Fusion frame and its alternative dual in tensor product of Hilbert spaces

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Abstract
We study fusion frame in tensor product of Hilbert spaces and discuss some of its properties. The resolution of the identity operator on a tensor product of Hilbert spaces is being discussed. An alternative dual of a fusion frame in tensor product of Hilbert spaces is being presented.

Keywords: Fusion frame, Resolution of identity operator, Canonical dual frame, Tensor product of Hilbert spaces, Tensor product of frames.

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1 Introduction

Fusion frame was introduced by P. Casazza and G. Kutyniok [2]. They define frames for closed subspaces of a given Hilbert spaces with respect to the orthogonal projections. Fusion frame is a natural generalization of the frame theory in Hilbert space and it has so many applications in data processing, coding theory, signal processing and many other fields. Infact, the fusion frame theory is more elegant due to complicated relations between the structure of the sequence of weighted subspace and the local frames in the subspace. In fusion frame, the atomic resolution of the identity operator on Hilbert space was studied by M. S Asgari and Amir Khosraki [1]. They present a reconstruction formula and establish some useful results about resolution of the identity operator.

The basic concepts of tensor product of Hilbert spaces were described by S. Rabinson [11]. Frames and Bases in Tensor Product of Hilbert spaces were introduced by A. Khosravi and M. S. Asgari [9]. Reddy et al. [12] also studied the frame in tensor product of Hilbert spaces and presented the frame operator on tensor product of Hilbert spaces. The concepts of fusion frames and g-frames in tensor product of Hilbert spaces were introduced by Amir Khosravi and M. Mirzae Azandaryani [10].

In this paper, fusion frame in tensor product of Hilbert spaces is developed and discuss a relationship among fusion frames in Hilbert spaces and their tensor
products. We shall verify that in tensor product of Hilbert spaces, an image of a fusion frame under a bounded linear operator will be a fusion frame if the operator is invertible and unitary. The resolution of the identity operator on a tensor product of Hilbert spaces is presented. In tensor product of Hilbert spaces, we study an alternative dual of a fusion frame and see that the canonical dual of a fusion frame is also an alternative dual. Finally, we establish that an alternative dual of a fusion frame is a fusion frame in tensor product of Hilbert spaces.

Throughout this paper, $H$ and $K$ are considered to be separable Hilbert spaces with associated inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. $I_H$ and $I_K$ are the identity operators on $H$ and $K$, respectively. $B(H, K)$ is the collection of all bounded linear operators from $H$ to $K$. In particular, $B(H)$ denote the space of all bounded linear operators on $H$. $P_V$ denote the orthogonal projection onto the closed subspace $V \subset H$. $\{V_i\}_{i \in I}$ and $\{W_j\}_{j \in J}$ are the collections of closed subspaces of $H$ and $K$, where $I, J$ are index sets. Define the space $l^2(\{V_i\}_{i \in I}) = \{\{f_i\}_{i \in I} : f_i \in V_i, \sum_{i \in I} \|f_i\|_1^2 < \infty\}$ with inner product is given by $\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle_1$. Clearly $l^2(\{V_i\}_{i \in I})$ is a Hilbert space with respect to the above inner product [1]. Similarly, we can define the space $l^2(\{W_j\}_{j \in J})$.

2 Preliminaries

**Theorem 2.1.** [6] Let $V \subset H$ be a closed subspace and $T \in B(H)$. Then $P_V T^* = P_V T^* P_{\overline{T V}}$. If $T$ is an unitary operator (i.e $T^* T = I_H$), then $P_{\overline{T V}} T = T P_V$.

**Theorem 2.2.** [7] The set $S(H)$ of all self-adjoint operators on $H$ is a partially ordered set with respect to the partial order $\leq$ which is defined as for $T, S \in S(H)$, $T \leq S \iff \langle T(f), f \rangle_1 \leq \langle S(f), f \rangle_1 \forall f \in H$.

**Definition 2.3.** [2] Let $\{v_i\}_{i \in I}$ be a collection of positive weights. A family of weighted closed subspaces $V = \{(V_i, v_i) : i \in I\}$ is called a fusion frame for $H$ if there exist constants $0 < A \leq B < \infty$ such that

\[ A \|f\|_1^2 \leq \sum_{i \in I} v_i^2 \|P_{V_i}(f)\|_1^2 \leq B \|f\|_1^2 \forall f \in H. \]  

The constants $A, B$ are called fusion frame bounds. If the family $V$ satisfies

\[ \sum_{i \in I} v_i^2 \|P_{V_i}(f)\|_1^2 \leq B \|f\|_1^2 \forall f \in H \]

then it is called a fusion Bessel sequence in $H$ with bound $B$. 
Theorem (2.2), we can write,

\[ T_V \{ f_i \}_{i \in I} = \sum_{i \in I} v_i f_i \quad \forall \{ f_i \}_{i \in I} \in l^2(\{ V_i \}_{i \in I}) \]

and the analysis operator is given by

\[ T_V^* : H \to l^2(\{ V_i \}_{i \in I}), \quad T_V^* (f) = \{ v_i P_{V_i} (f) \}_{i \in I} \quad \forall f \in H. \]

The fusion frame operator \( S_V : H \to H \) is defined as follows:

\[ S_V (f) = T_V T_V^* (f) = \sum_{i \in I} v_i^2 P_{V_i} (f) \quad \forall f \in H. \]

Note 2.5. Let \( V \) be a fusion frame with bounds \( A, B \). Then from (1),

\[ \langle A f, f \rangle_1 \leq \langle S_V (f), f \rangle_1 \leq \langle B f, f \rangle_1 \quad \forall f \in H. \]

The operator \( S_V \) is bounded, self-adjoint, positive and invertible. Now, according to the Theorem (2.2), we can write, \( A I_H \leq S_V \leq B I_H \) and this gives \( B^{-1} I_H \leq S_V^{-1} \leq A^{-1} I_H \).

Definition 2.6. Let \( V = \{ (V_i, v_i) \}_{i \in I} \) be a fusion frame for \( H \). Then \( \{ (S_V^{-1} V_i, v_i) \}_{i \in I} \) is called the canonical dual fusion frame of \( V \).

Theorem 2.7. Let \( V = \{ (V_i, v_i) \}_{i \in I} \) be a fusion frame for \( H \) with bounds \( A, B \) and \( S_V \) be the corresponding frame operator. Then the canonical dual fusion frame of \( V \) is a fusion frame with bounds

\[ \frac{A}{\| S_V \|^2 \| S_V^{-1} \|^2}, B \| S_V \|^2 \| S_V^{-1} \|^2. \]

Note 2.8. A reconstruction formula on \( H \) with the help of canonical dual fusion frame is given by

\[ f = \sum_{i \in I} v_i^2 P_{S_V^{-1} V_i} S_V^{-1} P_{V_i} (f) \quad \forall f \in H. \]

Definition 2.9. Let \( V = \{ (V_i, v_i) \}_{i \in I} \) be a fusion frame for \( H \) and \( S_V \) be the corresponding frame operator. Then a fusion Bessel sequence \( V' = \{ (V'_i, v'_i) \}_{i \in I} \) is said to be an alternative dual of \( V \) if

\[ f = \sum_{i \in I} v_i v'_i P_{V'_i} S_V^{-1} P_{V_i} (f) \quad \forall f \in H. \]

Definition 2.10. A family of bounded operators \( \{ T_i \}_{i \in I} \) on \( H \) is called a resolution of identity operator on \( H \) if for all \( f \in H \), we have \( f = \sum_{i \in I} T_i (f) \), provided the series converges unconditionally for all \( f \in H \).
There are several ways to introduced the tensor product of Hilbert spaces. The tensor product of Hilbert spaces $H$ and $K$ is a certain linear space of operators which was represented by Folland in [5], Kadison and Ringrose in [8].

**Definition 2.11.** [12] The tensor product of $H$ and $K$ is denoted by $H \otimes K$ and it is defined to be an inner product space associated with the inner product

$$
\langle f \otimes g, f' \otimes g' \rangle = \langle f, f' \rangle_1 \langle g, g' \rangle_2 \quad \forall \ f, f' \in H \& \ g, g' \in K.
$$

The norm on $H \otimes K$ is defined by

$$
\| f \otimes g \| = \| f \|_1 \| g \|_2 \quad \forall \ f \in H \& \ g \in K.
$$

(2)

The space $H \otimes K$ is complete with respect to the above inner product. Therefore, the space $H \otimes K$ is an Hilbert space.

For $Q \in B(H)$ and $T \in B(K)$, the tensor product of operators $Q$ and $T$ is denoted by $Q \otimes T$ and defined as

$$
(Q \otimes T) \ A = Q A T^* \quad \forall \ A \in H \otimes K.
$$

**Theorem 2.12.** [5, 13] Let $Q, Q' \in B(H)$ and $T, T' \in B(K)$. Then

(I) $Q \otimes T \in B(H \otimes K)$ and $\| Q \otimes T \| = \| Q \| \| T \|$. 

(II) $(Q \otimes T) (f \otimes g) = Q(f) \otimes T(g)$ for all $f \in H, g \in K$.

(III) $(Q \otimes T) (Q' \otimes T') = (QQ') \otimes (TT')$.

(IV) $Q \otimes T$ is invertible if and only if $Q$ and $T$ are invertible, in which case

$$(Q \otimes T)^{-1} = (Q^{-1} \otimes T^{-1})$$

(V) $(Q \otimes T)^* = (Q^* \otimes T^*)$.

(VI) Let $f, f' \in H \setminus \{0\}$ and $g, g' \in K \setminus \{0\}$. If $f \otimes g = f' \otimes g'$, then there exist constants $a$ and $b$ with $ab = 1$ such that $f = af'$ and $g = bg'$.

3 **Fusion frame in tensor product of Hilbert spaces**

In this section, fusion frame in the Hilbert space $H \otimes K$ have been discussed and some results associated to fusion frame in $H \otimes K$ are likely to be established. At the end, we discuss the relationship among the resolution of the identity operator on $H \otimes K$ and resolutions of the identity operators on $H$ and $K$, respectively.

**Definition 3.1.** Let $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$ be two families of positive weights i.e., $v_i > 0 \ \forall \ i \in I$ and $w_j > 0 \ \forall \ j \in J$ and $\{V_i \otimes W_j : (i, j) \in I \times J\}$ be
a family of closed subspaces of \( H \otimes K \). Then the family \( V \otimes W = \{ (V_i \otimes W_j, v_i w_j) \}_{i,j} \) is called a fusion frame for \( H \otimes K \), if there exist \( 0 < A \leq B < \infty \) such that

\[
A \| f \otimes g \|^2 \leq \sum_{i,j} v_i^2 w_j^2 \| P_{V_i \otimes W_j} (f \otimes g) \|^2 \leq B \| f \otimes g \|^2 \quad \forall f \otimes g \in H \otimes K,
\]

where \( P_{V_i \otimes W_j} \) is the orthogonal projection of \( H \otimes K \) onto \( V_i \otimes W_j \). The constants \( A \) and \( B \) are called the frame bounds. If \( A = B \) then it is called a tight fusion frame for \( H \otimes K \). If the family \( V \otimes W \) satisfies the inequality

\[
\sum_{i,j} v_i^2 w_j^2 \| P_{V_i \otimes W_j} (f \otimes g) \|^2 \leq B \| f \otimes g \|^2 \quad \forall f \otimes g \in H \otimes K,
\]

then it is called a fusion Bessel sequence in \( H \otimes K \) with bound \( B \).

**Definition 3.2.** For \( i \in I \) and \( j \in J \), define the space \( l^2 (\{ V_i \otimes W_j \}) \)

\[
= \left\{ \{ f_i \otimes g_j \} : f_i \otimes g_j \in V_i \otimes W_j, \text{ and } \sum_{i,j} \| f_i \otimes g_j \|^2 < \infty \right\}
\]

with inner product

\[
\langle \{ f_i \otimes g_j \}, \{ f_i' \otimes g_j' \} \rangle_{l^2} = \sum_{i,j} \langle f_i \otimes g_j, f_i' \otimes g_j' \rangle
\]

\[
= \sum_{i,j} \langle f_i, f_i' \rangle_1 \langle g_j, g_j' \rangle_2 = \left( \sum_{i \in I} \langle f_i, f_i' \rangle_1 \right) \left( \sum_{j \in J} \langle g_j, g_j' \rangle_2 \right)
\]

\[
= \left\langle \{ f_i \}_{i \in I}, \{ f_i' \}_{i \in I} \right\rangle_{l^2(\{ V_i \}_{i \in I})} \left\langle \{ g_j \}_{j \in J}, \{ g_j' \}_{j \in J} \right\rangle_{l^2(\{ W_j \}_{j \in J})}.
\]

It is easy to verify that the space \( l^2 (\{ V_i \otimes W_j \}) \) is an Hilbert space with respect to the above inner product.

**Note 3.3.** Since \( \{ V_i \}_{i \in I}, \{ W_j \}_{j \in J} \) and \( \{ V_i \otimes W_j \}_{i,j} \) are the families of closed subspaces of \( H, K \) and \( H \otimes K \) respectively, it is easy to verify that \( P_{V_i \otimes W_j} = P_{V_i} \otimes P_{W_j} \).

For the remaining part of this paper, we denote the collections \( \{ (V_i, v_i) \}_{i \in I}, \{ (W_j, w_j) \}_{j \in J}, \{ (V_i \otimes W_j, v_i w_j) \}_{i,j} \) and \( \{ (V'_{i'}, W'_{j'}, v'_{i'} w'_{j'}) \}_{i,j} \) by \( V, W, V \otimes W \) and \( V' \otimes W' \), respectively.

**Theorem 3.4.** Let \( V \) and \( W \) be the families of weighted closed subspaces in \( H \) and \( K \), respectively. Then \( V \) and \( W \) are fusion frames for \( H \) and \( K \) if and only if \( V \otimes W \) is a fusion frame for \( H \otimes K \).
\textbf{Proof.} First we suppose that $V$ and $W$ are fusion frames for $H$ and $K$. Then there exist positive constants $(A, B)$ and $(C, D)$ such that
\begin{equation}
A \| f \|_1^2 \leq \sum_{i \in I} v_i^2 \| P_{V_i} (f) \|_1^2 \leq B \| f \|_1^2 \quad \forall \ f \in H
\end{equation}
\begin{equation}
C \| g \|_2^2 \leq \sum_{j \in J} w_j^2 \| P_{W_j} (g) \|_2^2 \leq D \| g \|_2^2 \quad \forall \ g \in K.
\end{equation}
Multiplying (3) and (4), and using the definition of norm on $H \otimes K$, we get
\begin{equation}
AC \| f \|_1^2 \| g \|_2^2 \leq \left( \sum_{i \in I} v_i^2 \| P_{V_i} (f) \|_1^2 \right) \left( \sum_{j \in J} w_j^2 \| P_{W_j} (g) \|_2^2 \right) \leq BD \| f \|_1^2 \| g \|_2^2
\end{equation}
\begin{equation}
\Rightarrow AC \| f \otimes g \|_2^2 \leq \sum_{i,j} v_i^2 w_j^2 \| P_{V_i} (f) \|_1^2 \| P_{W_j} (g) \|_2^2 \leq BD \| f \otimes g \|_2^2
\end{equation}
\begin{equation}
\Rightarrow AC \| f \otimes g \|_2^2 \leq \sum_{i,j} v_i^2 w_j^2 \| P_{V_i} (f) \otimes P_{W_j} (g) \|_2^2 \leq BD \| f \otimes g \|_2^2.
\end{equation}
Therefore, for all $f \otimes g \in H \otimes K$, we have
\begin{equation}
AC \| f \otimes g \|_2^2 \leq \sum_{i,j} v_i^2 w_j^2 \| (P_{V_i} \otimes P_{W_j}) (f \otimes g) \|_2^2 \leq BD \| f \otimes g \|_2^2
\end{equation}
\begin{equation}
\Rightarrow AC \| f \otimes g \|_2^2 \leq \sum_{i,j} v_i^2 w_j^2 \| P_{V_i \otimes W_j} (f \otimes g) \|_2^2 \leq BD \| f \otimes g \|_2^2.
\end{equation}
This shows that $V \otimes W$ is a fusion frame for $H \otimes K$ with bounds $AC$ and $BD$.

Conversely, suppose that $V \otimes W$ is a fusion frame for $H \otimes K$ with bounds $A$ and $B$. Then, for each $f \otimes g \in H \otimes K - \{ \theta \otimes \theta \}$, we have
\begin{equation}
A \| f \otimes g \|_2^2 \leq \sum_{i,j} v_i^2 w_j^2 \| P_{V_i \otimes W_j} (f \otimes g) \|_2^2 \leq B \| f \otimes g \|_2^2
\end{equation}
\begin{equation}
\Rightarrow A \| f \|_1^2 \| g \|_2^2 \leq \sum_{i,j} v_i^2 w_j^2 \| P_{V_i} (f) \otimes P_{W_j} (g) \|_2^2 \leq B \| f \|_1^2 \| g \|_2^2
\end{equation}
\begin{equation}
\Rightarrow A \| f \|_1^2 \| g \|_2^2 \leq \left( \sum_{i \in I} v_i^2 \| P_{V_i} (f) \|_1^2 \right) \left( \sum_{j \in J} w_j^2 \| P_{W_j} (g) \|_2^2 \right) \leq B \| f \|_1^2 \| g \|_2^2.
\end{equation}
Since $f \otimes g$ is non-zero vector, $f$ and $g$ are also non-zero vectors and therefore
\begin{equation}
\sum_{i \in I} v_i^2 \| P_{V_i} (f) \|_1^2 \quad \text{and} \quad \sum_{j \in J} w_j^2 \| P_{W_j} (g) \|_2^2
\end{equation}
are non-zero.
\begin{equation}
\Rightarrow \frac{A \| g \|_2^2}{\sum_{j \in J} w_j^2 \| P_{W_j} (g) \|_2^2} \| f \|_1^2 \leq \sum_{i \in I} v_i^2 \| P_{V_i} (f) \|_1^2 \leq \frac{B \| g \|_2^2}{\sum_{j \in J} w_j^2 \| P_{W_j} (g) \|_2^2} \| f \|_1^2.
\end{equation}
\[ \Rightarrow A_1 \| f \|_1^2 \leq \sum_{i \in I} v_i^2 \| P_{V_i} (f) \|_1^2 \leq B_1 \| f \|_1^2 \quad \forall f \in H, \]

where \( A_1 = \frac{A \| g \|_2^2}{\sum_{j \in J} w_j^2 \| P_{W_j} (g) \|_2^2} \) and \( B_1 = \frac{B \| g \|_2^2}{\sum_{j \in J} w_j^2 \| P_{W_j} (g) \|_2^2} \). This shows that \( V \) is a fusion frame for \( H \). Similarly, it can be shown that \( W \) is a fusion frame for \( K \). \( \square \)

**Note 3.5.** Let \( V \otimes W \) be a fusion frame for \( H \otimes K \). According to the definition [2.4], the corresponding frame operator \( S_{V \otimes W} : H \otimes K \rightarrow H \otimes K \) is given by

\[ S_{V \otimes W} (f \otimes g) = \sum_{i,j} v_i^2 w_j^2 P_{V_i \otimes W_j} (f \otimes g) \quad \forall f \otimes g \in H \otimes K. \]

**Theorem 3.6.** Let \( S_V, S_W \) and \( S_{V \otimes W} \) be the corresponding frame operators for the fusion frames \( V, W \) and \( V \otimes W \), respectively. Then \( S_{V \otimes W} = S_V \otimes S_W \) and \( S_{V \otimes W}^{-1} = S_V^{-1} \otimes S_W^{-1} \).

**Proof.** For each \( f \otimes g \in H \otimes K \), we have

\[ S_{V \otimes W} (f \otimes g) = \sum_{i,j} v_i^2 w_j^2 P_{V_i \otimes W_j} (f \otimes g) \]

\[ = \sum_{i,j} v_i^2 w_j^2 (P_{V_i} \otimes P_{W_j}) (f \otimes g) = \sum_{i,j} v_i^2 w_j^2 (P_{V_i} (f) \otimes P_{W_j} (g)) \]

\[ = \left( \sum_{i \in I} v_i^2 P_{V_i} (f) \right) \otimes \left( \sum_{j \in J} w_j^2 P_{W_j} (g) \right) = S_V (f) \otimes S_W (g) = S_{V \otimes W} (f \otimes g). \]

This implies that \( S_{V \otimes W} = S_V \otimes S_W \). Since \( S_V \) and \( S_W \) are invertible, by (IV) of the Theorem 3.7, it follows that \( S_{V \otimes W}^{-1} = S_V^{-1} \otimes S_W^{-1} \). \( \square \)

**Theorem 3.7.** Let \( V \) and \( W \) be fusion frames for \( H \) and \( K \) with frame bounds \((A, B)\) and \((C, D)\) having their associated frame operators \( S_V \) and \( S_W \), respectively. If \( T_1 \) and \( T_2 \) are invertible and unitary operators on \( H \) and \( K \), then \( \Delta = \{(T_1 \otimes T_2) (V_i \otimes W_j), v_i w_j \}_{i,j} \) is a fusion frame for \( H \otimes K \).

**Proof.** Since \( T_1 \) and \( T_2 \) are invertible, by (IV) of the Theorem 2.12, \( T_1 \otimes T_2 \) is invertible and \((T_1 \otimes T_2)^{-1} = (T_1^{-1} \otimes T_2^{-1})\). Also, by Theorem 2.1, for any \( i \in I \) and \( j \in J \), we get

\[ \| P_{V_i} T_1^* (f) \|_1 \leq \| T_1^* \| \| P_{T_1 V_i} (f) \|_1 \quad \forall f \in H, \quad (5) \]

\[ \| P_{W_j} T_2^* (g) \|_2 \leq \| T_2^* \| \| P_{T_2 W_j} (g) \|_2 \quad \forall g \in K. \quad (6) \]

Again, since \( T_1 \) and \( T_2 \) are invertible, for each \( f \in H \) and \( g \in K \), we obtain

\[ \| f \|_1 \leq \| (T_1^{-1})^* \| \| T_1^* (f) \|_1 \quad \& \quad \| g \|_2 \leq \| (T_2^{-1})^* \| \| T_2^* (g) \|_2. \quad (7) \]
Now, for each $f \otimes g \in H \otimes K$, using Theorem 2.12, we get
\[
\sum_{i,j} v_i^2 w_j^2 \left\| P_{(T_1 \otimes T_2)(V_i \otimes W_j)} (f \otimes g) \right\|^2 = \sum_{i,j} v_i^2 w_j^2 \left\| P_{(T_1 V_i \otimes T_2 W_j)} (f \otimes g) \right\|^2
\]
\[
= \sum_{i,j} v_i^2 w_j^2 \left\| (P_{T_1 V_i \otimes P_{T_2 W_j}}) (f \otimes g) \right\|^2 \quad [\text{by note (3.3)}]
\]
\[
= \left( \sum_{i \in I} v_i^2 \left\| P_{T_1 V_i} (f) \right\|_1^2 \right) \left( \sum_{j \in J} w_j^2 \left\| P_{T_2 W_j} (g) \right\|_2^2 \right) \quad [\text{using (2)}] \quad (8)
\]
\[
\geq \frac{AC}{\left\| T_1 \right\|^2 \left\| T_2 \right\|^2} \left( \sum_{i \in I} v_i^2 \left\| P_{V_i} (T_1^* f) \right\|_1^2 \right) \left( \sum_{j \in J} w_j^2 \left\| P_{W_j} (T_2^* g) \right\|_2^2 \right) \quad [\text{by (5) & (6)}]
\]
\[
\geq \frac{AC}{\left\| T_1 \right\|^2 \left\| T_2 \right\|^2} \left( \sum_{i \in I} v_i^2 \left\| P_{V_i} (T_1^{-1} f) \right\|_1^2 \right) \left( \sum_{j \in J} w_j^2 \left\| P_{W_j} (T_2^{-1} g) \right\|_2^2 \right) \quad [\text{since } V, W \text{ are fusion frames}]
\]
\[
\geq \frac{AC}{\left\| T_1 \otimes T_2 \right\|^2} \left( \sum_{i \in I} v_i^2 \left\| P_{V_i} (T_1^{-1} f) \right\|_1^2 \right) \left( \sum_{j \in J} w_j^2 \left\| P_{W_j} (T_2^{-1} g) \right\|_2^2 \right) \quad [\text{by (7)}]
\]
\[
= \frac{AC}{\left\| T_1 \otimes T_2 \right\|^2} \left( \sum_{i \in I} v_i^2 \left\| P_{V_i} (T_1^{-1} f) \right\|_1^2 \right) \left( \sum_{j \in J} w_j^2 \left\| P_{W_j} (T_2^{-1} g) \right\|_2^2 \right)
\]
\[
= \frac{AC}{\left\| T_1 \otimes T_2 \right\|^2} \left( \sum_{i \in I} v_i^2 \left\| P_{V_i} (T_1^{-1} f) \right\|_1^2 \right) \left( \sum_{j \in J} w_j^2 \left\| P_{W_j} (T_2^{-1} g) \right\|_2^2 \right)
\]
\[
\leq \left\| T_1 \right\|^2 \left\| T_2 \right\|^2 \left( \sum_{i \in I} v_i^2 \left\| P_{V_i} (T_1^{-1} f) \right\|_1^2 \right) \left( \sum_{j \in J} w_j^2 \left\| P_{W_j} (T_2^{-1} g) \right\|_2^2 \right)
\]
\[
\leq BD \left\| T_1 \right\|^2 \left\| T_2 \right\|^2 \left\| T_1^{-1} f \right\|_1^2 \left\| T_2^{-1} g \right\|_2^2 \quad [\text{since } V, W \text{ are fusion frames}]
\]
\[
\leq BD \left\| T_1 \right\|^2 \left\| T_2 \right\|^2 \left\| T_1^{-1} f \right\|_1^2 \left\| T_2^{-1} g \right\|_2^2 \| f \otimes g \|_2^2
\]
\[
= BD \left\| T_1 \otimes T_2 \right\|^2 \left\| (T_1 \otimes T_2)^{-1} \right\|^2 \| f \otimes g \|_2^2
\]
Hence, $\Delta$ is a fusion frame for $H \otimes K$. \hfill \Box
Tensor Product of fusion frames

**Theorem 3.8.** The corresponding fusion frame operator for the fusion frame \( \Delta \) is \( (T_1 \otimes T_2) \, S_{V \otimes W} \, (T_1 \otimes T_2)^{-1} \).

**Proof.** For each \( f \otimes g \in H \otimes K \), we have

\[
\sum_{i,j} v_i^2 w_j^2 P_{(T_1 \otimes T_2)}(V_i \otimes W_j)(f \otimes g) = \sum_{i,j} v_i^2 w_j^2 (P_{T_1 V_i} \otimes P_{T_2 W_j})(f \otimes g)
\]

\[
= \left( \sum_{i \in I} v_i^2 P_{T_1 V_i}(f) \right) \otimes \left( \sum_{j \in J} w_j^2 P_{T_2 W_j}(g) \right)
\]

\[
= \left( \sum_{i \in I} v_i^2 T_1 P_{V_i} (T_1^{-1} f) \right) \otimes \left( \sum_{j \in J} w_j^2 T_2 P_{W_j} (T_2^{-1} g) \right) \quad [\text{by Theorem } (2.1)]
\]

\[
= T_1 S_V (T_1^{-1} (f)) \otimes T_2 S_W (T_2^{-1} (g))
\]

\[
= (T_1 \otimes T_2) \, (S_V \otimes S_W) \, (T_1^{-1} \otimes T_2^{-1}) \, (f \otimes g) \quad [\text{by Theorem } (2.12)]
\]

\[
= (T_1 \otimes T_2) \, S_{V \otimes W} \, (T_1 \otimes T_2)^{-1} \, (f \otimes g).
\]

This shows that \( (T_1 \otimes T_2) \, S_{V \otimes W} \, (T_1 \otimes T_2)^{-1} \) is the corresponding fusion frame operator for \( \Delta \). \( \square \)

**Definition 3.9.** A family of bounded operators \( \{T_i \otimes U_j\}_{i,j} \) on a tensor product of Hilbert space \( H \otimes K \) is called a resolution of the identity operator on \( H \otimes K \), if for all \( f \otimes g \in H \otimes K \), we have

\[
f \otimes g = \sum_{i,j} (T_i \otimes U_j) \, (f \otimes g),
\]

provided the series converges unconditionally for all \( f \otimes g \in H \otimes K \).

**Proposition 3.10.** If the families of bounded operators \( \{T_i\}_{i \in I} \) and \( \{U_j\}_{j \in J} \) on \( H \) and \( K \) are the resolutions of the identity operator on \( H \) and \( K \), then \( \{T_i \otimes U_j\}_{i,j} \) is a resolution of the identity operator on \( H \otimes K \).

**Proof.** Since \( \{T_i\}_{i \in I} \) and \( \{U_j\}_{j \in J} \) are the resolutions of the identity operator on \( H \) and \( K \), respectively, we have

\[
f = \sum_{i \in I} T_i (f) \quad \forall \ f \in H \quad \text{and} \quad g = \sum_{j \in J} U_j (g) \quad \forall \ g \in K.
\]

Then, for all \( f \otimes g \in H \otimes K \), we have

\[
f \otimes g = \left( \sum_{i \in I} T_i (f) \right) \otimes \left( \sum_{j \in J} U_j (g) \right) = \sum_{i,j} (T_i \otimes U_j) \, (f \otimes g).
\]

This completes the proof. \( \square \)
Note 3.11. Let $V$ and $W$ be fusion frames for $H$ and $K$ with their associated frame operators $S_V$ and $S_W$, respectively. By reconstruction formula we can write
\[ f = \sum_{i \in I} v_i^2 S_V^{-1} P_{V_i} (f) \quad \forall f \in H \] and \[ g = \sum_{j \in J} w_j^2 S_W^{-1} P_{W_j} (g) \quad \forall g \in K. \]
Then it is easy to verify that
\[ f \otimes g = \sum_{i, j} v_i^2 w_j^2 S_V^{-1} \otimes S_W^{-1} P_{V_i \otimes W_j} (f \otimes g) \quad \forall f \otimes g \in H \otimes K. \]
This shows that the family of operators \( \left\{ v_i^2 w_j^2 S_V^{-1} \otimes S_W^{-1} P_{V_i \otimes W_j} \right\}_{i, j} \) is resolution of the identity operator on \( H \otimes K \).

Theorem 3.12. Let $V$ and $W$ be fusion frames for $H$ and $K$ with frame bounds $(A, B)$ and $(C, D)$ having their associated frame operators $S_V$ and $S_W$, respectively. Then \( \left\{ v_i^2 w_j^2 (T_i \otimes U_j) \right\}_{i, j} \) is a resolution of the identity operator on $H \otimes K$, where $T_i \otimes U_j = P_{V_i \otimes W_j} S_V^{-1} \otimes S_W^{-1}$ for $i \in I$ and $j \in J$. Furthermore,
\[ \frac{AC}{B^2 D^2} a^2 b^2 \| f \otimes g \|^2 \leq \sum_{i, j} v_i^2 w_j^2 \| (T_i \otimes U_j) (f \otimes g) \|^2 \leq \frac{BD}{A^2 C^2} a^2 b^2 \| f \otimes g \|^2, \]
for all $f \otimes g \in H \otimes K$, where $a$ and $b$ are constants with $ab = 1$.

Proof. Since $S_V$ and $S_W$ are frame operator for $V$ and $W$, for all $f \in H, g \in K$,
\[ f = \sum_{i \in I} v_i^2 P_{V_i} \left( S_V^{-1} f \right) \quad \text{and} \quad g = \sum_{j \in J} w_j^2 P_{W_j} \left( S_W^{-1} g \right). \]
Now, for all $f \otimes g \in H \otimes K$, we have
\[ f \otimes g = \left( \sum_{i \in I} v_i^2 P_{V_i} \left( S_V^{-1} f \right) \right) \otimes \left( \sum_{j \in J} w_j^2 P_{W_j} \left( S_W^{-1} g \right) \right) \]
\[ = \sum_{i, j} v_i^2 w_j^2 \left( P_{V_i} S_V^{-1} (f) \otimes P_{W_j} S_W^{-1} (g) \right) \]
\[ = \sum_{i, j} v_i^2 w_j^2 \left( P_{V_i} \otimes P_{W_j} \right) \left( S_V^{-1} \otimes S_W^{-1} \right) (f \otimes g) \]
\[ = \sum_{i, j} v_i^2 w_j^2 P_{V_i \otimes W_j} S_V^{-1} \otimes S_W^{-1} (f \otimes g). \]
This shows that \( \left\{ v_i^2 w_j^2 (T_i \otimes U_j) \right\}_{i, j} \) is a resolution of the identity operator on $H \otimes K$, where $T_i \otimes U_j = P_{V_i \otimes W_j} S_V^{-1} \otimes S_W^{-1}$. Now, by (V I) of the Theorem 2.12, there exist constants $a$ and $b$ with $ab = 1$ such that
\[ T_i (f) = a P_{V_i} S_V^{-1} (f) \quad \forall f \in H, \quad \text{and} \quad U_j (g) = b P_{W_j} S_W^{-1} (g) \quad \forall g \in K. \]
Then, for all $f \otimes g \in H \otimes K$, we have

$$\sum_{i,j} v_i^2 w_j^2 \| (T_i \otimes U_j) (f \otimes g) \|^2 = \sum_{i,j} v_i^2 w_j^2 \| T_i (f) \otimes U_j (g) \|^2$$

$$= \sum_{i,j} v_i^2 w_j^2 \| T_i (f) \|_1 \| U_j (g) \|_2^2 = \left( \sum_{i \in I} v_i^2 \| T_i (f) \|_1^2 \right) \left( \sum_{j \in J} w_j^2 \| U_j (g) \|_2^2 \right)$$

$$= \left( \sum_{i \in I} v_i^2 \| a P_V, S_V^{-1} (f) \|_1^2 \right) \left( \sum_{j \in J} w_j^2 \| b P_W, S_W^{-1} (g) \|_2^2 \right) \geq \frac{AC}{BD} a^2 b^2 \| f \|_1 \| g \|_2^2 \geq \frac{AC}{BD} a^2 b^2 \| f \otimes g \|^2.$$ 

On the other hand, using (9)

$$\sum_{i,j} v_i^2 w_j^2 \| (T_i \otimes U_j) (f \otimes g) \|^2 \geq \sum_{i,j} v_i^2 w_j^2 \| T_i (f) \|_1 \| U_j (g) \|_2^2 = \left( \sum_{i \in I} v_i^2 \| T_i (f) \|_1 \right) \left( \sum_{j \in J} w_j^2 \| U_j (g) \|_2 \right)$$

$$\geq \frac{AC}{BD} a^2 b^2 \| f \|_1 \| g \|_2^2 = \frac{AC}{BD} a^2 b^2 \| f \otimes g \|^2.$$ 

This completes the proof. \qed

4 Alternative dual fusion frame in tensor product of Hilbert spaces

In this section, an alternative dual of a fusion frame in $H \otimes K$ is discussed.

Theorem 4.1. Let $V$ and $W$ be fusion frames for $H$ and $K$ with frame bounds $A$, $B$, and $C$, $D$ having their corresponding fusion frame operators $S_V$ and $S_W$, respectively. Then the family $\Lambda = \{ S_V^{-1} (V_i \otimes W_j), v_i w_j \}_{i,j}$ is a fusion frame for $H \otimes K$.

Proof. By Theorem 2.7, for all $f \in H$ and $g \in K$, we have

$$\frac{A}{\| S_V \|^2 \| S_V^{-1} \|^2} \leq \sum_{i \in I} v_i^2 \| P_{S_V^{-1} V_i} (f) \|_1^2 \leq B \| S_V \|^2 \| S_V^{-1} \|^2 \| f \|_1^2 \leq B \| S_V \|^2 \| S_V^{-1} \|^2 \| f \|^2 \|_1, (10)$$

$$\frac{C}{\| S_W \|^2 \| S_W^{-1} \|^2} \leq \sum_{j \in J} w_j^2 \| P_{S_W^{-1} W_j} (g) \|_2^2 \leq D \| S_W \|^2 \| S_W^{-1} \|^2 \| g \|^2 \|_2 \leq D \| S_W \|^2 \| S_W^{-1} \|^2 \| g \|_2^2 (11)$$

$$\leq \frac{AC}{BD} a^2 b^2 \| f \|_1 \| g \|_2^2 = \frac{AC}{BD} a^2 b^2 \| f \otimes g \|^2.$$
Multiplying the inequalities (10) and (11) and using (2), we get
\[
\frac{AC \| f \otimes g \|^2}{\| S_V \|^2 \| S_W \|^2 \| S_V^{-1} \|^2 \| S_W^{-1} \|^2} \leq \sum_{i,j} v_i^2 w_j^2 \left\| P_{S_V^{-1} V_i} (f) \otimes P_{S_W^{-1} W_j} (g) \right\|^2 \\
\leq B D \| S_V \|^2 \| S_W \|^2 \| S_V^{-1} \|^2 \| S_W^{-1} \|^2 \| f \otimes g \|^2.
\]
Therefore, for each \( f \otimes g \in H \otimes K \), we get
\[
\Rightarrow \frac{AC \| f \otimes g \|^2}{\| S_V \otimes S_W \|^2 \| S_V^{-1} \otimes S_W^{-1} \|^2} \leq \sum_{i,j} v_i^2 w_j^2 \left\| P_{S_V^{-1} V_i \otimes S_W^{-1} W_j} (f \otimes g) \right\|^2 \\
\leq B D \| S_V \otimes S_W \|^2 \| S_V^{-1} \otimes S_W^{-1} \|^2 \| f \otimes g \|^2
\]
\[
\Rightarrow \frac{AC \| f \otimes g \|^2}{\| S_V \otimes W \|^2 \| S_V^{-1} \otimes W \|^2} \leq \sum_{i,j} v_i^2 w_j^2 \left\| P_{S_{V \otimes W}^{-1} (V_i \otimes W_j)} (f \otimes g) \right\|^2 \\
\leq B D \| S_V \otimes W \|^2 \| S_V^{-1} \otimes W \|^2 \| f \otimes g \|^2.
\]
This shows that \( \Lambda \) is a fusion frame for \( H \otimes K \) with bounds \( \frac{AC}{\| S_V \otimes W \|^2 \| S_V^{-1} \otimes W \|^2} \) and \( B D \| S_V \otimes W \|^2 \| S_V^{-1} \otimes W \|^2 \).

\[\square\]

**Definition 4.2.** Let \( V \otimes W \) be a fusion frame for \( H \otimes K \) and \( S_{V \otimes W} \) be the corresponding fusion frame operator. Then a fusion Bessel sequence \( V' \otimes W' \) in \( H \otimes K \) is said to be an alternative dual of \( V \otimes W \) if for all \( f \otimes g \in H \otimes K \),
\[
f \otimes g = \sum_{i,j} v_i w_j v_i' w_j' P_{V_i \otimes W_j} S_{V \otimes W}^{-1} P_{V_i \otimes W_j} (f \otimes g).
\]

**Note 4.3.** According to note [2,3], a reconstruction formula on \( K \) is also described by
\[
g = \sum_{j \in J} w_j^2 P_{S_W^{-1} W_j} S_W^{-1} P_{W_j} (g) \quad \forall g \in K.
\]

Therefore, for all \( f \otimes g \in H \otimes K \), we have
\[
f \otimes g = \left( \sum_{i \in I} v_i^2 P_{S_V^{-1} V_i} S_V^{-1} P_{V_i} (f) \right) \otimes \left( \sum_{j \in J} w_j^2 P_{S_W^{-1} W_j} S_W^{-1} P_{W_j} (g) \right)
\]
\[
= \sum_{i,j} v_i^2 w_j^2 \left( P_{S_V^{-1} V_i} \otimes P_{S_W^{-1} W_j} \right) \left( S_V^{-1} \otimes S_W^{-1} \right) \left( P_{V_i} \otimes P_{W_j} \right) (f \otimes g)
\]
\[
= \sum_{i,j} v_i^2 w_j^2 P_{S_{V \otimes W}^{-1} (V_i \otimes W_j)} S_{V \otimes W}^{-1} P_{V_i \otimes W_j} (f \otimes g)
\]
\[
= \sum_{i,j} v_i^2 w_j^2 P_{S_{V \otimes W}^{-1} (V_i \otimes W_j)} S_{V \otimes W}^{-1} P_{V_i \otimes W_j} (f \otimes g).
\]
Thus, we see that the canonical dual frame \( \{ S_{V \otimes W}^{-1} (V_i \otimes W_j), \; v_i w_j \}_{i,j} \) is an alternative dual fusion frame for \( H \otimes K \).

**Theorem 4.4.** Let \( V \) and \( W \) be fusion frames for \( H \) and \( K \) with their alternative dual \( V' = \{ (V'_i, v'_i) \}_{i \in I} \) and \( W' = \{ (W'_j, w'_j) \}_{j \in J} \), respectively. Then \( V' \otimes W' \) is an alternative dual of the fusion frame \( V \otimes W \) for \( H \otimes K \).

**Proof.** By Theorem 4.4, \( V \otimes W \) is a fusion frame for \( H \otimes K \) and \( V' \otimes W' \) is a fusion Bessel sequence in \( H \otimes K \). Since \( V' \) and \( W' \) are alternative dual sequences of \( V \) and \( W \), for all \( f \in H \) and \( g \in K \),

\[
    f = \sum_{i \in I} v_i v'_i P_{V_i} S_{V_i}^{-1} P_{V'_i} (f) \quad \text{and} \quad g = \sum_{j \in J} w_j w'_j P_{W_j} S_{W_j}^{-1} P_{W'_j} (g).
\]

Then, for all \( f \otimes g \in H \otimes K \), using the Theorem 2.12, we get

\[
    f \otimes g = \left( \sum_{i \in I} v_i v'_i P_{V'_i} S_{V'_i}^{-1} P_{V_i} (f) \right) \otimes \left( \sum_{j \in J} w_j w'_j P_{W'_j} S_{W'_j}^{-1} P_{W_j} (g) \right)
\]

\[
    = \sum_{i,j} v_i w_j v'_i w'_j \left( P_{V'_i} \otimes P_{W'_j} \right) \left( S_{V'_i}^{-1} \otimes S_{W'_j}^{-1} \right) \left( P_{V_i} \otimes P_{W_j} \right) (f \otimes g)
\]

\[
    = \sum_{i,j} v_i w_j v'_i w'_j P_{V'_i \otimes W'_j} S_{V'_i \otimes W'_j}^{-1} P_{V_i \otimes W_j} (f \otimes g).
\]

This completes the proof. \( \square \)

**Theorem 4.5.** Let \( V \) and \( W \) be fusion frames for \( H \) and \( K \) with bounds \( (B_1, D_1) \) and \( (B_2, D_2) \), respectively, by Theorem 3.4, \( V' \otimes W' \) is a fusion Bessel sequence in \( H \otimes K \). Also, since \( V' \) and \( W' \) are alternative dual sequences of \( V \) and \( W \), respectively, by Theorem 4.4, \( V' \otimes W' \) is an alternative dual of the fusion frame \( V \otimes W \) for \( H \otimes K \). Now, for each \( f \otimes g \in H \otimes K \), we have \( \| f \otimes g \|^2 = \langle f \otimes g, f \otimes g \rangle \)

\[
    = \left\langle \sum_{i,j} v_i w_j v'_i w'_j P_{V'_i \otimes W'_j} S_{V'_i \otimes W'_j}^{-1} P_{V_i \otimes W_j} (f \otimes g), \; f \otimes g \right\rangle
\]

\[
    = \sum_{i,j} v_i w_j v'_i w'_j \left\langle S_{V'_i \otimes W'_j}^{-1} P_{V_i \otimes W_j} (f \otimes g), \; P_{V'_i \otimes W'_j} (f \otimes g) \right\rangle
\]

\[
    = \sum_{i,j} v_i w_j v'_i w'_j \left\langle S_{V'_i}^{-1} P_{V_i} (f) \otimes S_{W'_j}^{-1} P_{W_j} (g), \; P_{V'_i} (f) \otimes P_{W'_j} (g) \right\rangle
\]
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\[ \left( \sum_{i \in I} v_i v_i' \left\langle S_V^{-1} P_{V_i}(f), P_{V_i'}(f) \right\rangle_1 \right) \left( \sum_{j \in J} w_j w_j' \left\langle S_W^{-1} P_{W_j}(g), P_{W_j'}(g) \right\rangle_2 \right) \]

\[ \leq \left( \sum_{i \in I} v_i v_i' \left\| S_V^{-1} P_{V_i}(f) \right\|_1 \left\| P_{V_i'}(f) \right\|_1 \right) \left( \sum_{j \in J} w_j w_j' \left\| S_W^{-1} P_{W_j}(g) \right\|_2 \left\| P_{W_j'}(g) \right\|_2 \right) \]

\[ \leq \left( \sum_{i \in I} v_i^2 \left\| S_V^{-1} P_{V_i}(f) \right\|_1^2 \right) ^{1/2} \left( \sum_{i \in I} (v_i')^2 \left\| P_{V_i'}(f) \right\|_1^2 \right) ^{1/2} \left( \sum_{j \in J} w_j^2 \left\| S_W^{-1} P_{W_j}(g) \right\|_2^2 \right) ^{1/2} \times \]

\[ \left( \sum_{j \in J} (w_j')^2 \left\| P_{W_j'}(g) \right\|_2^2 \right) ^{1/2} \quad \text{[by C-S inequality]} \]

\[ \leq \sqrt{D_1 D_2} \left\| S_V^{-1} \right\| \left\| S_W^{-1} \right\| \left\| f \otimes g \right\|_2 \left( \sum_{i \in I} (v_i')^2 \left\| P_{V_i'}(f) \right\|_1^2 \right) ^{1/2} \left( \sum_{j \in J} (w_j')^2 \left\| P_{W_j'}(g) \right\|_2^2 \right) ^{1/2} \]

\[ \quad \text{[since } V, W \text{ are fusion frames]} \]

\[ = \sqrt{D_1 D_2} \left\| S_V^{-1} \right\| \left\| f \right\| \left\| g \right\| \left( \sum_{i,j} (v_i')^2 (w_j')^2 \left\| P_{V_i'} \otimes P_{W_j'}(f \otimes g) \right\|_2^2 \right) ^{1/2} \]

\[ \Rightarrow \frac{1}{D_1 D_2} \left\| S_V^{-1} \right\|_2^2 \left\| f \otimes g \right\|_2^2 \leq \sum_{i,j} (v_i')^2 (w_j')^2 \left\| P_{V_i'} \otimes P_{W_j'}(f \otimes g) \right\|_2^2 . \]

This completes the proof. \[ \square \]

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