Symmetries and the subregion bulk reconstruction paradoxes

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It is known that different subregion reconstructions of the same bulk field act identically, at least for states which have an interpretation in bulk effective field theory. In this paper we argue that for bulk subregions isometric to AdS/Rindler wedges this is a consequence of the transformation properties of the subregion representations under conformal symmetries, and the isometries of the bulk theory.

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I. INTRODUCTION.

The bulk reconstruction program aims to find operators in the boundary CFT that represent the fields in the bulk\[^{[1–6]}\]. Boundary representations of bulk fields are found by solving the equation of motion. They turn out to be non local CFT operators of the form:

\[
\phi_{\text{CFT}}(x) = \int dy K(x; y) \mathcal{O}(y)
\]  

(1)

Where \(y\) is the boundary coordinate and \(\mathcal{O}(y)\) is the CFT primary dual to the bulk field \(\phi\). The function \(K(x; y)\) is called the smearing function.

One can also solve the equation of motion in an AdS/Rindler wedge. This gives the CFT representation of a bulk field at any point inside a AdS/Rindler wedge as an operator which smeared on the boundary of that AdS/Rindler wedge.

This above result gives rise to an apparent paradox. Because if a bulk point lies in the overlap region of two different AdS/Rindler wedges it will have two different boundary representations, smeared on the corresponding boundaries. These are two different CFT operators, but they both represent the same bulk field, at least for bulk fields. There is thus an apparent redundancy of description of the bulk from the boundary – the same bulk field can be described by more than one boundary operator.

This appears paradoxical, because it seems one has multiple CFT operators whose actions are identical. It would thus seem that the same information is somehow stored in the boundary CFT in different regions. This paradox was pointed out in \[^{[7]}\]. In this paper we will argue that this paradox can be resolved by noting how subregion representations transform under conformal symmetries.

A different approach to bulk reconstruction is based on symmetries (introduced in \[^{[8]}\], see also \[^{[9–11]}\]). In this approach one uses the fact that for pure AdS there is a one-to-one correspondence between bulk isometries and boundary conformal symmetries. The boundary representation of a bulk field is obtained by demanding that it transforms under any given boundary conformal symmetry in the same way as the bulk field would transform under the corresponding bulk isometry. That is to say, the boundary representation of a scalar field will be given by:

\[
U(\Lambda) \phi_{\text{CFT}}(x) U^{-1}(\Lambda) = \phi_{\text{CFT}}(\Lambda_{\text{bulk}}^{-1} x)
\]  

(2)

where \(\Lambda\) is any boundary conformal symmetry while \(\Lambda_{\text{bulk}}\) denotes the corresponding bulk isometry. In what follows we will drop the suffix ‘CFT’ and represent these operators as \(\phi\).

It is interesting to ask, how does an AdS/Rindler wedge representation transform under conformal symmetries. The boundary of an AdS/Rindler wedge is not necessarily preserved under a conformal transformation, it will generally get mapped to the boundary of a different wedge. For symmetries that preserve the wedge-boundary (or equivalently, isometries that preserve the wedge) we may expect the above relation to hold, and we will check this explicitly.

Let us consider an isometry \(\Lambda_{\text{bulk}}\) that maps one wedge to a different wedge. Let the boundary of one wedge be denoted as \(a\). Under the corresponding conformal symmetry \(\Lambda\) any operators smeared over the region the boundary \(a\) is mapped to an operators smeared over the region \(\Lambda_{\text{bulk}}^{-1} a\), which is the boundary of a different wedge.

Then an immediate conjecture would be to expect that the following transformation property holds:

\[
U(\Lambda) \phi_a(x) U^{-1}(\Lambda) = \phi_{\Lambda_{\text{bulk}}^{-1}}(\Lambda_{\text{bulk}}^{-1} x)
\]  

(3)

Here \(\phi_a\), \(\phi_{\Lambda a}\) denote the CFT representations in the two wedges.

In this paper we will posit that this transformation rule is true. We will not prove it in this paper, but we will provide evidence for it in the next section for the case where \(\Lambda\) are boosts.

We will then show that (3) along with bulk isometry (which is encoded in the smearing function) explains, for all subregion isometric to AdS/Rindler wedges (that is, all wedges whose boundary is a causal diamond of a spherical spatial region), the identical action of the different subregion representations for all boundary states which admit a bulk intrepretation (in other words states in the code subspace). This resolves the apparent paradox for this class of wedges. It shows that the reason the same information was present in more than one location is because they are mapped by symmetry, and symmetry preserves information.

In the next section we present evidence for (3). In the third section we show how the apparent paradox is resolved by this equation. We conclude with a summary.
II. EVIDENCE FOR THE TRANSFORMATION RULE

Let us consider two overlapping boundary Rindler wedges $a$ and $b$ which correspond to accelerated observers moving in different directions. They are boundaries of two overlapping AdS/Rindler wedges $A$ and $B$ in the bulk. The corresponding CFT representations of a scalar field are denoted by $\phi_a$ and $\phi_b$.

Let $X$ be a point in the region of overlap between the two AdS/Rindler wedges. Let its co-ordinates in the two Rindler charts be given by $(z_a, \eta_a, \xi_a)$ and $(z_b, \eta_b, \xi_b)$. Let us consider another point $Y$ in the region of overlap whose coordinates are given in $A$ by $(z_a, \eta_a + \alpha, \xi_a)$.

Since the subregion reconstructions act identically, we have that

$$\phi_a(X)|0\rangle = \phi_b(X)|0\rangle$$ (4)

$$\phi_a(Y)|0\rangle = \phi_b(Y)|0\rangle$$ (5)

Now note that the region $a$ is preserved under any boost in the direction of its acceleration. In fact such a boost acts as time translation in the Rindler wedge. The corresponding bulk isometry is time translation in the AdS-Rindler wedge.

However the region $b$ is not preserved under this boost, by the assumption that it corresponds to an accelerated observer in a different direction. So the CFT representation of a bulk field, which is smeared over $b$ would get mapped to an operator smeared over different Rindler wedge $b' = \Lambda^{-1}b$. Using suggestive notation we define

$$\phi_{b'}(Y) := U(\Lambda)\phi_b(\Lambda_{bulk}Y)U(\Lambda^{-1})$$ (6)

This is just notation, $\phi_{b'}(Y)$ does not yet have the interpretation as the CFT representation of the bulk field at $Y$. However we will show that if both (4) and (5) are to be true, $\phi_{b'}(Y)$ will have the interpretation of being a CFT representation smeared over the region $b'$.

To see this first we consider a boundary boost $\Lambda$ that acts as a time translation in the wedge $a$ which takes $\eta$ to $\eta - \alpha$. Therefore the corresponding bulk isometry takes $X$ to look at how $\phi_a$ transforms under a boundary boost.

$\phi_a$ is given by [7]:

$$\phi_a(X) = \int d\eta'_a d\xi'_a K(z_a, \eta_a; \xi'_a)\mathcal{O}(\eta'_a, \xi'_a)$$ (7)

where

$$K(z_a, \eta_a; \eta'_a, \xi'_a) = \frac{e^{-2\xi_a}}{2\pi} \int d\omega dk V_{\omega,k}(z_a) e^{-i\omega(\eta_a - \eta'_a) + ik(\xi_a - \xi'_a)}.$$ (8)

Using the fact that

$$U(\Lambda)\mathcal{O}(\eta'_a, \xi'_a)U(\Lambda^{-1}) = \mathcal{O}(\eta'_a + \alpha, \xi'_a)$$ (9)

Then one finds

$$U(\Lambda)\phi_a(X)U(\Lambda^{-1}) = \phi_a(Y)$$ (10)

It follows that

$$U(\Lambda)\phi_a(X)|0\rangle = \phi_a(Y)|0\rangle$$ (11)

From (4), (5) we should have then that

$$U(\Lambda)\phi_b(X)|0\rangle = \phi_b(Y)|0\rangle$$ (12)
However we saw that the region $b$ is not preserved under the transformation and is mapped to a different operator. In fact we have

$$U(\Lambda)\phi_b(X)|0\rangle = \phi_{b'}(Y)|0\rangle$$

(13)

So for (14) and (15) to be both true it must be that

$$\phi_{b'}(Y)|0\rangle = \phi_b(Y)|0\rangle$$

(14)

It follows then $\phi_{b'}$ is the boundary representation of the bulk field smeared over the region $b'$. It represents the bulk fields in the AdS/Rindler wedge whose boundary is $b'$.

Thus we have shown that for boosts it is true that

$$U(\Lambda)\phi_b(X)U^{-1}(\Lambda)|0\rangle = \phi_{\Lambda^{-1}b}(\Lambda_{\text{bulk}}^{-1}X)|0\rangle$$

(15)

The above argument can be generalized to hold not just for the vacuum but for boundary states which can be interpreted as small excitations on pure AdS.

This provides evidence for (3).

III. SYMMETRY EXPLANATION OF APPARENT PARADOX

In the previous section we started from the fact that different subregion representations act identically in low energy states and provided evidence for (3). In this section we will go the other way and show that if we assume (3), it follows with the additional input of bulk isometry that the correlators of different representations match in the states which belong to the code subspace.

First let us do this for the vacuum. We would like to explain the following statement:

$$\langle 0|\phi_b(x_1)\ldots\phi_b(x_n)|0\rangle = \langle 0|\phi_{b'}(x_1)\ldots\phi_{b'}(x_n)|0\rangle.$$  

(16)

where $b$ can be mapped to $b'$ by some conformal transformation $\Lambda$.

This follows straightforwardly. Using (3) we have that:

$$\langle 0|\phi_{b'}(x_1)\ldots\phi_{b'}(x_n)|0\rangle = \langle 0|\phi_b(\Lambda_{b}^{-1}x_1)\ldots\phi_b(\Lambda_{b}^{-1}x_n)|0\rangle$$

(17)

Now the requirement of bulk isometry is encoded in the CFT representation via the smearing functions. So it is true by construction that

$$\langle 0|\phi_b(\Lambda_{b}^{-1}x_1)\ldots\phi_b(\Lambda_{b}^{-1}x_n)|0\rangle = \langle 0|\phi_b(x_1)\ldots\phi_b(x_n)|0\rangle$$

(18)

From (17) and (18) (16) follows directly.

We can show this not just for the vacuum but also for states which can be interpreted as a bulk configuration. We will show this for "one particle states".

Consider the boundary state obtained by action on the vacuum by the following operator:

$$\mathcal{O}_{\omega,k} = \int dt \, dk_i \mathcal{O}(t, x_i) e^{-i\omega t + ik^i x_i}$$

(19)

Here the integration is over the full boundary, not just a wedge. These operators are the boundary representations of creation operators. It is easy to check that the corresponding state $|k\rangle = \mathcal{O}_{\omega,k}|0\rangle$ transforms under conformal transformations just as a one particle state in the bulk theory would under bulk isometries.

Symbolically we may write:

$$U(\Lambda)|k\rangle = |\Lambda_{\text{bulk}}k\rangle$$

(20)

Now we would like to show that

$$\langle 0|\phi_b(Y)|k\rangle = \langle 0|\phi_{b'}(Y)|k\rangle$$

(21)
It is straightforward to see that this follows from an identical argument as before using (3), (20) and the requirement that boundary representations must encode bulk isometry.

This can be extended to multi-particle states. The only requirement is that the state have a bulk interpretation as an excitation on a pure AdS background so that the requirement of bulk isometry can be imposed on boundary representations.

Thus we see that (3) indeed explains, for AdS/Rindler wedges, why the different representations act identically. This can be extended to multi-particle states. The only requirement is that the state have a bulk interpretation as an excitation on a pure AdS background so that the requirement of bulk isometry can be imposed on boundary representations.

Thus one might say that the apparent paradox is, for these wedges, resolved by symmetry. The reason the same information was present in different regions was that it was preserved by symmetry.

Finally as an aside we note the resolution of another apparent paradox that was suggested in [7]. The paradox was that bulk microcausality dictates that the CFT representation of a bulk field at the origin commutes with the CFT representations of scalar fields on the boundary at $t = 0$. This would imply the existence of a CFT operators (representation of the bulk field at the origin) which commutes with all CFT operators at a given time which contradicts the time slice axiom of QFT.

However, a bulk field at a boundary point is given by:

$$\lim_{r \to \infty} \phi(r, t, \vec{x}) = r^{-\Delta} O(t, \vec{x})$$

(22)

So from bulk microcausality we have, denoting the CFT representation at the origin as $\phi(0)$

$$\lim_{r \to \infty} r^{-\Delta} [\phi(0), O(t = 0, \theta)] = 0$$

(23)

But this does not imply that $\phi(0)$ commutes with all $O(t = 0, \theta)$ because the $r^{-\Delta}$ term ensures that the expression always vanishes in the boundary limit.

IV. SUMMARY

In this paper we posited that the transformation of AdS/Rindler representations under boundary conformal symmetries is given by (3). Assuming that different boundary representations act equivalently on certain states we provided some evidence for this equation.

Then we assumed (3) to be true and showed that the matching of correlators within boundary states which can be interpreted as low energy excitations on pure AdS follows.

However we have not proven (3). This is left for future work. We also did not explain the apparent paradox for all causal wedges, but only for the ones which are isometric to AdS/Rindler wedges.

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