GRAVITY-MATTER COUPLINGS
FROM LIOUVILLE THEORY

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Abstract
The three-point functions for minimal models coupled to gravity are derived in the operator approach to Liouville theory which is based on its $U_q(sl(2))$ quantum group structure. The gravity-matter coupling is formulated by treating the latter as a continuation of the former. The result is very simple, and shown to agree with matrix-model calculations on the sphere. The precise definition of the corresponding cosmological constant is given in the operator solution of the quantum Liouville theory. It is shown that the symmetry between quantum-group spins $J$ and $-J-1$ previously put forward by the author is the explanation of the continuation in the number of screening operators discovered by Goulian and Li. Contrary to the previous discussions of this problem, the present approach clearly separates the emission operators for each leg. This clarifies the structure of the dressing by gravity. It is shown, in particular that the end points are not treated on the same footing as the mid point. Since the outcome is completely symmetric this suggests the possibility of a picture-changing mechanism.
1. Introduction

Matrix models have shown that minimal models become much simpler when they are coupled to 2D gravity. This was understood\cite{1, 11, 12} in the Coulomb-gas formulation by performing a continuation to negative numbers of screening operators. It is the purpose of the present article to study this phenomenon in the operator treatment of the Liouville theory which is based on the quantum group structure displayed\cite{4, 5, 6, 7, 8, 9, 10}, in recent years. The results discussed here were already reported some time ago\cite{11}, in an abbreviated form.

It should be stressed that the tools needed for deriving the three-point function are already present in earlier papers of mine\cite{6, 7, 12}. In particular, the building of operators with positive weights in the strong-coupling regime led to the unravelling of a symmetry between quantum group spins $J$ and $-J - 1$ some time ago\cite{6, 7}. This symmetry is basically why the Goulian and Li continuation\cite{1} is valid. Moreover, the construction of a local Liouville field was already done before\cite{8}. Finally, the universal chiral family associated with $U_q(sl(2))$ that comes out by quantizing Liouville theory, was completely determined in\cite{3}, including the detailed normalisation of the operators, so that their matrix elements between highest-weight states may be read off from\cite{3}. This is enough to determine the three-point functions, as we shall show below.

The outcome of the present study is threefold. First, we shall spell out the drastic simplifications that occur in the present approach when gravity and matter are coupled. This is useful, in particular, since the quantum group structure mentioned above\cite{4, 5, 6, 7, 8, 9, 10}, gives general formulae which remain rather complicated, compact as they may be. Second, the other discussions\cite{1, 2, 3} deal with each correlation function separately, and thus remain at the level of conformal blocks, where the screening operators are not specifically attributed to any particular external leg. Thus they do not clarify how world-sheet operators such as powers of the Liouville field are given as sums over products of holomorphic and antiholomorphic vertex operators with specified screening charges. Expanding ref.\cite{3}, we shall first (sections 2, and 3) construct the needed operators once for all by imposing the physical requirement that the exponentials of the Liouville field be local and possess a consistent restriction to the space of states with equal right and left momenta. Then, (sections 4, and 5) we shall show that appropriate matrix elements agree with the result of matrix-models. The operator content of the world-sheet fields is clear in the present discussion, contrary to the Coulomb gas picture. In particular we shall see that, in the three-point function, the endpoints are treated on a different footing even if the result is completely symmetric. Last, we shall show how the cosmological constant should be introduced in the present scheme. This will give us an operator derivation of the DDK argument\cite{12, 13}, which may be immediately applied on the sphere and on the torus (see section 5).

In order to set the stage, let us first recall some feature of the classical Liouville
1. Introduction

In the conformal gauge, it is governed by the action:

\[ S = \frac{1}{4\pi} \int d^2x \sqrt{\hat{g}} \left\{ \frac{1}{2} \hat{g}^{ab} \partial_a \Phi \partial_b \Phi + e^2 \sqrt{\gamma} \Phi + \frac{1}{2\sqrt{\gamma}} R_0 \Phi \right\} \]

(1.1)

\( \hat{g}_{ab} \) is the fixed background metric. We work for fixed genus, and do not integrate over the moduli. As is well known, one can choose a local coordinate system such that \( \hat{g}_{ab} = \delta_{ab} \). Thus we are reduced to the action

\[ S = \frac{1}{4\pi} \int d\sigma d\tau \left( \frac{1}{2} \left( \frac{\partial \Phi}{\partial \sigma} \right)^2 + \frac{1}{2} \left( \frac{\partial \Phi}{\partial \tau} \right)^2 + e^2 \sqrt{\gamma} \Phi \right) \]

(1.2)

where \( \sigma \) and \( \tau \) are the local coordinates. The complex structure is assumed to be such that the curves with constant \( \sigma \) and \( \tau \) are everywhere tangent to the local imaginary and real axis respectively. In a typical situation, one may work on the cylinder \( 0 \leq \sigma \leq 2\pi, -\infty \leq \tau \leq \infty \) obtained by an appropriate mapping from one of the handles of a general Riemann surface, and we shall do so in the present article. The action (1.2) corresponds to a conformal theory such that \( \exp(2\sqrt{\gamma} \Phi) d\sigma d\tau \) is invariant. The classical equivalent of the chiral vertex operators may be obtained very simply [15, 16, 8] by using the fact that the field \( \Phi(\sigma, \tau) \) satisfies the equation

\[ \frac{\partial^2 \Phi}{\partial \sigma^2} + \frac{\partial^2 \Phi}{\partial \tau^2} = 2\sqrt{\gamma} e^{2\sqrt{\gamma} \Phi} \]

(1.3)

if and only if

\[ e^{-\sqrt{\gamma} \Phi} = i \sqrt{\frac{\gamma}{2}} \sum_{j=1,2} f_j(x_+) g_j(x_-); \quad x_\pm = \sigma \mp i\tau \]

(1.4)

where \( f_j \) (resp. \( g_j \)), which are functions of a single variable, are solutions of the same Schrödinger equation

\[ -f_j'' + T(x_+) f_j = 0, \quad \text{resp.} -g_j'' + \bar{T}(x_-) g_j = 0 \]

(1.5)

The solutions are normalized so that their Wronskians \( f_1'f_2 - f_1 f_2' \) and \( g_1'g_2 - g_1 g_2' \) are equal to one. The proof of this basic fact is straightforward [13, 16, 8]. The potentials \( T(x_+) \) and \( \bar{T}(x_-) \) are the two components of the stress-energy tensor, and, after quantization, Eqs. (1.3) become the Virasoro Ward-identities associated with the vanishing of the singular vector at the second level. As a result the Liouville theory also describes minimal models provided the coupling constant \( \gamma \) is taken to be negative. This is how, we shall treat the matter fields. For the dynamics associated with the action Eq. (1.2), \( \tau \) is the time variable, and the canonical Poisson brackets are

\[ \{ \Phi(\sigma_1, \tau), \frac{\partial}{\partial \gamma} \Phi(\sigma_2, \tau) \}_{\text{P.B.}} = 4\pi \delta(\sigma_1 - \sigma_2), \quad \{ \Phi(\sigma_1, \tau), \Phi(\sigma_2, \tau) \}_{\text{P.B.}} = 0 \]

(1.6)

The cylinder \( 0 \leq \sigma \leq 2\pi, -\infty \leq \tau \leq \infty \) may be mapped on the complex plane of \( z = e^{\tau+i\sigma} \), and the above Poisson brackets lead to the usual radial quantization.
A priori, any two pairs \( f_j \) and \( g_j \) of linearly independent solutions of Eq.1.5 are suitable. In this connection, it is convenient to rename the functions \( g_j \) by letting \( \bar{f}_1 = -g_2 \), \( \bar{f}_2 = g_1 \). Then one easily sees that Eq.1.4 is left unchanged if \( f_j \) and \( \bar{f}_j \) are replaced by \( \sum_k M_{jk} f_k \) and \( \sum_k M_{kj} \bar{f}_k \), respectively, where \( M_{jk} \) is an arbitrary constant matrix with determinant equal to one. Eq.1.4 is \( sl(2,C) \)-invariant with \( f_j \) transforming as a representation of spin \( 1/2 \). At the quantum level, the \( f_j \)'s and \( \bar{f}_j \)'s become operators that do not commute, and the group \( sl(2) \) is deformed to become the quantum group \( U_q(sl(2)) \). This structure plays a crucial role at the quantum level, and we now elaborate upon the classical \( sl(2) \) structure where the calculations are simple.

At the classical level, it is trivial to take Eq.1.4 to any power. For positive integer powers \( 2J \), one gets (letting \( \beta = i\sqrt{2} \))

\[
e^{-2J\sqrt{\gamma}\Phi} = \sum_{M=-J}^{J} \frac{\beta^{2J}}{(J+M)!(J-M)!} \left( f_1(x+) \bar{f}_2(x-) \right)^{J-M} \left( f_2(x+) \bar{f}_1(x-) \right)^{J+M}.
\]

It is convenient to put the result under the form

\[
e^{-2J\sqrt{\gamma}\Phi} = \beta^{2J} \sum_{M=-J}^{J} (-1)^{J-M} f_{M}^{(J)}(x+) \bar{f}_{-M}^{(J)}(x-).
\]  (1.7)

where \( J \pm M \) run over integer. The \( sl(2) \)-structure has been made transparent by letting

\[
f_{M}^{(J)} = \sqrt{\frac{2J}{J+M}} (f_1)^{J-M} (f_2)^{J+M}, \quad \bar{f}_{-M}^{(J)} = \sqrt{\frac{2J}{J+M}} (\bar{f}_1)^{J-M} (\bar{f}_2)^{J+M}.
\]  (1.9)

The notation anticipates that \( f_{M}^{(J)} \) and \( \bar{f}_{-M}^{(J)} \) form representations of spin \( J \). This is indeed true since \( f_1, f_2 \) and \( \bar{f}_1, \bar{f}_2 \) span spin \( 1/2 \) representations, by construction. Explicitly one finds

\[
I_{\pm} f_{M}^{(J)} = \sqrt{(J \mp M)(J \mp M+1)} f_{M \pm 1}^{(J)}, \quad I_{3} f_{M}^{(J)} = M f_{M}^{(J)}
\]

\[
\bar{I}_{\pm} \bar{f}_{-M}^{(J)} = \sqrt{(J \mp M)(J \mp M+1)} \bar{f}_{-M \pm 1}^{(J)}, \quad \bar{I}_{3} \bar{f}_{-M}^{(J)} = M \bar{f}_{-M}^{(J)},
\]  (1.10)

where \( I_{\ell} \) and \( \bar{I}_{\ell} \) are the infinitesimal generators of the \( x_+ \) and \( x_- \) components respectively. Moreover, one sees that

\[
(I_{\ell} + \bar{I}_{\ell}) e^{-2J\sqrt{\gamma}\Phi} = 0
\]  (1.11)

so that the exponential of the Liouville field are group invariants.

At this simple classical level one may continue trivially to negative spins. This gives exponentials of \( \Phi \) with a positive exponent, and in particular defines the cosmological term \( e^{2\sqrt{\gamma}\Phi} \) which is the potential term of the action Eq.1.2. These terms may be written in two equivalent ways:

\[
e^{2J\sqrt{\gamma}\Phi} = \beta^{-2J} \sum_{M>J \text{ or } M<J} \bar{f}_{M}^{(-J)}(x+) \bar{f}_{-M}^{(-J)}(x-).
\]  (1.12)
where

\[ f_{i M}^{(-J)} \equiv \sqrt{(-1)^{J+M}} \left(\frac{-2J}{-J+M}\right) (f_1)^{J-M} (f_2)^{J+M}, \]

\[ \bar{f}_{i M}^{(-J)} \equiv \sqrt{(-1)^{J-M}} \left(\frac{-2J}{-J+M}\right) (\bar{f}_1)^{J+M} (\bar{f}_2)^{J-M}. \]

(1.13)

The binomial factors are defined by the usual continuation of the Gamma-functions. This is similar to Eq.1.8, but here \( M \) is smaller than \(-J\) or larger than \( J\). This case of negative spin is the classical equivalent of the quantum case with negative number of screening operators. We shall discuss the latter in section 3, at the quantum level.

## 2 THE LIOUVILLE FIELDS WITH POSITIVE QUANTUM-GROUP SPINS

In this section, we show how the negative powers of the metric are reconstructed from the chiral fields whose properties were extensively studied recently[5, 6, 7]. We are aiming at quantum versions of Eq.1.9. As mentioned above, this discussion was already summarized in refs.[8, 11]. Denote by \( C \) the central charge of gravity. The standard screening charges \(-\alpha_{\pm}\) of the Liouville theory[17, 18] are such that

\[ \alpha_{\pm} = \frac{1}{2} \left( \sqrt{\frac{C-1}{3}} \pm \sqrt{\frac{C-25}{3}} \right), \]

\[ \alpha_{\pm} = \frac{Q}{2} \pm \alpha_0, \quad Q = \sqrt{\frac{C-1}{3}}, \quad \alpha_0 = \frac{1}{2} \sqrt{\frac{C-25}{3}} \]

(2.1)

\( Q \), and \( \alpha_0 \) are introduced so that they will agree with the standard notation, when we couple with matter. Kac’s formula in the present notation is recalled in the appendix. It may be written as

\[ \Delta_{Kac}(J, \hat{J}; C) = -\frac{1}{2} \beta(J, \hat{J}; C) \left( \beta(J, \hat{J}; C) + Q \right), \quad \beta(J, \hat{J}; C) = J\alpha_- + \hat{J}\alpha_+, \]

(2.2)

where \( 2J \) and \( 2\hat{J} \) are positive integers. Thus the most general Liouville field is to be written as \( \exp(-(J\alpha_- + \hat{J}\alpha_+)\Phi) \). At first we consider the operators \( \exp(-J\alpha_-\Phi) \). Their chiral components have been extensively studied[3, 7]. Their fusion and braiding are determined by the Clebsch-Gordan coefficients and universal R matrix of \( U_q(sl(2)) \) respectively, with deformation parameter \( h = \pi(\alpha_-)^2/2 \), and \( q = e^{ih} \). It may be appropriately called the Universal Chiral Family (UCF) associated with \( U_q(sl(2)) \) (There is of course a parallel discussion for \( \exp(-\hat{J}\alpha_+\Phi) \), to which we shall come below).

In refs.[1, 3, 8, 11], the quantum group properties are discussed on the example of the \( x_+ \) components—their characteristic feature is that they are holomorphic functions of \( \tau + i\sigma \). A summary of the relevant results is given in appendix A for completeness. As long as one deals with functions of a single variable, one may
describe the whole operator-structure, without loss of generality, at $\tau = 0$, that is on the unit circle $u = e^{i\sigma}$. The central charge $C$ and the quantum-group parameter $h$ are related by

$$C = 1 + 6\left(\frac{h}{\pi} + \frac{\pi}{h} + 2\right).$$

(2.3)

We work for generic value of $h$, and thus consider irrational theories. For the rational case, $q$ is a root of unity, and the quantum-group structure becomes much more complicated. We shall come back to this point at the end, but we shall see that, although the formulae we are using at intermediate stages may become meaningless for rational theories, the final three-point coupling does make sense. It is thus the correct answer in that case as well.

The UCF is conveniently described using two different basis of chiral operators of the type $(1, 2J + 1)$, with $2J$ a positive integer. The first one, called the Bloch-wave basis, is made up with fields denoted $\psi_m^{(J)}$, $-J \leq m \leq J$, that are periodic up to a phase:

$$\psi_m^{(J)}(\sigma + 2\pi) = e^{2ihm\varpi} e^{2ihm^2} \psi_m^{(J)}(\sigma),$$

(2.4)

where the quasi momentum $\varpi$ is an operator such that

$$\psi_m^{(J)}(\sigma) \varpi = (\varpi + 2m) \psi_m^{(J)}(\sigma).$$

(2.5)

We shall work in a basis where $\varpi$ is diagonal. $|\varpi>$ denotes the corresponding highest-weight vectors. The second basis, is made up with operators of the form

$$\xi_M^{(J)}(\sigma) := \sum_{-J \leq m \leq J} |J, \varpi\rangle_M^n m \psi_m^{(J)}(\sigma), -J \leq M \leq J;$$

where the $|J, \varpi\rangle_M^n$ are polynomials of $e^{ih(\varpi + m)}$ whose expression is recalled in appendix A. It may be called the quantum group basis, since the braiding and fusion relations of the $\xi$ are given by the universal R matrix and Clebsch-Gordan coefficients of the quantum group $\tilde{U}_q(sl(2))$ in the standard mathematical form$^2$.

Let us now turn to the $x_-$ components following ref.[8] (a similar discussion was independently made in ref.[14]). The corresponding quantities will be distinguished with a bar: we shall write $\bar{\xi}_M^{(J)}(x_-), \bar{\psi}_M^{(J)}(x_-), \bar{\varpi}$ and so on. The properties of the $x_-$ components are similar to the ones recalled above and in appendix A for the $x_+$ components, with a crucial difference: they are functions of $\tau + i\sigma$, that is anti-analytic functions of $\tau + i\sigma$. In going from $x_+$ to $x_-$ components, one has to reverse the orientation of the imaginary axis, which indeed changes the chirality. The simplest way to avoid going through the whole argument again, is to remark that we only need to replace $i$ by $-i$ everywhere in the above formulae, that is, to use the other root of $-1$. The whole discussion applies again since it only used the fact that $i^2 + 1 = 0$. Thus the quantum-group properties of $\bar{\xi}_M^{(J)}(\sigma), \bar{\psi}_M^{(J)}(\sigma)$ are the

$^2$The fusion property was not really proven so far, but rather made very plausible from quantum group invariance[8]. Its complete derivation will be given elsewhere[19].
2. Positive spins

same as those of \((\xi_M^{(J)}(\sigma))^*, (\psi_M^{(J)}(\sigma))^*\), where the star means the transposed of the Hermitian conjugate. The appropriate definition of \(\xi_M^{(J)}(\sigma)\) is for instance

\[
\xi_M^{(J)}(\sigma) := \sum_{-J \leq m \leq J} (|J, \varpi\rangle_M^m)^* \psi_M^{(J)}(\sigma), \quad -J \leq M \leq J; \tag{2.6}
\]

moreover, the shift properties of the \(\psi\)-fields are given by

\[
\psi_m^{(J)}(\sigma) \varpi^* = (\varpi^* + 2m) \psi_m^{(J)}(\sigma). \tag{2.7}
\]

As is usual in conformally invariant field theory we assume that the right- and left-movers commute. Thus we take the \(\psi\)- and \(\xi\)-fields to commute with the \(\psi\)- and \(\xi\)-fields. (more about this below). The quantum-group structure is thus a tensor product of the UCF’s, which we denote by \(U_q(sl(2)) \otimes U_q(sl(2))\).

Let us now begin the reconstruction of the Liouville field. There are two basic requirements that determine \(\exp(-J\alpha_\varphi)\). The first one is locality, that is, that it commutes with any other power of the metric at equal \(\tau\). This condition is required from the consistency of the quantization according to the scheme recalled in the introduction, since \(\sigma\) is the space variable. In practice it is also needed so that the correlators of the Liouville field be single valued around their short-distance singularities. The second requirement concerns the Hilbert space of states where the physical operator algebra is realized. The point is that, since we took the fields \(\xi_M^{(J)}\) and \(\xi_M^{(J)}\) to commute, the quasi momenta \(\varpi\) and \(\varpi\) of the left- and right-movers are unrelated, which cannot be true physically, as is well known. For reasons that will become clearer later on, we shall require that \(\exp(-J\alpha_\varphi)\) leave the subspace of states with \(\varpi = (\varpi)^*\) invariant. The latter condition defines the restricted Hilbert \(\mathcal{H}_r\) where it must be possible to restrict the operator-algebra consistently. The restricted Hilbert space is actually larger than the physical Hilbert space where \(\varpi\) is real, but it is more handy at first, in order to deal with possible cuts in the \(\varpi\)-plane. Remarkably the two requirements just stated determine \(\exp(-J\alpha_\varphi)\) almost uniquely. The appropriate ansatz is:

\[
e^{-J\alpha_\varphi}(\sigma, \tau) = \tilde{c}_J a(\varpi) \sum_{M=-J}^{J} (-1)^{J-M} e^{i\hbar(J-M)} \xi_M^{(J)}(x_+) \xi_{-M}^{(J)}(x_-)/a(\varpi) \tag{2.8}
\]

where \(\tilde{c}_J\) is a normalization constant, and \(a(\varpi)\) will be determined below. Before starting the derivation, we remark that this expression is very natural from the quantum group viewpoint. Indeed the \(\xi\) fields transform according to

\[
J_3 \xi_M^{(J)} = M \xi_M^{(J)}, \quad J_\pm \xi_M^{(J)} = \sqrt{[J + M][J \pm M + 1]} \xi_{M \pm 1}^{(J)}, \tag{2.9}
\]

so that the generators \(J_\pm, J_3\) obey the \(U_q(sl(2))\) quantum group algebra

\[
[J_+, J_-] = [2J_3], \quad [J_3, J_\pm] = \pm J_\pm \tag{2.10}
\]
Similarly, the transformation of the $\zeta$ fields is given by

$$
\mathcal{J}_3 \xi_M^{(j)} = M \xi_M^{(j)}, \quad \mathcal{J}_\pm \xi_M^{(j)} = \sqrt{[J \mp M][J \pm M + 1]} \xi_{M \pm 1}^{(j)}. \tag{2.11}
$$

On the other hand, and as is well known, naive tensor products of quantum-group representations do not form representations, since the algebra is non-linear. It is necessary to use the co-product. Indeed define

$$
\mathcal{J}_\pm = J_\pm e^{-ihJ_3} + e^{ihJ_3} \otimes \mathcal{J}_\pm, \quad \mathcal{J}_3 = J_3 + \mathcal{J}_3, \tag{2.12}
$$

which does give a representation of Eq.(2.9). Then one easily checks that

$$
\mathcal{J}_\pm \exp(-J\alpha - \Phi) = \mathcal{J}_3 \exp(-J\alpha - \Phi) = 0, \tag{2.13}
$$

so that the quantized Liouville field is a quantum-group invariant. This is the quantum version of the classical $sl(2)$ invariance recalled in the introduction (see Eq.(1.11)). Next, locality is checked by making use of Eqs.A.26-A.28 which remain true at equal $\tau \neq 0$ since

$$
\xi_M^{(j)}(x_+) = e^{\tau L_0} \xi_M^{(j)}(\sigma) e^{-\tau L_0}, \quad \xi_M^{(j)}(x_-) = e^{-\tau T_0} \xi_M^{(j)}(\sigma) e^{\tau T_0}. \tag{2.14}
$$

At the present stage, we follow the earlier conventions \[3, 4, 7\] and define primary fields on the cylinder $0 \leq \sigma \leq \pi, -\infty < \tau < \infty$ in such a way that $\sigma$ and $\tau$ translations are unbroken. This allows in principle to deal with arbitrary Riemann surface. When we specialize to the Riemann sphere, we shall change to the more standard definition (see below). Choose $\pi > \sigma_1 > \sigma_2 > 0$. In agreement with the above discussion we have, for instance,

$$
\xi_{M_1}^{(j_1)}(\sigma_1 + i\tau) \xi_{M_2}^{(j_2)}(\sigma_2 + i\tau) = \sum_{-L \leq N_1 \leq L; -L \leq N_2 \leq L} ((J_1, J_2)^{N_2 N_1}_{M_1 M_2})^* \xi_{N_2}^{(j_2)}(\sigma_2 + i\tau)) \xi_{N_1}^{(j_1)}(\sigma_1 + i\tau)), \tag{2.15}
$$

In checking locality, one encounters the product of two $R$ matrices. It is handled by means of the identities

$$
((J_1, J_2)^{-N_2}{N_1}_{M_1 - M_2})^* = ((J_2, J_1)^{M_1}{M_2}_{N_2 - N_1})^* = (J_2, J_1)^{M_1}{M_2}_{N_2 N_1} \tag{2.16}
$$

that follow from the explicit expressions Eqs.A.27, and A.29. In this way one deduces the equation

$$
\sum_{M_1 M_2} (J_1, J_2)^{P_1 P_2}_{M_1 M_2} ((J_1, J_2)^{-N_2}{N_1}_{M_1 - M_2})^* = \delta_{P_1 N_1} \delta_{P_2 N_2} \tag{2.17}
$$

from the inverse relation Eq.A.30, and the desired locality relation follows ($a(\varpi)$ does not play any role so far), that is,

$$
e^{-J_1 \alpha + \Phi(\sigma_1, \tau)} e^{-J_2 \alpha - \Phi(\sigma_2, \tau)} = e^{-J_2 \alpha - \Phi(\sigma_2, \tau)} e^{-J_1 \alpha + \Phi(\sigma_1, \tau)} \tag{2.18}$$
Another requirement is that the restricted Hilbert space $\mathcal{H}_r$ be left invariant. This is verified by re-expressing Eq. 2.8 in terms of $\psi$ fields. One gets, at first

$$e^{-J\alpha_-\Phi(\sigma, \tau)} = c_J a(\varpi) \times \sum_{M=-J}^{J} (-1)^{J-M} e^{ih(J-M)} |J, \varpi\rangle_M^m (|J, \varpi\rangle_{-M}^p)^* \psi_m^{(J)}(x_+) \overline{\psi}_p^{(J)}(x_-)/a(\varpi) \quad (2.19)$$

Using Eq. A.31, one writes

$$\sum_{M=-J}^{J} (-1)^{J-M} e^{ih(J-M)} |J, \varpi\rangle_M^m (|J, \varpi\rangle_{-M}^p)^* = \sum_{M=-J}^{J} (-1)^{J-M} e^{ih(J-M)} |J, \varpi\rangle_M^m |J, \varpi - 2p\rangle_{-M}^{-p} \quad (2.20)$$

If $\varpi = \varpi^*$, this becomes, according to Eqs. A.18, A.22, and A.23,

$$\sum_{M=-J}^{J} (-1)^{J-M} e^{ih(J-M)} |J, \varpi\rangle_M^m |J, \varpi + 2p\rangle_{-M}^{-p} = (-1)^{J-m} \left(2i \sin(h)e^{ih/2}\right)^{2J} \delta_{m,p} \frac{\lambda_m^{(J)}(\varpi)}{[\varpi + 2m]} \quad (2.21)$$

As a consequence, and when it is restricted to $\mathcal{H}_r$, Eq. 2.8 is equivalent to

$$e^{-J\alpha_-\Phi(\sigma, \tau)} = c_J a(\varpi) \sum_{m=-J}^{J} (-1)^{J-m} \frac{\lambda_m^{(J)}(\varpi)}{[\varpi + 2m]} \psi_m^{(J)}(\sigma) \overline{\psi}_m^{(J)}(\sigma)/a(\varpi), \quad (2.22)$$

where $c_J = \tilde{c}_J \left(2i \sin(h)e^{ih/2}\right)^{2J}$. The condition $\varpi^* = \varpi$ is indeed left invariant, in view of Eqs. [2.34] and [2.7]. Finally, and for reasons that will become clear in section 4, we choose $a(\varpi) = 1/\sqrt{[\varpi]}$. This gives

$$e^{-J\alpha_-\Phi(\sigma, \tau)} = c_J \sum_{m=-J}^{J} \frac{(-1)^{J-m} \lambda_m^{(J)}(\varpi)}{\sqrt{[\varpi]} \sqrt{[\varpi + 2m]}} \psi_m^{(J)}(x_+) \overline{\psi}_m^{(J)}(x_-) \quad (2.23)$$

which is the expression we shall use later on. This choice of $a(\varpi)$ will ensure the symmetry of the three-point functions, as we shall see. At this point a parenthesis is in order. The physical meaning of the condition $\varpi^* = \varpi$ is as follows. When we take $\varpi$ real at the end, it will mean that left and right movers have equal momenta. This will ensure that the Liouville field is periodic in $\sigma$, as it should be, if the winding number vanishes. In this connection, let us note that it is easy to see, from the explicit expression Eq. A.19 of $|J, \varpi\rangle_M^m$, that a similar discussion applies to arbitrary winding number. Then one would have $\varpi^* = \varpi + r\pi/h$ with $r$ integer. For simplicity, we only consider the case $r = 0$ in the following.
2. Positive spins

There remains to show that the operators just constructed are closed by fusion. For this purpose, the expression Eq.2.8 is more handy, since the fusion properties of the \( \xi \) fields are determined by their quantum group structure[7, 19]. One has, on the unit circle,

\[
\xi^{(J)}_{M_1}(\sigma_1) \xi^{(J)}_{M_2}(\sigma_2) = \sum_{J=|J_1-J_2|}^{J_1+J_2} \left\{ (d(\sigma_1 - \sigma_2))^{\Delta(J_1)-\Delta(J_1)-\Delta(J_2)} \right\} g^J_{J_1 J_2} \left( J_1, M_1; J_2, M_2 | J_1, J_2; J, M_1 + M_2 \right) \left( \xi^{(J)}_{M_1+M_2}(\sigma_1) + \text{descendants} \right),
\]

(2.24)

where \( d(\sigma - \sigma') \equiv 1 - e^{-i(\sigma - \sigma')} \), \((J_1, M_1; J_2, M_2 | J_1, J_2; J, M_1 + M_2)\) denotes the Clebsch-Gordan coefficients of \( U_q(\text{SL}(2)) \) (see, e.g. appendix C of [7]), \( g^J_{J_1 J_2} \) are numerical constants, and \( \Delta(J) := -h(J + 1) / \pi - J \) is the Virasoro-weight of \( \xi^{(J)}_{M}(\sigma) \). The argument given above shows that the fusion of the \( x_- \) components, is given by a similar formula the Clebsch-Gordan (CG) coefficients are replaced by their complex conjugate. Closure by fusion will follow from the following identity

\[
(J_1, -M_1; J_2, -M_2 | J_1, J_2; J, -M_1 - M_2)^* = (J_1, M_1; J_2, M_2 | J_1, J_2; J, M_1 + M_2)
\]

(2.25)

The proof of this relation goes as follows: The explicit expression of the (CG) coefficients is of the form[7]

\[
(J_1, M_1; J_2, M_2 | J_1, J_2; J, M) = \delta_{M, M_1+M_2} e^{ih(J_1+J_2-J)(J_1+J_2+J+1)/2} K(J_1, J_2, J) \times \\
e^{-h(\mu+2)(J_1+J_2-J)/2} \sum_{\mu=0}^{J_1+J_2-J} \left\{ \frac{1}{[\mu!] [J_1 + J_2 - J - \mu]!} \right\}
\]

(2.26)

where \( K(J_1, J_2, J) \) is real. One takes the complex conjugate of this expression and change \( \mu \) into \( J_1 + J_2 - J - \mu \) in the summation, and the result follows.

Closure by fusion is verified on the unit circle by writing, according to Eq. 2.24,

\[
e^{-J_1 \alpha \Phi(\sigma_1, 0)} e^{-J_2 \alpha \Phi(\sigma_2, 0)} = a(\varpi)c_{J_1}c_{J_2} \times \\
\sum_{J=|J_1-J_2|}^{J_1+J_2} \sum_{M_1}^{J_1-J_1} \sum_{M_2}^{J_2-J_2} (-1)^{J_1-M_1} e^{ih(J_1-M_1)} \sum_{M_2=-J_2}^{J_2} (-1)^{J_2-M_2} e^{ih(J_2-M_2)} \left\{ (d(\sigma_1 - \sigma_2))^{\Delta(J)-\Delta(J_1)-\Delta(J_2)} (d(\sigma_1 - \sigma_2)^*)^{\Delta(J)-\Delta(J_1)-\Delta(J_2)} g^J_{J_1 J_2} (g^J_{J_1 J_2})^* \right\} \left( \xi^{(J)}_{M_1+M_2}(\sigma_1) + \text{descendants} \right) \left( \xi^{(J)}_{M_1-M_2}(\sigma_1) + \text{descendants} \right) \right\}
\]

(2.27)
The summation over \(M_1, M_2\) may be explicitly carried out for fixed \(M = M_1 + M_2\), since it takes the form

\[
\sum_{M_1+M_2=M} (J_1, M_1; J_2, M_2|J_1, J_2; J, M) (J_1, -M_1; J_2, -M_2|J_1, J_2; \overline{J}, -M)^*.
\]

According to Eq. 2.24 one gets

\[
\sum_{M_1+M_2=M} (J_1, M_1; J_2, M_2|J_1, J_2; J, M) (J_1, M_1; J_2, M_2|J_1, J_2; \overline{J}, M) = \delta_{J,\overline{J}} \tag{2.28}
\]

where the last equality follows from the orthogonality of the q CG coefficients\(^7\). Thus Eq. 2.27 becomes

\[
e^{-J_1\alpha_- \Phi(\sigma_1,0)} e^{-J_2\alpha_- \Phi(\sigma_2,0)} = \sum_{J=|J_1-J_2|}^{J_1+J_2} \{|d(\sigma_1 - \sigma_2)|^2(\Delta(J) - \Delta(J_1) - \Delta(J_2))
\]

\[
\frac{c_J c_{J_2}}{c_J} |g_{J_1, J_2}|^2 \left( e^{-J\alpha_- \Phi(\sigma_1,0)} + \text{descendants} \right). \tag{2.29}
\]

This shows the closure of the operator-product expansion at the level of primaries.

Finally, we come to the most general Liouville field \(\exp(-J\alpha_- + J\alpha_+)\). It is most simply obtained by fusion from \(\exp(-J\alpha_- \Phi)\) and \(\exp(-\overline{J}\alpha_+ \Phi)\). From the quantum-group viewpoint, the existence of two screening charges comes from the fact that the relation between \(h\) and \(C\) (Eq. 2.3) is quadratic and has two solutions. Thus there are two quantum-group parameters \(h\) and \(\hat{h}\) which are such that

\[
C = 1 + 6 \left( \frac{h}{\pi} + \frac{\pi}{h} + 2 \right) = 1 + 6 \left( \frac{\hat{h}}{\pi} + \frac{\pi}{\hat{h}} + 2 \right), \quad \text{with} \quad h\hat{h} = \pi^2, \tag{2.30}
\]

and \(\alpha_+\) is such that \(\hat{h} = \pi(\alpha_+)^2\). The UCF associated with \(\hat{h}\) is distinguished by hats, and the hatted counterparts of Eqs. 2.8 and 2.23 are, respectively,

\[
e^{-\hat{J}\alpha_+ \Phi(\sigma, \tau)} = \left( \frac{2i \sin(\hat{h}) e^{i\hat{h}/2}}{\sqrt{\hat{\omega}}} \right)^{-\hat{J}} \sum_{n=-\hat{J}}^{\hat{J}} \xi_M^\hat{J}(x_+) \xi_M^{-\hat{J}}(x_-) \sqrt{\hat{\omega}} \tag{2.31}
\]

\[
e^{-\overline{J}\alpha_+ \Phi(\sigma, \tau)} = \hat{c}_{\overline{J}} \sum_{\overline{m}=-\overline{J}}^{\overline{J}} \overline{\xi}_{\overline{M}}^{\overline{J}}(\overline{\omega}) \overline{\psi}_{\overline{m}}^{\overline{J}}(x_+) \overline{\psi}_{\overline{m}}^{-\overline{J}}(x_-) \tag{2.32}
\]

The momentum \(\overline{\omega}\) is defined in complete parallel to \(\omega\), in order to keep the symmetry between the two quantum group-parameters. There is really one momentum only and they are proportional\(^3\) (see Eq. A.10): \(\pi\overline{\omega} = h\omega\). The two UCF’s have simple braiding relations: for \(\pi > \sigma_1 > \sigma_2 > 0\) one has\(^3\)

\[
\psi_{\overline{m}}^{(J)}(\sigma_1) \overline{\psi}_{\overline{m}}^{(J)}(\sigma_2) = e^{-2i\pi J\hat{J}} \psi_{\overline{m}}^{(J)}(\sigma_2) \psi_{\overline{m}}^{(J)}(\sigma_1), \tag{2.33}
\]
where it immediately follows from the above formulae that, if we define,

\[
\xi_m^{(j)}(\sigma_1) \xi_m^{(\hat{j})}(\sigma_2) = e^{-2i\pi \hat{J}_J} e^{2i\pi (M \hat{J} - \hat{M} J)} \xi_m^{(\hat{j})}(\sigma_2) \xi_m^{(j)}(\sigma_1). \tag{2.34}
\]

Thus it follows trivially that

\[
e^{-J_1 \alpha - \Phi(\sigma_1, \tau)} e^{-\hat{J}_2 \alpha + \Phi(\sigma_2, \tau)} = e^{-4i\pi J_1 \hat{J}_2} e^{-\hat{J}_2 \alpha + \Phi(\sigma_2, \tau)} e^{-J_1 \alpha - \Phi(\sigma_1, \tau)}. \tag{2.35}
\]

Since \(2J_1\) and \(2\hat{J}_2\) are integers, the factor \(\exp(-4i\pi J_1 \hat{J}_2)\) is equal to 1 and the two set of fields are mutually local. It is interesting to note at this point that the condition that \(4i\pi J_1 \hat{J}_2\) be an integer is similar to Dirac's quantization condition for the product of electric and magnetic charges. There remains to consider the fusion of the fields with the two values of the screening charges. The hatted and unhatted fields have simple fusion properties [5]:

\[
\xi_m^{(j_1)}(\sigma_1) \xi_m^{(\hat{j}_2)}(\sigma_2) \sim (d(\sigma_1 - \sigma_2))^\Delta_{Kac}(J_1, \hat{J}_2) - \Delta(J_1) - \Delta(\hat{J}_2) e^{i\pi (M_1 \hat{J}_2 - \hat{M}_2 J_1)} \xi_m^{(j_1, \hat{j}_2)}(\sigma_1), \tag{2.36}
\]

where \(\xi_m^{(j_1, \hat{j}_2)}\) is the most general chiral field whose weight is given by Kac's formula. It immediately follows from the above formulae that, if we define,

\[
e^{-(J \alpha_+ + \hat{J} \alpha_-)} \Phi(\sigma, \tau) = \frac{c_{J \hat{J}}}{\omega} \times
\]

\[
\sum_{M, \hat{M}} (-1)^{J - M + \hat{J} - \hat{M}} e^{ih(J - M) + i\hat{h}(\hat{J} - \hat{M})} \xi_M^{(j_1 \hat{j})}(x_+) \xi_m^{(j_1 \hat{j})}(x_-) e^{\Delta_{Kac}(J_1, \hat{J}_2) - \Delta(J_1) - \Delta(\hat{J}_2)} e^{i\pi (M_1 \hat{J}_2 - \hat{M}_2 J_1)} \xi_m^{(j_1, \hat{j}_2)}(\sigma_1), \tag{2.37}
\]

with

\[
c_{J \hat{J}} = c_J c_{\hat{J}} \left(2i \sin(h) e^{ih/2}\right)^{-2J} \left(2i \sin(\hat{h}) e^{i\hat{h}/2}\right)^{-2\hat{J}},
\]

we have

\[
e^{-J \alpha_- \Phi(\sigma_1, \tau)} e^{-\hat{J} \alpha_+ \Phi(\sigma_2, \tau)} \sim \nonumber
\]

\[
|d(\sigma_1 - \sigma_2)|^{2\Delta_{Kac}(J_1, \hat{J}_2) - \Delta(J_1) - \Delta(\hat{J}_2)} e^{-(J \alpha_+ + \hat{J} \alpha_-) \Phi(\sigma_2, \tau)}. \tag{2.38}
\]

Closure by fusion and braiding of the most general field follows from this last equality if one assumes that the order between fusions or fusion and braiding is irrelevant. It may be directly verified. This is left to the reader. Finally, another useful expression for the most general Liouville field may be derived in the Bloch-wave basis. The field \(\psi_{m_1 \hat{m}_2}^{(j_1 \hat{j}_2)}\) is such that [6]

\[
\psi_{m_1 \hat{m}_2}^{(j_1 \hat{j}_2)}(\sigma_1) \psi_{m_2 \hat{m}_1}^{(j_2 \hat{j}_1)}(\sigma_2) \sim (d(\sigma_1 - \sigma_2))^\Delta_{Kac}(J_1, \hat{J}_2) - \Delta(J_1) - \Delta(\hat{J}_2) (-1)^{2(J_1 \hat{J}_2 - m_1 \hat{m}_2)} \psi_{m_1 \hat{m}_2}^{(j_1 \hat{j}_2)}(\sigma_1), \tag{2.39}
\]

and it follows from Eqs.2.23, 2.32 2.37 that

\[
e^{-(J \alpha_+ + \hat{J} \alpha_-) \Phi(\sigma, \tau)} = c_{J \hat{J}} \sum_{m \hat{m}}
\]
The shift of momentum for the general $\psi$ field is given by

$$
\psi^{(J \hat{J})}_{m \hat{m}}(\sigma) \varpi = (\varpi + 2m + 2\hat{m}\pi/h) \psi^{(J \hat{J})}_{m \hat{m}}(\sigma).
$$

Thus the spectrum of eigenvalues of $\varpi$ is of the form $\varpi^{(0)} + n + \hat{n}\pi/h$ where $n$ and $\hat{n}$ are integers, and $\varpi^{(0)}$ is arbitrary. The $sl(2,C)$ invariant vacuum $|\varpi_0>$, which is such that $L_0|\varpi_0> = 0$ has a momentum $\varpi_0 = 1 + \pi/h$ (see Eq.A.11). Physically we should choose $\varpi^{(0)} = \varpi_0$, and the spectrum of $\varpi$ eigenvalues is real. At an intermediate stage, however, it is useful to give a small imaginary part, say positive, to $\varpi$ and to choose $\varpi^{(0)} = \varpi_0 + i\epsilon$. This allows us to handle possible branch points in $\varpi$ on the real axis, such as one sees in Eqs. (2.23, 2.32, and 2.40). The monodromy factor Eq. (2.4) is such that

$$
\exp(2ihm(\varpi_0 + n + \hat{n}\pi/h)) = \exp(2iJ\hat{n}) \exp(2ihm(\varpi_0 + n)).
$$

Thus $\hat{n}$ describes winding (fermionic or bosonic) with respect to the UCF with parameter $h$. The general fields $\psi^{(J \hat{J})}_{m \hat{m}}(\sigma)$, or equivalently $\xi^{(J \hat{J})}_{m \hat{m}}(\sigma)$, form a general chiral family (GCF) with a quantum-group structure $U_q(sl(2)) \otimes \overline{U}_q(sl(2))$ where the sign $\otimes$ means that the two UCF do not commute but satisfy Eqs. (2.33, 2.34) instead. The quantum-group structure of the most general Liouville field is of the type $U_q(sl(2)) \otimes \overline{U}_q(sl(2)) \otimes \overline{U}_q(sl(2)) \otimes \overline{U}_q(sl(2))$, where the overlined part correspond to the minus-components that we took to commute with the plus-components.

This completes the construction of the Liouville field on the cylinder. In the coming sections, we shall determine the three-point functions on the Riemann sphere. Some additional points are to be made for this purpose, before closing the present section. The Riemann sphere, is described by the complex variable $z = \exp(\tau + i\sigma)$. In this connection, let us recall that a primary field $A(z, z^*)$ of weight $\Delta, \overline{\Delta}$ transforms so that $A(z, z^*)(dz)^\Delta(dz^*)^{\overline{\Delta}}$ is invariant [20]. Given $A(z, z^*)$, and a conformal map $z \rightarrow Z(z)$, it is thus convenient to define

$$
A(Z, Z^*) := A(z, z^*) \left( \frac{dz}{dZ} \right)^\Delta \left( \frac{dz^*}{dZ^*} \right)^{\overline{\Delta}}.
$$

Thus one has

$$
A(z, z^*) = e^{-(\tau + i\sigma)\Delta - (\tau - i\sigma)\overline{\Delta}} A(\sigma, \tau)
$$

$A(\sigma, \tau)$, which we had been using before, was defined on the cylinder where the Fourier expansion in $\sigma$ has integer coefficients (see, e.g. Eq.A.1). This rule gives

\footnote{We do not write down the operators that realize the transformation of the states on which $A$ acts. This is an abuse of notation made to avoid clumsy formulae.}
the standard definition for the Laurent expansion of $A(z, z^*)$ as series in $z^{n+\Delta}$, and $(z^*)^{m+\Delta}$, $n$, $m$ integers. Moreover, $A(z, z^*)$ now transforms covariantly under $z$ and $z^*$ translations. Thus OPE’s take the usual form; for instance Eq.2.28 becomes

$$e^{-J_1\alpha_-\Phi(z_1, z_1^*)}e^{-J_2\alpha_-\Phi(z_2, z_2^*)} = \sum_{J=|J_1-J_2|}^{J_1+J_2} \{|z_1-z_2|\}^{2(\Delta(J)-\Delta(J_1)-\Delta(J_2))} \frac{c_{J_1}c_{J_2}}{c_J} |g_{J_1J_2}|^2(e^{-J\alpha_-\Phi(z_1, z_1^*)} + \text{descendants})\right\}. \quad (2.45)$$

The last point to discuss is the behaviour of the fields as $z$ goes to zero or $\infty$. The situation is not standard, since we have several fields with the same weights, so that the connection between primaries and highest-weight states is not one-to-one. Concerning the $\psi$ fields we are going to show that

$$\lim_{z \to 0} \psi_{m\hat{m}}^{(J\hat{J})}(z)|\omega_0 > \propto \delta_{m+J,0} \delta_{\hat{m}+\hat{J},0}|\omega_{J\hat{J}} >$$

$$\lim_{z \to \infty} < -\omega_0|\psi_{m\hat{m}}^{(J\hat{J})}(1/z) \propto \delta_{m+J,0} \delta_{\hat{m}+\hat{J},0} < -\omega_{J\hat{J}},$$

where

$$\omega_{J\hat{J}} = \omega + 2J + 2\hat{J}\pi/h. \quad (2.46)$$

First, the value of $\omega_{J\hat{J}}$ was to be expected since Eqs.A.11 and A.33 show that

$$(L_0 - \Delta_{Kac}(J, \hat{J}, C))|\omega_{J\hat{J}} > = -\omega_{J\hat{J}}(L_0 - \Delta_{Kac}(J, \hat{J}, C)) = 0, \quad (2.48)$$

where $\Delta_{Kac}(J, \hat{J}, C)$ is the weight of $\psi_{m\hat{m}}^{(J\hat{J})}$. Thus the eigenvalue of $L_0$ is indeed equal to the weight of the primary field $\psi_{m\hat{m}}^{(J\hat{J}}(z)$. The other primaries $\psi_{m\hat{m}}^{(J\hat{J})}(z)$, $m + J \neq 0$, or $\hat{m} + \hat{J} \neq 0$ have the same weight but their shift are different. Thus, if the limits were not zero, states with wrong eigenvalues of $L_0$ would come out. This is basically why Eq.2.46 gives zero unless $m + J = 0$, and $\hat{m} + \hat{J} = 0$. These relations may be verified as follows. By operator-product expansions, the powers of the field $\psi_{-1/2}(z) \equiv \psi_1$ generate the fields $\psi_{-j}(z)$. Since the former is a simple exponential of the free field $\phi_1$ this is also true for the latter, and one has, according to Eq.A.6 with $j = 1$,

$$\psi_{-j}(z) \propto N^{(1)}(e^{2J\sqrt{h/2\pi} \phi_1}) = e^{2J\sqrt{h/2\pi} q_0^{(1)}(z)2J(\omega - \omega_0)h/2\pi} \times \exp(2J\sqrt{h/2\pi}i \sum_{n<0} e^{-in\sigma}p_n^{(1)}/n) \exp(2J\sqrt{h/2\pi}i \sum_{n>0} e^{-in\sigma}p_n^{(1)}/n) \quad (2.49)$$

This immediately leads to the desired relations with $m + J = 0$, and $\hat{m} = \hat{J} = 0$, using the fact that for any highest-weight state $|\omega>$,

$$p_n^{(j)}|\omega > = < \omega|p_n^{(j)} = 0, \quad n > 0, \quad j = 1, 2. \quad (2.50)$$
3. Dressing by gravity

Next, and by the same reasoning, Eq. (A.6) with \( j = 2 \) gives

\[
\psi^{(J)}_{J}(z) = N^{(2)}(e^{2J\sqrt{h/2\pi}}) = e^{2J\sqrt{h/2\pi}q_{(2)}^{(2)}}z^{2J(-\omega - \omega_0)h/2\pi} \\
\times \exp\left(2J\sqrt{h/2\pi i} \sum_{n<0} e^{-\text{in}\sigma p_{n}^{(2)}/n}\right) \exp\left(2J\sqrt{h/2\pi i} \sum_{n>0} e^{-\text{in}\sigma p_{n}^{(2)}/n}\right)
\]

(2.51)

Using Eq.2.50, with \( j = 2 \), one sees that the limit Eq.2.46 vanishes. This is the desired result for \( J - m = 0, \hat{J} = \hat{m} = 0 \). A similar discussion obviously deals with the cases \( J = 0, m = 0, \hat{J} \pm \hat{m} = 0 \). Finally, the result is easily extended to the other values of \( m, \hat{m} \), and to the general case \( J \neq 0, \hat{J} \neq 0 \) by fusion.

It follows from Eqs 2.41 that

\[
e^{-(J\alpha_+ + \hat{J}\alpha_+)\Phi(z, z^*)}|\omega_0 > \sim z \to 0
\]

\[
|\omega_{J, \hat{J}} > < \omega_{J, \hat{J}} | e^{-(J\alpha_+ + \hat{J}\alpha_+)\Phi(1, 1)|\omega_0 >, \quad (2.52)
\]

\[
< -\omega_0|e^{-(J\alpha_+ + \hat{J}\alpha_+)\Phi(1/z, 1/z^*)} \sim z \to \infty
\]

\[
< -\omega_0|e^{-(J\alpha_+ + \hat{J}\alpha_+)\Phi(1, 1)} | - \omega_{J, \hat{J}} > < -\omega_{J, \hat{J}} |
\]

(2.53)

where the matrix elements on the right-hand side are given by the expansion Eq.2.40 restricted to \( m = -J \), and \( \hat{m} = -\hat{J} \). One sees that the treatment of the limits breaks the symmetry between the two free fields \( \phi_1 \) and \( \phi_2 \) which is the basis of the quantum group symmetry. It is thus spontaneously broken by the choice of matrix elements. One may exchange the role of \( \phi_1 \) and \( \phi_2 \), by replacing \(|\omega_0 >\), and \(< -\omega_0|\) by \(| - \omega_0 >\), and \(< \omega_0 |\) respectively.

3 THE DRESSING BY GRAVITY

In this section we study the dressing of conformal models with central charge \( D \) by the Liouville field with central charge \( C \) so that

\[
C + D = 26. \quad (3.1)
\]

We shall be concerned with the case \( D < 1 \), where the Liouville theory is in its weakly coupled regime \( C > 25 \). As is recalled in the appendix A, the existence of the UCF’s is basically a consequence of the operator differential equations A.8, A.9. These are equivalent to the Virasoro Ward-identities that describe the decoupling of null vectors. Thus the UCF’s, with appropriate quantum deformation parameters also describe the matter with \( D < 1 \). We will thus have another copy of the quantum-group structure recalled above. It will be distinguished by primes. Thus we let

\[
D = 1 + 6(\frac{h'}{\pi} + \frac{\pi}{h' + 2}) = 1 + 6(\frac{\hat{h}'}{\pi} + \frac{\pi}{\hat{h}' + 2}), \quad \text{with} \quad h'\hat{h}' = \pi^2, \quad (3.2)
\]
$h' = \frac{\pi}{12}(D - 13 - \sqrt{(D - 25)(D - 1)}), \quad \hat{h}' = \frac{\pi}{12}(D - 13 + \sqrt{(D - 25)(D - 1)}), \quad (3.3)$

and Eq.3.1 gives

$$h + \hat{h}' = \hat{h} + h' = 0 \quad (3.4)$$

Of course we choose the matter and gravity fields to commute. Using the notation of last section, the complete quantum group structure is thus of the type:

$$\left\{ \left[ U_q(sl(2)) \otimes U_q(sl(2)) \right] \otimes U_q(sl(2)) \right\} \otimes \left\{ \left[ U_{q^{-1}}(sl(2)) \otimes U_{q^{-1}}(sl(2)) \right] \otimes U_q^{-1}(sl(2)) \right\}, \quad (3.5)$$

where the first (second) line displays the quantum-group structure of gravity (matter). According to the above results, the spectrum of weights of the gravity and matter are respectively given by

$$\Delta_g(J, \tilde{J}) = \frac{C - 1}{24} - \frac{1}{24} \left( (J + \tilde{J} + 1)\sqrt{C - 1} \right. \left. - (J - \tilde{J})\sqrt{C - 25} \right)^2, \quad (3.6)$$

$$\Delta_M(J', \tilde{J}') = \frac{D - 1}{24} + \frac{1}{24} \left( (J' + \tilde{J}' + 1)\sqrt{1 - D} \right. \left. + (J' - \tilde{J}')\sqrt{25 - D} \right)^2. \quad (3.7)$$

The connection with Kac’s table will be spelled out in section 5. From the standpoint we are taking, the most general matter field is described by an operator of the form exp$(- (J'\alpha_+ + \tilde{J}'\alpha_+)X)$, where $X(\sigma, \tau)$ is a local field that commutes with the Liouville-field and whose properties are derived from those of $\Phi$ by continuation to central charges smaller than one. Following our general conventions we let $h' = \pi(\alpha_-')^2/2$, and $\hat{h}' = \pi(\alpha_+')^2/2$. The correct screening operator is the field exp$(- (J\alpha_- + \tilde{J}\alpha_+)\Phi)$ with spins $J$ and $\tilde{J}$ that $\Delta_g(J, \tilde{J}) + \Delta_M(J', \tilde{J}') = 1$. In this connection, it is an easy consequence of Eq.3.1 that

$$\Delta_g(\tilde{J}', -J' - 1) + \Delta_M(J', \tilde{J}') = 1 \quad (3.8)$$

$$\Delta_g(-\tilde{J}' - 1, J') + \Delta_M(J', \tilde{J}') = 1 \quad (3.9)$$

These two choices correspond to the existence of two cosmological terms. Indeed, the unity-operator of matter ($J' = \tilde{J}' = 0$) is dressed by operators of the type exp$(\alpha_+ \Phi)$ and exp$(\alpha_- \Phi)$, with the choices Eqs.3.8, 3.9, respectively. As is usual, we choose the latter as cosmological term so that the spins of gravity and matter fields will be related by Eq.3.3. Thus we shall be concerned with matrix elements of operators of the type

$$\mathcal{V}_{J', \tilde{J}'}(\sigma, \tau) \equiv e((\tilde{J}' + 1)\alpha_- - J'\alpha_+)\Phi(\sigma, \tau) \quad e^{-(J'\alpha_+ + \tilde{J}'\alpha_')X(\sigma, \tau)}. \quad (3.10)$$

Next consider the Hilbert space in which we are working. The UCF’s which appear in Eq.3.3 live in spaces with highest-weight vectors of the form

$$|\varpi > \otimes |\varpi > \otimes |\varpi > \otimes |\varpi >. \quad (3.11)$$
Indeed, as is hopefully clear from the appendix A (and ref.[5]), the two UCF’s of a product of the type $U_q(sl(2)) \otimes U_{\hat{q}}(sl(2))$ are realized in the same Hilbert space. In the restricted Hilbert space, one has $\varpi^* = \varpi$, and $\varpi'^* = \varpi'$. Thus we introduce states of the form

$$|\varpi, \varpi' > \equiv |\varpi > \otimes |\varpi^* > \otimes |\varpi' > \otimes |\varpi'^* > .$$ (3.12)

The next point concerns the physical on-shell states. They should satisfy the condition

$$(L_0 + \overline{L}_0 + L'_0 + \overline{L}'_0 - 2)|\varpi, \varpi' > = 0,$$ (3.13)

where the notation is self-explanatory. According to Eq.A.11, this is satisfied if

$$\frac{\hbar}{4\pi}(1 + \frac{\pi}{\hbar})^2 - \frac{\hbar}{4\pi}(1 + \frac{\pi}{\hbar'})^2 = 1.$$ (3.14)

It follows from Eqs.2.30, 3.1, 3.2, that

$$\frac{\hbar}{4\pi}(1 + \frac{\pi}{\hbar})^2 + \frac{\hbar'}{4\pi}(1 + \frac{\pi}{\hbar'})^2 \equiv \frac{C + D - 2}{24} = 1.$$ (3.15)

Moreover, according to Eqs.A.10, and 3.4,

$$\hbar' \varpi'^2 = \hat{\hbar}' \varpi'^2 = -\hbar \varpi'^2$$

so that, according to Eq.3.4, the on-shell condition is $\varpi^2 = \varpi'^2$. The sign ambiguity is related with the two possibilities of cosmological term. It will be seen below that the choice, which is consistent with the above definition of gravity-dressing is

$$\varpi = -\varpi', \text{ or, equivalently, } \varpi = \varpi'.$$ (3.16)

Our next topic is the precise definition of the operators $\mathcal{V}_{J', \hat{J}'}(\sigma, \tau)$. Concerning matter, choosing $J' > 0$ and $\hat{J}' > 0$ gives $\Delta_M > 0$ and the formulae of last section may be directly used, obtaining,

$$e^{-(J'\alpha'_- + \hat{J}'\alpha'_+)}X(\sigma, \tau) = e^{J'\alpha'_- + \hat{J}'\alpha'_+} \sum_{m' \hat{m}'} (-1)^{J'-m'+\hat{J}'-m'+2(J'-m')} \chi^{(J')}_{mm'}(\varpi') \chi^{(\hat{J}')}_{\hat{m}'\hat{m}'}(\varpi') \times$$

$$\frac{\chi^{(J')}_{m'\hat{m}'}(\varpi') \chi^{(\hat{J}')}_{\hat{m}'\hat{m}'}(\varpi')}{\sqrt{\varpi'} \sqrt{\varpi' + 2m' + 2\hat{m}'\pi/\hbar}}$$

$$\times \sqrt{\varpi'} \sqrt{\varpi' + 2m' + 2\hat{m}'\pi/\hbar} \times \chi^{(J')}_{m'\hat{m}'}(x+) \chi^{(\hat{J}')}_{\hat{m}'\hat{m}'}(x-),$$ (3.17)

where $\chi$ and $\overline{\chi}$ are the Bloch-wave operators for matter, and where we have let, in general,

$$[y] \equiv \frac{\sin(h'y)}{\sin h'}, \quad [\hat{y}] \equiv \frac{\sin(\hat{h}'y)}{\sin \hat{h}'}.$$ (3.18)
According to Eq.3.4 one has
\[ [x] = [\hat{x}], \quad [\hat{x}] = [x]. \tag{3.19} \]

Concerning gravity, it has been already emphasized[3, 4], that the dressing of matter-operators with positive spins requires the use of gravity fields with negative spins. For example, the cosmological term has \( J = -1, \tilde{J} = 0 \). Thus the gravity-UCF with parameter \( h \) must be extended. This problem was overcome in [3, 4] as follows. First, the quantum-group structure has an obvious symmetry in \( J \to -J - 1 \) with fixed \( M \), so that the universal R-matrix and Clebsch-Gordan coefficients are left unchanged. Second, the coefficients \( | J, \varpi \rangle^m_M \) also have a natural continuation to negative spin. For positive \( J \), one has
\[ | -J - 1, \varpi \rangle^m_M = | J, \varpi \rangle^m_M (-1)^{J + m} \sqrt{\left[(2i\sin(h))^{1+2J} \lambda^{(J)}_m(\varpi) \right]} \tag{3.20} \]

It follows that there exist fields \( \xi^{(-J-1)}_M \) with fusion and braiding similar to \( \xi^{(J)}_M \), and fields \( \psi^{(-J-1)}_m \) similar to \( \psi^{(J)}_m \) so that
\[ \xi^{(-J-1)}_M(\sigma) = \sum_{m=-J}^{J} \frac{(-1)^{J + m} | J, \varpi \rangle^m_M}{(2i\sin(h))^{1+2J} \lambda^{(J)}_m(\varpi)} \psi^{(-J-1)}_m(\sigma) \tag{3.21} \]

This motivates us to introduce a field of the form
\[ e^{(J + 1)\alpha_- \tilde{\Phi}(\sigma, \tau)} = \frac{1}{\sqrt{\varpi}} \xi^{(-J-1)}_M \sum_{M=-J}^{J} (-1)^{J-M} e^{ih(J-M)} \xi^{(-J-1)}_M(x_+) \xi^{(-J-1)}_M(x_-) \sqrt{\varpi}. \tag{3.22} \]

We write it as an exponential of a new field \( \tilde{\Phi} \), since it differs from what one expects upon quantization of the classical expressions Eqs.1.12. This field satisfies all the necessary requirements. First it is local since the exchange properties of the fields \( \xi^{(-J-1)}_M \) and \( \xi^{(-J-1)}_M \) are the same as those of the fields \( \xi^{(J)}_M \), and \( \xi^{(J)}_M \), respectively. Second it may also be re-expressed in the Bloch-wave basis, making use of Eq.3.20. One finds, taking Eq.2.20 into account,
\[ e^{(J + 1)\alpha_- \tilde{\Phi}(\sigma, \tau)} = c_J \sum_{m=-J}^{J} \frac{1}{\lambda^{(J)}_m(\varpi) \sqrt{\varpi} \sqrt{\varpi + 2m}} \psi^{(-J-1)}_m(x_+) \bar{\psi}^{(-J-1)}_m(x_-), \tag{3.23} \]

which shows that this field leaves the condition \( \varpi = \varpi \) invariant. Remarkably, we will find that the range of \( m \) which appears in this last equation is precisely what is needed to compute matrix elements between physical states satisfying condition 3.16. This fully motivates the introduction of the field \( \tilde{\Phi} \). By fusion, one finally arrives at the expression of the dressing-operator
\[ e^{((J + 1)\alpha_- \tilde{\Phi}(\sigma, \tau) - \hat{J} \alpha_+ \Phi(\sigma, \tau))} = c_{-J-1} \sum_{m \tilde{m}} \frac{\lambda^{(J)}_m(\varpi) \sqrt{\varpi} \sqrt{\varpi + 2m + 2m\pi/h} \sqrt{\varpi} \sqrt{\varpi + 2\tilde{m} + 2\tilde{m}\pi/h}}{\lambda^{(J)}_{\tilde{m}}(\varpi) \sqrt{\varpi} \sqrt{\varpi + 2\tilde{m} + 2m\pi/h} \sqrt{\varpi} \sqrt{\varpi + 2m + 2m\pi/h}} \psi^{(-J-1)\hat{J}}_{m \tilde{m}}(x_+) \overline{\psi}^{(-J-1)\hat{J}}_{m \tilde{m}}(x_-). \tag{3.24} \]
Next we want to compute the matrix element of the operator \( \mathcal{V}_{m,n}^{J,\hat{J}}(\sigma, \tau) \), given by Eq.3.10, between highest-weight states. For this purpose, we shall make use of the expressions in terms of \( \psi \)-fields Eqs.3.17, 3.24. The matrix element of a typical operator \( \psi^{(j,m)} \) between highest weight states was fully determined in [3] for positive \( J \). In Appendix E of this reference, quantities denoted \( C^{\mu,\nu;\hat{\mu},\hat{\nu}} \) and \( D^{\mu,\nu;\hat{\mu},\hat{\nu}}(\varpi) \) were defined so that

\[
\psi^{(j,m)}(\sigma) \equiv C^{\mu,\nu;\hat{\mu},\hat{\nu}} D^{\mu,\nu;\hat{\mu},\hat{\nu}}(\varpi) V^{\mu,\nu;\hat{\mu},\hat{\nu}}(\sigma),
\]

(3.25)

where

\[
2J = \mu + \nu, \quad 2m = \nu - \mu, \quad 2\hat{J} = \hat{\mu} + \hat{\nu}, \quad 2\hat{m} = \hat{\nu} - \hat{\mu}.
\]

(3.26)

\( V^{\mu,\nu;\hat{\mu},\hat{\nu}}(\sigma) \) is an operator whose normalization does not depend upon its indices: its only non-vanishing matrix element is

\[
<\varpi|V^{\mu,\nu;\hat{\mu},\hat{\nu}}(\sigma)|\varpi - \mu + \nu(\hat{\mu} + \hat{\nu})\pi/h > = e^{i\sigma(\varpi^2 - (\varpi - \mu + \nu(\hat{\mu} + \hat{\nu})\pi/h)^2)/4h}.
\]

(3.27)

The fact that the shift of \( \varpi \) can take only one value follows from Eq.A.12. According to Eqs.A.16 and E.3 of [3], \( C^{\mu,\nu;\hat{\mu},\hat{\nu}} \) is given by

\[
C^{\mu,\nu;\hat{\mu},\hat{\nu}} = C^{\mu,\nu;\hat{\mu},\hat{\nu}} \nabla^\mu \nabla^\nu \prod_{r=1}^{\mu+\nu} \prod_{s=1}^{\hat{\mu}+\hat{\nu}} \left( r \sqrt{h/\pi + \hat{r} \sqrt{\pi/h}} \right) \prod_{s=1}^{\mu} \prod_{t=1}^{\nu} \left( t \sqrt{h/\pi + \hat{t} \sqrt{\pi/h}} \right)
\]

(3.28)

\[
C^{\mu,\nu} = \prod_{r=1}^{\mu+\nu} \Gamma(1 + rh/\pi)/\left( \prod_{s=1}^{\mu} \Gamma(1 + sh/\pi) \prod_{t=1}^{\nu} \Gamma(1 + th/\pi) \right)
\]

(3.29)

with similar formulae for \( \nabla^{\hat{\mu},\hat{\nu}} \). The \( \varpi \) dependent part \( D^{\mu,\nu;\hat{\mu},\hat{\nu}}(\varpi) \) is given by Eqs.26a,b and E.4 of [3]:

\[
D^{\mu,\nu;\hat{\mu},\hat{\nu}}(\varpi) = D^{\mu,\nu}(\varpi) \nabla^{\hat{\mu},\hat{\nu}}(\varpi) M^{\mu,\nu;\hat{\mu},\hat{\nu}}(\varpi)
\]

(3.30)

\[
D^{\mu,\nu}(\varpi) = \prod_{t=1}^{\mu-\nu} \sqrt{\Gamma((-\varpi - t + 1)h/\pi)} \prod_{r=1}^{\mu} \Gamma(\varpi - rh/\pi) \prod_{s=1}^{\nu} \Gamma((\varpi - s)h/\pi),
\]

(3.31)

if \( \mu > \nu \), and

\[
D^{\nu,\mu}(\varpi) = \prod_{t=1}^{\nu-\mu} \sqrt{\Gamma((-\varpi - t + 1)h/\pi)} \prod_{r=1}^{\mu} \Gamma((\varpi - r)h/\pi) \prod_{s=1}^{\nu} \Gamma((\varpi - s)h/\pi),
\]

(3.32)

if \( \nu > \mu \). Moreover, if for instance, \( \mu > \nu, \hat{\mu} > \hat{\nu} \),

\[
M^{\mu,\nu;\hat{\mu},\hat{\nu}}(\varpi) = \]

\[
\]

\[\text{up to a few misprints,}\]
3. Dressing by gravity

\[
\prod_{r=1}^{\mu} \prod_{s=1}^{\nu} (-z' - t + 1) \sqrt{h/\pi} + (\tilde{t} - 1) \sqrt{\pi/h} \sqrt{(z' - t) \sqrt{h/\pi} - \tilde{t} \sqrt{\pi/h}} \prod_{t=1}^{\mu} \prod_{\tilde{t}=1}^{\nu} \prod_{\tilde{r}=1}^{\mu} (-z + s) \sqrt{h/\pi} - \tilde{s} \sqrt{\pi/h}). \tag{3.33}
\]

Eqs. 3.31 and 3.32 correspond to a situation which one encounters repeatedly, namely, one goes from one to the other by continuing a product of the form \( \prod_{j=1}^{N} f_j \) to negative values of \( N \). The answer, which is well known in mathematics, is obtained by writing (this trick was already used in [7])

\[
\prod_{j=1}^{N} f_j = \prod_{j=1}^{A} f_j / \prod_{k=N+1}^{A} f_k,
\]

where \( A \) is an integer larger than \( N \). This leads to the rule

\[
\prod_{j=1}^{N} f_j \implies \prod_{j=N+1}^{0} \frac{1}{f_j} \equiv \prod_{j=1}^{N} \frac{1}{f_{1-j}}, \quad \text{for} \quad N \leq -1. \tag{3.34}
\]

Eqs. 3.31 and 3.32 are an example of this rule. It also applies, as one may verify from the derivation of [3], in order to define \( M_{\mu,\nu}^{\hat{\mu},\hat{\nu}} \) when \( \mu - \nu \), and/or \( \hat{\mu} - \hat{\nu} \) are not positive. In the following, we shall freely write products from 1 to negative numbers, with the understanding that they are defined from the rule Eq. 3.34.

At this point, a technical point must be straightened out: The expressions we wrote, such as Eqs. 3.31-3.33 involve square roots of functions of \( z' \), and we have to take account of the branch-points. They are for real eigenvalues of \( z' \). This is why we gave to \( z' \) an imaginary part to be removed at the end of the calculation. The derivation of Eqs. 3.31-3.33 of [3] did not explicitly specify the definitions of the square roots. In order to do so, as simply as possible, we remark that it is based on the solution of recurrence relations which may be reduced to the basic identity

\[
\sqrt{\Gamma(z+1)} = \sqrt{z} \sqrt{\Gamma(z)}, \tag{3.35}
\]

for an arbitrary complex variable \( z \). By convention, we choose the usual definition \( \sqrt{z} = \sqrt{\rho} \exp(i\theta/2) \), where \( zz^* = \rho^2 \), and where \( -\pi < \theta < \pi \) is the argument of \( z \). In this way, Eq. 3.35 allows us to uniquely define \( \sqrt{\Gamma(z)} \) in the whole complex plane with cuts between \( -2N - 2 \) and \( -2N - 1 \), \( N \) positive integer. Then the function \( \sqrt{\sin(\pi z)} \) is defined so that

\[
\sqrt{\Gamma(z)} \sqrt{\Gamma(1-z)} = \frac{\sqrt{\pi}}{\sqrt{\sin(\pi z)}}. \tag{3.36}
\]

The imaginary part of \( z' \) is unchanged by the \( \psi \)-fields which shift \( z' \) by real amounts. It removes the sign ambiguity for square roots of negative real numbers that may appear. This precaution is necessary at the present intermediate stage only, however,
since we shall see that all cuts disappear from the final answer. On the other hand, the appearance of cuts spoils the general argument, recalled in the appendix (Eqs.A.14, and A.32), that formally shows that the exponentials of the Liouville field are hermitian operators. This point already made briefly in ref.[2] is related to the objections to the construction of the Liouville field which were raised in ref.[21].

Formulæ Eqs.3.25 – 3.33 directly apply to the matter-field Eq.5.17, after making the suitable replacements by primed quantities. Since \( h' < 0 \), one may be worried at first sight by the factors \( \sqrt{h/\pi} \) that occur in Eqs.3.28 and 3.33. An easy computation shows that the product \( C^{\mu,\nu,\hat{\rho},\hat{\sigma}} M^{\mu,\nu,\hat{\rho},\hat{\sigma}}(\varpi) \) is a rational function of \( h \). There is thus no problem in replacing \( h \) by \( h' \). It is convenient to use expressions similar to the above that are symmetric between unhatted and hatted quantities. We adopt the following definitions

\[
C^{\mu,\nu,\hat{\rho},\hat{\sigma}} = \frac{C^{\mu',\nu',\hat{\rho}',\hat{\sigma}'} C^{\mu',\nu,\hat{\rho}',\hat{\sigma}'} \prod_{r=1}^{\mu'} \prod_{r=1}^{\nu'} (r \sqrt{\pi/h} - \tilde{r} \sqrt{h/\pi})}{\prod_{s=1}^{\mu'} \prod_{s=1}^{\nu'} (s \sqrt{\pi/h} - \tilde{s} \sqrt{h/\pi}) \prod_{t=1}^{\mu'} \prod_{t=1}^{\nu'} (t \sqrt{\pi/h} - \tilde{t} \sqrt{h/\pi})},
\]

(3.37)

\[
M^{\mu,\nu,\hat{\rho},\hat{\sigma}}(\varpi') = \frac{\prod_{r=1}^{\mu} \prod_{r=1}^{\nu} \sqrt{-(\omega' - t + 1) \sqrt{\pi/h} - (\tilde{t} - 1) \sqrt{\pi/h} \sqrt{(\omega' - t) \sqrt{\pi/h} + \tilde{t} \sqrt{h/\pi}}}}{\prod_{s=1}^{\mu} \prod_{s=1}^{\nu} \sqrt{-(\omega' - s) \sqrt{\pi/h} + \tilde{s} \sqrt{h/\pi}}},
\]

(3.38)

whose product is the correct continuation of Eqs.3.28 and 3.33.

For gravity we need to continue to negative \( J \). This was the aim of theorem (5.1) of [7]. The basic point is that the derivation of Eqs. 3.28–3.33 carried out in [7] is based on the hypergeometric differential equation which is obeyed by the two-point functions \( < \varpi_{2}\mid \psi_{m,0}^{(1/2)}(\sigma_{1}) \psi_{m,0}^{(J \hat{J})}(\sigma_{2}) \mid \varpi_{1} > \), and \( < \varpi_{2}\mid \psi_{m,0}^{(1/2)}(\sigma_{1}) \psi_{m,0}^{(J \hat{J})}(\sigma_{2}) \mid \varpi_{1} > \). This differential equation is a consequence of Eqs.A.8 and A.9 valid for any sign of \( J \) and \( \hat{J} \). Thus the coefficients \( C^{\mu,\nu,\hat{\rho},\hat{\sigma}} \), and \( D^{\mu,\nu,\hat{\rho},\hat{\sigma}}(\varpi) \) obey the same recursion relations (solved in appendix E of [7]) after continuation in \( J \) or \( \hat{J} \). With an appropriate choice of their overall normalisations, they are thus given by the continuation of Eqs.3.28 – 3.33 according to the rule Eq.3.34.

Our aim is to compute the three-point function of the form

\[
\langle \prod_{\ell=1}^{3} \mathcal{V}_{\ell'}(z_{\ell}, z'_{\ell}) \rangle = C_{1,2,3}/\left( \prod_{k,l} |z_{k} - z_{l}|^{2} \right).
\]

(3.39)

where \( C_{1,2,3} \) is the coupling constant. Since the fields \( \mathcal{V}_{\ell'}(z_{\ell}, z'_{\ell}) \) are local, their order should be irrelevant on the left-hand side. We nevertheless fix it temporarily in order to perform the calculation. As usual, we shall take \( z_{3} \to \infty, z_{2} = 1, z_{1} = 0 \), and consider the matrix element between the \( sl(2, C) \)-invariant states \( < -\varpi_{0}, -\varpi'_{0} > \) and \( |\varpi_{0}, \varpi'_{0} > \). Then \( C_{1,2,3} \) will be computed from the matrix elements between highest-weight states that come out following a reasoning similar to the one that
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The part of gravity with negative spin introduces a new subtlety in this connection, since the \( sl(2, C) \)-invariant states do not satisfy the physical condition Eq.3.10 (this is obvious since they are annihilated by \( L_0, L'_0, \overline{T}_0 \), and \( \overline{T}_0 \)). The derivation of Eqs.2.52 and 2.53 showed that the limits only involve the terms that are simple exponentials of the field \( \phi_1 \), that is, those for which \( m, \hat{m} \) are equal to minus the unhatted and the hatted spins, respectively. Those terms are absent from \( \exp[(J+1)\alpha_{-}\Phi] \), as defined by Eq.3.22, and thus we conclude that

\[
\lim_{z\to0} e^{(J+1)\alpha_{-}\Phi(z, z^*)}|\omega_0> = \lim_{z\to\infty} -e^{(J+1)\alpha_{-}\Phi(1/z, 1/z^*)} = 0. \quad (3.40)
\]

We need to define another field to create the highest-weight states of the spectrum from the \( sl(2, C) \)-invariant left and right vacua. Denoted by \( e^{(J+1)\alpha_{-}\Phi(z, z^*)} \), it is such that the limit is finite:

\[
e^{(J+1)\alpha_{-}\Phi(z, z^*)}|\omega_0> \sim_{z\to0} \omega_0 \quad \omega_{-J,0} <\omega_{-J,0}|e^{(J+1)\alpha_{-}\Phi(1, 1)}|\omega_0>, \quad (3.41)
\]

\[
<\omega_{-J,0}|e^{(J+1)\alpha_{-}\Phi(1/\omega, 1/z^*)} \sim_{z\to\infty} \omega_{-J,0} \quad <\omega_{-J,0} > <\omega_{-J,0} > \quad (3.42)
\]

We define matrix elements between the highest-weight states that appear in the last equations, as given by the continuation of Eq.2.40 to \( J = \pm m \). It is appropriate to write these fields as exponentials of the original Liouville field \( \Phi \) since they correspond to the quantum versions of the classical expression Eq.1.13. Finally the coupling constants are defined by

\[
C_{1,2,3} = <\omega_0, -\omega_0|\tilde{V}_{J_2, \tilde{J}_3} - \omega_3, -\omega_3' > \times \\
<\omega_3, -\omega_3'|\tilde{V}_{J_2, \tilde{J}_3} - \omega_1, \omega_1' > <\omega_1, \omega_1'|\tilde{V}_{J_1, \tilde{J}_1} - \omega_0, \omega_0' >, \quad (3.43)
\]

where the notation is such that the complete symmetry between the three operators will be manifest at the end. The operators denoted with a tilde are given by

\[
\tilde{V}_{J, \tilde{J}}(\sigma, \tau) \equiv e^{(\hat{J} + 1)\alpha_{-}\Phi(\sigma, \tau) - J'\alpha_{+}\Phi(\sigma, \tau)} e^{-(J'\alpha_{-}' + \hat{J}\alpha_{+}')}X(\sigma, \tau), \quad (3.44)
\]

By definition they are such that Eqs.3.41 and 3.42 do not give zero. By construction, the corresponding matrix elements are given by the same formulae as for \( \tilde{V}_{J, \tilde{J}} \), after continuation to the appropriate quantum numbers (more about this below).

It is convenient to introduce \( \mu \) and \( \nu \) variables similar to Eq.3.26 for each leg. According to Eqs.2.53 and 2.47, the parameters relevant for the first and third
operators satisfy, for \( l = 1, \) and 3,
\[
\begin{align*}
2J_l &= \mu_l, & 2\hat{J}_l &= \hat{\mu}_l, & 2J'_l &= \mu'_l, & 2\hat{J}'_l &= \hat{\mu}'_l \\
\mu_l &= -\hat{\mu}'_l - 2, & \hat{\mu}_l &= \mu'_l, & \nu_l &= \nu'_l = \hat{\mu}_l = \mu'_l = 0,
\end{align*}
\]
\[
\begin{align*}
\varpi_l &= \varpi_0 + \mu_l + \hat{\mu}_l \frac{\pi}{h}, & \varpi'_l &= \varpi'_0 + \mu'_l - \hat{\mu}'_l \frac{h}{\pi}, & \varpi_l &= -\varpi'_l, & \varpi'_l &= \varpi'_l (3.45)
\end{align*}
\]
For \( \mathcal{V}_{J_2', \hat{J}_2} \) the only possible values of \( m_2, \hat{m}_2, m'_2, \) and \( \hat{m}_2 \) are such that
\[
\begin{align*}
\varpi_1 + \varpi_3 &= 2m_2 + 2\hat{m}_2 \pi/h, & \varpi'_1 + \varpi'_3 &= 2m'_2 - 2\hat{m}'_2 h/\pi (3.46)
\end{align*}
\]
Since \( h \) is not rational, this leads to
\[
\begin{align*}
\nu_2 &= 1 + J_1 + J_2 + J_3, & \nu'_2 &= 1 + \hat{J}_1 + \hat{J}_2 + \hat{J}_3, \\
\nu_2' &= 1 + J'_1 + J'_2 + J'_3, & \nu'_2' &= 1 + \hat{J}'_1 + \hat{J}'_2 + \hat{J}'_3. (3.47)
\end{align*}
\]
\( \nu_2, \ldots \nu'_2 \) are equal to the number of screening operators in the Coulomb gas language.

According to Eq.3.10, one also has
\[
J_l = -\hat{J}'_l - 1, & \hat{J}_l = J'_l, & l = 1, 2, 3. (3.48)
\]
Combined with the previous relations, this gives
\[
\mu_2 = -\hat{\mu}'_2 - 1, & \hat{\mu}_2 = \mu'_2, & \nu_2 = -\hat{\nu}'_2 - 1, & \hat{\nu}_2 = \nu'_2 (3.49)
\]
It is easy to see that, as a consequence of this last equation, the choice
\[
-\hat{J}'_2 \leq \hat{m}'_2 \leq \hat{J}'_2
\]
gives \( J_2 + 1 \leq m_2 \leq -J_2 + 1 \) which is precisely the range of values which appear in the definition Eq.3.23. Thus the field \( \hat{\Phi} \) obtained from the symmetry between spins \( J \) and \(-J - 1\) is precisely the one needed for the proper dressing by gravity, when dealing with the mid point.

## 4  HIGHEST-WEIGHT MATRIX ELEMENTS

Our aim in this section is to compute the three matrix elements that appear in Eq.4.43. We shall temporarily use a simplified notation. These matrix elements are of the form
\[
<\varpi, \varpi'| \mathcal{V}_{J, \hat{J}} | \varpi_i, \varpi_i >, \text{ or } <\varpi, \varpi'| \tilde{\mathcal{V}}_{J, \hat{J}} | \varpi_i, \varpi_i > (4.1)
\]
According to Eq.3.48, the spins of the gravity part is given by
\[
\hat{J} = J', \text{ and } J = -\hat{J}' - 1 (4.2)
\]
The calculation cannot be done for all three terms simultaneously, since they are of different types. It is convenient to compute the middle one first, since the other two will be obtained after simple modifications.
4. Matrix elements

4.1 Matrix elements between on-shell states

The characteristic feature of the matrix element \( < -\varpi_3, -\varpi'_3|\mathcal{V}_{J_2, \hat{J}_2}|\varpi_1, \varpi'_1 > \) is that it is computed between states that both satisfy condition Eq.3.16, and are thus on-shell states satisfying Eq.3.13. Thus our first problem is to compute Eq.4.1, with

\[
\varpi = -\hat{\varpi}, \quad \varpi_i = -\hat{\varpi}'_i, \quad \hat{\varpi} = \varpi', \quad \hat{\varpi}_i = \varpi'_i.
\] (4.3)

In this subsection, we write \( \mathcal{V}_{J_2, \hat{J}_2} \) instead of \( \mathcal{V}_{J_2', \hat{J}_2'} \), and Eq.3.48. becomes

\[
\mu = -\hat{\mu}' - 1, \quad \hat{\mu} = \mu', \quad \nu = -\hat{\nu}' - 1, \quad \hat{\nu} = \nu'.
\] (4.4)

In performing the calculation, one makes use of Eqs. 3.28-3.33, and 3.37, 3.38. It is convenient to put together the corresponding gravity and matter pieces, since striking simplifications take place. For instance, one has:

\[
D^{\mu, \nu}(\varpi)\hat{D}^{\mu', \nu'}(\hat{\varpi}') = D^{\mu, \nu}(\varpi)\hat{D}^{\mu-1, -\nu-1}(-\varpi) = \\
\prod_{t=1}^{\mu-\nu} \frac{\sqrt{\Gamma((-\varpi - t + 1)h/\pi)}}{\sqrt{\Gamma((\varpi - t)h/\pi)}} \prod_{t=1}^{-\mu+\nu} \frac{\sqrt{\Gamma((-\hat{\varpi}' - t + 1)\hat{h}/\pi)}}{\sqrt{\Gamma((\hat{\varpi}' - t)\hat{h}/\pi)}} \\
\prod_{r=1}^{\mu} \Gamma((\varpi - r)h/\pi) \prod_{r=1}^{-\nu-1} \Gamma((\hat{\varpi}' - r)\hat{h}/\pi) \\
\prod_{s=1}^{\nu} \Gamma((-\varpi - s)h/\pi) \prod_{s=1}^{-\mu-1} \Gamma((-\hat{\varpi}' - s)\hat{h}/\pi)).
\] (4.5)

The terms are combined using rule Eq.3.34. For instance, the first line gives, according to Eqs.3.35, and 3.36,

\[
\prod_{t=1}^{\mu-\nu} \frac{\sqrt{\Gamma((-\varpi - t + 1)h/\pi)}}{\sqrt{\Gamma((\varpi - t)h/\pi)}} = \frac{\sqrt{-\varpi - \mu + \nu}}{\sqrt{-\varpi}} \frac{\sin(-\varpi)}{\sqrt{\sin(-\varpi)}}
\] (4.6)

and the first term of the second line becomes

\[
\prod_{r=1}^{\mu} \Gamma((\varpi - r)h/\pi) \prod_{r=1}^{\mu+1} \frac{1}{\Gamma((\varpi - r + 1)h/\pi))} = \frac{1}{\Gamma(\varpi h/\pi)},
\] (4.7)

and so on. Altogether, one finds

\[
D^{\mu, \nu}(\varpi)\hat{D}^{\mu', \nu'}(\hat{\varpi}') = \frac{h \sin h}{\pi^2} \sqrt{-\varpi} \sqrt{-\varpi - \mu + \nu} \sqrt{|\varpi|} \sqrt{|\varpi - \mu + \nu|}
\] (4.8)
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\[
\left[D^{\mu \nu} (\varpi) D^{\mu' \nu'} (\varpi') \right] = \left( \frac{-\pi^2}{h \sin h} \right) \frac{\hat{\mu} + \hat{\nu}}{\sqrt{\varpi - (\hat{\mu} + \hat{\nu}) \pi / h}} \sqrt{\varpi - (\hat{\mu} - \hat{\nu}) \pi / h}^{\mu} \prod_{s=1}^{\hat{\nu}+1} \frac{1}{(\varpi + \hat{s})} \prod_{s=1}^{\hat{\mu}+1} \frac{1}{(\varpi - \hat{s})} \ 
\]

(4.9)

\[
M^{\mu, \nu} (\varpi) \hat{M}^{\mu', \nu'} (\varpi') = e^{i \varphi_1} \left( \frac{\pi}{h} \right)^{(\hat{\mu} + \hat{\nu})/2} \frac{\sqrt{\varpi \sqrt{\varpi_i} \prod_{i=1}^{\hat{\mu}} (\varpi - \hat{\mu}) \prod_{i=1}^{\hat{\nu}} (\varpi + \hat{s})}}{\sqrt{\varpi - (\hat{\mu} - \hat{\nu}) \pi / h \sqrt{\varpi - \mu + \nu}}} 
\]

(4.10)

\[
e^{i \varphi_1} = \prod_{t=1}^{\mu-\nu} \prod_{i=1}^{\nu} \sqrt{-(\varpi - t + 1) \sqrt{h/\pi} + (\hat{\nu} - i - 1) \sqrt{\pi/h}} \sqrt{-(\varpi - t) \sqrt{h/\pi} - (\hat{\nu} - i) \sqrt{\pi/h}} 
\]

(4.11)

\[
D^{\mu_{\hat{\nu}}, \mu, \nu} (\varpi) D^{\mu', \nu'} (\varpi') = 
\]

\[
e^{i \varphi_2} \frac{h \sin h}{\pi^2} \left( \frac{\pi}{h} \right)^{(\hat{\mu} + \hat{\nu})/2} \frac{\sqrt{\varpi \sqrt{\varpi_i} \prod_{i=1}^{\hat{\mu}} (\varpi - \hat{\mu}) \prod_{i=1}^{\hat{\nu}} (\varpi + \hat{s})}}{\sqrt{\varpi - (\hat{\mu} - \hat{\nu}) \pi / h \sqrt{\varpi - \mu + \nu}}} 
\]

(4.12)

\[
e^{i \varphi_2} = e^{i \varphi_1} \sqrt{-(\varpi - (\hat{\mu} + \hat{\nu}) \pi / h)} \sqrt{-(\varpi - \mu + \nu)} \sqrt{\varpi - \mu + \nu} \sqrt{\varpi - \mu + \nu} \frac{1}{\sqrt{\varpi - \mu + \nu} \Gamma[1 + \hat{\mu} + \hat{\nu} + (1 + \mu + \nu) h / \pi]} 
\]

(4.13)

\[
C^{\mu_{\hat{\nu}}, \mu, \nu} C^{\mu', \nu'} (\varpi) (\varpi') = \left( \frac{h}{\pi} \right)^{(\hat{\mu} + \hat{\nu})/2} \frac{\hat{\mu}! \hat{\nu}!}{\hat{\mu}! \hat{\nu}! \Gamma[1 + \hat{\mu} + \hat{\nu} + (1 + \mu + \nu) h / \pi]} 
\]

(4.14)

Making use of Eqs. 3.23 and 3.20, this leads to

\[
< \varpi | \psi_{m, \hat{m}}^{(j, \hat{j})} (1) | \varpi_i > = < \varpi' | \psi_{m', \hat{m}'}^{(j', \hat{j}')} (1) | \varpi' > = h \sin h \left( \frac{h}{\pi} \right)^{(\hat{\mu} + \hat{\nu})/2} e^{i \varphi_2} \frac{\hat{\mu}! \hat{\nu}!}{\hat{\mu}! \hat{\nu}! \Gamma[1 + \hat{\mu} + \hat{\nu} + (1 + \mu + \nu) h / \pi]} 
\]

(4.15)

Finally, one combines Eqs. 3.10, 3.17, 3.24 with the last equation, and writes

\[
< \varpi, \varpi' | \mathcal{V}_{j', \hat{j}'} | \varpi_i, \varpi_i > = c_{j, \hat{j}} c_{j', \hat{j}'} \times 
\]

\[
\left( \frac{-\pi^2}{h \sin h} \right) \frac{\hat{\mu} + \hat{\nu}}{\sqrt{\varpi - (\hat{\mu} + \hat{\nu}) \pi / h}} \sqrt{\varpi - (\hat{\mu} - \hat{\nu}) \pi / h}^{\mu} \prod_{s=1}^{\hat{\nu}+1} \frac{1}{(\varpi + \hat{s})} \prod_{s=1}^{\hat{\mu}+1} \frac{1}{(\varpi - \hat{s})} 
\]

(4.16)
4. Matrix elements

It follows from the discussions carried out earlier that

\[ m = -\mu + \nu, \quad \tilde{m} = -\tilde{\mu} + \tilde{\nu}, \quad m' = -\mu' + \nu', \quad \tilde{m}' = -\tilde{\mu}' + \tilde{\nu}', \quad (4.17) \]

so that one has, according to Eq.4.12

\[ m = -\tilde{m}', \quad \tilde{m} = m'. \quad (4.18) \]

This gives

\[ \lambda'(j') (\varpi') = \tilde{\lambda}(j) (\tilde{\varpi}), \quad \lambda^{-(j-1)} (\varpi) = (-1)^2 \tilde{\varpi'} \tilde{\lambda'}(j) (\tilde{\varpi'}), \quad (4.19) \]

and Eq.4.10 becomes

\[
< \varpi, \varpi' | V_{j', \tilde{j}'}, (\varpi, \varpi') | \varpi_i, \varpi_i > = \frac{c_{j', \tilde{j}'}}{[\varpi] [\varpi_i] [\tilde{\varpi}] [\tilde{\varpi}_i]} \cdot (-1)^{2j'+1} \times \\
< \varpi, \varpi' | \{ \psi_{m, \tilde{m}}^{(-j'-1)}(1) \tilde{\psi}_{m, \tilde{m}}^{(-j'-1)}(1) \psi_{m', \tilde{m}'}^{(j', \tilde{j}')} (1) \tilde{\psi}_{m', \tilde{m}'}^{(j', \tilde{j}')} (1) \} | \varpi_i, \varpi_i' >, \quad (4.20) \\
\]

where

\[ e^{i\varphi(\varpi, \varpi_i)} = \sqrt{[\varpi]} \sqrt{[\varpi_i]} \quad (4.21) \]

Substituting Eq.4.14 and the analogous expression with bars, one arrives at a very simple expression

\[
< \varpi, \varpi' | V_{j', \tilde{j}'} (\varpi, \varpi') | \varpi_i, \varpi_i > = B_{j', \tilde{j}'}, e^{i\varphi(\varpi, \varpi_i)} \frac{\varpi \varpi_i}{\Gamma[1 + \tilde{j} + \tilde{\nu} + (1 + \mu + \nu) h/\pi]} \bigg[ (4.22) \bigg]
\]

\[ B_{j', \tilde{j}'} = (-1)^{2j'+1} c_{j', \tilde{j}'}, \tilde{j}' \left( \frac{h \sin h}{\pi^2} \right)^2 \left( \frac{h}{\sin h} \right)^{\tilde{j}'}, \quad (4.23) \]

One sees that the final result is completely symmetric between \( \varpi \) and \( \varpi_i \). This justifies a posteriori our choice of \( a(\varpi) \) in defining the Liouville field (Eqs.2.22, 2.23).

4.2 Matrix element with the right vacuum

There are two differences between the matrix element \( < \varpi, \varpi' | V_{j', \tilde{j}'} | \varpi_0, \varpi'_0 > \) and the one we just computed. First, the state \( | \varpi_0, \varpi'_0 > \) has weights \( \Delta(\varpi_0) = \Delta(\varpi'_0) = 0 \) so that it does not satisfy the on-shell condition Eq.3.13. Second, the \( \nu' \)’s vanish. The problem is now to compute Eq.4.11, with,

\[ \varpi = -\varpi', \quad \varpi_i = 1 + \pi/\hbar, \quad \tilde{\varpi} = \varpi', \quad \tilde{\varpi}_i = 1 - \hbar/\pi, \quad (4.24) \]
and, according to Eq.3.48,

\[
\mu = -\tilde{\mu}' - 2, \quad \tilde{\mu} = \mu', \quad \nu = \tilde{\nu}' = 0, \quad \tilde{\nu} = \nu' = 0. \tag{4.25}
\]

One again computes \(\langle \varpi | \psi^\dagger_{\mu', \tilde{\mu}}(1) | \varpi' \rangle \) and \(\langle \varpi' | \psi_{\nu', \tilde{\nu}}(1) | \varpi \rangle \). It is possible to separate terms that are identical to the previous calculation, with the \(\nu\)'s set to zero, from new factors which must be computed. The result are most simply presented in the same way. Eqs.4.8, 4.9, and 4.10 are respectively replaced by

\[
D^\mu,0(\varpi)\tilde{D}'^{\tilde{\mu}},0(\varpi') = \left[ \text{expression (4.8)} \right] \times \\
\frac{\sqrt{\Gamma((\varpi - \mu)h/\pi)} \sqrt{\Gamma((\varpi - \mu - 1)h/\pi)} \Gamma(-\varpi h/\pi)}{\sqrt{\Gamma(-\varpi - \mu - 1)h/\pi)} \sqrt{\Gamma(-\varpi - \mu - 2)h/\pi) \Gamma((\varpi - \mu - 1)h/\pi)}} \tag{4.26}
\]

\[
\tilde{D}^{\tilde{\mu},0}(\varpi)\tilde{D}'^{\tilde{\mu}},0(\varpi') = \left[ \text{expression (4.9)} \right] \tag{4.27}
\]

\[
M^{\mu,0}(\varpi)\tilde{M}'^{\tilde{\mu}},0(\varpi') = \left[ \text{expression (4.10)} \right] \times \\
\frac{\sqrt{\Gamma(-\tilde{\varpi} + (\mu + 1)h/\pi)} \sqrt{\Gamma(-\tilde{\varpi} + (\mu + 2)h/\pi)} \Gamma(\tilde{\varpi} - \tilde{\mu} - \mu h/\pi)}{\sqrt{\Gamma(-\tilde{\varpi} + \tilde{\mu} + (\mu + 1)h/\pi)} \sqrt{\Gamma(-\tilde{\varpi} + \tilde{\mu} + (\mu + 2)h/\pi)} \Gamma(\tilde{\varpi} - \tilde{\mu} - \mu h/\pi) \times \\
\frac{\sqrt{\Gamma(\tilde{\varpi} - \tilde{\mu} - (\mu + 1)h/\pi)} \Gamma(\tilde{\varpi} - (\mu + 1)h/\pi)}{\sqrt{\Gamma(\tilde{\varpi} - (\mu + 1)h/\pi)} \Gamma(\tilde{\varpi} - \tilde{\mu} - (\mu + 1)h/\pi) \left( \frac{h}{\pi} \right)^{\tilde{\mu}/2}}, \tag{4.28}
\]

and, combining the last three relations

\[
D^{\mu,0;\tilde{\mu}},0(\varpi)\tilde{D}'^{\mu',0;\tilde{\mu}},0(\varpi') = \left[ \text{expression (4.12)} \right] \left( \frac{h}{\pi} \right)^{\tilde{\mu}/2} \times \\
\frac{\Gamma(-\tilde{\varpi}) \sqrt{\Gamma(\tilde{\varpi} - \tilde{\mu} - \mu h/\pi)} \sqrt{\Gamma(-\tilde{\varpi} + \tilde{\mu} + (\mu + 1)h/\pi) \sqrt{\Gamma(-\tilde{\varpi} + \tilde{\mu} + (\mu + 2)h/\pi) \Gamma(\tilde{\varpi} - \tilde{\mu} - (\mu + 1)h/\pi)}} \tag{4.29}
\]

Using the fact that \(\tilde{\varpi} - \tilde{\mu} - \mu h/\pi = \tilde{\varpi}_0\), one finds

\[
D^{\mu,0;\tilde{\mu}},0(\varpi)\tilde{D}'^{\mu',0;\tilde{\mu}},0(\varpi') = \left[ \text{expression (4.12)} \right] \times \\
\left( \frac{h}{\pi} \right)^{\tilde{\mu}/2} \frac{\Gamma(-\tilde{\varpi}) \sqrt{\Gamma(\tilde{\varpi}_i)}}{\sqrt{\Gamma(-\tilde{\varpi}_i + h/\pi) \sqrt{\Gamma(\tilde{\varpi}_i - h/\pi) \sqrt{\Gamma(-\tilde{\varpi}_i + 2h/\pi)}}} \tag{4.30}
\]

On the other hand, Eqs.3.28 and 3.29 immediately show that

\[
C^{\mu,0;\tilde{\mu}},0C'^{\mu',0;\tilde{\mu}},0 = 1, \tag{4.31}
\]
4. Matrix elements

and this replaces Eq.4.14 of the previous calculation. The matrix element of the \( \psi \) fields is now given by

\[
\langle \varpi | \psi^{(J, \hat{J})}_{m, \hat{m}}(1) | \varpi_i \rangle = \langle \varpi' | \psi^{(J', \hat{J}')}_{m', \hat{m}'}(1) | \varpi' \rangle = \frac{i \sin h}{\pi} \left( \frac{\hbar}{\sin h} \right)^{1/2} e^{i \varphi_2} \frac{\Gamma(-\varpi)}{\Gamma(-\varpi + h/\pi)} \times \\
\sqrt{\Gamma(\varpi_i) / \Gamma(\varpi)} \sqrt{\Gamma(\varpi) / \Gamma(\varpi_i)} \sqrt{\varpi_i} \sqrt{\varpi} \sqrt{\Gamma(\varpi)} \sqrt{\Gamma(\varpi_i)} \sqrt{\Gamma(-\varpi_i + 2h/\pi)}
\]

(4.32)

Assuming that the field \( \tilde{\Phi} \) is given by the same expression as \( \Phi \), one again makes use of Eq.4.16 (there is a change of normalisation which will appear soon). The choice of \( m \) variables is now

\[
m = -J, \quad \hat{m} = -\hat{J}, \quad m' = -J', \quad \hat{m}' = -\hat{J}'.
\]

(4.33)

According to Eq.4.24, and this last relation, one has

\[
\lambda^{(J', \hat{J}')}(\varpi') = \lambda^{(J, \hat{J})}(\varpi), \quad \lambda^{(J-1)}(\varpi) = \frac{[\varpi - 1]}{[\varpi]^{2}} (-1)^{2 \hat{J} + 1} \lambda^{(J', \hat{J}')}(\varpi').
\]

(4.34)

The derivation of the second relation needs some explanation. One first writes, according to Eqs.A.23, and 4.33,

\[
\lambda^{(J-1)}(\varpi) = \lambda^{(J, \hat{J})}_{\hat{J}+1}(\varpi) = \frac{[2 \hat{J} + 1]}{[2 \hat{J}' + 1][1]} [\varpi + 1]_{2 \hat{J}+1}
\]

(4.35)

Since in general \( |N!| \) is defined as the solution of \( |N+1!| = |N+1| |N!| \), it follows that \( |-1!| = 0 \), and we remove it once for all by changing the overall normalisation. On the other hand,

\[
\hat{\lambda}^{(J', \hat{J}')}_{\hat{J}+1}(\varpi') = [\varpi' - 2 \hat{J}']_{2 \hat{J}+1} = [-\varpi - 2 \hat{J}']_{2 \hat{J}+1},
\]

and, comparing the last two equalities,

\[
\lambda^{(J-1)}(\varpi) = \hat{\lambda}^{(J', \hat{J}')}(\varpi') (-1)^{2 \hat{J} + 1} \frac{[\varpi + 2 \hat{J} + 1]}{[2 \hat{J}' + 1][\varpi]}
\]

It is easy to see that

\[
[\varpi + 2 \hat{J} + 1] = [\varpi - 1] (-1)^{2 \hat{J}}, \quad [2 \hat{J}' + 1] = [\varpi] (-1)^{2 \hat{J}},
\]

and the second relation Eq.4.34 follows. From this point, the calculations proceeds in the same way as above, and one gets

\[
\langle \varpi, \varpi' | \Psi_{J', \hat{J}'}, \varpi_i, \varpi_i \rangle =
\]
4. Matrix elements

\[ B_{J', \hat{J}} e^{i\varphi(\varpi, \varpi_i)} \Gamma(\hat{\varpi}_i) \]
\[ \frac{1}{[\varpi - 1] \Gamma(-\hat{\varpi}_i + h/\pi) \Gamma(\hat{\varpi}_i - h/\pi) \Gamma(-\hat{\varpi}_i + 2h/\pi)} \varpi \varpi_i \Gamma(-\hat{\varpi}) [\varpi] \Gamma(\varpi) \]  \text{(4.36)}

One writes
\[ [\varpi - 1] \Gamma(-\hat{\varpi}_i + h/\pi) = \frac{\sin(\pi(\varpi_i - h/\pi)) \Gamma(-\hat{\varpi}_i + h/\pi)}{\sin h} \]
and it follows from the relations \( \Gamma(z) \Gamma(1 - z) = \pi/\sin(\pi z) \) and \( \hat{\varpi}_i = 1 + h/\pi \) that the above expression is simply equal to \( \pi/\sin h \). Using moreover the identity
\[ \Gamma(-\hat{\varpi}_i) [\varpi_i - 1] \Gamma(\varpi_i) = -\pi \hat{\varpi}_i \Gamma(\hat{\varpi}_i) \sin h, \]
one finally derives the desired expression
\[ <\varpi, \varpi' | \tilde{\mathcal{V}}_{J', \hat{J}} | \varpi_i, \varpi_i > = B_{J', \hat{J}} e^{i\varphi(\varpi, \varpi_i)} \Gamma(1 + h/\pi) \Gamma(2 - h/\pi) \frac{\varpi_i}{\varpi} \Gamma(\hat{\varpi}) [\varpi] \]  \text{(4.37)}

This result is perfectly finite, but this is not true at intermediate stages of the calculation: in Eq.4.36, \( [\varpi - 1] \) is actually equal to zero for \( \varpi_i = 1 + \pi/h \), while \( \Gamma(-\hat{\varpi}_i + h/\pi) = \Gamma(-1) \) diverges. One may bypass the ambiguity by first giving a small imaginary part to \( \varpi_0 \), as needed for the treatment of the possible cuts.

4.3 Matrix element with the left vacuum

This last case is relevant for the computation of \( < -\varpi_0, -\varpi'_0 | \mathcal{V}_{J', \hat{J}} | -\varpi_3, -\varpi'_3 > \) in Eq.4.43. The calculation is performed in the same way as for the case of subsection 4.2. One now has, instead of Eq.4.24,
\[ \varpi = -1 - \pi/h, \quad \varpi_i = -\varpi'_i, \quad \hat{\varpi} = -1 - h/\pi, \quad \hat{\varpi}_i = \varpi. \]  \text{(4.38)}

Since the \( \nu \) variables are zero, Eq.4.25 remains valid. Computing again the modifications, with respect to the case 4.1, one has to take account of the fact that the relationship between \( \varpi \) and \( \varpi' \) is modified to
\[ \varpi' = -\varpi - 2, \quad \varpi' = \hat{\varpi} + 2h/\pi. \]  \text{(4.39)}

Eqs.4.8, 4.9, and Eqs.4.10 are respectively replaced by
\[ D^{\mu, 0}(\varpi) \hat{\mathcal{D}}^{\mu, 0}(\varpi') = [\text{expression (4.8)}] \frac{\sqrt{\Gamma((\varpi + 2)h/\pi)}}{\sqrt{\Gamma(-(\varpi + 1)h/\pi)}} \frac{\Gamma((\varpi - \mu)h/\pi)}{\Gamma((\varpi + 1)h/\pi) \Gamma((\varpi + 2)h/\pi)} \quad \]  \text{(4.40)}

\[ \prod_{t=1}^{\hat{\mu}} \frac{\sqrt{\Gamma((\hat{\varpi} - t + 1)\pi/h + 2)}}{\sqrt{\Gamma((\hat{\varpi} - t + 1)\pi/h)}} \frac{\sqrt{\Gamma(-(\hat{\varpi} - t)\pi/h - 2)}}{\sqrt{\Gamma(-(\hat{\varpi} - t)\pi/h)}} \quad \]  \text{(4.41)}
The end of the calculations is exactly as in 4.2. Using Eq.4.39, one verifies that Eqs.4.34 are replaced by

\[
\frac{\Gamma((\omega - \mu)h/\pi - \bar{\mu})\Gamma((\omega + 1)h/\pi)}{\Gamma((\omega - \mu)h/\pi)\Gamma(\omega + 2)h/\pi} = \frac{\Gamma((\omega - \mu)h/\pi - \bar{\mu})\Gamma((\omega + 1)h/\pi)}{\Gamma((\omega - \mu)h/\pi)\Gamma(\omega + 2)h/\pi - \bar{\mu}}.
\]

(4.42)

where

\[
e^{i\varphi_4} = \prod_{t=1}^{\hat{\mu}} \left\{ \frac{\sqrt{\omega + 1 + (t - 1)\pi/h}}{\sqrt{-\omega - 1 + (t - 1)\pi/h}} \times \frac{\sqrt{\omega + (t - 1)\pi/h}}{\sqrt{-\omega + 2 + t\pi/h}} \right\}.
\]

(4.43)

Combining Eqs.4.40, 4.41, and 4.42, one gets, after a calculation similar to the one that lead to Eq.4.30

\[
D_{\mu,0;\hat{\mu},0}(\omega)D_{\mu',0;\hat{\mu}',0}(\omega') = \left[ \text{expression (4.12)} \right] \times \left[ \text{expression (4.10)} \right]
\]

\[
\left( \frac{h}{\pi} \right)^{\hat{\mu}/2} e^{i\varphi_4} \Gamma(\omega_i) \frac{\sqrt{\Gamma(-\omega h/\pi)}}{\sqrt{\Gamma((\omega + 1)h/\pi)} \sqrt{\Gamma((\omega + 2)h/\pi)}}
\]

(4.44)

The end of the calculations is exactly as in 4.2. Using Eq.4.39, one verifies that Eqs.4.34 are replaced by

\[
\lambda'_{-j'}(\omega') = \hat{\lambda}_{-j}(\omega), \quad \lambda^{(j-1)}(\omega) = \frac{1}{[\omega_i]^2}(-1)^{2j+1} \hat{\lambda}_{-j'}(\omega').
\]

(4.45)

The final result is

\[
<\omega, \omega' | \hat{\nu}_{j', \hat{\mu}} | \omega_i, \omega_i > = B_{j', \hat{\mu}} e^{i\varphi(\omega, \omega_i)} \Gamma(1 + h/\pi) \Gamma(2 - h/\pi) \frac{1}{\omega_i [\Gamma(\omega_i)]^2}
\]

(4.46)

### 5 THE THREE-POINT COUPLING

The three-point function is finally derived by substituting Eq.4.22 (with \(\omega = -\omega_3\), \(\omega_i = \omega_3\), and \(\mu + \nu = J_2, \cdots, \hat{\mu}' + \hat{\nu}' = \hat{J}_2\)), Eq.4.37 (with \(\omega = \omega_1\), \(\omega_i = \omega_0\), and \(J = J_2, \cdots, \hat{J}_i = \hat{J}_2\), and Eq.4.40 (with \(\omega = -\omega_0\), \(\omega_i = -\omega_3\), and \(J = J_3, \cdots, \hat{J}_i = \hat{J}_3\)), into Eq.8.43. One gets

\[
C_{1,2,3} = -e^{i\varphi(\omega_0, -\omega_3)} e^{i\varphi(-\omega_3, \omega_1)} e^{i\varphi(\omega_1, \omega_0)} (\Gamma(1 + h/\pi) \Gamma(2 - h/\pi))^2 \times \prod_l B_{J_l, \hat{J}_l} \left[ \frac{\omega_0}{\Gamma(\omega_3) \Gamma(1 + \hat{\mu}_2 + \hat{\nu}_2 + (1 + \mu_2 + \nu_2)h/\pi)} \Gamma(\omega_1) \right]^2.
\]

(5.1)
3. Three-point

It immediately follows from Eq. [4.21] that
\[ e^{i\varphi(-\omega_0,-\omega_3)}e^{i\varphi(-\omega_3,\omega_1)}e^{i\varphi(\omega_1,\omega_0)} = 1. \] (5.2)

One arrives at the completely symmetric expression
\[ C_{1,2,3} = -\left( \frac{\pi}{h} \Gamma(2 + h/\pi) \Gamma(2 - h/\pi) \right)^2 \prod \frac{B_{\hat{J}_l,\hat{J}_l'}}{\Gamma(1 + 2\hat{J}_l + (1 + 2J)h/\pi)^2}. \] (5.3)

Next, we make contact with other approaches to the same problem. For this we first explicitly connect the present group-theoretic notations with more standard conventions. Eqs. 2.1, and 3.2, correspond to the usual notations \( C = 1 + 3Q^2 \), and \( D = 1 - 12\alpha_0^2 \), for the gravity and matter central charges. The screening charges are given by (we choose \( \alpha_0 > 0 \))
\[ \alpha_{\pm} = \frac{Q}{2} \pm \alpha_0, \quad \alpha_{\pm}' = i(\alpha_0 \mp \frac{Q}{2}). \] (5.4)

The dressed vertex operator Eq. [3.10] may be rewritten as
\[ V_{J,J',\hat{J}} \equiv e^{((\hat{J}' + 1)\alpha_- - J'\alpha_+ + \hat{J}'\alpha_+ + J\alpha_- + \hat{J}'\alpha_+ + \hat{J}\alpha_- + J\alpha_- + \hat{J}'\alpha_+ + \hat{J}\alpha_-)X} = e^{\beta(k)}\Phi - ikX/\alpha_-, \] (5.5)

where
\[ \beta = (\hat{J}' + 1)\alpha_- - J'\alpha_+ , \quad ik = \alpha_- \left( J'\alpha_+ + \hat{J}'\alpha_+ \right). \] (5.6)

A simple calculation, using the formulae just given leads to
\[ k = 2J' - 2\hat{J}'h/\pi = 2\hat{J} + 2(J + 1)h/\pi, \] (5.7)
\[ \beta(k) + Q/2 = (k + k_0)/\alpha_- , \quad k_0 = \alpha_0\alpha_- = 1 - h/\pi \] (5.8)

According to Eq. [3.7], the weights are given by
\[ \Delta_G = -\frac{1}{2}\beta(\beta + Q), \quad \Delta_M = \frac{\pi}{4h}k(k + 2k_0). \] (5.9)

For rational theories \( D = 1 - 6(p - p')^2/pp', \ p > p' > 0 \),
\[ h = -\hat{h}', \quad \hat{h} = -h', \quad \hat{h} = \pi p'/p, \quad k_0 = (p - p')/p, \] (5.10)
\[ \Delta_M \text{ reduces to Kac's table} \]
\[ \Delta_M = \frac{1}{4pp'} \left[ (rp - sp')^2 - (p - p')^2 \right], \quad r = 2J' + 1, \quad s = 2\hat{J}' + 1, \quad k + k_0 = r - sh/\pi. \] (5.11)

The momentum \( k \) is defined so that it takes rational values for rational theories. For critical bosonic string with Regge slope \( \alpha' \), the tachyon vertex is \( \exp(ikX\sqrt{\alpha'}) \).
Eq.5.3 corresponds to $\alpha' = 1/(\alpha_-)^2$. With the conventions just introduced, the result Eq.5.3 takes the form

$$C_{1,2,3} \equiv A(k_1, k_2, k_3) = \prod_l \frac{B(k_l)}{[\Gamma(k_l + k_0)]^2},$$

(5.12)

where we left out an overall factor which does not depend upon the momenta.

Our next task is to re-establish the cosmological constant. In the present approach, it comes out as follows. Our basic guideline was to write down the most general local operators, as was discussed in section 2. We have not yet really done so, since we may multiply the right-hand sides of Eq.2.19 (or more generally of Eq.2.37) by $\mu^{-\varpi/2}$ on the left, and by $\mu^{\varpi/2}$ on the right, without breaking locality. This constant $\mu_\infty$ is arbitrary, and will play the role of the cosmological constant. According to Eq.2.41 one has

$$e^{-(J\alpha_- + \tilde{J}\alpha_+)}\Phi(\sigma, \tau) = \mu_\infty^{J\pi/\hbar} \mu^{\varpi/2} e^{-(J\alpha_- + \tilde{J}\alpha_+)}\Phi(\sigma, \tau) \mu^{\varpi/2}.$$  

(5.13)

In the Coulomb-gas picture, the power in $\mu_\infty$ is equal to the total number of screening charges. We shall agree with this definition if we let

$$V^{(\mu_\infty)}_{J', J}(\sigma, \tau) = \mu_\infty^{(J+1)} J\pi/\hbar \mu^{\varpi/2} V_{J', J}(\sigma, \tau) \mu^{\varpi/2}.$$  

(5.15)

The first factor coincides with the KPZ-DDK scaling factor. In particular, one has

$$V^{(\mu_\infty)}_{0, 0}(\sigma, \tau) = \mu_\infty^{-1} \mu^{\varpi/2} V_{0, 0}(\sigma, \tau) \mu^{\varpi/2},$$  

(5.16)

which is the expected scaling behaviour of the cosmological constant. The operators $\mu^{\varpi/2}$ induce a translation on the variable $x$ conjugate to the Liouville momentum. This degree of freedom already appears in the double free field representation recalled in appendix A. With the rescaled momentum $\varpi$, the natural definition of $x$ is such that $[x, \varpi] = i$, and $\psi^{(J\tilde{J})}_{m m}$ is proportional to $\exp[-2ix(m + \tilde{m}\pi/\hbar)]$, in agreement with Eq.5.13. This Liouville position-operator is equal to $-i\sqrt{h/2\pi q_0^{(1)}}$, where $q_0^{(1)}$ is the zero mode of the free field $\phi_1$ whose properties are summarized in the appendix. This translation, which is actually a global Weyl transformation, is analogous to the translation of the Liouville field in the work of DDK. This is seen by computing next the $\mu_\infty$-dependence of the three-point function. One gets immediately

$$A_{(\mu_\infty)}(k_1, k_2, k_3) = A(k_1, k_2, k_3) \mu^{\varpi_0 + \sum_{l}(J_l + \tilde{J}_l\pi/\hbar)}.$$  

(5.17)

The term $\mu^{\varpi_0}$ arises when $\mu^{\varpi/2}$, and $\mu^{\varpi/2}$ hit the left and right vacua respectively. In the DDK discussion, it comes from the term of the effective action which
6. Conclusion

is linear in the field\[^5\]. According to Eq.3.47, the power of $\mu_c$ is equal to $\nu_2 + \bar{\nu}_2 \pi/h$ where $\nu_2$ and $\bar{\nu}_2$ are the gravity screening-numbers. This agrees with the usual definition (see, e.g. ref.\[^2\]).

Finally we compare Eq.5.12 with the result of the matrix model. This part follows ref.\[^2\] closely. The two-point function $A_{(\mu_c)}(k,k)$ is determined by starting from the three-point function with one cosmological term $A_{(\mu_c)}(0,k,k)$, and writing

$$\frac{d}{d\mu_c} A_{(\mu_c)}(k,k) = A_{(\mu_c)}(0,k,k).$$

(5.18)

According to Eq.5.16, this gives

$$A_{(\mu_c)}(k,k) = \frac{\mu_c}{k + k_0} A_{(\mu_c)}(0,k,k).$$

(5.19)

Similarly, the partition function satisfies

$$\frac{d^3}{d\mu_c^3} Z_{(\mu_c)} = A_{(\mu_c)}(0,0,0),$$

(5.20)

$$Z_{(\mu_c)} = \frac{(\mu_c)^3}{(\varpi_0)(\varpi_0 - 1)(\varpi_0 - 2)} A_{(\mu_c)}(0,0,0).$$

(5.21)

We finally obtain the rescaled three point function

$$D(k_1,k_2,k_3) \equiv \left( A_{(\mu_c)}(k_1,k_2,k_3) \right)^2 Z_{(\mu_c)} = \frac{\prod_l (k_l + k_0)}{(\pi/h + 1)(\pi/h)(\pi/h - 1)},$$

(5.22)

which agrees with the results of the matrix models. Clearly, the key point in this final verification is that the final expression Eq.5.12 for the three-point function factorises. On the other hand, Eq.5.12 vanishes whenever $k_l + k_0 = 0$ for any of the three legs. According to Eq.5.11, this happens for rational theories, at the border of Kac’s table, where $r = p'$, and $s = p$. Thus formula Eq.5.22 holds only when the branching rules are satisfied.

6 CONCLUSION

Our starting point was the operator expressions of the Liouville and matter fields which are derived by imposing locality and consistency of the restriction to the Hilbert space of states with equal left and right momenta. These requirements uniquely determine these fields, the only arbitrariness being the choice of cosmological constant. We have seen how the three-point functions tremendously simplify when gravity and matter are coupled, so that the matrix-model results on the sphere come out.

\[^5\] It would not be there on the torus, since one would take a trace. This agrees with the DDK result where its contribution is shown to be proportional to the Euler characteristic.
The present operator viewpoint has given us an insight into the treatment of negative spins (positive powers of the metric) necessary for the proper dressing by gravity. Two kinds of operators of this type were found to appear. On the one hand, the symmetry between spin $J$ and spin $-J-1$ showed the existence of the local field $\exp((J+1)\alpha-\tilde{\Phi})$. It is similar to $\exp(-J\alpha-\Phi)$, since its expansion has an index running from $-J$ to $J$. It is appropriate for matrix elements between on-shell physical states. On the other hand, the treatment of the end points, requires the use of a field which is really the quantum analogue of the classical expression Eq.1.12. In spite of the lack of explicit symmetry in our definition of the three-point functions, the three legs play the same role in our final formulae. This situation is typical of a picture changing mechanism as the one of critical superstrings[23]. It might be possible to redefine the Fock spaces so that the complete symmetry becomes manifest. The situation is similar to the Neveu-Schwarz model in the old formulation, with the wrong vacuum. The possible change of picture is left for further studies.

In any case, since $C_{1,2,3} = C_{2,1,3}$, and $C_{1,2,3} = C_{1,3,2}$, it follows that

$$< -\omega_3, -\omega'_3 | \mathcal{V}_{j'_2, \tilde{j}'_2}(\sigma, \tau) \mathcal{\tilde{V}}_{j'_1, \tilde{j}'_1}(\sigma', \tau) | \omega_0, \omega'_0 > =$$

$$< -\omega_3, -\omega'_3 | \mathcal{V}_{j'_1, \tilde{j}'_1}(\sigma', \tau) \mathcal{\tilde{V}}_{j'_2, \tilde{j}'_2}(\sigma, \tau) | \omega_0, \omega'_0 >$$

(6.1)

$$< -\omega_0, -\omega'_0 | \mathcal{\tilde{V}}_{j'_3, \tilde{j}'_3}(\sigma, \tau) \mathcal{V}_{j'_2, \tilde{j}'_2}(\sigma', \tau) | \omega_1, \omega'_1 > = < -\omega_0, -\omega'_0 | \mathcal{\tilde{V}}_{j'_2, \tilde{j}'_2}(\sigma', \tau) \mathcal{V}_{j'_3, \tilde{j}'_3}(\sigma, \tau) | \omega_1, \omega'_1 > .$$

(6.2)

Thus the fields $\mathcal{V}$ and $\mathcal{\tilde{V}}$ are mutually local in a generalised sens.

It is rather remarkable that the symmetry between spins $J$ and $-J-1$ precisely gave us the operator $\tilde{\Phi}$ which is needed to treat the middle point, that is, to define the emission operators between physical states. For its exponentials, Eq.3.23 only involve a finite number of terms, contrary to what one could expect in view of the classical structure. In particular, the cosmological term $\exp(\alpha-\tilde{\Phi})$ corresponds to $J = 0$, so that it only involves a single term on the right-hand side! This is in sharp contrast with the direct quantization of the classical cosmological term which has an expansion over an infinite number of conformal blocks with different screening charges. This tremendous simplification may be the deep reason why matrix models give results to all orders in perturbation.

Since the present operator formalism is not restricted to the sphere, higher genus may certainly be considered. The case of the torus, should be straightforward.

Many future developments of the present discussion may be forseen. In particular, the present quantum-group approach is, so far, the only one that has been able to overcome the $D = 1$ barrier[3, 7]. I hope to return to this in latter publications.
7 Acknowledgements

This work was started when I was visiting the Mathematical Sciences Research Institute, in Berkeley, California, during March 1991. It is a pleasure to acknowledge the generous financial support, the very stimulating surrounding and the warm hospitality from which I benefitted very much during my participation in the program “Strings in Mathematical Physics”.

A Appendix

In this appendix we recall the basic properties of the $x_+$ components which are holomorphic functions of $z = \tau + i\sigma$. Since one deals with functions of a single variable, one may work at $\tau = 0$ without loss of generality that is on the unit circle $u = e^{i\sigma}$. The starting point is that, for generic $\gamma$, there exist two equivalent free fields:

$$\phi_j(\sigma) = q_0^{(j)} + p_0^{(j)}\sigma + i\sum_{n\neq 0} e^{-i\sigma} p_n^{(j)}/n, \quad j = 1, 2, \quad (A.1)$$

such that

$$\left[\phi'_1(\sigma_1), \phi'_2(\sigma_2)\right] = \left[\phi'_2(\sigma_1), \phi'_2(\sigma_2)\right] = 2\pi i \delta'(\sigma_1 - \sigma_2), \quad p_0^{(1)} = -p_0^{(2)}, \quad (A.2)$$

$$N^{(1)}(\phi'_1)^2 + \phi''_1/\sqrt{\gamma} = N^{(2)}(\phi'_2)^2 + \phi''_2/\sqrt{\gamma}. \quad (A.3)$$

$N^{(1)}$ (resp. $N^{(2)}$) denote normal orderings are with respect to the modes of $\phi_1$ (resp. of $\phi_2$). Eq. A.3 defines the stress-energy tensor and the coupling constant $\gamma$ of the quantum theory. The former generates a representation of the Virasoro algebra with central charge $C = 3 + 1/\gamma$. At an intuitive level, the correspondence between $\phi_1$ and $\phi_2$ may be understood from the fact that the Verma modules, which they generate, coincide since the highest weights only depend upon $(p_0^{(1)})^2 = (p_0^{(2)})^2$. The chiral family is built up\cite{17, 25, 26, 3, 27} from the following operators

$$\psi_j = d_j N^{(j)}(e^{\sqrt{\hbar/2\pi} \phi_j}), \quad \tilde{\psi}_j = \tilde{d}_j N^{(j)}(e^{\sqrt{\hbar/2\pi} \phi_j}), \quad j = 1, 2, \quad (A.4)$$

$$h = \frac{\pi}{12} \left(C - 13 - \sqrt{(C - 25)(C - 1)}\right), \quad \tilde{h} = \frac{\pi}{12} \left(C - 13 + \sqrt{(C - 25)(C - 1)}\right), \quad (A.5)$$

where $d_j$ and $\tilde{d}_j$ are normalization constants. In Eq. A.4 the zero modes are ordered, as is standard in string vertices, so that the operators are primary. Explicitly one has

$$N^{(j)}(e^{\sqrt{\hbar/2\pi} \phi_j}) = e^{\sqrt{\hbar/2\pi} q_0^{(j)}} e^{\sqrt{\hbar/2\pi} p_0^{(j)}} e^{-i\sigma h/4\pi} \times \exp \left(\sqrt{\hbar/2\pi i} \sum_{n<0} e^{-i\sigma} p_n^{(j)}/n\right) \exp \left(\sqrt{\hbar/2\pi i} \sum_{n>0} e^{-i\sigma} p_n^{(j)}/n\right) \quad (A.6)$$
with similar definitions for the hatted fields. The relation between \( h \) or \( \hat{h} \) and \( C \) which is equivalent to

\[
C = 1 + 6\left(\frac{h}{\pi} + \frac{\pi}{h} + 2\right) = 1 + 6\left(\frac{\hat{h}}{\pi} + \frac{\pi}{\hat{h}} + 2\right), \quad \text{with} \quad \hat{h}\pi = \pi^2,
\]

is such that \( \psi_j \) and \( \hat{\psi}_j \) are solutions of the equations\[12, 17]\n
\[
-\frac{d^2 \psi_j(\sigma)}{d\sigma^2} + \left(\frac{h}{\pi}\right)\left(\sum_{n<0} L_n e^{-in\sigma} + \frac{L_0}{2} + \left(\frac{h}{16\pi} - \frac{C - 1}{24}\right)\right)\psi_j(\sigma) \\
+ \left(\frac{h}{\pi}\right)\psi_j(\sigma)\left(\sum_{n>0} L_n e^{-in\sigma} + \frac{L_0}{2}\right) = 0 \tag{A.8}
\]

\[
-\frac{d^2 \hat{\psi}_j(\sigma)}{d\sigma^2} + \left(\frac{\hat{h}}{\pi}\right)\left(\sum_{n<0} L_n e^{-in\sigma} + \frac{L_0}{2} + \left(\frac{h}{16\pi} - \frac{C - 1}{24}\right)\right)\hat{\psi}_j(\sigma) \\
+ \left(\frac{\hat{h}}{\pi}\right)\hat{\psi}_j(\sigma)\left(\sum_{n>0} L_n e^{-in\sigma} + \frac{L_0}{2}\right) = 0 \tag{A.9}
\]

These are operator Schrödinger equations equivalent to the decoupling of Virasoro null-vectors\[17, 23, 26\]. Since there are two possible quantum modifications \( h \) and \( \hat{h} \), there are four solutions. By operator product \( \psi_j, j = 1, 2, \) and \( \hat{\psi}_j, j = 1, 2, \) generate two infinite families of chiral fields \( \psi^{(J)}_m, -J \leq m \leq J, \) and \( \hat{\psi}^{(J)}_m, -\hat{J} \leq \hat{m} \leq \hat{J}; \) with \( \psi_{-1/2} = \psi_1, \psi_{1/2}^{(1/2)} = \psi_2, \) and \( \hat{\psi}_{-1/2} = \hat{\psi}_1, \hat{\psi}_{1/2}^{(1/2)} = \hat{\psi}_2. \) The fields \( \psi^{(J)}_m, \hat{\psi}^{(J)}_m, \) are of the type \((1, 2J + 1)\) and \((2\hat{J} + 1, 1),\) respectively, in the BPZ classification. For the zero-modes, it is simpler\[5\] to define the rescaled variables

\[
\varpi = ip_{(1)0} \sqrt{\frac{2\pi}{h}}, \quad \hat{\varpi} = ip_{(1)0} \sqrt{\frac{2\pi}{\hat{h}}}, \quad \hat{\varpi} = \varpi = \hat{\varpi} = \hat{\varpi} \sqrt{\frac{\hat{h}}{\pi}}. \tag{A.10}
\]

The Hilbert space in which the operators \( \psi \) and \( \hat{\psi} \) live, is a direct sum\[3, 6, 7, 27\] of Fock spaces \( F(\varpi) \) spanned by the harmonic excitations of highest-weight Virasoro states noted \( |\varpi >. \) They are eigenstates of the quasi momentum \( \varpi, \) and satisfy \( L_n|\varpi > = 0, n > 0; (L_0 - \Delta(\varpi))|\varpi > = 0. \) The corresponding highest weights \( \Delta(\varpi) \) may be rewritten as

\[
\Delta(\varpi) \equiv \frac{1}{8\gamma} + \frac{(p_{(1)}^{(1)})^2}{2} = \frac{h}{4\pi}(1 + \frac{\pi}{h})^2 - \frac{h}{4\pi}\varpi^2. \tag{A.11}
\]

The commutation relations Eq.\[A.2\] are to be supplemented by the zero-mode ones:

\[
[q_{(1)}^{(1)}, p_{(1)}^{(1)}] = [q_{(2)}^{(2)}, p_{(2)}^{(2)}] = i.
\]
It thus follows (see in particular Eq\[A.6\]), that the fields \(\psi\) and \(\hat{\psi}\) shift the quasi momentum \(p_0(1) = -p_0(2)\) by a fixed amount. For an arbitrary c-number function \(f\) one has

\[
\psi_m^{(J)} f(\varpi) = f(\varpi + 2m) \psi_m^{(J)}, \quad \hat{\psi}_m^{(J)} f(\varpi) = f(\varpi + 2\hat{m} \pi / h) \hat{\psi}_m^{(J)}. \tag{A.12}
\]

The fields \(\psi\) and \(\hat{\psi}\) together with their products may be naturally restricted to discrete values of \(\varpi\). They thus live in Hilbert spaces of the form

\[
\mathcal{H}(\varpi_0) \equiv \bigoplus_{n, \hat{n}=-\infty}^{+\infty} \mathcal{F}(\varpi_0 + n + \hat{n} \pi / h). \tag{A.13}
\]

\(\varpi_0\) is a constant which is arbitrary so far. The \(sl(2,C)\)–invariant vacuum corresponds to \(\varpi_0 = 1 + \pi / h\). For \(h\) real, it is thus natural that the eigenvalues of \(\varpi\) be real. Thus \((p_0^{(1)})^\dagger = p_0^{(2)}\). As discussed several times, the natural hermiticity relation is \((\phi^{(1)}(\sigma))^\dagger = \phi^{(2)}(\sigma)\). Eq\[A.3\] shows that this is consistent with the usual hermiticity relation \(L_n^\dagger = L_{-n}\). As shown in \[5\], one has

\[
(\psi_m^{(J)})^\dagger = \psi_{-m}^{(J)}. \tag{A.14}
\]

For latter reference, we note that, besides their simple branching rules between Fock spaces, the \(\psi\) fields enjoy another important property: they are periodic up to a constant. This may be seen as follows: First it is obvious from the definitions Eqs\[A.4\], \[A.6\], \[A.10\] that

\[
\psi_{\pm 1/2}^{(1/2)}(\sigma + 2\pi) = e^{\pm i\varpi \sigma} e^{ih/2} \psi_{\pm 1/2}^{(1/2)}(\sigma) \tag{A.15}
\]

Next this is extended to the other \(\psi\) fields by fusion. To leading order at \(\sigma_1 \to \sigma_2\) one has\[6\]

\[
\psi_{\pm 1}^{(1/2)}(\sigma_1)\psi_{m}^{(J)}(\sigma_2) \sim (1 - e^{-i(\sigma_1 - \sigma_2)})^{-hJ/\pi} \frac{\sin[h(\varpi + J + m)]}{\sin h} \psi_{m+1/2}^{(J+1/2)}(\sigma_1), \tag{A.16}
\]

from which is follows that the fields \(\psi_m^{(J)}\) satisfy

\[
\psi_m^{(J)}(\sigma + 2\pi) = e^{2ihm\varpi} e^{2ihm^2} \psi_m^{(J)}(\sigma) \tag{A.17}
\]

The fields \(\psi_m^{(J)}\) are quantum Bloch waves. The second term in Eq\[A.17\] is a quantum modification. Its role is easily seen when one checks that Eq\[A.17\] is compatible with the hermiticity condition Eq\[A.14\], if one takes Eq\[A.12\] into account. The properties of the fields \(\hat{\psi}\) are of course similar, and we need not elaborate upon them.

\[6\] Mathematically they are not really Hilbert spaces since their metrics are not positive definite.

\[7\] There are subtleties which we are discussed in section 3.
An important point is that the $U_q(sl(2))$-quantum-group structure is best seen by changing basis to new chiral operators which obey a much simpler operator algebra. Following my recent work,[5] let us introduce

\[ \xi_m^{(J)}(\sigma) := \sum_{-J \leq m \leq J} |J, \bar{\omega} \rangle^m_M \psi_m^{(J)}(\sigma), \quad -J \leq M \leq J; \quad (A.18) \]

\[ |J, \bar{\omega} \rangle^m_M = \sqrt{\binom{2J}{J+M}} e^{ihm/2} \times \sum_{(J-M+m-t)/2} \text{integer} e^{iht(\bar{\omega}+m)} \left( \frac{J-M}{(J-M+m-t)/2} \right) \left( \frac{J+M}{(J+M+m+t)/2} \right); \quad (A.19) \]

The last equation introduces $q$-deformed factorials and binomial coefficients. The other fields $\hat{\xi}_m^{(\hat{J})}$ are defined in exactly the same way replacing $h$ by $\hat{h}$ everywhere.

The above transformation may be explicitly inverted. One has

\[ \psi_m^{(J)}(\sigma) = \sum_{M=-J}^J \xi_m^{(J)}(\sigma) \langle J, \bar{\omega} |^M_m \quad (A.22) \]

\[ (J, \bar{\omega})^M_m = \left( 2i \sin h e^{ih/2} \right)^{2J} (-1)^{J+M} e^{ih(J+M)} |J, \bar{\omega} \rangle^m_{-M} \frac{|\omega - 2m|}{\lambda_m^{(J)}(\omega)} \quad (A.23) \]

\[ \lambda_m^{(J)}(\omega) := \left( \frac{2J}{J-m} \right) |\omega - J + m|_{2J+1}. \quad (A.24) \]

In general, we define

\[ [x]_N = [x][x+1] \cdots [x+N-1]. \quad (A.25) \]

In [5, 7] the operator-algebra of the $\xi$ fields was completely determined. In particular, it was shown that for $\pi > \sigma_1 > \sigma_2 > 0$, these operators obey the exchange algebra

\[ \xi_m^{(J_1)}(\sigma_1) \xi_m^{(J_2)}(\sigma_2) = \sum_{-J_1 \leq N_1 \leq J_1; -J_2 \leq N_2 \leq J_2} (J_1, J_2)_m^{N_1 N_2} \xi_m^{(J_2)}(\sigma_2) \xi_m^{(J_1)}(\sigma_1), \quad (A.26) \]

\[ \text{Footnote 8: The fusion properties were not really proven so far, but rather made very plausible from quantum group invariance. Its complete derivation will be given elsewhere.} \]
\[ (J_1, J_2)^{N_2 N_1}_{M_1 M_2} = \delta(M_1 + M_2 - N_1 - N_2) e^{-2ihM_1M_2} (1 - e^{2ih})^n \times \]
\[ \frac{e^{ih(n-1)/2}}{n!} e^{-ih(M_1-M_2)} \left\langle \frac{|J_1 + M_1| |J_1 - N_1| |J_2 + M_2| |J_2 + N_2|}{|J_1 - M_1| |J_1 + N_1| |J_2 - M_2| |J_2 - N_2|} \right\rangle. \] (A.27)

where \( n = M_1 - N_1 = N_2 - M_2 \). For \( \pi > \sigma_2 > \sigma_1 > 0 \), on the other hand, one has
\[ \tilde{\xi}_{M_1}^{(J_1)}(\sigma_1) \tilde{\xi}_{M_2}^{(J_2)}(\sigma_2) = \sum_{-J_1 \leq N_1 \leq J_1; -J_2 \leq N_2 \leq J_2} (J_1, J_2)^{N_2 N_1}_{M_1 M_2} \tilde{\xi}_{M_2}^{(J_2)}(\sigma_2) \tilde{\xi}_{M_1}^{(J_1)}(\sigma_1), \] (A.28)

\[ \frac{e^{-ih(m-1)/2}}{m!} e^{ih(M_2-M_1)} \left\langle \frac{|J_1 - M_1| |J_1 + N_1| |J_2 + M_2| |J_2 - N_2|}{|J_1 + M_1| |J_1 - N_1| |J_2 - M_2| |J_2 + N_2|} \right\rangle. \] (A.29)

where \( m = M_2 - N_2 = N_1 - M_1 \). The two braiding matrices are related by the inverse relation
\[ \sum_{-J_1 \leq N_1 \leq J_1; -J_2 \leq N_2 \leq J_2} (J_1, J_2)^{N_2 N_1}_{M_1 M_2} (J_2, J_1)^{N_2 N_1}_{M_1 M_2} = \delta_{M_1, P_1} \delta_{M_2, P_2}. \] (A.30)

\((J_1, J_2)^{N_2 N_1}_{M_1 M_2}\) is given\[5\] by the corresponding matrix element of the universal \( R \) matrix of \( U_q(sl(2)) \). Concerning the hermiticity of the \( \xi \) fields, it was shown in \[5\] that, for \( \varpi \) real,
\[ (|J, \omega + 2m\rangle \tilde{\xi}^{m}_{M})^\dagger = |J, \omega\rangle \tilde{\xi}^{m}_{M}, \] (A.31)
so that
\[ \xi^{(J)}_{M} (\sigma)^* = \xi^{(J)}_{M} (\sigma) \] (A.32)
is a hermitian field. Obviously the same structure holds for the hatted fields. One replaces \( h \) by \( \tilde{h} \) everywhere. Moreover the hatted and unhatted fields have simple braiding and fusions\[5\]. The most general (2\( J + 1 \), 2\( J + 1 \)) field \( \xi^{(J, \tilde{J})}_{M, \tilde{M}} \sim \xi^{(J)}_{M} \tilde{\xi}^{(\tilde{J})}_{\tilde{M}} \) has weight
\[ \Delta_{Kac}(J, \tilde{J}; C) = \frac{C - 1}{24} - \frac{1}{24} ((J + \tilde{J} + 1)\sqrt{C - 1} - (J - \tilde{J})\sqrt{C - 25})^2, \] (A.33)
in agreement with Kac’s formula.

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