RELATIONSHIPS BETWEEN THE BERGMAN KERNELS OF HARTOGS DOMAINS AND THEIR BASE*

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Abstract. We explore the relationship between the Bergman kernel of a Hartogs domain and weighted Bergman kernels over its base domain. In particular we develop a representation of the Bergman kernel of a Hartogs domain as a series involving weighted Bergman kernels over its base, as well as a “transformation” formula for some weighted Bergman kernels. Other relationships of this type are presented.

1. Introduction

Suppose we are given a Hartogs domain of the form \( D = \{(z, w) \in G \times \mathbb{C}^N : \|w\| < e^{-\varphi(z)}\} \) over some base domain \( G \subseteq \mathbb{C}^M \). What kind of relationship is there between the holomorphic function theory on \( D \) and the holomorphic function theory on \( G \)? This is the question which we will investigate throughout this note.

Holomorphic function theory of several complex variables is much more delicate than that of one complex variable, so it is natural that the first domains studied in \( \mathbb{C}^M, M \geq 2 \), were domains with large amounts of symmetry, e.g. the unit ball and polydisk. A logical next step in this progression are Hartogs domains \( D \) as described above. These can be viewed as a collection of balls fibred over a base domain \( G \) whose radii are given by a function \( e^{-\varphi(z)} \), where \( \varphi \) is upper semicontinuous (so that \( D \) is indeed a domain). In this context, a Hartogs domain is a natural extension of the notion of a bidisk, which is, loosely speaking, a “trivial” collection of disks fibred over a disk.

The extra symmetry in one of the variables often allows for computations on \( D \) to be reduced to computations on \( G \), a domain with lower dimension. This strategy was used to great success in the work of Jucha [13]. For instance, given a holomorphic function \( f(z, w) \) on \( D \), fix \( z \in G \) and view \( w \mapsto f(z, w) \) as a function on the ball centred at the origin with radius \( e^{-\varphi(z)} \). Consequently \( f \) admits a power series expansion \( f(z, w) = \sum_n f_n(z)w^n \), where the \( f_n(z) \) are just the coefficients of the Maclaurin series for a fixed \( z \in G \). It is not difficult to see that the \( f_n \) must indeed be holomorphic functions of \( z \), and therefore to understand the holomorphic function theory of \( D \) it suffices in many cases to understand the holomorphic function theory \( G \) (for more details see Jakóbczak–Jarnicki [12]). Since the Bergman kernel of a domain is essentially the embodiment of its holomorphic function theory, our goal is to find relationships between the Bergman kernel of \( D \) and the Bergman kernel of \( G \).

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It is known that $f_n(z)$ is square-integrable with respect to the weight $e^{-2(N+|n|)\varphi}$ whenever $f(z,w)$ is square-integrable. In fact, the norms are proportional. In §3, we compute this constant of proportionality explicitly and use it to derive an expression of the Bergman kernel of $D$ as an infinite series involving weighted Bergman kernels on $G$. In §4, this relationship is used to determine explicit representations of the Bergman kernel for certain types of Hartogs domains, as well as an alternative proof of the “inflation” identity of Boas–Fu–Straube. A “transformation” formula for weighted Bergman spaces is developed in §5, and is applied to Hartogs domains of an appropriate type. The ideas of §5 are then extended to the context of Borichev–Le–Youssfi [4] using tools of analytic geometry in §6.

2. Notations and Preliminaries

Unless otherwise specified $G \subseteq \mathbb{C}^M$ is a domain, that is, $G$ is a connected open set in $\mathbb{C}^M$.

We write USC($G$) to denote the set of upper-semicontinuous functions on $G$. Likewise, we write PSH($G$) to denote the set of plurisubharmonic functions on $G$. The function identically equal to negative infinity is not admitted to be plurisubharmonic.

In a similar manner, $\mathcal{O}(G)$ denotes the set of holomorphic functions with domain $G$. If $\varphi \in USC(G)$, we set

$$L^2_{h}(G, \varphi) := \left\{ f \in \mathcal{O}(G) : \| f \|^2_{G, \varphi} := \int_{G} |f|^2 e^{-2\varphi} dV_M < \infty \right\},$$

where $dV_M$ denotes Lebesgue measure on $\mathbb{C}^M$. (We will often drop the subscript on $dV_M$ whenever there is no chance of confusion.) Under very mild assumptions on $\varphi$ (e.g. whenever $e^\varphi \in L^1_{loc}(G)$ [16, Corollary 3.1]), it can be shown that $L^2_{h}(G, \varphi)$ is a Hilbert space, known as the Bergman space of $G$ with weight $\varphi$, and admits a unique reproducing kernel $K_{G, \varphi}(z, \bar{\zeta}) \in \mathcal{O}(G \times G)$ which is likewise known as the Bergman kernel of $G$ with weight $\varphi$. This reproducing kernel has the property that

$$f(z) = \int_{G} f(\zeta) K_{G, \varphi}(z, \zeta) e^{-2\varphi(\zeta)} dV(\zeta)$$

for all $f \in L^2_{h}(D, \varphi)$. We adopt the common convention that $L^2_{h}(G) := L^2_{h}(G, 0)$, $\| f \|_G := \| f \|_{G, 0}$, and $K_{G, 0}(z, \zeta) := K_G(z, \zeta)$. The latter is referred to as the (unweighted) Bergman kernel of $G$.

In the last section of this note we will briefly make use of the notion of currents. A current $\Theta$ of bidegree $(p, q)$ on $G$ is simply a differential $(p, q)$-form $G$ with distributional coefficients. Another way to view bidegree $(p, q)$ currents is as the dual to the space of smooth differential forms of bidegree $(M-p, M-q)$ having compact support, with the pairing written as

$$\langle \Theta, \alpha \rangle = \int_{G} \Theta \wedge \alpha,$$

where $\alpha$ is a smooth $(M-p, M-q)$-form with compact support in $G$. Note that the integration above is merely formal whenever $\Theta$ is not an honest differential form. If $S$ is
RELATIONSHIPS BETWEEN THE BERGMAN KERNELS OF HARTOGS DOMAINS AND THEIR BASE

a complex hypersurface in $G$, then there is an associated $(1,1)$-current $[S]$ given by the pairing

$$\langle [S], \alpha \rangle = \int_S \alpha, \quad \alpha \text{ is a smooth } (M-1, M-1)-\text{form on } G.$$ (1)

Currents of this type are known as \textit{currents of integration}. This should be viewed as the analogue of the Dirac delta “function” from the theory of distributions. If $S$ is more generally an analytic set of codimension one (i.e. locally the zero set of a holomorphic function), then (1), integrated over the regular part of $S$, can also be used to define an analogous $(1,1)$-current associated to $[S]$ in this context—the only difficulty is ensuring that $S$ has locally finite area near its singular set. For more information on currents, the reader can consult Demailly [5, §1.C.] for a concise introduction, or Federer [8, §4.1] for a thorough treatment.

The set of nonnegative integers is denoted by $\mathbb{Z}_+$. We also write $\mathbb{Z}_+^N := (\mathbb{Z}_+)^N$ and $|n| = |n_1| + \cdots + |n_N|$ for a multiindex $n \in \mathbb{Z}_+^N$. We will frequently denote the unit disk and punctured unit disk in the complex plane as $\mathbb{D}$ and $\mathbb{D}^*$, respectively.

3. $p$-Hartogs Domains with Complete $N$-circled Fibres

Often when discussing Hartogs domains, the norm in which the circular symmetry takes place is assumed to be the $\ell^2$-norm. However many of the results discussed in this note hold in the more general setting where instead the “circular” symmetry takes place in the $\ell^p$-norm, $p \in [1, \infty]$.

In this section we define Hartogs domains in a way which makes the $\ell^p$-norm used explicit. Write $\|w\|_p$ for $p \in [1, \infty]$ and $w \in \mathbb{C}^N$.

**Definition 3.1.** Let $G \subseteq \mathbb{C}^M$ be a domain, $\varphi \in \text{USC}(G)$, and $p \in [1, \infty]$. Define

$$D^p_\varphi(G) = D^p = \{ (z, w) \in G \times \mathbb{C}^N \subseteq \mathbb{C}^{M+N} : \|w\|_p < \exp(-\varphi(z)) \}.$$ 

We call $D^p_\varphi(G)$ a \textit{complete} $N$-circled $p$-Hartogs domain with base $G$.

We remark here that $D^p_\varphi(G)$ is defined by the inequality $\|w\|_p < \exp(-\varphi(z))$ as opposed to $\|w\|_p < \exp(2\varphi(z))$ which one might expect based on the notation so far. This is done to clean up computations.

The following lemma shows that in order to understand the $L^2$-function theory of $p$-Hartogs domains with complete $N$-circled fibres it suffices to understand the function theory of certain weighted $L^2$-spaces over the base domain.

**Lemma 3.2.** Let $D^p_\varphi = D^p \subseteq G \times \mathbb{C}^N$ be a $p$-Hartogs domain over $G \subseteq \mathbb{C}^M$ with complete $N$-circled fibres.

(a) If $f \in \mathcal{O}(D^p_\varphi)$, then there exist $f_n \in \mathcal{O}(G)$, $n \in \mathbb{Z}_+^N$, such that

$$f(z, w) = \sum_{n \in \mathbb{Z}_+^N} f_n(z)w^n, \quad (z, w) \in D^p_\varphi,$$ (2)

and the series is uniformly convergent on compact subsets of $D^p_\varphi$. 

(b) If $f \in L^2_{p}(D^p_{\varphi})$, then $f_n(z)w^n \in L^2_{p}(D)$ for every $n \in \mathbb{Z}_+^N$ and the series \(^2\) is convergent in $L^2_{p}(D^p_{\varphi})$.

(c) If $f(z, w) = \sum_{n \in \mathbb{Z}_+^N} f_n(z)w^n \in L^2_{p}(D^p_{\varphi})$, then there exists a real number $C = C(p; n)$ so that $\|f_n w^n\|_{D^p_{\varphi}} = C\|f_n\|_{G,(|n|+N)\varphi}$. Furthermore,

$$
C(p; n) = \left(\frac{2^N}{p}\right)^{N-1} \frac{\pi^n \prod_{k=1}^N \Gamma\left(\frac{2n_k+2}{p}\right)}{(N + |n|) \cdot \Gamma\left(\frac{2N+2|n|}{p}\right)},
$$

whenever $p \in [1, \infty)$, and

$$
C(\infty; n) = \frac{\pi^n}{\prod_{k=1}^N (n_k + 1)}.
$$

Before beginning the proof of Lemma 3.2, it is useful to define generalized polar coordinates on $\mathbb{C}^N$. Let $S^N_p$ denote the unit $\ell^p$-sphere $\{w \in \mathbb{C}^N : \|w\|_p = 1\}$. For $w \neq 0$, the $p$-polar coordinates of $w$ are

$$
r_p = \|w\|_p \in (0, \infty) \quad \text{and} \quad w'_p = \frac{w}{\|w\|_p} \in S^N_p.
$$

The subscript $p$ will often be omitted when there is no chance of confusion.

Following classical theory (e.g. Folland [9, §2.7]), there is a unique Borel measure $\sigma_p = \sigma_{p,N}$ on $\mathbb{C}^N$ such that if $f$ is Borel measurable on $\mathbb{C}^N$ and $f \geq 0$ or $f \in L^1(\mathbb{C}^N)$, then

$$
\int_{\mathbb{C}^N} f(w) \mathrm{d}V = \int_0^\infty \int_{S^N_p} f(rw')r^{2N-1} d\sigma_p(w') \mathrm{d}r.
$$

**Proof of Lemma 3.2.** The proofs of (a) and (b) are the same as in the case $p = 2$ (see e.g. Jakóbczak–Jarnicki [12, Proposition 1.6.5] and Jucha [13, Lemma 3.1]), so we need only to show (c).

We will abuse notation slightly and write $(z, w) = (z, w_1, \ldots, w_N)$ for a point in $D^p_{\varphi}$; this should not be confused with the notation of $p$-polar coordinates introduced above. Indeed, we will be omitting the subscript “$p$” from the use of these $p$-polar coordinates here. By [13],

$$
\|f_n(z)w^n\|_{D^p_{\varphi}}^2 = \int_{D^p_{\varphi}} |f_n(z)|^2 |w^n|^2 \mathrm{d}V_{M+N}(z, w)
$$

$$
= \int_G |f_n(z)|^2 \int_{|w| < e^{-\varphi(z)}} \left(|w_1|^{n_1} \cdot |w_2|^{n_2} \cdot \ldots \cdot |w_N|^{n_N}\right) \mathrm{d}V_N(w) \mathrm{d}V_M(z)
$$

$$
= \int_G |f_n(z)|^2 \left(\int_0^{\exp(-\varphi(z))} r^{2N+2|n|-1} \mathrm{d}r\right) \cdot \left(\int_{S^N_p} \prod_{k=1}^N \left(\frac{|w_k|}{\|w\|_p}\right)^{n_k} d\sigma_p \right) \mathrm{d}V_M(z)
$$

$$
= C(p; n) \int_G |f_n(z)|^2 e^{-2(N+|n|)\varphi(z)} \mathrm{d}V_M(z),
$$
RELATIONSHIPS BETWEEN THE BERGMAN KERNELS OF HARTOGS DOMAINS AND THEIR BASIS

where

\[ C(p; n) = \frac{1}{2N + 2|n|} \int_{S_p^N} \left( \frac{|w_1|}{\|w\|_p} \right)^{2n_1} \cdots \left( \frac{|w_N|}{\|w\|_p} \right)^{2n_N} \, d\sigma_p. \]  \hspace{1cm} (4)

To find \( C(p; n) \) explicitly when \( 1 \leq p < \infty \), first note that (3) yields

\[ \int_{C^N} e^{-\|w\|_p^p} \prod_{k=1}^N |w_k|^{2n_k} dV(w) = \int_{S_p^N} \prod_{k=1}^N \left( \frac{|w_k|}{\|w\|_p} \right)^{2n_k} \int_0^\infty e^{-r_p r^{2N-1} \sum_{k=1}^N n_k} dr \, d\sigma_p. \]  \hspace{1cm} (5)

By a change of variables \( s = r^p \), the inner integral on the right side of (5) is

\[ \int_0^\infty e^{-r_p s^{2N+2|n|-1} \frac{1}{p}} dr = \frac{1}{p} \int_0^\infty e^{-s \frac{2N+2|n|}{p}} \frac{1}{p} ds = \frac{1}{p} \Gamma \left( \frac{2N+2|n|}{p} \right). \]  \hspace{1cm} (6)

On the other hand,

\[ \int_{C^N} e^{-\|w\|_p^p} \prod_{k=1}^N |w_k|^{2n_k} dV(w) = \prod_{k=1}^N \int_C e^{-|w_k|^p} |w_k|^{2n_k} dV_1(w_k) \]
\[ = (2\pi)^N \prod_{k=1}^N \int_0^\infty e^{-r_p r^{2n_k+1}} dr \]
\[ = \left( \frac{2\pi}{p} \right)^N \prod_{k=1}^N \int_0^\infty e^{-s \frac{2n_k+2}{p}} ds \]
\[ = \left( \frac{2\pi}{p} \right)^N \prod_{k=1}^N \Gamma \left( \frac{2n_k+2}{p} \right). \]  \hspace{1cm} (7)

Using equation (5) to compare equations (6) and (7) yields

\[ \int_{S_p^N} \left( \frac{|w_1|}{\|w\|_p} \right)^{2n_1} \cdots \left( \frac{|w_N|}{\|w\|_p} \right)^{2n_N} \, d\sigma_p = \frac{(2\pi)^N \prod_{k=1}^N \Gamma \left( \frac{2n_k+2}{p} \right)}{p^{N-1} \Gamma \left( \frac{2N+2|n|}{p} \right)} \]

and applying this to equation (4) shows the desired formula for \( C(p; n) \) above for \( p \in [1, \infty) \).
Lastly, since
\[
\int_{\|w\|_{\infty} < e^{-\phi(z)}} |w_1|^{2n_1} \cdots |w_N|^{2n_N} \, dV(w) = \prod_{k=1}^{N} \int_{|w_k| < e^{-\phi(z)}} |w_k|^{2n_k} \, dV_1(w_k)
\]
\[
= (2\pi)^N \prod_{k=1}^{N} \int_{0}^{e^{-\phi(z)}} r^{2n_k+1} \, dr
\]
\[
= \pi^N \prod_{k=1}^{N} \frac{1}{n_k + 1} e^{-2(n_k+1)\phi(z)}
\]
\[
= \pi^N e^{-2\phi(z)} \sum_{k=1}^{N} \frac{n_k + 1}{n_k + 1}
\]
\[
= \pi^N e^{-2(|n|+N)\phi(z)} \prod_{k=1}^{N} \frac{1}{n_k + 1},
\]
we see the desired formula for $C(\infty; n)$.

Lemma 3.2 has been known for quite some time in various forms, but we have not seen the following immediate consequence stated explicitly, so we mention it here.

**Corollary 3.3.** Let $D = D^p_{\phi} \subseteq \mathbb{C}^M \times \mathbb{C}^N$ be a pseudoconvex $p$-Hartogs domain. Suppose that there exists a point $z_0 \in G$ for which the Lelong number
\[
\nu(\phi, z_0) := \lim_{r \to 0} \frac{(2\pi)^{-1} \Delta \phi(z_0 + rB^M)}{dV_{M-1}(rB^M \cap \mathbb{C}^{M-1})},
\]
where $B^M \subset \mathbb{C}^M$ is the unit ball

is positive. Then the Bergman kernel of $D$ has zeroes.

**Proof.** By Lemma 3.2 any $f \in L^2_h(D)$ can be written as
\[
f(z, w) = \sum_{n \in \mathbb{Z}_+^N} f_n(z) w^n
\]
for some $f_n \in L^2_h(G, (|n| + N)\phi)$. By assumption $\nu(\phi, z_0) > 0$, so for sufficiently large $n \in \mathbb{Z}_+^N$ we have $\nu((|n| + N)\phi, z_0) \geq M$. Since $D^p_{\phi}$ is pseudoconvex, we see that $\phi \in \text{PSH}(G)$ [12 Proposition 4.1.14(b)], and so the work of Kiselman [14 Theorem 3.2] shows that $e^{-2(|n|+N)\phi}$ is not integrable in any neighbourhood of $z_0$ for large $n \in \mathbb{Z}_+^N$. Consequently $f_n(z_0, w) = 0$ for infinitely many $n$, and hence $w \mapsto f(z_0, w)$ is a polynomial on $\mathbb{C}^N$. Now, the fundamental theorem of algebra shows that $f(z_0, \cdot)$ has zeroes, and since $K_D$ is a member of $L^2_h(D)$ when one of its variables is fixed, this completes the proof. □

Our first theorem relates the Bergman kernel of $D^p$ to weighted Bergman kernels on $G$. This was shown in [15 Proposition 0.1], however only the case $p = 2$ is discussed, and $C(2; n)$ is not computed explicitly.
RELATIONSHIPS BETWEEN THE BERGMAN KERNELS OF HARTOGS DOMAINS AND THEIR BASIS

**Theorem 3.4.** If $G \subseteq \mathbb{C}^M$ is a domain and $\varphi \in \text{USC}(G)$, then the Bergman kernel $K_D(z, \zeta, w, \eta)$ of $D = D^p\varphi(G)$ can be written as

$$
\begin{cases}
\left(\frac{2}{p}\right)^{N-1} \pi^N \sum_{n \in \mathbb{Z}_+^N} \left(\frac{2N+2|n|}{p}\right) \prod_{k=1}^N \Gamma\left(\frac{2N+2|n|}{p}\right) K_{G,(N+|n|)\varphi}(z, \zeta)(w\bar{\eta})^n, & \text{whenever } p \in [1, \infty) \\
\frac{1}{\pi^N} \sum_{n \in \mathbb{Z}_+^N} \prod_{k=1}^N (n_k + 1) K_{G,(N+|n|)\varphi}(z, \zeta)(w\bar{\eta})^n, & \text{whenever } p = \infty,
\end{cases}
$$

with uniform convergence on compact sets of $D \times D$.

Note that in the particular case of $N = 1$, the domain $D^p\varphi(G)$ is not dependent on $p$. Indeed, all of the $\ell^p$-norms agree in one dimension.

This special case of Theorem 3.4 will be stated separately as it will be used often.

**Corollary 3.5.** Let $G$ and $\varphi$ be as above and let $N = 1$. Then

$$K_{D\varphi(G)}(z, \zeta, w, \eta) = \frac{1}{\pi} \sum_{n=0}^\infty (1 + n) K_{G,(1+n)\varphi}(z, \zeta)(w\bar{\eta})^n.$$

**Proof of Theorem 3.4.** Let $\{\chi_{n,j}\}_{j=0}^\infty$ be an orthonormal basis for $L^2_h(G, (N+|n|)\varphi)$ for each multiindex $n$. Then each $g_n \in L^2_h(G, (N+|n|)\varphi)$ has a unique representation $g_n = \sum_{j=0}^\infty c_{n,j} \chi_{n,j}$ and hence each $f \in L^2_h(D)$ has a decomposition

$$f(z, w) = \sum_{n \in \mathbb{Z}_+^N} f_n(z)w^n = \sum_{j=0}^\infty \sum_{n \in \mathbb{Z}_+^N} c_{n,j} \chi_{n,j}(z)w^n;$$

thus $\{\chi_{n,j}(z)w^n\}_{(n,j) \in \mathbb{Z}_+^N}$ is an orthogonal basis for $D$. By Lemma 3.2.

$$||\chi_{n,j}(z)w^n||_D^2 = \left\{ \left(\frac{2}{p}\right)^{N-1} \pi^N \prod_{k=1}^N \Gamma\left(\frac{2N+2|n|}{p}\right), \frac{1}{\pi^N} \prod_{k=1}^N (n_k + 1) \right\}, \quad \text{whenever } p \in [1, \infty)$$

so dividing by this quantity and taking a square root provides an orthonormal basis for $L^2_h(D)$.

The representation of the Bergman kernel by an orthonormal basis (see e.g. Hedenmalm–Korenblum–Zhu [11, §9.1]) yields

$$K_{D^p\varphi}(z, \zeta, w, \eta) = \frac{1}{2^{N-1}\pi^N} \sum_{j=0}^\infty \sum_{n \in \mathbb{Z}_+^N} \frac{p^{N-1}(N+|n|) \cdot \Gamma\left(\frac{2N+2|n|}{p}\right)}{\prod_{k=1}^N \Gamma\left(\frac{2n_k+2}{p}\right)} \chi_{n,j}(z) \chi_{n,j}(\zeta) (w\bar{\eta})^n$$

$$= \frac{1}{2^{N-1}\pi^N} \sum_{n \in \mathbb{Z}_+^N} p^{N-1}(N+|n|) \cdot \Gamma\left(\frac{2N+2|n|}{p}\right) K_{G,(N+|n|)\varphi}(z, \eta)(w\bar{\eta})^n$$
whenever $p \in [1, \infty)$, and

\[
K_{D^p}(z, \zeta, w, \eta) = \frac{1}{\pi^N} \sum_{n \in \mathbb{Z}_+^N} \sum_{j=0}^{\infty} \prod_{k=1}^{N} (n_k + 1) \chi_{n,j}(z) \chi_{n,j}(\zeta)(w \eta)^n
\]

\[
= \frac{1}{\pi^N} \sum_{n \in \mathbb{Z}_+^N} \prod_{k=1}^{N} (n_k + 1) K_{G,(N+|n|)\varphi}(z, \zeta)(w \eta)^n. \quad \square
\]

Theorem 3.4 gives an alternative proof of the well-known “inflation” identity of Boas–Fu–Straube [3, §2.2].

**Corollary 3.6** (Boas–Fu–Straube [3]). Let $G \subseteq \mathbb{C}^M$ be a domain. Consider two Hartogs domains:

\[
D = \{(z, w) \in G \times \mathbb{C} : |w| < \exp(-\varphi(z))\}
\]

and

\[
\tilde{D} = \{(z, W) \in G \times \mathbb{C}^N : ||W||_2 < \exp(-\varphi(z))\}.
\]

Let $K_D(z, \zeta, w, \eta)$ and $K_{\tilde{D}}(z, Z, w, W)$ be the Bergman kernels for $D$ and $\tilde{D}$, respectively. Because of the circular symmetry in the one-dimension variable (e.g. by observing the representation in Corollary 3.4), $K_D(z, \zeta, w, \eta)$ can be written as $L(z, \zeta, w \eta)$. Then

\[
K_{\tilde{D}}(z, Z, w, W) = \frac{1}{\pi^{N-1}} \frac{\partial^{N-1}}{\partial t^{N-1}} \left. L(z, w, t) \right|_{t=(Z,W)}.
\]

**Proof.** Using Corollary 3.5, we see that

\[
L(z, \zeta, t) = \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1) K_{G,(1+n)\varphi}(z, \zeta) t^n,
\]

from which it follows

\[
\frac{\partial^{N-1}}{\partial t^{N-1}} L(z, \zeta, t) = \frac{1}{\pi} \sum_{n=N-1}^{\infty} \frac{(n+1)!}{(n-N+1)!} K_{G,(1+n)\varphi}(z, \zeta) t^{n-N+1}
\]

\[
= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(k+N)!}{k!} K_{G,(N+k)\varphi}(z, \zeta) t^k;
\]
RELATIONSHIPS BETWEEN THE BERGMAN KERNELS OF HARTOGS DOMAINS AND THEIR BASIS

Here we made the change of variables $k = n - N + 1$. By the multinomial theorem and changing into multiindex notation, we therefore have

$$\frac{\partial^{N-1}}{\partial t^{N-1}} L(z, \zeta, t) \bigg|_{t=(Z,W)} = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(k + N)!}{k!} K_{G,(N+k)\varphi}(z, \zeta) \left( \sum_{j=1}^{N} Z_j \bar{W}_j \right)^k$$

$$= \frac{1}{\pi} \sum_{\alpha \in \mathbb{Z}_+^N} \frac{(|\alpha| + N)!}{\alpha!} K_{G,(N+|\alpha|)\varphi}(z, \zeta)(Z\bar{W})^\alpha$$

$$= \frac{1}{\pi} \sum_{\alpha \in \mathbb{Z}_+^N} \frac{(|\alpha| + N)\Gamma(|\alpha| + N)}{\prod_{k=1}^{N} \Gamma(\alpha_k + 1)} K_{G,(N+|\alpha|)\varphi}(z, \zeta)(Z\bar{W})^\alpha$$

$$= \pi^{N-1} K_{D}(z, \zeta, Z, W).$$

The last equality comes from a final use of Theorem 3.4. □

**Remark 3.7.** It would be interesting to see if one could develop a generalized version of the inflation identity (Corollary 3.6) for a general $p \in [1, \infty]$, rather than just when $p = 2$. It seems however that such an identity would involve a more complicated object than a differential operator.

4. Explicit Formulae for Bergman Kernels of Certain Hartogs Domains

Here we apply the techniques developed to determine the Bergman kernel of some Hartogs domains.

4.1. Generalization of a Domain of Bergman. Bergman [1, Formula (2,3)] found that the Bergman kernel of the domain $D_q = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^{2q} < 1\}, q > 0$, is given by

$$K_{D_q}(z, \zeta, w, \eta) = \frac{(q - 1)w\bar{\eta} - (q + 1)(1 - z\bar{\zeta})^{1/q}}{q\pi^2(1 - z\bar{\zeta})^{(2q-1)/q}(w\bar{\eta} - (1 - z\bar{\zeta})^{1/q})^3}.$$  

The explicit formula above can also be found by use of Corollary 3.5 and some known representations of weighted Bergman kernels.

As an example of the utility of what was done so far, we will find the Bergman kernel of the analogous domain with second variable “inflated” with respect to the $\ell^\infty$-norm.

**Corollary 4.1.** The Bergman kernel $K_{D_p^\infty}(z, \zeta, w, \eta)$ of the domain $D_p^\infty = \{(z, w) \in \mathbb{C} \times \mathbb{C}^2 : |z|^2 + \|w\|_{\ell^\infty}^2 < 1\}, q > 0$, where $w, \eta \in \mathbb{C}^2$, is given by
From Hedenmalm–Korenblum–Zhu [11], we know that

\[
\frac{1}{2\pi q(1 - z\zeta)^{2+\frac{k}{q}}(1 - z\zeta)^{(k+1)}(w_1\bar{\eta}_1 - (1 - z\zeta)^{\frac{1}{2q}})^3}\]

\[
\times \left\{ w_1\bar{\eta}_1 \left( \frac{w_2\bar{\eta}_2(2q - 2)(1 - z\zeta)^{\frac{3}{2q}} - 2q(1 - z\zeta)^{\frac{1}{2q}}}{(w_2\bar{\eta}_2 - (1 - z\zeta)^{\frac{1}{2q}})^3} \right) \right. \\
\left. - \left( \frac{2qw_2\bar{\eta}_2(1 - z\zeta)^{\frac{2}{2q}} - (2q + 2)(1 - z\zeta)^{\frac{1}{2q}}}{(w_2\bar{\eta}_2 - (1 - z\zeta)^{\frac{1}{2q}})^3} \right) \right\}.
\] (8)

**Proof.** Observe first that we can rewrite \(D_q^\infty\) as

\[
\left\{ (z, w) \in \mathbb{D} \times \mathbb{C}^2 : \|w\|_\infty < e^{\frac{1}{2q} \log(1 - |z|^2)} \right\}.
\]

So by Theorem 3.4

\[
K_{D_q^\infty}(z, \zeta, w, \eta) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} (n_1 + 1)(n_2 + 1)K_{D, -(2q)^{-1}(2 + |n|)\log(1 - |z|^2)}(z, \zeta)(w\eta)^n.
\]

From Hedenmalm–Korenblum–Zhu [11], we know that

\[
K_{D, -(2q)^{-1}(2 + |n|)\log(1 - |z|^2)}(z, \zeta) = \frac{2 + |n| + 2q}{2\pi q} \frac{1}{(1 - z\zeta)^{2 + |n| + 4q/2q}},
\]

so it suffices to find a closed-form expression for

\[
\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (j + 1)(k + 1) \frac{2 + j + k + 2q}{2\pi q} \frac{(w_1\bar{\eta}_1)^j(w_2\bar{\eta}_2)^k}{(1 - z\zeta)^{2 + j + k + 4q/2q}}.
\] (9)

Now we may write the above as

\[
\frac{1}{2\pi q(1 - z\zeta)^{2+\frac{k}{q}}} \sum_{k=0}^{\infty} \frac{(k + 1)(w_2\bar{\eta}_2)^k}{(1 - z\zeta)^{\frac{1}{2q}}} \sum_{j=0}^{\infty} (j + 1)(2 + j + k + 2q) \left( \frac{w_1\bar{\eta}_1}{(1 - z\zeta)^{\frac{1}{2q}}} \right)^j,
\]

and since

\[
\sum_{j=0}^{\infty} (j + 1)(2 + j + k + 2q)x^j = \frac{kx - k + 2qx - 2q - 2}{(x - 1)^3}
\]

whenever \(|x| < 1\), we see that (9) can be written as

\[
\frac{1}{2\pi q(1 - z\zeta)^{2+\frac{k}{q}}} \sum_{k=0}^{\infty} \frac{(k + 1)(w_2\bar{\eta}_2)^k}{(1 - z\zeta)^{\frac{1}{2q}}} \left( \frac{k}{(1 - z\zeta)^{\frac{1}{2q}}} - k + 2q \left( \frac{w_1\bar{\eta}_1}{(1 - z\zeta)^{\frac{1}{2q}}} - 2 \right) \right)\left( \frac{w_1\bar{\eta}_1}{(1 - z\zeta)^{\frac{1}{2q}}} - 1 \right)^j.
\]
RELATIONSHIPS BETWEEN THE BERGMAN KERNELS OF HARTOGS DOMAINS AND THEIR BASIS

or

\[
\frac{1}{2\pi q(1 - z\bar{\zeta})^{2 + \frac{1}{q}}} \sum_{k=0}^{\infty} \frac{(k + 1)(w_2\bar{\eta}_2)^k}{(1 - z\bar{\zeta})^{\frac{k}{2q}}} \left( \frac{w_1\bar{\eta}_1(2q + k)(1 - z\bar{\zeta})^{\frac{1}{q}} - (k + 2q + 2)(1 - z\bar{\zeta})^{\frac{3}{2q}}}{(w_1\bar{\eta}_1 - (1 - z\bar{\zeta})^{\frac{1}{2q}})^3} \right).
\]

Reorganizing yields

\[
\frac{1}{2\pi q(1 - z\bar{\zeta})^{2 + \frac{1}{q}}} \left( w_1\bar{\eta}_1 - (1 - z\bar{\zeta})^{\frac{1}{2q}} \right)^3
\]

\[
\times \left\{ w_1\bar{\eta}_1(1 - z\bar{\zeta})^{\frac{1}{2q}} \sum_{k=0}^{\infty} (k + 1)(2q + k) \left( \frac{w_2\bar{\eta}_2}{(1 - z\bar{\zeta})^{\frac{k}{2q}}} \right)^k - (1 - z\bar{\zeta})^{\frac{3}{2q}} \sum_{k=0}^{\infty} (k + 1)(k + 2q + 2) \left( \frac{w_2\bar{\eta}_2}{(1 - z\bar{\zeta})^{\frac{k}{2q}}} \right)^k \right\}.
\tag{10}
\]

Now

\[
\sum_{k=0}^{\infty} (k + 1)(2q + k)x^k = \frac{2qx - 2q - 2x}{(x - 1)^3}
\]

and

\[
\sum_{k=0}^{\infty} (k + 1)(k + 2q + 2)x^k = \frac{2qx - 2q - 2}{(x - 1)^3}
\]

whenever \(|x| < 1\), so applying this to (10) shows the Bergman kernel \(K_D\infty(z, \zeta, w, \eta)\) explicitly as

\[
\frac{1}{2\pi q(1 - z\bar{\zeta})^{2 + \frac{1}{q}}} \left( w_1\bar{\eta}_1 - (1 - z\bar{\zeta})^{\frac{1}{2q}} \right)^3
\]

\[
\times \left\{ w_1\bar{\eta}_1(1 - z\bar{\zeta})^{\frac{1}{2q}} \left( \frac{2q \left( \frac{w_2\bar{\eta}_2}{(1 - z\bar{\zeta})^{\frac{k}{2q}}} \right)}{\left( \frac{w_2\bar{\eta}_2}{(1 - z\bar{\zeta})^{\frac{k}{2q}}} \right)^3 - 1} \right) - (1 - z\bar{\zeta})^{\frac{3}{2q}} \left( \frac{2q \left( \frac{w_2\bar{\eta}_2}{(1 - z\bar{\zeta})^{\frac{k}{2q}}} \right)}{\left( \frac{w_2\bar{\eta}_2}{(1 - z\bar{\zeta})^{\frac{k}{2q}}} \right)^3 - 1} \right) \right\}.
\]

Simplifying this expression yields (8). \(\square\)
Remark 4.2. It should be clear that in principle this argument can be extended to the analogue of $D^\infty_q$ in $\mathbb{C} \times \mathbb{C}^N$, $N > 2$, although the computations involved become increasingly more complicated as $N$ increases.

4.2. Generalized Hartogs Triangles. For another application, we consider generalized Hartogs triangles

$$\mathbb{H}_q = \{(z, w) \in \mathbb{C}^2 : |w|^q < |z| < 1\},$$

where $q \in \mathbb{Q}_+$. Explicit formulae for the Bergman kernel of $\mathbb{H}_k$ and $\mathbb{H}_{1/k}$, $k \in \mathbb{N}$, were exhibited by Edholm [6]. More generally, we present explicit formulae for $\mathbb{H}_q$ when $q$ is a positive rational number. This has been done previously by Edholm–McNeal [7] through a different method.

Theorem 4.3. The Bergman kernel $K_{\mathbb{H}_q}(z, \zeta, w, \eta)$ of the domain $\mathbb{H}_q$, $q \in \mathbb{Q}_+$, is given by

$$K_{\mathbb{H}_q}(z, \zeta, w, \eta) = \left\{ \begin{array}{l}
\sum_{r=0}^{a-2} \left\{ \frac{((a - b(r + 1))z\zeta + b(r + 1))(w\eta)^r}{a\pi^2 z\zeta(z\zeta - 1)^2} \right. \\
\times \left. \left( \frac{a(w\eta)^a(z\zeta)^b + (1 + r)(z\zeta)^b ((z\zeta)^b - (w\eta)^a)}{(z\zeta)^b - (w\eta)^a} \right) \right}\right. \\
+ \frac{a(w\eta)^{a-1}(z\zeta)^b}{\pi^2(z\zeta - 1)^2 ((w\eta)^a - (z\zeta)^b)^2},
\end{array} \right. \quad (11)$$

where $q = a/b$ is written in lowest terms. If $a = 1$, then the sum on the left is taken to be identically zero.

It can be shown that this representation is equivalent to the one presented by Edholm–McNeal [7]. Indeed, by uniqueness of the Bergman kernel, they are necessarily equivalent. In the particular case of $q = 1/k$, $a = 1$, the sum of the left vanishes, and one is left with the formula in Theorem 1.4 of Edholm [7].

The proof of Theorem 4.3 is similar to that of Corollary 4.1 before it: we find explicit representations of the various weighted Bergman kernels that appear and compute the resulting infinite series.

$\mathbb{H}_q$ can be rewritten as

$$\{(z, w) \in \mathbb{D}^* \times \mathbb{C} : |w| < \exp\left(\frac{1}{q} \log |z|\right)\},$$

so we must first investigate the Bergman kernel of the weighted space $L^2_h(\mathbb{D}^*, -\alpha \log |z|)$, $\alpha > 0$.

Lemma 4.4. The Bergman kernel of the weighted space $L^2_h(\mathbb{D}^*, -\alpha \log |z|)$, $\alpha > 0$, is given by

$$K_{\mathbb{D}^*, -\alpha \log |z|}(z, \zeta) = \frac{(z\zeta)^{-\lfloor \alpha \rfloor - 1} ((1 - \text{frac}(\alpha))z\zeta + \text{frac}(\alpha))}{\pi(z\zeta - 1)^2}.$$

In particular,

$$K_{\mathbb{D}^*, -\alpha \log |z|}(z, \zeta) = \frac{1}{\pi(z\zeta)^\alpha(z\zeta - 1)^2}.$$
RELATIONSHIPS BETWEEN THE BERGMAN KERNELS OF HARTOGS DOMAINS AND THEIR BASE

whenever \( \alpha \in \mathbb{N} \). Here \( \lfloor x \rfloor \) represents the greatest integer less than or equal to \( x \) and \( \text{frac}(x) := x - \lfloor x \rfloor \) is the fractional part of \( x \).

**Proof.** Since the weight is radial, the monomials are orthogonal in \( L^2_h(\mathbb{D}^*, -\alpha \log |z|) \). Furthermore, for integers \( n \) such that \( n > -\alpha - 1 \) we have

\[
\int_{\mathbb{D}^*} |z|^{2n} |\bar{z}|^{2\alpha} \, dV = 2\pi \int_0^1 r^{2n+2\alpha+1} \, dr = \frac{\pi}{\alpha + n + 1},
\]

and so

\[
K_{\mathbb{D}^*, -\alpha \log |z|}(z, \zeta) = \frac{1}{\pi} \sum_{n > -\alpha - 1} (\alpha + n + 1)(z\bar{\zeta})^n.
\]

We now separate into two cases. If \( \alpha \) is an integer, then

\[
K_{\mathbb{D}^*, -\alpha \log |z|}(z, \zeta) = \frac{1}{\pi(z\bar{\zeta})^{\alpha}(z\bar{\zeta} - 1)^2}.
\]

If \( \alpha \) is not an integer, then

\[
K_{\mathbb{D}^*, -\alpha \log |z|}(z, \zeta) = \frac{1}{\pi} \sum_{n = -[\alpha] - 1}^{\infty} (\alpha + n + 1)(z\bar{\zeta})^n
\]

\[
= \frac{(z\bar{\zeta})^{-[\alpha] - 1}(z\bar{\zeta}[\alpha] - \lfloor \alpha \rfloor - \alpha z\bar{\zeta} + \alpha + z\bar{\zeta})}{\pi(z\bar{\zeta} - 1)^2}
\]

\[
= \frac{(z\bar{\zeta})^{-[\alpha] - 1}(1 - \frac{\alpha}{q})z\bar{\zeta} + \frac{\alpha}{q})}{\pi(z\bar{\zeta} - 1)^2}.
\]

**Proof of Theorem 4.3.** Suppose that \( q \in \mathbb{Q}^+ \). Write \( q \) in lowest terms as \( a/b \), and notice that \( q^{-1}(n + 1) \in \mathbb{N} \) if and only if \( a \) divides \( (n + 1) \). Set \( \varphi_{n,q} = \frac{(n+1)}{q} \log |\cdot| \). Using Corollary 3.5 and the division algorithm, we see that

\[
K_{\mathbb{H},q}(z, \zeta, w, \eta) = \frac{1}{\pi} \sum_{n=0}^{\infty} (1 + n)K_{\mathbb{D}^*, \varphi_{n,q}}(w\eta)^n
\]

\[
= \frac{1}{\pi} \sum_{k=0}^{\infty} \sum_{r=0}^{a-1} (1 + ka + r)K_{\mathbb{D}^*, \varphi_{ka+r,q}}(z, \zeta)(w\eta)^{ka+r}
\]

\[
= \frac{1}{\pi} \sum_{k=0}^{\infty} \sum_{r=0}^{a-2} (1 + ka + r)K_{\mathbb{D}^*, \varphi_{ka+r,q}}(z, \zeta)(w\eta)^{ka+r}
\]

\[
+ \frac{1}{\pi} \sum_{k=0}^{\infty} (ka + a)K_{\mathbb{D}^*, \varphi_{ka+a-1,q}}(z, \zeta)(w\eta)^{ka+a-1}.
\]  

(12)
Lemma 4.4 to write it explicitly as

\[ \frac{1}{\pi^2} \sum_{k=0}^{\infty} \frac{ka + a}{(z\zeta)^{b(k+1)}(z\zeta - 1)^2} (w\bar{\eta})^{ka+a-1} = \frac{a(w\bar{\eta})^{a-1}}{\pi^2(z\zeta)^b(z\zeta - 1)^2} \sum_{k=0}^{\infty} (k+1) \left( \frac{(w\bar{\eta})^a}{(z\zeta)^{b}} \right)^k \]

Now we turn our attention to the first sum in (12). Fix 0

weights is the modulus of a meromorphic function.

above is equal to

\[ \frac{a(w\bar{\eta})^{a-1}(z\zeta)^b}{\pi^2(z\zeta - 1)^2 ((w\bar{\eta})^a - (z\zeta)^b)^2}. \] (13)

Finally, combining (12), (13), and (14) yields the desired formula. □

Remark 4.5. At this point it should be clear how one can apply Corollary 3.6 to find explicit representations for even more general Hartogs triangles with inflated w-variable.

5. Other Relationships Between Bergman Kernels of Hartogs Domains and Weighted Bergman Kernels Over Their Base

There is a well-known relationship between the (unweighted) Bergman kernels of domains which are biholomorphic [10] [17]. In fact, there is even a transformation formula between the Bergman kernels of domains that are related via proper holomorphic mappings [2]. This motivates the following question: What if the domain remains the same but the weight changes? It turns out there is an analogous formula in the case the quotient of the two weights is the modulus of a meromorphic function.
We claim that the map $T$ is well-defined. Now a theorem of Bell [2, 687] shows that it is bounded from above on compact sets, and hence $T$ is locally square-integrable with respect to no weight. Now a theorem of Bell [2, 687] shows that $T(\lambda) \in L^2_h(D, \varphi_1)$ and indeed $T$ is well-defined.

$T$ has inverse $k \mapsto \frac{1}{k}$, and from the computation above we see that $T$ is an isometry. It follows that $\{T(\chi_j)\}_{j=0}^\infty$ is an orthonormal basis of $L^2_h(D, \varphi_1)$ whenever $\{\chi_j\}_{j=0}^\infty$ is an orthonormal basis of $L^2(D, \varphi_2)$. By the representation of the Bergman kernel via orthonormal basis [11 §9.1], we have

$$K_{D, \varphi_1}(z, \zeta) = \sum_{j=0}^\infty \frac{f(z)}{g(z)} \chi_j(z) \overline{\chi_j(\zeta)} \frac{f(\zeta)}{g(\zeta)} = \frac{f(z)}{g(z)} K_{D, \varphi_2}(z, \zeta) \frac{f(\zeta)}{g(\zeta)}.$$  

Proposition 5.1 in combination with Theorem 3.4 yields a relationship between the Bergman kernel of certain types of Hartogs domains and Bergman kernels of their base.

**Theorem 5.2.** Let $f$ and $g$ be nontrivial holomorphic functions on a domain $G \subseteq \mathbb{C}^M$, and let $D$ be the Hartogs domain

$$D = \{(z, w) \in G \times \mathbb{C}^N : \|wf(z)\| < |g(z)|\}.$$  

Then

$$K_D(z, \zeta, w, \eta) = \frac{N!f(z)^{N-1}g(z)^2K_{G, \log |f/g|}(z, \zeta) \overline{f(\zeta)}^{N-1}g(\zeta)}{\pi^N \left( g(z)g(\zeta) - \langle w, \eta \rangle f(z) \overline{f(\zeta)} \right)^{N+1}}$$  

and

$$K_D(z, \zeta, w, \eta) = \frac{N!f(z)^N g(z)K_G(z, \zeta)f(\zeta)^Ng(\zeta)}{\pi^N \left( g(z)g(\zeta) - \langle w, \eta \rangle f(z) \overline{f(\zeta)} \right)^{N+1}}.$$  

**Proof.** Let us first suppose that $N = 1$. By Theorem 3.4 (or Corollary 3.5), we then know that

$$K_D(z, \zeta, w, \eta) = \frac{1}{\pi} \sum_{n=0}^\infty (1 + n)K_{G,(1+n)\log |f/g|}(z, \zeta)(w\bar{\eta})^n.$$  

However, $(1 + n) \log |f/g| - \log |f/g| = n \log |f| - n \log |g|$ for each $n$, so Proposition 5.1 yields

$$g(z)^nK_{G,(1+n)\log |f/g|}(z, \zeta)\overline{f(\zeta)}^n = f(z)^nK_{G,\log |f/g|}(z, \zeta)\overline{f(\zeta)}^n.$$
Therefore

\[ K_D(z, \zeta, w, \eta) = \frac{1}{\pi} K_{G, \log |f/g|}(z, \zeta) \sum_{n=0}^{\infty} (n+1) \left( \frac{w \bar{\eta} f(z) f(\zeta)}{g(z) g(\zeta)} \right)^n \]

\[ = \frac{g(z)^2 K_{G, \log |f/g|}(z, \zeta) \overline{g(\zeta)}^2}{\pi \left( g(z) \overline{g(\zeta)} - w \overline{\eta} f(z) \overline{f(\zeta)} \right)^2}. \]

Now suppose \( N > 1 \). By the inflation identity (Corollary 3.6), we see that

\[ K_D(z, \zeta, w, \eta) = \frac{1}{\pi N} \partial^{N-1} \left( \frac{g(z)^2 K_{G, \log |f/g|}(z, \zeta) \overline{g(\zeta)}^2}{\pi \left( g(z) \overline{g(\zeta)} - t f(z) \overline{f(\zeta)} \right)^2} \right) \bigg|_{t=(w, \eta)} \]

\[ = \frac{N! f(z)^{N-1} g(z)^2 K_{G, \log |f/g|}(z, \zeta) \overline{f(\zeta)}^{N-1} \overline{g(\zeta)}^2}{\pi^N \left( g(z) \overline{g(\zeta)} - \langle w, \eta \rangle f(z) \overline{f(\zeta)} \right)^{N+1}}. \]

This shows (15).

(16) can be shown by applying Proposition 5.1 with \( \varphi_1 = \log |f/g| \) and \( \varphi_2 \equiv 0 \) to (15).

The following is a stronger version of Corollary 3.3 in this context.

**Corollary 5.3.** Let \( f, g, G, \) and \( D \) be as above. Then \( K_D(z, \zeta, w, \eta) \) has a zero whenever \( z \) or \( \zeta \) is a zero of \( f \).

Setting \( g(z) = z^k \) for some \( k \in \mathbb{N} \) yields an analogous relationship for a “twisted” Hartogs triangle with possibly inflated second coordinate.

**Corollary 5.4.** Let \( f \) be a holomorphic function on the punctured unit disc \( \mathbb{D}^* \) and let \( k \) be a natural number. Set

\[ D = \{(z, w) \in \mathbb{D}^* \times \mathbb{C}^N : \|w f(z)\|^{1/k} < |z| < 1\}. \]

Then

\[ K_D(z, \zeta, w, \eta) = \frac{N! f(z)^{N-1} z^{2k} K_{g \cdot |f/g|}(z, \zeta) \overline{g(\zeta)}^{N-1} \overline{\zeta}^{2k}}{\pi^N \left( z^k \overline{\zeta}^k - \langle w, \eta \rangle f(z) \overline{f(\zeta)} \right)^{N+1}} \]

and

\[ K_D(z, \zeta, w, \eta) = \frac{N! f(z)^N z^k K_{g \cdot |f/g|}(z, \zeta) \overline{f(\zeta)}^{N-1} \overline{\zeta}^{2k}}{\pi^N \left( z^k \overline{\zeta}^k - \langle w, \eta \rangle f(z) \overline{f(\zeta)} \right)^{N+1}} \]

\[ = \frac{N! z^k f(z)^N \overline{f(\zeta)}^{N-1} \overline{\zeta}^{2k}}{\pi^N \left( z^k \overline{\zeta}^k - \langle w, \eta \rangle f(z) \overline{f(\zeta)} \right)^{N+1}}. \]
RELATIONSHIPS BETWEEN THE BERGMAN KERNELS OF HARTOGS DOMAINS AND THEIR BASIS

In the last equality we have used that $K_{D^*} = K_D$ on $D^* \times D^*$, which is a consequence of $L^2_h(D^*) = L^2_h(D)$ via the work of Bell [20, 687], in addition to the well-known explicit formula for the Bergman kernel of the unit disk [10, 17]. Note that when $q \equiv 1$, (17) agrees with (11) of Theorem 4.3 whenever $q = 1/k$.

6. Generalizations of Proposition 5.1

Proposition 5.1 can, in a sense, be generalized by adapting terminology similar to that of Borichev–Le–Yousfi [4] and using some tools of analytic geometry.

For the following, let $\varphi \in \text{PSH}(G)$ on some domain $G \subset \mathbb{C}^M$. By the work of Siu [18], we have the unique decomposition

$$\text{dd}^c \varphi = \sum_{j=1}^{\infty} \lambda_j [Z_j] + R,$$

where $\lambda_j > 0$, $[Z_j]$ is a current of integration over an irreducible analytic set of complex dimension $M - 1$, and $R$ is a residual current. Here we are interpreting $\text{dd}^c \varphi$ as a positive $(1,1)$-current, and hence the convergence that takes place is in the sense of currents. This in turn implies that the collection of analytic sets in the series is locally finite. Define the $(1,1)$-current $\mu^d_{\varphi}$ by

$$\mu^d_{\varphi} := \sum_{j=1}^{\infty} |\lambda_j| [Z_j],$$

and set $\mu^c_{\varphi} := \text{dd}^c \varphi - \mu^d_{\varphi}$. Note that, if $\mu^c_{\varphi_1} = \mu^c_{\varphi_2}$ for $\varphi_1, \varphi_2 \in \text{PSH}(G)$, then $\text{dd}^c (\varphi_2 - \varphi_1)$ is a linear combination of currents of integration with coefficients in $\mathbb{Z}$ and consequently $\text{dd}^c (\varphi_2 - \varphi_1) = \text{dd}^c (\log |m|)$ for some meromorphic function $m$.

**Proposition 6.1.** Let $G \subset \mathbb{C}^M$ be a domain and $\varphi_1, \varphi_2 \in \text{PSH}(G)$. Suppose $\mu^c_{\varphi_1} = \mu^c_{\varphi_2}$ and

$$\int \gamma \, d^c (\varphi_2 - \varphi_1 - \log |m|) \in 2\pi \mathbb{Z}$$

for all $\gamma \in H_1(G, \mathbb{Z})$, where $m$ is a meromorphic function as described above. Then

$$m(z) F(z) K_{G, \varphi_2}(z, \zeta) \overline{m(\zeta) F(\zeta)} = K_{G, \varphi_1}(z, \zeta).$$

**Proof.** Let $\{\chi_k\}_{k=1}^{\infty}$ be an orthonormal basis for $L^2_h(G, \varphi_1)$. Then

$$\delta_{jk} = \int_G \chi_j(z) \overline{\chi_k(z)} e^{-2\varphi_1(z)} dV(z)$$

$$= \int_G \chi_j(z) \overline{\chi_k(z)} e^{-2\varphi_2(z) + 2\varphi_2(z) - 2\varphi_1(z)} dV(z)$$

$$= \int_G |m(z)|^2 \chi_j(z) \overline{\chi_k(z)} e^{-2\varphi_2(z) + 2\varphi_2(z) - 2\varphi_1(z) - 2 \log |m(z)|} dV(z).$$

By assumption, $d^c (\varphi_2 - \varphi_1 - \log |m|)$ is a closed 1-form with periods in $2\pi \mathbb{Z}$, so there exists a smooth function $v : G \to \mathbb{R}/2\pi \mathbb{Z}$ so that $dv = d^c (\varphi_2 - \varphi_1 - \log |m|)$ and hence
$F := e^{\varphi_2 - \varphi_1 - \log |m|e^{iv}}$ is a well-defined holomorphic function on $G$ with
\[
\delta_{jk} = \int_G |m(z)F(z)|^2\chi_j(z)\chi_k(z)e^{-2\varphi_2(z)}
\]
Therefore we can again apply a theorem of Bell [2, p. 687] to see that $f \mapsto (mF) \cdot f$ is an isometric isomorphism from $L^2(G, \varphi_1)$ to $L^2(G, \varphi_2)$, and by the same reasoning as in the proof of Proposition 5.1 we conclude that
\[
K_{G,\varphi_2}(z, \zeta) = \sum_{k=1}^{\infty} \left\{ m(z)F(z)\chi_k(z) \right\} \left\{ m(\zeta)F(\zeta)\chi_k(\zeta) \right\}
\]
\[
= m(z)F(z) \left( \sum_{k=1}^{\infty} \chi_k(z)\chi_k(\zeta) \right) m(\zeta)F(\zeta)
\]
\[
= m(z)F(z)K_{G,\varphi_1}(z, \zeta)m(\zeta)F(\zeta).
\]
\[\square\]
**Corollary 6.2.** Let $\varphi_1, \varphi_2 \in PSH(G)$. Then $\dim L^2_G(G, \varphi_1) = \dim L^2_G(G, \varphi_2)$ whenever $\varphi_1, \varphi_2$ satisfy the hypotheses of Proposition 6.1.

Interpreting Theorem 5.2 in this context yields the following.

**Corollary 6.3.** Let $\varphi \in PSH(G)$ with $dd^c \varphi = \sum_j \lambda_j[Z_j]$, (18) where $\lambda_j \in \mathbb{N}$ for each $j$, and $D = \{(z, w) \in G \times \mathbb{C}^N : \|w\| < e^{-\varphi(z)}\}$. Suppose $\varphi - \log |g|$ is pluriharmonic for $g \in \mathcal{O}(G)$ (such a $g$ is guaranteed to exist) with
\[
\int_{\gamma} d^c(\varphi - \log |g|) \in 2\pi \mathbb{Z}
\]
for all $\gamma \in H_1(G, \mathbb{Z})$. Then
\[
K_D(z, \zeta, w, \eta) = \frac{N!g(z)^2K_{G,\varphi}(z, \zeta)g(\zeta)^2}{\pi^N \left( g(z)g(\zeta) - \langle w, \eta \rangle \right)^{N+1}}
\]
and
\[
K_D(z, \zeta, w, \eta) = \frac{N!g(z)K_{G}(z, \zeta)g(z)}{\pi^N \left( g(z)g(\zeta) - \langle w, \eta \rangle \right)^{N+1}}
\]
for some entire function $g$.

In the particular case that $L^2_h(G) = \{0\}$ (e.g. $G = \mathbb{C}^M$), (20) of Corollary 6.3 implies that $L^2_h(D) = \{0\}$ whenever $\varphi$ satisfies (18) (condition (19) is trivially satisfied in view of the simple-connectivity of $\mathbb{C}^M$). This can be thought of as a generalization of one direction of the work of Jucha [13 Theorem 4.1] to higher dimensions.
RELATIONSHIPS BETWEEN THE BERGMAN KERNELS OF HARTOGS DOMAINS AND THEIR BASIS

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