Constructing Well-bounded Operators not of type (B) on a Class of Inductive Limits

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Abstract

Well-bounded operators are linear operators on a Banach space $X$ that have an $AC[a, b]$ functional calculus for some interval $[a, b]$. A well-bounded operator is of type (B) if it can be written as an integral against a spectral family of projections, and this is always the case when $X$ is reflexive. There are many examples of well-bounded operators on non-reflexive spaces that are not of type (B), and it is open whether there is a non-reflexive Banach space upon which every well-bounded operator is of type (B). It was suggested in [2] that the spaces constructed in [8] could provide an example of such a space. In this paper, it will be shown that on a class of Banach spaces containing the spaces from [8], there is always a well-bounded operator not of type (B).

1 Introduction

The class of well-bounded operators was introduced by Smart to generalise the spectral theory for self-adjoint operators to a theory which could accommodate operators on Banach spaces whose spectral expansions converged conditionally. Well-bounded operators are bounded linear operators that possess a functional calculus for the absolutely continuous functions on some compact interval $[a, b]$. It was shown by Smart and Ringrose [12] [10] that a well-bounded operator on a reflexive Banach space $X$ could always be written as an integral with respect to a unique ‘spectral family of projections’ on $X$. However, on non-reflexive Banach spaces, there are simple examples of well-bounded operators which could not be written as such an integral, such as $T$ acting on $C[0, 1]$ defined by $Tx(t) = tx(t)$. Ringrose [11] later extended this theory to the non-reflexive setting by considering projections on the dual space $X^*$ rather than $X$, and here the family of projections may no longer be unique.

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Berkson and Dowson [1] considered classes of well-bounded operators whose projections possessed certain properties. They said that a well-bounded operator is of type (B) if it possessed a spectral family of projections, and they with Spain [13] classified the type (B) well-bounded operators as those whose functional calculus is weakly compact. It is still an open problem whether there is a non-reflexive Banach space where all well-bounded operators are of type (B). Current results, which will be discussed in Section 2, show that such a Banach space would have to be rather exotic.

Much like the spectral theorem for compact self-adjoint operators, it is known from [3] that a compact well-bounded operator $T$ can always be expressed as

$$T = \sum_{j=1}^{\infty} \lambda_j P_j$$

(1.1)

for a sequence of real numbers $\{\lambda_j\}_{j=1}^{\infty}$ converging monotonely in absolute value to 0 and $\{P_j\}_{j=1}^{\infty}$ a sequence of uniformly bounded, mutually disjoint projections of finite rank. This identification provides a simple way to construct well-bounded operators with various properties. An in-depth discussion of compact well-bounded operators was undertaken by Cheng and Doust [2] where compact well-bounded operators of type (B) were classified. Moreover, they showed that from a compact well-bounded operator of infinite rank, one can find an increasing sequence of uniformly bounded finite rank projections. That is, a sequence of finite rank projections $\{Q_n\}_{n=1}^{\infty}$ such that

$$\sup_n \|Q_n\| < \infty, \quad Q_m \neq Q_n \quad \text{and} \quad Q_m Q_n = Q_n Q_m = Q_{\min(m,n)}$$

for all $m, n \in \mathbb{Z}^+$. In [8], Pisier showed that every Banach space $E$ of cotype 2 is isometric to a subspace of a Banach space $X_E$ also of cotype 2 satisfying, among other things, the following property:

there is a constant $C_E > 0$ such that if $P$ is a finite rank projection on $X_E$, then

$$\|P\| \geq C_E \sqrt{\text{rank}(P)}.$$  

(1.2)

It is well-known (for example, [9] Theorem 1.14) that given an $n$-dimensional subspace $Y$ of a Banach space $X$, there is a finite rank projection $P : X \to Y$ with $\|P\| \leq \sqrt{n}$, and so Pisier’s spaces exhibit extreme behaviour concerning finite rank projections. As a consequence of (1.2), there can be no increasing sequences of uniformly bounded finite rank projections. In particular, the only compact well-bounded operators are the ones of finite rank, which are all of type (B). Hence it was suggested in [2] that it may be possible for all well-bounded operators on these spaces to be of type (B).
The space $X_E$ is constructed by taking a Banach space $E = E_0$ and forming a particular sequence of Banach spaces $E_0, E_1, E_2, \ldots$ such that each space is isometric to a subspace of the next, the construction of which is detailed in Section 2.2. The inductive limit of this sequence of Banach spaces is $X_E$, upon which (1.2) is obtained when $E$ is of cotype 2. In Section 3, it will be shown that for every Banach space $E$, there is a well-bounded operator on $X_E$ not of type (B).

2 Background

In this section, the necessary definitions and results for this article regarding well-bounded operators will be presented, followed by a detailing of the construction of the space $X_E$ from a Banach space $E$. Throughout the following, $X$ and $E$ will denote Banach spaces and $B(X, E)$ will denote the set of bounded linear operators from $X$ to $E$ with $B(X) = B(X, X)$. The continuous dual of $X$ will be denoted by $X^*$ and the adjoint of $T : X \to E$ denoted by $T^*$. A ‘projection’ $P$ will refer to a bounded linear operator acting on a Banach space such that $P^2 = P$. Lastly, SOT will refer to the strong operator topology.

2.1 Well-bounded Operators

An operator $T \in B(X)$ is well-bounded if there exists an interval $[a, b]$ and a $K > 0$ such that

$$\|p(T)\| \leq K \left( |p(a)| + \int_a^b |p'(t)| \, dt \right)$$

for all polynomials $p$. As the polynomials are dense in $AC[a, b]$, the expression $f(T)$ can be made sense of for $f \in AC[a, b]$, and the mapping $f \mapsto f(T)$ defines a continuous Banach algebra homomorphism. That is, the operator $T$ has a functional calculus for $AC[a, b]$. A well-bounded operator is of type (B) if it can be written as an integral against a spectral family of projections.

**Definition 2.1.** A spectral family of projections is a projection-valued function $E : \mathbb{R} \to B(X)$ satisfying the following five conditions:

(i) there exist $a, b \in \mathbb{R}$ such that $E(\lambda) = 0$ for all $\lambda < a$ and $E(\lambda) = I$ for all $\lambda \geq b$;

(ii) $\sup_\lambda \|E(\lambda)\| < \infty$;

(iii) $E(\lambda) = E(\mu)$ for all $\lambda, \mu \in (a, b)$;

(iv) $E(\lambda^+) = E(\lambda^-) = E(\lambda)$ for all $\lambda \in (a, b)$;

(v) $E(\lambda) = I$ for all $\lambda \in (a, b)$.

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(iii) $E(a) = 0$ and $E(b) = I$;
(iv) $E(\lambda)E(\mu) = E(\min(\lambda, \mu))$ for all $\lambda, \mu \in \mathbb{R}$;
(v) $E$ is right-continuous and has left limits in the SOT. That is, for all $\mu \in \mathbb{R}$ and $x \in X$, 
\[ \lim_{\lambda \to \mu^+} E(\lambda)x = E(\mu)x \quad \text{and} \quad \lim_{\lambda \to \mu^-} E(\lambda)x \text{ exists.} \]

It was shown in [1] that a well-bounded operator $T$ acting on $X$ being of type (B) is equivalent to $f \mapsto f(T)x$ being a weakly compact mapping of $AC[a,b]$ into $X$ for each $x \in X$. Hence on reflexive spaces, all well-bounded operators are of type (B).

One can integrate a function of bounded variation $g \in BV[a,b]$ against a spectral family $E$ via a Riemann-Stieltjes integral. For a partition $P = \{t_{j}\}_{j=0}^{n}$ of $[a,b]$, partial sums of the form 
\[ g(a)E(a) + \sum_{j=1}^{n} g(t_{j})(E(t_{j}) - E(t_{j-1})) \]
converge in the SOT with respect to refinement of $P$, and the resulting operator is usually denoted in literature by 
\[ \int_{[a,b]}^{\oplus} g(\lambda) \, dE(\lambda). \]

The spectral theorem for well-bounded operators of type (B) (see Ch. 17 of [6]) states that there is a bijective correspondence between well-bounded operators $T$ of type (B) and spectral families $E$ given by
\[ T = \int_{[a,b]}^{\oplus} \lambda \, dE(\lambda). \]

One of the simplest ways to construct well-bounded operators is via increasing sequences of uniformly bounded projections. Moreover, it is simple to identify when these well-bounded operators are of type (B).

**Definition 2.2.** A sequence of projections $\{P_{k}\}_{k=1}^{\infty}$ on a Banach space is said to be **increasing** if $P_{k}P_{\ell} = P_{\ell}P_{k} = P_{\min(k,\ell)}$ and $P_{k} \neq P_{\ell}$ for all $k, \ell \in \mathbb{Z}^{+}$.

**Theorem 2.3.** ([3], Theorem 3.1 and Proposition 4.1.) Suppose that $\{\lambda_{n}\}_{n=1}^{\infty}$ is a convergent sequence of increasing real numbers with limit $L$ and $\{P_{n}\}_{n=1}^{\infty}$ is an increasing sequence of uniformly bounded projections on a Banach space $X$. Then 
\[ T = I - \sum_{k=1}^{\infty} (\lambda_{k+1} - \lambda_{k})P_{k} \]
defines a well-bounded operator on $X$. Moreover $T$ is of type (B) if and only if $\lim_{n \to \infty} P_{n}$ exists in the SOT.
It is currently an open problem whether or not there is a non-reflexive Banach space upon which every well-bounded operators is of type (B). Such a space must be rather unusual as most classical spaces are known to have a well-bounded operator not of type (B) due to the following results.

**Theorem 2.4.** ([4], Theorem 4.4) Suppose that a Banach space $X$ contains a subspace isomorphic to $c_0$. Then there is a well-bounded operator on $X$ not of type (B).

**Theorem 2.5.** ([3], Theorem 4.2) If a Banach space $X$ contains a complemented non-reflexive subspace with a Schauder basis, then there is a well-bounded operator on $X$ not of type (B).

Due to the property stated in (1.2), the spaces constructed by Pisier cannot have a Schauder basis and hence the second of the above theorems does not apply directly. Moreover, in the case of $E$ being of cotype 2, the corresponding space $X_E$ is also of cotype 2 and hence cannot possibly contain a subspace isomorphic to $c_0$. However, it will be shown that $X_E$ always contains an increasing sequence of uniformly bounded projections of infinite rank that do not converge in the SOT, and hence there is a well-bounded operator on $X_E$ not of type (B) by Theorem 2.3. To construct these projections, it will be necessary to understand how $X_E$ is constructed.

### 2.2 The Construction of $X_E$

The first ingredient for the construction of $X_E$ is inductive limits of Banach spaces. Given a sequence of Banach spaces $E_0, E_1, E_2, \ldots$ and linear isometries $j_n : E_n \to E_{n+1}$ for each $n \in \mathbb{N}$, one may construct the (Banach) inductive limit $\text{Ind}(E_n, j_n)$. Consider the subspace of $\prod_{n=0}^\infty E_n$ consisting of sequences $(x_n)_{n=0}^\infty$ such that $j_n x_n = x_{n+1}$ for all $n$ large enough, which will be denoted by $\mathcal{X}$. (The subspace $\mathcal{X}$ may be thought of as the set of sequences that are eventually ‘constant’.) A semi-norm $\rho$ may be defined on $\mathcal{X}$ by $\rho(x) = \lim_{n \to \infty} \|x_n\|$ and the inductive limit is then the completion of $\mathcal{X}/\ker\rho$. Note that one may think of $\mathcal{X}/\ker\rho$ as the collection of cosets of sequences which are eventually the same.

For each $n \in \mathbb{N}$, there is a linear isometry $i_n$ from $E_n$ to the inductive limit given by mapping $x \in E_n$ to

$$(0, \ldots, 0, x, j_n x, j_{n+1} j_n x, \ldots) + \ker\rho, \quad (2.1)$$

where there are $n$ zeros before $x$ in the sequence. As these maps satisfy $i_{n+1} j_n = i_n$ for each $n$, $E_n$ is isometric to the subspace $i_n(E_n)$ of the inductive limit. Hence $\mathcal{X}/\ker\rho$ may be identified with $\bigcup_{n=0}^\infty i_n(E_n)$. (For this reason, it is often easiest to think of the inductive limit as being $\bigcup_{n=0}^\infty E_n$.)
The next ingredient is to construct the particular sequence of Banach spaces from some base
space $E = E_0$. The construction involves iterating the following idea which Pisier credits to
Kisljakov [7]. Let $E$ and $B$ be Banach spaces, $S$ a closed subspace of $B$, and $u : S \to E$ a bounded
linear operator with $\|u\| \leq 1$. Then there exists a Banach space $\tilde{E}$, a linear isometry $j : E \to \tilde{E}$
and a bounded linear operator $\tilde{u} : B \to \tilde{E}$ with $\|\tilde{u}\| \leq 1$ and $\tilde{u}|_S = ju$.

Figure 1: A commutative diagram illustrating the construction of $\tilde{E}$ from $E$.

The space $\tilde{E}$ is constructed by taking the quotient of $B \oplus E$, equipped with the norm $\|(x, e)\| =
\|x\| + \|e\|$, by the subspace $\{(s, -us)\}_{s \in S}$. If $\pi$ denotes the quotient map from $B \oplus E$ to $\tilde{E}$, then $\tilde{u}$
is defined by $\tilde{u}x = \pi(x, 0)$ and the linear isometry $j$ of $E$ into $\tilde{E}$ is given by $je = \pi(0, e)$. Indeed,
since $\|us\| \leq \|s\|$ for all $s \in S$,

$$\|je\| \leq \|e\| \leq \inf_{s \in S} (\|e\| + \|s\| - \|us\|) \leq \inf_{s \in S} (\|s\| + \|e - us\|) = \|je\|,$$

and so $\|je\| = \|e\|$.

**Remark 2.6.** A property of the construction is that $B/S$ is isometric to $\tilde{E}/jE$ under the mapping
$x + S \mapsto \pi(x, 0) + jE$. Firstly, this map is well-defined since for all $s \in S$, $\pi(s, 0) = \pi(0, us)$ and

$$\pi(x + s, 0) + jE = \pi(x, 0) + jus + jE = \pi(x, 0) + jE.$$

Secondly, it is surjective since $\pi(x, e) + jE = \pi(x, 0) + jE$ for all $e \in E$. Lastly, it is isometric since

$$\|\pi(x, 0) + jE\| = \inf_{e \in E} \|\pi(x, e)\| = \inf_{e \in E, s \in S} \|(x + s, e - us)\| = \inf_{s \in S} \|x + s\| = \|x + S\|.$$

The final ingredients for Pisier’s construction involve specific choices of $B$, $S$ and $u$. These will
be built from the following three Banach spaces and choices of closed subspaces. Let $\mathbb{T}$ denote the
complex unit circle, $D = \{-1, 1\}^\mathbb{N}$, and $\mu$ be the normalised Haar measure on $D$ induced by the
group structure given by entrywise multiplication. Set

- $B_1 = L^1(D, \mu)$ and $S_1 = \overline{\text{span}}\{\varepsilon_k\}_{k=0}^\infty$ where $\varepsilon_k$ is the $k$th coordinate functional on $D$ defined
  by $\varepsilon_k((d_i)_{i=0}^\infty) = d_k$;
• $B_2 = L^1(\mathbb{T})$ equipped with the normalised Lebesgue measure and $S_2 = H^1(\mathbb{T})$, where $H^1(\mathbb{T})$ is the subspace of $L^1(\mathbb{T})$ consisting of functions whose negative Fourier coefficients are zero;

• $B_3 = L^1(\mathbb{T})/H^1(\mathbb{T})$ and $S_3 = \overline{\text{span}}\{\xi_k\}_{k=1}^\infty$, where $\xi_k$ is the coset of $z \mapsto z^{-k}$.

Now, for a collection of Banach spaces $(X_i)_{i \in I}$, define the notation

$$\ell^1(X_i)_{i \in I} = \left\{(x_i)_{i \in I} \in \prod_{i \in I} X_i : \| (x_i)_{i \in I} \| = \sum_{i \in I} \| x_i \| < \infty \right\}.$$ 

Let $F(S_i, E)$ denote the set of non-zero finite rank operators from $S_i$ to $E$ and let $A = \bigcup_{i=1}^3 F(S_i, E)$. For each $\alpha \in A$, set $B_\alpha = B_i$ and $S_\alpha = S_i$ whenever $\alpha \in F(S_i, E)$. The spaces $B$ and $S$ are defined as $B = \ell^1(B_\alpha)_{\alpha \in A}$, $S = \ell^1(S_\alpha)_{\alpha \in A}$, and the operator $u : S \rightarrow E$ defined by

$$u((s_\alpha)_{\alpha \in A}) = \sum_{\alpha \in A} \frac{\alpha s_\alpha}{\| \alpha \|},$$

which satisfies $\| u \| \leq 1$.

By iterating Kisljakov’s construction from Figure 11 one obtains a sequence of Banach spaces $E_0, E_1, E_2, \ldots$ with $E_0 = E$, $E_{n+1} = \overline{E_n}$ and linear isometries $j_n : E_n \rightarrow E_{n+1}$ for all $n \in \mathbb{N}$. The space $X_E$ is the inductive limit $\text{Ind}(E_n, j_n)$.

Obviously $X_E$ is non-reflexive if $E$ is non-reflexive, and comments made by Pisier in chapter 10e of his monograph [9] imply that whenever $X_E$ contains a subspace $M$ that is isomorphic to $\ell^2$ (so that $X_E^*$ admits $\ell^2$ as a quotient via $X_E^*/M^\perp \cong M^* \cong \ell^2$), the properties of $X_E$ and $X_E^*$ ensure that $X_E^*$ contains a subspace isomorphic to $\ell^1$, and hence cannot be reflexive. To conclude this section, it will be demonstrated that $X_E$ is always non-reflexive.

**Lemma 2.7.** Let $(B_i)_{i \in I}$ be a collection of Banach spaces and, for each $i \in I$, let $S_i$ be a closed subspace of $B_i$. Then $\ell^1(B_i)_{i \in I}/\ell^1(S_i)_{i \in I}$ is isometrically isomorphic to $\ell^1(B_i/S_i)_{i \in I}$ under the mapping

$$(x_i)_{i \in I} + \ell^1(S_i)_{i \in I} \mapsto (x_i + S_i)_{i \in I}.$$ 

The first isomorphism theorem yields that the above mapping is a linear isomorphism, after which it is not difficult to show that it is also an isometry.

Recall that reflexivity is a three-space property (TSP). That is, for a closed subspace $Y$ of a Banach space $X$, one has that $X$ is reflexive if and only if both $Y$ and $X/Y$ are reflexive.
Theorem 2.8. For all Banach spaces $E$, the space $X_E$ is non-reflexive.

Proof. Without loss, suppose that $E$ is a reflexive Banach space. By Remark 2.6, $\widetilde{E}/jE$ is isometric to $B/S$, which is isometric to $\ell^1(B_\alpha/S_\alpha)_{\alpha \in A}$ by Lemma 2.7 which clearly contains $B_1/S_1$ as a closed subspace. It is well-known that the Khintchine inequalities imply that $S_1$ is isomorphic to $\ell^2$. As reflexivity is a TSP, it must be that $B_1/S_1$, and hence $\widetilde{E}/jE$, are non-reflexive. Again appealing to reflexivity being a TSP yields that $\widetilde{E}$, and hence $X_E$, are non-reflexive. 

3 Main Results

3.1 Extending Projections from $E$ to $\widetilde{E}$

The core idea behind constructing projections on $X_E$ is being able to lift a projection $P$ defined on $E$ to a projection $\widetilde{P}$ defined on $\widetilde{E}$.

Lemma 3.1. Let $E$ be a Banach space and $P$ a projection on $E$. Then there is a projection $\widetilde{P}$ on $\widetilde{E}$ such that $\widetilde{P}|_{jE} = jP$ and $\|\widetilde{P}\| = \|P\|$. 

Proof. Recalling that $B = \ell^1(B_\alpha)_{\alpha \in A}$, define $Q : B \to B$ by

$$(Qx)_\alpha = \sum_{\beta : \alpha = P\beta} x_\beta \frac{\|P\beta\|}{\|\beta\|} \quad \forall \alpha \in A,$$

where an empty sum is interpreted to be zero. The definition of $Q$ is worth some elaboration. To obtain the $\alpha$th coordinate of $Qx$, one looks at the indices/operators $\beta \in F(S_\alpha, E)$ such that $\alpha = P\beta$ and adds the corresponding coordinates $x_\beta$ weighted by $\|P\beta\|/\|\beta\|$. Consequently, $(Qx)_\alpha = 0$ unless $\alpha \in PA = \{P\beta : \beta \in A\}$. It is now simple to check that $Q$ is linear and $\|Q\| \leq \|P\|$. To show that $Q$ is a projection, first identify $B^*$ with $\ell^\infty(B_\alpha^*)_{\alpha \in A}$, where

$$\ell^\infty(B_\alpha^*)_{\alpha \in A} = \left\{ (\psi_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} B_\alpha^* : \sup_{\alpha \in A} \|\psi_\alpha\| < \infty \right\}.$$ 

Then one sees that $Q^*$, and hence $Q$, are projections since for all $\psi \in \ell^\infty(B_\alpha^*)_{\alpha \in A}$,

$$(Q^*\psi)_\alpha = \begin{cases} \frac{\|P\alpha\|}{\|\alpha\|} \psi_{P\alpha} & \text{if } P\alpha \neq 0, \\ 0 & \text{if } P\alpha = 0. \end{cases}$$

(3.1)
Now it will be shown that \( \{(s, -us) : s \in S\} \) is an invariant subspace for \( Q \oplus P \). If \( s \in S \), then

\[
Pus = \sum_{\beta \in A} \frac{P_{\beta}s_{\beta}}{\|\beta\|} = \sum_{\alpha \in A} \frac{\alpha}{\|\alpha\|} \sum_{\beta, \alpha = P_{\beta}} s_{\beta} \frac{\|P_{\beta}\|}{\|\beta\|} = \sum_{\alpha \in A} \frac{\alpha(Qs)}{\|\alpha\|} = uQs,
\]

and so

\[
(Q \oplus P)(s, -us) = (Qs, -uQs) \in \{(s, -us) : s \in S\}.
\]

Thus the projection \( \tilde{P} \) on \( \tilde{E} \) is defined to be the corresponding quotient operator of \( Q \oplus P \). Lastly, if \( e \in E \), then

\[
\tilde{P}je = \tilde{P}\pi(0, e) = \pi(0, Pe) = jPe. \tag{3.2}
\]

That is, \( \tilde{P}|_{jE} = jP \), and so \( \|\tilde{P}\| = \|P\| \).

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**Lemma 3.2.** Let \( E \) be a Banach space and \( P : E \to E \) a projection. Then there is a projection \( \hat{P} \) on \( X_E \) such that \( \|\hat{P}\| = \|P\| \) and \( \hat{P}i_n e = i_n Pe \) for all \( n \in \mathbb{N} \) and \( e \in E_n \).

**Proof.** By inductively using Lemma 3.1 with \( P = P_0 \), one can construct for each \( n \in \mathbb{N} \) a projection \( P_{n+1} \) on \( E_{n+1} \) such that \( \|P_{n+1}\| = \|P\| \) and

\[
P_{n+1}|_{j_n E_n} = j_n P_n. \tag{3.3}
\]

Recalling that the maps \( i_n \) from (2.1) satisfy \( i_{n+1}j_n = i_n \) for all \( n \in \mathbb{N} \), define \( \hat{P} \) on \( \bigcup_{n=0}^{\infty} i_n(E_n) \) by

\[
\hat{P}i_n e = i_n P_n e
\]

whenever \( e \in E_n \), which is well-defined by (3.3). Observe that \( \hat{P} \) is a projection with \( \|\hat{P}\| = \|P\| \), which can then be extended to a projection with the same norm, also denoted by \( \hat{P} \), on the inductive limit \( X_E \) via density.

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This construction can be used to lift an increasing sequence of projections on \( E \) to one on \( \tilde{E} \), and hence also to \( X_E \).

**Lemma 3.3.** Suppose that \( E \) is a Banach space and \( \{P_k\}_{k=1}^{\infty} \) is an increasing sequence of uniformly bounded projections on \( E \) with \( \sup_k \|P_k\| = K \). Then there is an increasing sequence of projections \( \{\tilde{P}_k\}_{k=1}^{\infty} \) on \( \tilde{E} \) with \( \sup_k \|\tilde{P}_k\| = K \). Moreover, \( \tilde{P}_k|_{jE} = jP_k \) for all \( k \in \mathbb{Z}^+ \).
Proof. Suppose that \( \{P_k\}_{k=1}^{\infty} \) is an increasing sequence of uniformly bounded projections on a Banach space \( E \). For each \( k \in \mathbb{Z}^+ \), let \( \tilde{P}_k \) be the projection obtained from \( P_k \) and Lemma 3.1 so that \( \tilde{P}_k \) is the quotient operator induced by \( Q_k \oplus P_k \). Then it remains to show that \( \{\tilde{P}_k\}_{k=1}^{\infty} \) is increasing, which would follow from \( \{Q_k\}_{k=1}^{\infty} \) being increasing. From (3.1), one has that

\[
(Q_k^* \psi)_\alpha = \begin{cases} 
\frac{\|P_k \alpha\|}{\|\alpha\|} \psi_{P_k \alpha} & \text{if } P_k \alpha \neq 0, \\
0 & \text{if } P_k \alpha = 0 
\end{cases}
\]

for each \( k \in \mathbb{Z}^+ \). Now suppose that \( \ell > k \) and \( \psi \in B^* \). If \( P_k \alpha = 0 \), then it easily follows that 

\[
(Q_k^* Q_\ell^* \psi)_\alpha = (Q_\ell^* Q_k^* \psi)_\alpha = 0 = (Q_k^* \psi)_\alpha.
\]

If \( P_k \alpha \neq 0 \), then

\[
(Q_k^* Q_\ell^* \psi)_\alpha = \frac{\|P_k \alpha\|}{\|\alpha\|} (Q_\ell^* \psi)_{P_k \alpha} = \frac{\|P_k \alpha\|}{\|\alpha\|} \frac{\|P_\ell \alpha\|}{\|P_k \alpha\|} \psi_{P_\ell P_k \alpha} = \frac{\|P_k \alpha\|}{\|\alpha\|} \psi_{P_k \alpha} = (Q_k^* \psi)_\alpha
\]

and

\[
(Q_\ell^* Q_k^* \psi)_\alpha = \frac{\|P_\ell \alpha\|}{\|\alpha\|} (Q_k^* \psi)_{P_\ell \alpha} = \frac{\|P_\ell \alpha\|}{\|\alpha\|} \frac{\|P_k \alpha\|}{\|P_\ell \alpha\|} \psi_{P_k P_\ell \alpha} = \frac{\|P_k \alpha\|}{\|\alpha\|} \psi_{P_k \alpha} = (Q_\ell^* \psi)_\alpha.
\]

Moreover, \( Q_\ell \neq Q_k \) since there are operators \( \alpha \in A \) for which \( P_\ell \alpha \neq 0 \) but \( P_k \alpha = 0 \). So \( \{Q_k^*\}_{k=1}^{\infty} \) and hence \( \{Q_k\}_{k=1}^{\infty} \), are increasing sequences of uniformly bounded projections. Hence \( \{\tilde{P}_k\}_{k=1}^{\infty} \) is an increasing sequence of uniformly bounded projections on \( \tilde{E} \).

**Theorem 3.4.** Let \( E \) be a Banach space and \( \{P_k\}_{k=1}^{\infty} \) an increasing sequence of uniformly bounded projections on \( E \). Then there is an increasing sequence of uniformly bounded projections \( \{\tilde{P}_k\}_{k=1}^{\infty} \) on \( X_E \) with \( \sup_k \|\tilde{P}_k\| = \sup_k \|P_k\| \).

**Proof.** Suppose that \( \{P_k\}_{k=1}^{\infty} \) is an increasing sequence of uniformly bounded projections on a Banach space \( E \). Lemmas 3.2 and 3.3 yield an increasing sequence of projections \( \{\tilde{P}_k\}_{k=1}^{\infty} \) on \( X_E \) with \( \sup_k \|\tilde{P}_k\| = \sup_k \|P_k\| \).

### 3.2 Main Result

In this section, it will be shown that no matter the Banach space \( E \), there is a well-bounded operator on \( X_E \) that is not of type (B).
Lemma 3.5. Suppose that \( \{P_k\}_{k=1}^{\infty} \) and \( \{\tilde{P}_k\}_{k=1}^{\infty} \) are as in Lemma 3.3. Then \( \lim_{k \to \infty} \tilde{P}_k \) does not exist in the SOT.

Proof. As \( \{P_k\}_{k=1}^{\infty} \) is an increasing sequence of projections, let \( \{e_k\}_{k=1}^{\infty} \) be a set of unit vectors in \( E \) satisfying \( e_1 \in P_1E \) and \( e_k \in P_kE \cap \ker P_{k-1} \) for all \( k \geq 2 \). Further let \( e = \sum_{k=1}^{\infty} 2^{-k}e_k \in E \) and \( \gamma \in \mathcal{A} \) be a rank 1 operator whose range is \( \text{span}\{e\} \) with \( \|\gamma\| = 1 \), so that \( P_\ell \gamma \neq P_k \gamma \) for all \( \ell \neq k \).

Let \( b \in B_\gamma \setminus S_\gamma \) with \( d(b, S_\gamma) = 1 \), and \( x = (x_\alpha)_{\alpha \in A} \in B \) be defined by

\[
x_\alpha = \begin{cases} b & \text{if } \alpha = \gamma, \\ 0 & \text{otherwise.} \end{cases}
\]

We will show that \( \{\tilde{P}_k \pi(x, 0)\}_{k=1}^{\infty} \) does not converge. This is essentially because for \( \ell \neq k \), \( P_\ell \gamma \neq P_k \gamma \) means that \( b \) will always occupy a different coordinate in \( Q_\ell x \) to \( Q_k x \). So, if \( s \in S \) and \( \ell > k \), we have that

\[
\|(Q_\ell \oplus P_\ell - Q_k \oplus P_k)(x - s, us)\| \geq \|(Q_\ell - Q_k)(x - s)\|
\]

\[
= \sum_{\alpha \in A} \left| \sum_{\beta: \alpha = P_\ell \beta} (x - s)_\beta \frac{\|P_\ell \beta\|}{\|\beta\|} - \sum_{\beta: \alpha = P_k \beta} (x - s)_\beta \frac{\|P_k \beta\|}{\|\beta\|} \right|.
\]

By considering only the \( \alpha = P_\ell \gamma \) coordinate, we have that

\[
\|(Q_\ell - Q_k)(x - s)\| \geq \|x_\gamma P_\ell \gamma\| - \sum_{\beta: P_\ell \gamma = P_\ell \beta} s_\beta \frac{\|P_\ell \beta\|}{\|\beta\|} + \sum_{\beta: P_\ell \gamma = P_k \beta} s_\beta \frac{\|P_k \beta\|}{\|\beta\|}
\]

\[
\geq \|P_\ell \gamma\| d(b, S_\gamma)
\]

\[
= \|\tilde{P}_k \gamma\|.
\]

Since \( P_n e \to e \) as \( n \to \infty \), there is some \( N \in \mathbb{Z}^+ \) such that \( 2\|P_n e\| > \|e\| \) whenever \( n > N \). Let \( s_e \in S_\gamma \) satisfy \( \gamma s_e = e \) so that

\[
\|P_n \gamma\| \geq \frac{\|P_n \gamma s_e\|}{\|s_e\|} = \frac{\|P_n e\|}{\|s_e\|} > \frac{\|e\|}{2\|s_e\|}
\]

for all \( n > N \). Hence

\[
\|(\tilde{P}_\ell - \tilde{P}_k) \pi(x, 0)\| \geq \inf_{s \in S} \|(Q_\ell - Q_k)(x - s)\| \geq \frac{\|e\|}{2\|s_e\|}
\]

for all \( \ell > k > N \). That is, \( \{\tilde{P}_k\}_{k=1}^{\infty} \) does not converge in the SOT. \( \square \)
Consequently, whenever a Banach space $E$ has an increasing sequence of uniformly bounded projections, Lemma 3.2 and Theorem 2.3 yield a well-bounded operator on $X_E$ not of type (B). Now, it is not known in general whether one can always find such a sequence of projections on an infinite dimensional Banach space $E$, and certainly it is not possible when $E$ is finite dimensional. However, it will be shown that one can always find such a sequence of projections on $\tilde{E}$.

**Lemma 3.6.** If $E$ is finite dimensional, then there is an increasing sequence of uniformly bounded projections on $\tilde{E}$.

**Proof.** As $E$ is finite dimensional, $jE$ is complemented in $\tilde{E}$ and so one makes the identification

$$\tilde{E} = jE \oplus (\tilde{E}/jE)$$

as a topological direct sum. Recall that $\tilde{E}/jE$ is isometric to $B/S$, which can be identified isometrically by Lemma 2.7 as $\ell^1(B/\alpha \in A)$. As $A$ is infinite, one can easily construct an increasing sequence of uniformly bounded projections on $\tilde{E}/jE$, and hence also on $\tilde{E}$.

For the case of infinite dimensional $E$, one can construct a uniformly bounded sequence of increasing projections on $\tilde{E}$ in the following way. Recall from the construction of $\tilde{E}$ that $B_2 = L^1(\mathbb{T})$ and $S_2 = H^1(\mathbb{T})$. Define the projection $p : B_2 \to B_2$, whose range is in $S_2$, by

$$p(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \, d\theta.$$ 

That is, $p(f)$ is the constant function whose value is the mean of $f$. Now let $\{e_m\}_{m=1}^\infty$ be a set of linearly independent elements of the unit sphere of $E$. For each $m \in \mathbb{Z}^+$, define $\alpha_m \in F(S_2, E)$ by

$$\alpha_m(f) = \left( \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})(1 + e^{-i\theta}) \, d\theta \right) e_m.$$ 

As $\{e_m\}_{m=1}^\infty$ is linearly independent, we have that $\alpha_m p \neq \alpha_n p$ and $\alpha_m \neq \alpha_n$, for all $m \neq n$. Also, for all $m \in \mathbb{Z}^+$, one can easily verify that $\alpha_m \neq \alpha_m p$, $\|\alpha_m\| \leq 2$ and $\|\alpha_m p\| = 1$. Recall that $A = \bigcup_{i=1}^3 F(S_i, E)$ and $B = \ell^1(B_\alpha)_{\alpha \in A}$. For each $n \in \mathbb{Z}^+$, define $Q_n : B \to B$ by

$$(Q_n x)_\alpha = \begin{cases} x_\alpha & \text{if } \alpha = \alpha_m p \text{ for some } 1 \leq m \leq n, \\ -\|\alpha_m\| p x_{\alpha_m} & \text{if } \alpha = \alpha_m \text{ for some } 1 \leq m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Elaborating upon this definition, for each $1 \leq m \leq n$, the map $Q_n$ preserves the $(\alpha_m p)$th coordinate, replaces the $(\alpha_m)$th coordinate by $-\|\alpha_m\| p x_{\alpha_m}$, and deletes all the other coordinates.

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Lemma 3.7. \( \{Q_n\}_{n=1}^{\infty} \) is an increasing sequence of uniformly bounded projections on \( B \).

Proof. Clearly each \( Q_n \) is linear and \( \sup_n \|Q_n\| \leq 3 \). Now, for \( k \geq n \) and \( \alpha \in A \setminus \{\alpha_m\}_{m=0}^{n} \), one has that \( (Q_nQ_kx)_{\alpha} = (Q_kQ_nx)_{\alpha} = (Q_nx)_{\alpha} \) as these are all of the coordinates of \( x \) that are unchanged or deleted. Otherwise, if \( \alpha = \alpha_m \) for some \( 1 \leq m \leq n \), we have that

\[
(Q_nQ_kx)_{\alpha} = -\|\alpha_m\|p(\alpha_m) = (Q_nx)_{\alpha}.
\]

Thus we have that \( Q_kQ_n = Q_nQ_k = Q_n \) whenever \( k \geq n \). Moreover, \( Q_k \neq Q_n \) for all \( k \neq n \), and so \( \{Q_n\}_{n=1}^{\infty} \) is an increasing sequence of uniformly bounded projections.

\[ \square \]

Remark 3.8. One can construct \( \{Q_n\}_{n=1}^{\infty} \) more generally with any operator \( p \in B(X_i, S_i) \) and collection of operators \( \{\alpha_m\}_{m=1}^{\infty} \subset F(S_i, E) \) satisfying \( \sup_m \|\alpha_m\| < \infty \), \( \alpha_m \neq \alpha_n \) for all \( m \neq n \).

The projections in Lemma 3.7 may now be used to defined an increasing sequence of uniformly bounded projections on \( \tilde{E} \).

Lemma 3.9. If \( E \) is an infinite dimensional Banach space, then there is an increasing sequence of uniformly bounded projections \( \{P_n\}_{n=1}^{\infty} \) on \( \tilde{E} \).

Proof. Let \( \{Q_n\}_{n=1}^{\infty} \) be as in Lemma 3.7 and consider the increasing sequence of uniformly bounded projections \( \{Q_n \oplus 0\}_{n=1}^{\infty} \) on \( B \oplus \tilde{E} \). It will be shown that \( \{(s, -us)\}_{s \in S} \) is invariant under \( Q_n \oplus 0 \) for each \( n \in \mathbb{Z}^+ \). Recalling that \( \|\alpha_m p\| = 1 \) for all \( m \in \mathbb{N} \), we have that

\[
u Q_n s = \sum_{\alpha \in A} \frac{\alpha(Q_n s)_{\alpha}}{\|\alpha\|} = \sum_{m=0}^{n} \frac{\alpha_m p s_{\alpha_m p}}{\|\alpha_m p\|} - \sum_{m=0}^{n} \frac{\alpha_m (\|\alpha_m\| p s_{\alpha_m p})}{\|\alpha_m\|} = 0,
\]

and so \( (Q_n \oplus 0)(s, -us) = (Q_n s, 0) = (Q_n s, -uQ_n s) \) for all \( s \in S \). Thus setting \( P_n \) to be the quotient of \( Q_n \) for each \( n \in \mathbb{Z}^+ \), which will satisfy \( P_n \neq P_m \) for all \( n \neq m \), defines an increasing sequence of uniformly bounded projections \( \{P_n\}_{n=1}^{\infty} \) on \( \tilde{E} \). \[ \square \]
Corollary 3.10. For any Banach space $E$, there is an increasing sequence of uniformly bounded projections on $X_E$.

**Proof.** Suppose that $E$ is a Banach space. By Lemmas 3.6 and 3.9, there is a uniformly bounded sequence of increasing projections $\{P_n\}_{n=1}^\infty$ on $\tilde{E}$. As $X_E = X_{\tilde{E}}$, Lemma 3.2 yields a uniformly bounded sequence of increasing projections on $X_E$. □

The main result can now be easily concluded.

**Theorem 3.11.** Let $E$ be a Banach space. Then there is a well-bounded operator on $X_E$ that is not of type (B).

**Proof.** By Corollary 3.10 there is a uniformly bounded sequence of increasing projections $\{P_n\}_{n=1}^\infty$ on $X_E$ for any Banach space $E$. By Lemma 3.5, $\lim_{n \to \infty} P_n$ does not exist in the SOT, and hence the well-bounded operator on $X_E$ obtained via Theorem 2.3 is not of type (B). □

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