^1(Y, EndE|_Y) = 0$ for any $E \in \mathcal{M}_X$, then the relative $DT_4$ invariant is a map
\[
v(\mathcal{M}_X) : Sym^* \left( H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, \ldots] \right) \rightarrow H^*(\mathcal{M}_Y, \mathcal{P}^*_{\mathcal{M}_Y})
\]
such that
\[
v(\mathcal{M}_X)((\gamma_1, P_1), (\gamma_2, P_2), \ldots) = \sum_{i=1}^m [\mathcal{M}^{rel}_{X,E_i}]^{vir} \cup \mu(\gamma_1, P_1) \cup \mu(\gamma_2, P_2) \cup \ldots, E_i,
\]
where $\langle, \rangle$ denotes the natural pairing between homology and cohomology classes and $\{E_i\}_{1 \leq i \leq m}$ are taken as a basis of $H^*(r(\mathcal{M}_X))$.

**Case II: when $\mathcal{M}_X$ and $\mathcal{M}_Y$ are smooth and $r$ is surjective.** We assume $\mathcal{M}_X$ and $\mathcal{M}_Y$ are smooth (i.e. all Kuranishi maps are zero) and the restriction map $r$ is surjective. By Lemma 3.3 we get a canonical isomorphism
\[
H^2(X, END)^* \cong H^2(X, END)
\]
which endows $H^2(X, \text{End}E)$ a non-degenerate quadratic form, and a short exact sequence
\[ 0 \to H^1(X, \text{End}E)^* \to H^1(X, \text{End}E) \to H^1(Y, \text{End}E|_Y) \to 0. \]
Counting dimensions, we have
\[ 2h^1(X, \text{End}E) - h^2(X, \text{End}E) = h^1(Y, \text{End}E|_Y) - \chi(X, \text{End}E) + 1, \]
which is a constant on bundles of $\mathcal{M}_X$ by assumptions. Similar to \[\text{(2)}\], the self-dual subbundle of the obstruction bundle $\text{Ob}_{\mathcal{M}_X}$ exists. If it is also orientable ($\Leftrightarrow c_1(\text{Ob}_{\mathcal{M}_X}) = 0$), we define the relative $DT_4$ virtual cycle $[\mathcal{M}_X^\text{red}|^\text{vir} \subseteq H_n(\mathcal{M}_X, \mathbb{Z})$ to be the Euler class of the self-dual obstruction bundle, where $n = h^1(Y, \text{End}E|_Y) - \chi(X, \text{End}E) + 1$.

**Definition 3.6.** Let $\mathcal{M}_X$ be a Gieseker moduli space of semi-stable sheaves which is admissible with respect to $(X, Y)$ (see Definition \[\text{I.7}\]), and $r : \mathcal{M}_X \to \mathcal{M}_Y$ be the restriction morphism.

We assume $r$ is surjective between smooth moduli spaces, then the relative $DT_4$ invariant is a map
\[ v(\mathcal{M}_X) : \text{Sym}^* (H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]) \to H^*(\mathcal{M}_Y) \]
such that
\[ v(\mathcal{M}_X)((\gamma_1, P_1), (\gamma_2, P_2), ...) = \langle [\mathcal{M}_X^\text{red}|^\text{vir}], (r^*\alpha) \cup \mu(\gamma_1, P_1) \cup \mu(\gamma_2, P_2) \cup ..., \rangle, \]
where $\alpha \in H^*(\mathcal{M}_Y)$ and we identify $H^*(\mathcal{M}_Y) \cong H^*(\mathcal{M}_Y)^*$ via Poincaré pairing, $\langle , \rangle$ denotes the natural pairing between homology and cohomology classes.

**Case III: when $\mathcal{M}_X$ and $\mathcal{M}_Y$ are smooth and $r$ is injective.** We assume $\mathcal{M}_X$ and $\mathcal{M}_Y$ are smooth (i.e. all Kuranishi maps are zero) and the restriction map $r$ is injective. By Lemma \[\text{I.3}\], we get $H^3(X, \text{End}E) = 0$ and an exact sequence
\[ 0 \to H^1(X, \text{End}E) \to H^1(Y, \text{End}E|_Y) \to H^2(X, \text{End}E)^* \to \]
This determine a surjective map
\[ s : \text{Ob}_{\mathcal{M}_X} \to N^*_X/\mathcal{M}_Y, \]
and a non-degenerate quadratic form on the reduced bundle $\text{Ob}_{\mathcal{M}_X}^\text{red} \triangleq \ker(s)$, where $\text{Ob}_{\mathcal{M}_X}$ is the obstruction bundle of $\mathcal{M}_X$ with $\text{Ob}_{\mathcal{M}_X}|_E = H^2(X, \text{End}E)$ and $N^*_X/\mathcal{M}_Y$ is the conormal bundle of $\mathcal{M}_X$ inside $\mathcal{M}_Y$.

Then if the self-dual subbundle of $\text{Ob}_{\mathcal{M}_X}^\text{red}$ is orientable ($\Leftrightarrow c_1(\text{Ob}_{\mathcal{M}_X}^\text{red}) = 0$), we define the relative $DT_4$ virtual cycle $[\mathcal{M}_X^\text{red}|^\text{vir} \subseteq H_n(\mathcal{M}_X, \mathbb{Z})$ to be the Euler class of it, where the virtual dimension is $n = 2h^1(X, \text{End}E) - h^2(X, \text{End}E) - \text{codim}_{\mathcal{M}_X}(\mathcal{M}_X) = h^1(Y, \text{End}E|_Y) - \chi(X, \text{End}E) + 1$. Note that when $\mathcal{M}_X$ is smooth, $r$ is injective and a neighbourhood of $r(\mathcal{M}_X) \subseteq \mathcal{M}_Y$ is smooth, we could also define $[\mathcal{M}_X^\text{red}|^\text{vir}$ in a similar way.

**Definition 3.7.** Let $\mathcal{M}_X$ be a Gieseker moduli space of semi-stable sheaves which is admissible with respect to $(X, Y)$ (see Definition \[\text{I.7}\]), and $r : \mathcal{M}_X \to \mathcal{M}_Y$ be the restriction morphism.

We assume $r$ is injective between smooth moduli spaces (at least when restricted to a neighbourhood of $r(\mathcal{M}_X)$ in $\mathcal{M}_Y$), then the relative $DT_4$ invariant is a map
\[ v(\mathcal{M}_X) : \text{Sym}^* (H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]) \to H^*(\mathcal{M}_Y) \]
such that
\[ v(\mathcal{M}_X)((\gamma_1, P_1), (\gamma_2, P_2), ...) = \langle [\mathcal{M}_X^\text{red}|^\text{vir}], (r^*\alpha) \cup \mu(\gamma_1, P_1) \cup \mu(\gamma_2, P_2) \cup ..., \rangle, \]
where $\alpha \in H^*(\mathcal{M}_Y)$ and we identify $H^*(\mathcal{M}_Y) \cong H^*(\mathcal{M}_Y)^*$ via Poincaré pairing, $\langle , \rangle$ denotes the natural pairing between homology and cohomology classes.

**Remark 3.8.** It is easy to check Definition \[\text{I.3}, \text{I.6}, \text{I.7}\] are all compatible.

In general, one could consider moduli spaces of complexes of simple sheaves \[\text{III}\] on a complex projective 4-fold $X$ and resolve complexes of sheaves by complexes of holomorphic bundles, then there will be a natural restriction morphism to a moduli of simple bundles on an anti-canonical divisor of $X$. A similar long exact sequence in Lemma \[\text{I.3}\] still works and we could study virtual cycle constructions as in Cases I-III.

**Endomorphisms of $DT_3$ cohomologies from relative $DT_4$ invariants.** By considering the trace-free version of Lemma \[\text{I.3}\], the above definitions extend to any disconnected divisor $Y \subseteq X$. We are particularly interested in the case when $X = Y_1 \times \mathbb{P}^1$, where $Y_1$ is a compact Calabi-Yau
3-fold. Then $Y = (Y_1 \times 0) \cup (Y_1 \times \infty)$ will be a smooth anti-canonical divisor of $X$ and the relative $DT_4$ invariant in general is a map
\[ v(\mathcal{M}_X) : \text{Sym}^* (H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]) \to \mathbb{H}^*(\mathcal{M}_Y, P_{3\mathbb{R}_Y}) \cong \mathbb{H}^*(\mathcal{M}_{Y_1}, P_{3\mathbb{R}_{Y_1}}) \otimes \mathbb{H}^*(\mathcal{M}_{Y_1}, P_{3\mathbb{R}_{Y_1}}). \]

By the Verdier duality and $\mathbb{D}^{\mathbb{R}_{Y_1}} (P_{3\mathbb{R}_{Y_1}}) \cong P_{3\mathbb{R}_{Y_1}}$, we have
\[ \mathbb{H}^*(\mathcal{M}_{Y_1}, P_{3\mathbb{R}_{Y_1}}) \cong \mathbb{H}^*(\mathcal{M}_{Y_1}, P_{3\mathbb{R}_{Y_1}})^*. \]

Thus a relative $DT_4$ invariant of $(Y_1 \times \mathbb{P}^1, Y_1 \times \{0, \infty\})$ determines some endomorphisms of the $DT_3$ cohomology
\[ v(\mathcal{M}_X) : \text{Sym}^* (H_*(Y_1)[t]/(t^2) \otimes \mathbb{Z}[x_1, x_2, ...]) \to \text{End}_C(\mathbb{H}^*(\mathcal{M}_{Y_1}, P_{3\mathbb{R}_{Y_1}})) \]
for any Calabi-Yau 3-fold $Y_1$. In general, the above endomorphisms should be used to establish the gluing formula mentioned in Question 2.9.

3.2. Li-Qin’s examples. We have Li-Qin’s examples when conditions in Case II, III are satisfied [40]. Let $Y$ be a generic smooth hyperplane section in $X = \mathbb{P}^1 \times \mathbb{P}^3$ of bi-degree $(2, 4)$,
\[ c = [1 + (1, 1)] \cdot [1 + (\epsilon_1 + 1, \epsilon_2 - 1)], \]
\[ c|_{Y} = [1 + (-1, 1)]|_{Y} \cdot [1 + (\epsilon_1 + 1, \epsilon_2 - 1)|_{Y}], \]
\[ k = (1 + \epsilon_1) \left( \frac{5 - \epsilon_2}{3} \right), \quad \epsilon_1, \epsilon_2 = 0, 1, \quad L_r = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, r). \]

We denote $\mathcal{M}_c(L_r)$ to be the moduli space of $L_r$-slope stable rank-2 bundles on $X$ with a Chern class $c$ and $\mathcal{M}_{c|_{Y}}(L_r|_{Y})$ to be the moduli space of Gieseker $L_r|_{Y}$-semistable rank-2 torsion-free sheaves on $Y$ with Chern class $c|_{Y}$.

**Lemma 3.9.** (Li-Qin [40])

(i) If
\[ \frac{4(2 - \epsilon_2)}{2 + 2\epsilon_1 + \epsilon_2} < r < \frac{4(2 - \epsilon_2)}{\epsilon_1 \epsilon_2}, \]
then $\mathcal{M}_{c|_{Y}}(L_r|_{Y}) \cong \mathbb{P}^k$ and consists of rank-2 stable bundles. Furthermore, the restriction map
\[ r : \mathcal{M}_c(L_r) \to \mathcal{M}_{c|_{Y}}(L_r|_{Y}) \]
is well-defined and an isomorphic between projective varieties.

(ii) If $0 < r < \frac{4(2 - \epsilon_2)}{2 + 2\epsilon_1 + \epsilon_2}$, then $\mathcal{M}_c(L_r)$ and $\mathcal{M}_{c|_{Y}}(L_r|_{Y})$ are empty.

**Proposition 3.10.** In the above example, for any stable bundle $E \in \mathcal{M}_c(L_r)$ on $X = \mathbb{P}^1 \times \mathbb{P}^3$,
\[ \text{Ext}^1_X(E, E) \cong \text{Ext}^1_Y(E|_Y, E|_Y), \]
\[ \text{Ext}^2_X(E, E) \cong \text{Ext}^2_Y(E|_Y, E|_Y) \cong \mathbb{C}^k, \]
\[ \text{Ext}^i_X(E, E) = 0, \quad \text{if } i \geq 2. \]

The relative $DT_4$ virtual cycle $[\mathcal{M}_c(L_r)]_{vir} = [\mathcal{M}_c(L_r)] \in H_{2k}(\mathcal{M}_c(L_r), \mathbb{Z})$.

4. Computational examples of relative $DT_4$ invariants for ideal sheaves

In the above section, we studied relative $DT_4$ invariants for holomorphic bundles. To formulate the gluing formula, we need to have a good understanding of how stable sheaves could be degenerated into union of stable sheaves on irreducible components of degenerated varieties. At this moment, we will restrict ourselves to the ideal sheaves case where degenerations have simpler behavior.

We take a simple degeneration $\pi : X \to C$ of projective manifolds over a pointed smooth curve $(C, 0 \in C)$, i.e. (1) $X$ is smooth, $\pi$ is projective and smooth away from the central fiber $X_0 = \pi^{-1}(0)$, (2) $X_0$ is a union of two smooth irreducible components $X_+$, $X_-$ intersecting transversally along a smooth divisor $Y$. When generic fibers $X_t$’s are Calabi-Yau 4-folds and $Y$ is an anti-canonical divisor of $X_+$, $X_-$, we will study relative $DT_4$ invariants of ideal sheaves for pairs $(X_t, Y)$ and discuss their relations with $DT_4$ invariants of $X_t$, $t \neq 0$. The basic technique is the degeneration method developed by J. Li and B. Wu [36], [37], [39], [50].

Li-Wu’s construction will be recalled in the appendix and the associated obstruction theory is studied there. In this section, we concentrate on computational examples of relative $DT_4$ virtual cycles for ideal sheaves based on the extension (Definition 5.4) of constructions for bundles (Cases I-III).
Example 4.1. (Ideal sheaves of one point)

We take a compact simply connected 4-fold $X_+$ which contains a smooth Calabi-Yau 3-fold $Y$ as its anti-canonical divisor. We consider the moduli space $I_1(X_+, 0)$ of structure sheaves of one point (it is equivalent to consider ideal sheaves of one point) which has a well-defined restriction map to $Y$ if points sit inside $X_+ \setminus Y$.

To extend the map to the whole moduli space, we introduce Li-Wu’s expanded pair, i.e. we consider $X_+[1]_0 = X_+ \cup_Y \Delta$, $Y[1]_0 \subseteq \Delta$, where $\Delta \cong \mathbb{P}(O_Y \oplus N_{Y/X_+})$ and form the moduli space $I_1(X_+[1]_0, 0) \cong X_+$ of relative structure sheaf of one point, which is the union of $X_+ \setminus Y$ with the $\mathbb{C}^*$-equivalence classes of points in $\Delta \setminus \{(Y \times 0) \cup (Y \times \infty)\}$. By the Koszul resolution and Serre duality, we have canonical isomorphism

$\text{Ext}^i(\mathcal{O}_p, \mathcal{O}_p) \cong \wedge^i (TX_+|_p)$.

Then the obstruction bundle $\text{Ob} = \wedge^2 TX_+$ has a non-degenerate quadratic form only when it is restricted to $X_+ \setminus Y$ and $\Delta \setminus \{(Y \times 0) \cup (Y \times \infty)\}$, but they do not glue to become a non-degenerate quadratic form on $\text{Ob} \to X_+$ as $c_1(\text{Ob}) \neq 0$.

If we assume $K_{X_+}$ has a square root $K_{X+}^{\frac{1}{2}}$ (see the appendix), and form $\tilde{\text{Ob}} \cong \wedge^2 TX_+ \otimes K_{X+}^{\frac{1}{2}}$, then there exists a non-degenerate quadratic form

$$(\wedge^2 TX_+ \otimes K_{X+}^{\frac{1}{2}}) \otimes (\wedge^2 TX_+ \otimes K_{X+}^{\frac{1}{2}}) \to \mathcal{O}_{X+},$$

as $c_1(\tilde{\text{Ob}}) = 0$, the self-dual obstruction bundle $\tilde{\text{Ob}}_{+}$ exists and is orientable, and Conjectures 5.8, 5.9 hold for this case. To calculate the Euler class $e(\tilde{\text{Ob}}_{+})$, we consider the case when $X_+ = Y_1 \times \mathbb{P}^1$ and $Y = (Y_1 \times 0) \cup (Y_1 \times \infty)$, where $Y_1$ is a smooth Calabi-Yau 3-fold. Then

$$\wedge^2 TX_+ \otimes K_{X+}^{\frac{1}{2}} \cong (\wedge^2 TY_1 \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \oplus (TY_1 \otimes \mathcal{O}_{\mathbb{P}^1}(1)),$$

where both factors are maximal isotropic subbundles of $\tilde{\text{Ob}}_{+}$. By [19] or Lemma 5.13 [13],

$$e(\tilde{\text{Ob}}_{+}) = \pm (c_1(Y_1) + c_2(Y_1) \cdot c_1(\mathcal{O}_{\mathbb{P}^1}(1))) = \pm (c_1(X_+) - \frac{1}{2} c_2(X_+) \cdot c_1(Y_1)),$$

where the sign depends on the orientation of $\tilde{\text{Ob}}_{+}$. The relative $\text{DT}_4$ virtual cycle for structure sheaves of one point is the Poincaré dual of $e(\tilde{\text{Ob}}_{+})$ [23].

We then consider examples of ideal sheaves of curves.

Example 4.2. Let $Q \subseteq \mathbb{P}^4$ be a smooth generic quintic 3-fold. We take $X_+ = Q \times \mathbb{P}^1$ with an anti-canonical divisor $Y = (Q \times 0) \cup (Q \times \infty)$. Then $H_2(X_+, \mathbb{Z}) \cong H_2(Q, \mathbb{Z}) \oplus H_2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(i) We first fix the curve class to be the generator $[H] \in H_2(\mathbb{P}^1, \mathbb{Z}) \subseteq H_2(X_+, \mathbb{Z})$. The moduli space $I_0(X_+, [H]) \cong Q$ of ideal sheaves of curves with Chern character $c = (1, 0, 0, -PD([H]), 0)$ consists of ideal sheaves of curves of type $[pt] \times \mathbb{P}^1, pt \in Q$. We have a restriction morphism

$$r : I_0(X_+, [H]) \to I_1(Q, 0) \times I_1(Q, 0) \cong Q \times Q,$$

$$I_{[pt] \times \mathbb{P}^1} \Rightarrow (I_{pt}, I_{pt}),$$

to the moduli space of ideal sheaves of one point in $Y$. $r$ is the diagonal embedding if we identify $I_0(X_+, [H]) \cong Q$. By direct calculations, for any $I_C \in I_0(X_+, [H])$, we have

$$\text{Ext}^1_{X_+}(I_C, I_C) \cong \mathbb{C}^i, \ i = 1, 2, \ \text{Ext}^2_{X_+}(I_C, I_C) \cong \mathbb{C}, \ \text{Ext}^3_{X_+}(I_C, I_C) = 0.$$

Analogously to Lemma 5.3, we have a long exact sequence

$$0 \to \text{Ext}^1_{X_+}(I_C, I_C) \to \text{Ext}^1_Y(I_C|_Y, I_C|_Y) \to \text{Ext}^1_{X_+}(I_C, I_C \otimes K_{X_+})$$

$$\to \text{Ext}^2_{X_+}(I_C, I_C) \to \text{Ext}^2_Y(I_C|_Y, I_C|_Y) \to \text{Ext}^2_{X_+}(I_C, I_C \otimes K_{X_+}) \to 0.$$
Curves in $\bigcup_{2875}^\ast \mathbb{C}$ have well-defined restriction to trivial line bundles on $(Q \times 0) \cup (Q \times \infty)$. For curves in $\bigcup_{2875}^\ast [0, \infty) \subseteq I_1(X_+, [H_Q])$, we introduce Li-Wu’s expanded pair to define the restriction map. We denote $X_+^{[1]} = \Delta_1 \cup X_+ \cup \Delta_1$, where $\Delta_1 \cong Q \times \mathbb{P}^1$, and consider the moduli space $I_1^{(X_+^{[1]}}, [H_Q])$ of relative ideal sheaves of curves (normal to the divisor $(Q \times 0) \cup (Q \times \infty)$). $I_1^{(X_+^{[1]}}, [H_Q])$ is the union of $\bigcup_{2875}^\ast \mathbb{C}$ with $\mathbb{C}^\ast$-equivalence classes of curves inside $\Delta_1\setminus((Q \times 0) \cup (Q \times \infty))$, i.e.

$$I_1^{(X_+^{[1]}}, [H_Q]) \cong \bigcup_{2875}^\ast \mathbb{P}^1.$$  

We then have a restriction map $I_1^{(X_+^{[1]}}, [H_Q]) \to \{O_{Q \times 0} \} \cup \{O_{Q \times \infty}\}$, $I_C \mapsto I_C^{[(Q \times 0) \cup (Q \times \infty)]} = \{O_{Q \times 0} \cup O_{Q \times \infty}\}$

to the moduli space of trivial line bundles on $(Q \times 0) \cup (Q \times \infty) \subseteq \Delta_1 \cup \Delta_1$. By direct calculations, for any $I_C \in I_1^{(X_+^{[1]}}, [H_Q])$, we have

$$\text{Ext}_{X_+^{[1]}}, [I_C, I_C] \cong \mathbb{C}, \quad \text{Ext}_{X_+^{[1]}}, [I_C, I_C] \cong \mathbb{C}^2, \quad \text{Ext}_{X_+^{[1]}}, [I_C, I_C] = 0, \text{ i.e.}$$

Conditions in Definition [5.4] are satisfied and the relative $DT_4$ virtual cycle is the usual fundamental class of the moduli space, i.e. $[I_1^{(X_+^{[1]}}, [H_Q])] \in \mathcal{H}_3(I_1^{(X_+^{[1]}}, [H_Q]), \mathbb{Z})$.

We give a gluing formula of relative $DT_4$ invariants for the above example.

**Example 4.3.** In Example [5.2] (ii), we consider $X_0 = X_+ \cup_{Y} X_-$ and its smoothing $X_t = Q \times \mathbb{T}^2$, where $X_\pm \cong Q \times \mathbb{P}^1$ and $Y \subseteq (Q \times 0) \cup (Q \times \infty)$. The moduli space $I_1^{(X_\pm^{[1]}}, [H_Q])$ of relative ideal sheaves of curves satisfies

$$I_1^{(X_\pm^{[1]}}, [H_Q]) \cong \bigcup_{2875}^\ast \mathbb{P}^1.$$  

The relative $DT_4$ virtual cycle is usual fundamental class of $I_1^{(X_\pm^{[1]}}, [H_Q])$. Meanwhile, the moduli space $I_1(X_t, [H_Q])$ of ideal sheaves of curves in $X_t = Q \times \mathbb{T}^2$ with curve class $[H_Q]$ satisfies

$$I_1(X_t, [H_Q]) \cong \mathbb{T}^2,$$  

and its $DT_4$ virtual cycle is the usual fundamental class of $I_1(X_t, [H_Q])$ [13]. Under the homological relation $X_0 \sim X_t$, we have $\mathbb{P}^1 \cup_{(0, \infty)} \mathbb{P}^1 \sim \mathbb{T}^2$. Then the gluing formula is expressed by pairing $\mu$-map with these cycles (see Theorem 5.13 for such a formula).

**Example 4.4.** (Generic quintic in $\mathbb{P}^4$)

We take $X_+ = \mathbb{P}^4$ which contains a smooth generic quintic 3-fold $Y = Q$ as its anticanonical divisor, and then $H_2(X_+, \mathbb{Z}) \cong H_2(Q, \mathbb{Z}) \cong \mathbb{Z}$. We consider the primitive curve class $[H] \in H_2(X_+, \mathbb{Z})$ and ideal sheaves of curves representing this class have Chern character $c = (1, 0, 0, -PD([H]), \frac{3}{2})$. We denote their moduli space by $I_2^{(X_+}, [H]) \cong \text{Gr}(2, 5)$. The generic quintic $Q$ contains 2875 rigid degree 1 rational curves and $I_2^{(X_+}, [H])$ contains a finite subset $S$ with 2875 points. $I_2^{(X_+}, [H]) \setminus S$ has a well-defined restriction morphism to $\text{Hilb}^5(Q)$. To extend the morphism across those 2875 points, we introduce Li-Wu’s expanded pair.

We denote $X_+^{[1]} = X_t \cup \Delta_1, Y^{[1]} = \{Q\} \subseteq \Delta_1$, where $\Delta_1 \cong \mathbb{P}(O_Q \oplus O_Q(5))|_Q$, and consider the moduli space $I_2^{(X_+^{[1]}}, [H])$ of relative ideal sheaves of curves (normal to the divisor $Q$). Geometrically, it is the blow up of $I_2^{(X_+}, [H])$ along those 2875 points, i.e.

$$I_2^{(X_+^{[1]}}, [H]) \cong B_{\mu}(\text{Gr}(2, 5)), \quad \text{and each exceptional divisor corresponds to a } \text{Hilb}^5(\mathbb{P}^1) \cong \mathbb{P}^5 \text{ for each } \mathbb{P}^1 \subseteq Q.$$  

We then have a restriction morphism

$$I_2^{(X_+^{[1]}}, [H]) \to \text{Hilb}^5(Y^{[1]}), \quad I_C \mapsto I_C|_{Y^{[1]}},$$  

which is injective with smooth image. By direct calculations, we have

$$\text{Ext}_1^{(X_+^{[1]}}, [H], I_C) \cong \mathbb{C}^6, \quad \text{Ext}_2^{(X_+^{[1]}}, [H], I_C) \cong \mathbb{C}^9, \quad \text{Ext}_3^{(X_+^{[1]}}, [H], I_C) = 0,$$
and a long exact sequence

\[ 0 \to \Ext^1_{X_+}[1]_0(I_C, I_C) \to \Ext^1_{Y}[1]_0(I_{C|Y}, I_{C|Y}) \to \Ext^2_{X_+[1]_0}(I_C, I_C)^* \to \]

\[ \to \Ext^2_{X_+}[1]_0(I_C, I_C) \to \Ext^1_{Y}[1]_0(I_{C|Y}, I_{C|Y})^* \to \Ext^1_{X_+[1]_0}(I_C, I_C)^* \to 0. \]

As \( I_{C|Y} \in \text{Hilb}^2(Y[1]_0) \) is a smooth point with \( \text{Ext}^1_{Y}[1]_0(I_{C|Y}, I_{C|Y}) \cong \mathbb{C}^{15} \), conditions in Definition 5.4 are satisfied. The relative \( DT_4 \) virtual cycle is the usual fundamental class of the moduli space \( I_2^X(X_+[1]_0, [H]) \cong \text{Bl}_S(\text{Gr}(2, 5)) \).

We adapt Li-Wu's expanded degenerations to torsion sheaves and consider the following extension of Example 1.2.

**Example 4.5.** (Relative \( DT_3/DT_3 \))

Let \( X_+ = Y_1 \times \mathbb{P}^1 \) which contains \( Y = (Y_1 \times 0) \cup (Y_1 \times \infty) \) as an anti-canonical divisor, where \( Y_1 \) is a compact Calabi-Yau 3-fold. We denote \( \mathcal{M}_c(Y_1) \) to be a Gieseker moduli space of torsion-free semi-stable sheaves on \( Y_1 \) with Chern character \( c \in H^{even}(Y_1, \mathbb{Q}) \) (we assume there is no strictly semi-stable sheaf), and denote \( \mathcal{M}_c(X_+) \) to be the moduli space of sheaves on \( X_+ \) which are push-forward of stable sheaves in \( \mathcal{M}_c(Y_1 \times t) \) for some \( t \) \( \mathcal{M}_c(X_+) \cong \mathcal{M}_c(Y_1) \times \mathbb{P}^1 \). Let \( t : Y_1 \times t \hookrightarrow X_+ \) be the natural embedding. For any stable sheaf \( F \in \mathcal{M}_c(Y_1) \), as in Lemma 6.4 of [13] we have

\[ \Ext^1_{X_+}(t_* F, t_* F) \cong \Ext^1_{Y_1}(F, F) \oplus \Ext^0_{Y_1}(F, F), \]

\[ \Ext^2_{X_+}(t_* F, t_* F) \cong \Ext^2_{Y_1}(F, F) \oplus \Ext^1_{Y_1}(F, F)^*. \]

Furthermore, under the above identifications, a Kuranishi map \( \kappa_{t_* F} : \Ext^1_{X_+}(t_* F, t_* F) \to \Ext^2_{X_+}(t_* F, t_* F) \)

satisfies

\[ \kappa_{t_* F}(a, b) = (\kappa_F(a), 0), \]

for some Kuranishi map \( \kappa_F : \Ext^1_{Y_1}(F, F) \to \Ext^2_{Y_1}(F, F) \) of \( \mathcal{M}_c(Y_1) \) at \( F \).

To have a well-defined restriction map, we introduce \( X_+[1]_0 = \Delta_1 \cup X_+ \cup \Delta_1 \), where \( \Delta_{\pm} \cong Y_1 \times \mathbb{P}^1 \), and consider \( \mathcal{M}_c(X_+[1]_0) \) to be the union of \( \mathcal{M}_c(Y_1 \times \mathbb{P}^1) \) with the \( \mathbb{C}^* \)-equivalence classes of \( \mathcal{M}_c(Y_1 \times 0) \times \mathbb{C}^* \cup \mathcal{M}_c(Y_1 \times \infty) \times \mathbb{C}^* \), i.e.

\[ \mathcal{M}_c(X_+[1]_0) \cong \mathcal{M}_c(Y_1) \times \mathbb{P}^1. \]

As the supports are disjoint, we have \( \text{Tor}^{\mathcal{O}_Y}_{\geq 1}(t_* F, \mathcal{O}_Y) = 0 \) which implies a LES similar to the one in Lemma 5.3, and get a canonical isomorphism \( \Ext^2_{X_+[1]_0}(t_* F, t_* F) \cong \Ext^2_{Y_1}[1]_0(t_* F, t_* F)^* \) as [10]. In this specific case, there exists a canonical maximal isotropic subspace \( \Ext^2_{X_+[1]_0}(t_* F, t_* F) \) \( \cong \Ext^2_{Y_1}[1]_0(t_* F, t_* F) \) \( \cong \Ext^2_{X_+[1]_0}(t_* F, t_* F) \) and \( \Ext^2_{X_+[1]_0}(F, F) \) is a form of sheaf over \( \mathcal{M}_c(X_+[1]_0) \), thus Conjectures 5.8.9 hold. Analogously to Theorem 6.5 of [13], the relative \( DT_3 \) virtual cycle satisfies

\[ \text{[M}_c^{cr}(X_+[1]_0) \cap \text{vir} = DT_3(\mathcal{M}_c(Y_1)) \cdot [\mathbb{P}^1] \in H_2(\mathcal{M}_c(X_+[1]_0), \mathbb{Z}), \]

where \( DT_3(\mathcal{M}_c(Y_1)) \) is the Donaldson-Thomas invariant defined by Thomas [39].

5. Appendix on relative \( DT_4 \) invariants for ideal sheaves and gluing formulas

5.1. Li-Wu’s good degeneration of Hilbert schemes. In this subsection, we recall some basic notions and facts of Li-Wu’s good degeneration of Hilbert schemes. The precise definitions are left to their papers [39], [56].

The stack of expanded degenerations. We first introduce the stack of expanded degenerations for pairs \((X_+, Y)\). We replace a pair \((X_+, Y)\) by expanded pairs of length \(n\), \((X_+[n]_0, Y[n]_0)\), i.e.

\[ X_+[n]_0 = X_+ \cup \Delta_1 \cup \cdots \cup \Delta_n, \]

which is a chain of smooth irreducible components intersecting transversally with \( \Delta_i \) to be the \( i^{th} \) copy of \( \Delta \cong \mathbb{P}(N_{Y/X_+} \oplus \mathcal{O}_Y) \). \( \Delta \) is a \( \mathbb{P}^1 \) bundle over \( Y \) with two canonical divisors \( Y_{\pm} \) such that \( N_{Y_{\pm}/X_+} \cong N_{Y/X_+} \) and \( N_{Y_{\pm}/X_+} \cong N_{Y/X_+} \). We denote \( Y[n]_0 = Y_+ \) to be the divisor in the last component \( \Delta_n \). In fact, we can consider families of expanded pairs, \((X_+[n], Y[n])\) over affine space \( \mathbb{A}^n \) such that over \( 0 \in \mathbb{A}^n \) it coincides with \((X_+[0], Y[0])\). Then there exists a pair of Artin stacks \((\mathcal{X}_+, \mathcal{Y})\) as the direct limit of stack quotients of \((X_+[n], Y[n])\) by certain group actions. The projection of \((X_+[n], Y[n])\) to the affine space \( \mathbb{A}^n \) induces a morphism \( \mathcal{Y} \subseteq \mathcal{X}_+ \to \mathcal{A}_n \), where \( \mathcal{A}_n \) is the direct limit of some stack quotients of the affine space \( \mathbb{A}^{n+1} \).
To formula the gluing formula, we also need to replace the family \( X \to C \) by its expanded degeneration \( \mathcal{X} \to \mathcal{C} \), where \( \mathcal{X} \) is the direct limit of stack quotients of \( X[n] \) and \( X[n] \) is a family over \( C[n] \triangleq C \times \mathbb{A}^{n+1} \), \( \mathcal{C} \triangleq C \times \mathbb{A}^{n+1} \) and \( \mathfrak{A} \) is another stack quotient of the affine space \( \mathbb{A}^{n+1} \).

A smooth chart of \( X_0 \triangleq \mathcal{X} \times_{\mathcal{C}} 0 \) is

\[
X[n]_0 = X_+ \cup \Delta_1 \cup \cdots \cup \Delta_n \cup X_-,
\]

which is a chain of smooth irreducible components intersecting transversally with \( \Delta_i \) to be the \( i \)th copy of \( \Delta \triangleq P(N_Y/X_+ \oplus \mathcal{O}_Y) \simeq P(N_Y/X_- \oplus \mathcal{O}_Y) \). We also denote \( \Delta_0 = X_+ \cup \Delta_{n+1} = X_- \).

If we consider \( X[n]_0 \) as the gluing of \( (X_+|n_0, Y|n_0) \), we need to specify one of its divisor in some \( \Delta_i \). This is called a node-marking and there exists an Artin stack \( X_0 \) which is the collection of families in \( X_0 \) with node-markings. One can construct a stack \( \mathcal{C}_0^\dagger \) and an arrow \( \mathcal{C}_0^\dagger \to \mathcal{C} \) that fits into a Cartesian product

\[
\begin{array}{ccc}
X_0^\dagger & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathcal{C}_0^\dagger & \longrightarrow & \mathcal{C}
\end{array}
\]

By Proposition 2.13 [39], there exists a canonical isomorphism \( \mathcal{C}_0^\dagger \simeq \mathfrak{A}_0 \).

To fix Hilbert polynomials of ideal sheaves over \( X[n]_0 \) and decompose them into ideal sheaves of fixed Hilbert polynomials on \( (X_+|n_0, Y|n_0) \), we introduce

\[
\Lambda^\pl_0 \triangleq \{ \delta = (\delta_+, \delta_0) \mid \delta_+ + \delta_- - \delta_0 = P \},
\]

where \( \delta_+, \delta_0, P \) are polynomials in \( \mathbb{A} \triangleq \mathbb{A}^* \cup \{ 0 \} \), and \( \mathbb{A}^* \) is the set of \( \mathbb{Q} \)-coefficient polynomials whose leading terms are of the form \( a_\tau \frac{\tau}{\tau} \) with \( a_\tau \in \mathbb{Z}_+ \).

We define the stack \( X_0^\dagger, \delta \) whose closed points are \( (X[n]_0, Y_k, w) \), where \( w \) is a function such that

\[
w(\Delta_{[0,k-1]}) = \delta_-, \quad w(\Delta_{[k,n+1]}) = \delta_+,
\]

\( w(Y_k) = \delta_0 \)

We similarly assign functions \( w_\pm \) to \( (X_\pm|n_0, Y|n_0) \) with

\[
w_\pm(\Delta_{[0,n]}) = \delta_\pm, \quad w_\pm(Y|n_0) = \delta_0
\]

and define stacks \( X_\pm, \delta \). Then there exists stacks \( \mathfrak{A}_0^{\pm, \delta} \) so that we have Cartesian product

\[
\begin{array}{ccc}
X_0^{\pm, \delta} & \longrightarrow & X_0^\dagger \\
\downarrow & & \downarrow \\
\mathfrak{A}_0^{\pm, \delta} & \longrightarrow & \mathfrak{A}_0
\end{array}
\]

By gluing two components, we obtain the following commutative diagram

\[
\begin{array}{ccc}
X_0^{\pm, \delta} \times X_0^{\pm, \delta} & \longrightarrow & X_0^\dagger, \delta \\
\downarrow & & \downarrow \\
\mathfrak{A}_0^{\pm, \delta} \times \mathfrak{A}_0^{\pm, \delta} \overset{\sim}{\longrightarrow} & \mathcal{C}_0^\dagger, \delta
\end{array}
\]

We denote \( \mathcal{C}_0^{\dagger, P} = \bigsqcup_{\delta \in \Lambda^\pl_0} \mathcal{C}_0^{\dagger, \delta} \) and then have a natural morphism \( \Phi_\delta : \mathcal{C}_0^{\dagger, \delta} \to \mathcal{C}^P \) as the composition of the imbedding \( \mathcal{C}_0^{\dagger, \delta} \to \mathcal{C}_0^{\dagger, P} \) with forgetful map \( \mathcal{C}_0^{\dagger, P} \to \mathcal{C}^P \).

**Lemma 5.1.** (Li-Wu, Proposition 2.19 [39])

There are canonical line bundles with sections \( (L_\delta, s_\delta) \) on \( \mathcal{C}^P \) indexed by \( \delta \in \Lambda^\pl_0 \), such that

1. let \( t \) be the standard coordinate function on \( \mathbb{A}^1 \) and \( \pi : \mathcal{C}^P \to \mathbb{A}^1 \) be the tautological projection, then

\[
\bigotimes_{\delta \in \Lambda^\pl_0} L_\delta \cong \mathcal{O}_{\mathcal{C}^P}, \quad \prod_{\delta \in \Lambda^\pl_0} s_\delta = \pi^* t;
\]

2. \( \Phi_\delta \) factors through \( s_\delta^{-1}(0) \subseteq \mathcal{C}^P \) and there exists an isomorphism \( s_\delta^{-1}(0) \cong \mathcal{C}_0^{\dagger, \delta} \).

This states that \( \mathcal{C}_0^{\dagger} \subseteq \mathcal{C}^P \) is a complete intersection substack with \( \bigsqcup_{\delta \in \Lambda^\pl_0} \mathcal{C}_0^{\dagger, \delta} \) as its normalization.

**Moduli stacks of stable ideal sheaves.** By Theorem 4.14 [39], there exists a Deligne-Mumford
stack \( \mathcal{T}_{X/C}^P \) which is finite type, separated and proper over \( C \). It is a good degeneration of Hilbert scheme of subschemes of \( X/C \) with fixed Hilbert polynomial \( P \) in the sense that

\[
\mathcal{T}_{X/C}^P \times_C t \cong \text{Hilb}^P (X_t), \quad t \neq 0
\]

and the central fiber \( \mathcal{T}_{X_0/C_0}^P \cong \mathcal{T}_{X/C}^P \times_C 0 \) has a good obstruction theory. We recall that any closed point of \( \mathcal{T}_{X_0/C_0}^P \) is an ideal sheaf \( I_Z \) in some \( X[n]_0 \) such that

1. \( \mathcal{O}_Z \) is normal to all \( Y_i \subseteq X[n]_0 \), i.e. \( \text{Tor}_{-1}^{\mathcal{O}_X} (\mathcal{O}_Z, \mathcal{O}_{Y_i}) = 0 \);
2. \( \text{Aut}_X (I_Z) \) is finite.
3. The Hilbert polynomial of \( \mathcal{O}_Z \) is \( P \).

We define

\[
\mathcal{T}_{X_0/C_0}^S = \mathcal{T}_{X/C}^P \times_C \mathcal{E}^1_{0, \delta}.
\]

It parameterizes ideal sheaves \( I_Z \)'s on \( X[n]_0 \) with a node-marking \( Y_k \subseteq X[n]_0 \) so that the Hilbert polynomials of \( \mathcal{O}_Z \) restricted to \( \bigcup_{i<k} \Delta_i \), to \( \bigcup_{i \geq k} \Delta_i \) and to \( Y_k \) are \( \delta_- \), \( \delta_+ \) and \( \delta_0 \) respectively.

We can similarly define the moduli stack of stable relative ideal sheaves for \( \mathcal{Y} \subseteq X \) with Hilbert polynomial \( (\delta_\pm, \delta_0) \), denoted by \( \mathcal{T}_{X/\mathcal{A}_0}^{\delta_\pm, \delta_0} \), which are finite type, separated and proper Deligne-Mumford stacks (Theorem 4.15 [39]). The relations between \( \mathcal{T}_{X/C}^P \), \( \mathcal{T}_{X_0/C_0}^S \) and \( \mathcal{T}_{X/\mathcal{A}_0}^{\delta_\pm, \delta_0} \) are described as follows.

**Lemma 5.2.** (Li-Wu, Theorem 5.27 [39])

1. There exists natural restriction morphisms \( \mathcal{T}_{X_0/\mathcal{A}_0}^{\delta_\pm, \delta_0} \rightarrow \text{Hilb}^P_Y \), where \( \text{Hilb}^P_Y \) is the Hilbert scheme on \( Y \) with fixed Hilbert polynomial \( \delta_0 \), and an isomorphism

\[
\mathcal{T}_{X_0/\mathcal{A}_0}^{-\delta_- \delta_0} \cong \mathcal{T}_{X_0/\mathcal{A}_0}^S \rightarrow \mathcal{T}_{X_0/\mathcal{A}_0}^{-\delta_- \delta_0} \rightarrow \mathcal{T}_{X_0/\mathcal{A}_0}^{\delta_\pm, \delta_0} \rightarrow \mathcal{T}_{X_0/\mathcal{A}_0}^S.
\]

2. Let \( (L_S, s_S) \) be as in Lemma 5.1 and \( \pi_P : \mathcal{T}_{X/C}^P \rightarrow \mathcal{E}^P \) be the natural projection. Then

\[
\bigotimes_{\delta \in \Lambda_{P}} \pi_P L_\delta \cong \mathcal{O}_{X/C}^\ast, \quad \prod_{\delta \in \Lambda_{P}} \pi_P s_\delta = \pi_P t \ast t;
\]

As closed substacks of \( \mathcal{T}_{X/C}^P \), we have \( \mathcal{T}_{X/C}^S_{\pi_P s_\delta} \cong (\pi_P s_\delta) \).

**5.2. Relative DT \( * \) virtual cycles.** We study obstruction theories of Deligne-Mumford stacks \( \mathcal{T}_{X/C}^S_{\pi_P s_\delta} \) and \( \mathcal{T}_{X/\mathcal{A}_0}^{\delta_\pm, \delta_0} \).

**Lemma 5.3.** We take a closed point \( [I_Z] \in \mathcal{T}_{X_0/\mathcal{A}_0}^{\delta_\pm, \delta_0} \) with \( Z \subseteq X_+, Y \), then for \( Y = Y[n]_0 \), we have a short exact sequence

\[
0 \rightarrow \mathcal{O}_{X,+}[n]_0 (-Y) \rightarrow I_Z \rightarrow \mathcal{O}_{Y} \rightarrow 0
\]

and a long exact sequence

\[
\cdots \rightarrow \text{Ext}^1 (I_Z, \mathcal{O}_{X,+}[n]_0 (-Y)) \rightarrow \text{Ext}^1 (I_Z, I_Z) \rightarrow \text{Ext}^1 (I_Z, \mathcal{O}_{Y}) \rightarrow \text{Ext}^1 (I_Z, \mathcal{O}_{Y}) \rightarrow \text{Ext}^2 (I_Z, I_Z) \rightarrow \text{Ext}^2 (I_Z, \mathcal{O}_{Y}) \rightarrow \text{Ext}^2 (I_Z, \mathcal{O}_{Y}) \rightarrow \cdots
\]

Proof. We tensor 0 \( \rightarrow I_Z \rightarrow \mathcal{O}_{X,+}[n]_0 \rightarrow \mathcal{O}_{Y} \rightarrow 0 \) with \( \mathcal{O}_Y \) and get

\[
\text{Tor}_{i}^{\mathcal{O}_{X,+}[n]_0} (\mathcal{O}_Y, \mathcal{O}_Y) \cong \text{Tor}_{i}^{\mathcal{O}_{X,+}[n]_0} (I_Z, \mathcal{O}_Y), \quad i \geq 1.
\]

Applying tensor product with \( \mathcal{O}_Z \) to 0 \( \rightarrow \mathcal{O}_{X,+}[n]_0 (-Y) \rightarrow \mathcal{O}_{X,+}[n]_0 \rightarrow \mathcal{O}_Y \rightarrow 0 \) with \( I_Z \). We then take \( \text{Hom} (I_Z, \mathcal{O}_Y) \) to (7) and are left to show \( \text{Ext}^*(I_Z, I_Z \otimes \mathcal{O}_Y) \cong \text{Ext}^*(I_Z, I_Z \otimes \mathcal{O}_Y) \). We have a spectral sequence

\[
H^*(X_[n]_0, \text{Ext}^*(I_Z, I_Z \otimes \mathcal{O}_Y)) \Rightarrow \text{Ext}^*(I_Z, I_Z \otimes \mathcal{O}_Y).
\]

By Corollary 2.9 [50], we can take a finite length locally free resolution \( E^* \rightarrow I_Z \rightarrow 0 \). Then

\[
H^*(X_[n]_0, \text{Ext}^*(I_Z, I_Z \otimes \mathcal{O}_Y)) \cong H^*(X_[n]_0, \text{Ext}^*(\mathcal{O}_{X,+}[n]_0, \mathcal{O}_Y) \otimes \text{End}(E^*))
\]

\[
\cong H^*(X_[n]_0, \text{End}(E^*) \otimes \mathcal{O}_Y) \cong H^*(X_[n]_0, t, \text{End}(E^*) \otimes \mathcal{O}_Y) \cong \text{Ext}^*_Y (E^*_Y, E^*_Y),
\]
where \( \iota: Y \hookrightarrow X_+[n_0] \) is the closed imbedding. By \((8), (9)\), we have \( Tor_{X_+}^{i>1}(I_Z, \mathcal{O}_Y) = 0 \) which implies that \( E^*|_Y \to I_Z|_Y \to 0 \) is still an resolution. Thus \( Ext^*_Y(I_Z|_Y, I_Z|_Y) \cong Ext^*(I_Z, I_Z \otimes \mathcal{O}_Y) \).

The above long exact sequence is the ideal sheaf version of the long exact sequence in Lemma 3.3. We consider the following extension of virtual cycles for bundles (Cases I-III) to ideal sheaf cases.

**Definition 5.4.** Let \( Y \) be an anti-canonical divisor of a complex projective 4-fold \( X_+ \), and

\[
I \xrightarrow{r: \mathcal{I}^{X_+}[\delta_0]} \mathcal{Hilb}_Y^0
\]

be Li-Wu’s restriction morphism in Lemma 5.2. We assume \( \mathcal{I}^{X_+}[\delta_0] \) is a smooth moduli scheme (all Kuranishi maps vanish).

The relative DT virtual cycle for \( \mathcal{I}^{X_+}[\delta_0] \) is its usual fundamental class provided that any one of the following conditions holds,

1. \( r \) is surjective between smooth moduli spaces, and the obstruction bundle \( Ob(\mathcal{I}^{X_+}[\delta_0]) = 0 \),
2. \( r \) is injective between smooth moduli spaces (at least when restricted to a neighbourhood \( U(r(\mathcal{I}^{X_+}[\delta_0])) \) of \( r(\mathcal{I}^{X_+}[\delta_0]) \) in \( \mathcal{Hilb}_Y^0 \), and \( rk(Ob(\mathcal{I}^{X_+}[\delta_0])) = \text{codim}(\mathcal{I}^{X_+}[\delta_0], U(r(\mathcal{I}^{X_+}[\delta_0]))) \).

**A modification by twisting \( K_{X_+}^\frac{1}{2} \).** In general, extensions of virtual cycles for bundles (Cases I-III) to ideal sheaves are not straightforward. We study the extension for Case I.

**Proposition 5.5.** We take a smooth Calabi-Yau 3-fold \( Y \) in complex projective 4-folds \( X_\pm \) as their anti-canonical divisors. We assume any \( I_C \in \mathcal{Hilb}^0(Y) \) satisfies \( Ext^1(I_C, I_C) = 0 \). Then for any closed point of \( \mathcal{I}^{X_\pm}[\delta_0]_{X_+/\mathcal{A}_0} \), say \( I_{Z_\pm} \) with \( Z_\pm \subseteq X_\pm[n_0] \), we have canonical isomorphisms

\[
Ext^i_{X_\pm}[n_0](I_{Z_\pm}, I_{Z_\pm}) \cong Ext^i_{X_\pm}[n_0](I_{Z_\pm}, I_{Z_\pm}), \quad i = 1, 2.
\]

Furthermore, under the isomorphism in Lemma 5.2,

\[
\mathcal{J}^\delta_{X_+/\mathcal{A}_0} \cong \mathcal{J}^\delta_{X_+/\mathcal{A}_0} \times \mathcal{Hilb}^0_{X_+/\mathcal{A}_0} \mathcal{J}^\delta_{X_+/\mathcal{A}_0},
\]

where a closed point of \( \mathcal{J}^\delta_{X_+/\mathcal{A}_0} \) is written as \( I_Z = I_{Z_+} \cup I_{Z_-} \), with \( Z \subseteq X[n_+ + n_-] \), \( Z_\pm \subseteq X_\pm[n_\pm] \), we have canonical isomorphisms of trace-free extension groups

\[
Ext^i_{X_\pm}[n_+ + n_-](I_Z, I_Z) \cong Ext^i_{X_\pm}[n_+ + n_-](I_{Z_+}, I_{Z_+}) \oplus Ext^i_{X_\pm}[n_+ + n_-](I_{Z_-}, I_{Z_-}),
\]

under which the non-degenerate quadratic forms on \( Ext^2_{X_\pm}[n_+ + n_-](I_Z, I_Z) \) and \( Ext^2_{X_\pm}[n_0](I_{Z_\pm}, I_{Z_\pm}) \) (10) are preserved.

**Proof.** We apply the trace-free version of Lemma 5.3 to the case when \( Y \) is the Cartier divisor associated with the dualizing sheaf of \( X_\pm[n_0] \) and get canonical isomorphisms by Serre duality.

We take a closed point \( I_Z \in \mathcal{J}^\delta_{X_+/\mathcal{A}_0} \) with \( Z \subseteq X[n_+ + n_-] = X_\pm[n_0] \cap Y X_-[n_-] \) and restrict to get \( I_{Z_\pm} \subseteq \mathcal{O}_{X_\pm[n_\pm]} \). As \( [15], [39] \), we then get an exact triangle

\[
\text{RHom}_{X_\pm[n_+ + n_-]}(I_Z, I_Z) \to \bigoplus \text{RHom}_{X_\pm[n_\pm]}(I_Z|_{X_\pm[n_\pm]}), I_Z|_{X_\pm[n_\pm]} \to \text{RHom}_{X_\pm}(I_Z|_Y, I_Z|_Y) = 0.
\]

We take cohomology, use the assumption to deduce \( H^*(\text{RHom}_{X_\pm}(I_Z|_Y, I_Z|_Y)) = 0 \).

From Proposition 5.5, one may expect to get a \((−2)\)-shifted symplectic structure on \( \mathcal{J}^\delta_{X_\pm}[\delta_0] \) if any \( I_C \in \mathcal{Hilb}^0(Y) \) satisfies \( Ext^1(I_C, I_C) = 0 \). Then as Borisov-Joyce did in \( [6] \), one expects to use BBJ’s type local Darboux charts, partition of unity and homotopical algebra to obtain D-orbifolds associated with \( \mathcal{J}^\delta_{X_\pm}[\delta_0] \) and \( \mathcal{J}^\delta_{X_\pm}[\delta_0] \). In particular, analogs to Theorem 2.6, there should exist homology classes \( [\mathcal{J}^\delta_{X_\pm}[\delta_0]]_{\text{vir}} \in H_*(\mathcal{J}^\delta_{X_\pm}[\delta_0], \mathbb{Q}) \), \( [\mathcal{J}^\delta_{X_\pm}[\delta_0]]_{\text{vir}} \in H_*(\mathcal{J}^\delta_{X_\pm}[\delta_0], \mathbb{Q}) \) if the associated D-orbifolds of \( \mathcal{J}^\delta_{X_\pm}[\delta_0] \) and \( \mathcal{J}^\delta_{X_\pm}[\delta_0] \) are orientable [29].

However, under the assumption: any \( I_C \in \mathcal{Hilb}^0(Y) \) satisfies \( Ext^1(I_C, I_C) = 0 \), the non-degenerate quadratic forms on \( Ext^2_{X_\pm}[n_0](I_{Z_\pm}, I_{Z_\pm}) \)’s (see Proposition 5.5) don’t have to glue together, and the D-orbifold associated with \( \mathcal{J}^\delta_{X_\pm}[\delta_0] \) might not exist (see Example 4.1). For gluing, we discuss the case when there exist a square root \( K_{X_\pm}^\frac{1}{2} \) of \( X_\pm \) with \( K_{X_\pm}^\frac{1}{2} \otimes K_{X_\pm}^\frac{1}{2} \cong K_{X_\pm} \).
Lemma 5.6. Let $X_+$ be a complex projective 4-fold with a square root $K_{X_+}^{\frac{1}{2}}$, $Y_i$ ($i = 1, 2$) be two smooth zero loci of sections of $K_{X_+}^{-\frac{3}{2}}$ with $K_{Y_i} = 0$ ($\Leftrightarrow N_{Y_i/X_+} \cong \mathcal{O}_{Y_i}$) and $Y_1 \cap Y_2 = \emptyset$. We take $Y = Y_1 \sqcup Y_2$ which is a smooth anti-canonical divisor of $X_+$. Then for any closed point $[I_Z] \in \mathcal{F}^{3, 0}_{X_+/\mathbb{A}_n}$ with $Z \subseteq X_+[n]_0$ and $Y[n]_0 = Y_1 \sqcup Y_2$, we have a short exact sequence

$$0 \to I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0 \to I_Z \to I_Z \otimes \mathcal{O}_{Y_i} \to 0$$

and a long exact sequence

$$0 \to \text{Ext}^1_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0) \to \text{Ext}^1_{X_+[n]_0}(I_Z, I_Z) \to \text{Ext}^1_{Y_1}(I_Z \otimes \mathcal{O}_{Y_1}, I_Z \otimes \mathcal{O}_{Y_1}) \to \cdots,$n

where $K_{X_+}^{\frac{1}{2}}[n]_0$ is a square root of the dualizing sheaf of $X_+[n]_0$.

Furthermore, if any $I_C \in \text{Hilb}^0(Y)$ satisfies $\text{Ext}^1_C(I_C, I_C) = 0$, we have canonical isomorphisms

$$\phi_1 : \text{Ext}^2_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0) \cong \text{Ext}^2_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0);$$

$$\phi_2 : \text{Ext}^2_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0) \cong \text{Ext}^2_{X_+[n]_0}(I_Z, I_Z)_0.$n

Proof. It is similar to the proof of Lemma 5.3. The isomorphism $\phi_1$ is derived by tensoring $0 \to I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0 \to I_Z \to I_Z \otimes \mathcal{O}_{Y_i} \to 0$ with $K_{X_+}^{\frac{1}{2}}[n]_0$ and taking the long exact sequence. \hfill \Box

The reason of introducing $\text{Ext}^2_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0)$ is that Serre duality pairing defines a natural non-degenerate quadratic form on it. If $\text{Ext}^2_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0)$'s are glued to be a sheaf over the moduli space, the Serre duality pairing will probably glue. By Proposition 5.5 there is a non-degenerate quadratic form on $\text{Ext}^2_{X_+[n]_0}(I_Z, I_Z)_0$. We make a comparison between them.

Proposition 5.7. Let $X_+$ be a complex projective 4-fold with a square root $K_{X_+}^{\frac{1}{2}}$, $Y_i$ ($i = 1, 2$) be two smooth zero loci of sections of $K_{X_+}^{-\frac{3}{2}}$ with $K_{Y_i} = 0$ ($\Leftrightarrow N_{Y_i/X_+} \cong \mathcal{O}_{Y_i}$) and $Y_1 \cap Y_2 = \emptyset$. We take $Y = Y_1 \sqcup Y_2$ which a smooth anti-canonical divisor of $X_+$ and assume any $I_C \in \text{Hilb}^0(Y)$ satisfies $\text{Ext}^1_{X_+}(I_C, I_C) = 0$. Then for any closed point $[I_Z] \in \mathcal{F}^{3, 0}_{X_+/\mathbb{A}_n}$ with $Z \subseteq X_+[n]_0$ and $Y[n]_0 = Y_1 \sqcup Y_2$, we have a commutative diagram

$$\begin{array}{ccc}
\text{Ext}^2_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0) & \xrightarrow{\phi_1} & \text{Ext}^2_{X_+[n]_0}(I_Z, I_Z)_0 \\
\phi_1 \cong & & \phi_3 \\
\text{Ext}^2_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0) \xrightarrow{\phi_2} & & \text{Ext}^2_{X_+[n]_0}(I_Z, I_Z)_0,
\end{array}$$

where $\phi_1$, $\phi_2$ are defined in Lemma 5.6 and $\phi_3$ is the isomorphism induced from the long exact sequence in Lemma 5.3. Furthermore, $\phi_2$ is an isometry with respect to the Serre duality pairing on $\text{Ext}^2_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+}^{\frac{1}{2}}[n]_0)$ and the quadratic form on $\text{Ext}^2_{X_+[n]_0}(I_Z, I_Z)_0$ defined in Proposition 5.5.

Proof. The commutativity is because isomorphisms $\phi_1$, $\phi_2$ are pairings with sections in $H^0(X_+[n]_0, K_{X_+}^{-\frac{3}{2}}[n]_0)$ corresponding to Cartier divisors $Y_1$, $Y_2$, and $\phi_3$ is the pairing with a section in $H^0(X_+[n]_0, K_{X_+}^{-\frac{3}{2}}[n]_0)$ corresponding to $Y = Y_1 \sqcup Y_2$. Then it is easy to check $\phi_2$ is an isometry.

Conjecture 5.8. Let $Y$ be a smooth Calabi-Yau 3-fold in complex projective 4-folds $X_+$ as their anti-canonical divisors. We assume any $I_C \in \text{Hilb}^0(Y)$ satisfies $\text{Ext}^1(I_C, I_C) = 0$. Then there exists a D-orbifold associated with Deligne-Mumford stack $\mathcal{F}^{4, 0}_{X_+/\mathbb{A}_n}$, i.e. they have the same underlying topological structures.

Furthermore, if canonical bundles of $X_+$ admit square roots $K_{X_+}^{\frac{1}{2}}$, and there exist $Y_i$ ($i = 1, 2$) which are smooth zero loci of sections of $K_{X_+}^{-\frac{3}{2}}$ with $K_{Y_i} = 0$ ($\Leftrightarrow N_{Y_i/X_+} \cong \mathcal{O}_{Y_i}$) such that $Y = Y_1 \sqcup Y_2$, $Y_1 \cap Y_2 = \emptyset$, then there exist D-orbifolds associated with Deligne-Mumford stacks $\mathcal{F}^{4, 0}_{X_+/\mathbb{A}_n}$.\hfill \Box
Conjecture 5.9. We take a smooth Calabi-Yau 3-fold $Y$ in complex projective 4-folds $X_\pm$ as their anti-canonical divisors. We assume any $I_C \in \text{Hilb}^0(Y)$ satisfies $\text{Ext}^1(I_C, I_C) = 0$ (we assume $\text{Hilb}^0(Y)$ consists of one point without loss of generality), then we have

$$[\mathcal{F}_{X_\pm}^{\delta_0}]^{\text{vir}} \in H_*(\mathcal{F}_{X_\pm}^{\delta_0, 0}, \mathbb{Q}), \quad [\mathcal{F}_{X_\pm}^\delta]^{\text{vir}} \in H_*(\mathcal{F}_{X_\pm}^\delta, \mathbb{Q})$$

if there exist orientable $D$-orbifolds $\mathbb{C}^{\delta_0}$ associated with $\mathcal{F}_{X_\pm}^{\delta_0, 0}$ and $\mathcal{F}_{X_\pm}^\delta$. Furthermore, under the isomorphism

$$\mathcal{F}_{X_-}^{\delta_0, 0} \times \text{Hilb}^0 \mathcal{F}_{X_+}^{\delta_0, 0} \cong \mathcal{F}_{X_-}^\delta$$

in Lemma 5.2, we have an identification of virtual cycles

$$[\mathcal{F}_{X_\pm}^\delta, \mathcal{F}_{X_\pm}^{\delta_0, 0}]^{\text{vir}} = [\mathcal{F}_{X_\pm}^{\delta_0, 0}]^{\text{vir}} \times [\mathcal{F}_{X_\pm}^\delta]^{\text{vir}},$$

if we choose appropriate orientations on $D$-orbifolds associated with $\mathcal{F}_{X_-}^{\delta_0, 0}$ and $\mathcal{F}_{X_+}^\delta$.

5.3. A conjectural gluing formula. We state a conjectural gluing formula of $DT_4$ invariants for a simple degeneration $X \to C$ of projective CY 4’s. We assume $\omega_X/C = 0$ and $X_0 = X_+ \cup_Y X_-$ with $Y$ as an anti-canonical divisor of $X_\pm$.

We first consider the virtual cycle of $\mathcal{F}_{X/-}\mathbb{C}$. Because of the triviality of the relative canonical bundle $\omega_X/C = 0$, as Conjecture 5.8, 5.9, there should exist a $D$-orbifold associated with $\mathcal{F}_{X/-}\mathbb{C}$ and a Borel-Moore homology class $[\mathcal{F}_{X/-}\mathbb{C}]^{\text{vir}} \in H_*^{BM}(\mathcal{F}_{X/-}\mathbb{C}, \mathbb{Q})$ if the $D$-orbifold is orientable.

The comparison of $[\mathcal{F}_{X/-}\mathbb{C}]^{\text{vir}}$ and $[\mathcal{F}_{X_\pm}^{\delta_0, 0}]^{\text{vir}}$. By Lemma 5.2, $\mathcal{F}_{X_\pm}^{\delta_0, 0}$ is the zero loci of a section $\pi_p \delta_0$ of a complex line bundle $\pi_p L_0$ on $\mathcal{F}_{X/-}\mathbb{C}$. Meanwhile, by [29], the obstruction theory of $\mathcal{F}_{X_\pm}^{\delta_0, 0}$ is the pull-back of the obstruction theory of $\mathcal{F}_{X/-}\mathbb{C}$, thus we should have

$$(11) \quad [\mathcal{F}_{X_\pm}^{\delta_0, 0}]^{\text{vir}} = c_1(\pi_p L_0, \pi_p \delta_0) \cap [\mathcal{F}_{X/-}\mathbb{C}]^{\text{vir}},$$

where $c_1(\pi_p L_0, \pi_p \delta_0) \in H^2(\mathcal{F}_{X/-}\mathbb{C}, \mathbb{R})$ is the localized first Chern class (Proposition 19.1.2 [22]) and $c_1(\pi_p L_0, \pi_p \delta_0) : H_*^{BM}(\mathcal{F}_{X/-}\mathbb{C}) \to H_*-2(\mathcal{F}_{X_\pm}^{\delta_0, 0}, \mathcal{F}_{X_\pm}^{\delta_0, 0})$ is the cap product [28].

Summing over all splitting of $\hat{D}$, we get

$$(12) \quad \sum_{\delta \in \Lambda^{\mathbb{R}_p}} [\mathcal{F}_{X_\pm}^{\delta_0, 0}]^{\text{vir}} = c_1(\bigotimes_{\delta \in \Lambda^{\mathbb{R}_p}} \pi_p L_0) \cap [\mathcal{F}_{X/-}\mathbb{C}]^{\text{vir}}$$

The comparison of $[\mathcal{F}_{X_\pm}^\delta]^{\text{vir}}$ and $[\mathcal{F}_{X_\pm}^{\delta_0, 0}]^{\text{vir}}$. For $t \neq 0$, we have $\mathcal{F}_{X_\pm}^\delta \times_C t \cong \mathcal{F}_{X_\pm}^\delta$. The obstruction theory of $\mathcal{F}_{X_\pm}^\delta$ is the pull back of the obstruction theory of $\mathcal{F}_{X_\pm}^{\delta_0, 0}$. Without loss of generality, we assume $C = \mathbb{A}^1$ and similarly obtain

$$(13) \quad [\mathcal{F}_{X_\pm}^{\delta_0, 0}]^{\text{vir}} = c_1(\mathcal{O}_{\mathcal{F}_{X/-}\mathbb{C}}, \pi_p \pi^* t) \cap [\mathcal{F}_{X/-}\mathbb{C}]^{\text{vir}},$$

where $[\mathcal{F}_{X_\pm}^{\delta_0, 0}]^{\text{vir}}$ is the $DT_4$ virtual cycle mentioned in Theorem 2.6.

Conjecture 5.10. We take a simple degeneration $X \to C$ of projective CY 4’s with $\omega_X/C = 0$ such that $X_0 = X_+ \cup_Y X_-$ and $Y$ is an anti-canonical divisor of $X_\pm$. Then there exists a $D$-orbifold associated with $\mathcal{F}_{X/-}\mathbb{C}$, and a Borel-Moore homology class

$$[\mathcal{F}_{X_\pm}^{\delta_0, 0}]^{\text{vir}} \in H_*^{BM}(\mathcal{F}_{X/-}\mathbb{C}, \mathbb{Q})$$

if the corresponding $D$-orbifold is orientable.

Furthermore, equalities (11), (12) hold if we choose appropriate orientations for $D$-orbifolds associated with $\mathcal{F}_{X_\pm}^{\delta_0, 0}$ and $\mathcal{F}_{X_\pm}^\delta$.

To introduce the gluing formula, we make the following definition.

Definition 5.11. Let $X \to C$ be a simple degeneration of projective CY 4’s such that $X_0 = X_+ \cup_Y X_-$ with $Y$ as an anti-canonical divisor of $X_\pm$. $P$ is a polynomial and $\mathcal{F}_{X/-}\mathbb{C} \to C$ is Li-Wu’s good degeneration of $\text{Hilb}^p(P_0), t \neq 0$. We assume for any $\delta = (\delta_0, 0) \in \Lambda^{\mathbb{R}_p}$, any closed point $I_C \in \text{Hilb}^0(Y)$ satisfies $\text{Ext}^1(I_C, I_C) = 0$. Then the family version $DT_4$ invariant of $\text{Hilb}^p(P_0), t \neq 0$ is a map

$$DT_4(\mathcal{F}_{X/-}\mathbb{C}) : \text{Sym}^* (H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, \ldots]) \to \mathbb{Z}$$
such that
\[ DT_4(\mathcal{F}_{\gamma}^\mu)((\gamma_1, P_1), (\gamma_2, P_2), \ldots) = \int_{[\mathcal{F}_{\gamma}^\mu]_{vir}} \mu(\gamma_1, P_1) \cup \mu(\gamma_2, P_2) \cup \ldots, \]
where \( \gamma_i \in H_*(X, \mathbb{Z}) \), \( \mu(\cdot) \) is the \( \mu \)-map defined in Definition 3.4 with respect to the universal sheaf of \( \mathcal{F}_{\gamma}^\mu \) and we view \( \mathcal{F}_{\gamma}^\mu \to \mathcal{F}_{\gamma}^{P} \) as a closed substack to do integration.

We take a Kähn type decomposition of the cohomology class
\[ (\mu(\gamma_1, P_1) \cup \mu(\gamma_2, P_2) \cup \ldots)_{\gamma_{x_i}^{\mu}} = \sum_{\gamma_{x_i}^{\mu}} \tau_{\pm, \delta, i} \otimes \tau_{-, \delta, i} \in H^*(\mathcal{F}_{\gamma_{x_i}^{\mu}} \times \mathcal{F}_{\gamma_{x_i}^{\mu}}). \]
Then the relative \( DT_4 \) invariant of \( \mathcal{F}_{\gamma_{x_i}^{\mu}} \) with respect to \( \tau_{\pm, \delta, i} \) is
\[ DT_4(\mathcal{F}_{\gamma_{x_i}^{\mu}})(\tau_{\pm, \delta, i}) = \int_{[\mathcal{F}_{\gamma_{x_i}^{\mu}}]_{vir}} \tau_{\pm, \delta, i} \in \mathbb{Q}. \]

We state a gluing formula of \( DT_4 \) invariants on Calabi-Yau 4-folds based on previous conjectures.

**Theorem 5.12.** Let \( X \to C \) be a simple degeneration of projective CY_4 such that \( \omega_{X/C} = 0 \) and \( X_0 = X_+ \cup X_- \) with \( Y \) as an anti-canonical divisor of \( X_+ \). \( P \) is a polynomial and \( \mathcal{F}_{\gamma}^\mu \to C \) is Li-Wu’s good degeneration of \( \text{Hilb}^P(X_1) \), \( t \neq 0 \). We assume for any \( \delta = (\delta_\pm, \delta_0) \) is a closed point \( I_C \in \text{Hilb}^b(Y) \) satisfies \( \text{Ext}_Y^1(I_C, I_C) = 0 \) (without loss of generality, we assume \( \text{Hilb}^b(Y) \) consists of one point for simplicity), then for \( t \neq 0 \in C \),
\[ DT_4(\mathcal{F}_{\gamma}^\mu)((\gamma_1, P_1), (\gamma_2, P_2), \ldots) = \sum_{\delta \in \Lambda_{\text{spl}}^b, i} DT_4(\mathcal{F}_{\gamma_{x_i}^{\mu}})(\tau_{\pm, \delta, i}) \cdot DT_4(\mathcal{F}_{\gamma_{x_i}^{\mu}})(\tau_{-, \delta, i}), \]
where \( \gamma_i \in H_*(X, \mathbb{Z}) \), \( \mu(\cdot) \) is the \( \mu \)-map defined in Definition 3.4 with respect to the universal sheaf of \( \mathcal{F}_{\gamma}^\mu \); \( (\pm, \delta, i) \) is the factor in a Kähn type decomposition as in Definition 5.11.

**Proof.** By Lemma 5.2, Conjecture 5.9, 5.10 and 12. \( \square \)

**Remark 5.13.** By Corollary 2.16, if \( P \) is the Hilbert polynomial associated with structure sheaves of points. The condition which says for any \( \delta = (\delta_{\pm}, \delta_0) \) is \( \Lambda_{\text{spl}}^b \) and any closed point \( I_C \in \text{Hilb}^b(Y) \), we have \( \text{Ext}_Y^1(I_C, I_C) = 0 \) is satisfied.

6. **Appendix on the orientability of relative \( DT_4 \) theory**

In this section, we give a coherent description of orientability issues involved in definitions of relative \( DT_4 \) virtual cycles (in Section 3) and then prove the existence of orientations.

We take a smooth (Calabi-Yau) 3-fold \( Y \) in a complex projective 4-fold \( X \) as its anti-canonical divisor, and denote \( \mathfrak{M}_X \) to be a moduli space of stable bundles on \( X \) with fixed Chern classes. Assuming conditions in Theorem 5.1 are satisfied, we obtain a morphism
\[ r : \mathfrak{M}_X \to \mathfrak{M}_Y \]
to a Gieseker moduli space of stable sheaves on \( Y \). We denote the determinant line bundle of \( \mathfrak{M}_X \) by \( \mathcal{L}_X \) with \( \mathcal{L}_X|_E = \det(H^{odd}(X, EndE)) \otimes \det(H^{even}(X, EndE))^{-1} \) (similarly for \( \mathcal{L}_Y \to \mathfrak{M}_Y \)).

**Proposition 6.1.** We further assume \( \mathfrak{M}_Y \) is smooth as an abstract scheme (Kuranishi maps not necessarily vanish). Then \( c_1(\mathcal{L}_X) = r*c_1(\mathfrak{M}_Y) \) in Cases I-III individually is equivalent to the existence of orientations in each corresponding case, i.e.
(i) the D-manifold associated with \( \mathfrak{M}_X \) is orientable in Case I;
(ii) the self-dual obstruction bundle is orientable in Case II;
(iii) the self-dual reduced obstruction bundle is orientable in Case III.

**Proof.** In Case I, \( \mathfrak{M}_Y \) consists of finite number of points and \( c_1(\mathcal{L}_X) = 0 \) is equivalent to the orientability of the D-manifold associated with \( \mathfrak{M}_X \) (see also Theorem 2.7).

In Case II, \( H^*(X, EndE)^* \)’s and \( H^1(Y, EndE|_Y)^* \)’s are locally constant. We abuse them as notations for corresponding bundles. Then \( c_1(\mathcal{L}_X) = r*c_1(\mathfrak{M}_Y) \) is equivalent to
\[ c_1(H^1(X, EndE)^*) = r*c_1(H^1(Y, EndE|_Y)^*) + c_1(H^2(X, EndE)) = c_1(H^2(X, EndE)). \]
By the short exact sequence
\[ 0 \to H^1(X, EndE)^* \to H^1(X, EndE) \to r^*H^1(Y, EndE|_Y) \to 0 \]
and equality 14, we get \( c_1(H^2(X, EndE)) = 0 \) which reduces the structure group of the obstruction bundle to be \( SO(N, \mathbb{C}) \). Thus the self-dual obstruction bundle is orientable.
In Case III, \( H^2(X, \text{End}E) = 0 \), and \( c_1(\mathcal{L}_X) = r^*c_1(\mathcal{M}_Y) \) is equivalent to
\[
\begin{equation}
(\text{c}_1(H^2(X, \text{End}E)) - (\text{c}_1(\text{End}E)) - r^*c_1(\text{H}^2(Y, \text{End}E|_Y))) = 0,
\end{equation}
\]
which is the same as \( c_1(\text{Ob}_{\mathcal{M}_X}^\text{red}) = 0 \).

\begin{remark}
The above result can be easily extended to ideal sheaves cases.
\end{remark}

In general, the determinant line bundle \( \mathcal{L}_Y \) satisfies
\[
\mathcal{L}_Y|_\mathcal{F} = \text{det}(\text{Ext}^1(\mathcal{F}, \mathcal{F})) \otimes \text{det}(\text{Ext}^2(\mathcal{F}, \mathcal{F}))^{-1} = (\text{det}(\text{Ext}^1(\mathcal{F}, \mathcal{F})))^2.
\]
When \( \mathcal{M}_Y \) is smooth, \( \mathcal{L}_Y \) has a canonical square root given by \( \text{det}(\mathcal{T}\mathcal{M}_Y) \), the equality \( c_1(\mathcal{L}_X) = r^*c_1(\mathcal{M}_Y) \) can be reformulated as \( c_1(\mathcal{L}_X) = r^*c_1(\mathcal{L}_Y^\frac{1}{2}) \) for some square root \( \mathcal{L}_Y^\frac{1}{2} \) of \( \mathcal{L}_Y \). We thus make the following definition.

\begin{definition}
Let \( Y \) be a smooth anti-canonical divisor of a complex projective 4-fold \( X \), \( \mathcal{M}_X \) be a moduli space of stable bundles on \( X \) with fixed Chern classes which has a well-defined restriction morphism \( r: \mathcal{M}_X \to \mathcal{M}_Y \) to a Gieseker moduli space of stable sheaves on \( Y \).

The morphism \( r: \mathcal{M}_X \to \mathcal{M}_Y \) is orientable in relative DT\(_4\) theory if there exists a square root \( \mathcal{L}_Y^\frac{1}{2} \) of \( \mathcal{L}_Y \) ( \( \mathcal{L}_Y^\frac{1}{2} \otimes \mathcal{L}_Y^\frac{1}{2} \cong \mathcal{L}_Y \) ) such that
\[
c_1(\mathcal{L}_X) = r^*c_1(\mathcal{L}_Y^\frac{1}{2}).
\]

From the long exact sequence in Lemma 3.3, we have \( 2c_1(\mathcal{L}_X) = r^*c_1(\mathcal{L}_Y) \). Meanwhile, if \( \mathcal{L}_Y^\frac{1}{2} \) exists with \( c_1(\mathcal{L}_Y) = 2c_1(\mathcal{L}_Y^\frac{1}{2}) \), then \( c_1(\mathcal{L}_X) = r^*c_1(\mathcal{L}_Y^\frac{1}{2}) \) provided \( H^2(\mathcal{M}_Y) \) is free of even torison. In general, we have

\begin{theorem}
With the same set-up as Definition 6.3, we further assume \( \text{Tor}(H_1(X, \mathbb{Z})) = 0 \), then the morphism \( r: \mathcal{M}_X \to \mathcal{M}_Y \) is orientable in relative DT\(_4\) theory, i.e. there exists a square root \( \mathcal{L}_Y^\frac{1}{2} \) of \( \mathcal{L}_Y \) such that
\[
c_1(\mathcal{L}_X) = r^*c_1(\mathcal{L}_Y^\frac{1}{2}).
\]

\end{theorem}

\begin{proof}
See the proof of Theorem 4.2 [14].
\end{proof}

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