Invariance of Stationary and Ergodic Properties of a Quantum Source under Memoryless Transformations

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Abstract

We prove that the stationarity and the ergodicity of a quantum source \( \{ \rho_m \}_{m=1}^{\infty} \) are preserved by any trace-preserving completely positive linear map of the tensor product form \( \mathcal{E} \otimes m \), where a copy of \( \mathcal{E} \) acts locally on each spin lattice site. We also establish ergodicity criteria for so called classically-correlated quantum sources.

I. INTRODUCTION

The quantum ergodicity proves to be as instrumental in studying quantum information systems as is the classical ergodicity in studying classical systems. To give a rough idea of the role that quantum ergodicity plays in quantum information theory, one may name just one result, the quantum extension[3] of Shannon-McMillan theorem.

In this paper we are concerned with stationary and ergodic properties of quantum sources. Specifically, we study the case when a stationary and ergodic (weakly mixing or strongly mixing, respectively) quantum source \( \{ \rho_m \}_{m=1}^{\infty} \) is subjected to a trace-preserving completely positive linear transformation (map) of the tensor product form \( \mathcal{E} \otimes m \), where a copy of \( \mathcal{E} \) locally acts on each spin lattice site. We present several technical lemmas and prove that the map preserves all the listed source properties. Such maps describe the effect of a transmission via a memoryless
channel as well as the effect of memoryless coding, both lossless and lossy ones. As a corollary of our main result, we also establish ergodicity criteria for so called classically-correlated quantum sources.

II. QUANTUM SOURCES: MATHEMATICAL FORMALISM AND NOTATION

Before we define a general quantum source, we give an informal, intuitive definition of a so-called classically correlated quantum source as a triple[7] consisting of quantum messages, a classical probability distribution for the messages, and the time shift. Such the triple uniquely determines a state of a one-dimensional quantum lattice system. If quantum-mechanical correlation between the messages exists, one gets the notion of a general quantum source. While any given state corresponds to infinitely many different quantum sources, the quantum state formalism essentially captures all the information-theoretic properties of a corresponding quantum source. Thus, the notion of ”quantum source” is usually identified with the notion of ”state” of the corresponding lattice system and used interchangeably.

Let $Q$ be an infinite quantum spin lattice system over lattice $\mathbb{Z}$ of integers. To describe $Q$, we use the standard mathematical formalism introduced in [5, Sec. 6.2.1] and [14, Sec. 1.33 and Sec. 7.1.3] and borrow notation from [3] and [11]. Let $\mathcal{A}$ be a $C^*$-algebra\footnote{The algebra of all bounded linear operators may be simply thought of as the algebra of all square matrices with the standard matrix operations including conjugate-transpose.} with identity of all bounded linear operators $\mathcal{B}(\mathcal{H})$ on a $d$-dimensional Hilbert space $\mathcal{H}$, $d < \infty$. To each $x \in \mathbb{Z}$ there is associated an algebra $\mathcal{A}_x$ of observables for a spin located at site $x$, where $\mathcal{A}_x$ is isomorphic to $\mathcal{A}$ for every $x$. The local observables in any finite subset $\Lambda \subset \mathbb{Z}$ are those of the finite quantum system

$$\mathcal{A}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{A}_x$$

The quasilocal algebra $\mathcal{A}_\infty$ is the operator norm completion of the normed algebra $\bigcup_{\Lambda \subset \mathbb{Z}} \mathcal{A}_\Lambda$, the union of all local algebras $\mathcal{A}_\Lambda$ associated with finite $\Lambda \subset \mathbb{Z}$. A state of the infinite spin system is given by a normed positive functional

$$\varphi : \mathcal{A}_\infty \rightarrow \mathbb{C}.$$
We define a family of states \( \{ \varphi^{(A)} \}_{A \subset \mathbb{Z}} \), where \( \varphi^{(A)} \) denotes the restriction of the state \( \varphi \) to a finite-dimensional subalgebra \( \mathfrak{A}_A \), and assume that \( \{ \varphi^{(A)} \}_{A \subset \mathbb{Z}} \) satisfies the so called consistency condition\([3], [7]\), that is

\[
\varphi^{(A)} = \varphi^{(A')} | \mathfrak{A}_A
\]

for any \( \Lambda \subset \Lambda' \). The consistent family \( \{ \varphi^{(A)} \}_{A \subset \mathbb{Z}} \) can be thought of as a quantum-mechanical counterpart of a consistent family of cylinder measures. Since there is one-to-one correspondence between the state \( \varphi \) and the family \( \{ \varphi^{(A)} \}_{A \subset \mathbb{Z}} \), any physically realizable transformation of the infinite system \( Q \), including coding and transmission of quantum messages, can be well formulated using the states \( \varphi^{(A)} \) of finite subsystems. Where the subset \( \Lambda \in \mathbb{Z} \) needs to be explicitly specified, we will use the notation \( \Lambda(n) \), defined as

\[
\Lambda(n) := \{ x \in \mathbb{Z} : x \in \{1, \ldots, n\} \}
\]

Let \( \gamma \) (or \( \gamma^{-1} \), respectively) denote a transformation on \( \mathfrak{A}_\infty \) which is induced by the right (or left, respectively) shift on the set \( \mathbb{Z} \). Then, for any \( l \in \mathbb{N} \), \( \gamma^l \) (or \( \gamma^{-l} \), respectively) denotes a composition of \( l \) right (or left, respectively) shifts. Now we are equipped to define the notions of stationarity and ergodicity of a quantum source.

**Definition 2.1:** A state \( \varphi \) is called \( N \)-stationary for an integer \( N \) if \( \varphi \circ \gamma^N = \varphi \). For \( N = 1 \), an \( N \)-stationary state is called stationary.

**Definition 2.2:** A \( N \)-stationary state is called \( N \)-ergodic if it is an extremal point in the set of all \( N \)-stationary states. For \( N = 1 \), \( N \)-ergodic state is called ergodic.

The following lemma which provides a practical method of demonstrating the ergodicity of a state is due to [14, Propos. 6.3.5, Lem. 6.5.1].

**Lemma 2.1:** The following conditions are equivalent:

(a) A stationary state \( \varphi \) on \( \mathfrak{A}_\infty \) is ergodic.

(b) For all \( a, b \in \mathfrak{A}_\infty \), it holds

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(a \ \gamma^i(b)) = \varphi(a) \ \varphi(b).
\]

(c) For every self-adjoint \( a \in \mathfrak{A}_\infty \), it holds

\[
\lim_{n \to \infty} \varphi \left( \left( \frac{1}{n} \sum_{i=1}^{n} \gamma^i(a) \right)^2 \right) = \varphi^2(a).
\]

Now we state a series of definitions\([4]\) which provide ”stronger” notions of ergodicity:

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Definition 2.3: A state is called completely ergodic if it is $N$-ergodic for every integer $N$.

Definition 2.4: A stationary state $\varphi$ on $\mathfrak{A}_\infty$ is called weakly mixing if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |\varphi(a \gamma^i(b)) - \varphi(a) \varphi(b)| = 0, \quad \forall a, b \in \mathfrak{A}_\infty. \quad (3)$$

Definition 2.5: A stationary state $\varphi$ on $\mathfrak{A}_\infty$ is called strongly mixing if

$$\lim_{i \to \infty} \varphi(a \gamma^i(b)) = \varphi(a) \varphi(b), \quad \forall a, b \in \mathfrak{A}_\infty. \quad (4)$$

It is straightforward to see that $(4) \Rightarrow (3) \Rightarrow (2)$.

Let $tr_{\mathfrak{A}_\Lambda}(\cdot)$ denote the canonical trace on $\mathfrak{A}_\Lambda$ such that $tr_{\mathfrak{A}_\Lambda}(e) = 1$ for all one-dimensional projections $e$ in $\mathfrak{A}_\Lambda$. Where an algebra on which the trace is defined is clear from the context, we will omit the trace’s subscript and simply write $tr(\cdot)$. For each $\varphi(\Lambda)$ there exists a unique density operator $\rho_{\Lambda} \in \mathfrak{A}_\Lambda$, such that $\varphi(\Lambda)(a) = tr(\rho_{\Lambda} a), a \in \mathfrak{A}_\Lambda$. Thus, any stationary state $\varphi$ is uniquely defined by the family of density operators $\{\rho_{\Lambda(m)}\}_{m=1}^{\infty}$. Where no confusion arises, we will use the following abbreviated notation for the rest of the paper. For all $n \in \mathbb{N}$,

$$\mathfrak{A}(n) := \mathfrak{A}_{\Lambda(n)}$$

$$\psi(n) := \psi^{\Lambda(n)}$$

$$\rho_n := \rho_{\Lambda(n)}$$

III. Main Result

In this section we present a sequence of lemmas and a theorem which help to establish the ergodicity of a state. But first we shall reformulate the stationary ergodic properties of an infinite spin lattice system in terms of its finite subsystems. By rewriting the consistency condition (1), Definition 2.1, and the equations (2–4) in terms of density operators, we obtain the following three elementary lemmas.

Lemma 3.1: A family $\{\rho_m\}_{m=1}^{\infty}$ on $\mathfrak{A}_\infty$ is consistent if and only if, for all positive integers $m, i < \infty$ and every $a \in \mathfrak{A}(m)$, the following holds:

$$tr(\rho_m a) = tr(\rho_{m+i} (a \otimes I^{\otimes i})), \quad (5)$$

In what follows we abusively use the same symbol to denote both an operator (or superoperator), confined to a lattice box $\Lambda(m)$, and its "shifted" copy, confined to a box $\{1+j, \ldots, m+j\}$, where the value of integer $j$ will be understood from the context.
where $I^{\otimes i}$ stands for the $i$-fold tensor product of the identity operators acting on respective spins.

**Lemma 3.2:** A quantum source \( \{\rho_m\}_{m=1}^{\infty} \) on \( \mathcal{A}_\infty \) is stationary if and only if, for all positive integers $m, i < \infty$ and every $a \in \mathcal{A}^{(m)}$, the following equality is satisfied:

\[
\text{tr}(\rho_m a) = \text{tr}(\rho_{m+i} (I^{\otimes i} \otimes a)),
\]

(6)

**Lemma 3.3:** A stationary quantum source \( \{\rho_m\}_{m=1}^{\infty} \) on \( \mathcal{A}_\infty \) is ergodic (weakly mixing or strongly mixing, respectively) if and only if, for every positive integer $m < \infty$ and all $a, b \in \mathcal{A}^{(m)}$, the equality (7) (8) or (9), respectively) holds:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=m}^{n} \text{tr}(\rho_{m+i} (a \otimes I^{\otimes (i-m)} \otimes b)) = \text{tr}(\rho_m a) \text{ tr}(\rho_m b),
\]

(7)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=m}^{n} \left| \text{tr}(\rho_{m+i} (a \otimes I^{\otimes (i-m)} \otimes b)) - \text{tr}(\rho_m a) \text{ tr}(\rho_m b) \right| = 0,
\]

(8)

\[
\lim_{i \to \infty} \text{tr}(\rho_{m+i} (a \otimes I^{\otimes (i-m)} \otimes b)) = \text{tr}(\rho_m a) \text{ tr}(\rho_m b),
\]

(9)

We now need to fix some additional notation. Let $E$ be an arbitrary trace-preserving quantum operation that has the input space $\mathcal{B}(\mathcal{H})$. Without loss of generality we assume that the output space for $E$ is also $\mathcal{B}(\mathcal{H})$. It is known[9] that $E$ is a trace-preserving completely positive linear (TPCPL) map. Next, we define a composite map

\[
E^{\otimes m} : \mathcal{A}^{(m)} \to \mathcal{A}^{(m)}, \quad \forall m > 0.
\]

We point out that such a tensor product map is the most general description of a quantum memoryless channel[1].

**Theorem 3.1:** If \( \{\rho_m\}_{m=1}^{\infty} \) is a stationary and ergodic (weakly mixing or strongly mixing, respectively) source, then so is \( \{E^{\otimes m}(\rho_m)\}_{m=1}^{\infty} \). The proof of this theorem is given in the appendix [II].

**Remark 1:** We note that any weakly or strongly mixing quantum source is also completely ergodic. Then, for such sources, the theorem trivially extends to cover TPCPL maps of the form \( (E^k)^{\otimes (m/k)} \), \( (m/k) \in \mathbb{Z} \), where $E^k$ acts on $k$-blocks of lattice, in direct analogy with a $k$-block classical coding. Thus, our work is the quantum generalization of a well-known classical information-theoretic result[2, chap. 7] for memoryless- and block-coding and channel transmission.
**Definition 3.1:** We define a classically correlated quantum source \( \{ \rho_{m}^{\text{cls}} \}_{m=1}^{\infty} \) by an equation

\[
\rho_{m}^{\text{cls}} := \sum_{x_{1},x_{2},\ldots,x_{m}} p(x_{1},x_{2},\ldots,x_{m}) |x_{1}\rangle \langle x_{1}| \otimes |x_{2}\rangle \langle x_{2}| \otimes \cdots \otimes |x_{m}\rangle \langle x_{m}|, \tag{10}
\]

where \( p(\cdot) \) stands for a probability distribution, and for every \( i \), \(|x_{i}\rangle\) belongs to some fixed linearly-independent set \( S := \{|\psi_{1}\rangle, |\psi_{2}\rangle, \ldots, |\psi_{d}\rangle\} \) of vectors in the Hilbert space \( \mathcal{H} \). We recall that \( \mathcal{H} \) is the support space for the operators in \( \mathfrak{A} \). The set \( S \) is sometimes called a quantum alphabet.

**Corollary 3.2:** If a classical probability distribution \( p(\cdot) \) in Definition 3.1 is a stationary and ergodic (weakly mixing or strongly mixing, respectively), then so is the quantum source \( \{ \rho_{m}^{\text{cls}} \}_{m=1}^{\infty} \). The proof of this corollary is given in the appendix III.

**APPENDIX I**

**CONDITIONAL EXPECTATION**

Let \( \tilde{\mathfrak{A}} \) be a \( C^{*} \)-subalgebra of \( \mathfrak{A} \), and let \( E : \mathfrak{A} \to \tilde{\mathfrak{A}} \) be a linear mapping which sends the density of every state \( \varphi \) on \( \mathfrak{A} \) to the density of the state \( \varphi | \tilde{\mathfrak{A}} \). Such a mapping is usually called a conditional expectation and has the following properties[10, Propos. 1.12]:

(a) if \( a \in \mathfrak{A} \) is positive operator, then so is \( E(a) \in \tilde{\mathfrak{A}} \);

(b) \( E(b) = b \) for every \( b \in \tilde{\mathfrak{A}} \);

(c) \( E(ab) = E(a)b \) for every \( a \in \mathfrak{A} \) and \( b \in \tilde{\mathfrak{A}} \);

(d) for every \( a \in \mathfrak{A} \), it holds

\[
\text{tr}_{\mathfrak{A}}(a) = \frac{\text{tr}_{\tilde{\mathfrak{A}}}(I)}{\text{tr}_{\tilde{\mathfrak{A}}}(I)} \text{tr}_{\tilde{\mathfrak{A}}}(E(a)),
\]

where \( I \) stands for identity operator.

**APPENDIX II**

**PROOF OF THEOREM 3.1**

For any TPCPL map there exists a so-called ”operator-sum representation”[1],[9]. Then, an \( m \)-fold tensor product map \( \mathcal{E}^{\otimes m} \) has the following representation:

\[
\mathcal{E}^{\otimes m}(\rho_{m}) = \sum_{j_{1},j_{2},\ldots,j_{m}} (A_{j_{1}} \otimes A_{j_{2}} \otimes \cdots \otimes A_{j_{m}}) \rho_{[1,m]}(A_{j_{1}} \otimes A_{j_{2}} \otimes \cdots \otimes A_{j_{m}})^{\dagger} \tag{11}
\]

with

\[
\sum_{i} A_{i}^{\dagger}A_{i} = I, \quad A_{i}, I \in \mathfrak{A}, \tag{12}
\]
where \( I \) stands for identity operator.

Due to (11) and (12), the following three equalities hold for all positive integers \( m < i < \infty \) and all \( a, b \in \mathcal{A}(m) \)

\[
\text{tr}(\mathcal{E}^{\otimes (m+i)}(\rho_{m+i})(a \otimes I^{\otimes (i-m)} \otimes b)) = \text{tr}(\rho_{m+i}(\tilde{a} \otimes I^{\otimes (i-m)} \otimes \tilde{b})),
\]

\[
\text{tr}(\mathcal{E}^{\otimes m}(\rho_m)a) = \text{tr}(\rho_m\tilde{a}),
\]

\[
\text{tr}(\mathcal{E}^{\otimes m}(\rho_{\Lambda(m)})b) = \text{tr}(\rho_m\tilde{b}),
\]

where \( a, b \in \mathcal{A}(m) \) and \( \tilde{a} \) and \( \tilde{b} \) are defined as follows:

\[
\tilde{a} := \sum_{j_1, j_2, \ldots, j_m} (A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m})^\dagger a (A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m}),
\]

\[
\tilde{b} := \sum_{j_1, j_2, \ldots, j_m} (A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m})^\dagger b (A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_m}).
\]

Combining (13) with Lemma 3.3, we obtain the ergodicity (weakly mixing or strongly mixing, respectively) of \( \{\mathcal{E}^{\otimes m}(\rho_m)\}_{m=1}^{\infty} \). In a similar manner, the application of Lemma 3.1 establishes the consistency of \( \{\mathcal{E}^{\otimes m}(\rho_m)\}_{m=1}^{\infty} \), and the application of Lemma 3.2 establishes the stationarity of \( \{\mathcal{E}^{\otimes m}(\rho_m)\}_{m=1}^{\infty} \). ■

**APPENDIX III**

**PROOF OF COROLLARY 3.2**

Let \( S_\perp := \{|e_1\rangle, |e_2\rangle, \ldots, |e_d\rangle\} \) be any orthonormal basis in \( \mathcal{H} \), and let \( \{\tilde{\rho}^{cls}_m\}_{m=1}^{\infty} \) be the source with alphabet \( S_\perp \) and distribution \( p(\cdot) \). For \( i = 1, \ldots, d \), we define a set \( \{A_i\} \) of linear operators as follows

\[
A_i := |\psi_i\rangle\langle e_i|.
\]

Then, set \( \{A_i\} \) satisfies (12), and we define a TPCPL map \( \mathcal{E}^{\otimes m} \) as in (11). Consequently, we have \( \tilde{\rho}^{cls}_m = \mathcal{E}^{\otimes m}(\tilde{\rho}^{cls}_m) \). Thus, to complete the proof, we need to show that \( \{\tilde{\rho}^{cls}_m\}_{m=1}^{\infty} \) on \( \mathcal{A}_\infty \) is ergodic (weakly mixing or strongly mixing, respectively). Let \( \mathcal{C} \) be a subalgebra of \( \mathcal{A} \) spanned by the set \( \{|e_i\rangle\langle e_i| : |e_i\rangle \in S_\perp\} \). We extend \( \mathcal{C} \) to a quasilocal algebra \( \mathcal{C}_\infty \subset \mathcal{A}_\infty \) over lattice \( \mathbb{Z} \) in the same way we did for \( \mathcal{A}_\infty \). The algebra \( \mathcal{C}_\infty \) is abelian due to the orthogonality of the set \( S_\perp \). For any integer \( m > 1 \), let \( E_m : \mathcal{A}(m) \rightarrow \mathcal{C}(m) \) denote the conditional expectation.
Since $\mathfrak{C}(m)$ is a maximal abelian subalgebra of $\mathfrak{A}(m)$, we have $\text{tr}_{\mathfrak{C}(m)}(I) = \text{tr}_{\mathfrak{A}(m)}(I)$. Moreover, by our construction, $\tilde{\rho}^{cls}_m$ is an element of algebra $\mathfrak{C}(m) \subset \mathfrak{A}(m)$ for every $m$. Then, the following equalities hold by the properties of conditional expectation for all positive integers $m < i < \infty$ and all $a, b \in \mathfrak{A}(m)$:

$$\text{tr}_{\mathfrak{A}(m+i)}\left(\tilde{\rho}^{cls}_{m+i}(a \otimes I^{\otimes(i-m)} \otimes b)\right) = \text{tr}_{\mathfrak{A}(m+i)}\left(E_{m+i}\left(\tilde{\rho}^{cls}_{m+i}(a \otimes I^{\otimes(i-m)} \otimes b)\right)\right)$$

$$= \text{tr}_{\mathfrak{A}(m+i)}\left(\tilde{\rho}^{cls}_{m+i}E_{m+i}(a \otimes I^{\otimes(i-m)} \otimes b)\right),$$

$$\text{tr}_{\mathfrak{A}(m)}(\tilde{\rho}^{cls}_m a) = \text{tr}_{\mathfrak{A}(m)}\left(E_m(\tilde{\rho}^{cls}_m a)\right) = \text{tr}_{\mathfrak{A}(m)}\left(\tilde{\rho}^{cls}_m E_m(a)\right),$$

$$\text{tr}_{\mathfrak{A}(m)}(\tilde{\rho}^{cls}_m b) = \text{tr}_{\mathfrak{A}(m)}\left(E_m(\tilde{\rho}^{cls}_m b)\right) = \text{tr}_{\mathfrak{A}(m)}\left(\rho^{cls}_m E_m(b)\right).$$

Thus, if $\{\tilde{\rho}^{cls}_m\}_{m=1}^{\infty}$ is consistent, stationary, and ergodic (weakly mixing or strongly mixing, respectively) on $\mathfrak{C}_\infty$, then it also holds on $\mathfrak{A}_\infty$ by the lemmas 3.1 3.2 and 3.3 Finally, we note that since $\mathfrak{C}_\infty$ is abelian, $\{\tilde{\rho}^{cls}_m\}_{m=1}^{\infty}$ on $\mathfrak{C}_\infty$ is ergodic (weakly mixing or strongly mixing, respectively) if and only if so is $\rho(\cdot)$ by Proposition 4.1 from the appendix. ■

**APPENDIX IV**

**STATES ON QUASILOCAL COMMUTATIVE $C^*$-ALGEBRAS**

Let $\mathcal{B}$ be an arbitrary commutative $k$-dimensional $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$, and let $\mathcal{B}_\infty$ be a quasilocal algebra $\mathcal{B}_\infty$ over lattice $\mathbb{Z}$ with local algebras $\mathcal{B}_x$ isomorphic to $\mathcal{B}$ for every $x \in \mathbb{Z}$, i.e., $\mathcal{B}_\infty$ is constructed in the same way as is $\mathcal{A}_\infty$ in Section II Then, for any $\Lambda \subset \mathbb{Z}$, every minimal projector in $\mathcal{B}_\Lambda$ is necessarily one-dimensional, and the density operator for every pure state $\varphi^{(\Lambda)}$ on $\mathcal{B}_\Lambda$ is exactly a one-dimensional projector. Let $\{|z_i\rangle\langle z_i|\}_{i=1}^{k}$ be a collection of the density operators for all the distinct pure states on $\mathcal{B}$. We then define an abstract set $\mathcal{Z} := \{z_i\}_{i=1}^{k}$, where every element $z_i$ symbolically corresponds to the operator $|z_i\rangle\langle z_i|$, and $z_i \neq z_j$ for all $i \neq j$. For every finite lattice subset $\Lambda \in \mathbb{Z}$, we define the Cartesian product

$$\mathcal{Z}^\Lambda := \times_{x \in \Lambda} \mathcal{Z}_x,$$

i.e., the elements $\omega$ of $\mathcal{Z}^{\Lambda(n)}$ have the form $\omega = \omega_1 \ldots \omega_n$, $\omega_i \in \mathcal{Z}$. It is easy to see that, for every $\Lambda \in \mathbb{Z}$, the set $\mathcal{Z}^\Lambda$ and the set of all one-dimensional projectors in $\mathcal{B}_\Lambda$ are in one-to-one correspondence: $\omega \longleftrightarrow |\omega\rangle\langle \omega|$. Consequently, there is one-to-one correspondence between the set of all projectors in $\mathcal{B}_\Lambda$ and $\mathcal{P}^\Lambda(\mathcal{Z})$, the Cartesian product of the power sets of $\mathcal{Z}$.
particular, every projector \( p \in \mathcal{B}_\Lambda \) corresponds to a set \( \{ \omega : \omega \in \mathcal{Z}^\Lambda, |\omega\rangle\langle\omega| \leq p \} \). We note that, equipped with the product of the discrete topologies of the sets \( \mathcal{Z}_\Lambda, \mathcal{Z}^\Lambda \) is a compact space, and the pair \( (\mathcal{Z}^\Lambda, \mathcal{P}(\mathcal{Z})) \) defines a measurable space. Thus, by Gelfand-Naimark theorem[13, Chap. 11] and Riesz representation theorem[12, Sec. 2.14], for any pure or mixed state \( \varphi(\Lambda) \) on \( \mathcal{B}_\Lambda \), there exists a unique positive measure on \( (\mathcal{Z}^\Lambda, \mathcal{P}(\mathcal{Z})) \), denoted by \( \mu_\Lambda \), such that the following equality holds for any projector \( p \in \mathcal{B}_\Lambda \):

\[
\varphi(\Lambda)(p) = \sum_{|\omega\rangle\langle\omega| \leq p} \mu_\Lambda(\omega) \tag{15}
\]

Combining (15) and (5) and setting \( a := |\omega_1 \ldots \omega_m\rangle\langle\omega_m \ldots \omega_1| \) in the latter, we obtain, for any \( m, i \in \mathbb{N} \) and any \( \omega_1 \ldots \omega_m \in \mathcal{Z}^{\Lambda(m)} \),

\[
\mu_{\Lambda(m)}(\omega_1 \ldots \omega_m) = \sum_{\omega_{m+1} \ldots \omega_{m+i}} \mu_{\Lambda(m+i)}(\omega_1 \ldots \omega_m \omega_{m+1} \ldots \omega_{m+i}) \tag{16}
\]

The equality (16) is called the (classical) consistency condition. Thus, \( \{ \mu_\Lambda \}_{\Lambda \in \mathcal{Z}} \) is a consistent family of probability measures, and \( \mu_\Lambda \) extends to a probability measure on \( (\mathcal{Z}^\infty, \mathcal{P}^\infty(\mathcal{Z})) \) by the Kolmogorov extension theorem[8]. The extended measure is denoted by \( \mu \). In fact, the tuple \( (\mathcal{B}_\infty, \varphi) \) and the triple \( (\mathcal{Z}^\infty, \mathcal{P}^\infty(\mathcal{Z})) \) are just two equivalent descriptions[14] of a given classical stochastic process. This particularly implies the following proposition.

**Proposition 4.1:** If a state \( \varphi \) on \( \mathcal{B}_\infty \) is stationary and ergodic (weakly mixing or strongly mixing, respectively), then so is the corresponding measure \( \mu \) on \( (\mathcal{Z}^\infty, \mathcal{P}^\infty(\mathcal{Z})) \). The converse is also true.

**Proof:** The result follows immediately from Lemma 3.2, Lemma 3.3 and the equality (15).

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