The Tutte Polynomial of a Morphism of Matroids.

6. A Multi-Faceted Counting Formula for Hyperplane Regions and Acyclic Orientations

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Abstract

We show that the 4-variable generating function of certain orientation related parameters of an ordered oriented matroid is the evaluation at \((x + u, y + v)\) of its Tutte polynomial. This evaluation contains as special cases the counting of regions in hyperplane arrangements and of acyclic orientations in graphs. Several new 2-variable expansions of the Tutte polynomial of an oriented matroid follow as corollaries.

This result hold more generally for oriented matroid perspectives, with specific special cases the counting of bounded regions in hyperplane arrangements or of bipolar acyclic orientations in graphs.

In corollary, we obtain expressions for the partial derivatives of the Tutte polynomial as generating functions of the same orientation parameters.

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Introduction

Let $M$ be a matroid on a set $E$. The Tutte polynomial $t(M; x, y)$ of $M$, equivalent to the generating function for cardinality and rank of subsets of $E$, can be defined by the closed formula

$$t(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(M)-r_M(A)}(y-1)^{|A|-r_M(A)}$$

where $r_M(A)$ denotes the rank of $A$ in $M$.

As well-known, the Tutte polynomial of a matroid on a linearly ordered set can also be expressed as the generating function of Tutte activities - internal and external - of bases, providing a state model with numerous applications [3][4][11][25].

$$t(M; x, y) = \sum_{B \subseteq E \text{ basis of } M} x^{\iota_M(B)}y^{\epsilon_M(B)}$$

We have introduced in [20] a state model for the Tutte polynomial of an oriented matroid on a linearly ordered set as a generating function of orientation activities.

$$t(M; x, y) = \sum_{A \subseteq E} \left(\frac{x}{2}\right)^{o^*M(A)}\left(\frac{y}{2}\right)^{o_M(A)}$$

Basic definitions and properties of oriented matroids can be found in [1].

We point out that formula (3) contains as special cases, quoted here in increasing order of generality, the counting $t(2, 0)$ of acyclic orientations of graphs by R. Stanley [24], of acyclic orientations of regular matroids by T. Brylawski and D. Lucas [2], of regions in hyperplane arrangements by R.O. Winder [26] and T. Zaslavsky [27], of acyclic reorientations of oriented matroids by the author [15].

A comparison of the state models (2) and (3) for the Tutte polynomial - one in terms of Tutte activities of bases, the other in terms of orientation activities of subsets - has been a motivation for a series of papers by E. Gioan and the author on the so-called active bijection [5][7][8][9][10]. The active bijection relates the two types of activities by means of activity preserving mappings with prescribed multiplicities.
Surprisingly enough, the active bijection establishes a relationship between the Tutte polynomial and linear programming [10].

In the paper [12], G. Gordon and L. Traldi exhibit an expansion of the Tutte polynomial of an ordered matroid in terms of Tutte activities similar to (4) (see Example 3.3).

\[
t(M; x, y) = \sum_{A \subseteq E} \left( \frac{x}{2} \right)^{\epsilon_M(A)} \left( \frac{y}{2} \right)^{\epsilon_M(A)}
\]  

(4)

In [12], the formula (4) is derived from a 4-variable expansion for the Tutte polynomial in terms of (generalized) Tutte activities. One may wonder whether an analogous 4-variable expansion also exists in terms of orientation activities.

It turns out that such an expansion does exist. Its existence and properties constitute the object of the present note.

Actually, the formula in [20] is given for objects more general than Tutte polynomials of oriented matroids, namely for certain 2-variable specializations of the 3-variable Tutte polynomials of oriented matroid perspectives. The same level of generality holds here.

The 3-variable Tutte polynomial of a matroid perspective has been introduced by the author in 1975 [15]. We have studied its properties in a series of papers: fundamental properties in [17] [19], Eulerian partitions in surfaces [18], activities of orientations in [20] [22], vectorial matroids in [6], computational complexity in [21].

The present 4-variable expansion - see below Theorem 3.1 - refines the 2-variable expansion of [20] Theorem 3.1.

Let \( M \to M' \) be an oriented matroid perspective on a linearly ordered set \( E \). We have

\[
t(M, M'; x, y, 1) = \sum_{A \subseteq E} \left( \frac{x}{2} \right)^{o_{M'}(A)} \left( \frac{y}{2} \right)^{o_M(A)}
\]  

(5)

Among applications of (5) specific to perspectives are self-dual forms of the counting of bounded regions in hyperplane arrangements, or of bipolar acyclic orientations in graphs, generalizing to oriented matroids results of [13] - see [16] [20].
In [22], the expansion of $t(x + u, y + v)$ in terms of Tutte activities is obtained from expressions of the partial derivatives as generating functions. Here, we go the reverse way. In Section 3, we obtain expressions of partial derivatives from the 4-variable expansion.

We show on an example how this computation is related to the active partitions discussed in [5][7][9]. Details will be published later.

**Matroid Perspectives**

Matroid perspectives generalize linear mappings of vector spaces. For the convenience of the reader, we recall here the relevant definitions and main properties.

Let $M, M'$ be two matroids on a same set $E$. We say that they constitute a matroid perspective, denoted by $M \rightarrow M'$, when the identity map on $E$ is a matroid strong map, that is, if at least one, hence all, of the following equivalent properties (i)-(iv) holds

(i) Any circuit of $M$ is a union of circuits of $M'$.
(ii) Any cocircuit of $M'$ is a union of cocircuits of $M$.
(iii) No circuit of $M$ and cocircuit of $M'$ meet in exactly one element.
(iv) For all $Y \subseteq X \subseteq E$ we have

$$r_M(Y) - r_{M'}(Y) \leq r_M(X) - r_{M'}(X)$$

As well-known, a matroid perspective factorizes: there is a matroid $N$ on $F$, $E \subseteq F$, such that $M = N \setminus (E \setminus F)$ and $M' = N/(E \setminus F)$.

When $M M'$ are two oriented matroids, we say that they constitute an oriented matroid perspective if at least one, hence all, of the following equivalent properties (i')-(iii') holds.

(i') Any circuit of $M$ is a conformal union of (signed) circuits of $M'$.
(ii') Any cocircuit of $M'$ is a conformal union of (signed) cocircuits of $M$.
(iii') No (signed) circuit of $M$ and (signed) cocircuit of $M'$ have a non empty conformal intersection.
Two signed sets $Y \subseteq X$ are *conformal* if $Y^+ \subseteq X^+$ and $Y^- \subseteq X^-$.  

From a topological point of view, property (ii') expresses that the vertices of a pseudohyperplane arrangement representing $M'$ belong to faces of a pseudohyperplane arrangement representing $M$.

Oriented matroid perspectives do not factorize in general as shown by J. Richter-Gebert in [23] Corollary 3.5.

The Tutte polynomial of a matroid perspective $M \to M'$ is defined in [19] by the closed formula

$$t(M, M'; x, y, z) = \sum_{A \subseteq E} (x - 1)^{r(M') - r_{M'}(A)} (y - 1)^{|A| - r - M(A)} z^{r_{M}(A) - r_{M'}(A) - r_{M'}(A)}$$

Property (iii) ensures that we have indeed non-negative powers of $z$.

## 1 O-activities and Θ-activities

Let $M$ be an oriented matroid of a linearly ordered set $E$.

The following definitions have been introduced in [20]. An element of $E$ is *orientation active* resp. *orientation dually-active* in $M$ if it is smallest in some positive circuit resp. positive cocircuit of $M$. We denote by $O(M)$ the set of orientation active elements of $M$, and by $O^*(M)$ its set of orientation dually-active elements. We have $O^*(M) = O(M^*)$.

We set

$$o(M) = |O(M)|$$

$$o^*(M) = |O^*(M)|$$

For short, we say that $o(M)$ and $o^*(M)$ are the *o-activities* of $M$.

For $A \subseteq E$, we denote by $-A M$ the oriented matroid obtained from $M$ by reorientation on $A$. Note that $-E \setminus A M = -A M$.

We refine the o-activities as follows. Set

$$\Theta_M(A) = O(-A M) \setminus A$$
\[ \Theta_M(A) = \Theta_M(E \setminus A) = O(-A_M) \cap A \]
\[ \theta_M(A) = |\Theta_M(A)| \]
\[ \bar{\Theta}_M(A) = |\bar{\Theta}_M(A)| \]

We have
\[ o_M(A) = \theta_M(A) + \bar{\Theta}_M(A) \]

Dually, we set
\[ \Theta^*_M(A) = \Theta^*_M(A) = O^*(-A_M) \setminus A \]
\[ \bar{\Theta}^*_M(A) = \bar{\Theta}^*_M(A) = O^*(-A_M) \cap A \]
\[ \theta^*_M(A) = |\Theta^*_M(A)| \]
\[ \bar{\theta}^*_M(A) = |\bar{\Theta}^*_M(A)| \]

We have
\[ o^*_M(A) = \theta^*_M(A) + \bar{\theta}^*_M(A) \]

The 4 parameters \( \theta_M(A), \bar{\Theta}_M(A), \theta^*_M(A), \bar{\theta}^*_M(A) \), depending on \( A \), are the 4 \( \theta \)-activities of \( M \).

2 A 4-Expansion for the Tutte Polynomial

We have shown in [20] Theorem 3.1 that the generating function of the two \( o \)-activities of the reorientations of an oriented matroid is the evaluation of its Tutte polynomial at \( (2x, 2y) \). Our main result here is that the generating function of the four \( \theta \)-activities is also an evaluation of the Tutte polynomial.

**Theorem 2.1.** Let \( M \) be an oriented matroid on a linearly ordered set \( E \). We have
\[ t(M; x + u, y + v) = \sum_{A \subseteq E} x^{\theta_M(A)} u^{\bar{\Theta}_M(A)} y^{\theta^*_M(A)} v^{\bar{\theta}^*_M(A)} \]
where \( t(M; x, y) \) denotes the Tutte polynomial of \( M \).
Example 1

The Tutte polynomial of the cycle matroid of the above 4-edge directed graph is

\[ t(x, y) = x^2 + xy + y^2 + x + y. \]

A

\( \Theta(A) \)

\( \Theta^\star(A) \)

\( \Theta^\delta(A) \)

\( \Theta^\delta_\emptyset(A) \)

\( x^{\Theta^\delta(A)/y^{\Theta^\delta_\emptyset(A)/y^{\Theta(A)}}} \)

| A   | \( O^\star(A) \) | \( O(A) \) | \( \Theta^\star(A) \) | \( \Theta(A) \) | \( \Theta^\delta(A) \) | \( \Theta^\delta_\emptyset(A) \) |
|-----|----------------|---------|----------------|---------|----------------|----------------|
| \( \emptyset \) | 13 | 13 | | | | |
| 4 | 1 | 1 | | | | |
| 3 | 12 | 12 | | | | |
| 34 | 13 | 1 | 3 | | | |
| 2 | 3 | 1 | 3 | 1 | | |
| 23 | 1 | 1 | | | | |
| 24 | 12 | 1 | 2 | | | |
| 234 | 3 | 1 | 3 | 1 | | |
| 1 | 3 | 1 | 3 | 1 | | |
| 14 | 1 | 1 | | | | |
| 13 | 12 | 1 | 2 | | | |
| 134 | 3 | 1 | 3 | 1 | | |
| 12 | 3 | 1 | 3 | 1 | | |
| 124 | 12 | 12 | | | | |
| 123 | 1 | 1 | | | | |
| 1234 | 13 | 13 | | | | |

In accordance with Theorem 2.1, the last column sums up to

\[ t(x + u, y + v) = (x + u)^2 + (x + u)(y + v) + (y + v)^2 + (x + u) + (y + v). \]

n.b. Replacing \( A \) by \( E \setminus A \) exchanges \( x \) and \( u \) on one hand, and \( y \) and \( v \) on the other. Hence it would suffice to compute the lines \( A \) such that \( 1 \notin A \).

Table 1

Theorem 2.1 will be proved in Section 3, as a special case of the more general Theorem 3.1.
Theorem 2.1 is the orientation counterpart of the result of G. Gordon and L. Traldi for Tutte activities [12], quoted in the introduction as (4) (see also [22] Theorem 2.9). The relationship between orientation activities and Tutte activities is the object of a series of papers by E. Gioan and the author [5][7][8][9]: the active mapping from reorientations to bases preserves pair of activities. Theorem 2.1 allows to precise the properties of the active mapping: the associated active bijection from reorientations to subsets preserves 4-uplets of activities.

Theorem 2.1 is closely related to the active partition associated with a basis of an ordered oriented matroid. The notion of active partition has been introduced by E. Gioan and the author in terms of graphs in [5]. The case of general oriented matroids will appear in detail in [9] (an extended abstract can be found in [7]).

With a basis $B$ of $M$ is associated the term $t(M; B; x, y) = x^{\ell_M(B)} y^{\epsilon_M(B)}$ of the basis expansion of its Tutte polynomial. The active partition associated with $B$ is a partition of $E$ into $\ell_M(B) + \epsilon_M(B)$ classes. Each class of the active partition is associated with one $B$-active element, either internally or externally. By reorienting any orientation associated with $B$ on arbitrary unions of classes of the active partition, we get all the $2^{\ell_M(B)}+\epsilon_M(B)$ reorientations associated with $B$ by the active mapping. The reorientations associated in this way with $B$ are exactly those yielding the terms of the expansion $t(M; B; x + u, y + v)$ in Theorem 2.1 associated with the subsets in the Dawson interval defined by $B$.

A proof of Theorem 2.1 in terms of active partition will appear in [9].

We recover immediately the matroid case of [20] by specializing $x = u$ $y = v$ in Theorem 2.1.

Corollary 2.2 ([20] Theorem 3.1). Let $M$ be an ordered oriented matroid on a set $E$. We have

$$t(M; 2x, 2y) = \sum_{A \subseteq E} x^{\ell(-A_M)} y^{\epsilon(-A_M)}$$

As well-known, the evaluation $t(2, 0)$ of the Tutte polynomial counts the number acyclic orientations of graph resp. of regions of a hyperplane arrangement, of acyclic reorientations of an oriented matroid [15][24][26][27].
Setting \( x = 1 \) \( u = 1 \) in the formula of Theorem 2.1, we get indeed that \( t(2, 0) \) is the number of subsets \( A \subseteq E \) such that \( -A \) \( M \) has orientation activity 0, i.e. is acyclic. Setting \( x = 2 \) \( u = 0 \) resp. \( x = 0 \) \( u = 2 \), we get two alternate expansions of \( t(2, 0) \), in terms of orientation \( \theta \)-activities.

**Corollary 2.3.** Let \( M \) be an oriented matroid on a linearly ordered set \( E \). We have

\[
    t(M; 2, 0) = \sum_{A \subseteq E} 2^{\theta_M(A)} = \sum_{A \subseteq E} 2^{\bar{\theta}_M(A)}
\]

where \( t(M, M'; x, y, z) \) denotes the 3-variable Tutte polynomial of \( M \to M' \).

### 3 Generalization to Perspectives

Theorem 2.1 is a special case of a more general theorem dealing with Tutte polynomials of oriented matroid perspectives [15][19].

**Theorem 3.1.** Let \( M \to M' \) be an oriented matroid perspective on a linearly ordered set \( E \). We have

\[
    t(M, M'; x + u, y + v, 1) = \sum_{A \subseteq E} x^{\theta_{M'}(A)} y^{\bar{\theta}_{M'}(A)} v^{\theta_M(A)}
\]

where \( t(M, M; x, y, z) \) denotes the 3-variable Tutte polynomial of \( M \to M' \).

**Example 2**

![Diagram](image)

Let us consider the oriented matroid perspective given by the cycle matroids of the above graphs. The evaluation at \( z = 1 \) of its Tutte polynomial is

\[
    t(x, y, 1) = x^2 + 5x + 4y + 10.
\]

The \( \theta \)-activities are shown on the following table, together with the contributed terms.
In accordance with Theorem 3.1, the columns $t(A)$ sum up to 
$t(x + u, y + v, 1) = (x + u)^2 + 5(x + u) + 4(y + v) + 10$.

Table 2

| $A$ | $O^+(A)$ | $O(A)$ | $\theta^+(A)$ | $\theta(A)$ | $t(A)$ | $A$ | $O^+(A)$ | $O(A)$ | $\theta^+(A)$ | $\theta(A)$ | $t(A)$ |
|-----|----------|--------|----------------|-------------|--------|-----|----------|--------|----------------|-------------|--------|
| $\emptyset$ | 12 | 12 | $x^2$ | 1 | 12 | 2 | 1 | $xu$ |
| 5 | 1 | 1 | $x$ | 15 | 1 | | | |
| 45 | 1 | 145 | 1 | | | | | |
| 3 | 1 | 1 | $x$ | 13 | 1 | 1 | $y$ |
| 35 | 1 | 1 | $x$ | 135 | 1 | | $y$ |
| 34 | 1 | 1 | $x$ | 134 | 1 | 1 | $y$ |
| 345 | 1 | 1 | $x$ | 1345 | 1 | 1 | $y$ |
| 2 | 1 | 1 | $y$ | 12 | 1 | 1 | $y$ |
| 25 | 1 | 1 | $y$ | 125 | 1 | | | |
| 24 | 1 | 1 | $y$ | 124 | 1 | 1 | $y$ |
| 245 | 1 | 1 | $y$ | 1245 | 1 | 1 | $y$ |
| 245 | 1 | 1 | $y$ | 12345 | 1 | 1 | $y$ |
| 245 | 12 | 1 | 2 | $xu$ | 12345 | 12 | 12 | $xu^2$ |

In accordance with Theorem 3.1, the columns $t(A)$ sum up to 
$t(x + u, y + v, 1) = (x + u)^2 + 5(x + u) + 4(y + v) + 10$.

Table 2

As above in Section 2, we recover the main result of [20] in the case of matroid perspectives by the specialization $x = u$ $y = v$

**Corollary 3.2 (20) Theorem 3.1.** Let $M \rightarrow M'$ be an ordered oriented matroid perspective on a set $E$. We have 
\[
t(M, M'; 2x, 2y, 1) = \sum_{A \subseteq E} x^{\theta^+(A)} y^{\theta^+(A)}
\]

We recall that given an oriented matroid perspective on a set $E$, the evaluation $t(M, M'; 0, 0, 1)$ counts the number of subsets $A \subseteq E$ such that $-A$ is acyclic and $-A$ is totally cyclic [16]. This statement can be considered as self dual form of the counting of acyclic reorientations by $t(M; 2, 0)$. In the particular case $M' = M/e \oplus \emptyset(e)$, where $e$ is a non factor element of $M$ - neither loop or isthmus, and $\emptyset(e)$ the rank 0 matroid on $e$, then $t(M, M'; 0, 0, 1)$ counts the number of bounded regions of a hyperplane arrangement with infinity at $e$, or of bipolar acyclic orientations defined by the edge $e$ in a graph [16] [20].

Evaluating Theorem 3.1 at $x = y = -1$ and $u = v = 1$, we get alternate expansions of $t(M, M'; 0, 0, 1)$.
Corollary 3.3.

\[ t(M, M'; 0, 0, 1) = \sum_{A \subseteq E} (-1)^{\theta_{M'}(A) + \theta_{M}(A)} \]

Three similar formulas are obtained for the other different choices of -1 and 1 for the variables.

As of today, the generalization to oriented matroid perspectives of the active bijection and of active partitions by E. Gioan and the author is still in progress. Hence, the possible proof of Theorem 2.1 by methods along these lines mentioned in Section 2 cannot be used for Theorem 3.1. It turns out that the proof by deletion/contraction of [20] requires only slight adjustments to establish Theorem 3.1.

For \( a \in E \), set \( o^*(M; a) = 1 \) if \( a \) is internally active in \( M \) and \( o^*(M; a) = 0 \) otherwise, \( o(M; a) = 1 \) if \( a \) is externally active in \( M \) and \( o(M; a) = 0 \) otherwise.

The main step of the proof of Theorem 3.1 is the following lemma from [20] (see Lemma 3.2).

**Lemma 3.4.** Let \( M \to M' \) be an oriented matroid on a linearly ordered set \( E \), with greatest element \( e \). Then (i) or (ii) holds, where

(i) \( o^*(M'; a) = o^*(M' \setminus e; a) \), \( o(M; a) = o(M \setminus e; a) \), \( o^*(-eM'; a) = o^*(M'/e; a) \)
and \( o(-eM; a) = o(M/e; a) \) for all \( a \in E \setminus \{e\} \),

(ii) \( o^*(M'; a) = o^*(M'/e; a) \), \( o(M; a) = o(M/e; a) \), \( o^*(-eM'; a) = o^*(M' \setminus e; a) \)
and \( o(-eM; a) = o(M \setminus e; a) \) for all \( a \in E \setminus \{e\} \).

For the convenience of the reader, we reproduce the proof of Lemma 3.4.

**Proof.** The proof is broken into several steps.

Let \( a \in E \setminus \{e\} \).

(1) \( o(M; a) = 1 \) or \( -e o(M; a) = 1 \) implies \( o(M/e; a) = 1 \).

Let \( X \) be a positive circuit of \( M \) resp. \( -eM \) with least element \( a \). By a property of contraction in oriented matroids, there is a positive circuit \( X' \) of \( M/e = (-eM)/e \) such that \( a \in X' \subseteq X \), hence \( o(M/e; a) = 1 \).
(2) \( o(M; a) = 1 \) and \( -e o(M; a) = 1 \) implies \( o(M \setminus e; a) = 1 \).

There exist signed circuits \( X \) and \( Y \) of \( M \) both with least element \( a \) such that \( X^- = \emptyset \) and \( Y^- \subseteq \{e\} \). Suppose \( o(M \setminus e; a) = 0 \): Then necessarily \( e \in X^+ \cap Y^- \), hence by the elimination property in oriented matroids, there exists a positive circuit \( Z \) such that \( a \in Z \subseteq (X \cup Y) \setminus \{e\} \). We get \( o(M \setminus e; a) = 1 \), a contradiction.

(3) \( o(M \setminus e; a) = 1 \) implies \( o(M; a) = -e o(M; a) = 1 \).

The proof is immediate.

(4) \( o(M/e; a) = 1 \) implies \( o(M; a) = 1 \) or \( -e o(M; a) = 1 \).

Let \( X' \) be a positive circuit of \( M/e \) with least element \( a \). There exists a signed circuit \( X \) of \( M \) such that \( X' = X \setminus \{e\} \). Since \( e \) is the greatest element of \( E \), the element \( a \) also smallest in \( X \). We have \( X^- \subseteq \{e\} \), hence \( X \) is a positive circuit of \( M \) or \( -eX \) is a positive circuit of \( -eM \).

(5) \( o(M; a) = o(M \setminus e; a) = 1 \) if and only if \( o(-eM; a) = o(M/e; a) \).

Suppose \( o(M; a) = o(M \setminus e; a) = 1 \), but \( o(-eM; a) \neq o(M/e; a) \). We have \( o(-eM; a) = 0 \) since \( o(-eM; a) = 1 \) implies \( o(M/e; a) = 1 \) by (1), and \( o(M/e; a) = 1 \). Then \( o(M; a) = 1 \) by (4), hence \( o(M \setminus e; x) = a \), which contradicts (3) since \( o(-eM; a) = 0 \).

Conversely, suppose \( o(-eM; a) = o(M/e; a) \), but \( o(M; a) \neq o(M \setminus e; a) \). We have \( o(M \setminus e; a) = 0 \) since \( o(M \setminus e; a) = 1 \) implies \( o(M; a) = 1 \) by (3), and \( o(M; a) = 1 \). Then \( o(M/e; a) = 1 \) by (1), hence \( o(-eM; a) = 1 \). But this, together with \( o(M; a) = 1 \) contradicts (2) since \( o(M \setminus e; a) = 0 \).

(6) If, for some \( b \in E \setminus \{e\} \), we have \( o(M; b) \neq o(-eM; b) \) and \( o(M; b) = o(M \setminus e; b) \), then \( o(M; a) = o(M \setminus e; a) \) for all \( a \in E \setminus \{e\} \).

Suppose \( o(M; a) \neq o(M/e; a) \) for some \( a \in E \setminus \{e\} \). Then \( o(M \setminus e; x) = 0 \) and \( o(M; x) = 1 \) (since \( o(M \setminus e; x) = 1 \) implies \( o(M; a) = 1 \) by (3)). Hence there is a positive circuit \( X \) of \( M \) with least element \( a \), and necessarily \( e \in X \). On the other hand, \( o(M \setminus e; b) = 0 \) since \( o(M \setminus e; b) = 1 \) implies \( o(M; b) = o(-eM; b) \) by (3). Hence \( o(M; b) = 0 \), \( o(-eM; b) = 1 \), and there is a signed circuit \( Y \) of \( M \) with least element \( b \) such that \( Y^- \subseteq \{e\} \). Necessarily, \( e \in X^+ \cap Y^- \), since \( o(M \setminus e; a) = o(M \setminus e; b) = 0 \). Set \( c = \min(a, b) \). By the elimination property in oriented matroids, there is a positive circuit \( Z \) of \( M \) such that \( c \in Z \subseteq (X \cup Y) \setminus \{e\} \). Clearly, \( c \) is the least element of \( Z \). Hence
\[ o(M \setminus e; c) = 1, \text{ a contradiction.} \]

(7) \( o(M; a) = o(-_eM; a) \) implies \( o(M; a) = o(-_eM; a) = o(M \setminus e; a) = o(M/e; a) \).

Suppose \( o(M; a) = o(-_eM; a) \). If \( o(M; a) = 0 \), then \( o(M \setminus e; a) = 0 \) by (3). If \( o(M; a) = 1 \), then \( o(M \setminus e; a) = 1 \) by (2). In both cases, we have \( o(M; a) = o(M \setminus e; a) \). It follows that \( o(-_eM; a) = o(M/e; a) \) by (5).

(8) We have \( o(M; a) = o(M \setminus e; x) \) and \( o(-_eM; x) = o(M/e; a) \) for all \( a \in E \setminus \{e\} \), or \( o(M; a) = o(M/e; a) \) and \( o(-_eM; a) = o(M \setminus e; a) \) for all \( a \in E \setminus \{e\} \).

If \( o(M; a) = o(-_eM; a) \) for all \( a \in E \setminus \{e\} \), then \( o(M; a) = o(M \setminus e; a) = o(M/e; a) \) for all \( a \in E \setminus \{e\} \) by (7). Suppose there is \( b \in E \setminus \{e\} \) such that \( o(M; b) \neq o(-_eM; b) \). We have \( o(M; y) = o(M \setminus e; y) \) or \( o(-_eM; b) = o(M \setminus e; b) \). Suppose for instance \( o(M; b) = o(M \setminus e; b) \). We have \( o(M; a) = o(M \setminus e; a) \) for all \( a \in E \setminus \{e\} \) by (6), hence \( o(M; a) = o(M/e; a) \) for all \( a \in E \setminus \{e\} \) by (5).

(9) \( o(M; a) = 1 \) implies \( o^*(M'; a) = 0 \).

Suppose \( o(M; a) = 1 \): there is a positive circuit \( X \) of \( M \) with least element \( X \). Since \( M \rightarrow M' \) is an oriented matroid perspective, there is a positive circuit \( X' \) of \( M' \) such that \( a \in X' \subseteq X \), hence there is no positive cocircuit of \( M' \) containing \( a \) by the orthogonality property in oriented matroids, implying \( o^*(M'; a) = 0 \).

(10) If \( o(M; a) \neq o(-_eM; a) \), then \( o(M; a) = o(M \setminus e; a) \) implies \( o^*(M'; a) = o^*(M'/e; a) \).

Suppose \( o^*(M'; a) \neq o^*(M' \setminus e; a) \) under the hypothesis of (10). By (1*) (i.e. by (1) applied to \( M^* \)) \( o^*(M'; a) = 1 \) implies \( o^*(M' \setminus e; a) = 1 \). Hence \( o^*(M'; a) = 0 \) and \( o^*(M' \setminus e; a) = 1 \). Now by (4*) \( o^*(M'; a) = 1 \) or \( o^*(M'/e; a) = 1 \). Hence \( o^*(-_eM'; a) = 1 \). Then, by (9) \( o(-_eM'; a) = 0 \). It follows that \( o(M; a) = 1 \), hence \( o(M \setminus e; a) = 1 \), by hypothesis, contradicting \( o(-_eM; a) = 0 \) by (3).

(11) We now prove Lemma 3.4.

If \( o^*(M'; a) = o^*(-_eM'; a) \) for all \( a \in E \setminus \{e\} \), or \( o(M; a) = o(-_eM; a) \) for \( a \in E \setminus \{e\} \), then (11) follows clearly from (7), (7*) and (8), (8*). If there
is \( a \in E \setminus \{e\} \) such that \( o^*(M';a) \neq o^*(-eM';a) \) and \( o(M;a) \neq o(-eM;a) \), then Lemma 3.4 follows from (8), (8*) and (10).

The remaining possibility is that for all \( a \in E \setminus \{e\} \) we have either \( o^*(M';a) \neq o^*(-eM';a) \), \( o(M;a) = o(-eM;a) \), or \( o^*(M';a) = o^*(-eM';a) \), \( o(M;a) \neq o(-eM;a) \), and both cases occur. Replacing if necessary, \( M \to M' \) by \(-eM \to -eM'\), we may suppose notation such that \( o(M; b) = o(M \setminus e; b) \) for some \( b \in E \setminus \{e\} \) with \( o(M; b) \neq o(-eM; b) \). Then by (6) \( o(M; a) = o(M \setminus e; a) \) for all \( a \in E \setminus \{e\} \).

If \( a \in E \setminus \{e\} \) is such that \( o(M; a) \neq o(-eM; a) \) we have \( o^*(M';a) = o^*(M' \setminus e; a) \) by (10).

Consider now \( a \in E \setminus \{e\} \) such that \( o(M; a) = o(-eM; a) \). We have \( o^*(M';a) \neq o^*(-eM';a) \) by our hypothesis. If \( o^*(M';a) = 0 \), we have \( o^*(-eM';a) = 1 \), hence there is a cocircuit \( X' \) of \( M' \) with \( X^- = \{e\} \). By our hypothesis there is \( b \in E \setminus \{e\} \) with \( o(M; b) = 0 \), \( o(-eM; b) = 1 \), implying the existence of a circuit \( Y \) of \( M \) with \( Y^- = \{e\} \). Since \( M \to M' \) there is a circuit \( Y' \) of \( M' \) with \( Y'^- = \{e\} \). Then \( X' \) and \( Y' \) contradict the orthogonality property. Therefore \( o^*(M';a) = 1 \) and \( o^*(-eM';a) = 0 \). Now by (1*) \( o^*(M' \setminus e; a) = 1 \), hence \( o^*(M';a) = o^*(M' \setminus e; a) \).

Thus, for all \( a \in E \setminus \{e\} \) we have \( o^*(M';a) = o^*(M' \setminus e; a) \) and \( o(M; a) = o(M \setminus e; a) \). Therefore, Lemma 3.4 follows from (5) and (5*), or from (8) and (8*).

\[ \blacksquare \]

Set

\[ f(M, M'; A; x, u, y, v) = x^{\theta_{M'}(A)} u^{\bar{\theta}_{M'}(A)} y^{\theta_{M}(A)} v^{\bar{\theta}_{M}(A)} \]

and

\[ f(M, M'; x, u, y, v) = \sum_{A \subseteq E} f(M, M'; A; x, u, y, v) \]

Proof of Theorem 3.1. We distinguish several cases

(i) \( e \) is neither an isthmus of \( M' \) nor a loop of \( M \)

Summing up the equalities of Lemma 3.4 for all \( a \in A \) resp. \( a \in E \setminus \{e\} \setminus A \), and observing that \( o^*(M'; e) = o^*(-eM'; e) = o(M; e) = o(-eM; e) = 0 \), since \( e \) is the greatest element of \( E \), we get that

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\[ \theta^*_M(A) = \theta^*_{M\setminus e}(A) \quad \overline{\theta}_M(A) = \overline{\theta}_{M\setminus e}(A) \]
\[ \theta_M(A) = \theta_{M\setminus e}(A) \quad \overline{\theta}_M(A) = \overline{\theta}_M(A) \]
\[ \theta^*_{-eM}(A) = \theta^*_{M/e}(A) \quad \overline{\theta}^*_{-eM}(A) = \overline{\theta}^*_{M/e}(A) \]
\[ \theta_{-eM}(A) = \theta_{M/e}(A) \quad \overline{\theta}_{-eM}(A) = \overline{\theta}_{M/e}(A) \]

or
\[ \theta^*_M(A) = \theta^*_{M'/e}(A) \quad \overline{\theta}_M(A) = \overline{\theta}_{M'/e}(A) \]
\[ \theta_M(A) = \theta_{M/e}(A) \quad \overline{\theta}_M(A) = \overline{\theta}_{M/e}(A) \]
\[ \theta^*_{-eM}(A) = \theta^*_{M'/e}(A) \quad \overline{\theta}^*_{-eM}(A) = \overline{\theta}^*_{M'/e}(A) \]
\[ \theta_{-eM}(A) = \theta_{M/e}(A) \quad \overline{\theta}_{-eM}(A) = \overline{\theta}_{M/e}(A) \]

It follows that
\[
\theta(M, M'; A; x, u, y, v) + \theta(-eM, -eM'; A; x, u, y, v) = \\
= \theta(M \setminus e, M' \setminus e; A; x, u, y, v) + \theta(M/e, M'/e; A; x, u, y, v)
\]

Summing up for \( A \subseteq E \setminus \{e\} \) we get
\[
\theta(M, M'; x, u, y, v) = \theta(M \setminus e, M' \setminus e; x, u, y, v) + \theta(M/e, M'/e; x, u, y, v)
\]

(ii) \( e \) is an isthmus of \( M' \) (hence also an isthmus of \( M \))

For \( A \subseteq E \setminus \{e\} \), we have readily
\[ \theta^*_M(A) = \theta^*_M(A) + 1 \quad \overline{\theta}^*_M(A) = \overline{\theta}^*_M(A) \]
\[ \theta_M(A) = \theta_M(A) \quad \overline{\theta}_M(A) = \overline{\theta}_M(A) \]

and
\[ \theta^*_M(A \cup \{e\}) = \theta^*_M(A) \quad \overline{\theta}^*_M(A \cup \{e\}) = \overline{\theta}^*_M(A) + 1 \]
\[ \theta_M(A \cup \{e\}) = \theta_M(A) \quad \overline{\theta}_M(A \cup \{e\}) = \overline{\theta}_M(A) \]

It follows that
\[
\theta(M, M'; A; x, u, y, v) = x \theta(M \setminus e, M' \setminus e; A; x, u, y, v)
\]

and
\[
\theta(M, M'; A \cup \{e\}; x, u, y, v) = u \theta(M \setminus e, M' \setminus e; A; x, u, y, v)
\]

Therefore,
\[
\theta(M, M'; x, u, y, v) = \sum_{A \subseteq E} \theta(M, M'; A; x, u, y, v) = \\
\sum_{A \subseteq E} \theta(M, M'; A; x, u, y, v) + \sum_{e \in A \subseteq E} \theta(M, M'; A; x, u, y, v) = \\
\]

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\[ \sum_{A \subseteq E \setminus \{e\}} x f(M \setminus e, M' \setminus e; A; x, u, y, v) + \sum_{A \subseteq E \setminus \{e\}} u f(M \setminus e, M' \setminus e; A; x, u, y, v) = \\
= x f(M \setminus e, M' \setminus e; x, u, y, v) + u f(M \setminus e, M' \setminus e; x, u, y, v) = \\
= (x + u) f(M \setminus e, M' \setminus e; x, u, y, v) \\
\]

(iii) \( e \) is a loop of \( M \) (hence also a loop of \( M' \))

As in (ii), we have, dually

\[ f(M, M'; x, u, y, v) = (y + v) f(M \setminus e, M' \setminus e; x, u, y, v) \]

(iv) We have readily

\[ f(\emptyset, \emptyset; x, u, y, v) = 1 \]

Properties (i)-(iv) show that \( f(M, M'; x, u, y, v) \) verifies the deletion/contraction inductive relations satisfied by \( t(M, M'; x+u, y+v, 1) \). Therefore, by [19] Theorem 5.3, we have

\[ f(M, M'; x, u, y, v) = t(M, M'; x + u, y + v, 1) \]

As first shown by G. Gordon and L. Traldi (see [12] Examples 3.1-3.5), and extended by the author to matroid perspectives (see [22] Proposition 2.9), many 2-variable expansions of the Tutte polynomial follow readily from an expansion as a 4-variable generating function similar to Theorem 3.1, but in terms of Tutte activities.

The most remarkable are obtained by setting some of \( x, u, y, v \) to either 0 or 1, and/or replacing by \( x/2 \) \( y/2 \), and performing an appropriate change of variables. A total of 25 expansions, 9 different up to reordering, could be thus obtained from Theorem 3.1.

Here, we limit ourselves to three of them, referring the reader to [22] for a complete list in the case of Tutte activities.

**Corollary 3.5.** M. Las Vergnas 1984 [20] Let \( M \) be an oriented matroid on a linearly ordered set \( E \). We have

\[ t(M; x, y) = \sum_{A \subseteq E} \left( \frac{x}{2} \right)^{\sigma^*_M(A)} \left( \frac{y}{2} \right)^{\omega_M(A)} \]
Proof. Replace $x$ and $u$ by $\frac{x}{2}$, and $y$ and $v$ by $\frac{y}{2}$ in Theorem 2.1.

**Corollary 3.6.** Let $M$ be an oriented matroid on a linearly ordered set $E$. We have

$$t(M; x, y) = \sum_{A \subseteq E} (x - 1)^{\theta_{M}^*(A)}(y - 1)^{\theta_{M}(A)}$$

Proof. Replace $x$ by $x - 1$, $u$ by 1, $y$ by $y - 1$, and $v$ by 1 in Theorem 2.1.

**Corollary 3.7.** Let $M$ be an oriented matroid on a linearly ordered set $E$. We have

$$t(M; x, y) = \sum_{\substack{A \subseteq E \ni \theta_{M}^*(A) = 0 \
\theta_{M}(A) = 0}} x^{\theta_{M}^*(A)}y^{\theta_{M}(A)}$$

Proof. Replace $u$ and $v$ by 0 in Theorem 2.1.

**Corollary 3.8.** Let $M$ be an oriented matroid perspective on a linearly ordered set $E$. The number of subsets $A \subseteq E$ such that $\theta_{M}^*(A) = \theta_{M}(A) = 0$ is equal to the number of bases of $M$.

The same property holds for the three other classes of subsets obtained by duality and exchange of $A$ and $E \setminus A$.

The above corollaries generalize straightforwardly to oriented matroid perspectives.

### 4 Derivatives

In [22], we have used state models of Tutte polynomial partial derivatives in terms of internal and external activities to obtain an expansion of $t(M, M'; x + u, y + v, z)$. Here, we go the reverse way, using the expansion of $t(M, M'; x + u, y + v, 1)$ in terms of orientations given by Theorem 3.1 to obtain expansions of partial derivatives.

**Theorem 4.1.** Let $M \rightarrow M'$ be an oriented matroid perspective on a linearly ordered set $E$. We have

$$\frac{\partial^{p+q} t(M, M'; x, y, 1)}{\partial x^p \partial y^q} = p!q! \sum_{\substack{A \subseteq E \ni \theta_{M}^*(A) = p \
\theta_{M}(A) = q}} x^{\theta_{M}^*(A)}y^{\theta_{M}(A)}$$
Proof. By Taylor formula, we have

\[ t(M, M'; x + u, y + v, 1) = \sum_{p \geq 0, q \geq 0} \frac{1}{p!q!} u^p v^q \frac{\partial^{p+q} t}{\partial x^p \partial y^q} (M, M'; x, y, 1) \]

Theorem 4.1 follows readily from the expansion of \( t(M, M'; x + u, y + v, 1) \) given by Theorem 3.1.

We can easily obtain three alternative expansions. It suffices to apply differently Taylor formula, for instance with respect to \( x \) and \( v \) instead of \( u \) and \( v \).

Corollary 4.2. Let \( M \rightarrow M' \) be a matroid perspective on a linearly ordered set \( E \), and \( p \) be a non negative integer. Then

\[ \frac{d^p t}{dx^p} (M, M'; x, x, 1) = p! \sum_{A \subseteq E} x^{\theta^*_M(A) + \theta_M(A)} \sum_{\substack{\ell \subseteq E \\ \theta^*_M(A) + \theta_M(A) = p}} \theta^*_M(A) = p \]

Proof. By Taylor formula, and Theorem 4.1 we have

\[ t(M, M'; x + u, y + v, 1) = \sum_{p \geq 0, q \geq 0} u^p v^q \sum_{\substack{A \subseteq E \\ \theta^*_M(A) + \theta_M(A) = p \atop \theta_M(A) = q}} x^{\theta^*_M(A) + \theta_M(A)} \]

Hence

\[ t(M, M'; x + u, x + u, 1) = \sum_{p \geq 0, q \geq 0} u^{p+q} \sum_{\substack{A \subseteq E \\ \theta^*_M(A) + \theta_M(A) = p \atop \theta_M(A) = q}} x^{\theta^*_M(A) + \theta_M(A)} \]

By Taylor formula again

\[ \frac{d^p t}{dx^p} (M, M'; x + u, x + u, z) = \sum_{k \geq 0} \frac{1}{k!} u^k \frac{d^p t}{dx^p} (M, M'; x, x, z) \]

Corollary 4.2 follows.
It turns out that the subsets yielding the expansion of the partial derivatives, that is the subsets $A$ such that $\overline{\theta}_M(A) = p$ and $\overline{\theta}_M(A) = q$, can be simply described. In [22], an analogous description for internal and external activities was provided by the Dawson partitions associated to bases or independent/spanning sets. Here, the key tool are the active partitions, already mentioned in Section 2.

As of today, active partitions are fully available only for oriented matroids, the generalization to oriented matroid perspectives being still a work in progress. Details of the construction for oriented matroids will be given in [9]. We present it briefly on the example of Section 2.

**Example 1** (continued)

| basic orientations | 0   | 2   | 3   | 4   | 23  |
|--------------------|-----|-----|-----|-----|-----|
| active edges       | 13 *| 3 * 1| * 12| 1 * | * 1 |
| active partition   | 12 + 34 * | 34 * 12 | * 1 + 234 | 1234 * | * 1234 |
| active orientation partition | $\emptyset$ 12 34 1234 | 2 234 1 134 | 3 13 24 124 | 4 123 14 23 |
| $t(x, y) = x^2 + xy + y^2 + x + y$ | $x^2$ | $xy$ | $y^2$ | $x$ | $y$ |
| $\frac{\partial t}{\partial x}(x, y) = 2x + y + 1$ | 12 | 34 | 1 | 123 | $u$ |
| $\frac{\partial t}{\partial y}(x, y) = x + 2y + 1$ | 234 | 13 | 24 | 14 | $v$ |
| $\frac{\partial^2 t}{\partial x^2}(x, y) = 1$ | $u^2$ | 1234 | |
| $\frac{\partial^2 t}{\partial x\partial y}(x, y) = 1$ | 134 | $uv$ | |
| $\frac{\partial^2 t}{\partial y^2}(x, y) = 1$ | 124 | $v^2$ | |

Table 3 here has to be compared with Table 1 of [22]. Similarity is obvious. The precise relationship made explicit by the active bijection, a 1-1 mapping $2^E \rightarrow 2^E$, which provides theorems and algorithms relating the 4 $\theta$-activities to the 4 Tutte activities.

**Table 3**

We read on Table 1 the 5 orientations $A$ with $\overline{\theta}(A) = \overline{\theta}(A) = 0$, namely $\emptyset$ 4 3 2 23. These orientations have the role played by bases in the Dawson
partitions considered in [22]. We call them basic orientations.

The pairs of dual and primal orientation activities are respectively (2,0) (1,0) (0,2) (1,1) (0,1). We compute the active partitions as in [5], obtaining $12 + 34\ast$, $1234\ast$, $*1 + 234$, $12\ast 34$, $*1234$. The symbol $\ast$ separates the dually-active classes (on its left) from the primally-active classes (on its right). We obtain the orientations in each case by reversing unions of classes of the active partition, i.e. taking symmetric differences, in all possible ways. The active orientation partition is the partition of the set of $2^{|E|}$ orientations obtained in this way.

Knowing the activities of the basic orientations and which classes have been reversed yields the values of the 4 $\theta$-activities. When we reverse a dually-active resp. primally-active class, $\theta^\ast$ decreases by 1 and $\overline{\theta}^\ast$ increases by 1 resp. $\theta$ decreases by 1 and $\overline{\theta}$ increases by 1. Therefore, the monomial $x^iu^jy^kv^\ell$ associated with the basic orientation is multiplied by $x^{-1}u$ as many times as the number of reversed dually-active classes and by $y^{-1}v$ as many times as the number of reversed primally-active classes.

More precisely, let $B$ be a basis of $M$, and $X \subseteq \text{Int}_M(B)$, $Y \subseteq \text{Ext}_M(B)$. Let $A_B$ be any reorientation in the inverse image of $B$ by the active mapping - produced by an algorithm [5][7][8][9]. Let $A$ be the reorientation obtained from $A_B$ by reversing all elements in the union of the classes of the active partition activated by $X \cup Y$. Then $A' = B \setminus X \cup Y$ is the subset in the Dawson interval defined by $B$ associated with $A$ by the active bijection. Using notation of [22], $\theta$-activities of $A$ and Tutte activities of $A'$ are related as follows: $\theta^M_M(A) = cr_M(A')$ $\overline{\theta}^M_\overline{\theta}(A) = \iota_M(A')$ $\theta^M_\theta(A) = nl_M(A')$ $\overline{\theta}^M_\overline{\theta}(A) = \epsilon_\epsilon(A')$.

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