Strong solutions of a class of SDEs with jumps

Juan Zhao

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People’s Republic of China

Finalized October, 2008

Abstract

We study a class of stochastic integral equations with jumps under non-Lipschitz conditions. We use the method of Euler approximations to obtain the existence of the solution and give some sufficient conditions for the strong uniqueness.

Mathematics Subject Classification (2000): Primary 60H20; secondary 60H10.

Key words and Phrases: stochastic equations; jump; non-Lipschitz; Euler approximation; existence; uniqueness; strong solution.

1. Introduction

Modeling interest rate fluctuations is one of the major concerns of both practitioners and academics. There are many prominent interest rate models such as Vasicek model and Cox-Ingersoll-Ross model, see Lamberton and Lapeyre (1996) for more details. Suppose that \( \{B(t)\} \) is a Brownian motion and \( \{b(t)\} \) is a non-negative measurable stochastic process. Let \( \beta < 0 \) be a constant and \( \sigma \) be a \( 1/2 \)-Hölder continuous function on \( \mathbb{R}_+ \) vanishing at the origin. Deelstra and Delbaen (1995) introduced the so-called extended CIR model \( x(t) \) which is the solution of the stochastic differential equation

\[
    dx(t) = (b(t) + \beta x(t))dt + \sigma(x(t))dB(t)
\]

with \( x(0) \geq 0 \). Deelstra and Delbaen (1998) used the method of Euler approximations to prove the existence of the above stochastic equation. In this paper, we extend the model by considering some stochastic equations with jumps.

We consider a class of stochastic processes for the purpose of modeling interest rates. Suppose that \( U \) is a separable and complete metric space. Let \( \mu(du) \) be a \( \sigma \)-finite measure on \( U \). Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) be a filtered probability space satisfying the usual hypotheses. Let \( \{B(t)\} \) be a \( (\mathcal{F}_t) \)-Brownian motion and let \( \{p(t)\} \) be a \( (\mathcal{F}_t) \)-Poisson point process on \( U \) with characteristic measure \( \mu(du) \). Suppose that \( \{B(t)\} \) and \( \{p(t)\} \) are independent of each other. Let \( \{b(t)\} \) be a non-negative measurable and adapted process and let \( \{N(ds, du)\} \) be the Poisson random measure associated with \( \{p(t)\} \). Suppose that

1Supported by NSFC (No. 10721091)
2 E-mail address: zhaojuan@mail.bnu.edu.cn
(i) $\beta < 0$ is a constant and $x \mapsto \sigma(x)$ is a continuous function on $\mathbb{R}$ satisfying $\sigma(x) = 0$ for $x \leq 0$;

(ii) $(x,u) \mapsto g(x,u)$ is a Borel function on $\mathbb{R} \times U$ such that $g(x,u) + x \geq 0$ for $x > 0$ and $g(x,u) = 0$ for $x \leq 0$.

Given a non-negative $\mathcal{F}_0$ measurable random variable $x(0)$, we consider the following stochastic integral equation

$$x(t) = x(0) + \int_0^t (b(s) + \beta x(s))ds + \int_0^t \sigma(x(s))dB(s) + \int_0^t \int_U g(x(s-), u)\tilde{N}(ds, du)$$

with $\tilde{N}(ds, du) = N(ds, du) - ds\mu(du)$. We are interested in the existence and uniqueness of the solution for the above stochastic equation. The coefficients of (1.1) we are considering are non-Lipschitz. Many authors studied the stochastic equations which are closely related to the above equation. In particular, Dawson and Li (2006, pp.1122-1131) gave a characterization of continuous state branching processes with immigration as strong solutions of some stochastic integral equations. They used the tightness and the Skorokhod representation to obtain the existence. Fu and Li (2008) studied a more general class of stochastic equations with jumps. Under very weak conditions, they established the existence and uniqueness of strong solutions of those equations. The present work differs from that of Fu and Li (2008) in that our drift term is given by a stochastic process.

The remainder of the paper is organized as follows. In next section, we state some results on the pathwise uniqueness of solutions to (1.1). In section 3, we discuss the Euler scheme for the equation and show that the approximating solution converges in $L^1$-supnorm towards the solution of (1.1). Some criteria on the existence and uniqueness of strong solutions are established in the last section.

For some preliminary results concerning the stochastic differential equations with jumps, the reader is referred to Bass (2004). We refer to Ikeda and Watanabe (1989) and Protter (2004) for the theory of stochastic analysis.

2. Pathwise uniqueness

In this section, we give some results on stochastic equations and on the pathwise uniqueness of solutions to (1.1). Because these results can be obtained using essentially the same arguments as the corresponding results of Fu and Li (2008), we omitted their proofs here. Since the coefficients of (1.1) satisfy the above conditions, we have the following proposition.

**Proposition 2.1.** If $\{x(t)\}$ satisfies (1.1) and $P\{x(0) \geq 0\} = 1$, then $P\{x(t) \geq 0 \text{ for all } t \geq 0\} = 1$.

In the sequel, we shall always assume the initial variable $x(0)$ is non-negative, so Proposition 2.1 implies that any solution of (1.1) is non-negative. Then we can assume the ingredients are defined only for $x \geq 0$. In addition, for the convenience of the statements of the results, we introduce the following conditions.

(2.a) The measurable and adapted process $b(\cdot)$ satisfying $\int_0^t E b(s)ds < \infty$ for all $t \geq 0$;
There is a constant \( K \geq 0 \) such that \( \sigma^2(x) + \int_U \sup_{0 \leq y \leq x} g^2(x, u) \mu(du) \leq K(1 + x) \) for all \( x \geq 0 \);

For every fixed \( u \in U \), the function \( x \mapsto g(x, u) \) is non-decreasing, and for each integer \( m \geq 1 \), there is a non-negative and non-decreasing function \( z \mapsto \rho_m(z) \) on \( \mathbb{R}_+ \) so that \( \int_{0^+} \rho_m^{-2}(z) dz = \infty \) and \( |\sigma(x) - \sigma(y)|^2 + \int_U |l(x, y; u)| \land I^2(x, y; u) | \mu(du) \leq \rho_m^2(|x - y|) \) for all \( 0 \leq x, y \leq m \), where \( l(x, y; u) = g(x, u) - g(y, u) \);

For every fixed \( u \in U \), the function \( x \mapsto g(x, u) \) is non-decreasing, and for each integer \( m \geq 1 \), there is a non-negative and non-decreasing function \( z \mapsto \rho_m(z) \) on \( \mathbb{R}_+ \) so that \( \int_{0^+} \rho_m^{-2}(z) dz = \infty \), \( |\sigma(x) - \sigma(y)| \leq \rho_m(|x - y|) \) and \( |g(x, u) - g(y, u)| \leq \rho_m(|x - y|) f_m(u) \) for all \( 0 \leq x, y \leq m \) and \( u \in U \), where \( u \mapsto f_m(u) \) is a non-negative function on \( U \) satisfying \( \int_U [f_m(u) \land f_m^2(u)] \mu(du) < \infty \).

We close this section with two theorems on the pathwise uniqueness of solutions to (1.1).

**Theorem 2.1.** Suppose that conditions (2.a, b, c) hold. Then the pathwise uniqueness of solution holds for (1.1).

**Theorem 2.2.** Suppose that conditions (2.a, b, d) hold. Then the pathwise uniqueness for (1.1) holds.

### 3. Existence

In this section, we prove a strong convergence of the Euler approximations of the equation (1.1), giving a construction of the solution. A similar analysis was carried out in Yamada (1976, 1978) for continuous type equations, in Fu (2007, pp. 30-36) and Fu and Li (2008) for two classes of jump-type equations. We refer the reader to Mao et al. (2006, 2007) for recent results on related topics.

For a fixed time \( T > 0 \), we remark that \( \int_0^T Eb(s) ds < \infty \). Let us define the function \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
\gamma(\nu) = \sup_{0 \leq s \leq t \leq s + \nu \leq T} \int_s^t Eb(u) du, \quad \nu \geq 0.
\]

Since the function \( t \mapsto Eb(t) \) is integrable over the interval \([0, T]\), we have that \( \gamma(\nu) \) converges to zero as \( \nu \) tends to zero.

We divide the interval \([0, T]\), known as the Euler discretization method. For each \( n \geq 1 \), we take a subdivision

\[
0 = t_0^n \leq t_1^n \leq \cdots \leq t_{N_n}^n = T
\]

and denote this net by \( \Delta_n \). For notational use, we drop the index \( n \) of the discretization times and write \( N \) instead of \( N_n \).

The mesh of the net is defined as \( \| \Delta_n \| = \sup_{1 \leq k \leq N} |t_k - t_{k-1}| \). We are working with a sequence of nets \( (\Delta_n)_n \) such that the meshes are tending to zero. There is no need to suppose that \( \Delta_n \subset \Delta_{n+1} \).

The solutions of (1.1) turns out to be non-negative but the approximations we will need may take negative values. We therefore put \( \sigma'(x) = \sigma(x)I_{\{x \geq 0\}} \) and \( g'(x, u) = g(x, u)I_{\{x \geq 0\}} \). Note that \( \sigma'(\cdot) \) and \( g'(\cdot, \cdot) \) also satisfy conditions (2.b, c, d).
If we are working with the net $\Delta_n$, we look at $x_{\Delta_n}(t)$, which we denote by $x_n(t)$. We put $x_n(0) = x(0)$. Let $\eta_n(t) = \sum_{k=0}^{N-1} t_k I_{[t_k, t_{k+1})}(t)$, we define a process $\{x_n(t)\}$ by

$$x_n(t) = x(0) + \int_0^t (b(s) + \beta x_n(\eta_n(s)))ds + \int_0^t \sigma'(x_n(\eta_n(s)))dB(s)$$

$$+ \int_0^t \int_U g'(x_n(\eta_n(s) -), u)\tilde{N}(ds, du).$$

This is called an Euler approximation of (1.1).

In the next conclusions, we need the following conditions:

(3.a) For every fixed $u \in U$, the function $x \mapsto g(x, u)$ is non-decreasing, and for each integer $m \geq 1$, there is a non-negative and non-decreasing function $z \mapsto \rho_m(z)$ on $\mathbb{R}_+$ so that

$$\int_0^\infty \rho_m^{-2}(z)dz = \infty, \quad z \mapsto \rho_m^2(z)$$

is concave and $|\sigma(x) - \sigma(y)|^2 + \int_U l^2(x, y; u)\mu(du) \leq \rho_m^2(|x - y|)$ for all $0 \leq x, y \leq m$, where $l(x, y; u) = g(x, u) - g(y, u)$;

(3.b) For every fixed $u \in U$, the function $x \mapsto g(x, u)$ is non-decreasing, and for each integer $m \geq 1$, there is a non-negative and non-decreasing function $z \mapsto \rho_m(z)$ on $\mathbb{R}_+$ so that

$$\int_0^\infty \rho_m^{-2}(z)dz = \infty, \quad z \mapsto \rho_m^2(z)$$

is concave, $|\sigma(x) - \sigma(y)| \leq \rho_m(|x - y|)$ and $|g(x, u) - g(y, u)| \leq \rho_m(|x - y|)f_m(u)$ for all $0 \leq x, y \leq m$ and $u \in U$, where $u \mapsto f_m(u)$ is a non-negative function on $U$ satisfying $\int_U f_m^2(u)\mu(du) < \infty$.

It is easy to show that $\sigma'(\cdot)$ and $g'(\cdot, \cdot)$ also satisfy conditions (3.a, b).

**Remark 3.1.** The functions $\rho(z) = \sqrt{z}$, $\rho(z) = z^{\frac{1}{2}}(\log \frac{1}{z})^\frac{1}{2}$, $\rho(z) = z^{\frac{1}{2}}(\log \frac{1}{z})^\frac{1}{2}(\log \log \frac{1}{z})^\frac{1}{2}, \cdots$ satisfy conditions (3.a, b).

**Theorem 3.1.** Suppose that conditions (2.a, b) and (3.a) hold. Then the Euler scheme (3.1) with $t_k \leq t < t_{k+1}, \ k = 0, 1, \cdots, N - 1$ converges to the solution of (1.1) in $L^1$-supnorm.

**Remark 3.2.** If the intensity of the Poisson random measure is zero and $\rho(z) = \sqrt{z}$, the results are degenerated to those of Deelstra and Delbaen (1998).

Next, we prove $x_n(t)$ converges to the solution of (1.1) in $L^1$-supnorm.

**Proposition 3.1.** Suppose that condition (2.a, b) hold. Then for all $0 \leq t \leq T$, there exist constants $G_T \geq 0$ and $H_T \geq 0$ such that the following hold:

$$E[|x_n(\eta_n(t))|] \leq G_T; \quad \text{(3.2)}$$

$$E[|x_n(t)|] \leq H_T; \quad \text{(3.3)}$$

$$E[|x_n(t) - x_n(\eta_n(t))|] \leq \gamma(\|\Delta_n\|) - \beta G_T \|\Delta_n\| + 2\sqrt{K(G_T + 1)}\|\Delta_n\|. \quad \text{(3.4)}$$
Proof. From (3.1), we obtain

\[
E[|x_n(\eta_n(t))|] \leq E[x(0)] + \int_0^t E(b(s)ds + E^{1/2} [\int_0^t \sigma'(x_n(\eta_n(s)))dB(s)]^2 \\
+ |\beta| \int_0^t E[|x_n(\eta_n(s))|]ds + E^{1/2} \int_0^t \int_U g'(x_n(\eta_n(s)-), u)\tilde{N}(ds, du)]^2 \\
\leq E[x(0)] + \int_0^t E(b(s)ds + |\beta| \int_0^t E[|x_n(\eta_n(s))|]ds \\
+ 2 + E[\int_0^t \sigma'^2(x_n(\eta_n(s)))ds] + E[\int_0^t ds \int_U g'^2(x_n(\eta_n(s)-), u)\mu(du)] \\
\leq E[x(0)] + \int_0^t E(b(s)ds + |\beta| \int_0^t E[|x_n(\eta_n(s))|]ds \\
+ 2 + Kt + K \int_0^t E[|x_n(\eta_n(s))|]ds \\
\leq (E[x(0)] + \int_0^t E(b(s)ds + 2 + Kt) + (K - \beta) \int_0^t E[|x_n(\eta_n(s))|]ds.
\]

The first and the third inequalities follow by Cauchy-Schwarz inequality and condition (2.b) respectively. By Gronwall’s lemma, we get

\[
E[|x_n(\eta_n(t))|] \leq (E[x(0)] + \int_0^t E(b(s)ds + 2 + Kt) \exp\{(K - \beta)t\} \\
=: G_t \leq G_T.
\]

After similar calculations, from (3.1) and (3.2), we get

\[
E[|x_n(t)|] \leq E[x(0)] + \int_0^t E(b(s)ds + 2 + Kt + (K - \beta)tG_t \\
=: H_t \leq H_T.
\]

The above two bounds are independent of \( n \) and \( t \).

From (3.1), (3.2), (3.3) and Cauchy-Schwarz inequality, we get (3.4) immediately. \( \square \)

Given a function \( f \) defined on a subset of \( \mathbb{R} \), we note

\[
\Delta_z f(x) = f(x + z) - f(x) \quad \text{and} \quad D_z f(x) = \Delta_z f(x) - f'(x)z
\]

if the right hand sides are meaningful.

**Proposition 3.2.** Suppose that conditions (2.a, b) and (3.a) hold. Then there exists a progressive process \( \{y(t)\} \) such that the following convergence hold:

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} E[|x_n(t) - y(t)|] = 0; \quad (3.5)
\]

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} E[|x_n(\eta_n(t)) - y(t)|] = 0. \quad (3.6)
\]
Proof. Let $\zeta(t) = x_n(t) - x_{n'}(t)$ for fixed $n$, $n' \geq 1$. Following from (3.1), we get

$$
\zeta(t) = \beta \int_0^t [x_n(\eta_n(s)) - x_{n'}(\eta_{n'}(s))]ds + \beta \int_0^t [\sigma'(x_n(\eta_n(s))) - \sigma'(x_{n'}(\eta_{n'}(s)))]dB(s) + \int_0^t \int_U [g'(x_n(\eta_n(s)-), u) - g'(x_{n'}(\eta_{n'}(s)-), u)]\hat{N}(ds, du).
$$

Let $a_0 = 1$ and choose $a_k \to 0+$ decreasingly so that $\int_{a_k}^{a_{k-1}} \rho_{n-2}(z)dz = k$ for $k \geq 1$. Let $z \mapsto \psi_k(z)$ be a non-negative continuous function on $\mathbb{R}$ which has support in $(a_k, a_{k-1})$ and satisfies $\int_{a_k}^{a_{k-1}} \psi_k(z)dz = 1$ and $0 \leq \psi_k(z) \leq 2k^{-1}\rho_{n-2}(z)$ for $a_k < z < a_{k-1}$. For each $k \geq 1$ we define the non-negative and twice continuously differentiable function

$$
\phi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x)dx, \quad z \in \mathbb{R}.
$$

Clearly, the sequence $\{\phi_k\}$ satisfies

(i) $\phi_k(x) \to |x|$ non decreasingly as $k \to \infty$;

(ii) $0 \leq \phi_k(x) \leq 1$ for $x \geq 0$ and $-1 \leq \phi_k(x) \leq 0$ for $x \leq 0$.

Let $\tau_m = \inf \{t \geq 0, |x_n(t)| \geq m \text{ or } |x_{n'}(t)| \geq m\}$ for $m \geq 1$. Applying Itô’s formula, we get

$$
\phi_k(\zeta(t \wedge \tau_m)) = \beta \int_0^{t \wedge \tau_m} \phi'_k(\zeta(s))[x_n(\eta_n(s)) - x_{n'}(\eta_{n'}(s))]ds + \text{mart.} + \frac{1}{2} \int_0^{t \wedge \tau_m} \phi''_k(\zeta(s))[\sigma'(x_n(\eta_n(s))) - \sigma'(x_{n'}(\eta_{n'}(s)))]^2 ds
$$

$$
+ \int_0^{t \wedge \tau_m} ds \int_U[D_l(n, n'; u)]\phi_k(\zeta(s-))\mu(du)
$$

$$
=: I_1(t \wedge \tau_m) + \text{mart.} + I_2(t \wedge \tau_m) + I_3(t \wedge \tau_m),
$$

(3.8)

where $l(n, n'; u) = g'(x_n(\eta_n(s)-), u) - g'(x_{n'}(\eta_{n'}(s)-), u)$.

According to $\beta < 0$ and $\{\phi_k\}$ satisfies property (ii), we get

$$
I_1(t \wedge \tau_m)
$$

$$
= \beta \int_0^{t \wedge \tau_m} \phi'_k(\zeta(s))[x_n(\eta_n(s)) - x_n(s)]ds + \beta \int_0^{t \wedge \tau_m} \phi'_k(\zeta(s))[x_n(s) - x_{n'}(s)]ds
$$

$$
+ \beta \int_0^{t \wedge \tau_m} \phi'_k(\zeta(s))[x_{n'}(s) - x_{n'}(\eta_{n'}(s))]ds
$$

$$
\leq \beta \int_0^{t \wedge \tau_m} \phi'_k(\zeta(s))[x_n(\eta_n(s)) - x_n(s)]ds + \beta \int_0^{t \wedge \tau_m} \phi'_k(\zeta(s))[x_{n'}(s) - x_{n'}(\eta_{n'}(s))]ds.
$$

Consequently,

$$
E[I_1(t \wedge \tau_m)]
$$

$$
\leq |\beta|E\left[\int_0^{t \wedge \tau_m} |x_n(\eta_n(s)) - x_n(s)|ds\right] + |\beta|E\left[\int_0^{t \wedge \tau_m} |x_{n'}(s) - x_{n'}(\eta_{n'}(s))|ds\right]
$$

$$
=: |\beta|A(n, m, t) + |\beta|A(n', m, t),
$$

6
where \( A(n, m, t) = E[\int_0^{t \land \tau_m} |x_n(\eta_t(s)) - x_n(s)| ds] \).

Since \( \int_{a_k}^{a_{k-1}} \rho_m^2(z) dz = k \) and by the monotonicity of \( z \mapsto \rho_m(z) \), we have \( k^{-1} \rho_m^{-2}(a_k) \leq 2 \).

Note that \( 0 \leq \phi_k''(z) = \psi_k(|z|) \leq 2k^{-1} \rho_m^{-2}(|z|) I_{(a_k, a_{k-1})}(|z|) \leq 2k^{-1} \rho_m^{-2}(a_k) \leq 4 \) and \( \sigma'(x) \) satisfies condition (3.a), we have

\[
E[I_2(t \land \tau_m)] \leq \frac{3}{2} E\left[ \int_0^{t \land \tau_m} \phi_k''(\zeta(s))(\sigma'(x_n(\eta_t(s)))) - \sigma'(x_n(s)))^2 ds \right]
\]

\[
+ \frac{3}{2} E\left[ \int_0^{t \land \tau_m} \phi_k''(\zeta(s))(\sigma'(x_n(s)))^2 ds \right]
\]

\[
+ \frac{3}{2} E\left[ \int_0^{t \land \tau_m} \phi_k''(\zeta(s))(\sigma'(x_{n'}(s)))^2 ds \right]
\]

\[
\leq 6E\left[ \int_0^{t \land \tau_m} \rho_m^2(|x_n(\eta_t(s)) - x_n(s)| ds) \right] + \frac{3t}{k}
\]

\[
+ 6E\left[ \int_0^{t \land \tau_m} \rho_m^2(|x_{n'}(s) - x_{n'}(\eta_{n'}(s))| ds) \right]
\]

\[
= 6B(n, m, t) + \frac{3t}{k} + 6B(n', m, t),
\]

where \( B(n, m, t) = E[\int_0^{t \land \tau_m} \rho_m^2(|x_n(\eta_t(s)) - x_n(s)| ds) \] .

By Taylor’s expansion and the definition of \( \phi_k \), for all \( h, \zeta \in \mathbb{R} \) it is easy to show that

\[
D_h \phi_k(\zeta) = h^2 \int_0^1 \phi_k''(\zeta + th)(1 - t) dt = h^2 \int_0^1 \psi_k(|\zeta + th|)(1 - t) dt
\]

\[
\leq 2k^{-1} h^2 \int_0^1 \rho_m^{-2}(|\zeta + th|) \{ I_{(a_k, a_{k-1})}(|\zeta + th|) \}(1 - t) dt
\]

\[
\leq 2k^{-1} h^2 \rho_m^{-2}(a_k) \int_0^1 (1 - t) dt \leq 2h^2.
\]

(3.9)

Note also that \( \zeta(s-) \neq \zeta(s) \) for at most countably many \( s \geq 0 \). From (3.8), (3.9) and \( g'(x, u) \) satisfies condition (3.a), we have

\[
E[I_3(t \land \tau_m)] \leq 2E[\int_0^{t \land \tau_m} ds \int_U (u_1 + u_2 + u_3)^2 \mu(du)] \leq 6 \sum_{i=1}^3 E[\int_0^{t \land \tau_m} ds \int_U u_i^2 \mu(du)]
\]

\[
\leq 6E[\int_0^{t \land \tau_m} \rho_m^2(|x_n(\eta_t(s)) - x_n(s)| ds) + 6E[\int_0^{t \land \tau_m} \rho_m^2(|x_n(s) - x_{n'}(s)| ds) + 6E[\int_0^{t \land \tau_m} \rho_m^2(|x_{n'}(s) - x_{n'}(\eta_{n'}(s))| ds)\]

\[
= 6B(n, m, t) + 6E[\int_0^{t \land \tau_m} \rho_m^2(|\zeta(s)| ds) + 6B(n', m, t),
\]

where

\[
u_1 = g'(x_n(\eta_t(s)-), u) - g'(x_n(s-), u),
\]

\[
u_2 = g'(x_n(s-), u) - g'(x_{n'}(s-), u),
\]

\[
u_3 = g'(x_{n'}(s-), u) - g'(x_{n'}(\eta_{n'}(s)-), u).
\]
Consequently,
\[ E[\phi_k(\zeta(t \wedge \tau_m))] \leq -\beta A(n, m, t) - \beta A(n', m, t) + 12B(n, m, t) + 12B(n', m, t) \]
\[ + \frac{3t}{k} + 6E[\int_0^{t \wedge \tau_m} \rho_m^2(\zeta(s))ds]. \tag{3.10} \]

By Proposition 3.1, the assumption on \( z \mapsto \rho_m^2(z) \) and the dominated convergence theorem, it is easy to see that
\[ \lim_{n \to \infty} A(n, m, t) = \lim_{n \to \infty} B(n, m, t) = 0. \]

From the definition of \( \phi_k(\cdot) \), we remark that \(|z| \leq a_{k-1} + \phi_k(z)\) for every \( z \in \mathbb{R} \). For given \( T \geq 0 \) and \( \varepsilon > 0 \), we first take an integer \( k_0 \geq 1 \) such that \( a_{k_0-1} + 3T/k_0 < \varepsilon/2 \). Then we choose sufficiently large \( N = N(k_0) \geq 1 \) so that \( 12B(n, m, t) - \beta A(n, m, t) < \varepsilon/4 \) for every \( n \geq N \). By (3.10), we have
\[ E[|\zeta(t \wedge \tau_m)|] \leq \varepsilon + 6E[\int_0^{t \wedge \tau_m} \rho_m^2(\zeta(s))ds] \]
for \( 0 \leq t \leq T \) and \( n, n' \geq N \). Since \( \zeta(s) < 2m \) for all \( 0 < s \leq \tau_m \), we infer that \( t \mapsto E[|\zeta(t \wedge \tau_m)|] \)
is locally bounded. Then the concaveness of \( z \mapsto \rho_m^2(z) \) implies that
\[ E[|\zeta(t \wedge \tau_m)|] \leq \varepsilon + 6E[\int_0^t \rho_m^2(E(|\zeta(s \wedge \tau_m)|))ds] \]
\[ \leq \varepsilon + 6E[\int_0^t \rho_m^2(E(|\zeta(s \wedge \tau_m)|))ds] \tag{3.11} \]
for \( 0 \leq t \leq T \) and \( n, n' \geq N \). Let
\[ R_n(t) = \sup_{n' \geq n} \sup_{0 \leq s \leq t} E[|x_n(s \wedge \tau_m) - x_{n'}(s \wedge \tau_m)|], \quad n \geq 1, \ 0 \leq t \leq T. \]

In view of (3.11), the monotonicity of \( z \mapsto \rho_m^2(z) \) gives
\[ R_n(t) \leq \varepsilon + 6 \int_0^t \rho_m^2(R_n(s))ds, \quad n \geq N, \ 0 \leq t \leq T. \]

By letting \( n \to \infty \) and \( \varepsilon \to 0 \) we obtain
\[ \lim_{n \to \infty} R_n(t) \leq 6 \int_0^t \rho_m^2(\lim_{n \to \infty} R_n(s))ds, \quad 0 \leq t \leq T. \]

Thus \( \lim_{n \to \infty} R_n(t) = 0 \) for every \( 0 \leq t \leq T \). Since \( \tau_m \to \infty \) as \( m \to \infty \) by Proposition 3.1 now letting \( m \to \infty \), it is easy to find a progressive process \( \{y(t)\} \) such that (3.5) holds. Moreover, by (3.2) and (3.5), (3.6) is also obtained. \( \Box \)

**Proposition 3.3.** Suppose that conditions (2.a, b) and (3.a) hold. Then there exists a càdlàg process \( \{x(t)\} \) such that:
\[ \lim_{n \to \infty} E[\sup_{0 \leq t \leq T} |x_n(t) - x(t)|] = 0; \]
\[ \lim_{n \to \infty} E[|x_n(\eta_n(t)) - x(t)|] = 0 \]
hold. Moreover, \( \{x(t)\} \) is a non-negative solution of (1.1).
Proof. Let $\tau_m = \inf\{t \geq 0, x_n(t) \geq m \text{ or } x_{n'}(t) \geq m\}$ for $m \geq 1$. Applying Doob’s martingale inequality to \eqref{3.7}, we get

\[
E\left[ \sup_{0 \leq t \leq T} |x_n(t \wedge \tau_m) - x_{n'}(t \wedge \tau_m)| \right] \leq |\beta| \int_0^{T \wedge \tau_m} E[|x_n(\eta_n(s)) - x_{n'}(\eta_{n'}(s))|]ds
\]

\[
+ 4E^{\frac{1}{2}}\left[ \int_0^{T \wedge \tau_m} (\sigma'(x_n(\eta_n(s))) - \sigma'(x_{n'}(\eta_{n'}(s))))^2 ds \right]
\]

\[
+ 4E^{\frac{1}{2}}\left[ \int_0^{T \wedge \tau_m} ds \int_U l^2(n, n'; u)\mu(du) \right].
\]

Letting $m \to \infty$, by condition (3.a), Proposition 3.1 and 3.2 and dominated convergence theorem, we get

\[
\lim_{n, n' \to \infty} E\left[ \sup_{0 \leq t \leq T} |x_n(t) - x_{n'}(t)| \right] = 0.
\]

Consequently, \{y(t)\} has a c\`adl\`ag modification \{x(t)\} satisfying the first equality. The second equality then follows by Proposition 3.3.

Next, we will show that

\[
x(t) = x(0) + \int_0^t (b(s) + \beta x(s)) ds + \int_0^t \sigma'(x(s)) dB(s) + \int_0^t \int_U g'(x(s), u)\tilde{N}(ds, du).
\]

Indeed, from \eqref{3.1}

\[
E\left[ \sup_{0 \leq t \leq T} |x(t) - x(0) - \int_0^t (b(s) + \beta x(s)) ds - \int_0^t \sigma'(x(s)) dB(s)
\right.
\]

\[
- \int_0^t \int_U g'(x(s), u)\tilde{N}(ds, du)|\right] = E\left[ \sup_{0 \leq t \leq T} |x(t) - x_n(t) + \beta \int_0^t (x_n(\eta_n(s)) - x(s)) ds + \int_0^t (\sigma'(x_n(\eta_n(s))) - \sigma'(x(s))) dB(s)
\right.
\]

\[
+ \int_0^t \int_U (g'(x_n(\eta_n(s)), u) - g'(x(s), u))\tilde{N}(ds, du)|\right]
\]

and the result follows by the triangular inequality, Doob’s martingale inequality and the previous calculations.

By Proposition 2.1 and the definitions of $\sigma'(x)$ and $g'(x, u)$, we remark that $x(t)$ is a non-negative process. Therefore, we can replace $\sigma'(x)$ and $g'(x, u)$ by $\sigma(x)$ and $g(x, u)$ respectively. Consequently, $x(t)$ satisfies

\[
x(t) = x(0) + \int_0^t (b(s) + \beta x(s)) ds + \int_0^t \sigma(x(s)) dB(s) + \int_0^t \int_U g(x(s), u)\tilde{N}(ds, du).
\]

Then we complete the proof.\[\square\]

We prove that the Euler scheme \eqref{3.1} converges to the unique solution of \eqref{1.1} in $L^1$-supnorm. The conclusion of Theorem 3.1 holds immediately.

After similar analysis to the previous results, we have the following theorem.
Theorem 3.2. Suppose that conditions (2.a, b) and (3.b) are satisfied. Then the Euler scheme (3.1) with \( t_k \leq t < t_{k+1}, \ k = 0, 1, \cdots, N - 1 \) converges in \( L^1 \)-supnorm towards the solution of (1.1).

4. Strong solutions

In this section, we give some criteria on the existence and uniqueness of the strong solution of equation (1.1) and illustrate a simple application of the results to stochastic differential equations driven by one-sided Lévy processes.

Theorem 4.1. Suppose that conditions (2.a, b) and (3.a) are satisfied. Then there exists a unique non-negative strong solution to (1.1).

Proof. By applying Theorem 3.1 we infer that (1.1) has a non-negative solution. In addition, the ingredients of (1.1) satisfy condition (2.c). Then the pathwise uniqueness of the equation follows from Theorem 2.1. \(\square\)

Based on the pathwise uniqueness stated in Theorem 2.2, the following result can be proved similarly as the above.

Theorem 4.2. Suppose that conditions (2.a, b) and (3.b) are satisfied. Then there exists a unique non-negative strong solution to (1.1).

At last, we give a simple application of Theorem 4.1 and Theorem 4.2. Now we consider stochastic equations driven by one-sided Lévy processes. Let \( \mu(dz) \) be a \( \sigma \)-finite measure on \((0, \infty)\). We assume that \( \int_0^\infty z^2 \mu(dz) < \infty \). Let \( \{B(t)\} \) be a standard \((\mathcal{F}_t)\)-Brownian motion. Let \( \{z(t)\} \) be a \((\mathcal{F}_t)\)-Lévy process with exponent \( u \mapsto \int_0^\infty (e^{izu} - 1 - iuz) \mu(dz) \). Therefore \( \{z(t)\} \) is centered. Suppose that those processes are independent of each other. In addition, suppose that \( \beta < 0 \) is a real constant and

(i) a measurable and adapted process \( b : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+ \) satisfying \( \int_0^t Eb(s)ds < \infty \) for all \( t \geq 0 \).

(ii) \( x \mapsto \sigma(x) \) is a continuous function on \( \mathbb{R}_+ \) satisfying \( \sigma(0) = 0 \);

(iii) \( x \mapsto \phi(x) \) is a continuous non-negative function on \( \mathbb{R}_+ \) satisfying \( \phi(0) = 0 \).

We assume the following condition on the ingredients:

(4.a) The function \( x \mapsto \phi(x) \) is non-decreasing and for each \( m \geq 1 \) there is a constant \( K_m \geq 0 \) so that

\[
|\sigma(x) - \sigma(y)|^2 + |\phi(x) - \phi(y)|^2 \leq K_m |x - y|
\]

for all \( 0 \leq x, y \leq m \).

Theorem 4.3. Under condition (4.a), there is a unique non-negative strong solution to

\[
dx(t) = \sigma(x(t))dB(t) + \phi(x(t-))dz(t) + (b(t) + \beta x(t))dt. \tag{4.1}
\]
**Proof.** By the general result on Lévy-Itô decompositions, see, e.g., Sato (1999, p.120, Theorem 19.2), we have

\[ z(t) = \int_0^t \int_0^\infty z \tilde{N}(ds, dz), \]

where \( N(ds, dz) \) is a poisson random measure with intensity \( ds\mu(dz) \). By Theorem 4.2 there is a unique strong solution to

\[ x(t) = x(0) + \int_0^t \sigma(x(s))dB(s) + \int_0^t \int_0^\infty \phi(x(s-))z \tilde{N}(ds, dz) + \int_0^t (b(s) + \beta x(s))ds, \]

which is just another form of (4.1). The conclusion holds immediately. \( \square \)

**Acknowledgements**

I would like to give my sincere thanks to my supervisor Professor Zenghu Li for his encouragement and helpful suggestions. This research is supported by National Science Foundation of China (No. 10721091).

**References**

[1] Bass, R. F., 2004. Stochastic differential equations with jumps. Probability Survey. 1. 1-19.

[2] Dawson, D.A. and Li, Z.H., 2006. Skew convolution semigroups and affine Markov processes. The Annals of Probability 34 (3), 1103-1142.

[3] Deelstra, G. and Delbaen, F., 1995. Long-term returns in stochastic interest rate models. Insurance: Mathematics and Economics 17, 163-169.

[4] Deelstra, G. and Delbaen, F., 1998. Convergence of discretized stochastic (interest rate) processes with stochastic drift term. Applied Stochastic Models and Data Analysis. 14. no. 1. 77-84.

[5] Fu, Z.F., 2007. Stochastic differential equations of non-negative processes with non-negative jumps. Ph.D Thesis of Beijing Normal University.

[6] Fu, Z.F. and Li, Z.H., 2008. Stochastic equations of non-negative processes with jumps. Submitted to Stochastic Processes and Their Applications. [Preprint from: http://math.bnu.edu.cn/~lizh]

[7] Ikeda, N. and Watanabe, S., 1989. Stochastic differential equations and diffusion processes. North-Holland/Kodansha, Amsterdam/Tokyo.

[8] Lamberton, D. and Lapeyre, B., 1996. Introduction to stochastic calculus applied to finance. Chapman and Hall, London.

[9] Mao, X.R., Truman, A. and Yuan, C.G., 2006. Euler-Maruyama approximations in mean-reverting stochastic volatility model under regime-switching. Journal of Applied Mathematics and Stochastic Analysis. Volume 2006, Article ID 80967.

[10] Mao, X.R., Yuan, C.G. and Yin, G., 2007. Approximations of Euler-Maruyama type for stochastic differential equations with Markovian switching, under non-Lipschitz conditions. Journal of Computational and Applied Mathematics. 205, 936–948.
[11] Protter, P.E., 2004. Stochastic integration and differential equations. Springer-Verlag, Berlin, Heidelberg, New York.

[12] Sato, K., 1999. Lévy processes and infinitely divisible distributions. Cambridge University Press, Cambridge.

[13] Yamada, T., 1976. Sur l'approximation des solutions d’équation différentielles stochastiques. Z. Wahrscheinlichkeitstheorie verw. Gebiete. 36, 153-164.

[14] Yamada, T., 1978. Sur une construction des solutions d’équation différentielles stochastiques dans le cas non-lipschitzien. Lecture Notes in Mathematics. 649, 114-131. Springer, Berlin.