Distribution of eigenvalues of ensembles of asymmetrically diluted Hopfield matrices

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Abstract. Using Grassmann variables and an analogy with two dimensional electrostatics, we obtain the average eigenvalue distribution $\rho(\omega)$ of ensembles of $N \times N$ asymmetrically diluted Hopfield matrices in the limit $N \to \infty$. We found that in the limit of strong dilution the distribution is uniform in a circle in the complex plane.

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Random matrix theory has become an active field of research in mathematics and physics in the last decades. In physics, since the now classical work of Metha[1] on the statistical description of the energy levels of atomic nuclei, random matrices have emerged as an important tool in the study of the localization transition[2, 3], quantum chaos[4], spin glasses[5], neural networks[6, 7], and disordered systems in general. Most of the work deal with ensembles of hermitian or symmetric matrices whose individual properties are well known and can be exploited in more complex situations. In the last years the interest in asymmetric matrices has grown, motivated, besides its intrinsic mathematical value, by problems of dissipative quantum dynamics[8] and the dynamics of neural networks[9].

In a recent paper, Sommers et.al.[10] calculated the average density of eigenvalues $\rho(\omega)$ of $N \times N$ random asymmetric matrices in the limit $N \rightarrow \infty$, with elements $J_{ij}$, given by a Gaussian distribution with zero mean and correlations

$$N\langle\langle J_{ij}^2\rangle\rangle_{J} = 1,$$

$$N\langle\langle J_{ij} J_{ji}\rangle\rangle_{J} = \tau$$

They found that the eigenvalues are uniformly distributed inside an ellipse in the complex plane, whose semi axes depend on the degree of asymmetry of the ensemble $\tau$. Generalizing this result, Lehmann et. al. [11] calculated the joint probability distribution of eigenvalues in Gaussian ensembles of real asymmetric matrices, recovering the elliptic law in the large $N$ limit.

In this paper we calculate the average eigenvalue distribution $\rho(\lambda)$ of an ensemble of asymmetrically diluted Hopfield matrices, whose elements are given by:

$$J_{ij} = \frac{c_{ij}}{N} \sum_{\mu=1}^{p} \xi_{i}^{\mu} \xi_{j}^{\mu} \quad i, j = 1 \ldots N ,$$

where $\{\xi_{i}^{\mu} i = 1 \ldots N, \mu = 1 \ldots p\}$ represents a set of $p$ random patterns. The $\xi_{i}^{\mu}$ are random independent variables that can take the values $\pm 1$ with the same probability and the $c_{ij}$ are random variables chosen accordingly to the following distribution

$$P(c_{ij}) = \gamma\delta(c_{ij} - 1) + (1 - \gamma)\delta(c_{ij}).$$
0 ≤ γ ≤ 1 measures the degree of dilution of the matrices. γ = 1 corresponds to symmetric Hopfield matrices whose eigenvalue distribution is known [12].

In order to obtain the distribution ρ(ω) we use an analogy with a two dimensional electrostatic problem introduced in ref. [10]. Let us define the Green function associated to the matrix J

\[ G(\omega) = \frac{1}{N} \langle \langle Tr \frac{1}{I\omega - J} \rangle \rangle_J, \]  

where ω = x + iy is a complex variable, I the identity matrix and \( \langle \langle \ldots \rangle \rangle_J \) denotes an average over the random variables \( \xi^\mu_i \). If \( \lambda_i, i = 1, \ldots, N \) are the eigenvalues of J, then

\[ Tr \frac{1}{I\omega - J} = \sum_i \frac{1}{\omega - \lambda_i} \]  

(5)

For large N the sum can be approximated by an integral and the Green function becomes

\[ G(\omega) = \int d^2\lambda \frac{\rho(\lambda)}{\omega - \lambda} \]  

(6)

where ρ(λ) is the density of eigenvalues in the plane. The last equation suggests an analogy with a two-dimensional classical electrostatics problem in which ρ(λ) represents the density of charge in the plane. It can be demonstrated [10] that an electrostatic potential \( \Phi \) exists, satisfying

\[ 2 \text{Re} G = -\frac{\partial \Phi}{\partial x}, \quad -2 \text{Im} G = -\frac{\partial \Phi}{\partial y} \]  

(7)

and which obeys Poisson’s equation:

\[ \nabla^2 \Phi = -4\pi \rho \]  

(8)

Thus, in order to determine ρ(ω) we may calculate the potential \( \Phi \). Using that \( \text{det}(AB) = \text{det}A \text{det}B \) and \( \text{det}A^T = \text{det}A \) one can prove that a good definition for \( \Phi \) can be

\[ \Phi(\omega) = -1/N \langle \langle \ln \det \left( (I\omega^* - J^T)(I\omega - J) \right) \rangle \rangle_J \]  

(9)
with $\omega^*$ the complex conjugate of $\omega$ and $J^T$ the transpose of $J$. In what follows we will consider the case $N \to \infty$ and assume that in this limit the average and the ln operations commute \[10\]. By using a Grassmannian representation [13] of the determinant and adding a matrix $\epsilon \delta_{ij}$, with $\epsilon$ positive and infinitesimal in order to avoid zero eigenvalues, we get:

$$\exp \left[ -N \Phi(\omega) \right] = \langle \langle \int_{-\infty}^{\infty} \left( \prod_{i=1}^{N} d\eta_i^* d\eta_i \right) \exp \left\{ -\sum_{i,j,k} \eta_i^* (\omega^* \delta_{ik} - J_{ik}^T (\omega \delta_{kj} - J_{kj}) \eta_j - \epsilon \sum_i \eta_i^* \eta_i \right\} \rangle \rangle$$  \hspace{1cm} (10)

After performing the average over the $c_{ij}$ and over the random patterns $\{\xi_{\mu}i\}$, we arrive at the following expression:

$$\exp \left[ -N \Phi(\omega) \right] = \int_{-\infty}^{\infty} \left( \prod_{i=1}^{N} d\eta_i d\eta_i^* \right) \exp \left\{ (\epsilon + |\omega|^2)Nq - \alpha N \ln t + \alpha \gamma (\omega + \omega^*) Nq - \alpha \gamma (1 - \gamma) |\omega|^2 Nq^2 \right\}$$

$$\times \int_{-\infty}^{+\infty} \left( \prod_{i=1}^{N} d\chi_i d\chi_i^* \right) \exp \left\{ -\sum_i \chi_i^* \chi_i [1 + \alpha \gamma (1 - \gamma) q] + \right.$$

$$\left. \sum_i \chi_i^* \eta_i [\alpha \gamma - \alpha \gamma (1 - \gamma) \omega q] + \sum_i \eta_i^* \chi_i [\alpha \gamma - \alpha \gamma (1 - \gamma) \omega^* q] \right\}$$

$$\times \exp \alpha N \ln \left\{ 1 - \gamma^2 q t \sum_i \chi_i^* \chi_i / N - \gamma t (\sum_i \chi_i^* \eta_i + \sum_i \eta_i^* \chi_i) / N + \right.$$

$$\left. \gamma^2 t (\sum_i \eta_i^* \chi_i \sum_j \eta_j \chi_j + \sum_i \eta_i^* \chi_i \sum_j \chi_j^* \eta_j) / N^2 \right\}$$  \hspace{1cm} (11)

where

$$q = \frac{1}{N} \sum_i \eta_i^* \eta_i$$  \hspace{1cm} (12)

$$t = \frac{1}{1 - \frac{\gamma}{N} (\omega + \omega^*) \sum_i \eta_i^* \eta_i}$$  \hspace{1cm} (13)

respectively. Next we define the following parameters:

$$q = \frac{1}{N} \sum_{i=1}^{N} \eta_i^* \eta_i \quad \quad \quad z = \frac{1}{N} \sum_{i=1}^{N} \chi_i^* \chi_i$$
\[ r = \frac{1}{N} \sum_{i=1}^{N} \eta_i^* \chi_i \quad r^* = \frac{1}{N} \sum_{i=1}^{N} \chi_i^* \eta_i \quad \] (14)

and introduce them into Eq. (11) by using delta functions. After integrating over the Grassmann variables we get:

\[ \exp \left[ -N \Phi(\omega) \right] = \left( \frac{N}{2\pi} \right)^4 \int_{-\infty}^{\infty} dq \, dQ \, dz \, dZ \, dr \, dR \, dr^* \, dR^* \times \exp \left\{ qQ + zZ + rR + r^* R^* + \left[ -\epsilon + |\omega|^2 + \alpha \gamma (\omega + \omega^*) q \right] + \alpha \ln \left[ 1 - \gamma (\omega + \omega^*) q \right] + \ln \left( RR^* - ZQ \right) - z + \alpha \gamma (r + r^*) - \alpha \gamma (1 - \gamma) \left[ 2q^2 + 4q + r^* q + \omega r^* q \right] + \alpha \ln \left[ 1 - \frac{\gamma^2 qz}{\gamma^2 qz - \gamma (r + r^*)} - \frac{\gamma (r + r^*)}{\gamma (r + r^*)} + \frac{\gamma^2 r r^*}{\gamma^2 r r^* - \gamma (r + r^*)} \right] \right\} \] (15)

In the large \( N \) limit this multiple integral can be evaluated by the saddle point method. Up to now the calculation is exact for arbitrary \( \gamma \). Since the resulting saddle point equations are difficult to solve analytically, in this work we present the results for the strong dilution limit (\( \gamma \ll 1 \)). Expanding the exponent in powers of \( \gamma \) and keeping terms up to \( O(\gamma) \) we obtain, after some calculations:

\[ \exp \left[ -N \Phi(\omega) \right] \propto \int_{-\infty}^{+\infty} dq \exp -N \left[ \ln |q| - (\epsilon + |\omega|^2) q + \alpha \gamma |\omega|^2 q^2 - \ln(1 + \alpha \gamma q) \right] \] (16)

After a change of variables \( \sigma = 1/q \) we arrive at the following saddle-point equation:

\[ \frac{\epsilon}{\sigma^2} = \frac{1}{\sigma + 2 \gamma} - \frac{x^2}{(\sigma + 2 \gamma)^2} - \frac{y^2}{(\sigma + 2 \gamma)^2} \] (17)

Expanding \( \epsilon \) in powers of \( \sigma \), the solution of the saddle-point equation in the limit \( \epsilon \to 0 \) is \( \sigma = 0 \) inside the circle \( x^2 + y^2 = \alpha \gamma \). In this region \( G(\omega) = \omega^*/(\alpha \gamma) \) (non analytic) and
\[ \nabla^2 \Phi = -4/(\alpha \gamma) \]. This implies that the density of eigenvalues is uniform inside a circle of radius \( \sqrt{\alpha \gamma} \) in the complex plane. Outside the circle the solution to Eq. (17) becomes \( \sigma = x^2 + y^2 - \alpha \gamma \), the Green function is \( G(\omega) = 1/\omega \) \((\text{analytic})\), and the density \( \rho = 0 \).

The density of eigenvalues in the whole complex plane is:

\[
\rho(\omega) = \begin{cases} 
1/\pi \alpha \gamma & \text{if } x^2 + y^2 \leq \alpha \gamma \\
0 & \text{otherwise} 
\end{cases}
\]

(18)

It is important to note that \( \langle \langle J_{ij} J_{ji} \rangle \rangle \propto \gamma^2 \) and consequently, in this limit of strong dilution, the matrix elements become effectively uncorrelated and we obtain a "circular law". This result has to be compared with the similar result of ref. [10] for the case \( \tau = 0 \), i.e., a Gaussian ensemble of completely asymmetric random matrices. It is expected that this circle deforms into an ellipse as the asymmetry parameter \( \gamma \) increases and permits the appearance of random correlations between the patterns.

In figures (1-3) we show the results of numerical diagonalization of sets of \( N \times N \) matrices, with linear sizes \( N \) ranging from 512 to 2048 for \( \alpha = 0.25 \) and \( \gamma = 0.01 \). The figures show the projection of the distribution of complex eigenvalues in the real axis. The full curves represent the analytical solution:

\[
\rho_x = \int \rho(x, y) dy \\
= \frac{2}{\pi \alpha \gamma} (\alpha \gamma - x^2)^{1/2}, \quad |x| \leq \sqrt{\alpha \gamma}.
\]

(19)

We found that the numerical results present a peak at the origin that becomes smaller as the size \( N \) increases. Assuming that it is a finite size effect, and that the weight of the peak might be uniformly distributed on the whole support of the distribution, we renormalized the distributions. After renormalizing the numerical data the agreement with the analytic curves becomes very good as the size increases. Figure (4) shows the dependence of the peak at the origin with the system size. We have fitted the data at
the origin with an exponential function in $1/N$, $\rho_x(0) = a \exp(b/N)$. The extrapolation to $N \to \infty$ coincides with an error of 10% with the analytic result at the origin.

Concluding, in this paper we have obtained analytically the distribution of eigenvalues of an ensemble of asymmetrically diluted Hopfield matrices in the limit of strong dilution. The eigenvalues are uniformly distributed inside a circle in the complex plane. Our results are supported by numerical diagonalization of the ensemble considered. Although we presented only the results for the strong dilution limit, the saddle point equations are valid for any amount of dilution. Work for solving the general case is in progress.

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Caption for figures

**Figure 1** Projection of the eigenvalue distribution on the real axis. The full line correspond to the analytical results and the histogram to the numerical diagonalization performed with $\gamma = 0.01$, $\alpha = 0.25$ and $N = 512$ and averaged over 20 realizations of the matrices.

**Figure 2** Same as figure (1) with $N = 1024$ and averaged over 10 realizations of the matrices.

**Figure 3** Same as previous figures with $N = 2048$ averaged over 8 realizations of the matrices.

**Figure 4** Finite size scaling of the peak at the origin in the complex plane $\ln \rho(0)$ versus $1/N$. 
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FIGURE 3
FIGURE 4