Threshold resummation for Drell-Yan production: theory and phenomenology

Marco Bonvini

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In collaboration with:
Stefano Forte, Giovanni Ridolfi
Plan of the talk:

- For which values of $\tau = Q^2 s$ is resummation important?
- Two prescriptions for resummation:
  - Minimal prescription
  - Borel prescription
- Subleading terms
- New phenomenological results:
  - Rapidity distributions at NNLO + NNLL
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  - rapidity distributions at NNLO + NNLL
For which $\tau$ is resummation important?

$z \sim 1$: logarithmic enhancement $\rightarrow$ resummation of $\frac{\log^k(1-z)}{1-z}$

\[\sigma(\tau) = \int_{\tau}^{1} \frac{dz}{z} L \left( \frac{\tau}{z} \right) \hat{\sigma}(z), \quad \tau = \frac{Q^2}{s}, \quad z = \frac{Q^2}{\hat{s}}.\]
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$z \sim 1$ always contained in the integration region
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Standard argument*: resummation is relevant at a given $\tau$ when the region of partonic $z \sim 1$ is enhanced by PDFs.

* S.Catani, D.de Florian, M.Grazzini (hep-ph/0102227)
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$N$–space analysis
and saddle point argument

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Drell-Yan $q\bar{q}$ at NLO in $N$–space

$$\frac{\alpha_s}{\pi} 4C_F \left\{ \left[ \frac{\log(1-z)}{1-z} \right]_+ - \frac{\log \sqrt{z}}{1-z} - \frac{1+z}{2} \log \frac{1-z}{\sqrt{z}} + \left( \frac{\pi^2}{12} - 1 \right) \delta(1-z) \right\}$$

For $N \gg 2$ more than 50% of the NLO is given by the log term.
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For $N \gtrsim 2$ more than 50% of the NLO is given by the log term
The Mellin inversion integral is dominated by the values of $N$ in the proximity of the saddle point $N = N_0$:

$$\log \frac{1}{\tau} = -\frac{d}{dN} \log L(N) - \frac{d}{dN} \log \hat{\sigma}(N)$$

RHS: monotonically decreasing function, with singularity at small $N \geq 0$ saddle $N_0$ real, positive and unique.

$\tau \sim 1 \Rightarrow \log \frac{1}{\tau} \to 0 \Rightarrow N_0$ large

$\tau \ll 1 \Rightarrow \log \frac{1}{\tau}$ large $\Rightarrow N_0$ small

How small?
Saddle point argument

\[ \sigma(\tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dN \exp \left[ N \log \frac{1}{\tau} + \log \mathcal{L}(N) + \log \hat{\sigma}(N) \right] \]

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How small?
The figure shows the saddle point $N_0$ vs $\tau$ for different collider scenarios. For p-p collider (LHC), $\tau\gtrsim 0.003$, while for p-pbar collider (Tevatron), $\tau\gtrsim 0.02$. The curves are labeled as p-p and p-pbar, respectively. The plot represents the Drell-Yan production with $Q = 100$ GeV, using NNPDF 2.0 with $\alpha_s(m_Z) = 0.118$. The log contribution is dominant for $\tau\gtrsim 2$. The graph is part of Marco Bonvini's presentation on threshold resummation for Drell-Yan production: theory and phenomenology.
Saddle point $N_0$ vs $\tau$

![Graph showing $N_0$ vs $\tau$](image)

$N_0 \gtrsim 2 \implies$ the log contribution is dominant
Saddle point $N_0$ vs $\tau$

$N_0 \gtrsim 2 \Rightarrow$ the log contribution is dominant

$\tau \gtrsim \begin{cases} 
0.003 & \text{for } pp \text{ colliders (LHC)} \\
0.02 & \text{for } p\bar{p} \text{ colliders (Tevatron)}
\end{cases}$
To summarize:

resummation is relevant when log contribution is dominant (at hadron level)
log contribution is dominant (at parton level) for $N \gtrsim 2$
the Mellin inversion integral is dominated by the saddle point $N = N_0$
log contribution is dominant (at hadron level) when $N_0 \gtrsim 2$
resummation is relevant for $\tau \gtrsim \{0.003 \text{ for } pp \text{ colliders (LHC)}, 0.02 \text{ for } p\bar{p} \text{ colliders (Tevatron)} \}$

Much smaller than expected!
To summarize:

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\[ N \gtrsim 2 \]

\[ \tau \gtrsim \begin{cases} 0.003 & \text{for } pp \text{ colliders (LHC)} \\ 0.02 & \text{for } p\bar{p} \text{ colliders (Tevatron)} \end{cases} \]

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Much smaller than expected!
Resummation is performed in $N$–space \((L = 2\beta_0\alpha_s \log \frac{1}{N})\)

\[
\hat{\sigma}^{\text{res}}(N) = g_0(\alpha_s) \exp \left[ \frac{1}{\alpha_s} g_1(L) + g_2(L) + \alpha_s g_3(L) + \alpha_s^2 g_4(L) + \ldots \right]
\]

known up to $g_4$ (N$^3$LL): S.Moch, J.A.M.Vermaseren, A.Vogt (hep-ph/0506288)
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Branch cut due to the Landau singularity for \( N > N_L = \exp \frac{1}{2\beta_0\alpha_s} \)

\[\begin{array}{c}
\text{N space} \\
N_L
\end{array}\]
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Branch cut due to the Landau singularity for $N > N_L = \exp \frac{1}{2\beta_0 \alpha_s}$

The Mellin inverse does not exist
Minimal prescription

S.Catani, M.L.Mangano, P.Nason, L.Trentadue  (hep-ph/9604351)

\[
\sigma_{\text{MP}}(\tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dN \, \tau^{-N} \mathcal{L}(N) \, \hat{\sigma}_{\text{res}}(N)
\]

with \( c < N_L = \exp \frac{1}{2\beta_0 \alpha_s} \), as in the figure.
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Good properties:
- well defined for all \( \tau < 1 \)
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Good properties:

- well defined for all \( \tau < 1 \)
- exact for invertible functions
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\[ N \text{ space} \]

\[ N_L \]

\[ c \]
Minimal prescription

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But...
- a non-physical region of the parton cross-section contributes
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Good properties:
- well defined for all \( \tau < 1 \)
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But...
- a non-physical region of the parton cross-section contributes
- difficult numerical implementation
Minimal prescription: non-physical contribution

\[ \sigma_{MP}(\tau) = \int_{\tau}^{+\infty} \frac{dz}{z} \mathcal{L} \left( \frac{\tau}{z} \right) \hat{\sigma}_{MP}(z) \]

The integral extends to \(+\infty\), not to 1!
Minimal prescription: non-physical contribution

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\( \hat{\sigma}_{\text{MP}}(z > 1) \) suppressed by powers of \( \frac{\Lambda}{Q} \), but huge oscillations near \( z = 1 \)

![Graph showing the behavior of \( \hat{\sigma}_{\text{MP}}(z) \) for \( Q = 8 \text{ GeV} \)]
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The MP is more conveniently used in the $N$–space formulation.
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The MP is more conveniently used in the \( N \)-space formulation

Need for \( \mathcal{L}(N) \), for values of \( N \) where the Mellin transform of \( \mathcal{L}(x) \) does not converge
Borel prescription (1)

$$\hat{\sigma}^{\text{res}}(N)$$

Treat the divergent series $M^{-1}\hat{\sigma}^{\text{res}}(N)$ with Borel method:

$$\sum_{k=1}^{\infty} b_k \left[ \frac{1}{k!} \int_{0}^{\infty} dw e^{-w} w^k \right]$$

Borel = $\int_{0}^{\infty} dw e^{-w} \sum_{k=1}^{\infty} b_k \frac{1}{k!} w^k$

the inner sum converges

the integral diverges (the series is not Borel-summable)

proposed solution: cut-off

S.Forte, G.Ridolfi, J.Rojo, M.Ubiali (hep-ph/0601048); R.Abbate, SF, GR (hep-ph/0707.2452); MB, SF, GR (hep-ph/0807.3830); MB, SF, GR (coming soon)
Borel prescription (1)

\[ \hat{\sigma}^{\text{res}}(N) = \sum_{k=1}^{\infty} h_k(\bar{\alpha}) \bar{\alpha}^k \] \[ \log^k \frac{1}{N} , \quad \bar{\alpha} = 2\beta_0 \alpha_s \]
Borel prescription (1)

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Treat the divergent series \( M^{-1}(\hat{\sigma}^{\text{res}}(N)) \) with Borel method:

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Borel prescription (1)

\[ M^{-1}[\hat{\sigma}^{\text{res}}(N)] = \sum_{k=1}^{\infty} h_k(\hat{\alpha}) \, \bar{\alpha}^k \, M^{-1} \left[ \log^k \frac{1}{N} \right], \quad \bar{\alpha} = 2\beta_0 \alpha_s \]

Treat the divergent series \( M^{-1}(\hat{\sigma}^{\text{res}}(N)) \) with Borel method:

\[ \sum_{k=1}^{\infty} b_k \left[ \frac{1}{k!} \int_{0}^{+\infty} dw \, e^{-w} \, w^k \right] \quad \text{Borel} \quad \int_{0}^{+\infty} dw \, e^{-w} \sum_{k=1}^{\infty} \frac{b_k}{k!} \, w^k \]

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Treat the divergent series \( M^{-1}(\hat{\sigma}_{\text{res}}(N)) \) with Borel method:* 

\[ \sum_{k=1}^{\infty} b_k \left[ \frac{1}{k!} \int_0^{+\infty} dw \ e^{-w} w^k \right] \xrightarrow{\text{Borel}} \int_0^{+\infty} dw \ e^{-w} \sum_{k=1}^{\infty} \frac{b_k}{k!} w^k \]

- the inner sum converges

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Treat the divergent series \( \mathcal{M}^{-1} (\hat{\sigma}^{\text{res}}(N)) \) with Borel method:

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\sum_{k=1}^{\infty} b_k \left[ \frac{1}{k!} \int_{0}^{+\infty} dw \, e^{-w} \, w^k \right] \quad \text{Borel} = \int_{0}^{+\infty} dw \, e^{-w} \sum_{k=1}^{\infty} \frac{b_k}{k!} \, w^k
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- the inner sum converges
- the integral diverges (the series is not Borel-summable)
- proposed solution: cut-off \( C \) in the integral

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Borel prescription (2)

\[ \hat{\sigma}_{BP}(z, C) = \frac{1}{2\pi i} \oint_{C} \frac{d\xi}{\Gamma(\xi + 1)} \left[ \log^{-1} \frac{1}{z} \right] + \int_{0}^{C} \frac{dw}{\bar{\alpha}} e^{-\frac{w}{\bar{\alpha}}} \Sigma \left( \frac{w}{\xi} \right) \]

where \( \Sigma(\bar{\alpha} \log \frac{1}{N}) \equiv \hat{\sigma}_{\text{res}}(N) \)
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\hat{\sigma}_{\text{BP}}(z, C) = \frac{1}{2\pi i} \oint_{C} \frac{d\xi}{\Gamma(\xi + 1)} \left[ \log \xi^{-1} \frac{1}{z} \right] + \int_{0}^{C} \frac{dw}{\bar{\alpha}} e^{-\frac{w}{\bar{\alpha}}} \Sigma \left( \frac{w}{\xi} \right)
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Remarks

- resummed expression at parton level \( \rightarrow \) easier numerical implementation
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\[ \frac{\log \log \frac{1}{z}}{\log \frac{1}{z}} \]
Borel prescription (2)

\[ \hat{\sigma}_{BP}(z, C) = \frac{1}{2\pi i} \oint_C \frac{d\xi}{\Gamma(\xi + 1)} \left[ (1 - z)^{\xi-1} \right] + \int_0^C \frac{dw}{\bar{\alpha}} e^{-\frac{w}{\bar{\alpha}}} \Sigma \left( \frac{w}{\xi} \right) \]

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\[ \frac{\log^k \log \frac{1}{z}}{\log \frac{1}{z}} \quad \frac{\log^k (1 - z)}{1 - z} \]
Borel prescription (2)

\[ \hat{\sigma}_{BP}(z, C) = \frac{1}{2\pi i} \oint_C \frac{d\xi}{\Gamma(\xi + 1)} \frac{(1 - z)^{\xi-1}}{\sqrt{z}^\xi} \left[ \int_0^C \frac{dw}{\bar{\alpha}} e^{-\frac{w}{\bar{\alpha}}} \Sigma \left( \frac{w}{\xi} \right) \right] + \sqrt{z} \int_C d\xi \Gamma(\xi + 1) \]

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\[
\begin{align*}
\frac{\log^k \log \frac{1}{z}}{\log \frac{1}{z}} & \quad \frac{\log^k (1 - z)}{1 - z} & \quad \frac{\log^k \frac{1-z}{\sqrt{z}}}{1 - z}
\end{align*}
\]
Comparison with fixed order: Drell-Yan $q\bar{q}$ at NLO

$$\frac{\alpha_s}{\pi} 4C_F \left\{ \left[ \log \frac{1 - z}{1 - z} \right] + \frac{\log \sqrt{z}}{1 - z} - \frac{1 + z}{2} \log \frac{1 - z}{\sqrt{z}} + \left( \frac{\pi^2}{12} - 1 \right) \delta(1 - z) \right\}$$
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\[
\frac{\alpha_s}{\pi} 4C_F \left\{ \left[ \frac{\log(1-z)}{1-z} \right]_+ - \frac{\log \sqrt{z}}{1-z} - \frac{1+z}{2} \log \frac{1-z}{\sqrt{z}} + \left( \frac{\pi^2}{12} - 1 \right) \delta(1-z) \right\}
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\left[ \log \log \frac{1}{z} \right]
\left[ \log \frac{1}{z} \right]_+
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Comparison with fixed order: Drell-Yan $q \bar{q}$ at NNLO

Discrepancy due to terms like $\log k (1 - z)$ → $\log k N$ ⇒ subleading
Comparison with fixed order: Drell-Yan $q\bar{q}$ at NNLO

Discrepancy due to terms like $\log^k(1-z) \rightarrow \frac{\log^k N}{N}$ $\Rightarrow$ subleading
the BP can produce the same logs of MP

- indistinguishable at hadron level for $\tau \ll 1$ (always in phenomenological applications)
-
BP has an easier and faster numerical implementation
State of the art

- the BP can produce the same logs of MP
  - indistinguishable at hadron level for $\tau \ll 1$ (always in phenomenological applications)
  - BP has an easier and faster numerical implementation
- the BP can produce “more physical” logs
  - include some classes of subleading terms
  - better small–$z$ behaviour
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there are subleading terms which are important
\[
\log^k(1 - z) \quad \text{and similar}
\]
and which are not included in the resummed expressions
State of the art

- the BP can produce the same logs of MP
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- there are subleading terms which are important

$$\log^k(1 - z) \quad \text{and similar}$$

and which are not included in the resummed expressions
- the difference in the included subleading terms is useful to estimate the importance of these terms
\[
\frac{1}{\tau} \frac{d\sigma}{dQ^2 dY} = \int_{\sqrt{\tau} e^Y}^1 \frac{dx_1}{x_1} \int_{\sqrt{\tau} e^{-Y}}^1 \frac{dx_2}{x_2} \ f_1(x_1) f_2(x_2) \ C \left( \frac{\tau}{x_1 x_2}, Y - \frac{1}{2} \log \frac{x_1}{x_2} \right)
\]
\[
\frac{1}{\tau} \frac{d\sigma}{dQ^2 dY} = \int_{\sqrt{\tau}e^Y}^1 \frac{dx_1}{x_1} \int_{\sqrt{\tau}e^{-Y}}^1 \frac{dx_2}{x_2} f_1(x_1) f_2(x_2) C\left(\frac{\tau}{x_1 x_2}, Y - \frac{1}{2} \log \frac{x_1}{x_2}\right)
\]

Fourier transform of \( C(z, y) \) wrt \( y \)

\[
\tilde{C}(z, M) = \int_{-\infty}^{+\infty} dy \ C(z, y) \ e^{iMy}
\]
\[
\frac{1}{\tau} \frac{d\sigma}{dQ^2 dY} = \int_{\sqrt{\tau} e^y}^1 \frac{dx_1}{x_1} \int_{\sqrt{\tau} e^{-y}}^1 \frac{dx_2}{x_2} \ f_1(x_1) f_2(x_2) \ C\left(\frac{\tau}{x_1 x_2}, Y - \frac{1}{2} \log \frac{x_1}{x_2}\right)
\]

Fourier transform of \( C(z, y) \) wrt \( y \)

\[
\tilde{C}'(z, M) = \int_{\log \sqrt{z}}^{\log \sqrt{z}} dy \ C(z, y) \ e^{i M y}
\]
Impact in phenomenology: rapidity distributions (1)

\[ \frac{1}{\tau} \frac{d\sigma}{dQ^2 dY} = \int_{\sqrt{\tau}e^Y}^{1} \frac{dx_1}{x_1} \int_{\sqrt{\tau}e^{-Y}}^{1} \frac{dx_2}{x_2} f_1(x_1) f_2(x_2) C \left( \frac{\tau}{x_1 x_2}, Y - \frac{1}{2} \log \frac{x_1}{x_2} \right) \]

Fourier transform of \( C(z, y) \) wrt \( y \)

\[ \tilde{C}(z, M) = \int_{\log \sqrt{z}}^{\log \sqrt{z}} dy \ C(z, y) \left[ 1 + \mathcal{O}(y) \right] \]
\[ \frac{1}{\tau} \frac{d\sigma}{dQ^2 dY} = \int_{\sqrt{e^{Y}}}^{1} \frac{dx_1}{x_1} \int_{\sqrt{e^{-Y}}}^{1} \frac{dx_2}{x_2} f_1(x_1) f_2(x_2) C \left( \frac{\tau}{x_1 x_2}, Y - \frac{1}{2} \log \frac{x_1}{x_2} \right) \]

**Fourier transform of** \( C(z, y) \) **wrt** \( y \)

\[ \tilde{C}'(z, M) = \int_{-\log \sqrt{z}}^{\log \sqrt{z}} dy C(z, y) \left[ 1 + \mathcal{O}(y) \right] \]

**Since** \( |\log z| \simeq 1 - z \) **we have**

\[ \tilde{C}'(z, M) = C(z) \left[ 1 + \mathcal{O}(1 - z) \right] \]
\[
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Fourier transform of \( C(z, y) \) wrt \( y \)

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\tilde{C}'(z, M) = \int_{-\log \sqrt{z}}^{\log \sqrt{z}} dy \ C(z, y) \ [1 + O(y)]
\]

Since \( |\log z| \simeq 1 - z \) we have

\[
\tilde{C}'(z, M) = C(z) \ [1 + O(1 - z)]
\]

or, back to \( y \) space,

\[
C(z, y) = C(z) \delta(y) \ [1 + O(1 - z)]
\]
After changing variables we get the compact expression

\[ \frac{1}{\tau} \frac{d\sigma^{\text{res}}}{dQ^2dY} = \int_{\tau e^2|Y|}^{1} \frac{dz}{z} C^{\text{res}}(z) f_1 \left( \sqrt{\frac{\tau}{z} e^Y} \right) f_2 \left( \sqrt{\frac{\tau}{z} e^{-Y}} \right) \]

depends on \( C^{\text{res}}(z) = M - 1 \left[ \hat{\sigma}^{\text{res}}(N) \right] \), the well-known rapidity-integrated resummed coefficient has the form of a convolution product→ both Borel and minimal prescriptions are applicable!

Results at NNLO + NNLL

\( \text{C++ code:} \)

NNLO:
C.Anastasiou, L.Dixon, K.Melnikov, F.Petriello (hep-ph/0312266)

extension with NNLL resummation (Borel and minimal)

interface to LHAPDF library
After changing variables we get the compact expression

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\frac{1}{\tau} \frac{d\sigma^{\text{res}}}{dQ^2 dY} = \int_{\tau e^2 |Y|}^{1} \frac{dz}{z} C^{\text{res}}(z) f_1 \left( \sqrt{\frac{\tau}{z} e^Y} \right) f_2 \left( \sqrt{\frac{\tau}{z} e^{-Y}} \right)
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depends on \( C^{\text{res}}(z) = \mathcal{M}^{-1} [\hat{\sigma}^{\text{res}}(N)] \), the well-known rapidity-integrated resummed coefficient.
Impact in phenomenology: rapidity distributions (2)

After changing variables we get the compact expression

\[
\int \frac{d\sigma_{\text{res}}}{\tau dQ^2 dY} = \int_{\tau e^{2|Y|}}^{1} \frac{dz}{z} C_{\text{res}}(z) f_1 \left( \sqrt{\frac{\tau}{z}} e^Y \right) f_2 \left( \sqrt{\frac{\tau}{z}} e^{-Y} \right)
\]

- depends on \( C_{\text{res}}(z) = \mathcal{M}^{-1} [\hat{\sigma}_{\text{res}}(N)] \), the well-known rapidity-integrated resummed coefficient
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**Results at NNLO + NNLL**

**C++ code:**
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$W$ asymmetry at Tevatron with NNPDF2.0

$\sqrt{s} = 1.96$ TeV
$Q = \mu_R = \mu_F = M_W$
$\tau = 0.00168$

M. Bonvini, S. Forte, G. Ridolfi - preliminary
\[ \tau \approx 0.04 \]

M. Bonvini, S. Forte, G. Ridolfi
preliminary

NPDF2.0

T. Becher, M. Neubert, G. Xu
(hep-ph/0710.0680)

MRST04NNLO
Rapidity distribution: DY (8 GeV) at NuSea

\[ \tau \simeq 0.04 \]

M. Bonvini, S. Forte, G. Ridolfi
preliminary

\[ \frac{d^2 \sigma}{dY} \text{ [pb/GeV]} \]

\[ M = 8 \text{ GeV} \]

\[ \frac{d^2 \sigma}{dY} \text{ [pb/GeV]} \]

\[ \frac{d^2 \sigma}{dY} \text{ [pb/GeV]} \]

T. Becher, M. Neubert, G. Xu
(hep-ph/0710.0680)

NNPDF2.0

MRST04NNLO
DY rapidity distribution. Collider: pp Subprocess: Z+gamma

\[ \sqrt{s} = 7.00 \text{ TeV} \]
\[ Q = 1000 \text{ GeV} \]
\[ \tau = 0.02041 \]
\[ 0.5 < \mu_R/Q < 2 \]
\[ 0.5 < \mu_F/Q < 2 \]

M. Bonvini, S. Forte, G. Ridolfi - preliminary
Rapidity distribution: DY (1 TeV) at LHC with NNPDF2.0

DY rapidity distribution. Collider: pp Subprocess: Z+gamma

$\sqrt{s} = 7.00$ TeV
$Q = 1000$ GeV
$\tau = 0.02041$
$0.5 < \mu_R/Q < 2$
$0.5 < \mu_F/Q < 2$

M. Bonvini, S. Forte, G. Ridolfi - preliminary
Rapidity distribution: $Z$ at LHC with NNPDF2.0

DY rapidity distribution. Collider: pp Subprocess: $Z+\gamma$

- $\sqrt{s} = 7.00$ TeV
- $Q = M_Z$
- $\tau = 0.00017$
- $0.5 < \mu_R/Q < 2$
- $0.5 < \mu_F/Q < 2$

M. Bonvini, S. Forte, G. Ridolfi - preliminary
DY rapidity distribution. Collider: pp Subprocess: Z+gamma

\[ \sqrt{s} = 7.00 \text{ TeV} \]
\[ Q = M_Z \]
\[ \tau = 0.00017 \]
\[ 0.5 < \mu_R/Q < 2 \]
\[ 0.5 < \mu_F/Q < 2 \]

M.Bonvini, S.Forte, G.Ridolfi - preliminary
DY rapidity distribution. Collider: pp Subprocess: W⁺

\( \sqrt{s} = 7.00 \text{ TeV} \)
\( Q = M_W \)
\( \tau = 0.00013 \)

\( 0.5 < \mu_R/Q < 2 \)
\( 0.5 < \mu_F/Q < 2 \)

M.Bonvini, S.Forte, G.Ridolfi - preliminary

Threshold resummation for Drell-Yan production: theory and phenomenology
DY rapidity distribution. Collider: pp Subprocess: $W^+$

- $\sqrt{s} = 7.00$ TeV
- $Q = M_W$
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M. Bonvini, S. Forte, G. Ridolfi - preliminary
Rapidity distribution: $W^-$ at LHC with NNPDF2.0

DY rapidity distribution. Collider: pp Subprocess: W-$\sqrt{s} = 7.00\text{ TeV}$

$Q = M_W$

$\tau = 0.00013$

$0.5 < \mu_R/Q < 2$

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M. Bonvini, S. Forte, G. Ridolfi - preliminary
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Conclusions

New results

Quantitative evaluation of $\tau$ for which resummation is important: much smaller than expected

Improved Borel prescription

New phenomenological results: rapidity distributions

Outlook

Include subdominant $1/N$ contributions (S.Moch, A.Vogt: hep-ph/0909.2124 and today talk)

Apply to other processes such as Higgs production
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Backup slides
Expand the function

\[ \frac{z^\alpha}{(1-z)\beta} \mathcal{L}(z) \]

on a polynomial basis (with suitable \( \alpha, \beta > 0 \))

- Compute the Mellin transform of \( \mathcal{L}(z) \) analytically
- Compute the complex Mellin inversion integral numerically
Borel prescription: practical implementation

- Compute the convolution integral

\[ \int_{\tau}^{1} \frac{dz}{z} \mathcal{L} \left( \frac{\tau}{z} \right) \left[ (1 - z)^{\xi - 1} \right]_+ \]

It is convenient to expand on a polynomial basis the function

\[ \frac{1}{1 - z} \left[ \frac{1}{z} \mathcal{L} \left( \frac{\tau}{z} \right) - \mathcal{L}(\tau) \right] \]

and compute the integral analytically

- Compute the complex \( \xi \) integral numerically
How BP works

Apply the BP to a power of $\log \frac{1}{N}$

$$M^{-1} \left( \log^k \frac{1}{N} \right) \bigg|_{BP} = \frac{\gamma(k + 1, C/\bar{\alpha})}{\Gamma(k + 1)} M^{-1} \left( \log^k \frac{1}{N} \right)$$

The BP essentially truncates the divergent sum