Landau-Zener Tunnelling in a Nonlinear Three-level System

Guan-Fang Wang 1,2, Di-Fa Ye 1, Li-Bin Fu 1, Xu-Zong Chen 3, and Jie Liu 1 *

1 Institute of Applied Physics and Computational Mathematics, P.O. Box 8009 (28), 100088 Beijing, China
2 Institute of Physical Science and Technology, Lanzhou University, 730000 Lanzhou, China
3 Key Laboratory for Quantum Information and Measurements, Ministry of Education, School of Electronics Engineering and Computer Science, Peking University, Beijing 100871, China

We present a comprehensive analysis of the Landau-Zener tunnelling of a nonlinear three-level system in a linearly sweeping external field. We find the presence of nonzero tunnelling probability in the adiabatic limit (i.e., very slowly sweeping field) even for the situation that the nonlinear term is very small and the energy levels keep the same topological structure as that of linear case. In particular, the tunnelling is irregular with showing an unresolved sensitivity on the sweeping rate. For the case of fast-sweeping fields, we derive an analytic expression for the tunnelling probability with stationary phase approximation and show that the nonlinearity can dramatically influence the tunnelling probability when the nonlinear "internal field" resonate with the external field. We also discuss the asymmetry of the tunnelling probability induced by the nonlinearity. Physics behind the above phenomena is revealed and possible application of our model to triple-well trapped Bose-Einstein condensate is discussed.

PACS numbers: 03.75.-b, 05.45.-a, 03.75.Kk, 42.50.Vk

I. INTRODUCTION

Avoiding crossing of energy levels is a universal phenomenon for the quantum non-integrable systems where the symmetry break leads to the splitting of degenerate energy levels forming a tiny energy gap. Around the avoided crossing point of the two levels the Landau-Zener tunnelling (LZT) model provides an effective description for the tunnelling dynamics under assumption that the energy bias of two levels undergoes a linear change with time [1]. It is a basic model in quantum mechanics and has versatile applications in quantum chemistry [2], collision theory [3], and more recently in the spin tunnelling of nanomagnets [4, 5], Bose-Einstein condensates (BEC) [6] and quantum computing [7], to name only a few.

LZT model has been extended to many versions taking diverse physical conditions into account: LZT problem with a time-varied sweeping rate [8], LZT model with a fast noise from the outer environment [9], LZT model with periodic modulation [10, 11], and so on. Among them, LZT in a nonlinear two-level system is one of most interesting models and attracts much attention recently [12, 13, 14, 15]. In this model, the level energies depend on the occupation of the levels, may arise in a meanfield treatment of a many-body system where the particles predominantly occupy two energy levels. The nonlinear LZT model not only demonstrate many novel behavior of great interest in theory but also has important applications in spin tunnelling of nanomagnets [16] and a Bose-Einstein condensate in a double-well potential [17, 18, 19] or in an optical lattice [20, 21]. However, since most of the problems of interests involve more than two energy levels, with transitions between several levels happening simultaneously [18, 19, 20, 21], for example, BECs trapped in multiple wells [22, 23, 24, 25], spin tunnelling of nanomagnets with large spin, etc. It is naturally desirable to extend the above nonlinear tunnelling to the multi-level situation.

In present paper, we consider the simplest multi-level system—three-level system, to investigate its complicated tunnelling dynamics in the presence of nonlinearity. Because quantum transitions may happen between several levels simultaneously, the LZT in the nonlinear three-level model show many striking properties distinguished from that of the two-level case. In the adiabatic limit we will show that, for a very small nonlinear parameter that the energy levels still keep the same topological structure as its linear counterpart, the adiabaticity breaks down manifesting the presence of a nonzero tunnelling probability. This is quite different from the two-level case, where the break down of the adiabaticity is certainly accompanied by a topological change on the energy levels. More interestingly, the tunnelling is irregular with showing an unresolved sensitivity on the sweeping rate, a phenomenon attributed to the existence of chaotic state. In the sudden limit, we derive an analytic expression for the tunnelling probability under stationary phase approximation and show that the nonlinearity can dramatically influence the tunnelling probability at the resonance between the nonlinear "internal field" and the external field. We also discuss the asymmetry of the tunnelling probability induced by the nonlinearity. The physical mechanism behind these phenomena is revealed and possible application of our model to triple-well trapped Bose-Einstein condensate is discussed.

The paper is organized as follows. In Sec.II we introduce our nonlinear three-level LZT model and calculate its adiabatic levels. Section III discusses LZT among the levels. Section IV gives a possible application of the model to triple-well trapped BEC.
II. THE MODEL AND ADIABATIC LEVELS

We consider following dimensionless Schrödinger equation

\[
\begin{align*}
\frac{d}{dt} & \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = H \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\
& \text{with the Hamiltonian given by}
\end{align*}
\]

\[
H = \begin{pmatrix} \frac{v}{2} + \frac{\xi}{4} |a_1|^2 & -\frac{v}{2} & 0 \\
-\frac{v}{2} & \frac{v}{2} |a_2|^2 & -\frac{v}{2} \\
0 & -\frac{v}{2} + \frac{\xi}{4} |a_3|^2 & 0
\end{pmatrix}
\]

(2)

where \( v \) is the coupling constant between the neighboring levels; \( c \) is the nonlinear parameter; the energy bias \( \gamma \) is supposed to be adjusted by a linearly external field, i.e., \( \gamma = \alpha t \), \( \alpha \) is the sweeping rate; \( a_1, a_2, a_3 \) is probability amplitude in each level and the total probability \( |a_1|^2 + |a_2|^2 + |a_3|^2 \) is conserved and set to be unit.

When the nonlinear parameter vanishes, our model reduces to the linear case and the adiabatic energy levels \( \varepsilon(\gamma) = 0, \pm \frac{1}{2} \sqrt{\gamma^2 + 2v^2} \) (Fig.1(a)) derived by diagonalizing the Hamiltonian (2). Tunnelling probability \( \Gamma_{nm} \) \((n, m = 1, 2, 3)\) is defined as the occupation probability on the \( n \)-th level at \( \gamma \rightarrow +\infty \) for the state initially on the \( n \)-th level at \( \gamma \rightarrow -\infty \). For the linear case, the above system is solvable analytically and the tunnelling probabilities can be explicitly expressed as \[21\]

\[
\Gamma_{11} = \left[1 - \exp\left(-\frac{\pi v^2}{2\alpha}\right)\right]^2
\]

(3)

\[
\Gamma_{12} = 2 \exp\left(-\frac{\pi v^2}{2\alpha}\right) \left[1 - \exp\left(-\frac{\pi v^2}{2\alpha}\right)\right]
\]

(4)

\[
\Gamma_{13} = \exp\left(-\frac{\pi v^2}{\alpha}\right)
\]

(5)

\[
\Gamma_{22} = \left[1 - 2 \exp\left(-\frac{\pi v^2}{2\alpha}\right)\right]^2
\]

(6)

The others are \( \Gamma_{21} = \Gamma_{23} = \Gamma_{32} = \Gamma_{12}, \Gamma_{31} = \Gamma_{13}, \Gamma_{33} = \Gamma_{11} \) due to the symmetry of the levels.

With the presence of the nonlinear terms, we want to know how the tunnelling dynamics in the above system is affected. In our discussions, the coupling parameter is set to be unit, i.e., \( v = 1 \). Therefore, weak nonlinear case and strong nonlinear case mean that \( c << 1 \) and \( c >> 1 \), respectively.

As to the external fields, we will consider three cases, namely, adiabatic limit, sudden limit, and moderate case, corresponding to \( \alpha << 1, \alpha >> 1 \) and \( \alpha \sim 1 \), respectively.

Similar to the linear case, we need to analyze the adiabatic levels of the nonlinear model first. With \( a_1 = \sqrt{s_1} e^{i\theta_1}, a_2 = \sqrt{1 - s_1 - s_2} e^{i\theta_2}, a_3 = \sqrt{s_2} e^{i\theta_3} \), we introduce the relative phase \( \theta_1 = \theta_2, \theta_2 = \theta_3 - \theta_1 \). In terms of \( s_1, \theta_1 \) and \( s_2, \theta_2 \), the nonlinear three-level system is casted into a classical Hamiltonian system,

\[
H_c = \left(\frac{\gamma}{2} + \frac{c}{8} \right) s_1 + \frac{c}{8} (1 - s_1 - s_2)^2 + \left(-\frac{\gamma}{2} + \frac{c}{8} s_2\right) s_2
\]

(7)

\[
\begin{align*}
s_1, \theta_1 \text{ and } s_2, \theta_2 \text{ are two pairs of canonically conjugate variables of the classical Hamiltonian system. The fixed points of the nonlinear classical Hamiltonian correspond to the eigenstates of the nonlinear three-level system, and are given by the following equations:}
\end{align*}
\]

\[
\begin{align*}
\dot{s}_1 &= -v \sqrt{(1 - s_1 - s_2)s_1} \sin \theta_1 \\
\dot{\theta}_1 &= \frac{\gamma}{2} \left(1 - s_1 - s_2\right) - \frac{1 - 2s_1 - s_2}{2v(1 - s_1 - s_2)s_1} v \cos \theta_1 \cr &\quad + \frac{s_2}{2v(1 - s_1 - s_2)s_2} v \cos \theta_2
\end{align*}
\]

(8) (9)
In this section we study LZT in the nonlinear three-level system both numerically and analytically. First, we consider two limit cases: adiabatic limit, sudden limit, respectively. Then we will discuss the tunnelling probability in general case and investigate the symmetry of the tunnelling probability.

### A. Adiabatic limit \((\alpha \ll 1)\)

In adiabatic limit, the characters of the tunnelling probabilities should be entirely determined by the topology of the energy levels and the eigenstates' properties (corresponding to the stability of the fixed points in classical Hamiltonian system), according to the adiabatic theorem\[^{21,27}\]. So, we expect that, for the weak nonlinearity case, an initial state started from any levels (upper, mid or lower) will follow the levels and evolves adiabatically, as a result, no quantum transition between levels occurs; for the strong nonlinearity, an initial state from the lower level is expected to evolve adiabatically keeping stay on the ground state, leading to zero adiabatic tunnelling probability, whereas for the state initially from the mid or upper level, due to the topological change of the level, it can not move smoothly from left-side to the right-side. Transition to other levels happens at the tip of the loop or butterfly. Consequently, the adiabatic tunnelling probability is expected to be nonzero.

However, the above picture is only partly corroborated by our directly solving the Schrödinger equation using forth-fifth order Runge-Kutta adaptive-step algorithm, as shown in Fig.2.

On the one hand, Fig.2 clearly shows that, for the strong nonlinearity case, as we expect, no tunnelling for the state from the lower level, but a serious adiabatic tunnelling is observed for the states from upper two levels. In particular, we find that the tunnelling probability as a function of the sweeping rate shows an irregular oscillation. We associate this irregularity to the chaotic state. To demonstrate it, we plot in Fig.3 the Poincare section of the trajectories for \(c = 10\) before and after the tip of the butterfly structure of the upper level in Fig.1c. It shows that, before the tip, the eigenstate corresponds to the fixed point surrounded by quasi-periodic orbit, therefore is stable. As the state evolves to the right tip of the butterfly, it contact with chaotic sea, after that the state become chaotic. The characteristics of the chaos is sensitive on the parameters, therefore the chaotic state is responsible for the irregular tunnelling probability exposed by Fig.2h,i.

On the other hand, Fig.2 also shows that for the weak nonlinearity, even though the adiabatic levels keeps the same topological structure as the linear case, there is still nonzero tunnelling probability for the state started from the mid-level. The tunnelling also shows some kind of irregularity. This phenomenon counter to our naive conjecture from observing the topological structure of the adiabatic levels.

To explain this unusual phenomenon, we need make detailed analysis on the property of the fixed points of

\[
\dot{s}_2 = -v \sqrt{(1 - s_1 - s_2)s_2 \sin \theta_2} \quad (10)
\]

\[
\dot{\theta}_2 = -\frac{\gamma}{2} - \frac{c}{4}(1 - s_1 - 2s_2) + \frac{s_1}{2\sqrt{(1 - s_1 - s_2)s_1}}v \cos \theta_1 \\
- \frac{1 - s_1 - 2s_2}{2\sqrt{(1 - s_1 - s_2)s_2}}v \cos \theta_2 \quad (11)
\]

By solving the equations (10)-(11) the eigenstates of the system are obtained. Accordingly, the eigenenergy is obtained by \(\epsilon = H_c\) i.e., the energy levels are gained as shown in Fig.1.

For weak nonlinearity, the levels' structure is similar to its linear counterpart (fig.1(b)). For strong nonlinearity (fig.1(c)), in the mid-level a double-loop topological structure emerges and in the upper-level a butterfly structure appears. Because of these topological distortions on the energy levels, we expect that the tunnelling dynamics will dramatically change.

### III. LANDAU-ZENER TUNNELLING

FIG. 2: The tunnelling probability \(\Gamma_{11}, \Gamma_{22}, \Gamma_{33}\) (full circles) as functions of \(\alpha\) for different nonlinear parameter at \(v = 1.0\). The dash lines represent the results from the linear Landau-Zener model for comparison.
the classical system Hamiltonian (7), corresponding to the eigenstates of the mid-level.

We plot quantity \( s_1 \) as the function of \( \gamma \) in Fig.4 (a,b), we see the adiabatic evolution of the eigenstate breaks down around \( \gamma = -2 \) due to the nonlinearity (Fig.4 (b)). This adiabaticity breakage is caused by the change on the property of the fixed point corresponding to the eigenstate of the mid-level. This is revealed by investigating the Hamiltonian-Jacobi matrix obtained by linearizing the nonlinear equations \( \mathbf{S}_1 - \mathbf{1} \) at fixed points,

\[
H_J = \begin{pmatrix}
-\frac{\partial^2 H_1}{\partial s_1 \partial s_1} & -\frac{\partial^2 H_1}{\partial s_1 \partial s_2} & -\frac{\partial^2 H_1}{\partial s_1 \partial \theta_1} & -\frac{\partial^2 H_1}{\partial s_1 \partial \theta_2} \\
-\frac{\partial^2 H_2}{\partial s_1 \partial s_1} & -\frac{\partial^2 H_2}{\partial s_1 \partial s_2} & -\frac{\partial^2 H_2}{\partial s_1 \partial \theta_1} & -\frac{\partial^2 H_2}{\partial s_1 \partial \theta_2} \\
-\frac{\partial^2 H_1}{\partial s_2 \partial s_1} & -\frac{\partial^2 H_1}{\partial s_2 \partial s_2} & -\frac{\partial^2 H_1}{\partial s_2 \partial \theta_1} & -\frac{\partial^2 H_1}{\partial s_2 \partial \theta_2} \\
-\frac{\partial^2 H_2}{\partial s_2 \partial s_1} & -\frac{\partial^2 H_2}{\partial s_2 \partial s_2} & -\frac{\partial^2 H_2}{\partial s_2 \partial \theta_1} & -\frac{\partial^2 H_2}{\partial s_2 \partial \theta_2}
\end{pmatrix}
\] (12)

We solve the eigenvalues of \( H_J \) for different \( \gamma \) and plot our results in Fig.4c. These eigenvalues can be real, complex or pure imaginary. Only pure imaginary eigenvalues correspond to the stable fixed point, others indicate the unstable ones. In Fig.4c we can see the eigenvalues are complex number (i.e., their real parts are not zero) around \( \gamma = 0, \pm 2 \). The corresponding fixed points are unstable. For other regions, the eigenvalues of \( H_J \) are pure imaginary. Therefore, even though no topological structure changes on the level structures, the instability of the fixed point corresponding to the mid-level leads to the breakdown of the adiabaticity manifesting the irregular nonzero tunnelling probability exposed by Fig.2e in the adiabatic limit.

The above instability mechanism occurs for any smaller nonlinear perturbation. Let us make some analytic deduction as follows. Note that the fixed points of equations \( \mathbf{S}_1 - \mathbf{1} \) can be accurately calculated if \( c = 0 \) : 

\[
s_1^0 = s_2^0 = \frac{1}{2 + \gamma}, \quad \theta_1^0 = 0, \quad \theta_2^0 = \pi \text{ for } \gamma > 0, \text{ and } \theta_1^0 = \pi, \quad \theta_2^0 = 0 \text{ for } \gamma < 0.
\]

By employing the perturbation theory using \( c \) as small parameter, we can get the fixed points for small \( c : s_1^0 = \frac{1}{2 + \gamma^2} - \frac{(1 - \gamma^2)^2}{4(2 + \gamma^2)} \gamma, \quad s_2^0 = \frac{1}{2 + \gamma^2} + \frac{(1 - \gamma^2)^2}{4(2 + \gamma^2)} \gamma, \quad \theta_1^0 = 0, \quad \theta_2^0 = \pi \) for nonlinear case. Substituting them into equation (12), we can obtain the eigenvalues of \( H_J \) by solving the following quartic equation:

\[
(64 + 1280 \gamma^4) x^4 + (64 + c^2 + 1344 \gamma^2) x^2 + (16 + c^2 + 352 \gamma^2) = 0
\]

The useful quadratic discriminant is \( \Delta = 4096 \gamma^4 - 2432 c^2 \gamma^2 + (c^4 - 128 c^2) \). In linear case, \( c = 0, \Delta = 4096 \gamma^4 \) is always larger than zero, which means that the solutions for \( x \) are pure imaginary, thus the fixed points are stable. For small \( c, \lim_{\gamma \to 0} \Delta < 0 \), the real part of the solutions is cubic, while the imaginary part is \( \sqrt{2}/2 \). As a result, the fixed point corresponding to mid-level becomes unstable around \( \gamma = 0 \) for any small nonlinearity, implying the breakdown of the adiabatic evolution of states on the mid-level.

**B. Sudden limit (\( \alpha \gg 1 \))**

The sudden limit corresponds to nonadiabatic LZT. The tunnelling probability does not relate much to the structure of the levels. In this limit a weak nonlinearity does not affect the tunnelling probability, however, a strong nonlinearity can dramatically influence the tunnelling dynamics.

In this limit, we can derive the analytical expression of the tunnelling probabilities using the stationary phase
approximation (SPA). As a demonstration, we concentrate on the mid-level, i.e. to calculate $\Gamma_{22}$ which equals to $1 - \Gamma_{21} - \Gamma_{23}$. Because of the large sweeping rate $\alpha$, a quantum state would stay on the initial level most of the time. Thus the amplitudes $a_2$ and $a_3$ in the Schrödinger equation (1) remain small and $|a_2| \sim 1$ all the time. A perturbation treatment of the problem becomes adequate.

We begin with the variable transformation,

$$a_1 = a_1' \exp[-i \int_0^t \left(\frac{\gamma}{2} + \frac{c}{4} |a_1|^2 \right) dt],$$ (13)

$$a_2 = a_2' \exp[-i \int_0^t \left(\frac{\gamma}{2} + \frac{c}{4} |a_2|^2 \right) dt],$$ (14)

$$a_3 = a_3' \exp[-i \int_0^t \left(-\frac{\gamma}{2} + \frac{c}{4} |a_3|^2 \right) dt].$$ (15)

As a result, the diagonal terms in Hamiltonian are transformed away, and the evolution equations of $a_1'$, $a_2'$, $a_3'$ become:

$$\frac{da_1'}{dt} = -\frac{v}{2i} a_2' \exp[ i \int_0^t \left(\frac{\gamma}{2} + \frac{c}{4} (|a_1|^2 - |a_2|^2) \right) dt],$$

$$\frac{da_2'}{dt} = -\frac{v}{2i} a_1' \exp[ i \int_0^t \left(-\frac{\gamma}{2} + \frac{c}{4} (|a_2|^2 - |a_1|^2) \right) dt] - \frac{v}{2i} a_3' \exp[ i \int_0^t \left(\frac{\gamma}{2} + \frac{c}{4} (|a_2|^2 - |a_3|^2) \right) dt],$$

$$\frac{da_3'}{dt} = -\frac{v}{2i} a_1' \exp[ i \int_0^t \left(-\frac{\gamma}{2} + \frac{c}{4} (|a_3|^2 - |a_2|^2) \right) dt].$$

We need to calculate the above integrals self-consistently. Due to the large $\alpha$, the non-linear term in the exponent generally gives a rapid phase oscillation, which makes the integral small. The dominant contribution comes from the stationary point $t_0$ of the phase around which we have

$$a_1' = -\frac{v}{2i} \int_{-\infty}^t dt \exp[ i \int_0^t \left(\frac{\gamma}{2} + \frac{3c}{4} |a_1|^2 - \frac{c}{4} \right) dt],$$ (16)

$$\frac{\gamma}{2} + \frac{3c}{4} |a_1|^2 - \frac{c}{4} = \alpha_1 (t - t_0),$$ (17)

with

$$\alpha_1 = \frac{\alpha}{2} + \frac{3c}{4} \left[ \frac{d|a_1|^2}{dt} \right]_{t_0}. $$ (18)

Since $|a_1|^2 = |a_1'|^2$, then we have

$$|a_1|^2 = \left(\frac{v}{2} \right)^2 \left| \int_{-\infty}^t dt \exp \left[ i \frac{\gamma}{2} (t - t_0)^2 \right] \right|^2 $$ (19)

This expression can be differentiated and evaluated its result at time $t_0$. A standard Fresnel integral with the result $\left[ \frac{d|a_1|^2}{dt} \right]_{t_0} = \frac{(\frac{v}{2})^2}{\sqrt{\alpha_1}}$ is obtained. Combining this with the relation (15), we come to a closed equation for $\alpha_1$,

$$\alpha_1 = \frac{\alpha}{2} + \frac{3c}{4} \left( \frac{v}{2} \right)^2 \sqrt{\frac{\pi}{\alpha_1}} $$ (20)

For given $\alpha$, $c$, $v$, $\alpha_1$ in the above equation can be obtained. The tunnelling probability

$$\Gamma_{23} = |a_1|^2 = \frac{\pi v^2}{2 \alpha_1} $$ (21)

The alliance of Eq.(20) and (21) gives the analytic expression on the tunnelling probability $\Gamma_{23}$ in the sudden limit. Compared with our numerical simulation it shows good agreement at $c/v < 130$, $c/v > 130$ a clear deviation is observable (Fig.5a). It is due to the resonance between the "internal field" and the external field leads to the invalidity of our assumption $|a_2| \sim 1$, as we show latter.

Similarly, to calculate $\Gamma_{21}$, we consider following equation,

$$a_3' = -\frac{v}{2i} \int_{-\infty}^t dt \exp[ i \int_0^t \left(-\frac{\gamma}{2} + \frac{3c}{4} |a_3|^2 - \frac{c}{4} \right) dt]$$

$$\frac{\gamma}{2} + \frac{3c}{4} |a_3|^2 - \frac{c}{4} = \alpha_3 (t - t_0)$$ (23)

$$\alpha_3 = -\frac{\alpha}{2} + \frac{3c}{4} \left( \frac{v}{2} \right)^2 \sqrt{\frac{\pi}{|a_3|}} $$ (24)

Differently, in this case we may have three stationary phase points that are solutions of equation (24) when $c < \frac{8}{27} \sqrt{\frac{\beta}{2} \alpha_3^{3/2}}$, but only one solution otherwise, as demonstrated in Fig.6. We denote them as $\alpha_{31}$, $\alpha_{32}$, $\alpha_{33}$ from smallest to largest. For small $c$, $\alpha_{31}$ is around $-\alpha/2$, and the other two solutions locates at the two sides of the origin. In this case, we simply take $\alpha_3 = \alpha_{31} + \alpha_{32} + \alpha_{33}$.

Then

$$a_3' = -\frac{v}{2i} \int_{-\infty}^t dt \exp[ i \int_0^t \frac{\alpha_3}{2} (t - t_0)^2 dt].$$ (25)

The tunnelling probability

$$\Gamma_{21} = |a_3|^2 = \frac{\pi v^2}{2 \alpha_3} $$ (26)

The alliance of the Eq.(24,26) will give the approximate solution of the $\Gamma_{21}$. Compared with our numerical simulation it shows a good agreement at $c/v < 110$, whereas for $c/v > 110$ a clear deviation is observed (Fig.5b).
FIG. 5: Comparison between our analytic results using SPA (full circles and crosses) and the numerical integration of the Schrödinger equation (solid lines). A cross is used to denote the invalidation of SPA.

What happens around $c/v = 110$ that leads to the break down of our stationary phase approximation? The reason is the resonance between the "internal field" and the external field. Let us recall the exponent in the integrand of Eq.(22), we find the effective sweeping rate should be the difference between the change rate of the "internal field" (i.e., $|a_3|$) and the sweeping rate of the external field. At $c/v = 110$, we find the two frequencies become almost identical, leading to the invalidity of SPA assumption of rapid phase oscillation. This resonance accompanied by the bifurcation of the stationary phase points. Crossing $c/v = 110$ we observe the number stationary phase points changes from three to one, as shown in Fig.6. The resonance breaks the SPA leading to serious transition from level 2 to level 1, consequently, at $c/v > 130$, our assumption $|a_2| \sim 1$ become invalid, and our approximation on the $\Gamma_{23}$ from SPA is no longer good as shown in Fig.5a.

FIG. 6: The plot of function $f(x) = x + \frac{\alpha}{v^2} - \frac{2\alpha^2}{mv^4} \sqrt{1}$

FIG. 7: The contour plot of tunnelling probability $\Gamma_{22}$ as the functions of the scaled sweeping rate and nonlinearity.

C. General property of the nonlinear tunnelling probability

The nonlinear tunnelling probability as the function of the two scaled quantities $\alpha/v^2$ and $c/v$, show many unusual properties. Taking mid-level tunnelling $\Gamma_{22}$ for example, we make a large numerical exploration for a wide range of parameters, to demonstrate the general property of the nonlinear tunnelling probability in Fig.7. In general, increasing the sweeping rate will reduce the probability of tunnelling to upper or lower level and the positive nonlinearity usually suppresses the probability of the state's staying in the mid-level, because that the nonlinearity with positive $c$ can be regarded as a kind of repulsive potential. This repulsive self-interaction make particle tend to transition to lower level more easily, and this transition becomes more serious at the occurrence of the resonance between the "internal field" and external field. The occurrence of the resonance is clearly exposed by the boundary between the white regime and dark regime in Fig.7. In the white regime, due to the resonance, the nonlinearity dramatically changes the tunnelling probability.

The other issue we want to address is the symmetry. The nonlinearity makes levels deform and therefore break the symmetry between upper level and lower
FIG. 8: $\Gamma_{nm}$ as the function of $\alpha$ for $c = 10$ (open pentacles), $c = 40$ (open circles) at $v = 1.0$. Dashed line denotes the linear case for comparison.

level, consequently, the relations $\Gamma_{21} = \Gamma_{23} = \Gamma_{32} = \Gamma_{12}, \Gamma_{31} = \Gamma_{13}, \Gamma_{33} = \Gamma_{11}$ hold in linear case break in presence of the nonlinearity. For our three-level system, the symmetry breaking is clearly exposed by Fig. 8 of showing the tunnelling probability $\Gamma_{nm}$ as the functions of $\alpha/v^2$ for $c = 10, c = 40$. In the linear case, we have $\Gamma_{21} = \Gamma_{23} = \Gamma_{32} = \Gamma_{12},$ however, with the presence of the nonlinear, $\Gamma_{12}, \Gamma_{21}$ increases whereas the $\Gamma_{23}, \Gamma_{32}$ decreases. The similar things happen for the $\Gamma_{31}, \Gamma_{13}$ and $\Gamma_{33}, \Gamma_{11}$. The above symmetry breaking may be observed experimentally [23].

IV. CONCLUSION AND APPLICATION

In conclusion, we have made a comprehensive analysis of the Landau-Zener tunnelling in a nonlinear three-level system, both analytically and numerically. Many novel tunnelling properties are demonstrated and behind dynamical mechanism is revealed.

Our model can be directly applied to the triple-well trapped BEC and to explain the tunnelling dynamics between the traps [24, 25]. In a triple-trap $v(r)$, a BEC is described by Gross-Pitaevskii equation (GPE) $i\hbar \partial \psi(r,t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(r,t) + \left[ v(r) + g_0 |\psi(r,t)|^2 \right] \psi(r,t)$ under the mean-field approximation, where $g_0 = \frac{4\pi \hbar^2 a_N}{m}$, $m$ the atomic mass and $a$ the scattering length of the atom-atom interaction. The wave function $\Psi(r,t)$ of GPE is the superposition of three wave functions describing the condensate in each trap [12], i.e., $\Psi(r,t) = \psi_1(t)\phi_1(r) + \psi_2(t)\phi_2(r) + \psi_3(t)\phi_3(r)$. When we study the tunnelling of three weakly coupled BEC in traps 1, 2 and 3, the dynamics of the system is described by the nonlinear Schrödinger equation with the Hamiltonian,

$$H = \begin{pmatrix}
E_1^0 + c_1 |\psi_1|^2 & -K_{12} & 0 \\
-K_{12} & E_2^0 + c_2 |\psi_2|^2 & -K_{23} \\
0 & -K_{23} & E_3^0 + c_3 |\psi_3|^2
\end{pmatrix}, \quad (27)
$$

where $E_{\alpha}^0 = \int (\frac{\hbar^2}{2m} |\nabla \phi_{\alpha}|^2 + v(r) |\phi_{\alpha}|^2) \, dr$ ($\alpha = 1, 2, 3$) is the ground state energy for each trap. $c_\alpha = \int g_0 |\phi_{\alpha}|^4 \, dr$ ($\alpha = 1, 2, 3$) stands for atom-atom interaction, i.e., nonlinear parameter. $K_{12} = -\int (\frac{\hbar^2}{2m} \nabla \phi_1 \nabla \phi_2 + v(r) \phi_2 \phi_1) \, dr$ is the coupling matrix element between trap 1 and 2. $K_{23} = -\int (\frac{\hbar^2}{2m} \nabla \phi_2 \nabla \phi_3 + v(r) \phi_2 \phi_3) \, dr$ is the coupling matrix element between trap 2 and 3. For simplicity, we only consider the case that these two coupling matrix elements are the same and there is no coupling between trap 1 and 3, i.e., $K_{12} = K_{23} = K$, $K_{13} = 0$. The energy bias can be adjusted by tilting the trapping well and the nonlinearity can be adjusted by the Feshbach resonance technique. We hope our theory will stimulate the experiment in the direction.

V. ACKNOWLEDGMENTS

This work was supported by National Natural Science Foundation of China (No.10474008, 10445005), Science and Technology fund of CAEP, the National Fundamental Research Programme of China under Grant No. 2005CB324503, the National High Technology Research and Development Program of China (863 Program) international cooperation program under Grant No.2004AA1Z1220.

[1] D. Landau, Phys. Z. Sowjetunion 2, 46(1932); C. Zener,
[2] V. May, O. Kuhn, Charge and Energy Transfer Dynamics in Molecular Systems, Wiley-VCH Verlag, Berlin, 2000.
[3] D. A. Harmin, P. N. Price, Phys. Rev. A 49, 1933(1994).
[4] W. Wernsdorfer and R. Sessoli, Science 284, 133 (1999).
[5] W. Wernsdorfer, S. Bhaduri, C. Boskovic, G. Christou, and D. N. Hendrickson, Phys. Rev. B 65, 180403 (2002);
W. Wernsdorfer, R. Sessoli, A. Caneschi, D. Gatteschi, and A. Cornia, Europhys. Lett. 50, 552 (2000).
[6] V. A. Yurovsky, A. Ben-Reuven, Phys. Rev. A 63, 043404(2001).
[7] A. V. Shytov, D. A. Ivanov, M. V. Feigel’man, cond-mat/0110490.
[8] D. A. Garanin, R. Schilling, Phys. Rev. B. 66, 174438(2002).
[9] V. L. Pokrovsky, N. A. Sinitsyn, Phys. Rev. B. 67, 144303(2003).
[10] Duan Suqing, Li-Bin Fu, Jie Liu, Xian-Geng Zhao, Phys. Lett. A 346 315(2005).
[11] Guan-Fang Wang, Li-Bin Fu, and Jie Liu, Phys. Rev. A 73, 013619(2006).
[12] S. Raghavan, A. Smerzi, S. Fantoni, S. R. Shenoy, Phys. Rev. A 59, 620(1999); A. Smerzi, S. Fantoni, S. Giovanazzi and S. R. Shenoy, Phys. Rev. Lett. 79, 4950(1997).
[13] Biao Wu and Qian Niu, Phys. Rev. A. 61, 023402(2000).
[14] O. Zobay and B. M. Garraway, Phys. Rev. A 61, 033603(2000).
[15] Jie Liu, et. al., Phys. Rev. A 66, 023404(2002).
[16] Jie Liu, B. Wu, L. B.Fu, R. B. Diener, Q. Niu, Phys. Rev. B 65, 224401(2002)
[17] Michael Albiez et. al., Phys. Rev. Lett 95, 010402(2005).
[18] A. V. Shytov, Phys. Rev. A 70, 052708(2004).
[19] Valentine N Ostrovsky and Hiroki Nakamura, J. Phys. A: Math. Gen. 30, 6939(1997).