MULTIFRACTAL ANALYSIS OF THE DIVERGENCE OF
FOURIER SERIES: THE EXTREME CASES

By

FRÉDÉRIC BAYART AND YANICK HEURTEAUX

Abstract. We study the size, in terms of the Hausdorff dimension, of the
subsets of \( \mathbb{T} \) such that the Fourier series of a generic function in \( L^1(\mathbb{T}) \), \( L^p(\mathbb{T}) \), or \( C(\mathbb{T}) \) may behave badly. Genericity is related to the Baire Category Theorem or
the notion of prevalence. This paper is a continuation of [3].

1 Introduction

This paper, which can be seen as a continuation of [3], deals with the divergence
of Fourier series of functions in \( L^p(\mathbb{T}) \), \( p \geq 1 \), where \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \), or in \( C(\mathbb{T}) \), from
the multifractal point of view. More precisely, let \( f \) be in \( L^p(\mathbb{T}) \), or in \( C(\mathbb{T}) \), and
let \( (S_n f)_{n \geq 0} \) be the sequence of partial sums of its Fourier series. We are interested
in the size of the sets of the real numbers \( x \) such that \( (S_n f(x))_{n \geq 0} \) diverges with a
prescribed growth.

We measure the size of subsets of \( \mathbb{T} \) using the Hausdorff dimension. Let us
recall the relevant definitions (we refer to [6] and to [9] for more on this subject).
If \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a nondecreasing continuous function satisfying \( \phi(0) = 0 \) (\( \phi \) is
called a dimension function or a gauge function), the \( \phi \)-Hausdorff outer
measure of a set \( E \subset \mathbb{R}^d \) is

\[
\mathcal{H}^\phi(E) = \lim_{\varepsilon \to 0} \inf_{r \in R_\varepsilon(E)} \sum_{B \in r} \phi(|B|),
\]

where \( R_\varepsilon(E) \) is the set of (countable) coverings of \( E \) with balls \( B \) of diameter
\( |B| \leq \varepsilon \). When \( \phi(x) = \phi_s(x) = x^s \), we write \( \mathcal{H}^s \) for short instead of \( \mathcal{H}^{\phi_s} \). The
Hausdorff dimension of a set \( E \) is defined by

\[
\dim_{\mathcal{H}}(E) := \sup\{ s > 0; \mathcal{H}^s(E) > 0 \} = \inf\{ s > 0; \mathcal{H}^s(E) = 0 \}.
\]

The first result on the Hausdorff dimension of the divergence sets of Fourier
series is due to J-M. Aubry [2].

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Theorem 1.1. Let $f \in L^p(\mathbb{T})$, $1 < p < +\infty$. If $\beta \geq 0$, define

$$E(\beta, f) = \left\{ x \in \mathbb{T} : \limsup_{n \to +\infty} n^{-\beta}|S_n f(x)| > 0 \right\}.$$ 

Then $\dim_H(E(\beta, f)) \leq 1 - \beta p$. Conversely, given a set $E$ such that $\dim_H(E) < 1 - \beta p$, there exists a function $f \in L^p(\mathbb{T})$ such that $\limsup_{n \to +\infty} n^{-\beta}|S_n f(x)| = +\infty$ for all $x \in E$.

This result motivated us to introduce, in [3], the notion of divergence index. For a given function $f \in L^p(\mathbb{T})$ and a given point $x_0 \in \mathbb{T}$, we define the divergence index $\beta(x_0)$ of the Fourier series of $f$ at point $x_0$ as the infimum of the nonnegative real numbers $\beta$ such that $|S_n f(x_0)| = O(n^\beta)$. An easy consequence of Nikolsky’s inequality [10] and F. Riesz’s theorem [12, Vol. I, p. 266] is that $0 \leq \beta(x_0) \leq 1/p$ for every $f \in L^p(\mathbb{T})$ ($1 \leq p < +\infty$) and $x_0 \in \mathbb{T}$. Moreover, Carleson’s theorem implies that when $p > 1$, $\beta(x_0) = 0$ almost surely. In [3], we gave precise estimates on the size of the level sets of the function $\beta$. These are defined as

$$E(\beta, f) = \{ x \in \mathbb{T} : \beta(x) = \beta \} = \{ x \in \mathbb{T} : \limsup_{n \to +\infty} \frac{\log|S_n f(x)|}{\log n} = \beta \}. $$

Theorem 1.2 ([3]). Let $1 < p < +\infty$. For quasi-all functions $f \in L^p(\mathbb{T})$, $\dim_H(E(\beta, f)) = 1 - \beta p$ for all $\beta \in [0, 1/p]$.

The terminology "quasi-all" used here is relative to the Baire Category Theorem. It means that the property holds for a residual set of functions in $L^p(\mathbb{T})$, and we say that the behaviour of the Fourier series of every function of this residual set is multifractal.

In the case of continuous functions, the situation breaks down dramatically. Letting $(D_n)_{n \geq 0}$ denote the Dirichlet kernel, we first observe that when $f \in \mathcal{C}(\mathbb{T})$,

$$\|S_n f\|_{\infty} \leq \|D_n\|_1 \|f\|_{\infty} \leq C \|f\|_{\infty} \log n.$$ 

This motivated us, in [3], to introduce the level sets

$$F(\beta, f) = \left\{ x \in \mathbb{T} : \limsup_{n \to +\infty} (\log n)^{-\beta}|S_n f(x)| > 0 \right\},$$

$$F(\beta, f) = \left\{ x \in \mathbb{T} : \limsup_{n \to +\infty} \frac{\log|S_n f(x)|}{\log \log n} = \beta \right\}.$$ 

Whereas on $L^p(\mathbb{T})$, $1 < p < +\infty$, the divergence index takes its largest value ($\beta(x) = 1/p$) on small sets, this is far from being the case on $\mathcal{C}(\mathbb{T})$, as the following very surprising result indicates.
Theorem 1.3 ([3]). For quasi-all functions $f \in \mathcal{C}(\mathbb{T})$, $F(\beta, f)$ is non-empty and has Hausdorff dimension 1 for every $\beta \in [0, 1]$.

However, several questions were left open in [3].

Question 1. What happens on $L^1(\mathbb{T})$?

In view of the differences between $L^p(\mathbb{T})$, $p \in (1, +\infty)$, and $\mathcal{C}(\mathbb{T})$, it is not clear what to expect on $L^1(\mathbb{T})$. Moreover, Carleson’s theorem is false on $L^1(\mathbb{T})$ and Kolmogorov’s theorem ensures that there exist functions in $L^1(\mathbb{T})$ with everywhere divergent Fourier series.

The proof of Theorem 1.2 proceeds in two steps. First we build a residual set of functions in $L^p(\mathbb{T})$ such that if $f$ lies in this residual set and if $0 \leq \beta \leq 1/p$, $\dim_H(E(\beta, f)) \geq 1 - \beta p$. Then we use Theorem 1.1 to conclude that $\dim_H(E(\beta, f)) = 1 - \beta p$. The first step works as well in $L^1(\mathbb{T})$; the trouble comes from Aubry’s result, which uses the Carleson-Hunt maximal inequality. In Section 2, we overcome this difficulty by proving a (very weak!) version of Carleson’s maximal inequality in $L^1(\mathbb{T})$ which, nevertheless, is sufficient for the proof of an analogue of Theorem 1.1. In particular, we prove the following result.

Theorem 1.4. For quasi-all functions $f \in L^1(\mathbb{T})$,

$$\dim_H(E(\beta, f)) = 1 - \beta$$

for all $\beta \in [0, 1]$

Question 2. What is the size of the set of functions satisfying the conclusion of Theorem 1.2 and Theorem 1.4?

Theorem 1.2 and Theorem 1.4 say that in $L^p(\mathbb{T})$ ($p \geq 1$), the set of functions for which the Fourier series has a multifractal behaviour is large in a topological sense. One can ask if it remains large from other points of view. We deal here with an infinite-dimensional version of the notion of “almost-everywhere”. This notion, called prevalence, was introduced by J. Christensen in [5] and has been widely studied since then. In multifractal analysis, some properties which hold on a dense $G_\delta$-set are also prevalent (see, for instance, [8] or [7]), whereas others are not (see, for instance, [11] or [8]). This motivated us to examine Theorems 1.2 and 1.4 from this point of view.

Definition 1.5. Let $E$ be a complete metric vector space. A Borel set $A \subset E$ is called Haar-null if there exists a compactly supported probability measure $\mu$ such that $\mu(x + A) = 0$ for every $x \in E$. If this property holds, the measure $\mu$ is said to be transverse to $A$. 
A subset of $E$ is called **Haar-null** if it is contained in a Haar-null Borel set. The complement of a Haar-null set is called a **prevalent** set.

The following statements enumerate important properties of prevalence and show that this notion supplies a natural generalization of “almost every” to infinite-dimensional spaces.

- If $A$ is Haar-null, then $x + A$ is Haar-null for every $x \in E$.
- If $\dim(E) < +\infty$, $A$ is Haar-null if and only if it is negligible with respect to Lebesgue measure.
- Prevalent sets are dense.
- The intersection of a countable collection of prevalent sets is prevalent.
- If $\dim(E) = +\infty$, compact subsets of $E$ are Haar-null.

In Section 3, we prove the following result.

**Theorem 1.6.** Let $1 \leq p < +\infty$. The set of functions $f \in L^p(\mathbb{T})$ such that $\dim_H(E(\beta, f)) = 1 - \beta p$ for all $\beta \in [0, 1/p]$, is prevalent.

Thus, almost every function in $L^p(\mathbb{T})$ is multifractal with respect to the summation of its Fourier series.

**Question 3.** Can we say more about the case $C(\mathbb{T})$?

Theorem 1.3 implies that there exists a residual subset $A \subset C(\mathbb{T})$ such that for $f \in A$ and $\beta < 1$, there exists some $E \subset \mathbb{T}$ with Hausdorff dimension 1 satisfying

$$\limsup_{n \to +\infty} \frac{|S_nf(x)|}{(\log n)^\beta} = +\infty \text{ for any } x \in E.$$ (1)

On the other hand, we know that for any fixed $f \in C(\mathbb{T})$, $\|S_nf\|_\infty$ is negligible compared to $\log n$ and that, conversely, given a sequence $(\delta_n)_{n \geq 2}$ of positive real numbers tending to 0, there exists $f \in C(\mathbb{T})$ such that

$$\limsup_{n \to +\infty} \frac{|S_nf(0)|}{\delta_n \log n} = +\infty.$$ (2)

These last two statements can be found, for example, in [12, Vol I, p. 298]. It thus seems natural to ask whether this last property holds on a set of Hausdorff dimension 1. (By (1), this is the case when $\delta_n = (\log n)^{\beta - 1}$, $0 < \beta < 1$). That it does is the content of the following result.

**Theorem 1.7.** Let $(\delta_n)_{n \geq 2}$ be a sequence of positive real numbers tending to 0. For quasi-all functions $f \in C(\mathbb{T})$, there exists $E \subset \mathbb{T}$ of Hausdorff dimension 1 such that

$$\limsup_{n \to +\infty} \frac{|S_nf(x)|}{\delta_n \log n} = +\infty$$

for all $x \in E$. 

The same result also holds in a prevalent subset of $\mathcal{C}(\mathbb{T})$.

**Theorem 1.8.** Let $\{\delta_n\}_{n \geq 2}$ be a sequence of positive real numbers tending to 0. For almost every function $f \in \mathcal{C}(\mathbb{T})$, there exists $E \subset \mathbb{T}$ of Hausdorff dimension 1 such that

$$\limsup_{n \to +\infty} \frac{|S_n f(x)|}{\delta_n \log n} = +\infty$$

for all $x \in E$.

The proofs of Theorems 1.7 and 1.8 are presented in Section 4.

## 2 Multifractal analysis of the divergence of Fourier series in $L^1(\mathbb{T})$

We first recall some basic facts on Fourier series and Fourier transforms. A function $f \in L^1(\mathbb{T})$ is identified with a 1-periodic function on $\mathbb{R}$. Its **Fourier coefficients** are defined by

$$\hat{f}(k) = \langle f, e_k \rangle = \int_{\mathbb{T}} f(t) \overline{e_k(t)} dt = \int_{\mathbb{T}} f(t) e^{-2i\pi k t} dt,$$

where $e_k(t) = e^{2\pi i kt}, k \in \mathbb{Z}$. The partial sums of its Fourier series are given by

$$S_n f : t \mapsto \sum_{k=-n}^{n} \langle f, e_k \rangle e_k(t).$$

We also write $S_n f = D_n * f$, where

$$D_n(t) = \sum_{k=-n}^{n} e_k(t) = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}$$

is the **Dirichlet kernel**. The Riesz Theorem (see [12, Vol. I, p. 266]) says that when $p > 1$, the projections $\{S_n\}_{n \geq 1}$ are uniformly bounded on $L^p(\mathbb{T})$. This is not necessarily the case for $f \in L^1(\mathbb{T})$. However, it is well known that the $L^1$ norm of the Dirichlet kernel is estimated by $\log n$. It follows that there exists some absolute constant $C > 0$ such that $\|S_n f\|_1 \leq C \log n \|f\|_1$ for all $n \geq 2$ and $f \in L^1(\mathbb{T})$.

We next recall the definition of the $n$-th **Féjer sum** of $f$

$$\sigma_n(f) = \frac{1}{n}(S_0 f + \cdots + S_{n-1} f).$$

Let $\mathcal{E}_n(\mathbb{T}) := S_n(L^1(\mathbb{T}))$ be the set of trigonometric polynomials of degree less than $n$. The classical Nikolsky inequality (see, for example, [10]) says that if $P \in \mathcal{E}_n(\mathbb{T})$ and $1 \leq p \leq q \leq \infty$, then

$$\|P\|_q \leq n^{\frac{1}{p} - \frac{1}{q}} \|P\|_p.$$
Finally, recall that if $f \in L^p(\mathbb{T})$, $p \in [1, +\infty)$, and $q$ is the conjugate exponent (i.e., $\frac{1}{p} + \frac{1}{q} = 1$), then

$$\|S_nf\|_\infty = \|D_n \ast f\|_\infty \leq \|D_n\|_q \times \|f\|_p \leq C_p n^{1/p} \|f\|_p. \quad (3)$$

The **Fourier transform** of a (non-periodic) function $f \in L^1(\mathbb{R})$ is the continuous function

$$\hat{f} : \xi \in \mathbb{R} \mapsto \int_{\mathbb{R}} f(x)\bar{e}_\xi(x)dx,$$

where $e_\xi(t) = e^{2\pi i \xi t}$.

Our first lemma gives bounds on a function which is locally a Dirichlet kernel.

**Lemma 2.1.** There exists a constant $A > 0$ such that

$$\int_T |D_n(x - t)|dx \leq A \log N$$

for all $N \geq 2$, measurable functions $n : \mathbb{T} \rightarrow \{1, \ldots, N\}$, and $t \in \mathbb{T}$.

**Proof.** It is obvious from the above expression for $D_n$ that if $k \leq N$ and $u \in [-1/2, 1/2]$, then

$$|D_k(u)| \leq \begin{cases} CN, \\ C/|u| \end{cases}$$

for some absolute constant $C > 0$. We then split the integral into two parts, obtaining

$$\int_{|x-t| \leq 1/N} |D_n(x - t)|dx \leq 2CN \frac{1}{N}$$

and

$$\int_{1/N < |x-t| \leq 1/2} |D_n(x - t)|dx \leq C \int_{1/N < |x-t| \leq 1/2} \frac{dx}{|x-t|} \leq 2C \log N. \quad \square$$

Writing $S_{n(x)}f(x) = (f \ast D_{n(x)})(x)$ and using Fubini's theorem, we easily deduce the following inequality on partial sums of Fourier series of $L^1$-functions.

**Lemma 2.2.** There exists a constant $A > 0$ such that

$$\int_T |S_{n(x)}f(x)|dx \leq A \log N \|f\|_1$$

for every $N \geq 2$, measurable function $n : \mathbb{T} \rightarrow \{1, \ldots, N\}$, and $f \in L^1(\mathbb{T})$. 

We are now ready to prove the following weak version of the maximal inequality of Carleson and Hunt on $L^1(\mathbb{T})$ [1].

**Corollary 2.3.** For $\alpha > 0$, there exists $C := C_\alpha > 0$ such that

$$\int \sup_{n \geq 2} \frac{|S_n f(x)|}{(\log n)^{1+\alpha}} \, dx \leq C \|f\|_1$$

for all $f \in L^1(\mathbb{T})$.

**Proof.** Using the Monotone Convergence Theorem, we first observe that it is suffices to prove that

$$\int \sup_{2 \leq n \leq N} \frac{|S_n f(x)|}{(\log n)^{1+\alpha}} \, dx \leq C \|f\|_1$$

for all $N \geq 2$, where, of course, $C$ does not depend on $N$. Now we take a measurable function $n : \mathbb{T} \to \mathbb{N}\setminus\{0, 1\}$, not necessarily bounded, and observe that (4) is proved once we show that

$$\int \frac{|S_n f(x)|}{(\log n(x))^{1+\alpha}} \, dx \leq C \|f\|_1$$

for some constant $C$ independent of the function $n$.

If $k \geq 0$, let $A_k = \{x \in \mathbb{T} : 2^k \leq n(x) < 2^{k+1}\}$. Lemma 2.2 ensures that

$$\int |S_{n(x)} f(x)| \, dx = \sum_{k \geq 0} \int_{A_k} |S_{n(x)} f(x)| \, dx$$

$$\leq \sum_{k \geq 0} \frac{1}{(2^k \log 2)^{1+\alpha}} \int_{A_k} |S_{n(x)} f(x)| \, dx$$

$$\leq \sum_{k \geq 0} C \frac{2^{k+1} \log 2}{2^{k(1+\alpha)}(\log 2)^{1+\alpha}} \|f\|_1$$

$$= C_\alpha \|f\|_1.$$ 

□

We also need a periodic function that is well-localized and has rapidly decreasing Fourier coefficients. The following lemma ensures its existence.

**Lemma 2.4.** Let $\gamma \in (0, 1)$. There exist $A, B > 0$ such that given an interval $I$ of length less than 1, there exists a 1-periodic $C^\infty$-function $w_I$ with support in $I + \mathbb{Z}$ satisfying $w_I(0) = 1$, $0 \leq w_I \leq 1$, and $\sum_{|n| \geq \lambda |I|^{-1}} |\hat{w}_I(n)| \leq Ae^{-B\lambda^\gamma}$ for all $\lambda \geq 2$. 

Proof. A classical result in Fourier analysis (see, e.g., [2, Lemma 6]), there exists a $C^\infty$-function $w : \mathbb{R} \to \mathbb{R}$ supported in $[-1, 1]$ and satisfying $0 \leq w \leq 1$, $w(0) = 1$ and

$$|\hat{w}(\xi)| \leq De^{-E|\xi|^\gamma} \quad \text{for all } \xi \in \mathbb{R},$$

for some $D, E > 0$. Define the 1-periodic function $w_I$ by

$$w_I(x) = \sum_{k \in \mathbb{Z}} w\left(\frac{x - k}{|I|}\right).$$

A familiar calculation shows that the Fourier coefficients of the periodic function $w_I$ are given by $\hat{w}_I(n) = |I|\hat{w}(n|I|)$. Then

$$\sum_{|n| \geq \lambda|I|^{-1}} |\hat{w}_I(n)| \leq 2|I| \sum_{n \geq \lambda|I|^{-1}} De^{-En^\gamma|I|^\gamma} \leq 2D \int_{\lambda}^{+\infty} e^{-Et} dt.$$

Observe that

$$\int_{X}^{+\infty} e^{-Et} dt = \frac{1}{\gamma} \int_{X^\gamma}^{+\infty} e^{-Eu} u^{(1/\gamma)-1} du \leq Ce^{-(E/2)X^\gamma}.$$

Theis yields the result easily. □

The next lemma is inspired by Aubry’s paper. It asserts that if a trigonometric polynomial is large at $a \in \mathbb{T}$, it is also large in small intervals around $a$, and with rather good control of the $L^p$-norm of the polynomial.

Lemma 2.5. Let $p \geq 1$ and $\varepsilon > 0$. There exists $\delta > 0$ such that for $P \in E_n(\mathbb{T})$ for sufficiently large $n$ and $a \in \mathbb{T}$ such that $|P(a)| \geq \|P\|_p$,

$$\|P\|_{L^p(I)} \geq \delta |P(a)| \times |I|^{1/p} \times (\log n)^{-(1+\varepsilon)/p}$$

on every interval $I$ centered at $a$ with length $|I| \leq 1/n$.

Remarks. - Such an $a$ exists because $P$ is continuous.

- In the context of the $L^\infty$-norm, Bernstein’s inequality says if $a$ is such that $|P(a)| \geq \|P\|_\infty$ and $x \in I$, then

$$|P(x)| \geq |P(a)| - n\|P\|_\infty |x - a| \geq \frac{1}{2}|P(a)|.$$ 

In Lemma 2.5, we try to find a similar bound in terms of the $L^p$-norm.

- In fact, we need Lemma 2.5 only in the case $p = 1$; but, for completeness, we prove it for the general case.
\textbf{Proof.} Without loss of generality, we may assume that \( a = 0 \). The idea is to localize \( P \) around \( 0 + \mathbb{Z} \) and to use the Nikolsky inequality to estimate the \( L^p \)-norm from the \( L^\infty \)-norm.

Let \( \gamma \in (0, 1) \) be such that \( \gamma (1 + \varepsilon) > 1 \). Let \( w_I \) be given by Lemma 2.4 for this value of \( \gamma \). Decompose \( P w_I \) as \( f_1 + f_2 \), where \( f_1 = S_N P w_I \) and \( N = \lfloor |I|^{-1} (\log n)^{1+\varepsilon} \rfloor \), (the integer part of \( |I|^{-1} (\log n)^{1+\varepsilon} \)).

On the one hand, if \( p \geq 1 \),
\[
\| f_1 \|_\infty \leq C_p |I|^{-1/p} (\log n)^{(1+\varepsilon)/p} \| P w_I \|_p \quad \text{(Inequality (3))}
\leq C_p |I|^{-1/p} (\log n)^{(1+\varepsilon)/p} \| P \|_{L^p(I)} \quad (0 \leq w_I \leq 1).
\]

On the other hand,
\[
f_2(x) = \sum_{|k| > N} \hat{P} w_I (k) e^{2i\pi k x}.
\]
To obtain an upper bound for \( \| f_2 \|_\infty \), we need to estimate the Fourier coefficients of \( P w_I \). We know that
\[
\hat{P} w_I (k) = \hat{P} \ast \hat{w}_I (k) = \sum_{j=-n}^{n} \hat{P} (j) \hat{w}_I (k-j),
\]
so that
\[
\| f_2 \|_\infty \leq \sum_{j=-n}^{n} |\hat{P} (j)| \sum_{|k| > N} |\hat{w}_I (k-j)|.
\]
Now, since \( |I| \leq 1/n \), for large enough \( n \) and \( |j| \leq n \),
\[
\sum_{|k| > N} |\hat{w}_I (k-j)| \leq \sum_{|k| > |I|^{-1} (\log n)^{1+\varepsilon}/2} \hat{w}_I (k) \leq Ae^{-B (\log n)^{1+\varepsilon}/2} \leq C n^{-2}.
\]
This implies (for large enough \( n \))
\[
\| f_2 \|_\infty \leq C n^{-2} \sum_{j=-n}^{n} |\hat{P} (j)| \leq C n^{-2} (2n+1) \| P \|_1 \leq C n^{-2} (2n+1) \| P \|_p \leq \frac{1}{2} \| P \|_p.
\]

Recalling that \( |P(0)| \geq \| P \|_p \) and \( P(0) = f_1(0) + f_2(0) \), we get
\[
\| f_1 \|_\infty \geq |P(0)| - \| f_2 \|_\infty \geq \frac{1}{2} |P(0)|,
\]
and the result follows from the above estimates of \( \| f_1 \|_\infty \).

We conclude by proving the following result (Proposition 2.6) and its corollary (Corollary 2.7) on the Hausdorff dimension of \( E(\beta, f) \). This is all that is needed to obtain Theorem 1.4 since the construction in [3] is always valid when \( p = 1 \) and shows that there exists a residual set of functions \( f \in L^1 (\mathbb{T}) \) with \( \dim_H (E(\beta, f)) \geq 1 - \beta \) for all \( \beta \in [0, 1] \).
Proposition 2.6. Let \( f \in L^1(\mathbb{T}) \) and \( \tau : (0, +\infty) \to (0, +\infty) \) be an increasing function. Define

\[
E(\tau, f) := \left\{ x \in \mathbb{T} : \limsup_{n \to +\infty} \frac{|S_nf(x)|}{\tau(n)} = +\infty \right\}.
\]

Then

\[
\mathcal{H}^{\phi}(E(\tau, f)) = 0
\]

for all \( \nu > 2 \) and dimension functions \( \phi \) satisfying

\[
c_1 s \leq \phi(s) \leq c_2 s \tau(s^{-1}) \log(s^{-1})^\nu.
\]

Proof. Let \( M > 0 \) and \( \epsilon = \nu - 2 \). Define

\[
E_M(\tau, f) = \left\{ x \in \mathbb{T} : \limsup_{n \to +\infty} \frac{|S_nf(x)|}{\tau(n)} > M \right\}.
\]

If \( x \in E_M(\tau, f) \), there exist arbitrarily large \( n_x \) such that \( |S_{n_x}f(x)| \geq M\tau(n_x) \). Set

\[
I_x = \left[ x - \frac{1}{2n_x}, x + \frac{1}{2n_x} \right]
\]

and observe that \( \|S_{n_x}f\|_1 \leq C(\log n_x) \). The hypothesis on the function \( \tau \) implies that \( \tau(n) \gg \log n \). It follows that \( \|S_{n_x}f\|_1 \leq |S_{n_x}f(x)| \) for sufficiently large \( n_x \). We can then apply Lemma 2.5 and get

\[
\|S_{n_x}f\|_{L^1(I_x)} \geq \frac{\delta}{n_x(\log n_x)^{1+\epsilon/2}} \cdot
\]

Now \((I_x)_{x \in E_M(\tau, f)}\) is a covering of \( E_M(\tau, f) \), and we can extract a Vitali covering, i.e., a countable family of disjoint intervals \((I_i)_{i \in \mathbb{N}}\) of length \( 1/n_i \) such that \( E_M(\tau, f) \subset \bigcup_{i \in \mathbb{N}} 5B_i \). Corollary 2.3 then implies

\[
C\|f\|_1 \geq \int \sup_{n \geq 2} \frac{|S_nf(x)|}{(\log n)^{1+\epsilon/2}} dx \geq \sum_i \int \frac{|S_{n_i}f(x)|}{(\log n_i)^{1+\epsilon/2}} dx \geq \delta M \sum_i \frac{|I_i|\tau(1/|I_i|)}{(\log(1/|I_i|))^{2+\epsilon}}.
\]

This yields \( \sum_i \phi(5|I_i|) \leq C\|f\|_1/\delta M \) (recall that \( \tau \) is increasing), where \( C \) is another absolute constant and \( M > 0 \) arbitrarily large. Hence, \( \mathcal{H}^{\phi}(E_M(\tau, f)) \leq C\|f\|_1/\delta M \) (the length of the intervals of the covering can be arbitrarily small). This, in turn, implies \( \mathcal{H}^{\phi}(E(\tau, f)) = 0 \), since \( E(\tau, f) = \bigcap_{M>0} E_M(\tau, f) \). \( \square \)

Applying the previous proposition to \( \tau(s) = s^\beta \) and \( \phi(s) = s^{1-\beta}/\log(s^{-1})^3 \), we get the following corollary.

Corollary 2.7. For all \( f \in L^1(\mathbb{T}) \) and \( \beta \in [0, 1] \),

\[
\dim_{\mathcal{H}}(E(\beta, f)) \leq 1 - \beta.
\]

3 Prevalence of multifractal behaviour

Throughout this section, \( p \) is a fixed real number greater than or equal to 1.
3.1 Strategy. In the proof that a certain set $A \subset E$ is Haar-null, Lebesgue measure on the unit ball of a finite-dimensional subspace $V$ often plays the role of the transverse measure. More precisely, if there exists a finite-dimensional subspace $V$ of $E$ such that, for all $x \in E$, $V \cap (x + A)$ has full Lebesgue-measure, then $A$ is prevalent. Such a finite-dimensional subspace $V$ is called a probe for $A$. Equivalently, $V$ is a probe for $A$ if and only if for all $x \in E$, $(x + V) \cap A$ has full Lebesgue-measure.

This last definition of a probe is what we use to prove prevalence. More precisely, we first prove that for fixed $\beta \in [0, 1/p]$, the set of functions $f$ in $L^p(\mathbb{T})$ satisfying $\dim_H(E(\beta, f)) = 1 - \beta p$ is prevalent. Then, because a countable intersection of prevalent sets is prevalent, we obtain the desired conclusion.

3.2 The construction of saturating functions with disjoint spectra. In this subsection, $\alpha > 1$ is fixed. For $j \geq 1$, define $J = \lfloor j/\alpha \rfloor + 1$, which, for large enough $j$, (say, $j \geq j_\alpha$), is smaller than $j - 2$. For $0 \leq K \leq 2^J - 1$, define the dyadic intervals

$$I_{K,j} := \left[ \frac{K}{2^j} - \frac{1}{2^j}, \frac{K}{2^j} + \frac{1}{2^j} \right].$$

Also define

$$I_j := \bigcup_{K=0}^{2^j-1} I_{K,j} \quad \text{and} \quad I'_j := \bigcup_{K=0}^{2^j-1} 2I_{K,j}.$$

The condition $j \geq j_\alpha$ ensures that the intervals $2I_{K,j}$ do not overlap. Finally, we introduce the set of real numbers $D_\alpha$ in $[0, 1]$ which are $\alpha$-approximable by dyadics, i.e., $x \in [0, 1]$ belongs to $D_\alpha$ if there exist two sequences of integers $(k_n)_{n \geq 0}$ and $(j_n)_{n \geq 0}$ such that $|x - k_n/2^{j_n}| \leq 1/2^{\alpha j_n}$. It is easy to verify that $D_\alpha \subset \lim \sup_{j \to +\infty} I_j$. Indeed, let $x \in D_\alpha$. There exist $J$ as large as we like and $K$ such that $|x - K/2^J| \leq 1/2^{\alpha J}$. Let $j$ be an integer such that $J - 1 = [j/\alpha]$ (such an integer exists because $\alpha \geq 1$). We get

$$\left| x - \frac{K}{2^J} \right| \leq \frac{1}{2^j}.$$ 

Finally, $x \in I_j$. Furthermore, it is well known (see, for instance, [4] and the mass transference principle) that $\dim_H(D_\alpha) = 1/\alpha$ and even that $\mathcal{H}^{1/\alpha}(D_\alpha) = +\infty$. It follows that

$$\dim_{\mathcal{H}} \left( \lim \sup_{j \to +\infty} I_j \right) \geq \frac{1}{\alpha}.$$

Our goal is to build finite families of functions which behave badly on each $I_j$ and have disjoint spectra. The starting point is a modification of the basic construction of [3].
Lemma 3.1. Let \( j \geq j_\alpha \) and \( J = [j/\alpha] + 1 \). There exists a trigonometric polynomial \( P_j \) with spectrum contained in \((0, 2^{j+1} - 1]\) such that

- \( \|P_j\|_p \leq 1 \),
- \( |P_j(x)| \geq C 2^{-(J-j)/p} \) for any \( x \in I_j \),

where the constant \( C \) is independent of \( j \).

**Proof.** Let \( \chi_j \) be a continuous piecewise linear function equal to 1 on \( I_j \), equal to 0 outside \( I_j' \) and satisfying \( 0 \leq \chi_j \leq 1 \) and \( \|\chi'_j\|_\infty \leq 2 \). Define \( P_j \) by

\[
P_j := 2^{-(J-j+2)/p} e^{2\sigma_j \chi_j}.
\]

Clearly, the \( L^p \)-norm of \( P_j \) is less than or equal to 1 (observe that the measure of \( I_j' \) is \( 2^{J-j+2} \)). Applying [3, Lemma 1.7] to \( 1 - \chi_j \), we find that \( \sigma_j \chi_j(x) \geq 1/4 \) for any \( x \in I_j \). This gives the second assertion of the lemma. \( \square \)

We now collapse these polynomials to get as many saturating functions as necessary with disjoint spectra.

Lemma 3.2. Let \( s \geq 1 \). There exist functions \( g_1, \ldots, g_s \) in \( L^p(\mathbb{T}) \), sequences of integers \( (n_{j,r})_{j \geq j_\alpha, 1 \leq r \leq s}, (m_{j,r})_{j \geq j_\alpha, 1 \leq r \leq s} \) and a constant \( C > 0 \) satisfying

- \( 1 \leq m_{j,r} < n_{j,r} \leq C2^j \) for all \( j \) and \( r \);
- for all \( j \geq j_\alpha, x \in I_j \), and \( r \in \{1, \ldots, s\} \),

\[
|S_{n_{j,r}} g_r(x) - S_{m_{j,r}} g_r(x)| \geq C \frac{1}{j^2} 2^{j-j/p};
\]

- for all \( r \in \{1, \ldots, s\} \), the spectrum of \( g_r \) is included in \( \bigcup_{j \geq j_\alpha} (m_{j,r}, n_{j,r}] =: G_r \), where the intervals \( (m_{j,r}, n_{j,r}] \) are disjoint.
- if \( r_1 \neq r_2 \), \( G_{r_1} \cap G_{r_2} = \emptyset \).

**Proof.** For \( r \in \{1, \ldots, s\} \), \( g_r := \sum_{j \geq j_\alpha} \frac{1}{j} e^{(s+r)2j/j} P_j \). Define

\[
m_{j,r} := (s+r)2^{j+1} - (2^{j+1} - 1),
\]

\[
n_{j,r} := (s+r)2^{j+1} + (2^{j+1} - 1),
\]

so that each \( g_r \) belongs to \( L^p \) with spectrum included in \( \bigcup_{j \geq j_\alpha} (m_{j,r}, n_{j,r}] \). Moreover, the intervals \( (m_{j,r}, n_{j,r}] \) are disjoint, and so

\[
|S_{n_{j,r}} g_r - S_{m_{j,r}} g_r| = \frac{1}{j^2} |P_j|.
\]

To complete the proof, observe that \( n_{j,r} < m_{j+1,r} \) and \( n_{j,s} < m_{j+1,1} \) for all \( j \geq j_\alpha, r < s \), and so the sets \( G_1, \ldots, G_s \) are disjoint. \( \square \)
It is easy to show that if \( x \in \limsup I_r \), \( r \in \{1, \ldots, s\} \) and \( \beta < \frac{1}{p} (1 - \frac{1}{a}) \), then
\[
\limsup_{n \to +\infty} \frac{|S_n g_r(x)|}{n^\beta} = +\infty.
\]
In some sense, the functions \( g_r \) have the worst possible behaviour on \( I_j \); keep in mind that they have to belong to \( L^p(\mathbb{T}) \). We now show that this property holds almost everywhere (in the sense of the Lebesgue measure) on every affine subspace \( f + \text{span}(g_1, \ldots, g_s) \), provided that \( s \) is large enough. This is the main step toward the proof of Theorem 1.6.

### 3.3 Prevalence of divergence for a fixed divergence index.
We retain the notation of the previous subsection.

**Proposition 3.3.** Let \( 0 < \beta < \frac{1}{p} (1 - \frac{1}{a}) \). There exists \( s \geq 1 \) such that for every \( f \in L^p(\mathbb{T}) \), for almost every \( c = (c_1, \ldots, c_s) \in \mathbb{R}^s \), the function \( g = f + c_1 g_1 + \cdots + c_s g_s \) satisfies
\[
\limsup_{n \to +\infty} \frac{|S_n g(x)|}{n^\beta} = +\infty
\]
for every \( x \in D_\alpha \).

**Proof.** Set \( \varepsilon = \frac{1}{p} (1 - \frac{1}{a}) - \beta \). Let \( s > 4/\varepsilon \), and let \( f \) be an arbitrary function in \( L^p(\mathbb{T}) \). We prove the conclusion of the proposition for \( s \) and every \( x \in \limsup I_j \) (recall that \( D_\alpha \subset \limsup I_j \)).

Let \( M > 0 \) and
\[
S_M := \left\{ g \in L^p(\mathbb{T}) : \exists x \in \limsup I_j \ni \forall n \geq 1, \ |S_n g(x)| \leq Mn^\beta \right\}.
\]
It suffices to show that for every \( R > 0 \), the set of \( c \in \mathbb{R}^s \) satisfying \( \|c\|_\infty \leq R \) and such that \( f + c_1 g_1 + \cdots + c_s g_s \) belongs to \( S_M \) has Lebesgue measure 0. In the sequel, we fix the values of \( M \) and \( R \).

For \( j \geq 1 \), we split each interval \( I_{K,j} \) into \( 2^j \) subintervals. Each interval has length \( 2^{-2j+1} \), and we get \( 2^{j+1} \) intervals \( O_{l,j} \) with \( \bigcup_{l=1}^{2^{j+1}} O_{l,j} = I_j \). For \( j \geq 1 \), \( l \in \{1, \ldots, 2^{j+1}\} \), set
\[
S_M^{(l,j)} := \left\{ g \in L^p(\mathbb{T}) : \exists x \in O_{l,j} \ni \forall n \geq 1, \ |S_n g(x)| \leq Mn^\beta \right\}.
\]
Clearly, \( S_M \subset \limsup_{j \to +\infty} \bigcup_{l=1}^{2^{j+1}} S_M^{(l,j)} \). We first control the size of the set
\[
V = \{ c = (c_1, \ldots, c_s) \in \mathbb{R}^s : \|c\|_\infty \leq R \text{ and } f + c_1 g_1 + \cdots + c_s g_s \in S_M^{(l,j)} \}.
\]
Denote by $\lambda_\varepsilon$ Lebesgue measure on $\mathbb{R}^\varepsilon$ and fix $j \geq j_a$, $l$ in $\{1, \ldots, 2^{j+1}\}$ and $c$, $c_0$ in $\mathbb{R}^\varepsilon$ such that $\|c\|_\infty \leq R$, $\|c_0\|_\infty \leq R$ and 
\[
\begin{align*}
  f + c_1 g_1 + \cdots + c_s g_s &\in S_M^{(l, j)}, \\
  f + c_0 g_1 + \cdots + c_0^0 g_s &\in S_M^{(l, j)}.
\end{align*}
\]

The goal is to find an upper bound for $\|c - c_0\|_\infty$.

Let $r \in \{1, \ldots, s\}$ and apply the definition of $S_M^{(l, j)}$ with $n = n_{j, r}$ and $n = m_{j, r}$.
Since the sets $(G_i)_{i \neq r}$ are disjoint from $G_r$, there exists $x \in O_{l, j}$ such that
\[
|S_{n_{j, r}} f(x) - S_{m_{j, r}} f(x) + c_r (S_{n_{j, r}} g_r(x) - S_{m_{j, r}} g_r(x))| \leq M n_{j, r}^\beta + M m_{j, r}^\beta \leq 2CM 2^{\beta j}.
\]

Similarly, there exists $y \in O_{l, j}$ such that
\[
|S_{n_{j, r}} f(y) - S_{m_{j, r}} f(y) + c_0 (S_{n_{j, r}} g_r(y) - S_{m_{j, r}} g_r(y))| \leq 2CM 2^{\beta j}.
\]

Using the triangle inequality, we get
\[
|c_r (S_{n_{j, r}} g_r(x) - S_{m_{j, r}} g_r(x)) - c_0^0 (S_{n_{j, r}} g_r(y) - S_{m_{j, r}} g_r(y))| \leq 4CM 2^{\beta j} + |S_{n_{j, r}} f(x) - S_{n_{j, r}} f(y)| + |S_{m_{j, r}} f(x) - S_{m_{j, r}} f(y)|.
\]

Now, combining the norm of the Riesz projection, Nikolsky’s inequality, and Bernstein’s inequality, we have
\[
\|(S_n f)'\|_\infty \leq C (\log n) n^{1+1/p} \|f\|_p
\]
(the factor $\log n$ disappears when $p > 1$). This yields
\[
|S_{n_{j, r}} f(x) - S_{n_{j, r}} f(y)| \leq C \log(n_{j, r}) n_{j, r}^{1+1/p} |x - y| \|f\|_p \leq C j 2^{j(1+1/p)} 2^{-2j+1} \|f\|_p \ll 2^{\beta j}.
\]

Similarly, $|S_{m_{j, r}} f(x) - S_{m_{j, r}} f(y)| \ll 2^{\beta j}$, and we get
\[
|c_r (S_{n_{j, r}} g_r(x) - S_{m_{j, r}} g_r(x)) - c_r^0 (S_{n_{j, r}} g_r(y) - S_{m_{j, r}} g_r(y))| \leq \kappa 2^{\beta j}
\]
for some constant $\kappa$ depending on $M$ and $\|f\|_p$ but not on $j$. Similarly,
\[
\|(S_n g_r)'\|_\infty \leq C (\log n) n^{1+1/p} \|g_r\|_p \leq C (\log n) n^{1+1/p}.
\]

It follows that
\[
|c_r^0 ((S_{n_{j, r}} g_r(x) - S_{m_{j, r}} g_r(x)) - (S_{n_{j, r}} g_r(y) - S_{m_{j, r}} g_r(y)))| \leq CR j 2^{j(1+1/p)} 2^{-2j+1} \ll 2^{\beta j}.
\]
Combining this with (6), we obtain a new constant $\kappa$, depending on $M$, $\|f\|_p$ and $R$ but not on $j$, such that

\[
(c_r - c_r^0)(S_{n,j}g_r(x) - S_{m,j}g_r(x)) \leq \kappa 2^{\beta j}.
\]

Dividing (7) by $|S_{n,j}g_r(x) - S_{m,j}g_r(x)|$ (which is not equal to 0), we find

\[
|c_r - c_r^0| \leq \frac{\kappa 2^{\beta j} |S_{n,j}g_r(x) - S_{m,j}g_r(x)|^{-1}}{C} 2^{\beta j} j^2 2^{-(j-j)/p} \leq \frac{\kappa 2^{1/p}}{C} j^2 2^{-\epsilon j} \leq 2^{-\epsilon j/2},
\]

provided $j$ is large enough. Thus, $V$ is contained in a ball (for the $l_\infty$ norm) of radius $2^{-\epsilon j/2}$. Taking the $s$-dimensional Lebesgue measure, we obtain $\Lambda(V) \leq 2^s 2^{-\epsilon s j/2}$. This, in turn, gives

\[
\lambda_s \left( \{ c \in \mathbb{R}^s : \|c\|_\infty \leq R \text{ and } f + c_1g_1 + \cdots + c_sg_s \in \bigcup_{l=1}^{2^j} S_{M}^{(l,j)} \} \right) \leq 2^s 2^s j^2 2^{-\epsilon s j/2}.
\]

Since $\epsilon s/2 > 2$, this last quantity is the general term of a convergent series. Recall that $S_M \subset \lim sup_{j \to +\infty} \bigcup_{l=1}^{2^j} S_{M}^{(l,j)}$. The conclusion of Proposition 3.3 then follows from the Borel-Cantelli Lemma.

\[ \square \]

**Corollary 3.4.** Let $\alpha > 1$. For almost every function $f$ in $L^p(\mathbb{T})$, for every $x \in D_\alpha$,

\[
\limsup_{n \to +\infty} \frac{\log |S_nf(x)|}{\log n} \geq \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right).
\]

**Proof.** This follows immediately from Proposition 3.3 by taking a sequence $(\beta_n)$ increasing to $\frac{1}{p} \left( 1 - \frac{1}{\alpha} \right)$ and using the fact that a countable intersection of prevalent sets remains prevalent. \[ \square \]

### 3.4 The general case.

We are now able to complete the proof of Theorem 1.6, i.e., to prove that almost every function $f \in L^p(\mathbb{T})$ in the sense of prevalence has a multifractal behaviour with respect to the summation of its Fourier series. Indeed, let $\{a_k\}_{k \geq 0}$ be a dense sequence in $(1, +\infty)$. By Corollary 3.4, for almost every function $f \in L^p(\mathbb{T})$, for every $k \in \mathbb{N}$ and every $x \in D_{a_k}$,

\[
\limsup_{n \to +\infty} \frac{\log |S_nf(x)|}{\log n} \geq \frac{1}{p} \left( 1 - \frac{1}{a_k} \right).
\]

Now, let $\alpha > 1$, and consider a subsequence $\{a_{\phi(k)}\}_{k \geq 0}$ which increases to $\alpha$. Then $D_\alpha \subset \bigcap_{k \geq 0} D_{a_{\phi(k)}}$, and

\[
\limsup_{n \to +\infty} \frac{\log |S_nf(x)|}{\log n} \geq \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right).
\]
for every \( x \in D_\alpha \). The rest of the proof is now exactly as in [3]. For completeness, we provide full details. Define

\[
D_\alpha^1 = \left\{ x \in D_\alpha : \limsup_{n \to +\infty} \frac{\log |S_n f(x)|}{\log n} = \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right) \right\}
\]

\[
D_\alpha^2 = \left\{ x \in D_\alpha : \limsup_{n \to +\infty} \frac{\log |S_n f(x)|}{\log n} > \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right) \right\},
\]

so that \( \mathcal{H}^{1/\alpha}(D_\alpha^1 \cup D_\alpha^2) = \mathcal{H}^{1/\alpha}(D_\alpha) = +\infty \). It suffices to prove that \( \mathcal{H}^{1/\alpha}(D_\alpha^2) = 0 \).

Let \( \{ \beta_n \}_{n \geq 0} \) be a sequence such that

\[
\beta_n > \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right) \quad \text{and} \quad \lim_{n \to +\infty} \beta_n = \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right).
\]

Observe that \( D_\alpha^2 \subset \bigcup_{n \geq 0} \mathcal{E}(\beta_n, f) \). Moreover, Theorem 1.1 for \( p > 1 \) and Corollary 2.7 for \( p = 1 \) give \( \mathcal{H}^{1/\alpha}(\mathcal{E}(\beta_n, f)) = 0 \) for all \( n \). Hence, \( \mathcal{H}^{1/\alpha}(D_\alpha^2) = 0 \) and \( \mathcal{H}^{1/\alpha}(D_\alpha^1) = +\infty \), which proves that

\[
\dim_{\mathcal{H}} \left( \mathcal{E} \left( \frac{1}{p} \left( 1 - \frac{1}{\alpha} \right), f \right) \right) \geq \frac{1}{\alpha}.
\]

By Theorem 1.1 and Corollary 2.7 again, this inequality is an equality. Finally, such an \( f \) satisfies the conclusion of Theorem 1.6 with \( 1/\alpha = 1 - \beta p \).

### 4 Rapid divergence on big sets for Fourier series of continuous functions

This section is devoted to the proof of Theorems 1.7 and 1.8. We need to construct functions in \( C(T) \) for which the Fourier series behave badly on a set with Hausdorff dimension 1. We construct these functions by blocks. For \( k \geq 1 \) and \( \omega > 1 \), set

\[
J^\omega_k := \bigcup_{j=0}^{k-1} \left[ \frac{j}{k} - \frac{1}{2\omega k}, \frac{j}{k} + \frac{1}{2\omega k} \right],
\]

which is viewed as a subset of \( T = \mathbb{R}/\mathbb{Z} \). The construction makes use of holomorphic functions; so we also view \( T \) as the boundary of the unit disk \( D \) and \( J^\omega_k \) as a part of \( \partial D \).

**Lemma 4.1.** There exist three absolute constants \( C_1, C_2, C_3 > 0 \) such that for each \( k \geq 3 \) and \( \omega \geq \log k \), there exists a function \( f \) which is holomorphic in a
neighbourhood of \( \overline{\mathbb{D}} \) and satisfies

\[
\Re f(z) \geq \frac{C_1}{\omega k}
\]

for all \( z \in \overline{\mathbb{D}} \),

\[
|f(z)| \geq C_2 \omega
\]

for all \( z \in J^o_k \),

\[
|f(z)| \leq C_3 \omega
\]

for all \( z \in \mathbb{T} \),

\[
\frac{|f'(z)|}{f(z)} \leq \omega k
\]

for all \( z \in \mathbb{T} \).

**Proof.** Set

\[
\varepsilon = \frac{1}{\omega k}; \quad z_j = e^{\frac{2\pi i j}{k}}, \quad j = 0, \ldots, k - 1;
\]

\[
f(z) = \frac{1}{k} \sum_{j=0}^{k-1} \frac{1 + \varepsilon}{1 + \varepsilon - z_j z}.
\]

We claim that \( f \) is the function we are seeking.

Indeed, for \( z \in \overline{\mathbb{D}} \) and \( j \in \{0, \ldots, k - 1\} \),

\[
\Re \left( \frac{1 + \varepsilon}{1 + \varepsilon - z_j z} \right) = \frac{1 + \varepsilon}{|1 + \varepsilon - z_j z|^2} \Re(1 + \varepsilon - z_j z) \geq \frac{1 + \varepsilon}{(2 + \varepsilon)^2} \times \varepsilon \geq C_1 \varepsilon,
\]

which proves (8). To prove (9), we may assume that \( z = e^{2\pi i \theta} \) with \( \theta \in \left[ -\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right] \).

Then

\[
\Re \left( \frac{1 + \varepsilon}{1 + \varepsilon - z_0 z} \right) = \frac{1 + \varepsilon}{|1 + \varepsilon - z|^2} \Re(1 + \varepsilon - z) \geq \frac{C_2}{\varepsilon}.
\]

Moreover, by (12), \( \Re ((1 + \varepsilon)(1 + \varepsilon - z_j z)^{-1}) \geq 0 \) for all \( j \). It follows that

\[
\Re f(z) \geq \frac{C_2}{k \varepsilon} = C_2 \omega.
\]

In order to prove (10), pick arbitrary \( z = e^{2\pi i \theta} \in \mathbb{T} \). In what follows, \( C > 0 \) is a constant whose value in different lines may be different. By symmetry, we may assume that \( |\theta| \leq 1/2k \). Then

\[
\left| \frac{1 + \varepsilon}{1 + \varepsilon - z_0 z} \right| \leq \frac{C}{\varepsilon}.
\]

Now, for each \( j \in \{1, \ldots, k/4\} \), we can write

\[
|1 + \varepsilon - z_j z| \geq |\Im(z_j z)| \geq \sin \left( \frac{2\pi j}{k} - 2\pi \theta \right) \geq \frac{2}{\pi} \times 2\pi \left( \frac{j}{k} - \theta \right) \geq \frac{4}{k} \left( j - \frac{1}{2} \right).
\]
Hence
\[
\left| \sum_{j=1}^{k/4} \frac{1 + \varepsilon}{1 + \varepsilon - \frac{z}{j^2}} \right| \leq k(1 + \varepsilon) \sum_{j=1}^{k/4} \frac{1}{j - 1/2} \leq Ck \log k.
\]
Similarly,
\[
\left| \sum_{j=3k/4}^{k-1} \frac{1 + \varepsilon}{1 + \varepsilon - \frac{z}{j^2}} \right| \leq Ck \log k.
\]
If \( j \in [k/4, 3k/4] \), then \( |1 + \varepsilon - \frac{z}{j^2}| \geq C \); thus
\[
\left| \sum_{j=k/4}^{3k/4} \frac{1 + \varepsilon}{1 + \varepsilon - \frac{z}{j^2}} \right| \leq Ck,
\]
Putting this all together, we get
\[
|f(z)| = \left| \frac{1}{k} \sum_{j=0}^{k-1} \frac{1 + \varepsilon}{1 + \varepsilon - \frac{z}{j^2}} \right| \leq C \left( \frac{1}{k \varepsilon} + \log k + 1 \right) \leq C\varepsilon \omega,
\]
which proves (10). (This is where we need \( \omega \geq \log k \).)

Finally, it remains to prove (11). Observe that
\[
\frac{f'(z)}{f(z)} = \frac{\sum_{j=0}^{k-1} \frac{1}{(1 + \varepsilon - \frac{z}{j^2})^2}}{\sum_{j=0}^{k-1} \frac{1}{1 + \varepsilon - \frac{z}{j^2}}}.\n\]
We deduce that
\[
\left| \frac{f'(z)}{f(z)} \right| \leq \frac{\sum_{j=0}^{k-1} \frac{1}{(1 + \varepsilon - \frac{z}{j^2})^2}}{\sum_{j=0}^{k-1} \frac{\Re(1 + \varepsilon - \frac{z}{j^2})}{1 + \varepsilon - \frac{z}{j^2}}^2} \leq \frac{\sum_{j=0}^{k-1} \frac{1}{(1 + \varepsilon - \frac{z}{j^2})^2}}{\sum_{j=0}^{k-1} \frac{\varepsilon}{1 + \varepsilon - \frac{z}{j^2}}^2} \leq \frac{1}{\varepsilon} = \omega k.
\]

The crucial step for Theorems 1.7 and 1.8 is given by the following lemma.

**Lemma 4.2.** Let \( \{\varepsilon_n\}_{n \geq 1} \) be a sequence of positive real numbers decreasing to 0. Then, for sufficiently large \( n \), there exist an integer \( k_n \), a real number \( \omega_n > 1 \), and a trigonometric polynomial \( P_n \) with spectrum in \( [1, 2n - 1] \) such that

- \( ||P_n||_{\infty} \leq 1; \)
- \( \text{For all } x \in J_n^{\omega_n}, \ |S_nP_n(x)| \geq \varepsilon_n \log n. \)

Moreover, \( \{k_n\} \) and \( \omega_n \) can be chosen in such a way that \( k_n \to +\infty \) as \( n \to \infty \) and \( \omega_n = o(k_n^\alpha) \) for all \( \alpha > 0 \).

**Proof.** It is clear that the conclusion of the lemma is more difficult to obtain when the sequence \( \{\varepsilon_n\} \) consists of large terms. Thus, we may assume that
\[
\varepsilon_n \geq \frac{\log \log n}{4\pi \log n};
\]
in particular, \( \varepsilon_n \log n \to \infty \) as \( n \to \infty \). Define \( k_n \) and \( \omega_n \) by
\[ \omega_n = \exp(4\pi (\log n) \varepsilon_n), \]

- \( k_n \) is the largest integer \( k \) satisfying \( 2\pi k \omega_n \leq n \).

Observe that \( \omega_n \geq \log n \) and \( \omega_n = o(n^\alpha) \) for all \( \alpha > 0 \). Thus the inequalities

\[ 2\pi k_n \omega_n \leq n \leq 2\pi (k_n + 1) \omega_n \]

ensure that \( k_n \leq n \leq Ck_n n^{1/2} \) for sufficiently large \( n \). It follows that \( k_n \to +\infty \) as \( n \to \infty \), \( \omega_n \geq \log k_n \), and \( \omega_n = o(k_n^\alpha) \) for all \( \alpha > 0 \).

Let \( f_n \) be the holomorphic function given by Lemma 4.1 for the values \( k = k_n \) and \( \omega = \omega_n \). Take \( h_n(z) = \log(f_n(z)) \), which defines a holomorphic function in a neighbourhood of \( \overline{D} \); see (8). Then \( |\Im m(h_n(z))| \leq \pi/2 \) for all \( z \in \overline{D} \) and \( h_n(0) = 0 \). Now consider at the function \( h_n \) on the boundary of the unit disk \( D \), i.e., introduce the function \( g_n(x) = h_n(e^{2i\pi x}) \) defined on the circle \( T = \mathbb{R}/\mathbb{Z} \). The conditions satisfied by \( f_n \) imply that

\[
\begin{align*}
|g_n(x)| &\geq \log \omega_n + \log C_2 \quad \text{for all } x \in J_{k_n}^{\omega_n}, \\
|g_n(x)| &\leq \log \omega_n + \log C_3 \quad \text{for all } x \in T, \\
|g'_n(x)| &\leq 2\pi k_n \omega_n \leq n \quad \text{for all } x \in T.
\end{align*}
\]

Apply [3, Lemma 1.7], which is a precise version of Féjer’s theorem, to the function \( \theta_x(t) = g_n(t) - g_n(x) \) for \( x \in T \). Since \( \|\theta_x\|_\infty \leq 2 \log \omega_n + 2 \log C_3 \), \( \|\theta'_x\|_\infty \leq n \), and \( \theta_c(x) = 0 \), the \( n \)-th Féjer sum of \( \theta_x \) satisfies

\[ |\sigma_n \theta_x(x)| \leq \frac{1}{2} \log \omega_n + C_4 \]

for some absolute constant \( C_4 \). If \( x \in J_{k_n}^{\omega_n} \), we deduce that

\[ |\sigma_n g_n(x)| \geq \frac{1}{2} \log \omega_n - C_5. \]

Finally set

\[ P_n = \frac{2}{\pi} e_n \sigma_n(\Im m g_n) = \frac{2}{\pi} e_n \Im m(\sigma_n g_n), \]

so that \( \|P_n\|_\infty \leq 1 \). Now, recall that \( g_n \) is the restriction to the circle of the holomorphic function \( h_n \), which satisfies \( h_n(0) = 0 \). Thus we can write \( \sigma_n g_n = \sum_{j=1}^{n-1} a_j e_j \), so that

\[ 2i \Im m \sigma_n g_n = -\sum_{j=1}^{n-1} \bar{a}_j e_{-j} + \sum_{j=1}^{n-1} a_j e_j. \]
It follows that the spectrum of $P_n$ is contained in $[1, 2n - 1]$. Moreover, for each $x \in J_{k_n}^{\omega}$,

$$|S_n P_n(x)| = \frac{1}{\pi} \left| \sum_{j=1}^{n-1} a_j e^{-j+n} \right| = \frac{1}{\pi} |\sigma_n g_n(x)| \geq \frac{1}{2\pi} \log \omega_n - C_6 = 2\epsilon_n \log n - C_6$$

$$\geq \epsilon_n \log n$$

for large enough $n$.

We are now ready to construct the dense $G_\delta$-set of functions required in Theorem 1.7.

**Proof of Theorem 1.7.** Let $\{\delta_n\}_{n \geq 2}$ be a sequence tending to 0. First consider an auxiliary sequence $\{\delta'_n\}_{n \geq 1}$ satisfying

$$\lim_{n \to +\infty} \delta'_n = 0, \quad \lim_{n \to +\infty} \frac{\delta'_n}{\delta_n} = +\infty, \quad \text{and} \quad \lim_{n \to +\infty} \delta'_n \log n = +\infty.$$ 

Let $\{g_n\}_{n \geq 1}$ be a dense sequence in $\mathcal{C}(\mathbb{T})$, such that the spectrum of $g_n$ is contained in $[-n, n]$. Then set $\eta_n = \max(\delta'_k : n \leq k)$. The sequence $\{\eta_n\}_{n \geq 1}$ decreases to 0. Fix a sequence $\{\epsilon_n\}_{n \geq 1}$ tending to 0 such that $\lim_{n \to +\infty} \epsilon_n / \eta_n = +\infty$. Lemma 4.2 gives an integer $N$, a sequence $(P_j)_{j \geq N}$ of trigonometric polynomials with spectrum contained in $[1, 2j - 1]$, a sequence $\{k_j\}_{j \geq N}$ of integers tending to $+\infty$, and a sequence $\{\omega_j\}_{j \geq N}$ satisfying $\omega_j > 1$, such that $|S_j P_j(x)| \geq \epsilon_j \log j$ for all $x \in J_{k_j}^{\omega}$. Moreover, we can choose $\omega_j$ such that $\omega_j = o(k_j^\alpha)$ for all $\alpha > 0$.

For $j \geq N$, define

$$h_j := g_j + \frac{\eta_j}{\epsilon_j} e_j P_j.$$ 

The sequence $\{h_j\}_{j \geq N}$ is also dense in $\mathcal{C}(\mathbb{T})$. Observe also that the spectra of $g_j$ and $(\eta_j / \epsilon_j) e_j P_j$ are disjoint. It follows that if $x \in J_{k_j}^{\omega}$,

$$|S_{2j} h_j(x) - S_j h_j(x)| = \left| \frac{\eta_j}{\epsilon_j} S_j P_j(x) \right| \geq \eta_j \log j.$$ 

Thus, for each $x \in J_{k_j}^{\omega}$, there exists $n \in \{j, 2j\}$ such that

$$|S_n h_j(x)| \geq \frac{1}{2} \eta_j \log j \geq \frac{1}{2} \delta'_n (\log n - \log 2).$$

Let $r_j > 0$ be so small that $|S_n h(x)| \geq |S_n h_j(x)| - 1$ for all $h \in B(h_j, r_j)$ and $n \in \{j, 2j\}$ (the open balls are defined by the norm $\| \|$).

We now claim that the dense $G_\delta$-set of $\mathcal{C}(\mathbb{T})$ $G := \bigcap_{p \geq N} \bigcup_{j \geq p} B(h_j, r_j)$ fulfills all the requirements needed to complete the proof of Theorem 1.7. Indeed, pick
any \( h \in G \) and an increasing sequence \( \{ j_p \} \) such that \( h \in B(h_{j_p}, r_{j_p}) \). Setting \( \rho_p = \omega_{j_p} \) and \( s_p = k_{j_p} \), we can show without difficulty that

\[
E := \limsup_{p \to +\infty} E_p, \quad \text{with} \quad E_p = f_{s_p}^{\rho_p}
\]

has Hausdorff dimension 1. Indeed, recall that \( \omega_j = o(k_j^\alpha) \) for all \( \alpha > 0 \). It follows for any \( \alpha > 0 \) and large enough \( p \) that \( E_p \) contains

\[
E_p \supset F_p = s_p - 1 \bigcup_{j=0}^{s_p-1} \left[ j s_p - \frac{1}{2 s_p^{1+\alpha}}, j s_p + \frac{1}{2 s_p^{1+\alpha}} \right].
\]

Now, it is well known that the Hausdorff dimension of \( \limsup_p F_p \) equals \( 1/(1 + \alpha) \) (this follows, for instance, from the mass transference principle of [4]). Finally, \( \dim_H(E) \geq 1/(1 + \alpha) \). Moreover, the work done before and the fact that \( \lim_{n \to \infty} \delta_n \log n = +\infty \) show that for all \( x \in E \),

\[
|S_n h(x)| \geq \frac{1}{2} \delta_n \log n - 1 \geq \frac{1}{4} \delta_n \log n
\]

for infinitely many values of \( n \). We then get

\[
\frac{|S_n h(x)|}{\delta_n \log n} \geq \frac{\delta_n}{4 \delta_n}
\]

for infinitely many values of \( n \). This proves Theorem 1.7. \( \square \)

We can finally construct the prevalent set of functions required in Theorem 1.8.

**Proof of Theorem 1.8.** Let \( \{ \delta_n \}_{n \geq 2} \) be a sequence tending to 0 and

\[
A = \left\{ f \in \mathcal{C}(\mathbb{T}) : \dim_{\mathcal{L}} \left( \left\{ x \in \mathbb{T} : \limsup_{n \to +\infty} \frac{|S_n f(x)|}{\delta_n \log n} = +\infty \right\} \right) < 1 \right\}.
\]

We have to prove that \( A \) is Haar-null in \( \mathcal{C}(\mathbb{T}) \).

Let \( f_0 \) be a fixed function in the complement of \( A \) (such a function exists by Theorem 1.7), and let \( g \) be an arbitrary function in \( \mathcal{C}(\mathbb{T}) \). Suppose that \( t_1 \) and \( t_2 \) are two real numbers such that \( t_1 f_0 \in (g + A) \) and \( t_2 f_0 \in (g + A) \). There then exists \( f_1 \in A \) and \( f_2 \in A \) such that \( (t_1 - t_2) f_0 = f_1 - f_2 \). It is clear that \( f_1 - f_2 \in A \) (\( A \) is a vector subspace of \( \mathcal{C}(\mathbb{T}) \)). It follows that \( t_1 = t_2 \), and so

\[
\#(\text{span}(f_0) \cap (g + A)) \leq 1.
\]

In particular, the Lebesgue-measure in \( \text{span}(f_0) \) is transverse to \( A \), and \( A \) is Haar-null in \( \mathcal{C}(\mathbb{T}) \). \( \square \)

**Remark.** We proved only that a proper subspace in a complete metric vector space is Haar-null. This fact is probably well known.
Acknowledgments. We thank the referee for a careful reading.

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Frédéric Bayart and Yanick Heurteaux

Laboratoire de Mathématiques
Clermont Université, Université Blaise Pascal
BP 10448, F-63000 Clermont-Ferrand, France

and

Laboratoire de Mathématiques
CNRS, UMR 6620
F-63177 Aubière Cedex, France

email: Frederic.Bayart@math.univ-bpclermont.fr, Yanick.Heurteaux@math.univ-bpclermont.fr

(Received December 13, 2012 and in revised form June 5, 2013)