Decoherence and entropy of primordial fluctuations

II. The entropy budget

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We calculate the entropy of adiabatic perturbations associated with a truncation of the hierarchy of Green functions at the first non trivial level, i.e. in a self-consistent Gaussian approximation. We give the equation governing the entropy growth and discuss its phenomenology. It is parameterized by two model-dependent kernels. We then examine two particular inflationary models, one with isocurvature perturbations, the other with corrections due to loops of matter fields. In the first model the entropy grows rapidly, while in the second the state remains pure (at one loop).

I. INTRODUCTION

This is the second part of a series of notes on the decoherence and entropy of primordial fluctuations predicted by inflation. The first part, subsequently called I [1], was devoted to the operational formulation of the notion of decoherence of metric perturbations. Here we turn to the calculation of the entropy when truncating the hierarchy of Green functions at the first non-trivial level, first in general and then in two particular models. As an introduction, we briefly review the present state-of-the-art concerning the entropy growth during inflation.

Up to now, the time dependence of the entropy as well as the main source(s) of entropy remain undetermined. Part of the difficulties arises from the definition of the reduced density matrix. Concerning the methodology, previous studies either work with a master equation for the reduced density matrix [2, 3], or directly calculate the reduced density matrix [4, 5], or use analogies with other quantum mechanical situations [6].

The main problem with the first two approaches comes from the actual calculation of the trace. While this poses no difficulty for Gaussian models as in [4, 5], it is hampered by the infinities ubiquitous in interacting Quantum Field Theories. In previous attempts where this calculation was undertaken [2, 3], unjustified assumptions were made and lead to contradictory results. We argued in paper I that the only way to deal properly with these infinities is to work with Green functions. The approach of [2, 3] is different as it assumes a general Gaussian and Markovian Ansatz for the master equation. The limitation of this approach is that it does not make predictions. Indeed, first the parameters of the master equation are undetermined unless they are calculated in field theoretical settings. Second, the Markovian hypothesis is unjustified for superhorizon perturbations at any epoch, as well as for perturbations of any scale during inflation.

In this paper, we present an approach which combines the advantages of the previous ones, but without their shortcomings. We work with Green functions from the onset. Infinities are handled by the standard renormalization techniques, and the general evolution equation for the entropy follows straightforwardly from the knowledge of these Green functions. This emphasis on Green functions is not new [7], but to our knowledge it has not been applied to the calculation of the entropy of primordial fluctuations in realistic models of inflation.
After presenting the settings in Sec. II, the evolution equation of the entropy of the reduced density matrix is derived in Sec. III, followed by a general discussion of this equation. We then examine two specific models. A model of two field inflation is presented in Sec. IV. It confirms and generalizes the analysis of [5]. In particular we identify the conditions under which the entropy grows with the number of e folds. In Sec. V we compute the entropy associated to one-loop corrections from matter fields in a class of semi-realistic theories analyzed by Weinberg in [8]. In these models we show that no significant decoherence occurs during inflation.

II. THE GAUSSIAN APPROXIMATION

We work in the coordinate system in which the inflaton field is homogeneous on each space-like hypersurface,
\[
\varphi(t, x) = \varphi_0(t).
\]
We focus on the curvature perturbations \( \zeta \). In the linear approximation, their evolution is described by the action
\[
S = \frac{1}{8\pi G} \int dt \, d^3x \, a^3 \epsilon \left( \dot{\zeta}^2 - \frac{1}{a^2} (\nabla \zeta)^2 \right),
\]
where \( a \) is the scale factor and \( \epsilon = -\dot{H}/H^2 \) is the first slow roll parameter, see [9] for a comprehensive derivation. A dot stands for the derivation w.r.t. the cosmological time \( t \).

The Fourier mode labeled with the conserved comoving wave vector \( q \) obeys
\[
\ddot{\zeta}_q + \frac{d \ln (a^3 \epsilon)}{dt} \dot{\zeta}_q + \frac{q^2}{a^2} \zeta_q = 0,
\]
where \( q \) is the norm of the vector \( q \).

The free vacuum is taken to be the Bunch-Davis vacuum, defined from the positive frequency solutions of (3) for infinite physical momentum,
\[
(i \partial_\tau - q) \left( a\sqrt{\epsilon} \zeta_{q}^{\text{in}} \right) \to 0, \quad \frac{q}{aH} \to \infty.
\]
\( \tau \) is the conformal coordinate time \( dt = ad\tau \). Using this solution, the mode operator can be decomposed as
\[
\zeta_q(t) = a_q^{\text{BD}} \zeta_q^{\text{in}}(t) + a_{-q}^{\text{BD}} \zeta_q^{\text{in}*}(t),
\]
where the destruction operator \( a_q^{\text{BD}} \) annihilates the Bunch-Davis vacuum.

We shall work in a quasi-de Sitter approximation. It can be shown that the results are still valid in slow roll inflation. In this approximation, the \( in \) mode is
\[
\zeta_q^{\text{in}}(t) = \zeta_q^0(1 + iq\tau)e^{-i q \tau}.
\]
The constant \( \zeta_q^0 \) is fixed by the equal time commutator \([\zeta_q, \pi_{q'}] = i\delta^3(q - q')\), where
\[
\pi_q = \frac{a^3 \epsilon}{4\pi G} \dot{\zeta}_q,
\]
is the Fourier component of the momentum conjugate to \( \zeta \). One finds
\[
|\zeta_q^0|^2 = \frac{4\pi G}{\epsilon} \frac{H^2}{2q^3}.
\]
In the linearized description based on Eq. (2), the state of $\zeta$ stays Gaussian and its properties are characterized by a single function, the power spectrum $\propto |q_0^\text{in}|^2$. In interacting field theories, the state is characterized by the full hierarchy of connected Green functions. Such knowledge is out-of-reach, so that in practice one resorts to a (self-consistent) truncation of this hierarchy. This coarse graining defines a reduced density matrix with a non vanishing entropy, see Sec. III in paper I. The first non trivial level of truncation is the c-number function $G(\tau, \tau'; q)$ given by

$$G(\tau, \tau'; q) \propto e^{-\frac{1}{2} \text{Tr} \left( \rho_{\text{red}} \{ \zeta_q(\tau), \zeta_{-q}'(\tau') \} \right)}.$$  

Since the different sectors do not mix, we shall work at fixed $q$ and no longer write the trivial $\delta^3(q - q')$ coming from the plane wave normalization.

To calculate the entropy carried by $\rho_{\text{red}}$ it is convenient to recast the information contained in $G(\tau, \tau'; q)$ into the covariance matrix $C$ defined by

$$C \equiv \frac{1}{2} \text{Tr} \left( \rho \left\{ V, V^\dagger \right\} \right) = \left( \begin{array}{cc} \mathcal{P}_\zeta & \mathcal{P}_{\zeta \pi} \\ \mathcal{P}_{\zeta \pi} & \mathcal{P}_\pi \end{array} \right), \quad V = \left( \begin{array}{c} \zeta_q \\ \pi \end{array} \right).$$

As shown in Appendix A of the Gaussian approximation it is always possible to make a canonical transformation $(\zeta, \pi) \mapsto (\zeta', \pi')$ where the new momentum is related to $\zeta$ as in (7). This canonical transformation leaves invariant the entropy [1]. Using Eq. (7), the three moments $\mathcal{P}$ are then related to $G$ by

\begin{align}
\mathcal{P}_\zeta(q, t) &= G(t, t; q), \\
\mathcal{P}_{\zeta \pi}(q, t) &= \frac{a^3 \epsilon}{4\pi G} \partial_t G(t, t'; q)|_{t=t'}, \\
\mathcal{P}_\pi(q, t) &= \left( \frac{a^3 \epsilon}{4\pi G} \right)^2 \partial_{\pi} G(t, t'; q)|_{t=t'}.
\end{align}

For the linear perturbations in the Bunch-Davies vacuum, since $G(\tau, \tau'; q) = \mathcal{R}(\zeta^\text{in}_q(\tau) \zeta^\text{in}_q(\tau'))$ where $\zeta^\text{in}_q$ is given in Eq. (3), the (unperturbed) covariances are

\begin{align}
\mathcal{P}^0_\zeta(q, \tau) &= |\zeta^\text{in}_q(\tau)|^2 = |q_0^\text{in}|^2 \left( 1 + x^2 \right), \\
\mathcal{P}^0_{\zeta \pi}(q, \tau) &= \frac{a^3 \epsilon}{4\pi G} \mathcal{R}(\zeta^\text{in}_q(\tau) \partial_t \zeta^\text{in}_q) = -H x^2 |q_0^\text{in}|^2 \left( \frac{a^3 \epsilon}{4\pi G} \right), \\
\mathcal{P}^0_\pi(q, \tau) &= \left( \frac{a^3 \epsilon}{4\pi G} \right)^2 |\partial_{\pi} \zeta^\text{in}_q|^2 = H^2 x^4 |q_0^\text{in}|^2 \left( \frac{a^3 \epsilon}{4\pi G} \right)^2,
\end{align}

where we used the relation $\tau \simeq -1/aH$ and introduced

$$x = \frac{q}{aH} = e^{-N}.$$  

$N = -\ln x$ is the number of e-folds with respect to horizon exit.

The entropy of $\rho_{\text{red}}$ is related to the determinant of the covariance matrix by

$$S = 2 \left[ (\bar{n} + 1) \ln(\bar{n} + 1) - \bar{n} \ln(\bar{n}) \right],$$

where the parameter $\bar{n}$ is defined by

$$\left( \frac{\bar{n} + 1}{2} \right)^2 \equiv \text{det}(C) = \mathcal{P}_\zeta \mathcal{P}_\pi - \mathcal{P}^2_{\zeta \pi}.$$
The prefactor 2 in (14) accounts for the fact that $\rho_{\text{red}}^{(q,-q)}$ is the state of two modes. The pure states, $S = \tilde{n} = 0$, correspond to $\det(C) = 1/4$. At any time, one verifies that Eqs. (12) exactly give this constant value of the determinant. Notice also that, had we neglected the terms $O(q^2/a^2H^2)$ in (12a), we would have instead obtained $\det(C) = 0$ which makes no sense in quantum mechanical settings, since the inequality $\det(C) \geq 1/4$ is nothing but the Heisenberg uncertainty relations.

We will also employ another useful parameterization of the covariance matrix

$$n_\zeta(t) \equiv \text{Tr} \left( \rho_{\text{red}}^{(q,-q)}(t) a^{\text{BD}}_q a^{\text{BD}}_q \right),$$

$$c_\zeta(t) \equiv \text{Tr} \left( \rho_{\text{red}}^{(q,-q)}(t) a^{\text{BD}}_q a^{\text{BD}}_{-q} \right),$$

$n_\zeta$ is real while $c_\zeta$ is complex. We recall that these parameters depend on the choice of canonical variables, that is on the choice of $(a_q, a_q^\dagger)$, see [1] for more details. So does the parameter $\delta_\zeta$ defined in the next equation. This parameter characterizes the correlations between the modes $q$ and $-q$ in the state $\rho_{\text{red}}$:

$$|c_\zeta|^2 \equiv n_\zeta(n_\zeta + 1 - \delta_\zeta), \quad 0 \leq \delta_\zeta \leq n_\zeta + 1.$$ (18)

In this parameterization, the determinant of $C$ is

$$\det(C) = \left(n_\zeta + \frac{1}{2}\right)^2 - |c_\zeta|^2 = \frac{1}{4} + n_\zeta \delta_\zeta.$$ (19)

The entropy is a monotonically growing function of $\delta_\zeta$. This parameter quantifies the residual coherence in the state: the state is pure when $\delta_\zeta = 0$ and thermal when $\delta_\zeta = n_\zeta + 1$. The threshold value $\delta_\zeta = 1$ separates entangled states from decohered states which cannot be distinguished from statistical ensembles [1]. At the threshold value, in the radiation dominated era, the entropy is

$$S_{\text{sep}} = \ln n_\zeta^{\text{end}} \simeq -\ln \left(x_\text{end}^4\right) = 4 N_{\text{end}},$$ (20)

where $N_{\text{end}} = -\ln x_\text{end}$ is the number of e-folds from horizon exit to the end of inflation. We have used the linearized treatment to estimate $n_\zeta^{\text{end}}$ (i.e. $n_\zeta = 1/4 x_\text{end}^4$) [11]. The maximal (thermal) entropy (per two-modes) is $S_{\text{max}} = 2S_{\text{sep}} \simeq 2 \ln n_{\text{end}} \simeq 8N_{\text{end}}$.

### III. EVOLUTION OF THE ENTROPY

#### A. Equation of evolution

As seen from Eqs. (14) and (15), $S$ is a monotonically growing function of $\det(C)$. In particular, for $n\delta \gg 1$, the entropy is simply given by $S = \ln(n\delta) = \ln(\det C)$. Hence, in this regime one has

$$n\delta \gg 1, \quad \dot{S} = \frac{d}{dt} \ln(\det C).$$ (21)

The evolution of $\det(C)$ follows from that of the three expectation values of Eqs. (11) which are all determined by the anticommutator $G$. Therefore the time dependence of $\det(C)$ is governed by the equation obeyed by $G$. The latter is a linear second order integro-differential equation. It is derived in Appendix A1. But, as explained in A2, it is simpler to exploit the Gaussianity and derive $G$ from an equivalent quantum Langevin
equation. That is, $G$ is the anticommutator of the operator $\zeta_q(t)$ which verifies the effective equation

$$\ddot{\zeta}_q + \frac{d\ln(a^3\epsilon)}{dt} \dot{\zeta}_q + \frac{q^2}{a^2} \zeta_q(t) + \int_{-\infty}^{t} dt' D_q(t, t') \zeta_q(t') = \xi_q(t). \quad (22)$$

In the Gaussian approximation, the effects of the interactions are summarized in two kernels, namely the "dissipation" kernel $D_q$ and the so-called noise kernel

$$N_q(t, t') = \frac{1}{2} \langle \{\xi_q(t), \xi_{-q}(t')\} \rangle. \quad (23)$$

$N$ and $D$ appear in the effective action of the curvature perturbation as the real and imaginary parts of the renormalized self-energy $\langle A_{15} \rangle$. For the differential part of $(22)$ we assumed the same structure as in the free mode equation $(3)$. Equation $(22)$ could be generalized to the case where the renormalized frequency differs from $q^2/a^2$. This would only affect the retarded Green function and $G$ but not the form of the equation $(28)$ governing $\det(C)$. However our result does not generalize to time dependent "wave function renormalization" since, should they occur, they would invalidate the use of Eqs. $(24)$ and $(25)$.

We now proceed assuming that the kernels $D$ and $N$ are known, and that $G$ and its time derivatives possess well-defined coincidence point limits. We derive an equation for the covariances of $\zeta$. For this we need the identities

$$\frac{dP_{\zeta}}{dt} = 2P_{\zeta\zeta}, \quad (24)$$

$$\frac{dP_{\zeta\zeta}}{dt} = P_{\zeta\zeta} + \frac{1}{2} \langle \{\dot{\zeta}_q(t), \zeta_{-q}(t)\} \rangle. \quad (25)$$

The equations can be easily derived in the Heisenberg picture, since time derivation and taking the expectation value, being both linear operations, commute. These equations are easily verified for $(12)$ and $(3)$. Inserting $(22)$ into $(25)$ we get

$$P_{\zeta\zeta} = \frac{dP_{\zeta}}{dt} - \frac{1}{2} \langle \{\zeta_q(t), \zeta_{-q}(t)\} \rangle$$

$$= \left( d\frac{\ln(a^3\epsilon)}{dt} \right) P_{\zeta\zeta} + \frac{q^2}{a^2} P_{\zeta} \quad (26a)$$

$$+ \int_{-\infty}^{t} dt' D_q(t, t') \frac{1}{2} \langle \{\zeta_q(t), \zeta_{-q}(t')\} \rangle - \frac{1}{2} \langle \{\xi_q(t), \zeta_{-q}(t)\} \rangle. \quad (26b)$$

The three terms in $(26a)$ are local in time and do not contribute to the time evolution of $\det(C)$, as will be seen in Eq. $(25)$. Instead, the terms in $(26b)$ are non local and absent in the free evolution. Without them, the time evolution of $P_{\zeta\zeta}$ is entirely determined by that of $P_{\zeta\zeta}$ and $P_{\zeta\zeta}$, as shown in Eq. $(25)$.

To compute the growth in entropy, we also need the time derivative of $P_{\zeta\zeta}$, obtained from using again $(22)$,

$$\frac{dP_{\zeta\zeta}}{dt} = \langle \dot{\zeta}_q(t), \zeta_{-q}(t) \rangle = -2 \frac{d\ln(a^3\epsilon)}{dt} P_{\zeta\zeta} - 2 \frac{q^2}{a^2} P_{\zeta\zeta} \quad (27)$$

$$- \int_{-\infty}^{t} dt' D_q(t, t') \left( \{\zeta_q(t), \zeta_{-q}(t')\} \right) + \left( \{\xi_q(t), \xi_{-q}(t)\} \right).$$
Using the equations (7), (26) and (27), we find

\[
\frac{1}{2} \frac{d \det(C)}{dt} = \left( \frac{a^3 \epsilon}{4\pi G} \right)^2 \left\{ \mathcal{P}_\zeta(t) \int_{-\infty}^t dt' \left[ D_q(t,t') G(t,t') - N_q(t,t') G_{ret}(t,t') \right] 
\right.
\]

\[
+ \mathcal{P}_\zeta(t) \int_{-\infty}^t dt' \left[ N_q(t,t') \partial_t G_{ret}(t,t') - D_q(t,t') \partial_t G(t,t') \right] \right\},
\]

where \( G_{ret} \) is the retarded propagator associated with (22). The prefactor \( (a^3 \epsilon/4\pi G)^2 \) comes from the relation (7). We will verify on several examples in Sec. IV that this prefactor yields a rapid growth in \((aH/q)^6\).

Equation (28) governs the rate of change of the entropy defined by reducing the state as explained in Section II. It is generic and exact. The explicit expressions of the noise and dissipation kernels are of course model dependent. The important point is that they can be calculated using standard QFT techniques.

This equation is equivalent to a master equation, but has several advantages over it. The master equation is the evolution equation for the reduced density matrix. It is obtained by calculating the partial trace on the r.h.s. of the Heisenberg equation

\[
i \frac{d}{dt} \rho^\text{red}_{q,-q} = \text{Tr}_{q' \neq q} \left[ [H, \rho] \right],
\]

where \( H \) and \( \rho \) are the Hamiltonian and density matrix of the entire system. The trace is performed on all field configurations except \( \zeta_q \). The difficulty with (29) (or the equivalent equation for its Wigner representation) is to actually calculate the trace. This is relatively straightforward in non relativistic quantum mechanics where the master equation has indeed proven to be a fruitful approach, see [11] and [12, 13] for recent reviews. For quantum fields however, the first nontrivial contributions to the r.h.s. of (29) arise from loop corrections. Therefore, the calculation can only be reliably done at the level of Green functions. Previous attempts to calculate directly (29) involved dubious assumptions and approximations which have lead to contradictory results, compare e.g. [2, 3, 6]. Moreover, Eq. (29) does not bring us anywhere nearer to the growth of entropy since one still has to calculate the covariances. In contrast, Eq. (28) involves the renormalized expectations values of the relevant observables.

In the remainder of this section we adopt a phenomenological approach and discuss how \( \dot{S} \) depends on the properties of \( N \) and \( D \). We start with two general remarks.

As a consistency check we verify that \( \det(C) \) is constant in an equilibrium state and in a static universe. In that case, \( N \) and \( D \) and \( G \) and the commutator \( G[.] \) are related by the fluctuation-dissipation (or KMS) relations (for each \( q \)),

\[
N(\omega) = \coth \left( \frac{\omega}{2k_B T} \right) D(\omega),
\]

\[
G(\omega) = \coth \left( \frac{\omega}{2k_B T} \right) G_{ret}(\omega),
\]

where \( T \) is the temperature. Inserting these identities into the r.h.s. of (28) yields, as expected, \( \det(C) = \text{cte} \) through a detailed balance.

We also note that out of equilibrium, the sign on the r.h.s. of (28) is a priori undetermined. However, one expects on physical grounds that the entropy averaged over sufficient time scales does not decrease.\(^1\)

\(^1\) This is indeed what happens, for instance, during the early stages of the far-from-equilibrium evolution of a quantum field [14]. Then the value of \( \det(C) \) oscillates around a monotonously growing mean value.
We now discuss two limiting cases where $N$ and $D$ are approximately local in time or not. We then return to the general discussion of (28) in light of these results.

**B. Markov approximation**

This is by definition the regime where the correlation time of $\xi$ is short compared to the characteristic time scale(s) of evolution of the cosmological perturbations. In this limit, both $N$ and $D$ are local in time. Let us assume that the environment resembles a thermal bath (contrary to the discussion at the end of the previous section we do not assume here that $\zeta$ is at equilibrium). We write the Ansatz

$$N(t, t') = N(t)\delta(t - t'), \quad D(t, t') = D(t)\frac{\partial}{\partial t'}\delta(t - t'),$$

(31)

where $N(t)$ and $D(t)$ are slow functions of time compared to the correlation time scale of the environment, caused for instance by the adiabatic expansion of the universe. According to Eq. (22) dissipation corresponds to $D > 0$. Using the identities $G_{\text{ret}}(t, t) = 0$, $\partial_t G_{\text{ret}}|_{t'=t} = 1$, and the definitions (11), Eq. (28) yields

$$\frac{d\det(C)}{dt} = -2D(t)\det(C) + 2\left(\frac{a^3\epsilon}{4\pi G}\right)^2 N(t)\mathcal{P}_\zeta.$$

(32)

It is interesting that in this limit, the dissipation kernel acts on the determinant of $C$, whereas the noise kernel acts only on the power spectrum times the factor $a^6$. To continue the discussion, we simplify Eq. (32) by dropping the term proportional to $D$. This is a bad approximation to describe the approach to equilibrium, since (32) would not lead to $\det(C) \to \text{cte}$. Far from equilibrium, it leads to an upper bound on the entropy (since $D$ and $\det(C)$ are both positive). Then, for $n \gg 1$, the rate of growth of the entropy per efold is

$$\frac{dS}{dN} \simeq \frac{a^6(t)c^2H^{-1}N(t)\mathcal{P}_\zeta(t)}{\int_0^N dN' a^6(t)c^2H^{-1}N'\mathcal{P}_\zeta}.$$

(33)

The relevance of the Markov limit for cosmological perturbations is on the contrary disputable. The typical context where the Markovian limit emerges is that of an environment in a thermal state at high temperature [11]. This is irrelevant for inflation (both single and multi-fields) since the perturbations are in a squeezed state. This is also unlikely to be a realistic model after horizon reentry but before decoupling, because the scales entering the horizon are not in thermal equilibrium (since we observe acoustic peaks). We will come back to this point in the conclusions.

**C. Non Markovian regime**

At low and vanishing temperatures the Markovian approximation fails and all the terms in the r.h.s. of (28) are a priori of equal importance since $D \sim N$. Most likely this is the case relevant for (single field) inflation since all energy densities associated to particle excitations are redshifted away (with the exception of multifield inflation treated in Sec. [IV]).

When $N$ and $D$ are of the same order, we see on (28) the possibility of a partial cancellation between the terms proportional to $\mathcal{P}_\zeta$ and $\mathcal{P}_{\zeta\zeta}$. It is perhaps easier to understand
this from the decomposition of the covariance matrix into the sum of the free contributions Eqs. (12) and a remainder $M$

$$C = C_{\text{free}} + M.$$  \hfill (34)

This separation follows naturally from the perturbative calculation of $G$ in (11) where $M$ regroups the corrections due to interactions, see Appendix A1. In the representation (16-17), the decomposition (34) corresponds to

$$n = n_{\text{free}} + \alpha, \quad c = c_{\text{free}} + \alpha \chi.$$  \hfill (35)

where $|c_{\text{free}}|^2 = n_{\text{free}}(n_{\text{free}} + 1)$, where $\alpha$ is real and $\chi$ is a complex number with norm smaller than 1. The quantity $\alpha$ is essentially governed by the power of the fluctuations of the environment, while $\chi$ accounts for a possible squeezing of the state of the environment (which implies in particular $N(t, t') \neq N(t)\delta(t - t')$). The corresponding value of $\delta$ as defined in (18) is

$$\delta(\alpha, \chi) = 2\alpha \left( 1 + \frac{1 - \alpha}{2n_{\text{free}}} \right) \left[ 1 - \text{Re} \left( \chi e^{-i\arg(c_{\text{free}})} \right) \right] + \frac{\alpha^2}{n_{\text{free}}} (1 - |\chi|^2) + O \left( \frac{1}{n_{\text{free}}^2} \right).$$  \hfill (36)

This equation clearly shows the competition between the contribution $\alpha$, that tends to increase the entropy, and the contribution of $\chi$ when the latter has a phase similar to $\arg(c_{\text{free}})$. In particular, when $\arg(\chi) = \arg(c_{\text{free}})$, the first term in (36) vanishes, $\delta$ is quadratic in $\alpha$, and will thus still obey $\delta \ll 1$. Equation (36) shows that due to subtle interference effects, decoherence may be largely suppressed when the perturbed state is squeezed as the unperturbed one. We will have an example of this phenomenon in Sec. V where the noise is due to matter fields in the interacting vacuum. Given the fine tuned nature of the condition $\arg(\chi) = \arg(c_{\text{free}})$, it is likely that this is the only case where it occurs. We notice finally that a similar mechanism has long been envisaged to circumvent the fundamental limit of quantum noise (i.e. the shot noise and pressure noise of the laser) in interferometric detectors of gravitational waves.\footnote{In the third generation of those detectors, the numerous sources of noise could possibly be reduced enough so that the dominant one would be the quantum noise of the laser monitoring the position of the mirrors, see for instance [15]. In conventional interferometers, the laser produces two types of noise, the shot noise proportional to the intensity and the radiation pressure noise inversely proportional to the intensity. The total quantum noise can therefore not be arbitrarily small. This minimal value is called the Standard Quantum Limit. Noticing that light is squeezed during its travel in the arms of the interferometer, Unruh [16] proposed that judiciously squeezing the light send in the so-called dark port of the interferometer could beat the Standard Quantum Limit.}

\section*{D. Constant rate of entropy growth}

One can easily show that the entropy rate depends crucially on how fast the solution of (22) asymptotes to a constant (when it does) outside the horizon. To do this, we use the equivalence between the quantum and stochastic versions of Eq. (22) as explained in Appendix A2 and we consider the Ansatz

$$\zeta_q(t) = \zeta_0^q \left( 1 + \alpha x + \beta \frac{x^2}{2} + \gamma \frac{x^3}{3} + O(x^4) \right).$$  \hfill (37)

written with the variable $x = q/aH$. The coefficients $\alpha, \ldots$ of the expansion are the sum of two terms, $\alpha = \bar{\alpha} + \tilde{\alpha}$: a constant term $\bar{\alpha} = \langle \alpha \rangle$, corresponding to the positive
frequency solution of the free mode equation, given by $\alpha = 0$, $\beta = 1/2$ and $\gamma = i/3$; and a stochastic component $\tilde{\alpha}$, ... which depend linearly on the stochastic source $\xi$ and the dissipation kernel $D$, so that $\langle \langle \tilde{\alpha} \rangle \rangle = 0$, .... Of course this Ansatz does not cover all the cases possible. We will consider another behaviour below.

From (37) we deduce

$$\det(C) = \frac{1}{4} \left( \frac{aH}{q} \right)^4 \left[ c_0 + 2c_1 x + c_2 x^2 + c_3 x^3 + O(x^4) \right]$$

where the coefficients are

$$c_0 = \langle \langle |\alpha|^2 \rangle \rangle$$
$$c_1 = \langle \langle \text{Re}(\alpha^* \beta) \rangle \rangle$$
$$c_2 = \langle \langle |\beta|^2 - \frac{|\alpha|^2}{2} + 2 \langle \text{Re}(\alpha^* \gamma) \rangle \rangle - \frac{1}{4}$$
$$c_3 = \langle \langle 2\text{Re}(\beta^* \gamma) - \frac{1}{2} \text{Re}(\alpha^* \beta) \rangle \rangle$$

The following cases may then occur:

- If $\langle \langle |\alpha|^2 \rangle \rangle \neq 0$, then $c_0 \neq 0$ and $dS/dN = 4$.
- If $\langle \langle |\alpha|^2 \rangle \rangle = 0$, then $c_0 = c_1 = 0$ (since $\langle \langle \alpha \rangle \rangle = 0$). If in addition $\langle \langle |\beta|^2 \rangle \rangle \neq 1/4$, then $c_2 \neq 0$ and $dS/dN = 2$.
- If $\langle \langle |\alpha|^2 \rangle \rangle = 0$ and $\langle \langle |\beta|^2 \rangle \rangle = 1/4$, then $\beta = \bar{\beta}$ and $c_0 = c_1 = c_2 = c_3 = 0$. Hence $dS/dN = 0$.

The threshold of separability, Eq. (20), can be reached by the end of inflation only in the first case.

Let us consider a second Ansatz where the conservation of $\zeta$ on superhorizon scales is violated by a logarithmic term,

$$\zeta(t) = \zeta_q^0 \left( 1 + \alpha \ln(x) + \frac{x^2}{2} + \frac{i x^3}{3} + O(x^4) \right),$$

where $\langle \langle \alpha \rangle \rangle = 0$. The covariances are given by

$$\mathcal{P}_{\zeta \zeta} = |\zeta_q^0|^2 \{ 1 + x^2 + \langle \langle |\alpha|^2 \rangle \rangle \ln^2(x) \},$$
$$\mathcal{P}_{\zeta \dot{\zeta}} = -H |\zeta_q^0|^2 \{ x^2 + \langle \langle |\alpha|^2 \rangle \rangle \ln(x) \},$$
$$\mathcal{P}_{\dot{\zeta} \dot{\zeta}} = H^2 |\zeta_q^0|^2 \{ x^2 + \langle \langle |\alpha|^2 \rangle \rangle \},$$

and the covariance matrix has the determinant

$$\det(C) = \frac{1}{4} + \left( \frac{aH}{q} \right)^6 \langle \langle |\alpha|^2 \rangle \rangle + O \left( \left( \frac{aH}{q} \right)^4 \ln \left( \frac{aH}{q} \right) \right).$$

The entropy grows with the rate $dS/dN \simeq 6$. Higher rates necessitate a stronger violation of the constancy of $\zeta$. This behaviour will be observed below in multifield inflation.

**IV. COUPLING TO ISOCURVATURE PERTURBATIONS DURING MULTIFIELD INFLATION**

In multifield inflation, adiabatic and isocurvature linear perturbations can be coupled on scales larger than the Hubble radius. As a result, even after their decay, isocurvature
perturbations have affected the primordial curvature spectra in an irreversible way. In this case, tracing over the isocurvature perturbations furnishes a non zero entropy at tree level. This source of entropy was considered in \[3\]. We generalize the analysis to a wider class of models while using the method of Section III. We clarify the conditions leading to a linear growth of the entropy with the number of efolds. In particular we consider the (non-intuitive) limiting case where the curvature of the background trajectory in field space is very weak.

A. The model

This subsection contains review material and may be skipped by the learned reader. We consider the following class of two-field models \[17\]

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{2} (\partial \varphi)^2 - \frac{e^{2b\varphi}}{2} (\partial \chi)^2 - V(\varphi, \chi) \right],
\]

with a non standard kinetic term for the field $\chi$, thereby generalizing the action considered in \[5\]. The Klein-Gordon equations for the homogeneous background fields are

\[
\ddot{\varphi} + 3H \dot{\varphi} + V_{,\varphi} = b_{,\varphi} e^{2b\varphi} \chi^2, \quad \ddot{\chi} + (3H + 2b_{,\varphi}\dot{\varphi}) \dot{\chi} + e^{-2b} V_{,\chi} = 0,
\]

where the subscripts $,i$ designate a partial derivative with respect to the field indicated. The Einstein equations are

\[
H^2 = \frac{8\pi G}{3} \left[ \frac{\dot{\sigma}^2}{2} + V \right], \quad \dot{H} = -4\pi G \sigma^2,
\]

where we introduced the field

\[
\dot{\sigma}^2 \equiv \dot{\varphi}^2 + e^{2b} \dot{\chi}^2.
\]

The linear cosmological perturbations are perhaps most transparently written in the instantaneous basis $\delta \sigma$ and $\delta s$ of perturbations respectively tangent and orthogonal to the background trajectory in the field space $(\varphi, \chi)$ \[18\]

\[
\delta \sigma \equiv \cos(\theta) \delta \varphi + \sin(\theta) e^{b\chi} \delta \chi, \quad \delta s \equiv -\sin(\theta) \delta \varphi + \cos(\theta) e^{b\chi} \delta \chi,
\]

where

\[
\cos \theta \equiv \frac{\dot{\varphi}}{\dot{\sigma}}, \quad \sin \theta \equiv \frac{e^{b\chi} \dot{\chi}}{\dot{\sigma}}.
\]

$\delta \sigma$ is the adiabatic component of the vector of linear perturbations, and $\delta s$ is called the entropy component. In the present class of models, the anisotropic stress vanishes and the line element in the longitudinal gauge simplifies to

\[
ds^2 = -(1 + 2\Phi) dt^2 + a^2 (1 - 2\Phi) \delta_{ij} dx^i dx^j,
\]

where $\Phi$ is the gravitational potential. The gauge invariant curvature perturbation (in the comoving gauge) is

\[
\zeta \equiv \Phi - \frac{H}{\dot{H}} \left( \dot{\Phi} + H \Phi \right) = \Phi + \frac{H}{\dot{\sigma}} \delta \sigma,
\]
where the second expression is obtained using the perturbed momentum constraint equation $\dot{\Phi} + H\dot{\Phi} = 4\pi G\dot{\sigma}\delta\sigma$. One also introduces the dimensionless isocurvature perturbation during inflation

$$S \equiv \frac{H}{\dot{\sigma}}\delta s,$$

(51)

which is gauge invariant by construction.

The linearized equation for $\zeta$ is remarkably simple,

$$\ddot{\zeta} + \frac{d\ln(a^3\epsilon)}{dt}\dot{\zeta} + \frac{q^2}{a^2}\zeta = \frac{1}{a^3\epsilon} \frac{d}{dt}\left(-2a^3\epsilon \frac{V_s}{\dot{\sigma}} S\right) = \xi(t),$$

(52)

where $\epsilon = -\dot{H}/H^2$. The equation for the isocurvature perturbations is a little bit more involved

$$\ddot{S} + \frac{d\ln(a^3\epsilon)}{dt}\dot{S} + \left[\frac{q^2}{a^2} + C_{SS}\right]S = -2V_s \left(\frac{q}{aH}\right)^2 \frac{\dot{\Phi}}{\epsilon}.$$

(53)

The time dependent function $C_{SS}$,

$$C_{SS} = V_{ss} + 3\dot{\theta}^2 - b_{i\varphi}\dot{\sigma}^2 + b_{i\varphi}^2g(t) + b_{i\varphi}f(t) + \frac{3}{2}H\frac{\dot{\epsilon}}{\epsilon} + \frac{1}{4}\left(\frac{\dot{\epsilon}}{\epsilon}\right)^2,$$

$$f(t) = V_{i\varphi}(1 + \sin^2\theta) - 4V_s\sin\theta,$$

$$g(t) = -\dot{\sigma}^2(1 + 3\sin^2\theta),$$

(54)

depends only on background quantities. Finally, $\zeta$ and $S$ are coupled via the function

$$g \equiv -2\frac{V_s}{H\dot{\sigma}} = \frac{2}{H} \left(\dot{\theta} + b_{i\varphi}\dot{\sigma}\sin(\theta)\right)$$

$$\simeq 2\eta_{s\sigma} - 2b_{i\varphi}\frac{\dot{\sigma}}{H}\dot{\sin}(\theta).$$

(55)

The last expression is valid in the slow-roll approximation, and $\eta_{s\sigma}$ is the slope parameter of the potential

$$\eta_{s\sigma} \equiv \frac{V_{s\sigma}}{3H^2}.$$

(56)

In brief, $\zeta$ and $S$ are correlated when the background trajectory is curved (in field space).

### B. Qualitative discussion

Equations (52) and (53) are sufficient to understand qualitatively the possible behaviours of the entropy. These equations differ in two qualitative ways, through their effective mass and the source. Let us first consider the r.h.s. of the equations. The source $\xi$ of the adiabatic perturbations is a linear combination of $S$ and its time derivative, but does not contain spatial derivatives. It cannot in general be neglected on superhorizon scales. The source of $S$ on the contrary is second order in the gradient of $\Phi$ and decays exponentially fast. It can therefore be neglected in the long-wavelength approximation.

Let us now examine the homogeneous equations. While $\zeta$ is massless, and therefore reaches a constant value outside the horizon, isocurvature perturbations tend to decay
because of the effective mass $C_{SS}$. More precisely, on superhorizon scales (in the slow roll approximation), equation (53) gives

$$P_S = P_0^S \left( \frac{H}{H_*} \right)^{2\text{Re}(\nu)} \left( \frac{q}{aH} \right)^{3-2\text{Re}(\nu)},$$

where

$$\nu^2 = \frac{9}{4} - \frac{C_{SS}}{H^2}. \quad (58)$$

For $C_{SS} > 9H^2/4$, $P_S$ decreases like $1/a^3$ which is the damping factor of a massive field.

The entropy gained by $\zeta$ after tracing over $S$ follows therefore one of two behaviours. If $C_{SS} \geq 9H^2/4$, the isocurvature component remains unexcited (no parametric amplification). The entropy gain is therefore negligible, frozen at its value at horizon exit,

$$S \left( \frac{q}{aH} \ll 1 \right) \simeq S \left( \frac{q}{aH} = 1 \right) \equiv S_*.$$

We have not attempted to calculate this value. In general, it depends on the integrated effect of $\epsilon, b, \varphi$, and the slope parameters $\eta_{ij}$. The analysis of the last reference in [17] shows that in a wide class of models, the slow roll approximation yields good estimates of the power spectra at horizon exit. In this class, they depend only on the values of the parameters at horizon exit. Unless some fine tuning of these parameters, we expect the amplitude of $S_*$ to be

$$S_* = O(\eta_{ss}^2, \epsilon^2, b_{\varphi}^2).$$

In this case, the reduced state of $\zeta$ remains quantum mechanically entangled since $S_* \ll S_{\text{sep}}$, see Eq. (20).

The other case where $C_{SS} < 9H^2/4$ is much more interesting. Below we now show that the entropy grows linearly with the number of efolds as long as the product $gS$ decays slower than $1/a^3$.

### C. Entropy growth when isocurvature modes are excited

The analysis simplifies considerably in the long wavelength limit as the damping kernel is negligible. Indeed, in order to get the equation (22) for this model, we write the solution of (53) as

$$S(t) = S_h(t) - 2 \int_{-\infty}^t dt' G_{ret}^S(t, t') \left( V_s \left( \frac{q}{aH} \right)^2 \frac{\Phi}{\epsilon} \right),$$

$G_{ret}^S$ is the retarded Green function of (53). $S_h$ is solution of the homogeneous equation, fixed by choosing the adiabatic vacuum for $q/aH \gg 1$. Substituting this solution into (52), one has

$$\ddot{\zeta} + \frac{d \ln(a^3 \epsilon)}{dt} \dot{\zeta} + \frac{q^2}{a^2} \zeta - \frac{4}{a^3 \epsilon} \frac{d}{dt} \left( \frac{q^2 V_s}{\sigma} \int_{-\infty}^t dt' G_{ret}^S(t, t') \left( V_s \left( \frac{q}{aH} \right)^2 \frac{\Phi}{\epsilon} \right) \right) = \xi(t),$$

where $\xi$ is given in (52) with the substitution $S \mapsto S_h$. By identification with (22), we see that $D$ is suppressed by a factor $\left( \frac{q}{aH} \right)^2$. We neglect it in the following with the resulting simplification that (62) reduces to (52). Therefore all the expressions below are only valid for $q/aH \ll 1$. 
The solution of (52) has the following form. Before horizon exit, it is given by Eq. (6). Then it is given by

$$\zeta(t) = \zeta^{\text{in}}(t) + \int_{t_*}^{t} dt' G_{\text{ret}}(t, t') \xi(t'),$$  \hspace{1cm} (63)

where $\zeta^{\text{in}}(t)$ is given by (6) and $G_{\text{ret}}$ is the retarded Green function of (3). To evaluate the integral of (63), it is sufficient to use the long wavelength approximation (B6) of the retarded Green function. More detailed expressions are given in Appendix B2.

To estimate qualitatively the impact of the source $\xi$, we first consider a simplified model where $\xi$ is constant during $N_\xi$ e-folds after horizon exit and where it vanishes afterwards. This assumption may appear unrealistic at first, but in the light of the analytical solution of the next section, we will see that it contains the essential physics to understand the evolution of the entropy. We therefore consider the source term

$$\xi(t) = \theta(t_* - t)\theta(t - t_*) \xi,$$  \hspace{1cm} (64)

where the constant value $\xi$ is given by

$$\xi \simeq 3H^2 g S_0.$$  \hspace{1cm} (65)

Then, for $t_* \leq t \leq t_\xi$, $\zeta$ of (63) tends to

$$\zeta(t) = \zeta^{\text{in}}(t) + g S_0 \ln \left( \frac{a}{a_*} \right),$$

$$\dot{\zeta}(t) = \dot{\zeta}^{\text{in}}(t) + g H S_0.$$  \hspace{1cm} (66)

(Notice that this could have been directly derived from the evolution equation)

$$\frac{d\zeta}{dN} = -\left( \frac{q}{aH} \right)^2 \frac{\Phi}{\epsilon} + g S$$  \hspace{1cm} (67)

assuming $S = \text{cte} = S_0$ and neglecting of the first term.)

Taking into account that at horizon exit, the power spectra are

$$P^0_S \simeq P^0_\zeta \simeq \frac{4\pi G H^2}{e} \frac{H^2}{2q^3},$$  \hspace{1cm} (68)

the covariances for $t_\xi \geq t \gg t_*$ are given by

$$P_{\zeta} = P^0_\zeta \left[ 1 + g^2 \ln^2 (x) \right],$$

$$P_{\zeta \dot{\zeta}} = P^0_{\zeta \dot{\zeta}} - g^2 H P^0_\zeta \ln (x),$$

$$P_{\dot{\zeta} \dot{\zeta}} = P^0_{\dot{\zeta} \dot{\zeta}} + g^2 H^2 P^0_\zeta.$$  \hspace{1cm} (69a-c)

Recalling the relation (7) between $\pi$ and $\dot{\zeta}$, we get

$$\det(C) \simeq \frac{1}{4} + g^2 \left( \frac{aH}{q} \right)^6 \left[ 1 + O \left( g^2 x^2, g^2 \ln^2 (x) \right) \right].$$  \hspace{1cm} (70)

We left the subdominant $1/4$ to remind that $\det(C) \geq 1/4$ by Heisenberg uncertainty relations. Hence, within the interval

$$H t_* + \ln 4 g^2 \leq H t \leq H t_\xi,$$  \hspace{1cm} (71)
the entropy grows with a rate
\[ \frac{dS}{dN} = 6. \] (72)

This expression is rather remarkable since it is independent of the parameters of the model and of the initial conditions of \( \varphi \) and \( \chi \). These appear in the expression of the entropy only as logarithmic additive constant, e.g. \( 2 \ln(g) \), and in the boundary of the domain of \( \ref{71} \). These features seem to be a generic property of classically unstable systems.

After the decoupling of curvature and isocurvature perturbations, we verify that the entropy is again constant. The expressions are given in Appendix 13.

D. Canonical kinetic term

To confirm the physical relevance of the above result, we study in more details the dependence of the entropy on \( g \) and on the properties of the background trajectory. To this end, we turn to the case
\[ b_1 \varphi = 0, \quad V = \frac{1}{2} \left( m_\chi^2 \chi^2 + m_\varphi^2 \varphi^2 \right), \] (73)
for which analytical solutions exist in the slow roll approximation [19]. We shall also explain the counter-intuitive result of [19], namely that in the limit of a small mass difference, the entropy grows linearly till the end of inflation with the rate (72) while the background trajectory is almost straight.

In the slow roll regime, the background trajectory can be written in the parametric form
\[ \chi = 2 M_{Pl} \sqrt{s} \sin \alpha, \quad \varphi = 2 M_{Pl} \sqrt{s} \cos \alpha, \quad 0 \leq \alpha < \frac{\pi}{2}, \] (74)
where
\[ s = \ln \left( \frac{a_{\text{end}}}{a} \right) = - \ln \left( \frac{x_{\text{end}}}{x} \right). \] (75)
Forward time propagation corresponds to decreasing values of \( s \). The slow roll regime corresponds to \( s \gg 1 \) and ends at \( s \simeq 1 \). We call \( \chi \) the heavy field and introduce the relative mass difference
\[ \lambda = \frac{m_\chi^2 - m_\varphi^2}{m_\varphi^2} > 0. \] (76)

The evolution equation
\[ \frac{d\alpha}{d \ln s} = \frac{\lambda}{4} \sin 2\alpha \left( 1 + \lambda \sin^2 \alpha \right), \] (77)
is obtained from the Klein-Gordon equations (14). Its solution can be written
\[ s = s_0 \left( \frac{t}{t_0} \right)^{1 + \frac{t^2}{1 + t_0^2}} = s \left( t^2 \right)^{1 + \frac{t^2}{1 + t_0^2}}, \] (78)
where \( t \equiv \tan(\alpha) \). The index 0 refers to the initial conditions. From the solution written in this form we immediately see that \( \alpha \) is a monotonously growing function of \( s \). Omitting the contribution of \( \sigma^2/2 \), the Hubble parameter is in turn given by
\[ H^2(s) = \frac{2}{3} m_\varphi^2 s \left( 1 + \lambda \sin^2 \alpha \right). \] (79)
The coupling parameter $g$ has the form
\[ g = -2 \frac{d\theta}{ds} = -\frac{\lambda m_{\phi}^2}{3 H^2} \sin 2\theta, \] (80)
where $\theta$ and $\alpha$ are related by
\[ \tan \theta = \tan \alpha \frac{1 + 2 \frac{d\alpha}{dn s} \frac{1}{\tan \alpha}}{1 - 2 \frac{d\alpha}{dn s} \tan \alpha}, \] (81)
Substituting (77) into (81) yields
\[ \tan \theta = (1 + \lambda) \tan \alpha, \] (82)
which is exact in the slow-roll approximation. In the limit of small mass difference $\lambda \ll 1$, the velocity vector is almost aligned with the position vector, hence the background trajectory is almost straight in this case.

After horizon exit, the isocurvature perturbations are given by
\[ S = S_0 \frac{H^2}{H^2_{\phi}} \sin 2\theta \equiv S_0 T_S(s, s_*) , \] (83)
where $S_0$ is as previously the amplitude at horizon crossing given in the first approximation by (68). The gravitational potential is determined by the constraint $\dot{\Phi} + H\Phi = 4\pi G \dot{\sigma} \delta \sigma$. Together with (50) it implies that $\zeta$ still reaches a constant and that
\[ \frac{\Phi}{G} \approx \zeta (1 + O(\epsilon)) . \] (84)
Inserting this solution into (67), one gets
\[ \frac{d\zeta}{ds} = \left( \frac{q}{aH} \right)^2 \zeta - gS . \] (85)
The first term on the r.h.s. can therefore be neglected. Since $gS$ decays like a power of $s$, which is integrable at $s = 0$, the amplitude $\zeta(s)$ indeed asymptotes to a constant. In the limit $\lambda \ll 1$, integrating equation (85) yields
\[ \zeta(s) - \zeta_0 = S_0 F(s, s_*) = -2S_0 \int_{\theta_*}^{\theta(s)} d\theta T_S(s(\theta), s_*) , \] (86)
where $\zeta_0$ is the amplitude at horizon exit and we used (80) and (83). This integral is manifestly convergent but its expression will not be needed here. Therefore the three covariances are given by
\[ P_{\zeta\zeta} = P_0^0 \left( 1 + x^2 + F^2(s) \right) , \] (87a)
\[ P_{\zeta\dot{\zeta}} = P_{\zeta\dot{\zeta}}^0 + P_0 gT_S F , \] (87b)
\[ P_{\dot{\zeta}\dot{\zeta}} = P_{\dot{\zeta}\dot{\zeta}}^0 + P_0 (gT_S)^2 . \] (87c)
$P_{\zeta\zeta}^0$ and $P_{\zeta\dot{\zeta}}^0$ are given by Eqs. (12b) and (c). The part of $\dot{\zeta}$ driven by $S$ decays polynomially in $s$ whereas the homogeneous solution decays as $(s/s_*)^2 \exp(2(s - s_*))$. In consequence, in the determinant the term $P_{\dot{\zeta}\dot{\zeta}}^0 (gT_S)^2$ rapidly overtakes the others, and we have
\[ \det(C) \approx \frac{1}{4} \left[ 1 + \left( \frac{aH}{q} \right)^6 (gT_S)^2 \right] . \] (88)
Combining these two inequalities, we get

\[ ξ_{\text{source}} \]

coherence of the reduced state is lost since the resulting value of the entropy is too large. Instead, in the presence of a source \( C \), the change in \( \text{det}(C) \) is given by \( a^6 \) times \( \Pi_0^\xi \delta P_\xi + P_0^\xi \delta P_\xi - 2P_0^\xi \delta P_\xi \delta P_\xi \). This product rapidly asymptotes to \( a^6 P_0^\xi \delta P_\xi \delta P_\xi \) unless the source \( \xi \) decays faster than \( 1/a^3 \). In other words, the conditions of an efficient decoherence are provided by the same mechanism producing the highly non-classical state in the first place.

When the growth of entropy follows Eq. (72) till the end of inflation, the quantum coherence of the reduced state is lost since the resulting value of the entropy is \( S_{\text{end}} \sim 6 \ln N_{\text{end}} \) which is higher than the threshold of separability given in Eq. (20). We recall that this large entropy is however compatible with the classical coherence of the distribution (which implies the presence of the acoustic peaks) because it is much smaller than the thermal entropy \( = 8 \ln N_{\text{end}} \) which characterizes the incoherent distribution (with no peak) at \( \eta = 1 \).

In the model studied in Sec. [IV.D], the law (72) is 'generic' in that lower rates \( 0 < dS/dN < 6 \) are only found when \( 4C_{SS}/9H^2 \) approaches one before the end of inflation. Once \( 4C_{SS}/9H^2 \geq 1 \), the entropy saturates at \( S \sim S_* + 6N_\xi \), with \( S_* = O(n^2_{ss}, c^2, b^2_\phi) \) and \( N_\xi \) is the number of e-folds from horizon exit till that moment. Hence the threshold of separability is reached when \( N_\xi \geq N_{\text{end}}/2 \). From the inequality (90), this translates into an upper bound on the relative mass difference \( \lambda \). To get a rough estimate of this bound,
we simplify $4C_{SS}/9H^2$ by its upper bound \((90)\). The condition $2(1 + \lambda)/3s_\xi = 1$ with $s_\xi = N_{\text{end}} - N_\xi$ then gives the upper bound

$$\frac{N_{\text{end}}}{2} e^{-3N_\xi} \leq \lambda \leq \frac{N_{\text{end}}}{2} - 1.$$  \hspace{1cm} (91)

The lower bound on $\lambda$ comes from \((71)\) and $g_* \sim \lambda/s_* = \lambda/N_{\text{end}}$. It is easily satisfied when the number of e-folds $N_\xi$ is not fine tuned (e.g. by the choice of initial conditions) to be small.

It is simple to extend our conclusion to inflationary models with $\mathcal{N}$ fields. Then, $\xi$ is the sum of the contributions of the $\mathcal{N} - 1$ isocurvature modes \([21]\). Hence, the entropy grows steadily as long as at least one isocurvature mode is excited.

It is harder to reach conclusions in the case of multifield inflation with a non-canonical term. In this case, the derivatives of $b(\varphi)$ can contribute significantly to $C_{SS}$ so as to increase or lower the effective mass. Moreover, the coupling can be strong enough to lead to a significant change of $\zeta$ on superhorizon scales and to invalidate a perturbative treatment. A numerical treatment is probably required to settle the question.

V. COUPLING TO MATTER FIELDS

In a non stationary background, there are a priori two distinct sources of entropy from loop corrections. The first is dissipation, as in non vacuum states in Minkowski space. The second is a possible non trivial time dependence of radiative corrections that would frustrate the cancellation between the variances which at tree level lead to $\det(C) = 1/4$. Weinberg \([8]\) identified the conditions such that the corrections depend only on the values of background quantities at horizon crossing (instead of the whole history after horizon crossing). These conditions are satisfied by a wide class of semi-realistic theories \([22, 23]\). We verify that in these theories, the entropy vanishes at one loop approximation.

A. The model

We consider theories of one inflaton field $\varphi(t, x)$. Matter is modeled by $\mathcal{N}$ copies of free scalar field $\sigma$. More precisely, we consider the vector field $\vec{\sigma}(t, x) = (\sigma_1, ..., \sigma_\mathcal{N})$ in the fundamental representation of $O(\mathcal{N})$. We assume no symmetry breaking otherwise isocurvature perturbation are excited. The gravitational sector was described in Sec. II. The gravitons have a similar action but as explained below we will not need to take them into account. The quadratic part of the action describing the free evolution of the matter fields is

$$S[\sigma_n] = \frac{1}{2} \sum_{n=1}^{\mathcal{N}} \int dt d^3x a^3 \left[ \dot{\sigma}_n^2 - \frac{1}{a^2} (\nabla \sigma_n)^2 - 12 \xi H^2 \sigma_n^2 \right]$$  \hspace{1cm} (92)

Minimal (conformal) coupling to gravity corresponds to $\xi = 0$ (resp. $\xi = 1/6$).

$\zeta$ and the matter fields are canonically quantized. We work in the interacting picture. The momenta conjugate to $\sigma_n$ are

$$\pi_i(t, x) = a^3 \dot{\sigma}_i(t, x)$$  \hspace{1cm} (93)

and the mode equations are

$$\ddot{\sigma}_q + \frac{d \ln(a^3)}{dt} \dot{\sigma}_q + \left( \frac{q^2}{a^2} + 12 \xi H^2 \right) \sigma_q = 0$$  \hspace{1cm} (94)
The free vacuum is the Bunch-Davis vacuum defined by
\[(i\partial_\tau - q)(a\sigma_q) \to 0 \quad \text{for} \quad \frac{q}{aH} \to \infty\] (95)

In a quasi-de Sitter approximation, the mode functions are
\[
\sigma_q(t) = \sigma_q^{0\min}(1 + iq\tau)e^{-iq\tau},
\]
\[
\sigma_q(t) = \sigma_q^{0\conf}e^{-iq\tau},
\]
for minimally and conformally coupled scalars where the normalization constants are
\[
|\sigma_q^{0\min}|^2 = \frac{H^2}{2q^3}, \quad |\sigma_q^{0\conf}|^2 = \frac{1}{2q}.
\] (98)

We work at the leading order of the large-$\mathcal{N}$ limit. At this order, inspection of the diagrammatic expansion reveals the following elements. First, gravitational self-interactions are irrelevant. It means that the gravitons decouple from the scalar perturbations, and that scalar self-interactions do not contribute. Second, since each matter loop is enhanced by a factor $\mathcal{N}$, matter fields propagate freely. Therefore the only loop corrections to consider are matter loops in the two-point function of $\zeta$. Moreover, only the trilinear vertex $\zeta\sigma\sigma$ contributes to the logarithmic part of the one loop correction (the local-regular part of the loop correction does not contribute to the entropy [1], so that the one-loop correction coming from the vertex $\zeta\zeta\sigma\sigma$ need not be considered here). Finally, an explicit calculation [8] shows that the relevant part of the trilinear vertex responsible for the logarithm is
\[
S_{\zeta\sigma\sigma} = -\int dt H_{\text{int}} = -\int dt d^3x a^3 \left(T^{00} + a^2\delta_{ij}T^{ij}\right) \left(-\epsilon H a^2 \nabla^2 \zeta\right),
\] (99)
where $T_{\mu\nu}$ is the energy momentum tensor of matter. For minimally and conformally coupled scalars, the linear combination in (99) is respectively
\[
(T^{00} + a^2\delta_{ij}T^{ij})^{\min} = 2\sigma^2,
\]
\[
(T^{00} + a^2\delta_{ij}T^{ij})^{\conf} = \dot{\sigma}^2 + \frac{1}{3} \left(\frac{1}{a^2}(\nabla\sigma)^2 - 2\sigma\ddot{\sigma} - H^2\sigma^2\right).
\] (101)

**B. Outline of the calculation**

The expectation value of a local (possibly composite) operator $Q(t,x)$ is given by
\[
Q(t,x) = Q_I(t,x) + i\int_{-\infty}^{t} dt_2 \left[ H_I(t_2), Q_I(t,x) \right]
-\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \left[ H_I(t_1), [H_I(t_2), Q_I(t,x)] \right] + ...
\] (102)

The subscript $I$ refers to the interaction picture and will be omitted in the following. In this expression, $H_I$ is interacting Hamiltonian, e.g. $H_{\text{int}}$ in (99). The dots stand for higher order corrections and counterterms (whose explicit form will not be needed here). The Fourier transform of the one loop correction to covariances $\delta P_{\xi\xi'}$ can then be written
\[
\int d^3x e^{iqx} \delta(\xi(t,x)\xi'(t,0)) = -4N \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \delta^{(3)}(q + p + p')
\times \int_{-\infty}^{t} dt_2 V_2 \int_{-\infty}^{t_2} dt_1 V_1 \text{Re} \left( Z_{\xi\xi'}^{\xi\xi'} M_{\mu\nu}^{\sigma} \right) + ...
\] (103)
where $\xi$ and $\xi'$ can be either $\zeta$ of $\dot{\zeta}$, and $V(t) = -\epsilon Ha^5$. The same function $\mathcal{M}$ appears in the calculation of the three covariances. It depends only on the matter fields $\sigma_n$ and the loop variables $p$, $p'$, $t_1$ and $t_2$. If the $\sigma$'s are conformally coupled, one finds

$$\mathcal{M}_{pp'}^{\text{conf}} = \frac{N}{36a_1^4a_2^4} \frac{(p^2 + p'^2 - 4pp')^2}{pp'} e^{-i(p+p')(\tau_1-\tau_2)}.$$  

(104)

For minimally coupled scalar fields one gets

$$\mathcal{M}_{pp'}^{\text{min}} = \frac{N}{a_1^4a_2^4} (pp')^2 e^{-i(p+p')(\tau_1-\tau_2)}.$$  

(105)

Notice that the time dependence of these two expressions is the same. This remark is essential to understand why the entropy from minimally coupled fields is not much larger as might have been naively expected (since they are not in the conformal vacuum but are parametrically amplified). The function $Z_q^\xi$ depends only of the modes $\zeta_q$ and/or $\dot{\zeta}_q$. Using the solution (6) of the mode equation, we get

$$Z_q^{\zeta\zeta} = \frac{|\zeta_q|^4}{H_1H_2a_1^2a_2^2} \left[ e^{-i\zeta(\tau_1+\tau_2-2\tau)} (1-i\beta\tau)^2 - e^{-i\zeta(\tau_1-\tau_2)} (1+q^2\beta^2) \right],$$  

(106a)

$$Z_q^{\zeta\dot{\zeta}} = -\frac{q^2}{a^2(t)H} \frac{|\zeta_q|^4}{H_1H_2a_1^2a_2^2} \left[ e^{-i\zeta(\tau_1+\tau_2-2\tau)} (1-i\beta\tau) - e^{-i\zeta(\tau_1-\tau_2)} \right],$$  

(106b)

$$Z_q^{\dot{\zeta}\dot{\zeta}} = \frac{q^4}{a^2(t)H^2} \frac{|\zeta_q|^4}{H_1H_2a_1^2a_2^2} \left[ e^{-i\zeta(\tau_1+\tau_2-2\tau)} - e^{-i\zeta(\tau_1-\tau_2)} \right].$$  

(106c)

In each line, the first term in the brackets comes from the term $\langle \zeta_1\zeta_2Q \rangle$ of the perturbative expansion (102), and the second from $\langle \zeta_1Q\zeta_2 \rangle$. It is tempting to dismiss the terms $O(q\tau)$ since for the power spectrum of super Hubble scales, they represent subleading terms. These must however be kept to calculate the entropy. Indeed, we remarked already below Eq. (15) that at tree level, three powers of $1/q\tau$ cancel in the expression of $\det(C)$ so as to ensure that the entropy vanishes, see Sec. 11.

Then, the key observation is that all the factors of $a$ coming from $V(t_{1,2})$, $Z$ and $\mathcal{M}$ cancel in the integrand of (102). Thus, the dummy variables $\tau_1$ and $\tau_2$ appear only in the phase of exponentials. Explicitly, they are the phase factors in the brackets of Eqs. (106) and from $\mathcal{M}$ of Eqs. (104) and (105). The singular logarithm of the one loop correction can be calculated by exchanging the order of integration. The integration of the first phase in Eqs. (106) gives

$$I_1(q,p,p') = e^{i2q\tau} \int_{-\infty}^{\tau} d\tau_2 e^{i(p+p'-q)\tau_2} \int_{-\infty}^{\tau_2} d\tau_1 e^{-i(p+p'+q)\tau_1} = \frac{1}{2q(q+p+p')}.$$  

(107)

The remarkable property of this term is its $\tau$-independence. This is a consequence of the stationarity of de Sitter space, but it can be shown that the integral is also finite in the limit $t \to +\infty$ in power law inflation for $\epsilon < 1/3$. For the second term, we introduce $Q = p + p' + q$, and make the change of variables $(\tau_2,\tau_1) \leftrightarrow (\tau_2,\delta\tau = \tau_1 - \tau_2)$. We get

$$I_2(q,p,p') = \int_{-T}^{\tau} d\tau_2 \frac{i}{Q + i\epsilon} = \Delta\tau \left[ iP \frac{1}{Q} + \pi\delta(Q) \right],$$  

(108)

where $\Delta\tau = \tau + T$.

The integrals $I_1$ and $I_2$ times the kernel $\mathcal{M}$ must then be integrated. Since $p, p', q \geq 0$, the Dirac $\delta$ in (108) gives a vanishing contribution for any finite $q$, while the principal
value gives a purely imaginary contribution. Since this second term is only multiplied by real functions in Eqs. (106), it therefore does not contribute to (103). Hence, only the interfering term \(\langle \zeta_1 \zeta_2 Q \rangle\) contributes. Because of the curvature perturbation couples to matter via its time derivative \(\dot{\zeta}\), its conjugate momentum is not \(\pi = \frac{a^3}{4\pi G} + F(\partial \sigma, \dot{\sigma})\) where \(F\) is a functional quadratic in \(\sigma\). These additional terms do not contribute at one loop to the logarithmic part of the variance \(P_{\zeta\pi}\) and \(P_{\pi\pi}\) so that we ignore them. Therefore the 1-loop modifications of the variances are

\[
\langle \zeta_q(t)\zeta_{-q}(t) \rangle_{1\text{-loop}} = \mathcal{N} \left( 8\epsilon_q^2 |q|^{4} \right) \left( 1 - \frac{q^2}{a^2 H^2} \right) \mathcal{J} \left( \frac{q}{\mu} \right), \tag{109a}
\]
\[
\frac{1}{2} \langle \{\zeta_q(t), \pi_{-q}(t)\} \rangle_{1\text{-loop}} = -\mathcal{N} \left( 8\epsilon_q^2 |q|^{4} \right) \left( -\frac{q^2}{a^2 H} \frac{\epsilon a^3}{4\pi G} \right) \mathcal{J} \left( \frac{q}{\mu} \right), \tag{109b}
\]
\[
\langle \pi_q(t), \pi_{-q}(t) \rangle_{1\text{-loop}} = \mathcal{N} \left( 8\epsilon_q^2 |q|^{4} \right) \left( -\frac{q^2}{a^2 H} \frac{\epsilon a^3}{4\pi G} \right)^2 \mathcal{J} \left( \frac{q}{\mu} \right), \tag{109c}
\]

where \(\mathcal{J}\) contains the (ultra-violet divergent) integral over loop momenta. For instance, for a minimally coupled field its un-subtracted expression is

\[
\mathcal{J}(q) \equiv \frac{1}{q} \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(q + p + p') \frac{pp'}{q + p + p'}. \tag{110}
\]

The new scale \(\mu\) in Eq. (109) is an a priori arbitrary scale associated with regularization procedure, e.g. dimensional regularization. The occurrence of \(q > 0\) in (107) regulates the momentum integral in the infrared. This is the advantage of having inverted the order of the integrals. Note that the expressions (109) hold for both minimally and conformally coupled fields. We call \(A\) the coefficient of the logarithm of \(\mathcal{J}\). Its value depends of course on whether \(\sigma\) is minimally or conformally coupled. It is given in [3, 23] but we shall not need its explicit expression here. Taking into account the normalization [3], the relative change of the power spectrum is

\[
\frac{\delta P_\zeta}{P_\zeta} = AN\epsilon GH^2 \ln \left( \frac{q}{\mu} \right), \tag{111}
\]

up to terms \((q/aH)^2\).

The covariance matrix does not grow with a certain power of \(a\) as it could have done a priori. It is given by

\[
\text{det}(C) = \frac{1}{4} + O \left( \epsilon_q GH^2 N A \ln(q/\mu) \right)^2. \tag{112}
\]

In conclusion at one loop approximation,

\[
S = O(\epsilon_q \times 10^{-10} \times N)^2, \tag{113}
\]
even though the power spectrum is modified to linear order in \(\epsilon_q \times GH^2 \times N\). Comparing with the entropy of a separable state in the radiation dominated era \(S_{\text{sep}} = 100 \ln(10)\), the reduced density matrices remain entangled (not separable) at the end of inflation, unless the number of fields is larger than \(1/\epsilon_q \times 10^{10}\) but in this case the whole perturbative treatment is no longer valid.

The reduced density matrix of modes of opposite wave vectors remains very pure, and can thus be interpreted as a "dressed squeezed vacuum state". This is an example of the case discussed in Sec. IIIIC. We stress that this conclusion could not be expected \(a\ priori\) on the basis of an analogy with field theories in the Minkowski vacuum. In
inflation, matter fluctuations of non conformal fields are parametrically amplified just as the curvature perturbations, so that they could have been responsible for a strong decoherence. As we saw, this is not the case and mainly follows from the fact that matter couples to $\zeta$ via its energy-momentum tensor, see (99).

VI. CONCLUSION AND OUTLOOK

We found that in multi-field inflation scenarios, the entropy grows at a high rate, typically $dS/dN = 6$, as long as some isocurvature mode decay polynomial in the number of efolds after horizon exit. By contrast, in single field inflation we found (at one-loop) no evidence for decoherence, so that the state remains essentially pure. Intermediate rates between these two cases seem hard to find because of the efficiency of the mechanism of parametric amplification (in amplifying or canceling the action of a noise).

We concentrated on the entropy of curvature perturbations, but our method may as well be applied to tensor modes. We conclude on the perspectives to apply this method to the regimes of (p)reheating and horizon reentry. One should distinguish between sub- and superhorizon scales. The evolution of the former is described by a regime of broad parametric resonance. This is a process characterized by its efficiency. Decoherence should therefore be achieved very rapidly. The evolution of super-Hubble modes depends on whether isocurvature perturbations are excited or not. The discussion of this case can be incorporated in the model of two field inflation of Sec IV (see [24] as well as references therein).

After reheating but before decoupling, it is expected that the coupling of curvature modes entering the horizon with the radiation density perturbations in quasi-thermal equilibrium will erase the possible remaining quantum features after typically one oscillation (this should not be mistaken with the thermalization of the perturbations which occurs much later). We have little to say about this regime of horizon reentry. A field theoretic proof remains a formidable task because long wavelength radiation perturbations are not thermal fluctuations since we observe acoustic oscillations. Since this phenomenon is reliably described by a hydrodynamic model, it means that scales decouple. As a result, we expect that the curvature perturbations entering the horizon dominantly couple to the same scales of the plasma. A proper investigation of the regime of horizon reentry therefore requires first to write an effective field theoretic model of these out-of-equilibrium long wavelength modes. This is arguably an academic question, since we saw in [1] that the loss of quantum coherence occurs on time scales much shorter than the characteristic time of thermalization.

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APPENDIX A: DERIVATION OF EQ. (22)

1. Equation for the anti-commutator

When the Hamiltonian depends explicitly on time, there is no stable ground state. In this case, Green functions, e.g.

\[ G(x, y) = \langle 0 | T \varphi(x) \varphi(y) | 0 \rangle, \quad (A1) \]

are expectation values in the 'in'-vacuum \(|0\rangle\) which cannot be expressed in terms of Feynman graphs with internal lines corresponding to time-ordered propagators. Rather, their generating functional is given by the transition amplitude of two 'in'-vacua in the presence of external sources \(J_+\) and \(J_-\) (see for instance [25]),

\[ e^{iW[J^+, J^-; \rho_{in}]} = \text{Tr} \left\{ T e^{i \int_{-\infty}^{\infty} dt \varphi(x) J_+} \rho_{in} T e^{-i \int_{-\infty}^{\infty} dt \varphi(x) J_-} \right\}, \quad (A2) \]

where \(T\) is the reversed-time ordered product and \(J_+\) and \(J_-\) are the two classical sources associated with the two branches of evolution, forward and backward in time respectively. Note that the operation of taking the trace couples the forward and backward time evolutions. As a result, \(W[J_+, J_-]\) generates four types of connected two-point functions: the time-ordered propagator

\[ G_{++}(x, y) \equiv i \langle T \varphi(x) \varphi(y) \rangle = \frac{\delta W}{\delta J^+_\infty(x) \delta J^-_\infty(y)} \bigg|_{J_+=J_-=0}, \quad (A3) \]

the reverse times ordered propagator

\[ G_{--}(x, y) \equiv i \langle \bar{T} \varphi(x) \varphi(y) \rangle = \frac{\delta W}{\delta J^-_\infty(x) \delta J^+_\infty(y)} \bigg|_{J_+=J_-=0}, \quad (A4) \]

and the two on-shell two-point functions

\[ G_{-+}(x, y) \equiv i \langle \varphi(x) \varphi(y) \rangle = \frac{\delta W}{\delta J^-_\infty(x) \delta J^+_\infty(y)} \bigg|_{J_+=J_-=0}, \quad (A5) \]

\[ G_{+-}(x, y) \equiv i \langle \varphi(y) \varphi(x) \rangle = \frac{\delta W}{\delta J^+_\infty(x) \delta J^-_\infty(y)} \bigg|_{J_+=J_-=0}. \quad (A6) \]

The latters are not time ordered because they are build from operators coming from different branches of time evolution. Eqs. (A3, A4) are easily derived using the second equality in (A2). The path integral representation of (A2) is

\[ e^{iW[J^+, J^-; \rho_{in}]} = \int d\varphi^+ \int d\varphi^- \delta \left( \varphi^+ - \varphi^- \right) \int d\varphi^\infty d\varphi^- \langle \varphi^+_\infty \rho_{in} \varphi^- \rangle \times \int \mathcal{D} \phi^+ \mathcal{D} \phi^- e^{iS[\varphi_x^+, \varphi_y^+](\phi^+)+iJ^+\phi^+ - i\varphi_x^-](\phi^-)-iJ^-\phi^-}, \quad (A7) \]

where \(S[\varphi_x^+, \varphi_y^+](\phi^+)=\) the classical action evaluated for the paths \(\phi^+(z^\infty, z^-)\) with fixed end-points \(\phi^+(\infty, x) = \varphi_x^+\) and \(\phi^+(-\infty, y) = \varphi_y^+\).

In the above equations, \(\varphi\) denotes the collection of fields. In cosmological settings, we decompose these fields into the adiabatic perturbations \(\zeta\) and the rest we call \(\sigma\). Then, the
so-called influence-functional (IF) encodes the effective dynamics of \( \zeta \) when these extra fields have been integrated:

\[
e^{iS_{IF}[\zeta_+ , \zeta_-]} = \int_{-\infty}^{+\infty} d\sigma_+^+ d\sigma_-^+ \delta[\sigma_+^+ - \sigma_-^-] \int_{-\infty}^{+\infty} d\sigma_+^- d\sigma_-^- \langle \sigma_+^+ | \rho_{in}^0 | \sigma_-^- \rangle 
\times \int D\sigma^+ D\sigma^- e^{i(S[\sigma_+] - S[\sigma_-] + S_{int}[\sigma_+ , \sigma_-] - S_{int}[\sigma_- , \sigma_-])}.
\]

We can evaluate perturbatively the IF by expanding the terms \( e^{i(S_{int}[\Phi_+] - S_{int}[\Phi_-])} \) to a given order in \( \hbar \), then by taking the expectation value for the field \( \sigma \), and finally by re-exponentiating the result. Hence we get

\[
S_{IF}[\zeta_+ , \zeta_-] = i \langle S_{int}[\zeta_+] - S_{int}[\zeta_-] \rangle 
- \frac{1}{2} \{ \langle S_{int}[\zeta_+] S_{int}[\zeta_+] \rangle_{con} + \langle S_{int}[\zeta_-] S_{int}[\zeta_-] \rangle_{con} 
- \langle S_{int}[\zeta_+] S_{int}[\zeta_-] \rangle_{con} - \langle S_{int}[\zeta_-] S_{int}[\zeta_+] \rangle_{con} \} + \ldots
\]

where \( \langle \, \rangle_{con} \) means the (connected part of) expectation value in \( \rho_{in}^0 \), the initial state of the field \( \sigma \).

The Gaussian approximation consists in keeping only the part quadratic in \( \zeta_\pm \) of this functional. Then the generating functional is given by

\[
e^{iW_{gauss}[J^+, J^-; \rho_{in}]} = \int D\zeta^+ D\zeta^- \exp\left\{-\frac{1}{2} tZ MZ + tJ \right\}
\times \exp\left(-\frac{1}{2} tJ M^{-1} J \right),
\]

where \( tZ = (\zeta_+ \zeta_-) \) and \( tJ = i (J_+ , -J_-) \). The quadratic form \( M \) is by definition

\[
\frac{1}{2} tZ MZ = S[\zeta_+] - S[\zeta_-] + S_{IF}^{gauss}[\zeta_+, \zeta_-].
\]

The action \( S \) is given by eq. (2) whereas the Gaussian approximation of the IF is

\[
S_{IF}^{gauss} = \frac{1}{8\pi G} \int dt d^3 x (a^3(t) \epsilon) \int dt' d^3 y (a^3(t') \epsilon) \zeta_a(x) \Sigma_{ab}(x, y) \zeta_b(y),
\]

where the indices \( a, b \) = \( \pm \). This equation defines the self-energy matrix \( \Sigma_{ab} \). We have introduced twice the factor \( a^3 \epsilon \), which it is already present in the kinetic action \( \mathcal{K} \), because it simplifies the forthcoming equations.

Taking the functional derivatives with respect to the sources, we arrive at a system of two linear coupled equations. The system decouples when using the (odd) commutator (also called the spectral function) \( \rho = i(t(G_++ - G_-^-)) \) and the (even) anticommutator \( G = (G_++ + G_-^-)/2 \) = \( (G_+^- + G_-^+)/2 \). Indeed, using the definition of the free propagators \( \delta_2 S = D_x = G_0^{-1} \), one gets

\[
D_x \rho(x, y) + \int_{y_0}^x d^4 z D(x, z) \rho(z, y) = 0,
\]

\[
D_x G(x, y) + \int_{-\infty}^{y_0} d^4 z D(x, z) G(z, y) = \int_{-\infty}^{y_0} d^4 z N(x, z) \rho(z, y).
\]

Straightforward algebra gives the odd and the even kernels

\[
D(x, x') = i a^3(t') \left[ \Sigma_{-+}(x, x') - \Sigma_{+-}(x, x') \right],
\]

\[
N(x, x') = \frac{a^3(t') \epsilon}{2} \left[ \Sigma_{-+}(x, x') + \Sigma_{+-}(x, x') \right],
\]

\[
(A15)
\]
in terms of $\Sigma_{ab}$, the self-energy matrix of $\zeta$.

We conclude this part by a comment. In the Gaussian approximation it is always possible to write the effective action with only the field (here $\zeta$) but no field derivatives. Should one find, after computation of the effective action, field derivatives (e.g. because in the total action the curvature perturbation couples to other fields via $\dot{\zeta}$), one would simply do an integration by parts in order to bring the effective action into the form (A12). This operation is a linear canonical transformation which changes neither the equations (A13, A14) nor the value of the determinant of the covariance matrix $[1]$. Moreover, with the action written in the form (A12), the canonical momentum is still given by $\pi = (a^3 \epsilon/4\pi G) \dot{\zeta}$ which justifies the identities $[11]$.

2. The effective quantum source

Since the unknown of Eqs. (A13) and (A14) are real functions, they are apt for a numerical analysis. Alternately, for an analytical treatment it is simpler to go one step backwards and consider the operator $\zeta_q$ coupled to a quantum mechanical source $\xi_q$ whose statistical properties are such that the anti-commutator of $\zeta_q$ obeys by construction Eq. (A14). This effective source is not just an artificial trick, because it coincides with the true fluctuating source operator when non-Gaussianities are neglected, as it is the case for the two models considered in the body of the paper.

Let us consider the following Heisenberg equation

$$D_t \Phi_q(t) + \int_{-\infty}^{t} ds \, D(t, s, q) \Phi_q(s) = \xi_q(t),$$

(A16)

where $\Phi_q$ and $\xi_q$ are two operators. We also introduce the anti-commutator

$$G_q(t, t') \delta^3(q - q') = \frac{1}{2} \text{Tr} \left[ \rho_\Phi \rho_\xi \left\{ \Phi_q(t), \Phi^\dagger_{-q'}(t') \right\} \right].$$

(A17)

For an appropriately chosen anti-commutator of $\xi_q$, we now show that given some initial conditions, the function $G_q(t, t')$ solves for the (Fourier transform in space) of (A14) with the same initial conditions. Hence it is equal to the anti-commutator $G(t, t', q)$.

To determine the statistical properties of the noise $\xi_q(t)$ that reproduce the action of the r.h.s. of (A14), we let the integro-differential operator on the l.h.s. of (A13) act on $G_q$, and calculate the result using (A16). Since the action of taking the trace commutes with the partial derivation and integration, we have

$$\delta^3(q - q') \left[ D_t G_q(t, t') + \int_{-\infty}^{t} ds \, D(t, s, q) G_q(t, t') \right] = \langle \{ \xi_q(t), \Phi^\dagger_{-q'}(t') \} \rangle.$$ 

(A18)

To calculate the anticommutator on the r.h.s., we need the solution of (A16). Remembering that the exact retarded Green function of $\Phi_q$ is the spectral function multiplied by a theta function, i.e. $\theta(t - t') \rho_q(t, t')$, the general solution of (A16) is

$$\Phi_q(t) = \Phi_q^0(t) + \int_{-\infty}^{t} ds \, \rho_q(t, s) \xi_q(s)$$

(A19)

where $\Phi_q^0(t)$ is the solution of the homogeneous equation. Since it is independent of $\xi(t)$, upon substitution of this solution into the r.h.s. of (A18) one finds,

$$\delta^3(q - q') \left[ D_t G_q(t, t') + \int_{-\infty}^{t} ds \, D(t, s, q) G_q(t, t') \right] =$$

$$\frac{1}{2} \int_{-\infty}^{t'} ds \, \langle \{ \xi_q(t), \xi_{-q'}(s) \} \rangle \rho_q(s, t')$$

(A20)
Identification with (A14) finally yields
\[
\frac{1}{2} \left\langle \{ \xi_q(t), \xi_{-q'}(s) \} \right\rangle = N(t, s, q) \delta^3(q - q')
\] (A21)
where \( N(t, s, q) \) is the (Fourier transform of the) kernel appearing in Eq. (A14). In conclusion, the dynamics of \( \zeta \) is equivalent to the "open dynamics" (A16) of \( \Phi \) with the same spectral function \( \rho \) and subject to the Gaussian source \( \xi \) characterized by the spectrum (A21).

This result is general because we are only considering the two-point functions. Because of the linearity of the equations and the Gaussianity of the noise, this approach can be also phrased in terms of a stochastic (commuting) c-number source \( \xi_q \), see the stochastic approach of quantum gravity in [26]. This equivalence is used in Sec. III D.

Finally, from the definitions (11) and the identification
\[
G = G_r
\]
the covariances can always be written as the sum of a free (and decaying) part and a driven part:
\[
\mathcal{P}_{\zeta\zeta} = \langle \Phi_q^0(t) \Phi_q^0(t) \rangle + \int dt_1 dt_2 G_{\text{ret}}(t, t_1)G_{\text{ret}}(t, t_2)N(t_1, t_2),
\]
\[
\mathcal{P}_{\zeta\dot{\zeta}} = \frac{1}{2} \left\langle \{ \dot{\Phi}_q^0(t), \Phi_q^0(t) \} \right\rangle + \int dt_1 dt_2 \partial_t G_{\text{ret}}(t, t_1)\partial_t G_{\text{ret}}(t, t_2)N(t_1, t_2),
\]
\[
\mathcal{P}_{\dot{\zeta}\dot{\zeta}} = \langle \dot{\Phi}_q^0(t) \dot{\Phi}_q^0(t) \rangle + \int dt_1 dt_2 \partial_t G_{\text{ret}}(t, t_1)\partial_t G_{\text{ret}}(t, t_2)N(t_1, t_2).
\] (A22a, b, c)

**APPENDIX B: INTERMEDIATE RESULTS FOR SEC. IV**

1. The free retarded propagator in the long wavelength limit

The free retarded propagator of (3) is
\[
G_{\text{ret}}(t, t', q) = 2 \theta(t - t') \frac{\zeta_q(t)\zeta_q(t') - \zeta_q(t)\zeta_q(t')}{W(t')}
\] (B1)
where \( \zeta_q \) and \( \zeta_d \) are the homogeneous growing and decaying solutions of (52), and \( W(t) \) is their Wronskian. For our calculation, we only need to retain the leading terms of their expansion in powers of \((q/aH)^2\). In the limit \( q \to 0 \), the Wronskian solves for the equation
\[
\dot{W} + \frac{d \ln(a^3 \epsilon)}{dt} W = 0
\] (B2)
whose solution can be conveniently written as
\[
W(t) = W(t_*) \frac{a^3 \epsilon_s}{a^3(t) \epsilon(t)}
\] (B3)
where \( t_* \) is the time of horizon crossing \( q = a_* H_* \). The solution of (3) is
\[
\zeta_q(t) = \zeta_q^0 \left[ 1 + O \left( \frac{q^2}{a^2 H^2} \right) \right] + A_q \left[ \int_{t_*}^t \frac{dt'}{a^3(t') \epsilon(t')} + O \left( \frac{q^2}{a^2 H^2} \right) \right]
\] (B4)
The coefficients \( \zeta_q^0 \) and \( A_q \) of the growing and decaying solutions are related by the Wronskian condition
\[
i \left( \zeta_q \zeta_{q*} - c.c. \right) = \frac{2 \text{Im} \left( \zeta_q^0 A_q \right)}{a^3(t) \epsilon(t)} = \frac{4 \pi G}{a^3(t) \epsilon(t)}
\] (B5)
Combining these results, we get

\[ G_{\text{ret}}(t, t', q) = \theta(t - t')a^3(t')e(t') \int_t^{t'} \frac{dt_1}{a^3(t_1)\epsilon(t_1)} + O \left( \frac{q^2}{a^2 H^2} \right) \]

\[ \simeq \frac{\theta(t - t')}{3H} \left[ 1 - \left( \frac{a(t')}{a(t)} \right)^3 \right] \simeq \frac{\theta(t - t')}{3H} \]  

To get the second line, we substituted the solution (B4) of the mode equation in the slow-roll approximation. We also give the first time derivative of \( G_{\text{ret}} \) in that approximation,

\[ \partial_t G_{\text{ret}}(t, t', q) \simeq \theta(t - t') \left( \frac{a(t')}{a(t)} \right)^3 \]  

2. The covariances of Sec. 4B

We use the solution (6) of the mode equation. The retarded Green function is

\[ G_{\text{ret}}(t, t_1) = -2\theta(t - t_1) \frac{Im \{ \zeta(t)\zeta^*(t_1) \}}{W(t_1)} = \frac{\theta(t - t_1)}{H^2} Im \left\{ (1 - ix)(1 + ix_1)e^{i(x-x_1)} \right\} \]  

We use again the variable

\[ x = \frac{q}{aH} = -q\tau \] 

The noise kernel is

\[ N(t_1, t_2) = 9H^2 g^2 P_0 \text{ for } t_\xi \geq t_1, t_2 \geq t_\ast \] 

The covariances calculated from (A22) are given by

\[ P_{\zeta\zeta} = P_0 \left\{ 1 + x^2 + g^2 f^2(x; x_\xi) \right\} \]  

\[ P_{\zeta\tilde{\zeta}} = -H x^2 P_0 \left\{ 1 - g^2 f(x; x_\xi)h(x; x_\xi) \right\} \]  

\[ P_{\tilde{\zeta}\tilde{\zeta}} = H^2 x^4 P_0 \left\{ 1 + g^2 h^2(x; x_\xi) \right\} \] 

and the determinant of the covariance matrix is

\[ \det(C) = \frac{1}{4 \pi^2} \left\{ x^2 \left[ 1 + g^2 h^2(x; x_\xi) \right] + g^2 (f - h)^2 \right\} \] 

The functions \( f \) and \( h \) come from the integral of \( G_{\text{ret}}(t, t_1) \) and \( \partial_t G_{\text{ret}}(t, t_1) \) respectively. Their expressions are

\[ f(x; x_\xi) = Im \left\{ (1 - ix)e^{ix} J(x; x_\xi) \right\} = h(x; x_\xi) - x Re \left\{ e^{ix} J(x; x_\xi) \right\} \]  

\[ h(x; x_\xi) = Im \left\{ e^{ix} J(x; x_\xi) \right\} \] 

and the function \( J(x; x_\xi) \) is

\[ J(x; x_\xi) = 3 \int_{x_1}^{x} \frac{dx_1}{x_1} \theta(x_\xi - x) (1 + ix_1)e^{-ix_1} \]
a. Entropy growth for $t \leq t_\xi$

In that case

$$J(x; x_\xi) = -i\mathcal{L}(x) - e^{-ix}\left(\frac{1}{x^3} + \frac{i}{x^2} + \frac{1}{x}\right) + e^{-i}(2 + i)$$

(B16)

where the first term is a logarithm

$$\mathcal{L}(x) = \int_1^x e^{-ix_1} = E_1(i) - E_1(ix) = \ln(x) + O(x)$$

(B17)

Combining these expressions, we get

$$f = -\ln(x) + C_1 + O(x)$$
$$h = -\frac{1}{x^2} - \ln(x) + C_2 + O(x)$$

(B18)

from which we obtain the expressions \(69 \pm 69\) as well as

$$\det(C) = \frac{g^2}{4x^6} \{1 + O(x^2)\}$$

(B19)

b. Constant entropy after $t_\xi$

We see on (B12) that the entropy is constant provided

$$x^2h^2 + (f - h)^2 = \lambda x^2$$

(B20)

where $\lambda$ is a constant. For $t \geq t_\xi$, the integral $J$ is equal to its value at $t_\xi$

$$J(x; x_\xi) = cte = |J_\xi| e^{i\varphi}$$

(B21)

Substituting into the expressions of $f$ and $h$, we find

$$f - h = -x\Re\{e^{ix}J_\xi\} = -x|J_\xi| \cos(x + \varphi)$$
$$h = x\Im\{e^{ix}J_\xi\} = x|J_\xi| \sin(x + \varphi)$$

(B22)

The condition (B20) is realized and we have

$$\det(C) = \frac{1}{4} \{1 + g^2|J_\xi|^2\}$$

(B23)

with $|J_\xi| = x_\xi^{-3}$.

One also checks that the covariances (B11) verify the identities (24) and (25) together with the free equation of motion.

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[1] D. Campo and R. Parentani, Decoherence and entropy of primordial fluctuations. I. Formalism and interpretation.
[2] C. P. Burgess, R. Holman, D. Hoover, Phys. Rev. D 77, 063534 (2008).
[3] P. Martineau, Class. Quant. Grav. 24, 5817 (2007).
[4] R. Brandenberger, V. Mukhanov, and T. Prokopec, Phys. Rev. Lett. 69, 3606 (1992); Phys. Rev. D 48, 2443 (1993).
[5] T. Prokopec and G. I. Rigopoulos, JCAP 0711, 029 (2007).
[6] C. Kiefer, I. Lohmar, D. Polarski, and A. A. Starobinsky, Class. Quant. Grav. 24, 1699 (2007).
[7] E. Calzetta and B. L. Hu, Phys. Rev. D 37, 2878 (1988).
[8] S. Weinberg, Phys. Rev. D 72, 043514 (2005); ibid. 74, 023508 (2006).
[9] J. Maldacena, JHEP 0305, 013 (2003).
[10] D. Campo and R. Parentani, Phys. Rev. D 72, 045015 (2005).
[11] E. Joos and H. D. Zeh, Z. Phys. B 59, 223 (1985); A. O. Caldeira and A. J. Leggett, Phys. Rev. A 31, 1059 (1985); W. G. Unruh and W. H. Zurek, Phys. Rev. D 40, 1071 (1989); B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D 45, 2843 (1992);
[12] E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, I.-O. Stamatescu, Decoherence and the Appearance of a Classical World in Quantum Theory, Lecture Notes in Physics Vol. 538, Springer (2003).
[13] W. H. Zurek, Rev. Mod. Phys. 75, 715 (2003).
[14] J. Berges, AIP Conf.Proc. 739, 3 (2005).
[15] H.J. Kimble, Y. Levin, A.B. Matsko, K.S. Thorne, and S.P. Vyatchanin, Phys. Rev. D 65, 022002 (2002).
[16] W. G. Unruh, in Quantum Optics, Experimental Gravitation, and Measurement Theory, edited by P. Meystre and M. O. Scully (Plenum, New York, 1982), p. 647.
[17] S. G. Nibbelink and B.J.W. van Tent, Class. Quant. Grav. 19 613 (2002); F. Di Marco, F. Finelli, and R. Branderberger, Phys. Rev. D 67, 063512 (2003); F. Di Marco and F. Finelli, Phys. Rev. D 71, 123502 (2005); Z. Lalak, D. Langlois, S. Pokorski, K. Turzynski, JCAP 0707, 014 (2007).
[18] C. Gordon, D. Wands, B. A. Bassett, and R. Maartens, Phys. Rev. D 63, 023506 (2005).
[19] D. Polarski and A.A. Starobinsky, Nucl. Phys. B 385, 623 (1992).
[20] C. Kiefer, D. Polarski, and A. A. Starobinsky, Phys. Rev. D 62, 043518 (2000).
[21] D. Polarski and A. A. Starobinsky, Phys. Rev. D 50, 6123 (1994).
[22] S. Weinberg, Phys. Rev. D 74, 023508 (2006).
[23] K. Chaicherd sakul, Phys. Rev. D 75, 063522 (2007).
[24] B. A. Bassett, S. Tsujikawa, D. Wands, Rev. Mod. Phys. 78, 537 (2006).
[25] R.D. Jordan, Phys. Rev. D 33, 444 (1986); E. Calzetta and B. L. Hu, Phys. Rev. D 37, 2878 (1988).
[26] B.L. Hu and E. Verdaguer, Living Rev. Rel. 7, 3 (2004).