Fuglede–Putnam type theorems via the Aluthge transform

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Abstract Let \( A = U|A| \) and \( B = V|B| \) be the polar decompositions of \( A \in \mathcal{B}(\mathcal{H}_1) \) and \( B \in \mathcal{B}(\mathcal{H}_2) \) and let \( \text{Com}(A, B) \) stand for the set of operators \( X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \) such that \( AX = XB \). A pair \((A, B)\) is said to have the FP-property if \( \text{Com}(A, B) \subseteq \text{Com}(A^*, B^*) \). Let \( \tilde{C} \) denote the Aluthge transform of a bounded linear operator \( C \). We show that (i) if \( A \) and \( B \) are invertible and \((A, B)\) has the FP-property, then so is \((\tilde{A}, \tilde{B})\); (ii) if \( A \) and \( B \) are invertible, the spectrums of both \( U \) and \( V \) are contained in some open semicircle and \((\tilde{A}, \tilde{B})\) has the FP-property, then so is \((A, B)\); (iii) if \((A, B)\) has the FP-property, then \( \text{Com}(A, B) \subseteq \text{Com}(\tilde{A}, \tilde{B}) \), moreover, if \( A \) is invertible, then \( \text{Com}(A, B) = \text{Com}(\tilde{A}, \tilde{B}) \). Finally, if \( \text{Re}(|A|^{\frac{1}{2}}) \geq a > 0 \) and \( \text{Re}(|B|^{\frac{1}{2}}) \geq a > 0 \) and \( X \) is an operator such that \( U^*X = XV \), then we prove that \( \|\tilde{A}^*X - XB\|_p \geq 2a \| |B|^{\frac{1}{2}}X - X|B|^{\frac{1}{2}} \|_p \) for any \( 1 \leq p \leq \infty \).

Keywords Fuglede–Putnam theorem · Aluthge transform · Polar decomposition · Normal operator · Schatten \( p \)-norm · Norm inequality

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1 Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be the algebra of all bounded linear operators between (separable) complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, let $\mathcal{B}(\mathcal{H})$ denote $\mathcal{B}(\mathcal{H}, \mathcal{H})$ and let $I \in \mathcal{B}(\mathcal{H})$ be the identity operator. A subspace $\mathcal{K} \subseteq \mathcal{H}$ is said to reduce $A \in \mathcal{B}(\mathcal{H})$ if $A \mathcal{K} \subseteq \mathcal{K}$ and $A^* \mathcal{K} \subseteq \mathcal{K}$. Let $\mathcal{K}(\mathcal{H})$ denote the two-sided ideal of all compact operators on $\mathcal{H}$. For any compact operator $A$, let $s_1(A), s_2(A), \ldots$ be the singular values of $A$, i.e., the eigenvalues of $|A| = (A^*A)^{1/2}$ in decreasing order and repeated according to the multiplicity. If $\sum_{i=1}^{\infty} s_i(A)^p < \infty$, for some $1 \leq p < \infty$, we say that $A$ is in the Schatten class $C_p$ and $\|A\|_p = (\sum_{i=1}^{\infty} s_i(A)^p)^{1/p}$ is called the Schatten $p$-norm of $A$. This norm makes $C_p$ into a Banach space. Note that $C_1$ is the trace class and $C_2$ is the Hilbert-Schmidt class. It is convenient to put $C_\infty = \mathcal{K}(\mathcal{H})$ and to denote the usual operator norm $\|\cdot\|$ by $\|\cdot\|_\infty$. If $\{e_i\}_{i=1}^\infty$ and $\{f_i\}_{i=1}^\infty$ are two orthonormal families in $\mathcal{H}$, then for $A \in C_p$, $\|A\|_p \geq \sum_{i=1}^\infty |\langle Ae_i, f_i \rangle|^p$. If $A, B \in C_p$, then

$$\left\| \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right\|_p^p = \|A\|_p^p + \|B\|_p^p \quad (0 \leq p < \infty),$$

$$\left\| \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right\|_{\infty} = \max(\|A\|_{\infty}, \|B\|_{\infty}).$$

We refer the reader to [24] for further properties of the Schatten $p$-classes.

For $p > 0$, an operator $A$ is called $p$-hyponormal if $(A^*A)^{1/p} \geq (AA^*)^{1/p}$. If $A$ is an invertible operator satisfying $\log(A^*A) \geq \log(AA^*)$, then it is called log-hyponormal. If $p = 1$, then $A$ is said to be hyponormal. If $A$ is invertible and $p$-hyponormal then $A$ is log-hyponormal.

Let $A = U|A|$ be the polar decomposition of $A$. It is known that if $A$ is invertible then $U$ is unitary and $|A|$ is also invertible. The Aluthge transform $\tilde{A}$ of $A$ is defined by $\tilde{A} := |A|^{1/2} U |A|^{1/2}$. This notion was first introduced by Aluthge [1] and is a powerful tool in the operator theory. There are some significant evidences for this assertion, for instance, it is proved in [14] that any operator $A$ has a non-trivial invariant subspace if and only if so does $\tilde{A}$. Another interesting application deals with an application of the Aluthge transform for generalizing the Fuglede–Putnam theorem [12]. It indeed is a motivation for our work in this paper. Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. For such pair $(A, B)$, denote by $\Com(A, B)$ the set of operators $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $AX = XB$. A pair $(A, B)$ is said to have the FP-property if $\Com(A, B) \subseteq \Com(A^*, B^*)$. The Fuglede–Putnam theorem is well-known in the operator theory. It asserts that for any normal operators $A$ and $B$, the pair $(A, B)$ has the FP-property. First Fuglede [6] proved it in the case when $A = B$ and then Putnam [22] proved it in a general case. There exist many generalizations of this theorem which most of them go into relaxing the normality of $A$ and $B$; see [4,5,8,13,17,21,23] and references therein. The two next lemmas are concerned with the Fuglede–Putnam theorem and we need them in the future.
Lemma 1.1 [25] Let \( A \in \mathbb{B}(\mathcal{H}_1) \) and \( B \in \mathbb{B}(\mathcal{H}_2) \). Then the following assertions are equivalent

(i) The pair \((A, B)\) has the FP-property.
(ii) If \( X \in \text{Com}(A, B) \), then \( \tilde{R}(X) \) reduces \( A \), \( (\ker X)^\perp \) reduces \( B \), and \( A|\tilde{R}(X)^\perp B|_{(\ker X)^\perp} \) are unitarily equivalent normal operators.

Lemma 1.2 [12] Let \( A \in \mathbb{B}(\mathcal{H}_1) \) and \( B^* \in \mathbb{B}(\mathcal{H}_2) \) be either log-hyponormal or \( p \)-hyponormal operators. Then the pair \((A, B)\) has the FP-property.

Recently some investigation on the operator theory have been related to relationship between operators and their Aluthge transform; see [2,9,10,19,26,27]. In this paper we present some results that are in the same direction but of some new views of points via the Fuglede–Putnam theorem. For instance, one of our problems is as follows: Under what conditions on operators \( A \) and \( B \) does the FP-property for the pair \((A, B)\) imply that for \((\tilde{A}, \tilde{B})\)? Another question is related to the converse. In Sect. 2 we try to answer these questions.

The iterated Aluthge transforms of \( A \) are the operators \( \Delta_n(A) \) defined by \( \Delta_1(A) := \tilde{A} \) and \( \Delta_n(A) := \Delta_1(\Delta_{n-1}(A)) \) for \( n > 1 \). A surprising fact about these operators is the convergence of their norms to the spectral radius of \( A \); cf. [27]. Also the convergence of the sequence of iterates is an interesting question, which is recently investigated in [3]. In Sects. 2 and 3, we provide some results concerning these operators as well.

Another interesting problem is that under what conditions on \( A, B, X \), any one of \( AX = XB \) and \( AX = X\tilde{B} \) implies the other. In Sect. 3 we try to provide some results concerning this problem that we call it the Fuglede–Putnam–Aluthge problem. More precisely, we prove that if \((A, B)\) has the FP-property, then \( \text{Com}(A, B) \subseteq \text{Com}(\tilde{A}, \tilde{B}) \) and \( (\text{Com}(A, B))^{-1} = \text{Com}(\tilde{A}, \tilde{B}) \). We also study Fuglede–Putnam–Aluthge problem modulo trace ideals and give several Schatten \( p \)-norm inequalities in Sect. 4; see also [16,20]. The reader is referred to [7] for undefined notions and terminology.

2 Fuglede–Putnam theorem for the Aluthge transforms

In this section we assume that \( A \in \mathbb{B}(\mathcal{H}_1) \) and \( B \in \mathbb{B}(\mathcal{H}_2) \) are invertible operators with the polar decompositions \( A = U|A| \) and \( B = V|B| \), where \( U \) and \( V \) are unitaries.

Lemma 2.1

(i) \( X \in \text{Com}(A, B) \iff |A|X|B|^{-1} = U^* XV; \)
(ii) \( X \in \text{Com}(A, B) \cap \text{Com}(A^*, B^*) \iff |A|X|B|^{-1} = U^* XV = X. \)

Proof (i) is just the definition itself.

(ii) Let \( X \in \text{Com}(A, B) \cap \text{Com}(A^*, B^*) \). Then \( |A|^2 X = X|B|^2 \). Utilizing a sequence of polynomials uniformly converging to \( f(t) = \sqrt{t} \) on \( \text{sp}(|A|^2) \cup \text{sp}(|B|^2) \) and the functional calculus we get \( |A|X = X|B| \), that is \( |A|X|B|^{-1} = X \). Hence from (i) we have \( U^* XV = X \). The reverse direction is trivial. \( \square \)
Remark 2.2 The proof of Lemma 2.1 shows that if \( X \in \text{Com}(A, B) \cap \text{Com}(A^*, B^*) \) and \( p \) be a positive number, then \( |A|^p X = X|B|^p \).

Lemma 2.3

(i) \( X \in \text{Com}(A, B) \iff |A|^\frac{1}{2}X|B|^{-\frac{1}{2}} \in \text{Com}(\bar{A}, \bar{B}) \);
(ii) \( X \in \text{Com}(A^*, B^*) \iff |A|^{-\frac{1}{2}}X|B|^\frac{1}{2} \in \text{Com}((\bar{A})^*, (\bar{B})^*) \).

Proof

(i) Let \( AX = XB \) for some \( X \in \mathcal{B}(\mathcal{H}) \). Then

\[
U|A|X = XV|B|.
\]

Hence

\[
\bar{A}(|A|^\frac{1}{2}X|B|^{-\frac{1}{2}}) = |A|^\frac{1}{2}(U|A|^\frac{1}{2}|A|^\frac{1}{2}X)|B|^{-\frac{1}{2}}
\]

\[
= |A|^\frac{1}{2}(X|B|^{-\frac{1}{2}}|B|^\frac{1}{2}V|B|)|B|^{-\frac{1}{2}}
\]

\[
= (|A|^\frac{1}{2}X|B|^{-\frac{1}{2}})\bar{B}.
\]

The converse obviously holds.

(ii) It can be proved in a similar way to (i).

\[\Box\]

Theorem 2.4 The pair \((\bar{A}, \bar{B})\) has the FP-property, i.e. \(\text{Com}(\bar{A}, \bar{B}) \subseteq \text{Com}((\bar{A})^*, (\bar{B})^*)\) if and only if \(U^2X = XV^2\) for any \(X \in \text{Com}(A, B)\).

Proof First we show that the FP-property for \((\bar{A}, \bar{B})\) is equivalent to the following requirement

\[
|A|X|B|^{-1} \in \text{Com}(A^*, B^*) \quad (X \in \text{Com}(A, B)). \tag{1}
\]

Let \((\bar{A}, \bar{B})\) have the FP-property and \(X \in \text{Com}(A, B)\). By Lemma 2.3(i), \(|A|^\frac{1}{2}X|B|^{-\frac{1}{2}} \in \text{Com}(\bar{A}, \bar{B})\). Since \((\bar{A}, \bar{B})\) has the FP-property we have \(|A|^\frac{1}{2}X|B|^{-\frac{1}{2}} \in \text{Com}((\bar{A})^*, (\bar{B})^*)\). By Lemma 2.3(ii) we have \(|A|X|B|^{-1} \in \text{Com}(A^*, B^*)\), so we reach (1). To prove the reverse, assume the assertion (1) and let \(X \in \text{Com}(\bar{A}, \bar{B})\). It follows from Lemma 2.3(i) that \(|A|^{-\frac{1}{2}}X|B|^\frac{1}{2} \in \text{Com}(A, B)\). Hence by (1) we have \(|A|^\frac{1}{2}X|B|^{-\frac{1}{2}} \in \text{Com}(A^*, B^*)\) which in turn implies that \(X \in \text{Com}((\bar{A})^*, (\bar{B})^*)\). Thus \((\bar{A}, \bar{B})\) has the FP-property.

Let (1) hold. For any \(X \in \text{Com}(A, B)\) it follows from Lemma 2.1(i) that \(|A|X|B|^{-1} = U^*XV\). Using (1) we obtain

\[
|A|U^*U^*XV = |A|X|B|^{-1}|B|V^* \quad (X \in \text{Com}(A, B)),
\]

which simply becomes \(U^2X = XV^2\) for any \(X \in \text{Com}(A, B)\). The converse can be proved in a similar fashion.

\[\Box\]

Corollary 2.5 If \((A, B)\) has the FP-property, then so is \((\bar{A}, \bar{B})\).
Proof If \((A, B)\) has the FP-property, then by Lemma 2.1(ii) \(UX = XV\) for any \(X \in \text{Com}(A, B)\). Hence \(U^2X = XV^2\). Applying Theorem 2.4 we observe that \((\tilde{A}, \tilde{B})\) has the FP-property. \(\square\)

Corollary 2.6 If \((A, B)\) has the FP-property, then so is \((\Delta_n(A), \Delta_n(B))\) for any positive integer \(n\).

Corollary 2.7 If the spectrums of both \(U\) and \(V\) are contained in some open semicircle, then the FP-property for \((A, B)\) is equivalent to the FP-property for \((\tilde{A}, \tilde{B})\).

Proof Let \((\tilde{A}, \tilde{B})\) have the FP-property and \(X \in \text{Com}(A, B)\). Then \(U^2X = XV^2\) by Theorem 2.4. Under the spectral conditions on \(U\) and \(V\) the unitary operator \(U\) (resp. \(V\)) can be approximated by polynomials of \(U^2\) (resp. \(V^2\)), therefore \(U^2X = XV^2\) implies \(UX = XV\), that is, \(U^*XV = X\) and this by Lemma 2.1 implies that \(X \in \text{Com}(A^*, B^*)\). The rest follows from Corollary 2.5. \(\square\)

Remark 2.8 Note that if the conditions on \(U\) and \(V\) in Corollary 2.7 are replaced by the condition that \(U^{2n_0+1} = V^{2n_0+1} = I\) for some positive integer \(n_0\), then we obtain the same result. In fact from \(U^2X = XV^2\) we get \(U^{2n_0}X = XV^{2n_0}\) that under our assumption implies that \(UX = XV\).

An interesting problem is that under what conditions on operator \(A\), \(A^n = I\) implies that \(U^n = I\), where \(A = U|A|\) is the polar decomposition of \(A\) and \(n \geq 1\). It is known that for a normaloid operator \(A\), \(A^n = I\) implies that \(A\) is unitary [7, Corollary 3.7.3.6]. The next result is related to this problem.

Proposition 2.9 Let \(A = U|A|\) be the polar decomposition of \(A\) and \(A^2 = I\) then \(U^2 = I\).

Proof Since \(A^2 = I\) we have

\[U|A|U|A| = I.\]

We multiply both side of (2) by \(|A|^{-1}\) to obtain

\[|A|^{-1} = U|A|U = U^2U^*|A|U.\]

Since \(U^2\) is unitary and \(U^*|A|U \geq 0\) and view of uniqueness of polar decomposition of \(|A|^{-1}\), the unitary operator \(U^2\) should coincide with the angular part \(I\) of positive definite \(|A|^{-1}\). \(\square\)

Remark 2.10 The above proposition is a consequence of \([11, \text{Theorem 2.1}]\), which states that if \(T = U|T|\), \(S = V|S|\) and \(|T||S^*| = W||T||S^*||\) are the polar decompositions, then \(TS = UWV|TS|\) is also the polar decomposition.

Example 2.11 Proposition 2.9 is not valid when the power 2 is replaced by 3. Indeed there exists an operator \(A\) with the polar decomposition \(A = U|A|\) such that \(A^3 = I\) but \(U^3 \neq I\). To see this let \(A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\). Then \(|A| = \begin{pmatrix} \frac{3\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{3\sqrt{5}}{5} \end{pmatrix}\) and \(U = A|A|^{-1} = \begin{pmatrix} -\frac{\sqrt{5}}{5} & 2\frac{\sqrt{5}}{5} \\ -\frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{pmatrix}\). It is easy to verify that \(A^3 = I\) and \(U^3 = \begin{pmatrix} \frac{11\sqrt{5}}{25} & -\frac{2\sqrt{5}}{25} \\ -\frac{2\sqrt{5}}{25} & \frac{11\sqrt{5}}{25} \end{pmatrix} \neq I\).
Now we present an example to show that in Corollary 2.7 and Remark 2.8 the conditions are essential.

**Example 2.12** Let \( A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \) and \( X = \begin{pmatrix} 0 & -3 \\ 1 & -4 \end{pmatrix} \). It is easy to verify that \( AX = XA \) and \( A^*X \neq XA^* \). On the other hand, an easy computation shows that \( A^2 = I \). Hence by Proposition 2.9, \( U^2 = I \) in which \( A = U|A| \) is the polar decomposition of \( A \). Hence \( U = U^* \), so that \( \tilde{A} = |A|^\frac{1}{2} U |A|^\frac{1}{2} \) is self adjoint. Thus \( \tilde{A}X = X\tilde{A} \) implies \( \tilde{A}^*X = X\tilde{A}^* \) for any \( X \).

### 3 The Fuglede–Putnam–Aluthge problem

In this section, we present some results concerning the Fuglede–Putnam–Aluthge problem without assumption of invertibility of \( A \) and \( B \), in general.

**Theorem 3.1** Let \( A \in \mathcal{B}(\mathcal{H}_1) \), \( B \in \mathcal{B}(\mathcal{H}_2) \) and \((A, B)\) have the FP-property. Then \( \text{Com}(A, B) \subseteq \text{Com}(\tilde{A}, \tilde{B}) \).

**Proof** Let \( A = U|A| \) and \( B = V|B| \) be the polar decompositions of \( A \) and \( B \), respectively. Let \( \{p_n\} \) be a sequence of polynomials with no constant term such that \( p_n(t) \to t^\frac{1}{2} \) uniformly on a certain compact set as \( n \to \infty \). Let \( X \in \text{Com}(A, B) \).

By our hypothesis, \( A^*X = XB^* \). Hence \( |A|^2X = |X|B|^2 \) and so \( p_n(|A|^2)X = Xp_n(|B|^2) \), hence \( |A|X = X|B| \). Using the same argument we get \( |A|^\frac{1}{2}X = X|B|^\frac{1}{2} \), \( \tilde{U}|A|^nX = XV|B|^n \) for \( n \in \mathbb{N} \). We can use the argument above to show that \( U p_n(|A|X)X = XVp_n(|B|) \) and conclude that \( U|A|^\frac{1}{2}X = XV|B|^\frac{1}{2} \). Hence we have

\[
\tilde{A}X = |A|^\frac{1}{2}U|A|^\frac{1}{2}X = |A|^\frac{1}{2}XV|B|^\frac{1}{2} = X|B|^\frac{1}{2}V|B|^\frac{1}{2} = X\tilde{B}.
\]

Corollary 3.2 Let \( A \in \mathcal{B}(\mathcal{H}_1) \) and \( B^* \in \mathcal{B}(\mathcal{H}_2) \) be either log-hyponormal or \( p \)-hyponormal operators. Then \( \text{Com}(A, B) \subseteq \text{Com}(\tilde{A}, \tilde{B}) \).

**Proof** It follows from Lemma 1.2 and Theorem 3.1.

Let \( A = U|A| \) be the polar decomposition of \( A \). \( A_{(s, t)} = |A|^sU|A|^t \), for \( s, t \geq 0 \) is called \((s, t)\)-Aluthge transform of \( A \). Note that we can use the proof of Theorem 3.1 for \( A_{(s, t)}, B_{(s, t)} \) instead of \( \tilde{A}, \tilde{B} \), respectively.

Using some ideas of [12, Theorem 8] we prove the next result.

**Theorem 3.3** Let \( A \in \mathcal{B}(\mathcal{H}_1) \) be invertible and \( B \in \mathcal{B}(\mathcal{H}_2) \) be arbitrary. If \((A, B)\) has the FP-property, then \( \text{Com}(A, B) = \text{Com}(\tilde{A}, \tilde{B}) \).

**Proof** It is sufficient to prove that \( \text{Com}(A, B) \supseteq \text{Com}(\tilde{A}, \tilde{B}) \). Let \( A = U|A| \) and \( B = V|B| \) be the polar decompositions of \( A \) and \( B \), respectively and \( X \in \text{Com}(\tilde{A}, \tilde{B}) \).

Let \( W = |A|^\frac{1}{2} X|B|^\frac{1}{2} \). Since \( \tilde{A}X = X\tilde{B} \), we have

\[
|A|^\frac{1}{2} \tilde{A}X|B|^\frac{1}{2} = |A|^\frac{1}{2} X\tilde{B}|B|^\frac{1}{2}.
\]
with respect to $H$ and $F$ and Fuglede–Putnam type theorems

and Remark 2.2

where $N$ and so are either p-hyponormal or log-hyponormal operator, then Com

Hence by hypothesis and Lemma 1.1, $R(W)$ reduces $A$, $N(W) \perp$ reduces $B$ and $A|_{R(W)}$ and $B|_{N(W)\perp}$ are normal operators. Therefore

$$A = N \oplus S \text{ on } R(W) \oplus R(W)\perp$$

and

$$B = M \oplus T \text{ on } N(W)\perp \oplus N(W),$$

where $N$ and $M$ are unitarily equivalent normal operators. Operator $A$ is invertible and so are $N$ and $S$. Since $N$ and $M$ are unitarily equivalent, $M$ is invertible. Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \text{ and } W = \begin{pmatrix} W_1 & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to $\mathcal{H}_1 = R(W) \oplus R(W)\perp$ and $\mathcal{H}_2 = N(W)\perp \oplus N(W)$. Clearly $|A|^{-1} = |N|^{-1} \oplus |S|^{-1}$. It follows from $W = |A|^{-1} X |B|^\frac{1}{2}$ that

$$\begin{pmatrix} W_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |N|^{-\frac{1}{2}} X_1 |M|^\frac{1}{2} & |N|^{-\frac{1}{2}} X_2 |T|^\frac{1}{2} \\ |S|^{-\frac{1}{2}} X_3 |M|^\frac{1}{2} & |S|^{-\frac{1}{2}} X_4 |T|^\frac{1}{2} \end{pmatrix}.$$ 

Hence $X_2 |T|^\frac{1}{2} = 0$, $X_3 = 0$, $X_4 |T|^\frac{1}{2} = 0$ so $X_2 \tilde{T} = 0$ and $X_4 \tilde{T} = 0$. Then $\tilde{A} X = X \tilde{B}$ implies that

$$\begin{pmatrix} NX_1 & NX_2 \\ 0 & \tilde{S} X_4 \end{pmatrix} = \begin{pmatrix} X_1 M & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Hence $X_2 = 0$ and $X_4 = 0$. Since $\tilde{A} = N \oplus \tilde{S}$ and $\tilde{B} = M \oplus \tilde{T}$ and $\tilde{A} X = X \tilde{B}$ and $X = X_1 \oplus 0$, we have $NX_1 = X_1 M$ and this, in turn, implies that $AX = XB$. \qed

Remark 3.4 In the preceding theorem if we assume that both $A$ and $B$ are invertible, then we can easily prove the theorem. To see this, let $Y \in \text{Com}(\tilde{A}, \tilde{B})$. Then by Lemma 2.3(i) $|A|^{-\frac{1}{2}} Y |B|^\frac{1}{2} \in \text{Com}(A, B)$. It follows from the FP-property for $(A, B)$ and Remark 2.2

$$Y = |A|^\frac{1}{2} |A|^{-\frac{1}{2}} Y |B|^\frac{1}{2} |B|^{-\frac{1}{2}} = |A|^{-\frac{1}{2}} Y |B|^\frac{1}{2} \in \text{Com}(A, B).$$

Corollary 3.5 Let $A \in \mathcal{B}(\mathcal{H}_1)$ be log-hyponormal operator and $B^* \in \mathcal{B}(\mathcal{H}_2)$ be either p-hyponormal or log-hyponormal operator, then $\text{Com}(A, B) = \text{Com}(\tilde{A}, \tilde{B})$. 
**Proof** It follows from Lemma 1.2 and Theorem 3.3. □

**Corollary 3.6** Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$ be invertible operators. If $(A, B)$ has the FP-property and $n \in \mathbb{N}$, then $\text{Com}(\Delta_n(A), \Delta_n(B)) = \text{Com}(A, B)$.

**Proof** Let $\Delta_n(A)Y = Y \Delta_n(B)$ for some $Y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. By Corollary 2.6, we see that $(\Delta_{n-1}(A), \Delta_{n-1}(B))$ has the FP-property, so by Theorem 3.3 we have $\Delta_{n-1}(A)Y = Y \Delta_{n-1}(B)$. Repeating this process we conclude the result as desired. For the reverse similar argument can be applied. □

### 4 Fuglede–Putnam–Aluthge problem modulo trace ideals

In this section, we present some results about the Fuglede–Putnam–Aluthge problem modulo trace ideals. We obtain some inequalities related to this problem by using some ideas of [15].

**Lemma 4.1** Let $A = U|A|$ be the polar decomposition of $A$ and $X \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator such that $\text{Re}(U|A|^\frac{1}{2}) \geq a > 0$ and $U^*X = XU$. Then

$$
\|\tilde{A}^*X - X\tilde{A}\|_p \geq 2a\||A|^\frac{1}{2}X - X|A|^\frac{1}{2}\|_p
$$

for $1 \leq p \leq \infty$.

**Proof** We consider two cases:

**Case (i) $p = \infty$**. Clearly $(|A|^\frac{1}{2}X - X|A|^\frac{1}{2})^* = -(|A|^\frac{1}{2}X - X|A|^\frac{1}{2})$. It follows from [7, Theorem 2.4.1.16] that there exist a sequence $\{f_n\}_{n \in \mathbb{N}}$ of unit vectors in $\mathcal{H}$ and number $t \in \text{sp}(|A|^\frac{1}{2}X - X|A|^\frac{1}{2})$ such that $\tilde{t} = -t$, $(|A|^\frac{1}{2}X - X|A|^\frac{1}{2} - t)f_n \to 0$ as $n \to \infty$ and $|t| = \||A|^\frac{1}{2}X - X|A|^\frac{1}{2}\|_\infty$. Now

$$
\|\tilde{A}^*X - X\tilde{A}\|_\infty \geq |\langle \tilde{A}^*X - X\tilde{A}; f_n, f_n \rangle|
$$

$$
= |\langle |A|^\frac{1}{2}U^*|A|^\frac{1}{2}X - X|A|^\frac{1}{2}U|A|^\frac{1}{2}f_n, f_n \rangle| + |\langle |A|^\frac{1}{2}X - X|A|^\frac{1}{2}U|A|^\frac{1}{2}f_n, f_n \rangle| + |\langle |A|^\frac{1}{2}X - X|A|^\frac{1}{2} f_n, f_n \rangle| + |\langle |A|^\frac{1}{2}X - X|A|^\frac{1}{2} - t)^U|A|^\frac{1}{2}f_n, f_n \rangle| - |\langle |A|^\frac{1}{2}X - X|A|^\frac{1}{2} - t)U|A|^\frac{1}{2}f_n, f_n \rangle|
$$

$$
\geq |t|\||A|^\frac{1}{2}U^* + U|A|^\frac{1}{2}f_n, f_n \rangle - |\langle |A|^\frac{1}{2}X - X|A|^\frac{1}{2} f_n, f_n \rangle| + |\langle |A|^\frac{1}{2}X - X|A|^\frac{1}{2} - t)U|A|^\frac{1}{2}f_n, f_n \rangle|.
$$

We observe that

$$
|\langle |A|^\frac{1}{2}U^*|A|^\frac{1}{2}X - X|A|^\frac{1}{2} f_n, f_n \rangle| + |\langle |A|^\frac{1}{2}X - X|A|^\frac{1}{2} - t)U|A|^\frac{1}{2}f_n, f_n \rangle| \to 0
$$
as $n \to \infty$. Hence

$$\| \tilde{A}^* X - X \tilde{A} \|_\infty \geq 2a \| A |^{\frac{1}{2}} X - X |A|^{\frac{1}{2}} \|_\infty.$$ 

**Case (ii)** $1 \leq p < \infty$. We can assume that $\tilde{A}^* X - X \tilde{A} \in \mathcal{C}_p$ and hence it is compact. If $\pi : \mathbb{B}(\mathcal{H}) \to \frac{\mathbb{B}(\mathcal{H})}{\mathcal{C}_\infty}$ is the quotient map then we have $\pi(\tilde{A}^* X - X \tilde{A}) = 0$. It is obvious that $\pi(A) = \pi(U)\pi(|A|)$ is the polar decomposition of $\pi(A)$. Since $U^* X = X U$ we have $\pi(U^*)\pi(X) = \pi(X)\pi(U)$. Hence $\pi(|A|^{\frac{1}{2}} X - X |A|^{\frac{1}{2}}) = 0$ by Case (i). So $|A|^{\frac{1}{2}} X - X |A|^{\frac{1}{2}}$ is a compact normal operator. It is therefore diagonalizable and hence there exist an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $\mathcal{H}$ and numbers $t_n$ such that $(|A|^{\frac{1}{2}} X - X |A|^{\frac{1}{2}})e_n = t_n e_n$. Thus the $|t_n|$'s are the singular values of $|A|^{\frac{1}{2}} X - X |A|^{\frac{1}{2}}$ and

$$\| \tilde{A}^* X - X \tilde{A} \|_p \geq \sum_{n=1}^{\infty} |\langle \tilde{A}^* X - X \tilde{A} e_n, e_n \rangle|^p$$

$$= \sum_{n=1}^{\infty} |\langle |A|^{\frac{1}{2}} U^* |A|^{\frac{1}{2}} X - X |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}} e_n, e_n \rangle|^p$$

$$= \sum_{n=1}^{\infty} |\langle (|A|^{\frac{1}{2}} U^* (|A|^{\frac{1}{2}} X - X |A|^{\frac{1}{2}}) + (|A|^{\frac{1}{2}} X - X |A|^{\frac{1}{2}}) U |A|^{\frac{1}{2}} + |A|^{\frac{1}{2}} (U^* X - X U) |A|^{\frac{1}{2}} e_n, e_n \rangle|^p$$

$$= \sum_{n=1}^{\infty} |t_n|^p |\langle |A|^{\frac{1}{2}} U^* + U |A|^{\frac{1}{2}} e_n, e_n \rangle|^p$$

$$\geq \left( \sum_{n=1}^{\infty} |t_n|^p \right) (2a)^p = (2a)^p \| A |^{\frac{1}{2}} X - X |A|^{\frac{1}{2}} \|_p.$$

Thus

$$\| \tilde{A}^* X - X \tilde{A} \|_p \geq 2a \| A |^{\frac{1}{2}} X - X |A|^{\frac{1}{2}} \|_p.$$

Our next result reads as follows.

**Theorem 4.2** Let $A = U |A|$ and $B = V |B|$ be the polar decompositions of $A$ and $B$, respectively, and $X \in \mathbb{B}(\mathcal{H})$ such that $\text{Re}(U |A|^{\frac{1}{2}}) \geq a > 0$ and $\text{Re}(V |B|^{\frac{1}{2}}) \geq a > 0$ and $U^* X = X V$. Then

$$\| \tilde{A}^* X - X \tilde{B} \|_p \geq 2a \| A |^{\frac{1}{2}} X - X |B|^{\frac{1}{2}} \|_p$$

for $1 \leq p \leq \infty$.  

Proof Let \( T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) and \( Y = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \). Then \( Y \) is self-adjoint. Let \( T = W|T| \) be the polar decomposition of \( T \). Note that \( W = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \) and hence \( W^*Y = YW \) by the assumption \( U^*X = XV \).

Also \( W|T|^{\frac{1}{2}} = \begin{pmatrix} |U|A^{\frac{1}{2}} & 0 \\ 0 & |V||B|^{\frac{1}{2}} \end{pmatrix} \geq a \geq 0 \) so we have \( \|\tilde{T}^*Y - Y\tilde{T}\|_p \geq 2a\|T|^{\frac{1}{2}}Y - Y|T|^{\frac{1}{2}}\|_p \) by Lemma 4.1. Since \( \tilde{T} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) and \( |T|^{\frac{1}{2}} = \begin{pmatrix} |A|^{\frac{1}{2}} & 0 \\ 0 & |B|^{\frac{1}{2}} \end{pmatrix} \), a simple computation shows that

\[
\left\| \begin{pmatrix} 0 & \tilde{A^*}X - X\tilde{B} \\ \tilde{B}^*X^* - X^*\tilde{A} & 0 \end{pmatrix} \right\|_p^p \\
\geq 2^p a^p \left\| \begin{pmatrix} 0 & |A|^{\frac{1}{2}}X - X|B|^{\frac{1}{2}} \\ |B|^{\frac{1}{2}}X^* - X^*|A|^{\frac{1}{2}} & 0 \end{pmatrix} \right\|_p
\]

Utilizing (1) we obtain

\[
\|\tilde{A^*}X - X\tilde{B}\|_p \geq 2a\| |A|^{\frac{1}{2}}X - X|B|^{\frac{1}{2}} \|_p.
\]

\[
\square
\]

**Corollary 4.3** Let \( A = U|A| \) and \( B = V|B| \) be the polar decompositions of \( A \) and \( B \), respectively, and \( X \in \mathbb{B}(\mathcal{H}) \) such that \( \text{Re}(U|A|^{\frac{1}{2}}) \geq a > 0 \) and \( \text{Re}(V|B|^{\frac{1}{2}}) \geq a > 0 \) and \( U^*X = XV \) and \( \tilde{A^*}X - X\tilde{B} \in \mathcal{C}_p \) for some \( 1 \leq p \leq \infty \). Then \( |A|^{\frac{1}{2}}X - X|B|^{\frac{1}{2}} \in \mathcal{C}_p \).

**Corollary 4.4** Let \( A = U|A| \) and \( B = V|B| \) be the polar decompositions of \( A \) and \( B \), respectively, and \( X \in \mathbb{B}(\mathcal{H}) \) such that \( \text{Re}(U|A|^{\frac{1}{2}}) \geq a > 0 \) and \( \text{Re}(V|B|^{\frac{1}{2}}) \geq a > 0 \) and \( U^*X = XV \) and \( \tilde{A^*}X = X\tilde{B} \), then \( |A|X = X|B| \).

**Remark 4.5** Under the conditions of Corollary 4.4 we have

\[
A^*A|X = |A|U^*A|X = |A|U^*X|B| = |A|XV|B| = |A|XB.
\]

Hence there exists an operator \( Y(=|A|X) \) such that \( A^*Y = YB \).

**Remark 4.6** For \( \delta > 0 \), let \( \text{Com}_\delta(A, B) \) be the set of all operators \( X \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1) \) such that \( \|AX - XB\| \leq \delta \). Moore [18] proved that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\text{Com}_\delta(A, B) \cap \mathbb{B}(\mathcal{H}_1) \subseteq \text{Com}_\varepsilon(A^*, B^*),
\]

where \( \mathbb{B}(\mathcal{H}_1) \) denotes the closed norm-unit ball of \( \mathbb{B}(\mathcal{H}) \). Let \( A \) be an operator with the polar decomposition \( U|A| \) and \( X \in \text{Com}_\delta(|A|^{\frac{1}{2}}, |A|^{\frac{1}{2}}) \cap \text{Com}_\delta(U^*, U) \) for some \( \delta > 0 \). From
\[
\| \tilde{A}^*X - X\tilde{A} \| = \| |A|^{\frac{1}{2}} U^* (|A|^{\frac{1}{2}} X - X|A|^{\frac{1}{2}}) + (|A|^{\frac{1}{2}} X - X|A|^{\frac{1}{2}}) U|A|^{\frac{1}{2}} + |A|^{\frac{1}{2}} (U^*X - XU)|A|^{\frac{1}{2}} \| \\
\leq (\| |A|^{\frac{1}{2}} U^* \| + \| U|A|^{\frac{1}{2}} \| + \| A \|) \delta \\
= (2\| A \|^{\frac{1}{2}} + \| A \|) \delta
\]

we can see that \( X \in \text{Com}_{\psi_A(\delta)}(\tilde{A}^*, \tilde{A}) \) for some positive increasing function \( \psi_A(t) \) on \((0, \infty)\). Thus

\[
\text{Com}_\delta(|A|^{\frac{1}{2}}, |A|^{\frac{1}{2}}) \cap \text{Com}_\delta(U^*, U) \subseteq \text{Com}_{\psi_A(\delta)}(\tilde{A}^*, \tilde{A}).
\]

Now by Lemma 4.1 if \( \text{Re}(U|A|^{1/2}) \geq a > 0 \) it is easy to see that

\[
\text{Com}_{\psi_A(\delta)}(\tilde{A}^*, \tilde{A}) \cap \text{Com}(U^*, U) \subseteq \text{Com}_{\psi(\delta)}(|A|^{\frac{1}{2}}, |A|^{\frac{1}{2}}),
\]

where \( \psi(t) = \frac{\psi_A(t)}{2a} \).

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