COMPARING THE CARATHÉODORY PSEUDO-DISTANCE AND
THE KÄHLER-EINSTEIN DISTANCE ON COMPLETE
REINHARDT DOMAINS

GUNHEE CHO

Abstract. We show that on a certain class of bounded, complete Reinhardt domains in $\mathbb{C}^n$ that enjoy a lot of symmetries, the Carathéodory pseudo-distance and the geodesic distance of the complete Kähler-Einstein metric with Ricci curvature $-1$ are different.

1. Introduction and main results

In this paper, we compare the Carathéodory pseudo-distance and the geodesic distance of the complete Kähler–Einstein metric with Ricci curvature $-1$ on certain complete Reinhardt domains in $\mathbb{C}^n$, $n \geq 2$. Throughout the paper, we will call the geodesic distance of the complete Kähler–Einstein metric with Ricci curvature $-1$ by the Kähler–Einstein distance. The Carathéodory pseudo-distance $c_M$ on a complex manifold $M$ is defined by

$$c_M(x, y) := \sup_{f \in \text{Hol}(M, \mathbb{D})} \rho_\mathbb{D}(f(x), f(y)).$$

Here, $\rho_\mathbb{D}$ denotes the Poincaré distance on the unit disk $\mathbb{D}$ in $\mathbb{C}^1$ and $\text{Hol}(M, \mathbb{D})$ is the collection of all holomorphic functions from $M$ to $\mathbb{D}$. We call $c_M$ the pseudo-distance instead of the distance because there could be distinct points $x, y \in M$ such that $c_M(x, y) = 0$.

The Carathéodory–Reiffen (pseudo–) metric and the Carathéodory pseudo–distance are objects of interest in the long-standing conjecture on the existence of a bounded, non-constant, holomorphic function on a simply connected, complete Kähler manifold whose sectional curvature is bounded from above by a negative constant. Recently, Wu and Yau proved that on a manifold with negatively pinched sectional curvature, the Kähler–Einstein, Bergman, and Kobayashi–Royden metrics exist and are uniformly equivalent to one another [33]. Thus, it is natural to compare the three invariant metrics above to Carathéodory-Reiffen metric [32,34]. The existence of the complete Kähler–Einstein metric and distance on wide classes of negatively curved complex manifolds that are invariant under biholomorphic mappings have

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been studied; especially for various classes of pseudoconvex domains in $\mathbb{C}^n$. In recent years a number of results devoted to the study of the relation between the Kähler–Einstein metric and other invariant objects were obtained (see, for example, [1–10, 13, 16, 18, 21–27, 31–34, 36] and references therein). The results involved also the smallest invariant metric and the invariant distance among the possible invariant metrics and invariant distances, i.e., the Carathéodory-Reiffen metric and Carathéodory pseudo–distance ([15, Proposition 3.1.7],[14, Chapter 2]).

As an invariant pseudo–distance, the Carathéodory pseudo–distance is less than or equal to the Carathéodory inner–distance [14, Remark 2.7.5], Bergman distance [11, 12, 19], and Kobayashi-Royden distance [14, 15] and specific examples in which Carathéodory pseudo–distance is strictly smaller than each of these distances are known (for example, [24, 25, 30]). As a consequence of the Schwarz–Yau Lemma (Section 2), the Carathéodory pseudo–distance is less than or equal to the Kähler–Einstein distance but a specific example was not investigated. In this paper, we provide a class of complete Reinhardt domains in $\mathbb{C}^n$, $n \geq 2$ that distinguish the Kähler–Einstein distance from the Carathéodory pseudo–distance.

The difficulty of distinguishing the Kähler–Einstein distance from the Carathéodory pseudo–distance is that even in the case of strictly pseudoconvex domains in $\mathbb{C}^n$, the concrete formulas of the Kähler–Einstein distance and the Carathéodory pseudo–distance are unknown. Thus, a concrete comparison of the two distances is non-trivial (see [17, 29] and [4, Proposition 5.5, Theorem 7.5] for the case of smoothly bounded, strictly pseudoconvex domains). Vigue’s Work [30] inspired one approach to compare the Carathéodory pseudo–distance and the Carathéodory inner–distance. After a clever choice of one point with origin on the diagonal entries on

$$\left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1, |z_1z_2| < \frac{1}{16}\right\},$$

(1.1)

Vigue distinguished the Carathéodory inner-distance and the Carathéodory pseudo–distance. By extending this idea, we could distinguish the Carathéodory-distance from the Kähler–Einstein distance. We say a domain $\Omega$ in $\mathbb{C}^n$ is a complete Reinhardt domain if for any $(z_1, \ldots, z_n) \in \Omega$ and $(\lambda_1, \ldots, \lambda_n) \in \overline{\mathbb{D}}^n$, the point $(\lambda_1z_1, \ldots, \lambda_nz_n)$ belongs to $\Omega$ [14, Remark 2.2.1]. We say $\Omega \subset \mathbb{C}^n$ is compatible with the symmetry by all permutations if $\Omega$ satisfies the following: $(z_1, \ldots, z_a, \ldots, z_b, \ldots, z_n) \in \Omega$ if and only if $(z_1, \ldots, z_{\phi(a)}, \ldots, z_{\phi(b)}, \ldots, z_n) \in \Omega$ for any $a, b \in \{1, \ldots, n\}$ and any permutation $\phi : \{1, \ldots, n\} \to \{1, \ldots, n\}$. We denote the ball of radius $R > 0$, centered at the origin in $\mathbb{C}^n$ by $B_R$. We denote the complete Kähler–Einstein metric with Ricci curvature $-1$ on $\Omega$ by $\omega_{KE}$, and let $d_{\Omega}^{KE}$ be the distance induced by $\omega_{KE}$.

**Theorem 1.** Let $\Omega$ be a bounded, complete Reinhardt domain in $\mathbb{C}^n$, $n \geq 2$. Suppose that $\Omega$ is compatible with the symmetry by all permutations and there exists $R > 0$ such that $\Omega$ is contained in $B_R$ and $\sum_{k=1}^n p_k = R$ for some $(p_1, \ldots, p_n) \in \Omega$. Then

$$\epsilon_{\Omega} \leq d_{\Omega}^{KE}.$$  

(1.2)
Here, (1.2) means for any \(a, b \in \Omega\), and there exist \(p, q \in \Omega\) such that
\[
c_\Omega(a, b) \leq d_{\Omega}^{KE}(a, b),
\]
\[
c_\Omega(p, q) < d_{\Omega}^{KE}(p, q).
\]

We apply the theorem of the existence of the complete Kähler–Einstein metric of a negative Ricci curvature on a bounded pseudoconvex domain in \(\mathbb{C}^n\) as given in the main theorem in [20]. The hypothesis of Theorem 1 is satisfied in several examples including (1.1) complex ellipsoids [5] and symmetrize polydisks of arbitrary complex dimensions [8, 24] and also others [6]. The proof of Theorem 1 will be presented in Section 3.

Theorem 1 holds on weakly pseudoconvex domains with very nice symmetries. In particular, if \(\Omega\) is selected as in Theorem 2, and it would be particularly beneficial to obtain a distance comparison between the origin \((0, \ldots, 0)\) and the diagonal entry \((x, \ldots, x)\). Especially, the following theorem is implicitly related to [14, Problem 2.10].

Theorem 2. Let \(n \geq 2\). For each \(0 < \epsilon < \frac{1}{n!}\), define
\[
\mathbb{D}_\epsilon^n := \{(z_1, \ldots, z_n) \in \mathbb{D}^n : |\prod_{i=1}^n z_i| < \epsilon\}.
\]
Then for any \((x, \cdots, x) \in \mathbb{D}_\epsilon^n\) satisfying \(|x| > (en)^{\frac{1}{n-1}}\), we have
\[
c_{\mathbb{D}_\epsilon^n}((0, \ldots, 0), (x, \ldots, x)) < d_{\mathbb{D}_\epsilon^n}^{KE}((0, \ldots, 0), (x, \ldots, x)).
\]

Note that \(\epsilon > 0\) must satisfy \(\epsilon < \frac{1}{n!}\) in order to take \((x, \cdots, x) \in \mathbb{D}_\epsilon^n\) with \(|x| > (en)^{\frac{1}{n-1}}\). The proof of Theorem 2 will be presented in Section 4 with some additional remarks.

Lastly, one can easily extend the proof of Theorem 1 with the complete Kähler–Einstein metric of Ricci curvature \(-\lambda, \lambda > 0\).

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2. Schwarz–Yau lemma

The following generalized Schwarz lemma due to Yau will be used to compare the Carathéodory-distance and the Kähler–Einstein distance.

Theorem 3 (the Schwarz-Yau lemma, [28, 35]). Let \((M, g)\) be a complete Kähler manifold with Ricci curvature bounded from below by a negative constant \(K_1\). Let \((N, h)\) be another Hermitian manifold with holomorphic bisectional curvature bounded...
from above by a negative constant $K_2$. If there is a non-constant holomorphic map $f$ from $M$ to $N$, we have

$$f^*h \leq \frac{K_1}{K_2} g.$$ 

We can use the upper bound of holomorphic sectional curvature of $(N, h)$ instead of the upper bound of bisectional curvature if $N$ is a Riemann surface [28].

$3. \ c_\Omega \leq d_{\Omega}^{KE}$ for some complete Reinhardt domains $\Omega$ in $\mathbb{C}^n, n \geq 2$

The proof of Theorem 1 can be reduced to Lemma 4 which gives the comparison of the Carathéodory pseudo–distance and the Kähler–Einstein distance. Note that by Montel’s theorem, for any two points in a complex manifold $M$, we can always achieve the extremal map $f \in \text{Hol}(M, \mathbb{D})$ with respect to the Carathéodory pseudo–distance. In Lemma 4, $\gamma_\mathbb{D}$ is the Poincaré metric on the unit disk and $\omega_{KE}(a)$ is the hermitian inner product on the holomorphic tangent space at $a \in \Omega$.

**Lemma 4.** Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, and $a, b \in \Omega$, $a \neq b$. Suppose there exists $f \in \text{Hol}(\Omega, \mathbb{D})$ that is extremal for $c_\Omega(a, b)$ such that

$$\gamma_\mathbb{D}(f(a), f'(a)X) < \sqrt{(\omega_{KE}(a)(X, X))}, X \in \mathbb{C}^n - \{0\}. \quad (3.1)$$

Then

$$c_\Omega(a, b) < d_{\Omega}^{KE}(a, b).$$

**Proof.** By assumption, continuity of metrics gives $\epsilon, \delta$ such that

$$\gamma_\mathbb{D}(f(z), f'(z)X) + \epsilon\|X\| \leq \sqrt{(\omega_{KE}(z)(X, X))}, z \in B(a, \delta) \subseteq \Omega, X \in \mathbb{C}^n.$$

On the other hand, by Theorem 3, we have

$$\gamma_\mathbb{D}(f(z), f'(z)X) \leq \sqrt{(\omega_{KE}(z)(X, X))}, z \in \Omega, X \in \mathbb{C}^n.$$

Let $\alpha : [0, 1] \to \Omega$ be any piecewise $C^1$-curve joining $a$ and $b$. Denote by $t_0$ the maximal $t \in [0, 1]$ such that $\alpha([0, t]) \subset B(a, \delta)$. Denote the arc-length of $\alpha$ with respect to the Kähler–Einstein metric by $L_{KE}(\alpha)$. Then

$$L_{KE}(\alpha) = \int_0^{t_0} \sqrt{\omega_{KE}(\alpha(t))\langle \alpha'(t), \alpha'(t) \rangle}dt + \int_{t_0}^1 \sqrt{\omega_{KE}(\alpha(t))\langle \alpha'(t), \alpha'(t) \rangle}dt$$

$$\geq \int_0^1 \gamma_\mathbb{D}(f(\alpha(t)), f'(\alpha(t))dt + \epsilon \int_0^{t_0} \|\alpha'(t)\|dt$$

$$\geq L_{\mathbb{D}}(f \circ \alpha) + \epsilon \delta \geq \rho_\mathbb{D}(f(a), f(b)) + \epsilon \delta = c_\Omega(a, b) + \epsilon \delta.$$

Hence $d_{\Omega}^{KE}(a, b) \geq c_\Omega(a, b) + \epsilon \delta$. 

$\square$
Proof of Theorem 1. The comparison between $c_{\Omega}$ and $d^{KE}_{\Omega}$ can be achieved once some extremal map $f \in \text{Hol}(\Omega, \mathbb{D})$ with respect to the Carathéodory pseudo–distance satisfies the assumption of Lemma 4.

Since $\Omega$ is a bounded domain in $\mathbb{C}^n$, we may assume that $\Omega$ is contained in the unit ball $B \subset \mathbb{C}^n$ centered at the origin due to the fact that the scaling transformation is a biholomorphism and the Carathéodory pseudo–distance and the Kähler–Einstein distance are preserved under any biholomorphism.

With the global coordinate $(z_1, \ldots, z_n) \in \Omega$ in $\mathbb{C}^n$, let $\{\frac{\partial}{\partial z_i} | i = 1, \ldots, n\}$ be the basis on the holomorphic tangent bundle $T_{0,0}^1, \Omega$. Without loss of generality, we may assume that $\{\frac{\partial}{\partial z_i} | i = 1, \ldots, n\}$ are orthonormal with respect to the Euclidean metric and orthogonal with respect to $\omega_{KE}(0)$. With the usual global coordinates $(z_1, \cdots, z_n) \in \Omega$, denote $\frac{\partial}{\partial z_i} = X_i, i = 1, \cdots, n$. We may assume that $0 < \omega_{KE}(0)(X_1, \overline{X}_1) \leq \omega_{KE}(0)(X_i, \overline{X}_i), i = 1 \cdots, n$.

We will show there exists a local hypersurface in $\Omega$ which is defined by
\[
\sqrt{\omega_{KE}(X_1, \overline{X}_1)(0)} \sum_{k=1}^{n} z_k = 1.
\]

From the assumption of Theorem 1, we can take $0 < R \leq 1$ such that $\Omega$ is contained in $B_R$ and the boundary of $\Omega$ touches the boundary of $B_R$. Then by using a projection map with respect to the first coordinate, we can define the holomorphic map from $(\Omega, \omega_{KE})$ to $(\mathbb{D}_R, h)$, where $h$ is the Poincaré metric on $\mathbb{D}_R$ the ball of radius $R$ in $\mathbb{C}^1$. Then by applying Theorem 3, $h(0) \leq R^2 \omega_{KE}(0)$. This implies
\[
1 \leq \frac{1}{R} \leq R \sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}. \tag{3.2}
\]

In particular, $R \leq 1 \leq R \sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}$ and the assumption of the existence of $(p_1, \cdots, p_n) \in \Omega$ satisfying $\sum_{k=1}^{n} p_k = 1$ implies the points $(z_1, \cdots, z_n) \in \Omega$ satisfying $\sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)} \sum_{k=1}^{n} z_k = 1$. Here, the rescaling of the domain $\Omega \subset B$ and the fact that $\Omega$ is a complete Reinhardt domain are applied.

We will control the extremal map with respect to the Carathéodory pseudo–distance $f : \Omega \rightarrow \mathbb{D}$ such that
\[
c_{\Omega}((0, \cdots, 0), (x, \cdots, x)) = c_{\mathbb{D}}(f(0, \cdots, 0), f(x, \cdots, x)). \tag{3.3}
\]

After acting on the unit disk by an automorphism, we may assume that $f(0, \cdots, 0) = 0$. Also we can replace $f$ by the symmetrization map $\frac{1}{n!} \sum_{\sigma} f(z_{\sigma(1)}, \cdots, z_{\sigma(n)})$ with all permutations $\sigma : \{1, \cdots, n\} \rightarrow \{1, \cdots, n\}$. Around the origin, one can write $f$ as a power series $\sum_{m=0}^{\infty} P_m(z_1, \cdots, z_n) = \sum_{i=1}^{n} a_i z_i + f_2(z_1, \cdots, z_n)$, where each $P_m(z_1, \cdots, z_n)$ is a homogeneous polynomial of degree $m$. Since we use $\frac{1}{n!} \sum_{\sigma} f(z_{\sigma(1)}, \cdots, z_{\sigma(n)})$, we can replace $\sum_{i=1}^{n} a_i z_i$ by $a \sum_{k=1}^{n} z_k$ for some real number $a$ (after acting on the unit disk by an automorphism if necessary). Then by
Theorem 3, \( f^* \gamma_\mathbb{D} \leq \sqrt{\omega_{KE}} \). In particular, we get
\[
a \leq \sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}.
\]

Now let’s suppose \( a = \sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)} \). Since there exists a hypersurface in \( \Omega \) given by \( a \sum_{k=1}^{n} z_k = \sqrt{\omega_{KE}(X_1, \overline{X}_1)(0)} \sum_{k=1}^{n} z_k = 1 \), we can choose \( (u_1, \ldots, u_n) \in \Omega \) such that \( \sum_{k=1}^{n} u_k = \frac{1}{\sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)}} \). Define \( g : \mathbb{D} \rightarrow \Omega \) by \( g(\lambda) := (\lambda u_1, \ldots, \lambda u_n) \).

Since \( \Omega \) is a complete Reinhardt domain, \( g \) is a well-defined holomorphic function. Furthermore, \( f \circ g(0) = 0 \) and \( (f \circ g)'(0) = 1 \). Thus \( f \circ g = id_\mathbb{D} \) by the classical Schwarz’s lemma that \( f_2(\lambda u_1, \ldots, \lambda u_n) = 0 \). Since \( \lambda \in \mathbb{D} \) is arbitrary and with other choices of \( (u_1, \ldots, u_n) \), we can take a small open set in \( \Omega \) such that \( f_2 = 0 \) on this open set. Thus by the identity theorem, \( f_2 \equiv 0 \) on \( \mathbb{D} \). Hence \( f(z_1, \ldots, z_n) = a \sum_{k=1}^{n} z_k \), which is impossible since the image of \( f \) can’t hit the boundary of \( \mathbb{D} \). Hence \( a < \sqrt{\omega_{KE}(0)(X_1, \overline{X}_1)} \). In particular, this implies
\[
|f'(0)X_i| < \sqrt{\omega_{KE}(0)(X_i, \overline{X}_i)}, i = 1, \ldots, n. \tag{3.4}
\]

Since we showed the above inequality for \( X_i \)’s, that are orthogonal to the Euclidean metric and \( \omega_{KE}(0) \) at the same time, the same inequality holds for arbitrary tangent vectors \( X \in \mathbb{C}^n - \{0\} \). Consequently, (3.4) implies the assumption in Lemma 4, and the proof is over.

**Remark 5.** A separation between the Carathéodory pseudo–distance and the inner Carathéodory pseudo–distance was made from similar argument (see [30], [14, Lemma 2.7.8]).

4. \( c_{\mathbb{D}^n}(0, \ldots, 0, (x, \ldots, x)) \leq d_{\mathbb{D}^n}^{KE}((0, \ldots, 0), (x, \ldots, x)) \)

Although the proof of the Theorem 2 contains the similar argument in the proof of the Theorem 1, we will provide the proof in detail, because the role of two fixed points \((x, \ldots, x)\) and \((0, \ldots, 0)\) in \( \mathbb{D}^n \) should be justified.

**Proof of Theorem 2.** As we did in the proof of the Theorem 1, we establish the basic setting first. Fix \( \epsilon > 0 \), with the global coordinate \((z_1, \ldots, z_n) \in \mathbb{D}_\epsilon^n \) in \( \mathbb{C}^n \), let \( \{\frac{\partial}{\partial z_i} | i = 1, \ldots, n\} \) be the basis on the holomorphic tangent bundle \( T_0^{1,0}\mathbb{D}_\epsilon^n \).

Without loss of generality, we may assume that \( \{\frac{\partial}{\partial z_i} | i = 1, \ldots, n\} \) are orthonormal with respect to the Euclidean metric and orthogonal with respect to \( \omega_{KE}(0) \). With the global coordinates \((z_1, \ldots, z_n) \in \mathbb{D}^n \), denote \( \frac{\partial}{\partial z_i} = \partial_i, i = 1, \ldots, n \). Also, we may assume that \( 0 < \omega_{KE}(0)(X_1, \overline{X}_1) \leq \omega_{KE}(0)(X_i, \overline{X}_i), i = 1, \ldots, n \). For \( z = (z_1, \ldots, z_n) \), define the holomorphic function \( h : \mathbb{D}^n \rightarrow \mathbb{D} \) by \( h(z) := \frac{1}{\epsilon} \Pi_{k=1}^{n} z_k \).

By the distance-decreasing property of the Carathéodory pseudo–distance,
\[
c_{\mathbb{D}}(h(0, \ldots, 0), h(x, \ldots, x)) \leq c_{\mathbb{D}^n}(0, \ldots, 0, (x, \ldots, x)).
\]
Thus
\[ c_{D}(0, \frac{1}{\varepsilon}x^n) \leq c_{D}(0, \cdots, 0, (x, \cdots, x)). \] (4.1)

From the hypothesis, take \((x, \cdots, x) \in D_{\varepsilon}^n\) satisfying \((en)^{\frac{1}{n-1}} < |x| < 1\). We may assume \(x\) is a positive real number so that \(x \in D\) satisfies
\[ x > (en)^{\frac{1}{n-1}}. \] (4.2)

Let \(f : D_{\varepsilon}^n \to D\) be the extremal map with respect to the Carathéodory pseudo-distance so that
\[ c_{D}(0, \cdots, 0, (x, \cdots, x)) = c_{D}(f(0, \cdots, 0), f(x, \cdots, x)). \] (4.3)

We may assume that \(f(0, \cdots, 0) = 0\). Since \(D_{\varepsilon}^n\) is compatible with the symmetry by all permutations, we may assume that \(f\) itself is a symmetrization map so that
\[ f(z) = a \sum_{k=1}^{n} z_k + \sum_{m=2}^{\infty} P_m(z), \]
where each \(P_m\) is a homogeneous polynomial of degree \(m\) and as in the same argument of the proof of Theorem 1, we get
\[ a \leq \sqrt{\omega_{KE}(0)(X_1, X_1)}. \]

From the description of \(D_{\varepsilon}^n\), we can find \((p_1, \cdots, p_n) \in D_{\varepsilon}^n\) satisfying \(\sum_{k=1}^{n} z_k = 1\) by taking one component with the magnitude almost one and the magnitudes of the other components almost zero. Then as in following the same argument of the proof of Theorem 1, there exists a local hypersurface of \(D_{\varepsilon}^n\) given by
\[ \sqrt{\omega_{KE}(0)(X_1, X_1)} \sum_{k=1}^{n} z_k = 1. \]

Now we will show \(a < \sqrt{\omega_{KE}(0)(X_1, X_1)}\). Assume \(a = \sqrt{\omega_{KE}(0)(X_1, X_1)}\). By (4.1) and (4.3),
\[ c_{D}(0, n\sqrt{\omega_{KE}(0)(X_1, X_1)}x) = c_{D}(0, 0, (x, x)) \geq c_{D}(0, \frac{1}{\varepsilon}x^n). \]

Thus we obtain \(x \leq \left( n\varepsilon\sqrt{(\omega_{KE}(0)(X_1, X_1))} \right)^{1/n} \). On the other hand, since the projection of \(f : D_{\varepsilon}^n \to D\) from \(D_{\varepsilon}^n\) to the first coordinate induces the holomorphic function from \(D\) to \(D\), the classical Schwarz lemma implies \(\sqrt{(\omega_{KE}(0)(X_1, X_1))} = a \leq 1\). In particular, we have
\[ n\varepsilon\sqrt{(\omega_{KE}(0)(X_1, X_1))} = n\varepsilon a \leq n\varepsilon < 1. \]

However, by (4.2), we also have \(x > (en)^{\frac{1}{n-1}} \geq \left( n\varepsilon\sqrt{(\omega_{KE}(0)(X_1, X_1))} \right)^{1/n} \), which is impossible. Hence \(a < \sqrt{\omega_{KE}(0)(X_1, X_1)}\).

Then the rest of the proof follows as in the proof of Theorem 1. \(\square\)
References

[1] John S. Bland, *The Einstein-Kähler metric on \(|z|^2 + |w|^2 < 1\)*, Michigan Math. J. 33 (1986), no. 2, 209–220. MR837579

[2] Brian E. Blank, Da Shan Fan, David Klein, Steven G. Krantz, Daowei Ma, and Myung-Yull Pang, *The Kobayashi metric of a complex ellipsoid in \(\mathbb{C}^2\)*, Experiment. Math. 1 (1992), no. 1, 47–55. MR1181086

[3] David W. Catlin, *Estimates of invariant metrics on pseudoconvex domains of dimension two*, Math. Z. 200 (1989), no. 3, 429–466. MR978601

[4] Shiu Yuen Cheng and Shing Tung Yau, *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman’s equation*, Comm. Pure Appl. Math. 33 (1980), no. 4, 507–544. MR575736

[5] Gunhee Cho, *Invariant metrics on the complex ellipsoid*, J. Geom. Anal. 31 (2021), no. 2, 2088–2104. MR4215285

[6] Gunhee Cho, *Quasi-bounded geometry of the Bergman metric and equivalence of invariant metrics*, arXiv:2009.13027 [math.DG], 2020.

[7] Gunhee Cho and Junqing Qian, *The Kobayashi-Royden metric on punctured spheres*, Forum Math. 32 (2020), no. 4, 911–918. MR4116646

[8] Gunhee Cho and Yuan Yuan, *Bergman metric on the symmetrized bidisc and its consequences*, arXiv:2004.04637 [math.DG], 2020.

[9] Siqi Fu, *Estimates of invariant metrics on pseudoconvex domains near boundaries with constant Levi ranks*, J. Geom. Anal. 24 (2014), no. 1, 32–46. MR3145913

[10] Ian Graham, *Intrinsic measures and holomorphic retracts*, Pacific J. Math. 130 (1987), no. 2, 299–311. MR914103

[11] Kyong T. Hahn, *On completeness of the Bergman metric and its subordinate metric*, Proc. Nat. Acad. Sci. U.S.A. 73 (1976), no. 12, 4204. MR417459

[12] Kyong T. Hahn, *On completeness of the Bergman metric and its subordinate metrics. II*, Pacific J. Math. 68 (1977), no. 2, 437–446. MR486653

[13] Gregor Herbort, *Estimation on invariant distances on pseudoconvex domains of finite type in dimension two*, Math. Z. 251 (2005), no. 3, 673–703. MR2190351

[14] Marek Jarnicki and Peter Pflug, *Invariant distances and metrics in complex analysis*, extended, De Gruyter Expositions in Mathematics, vol. 9, Walter de Gruyter GmbH & Co. KG, Berlin, 2013. MR3114789

[15] Shoshichi Kobayashi, *Hyperbolic complex spaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 318, Springer-Verlag, Berlin, 1998. MR1635983

[16] Łukasz Kosiński, *Comparison of invariant functions and metrics*, Arch. Math. (Basel) 102 (2014), no. 3, 271–281. MR3181717

[17] John M. Lee and Richard Melrose, *Boundary behaviour of the complex Monge-Ampère equation*, Acta Math. 148 (1982), 159–192. MR666109

[18] László Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France 109 (1981), no. 4, 427–474. MR660145

[19] K. H. Look, *Schwarz lemma in the theory of functions of several complex variables*, Acta Math. Sinica 7 (1957), 370–420. MR121503

[20] Ngaiming Mok and Shing-Tung Yau, *Completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions*, The mathematical heritage of Henri Poincaré, Part 1 (Bloomington, Ind., 1980), 1983, pp. 41–59. MR720056

[21] Nikolai Nikolov and Peter Pflug, *On the derivatives of the Lempert functions*, Ann. Mat. Pura Appl. (4) 187 (2008), no. 3, 547–553. MR2393147

[22] Nikolai Nikolov and Peter Pflug, *Remarks on Lempert functions of balanced domains*, Monatsh. Math. 156 (2009), no. 2, 159–165. MR2488860
[23] , Kobayashi-Royden pseudometric versus Lempert function, Ann. Mat. Pura Appl. (4) 190 (2011), no. 4, 589–593. MR2861060

[24] Nikolai Nikolov, Peter Pflug, Pascal J. Thomas, and Wlodzimierz Zwonek, Estimates of the Carathéodory metric on the symmetrized polydisc, J. Math. Anal. Appl. 341 (2008), no. 1, 140–148. MR2394071

[25] Nikolai Nikolov, Peter Pflug, and Wlodzimierz Zwonek, The Lempert function of the symmetrized polydisc in higher dimensions is not a distance, Proc. Amer. Math. Soc. 135 (2007), no. 9, 2921–2928. MR2317970

[26] Peter Pflug and Wlodzimierz Zwonek, Effective formulas for invariant functions—case of elementary Reinhardt domains, Ann. Polon. Math. 69 (1998), no. 2, 175–196. MR1641884

[27] Junqing Qian, Hyperbolic metric, punctured Riemann sphere, and modular functions, Trans. Amer. Math. Soc. 373 (2020), no. 12, 8751–8784. MR4177275

[28] H. L. Royden, The Ahlfors-Schwarz lemma in several complex variables, Comment. Math. Helv. 55 (1980), no. 4, 547–558. MR604712

[29] Sergio Venturini, Comparison between the Kobayashi and Carathéodory distances on strongly pseudoconvex bounded domains in \( C^n \), Proc. Amer. Math. Soc. 107 (1989), no. 3, 725–730. MR984819

[30] Jean-Pierre Vigué, La distance de Carathéodory n’est pas intérioré, Results Math. 6 (1983), no. 1, 100–104. MR714663

[31] Wei Wang, Estimate of the Einstein-Kähler metric on a weakly pseudoconvex domain in \( C^2 \), Math. Z. 223 (1996), no. 3, 535–545. MR1417859

[32] Damin Wu and Shing-Tung Yau, Some negatively curved complex geometry, ICCM.

[33] , Invariant metrics on negatively pinched complete Kähler manifolds, J. Amer. Math. Soc. 33 (2020), no. 1, 103–133. MR4066473

[34] H. Wu, Old and new invariant metrics on complex manifolds, Several complex variables (Stockholm, 1987/1988), 1993, pp. 640–682. MR1207887

[35] Shing Tung Yau, A general Schwarz lemma for Kähler manifolds, Amer. J. Math. 100 (1978), no. 1, 197–203. MR486659

[36] Sai-Kee Yeung, Geometry of domains with the uniform squeezing property, Adv. Math. 221 (2009), no. 2, 547–569. MR2508930

Department of Mathematics, University of California, Santa Barbara, Santa Barbara, CA 93106

Email address: gunhee.cho@math.ucsb.edu