On quantum determinants in integrable quantum gravity

B Runov

Centre de recherches mathématiques, Université de Montréal, C. P. 6128, succursale Centre ville, Montréal, QC H3C 3J7, Canada
Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. W. Montreal, QC H3G 1M8, Canada

E-mail: b.runov@spbu.ru

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Abstract
Einstein–Rosen waves with two polarizations are cylindrically symmetric solutions to vacuum Einstein equations. Einstein equations in this case reduce to an integrable system. In 1971, Geroch has shown that this system admits an infinite-dimensional group of symmetry transformations known as the Geroch group. The phase space of this system can be parametrized by a matrix-valued function of spectral parameter, called monodromy matrix. The latter admits the Riemann–Hilbert factorization into a pair of transition matrices, i.e. matrix-valued functions of spectral parameter such that one of them is holomorphic in the upper half-plane, and the other is holomorphic in the lower half-plane. The classical Geroch group preserves the determinants of transition and monodromy matrices by construction. The algebraic quantization of the quadratic Poisson algebra generated by transition matrices of Einstein–Rosen system was proposed by Korotkin and Samtleben in 1997. Paternain, Peraza and Reisenberger have recently suggested a quantization of the Geroch group, which can be considered as a symmetry of the quantum algebra of observables. They have shown that commutation relations involving quantum monodromy matrices are preserved by the action of the quantum Geroch group. In present paper we introduce the notion of the determinant of the quantum monodromy matrix. We derive a factorization formula expressing the quantum determinant of the monodromy matrix as a product of the quantum determinants of the transition matrices. The action of the quantum Geroch group is extended from the subalgebra generated by the monodromy matrix onto the full algebra of observables.

* Author to whom any correspondence should be addressed.

Present address: Euler International Mathematical Institute, St. Petersburg State University, Universitetskaya nab. 7–9, 199034 St. Petersburg, Russia
This extension is used to prove that the quantum determinant of the quantum monodromy matrix is invariant under the action of quantum Geroch group.

Keywords: integrable quantum gravity, integrable systems, Einstein–Rosen waves, midi-superspace models, quantum determinants, Yangians, quantum double

1. Introduction

The Einstein–Rosen model with two polarizations describes cylindrically symmetric vacuum solutions in general relativity. We shall refer to this model simply as Einstein–Rosen model throughout this paper, using the term reduced Einstein–Rosen model for the system with one polarization originally discussed by Einstein and Rosen. The reduced Einstein–Rosen model is an exactly solvable midi-superspace model of quantum gravity [1–3]. Both reduced and full Einstein–Rosen model are useful toy models to study the effects of quantum gravity in two dimensions [4–15]. The full Einstein–Rosen model has recently attracted renewed interest [16–18] as it is Poisson structure was demonstrated to be closely related to Poisson brackets of null initial data for the full vacuum general relativity in four dimensions. Therefore the quantization of the Einstein–Rosen model might lead to insights into more general quantum general relativity.

On the classical level, integrability of the Einstein–Rosen model was established by Belinsky and Zakharov in [19], and independently by Maisón [20]. The group of non-commutative symmetries of the model was discovered by Geroch [21,22] and further explored Breitenlohner and Maisón [23] and other authors. The Poisson structure of Einstein–Rosen model was explored in a series of works [10,24–29]. The phase space of the system can be parametrized by so-called transition matrices, i.e. two one-parametric families of $SL(2, \mathbb{C})$-valued constants of motion $T_{\pm}(u)$, where complex parameter $u$ is called the constant spectral parameter. The physical content of the theory is encoded into the monodromy matrix originally introduced in [30] (referred to as ‘deformed metric’ in [18]), which is the product of transition matrices computed at real values of spectral parameter. It can be shown that the monodromy matrix can be identified with a submatrix of the metric of the four-dimensional spacetime at the symmetry axis [30].

In [24,25] the authors suggested a quantum deformation of the Poisson structure by proposing a set of exchange relations defining the quantum algebra of observables which quantize the classical Poisson algebra generated by transition matrices.

To complete the quantization program one has to specify a 'physically acceptable' (in the main body of the paper we discuss the meaning of the term) representation of the algebra. So far such a representation is unknown. The representation originally proposed by Korotkin and Samtleben [25] does not possess a positive-definite scalar product. On the other hand, in the alternative representation suggested in the subsequent work by Niedermaier and Samtleben [15] the quantum metric becomes non-symmetric, while the problem of positive-definitness remains unsolved.

Another approach to the problem of constructing the physically meaningful representation was suggested by Paternain, Peraza and Reisenberger [18]. Their approach is based on the fact that the Geroch group, which is a group of dressing transformations of the classical model, acts transitively on the space of solutions. The transformations from the Geroch group are not canonical. Instead, the Geroch group is a Lie–Poisson group. Therefore, the elements of the Geroch group belong to an auxiliary phase space equipped with its own non-trivial Poisson bracket. In accordance with the general theory of the quantum groups [31–33], the quantization of the Geroch group should be a quasitriangular Hopf algebra. In the approach proposed in [18] one defines the representation of the original algebra using the orbits of quantum Geroch
group. The classical Geroch group is known to act transitively on the space of solutions with appropriate asymptotic behaviour \cite{23}. Therefore it is natural to assume that the quantum analogue of the Geroch group would have the entire hypothetical Hilbert space as its orbit. The quantum analogue of the Lie–Poisson action constructed in \cite{18} is an isomorphism of the algebra of observables. However, to be a genuine symmetry the quantum Geroch group must also preserve the essential properties of the representation which arise from the classical limit. In particular, on the real line the quantum monodromy matrix should be symmetric and positive-definite and its components must be Hermitian. Furthermore, the quantum determinants of the quantum transition matrices $T_{\pm}(u)$ must coincide with the identity operators for all $u$ in $\mathbb{C}$. Finally, the transition matrix $T_{\pm}(u)$ must me holomorphic in upper (resp. lower) half-plane.

The authors of \cite{18} have proven that their candidate for the quantum Geroch transformation respects most of the key properties of the quantum monodromy matrix, such as its exchange relation with itself, positive-semidefiniteness, invariance with respect to conjugation and the vanishing of antisymmetric part. However, the paper \cite{18} does not contain a definition of the action of the quantum Geroch group on individual transition matrices $T_{\pm}$. Therefore, the compatibility of the quantum Geroch group symmetry with the requirement that the quantum determinants of $T_{\pm}$ coincide with the identity operator (which is an important property of the hypothetical physically meaningful representation) remains unchecked. In this paper we address this issue.

We introduce the quantum analogue of determinant of the monodromy matrix as follows.

**Definition 1.1.** The regularized quantum determinant of $M(u)$ is defined as

$$q\text{det} M(u) = \frac{2}{3} \text{Res}_{s=0} \epsilon^{ab} M_{1a}^{1b} (u + i\hbar + is),$$  \hspace{1cm} (1.1)

where $\epsilon^{ab}$ is the completely antisymmetric tensor and the summation over repeated indices is assumed.

The main result of the paper is the following factorization formula for $q\text{det} M(u)$:

$$q\text{det} M(u) = q\text{det} T_{+}(u) q\text{det} T_{-}(u).$$  \hspace{1cm} (1.2)

The quantum determinants of the transition matrices in the right-hand side are central elements of the algebra of observables defined as

$$q\text{det} T_{\pm}(u) = T_{\pm}^{11}(u) T_{\pm}^{22}(u + i\hbar) - T_{\pm}^{12}(u) T_{\pm}^{21}(u + i\hbar).$$  \hspace{1cm} (1.3)

The identity (1.2) implies that the quantum determinant of the monodromy matrix belongs to the center of the algebra of observables. In physically relevant representation one must have

$$q\text{det} M(u) = 1.$$  \hspace{1cm} (1.4)

The relation (1.2) also means that the invariance of the regularized quantum determinant of the monodromy matrix under the action of the quantum Geroch group is a necessary condition of the invariance of the quantum determinants of the transition matrices. The latter statement partially answers the question posed in paper \cite{18}.

We extend the action of the quantum Geroch group of \cite{18} to the full algebra of observables as follows:

$$T_{\pm}(u) \mapsto \tilde{T}_{\pm}(u) := S \left( u + \frac{i}{2} \hbar \right) T_{\pm}(u) \tilde{U} \left( -u + \frac{i}{2} \hbar \right),$$  \hspace{1cm} (1.5)
\[ T_- (u) \mapsto \tilde{T}_- (u) := S \left( u - \frac{i \hbar}{2} \right) T_- (u) U^T \left( -u - \frac{i \hbar}{2} \right). \]  

This transformation quantizes the action of the classical Geroch group, which is given by

\[ T_+ (u) \mapsto \tilde{T}_+ (u) := S(u) T_+ (u) U^{-1} \left( S, T_\pm |u \right), \] (1.7)

\[ T_- (u) \mapsto \tilde{T}_- (u) := S(u) T_- (u) U^T \left( S, T_\pm |u \right). \] (1.8)

In the formulae (1.7) and (1.8) symbol \( S(u) \) denotes an element of the classical Geroch group parametrized by a \( SL(2, \mathbb{R}) \)-valued function of the constant spectral parameter on the real line. The symbol \( U(S, T_\pm |u) \) denotes another \( SL(2, \mathbb{R}) \)-valued function of the spectral parameter, which is also a functional of the transition matrices and the Geroch group element. The superscript \( \mathbb{T} \) denotes the transposition. The matrix-valued function \( U(S, T_\pm |u) \) is fixed by the requirement that the transformed transition matrices \( \tilde{T}_\pm (u) \) are analytical in the same half-planes as the untransformed ones and have the same leading term of the asymptotic expansion at large values of the spectral parameter.

The quantum matrices \( S(u) \) and \( U(u) \) in the formulae (1.5) and (1.6) obey the exchange relations of \( sl(2) \) Yangian:

\[ R(u-v)^{1 \over 2} S(u) S(v) = S(v) S(u) R(u-v), \] (1.9)

\[ R(u-v)^{1 \over 2} U(u) U(v) = U(v) U(u) R(u-v). \] (1.10)

The check denotes the ‘quantum comatrix’, i.e.

\[ \tilde{U}^{ij}(u) = e^{ia \epsilon^{jk} a^{mb}(u)} = q\text{det} U(u)(U^{-1}(u - i\hbar))^{ij}. \] (1.11)

The entries of \( S(u) \) and \( U(u) \) act not on the Hilbert space \( \mathcal{H} \) but on their own respective phase spaces \( \mathcal{G} \) and \( \mathcal{K} \) so that the following commutation relations are respected

\[ [S^{ij}(u), T^{kl}_{\pm}(v)] = 0, \] (1.12)

\[ [S^{ij}(u), U^{kl}(v)] = 0, \] (1.13)

\[ [U^{ij}(u), T^{kl}_{\pm}(v)] = 0. \] (1.14)

The operator \( U(u) \) is determined by the operators \( S(u) \) and \( T(u) \). However, the explicit relation between them is unknown to the author. Unlike the classical matrices \( T_\pm (u) \), the quantum operators \( \tilde{T}_\pm (u) \) cannot be made analytical in their respective half-planes (see proposition 5.5). The best we can hope for is that the representation of the algebra generated by transformed transition matrices on the space \( \mathcal{G} \otimes \mathcal{H} \otimes \mathcal{K} \) is decomposable into a direct sum of several other representations, of which at least one is ‘physically acceptable’. In other words, in some appropriate basis the transformed transition matrices are block-diagonal, and some blocks possess the required analytical properties. We hypothesize that the operator \( U(u) \) can be fixed completely by requiring that such decomposition is possible.

We show that the transformation (1.5) and (1.6) preserves the quantum determinants of the transition matrices. Then the factorization formula (1.2) implies that the quantum Geroch group also preserves the quantum determinant of the monodromy matrix. However, the derivation of the identity (1.2) relies upon the analytical properties of the quantum transition matrices and their products in ‘physically acceptable’ representations. To prove that there exists a representation of the Yangian such that with the operator \( U(u) \) in this representation the transformed...
transition matrices $T_{\pm}(u)$ have regular blocks described above one needs to know more about representations of the algebra of observables $D$ and the Yangian generated by the operator $S(u)$. We hope to address this problem in future works.

The paper is organized as follows. In section 2 we review the integrable structure of the classical Einstein–Rosen model following [18, 24]. In section 3 we describe the quantum algebra of observables and formulate restrictions on its physically relevant representation. In section 4 we introduce the quantum determinant of the monodromy matrix and derive the factorization formula (1.2). In section 5 a generalization of the action of the quantum Geroch group of Paternain, Peraza and Reisenberger [18] to the full algebra of observables is proposed.

2. Integrable structure of the classical Einstein–Rosen model

2.1. Equations of motion and Poisson structure

Let us consider the metric of the form
\[ ds^2 = e^{\Gamma(\rho, \tau)} \left( -d\tau^2 + d\rho^2 \right) + g_{ab}(\rho, \tau) dx^a dx^b, \quad a, b = 1, 2 \]
(2.1)
where the function $\Gamma$ and the $SL(2, \mathbb{R})$-valued symmetric matrix $g$ are independent of $x_1, x_2$. Then the vacuum Einstein equations imply that the matrix $g$ satisfies the following equation
\[ \partial_\rho \left( \rho g^{-1} \partial_\rho g \right) - \partial_\tau \left( \rho g^{-1} \partial_\tau g \right) = 0, \]
(2.2)
and that the conformal factor $\Gamma(\rho, \tau)$ is given by
\[ \Gamma(\rho, \tau) = \frac{1}{2} \int_0^\rho \rho' d\rho' \text{Tr} \left( \left( g^{-1} \partial_\rho g \right)^2 + \left( g^{-1} \partial_\tau g \right)^2 \right). \]
(2.3)

The solutions of the equations (2.2) and (2.3) are known as the Einstein–Rosen waves with two polarizations. The equation (2.2) can be derived from the following action:
\[ A = \frac{1}{2} \int_0^\infty \rho d\rho \text{Tr} \left( \left( g^{-1} \partial_\rho g \right)^2 - \left( g^{-1} \partial_\tau g \right)^2 \right). \]
(2.4)

The action (2.4) differs from the action of the non-linear sigma model with $PSL(2, \mathbb{R})$ symmetry only by the coordinate-dependent factor $\rho$. Therefore one can think of this model as of a non-autonomous generalization of the sigma model. We will refer to this theory as the Einstein–Rosen model. A similar metric with a different signature describes the cosmological Gowdy model. Integrable structure outlined below is also present there with minor modifications [24].

While the equation (2.2) is written in terms of the matrix $g$, which is a submatrix of the metric, it is convenient to rewrite this equation in terms of the zweibein to stress the similarity to coset sigma models. Let us parametrize the fields by an $SL(2, \mathbb{R})$-valued zweibein $V(\rho, \tau)$, which is a submatrix of the vierbein of the original four-dimensional problem (2.1). The components of the metric $g_{ab}$ can be expressed in terms of the zweibein $V$ as
\[ g_{ab} = V_a^i V_b^j \delta_{ij}. \]
(2.5)
The fields $V$ exhibit gauge freedom with respect to the right multiplication by a matrix from the group $SO(2)$. The components of the current $J$
\[ J_\mu = V^{-1} \partial_\mu V. \]
(2.6)
take values in the algebra $g = \mathfrak{sl}_2(\mathbb{R})$. By construction, the current $J$ is a flat connection. The algebra $g$ can be decomposed into a direct sum of the algebra $h = \mathfrak{so}_2(\mathbb{R})$ and the orthogonal (i.e. symmetrized) component $l$

$$g = h \oplus l. \quad (2.7)$$

Then the components of the current $J$ can be decomposed as

$$J_\mu = P_\mu + Q_\mu, \quad P_\mu = \frac{1}{2} (J_\mu + J_\mu^T) \in l, \quad Q_\mu = \frac{1}{2} (J_\mu - J_\mu^T) \in h. \quad (2.8)$$

The equation (2.2) then takes the form

$$D_\mu (\rho P_\mu) = 0, \quad (2.9)$$

where the covariant derivative $D$ is defined as

$$D_\mu X = \partial_\mu X + [Q_\mu, X]. \quad (2.10)$$

The action (2.4) in this parametrization looks as follows:

$$A = \int d\tau \int_0^\infty \rho d\rho \left( P_0^2 - P_1^2 \right). \quad (2.11)$$

Expressing the components of the current $J$ in terms of the zweibein one finds

$$A = \frac{1}{2} \int_0^\infty \rho d\rho \ Tr \left[ (D_\mu V)V^{-1} (D_\nu V)V^{-1} \right]. \quad (2.12)$$

The action written in terms of the zweibein is invariant under local $SO(2)$ transformations. The physical fields therefore take values in the coset space $SL(2, \mathbb{R})/SO(2)$.

As it was discovered in [19, 30] the equation (2.2) is integrable in the sense of theory of solitons (i.e. it possesses a zero curvature representation). Namely, consider the following deformation of the connection $J$ which also turns out to be flat on solutions of equations of motion

$$\hat{J}_\mu = Q_\mu + \frac{1 + \gamma^2}{1 - \gamma^2} P_\mu + \frac{2\gamma}{1 - \gamma^2} \epsilon_{\mu\nu} P_\nu \quad (2.13)$$

provided the parameter $\gamma$, which is called the dynamical spectral parameter, depends on spacetime coordinates and the constant spectral parameter $u$ as follows

$$\gamma(\rho, \tau, u) = \frac{1}{\rho} \left( u + \tau + \sqrt{(u + \tau)^2 - \rho^2} \right). \quad (2.14)$$

It means that the equations of motion now can be written in zero curvature form, i.e. as

$$[\partial_\mu - \hat{J}_\mu, \partial_\nu - \hat{J}_\nu] = 0. \quad (2.15)$$

For a fixed point in the spacetime, the dynamical spectral parameter is in fact a function on a two-sheeted covering of the Riemann sphere. Either choice of the branch of the square root in the definition (2.14) results in a flat connection $\hat{J}$ as long as the choice is consistent, i.e. the dynamical spectral parameter is a smooth function of the spacetime coordinates for $\rho > 0$.

**Remark 2.1.** The equations of motion of $PSL(2, \mathbb{R})$ nonlinear sigma model can also be written in zero curvature form (2.15) with the same flat connection (2.13) provided we treat the spectral parameter $\gamma$ as a constant [34, 35].
The solution $\hat{V}(\rho, \tau, u)$ of the linear problem
\[ \partial_\mu \hat{V} = \hat{V} \hat{J} \] (2.16)
can be considered the ‘deformed zweibein’. We normalize the matrix $\hat{V}$ as follows:
\[ \hat{V}(x, u) = V(0) P \exp \left[ \int_0^x J_\mu(y, u) dy^\mu \right], \] (2.17)
where
\[ x = (\rho, \tau), \quad 0 = (0, 0). \] (2.18)
The matrix $\hat{V}(x, u)$ does not depend on the choice of the contour in the equation (2.17) due to the flatness of the connection $\hat{J}$. Similarly to (2.5) one can construct the ‘deformed metric’ (borrowing the terminology of [18]) from the ‘deformed zweibein’:
\[ M(x, u) = \hat{V}(x, u) \kappa(\hat{V}(x, u)) \] (2.19)
where the involution $\kappa$ interchanges the sheets of the Riemann surface on which the dynamical spectral parameter $\gamma$ is defined, i.e.
\[ \kappa(\gamma(x, u)) = \frac{1}{\rho} \left( u + \tau - \sqrt{(u + \tau)^2 - \rho^2} \right) = \gamma^{-1}(x, u). \] (2.20)
As it is explained in [18], one can prove that $M(x, u)$ does not in fact depend on $x$. So we will call it the monodromy matrix and denote it by $M(u)$.

Korotkin and Samtleben [24, 25] found two families of conserved quantities $T_{\pm}(u, \tau)$
\[ T_{\pm}(u, \tau) = \lim_{\epsilon \to 0} P \exp \left\{ 2 \int_0^\infty d\rho \left( \frac{\gamma_+^{-1} \partial_\rho g}{1 - \gamma_+^{-1}} - \frac{\gamma_-^{-1} \partial_\tau g}{1 - \gamma_-^{-1}} \right) \right\}, \] (2.21)
with
\[ \gamma_{\pm} = \gamma(\rho, \tau, u \pm i\epsilon). \] (2.22)
Since the total derivative of $T_{\pm}(u, \tau)$ with respect to the time $\tau$ vanishes due to the flatness of $\hat{J}$ [24], we shall denote these matrices by $T_{\pm}(u)$. The matrices $T_{\pm}(u)$ solve the Riemann–Hilbert problem for the jump matrix $M(u)$ on the real line [24], i.e. they provide a factorization of the matrix $M(u)$
\[ M(u) = T_+(u)T_-^T(u) \] (2.23)
such that $T_+$ is holomorphic in the upper half-plane, and $T_-$ is holomorphic in the lower half-plane. The set of matrices $T_{\pm}(u)$ completely determines the metric. The equations of motion (2.2) imply that
\[ \frac{d}{d\tau} T_{\pm}(u) = 0. \] (2.24)

The Poisson brackets derived from the action (2.4) ignoring the constraint $\det g = 1$ look like
\[ \{ g_{ab}(\rho), \pi_{cd}(\rho') \} = \delta_{ac} \delta_{bd} (\rho - \rho') \] (2.25)
with canonical momenta given by
\[ \pi_{ab}(\rho) = \rho \left( g^{-1} \partial_\tau g g^{-1} \right)_{ab}. \]  
(2.26)

Taking the constraints into account one can compute the Poisson brackets between the components of the current. The non-vanishing brackets are given by

\[ \{ P_0(\rho), P_1(\rho') \} = \rho^{-1} \left[ \Omega_1, \frac{2}{\rho} \Omega_2 \right] \delta(\rho - \rho') \Omega_1 \]  
(2.27)

\[ \{ Q_0(\rho), Q_1(\rho') \} = \rho^{-1} \left[ \Omega_1, \frac{2}{\rho} \Omega_2 \right] \delta(\rho - \rho'), \]  
(2.28)

where the following notation was adopted for the Poisson brackets between matrices

\[ \{ A_{ab}, B_{cd} \} = \{ A_{ab}, B_{cd} \}, \]  
(2.29)

and the symbol \( \Omega_1 \) is given by:

\[ \Omega_1 = 1 \otimes 1 + \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z, \]  
(2.30)

where \( 1 \) is a \( 2 \times 2 \) identity matrix and the symbols \( \sigma_x \) and \( \sigma_z \) denote Pauli matrices

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  
(2.31)

From the Poisson brackets (2.27) we can derive the Poisson brackets between the transition matrices:

\[ \{ T_\pm(u), T_\pm(v) \} = \left[ r(u - v), \frac{2}{\rho} T_\pm(u) T_\pm(v) \right], \]  
(2.32)

\[ \{ T_\pm(u), T_\pm(v) \} = \left( r(u - v) \frac{2}{\rho} T_\pm(u) T_\pm(v) - \frac{1}{\rho} T_\pm(u) T_\pm(v) r^\tau(u - v) \right). \]  
(2.33)

The brackets (2.32) and (2.33) have \( r \)-matrix structure with rational classical \( r \)-matrix

\[ r(u) = \frac{1}{u} \sum_{a,b=1}^2 E_{ab} \otimes E_{ba}, \quad r^\tau = \frac{1}{u} I - r^\tau(u), \quad (E_{ab})_{ij} = \delta_{ia} \delta_{jb}. \]  
(2.34)

satisfying the classical Yang–Baxter equation

\[ \left[ \frac{1}{2} r(u_1 - u_2), \frac{1}{2} r(u_1 - u_3) \right] + \left[ \frac{1}{2} r(u_2 - u_3), \frac{1}{2} r(u_1 - u_3) \right] + \left[ \frac{1}{2} r(u_1 - u_2), \frac{1}{2} r(u_2 - u_3) \right] = 0. \]  
(2.35)

The equation (2.35) is defined on the tensor product of three identical spaces \( V \otimes V \otimes V \) with \( V = \mathbb{C}^2 \), and the indices on top of the symbols \( r \) denote the pairs of spaces on which the respective matrix acts non-trivially. The symbol \( T \) in (2.34) denotes the transposition with respect to one of the spaces.
2.2. Geroch group

The space of solutions of (2.2) is isomorphic to the space of admissible boundary values of metric on the symmetry axis, which within the construction of [24] are given by

\[ g(0, u) = M(u). \]  

(2.36)

The admissible boundary values are regular functions on the real axis taking values in $2 \times 2$ symmetric real matrices with unit determinant. Any admissible boundary value can be obtained from the trivial with help of the following transformation

\[ g(0, u) \mapsto \tilde{g}(0, u) := S(u)g(0, u)S^T(u), \]  

(2.37)

where the parameter of the transformation $S(u)$ is matrix-valued function of a real argument. The function $S(u)$ takes values in $SL(2, \mathbb{R})$ and tends to the identity matrix for large absolute values of $u$. The transformations (2.37) with different choices of the function $S(u)$ obviously form a group. The action of the transformation (2.37) on boundary values can be obviously extended to the action on fields such that any solution maps to a solution. This group of transformations is known as the Geroch group, and it is a particular example of a dressing transformation commonly found in infinite-dimensional classical integrable models. It is not a symmetry in the usual sense of the Hamiltonian mechanics: it does not preserve the Hamiltonian. Contrarily, it is a Lie–Poisson group.

The infinitesimal action of the Geroch group on an arbitrary observable can be represented as follows. Let the contour $l_+$ be the line $\mathcal{I}(u) = \varepsilon$, and the contour $l_-$ be the line $\mathcal{I}(u) = -\varepsilon$ for some small positive value of $\varepsilon$. Let us define the adjoint action of the transition matrices by

\[ \text{ad}_{T^\pm(w)} X = \{ T^\pm(w), X \}. \]  

(2.38)

Let $\Lambda(w)$ be a regular $sl(2, \mathbb{R})$-valued function of a real argument representing an element of the Lie algebra of the Geroch group. Then the action of the algebra element $\Lambda(w)$ on an arbitrary observable is given by

\[ G[\Lambda] = \lim_{\varepsilon \to 0^+} \text{Tr} \left( \int_{l_+} T^{-1}_{+}(w)\Lambda(w)\text{ad}_{T^+}(w) \, dw + \int_{l_-} T^{-1}_{-}(w)\Lambda(w)\text{ad}_{T^-}(w) \, dw \right). \]  

(2.39)

The formula (2.39) implies [24] that classically the Geroch group acts on the monodromy matrix $M(u)$ as

\[ G[\Lambda]M(u) = \Lambda(u)M(u) + M(u)\Lambda^T(u). \]  

(2.40)

The action (2.40) can obviously be exponentiated to

\[ M(u) \mapsto \tilde{M}(u) := S(u)M(u)S^T(u), \]  

(2.41)

where the matrix $S(u)$ belongs to $SL(2, \mathbb{R})$. The action of the Geroch group on the transition matrices is then given by equations (1.7) and (1.8). The ‘gauge’ factor $U(S, T^\pm | u)$ appearing in the equations (1.7) and (1.8) is a functional of the transition matrices and the element of the Geroch group. The matrix $U(S, T^\pm | u)$ is implicitly fixed by the following requirements. Firstly, it is a function of a real argument which takes values in the group $SL(2, \mathbb{R})$. Secondly, the transformed transition matrices $\tilde{T}^\pm(u)$ admit analytical continuation into the same respective
half-planes as the original ones. Finally, the matrix-valued function $U(S, T_{\pm}|u)$ tends to the identity matrix at large values of the spectral parameter $u$.

**Remark 2.2.** The problem of constructing the function $U(S, T_{\pm}|u)$ is equivalent to the Riemann–Hilbert problem for the jump matrix $\tilde{M}(u)$. This problem has unique solution up to a left multiplication by an entire $SL(2, \mathbb{C})$ valued function. The ambiguity is fixed by imposing conditions on the asymptotic behaviour of the solution at infinity. One can also consider slightly different Riemann–Hilbert problem formulated in terms of dynamical spectral parameter $\gamma$. The latter problem is solved by the ‘deformed zweibein’ $\hat{v}(x, u)$, which is unique up to an $SO(2)$ gauge transformation [18, 30]. However, the definition (2.21) of the transition matrices $T_{\pm}(u)$ involves only gauge invariant quantities. In other words, the identity

$$\hat{v}(\infty, 0, u \pm i\varepsilon) = T_{\pm}(u \pm i\varepsilon), \quad u \in \mathbb{R}, \quad \varepsilon > 0$$

(2.42)

holds only for a particular choice of gauge. Therefore, the matrices $\tilde{T}_{\pm}(u)$ and $U(S, T_{\pm}|u)$ can be determined unambiguously from the requirements above.

The action of the Geroch group preserves the Poisson structure in the following sense: if we consider independent elements of the matrices $S(u)$ to be dynamical variables with a nontrivial Poisson structure

$$\{1_{S(u)}, 2_{S(v)}\} = [r(u - v), 1_{S(u)}2_{S(v)}], \quad \{1_{S(u)}, 2_{T_{\pm}(v)}\} = 0$$

(2.43)

then the Poisson bracket between the transformed matrices $\tilde{T}_{\pm}$ would be given by the same formulae (2.32) and (2.33).

### 3. Operator products and properties of representation

In [24] the authors suggested a consistent way to quantize the Einstein–Rosen model. The Poisson brackets (2.32) and (2.33) should be replaced by the following exchange relations:

$$R(u - v)1_{T_{\pm}(u)}2_{T_{\pm}(v)} = 2_{T_{\pm}(v)}1_{T_{\pm}(u)}R(u - v),$$

(3.1)

$$R(u - v - i\hbar)1_{T_{\pm}(u)}2_{T_{\pm}(v)} = 2_{T_{\pm}(v)}1_{T_{\pm}(u)}R^\hbar(u - v + i\hbar)\chi(u - v),$$

(3.2)

where the scalar factor $\chi$ reads

$$\chi(u) = \frac{u(u - 2i\hbar)}{u^2 + \hbar^2},$$

(3.3)

and the arguments in (3.2) satisfy

$$u \notin \{v - i\hbar, v, v + i\hbar, v + 2i\hbar\}.$$

(3.4)

The indices 1, 2 on top of the symbols $T_{\pm}$ above denote the space on which the corresponding matrix acts non-trivially:

$$1_{T_{\pm}(u)} = T_{\pm}(u) \otimes 1, \quad 2_{T_{\pm}(u)} = 1 \otimes T_{\pm}(u).$$

(3.5)

The quantum $R$-matrix and its twisted version $R^\hbar$ are given by

$$R(u, v) = (u - v)(1 - i\hbar r(u - v)), \quad R^\hbar(u, v) = -R^\hbar(i\hbar - v, u).$$

(3.6)
where the superscript $T$ denotes the transposition with respect to one of the spaces in the tensor product, i.e.

$$R_{ijkl}^T = R_{jikl} = R_{jilk}.$$  \hfill (3.7)

The operators $T_{ij}^+(u)$ and $T_{ij}^-(u)$ are related via the Hermitian conjugation

$$
\left( T_{ij}^+(u) \right)^\dagger = T_{ij}^-(\bar{u}).
$$  \hfill (3.8)

The algebra of observables is generated by the evaluations of the operator-valued functions $T_{ij}^\pm(u)$ on the real axis subject to the relations (3.1) and (3.2). Any polynomial in $T_{ij}^\pm(u)$, in particular the bilinear expressions of the form $T_{ij}^+(u)T_{kl}^-(v)$, is well-defined for any real value of $u$ and $v$, which leads to the following identities

$$
T_{ai}^-(u)T_{aj}^+(u) + (u) = 0, \quad (3.9)
$$

$$
T_{1i}^-(u)T_{2j}^+(u) + T_{2i}^-(u)T_{2j}^+(u) = 0. \quad (3.10)
$$

The expressions in the left-hand sides of the identities (3.9) and (3.10) form an ideal and can be factored out. Within the quotient algebra the relation (3.2) can be extended to the point $u = v$.

We shall call this quotient algebra, following the paper [24], a ‘twisted sl(2) Yangian double’ and denote it by $\mathcal{D}$. The parameter $u$ will be referred to as the spectral parameter.

Similarly to the classical case one defines the quantum monodromy matrix as follows:

$$
M(u) = T_{ij}^+(u)T_{ij}^-(u).
$$  \hfill (3.11)

The operators $M(u)$ form a subalgebra of the algebra $\mathcal{D}$, with the following exchange relations:

$$
R(u-v)M(u)R^0(v-u + 2i\hbar)M(v) = 2M(v)R^0(u-v + 2i\hbar)M(u)R^0(v-u)\frac{\chi(u-v)}{\chi(v-u)}.
$$  \hfill (3.12)

The following property of the algebra $\mathcal{D}$ was established in the work [25].

**Proposition 3.1.** The quantum determinants of the transition matrices

$$
\text{qdet} T_{ij}^+(u) = T_{ij}^{11}(u)T_{ij}^{22}(u + i\hbar) - T_{ij}^{12}(u)T_{ij}^{21}(u + i\hbar)
$$  \hfill (3.13)

and the antisymmetric part of the quantum monodromy matrix

$$
M^{12}(u) - M^{21}(u)
$$  \hfill (3.14)

belong to the center of the algebra $\mathcal{D}$.

The RTT relation (3.1) defines two subalgebras: $Y_+$ generated by the operators $T_{ij}^+(u)$, and $Y_-$ generated by the operators $T_{ij}^-(u)$. The equation (3.1) coincides with the defining relation of the Yangian $Y(sl(2))$ up to a rescaling of the spectral parameter:

$$
R^{12}(u-v)T(u)T(v) = 2T(v)T(u)R^{12}(u-v),
$$  \hfill (3.15)

$$
R(u) = uR^{12}(-u^{-1}).
$$  \hfill (3.16)

According to the textbook definition [36], the Yangian relation (3.15) should be interpreted as an equality of two bivariate formal power series in inverse powers of $u$ and $v$. The symbol
$T(u)$ should be understood as a formal generating function of the Yangian generators, which are given by the coefficients of the expansion of $T(u)$ in $u^{-1}$:

$$T^{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} t^{(k)}_{ij} u^{-k}. \quad (3.17)$$

In particular representations of Yangian the evaluations $T^{ij}(u)$ are allowed to be ill-defined (i.e. not be a part of the algebra) for some (or even all) values of $u$, but the generators $t^{(k)}_{ij}$ are always assumed to be well-defined (i.e. must have finite matrix elements).

The representation of the algebra $\mathcal{D}$ on the Hilbert space of the quantum Einstein–Rosen model is unknown. However, one can expect that in a hypothetical representation analytical and other properties of the quantum operators $T^{\pm}(u)$ reproduce respective properties of their classical analogues (discussed in the previous section) in the limit $\hbar \to 0$. It is therefore natural to look for representations satisfying the following conditions.

(a) All matrix elements of the operators $T^{\pm}(u)$ in this representation are smooth complex-valued functions on the real line.

(b) The Fourier transforms of the operators $T^{\pm}(u)$

$$\hat{T}^{\pm}(k) = \int_{-\infty}^{\infty} e^{iku} T^{\pm}(u) du \quad (3.18)$$

are convergent for real $k \neq 0$.

(c) The operators $T^{+}(u)$ is analytic in the lower half-plane, and the operators $T^{-}(u)$ must be analytic in the upper half-plane.

(d) On the imaginary axis the following asymptotic formulae hold true:

$$T^{+}(is) = \delta_{ij} + \sum_{n=1}^{\infty} (-1)^{n} t^{(n)}_{ij} \hbar^{n} s^{-n} \quad s \in \mathbb{R}, \quad s \to +\infty \quad (3.19)$$

$$T^{-}(is) = \delta_{ij} + \sum_{n=1}^{\infty} (t^{(-n)}_{ij} \hbar^{n} s^{-n} \quad s \in \mathbb{R}, \quad s \to +\infty. \quad (3.20)$$

Note that this requirement does not imply the existence of an open neighbourhood where both expansions are valid simultaneously.

(e) The Fourier modes $\hat{T}^{\pm}(k)$ have the following expansion

$$\hat{T}^{+}(k) = \delta_{ij} \hat{\rho}(k) + \theta(k) \sum_{n=0}^{\infty} \frac{t^{(n+1)}_{ij} \hbar^{n+1} k^{n}}{n!} \quad (3.21)$$

$$\hat{T}^{-}(k) = \delta_{ij} \hat{\rho}(k) + \theta(-k) \sum_{n=0}^{\infty} (-1)^{n} t^{(-n-1)}_{ij} \hbar^{n+1} k^{n} \quad (3.22)$$

(f) The operators $T^{+}(u)$ and $T^{\pm}(u)$ are analytic in the strip $-\frac{1}{\hbar} < \Im(u) < \frac{1}{\hbar}$, and all normal-ordered products (i.e. the products such that all the instances of $T^{-}$ are to the right of all $T^{+}$) are regular in all arguments as long as all the arguments lie within this strip. However, the products with a different ordering, p.e. $T^{\pm}(u)T^{\pm}(u \pm i\hbar)$, might be singular.

(g) In order to achieve the consistency with the classical expressions the central elements are fixed as follows:

$$q\text{det} T^{+}(u) = q\text{det} T^{-}(u) = 1, \quad (3.23)$$
We shall refer to representations satisfying the conditions (a)–(g) as ‘physically acceptable’ representations. Let us summarize the desired properties of the hypothetical Hilbert space representation:

- The Fourier transforms $\hat{T}_\pm(k)$ are well-defined for $k \neq 0$.
- The operator $T_+(u)$ is analytic for $\Im(u) > -\frac{1}{2}\hbar$ (negative frequencies vanish).
- The operator $T_-(u)$ is analytic for $\Im(u) < \frac{1}{2}\hbar$ (positive frequencies vanish).
- $\det T_+(u) = \det T_-(u)$.
- All normal-ordered products of the operators $T_\pm(u)$ (i.e. such that all instances of $T_-(u)$ are to the right of all instances of $T_+(u)$) are regular functions in all arguments in the strip.

At this point we cannot prove the existence of such a representation. We proceed on the assumption that such a representation exists, leaving the construction for the future.

The following proposition shows that a ‘physically acceptable’ representation cannot be finite-dimensional.

**Proposition 3.2.** There is no finite-dimensional representation satisfying all the constraints above.

**Proof.** For a finite-dimensional representation all products of regular operator-valued functions must be regular. Multiplying both sides of the identity (3.2) by a factor $(u - v)^2 + \hbar^2$ and evaluating both sides of the resulting equation at the points $u - v = \pm i\hbar$ one gets the following identity:

$$T_+^{ij}(u)T_-^{kl}(u \pm i\hbar) = 0.$$  \hspace{1cm} (3.25)

However, the product

$$\det T_+(u)\det T_-(u) = 1,$$  \hspace{1cm} (3.26)

is, per condition (g), non-zero. Let us rewrite (3.26) in terms of the matrix entries of the matrices $T_\pm(u)$ using the definition (3.13). Then it is clear that this product is a linear combination of expressions proportional to quadratic expressions of the form (3.25), and therefore must vanish in any finite-dimensional representation. Thus we have demonstrated that a finite-dimensional representation cannot satisfy all the requirements simultaneously. \( \Box \)

### 4. Quantum determinant of the deformed metric

The quantum comatrix $\hat{K}(u)$ of a $2 \times 2$ matrix $K(u)$ with entries $K^{ij}(u)$ being formal series with coefficients in Yangian is defined as

$$K^{ij}(u) = c^a e_{[ab]} K^{ba}(u).$$  \hspace{1cm} (4.1)

Due to the RTT relation (3.1) the quantum transition matrices $T_\pm(u)$ have the following property:

$$T_\pm(u)T_\pm(u + i\hbar) = \det T_\pm(u)I,$$  \hspace{1cm} (4.2)
where \( \text{qdet} T_\pm(u) \) lies in the center of algebra due to the double being at the critical level and \( 1 \) is a \( 2 \times 2 \) identity matrix. Since for the classical Einstein–Rosen model we know that
\[
\text{det} T_\pm(u) = 1,
\]
we are primarily interested in the representations such that
\[
\text{qdet} T_\pm(u) = 1. \tag{4.3}
\]

In order to verify that the transformation suggested in the work [18] as a quantum analogue of the Geroch group (see section 5 for details) preserves the vacuum expectation values of the elements of the algebra \( \mathcal{D} \) (and therefore preserves the physical content of the model) one has to ensure that the condition (4.3) is preserved by the transformation. Note that the authors of [18] defined the quantum Geroch transformation only for the quantum monodromy matrix \( M(u) \). Therefore, we must relate (4.3) to some constraint written in terms of the quantum monodromy matrix.

Classically, the monodromy matrix \( M(u) \) must have unit determinant. We suggest a quantum analogue of this constraint. Let us consider the following expression:
\[
\mathfrak{M}(u, s) = M(u)M(u + i\hbar + is). \tag{4.4}
\]

In any representations satisfying requirements listed in the section 3 the operator \( \mathfrak{M}(u, s) \) must have a pole at \( s = 0 \) as long as both \( u \) and \( u + i\hbar \) are within the intersection of the domains of analyticity of the quantum transition matrices \( T_+(u) \) and \( T_-(u) \). The appearance of the pole of \( \mathfrak{M}(u, s) \) at \( s = 0 \), which does not appear in the classical case, can be informally understood as a manifestation of an infinite multiplicative renormalization factor.

We define the regularized quantum determinant of the quantum monodromy matrix as the residue of the top left component of \( \mathfrak{M} \):
\[
\text{qdet} M(u) := \frac{2}{3} \text{Res}_{s=0} \mathfrak{M}^{11}(u, s). \tag{4.5}
\]

Then the following proposition holds.

**Proposition 4.1.** Let both quantum transition matrices \( T_\pm(u) \) be regular functions of the spectral parameter within the strip \(-\frac{1}{2}\hbar - \epsilon < \Im(u) < \frac{1}{2}\hbar + \epsilon \) for some real positive \( \epsilon \). Let all normal-ordered monomials composed of several instances of \( T_+ \) and \( T_- \), i.e. expressions of the form
\[
T_+^{i_1}(u_1)T_+^{i_2}(u_2) \ldots T_+^{i_k}(u_k)T_-^{j_1}(v_1)T_-^{j_2}(v_2) \ldots T_-^{j_m}(v_m) \tag{4.6}
\]
be regular as functions of all spectral parameters \( u_i, v_j \) involved provided all of these spectral parameters belong to the joint strip of analyticity. Then exchange relations imply the following identities:
\[
\text{qdet} M(u) = \text{qdet} T_+(u)\text{qdet} T_-(u), \tag{4.7}
\]
\[
\frac{2}{3} \text{Res}_{s=0} \mathfrak{M}^{11}(u, s) = \text{qdet} M(u)\delta_{ij}. \tag{4.8}
\]

The conditions of the proposition are fulfilled if one adopts the set of assumptions outlined in the section 3. The detailed derivation of the formulae (4.7) and (4.8) can be found in the appendix.

The factorization formula (4.7) implies that the regularized quantum determinant of the monodromy matrix belongs to the center of the algebra \( \mathcal{D} \).
5. Quantum Geroch group

The authors of [18] proposed the following quantization of the transformation (2.41): consider a quantum group defined by the relation

\[ R(u - v)S(u)S(v) = S(v)S(u)R(u - v), \]

and let it act on the quantum monodromy matrix \( M(u) \) as

\[ M(u) \rightarrow \tilde{M}(u) = S(u + \frac{i}{2} \hbar) M(u) S(u - \frac{i}{2} \hbar), \]

assuming

\[ [M^{ij}(u), S^{kl}(v)] = 0, \quad \forall \ u, v \in \mathbb{C}, \quad i, j, k, l \in \{1, 2\}. \]

(5.3)

Under the Hermitian conjugation the operator \( S(u) \) transforms as follows:

\[ (S^{ij}(u))^{\dagger} = S^{ij}(\bar{u}). \]

(5.4)

The representation of Yangian in which the operator \( S(u) \) is evaluated is not specified. The work [18] contains the proof that the transformed quantum monodromy matrix \( \tilde{M}(u) \) is symmetric and positive-definite for all real values of \( u \).

The quantum Geroch group is not a symmetry group in the usual sense of the word. More precisely, the transformation (5.2) is an isomorphism of the algebras generated by the operators \( M(u) \) and \( \tilde{M}(u) \) respectively, but if the operator \( M(u) \) acts on the Hilbert space \( \mathcal{H} \) and the operator \( S(u) \) acts on some representation \( \mathcal{G} \) of the Yangian then the operator \( \tilde{M}(u) \) acts on the tensor product \( \mathcal{G} \otimes \mathcal{H} \). The repeated application of the transformation (5.2) yields an operator acting on the tensor product \( \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{H} \), i.e. each copy of \( S(u) \) acts on its own space.

The authors of [18] define the action of the quantum Geroch group on the subalgebra of the algebra \( \mathcal{D} \) generated by \( M \).

We suggest the following extension of the transformation (5.2) to the full algebra \( \mathcal{D} \):

\[ T_+(u) \rightarrow \tilde{T}_+(u) := S(u + \frac{i}{2} \hbar) T_+(u) U\left(-u + \frac{i}{2} \hbar\right), \]

(5.5)

\[ T_-(u) \rightarrow \tilde{T}_-(u) := S(u - \frac{i}{2} \hbar) T_-(u) U^\dagger\left(-u - \frac{i}{2} \hbar\right), \]

(5.6)

where the operator \( U(u) \) generates another copy of Yangian. In other words, the operator \( U(u) \) is an arbitrary operator satisfying the RTT relation:

\[ R(u - v)U(u)U(v) = \frac{1}{2} U(v)U(u)R(u - v). \]

(5.7)

The symbols \( S(u) \) and \( U(u) \) are assumed to commute with the elements of the algebra \( \mathcal{D} \) and between each other:

\[ [S^{ij}(u), T^{kl}_+(v)] = 0, \]

(5.8)

\[ [S^{ij}(u), U^{kl}(v)] = 0, \]

(5.9)

\[ [U^{ij}(u), T^{kl}_+(v)] = 0. \]

(5.10)
The Hermitian conjugate of $U(u)$ is given by

$$(U^\dagger(u))^\dagger = e^{u}_a e^{b}_b U^{ab}(\bar{u}).$$

(5.11)

The following properties can be established immediately.

**Proposition 5.1.** The quantum monodromy matrix transforms according to the rule (5.2) under the transformation (5.5) and (5.6).

**Proposition 5.2.** The transformed algebra $\tilde{D}$ generated by the transformed transition matrices $\tilde{T}_\pm$ is isomorphic to the original algebra $D$, i.e. the exchange relations are preserved:

$$R(u - v)\tilde{T}_\pm(u)\tilde{T}_\pm(v) = \tilde{T}_\pm(v)\tilde{T}_\pm(u)R(u - v),$$

(5.12)

$$R(u - v - ih)\tilde{T}_-(u)\tilde{T}_+(v) = \tilde{T}_+(v)\tilde{T}_-(u)R^0(u - v + ih)\chi(u - v).$$

(5.13)

**Proposition 5.3.** The quantum determinant of the transformed quantum transition matrices factorizes as follows:

$$q\text{det} \tilde{T}_\pm(u) = q\text{det} S \left( u \pm \frac{i}{2}\hbar \right) q\text{det} T_\pm(u) q\text{det} U \left( -u - \frac{1}{2}i\hbar \right).$$

(5.14)

**Proposition 5.4.** The transformed transition matrices $\tilde{T}_\pm(u)$ are related via Hermitian conjugation:

$$(\tilde{T}_\pm^\dagger(u))^\dagger = \tilde{T}_\pm(u).$$

(5.15)

The commutation relations (5.8)–(5.10) imply that the operator $S(u)$ acts on some space $G$, the operator $U(u)$ acts on another space $K$, and the transformed transition matrices $\tilde{T}_\pm(u)$ act on the tensor product $G \otimes H \otimes K$, where $H$ is the Hilbert space of the untransformed model. The propositions 5.1–5.3 hold for any choice of the representations $G$ and $K$ of the Yangian, and the proposition 5.4 is valid for any pair of representations satisfying identities (5.4) and (5.11) respectively.

However, the resulting representation will not be ‘physically acceptable’ for a generic choice of $G$ and $K$. In order to get a ‘physically acceptable’ representation, we must impose on the operator $U(u)$ some constraint written in terms of the operators $S(u)$ and $T_\pm(u)$. The explicit formulation of this constraint is not known at the moment.

We can make the following observation, which helps to understand the difficulty with the formulation of this constraint.

**Proposition 5.5.** The transformed transition matrices $\tilde{T}_\pm(u)$ can’t be analytical in the same half-planes as the original matrices.

**Proof.** Suppose that for some positive number $\varepsilon$ the operator $\tilde{T}_+(u)$ is holomorphic in the half-plane $\Im(u) > -\frac{1}{2}\hbar - \varepsilon$, and the operator $\tilde{T}_-(u)$ is holomorphic in the half-plane $\Im(u) < \frac{1}{2}\hbar + \varepsilon$. Then any matrix element of any of these operators must be a holomorphic function in the corresponding half-plane. Let us consider a pair of states $|\Omega\rangle$ and $|\Omega'\rangle$

$$|\Omega\rangle = |\phi\rangle \otimes |\psi\rangle \otimes |\chi\rangle,$$

(5.16)

$$|\Omega'\rangle = |\phi'\rangle \otimes |\psi'\rangle \otimes |\chi'\rangle,$$

(5.17)
The matrix elements \( \langle \Omega\mid T_{ij}^{(u)}(u)\mid \Omega' \rangle \) then read
\[
\langle \Omega\mid T_{ij}^{(u)}(u)\mid \Omega' \rangle = \langle \phi\mid S^a \left( u - \frac{i}{2} \hbar \right) \mid \phi' \rangle \langle \psi\mid T_{ij}^{(u)}(u)\mid \psi' \rangle \langle \chi\mid U^{ab} \left( u - \frac{i}{2} \hbar \right) \mid \chi' \rangle, \tag{5.21}
\]
\[
\langle \Omega\mid T_{ij}^{(u)}(u)\mid \Omega' \rangle = \langle \phi\mid S^a \left( u + \frac{i}{2} \hbar \right) \mid \phi' \rangle \langle \psi\mid T_{ij}^{(u)}(u)\mid \psi' \rangle e^{\hbar \epsilon \epsilon' \epsilon''} \langle \chi\mid U^{cd} \left( u - \frac{i}{2} \hbar \right) \mid \chi' \rangle. \tag{5.22}
\]

The problem of finding the matrix elements \( \langle \chi\mid U^{ij}(u)\mid \chi' \rangle \) as functions of the spectral parameter is similar to the problem we encounter in the classical case when we solve for the matrix \( U(S, T_\pm u) \). The only difference is that the determinant of the matrix \( \langle \chi\mid U^{ij}(u)\mid \chi' \rangle \) is not fixed. Still, the requirement that the matrix elements \( \langle \Omega\mid T_{ij}^{(u)}(u)\mid \Omega' \rangle \) should be holomorphic in their respective half-plane significantly constraints possible choices for the matrix elements of the operator \( U(u) \). In particular, they can have singularities only at the points where the matrix element \( \langle \phi\mid S^{(u)}\mid \phi' \rangle \) is singular or degenerate.

Since the states \( |\phi\rangle, |\phi'\rangle, |\psi\rangle, |\psi'\rangle, |\chi\rangle \) and \( |\chi'\rangle \) can be chosen independently, any matrix element of \( U(u) \) should be a common solution to the problems with all pairs of states \( |\phi\rangle, |\phi'\rangle \) from \( \mathcal{G} \). It is possible only if all matrix elements of the operator \( S(u) \) have their singularities at the same points. But that is incompatible with the classical limit, as almost any meromorphic function with unit determinant and an appropriate asymptotic behaviour represents an element of the classical Geroch group.

The proposition 5.5 means that the operator \( U(u) \) must be fixed by some weaker condition. This hypothetical condition should still give rise to a ‘physically acceptable’ representation, albeit in an indirect way. We suggest the following scenario, which is not forbidden by the proposition 5.5: the representation of \( D \) on the space \( \mathcal{G} \otimes \mathcal{H} \otimes \mathcal{K} \) decomposes into a direct sum of multiple representations. The operator \( U(u) \) should be chosen in such a way that at least one of these subrepresentations is ‘physically acceptable’. In this scenario the quantum Geroch transformation (5.5) and (5.6) is not an automorphism of the ‘physically acceptable’ representation, but it can induce such automorphisms. The latter are constructed by identifying the vacuum state from the space \( \mathcal{H} \) with some vector from one of the ‘physically acceptable’ subrepresentations.

The extension of the quantum Geroch group described above may be used to prove that the transformed monodromy matrix satisfies the identity (4.7). Indeed, by construction the transformed monodromy matrix admits factorization
\[
\tilde{M} = \tilde{T}_+(u)\tilde{T}_-(u) \tag{5.23}
\]
such that the transformed transition matrices \( \tilde{T}_\pm \) give a representation of the deformed quantum double (3.1), (3.2) and
\[
q\det \tilde{T}_\pm(u) = 1. \tag{5.24}
\]

The derivation of the factorization formula (4.7) remains valid for the transformed transition matrices provided that the assumptions about analyticity of the transition matrices and
their normal-ordered products hold for the transformed algebra. These assumptions would be fulfilled for the restriction of the transformed transition matrices onto one of the ‘physically acceptable’ subrepresentations in the hypothetical scenario outlined above. If that is the case, the regularized quantum determinant of the quantum monodromy matrix is an invariant of the quantum Geroch group.

6. Conclusion

In this paper we constructed a regularization of the quantum determinant of the quantum monodromy matrix which deforms the determinant of the classical monodromy matrix. We expressed it as a product of quantum determinants of the transition matrices. We gave arguments supporting the conjecture that the regularized quantum determinant of the quantum monodromy matrix is preserved by the action of the quantum Geroch group. We also suggested the extension of the action of the quantum dressing transformation onto the full algebra of observables which preserves the quantum determinants of the quantum transition matrices. To prove that the action of the quantum Geroch group is an automorphism of the representation it remains to verify the analyticity assumptions for both original and transformed transition matrices.

The quantum integrable structure of Einstein–Rosen model merits further investigation. To begin with, the complete description of the Hilbert space of the system is still unknown, as well as the explicit action of the quantum Geroch group on the space of states. Furthermore, there is no known infinite family of commuting integrals of motion for the Einstein–Rosen model, even though the presence of such families is a common feature of integrable quantum models. Finally, an explicit expression of Hamiltonian in terms of quantum transition matrices is also missing.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Proof of the factorization formula for residue of quantum determinant of $M(u)$

In components, the matrix $M(u, s)$ can be written as

$$
M^{ij}(u, s) = \epsilon^j_\nu \epsilon^{\alpha \beta} M^i_\alpha(u) M^\nu_\beta(\hat{u} + is),
$$

(A.1)

where we use the abbreviated notation

$$
\hat{u} = u + i\hbar.
$$

(A.2)
Using the exchange relation (3.2) we rewrite the operator $\mathcal{M}(u, s)$ in its normal-ordered form:

$$
\mathcal{M}^{\pm}(u, s) = e^{\nu_{s}^{\alpha_{s}}} T^\nu_+(u) T^{\alpha}_{-}(u) (\hat{u} + is) T^{\beta}_{-}(\hat{u} + is) \\
= e^{\nu_{s}^{\alpha_{s}}} T^\nu_+(u) R_+^{-1}( -2\hbar - is) T^{\beta}_{-}(\hat{u} + is) \\
\times T^{\nu}_{-}(u) R_u^{-1}( -is) T^{\beta}_{-}(\hat{u} + is) \chi(-i\hbar - is). \quad (A.3)
$$

Substituting the explicit expressions for $\chi$, $R$ and $R^n$ we obtain

$$
\mathcal{M}^{\pm}(u, s) = \frac{e^{\nu_{s}^{\alpha_{s}}}}{s(s + 2\hbar)} T^\nu_+(u) ((2\hbar + s)\delta_{\alpha\beta} - \hbar \delta_{\alpha\beta}) \\
\times T^{\nu}_{-}(u) (\delta_{\nu\nu} \delta_{\delta\gamma} - \delta_{\nu\gamma} \delta_{\nu\delta}) T^{\beta}_{-}(\hat{u}). \quad (A.4)
$$

Assuming the representation satisfies the conditions outlined in the section 3 we compute the limit of operator $s\mathcal{M}(u, s)$ as $s$ tends to zero:

$$
\lim_{s \to 0} s\mathcal{M}^{\pm}(u, s) = \frac{\hbar e^{\nu_{s}^{\alpha_{s}}}}{2} T^\nu_+(u) (2\delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\gamma} \delta_{\beta\delta}) T^{\beta}_{-}(\hat{u}) \\
\times T^{\nu}_{-}(u) (\delta_{\nu\nu} \delta_{\delta\gamma} - \delta_{\nu\gamma} \delta_{\nu\delta}) T^{\beta}_{-}(\hat{u}). \quad (A.5)
$$

In the above expression we relabeled the contracted indices to make use of the following identity for $2 \times 2$ matrices:

$$
e^{\nu_{s}^{\alpha_{s}}} e^{\nu_{s}^{\alpha_{s}}} T^\nu_+(u) T^{\alpha}_{-}(u) T^{\beta}_{-}(v) = T^\nu_+(u) T^{\beta}_{-}(v) T^{\alpha}_{-}(u) T^{\beta}_{-}(v) \\
- T^{\nu}_{-}(u) T^{\alpha}_{-}(v) T^{\beta}_{-}(u) T^{\beta}_{-}(v). \quad (A.6)
$$

The identity (A.6) is verified by explicitly calculating both sides. Combined with the definition of the quantum determinants of the quantum transition matrices $T_{\pm}$ in the following form

$$
qdet T_{\pm}(u) e^{ij} = e^{\nu_{s}^{\alpha_{s}}} T^\nu_+(u) T^{\beta}_{-}(\hat{u}) \quad (A.7)
$$

it allows to simplify the expression (A.5) further to get

$$
\lim_{s \to 0} s\mathcal{M}^{\pm}(u, s) = \frac{3}{2} qdet T_{+}(u) qdet T_{-}(u) \delta_{ij} \quad (A.8)
$$

ORCID iDs

B Runov  https://orcid.org/0000-0002-3473-4534

References

[1] Ashtekar A and Pierri M 1996 Probing quantum gravity through exactly soluble midi-superspaces
\textit{J. Math. Phys.} \textbf{37} 6250–70

[2] Kuchař K 1971 Canonical quantization of cylindrical gravitational waves \textit{Phys. Rev. D} \textbf{4} 955–86

[3] Allen M 1987 Canonical quantisation of a spherically symmetric, massless scalar field interacting
with gravity in $(2 + 1)$ dimensions \textit{Class. Quantum Grav.} \textbf{4} 149
[4] Ashtekar A 1996 Large quantum gravity effects: unforeseen limitations of the classical theory Phys. Rev. Lett. 77 4864
[5] Varadarajan M 2000 On the metric operator for quantum cylindrical waves Class. Quantum Grav. 17 189
[6] Gambini R and Pullin J 1997 Large quantum gravity effects: backreaction on matter Mod. Phys. Lett. A 12 2407–13
[7] Dominguez A E and Tiglio M H 1999 Large quantum gravity effects and nonlocal variables Phys. Rev. D 60 064001
[8] Angulo M E and Mena Marugán G A 2000 Large quantum gravity effects: cylindrical waves in four dimensions Int. J. Mod. Phys. D 9 669
[9] Fernando Barbero G I, Mena Marugán G A and Villaseñor E J S 2003 Microcausality and quantum cylindrical gravitational waves Phys. Rev. D 67 142006
[10] Korotkin D and Nicolai H 1996 Isomonodromic quantization of dimensionally reduced gravity Nucl. Phys. B 475 397–439
[11] Cruz J, Miković A and Navarro-Salas J 1998 Free field realization of cylindrically symmetric Einstein gravity Phys. Lett. B 437 273
[12] Husain V and Smolin L 1989 Exactly solvable quantum cosmologies from two Killing field reductions of general relativity Nucl. Phys. B 327 205
[13] Husain V 1996 Einstein’s equations and the chiral model Phys. Rev. D 53 4327
[14] Niedermaier M 2003 Dimensionally reduced gravity theories are asymptotically safe Nucl. Phys. B 673 131
[15] Niedermaier M and Samtleben H 2000 An algebraic bootstrap for dimensionally reduced quantum gravity Nucl. Phys. B 579 437–91
[16] Fuchs A and Reisenberger M P 2017 Integrable structures and the quantization of free null initial data for gravity Class. Quantum Grav. 34 185003
[17] Reisenberger M P 2018 The Poisson brackets of free null initial data for vacuum general relativity Class. Quantum Grav. 35 185012
[18] Peraza J, Paternain M and Reisenberger M 2019 On the quantum Geroch group (arXiv:1906.04856 [gr-qc])
[19] Belinsky V A and Zakharov V E 1978 Integration of the Einstein equations by the inverse scattering problem technique and the calculation of the exact soliton solutions Sov. Phys. JETP 48 985–94
[20] Maison D 1978 Are the stationary, axially symmetric Einstein equations completely integrable? Phys. Rev. Lett. 41 521–2
[21] Geroch R 1971 A method for generating solutions of einstein’s equations J. Math. Phys. 12 918–24
[22] Geroch R 1972 A method for generating new solutions of Einstein’s equation. II J. Math. Phys. 13 394–404
[23] Breitenlohner P, Maison D and Gibbons G 1988 Four-dimensional black holes from Kaluza–Klein theories Commun. Math. Phys. 120 295–333
[24] Korotkin D and Samtleben H 1998 Yangian symmetry in integrable quantum gravity Nucl. Phys. B 527 657–89
[25] Korotkin D and Samtleben H 1998 Canonical quantization of cylindrical gravitational waves with two polarizations Phys. Rev. Lett. 80 14–7
[26] Korotkin D and Samtleben H 1997 Poisson realization and quantization of the Geroch group Class. Quantum Grav. 14 151–6
[27] Korotkin D and Samtleben H 1997 Quantization of coset space σ-models coupled to two-dimensional gravity Commun. Math. Phys. 190 411–57
[28] Nikolai H, Korotkin D and Samtleben H 1997 Integrable classical and quantum gravity ed G ’ t Hooft, A Jaffe, G Mack and P K Mitter Quantum Fields and Quantum Space Time (NATO ASI Series (Series B: Physics)) vol 364 (Berlin: Springer)
[29] Korotkin D, Nicolai H and Samtleben H 1996 On 2D quantum gravity coupled to a σ-model Nucl. Phys. B 491 1–9
[30] Breitenlohner P and Maison D 1987 On the Geroch group Ann. Inst. H. Poincare Phys. Theor. 46 215–246 On Geroch Group
[31] Bernard D 1991 Quantum symmetries in 2D massive field theories New Symmetry Principles in Quantum Field Theory, Proc. NATO Advanced Study Institute (Cargese, France) ed J Froehlich, G ’ t Hooft, A Jaffe, G Mack, P K Mitter and R Stora
[32] Babelon O and Bernard D 1991 Dressing transformations and the origin of the quantum group symmetries Phys. Lett. B 260 81–6
[33] Babelon O and Bernard D 1992 Dressing symmetries Commun. Math. Phys. 149 279–306
[34] Zakharov V E and Mikhailov A V 1978 Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method Sov. Phys. JETP 47 1017–27
[35] Pohlmeyer K 1976 Integrable Hamiltonian systems and interactions through quadratic constraints Commun. Math. Phys. 46 207–21
[36] Molev A 2007 Yangians and Classical Lie Algebras (Providence, RI: American Mathematical Society)