ON AFFINE HYPERSURFACES WITH EVERYWHERE NONDEGENERATE SECOND QUADRATIC FORM

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Abstract. Consider a closed connected hypersurface in $\mathbb{R}^n$ with constant signature $(k,l)$ of the second quadratic form, and approaching a quadratic cone at infinity. This hypersurface divides $\mathbb{R}^n$ into two pieces. We prove that one of them contains a $k$-dimensional subspace, and another contains a $l$-dimensional subspace, thus proving an affine version of Arnold hypothesis. We construct an example of a surface of negative curvature in $\mathbb{R}^3$ with slightly different asymptotical behavior for which the previous claim is wrong.

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1. Introduction

In this paper we prove three results connected to an Arnold hypothesis formulated in \cite{1}. Recall first this hypothesis.

Definition 1. A smooth hypersurface in $\mathbb{R}^{n+1}$ is called $(k,l)$-hyperbolic if its second quadratic form has signature $(k,l)$. In other words, in some affine system of coordinates $(x,y,z)$ its local equation is $z = \sum_{i=1}^{k} x_i^2 - \sum_{j=1}^{l} y_i^2 + \text{higher order terms}$. 

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One class of examples of such hypersurfaces is the class of quadrics. Namely, let $Q(x, y)$ be a non-degenerate symmetric bilinear form in $\mathbb{R}^{n+1}$ of signature $(k + 1, l + 1)$ (i.e. its restriction to some $(k + 1)$-dimensional subspace $L_+$ is positively defined, and its restriction to some $(l + 1)$-dimensional subspace $L_-$ is negatively defined, and $k + l = n - 1$). Then a hypersurface $S_Q$ in $\mathbb{R}P^n$ given by equation $Q(x, x) = 0$ is smooth and $(k, l)$-hyperbolic.

The hypersurface $S_Q$ has the following remarkable property: the domain bounded by $S_Q$ (i.e. the domain $(Q(x, x) \leq 0)$) contains a $l$-dimensional projective subspace $P(L_+)$ - a projectivization of $L_+$, and its complement contains a $k$-dimensional projective subspace $P(L_-)$ - a projectivization of $L_-$.

Arnold’s hypothesis claims existence of such subspaces for any $(k, l)$-hyperbolic hypersurface.

**Conjecture 1** (Arnold Conjecture). 1. For any domain $U \subset \mathbb{R}P^n$ bounded by a connected smooth $(k, l)$-hyperbolic hypersurface $B$ there exist a projective subspace $L_-$ of dimension $k$ not intersecting $U$ and a projective subspace $L_+$ of dimension $l$ contained in $U$.

2. Any projective line joining a point of $L_+$ and a point $L_-$ intersects $B$ at exactly 2 points.

Apart from the case of quadrics discussed above, another fact justifying this conjecture is the following well-known fact. Consider a locally convex connected surfaces $S$ in $\mathbb{R}P^n$. Then $S$ is a boundary of some domain in $\mathbb{R}P^n$ and this domain doesn’t intersect any hyperplane, see [1]. This case corresponds to the case of $k = 0$ or $l = 0$ of the Arnold Conjecture.

We prove in [2] the first case $k = l = 1$ of the Arnold Conjecture in some additional assumptions. Namely, we consider a smaller then class of (1,1)-hyperbolic surfaces class of projective $L$-convex-concave subsets of $\mathbb{R}P^3$. This class is a close relative of the class convex-concave sets considered in [1]. We prove that any $L$-convex-concave subset contains a line inside.

In this paper we deal with affine version of Arnold’s hypothesis. Namely, we consider $(k-1,l)$-hyperbolic hypersurfaces in $\mathbb{R}^{k+l}$ with some prescribed asymptotic behavior at infinity. Our results can be roughly summarized as follows: if the asymptotic at infinity forces the closure in $\mathbb{R}^{p+k+l}$ of the hypersurface to be $(k-1,l)$-hyperbolic, then the domain bounded by the hypersurface contains a line. And if the closure is not $(k-1,l)$-hyperbolic, then one can construct a domain bounded by such hypersurface and not containing any line inside.

Here is more exact description of the results. Consider $(k-1,l)$-hyperbolic hypersurfaces $M$ in $\mathbb{R}^n$, $k + l = n$. We say that $M$ approaches a surface $L$ at infinity if $M$ and $L$ are arbitrarily $C^2$-close outside a big enough ball (see [3.1] for an exact $\epsilon, \delta$-definition). For example, the quadric $\{ \sum_{i=1}^{k} x_i^2 - \sum_{j=1}^{l} x_{k+j}^2 \} = A$ approaches the cone $K = \{ \sum_{i=1}^{k} x_i^2 = \sum_{j=1}^{l} x_{k+j}^2 \} = \mathbb{R}^n$ at infinity for any $A$.

We prove the following theorem:

**Theorem 1.** The first claim of the Arnold hypothesis is true for any $(k-1,l)$-hyperbolic hypersurface $M$ approaching a quadratic cone $K = \{ \sum_{i=1}^{k} x_i^2 = \sum_{j=1}^{l} x_{k+j}^2 \} \subset \mathbb{R}^n$ at infinity.

We also prove the following result, strengthening the second part of the Arnold Conjecture.
Theorem 2. Any \((1,1)\)-hyperbolic surface \(M\) in \(\mathbb{R}^3\) approaching a quadratic cone \(K = \{ x^2 + y^2 = z^2 \}\) intersects any ray from the origin in at most one point. In other words, the projection \(M \to S^2 = \{ x^2 + y^2 + z^2 = 1 \}\) is embedding.

Note that the surface described in Theorem 2 has a \(C^1\)-closure in \(\mathbb{R}P^n\). Consider the simplest case \(k = l = 1\), and denote \(K_- = K \cap \{ z \leq 0 \}\) and \(K_+ = K \cap \{ z \geq 0 \}\). Will the result remain true if we consider surface \(B\) approaching union of translated \(K_-\) and \(K_+\)? It turns out that if the translates intersect, then the domain bounded by the surface \(B\) still contains a line, and if the intersection is empty, then this is not necessarily true. Note the different behavior of the projective closure \(\overline{B}\) of the surface in these two cases. In both cases the points of \(\overline{B} \setminus B\) are not smooth points of the closure. However, in the first case the \(\overline{B}\) is \((1,1)\)-hyperbolic after arbitrarily small perturbation, and in the second case the surface \(\overline{B}\) is locally convex at these points. This coincides with what the Arnold Conjecture prescribes, further strengthening it.

Theorem 3. Let \(K' = \{ (x,y,z) \mid x^2 + y^2 = (|z| - 1)^2, |z| \geq 1 \}\) be a union of non-intersecting translates of \(K_-\) and \(K_+\). There are \((1,1)\)-hyperbolic surfaces approaching \(K'\) at infinity and not containing any lines.

All these result can be considered in more general context of existence and properties of a solution of some boundary problem. A natural boundary problem is to find a compact smooth \((1,1)\)-hyperbolic surface which boundary is a given tuple of non-intersecting closed smooth curves and which is tangent at this boundary to the given set of planes tangent to the curves. The Theorem 2 follows from the fact that solution of some boundary problem of this type cannot intersect an open domain – the interior of the cone \(K\).

2. Preliminaries: Quadrics

The hypersurface \(M\) of the Theorem 2 is approaching at infinity a cone given by a quadric. In this paragraph we collect some standard facts about quadrics we will need later.

Theorem 4. Let \(Q(x) = \sum_{i=1}^k x_i^2 - \sum_{j=1}^l \epsilon_{k+j}^2\), and let \(Q_\epsilon = \{ f = \epsilon \}\) be its level hypersurfaces.

1. restriction of \(Q\) to a tangent plane \(T_xQ_\epsilon \subset \mathbb{R}^{k+l}\) at point \(x \in Q_\epsilon\) is a polynomial of second degree having signatures \((k,1-1), (k-1,1-1)\) or \((k-1,1)\) according to \(\epsilon < 0, \epsilon = 0\) or \(\epsilon > 0\) correspondingly;
2. \(Q_\epsilon\) is \((k,1-1)\)-hyperbolic if \(\epsilon < 0\) and \((k-1,l)\)-hyperbolic if \(\epsilon > 0\);
3. Projectivization of \(Q_0\) is a \((k-1,l-1)\)-hyperbolic hypersurface in \(\mathbb{R}P^{k+l-1}\).

Proof. First we prove a general Lemma. To a nondegenerate bilinear form \(q\) on a linear space \(L\) corresponds an isomorphism \(\tilde{q}\) between \(L\) and its dual \(L^*\), defined by the condition \(\ell(x) = q(\tilde{q}(\ell), x)\) for \(\ell \in L^*\) and all \(x \in L\). Using this form one can define a bilinear form \(q^*(\ell_1, \ell_2) = q(\tilde{q}(\ell_1), \tilde{q}(\ell_2))\) on \(L^*\).

Lemma 1. Restriction of \(q\) on a hyperplane \(H = \{ \ell = 0 \}\subset L\) has signature \((k,l-1), (k-1,l-1), (k-1,l)\) if \(q^*(\ell, \ell) < 0, q^*(\ell, \ell) = 0\) or \(q^*(\ell, \ell) > 0\) correspondingly.

Proof. By definition \(q(\tilde{q}(\ell), x) = 0\) is equivalent to \(x \in H\). So \(q^*(\ell, \ell) = 0\) means \(y = \tilde{q}(\ell) \in H\), i.e. the restriction of \(q\) on \(H\) is degenerate. From the other hand,
$H$ should intersect subspaces where $q$ is positive/negative definite by subspaces of dimensions $k-1$ and $l-1$ at least, so the only possible signature is $(k-1, l-1)$. Vice versa, if for some $y \in H$ and for all $x \in H$ we have $q(y, x) = 0$, then $\ell$ is proportional to $q^{-1}(y)$ and so $q^*(\ell, \ell) = q(y, y) = 0$.

So suppose $q^*(\ell, \ell) \neq 0$. Then Gramm-Schmidt procedure starting from $y = q(\ell)$ is possible and results in a basis which first vector is $y$ and all the rest span a hyperplane of vectors $q$-orthogonal to $y$, i.e. $H$. Since in this basis $q$ is diagonal, we easily see that the signature of the restriction of $q$ to $H$ is as required.

To prove the Theorem 4 apply Lemma 2 to $Q(x) = q(x, x)$ in $\mathbb{R}^n$, where $n = k+l$. The first claim of the Theorem 4 follows from the Lemma 2 and the fact that $q(dQ(x)) = 2x$, so $q^*(dQ(x), dQ(x)) = 4q(x, x) = 4\varepsilon$.

To prove the second claim of the Theorem 4 change coordinates in such a way that $x \in Q(\varepsilon)$ is an origin and $T_xQ(\varepsilon) = \{x_n = 0\}$. The restriction $Q_1$ of $Q$ to $T_xQ(\varepsilon)$ is a quadratic polynomial, without linear and free terms in these coordinates. So $Q = \varepsilon + \lambda x_n + Q_1(x_1, ..., x_{n-1}) + x_n\ell(x_1, ..., x_{n-1})$, where $Q_1$ is a homogeneous polynomial of degree 2 and $\ell(x_1, ..., x_{n-1})$ is linear. An easy computation shows that $x_n = -\lambda^{-1}Q_1(x_1, ..., x_{n-1}) + \cdots$ on $\{Q = \varepsilon\}$. The second claim follows now from the first claim of the Theorem 4.

The third claim follows since the kernel of the bilinear form $q_1$ on $T_xQ(\varepsilon)$ corresponding to the quadratic form $Q_1$ is exactly the kernel of the projection $\mathbb{R}^n \to \mathbb{R}^{n-1}$.

Denote by $\mathbb{S}^{k+l-1}$ the standard sphere $\{\sum_{i=1}^k x_i^2 + \sum_{j=1}^l x_{k+j}^2 = 1\} \subset \mathbb{R}^{k+l}$. The standard scalar product $(x, x') = \sum_i x_i x'_i$ in $L$ defines, as above, a scalar product in $L^*$ and also an isomorphism of $L$ and $L^*$. For a smooth cooriented hypersurface $M \subset \mathbb{R}^n$ the Gauss map $G : M \to \mathbb{S}^{n-1}$ maps a point $x \in M$ to the vector normal to $M$ at $x$. If $M = \{P = 0\}$ and $dP \neq 0$ on $M$, then the Gauss map is a composition of the map $x \to \frac{dP}{dP}$ and the isomorphism between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ provided by a standard scalar product. A classical computation shows that the Jacobian of the Gauss map is exactly the Gaussian curvature of the hypersurface.

**Lemma 2.** Let $Q = \sum_{i=1}^k x_i^2 - \sum_{j=1}^l x_{k+j}^2$ be as in Theorem 4. Then the pull-back of $Q^*$ from $(\mathbb{R}^n)^*$ to $\mathbb{R}^n$ by the isomorphism given by the standard scalar product coincides with $Q$. The Gauss map provides diffeomorphisms between $\{Q = \varepsilon > 0\}$ and $\mathbb{S}^n \cap \{Q > 0\}$, between $\{Q = \varepsilon < 0\}$ and $\mathbb{S}^n \cap \{Q < 0\}$, and maps $\mathbb{S}^n \cap \{Q = 0\}$ diffeomorphically onto itself.

This follows immediately from the explicit formulae for the Gauss mapping $G$ of the quadric $\{Q = \varepsilon\}$. Namely,

$$G(x_1, ..., x_k, x_{k+1}, ..., x_{k+l}) = \frac{1}{\|x\|}(x_1, ..., x_k, -x_{k+1}, ..., -x_{k+l}).$$

The topology of the domains, and therefore of level hypersurfaces of $Q$ is quite simple.

**Corollary 1.** $\{Q = \varepsilon > 0\}$ is diffeomorphic to $\mathbb{S}^{k-1} \times B^l$, $\{Q = \varepsilon < 0\}$ is diffeomorphic to $\mathbb{S}^{l-1} \times B^k$ and $\mathbb{S}^n \cap \{Q = 0\}$ is diffeomorphic to $\mathbb{S}^{k-1} \times \mathbb{S}^{l-1}$.

For example, $\mathbb{S}^n \cap \{Q = 0\} = \{\sum_{i=1}^k x_i^2 = \sum_{j=1}^l x_{k+j}^2 = \frac{1}{2}\} = \mathbb{S}^{k-1} \times \mathbb{S}^{l-1}$. 
3. Surface doesn’t enter the half-cones

3.1. Exact formulation of the result. In this section we prove the Theorem.

We start with a definition of an affine version of \((k,l)\)-hyperbolicity and state more precisely the asymptotic conditions on the hypersurface \(M\).

**Definition 2.** A smooth connected hypersurface \(M\) lying in \(\mathbb{R}^n\) equipped with a standard Euclidean metric is \((k,l)\)-hyperbolic if its second quadratic form is everywhere nondegenerate and have signature \((k,l)\).

**Definition 3.** We say that a hypersurface \(M\) approaches a hypersurface \(L\) at infinity if for any \(\epsilon > 0\) there exists an \(R > 0\) such that

1. there exists a diffeomorphism \(\phi : L \setminus B_R \to M \setminus B_R\), such that \(\|\phi(x) - x\| < \epsilon\) for any \(x \in L \setminus B_{2R}\) and
2. there is a diffeomorphism \(\psi\) of the Gauss images of \(L \cap S_{2R}^{n-1}\) onto the Gauss image of \(\phi(M \cap S_{2R}^{n-1}) \subset L\) such that \(\text{dist}(\psi(x), x) < \epsilon\) in standard metric on \(S^{n-1}\), where \(S_{2R}^{n-1}\) denotes a sphere of radius \(2R\) with center at the origin.

The proof of the Theorem goes as follows. First, using simple topological arguments, we prove that image under the Gauss map of the hypersurface \(M\) does not intersect the image under the Gauss map of the quadric \(\{Q = \epsilon > 0\}\) for any positive \(\epsilon\). Second, we note that the interior \(U = \{Q > 0\}\) of the cone \(K\) can be exhausted by these quadrics. If we suppose that \(M \cap U\) is nonempty and compact, then an analogue of the Rolle theorem says that there should be a level curve of \(Q\) lying in \(U\) and tangent to \(M\), namely the level curve \(\{Q = \max_{x \in M} Q(x)\}\). This contradicts to the properties of the Gauss map of \(M\) mentioned above. The assumption of compactness can be dropped by a slight modification of these arguments.

3.2. The topological Lemma. The proof of the Theorem starts from a simple topological lemma. This lemma will be applied further to the Gauss map of \((k,l)\)-hyperbolic hypersurfaces.

**Lemma 3.** Let \(M\) be a compact connected manifold with boundary \(\partial M\) and \(f : M \to N\) be a local diffeomorphism to a compact simply-connected manifold \(N\), \(\pi_1(N) = 0\). Suppose that the restriction \(f|_{\partial M}\) of \(f\) to each connected component of the boundary of \(M\) is also a diffeomorphism (i.e. also embedding). Then \(f\) is a diffeomorphism of \(M\) to \(f(M)\).

**Proof.** An easy case is \(\partial M = \emptyset\). In this case \(f\) is a covering, so should be a trivial one (since \(\pi_1(N)\), being trivial, has no nontrivial subgroups).

The general case will be reduced to this case by gluing "hats" to \(M\), thus eliminating the boundary components one-by-one.

Namely, consider a connected component of \(\partial M\) (denote it by \(B\)). Its image \(f(B)\) is a cooriented hypersurface in \(N\). Indeed, \(f(B)\) divides any sufficiently small neighborhood of any of its point \(f(b)\) into two parts, and one of them is distinguished, since it is an image of a small neighborhood of \(b\) in \(M\). Therefore \(N \setminus f(B)\) consists of two open parts (having even and odd number of preimages correspondingly). Call the part which doesn’t intersect the image of a sufficiently small neighborhood of \(B\) by "hat". We can glue the "hat" to \(M\) along \(B \cong f(B)\): the new manifold will be the union of \(B\) and the "hat" with the neighborhood of \(b \in B\) being defined as union of the connected component of \(f^{-1}(U)\) containing \(b\) and the intersection of \(U\) and the "hat" (where \(U \subset N\) is a small open ball containing \(f(b)\)).
Repeating this operation with all components of the boundary, we get a new manifold \( \tilde{M} \) without a boundary and \( M \subset \tilde{M} \). The map \( f \) extends to \( \tilde{f} : \tilde{M} \to N \) in a trivial manner: we define it to be an identity on "hats". The map \( f \) satisfies conditions of the Lemma 3, so it is a global diffeomorphism by the first part of the proof. So \( f \) is also a diffeomorphism, since \( f = \tilde{f}|_M \).

**Corollary 2.** Let \( M \) be a compact connected oriented \((n-1)\)-dimensional submanifold with boundary of \( \mathbb{R}^n \), and suppose that its second fundamental form is everywhere nondegenerate, including the boundary. Assume that the restriction of the Gauss map of \( M \) to each connected component of the boundary \( \partial M \) is one-to-one. Then the Gauss map of \( M \) itself is one-to-one.

3.3. **Gauss image of \((k-1,l)\)-hyperbolic hypersurface.** Consider a \((k-1,l)\)-hyperbolic connected closed hypersurface \( M \subset \mathbb{R}^{k+l+1} \) approaching the quadratic cone \( K = \{Q = 0\} \) at infinity, where, as before, \( Q(x) = \sum_{i=1}^{k} x_i^2 - \sum_{j=1}^{l} x_{k+j}^2 \) is a quadratic form on \( \mathbb{R}^n \), \( n = k + l \). Similarly to the previous Corollary, we prove that the Gauss image of \( M \) coincides with the Gauss image of a \((k-1,l)\)-hyperbolic level hypersurface of \( Q \).

We will consider later the case of a surface \( M \) approaching at infinity the cone \( K' = \{(|z| - 1)^2 = x^2 + y^2\} \subset \mathbb{R}^3 \) (i.e. \( k - 1 = l = 1 \)). The Lemma 3 is true for this case as well.

**Lemma 4.** Let \( M \) be a \((k-1,l)\)-hyperbolic closed connected hypersurface approaching \( K \). Then the Gauss mapping maps \( M \) diffeomorphically onto the Gauss image of a \((k-1,l)\)-hyperbolic level hypersurface of \( Q \).

**Proof.** The Jacobian of the Gauss map is equal to the Gaussian curvature, i.e is non-vanishing. Therefore the Gauss map is a local diffeomorphism. \( M \) is not a compact manifold with boundary, so the Lemma 3 is not directly applicable.

Consider the compact \( M_R = M \cap B_R \), where \( B_R \subset \mathbb{R}^n \) is a big closed ball of radius \( R \) centered at the origin, and denote by \( \partial M_R \) the boundary of \( M_R \). Its image \( G(\partial M_R) \) under the Gauss map \( G^1 \)-converge, as \( R \to \infty \), to the Gauss image of \( K \cap \partial B_R \), which coincides with the Gauss image of the whole cone \( K \).

**Lemma 5.** \( G(M_R) \cap G(\partial M_R') = \emptyset \) for any \( R \) and any \( R' > R \).

For big enough \( R' \) this follows immediately from Lemma 3 applied to \( M_R' \) and its Gauss map. The Lemma 3 is applicable since \( M_R \) is compact and the restriction of the Gauss map to its boundary is a diffeomorphism, due to the condition "approaching at infinity". Therefore this is true for any \( R \).

**Corollary 3.** The Gauss image of \( M_R \) doesn’t intersect the \( G(K) \).

Indeed, if the intersection is non-empty, then there is a point of \( G(k) \) which lies in the interior of \( G(M_R') \), where \( R' \) is any number greater than \( R \). Therefore this point cannot be a limit point of \( G(\partial M_R) \) as \( R \to \infty \). This contradicts to the condition that \( M \) approaches \( K \) at infinity.

Therefore the \( G(M_R) \) should lie entirely in one of the connected components into which the \( G(K) \) divides the sphere \( \mathbb{S}^{n-1} \). Since \( \partial G(M_R) \) converge uniformly to \( G(K) \), we conclude that \( G(M) \) is exactly one of them and the Gauss mapping is a diffeomorphism.

So the last part is to prove that \( G(M) \) fall into the right connected component.

If \( k = l \), then all nonsingular level hypersurfaces of \( Q \) are \((k-1,l)\)-hyperbolic.
(since \((k-1,l)-\)hyperbolicity and \((k,l-1)-\)hyperbolicity are the same), and there is nothing to prove. So we suppose that \(k \neq l\). Then the Gauss image of \(K\) divides the sphere \(S^{n-1}\) into two domains of different topological type: the Gauss image of a \((k-1,l)-\)hyperbolic level surface is diffeomorphic to \(S^{k-1} \times B^l\), and the Gauss image of a \((k,l-1)-\)hyperbolic level hypersurface is diffeomorphic to \(S^{l-1} \times B^k\).

Denote these domains by \(D_+\) and \(D_-\) correspondingly, so that \(D_+ = G(Q = +1)\) and \(D_- = G(Q = -1)\). Since the Gauss mapping of \(M\) is a diffeomorphism, we get that \(M\) is diffeomorphic to \(D_+\) or \(D_-\), and our goal is to exclude the last possibility.

We will prove that \(M\) is topologically different from \(D_-\). We will apply the Morse theory to \(M\) and the restriction of the linear functional \(f = x_n|_M\) to \(M\).

Since \(\nabla(x_n) = e_n = (0,0,...,0,1) \in D_-\), the function \(f\) has exactly two critical points on \(M\), namely the preimages of \(-e_n\) and \(e_n\) under the Gauss map of \(M\) (which is a diffeomorphism). The condition of \((k-1,l)-\)hyperbolicity means that both critical points are non-degenerate and their indices are equal to \(k-1\) or \(l\).

Let \(R\) be a big number and denote by \(\hat{M} = M \cap \{x_n < -R\}\). Although the level sets \(\{f = c\} = M \cap \{x_n = c\}\) are not compact, their behavior at infinity is trivial. Indeed, it the same as of the sections of cone \(K\) by the hyperplanes \(\{x_n = c\}\), due to the condition at infinity. Therefore the standard results of the Morse theory still hold, so the dimension \(h_i(M,\hat{M})\) of the group of relative homologies \(H_i(M,\hat{M})\) is less than the number of critical points of index \(i\) of the function \(f\). The pair \((M,\hat{M})\) is diffeomorphic (through the Gauss maps) to the pair \((Q_1,\hat{Q}_1)\), where \(Q_1 = \{Q = -1\}\), and \(\hat{Q}_1 = Q_1 \cap \{x_n < -R\}\), so \(H_1(M,\hat{M}) = H_1(Q_1,\hat{Q}_1)\).

The latter can be easily computed to be equal to 1 for \(i = k\) (since \(Q_1/\hat{Q}_1 \cong S^{k-1} \cup S^l\)), and this contradicts to the fact that \(f\) has no critical points of index \(k\) (recall that \(k \neq l\)).

3.4. **Rolle Lemma.** Let \(Q(x) = x_1^2 + ... + x_k^2 - x_{k+1}^2 - ... - x_n^2\) be a quadratic form in \(\mathbb{R}^n\). The cone \(K = \{Q = 0\}\) divides \(\mathbb{R}^n\) into two parts, \(\{Q > 0\}\) and \(\{Q < 0\}\). We prove in this paragraph that \(M\) does not intersect one of these domains. In fact the assumption are weaker than before.

**Theorem 5.** Let \(M\) be a smooth connected hypersurface \(M\) such that \(\text{dist}(x,K) \to 0\) as \(x \to \infty\). Suppose that the Gauss image of \(M\) is disjoint from the Gauss image of \(\{Q = -1\}\). Then \(M\) does not intersect the whole domain \(\{Q < 0\}\).

The proof is a specialization of the following general lemma.

**Lemma 6.** Let \(M \subset \mathbb{R}^n\) be a smooth closed embedded hypersurface without boundary and suppose that its image under the Gauss mapping \(G_M : M \to S^{n-1}\) does not intersect a domain \(U \subset S^{n-1}\). Suppose that \(U\) is symmetric with respect to antipodal map \(x \to -x\) of \(S^{n-1}\).

Suppose that on \(\mathbb{R}^n\) we are given a function \(f\) with nonnegative only critical values. Suppose that \(\frac{\nabla f(x)}{\|\nabla f(x)\|} \in U\) if \(f(x) < 0\) (in other words, \(G(\{f = t\}) \subset U\) for any \(t < 0\)). If \(f(x) \to 0\) as \(M \ni x \to \infty\), then \(f\) is nonnegative on \(M\).

**Proof.** Suppose that \(f(x) < 0\) for some \(x \in M\). Let \(x_0 \in M\) the point of minimum of the restriction of \(f\) to \(M\). It exists since \(M \cap \{f \leq f(M)\}\) is compact and nonempty. \(x_0\) is a critical point of the restriction of \(f\) to \(M\). Equivalently, \(T_{x_0}M\) is
perpendicular to the nonzero vector $\nabla f(x_0)$. This means that $\Gamma_S(x_0) \in U$, which is forbidden.

Apply this Lemma to the proof of the theorem. A first candidate for the function $f$ is the $Q$ itself: Gauss images of $\{Q = t < 0\}$ are all equal and do not intersect the Gauss image of $M$. However, $Q$ itself do not satisfy the conditions of the Lemma 3; one should slightly adjust $Q$ to ensure that the restriction of $f$ to $M$ tends to zero at infinity.

Denote $\sqrt{x_1^2 + \ldots + x_k^2}$ by $a$ and $\sqrt{x_{k+1}^2 + \ldots + x_n^2}$ by $b$. Suppose that $Q(x) = a^2 - b^2$ takes a negative value $-2\varepsilon$ at some point $x' \in M$, i.e. that $M$ intersects the domain $\{Q < -\varepsilon\}$. Consider the function $f_1 = \sqrt{a^2 + \varepsilon} - b$ and denote by $f$ its smoothing: the $f_1$ is not smooth at $b = 0$, but one can smoothen $f_1$ without changing it on $\{f_1 < 0\} = \{Q < -\varepsilon\}$, the only domain interesting for us. The main point is that the Gauss image of the negative level hypersurfaces of $f$ is the same as the Gauss image of $\{Q = -\varepsilon\}$. This is most evident in the planar $k = l = 1$ case, where the level curves of $f$ are just translates of $\{Q = -\varepsilon\}$. Indeed, take a point $(x_1, \ldots, x_n)$ such that $t = f(x) < 0$. A simple computation shows that, first, $df(x_1, \ldots, x_n)$ is proportional to $dQ(x_1, \ldots, x_k, \lambda x_{k+1}, \ldots, \lambda x_n)$ for $\lambda = \frac{b + \varepsilon}{a}$ and, second, $Q(x_1, \ldots, x_n, \lambda x_{k+1}, \ldots, \lambda x_n) = -\varepsilon$. This computation proves that the Gauss images of $\{f = t < 0\}$ and $\{Q = -\varepsilon\}$ are equal, and also shows that $f$ has only positive critical values.

One can easily see that different level curves of $f$ lie on positive distance one from another, so the only one approaching $K$ at infinity is the zero level curve $\{f = 0\} = \{Q = -\varepsilon\}$. Therefore, since $\text{dist}(x, K) \to 0$ as $M \ni x \to \infty$, the restriction of $f$ to $M$ tends to zero at infinity.

So $f$ satisfies the conditions of the Lemma 3 and therefore $M \cap \{Q < -\varepsilon\} = M \cap \{f < 0\} = \emptyset$, a contradiction with the choice of $x'$.

3.5. End of the proof of the Theorem 3. The rest of the proof of the Theorem 3 is just application of the two results proven above, the Lemma 3 and the Theorem 3.

First prove existence of an $l$-dimensional subspace in one of the domains into which $M$ divides $\mathbb{R}^n$. Suppose first that $k \neq l$. In this case the quadrics $Q_1 = \{Q = 1\}$ and $Q_{-1} = \{Q = -1\}$ have different signatures of the second quadratic forms: the first one is $(k - 1, l)$-hyperbolic, and the second is $(k, l - 1)$-hyperbolic. By Lemma 3 the Gauss image of the $(k - 1, l)$-hyperbolic hypersurface $M$ coincide with the Gauss image of $Q_1$, and is therefore disjoint from the Gauss image of $Q_{-1}$. So, by Theorem 3 $M$ does not intersect the domain $\{Q < 0\}$, which contains the $l$-dimensional subspace $\{x_1 = \ldots = x_k = 0\}$.

If $k = l$, then both $Q_1$ and $Q_{-1}$ are $(k - 1, l)$-hyperbolic, and Lemma 3 claims that the Gauss image of $M$ coincides with the Gauss image of one of them. Taking $-Q$ instead of $Q$ if necessary, we can assume that $G(M)$ coincides with $G(Q_1)$, and the same arguments hold.

The existence of a $(k - 1)$-dimensional affine subspace in the second part of $\mathbb{R}^n \setminus M$ is evident. By assumption $M$ approaches the cone $K$ at infinity, so the distance between $M \setminus B_R$ and $K \setminus B_R$ is less than distance between $K \setminus B_R$ and $L = \{x_{k+1} = \ldots = x_n = 0\}$ for big enough ball $B_R$. So take any $(k - 1)$-dimensional affine subspace of $L$ lying outside $B_R$, and it will not intersect $M$. 

4. Projection from the origin

Starting from this moment we will deal with \((1, 1)\)-hyperbolic surfaces in \(\mathbb{R}^3\) only. So we will omit the \((1, 1)\) and will call \((1, 1)\)-hyperbolic surfaces hyperbolic surfaces.

Theorem 6. Let \(M \subset \mathbb{R}^3\) be a hyperbolic surface approaching the standard cone \(K\) at infinity.

Then the restriction to \(M\) of the projection \(\pi : \mathbb{R}^3 \to S^2 = \{\|x\| = 1\}\) is embedding.

4.1. Arnold’s formula. The Theorem 6 follows from a remarkable formula due to Arnold, see [1]. Consider a generic smooth hypersurface \(M \subset \mathbb{R}P^3\). Denote by \(#\{M \cap \ell\}\) number of its points of intersections with a line \(\ell\) and by \(\text{sign}(M, \ell)\) the number of point \(x \in M\) containing the line \(\ell\) in their tangent planes counted with multiplicities. The multiplicity is equal to “+ 1” if the Gaussian curvature of \(M\) is positive at \(x\) and to “− 1” if it is negative at \(x\) (if the curvature at \(x\) is zero then the formula for multiplicity is more complicated).

Lemma 7 (Arnold, 88). For a generic smooth hypersurface \(M \subset \mathbb{R}P^3\) the sum \(#\{M \cap \ell\}\) + \(\text{sign}(M, \ell)\) is the same for all \(\ell\) and is equal to the Euler characteristic of \(M\).

Sketch of the proof for a semialgebraic \(M\) (due to O.Viro). Take out from \(M\) its points of intersection with \(\ell\) and compute the Euler characteristic of the result using Fubini theorem for Euler characteristic. Namely, the Euler characteristic of \(M \setminus \ell\) is equal to the integral over the space of all planes \(L_t, t \in \mathbb{R}P^1\), containing \(\ell\) of the Euler characteristic of \(M_t = \{M \setminus \ell\} \cap L_t\). For simplicity, suppose that each section \(M_t\) has at most one singular point (if not, perturb \(\ell\) slightly). Each nonsingular section \(M_t\) is a one-dimensional manifold, so is a union of circles and open intervals with ends at removed points. Therefore its Euler characteristic is equal to \(-\#\{M \cap \ell\}\) (since Euler characteristic of a circle is equal to zero). Euler characteristic of a singular section differs by +1 or by −1, depending on the sign of the curvature of \(M\) at the singular point. Indeed, if the curvature at the singular point is negative, then the section has a self-intersection, so the Euler characteristic drops by 1. If the curvature at the singular point is positive, then the section has an isolated point, and Euler characteristic increases by 1.

Since the Euler characteristic of \(\mathbb{R}P^1\) is equal to zero, the integration of \(-\#\{M \cap \ell\}\) over \(\mathbb{R}P^1\) gives zero. So the Euler characteristic of \(M \setminus \ell\), being equal to the integral of the Euler characteristic of \(M_t\) over \(\mathbb{R}P^1\), is equal to \(\text{sign}(M, \ell)\), and the result follows.

4.2. Compactification of \(M\) and end of the proof of the Theorem 2. We apply Lemma 7 to the closure of \(M\) in \(\mathbb{R}P^3\). First, we have to show that the closure of \(M\) in \(\mathbb{R}P^3\) is a smooth surface.

Lemma 8. The closure \(\bar{M}\) of \(M\) in \(\mathbb{R}P^3\) is smooth.

Proof. Take affine coordinates \(\bar{x} = \frac{x}{z}, \bar{y} = \frac{y}{z}, \bar{w} = \frac{1}{z}\). We are interested in the points of \(\bar{M} \cap \{\bar{w} = 0\}\). The first part of the condition “\(M\) approaches \(K\) at
infinity” implies that \( \tilde{M} \) approaches \( \{ \tilde{x}^2 + \tilde{y}^2 = 1 \} \) faster than \( |\tilde{w}| \), so \( \tilde{M} \) is smooth at these points. The second part means that as \( x \in M \) tends to \( x_0 \in \tilde{M} \cap \{ \tilde{w} = 0 \} \) the limit of tangent planes \( T_x M \) exists and is equal to the tangent plane at \( x_0 \). This means \( C^1 \)-smoothness of \( \tilde{M} \).

For the hyperbolic surface \( M \) the curvature is always negative. Therefore the sign in Lemma 7 is always “-”. The Euler characteristic of \( \tilde{M} \) is equal to the Euler characteristic of \( \{ x^2 + y^2 = z^2 + w^2 \} \) (essentially Lemma 3), i.e. is equal to zero. So the Lemma 7 claims in this case that for generic \( \ell \)
\[
\# \{ M \cap \ell \} = \# \{ x \in M \mid \ell \subset T_x M \}.
\] (4.1)

We want to prove that projection of \( M \) to \( \mathbb{S}^2 \) has no folds, i.e. is a local diffeomorphism. In other words, we have to show that no tangent plane to \( M \) passes through the vertex \( O \) of the cone. Suppose otherwise and take a plane tangent to \( M \) and passing through \( O \). The normal to this plane lies in the image of \( M \) under the Gauss map, i.e. in \( \mathbb{S}^2 \cap \{ |z| < \frac{1}{\sqrt{2}} \} \). Equivalently, this plane should intersect the domain \( \{ x^2 + y^2 < z^2 \} \). Since this plane contains also the vertex of the cone, so contains a line \( \ell \subset \{ x^2 + y^2 < z^2 \} \cup \{0\} \). By Lemma 5 \( \ell \cap M = \emptyset \). Therefore, by compactness of \( \tilde{M} \), it is true for all lines close enough to \( \ell \). Moreover, if \( \ell \) is contained in a plane \( T_x M \), then, due to the nonzero curvature of \( M \) at \( x \), any line close enough to \( \ell \) is also contained in some tangent plane to \( M \).

So in (4.1) applied to a small perturbation of \( \ell \) the left side is equal to zero, and the right side is at least one, which is impossible.

The Theorem 6 now follows from Lemma 3 applied to the restriction of the projection to \( M_R = M \cap B_R \) – intersection of \( M \) with a big enough ball \( B_R \). Indeed, we just proved that the projection is a local diffeomorphism. Also, the restriction of the projection to the boundary of \( M_R \) is diffeomorphism since the boundary of \( M_R \) is \( C^1 \)-close to \( K \cap \{ x^2 + y^2 + z^2 = R^2 \} \), so is embedded by projection.

5. Example

In this section we provide an example of a hyperbolic closed connected surface without boundary in \( \mathbb{R}^3 \) bounding a domain without lines inside. This surface has an asymptotic behavior similar to those considered above (namely it approaches the pseudo-cone \( K' \) at infinity), but unlike the surfaces before, its closure in \( \mathbb{R}P^3 \) is not smooth and is not hyperbolic after smoothening.

Construction starts by definition of an affine convex-concave sets. Consider a hyperbolic surface bounding some domain in \( \mathbb{R}^3 \). At each point it has a direction of positive sectional curvature and an orthogonal direction of negative sectional curvature. The affine convex-concave sets come from a requirement that these directions should not be far from a vertical (=parallel to \( z \)-axis) and horizontal (=perpendicular to \( z \)-axis) respectively. In other words, we want the horizontal sectional curvature to be always negative and the vertical sectional curvature to be always positive. The first requirements implies that the horizontal sections of a domain bounded by the hyperbolic surface are convex, and then the second one gives a concavity-type condition on these sections. We introduce the affine convex-concave sets as sets satisfying these two properties, i.e. using only the notion of convexity. This class is an affine relative of the class of \( L \)-convex-concave subsets of \( \mathbb{R}P^n \) defined in [3], and is similarly closed under surgeries considered there. An
analogue of the first part of the Arnold conjecture can be formulated for convex-concave sets and for L-convex-concave sets (the second part then follows trivially). We prove it to be true for the first nontrivial case of L-convex-concave sets in [2], by a rather complicated considerations.

The first step of our construction in this section is to construct a counterexample to an analogue of Arnold conjecture for convex-concave subsets of $\mathbb{R}^3$. This counterexample is a so-called strip - a piece of a two-dimensional surface which is at the same time a convex-concave set. A strip in no way can be interpreted as a domain bounded by a hyperbolic surface. However, one can think about it as an interior-less limit of certain domains bounded by hyperbolic surfaces. The condition of absence of lines inside a convex-concave body is an open one, so any convex-concave body close enough to the strip also does not contains a line inside.

The second step consists of a small perturbation of the set $E$ - the cone $K'$ with attached strip - in order to get a convex-concave set which is a genuine domain bounded by a hyperbolic surface.

This perturbation is based on the fact that the class of convex-concave sets is closed under taking the fiberwise affine linear combinations (by Minkowski). We replace $E$ by an affine linear combination of $E$ and the result is a genuine domain bounded by a hyperbolic surface (with an additional property that all its sections are convex).

5.1. Affine convex-concave sets. We will call by horizontal everything in $\mathbb{R}^3$ which is parallel to the coordinate $(x,y)$-plane. For example,
1. planes $\{z = c\}$ are called horizontal,
2. directions $(a,b,0)$ are called horizontal.

Definition 4. We say that a set $A \subset \mathbb{R}^3$ is convex-concave if
1. its sections by horizontal planes are nonempty, convex and compact and
2. the sections $S_t = A \cap \{z = t\}$ depends in concave way (in Minkowski sense) on $t$.

The second condition means that for any $t_1 < t_2 < t_3$ the section $S_{t_2}$ is contained inside the linear (in Minkowski sense) combination of $\frac{t_2-t_1}{t_3-t_1}S_{t_1} + \frac{t_3-t_2}{t_3-t_1}S_{t_3}$, i.e. the convex hull of the union $S_{t_1} \cup S_{t_3}$. It can be reformulated in several possible ways.

The first equivalent reformulation is that

(2') for any $t_1 < t_2 < t_3$ any point of the section $S_{t_2}$ lies on a line intersecting both $S_{t_1}$ and $S_{t_3}$.

Form this follows another reformulation of the same condition. Namely, it is the requirement that the complement to the projection of $A$ along any horizontal direction is locally convex. In other words, if we introduce a coordinate $(w,z)$ on the plane of projection, then

(2'') the projection should be given by $\pi(A) = \{-\phi_1(z) \leq w \leq \phi_2(z)\}$ with both $\phi_1(z)$ and $\phi_2(z)$ being convex and $-\phi_1(z) \leq \phi_2(z)$ for all $z$.

Here "$f$ is a convex function" means that $\frac{f(x+a)-f(x)}{a} - \frac{f(x)-f(x+b)}{b} \geq 0$ for all $a, b > 0$. For $C^2$-smooth functions this is equivalent to $f'' \geq 0$, and for continuous functions it can be defined in more distribution-like spirit: $f$ is convex if $\int f g''dz \geq 0$ for any smooth nonnegative function $G$ with compact support or tending fast enough to zero as $|t| \to \infty$. 


5.1.1. **Support function.** Let recall the definition and basic properties of the support functions of a convex set in $\mathbb{R}^n$ (we will need the case of $n = 2$ only). Let $S \subset \mathbb{R}^n$ be a compact convex set. Then one can define a support function $F_S(\ell) = \max_{x \in S} \ell(x)$ on $(\mathbb{R}^n)^*$. This function is clearly $\mathbb{R}_+^*$. Let $F_S(\alpha \ell) = \alpha F_S(\ell)$ for any $\alpha > 0$ and $\ell \in (\mathbb{R}^n)^*$. Since $S \subset \{\ell_1(x) \leq F_S(\ell_1)\} \cap \{\ell_2(x) \leq F_S(\ell_2)\}$ for any $\ell_1, \ell_2 \in (\mathbb{R}^n)^*$, we get that $F_S(\alpha \ell_1 + \beta \ell_2) \leq \alpha F_S(\ell_1) + \beta F_S(\ell_2)$ for any $\alpha, \beta \geq 0$. If $S'$ is another compact convex subset of $\mathbb{R}^n$, then $F_S \neq F_{S'}$. Vice versa, for any $F : (\mathbb{R}^2)^* \to \mathbb{R}$ satisfying the previous conditions one can construct a convex compact figure $S = \cap_{\ell \in (\mathbb{R}^2)}, \{\ell(x) \leq F(\ell)\}$. One can check that $F_S = F$.

If we define the Minkowski sum of two sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ as $A + B = \{a + b | a \in A, b \in B\}$, then the support function of $A + B$ is equal to the sum of support functions of $A$ and $B$.

The boundaries of projections in the Definition 5.1 are exactly the values of the support function of $S_1$ on this direction. Namely, if the projection is defined by $\pi : (x, y, z) \to (ax + by, z)$, then $\phi_1(z) = \min_{(x, y, z) \in S_1} (-a)x + (-b)y$ and $\phi_2(z) = \max_{(x, y, z) \in S_1} ax + by$. So the second condition of the Definition 5.1 means that the support functions of $S_1$ depend concavely on $t$.

We will further need the case when the boundary of projection (=support function) is piecewise.

5.2. **Strips.** First we construct an unexpected object: a convex-concave set all horizontal section of which are segments.

**Definition 5.** A surface with boundary $\{(x, y, z) \in \mathbb{R}^3 | x = u_1(z) + tf_1(z), y = u_2(z) + tf_2(z), \|t\| \leq 1\}$ which is also a convex-concave set will be called a strip.

In other words, we parameterize the strip by two curves: one formed by the middle-points of the segments $M = (u_1(z), u_2(z), z)$ and another formed by the ends of the segments, $M_1 = (u_1(z) + f_1(z), u_2(z) + f_2(z), z)$.

An evident example of a strip corresponds to linear $f_1$ and $f_2$ and $u_i \equiv 0$ (so-called **degenerate** strip). In this case the strip is simply a piece of a quadric and contains two one-parametric families of lines (one of them consists of horizontal lines). So any point of a degenerate strip lies on a line intersecting all sections and convex-concavity follows by Definition 5.1.

It turns out that there exist non-degenerate strips, and they survive some small perturbations, whereas the property to contain a line doesn’t. So we construct a strip not containing a line inside.

5.2.1. **An unperturbed strip with exactly one line inside.** Consider first unperturbed strips, i.e. strips with $u_i(z) \equiv 0$.

We suppose further that $f_j(z)$ are two linearly independent solutions of a second order linear differential equation of the type $y'' = g(z)y$. This is not very restrictive. Indeed, any two functions are solutions of a differential equation of second order as soon as their Wronskian is nonzero. But if their Wronskian is zero at $z = t$, then, assuming some genericity, its sign will change at this point. It follows (after some computations) that the projection along direction $(f_1(t), f_2(t), 0)$ doesn’t satisfy condition 5.1.

Our first step is to build a strip containing exactly one line. More exact, we will prove that almost all unperturbed strips are like this.
Lemma 9. Suppose that $f_1, f_2$ are linearly independent solutions of a second order linear differential equation of the type $y'' = g(z)y$. Suppose that the set $\{x = tf_1(z), y = tf_2(z), |t| \leq 1\}$ contains another line $\ell$ (apart from the $z$-axis). Then $g(z) \equiv 0$.

Proof. First, $\ell$ cannot be parallel or intersect the $z$-axis. Indeed, in this case our solutions are linearly dependent (since the equation of a vertical plane is $y = kx$).

So these two lines ($\ell$ and the $z$-axis) are not in the same plane. After a rotation in $(x, y)$ plane we can assume that $\ell \in \{x = A\}$, so the line $\ell$ is defined by equations $x = A, y = az + b$. Consider the quotient $k(z) = \frac{f_2(z)}{f_1(z)}$. Then $k'(z) = \frac{W(f_1, f_2)}{f_1^2}$ since $W(f_1, f_2) \equiv \text{const}$ (the equation $y'' = g(z)y$ has no term with $y'$). From the other side, $k(z) = \frac{2az+b}{A}$ is a linear function, so its derivative is a constant. So $f_1(z) \equiv \text{const}$ and therefore $g(z) \equiv 0$. □

Lemma 10. Let $f_i$ be two linearly independent solutions of $y'' = g(z)y$. If, $g(z) \geq 0$ for all $z \in \mathbb{R}$ then the set $S = \{x = tf_1(z), y = tf_2(z), |t| \leq 1\}$ is a strip (i.e. is convex-concave).

Proof. Projection of the strip in the direction $(-b, a, 0)$ is given by $\pi(A) = \{-|\phi(z)| \leq w \leq |\phi(z)|\},$ where $\phi(z) = af_1(z) + bf_2(z)$ is again a solution of the same equation $y'' = g(z)y$. We have to prove that $|\phi(z)|$ is convex, or, equivalently, that $\phi(z)'' \geq 0$ when $\phi(z) > 0$ and that $\phi(z)'' \leq 0$ when $\phi(z) < 0$. But this follows immediately from the equation and positivity of $g(z)$. □

5.2.2. Perturbation of a strip. Take any strip from the Lemma 10. We want to perturb our strip between two levels in such a way that the perturbed set will be still a strip but will not contain any lines. There is only one line passing through the unperturbed part and our goal is to make sure that

1. the perturbed part will not contain this line and
2. that the perturbed strip will remain convex-concave.

Here is the construction. Take any strip $S$ described in the Lemma 10. Take any $\rho(t)$ such that, first, $|\rho(z)| \leq g(z)$ and, second, $\rho(z) \not\equiv \text{const} \cdot g(z)$. Take $u_i(z)$ such that $u_i''(z) = \rho(z)f_i(z)$ for $i = 1, 2$. Consider the strip $\tilde{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x = u_1(z) + tf_1(z), y = u_2(z) + tf_2(z), |t| \leq 1\}$. In other words, we shift the segment $S_t$ – the horizontal section of $S$ – by vector $(u_1(t), u_2(t), 0)$. We prove that the first condition on $\rho(z)$ implies concave-convexity of $\tilde{S}$ and the second implies that $z$-axis $\not\subset S$.

Lemma 11. $\tilde{S}$ is convex-concave.

Proof. The horizontal sections of $S$ are segments, so the first condition of the Definition 4 is satisfied. We check the second condition in the form of the Definition 5.2. A horizontal projection of $S$ along the direction $(-b, a, 0)$ is given by $\pi(A) = \{(\psi(z) - |\phi(z)| \leq w \leq \psi(z) + |\phi(z)|), \text{where } \psi(z) = au_1(z) + bu_2(z) \text{ and } \phi(z) = af_1(z) + bf_2(z)$.

We have to check that the boundary of any horizontal projection is given by convex functions, i.e. that

- $(\psi(z) + \phi(z))'' \geq 0$ and $(\psi(z) - \phi(z))'' \leq 0$ when $\phi(z) \geq 0$ and that
- $(\psi(z) + \phi(z))'' \leq 0$ and $(\psi(z) - \phi(z))'' \geq 0$ when $\phi(z) \leq 0$
In other words, we have to prove that \((\phi(z) \pm \psi(z))''\) has the same sign as \(\phi(z)\). This is evident since their ratio is equal to \(g \pm \rho\) which is always nonnegative.

We have to check is that the perturbation is not directed along the segments, i.e. that \(z\)-axis does not lie in \(\tilde{S}\).

**Lemma 12.** If \(\rho \neq cg\) for a \(|c| < 1\) then for some \(t \in \mathbb{R}\) the section \(\tilde{S} \cap \{z = t\}\) of the perturbed strip doesn’t intersect the section \(S \cap \{z = t\}\) of the unperturbed one.

**Proof.** Suppose opposite, i.e. that \(u_i = \lambda(z)f_i(z)\). Then \(\rho(z)f_i(z) = u_i'' = (\lambda'' + \lambda g)f_i + 2\lambda'f_i'\). In other words, the vector \(2\lambda'(f_1', f_2')\) should be proportional to the vector \((f_1, f_2)\). This is possible if and only if \(\lambda' \equiv 0\) or the vector \((f_1', f_2')\) is proportional to the vector \((f_1, f_2)\). The second possibility contradicts to the linear independence of \(f_1\) and \(f_2\). The first one means that \(\lambda \equiv \text{const}\), so \(\rho\) and \(g\) are proportional.

The perturbation can be made local, i.e. between two levels.

**Lemma 13.** We can find an even \(\rho(z)\) satisfying all previous conditions and such that \(u_i(z) \equiv 0\) for \(|z| \geq 1/2\).

**Proof.** Indeed, consider the space \(L\) of even \(C^2\)-smooth functions \(\rho(z)\) vanishing identically for \(|z| \geq 1/2\). The functions \(u_i\) solving \(u_i'' = \rho(z)f_i\) with \(\rho \in L\) and initial conditions \(u_i(-1) = u_i'(-1) = 0\) are identical zero on \(z \leq -1/2\) and are linear on \(z \geq 1/2\), i.e. \(u_i(z) = a_{i1}z + a_{i0}\) for \(z > 1/2\). Evidently, \(a_{ij}\) depend linearly on \(\rho\), so in the infinite-dimensional space \(L\) there is a subspace \(L', \text{codim } L' \leq 4\), of functions corresponding to \(a_{ij} = 0\).

**5.2.3. Specification of the strip.** The examples of strips constructed above depend essentially on \(g(z)\) and \(\rho(z)\) only (different choices of \(f_i\) and \(u_i\) result in strips differing by a linear transformation of \(\mathbb{R}^3\)). There is a big degree of freedom in their choice. Here we impose some additional restrictions on these two functions and on the choice of \(f_i\) and \(u_i\), in order to facilitate the following constructions – transforming of the strip to a domain bounded by a hyperbolic surface not containing a line.

**Corollary 4** (of the constructions above). Take \(g(z)\) be an even smooth function identically equal to 0 for \(|z| \geq 1\) and strictly positive otherwise. There is an even nonzero function \(\rho(z)\) vanishing identically for \(|z| \geq 1/2\) and functions \(u_1(z), u_2(z), f_1(z), f_2(z)\) such that

1. \(f_i(z)\) are two linearly independent solutions of \(f''(z) = g(z)f(z)\) and \(u_i''(z) = \rho(z)f_i(z)\);
2. the perturbed strip

\[S = \{(x, y, z) \in \mathbb{R}^3 \mid x = u_1(z) + tf_1(z), \quad y = u_2(z) + tf_2(z), \quad \|t\| \leq 1\} \]

does not contain lines and is symmetric with respect to the rotation \((x, y, z) \rightarrow (x, -y, -z)\) of \(\mathbb{R}^3\).
3. the part of the strip \(S\) lying in \(|z| \geq 1\) is bounded by four rays, and directions of these rays lie inside the cone \(z^2 > x^2 + y^2\).
4. \(f_1^2(\pm 2) + f_2^2(\pm 2) < 1\)

**Proof.** Take a \(\rho(z)\) as in the Lemma 13. Take \(f_1(z)\) and \(f_2(z)\) to be any even and odd correspondingly solutions of \(f''(z) = g(z)f(z)\). Then one can take an even
$u_1(z)$ and an odd $u_2(z)$ solutions of equations $u_1'' = \rho(z) f_i(z)$. Since $\rho(z)$ and $g(z)$ are not proportional, the strip $S$ does not contain lines. Together this means that the strip $S$ is symmetric with respect to the rotation $(x, y, z) \to (x, -y, -z)$ of $\mathbb{R}^3$.

Since $f_1''(z) = u_1(z) = 0$ for $|z| > 1$, the boundary of the part of $S$ lying in $\{|z| > 2\}$ is just four rays $\{(x, y, z)|x = \pm(a_0 + a_1|z|), y = \pm b_1|z|, |z| > 1\}$. Multiplying $f_i(z)$ and $u_i(z)$ by a small number (i.e. after a dilatation of $(x, y)$-plane), we can assume that $a_i^2 + b_i^2 < 1$ and $f_i^2(\pm 2) + f_i^2(\pm 2) < 1$, as required. \hfill \Box

5.3. Gluing to the quasi-cone and smoothening. Here we glue the strip $S$ of the Corollary \ref{cor1.2} to the quasi-cone $K' = \{(x, y, z) \mid x^2 + y^2 = (|z| - 1)^2, |z| \geq 1\}$. In other words, we construct a convex-concave set $E$ with horizontal sections coinciding with sections of $K'$ for $|z| \geq 2$ and with sections of $S$ for $|z| \leq 1$.

Here is how $E$ is constructed. Take the union $E_1$ of $S$ and $K'$. Horizontal sections of $E_1$ are sometimes segments, sometimes closed discs and sometimes their unions. Denote by $E$ a set whose horizontal sections are the convex hulls of the corresponding horizontal sections of $E_1$. $E$ coincides with $S$ for $|z| \leq 1$, so in particular doesn’t contain a line. Also, the last two conditions of the Corollary \ref{cor1.2} together guarantee that the part of $S$ lying in $\{|z| \geq 2\}$ lies inside $K'$, i.e. $E$ coincide with $K'$ outside $\{|z| \leq 2\}$.

$E$ turns out to be a convex-concave set since its support function is a maximum of support function of $S$ and a linear function – a support function of $K'$ – overtaking it as $z \to \infty$, thus convex. Here are the details.

Lemma 14. $E$ is a convex-concave set.

Proof. All horizontal sections of $E$ are nonempty and convex by definition. So we have to check that all projections of $E$ are bounded by graphs of a convex functions, as in Definition \ref{def1.1}. Let a projection of $E$ be given by $\pi(E) = \{-\phi_2(z) \leq w \leq \phi_1(z)\}$. We have to prove that both $\phi_1(z)$ and $\phi_2(z)$ are convex.

The proof is the same for both $\phi_1(z)$ and $\phi_2(z)$, so we consider only $\phi_1(z)$. Taking convex hull of sections doesn’t change projection, so $\pi(E) = \pi(E_1)$. Let projections of $S$ and $K'$ be defined by $\pi(S) = \{-\phi_2^S(z) \leq w \leq \phi_1^S(z)\}$ and $\pi(K') = \{-\phi_2^{K'}(z) \leq w \leq \phi_1^{K'}(z), |z| \geq 1\}$. Then $\phi_1(z) = \max(\phi_1^S(z), \phi_1^{K'}(z))$ for $|z| \geq 1$ and $\phi_1(z) = \phi_1^S(z)$ for $|z| < 1$.

Let $\phi_1^{K'}(z)$ be a piece-wise linear function equal to $\phi_1^{K'}(z)$ for $|z| \geq 1$ and equal to 0 for $|z| \leq 1$. Trivially $\phi_1^{K'}(z)$ is a convex function.

Note that by choice of $\rho(z)$ in the Corollary \ref{cor1.2} the middle point of the intervals $S \cap \{z = t\}$ lie on the $z$-axis for $|t| \geq 1/2$, so $\phi_1^S(z) = \phi_1^{K'}(z) \geq 0$ for $|z| \geq 1/2$. Therefore $\phi_1(z) = \max(\phi_1^S(z), \phi_1^{K'}(z))$ for $z \in [1/2, \infty)$, so is convex on this interval as a maximum of two convex functions. Similarly $\phi_1(z)$ is convex on $(-\infty, -1/2]$. By definition $\phi_1(z)$ is a convex function on $[-1, 1]$. Therefore $\phi_1(z)$ is convex on the whole real line. \hfill \Box

5.4. Smoothening. The convex-concave body $E$ built in the previous section doesn’t contain a line but still is not a domain bounded by a hyperbolic surface. To finish the construction of an example we will smoothen $E$ and will get a convex-concave domain $D$ bounded by a smooth hyperbolic surface.

The further constructions are based on the fact that even after small enough deformations $E$ still doesn’t contain a line. In fact, the deformation should be small enough only near the piece of the strip $S$. 


Lemma 15. Let \( E \subset \mathbb{R}^3 \) be some set. Suppose that \( E' \cap \{ z = t \} \) is compact for all \( t \) and is in an \( \varepsilon \)-neighborhood of \( E \cap \{ z = t \} \) for all \(-10 \leq t \leq 10\). If \( \varepsilon \) is less than some number \( c \) depending on \( E \), then the set \( E' \) doesn’t contain lines.

Proof. It is enough to check only non-horizontal lines, i.e. the lines given by \( \ell = \{(a + bz, c + dz, z) | z \in \mathbb{R} \} \). It is easy to see that the function \( \text{maxdist}(\ell, E) = \max_{x \in \ell \cap \{|z| \leq 10\}} \text{dist}(x, E) \) achieves its nonzero minimum \( c \) on the set of all non-horizontal lines. Indeed, \( \text{maxdist}(\ell, E) \to \infty \) as \((a, b, c, d) \to \infty\), so there is a global minimum of \( \text{maxdist}(\ell, E) \). Moreover, this minimum is non-zero since \( \text{maxdist}(\ell, E) = 0 \) would imply that \( \ell \cap \{|z| \leq 1\} \subset E \cap \{|z| \leq 1\} = S \cap \{|z| \leq 1\}, \) which is impossible.

If \( \varepsilon < c \), then \( \text{maxdist}(\ell, E') > \text{maxdist}(\ell, E) - \varepsilon \geq 0 \) for any line \( \ell \). This means that \( \ell \not\in E' \cap \{|z| \leq 1\}, \) so \( \ell \not\in E' \).

5.4.1. Convolution. The procedure described below are in fact a particular case of a general method of smoothing of convex-concave sets: though all considerations are done for the set \( E \) constructed in the previous section, the constructions can be easily generalized for any convex-concave set with moderate growth of support function as \( |z| \to \infty \). The procedure is a generalization of the well-known fact that convolution of an integrable function with a \( C^\infty \)-smooth function is a \( C^\infty \)-smooth function.

There is a well-known operation of taking an affine combination by Minkowski of two convex sets (by affine combination we mean a linear combination with positive coefficients sum of which is equal to 1). The result of this operation can be described in two ways:

1. it is a set of all points which are a convex combination with the same coefficients of a point in the first set and a point in the second set.
2. this is a convex set with the support function equal to the linear combination with the same coefficients of the support functions of the first and the second sets.

We can apply this operation fiberwise to convex-concave sets.

Lemma 16. Let \( A \) and \( B \) be two convex-concave sets, and let \( \lambda_1 \) and \( \lambda_2 \) be two positive numbers, \( \lambda_1 + \lambda_2 = 1 \). Then the set \( C \) whose horizontal sections \( C_z \) are equal to the \( \lambda_1 A_z + \lambda_2 B_z \) is convex-concave.

Indeed, the sections are convex by definition and support function of \( C_z \), which define boundaries of projections of \( C \), are convex in \( z \) as a sum of two convex in \( z \) functions – the support functions of \( A_z \) and \( B_z \).

We will apply a generalization of this operation to the set \( E \) of the previous section. Let take any line \( \ell \) perpendicular to the horizontal planes. Consider a group of affine transformations of \( \mathbb{R}^3 \) generated by translations in vertical directions and rotations with \( \ell \) as an axis. This group \( \Gamma \) is just a cylinder \( \Gamma \cong \mathbb{R} \times S^1 \): to \((z, \phi)\) corresponds a composition \( g_{z, \phi} \) of a shift by \( z \) along \( \ell \) and rotation by angle \( \phi \) around \( \ell \). Fix a standard Lebesgue measure \( \mu = dz d\phi \) on \( \Gamma \).

This group acts on the class of convex-concave sets: evidently, the shifts and rotations around vertical axis of a convex-concave set gives again a convex-concave set.

As soon as we chose \( \ell \), arises a one-to-one correspondence between convex-concave sets and their support functions \( F_A(z, \ell) : \mathbb{R} \times \mathbb{R}^2 \): for each fixed \( z \) \( F_A(z, \ell) \) is just a support function of the section \( A_z \), i.e. \( F_A(z, \ell) = \max_{z \in A_z} \ell(x) \). This
function is convex in $z$, is positively homogeneous and is also convex for any fixed $z$. Vice versa, any such function defines a convex-concave set (just reconstruct each section separately by its support function).

Take a $\delta$-like function on $\Gamma$ concentrated near the identity element of $\Gamma$. More exact, take a function $K_\varepsilon(z, \phi) : \Gamma \to \mathbb{R}$ with a following properties:

1. $\int_\Gamma K_\varepsilon(z, \phi) d\mu = 1$, and $\int_{|z|,|\phi| \leq \varepsilon} K_\varepsilon(x, \phi) d\mu \geq 1 - \varepsilon$,
2. $K_\varepsilon(z, \phi)$ is $C^\infty$-smooth and strictly positive,
3. $K_\varepsilon(z, \phi)$ and all its partial derivatives in $z$ decrease exponentially as $|z| \to \infty$
4. $K_\varepsilon(z, \phi)$ is even function of $z$.

For example, one can take the $K_\varepsilon(z, \phi) = C(\varepsilon) \exp(\cos \phi - z^2)/\varepsilon$ with a suitable choice of the constant $C(\varepsilon)$.

We take shifts and rotations of $E$ and define $D_\varepsilon$ as an affine combination of the results with weight $K_\varepsilon$. In other words, we define the convex-concave set $D_\varepsilon$ as a convolution of $E$ with $K_\varepsilon(x, \phi)$: $D = \int_\Gamma K_\varepsilon(z, \phi) g_{z, \phi}(E) d\mu$. Alternatively, the support function of the set $D$ is defined by the convolution $F_D(z, \phi) = \int_\Gamma K_\varepsilon(t, \psi) F_E(z - t, \phi - \psi) d\mu d\psi$.

These integrals converge since the support function of $E$ grows as a linear function of $|z|$: as soon as $|z| \geq 2$, the sections of $E$ are just circles of radius $|z| - 1$.

We claim that

**Theorem 7.**

1. Sections of $D_\varepsilon$ are strictly convex with nonempty interior. Moreover, boundaries of sections of $D_\varepsilon$ are smooth and have everywhere non-vanishing curvature.
2. $D_\varepsilon$ is convex-concave domain. Moreover, boundaries of projections of $D_\varepsilon$ are smooth and have everywhere non-vanishing curvature.
3. $D_\varepsilon$ is close to $E$ in the sense of the Lemma [17]. It means that for any $\delta > 0$ we can find an $\epsilon > 0$ such that the sections of domain $D_\varepsilon$ are in the $\delta$-neighborhoods of the corresponding sections of $E$. Therefore $D_\varepsilon$ does not contain lines for a sufficiently small $\epsilon$.
4. Boundary of $D_\varepsilon$ is a smooth hyperbolic surface. Moreover, $D_\varepsilon$ approaches $K'$ at infinity.

In other words, the Theorem says that the $D_\varepsilon$ is an example we are looking for.

**Proof.** Evidently, the support function of $D_\varepsilon$ is infinitely smooth as a result of convolution with an infinitely smooth function. Moreover, since the support function of $E$ convexly and non-linearly depends on $z$, the support function of $D_\varepsilon$ will be strictly convex in $z$ and will have everywhere positive second derivative on $z$.

To prove non-degeneracy of the curvature of the sections of $D_\varepsilon$, one should look on the way to restore a (planar) convex figure from its support function. Let $F(\ell)$ be a support function of a planar convex figure $B$, and suppose that $F(\ell)$ is smooth. Then the gradient $\nabla F(\ell)$ is constant on rays beginning at the origin (i.e. is in fact a mapping $\nabla F : S^1 \to \mathbb{R}^2$), and its image is exactly the boundary of $B$. One can easily check that the boundary of $B$ is smooth and have a nonzero curvature at $\nabla F(\ell_0)$ if the kernel of the Hessian $H(F)(\ell_0) = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{pmatrix}(\ell_0)$ is one-dimensional, i.e. coincides with the line joining the origin and the point.

We apply this construction to the support functions of sections of $D_\varepsilon$. The Hessian of the support function of a section of $D_\varepsilon$ is a convolution of $K_\varepsilon$ with the Hessian in coordinates $(x, y)$ of the support function of $E$. Since the latter is
somewhere nonzero and everywhere positively semi-definite as a quadratic form, the convolution will be everywhere nonzero and positive semi-definite. Thus the boundary of sections of $D_\varepsilon$ are smooth with nonzero curvature.

Moreover, the gradient mapping $\nabla_{x,y} F_{D_\varepsilon}(z,\ell) : \mathbb{R} \times S^1 \to \partial D_\varepsilon$ is a smooth parameterization of the boundary of $D_\varepsilon$. Taken together, this means that $D_\varepsilon$ is convex-concave and its boundary is smooth with everywhere nonzero curvature.

By standard arguments one can prove that as $\varepsilon \to 0$, the result of a convolution of a function $F_E$ with $K_\varepsilon$ converges to $F_E$ itself. This implies that the set $D_\varepsilon$ lies in a $\delta$-neighborhood of $E$, as required.

The last claim is that $D_\varepsilon$ approaches $E$ at infinity. First, the sections $E_z$ of $E$ are circles of radius $|z| - 1$ for $|z| \geq 2$, and therefore $F_E(z,\ell) = (|z| - 1)\|\ell\|$ for $|Z| \geq 2$. Since $K_\varepsilon$ is even as a function of $z$, the convolution with $K_\varepsilon$ do not changes linear functions of $z$. Therefore the difference $|F_{D_\varepsilon}(z,\ell) - F_E(z,\ell)|$ decreases exponentially together with all its derivatives as $|z| \to \infty$. Therefore the parameterizations of boundaries of $E$ and $D_\varepsilon$ by the gradient of their support functions as before are exponentially close, which proves the claim.

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