Stability of undercompressive viscous shock profiles of hyperbolic–parabolic systems.

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October 4, 2018

Abstract

Extending to systems of hyperbolic–parabolic conservation laws results of Howard and Zumbrun for strictly parabolic systems, we show for viscous shock profiles of arbitrary amplitude and type that necessary spectral (Evans function) conditions for linearized stability established by Mascia and Zumbrun are also sufficient for linearized and nonlinear phase-asymptotic stability, yielding detailed pointwise estimates and sharp rates of convergence in $L^p$, $1 \leq p \leq \infty$.

1 Introduction

Consider a (possibly) large-amplitude viscous shock profile, or traveling-wave solution

\begin{equation}
\bar{u}(x-st); \quad \lim_{x \to \pm \infty} \bar{u}(x) = u_{\pm},
\end{equation}

of a system of partially or fully parabolic conservation laws

\begin{equation}
u_t + F(u)_x = (B(u)u_x)_x,\end{equation}
\( x \in \mathbb{R}, \ u, \ F \in \mathbb{R}^n, \ B \in \mathbb{R}^{n \times n}, \) where

\[
(1.3) \quad u = \begin{pmatrix} u^I \\ u^{II} \end{pmatrix}, \quad F = \begin{pmatrix} F^I \\ F^{II} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix},
\]

\( u^I \in \mathbb{R}^{n-r}, \ u^{II} \in \mathbb{R}^r, \) \( r \) some positive integer, possibly \( n \) (full regularization), and

\[ \Re \sigma(b_2) \geq \theta > 0. \]

Here and elsewhere, \( \sigma \) denotes spectrum of a matrix or other linear operator. Working in a coordinate system moving along with the shock, we may without loss of generality consider a standing profile \( \bar{u}(x), \ s = 0; \) we take \( s = 0 \) from now on.

Following [Z2], we assume that, by some invertible change of coordinates \( u \rightarrow w(u) \), followed if necessary by multiplication on the left by a nonsingular matrix function \( S(w) \), equations (1.2) may be written in the quasilinear, partially symmetric hyperbolic-parabolic form

\[
(1.4) \quad \tilde{A}^0 w_t + \tilde{A} w_x = (\tilde{B} w_x)_x + G, \quad w = \begin{pmatrix} w^I \\ w^{II} \end{pmatrix},
\]

\( w^I \in \mathbb{R}^{n-r}, \ w^{II} \in \mathbb{R}^r, \ x \in \mathbb{R}, \ t \in \mathbb{R}_+ \), where, defining \( w_\pm := w(u_\pm) \):

(A1) \( \tilde{A}(w_\pm), \tilde{A}_{11}, \tilde{A}^0 \) are symmetric, \( \tilde{A}^0 > 0 \).

(A2) Dissipativity: no eigenvector of \( dF(u_\pm) \) lies in the kernel of \( B(u_\pm) \).

(Equivalently, no eigenvector of \( \tilde{A}(\tilde{A}^0)^{-1}(w_\pm) \) lies in the kernel of \( \tilde{B}(\tilde{A}^0)^{-1}(w_\pm) \)).

(A3) \( \tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b} \end{pmatrix}, \tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix} \), with \( \Re \tilde{b}(w) \geq \theta \) for some \( \theta > 0 \), for all \( w \), and \( \tilde{g}(w_x, w_x) = O(|w_x|^2) \).

Here, the coefficients of (1.4) may be expressed in terms of the original equation (1.2), the coordinate change \( u \rightarrow w(u) \), and the approximate symmetrizer \( S(w) \), as

\[
(1.5) \quad \tilde{A}^0 := S(w)(\partial u / \partial w), \quad \tilde{A} := S(w)dF(u(w))(\partial u / \partial w),
\]

\[
\tilde{B} := S(w)B(u(w))(\partial u / \partial w), \quad G = -(dSw_x)B(u(w))(\partial u / \partial w)w_x.
\]

Alternatively, we assume, simply,

(B1) Strict parabolicity: \( n = r \), or, equivalently, \( \Re \sigma(B) > 0. \)
Along with the above structural assumptions, we make the technical hypotheses:

(H0) $F, B, w, S \in C^5$.

(H1) The eigenvalues of $\tilde{A}_*: \tilde{A}_{11}(\tilde{A}_{11}^0)^{-1}$ are (i) distinct from 0; (ii) of common sign; and (iii) of constant multiplicity with respect to $u$.

(H2) The eigenvalues of $dF(u_{\pm})$ are real, distinct, and nonzero.

(H3) Nearby $\bar{u}$, the set of all solutions of (1.1)–(1.2) connecting the same values $u_{\pm}$ forms a smooth manifold $\{\bar{u}^\delta\}$, $\delta \in U \subset \mathbb{R}^\ell$, $\bar{u}^0 = \bar{u}$.

Remark 1.1. Structural assumptions (A1)–(A3) [alt. (B1)] and technical hypotheses (H0)–(H2) admit such physical systems as the compressible Navier–Stokes equations, the equations of magnetohydrodynamics, and Slemrod’s model for van der Waal gas dynamics [Z2, Z3]. Moreover, existence of waves $\bar{u}$ satisfying (H3) has been established in each of these cases.

Definition 1.2. An ideal shock

(1.6)

$$u(x, t) = \begin{cases} 
    u_- & x < st, \\
    u_+ & x > st,
\end{cases}$$

is classified as undercompressive, Lax, or overcompressive type according as $i - n$ is less than, equal to, or greater than 1, where $i$, denoting the sum of the dimensions $i_-$ and $i_+$ of the center–unstable subspace of $dF(u_-)$ and the center–stable subspace of $dF(u_+)$, represents the total number of characteristics incoming to the shock.

A viscous profile (1.1) is classified as pure undercompressive type if the associated ideal shock is undercompressive and $\ell = 1$, pure Lax type if the corresponding ideal shock is Lax type and $\ell = i - n$, and pure overcompressive type if the corresponding ideal shock is overcompressive and $\ell = i - n$, $\ell$ as in (H3). Otherwise it is classified as mixed under–overcompressive type; see [ZH].

Pure Lax type profiles are the most common type, and the only type arising in standard gas dynamics, while pure over- and undercompressive type profiles arise in magnetohydrodynamics (MHD) and phase-transitional models. Mixed under–overcompressive profiles are also possible, as described in [LZ2, ZH], though we do not know a physical example. In the pure Lax
or undercompressive case, \( \{ \bar{u}^\delta \} = \{ \bar{u}(\cdot - \delta) \} \) is just the set of all translates of the base profile \( \bar{u} \), whereas in other cases it involves also deformations of \( \bar{u} \). For further discussion of existence, structure, and classification of viscous profiles, see, e.g., [LZ2, ZH, MZ2, MZ3, MZ4, Z1, Z2, Z3], and references therein.

**Definition 1.3.** The profile \( \bar{u} \) is said to be nonlinearly orbitally stable if \( \tilde{u}(\cdot, t) \) approaches \( \bar{u}^\delta(t) \) as \( t \to \infty \), \( \bar{u}^\delta \) as defined in (H3), for any solution \( \tilde{u} \) of (1.2) with initial data sufficiently close in some norm to the original profile \( \bar{u} \). If, also, the phase \( \delta(t) \) converges to a limiting value \( \delta_* \), the profile is said to be nonlinearly phase-asymptotically orbitally stable.

An important result of [MZ3] was the identification of the following stability criterion equivalent to \( L^1 \to L^p \) linearized orbital stability of the profile, \( p > 1 \), where \( D(\lambda) \) as described in [GZ, ZH] denotes the Evans function associated with the linearized operator \( L \) about the profile: an analytic function analogous to the characteristic polynomial of a finite-dimensional operator, whose zeroes away from the essential spectrum agree in location and multiplicity with the eigenvalues of \( L \).

\[(D)\] There exist precisely \( \ell \) zeroes of \( D(\cdot) \) in the nonstable half-plane \( \mathbb{R}_\lambda \geq 0 \), necessarily at the origin \( \lambda = 0 \).

As discussed, e.g., in [MZ4, Z2, Z3], under assumptions (A1)–(A3) and (H0)–(H3), \((D)\) is equivalent to (i) strong spectral stability, \( \sigma(L) \subset \{ \mathbb{R}_\lambda < 0 \} \cup \{0\} \), (ii) hyperbolic stability of the associated ideal shock, and (iii) transversality of \( \bar{u} \) as a solution of the connection problem in the associated traveling-wave ODE, where hyperbolic stability is defined for Lax and undercompressive shocks by the Lopatinski condition of [M1, M2, M3, Fre] and for overcompressive shocks by an analogous long-wave stability condition [Z1, Z2]. Here and elsewhere \( \sigma \) denotes spectrum of a linearized operator or matrix.

The stability condition holds always for small-amplitude Lax profiles [HuZ, PZ, FreS], but may fail for large-amplitude, or nonclassical over- or undercompressive profiles [AMPZ1, GZ, FreZ, ZS, Z2]. It may be readily checked numerically, as described, e.g., in [Br1, Br2, Br3, BrZ, BDG]. It was shown by various techniques in [MZ1, MZ2, MZ3, MZ4, MZ5, Ra, HR, HRZ] that the linearized stability condition \((D)\) is also sufficient for nonlinear orbital stability of Lax or overcompressive profiles of arbitrary amplitude. In the strictly parabolic case (B1), this result was extended in [HZ] to shocks
of arbitrary amplitude and type, in particular to shocks of under- or mixed over–undercompressive type. However, up to now, it had not been verified for under- or under–overcompressive profiles of systems with real viscosity.

In this paper, we establish for shocks of any type and general systems satisfying (A1)–(A3) [alt. (B1)] and (H0)–(H3) that (D) is sufficient for nonlinear phase-asymptotic orbital stability. More precisely, denoting by

\[ a_1^± < a_2^± < \cdots < a_n^± \]

the eigenvalues of the limiting convection matrices \( A_± := df(u_±) \), define

\[ \theta(x,t) := \sum_{a_j^< < 0} (1 + |x| + t)^{-1/2}(1 + |x - a_j^-t|)^{-1/2} \]
\[ + \sum_{a_j^+ > 0} (1 + |x| + t)^{-1/2}(1 + |x - a_j^+t|)^{-1/2}, \]

\[ \psi_1(x,t) := \chi(x,t) \sum_{a_j^< < 0} (1 + |x| + t)^{-1/2}(1 + |x - a_j^-t|)^{-1/2} \]
\[ + \chi(x,t) \sum_{a_j^+ > 0} (1 + |x| + t)^{-1/2}(1 + |x - a_j^+t|)^{-1/2}, \]

and

\[ \psi_2(x,t) := (1 - \chi(x,t))(1 + |x - a_1^-t| + t^{1/2})^{-3/2} \]
\[ + (1 - \chi(x,t))(1 + |x - a_n^+t| + t^{1/2})^{-3/2}, \]

where \( \chi(x,t) = 1 \) for \( x \in [a_1^-t, a_n^+t] \) and zero otherwise, and \( L > 0 \) is a sufficiently large constant.

Then, we have the following main theorem.

Theorem 1.4. Under assumptions (A1)–(A3) [alt. (B1)], (H0)–(H3), and (D), the profile \( \bar{u} \) is nonlinearly phase-asymptotically orbitally stable with respect to \( H^4 \) initial perturbations \( u_0 \), with \( \|(1 + |x|^2)^{-3/4}u_0(x)\|_{H^5} \leq E_0 \) sufficiently small. More precisely, there exist \( \delta(\cdot) \) and \( \delta_\ast \) such that

\[ |\bar{u}(x,t) - \bar{\bar{u}}^{\delta(\cdot)}(x)| \leq CE_0(\theta + \psi_1 + \psi_2)(x,t), \]
\[ |\partial_x(\bar{u}(x,t) - \bar{\bar{u}}^{\delta(\cdot)}(x))| \leq CE_0(\theta + \psi_1 + \psi_2)(x,t), \]
\[ |\delta(\cdot)| \leq CE_0, \]
\[ |\dot{\delta}(t)| \leq CE_0(1 + t)^{-1}, \]
\[ |\delta(t)| \leq CE_0(1 + t)^{-1/2}, \]

where \( \bar{u} \) denotes the solution of (1.2) with initial data \( \bar{u}_0 = \bar{u} + u_0 \).
In particular, Theorem 1.4 yields the desired result of nonlinear stability in the undercompressive or mixed case, effectively completing the one-dimensional stability analysis initiated in [ZH, MZ3].

**Remark 1.5.** Pointwise bound (1.11) yields as a corollary the sharp $L^p$ decay rate

\[
|\tilde{u}(x, t) - \tilde{u}^{\delta, \delta}(x)|_{L^p} \leq CE_0 (1 + t)^{-\frac{1}{2}(1 - \frac{1}{p})}, \quad 1 \leq p \leq \infty.
\]

The main new difficulty in the analysis beyond those faced in the strictly parabolic case of [HZ] is to control higher derivatives in the absence of parabolic smoothing. We accomplish this by a modification of the fixed-point iteration scheme introduced in [HZ], changing to an *implicit* iteration scheme in order to avoid loss of derivatives as discussed in Remark 6.2.

This is a standard device in situations of limited regularity, especially for quasilinear hyperbolic equations. However, here the situation is complicated by the nonlocal character of the defining integral equations, which appears to prevent the standard treatment of regularity by energy estimates. The simple resolution to this problem is that, by appropriate choice of implicit scheme, we obtain a nonlocal system of integral equations that admits also a *local* description in terms of a symmetric hyperbolic–parabolic system amenable to the same type of energy estimates used to control regularity in the study of Lax and overcompressive shocks in [MZ4, Z2, Ra].

New physical applications beyond those of [HZ] are to undercompressive waves in MHD and, with slight modification following [LRTZ], to weak detonation waves in reactive compressible Navier–Stokes equations. The latter we intend to treat in a future work.

**Plan of the paper.** In Sections 2 and 3, we recall the basic profile bounds and linearized estimates obtained in [MZ3, ZH, HRZ], and in Section 4 the convolution estimates established in [HZ]. In Section 5, we recall (a slight modification of) an auxiliary energy, or “hyperbolic–parabolic damping” estimate, established in [MZ4, Z2, Ra], along with a more standard weighted $H^5$ estimate. In Sections 6, 7, and 8, we carry out the main work of the paper, introducing a crucial implicit version of the iteration scheme described in [HZ], establishing local existence by $H^5$ energy estimate, and then showing by a combination of estimates like those of [HZ] and [Z2] that this is contractive in an appropriate norm encoding the claimed rates of decay.
2 Profile facts

We first recall the profile analysis carried out in [MZ3, HRZ], a slight generalization of Corollary 1.2, [ZH], which in turn generalizes results of [MP] in the strictly parabolic case. Profile $\bar{u}(x)$ satisfies the standing-wave ordinary differential equation (ODE)

$$B(\bar{u})\bar{u}' = F(\bar{u}) - F(u_-).$$

Considering the block structure of $B$, this can be written as:

$$F^I(u^I, u^{II}) \equiv F^I(u^I_-, u^{II})$$

and

$$b_1(u^I)' + b_2(u^{II})' = F^{II}(u^I, u^{II}) - F^{II}(u^I_-, u^{II}_-).$$

**Lemma 2.1** ([MZ3, HRZ]). Given (H1)–(H3), the endstates $u_\pm$ are hyperbolic rest points of the ODE determined by (2.3) on the $r$-dimensional manifold (2.2), i.e., the coefficients of the linearized equations about $u_\pm$, written in local coordinates, have no center subspace. In particular, under regularity (H0),

$$D^j_xD^i_\delta(\bar{u}^\delta(x) - u_\pm) = O(e^{-\alpha|x|}), \quad \alpha > 0, \ 0 \leq j \leq 5, \ i = 0, 1,$$

as $x \to \pm \infty$.

**Proof.** By (H1), (2.2) may be solved for $u^I = h(u^{II})$, reducing the problem to an ODE in $u^{II}$. Under assumptions (H1)–(H3), this is a nondegenerate ODE of which $u^{II}_\pm$ are hyperbolic rest points; see [MZ3, Z2]. The family $\bar{u}^\delta$ is thus the intersection of the unstable manifold at $u_-$ with the stable manifold at $u_+$, both of which are $C^5$ by (H0) and standard invariant manifold theory. This intersection is transversal as a consequence of (D), [MZ3], hence $\bar{u}^\delta$ is $C^5$ in $\delta$ by the Implicit Function Theorem, and $C^6$ in $x$ by (H0) and the defining ODE. Finally, (2.4) follows from hyperbolicity of $u_\pm$, by standard ODE estimates on the ODE and its variations about $\bar{u}^\delta$. For further details, see [MZ3, Z2].
3 Linearized estimates

We next recall some linear theory from [MZ3, ZH]. Linearizing (1.2) about $\bar{u}^\delta(\cdot)$, $\delta_*$ to be determined later, gives

$$v_t = L^\delta v := -(A^\delta v)_x + (B^\delta v)_x,$$

with

$$B^\delta(x) := B(\bar{u}^\delta(x)), \quad A^\delta(x)v := dF(\bar{u}^\delta(x))v - dB(\bar{u}^\delta(x))v\bar{u}_x^\delta.$$

Denoting $A^\pm := A(\pm \infty)$, $B^\pm := B(\pm \infty)$, and considering Lemma 2.1, it follows that

$$|A^\delta(x) - A^-| = O(e^{-\eta|x|}), \quad |B^\delta(x) - B^-| = O(e^{-\eta|x|})$$

as $x \to -\infty$, for some positive $\eta$. Similarly for $A^+$ and $B^+$, as $x \to +\infty$. Also $|A^\delta(x) - A^\pm|$ and $|B^\delta(x) - B^\pm|$ are bounded for all $x$.

Define the \textit{(scalar) characteristic speeds} $a_1^\pm < \cdots < a_n^\pm$ (as above) to be the eigenvalues of $A^\pm$, and the left and right \textit{(scalar) characteristic modes} $l_j^\pm$, $r_j^\pm$ to be corresponding left and right eigenvectors, respectively (i.e., $A^\pm r_j^\pm = a_j^\pm r_j^\pm$, etc.), normalized so that $l_j^+ \cdot r_k^- = \delta_j^k$ and $l_j^- \cdot r_k^+ = \delta_j^k$. Following Kawashima [Kaw], define associated \textit{effective scalar diffusion rates} $\beta_j^\pm : j = 1, \cdots, n$ by relation

$$\begin{pmatrix} \beta_1^\pm & 0 \\ \vdots & \ddots \\ 0 & \beta_n^\pm \end{pmatrix} = \text{diag} L^\pm B^\pm R^\pm,$$

where $L^\pm := (l_1^\pm, \ldots, l_n^\pm)^t$, $R^\pm := (r_1^\pm, \ldots, r_n^\pm)$ diagonalize $A^\pm$.

Assume for $A$ and $B$ the block structures:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix}.$$

Also, let $a_j^*(x), j = 1, \ldots, (n - r)$ denote the eigenvalues of

$$A_* := A_{11} - A_{12}B_{22}^{-1}B_{21},$$

where $L^\pm := (l_1^\pm, \ldots, l_n^\pm)^t$, $R^\pm := (r_1^\pm, \ldots, r_n^\pm)$ diagonalize $A^\pm$. 

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with \( l_j^r(x), r_j^r(x) \in \mathbb{R}^{n-r} \) associated left and right eigenvectors, normalized so that \( l_j^r \equiv \delta_j^k \). More generally, for an \( m_j^r \)-fold eigenvalue, we choose \( (n-r) \times m_j^r \) blocks \( L_j^r \) and \( R_j^r \) of eigenvectors satisfying the dynamical normalization

\[
L_j^r \partial_x R_j^r = 0,
\]

along with the usual static normalization \( L_j^r R_j^r \equiv \delta_j^k \). As shown in Lemma 4.9, [MZ1], this may always be achieved with bounded \( L_j^r, R_j^r \). Associated with \( L_j^r, R_j^r \), define extended, \( n \times m_j^r \) blocks

\[
L_j^* := \begin{pmatrix} L_j^r \\ 0 \end{pmatrix}, \quad R_j^* := \begin{pmatrix} R_j^r \\ -B_{22}^{-1}B_{21}R_j^r \end{pmatrix}.
\]

Eigenvalues \( a_j^* \) and eigenmodes \( L_j^*, R_j^* \) correspond, respectively, to short-time hyperbolic characteristic speeds and modes of propagation for the reduced, hyperbolic part of degenerate system (1.2).

Define local, \( m_j \times m_j \) dissipation coefficients

\[
\eta_j^r(x) := -L_j^r D_\ast R_j^r(x), \quad j = 1, \ldots, J \leq n - r,
\]

where

\[
D_\ast(x) := A_{12}B_{22}^{-1} \left[ A_{21} - A_{22}B_{22}^{-1}B_{21} + A_\ast B_{22}^{-1}B_{21} + B_{22}\partial_x (B_{22}^{-1}B_{21}) \right]
\]

is an effective dissipation analogous to the effective diffusion predicted by formal, Chapman–Enskog expansion in the (dual) relaxation case.

The Green distribution (fundamental solution) associated with (3.1) is defined by

\[
G(x, t; y) := e^{t L_\ast \delta_y(x)}.
\]

or, equivalently,

\[
G_t - L_\ast G = 0, \quad \lim_{t \to 0^+} G(x, t; y) = \delta_y(x).
\]

Recalling the standard notation \( \text{errfn}(z) := \frac{1}{2\pi} \int_{-\infty}^{z} e^{-\xi^2} d\xi \), we have the following pointwise description.
Proposition 3.1 ([MZ3]). Under the assumptions of Theorem 1.4, the Green distribution $G(x,t;y)$ associated with the linearized equations (3.1) may be decomposed as $G = H + E + \tilde{G}$, where, for $y \leq 0$: 

\begin{equation}
H(x,t;y) := \sum_{j=1}^{J} a_j^*(x) a_j^*(y) R_j^*(x) \zeta_j^*(y,t) \delta_{x-a_j^*t}(-y) L_j^*(y)
\end{equation}

where the averaged convection rates $\bar{a}_j^* = \bar{a}_j^*(x,t)$ in (3.6) denote the time-averages over $[0,t]$ of $a_j^*(x)$ along backward characteristic paths $z_j^* = z_j^*(x,t)$ defined by 

$$dz_j^*/dt = a_j^*(z_j^*), \quad z_j^*(t) = x,$$

the dissipation matrix $\zeta_j^* = \zeta_j^*(x,t) \in \mathbb{R}^{m_j^* \times m_j^*}$ is defined by the dissipative flow 

$$d\zeta_j^*/dt = -\eta_j^*(z_j^*) \zeta_j^*, \quad \zeta_j^*(0) = I_{m_j},$$

and $\delta_{x-a_j^*t}$ denotes Dirac distribution centered at $x - a_j^*t$.

\begin{equation}
E(x,t;y) = \sum_{j=1}^{J} \frac{\partial \bar{a}_j^*(x)}{\partial \delta_j} |_{\delta=\delta} e_j(y,t),
\end{equation}

\begin{equation}
e_j(y,t) = \sum_{a_k > 0} \left( \text{erf} \left( \frac{y + a_k^* t}{\sqrt{4\beta_k^* t}} \right) - \text{erf} \left( \frac{y - a_k^* t}{\sqrt{4\beta_k^* t}} \right) \right) l_{jk}^*(y)
\end{equation}

for $y \leq 0$ and symmetrically for $y \geq 0$, with 

\begin{equation}
|l_{jk}^\pm| \leq C, \quad |(\partial/\partial y)l_{jk}^\pm| \leq Ce^{-\eta|y|},
\end{equation}
and

\begin{equation}
|\partial^\alpha_{x,y} \tilde{G}(x,t;y)| \leq C e^{-\eta(|x-y|+t)}
\end{equation}

\[ + C(t^{-|\alpha|/2} + |\alpha_y| e^{\eta |y|} + |\alpha_x| e^{-\eta |x|}) \left( \sum_{k=1}^{n} t^{-1/2} e^{-(x-y-a_k t)^2/M t e^{-\eta x^+}} \right) \]

\[ + \sum_{a_k > 0, a_j < 0} X_{\{|a_k t| \geq |y|\}} t^{-1/2} e^{-(x-a_j^-(t-|y/a_k^-|))^2/M t e^{-\eta x^+}}, \]

\[ + \sum_{a_k > 0, a_j > 0} X_{\{|a_k t| \geq |y|\}} t^{-1/2} e^{-(x-a_j^+(t-|y/a_k^+|))^2/M t e^{-\eta x^-}} \],

0 \leq |\alpha| \leq 2 for \( y \leq 0 \) and symmetrically for \( y \geq 0 \), for some \( \eta, C, M > 0 \), where \( a_j^\pm \) are as in Theorem 1.4, \( \beta_k^\pm > 0 \), \( x^\pm \) denotes the positive/negative part of \( x \), indicator function \( X_{\{|a_k t| \geq |y|\}} \) is 1 for \( |a_k t| \geq |y| \) and 0 otherwise. Moreover, all estimates are uniform in the supressed parameter \( \partial \).

**Proof.** This is a restatement of the bounds established in [MZ3] for pure undercompressive, Lax, or overcompressive type profiles; the same argument applies also in the mixed under–overcompressive case. Also, though it was not explicitly stated, uniformity with respect to \( \partial \) is a straightforward consequence of the argument. \( \square \)

**Corollary 3.2.** Under the assumptions of Theorem 1.4 and the notation of
Proposition 3.1,

\[ |e_j(y, t)| \leq C \sum_{a_k > 0} \left( \text{erf} \left( \frac{y + a_k t}{\sqrt{4\beta_k t}} \right) - \text{erf} \left( \frac{y - a_k t}{\sqrt{4\beta_k t}} \right) \right), \]

\[ |e_j(y, t) - e_j(y, +\infty)| \leq C \text{erf} \left( \frac{|y| - at}{M \sqrt{t}} \right), \text{ some } a > 0 \]

\[ |\partial_t e_j(y, t)| \leq C t^{-1/2} \sum_{a_k > 0} e^{-|y + a_k t|^2/Mt}, \]

(3.11) \[ |\partial_y e_j(y, t)| \leq C t^{-1/2} \sum_{a_k > 0} e^{-|y + a_k t|^2/Mt} \]

\[ + Ce^{-\eta|y|} \left( \text{erf} \left( \frac{y + a_k t}{\sqrt{4\beta_k t}} \right) - \text{erf} \left( \frac{y - a_k t}{\sqrt{4\beta_k t}} \right) \right), \]

\[ |\partial_y e_j(y, t) - \partial_y e_j(y, +\infty)| \leq C t^{-1/2} \sum_{a_k > 0} e^{-|y + a_k t|^2/Mt} \]

\[ |\partial_y e_j(y, t)| \leq C(t^{-1} + t^{-1/2} e^{-\eta|y|}) \sum_{a_k > 0} e^{-|y + a_k t|^2/Mt} \]

for \( y \leq 0 \), and symmetrically for \( y \geq 0 \).

Proof. Straightforward calculation using (3.8) and (3.9); see [MZ3].

From now on, let

(3.12) \[ e := \begin{pmatrix} e_1 \\ \vdots \\ e_\ell \end{pmatrix}. \]

Corollary 3.3 ([HZ]). Under the assumptions of Theorem 1.4 and the notation of Proposition 3.1,

(3.13) \[ \int_{-\infty}^{\infty} e(y, +\infty) (\partial \bar{u}^\delta / \partial \delta)_{|\delta_\ast}(y) \, dy = I_\ell \]

Proof. This follows from the standard fact that \( L^{\delta_\ast} (\partial \bar{u}^\delta / \partial \delta)_{|\delta_\ast} = 0 \), hence

\[ \int_{-\infty}^{+\infty} G(x, t; y) (\partial \bar{u}^\delta / \partial \delta)_{|\delta_\ast}(y) \, dy \equiv (\partial \bar{u}^\delta / \partial \delta)_{|\delta_\ast}(x) \]
which, together with the fact that \( E = (\partial \bar{u}^\delta / \partial \delta)_{|\delta_*}(x)e(y, t) \) represents the only nondecaying part of \( G(x, t; y) \) under stability criterion \((D)\), yields
\[
(\partial \bar{u}^\delta / \partial \delta)_{|\delta_*}(x) \int_{-\infty}^{+\infty} e(x, +\infty; y)(\partial \bar{u}^\delta / \partial \delta)_{|\delta_*}(y) dy = (\partial \bar{u}^\delta / \partial \delta)_{|\delta_*}(x)
\]
in the limit as \( t \to +\infty \).

**Proposition 3.4** (Parameter-dependent bounds). Under the assumptions of Theorem 1.4 and the notation of Proposition 3.1,

\begin{equation}
\partial_{\delta_*} H(x, t; y) = O(H) + \partial_y O(tH),
\end{equation}

\begin{align}
|\partial_{\delta_*} e| & \leq C|e|, \\
|\partial_{\delta_*} e_t| & \leq C|e_t|,
\end{align}

\begin{equation}
|\partial_{\delta_*} (e(y, t) - e(y, +\infty))| \leq C|(e(y, t) - e(y, +\infty))|,
\end{equation}

\begin{align}
|\partial_{\delta_*} e_y| & \leq C|e_y|,
|\partial_{\delta_*} (e_y(y, t) - e_y(y, +\infty))| \leq C|(e_y(y, t) - e_y(y, +\infty))|,
\end{align}

\begin{equation}
|\partial_{\delta_*} e_{yt}| \leq C|e_{yt}|.
\end{equation}

and

\begin{equation}
|\partial_{\delta_*} \partial_{x,y}^\alpha \tilde{G}(x, t; y)| \leq C|\partial_{x,y}^\alpha \tilde{G}(x, t; y)|,
\end{equation}

\(0 \leq |\alpha| \leq 2\) for \( y \leq 0\) and symmetrically for \( y \geq 0\), for some \( C > 0\).

**Proof.** These follow by the same argument used to establish the parameter-dependent bounds of Proposition 3.11, [TZ1], using the additional fact that neither speeds \( s\) nor endstates \( u_\pm\) depend on the choice of \( \delta_*\) to obtain better decay estimates on certain terms.

Bounds (3.14) and (3.15) follow by direct calculation, together with the observations (obtained similarly as bounds established in the proof of Proposition 3.11, [TZ1], using parameter-dependent asymptotic ODE bounds) that

\[
\partial_{\delta_*} a_j^*, \partial_{\delta_*} \bar{a}_j^*, \partial_{\delta_*} \mathcal{R}_j^*, \partial_{\delta_*} \mathcal{L}_j^* = O(1)
\]

and

\begin{equation}
\partial_{\delta_*} \partial_{y}^\alpha l_j^\pm(y) = O(e^{-|y|}), \quad 0 \leq |\alpha| \leq 2.
\end{equation}

Bounds (3.16) follow by the argument of [TZ1], but using the fact that \( \alpha = a_k^\pm, \beta_k^\pm \) (since \( u_\pm\)) do not depend on \( \delta_*\), hence “bad” factors \( t\partial_{\delta_*} \alpha = O(t)\) do not appear, but only factors \( O(1)\) or better. \(\square\)
Remark 3.5. The additional factor $t$ in the righthand side of (3.14) may be absorbed in time-exponential decay of $H$, i.e., $tH$ obeys the same decay bounds as $H$, but with slightly smaller time-exponential decay rate.

4 Convolution lemmas

We shall make use of the following technical lemmas proved in [HZ, HR].

**Lemma 4.1 (Linear estimates I).** *Under the assumptions of Theorem 1.4,*

\[
\int_{-\infty}^{+\infty} |\tilde{G}(x, t; y)|(1 + |y|)^{-3/2} dy \leq C(\theta + \psi_1 + \psi_2)(x, t),
\]

\[
\int_{-\infty}^{+\infty} |e_t(y, t)|(1 + |y|)^{-3/2} dy \leq C(1 + t)^{-3/2},
\]

\[
\int_{-\infty}^{+\infty} |e(y, t)|(1 + |y|)^{-3/2} dy \leq C,
\]

\[
\int_{-\infty}^{+\infty} |e(y, t) - e(y, +\infty)|(1 + |y|)^{-3/2} dy \leq C(1 + t)^{-1/2},
\]

for $0 \leq t \leq +\infty$, some $C > 0$, where $\tilde{G}$ and $e$ are defined as in Proposition 3.1.

**Lemma 4.2 (Nonlinear estimates I).** *Under the assumptions of Theorem 1.4,*

\[
\int_0^t \int_{-\infty}^{+\infty} |\tilde{G}_y(x, t - s; y)|\Psi(y, s) dy ds \leq C(\theta + \psi_1 + \psi_2)(x, t),
\]

\[
\int_0^{t-1} \int_{-\infty}^{+\infty} |\tilde{G}_{xy}(x, t - s; y)|\Psi(y, s) dy ds \leq C(\theta + \psi_1 + \psi_2)(x, t),
\]

\[
\int_0^t \int_{-\infty}^{+\infty} |e_{yt}(y, t - s)|\Psi(y, s) dy ds \leq C(1 + t)^{-1},
\]

\[
\int_t^{+\infty} \int_{-\infty}^{+\infty} |e_y(y, +\infty)|\Psi(y, s) dy \leq C(1 + t)^{-1/2},
\]

\[
\int_0^t \int_{-\infty}^{+\infty} |e_y(y, t - s) - e_y(y, +\infty)|\Psi(y, s) dy ds \leq C(1 + t)^{-1/2},
\]
for $0 \leq t \leq +\infty$, some $C > 0$, where $\tilde{G}$ and $e$ are defined as in Proposition 3.1 and

\begin{equation}
\Psi(y, s) := (1 + s)^{1/2} s^{-1/2} (\theta + \psi_1 + \psi_2)^2 (y, s) + (1 + s)^{-1} (\theta + \psi_1 + \psi_2)(y, s). \tag{4.3}
\end{equation}

**Lemma 4.3** (Nonlinear estimates II). Under the assumptions of Theorem 1.4, \n
\begin{align*}
\int_0^t \int_{-\infty}^{+\infty} |\tilde{G}_y(x, t - s; y)| \Phi_1(y, s) \, dy \, ds &\leq C(\theta + \psi_1 + \psi_2)(x, t), \\
\int_0^{t-1} \int_{-\infty}^{+\infty} |\tilde{G}_{xy}(x, t - s; y)| \Phi_1(y, s) \, dy \, ds &\leq C(\theta + \psi_1 + \psi_2)(x, t), \\
\int_0^t \int_{-\infty}^{+\infty} |e_{yt}(y, t - s)| \Phi_1(y, s) \, dy \, ds &\leq C(1 + t)^{-1}, \\
\int_0^t \int_{-\infty}^{+\infty} |e_y(y, t - s)| \Phi_1(y, s) \, dy \, ds &\leq C(1 + t)^{-1/2}
\end{align*}

and

\begin{align*}
\int_0^t \int_{-\infty}^{+\infty} |\tilde{G}(x, t - s; y)| \Phi_2(y, s) \, dy \, ds &\leq C(\theta + \psi_1 + \psi_2)(x, t), \\
\int_0^t \int_{-\infty}^{+\infty} \tilde{G}_x \Phi_2(y, s) \, dy \, ds &\leq C(\theta + \psi_1 + \psi_2)(x, t), \\
\int_0^t \int_{-\infty}^{+\infty} |e_t(y, t - s)| \Phi_2(y, s) \, dy \, ds &\leq C(1 + t)^{-3/2}, \\
\int_0^t \int_{-\infty}^{+\infty} |e(y, t - s) - e(y, +\infty)| \Phi_2(y, s) \, dy \, ds &\leq C(1 + t)^{-3/2}
\end{align*}

for $0 \leq t \leq +\infty$, some $C > 0$, where $\tilde{G}$ and $e$ are defined as in Proposition 3.1 and

\begin{align*}
\Phi_1(y, s) := e^{-\eta|y|} s^{-1/2}(\theta + \psi_1 + \psi_2)(y, s) &\leq Ce^{-\eta|y|/2} s^{-1/2}(1 + s)^{-1}, \\
\Phi_2(y, s) := e^{-\eta|y|}(1 + s)^{-3/2}.
\end{align*}

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Lemma 4.4 (Linear estimates II). Under the assumptions of Theorem 1.4, if $|v_0(x)|, |\partial_x v_0(x)| \leq E_0(1 + |x|)^{-\frac{3}{2}}$, $E_0 > 0$, then, for some $\theta > 0$,

$$
\int_{-\infty}^{+\infty} H(x, t; y)v_0(y)dy \leq CE_0e^{-\theta t}(1 + |x|)^{-\frac{3}{2}}
$$

(4.7)

$$
\int_{-\infty}^{+\infty} H_x(x, t; y)v_0(y)dy \leq CE_0e^{-\theta t}(1 + |x|)^{-\frac{3}{2}},
$$

and so both expressions are dominated by $E_0(\psi_1 + \psi_2)$.

**Proof.** See [HR, HRZ].

Lemma 4.5 (Nonlinear estimates III). Under the assumptions of Theorem 1.4, if $|\Upsilon(y, s)| \leq s^{-1/4}(\psi_1 + \psi_2 + \theta)(y, s) + s^{-1/2}e^{-\eta|y|}$, then

$$
\left| \int_0^t \int_{-\infty}^{+\infty} H(x, t - s; y)\Upsilon(y, s)dyds \right| \leq C(\psi_1 + \psi_2)(x, t),
$$

(4.8)

$$
\left| \int_0^t \int_{-\infty}^{+\infty} (H_x - H_y)(x, t - s; y)\Upsilon(y, s)dyds \right| \leq C(\psi_1 + \psi_2)(x, t),
$$

$$
\int_{t-1}^{t} \int_{-\infty}^{+\infty} |\tilde{G}_x(x, t - s; y)|\Upsilon(y, s)dyds \leq C(\psi_1 + \psi_2)(x, t).
$$

**Proof.** The proof of the first inequality is very much similar to that of the similar estimates proved in [HR]. Here we prove only the part which contains the convolution of $H$ against $s^{-1/4}\psi_1$. A typical term in (4.8) coming from $\psi_1$ would be dominated by a term of the form

$$
C \int_0^t e^{-\eta_0(t-s)}\chi(x - \bar{a}^*_j(t-s), s)s^{-\frac{3}{2}}(1 + s)^{-\frac{3}{2}}
$$

$$
\times (1 + |x - \bar{a}^*_j(t-s) - a^-_i s|)^{-\frac{3}{2}}ds
$$

$$
= C \int_0^t e^{-\eta_0(t-s)}\chi(x - \bar{a}^*_j(t-s), s)s^{-\frac{3}{2}}(1 + s)^{-\frac{3}{2}}
$$

$$
\times (1 + |x - a^-_i t - (\bar{a}^*_j - a^-_i)(t-s)|)^{-\frac{3}{2}}ds;
$$

now, using $\frac{1}{1+|a+b|} \leq \frac{1+|b|}{1+|a|}$ the above would be smaller than

$$
C \int_0^t e^{-\eta_0(t-s)}\chi(x - \bar{a}^*_j(t-s), s)s^{-\frac{3}{2}}(1 + s)^{-\frac{3}{2}}
$$

$$
\times (1 + |x - a^-_i t|)^{-\frac{3}{2}}(|1 + (\bar{a}^*_j - a^-_i)(t-s)|)^{\frac{1}{2}}ds.
$$
Notice that \((\bar{a}_j - a^-_i) \leq C\). Now we use half of \(\eta\) to neutralize \(t - s\), and to get

\[
C(1 + |x - a^-_i t|)^{-\frac{1}{2}} \int_0^t e^{-\frac{2\eta}{3}(t-s)} \chi(x - \bar{a}_j^+(t-s), s)s^{-\frac{1}{2}}(1 + s)^{-\frac{1}{2}} ds
\]

On the one hand, this is obviously dominated by

\[
C(1 + |x - a^-_i t|)^{-\frac{1}{2}}(1 + t)^{-\frac{1}{2}}.
\]

which is absorbed by a factor of \(\psi_1\), if \(x \in [a^-_1 t, a^+_n t]\). On the other hand, \(\chi(x - \bar{a}_j^+(t-s), s)\) in the integral means that the above expression would vanish if \(x > Mt\), for some fixed \(M\). Therefore the above expression is always less than a factor of \(\psi_1 + \psi_2\).

The second inequality follows by an identical proof, using \(H_x - H_y \sim H\).

The proof of the third inequality is similar, using the fact that

\[
C|\hat{G}_y(t-s)| \leq e^{-\theta(t-s)}(t-s)^{-1}e^{-|x-y|^2/C(t-s)}
\]

for \(t - 1 \leq s \leq t\); see [MZ2] for similar calculations.

\[\square\]

## 5 Auxiliary energy estimate

We shall require the following auxiliary energy estimate adapted essentially unchanged from [MZ4, Z2, Ra]. Let

\[
\tilde{u}_t + f(\tilde{u})_x - (B(\tilde{u})\tilde{u}_x)_x = (\partial \tilde{u})/\partial \delta)|\delta_\ast(x)\gamma(t),
\]

\[
u := \tilde{u} - \tilde{u}^{\delta_\ast + \delta(t)}.
\]

**Lemma 5.1** ([MZ4, Z2, Ra]). Under the hypotheses of Theorem 1.4 let \(u_0 \in H^5\), and suppose that, for \(0 \leq t \leq T\), the suprema of \(|\delta|\) and \(|\gamma|\) and the \(W^{3,\infty}\) norm of the solution \(u = (u^I, u^{II})^t\) of (5.1), (5.2) each remain bounded by a sufficiently small constant \(\zeta > 0\). Then, for all \(0 \leq t \leq T\),

\[
|u(t)|^2_{H^5} \leq C e^{-\theta t}|u(0)|^2_{H^5} + C \int_0^t e^{-\theta_2(t-\tau)}(|u|^2_{L^2} + |\delta|^2 + |\gamma|^2)(\tau) d\tau.
\]
Proof. This follows exactly as in the $\gamma \equiv 0$ case treated in [MZ4, Z2, Ra], observing that term

$$(\partial \bar{u}/\partial \delta)|_{\delta}(x) \gamma(t)$$

is of the same form as terms

$$(\partial \bar{u}/\partial \delta)|_{\delta}(x) \cdot \delta(t)$$

already arising in the nonlinear perturbation equations in the former case. □

We require also the following much cruder estimate adapted from [HR].

**Lemma 5.2 ([HR]).** Under the hypotheses of Theorem 1.4 let $E_0 := \|(1 + |x|^2)^{-3/4}u_0(x)\|_{H^5} < \infty$, and suppose that, for $0 \leq t \leq T$, the suprema of $|\delta|$ and $|\gamma|$ and the $W^{3,\infty}$ norm of the solution $u = (u^I, u^{II})^t$ of (5.1), (5.2) each remain bounded by some constant $C > 0$. Then, for all $0 \leq t \leq T$, some $M = M(C) > 0$,

$$(5.4) \quad \|(1 + |x|^2)^{-3/4}u(x,t)\|_{H^5}^2 \leq Me^{Mt} \left( E_0 + \int_0^t (|\delta|^2 + |\gamma|^2(\tau)) \, d\tau \right).$$

Proof. This follows by standard Friedrichs symmetrizer estimates carried out in the weighted $H^5$ norm. (Recall, these plus several more complicated estimates are used in the proof of Lemma 5.1.) □

**Remark 5.3.** An immediate consequence of Lemma 5.2, by Sobolev embedding and equation (5.1), is that, if $E_0 := \|(1 + |x|^2)^{-3/4}u_0(x)\|_{H^5}$, $\|u\|_{H^5}$, $|\delta(\cdot)|$ and $|\gamma(\cdot)|$ are uniformly bounded on $0 \leq t \leq T$, then

$$|(1 + |x|^2)^{-3/4}u(x,t)|, \quad |(1 + |x|^2)^{-3/4}u_t(x,t)|$$

are uniformly bounded on $0 \leq t \leq T$ as well.

## 6 Fixed-point iteration scheme

We now introduce the fixed-point iteration scheme by which we shall simultaneously construct and estimate the solution of the perturbed shock problem.

Our starting point, similarly as in [HZ], is the observation that

$$(6.1) \quad u(x,t) := \bar{u}(x,t) - \bar{u}^{\delta_0 + \delta(t)}(x)$$
satisfies

\begin{equation}
(6.2) \quad u_t - L^{δ*} u = Q^{δ*}(u) + \hat{δ}(t)(\partial u^{δ}/\partial δ)|_{δ*} + R^{δ*}(δ, u) + S^{δ*}(δ, δ),
\end{equation}

where

\begin{equation}
(6.3) \quad Q^{δ*}(u, u) := \left( f(\tilde{u}^{δ*}) + A(\tilde{u}^{δ*})u - f(\tilde{u}^{δ*} + u) \right) + \left( B(\tilde{u}^{δ*} + u) - B(\tilde{u}^{δ*}) \right) u_x
\end{equation}

\[ = \mathcal{O}(|u|^2 + |u||u_x|), \]

\[ Q^{δ*}(u, u)_x = \mathcal{O}(|u||u_x| + |u|^2 + |u||u_{xx}|), \]

\[ Q^{δ*}(u, u)_{xx} = \mathcal{O}(|u||u_{xx}| + |u|^2 + |u||u_{xxx}| + |u_x||u_{xx}|), \]

\[ R^{δ*} := \left( A(\tilde{u}^{δ*}(x)) - A(\tilde{u}^{δ^{+δ}(t)}(x)) \right) u = \mathcal{O}(e^{-η|x|}||δ||u||), \]

\begin{equation}
(6.4) \quad R^{δ*}_x = \mathcal{O}(e^{-η|x|}||δ||u_x||), \quad R^{δ*}_{xx} = \mathcal{O}(e^{-η|x|}||δ||u_{xx}||),
\end{equation}

equation and

\begin{equation}
(6.5) \quad S^{δ*} := \hat{δ} \left( (\partial u^{δ}/\partial δ)|_{δ^{+δ}(t)} - (\partial \tilde{u}^{δ}/\partial δ)|_{δ^*} \right) = \mathcal{O}(e^{-η|x|}||δ||δ)||
\end{equation}

so long as \(|u|\) remains bounded, by Taylor’s Theorem together with (2.4).

Accordingly, for given \(δ^{n-1}, n^{n-1}(\cdot)\), define \(u^n\) to be the solution of

\begin{equation}
(6.6) \quad u^n(x, t) = \int_{-\infty}^{+\infty} \left( H^{n-1} + \tilde{G}^{n-1} \right)(x, t; y) u_0^{n-1}(y) dy
\end{equation}

\[ + \int_0^t \int_{-\infty}^{+\infty} \left( H^{n-1} + \tilde{G}^{n-1} \right)(x, t - s; y) \]

\[ \times S^{δ^{n-1}}(δ^{n-1}, n^{n-1})(y, s) dy ds \]

\[ - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}^{n-1}(x, t - s; y) \]

\[ \times \left( Q^{δ^{n-1}}(u^n, u^n_x) + R^{δ^{n-1}}(δ^{n-1}, u^n) \right)(y, s) dy ds \]

\[ + \int_0^t \int_{-\infty}^{+\infty} H^{n-1}(x, t - s; y) \]

\[ \times \left( Q^{δ^{n-1}}(u^n, u^n_x) + R^{δ^{n-1}}(δ^{n-1}, u^n) \right)(y, s) dy ds \]

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where $H^{n-1}, \tilde{G}^{n-1}$ are the parts of Green distribution $G^{n-1}$ of linearized equation around $\tilde{u}^{\delta_\tau^{n-1}}$, and

\begin{equation}
(6.7) \quad u_0^{n-1} = \tilde{u}_0 - \tilde{u}^{\delta_\tau^{n-1}}.
\end{equation}

Further, set

\begin{equation}
\delta^n(t) :=
\begin{multline}
- \int_{-\infty}^{\infty} (e^{n-1}(y, t) - e^{n-1}(y, +\infty)) u_0^{n-1}(y) \, dy \\
- \int_{0}^{t} \int_{-\infty}^{+\infty} (e^{n-1}(y, t - s) - e^{n-1}(y, +\infty)) \times S^{\delta_\tau^{n-1}}(\delta^{n-1}, \dot{\delta}^{n-1})(y, s) \, dy \, ds \\
+ \int_{0}^{t} \int_{-\infty}^{+\infty} (e_y^{n-1}(y, t - s) - e_y^{n-1}(y, +\infty)) \times (Q^{\delta_\tau^{n-1}}(u^n, u^n_x) + R^{\delta_\tau^{n-1}}(\delta^{n-1}, u^n))(y, s) \, dy \, ds \\
+ \int_{t}^{+\infty} \int_{-\infty}^{+\infty} e^{n-1}(y, +\infty) \times S^{\delta_\tau^{n-1}}(\delta^{n-1}, \dot{\delta}^{n-1})(y, s) \, dy \, ds,
\end{multline}

\begin{equation}
(6.8)
\end{equation}

and

\begin{equation}
\delta_\tau^n := \delta_\tau^{n-1} + \int_{-\infty}^{+\infty} e^{n-1}(y, +\infty) u_0^{n-1}(y) \, dy \\
+ \int_{0}^{+\infty} \int_{-\infty}^{+\infty} e^{n-1}(y, +\infty) S^{\delta_\tau^{n-1}}(\delta^{n-1}, \dot{\delta}^{n-1})(y, s) \, dy \, ds \\
- \int_{0}^{+\infty} \int_{-\infty}^{+\infty} e_y^{n-1}(y, +\infty) \times (Q^{\delta_\tau^{n-1}}(u^n, u^n_x) + R^{\delta_\tau^{n-1}}(\delta^{n-1}, u^n))(y, s) \, dy \, ds.
\end{equation}

\begin{equation}
(6.9)
\end{equation}

Define an associated iteration map $T$, formally, by

\begin{equation}
(6.10) \quad (\delta^n, \delta_\tau^n) = T(\delta^{n-1}, \delta_\tau^{n-1}).
\end{equation}
Lemma 6.1. Under (6.6)–(6.9), \( \tilde{u}^n := u^n + \tilde{u}^{\delta_{n-1} + \delta_{n-1}} \) satisfies

\begin{equation}
\tilde{u}_t^n + f(\tilde{u}^n)_x - (B(\tilde{u}^n)\tilde{u}^n)_x = \left( \delta^n(t) - \delta^{n-1}(t) \right) \frac{\partial u_t^\delta}{\partial \delta} |_{\delta^{n-1}},
\end{equation}

(6.11)

\[
\delta^n(t) = -\int_{-\infty}^{\infty} e^{n-1}_t(y, t)u_0^{n-1}(y)dy \\
- \int_0^t \int_{-\infty}^{+\infty} e^{n-1}_t(y, t - s)S^{\delta_{n-1}}(\delta^{n-1}, \delta^{n-1})(y, s)dyds \\
+ \int_0^t \int_{-\infty}^{+\infty} e^{n-1}_t(y, t - s) \\
\times (Q^{\delta_{n-1}}(u^n, u^n_x) + R^{\delta_{n-1}}(\delta^{n-1}, u^n))(y, s)dyds,
\]

(6.12)

with initial data \( \tilde{u}^n(\cdot, 0) = \tilde{u}_0 + (u^{\delta_{n-1} + \delta_{n-1}(0)} - \tilde{u}^{\delta_{n-1}}) \), and therefore satisfies (1.2) with initial data \( \tilde{u}_0 \) if and only if

\[
(\delta^n, \delta^n_\ast) = (\delta^{n-1}, \delta^{n-1}_\ast),
\]

(6.13)

i.e., \( (\delta^n, \delta^n_\ast) \) is a fixed point of \( T \), in which case also \( \delta(0) = \delta(+\infty) = 0 \).

Proof. Equation (6.12) follows immediately upon differentiation of (6.8). From (6.8), we obtain, further, that \( \delta^n(+\infty) = 0 \), and

\[
\delta^n(t) - \delta^n(0) = -\int_{-\infty}^{+\infty} e^{n-1}_t(y, t)u_0^{n-1}(y)dy \\
- \int_0^t \int_{-\infty}^{+\infty} e^{n-1}_t(y, t - s)S^{\delta_{n-1}}(\delta^{n-1}, \delta^{n-1})(y, s)dyds \\
- \int_0^t \int_{-\infty}^{+\infty} e^{n-1}_t(y, t - s) \\
\times (Q^{\delta_{n-1}}(u^n, u^n_x) + R^{\delta_{n-1}}(\delta^{n-1}, u^n))(y, s)dyds.
\]

(6.14)

Setting \( t = +\infty \) in (6.14), and comparing with (6.9), we find, therefore, that \( \delta^n(0) = \delta^n(+\infty) = 0 \) if and only if \( \delta^n_\ast = \delta^{n-1}_\ast \).
From (6.6) and (6.8) we conclude that

\[
\begin{align*}
    u^n(x, t) &= \int_{-\infty}^{+\infty} G^{n-1}(x, t; y) u_0^{n-1}(y) dy \\
    &+ \int_0^t \int_{-\infty}^{+\infty} G^{n-1}(x, t-s; y) S^{\delta, n-1}(\delta^{n-1}, \dot{\delta}^{n-1})(y, s) dy ds \\
    &\quad + \int_0^t \int_{-\infty}^{+\infty} G^{n-1}(x, t-s; y) \\
    &\quad \times (Q^{\delta, n-1}(u^n, u^n_x) + R^{\delta, n-1}(\delta^{n-1}, u^n) \big)_{y,s} dy ds \\
    &\quad + (\delta^n(t) - \delta^n(0)) \frac{\partial \bar{u}^{\delta} \delta}{\partial \delta} |_{\delta^{n-1}}
\end{align*}
\]  

(6.15)

and thus, by Duhamel's Principle,

\[
\begin{align*}
    u^n(t) - L^{\delta, n-1} u^n &= Q^{\delta, n-1}(u^n, u^n_x) + R^{\delta, n-1}(\delta^{n-1}, u^n) \\
    &\quad + \dot{\delta}^{n}(t) \frac{\partial \bar{u}^{\delta} \delta}{\partial \delta} |_{\delta^{n-1}}.
\end{align*}
\]  

(6.16)

Setting \( \tilde{u}^n = u^n + \bar{u}^{\delta, n-1} + \delta^{n-1} \), we then obtain (6.11) by a straightforward calculation comparing with (6.2), with the claimed initial data

\[
\begin{align*}
    \tilde{u}^n(\cdot, 0) &= \bar{u}^{\delta, n-1} + \delta^{n-1}(0) + u^{n-1}_0 = \bar{u}_0 + (\bar{u}^{\delta, n-1} + \delta^{n-1}(0) - \bar{u}^{\delta, n-1}).
\end{align*}
\]  

(6.17)

Note that the righthand side is equal to \( \bar{u}_0 \) if and only if \( \delta^{n-1}(0) = 0 \), or, in case \( \delta^n \equiv \delta^{n-1} \) (a fixed point), if \( \delta^n(0) = 0 \), or equivalently \( \delta^* = \delta^{n-1} \).

Remark 6.2. Other than a slight notational change \( \delta \rightarrow \delta^* + \delta \) made to simplify the exposition, the difference between this iteration scheme and the one used in [HZ] in the strictly parabolic case is that we have made it implicit in \( u^n \), i.e., \( u \) appears everywhere on the righthand side of the integral equations with index \( n \) rather than \( n-1 \). By this change we preserve regularity properties, as encoded in the nonlinear structure of equation (6.11); see Lemma 5.1. By contrast, the explicit version of [HZ] is not associated with a (favorable) nonlinear equation, and so would lose derivatives, preventing the iteration scheme from closing.

Remark 6.3. Note that (6.11)–(6.12) form a closed system for \( (u^n, \dot{\delta}^n) \), in the form of a true Cauchy problem; that is, the values of \( (u^n, \dot{\delta}^n) \) at time \( T \)
depend only on values for $0 \leq t \leq T$, and not on future times. By (6.6), we have, evidently,

$$u^n(\cdot, 0) = u_0^{n-1}.$$  

## 7 Local existence

**Lemma 7.1** ($H^5$ local theory). Under the hypotheses of Theorem 1.4 let

$$E_1 := \|u_0(x)\|_{H^5} + \|\delta^{n-1}\|_{B_1} + |\delta_0^{n-1}| < \infty.$$  

Then, for $T = T(E_1) > 0$ sufficiently small and $C = C(E_1, T) > 0$ sufficiently large, there exists on $0 \leq t \leq T$ a unique solution

$$(u^n, \dot{\delta}^n) \in L^\infty(H^5(x); t) \times C^0(t)$$

of (6.11)–(6.12), satisfying

$$\|u^n\|_{H^5(t)}, |\dot{\delta}^n|(t) \leq CE_1.$$  

**Proof.** Short-time existence, uniqueness, and stability follow by (unweighted) energy estimates in $u^n$ similar to (5.4) combined with more straightforward estimates on $\dot{\delta}^n$ carried out directly from integral equation (6.12), using a standard (bounded high norm, contractive low norm) contraction mapping argument like those described in [Z2, Z3]. We omit the details. \qed

**Remark 7.2.** A crucial point is that equations (6.11)–(6.12) depend only on values of $(u^n, \dot{\delta}^n)$ on the range $t \in [0, T]$; see Remark 6.3.

## 8 Proof of the Main Theorem

We are now ready to prove the main theorem. Define norms

$$|h|_{B_1} := |h(t)(1 + t)^{1/2}|_{L^\infty(t)} + |\dot{h}(t)(1 + t)|_{L^\infty(t)},$$

and

$$|g|_{B_2} := |f(\theta + \psi_1 + \psi_2)^{-1}|_{W^{1,\infty}(x,t)}$$

and Banach spaces

$$B_1 := \{h : |h|_{B_1} < +\infty\}, \quad B_2 := \{g : |g|_{B_2} < +\infty\}.$$
Lemma 8.1. Under the hypotheses of Theorem 1.4, let

\((u^n, \dot{\delta}^n) \in L^\infty(t, H^5(x)) \times L^\infty(t)\)

satisfy (6.11)–(6.12) on \(0 \leq t \leq T\), and define

(8.4) \(\zeta(t) := \sup_{x, 0 \leq s \leq t} \left( \left| |u^n| + |u^n_x| \right| (\theta + \psi_1 + \psi_2)^{-1}(x, s) + |\dot{\delta}^n(s)(1 + s)| \right)\).

If \(\zeta(T), \|u_0^{n-1}\|_{H^5}, \text{ and } |\delta_0^{n-1}|_{B_1}\) are bounded by \(\zeta_0 > 0\) sufficiently small, then, for some \(\varepsilon > 0\), (i) the solution \((u^n, \dot{\delta}^n)\), and thus \(\zeta\) extends to \(0 \leq t \leq T + \varepsilon\), and (ii) \(\zeta\) is bounded and continuous on \(0 \leq t \leq T + \varepsilon\).

Proof. By (6.18) and Lemma 5.1, smallness of \(\zeta(T), |\delta_0^{n-1}(s)|_{B_1}\), and \(\|u_0^{n-1}\|_{H^5}\) together imply boundedness (and smallness, though we don’t need this) of \(\|\dot{u}\|_{H^5}\) and \(\|\dot{\delta}\|_{L^\infty}\) on \(0 \leq t \leq T\). By Lemma 7.1, this implies existence, boundedness of \(\|u\|_{H^5}, \|\delta\|_{W^{1,\infty}}\) on \(0 \leq t \leq T + \varepsilon\) for \(\varepsilon > 0\), and thus, by Remark 5.3, boundedness and continuity of \(\zeta\) on \(0 \leq t \leq T + \varepsilon\). \(\Box\)

Lemma 8.2. For \(M > 0\) and \(C_1 \geq C >> M\) sufficiently large, for

\[ E_0 := \|((1 + |x|^2)^{-3/4}(\tilde{u}_0 - \tilde{u}))\|_{H^5} \]

sufficiently small, and \(|\delta_0^{n-1}|_{B_1} + M|\delta^{n-1}_*| \leq 2CE_0\), there exist solutions \((u^n, \delta^n, \tilde{\delta}^n)\) of (6.6)–(6.9) for all \(t \geq 0\), satisfying

(8.5) \(|u^n|_{H^5} \leq C_1E_0\)

and

(8.6) \(|u^n|_{B_2} + |\delta^n|_{B_1} + M|\tilde{\delta}^n| \leq 2CE_0\).

Proof. Define \(\zeta\) as in (8.4). Then, it is sufficient to show that

(8.7) \(\zeta(t) \leq CE_0 + C_\ast(E_0 + \zeta(t))^2\)

for fixed \(C, C_\ast > 0\), so long as the solution \((u^n, \dot{\delta}^n) \in L^\infty(t, H^5(x)) \times C^0(t)\) of (6.11)–(6.12) exists and

(8.8) \(\zeta(t) \leq (3/2)CE_0\).
in order to conclude that solution \((u^n, \dot{\delta}^n)\) exists and satisfies (8.8) for all \(t \geq 0\), provided
\[
E_0 < \frac{25}{2CC_*}
\]
is sufficiently small.

For, by (6.18) and (6.7), and (2.4),
\[
\|u^n(\cdot, 0)\|_{H^5} = \|u_0^{n-1}\|_{H^5}
= \|\bar{u}_0 - \bar{u}^{\delta^{n-1}}\|_{H^5}
\leq \|\bar{u}_0 - \bar{u}\|_{H^5} + \|\bar{u}^{\delta^{n-1}} - \bar{u}\|_{H^5}
\leq E_0 + c_1|\delta^{n-1}_*|
\leq E_0(1 + 2c_1C/M)
\]
is small, for \(E_0\) sufficiently small. Letting \(T\) be the maximum time up to which a solution \((u^n, \dot{\delta}^n)\) exists and \(\zeta \leq \zeta_0\) sufficiently small (note: \(T \geq 0\) by the weighted version of (8.9), together with (6.12)), we find by Lemma 8.1, therefore, and the assumed bounds on \(\delta^{n-1}_*\) and \(\dot{\delta}^{n-1}_*\), that \((u^n, \dot{\delta}^n)\) exists up to \(T + \varepsilon, \varepsilon > 0\), and that \(\zeta\) remains bounded and continuous up to \(T + \varepsilon\) as well. Observing that (8.7) together with \(E_0 < 2/(9C^2 + 6C)\) implies that \(\zeta(t) < (3/2)CE_0\) whenever \(\zeta(t) \leq (3/2)CE_0\), we find by continuity that \(\zeta(t) \leq (3/2)CE_0\) up to \(t = T + \varepsilon\) as claimed.

By the definition of \(\zeta\), we obtain therefore
\[
|u^n|_{B^2} + |\dot{\delta}^n(t)(1 + t)|_{L^\infty} \leq (3/2)CE_0.
\]
Thus, it is sufficient to establish first (8.7), then afterward, assuming (8.8),
\[
|\delta^n(t)| \leq (CE_0/4)(1 + t)^{-1/2}
\]
and
\[
|\delta_*^n| \leq (CE_0/4M),
\]
from which we obtain (8.6) by summation with (8.10); noting that (8.11) and (8.12) include the information that the integral equations for \(\delta^n\) and \(\delta_*^n\) converge, we obtain also, by Lemma 6.1 and the fact that \((u^n, \dot{\delta}^n)\) satisfies (6.11)–(6.12) for all \(t \geq 0\), that \((u^n, \delta^n, \delta_*^n)\) satisfies (6.6)–(6.9) as claimed. Finally, recalling (6.11) and applying Lemma 5.1 with \(\gamma := \dot{\delta}^n - \dot{\delta}^{n-1}_*\), we obtain (8.5) so long as (8.7) remains valid, controlling \(\|u^n\|_{H^5}\) by integrating.
the righthand side of (5.3) and using (8.8), the definition of \( \zeta \), and the assumed bounds on \( \dot{\delta}^{n-1} \). (We carry out this last calculation in detail in the following paragraph, in the course of proving (8.7)).

We now establish (8.7) assuming (8.8). By Lemma 5.1, and the one-dimensional Sobolev bound \( |u^n|_{W^{3,\infty}} \leq c |u^n|_{H^3} \), we have

\[
|u^n(t)|_{H^3}^2 \leq c |u^n(0)|_{H^3}^2 e^{-\theta t} + c \int_0^t e^{-\theta(t-\tau)}(|u^n|_{L^2}^2 + |\dot{\delta}^n|^2 + |\dot{\delta}^n - \dot{\delta}^{n-1}|^2)(\tau) \, d\tau
\]

(8.13)

\[
\leq c |u^n(0)|_{H^3}^2 e^{-\theta t} + c \int_0^t e^{-\theta(t-\tau)}(|u^n|_{L^2}^2 + \max\{|\dot{\delta}^n|^2, |\dot{\delta}^{n-1}|^2\})(\tau) \, d\tau
\]

\[
\leq c_2 (|u^n(0)|_{H^3}^2 + \zeta(t)^2)(1 + t)^{-1/2}
\]

\[
\leq c_2 (E_0^2 (1 + 2C/M)^2 + (3CE_0/2)^2)(1 + t)^{-1/2}
\]

\[
\leq (C_1 E_0)^2 (1 + t)^{-1/2},
\]

for \( C_1 > 0 \) sufficiently large \( E_0 \) sufficiently small, by (8.9), (8.8), and the definition of \( \zeta \). This verifies (8.5), assuming (8.8).

With (6.2), (8.5) and the resulting Sobolev estimate \( \|u^n\|_{W^{3,\infty}} \leq c C_1 E_0 \), assumption \( |\delta|_{B_1} \leq 2CE_0 \), and the definitions of \( \zeta \) and \( |\cdot|_{B_1} \), we obtain readily

\[
|Q_{\delta^*} + R_{\delta^*}| \leq c (\zeta^2 + 4C^2 E_0^2)(\Psi + \Phi_2),
\]

(8.14)

\[
|Q_y^\delta + R_y^\delta|, |Q_{yy}^\delta + R_{yy}^\delta| \leq c (\zeta^2 + 4C^2 E_0^2) \Upsilon
\]

and

\[
|S_{\delta^*}|, |S_y^\delta| \leq c (\zeta^2 + 4C^2 E_0^2) \Phi_2,
\]

(8.15)

where \( \Phi, \Psi, \) and \( \Upsilon \) are as defined in Lemmas 4.1–4.5.
Expressing $u^n_x$ using (6.6) as

$$u^n_x(x,t) = \int_{-\infty}^{+\infty} (H^{n-1}_x + \bar{G}^{n-1}_x)(x,t; y)u^n_{0-1}(y)dy$$

$$+ \int_0^t \int_{-\infty}^{+\infty} (H^{n-1}_x - H^{n-1}_y + \bar{G}^{n-1}_x)(x,t-s; y) \times \delta^{n-1}(\delta^{n-1}, \bar{\delta}^{n-1})(y,s)dyds$$

$$- \int_0^t \int_{-\infty}^{+\infty} H^{n-1}_x(x,t-s; y)S^{\delta^n}_{gy}(\delta^{n-1}, \bar{\delta}^{n-1})(y,s)dyds$$

$$- \int_{t-1}^t \int_{-\infty}^{+\infty} \bar{G}^{n-1}_x(x,t-s; y) \times (Q^{\delta^n-1}(u^n, u^n_x) + R^{\delta^n-1}(\delta^{n-1}, u^n))(y,s)dyds$$

and applying Lemmas 4.1–4.5 to (6.6), (8.16), and (6.12), we thus obtain (8.7) as claimed.

Likewise, we obtain easily (8.11) from (6.8) and (8.9), using Lemmas 4.1–4.5 and the definitions of $\zeta$ and $\| |_{B_1}$.

Thus, it remains only to establish (8.12). This is more delicate, due to the appearance of $M$ in the denominator of the righthand side, and depends on the key fact that estimate $\tilde{\delta}^n$ of the asymptotic shock location is to linear order insensitive to the initial guess $\delta^n$. To see this, decompose the
expression (6.9) for $\delta_*^n$ into linear and nonlinear parts

$$I := \delta_*^{n-1} - \int_{-\infty}^{+\infty} e^{n-1}(y, +\infty)u_0^{n-1}(y)dy$$

$$= \left(\delta_*^{n-1} - \int_{-\infty}^{+\infty} e|\delta_*=0(y, +\infty)(\bar{u} - \bar{u}^{\delta_*^{n-1}})(y)dy\right)$$

(8.17)

$$- \int_{-\infty}^{+\infty} e|\delta_*=0(y, +\infty)(\bar{u}_0 - \bar{u})(y)dy$$

$$- \int_{-\infty}^{+\infty} (e^{n-1} - e|\delta_*=0)(y, +\infty)u_0^{n-1}(y)dy$$

$$=: I_a + I_b + I_c$$

and

$$II := -\int_0^{+\infty} \int_{-\infty}^{+\infty} e^{n-1}(y, +\infty)S^{\delta_*^{n-1}}(\delta^{n-1}, \dot{\delta}^{n-1})(y, s)dyds$$

(8.18)

$$- \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{n-1}(y, +\infty)$$

$$\times \left(Q^{\delta_*^{n-1}}(u^n, u^n_x) + R^{\delta_*^{n-1}}(\delta^{n-1}, u^n_x)(y, s)dyds,\right.$$ 

respectively.

By estimates like the previous ones, we readily obtain

$$|II| \leq 2c(2CE_0)^2,$$

which is $<< CE_0/4M$ for $E_0$ sufficiently small. Likewise, $|I_c| \leq c|\delta_*|E_0$, by (8.9), (3.15), and the Mean Value Theorem, hence is $<< CE_0/4M$ for $E_0$ sufficiently small (recall that we assume $|\delta_*| \leq 2CE_0$), and

$$|I_b| \leq c\|\bar{u}_0 - \bar{u}\|_{L^1}$$

$$\leq c_2\|(1 + |x|^2)^{-3/4}(\bar{u}_0 - \bar{u})\|_{H^4}$$

$$\leq c_2E_0,$$

(8.19)

hence is $<< CE_0/4M$ for $C > 0$ sufficiently large.

Finally, Taylor expanding, and recalling (2.4) and (3.13), we obtain

$$I_a = \delta_*^{n-1} - \delta_*^{n-1} \int_{-\infty}^{+\infty} e|\delta_*=0(y, +\infty)(\partial \bar{u}^{\delta_*}/\partial \delta_*)|\delta_*=0(y)dy$$

(8.20)

$$+ O(|\delta_*|^2)$$

$$= O(|\delta_*|^2),$$

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which is also \( \ll CE_0/4M \) for \( E_0 \) sufficiently small (recall that we assume \(|\delta_*| \leq 2CE_0|\). Summing, we obtain (8.12) for \( E_0 \) sufficiently small and \( C > 0 \) sufficiently large, as claimed. This completes the proof.

\[ \Box \]

**Proof of Theorem 1.4.** Define now

\[(8.21)\]

\[ |(i,j)_*| := |i|_{B_1} + M|j|, \quad (i,j) \in B_1 \times \mathbb{R}. \]

By Lemma 8.2, for \( M > 0 \) sufficiently large, and \( E_0 := \|(1 + |x|^2)^{-3/4} (\tilde{u}_0 - \bar{u})\|_{H^5} \) and \( \bar{r} > 0 \) sufficiently small, \( T = (T_{\delta}, T_{\delta^*}) \) is a well-defined mapping from \( B(0, \bar{r}) \subset B_1 \times \mathbb{R} \rightarrow B_1 \times \mathbb{R} \).

To establish the theorem, therefore, it suffices to establish that \( T \) is a contraction on \( B(0, \bar{r}) \) in the norm \( |\cdot|_* \). For, then, applying Contraction Mapping Theorem, we find that \( (T_{\delta}, T_{\delta^*})(\delta, \delta^*) = (\delta, \delta^*) \) has a unique solution \((\delta^n, \delta^*_n) \in B(0, \bar{r}) \subset B_1 \times \mathbb{R}, \) for which the associated \((u^n, \delta^n, \delta^*_n) \) by Lemma 6.1 satisfy \( u^n = \hat{u} - \bar{u} \delta_n \delta^*_n \) with \( \hat{u} \) a solution of (1.2) with initial data \( \tilde{u}_0 \), and the stated decay estimates follow by (8.5) and (8.6).

That \( T \) is a contraction follows, provided we can establish on \( B(0, \bar{r}) \) the Lipschitz bounds

\[(8.22)\]

\[ |T(\delta, \delta^*) - T(\hat{\delta}, \hat{\delta}^*)|_* \leq \alpha |(\delta, \delta^*) - (\hat{\delta}, \hat{\delta}^*)|_* \]

for some \( \alpha < 1 \).

Letting \((u^n, \delta^n, \delta^*_n) \) satisfy (6.6)–(6.9) for \( \delta^{n-1}, \delta_*^{n-1} \), and \((\tilde{u}^n, \hat{\delta}^n, \hat{\delta}^*_n) \) satisfy (6.6)–(6.9) with \( \delta^{n-1}, \delta_*^{n-1} \) replaced by \( \hat{\delta}^{n-1}, \hat{\delta}^*_1, \), define variations

\[(8.23)\]

\[ \Delta u^n := \hat{u}^n - u^n, \quad \Delta \delta^n := \hat{\delta}^n - \delta^n, \quad \Delta \delta^*_n := \hat{\delta}^*_* - \delta^*_n \]

and

\[(8.24)\]

\[ \Delta \delta^{n-1} := \hat{\delta}^{n-1} - \delta^{n-1}, \quad \Delta \delta_*^{n-1} := \hat{\delta}^*_* - \delta^*_*^{n-1}. \]

Likewise, define \( \Delta \tilde{G}^{n-1}, \Delta H^{n-1}, \Delta e^{n-1} \) in the obvious way.
**Differential variational equation.** From (6.11), we find after a brief calculation that \( \Delta \tilde{u}^n \) defined by

\[
\Delta u^n = \Delta \tilde{u}^n - \left( \tilde{u}^{\delta_n^{-1} + \delta_n^{-1}}(t) - \tilde{u}^{\delta_n^{-1} + \delta_n^{-1}}(t) \right)
\]

satisfies the variational equations associated with

\[
(8.25) \quad \tilde{u}_t^n + f(\tilde{u}^n)_x - (B(\tilde{u}^n)\tilde{u}^n_x)_x = \left( \delta^n(t) - \delta_n^{-1}(t) \right) \frac{\partial \tilde{u}^\delta}{\partial \delta}|_{\delta_n^{-1}},
\]

from which we obtain by an energy estimate similar to that of Lemma 5.1 and the observation

\[
|\Delta u^n(0)|^2_{H^4} = |\tilde{u}^{\delta_n^{-1}} - \tilde{u}^{\delta_n^{-1}}|_{H^4}^2 \leq C|\Delta \delta_n^{-1}|^2
\]

the bound

\[
|\Delta u^n(t)|^2_{H^4} \leq Ce^{-\theta t}|\Delta u^n(0)|^2_{H^4}
\]

\[
+ C \int_0^t e^{-\theta(t-\tau)} \left( |\Delta u^n|_{L^2}^2 + \max\{|\Delta \delta^n|^2, |\Delta \delta_n^{-1}|^2\} \right) d\tau
\]

\[
+ |\Delta \delta_n^{-1}|^2 \max\{|\delta^n|^2, |\delta_n^{-1}|^2\}(\tau) \right) d\tau
\]

\[
(8.26) \quad \leq C \int_0^t e^{-\theta(t-\tau)} \left( |\Delta u^n|_{L^2}^2 + |\Delta \delta^n|^2 \right) d\tau
\]

\[
+ C \left( |\Delta \delta_n^{-1}, \Delta \delta_n^{-1}|_{1/2}^2(1 + t)^{-1/2}, \right)
\]

provided \( r \) (and so \( \sup \|\tilde{u}^n\|_{H^5} \) and \( \sup \|u^n\|_{H^5} \)) is sufficiently small, so long as \( \|\Delta u^n\|_{H^4} \) remains sufficiently small. We omit the (standard) details.

**Integral variational equations.** Applying the quadratic Leibnitz formula
\( \Delta (fg) = f \Delta g + \Delta fg \), we obtain

\[
\Delta u^n(x, t) = \int_{-\infty}^{+\infty} (\Delta H^{n-1} + \Delta \tilde{G}^{n-1})(x, t; y) u_0^{n-1}(y) dy \\
+ \int_{-\infty}^{+\infty} (H^{n-1} + \tilde{G}^{n-1})(x, t; y) \Delta u_0^{n-1}(y) dy \\
+ \int_0^t \int_{-\infty}^{+\infty} (\Delta H^{n-1} + \Delta \tilde{G}^{n-1})(x, t - s; y) \\
\times S^{\delta_{n-1}}(\delta^{n-1}, \delta^{n-1})(y, s) dy ds \\
+ \int_0^t \int_{-\infty}^{+\infty} (H^{n-1} + \tilde{G}^{n-1})(x, t - s; y) \\
\times \Delta S(y, s) dy ds \\
+ \int_0^t \int_{-\infty}^{+\infty} \tilde{G}^{n-1}(x, t - s; y) \\
\times (\Delta Q + \Delta R)(y, s) dy ds \\
- \int_0^t \int_{-\infty}^{+\infty} \Delta \tilde{G}^{n-1}(x, t - s; y) \\
\times (Q^{\delta_{n-1}}(u^n, u^n_x) + R^{\delta_{n-1}}(\delta_{n-1}, u^n))(y, s) dy ds \\
+ \int_0^t \int_{-\infty}^{+\infty} \Delta H^{n-1}(x, t - s; y) \\
\times (Q^{\delta_{n-1}}(u^n, u^n_x) + R^{\delta_{n-1}}(\delta_{n-1}, u^n))(y, s) dy ds \\
+ \int_0^t \int_{-\infty}^{+\infty} H^{n-1}(x, t - s; y) \\
\times (\Delta Q + \Delta R)(y, s) dy ds,
\]
\[
\Delta \dot{\delta}^n(t) = - \int_{-\infty}^{\infty} \Delta e_t^{n-1}(y, t) u_0^{n-1}(y) \, dy \\
- \int_{-\infty}^{\infty} e_t^{n-1}(y, t) \Delta u_0^{n-1}(y) \, dy \\
- \int_{0}^{t} \int_{-\infty}^{+\infty} \Delta e_t^{n-1}(y, t - s) S^{\delta^{n-1}_y, \dot{\delta}^{n-1}}(y, s) \, dy \, ds \\
- \int_{0}^{t} \int_{-\infty}^{+\infty} e_t^{n-1}(y, t - s) \Delta S(y, s) \, dy \, ds \\
+ \int_{0}^{t} \int_{-\infty}^{+\infty} \Delta e^{n-1}_y(y, t - s) \\
\times (Q^{\delta^{n-1}_y, u^n} + R^{\delta^{n-1}_y, u^n})(y, s) \, dy \, ds, \\
+ \int_{0}^{t} \int_{-\infty}^{+\infty} e^{n-1}_y(y, t - s) \\
\times (\Delta Q + \Delta R)(y, s) \, dy \, ds,
\]

(8.28)

\[
\Delta \delta^{n}_y := \Delta \delta^{n-1}_y + \int_{-\infty}^{+\infty} \Delta e^{n-1}(y, +\infty) u_0^{n-1}(y) \, dy \\
+ \int_{-\infty}^{+\infty} e^{n-1}(y, +\infty) \Delta u_0^{n-1}(y) \, dy \\
+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Delta e^{n-1}(y, +\infty) S^{\delta^{n-1}_y, \dot{\delta}^{n-1}}(y, s) \, dy \, ds \\
+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{n-1}(y, +\infty) \Delta S(y, s) \, dy \, ds \\
- \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \Delta e_{y}^{n-1}(y, +\infty) \\
\times (Q^{\delta^{n-1}_y, u^n} + R^{\delta^{n-1}_y, u^n})(y, s) \, dy \, ds \\
- \int_{0}^{+\infty} \int_{-\infty}^{+\infty} e_{y}^{n-1}(y, +\infty) \\
\times (\Delta Q + \Delta R)(y, s) \, dy \, ds,
\]

(8.29)
and similarly for $\Delta \delta^n$, where

$$\Delta u_0^{n-1} = \bar{u}\delta^{n-1} - \bar{u}\hat{\delta}^{n-1}$$

(8.30)

$$= \left(\frac{\partial \bar{u}\delta}{\partial \bar{\delta}}\right)|_{\bar{\delta} = \delta^{n-1}} \Delta \delta^n - O(|\Delta \delta^n| e^{-|x|})$$

$$= O(|\Delta \delta^n| e^{-|x|}).$$

Now define

$$\xi(t) := \sup_{x, 0 \leq s \leq t} \left( (|\Delta u^n| + |\Delta u^n|)(\theta + \psi_1 + \psi_2)^{-1}(x, s) + |\Delta \delta^n(s)(1 + s)| \right).$$

(8.31)

Let $r' := |(\Delta \delta^n - \Delta \delta_1^n)|_s = |\Delta \delta^n - \Delta \delta_1^n|_{H^1} + M|\Delta \delta^n - \Delta \delta_1^n|$, $r'$ sufficiently small. From (8.26) and smallness of $|u^n|_{H^5}$ and $|\hat{u}^n|_{H^5}$, and the fact that $r, r' << 1$, we obtain

$$|\Delta u^n(t)|_{H^4} \leq C(r' + \xi(t))(1 + t)^{-\frac{2}{3}},$$

(8.32)

which gives us a bound on $L^\infty$-norm of $\Delta u^n_{xxx}$, providing us, therefore, with the bounds,

$$|\Delta Q + \Delta R| \leq C(r\xi(t) + rr')(\Psi + \Phi_1),$$

(8.33)

$$|\Delta Q_y + \Delta R_y|, |\Delta Q_{yy} + \Delta R_{yy}| \leq C(r\xi(t) + rr')\Upsilon,$$

$$|\Delta S|, |\Delta S_y| \leq C(r\xi(t) + rr')\Phi_2,$$

Also, (8.14) and (8.15) hold with $c(\zeta^2 + 4C^2E_0^2)$ replaced with $Cr^2$. We use (3.14), (3.15) and (3.16) to obtain

$$\Delta e^n \sim e\delta^n \leq r'e,$$

(8.34)

and similar appropriate bounds for $\Delta H^n, \Delta G^n$ and their derivatives (of course, $e$ in (8.34) is defined at a point between $\delta^n_1$ and $\hat{\delta}^{n-1}$). Next, using lemmas 4.1–4.5 in a procedure parallel to the one used in the proof of Lemma 8.2, we obtain

$$\xi(t) \leq C(r' + r\xi(t)),$$

from which we conclude that

$$\xi(t) \leq \frac{Cr'}{1 - Cr}.$$
with constant $C$ independent of $r$ and $r'$. Now, replacing $\xi$ in (8.33) with this bound, we plug back the result into (8.28) and into the similar formula for $\Delta \delta^n$. Notice that, with the exception of the first two terms, the other term in (8.28) have quadratic terms in their source term, so giving us small enough bounds. Hence, using one again lemmas 4.1–4.5, we obtain

\begin{equation}
|\Delta \delta^n| \leq (C E_0 r' + \frac{C}{M} r' + C r' r)(1 + t)^{-1},
\end{equation}

of which the two first terms in the right hand side come from the first two terms of (8.28). Similarly we obtain

\begin{equation}
|\Delta \delta^n| \leq (C E_0 + \frac{C}{M} + C r)r'(1 + t)^{-\frac{1}{2}}
\end{equation}

We notice that $(C E_0 + \frac{C}{M} + C r)$ can be made arbitrarily small, provided that $E_0$, $r$ are small enough and $M$ is large enough. Next, we use (8.29) to bound $\Delta \delta^{n}_\ast$, using basically the same method used in (8.17)–(8.20), and therefore obtaining

\begin{equation}
M|\Delta \delta^n_\ast| \leq (C E_0 + C r)r'.
\end{equation}

This, together with (8.35) and (8.36), gives us (8.22) with $\alpha < 1$, finishing the proof of the (main) Theorem 1.4.

\[ \Box \]

**Remark 8.3.** In order to control the $H^4$ norm of the variational problem, as in (8.26), we indeed need regularity $C^5$ for the coefficients in hypothesis (H0), since one derivative is lost in variational energy estimate (5.3).

**Remark 8.4.** HERE

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