The Relevant Scale Parameter in the High Temperature Phase of QCD

Suzhou Huang\(^{(1,2)}\) and Marcello Lissia\(^{(1,3)}\)

\(^{(1)}\)Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
\(^{(2)}\)Department of Physics, FM-15, University of Washington, Seattle, Washington 98195
\(^{(3)}\)Istituto Nazionale di Fisica Nucleare, via Negri 18, I-09127 Cagliari, Italy and Dipartimento di Fisica dell’Università di Cagliari, I-09124 Cagliari, Italy

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Abstract

We introduce the running coupling constant of QCD in the high temperature phase, \(\tilde{g}^2(T)\), through a renormalization scheme where the dimensional reduction is optimal at the one-loop level. We then calculate the relevant scale parameter, \(\Lambda_T\), which characterizes the running of \(\tilde{g}^2(T)\) with \(T\), using the background field method in the static sector. It is found that \(\frac{\Lambda_T}{\Lambda_{\overline{\text{MS}}}} = e^{(\gamma_E+1)/22}/(4\pi) \approx 0.148\). We further verify that the coupling \(\tilde{g}^2(T)\) is also optimal for lattice perturbative calculations. Our result naturally explains why the high temperature limit of QCD sets in at temperatures as low as a few times the critical temperature. In addition, our \(\Lambda_T\) agrees remarkably well with the scale parameter determined from the lattice measurement of the spatial string tension of the SU(2) gauge theory at high \(T\).
I. INTRODUCTION

At high temperatures QCD is expected to undergo a partial dimensional reduction \[1,2\], namely static correlations at distances larger than the thermal wavelength \((1/T)\) can be reproduced by a three dimensional Lagrangian, where only the static modes of the original theory are present. This reduced Lagrangian can be computed perturbatively up to a specific order in the QCD running coupling constant. In fact, non-perturbative infrared phenomena (e.g. thermo-mass generation) prevent the complete reduction, i.e. reduction to all orders in the QCD running coupling, from taking place \[2\]. Consequently, observables can be reproduced only up to corrections of a specific order, before non-perturbative physics begins to dominate.

Even though a complete dimensional reduction is not possible, the partial dimensional reduction of QCD still provides a simplified physical picture. However, phenomenological applications of this picture depends crucially on how high is the temperature above which this picture begins to take place. Since we expect corrections to vanish with some power of the QCD coupling, and since at zero temperature the asymptotic freedom starts dominating QCD physics at typical scales of about 10 to 20 times \(\Lambda_{\text{MS}}\), one might anticipate that the reduced theory should become effective only for \(T \gg \Lambda_{\text{MS}}\).

Contrary to this expectation, there are strong evidences \[3–5\] that the dimensional reduction picture is already valid at temperatures as low as two or three times the critical temperature \(T_c\) (the deconfining transition in the pure Yang-Mills case or the chiral restoration in full QCD). Since \(T_c\) is numerically not very different from \(\Lambda_{\text{MS}}\), and considering that the QCD coupling constant only runs logarithmically, it is very surprising to find that the high-\(T\) regime of QCD starts at temperatures this low.

This apparent puzzle can be solved with the observation that the effective scale parameter for the reduced theory, \(\Lambda_T\), is renormalized after integrating out the non-static modes and becomes drastically smaller than \(\Lambda_{\text{MS}}\). In fact, the definition of a scale parameter that characterizes the approach to the dimensional reduction regime implies the definition of a suitable coupling constant, \(\tilde{g}^2(T)\), that yields a sensible perturbative expansion at high temperature, i.e. an expansion whose coefficients contain minimal contribution from non-static modes.

In this paper, we use the background field approach to define and compute the relevant coupling constant, and hence the scale parameter \(\Lambda_T\). More specifically, we calculate the one-loop effective action for the background field in the static sector, and define the renormalization scheme by requiring that dimensional reduction be optimal for this gauge-invariant quantity. Furthermore, we verify by an explicit computation that this same renormalization scheme is also optimal for lattice perturbative calculations at high \(T\), and therefore it provides a natural scale also for lattice simulations.

In section II we introduce the renormalization scheme that defines the scale parameter \(\Lambda_T\) within the background field approach. In section III, we apply this definition and calculate \(\Lambda_T/\Lambda_{\text{MS}}\). First we perform the calculation for the SU\((N)\) gauge theory in the continuum using dimensional regularization; the effect of light quarks is also considered. Then we repeat the calculation for the pure SU\((N)\) gauge theory on the lattice in the Wilson formulation. In section IV, we compare our result with numerical determinations of \(\Lambda_T\) from a lattice measurement of the spatial string tension at high \(T\). Section V is reserved for
the conclusions. Several technical points pertinent to the lattice perturbative calculation at high \( T \) are discussed in the Appendix.

II. DIMENSIONAL REDUCTION AND OPTIMAL RENORMALIZATION SCHEME

The standard \( SU(N) \) Yang-Mills gauge theory reduces at the tree level to the three dimensional Yang-Mills theory with adjoint Higgs \( (\phi^a \equiv Q^0_a) \)

\[
\mathcal{L}_{\text{RD}} = -\frac{1}{4} F^a_{ij} F^a_{ij} - \frac{1}{2} (D_i \phi)^a (D^i \phi)^a , \tag{1}
\]

where \( F^a_{ij} = \partial_i Q^a_j - \partial_j Q^a_i - g_3 f^{abc} Q^b_i Q^c_j \) and \( (D_i \phi)^a = \partial_i \phi^a - g_3 f^{abc} Q^b_i \phi^c \). The coupling \( g_3 \) is related to the four dimensional coupling through \( g_3^2 = g_2^2 T \). Since \( \mathcal{L}_{\text{RD}} \) is a super-renormalizable theory in three dimensions and there is no other dimensionful scale around, all the dynamical scales must be set by the coupling constant \( g_3^2 = g^2 T \).

Of course, once loop corrections are included the reduced theory in Eq. (1) would acquire new vertices and the coupling constant \( g_3^2 \) would depend on the original coupling \( g_2^2 \) in a more complicated way. For example, \( g_3^2 \) would receive corrections, such as \( g^4 T \) and so on. However, due to the asymptotic freedom of QCD \( (g^2 \sim 1/\ln T) \) we still expect that dynamical scales are set by \( g_3^2 \approx g^2 T \), provided the scale parameter is chosen in a suitable way.

Therefore, we believe that the concept of dimensional reduction involves two equally important aspects. On one hand, there is the possibility of a simplified description by using a theory \( \mathcal{L}_{\text{RD}} \) with less degrees of freedom in lower dimensions. On the other hand, the evolution of the parameters of \( \mathcal{L}_{\text{RD}} \) as a function of temperature should be dictated by the original theory. The main concern of our present work is to determine this evolution, which in turn determines the temperature dependence of the relevant physical observables.

A. Background Field Method in the Static Sector

It is well known that the effective action calculated using the background field method \( [6] \) is gauge invariant for the background gauge field at \( T = 0 \). This gauge invariance guarantees that the coupling constant renormalization is related to the wavefunction renormalization of the background field through \( Z_g = Z_A^{-1/2} \). Hence, the calculation of the quadratic part of the effective action, i.e. the two-point function for the background field, is sufficient to renormalize the coupling \( g_3^2 \).

Moreover, to the leading order, there is no magnetic mass generation at finite \( T \). Therefore, the one-loop effective action for the magnetic sector is invariant under time-independent gauge transformations also at finite \( T \), insuring that the relation \( Z_g = Z_A^{-1/2} \) still holds for the static background field

\[
A^0_0(\tau, \mathbf{x}) = 0, \quad A^0_i(\tau, \mathbf{x}) = A^0_i(\mathbf{x}) . \tag{2}
\]

The same conclusion can also be reached more formally by applying, for instance, the methods of Ref. \( [6] \) to the background field of Eq. (2). The residual gauge invariance in the static
sector implies that, in order to compute the coupling constant renormalization at finite $T$, we only need to compute the two-point function of the background field $A^a_i$ in the static sector.

We can still use the zero temperature Feynman rules, as given for instance by Abbott [6]. The only difference in the calculation is that time-components of all momenta become discrete Matsubara frequencies $(2\pi n T)$, and the corresponding integrals become discrete sums.

### B. Subtraction Scale

As exhaustively discussed by Landsman [2], the decoupling of the non-zero modes at high-$T$ is maximal only in some specific renormalization schemes, such as the BPHZ scheme. In the background field method we only need to fix one renormalization condition: we demand that the two-point function for the background field in the low external momentum (relative to $T$) limit coincides with the contribution solely from zero modes.

Landsman [2] uses a finite temperature renormalization group approach, since he discusses thermal reduction in a more general context where several couplings are present. Thanks also to the background field approach, we deal with a simpler situation where only one coupling needs renormalization.

Therefore, we can directly implement the renormalization condition by using the freedom in the choice of the subtraction scale, $\mu$, which becomes a function of $T$. Intuitively, we expect $\mu$ to be of order of $T$. The purpose of our paper is to find out what is the proportionality constant.

Then the reduced theory, Eq. (1), with the $T$-dependent coupling $g_3^2 = g^2(\mu(T)) T$, reproduces the full two-point function up to corrections of order of $p^2/T^2$ at the one-loop level. Due to the gauge invariance, the two-point function for the static background field $A^a_i$ must have the form

$$
(\delta_{ij} p^2 - p_i p_j) \delta_{ab} \Pi_M(p^2, T, \mu). 
$$

Specifically, we choose $\mu$ by requiring the following renormalization condition for the non-static contribution to $\Pi_M(p^2, T, \mu)$:

$$
\Pi^M_{NS}(p^2 = 0, T, \mu(T)) = 0. 
$$

The procedure is best explained by directly going through the calculation in the next section.

### III. CALCULATION OF THE SCALE PARAMETER

#### A. In the Continuum

In the continuum calculation we use dimensional regularization in the spatial dimensions, that is

$$
\int \frac{d^4 k}{(2\pi)^4} \rightarrow T \sum_{n=-\infty}^{\infty} \mu^{2\epsilon} \int \frac{d^{3-2\epsilon} k}{(2\pi)^{3-2\epsilon}},
$$
and the $\overline{\text{MS}}$ subtraction scheme.

At the one-loop level there are four graphs that contribute to $\Pi_M$ in the full theory: the bubble and tadpole graphs for both the quantum gauge fields and the ghost fields. The resulting total contribution is

$$\Pi_M(p^2, T, \mu) = \frac{1}{g^2(\mu)} - \left[\frac{21}{64} + \frac{3}{32} \alpha + \frac{1}{64} \alpha^2\right] \frac{NT}{\sqrt{p^2}}$$

$$- \beta_0 \left[ \ln(\mu^2/T^2) + 2\gamma_E - 2 \ln 4\pi + \frac{1}{11} \right] + O(p^2/T^2),$$

where $g^2(\mu)$ is the running coupling defined in the $\overline{\text{MS}}$ scheme, $\beta_0 = 11N/(48\pi^2)$, $\alpha$ is the gauge parameter, and $\gamma_E$ is the Euler constant.

The first term in Eq. (6) is obviously the classical contribution. The second term is the contribution of the static modes, and one can easily check that it can be reproduced by the reduced theory, Eq. (1), with coupling constant $g_3^2 = g^2(\mu)T$ and the same gauge parameter. The third term is the one that must be eliminated according to our renormalization prescription, which accomplishes maximal decoupling \[2\]. We obtain this result by choosing

$$\mu(T) = 4\pi T e^{-(\gamma_E + c)},$$

where $c = 1/22$. It is very reassuring to find that this optimal choice of the subtraction scale $\mu$ is independent of the gauge parameter $\alpha$. The remaining contributions in Eq. (6) are suppressed by powers of $1/T^2$.

In summary, to achieve maximal decoupling and hence the optimal dimensional reduction, the effective coupling in the reduced theory must be

$$\tilde{g}^2(T) \equiv \frac{1}{\beta_0 \ln(T^2/\Lambda_T^2)} = g^2(\mu) \bigg|_{\mu=4\pi T e^{-(\gamma_E + c)}},$$

which defines the scale parameter

$$\Lambda_T = \frac{e^{(\gamma_E + c)}}{4\pi} \Lambda_{\overline{\text{MS}}}.$$

This result has a clear physical interpretation. The non-static modes decouple in the high-$T$ limit, but their presence is nevertheless revealed by the appearance of the new scale $\Lambda_T$ in the reduced theory (without any reference to the original theory, the only scale would be $T$). While this new scale is obviously related to the scale $\Lambda_{\overline{\text{MS}}}$ that governs the full theory at zero temperature, the two scales do not coincide. Only with the coupling constant of Eq. (8), whose temperature evolution is set by the scale $\Lambda_T$ in Eq. (9), the reduced theory is capable of reproducing the full four-dimensional one-loop corrections up to terms suppressed by $1/T^2$. Incidentally, it is interesting to notice that, if one intuitively identifies $2\pi T$ (rather than $T$) as the relevant frequency unit, one gets an answer numerically close to the right one in Eq. (7).

At this point, we must point out that the optimal perturbative dimensional reduction criterion alone does not uniquely determine the scale $\Lambda_T$. In general, $\Lambda_T$ also depends on the specific Green’s functions for which we demand optimal reduction. The use of a different
process, represented by a different set of Feynman graphs, yields a different $c$ in Eq. (9). However, we believe that typically $|c| \lesssim 1$, and a different choice should not modify the scale ratio in Eq. (9) in an essential way. For example, Landsman [2] calculated the temperature dependent coupling renormalization factor $Z_g$ by imposing maximal dimensional reduction on the two- and three-point functions in the conventional effective action (where the relationship $Z_g = Z^{-1/2}_A$ no longer holds). He did not express his result explicitly in terms of the scale ratio. But if we do it, we find that his result is quite close to ours, i.e. Eq. (9) with $c = 0$.

Another example that clearly shows the necessity of using an optimal dimensional reduction scheme for defining the relevant scale at high-$T$ can be found in the Gross-Neveu model [7]. In that model a similar strategy makes the sub-leading correction to the screening mass of order of $\tilde{g}^6(T)$, rather than $\tilde{g}^4(T)$, demonstrating that $\tilde{g}^2(T)$ is a sensible expansion parameter.

Of course, the optimal dimensional reduction criterion is not the only way to define a temperature dependent coupling constant. For example, the quark-antiquark potential at a distance of order of $1/T$ is used to define $\tilde{g}^2(T)$ in Ref. [3]. While it is certainly legitimate to make such a choice, it is also true that, because the reduced theory is meant to reproduce the full theory only at distances much larger than $1/T$ (spatial momenta small compared to $T$), definitions of the couplings made by matching short distance properties of the full and reduced theories do not necessarily define a scale that correctly characterizes the approach to the asymptotic high-$T$ regime.

At last, let us consider the effect of quarks on our result. If $N_f$ light quarks are present in the theory, results of Eqs. (7), (8) and (9) still apply, but with $\beta_0 = (11N - 2N_f)/(48\pi^2)$ and $c = (N/2 - 2N_f\ln 4)/(11N - 2N_f)$, where we have adopted the convention for the trace of the Dirac-matrix: $\text{Tr}\gamma_\mu\gamma_\nu = -4\delta_{\mu\nu}$.

For the phenomenologically relevant case of $N = N_f = 3$, we get the value $c = -0.2525$, which corresponds to $\Lambda_T/\Lambda_{\overline{MS}} \approx 0.110$. Therefore, additional flavors further decrease the scale ratio until $N_f > 16$ (for $N = 3$), where asymptotic freedom is lost.

### B. On the Lattice

The determination of the scale parameter $\Lambda_T$ that governs the temperature dependence of the coupling in the reduced theory only involves an one-loop calculation. Nevertheless, the reduced theory is in general still non-perturbative, and non-perturbative methods are necessary to extract information from it. The standard approach is of course the lattice formulation.

It should be clear from its definition (see section II and Ref. [4]) that the concept of maximal decoupling scheme, along with the associated scale parameter, is independent of how the theory is regularized, and we expect the same $\Lambda_T$ on the lattice, as long as we use the same renormalization condition, Eq. (4).

On the other hand, since the lattice theory is usually defined in terms of the bare lattice coupling, $g_0(a)$, without introducing additional renormalization scale other than the lattice constant $a$, it is quite interesting to see with an explicit calculation how this scale emerges on the lattice at high $T$. In this respect, there are close analogies between our choice of $\tilde{g}^2(T)$ as...
a suitable expansion parameter for lattice perturbative calculation at high temperature, and
the necessity of using expansion parameters different from the bare lattice coupling \(g_0^2(a)\)
for perturbative calculations at zero temperature [8].

In the following we verify that optimal dimensional reduction for the lattice effective
action computed in the background field method defines indeed the same scale parameter
we have found in the continuum calculation. For the sake of concreteness, we perform the
calculation for the pure SU(\(N\)) Wilson action, but the same result is expected to hold for
other actions as well.

In general, the coupling defined in the lattice background field method should have the
following dependence on the bare lattice coupling up to one-loop

\[
g_L^2(T) \equiv g_0^2(a) + g_0^4(a) \beta_0 \left[ -\ln(a^2 T^2) + c_L^T \right]. \tag{10}
\]

We want to show that \(c_L^T\) is such that \(g_L^2(T) = \tilde{g}^2(T)\). Since we have expressed \(\tilde{g}^2(T)\) in
terms of \(g^2(\mu)\) in the \(\overline{\text{MS}}\) scheme, see Eq. (8), we use the known relation between
\(g_0^2(a)\) and \(g^2(\mu)\) in the \(\overline{\text{MS}}\) scheme [9,10]

\[
g_0^2(a) = g^2(\mu) - g^4(\mu) \beta_0 \left[ -\ln(\mu^2 a^2) + c_0^L \right], \tag{11}
\]

and express also \(g_L^2(T)\) in Eq. (10) in terms of \(g^2(\mu)\)

\[
g_L^2(T) = g^2(\mu) - g^4(\mu) \beta_0 \left[ -\ln(\mu^2 / T^2) - c_L^T + c_0^L \right] + O(g^6(\mu)). \tag{12}
\]

By comparing Eq. (8) and Eq. (12) we see that to show \(g_L^2(T) = \tilde{g}^2(T)\) is equivalent to show that

\[
c_L^T = c_0^L + 2 \gamma_E - 2 \ln(4\pi) + \frac{1}{11}, \tag{13}
\]

where the explicit expression of \(c_0^L\), which has been calculated by several authors [9,10], is

\[
c_0^L = \frac{1}{11} \left( -11 \gamma_E + 2 f_{11} + 3 f_{00} + 6 f_{10} - 1 + 24 \pi^2 z_{10} + 6 \pi^2 - 6 \pi^2 / N^2 \right). \tag{14}
\]

The constants \(f_{ij}\) and \(z_{ij}\) are defined as

\[
f_{ij} \equiv (4\pi)^2 \int_0^\infty dx \left[ e^{-8x I_0^2(2x) I_i(2x) I_j(2x)} - \frac{\theta(x - 1)}{(4\pi x)^2} \right] \tag{15}
\]

and

\[
z_{ij} \equiv \int_0^\infty dx e^{-8x I_0^2(2x) I_i(2x) I_j(2x)}. \tag{16}
\]

Here and in the following we have closely followed the notation of Ref. [10].

Since most of the calculation of the finite constant \(c_L^T\) is closely parallel to the calculation
of \(c_0^L\), we only report the final result. In the Appendix, however, we illustrate the only new
ingredient that is not a trivial extension of the calculation of \(c_0^L\): the high temperature
expansion on the lattice. The lattice correspondent of the continuum result of Eq. (6) in the Feynman gauge ($\alpha = 1$) is

$$\Pi_M^L(p^2, T, a) = \frac{1}{g_0^2(a)} - \frac{7}{16} \frac{NT}{\sqrt{p^2}} \beta_0 \left[ - \ln(a^2 T^2) + c_L^T \right] + O(p^2/T^2, a|p|, aT), \quad (17)$$

with $c_L^T$ given by

$$c_L^T = \frac{1}{11} \left( 22\gamma_E + 11 \ln(4/\pi^2) + 2f_{11}' + 3f_{00}' + 6f_{10}' + 24\pi^2 z_{10} + 6\pi^2 - 6\pi^2/N^2 \right), \quad (18)$$

and

$$f_{ij}' \equiv (4\pi)^2 \int_0^\infty dx x e^{-2x} I_0(2x) e^{-6x} I_0(2x) I_i(2x) I_j(2x) - \frac{1}{(4\pi x)^{3/2}}. \quad (19)$$

Since, as shown in the Appendix, $f_{ij}' = f_{ij} - \gamma_E - 3 \ln 4$, this complete the proof of Eq. (13) and, therefore, of the fact that $g_2^L(T) = \tilde{g}^2(T)$.

In other words, if we use $\tilde{g}^2(T)$ as the expansion parameter, the lattice effective action in the high-$T$ limit takes the following form

$$\Pi_M^L(p^2, T, a) = \frac{1}{\tilde{g}^2(T)} - \frac{7}{16} \frac{NT}{\sqrt{p^2}} + O(p^2/T^2, a|p|, aT), \quad (20)$$

which is the same as its continuum counterpart, if we use the same coupling $\tilde{g}^2(T)$ (see Eq. (6) with $\alpha = 1$ and $\mu$ given by Eq. (7) ). In both cases we have been able to absorb in the coupling constant all leading local corrections due to non-static modes, while the non-local ones are reproduced by the reduced theory.

**IV. COMPARISON TO LATTICE RESULT**

In the preceding section, we have demonstrated that $\Lambda_T$ is the relevant scale parameter in the high-$T$ limit. Our argument is yet only perturbative in nature. However, as we emphasized earlier, the determination of the scale parameter is largely one-loop effect. Now let us compare our result with the scale parameter determined from a non-perturbative method: lattice measurement of the spatial string tension at high $T$. The primary reason for choosing the spatial string tension [4] rather than the heavy quark potential at distances of order of $1/T$ [3] is that the concept of dimensional reduction only makes sense for large distance (low momentum) quantities.

Bali et al. [4] measured the spatial string tension in SU(2) gauge theory as a function of temperature $\sigma_s(T)$. Then they fitted their result to the expected form of the string tension in the three-dimensional SU(2) Yang-Mills theory

$$\sqrt{\sigma_s(T)} \propto g^2(T) T, \quad (21)$$

where the running of $g^2(T)$ with temperature is determined by the SU(2) $\beta$-function

$$g^{-2}(T) = \frac{11}{12\pi^2} \ln(T/\Lambda_T) + \frac{17}{44\pi^2} \ln \left[ 2 \ln(T/\Lambda_T) \right]. \quad (22)$$
Even though the simulation process knows nothing about the dimensional reduction, the fitting formula Eq. (21) in fact defines the optimal three-dimensional coupling $g_2^3 = g_2(T)T$ through the string tension, similar in spirit to what we have done for the background field effective action. As a result, their fitted value of $\Lambda_T^2 = (0.076 \pm 0.013)T_c$ is the first, to our knowledge, non-perturbative determination of the scale that characterizes the high-$T$ regime for the SU(2) gauge theory.

In the scaling regime we expect that the critical temperature behaves like

$$T_c = \frac{\Lambda_L}{N} \left( \frac{11N^2}{24\pi^2\beta_c} \right)^{\frac{11}{12}} \exp \left( \frac{12\pi^2}{11N^2\beta_c} \right),$$

where $\beta_c = 2N/g_0^2(a)$. From their critical coupling $\beta_c = 2.74$ at $N = 16$, and the known ratio $\Lambda_{\text{MS}}/\Lambda_L = 38.85 \exp[-3\pi^2/(11N^2)]$, it is straightforward to express $T_c$ in terms of $\Lambda_{\text{MS}}$: $T_c = 1.62\Lambda_{\text{MS}}$. Then their numerical measurement yields

$$\Lambda_T^2 = (0.123 \pm 0.021)\Lambda_{\text{MS}},$$

which is remarkably close to our result

$$\Lambda_T = e^{\gamma_E+1/22}/4\pi \Lambda_{\text{MS}} \approx 0.148\Lambda_{\text{MS}},$$

in spite of the different renormalization conditions.

The smallness of $\Lambda_T/T_c$ gives a natural explanation of why their spatial string tension already at temperatures around $2T_c$ is numerically so close to the string tension of the three dimensional SU(2) Yang-Mills gauge theory.

The result of Ref. [4] implies a minor role for the Higgs sector in the reduced theory, whereas the result of Ref. [3] seemingly implies the contrary. It would be interesting to see whether the use of an optimal coupling in calculations such as those in Ref. [3] and focusing only on long distance quantities could resolve the disagreement. For example, the gluonic Debye-screening mass, $\mu_D \sim g(T)T$, could be used to define yet another optimal coupling. It is rather unfortunate that the numerical results in Refs. [3,11] are not accurate enough to determine a meaningful $\Lambda_T$.

V. CONCLUSIONS

We have defined a temperature dependent running coupling constant $\tilde{g}^2(T)$ in the spirit of the maximal decoupling of non-static modes of Landsman [2] for SU($N$) gauge theories. More specifically, this coupling is such that the static modes reproduce the quadratic part of the one-loop effective action for the background field in the low momentum limit. Furthermore, $\tilde{g}^2(T)$ provides a meaningful expansion parameter in the high-$T$ limit, and its dependence on temperature defines a typical scale $\Lambda_T$ for the high-$T$ regime.

We have calculated this coupling constant, and the related scale parameter, first in the continuum with dimensional regularization, where we verified its independence from the gauge fixing parameter $\alpha$, and then showed that the same coupling is also optimal for the
lattice perturbative calculation at high $T$. Our results are $\Lambda_T = 0.148 \Lambda_{\overline{MS}}$ in the pure Yang-Mills or quenched cases and $\Lambda_T = 0.110 \Lambda_{\overline{MS}}$ for $N = N_f = 3$.

We have argued that this scale is typical in the high-$T$ regime, even if its precise value depends on the specific definition. The consequence of our result is that the high-$T$ regime of QCD, where the dimensional reduction picture appears to take place, sets in at temperatures as low as a few times of the critical temperature.

Our calculation is in very good agreement with the non-perturbative determination of the scale parameter in the lattice simulations \[4\] in the SU(2) Yang-Mills theory, therefore reinforcing the advantage of the renormalization scheme based on the optimal dimensional reduction criterion.

It would be of great interest to have other lattice measurements of the scale parameter using other observables, such as the ones related to the gluonic Debye-screening mass and and the deviations of the mesonic and baryonic screening masses from their free values.

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**APPENDIX:**

In this appendix we discuss several points of the high temperature expansion in perturbative lattice calculations. First we use the ghost bubble-graph to illustrate the general method, then we prove that $f'_{ij} - f_{ij} = -\gamma_E - 3 \ln 4$, and finally discuss the convergence of the frequency sums to the corresponding zero temperature integrals.

From the lattice action, see for instance Ref. \[10\], we derive the following expression for the ghost bubble-graph

\[ B_{\mu\nu}(p) = \frac{N}{4a^2 \Omega} \sum_{k} \frac{(e^{-ik_{\mu}a} - e^{i(k_{\mu}-p_{\mu})a})(e^{-i(k_{\nu}-p_{\nu})a} - e^{ik_{\nu}a})}{\sum_{\lambda}(1 - \cos k_{\lambda}a)\sum_{\rho}(1 - \cos(k_{\rho} - p_{\rho})a)}, \quad (A1) \]

where $\Omega$ is the space-time volume and $p = (0, \mathbf{p})$. Exponentiating the denominator and converting the spatial momentum sums into integrals (we work in the infinite spatial volume limit), we obtain

\[ B_{\mu\nu}(p) = \frac{N}{4a^2 N_T} \sum_{n=0}^{N_T - 1} \int_{-\pi}^{\pi} \frac{d^3 \mathbf{k}}{(2\pi)^3} \int_{0}^{\infty} d\alpha d\beta e^{-(\alpha + \beta)(4 - \cos \frac{2\pi n}{N_T})} \prod_{\lambda=1}^{3} e^{\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos p_{\lambda}a \cos(k_{\lambda} - \phi_{\lambda})}} \times \left( e^{-i(k_{\mu} + k_{\nu} - p_{\mu}a)} + e^{i(k_{\mu} + k_{\nu} - p_{\mu}a)} - e^{i(k_{\mu} - k_{\nu} - p_{\mu}a)} - e^{-i(k_{\mu} - k_{\nu})} \right), \quad (A2) \]

where $\phi_{\lambda}$ is implicitly defined by $\tan \phi_{\lambda} = \beta \sin(p_{\lambda}a)/[\alpha + \beta \cos(p_{\lambda}a)]$. Now we perform the spatial momentum integrals, yielding the modified Bessel functions. For the sake of concreteness, let us consider the component $\mu = 1$ and $\nu = 2$

\[ B'_{12}(p) = \frac{N}{4a^2 N_T} \sum_{n=1}^{N_T - 1} \int_{0}^{\infty} d\alpha d\beta e^{-(\alpha + \beta)(4 - \cos \frac{2\pi n}{N_T})} \prod_{\lambda=1}^{2} I_1(\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos p_{\lambda}a}) \times I_0(\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos p_{3}a}) \left( e^{-i(\phi_1 + \phi_2 - p_{2}a)} + e^{i(\phi_1 + \phi_2 - p_{1}a)} - e^{i(\phi_1 - \phi_2 - p_{1}a + p_{2}a)} - e^{-i(\phi_1 - \phi_2)} \right). \quad (A3) \]
In Eq. (A3) $B'$ is just $B$ without the $n = 0$ term in the frequency sum. This static term is in fact the one that is directly reproduced by the reduced theory, and should be excluded from the contribution due to non-static modes.

In the limit of $|p| n \ll 1$ and $|p| \ll T$ (we are interested in the small lattice spacing and high-$T$ limit), Eq. (A3) further simplifies

$$B'_{12}(p) = -N^2 p_1 p_2 \frac{12}{12} \int_0^\infty dx x \left[ \frac{1}{N_\tau} \sum_{n=1}^{N_\tau-1} e^{-x(1-\cos \frac{2\pi n}{N_\tau})} \right] e^{-3x I_1(x) I_0(x)} + O \left( p^2 / T^2, a |p| \right). \tag{A4}$$

The expression in Eq. (A4) diverges in the limit $N_\tau \to \infty$. We explicitly isolate its divergent part with the following subtraction

$$B'_{12}(p) = -N^2 p_1 p_2 \frac{12}{12} \int_0^\infty dx x \left[ \frac{1}{N_\tau} \sum_{n=1}^{N_\tau-1} e^{-x(1-\cos \frac{2\pi n}{N_\tau})} \right] \left( e^{-3x I_1(x) I_0(x)} - \frac{1}{(2\pi x)^{3/2}} \right)$$

$$- N^2 p_1 p_2 \frac{12}{12} \int_0^\infty dx x \left[ \frac{1}{N_\tau} \sum_{n=1}^{N_\tau-1} e^{-x(1-\cos \frac{2\pi n}{N_\tau})} \right] \frac{1}{(2\pi x)^{3/2}} + O \left( p^2 / T^2, a |p| \right). \tag{A5}$$

Now the first term is finite in the limit $N_\tau \to \infty$, i.e. the limit $a \to 0$ with $a N_\tau = 1 / T$ fixed, and it is equal to $-N^2 p_1 p_2 f'_{11} / (48 \pi^2)$. We then use in the second term the expansion

$$\pi \sum_{n=1}^{N_\tau-1} \frac{1}{\sin(\pi n / N_\tau)} = 2 \gamma_E + \ln \left( \frac{4}{\pi^2 a^2 T^2} \right) + O(a T), \tag{A6}$$

and obtain

$$B'_{12}(p) = -N^2 p_1 p_2 \frac{12}{48 \pi^2} \left[ f'_{11} + 2 \gamma_E + \ln(4 / \pi^2) - \ln(a^2 T^2) \right] + O \left( p^2 / T^2, a |p|, a T \right). \tag{A7}$$

Next we want to relate the finite integrals that are found in the lattice high-$T$ expansion $f'_{ij}$, defined in Eq. (13), to the corresponding integrals that are found in the zero temperature calculation $f_{ij}$, defined in Eq. (13). Directly from their definitions, we find

$$f'_{ij} - f_{ij} = \lim_{\epsilon \to 0} \int_0^\infty dx e^{-\epsilon x} \left[ \frac{\theta(x-2)}{x} - \sqrt{2\pi} e^{-x I_0(x)} \right]$$

$$= \lim_{\epsilon \to 0} \left[ -Ei(-2\epsilon) - 2 Q_{-1/2}(1 + \epsilon) \right]. \tag{A8}$$

The expression is finite, and we have introduced a convergence factor $e^{-\epsilon x}$ in the integral only to be able to integrate separately the two terms. At last we obtain the desired result $f'_{ij} - f_{ij} = -\gamma_E - 3 \ln 4$ by using the small epsilon expansions of the exponential-integral function $Ei(-2\epsilon) = \gamma_E + \ln(2\epsilon) + O(\epsilon)$ and of the Legendre function of the second kind $2 Q_{-1/2}(1 + \epsilon) = - \ln(2\epsilon) + 3 \ln 4 + O(\epsilon)$.

The last issue we would like to address in this appendix is the high-$T$ expansion of those terms independent of external momentum, such as the tadpole graphs. Physically we expect that these terms cannot contain $\ln(a T)$, since they do not contribute to the renormalization, and in fact they should eventually cancel out due to the gauge invariance or the lack of magnetic mass generation at the one-loop level. Therefore we should be able to factorize...
any power dependence on $T$ trivially, and take the continuum limit of the frequency sums. Mathematically, this is guaranteed by the fact that the convergence of the limit

$$
\lim_{N \tau \to \infty} \frac{1}{N \tau} \sum_{n=0}^{N \tau - 1} f \left( \cos \frac{2 \pi n}{N \tau} \right) = \int_0^1 dx \ f (\cos 2 \pi x)
$$

is exponential, at least when $f(z)$ can be expanded as a power series in $z$, which includes the cases we are concerned with. Note that the terms with $n = 0$ should be included in these tadpole-like graphs, since they are not reproducible by the reduced theory.
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*   E-mail: shuang@pierre.mit.edu and lissia@pierre.mit.edu
†  Present address.

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