On supersymmetric Dirac delta interactions

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Abstract

In this paper we construct $\mathcal{N} = 2$ supersymmetric quantum mechanics over several configurations of Dirac delta potentials from one single delta to a Dirac “comb”. We show in detail how the building of supersymmetry on potentials with delta interactions placed in two or more points on the real line requires the inclusion of quasi-square wells. Therefore, the basic ingredient of a supersymmetric Hamiltonian containing two or more Dirac $\delta$s is the singular potential formed by a Dirac $\delta$ plus a step at the same point. In this $\delta/\theta$ SUSY Hamiltonian there is only one singlet ground state of zero energy annihilated by the two supercharges, or, a doublet of ground states paired by supersymmetry of energy positive depending on the relation between the Dirac well strength and the height of the step potential. We find an scenario of either unbroken supersymmetry with Witten index one or supersymmetry breaking when there is one “bosonic” and one “fermionic” ground state such that the Witten index is zero. We explain next the different structure of the scattering waves produced by three $\delta/\theta$ potentials with respect to the the eigenfunctions arising in the non-SUSY case. In particular, much more bound states paired by supersymmetry exist within the supersymmetric framework as compared with the non-SUSY problem. An infinite array of equally spaced $\delta$-interactions of the same strength but alternatively attractive and repulsive are susceptible of being promoted to a $\mathcal{N} = 2$ supersymmetric system. The Bloch’s theorem for wave functions in periodic potentials prompts a band spectrum also paired by supersymmetry. Self-isospectrality between the two partner Hamiltonians is thus found. Zero energy ground states are the non propagating band lower edges, which exist in the spectra of both the two diagonal operators forming the SUSY Hamiltonian. We find that for the SUSY Dirac “comb” the naif Witten index is zero but supersymmetry is unbroken.

1. Introduction

Point or contact interactions modelled by “Dirac $\delta$-potentials” give rise to self-adjoint extensions of free quantum Hamiltonians prompting solvable spectral problems [1][2]. Quantum systems with contact interactions between particles are particularly important in one-dimensional physics giving rise to many-body integrable problems with a very interesting mathematical structure [3][4]. In this paper we focus on the analysis of the quantum dynamics of one-body moving on the real line under the influence of several arrays of Dirac potential attractive or repulsive interactions, the one-particle problem behind the dynamics of many-body Yang-Lieb-Liniger systems. Dirac $\delta$ potentials are interesting also in solid state physics. A Dirac comb formed for infinite $\delta$-walls of identical strength equally spaced, for instance, is the idealization [5] of the Kronig-Penney model [6][7] describing electric conductivity in a periodic crystal.

Rather recently self-adjoint extensions of free quantum Hamiltonians defined over bounded domains have been studied in the context of the vacuum properties of quantum field theories defined over bounded domains.
In this spirit the contact interactions represent quantum boundary conditions in the formalism developed in reference [12] and generalised to quantum field theory in references [8, 9, 10]. Quantum fields interacting with classical backgrounds defined by potentials with compact support play a central role in Casimir physics [13]. Classical background potentials with compact support represent objects with arbitrary shape [14] that distort free wave propagation. Thus, the classical backgrounds provide mathematical idealizations in order to compute the quantum vacuum interaction energy between objects with arbitrary shape from non-relativistic quantum mechanical scattering theory, see e.g. Reference [17] for a review. It is our purpose to analyse the scattering processes induced by several $\delta$ configurations as extreme cases of compact objects. Classical $\delta$ backgrounds offer analytical treatment of scattering data because the good integration properties of Dirac’s $\delta$ distributions.

In particular, we shall study scattering processes of one-particle systems with one degree of freedom moving in several configurations of Dirac $\delta$-potentials in the framework of supersymmetric quantum mechanics. The idea is the posterior use of the results collected in the calculation of quantum vacuum interactions induced by two kinds of fluctuations between two or more compact $\delta$ objects. Wave functions in one-dimensional supersymmetric quantum mechanics are Pauli two-component spinors, offering room to describe two kinds of fluctuations. There is more structure in this kind of systems, particularly the existence of supercharges. Supersymmetric Quantum Mechanics was invented by Witten in the seminal reference [18] with the aim of distinguishing in a framework as simple as possible between systems having singlet ground states of zero energy annihilated by the supercharges and those exhibiting doublet ground states of positive energy paired by the supercharges. The motivation came from supersymmetric field theory models of the fundamental interactions. Elementary particle phenomenology demands that the conjectural supersymmetry in the leptonic and hadronic microscopic world must be be spontaneously broken, see e.g. [19]. This concept is rather well understood in a non-supersymmetric context but in the early steps of supersymmetry it was rather unclear how to interpret supersymmetry breaking in quantum field theory requiring such awkward particles as Goldstone fermions. An important breakthrough, with far reaching consequences both in physics and mathematics, came from the bold idea by Witten [18], further analyzed in [20], and just mentioned above: address the supersymmetry breaking issue in $(0+1)$-dimensional quantum field theory, i.e., quantum mechanics. An enormous quantity of activity in this new field of quantum physics started at the early 90s and new methods of quantum spectral design were developed, see e.g. the textbooks/reviews [19, 21, 22].

The structure of a $\mathcal{N} = 2$ supersymmetric quantum mechanical system is constructed from a pair of “bosonic” canonically conjugated variables, typically the position and momentum operators, supplemented by one “creation” and one “annihilation” fermionic operators space-time independents, see e.g. [23-24]. The key ingredient is the supersymmetry algebra closed by the Hamiltonian and two nilpotent operators called the supercharges:

$$\hat{H} = \hat{Q}^\dagger \hat{Q} + \hat{Q} \hat{Q}^\dagger = \left( \hat{Q}^\dagger + \hat{Q} \right)^2, \quad \hat{Q}^2 = 0 = \left( \hat{Q}^\dagger \right)^2$$

$$\left[ \hat{H}, \hat{Q} \right] = \left[ \hat{H}, \hat{Q}^\dagger \right] = 0,$$

which obviously generates “super”-symmetry transformations of the system. Interactions compatible with this algebra are introduced by means of an auxiliary function of the position called the “superpotential” that determines the potential energy in the supersymmetric Hamiltonian. In this paper we shall address the inverse problem: demanding a certain potential energy in the Hamiltonian, what is the appropriate superpotential? It is well known that one must solve a Riccati equation to build supersymmetric quantum mechanical systems of one degree of freedom encompassing a given interaction.

In particular, the literature on this problem starting from a Dirac $\delta$-interaction abound. We mention, for instance, the papers [25-26, 27, 28], although SUSY Dirac potentials have been selected as pedagogical examples in supersymmetric quantum mechanics in many other references. Quite recently, a curious mixture of polynomial and Dirac deltas has been extended to supersymmetric quantum mechanics in [29], whereas the intriguing concept of hidden bosonic supersymmetry has been shown to arise also for Dirac $\delta$-interactions, [30, 31]. Our goal in this paper is the construction of $\mathcal{N} = 2$ supersymmetric quantum mechanics for configurations of $N \delta$-interactions on a line. The number of bound states in a configuration of $N \delta$ wells is potentially greater than the unique ground state existing in one $\delta$-well depending the strengths and the separation between $\delta$s (see reference [11]). Therefore, it will be convenient to analyse first the bound state dependence on the couplings and the well interdistances in a configuration of $N \delta$s. The reason is that in SUSY Quantum Mechanics the
bound states arise in degenerate in energy pairs linked through the supercharges. Remarkably, we shall show that the bound state pairing requires the appearance of quasi-square wells between the attractive \( \delta \)-interactions such that the number of ground states augments with respect to the non-SUSY case. The general pattern when the \( N \delta \)-interactions are attractive is technically more involved but qualitatively clear: there is always a zero mode, either bosonic or fermionic, and one ever increasing with \( N \) number of bosonic/fermionic pairs of degenerate bound states.

A very specific arrangement of \( \delta \)s is simpler to deal with but rich enough to provide interesting information. An array of \( N \) equally spaced \( \delta \)s of the same strength but alternatively attractive and repulsive admit a supersymmetric formulation without additional wells giving rise to an spectral problem as close as possible to twice the spectrum found in identical non-SUSY array. Only the threshold of the continuous spectrum is displaced to push the ground state energy to zero and in one of the two partner Hamiltonians wells are replaced by walls and viceversa. In this paper we shall study this array in the \( N \to +\infty \) limit. We shall thus address a Dirac comb with alternating white and black teeth, which is accordingly a periodic potential causing a band spectrum that is easily computable. By comparison, the supersymmetric extension of the ordinary Dirac comb, teeth of a single color, is much more complicated. Knowledge of the ground state allows the identification of a superpotential which produces a partner Hamiltonian piecewise formed by Pösch-Teller potentials with \( \delta \)-interactions at the junctions, see [32]. The eigenfunctions of the the partner operators clearly differ. The supersymmetric Lamé system, however, is realized through two partner Hamiltonians exactly isospectral, a situation described in the references [33, 34, 35, 36, 37] and it is precisely this kind of SUSY spectrum what arises in the special array of infinite alternating \( \delta \)-interactions mentioned above. It is curious that the same spectrum emerges in a second-order SUSY extension of the Lamé equation, see [37], and even a two-dimensional version of the same construction, see [38].

The organization of this paper is as follows: Section 2 is devoted to the general formulation of the structure of a dynamical system in \( \mathcal{N} = 2 \) extended supersymmetric quantum mechanics. The definition of the supercharges as first-order differential operators times the Clifford algebra of \( \mathbb{R}^2 \) operators is the chosen representation of the supersymmetry algebra generators. In Section 3 the spectrum of the ensuing supersymmetric Hamiltonian is described for a general superpotential. The relationships are described between the scattering amplitudes of the of the continuous spectrum of the superpartner Hamiltonians. It is also explained how the positive energy bound states are paired through the supercharges but the zero energy ground states are singlets. A regularization of the Witten index due to the continuous spectra is offered. In Section 4 the supersymmetric generalization is studied of one \( \delta \)-potential together with an step potential at the same point as the basic building block of other models to come in the following Sections. It is explained in which circumstances this simple system exhibits spontaneous supersymmetry breaking. Supersymmetric quantum systems of finite arrays of \( \delta \)s and steps are addressed in Sections 5, and 6. Section 7 extend the former calculations to an array of infinite \( \delta \)-potentials such that the subtleties of the interplay between band spectra and supersymmetry are discussed. Finally, we offer a summary and outlook in Section 8.

2. \( \mathcal{N} = 2 \) extended supersymmetric quantum mechanics: Systems with one degree of freedom

2.1. The \( \mathcal{N} = 2 \) supersymmetry algebra

\( \mathcal{N} = 2 \) extended supersymmetric quantum mechanical systems are built from two symmetric quantum operators \( \hat{Q}_i, i = 1, 2 \), referred to as quantum supercharges. The quantum Hamiltonian operator \( \hat{H} \) determining the dynamics of the system is defined from the supercharges through the Heisenberg superalgebra [19]:

\[
\left\{ \hat{Q}_i, \hat{Q}_j \right\} = \delta_{ij} \hat{H}, \quad i, j = 1, 2, \quad [\hat{H}, \hat{Q}_i] = 0, \quad i = 1, 2.
\]

Here we denote respectively by \( \{ \hat{A}, \hat{B} \} = \hat{A}\hat{B} + \hat{B}\hat{A} \) and \( [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \) the anticommutator and the commutator between any two operators \( \hat{A}, \hat{B} \). We stress from these relations that:

1. The \( \hat{Q}_i = \hat{Q}_i^\dagger \) operators are “square roots” of the Hamiltonian.
2. These operators are conserved charges of the dynamics and the generators of a symmetry in the system called supersymmetry. Henceforth the name: supercharges.
It is convenient to combine the two hermitian supercharges in a non-hermitian supercharge and its adjoint in the form: $\hat{Q} = \frac{1}{\sqrt{2}} \left( \hat{Q}_1 + i\hat{Q}_2 \right)$, $\hat{Q}^\dagger = \frac{1}{\sqrt{2}} \left( \hat{Q}_1 - i\hat{Q}_2 \right)$. The supersymmetry algebra, see [39], looks simpler in terms of the new supercharges:

$$\{\hat{Q}, \hat{Q}^\dagger\} = 2\hat{H},$$

$$[\hat{H}, \hat{Q}] = 0, \quad [\hat{H}, \hat{Q}^\dagger] = 0, \quad \{\hat{Q}, \hat{Q}\} = 0 = \{\hat{Q}^\dagger, \hat{Q}^\dagger\},$$

and obviously $\hat{Q}$ and $\hat{Q}^\dagger$ are linear combinations of conserved charges that also generate the supersymmetry algebra. We point out that the important point is the appearance of nilpotent operators: the supercharges.

### 2.2. $\mathcal{N} = 2$ supersymmetric quantum mechanical systems with one “bosonic” degree of freedom

The implementation of the previously described algebraic structure starting from ordinary quantum Hamiltonian systems of one degree of freedom is as follows:

1. In coordinate representation, the position operator $\hat{x} = x$ acts by multiplication on the square integrable functions $f(x) \in L^2(\mathbb{R})$, which form the Hilbert space of the system. The canonically conjugate momentum operator $\hat{p} = -i\hbar \frac{d}{dx}$ acts on the same space of states by derivation. From standard quantum mechanics this irreducible representation of the canonical quantization commutation rules,

$$[\hat{x}, \hat{p}] = 0 = [\hat{p}, \hat{p}] = i\hbar,$$

is unitarily equivalent to any other irreducible representation (Stone-von Neumann theorem). We recall that there are no finite dimensional representations of the Heisenberg algebra [3]. The proof of this statement is easy: if we suppose there is one finite dimensional representation and take traces in both members of the identity on the right, by using the cyclic property of the trace we will find the contradiction $0 = \text{constant} \neq 0$.

2. To build $\mathcal{N} = 2$ supersymmetry on this quantum mechanical Hamiltonian system one thinks of $\hat{x}$ as a “bosonic” operator in a QFT in 0-spatial dimensions. The position operator of a quantum particle moving on a line is accordingly understood as describing a bosonic degree of freedom. The promotion to the $\mathcal{N} = 2$ supersymmetric category of this set up is achieved by adding one “fermionic” degree of freedom characterized by the pair of nilpotent operators $\hat{\psi}$ and $\hat{\psi}^\dagger$: $(\hat{\psi})^2 = (\hat{\psi}^\dagger)^2 = 0$ having physical dimensions of $[\hat{\psi}_k] = M^{-\frac{1}{2}}$ [2]. Nilpotency demands the anticommutation rules

$$\{\hat{\psi}, \hat{\psi}\} = 0 = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\}, \quad \{\hat{\psi}, \hat{\psi}^\dagger\} = \frac{1}{m},$$

between the $\hat{\psi}$ and $\hat{\psi}^\dagger$ operators as the canonical quantization rules, together with $[\hat{x}, \hat{\psi}] = [\hat{x}, \hat{\psi}^\dagger] = 0$, involving the fermionic variables.

From these ingredients the supercharges and the supersymmetric Hamiltonian are defined as follows:

$$\hat{Q} = i\hat{\psi} \left( \hbar \frac{d}{dx} + \frac{dW}{dx} \right) = i\hat{\psi} \hat{D}, \quad \hat{Q}^\dagger = i\hat{\psi}^\dagger \left( \hbar \frac{d}{dx} - \frac{dW}{dx} \right) = i\hat{\psi}^\dagger \hat{D}^\dagger,$$

$$\hat{H} = -\frac{1}{2m} \left( \hbar \frac{d}{dx} + \frac{dW}{dx} \right) \left( \hbar \frac{d}{dx} - \frac{dW}{dx} \right) - m\hbar \hat{\psi}^\dagger \hat{\psi} \frac{d^2W}{dx^2}.$$

Here $W(x): \mathbb{R} \to \mathbb{R}$, a function from the real line to the reals, is the “superpotential”. The reason for the name is obvious: the potential energies, responsible for the different interactions, in $\hat{H}$ are determined from $W(x)$. One can easily check from these definitions and the (3)-(4) quantization rules that $\hat{Q}$, $\hat{Q}^\dagger$, and $\hat{H}$ close the superalgebra [1]-[2].

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1. Of course this terminology has nothing to do with the statistics of an ensemble of several particles. Rather, only the commutation properties of the operators are referred to. Similar considerations apply to the fermionic degree of freedom to come.

2. $[\mathcal{O}]$ denotes the physical dimensions of the $\mathcal{O}$ observable.

3. Dealing with supersymmetric quantum systems with delta interactions we shall use de facto continuous functions having no continuous first derivatives as superpotentials. Their first derivatives will have finite discontinuities at a discrete set of points in such a way that the second derivatives will be distributions of Dirac delta type.
2.3. The fermionic Fock space

The extension of the space of states required by the addition of the fermionic degree of freedom is provided by the two dimensional fermionic Fock space. One starts from the fermionic vacuum state abstractly defined as the state \(|0\rangle\) annihilated by \(\hat{\psi}: \hat{\psi}|0\rangle = 0\). It is thus an eigenstate of the Fermi number operator \(\hat{F} = \hat{\psi}^\dagger \hat{\psi}\) belonging to its kernel: \(\hat{F}|0\rangle = |0\rangle\). In this state the fermionic degree of freedom is unoccupied. The action of the creation operator \(\hat{\psi}^\dagger\) on the vacuum state produces a new state \(|1\rangle = \hat{\psi}^\dagger|0\rangle\) where the fermionic degree of freedom is occupied: \(\hat{F}|1\rangle = 1|1\rangle\). These states are orthogonal to each other and can be properly normalized: \(\langle 0|0\rangle = 1, \langle 0|1\rangle = 0, \langle 1|0\rangle = 0, \) and \(\langle 1|1\rangle = 1\). The Fermionic Fock space, spanned by the \(|0\rangle\) and \(|1\rangle\) states, is isomorphic to \(\mathbb{C}^2 : \mathcal{F} = \{\langle \Psi | = f_0|0\rangle + f_1|1\rangle : f_0, f_1 \in \mathbb{C}\}\).

There is a \(\mathbb{Z}_2\) grading in the Fermionic Fock space determined by the fermionic Klein operator \(\hat{K}_F = (-1)^F\): \(\hat{K}_F |0\rangle = +|0\rangle\) and \(\hat{K}_F |1\rangle = -|1\rangle\). This grading induces a direct sum structure in the Fock space in the form: \(\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1\). Here, the one-dimensional subspace \(\mathcal{F}_0\) spanned by \(|0\rangle\), respectively \(\mathcal{F}_1\) spanned by \(|1\rangle\), is the subspace where the fermionic degree of freedom is unoccupied, respectively occupied. For completeness we also define the bosonic number operator \(\hat{B} = \hat{\psi}^\dagger \hat{\psi}\) acting on the basis states in the form: \(\hat{B}|0\rangle = |0\rangle, \hat{B}|1\rangle = 0|1\rangle\). The corresponding bosonic Klein operator \(\hat{K}_B = (-1)^B\) accordingly labels the basis states: \(\hat{K}_B |0\rangle = -|0\rangle, \hat{K}_B |1\rangle = |1\rangle\).

The total Hilbert space of states of the supersymmetric quantum mechanical system is obtained by tensoring the ordinary space of square integrable functions with the Fermionic Fock space \(\mathcal{S}\mathcal{H} = L_2(\mathbb{R}) \otimes \mathcal{F}\). The structure induced by the grading is inherited by the total space of states \(\mathcal{S}\mathcal{H} = \mathcal{S}\mathcal{H}_0 \oplus \mathcal{S}\mathcal{H}_1\), where \(\mathcal{S}\mathcal{H}_0 = L_2(\mathbb{R}) \otimes \mathcal{F}_0\) and \(\mathcal{S}\mathcal{H}_1 = L_2(\mathbb{R}) \otimes \mathcal{F}_1\). Thus, the wave functions in the supersymmetric system have the general form:

\[
\Psi(x) = \langle f|0\rangle + f_1(x)|1\rangle
\]

where the fermionic degree of freedom is occupied: \(\langle f|0\rangle = 0, \) and \(\langle f|1\rangle = 1\). The equations (4) are realized by the \(2 \times 2\) matrices:

\[
\hat{\psi} = \frac{1}{\sqrt{m}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\psi}^\dagger = \frac{1}{\sqrt{m}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

The Fermi/Bose number operators and their Klein counterparts are respectively:

\[
\hat{F} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{K}_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{K}_B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

The supercharges in (5) become \(2 \times 2\) matrix first-order differential operators and the Hamiltonian is a \(2 \times 2\) diagonal matrix of Schrödinger operators:

\[
\hat{Q} = \frac{i}{\sqrt{m}} \begin{pmatrix} 0 & \hat{D} \\ 0 & 0 \end{pmatrix}, \quad \hat{Q}^\dagger = \frac{i}{\sqrt{m}} \begin{pmatrix} 0 & 0 \\ \hat{D}^\dagger & 0 \end{pmatrix}, \quad \hat{H} = \begin{pmatrix} \hat{H}_0 & 0 \\ 0 & \hat{H}_1 \end{pmatrix}
\]

\[
\hat{H}_0 = -\frac{\hat{D}\hat{D}^\dagger}{2m} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2m} W'' + \frac{\hbar}{2m} W',
\]

\[
\hat{H}_1 = -\frac{\hat{D}^\dagger\hat{D}}{2m} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2m} W'' - \frac{\hbar}{2m} W'.
\]

Finally, the supersymmetric states become simply Pauli spinor wave functions:

\[
\Psi(x) = \langle x|\Psi\rangle = \begin{pmatrix} \psi_0(x) \\ \psi_1(x) \end{pmatrix}.
\]
3. The spectrum of the supersymmetric Hamiltonian

The spectrum of a supersymmetric Hamiltonian $\hat{H}|\Psi_E\rangle = E|\Psi_E\rangle$ is non-negative:

$$E = \frac{\langle \Psi_E|\hat{H}|\Psi_E\rangle}{\langle \Psi_E|\Psi_E\rangle} = \frac{1}{2}\left(\langle \hat{Q}\Psi_E|\hat{Q}\Psi_E\rangle + \langle \hat{Q}^\dagger\Psi_E|\hat{Q}^\dagger\Psi_E\rangle\right) \geq 0 .$$

The energy eigenstates are also eigenstates of the Fermi number operator of two types

$$\hat{F}|\Psi_{E_0}\rangle = \hat{F}(f_{0\Psi(0)}) = 0|\Psi_{E_0}\rangle , \quad \hat{F}|\Psi_{E_1}\rangle = \hat{F}(f_{1\Psi(1)}) = 1|\Psi_{E_1}\rangle , \quad \hat{K}_E|\Psi_{E_0}\rangle = |\Psi_{E_0}\rangle , \quad \hat{K}_E|\Psi_{E_1}\rangle = -|\Psi_{E_1}\rangle$$

that we shall call “bosonic” and “fermionic”.

3.1. Bound states: singlets and doublets

Classified by energies there are two categories of bound states:

1. Zero modes. These states belong to the kernel of the supercharges:

$$\hat{Q}^\dagger|\Psi_{E_0}\rangle = 0 \text{ and/or } \hat{Q}|\Psi_{E_0}\rangle = 0 , \quad E_0 = 0 .$$

It is clear that $\hat{Q}$ also annihilates the bosonic zero modes and $\hat{Q}^\dagger$ does the same with the fermionic zero modes. With only one degree of freedom there are at most two states of this type, each of them forming one singlet state (short multiplet) of the supersymmetry algebra.

2. Pairs of positive energy bound states. The energy potentials

$$V_0(x) = \frac{1}{2m}W'^2 + \frac{\hbar^2}{2m}W'' , \quad \lim_{x \to \pm \infty} V_0(x) = v_\pm^2$$

$$V_1(x) = \frac{1}{2m}W'^2 - \frac{\hbar^2}{2m}W'' , \quad \lim_{x \to \pm \infty} V_1(x) = v_\pm^2$$

may give rise to $j = 1, 2, \cdots , n_b$ positive energy bound eigenstates in the supersymmetric Hamiltonian that come in pairs and form doublets (long multiplets) of the supersymmetry algebra:

$$\hat{H}|\Psi_{E_j}\rangle_0 = \frac{1}{2}\hat{Q}\hat{Q}^\dagger|\Psi_{E_j}\rangle_0 = E_j|\Psi_{E_j}\rangle_0 , \quad |\Psi_{E_j}\rangle_0 = \hat{Q}|\Psi_{E_j}\rangle_1$$

$$\hat{H}|\Psi_{E_j}\rangle_1 = \frac{1}{2}\hat{Q}\hat{Q}^\dagger|\Psi_{E_j}\rangle_1 = E_j|\Psi_{E_j}\rangle_1 , \quad |\Psi_{E_j}\rangle_1 = \hat{Q}^\dagger|\Psi_{E_j}\rangle_0$$

$E_0 = 0 < E_1 < E_2 < \cdots < E_{n_b} , \quad n_b \in \mathbb{N} .$

If $v_\pm^2 = \infty$ the natural number $n_b$ goes to $\infty$, i.e., the spectrum of $\hat{H}$ is purely discrete.

3.2. Supersymmetric scattering amplitudes

If $v_\pm^2 < +\infty$ there is a threshold for scattering eigenstates at $E = \min(v_-, v_+).$ Let us consider a plane wave basis in the completion of $L^2(\mathbb{R})$: $\hat{p}|k\rangle = \hbar k|k\rangle$, $f_k(x) = \langle x|k\rangle = \frac{1}{\sqrt{2\pi}} e^{ikx}$. Defining $\hbar k_\pm = \sqrt{E - v_\pm^2}$ and considering energies $E \geq \max(v_-, v_+)$ the scattering eigenstates of the Hamiltonian have the asymptotic form:

$$|\Psi_{E, 0}\rangle = \begin{cases} |k_-\rangle + \sigma_0(E)|k_+\rangle & , \quad |\Psi_{E, 1}\rangle = \begin{cases} |k_+\rangle + \sigma_1(E)|k_-\rangle & , \quad x < 0 \\ \sigma_0(E)|k_+\rangle & , \quad x > 0 \end{cases} \\
|\Psi_{E, 1}\rangle = \begin{cases} \sigma_1(E)|k_-\rangle & , \quad x < 0 \\ |k_+\rangle + \sigma_0(E)|k_+\rangle & , \quad x > 0 \end{cases}
$$

Corresponding to the right moving waves in the first row and the left moving ones in the second row, both bosonic and fermionic. Except when the threshold lies exactly at $0$, $v_\pm = 0$, these states also form doublets of the supersymmetry algebra:

$$|\Psi_{E, 1}\rangle = \hat{Q}^\dagger|\Psi_{E, 0}\rangle , \quad |\Psi_{E, 1}\rangle = \hat{Q}^\dagger|\Psi_{E, 0}\rangle .$$

(6)
Recalling that the supercharge has the form $\hat{Q}^I = -\hat{\psi}^I (\hat{p} + iW'(x))$ and taking into account \(^\text{[6]}\) the identities between the scattering amplitudes hold \(^\text{[21, 23]}\):

$$
\sigma_1^I(E) = \frac{ihk_+ - v_+}{ihk_- + v_-} \sigma_0^I(E), \quad \rho_1^I(E) = \frac{ihk_- - v_-}{ihk_- + v_-} \rho_0^I(E)
$$

\((7)\)

$$
\sigma_1^I(E) = \frac{ihk_- + v_-}{ihk_+ - v_+} \sigma_0^I(E), \quad \rho_1^I(E) = \frac{ihk_+ + v_+}{ihk_+ - v_+} \rho_0^I(E)
$$

\((8)\)

In the derivation of the formulas \((7)\) and \((8)\) we have assumed very mild conditions for $W(x)$:

$$
\lim_{x \to +\infty} W'(x) = v_+, \quad \lim_{x \to -\infty} W'(x) = -v_-, \quad \lim_{x \to \pm\infty} W''(x) = 0
$$

Therefore, supersymmetry shows itself in the fact that the moduli of the scattering amplitudes are invariant:

$$
|\sigma_0^I(E)| = |\sigma_1^I(E)|; |\rho_0^I(E)| = |\rho_1^I(E)|; |\sigma_0^I(E)| = |\sigma_1^I(E)|; |\rho_0^I(E)| = |\rho_1^I(E)|.
$$

Moreover, as in the non supersymmetric systems there is conservation of probability density fluxes:

$$
\frac{k_+}{k_-} |\sigma_0^I(E)|^2 + |\rho_0^I(E)|^2 = \frac{k_+}{k_-} |\sigma_1^I(E)|^2 + |\rho_1^I(E)|^2 = 1
$$

$$
\frac{k_-}{k_+} |\sigma_0^I(E)|^2 + |\rho_0^I(E)|^2 = \frac{k_-}{k_+} |\sigma_1^I(E)|^2 + |\rho_1^I(E)|^2 = 1
$$

3.3. Spontaneous supersymmetry breaking

The lack of experimental evidence for the existence of degenerate pairs of particle superpartners in nature makes the research of supersymmetry breaking, compelling either explicit or spontaneous, in any supersymmetric field theory candidate to embrace particle physics phenomenology, let us say the supersymmetric standard model or any supersymmetric grand unified model. It was the extremely fertile idea of Witten with the greatest importance in Mathematical Physics to investigate this problem in supersymmetric field theory in zero spatial dimensions, i.e., supersymmetric Quantum Mechanics \(^\text{[18, 20]}\). Witten’s bold observation is the following:

“Spontaneous supersymmetry breaking occurs in any supersymmetric quantum mechanical system having no zero modes in its spectrum”.

If the spectrum of $\hat{H}$ is purely discrete the Witten index $I_W$ is defined as the difference between the number of bosonic zero modes $z_0 = \dim \ker(\hat{H}_0)$ and the number of fermionic zero modes $z_1 = \dim \ker(\hat{H}_1)$:

$I_W = z_0 - z_1$ (see references \(^\text{[20, 19]}\)). When there is spontaneous supersymmetry breaking $I_W = 0^\text{[4]}$. In the case of mixed spectrum -discrete plus continuous- it is convenient to introduce the following regularization of the Witten index:

$$
I_W = \lim_{t \to 0} \text{Tr}_{L^2} \hat{K}_F e^{-i\frac{t}{H}}
$$

where $\hat{K}_F$ is the the fermionic Klein operator and $t$ can be understood as Euclidean time. In particular, if $v_+ = v_- = v$ such that $k_+ = k_- = k$ and $\sigma^I(k) = \sigma^I(k)$, setting $\hbar = 1$, we obtain:

$$
I_W = z_0 - z_1 + \lim_{t \to 0} \int_{-\infty}^{\infty} dk \, (\varrho_0(k) - \varrho_1(k)) \exp \left[-i(t(k^2 + v^2))\right].
$$

The spectral densities are defined from the phase shifts when the system is restricted to a finite interval of very long length $L$ and subjected to periodic boundary conditions in the form:

$$
\varrho_0(k) = \frac{L}{2\pi} + \frac{1}{2\pi} \frac{d\delta_0}{dk}(k), \quad \delta_0(k) = \delta_{0+}(k) + \delta_{0-}(k)
$$

$$
\varrho_1(k) = \frac{L}{2\pi} + \frac{1}{2\pi} \frac{d\delta_1}{dk}(k), \quad \delta_1(k) = \delta_{1+}(k) + \delta_{1-}(k)
$$

\(^4\)The reverse is not true: $I_W = 0$ does not imply that there is spontaneous supersymmetry breaking if $z_0, z_1 \neq 0$. 

The supersymmetric phase shifts are:
\[ \delta_{s\pm}(k) = \frac{1}{2i} \log \left( \sigma_s(k) \pm \sqrt{\rho_s^r(k)\rho_s^l(k)} \right) ; \quad s = 0, 1 \]
whereas the supersymmetric scattering amplitudes are related as follows:
\[ \sigma_1(k) = \frac{ik - v}{ik + v} \sigma_0(k) , \quad \rho_1^r(k) = -\frac{ik - v}{ik + v} \rho_0^l(k). \]

Therefore,
\[ \rho_0(k) - \rho_1(k) = -\frac{1}{2\pi i} \frac{d}{dk} \left( \log \left( \frac{ik - v}{ik + v} \right) \right) = \frac{v}{\pi} \frac{1}{k^2 + v^2} \]

\[ \lim_{t \to 0} \left( \frac{v}{\pi} \int_{-\infty}^{\infty} \exp \left[ -t(k^2 + v^2) \right] \frac{dk}{k^2 + v^2} \right) = \lim_{t \to 0} \left( 1 - \text{Erf}[v\sqrt{t}] \right) = 1 \]

and finally we obtain: \( I_W = z_0 - z_1 + 1 \). In scattering problems the Witten index is shifted by 1 due to the different spectral densities. For periodic potentials, however, the situation is even more intricate.

4. The Dirac \( \delta \)/step potential

4.1. Analysis of the non supersymmetric problem

In ordinary quantum mechanics the Hamiltonian (in a system of units such that \( \hbar = 1, m = \frac{1}{2} \)) for a particle moving on the real line and confronting one step plus a Dirac \( \delta \) potential at the same point (the origin) is:
\[ \hat{H} = -\frac{d^2}{dx^2} + \mu\delta(x) + \frac{g}{2} \varepsilon(x) + \frac{g}{2} \]

Here \( \delta(x) \) and \( \varepsilon(x) \) are respectively the Dirac \( \delta \) and sign distributions, whereas \( \mu \) is a parameter of dimensions \( L^{-1} \) setting the strength of the \( \delta \) potential and \( g \), with dimensions \( L^{-2} \), measures the height of the step. The spectral problem is tantamount to solve the Schrödinger equation
\[ -\psi'' + \left[ \mu\delta(x) + \frac{g}{2} \varepsilon(x) + \frac{g}{2} \right] \psi(x) = E\psi(x), \quad \text{(9)} \]

together with the matching conditions at the origin,
\[ \lim_{x \to 0^+} \psi(x) = \lim_{x \to 0^-} \psi(x) , \quad \lim_{x \to 0^+} \psi'(x) - \lim_{x \to 0^-} \psi'(x) = \mu \lim_{x \to 0^-} \psi(x) \]
\[ \text{(10)} \]

to be held by the scattering solutions.

4.1.1. Scattering waves

The continuity/discontinuity conditions \[ 10 \] define the self-adjoint extension of the free particle Hamiltonian equivalent to the Dirac \( \delta \) potential and the fact that \( k_- = k = \sqrt{E} \) on the negative half-line whereas \( k_+ = p = \sqrt{E - g} \) on the positive half-line is due to the jump in energy at the step \( \varepsilon \)-potential. Together with the scattering wave ansatz (waves incoming towards the \( \delta/\varepsilon \)-potential from the left)
\[ \psi^r(x, E) = \begin{cases} e^{ikx} + \rho^r(E)e^{-ikx} , & x \to -\infty \\ \sigma^r(E)e^{ipx} , & x \to \infty \end{cases} \]

(waves incoming towards the \( \delta/\varepsilon \)-potential from the right)
\[ \psi^l(x, E) = \begin{cases} \sigma^l(E)e^{-ikx} e^{-ipx} + \rho^l(E)e^{ipx} , & x \to -\infty \\ e^{-ikx} , & x \to \infty \end{cases} \]
the matching conditions \([10]\) allow to identify the scattering amplitudes

\[
\begin{align*}
\sigma^r(E) &= \frac{2k}{k + p + i\mu}, & \rho^r(E) &= \frac{k - p - i\mu}{k + p + i\mu} \\
\sigma^l(E) &= \frac{2p}{k + p + i\mu}, & \rho^l(E) &= \frac{-k + p - i\mu}{k + p + i\mu}
\end{align*}
\]

by solving an algebraic linear system of two equations in two unknowns. The conservation of the total probability density flux is manifest:

\[
\frac{p}{k} |\sigma^r(E)|^2 + |\rho^r(E)|^2 = \frac{k}{p} |\sigma^l(E)|^2 + |\rho^l(E)|^2 = 1
\]

4.1.2. Bound/anti-bound states

The poles of the transmission amplitudes \(\sigma^r\) and \(\sigma^l\), the purely imaginary solutions of the equation \(k + p + i\mu = 0 = i(\kappa + \pi + \mu)\), correspond to either bound or anti-bound solutions of the Schrödinger equation \([9]\) of the form

\[
\psi(x) = N \left[e^{i\kappa x} \theta(-x) + e^{-\pi x} \theta(x)\right]
\]

where \(\theta(x)\) is the step Heaviside function if \(k = ik, p = i\pi\) and \(\kappa \in \mathbb{R}, \pi \in \mathbb{R}\). The existence of the bound state is a more subtle problem than in the \(g = 0\) case. Setting, e.g., \(g > 0\) \((g < 0\) yields a completely analogous situation exchanging \(\kappa\) and \(\pi\), there are several possibilities:

1. \(\mu > 0\): The \(\delta\)-potential is repulsive and the right-to-left and left-to-right transmission amplitudes \(\sigma^r\) and \(\sigma^l\) have a unique purely imaginary pole in \(k + p = i(\kappa + \pi) = -i\mu\). From the relation \(\kappa^2 - \pi^2 = -g\) the pole in terms of \(\kappa\) and \(\pi\) can be separately identified: \(\kappa = -\frac{1}{2}(\mu - \frac{g}{p})\) and \(\pi = -\frac{1}{2}(\mu + \frac{g}{p})\). Given that \(\pi < 0\) the associated wave function \([11]\) is non normalizable. Accordingly, this pole corresponds to an anti-bound state.

2. \(\mu < 0, |\mu|^2 > g\): The \(\delta\)-potential is attractive and the pole of \(\sigma^r\) and \(\sigma^l\) is found at: \(\kappa = \frac{1}{2}(|\mu| - \frac{\mu}{|\mu|})\) and \(\pi = \frac{1}{2}(|\mu| + \frac{\mu}{|\mu|})\). Thus, both \(\kappa > 0\) and \(\pi > 0\) are positive and a true (normalizable) bound state corresponds to the pole in this range of the parameters.

If \(\mu < 0\) and \(|\mu|^2 > g\) there exists a bound state of energy \(E = -\kappa^2 = -\frac{1}{4}(|\mu| - \frac{\mu}{|\mu|})^2\) and normalized wave function:

\[
\psi(x) = \sqrt{\frac{|\mu|^2 - g^2}{2|\mu|^3}} \left(e^{\frac{|\mu|^2 - g^2}{2|\mu|}x} \theta(-x) + e^{-\frac{|\mu|^2 - g^2}{2|\mu|}x} \theta(x)\right)
\]

3. \(\mu < 0, |\mu|^2 < g\): The difference is that now \(\kappa = \frac{|\mu|^2 - g}{2|\mu|} < 0\) and the wave function \([12]\) becomes non normalizable (the norm is imaginary). The pole in this range of the parameters gives an anti-bound state.

In summary, there is one bound state if and only if the \(\delta\) potential is an attractive well, \(\mu < 0\), strong enough, \(|\mu|^2 > g\), to overcome the repulsion at the step \(g\).

4.2. \(N = 2\) supersymmetric Dirac \(\delta/\varepsilon\) Hamiltonian

Let us choose

\[
W(x) = \frac{\mu}{2} |x| + \frac{g}{2\mu} x,
\]

as the superpotential. The partner scalar Hamiltonians acting respectively on the sub-spaces of zero \((s = 0)\) and one \((s = 1)\) Fermi number are:

\[
\hat{H}_s \equiv \hat{H}
\]

Assuming \(\mu > 0\) and \(g > 0\) \(V_0\) and \(V_1\) correspond to respectively repulsive and attractive Dirac delta potentials of strength \(\mu\) plus one step potential of height \(g\) shifted over zero energy by the quantity \(\frac{1}{4}(\mu - \frac{g}{\mu})^2\). Changing
the sign of \( \mu \) merely exchanges the repulsive/attractive character of the \( \delta \) potential between \( V_0 \) and \( V_1 \). For fixed \( \mu \) changing the sign of \( g \) the step is reversed from right to left or vice versa. The key point is that the shift in the energy of the scattering threshold with respect to the non-SUSY threshold exactly pushes the energy of the ground state to zero. This is necessary: the spectrum of a super-symmetric Hamiltonian is non-negative. The Schrödinger equations to be solved are:

\[
-\frac{d^2}{dx^2} \psi^{(s)}(x) + V_s \psi^{(s)}(x) = E^{(s)} \psi^{(s)}(x); \quad s = 0, 1,
\]

together with the continuity of \( \psi^{(s)} \) and the discontinuity of \( \frac{d\psi^{(s)}}{dx} \), \( \iota = 0, 1 \), at the \( x = 0 \) point, according to the matching conditions \[10\].

### 4.2.1 Scattering waves

The scattering solutions of these equations are similar to those of the non-SUSY problem. The only difference is in the relations between momenta and energies, which in this case are:

\[
k^2 = E^{(0)} - \frac{1}{4} \left( \mu - \frac{g}{\mu} \right)^2, \quad p^2 = E^{(0)} - \frac{1}{4} \left( \mu + \frac{g}{\mu} \right)^2.
\]

\[
k^2 = E^{(1)} - \frac{1}{4} \left( \mu - \frac{g}{\mu} \right)^2, \quad p^2 = E^{(1)} - \frac{1}{4} \left( \mu + \frac{g}{\mu} \right)^2.
\]

From these identities it is clear that \( E^{(0)} = E^{(1)} = E \) for the scattering solutions as the SUSY algebra demands. The matching conditions \[10\], however, on the the left-to-right and right-to-left waves:

\[
\psi^r_{\iota}(x, E) = (e^{ikx} + \rho^r_{\iota}(E)e^{-ikx}) \theta(-x) + \sigma^r_{\iota}(E)e^{ipx} \theta(x)
\]

\[
\psi^l_{\iota}(x, E) = (e^{-ipx} + \rho^l_{\iota}(E)e^{ipx}) \theta(x) + \sigma^l_{\iota}(E)e^{-ikx} \theta(-x)
\]

\[s = 0, 1\]

give rise to the different scattering amplitudes [7]

\[
\sigma^r_{\iota}(E) = \frac{2k}{k + p + i(-1)^s\mu}, \quad \rho^r_{\iota}(E) = \frac{k - p - i(-1)^s\mu}{k + p + i(-1)^s\mu};
\]

\[
\sigma^l_{\iota}(E) = \frac{2p}{k + p + i(-1)^s\mu}, \quad \rho^l_{\iota}(E) = \frac{-k + p - i(-1)^s\mu}{k + p + i(-1)^s\mu}.
\]

These scattering waves are super-symmetric in the sense that, for example, the left-to-right waves in the two sectors are connected in the form:

\[
\begin{pmatrix} 0 \\ \psi^l_{\iota}(x; E) \end{pmatrix} \propto \hat{Q}^l \begin{pmatrix} \psi^0_{\iota}(x; E) \\ 0 \end{pmatrix} \quad \text{with} \quad \hat{Q}^l = \begin{pmatrix} \frac{d}{dx} - \mu \frac{\varepsilon(x)}{2\mu} - \frac{g}{2\mu} & 0 \\ 0 & \frac{d}{dx} - \mu \frac{\varepsilon(x)}{2\mu} - \frac{g}{2\mu} \end{pmatrix}.
\]

Because,

\[
\frac{d}{dx} - \mu \frac{\varepsilon(x)}{2\mu} - \frac{g}{2\mu} \psi^0_{\iota}(x; E) = \left( \frac{i k + \mu}{2} - \frac{g}{2\mu} \right) \times \\
\times \left[ e^{ikx} + \frac{-ik + \mu}{2k} - \frac{g}{2\mu} \rho^0_{l}(E) e^{-ikx} \right] \theta(-x) + \frac{ip - \mu}{2} - \frac{g}{2\mu} \sigma^0_{l}(E) e^{ipx} \theta(x)
\]

\(\psi^l_{\iota}(x; E)\) and \(\psi^0_{\iota}(x; E)\) are paired by supersymmetry if the scattering amplitudes satisfy:

\[
\rho^r_{\iota} = \frac{-ik + \mu}{2k} - \frac{g}{2\mu} \rho^0_{l}, \quad \sigma^r_{\iota} = \frac{ip - \mu}{2} - \frac{g}{2\mu} \sigma^0_{l}, \quad (14)
\]

[7] The bosonic and fermionic amplitudes only differ in the sign of \( \mu \).
From the explicit form of the scattering amplitudes above it is not difficult to check that the identities \[14\] hold. Even though \(\sigma^0 \neq \sigma^1 \) \( (\sigma^0 \neq \sigma^1) \) and \(\rho^0 \neq \rho^1 \) \( (\rho^0 \neq \rho^1) \) the scattering coefficients of the SUSY partner Hamiltonians are identical:

\[
|\rho_0|^2 = |\rho_1|^2 = 1 - \frac{4kp}{\mu^2 + (k + p)^2} = |\rho^0|^2 = |\rho^1|^2
\]

\[
|\sigma^0_0|^2 = |\sigma^1_0|^2 = \frac{4k^2}{\mu^2 + (k + p)^2} , \quad |\sigma^0_1|^2 = |\sigma^1_1|^2 = \frac{4p^2}{\mu^2 + (k + p)^2} .
\]

We remark that despite \(\sigma^* \neq \sigma^t\) there is probability density flux conservation: \(\frac{k \exp(\pm g \sigma^0)}{k^*} |\sigma|^2 + |\rho|^2 = 1\) in the two sectors.

4.2.2. Bound/anti-bound states

Like in the non SUSY system the pole in the transmission amplitudes of \(\hat{H}_0 \) \(\sigma^0 \) and \(\sigma^1 \) appears in:

\[
\kappa_0 = -\frac{1}{2} \left( \mu - \frac{g}{\mu} \right) , \quad \pi_0 = -\frac{1}{2} \left( \mu + \frac{g}{\mu} \right) .
\]

Because \(\kappa_1 + \pi_1 = \mu\) is the pole of \(\sigma^1 \) and \(\sigma^t\) but still \(\kappa_1^2 - \pi_1^2 = -g\) the separate values of \(\kappa_1\) and \(\pi_1\) at the pole are:

\[
\kappa_1 = \frac{1}{2} \left( \mu - \frac{g}{\mu} \right) , \quad \pi_1 = \frac{1}{2} \left( \mu + \frac{g}{\mu} \right) .
\]

The bound/anti-bound wave functions are respectively:

\[
\psi^{(0)}(x) = \sqrt{\frac{g^2 - \mu^4}{2\mu^3}} \left( \exp \left[ \frac{g - \mu^2}{2\mu} x \right] \theta(-x) + \exp \left[ \frac{\mu^2 + g}{2\mu} x \right] \theta(x) \right)
\]

\[
\psi^{(1)}(x) = \sqrt{\frac{\mu^4 - g^2}{2\mu^3}} \left( \exp \left[ \frac{\mu^2 - g}{2\mu} x \right] \theta(-x) + \exp \left[ -\frac{\mu^2 + g}{2\mu} x \right] \theta(x) \right).
\]

There are four possibilities if we set (with no loss of generality) \(g > 0\):

1. \(\mu^2 > g\).
   - \(\mu < 0\). \(\psi^{(0)}(x)\) is normalizable but \(\psi^{(1)}(x)\) is not. There is a unique ground state that in this case is “bosonic”.
   - \(\mu > 0\). \(\psi^{(1)}(x)\) is normalizable but \(\psi^{(0)}(x)\) is not. There is a unique ground state that in this case is “fermionic”.

The Witten index, the difference between the number of bosonic and fermionic ground states, is one and supersymmetry is not spontaneously broken in these cases.

2. \(\mu^2 < g\).
   - \(\mu < 0\). Neither \(\psi^{(0)}(x)\) nor \(\psi^{(1)}(x)\) are normalizable. All the states in the spectrum of \(\hat{H}_S\) are degenerated in energy and paired through the supercharges.
   - \(\mu > 0\). Again neither \(\psi^{(0)}(x)\) nor \(\psi^{(1)}(x)\) are normalizable. All the states in the spectrum of \(\hat{H}_S\) are degenerated in energy and paired through the supercharges.

There are no zero modes and supersymmetry is spontaneously broken.

Finally, it is obvious that in the \(g \to 0\) limit we recover the Dirac \(\delta\) potential in the supersymmetric version \[26\] \[27\] \[28\] \[30\] \[31\] \[29\].

\[6\] Analogous relations for the left-going amplitudes also hold.
5. The SUSY double Dirac delta potential

In this section we analyse the the supersymmetric double Dirac-δ potential. The analysis to be carried out is the exactly the one carried out for the non-supersymmetric double Dirac-δ in reference [11]. In this reference the hamiltonian

\[ \hat{H} = -\frac{d^2}{dx^2} + V(x) = -\frac{d^2}{dx^2} + \alpha \delta(x + a) + \beta \delta(x - a) \]  

is analysed in order to study the quantum vacuum interaction energy between two Dirac-δ potentials. The steps to follow in the study of the double supersymmetric Dirac-δ potential are:

- Characterise the scattering states of the system, and the corresponding scattering coefficients.
- Compute the scattering matrix and the analytic function that characterises the poles of the transmission coefficient.
- A study of the poles of the transmission coefficient allows to investigate the existence of bound and anti-bound states in terms of the parameters that characterise each of the supersymmetric Dirac-δs. For the case of two non-supersymmetric Dirac-δ potentials reference [11] shows that depending on the values of the parameters \( \alpha a \) and \( \beta a \) defined in the Hamiltonian (15) one can have regions where there are one or two bound states or regions where there are none. Figure 1 shows how these three regions are distributed in the \( \alpha a - \beta a \) plane.

5.1. \( \mathcal{N} = 2 \) supersymmetric double Dirac-δ Hamiltonian: two δs of the same strength

To work the \( \mathcal{N} = 2 \) super-symmetric extension of this system it is convenient to deal separately with the cases of equal and different strength. In the symmetric case we choose the following super-potential:

\[ W(x) = \frac{\alpha}{2} |x + a| + \frac{\alpha}{2} |x - a| , \]

The SUSY partner scalar Hamiltonians are:

\[ \hat{H}_s = -\frac{d^2}{dx^2} + (-1)^s \alpha \delta(x + a) + (-1)^s \alpha \delta(x - a) + \frac{\alpha^2}{2} \epsilon(x + a)\epsilon(x - a) + \frac{\alpha^2}{2} , \] (16)
being $s = 0, 1$ the Fermi number. The Schrödinger equations

$$\hat{H}_0 \psi^{(0)}(x) = E^{(0)} \psi^{(0)}(x); \quad \hat{H}_1 \psi^{(1)}(x) = E^{(1)} \psi^{(1)}(x)$$

must be solved together with the continuity of $\psi^{(i)}$ and the discontinuity of $\frac{d\psi^{(i)}}{dx}$, $i = 0, 1$, at the points $x = \pm a$. These are the same matching conditions described in references [11,2].

### 5.1.1. Scattering waves

Besides of the two $\delta s$ at the points $x = \pm a$, with opposite signs of $\alpha$ respectively for $V_0$ and $V_1$, the partner potentials $V_0(x) = \frac{dV_0}{dx} \frac{d\psi}{dx} + \frac{d^2V_0}{dx^2}$ and $V_1(x) = \frac{dV_1}{dx} \frac{d\psi}{dx} - \frac{d^2V_1}{dx^2}$ exhibit an identical square well: $V_0(x) = V_1(x) = \alpha^2 |x| > a$ (in zones II and III), but $V_0(x) = V_1(x) = 0, |x| < a$ (in zone I). The “right-going” and “left-going” scattering wave functions, both in the bosonic and fermionic sectors, are of the form:

$$\psi^{(i)}(x, E) = \begin{cases}
\frac{e^{ikx} + \rho_i(E)e^{-ikx}}{\sqrt{\alpha}}, & x \in \Pi \\
\frac{A_i(E)e^{iqx} + B_i(E)e^{-iqx}}{\sqrt{\alpha}}, & x \in \Pi \\
\frac{e^{-ikx} + \rho_i(E)e^{ikx}}{\sqrt{\alpha}}, & x \in \Pi
\end{cases}$$

where $k = \sqrt{E^{(i)} - \alpha^2}$ and $q = \sqrt{E^{(i)}}$.

As in the non SUSY case, the system of four linear equations in the four unknowns $\sigma$, $A$, $B$, $\rho$ set by the matching conditions is easy to solve, and we find, e.g., in the bosonic sector the following scattering matrix:

$$S = \begin{pmatrix}
e^{-2ia[(k-q)\alpha]} & e^{-2ia[(k-q)\alpha]} \\
e^{-2ia[(k+q)\alpha]} & e^{-2ia[(k+q)\alpha]}
\end{pmatrix}$$

encoding the scattering amplitudes $\sigma_0 = \sigma_1 = \sigma_0$ and $\rho_0 = \rho_1 = \rho_0$.

The eigenvalues of the unitary $S$ matrix

$$\lambda_{0+} = e^{2i\delta_0+} = \frac{e^{-2iak[(k - i\alpha) \cos qa + i\alpha \sin qa]}}{(k + i\alpha) \cos qa - i\alpha \sin qa}$$

$$\lambda_{0-} = e^{2i\delta_0-} = -\frac{e^{-2iak[(k - i\alpha) \sin qa - i\alpha \cos qa]}}{(k + i\alpha) \sin qa + i\alpha \cos qa}$$

provide the total phase shift in the bosonic sector:

$$\delta_0 = \delta_{0+} + \delta_{0-} = \frac{1}{2i} \ln \left[ \frac{e^{-4i[\alpha + ik]}(q \cos 2qa + ik \sin 2qa)}{(k + i\alpha) \sin 2qa + i\alpha \cos 2qa} \right]$$

where $q = \sqrt{E^{(0)}} = \sqrt{k^2 + \sigma^2}$ for the scattering solutions. Thus, the spectral density for the bosonic scattering waves with periodic boundary conditions on a line of very long length $L$ becomes

$$g_0(k) = \frac{L}{2\pi} \frac{1}{2\pi} \frac{d\delta_0}{dk} = \frac{L}{2\pi} \frac{aq[\alpha^2 - 2a\alpha^3 + k^2(2 - 2a)\alpha] + \alpha^2 \alpha^2 \sin 4qa + \alpha^2 q(\alpha - 2a\alpha^2) \cos 4qa}{2\pi q^3(2\alpha + 2\alpha^2 \cos 4qa + 2k^2)}$$

The scattering amplitudes in the fermionic sector driven by $V_1$ are the same changing $\alpha$ by $-\alpha$. The analogous identities to [14] between the scattering amplitudes in the fermionic and bosonic sectors of the SUSY two-$\delta$ problem are:

$$\rho_1 = \frac{-i\alpha + \alpha}{ik + \alpha} \rho_0, \quad \sigma_1 = \frac{i\alpha - \alpha}{ik + \alpha} \sigma_0$$

meaning that the scattering wave functions in the sectors with different Fermi number are paired through the supercharges. It is easy to check that the bosonic and fermionic right-to-left waves satisfy the same relations changing $\alpha$ by $-\alpha$. Even though $\rho_0 \neq \rho_1$ and $\sigma_0 \neq \sigma_1$, the reflection and transmission coefficients are equal in the bosonic and fermionic sectors: $|\rho_0|^2 = |\rho_1|^2$, $|\sigma_0|^2 = |\sigma_1|^2$. Moreover, there is conservation of the probability density flux: $|\sigma_0|^2 + |\rho_0|^2 = |\sigma_1|^2 + |\rho_1|^2 = 1$ as one can explicitly (but painfully) check with Mathematica.

---

7There would be different labels in each sector: $\psi_0$, $\psi_1$, $\psi_0'$, $\psi_1'$, $E^{(0)}$, $E^{(1)}$, and so on.

8$V_0$ (and $V_1$) is even in the $x \rightarrow -x$ exchange and respect time reversal symmetry.
5.1.2. Bound/anti-bound states

The bound states, the wave functions of energy $0 < E < \alpha^2$, are solutions of the form given by equation (53) in reference [11] with $iq$ instead of $\kappa$ in the zone I ($-a < x < a$). The values of $\kappa$ providing bound/anti-bound solutions are the positive/negative imaginary part of the purely imaginary poles of the transmission amplitudes. Starting with the bosonic sector the purely imaginary poles of $\sigma_0$ are the solutions of the transcendent equation

$$e^{4iaq} (i(\kappa + \alpha) - q)^2 = (i(\kappa + \alpha) + q)^2 , \quad \kappa = \sqrt{\alpha^2 - E(0)}$$

$$e^{4iaq} = \left[ \frac{q^2 - (\alpha + \kappa)^2 + 2iq(\kappa + \alpha)}{q^2 + (\kappa + \alpha)^2} \right] \equiv \tan2aq = \frac{2q(\alpha + \kappa)}{q^2 - (\kappa + \alpha)^2} . \quad (17)$$

Because $\tan(2z) = \frac{2}{\text{cosh}2z - \sin^2z}$ this equation (17) is equivalent to: $\cot qa - \tan qa = \frac{q}{\alpha + \kappa} - \frac{\alpha + \kappa}{q}$. The bound state spectral condition (17) decomposes into two kinds of alternative (simpler) spectral conditions:

$$(e) \quad \sin qa = \frac{\kappa + \alpha}{q} \cos qa , \quad (o) \quad \cos qa = -\frac{\kappa + \alpha}{q} \sin qa . \quad (18)$$

This factorization is due to the fact that there is parity invariance when $\alpha = \beta$ and the wave eigen-functions are either even or odd functions of $x$.

- **Even bound states.** The coefficients of the even bound states solutions in zones II and III must be equal: $A = D$. The wave function in zone I must be even: $C = B$. It is necessary to consider only the Dirac-$\delta$ matching conditions at $x = -a$ (see references [11][2]), i.e., the homogeneous linear system:

$$\begin{pmatrix}
  e^{-\kappa a} & -2q \cos a \\
  (\kappa + \alpha) e^{-\kappa a} & -2q \sin a
\end{pmatrix}
\begin{pmatrix}
  A \\
  B
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0
\end{pmatrix} .$$

It is easy to check that the annihilation of the determinant of this matrix is precisely the transcendent equation (18)(e):

$$\sin \sqrt{E(0)} a = \alpha + \sqrt{\alpha^2 - E(0)} \cos \sqrt{E(0)} a$$

The positive solutions $\kappa_B > 0$ of this equation are the even bound states of $\hat{H}_0$ (bosonic):

$$\psi_{B+1I}^{(0)}(x) = A e^{\kappa_B x} , \quad \psi_{B+1I}^{(0)}(x) = A \left( \frac{e^{-\kappa_B a}}{\cos \kappa_B a} \right) \cos q_B x , \quad \psi_{B+1II}^{(0)}(x) = A e^{-\kappa_B x}$$

- **Odd bound states.** The coefficients of the odd bound states in zones II and III are opposite: $A = -D$. The wave function in zone I must be odd: $C = -B$. The homogeneous linear system becomes:

$$\begin{pmatrix}
  e^{-\kappa a} & -2q \sin a \\
  (\kappa + \alpha) e^{-\kappa a} & 2q \cos a
\end{pmatrix}
\begin{pmatrix}
  A \\
  B
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0
\end{pmatrix} .$$

The annihilation of the determinant of the odd states matrix is the other transcendent equation (18)(o):

$$\cos \sqrt{E(0)} a = -\alpha + \sqrt{\alpha^2 - E(0)} \sin \sqrt{E(0)} a$$

The positive solutions $\kappa_B > 0$ of this equation are the odd bound states of $\hat{H}_0$ (bosonic):

$$\psi_{B-1I}^{(0)}(x) = A e^{\kappa_B x} , \quad \psi_{B-1I}^{(0)}(x) = A \left( \frac{e^{-\kappa_B a}}{\sin \kappa_B a} \right) \sin q_B x , \quad \psi_{B-1II}^{(0)}(x) = -A e^{-\kappa_B x}$$
Table 1: Graphics of the curves in the two members of the spectral equations (18). The particular values $\alpha = 7$ and $\alpha = 2$ have been selected for the plots. In the left column the curves describing the two members of (18)(e) for $\alpha = 2$ (upper box, $F = (\sqrt{4 - E} + 2) \cos \sqrt{E}$, $G = \sqrt{E} \sin \sqrt{E}$, henceforth identifying the bosonic even bound states) and (18)(o) for $\alpha = -2$ (lower box, $F = (\sqrt{4 - E} - 2) \sin \sqrt{E}$, $G = \sqrt{E} \cos \sqrt{E}$, henceforth corresponding to the odd fermionic bound states) are depicted as functions of $E$. Therefore, the values of $E$ for which the curves intersect are the even bound state eigenvalues of $\hat{H}_0$ and the odd bound state eigenvalues of $\hat{H}_1$. Because the cuts happens between 0 and 4 the corresponding values of $\kappa_B^{(0)}$ and $\kappa_B^{(1)}$ are between 0 and 2. Observe that in all the cases $\kappa_B^{(0)} = \kappa_B^{(1)} > 0$ as it should be in a supersymmetric spectrum. In the right column we plot the curves in (18)(e) for $\alpha = -2$ (lower box, $F = (\sqrt{4 - E} - 2) \cos \sqrt{E}$, $G = \sqrt{E} \sin \sqrt{E}$, henceforth describing the fermionic even bound states) and (18)(o) for $\alpha = 2$ (upper box, $F = (\sqrt{4 - E} - 2) \sin \sqrt{E}$, $G = \sqrt{E} \cos \sqrt{E}$, henceforth identifying the bosonic odd bound states). We check that the only zero mode is fermionic, see the intersection of the two curves at $E^{(1)}_\lambda = 0$ in the lower right box. On physical grounds, recall that in this case we assigned $\alpha = 2 > 0$ to the repulsive two-$\delta$ potential in the $\hat{H}_0$ Hamiltonian.
The spectral equations only can be solved by graphic methods (see Figures in Table 1).

The positive values of $\kappa$ where the curves in the left and right members of (18)(e) and (18)(o) intersect provide the bosonic even and odd bound states respectively. It is clear that the even spectral condition admit a solution $\kappa_0 = -\alpha$, $q_0 = 0 = E_{0+}^{(0)}$ that is a bona fide ground state if $\alpha < 0$. This (always even) bound state is the bosonic ground state. Regardless the sign of $\alpha$, the odd spectral condition presents no zero energy bound state because the cotangent at zero angle is infinity. Nevertheless, the first intersection in (18)(o) is the next bound state in energy $E_{1-}^{(0)}$, there is then an even bound state and so on: $0 = E_{0+}^{(0)} < E_{1-}^{(0)} < E_{2+}^{(0)} < \cdots < \alpha^2$.

Regarding the positive fermionic bound states of $\hat{H}_1$ everything is the same if one changes $\alpha$ by $-\alpha$. Because

$$
\tan \sqrt{E^{(1)}_a} = -\frac{-\alpha + \sqrt{\alpha^2 - E^{(1)}_a}}{\sqrt{E^{(1)}_a}} = \cot \sqrt{E^{(1)}_a} = -\frac{\alpha + \sqrt{\alpha^2 - E^{(1)}_a}}{\sqrt{E^{(1)}_a}}
$$

$$
\cot \sqrt{E^{(1)}_a} = -\frac{-\alpha + \sqrt{\alpha^2 - E^{(1)}_a}}{\sqrt{E^{(1)}_a}} = \tan \sqrt{E^{(1)}_a} = -\frac{\alpha + \sqrt{\alpha^2 - E^{(1)}_a}}{\sqrt{E^{(1)}_a}}
$$

$E^{(1)}_{n+1} = E^{(0)}_{(n+1)-}$, $E^{(1)}_{n+2} = E^{(0)}_{(n+2)+}$, $n = 0, 1, 2, \cdots$ and the fermionic and bosonic bound states are paired by supersymmetry. Note that the supercharges necessarily change the parity:

$$
\hat{Q}^\dagger \begin{pmatrix} \psi_B^{(0)}(x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_B^{(1)}(x) \end{pmatrix}, \quad \hat{Q} \begin{pmatrix} 0 \\ \psi_B^{(1)}(x) \end{pmatrix} = \begin{pmatrix} \psi_B^{(0)}(x) \\ 0 \end{pmatrix}.
$$

$\hat{H}_0$ and $\hat{H}_1$, however, are not completely isospectral. The zero energy ground state is either bosonic, if $\alpha < 0$, or fermionic, when $\alpha$ is positive. Note that $\kappa_0 = -\alpha \equiv E_0^{(0)} = 0$ whereas $\kappa_1 = \alpha \equiv E_0^{(1)} = 0$. Thus, one of the two zero modes is an anti-bound state because either $\kappa_0$ or $\kappa_1$ is negative. The Witten index is one\footnote{In fact, there is an infinite contribution to the Witten index due to the continuous spectrum. Conveniently regularized, the difference between the bosonic and fermionic spectral densities induces the result: $I_W = 2$.} and supersymmetry is not spontaneously broken. The ground states (zero energy) belong simultaneously to the kernels of $\hat{Q}$ and $\hat{Q}^\dagger$ being the exponential of either minus (bosonic) or plus (fermionic) the superpotential. The key fact is that the ground states are singlets, not paired states, of the supersymmetry algebra. We choose now $\alpha > 0$ to describe explicitly stationary wave functions. The unique zero energy $q_0 = E^{(1)} = 0$, $\kappa_0 = \alpha$ (normalized) ground state is fermionic and even (see Figure 2):

$$
\psi_0^{(1)}(x) = N \exp \left[ -\frac{\alpha}{2} |x - a| - \frac{\alpha}{2} |x + a| \right] = \sqrt{\frac{\alpha}{1 + 2\alpha a}} \begin{cases} 
 1 & \text{if } x \in \Pi \\
 e^\alpha(x+a) & \text{if } x \in \Pi \\
 e^{-\alpha(x-a)} & \text{if } x \in \Pi 
\end{cases},
$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fermionic_ground_state.png}
\caption{The fermionic ground state wave function for $a = 7$ and $\alpha = 2$.}
\end{figure}

The remaining non zero energies of the bosonic and fermionic bound states $E^{(0)} = E^{(1)} \neq 0$ cannot be given analytically because they are graphically determined as the intersections of the curves in the two members of the transcendent equations (18)(e)-(o). From the $n$-th intersection values $\kappa_n = \sqrt{\alpha^2 - E_n^{(0)}}$, $q_n = \sqrt{E_n^{(0)}}$, one
obtains:
\[
\psi_{n+}^{(0)}(x) = \sqrt{\frac{\kappa_n}{1 + \kappa_n a + \frac{\alpha}{q_n} \sin q_n a \cos q_n a}} \begin{cases} 
\cos q_n a \ e^{\kappa_n (x+a)} , & x \in \text{II} \\
\cos q_n x , & x \in \text{I} \\
\cos q_n a \ e^{-\kappa_n (x-a)} , & x \in \text{III}
\end{cases}
\]
\[
\psi_{n-}^{(0)}(x) = \sqrt{\frac{\kappa_n}{1 + \kappa_n a + \frac{\alpha}{q_n} \sin q_n a \cos q_n a}} \begin{cases} 
-\sin q_n a \ e^{\kappa_n (x+a)} , & x \in \text{II} \\
\sin q_n x , & x \in \text{I} \\
\sin q_n a \ e^{-\kappa_n (x-a)} , & x \in \text{III}
\end{cases}
\]

The corresponding fermionic bound state wave functions are easily obtained: \( \psi_{n}^{(1)}(x) = \hat{Q}^{\dagger} \psi_{n+}^{(0)}(x), \psi_{n}^{(1)}(x) = \hat{Q}^{\dagger} \psi_{n-}^{(0)}(x) \). We recall that the hierarchy of bound states is: \( E_{0+}^{(1)} = 0 < E_{1+}^{(1)} < E_{2-}^{(1)} = E_{2+}^{(1)} < \cdots \).

The secret reason of the need of one square well to be added to the two \( \delta s \) in order to achieve a supersymmetric quantum mechanical system can be explained as follows. The analogous factorization of the spectral condition into the two (18)-(e)-(o) equations in the non-SUSY two \( \delta \) problem is:

\[
e^{-2a\sqrt{|E|}} = 1 - \frac{2\sqrt{|E|}}{\alpha}, \quad \text{and} \quad e^{-2a\sqrt{|E|}} = 1 + \frac{2\sqrt{|E|}}{\alpha}.
\]

If \( |E_0| \) and \( |E_1| \) are respectively the solutions of (21)-(e) and (21)-(o) then \( |E_1| < |E_0| \) which means that the even bound state has less energy than the odd one because both bound states are of negative energy. Therefore, the role of the well is to push the deepest bound state energy to zero.

In Table 2 several graphics of the wave eigenfunctions of \( \hat{H}_0(x) \) and \( \hat{H}_1(x) \) as well as their eigenvalues identified graphically are shown.

| Table 2: The three lower bosonic and fermionic paired bound state wave functions for \( a = 7 \) and \( \alpha = 2 \) |
|---|---|---|
| \( E_{1+}^{(0)} \) | \( E_{2+}^{(0)} \) | \( E_{3+}^{(0)} \) |
| 0.0469 | 0.187 | 0.421 |

5.2. \( \mathcal{N} = 2 \) supersymmetric two Dirac-\( \delta \) Hamiltonian: two \( \delta s \) of different strength

If the two \( \delta s \) are of different strengths we choose the superpotential:

\[ W(x) = \frac{\alpha}{2} |x + a| + \frac{\beta}{2} |x - a| - \frac{1}{2}(\alpha - \beta)x; \quad v_- = -\alpha, \quad v_+ = \beta \]

the reason for this selection will be clear later. The intertwined SUSY partner Hamiltonians are

\[
\hat{H}_s = -\frac{d^2}{dx^2} + (-1)^s \alpha \delta(x + a) + (-1)^s \beta \delta(x - a) + \frac{\alpha \beta}{2}(x + a)(x - a) \]
\[
- (\alpha - \beta) \left[ \frac{\alpha}{2} \delta(x + a) + \frac{\beta}{2} \delta(x - a) \right] + \frac{\alpha^2}{2} + \frac{\beta^2}{2} - \frac{\alpha \beta}{2}
\]
where $s = 0, 1$ is the Fermi number. The potential energies $V_0(x)$ and $V_1(x)$ are composed of the two $\delta s$ plus one quasi-square well:

$$V_s(x) = (-1)^s \alpha \delta(x + a) + (-1)^s \beta \delta(x - a) + \begin{cases} 
\alpha^2, & x \in \Pi \\
0, & x \in \Omega \\
\beta^2, & x \in \Omega
\end{cases}, \quad s = 0, 1.$$  

The momenta of the wave functions in the three zones are accordingly different: Zone II, $k = \sqrt{E^{(i)} - \alpha^2}$. Zone I, $q = \sqrt{E^{(i)}}, E^{(i)}$. Zone III, $p = \sqrt{E^{(i)} - \beta^2}, E^{(i)}$. We assume without loss of generality that $\alpha^2 < \beta^2$. Note the similarity with the $\delta$-step potential. There are three ranges of energies:

5.2.1. Double degenerate scattering waves

If $E^{(i)} > \beta^2$ there are left-to-right and right-to-left scattering wave solutions of the bosonic and fermionic Schrödinger equations:

$$\psi^{(i)}(x, E) = \begin{cases}
e^{ikx} + \rho_1^i(E)e^{-ikx}, & x \in \Pi \\
A_1^i(E)e^{iqx} + B_1^i(E)e^{-iqx}, & x \in \Omega \\
\sigma_1^i(E)e^{ipx}, & x \in \Omega
\end{cases}, \quad \psi^{(i)}(x, E) = \begin{cases}
\sigma_1^i(E)e^{-ikx}, & x \in \Pi \\
A_1^i(E)e^{iqx} + B_1^i(E)e^{-iqx}, & x \in \Omega \\
e^{-ipx} + \rho_1^i(E)e^{ipx}, & x \in \Omega
\end{cases}$$

Solving the linear system which arise from imposing the matching conditions on the scattering waves we find the scattering amplitudes:

$$\rho_0^i(E) = \frac{e^{-2iak} [e^{4iaq(k + q - i\alpha)}(-p + q - i\beta) + (k - q - i\alpha)(p + q + i\beta)]}{\Delta_0(E, \alpha, \beta, a)}$$

$$\rho_1^i(E) = \frac{e^{-2iap} [e^{4iaq(-k + q + i\alpha)}(p + q - i\beta) + (k + q + i\alpha)(p - q - i\beta)]}{\Delta_0(E, \alpha, \beta, a)}$$

$$\sigma_0^i(E) = \frac{4ke - ia(k + p - 2q)}{\Delta_0(E, \alpha, \beta, a)}, \quad \sigma_1^i(E) = \frac{4pqe - ia(k + p - 2q)}{\Delta_0(E, \alpha, \beta, a)}$$

$$\Delta_0(E, \alpha, \beta, a) = (k + q + i\alpha)(p + q - i\beta) - e^{4iaq}(k - q + i\alpha)(p - q - i\beta)$$

We stress that the zone III momentum is $p$ as the only difference with respect to the symmetric case where it is identical to the zone I momentum $k$.

Since the fermionic scattering amplitudes are obtained from the bosonic ones merely changing $\alpha$ by $-\alpha$ and $\beta$ by $-\beta$, the bosonic scattering coefficients (of $\hat{H}_0$) are identical to the fermionic scattering coefficients (of $\hat{H}_1$). In fact, the following identities between the bosonic and fermionic scattering amplitudes hold:

$$\rho_1^i(E) = \frac{-ik + \alpha}{ik + \alpha} \rho_0^i(E), \quad \sigma_1^i(E) = \frac{ip - \beta}{ik + \alpha} \sigma_0^i(E),$$

and the corresponding ones between the right-to-left amplitudes. Accordingly, the transmission and reflection bosonic and fermionic probabilities are equal: $|\rho_0^i|^2 = |\rho_1^i|^2$ and $|\sigma_0^i|^2 = |\sigma_1^i|^2$, $|\rho_1^i|^2 = |\rho_1^i|^2$ and $|\sigma_0^i|^2 = |\sigma_1^i|^2$. Also, we have: $\Delta_1(E, \alpha, \beta, a) = \Delta_0(E, -\alpha, -\beta, a)$.

Only left-to-right scattering waves. When $\alpha^2 < E^{(i)} < \beta^2$ the momentum in the zone III becomes purely imaginary: $p = i\sqrt{\beta^2 - E^{(i)}}$. Hence there are no incoming right-to-left waves but the left-to-right waves are partly reflected and the transmitted waves decay exponentially.

5.2.2. Bound/anti-bound states

For energies such that $0 \leq E^{(i)} \leq \alpha^2$, corresponding to purely imaginary momenta $k^{(i)} = i\kappa^{(i)} = i\sqrt{\alpha^2 - E^{(i)}}$ and $\rho^{(i)} = i\kappa^{(i)} = i\sqrt{\beta^2 - E^{(i)}}$ (positive imaginary part: $\kappa^{(i)}, \kappa^{(i)} > 0$), the poles of the scattering amplitudes occur at the bound state energies. The poles are obviously the zeroes of $\Delta_1(E, \alpha, \beta, a)$. Henceforth in the bosonic sector, solving for $E$ the spectral equation

$$\Delta_0(E^{(0)}, \alpha, \beta, a) = 0 \equiv \frac{q(\kappa^{(0)} + \pi^{(0)} + \alpha + \beta)}{q^2 - (\kappa^{(0)} + \alpha)(\pi^{(0)} + \beta)} \cos 2qa = \sin 2qa$$

(22)
we obtain the bound state energies of $\hat{H}_0$. With the required changes, in the fermionic sector the solutions in $E$ of the spectral equation

$$\Delta_1(E^{(1)}, \alpha, \beta, a) = 0 \equiv \frac{q(\kappa^{(1)} + \pi^{(1)} - \alpha - \beta)}{q^2 - (\kappa^{(1)} - \alpha)(\pi^{(1)} - \beta)} \cos 2qa = \sin 2qa.$$ (23)

are the bound state eigenvalues.

Since

$$\frac{q(\sqrt{\alpha^2 - E} + \sqrt{\beta^2 - E} \pm (\alpha + \beta))}{q^2 - (\sqrt{\alpha^2 - E} \pm \alpha)(\sqrt{\beta^2 - E} \pm \beta)} = -\frac{\beta \sqrt{\alpha^2 - E} - \alpha \sqrt{\beta^2 - E}}{(\alpha - \beta)q}$$

for arbitrary $\alpha$ and $\beta$, the solutions for $E^{(0)}$ and $E^{(1)}$ of the transcendent equations (22) and (23) are the same. As expected, there is pairing between the bound states of the intertwined operators $\hat{H}_0(x)$ and $\hat{H}_1(x)$, see the Table 3.

Table 3: Graphics of the curves in the spectral equations (22) (left) and (23) (right) for $a = 7$, $\alpha = 2$ and $\beta = 4$. The intersection points give the bound state eigenvalues for $\hat{H}_0$ (left, $F = [x - (\sqrt{4 - x} - 2)(\sqrt{16 - x} + 4)] \sin 14\sqrt{x}$, $G = (\sqrt{4 - x} + \sqrt{16 - x} - 6) \sqrt{x} \cos 14\sqrt{x}$) and $\hat{H}_1$ (right, $F = [x - (\sqrt{4 - x} - 2)(\sqrt{16 - x} - 4)] \sin 14\sqrt{x}$, $G = (\sqrt{4 - x} + \sqrt{16 - x} - 6) \sqrt{x} \cos 14\sqrt{x}$). We remark that $E_B^{(0)} = E_B^{(1)} = x$.

![Graphs of spectral equations](image)

Both $\hat{H}_0$ and $\hat{H}_1$ have a zero energy eigenstate, as one can check taking the $E \to 0$ limit of the spectral equations. Nevertheless, for the particular choice where $\alpha$ and $\beta$ are positive, the wave eigenfunction in the kernel of $\hat{H}_0$ is not normalizable because $W \to +\infty$ in the $x \to \pm \infty$ limits. The zero energy wave function of $\hat{H}_1$, however, is normalizable: $\psi^{(1)}_0(x) \propto e^{-W(x)}$ tends to zero at both ends of the real line.

In fact, the fermionic zero mode wave function properly normalized is\(^{10}\)

$$\psi^{(1)}_0(x) = \sqrt{\frac{2\alpha}{\alpha + 4\alpha \beta + \beta^2}} \begin{cases} e^{\alpha(x+a)} & x \in \Pi \\ 1 & x \in \Pi \\ e^{-\beta(x-a)} & x \in \Pi \end{cases},$$

see the Figure 3 where the graphic of $\psi^{(1)}_0$ for $a = 7$, $\alpha = 2$, $\beta = 4$ is plotted. For energies in the range $0 < E^{(0)} < \alpha^2$ we label as $\kappa_n$, $\pi_n$, $q_n$ the momenta in the three zones corresponding to the bound state energies that solve the transcendent bosonic spectral equation. The normalized bosonic wave functions are:

$$\psi^{(0)}_n(x) = \sqrt{\frac{\kappa_n}{\kappa_n^2 + \kappa_n(m_n + \frac{2\pi n}{\kappa_n})}} \begin{cases} (\cos 2q_n a + \frac{\pi_n - \beta}{q_n} \sin 2q_n a) e^{\kappa_n(x+a)} & x \in \Pi \\ \cos q_n(x-a) - \frac{\pi_n - \beta}{q_n} \sin q_n(x-a) & x \in \Pi \\ e^{-\pi_n(x-a)} & x \in \Pi \end{cases}$$

\(^{10}\)Recall the redefinitions: $2m E \to E$ and $2m V(x) \to V(x)$
where \( r_n \) and \( m_n \) are defined as:

\[
r_n = \left( \cos 2q_n a + \frac{\pi_n - \beta}{q_n} \sin 2q_n a \right)^2,
\]

\[
m_n = \frac{2q_n a \left[ q_n^2 + (\pi_n - \beta)^2 \right]}{2q_n^3} + \sin 2q_n a \left\{ \left[ q_n^2 - (\pi_n - \beta)^2 \right] \cos 2q_n a + 2(\pi_n - \beta)q_n \sin 2q_n a \right\}.
\]

Again, the fermionic bound state wave functions are obtained by applying the supercharge to the bosonic ones:

\[\psi_n^{(1)}(x) = \hat{Q}^\dagger \psi_n^{(0)}(x).\]

In the Table 4 plots of the lower bound state wave functions of the Hamiltonians \( \hat{H}_0 \) and \( \hat{H}_1 \) are shown. The bound state energy eigenvalues are determined by graphically solving respectively (22) and (23).

### Table 4: The lower three paired bound state wave functions of \( \hat{H}_0 \) and \( \hat{H}_1 \) with positive energy for \( a = 7, \alpha = 2 \) and \( \beta = 4 \)

|     | \( E_n^{(0)} \) | \( E_n^{(1)} \) |
|-----|-----------------|-----------------|
| \( E_1^{(0)} \) | 0.0478          | 0.0478          |
| \( E_2^{(0)} \) | 0.191           | 0.191           |
| \( E_3^{(0)} \) | 0.429           | 0.429           |

### 6. The triple Dirac-\( \delta \) potential

In this section we will study in detail the SUSY quantum mechanical system of three Dirac \( \delta \)s. This system provides the necessary information to understand any SUSY configuration with a finite number of Dirac \( \delta \)s and also establishes basis to deal with a system of infinite \( \delta \)s. From a quantum field theoretical point of view the system of three supersymmetric Dirac \( \delta \)s can be interpreted as a SUSY version of the piston geometries modelled by point interactions, see refs. [40, 41], as well as supersymmetric Kaluza-Klein field theories when these theories are interpreted as field theories defined over piston geometries as it was done in ref. [42].
6.1. The non-supersymmetric triple \( \delta \) potential

The Hamiltonian is:

\[
\hat{H} = -\frac{d^2}{dx^2} + V(x) , \quad V(x) = \alpha \delta(x + a) + \mu \delta(x) + \beta \delta(x - a) .
\]

Scattering and bound state eigenfunctions of the spectral problem \( \hat{H}\psi(x; E) = E\psi(x; E) \) complying with the three pairs of matching conditions

\[
\begin{align*}
\psi(-a-; E) &= \psi(-a+; E) , & \psi'(-a+; E) - \psi'(-a-; E) &= \alpha \psi(-a+; E) \\
\psi(0-; E) &= \psi(0+; E) , & \psi'(0+; E) - \psi'(0-; E) &= \mu \psi(0+; E) \\
\psi(a-; E) &= \psi(a+; E) , & \psi'(a+; E) - \psi'(a-; E) &= \beta \psi(a+; E)
\end{align*}
\]

(24)

are the zeroes of the determinant characterized as the solutions of the transcendental equation

\[
\Delta(x; \alpha, \beta, \mu) = 0 \equiv g(\kappa; \alpha, \beta, \mu, a) = f(\kappa; \alpha, \beta, \mu) , \quad \kappa = \sqrt{-E} ,
\]

(25)

where the families of curves in \( \mathbb{R}^3 \) entering in the equation \( [25] \) are:

\[
\begin{align*}
f(\kappa; \alpha, \beta, \mu) &= \left( \frac{2}{\alpha} \kappa + 1 \right) \left( \frac{2}{\beta} \kappa + 1 \right) \mu , & g(\kappa; \alpha, \beta, \mu, a) &= e^{-4\alpha} (2\kappa - \mu) + 2\mu e^{-2\alpha} \left( \frac{\kappa}{\alpha} + \frac{\kappa}{\beta} + 1 \right).
\end{align*}
\]
We stress that both curves intersect at the origin:

\[ f(0; \alpha, \beta, \mu) = g(0; \alpha, \beta, \mu, a) = \mu, \forall \alpha, \beta, \mu, a. \]

As in the two \( \delta \) problem, in order to study the intersections of the families of curves in equation (25) it is convenient to compare their derivatives at the origin:

\[
\begin{align*}
    g'(0; \alpha, \beta, \mu, a) &= \left( \frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\mu} \right) \mu = f'(0; \alpha, \beta, \mu) \\
    g''(0; \alpha, \beta, \mu, a) &= -8\mu \left[ a^2 + a \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{\mu} \right) \right], \quad f''(0; \alpha, \beta, \mu) = \frac{8\mu}{\alpha\beta}(\alpha + \beta + \mu).
\end{align*}
\]

The number of solutions of (25) on the positive \( \kappa \)-axis depends on the inequalities: \( g''(0; \alpha, \beta, \mu, a) > f''(0; \alpha, \beta, \mu) \) and \( g''(0; \alpha, \beta, \mu, a) < f''(0; \alpha, \beta, \mu) \).

Thus, the equation

\[
g''(0; \alpha, \beta, \mu, a) = f''(0; \alpha, \beta, \mu) \equiv a^2 + \frac{\mu(\alpha + \beta) + 2\alpha\beta}{\alpha\beta} a + \frac{\alpha + \beta + \mu}{\alpha\beta} = 0 \quad (26)
\]

determines a three branched surface dividing the \((\alpha, \beta, \mu)\) parameter space \(\mathbb{R}^3\) in four regions: no bound states, one bound state, two bound states, and three bound states, according to the signs and strengths of the \(\delta\)s and the distance between them. In fact, fixing a value of \(a\), the two surfaces in \(\mathbb{R}^3\)

\[
a_\pm = -\left( \frac{\alpha + \beta}{2\alpha\beta} + \frac{1}{\mu} \right) \pm \sqrt{\frac{1}{\mu^2} + \frac{(\alpha - \beta)^2}{4\alpha^2\beta^2}} \quad (27)
\]

corresponding to the two roots \(a_\pm\) of the quadric curve (26), are the frontier between the different regimes.

6.1.3. **Bound states of three equal \( \delta \)-wells**

A completely explicit description of the discrete spectrum is possible in the case of three equal \( \delta \)-wells. Because the Hamiltonian is invariant under the \( x \to -x \) reflection the eigenfunctions are either even or odd function of \( x \). The bound state wave functions

\[
\psi_B(x) = \begin{cases} 
    A(\kappa)e^{\kappa x} & , \ x \in \text{III} \\
    B(\kappa)e^{\kappa x} + C(\kappa)e^{-\kappa x} & , \ x \in \text{I} \\
    D(\kappa)e^{\kappa x} + F(\kappa)e^{-\kappa x} & , \ x \in \text{II} \\
    G(\kappa)e^{-\kappa x} & , \ x \in \text{IV}
\end{cases}
\]

are thus of two classes.

1. **Even bound states**: \( A = G, B = F, \) and \( C = D \). The homogeneous linear system implementing the matching conditions is given by:

\[
\begin{pmatrix}
    e^{-\kappa a} & -e^{-\kappa a} & -e^{\kappa a} \\
    (\kappa + \mu)e^{-\kappa a} & -\kappa e^{-\kappa a} & \kappa e^{\kappa a} \\
    0 & -2\kappa + \mu & -2\kappa + \mu
\end{pmatrix}
\begin{pmatrix}
    A \\
    B \\
    C
\end{pmatrix}
= \begin{pmatrix}
    0 \\
    0 \\
    0
\end{pmatrix} \quad (28)
\]

There are non-trivial solutions to (28) only if the determinant of this matrix

\[
\det \Delta (i\kappa) = \mu e^{-2\kappa}(\mu - 2\kappa) - (2\kappa + \mu)^2
\]

is zero, i.e., for the solutions of the transcendent equation:

\[
\mu e^{-2\kappa}(\mu - 2\kappa) = (2\kappa + \mu)^2.
\]

The solutions of this equation are the intersections of the curves \( g(\kappa, \mu, a) = \mu(\mu - 2\kappa)e^{-2\kappa a} \) and \( f(\kappa, \mu) = (2\kappa + \mu)^2 \) for \( \kappa > 0 \). Because \( g'(0, \mu, a) = -2\mu(1 + a\mu) \) and \( f'(0, \mu) = 4\mu \) the tangents at the origin of these two curves are equal iff: \( \mu = \frac{-3}{a} \). Thus, there are no \( \kappa > 0 \) intersection if \( \mu > 0 \), one \( \kappa > 0 \) intersection if \( \mu > \frac{-3}{a} \), and two positive intersections if \( \mu < -\frac{3}{a} \). In summary, there are no even bound state on the positive half-line \( 0 < \mu < +\infty \), one even bound state in the interval \( -\frac{3}{a} < \mu < 0 \), and two even bound states in the negative range \( -\infty < \mu < -\frac{3}{a} \).
2. **Odd bound states** \( A = -G, B = -F; \) and \( C = -D. \) The \(^{\text{odd}}\) homogeneous linear system reads:

\[
\begin{pmatrix}
  e^{-\kappa a} & -e^{-\kappa a} & -e^{\kappa a} \\
  (\kappa + \mu)e^{-\kappa a} & -\kappa e^{-\kappa a} & \kappa e^{\kappa a} \\
  0 & 2\kappa + \mu & -2\kappa + \mu
\end{pmatrix}
\begin{pmatrix}
  A \\
  B \\
  C
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}.
\]

The spectral equation is now the transcendent equation

\[
e^{-2a\kappa} = \frac{2}{\mu} + 1.
\]

It is clear that only if \( \mu < -\frac{1}{a} \) there is one intersection. Thus, there is one odd bound state if \( -\infty < \mu < -\frac{1}{a} \). The odd bound state has more energy than the even bound state (always the ground state is even with no nodes). If \( -\infty < \mu < -\frac{3}{a} \), \( \kappa^- < \kappa^+ \), the second even bound state is the more energetic.

In the Table (5) we show the bound state wave functions of three equal \( \delta \)-wells in a region of three bound states: \( \mu < -\frac{3}{a} \). For the values of \( a = 2 \) and \( \mu = -2 \) chosen in this region the intersection values of the imaginary momentum are:

- \( \kappa_0^+ = \sqrt{-E_0^+} = 1.147 \) > \( \kappa_1^- = \sqrt{-E_1^-} = 0.982 \) > \( \kappa_2^+ = \sqrt{-E_2^+} = 0.649 \).

From these values the solution of the homogeneous system forced by the matching conditions leads us to obtain the normalized bound state wave functions.

**Table 5:** The three bound state wave functions of \( \hat{H} \), successively even, odd, even, for \( \mu = -2 < -\frac{3}{a} = -\frac{3}{2} \). The number of nodes is respectively 0, 1, and 2.

6.1.4. **Bound states if the two external \( \delta \)-wells are equal**

In this case the Hamiltonian is also invariant under the \( x \rightarrow -x \) reflection and the eigenfunctions are either even or odd function of \( x \).

1. **Even bound states** \( A = G, \ B = F, \) and \( C = D. \) The homogeneous linear system implementing the matching conditions now is:

\[
\begin{pmatrix}
  e^{-\kappa a} & -e^{-\kappa a} & -e^{\kappa a} \\
  (\kappa + \alpha)e^{-\kappa a} & -\kappa e^{-\kappa a} & \kappa e^{\kappa a} \\
  0 & 2\kappa + \mu & -2\kappa + \mu
\end{pmatrix}
\begin{pmatrix}
  A \\
  B \\
  C
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}.
\]

The spectral equation is the transcendent equation

\[
\alpha e^{-2a\kappa}(\mu - 2\kappa) = (2\kappa + \alpha)(2\kappa + \mu).
\]

The solutions of this equation are the intersections of the curves \( g(\kappa, \alpha, \mu, a) = \alpha e^{-2a\kappa}(\mu - 2\kappa) \) and \( f(\kappa, \alpha, \mu) = (2\kappa + \alpha)(2\kappa + \mu) \) for \( \kappa > 0 \). Because \( g'(0, \mu, a) = -2\alpha(a\mu + 1) \) and \( f'(0, \mu) = 2(\alpha + \mu) \) the tangents at the origin of these two curves are equal on the hyperbola:

\[
a = -\frac{2\alpha + \mu}{\alpha\mu} \equiv \left\{ \begin{array}{c}
  \mu = -\frac{2\alpha}{1 + a}\alpha \\
  \alpha = -\frac{\mu}{2 + \mu}a
\end{array} \right\}.
\]
The asymptotes of this quadric curve are the straight lines: \( \alpha = -\frac{1}{a} \) and \( \mu = -\frac{2}{a} \), the vertices lie at the points \((-\frac{1}{a}, \frac{\sqrt{2}}{a})\), \((-\frac{1}{a}, -\frac{\sqrt{2}}{a})\) and \((-\frac{1}{a}, \frac{\sqrt{2}}{-a})\), whereas the center is the point: \((-\frac{1}{a}, -\frac{2}{a})\) in the \( \alpha : \mu : \alpha \)-plane.

Identification of the domains in the parameter space \( \mathbb{R}^3 \) where there exist 0, 1, 2 or 3 bound states requires to perform the following analysis:

(a) Zero even bound states.

- If \( \alpha + \mu > 0 \) and \( \text{sgn} \alpha = \text{sgn} \mu \) all the parabolas \( f(\alpha, \mu, \lambda) \) have a unique critical point at \( \kappa = -\frac{a + \mu}{4} < 0 \) that is a minimum on the negative \( \kappa \) half-line: \( f''(\kappa) = 8 > 0 \). Moreover, \( f(\kappa, \alpha, \lambda, \mu) = -\frac{(\alpha - \mu)^2}{4} < 0 \) and because \( f(0, \alpha, \mu) = \alpha \mu \) these parabolas are monotonically increasing functions all along the positive \( \kappa \) half-line from their value at the origin. Also the transcendent curves \( g(\kappa, \alpha, \beta, \lambda) \) have a unique critical point: \( \kappa = \frac{1 + \alpha}{2} > 0 \). This point is a minimum, \( g''(\kappa, \alpha, \beta, \lambda) = 4 \alpha \mu e^{-1 + \alpha \mu} > 0 \) and \( g(\kappa, \alpha, \beta, \lambda) = -\frac{\alpha}{\mu} e^{-(1 + \alpha \mu)} < 0 \). Thus, in the interval \( 0 < \kappa < \kappa \) the functions \( g \) decrease from their maximum value at the origin \( g(0, \alpha, \mu, \lambda) = \alpha \mu \) up to their minimum values at \( \kappa \). After that point, \( \kappa > \kappa \), these functions monotonically increase and approach to \( 0 \) at \( \kappa = \infty \) from below. Therefore, there are no intersections between \( f \) and \( g \) in this regime of parameters for \( \kappa > 0 \) and no bound states exist for three repulsive walls of this type. Since the inequality \( \alpha > \frac{1}{\alpha} - \frac{2}{\mu} \) holds in this range of parameters the subsequent inequality \( f''(0, \alpha, \mu) = 2 + 2a > g''(0, \alpha, \mu, \lambda) = -2(\alpha(\mu + 1)) \) between the derivatives at the origin characterize the non existence of bound states, i.e., in all the positive quadrant of the \( \alpha : \mu \) plane, regardless the value of \( a \).

- Even if one of the two \( s \) are attractive, e.g. \( \alpha < 0 \), (a wall), and the other repulsive, e.g. \( \mu > 0 \), (a wall), there may be no bound state (no \( \kappa > 0 \) intersection of \( f \) and \( g \)). The tangents to the \( g \) curves at the origin \( g'(0, \alpha, \mu, \lambda) = 2(\alpha(\mu + 1)) \) become positive but only if \( g'(0, \alpha, \mu, \lambda) > f'(0, \alpha, \mu) = 2(\mu - |\lambda|) \) there will be an intersection between \( f \) and \( g \) for \( \kappa > 0 \). Thus, the inequality \( \alpha < \frac{1}{\alpha} - \frac{2}{\mu} \) characterizes the non existence of even bound states in this regime. This requires \( \frac{1}{\alpha} < \alpha < 0 \) and we are in the area between the upper branch of the hyperbola in the second quadrant and the positive \( \mu \)-half-axis, see Figure 3. If now we consider \( \alpha > 0 \) and \( -\frac{2}{a} < \mu < 0 \) the inequality \( \alpha < \frac{1}{\alpha} - \frac{2}{\mu} \) guarantees that \( g''(0, a, \alpha, \mu, \lambda) = -2(\alpha(\mu + 1)) \) and there are no \( \kappa > 0 \) intersections between \( f \) and \( g \) and no even bound states exist in the area enclosed by the upper branch of the hyperbola in the third quadrant and the positive \( \alpha \)-half-axis.

(b) One even bound state

- If the inequalities above change the sense, i.e. \( \alpha > \frac{1}{|\alpha|} - \frac{2}{\mu} \) or \( \alpha < -\frac{1}{\alpha} + \frac{2}{|\mu|} \), there is one \( \kappa > 0 \) intersection of the curves \( g \) and \( f \) and therefore there exits one even bound state. This situation happens in the area enclosed by the upper branch of the hyperbola and the asymptotae in the second and fourth quadrants, see Figure 3.

- When \( \text{sgn} \alpha \neq \text{sgn} \mu \) and either \( \alpha < -\frac{1}{\alpha} \) or \( \mu < -\frac{2}{\mu} \) the inequalities \( \alpha > \frac{1}{|\alpha|} - \frac{2}{\mu} \) or \( \alpha < -\frac{1}{\alpha} + \frac{2}{|\mu|} \) respectively hold. Therefore \( f'(0, \alpha, \mu) < g'(0, a, \alpha, \mu) \), there is one \( \kappa > 0 \) intersection of the curves \( g \) and \( f \) and these are one even bound state areas.

- In the case when the two \( s \) are attractive i.e. \( \alpha < 0 \), (a well) and \( \beta < 0 \), (another well) there is only one \( \kappa > 0 \) intersection if the inequality \( \alpha < \frac{1}{\alpha} + \frac{2}{|\mu|} \) is satisfied. The area in the third quadrant enclosed by the lower branch of the hyperbola and the negative \( \alpha \) and \( \mu \)-axes is a one even bound state zone.

(c) Two even bound states

- There are two \( \kappa > 0 \) intersections if \( \alpha + \mu < 0 \), \( \text{sgn} \alpha = \text{sgn} \mu \), and the inequality \( \alpha > \frac{1}{|\alpha|} + \frac{2}{|\mu|} \) is satisfied. This area beyond the lower branch of the hyperbola in the third quadrant is a zone of two even bound states.

2. Odd bound states: \( A = -G, B = -F; \) and \( C = -D \). The "odd" homogeneous linear system reads:

\[
\begin{pmatrix}
e^{-\kappa \alpha} & -e^{-\kappa \alpha} & -e^{\kappa \alpha} \\
(\kappa + \alpha)e^{-\kappa \alpha} & -\kappa \alpha & \kappa e^{\kappa \alpha} \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]
The spectral equation is now the transcendent equation
\[ e^{-2\alpha \kappa} = \frac{2}{\alpha} - \kappa + 1. \]
The solutions of this spectral equation are the intersections of the curves \( g(\kappa, \alpha) = e^{-2\alpha \kappa} \) and \( f(\kappa, \alpha) = \frac{2\alpha}{\alpha} + 1 \) for \( \kappa > 0 \). Now \( g'(0, \alpha) = -2\alpha \) and \( f'(0, \alpha) = \frac{2}{\alpha} \) the tangents at the origin are equal on the straight line: \( \alpha = -\frac{1}{2} \).

(a) Zero odd bound states
- If \( \alpha > 0 \) then \( f'(0, \alpha) > g'(0, \alpha) \), or \( a > -\frac{2}{\alpha} \). In this case there is no \( \kappa > 0 \) intersection of the curves \( g \) and \( f \) and therefore there is no odd bound state.
- If \( \alpha < 0 \) and \( a < \frac{1}{|\alpha|} \) such that \( f'(0, \alpha) < g'(0, \alpha) \), there is no \( \kappa > 0 \) intersection of the curves \( g \) and \( f \).

(b) One odd bound state
- If the inequality changes the sense, i.e., \( a > \frac{1}{|\alpha|} \), there is one \( \kappa > 0 \) intersection of the curves \( g \) and \( f \) and therefore there is one odd bound state.

In sum, in this \( \alpha = \beta \) (the two external potentials have the same strength) “tomographic ”vision, where we restrict ourselves to the \((\alpha, \mu, \alpha)\) plane embedded in \( \mathbb{R}^3 \), the surfaces \([27]\) are both projected onto the straight line \( \alpha = -\frac{1}{a} \), which is one separatrix, and the hyperbola \([29]\), whose branches are another separatrices between regions of different number of bound states. The situation is described in the Figure 4: the upper branch of the hyperbola \([29]\) is the frontier of the region of no bound states. The area enclosed by the asymptote \( \alpha = -\frac{1}{a} \), and the lower branch of the hyperbola is a region of one even bound state. There are one even and one odd bound states between the lower branch of the hyperbola and this asymptote. Below the lower branch of the hyperbola there exist two even and one odd bound states. In the Table 6 the wave functions in a parameter region of three bound states are shown when the two external \( \delta \)-wells are of the same strength: \( a = \frac{1}{|\alpha|} + \frac{2}{|\mu|} \). Choosing, e. g., \( a = 3 \), \( \alpha = \beta = -2 \), and \( \mu = -2.5 \) within this region, the three intersections of the curves occur at the imaginary momentum values: \( \kappa_0^+ = \sqrt{-E_0^+} = 1.255 > \kappa_1^- = \sqrt{-E_1^-} = 0.997 > \kappa_2^+ = \sqrt{-E_2^+} = 0.976 \). Given these values, the solutions of the homogeneous system forced by the matching conditions provide the coefficients of the three (normalized) bound state wave functions.

### 6.1.5. Bound state analysis in the case that the central and right \( \delta \)-potentials have the same strength: \( \mu = \beta \)

If \( \beta = \mu \), the central and right \( \delta \)-potentials have the same strength. The three parametric variety of systems is restricted to the \((\alpha, \mu, \mu)\) plane enclosed within the \((\alpha, \mu, \beta)\) 3D-space homeomorphic to \( \mathbb{R}^3 \). The Hamiltonian is not invariant (unless there is further restriction to the \( \alpha = \mu \) real line) under the \( x \to -x \) reflection. Thus, away from the diagonal \( \alpha = \mu \), the eigenfunctions are neither even nor odd function of \( x \).

The spectral equation is the transcendent equation:
\[ \alpha \mu e^{-4\alpha \kappa}(2\kappa - \mu) + 2\mu e^{-2\alpha \kappa}[\alpha \mu + \kappa(\alpha + \mu)] = (2\kappa + \alpha)(2\kappa + \mu)^2, \]
and the surfaces \([27]\) are projected into the curves:
\[ a_{\pm} = \frac{3}{2\mu} - \frac{1}{2\alpha} \pm \frac{\sqrt{5\alpha^2 - 2\alpha \mu + \mu^2}}{2\alpha \mu}. \]
Figure 4: The distribution of bound states in the parameter space of the triple delta Dirac potential with equal strength of the external δs potentials: α = β.

These curves are plotted in the Figure for the special choice of a = 1. The curve for the a− root is the frontier between the areas of one and two bound states, whereas the two branches of the a+- curve respectively divide the zero/one and the two/three bound state areas. The areas with fixed number of bound states are distributed according to the following pattern, see Figure 5:

1. **Zero bound states** There are three zones of no bound states enclosed between the upper branch of the curve a+=30 and infinity.
   - The two couplings are positive, α > 0, μ > 0, and the inequality a > a+ is satisfied.
   - The δ on the left is attractive, α < 0, and the other two are repulsive: μ < 0. There are no bound states if the inequality a < a− holds.
   - In the opposite configuration, α > 0, μ < 0, there are no bound states when the inequality a < a+ is satisfied.

2. **One bound state** The one-bound state area is divided also into three zones.
   - The first one is characterized by the signs of the couplings α < 0, μ > 0, and the inequality: a > a−.
   - In the second zone the coupling signs are exchanged, α > 0, μ < 0, and the double inequality a− < a < a+ holds.
   - Even when the three δs are attractive, α < 0, μ < 0, there is only one bound state if the inequality a < a− is satisfied.

3. **Two-bound states** The two-bound state area is only composed of two zones.
   - The first two-bound state zone arises when α > 0, μ < 0, and a < a−.
   - There are also two-bound states if the three δs are attractive, α < 0, μ < 0, but the double inequality a− < a < a+ holds.

4. **Three bound states** Finally, there is only one zone of three bound states enclosed between the lower branch of the curve a = a+ and infinity.
   - The three δs are attractive, α < 0, μ < 0, and the inequality a < a− is satisfied.
6.1.6. Bound states of three different δ-wells

A quite explicit description of the discrete spectrum is also possible in the case of three different δ-potentials. Because the Hamiltonian is not invariant under the $x \rightarrow -x$ reflection the eigenfunctions are neither even nor odd functions of $x$. In the Table (7) we plot the bound state wave functions of three different δ-wells. The negative couplings are chosen in a region of three bound states: $a > - \left( \frac{\alpha + \beta}{2\alpha \beta} + \frac{1}{\mu} \right) - \sqrt{\frac{1}{\mu^2} + \left( \frac{(\alpha - \beta)^2}{4\alpha^2\beta^2} \right)}$. Specifically, we set $a = 2$, $\alpha = -1$, $\beta = -2$ and $\mu = -3$ in this region. The three intersections of the spectral curves $g$ and $f$ occur at the imaginary momentum values: $\kappa_0^+ = \sqrt{-E_0^+} = 1.508 > \kappa_1^- = \sqrt{-E_1^-} = 0.939 > \kappa_2^- = \sqrt{-E_2^-} = 0.307$. From these values, the non-trivial solutions of the homogeneous linear system in six unknowns, complying with the matching conditions, provide us with the normalizable bound state wave functions

Table 7: The three bound state wave functions of $\hat{H}$ for $\alpha = -1$, $\beta = -2$ and $\mu = -3$. The number of nodes is respectively 0, 1, and 2.

We end this Section with a brief summary of the spectrum of stationary wave functions in one array of three δ-potentials.

- If the central potential is switched off, $\mu = 0$, the surface $a_- \rightarrow \{27\}$ reduces to the hyperbola $a = - \frac{\alpha + \beta}{2\alpha \beta}$, whereas the surface $a_+ \rightarrow \{27\}$ disappears at infinity. The problem of two δ potentials is recovered in the $(\alpha,0,\beta)$ plane. The planar zones of 0,1, and two bound states separated by the two branches of the hyperbola reappear.

- This tomography, together with the other two described before in the $(\alpha,\mu,\alpha)$ and $(\alpha,\mu,\mu)$ planes, illuminates the three dimensional problem. The branches of the separatrix surfaces divide the parameter
space in four three-dimensional subspaces with zero, one, two, and three bound states. For instance, the existence of three bound states require not only that the three \( \delta \)-potentials are attractive wells but also that the distance between them must fit with their relative strengths.

6.2. \( \mathcal{N} = 2 \) supersymmetric three Dirac \( \delta \) Hamiltonian: three \( \delta \)s of different strengths

The supersymmetric quantum mechanics of three \( \delta \) configurations of different strength is determined from the superpotential

\[
W(x) = \frac{\alpha}{2} |x + a| + \frac{\mu}{2} |x| + \frac{\beta}{2} |x - a| - \frac{1}{2}(\alpha - \beta)x,
\]

where \( \alpha \neq \beta, \alpha \neq \mu, \beta \neq \mu \) are three different positive constants. It is clear that

\[
W'(x) = \frac{\alpha}{2} \varepsilon(x + a) + \frac{\mu}{2} \varepsilon(x) + \frac{\beta}{2} \varepsilon(x - a) - \frac{\alpha - \beta}{2}, \quad v_+ = \beta + \frac{\mu}{2}, \quad v_- = -\alpha - \frac{\mu}{2}
\]

\[
\lim_{x \to a_{\pm}} W'(x) = \pm \frac{\mu}{2}, \quad \pm W''(x) = \pm [\alpha \delta(x + a) + \mu \delta(x) + \beta \delta(x - a)]
\]

The bosonic scalar Hamiltonian is the Schrödinger operator for three repulsive \( \delta \)s plus one quasi-square well whereas the fermionic Hamiltonian is obtained replacing the repulsive \( \delta \)-walls by attractive \( \delta \)-wells:

\[
\begin{align*}
\hat{H}_0 &= -\frac{d^2}{dx^2} + V_0(x) \quad , \quad \hat{H}_1 = -\frac{d^2}{dx^2} + V_1(x) \\
V_0(x) &= \alpha \delta(x + a) + \mu \delta(x) + \beta(x - a) + \begin{cases} 
(\alpha + \frac{\mu}{2})^2, & x \in \text{III} \\
\frac{a^2}{4}, & x \in \text{I, II} \\
(\beta + \frac{\mu}{2})^2, & x \in \text{IV} 
\end{cases} \\
V_1(x) &= -\alpha \delta(x + a) - \mu \delta(x) - \beta(x - a) + \begin{cases} 
(\alpha + \frac{\mu}{2})^2, & x \in \text{III} \\
\frac{a^2}{4}, & x \in \text{I, II} \\
(\beta + \frac{\mu}{2})^2, & x \in \text{IV} 
\end{cases}
\end{align*}
\]

Like in the case of two different \( \delta \)s we introduce a quasi-square well, in this case different from zero also in the middle region, to make the problem compatible with supersymmetry.

6.2.1. Scattering waves

There are four zones separated by the three \( \delta \) walls/wells. There are, however, only three different momenta: Zone III, \( k = \sqrt{E^{(i)} - (\alpha + \frac{\mu}{2})^2} \). Zone I, Zone II: \( q = \sqrt{E^{(i)} - \frac{a^2}{4}} \). Zone IV, \( p = \sqrt{E^{(i)} - (\beta + \frac{\mu}{2})^2} \). We assume without loss of generality that \( \alpha^2 < \beta^2 \). There are three ranges of energies:

6.2.2. Double degenerate scattering waves

If \( E^{(i)} > (\beta + \frac{\mu}{2})^2 \) there are right-going and left-going scattering wave solutions of the bosonic and fermionic Schrödinger equations:

\[
\psi^{(r)}(x, E) = \begin{cases} 
eq 0 & x \in \text{III} \\
eq 0 & x \in \text{I} \\
eq 0 & x \in \text{II} \\
eq 0 & x \in \text{IV} 
\end{cases}
\]

\[
\psi^{(l)}(x, E) = \begin{cases} 
eq 0 & x \in \text{III} \\
eq 0 & x \in \text{I} \\
eq 0 & x \in \text{II} \\
eq 0 & x \in \text{IV} 
\end{cases}
\]

The solution of the linear system of six equations in six unknowns arising from the matching conditions at the points \( x = \pm a \) and \( x = 0 \) on the scattering waves, e.g., for the right- and left-movers in the bosonic sector,
gives the scattering amplitudes:

\[
\sigma_0^r(E) = \frac{8kq^2e^{-iak(k-3q)}}{\Delta_0^r} , \quad \sigma_0^l(E) = \frac{8pq^2e^{-ia(p-3q)}}{\Delta_0^l}
\]

\[
\rho_0^r(E) = e^{-2iak}\frac{k + q - i\alpha}{k - q + i\alpha} - \frac{4kqe^{ia(p+q-2k)}}{(k - q + i\alpha)\Delta_0^r} \left[ \mu e^{2iak}(\beta - ip + iq) + (2q + i\mu)(i\beta + p + q) \right]
\]

\[
\rho_0^l(E) = -e^{2iap} + \frac{4pe^{ia(k+3q)-2p}}{\Delta_0^l} \left[ \cos(2qa) (-\alpha + i\mu + 2q^2) + q\sin(2qa)(2\alpha - 2ik + \mu) + \mu(\alpha - ik) \right]
\]

\[
\Delta_0^r(E, \alpha, \mu, \beta, a) = 2\mu e^{ia(p+3q)} [(\alpha - ik)(p + i\beta) + iq^2] + e^{ia(p+q)}(2q + i\mu)(k + q + i\alpha)(p + q + i\beta) + e^{ia(p+5q)}(2q - i\mu)(k - q + i\alpha)(-p + q + i\beta)
\]

\[
\Delta_0^l(E, \alpha, \mu, \beta, a) = 2\mu e^{ia(k+3q)} [(\alpha - ik)(p + i\beta) + iq^2] + e^{ia(k+q)}(2q + i\mu)(k + q + i\alpha)(p + q + i\beta) + e^{ia(k+5q)}(2q - i\mu)(k - q + i\alpha)(-p + q - i\beta)
\]

The scattering amplitudes in the fermionic sector are then, e.g., for the right-movers \(\sigma_1^r(E), \rho_1^r(E)\), determined from supersymmetry:

\[
\sigma_1^r(E) = \frac{ip - \beta - \frac{\mu}{2}}{ik + \alpha + \frac{\mu}{2}} \sigma_0^r(E) , \quad \rho_1^r(E) = \frac{-ik + \alpha + \frac{\mu}{2}}{ik + \alpha + \frac{\mu}{2}} \rho_0^r(E)
\]

6.2.3. Bound state wave functions

For energies such that \((\alpha + \frac{\mu}{2})^2 < E^{(i)} < (\beta + \frac{\mu}{2})^2\) there are no left-going incoming scattering waves and only exponentially decaying right-going transmitted waves. If \(\frac{\mu^2}{4} < E^{(i)} < (\alpha + \frac{\mu}{2})^2, \) however, the two asymptotic momenta become purely imaginary: \(k^{(i)} = ik^{(i)} = i\sqrt{(\alpha + \frac{\mu}{2})^2 - E^{(i)}}, \) \(p^{(i)} = ip^{(i)} = i\sqrt{(\beta + \frac{\mu}{2})^2 - E^{(i)}}\). The purely imaginary poles of \(\sigma_0^r(E)\) and \(\sigma_0^l(E)\) such that \(\kappa^{(0)} > 0\) and \(\pi^{(0)} > 0\) provide the bosonic bound state eigenvalues and eigenfunctions. Therefore, we need to identify the zeroes of the determinants \(\Delta_0^r(E, \alpha, \mu, \beta, a) = 0 = \Delta_0^l(E, \alpha, \mu, \beta, a)\) as functions of \(E\). A routine calculation shows that these zeroes are given by the spectral equation:

\[
(2q\cos 2aq + \mu \sin 2aq) \left( \frac{q(\kappa^{(0)} + \pi^{(0)} + \alpha + \beta)}{(\kappa^{(0)} + \alpha)(\pi^{(0)} + \beta) - q^2} \right) = \\
= (\mu \cos 2aq - 2q \sin 2aq) - \mu \left( \frac{q(\kappa^{(0)} + \alpha)(\pi^{(0)} + \beta) + q^2}{(\kappa^{(0)} + \alpha)(\pi^{(0)} + \beta) - q^2} \right)
\]

(31)

It is obvious in formula (31) that in the \(\mu = 0\) limit we recover the spectral equation for the bosonic bound state eigenvalues of two SUSY Hamiltonians. The fermionic bound state eigenvalues are determined from the same equation (31) replacing the \(\delta\) walls by wells, i.e., changing \(\alpha, \mu, \) and \(\beta\) by \(-\alpha, -\mu,\) and \(-\beta:\)

\[
(2q\cos 2aq - \mu \sin 2aq) \left( \frac{q(\kappa^{(1)} + \pi^{(1)} - \alpha - \beta)}{(\kappa^{(1)} - \alpha)(\pi^{(1)} - \beta) - q^2} \right) = \\
= (-\mu \cos 2aq + 2q \sin 2aq) + \mu \left( \frac{q(\kappa^{(1)} - \alpha)(\pi^{(1)} - \beta) + q^2}{(\kappa^{(1)} - \alpha)(\pi^{(1)} - \beta) - q^2} \right)
\]

(32)

A laborious test allows us to check that the solutions of the transcendental equations (31) and (32) are the same. Henceforth, the bound state eigenvalues of the Hamiltonians \(H_0(x)\) and \(H_1(x)\) are identical as required by supersymmetry. In Table 3 the two members of the spectral equations (31)-(32) for \(H_0\) (left) and \(H_1\) (right) are plotted as functions of \(E\). Our choice of signs for the couplings determines that only the zero energy eigenvalue \(E^{(0)}_0 = 0\) of \(H_1\) is a bound state whereas the zero energy eigenvalue \(E^{(0)}_0 = 0\) of \(H_0\) is a (non-normalizable) anti-bound state. The normalized wave function of the fermionic zero mode is:

\[
\psi^{(1)}_0(x) = \begin{cases} 
\frac{e^{\alpha(\alpha+\beta+\mu)}}{2(e^{\alpha(\alpha+\beta+\mu)} + \frac{1}{2\alpha+\mu} + \frac{1}{2\beta+\mu})} & , x \in \text{III} \\
\frac{e^{\alpha+\alpha+\beta+\mu}}{2(e^{\alpha+\alpha+\beta+\mu} + \frac{1}{2\alpha+\mu} + \frac{1}{2\beta+\mu})} & , x \in \text{I} \\
\frac{e^{-\alpha+\alpha+\beta+\mu}}{2(e^{-\alpha+\alpha+\beta+\mu} + \frac{1}{2\alpha+\mu} + \frac{1}{2\beta+\mu})} & , x \in \text{II} \\
\frac{e^{\beta+\beta+\alpha+\mu}}{2(e^{\beta+\beta+\alpha+\mu} + \frac{1}{2\alpha+\mu} + \frac{1}{2\beta+\mu})} & , x \in \text{IV}
\end{cases}
\]
Table 8: Graphics of the curves in the two members of the $\hat{H}_0$ and $\hat{H}_1$ spectral equations for $\alpha = 4$, $\beta = 6$ and $\mu = 1$. The curves on the left and on the right Figures intersect at the same values of $\sqrt{E}$ meaning that the bosonic and fermionic bound states form supersymmetry doublets.

It is plotted in the Figure 6 for the values $a = 4$, $\alpha = 2$, $\beta = 6$ and $\mu = 1$.

In sum, the unique ground state of the supersymmetric Hamiltonian $\hat{H}$ is fermionic:

$$\Psi_0^{(1)}(x) = \begin{pmatrix} \Psi_0^{(1)}(0) \\ \Psi_0^{(1)}(x) \end{pmatrix}.$$  

It is clear that $\hat{Q}^\dagger \Psi_0^{(1)} = 0$, but also it belongs to the kernel of the other supercharge: $\hat{Q} \Psi_0^{(1)} = 0$:

$$\hat{D}\Psi_0^{(1)}(x) = \left[ \frac{d}{dx} - \frac{\alpha}{2} \varepsilon(x + a) + \frac{\mu}{2} \varepsilon(x - a) - \frac{\alpha - \beta}{2} \right] \psi_0^{(1)}(x) = 0.$$  

The positive energy bosonic bound state, belonging to the discrete spectrum of $\hat{H}_0$, thus complying with the matching conditions, are built from the scattering solutions for the imaginary external momenta that are poles of the transmission amplitudes:

$$\psi_n^{(0)}(x) = N \cdot \begin{cases} 
\frac{1}{2} \left[ \frac{4\pi}{q_n} + \frac{4\beta}{q_n} + \frac{2\mu}{q_n} \right] \sin 2q_n a + \left( -\frac{2\mu\pi}{q_n^2} - \frac{2\beta\mu}{q_n^2} + 4 \right) \cos 2q_n a 
+ \frac{2\mu\pi}{q_n^2} + \frac{2\beta\mu}{q_n^2} \right] e^{\kappa_n x + \kappa_n a - \pi_n a} & , x \in \text{III} \\
\frac{1}{2} e^{-\pi_n a} \left[ \left( -\frac{4\pi}{q_n} - \frac{4\beta}{q_n} - \frac{2\mu}{q_n} \right) \sin q_n (x - a) - \frac{2\mu}{q_n} \sin q_n (x + a) 
+ \frac{2\mu\pi}{q_n^2} + \frac{2\beta\mu}{q_n^2} \right] \cos q_n (x - a) + \left( \frac{2\mu\pi}{q_n^2} + \frac{2\beta\mu}{q_n^2} \right) \cos q_n (x + a) & , x \in \text{I} \\
\frac{1}{2} e^{-\pi_n a} \left[ \cos q_n (x - a) - \left( \frac{\pi}{q_n} + \frac{\beta}{q_n} \right) \sin q_n (x - a) \right] & , x \in \text{II} \\
e^{-\pi_n x} & , x \in \text{IV} 
\end{cases}$$
where $N$ is a normalization constant. In the Table 9 several graphics of the bound state eigenfunctions of the Hamiltonians $\hat{H}_0$ and $\hat{H}_1$ together with the corresponding energy eigenvalues are shown. The eigenvalues are graphically obtained as the intersection points of the curves in the spectral equations (31)-(32). The fermionic supercharges provide in turn the fermionic transmission and reflection amplitudes $\sigma_0(k)$, $\rho_0(k)$:

| $E_n^{(0)}$ | $E_1^{(0)}$ = 0.557 | $E_2^{(0)}$ = 0.822 | $E_3^{(0)}$ = 1.752 |
|---|---|---|---|
| $\psi(x)$ | $\psi(x)$ | $\psi(x)$ |

6.3. Three equal $N = 2$ supersymmetric $\delta$ walls/wells: $\alpha = \beta = \mu$

6.3.1. Scattering waves

In this case many things simplify, specially the asymptotic momenta become equal: $k^{(i)} = p^{(i)} = \sqrt{E^{(i)} - \frac{q}{4} \mu^2}$, although the intermediate momentum remains unmodified: $q^{(i)} = \sqrt{E^{(i)} - \frac{1}{4} \mu^2} = \sqrt{k^{(i)2} + 2 \mu^2}$. For $E^{(i)} > \frac{9}{4} \mu^2$ the eigenfunctions are scattering waves. The transmission amplitudes for the right-movers and the left-movers also do not differ, $\sigma_0^r(k) = \sigma_0^l(k) = \sigma_0(k)$, $\sigma_1^r(k) = \sigma_1^l(k) = \sigma_1(k)$ due to time reversal invariance, and spatial reflection invariance do the same job as far as reflection amplitudes is concerned: $\rho_0^r(k) = \rho_0^l(k) = \rho_0(k)$, $\rho_1^r(k) = \rho_1^l(k) = \rho_1(k)$. In the bosonic sector, for instance, we find

$$\sigma_0(k) = \frac{8kq^2 e^{-i\alpha(k-q)}}{\Delta(k, q; \mu, a)}$$

where the denominator $\Delta(k, q; \mu, a)$ is defined as:

$$\Delta(k, q; \mu, a) = e^{i\alpha(k+q)}(k + q + i\alpha) \left[2\alpha e^{2i\alpha q}(\alpha - i k + i q) + (2q + i\alpha)(k + q + i\alpha)\right] - e^{i\alpha(k+5q)}(2q - i\alpha)(k - q + i\alpha)^2 .$$

The reflection amplitude, however, is more complicated:

$$\rho_0(k) = \frac{e^{-i\alpha(k-q)}}{\Delta} \left\{ 2i\alpha e^{2i\alpha q}(\alpha^2 + k^2 + q^2) + e^{4i\alpha q}(2q - i\alpha)[k^2 - (q - i\alpha)^2] \right\} .$$

The supercharges provide in turn the fermionic transmission and reflection amplitudes $\sigma_1(k)$, $\rho_1(k)$:

$$\sigma_1(k) = \frac{ik - \frac{3q}{2}}{ik + \frac{3q}{2}} \sigma_0(k) , \quad \rho_1(k) = \frac{-ik + \frac{3q}{2}}{ik + \frac{3q}{2}} \rho_0(k) .$$
Because the asymptotic momenta are equal the bosonic and fermionic $S_\tau$-matrices

$$S_0(k, \mu, a) = \left( \begin{array}{cc} \sigma_0(k) & \rho_0(k) \\ \rho_0(k) & \sigma_0(k) \end{array} \right), \quad S_1(k, \mu, a) = \left( \begin{array}{cc} \sigma_1(k) & \rho_1(k) \\ \rho_1(k) & \sigma_1(k) \end{array} \right)$$

are properly defined unitary matrices. Their eigenvalues, e.g. of $S_0$, are thus complex numbers of unity modulus:

$$
\begin{align*}
\lambda_+^0(k, \mu, a) &= \frac{e^{-2iak} \left[ k \left(1 - e^{2iaq} \right) + i\mu \left(-1 + e^{2iaq} \right) - q \left(1 + e^{2iaq} \right) \right]}{k \left(-1 + e^{2iaq} \right) + i\mu \left(-1 + e^{2iaq} \right) - q \left(1 + e^{2iaq} \right)} \\
\lambda_-^0(k, \mu, a) &= \frac{e^{-2iak} \left[e^{2iaq}(2q - i\mu)(k + q - i\alpha) + (2q + i\mu)(k - q - i\mu)\right]}{e^{2iaq}(2q - i\mu)(k - q + i\alpha) + (2q + i\alpha)(k + q + i\alpha)}
\end{align*}
$$

whereas the eigenvalues of the fermionic $S$-matrix are related through the supercharges: $\lambda_+^\tau(k, \mu, a) = \sigma_1(k) \pm \rho_1(k)$. The bosonic total phase shift is

$$
\delta_0(k; \mu, a) = \frac{1}{2i} \left\{ \log \left( \frac{e^{-2iak} \left[ k \left(1 - e^{2iaq} \right) + i\alpha \left(-1 + e^{2iaq} \right) - q \left(1 + e^{2iaq} \right) \right]}{k \left(-1 + e^{2iaq} \right) + i\alpha \left(-1 + e^{2iaq} \right) - q \left(1 + e^{2iaq} \right)} \right) \\
+ \log \left( \frac{e^{-2iak} \left[e^{2iaq}(2q - i\alpha)(k + q - i\alpha) + (2q + i\alpha)(k - q - i\alpha)\right]}{e^{2iaq}(2q - i\alpha)(k - q + i\alpha) + (2q + i\alpha)(k + q + i\alpha)} \right) \right\}
$$

from which one can calculate the spectral density.

### 6.3.2. Bound states

If $\frac{E^2}{\lambda} < E^{(c)} < \frac{1}{3} \mu^2$ bound state eigenfunctions arise both for the bosonic and fermionic Hamiltonians. In terms of the purely imaginary external momentum $\kappa^{(0)} = \sqrt{\frac{9}{4} \mu^2 - E^{(0)}}$ the spectral equation for the bosonic bound states is:

$$
(2\sqrt{2\mu^2 - \kappa^2} \cos 2a \sqrt{2\mu^2 - \kappa^2} + \mu \sin 2a \sqrt{2\mu^2 - \kappa^2}) \frac{2\sqrt{2\mu^2 - \kappa^2} (\kappa + \mu)}{(\kappa + \mu)^2 - 2\mu^2 + \kappa^2} = \\
= \mu \cos 2a \sqrt{2\mu^2 - \kappa^2} - 2\sqrt{2\mu^2 - \kappa^2} \sin 2a \sqrt{2\mu^2 - \kappa^2} - \mu \frac{(\kappa + \mu)^2 + 2\mu^2 - \kappa^2}{(\kappa + \mu)^2 - 2\mu^2 + \kappa^2} .
$$

The fermionic spectral equation is exactly (33) merely exchanging $\mu$ by $-\mu$. Needless to say the solutions of the transcendent equation (33) and its fermionic partner, the intersections between the curves in the two members of the bosonic and fermionic spectral equations, are identical, as requested by supersymmetry.

### 6.4. Alternatively attractive and repulsive triple $\delta$-potential of the same strength

We consider now the following superpotential:

$$W = \frac{\alpha}{2} |x| - \frac{\alpha}{2} |x - a| - \frac{\alpha}{2} |x + a|.$$ 

This choice exactly leads to the spectral problem of the general SUSY triple $\delta$-potential restricted to the straight line $(-\alpha, \alpha, -\alpha)$ in the space of parameters $(\alpha, \mu, \beta) \in \mathbb{R}^3$. In this case the potentials are even and the incoming from the left transmission and reflection amplitudes are equal to the analogous amplitudes for scattering waves incoming from the right:

$$
\begin{align*}
\sigma_+^1(k) &= \sigma_1(k), \quad \rho_+^1(k) = \rho_1(k), \quad E^{(1)} = k^2 + \frac{\alpha^2}{4}, \\
\sigma_1(k) &= \frac{i k + \frac{3}{2} \alpha}{i k - \frac{3}{2} \alpha} \cdot \sigma_0(k), \quad \rho_1(k) = \frac{-i k - \frac{3}{2} \alpha}{i k - \frac{3}{2} \alpha} \cdot \rho_0(k), \quad E^{(0)} = k^2 + \frac{\alpha^2}{4},
\end{align*}
$$

32
where the identities in the second row between the amplitudes in different sectors come from supersymmetry. From the scattering amplitudes derived in Section VI.B we easily read the $\hat{H}_0$ and $\hat{H}_1$ S-matrices:

$$S_s = \frac{1}{(2ik + (-1)^s \alpha) \left[ 4k^2 + (-1 + e^{2iak})^2 \alpha^2 \right]} \left( \frac{8i k^3}{(-1)^{s+1} 2 \alpha \left( (4k^2 + \alpha^2) \cos 2ka - \alpha^2 - 2k^2 \right)} \right),$$

where as usual $s = 0, 1$. The poles of the transmission amplitudes $\sigma_0(k)$ and $\sigma_1(k)$ on the positive imaginary half-axis $k = i\kappa$, $\kappa > 0$, in the $k$-complex plane give the bound states respectively of $\hat{H}_0$ and $\hat{H}_1$. The roots of the transcendent equations

$$(2\kappa - \alpha) (2\kappa + \alpha - \alpha e^{-2\alpha \kappa}) (2\kappa - \alpha + \alpha e^{-2\alpha \kappa}) = 0.$$  \hspace{1cm} (34)

$$(2\kappa + \alpha) (2\kappa + \alpha - \alpha e^{-2\alpha \kappa}) (2\kappa - \alpha + \alpha e^{-2\alpha \kappa}) = 0.$$  \hspace{1cm} (35)

characterize thus the bound state energies and wave functions of both $\hat{H}_0$ and $\hat{H}_1$. There are one or two solutions of (34) for positive $\kappa$ depending on the relative values of $\alpha$ and $a$. The ground state of the SUSY system is the lower energy bound state of $\hat{H}_0$:

1. $\kappa_0 = \frac{\alpha}{2}$ is a pole of $\sigma_0(k)$ corresponding to the eigenvalue $E_0^{(0)} = 0$ of $\hat{H}_0$. Because $\kappa > 0$ the bound state wave function $\psi_0(x)$ is normalizable and supersymmetry is unbroken.

2. The second factor in the left member of (34) is only null for $\kappa = 0$ giving an unacceptable physical state.

3. The third factor, however, presents a positive root $\kappa_1 > \kappa_0 > 0$ of (34) if and only if the distance between $\delta s$ is such that $a > \frac{1}{\alpha}$. There is a second bound state of $\hat{H}_0$ of energy $E_1^{(0)} = \frac{\alpha^2}{4} - \kappa_1^2 > 0 = E_0^{(0)}$.

The point spectrum of $\hat{H}_1$ is also easy to unveil. There is an antibound state of zero energy because the root $\kappa_0 = -\frac{\alpha}{2}$ gives a non-normalizable wave function. The other, positive, root of (35) exists provided that $a > \frac{1}{\alpha}$. This root is exactly the same as the second positive root in (34). Therefore, there might be only one positive bound state in the point spectrum of $\hat{H}_1$, paired with the second bound state of $\hat{H}_0$, $E_1^{(1)} = E_1^{(0)} > 0$, as requested by supersymmetry.

![Wave function of the ground state: the zero mode of $\hat{H}_0$, plotted for $a = 1$ and $\alpha = 2$](image)

The normalized wave function of the lower bound state of $\hat{H}_0$ is of the form:

$$\psi_0^{(0)}(x) = \sqrt{\frac{\alpha}{4 e^{-\alpha a} - 2 e^{-2\alpha a}}} \begin{cases} e^{\frac{\alpha}{2}x} & x < -a, \\ e^{-\alpha (x-a)} & -a < x < a, \\ e^{-\frac{\alpha}{2}x} & x > a. \end{cases}$$

A graphic of $\psi_0^{(0)}(x)$ is shown in Figure 7 for the special values $a = 1$ and $\alpha = 2$.

7. **Infinite Dirac $\delta$-interactions of the same strength alternatively attractive and repulsive**

It might be interesting to pursue the analysis of this kind of systems for superpotentials:

$$W(x, \alpha) = \frac{\alpha}{2} \sum_{n=-J}^{J} (-1)^n |x - na|, \hspace{1cm} 0 < J \in \mathbb{N}.$$  \hspace{1cm} (36)
The essential features of the spectra of these arrays of $N = 2J + 1 \delta s$, however, are captured by the $J = 1$ system just analyzed. The only possible differences are due to the number of bound states, up to a maximum of $J + 1$, depending on the strength $\alpha$ and the distance $a$ between the $\delta$-potentials.

We therefore jump to discuss the $N = +\infty$ array characterized by the superpotential ($\alpha > 0$)

$$W(x) = \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} (-1)^n |x - na|, \quad n \in \mathbb{Z}$$

that gives rise to the following Hamiltonians ($s = 0, 1$):

$$\hat{H}_s = -\frac{d^2}{dx^2} + W' + (-1)^s W'' = -\frac{d^2}{dx^2} + (-1)^s \alpha \sum_{n=-\infty}^{\infty} (-1)^n \delta(x - na) + \frac{\alpha^2}{4}$$

In order to solve the spectral equations for the Schrödinger operators $\hat{H}_0$ and $\hat{H}_1$

$$-\frac{d^2}{dx^2} + V_s(x)\psi(x) = -\frac{d^2}{dx^2} \psi(0) + \sum_{n=-\infty}^{\infty} (-1)^n \alpha \sum_{n=-\infty}^{\infty} (-1)^n \delta(x - na) + \frac{\alpha^2}{4}$$

we emphasize that $V_0$ and $V_1$ are periodic potentials of period $2a$: $V_0(x + 2a) = V_0(x)$, $V_1(x + 2a) = V_1(x)$. According to Bloch’s theorem the wave functions are quasi-periodic

$$\psi_q(x + 2a) = e^{i2qa}\psi_q(x), \quad -\frac{\pi}{2a} \leq q \leq \frac{\pi}{2a},$$

the quasi-momentum $q$ characterizing a particular wave function within each (allowed) band. It is enough to search for the solution in one primitive cell, e.g., in the interval $[2na, 2(n + 1)a]$, where the solution can be written as the linear combination ansatz

$$\psi_{qn}(x) = \begin{cases} A_n e^{ik(x - 2na)} + B_n e^{-ik(x - 2na)} & 2na < x < (2n + 1)a \\ C_n e^{ik(x - 2na)} + D_n e^{-ik(x - 2na)} & (2n + 1)a < x < 2(n + 1)a \end{cases}$$

provided that: $k^2 = E - \frac{\alpha^2}{4}$. Bloch’s theorem now relates the amplitudes between any pair of primitive cells, e.g., $[0, 2a]$ and $[2na, 2(n + 1)a]$, in such a way that, $\forall n$ and $k \neq q$,

$$A_n = e^{2i\pi q} A_0, \quad B_n = e^{2i\pi q} B_0.$$  

### 7.1. Propagating bands

In the $[0, 2a]$ primitive cell we thus write $\psi_{q0}(x)$ as

$$\psi_{q0}(x) = \begin{cases} A_0 e^{ikx} + B_0 e^{-ikx} & 0 < x < a \\ C_0 e^{ikx} + D_0 e^{-ikx} & a < x < 2a \end{cases}$$

whereas $\psi_{q1}(x)$ in the next $[2a, 4a]$ interval reads:

$$\psi_{q1}(x) = e^{2i\pi q} \begin{cases} A_0 e^{ik(x - 2a)} + B_0 e^{-ik(x - 2a)} & 2a < x < 3a \\ C_0 e^{ik(x - 2a)} + D_0 e^{-ik(x - 2a)} & 3a < x < 4a \end{cases}$$

The matching conditions at the point $x = a$, $\psi_{q0}(a^-) = \psi_{q0}(a^+$) and $\psi'_{q0}(a^-) - \psi'_{q0}(a^+) = \pm \alpha \psi_{q0}(a)$ force the two algebraic equations:

$$-e^{iak} A_0 - e^{-iak} B_0 + e^{iak} C_0 + e^{-iak} D_0 = 0, \quad (\mp \alpha e^{iak} - ike^{-iak}) A_0 + (ike^{-iak} \mp \alpha e^{-iak}) B_0 + ike^{-iak} C_0 - ike^{-iak} D_0 = 0,$$

where the sign $-$ responds to the $V_0$ potential and the sign $+$ arises in the $V_1$ comb. Two more algebraic equations come from the matching conditions at the boundary point between the two primitive cells:
\[ \psi_{q0}(2a_>) = \psi_{q1}(2a_>) \quad \text{and} \quad \psi_{q1}(2a_>) - \psi_{q0}^\prime(2a_<) = \mp \alpha \psi_{q0}(2a) . \]

With the signs \( \pm \) describing respectively the \( V_0 \) and \( V_1 \) potentials these equations are:

\[
- e^{2iq\alpha} A_0 - e^{2iq\alpha} B_0 + e^{2ik\alpha} C_0 + e^{-2ik\alpha} D_0 = 0 , \tag{42}
\]

\[
i e^{2iq\alpha} A_0 - i e^{2iq\alpha} B_0 + (\pm i e^{2ik\alpha} - i e^{2ik\alpha}) C_0 + (\pm i e^{-2ik\alpha} + i e^{-2ik\alpha}) D_0 = 0 . \tag{43}
\]

All together the four equations (40), (41), (42) and (43) form an homogeneous system of four algebraic equations. The four unknowns \( A_0, B_0, C_0, \) and \( D_0 \) give rise to non-trivial solutions of (38)-(39) if the determinant of the \( 4 \times 4 \)-matrix of this algebraic system is zero, i.e., if the spectral equation

\[
\cos 2qa = \frac{(4k^2 + \alpha^2) \cos 2ka - \alpha^2}{4k^2} , \tag{44}
\]

is satisfied.

**Remark:** The determinant of the system and consequently the equation (44) is invariant under the exchanges of \( \alpha \to -\alpha \) and \( k \to -k \). The intersection points of the curves \( f(q) = \cos 2qa \) and \( g(k) = (4k^2 + \alpha^2) \cos 2ka - \alpha^2 \) determine the correlative values of the quasi-momenta \( q \) and the wave momenta \( k \) from which solutions of the Schrödinger equations for both \( H_0 \) and \( H_1 \) can be built by extending the solutions in one primitive cell to any other primitive cell through use of the connection formulas (37).

In Figure 8 we show a graphic of the dispersion relation (44). Four energy bands are depicted with band lower edges given by the intersection of the \( g(k) \) curve with the straight line \( y(q) = -1 \) whereas the band upper edges are the intersections of \( g(k) \) with \( y(q) = 1 \). Because the invariance with respect to the sign of \( \alpha \) these bands are paired by supersymmetry and belongs to both the spectra of \( H_0 \) and \( H_1 \). We also mention that due to the invariance with respect to the sign of \( k \) all these wave functions are double degenerate corresponding to propagating waves moving either from left to right or viceversa.

In the allowed bands the solution of the homogeneous system (40)-(43) provides us with the amplitudes \( B_0(k), C_0(k) \) and \( D_0(k) \) as functions of \( A_0(k) \) for any pair \( (q, k) \) complying with (44). Plugging these solutions in (38) we obtain the propagating wave functions of the bosonic \( (\psi_{q0}^{(0)}) \) and fermionic sectors \( (\psi_{q0}^{(1)}) \) in the primitive cell \([0, 2a]\)

\[
\psi_{q0}^{(0)}(x) = A_0 \left\{ \begin{array}{ll}
\frac{(2k+i\alpha)e^{2ik+q}+i\alpha e^{2ik+q}-2k}{i\alpha(e^{2ik+q}-1)-2k+2\alpha e^{2ik+q}} e^{ikx} + \frac{2k-\alpha i e^{2ik+q}+i\alpha e^{2ik+q}+2k e^{2ik+q}}{i\alpha(e^{2ik+q}-1)-2k+2\alpha e^{2ik+q}} e^{-ikx}, & 0 < x < a \\
\frac{2k+i\alpha e^{2ik+q}+i\alpha e^{2ik+q}}{i\alpha e^{2ik+q}-1}-2k+\alpha e^{2ik+q} e^{ikx} + \frac{2k+\alpha i e^{2ik+q}+i\alpha e^{2ik+q}-2k e^{2ik+q}}{i\alpha(e^{2ik+q}-1)-2k+2\alpha e^{2ik+q}} e^{-ikx}, & a < x < 2a 
\end{array} \right.
\]

\[
\psi_{q0}^{(1)}(x) = A_0 \left\{ \begin{array}{ll}
\frac{(2k+i\alpha)e^{2ik+q}+i\alpha e^{2ik+q}-2k}{i\alpha(e^{2ik+q}-1)-2k+2\alpha e^{2ik+q}} e^{ikx} + \frac{2k-\alpha i e^{2ik+q}+i\alpha e^{2ik+q}+2k e^{2ik+q}}{i\alpha(e^{2ik+q}-1)-2k+2\alpha e^{2ik+q}} e^{-ikx}, & 0 < x < a \\
\frac{2k+i\alpha e^{2ik+q}+i\alpha e^{2ik+q}}{i\alpha e^{2ik+q}-1}-2k+\alpha e^{2ik+q} e^{ikx} + \frac{2k+\alpha i e^{2ik+q}+i\alpha e^{2ik+q}-2k e^{2ik+q}}{i\alpha(e^{2ik+q}-1)-2k+2\alpha e^{2ik+q}} e^{-ikx}, & a < x < 2a 
\end{array} \right.
\]
being $A_0 = A_0(q,k)$ a normalization constant.

7.2. Non-propagating band

There might exist also bands of non propagating states if the dispersion relation \((44)\) admits solutions with purely imaginary momenta: $k = i\kappa$, $\kappa \in \mathbb{R}$. In the right member of this transcendental equation trigonometric functions are traded by hyperbolic:

$$\cos 2qa = \left(\frac{4\kappa^2 - \alpha^2}{4\kappa^2}\right) \cosh 2\kappa a + \alpha^2. \quad (45)$$

Because the modulus of $g(\kappa) = \left(\frac{4\kappa^2 - \alpha^2}{4\kappa^2}\right) \cosh 2\kappa a + \alpha^2$ is greater than one for a relatively low value of $\kappa$ (depending on the values of $a$ and $\alpha$) there is at most one band of this kind of states. As requested by supersymmetry the energy of any state in this band is such that $0 \leq E_q < \frac{\alpha^2}{4}$, this last value being the threshold for propagating bands. The bosonic and fermionic non propagating wave functions in the band with energy less than $\frac{\alpha^2}{4}$ have the following form in the primitive cell $[0, 2a]$

$$\psi_{q0}^{(0)}(x) = A_0 \left\{ \begin{array}{ll}
e^{-\kappa x} + \frac{(2\kappa - \alpha)e^{-4\kappa a - 2\kappa e^{2a(q-\kappa)}} + \alpha e^{-2a\kappa} - 2\kappa}{\alpha(e^{-2\kappa a} - 1) - 2\kappa + 2\kappa e^{2a(q-\kappa)}} e^{\kappa x}, & 0 < x < a \\
\frac{2\kappa - \alpha}{\alpha(e^{-2\kappa a} - 1) - 2\kappa + 2\kappa e^{2a(q-\kappa)}} e^{-\kappa x} + \frac{\alpha e^{2a(q-\kappa)} - (2\kappa + \alpha)e^{2a(q-\kappa)} + 2\kappa e^{-4\kappa a}}{\alpha(e^{-2\kappa a} - 1) - 2\kappa + 2\kappa e^{2a(q-\kappa)}} e^{\kappa x}, & a < x < 2a 
\end{array} \right. \quad (46)$$

$$\psi_{q0}^{(1)}(x) = A_0 \left\{ \begin{array}{ll}
e^{-\kappa x} - \frac{(2\kappa + \alpha)e^{-4\kappa a - 2\kappa e^{2a(q-\kappa)}} - \alpha e^{-2a\kappa} - 2\kappa}{\alpha(e^{-2\kappa a} - 1) + 2\kappa - 2\kappa e^{2a(q-\kappa)}} e^{\kappa x}, & 0 < x < a \\
\frac{2\kappa + \alpha}{\alpha(e^{-2\kappa a} - 1) + 2\kappa - 2\kappa e^{2a(q-\kappa)}} e^{-\kappa x} - \frac{\alpha e^{2a(q-\kappa)} - (2\kappa - \alpha)e^{2a(q-\kappa)} - 2\kappa e^{-4\kappa a}}{\alpha(e^{-2\kappa a} - 1) + 2\kappa - 2\kappa e^{2a(q-\kappa)}} e^{\kappa x}, & a < x < 2a 
\end{array} \right. \quad (47)$$

being again $A_0 = A_0(q,\kappa)$ a normalization constant. In order to determine which are the relative values of $a$ and $\alpha$ such that a given member of the non propagating band characterized by the quasi-momentum $q$ is a truly solution of the Schrödinger equation we rewrite equation \((45)\) in the form

$$\frac{4\kappa^2}{\alpha^2} = \frac{1 - \cosh 2\kappa a}{\cos 2qa - \cosh 2\kappa a}. \quad (48)$$

We observe that \((48)\), like \((45)\), is also invariant under $\alpha \rightarrow -\alpha$ and $\kappa \rightarrow -\kappa$. This means that there are one bosonic and one fermionic non propagating band paired by supersymmetry. The wave functions obtained from \((46)\) and \((47)\) by applying Bloch’s theorem are, however, non degenerate. Despite the $\kappa \rightarrow -\kappa$ invariance of the spectral equation only solutions with $\kappa > 0$ are physically sensible. We list the critical relations between $a$ and $\alpha$ for the existence of this kind of solutions.

- Energy band lower edge solutions: $q = 0 \Rightarrow \cos 2qa = 1$. In this case \((48)\) reduces to

$$\frac{4\kappa^2}{\alpha^2} = 1.$$ 

There are two intersection points, $\kappa = \pm \frac{\alpha}{2}$, between the parabola on the left and the straight line on the right for any value of $a$ and $\alpha$. It is convenient to discuss these two solutions separately

1. $\kappa = \frac{\alpha}{2}$. This is the lowest (zero) energy state (being the lower edge in the non propagating band).

Setting $A_0(0, \frac{\alpha}{2}) = 1$ the ground state wave function in the primitive cell reads:

$$\psi^{(1)}_{00}(x) = \left\{ \begin{array}{ll}
e^{-\frac{\alpha}{2}x}, & 0 < x < a \\
\frac{\alpha}{2}x - \alpha a, & a < x < 2a 
\end{array} \right. \quad (49)$$

Accordingly, we plot the ground state wave function (of $\hat{H}_1$) by applying to \((49)\) Bloch’s theorem, as it is shown in Figure \([10]\). We realize that the corresponding wave function is a bona fide zero mode of the Hamiltonian $\hat{H}_1$ under the assumption $\alpha > 0$. Note that $V_1(x)$ describes wells at $x = na$ if $n$ is odd but walls for $n$ even. $\psi^{(1)}_{0n}(x)$ exhibits peaks at $x = na$ for both $n$ odd and even but the
peaks are maxima at the wells and minima at the walls as it should be. This state behaves badly, however, with respect to \( V_0(x) \) because the maxima are at the walls and the minima at the wells, a physically absurd situation that must be rejected due to the nature of this state as the analogous to an antibound state arising in the spectrum of \( \hat{H}_0 \).

2. \( \kappa = -\frac{\alpha}{2} \). This state also is the band lower edge of non propagating wave functions. It is exactly the (zero mode) ground state of \( \hat{H}_0 \). In fact, in the primitive cell this wave function reads:

\[
\psi^{(0)}_{00}(x) = \begin{cases} 
  e^{\frac{\kappa}{2} x}, & 0 < x < a \\
  e^{-\frac{\kappa}{2} x + \alpha}, & a < x < 2a
\end{cases}
\]  

(50)

The Bloch extension behaves appropriately on the wells (having maxima) and the walls (having minima) of \( V_0(x) \). We plot the ground state wave function (of \( \hat{H}_0 \)) by applying to (50) Bloch’s theorem, as it is shown in Figure 10. Therefore, the union of all the \( \psi^{(n)}_{0m}(x) \) cell wave functions is the ground state of zero energy of \( \hat{H}_0 \). This state, however, does not belong to the spectrum of \( \hat{H}_1 \) because it is physically nonsense just like \( \psi^{(1)}_{0n}(x) \) with respect to \( \hat{H}_0 \).

In sum, in the supersymmetric quantum mechanical system there are always one bosonic and one fermionic non propagating ground states, a paradoxical situation typical of supersymmetric periodic potentials.

Table 10: Graphics of \( \psi^{(0)}_{00}(x) \), \( V_0(x) \) (left) and \( \psi^{(1)}_{00}(x) \), \( V_1(x) \) (right) for \( a = 1 \) and \( \alpha = 2 \)

\[ \psi^{(0)}_{00}(x) \]

![Graph of \( \psi^{(0)}_{00}(x) \), \( V_0(x) \) (left) and \( \psi^{(1)}_{00}(x) \), \( V_1(x) \) (right) for \( a = 1 \) and \( \alpha = 2 \).]

- At upper edge band values of the quasi-momentum \( q = \pm \frac{\pi}{2a} \Rightarrow \cos 2qa = -1 \) the equation (48) reduces to the form:

\[
\frac{4\kappa^2}{\alpha^2} = \frac{\cosh 2\kappa a - 1}{\cosh 2\kappa a + 1}.
\]

(51)

Equation (51) is a transcendent one such that the solutions can be only identified by graphical methods as the intersections of the parabola \( f(\kappa) = \frac{4\kappa^2}{\alpha^2} \) with the transcendent curve \( h(\kappa) = \frac{\cosh 2\kappa a - 1}{\cosh 2\kappa a + 1} \). Both curves pass through the origin, \( f(0) = 0 = h(0) \), which is a critical point of \( f \) and \( h \) as well: \( f'(0) = 0 = h'(0) \).

Besides \( \kappa = 0 \), which is a solution that does not correspond to a physical acceptable state the wave function being trivial, the number of intersections depends on the second derivative of these functions at \( \kappa = 0 \): \( f''(0) = \frac{\alpha^2}{a^2}, h''(0) = 2a^2 \). There is a critical width \( a_c \), depending on the strength \( \alpha \), \( a_c = \frac{\alpha}{2} \) for which the curvature of the two curves at the origin coincide. If \( a > a_c \) there are one positive \( \kappa_+ > 0 \) and one negative \( \kappa_- < 0 \) solutions of (51). In this case the whole non propagating band belongs to the spectrum of the supersymmetric Hamiltonian, the positive intersection to \( \hat{H}_1 \) and the negative intersection to \( \hat{H}_0 \). When \( a < a_c \) there are no intersections between \( g(\kappa) \) and the straight line \( y = -1 \): the band upper edge disappears of the \( \hat{H}_1 \) and \( \hat{H}_0 \) spectra.

- The search for solutions of (48) inside the non propagating band, \(-1 < \cos 2qa < 1\), also requires the identification of a critical width. The second derivative of the curve \( h(q, \kappa) \) on the right member of this equation is equal to the curvature of \( f(\kappa) \) at the \( \kappa = 0 \) point if and only if:

\[
4\sin^2 a_c q = \alpha^2 a_c^2.
\]
Again, given a solution of this transcendent equation, if $a > a_c$ there are two intersections, one positive, the other negative, of the curves in the two members of (48), telling us that the corresponding wave function belongs to the spectrum of the supersymmetric pair of Hamiltonians. If $a < a_c$ there are no non null solutions and the non propagating state dissapears from the SUSY spectrum.

In Figure 9 we have depicted the non propagating band for three different weights $\alpha = 2, \alpha = 3, \alpha = 4$ and a single width: $a = 1$. Note that the last two cases are well within the $a > a_c$ regime even for the band upper edge. In the first case, however, the width is critical: $a_c = 1$. Thus, the band upper edge lies precisely at the boundary.

![Figure 9: Graphics of $g(\kappa) = -\frac{1}{4\kappa^2}(\alpha^2 - 4\kappa^2) \cosh 2\kappa a - \alpha^2$ for $a = 1$ and $\alpha = 2$ (blue), $\alpha = 3$ (brown) and $\alpha = 4$ (green). States such that $-1 \leq g(\kappa) \leq 1$ form respectively the blue, brown, and green bands.](image)

Table 11: Graphics of $g(\kappa) = -\frac{1}{4\kappa^2}(\alpha^2 - 4\kappa^2) \cosh 2\kappa a - \alpha^2$. In the upper row the $g(\kappa)$ curves are plotted for $a = 1, \alpha = 1.8$ (left) and $a = 0.75, \alpha = 2$ (right). In the lower row the $g(\kappa)$ curves are depicted for $a = 0.5, \alpha = 2.2$ (left) and the almost limiting case $a = 0.01, \alpha = 1$ (right).
7.3. Ground states: the problem of spontaneous supersymmetry breaking

Analysis of the possible spontaneous symmetry breaking in this system requires to focus on the characteristics of the ground states. If there are ground states of zero energy such that these states are singlets of the SUSY algebra supersymmetry is unbroken. The singlet ground states have necessarily the form \( \psi^{(0)}(x) = e^{W(x,\alpha,a)} \) or \( \psi^{(1)}(x) = e^{-W(x,\alpha,a)} \) respectively in the bosonic and fermionic sectors. In the \( J = 1 \) system \( \psi^{(0)}(x) = e^{W(x,\alpha,a)} \) is a normalizable state, whereas \( \psi^{(1)}(x) \notin L^2(\mathbb{R}) \). Thus, there is a bona fide bosonic singlet ground state and supersymmetry is unbroken. Note that the value of the wave function at the \( V_0 \) wells, \( \psi^{(0)}(\pm a) = e^{-\frac{\alpha}{2}a} \), is bigger than the value at the wall: \( \psi^{(0)}(0) = e^{-\alpha a} \). It is clear that the wave function exponentially decreases to zero at \( \pm \infty \): \( \psi^{(0)}(\pm \infty) = 0 \). The reason why we select \( \psi^{(0)}(x) = e^{W(x,\alpha,a)} \) is also clear. It is not only that changing \( W \) by \( -W \) makes the wave function not normalizable but there would be a maximum at the wall and minima at the wells contrarily to expectations from physics. In the next case, \( J = 2 \), we have: \( \psi^{(1)}(0) = \pm 2a ) = e^{-\alpha a}, \psi^{(1)}(0) = e^{-\frac{3}{2}a} \), and \( \psi^{(1)}(\pm \infty) = 0 \). The situation is identical to the former one in the \( J = 1 \) problem but the normalizable ground state \( \psi^{(1)}(x) \) belongs to the fermionic sector and \( \psi^{(0)}(x) \notin L^2(\mathbb{R}) \). One easily checks that there is also a single bosonic ground state \( \psi^{(0)}(x) \) for \( J = 3 \), whereas for \( J = 4 \) the ground state is also unique but fermionic: \( \psi^{(1)}(x) \). The general pattern is clear for finite \( N = 2J + 1 \): supersymmetry is always unbroken but the “regularized” Witten index is:

\[
I^N_W = (-1)^{J+1} + 1.
\]

It is thus two for \( J \) odd and zero for \( J \) even. We recall that the addend 1 comes from the difference between the spectral densities in the continuous spectra.

In order to find what happens in the periodic \( N \to +\infty \) limit we write in a compact form the superpotential for finite \( N = 2J + 1 \) in what is going to be the primitive cell of the periodic potential:

\[
W(x,\alpha,a) = \begin{cases} 
\frac{a}{2}(x + 2a \sum_{n=1}^{J} (-1)^{n}n) = \frac{a}{2} \left( x + \frac{a}{2} \left( (-1)^{J}(1 + 2J) - 1 \right) \right) & , \quad 0 < x < a \\
-\frac{a}{2}(x - 2a \sum_{n=2}^{J} (-1)^{n}n) = -\frac{a}{2} \left( x - \frac{a}{2} \left( (-1)^{J}(1 + 2J) + 3 \right) \right) & , \quad a < x < 2a.
\end{cases}
\]  

(52)

To take the \( J \to \infty \) limit in (52) requires to consider the Dirichlet eta function defined as the alternating series

\[
\eta(s) = -\sum_{n=1}^{\infty} (-1)^{n} \cdot \frac{1}{n^{s}} = -(1 - 2^{1-s})\zeta(s) , \quad s \in \mathbb{C}
\]

at the special point: \( s = -1 \). Although \( \eta(s) \) is strictly convergent only for \( \text{Re} s > 0 \) the relation shown above with the Riemann zeta function \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s}} \) allows us to assign via analytic continuation the value \( \eta(-1) = -\frac{1}{12} \) from the famous sum of all the positive integers: \( \zeta(-1) = -\frac{1}{12} \). Therefore, we obtain:

\[
W(x,\alpha,a) = \begin{cases} 
\frac{a}{2}(x - \frac{2a}{3}) & , \quad 0 < x < a \\
-\frac{a}{2}(x - \frac{2a}{3}) & , \quad a < x < 2a.
\end{cases}
\]

Neither \( \psi^{(0)}(x) = e^{W(x,\alpha,a)} \) nor \( \psi^{(1)}(x) = e^{-W(x,\alpha,a)} \) are normalizable. Nevertheless, both wave functions are bona fide ground states. In periodic potentials the wave functions are not normalizable and, moreover, the behaviour of \( \psi^{(0)} \) and \( \psi^{(1)} \) properly describes the physics of the problem:

\[
\psi^{(0)}(\pm 2a) = e^{-\frac{\alpha}{2}a} , \quad \psi^{(1)}(\pm 2a) = e^{\frac{\alpha}{2}a} \\
\psi^{(0)}(\pm (2n + 1)a) = e^{\frac{\alpha}{2}a} , \quad \psi^{(1)}(\pm (2n + 1)a) = e^{-\frac{\alpha}{2}a}.
\]

In the \( N = \infty \) limit there are two ground states, one bosonic and one fermionic. The Bloch condition \( \psi_{q}(x + 2a) = e^{2\pi qa} \psi_{q}(x) \) is satisfied by both

\[
\psi^{(0)}(x) = \exp \left[ \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} (-1)^{n} |x - na| \right] \quad \text{and} \quad \psi^{(1)}(x) = \exp \left[ -\frac{\alpha}{2} \sum_{n=-\infty}^{\infty} (-1)^{n} |x - na| \right]
\]
because $\psi_0^{(0)}(2a) = \psi_0^{(0)}(0)$ and $\psi_0^{(1)}(2a) = \psi_0^{(1)}(0)$. Note that

\[
\psi_0^{(0)}(2a) = \exp \left[ \frac{\alpha}{2} a \sum_{n=-\infty}^{\infty} (-1)^n |2-n| \right], \quad \psi_0^{(0)}(0) = \exp \left[ \frac{\alpha}{2} a \sum_{n=-\infty}^{\infty} (-1)^n |n| \right]
\]

\[
\psi_0^{(1)}(2a) = \exp \left[ -\frac{\alpha}{2} a \sum_{n=-\infty}^{\infty} (-1)^n |2-n| \right], \quad \psi_0^{(1)}(0) = \exp \left[ -\frac{\alpha}{2} a \sum_{n=-\infty}^{\infty} (-1)^n |n| \right],
\]

whereas the two series in the exponents are identical:

\[
\sum_{n=-\infty}^{\infty} (-1)^n |2-n| = 2 \sum_{j=1}^{\infty} (-1)^j j = 2\eta(-1) = -\frac{1}{2}, \quad j = |2-n|
\]

\[
\sum_{n=-\infty}^{\infty} (-1)^n |n| = 2 \sum_{n=1}^{\infty} (-1)^n n = 2\eta(-1) = -\frac{1}{2}.
\]

Here we have dealt with the Dirichlet eta function $\eta(s) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^s}$ which is finite at $s = -1$. Supersymmetry is thus unbroken but the contribution to the Witten index of the zero energy modes is zero.

8. Summary and outlook

We have analyzed in depth the quantum dynamics of one massive particle moving on the real line under the influence of a potential produced by several configurations of Dirac $\delta$-wells and/or $\delta$-walls. Particular attention has been paid to the study of bound state distribution as function of the $\delta$-strengths and the distances between them. This information has been used to construct supersymmetric extensions of this system that requires the addition of quasi-square wells. In the supersymmetric framework the dynamics is governed by two intertwined scalar Schrödinger operators with semi-definite spectra: the positive energy eigenfunctions form doublets of the SUSY algebra whereas the zero modes are singlets. We have found that in the supersymmetric systems with a finite number of $\delta$s there is always one zero energy ground state in the spectrum of one of the two partner Hamiltonians such that supersymmetry is unbroken. Letting the number of $\delta$-potentials go to infinity spectra of SUSY paired conducting bands arise in the two intertwined operators. The two partner Hamiltonians, however, admit also a non-conducting band that encompasses one ground state of zero energy. Supersymmetry is also unbroken but the Witten index is zero.

The concepts, ideas and techniques developed in this paper are prepared to be applied on systems of $N$ particles restricted to move on a line with contact interactions. In particular, the integrable models of Yang, Lieb, and Liniger, see [4]-[3], present the challenge of building over them supersymmetric extensions that preserve their integrability. An intriguing question to be answered is the possible existence of more than one ground state of zero energy. One may state, rather vaguely, that in Supersymmetric Quantum Mechanics of one degree of freedom systems only periodic potentials leave room for more than one ground state of zero energy. The $N$-body Yang-Lieb-Liniger systems have $N$ degrees of freedom and one might think in more than one zero energy ground state in the SUSY version of these systems. In fact, there are two zero energy ground states in the supersymmetric extension of the Euler-Coulomb problem, a charged particle moving on a plane under the action of two Coulombian centres of force, build in the Reference [43].

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