Martingale Nature and Laws of the Iterated Logarithm for Markov Processes of Pure-Jump Type

Yuichi Shiozawa 1 · Jian Wang 2

Received: 5 November 2019 / Revised: 29 July 2020 / Published online: 3 September 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
We present sufficient conditions, in terms of the jumping kernels, for two large classes of conservative Markov processes of pure-jump type to be purely discontinuous martingales with finite second moment. As an application, we establish the law of the iterated logarithm for sample paths of the associated processes.

Keywords  Feller process · Hunt process · Lower bounded semi-Dirichlet form · Martingale · Jumping kernel · Law of the iterated logarithm

Mathematics Subject Classification  60J75 · 47G20 · 60G52

1 Introduction
It is well known that any symmetric Lévy process with finite first moment possesses the martingale property because of the independent increments property. Apart from Lévy processes, the martingale property was studied for a one-dimensional diffusion process with natural scale (see [9,17] and references therein). Note that this process is a time-changed Brownian motion and thus possesses the local martingale property (see, e.g., [13, Proposition V.1.5]). In [9,17], a necessary and sufficient condition is given for this process to be a martingale by adopting the Feller theory.
To the best of our knowledge, except for these Markov processes mentioned above, answers are not available in the literature to the following “fundamental” question — when does a Markov process become a martingale? The aim of this paper is to present explicit sufficient conditions for two large classes of jump processes to be purely discontinuous martingales with finite second moment in terms of jumping kernels. As an application, we show Khintchine’s law of the iterated logarithm (LIL) for two classes of non-symmetric jump processes. We also provide examples of non-symmetric jump processes which are purely discontinuous martingales with finite second moment and satisfy the LIL.

To derive the martingale property for a jump process, we apply two different approaches: One is based on the infinitesimal generator along with the moments calculus of the process, and the other relies on the componentwise decomposition of the process with the aid of the semimartingale theory ([7, Chapter II, Section 2]). The assumptions of our paper are mild. For example, condition (2.3) (or (3.4)) means the existence of the second moment for the jumping kernel (which seems to be necessary for the LIL), while condition (2.6) (or (3.3)) roughly indicates that there is no drift arising from jumps of the process.

Our motivation lies in the fact that the LIL holds for Lévy processes with zero mean and finite second moment as proved by Gnedenko [6] (see also [14, Proposition 48.9]). J.-G. Wang [22] established this kind of result for locally square integrable martingales and obtained Gnedenko’s result as a corollary ([22, Corollary 2]). For a symmetric jump process generated by non-local Dirichlet form, we provided in [20] a sufficient condition, in terms of the jumping kernel, for the long-time behavior of the sample path being similar to that of the Brownian motion. This condition implies the existence of the second moment for the jumping kernel. Our approach in [20] was based on the long-time heat kernel estimate. Recently, it is proved in [1] that for a special symmetric jump process, the second moment condition on the jumping kernel is equivalent to the validity of the LIL. The approach of [1] is based on the two-sided heat kernel estimate for full times. See [8,19] for related discussions on this topic. In contrast with [1,8,19,20], our result is applicable to non-symmetric jump processes. Moreover, our approach is elementary in the sense that we use the martingale theory of stochastic processes.

The rest of this paper is organized as follows: In Section 2, we first consider the martingale property of a class of Feller processes of pure-jump type and then prove the LIL. Some new examples including jump processes of variable order are also presented. The corresponding discussions for non-symmetric Hunt processes generated by lower bounded semi-Dirichlet forms of pure-jump type are considered in Section 3.

Throughout this paper, the letters $c$ and $C$ (with subscript) denote finite positive constants which may vary from place to place. For $x \in \mathbb{R}^d$, let $x^{(i)}$ be its $i$th coordinate; that is, $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d$. We denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}^d$. Let $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_b(\mathbb{R}^d)$ denote, respectively, the family of Borel measurable sets on $\mathbb{R}^d$ and the set of bounded Borel measurable functions on $\mathbb{R}^d$. Let $C^\infty_c(\mathbb{R}^d)$ (resp. $C^2_c(\mathbb{R}^d)$) be the set of smooth (resp. twice continuously differentiable) functions with compact support in $\mathbb{R}^d$, and let $C^2_b(\mathbb{R}^d)$ be the set of twice continuously differentiable
functions on \( \mathbb{R}^d \) with all bounded derivatives. Let \( C_\infty(\mathbb{R}^d) \) be the set of continuous functions on \( \mathbb{R}^d \) vanishing at infinity.

2 Martingale Nature for Feller Processes

2.1 Preliminaries

Let \( X := \{ \Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d} \} \) be a time-homogeneous Markov process on \( \mathbb{R}^d \). Let \((T_t)_{t \geq 0}\) be the Markov semigroup associated with the process \( X \), i.e.,

\[
T_t u(x) = \mathbb{E}_x u(X_t), \quad u \in B_b(\mathbb{R}^d), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d.
\]

According to [3, Definition 1.16], we call \( X \) a Feller process if \((T_t)_{t \geq 0}\) is a Feller semigroup on \( C_\infty(\mathbb{R}^d) \); that is, it satisfies the following properties:

- (Feller property) for any \( u \in C_\infty(\mathbb{R}^d) \) and \( t > 0 \), \( T_t u \in C_\infty(\mathbb{R}^d) \);
- (strong continuity) for any \( u \in C_\infty(\mathbb{R}^d) \), \( \| T_t u - u \|_\infty \rightarrow 0 \) as \( t \rightarrow 0 \).

In what follows, we suppose that \( X \) is a Feller process on \( \mathbb{R}^d \). Let \( (\mathbb{R}^d)_\Delta := \mathbb{R}^d \cup \{ \Delta \} \) be a one-point compactification of \( \mathbb{R}^d \). Then, by [3, Theorems 1.19 and 1.20], we may and do assume that \( X \) satisfies the next properties:

- \( X \) has a càdlàg modification; that is, for every \( x \in \mathbb{R}^d \), a map \( t \mapsto X_t(\omega) \) is right continuous with left limits in \((\mathbb{R}^d)_\Delta\) for \( \mathbb{P}_x \)-a.s. \( \omega \in \Omega \);
- the filtration \((\mathcal{F}_t)_{t \geq 0}\) is complete and right continuous, and \( X \) is a strong Markov process with this filtration.

Define

\[
D(L) = \left\{ u \in C_\infty(\mathbb{R}^d) \left| \lim_{t \rightarrow 0} \frac{T_t u - u}{t} \text{ exists in } C_\infty(\mathbb{R}^d) \right. \right\}
\]

and

\[
Lu = \lim_{t \rightarrow 0} \frac{T_t u - u}{t}, \quad u \in D(L).
\]

The pair \((L, D(L))\) is called a Feller generator of the Feller semigroup \((T_t)_{t \geq 0}\). If \( C_\infty^c(\mathbb{R}^d) \subset D(L) \), then the general form of \( L \) is known (see, e.g., [3, Theorem 2.21]), and \( X \) enjoys an analogous Lévy-Itô decomposition (see, e.g., [15, Theorem 3.5] or [3, Theorem 2.44]). Furthermore, by [3, Theorem 1.36], for any \( u \in D(L) \),

\[
M_t^{[u]} := u(X_t) - u(X_0) - \int_0^t Lu(X_s) \, ds, \quad t \geq 0,
\]

is a martingale with respect to \((\mathcal{F}_t)_{t \geq 0}\).

Throughout this section, we impose the following conditions on the Feller generator \((L, D(L))\).
Assumption 2.1 Let \((L, D(L))\) be a Feller generator of the Feller semigroup associated with \(X\) so that the next two conditions are satisfied:

(i) \(C^\infty_c(\mathbb{R}^d) \subset D(L)\);
(ii) for any \(u \in C^\infty_c(\mathbb{R}^d)\),
\[
Lu(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left( u(x + z) - u(x) - \langle \nabla u(x), z \rangle 1_{\{|z| < 1\}}(z) \right) N(x, dz), \tag{2.2}
\]
where for any fixed \(x \in \mathbb{R}^d\), \(N(x, dz)\) is a nonnegative deterministic measure on \(\mathbb{R}^d \setminus \{0\}\) such that
\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 N(x, dz) < \infty. \tag{2.3}
\]

We comment on Assumption 2.1. Since the generator \(L\) in (2.2) consists of the jump part only, Assumption 2.1 implies that \(X\) is a semimartingale of pure-jump type; that is, there is no continuous part in the semimartingale decomposition of \(X\) (i.e., no diffusion term involved) (see, e.g., [3, Theorem 2.44]). The kernel \(N(x, dz)\) is called the jumping kernel of \(X\). Note that by (2.3),
\[
\sup_{x \in \mathbb{R}^d} \int_{\{|z| \geq 1\}} |z|^2 N(x, dz) < \infty, \tag{2.4}
\]
and so
\[
\sup_{x \in \mathbb{R}^d} \int_{\{|z| \geq 1\}} |z| N(x, dz) < \infty, \quad 1 \leq i \leq d. \tag{2.5}
\]

Under the full conditions of Assumption 2.1, \(X\) is conservative, i.e., \(T_t 1 = 1\) for any \(t \geq 0\) ([3, Theorem 2.33]). According to [3, Theorem 2.37 c) and a)], \(C^\infty_c(\mathbb{R}^d) \subset D(L)\) and the operator \((L, C^\infty_c(\mathbb{R}^d))\) has a unique extension to \(C^2_b(\mathbb{R}^d)\), which is still denoted by \((L, C^2_b(\mathbb{R}^d))\), such that the representation (2.2) remains true for this extension. We see further by [3, Theorems 2.37 i) and 1.36] that for any \(u \in C^2_b(\mathbb{R}^d)\), (2.1) is also a martingale with respect to \((\mathcal{F}_t)_{t \geq 0}\).

2.2 Martingale Property of Feller Processes

In this subsection, we present a sufficient condition on the jumping kernel \(N(x, dz)\) such that \(X\) is a purely discontinuous martingale with finite second moment.

**Theorem 2.2** Let Assumption 2.1 hold. Assume also that for any \(x \in \mathbb{R}^d\),
\[
\int_{\{|z| \geq 1\}} z^{(i)} N(x, dz) = 0, \quad 1 \leq i \leq d. \tag{2.6}
\]
Then, \(X\) is a purely discontinuous martingale such that for each \(t > 0\) and \(i = 1, \ldots, d\), \(X_t^{(i)}\) has finite second moment and the predictable quadratic variation of \(X\)
is given by
\[
\langle X^{(i)}, X^{(j)} \rangle_t = \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} z^{(i)} z^{(j)} N(X_s, \, dz) \right) \, ds, \quad 1 \leq i, j \leq d, \quad t > 0.
\] (2.7)

In the following, we will show two different approaches to prove Theorem 2.2. The first one relies on the expression of the generator \( L \) via moment calculus, and the second one is based on the canonical representation of the semimartingale. We think that the generator approach is more self-contained and does not refer to the [7] much. In particular, we believe that the generator approach works for more general Markov processes beyond Feller processes; for instance, Lévy type processes. Roughly speaking, the generator approach yields that a strong Markov process on \( \mathbb{R}^d \) is a martingale, if \( Lf(x) = 0 \), where \( f(x) = x^{(i)} \) for all \( 1 \leq i \leq d \). On the other hand, we also think that the approach based on the canonical representation of the semimartingale is quite interesting in itself.

2.2.1 Generator Approach

The key ingredient of the generator approach to establish Theorem 2.2 is the following statement for the first and second moments of \( X_t \).

**Proposition 2.3** Let Assumption 2.1 hold. Then, for any \( t > 0 \) and \( i = 1, \ldots, d \), \( X^{(i)}_t \) has finite second moment, and, for any \( x_0 \in \mathbb{R}^d \),

\[
\mathbb{E}_{x_0} \left[ (X^{(i)}_t - x^{(i)}_0)^2 \right] = \mathbb{E}_{x_0} \left[ \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} (z^{(i)})^2 N(X_s, \, dz) \right) \, ds \right] + 2 \mathbb{E}_{x_0} \left[ \int_0^t (X^{(i)}_s - x^{(i)}_0) \left( \int_{\{|z| \geq 1\}} z^{(i)} N(X_s, \, dz) \right) \, ds \right] < +\infty.
\] (2.8)

Moreover, it also holds that

\[
\mathbb{E}_{x_0} \left[ X^{(i)}_t - x^{(i)}_0 \right] = \mathbb{E}_{x_0} \left[ \int_0^t \left( \int_{\{|z| \geq 1\}} z^{(i)} N(X_s, \, dz) \right) \, ds \right].
\] (2.9)

We postpone the proof of Proposition 2.3 to the end of this part. Using this proposition, we can present the

**Proof of Theorem 2.2** Let Assumption 2.1 and (2.6) hold. We first show that \( X \) is a martingale having finite second moment and satisfying (2.7). That \( X \) has finite second moment has been claimed in Proposition 2.3. According to (2.9) and (2.6),

\[
\mathbb{E}_{x_0} [X^{(i)}_t - x^{(i)}_0] = \mathbb{E}_{x_0} \left[ \int_0^t \left( \int_{\{|z| \geq 1\}} z^{(i)} N(X_s, \, dz) \right) \, ds \right] = 0, \quad t > 0, \quad 1 \leq i \leq d.
\]

Then, by the Markov property, for any \( 0 < s \leq t \),

\[
\mathbb{E}_{x_0} [X^{(i)}_t \mid \mathcal{F}_s] = \mathbb{E}_{X_s} [X^{(i)}_{t-s}] = X^{(i)}_s,
\]
whence \((X^{(i)}_t)_{t \geq 0}\) is a martingale for all \(1 \leq i \leq d\).

Since
\[
\mathbb{E}_{x_0}[ (X^{(i)}_t - x^{(i)}_0)^2 ] = \mathbb{E}_{x_0}[(X^{(i)}_t)^2] - (x^{(i)}_0)^2,
\]
we see by (2.8), (2.6) and the Markov property that
\[
L_t := (X^{(i)}_t)^2 - \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} (z^{(i)})^2 N(X_s, dz) \, ds,
\]
is a martingale. This implies that
\[
\langle X^{(i)} \rangle_t = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} (z^{(i)})^2 N(X_s, dz) \, ds
\]
and thus, we obtain (2.7).

We next show that the martingale \(X\) is purely discontinuous. Let \(X^{(i),c}\) and \(X^{(i),d}\) be the continuous and purely discontinuous parts of \(X^{(i)}\), respectively (see [7, Theorem I.4.18]). Let \(\Delta X^{(i)}_s = X^{(i)}_s - X^{(i)}_s\) for all \(s \geq 0\). Denote by \((\langle X^{(i)} \rangle_t)_{t \geq 0}\) the quadratic variation of \(X^{(i)}\). Then, by [7, Theorem I.4.52],
\[
[X^{(i)}]_t = \langle X^{(i),c} \rangle_t + \sum_{s \in (0, t]: \Delta X_s \neq 0} (\Delta X^{(i)}_s)^2. \tag{2.10}
\]
Since \(\mathbb{E}_{x_0} [[X^{(i)}]_t] = \mathbb{E}_{x_0} [\langle X^{(i)} \rangle_t]\) and the Lévy system formula (see, e.g., [3, Remark 2.46] and references therein) implies that
\[
\mathbb{E}_{x_0} [\langle X^{(i)} \rangle_t] = \mathbb{E}_{x_0} \left[ \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} (z^{(i)})^2 N(X_s, dz) \right) \, ds \right]
= \mathbb{E}_{x_0} \left[ \sum_{s \in (0, t]: \Delta X_s \neq 0} (\Delta X^{(i)}_s)^2 \right],
\]
we see from (2.10) that \(\mathbb{E}_{x_0} [\langle X^{(i),c} \rangle_t] = 0\) for any \(t > 0\). Therefore, \(\mathbb{P}_{x_0}(\langle X^{(i),c} \rangle_t = 0) = 1\) for any \(t > 0\). Noticing that \((\langle X^{(i),c} \rangle_t)_{t \geq 0}\) is right continuous, we further obtain
\[
\mathbb{P}_{x_0} \left( \langle X^{(i),c} \rangle_t = 0 \text{ for any } t > 0 \right) = 1,
\]
and so, by [7, Lemma I.4.13 a])
\[
\mathbb{P}_{x_0} \left( X^{(i),c}_t = 0 \text{ for any } t > 0 \right) = 1.
\]
Hence, the martingale \(X\) is purely discontinuous. \(\square\)
Next, we are back to the proof of Proposition 2.3. Let $X$ be a Feller process on $\mathbb{R}^d$ such that the associated Feller generator $(L, D(L))$ fulfills Assumption 2.1. For any differentiable function $u$ on $\mathbb{R}^d$ such that

$$\int_{|z|<1} |u(x+z) - u(x) - \langle \nabla u(x), z \rangle| N(x, dz) + \int_{|z|\geq 1} |u(x+z) - u(x)| N(x, dz) < \infty$$

for any $x \in \mathbb{R}^d$, we will define $Lu$ by (2.2) in the pointwise sense. This definition is clearly consistent with $(L, C^2_b(\mathbb{R}^d))$. Moreover, under (2.3), $Lu$ can be well defined by (2.2) even for unbounded twice continuously differentiable functions $u$; for example, $u(x) = x^{(i)} - x_0^{(i)}$ and $u(x) = (x^{(i)} - x_0^{(i)})^2$ for any $x_0 \in \mathbb{R}^d$.

To show Proposition 2.3, we start with the following simple lemma. For any $x_0 \in \mathbb{R}^d$ and any $l \in \mathbb{N}$, let $\phi_l(x) \in C^\infty_c(\mathbb{R}^d)$ satisfy

$$\phi_l(x) = \begin{cases} 1, & 0 \leq |x - x_0| \leq l, \\ \in [0, 1], & l < |x - x_0| < l + 1, \\ 0, & |x - x_0| \geq l + 1. \end{cases} \quad (2.11)$$

**Lemma 2.4** Under Assumption 2.1, the following two statements hold.

1. The function $f(x) = (x^{(i)} - x_0^{(i)})^2$ for any fixed $x_0 \in \mathbb{R}^d$ satisfies

$$\lim_{l \to \infty} L(f \phi_l)(x) = Lf(x), \quad x \in \mathbb{R}^d, \quad (2.12)$$

where $\phi_l$ is defined by (2.11). Moreover, there is a constant $C_1 > 0$ such that for all $l \in \mathbb{N}$ and $x_0, x \in \mathbb{R}^d$,

$$|L(f \phi_l)(x)| \leq C_1(1 + f(x)), \quad (2.13)$$

and so

$$|Lf(x)| \leq C_1(1 + f(x)).$$

2. The function $f(x) = x^{(i)} - x_0^{(i)}$ for any fixed $x_0 \in \mathbb{R}^d$ also satisfies (2.12), and there is a constant $C_2 > 0$ such that for all $x_0, x \in \mathbb{R}^d$,

$$|L(f \phi_l)(x)| \leq C_2(1 + |f(x)|)$$

and

$$|Lf(x)| \leq C_2(1 + |f(x)|).$$
Proof We only prove (1), since (2) can be verified similarly. Since \( f \phi_l \in C_c^2(\mathbb{R}^d) \subset D(L) \),

\[
L(f \phi_l)(x) = \int_{|z|<1} (f \phi_l(x + z) - f \phi_l(x) - \langle \nabla (f \phi_l)(x), z \rangle) N(x, dz) + \int_{|z|\geq1} (f \phi_l(x + z) - f \phi_l(x)) N(x, dz)
\]

(2.14)

=: (I) + (II).

For any \( x, z \in \mathbb{R}^d \) with \( 0 < |z| < 1 \),

\[
(\phi_l(x + z) - \phi_l(x) - \langle \nabla (f \phi_l)(x), z \rangle
= f(x)(\phi_l(x + z) - \phi_l(x) - \langle \nabla \phi_l(x), z \rangle) + \phi_l(x + z)(f(x + z) - f(x) - \langle \nabla f(x), z \rangle)
\]

Note that, by the Taylor theorem, there exist positive constants \( C_* \) and \( C_{**} \) such that for any \( l \in \mathbb{N} \) and \( x, z \in \mathbb{R}^d \),

\[
|\phi_l(x + z) - \phi_l(x)| \leq C_*|z|, \quad |\phi_l(x + z) - \phi_l(x) - \langle \nabla \phi_l(x), z \rangle| \leq C_{**}|z|^2.
\]

This yields that there exists \( c_1 > 0 \) such that for any \( l \in \mathbb{N} \) and \( x, z \in \mathbb{R}^d \),

\[
|f \phi_l(x + z) - f \phi_l(x) - \langle \nabla (f \phi_l)(x), z \rangle| \leq c_1(1 + f(x))(z^{(i)})^2.
\]

Hence, by (2.3),

\[
|(I)| \leq c_1(1 + f(x)) \int_{|z|<1} (z^{(i)})^2 N(x, dz) \leq c_2(1 + f(x)). \tag{2.15}
\]

Because

\[
f \phi_l(x + z) - f \phi_l(x) - \langle \nabla (f \phi_l)(x), z \rangle \rightarrow f(x + z) - f(x) - \langle \nabla f(x), z \rangle \quad \text{as} \quad l \rightarrow \infty,
\]

we get, by the dominated convergence theorem, that

\[
(I) \rightarrow \int_{|z|<1} (f(x + z) - f(x) - \langle \nabla f(x), z \rangle) N(x, dz) \quad \text{as} \quad l \rightarrow \infty. \tag{2.16}
\]

Since there exists \( c_3 > 0 \) such that for any \( l \in \mathbb{N} \) and \( x, z \in \mathbb{R}^d \),

\[
|f \phi_l(x + z) - f \phi_l(x)| \leq f(x + z) + f(x) \leq c_3(f(x) + (z^{(i)})^2),
\]

\( \square \) Springer
it also follows from (2.3) that for any \( x \in \mathbb{R}^d \),
\[
|II| \leq c_3 \left( \int_{|z| \geq 1} N(x, dz) + \int_{|z| \geq 1} (z^{(i)})^2 N(x, dz) \right) \leq c_4 (1 + f(x)).
\]

With this at hand, we get by the dominated convergence theorem again that
\[
(II) \to \int_{|z| \geq 1} (f(x + z) - f(x)) N(x, dz) \quad \text{as } l \to \infty.
\]

Combining this with (2.16), we complete the proof of (2.12). Furthermore, (2.13) follows from (2.15) and (2.17).

Using Lemma 2.4, we can now present the

**Proof of Proposition 2.3** Throughout the proof, we fix \( x_0 \in \mathbb{R}^d \). We first show that \( X_t^{(i)} \) has finite second moment and (2.8) holds. Let \( f(x) = (x^{(i)} - x_0^{(i)})^2 \), and \( \phi_t \) be the function defined by (2.11). Then, \( f \phi_t \in C_\mathcal{C}^2(\mathbb{R}^d) \subset D(L) \), and so
\[
M_t^{[f \phi_t]} = f \phi_t(X_t) - f \phi_t(X_0) - \int_0^t L(f \phi_t)(X_s) \, ds, \quad t \geq 0
\]
is a martingale as mentioned in (2.1) and remarks before Subsection 2.2. Hence,
\[
\mathbb{E}_{x_0}[f \phi_t(X_t)] = f \phi_t(x_0) + \mathbb{E}_{x_0} \left[ \int_0^t L(f \phi_t)(X_s) \, ds \right] = \mathbb{E}_{x_0} \left[ \int_0^t L(f \phi_t)(X_s) \, ds \right].
\]

For \( m \in \mathbb{N} \), let \( \tau_m := \inf \{ t > 0 : |X_t - x_0| \leq m \} \). Then, by the optional stopping theorem, for all \( l, m \in \mathbb{N} \) and \( t > 0 \),
\[
\mathbb{E}_{x_0}[f \phi_t(X_t \wedge \tau_m)] = \mathbb{E}_{x_0} \left[ \int_0^{t \wedge \tau_m} L(f \phi_t)(X_s) \, ds \right].
\]

According to (2.13), there is \( c_1 > 0 \) such that for all \( l, m \in \mathbb{N} \) and \( t > 0 \),
\[
\mathbb{E}_{x_0} \left[ \int_0^{t \wedge \tau_m} |L(f \phi_t)(X_s)| \, ds \right] \leq c_1 \mathbb{E}_{x_0} \left[ \int_0^{t \wedge \tau_m} [1 + f(X_s)] \, ds \right]
\]
\[
= c_1 \mathbb{E}_{x_0} \left[ \int_0^{t \wedge \tau_m} [1 + f(X_s - \cdot)] \, ds \right]
\]
\[
= c_1 \mathbb{E}_{x_0} \left[ \int_0^{t \wedge \tau_m} [1 + (m^2 \wedge f)(X_s - \cdot)] \, ds \right]
\]
\[
= c_1 \mathbb{E}_{x_0} \left[ \int_0^{t \wedge \tau_m} [1 + (m^2 \wedge f)(X_s)] \, ds \right]
\]
\[
\leq c_1 \mathbb{E}_{x_0} \left[ \int_0^t [1 + (m^2 \wedge f)(X_s \wedge \tau_m)] \, ds \right].
\]
Here, \( X_{t-} := \lim_{s \to t-} X_s \) for any \( t > 0 \); in the first and the third equalities above we used the fact that almost surely \( t \mapsto X_t \) is a càdlàg function which has at most countably many jumps on \([0, t]\), i.e., \( \{ s \in [0, t] : X_s \neq X_{s-} \} \) is almost surely a set of Lebesgue measure zero; in the second equality we used the fact that \( f(X_{s-}) \leq m^2 \) for all \( s \in [0, \tau_m] \); and the last inequality follows from the monotonicity of the Lebesgue integral.

On the other hand, the monotone convergence theorem yields that for all \( m \in \mathbb{N} \) and \( t > 0 \),

\[
\lim_{l \to \infty} \mathbb{E}_{x_0}[f \phi_l(X_t \wedge \tau_m)] \geq \lim_{l \to \infty} \mathbb{E}_{x_0}[(m^2 \wedge f)\phi_l(X_t \wedge \tau_m)] = \mathbb{E}_{x_0}[(m^2 \wedge f)(X_t \wedge \tau_m)].
\]

Combining this and (2.20) with (2.19), we have for all \( m \in \mathbb{N} \) and \( t > 0 \),

\[
\mathbb{E}_{x_0}[1 + (m^2 \wedge f)(X_t \wedge \tau_m)] \leq 1 + c_1 \int_0^t \mathbb{E}_{x_0}[1 + (m^2 \wedge f)(X_s \wedge \tau_m)] \, ds.
\]

Then, by the Gronwall inequality,

\[
\mathbb{E}_{x_0}[1 + (m^2 \wedge f)(X_t \wedge \tau_m)] \leq e^{c_1 t}.
\]

Since the process \( X \) is conservative, letting \( m \to \infty \), we have for all \( t > 0 \),

\[
\mathbb{E}_{x_0}[f(X_t)] \leq e^{c_1 t} - 1. \tag{2.21}
\]

Therefore, \( X_t^{(i)} \) has finite second moment.

Furthermore, by (2.21), Lemma 2.4 and the dominated convergence theorem, letting \( l \to \infty \) in (2.18), we obtain that for all \( t > 0 \),

\[
\mathbb{E}_{x_0}[f(X_t)] = \mathbb{E}_{x_0} \left[ \int_0^t Lf(X_s) \, ds \right].
\]

Note that, for any \( x \in \mathbb{R}^d \),

\[
Lf(x) = \int_{\{0 < |z| < 1\}} \left( f(x + z) - f(x) - \langle \nabla f(x), z \rangle \right) N(x, dz) + \int_{\{|z| \geq 1\}} \left( f(x + z) - f(x) \right) N(x, dz)
= \int_{\{0 < |z| < 1\}} (z^{(i)})^2 N(x, dz) + \int_{\{|z| \geq 1\}} \left( (z^{(i)})^2 + 2(x^{(i)} - x_0^{(i)})z^{(i)} \right) N(x, dz)
= \int_{\mathbb{R}^d \setminus \{0\}} (z^{(i)})^2 N(x, dz) + 2(x^{(i)} - x_0^{(i)}) \int_{\{|z| \geq 1\}} z^{(i)} N(x, dz),
\]

where all the integrals above are well defined by (2.3) and (2.4). With this at hand, we obtain (2.8).
We next show (2.9). We note that $E_{x_0}[^{\lfloor X \rfloor}] < \infty$ because $X^{(i)}$ has finite second moment. Let $f(x) = x^{(i)} - x_0^{(i)}$. Then, (2.18) still holds true. Hence, following the argument above and using Lemma 2.4(2) and the dominated convergence theorem, we can also obtain (2.9).

\[ \Box \]

### 2.2.2 Canonical Representation Approach

In this part, we present another approach to Theorem 2.2, which is based on the canonical representation of the semimartingale (see [3, Section 2.5] and [7, Chapter II, Section 2]).

**Proof of Theorem 2.2** Let $X$ satisfy Assumption 2.1 and (2.6). Let $\delta(s, w)$ be the Dirac measure at $(s, w) \in (0, \infty) \times (\mathbb{R}^d \setminus \{0\})$ and $\mu^X(dt, dz)$ a (random) jumping measure of $X$ defined by

$$\mu^X(dt, dz) = \sum_{s \in (0, \infty): \Delta X_s \neq 0} \delta(s, \Delta X_s)(dt, dz).$$

If $\nu^X(dt, dz)$ denotes the compensator (or the dual predictable projection) of $\mu^X$, then by the Lévy system formula,

$$\nu^X(dt, dz) = N(X_t, dz)dt.$$  \hfill (2.22)

Let $W^{(i)}_t(z) = z^{(i)} 1_{\{|z| < 1\}}(z)$ for $z \in \mathbb{R}^d$. Then, by [3, Theorem 2.44] and [7, Theorems II.2.34 and II.2.42], $X$ is a semimartingale and has the following componentwise decomposition:

$$X_t^{(i)} = X_0^{(i)} + W^{(i)}_t(\mu^X - \nu^X)_t + \sum_{s \in (0, t]: \Delta X_s \geq 1} \Delta X_s^{(i)}, \quad 1 \leq i \leq d.$$  \hfill (2.23)

Here $W^{(i)}_t(\mu^X - \nu^X)$ is a stochastic integration; that is, it is a locally square integrable and purely discontinuous martingale such that

$$\langle W^{(i)}_t(\mu^X - \nu^X) \rangle_t = \int_0^t \left( \int_{\{0 < |z| < 1\}} (z^{(i)})^2 N(X_s, dz) \right) ds$$

(see, e.g., [7, Definition II.1.27 and its subsequent comment, and Theorem II.1.33] for the definition and properties of stochastic integrals with respect to a random measure). Then, by [7, Proposition 4.50] and (2.3), we have for any $t \geq 0$,

$$\mathbb{E}_{x_0}\left[ W^{(i)}_t(\mu^X - \nu^X)_t \right] = \mathbb{E}_{x_0}\left[ \int_0^t \left( \int_{\{0 < |z| < 1\}} (z^{(i)})^2 N(X_s, dz) \right) ds \right]$$

$$\leq t \sup_{x \in \mathbb{R}^d} \int_{\{0 < |z| < 1\}} (z^{(i)})^2 N(x, dz) < \infty.$$
Hence, according to [12, Corollary II.6.3],
\[ (W_1^{(i)} * (\mu^X - \nu^X)_t)_{t \geq 0} \text{ is a martingale with finite second moment.} \quad (2.24) \]

Let \( W_2^{(i)}(z) = z^{(i)} 1_{|z| \geq 1}(z) \). Then, by (2.3), \( W_2^{(i)} * (\mu^X - \nu^X) \) is a locally square integrable and purely discontinuous martingale. Furthermore, since (2.6) yields that
\[ W_2^{(i)} * \nu^X_t = \int_0^t \left( \int_{|z| \geq 1} z^{(i)} N(X_s, dz) \right) ds = 0, \]
we obtain, by [7, Proposition II.1.28] and (2.5),
\[ W_2^{(i)} * (\mu^X - \nu^X)_t = W_2^{(i)} * \mu^X_t = \sum_{s \in (0,t]: \Delta X_s \geq 1} \Delta X_s^{(i)}. \]

Hence, by (2.23), \( X \) is a locally square integrable and purely discontinuous martingale such that
\[ X^{(i)}_t = X^{(i)}_0 + (W_1^{(i)} + W_2^{(i)} * (\mu^X - \nu^X)_t \quad (2.25) \]
and
\[ \langle X^{(i)} \rangle_t = \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} (z^{(i)})^2 N(X_s, dz) \right) ds. \]

Following the argument for (2.24), we can further claim that \((X^{(i)}_t)_{t \geq 0}\) is a martingale with finite second moment.

Similarly, we have
\[ \langle X^{(i)} \pm X^{(j)} \rangle_t = \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} (z^{(i)} \pm z^{(j)})^2 N(X_s, dz) \right) ds, \]
which implies that
\[
\langle X^{(i)}, X^{(j)} \rangle_t = \frac{1}{4} \left( \langle X^{(i)} + X^{(j)} \rangle_t - \langle X^{(i)} - X^{(j)} \rangle_t \right)
= \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} z^{(i)} z^{(j)} N(X_s, dz) \right) ds.
\]

Therefore, we obtain (2.7).

2.3 Application: Law of the Iterated Logarithm

Let \( X \) be a Feller process satisfying Assumption 2.1 and (2.6). Then, \( X \) is a conservative Markov process and is a purely discontinuous martingale with finite second moment by Theorem 2.2. Hence, for any unit vector \( r = (r^{(1)}, \ldots, r^{(d)}) \in \mathbb{R}^d \), \( X^r := (\langle X_t, r \rangle)_{t \geq 0} \)
is also a purely discontinuous martingale with finite second moment such that for any \( t \geq 0 \),

\[
\langle X^r \rangle_t = \sum_{i=1}^{d} (r^{(i)})^2 \langle X^{(i)} \rangle_t + 2 \sum_{1 \leq i < j \leq d} r^{(i)} r^{(j)} \langle X^{(i)}, X^{(j)} \rangle_t \\
= \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} \langle \mathbf{r}, \mathbf{z} \rangle^2 N(X_s, d\mathbf{z}) \right) \, ds
\]

(2.26)

where the last equality follows from Theorem 2.2.

In this subsection, we establish Khintchine’s law of the iterated logarithm for \( X \). To do so, we make use of the stochastic integral representation of \( X \). As in Subsection 2.2.2, let \( \mu^X(dt, dz) \) be the (random) jumping measure of \( X \) and \( \nu^X(dt, dz) \) the compensator of \( \mu^X \) given by (2.22). Let \( V^{(i)}(z) = z^{(i)} \) for \( z \in \mathbb{R}^d \). Then, by (2.25),

\[
X^{(i)}_t = X^{(i)}_0 + V^{(i)}(\mu^X - \nu^X)_t, \quad 1 \leq i \leq d.
\]

Here \( V^{(i)}(\mu^X - \nu^X) \) is a stochastic integration; moreover, by Theorem 2.2, it is defined as a locally square integrable and purely discontinuous martingale such that

\[
\langle V^{(i)}(\mu^X - \nu^X) \rangle_t = \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} (z^{(i)})^2 N(X_s, dz) \right) \, ds, \quad t > 0.
\]

We will further impose the following assumption on the jumping kernel \( N(x, dz) \).

**Assumption 2.5** The jumping kernel \( N(x, dz) \) satisfies the next two conditions:

(i) there exist nonnegative measures \( \nu_1(dz) \) and \( \nu_2(dz) \) on \( \mathbb{R}^d \setminus \{0\} \) such that for \( 1 \leq i \leq d \),

\[
\int_{\mathbb{R}^d \setminus \{0\}} (z^{(i)})^2 \nu_1(dz) > 0, \quad \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 \nu_2(dz) < \infty,
\]

and for any \( x \in \mathbb{R}^d \) and \( A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \),

\[
\nu_1(A) \leq N(x, A) \leq \nu_2(A);
\]

(ii) for \( x \in \mathbb{R}^d \), let

\[
a_{ij}(x) = \int_{\mathbb{R}^d \setminus \{0\}} z^{(i)} z^{(j)} N(x, dz), \quad 1 \leq i, j \leq d.
\]

Then, there exist constants \( 0 < \lambda \leq \Lambda < \infty \) such that for any \( x, \xi \in \mathbb{R}^d \),

\[
\lambda |\xi|^2 \leq \sum_{i, j=1}^{d} a_{ij}(x) \xi^{(i)} \xi^{(j)} \leq \Lambda |\xi|^2.
\]
Note that
\[ \sum_{i,j=1}^{d} a_{ij}(x) \xi^{(i)}(x) \xi^{(j)}(x) = \int_{\mathbb{R}^d \setminus \{0\}} \langle \xi, z \rangle^2 N(x, dz) \leq |\xi|^2 \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 N(x, dz) \]
\[ \leq |\xi|^2 \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 N(x, dz), \]
so it holds that
\[ \Lambda \leq \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 N(x, dz). \]

**Theorem 2.6** Let \( X \) be a Feller process such that Assumptions 2.1 and 2.5, and (2.6) hold.

(1) For every \( x \in \mathbb{R}^d \) and every unit vector \( r \in \mathbb{R}^d \),
\[ \mathbb{P}^x \left( \limsup_{t \to \infty} \frac{X^r_t}{\sqrt{2t} \log \log (X^r)_t} = 1 \right) = 1. \]

(2) For every \( x \in \mathbb{R}^d \),
\[ \mathbb{P}^x \left( \sqrt{\Lambda} \leq \limsup_{t \to \infty} \frac{|X_t|}{\sqrt{2t \log \log t}} \leq \sqrt{\Lambda} \right) = 1. \]

**Proof** Let \( X \) be a Feller process satisfying Assumptions 2.1 and 2.5, and (2.6). Then, by Theorem 2.2, \( X \) is a purely discontinuous martingale with finite second moment.

We first prove (1) by applying [22, Theorem 1] to \( X \). To do so, we see that \( X \) satisfies Assumption A (i) and (ii) of [22]. Let
\[ F(x) := \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 N(x, dz) \]
and
\[ c_1 := \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 v_1(dz), \quad c_2 := \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 v_2(dz). \]
Then, by Assumption 2.5 (i), \( c_1 \) and \( c_2 \) are finite positive constants and
\[ c_1 \leq F(x) \leq c_2, \quad x \in \mathbb{R}^d. \] (2.27)

Hence, if we define
\[ C_t := \sum_{i=1}^{d} \langle X^{(i)} \rangle_t = \int_0^t F(X_s) ds = \int_0^t F(X_{s-}) ds, \quad t \geq 0, \]
then \((C_t)_{t \geq 0}\) is a predictable increasing process such that

\[
P_x \left( \lim_{t \to \infty} C_t = \infty \right) = 1.
\]

Let

\[
N_t(dz) := \frac{1}{F(X_{t-})} \varepsilon(X_{t-}, dz).
\]

Then, by (2.22),

\[
\nu^X(dt, dz) = N_t(dz) \, dC_t.
\]

Define a \(d \times d\)-symmetric matrix \(S_t\) by

\[
S_t := \frac{1}{F(X_{t-})} \begin{pmatrix} a_{ij}(X_{t-}) \end{pmatrix}_{1 \leq i, j \leq d}.
\]

Note that each entry of \(S_t\) is a predictable density function of the predictable quadratic variation of \(X\) in (2.7) with respect to \(C_t\). If we let \(\Lambda_1 := \Lambda/c_1\) and \(\lambda_1 := \lambda/c_2\), then Assumption 2.5 implies that \(\Lambda_1 I - S_t\) and \(S_t - \lambda_1 I\) are nonnegative definite matrices, where \(I\) is a \(d \times d\)-unit matrix. Combining this with (2.28), we have verified Assumption A (i) of [22] for \(X\).

By (2.27), we have for any \(t > 0\) and \(A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})\),

\[
N_t(A) \leq \frac{1}{c_1} \nu_2(A).
\]

This implies that \(X\) satisfies also Assumption A (ii) of [22].

Let \(r\) be a unit vector in \(\mathbb{R}^d\). Since (2.26) and Assumption 2.5 (ii) yield that

\[
\langle X^r \rangle_t = \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} \langle r, z \rangle^2 N(X_s, dz) \right) ds \geq \lambda t,
\]

we obtain

\[
P_x \left( \lim_{t \to \infty} \langle X^r \rangle_t = \infty \right) = 1.
\]

Therefore, (1) follows by applying [22, Theorem 1] to \(X\).

We next prove (2) in the same way as for the law of the iterated logarithm for the multidimensional Brownian motion (see, e.g., [21, Exercise 1.5.17]). Let \(r\) be a unit vector in \(\mathbb{R}^d\). Since \(|X_t| \geq |X_r^r|\), we have, by (2.29), for large \(t\),

\[
\frac{|X_t|}{\sqrt{2t \log \log t}} \geq \frac{|X_r^r|}{\sqrt{2t \log \log t}} = \frac{\sqrt{2(\langle X^r \rangle_t \log \log(\langle X^r \rangle_t))}}{\sqrt{2t \log \log t}} \frac{|X_r^r|}{\sqrt{2(\langle X^r \rangle_t \log \log(\langle X^r \rangle_t))}} \geq \sqrt{\lambda} \frac{\log \log(\lambda t)}{\log \log t} \frac{|X_r^r|}{\sqrt{2(\langle X^r \rangle_t \log \log(\langle X^r \rangle_t))}}.
\]
Hence, according to (1), for every $x \in \mathbb{R}^d$,

$$
\mathbb{P}_x \left( \limsup_{t \to \infty} \frac{|X_t|}{\sqrt{2t \log \log t}} \geq \sqrt{\lambda} \right) = 1. \tag{2.30}
$$

Since (2.26) and Assumption 2.5 (ii) imply that for any unit vector $r \in \mathbb{R}^d$,

$$
\langle X_r \rangle_t = \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} \langle r, z \rangle^2 N(X_s, dz) \right) ds \leq \Lambda t,
$$

we have, by (2.29) again, for large $t$,

$$
\frac{|X^r_t|}{\sqrt{2t \log \log t}} = \frac{\sqrt{2\langle X^r \rangle_t \log \log \langle X^r \rangle_t}}{\sqrt{2t \log \log t}} \leq \frac{\sqrt{2\Lambda t \log \log (\Lambda t)}}{\sqrt{2t \log \log t}} \frac{|X^r_t|}{\sqrt{2\langle X^r \rangle_t \log \log \langle X^r \rangle_t}}.
$$

Then, due to (1),

$$
\limsup_{t \to \infty} \frac{|X^r_t|}{\sqrt{2t \log \log t}} \leq \sqrt{\Lambda}, \quad a.s. \tag{2.31}
$$

Noting that $|X_t| \leq \sqrt{d} \max_{1 \leq i \leq d} |X^{(i)}_t|$, we obtain from (2.30) and (2.31) that for almost surely there is a finite random variable $t_0 := t_0(\omega) > e$ so that

$$
\sup_{t \geq t_0} \frac{|X_t|}{\sqrt{2t \log \log t}} \in (0, \infty)
$$

being bounded by two positive (non-random) constants $C_1 \leq C_2$. Let $\{r_n\}_{n \geq 1}$ be a family of unit vectors in $\mathbb{R}^d$ forming a dense set in the unit sphere $S^{d-1}$. Then, for any $\varepsilon \in (0, 2C_2)$, there exists $l \geq 1$ such that

$$
S^{d-1} \subset \bigcup_{k=1}^l B(r_k, \varepsilon/(2C_2)).
$$

Hence, if we let

$$
R_t := \frac{X_t}{\sqrt{2t \log \log t}}.
$$

then almost surely, for any $\varepsilon > 0$ and $t \geq t_0$ there exists a random variable $j := j(\varepsilon, t, \omega) \in \{1, \ldots, l\}$ such that

$$
|R_t/|R_t| - \langle R_t/|R_t|, r_j \rangle r_j| \leq |R_t/|R_t| - r_j| + |1 - \langle R_t/|R_t|, r_j \rangle| = |R_t/|R_t| - r_j| + |R_t/|R_t| - r_j|^2/2 \leq \varepsilon/C_2,
$$

 Springer
which along with the fact that $|R_t| \leq C_2$ for all $t \geq t_0$ a.s. shows that

$$|R_t - \langle R_t, r_j \rangle r_j| \leq \varepsilon$$

for any $\varepsilon > 0$ and $t \geq t_0$, a.s.

Since this inequality implies that

$$|R_t| \leq \max_{1 \leq k \leq l} |\langle R_t, r_k \rangle| + \varepsilon$$

for any $\varepsilon > 0$ and $t \geq t_0$, a.s., we have, by (2.31),

$$\limsup_{t \to \infty} |R_t| \leq \limsup_{t \to \infty} \max_{1 \leq k \leq l} |\langle R_t, r_k \rangle| + \varepsilon \leq \sqrt{\Lambda} + \varepsilon$$

for any $\varepsilon > 0$, a.s.

Letting $\varepsilon \to 0$, we get

$$\limsup_{t \to \infty} \frac{|X_t|}{\sqrt{2t \log \log t}} \leq \sqrt{\Lambda}, \text{ a.s.}$$

Hence, the proof is complete. \hfill \Box

**Remark 2.7** We do not know the zero-one law for the tail events of $X$ (see [8, Theorem 2.10] and references therein for symmetric jump processes with heat kernel estimates). In particular, it is not clear whether there exists a positive non-random constant $c_0$ such that

$$P_{x_0} \left( \limsup_{t \to \infty} \frac{|X_t|}{\sqrt{2t \log \log t}} = c_0 \right) = 1.$$ 

On the other hand, since

$$\langle X^{(i)} \rangle_t \geq t \inf_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \langle z^{(i)} \rangle^2 N(x, dz), \quad 1 \leq i \leq d,$$

it follows from the same argument as for the proof of (2) that

$$P_x \left( \limsup_{t \to \infty} \frac{|X_t|}{\sqrt{2t \log \log t}} \geq \max_{1 \leq i \leq d} \left[ \inf_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \langle z^{(i)} \rangle^2 N(x, dz) \right] \right) = 1.$$ 

### 2.4 Examples

In this subsection, we provide a class of Feller processes which are purely discontinuous martingales with finite second moment, and satisfy Khintchine’s law of the iterated logarithm. For any $u \in C_c^\infty(\mathbb{R}^d)$, let

$$L u(x) = L_0 u(x) + B u(x),$$
where

\[ L_0 u(x) = \int_{|0 < |z| < 1} (u(x + z) - u(x) - \langle \nabla u(x), z \rangle) \frac{c(x)}{|z|^{d+\alpha(x)}} \, dz, \]

and

\[ B u(x) = \int_{|z| \geq 1} (u(x + z) - u(x)) n_0(x, z) \, dz. \]

Here, \( c(x) \) and \( \alpha(x) \) are positive measurable functions on \( \mathbb{R}^d \), and \( n_0(x, z) \) is a nonnegative Borel measurable function on \( \mathbb{R}^d \times \{ z \in \mathbb{R}^d : |z| \geq 1 \} \).

We will impose the following conditions on \( \alpha(x), c(x) \) and \( n_0(x, z) \), respectively.

**Assumption 2.8**

(i) The index function \( \alpha(x) \) satisfies the next conditions:

(i-1) \( 0 < \inf_{x \in \mathbb{R}^d} \alpha(x) \leq \sup_{x \in \mathbb{R}^d} \alpha(x) < 2; \)

(i-2) \( \lim_{r \to 0} |\log r| [\sup_{|x-y| \leq r} |\alpha(x) - \alpha(y)|] = 0; \)

(i-3) \( \int_0^1 \sup_{|x-y| \leq r} |\alpha(x) - \alpha(y)| \, dr^{-1} \, dr < \infty. \)

(ii) The coefficient \( c(x) \) is continuous and \( 0 < \inf_{x \in \mathbb{R}^d} c(x) \leq \sup_{x \in \mathbb{R}^d} c(x) < \infty. \)

(iii) The function \( n_0(x, z) \) satisfies the following conditions:

(iii-1) there is a nonnegative Borel measurable function \( \tilde{n}_0(x) \) on \( \{ x \in \mathbb{R}^d : |x| \geq 1 \} \) such that

\[ \int_{\{|z| \geq 1\}} |z|^2 \tilde{n}_0(z) \, dz < \infty \]

and for any \( x, z \in \mathbb{R}^d \) with \( |z| \geq 1, \)

\[ n_0(x, z) \leq \tilde{n}_0(z); \]

(iii-2) for almost every \( z \in \mathbb{R}^d \) with \( |z| \geq 1, \) the function \( x \mapsto n_0(x, z) \) is continuous on \( \mathbb{R}^d; \)

(iii-3) for any \( x \in \mathbb{R}^d, \)

\[ \int_{\{|z| \geq 1\}} z^{(i)} n_0(x, z) \, dz = 0, \quad 1 \leq i \leq d. \]

**Proposition 2.9** Under Assumption 2.8, \((L, C_c^\infty(\mathbb{R}^d))\) is closable on \( C_\infty(\mathbb{R}^d) \) and its closure is the generator of a Feller semigroup. Furthermore, let \( X := \{(X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d}\} \) be the Feller process on \( \mathbb{R}^d \) associated with the closure of \((L, C_c^\infty(\mathbb{R}^d)). \) Then, \( X \) is a purely discontinuous martingale with finite second moment and satisfies Khintchine’s law of the iterated logarithm.
To prove Proposition 2.9, we start from the following operator \((A, C_c^\infty(\mathbb{R}^d))\):

\[
Au(x) = \int_{\mathbb{R}^d \backslash \{0\}} \left( u(x + z) - u(x) - \langle \nabla u(x), z \rangle 1_{\{0 < |z| < 1\}} \right) \frac{C_\alpha(x)}{|z|^{d+\alpha(x)}} \, dz, \quad u \in C_c^\infty(\mathbb{R}^d),
\]

where

\[
C_\alpha(x) = \frac{\alpha(x)^{2\alpha(x)-1} \Gamma((\alpha(x) + d)/2)}{\pi^{d/2} \Gamma(1 - \alpha(x)/2)}.
\]

By [2, Theorem 2.2] (see also [3, Theorem 3.31] or [10, Theorem 5.2]), there exists a Feller process \(Y := \{(Y_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d}\}\) under Assumption 2.8 (i) such that

- \(Y\) is a unique solution to the \((A, C_c^\infty(\mathbb{R}^d))\)-martingale problem;
- \(C_c^\infty(\mathbb{R}^d)\) is the core of the Feller generator of \(Y\).

Indeed, as Bass [2] proved the existence and uniqueness of the \((A, C^2_b(\mathbb{R}^d))\)-martingale problem, \(Y\) is called the stable-like process in the sense of Bass in the literature.

Let \((A, D(A))\) be the Feller generator of \(Y\). Note that, by Assumption 2.8 (ii), \(m(x) = c(x)C^{-1}_{\alpha(x)}\) is a continuous function on \(\mathbb{R}^d\) bounded from below and above by positive constants. Hence, the operator \((L_1, D(L_1)) = (m(\cdot)A, D(A))\) is also a Feller generator (see, e.g., [3, Theorem 4.1]).

Next, we consider the following operator \((B_1, C_\infty(\mathbb{R}^d))\):

\[
B_1u(x) = Bu(x) - \int_{|z| \geq 1} (u(x + z) - u(x)) \frac{c(x)}{|z|^{d+\alpha(x)}} \, dz.
\]

Note that for \(u \in C_c^\infty(\mathbb{R}^d)\),

\[
B_1u = Bu - (L_1 - L_0)u.
\]

We can claim that

**Lemma 2.10** Under Assumption 2.8, the operator \(B_1\) is a bounded linear operator on \(C_\infty(\mathbb{R}^d)\).

**Proof** For \(u \in C_\infty(\mathbb{R}^d)\), let

\[
B_2u(x) = \int_{|z| \geq 1} (u(x + z) - u(x)) \frac{c(x)}{|z|^{d+\alpha(x)}} \, dz.
\]

Then, we will prove that both \(B\) and \(B_2\) are bounded linear operators on \(C_\infty(\mathbb{R}^d)\). If it holds, then we can prove the assertion. For simplicity, we verify the conclusion only for the operator \(B\).

First, by Assumption 2.8 (iii-1), there is a constant \(C > 0\) such that for any \(u \in C_\infty(\mathbb{R}^d)\), \(\|Bu\|_\infty \leq C\|u\|_\infty\); according to Assumption 2.8 (iii-1)-(iii-2) and the dominated convergence theorem, the function \(Bu\) is continuous on \(\mathbb{R}^d\). To complete the proof, it is enough to show that \(Bu(x) \to 0\) as \(|x| \to \infty\). For any \(\varepsilon > 0\), there
exists a constant \( R > 1 \) such that for any \( x \in \mathbb{R}^d \) with \( |x| \geq R \), \(|u(x)| \leq \varepsilon \). By Assumption 2.8 (iii-1), we can also assume that \[
\int_{|z| \geq R} n_0(x, z) \, dz \leq \varepsilon.
\]

Now, assume that \(|x| \geq 2R\). We write
\[
|Bu(x)| \leq \int_{|z| \geq 1} |u(x)|n_0(x, z) \, dz + \int_{|z| \geq 1} |u(x + z)|n_0(x, z) \, dz
\]
\[
\leq \varepsilon \int_{|z| \geq 1} n_0(x, z) \, dz + \int_{|z| \geq 1, |x + z| \geq R} |u(x + z)|n_0(x, z) \, dz
\]
\[
+ \int_{|z| \geq 1, |x + z| < R} |u(x + z)|n_0(x, z) \, dz
\]
\[
\leq 2\varepsilon \sup_{x \in \mathbb{R}^d} \int_{|z| \geq 1} n_0(x, z) \, dz + \|u\|_\infty \int_{|z| \geq R} n_0(x, z) \, dz
\]
\[
\leq \varepsilon \left( 2 \sup_{x \in \mathbb{R}^d} \int_{|z| \geq 1} n_0(x, z) \, dz + \|u\|_\infty \right),
\]

which yields the desired assertion. \(\square\)

**Proof of Proposition 2.9** Recall that \( L_1 = m(\cdot)A \) and \( m(x) = c(x)C^{-1}\alpha(x) \) is a continuous function on \( \mathbb{R}^d \) bounded from below and above by positive constants. Therefore, \((L_1, C^\infty_c(\mathbb{R}^d))\) is closable on \( C^\infty(\mathbb{R}^d) \). Since \( B_1 \) is bounded on \( C^\infty(\mathbb{R}^d) \) by Lemma 2.10 and \( L = L_1 + B_1 \), \((L, C^\infty_c(\mathbb{R}^d))\) is also closable on \( C^\infty(\mathbb{R}^d) \). As \( L \) satisfies the positive maximum principle, the closure of \((L, C^\infty_c(\mathbb{R}^d))\) is the generator of a Feller semigroup by [18, Proposition 2.1].

Let \( X := \{(X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d}\} \) be the Feller process on \( \mathbb{R}^d \) associated with the closure of \((L, C^\infty_c(\mathbb{R}^d))\), and let \( N(x, dz) \) be the jumping kernel of \( X \). Then, \( N(x, dz) \) is absolutely continuous with respect to the Lebesgue measure, and the density function \( n(x, z) \) is given by
\[
n(x, z) = c(x)\frac{1}{|z|^{d+\alpha(x)}}1_{[0<|z|<1]} + n_0(x, z)1_{[|z| \geq 1]}.
\]

Since \( N(x, dz) \) fulfills the condition (2.6) and Assumption 2.5 by Assumption 2.8, \( X \) is a purely discontinuous martingale with finite second moment and satisfies Khintchine’s law of the iterated logarithm, respectively, by Theorems 2.2 and 2.6. \(\square\)

At the end of this subsection, we present two examples of \( n_0(x, z) \) such that Assumption 2.8 (iii) is satisfied.

**Example 2.11** Let \( c_0(x) \) be a continuous function on \( \mathbb{R}^d \) which is bounded from below and above by positive constants.
(1) Let $\beta_1(x)$ be a positive continuous function on $\mathbb{R}^d$ such that $\inf_{x \in \mathbb{R}^d} \beta_1(x) > 2$. Then, the function

$$n_0(x, z) = \frac{c_0(x)}{|z|^{d+\beta_1(x)}}$$

satisfies Assumption 2.8 (iii).

(2) Let $\lambda$ be a positive constant and $\beta_2(x)$ a positive continuous function on $\mathbb{R}^d$ such that $\inf_{x \in \mathbb{R}^d} \beta_2(x) > 0$. Then, the function

$$n_0(x, z) = c_0(x)e^{-\lambda|z|^{\beta_2(x)}}$$

satisfies Assumption 2.8 (iii).

3 Martingale Nature and LIL of Hunt Processes

In this section, we discuss the martingale property and Khintchine’s law of the iterated logarithm for a class of jump-type Hunt processes generated by regular lower bounded semi-Dirichlet forms.

3.1 Regular Lower Bounded Semi-Dirichlet Forms

In this subsection, we recall the notion of regular lower bounded semi-Dirichlet forms by following [11, Section 1] and [5]. Let $M$ be a locally compact separable metric space and $\mu$ a positive Radon measure on $M$ with full support. Let $(\eta, \mathcal{F})$ be a bilinear form on $L^2(M; \mu)$, and $\eta_\alpha(u, u) = \eta(u, u) + \alpha \|u\|_{L^2(M; \mu)}^2$ for $\alpha \geq 0$. We say that $(\eta, \mathcal{F})$ is a lower bounded closed form, if there exists $\alpha_0 \geq 0$ such that the next three conditions hold:

(i) $\eta_\alpha_0(u, u) \geq 0$ for any $u \in \mathcal{F}$;
(ii) there exists $K \geq 1$ such that for any $u, v \in \mathcal{F}$, $|\eta(u, v)| \leq K \sqrt{\eta_\alpha_0(u, u)} \sqrt{\eta_\alpha_0(v, v)}$;
(iii) $\mathcal{F}$ is complete with respect to the norm $\|u\|_{\eta_\alpha} : = \sqrt{\eta_\alpha(u, u)}$ for some/any $\alpha > \alpha_0$.

Let $(\eta, \mathcal{F})$ be a lower bounded closed form on $L^2(M; \mu)$. Then, $(\eta, \mathcal{F})$ is called Markovian, if for any $u \in \mathcal{F}$, $Uu := 0 \lor u \land 1 \in \mathcal{F}$ and $\eta(Uu, u - Uu) \geq 0$. A lower bounded semi-Dirichlet form is by definition a lower bounded closed Markovian form. For a lower bounded semi-Dirichlet form $(\eta, \mathcal{F})$, there exists a strongly continuous Markovian semigroup $(T_t)_{t \geq 0}$ on $L^2(M; \mu)$ such that for any $\alpha > \alpha_0$, the $\alpha$-resolvent $G_\alpha f := \int_0^\infty e^{-\alpha t} T_t f \, dt$ satisfies $\eta_\alpha(G_\alpha f, g) = \int_M fg \, d\mu$ for any $f \in L^2(M; \mu)$ and $g \in \mathcal{F}$ ([11, Theorems 1.1.2 and 1.1.5]). We can further extend $(T_t)_{t \geq 0}$ and $(G_\alpha)_{\alpha > \alpha_0}$ to $L^\infty(M; \mu)$ so that $\|T_t\|_{\infty} \leq 1$ for all $t > 0$ and $\|G_\alpha\|_{\infty} \leq 1/\alpha$ for all $\alpha > \alpha_0$ (see the discussion after the proof of [11, Theorem 1.1.5]).

Let $C_0(M)$ be the set of continuous functions on $M$ with compact support. We say that a lower bounded semi-Dirichlet form $(\eta, \mathcal{F})$ is regular, if $\mathcal{F} \cap C_0(M)$ is dense.
both in $L^2(M; \mu)$ with respect to the norm $\| \cdot \|_{\alpha}$ for some/any $\alpha > \alpha_0$ and in $C_0(M)$ with respect to the uniform norm.

Let $(\eta, \mathcal{F})$ be a regular lower bounded semi-Dirichlet form on $L^2(M; \mu)$, and fix a constant $\alpha > \alpha_0$. For an open set $O \subset E$, let $\mathcal{L}_O := \{ u \in \mathcal{F} : u \geq 1\mu - \text{a.e. on } O \}$, and define the capacity of $O$ by

$$\text{Cap}(O) = \begin{cases} \inf \{ \eta_\alpha(u, u) : u \in \mathcal{L}_O \} & \text{if } \mathcal{L}_O \neq \emptyset, \\ \infty & \text{if } \mathcal{L}_O = \emptyset. \end{cases}$$

The capacity of a set $A \subset M$ is defined by

$$\text{Cap}(A) = \inf \{ \text{Cap}(O) : O \subset M \text{ is open and } A \subset O \}.$$ 

A set $A$ is called exceptional, if $\text{Cap}(A) = 0$; see [11, Theorem 3.4.4]. Then, there exist a Borel exceptional set $\mathcal{N}_0 \subset M$ and a Hunt process $X := \{(X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in M \setminus \mathcal{N}_0}, (\mathcal{F}_t)_{t \geq 0}\}$ properly associated with $(\eta, \mathcal{F})$. Namely, $X$ is a quasi-left continuous strong Markov process on $M \setminus \mathcal{N}_0$ with the quasi-left continuity on the time interval $(0, \infty)$ (see, e.g., [11, Subsection 3.1] for details), and for any $\alpha > \alpha_0$ and bounded Borel function $f \in L^2(M; \mu)$,

$$\mathbb{E}_x \left[ \int_0^\infty e^{-\alpha t} f(X_t) \, dt \right] = G_\alpha f(x), \quad \mu\text{-a.e. } x \in M$$

(see, e.g., [11, Theorem 3.3.4] or [5, Theorem 4.1]). Note that $X$ has càdlàg sample paths by definition, and the filtration $(\mathcal{F}_t)_{t \geq 0}$ can be assumed complete and right continuous (see, e.g., [11, Subsection 3.1]).

For a Borel set $B \subset M$, let

$$\sigma_B = \inf \{ t > 0 : X_t \in B \}, \quad \hat{\sigma}_B = \inf \{ t > 0 : X_t^- \in B \}.$$ 

A Borel set $\mathcal{N} \subset M$ is called properly exceptional, if $\mu(\mathcal{N}) = 0$ and $\mathbb{P}_x(\sigma_\mathcal{N} = \hat{\sigma}_\mathcal{N} = \infty) = 1$ for any $x \in M \setminus \mathcal{N}$. We can take a properly exceptional set $\mathcal{N}$ so that $\mathcal{N} \supset \mathcal{N}_0$ (see, e.g., [5, Theorem 4.2 (ii)]). In particular, $X|_{M \setminus \mathcal{N}} := \{(X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in M \setminus \mathcal{N}}, (\mathcal{F}_t)_{t \geq 0}\}$ is still a Hunt process properly associated with $(\eta, \mathcal{F})$.

### 3.2 Martingale Property and LIL of Hunt Processes

Let $\text{diag} = \{(x, x) : x \in \mathbb{R}^d\}$ be the diagonal set, and let $J(x, y)$ be a nonnegative Borel measurable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$ such that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{x\}} (1 \wedge |x - y|^2) J_s(x, y) \, dy < \infty$$

(3.1)

and

$$\sup_{x \in \mathbb{R}^d} \int_{\{J_s(x, y) \neq 0\}} \frac{J_a(x, y)^2}{J_s(x, y)} \, dy < \infty,$$

(3.2)
where

\[ J_s(x, y) = \frac{1}{2} (J(x, y) + J(y, x)), \quad J_a(x, y) = \frac{1}{2} (J(x, y) - J(y, x)). \]

For \( n \in \mathbb{N} \) and \( u, v \in C_c^\infty(\mathbb{R}^d) \), we define

\[ L_n u(x) = \int_{\{|y-x| \geq 1/n\}} (u(y) - u(x)) J(x, y) \, dy \]

and

\[ \eta_n(u, v) = -\int_{\mathbb{R}^d} L_n u(x) v(x) \, dx. \]

Note that by (3.1), the right-hand side of the equality above is absolutely convergent.

Let

\[ D = \left\{ u \in L^2(\mathbb{R}^d; dx) : \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}} (u(x) - u(y))^2 J(x, y) \, dx \, dy < \infty \right\} \]

and

\[ D(u, v) = \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y)) J(x, y) \, dx \, dy. \]

Then, by (3.1) again, \((D, D(D))\) is a symmetric Dirichlet form on \( L^2(\mathbb{R}^d; dx) \) such that \( C_c^\infty(\mathbb{R}^d) \subset D(D) \). Let \( F \) be the closure of \( C_c^\infty(\mathbb{R}^d) \) with respect to the norm \( \|u\| := \sqrt{D(u, u) + \|u\|_{L^2(\mathbb{R}^d; dx)}^2} \). Then, \((D, F)\) becomes a regular symmetric Dirichlet form on \( L^2(\mathbb{R}^d; dx) \). Furthermore, according to [16, Theorem 2.1] (see also [5, Proposition 2.1]), the limit

\[ \eta(u, v) := \lim_{n \to \infty} \eta_n(u, v) \]

exists for all \( u, v \in C_c^\infty(\mathbb{R}^d) \) such that

\[ \eta(u, v) = \frac{1}{2} D(u, v) + \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}} (u(x) - u(y)) v(y) J_a(x, y) \, dx \, dy. \]

In particular, \((\eta, F)\) becomes a regular lower bounded semi-Dirichlet form on \( L^2(\mathbb{R}^d; dx) \).

In what follows, let \( X := \{(X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d \setminus N_0}, (\mathcal{F}_t)_{t \geq 0}\} \) be a Hunt process on \( \mathbb{R}^d \) properly associated with \((\eta, F)\), where \( N_0 \) is an exceptional set as mentioned in Subsection 3.1. According to the Beurling–Deny type decomposition for semi-Dirichlet forms (see [11, Theorem 5.2.1]), there are no local part and no killing term in the lower bounded semi-Dirichlet form \((\eta, F)\) given above, and so the associated process
X is also of pure-jump type. In order to present sufficient conditions on the jumping kernel $J(x, y)$ such that $X$ itself is a purely discontinuous martingale with finite second moment, we will make use full of the expression for the generator associated with $(\eta, \mathcal{F})$. For this purpose, we impose the following assumption on $J(x, y)$.

**Assumption 3.1** The jumping kernel $J(x, y)$ satisfies the next three conditions:

(i) for any $\varepsilon > 0$, $x \in \mathbb{R}^d$ and $1 \leq i \leq d$,

$$
\int_{\{|x-y|\geq\varepsilon\}} (y - x)^{(i)} J(x, y) \, dy = 0;
$$

(ii) $J(x, y)$ has the second moment in the sense that

$$
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{x\}} |x - y|^2 J(x, y) \, dy < \infty;
$$

(iii) the function

$$
x \mapsto \int_{\{|y-x|\geq 1\}} J(x, y) \, dy
$$

belongs to $L^2(\mathbb{R}^d; dx) \cup L^\infty(\mathbb{R}^d; dx)$.

**Lemma 3.2** Let $(L, \mathcal{D}(L))$ be the $(L^2)$-generator of $(\eta, \mathcal{F})$. Under Assumption 3.1, we have the following two statements.

1. $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(L)$, and for any $u \in C_c^\infty(\mathbb{R}^d)$,

$$
Lu(x) = \int_{\mathbb{R}^d \setminus \{x\}} (u(y) - u(x) - \langle \nabla u(x), y - x \rangle) J(x, y) \, dy.
$$

Moreover, $(L, C_c^\infty(\mathbb{R}^d))$ extends to $C_b^2(\mathbb{R}^d)$, and the expression above remains valid for any $u \in C_b^2(\mathbb{R}^d)$.

2. There exists a Borel properly exceptional set $\mathcal{N} \supset \mathcal{N}_0$ such that for any $u \in C_b^2(\mathbb{R}^d)$,

$$
M_t^{[u]} = u(X_t) - u(X_0) - \int_0^t Lu(X_s) \, ds, \quad t \geq 0,
$$

is a $\mathbb{P}_x$-martingale for each $x \in \mathbb{R}^d \setminus \mathcal{N}$. Moreover, $X|_{M_t^{[u]}, \mathcal{N}}$ is conservative.

**Proof** According to (3.3), for any $n \geq 1$,

$$
L_n u(x) = \int_{\{|y-x|\geq 1/n\}} (u(y) - u(x) - \langle \nabla u(x), y - x \rangle) J(x, y) \, dy.
$$
Let $L$ be as in (3.5). It is obvious that, under Assumption 3.1 (ii), $Lu$ is pointwisely well defined for any $u \in C^\infty_c(\mathbb{R}^d)$. Moreover,

$$|Lu(x) - L_n u(x)| = \left| \int_{|y-x| < 1/n} (u(y) - u(x) - \langle \nabla u(x), y - x \rangle) J(x, y) \, dy \right| \leq \| \nabla^2 u \|_\infty \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{x\}} |y - x|^2 J(x, y) \, dy < \infty.$$  

Then, by Assumption 3.1 (ii) again and the dominated convergence theorem, for any $f, g \in C^\infty_c(\mathbb{R}^d)$,

$$\lim_{n \to \infty} \eta_n(f, g) = - \lim_{n \to \infty} \int_{\mathbb{R}^d} L_n f(x) g(x) \, dx = - \int_{\mathbb{R}^d} Lf(x) g(x) \, dx.$$  

In particular, the equality above shows that the operator $L$ is the generator of $(\eta, \mathcal{F})$. Following the argument in step 2 of [16, Theorem 2.2] and using (3.1) and Assumption 3.1 (iii), we know that $L$ maps $C^\infty_c(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d; dx)$. We also note that the operator $(L, C^\infty_c(\mathbb{R}^d))$ extends to $C^2_b(\mathbb{R}^d)$ in a similar way as in [5, Section 5] or [3, Theorem 2.37]. Hence we arrive at the assertion (1).

Applying [5, Theorem 4.3] to $(L, C^2_\mathcal{F}(\mathbb{R}^d))$, we can obtain the assertion (2). We note that, even though [5, Theorem 4.3] requires the continuity of $Lu$ for any $u \in C^\infty_c(\mathbb{R}^d)$, the proof of this theorem is still true without this assumption.

According to (3.3) and Lemma 3.2, we obtain the statement below by letting $N(x, dz) := J(x, x + z) \, dz$, and following the generator approach to Theorem 2.2 (see Subsection 2.2.1) and the proof of Theorem 2.6.

**Theorem 3.3** Let Assumption 3.1 hold. Then, we have

1. $X$ is a purely discontinuous martingale such that for each $t > 0$ and $i = 1, \ldots, d$, $X^{(i)}_t$ has finite second moment and the quadratic variation of $X$ is given by

$$\langle X^{(i)}, X^{(j)} \rangle_t = \int_0^t \left( \int_{\mathbb{R}^d \setminus \{0\}} z^{(i)} z^{(j)} J(X_s, X_s + z) \, dz \right) \, ds, \quad 1 \leq i, j \leq d, \quad t > 0.$$  

2. If the kernel $N(x, dz) := J(x, x + z) \, dz$ satisfies Assumption 2.5, then the assertion of Theorem 2.6 is valid for every $x \in \mathbb{R}^d \setminus \mathcal{N}$.

**3.3 Examples**

In this subsection, we provide a class of jump-type Hunt processes generated by regular lower bounded semi-Dirichlet forms such that they are purely discontinuous martingales with finite second moment, and satisfy Khintchine’s law of the iterated logarithm.
Example 3.4 Let \( J(x, y) \) be a nonnegative Borel function on \( \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag} \) given by

\[
J(x, y) = \frac{c(x)}{|x - y|^{d + 2\alpha(|x - y|)}}
\]

such that the following two conditions hold:

(i) \( \alpha(r) \) is a positive function on \((0, \infty)\) such that

\[
\int_0^\infty r^{1-\alpha(r)} \, dr < \infty; \tag{3.7}
\]

(ii) \( c(x) \) is a function on \( \mathbb{R}^d \) bounded from below and above by positive constants, and

\[
\int_0^1 \frac{g_c(r)^2}{r^{1+\alpha(r)}} \, dr < \infty,
\]

where

\[
g_c(r) = \sup_{x, y \in \mathbb{R}^d, |x - y| = r} |c(x) - c(y)|.
\]

Then, the jumping kernel \( J(x, y) \) above generates a regular lower bounded semi-Dirichlet form \((\eta, \mathcal{F})\) on \( L^2(\mathbb{R}^d; dx)\). Indeed, by definition,

\[
J_s(x, y) = \frac{1}{2} \frac{c(x) + c(y)}{|x - y|^{d + \alpha(|x - y|)}}, \quad J_a(x, y) = \frac{1}{2} \frac{c(x) - c(y)}{|x - y|^{d + \alpha(|x - y|)}}.
\]

Then,

\[
\int_{\mathbb{R}^d} (1 \wedge |x - y|^2) J_s(x, y) \, dy = \frac{1}{2} \int_{\mathbb{R}^d} (1 \wedge |x - y|^2) \frac{c(x) + c(y)}{|x - y|^{d + \alpha(|x - y|)}} \, dy \\
\leq c_1 \int_{\mathbb{R}^d} \frac{1 \wedge |x - y|^2}{|x - y|^{d + \alpha(|x - y|)}} \, dy = c_2 \int_0^\infty \int_0^r \frac{1 \wedge r^2}{r^{1+\alpha(r)}} \, dr,
\]

which implies (3.1). Since

\[
\frac{J_a(x, y)^2}{J_s(x, y)} = \frac{1}{2} \frac{(c(x) - c(y))^2}{c(x) + c(y)} \frac{1}{|x - y|^{d + \alpha(|x - y|)}} \\
\leq c_3 \frac{(c(x) - c(y))^2}{|x - y|^{d + \alpha(|x - y|)}} \leq c_4 \frac{g_c(|x - y|)^2}{|x - y|^{d + \alpha(|x - y|)}},
\]

we also obtain (3.2).

It is obvious that Assumption 3.1 (i) holds. By (3.7) and the calculations above, one can see that Assumptions 3.1 (ii) and (iii) are satisfied. Since the kernel \( N(x, dz) = \)
\( J(x, x + z) \, dz \) fulfills Assumption 2.5, the statement of Theorem 3.3 holds for a Hunt process \( X \) generated by \((\eta, \mathcal{F})\).

The concrete example for \( \alpha(r) \) and \( c(x) \) satisfying the conditions (i) and (ii) above is as follows. Let \( \alpha(r) \) be a locally bounded and positive measurable function on \((0, \infty)\) such that

\[
\lim_{r \to +0} \alpha(r) < 2, \quad \lim_{r \to \infty} \alpha(r) > 2.
\]

Let \( c(x) \) be a Lipschitz continuous function on \( \mathbb{R}^d \) bounded from below and above by positive constants.

**Remark 3.5** To the best of our knowledge, it is unknown in the literature whether the martingale problem is well posed or not for the operator \((L, C_c^\infty(\mathbb{R}^d))\) defined by (3.5) with the jumping kernel \( J(x, y) \) in (3.6). In particular, we do not know whether \((L, C_c^\infty(\mathbb{R}^d))\) can generate a Feller semigroup or not.

On the other hand, we can construct a Hunt process on \( \mathbb{R}^d \) associated with the jumping kernel \( J(x, y) \) by using the Dirichlet form theory. The price is to take into consideration the exceptional set restricting the initial points of the process.

We further present examples of the jumping kernels \( J(x, y) \) such that the statement of Theorem 3.3 is valid for the associated Hunt processes. These examples can be regarded as variants of the jumping kernels given in [4, Subsection 6.2, (9) and (13)].

**Example 3.6** (1) Let \( A \) be a Borel set of \( \mathbb{R}^d \setminus \{0\} \) with positive Lebesgue measure such that

(a) \( A = -A := \{ x \in \mathbb{R}^d \setminus \{0\} \mid -x \in A \} \);

(b) for any \( (x^{(1)}, \ldots, x^{(d)}) \in A \) and for any permutation \( \{i_1, \ldots, i_d\} \) of \( \{1, \ldots, d\} \), \( (x^{(i_1)}, \ldots, x^{(i_d)}) \in A \).

Let \( J(x, y) \) be a nonnegative Borel function on \( \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag} \) given by

\[
J(x, y) = \frac{c(x)}{|x - y|^{d + \alpha(|x - y|)}} 1_{\{y - x \in A\}}.
\]

Suppose that the functions \( \alpha(r) \) and \( c(x) \) satisfy (i) and (ii) as these in Example 3.4. Then, in the same manner as for (3.6), we can show that the statement of Theorem 3.3 is true for the Hunt process generated by a lower bounded semi-Dirichlet form with the jumping kernel \( J(x, y) \).

(2) Let \( n \in \mathbb{N} \). Let \( A_i \ (1 \leq i \leq n) \) be a Borel set of \( \mathbb{R}^d \setminus \{0\} \) with positive Lebesgue measure such that (a) and (b) in (1) are fulfilled. For each \( i \in \{1, \ldots, n\} \), let \( \alpha_i(r) \) be a positive function on \((0, \infty)\) and \( c_i(x) \) a function on \( \mathbb{R}^d \) such that (i) and (ii) in Example 3.4 are satisfied. Then, the same consequence as in (1) is valid for the jumping kernel

\[
J(x, y) = \sum_{i=1}^{n} \frac{c_i(x)}{|x - y|^{d + \alpha_i(|x - y|)}} 1_{\{y - x \in A_i\}}.
\]
Acknowledgements The authors would like to thank Professor Takashi Kumagai and Professor Masayoshi Takeda for their valuable comments on the draft of this paper. They are grateful to the referee and the associate editor for their valuable comments and suggestions, which improved the results and presentation of the manuscript. The research of Yuichi Shiozawa is supported in part by JSPS KAKENHI No. JP17K05299. The research of Jian Wang is supported by the National Natural Science Foundation of China (No. 11831014), the Program for Probability and Statistics: Theory and Application (No. IRT1704) and the Program for Innovative Research Team in Science and Technology in Fujian Province University (IRTSTFJ).

References

1. Bae, J., Kang, J., Kim, P., Lee, J.: Heat kernel estimates for symmetric jump processes with mixed polynomial growths. Ann. Probab. 47, 2830–2868 (2019)
2. Bass, R.F.: Uniqueness in law for pure jump Markov processes. Probab. Theory Relat. Fields 79, 271–287 (1988)
3. Böttcher, B., Schilling, R.L., Wang, J.: Lévy-Type Processes: Construction, Approximation and Sample Path Properties. Lecture Notes in Mathematics, vol. 2099. Lévy Matters III. Springer, Berlin (2014)
4. Felsinger, M., Kassmann, M., Voigt, P.: The Dirichlet problem for nonlocal operators. Math. Z. 279, 779–809 (2015)
5. Fukushima, M., Uemura, T.: Jump-type Hunt processes generated by lower bounded semi-Dirichlet forms. Ann. Probab. 40, 858–889 (2012)
6. Gnedenko, B.V.: Sur la croissance des processus stochastiques homogènes à accroissements indépendants. Izv. Akad. Nauk SSSR Ser. Mat. 7, 89–110 (1943). (in Russian with French summary)
7. Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes, 2nd edn. Springer-Verlag, Berlin (2003)
8. Kim, P., Kumagai, T., Wang, J.: Laws of the iterated logarithm for symmetric jump processes. Bernoulli 23, 2330–2379 (2017)
9. Kotani, S.: On a condition that one-dimensional diffusion processes are martingales. In: Memoriam Paul-Andr’e Meyer: Seminaire de Probabilit’es XXXIX, Lecture Notes in Mathematics, vol. 1874. Springer, pp. 149–156 (2006)
10. Kühn, F.: Lévy-Type Processes: Moments, Construction and Heat Kernel Estimates. Lecture Notes in Mathematics, vol. 2187. Lévy Matters IV. Springer, Cham (2017)
11. Oshima, Y.: Semi-Dirichlet Forms and Markov Processes, De Gruyter Studies in Mathematics, vol. 48. De Gruyter, Berlin (2013)
12. Protter, P.E.: Stochastic Integration and Differential Equations, 2nd edn. Springer-Verlag, Berlin (2004)
13. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, 3rd edn. Springer-Verlag, Berlin (1999)
14. Sato, K.: Lévy Processes and Infinitely Divisible Distributions, Revised edn. Cambridge University Press, Cambridge (2013)
15. Schilling, R.L.: Growth and Hölder conditions for the sample paths of Feller processes. Probab. Theory Relat. Fields 112, 565–611 (1998)
16. Schilling, R. L., Wang, J.: Lower bounded semi-Dirichlet forms associated with Lévy type operators. In: Chen, Z.-Q., Jacob, N., Takeda, M., Uemura T. (eds.) Festschrift Masatoshi Fukushima. In Honor of Masatoshi Fukushima’s Sanju. World Scientific, New Jersey, pp. 507–527 (2015)
17. Shimizu, Y., Nakano, F.: A remark on conditions that a diffusion in the natural scale is a martingale. Osaka J. Math. 55, 385–391 (2018)
18. Shiozawa, Y., Uemura, T.: Stability of the Feller property for non-local operators under bounded perturbations. Glas. Mat. Ser. III(45), 155–172 (2010)
19. Shiozawa, Y., Wang, J.: Rate functions for symmetric Markov processes via heat kernel. Potential Anal. 46, 23–53 (2017)
20. Shiozawa, Y., Wang, J.: Long-time heat kernel estimates and upper rate functions of Brownian motion type for symmetric jump processes. Bernoulli 25, 3796–3831 (2019)
21. Stroock, D.W.: Probability Theory, An Analytic View. Cambridge University Press, Cambridge (2011)
22. Wang, J.-G.: A law of the iterated logarithm for stochastic integrals. Stoch. Process. Appl. 47, 215–228 (1993)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.