REGULARIZATION BY NOISE FOR ROUGH DIFFERENTIAL EQUATIONS DRIVEN BY GAUSSIAN ROUGH PATHS

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Abstract. We consider the rough differential equation with drift driven by a Gaussian geometric rough path. Under natural conditions on the rough path, namely non-determinism, and uniform ellipticity conditions on the diffusion coefficient, we prove path-by-path well-posedness of the equation for poorly regular drifts. In the case of the fractional Brownian motion $B^{H}$ for $H > \frac{1}{4}$, we prove that the drift may be taken to be $\kappa > 0$ Hölder continuous and bounded for $\kappa > \frac{3}{2} - \frac{1}{2H}$. A flow transform of the equation and Malliavin calculus for Gaussian rough paths are used to achieve such a result.

1. Introduction

In this paper, we want to study a regularization by noise property for the following rough differential equation:

$$\text{d}x_t = b(x_t) \, \text{d}t + \sigma(x_t) \, \text{d}w_t,$$

where $w$ is a (weakly) geometric rough path. We intend to show that under suitable conditions on $\sigma$ and $w$ this equation is wellposed when $b$ has (very) poor regularity properties.

This phenomenon of regularization by noise is now well-studied in several situations. A lot of work has been devoted to the additive case $\sigma \equiv 1$ and for several kind of processes. One can think of the seminal work of [71, 52] for strong solutions of this equation when $w$ is a Brownian motion. In this additive case and when $w$ is a Brownian motion, Davie [28] exhibit a new and stronger notion of uniqueness and proved that the previous equation has a strong solution and enjoys "path-by-path" uniqueness whenever $b \in L^\infty(\mathbb{R}^d)$, whereas standard theory and counterexample require $b$ to be Lipschitz continuous. One can consult [66] for a deep discussion about notions of solutions in the additive case.

Since then, Davie's work has led to a certain number of results in several directions which are usually done in the additive cases. One can take $w$ to be a more general stochastic process and still has (even better) regularization by noise phenomenon. For example one can consider $w$ to be a fractional Brownian motion [19], a Lévy process, see [63, 3] and the references therein, or a more general stochastic process [43, 47, 30]. In a more general context some similar results when $w$ is non random and as general as possible can be exhibited [39, 19] and [38, 65]. This phenomenon of regularization by noise in a path-by-path manner was also study in other kind of problems, such as SPDEs [21, 22, 24, 23, 2, 20], mean field differential equations [41, 9], and in a mixed additive/multiplicative setting [10, 40].

A general strategy in this additive context is to look at the averaged field

$$t, x \mapsto \int_0^t b(x + w_r) \, \text{d}t$$

and to study its space-time regularity properties. To do so, one could use for example an Itô-Tanake trick (see [25] in the fractional Brownian motion setting) or the corresponding Kolmogorov equation [63]. More recently, techniques involving stochastic sewing lemma [53] and/or...
properties of the occupation measure/local time of the process [47] where used. Once a space-time regularity of the averaged field is exhibited, one can then use some pathwise non-linear calculus [37] to conclude. We will see in the following how this strategy of proof is implemented in our setting.

In the full multiplicative case, there are very few results (up to our knowledge only two). First of all, one must make sense of the equation in a pathwise sense. To do so, one usually relies on rough path theory [56, 27, 57, 5, 45, 35, 36].

Up to our knowledge in this setting, the first related result is due (again) to Davie [29], where he studies Equation (1) where \( w \) is the Statonovitch Brownian rough path. In this setting, when \( \sigma \) is an invertible \( C^3 \) matrix, he shows that uniqueness holds whenever \( b \in L^\infty \). The strategy of the previous work is to consider a strong solution \((X_t)_{t \in [0,T]}\) of the Stratonovitch SDE

\[
X_t = X_0 + \int_0^t b(X_r) \, dr + \int_0^t \sigma(X_r) \circ dB_r,
\]

to use a Girsanov transform on

\[
W_t = B_t + \int_0^t \sigma(X_r)^{-1} b(X_r) \, dr
\]

and to get back to the driftless Stratonovitch SDE on an equivalent probability measure \( \tilde{\mathbb{P}} \).

\[
Y_t = Y_0 + \int_0^t \sigma(Y_t) \circ dW_r.
\]

Then the use of the Kolmogorov PDE related to the previous equation enables to prove that \( Y \) enjoys some regularizing properties. Finally, one can write \( X = Y + Z \) and \( Z \) solve the following equation

\[
Z_t = X_0 + \int_0^t b(Z_r + Y_r) \, dr.
\]

Hence, under \( \tilde{\mathbb{P}} \), one can use the regularizing properties of \( Z \) and prove the result.

It is not directly possible to implement such a strategy in our context. Firstly, one lacks of a suitable Girsanov transform in a general (Gaussian) rough path setting. Secondly, there is no Kolmogorov equation. Nevertheless, one can see two ingredients of the proof: going back to the driftless equation, and considering an ODE where the drift and the solution of the driftless equation appears.

The second result about uniqueness of SDE in a poor regularity setting is due to Athreya, Bhar and Shekhar [1] when \( w \) is the fractional Brownian motion rough path (for \( H > \frac{1}{4} \)). There strategy is to make Lamperti transform to get back to an additive problem, and then to use the result of [19]. To do so, whereas in dimension \( d = 1 \) this strategy works pretty well, in dimension \( d \geq 2 \) one has to ask that \( \sigma^{-1} \) is conservative. This is a huge restriction, since one could expect that \( \sigma \) strictly-elliptic would be sufficient.

While we were finishing the writing of this paper, a paper by Dareiotis and Gerenscér with similar results [26] came up. Not that the techniques involved are quite different, and we are able to treat more general cases (general Gaussian rough paths instead of only fractional Brownian motion) than the results of the two authors. The price to pay in our case is a slightly worse condition on \( \sigma \) and on sigma, but a far better result on the flow of the equation.

Finally let us mention that in the rough path setting, some results have appeared considering non-Lipschitz drift [11, 64]. These work focus on growth (non-linear damping) of the coefficients and wellposedness in that context, and the local-Lipschitz continuity is always asked.

Our strategy to tackle the problem of uniqueness for poorly regular coefficients of Equation (1) may be summerize as follow, and is inspired by the work of Davie [29]. In order to compensate the lack of Girsanov transform in this setting, we rely on a flow transformation presented in the rough path setting in [64]. Indeed, it allows us to consider the regularization properties of the flow of the driftless equation. Furthermore, in order to replace the Kolmogorov equations
arising in a Brownian context, we will use some Malliavin calculus in a Gaussian rough path setting, which have been developed by several authors \[8, 14, 15, 55, 16, 44, 51]\]

1.1. **Difficulties and extended plan of the paper.** Let us emphasis the difficulties and the achievements of the paper. We have divided the study of the uniqueness of solutions of Equation (1) in three parts.

The first one, in Section 2, recalls the basic definitions and results about rough path theory. The main idea of this part is a development of an idea of \[64\], and consists of proving an equivalence for the uniqueness of solution (in the sense of Davie) of the rough differential equation (1) with the uniqueness of the solution to the standard ODE

\[ y'(t) = D\varphi_t(y(t))^{-1}b(\varphi_t(y(t))), \]

where \( \varphi \) is the flow arising from the rough differential equation when \( b \equiv 0 \). This is done in Theorem 2.8. Somehow this is a way of avoiding the Girsanov transform from Davie’s work \[29\], but still working with the solution of the driftless equation. However, this comes at price: we need to work with the flow instead of a trajectory of the driftless system. Furthermore, we can see the appearance of the inverse of the Jacobian \( (D\varphi)^{-1} \) that we need to handle. Another objective of this part is to exploit an idea from \[19\] and to exhibit a criterion for this averaged field

\[ t, x \mapsto \int_0^t (D\varphi_t(x))^{-1}b(\varphi_t(x)) \, dr \]

such that the ODE (and hence the RDE) has a unique solution.

The second part (Section 3) focuses on the regularity of the averaged field whenever the flow \( \varphi \) has some stochastic properties, this is done in Theorem 3.3 and Corollary 3.6. The idea here is to use Burkholder-Davis-Gundy (BDG) inequality for martingales in an infinite dimensional setting (namely in \( L^p(\mathbb{R}^d) \)). This is quite close to the use of the stochastic sewing lemma in infinite dimensional spaces \[54\]. Nevertheless, since those work where developed in parallel we keep here with our presentation. Furthermore, since a BDG inequality in Banach spaces is not a so standard topic, and in order to be as self contained as possible, we have included in the Appendix C a full proof of it. Note that in this section we have proved a general Kolmogorov criterion for regularity of the averaged field in Besov spaces which could be interesting in itself.

The third part (Section 4 and 5) focuses on a tool to avoid the use of Kolmogorov equations: the Malliavin calculus. Indeed, by using Section 3, the main idea to deduce the regularizing properties of the averaged field constructed thanks to the flow of the driftless RDE is to have a integration by part formula, such that, formally

\[ \mathbb{E}[D\varphi_t(x)\nabla b(\varphi_t(x))|\mathcal{F}_s] = \mathbb{E}[b(\varphi_t(x))H_{s,t}(x)|\mathcal{F}_s], \]

with

\[ \|H_{s,t}(x)\|_{L^q(\Omega)} \lesssim |t-s|^{-H}, \]

for some \( H > 0 \). This is precisely one of the key point of Malliavin calculus. Note that, in that setting, one has to focus on conditional Malliavin calculus. Hence, Section 4 recalls the standard facts and results about Malliavin calculus in a Gaussian context. In this section, some conditional integrations by part results are also proved. Section 5 focuses itself on Malliavin calculus for solutions of driftless rough differential equation driven by Gaussian rough paths.

Finally, in Section 6 we are able to prove the desired wellposedness result which can be stated as follow (see Theorem 6.3 and 6.1 for precise statements):

**Theorem.** Let \( 2 < p < 4 \). Let \( W \) be a \( p \)-geometric Gaussian rough path such that its first component \( (W_t)_{t \in [0,T]} = (W_{1,t}^1)_{t \in [0,T]} \) is \( \alpha \)-locally non-determinism, namely

\[ \inf_{0 \leq s \leq t \leq T} (t-s)^{-\alpha} \text{Var} [W_t - W_s|\mathcal{F}_{[0,s]} \vee \mathcal{F}_{[t,1]}] = c_W > 0. \]

Let us suppose that \( \sigma \in C^\infty_{\text{b}}(\mathbb{R}^d; \mathbb{R}^{d \times d}) \) is uniformly elliptic. Let \( b \in C^\alpha \) for \( \kappa + \frac{1}{a} > \frac{3}{2} \) and \( \kappa > 0. \) Then almost surely there is a unique solution of Equation (1). Furthermore the solutions of this equation are locally-Lipschitz continuous with respect to the initial condition.
Moreover, when $B^H$ is the geometric rough path above the fractional Brownian motion of Hurst parameter $\frac{1}{4} < H < \frac{1}{2}$, one can take $\alpha = 2H$.

Finally, in order to be self-contained as possible, we have included some multidimensional Garsia-Rodemich-Rumsey inequalities in the Appendix (Appendix D as well as a small topo on Besov spaces (Appendix A).

1.2. Notations and preliminary. We gather here some useful notations for the following.

Throughout this paper, we consider $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ to be a filtered probability space. We recall that $S$ is the Schwarz space (see [69]) defined by

$$S(\mathbb{R}^d) = \left\{ f \in C_0^\infty(\mathbb{R}^d) ; \forall k \in \mathbb{N}, \| f \|_{S,k} < +\infty \right\},$$

where

$$\| f \|_{S,k} = \max_{|\beta| + |\gamma| \leq k} \sup_{x \in \mathbb{R}^d} |x^\beta D^\gamma f(x)|.$$

It is a topological vector space that becomes a Fréchet space when it is equipped with the distance

$$d_S(\phi, \varphi) = \sum_{k=0}^{\infty} 2^{-k} \frac{\| \phi - \varphi \|_{S,k}}{1 + \| \phi - \varphi \|_{S,k}}.$$

Its continuous dual is $S'$ the space of tempered distributions and, moreover, we have the following Gelfand triple

$$S \subset L^2(\mathbb{R}^d) \subset S',$$

where $S$ is dense in $S'$ with respect to the weak topology of $L^2(\mathbb{R}^d)$. We denote $\langle \cdot, \cdot \rangle$ the duality product which is an extension of the scalar product in $L^2(\mathbb{R}^d)$. The Besov spaces $B^\alpha_{p,r}$ (associated to the Littlewood-Paley blocks) are Banach spaces constructed by a completion in $S'$

$$B^\alpha_{p,r} := \left\{ u \in S'(\mathbb{R}^d) ; \| u \|_{B^\alpha_{p,r}} := \left( \sum_{j=-1}^{+\infty} 2^{jd\alpha} \| \Delta_j u \|_{L^p(\mathbb{R}^d)}^r \right)^{1/r} < \infty \right\},$$

where $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$ (see Appendix A for details and some results).

We also denote, for any $k \geq 0$, $C^k(\mathbb{R}^d; \mathbb{R}^\ell)$ the set of functions from $\mathbb{R}^d$ to $\mathbb{R}^\ell$ that are continuous, $k$ times differentiable and whose derivatives are continuous. We denote $C^\infty(\mathbb{R}^d; \mathbb{R}^\ell) = \bigcap_{k=1}^{+\infty} C^k(\mathbb{R}^d; \mathbb{R}^\ell)$. We denote by $C^k_b$ (respectively $C^\infty_b$) the functions in $C^k$ (respectively in $C^\infty$) bounded together with all their derivatives.

**Definition 1.1.** A function from $\mathbb{R}_+ \to \mathbb{R}_+^*$ is called a weight. Let $f$ be a function from $\mathbb{R}^d$ to $\mathbb{R}$. Let us define the weighted sup norm of $f$ by

$$\| f \|_{\infty,w} = \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{w(|x|)}.$$

Furthermore, we define

$$L^\infty_w(\mathbb{R}^d; \mathbb{R}) = \left\{ f : \mathbb{R}^d \to \mathbb{R} ; \| f \|_{\infty,w} < +\infty \right\}.$$

When there exists $\gamma > 0$ such that $\sup_{x \in \mathbb{R}^d} \frac{w(x)}{(1+x)^\gamma} < +\infty$, we say that $w$ has $\gamma$-power growth. Let us now (with a slight abuse of notations) consider $w = (w_k)_{k \geq 0}$ to be a sequence of positive functions from $\mathbb{R}_+$ to $\mathbb{R}_+^*$. We say that $w$ has $\gamma$-power growth if each $w_k$ has a $\gamma$-power growth.

**Definition 1.2.** Let $w$ be a sequence of weights and $\alpha > 0$. We denote

$$\| f \|_{C^\alpha_w} := \sum_{k=0}^{[\alpha] - 1} \| D^k f \|_{\infty,w_k} + \sup_{x \neq y} \frac{|D^{[\alpha]-1} f(x) - D^{[\alpha]-1} f(y)|}{|x - y|^\alpha w^{[\alpha]}(|x| + |y|)},$$

and define the Banach space

$$C^\alpha_w(\mathbb{R}^d; \mathbb{R}^\ell) := \left\{ f : \mathbb{R}^d \to \mathbb{R}^\ell ; \| f \|_{C^\alpha_w} < +\infty \right\}.$$
To simplify notations, we denote $C^w = C^w_0$ when $w = (1, 1, \ldots, 1)$, and we have (see Appendix A) $C^w = B^w_{\infty, \infty}$ for all $\alpha \in \mathbb{R}^* \setminus \mathbb{N}$. We also denote

$$C^k_{\mathbb{R}^n} = C^k_{(1+, 1, \ldots, 1)}(\mathbb{R}^d)$$

**Remark 1.3.** Note that when $w = (w_0, 1)$, then $C^1_w$ denotes the set of Lipschitz continuous function with a growth rate $w_0$.

We finally recall some properties of linear operator. Let $T$ be a linear operator from $D(T) \supset S$ to $E$ where $E$ is a Banach space. For any Banach space $F \supset D(T)$, if we have, for any $x \in D(T)$,

$$\|Tx\|_E \leq C\|x\|_F,$$

for some constant $C > 0$ independent of $x$, then $T$ admits a closure from $F$ to $E$. The space of bounded linear operator from $E$ to $F$ is denoted $\mathcal{B}(F, E)$ and is equipped with the norm

$$\|T\|_{\mathcal{B}(F, E)} = \sup_{\|x\|_F \leq 1} \|Tx\|_E.$$

Finally, we write $a \lesssim b$ when there exists a constant $c > 0$ such that $a \leq cb$.

2. Rough differential equation with drift

In this section, we recall some basic facts about rough paths and rough differential equations. For more details, one can consult the seminal papers [56, 27, 45], but also the monographs [57, 35, 36]. Note that an interesting perspective for flow driven by rough differential equations is given by Bailleul in [5]. We will focus here on the Davie’s definition of solutions of rough differential equations driven by (weak-)geometric Hölder path, with $p \geq 2$.

2.1. Rough paths and differential equations in a nutshell. For any $N \in \mathbb{N}$, a truncated tensor algebra $T^N(\mathbb{R}^d)$ is defined by

$$T^N(\mathbb{R}^d) = \bigoplus_{k=0}^{N} (\mathbb{R}^d)^{\otimes k},$$

with the convention $(\mathbb{R}^d)^{\otimes 0} = \mathbb{R}$. Let $(e_1, \cdots, e_d)$ be the canonical basis of $\mathbb{R}^d$, then for any $k \in \{1, \cdots, N\}$,

$$(e_I)_{I \in \{1, \cdots, d\}^k} := (e_{i_1} \otimes \cdots \otimes e_{i_k})_{I = (i_1, \cdots, i_k) \in \{1, \cdots, d\}^k}$$

is the canonical basis of $(\mathbb{R}^d)^{\otimes k}$, and for any $x \in T^N(\mathbb{R}^d)$,

$$x = \sum_{k \in \{0, \cdots, N\}} x^k = x^0 + \sum_{k \in \{1, \cdots, N\}} \sum_{I \in \{1, \cdots, d\}^k} x^{k, I} e_I,$$

where $x^k$ is the projection of $x$ on the $k$-th tensor.

This space is equipped with a vector space structure as well as an operation $\otimes$ defined by

$$(x \otimes y)^k = \sum_{\ell=0}^{N} (x^{k-\ell} \otimes y^\ell), \quad \forall x, y \in T^N(\mathbb{R}^d),$$

In the end, $(T^N(\mathbb{R}^d), +, \otimes)$ is an associative algebra with unit element $1 \in (\mathbb{R}^d)^{\otimes 0}$.

For any $s < t$ and $k \geq 1$, we define the simplex

$$\Delta^k_{s,t} = \{(u_1, \ldots, u_k) \in [s, t]^k; u_1 < \ldots < u_k\}.$$

We also denote $\Delta^k_T := \Delta^k_{0,T}$. A multiplicative functional is a continuous map $w : \Delta^2_T \to T^N(\mathbb{R}^d)$ that verifies, for any $s < u < t$,

$$w_{s,t} = w_{s,u} \otimes w_{u,t}$$
A fundamental example of such a map are the iterated integrals of a smooth paths \( w : [0, T] \to \mathbb{R}^d \)
\[
\mathcal{W}^k_{s,t} = \sum_{1 \leq i_1, \ldots, i_k \leq d} \left( \int_{\Delta_{s,t}} du^{i_1} \cdots du^{i_k} \right) e_{i_1} \otimes \cdots \otimes e_{i_k},
\]
where \((e_1, \cdots, e_d)\) is the canonical basis of \( \mathbb{R}^d \) and \( k \geq 1 \). Then, we call the signature of \( w \) the mapping \( S_N(w) : \Delta^2 \to T^N(\mathbb{R}^d) \) given by
\[
(s, t) \mapsto 1 + \sum_{k=1}^N \mathcal{W}^k_{s,t}.
\]

It turns out that every signature is a multiplicative functional that belongs to \( G^N(\mathbb{R}^d) \) which is a subset of \( T^N(\mathbb{R}^d) \) of group-like elements given by
\[
G^N(\mathbb{R}^d) := \exp \otimes (L^N(\mathbb{R}^d)),
\]
where \( L^N(\mathbb{R}^d) \) is the linear span of elements that can be written as a commutator \( a \otimes b - b \otimes a \) with \( a, b \in T^N(\mathbb{R}^d) \). Furthermore, there is a Carnot-Caratheodory norm on \( G^N(\mathbb{R}^d) \), denoted \( \| \cdot \|_{CC} \), which is homogeneous with respect to the natural scaling operation on \( T^N(\mathbb{R}^d) \). We can now introduce the notion of rough paths.

**Definition 2.1.** The space of weakly geometric \( p \)-rough paths is the set of multiplicative functional \( \mathbf{x} : \Delta^2 \to G^{|p|}(\mathbb{R}^d) \) such that
\[
\| \mathbf{w} \|_{p\text{-var} [0,T]} := \sup_{\pi \in \Pi([0,T])} \left( \sum_{[u,v] \in \pi} \| \mathbf{w}_{u,v} \|_{CC}^p \right)^{1/p} < \infty,
\]
where \( \Pi([0,T]) \) is the set of all subdivisions of \([0,T]\). The space of geometric \( p \)-rough paths is the closure under \( \| \cdot \|_{p\text{-var} [0,T]} \) of smooth signatures \( \{ S_{|p|}(w) \mid w \in C^\infty([0,T]; \mathbb{R}^d) \} \).

In the following, we will also use the Hölder norm for rough paths that is given by, for \( \alpha \in (0,1) \),
\[
\| \mathbf{w} \|_{\alpha \text{-Hölder} [0,T]} := \sup_{[s,t] \subset [0,T]} \frac{\| \mathbf{w}_{s,t} \|_{CC}}{(t-s)^\alpha}.
\]

**2.2. Flow generated by driftless differential equation driven by rough path.** In this subsection, we give known and basic properties about solutions of rough differential equations of the form
\[
dy_t = \sigma(y_t) \, dw_t, \quad y_0 = x, \tag{2}
\]
where \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d} \) is such that
\[
\sigma(x)w = \sum_{i=1}^d \sigma_i(x)w^i,
\]
for any \( w = (w^i)_{i \in \{1, \ldots, d\}} \in \mathbb{R}^d \). Following [6] we identify \( \sigma_i \) to a first order differential operator, namely for any smooth function \( f : \mathbb{R}^d \to \mathbb{R}^d \),
\[
\sigma_i f(x) = Df(x) \sigma_i(x), \quad x \in \mathbb{R}^d.
\]
For \( k \geq 1 \) and \( I \in \{1, \cdots, d\}^k \) we define the \( k \)-th order differential operator
\[
\sigma_I = \sigma_{i_1} \cdots \sigma_{i_k}
\]
via \( \sigma_I f = \sigma_{i_1} \cdots (\sigma_{i_k} f) \), with the slight abuse of notations \( \sigma_I(x) = (\sigma_I id)(x) \).
Definition 2.2. Let \( p \geq 2 \) and \( T > 0 \). Let \( w \) be an Hölder \( \frac{1}{p} \)-Hölder weakly geometric rough path. Let \( \sigma \in C^{\beta} \) for some \( \beta \geq |p| \). A function \( x \in C^{\frac{1}{p}}([0, T]; \mathbb{R}^d) \) is a solution (in the sense of Davie) to the driftless Rough Differential Equation (RDE) (2) if there exists two constants \( C = C \left( \|w\|_{\frac{1}{p}-\text{Hölder};[0, T]}; \sigma \right) \) and \( a > 1 \) (independent of \( \sigma \) and \( w \)) such that for all \((s, t) \in \Delta_T^2\),
\[
\left| (y_t - y_s) - \left( \sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} w_{s, t}^{k, I} \sigma_I(y_s) \right) \right| \leq C|t - s|^a.
\]

The existence, uniqueness and flow properties of solution to driftless RDE is given in [46, 36, 35, 27, 7, 5, 6].

Note that using [6, 18], one can see that there is an equivalence between Davie’s notion of solutions to RDE and Bailleul’s one. Namely a path \( y \) is a solution to Equation (2) if and only if for any \( f \in C_{\text{lin}}^{[p]+1}(\mathbb{R}^d; \mathbb{R}^d) \) and for all \((s, t) \in \Delta_T^2\),
\[
\left| (f(y_t) - f(y_s)) - \left( \sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} w_{s, t}^{k, I} \sigma_I(f(y_s)) \right) \right| \leq C\|f\|_{C_{\text{lin}}^{[p]+1}}|t - s|^a,
\]
where \( C \) also depends on \( f \). Furthermore, using [7], one has the following result

Theorem 2.3. Let \( p \geq 2 \) and \( T > 0 \) and \( n \geq 0 \). Let \( w \) be a \( \frac{1}{p} \)-Hölder weakly geometric \( p \) rough path, and let \( \sigma \in C_b^{[p]+n+1}(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d) \). There is a unique
\[
\varphi : \Delta_{s, t}^2 \mapsto C_{\text{lin}}^{n}(\mathbb{R}^d; \mathbb{R}^d)
\]
and a unique
\[
\psi : \Delta_{s, t}^2 \mapsto C_{\text{lin}}^{n}(\mathbb{R}^d; \mathbb{R}^d)
\]
such that
1. For all \( s \in [0, T] \) and \( x \in \mathbb{R}^d \), \( t : [s, T] \mapsto \varphi_{s, t}(x) \) is the unique solution to
\[
dx_t = \sigma(x_t) \, dw_t, \quad x_s = x, \quad t \in [s, T].
\]
2. For all \( x \in \mathbb{R}^d \) and all \((s, u), (u, t) \in \Delta_T^2\),
\[
\varphi_{s, u}(\varphi_{u, t}(x)) = \varphi_{s, t}(x) \quad \text{and} \quad \psi_{s, u}(\psi_{u, t}(x)) = \psi_{s, t}(x).
\]
3. A constant \( C = C(\|w\|_{\frac{1}{p}-\text{Hölder};[0, T]}; \sigma) \) exists such that for all \((s, t) \in \Delta_T^2\), all \( x \in \mathbb{R}^d \) and all \( 0 \leq m \leq n \)
\[
\left| D^m \varphi_{s, t}(x) - \left( (D^m \text{id})(x) + \sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} w_{s, t}^{k, I} (D^m \sigma_I)(x) \right) \right| \leq C|t - s|^{\frac{|p|+1}{p}}.
\]
4. For all \((s, t) \in \Delta_T^2\) and all \( x \in \mathbb{R}^d \)
\[
\varphi_{s, t}(\psi_{s, t}(x)) = \psi_{s, t}(\varphi_{s, t}(x)) = x.
\]
5. There exists a constant \( C = C(\|w\|_{\frac{1}{p}-\text{Hölder};[0, T]}; \sigma) \) such that for all \((s, t) \in \Delta_T^2\), all \( x \in \mathbb{R}^d \) and all \( 0 \leq m \leq n \)
\[
\left| D^m \psi_{s, t}(x) - \left( (D^m \text{id})(x) + \sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} v_{s, t}^{k, I} (D^m \sigma_I)(x) \right) \right| \leq C|t - s|^{\frac{|p|+1}{p}}.
\]

where
\[
v_{s, t} = \sum_{k=0}^{|p|} (1 - w_{s, t})^\otimes k.
\]
(6) The maps from \( C^1([0, T]; G^{[p]}(\mathbb{R}^d)) \) to \( C^1([0, T]; C^m_{\text{lin}}(\mathbb{R}^d; \mathbb{R}^d)) \) given by
\[
\mathbf{w} \mapsto \varphi \quad \text{and} \quad \mathbf{w} \mapsto \psi
\]
are continuous.

**Remark 2.4.** Note that Theorem 2.3 (5) is equivalent to say that for all \( s \in [0, T) \) and for all \( x \in \mathbb{R}^d \), \( t \mapsto \psi_{s,t}(x) \) is the unique solution of the following ordinary differential equation:
\[
\frac{d y_t}{dt} = \sigma(y_t) \mathbf{d} v_t, \quad y_s = x, \quad t \in [s, T].
\]
In that setting, one can use the equivalent definition of solutions seen in equation (3).

**Proof.** The proof is somehow classical. For a similar proof without the full Euler scheme expansion, one can consult [64] Lemma 2.3 or [36], Theorem 10.14 and 10.26.

The proof here is an easy adaptation to the one of [7]. Indeed, one can combine Theorem 2.2, Corollary 3.5, Remark 3.6 and Theorem 4.2 from [7] to get the existence and uniqueness of \( \varphi \), and points (1), (2) and (3). Note also that the continuity of \( \varphi \) with respect to \( \mathbf{w} \) is also a consequence of the previous theorems.

To construct \( \psi \) and prove the wanted Euler expansion, let us remind the strategy of proof in [7] (see also [3]). Since \( \mathbf{w} \) is in \( G^{[p]}(\mathbb{R}^d) \), there exists an \( \mathbf{1} \in L^{[p]}(\mathbb{R}^d) \) such that
\[
\exp^\mathbf{1}(\mathbf{1}) = \mathbf{w}.
\]
Remark that \( \mathbf{1} \in L^{[p]}(\mathbb{R}^d) \) and a basis of this space is
\[
e_i = e_i, \quad e_{[i],t} = [e_i, e_{[t]}] = e_i \otimes e_{[t]} - e_{[t]} \otimes e_i.
\]
Hence
\[
\mathbf{1} = \sum_{k=1}^{[p]} \sum_{I \in \{1, \ldots, d\}^k} 1^{[k]}_{I} e_{[I]}
\]
and one can show that, in \( C^1([-1, 1]; C^m_{\text{lin}}(\mathbb{R}^d; \mathbb{R}^d)) \),
\[
\varphi_{s,t} = \lim_{\pi \in \Pi([s, t])} \mu_{N-1-tN} \circ \cdots \circ \mu_{t0, t1},
\]
where \( \Pi([s, t]) \) denotes the set of partitions of \([s, t]\), \( \pi = \{ s = t_0 < t_1 \cdots < t_{N-1} < t_N = t \} \) is a partition of \([s, t]\) and \( |\pi| = \sup_{\pi \in \Pi([s, t])} |t_{i+1} - t_i| \) is the radius of \( \pi \), and \( \mu_{s,t} = y(1) \) where \( y \) is the solution of the following ordinary differential equation:
\[
y'(r) = \sum_{I \in \{1, \ldots, d\}^k} 1^{[k]}_{I} \sigma_{[I]}(y(r)), \quad y(0) = x, \; r \in [0, 1].
\]
Here, for \( i \in \{1, \ldots, d\} \), \( \sigma_{[i]} = \sigma_i \) and for \( k \geq 1, i \in \{1, \ldots, d\} \) and \( I \in \{1, \ldots, d\}^k \), \( \sigma_{[i,I]} = \sigma_i \sigma_{[I]} - \sigma_{[I]} \sigma_i \) is a combination of two \( \mu_{s,t} \). Now, let us define the terminal value solution of the previous equation
\[
z'(r) = \sum_{I \in \{1, \ldots, d\}^k} 1^{[k]}_{I} \sigma_{[I]}(z(r)), \quad z(1) = x, \; r \in [0, 1]
\]
and
\[
\nu_{s,t}(x) = z(0).
\]
We have for all \( x \in \mathbb{R}^d \) and all \((s, t) \in \Delta^d_T \),
\[
\nu_{s,t}(\mu_{s,t})(x) = \mu_{s,t}(\nu_{s,t})(x) = x,
\]
Let us define for a partition \( \pi \in \Pi([s, t]) \)
\[
\nu_{s,t}^{\pi} = \mu_{t0, t1} \circ \cdots \circ \mu_{t_{N-1}, tN}
\]
and
\[
\mu_{s,t}^{\pi} = \mu_{t_{N-1}, tN} \circ \cdots \circ \mu_{t0, t1}.
\]
then
\[ \mu_{s,t}^\pi(\nu_{s,t}^\pi(x)) = \mu_{s,t}^\pi(\mu_{s,t}^\pi(x)) = x. \]

Hence, one only has to prove that \( \nu^\pi \) converges to a (backward) flow in the suitable space. Note that this follows from the same arguments as the proof of Theorem 2.2 in [7], which guarantees that \( \psi \) exists in \( C_{lin}^m \) is continuous with respect to \( w \) and that for \((s, t) \in \Delta^2\) small enough (with respect to \( w \)), one has for any \( 0 \leq m \leq n \),

\[ \sup_{x \in \mathbb{R}^d} |D^m \psi_{s,t} - D^m \nu_{s,t}(x)| \lesssim_w (t - s)^{-\frac{m+1}{p}}. \]

Furthermore, one also have (see Lemmas 3.3, 3.4 and Corollary 3.5 in [7]), for \( a = \frac{\lfloor p \rfloor + 1}{p} \)

\[ |D^m \nu_{s,t}(x) - \left( (D^m id)(x) + \sum_{j=1}^{\lfloor p \rfloor} (-1)^j \sum_{1 \leq k_1, \ldots, k_j \leq \lfloor p \rfloor} \prod_{i=1}^j V_i(x) \right) | \lesssim_w (t - s)^a. \]

Hence, if we define \( v = \exp^\oplus(-1) \) we have

\[ \left| D^m \nu_{s,t}(x) - \left( (D^m id)(x) + \sum_{k=1}^{\lfloor p \rfloor} \sum_{I \subseteq \{1, \ldots, d\}^k} v_k I D^m V_I(x) \right) \right| \lesssim_w (t - s)^a. \]

Note first that

\[ v_{s,t} \otimes w_{s,t} = \exp^\oplus(-1_{s,t}) \otimes \exp^\oplus(1_{s,t}) = \exp^\oplus(0) = 1 = w_{s,t} \otimes v_{s,t}. \]

Furthermore, note also that for \( k > \lfloor p \rfloor \),

\[ (1 - w_{s,t})^{\otimes k} = 0. \]

Hence

\[ v_{s,t} = \sum_{k=0}^{\lfloor p \rfloor} (1 - w_{s,t})^{\otimes k} \]

and the result follows. \( \Box \)

**Corollary 2.5.** Let \( n \geq 1 \) and \( p, T, w, \sigma \) as in the previous Theorem. Then

\[ (D\varphi)^{-1}, ((D\varphi)^{-1} \in C^\bar{m}_{\bar{b}}([0, T]; C_{lin}^{n-1}(\mathbb{R}^d; \mathcal{M}_d(\mathbb{R}))). \]

**Proof.** Let us remark that under the previous hypothesis, both \( \varphi \) and \( \psi \) are in \( C_{lin}^m \). Hence \( D\varphi_{s,t}(x) D\psi_{s,t}(\varphi_{s,t}(x)) = id. \) The previous equality gives \( (D\varphi_{s,t}(x))^{-1} = D\psi_{s,t}(\varphi_{s,t}(x)) \), and there exists a constant, depending on \( w, T, \sigma \) such that

\[ \sup_{(s, t) \in \Delta^2} \| (D\varphi_{s,t}(x))^{-1} \|_{C_{lin}^{n-1}} \lesssim_w 1, \]

which ends the proof. \( \Box \)

### 2.3. Solution of RDE with drift : flow transform.

We turn now to the study of rough differential equations with a drift:

\[ dx_t = b(x_t) dt + \sigma(x_t) dw_t, \quad x_0 = x, \quad t \in [0, T]. \quad (4) \]

In the spirit of Definition 2.2, let us define the solutions of the rough differential equation with drift as follow
Definition 2.6. Let \( p \geq 2, T > 0 \) and \( n \geq 0 \). Let \( b \in C^0_b(\mathbb{R}^d; \mathbb{R}^d) \) and let \( \sigma \in C^{(p)}(\mathbb{R}^d; \mathbb{R}^d) \) and \( w \) be a \( \frac{1}{p} \)-Hölder \( p \)-weakly geometric rough path. A path \((x_t)_{t \in [0,T]} \in C^p_T([0,T]; \mathbb{R}^d)\) is a solution to Equation (4) if \( x_0 = x \) and there exists a constant \( C = C(w, \sigma, b) > 0 \) and a constant \( a > 1 \) independent of \( w \) such that for all \( (s, t) \in \Delta_T^p \),
\[
| x_t - (x_s + b(x_s)(t-s)) + \sum_{k=1}^{[p]} \sum_{I \in \{1, \ldots, d\}^k} w^{k,I}_{s,t} \sigma_I(x_s) | \leq C | t-s |^a.
\]

Remark 2.7. As proved in [6] and [18] (we also refer to [7] for the precise value of the following constants), the following equivalent solution can be used: Let \( p, w, T, b, \sigma, \varepsilon \) as in the previous Definition. Then \((x_t)_{t \in [0,T]} \) is a solution to Equation (4) if and only if there exists a \( a > 1 \) independent of \( w, \sigma, b, \varepsilon = \varepsilon(w) > 0 \) and a constant \( C = C(w, \sigma, b) \) such that for all \( f \in C^{[p]+1}_b(\mathbb{R}^d; \mathbb{R}^d) \) and all \( (s, t) \in \Delta_T^p \) with \( | t-s | \leq \varepsilon \),
\[
| f(x_t) - (bf)(x_s)(t-s) + \sum_{k \in \{1, \ldots, [p]\}} \sum_{I \in \{1, \ldots, d\}^k} w^{k,I}_{s,t} (\sigma_I f)(x_s) | \leq C ||f||_{C^{[p]+1}_b} | t-s |^a.
\]
This point will be crucial in the following.

Several results concerning the (optimal) regularity of the drift are available in order to have existence and uniqueness for rough differential equations. Namely, in [36] one can see that whenever \( \sigma \in C^{[p]+1}_b \) and \( b \) is globally Lipschitz continuous with linear growth, there is a unique solution to equation (4). Furthermore, some improvement of this criteria appears in [64] (weak local Lipschitz condition and some control on the growth) and in [11] (Lyapunov conditions on \( b \)). In order to go beyond (using some stochasticity of the rough path), one needs to develop an other approach of the solution to the RDE with drift. Following a classical ideas of flow transform (see for example [64]) we give an equivalent formulation for the solutions of Equation (4).

Theorem 2.8. Let \( p \geq 2, T > 0, \kappa > 0, b \in C^\kappa(\mathbb{R}^d; \mathbb{R}^d), \sigma \in C^{2[p]+1}_b(\mathbb{R}^d; \mathbb{R}^d) \) and let \( w \) be a \( \frac{1}{p} \)-Hölder \( p \)-weakly geometric rough path. Then \((x_t)_{t \in [0,T]} \) is a solution to the rough differential equation with drift (4) if and only if
\[
x_t = \varphi_{0,t}(z(t))
\]
and \((z(t))_{t \in [0,T]} \) is a solution to the ordinary differential equation
\[
z'(t) = (D\varphi_{0,t}(z(t)))^{-1}b(\varphi_{0,t}(z(t))), \quad z_0 = x, \quad t \in [0,T],
\]
where \( \varphi \) is the flow constructed in Theorem 2.3 and Corollary 2.5.

Proof. First, we remark that since \( \sigma \in C^{[p]+1}_b(\mathbb{R}^d; \mathbb{R}^d) \), thanks to Theorem 2.3 and Corollary 2.5,
\[
\left( (t, x) \mapsto (D\varphi_{0,t}(x))^{-1}b(\varphi_{0,t}(x)) \right) \in C^0_b([0,T]; C^0(\mathbb{R}^d; \mathbb{R}^d)).
\]
Hence, it follows from Peano’s existence theorem that there exists a solution \((z(t))_{t \in [0,T]} \) to Equation (5). Let us define for all \( t \in [0,T], x_t = \varphi_{0,t}(z(t)) \). Note that, thanks to the hypothesis and Corollary 2.5,
\[
| z(t) - x | \lesssim_{w, \sigma, T} 1,
\]
and
\[
| z(t) - z(s) | \lesssim_{w, \sigma, T} | t-s |.
\]
Furthermore, note that for any \( s, t \in \Delta_T^p \),
\[
z(t) - z(s) = \int_s^t (D\varphi_{0,r}(z(r)))^{-1} b(\varphi_{0,r}(z(r))) \, dr
\]
\[
= (D\varphi_{0,s}(z(s)))^{-1} b(\varphi_{0,s}(z(s)))))
\]

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Thanks to the hypothesis and the previous computations, the re exists

\[ (D\varphi_{0,r}(z(r)))^{-1} - (D\varphi_{0,s}(z(r)))^{-1} \] b(\varphi_{0,r}(z(r))) dr \quad (6)

\[ + \int_s^t \left( (D\varphi_{0,s}(z(r)))^{-1} - (D\varphi_{0,s}(z(s)))^{-1} \right) b(\varphi_{0,r}(z(r))) dr \quad (7) \]

\[ (D\varphi_{0,s}(z(r))) \int_s^t b(\varphi_{0,r}(z(r))) - b(\varphi_{0,r}(z(r))) dr \quad (8) \]

Note that thanks to Corollary 2.5, \((t, x) \mapsto (D\varphi_{0,t}(x))^{-1} \in C^7([0, T]; C^6_b(\mathbb{R}^d; \mathbb{R}^d))\). Hence, since \(b\) is bounded there exists a constant \(C = C(w, \sigma, T, b) > 0\) such that

\[ |(6)| \leq C|t - s|^{1 + \frac{\kappa}{2}}, \]

\[ |(7)| \leq C|t - s|^2 \]

and since \(b \in C^0(\mathbb{R}^d; \mathbb{R}^d)\),

\[ |(8)| \leq C|t - s|^{1 + \kappa}. \]

Let us define \(R'_{s,t}(x) = (6) + (7) + (8)\).

Furthermore, let us remind that we have for all \((s, t) \in \Delta_T\),

\[ \varphi_{s,t}(x) = x + \sum_{k \in \{1, \ldots, |p| \}} w^{k,I}_{s,t} \sigma_I(x) + R_{s,t}(x) \]

with

\[ \sup_{x \in \mathbb{R}^d} |R_{s,t}(x)| \lesssim_{w, \sigma, T} |t - s|^{\frac{|x| + 1}{p}}. \]

Finally

\[ x_t - x_s = \left( \varphi_{s,t}(\varphi_{0,s}(z(t))) - \varphi_{0,s}(z(t)) \right) + \left( \varphi_{0,s}(z(t)) - \varphi_{0,s}(z(s)) \right) \]

\[ = \sum_{k \in \{1, \ldots, |p| \}} w^{k,I}_{s,t} \sigma_I(\varphi_{0,s}(z(s))) \]

\[ + \sum_{k \in \{1, \ldots, |p| \}} w^{k,I}_{s,t} \left( \sigma_I(\varphi_{0,s}(z(t))) - \sigma_I(\varphi_{0,s}(z(s))) \right) + R_{s,t}(\varphi_{0,s}(z(t))) \]

\[ + D\varphi_{0,s}(z(s)) \left( D\varphi_{0,s}(z(s))^{-1} b(\varphi_{0,s}(z(s))) + R'_{s,t}(x) \right) \]

\[ + \int_0^1 \int_0^1 \lambda \left( D^2\varphi_{0,s}(\lambda \mu(z(t) - z(s)) + z(s)) \right) (z(t) - z(s)) \otimes^2 d\mu d\lambda \]

\[ = b(\varphi_{0,s}(z(s))) + \sum_{k \in \{1, \ldots, |p| \}} w^{k,I}_{s,t} \sigma_I(\varphi_{0,s}(z(s))) + R''_{s,t}(x), \]

where

\[ R''_{s,t}(x) = R_{s,t}(\varphi_{0,s}(z(t))) + D\varphi_{0,s}(z(s)) R'_{s,t}(x) + \sum_{k \in \{1, \ldots, |p| \}} w^{k,I}_{s,t} \left( \sigma_I(\varphi_{0,s}(z(t))) - \sigma_I(\varphi_{0,s}(z(s))) \right) \]

\[ + \int_0^1 \int_0^1 \lambda \left( D^2\varphi_{0,s}(\lambda \mu(z(t) - z(s)) + z(s)) \right) (z(t) - z(s)) \otimes^2 d\mu d\lambda. \]

Thanks to the hypothesis and the previous computations, there exists \(a > 1\) and a constant \(C = C(w, \sigma, b, T)\) such that

\[ \sup_{x \in \mathbb{R}^d} |R''_{s,t}(x)| \leq C|t - s|^a, \]

which proves that whenever \((z(t))_{t \in [0, T]}\) is a solution of (5), then \((x_t)_{t \in [0, T]} = (\varphi_{0,t}(z(t)))_{t \in [0, T]}\) is a solution of Equation (4).
Now, let us take \((x_t)_{t \in [0,T]}\) a solution of Equation (4). Let us denote by \((z(t))_{t \in [0,T]} = (\psi_{0,t}(x_t))_{t \in [0,T]}\) and let us prove that \((z(t))_{t \in [0,T]}\) is a solution of (5). In the following, we will crucially use Remarks 2.4 and 2.7, and the fact that since \(\sigma \in C^2([p]+1)\), for all \(s \in [0,T]\),
\[
\|\psi_{0,s}\|_{C_{lin}^{[p]+1}} \lesssim w,T,\sigma 1.
\]
We have, for all \((s,t) \in \Delta^2_T\) small enough (depending on \(w\)),
\[
z(t) - z(s) = \psi_{0,s}(x_t) - \psi_{0,s}(x_s)
= \psi_{0,s}(\psi_s(x_t)) - \psi_{0,s}(x_t) + \psi_{0,s}(x_t) - \psi_{0,s}(x_s)
= \sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} \mathbf{v}_{s,t}^{k,I} ((\sigma_I \psi_{0,s})(x_t) - (\sigma_I \psi_{0,s})(x_s)) + R_{s,t}
+ \sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} \mathbf{v}_{s,t}^{k,I}(\sigma_I \psi_{0,s})(x_s)
+ (b \psi_{0,s})(x_s)(t-s) + \sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} \mathbf{w}_{s,t}^{k,I}(\sigma_I \psi_{0,s})(x_s) + R'_{s,t}
=
\sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} \mathbf{v}_{s,t}^{k,I}(\sigma_I \psi_{0,s})(x_s)
+ \sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} \mathbf{w}_{s,t}^{k,I}(\sigma_I \psi_{0,s})(x_s)
+ (b \psi_{0,s})(x_s)(t-s) + R_{s,t} + R'_{s,t}.
\]
Here, we have
\[
\sup_{x \in \mathbb{R}^d} |(R_{s,t} | + | R'_{s,t} | + | R''_{s,t} |) \lesssim w,\sigma,b,T |t-s|_a
\]
for a certain \(a > 1\). Furthermore, note that since \(b\) is bounded, \(\sigma \in C^2([p]+1)\) and \(\psi_{0,s} \in C_{lin}^{[p]+1}\), for all \(k' \in \{1, \ldots, |p| + 1\}\) and for all \(I', I'' \in \{1, \ldots, d\}^k\),
\[
\sup_{x \in \mathbb{R}^d} |(b \sigma_I \psi_{0,s})(x_s)(t-s)| \lesssim |t-s|^{k'}\quad \text{and} \quad \sup_{x \in \mathbb{R}^d} \mathbf{w}_{s,t}^{k',I'}(\sigma_{I'} \sigma_I \psi_{0,s})(x_s) \lesssim |t-s|^{k'},
\]
where the previous bounds depend on \(w, \sigma, T, b\). Hence, there exists \(a > 1\) and \(\bar{R}_{s,t}\) with \(\sup_{x \in \mathbb{R}^d} |\bar{R}_{s,t}| \lesssim |t-s|^{a}\) such that
\[
z(t) - z(s) = (b \psi_{0,s})(x_s) + \bar{R}_{s,t} + \sum_{k \in \{1, \ldots, |p|\}} \sum_{j \in \{1, \ldots, k-1\}} \sum_{I \in \{1, \ldots, d\}^j} \sum_{J' \in \{1, \ldots, d\}^{k-j}} \mathbf{v}_{s,t}^{k,j,I} \mathbf{w}_{s,t}^{k-j,J'} (\sigma_{J} \sigma_{J'} \psi_{0,s})(x_s)
+ \sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} \mathbf{v}_{s,t}^{k,I}(\sigma_I \psi_{0,s})(x_s)
+ (b \psi_{0,s})(x_s) + \bar{R}_{s,t} + \sum_{k \in \{1, \ldots, |p|\}} \sum_{I \in \{1, \ldots, d\}^k} (\mathbf{v}_{s,t} \otimes \mathbf{w}_{s,t})^{k,I}(\sigma_I \psi_{0,s})(x_s).
\]
Note that since \(\mathbf{v}_{s,t} \otimes \mathbf{w}_{s,t} = 1\), we have for all \(k \geq 1\), \((\mathbf{v}_{s,t} \otimes \mathbf{w}_{s,t})^{k} = 0\).
Furthermore \( D\psi_{0,s}(x) = (D\varphi_{0,s}(\psi_{0,s}(x)))^{-1} \), hence
\[
(b\psi_{0,s})(x_s) = (D\varphi_{0,s}(\psi_{0,s}(x_s)))^{-1}b(\varphi_{0,s}(\psi_{0,s}(x_s))) = (D\varphi_{0,s}(z(s)))^{-1}b(\varphi_{0,s}(z(s))).
\]
Finally, for \((s, t) \in \Delta^2_T\) small enough
\[
z(t) - z(s) = (D\varphi_{0,s}(z(s)))^{-1}b(\varphi_{0,s}(z(s))) + \tilde{R}_{s,t}.
\]
We also know that \( s \mapsto (D\varphi_{0,s}(z(s)))^{-1}b(\varphi_{0,s}(z(s))) \) is a continuous functions. Hence, thanks to a standard Riemann sum argument,
\[
z(t) - z(s) = \int_s^t (D\varphi_{0,r}(z(r)))^{-1}b(\varphi_{0,r}(z(r))) \, dr
\]
and \((z(t))_{t \in [0,T]}\) is a solution of Equation (5), and \((x_t)_{t \in [0,T]} = (\varphi_{0,t}(z(t)))_{t \in [0,T]}\), which ends the proof. \(\square\)

2.4. Rough differential equation with drift : the averaged field. Whenever \( \sigma \) is regular (and bounded), and \( b \) is a bit more than continuous, Theorem 2.8 allows us to focus, in order to prove existence and uniqueness of solutions to the Rough Differential Equations with drift
\[
dx_t = b(x_t) \, dt + \sigma(x_t) \, dw_t, \quad x_0 \in \mathbb{R}^d, \quad t \in [0,T],
\]
to existence and uniqueness of solutions to the ordinary differential equation
\[
z'(t) = (D\varphi_{0,t}(z(t)))^{-1}b(\varphi_{0,t}(z(t))), \quad z_0 = x_0 \in \mathbb{R}^d, \quad t \in [0,T],
\]
where \( \varphi \) is the flow generated by the driftless RDE (2).

The previous equation may be rewritten in its integral form as
\[
z(t) = x_0 + \int_0^t (D\varphi_{0,r}(z(r)))^{-1}b(\varphi_{0,r}(z(r))) \, dr, \quad t \in [0,T]. \tag{9}
\]
Using ideas from [19], to study wellposedness of the previous equation we will try to exhibit a general criterion of regularity of the space-time averaged fields defined in the following definition :

**Definition 2.9.** Let \( p \geq 2, \ T > 0, \ \sigma \in C_b^{[p]+2}(\mathbb{R}^d; \mathbb{R}^{d \otimes 2}) \) and \( b \in C_b^p(\mathbb{R}^d; \mathbb{R}^d) \). We define the averaged field \( T_b \) of \( b \) along the flow generated by the driftless rough differential equation driven by \( w \) has the following space-time vector field :
\[
(t,x) \in [0,T] \times \mathbb{R}^d \mapsto T_b(x) := \int_0^t (D\varphi_{0,r}(x))^{-1}b(\varphi_{0,r}(x)) \, dr. \tag{10}
\]

For all \((s, t) \in \Delta^2_T\) we denote by \( T_{b,s,t} = T_b - T_b \).

When \( \sigma = 1 \) and \( w \) is a Brownian motion (or fractional Brownian motion), one can see (in [28] and [19] for example) that the previous averaged field enjoys better regularity properties (in space) than \( b \). On can also consult [39] and [38] for a systematic study of the previous averaged field in a more general context.

In our context, one generally use non-linear Young integration theory (see [19] and [37]) to link the space-time regularity of the averaged field (10) and the wellposedness of the integral equation (9). Nevertheless the example that our techniques will allow us to treat (Gaussian rough path, see Section 4) will not allow us to go beyond \( b \in C^\kappa \) for \( \kappa > 0 \). In that setting, Equation (9) always makes sense, always has a solution (thanks to Peano existence theorem) and one only to focus on uniqueness. The following Theorem is the main result in a general context in this setting:

**Theorem 2.10.** Let \( p \geq 2, \ T > 0, \ \kappa > 0, \ b \in C_b^\kappa(\mathbb{R}^d; \mathbb{R}^d), \ \sigma \in C_b^{2([p]+1)}(\mathbb{R}^d; \mathbb{R}^{d \otimes 2}) \) and \( w \) be a \( \frac{1}{p} \)-Hölder \( p \) weakly-geometric rough path. Assume that there exists \( 1 \geq \nu > \frac{1}{2}, \ \alpha > 0 \) such that \( \kappa + \frac{1}{\alpha} > 2 - \nu \) and is a weight \( w \) (see Definitions 1.1 and 1.2) and a constant \( K > 0 \) such that
\[
\|Tb\|_{C_T^{p}\Gamma w^{\kappa+\frac{1}{p}}} < +\infty.
\]
Then there is a unique solution to Equation (9).

Furthermore, let \( x_1, x_2 \in \mathbb{R}^d \) and \( b_1, b_2 \) which satisfy the previous hypothesis, and let \( z_1 \) (respectively \( z_2 \)) be the unique solution to Equation (9) with \( b = b_1 \) and \( x = x_2 \) (respectively \( x = x_2 \) and \( b = b_2 \)). Let us suppose that \( T(b_1) - T(b_2) \in C^0_{\mathrm{loc}} \).

Then there is a non decreasing positive function \( K : \mathbb{R}^+ \to \mathbb{R}^+ \), which depends on \( \|Tb_1\|_{C^0_{\mathrm{loc}}} \), such that

\[
\|z_1 - z_2\|_{[0,T]} \leq K\|x_1 - x_2\| + \|Tb_1 - Tb_2\|_{C^0_{\mathrm{loc}}}.
\]

In this setting, there is a unique solution to the RDE with drift (4) in the sense of the Definition 2.6. Furthermore it generates a locally Lipschitz-continuous semi flow with respect to the initial condition.

The proof is an adaptation of [19] Theorem 2.21. We also refer to Section 5 (and in particular to Theorem 5.6) of [37]. In order to be self-contained, we give here another proof, relying on the standard sewing lemma (see for example [46, 31, 13] and the reference therein). We recall here its standard formulation:

**Lemma 2.11.** Let \( V_1, V_2 \) be two Banach spaces. Let \( A : \Delta_{2}^{\frac{d}{2}} \to V \) be such that there exists two constants \( C > 0 \) and \( a > 1 \) such that for all \( (s,u), (u,t) \in \Delta_{2}^{\frac{d}{2}} \),

\[
|A_{s,t} - A_{s,u} - A_{u,t}|_{V} \leq C|t - s|^a.
\]

Then there exists a unique function \( A : [0, T] \to V \) such that for all \( (s,t) \in \Delta_{2}^{\frac{d}{2}} \),

\[
A_{t} - A_{s} = \lim_{\pi \in \Pi([s,t])} \sum_{(u,v) \in \pi} A_{u,v}.
\]

Furthermore there exists a universal constant \( k > 0 \) such that for all \( (s,t) \in \Delta_{2}^{\frac{d}{2}} \),

\[
|A_{t} - A_{s} - A_{s,t}|_{V} \leq kC|t - s|^a.
\]

The end of this section is devoted to the proof of Theorem 2.10. We begin by the following lemma:

**Lemma 2.12.** Let us suppose that the hypothesis of Theorem 2.10 are satisfied. Let \( (z(t))_{t \in [0,T]} \) be a continuous functions. Then for all \( (s,t) \in \Delta_{2}^{\frac{d}{2}} \),

\[
\int_{s}^{t} (D\varphi_{0,r}(z(r)))^{-1} b(\varphi_{0,r}(z(r))) \, dr = \lim_{\pi \in \Pi([s,t])} \sum_{(u,v) \in \pi} Tb_{u,v}(z(u)).
\]

**Proof.** First, let us remark that thanks to Theorem 2.3, \( x \mapsto \varphi(x) \) and \( x \mapsto (D\varphi(x))^{-1} \) are Lipschitz continuous uniformly in time, and \( x \mapsto (D\varphi(x))^{-1} \) is bounded. Furthermore, since \( b \in C_{b}^{\alpha} \) for some \( \alpha > 0 \), we have

\[
\left| \int_{s}^{t} (D\varphi_{0,r}(z(r)))^{-1} b(\varphi_{0,r}(z(r))) \, dr - Tb_{s,t}(z(s)) \right|
\leq \left| \int_{s}^{t} (D\varphi_{0,r}(z(r)))^{-1} \left( b(\varphi_{0,r}(z(r))) - b(\varphi_{0,r}(z(s))) \right) \, dr \right|
+ \left| \int_{s}^{t} \left( (D\varphi_{0,r}(z(r)))^{-1} - D\varphi_{0,r}(z(s))^{-1} \right) b(\varphi_{0,r}(z(s))) \, dr \right|
\lesssim_{w} \|b\|_{C_{b}^{\alpha}} \int_{s}^{t} |z(r) - z(s)|^a \, dr.
\]

since \( z \) is uniformly continuous on \([0,T]\) the result follows by standard arguments. \( \square \)

Until the end of this Section, we write \( \kappa' = \kappa + \frac{1}{\alpha} \).
Lemma 2.13. Again, let us work in the setting of Theorem 2.10. Let $z_1, z_2 \in C^0_T([0, T]; \mathbb{R}^d)$ be two Lipschitz continuous paths. There exists a positive, non-decreasing locally bounded function $K_0 : \mathbb{R}_+ \to \mathbb{R}_+^*$ depending on $\|Tb_1\|_{C^T_w C^w_0'}, \|Tb_2\|_{C^T_w C^w_0'}$, $T$ such that for all $(s, t) \in \Delta_T^2$, we have the following bound

$$
\left| \int_s^t (D\varphi_{0,r}(z_1(r)))^{-1} b_1(\varphi_{0,r}(z_1(r))) \, dr - \int_s^t (D\varphi_{0,r}(z_2(r)))^{-1} b_2(\varphi_{0,r}(z_2(r))) \, dr \right|
$$

$$
\leq K_0 \left( \|z_1\|_{L^1_T^1} + \|z_2\|_{L^1_T^1} \right) \left( \|T(b_1 - b_2)\|_{C^T_w C^w_0} + \|z_1 - z_2\|_{C^T_{\infty,s,T}} + \|z_1 - z_2\|_{C^T_{\infty,s,T}} \right) \left| t - s \right|^\alpha,
$$

where $\tilde{w}_0(x) = (1 + x)w_0(x)$ and $a = \min \{2\nu, \kappa - 1 + \nu\}$. Here we can choose

$$
K_0(x) = c(1 + \|Tb_1\|_{C^T_w C^w_0'} + \|Tb_2\|_{C^T_w C^w_0'})(1 + x)w_0(x),
$$

where $c > 0$ is a universal constant.

Proof. Let us define for $(s, t) \in \Delta_T^2$,

$$
A_{s,t} = \left( (Tb_1)_{s,t}(z_1(s)) - (Tb_2)_{s,t}(z_2(s)) \right).
$$

Thanks to the previous lemma, we already know that each integral is the limit of the Riemann sum involving $Tb_1, Tb_2$, hence

$$
\int_s^t (D\varphi_{0,r}(z(r)))^{-1} b_1(\varphi_{0,r}(z_1(r))) \, dr - \int_s^t (D\varphi_{0,r}(z_2(r)))^{-1} b_2(\varphi_{0,r}(z_2(r))) \, dr
$$

$$
= \lim_{\pi \to \Pi([s, t])} \sum_{(u,v) \in \pi} A_{u,v}.
$$

Furthermore, we have for $(s, u), (u, t) \in \Delta_T^2$,

$$
A_{s,u} + A_{u,t} - A_{s,t} = \left( (Tb_1)_{s,t}(z_1(u)) - (Tb_2)_{s,t}(z_2(u)) \right) - \left( (Tb_1)_{s,t}(z_1(s)) - (Tb_2)_{s,t}(z_2(s)) \right)
$$

$$
= (Tb_1)_{s,t}(z_1(u)) - (Tb_1)_{s,t}(z_2(u))
$$

$$
- \left( (Tb_1)_{s,t}(z_1(u) - z_2(u) + z_2(s)) - (Tb_1)_{s,t}(z_2(s)) \right)
$$

$$
+ (Tb_1)_{s,t}(z_1(u) - z_2(u) + z_2(s)) - (Tb_1)_{s,t}(z_1(s))
$$

$$
+ T(b_1 - b_2)_{s,t}(z_2(u)) - T(b_1 - b_2)_{s,t}(z_2(s)).
$$

Hence, we have

$$
A_{s,u} + A_{u,t} - A_{s,t}
$$

$$
= \int_0^1 \left( D(Tb_1)_{u,t}(\lambda(z_1(u) - z_2(u)) + z_2(u)) - D(Tb_1)_{u,t}(\lambda(z_1(u) - z_2(u)) + z_2(s)) \right)
$$

$$
\times (z_1(u) - z_2(u)) \, d\lambda
$$

$$
+ \int_0^1 \left( D(Tb_1)_{u,t}(\lambda((z_1 - z_2)(u) - (z_1 - z_2))(s) + z_2(s)) \right)((z_1 - z_2)(u) - (z_1 - z_2)(s)) \, d\lambda
$$

$$
+ T(b_1 - b_2)_{u,t}(z_2(u)) - T(b_1 - b_2)_{u,t}(z_2(s)).
$$

Hence, we have

$$
|A_{s,u} + A_{u,t} - A_{s,t}| \leq \|D(Tb_1)\|_{C^T_w C^w_0'} \omega_0(\|z_1\|_{C^T_{\infty,u,s,T}} + \|z_2\|_{C^T_{\infty,u,s,T}})\|z_2\|_{C^T_{\infty,u,s,T}}^{s, t - s}
$$

$$
\times \|z_1 - z_2\|_{C^T_{\infty,s,t}} |t - s|^{\nu + \kappa - 1}
$$

$$
+ \|D(Tb_1)\|_{C^T_w C^w_0'} \omega_0(\|z_1\|_{C^T_{\infty,u,s,T}} + \|z_2\|_{C^T_{\infty,u,s,T}})\|z_1 - z_2\|_{C^T_{w}} |t - s|^{2r}
$$

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Since the previous computation is symmetric in \( z_1 \) and \( z_2 \), and by using the Sewing Lemma 2.11, we have the wanted result. \( \square \)

Proof of Theorem 2.10. Let us consider directly that we have \( b_1 \) and \( b_2 \) and \( x_1 \) and \( x_2 \) has in the theorem. Let us remark that thanks to Peano existence theorem, there exists \( z_1 \) and \( z_2 \) solution of the corresponding integral equations. Furthermore, since

\[
z_1(t) = x_1 + \int_s^t (D\varphi_{0,r}(z_1(r)))^{-1}b_1(\varphi_{0,r}(z_1(r))) \, dr.
\]

Since \( b_1 \) is bounded and \( D\varphi \) is also bounded, there exists a constant \( C_1 = C_1(w, b_1, \sigma, T) \) such that

\[
\|z_1\|_C^2 \leq C_1(|x_1| + 1),
\]

and the same holds for \( z_2 \) (with the corresponding constant \( C_2 \)). Let us called

\[
K = c \left( (1 + \|Tb_1\|_{C_T^\nu C_{w_0}^\nu} + \|Tb_2\|_{C_T^\nu C_{w_0}^\nu})\tilde{w}_0(2(C_1 + C_2)(1 + |x_1| + |x_2|)) \right),
\]

where \( c > 0 \) is the constant in the inequality of the previous lemma. Hence, thanks to the previous lemma, for all \( (s, t) \in \Delta_T^2 \),

\[
|z_1(t) - z_2(t)| - |z_1(s) - z_2(s)| \leq K(|z_1(s) - z_2(s)| + \|T(b_1 - b_2)\|_{C_T^\nu L_{w_0}^{\nu}}|t - s|^\nu
\]

\[+ K(\|T(b_1 - b_2)\|_{C_T^{\nu} C_{w_0}^{\nu}} + \|z_1 - z_2\|_{\infty, |s, t|} + \|z_1 - z_2\|_{\infty, |s, t|}) |t - s|^\alpha.
\]

Let \( h > 0 \) such that \((h^\nu + h^{\alpha-\nu}) K \leq \frac{1}{3} \) and suppose that \( t - s \leq h \), then

\[
\|z_1 - z_2\|_{\nu, |s, t|} \leq \frac{1}{3(t - s)^\nu}(\|z_1 - z_2\|_{\infty, |s, t|} + \|T(b_1 - b_2)\|_{C_T^{\nu} L_{w_0}^{\nu}}
\]

\[+ \frac{1}{3}(\|T(b_1 - b_2)\|_{C_T^{\nu} C_{w_0}^{\nu}} + \|z_1 - z_2\|_{\infty, |s, t|} + \|z_1 - z_2\|_{\infty, |s, t|}),
\]

and

\[
\|z_1 - z_2\|_{\nu, |s, t|} \leq \frac{1}{2} \left( 1 + \frac{1}{(t - s)^\nu} \right)(\|z_1 - z_2\|_{\infty, |s, t|} + \|T(b_1 - b_2)\|_{C_T^{\nu} C_{w_0}^{\nu}}),
\]

and

\[
K|t - s|^\alpha \|z_1 - z_2\|_{\nu, |s, t|} \leq \frac{4}{9}(\|z_1 - z_2\|_{\infty, |s, t|} + \|T(b_1 - b_2)\|_{C_T^{\nu} C_{w_0}^{\nu}}
\]

Finally, we have, by injecting this inequality into the previous one

\[
|z_1(t) - z_2(t)| \leq \frac{4}{3}|z_1(s) - z_2(s)| + \frac{5}{9}(\|T(b_1 - b_2)\|_{C_T^{\nu} C_{w_0}^{\nu}} + \|z_1 - z_2\|_{\infty, |s, t|}).
\]

This gives

\[
\|z_1 - z_2\|_{\infty, |s, t|} \leq 3 \left( |z_1(s) - z_2(s)| + \|T(b_1 - b_2)\|_{C_T^{\nu} C_{w_0}^{\nu}} \right).
\]

One can iterate the previous bound one small intervals to have wanted bound.

Note that when \( b_1 = b_2 = b \), we have directly the uniqueness of the solution.

Uniqueness and regularity of the flow for equation (4) in the a direct consequence of Theorems 2.8 and 2.3. \( \square \)

Remark 2.14. Theorem 2.10 requires that \( b \in C^\kappa \) for some \( \kappa > 0 \). This hypothesis is needed in view of Theorem (2.8) which allows us to states an equivalent notion of solutions for RDE with drift, and allow us to state Equation (5) as an actual ODE.

Nevertheless, for any \( b \in SP(\mathbb{R}^d) \) such that \( Tb \) exists and \( Tb \in C_T^{\nu} C_{w_0}^{\nu} \) for some sublinear weight \( w_0 \) and \( \nu > \frac{1}{2} \), there exists a solution \( (\theta_t)_{t \in [0, T]} \in C_T^{\nu} \) to the non linear Young differential Equation

\[
\theta_t = \theta_0 + \int_0^t ((Tb)_{\Delta r})(\theta_r),
\]

(11)
where the integral is constructed using the sewing lemma applied to
\[ A_{s,t} = Tb_{s,t}(\theta_s). \]

One can consult [19, 37] for more details.

Hence, one could use the following definition in order to extend the notion of solution to RDE with drift:

**Definition 2.15.** Let \( p \geq 2 \), \( T > 0 \) and \( \sigma \in C^{[p]+2}(\mathbb{R}^d; \mathbb{R}^{d \times d}). \) Let \( w \) be a \( \frac{1}{p} \)-Hölder weakly \( p \)-geometric rough path. Let \( b \in S(\mathbb{R}^d; \mathbb{R}^d) \) and \( w_0 \) be a sublinear weight such that \( Tb \in \mathcal{C}^\kappa_{2(\nu)} \) for some \( \nu > \frac{1}{2}. \) A path \((x_t)_{t \in [0,T]}\) is a solution to Equation (4) if \((x_t)_{t \in [0,T]} = (\varphi_0, \theta_t)_{t \in [0,T]},\)
where \( \varphi \) is the flow generated by the driftless RDE (2), and \((\theta_t)_{t \in [0,T]}\) is a solution to the non-linear Young differential equation (11) with \( \theta_0 = x_0. \)

An obvious remark is that whenever \( b \in C^\kappa \) for some \( \kappa > 0 \) and \( \sigma \in C^{2([p]+1)} \), thanks to Lemma 2.12 and Theorem 2.8, this definition is equivalent to Definition 2.6. The huge difference when \( b \) is not bounded, is the lack of a priori estimates for the solution \( \theta. \) In the setting, one could prove the following theorem:

**Theorem 2.16.** Let \( p \geq 2 \), let \( T > 0 \) and let \( \alpha > 0. \) Let \( b \in S(\mathbb{R}^d; \mathbb{R}^d) \) and \( \sigma \in C^\alpha_b(\mathbb{R}^{d}; \mathbb{R}^{d \otimes 2}). \) Let \( w \) be a \( \frac{1}{p} \)-Hölder \( p \)-weakly-geometric rough path.

Let us suppose that there exists \( 1 \geq \nu > \frac{1}{2}, \kappa > 2 \) and a sublinear weight \( w_0 \) (see Definitions 1.1 and 1.2) such that
\[ ||Tb||_{\mathcal{C}^\kappa_{2(\nu)}} < +\infty. \]

Then there is a unique solution, in the sense of Definition 2.15 to Equation (4).

**Proof.** The proof is a direct adaptation of the ideas of [19]. \( \square \)

The only difference comparing to the Theorem 2.10 is the requirement that \( Tb \in \mathcal{C}^\kappa_{2(\nu)} \) with \( \kappa > 2. \) In our context of application, when \( w \) is a Gaussian rough with \( p < 4 \) and \( w \) enjoy some non-determinism properties (see Section 4), one cannot expect to have a better regularizing effect than \( \frac{p}{4} - \varepsilon \) for some small \( \varepsilon > 0 \) (see Section 3 and Theorem 6.1). Namely, if \( b \in C^\alpha := B^\alpha_{\infty,\infty} \) for some \( \alpha \in \mathbb{R}, \) one may expect that \( Tb \in \mathcal{C}^\alpha_{2(\nu + \varepsilon)} \) for some small \( \varepsilon > 0 \) and some sublinear weight \( w_0. \) When applying Theorem 2.16, one should ask that \( \frac{p}{4} + \alpha > 2, \) ans since \( p < 4, \) this gives necessarily \( \alpha > 0, \) and we can apply the better result of Theorem 2.10.

Nevertheless, Definition 2.15 and Theorem 2.16 could be useful when \( b \in L^\infty \cap B^\alpha_{\infty,\infty} \) or when \( w \) enjoys better regularizing properties. We give those investigation for future works.

Finally, in view of [29], one could expect that in the setting of fractional Brownian motion, the only requirement should be \( Tb \in \mathcal{C}^\alpha_{2(\nu)} \) (this is the case when \( w \) is the Stratonovich Brownian rough path). This kind of result would require a use of Girsanov transform as in [27, 19, 29]. The bounds of Section 4 are not good enough, neither the Kolmogorov estimates of Section 3. Again, we leave this for future investigations.

3. Kolmogorov type theorem for averaged fields.

In the previous section, and especially in Theorem 2.10, we have exhibit a general criterion in terms of the averaged field \( Tb \) such that Equation (4) has a unique solution. Nevertheless, remark that without using anymore property of the flow \( \varphi, \) even in the case when \( b \) is Lipschitz continuous (and thus Equation (4) has a unique solution thanks to standard arguments) this does not gives uniqueness of solutions. An idea should be to use fine stochastic properties of the flow, when \( w \) is a rough random path.

Hence, in this section, we will focus ourselves on the action a the random flow \( \varphi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) as an averaging operator where \( T > 0 \) is fixed. Thanks to the flow decomposition technique if \( \varphi \) is a \( C^1 \) flow of diffeomorphism, we will investigate the mixed space time regularity of the following averaged field:

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Remark 3.1. For any $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, we denote, for any $0 \leq s \leq t \leq T$, $x \in \mathbb{R}^d$ and $f \in \mathcal{S}$,

$$T^{\phi,\varphi}f_t(x) = \int_0^t \phi(u,x) f(\varphi(u,x),du).$$

In order to be as general as possible, we want to take $b \in B^{p,r}_{p,r}$ as generic as possible. In order to do so, we rely on estimates of the type

$$\|T^{(D\varphi)^{-1},\varphi}b\|_{L_s([0,T];B^{p,r}_{p,r})} \leq C\|f\|_{B^{p,r}_{p,r}}$$

for some constant $C > 0$ and parameters $q \geq 2$, $\gamma > 0$, $0 < \alpha' < 1$, $p, r, \ell, m \in [1, \infty]$ and any $f \in \mathcal{S}$. Then, since $T^{(D\varphi)^{-1},\varphi}$ is a linear operator (on $\mathcal{S}$), we consider its closure from $B^{p,r}_{\ell,m}$ to $L^q(\omega; C^r([0,T]; B^{p,r}_{p,r}))$ (which is still denoted $T^{(D\varphi)^{-1},\varphi}$).

In the following, we will consider a general form of the averaging operator: for any $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and any $f \in \mathcal{S}$, we begin by proving a Lemma which allow us to transform the condition $(2)$ for all $q$ and $p,s$ into a useful one concerning Paley-Littlewood blocks. To do so, we use a trick which can be found in [4] Lemma 2.1.
Lemme 3.4. Let $f \in S$ and let $\phi$ and $\varphi$ as in Theorem 3.3. Then, for any $p \in [2, +\infty]$, for all $j \geq 0$ and all $\eta \in [0, 1]$, we have

$$
\left\| \mathbb{E}\left[\phi(r, \cdot) \Delta_j f(\varphi(\cdot, \cdot)) | \mathcal{F}_s \right] \right\|_{L^p(\mathbb{R}^d; \mathbb{R})} \lesssim |r - s|^{-(1-\eta)} 2^{-jd}\frac{1}{|B|} \left\| \Delta_j f \right\|_{L^p(\mathbb{R}^d; \mathbb{R})} 
$$

$$
\times G_{p,s} \left( 1 + \mathbb{E}\left[ \| \det(J_{p-1}) \|_{L^\infty([0,T] \times \mathbb{R}^d)} |\mathcal{F}_s \right] \right)^\frac{\gamma}{2} 1_{\{p < +\infty\}}.
$$

(14)

Proof. We remark that, for all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and all $k \in \mathbb{N}$,

$$
|\xi|^{2k} = \left( \sum_{j=1}^{d} \xi_j^2 \right)^k = \sum_{1 \leq j_1, \ldots, j_k \leq d} \xi_{j_1}^2 \ldots \xi_{j_k}^2 = \sum_{|\beta| = k} A_\beta (-i\xi)^\beta (i\xi)^\beta,
$$

where $A_\beta$ are non negative constants and for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and a multi-index $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$, $\xi^\beta = \xi_{\beta_1} \cdots \xi_{\beta_d}$. Let $\varrho \in C_0^\infty(\mathbb{R}^d)$ such that $\text{supp}(\varrho) \subset 2\mathcal{A}$ and $\varrho|\mathcal{A} = 1$. For any $\alpha \in \mathbb{N}^d$, such that $|\alpha| = k$, and $j \geq 0$, we define the function $m_{k,\beta,j}$ given by

$$
m_{k,\beta,j} = F^{-1}(A_\beta (-i\xi)^\beta |\xi|^{-2k} \varrho(2^{-j}\xi)).
$$

It follows by (15) that, for any $k \in \mathbb{N}$,

$$
\Delta_j f = \sum_{|\beta| = k} m_{k,\beta,j} \ast \partial^\beta \Delta_j f,
$$

(16)

where, by Lemma (A.5), we have

$$
\|m_{k,\beta,j}\|_{L^1(\mathbb{R}^d)} \lesssim 2^{-jk}.
$$

We have, for $p < +\infty$

$$
\left\| \mathbb{E}\left[\phi(r, \cdot) \Delta_j f(\varphi(\cdot, \cdot)) | \mathcal{F}_s \right] \right\|_{L^p(\mathbb{R}^d)} \leq \sum_{|\alpha| = k} \left\| \mathbb{E}\left[\phi(r, \cdot) (m_{k,\alpha,j} \ast \partial^\alpha \Delta_j f)(\varphi(\cdot, \cdot)) | \mathcal{F}_s \right] \right\|_{L^p(\mathbb{R}^d)}
$$

$$
\leq \sum_{|\alpha| = k} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m_{k,\alpha,j}(y) \mathbb{E}\left[\phi(r,x)(\partial^\alpha \Delta_j f)(\varphi(x, y)) | \mathcal{F}_s \right] dy \right)^{\frac{1}{p}},
$$

and, for $p = +\infty$,

$$
\left\| \mathbb{E}\left[\phi(r, \cdot) \Delta_j f(\varphi(\cdot, \cdot)) | \mathcal{F}_s \right] \right\|_{L^p(\mathbb{R}^d)} \leq \sum_{|\beta| = k} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} m_{k,\beta,j}(y) \mathbb{E}\left[\phi(r,x)(\partial^\beta \Delta_j f)(\varphi(x, y)) | \mathcal{F}_s \right] dy.
$$

We want to apply Young inequality for kernel operators (Theorem B.1) with

$$
K(x, y) = \mathbb{E}\left[\phi(r,x)(\partial^\beta \Delta_j f)(\varphi(x, y)) | \mathcal{F}_s \right].
$$

Note that, thanks to Equation (13), we know that

$$
|K(x, y)| \lesssim \begin{cases} 
|r-s|^{-|\beta|} H G_{p,s} |\Delta_j f(\varphi(x, y) - y)|^{\frac{1}{p}} & \text{if } 2 \leq p < +\infty, \\
|r-s|^{-|\beta|} H G_{\infty,s} |\Delta_j f|_\infty & \text{if } p = +\infty.
\end{cases}
$$

Hence, for all $p \in [2, +\infty]$,

$$
\sup_{x \in \mathbb{R}^d} \left\| K(x, \cdot) \right\|_{L^p(\mathbb{R}^d)} \lesssim |r-s|^{-|\beta|} H G_{p,s} |\Delta_j f|_{L^p(\mathbb{R}^d)}.
$$
Furthermore, we note that, if $p < +\infty$,  
\[
\int_{\mathbb{R}^d} \mathbb{E} \left[ |\Delta_j f(\varphi(r, x) - y)|^p \right] dF_s = \mathbb{E} \left[ \int_{\mathbb{R}^d} |\Delta_j f(z)|^p \mathbb{E} \left[ |\det(J_{\varphi^{-1}}(r, z + y))| |F_s| \right] dz \right] 
\lesssim |\Delta_j f|_{L^p(\mathbb{R}^d)} \mathbb{E} \left[ \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} |F_s| \right]
\]
and we obtain  
\[
\sup_{y \in \mathbb{R}^d} \|K(\cdot, y)\|_{L^p(\mathbb{R}^d)} 
\lesssim |r - s|^{-|\beta|H} G_{p,s} \|\Delta_j f\|_{L^p(\mathbb{R}^d)} \times \begin{cases} 
\mathbb{E} \left[ \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} |F_s| \right] \frac{1}{p} & \text{if } 1 < p < +\infty, \\
1 & \text{if } p = +\infty.
\end{cases}
\]

Applying Young inequality (Theorem B.1), we deduce that  
\[
\left\| \mathbb{E} \left[ \phi(r, \cdot)(m_{\kappa,\beta,j} \ast \partial^{\beta}_s \Delta_j f)(\varphi(r, \cdot)) \right] \right\|_{L^p(\mathbb{R}^d)} 
\lesssim 2^{-j \nu} |r - s|^{-|\beta|H} G_{p,s} \|\Delta_j f\|_{L^p(\mathbb{R}^d)} \left( 1 + \mathbb{E} \left[ \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} |F_s| \right] \frac{1}{p} \right). 
\]

It follows that, for any $k \in \mathbb{N}$,  
\[
\left\| \mathbb{E} \left[ \phi(r, \cdot) \Delta_j f(\varphi(r, \cdot)) \right] \right\|_{L^p(\mathbb{R}^d)} 
\lesssim 2^{-j} \sum_{|\beta| = k} |t - s|^{-|\beta|H} \|\Delta_j f\|_{L^p(\mathbb{R}^d)} G_{p,s} \left( 1 + \mathbb{E} \left[ \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} |F_s| \right] \frac{1}{p} \right). 
\]

By interpolating the previous inequalities, we obtain the estimate, for any $\nu \in \mathbb{R}^+$,  
\[
\left\| \mathbb{E} \left[ \phi(r, \cdot) \Delta_j f(\varphi(r, \cdot)) \right] \right\|_{L^p(\mathbb{R}^d)} 
\lesssim 2^{-\nu \nu |r - s|^{-|\beta|H} \|\Delta_j f\|_{L^p(\mathbb{R}^d)} G_{p,s} \left( 1 + \mathbb{E} \left[ \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} |F_s| \right] \frac{1}{p} \right), 
\]

which gives the desired result by taking $\nu = \frac{1}{4^j}$. \hfill \Box

**Proof of Theorem 3.3.** We split the proof in two parts.

The $p < +\infty$ case. Let $0 \leq s < t \leq T$ be fixed. We denote $j = \min\{j \in \mathbb{N}; 2^{-j} \leq (t - s)\}$. First, remark that  
\[
\left\| (T^{\phi, \nu} \Delta_j f)_{s,t} \right\|_{L^p(\mathbb{R}^d)} \leq \int_s^t \left\| \phi(r, \cdot) \Delta_j f(r, \varphi(r, \cdot)) \right\|_{L^p(\mathbb{R}^d)} dr 
\leq (t - s) \left\| \phi \right\|_{L^\infty([0,T] \times \mathbb{R}^d)} \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}.
\]

For any $0 \leq j < j$ and $\varepsilon \in [0, 1]$, this yields, since $(t - s) \leq 2^{-j}$,  
\[
\left\| (T^{\phi, \varepsilon} \Delta_j f)_{s,t} \right\|_{L^p(\mathbb{R}^d)} \leq (t - s)^{\frac{\nu}{1 - \varepsilon}} \left( 2^{-\varepsilon j} \left\| \phi \right\|_{L^\infty([0,T] \times \mathbb{R}^d)} \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}. \right.
\]

Furthermore, thanks to Lemma 3.4, we also deduce that, for any $\eta \in [0, 1]$,
\[
\left\| \mathbb{E}[T^{\phi, \varepsilon} \Delta_j f]_{s,t} |F_s| \right\|_{L^p(\mathbb{R}^d)} 
\lesssim (t - s)^{\eta} \|\Delta_j f\|_{L^p(\mathbb{R}^d)} \left( 2^{-\varepsilon j} |G_{p,s}| \left( 1 + \mathbb{E} \left[ \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} |F_s| \right] \frac{1}{p} \right) \right).
\]
We now assume that \( j \geq j \). Let \( n_j \geq 1 \) and let us define, for any \( k \in \{0, \ldots, n_j\} \), the following quantities:

\[
t^j_k := \frac{k}{n_j}(t-s) + s \quad \text{and} \quad M^j_k := \mathbb{E}[(T^{\phi,\varphi}\Delta_j f)_{t_k'}] .
\]

Hence, \((M^j_k)_{0 \leq k \leq n_j}\) is a martingale in \(L^p(\mathbb{R}^d)\) with respect to the filtration \((\mathcal{F}_{t'_k})_{0 \leq k \leq n_j}\). Furthermore, it follows from (19) that

\[
\|M^j_0\|_{L^p(\mathbb{R}^d)} \lesssim (t-s)^q \|\Delta_j f\|_{L^p(\mathbb{R}^d)} 2^{-j/2}\|g_{p,s}\|_1 + 1 + \mathbb{E} \left[ \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)}|\mathcal{F}_{t'_k}| \right]^{1/2},
\]

and, in particular, by setting \( \eta = (1+\varepsilon)/2 \), where \( \varepsilon \in [0,1] \), and for \( q \geq 1 \), we have

\[
\mathbb{E}[\|M^j_0\|_{L^p(\mathbb{R}^d)}^q] \lesssim (t-s)^{\frac{q}{2}} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}^q 2^{-j/2} \|\eta\|^{1/2}. \tag{20}
\]

Since we are working in \(L^p(\mathbb{R}^d), p \in [2, +\infty)\), which is a UMD space of type 2 (see for example [49, 50] and Appendix C.2), we know that the Burkholder-Davis-Gundy inequality holds, and we have, for all \( q \geq 1 \),

\[
\mathbb{E}[\|M^j_{n_j} - M^j_0\|_{L^p(\mathbb{R}^d)}^q] \lesssim \mathbb{E} \left[ \sum_{k=0}^{n_j-1} \|M^j_{k+1} - M^j_k\|_{L^p(\mathbb{R}^d)}^q \right]^{1/q} .
\]

Furthermore, we obtain that, for all \( 0 \leq k \leq n_j - 1 \),

\[
M^j_{k+1} = (T^{\phi,\varphi}\Delta_j f)_{t'_{k+1}} + \mathbb{E} \left[ (T^{\phi,\varphi}\Delta_j f)_{t'_{k+1}} |\mathcal{F}_{t_k'} \right] - \mathbb{E} \left[ (T^{\phi,\varphi}\Delta_j f)_{t'_{k+1}} |\mathcal{F}_{t_k'} \right].
\]

Using again (17) to estimate the first term on the right-hand-side and (19) for the second and third terms, we deduce, for any \( \varepsilon \in [0,1] \),

\[
\|M^j_{k+1} - M^j_k\|_{L^p(\mathbb{R}^d)} \lesssim \|\Delta_j f\|_{L^p(\mathbb{R}^d)} (t-s)^{\varepsilon} \left( \frac{(t-s)^{1-\varepsilon}}{n_j} + 2^{-j/2} \right) \kappa^j_k ,
\]

where

\[
\kappa^j_k := \|\phi\|_{L^\infty([0,T] \times \mathbb{R}^d)} \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} + G_{p,t'_k} \left( 1 + \mathbb{E} \left[ \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} |\mathcal{F}_{t'_k}| \right]^{1/2} \right)\]

\[
+ G_{p,t'_{k+1}} \left( 1 + \mathbb{E} \left[ \|\det(J_{\varphi^{-1}})\|_{L^\infty([0,T] \times \mathbb{R}^d)} |\mathcal{F}_{t'_{k+1}}| \right]^{1/2} \right).
\]

Note that thanks to the hypothesis,

\[
\sup_{j \geq 0} \sup_{k \in \{0, \ldots, n_j\}} \mathbb{E}[\kappa^j_k]^q < +\infty.
\]

Hence, we have, for any \( q \geq 2 \),

\[
\mathbb{E}[\|M^j_{n_j} - M^j_0\|_{L^p(\mathbb{R}^d)}^q] \lesssim \mathbb{E} \left[ \sum_{k=0}^{n_j-1} \|M^j_{k+1} - M^j_k\|_{L^p(\mathbb{R}^d)}^q \right]^{1/q} \]

\[
\lesssim (t-s)^{q\varepsilon} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}^q \left( \frac{(t-s)^{1-\varepsilon}}{n_j} + 2^{-j/2} \right) \mathbb{E} \left[ \sum_{k=0}^{n_j-1} \kappa^j_k^2 \right]^{1/2} \]

\[
\lesssim (t-s)^{q\varepsilon} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}^q \left( \frac{(t-s)^{1-\varepsilon}}{n_j} + 2^{-j/2} \right)^q \mathbb{E} \left[ \sum_{k=0}^{n_j-1} \mathbb{E}[\kappa^j_k]^q \right]^{1/2}.
\]
\[(t-s)^{\varepsilon q} \| \Delta_j f \|^q_{L^p(R^d)} \left( \frac{(t-s)^{1-\varepsilon}}{\sqrt{n_j}} + \sqrt{n_j} 2^{-\frac{1-\varepsilon}{p'j}} \right)^q.

We now optimize the estimate on \( n_j \geq 1 \). We choose
\[ n_j = \frac{(t-s)^{1-\varepsilon}}{2^{-\frac{1-\varepsilon}{p'j}} + \varepsilon}, \]
for a certain \( \varepsilon \in [0, 1] \). This yields, since \( 2^{-\frac{1-\varepsilon}{p'j}} \leq (t-s) \),
\[ \sqrt{n_j} 2^{-\frac{1-\varepsilon}{p'j}} = (t-s)^{1-\varepsilon} 2^{-\frac{1-\varepsilon}{p'j}} \sqrt{1 + \frac{2^{-\frac{1-\varepsilon}{p'j}}}{(t-s)^{1-\varepsilon}}} \leq \sqrt{2(t-s)^{1-\varepsilon}} 2^{-\frac{1-\varepsilon}{p'j}}, \]
and, also,
\[ \frac{(t-s)^{1-\varepsilon}}{\sqrt{n_j}} \leq (t-s)^{1-\varepsilon} 2^{-\frac{1-\varepsilon}{p'j}}. \]
Thus, we deduce that
\[ \mathbb{E} \left[ \left\| M_{n_j} f - M_0 f \right\|_{L^p(R^d)}^q \right]^\frac{1}{q} \lesssim (t-s)^{1+\varepsilon} \| \Delta_j f \|^q_{L^p(R^d)} 2^{-\frac{1-\varepsilon}{p'j}}. \]
Using the fact that \( (T^{\phi, \varphi} \Delta_j f)_{s,t} = M_{n_j} f \) as well as the estimate (20), we obtain, for any \( q \geq 2 \) and for all \( j \geq 1 \),
\[ \mathbb{E} \left[ \left\| (T^{\phi, \varphi} \Delta_j f)_{s,t} \right\|_{L^p(R^d)}^q \right]^\frac{1}{q} \lesssim (t-s)^{1+\varepsilon} \| \Delta_j f \|^q_{L^p(R^d)} 2^{-j \frac{1-\varepsilon}{p'j}}. \]
(21)
It follows from (18) that the previous estimate holds for all \( \varepsilon \in [0, 1] \) and any \( j \geq -1 \). Thus, we deduce that, for any \( p \in [2, +\infty) \), \( q \geq 2 \) and \( \varepsilon \in [0, 1] \),
\[ \mathbb{E} \left[ \left\| (T^{\phi, \varphi} f)_{s,t} \right\|_{L^p(R^d)}^q \right]^\frac{1}{q} \lesssim \sum_{j=-1}^{+\infty} \mathbb{E} \left[ \left\| (T^{\phi, \varphi} \Delta_j f)_{s,t} \right\|_{L^p(R^d)}^q \right]^\frac{1}{q} \lesssim (t-s)^{1+\varepsilon} \sum_{j=-1}^{+\infty} 2^{-j \frac{1-\varepsilon}{p'j}} \| \Delta_j f \|^q_{L^p(R^d)} = (t-s)^{1+\varepsilon} \| f \|_{B_{p,1}^{-1-\varepsilon}}. \]
It follows from Theorem D.1 that, for all \( 0 < \varepsilon' < \varepsilon \) and \( f \in B_{p,1}^{-1-\varepsilon} \), we have
\[ \| T^{\phi, \varphi} f \|_{L^q(\Omega; C^\nu([0,T];L^p(R^d)))} \lesssim \mathbb{E} \left[ \int_{[0,T]^2} \left\| (T^{\phi, \varphi} f)_{s,t} \right\|_{L^p(R^d)}^q ds \right]^\frac{1}{q} \lesssim \| f \|_{B_{p,1}^{-1-\varepsilon'}} \int_{[0,T]^2} |t-s|^{\frac{1}{2} - \nu} ds \lesssim \| f \|_{B_{p,1}^{-\nu}}, \]
with \( \nu = \frac{1+\varepsilon'}{2} - \frac{1}{q} \), which is exactly the theorem in the case \( p = +\infty \) when we remind that whenever \( f \in B_{p,p}^{\alpha} \), one has for \( \varepsilon'' > 0 \), thanks to Hölder inequality,
\[ \| f \|_{B_{p,1}^{\alpha-\varepsilon''}} = \sum_{j \geq 1} 2^{j(\alpha-\varepsilon'')j} \| \Delta_j f \|_{L^p(R^d)} \leq \left( \sum_{j \geq -1} 2^{-\varepsilon'' \frac{j}{p-1}} \| \Delta_j f \|_{L^p(R^d)} \right)^{1-\frac{1}{p}} \left( \sum_{j \geq -1} 2^{\alpha' r j} \| \Delta_j f \|_{L^p(R^d)}^r \right)^{\frac{1}{r}} \lesssim \varepsilon'' \| f \|_{B_{p,r}^\alpha}. \]
The $p = +\infty$ case. The proof is pretty much the same as the proof in the previous case. We provide the main arguments for completeness. Let $0 \leq s < t \leq T$ be fixed. We denote $j = \min\{j \in \mathbb{N}; 2^{-j} \leq (t-s)\}$. First, we can see that, for any $-1 \leq j < k$ and $\varepsilon \in [0, 1]$, this yields, since $(t-s) \leq 2^{-j}$, for any $x \in \mathbb{R}^d$,

$$\left| (T^{\phi,\varphi})_{s,t}(x) \right| \leq (t-s)^{\frac{1}{2}} 2^{-\frac{1}{2} j} \|\phi\|_{L^\infty([0,T] \times \mathbb{R}^d)} \|\Delta_j f\|_{L^\infty(\mathbb{R}^d)}. \quad (22)$$

Moreover, it follows from Lemma 3.4 that, for any $j \in [0, 1]$,

$$\left| \mathbb{E}[(T^{\phi,\varphi})_{s,t}(x)|\mathcal{F}_s]\right| \leq (t-s)^{\eta} \|\Delta_j f\|_{L^\infty(\mathbb{R}^d)} 2^{-\frac{1}{2} j} G_{\infty,s}. \quad (23)$$

We now assume that $j \geq 1$. Let $x \in \mathbb{R}^d$, $n_j \geq 1$ and let us define, for any $k \in \{0, \ldots, n_j\}$, the following quantities:

$$t_k^j := \frac{k}{n_j} (t-s) + s \quad \text{and} \quad M^j_k(x) := \mathbb{E}[(T^{\phi,\varphi})_{s,t}(x)|\mathcal{F}_{t_k^j}].$$

We can see that, $(M^j_k(x))_{0 \leq k \leq n_j}$ is a martingale with respect to the filtration $(\mathcal{F}_{t_k^j})_{0 \leq k \leq n_j}$. Thanks to (23), we obtain

$$|M^j_0(x)| \lesssim (t-s)^{\eta} \|\Delta_j f\|_{L^\infty(\mathbb{R}^d)} 2^{-\frac{1}{2} j} G_{\infty,s},$$

which yields, by setting $\eta = (1+\varepsilon)/2$, where $\varepsilon \in [0, 1]$, and for $q \geq 1$,

$$\mathbb{E}[|M^j_0(x)|^q] \lesssim (t-s)^{\frac{1}{2} q} \|\Delta_j f\|^q_{L^\infty(\mathbb{R}^d)} 2^{-\frac{1}{2} q j}. \quad (24)$$

It follows from (22) and (23) that, for any $\varepsilon \in [0, 1]$ and $1 \leq k \leq n_j$,

$$|M^j_{k+1}(x) - M^j_k(x)| \lesssim \|\Delta_j f\|_{L^\infty(\mathbb{R}^d)} (t-s)^\varepsilon \left(\frac{(t-s)^{1-\varepsilon}}{n_j} + 2^{-\varepsilon j}\right) \kappa^j_k,$$

where

$$\kappa^j_k := \|\phi\|_{L^\infty([0,T] \times \mathbb{R}^d)} + G_{\infty,t_k^j} + G_{\infty,t_{k+1}^j}.$$ 

The assumptions gives the bound

$$\sup_{j \geq 0} \sup_{k \in \{0, \ldots, n_j\}} \mathbb{E}[|M^j_k|^q] < +\infty.$$

From here, by following the same arguments as in the $p < +\infty$ case, we deduce, thanks to the BDG inequality and some optimization on $n_j$, that, for any $q \geq 2$, for any $\varepsilon \in [0, 1]$ and for all $j \geq 1$,

$$\mathbb{E} \left[ \left| (T^{\phi,\varphi})_{s,t}(x) \right|^q \right]^{\frac{1}{q}} \lesssim (t-s)^{\frac{1}{2} q} \|\Delta_j f\|^q_{L^\infty(\mathbb{R}^d)} 2^{-j \frac{1}{2} q}, \quad (25)$$

and we obtain the estimate, for any $x \in \mathbb{R}^d$,

$$\mathbb{E}[|(T^{\phi,\varphi})_{s,t}(x)|^q]^{\frac{1}{q}} \lesssim |t-s|^{\frac{1}{2} q} \|f\|_{B_{\infty,1}^q}^{-\frac{1}{2} q}.$$ 

Finally, Theorem D.5 gives the desired result.

We can extend the previous result with the help of the following Lemma.

**Lemma 3.5.** Let $\beta \in \mathbb{N}^d$, $\phi, \varphi \in L^q(\Omega; L^\infty([0,T]; C^{[\beta]}(\mathbb{R}^d)))$ and $f \in \mathcal{S}$. Then, we have, for any $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\partial^\beta (T^{\phi,\varphi})_t f(x) = \sum_{\gamma \leq \beta, 1 \leq |\mu| \leq |\gamma|} \sum_{\nu \in \Lambda_{\gamma,\mu}} C_{\beta,\gamma,\mu,\nu} \left( T^{\Lambda_{\gamma,\nu}(x)\partial^\beta - \gamma \phi, \varphi} (\partial^\mu f) \right)_t (x),$$

where $T^{\phi,\varphi}$ is defined as in Theorem 3.3.
where \( \Lambda_{\gamma,\upsilon}(\varphi) := \prod_{1 \leq j \leq d} (\partial^\beta \varphi_j(r,x))^{\upsilon_j} \) and \( C_{\beta,\gamma,\mu,\upsilon} > 0 \) and

\[
\mathcal{N}_{\gamma,\mu} := \left\{ v \in \mathbb{N}^d; \sum_{1 \leq |\delta| \leq |\gamma|} \upsilon_{\delta_j} = \mu_j, \text{ for any } 1 \leq j \leq d, \text{ and } \sum_{1 \leq |\delta| \leq |\gamma|} \delta_{\upsilon_{\delta_j}} = \gamma \right\},
\]

Proof. The proof of the lemma is a straightforward consequence of Leibniz’s rule of differentiation, which yields

\[
\partial^\beta \phi(r,x)b(\varphi(r,x)) = \sum_{\gamma \leq \beta} C_{\beta,\gamma}^1 \partial^{\beta - \gamma} \phi(r,x)\partial^{\gamma}(b(\varphi(r,x))),(\n
\]

for some constants \( \{C_{\beta,\gamma}^1\}_{\gamma \leq \beta} \subset \mathbb{R}^+ \) and Faà di Bruno’s formula, which gives

\[
\partial^{\gamma}(b(\varphi(r,x))) = \sum_{1 \leq |\mu| \leq |\gamma|} \sum_{v \in \mathcal{N}_{\gamma,\mu}} C_{\beta,\mu,v}^2 \partial^{\mu}b(\varphi(r,x)) \prod_{1 \leq j \leq d} (\partial^\delta \varphi_j(r,x))^{\upsilon_j},
\]

for some constants \( \{C_{\gamma,\mu,v}^2\}_{1 \leq |\mu| \leq |\gamma|, v \in \mathcal{N}_{\gamma,\mu}} \subset \mathbb{R}^+ \).

We can now state a by-product of Theorem 3.3.

Corollary 3.6. Let \( p \in [2, +\infty) \), \( r \in [1, +\infty) \), \( n \geq 1 \) and \((\phi, \varphi)\) be a couple of functions such that, for any \( \beta, \gamma, \mu, \upsilon \in \mathbb{N}^d \) that verify \( |\beta| \leq n \), \( \gamma \leq \beta \), \( 1 \leq |\mu| \leq |\gamma| \) and \( v \in \mathcal{N}_{\gamma,\mu} \), \((\Lambda_{\gamma,\upsilon}(\varphi)\partial^{\beta - \gamma} \phi, \varphi)\) satisfies Assumption 3.2. Then, for all \( \varepsilon \in (0,1] \), \( \varepsilon', \varepsilon'' \in (0, \epsilon) \), \( \kappa \in (0, n) \), \( f \in B^p_{r,\frac{d}{2}} \), \( \eta > d/q \) and \( q \geq 2 \) such that \( q \frac{\varepsilon' + \varepsilon''}{2} > 1 \), \( \nu = \frac{1 + \varepsilon'}{2} - \frac{1}{q} \), the following estimate holds

\[
\|T^{\phi,\varphi}f\|_{L^q(\Omega; C^\kappa([0,T]; E_{p,\kappa}))} \lesssim \|f\|_{B^p_{r,\frac{d}{2}}},
\]

with \( E_{p,\kappa} = B^p_{\rho,\kappa}(\mathbb{R}^d) \) if \( p < +\infty \) and \( E_{p,\infty} = C^0(\mathbb{R}^d) \) if \( p = +\infty \), where \( \omega \) is a weight with \( \eta \text{-polynomial growth} \). When \( p < +\infty \) and \( r = 1 \), one can take \( \varepsilon'' = \varepsilon \).

Proof. As usual, one needs to distinguish between \( p = +\infty \) and \( 1 < p < +\infty \). The \( p < +\infty \) case. Thanks to Lemma 3.5 and (21), we obtain that, for any \( \beta \in \mathbb{N}^d \), \( \varepsilon \in (0,1] \), \( q \geq 2 \) and \( j \geq -1 \),

\[
\mathbb{E} \left[ \left\| \partial^{\beta}(T^{\phi,\varphi} \Delta_j f)_{s,t} \right\|_{L^p(\mathbb{R}^d)}^q \right]^\frac{1}{q} \leq \sum_{\gamma \leq \beta} \sum_{1 \leq |\mu| \leq |\gamma|} C_{\beta,\gamma,\mu,\upsilon} \mathbb{E} \left[ \left\| T^{\Lambda_{\gamma,\upsilon}(\varphi)\partial^{\beta - \gamma} \phi, \varphi}(\partial^\mu \Delta_j f) \right\|_{L^p(\mathbb{R}^d)}^q \right]^\frac{1}{q} \leq (t - s)^{\frac{1 + \varepsilon'}{2} - j - \frac{j}{2n}} \sum_{\gamma \leq \beta} \left\| \partial^\gamma \Delta_j f \right\|_{L^p(\mathbb{R}^d)}^q \leq (t - s)^{\frac{1 + \varepsilon'}{2} - j - \frac{j}{2n} + |\beta|} \left\| \Delta_j f \right\|_{L^p(\mathbb{R}^d)}^q.
\]

In particular, we deduce that, for any \( k \in \mathbb{N} \), we have

\[
\mathbb{E} \left[ \left\| (T^{\phi,\varphi} \Delta_j f)_{s,t} \right\|_{B^k_{p,p}}^q \right]^\frac{1}{q} \simeq \mathbb{E} \left[ \left\| (1 - \Delta)^{2k}(T^{\phi,\varphi} \Delta_j f)_{s,t} \right\|_{L^p(\mathbb{R}^d)}^q \right]^\frac{1}{q} \leq (t - s)^{\frac{1 + \varepsilon'}{2} - j - \frac{j}{2n} + 2k} \left\| \Delta_j f \right\|_{L^p(\mathbb{R}^d)},
\]

and, thus, by interpolation, we deduce that, for any \( q \in \mathbb{R} \),

\[
\mathbb{E} \left[ \left\| (T^{\phi,\varphi} \Delta_j f)_{s,t} \right\|_{B^k_{p,p}}^q \right]^\frac{1}{q} \lesssim (t - s)^{\frac{1 + \varepsilon'}{2} - j - \frac{j}{2n} + \varepsilon} \left\| \Delta_j f \right\|_{L^p(\mathbb{R}^d)}.
\]
In particular, this yields
\[ \mathbb{E} \left[ \left\| (T^{\phi,\varphi} f)_{s,t} \right\|_{W^{2k,p}(\mathbb{R}^d)}^q \right]^{\frac{1}{q}} \lesssim (t - s)^{\frac{1+\xi}{2}} \| f \|_{B_{p,1}^{\frac{1+\xi}{2}}}^{1+\xi}, \]
which gives the desired result by using Kolmogorov’s continuity theorem. The \( p = +\infty \) case. For any \( \beta \in \mathbb{N}^d \), any \( x, y \in \mathbb{R}^d \), we have, for any \( \theta \in [0,1] \) and \( j \geq -1 \),
\[ \left| \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(x) - \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(y) \right| \]
\[ \leq \left| \nabla \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(z) \right| ^\theta \left( \left| \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(x) \right| + \left| \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(y) \right| \right)^{1-\theta} |x - y|^\theta, \]
for some \( z \in \{(1 - \zeta)x + \zeta y : \zeta \in [0,1] \} \). It follows from Lemma 3.5 and (25) that, for any \( q \geq 2 \),
\[ \mathbb{E} \left[ \left| \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(x) - \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(y) \right| ^q \right]^{\frac{1}{q}} \leq |x - y|^\theta \mathbb{E} \left[ \left| \nabla \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(z) \right| ^{2(1-\theta)q} \right]^{1/(2q)} \]
\[ \times \left( \mathbb{E} \left[ \left| \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(x) \right| ^{2(1-\theta)q} \right]^{1/(2q)} + \mathbb{E} \left[ \left| \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(y) \right| ^{2(1-\theta)q} \right]^{1/(2q)} \right)^{1-\theta} \]
\[ \lesssim |x - y|^\theta (t - s)^{\frac{1+\xi}{2} - \frac{j-1}{2\beta}} \left( \sum_{\gamma \leq \beta} \| \nabla \partial^\gamma \Delta_j f \|_{L^\infty(\mathbb{R}^d)} \right)^{1-\theta} \left( \sum_{\gamma \leq \beta} \| \partial^\gamma \Delta_j f \|_{L^\infty(\mathbb{R}^d)} \right)^{1-\theta} \]
\[ \lesssim |x - y|^\theta (t - s)^{\frac{1+\xi}{2} - \frac{j-1}{2\beta}} 2^{(\beta + \theta)j} \| \Delta_j f \|_{L^\infty(\mathbb{R}^d)}. \]
We also have, by Lemma 3.5 and (25),
\[ \mathbb{E} \left[ \left| \partial^\beta (T^{\phi,\varphi} \Delta_j f)_{s,t}(x) \right| ^q \right]^{\frac{1}{q}} \lesssim (t - s)^{\frac{1+\xi}{2} - \frac{j-1}{2\beta}} 2^{(\beta + \theta)j} \| \Delta_j f \|_{L^\infty(\mathbb{R}^d)}. \]
From here, we use Theorem D.4 and Theorem D.5 to deduce the desired result. \( \square \)

4. Malliavin Calculus

In order to be self-contain, and in the spirit of Section 2 we recall in this section some facts about Malliavin calculus. Most of them are well-known facts, but Subsection refsubsec:conditional yields to non-standard estimates that we will use in the following. In particular, we will derived a not so surprising conditional integration by part formula, which will allow us to check whenever a flow generated by a RDE driven by a Gaussian rough path satisfy the conditions of Theorem 3.3 or Corollary 3.6.

One may consult [58, 60] for more details.

4.1. Isonormal Gaussian processes.

**Definition 4.1.** (Isonormal Gaussian process) An Isonormal Gaussian process is the set of:

1. a real and separable Hilbert space \( \mathcal{H} \) whose scalar product is denoted as \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and its norm as \( \| \cdot \|_{\mathcal{H}} \),
2. a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \),
3. a real-valued Gaussian process \( W : h \in \mathcal{H} \rightarrow W(h) \), i.e. \( (W(h))_{h \in \mathcal{H}} \) is a family of centered Gaussian random variables such that \( \mathbb{E}[W(h)W(g)] = \langle h, g \rangle_{\mathcal{H}} \), for any \( h, g \in \mathcal{H} \).

**Remark 4.2.**

1. By Kolmogorov’s theorem, given only \( \mathcal{H} \), we can construct \( (\Omega, \mathcal{F}, \mathbb{P}) \) and \( W \) satisfying the above conditions.
2. The mapping \( h \mapsto W(h) \) is linear.

The two following examples will be of interest in the following. The first one is a classical construction linked to the standard Brownian motion.  

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We now consider\( H \) the restriction of\( W \). Remark 4.5.

For any\( n \) the following, we will treat the case\( K \). Then, we define \( \mathcal{H} \). We can extend the linear mapping \( \mathcal{K} : \mathcal{H} \to W^j(\Omega) \to L^2(\Omega) \) to an isometry \( \mathcal{K} : h \in \mathcal{H} \to W^j(h) \) since

\[
\mathbb{E} \left[ \mathcal{K}(\mathbb{1}_{[0,t]} \mathbb{1}_{[0,s]}) \mathcal{K}(\mathbb{1}_{[0,t]} \mathbb{1}_{[0,s]}) \right] = \mathbb{E} \left[ W^j_t W^j_s \right] = R_W(t,s) = \langle \mathbb{1}_{[0,t]} \mathbb{1}_{[0,s]} \rangle_H.
\]

Then, \( \mathcal{K} : h \in \mathcal{H} \to W^j(h) \) is an isonormal Gaussian process. Let us construct the vector-valued isonormal Gaussian process \( \mathcal{K} : h \in \mathcal{H}^{2d} \to W(h) = (W^1(h_1), \cdots, W^n(h_n)) \in L^2(\Omega; \mathbb{R}^d) \). In the following, we will treat the case \( n = 1 \) without loss of generality.

**Remark 4.5.**

1. An interesting instance is, for example, the case where \( W = B^H \) the fractional Brownian motion with Hurst parameter \( H \in (0,1) \) and covariance

\[
R_{BH}(t,s) = \mathbb{E}[B^H_t B^H_s] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).
\]

2. For \( W_0 = 0 \) (and \( R_W(0,0) = 0 \)), we have, by [15, Proposition 4],

\[
\langle h_1, h_2 \rangle_H = \int_0^1 \int_0^1 h_1(t) h_2(s) dR_W(t,s), \tag{26}
\]

whenever the 2d Young integral on the right-hand-side is well-defined.

Since \( \mathcal{H} \) is the closure of indicator functions, for any \( [a,b] \subset [0,1] \), we can also define \( \mathcal{H}([a,b]) \) the restriction of \( \mathcal{H} = \mathcal{H}([0,1]) \) to \( [a,b] \). Thus, for almost every \( s \in [a,b]^c \), \( h(s) = 0 \) for \( h \in \mathcal{H}([a,b]) \). Furthermore, we directly deduce that, for any \( h, g \in \mathcal{H} \),

\[
\langle h, g \rangle_{\mathcal{H}([a,b])} = \langle h 1_{[a,b]}, g 1_{[a,b]} \rangle_H.
\]

For any \( [a,b] \subset [0,1] \), we denote \( \mathcal{F}_{[a,b]} \) the \( \sigma \)-algebra generated by \( \{W(1_{[u,v]}), [u,v] \in [a,b]\} \).
4.2. The Malliavin derivative. We consider the set of smooth cylindrical fields \( S \) given by
\[
S = \{ F = F(W) = f(W(h_1), W(h_2), \ldots, W(h_n)) \mid f \in C_p^\infty(\mathbb{R}^d), (h_k)_{k \in \{1, \ldots, n\}} \in \mathcal{H}^n, n \geq 0 \},
\]
where \( C_p^\infty(\mathbb{R}^d) \) denotes the set of functions from \( \mathbb{R}^d \) to \( \mathbb{R} \) that are infinitely differentiable and with all their partial derivatives having polynomial growth. Furthermore, we denote \( C_b^\infty(\mathbb{R}^d) \) the subset of \( C_p^\infty(\mathbb{R}^d) \) where all the partial derivatives are bounded. The smooth random variables associated to this space \( C_b^\infty \) is denoted by \( S_b \). We note that \( S_b \) is dense in \( L^2(\Omega) \).

**Definition 4.6.** For any \( F = f(W(h_1) \cdots W(h_n)) \in S \) with \( f \in C_p^\infty \), we define the Malliavin derivative of \( F \) as
\[
DF = \sum_{k=1}^n \partial_k f(W(h_1), \ldots, W(h_n)) h_k,
\]
which takes values in \( \mathcal{H} \).

We can directly check that, for any \( F, G \in S \), we have the "chain rule" relation
\[
D(FG) = GDF + FDG.
\]

One can reformulate this definition in terms of of directional derivative in the so-called Cameron-Martin space. Indeed, for isonormal Gaussian process \( W \), any \( h \in \mathcal{H} \) and \( \varepsilon \in [0, 1] \), we defined the shifted process
\[
\tau_{\varepsilon,h}W : u \in \mathcal{H} \to W(u) + \varepsilon \langle u, h \rangle_{\mathcal{H}}.
\]
Then, the Malliavin derivative can be identified as
\[
\langle DF, h \rangle_{\mathcal{H}} = \lim_{\varepsilon \to 0} \varepsilon^{-1} (F(\tau_{\varepsilon,h}W) - F(W)), \quad \forall h \in \mathcal{H},
\]
that is an element in the dual space \( \mathcal{H}' \) (which is still \( \mathcal{H} \) here).

The shifted process \( \tau_{\varepsilon,h}W \) can also be interpreted for \( h \) in the so-called **Cameron-Martin space** denoted \( \mathcal{H}_1 \) which is defined as the completion of
\[
\tilde{\mathcal{E}} = \left\{ t \to \sum_{k=1}^m a_k R_W(t_k, t) ; \; m \in \mathbb{N}, (a_k)_{1 \leq k \leq m}, (t_k)_{1 \leq k \leq m} \subset [0, T] \right\},
\]
with respect to the scalar product
\[
\langle R_W(t, \cdot), R_W(s, \cdot) \rangle_{\mathcal{H}_1} = R_W(t, s).
\]
Then, the mapping defined by
\[
\mathcal{R} : 1_{[0, \varepsilon]} \in \mathcal{E} \to R_W(t, \cdot) \in \tilde{\mathcal{E}}
\]
can be extended to an isometry \( \mathcal{R} : \mathcal{H} \to \mathcal{H}_1 \) since
\[
\langle \mathcal{R}(1_{[0, \varepsilon]}), \mathcal{R}(1_{[0, \varepsilon]}) \rangle_{\mathcal{H}_1} = R_W(t, s) = \langle 1_{[0, \varepsilon]}, 1_{[0, s]} \rangle_{\mathcal{H}}.
\]

We have the following integration by parts formula.

**Lemme 4.7.** For any \( F, G \in S \) and \( h \in \mathcal{H} \), we have
\[
\mathbb{E}[G\langle DF, h \rangle_{\mathcal{H}}] = -\mathbb{E}[F\langle DG, h \rangle_{\mathcal{H}}] + \mathbb{E}[FGW(h)].
\]

Furthermore, we have the following result.

**Proposition 4.8.** For any \( p \geq 1 \), the linear operator \( D \) is closable from \( L^p(\Omega) \) to \( L^p(\Omega; \mathcal{H}) \).

We can iteratively define, for any \( m \geq 1 \),
\[
D^m F = \sum_{k_1, k_2, \ldots, k_m = 1}^n \partial_{k_1, k_2, \ldots, k_m} f(W(h_1), \ldots, W(h_n)) h_{k_1} \otimes \cdots \otimes h_{k_m}
\]
which takes values in \( \mathcal{H}^{\otimes m} \). Moreover, following Proposition 4.8, we can close \( D^m \).
Proposition 4.9. For any $m \geq 1$ and $p \geq 1$, the linear operator $D^m$ is closable from $S$ to $L^p(\Omega; \mathcal{H}^\otimes m)$.

By using the same notation for its extension, the domain of the operator $D^m$ is the space $\mathbb{D}^{m,p}$ which is the completion of $S$ with respect to the norm

$$\|F\|_{\mathbb{D}^{m,p}} := \left( \mathbb{E}[|F|^p] + \sum_{k=1}^{m} \mathbb{E}[|D^kF|^p]_{\mathcal{H}^\otimes k} \right)^{1/p}.$$ 

In the previous norm, in the case of multidimensional processes, we remark that $\|\cdot\|_{\mathcal{H}^\otimes k}$ is not an operator norm but the Hilbert-Schmidt norm. We also denote $\mathbb{L}$ by $\mathbb{L}$.

Lemma 4.12. For any $r, q \geq 1$ such that $1/r + 1/q = 1/p$. We now state a chain rule with respect to the Malliavin derivative.

Proposition 4.10. Let $g \in C^1(\mathbb{R}^d; \mathbb{R})$ with bounded derivatives, $p \geq 1$ and $F = (F_1, \ldots, F_n)$ be a random vector such that $F^k \in \mathbb{D}^{1,p}$ for any $k \in \{1, \ldots, n\}$. Then, $g(F) \in \mathbb{D}^{1,p}$ and

$$D(g(F)) = \sum_{k=1}^{n} \partial_k g(F) D F^k.$$ 

4.3. The divergence operator. The divergence operator $\delta$ is the adjoint of the derivative $D$. In fact, $\delta$ is an unbounded operator from $L^2(\Omega; \mathcal{H}) = \mathbb{D}^{0,2}(\mathcal{H})$ to $L^2(\Omega)$ such that:

1. its domain $\text{Dom}(\delta)$ is the set of random variables $u \in L^2(\Omega; \mathcal{H})$ such that, for all $F \in \mathbb{D}^{1,2},$

$$\|\mathbb{E}[(DF, u)_{\mathcal{H}}]\| \leq c_u\|F\|_{L^2(\Omega)}.$$

2. for any $u \in \text{Dom}(\delta)$, we have $\delta(u) \in L^2(\Omega)$ and the following duality relation holds, for all $F \in \mathbb{D}^{1,2},$

$$\mathbb{E}[(DF, u)_{\mathcal{H}}] = \mathbb{E}[F \delta(u)].$$

We have the following result concerning the continuity of $\delta$.

Theorem 4.11. For any $p > 1$ and $m \geq 1$, the operator $\delta$ is continuous from $\mathbb{D}^{m-1,p}(\mathcal{H})$ to $\mathbb{D}^{m,p}$. That is, the following inequality holds for any $u \in \mathbb{D}^{m,p}(\mathcal{H}),$

$$\|\delta(u)\|_{\mathbb{D}^{m-1,p}} \leq c_{m,p}\|u\|_{\mathbb{D}^{m,p}(\mathcal{H})}.$$ 

This yields in particular that $\mathbb{D}^{m,p}(\mathcal{H}) \subset \text{Dom}(\delta)$ for any $p > 1$ and $m \geq 1$.

For any random variable $F \in \mathbb{D}^{1,2}$, we know that $DF$ is a stochastic process in $\mathcal{H}$ that we can denote $(D_t F)_{t \in [0,1]}$ which is defined almost surely with respect to the measure $\lambda \times \mathbb{P}$ (where $\lambda$ is the usual Lebesgue measure). With this, we can deduce a local property of the derivative operator.

Lemme 4.12. Let $F \in \mathbb{D}^{1,2} \cap L^2(\Omega, F_{[a,b]}, \mathbb{P})$. Then we have

$$D_s F(\omega) = 0,$$

for $(\lambda \times \mathbb{P})$-almost every $(s, \omega) \in [a,b]^c \times \Omega$.

Proof. We use $S_k([a,b])$ the natural restriction of $S_k = S_k([0,1])$ to $[a,b]$. It is dense in $\mathbb{D}^{1,2} \cap L^2(\Omega, F_{[a,b]}, \mathbb{P})$. Then, for any $F \in S([a,b]),$

$$D_s F = \sum_{k=1}^{n} \partial_k f(W(h_1), \ldots, W(h_n)) h_k(s),$$

where $h_k \in \mathcal{H}([a,b])$ which is such that $h_k(s) = 0$ for $s \notin [a,b]$.

□
4.4. Conditional Integration by parts formula. We can now introduce the Skorokhod integral (which is essentially the divergence operator). For any \([a, b] \subset [0, 1]\), we denote \(\delta_{[a,b]}\) the Skorokhod integral on \([a, b]\) which is defined by

\[
\delta_{[a,b]}(u) = \delta(u 1_{[a,b]}). 
\]

That is, \(\delta_{[a,b]}\) is an unbounded operator from \(L^2(\Omega; \mathcal{H})\) to \(L^2(\Omega)\) such that, for any \(u \in \text{Dom}(\delta)\), we have, for all \(F \in \mathbb{D}^{1,2}\),

\[
\mathbb{E} \left[ \langle DF, u \rangle_{\mathcal{H}([a,b])} \right] = \mathbb{E} \left[ \langle DF, u 1_{[a,b]} \rangle \right] = \mathbb{E} \left[ F \delta_{[a,b]}(u) \right].
\]

Furthermore, for any \(G_{[a,b]} \in L^2(\Omega, \mathcal{F}_{[a,b]}^c, \mathbb{P}) \cap \mathbb{D}^{1,2}\), we can also see that, thanks to Lemma 4.12,

\[
\mathbb{E} \left[ \mathbb{E} \left[ \langle DF, u \rangle_{\mathcal{H}([a,b])} \mid \mathcal{F}_{[a,b]}^c \right] G_{[a,b]^c} \right] = \mathbb{E} \left[ \langle DF, u \rangle_{\mathcal{H}([a,b])} G_{[a,b]^c} \right] = \mathbb{E} \left[ \langle DF, u \rangle_{\mathcal{H}([a,b])} G_{[a,b]^c} \right].
\]

This yields in fact to the following duality relation under a conditional expectation since \(\mathbb{D}^{1,2}\) is dense in \(L^2(\Omega)\),

\[
\mathbb{E} \left[ \langle DF, u \rangle_{\mathcal{H}([a,b])} \mid \mathcal{F}_{[a,b]}^c \right] = \mathbb{E} \left[ F \delta_{[a,b]}(u) \mid \mathcal{F}_{[a,b]}^c \right]. \tag{28}
\]

We can now proceed to prove a conditional integration by parts formula.

**Proposition 4.13.** Let \(F \in \mathbb{D}^{1,2}\), \(G\) be a random variable and \(u\) be an \(L^2([a, b])\)-valued random variable such that

\[
\langle DF, u \rangle_{\mathcal{H}([a,b])} \neq 0 \quad \text{and} \quad G_{\langle DF, u \rangle_{\mathcal{H}([a,b])}} \in \text{Dom}(\delta).
\]

Then, for any function \(f \in C^1\) with bounded derivatives, we have that

\[
\mathbb{E} \left[ f'(F) G \mid \mathcal{F}_{[a,b]^c} \right] = \mathbb{E} \left[ f(F) H_{[a,b]}(F, G) \mid \mathcal{F}_{[a,b]^c} \right],
\]

where

\[
H_{[a,b]}(F, G) := \delta_{[a,b]} \left( G_{\langle DF, u \rangle_{\mathcal{H}([a,b])}} \right).
\]

**Proof.** We apply Proposition 4.10 to deduce that

\[
f'(F) = \left( D(f(F)), \frac{u_{\langle DF, u \rangle_{\mathcal{H}([a,b])}}} {\langle DF, u \rangle_{\mathcal{H}([a,b])}} \right) H_{[a,b]}. 
\]

Then, it follows from (28) that

\[
\mathbb{E} \left[ f'(F) G \mid \mathcal{F}_{[a,b]^c} \right] = \mathbb{E} \left[ (D(f(F)), u)_{\mathcal{H}([a,b])} \left( (DF, u)_{\mathcal{H}([a,b])} \right)^{-1} G \mid \mathcal{F}_{[a,b]^c} \right]
\]

\[= \mathbb{E} \left[ (D(f(F)), Gu)_{\mathcal{H}([a,b])} \left( (DF, u)_{\mathcal{H}([a,b])} \right)^{-1} H_{[a,b]} \mid \mathcal{F}_{[a,b]^c} \right]
\]

\[= \mathbb{E} \left[ f(F) \delta_{[a,b]} \left( Gu \left( (DF, u)_{\mathcal{H}([a,b])} \right)^{-1} \right) \mid \mathcal{F}_{[a,b]^c} \right],
\]

which is the desired result.

\[
\square
\]

We may now introduce the Malliavin matrix associated to a random vector.

**Definition 4.14.** Let \(p > 1\) and \(F = (F^1, \ldots, F^m)\) be a random vector such that \(F^k \in \mathbb{D}^{1,p}\) for all \(k \in \{1, \ldots, m\}\). The covariance matrix \(\gamma_F\) associated to \(F\) is defined as

\[
\gamma_F := \left( (DF^i, DF^j)_{\mathcal{H}} \right)_{(i, j) \in \{1, \ldots, m\}^2}.
\]

In the case \(\mathcal{H} = \mathcal{H}([a, b])\), we use the notation \(\gamma_{F, [a,b]}\). We then have the following Lemma [59, Lemma 7.2.3]:

\[
\square
\]
Thus, we obtain that

\[ D(\gamma^{-1})_{i,j} = - \sum_{k,\ell=1}^{m} (\gamma^{-1})_{i,k}(\gamma^{-1})_{\ell,j} D\gamma_{k,\ell}. \]

Thanks to the Malliavin matrix, we define a nondegeneracy condition for random vectors.

**Definition 4.16.** We say that a random vector \( F = (F^1, \ldots, F^m) \), whose components are in \( \mathbb{D}^\infty \), is nondegenerate if the Malliavin matrix \( \gamma_F \) is invertible almost surely and

\[ \det(\gamma_F)^{-1} \in \cap_{p \geq 1} L^p(\Omega). \]

We now state a more general conditional integration by parts result for nondegenerate random vectors.

**Proposition 4.17.** Let \( F = (F^1, \ldots, F^m) \) be a nondegenerate random vector, \( G \in \mathbb{D}^\infty \) and \( g \in C^\infty_p(\mathbb{R}^m) \). Then, for any multiindex \( \alpha \in \{1, \ldots, m\}^k \), \( k \geq 1 \), there exists an element \( H_{\alpha,[a,b]}(F,G) \in \mathbb{D}^\infty \) such that

\[ \mathbb{E} \left[ \partial^\alpha g(F)G|\mathcal{F}_{[a,b]} \right] = \mathbb{E} \left[ g(F)H_{\alpha,[a,b]}(F,G)|\mathcal{F}_{[a,b]} \right]. \]

The variable \( H_{\alpha,[a,b]}(F,G) \) is given recursively by

\[ H_{\alpha,[a,b]}(F,G) = H_{a_1,b_1}(F,H_{\alpha_1,\ldots,\alpha_{m-1},[a,b]}(F,G)) \]

where, for any \( i \in \{1, \ldots, m\} \),

\[ H_{i,[a,b]}(F,G) = \sum_{j=1}^{m} \delta_{a,b} \left( G(\gamma^{-1}_{F,[a,b]})_{i,j} DF^j \right). \]

**Proof.** It follows from the chain rule that, for any \( j \in \{1, \ldots, m\} \),

\[ \langle D(g(F)), DF^j \rangle_{H([a,b])} = \sum_{i=1}^{m} \partial_i g(F) \langle DF^i, DF^j \rangle_{H([a,b])} = \sum_{i=1}^{m} \partial_i g(F)(\gamma_{F,[a,b]})_{i,j}. \]

Thus, we obtain that

\[ \partial_i g(F) = \sum_{j=1}^{m} (\gamma^{-1}_{F,[a,b]})_{i,j} \langle D(g(F)), DF^j \rangle_{H([a,b])}. \]

By exploiting the duality relation (28), this gives

\[ \mathbb{E} \left[ \partial_i g(F)G|\mathcal{F}_{[a,b]} \right] = \mathbb{E} \left[ \left\langle D(g(F)), \sum_{j=1}^{m} G(\gamma^{-1}_{F,[a,b]})_{i,j} DF^j \right\rangle_{H([a,b])}|\mathcal{F}_{[a,b]} \right]. \]

Thanks to Lemma 4.15, we can deduce that \( \gamma_{F,[a,b]} \in \mathbb{D}^\infty \). Furthermore, by Theorem 4.11, this yields that \( H_{i,[a,b]}(F,G) \in \mathbb{D}^\infty \) if \( G \in \mathbb{D}^\infty \). From here, we can proceed by induction and deduce the result. \( \square \)

We finally give an estimate for the random variables \( H_{\alpha,[a,b]} \).

**Lemma 4.18.** Let \( k \geq 1 \) and \( p > 1 \). For any \( \alpha \in \{1, \ldots, m\}^k \), we have

\[ \|H_{\alpha,[a,b]}(F,G)\|_{L^p(\Omega)} \leq \|G\|_{\mathbb{D}^\infty} \prod_{\ell=1}^{k} \left( \sum_{j=1}^{m} (\gamma^{-1}_{F,[a,b]})_{\alpha_\ell,j} DF^j \right)_{\mathbb{D}^{k-\ell+1},r_\ell} \]

where \( 1/p = 1/q + \sum_{\ell=1}^{m} 1/r_\ell \).
Proof. By Theorem 4.11 and Hölder’s inequality, we have
\[ \|H_{\alpha,[a,b]}(F,G)\|_{L^p(\Omega)} \leq c_{p,m} \left\| H_{(\alpha_1,\ldots,\alpha_{k-1}),[a,b]}(F,G) \sum_{j=1}^{m} (\gamma_{F,[a,b]}^{-1})_{\alpha_k,j} DF \right\|_{D^{1,p}} \]
\[ \leq c_{p,m} \left\| H_{(\alpha_1,\ldots,\alpha_{k-1}),[a,b]}(F,G) \right\|_{D^{1,q}} \left\| \sum_{j=1}^{m} (\gamma_{F,[a,b]}^{-1})_{\alpha_k,j} DF \right\|_{\mathcal{D}^{1,r}}, \]
where \( 1/p = 1/q + 1/r \), which yields the desired result by induction. \( \square \)

5. Malliavin calculus and rough paths

The idea of this Section is to use Malliavin calculus (and especially the conditional bounds proved in the previous subsection) to derive some nice bounds on flow generated by rough differential equations. In order to do so, we will need some properties of the Cameron-Martin spaces \( \mathcal{H}_1 \)

5.1. Embedding of the spaces \( \mathcal{H} \) and \( \mathcal{H}_1 \). In this section, we give some embedding of \( \mathcal{H} \) and \( \mathcal{H}_1 \) into the space of continuous functions with finite \( p \)-variation, with \( p > 0 \).

Definition 5.1. Let \( f : [0,T] \to \mathbb{R}^d \) be a continuous function. For any \( p > 0 \), we define its \( p \)-variation on \([a,b] \subset [0,T]\) as
\[ \|f\|_{p\text{-var};[a,b]} = \sup_{\pi \in \Pi([a,b])} \left( \sum_{(u,v) \in \pi} |f(v) - f(u)|^p \right)^{1/p}, \]
where we remind that \( \Pi([a,b]) \) is the set of all subdivisions of \([a,b]\). The set of continuous functions with finite \( p \)-variation on \([a,b]\) is denoted \( \mathcal{C}^{p\text{-var}}([a,b]; \mathbb{R}^d) = \mathcal{C}^{p\text{-var}}([a,b]) \).

A, similar, mixed \((p,q)\)-variation notion also exists for continuous functions on \([0,T]^2\).

Definition 5.2. Let \( R : [0,T]^2 \to \mathbb{R}^d \) be a continuous function. For any \( p,q > 0 \), we define its \((p,q)\)-variation on the square \([a,b] \times [c,d]\) as
\[ \|R\|_{(p,q)\text{-var};[a,b] \times [c,d]} = \sup_{\pi \in \Pi([a,b] \times [c,d])} \left( \sum_{(u_1,v_1 \times u_2,v_2) \in \pi} \left( \sum_{\pi_1 \in \Pi([a,b], 1]} \left| \square_{[u_1,v_1] \times [u_2,v_2]} R \right|^p \right)^{q/p} \right)^{1/q}, \]
where \( \square_{[u_1,v_1] \times [u_2,v_2]} R = R(v_1, v_2) - R(v_1, u_2) - R(u_1, v_2) + R(u_1, u_2) \). The set of continuous functions with finite \((p,q)\)-variation on \([a,b] \times [c,d]\) is denoted \( \mathcal{C}^{(p,q)\text{-var}}([a,b] \times [c,d]; \mathbb{R}^d) = \mathcal{C}^{(p,q)\text{-var}}([a,b] \times [c,d]). \)

We also denote \( \|R\|_{p\text{-var};[a,b]} = \|R\|_{(p,p)\text{-var};[a,b]} \) as well as \( \mathcal{C}^{p\text{-var}}([a,b]^2) := \mathcal{C}^{(p,p)\text{-var}}([a,b]^2). \)

Remark 5.3. We can see that
\[ \|R\|_{p\text{-var};[a,b]} \leq \|R\|_{(p,q)\text{-var};[a,b]} \leq \|R\|_{p\text{-var};[a,b]}. \]

Furthermore, we have the following definition.

Definition 5.4. Let \( \rho \in [1,2) \) and \([a,b] \subset [0,T]\). We say that \( R \) has finite Hölder-controlled \( \rho \)-variation on \([a,b]\) if \( \|R\|_{p\text{-var};[a,b]} < +\infty \) and if the following estimate holds, for any \([s,t] \subset [a,b]\),
\[ \|R\|_{p\text{-var};[s,t]} \lesssim (t-s)^{1/p}. \]

We are now in position to state a first assumption on the process \( W \) that we will consider on the rest of the section.
Assumption 5.5. The process $W$ is a $\mathbb{R}^d$-valued continuous centered Gaussian process starting at 0 with iid components and covariance $R_W$ which belongs in $C^{(1,\rho)}(\var)$ for some $\rho \in [1,2)$. Furthermore, the following estimate holds, for any $[a,b] \subset [0,T]$,

$$\|R_W\|_{(1,\rho)\text{-var};[a,b]}^2 \lesssim (b-a)^{1/\rho}. \quad (29)$$

We immediately see that (29) is verified if the covariance has finite Hölder-controlled $\rho$-variation on $[0,T]$. This leads in particular to the following hypothesis.

Assumption 5.6. The process $W$ is a $\mathbb{R}^d$-valued continuous centered Gaussian process starting at 0 with iid components and covariance $R_W$ which has finite Hölder-controlled $\rho$-variation on $[0,T]$ for some $\rho \in [1,2)$.

In the following, we denote

$$\kappa_{a,b} := \|R_W\|_{(1,\rho)\text{-var};[a,b]}^{1/2}$$

and

$$\sigma_{a,b} := \mathbb{E}\left[\left(W_b^{(j)} - W_a^{(j)}\right)^2\right]^{1/2} = (\Box_{[a,b]}^2 R_W)^{1/2},$$

for any $j \in \{1, \ldots, n\}$.

We can now proceed to state our first embedding:

Theorem 5.7 ([34], Theorem 1.1). Let $W$ be a centered Gaussian process satisfying Assumption 5.5. Then, for any $h \in \mathcal{H}_1$, $[a,b] \subset [0,T]$, we have

$$\|h\|_{q\text{-var};[a,b]} \leq \kappa_{a,b}\|h\|_{\mathcal{H}_1},$$

where $q = 1/(1/2\rho + 1/2) < 2$. In particular, the embedding $\mathcal{H}_1 \hookrightarrow C^{q\text{-var}}([a,b])$ is continuous.

Furthermore, we also have the following results [44, Remark 2.16].

Proposition 5.8. Let $W$ be a centered Gaussian process satisfying Assumption 5.5 and $[a,b] \subset [0,T]$.

(1) Let $f \in C^{p\text{-var}}([a,b])$ with $1/p + 1/\rho > 1$. Then, $f \in \mathcal{H}$ and

$$\|f\|_{\mathcal{H}}^2 = \int_a^b \int_a^b f(s)f(t) \, dR_W(s,t),$$

where the right-hand side is well defined as a $2d$ Young integral. In particular, we have the continuous embedding $C^{p\text{-var}}([a,b]) \hookrightarrow \mathcal{H}([a,b])$ since

$$\|f\|_{\mathcal{H}([a,b])}^2 \lesssim \|f\|_{p\text{-var};[a,b]}^2 \|R_W\|_{p\text{-var};[a,b]}.$$

(2) Let $f_1 \in C^{p\text{-var}}([a,b])$ with $1/p + 1/\rho > 1$ and $f_2 \in \mathcal{H}([a,b])$. Then,

$$\langle f_1, f_2 \rangle_{\mathcal{H}([a,b])} = \int_a^b f_1 \, d\mathcal{R} f_2,$$

where the right-hand side is well defined as a Young integral and $\mathcal{R} : \mathcal{H} \to \mathcal{H}_1$ is the isomorphism defined by (27).

We now make an additional assumption on the process $W$.

Assumption 5.9. Let $W$ be an $\mathbb{R}^d$-valued Gaussian process with iid coordinates and covariance function $R_W$ such that

(1) it has non-positively correlated increments, that is, for all $(t_1,t_2,t_3,t_4) \in [0,T]^4$ with $t_1 < t_2 < t_3 < t_4$, we have

$$\Box_{[t_1,t_2] \times [t_3,t_4]} R_W \leq 0,$$

(2) its covariance function is diagonally dominant, that is, for all $(t_1,t_2,t_3,t_4) \in [0,T]^4$ with $t_1 < t_2 < t_3 < t_4$, we have

$$\Box_{[t_2,t_3] \times [t_1,t_4]} R_W \geq 0.$$
We deduce the following inequalities [44, Proposition 2.8].

**Proposition 5.10.** Let $W$ be a Gaussian process satisfying Assumption 5.5, $p \geq 1$ such that $1/p + 1/\rho > 1$.

1. For every $f \in C^p$-var$((a, b])$, we have
   \[ \|f\|_{H^p((a, b])} \leq \kappa_{a,b}^2 \left( \|f\|_{p-var,(a, b]}^2 + \|f\|_{\infty,(a, b]}^2 \right) \]

2. If $W$ satisfies Assumption 5.9, we have, for any $f \in C^\gamma((a, b])$ with $1/\rho + \gamma > 1$,
   \[ \|f\|_{H^\gamma((a, b])}^2 \geq \sigma_{a,b}^2 \min_{t \in [a,b]} |f(t)|. \]

To continue, we need some non-degeneracy concerning $W$.

**Assumption 5.11.** Let $W$ be a centered continuous $\mathbb{R}^d$-valued Gaussian process. We assume that there exists an $\alpha > 0$ such that
   \[ \inf_{0 \leq s \leq t \leq T} (t-s)^{-\alpha} \text{Var} \left[ W_t - W_s \mid \mathcal{F}_{[0,s]} \right] = c_W > 0. \]

The smallest $\alpha$ that satisfies the above condition is called the index of non-determinism of $W$.

Under the previous assumption, we deduce our last inequality which is taken from [16, Corollary 6.10].

**Proposition 5.12.** Let $W$ be a continuous Gaussian process satisfying Assumptions 5.5, 5.9 and 5.11. Then, for any $f \in C^\gamma((a, b])$ with $\gamma + 1/\rho > 1$, we have
   \[ \|f\|_{\infty,(a, b]} \leq 2 \max \left( \sigma_{a,b} \|f\|_{H^\gamma((a, b])}, \sqrt{c_W} \|f\|_{H^\gamma((a, b])}^{2\gamma/(2\gamma + \alpha)} \|f\|_{C^\gamma((a, b])}^{\alpha/(2\gamma + \alpha)} \right). \]

As a direct consequence, we have the following result.

**Corollary 5.13.** Let $W$ be a continuous Gaussian process satisfying Assumptions 5.5, 5.9 and 5.11. Then, for any $f \in C^\gamma((a, b])$ with $\gamma + 1/\rho > 1$, we have
   \[ \|f\|_{H^\gamma((a, b])} \geq \frac{\sigma_{a,b}}{2} \min \left( 1, \frac{2(c_W/2)^{(2\gamma + \alpha)/4\gamma} \|f\|_{C^\gamma((a, b])}^{\alpha/2\gamma}}{\sigma_{a,b} \|f\|_{\infty,(a, b]}^{\alpha/(2\gamma)}} \right). \]

### 5.2. Gaussian rough paths

Let us now focus ourselves on rough differential equations driven by Gaussian rough paths. Let us consider $W$ a continuous centered Gaussian process satisfying Assumption 5.5. It can therefore be lifted to a geometric rough path (see [36]) that we denote $W$.

**Proposition 5.14.** Let $W$ be a continuous Gaussian process satisfying Assumption 5.5. Then, almost surely, $W$ can be lifted to a geometric $p$-rough path $W$ with $p > 2\rho$ and verifies
   \[ \mathbb{E} \left[ \exp \left( \eta \|W\|_{1/p\text{-var},[0,T]} \right) \right], \text{ for some } \eta > 0. \]

We also have the following useful result (which is a direct consequence of the Proposition 5.14 and [36, Theorem 15.33]).

**Proposition 5.15.** Let $W$ be a Gaussian process satisfying Assumption 5.6. Then, $W$ can be lifted to a geometric $p$-rough path $W$ with $p > 2\rho$ which has $1/p$-Hölder sample paths and verifies
   \[ \mathbb{E} \left[ \exp \left( \eta \|W\|_{1/p\text{-Hölder},[0,T]} \right) \right], \text{ for some } \eta > 0. \]

We will need the following more precise statement of Theorem (2.3) in this setting:

**Theorem 5.16.** Let $p \geq 1$, $w$ be a weakly geometric $p$-rough path, $\sigma \in C^\rho_0(\mathbb{R}^d; \mathbb{R}^{d \times d})$ for some $\gamma > p$. Let $a \in [0,T)$ and $x \in \mathbb{R}^d$.

There exists a unique $p$-weakly geometric rough path $x \in C^{p\text{-var}}([a,T]; G[p](\mathbb{R}^d))$ such that
   \[ x_{a,}^1 = \phi_a(x) \]
where $\varphi$ is the flow constructed in Theorem 2.3.

Furthermore, if $(w^\varepsilon)_{\varepsilon > 0}$ is a family of smooth functions on $[0,T]$ such that

$$S_\varepsilon(w^\varepsilon) \to w,$$

in $p$-variation. Then, for any $0 \leq a \leq t \leq T$, the solutions $(x^\varepsilon)_{\varepsilon > 0} \subset C^1([s,T])$ of

$$x^\varepsilon_{t,a} = x + \sum_{k=1}^d \int_a^t \sigma_k(x^\varepsilon_{r,a}) \, dw_r^k,$$

are such that

$$S_\varepsilon(x^\varepsilon) \to x,$$

in $p$-variation, where $x$. Moreover, for any $[s,t] \subset [a,T]$, the following estimates hold

$$\|x\|_{p\text{-var};[s,t]} \lesssim_{p,\gamma} \left( \|\sigma\|_{C^\gamma_b} \|w\|_{p\text{-var};[s,t]} \vee \|\sigma\|_{C^\gamma_b} \|W\|_{p\text{-var};[s,t]} \right).$$

(30)

As a direct consequence of the previous result, we deduce.

**Corollary 5.17.** Let $W$ be a Gaussian process satisfying Assumption 5.5 and let $p > 2\rho$. Let $X$ be the unique solution of equation

$$dX_t = \sigma(X_t) \, dW_t, \quad X_a = x, \quad t \in [a,T],$$

in the sense of Theorem 5.16. Let us denote $X^1_{t,a} = X_{a,t}$.

Then, for any $[s,t] \subset [a,T]$,

$$\sup_{x \in \mathbb{R}^d} |X_{t,a}^x - x| \lesssim_{p,\gamma} \left( \|\sigma\|_{C^\gamma_b} \|W\|_{p\text{-var};[s,t]} \vee \|\sigma\|_{C^\gamma_b} \|W\|_{p\text{-var};[s,t]} \right).$$

(31)

5.3. **Malliavin calculus on rough differential equations.** The solution $X$ of a RDE driven by a Gaussian process $W$ is a random variable inheriting its randomness from $W$. One can thus try to differentiate $X$ in the Malliavin sense. The Malliavin derivative of $X$ can indeed be simply expressed thanks to the Jacobian $J$ of the solution of equation (2). It is given by $(J_t^{x,a})_{(i,j) \in \{1,\ldots,d\}^2} = \partial_{x,j}(X_{t,a})_i$ and solution the following linear RDE

$$J_t^{x,a} = \text{Id} + \sum_{k=1}^d \int_a^t \nabla \sigma_k(X_{s,a}) J_s^{x,a} \, dW_s^k,$$

(32)

where $\sigma_k$ is the $k$-th column of $\sigma$. We notice that the inverse Jacobian $J^{-1}$ solves

$$(J_t^{x,a})^{-1} = \text{Id} - \sum_{k=1}^d \int_a^t (J_s^{x,a})^{-1} \nabla \sigma_k(X_s^{x,a}) \, dW_s^k,$$

(33)

and that, for any $s \leq r \leq t$, we have, by the flow property of the Jacobian,

$$J_r^{x,a} = J_t^{x,a} (J_t^{x,a})^{-1}.$$

(34)

We have the following results concernant $J$ as well as the Malliavin derivative of $X$ [14, 17, 51].

**Proposition 5.18.** Let $W$ be a continuous centered Gaussian process satisfying Assumption 5.5 and $\sigma \in C^\infty_b(\mathbb{R}^d; \mathbb{R}^{d \times d})$.

(1) For any $q \geq 1$, there exists a constant $c_{q,a}$ such that the Jacobian $J$ defined by (32) satisfies

$$\mathbb{E} \left[ \sup_{x \in \mathbb{R}^d} \|J_t^{x,a}\|_{p\text{-var};[0,T]}^q \right] = c_{q,a}.$$

(35)

(2) For any $x \in \mathbb{R}^d$ and $t > 0$, we have that $X_t \in D^\infty$ and, furthermore, the Malliavin derivative $D_sX_t^{x,a}$ is such that

$$D_sX_t^{x,a} = J_t^{x,a} \sigma(X_s^{x,a}),$$

(36)

for any $a \leq s \leq t$, and $D_sX_t^{x,a} = 0$ for all $s > t$. 
Remark 5.19. We remark that Equation (35) can also be replaced by

\[
E \left[ \sup_{x \in \mathbb{R}^d} \|J^{x,a}_{t}||_1^{q-Ho\delta[0,T]} \right] = c_{q,a},
\]

We now state some estimates on the Malliavin derivative of \(X\) as well as the associated covariance matrix \(\gamma_X\). We follow the approach of [51] which uses some functionals \(\Xi_m\) that are equal to \(D_h^m X_t\) along a single path \(h \in \mathcal{H}([a,b])\) (the identification extends to any direction \((h_1, h_2, \ldots, h_m)\) since these are bounded symmetric \(m\)-multilinear maps). For any \(m \geq 2\), the Malliavin derivative of \(X\) along \(h^{\otimes m} \in \mathcal{H}^{\otimes m}([a,b])\) is given by

\[
D_h^m X_t^{x,a} := \langle D_h^m X_t^{x,a}, h^{\otimes m} \rangle_{\mathcal{H}^{\otimes m}([a,b])}
\]

\[
= \sum_{k=1}^{d} \sum_{i=2}^{m} \sum_{j \in K_{ilt}} \sum_{|i|=m} C_{i,j} \int_a^t J_{i,j}^{s,x} \nabla^E D_h^{i1} X_s^{x,a}, \ldots, D_h^{i \ell} X_s^{x,a} \sigma_k (X_s^{x,a}) \, dW_s^k
\]

\[
+ \sum_{k=1}^{d} \sum_{i=1}^{m-1} \sum_{j \in K_{ilt}} \sum_{|i|=m} C_{i,j} \int_a^t J_{i,j}^{s,x} \nabla^E D_h^{i1} X_s^{x,a}, \ldots, D_h^{i \ell} X_s^{x,a} \sigma_k (X_s^{x,a}) \, dh_s^k
\]

where \(K_{\ell} = \{j \in \mathbb{N}^d : 0 < j_1 \leq j_2 \leq \ldots \leq j_{\ell}\}\), for some constants \(\{C_{i,j}\}_{i \in K_{\ell}, 2 \leq \ell \leq m}\) and \(\{C_{i,j}\}_{i \in K_{\ell}, 1 \leq \ell \leq \ell_m - 1}\). Furthermore, by Equation (33), Leibniz’s differentiation rule and Faà di Bruno’s formula, the Malliavin derivative of \((J)^{-1}\) along \(h^{\otimes m} \in \mathcal{H}^{\otimes m}([a,b])\) is such that

\[
D_h^m (J^{x,a})^{-1} = \sum_{k=1}^{d} \sum_{i=2}^{m} \sum_{j \in K_{\ell}} \sum_{|i|=m} C_{i,j} \int_a^t (J_{i,j}^{s,x})^{-1} D_h^{m-i} (J^{s,x}_s)^{-1} \nabla^E D_h^{i1} X_s^{x,a}, \ldots, D_h^{i \ell} X_s^{x,a} (\nabla \sigma_k (X_s^{x,a})) \, dW_s^k
\]

\[
- \sum_{k=1}^{d} \sum_{i=1}^{m-1} \sum_{j \in K_{\ell}} \sum_{|i|=m} C_{i,j} \int_a^t (J_{i,j}^{s,x})^{-1} D_h^{m-i} (J^{s,x}_s)^{-1} \nabla^E D_h^{i1} X_s^{x,a}, \ldots, D_h^{i \ell} X_s^{x,a} (\nabla \sigma_k (X_s^{x,a})) \, dh_s^k
\]

for some constants \(\{C_{i,j}\}_{i \in K_{\ell}, 1 \leq o \leq \ell, 1 \leq \ell \leq m}\) and \(\{C_{i,j}\}_{i \in K_{\ell}, 1 \leq o \leq \ell, 1 \leq \ell \leq m}\).

**Proposition 5.20.** Let \(B\) be an independent copy of \(W\) and let \(B\) the correspondin rough path above \(B\). For any \(t \in [a, b]\), we denote by

\[
\mathcal{X}_{1,t}(W, B) = \sum_{k=1}^{d} \int_a^t J^{s,x}_k \sigma_k (X_s^{x,a}) \, dB_s^k,
\]

\[
\mathcal{Y}_{1,t}(W, B) = -\sum_{k=1}^{d} \left( \int_a^t (J^{s,x}_k)^{-1} \nabla \mathcal{X}_{1,s}(W, B) (\nabla \sigma_k (X_s^{x,a})) \, dW_s^k - \int_a^t (J^{s,x}_k)^{-1} \nabla \sigma_k (X_s^{x,a}) \, dB_s^k \right),
\]

and, for any \(m \geq 2\),

\[
\mathcal{X}_{m,t}(W, B) = \sum_{k=1}^{d} \left( \sum_{\ell=2}^{m} \sum_{j \in K_{\ell}} \sum_{|i|=m} C_{i,j} \int_a^t J^{s,x}_i \nabla^E \mathcal{X}_{1,s}(W, B), \ldots, \mathcal{X}_{1,s}(W, B) \sigma_k (X_s^{x,a}) \, dW_s^k \right)
\]
\[
+ \sum_{\ell=1}^{m-1} \sum_{|i|=m} C_{2,4} \int_a^t J_{\ell,s}^x \nabla_{X_{\ell,s}}(W, B) \sigma_k(X_s^{x,a}) \, dB_s^k, \quad (40)
\]

\[\mathcal{Y}_{m,t}(W, B) = \]
\[- \sum_{k=1}^m \sum_{\ell=1}^{m-1} \sum_{|i|=m} C_{3,m,\ell,i} \int_a^t (J_{\ell,s}^x)_{m-\ell, a}(W, B) \nabla_{X_{\ell,s}}(W, B) \sigma_k(X_s^{x,a}) \, dB_s^k \]
\[+ \sum_{\ell=1}^{m-1} \sum_{|i|=m} C_{4,m,\ell,i} \int_a^t (J_{\ell,s}^x)_{m-1-\ell, a}(W, B) \nabla_{X_{\ell,s}}(W, B) \sigma_k(X_s^{x,a}) \, dB_s^k \]
\[+ m \int_a^t (J_{\ell,s}^x)_{m-1, a}(W, B) \nabla_{X_{\ell,s}}(W, B) \sigma_k(X_s^{x,a}) \, dB_s^k, \quad (41)\]

Then, for any \( m \geq 1 \) and \( h \in \mathcal{H}([a, b]) \),
\[\hat{D}_h^m \mathcal{X}_{m,t}(\omega, \cdot) = m!D_h^m X_{t,a}^{x,a} \quad \text{and} \quad \hat{D}_h^m \mathcal{Y}_{m,t}(\omega, \cdot) = m!D_h^m (J_{t,a}^x)^{-1},\]
where \( \hat{D} \) is the Malliavin derivative with respect to \( B \) and the left-hand side does not depend on \( B \), and we have the estimates, for any \( r \geq 2 \),
\[\mathbb{E} \left[ \sup_x \| D^m X_{t,a}^x \|_{\mathcal{H}^m([a, b])}^r \right] \lesssim_{m,r} \mathbb{E} \left[ \sup_x |\mathcal{X}_{m,t}(W, B)|^r \right] \quad (42)\]

and
\[\mathbb{E} \left[ \sup_x \| D^m (J_{t,a}^x)^{-1} \|_{\mathcal{H}^m([a, b])}^r \right] \lesssim_{m,r} \mathbb{E} \left[ \sup_x |\mathcal{Y}_{m,t}(W, B)|^r \right]. \quad (43)\]

Proof. We only give the main arguments of the proofs (see [51, Proposition 3.3]). The fact that the Malliavin derivative with respect to \( B \) of \( \mathcal{X}_m \) (resp. \( \mathcal{Y}_m \)) is related to the Malliavin derivative of \( X \) (resp. \( J^{-1} \)) is a straightforward consequence of their expressions. Concerning the inequality, we remark that
\[\| D^m X_{t,a}^x \|_{\mathcal{H}^m([a, b])} = \frac{1}{m!} \| \hat{D}^m \mathcal{X}_{m,t}(W, \cdot) \|_{\mathcal{H}^m([a, b])} = \frac{1}{m!} \hat{\mathbb{E}} \left[ \| \hat{D}^m \mathcal{X}_{m,t}(W, \cdot) \|_{\mathcal{H}^m([a, b])}^2 \right]^{\frac{1}{2}}\]
where \( \hat{\mathbb{E}} \) is the expectation with respect to \( B \), as well as
\[\| D^m (J_{t,a}^x)^{-1} \|_{\mathcal{H}^m([a, b])} = \frac{1}{m!} \| \hat{D}^m \mathcal{Y}_{m,t}(W, \cdot) \|_{\mathcal{H}^m([a, b])} = \frac{1}{m!} \hat{\mathbb{E}} \left[ \| \hat{D}^m \mathcal{Y}_{m,t}(W, \cdot) \|_{\mathcal{H}^m([a, b])}^2 \right]^{\frac{1}{2}}.\]

Since \( \mathcal{X}_m \) (resp. \( \mathcal{Y}_m \)) belongs to the \( m \)-th order inhomogeneous Wiener chaos generated by \( B \), we know that all the \( \mathbb{D}^{2,m} \)-norms are equivalent for the right-hand-side term. Hence, we obtain
\[\mathbb{E} \left[ \sup_x \| D^m X_{t,a}^x \|_{\mathcal{H}^m([a, b])}^r \right] \lesssim_{m,r} \mathbb{E} \left[ \sup_x \hat{\mathbb{E}} \left[ \| \hat{D}^m \mathcal{X}_{m,t}(W, \cdot) \|_{\mathcal{H}^m([a, b])}^2 \right]^{r/2} \right]^{r/2},\]
and, similarly,
\[\mathbb{E} \left[ \sup_x \| D^m (J_{t,a}^x)^{-1} \|_{\mathcal{H}^m([a, b])}^r \right] \lesssim_{m,r} \mathbb{E} \left[ \sup_x \hat{\mathbb{E}} \left[ \| \hat{D}^m \mathcal{Y}_{m,t}(W, \cdot) \|_{\mathcal{H}^m([a, b])}^2 \right]^{r/2} \right]^{r/2},\]
for which gives the desired results. \(\square\)
In order to push further estimates (42) and (43), we need to estimate the rough integrals from (38), (40), (39) and (41). This is done by considering, for any \( m \geq 1 \), the function \((X, J, J^{-1}, \alpha_1, X_2, \ldots, X_m, \gamma_1, \gamma_2, \ldots, \gamma_m)\) as a solution of a (system of) RDE given by (2)-(32)-(33)-(38)-(40)-(39)-(41) driven by \( Z \) which is the lifted Gaussian process \( Z = (W, B) \) (which satisfies Assumption 5.5) in the geometric \( p \)-rough paths. It turns out that, for any \( q > 2 \) large enough, we have estimate [33, Theorem 35-(i) and Corollary 56] 
\[
\mathbb{E} \left[ \|Z\|_{p-\text{var};[a,t]}^q \right] \lesssim_q \kappa_{a,t}.
\]
We remark that we can not immediately apply the estimate (30) since the vector-fields in (32)-(33)-(38)-(40)-(39)-(41) are not bounded with respect to \((J, J^{-1}, \alpha_1, X_2, \ldots, X_m, \gamma_1, \gamma_2, \ldots, \gamma_m)\) but have (as well as their derivatives) polynomial growth.

To proceed, as in [44], we use an induction argument. We first consider \( V = (X, Y) \) where, for any \( 1 \leq k \leq d \),
\[
(Y)^k = \int_a^t \nabla \sigma_k(X_{s}^{x,a})dW_s^k.
\]
Then, \( V \) is the solution of (2)-(44) driven by \( Z \) and, thus, is a geometric \( p \)-rough path (denoted \( V \)). Since the vector-fields are smooth and bounded, we have, by Theorem 5.16,
\[
\|V\|_{p-\text{var};[a,t]} \lesssim_{p,\gamma,\sigma} \|Z\|_{p-\text{var};[a,t]}.
\]

**Remark 5.21.** Note here that when \( x \) is smooth, the definition of \( S_{[p]}(x) \) only involved increments of \( x \). Remark also that when \( W^\epsilon \) is a sequence of smooth paths such that \( S_{[p]}(W^\epsilon) \) converge to \( W \) in the rough path topology, then by Inequality (30),
\[
\sup_x \|S_{[p]}(X^{x,a})\|^p_{p-\text{var};[s,t]} \lesssim \left( \|\sigma\|^\gamma_1 \|S_{[p]}(W^\epsilon)\|^p_{p-\text{var};[s,t]} \right)
\]
and one has, by letting \( \epsilon \to 0 \),
\[
\sup_x \|X^{x,a}\|^p_{p-\text{var};[s,t]} \lesssim \left( \|\sigma\|^\gamma_1 \|W\|_{p-\text{var};[s,t]} \right).
\]

Now, let \( V_1 = (X, J, J^{-1}) \) be a solution of the RDE (2)-(32)-(33) driven by \( V \). This RDE can be solved and is a geometric \( p \)-rough path (denoted \( V_1 \)) that satisfies \([17, 7]\]
\[
\|V_1\|_{p-\text{var};[s,t]} \leq \kappa_1 \|V\|_{p-\text{var};[s,t]} e^{\kappa_2 N_{a,[s,t]}p(V)}.
\]
where \( \kappa_1, \kappa_2 > 0 \) depend on \( \sigma \) and \( N_{a,[s,t]}p(V) \) as finite moments of any order (see [32, 7] for details). Now, we can see that the integrand in (38) as a polynomial growth (as well as its derivatives) with respect to \( V_1 \). Thus, we have the following estimate from [44, Equation (52)] (see also [7] Subsection 4.3)
\[
|X_{1,t}(W, B)| \wedge \|X_{1}(W, B)\|_{p-\text{var};[a,t]} \lesssim (1 + \|V_1\|_{p-\text{var};[a,t]}^r) \|V_1\|_{p-\text{var};[a,t]}
\]
for some \( r > 0 \). This yields, in the end,
\[
|X_{1,t}(W, B)| \lesssim (1 + \|Z\|_{p-\text{var};[a,t]}^r) \|Z\|_{p-\text{var};[a,t]} e^{\kappa_2 N_{a,[a,t]}p(V)},
\]
which leads to the estimate
\[
\mathbb{E} \left[ \sup_x |X_{1,t}(W, B)|^q \right] \lesssim \kappa_{a,t}.
\]
Finally, if we assume that we have, for any \( 1 \leq \ell \leq m - 1 \),
\[
\|X_{\ell}(W, B)\|_{p-\text{var};[a,t]} \lesssim (1 + \|Z\|_{p-\text{var};[a,t]}^r) \|Z\|_{p-\text{var};[a,t]} e^{\kappa_2 N_{a,[a,t]}p(V)},
\]
for some \( r > 0 \), then one can consider the function \( V_{m-1} = (X, J, J^{-1}, X_1, \ldots, X_{m-1}) \) as a geometric \( p \)-rough path driven by \( V \) denoted by \( V_{m-1} \) and, since the integrand in (40) as a polynomial growth (as well as its derivatives), we deduce the estimate
\[
|X_{m,t}(W, B)| \wedge \|X_m(W, B)\|_{p-\text{var};[a,t]} \lesssim (1 + \|V_{m-1}\|_{p-\text{var};[a,t]}^r) \|V_{m-1}\|_{p-\text{var};[a,t]}.
\]
In particular, following the same arguments as for $\mathcal{X}_1$, we obtain that

$$E \left[ \sup_x |\mathcal{X}_{m,t}(W, B)|^q \right] \lesssim \kappa_{a,t}.$$  

We proceed then to obtain estimates on $(\mathcal{Y}_t)_{1 \leq t \leq m}$ by considering iteratively $V_{t+m} = (X, J, J^{-1}, X_1, \ldots, X_m, Y_1, \ldots, Y_t)$ taken as a geometric $p$-rough path driven by $V$ and denoted $V_{t+m}$. By considering $(39)-(41)$, we deduce that, for any $0 \leq t \leq m - 1$,

$$|\mathcal{Y}_{t+1}(W, B)| \wedge \|\mathcal{Y}_{t+1}(W, B)\|_{p \text{-var}; [a, t]} \lesssim (1 + \|V_{t+m}\|_{p \text{-var}; [a, t]})^r \|V_{t+m}\|_{p \text{-var}; [a, t]},$$

which leads to the estimate

$$E \left[ \sup_x |\mathcal{Y}_{m,t}(W, B)|^q \right] \lesssim \kappa_{a,t}.$$  

For the following result, we need a notation. Let $F : x \mapsto F(x) \in \mathbb{D}^{m,p}$ Let us define

$$\|F\|_{s, \mathbb{D}^{m,p}} = \left( E[\sup_{x \in \mathbb{R}^d} |F(x)|^p] + \sum_{k=1}^m E[\sup_{x \in \mathbb{R}^d} \|D^k F(x)\|_{\mathcal{H}^k}^p] \right)^{1/p}.$$  

What we just have proved is that

**Proposition 5.22.** We have the estimates, for any $m \geq 1$ and $p \geq 2$,

$$\|DX_t^{x,a}\|_{s, \mathbb{D}^{m,p}(H([a,b]))} \lesssim \kappa_{a,t} \quad \text{and} \quad \|(J_t^{x,a})^{-1}\|_{s, \mathbb{D}^{m,p}(H([a,b]))} \lesssim \kappa_{a,t}.$$  

**Remark 5.23.** Let us remark that if if we denote by $J^0 = J$ and $J^m = D_m J^{k-1}$, where $D_x$ stands for the derivative with respect to $x$. We have some constants such that following equation:

$$J^m_t = \sum_{\ell \in \{2, \ldots, m\}} \sum_{a \in \{1, \ldots, m-1\}} c_{\ell, a} \sum_{k=1}^d D^k \sigma_k(X_t^{x,a}) J_{t}^{a_1} \cdots J_{t}^{a_{\ell}} dW_t + \sum_{k=1}^d \int_a^t \nabla \sigma_k(X_r^{x,a}) J_{r}^{m} dW_r.$$  

By using exactly the same strategy as before, one can also prove that

$$\|J^m_t\|_{s, \mathbb{D}^{k,a}(H([a,b]))} \lesssim \kappa_{a,t}.$$  

Indeed one has just to reuse Equation $(41)$ in this situation.

We now turn to the covariance matrix $\gamma_{X_t^{x,a}, [a,b]}$ which is expressed, following Definition 4.14, as

$$\gamma_{X_t^{x,a}, [a,b]} = \left( (DX_t^{x,a}, DX_t^{x,a}) H([a,b]) \right)_{(i,j) \in \{1, \ldots, d\}^2}.$$  

We remark that $\gamma_{X_t^{x,a}, [a,b]}$ is a symmetric definite positive matrix. Furthermore, thanks to $(26)$, $(36)$ and $(34)$, we deduce the expression

$$\gamma_{X_t^{x,a}, [a,b]} = (DX_t^{x,a}, DX_t^{x,a})_{H([a,b])} = (DX_t^{x,a}, (DX_t^{x,a})^*)_{H([a,t])} = \int_a^t \int_a^t J_t^{x,s} \sigma(X_t^{x,a}) \sigma(X_t^{x,a})^* (J_t^{x,s})^* dR_W(s_1, s_2).$$  

**Proposition 5.24.** Under the uniform ellipticity condition

$$\|\sigma(x)z\|^2 \geq \varsigma \|z\|^2, \quad \forall z, x \in \mathbb{R}^d,$$

for some constant $\varsigma > 0$, we have the estimate, for any $m \geq 1$ and $p \geq 2$,

$$\|\gamma_{X_t^{x,a}, [a,b]}^{-1}\|_{s, \mathbb{D}^{m,p}(H([a,b]))} \lesssim \max(1, \vartheta_{a,t})^m \frac{\varsigma^{1/2}}{\sigma_{a,t}},$$

where $\vartheta_{a,t} := \kappa_{a,t}/\sigma_{a,t}$.  

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Proof. We follow the arguments from [44, Proposition 3.9]. We start with the estimate in the case \( m = 0 \). Since \( \gamma_{x,a}^{-1} [a,b] \) is symmetric definite positive and all the matrix norms are equivalent, an upper bound on \( \| \gamma_{x,a}^{-1} [a,b] \| \) can be deduced by estimating its lowest eigenvalue. In order to do so, we will identify a positive random variable \( \varrho_{a,t} \) admitting negative moments of any order such that, for any \( z \in \mathbb{R}^d \),

\[
zs' \gamma_{x,a}^{-1} [a,b] z \geq \varrho_{a,t} \| z \|^2.
\]

Thanks to (48) and by denoting \( f_s z, x, a, t = \sigma(X_s x, a)^* (J_t z, x, a)^* z, \) we obtain

\[
z^* \gamma_{x,a}^{-1} [a,b] z = \int_a^t \int_a^t z^* J_t^{s,1} \sigma(X_{s_{1}} x, a)^* \sigma(X_{s_{2}} x, a)^* J_t^{s,2} z \, dR_W(s_{1}, s_{2})
= \int_a^t \int_a^t (f_{s_{1}} z, x, a, t)^* f_{s_{2}} z, x, a, t \, dR_W(s_{1}, s_{2}) = \| f_s z, x, a, t \|^2_{H([a,b])}.
\]

By Corollary 5.13, we furthermore deduce that

\[
\| f_s z, x, a, t \|^2_{H([a,b])} \geq \frac{\sigma_{a,t}^2 \| f_s z, x, a, t \|^2_{L^2([a,b], \mathcal{A})}}{4} \min \left( 1, \frac{4(cW/2)^{2(\gamma+\alpha)/\gamma}}{\sigma_{a,t}^2} \int |f_s z, x, a, t|^2 \, d\gamma_{x,a}([a,b]) \right).
\]

Also, thanks to the uniform ellipticity condition, we obtain

\[
\| f_s z, x, a, t \|^2 \leq c \| (J_t^{x,s})^{-1} z \|^2 \leq c \| J_t^{x,s} \|^2 \| z \|^2,
\]

which, since \( \sup_{s \in [a,t]} \| J_t^{x,s} \|^2 \| \| \leq \| \text{Id} \|^2 = 1 \), writes as

\[
\| f_s z, x, a, t \|^2 \| \gamma_{x,a}([a,b]) \| \leq \sqrt{c} \| z \|.
\]

Moreover, we have

\[
|f_s z, x, a, t| \leq \| \sigma(X_s x, a)^* (J_t^{x,s})^* \gamma_{x,a}([a,b]) \| \| z \| = \| J_t^{x,s} \sigma(X_s x, a) \| \gamma_{x,a}([a,b]) \| \| z \|.
\]

This leads to the estimate

\[
zs' \gamma_{x,a}^{-1} [a,b] z \geq \frac{\sigma_{a,t}^2 \| f_s z, x, a, t \|^2_{L^2([a,b], \mathcal{A})}}{4} \min \left( 1, \frac{4(cW/2)^{2(\gamma+\alpha)/\gamma}}{\sigma_{a,t}^2} \int |f_s z, x, a, t|^2 \, d\gamma_{x,a}([a,b]) \right) \| z \|^2,
\]

and we identify

\[
\varrho_{a,t} = \frac{\sigma_{a,t}^2 \| f_s z, x, a, t \|^2_{L^2([a,b], \mathcal{A})}}{4} \min \left( 1, \frac{4(cW/2)^{2(\gamma+\alpha)/\gamma}}{\sigma_{a,t}^2} \int |f_s z, x, a, t|^2 \, d\gamma_{x,a}([a,b]) \right).
\]

We now have to prove that \( E[\sup_{x \in \mathbb{R}^d} \varrho_{a,t}] < +\infty \). We can see that

\[
\varrho_{a,t} \leq \frac{4}{\sigma_{a,t}^2} \sup_{x \in \mathbb{R}^d} \left( \min \left( 1, \frac{4(cW/2)^{2(\gamma+\alpha)/\gamma}}{\sigma_{a,t}^2} \int |f_s z, x, a, t|^2 \, d\gamma_{x,a}([a,b]) \right) \right),
\]

and, thus, we essentially have to see that \( \sup_{x \in \mathbb{R}^d} |J_t^{x,s} \sigma(X_s x, a) \| \gamma_{x,a}([a,b]) \| \) admits moments of any order. We have

\[
\sup_{x \in \mathbb{R}^d} |J_t^{x,s} \sigma(X_s x, a) \| \gamma_{x,a}([a,b]) \| \leq \| \sigma \|_{C^0} \sup_{x \in \mathbb{R}^d} |J_t^{x,s} \| \gamma_{x,a}([a,b]) \| + \sup_{s \in [a,t], x \in \mathbb{R}^d} |J_t^{x,s} | \sup_{x \in \mathbb{R}^d} |\sigma(X_s x, a) \| \gamma_{x,a}([a,b]) \|.
\]

From here, it is rather direct to prove that, for any \( p \in [2, +\infty) \),

\[
E \left( \sup_{x \in \mathbb{R}^d} |\sigma(X_s x, a) \| \gamma_{x,a}([a,b]) \|^p \right) \leq \| \sigma \|_{C^0} E \left( \sup_{x \in \mathbb{R}^d} |X_s x, a \| \gamma_{x,a}([a,b]) \|^p \right) < +\infty,
\]

thanks to (31) and Proposition 5.15. We use Equation (37) to handle the Jacobian. In the end, we obtain the bound

\[
\frac{\varrho_{a,t}^p}{\sigma_{a,t}^{2p}} \leq \frac{\varrho_{a,t}^p}{\sigma_{a,t}^{2p}}.
\]
which provides the desired result in the case $m = 0$. In the case $m \geq 1$, we use Lemma 4.15 to deduce that, for any $(i, j) \in \{1, \ldots, d\}^2$,

$$D(\gamma^{-1}_{X_t^r,a,[a,b]} i, j) = - \sum_{\ell_1, \ell_2=1}^{d} (\gamma^{-1}_{X_t^r,a,[a,b]} i, \ell_1) D(\gamma_{X_t^r,a,[a,b]} \ell_1, j)$$

which yields, thanks to Leibniz’s rule, for any $m \geq 1$,

$$D^m(\gamma^{-1}_{X_t^r,a,[a,b]} i, j) = - \sum_{\ell_1, \ell_2=1}^{d} \sum_{k_1, k_2, k_3 \in \mathbb{N}} \binom{m-1}{k_1, k_2, k_3} D^{k_1}(\gamma^{-1}_{X_t^r,a,[a,b]} i, j, \ell_1) D^{k_2+1}(\gamma_{X_t^r,a,[a,b]} \ell_1, \ell_2) D^{k_3}(\gamma^{-1}_{X_t^r,a,[a,b]} \ell_2, j). \quad (49)$$

From here, we can see that, thanks to (46),

$$D^{k_2+1}(\gamma_{X_t^r,a,[a,b]} \ell_1, \ell_2) = \sum_{k=0}^{k_2+1} \binom{k_2+1}{k} (D^{k_2+1}X_t^{\gamma, a}, (D^{k_2+1-k}X_t^{\gamma, a})^p)_{\mathcal{H}((a,b))},$$

which provides the estimate, thanks to Proposition 5.22 and Hölder’s inequality, for any $p \in [2, +\infty)$,

$$\mathbb{E} \left[ \sup_{x \in \mathbb{R}^d} \|D^{k_2+1}(\gamma_{X_t^r,a,[a,b]} \ell_1, \ell_2)_{\mathcal{H}((a,b))} \|^p \right]^{1/p} \leq \sum_{k=0}^{k_2+1} \binom{k_2+1}{k} \mathbb{E} \left[ \sup_{x \in \mathbb{R}^d} \|D^{k_2+1}X_t^{\gamma, a} \|^p_{\mathcal{H}((a,b))} \right]^{1/p_1} \times \mathbb{E} \left[ \sup_{x \in \mathbb{R}^d} \|D^{k_2+1-k}X_t^{\gamma, a} \|^p_{\mathcal{H}((a,b))} \right]^{1/p_2} \leq \kappa_{a,t}, \quad (50)$$

where $1/p = 1/p_1 + 1/p_2$. Proceeding by induction, we assume that, for any $p \in [2, +\infty)$,

$$\mathbb{E} \left[ \sup_{x \in \mathbb{R}^d} \|D^{m-1}(\gamma^{-1}_{X_t^r,a,[a,b]} \|^p_{\mathcal{H}((a,b))} \right]^{1/p} \leq \max(1, \Theta_{a,t})^{m-1} \sigma_{a,t}^{2(m-1)}.$$

From (49), (50) and Hölder’s inequality, we deduce that, for $1/p = 1/p_1 + 1/p_2 + 1/p_3$,

$$\mathbb{E} \left[ \sup_{x \in \mathbb{R}^d} \|D^m(\gamma^{-1}_{X_t^r,a,[a,b]} \|^p_{\mathcal{H}((a,b))} \right]^{1/p} \leq \sum_{k_1, k_2, k_3 \in \mathbb{N}} \binom{m-1}{k_1, k_2, k_3} \mathbb{E} \left[ \sup_{x \in \mathbb{R}^d} \|D^{k_1}(\gamma^{-1}_{X_t^r,a,[a,b]} \|^p_{\mathcal{H}((a,b))} \right]^{1/p_1} \times \mathbb{E} \left[ \sup_{x \in \mathbb{R}^d} \|D^{k_2+1}(\gamma_{X_t^r,a,[a,b]} \|^p_{\mathcal{H}((a,b))} \right]^{1/p_2} \mathbb{E} \left[ \sup_{x \in \mathbb{R}^d} \|D^{k_3}(\gamma^{-1}_{X_t^r,a,[a,b]} \|^p_{\mathcal{H}((a,b))} \right]^{1/p_3} \leq \max(1, \Theta_{a,t})^{m} \sigma_{a,t}^{2m},$$

which concludes our proof.\[\square\]
6. Proof of the main theorem

**Theorem 6.1.** Let $W$ be a Gaussian process which satisfies Assumption 5.5, 5.9 and 5.11. Let $2 < p < 4$ such that $W$ can be lift into a geometric $p$ rough path (from assumption (5.5)). Furthermore let $a$ its index of non-determinism from Assumption 5.11.

Let $\sigma \in C^\infty_b(\mathbb{R}^d; \mathbb{R}^{d \times d})$ be such there exists $\zeta > 0$ that for all $x, z \in \mathbb{R}^d$

$$|\sigma(x)z|^2 \geq \zeta |z|^2.$$ 

Let $b \in C^\infty(\mathbb{R}^d$ with $\kappa + \frac{1}{\alpha} > \frac{1}{2}$. Then, almost surely, there exists a solution flow to the equation

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t, \quad X_0 = x, \quad t \in [0, T].$$

Furthermore, this flow is locally Lipschitz continuous.

**Proof.** As seen in Section 2 and especially in Theorem 2.10, one only has to check that there exists $\nu > \frac{1}{2}$, $\kappa > 2 - \nu$ and a weight $w_0$ such that almost surely

$$T^{D\varphi, b} \in C^\nu_T w_0.$$

Furthermore, thanks to Section 3 and especially Corollary 3.6, it is enough to check that for all $i, j \in \{1, \cdots, d\}$, we have the following couples

$$(J^{-1}, X), (\partial_i J^{-1}, X), (\partial_i X \partial_j X^{-1}, X), (\partial_j \partial_i X \partial_j X^{-1}, X), \partial_j \partial_i X \partial_j X^{-1}, X$$

which satisfy Assumption 3.2 for some $H \in (0, 1)$. Remark that thanks to the flow properties, for $s \in [0, T)$ one can always consider that for $t \in [s, T]$,

$$\phi(\cdot, x) \in \{J^{x,s} \}_{i, j}^{-1}, \partial_i (J^{x,s} \}_{i, j}^{-1}, \partial_i X \partial_i X \partial_j X^{-1}, \partial_j \partial_i X \partial_j X^{-1}, \partial_j X \partial_i X \partial_j X^{-1}\}$$

and

$$\varphi(\cdot, x) = X^{x, s}.$$

Thanks to Proposition 4.17, for all $t \in [0, T]$, we take $F = X_t^{x, s}$ and $G = \phi(t, x)$ as previously. Remark that $F, G \in \mathcal{F}_t$. We have for all $\beta \in \mathbb{N}^d$ and all $f \in S$,

$$\mathbb{E}[[\partial^\beta f(F)G]_{\mathcal{F}_s}] = \mathbb{E}[[f(F)H_{\beta, [s,t]}(F, G)]_{\mathcal{F}_0, s}].$$

and for all $q \geq 0$,

$$|\mathbb{E}[\partial^\beta f(F)G]_{\mathcal{F}_s}| = |\mathbb{E}[[f(F)H_{\beta, [s,t]}(F, G)]_{\mathcal{F}_0, s}]| \leq \mathbb{E}[[f(S)]_{\mathcal{F}_s}]^\frac{q}{2} \mathbb{E}[[H_{\beta, [s,t]}(F, G)]^2_{\mathcal{F}_s}]^\frac{q}{2} \leq \mathbb{E}[[f(S)]_{\mathcal{F}_s}]^\frac{q}{2}.$$

Note that for $q \geq 2$, thanks to Lemma 4.18,

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}[H_{\beta, [s,t]}(F, G)]_{\mathcal{F}_s}^{\frac{q}{2}} \mathbb{E}[[f(S)]_{\mathcal{F}_s}]^\frac{q}{2} \leq \sup_{x \in \mathbb{R}^d} [H_{\beta, [s,t]}(F, G)]_{\mathcal{F}_s}^{\frac{q}{2}} \mathbb{E}[[f(S)]_{\mathcal{F}_s}]^\frac{q}{2} \leq \mathbb{E}[[f(S)]_{\mathcal{F}_s}]^\frac{q}{2}$$

with $\frac{q}{2} = \frac{1}{r} + \sum_{i} \frac{1}{r_i}$. Furthermore, thanks to Proposition 5.22 and Remark 5.23, we know that for every $\beta \in \mathbb{N}^d$ and every $r \geq 1$

$$||G||_{s, D_{\beta}}^\frac{q}{2} \leq \kappa_{s, t} = \mathbb{E}[[R_{\{1, \cdots, q\}}]_{\mathcal{F}_s}^{\frac{1}{2}} \leq 1.$$ 

Furthermore, thanks to Propositions 5.24 5.22 and Hölder inequality, , we know that for all $\ell \leq |\beta|$ and all $r \geq 2$,

$$\|\gamma^{-1}_{X^{s,t}, [a, b]}(D(X^{s,t})^2)\|_{s, D_{\beta - \ell - 1, r}} \leq \frac{1}{\sigma_{a,b}}.$$

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and finally
\[
\| \sup_{x \in \mathbb{R}^d} \mathbb{E}[H_{b,[s,t]}(F,G)^2 | \mathcal{F}_s]^{\frac{1}{2}} \|_{L^4(\Omega)} \lesssim \sigma_{s,t}^{-|\beta|}.
\]
Furthermore, note that since \( W \) is centered and has the local non-determinism property, we know that
\[
\sigma^2_{s,t} = \mathbb{E}[(W_t - W_s)^2] = \mathbb{E} \left[ \mathbb{E}[(W_t - W_s)^2 | \mathcal{F}_{[0,s]} \lor \mathcal{F}_t] \right] \gtrsim |t - s|^{\alpha},
\]
where \( \alpha \) is the index of non-determinism. Altogether, it gives
\[
\| \mathbb{E}[\partial^\beta f(F,G) | \mathcal{F}_s] \| \lesssim \| f \|_\infty |t - s|^{-\frac{\beta}{2}} G_{\infty,s},
\]
where
\[
G_{\infty,s} = (t - s)^{\frac{2}{\alpha}} \sup_{x \in \mathbb{R}^d} \mathbb{E}[H_{b,[s,t]}(F,G)^2 | \mathcal{F}_s]^{\frac{1}{2}}.
\]
-enjoys finite \( L^q(\Omega) \) moments which are bounded uniformly in \((s, t) \in \Delta_T^2\).

Hence, using Corollary 3.6, for all \( \kappa \in \mathbb{R} \) with \( 0 < \kappa + \frac{1}{\alpha} \leq 2 \) there exists \( \varepsilon > 0 \) small enough such that for all \( q \geq \frac{8}{\varepsilon} \lor d \) and for \( w_0(x) = (1 + |x|) \) there exists a positive random variable \( K(b) = K(b, \sigma, W) \) with \( K(b) \in L^q(\Omega) \) and such that
\[
\| T^{D_{\kappa,\sigma}b} \|_{C^\kappa + \frac{1}{\alpha} + \frac{1}{d} - \frac{8}{\varepsilon}} \leq K(b) \| b \|_{C^\kappa}.
\]
Hence, for \( \kappa + \frac{1}{\alpha} - \frac{8}{\varepsilon} + \frac{2}{d} + \frac{\varepsilon}{8} > 1 \), namely for
\[
\kappa > \frac{3}{2} + \varepsilon + \frac{\varepsilon}{4}.
\]
thanks to Theorem 2.10, there exists a unique solution, locally Lipschitz in the initial condition to Equation (4).

Finally we give a (standard) example that satisfies the conditions which are required for uniqueness of the solutions. We refer to [16] and [34] for the proofs of the following propositions.

**Proposition 6.2.** Let \( H \in (0,1) \). Let \( B, \tilde{B} \) be two independent \( d \)-dimensional Brownian motions. We define the \( d \)-dimensional fractional Brownian motion of Hurst parameter \( H \) as the process defined for all \( t \geq 0 \) by
\[
B^H_t = \frac{1}{c_H} \left( \int_0^t (t - r)^{H - \frac{1}{2}} dB_r + \int_0^{+\infty} (t + r)^{H - \frac{1}{2}} - r^{H - \frac{1}{2}} d\tilde{B}_r \right),
\]
where
\[
c_H = \sqrt{\frac{1}{2H} + \int_0^{+\infty} ((t + r)^{H - \frac{1}{2}} - r^{H - \frac{1}{2}})^2 dr}.
\]
The fractional Brownian motion is a centered continuous Gaussian process with stationary increments and covariance
\[
R_{B^H}(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).
\]
For \( \frac{3}{4} < H < \frac{1}{2} \) it satisfies Assumption 5.5 with \( \rho = \frac{1}{2H} \), Assumption 5.9 and Assumption 5.11 with \( \alpha = 2H \).

**Corollary 6.3.** Let \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \). Let \( B^H \) be a \( d \)-dimensional fractional Brownian motion and let \( B^H \) the geometric rough path above \( B^H \).

Let \( b \in C^\kappa \quad with \quad \kappa > 0 \lor \left( \frac{3}{2} - \frac{1}{2H} \right) \).

Let \( \sigma \in C^\infty_b(\mathbb{R}^d; \mathbb{R}^{d \times d}) \) which satisfies the strong ellipticity condition there exists \( c > 0 \) such that :
\[
|\sigma(x)z|^2 \geq c|z|^2, \quad x, z \in \mathbb{R}^d.
\]
Then, there exists $\mathcal{N} = \mathcal{N}(b) \in \mathcal{F}$ such that $P(\mathcal{N}) = 0$ and for all $\omega \notin \mathcal{N}$ and for all $x \in \mathbb{R}^d$, 
\[ dx_t = b(x_t) \, dt + \sigma(x_t) \, dB_t^H, \quad x_0 = x, \quad t \in [0, T] \]

admits a unique solution. Furthermore, for all $R > 0$ and all all $b, \tilde{b}$ which satisfies the above conditions and all $x_0, \tilde{x}_0 \in \mathbb{R}^d$ with $|x_0|, |\tilde{x}_0| \leq R$, there exists an almost surely positive and finite random variable $K = K(B^H, \sigma, b, \tilde{b}, R, T)$ such that 
\[ \sup_{t \in [0,T]} |x_t - \tilde{x}_t| \leq K(|x - \tilde{x}| + |b - \tilde{b}|_{C^\infty}), \]
where $x$ (respectively $\tilde{x}$) is the solution of the rough differential equation driven by the fractional Brownian motion with initial value $x_0$ (respectively $\tilde{x}_0$) and coefficient $b$ and $\sigma$ (respectively $\tilde{b}$ and $\sigma$).

**Proof.** One only has to use Theorem 6.1 and Proposition 6.2 and Theorem 2.10 to conclude. \hfill $\square$

**Remark 6.4.** Let us remark that we where not able to prove the usual continuity of the Itô map solution with respect to the driven rough path. Indeed, whenever $B^{\varepsilon,H}$ is a smooth (say piecewise linear) approximation of the fractional Brownian motion, one does not necessarily have Proposition 5.24. In the same manner, the continuity with respect to the coefficient $\sigma$ is neither clear, since it appears in a deeply non-linear fashion manner in the averaged field.

### Appendix A. Besov spaces

We first recall some of the Littlewood-Paley theory that is used in Besov spaces. There exist $\psi, \phi$ two functions valued in $[0, 1]$ such that

1. $\psi, \phi \in C^\infty_c(\mathbb{R}^d)$ with supp$(\psi) \subset B(0, 3/4)$ and supp$(\phi) \subset A$, where $A = \{ \xi \in \mathbb{R}^d; 3/4 \leq |\xi| \leq 8/3 \}$ is an annulus in $\mathbb{R}^d$,
2. $\forall \xi \in \mathbb{R}^d, \; \psi(\xi) + \sum_{j=0}^{\infty} \phi(2^{-j} \xi) = 1$,
3. supp$(\psi) \cap$ supp$(\phi(2^{-j} \cdot)) = \emptyset$ for any $j \geq 1$,
4. supp$(\phi(2^{-j} \cdot)) \cap$ supp$(\phi(2^{-k} \cdot)) = \emptyset$ for any $j, k \geq 0$ such that $|j - k| \geq 2$.

We define the associated Littlewood-Paley blocks as
\[ \Delta_{j-1} u = \mathcal{F}^{-1} (\psi \mathcal{F}(u)) \quad \text{and} \quad \Delta_j u = \mathcal{F}^{-1} (\phi(2^{-j} \cdot) \mathcal{F}(u)), \quad \forall j \geq 0, \tag{51} \]
for any $u \in S'$. We recall the Bernstein Lemmas in this context.

**Lemme A.1.** Let $k \in \mathbb{N}$. There exists a constant $C > 0$ such that, for all $1 \leq q \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^d)$, we have

1) $\sup_{|\alpha| = k} \| \partial^\alpha \Delta_j f \|_{L^q(\mathbb{R}^d)} \leq C^{k+1} 2^{j(k+1/q)} \| \Delta_j f \|_{L^p(\mathbb{R}^d)}$, for all $j \in \mathbb{N}$ \bigcup \{-1\},
2) $C^{-k-1} 2^{jk} \| \Delta_j f \|_{L^p(\mathbb{R}^d)} \leq \sup_{|\alpha| = k} \| \partial^\alpha \Delta_j f \|_{L^p(\mathbb{R}^d)} \leq C^{k+1} 2^{jk} \| \Delta_j f \|_{L^p(\mathbb{R}^d)}$, for all $j \in \mathbb{N}$.

The Besov spaces associated to the Littlewood-Paley blocks are defined as
\[ B^\alpha_{p,r} := \left\{ u \in S'(\mathbb{R}^d); \| u \|_{B^\alpha_{p,r}} := \left( \sum_{j=-\infty}^{+\infty} 2^{r \alpha j} \| \Delta_j u \|_{L^p(\mathbb{R}^d)}^r \right)^{1/r} < \infty \right\}, \]
where $s \in \mathbb{R}$ and $p, r \in [1, \infty]$. Those spaces are Banach spaces continuously embedded in $S'$ and they are of type $p \wedge r \wedge 2$ (see Corollary C.10 below) when $p, r \in (1, +\infty)$.

**Remark A.2.**

1) We have the following continuous embeddings
   a) for $\tilde{\alpha} < \alpha$ or $\tilde{\alpha} = \alpha$ and $\tilde{r} \geq r$
   \[ B^\alpha_{p,r} \hookrightarrow B^{\tilde{\alpha}}_{\tilde{p},\tilde{r}}, \]
   b) for $\tilde{p} \geq p$,
   \[ B^\alpha_{p,r} \hookrightarrow B^{\alpha - d(1/p - 1/)\tilde{p}}_{\tilde{p},r}, \]

   \[ \frac{43}{43} \]
c) for $p < \infty$, 
\[ B^{d/p}_{p,1} \to C^0 \]
where $C$ is the space of uniformly continuous bounded functions.

2) For $p = r = \infty$ and $s \in \mathbb{R}^+/\mathbb{N}$, we have that 
\[ \|u\|_{B^s_{\infty,\infty}} \cong \sum_{|\alpha| \leq |s|} \left( \|\partial^\alpha u\|_{L^\infty(\mathbb{R}^d)} + \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x - y|^{s - |\alpha|}} \right). \]

Thus, the spaces $B^s_{\infty,\infty}$ consist of $\alpha$-Hölder functions and are called Hölder-Besov spaces.

3) For $p = r$, we have that 
\[ \|u\|_{B^s_{p,p}} \cong \|u\|_{H^s_p(\mathbb{R}^d)}, \]
where 
\[ \|u\|_{H_p^s(\mathbb{R}^d)} = \left\| (1 - \Delta)^{s/2} u \right\|_{L^p(\mathbb{R}^d)} \]
and, thus, we recover the usual Sobolev spaces.

4) Let $k \in \mathbb{N}$. It follows from Bernstein’s lemma that, for any $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leq k$, 
\[ \|\partial^\alpha u\|_{B^k_{p,r}} \lesssim \|u\|_{B^{k+k}_{p,r}}. \]

It turns out that the space $C^\infty_{\infty,0}(\mathbb{R}^d)$ is dense in $B^s_{p,r}$ if and only if $p, r < \infty$. If we wish to build approximations of functions in the Hölder-Besov spaces, we can rely on the following result.

**Lemma A.3.** Let $s \in \mathbb{R}$ and $u \in B^s_{\infty,\infty}$. Then, the sequence $(u_n)_{n \geq 0}$ defined by 
\[ u_n = \sum_{q = -1}^{n-1} \Delta_q u, \]
converges to $u$ in $B^s_{\infty,\infty}$ for any $s < s$. Moreover, we have, $\forall n \in \mathbb{N}$ and $\forall \ell \in \mathbb{R}$, 
\[ \|u_n\|_{B^\ell_{\infty,\infty}} < +\infty. \]

Finally, we recall the following results for pointwise product of functions in Besov spaces.

**Lemma A.4.** Let $p \in [1, \infty], r_1, r_2 \in [1, \infty]$ and $s_1, s_2 \in \mathbb{R}^+$ such that 
\[ s_1 + s_2 > 0 \quad \text{and} \quad \max(s_1, s_2) < \frac{d}{p}. \]
Then, for any $u \in B^{s_1}_{p,r_1}$ and $v \in B^{s_2}_{p,r_2}$, we have 
\[ \|uv\|_{B^s_{p,r}} \lesssim \|u\|_{B^{s_1}_{p,r_1}} \|v\|_{B^{s_2}_{p,r_2}}, \] (52)
where 
\[ s = s_1 + s_2 - \frac{d}{p} \quad \text{and} \quad \frac{1}{r} = \max \left( 1, \frac{1}{r_1} + \frac{1}{r_2} \right). \]

We give a result concerning Fourier multiplier.

**Lemma A.5.** Let $m : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that there, $\gamma \in \mathbb{N}^d$ and $p \geq 0$, 
\[ \partial^\gamma m(\xi) \leq |\xi|^{-|\gamma| - p}, \quad \xi \in \mathbb{R}^d \setminus \{0\}, \]
and, for any $j$, define the function $m_j$ given by 
\[ m_j = \mathcal{F}^{-1}(m(\xi) \varphi(2^{-j} \xi)), \]
where $\varphi$ is a compactly supported function such that $\text{supp}(\varphi) \subset 2\mathcal{A}$ and $\varphi|\mathcal{A} \equiv 1$. Then, we have 
\[ \|m_j\|_{L^1(\mathbb{R}^d)} \lesssim_{\varepsilon,d} 2^{-jp}. \]
Proof. It turns out that, by a change of variable,  
\[ \|m_j\|_{L^1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}(m(2^j \xi)\varphi(\xi))\|_{L^1(\mathbb{R}^d)}. \]
Furthermore, we have, by integration by parts and Leibniz’s formula,  
\[
\mathcal{F}^{-1}(m(2^j \xi)\varphi(\xi))(x) = (1 + |x|^2)^{-d}(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi \cdot x}(1 - \Delta)^{d}(m(2^j \xi)\varphi(\xi)) \, d\xi \\
= (1 + |x|^2)^{-d}(2\pi)^{-d/2} \sum_{\alpha, \beta = 1}^{\|n\|^{2d}} C_{\alpha, \beta} 2^{j|\alpha|} \int_{\mathbb{R}^d} e^{i\xi \cdot x} \partial_\xi^\alpha m(2^j \xi)\varphi_{\alpha, \beta} \, d\xi.
\]
We can see that, by (53)  
\[
\left| \int_{\mathbb{R}^d} e^{i\xi \cdot x} \partial_\xi^\alpha m(2^j \xi)\varphi_{\alpha, \beta} \, d\xi \right| \lesssim 2^{-j|\alpha| - j\beta},
\]
which leads to the estimate  
\[ \|m_j\|_{L^1(\mathbb{R}^d)} \lesssim 2^{-j\beta} \]
\[ \square \]

**Appendix B. Young Inequality**

Let us recall the following Young inequality for Kernel (See [67], Theorem 0.3.1).

**Theorem B.1.** Let \( K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be a measurable function such that for some \( p \in [1, +\infty] \),  
\[ \sup_{x \in \mathbb{R}^d} \|K(x, \cdot)\|_{L^p(\mathbb{R}^d)} \leq C \quad \text{and} \quad \sup_{y \in \mathbb{R}^d} \|K(\cdot, y)\|_{L^p(\mathbb{R}^d)} \leq C. \]

Then \( p, q, r \in [1, +\infty] \) with \( \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \) and for all \( f \in L^q(\mathbb{R}^d) \), we have  
\[
\left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} K(x, y)f(y) \, dy \right)^r \, dx \right)^{\frac{1}{r}} \leq C \|f\|_{L^p(\mathbb{R}^d)}. 
\]

**Appendix C. Burkholder Davis Gundy Inequality in Lebesgue Spaces**

In this section, we recall some standard results about martingales in Banach spaces, and the corresponding inequalities. We refer to Pisier [62] chapter 4 and to Pinelis [61] and Veraar and coauthors [49, 50] for proofs.

The aim of this section is to give an almost self-contain proof of Burkholder-Davis-Gundy inequality in infinite dimensional Banach spaces, especially in \( L^p(\mathbb{R}^d) \) spaces. The idea of the proof is to use Kahane-Khinchine inequality, and UMD properties via smoothness of spaces.

**C.1. Discrete Fourier Transform and Kahane-Khinchine Inequality.** We recall some of standard feature about the Fourier transform on the hypercube.

Let \( \Omega_n = \{-1, 1\}^n \). We endowed \( \Omega_n \) with the uniform measure \( \mu_n \), such that for a function \( f : \Omega_n \to \mathbb{R} \) we have  
\[
\int_{\Omega_n} f \, d\mu_n = \mathbb{E}[f(X)] = \frac{1}{2^n} \sum_{x \in \Omega_n} f(x)
\]
where \( X = (X_1, \ldots, X_n) \) is a vector of i.i.d. random variables such that \( \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2} \).

Remark that for any \( Y \in L^2(\Omega_n, \mathcal{P}(\Omega_n), \mu_n) \) a unique function \( f : \Omega_n \to \mathbb{R} \) exists such that \( Y = f(X) \). Hence \( L^2(\Omega_n, \mathcal{P}(\Omega_n), \mu_n) \) is a finite dimensional vector space of dimension \( 2^n \). For \( Y_1 = f_1(X), Y_2 = f_2(X) \in L^2(\Omega_n, \mathcal{P}(\Omega_n), \mu_n) \) let us write the standard inner product as  
\[
\langle Y_1, Y_2 \rangle = \mathbb{E}[f_1(X)f_2(X)].
\]

Furthermore let us define for \( S \subset \{1, \ldots, n\} \), we define  
\[
X_S = \prod_{i \in S} X_i, \quad X_\emptyset = 1.
\]
Remark that \( \langle X_S, X_S \rangle = 1 \) and for \( S' \neq S \) we have \( \langle X_S, X_{S'} \rangle = 0 \). Hence \( X_S \) is a orthogonal basis of \( L^2(\Omega_n, \mathcal{P}(\Omega_n), \mu_n) \). Furthermore for \( Y \in L^2(\Omega_n, \mathcal{P}(\Omega_n), \mu_n) \) and \( S \subset \{1, \ldots, n\} \), we define
\[
\hat{Y}(S) = \langle Y, X_S \rangle = \mathbb{E}[Y X_S],
\]
such that
\[
Y = \sum_{S \subset \{1, \ldots, n\}} \hat{Y}(S) X_S,
\]
and
\[
\mathbb{E}[YZ] = \sum_{S \subset \{1, \ldots, n\}} \hat{Y}(S) \hat{Z}(S).
\]

Note that we have the two following identities:
\[
\mathbb{E}[Y^2] = \sum_{S \subset \{1, \ldots, n\}} \hat{Y}(S)^2
\]
and
\[
\mathbb{E}[Y] = \hat{Y}(\emptyset).
\]

**Theorem C.1** (Khinchine-Kahane Inequality, via Szarek’s proof, [12] Theorem 5.20). Let \((B, |\cdot|_B)\) be a normed vector space. Then for any \( p \in [1, +\infty] \), there exists a constant \( C_p \) such that for any \( n \geq 2 \) and for any \( b_1, \ldots, b_n \in B \), we have
\[
\mathbb{E} \left[ \left| \sum_{i=1}^n b_i X_i \right|^p_B \right] \leq C_p \mathbb{E} \left[ \left| \sum_{i=1}^n b_i X_i \right|_B^{k+1} \right]^{\frac{p}{k+1}},
\]
where \( X = (X_1, \ldots, X_n) \) is an iid random vector with \( \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2} \).

**Proof.** Let \( k \geq 1 \) such that \( k \leq p < k + 1 \).
\[
\mathbb{E} \left[ \left| \sum_{i=1}^n b_i X_i \right|^{\frac{p}{k+1}}_B \right] \leq \mathbb{E} \left[ \left| \sum_{i=1}^{n-k} b_i X_i \right|_B^{k+1} \right] \leq C_{k+1} \mathbb{E} \left[ \left| \sum_{i=1}^n b_i X_i \right|_B \right].
\]

It is enough to prove the Theorem for \( k + 1 \). Let us define
\[
Z = \left| \sum_{j=1}^n b_j X_j \right|_B.
\]

Let us define for \( i \in \{1, \ldots, n\} \),
\[
X^{(i)} = (X^{(i)}_1, \ldots, X^{(i)}_n) = (X_1, \ldots, X_{i-1}, -X_i, X_{i+1}, \ldots, X_n),
\]
such that \( X^{(i)} \cong X \), and
\[
Z^{(i)} = \left| \sum_{j=1, j \neq i}^n b_j X^{(i)}_j \right|_B.
\]

and finally
\[
Z = \sum_{i=1}^n Z^{(i)}.
\]

The core of the proof is then the following : we will give some lower and upper bounds for \( \mathbb{E}[Z^k Z] \) with respect to \( \mathbb{E}[Z], \mathbb{E}[Z^{k+1}] \). To do so, we will give a direct lower bound to \( Z \). For the upper bound, we will dramatically use the Fourier transform.

Let us focus first on the lower bound. We have
\[
Z = \sum_{i=1}^n \left| \sum_{j=1}^n X^{(i)}_j b_j \right|_B \geq \sum_{i=1}^n \sum_{j=1}^n X^{(i)}_j b_j \bigg|_B = (n - 2)Z.
\]
Since $Z_k$ is non-negative, we have  
$$(n - 2)\mathbb{E}[Z^{k+1}] \leq \mathbb{E}[Z^k \tilde{Z}].$$

Let us now focus on the upper bound. First note that if we define for $S \subset \{1, \cdots, n\}$ and $i \in \{1, \cdots, n\}$,

$$X^{(i)}_S = \prod_{l \in S} X^{(i)}_l, \quad X^{(i)}_\emptyset = 1,$$

then $(X^{(i)}_S)_{S \subset \{1, \cdots, n\}}$ is a orthonormal basis of $L^2$ and furthermore note that $(Z^{(i)}, X^{(i)}_S)$ has the same distribution as $(Z, X_S)$ for all $i$ and all $S$. Note also that

$$X^{(i)}_S = \begin{cases} X_S & \text{if } i \notin S \\ -X_S & \text{if } i \in S. \end{cases}$$

Hence,

$$\hat{Z}(S) = \sum_{i=1}^n \mathbb{E}[Z^{(i)} X_S]$$

$$= \sum_{i \notin S} \mathbb{E}[Z^{(i)} X^{(i)}_S] - \sum_{i \in S} \mathbb{E}[Z^{(i)} X^{(i)}_S]$$

$$= (n - 2|S|) \mathbb{E}[ZX_S]$$

$$= (n - 2|S|) \hat{Z}(S).$$

Not also that if there exists $m$ such that $|S| = 2m + 1$, since $X \overset{d}{=} -X$ we have

$$\hat{Z}(S) = \mathbb{E}[ZX_S] = \left[ \sum_{j=1}^n -X_j b_j \prod_{l \in S} (-X_l) \right] = -\hat{Z}(S),$$

and

$$\hat{Z}(S) = 0.$$

Hence we have

$$\mathbb{E}[Z^k \tilde{Z}] = \sum_{S \subset \{1, \cdots, n\}} \tilde{Z}^k(S) \hat{Z}(S)$$

$$= \sum_{S \subset \{1, \cdots, n\}} (n - 2|S|) \tilde{Z}^k(S) \hat{Z}(S)$$

$$= n \tilde{Z}^k(\emptyset) \hat{Z}(\emptyset) + \sum_{S \subset \{1, \cdots, n\}} (n - 2|S|) \tilde{Z}^k(S) \hat{Z}(S)$$

$$\leq 4\mathbb{E}[Z^k] \mathbb{E}[Z] + (n - 4) \sum_{S \subset \{1, \cdots, n\}} \tilde{Z}^k(S) \hat{Z}(S)$$

$$= 4\mathbb{E}[Z^k] \mathbb{E}[Z] + (n - 4) \mathbb{E}[Z^{k+1}].$$

Putting upper and lower bounds together, we get

$$\mathbb{E}[Z^{k+1}] \leq 2\mathbb{E}[Z^k] \mathbb{E}[Z].$$

By a direct induction, we have

$$\mathbb{E}[Z^k] \leq 2^{1 - \frac{k}{n}} \mathbb{E}[Z],$$

which is the wanted bound with $C_k = 2^{1 - \frac{k}{n}}$. \hfill \Box
C.2. UMD spaces.

**Definition C.2.** Let \((B, | \cdot |_B)\) be a Banach space and \(p \in (1, +\infty)\). We say that \(B\) as the Unconditionnal Martingale Difference-\(p\) (UMD\(_p\)) property if there exists a constant \(K_p(B) > 0\) such that for any filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})\) and any \(B\) value martingale \((M_n)_{n\geq 0}\) with \(p\) moments, any \(N \geq 1\) and any \(\varepsilon_1, \ldots, \varepsilon_N \in \{-1, 1\}\),

\[
\mathbb{E} \left[ \left| \sum_{k=0}^{N-1} \varepsilon_k(M_{k+1} - M_k) \right|^p \right] \leq K_p(B)^p \mathbb{E} \left[ |M_N - M_0|_B^p \right]
\]

We say that \(B\) as the UMD property if \(B\) is UMD\(_p\) for all \(p \in (0, +\infty)\).

The following theorem is a fundamental tool in the theory of UMD spaces. We refer to [70] for a proof of it (and much more).

**Theorem C.3.** Let \((B, | \cdot |_B)\) be a Banach space with the UMD\(_p\) property. Then \((B, | \cdot |_B)\) has the UMD property.

The following proposition emphasize the fact that UMD spaces and inequality for martingales are strongly linked:

**Proposition C.4.** The finite dimensional vector space \(\mathbb{R}^d\) endowed with its standard euclidean norm has the UMD property.

**Proof.** The proof relies on the classical discrete Burkolder Davis Gundy (BDG) inequality. Indeed let \((M_n)_{n}\) be an \(L^p\) martingale in \(\mathbb{R}^d\). Let \(\varepsilon_1, \ldots, \varepsilon_N \in \{-1, 1\}, M_0 = 0\) and for \(N \in \{1, \ldots, N\}\)

\[
M_n^\varepsilon = \sum_{k=0}^{n-1} \varepsilon_k(M_{k+1} - M_k).
\]

Then \((M_n^\varepsilon)_{n\in\{0,\ldots,N\}}\) is a martingale, and we have

\[
\mathbb{E} \left[ |M_N^\varepsilon|^p \right] \leq \mathbb{E} \left[ \left( \sum_{k=0}^{N-1} |M_{k+1}^\varepsilon - M_{k+1}^\varepsilon|^2 \right)^{\frac{p}{2}} \right] \leq \mathbb{E} \left[ |M_N - M_0|^p \right],
\]

were we have applied twice the BDG inequality.

**Theorem C.5.** Let \(p \in (1, +\infty)\), let \((B, | \cdot |_B)\) be an Banach space with the UMD and let \((A, \mathcal{A}, \mu)\) be a measured space. The space \(L^p(A; B)\) is a UMD space.

**Proof.** The proof follows easily from the previous results. Indeed, it is enough to prove that \(L^p(A; B)\) has the UMD\(_p\) property. Let \((M_n)_{n}\) be a \(L^p\) martingale with value in \(L^p(A; B)\). Remark that for \(\mu\)-almost all \(x \in A\), \((M_n(x))_{n}\) is a \(B\) value martingale. Furthermore, thanks to the Fubini property, we have

\[
\mathbb{E} \left[ \|M_n\|_{L^p(A; B)}^p \right] = \int_A \mathbb{E} \left[ |M_n(x)|_B^p \right] d\mu(x).
\]

Hence for \(\mu\)-almost all \(x\), \(\mathbb{E} \left[ |M_n(x)|_B^p \right] < +\infty\) and \((M_n(x))_{n}\) is a \(L^p\) martingale with value in \(B\). Hence for all \(\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}\), we have for almost all \(x \in A\) and all \(N \geq 1\)

\[
\mathbb{E} \left[ |M_N^\varepsilon(x)|_B^p \right] \leq K_p(B)^p \mathbb{E} \left[ |M_N(x) - M_0(x)|_B^p \right].
\]

Hence, we have
\[ \mathbb{E}[\|M_N^p\|^p_{L^p(A;B)}] = \int_A \mathbb{E}[|M_N^p(x)|^p_B] \, d\mu(x) \leq K_p(B)^p \int_A \mathbb{E}[|M_N(x) - M_0(x)|^p_B] \, d\mu(x) = K_p(B)^p \mathbb{E}[\|M_N - M_0\|^p_{L^p(A;B)}]. \]

Hence, \( L^p(A;B) \) is UMD and then UMD. \( \square \)

**Corollary C.6.** Let \( p, q \in (1, +\infty) \) and let \( \alpha \in \mathbb{R} \). Then \( B_{p,q}^\alpha(\mathbb{R}^d, \mathbb{R}) \) is UMD.

**Proof.** Is previously it is enough to prove that \( B_{p,q}^\alpha \) is UMD. Furthermore, since \( L^p(\mathbb{R}^d; \mathbb{R}) \) is UMD we have for all \( N \in \mathbb{N} \) and all \( \varepsilon_1, \ldots, \varepsilon_N \in \{-1, 1\}, \)

\[ \mathbb{E}[\|M_N^p\|^q_{B_{p,q}^\alpha}] = \sum_{j \geq -1} 2^{jq} \mathbb{E}[\|\Delta_j M_N^p\|^q_{L^p(\mathbb{R}^d; \mathbb{R})}] = K_q(L^p(\mathbb{R}^d; \mathbb{R}))^q \mathbb{E}[\|M_N - M_0\|^q_{B_{p,q}^\alpha}], \]

and the result follows easily. \( \square \)

**C.3. Banach space, smoothness and type.**

**Definition C.7.** Let \( p \in (1, 2] \) and \( B, | \cdot |_B \) be a Banach space. We say that \( B \) is of type \( p \) if there exists a constant \( M_p \) such that for every \( N \geq 0 \), every iid sequence \( X_1, \ldots, X_n \) with \( \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2} \) and every \( b_1, \ldots, b_N \in B \), we have

\[ \mathbb{E}\left[ \left( \sum_{k=1}^N X_k b_k \right)^p \right] \leq M_p \left( \sum_{k=1}^N |b_k|^p \right)^{\frac{1}{p}}. \]

Let us first give an example:

**Proposition C.8.** Let \( \mathbb{R}^d \) with its Euclidean norm. Then \( \mathbb{R}^d \) is of type 2. More generally any Hilbert space is of type 2.

**Proof.** The proof is quite straightforward. Let us denote by \( \langle \cdot, \cdot \rangle \) the inner product. We have for \( N \geq 1, b_1, \ldots, b_N \in \mathbb{R}^d \) and \( X_1, \ldots, X_n \) a sequence of iid random variables with \( \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2} \),

\[ \mathbb{E}\left[ \left( \sum_{k=1}^N X_k b_k \right)^2 \right] = \sum_{k,l=1}^N \mathbb{E}\left[ X_k X_l \right] \langle b_k, b_l \rangle = \sum_{k=1}^N |b_k|^2. \]

Applying the Khinchine-Kahane inequality, we have the result. \( \square \)

**Theorem C.9.** Let \( (A, \mathcal{A}, \mu) \) be a measured space and let \( (B, | \cdot |_B) \) be a type \( p \) Banach space with \( p \in (1, 2] \). Let \( q \in [1, +\infty) \). Then \( L^q(A; B) \) is of type \( p \wedge q \).

**Proof.** Take \( N \in \mathbb{N}, b_1, \ldots, b_N \in L^q(A; B) \) and \( X_1, \ldots, X_N \) a sequence of iid random variables with \( \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2} \).

Applying twice the Khinchine-Kahane inequality, we get

\[ \mathbb{E}\left[ \left\| \sum_{k=1}^n X_k b_k \right\|_{L^q(A;B)}^q \right] \leq \mathbb{E}\left[ \left( \sum_{k=1}^n X_k b_k(x) \right)^q \right] \, d\mu(x) \]

\[ \leq \left( \int_A \mathbb{E}\left[ \left( \sum_{k=1}^n X_k b_k(x) \right)^q \right] \, d\mu(x) \right)^{\frac{1}{q}} \]

\[ \leq \left( \int_A \mathbb{E}\left[ \left( \sum_{k=1}^n |X_k b_k(x)|^q \right) \, d\mu(x) \right] \right)^{\frac{1}{q}} \]

\[ \leq \left( \int_A \left( \sum_{k=1}^n |b_k(x)|^p \right)^{\frac{p}{q}} \, d\mu(x) \right)^{\frac{1}{p}} \]

\[ \leq \left( \int_A \left( \sum_{k=1}^n |b_k(x)|^p \right)^{\frac{p}{q}} \, d\mu(x) \right)^{\frac{1}{p}}. \]
where we have used the fact that \((B, \cdot | B)\) is of type \(p\) in the last inequality.

Now, let us suppose that \(q \leq p\). We have for every \(N \geq 1\) and \(y_1, \ldots, y_N \in \mathbb{N},
\[
\left( \sum_{k=1}^{N} |y_k|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{N} |y_k|^q \right)^{\frac{1}{q}}.
\]

Finally we have
\[
\mathbb{E} \left[ \left\| \sum_{k=1}^{N} X_k b_k \right\|_{L^q(A; B)} \right] \lesssim \left( \int \left( \sum_{k=1}^{N} |b_k(x)|_B^q \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \lesssim \left( \sum_{k=1}^{N} \|b_k\|_{L^q(A; B)}^q \right)^{\frac{1}{q}}.
\]

Hence, in that case we have proved \(L^q(A; B)\) which is of \(q\) type.

Let us suppose that \(p \leq q\). Note that we have
\[
\mathbb{E} \left[ \left\| \sum_{k=1}^{N} X_k b_k \right\|_{L^q(A; B)} \right] \lesssim \left( \sum_{k=1}^{N} \left( \int |b_k(x)|_B^q \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \lesssim \left( \sum_{k=1}^{N} \|b_k\|_{L^q(A; B)}^p \right)^{\frac{1}{q}}.
\]

In that case \(L^q(A; B)\) if of \(q\) type, which ends the proof. \(\square\)

**Corollary C.10.** Let \(1 \leq p, q < +\infty\) and \(\alpha \in \mathbb{R}\). Then \(B^\alpha_{p,q}(\mathbb{R}^d; \mathbb{R})\) is of type \(p \wedge q \wedge 2\).

**Proof.** The proof follows easily when we see that
\[
\|f\|_{B^\alpha_{p,q}} = \|2^{j\alpha}f\|_{L^p(\mathbb{R}^d; \mathbb{R})}.
\]

By the previous result, \(L^p(\mathbb{R}^d; \mathbb{R})\) is of type \(p \wedge 2\) and \(\ell^q(L^p(\mathbb{R}^d; \mathbb{R}))\) is of type \(q \wedge p \wedge 2\). \(\square\)

**C.4. Burkholder-Davis-Gundy in Banach spaces.** We now have all the tools to enounce and proof the following theorem. One can consult [62] chapter 4 for more details.

**Theorem C.11.** Let \((B, \cdot | B)\) be a UMD Banach space of type \(p \in (1, 2]\). Then for any \(r \in (1, +\infty)\) a constant \(C_r > 0\) exists such that for any \(N \geq 1\) and any \(L^r\) martingale with value in \(B\)
\[
\mathbb{E}||M_N - M_0||_B^r \leq C_r \mathbb{E} \left[ \left\{ \sum_{k=0}^{N-1} |M_{k+1} - M_k|_B^r \right\}^{\frac{r}{p}} \right].
\]  

**Proof.** Let \((M_n)\) as in the theorem and \(N \geq 1\). Let \((X_1, \ldots, X_N)\) be an iid random vector independent of \(M\) with \(\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}\).

First, remark that thanks to the UMD property, we have for all \(\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\},\n\]
\[
\mathbb{E}||M_N - M_0||_B^r \lesssim \mathbb{E}||M_N^\varepsilon||_B^r.
\]

Hence, we have
\[
\mathbb{E}||M_N - M_0||_B^r \lesssim \mathbb{E} \left[ \sum_{k=0}^{N-1} X_k (M_{k+1} - M_k) \right]_{B^r}\]
\[
\lesssim \mathbb{E} \left[ \left\{ \sum_{k=0}^{N-1} X_k (M_{k+1} - M_k) \right\}^{r}_B \right]_{(M_n)_{n \in \{0, \ldots, N\}}}.
\]

(UMD)
Using Corollaries C.10 and C.6 and Theorems C.9 and C.5, we have the following useful corollary:

**Corollary C.12.** Let $p, q \in (1, +\infty)$ and $\alpha \in \mathbb{R}$. The previous Inequality 54 holds for $L^p(\mathbb{R}^d; \mathbb{R})$ (respectively $B^p_{\alpha,q}(\mathbb{R}^d; \mathbb{R})$) with $p \wedge 2$ instead of $p$ (respectively with $p \wedge 2 \wedge q$ instead of $p$).

**Remark C.13.** Using the fact that $F^\alpha_{p,q}(\mathbb{R}^d; \mathbb{R})$ is endowed in $L^q(B^p)$, one could prove exactly the same theorem for Triebel-Lizorkin spaces.

**Appendix D. The multi-parameter and multi-dimensionnal Garsia-Rodemich-Rumsey inequality.**

Let us recall the standard Garsia-Rodemich-Rumsey inequality [42].

**Theorem D.1.** Let $p \geq 1$, $\alpha > p^{-1}$ and $f \in C([0,1])$. Then, there exists a constant $C_{\alpha,p} > 0$ such that

\[
|f(t) - f(s)| \leq C_{\alpha,p} \kappa_f |s - t|^{\alpha - 1/p}
\]

for all $t, s \in [0,1]^2$, where

\[
\kappa_f = \left( \int_{[0,1]^2} \frac{|f(u) - f(v)|^p}{|u - v|^{\alpha p + 1}} dudv \right)^{1/p}.
\]

We can extend the previous result to the case $f \in C([0,1] \times B(0,R))$ where $B(0,R) \subset \mathbb{R}^d$ is a closed ball of radius $R > 0$. By denoting, for any $x, y \in [0,1]^d$ and $s, t \in [0,1]$,

\[
\square_{(s,x)} f(t,y) = f(s,x) - f(s,y) - f(t,x) + f(t,y),
\]

we have the following result.

**Corollary D.2.** Let $p \geq 1$, $\alpha_1 > 1/p$, $\alpha_2 > d/p$ and $f \in C([0,1] \times B(0,R))$. Then, there exists a constant $C_{d,\alpha_1,\alpha_2,p} > 0$ such that

\[
\square_{(s,x)} f(t,y)^p \leq C_{d,\alpha_1,\alpha_2,p} \kappa_f |s - t|^{\alpha_1 p - 1} |x - y|^{\alpha_2 p - d}
\]

for all $t, s \in [0,1]$ and $x, y \in B(0,R)$, where

\[
\kappa_f = \int_{[0,1]^2 \times B(0,R)^2} \frac{|\square_{(u,z)} f(v,w)|^p}{|u - v|^{\alpha_1 p + 1} |z - w|^{\alpha_2 p + d}} dudvdzdw.
\]

**Proof.** We essentially use the arguments from [68] and [48]. Let $\{(E_j, d_j)\}_{1 \leq j \leq m}$ be a family of separable metric space each endowed with a finite doubling measure $\nu_j$ defined on the Borel sets of $E_j$ (we essentially need that Lebesgue’s differentiation theorem holds) and $\Psi$ a positive increasing convex function such that $\Psi(0) = 0$ (and, thus, $\Psi^{-1}$ is a positive increasing concave function). We define $\sigma_j(r) = \min_{x \in E_j} \nu_j(B(x, r))$ the volume (under $\nu_j$) of the smallest ball of radius $r > 0$ in $E_j$. For any metric space $(E, d)$ endowed with a measure $\nu$ defined on the Borel sets of $E$ and for any scalar-valued function $f \in C(E)$, we define

\[
\bar{f}(A) := \nu(A)^{-1} \int_A f(x)\nu(dx),
\]

for any Borel set $A \in E$. For any $(E, d, \nu) \in \{(E_j, d_j, \nu_j)\}_{1 \leq j \leq m}$, we have, by Lebesgue’s differentiation theorem

\[
\bar{f}(B(x,r)) \to f(x),
\]
Furthermore, for any function \( f \) defined on \( \mathcal{E} \), we denote \( V_{k,y}f(x) = f(V_{k,y}x) \). We proceed to define the joint increment of \( f \) as

\[
\square_y f(x) := \prod_{j=1}^{m} (\text{id} - V_{k,y}) f(x).
\]

From the previous definitions, we can see that, for any \( x, y \in \mathcal{E} \) and \( f \) defined on \( \mathcal{E} \)

\[
\square_y f(x) = \square_y f(x', x_m) - \square_y f(x', y_m).
\]

We can now state an intermediate result.

**Lemma D.3.** Let \( f \in \mathcal{C}(\mathcal{E}) \) be such that

\[
\kappa_f := \int_{\mathcal{E}} \Psi \left( \frac{\square_y f(x)}{\prod_{j=1}^{m} d_j(x, y_j)} \right) v(dx)v(dy) < +\infty.
\]

Then the following inequality holds for any \( x, y \in \mathcal{E} \)

\[
|\square_y f(x)| \leq 18^m \int_0^{d_1(x_1, y_1)/2} \cdots \int_0^{d_m(x_m, y_m)/2} \Psi^{-1} \left( \frac{\kappa_f}{\prod_{j=1}^{m} \sigma_j(r_j)^2} \right) dr_1 \cdots dr_m.
\]

**Proof.** We proceed by induction; the case \( m = 1 \) being the classical Garsia-Rodemich-Rumsey. We assume that the result holds for a certain \( m - 1 \geq 1 \). For any \( x, y \in \mathcal{E} \), we denote \( g_{x', y'}(z) = \square_y f(x', z) \). For any Borel sets \( A_1, A_2 \subset E_m \), we have

\[
|\bar{g}_{x', y'}(A_1) - \bar{g}_{x', y'}(A_2)| \leq \int_{A_1} \int_{A_2} |g_{x', y'}(x_m) - g_{x', y'}(y_m)| \frac{v_m(dx_m)v_m(dy_m)}{v_m(A_1)v_m(A_2)}
\]

\[
\leq d_m(A_1, A_2) \int_{A_1} \int_{A_2} \frac{g_{x', y'}(x_m) - g_{x', y'}(y_m)}{d_m(x_m, y_m)} \frac{v_m(dx_m)v_m(dy_m)}{v_m(A_1)v_m(A_2)},
\]

where \( d_m(A_1, A_2) = \sup_{x_m \in A_1, y_m \in A_2} d_m(x_m, y_m) \). By the induction hypothesis, we have that

\[
\frac{|g_{x', y'}(x_m) - g_{x', y'}(y_m)|}{d_m(x_m, y_m)} = |\square_y \delta_{x_m, y_m} f(x')| 
\]

\[
\leq 18^{m-1} \int_{D} \Psi^{-1} \left( \frac{\kappa_{x_m, y_m} f}{\prod_{j=1}^{m-1} \sigma_j(r_j)^2} \right) dr,
\]

where we denote \( D = [0, d_1(x_1, y_1)/2] \times \cdots \times [0, d_{m-1}(x_{m-1}, y_{m-1})/2] \), \( r = (r_1, \ldots, r_{m-1}) \) and \( \delta_{x_m, y_m} f(x') = (f(x', x_m) - f(x', y_m))/d_m(x_m, y_m) \). Thus, by Fubini’s theorem and Jensen’s inequality, we obtain that

\[
|\bar{g}_{x', y'}(A_1) - \bar{g}_{x', y'}(A_2)| 
\]

\[
\leq 18^{m-1} d_m(A_1, A_2) \int_{D} \int_{A_1} \int_{A_2} \Psi^{-1} \left( \frac{\kappa_{x_m, y_m} f}{\prod_{j=1}^{m-1} \sigma_j(r_j)^2} \right) \frac{v_m(dx_m)v_m(dy_m)}{v_m(A_1)v_m(A_2)} dr
\]

\[
\leq 18^{m-1} d_m(A_1, A_2) \int_{D} \Psi^{-1} \left( \int_{A_1} \int_{A_2} \frac{\kappa_{x_m, y_m} f}{\prod_{j=1}^{m-1} \sigma_j(r_j)^2} \frac{v_m(dx_m)v_m(dy_m)}{v_m(A_1)v_m(A_2)} \right) dr.
\]
We remark that
\[
\int_{A_1} \int_{A_2} \kappa \delta_{\mu} \cdot f \cdot \nu_m (dx) \cdot \nu_m (dy) \leq \int_{E_m} \int_{E_m} \kappa \delta_{\mu} \cdot f \cdot \nu_m (dx) \cdot \nu_m (dy) = \kappa_f,
\]
and, thus, we obtain
\[
|\tilde{g}_{x', y'} (A_1) - \tilde{g}_{x', y'} (A_2)| \leq 18^{m-1} d_m (A_1, A_2) \int_D \Psi^{-1} \left( \frac{\kappa_f}{\prod_{j=1}^{m-1} \sigma_j (r_j) \sigma_m (\lambda_0)^2} \right) \ dr. \quad (57)
\]
We now denote \( \tilde{g}_{x', y'} (x, m, r) = \tilde{g}_{x', y'} (B(x, m, r)) \) for any \( x_m \in E_m \) and
\[
B(x_m, r) = \{ x \in E_m; d_m (x, x_m) \leq r \}
\]
for a certain \( r > 0 \). We now fix \( x_m, y_m \in E_m \) and let \( \lambda_k = d_m (x_m, y_m) 2^{-k} \) for any \( k \geq 0 \). Since \( d_m (B(x, m, \lambda), B(y, m, \lambda)) = 3 \lambda_0, \sigma_m (\lambda_k) \geq \sigma_m (\tau) \) for any \( \tau \leq \lambda_k \) and \( \Psi^{-1} \) is increasing, it follows from (57) that
\[
|\tilde{g}_{x', y'} (x, m, \lambda_0) - \tilde{g}_{x', y'} (y, m, \lambda_0)| \\
\leq 3 \times 18^{m-1} \lambda_0 \int_D \Psi^{-1} \left( \frac{\kappa_f}{\prod_{j=1}^{m-1} \sigma_j (r_j) \sigma_m (\lambda_0)^2} \right) \ dr \ d \tau.
\]
Furthermore, by using (57) and since \( d_m (B(x, m, \lambda), B(x, m, \lambda+1)) = \lambda + 6 \lambda_2 \) for any \( \ell \geq 0, \sigma_m (\lambda_k) \geq \sigma_m (\tau) \) for any \( \tau \leq \lambda_k \) and \( \Psi^{-1} \) is increasing, we obtain that, for any \( k \geq 1, \)
\[
|\tilde{g}_{x', y'} (x, m, \lambda_0) - \tilde{g}_{x', y'} (x, m, \lambda_{\ell+1})| \\
\leq 6 \times 18^{m-1} \sum_{\ell=0}^{k-1} (\lambda_{\ell+1} - \lambda_{\ell+2}) \int_D \Psi^{-1} \left( \frac{\kappa_f}{\prod_{j=1}^{m-1} \sigma_j (r_j) \sigma_m (\lambda_{\ell+1}) \sigma_m (\lambda_{\ell+2})} \right) \ dr
\]
\[
\leq 6 \times 18^{m-1} \sum_{\ell=0}^{k-1} \int_{\lambda_{\ell+2}}^{\lambda_{\ell+1}} \int_D \Psi^{-1} \left( \frac{\kappa_f}{\prod_{j=1}^{m-1} \sigma_j (r_j) \sigma_m (\tau)^2} \right) \ dr \ d \tau
\]
\[
\leq 6 \times 18^{m-1} \int_{\lambda_{k+1}}^{\lambda_k} \int_D \Psi^{-1} \left( \frac{\kappa_f}{\prod_{j=1}^{m-1} \sigma_j (r_j) \sigma_m (\tau)^2} \right) \ dr \ d \tau.
\]
Letting \( k \to +\infty \), we deduce that
\[
|\tilde{g}_{x', y'} (x, m, \lambda_0) - \tilde{g}_{x', y'} (x, m)| \leq 6 \times 18^{m-1} \int_0^{d_m (x, m, y_m)^2 / 2} \int_D \Psi^{-1} \left( \frac{\kappa_f}{\prod_{j=1}^{m-1} \sigma_j (r_j) \sigma_m (\tau)^2} \right) \ dr \ d \tau,
\]  
and, similarly,
\[
|\tilde{g}_{x', y'} (y, m, \lambda_0) - \tilde{g}_{x', y'} (y, m)| \leq 6 \times 18^{m-1} \int_0^{d_m (x, m, y_m)^2 / 2} \int_D \Psi^{-1} \left( \frac{\kappa_f}{\prod_{j=1}^{m-1} \sigma_j (r_j) \sigma_m (\tau)^2} \right) \ dr \ d \tau.
\]
Thanks to (58), (59) and (60), we obtain the desired result. \( \square \)

Corollary D.2 follows directly from Lemma D.3 by choosing \( E_1 = [0, 1] \) endowed with \( d_1 (t, s) = |t - s|^{\alpha+1/p} \) and \( E_2 = B(0, R) \) endowed with \( d_2 (x, y) = |x - y|^{\alpha+2/d/p} \). We also use, for both,
Lebesgue’s measure and set $Ψ(t) = |t|^p$. It turns out that $σ_1(r) \simeq r^{1/(α_1+1/p)}$ and $σ_2(r) \simeq r^{1/(α_2+d+1/p)}$. With the previous choices at hand, we obtain

$$
\int_0^{d_1(s,t)}/2 \int_0^{d_2(x,y)/2} \Psi^{-1} \left( \frac{κ_f}{σ_1(r_1)^2σ_2(r_2)^2} \right) \, dr_1 \, dr_2
\simeq κ_f^{-1/p} \int_0^{d_1(s,t)/2} r_1^{-2/(α_1p+1)} \, dr_1 \int_0^{d_2(x,y)/2} r_2^{-2/(α_2p+d+1)} \, dr_2
\simeq κ_f^{-1/p} |t-s|^{α_1-1/p} |x-y|^{α_2-d/p},
$$

where

$$
κ_f = \int_{[0,1]^2} \int_B(0,R)^2 \frac{|[s,x]f(t,y)|^p}{|t-s|^{α_1p+1} |x-y|^{α_2p+d}} \, dt \, ds.
$$

We can finally state a Kolmogorov’s continuity theorem.

**Theorem D.4.** Let $p \geq 1$, $α_1 > 1/p$, $α_2 > d/p$ and $f \in C([0,1] \times \mathbb{R}^d)$ a random process. Assume that there exists $ε_1, ε_2 \in (0, 1)$ such that

$$
θ_f := \sup_{s,t \in [0,1]} E \left[ \frac{|[s,x]f(t,y)|^p}{|t-s|^{α_1p+1} |x-y|^{α_2p+ε}} \right]^{1/p} < +∞.
$$

(61)

Then, for any $ε_3 > ε_2$, there exists a positive random variable $Υ \in L^p(Ω)$ which depends on $f, d, α_1, α_2, p, ε_1, ε_2$ and $ε_3$ such that, $P$-a.s.,

$$
|[s,x]f(t,y)| \leq Υ |s-t|^{α_1-1/p} |x-y|^{α_2-d/p}(1 + |x| + |y|)^{d+ε_3}/p,
$$

(62)

for all $t, s \in [0,1]$ and $x, y \in \mathbb{R}^d$.

**Proof.** Let $A = \{ x \in \mathbb{R}^d ; 1 \leq |x| \leq 2 \}$ be an annulus in $\mathbb{R}^d$ such that $\mathbb{R}^d = B(0, 1) \cup \bigcup_{j=0}^{+∞} 2^j A$. It follows from Corollary D.2 and (61) that, for any $j ≥ 0$,

$$
E \left[ \sup_{t,s \in [0,1], x,y \in \mathbb{R}^d} \frac{|[s,x]f(t,y)|^p}{|s-t|^{α_1p+1} |x-y|^{α_2p+d}(1 + |x| + |y|)^{d+ε_3}} \right]
\leq C_d,α_1,α_2,p,θ_f \frac{1}{(1 + 2^{j+1})^{d+ε_3}} \int_{[0,1]^2 \times B(0,1+2^{j+2})^2} \frac{du_1 \, du_2 \, dz_1 \, dz_2}{|u_1 - u_2|^{1-ε_1}|z_1 - z_2|^{d-ε_2}}
\leq C_d,α_1,α_2,p,θ_f \frac{2^{d+ε_3}(j+2)}{1 + 2^{j+1})^{d+ε_3}} \int_{[0,1]^2 \times B(0,2)^2} \frac{du_1 \, du_2 \, dz_1 \, dz_2}{|u_1 - u_2|^{1-ε_1}|z_1 - z_2|^{d-ε_2}}
\leq 2^{j(ε_2-ε_3)}.
$$

Thus, we deduce that

$$
E \left[ \sup_{t,s \in [0,1], x,y \in \mathbb{R}^d} \frac{|[s,x]f(t,y)|^p}{|s-t|^{α_1p+1} |x-y|^{α_2p+d}(1 + |x| + |y|)^{d+ε_3}} \right]
\leq E \left[ \sup_{t,s \in [0,1], x,y \in B(0,1)} \frac{|[s,x]f(t,y)|^p}{|s-t|^{α_1p-1} |x-y|^{α_2p-d}} \right] + \sum_{j=0}^{+∞} E \left[ \sup_{t,s \in [0,1], x,y \in 2^j A} \frac{|[s,x]f(t,y)|^p}{|s-t|^{α_1p+1} |x-y|^{α_2p+d}(1 + |x| + |y|)^{d+ε_3}} \right].
$$

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\[ \lesssim 1 + \sum_{j=0}^{+\infty} 2^{(\varepsilon_2 - \varepsilon_3)} < +\infty, \]

which ends the proof. \( \square \)

We also recall another standard Kolmogorov theorem.

**Theorem D.5.** Let \( f \in C([0, 1]; L^\infty_{\text{loc}}(\mathbb{R}^d)) \) be a random process and \( p \geq 1 \) such that
\[
\sup_{t, s \in [0, 1]} \mathbb{E} \left[ \frac{|f(t, x) - f(s, x)|^p}{|t - s|^{\alpha p + \varepsilon}} \right] < +\infty,
\]
for a certain \( \alpha > 1/p \) and \( \varepsilon \in (0, 1) \). Then, there exists a positive random variable \( \Upsilon \in L^p(\Omega) \) which depends on \( f, d, \alpha \) and \( p \) such that
\[
|f(t, x) - f(s, x)| \leq \Upsilon |s - t|^\alpha (1 + |x|)^{(1 + \iota)/p},
\]
where \( \iota > 0 \).

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