The Field Theory of the $q \to 4^+$ Potts Model

G. Delfino$^a$ and John Cardy$^{b,c}$

$^a$ Laboratoire de Physique Théorique et Hautes Energies
Université Pierre et Marie Curie, Tour 16 1er étage, 4 place Jussieu
75252 Paris cedex 05, France

$^b$ Theoretical Physics, University of Oxford
1 Keble Road, Oxford OX1 3NP, United Kingdom

$^c$ All Souls College, Oxford

Abstract

The $q$-state Potts model in two dimensions exhibits a first-order transition for $q > 4$. As $q \to 4^+$ the correlation length at this transition diverges. We argue that this limit defines a massive integrable quantum field theory whose lowest excitations are kinks connecting $4 + 1$ degenerate ground states. We construct the $S$-matrix of this theory and the two-particle form factors, and hence estimate a number of universal amplitude ratios. These are in very good agreement with the results of extrapolated series in $q^{-1/2}$ as well as Monte Carlo results for $q = 5$. 
The $q$-state Potts model, defined by the lattice Hamiltonian

$$H = -J \sum_{\langle x,y \rangle} \delta(s(x),s(y)),$$

where the spin variable $s(x)$ assumes $q$ different values (colours), continues to be of fundamental importance in the description of a large variety of critical phenomena, ranging from ferromagnetism to percolation and adsorbed monolayers [2]. It was shown by Baxter that in two dimensions the ferromagnetic model undergoes a phase transition which is continuous for $q \leq 4$ and first order otherwise [3]. The Coulomb gas [4] and conformal field theory (CFT) [5] provided later a complete description of the second order phase transition line, which turned out to correspond to CFT’s with central charge $c \leq 1$, the value $c = 1$ corresponding to the end point $q = 4$. More recently, integrable field theory has lead to new results for the scaling limit of the off-critical model for $q \leq 4$ [6, 7, 8].

Concerning the first order transition for $q > 4$, several exact lattice results – internal energy [6], magnetisation [9], correlation length [10] – are known, but progress through field theoretic methods is generally prevented by the absence of a scaling limit. When $q$ approaches 4 from above, however, the correlation length at $T_c$ diverges and a continuum description in terms of a massive quantum field theory should be possible. The identification and solution of the quantum field theory describing the limit $q \to 4^+$ at $T_c$, as well as the determination of the universal critical quantities, are the subject of this note.

The correlation length at criticality of the lattice model is known to behave as

$$\xi \sim ae^{\pi^2/\sqrt{q-4}},$$

as $q \to 4^+$, where $a$ is proportional to the lattice spacing, so that the continuum limit corresponds to taking the limits $a \to 0$, $q \to 4^+$ in such a way that $\xi$ remains finite. The presence of an essential singularity rather than a power law divergence in Eq. (2) is characteristic of a perturbing operator which is only marginally relevant (scaling dimension 2) – except that in the original description of the Potts model $q$ is a parameter which should be invariant along RG trajectories and therefore such an interpretation needs to be treated with some care. In fact we know that at $q = 4$ there is such a marginal field $\psi$: it was shown by Nienhuis et al. [11] to correspond to the fugacity for vacancies in the lattice model. When $\psi < 0$ the transition is second order, with logarithmic modifications to scaling arising from the marginal irrelevance of $\psi$, but when $\psi > 0$, the transition is first order with a correlation length at the transition which diverges as $\psi \to 0^+$ like

$$\xi \sim ae^{1/\psi}.$$
where $b$ is a constant.

The main point is that the scaling limits in which $a \to 0$ with $\xi$ fixed are identical in these two cases. This may be seen, for example, from the general structure of the RG equations near $q = 4$ and $\psi = 0$. Based on the analysis of Nienhuis et al. [11], Cardy, Nauenberg and Scalapino [12] argued that these have, to lowest order, the general form
\[
\frac{d\psi}{dl} = b\psi^2 + a(q - 4) + O(\psi^3, (q - 4)\psi, (q - 4)^2)
\]
\[
\frac{dt}{dl} = (y_t + c_t\psi)t + O(\psi^2t, (q - 4)t, t^2)
\]
\[
\frac{dh}{dl} = (y_h + c_h\psi)h + O(\psi^2h, (q - 4)h, th)
\]

where $t$ is the deviation from the critical temperature and $h$ is a symmetry-breaking field. $a$, $b$, $c_t$ and $c_h$ are all constants, certain combinations of which are universal [12]. An important feature of (5,6) is that, to lowest nontrivial order, the $y$ do not involve $q$. This is because, as may be seen from the first equation, $q - 4$ is effectively $O(\psi^2)$. Integrating (4) up to a value $\tilde{l}$ such that $\psi(\tilde{l}) = O(1)$ then gives results for the correlation length $\xi \sim e^{\tilde{l}}$ in agreement with (2,3) in the two cases $q > 4$ and $q = 4, \psi > 0$. The various thermodynamic quantities are then found by integrating the other equations up to $\tilde{l} \sim \ln \xi$.

To the order stated, the results will be identical in the two cases, when expressed in terms of $\xi$. Therefore the scaling limits are identical.

Another way of understanding this is through the mapping of the lattice Potts model to a height model and thence to a Coulomb gas or sine-Gordon theory [4]. From this point of view, $q$ is merely a parameter identified with a certain function of the coupling constant conventionally called $\beta$. $q = 4$ corresponds to $\beta^2 = 8\pi$ at which point the operator corresponding to $\psi$ becomes marginal. Within the standard RG picture of the sine-Gordon model, both $\beta$ and $\psi$ have non-trivial marginal flows. However the scaling limit, corresponding to the massive sine-Gordon theory at $\beta^2 = 8\pi$, is unique, and therefore describes both the cases $q \to 4^+$ and $\psi \to 0^+$ at $q = 4$.

Having made this observation, we may take over the results of Ref. [13] in which it was pointed out that the scaling limit of the massive $q = 4$ theory is integrable, and in which the scattering theory and form factors were determined.

In our case, the construction of the scattering theory goes as follows. Along the first order phase transition line, $q$ ordered ground states are degenerate with the disordered ground state. The field theory describing the scaling limit $q \to 4^+$ has 4 ordered vacuum states $\Omega_i, i = 1, \ldots, 4$. Invariance under colour permutations implies that, in the order parameter space, they lie at the vertices of a tetrahedron having the disordered vacuum $\Omega_0$ at its center. The elementary excitations of the scattering theory are stable kinks
$K_{0i}(\theta), K_{i0}(\theta)$ interpolating between the center of the tetrahedron and the $i$-th vertex, and vice versa. We denote by $\theta$ the rapidity variable parameterising the on-shell momenta as $(p^0, p^1) = (m \cosh \theta, m \sinh \theta)$. The mass of the kinks $m \sim \xi^{-1}$ measures the deviation from the conformal point $q = 4$. The space-time trajectory of a kink on the plane draws a domain wall separating a coloured phase from the disordered one. The space of asymptotic states is made of multi-kink sequences in which adjacent vacuum indices belonging to different kinks have to coincide. For example, up to possible bound states, the lightest excitation interpolating between two ordered vacua is $K_{i0}(\theta_1)K_{0j}(\theta_2)$.

The factorisation of multi-kink processes reduces the scattering problem to the determination of the two-kink amplitudes. Colour permutation symmetry allows only for the four elementary processes depicted in Fig. 1. The four amplitudes can be determined as a solution of the requirements of unitarity (crucially simplified by the absence of particle production), crossing and factorisation. The scattering amplitudes of Fig. 1 are given in [13] and read

\begin{align}
A_0(\theta) &= e^{-i\gamma\theta} \frac{2i\pi - \theta}{i\pi - \theta} S_0(\theta), \\
A_1(\theta) &= e^{-i\gamma\theta} \frac{\theta}{i\pi - \theta} S_0(\theta), \\
B_0(\theta) &= e^{i\gamma\theta} \frac{i\pi + \theta}{i\pi - \theta} S_0(\theta), \\
B_1(\theta) &= e^{i\gamma\theta} S_0(\theta),
\end{align}

where $\theta$ is the rapidity difference of the two kinks, $\gamma = \frac{1}{\pi} \ln 2$, and

\begin{equation}
S_0(\theta) = \frac{\Gamma \left( \frac{1}{2} + \frac{\theta}{2\pi} \right) \Gamma \left( -\frac{\theta}{2\pi} \right)}{\Gamma \left( \frac{1}{2} - \frac{\theta}{2\pi} \right) \Gamma \left( \frac{\theta}{2\pi} \right)} = -\exp \left\{ i \int_0^\infty \frac{dx}{x} \frac{e^{-\frac{x}{2}}}{\cosh \frac{x}{2}} \sin \frac{x\theta}{\pi} \right\}.
\end{equation}

The absence of poles in the physical strip $\text{Im} \theta \in (0, i\pi)$ ensures that there are no bound states and that the four amplitudes above completely determine the scattering theory. This $S$-matrix shares evident analytic similarities with that of the $SU(2)$-invariant Thirring model. As a matter of fact, the latter is a realisation of the same perturbed CFT on a different particle basis (an $SU(2)$ doublet rather than our kinks). The possibility for a single perturbed CFT to be invariant under different symmetry groups and to describe different universality classes is discussed, for example, in Ref. [14].

Making contact with the thermodynamics requires the computation of correlation functions. In our $S$-matrix framework these are obtained as spectral series summing over all multi-kink intermediate states. Neglecting terms of order $e^{-4m|x|}$ in this large distance
expansion, we will approximate the (connected) two-point correlator of a scalar operator \( \Phi(x) \) as
\[
\langle \Omega_0 | \Phi(x) \Phi(0) | \Omega_0 \rangle \simeq \sum_{i=1}^{4} \int_{\theta_1 > \theta_2} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} |F^\Phi_{0i}(\theta_1 - \theta_2)|^2 e^{-|x|E_2} ,
\]
in the disordered phase, and
\[
\langle \Omega_i | \Phi(x) \Phi(0) | \Omega_i \rangle \simeq \int_{\theta_1 > \theta_2} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} |F^\Phi_{i0}(\theta_1 - \theta_2)|^2 e^{-|x|E_2} ,
\]
in the \( i \)-th ordered phase. Here \( E_2 = m(\cosh \theta_1 + \cosh \theta_2) \) is the energy of the two-kink asymptotic state and we introduced the two-kink form factors
\[
F^\Phi_{0i}(\theta_1 - \theta_2) = \langle \Omega_0 | \Phi(0) | K_{0i}(\theta_1) K_{i0}(\theta_2) \rangle ,
\]
\[
F^\Phi_{i0}(\theta_1 - \theta_2) = \langle \Omega_i | \Phi(0) | K_{i0}(\theta_1) K_{0i}(\theta_2) \rangle .
\]

The operators of interest for us are the spin \( \sigma_j(x) = \delta_{s(x),j} - 1/q \), and the energy \( \varepsilon(x) = \sum_y \delta_{s(x),s(y)} \), whose scaling dimensions around the \( q = 4 \) fixed point are \( X_{\sigma} = 1/8 \) and \( X_{\varepsilon} = 1/2 \) [4].

Some consequences for the physics of the coexisting phases follow immediately from the structure of the scattering theory. The ‘true’ correlation length \( \xi \) is determined by the large distance decay of the spin-spin correlator as \( \langle \sigma_j(x)\sigma_j(0) \rangle \sim e^{-|x|/\xi} \). Then it follows from (12,13) that
\[
\xi_o = \xi_d = 1/2m
\]
(here and below the subscript \( o \) (d) denotes quantities computed in the ordered (disordered) phase). Numerical simulations [15] and large \( q \) expansions [16] suggest that the phase independence of \( \xi \) holds true for all \( q > 4 \).

Since the interfacial tension between two coexisting phases is given by the total mass of the lightest excitation interpolating between them, we also have \( \sigma_{od} = m \) and \( \sigma_{od} = \sigma_{oo}/2 \). The latter result is known to hold for all for \( q > 4 \) [17].

The relation of the other interesting thermodynamic quantities (spontaneous magnetisation \( M \), latent heat \( L \), susceptibility \( \chi \), specific heat \( C \), second moment correlation length \( \xi_{2nd} \)) with the connected correlators of \( \sigma_j \) and \( \varepsilon \), and their behaviour as \( q \to 4^+ \) are
\[
M = \langle \Omega_j | \sigma_j | \Omega_j \rangle \simeq B \xi^{-1/8} ,
\]
\[
L = \langle \varepsilon \rangle_d - \langle \varepsilon \rangle_o \simeq L \xi^{-1/2} ,
\]
\[
\chi = \int d^2x \langle \sigma_j(x)\sigma_j(0) \rangle \simeq \Gamma_{o,d} \xi^{7/4} ,
\]
\[ C = \int d^2x \langle \varepsilon(x)\varepsilon(0) \rangle \simeq A_{o,d} \xi, \]
\[ \xi_{2nd}^2 = \frac{1}{4\chi} \int d^2x |x|^2 \langle \sigma_j(x)\sigma_j(0) \rangle \simeq (f_{o,d} \xi)^2. \]

(20)

For the ordered case, the two-point correlators of \( \sigma_j \) entering \( \chi \) and \( \xi_{2nd} \) are computed on the vacuum \( |\Omega_j \rangle \). Since \( \varepsilon(x) \) is odd under the duality transformation exchanging the low- and high-temperature phases, we have

\[ \langle \varepsilon \rangle_d = -\langle \varepsilon \rangle_o, \quad A_d = A_o. \]

(21)

The critical amplitudes are normalisation dependent but can be combined into a series of universal ratios characterising the scaling limit. We can evaluate the critical amplitudes by integrating the two-particle approximations (12,13) of the correlators. What we need to know are the two-kink form factors of the operators \( \varepsilon \) and \( \sigma_j \). Once again the result is contained in Ref. [13] and reads

\[ F_{\varepsilon_{0,0i}}^\varepsilon(\theta) = \mp iL \frac{e^{\pm \frac{i}{2}(\pi+i\theta)}}{\theta - i\pi} F_0(\theta), \]
\[ F_{\sigma_{0,0i}}^{\sigma_j}(\theta) = \mp M \frac{4\delta_{ij} - 1}{6\Upsilon_+(i\pi)} \frac{e^{\pm \frac{i}{2}(\pi+i\theta)}}{\cosh \frac{\theta}{2}} \Upsilon_-(\theta) F_0(\theta), \]

(22, 23)

with \( \Upsilon_-(\theta) = \Upsilon_+(\theta + 2i\pi) \),

\[ \Upsilon_+(\theta) = \exp \left\{ 2 \int_0^\infty \frac{dx}{x} \frac{e^{-x}}{\sinh 2x} \sin^2 \left[ (2i\pi - \theta) \frac{x}{2\pi} \right] \right\}, \]
\[ F_0(\theta) = -i \sinh \frac{\theta}{2} \exp \left\{ - \int_0^\infty \frac{dx}{x} \frac{e^{-x}}{\cosh \frac{x}{2}} \sin^2 \left[ (i\pi - \theta) \frac{x}{2\pi} \right] \right\}. \]

(24, 25)

The results we obtain for the universal amplitude ratios are given in Table 1 and compared with those following from the combination of the exact \([3,4,10]\) and series \([18]\) lattice results for the amplitudes.

|                | Field theory | Lattice  |
|----------------|--------------|----------|
| \( f_d \)     | 0.6744       | 0.673(8) |
| \( f_o/f_d \) | 0.9340       | 0.935(5) |
| \( \Gamma_d/\Gamma_o \) | 1.1406       | 1.19(5)  |
| \( A_d/\mathcal{L}^2 \) | 0.1047       | 0.105(3) |
| \( \Gamma_d/B^2 \) | 0.06607      | 0.0656(15) |
Table 1. Universal amplitude ratios for the q-state Potts model at \( T = T_c, q \to 4^+ \). The field theoretical results are obtained within the two-particle approximation.

The accuracy exhibited by the two-particle approximation does not come as a surprise since it is known as a common feature of this kind of computations within integrable field theory. In the present case the accuracy is enhanced by the low scaling dimensions of the spin and energy operators which lead to mild singularities for their correlators and then to a small contribution of short distances to the integrals. We estimate that the errors on our values for the amplitude ratios do not exceed order 0.1%.

The scaling limit we discussed so far corresponds to \( q \to 4^+ \). At \( q = 5 \), however, the correlation length is still some 2500 times the lattice spacing and this suggests that our results for \( q \to 4^+ \) could still provide the basis for an approximate description.

For a generic value of \( q > 4 \), the model has \( q + 1 \) degenerate ground states at the transition point and the elementary excitations are \( 2q \) kinks going from the disordered vacuum to the \( q \) ordered vacua, and vice versa. If the correlation length is sufficiently large, an approximate scaling should still hold and then it makes sense to keep for the physical quantities the parameterisations \((17–20)\), namely a power of the correlation length times an amplitude. It is easy to see, however, that in the present case we have to allow for a \( q \)-dependence of the amplitudes. Consider in fact the correlator

\[
G_{\alpha j}(x) = \langle \Omega_\alpha | \sigma_j(x) \sigma_j(0) | \Omega_\alpha \rangle, \quad \alpha = 0, i; \quad i, j = 1, \ldots, q. \tag{26}
\]

Its two-particle approximation in the disordered phase is

\[
G_{0j}(x) \simeq \sum_{i=1}^q |F_{0i}^{\sigma j}|^2 e^{-E_2|x|} = \frac{q}{q-1} |F_{0j}^{\sigma j}|^2 e^{-E_2|x|}, \tag{27}
\]

where integration over momenta is understood. The last equality follows from colour symmetry and \( \sum_j \sigma_j = 0 \) which imply

\[
F_{0i;0}^{\sigma_j} = (q\delta_{ij} - 1)/(q - 1)F_{0j;0}^{\sigma_j}. \tag{28}
\]

When integrating the correlator to obtain the amplitude of, say, the susceptibility in the disordered phase, the form factors computed at \( q = 4 \) should give a good approximation as long as \( \xi \) is large. The explicit factor \( q/(q - 1) \) dictated by the number of intermediate states and symmetry, however, has to be taken into account and is expected to determine the main deviation from the \( q \to 4^+ \) value of the amplitude. Following the same reasoning, the susceptibility amplitude in the ordered phase should be basically constant in \( q \) since
there is only one intermediate state. More generally, we are led to expect that the ratios listed in Table 2 are approximately constant in \( q \) for \( \xi \) large enough so that the continuum description is accurate. Their values determined from the results of the large \( q \) expansion and reported in the Table seem to confirm our picture. Our field theory results for \( R_1 \) and \( R_2 \) as \( q \to 4^+ \) are 0.855 and 0.0579, respectively.

| \( q \)       | 4\(^+\) | 5     | 10    |
|---------------|---------|-------|-------|
| \( \xi \) (lattice units) |         |       |       |
| \( \xi^{(d)}_{2nd}/\xi \) | 0.673(8) | 0.671(3) | 0.6587(1) |
| \( \xi^{(d)}_{2nd}/\xi^{(o)}_{2nd} \) | 0.935(5) | 0.934(7) | 0.9579(2) |
| \( R_1 = (q - 1)/q \chi_d/\chi_o \) | 0.89(4) | 0.810(5) | 0.80399(1) |
| \( R_2 = \chi_o/(M\xi)^2 \) | 0.0550(6) | 0.0589(2) | 0.05784(1) |

Table 2. Values obtained combining the exact results of Refs. [9, 10] and the large \( q \) expansions of Ref. [18].

Let us conclude this note by considering the ‘transverse’ susceptibility \( \chi_T \) obtained integrating \( G_{ij}(x) \) rather than \( G_{jj}(x) \). From (28), in the two-particle approximation

\[
\frac{\chi_T}{\chi_o} \simeq \frac{1}{(q - 1)^2}.
\]  

(29)

This result is basically a consequence of the nature of the elementary excitations and is expected to hold as a good approximation for all \( q > 4 \) at \( T_c \), as long as \( \xi \gg a \). We are not aware of lattice results on \( \chi_T \) for comparison.

In summary, we have shown that the limit \( q \to 4^+ \) in the Potts model defines an integrable massive field theory, whose \( S \)-matrix and form factors may be computed exactly. The results for integrated correlation functions are in excellent agreement with lattice-based numerical results. This shows how methods of continuum field theory are not restricted to the description of second-order transitions only.

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Figure 1. The two-kink scattering amplitudes $A_0$, $A_1$, $B_0$, $B_1 \ (i \neq j)$.

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