LOWER REGULARITY SOLUTIONS OF THE BIHARMONIC SCHRÖDINGER EQUATION IN A QUARTER PLANE

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ABSTRACT. This paper deals with the initial-boundary value problem of the biharmonic cubic nonlinear Schrödinger equation in a quarter plane with inhomogeneous Dirichlet-Neumann boundary data. We prove local well-posedness in the low regularity Sobolev spaces introducing Duhamel boundary forcing operator associated to the linear equation to construct solutions on the whole line. With this in hands, the energy and nonlinear estimates allow us to apply Fourier restriction method, introduced by J. Bourgain, to get the main result of the article. Additionally, adaptations of this approach for the biharmonic cubic nonlinear Schrödinger equation on star graphs are also discussed.

1. Introduction

1.1. Presentation of the model. Fourth-order cubic nonlinear Schrödinger (4NLS) equation or biharmonic cubic nonlinear Schrödinger (biharmonic NLS) equation

\[ i \partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda |u|^2 u, \]

have been introduced by Karpman [34] and Karpman and Shagalov [35] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Equation (1.1) arises in many scientific fields such as quantum mechanics, nonlinear optics and plasma physics, and has been intensively studied with fruitful references (see \([2, 18, 34, 40, 41]\) and references therein).

The past twenty years such 4NLS have been deeply studied from different mathematical viewpoints. For example, Fibich et al. [21] worked various properties of the equation in the subcritical regime, with part of their analysis relying on very interesting numerical developments. The well-posedness and existence of the solutions has been shown (see, for instance, \([40, 41, 44, 45, 46]\)) by means of the energy method, harmonic analysis, etc.

It is interesting to point out that there are many works related with the equations (1.2) not only dealing with well-posedness theory. For example, recently Natali and Pastor [38], considered the fourth-order dispersive cubic nonlinear Schrödinger equation on the line with mixed dispersion. They proved the orbital stability, in the \(H^2(\mathbb{R})\)-energy space by constructing a suitable Lyapunov function. Considering the equation (1.2) on the circle, Oh and Tzvetkov [45], showed that the mean-zero Gaussian measures on Sobolev spaces \(H^s(\mathbb{T})\), for \(s > \frac{3}{4}\), are quasi-invariant under the flow. For instance, in this spirit, there has been a significant progress over the recent years and the reader can have a great view in to nonlinear Schrödinger equation in \([8, 9]\).

Recently, the first and the second authors worked in this equation with the purpose to prove results of controllability. More precisely, they proved that on torus \(\mathbb{T}\), the solution of the associated linear system (1.1) is globally exponential stable, by using certain properties of propagation of compactness and regularity in Bourgain spaces. This property together with the local exact controllability ensures that fourth order nonlinear Schrödinger is globally exactly controllable, for more details see \([10]\).

In a recent work, Özsari and Yolcu [39] proposed that the system (1.1), without term \(\partial_x^2 u\), has an interesting physical point of view, more precisely, the model corresponds to a situation in which wave is generated from a fixed source such that it moves into the medium in one specific direction.
1.2. Setting of the problem. We mainly consider the biharmonic Schrödinger equation on the right half-line

\[
\begin{aligned}
&i\partial_t u - \partial_x^4 u + \lambda |u|^2 u = 0, & (t, x) \in (0, T) \times (0, \infty), \\
u(0, x) = u_0(x), & x \in (0, \infty), \\
u(t, 0) = f(t), & u_x(t, 0) = g(t) & t \in (0, T).
\end{aligned}
\]

(1.2)

With suitable choice of \(f(t)\) and \(g(t)\) on the equation (1.2), we are interested on the following initial-boundary value problem (IBVP):

Is the IBVP (1.2) local well-posedness in the low regularity Sobolev space, more precisely, in \(H^s(\mathbb{R}^+)\) for \(0 \leq s \leq \frac{1}{2}\)?

Before to present the answer for this question, let us to present a brief comments of the techniques to solve IBVPs on the half-line.

1.3. Comments about the techniques to solve IBVPs on the half-line. Different techniques have been developed in the last years in order to solve IBVPs associated to some dispersive models on the half-line. In [23], Fokas introduced an approach to solve IBVPs associated to integrable nonlinear evolution equations, which is known as the unified transform method (UTM) or as Fokas transform method. The UTM method provides a generalization of the Inverse Scattering Transform (IST) method from initial value problems to IBVPs. The classical method based on the Laplace transform was used successfully in the works [5, 6, 19, 17] developed by Bona, Sun, Zhang, Erdogan, Compaan and Tzirakis. Recently, a new approach was introduced by Colliander and Kenig [16] by recasting the IBVP on the half-line by a forced IVP defined in the line \(\mathbb{R}\). To see other applications of this technique, we refer the results established by Holmer, Cavalcante and Corcho on the works [13, 14, 30, 31]. On the other hand, in [20], Faminskii used an approach based on the investigation of special solutions of a “boundary potential” type for solution of linearized Korteweg–de Vries (KdV) equation in order to obtain global results for the IBVP associated to the KdV equation on the half-line with more general boundary conditions. More recently, Fokas et al. [24] introduced a method which combines the UTM method with a contraction mapping principle. We caution that this is only a small sample of the extant work on these techniques.

1.4. Biharmonic NLS equation. As mentioned in the beginning of this introduction 4NLS equation or biharmonic NLS equation

\[
i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda |u|^2 u,
\]

(1.3)

have been introduced by Karpman [34] and Karpman and Shagalov [35] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Huo and Jia [32] studied the Cauchy problem of one-dimensional fourth-order nonlinear Schrödinger equation. They proved the local well-posedness for initial data in \(H^s(\mathbb{R})\) for \(s \geq \frac{1}{2}\) by using the Fourier restriction norm method under certain coefficient condition. Concerning to local well-posedness of the nonlinear fourth order Schrödinger equations, we cite [29, 42]. With respect of the global well-posedness, in one dimensional case with some restriction in the initial data for various nonlinearities, we infer [25, 26, 27, 28] and, finally, for the study of the n-dimensional, the reader can see [3, 4].

In a recent work on IBVP of biharmonic Schrödinger equation on the half-line

\[
i\partial_t u + \partial_x^4 u = \lambda |u|^p u,
\]

(1.4)

Özsari and Yolcu [39], proved local well-posedness on the high regularity function spaces \(H^s(\mathbb{R}^+)\), for \(\frac{1}{2} < s < \frac{3}{2}\), with \(s \neq \frac{3}{2}\). The authors used the Fokas method [22, 23] combined with contraction arguments to archived the result.
1.5. Main result. Now, let us present the main result of this article. Consider the biharmonic Schrödinger equation on the right half-line
\begin{align*}
\begin{cases}
    i\partial_t u + \gamma \partial_x^4 u + \lambda |u|^2 u = 0, \\
    u(0, x) = u_0(x), \\
    u(t, 0) = f(t), \quad u_x(t, 0) = g(t)
\end{cases}
\quad (t, x) \in (0, T) \times (0, \infty),
\end{align*}
(1.5)
for \( \gamma, \lambda \in \mathbb{R} \). We said that the system (1.5) is to be focusing if \( \gamma \lambda < 0 \) and defocusing \( \gamma \lambda > 0 \). In this paper we will study the case when \( \gamma = -1 \), however the approach used here can be applied when \( \gamma \in \mathbb{R} \setminus \{0\} \).

The presence of two boundary conditions in (1.5) can be motivated by integral identities on smooth decaying solutions to the linear equation
\begin{align*}
    i\partial_t u - \partial_x^4 u = 0.
\end{align*}
Indeed, for a smooth decaying solution \( u \) to (1.6) and \( T > 0 \), we have
\begin{align*}
    \int_0^\infty |u(T, x)|^2 dx = \int_0^\infty |u(0, x)|^2 dx - \int_0^T \text{Im}(\partial_x^3 u(t, 0)\overline{\pi(t, 0)}) dt + \int_0^T \text{Im}(\partial_x^2 u(t, 0)\partial_x \pi(t, 0)) dt.
\end{align*}
(1.7)
Thus, we can conclude from (1.7) that if we assume \( u(0, x) = u(t, 0) = u_x(t, 0) = 0 \) the linear solution for (1.6) is the trivial one.

It is well-known by Kenig et al. [36] that the local smoothing effect for the fourth-order linear group operator \( e^{it\partial_x^4} \)
\begin{align*}
    \|e^{it\partial_x^4}\phi\|_{L_x^\infty H^{2+\frac{3}{8}}(\mathbb{R})} \leq c\|\phi\|_{L_x^2(\mathbb{R})},
\end{align*}
plays an important role in the low regularity local theory for the fourth-order equations. Moreover, it can be expressed as
\begin{align*}
    \|\partial_x^j e^{it\partial_x^4}\phi\|_{L_x^\infty H^{2+\frac{3}{8}-2j}(\mathbb{R})} \leq c\|\phi\|_{H^s(\mathbb{R})}, \text{ for } j = 0, 1,
\end{align*}
(1.8)
which motivates the relation of regularities among initial and boundary data.

Thus, we are able to present the main goal in the paper: To answer the problem cited on the beginning of this introduction, that is, to show the local well-posedness of (1.5) in the low regularity Sobolev spaces, more precisely in \( H^s(\mathbb{R}^+) \), for \( 0 \leq s < \frac{1}{2} \).

We state the main theorem for IBVP (1.5) as follows.

**Theorem 1.1.** Let \( s \in [0, \frac{1}{2}) \). For given initial-boundary data
\begin{align*}
    (u_0, f, g) \in H^s(\mathbb{R}^+) \times H^{2s+\frac{3}{8}}(\mathbb{R}^+) \times H^{2s+\frac{3}{8}}(\mathbb{R}^+),
\end{align*}
there exist a positive time
\begin{align*}
    T := T\left(\|u_0\|_{H^s(\mathbb{R}^+)}, \|f\|_{H^{2s+\frac{3}{8}}(\mathbb{R}^+)}, \|g\|_{H^{2s+\frac{3}{8}}(\mathbb{R}^+)}\right),
\end{align*}
and unique solution \( u(t, x) \in C((0, T); H^s(\mathbb{R}^+)) \) of the IBVP (1.5), when \( \gamma = -1 \), satisfying
\begin{align*}
    u \in C(\mathbb{R}^+; H^{2s+\frac{3}{8}}(0, T)) \cap X^{s,b}(0, T) \times \mathbb{R}^+ \quad \text{and} \quad \partial_x u \in C(\mathbb{R}^+; H^{2s+\frac{3}{8}}(0, T)),
\end{align*}
for some \( b(s) < \frac{1}{2} \). Moreover, the map \( (u_0, f, g) \mapsto u \) is analytic from \( H^s(\mathbb{R}^+) \times H^{2s+\frac{3}{8}}(\mathbb{R}^+) \times H^{2s+\frac{3}{8}}(\mathbb{R}^+) \) to \( C((0, T); H^s(\mathbb{R}^+)) \).

**Remarks 1.1.** Finally, the following comments are now given in order:

1. The proof of Theorem 1.1 is based on the Fourier restriction method for a suitable extension of solutions. We first convert the IBVP of (1.5) posed in \( \mathbb{R}^+ \times \mathbb{R}^+ \) to the initial value problem (IVP) of (1.5) (integral equation formula) in the whole space \( \mathbb{R} \times \mathbb{R} \) (see Section 3) by using the Duhamel boundary forcing operator. The energy and nonlinear estimates (will
be established in Sections 4) allow us to apply the Picard iteration method to IVP of (1.5), and hence we can complete the proof. The new ingredients here are the Duhamel boundary forcing operator for the fourth-order linear equation and its analysis.

2. Note that Theorem 1.1 completes the regularity obtained by [39] for the biharmonic nonlinear Schrödinger equation, in the following sense: In [39], the authors showed the local well-posedness in the Sobolev spaces $H^s(\mathbb{R}^+)$ with high regularity $\frac{1}{2} < s < \frac{3}{2}$, when $s \neq \frac{3}{2}$, by using Fokas approach. In addition, introducing the boundary forcing operator, proposed by Holmer, we showed the local well-posedness in the lower regularity, that is, when $s \in [0, \frac{1}{2})$.

3. The approach used in our result, together with some extension as did [13, 15, 30, 31] also guarantee the local well-posedness result in high regularity.

1.6. Notations. In all this paper, we will consider $\mathbb{R}^+$ as $(0, \infty)$. Moreover, for positive real numbers $x, y \in \mathbb{R}^+$, we mean $x \lesssim y$ by $x \leq Cy$ for some $C > 0$. Also, denote $x \sim y$ by $x \lesssim y$ and $y \lesssim x$. Similarly, $\lesssim_s$ and $\sim_s$ can be defined, where the implicit constants depend on $s$.

Our work is outlined in the following way: In Section 2, we introduce some function spaces defined on the half-line and construct the solution spaces. Section 3 is devoted to introduce the boundary forcing operator for the biharmonic Schrödinger equation. In section 4, we show the energy estimates and present the trilinear estimates, respectively. The main result of this article, Theorem 1.1, is proved in Section 5. Finally, Section 6, we present some open problems which seems to be of interest from the mathematical point of view.

2. Preliminaries

Throughout the paper, we fix a cut-off function $\psi(t) := \psi$

(2.1) \[ \psi \in C_0^\infty(\mathbb{R}) \text{ such that } 0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ on } [0, 1], \quad \psi \equiv 0, \text{ for } |t| \geq 2. \]

2.1. Sobolev spaces on the half-line. For $s \geq 0$, we define the homogeneous $L^2$-based Sobolev spaces $\dot{H}^s = H^s(\mathbb{R})$ by the norm $\|\phi\|_{\dot{H}^s} = \|\xi^s \hat{\phi}(\xi)\|_{L_x^2}$ and the $L^2$-based inhomogeneous Sobolev spaces $H^s = H^s(\mathbb{R})$ by the norm $\|\phi\|_{H^s} = \|(1 + |\xi|^2)^{\frac{s}{2}}\hat{\phi}(\xi)\|_{L_x^2}$, where $\hat{\phi}$ denotes the Fourier transform of $\phi$. Moreover, we say that $f \in H^s(\mathbb{R}^+)$, if there exists $F \in H^s(\mathbb{R})$ such that $f(x) = F(x)$ for $x > 0$, in this case we set

\[ \|f\|_{H^s(\mathbb{R}^+)} = \inf_F \|F\|_{H^s(\mathbb{R})}. \]

On the other hand, for $s \in \mathbb{R}$, $f \in H^s_0(\mathbb{R}^+)$ provided that there exists $F \in H^s(\mathbb{R})$ such that $F$ is the extension of $f$ on $\mathbb{R}$ and $F(x) = 0$ for $x < 0$. In this case, we set $\|f\|_{H^s_0(\mathbb{R}^+)} = \inf_F \|F\|_{H^s(\mathbb{R})}$.

For $s < 0$, we define $H^s(\mathbb{R}^+)$ as the dual space of $H^{-s}(\mathbb{R}^+)$.

Let us also define the sets $C^\infty_0(\mathbb{R}^+) = \{ f \in C^\infty(\mathbb{R}); \text{supp} f \subset [0, \infty) \}$ and $C^\infty_{0,c}(\mathbb{R}^+)$ as the subset of $C^\infty_0(\mathbb{R}^+)$, whose members have a compact support on $(0, \infty)$. We remark that $C^\infty_{0,c}(\mathbb{R}^+)$ is dense in $H^s_0(\mathbb{R}^+)$ for all $s \in \mathbb{R}$.

To finish this subsection, we give some elementary properties of the Sobolev spaces.

**Lemma 2.1.** [33, Lemma 3.5] For $-\frac{1}{2} < s < \frac{1}{2}$ and $f \in H^s(\mathbb{R})$, we have

(2.2) \[ \|\chi_{(0, \infty)}f\|_{H^s(\mathbb{R})} \leq c\|f\|_{H^s(\mathbb{R})}. \]

**Lemma 2.2.** [16, Lemma 2.8] If $0 \leq s < \frac{1}{2}$, then, for the cut-off function $\psi$ defined in (2.1), $\|\psi f\|_{H^s(\mathbb{R})} \leq c\|f\|_{\dot{H}^s(\mathbb{R})}$ and $\|\psi f\|_{H^{-s}(\mathbb{R})} \leq c\|f\|_{H^{-s}(\mathbb{R})}$, where the constant $c$ depends only on $s$ and $\psi$.

**Remark 2.1.** Lemma 2.2 is equivalent to

\[ \|f\|_{H^s(\mathbb{R})} \sim \|f\|_{\dot{H}^s(\mathbb{R})}, \]

for $-\frac{1}{2} < s < \frac{1}{2}$ where $f \in H^s(\mathbb{R})$ with suppf $\subset [0, 1]$.

The following two auxiliaries lemmas can be found in [16] and its proofs will be omitted.
Lemma 2.3. [16, Proposition 2.4] If $\frac{1}{2} < s < \frac{3}{2}$ the following statements are valid:
(a) $H_0^s(\mathbb{R}^+) = \{ f \in H^s(\mathbb{R}^+); f(0) = 0 \}$,
(b) If $f \in H^s(\mathbb{R}^+)$ with $f(0) = 0$, then $\|X(0,\infty) f\|_{H^s_0(\mathbb{R}^+)} \leq c\|f\|_{H^s(\mathbb{R}^+)}$.

Lemma 2.4. [16, Proposition 2.5] Let $-\infty < s < \infty$ and $f \in H^s_0(\mathbb{R}^+)$. For the cut-off function $\psi$ defined in (2.1), we have $\|\psi f\|_{H^s_0(\mathbb{R}^+)} \leq c\|f\|_{H^s_0(\mathbb{R}^+)}$.

2.2. Solution spaces. For $f \in \mathcal{S}(\mathbb{R}^2)$, we denote by $\tilde{f}$ or $\mathcal{F}(f)$ the Fourier transform of $f$ with respect to both spatial and time variables
$$\tilde{f}(\tau, \xi) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-i\tau t} f(t, x) \, dx \, dt.$$ Moreover, we use $\mathcal{F}_x$ and $\mathcal{F}_t$ to denote the Fourier transform with respect to space and time variable respectively (also use $\tilde{}$ for both cases).

In a celebrated paper in 90’s Bourgain [7], established a way to prove the well-posedness of a classes of dispersive systems. More precisely, on the Sobolev spaces $H^s$, for smaller values of $s$, Bourgain found a yet more suitable smoothing property for solutions of the Korteweg-de Vries equation.

In this spirit, for $s, b \in \mathbb{R}$, we introduce the classical Bourgain spaces $X^{s,b}$ associated to (1.2) as the completion of $\mathcal{S}'(\mathbb{R}^2)$ under the norm
$$\|f\|_{X^{s,b}}^2 = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} (\tau + \xi^4)^{2b} |\tilde{f}(\tau, \xi)|^2 \, d\xi \, d\tau,$$
where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

One of the basic property of $X^{s,b}$ can be read as follows:

Lemma 2.5. [43, Lemma 2.11] Let $\psi(t)$ be a Schwartz function in time. Then, we have
$$\|\psi(t)f\|_{X^{s,b}} \lesssim_{\psi,b} \|f\|_{X^{s,b}}.$$

As well-known, the space $X^{s,b}$ with $b > \frac{1}{2}$ is well-adapted to study the IVP of dispersive equations. However, in the study of IBVP, the standard argument cannot be applied directly. This is due to the lack of hidden regularity, more precisely, the control of (derivatives) time trace norms of the Duhamel parts requires to work in $X^{s,b}$-type spaces for $b < \frac{1}{2}$, since the full regularity range cannot be covered (see Lemma 4.2 (b)).

Therefore, to treat the solution of our problem, set the solution space denoted by $Z^{s,b}$ with the following norm
$$\|f\|_{Z^{s,b}(\mathbb{R}^2)} = \sup_{t \in \mathbb{R}} \|f(t, \cdot)\|_{H^s(\mathbb{R})} + \sum_{j=0}^{\infty} \sup_{\gamma \in \mathbb{R}} \|\partial_x^j f(\cdot, x)\|_{H^{s+\frac{4j+2}{4}(\mathbb{R})}} + \|f\|_{X^{s,b}}.$$

The spatial and time restricted space of $Z^{s,b}(\mathbb{R}^2)$ is defined by the standard way:
$$Z^{s,b}((0, T) \times \mathbb{R}^+) = Z^{s,b} \big|_{(0, T) \times \mathbb{R}^+}$$
equipped with the norm
$$\|f\|_{Z^{s,b}((0, T) \times \mathbb{R}^+)} = \inf_{g \in Z^{s,b}} \{ \|g\|_{Z^{s,b}} : g(t, x) = f(t, x) \text{ on } (0, T) \times \mathbb{R}^+ \}.$$

2.3. Riemann-Liouville fractional integral. Before to begin our study of the IBVP for (1.2), in this section, we just give a brief summary of the Riemann-Liouville fractional integral operator, the reader can see [16, 31] for more details.

Let us define the function $t_+$ as follows
$$t_+ = t \quad \text{if} \quad t > 0, \quad t_+ = 0 \quad \text{if} \quad t \leq 0.$$
The tempered distribution $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ is defined as a locally integrable function for $\text{Re} \, \alpha > 0$ by

$$
\left< \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \ f \right> = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f(t) \, dt.
$$

It follows that

$$(2.3) \quad \frac{t^{\alpha-1}}{\Gamma(\alpha)} = \partial_t^k \left( \frac{t^{\alpha+k-1}}{\Gamma(\alpha+k)} \right),$$

for all $k \in \mathbb{N}$. Expression (2.3) can be used to extend the definition of $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ to all $\alpha \in \mathbb{C}$ in the sense of distributions. In fact, a change of contour calculation shows the Fourier transform of $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ is the following one

$$
(2.4) \quad \left< \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \ e^{-i\alpha \tau} \right> = e^{-\frac{i}{2} \pi \alpha (\tau - i0)}^{-\alpha},
$$

where $(\tau - i0)^{-\alpha}$ is the distributional limit. For $\alpha \notin \mathbb{Z}$, let us rewrite (2.4) on the following way

$$
(2.5) \quad \left< \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \ e^{-\frac{i}{2} \pi \alpha \tau} \right> = e^{-\frac{i}{2} \alpha \pi i |\tau|^{-\alpha} \chi(0,\infty)} + e^{\frac{i}{2} \alpha \pi i |\tau|^{-\alpha} \chi(-\infty,0)}.
$$

Note that from (2.4) and (2.5), we have that

$$
(2.6) \quad (\tau - i0)^{-\alpha} = |\tau|^{-\alpha} \chi(0,\infty) + e^{\alpha \pi i |\tau|^{-\alpha} \chi(-\infty,0)}.
$$

For $f \in C_{0}^{\infty} (\mathbb{R}^+)\), define $I_\alpha f$ as

$$
I_\alpha f = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f.
$$

Thus, for $\text{Re} \, \alpha > 0$, we have

$$
(2.7) \quad I_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds.
$$

The following properties easily holds $I_0 f = f$, $I_1 f(t) = \int_0^t f(s) \, ds$, $I_{-1} f = f'$ and $I_\alpha I_\beta = I_{\alpha + \beta}$. Moreover, the lemmas below can be found in [31], and we will omitted the proofs.

**Lemma 2.6.** [31, Lemma 2.1] If $f \in C_{0}^{\infty} (\mathbb{R}^+)$, then $I_\alpha f \in C_{0}^{\infty} (\mathbb{R}^+)$, for all $\alpha \in \mathbb{C}$.

**Lemma 2.7.** [31, Lemma 5.3] If $0 \leq \text{Re} \, \alpha < \infty$ and $s \in \mathbb{R}$, then $\|I_{-\alpha} h\|_{H^c_0(\mathbb{R}^+)} \leq c \|h\|_{H^{c+\alpha}(\mathbb{R}^+)}$, where $c = c(\alpha)$.

**Lemma 2.8.** [31, Lemma 5.4] If $0 \leq \text{Re} \, \alpha < \infty$, $s \in \mathbb{R}$ and $\mu \in C_{0}^{\infty} (\mathbb{R})$, then $\|\mu I_{\alpha} h\|_{H^c_0(\mathbb{R}^+)} \leq c \|h\|_{H^{c+\alpha}(\mathbb{R}^+)}$, where $c = c(\mu, \alpha)$.

### 2.4. Oscillatory integral.

In this subsection, we will define the oscillatory integral which one is the key to define, in the next section, the Duhamel boundary forcing operator. Let

$$
B(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix \xi} e^{-i\xi^4} \, d\xi.
$$

We first calculate $B(0)$. A change of variable $(\eta = \xi^4)$, give us the following

$$
B(0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi^4} \, d\xi = \frac{1}{8\pi} \int_{\mathbb{R}} e^{-i\eta \eta^{-3/4}} \, d\eta = -\frac{i}{\pi} \Gamma \left(\frac{5}{4}\right).
$$

Now, let us define the Mellin transform of $B(x)$.

**Lemma 2.9.** For $\text{Re} \, \lambda > 0$ we have

$$
(2.9) \quad \int_0^\infty x^{\lambda-1} B(x) \, dx = \frac{\Gamma(\lambda) \Gamma \left(\frac{1}{4} - \frac{\lambda}{2}\right)}{8\pi} \left( e^{-i\pi(1+3\lambda)} + e^{-i\pi(1-5\lambda)} \right).
$$
Finally, as we can split by (2.15)

Once more, applying dominated convergence theorem and changing the contour, yields that

Using a change of contour, we get that (2.10)

By using dominated convergence and Fubini’s theorems we have

then we have

Set (0, 3/8).

We also refer to [13, 14, 30] for a well exposition about the topic.

Colliander and Kenig [16], in order to construct the solution to (1.2) forced by boundary conditions.

Proof. On the argument, by analyticity argument, we can assume that \( \lambda \) is a real number in the set \((0, 3/8)\).

Consider

\[
B_1(x) = \frac{1}{2\pi} \int_0^{+\infty} e^{ix\xi} e^{-i\xi^4} d\xi
\]

and

\[
B_2(x) = \frac{1}{2\pi} \int_0^0 e^{ix\xi} e^{-i\xi^4} d\xi = \frac{1}{2\pi} \int_0^\infty e^{-ix\xi} e^{-i\xi^4} d\xi,
\]

then we have \( B(x) = B_1(x) + B_2(x) \). Define

\[
B_{1,\epsilon}(x) = \frac{1}{2\pi} \int_0^{+\infty} e^{ix\xi} e^{-i\xi^4} e^{-\epsilon \xi} d\xi.
\]

By using dominated convergence and Fubini’s theorems we have

\[
\int_0^\infty x^{\lambda-1} B_1(x) dx = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_0^\infty e^{-\delta x} x^{\lambda-1} B_1(x) dx
\]

(2.10)

\[
= \lim_{\epsilon \to 0} \lim_{\delta \to 0} \frac{1}{2\pi} \int_0^{+\infty} e^{-i\xi^4} e^{-\epsilon \xi} \int_0^\infty e^{ix\xi} e^{-\delta x} x^{\lambda-1} dx d\xi.
\]

Using a change of contour, we get that

(2.11)

\[
\int_0^{+\infty} e^{ix\xi} e^{-\delta x} x^{\lambda-1} dx = \xi^{-\lambda} e^{i\lambda^2\frac{\pi}{4}} \Gamma \left( \lambda, \frac{\delta}{\epsilon} \right),
\]

where \( \Gamma(\lambda, z) = \int_0^{+\infty} r^{\lambda-1} e^{ir \xi} e^{-r} dr \). Again, thanks to dominated convergence theorem follows that

(2.12)

\[
\lim_{\delta \to 0} \int_0^{+\infty} e^{ix\xi} e^{-\delta x} x^{\lambda-1} dx = \xi^{-\lambda} e^{i\lambda^2\frac{\pi}{4}} \Gamma (\lambda).
\]

Once more, applying dominated convergence theorem and changing the contour, yields that

(2.13)

\[
\int_0^{+\infty} x^{\lambda-1} B_1(x) dx = \frac{\Gamma(\lambda)}{2\pi} e^{i\lambda^2\frac{\pi}{4}} \lim_{\epsilon \to 0} \int_0^{+\infty} e^{-i\xi^4} e^{-\epsilon \xi} \xi^{-\lambda} d\xi
\]

\[
= \frac{\Gamma(\lambda)}{2\pi} e^{i\lambda^2\frac{\pi}{4}} \lim_{\epsilon \to 0} \int_0^{+\infty} e^{-\epsilon \xi} \xi^{-\lambda} \frac{d}{d\eta} \left( \frac{\eta^{1/4}}{\eta} \right) d\eta
\]

\[
= \frac{\Gamma(\lambda)}{2\pi} e^{i\lambda^2\frac{\pi}{4}} \frac{1}{4} e^{-\frac{\lambda}{4} \left( 1 - \lambda \right)} \Gamma \left( \frac{1}{4} - \frac{\lambda}{4} \right)
\]

\[
= \frac{\Gamma(\lambda) \Gamma \left( \frac{1}{4} - \frac{\lambda}{4} \right)}{2\pi} e^{-i\lambda^2\frac{\pi}{4}} (1-5\lambda).
\]

In the similar way, by using the following

(2.14)

\[
\int_0^{+\infty} e^{-ix\xi} e^{-\delta x} x^{\lambda-1} dx = \xi^{-\lambda} e^{-i\lambda^2\frac{\pi}{4}} \Gamma \left( \lambda, \frac{\delta}{\epsilon} \right),
\]

we can obtain

(2.15)

\[
\int_0^{+\infty} x^{\lambda-1} B_2(x) dx = \frac{\Gamma(\lambda)}{2\pi} e^{-i\lambda^2\frac{\pi}{4}} e^{-\frac{\lambda}{2} \left( 1 - \lambda \right)} \Gamma \left( \frac{1}{4} - \frac{\lambda}{4} \right)
\]

\[
= \frac{\Gamma(\lambda) \Gamma \left( \frac{1}{4} - \frac{\lambda}{4} \right)}{2\pi} e^{-i\lambda^2\frac{\pi}{4}} (1+3\lambda).
\]

Finally, as we can split by \( B(x) = B_1(x) + B_2(x) \), (2.9) holds. \( \square \)

3. Duhamel boundary forcing operator

In this section, we study the Duhamel boundary forcing operator, which was introduced by Colliander and Kenig [16], in order to construct the solution to (1.2) forced by boundary conditions. We also refer to [13, 14, 30] for a well exposition about the topic.
3.1. Duhamel boundary forcing operator class. Let us introduce the Duhamel boundary forcing operator associated to the linearized biharmonic Schrödinger equation. Consider
\begin{equation}
M = \frac{1}{B(0)\Gamma(3/4)}.
\end{equation}
For \( f \in C_0^\infty(\mathbb{R}^+) \), define the boundary forcing operator \( \mathcal{L}^0 \) (of order 0) as
\begin{equation}
\mathcal{L}^0 f(t, x) := M \int_0^t e^{i(t-t')\partial^4_x} \delta_0(x) \mathcal{I}_{-\frac{3}{4}} f(t') dt',
\end{equation}
where \( e^{i\theta \partial_x} \) denotes the group associated to (1.6) given by
\begin{equation}
e^{i\partial^4_x \psi(x)} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-it\xi} \hat{\psi}(\xi) d\xi.
\end{equation}
Note that the property of convolution operator \( (\partial_x (f \ast g)) = (\partial_x f) \ast g = f \ast (\partial_x g) \) and the integration by parts in \( t' \) of (3.2) yields that
\begin{equation}
\partial_t \mathcal{L}^0 f(t, x) = iM \delta_0(x) \mathcal{I}_{-\frac{3}{4}} f(t) + \partial^4_x \mathcal{L}^0 f(t, x).
\end{equation}
By change of variable and using (2.8) gives
\begin{align}
\mathcal{L}^0 f(t, x) &= M \int_0^t e^{i(t-t')\partial^4_x} \delta_0(x) \mathcal{I}_{-\frac{3}{4}} f(t') dt' \\
&= M \int_0^t B\left( \frac{x}{(t-t')^{1/4}} \right) \mathcal{I}_{-\frac{3}{4}} f(t') dt'.
\end{align}
We are now in position to be precisely when the boundary forcing operator is continuous or discontinuous. More precisely, the following lemma holds.

**Lemma 3.1.** Let \( f \in C_0^\infty(\mathbb{R}^+) \).

(a) For fixed \( 0 \leq t \leq 1 \), \( \partial^k_x \mathcal{L}^0 f(t, x) \), \( k = 0, 1, 2 \), is continuous in \( x \in \mathbb{R} \) and has the decay property in terms of the spatial variable as follows:
\begin{equation}
|\partial^k_x \mathcal{L}^0 f(t, x)| \lesssim_N \|f\|_{H^{N+k}(x)^{-N}}, \quad N \geq 0.
\end{equation}
(b) For fixed \( 0 \leq t \leq 1 \), \( \partial^3_x \mathcal{L}^0 f(t, x) \) is continuous in \( x \) for \( x \neq 0 \) and is discontinuous at \( x = 0 \) satisfying
\begin{equation}
\lim_{x \to 0^-} \partial^3_x \mathcal{L}^0 f(t, x) = -i \frac{M}{2} \mathcal{I}_{-3/4} f(t), \quad \lim_{x \to 0^+} \partial^3_x \mathcal{L}^0 f(t, x) = i \frac{M}{2} \mathcal{I}_{-3/4} f(t).
\end{equation}
\( \partial^3_x \mathcal{L}^0 f(t, x) \) also has the decay property in terms of the spatial variable
\begin{equation}
|\partial^3_x \mathcal{L}^0 f(t, x)| \lesssim_N \|f\|_{H^{N+3}(x)^{-N}}, \quad N \geq 0.
\end{equation}

**Proof.** In fact, the continuity of \( \partial^k_x \mathcal{L}^0 f(t, x) \) follows from (3.4), for \( k = 0, 1, 2 \) and the proof of (3.5) exactly follows the idea introduced by Holmer in [30, Lemma 12]. Moreover, (3.5) and (3.3) yield that \( \partial^3_x \mathcal{L}^0 f(t, x) \) is discontinuous only at \( x = 0 \) of size \( MT_{-\frac{3}{4}} f(t) \) (where \( M \) is defined as in (3.1)), and the decay bounds (3.6) holds. \( \square \)

**Remark 3.1.** Lemma 3.1 ensures that \( \mathcal{L}^0 f(t, 0) = f(t) \).

We are now generalize the boundary forcing operator (3.2). Firstly, for \( \text{Re} \lambda > -4 \) and given \( g \in C_0^\infty(\mathbb{R}^+) \), we define
\begin{equation}
\mathcal{L}^\lambda g(t, x) = \left[ \frac{x^{\lambda-1}}{\Gamma(\lambda)} \ast \mathcal{L}^0(\mathcal{I}_{-\frac{3}{4}} g)(t, \cdot) \right](x),
\end{equation}
where \( \ast \) denotes the convolution operator and \( \frac{x^{\lambda-1}}{\Gamma(\lambda)} = \frac{(-x)^{\lambda-1}}{\Gamma(\lambda)} \). In particular, for \( \text{Re} \lambda > 0 \), we have
\begin{equation}
\mathcal{L}^\lambda g(t, x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (y-x)^{\lambda-1} \mathcal{L}^0(\mathcal{I}_{-\frac{3}{4}} g)(t, y) dy.
\end{equation}
Property of convolution operator $(\partial^j_x (f * g) = (\partial^j_x f) * g = f * (\partial^j_x g)$ and (3.3) give us

\[
\mathcal{L}^\lambda g(t, x) = \left[ \frac{\lambda^{(\lambda+4)-1}}{\Gamma(\lambda + 4)} \ast \partial^j_x \mathcal{L}^0(\mathcal{I}_{-\frac{j}{4}} g)(t, \cdot) \right](x)
\]

(3.9)

\[
= iM \frac{\lambda^{(\lambda+4)-1}}{\Gamma(\lambda + 4)} \mathcal{I}_{-\frac{j}{4}} g(t) + i \int_\mathbb{R} (y-x)^{(\lambda+4)-1} \frac{\lambda^{(\lambda+4)-1}}{\Gamma(\lambda + 4)} \mathcal{L}^0(\partial_t \mathcal{I}_{-\frac{j}{4}} g)(t, y)dy,
\]

for Re $\lambda > -4$, where $M$ is defined as (3.1). From (3.3) and (3.7), we have

\[
(i\partial_t - \partial^3_x)\mathcal{L}^\lambda g(t, x) = iM \frac{\lambda^{(\lambda+4)-1}}{\Gamma(\lambda + 4)} \mathcal{I}_{-\frac{j}{4}} g(t),
\]

in the distributional sense.

To finish this subsection, we will give two lemmas concerning the spatial continuity and decay properties of the $\mathcal{L}^\lambda g(t, x)$ and the explicit values for $\mathcal{L}^\lambda f(t, 0)$, respectively.

**Lemma 3.2.** Let $g \in C^\infty_0(\mathbb{R}^+)$ and $M$ be as in (3.1). Then, we have

\[
\mathcal{L}^{-k} g = \partial^k_x \mathcal{L}^0 \mathcal{I}_{\frac{j}{4}} g, \quad k = 0, 1, 2, 3.
\]

Moreover, $\mathcal{L}^{-3} g(t, x)$ is continuous in $x \in \mathbb{R} \setminus \{0\}$ and has a step discontinuity at $x = 0$. For real $\lambda$ satisfying $\lambda > -3$, $\mathcal{L}^\lambda g(t, x)$ is continuous in $x \in \mathbb{R}$. For $-3 \leq \lambda \leq 1$ and $0 \leq t \leq 1$, $\mathcal{L}^\lambda g(t, x)$ satisfies the following decay bounds:

\[
|\mathcal{L}^\lambda g(t, x)| \leq c_{\lambda, g}(x)^{\lambda-1}, \quad \text{for all} \quad x \geq 0
\]

and

\[
|\mathcal{L}^\lambda g(t, x)| \leq c_{m, \lambda, g}(x)^{-m}, \quad \text{for all} \quad x \geq 0 \quad \text{and} \quad m \geq 0.
\]

**Proof.** We give below the sketch of the proof. The detailed argument can be found in [31]. By using (3.9), we have immediately that (3.10) follows. Moreover, Lemma 3.1 with (3.10) guarantees the continuity (except for $x = 0$ when $\lambda = -3$) and discontinuity at $x = 0$ of $\mathcal{L}^\lambda g$ for $\lambda \geq -3$ and $\lambda = -3$, respectively. The proof of decay bounds can be obtained by using (3.9), (3.3) and Lemma 3.1. \hfill \Box

**Lemma 3.3.** For Re $\lambda > -4$ and $f \in C^\infty_0(\mathbb{R}^+)$, we have the following value of $\mathcal{L}^\lambda f(t, 0)$:

\[
\mathcal{L}^\lambda f(t, 0) = \frac{M}{8} f(t) \left( e^{-i\frac{\pi}{4}(1+3\lambda)} + e^{-i\frac{\pi}{4}(1-5\lambda)} \right) \sin \left( \frac{1-\lambda}{4\pi} \right).
\]

**Proof.** By using formula (3.9) we get

\[
\mathcal{L}^\lambda f(t, 0) = -i \int_0^\infty \frac{y^{(\lambda+4)-1}}{\Gamma(\lambda + 4)} \mathcal{L}^0(\partial_t \mathcal{I}_{-\frac{j}{4}} f)(t, y)dy.
\]

This show that $\mathcal{L}^\lambda f(t, 0)$ is analytic, in $\lambda$, for Re $\lambda > -4$.

By analyticity argument, it suffices to consider the case when $\lambda$ is a positive real number and (3.4), where $M$ is defined as in (3.1). In fact, we are taking $\lambda$ in $(0, 3/8)$ for use (3.8), in order to use (2.9). Thus, in the calculations, we use the representation (3.8) for $\lambda > 0$. Fubini’s Theorem,
the change of variable, (2.9) and (2.7), yields that
\[
\mathcal{L}^\lambda f(t, 0) = \frac{M}{\Gamma(\lambda)} \int_0^\infty y^{\lambda-1} \int_0^t B \left( \frac{y}{(t-t')^{1/4}} \right) \mathcal{I}_{-\frac{1}{4}} f(t') \left( \frac{y}{(t-t')^{1/4}} \right) dy dt
\]
\[
= \frac{M}{\Gamma(\lambda)} \int_0^t (t-t')^{\lambda+4-1} \mathcal{I}_{-\frac{1}{4}} f(t') \int_0^\infty y^{\lambda-1} B(y) dy dt'
\]
\[
= \frac{M}{\Gamma(\lambda)} \Gamma\left( \frac{\lambda}{4} + \frac{3}{4} \right) f(t) \Gamma\left( \frac{1}{4} - \frac{3}{4} \right) \left( e^{-i\frac{\pi}{8}(1+3\lambda)} + e^{-i\frac{\pi}{8}(1-5\lambda)} \right)
\]
\[
= \frac{M}{8} f(t) \left( e^{-i\frac{\pi}{8}(1+3\lambda)} + e^{-i\frac{\pi}{8}(1-5\lambda)} \right)
\]
where in the last equality we used that
\[
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},
\]
and the proof is complete.
\[
\square
\]

3.2. Construction of the solution. Let us describe how we can construct the solution for the linear fourth order Schrödinger equation
\[
i\partial_t u - \partial_x^4 u = 0.
\]

3.2.1. Linear version. First, we define the unitary group associated to (3.12) as
\[
e^{i\partial_x^4 t} \phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-it\xi} \hat{\phi}(\xi) d\xi,
\]
which allows
\[
\begin{cases}
(i\partial_t - \partial_x^4) e^{i\partial_x^4 t} \phi(x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\
e^{i\partial_x^4 t} \phi(x) \big|_{t=0} = \phi(x), & x \in \mathbb{R}.
\end{cases}
\]

Recall \( \mathcal{L}^\lambda \) in (3.9) for the right half-line problem. Let
\[
u(t, x) = \mathcal{L}^{\lambda_1} \gamma_1(t, x) + \mathcal{L}^{\lambda_2} \gamma_2(t, x),
\]
and
\[
\partial_x u(t, x) = \mathcal{L}^{\lambda_1} \mathcal{I}_{-\frac{1}{4}} \gamma_1(t, x) + \mathcal{L}^{\lambda_2} \mathcal{I}_{-\frac{1}{4}} \gamma_2(t, x),
\]
where \( \gamma_j \) (\( j = 1, 2 \)) will be chosen later in terms of given boundary data \( f \) and \( g \).

Let \( a_j \) and \( b_j \) be constants depending on \( \lambda_j \), \( j = 1, 2 \), given by
\[
a_j = \frac{M}{8} \left( \frac{e^{-i\frac{\pi}{8}(1+3\lambda_j)} + e^{-i\frac{\pi}{8}(1-5\lambda_j)}}{\sin\left(\frac{\lambda_j}{4}\pi\right)} \right) \text{ and } b_j = \frac{M}{8} \left( \frac{e^{-i\frac{\pi}{8}(-2+3\lambda_j)} + e^{-i\frac{\pi}{8}(6-5\lambda_j)}}{\sin\left(\frac{\lambda_j}{4}\pi\right)} \right).
\]

By Lemmas 3.2 and 3.3, we get
\[
f(t) = u(t, 0) = a_1 \gamma_1(t) + a_2 \gamma_2(t)
\]
and
\[
g(t) = \partial_x u(t, 0) = b_1 \mathcal{I}_{-\frac{1}{4}} \gamma_1(t) + b_2 \mathcal{I}_{-\frac{1}{4}} \gamma_2(t).
\]

By using (3.15) and (3.16), we can write a matrix in the following form
\[
\begin{bmatrix}
  f(t) \\
  \mathcal{I}_{-\frac{1}{4}} g(t)
\end{bmatrix} = A
\begin{bmatrix}
  \gamma_1(t) \\
  \gamma_2(t)
\end{bmatrix},
\]
where
\[
A(\lambda_1, \lambda_2) = \begin{bmatrix}
  a_1 & a_2 \\
  b_1 & b_2
\end{bmatrix}.
\]
Choosing appropriately \( \lambda_j, j = 1, 2 \), such that \( A \) is invertible, we have that \( u \) solves
\[
(i\partial_t - \partial_x^4)u(t,x) = iM^{\lambda_1-1}T_{-\frac{3}{4}} - \frac{1}{4} \gamma_1(t) + iM^{\lambda_2-1}T_{-\frac{3}{4}} - \frac{1}{4} \gamma_2(t), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R},
\]
(3.17)
\[
u(0,x) = 0, \quad u(t,0) = f(t), \quad \partial_x u(t,0) = g(t), \quad t \in \mathbb{R}^+.
\]
By restriction of the function \( u \) on the set \( \mathbb{R}^+ \times \mathbb{R}^+ \), we can construct a solution for the linear fourth order dispersive equation (3.12) posed on the right half-line.

3.2.2. Nonlinear version. Now we define the classical Duhamel inhomogeneous solution operator \( \mathcal{D} \) as follows
\[
\mathcal{D}w(t,x) = -i \int_0^t e^{i(t-t')\partial_x^4} w(t',x) dt'.
\]
It follows that
\[
\begin{cases}
(i\partial_t - \partial_x^4)\mathcal{D}w(t,x) = w(t,x), & (t,x) \in \mathbb{R} \times \mathbb{R}, \\
\mathcal{D}w(x,0) = 0, & x \in \mathbb{R}.
\end{cases}
\]
Let
\[
u(t,x) = \mathcal{L}^{\lambda_1} \gamma_1(t,x) + \mathcal{L}^{\lambda_2} \gamma_2(t,x) + e^{it\partial_x^4} \phi(x) + \mathcal{D}w.
\]
Similarly as on the construction of the Subsection 3.2, taking \( \gamma_1 \) and \( \gamma_2 \) appropriate depending of \( f, g, e^{it\partial_x^4} \phi(x) \) and \( \mathcal{D}w \), we see that \( u \) solves
\[
\begin{cases}
(i\partial_t - \partial_x^4)u(t,x) = w(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \\
u(0,x) = \phi(x), & x \in \mathbb{R}^+ \\
u(t,0) = f(t), \quad \partial_x u(t,0) = g(t), & t \in \mathbb{R}^+.
\end{cases}
(3.18)
This discussion about the structure of the system (3.18) can be found in section 5 and, in this moment, will be omitted.

4. Energy estimates

The main purpose of this section is to prove the energy estimate of the solutions of the fourth order nonlinear Schrödinger equation in the Bourgain spaces \( X^{s,b} \).

**Lemma 4.1.** Let \( s \in \mathbb{R} \) and \( b \in \mathbb{R} \). If \( \phi \in H^s(\mathbb{R}) \), then the following estimates holds
\[
\| \psi(t) e^{it\partial_x^4} \phi(x) \|_{C_s(\mathbb{R}; H^2_x(\mathbb{R}))} \lesssim \| \phi \|_{H^s(\mathbb{R})};
\]
(4.1)
\[
\| \psi(t) \partial_x^j e^{it\partial_x^4} \phi(x) \|_{C_{s,j}(\mathbb{R}; H^{\frac{3}{4} - j}_s(\mathbb{R}))} \lesssim \| \phi \|_{H^s(\mathbb{R})}, \quad j \in \{0, 1\};
\]
(4.2)
\[
\| \psi(t) e^{it\partial_x^4} \phi(x) \|_{X^{s,b}} \lesssim \| \phi \|_{H^s(\mathbb{R})}.
\]
(4.3)
Estimates (4.1), (4.2) and (4.3) are so called space traces, derivative time traces and Bourgain spaces estimates, respectively.

**Proof.** The proofs of (4.1) and (4.3) are standard and the proof of (4.2) follows from the smoothness of \( \psi \) and the local smoothing estimate (1.8), thus we will omit the details. \( \square \)

**Lemma 4.2.** Let \( 0 < b < \frac{1}{2} \) and \( j = 0, 1 \), we have the following inequalities
\[
\| \psi(t) \mathcal{D}w(t,x) \|_{C_s(\mathbb{R}; H^s(\mathbb{R}))} \lesssim \| w \|_{X^{s,-b}},
\]
for \( s \in \mathbb{R} \);
\[
\| \psi(t) \partial_x^j \mathcal{D}w(t,x) \|_{C_{s,j}(\mathbb{R}; H^{\frac{3}{4} - j}_s(\mathbb{R}))} \lesssim \| w \|_{X^{s,-b}},
\]
for \( -\frac{3}{2} + j < s < \frac{1}{2} + j \);
\[
\| \psi(t) \partial_x^j \mathcal{D}w(t,x) \|_{X^{s,b}} \lesssim \| w \|_{X^{s,-b}},
\]
(4.4)
(4.5)
(4.6)
for $s \in \mathbb{R}$.

Estimates (4.4), (4.5) and (4.6) are so called space traces, derivative time traces and Bourgain spaces estimates, respectively.

**Proof.** The idea to prove this lemma follows by a variation that was introduced in [36]. Here we give only the sketch of the proof for sake of completeness.

**Estimate (4.4):**

By using $2\chi_{(0,t)}(t') = \text{sgn}' + \text{sgn}(t-t')$, $\overline{\text{sgn}}(\tau) = \text{p.v.} \frac{2}{\tau}$ and $e^{ix\xi} \hat{f}(\tau) = \hat{f}(\tau + \xi^4)$ we have

$$\psi(t) Dw(t, x) = c \int e^{ix\xi} e^{-it\xi^4} \psi(t) \int \hat{w}(\tau', \xi) \frac{e^{it(\tau' + \xi^4)} - 1}{(\tau' + \xi^4)} d\tau' d\xi. \tag{4.7}$$

We denote by $w = w_1 + w_2$, where

$$\hat{w}_1(\tau, \xi) = \eta_0(\tau + \xi^4) \hat{w}(\tau, \xi),$$

and

$$\hat{w}_2(\tau, \xi) = (1 - \eta_0(\tau + \xi^4)) \hat{w}(\tau, \xi).$$

Here $\eta_0 : \mathbb{R} \rightarrow [0, 1]$ a smooth bump function supported in $[-2, 2]$ and equal to 1 in $[-1, 1]$. For $w_1$, we use the Taylor expansion of $e^x$ at $x = 0$. Then, we can rewrite (4.7) for $w_1$ as

$$\hat{F}_1^k(\xi) = \int \hat{w}_1(\tau, \xi)(\tau + \xi^4)^{k-1} d\tau. \tag{4.8}$$

Since

$$\|F_1^k\|_{H^s} = \left( \int \langle \xi \rangle^{2s} \left| \int \hat{w}_1(\tau, \xi)(\tau + \xi^4)^{k-1} d\tau \right|^2 \right)^{\frac{1}{2}} \lesssim \|w\|_{X^{s,b}}, \tag{4.9}$$

we have from (4.1) that

$$\|\psi(t) Dw(t, x)\|_{C_t H^s} \lesssim \sum_{k=1}^{\infty} \frac{1}{k!} \|F_1^k\|_{H^s_x} \lesssim \|w\|_{X^{s,b}}. \tag{4.10}$$

For $w_2$, a direct calculation gives

$$\mathcal{F}[\psi Dw](\tau, \xi) = c \int \hat{w}_2(\tau', \xi) \frac{\hat{\psi}(\tau - \tau') - \hat{\psi}(\tau + \xi^4)}{(\tau' + \xi^4)} d\tau'. \tag{4.11}$$

Since $\|\psi Dw\|_{C_t H^s} \lesssim \|\langle \xi \rangle^s \mathcal{F}[\psi Dw](\tau, \xi)\|_{L^1_t L^2_\xi}$, it suffices to bound the following term

$$\left( \int \langle \xi \rangle^{2s} \left| \int \hat{\psi}(\tau - \tau') - \hat{\psi}(\tau + \xi^4) \frac{d\tau d\tau'}{|\tau' + \xi^4|} \right|^2 \right)^{\frac{1}{2}},$$

due to (4.10). We use the $L^1$ integrability of $\hat{\psi}$, to bound (4.11) by

$$c \left( \int \langle \xi \rangle^{2s} \left| \int_{|\tau' + \xi^4| > 1} \frac{|\hat{w}_2(\tau', \xi)|}{|\tau' + \xi^4|} d\tau' \right|^2 \right)^{\frac{1}{2}} \lesssim \|w\|_{X^{s,b}}.$$
Estimate (4.5):

We only consider the case \( j = 0 \), since the estimate for \( j = 1 \) is a direct consequence of the case \( j = 0 \). Initially, take \( \theta(\tau) \in C^\infty(\mathbb{R}) \) such that \( \theta(\tau) = 1 \) for \( |\tau| < \frac{1}{2} \) and \( \text{supp} \, \theta \subset [-\frac{3}{2}, \frac{3}{2}] \). A standard calculation gives

\[
\mathcal{F}_x \left( \psi(t) \int_0^t e^{i(t-t')d^2_\tau} w(x, t') \right)(\xi) = c \psi(t) \int_\tau e^{it\tau} - e^{-it\xi^4} \hat{w}(\xi, \tau) d\tau = c \psi(t) e^{it\xi^4} \int_\tau \frac{-1}{\tau + \xi^4} \theta(\tau + \xi^4) \hat{w}(\xi, \tau) d\tau + c \psi(t) \int_\tau \frac{1}{\tau + \xi^4} \hat{w}(\xi, \tau) d\tau
\]

By using (4.2), it suffices to show that

\[
\psi \left| \left| \eta \right| \left| \left| \eta \right| \left( \hat{\psi}_k(\xi) - \hat{\phi}_k(\xi) \right) \right| \right|_{L_2(\mathbb{R}_x)} \leq c \left| \left| \eta \right| \left| \left| \eta \right| \left( \hat{\psi}_k(\xi) - \hat{\phi}_k(\xi) \right) \right| \right|_{L_2(\mathbb{R}_x)}
\]

and using the Cauchy-Schwarz inequality, we obtain

\[
\left| \left| \eta \right| \left| \left| \eta \right| \left( \hat{\psi}_k(\xi) - \hat{\phi}_k(\xi) \right) \right| \right|_{L_2(\mathbb{R}_x)}^2 \leq c \int_\tau (\tau + \xi^4)^{2c} |\hat{\psi}(\xi, \tau)|^2 d\tau.
\]

This completes the estimate of \( w_1 \). Now we treat \( w_2 \). Changing variables, \( \eta = \xi^4 \), and using the Cauchy-Schwarz inequality we obtain

\[
\left| \left| \eta \right| \left| \left| \eta \right| \left( \hat{\psi}_k(\xi) - \hat{\phi}_k(\xi) \right) \right| \right|_{L_2(\mathbb{R}_x)}^2 \leq c \int_\tau (\tau + \xi^4)^{2c} |\hat{\psi}(\xi, \tau)|^2 d\tau \]

where \( G(\tau) = c \int_\eta (\tau + \eta)^{-2-2c} |\eta|^{-\frac{3}{2} - s/2} d\eta \). To conclude the estimate of \( w_2 \), we need to prove that

\[
G(\tau) \leq c (\tau)^{\frac{2a+3}{4}}.
\]

We split in two cases. In the first case, we consider \( |\tau| < 1 \). For this, we use \( \tau + \eta \sim \eta \) to get

\[
G(\tau) \leq c \int (\eta)^{-2-2c} |\eta|^{-3/4} d\eta.
\]

The above integral is bounded on the case \( s > -\frac{7}{2} - 4c \), since \( c > -\frac{1}{2} \), this estimate is valid for \( s > -\frac{3}{2} \).

Now, the second case \( |\tau| \geq 1 \) can be estimated by separating the integral in three regions \( |\eta| \leq 1 \), \( 2|\eta| \leq |\tau| \), \( |\tau| \leq 2|\eta| \) and using that \( -\frac{3}{2} < s \leq -\frac{3}{2} \), (4.12) follows. This completes the estimate for \( w_2 \).

Finally, to obtain \( w_3 \), we rewrite \( w_3 = \psi(t) e^{it\xi^4} \phi(x) \), where

\[
\hat{\phi}(\xi) = \int \frac{1 - \theta(\tau + \xi^4)}{\tau + \xi^4} \hat{w}(\xi, \tau) d\tau.
\]
Thanks (4.2) and Cauchy-Schwarz inequality, we obtain
\[
\|w_3\|_{C\left(L^2_{\text{loc}}(\mathbb{R}^n)ight)} = \|\psi(t)\tilde{\phi}(x)\|_{C\left(L^2_{\text{loc}}(\mathbb{R}^n)ight)} \leq c\|\phi\|_{H^s(\mathbb{R})}.
\]
\[
\leq c \int_{\xi} (\xi)^{2\alpha} \left( \int_{\tau} |\tilde{w}(\xi, \tau)|^2 (\tau + \xi^4)^{2c} d\tau \right) d\xi.
\]
Since \(c > -\frac{1}{2}\), we have \(\int_{(\tau + \xi^4)^{-2c}} d\tau \leq c\). This completes the estimate of \(w_3\), and (4.5) holds.

**Estimate (4.6):**

Finally, we split \(w = w_1 + w_2\), similarly as in the proof of estimate of (4.4). For \(w_1\), (4.3) and (4.9), yields that
\[
\|\psi(t)Dw_1(t, x)\|_{X^{s,b}} \lesssim \sum_{k=1}^{\infty} \frac{1}{k!} \|F_k^1\|_{H^s} \lesssim \|w\|_{X^{s,-b}},
\]
where \(F_k^1\) is defined as in (4.8).

For \(w_2\),
\[
\psi \partial_t^j Dw(t, x) = c \int e^{i\xi \tau} e^{-i\xi^4} (i\xi)^j \psi(t) \int \frac{\tilde{w}(\tau', \xi)}{(\tau' + \xi^4)} \left( e^{it(\tau' + \xi^4)} - 1 \right) d\tau' d\xi
\]
\[
= c \int e^{i\xi \tau} e^{-i\xi^4} (i\xi)^j \psi(t) \int \frac{\tilde{w}(\tau', \xi)}{(\tau' + \xi^4)} e^{it(\tau' + \xi^4)} d\tau' d\xi
\]
\[
- c \int e^{i\xi \tau} e^{-i\xi^4} (i\xi)^j \psi(t) \int \frac{\tilde{w}(\tau', \xi)}{(\tau' + \xi^4)} d\tau' d\xi
\]
\[
= I - II.
\]
Let
\[
\tilde{W}(\xi) = \int \frac{\tilde{w}(\tau, \xi)}{(\tau + \xi^4)} d\tau.
\]
Therefore, we use (4.3) in \(I I\) to obtain
\[
\|\psi e^{i\xi^4} W\|_{X^{s,b}} \lesssim \|W\|_{H^s} \lesssim \|w\|_{X^{s,-b}},
\]
for \(b < \frac{1}{2}\).

Now, it remains to show
\[
\left( \int_{|\xi| > 1} |\xi|^{2s} \int (\tau + \xi^4)^{2b} \left| \int \frac{\tilde{w}(\tau', \xi)}{(\tau' + \xi^4)} \tilde{\psi}(\tau - \tau') d\tau' \right|^2 d\tau' d\xi \right)^{\frac{1}{2}} \lesssim \|w\|_{X^{s,-b}}.
\]
This follows using the argument as did in the proof of inequality (4.5). In fact, the proof of (4.13) is easier than the proof of (4.5), since \(L^2\) integral with respect to \(\xi\) is negligible and hence it is enough to consider the relation between \(\tau + \xi^4\) and \(\tau' + \xi^4\). Thus, we omit the details, and we have
\[
\|\psi Dw\|_{X^{s,b}} \lesssim \|w\|_{X^{s,-b}}.
\]
Therefore, Lemma 4.2 is archived. \(\square\)

**Lemma 4.3.** Let \(s \in \mathbb{R}\).

(a) For \(\frac{2s-7}{2} < \lambda < \frac{1+2s}{2}\) and \(\lambda < \frac{1}{2}\) the following inequality holds
\[
\|\psi(t)L^\lambda f(t, x)\|_{C\left(L^2_{\text{loc}}(\mathbb{R}^n)\right)} \leq c\|f\|_{H^s_0(\mathbb{R})};
\]

(b) For \(-4 + j < \lambda < 1 + j, j = 0, 1\), we have
\[
\|\psi(t)D^j L^\lambda f(t, x)\|_{C\left(L^2_{\text{loc}}(\mathbb{R}^n)\right)} \leq c\|f\|_{H^{2s}_{0}(\mathbb{R})};
\]

(c) If \(s < 4 - 4b, b < \frac{1}{2}, -5 < \lambda < \frac{1}{2}\) and \(s + 4b - 2 < \lambda < s + \frac{1}{2}\) yields that
\[
\|\psi(t)L^\lambda f(t, x)\|_{X^{s,b}} \leq c\|f\|_{H^{2s}_{0}(\mathbb{R})}.
\]
Estimates (4.14), (4.15) and (4.16) are so called space traces, derivative time traces and Bourgain spaces estimates, respectively.

Proof. Let us start to prove (4.14). By density, we may assume that $f \in C_{0,c}^\infty (\mathbb{R}^+)$. Moreover, from the definition of $L^\lambda$, it suffices to consider $L^\lambda f(t,x)$ (removing $\psi$) for $\text{supp} f \subset [0,1]$, thanks to Lemma 2.4.

From (2.4), (3.7) and (3.2), we see that

$$ F_x(L^\lambda f)(t,\xi) = M e^{-i\frac{\alpha}{2}\lambda} (\xi - i0)^{-\lambda} \int_0^t e^{-i(t-t')\xi^4 I_{\frac{3}{4} - \frac{3}{4}} f(t')} dt'. $$

Using the following change of variables

$$ \eta = \xi^4, $$

for (2.6) and the definition of the Fourier transform we have that

$$ \| L^\lambda f(t,\cdot) \|^2_{H^\sigma (\mathbb{R})} \leq c \int_\eta |\eta|^{-\lambda - \frac{3}{4}} (\eta)^{\frac{\delta}{2}} \left| \int_0^t e^{-i(t-t')\eta I_{\frac{3}{4} - \frac{3}{4}} f(t')} dt' \right|^2 d\eta $$

$$ = c \int_\eta |\eta|^{-\lambda - \frac{3}{4}} (\eta)^{\frac{\delta}{2}} \left| \langle \chi_{(-\infty,t)} I_{\frac{3}{4} - \frac{3}{4}} f \rangle (\eta) \right|^2 d\eta, $$

for a fixed $t$. Note that, by Lemma 2.2, we can replace $|\eta|^{-\lambda - \frac{3}{4}}$ by $|\eta|^{-\lambda - \frac{3}{4}}$, since

$$ -1 < -\frac{\lambda}{2} - \frac{3}{4} \Rightarrow \lambda < \frac{1}{2}. $$

Moreover, Lemma 2.1 (under the condition $-1 < -\frac{\lambda}{2} - \frac{3}{4} + \frac{\delta}{2} < 1$ for removing $\chi_{(-\infty,t)}$) and Lemma 2.7 (under the condition $-5 < \lambda$) yield that

$$ \int_\eta |\eta|^{-\lambda - \frac{3}{4}} (\eta)^{\frac{\delta}{2}} \left| \langle \chi_{(-\infty,t)} I_{\frac{3}{4} - \frac{3}{4}} f \rangle (\eta) \right|^2 d\eta \leq c \int_\eta |\eta|^{-\lambda - \frac{3}{4}} (\eta)^{\frac{\delta}{2} - \frac{3}{4}} \left| \langle \chi_{(-\infty,t)} I_{\frac{3}{4} - \frac{3}{4}} f \rangle (\eta) \right|^2 d\eta $$

$$ \leq c \|I_{\frac{3}{4} - \frac{3}{4}} f\|^2_{H^\sigma_{\frac{3}{4} - \frac{3}{4}}} \leq c \|f\|^2_{H^\sigma_{\frac{5}{14} + 1}}, $$

which proves (4.14) thanks to the definition of $H^\sigma_0 (\mathbb{R}^+)$-norm.

Now we prove (4.15). A direct calculation gives

$$ \partial_x^j L^\lambda f = L^{\lambda - j} (I_{\frac{3}{4} - \frac{3}{4}} f). $$

With the previous equality in hands and Lemma 2.7, it suffices to show that (4.15) for $j = 0$. Lemma 2.4 ensures us to ignore the cut-off function $\psi$. The change of variables $t \to t - t'$ gives

$$ (I - \partial_t^2_{\frac{2}{16}})^{2 + 3} \left( \frac{x_{\lambda - 1}}{\Gamma (\lambda)} * \int_{-\infty}^t e^{i(t-s)} \delta(x) h(t') dt' \right) = \left( \frac{x_{\lambda - 1}}{\Gamma (\lambda)} * \int_{-\infty}^t e^{i(t-s)} \delta(x) (I - \partial_t^2)^{2 + 3} h(t') dt' \right). $$

So, we just need to prove that

$$ \left\| \int_{-\infty}^t e^{ix\xi} (\xi - i0)^{-\lambda} \int_{-\infty}^t e^{-i(t-t')\xi^4 I_{\frac{3}{4} - \frac{3}{4}} f(t')} dt' \right\|_{L^{1+\epsilon}_x L^1_t (\mathbb{R}^+)} \leq c \|f\|_{L^2_t (\mathbb{R}^+)} $$

thanks to $\partial_t^\sigma (\partial_x f) = \partial_x (\partial_t^\sigma f)$. We use $\chi_{(-\infty,t)} = \frac{1}{2} \text{sgn} (t - t') + \frac{1}{2}$ to obtain

$$ \int_{-\infty}^t e^{ix\xi} (\xi - i0)^{-\lambda} \int_{-\infty}^t e^{-i(t-t')\xi^4 I_{\frac{3}{4} - \frac{3}{4}} f(t')} dt' dt' \xi $$

$$ = \frac{1}{2} \int_{-\infty}^t e^{ix\xi} (\xi - i0)^{-\lambda} \int_{-\infty}^t \text{sgn} (t - t') e^{-i(t-t')\xi^4 I_{\frac{3}{4} - \frac{3}{4}} f(t')} dt' dt' \xi $$

$$ + \frac{1}{2} \int_{-\infty}^t e^{ix\xi} (\xi - i0)^{-\lambda} \int_{-\infty}^t e^{-i(t-t')\xi^4 I_{\frac{3}{4} - \frac{3}{4}} f(t')} dt' dt' \xi $$

$$ := I(t, x) + II(t, x). $$
We will treat \( I(t, x) := I \) and \( II(t, x) := \tilde{I} \) separately. To estimate \( I \), we can rewrite it as

\[
I(t, x) = \frac{1}{2} \int_{\xi} e^{ix\xi} (\xi - i0)\lambda \left( (e^{-i\xi} \text{sgn}(\cdot)) * \mathcal{L}_{\frac{\lambda}{4}} \right)(t) \, d\xi.
\]

A direct calculation gives

\[
\mathcal{F}_\xi \left( (e^{-i\xi} \text{sgn}(\cdot)) * \mathcal{L}_{\frac{\lambda}{4}} \right)(\tau) = \frac{(\tau - i0)^{\frac{\lambda}{4}+\frac{\lambda}{4}}}{i(\tau + \xi^4)} \tilde{f}(\tau).
\]

Fubini’s theorem and dominated converge theorem, implies that

\[
I(t, x) = \int_{\tau} e^{it\tau} \lim_{\epsilon \to 0} \int_{|\tau + \xi^4| > \epsilon} \frac{e^{ix\xi}(\tau - i0)^{\frac{\lambda}{4}+\frac{\lambda}{4}}}{i(\tau + \xi^4)} \tilde{f}(\tau) \, d\xi \, d\tau.
\]

Thus, once we show that the function

\[
g(\tau) := \lim_{\epsilon \to 0} \int_{|\tau + \xi^4| > \epsilon} \frac{e^{ix\xi}(\tau - i0)^{\frac{\lambda}{4}+\frac{\lambda}{4}}}{i(\tau + \xi^4)} \tilde{f}(\tau) \, d\xi
\]

is bounded independently of \( \tau \) variable, the Plancherel’s theorem enables us to obtain (4.17). The change of variables \( \xi \mapsto |\tau|^\frac{1}{4}\xi \) and the fact that

\[
(|\tau|^\frac{1}{4}\xi - i0)^{-\lambda} = |\tau|^{-\frac{1}{4}}(\xi^+ - \lambda + e^{i\lambda\xi} \xi^-)\lambda
\]

gives

\[
g(\tau) = \int_{\xi} e^{ix|\tau|^\frac{1}{4}\xi} \frac{\xi^+ - \lambda + e^{i\lambda\xi} \xi^-}{1 + \xi^4} \, d\xi
\]

\[
= g_1 + e^{-\frac{i\pi(\lambda+1)}{4}} g_2.
\]

We only consider \( g_2 \), since \( g_1 \) is uniformly bounded in \( \tau \) for \(-3 < \lambda < 1 \). Let us define the following cut-off function \( \zeta \in C^\infty(\mathbb{R}) \) such that

\[
\zeta := \begin{cases} 
1, & \text{in } \left[ \frac{3}{4}, \frac{4}{3} \right] \\
0, & \text{outside } \left( \frac{1}{2}, \frac{3}{4} \right).
\end{cases}
\]

Then we obtain

\[
g_2 = \int_{\xi} e^{ix|\tau|^\frac{1}{4}\xi} \zeta(\xi) \frac{\xi^+ - \lambda + e^{i\lambda\xi} \xi^-}{1 + \xi^4} \, d\xi
\]

\[
= g_1 + g_2.
\]

It is clear that \( g_2 \) is bounded independently of \( \tau \) when \( \lambda > -3 \), and hence it remains to deal with \( g_2 \). Let

\[
\hat{\Theta}(\xi) = \frac{\zeta(\xi) \xi^-}{1 + \xi + \xi^2 + \xi^3} \quad \text{and} \quad \hat{\Psi}(\xi) = \frac{1}{i(\xi - 1)}.
\]

We remark that \( \hat{\Theta} \) is a Schwartz function, and hence \( \Theta \in \mathcal{S}(\mathbb{R}) \). Moreover, we immediately know from the fact \( \mathcal{F}_x[\text{sgn}(x)](\xi) = \text{v.p.} \frac{2}{\pi} \) that

\[
\Psi(x) = \frac{1}{2} e^{ix} \text{sgn}(x).
\]

Then, \( g_2 \) can be written as

\[
g_2(\tau) = -i \chi_{\{\tau < 0\}} \int_{\xi} e^{ix|\tau|^\frac{1}{4}\xi} \hat{\Theta}(\xi) \hat{\Psi}(\xi) \, d\xi = -2i\pi \chi_{\{\tau < 0\}} (\Theta * \Psi)(|\tau|^\frac{1}{4} x),
\]

which implies

\[
|g_2(\tau)| \lesssim \int |\Theta(y)\Psi(|\tau|^\frac{1}{4} x - y)\, dy | \lesssim \int |\Theta(y)| \, dy \lesssim 1.
\]
Now, we bounded $II$. By using the definition of Fourier transform and (2.6) we have, after the change of variables $\eta = \xi^4$ and contour, that

$$II(t, x) = \frac{1}{2} \int e^{i\xi \xi} e^{-it\xi^4} (\xi^4 - i0)^{-\frac{\lambda+3}{4}} \hat{f}(\xi^4)(\xi - i0)^{-\lambda} d\xi = \frac{1}{2} \int e^{it\eta} e^{-i\eta^4} (\eta - i0)^{-\frac{\lambda+3}{4}} (\eta^3 - i0)^{-\lambda} \eta^{-\frac{1}{2}} \hat{f}(\eta) d\eta = c f(t),$$

for some $c \in \mathbb{C}$, which implies $\|II(\cdot, x)\|_{L^2_t} \lesssim \|f\|_{L^4_x}$. Therefore, we complete the proof of (4.15).

Lastly, let us show (4.16). A direct calculation holds that

$$F_{\tau}(\psi(t)L^\lambda f)(t, \xi) = Me^{-\frac{in\lambda}{2} e^{it(n+4)x^4}(\xi - i0)^{-\lambda} \psi(t)e^{-it\xi^4} \int e^{it(\tau' + \xi^4)} - \frac{1}{i(\tau' + \xi^4)} (\tau' - i0)^{\frac{3}{2} + \frac{1}{2}} \hat{f}(\tau') d\tau',$$

which can be divided into the following quantities

$$\hat{f}_1(t, \xi) = Me^{-\frac{in\lambda}{2} e^{it(n+4)x^4}(\xi - i0)^{-\lambda} \psi(t) \int e^{it\tau'} - e^{-it\xi^4} \frac{1}{i(\tau' + \xi^4)} \theta(\tau' + \xi^4) (\tau' - i0)^{\frac{3}{2} + \frac{1}{2}} \hat{f}(\tau') d\tau',$$

$$\hat{f}_2(t, \xi) = Me^{-\frac{in\lambda}{2} e^{it(n+4)x^4}(\xi - i0)^{-\lambda} \psi(t) \int \frac{e^{it\tau'}}{i(\tau' + \xi^4)} (1 - \theta(\tau' + \xi^4)) (\tau' - i0)^{\frac{3}{2} + \frac{1}{2}} \hat{f}(\tau') d\tau'$$

and

$$\hat{f}_3(t, \xi) = Me^{-\frac{in\lambda}{2} e^{it(n+4)x^4}(\xi - i0)^{-\lambda} \psi(t) \int e^{-it\xi^4} \frac{1}{i(\tau' + \xi^4)} (1 - \theta(\tau' + \xi^4)) (\tau' - i0)^{\frac{3}{2} + \frac{1}{2}} \hat{f}(\tau') d\tau'.$$

Here $\theta \in S(\mathbb{R})$ such that

$$\theta(\tau) := \begin{cases} 1, & \text{for } |\tau| \leq 1 \\ 0, & \text{for } |\tau| \geq 2. \end{cases}$$

It follows that $\psi(t)L^\lambda f = f_1 + f_2 - f_3$.

For $f_1$, we use the same argument for $w_1$ in the proof of inequality (4.6). By the Taylor series expansion for $e^{it(\tau' + \xi^4)}$ at $it(\tau' + \xi^4) = 0$, we write

$$\psi(t)L^\lambda f_1(t, x) = e \sum_{k=1}^{\infty} \frac{ik}{k!} \psi^k(t)e^{it\xi^4} F^k_1(x),$$

for some constant $c \in \mathbb{C}$, where $\psi^k(t) = t^k\psi(t)$ and

$$\hat{F}^k_1(\xi) = (\xi - i0)^{-\lambda} \int \theta(\tau' + \xi^4)(\tau' + \xi^4)^{-\lambda} (\tau')^{\frac{k+1}{2}} \hat{f}(\tau') d\tau'.$$

By using (2.6), (6.4) and (4.18), it is enough to show that

$$\int |(\xi)^{2\lambda} |\xi|^{-2\lambda} \left| \int_{|\tau'| \leq 1} \left| \tau' + \xi^4k^{-1} |\tau'|^{\frac{\lambda+3}{2}} |\hat{f}(\tau')| d\tau' \right| d\xi \lesssim \|f\|^2_{H^\frac{2\lambda+3}{2}}.$$

Let us split $|\xi|$ in two regions: $|\xi| \leq 1$ and $|\xi| > 1$. For region $|\xi| \leq 1$ and $|\tau'| \leq 1$ (if $|\xi| \leq 1$ and $|\tau' + \xi^4| \leq 1$ imply $|\tau'| \leq 1$), we have that both $|\xi|^{-2\lambda}$ and $|\tau'|^{2\lambda} |\hat{f}(\tau')|$ are integrable, for $-\lambda < \frac{1}{2}$, respectively. So, we obtain (4.19) by using Cauchy-Schwarz inequality in $\tau'$.

Now, assume that $|\xi| > 1$, which in addition to $|\tau' + \xi^4| \leq 1$ implies $|\tau'| \sim |\xi^4| > 1$. Let $\hat{f}^*(\tau') = \frac{2\lambda + 3}{2} \hat{f}(\tau')$. Then, the change of variables $(\xi^4 \rightarrow \eta)$ gives that the left hand side of (4.19) is bounded by

$$\int_{|\xi| > 1} |\xi|^{2\lambda} |\hat{f}^*(\xi^4)|^2 d\xi \lesssim \int_{|\eta| > 1} |\hat{M}\hat{f}^*(\eta)|^2 d\eta \lesssim \|f^*\|^2_{L^2_{\eta}} = \|f\|^2_{H^\frac{2\lambda+3}{2}},$$

where $\hat{M}\hat{f}^*$ is the Hardy-Littlewood maximal function of $\hat{f}^*$, and $f_1$ is controlled.
For $f_2$, from (2.6), the definition of inverse Fourier transform and Lemma 2.5, follows that

$$\|f_2\|_{X^{s,b}}^2 \lesssim \int \int (\xi^{2\alpha} |\xi|^{-2\lambda} \tau + \xi^4)^{2b} \frac{(1 - \theta(\tau + \xi^4))^2 |\tau|^{\frac{\lambda + 3}{2}} |\hat{f}(\tau)|^2}{|\tau|^{\frac{2}{\alpha}}} \, d\tau d\xi \lesssim \int |\tau|^{\frac{\lambda + 3}{2}} \left( \int \frac{\xi^{2\alpha} |\xi|^{-2\lambda}}{|\tau + \xi^4|^{2-2b}} \, d\xi \right) |\hat{f}(\tau)|^2 \, d\tau.$$  

Thus, by the change of variables ($\eta = \xi^4$) and Lemma 2.2 for $-5 < \lambda$ (we may assume $supp f \subset [0,1]$, thanks to Lemma 2.4), it suffices to show

$$I(\tau) = \int \frac{|\eta|^{-\frac{2}{4} - \frac{3}{4}} \eta^\frac{5}{2}}{(\tau + \eta)^{2b-2}} \, d\eta \lesssim \langle \tau \rangle^{\frac{2b-\frac{3}{4}}{2} - \frac{3}{2}}.$$  

Here, we split $|\tau|$ in two regions: $|\tau| \leq 2$ and $|\tau| > 2$. When $|\tau| \leq 2$, we have $\langle \tau + \eta \rangle \sim \langle \eta \rangle$. For $s < 4 - 4b$ and $s + 4b - \frac{7}{2} < \lambda < \frac{1}{2}$, we get

$$I(\tau) \lesssim \int_{|\eta| \leq 1} |\eta|^{-\frac{3-2\lambda}{4}} + \int_{|\eta| \leq \frac{1}{4}} \frac{d\eta}{(|\eta|^{2-2b} - \frac{3}{4} + \frac{3}{2})} \lesssim 1.$$

Now, working in the region $|\tau| > 2$, we divide the integral region in $\eta$ into $|\eta| < \frac{|\tau|}{2}$ and $|\eta| \geq \frac{|\tau|}{2}$. In the first region, for $b < \frac{1}{4}$ and $\lambda < \min(\frac{1}{2}, s + \frac{1}{2})$, we bounded in the following way

$$\langle \tau \rangle^{2b-2} \left( \int_{|\eta| \leq \frac{1}{4}} |\eta|^{-\frac{2}{4} - \frac{3}{4}} \, d\eta \right) \lesssim \langle \tau \rangle^{-\frac{3-2\lambda+2\alpha}{4}}.$$  

On the other hand, in the second region, we have that $\langle \tau + \eta \rangle \gtrsim \frac{1}{2}|\tau| > 1$. Then, for $s - 2 < \lambda$ and $b < \frac{1}{2}$, holds that

$$I(\tau) \lesssim \langle \tau \rangle^{-\frac{3-2\lambda+2\alpha}{4}} \int_{|\eta| \leq \frac{1}{4}} \frac{d\eta}{(|\eta|^{2-2b} - \frac{3}{4} + \frac{3}{2})} \lesssim \langle \tau \rangle^{-\frac{3-2\lambda+2\alpha}{4}},$$  

so,

$$\|f_2\|_{X^{s,b}} \lesssim \|f\|_{H_0^{\frac{2\lambda+3}{8}}}.$$  

Finally, let us show that $f_3$ can be controlled. Similarly as for $f_1$, it suffices to show

$$\left( \int \left( 1 - \theta(\tau + \xi^4) \right) \langle \tau + \xi^4 \rangle^{-1} |\tau|^{\frac{\lambda + 3}{2}} |\hat{f}(\tau')| \, d\tau' \right)^2 \lesssim \|f\|_{H_0^{\frac{2\lambda+3}{8}}}^2.$$  

Again, split the region $|\xi|$ as follows: $|\xi| \leq 1$ and $|\xi| > 1$. Considering $|\xi| \leq 1$, since $|\xi|^{-2\lambda}$ is integrable, for $\lambda < \frac{1}{4}$, and we may ignore the integration in $\xi$. Let us work in the region $|\tau| \leq 1$. In this region $|\tau|^\frac{3-\lambda}{2}$ is integrable, for $\lambda > -5$, and hence we get (4.20).

On the region $|\tau'| > 1$, since $|\tau' + \xi^4| \sim |\tau'|$ and $|\tau'|^{-\frac{\lambda+3}{2}}$ are $L^2$ integrable, for $\lambda < s + \frac{1}{2}$, we also get (4.20) by using the Cauchy-Schwarz inequality in $\tau'$. Still looking on the region $|\tau'| > 1$, since $|\tau' + \xi^4| \sim |\tau'|$ we have that the left hand side of (4.20) is bounded by

$$c \left( \int_{|\tau'| > 1} \frac{|\tau'|^{\frac{\lambda+1}{2}} |\hat{f}|^2}{|\tau'|} \, d\tau' \right)^2 \sim \left( \int_{|\tau'| > 1} \frac{|\tau'|^{\frac{\lambda+1}{2}} |\hat{f}(\tau')|^2}{|\tau'|} \, d\tau' \right)^2 \lesssim \frac{|\tau|^{\frac{\lambda+3}{2}} |\hat{f}|^2}{|\tau'|} \, d\tau'$$  

(4.21)

where we have used that $\lambda < s + \frac{1}{2}$, and the result follows on $|\xi| \leq 1$. On the other hand, in the region $|\xi| > 1$ and $|\tau'| \leq 1$, since $|\tau' + \xi^4| \sim |\xi|^4$ and $|\xi|^{2+2\lambda-8}$ is integrable for $\lambda > -\frac{3}{2} + s$, we also get (4.20).

Considering the region $|\xi| > 1$ and $|\tau'| > 1$. There are two possibilities:

1. $|\tau'| \lesssim \frac{1}{4}|\xi|^4$;
II. \( \frac{1}{2}|\xi|^4 < |\tau'| \).

In view of the proof of [31, Lemma 5.8 (d)] (see also [30]), one can replace \( \frac{1-\theta(\tau'+\xi^4)}{\tau'+\xi^4} \) by \( \beta(\tau'+\xi^4) \) for some \( \beta \in \mathcal{S}(\mathbb{R}) \). Hence, the left-hand side of (4.20) is dominated by

\[
(4.22) \quad c \int_{|\xi|>1} |\xi|^{2s-2}\lambda \left| \int_{|\tau'|>1} |\tau' + \xi^4 - N| \frac{\lambda_{\frac{4\pi}{5}+\frac{8\pi}{15}}} {\lambda_{\frac{4\pi}{5}+\frac{8\pi}{15}}} \left| \tilde{f}(\tau') \right| d\tau' \right|^2 d\xi,
\]

for \( N \geq 0 \). By Cauchy-Schwarz inequality and choosing \( N = N(s, \lambda) \gg 1 \), we have (4.20) for both cases. Indeed, for the case I (in this case, \(|\tau' + \xi^4| \sim |\xi|^4\) (4.22) can be controlled by

\[
c\|f\|^2_{H^{\frac{2s+3}{8}}} \int_{|\xi|>1} |\xi|^{2s-2\lambda} \left| \int_{1<|\tau'|<\frac{1}{2}|\xi|^4} |\tau'|^{\frac{2\lambda+6-2s-3-4N}{4}} \left| \tilde{f}(\tau') \right| d\tau' \right|^2 d\xi \lesssim \|f\|^2_{H^{\frac{2s+3}{8}}},
\]

for \( N \geq 0 \). By Cauchy-Schwarz inequality and choosing \( N = N(s, \lambda) \gg 1 \), we have (4.20) for both cases. Indeed, for the case I (in this case, \(|\tau' + \xi^4| \sim |\xi|^4\) (4.22) can be controlled by

\[
c\|f\|^2_{H^{\frac{2s+3}{8}}} \int_{|\xi|>1} |\xi|^{2s-2\lambda} \left| \int_{1<|\tau'|<\frac{1}{2}|\xi|^4} |\tau'|^{\frac{2\lambda+6-2s-3-4N}{4}} \left| \tilde{f}(\tau') \right| d\tau' \right|^2 d\xi \lesssim \|f\|^2_{H^{\frac{2s+3}{8}}},
\]

thus

\[
\|f_3\|_{X^{s,\lambda}} \lesssim \|f\|_{H^{\frac{2s+3}{8}}},
\]

this finishes the estimate for \( f_3 \).

Remembering that \( \psi(t)\mathcal{L}f = f_1 + f_2 - f_3 \), and using the estimates of \( f_i \), \( i = 1, 2, 3 \), (4.16) follows and the proof is complete.

To close this section, let us enunciate the bilinear estimates associated to fourth order nonlinear Schrödinger equation (1.2). The proof of this estimate can be found in [45] (see also [10]), thus we will omit it.

**Proposition 4.1.** For \( s \geq 0 \), there exists \( b = b(s) < 1/2 \) such that we have

\[
(4.23) \quad \|u_1u_2\|_{X^{s,\lambda}} \leq c\|u_1\|_{X^{s,\lambda}}\|u_2\|_{Y^{s,\lambda}} \|u_3\|_{X^{s,\lambda}}.
\]

5. **Proof of Theorem 1.1**

Using a scaling argument, we can assume

\[
\|u_0\|_{H^s(\mathbb{R}^+)} + \|f\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|g\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} = \delta,
\]

for \( \delta \) sufficiently small. Pick an extension \( \tilde{u}_0 \in H^s(\mathbb{R}) \) of \( u_0 \) such that

\[
\|\tilde{u}_0\|_{H^s(\mathbb{R})} \leq 2\|u_0\|_{H^s(\mathbb{R})}.
\]

Let \( b = b(s) < \frac{1}{2} \) such that the estimates given in Proposition 4.1 are valid.

Together with arguments in Subsections 3.2 and 3.2.2, let

\[
(5.1) \quad u(t, x) = \mathcal{L}_{\gamma_1}^t \gamma_1(t, x) + \mathcal{L}_{\gamma_2}^t \gamma_2(t, x) + F(t, x),
\]

where \( \gamma_i \) \( (i = 1, 2) \) will be chosen in terms of given initial and boundary data \( u_0, f, g \) and \( F(t, x) = e^{it\alpha_2^2} \tilde{u}_0 + XD|u|^2u \).

Remember that \( a_j \) and \( b_j \) are defined by

\[
(5.2) \quad a_j = \frac{M}{8} \left( \frac{e^{-i\frac{2\pi}{5}(1+3\lambda_j)} + e^{-i\frac{2\pi}{5}(1-5\lambda_j)}}{\sin(\frac{1-\lambda_j}{4})} \right) \quad \text{and} \quad b_j = \frac{M}{8} \left( \frac{e^{-i\frac{2\pi}{5}(2+3\lambda_j)} + e^{-i\frac{2\pi}{5}(6-5\lambda_j)}}{\sin(\frac{2-\lambda_j}{4})} \right).
\]

By Lemmas 3.2 and 3.3, we get

\[
(5.3) \quad f(t) = u(t, 0) = a_1 \gamma_1(t) + a_2 \gamma_2(t) + F(t, 0)
\]

and

\[
(5.4) \quad g(t) = \partial_x u(t, 0) = b_1 \mathcal{L}_{-\frac{1}{4}}^t \gamma_1(t) + b_2 \mathcal{L}_{-\frac{1}{4}}^t \gamma_2(t) + \partial_x F(t, 0).
\]
Observe that putting together with (5.3) and (5.4), we can write a matrix in the following form
\[
\begin{bmatrix}
  f(t) - F(t,0) \\
  \mathcal{I}_{\frac{1}{4}}g(t) - \mathcal{I}_{\frac{1}{2}}\partial_x F(t,0)
\end{bmatrix} = A \begin{bmatrix}
  \gamma_1(t) \\
  \gamma_2(t)
\end{bmatrix},
\]
where
\[
A(\lambda_1, \lambda_2) = \begin{bmatrix}
  a_1 & a_2 \\
  b_1 & b_2
\end{bmatrix}.
\]
By using mathematical software, the determinant of matrix \(A(\lambda_1, \lambda_2)\) is given by
\[
\det A = 2(-1)^{\frac{15}{2}} e^{-\frac{i(6 + 3\lambda_1 + \lambda_2)\pi}{4}} \left(1 + e^{i\lambda_1\pi}\right) \left(-1 + e^{i\lambda_2\pi}\right) \sec \left(\frac{1 + \lambda_1}{4}\pi\right) \\
+ 4(-1)^{\frac{3}{2}} e^{-\frac{i(\lambda_1 + \lambda_2)\pi}{8}} \left(1 - e^{\frac{i\lambda_1}{2}}\right) \left(1 - ie^{-\frac{i\lambda_2}{2}}\right).
\]
Note that the following graphics, real and imaginary parts, of the determinant function \(A(\lambda_1, \lambda_2)\), helps us to see when the matrix \(A\) is invertible:

\[\text{Figure 1.}\ \lambda_1 = a\ \text{and}\ \lambda_2 = b\]

Thus, matrix \(A(\lambda_1, \lambda_2)\) is invertible if the following holds,
\[
\lambda_2 \neq \frac{2}{\pi} \left(2\pi n - i \log \left\{ \frac{-2(-1)^{\frac{1}{2}} e^{-\frac{i\pi\lambda_1}{4}} + 2(-1)^{\frac{1}{2}} e^{\frac{3i\pi\lambda_1}{4}} + (e^{i\pi\lambda_1} + 1) \sec \left(\frac{(1 + \lambda_1)\pi}{4}\right)}{\left(-2(-1)^{\frac{3}{2}} e^{-\frac{i\pi\lambda_1}{4}} + 2(-1)^{\frac{3}{2}} e^{\frac{3i\pi\lambda_1}{4}} + (e^{i\pi\lambda_1} + 1) \sec \left(\frac{(1 + \lambda_1)\pi}{4}\right)\right)} \right\},
\]
and
\[
\lambda_j \neq 1 - 4n, \ \lambda_j \neq 2 - 4n, \ j = 1, 2,
\]
for all \(n \in \mathbb{Z}\).

Figure 1 help us to see that there are infinities set of parameters which one satisfies the relations (5.5) and (5.6). In fact, for example, pick \(\lambda_1 \approx 0\) and \(\lambda_2 \approx 1/3\). Therefore, for \(0 \leq s < \frac{1}{2}\) the choice of parameters \(\lambda_1\) and \(\lambda_2\) satisfying the following conditions
\[
-3 < \lambda_j < \frac{1}{2}, \ s + 4b - 2 < \lambda_j < s + \frac{1}{2}, \ j = 1, 2
\]
holds Lemma 4.3.

Thus, for a fix \(s \in \left[0, \frac{1}{2}\right)\), we can choose \(\lambda_1\) and \(\lambda_2\) as before and define the forcing functions \(\gamma_1(t)\) and \(\gamma_2(t)\) for any \(\lambda_j, j = 1, 2\) given by
\[
\begin{bmatrix}
  \gamma_1(t) \\
  \gamma_2(t)
\end{bmatrix} = A^{-1} \begin{bmatrix}
  f(t) - F(t,0) \\
  \mathcal{I}_{\frac{1}{4}}g(t) - \mathcal{I}_{\frac{1}{2}}\partial_x F(t,0)
\end{bmatrix},
\]
which shows (5.1) satisfies \((\partial_t - \partial_t^3)u = \lambda|u|^2 u\).

Define the solution operator by
\[
Au(t, x) = \psi(t)\mathcal{L}^{\lambda_1}\gamma_1(t, x) + \psi(t)\mathcal{L}^{\lambda_2}\gamma_2(t, x) + \psi(t)F(t, x),
\]
where
\[
\begin{bmatrix}
\gamma_1(t) \\
\gamma_2(t)
\end{bmatrix} = A^{-1} \begin{bmatrix}
-f(t) - F(t, 0) \\
\mathcal{I}_4 g(t) - \mathcal{I}_4 \partial_x F(t, 0)
\end{bmatrix},
\]
\[
F(t, x) = e^{i\partial_x^4} \tilde{u}_0 + \lambda D(\|u\|^2 u) \quad \text{and} \quad \psi \text{ is defined by (2.1)}.
\]

We remark, in view of (5.3), (5.4) and (5.7), that it is necessary to check \(\gamma_i(t), \ i = 1, 2\) to be well-defined in \(H_{\frac{2s+3}{8}}(\mathbb{R}^+)\). It follows from Lemmas 4.1, 4.2 and 4.3, Propositions 4.1 and Lemmas 2.1 and 2.3.

Recall the solution space \(Z_{s,b}^*\) defined in Subsection 2.2 under the norm
\[
\|v\|_{Z_{s,b}} = \sup_{t \in \mathbb{R}} \|v(t, \cdot)\|_{H^s} + \sum_{j=0}^{1} \sup_{x \in \mathbb{R}} \|\partial_x^j v(\cdot, x)\|_{H_{\frac{2s+3-2j}{8}}} + \|v\|_{X_{s,b}},
\]

All estimates obtained in Section 2 together with estimates of Section 4 and (4.23) yields that
\[
\|\Lambda u\|_{Z_{s,b}} \leq c(\|u_0\|_{H^s(\mathbb{R}^+)} + \|f\|_{H_{\frac{2s+3}{8}}(\mathbb{R}^+)} + \|g\|_{H_{\frac{2s+1}{8}}(\mathbb{R}^+)} + C_1\|u\|_{Z_{s,b}}^3).
\]

Similarly,
\[
\|\Lambda u_1 - \Lambda u_2\|_{Z_{s,b}} \leq C_2(\|u_1\|_{Z_{s,b}}^2 + \|u_2\|_{Z_{s,b}}^2)\|u_1 - u_2\|_{Z_{s,b}},
\]

for \(u_1(0, x) = u_2(0, x)\).

Finally, choosing \(0 < \delta \ll 1\) satisfying
\[
4cC_1\delta < 1 \quad \text{and} \quad 4cC_2\delta < \frac{1}{2},
\]

follows then that \(\Lambda\) is a contraction map on \(\\{u \in Z_{s,b}^* : \|u\|_{Z_{s,b}} < 2\delta\}\), and it completes the proof of Theorem 1.1.

6. Further comments and open problems

In this section our plan is to present four interesting problem that can be treated with the approach used in this article.

6.1. Biharmonic NLS in star graphs. In the following forthcoming paper [11], we mainly consider the biharmonic Schrödinger equation on the star graphs, given by \(N\) edges \((0, \infty)\) connected with a common vertex \((0, 0, \cdots, 0)\) (see figure 2), namely

\[
\begin{cases}
    i\partial_t u_j - \partial_x^4 u_j + \lambda |u_j|^2 u_j = 0, & (t, x) \in (0, T) \times (0, \infty), \ j = 1, 2, \ldots, N \\
    u_j(0, x) = u_{j0}(x), & \ x \in (0, \infty),
\end{cases}
\]

with initial conditions \((u_1(0, x), u_2(0, x), \ldots, u_N(0, x)) \in H^s(\mathbb{R}^+)\)

Figure 2. Star graphs connected with a common vertex \((0, 0, \cdots, 0)\)
For a better understanding, we are interested to solve (6.1) with the following three classes boundary conditions:

\[
\begin{align*}
(6.2) & \quad \begin{cases} \partial_x^k u_1(t, 0) = \partial_x^k u_2(t, 0) = \cdots = u_N(t, 0), & k = 0, 1 \quad t \in (0, T) \\
\sum_{j=1}^n \partial_x^k u_j(t, 0) = 0, & k = 2, 3 \quad t \in (0, T)
\end{cases} \\
(6.3) & \quad \begin{cases} \partial_x^k u_1(t, 0) = \partial_x^k u_2(t, 0) = \cdots = u_N(t, 0), & k = 2, 3 \quad t \in (0, T) \\
\sum_{j=1}^n \partial_x^k u_j(t, 0) = 0, & k = 0, 1 \quad t \in (0, T)
\end{cases} \\
\text{and} & \quad \begin{cases} \partial_x^k u_1(t, 0) = \partial_x^k u_2(t, 0) = \cdots = u_N(t, 0), & k = 0, 3 \quad t \in (0, T) \\
\sum_{j=1}^n \partial_x^k u_j(t, 0) = 0, & k = 1, 2 \quad t \in (0, T)
\end{cases}
\end{align*}
\]

The motivation and how we can choose these boundary conditions follows the ideas contained in a paper of the second author [12], and will be detailed in [11].

6.2. Control theory. We split this section in two parts: Control theory of biharmonic NLS in star graphs and on unbounded domains.

6.2.1. Control theory of Biharmonic NLS in star graphs. First, let us consider the controllability problem associated to (6.1) with the three possibilities of boundary conditions, namely, (6.2), (6.3) and (6.4). Due of the recent development of graph theory for Korteweg-de Vries equation, in the following paragraph we present a few comments about this study.

In three interesting papers Ammari and Crepeau [1], Cavalcante [12] and Mugnolo et al. [37] deal with the study of the KdV and Airy equations in graphs. In summary, the first work, the authors proposed a model using the Korteweg-de Vries equation on a finite star-shaped network and proved the well-posedness of the system. Also, as the main result of the work, by using properties of the energy, they showed that the solutions of the system decays exponentially to zero (as \( t \to \infty \)) and studied an exact boundary controllability problem. In addition, Cavalcante showed local well-posedness for the Cauchy problem associated with Korteweg–de Vries equation on a metric star graph. More precisely, he used the Duhamel boundary forcing operator, in context of half-lines, introduced by Colliander and Kenig [16], Holmer [31] and Cavalcante [13] to archived his result. Finally, Mugnolo et al. obtained a characterization of all boundary conditions under which the Airy-type evolution equation \( u_t = \alpha u_{xxx} + \beta u_x \), for \( \alpha \in \mathbb{R} \setminus \{0\} \) and \( \beta \in \mathbb{R} \) on star graphs given by \( E = E_- \cup E_+ \), where \( E_+ \) and \( E_- \) are finite or countable collections of semi-infinite edges parametrized by \((-\infty, 0)\) or \((0, +\infty)\) respectively and the half-lines are connected at a vertex \( v \), generates a semigroup of contractions.

In this spirit, looking for the energy identity, namely the \( L^2 \)-energy, which one satisfies an equality given by

\[
E(u_1(T, x), u_2(T, x), \cdots, u_N(T, x)) = - \sum_{i=1}^N \int_0^T \text{Im}(\partial_x^2 u_j(t, 0)\bar{\pi}_j(t, 0))dt \\
+ \sum_{i=1}^N \int_0^T \text{Im}(\partial_x^2 u_j(t, 0)\partial_x \bar{\pi}_j(t, 0))dt \\
- E(u_1(0, x), u_2(0, x), \cdots, u_N(0, x)),
\]

\[
(6.5)
\]

where

\[
E(u_1(t, x), u_2(t, x), \cdots, u_N(t, x)) := \sum_{i=1}^N \int_0^{+\infty} |u(t, x)|^2 dx.
\]

Therefore, the following natural question arises.

**Problem A**: Which are the boundary conditions that we can impose in (6.2), (6.3) and (6.4) such that the energy is a non-increasing function of the time variable \( t \)?
If we can impose some boundary conditions such that the energy (6.5) is a non-increasing function of the time variable $t$, a interesting issue can be treated. 

**Problem B:** If we can impose some boundary conditions such that the energy (6.5) is a non-increasing function of the time variable $t$, is the system (6.1), with appropriated boundary conditions, asymptotically stable when the time tends to infinity?

Finally, the last problem in this section is the following one.

**Problem C:** Can we find appropriated boundary controls such that the system (6.1) is controllable in some sense?

6.2.2. Control theory of Biharmonic NLS on unbounded domains. On the context of control on unbounded domains, Faminskii [20], in a recent work, considered the initial-boundary value problems, posed on infinite domains for Korteweg de Vries equation. Precisely, he elected a function $f_0$ in the right-hand side of the equation as an unknown function, regarded as a control. Thus, the author proved the this function which must be chosen such that the corresponding solution should satisfy certain additional integral condition. 

Thus, this techniques, probably, work well for the following Biharmonic NLS system

\[
\begin{cases}
    i\partial_t u + \gamma \partial_x^4 u + \lambda |u|^2 u = f_0(t)v(x,t), & (t,x) \in (0,T) \times (0,\infty), \\
    u(0,x) = u_0(x), & x \in (0,\infty), \\
    u(t,0) = h(t), \quad u_x(t,0) = g(t) & t \in (0,T),
\end{cases}
\]

for $\gamma, \lambda \in \mathbb{R}$, where $v$ is a given function and $f_0$ is an unknown control function. Therefore, the issue here is:

**Problem D:** Is the (6.6) controllable in the sense of Faminskii’s work. That is, can we find a pair $\{f_0, u\}$, satisfying an appropriated additional integral conditions (for details see [20])?

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