On the size of the maximum of incomplete Kloosterman sums

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Abstract
Let \( t : \mathbb{F}_p \to \mathbb{C} \) be a complex valued function on \( \mathbb{F}_p \). A classical problem in analytic number theory is to bound the maximum of the absolute value of the incomplete sum

\[
M(t) := \max_{0 \leq H < p} \left| \frac{1}{\sqrt{p}} \sum_{0 \leq n < H} t(n) \right|.
\]

In this very general context one of the most important results is the Pólya-Vinogradov bound

\[
M(t) \leq \|K\|_\infty \log 3p.
\]

where \( K : \mathbb{F}_p \to \mathbb{C} \) is the normalized Fourier transform of \( t \). In this paper we provide a lower bound for incomplete Kloosterman sum, namely we prove that for any \( \varepsilon > 0 \) there exists some \( a \in \mathbb{F}_p^\times \) such that

\[
M(e(\frac{ax}{p} + x)) \geq \left( \frac{1 - \varepsilon}{\sqrt{2\pi}} + o(1) \right) \log \log p.
\]

Moreover we also provide some result on the growth of the moments of \( \{M(e(\frac{ax}{p} + x))\}_{a \in \mathbb{F}_p^\times} \).

1 Introduction
Let \( t : \mathbb{F}_p \to \mathbb{C} \) be a complex valued function on \( \mathbb{F}_p \). A classical problem in analytic number theory is to bound the incomplete sums

\[
S(t, H) := \frac{1}{\sqrt{p}} \sum_{0 \leq n < H} t(n),
\]

for any \( 0 \leq H < p \). In this very general context one of the most important results is the following:

**Theorem 1.1** (Pólya-Vinogradov bound, [Pol18], [Vin18]). For any \( 1 \leq H < p \) one has

\[
|S(t, H)| \leq \|K\|_\infty \log 3p,
\]

where \( K : \mathbb{F}_p \to \mathbb{C} \) is the normalized Fourier transform of \( t \)

\[
K(y) := -\frac{1}{\sqrt{p}} \sum_{0 \leq x < p} t(x)e\left(\frac{yx}{p}\right).
\]

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Notice that if \( \|K\|_\infty \) is bounded, then this bound is non-trivial as soon as \( H \gg \sqrt{p} \log p \).

If one defines
\[
M(t) := \max_{0 \leq H < p} |S(t, H)|,
\]
the Pólya-Vinogradov bound is equivalent to
\[
M(t) \leq \|K\|_\infty \log 3p.
\]

The first question which arises in this setting is the following: given a function \( t : \mathbb{F}_p \to \mathbb{C} \), is the Pólya-Vinogradov bound sharp for \( t \)? And if it is not, what is the best possible bound?

**Kloosterman sums, Birch Sums and main results**

The aim of this paper is to study the case of the Kloosterman sums and Birch sums. We recall here the definition of these two objects:

i) **Kloosterman sums.** For any \( a, b \in \mathbb{F}_p^\times \) one considers
\[
t : x \mapsto e\left(\frac{ax + b\overline{x}}{p}\right)
\]
where \( \overline{x} \) denotes the inverse of \( x \) modulo \( p \). The complete sum over \( \mathbb{F}_p^\times \) of the function above
\[
\text{Kl}(a, b; p) := \frac{1}{\sqrt{p}} \sum_{1 \leq x < p} e\left(\frac{ax + b\overline{x}}{p}\right)
\]
is called **Kloosterman sum associated to** \( a, b \). The Riemann hypothesis over curves for finite fields implies \( |\text{Kl}(a, b; p)| \leq 2 \) (Weil bound).

ii) **Birch sums.** For any \( a, b \in \mathbb{F}_p^\times \) one considers
\[
t : x \mapsto e\left(\frac{ax + bx^3}{p}\right).
\]
One defines the **Birch sum associated to** \( a, b \)
\[
\text{Bi}(a, b; p) := \frac{1}{\sqrt{p}} \sum_{1 \leq x < p} e\left(\frac{ax + bx^3}{p}\right).
\]

Also in this case an application of the Riemann hypothesis over curves for finite field leads to the bound \( |\text{Bi}(a, b; p)| \leq 2 \) (Weil bound).

It is known that \( M(e(\frac{ax + \overline{x}}{p})) \) and \( M(e(\frac{ax + x^3}{p})) \) can be arbitrarily large when \( a \) varies over \( \mathbb{F}_p^\times \) and \( p \) goes to infinity: as a consequence of [KS16, Proposition 4.1], one has that
\[
\lim_{p \to \infty} \max_a M(e(\frac{ax + \overline{x}}{p})) = \lim_{p \to \infty} \max_a M(e(\frac{ax + x^3}{p})) = \infty.
\]

We will prove the following lower bounds:

**Theorem 1.2.** Let \( 0 < \varepsilon < 1 \). For all \( p \), there exists \( S_p \subset \mathbb{F}_p^\times \) such that
Theorem 1.5. There exist two absolute positive constants $K_l$ and $K_o$ such that for any $a, b \in \mathbb{F}_p^\times$, one has
\[
M(e(ax + bx) / p) \geq \left( \frac{1 - \varepsilon}{\sqrt{2\pi}} + o(1) \right) \log \log p,
\]
for any $1 \leq n \leq (\log p)^{1-\varepsilon}$ odd, and
\[
K_l(a, 1; p) \geq \sqrt{n},
\]
for any $-(\log p)^{1-\varepsilon} \leq n \leq -1$ odd. Moreover $|S_p| \gtrsim p^{-\frac{\log p}{\log \log p}}$. The same is true if we replace $K_l$ by $K_o$.\n
Theorem 1.3. Let $0 < \varepsilon < 1$ and fix $m \geq 1$. For all $p$ such that $p \nmid m$, there exists $S_p \subset H_m, p$ such that
\[
\text{i) for any } (a, b) \in S_p \text{ one has } \quad M(e(\frac{ax + bx}{p})) \geq \left( \frac{1 - \varepsilon}{\sqrt{2\pi}} + o(1) \right) \log \log p, \quad \text{ii) } |S_p| \gtrsim p^{\frac{\log(4)}{\log \log p}}.
\]
The same is true if one replaces $M(e(\frac{ax + bx}{p}))$ by $M(e(\frac{ax + bx^2}{p}))$.\n
For all $p$ there exists $S_p \subset \mathbb{F}_p^\times$ such that for any $a \in S_p$
\[
K_l(a, 1; p) \geq \sqrt{n},
\]
for any $1 \leq n \leq (\log p)^{1-\varepsilon}$ odd, and
\[
K_l(a, 1; p) \leq -\sqrt{n},
\]
for any $-(\log p)^{1-\varepsilon} \leq n \leq -1$ odd. Moreover $|S_p| \gtrsim p^{\frac{\log(4)}{\log \log p}}$. The same is true if we replace $K_l$ by $K_o$.\n
In the second part of the paper, we focus our attention on the growth of the $2k$-th moments of $\{M(e(\frac{ax + bx}{p}))\}_{a \in \mathbb{F}_p}$ and $\{M(e(\frac{ax + bx^2}{p}))\}_{a \in \mathbb{F}_p}$ when $p \to \infty$, getting
\[
(c^{2k} + o(1))(\log k)^{2k} \leq \frac{1}{p-1} \sum_{a \in \mathbb{F}_p} M(e(\frac{ax + bx}{p}))^{2k} \leq ((Ck)^{2k} + o(1))(\log \log p)^{2k},
\]
and for any fixed $m \in \mathbb{Z} \setminus \{0\}$ and $p \to \infty$
\[
(c^{2k} + o(1))(\log k)^{2k} \leq \frac{1}{p-1} \sum_{(a, b) \in H_m, p} M(e(\frac{ax + bx}{p}))^{2k} \leq ((Ck)^{2k} + o(1))(\log \log p)^{2k}.
\]
Theorem 1.6. There exist two absolute constants $C > 1$ and $c < 1$ such that for any fixed $k \geq 1$ and $p \to \infty$ one has

$$(c^{2k} + o(1))(\log k)^{2k} \leq \frac{1}{p-1} \sum_{a \in \mathbb{F}_p^\times} M(e(\frac{ax+x^3}{p}))^{2k} \leq (C^{2k} + o(1))P(k),$$

and for any fixed $m \in \mathbb{Z} \setminus \{0\}$ and $p \to \infty$

$$(c^{2k} + o(1))(\log k)^{2k} \leq \frac{1}{p-1} \sum_{(a,b) \in \mathcal{C}_{m,p}} M(e(\frac{ax+bx^3}{p}))^{2k} \leq (C^{2k} + o(1))P(k),$$

where $P(k) := \exp(4k \log \log k + k \log \log \log k + o(k))$.

From this we get the following

Corollary 1.7. There exist two absolute constants $B, b > 0$ such that for $A \to \infty$ one has

$$\exp(-\exp(bA)) \leq \lim \inf_{p \to \infty} \frac{1}{p-1} |\{a \in \mathbb{F}_p^\times : M((e(\frac{ax+x^3}{p})) > A)\} | \leq \exp \left( -\exp \left( BA^{1/2-o(1)} \right) \right).$$

Remarks and related works

i) The upper bound in Theorem 1.6 can be improved conditionally on

Conjecture 1.8 (Short sums conjecture for Kloosterman sums). There exists an $\varepsilon > 0$ such that

$$\left| \sum_{N \leq x \leq N+H} e\left(\frac{ax+x^3}{p}\right) \right| \ll H^{1-\varepsilon},$$

uniformly for any $1 < N < p$, $p^{1/2-\varepsilon/2} < H < p^{1/2+\varepsilon/2}$ and $a \in \mathbb{F}_p^\times$.

Indeed, assuming this conjecture we will prove that

$$\frac{1}{p-1} \sum_{a \in \mathbb{F}_p^\times} M(e(\frac{ax+x^3}{p}))^{2k} \leq (C^{2k} + o(1))P(k).$$

Notice that Conjecture 1.8 is a (much) weaker form of Hooley’s $R^*$-assumption ([Hoo78, page 44]). In the case of the moments of maximum of incomplete Birch sums we get a better upper bound since the analogue of the (1) is known to be true for the function

$x \mapsto e\left(\frac{ax+bx^3}{p}\right)$ (Weyl’s inequality).

ii) Notice that for any $(a, b) \in \mathcal{C}_{m,p}$

$$\text{Bi}(a, b; p) = \text{Bi}(m, 1; p).$$

Combining Theorem 1.3 and Corollary 1.7 one proves that there exist $(a, b), (a', b') \in \mathcal{C}_{m,p}$ such that

$$M(e(\frac{ax+bx^3}{p})) \gg \log \log p, \quad M(e(\frac{a'x+b'x^3}{p})) \ll 1.$$
and

\[ \text{Bi}(a, b; p) = \text{Bi}(a', b'; p) = \text{Bi}(m, 1; p). \]

Assuming Conjecture 1.8, we can prove the same in the case of Kloosterman sums thanks to the fact that for any \((a, b) \in H_{m,p}\)

\[ \text{Kl}(a, b; p) = \text{Kl}(m, 1; p). \]

\(iii\) Lamzouri in [Lam18] has proved that there exist some (computable) constants \(C_0, C_1\) and \(\delta\) such that for any \(1 \ll A \leq \frac{2}{\pi} \log \log p - 2 \log \log \log p\) one has

\[
\frac{1}{p-1} | \{ a \in \mathbb{F}_p^\times : M(\{ e(\frac{ax^3}{p}) > A \}) \geq \exp \left( - C_0 \left( \frac{\pi}{2} A \right)^{1 + O\left( \frac{\sqrt{\pi e A}}{4} \right)} \right) \} | \geq \exp \left( - C_1 \exp \left( \left( \frac{\pi}{2} - \delta \right) A \right) \right).
\]  

(2)

He obtains the same result also for incomplete Kloosterman sums. For the family \(\{ M(e(\frac{ax^3}{p})) \}_{a \in \mathbb{F}_p^\times} \), he also proved that

\[
\frac{1}{p-1} | \{ a \in \mathbb{F}_p^\times : M(\{ e(\frac{ax^3}{p}) > A \}) \leq \exp \left( - C_1 \exp \left( \left( \frac{\pi}{2} - \delta \right) A \right) \right) \} | \leq \exp \left( - C_1 \exp \left( \left( \frac{\pi}{2} - \delta \right) A \right) \right).
\]  

(3)

Also in this case the difference between the incomplete Kloosterman sums and incomplete Birch sums depends on the cancellation of the short sums of Kloosterman sums (Conjecture 1.8). The proof of the lower bound in (2) implies that for at least \(p^{1 - \frac{1}{16\pi^2}}\) elements of \(\mathbb{F}_p^\times\) one has

\[
M(\text{Im}(t_a)) \geq \left( \frac{2}{\pi} + o(1) \right) \log \log p,
\]  

(4)

where \(t_a = e(\frac{ax^3}{p})\) or \(t_a = e(\frac{ax}{p})\).

\(iv\) One should compare our result with the case of incomplete character sums. Paley proved that the Pólya-Vinogradov bound is close to be sharp in this case: indeed in [Pal32] is shown that there exist infinitely many primes \(p\) such that

\[
M\left( \left( \frac{-}{p} \right) \right) \gg \log \log p,
\]

where \(\left( \frac{\cdot}{p} \right)\) is the Legendre symbol modulo \(p\). Similar results were achieved for nontrivial characters of any order by Granville and Soundararajan in [GS07], and by Goldmakher and Lamzouri in [GL12] and [GL14]. On the other hand Montgomery and Vaughan have shown under G.R.H. that

\[
M(\chi) \ll \log \log p,
\]  

(5)

for any \(\chi\) ([MV77]), which is the best possible bound up to evaluation of the constant.

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1.1 Notation and statement of the main results

In this section we recall some notion of the formalism of trace functions and state the general version of our main results. For a general introduction on this subject we refer to [FKM14]. Basic statements and references can also be founded in [FKM15a]. The main examples of trace functions we should have in mind are

i) For any \( f \in \mathbb{F}_p[T] \), the function \( x \mapsto e(f(x)/p) \): this is the trace function attached to the Artin-Schreier sheaf \( \mathcal{L}_{e(f/p)} \).

ii) The Birch sums: \( b \mapsto \text{Bi}(a, b; p) \) it can be seen as the trace function attached to the sheaf \( \mathcal{F}_T(\mathcal{L}_{e((aT^3)/p)}) \).

iii) The \( n \)-th Hyper-Kloosterman sums: the map \( x \mapsto \text{Kl}_n(x; q) := (-1)^{n-1} \sum_{y_1, \ldots, y_n \in \mathbb{F}_p^X} \psi(y_1 + \cdots + y_n). \)

Definition 1.1. Let \( \mathcal{F} \) be a middle-extension \( \ell \)-adic sheaf on \( \mathbb{A}^1_{\mathbb{F}_q} \). The conductor of \( \mathcal{F} \) is defined as
\[
c(\mathcal{F}) := \text{Rank}(\mathcal{F}) + |\text{Sing}(\mathcal{F})| + \sum_x \text{Swan}_x(\mathcal{F}).
\]

Definition 1.2. Let \( p, \ell > 2 \) be a prime numbers with \( p \neq \ell \) and let \( r \geq 1 \) be an integer. A middle-extension \( \ell \)-adic sheaf, \( \mathcal{F} \), is \( r \)-bountiful if

i) \( \mathcal{F} \) is pure of weight 0 and \( \text{Rank}(\mathcal{F}) \geq 2 \),

ii) the geometric and arithmetic monodromy groups of \( \mathcal{F} \) satisfy \( G_{\text{arith}}^\mathcal{F} = G_{\text{geom}}^\mathcal{F} \) and 

\( G_{\text{geom}}^\mathcal{F} \) is either \( \text{Sp}_r \) or \( \text{SL}_r \),

iii) the projective automorphism group
\[
\text{Aut}_0(\mathcal{F}) := \{ \gamma \in \text{PGL}_2(\mathbb{F}_p) : \gamma^* \mathcal{F} \cong \mathcal{F} \otimes \mathcal{L} \text{ for some rank 1 sheaf} \}
\]
of \( \mathcal{F} \) is trivial.

Definition 1.3. Let \( p, \ell > 2 \) be a prime numbers and let \( r \geq 1 \) be an integer. A \( r \)-family \( (\mathcal{F}_a)_{a \in \mathbb{F}_p^X} \) is \( r \)-acceptable if the following conditions are satisfied:

i) for any \( a \in \mathbb{F}_p^X \), \( \mathcal{F}_a \) is an irreducible middle-extension \( \ell \)-adic Fourier sheaf on \( \mathbb{A}^1_{\mathbb{F}_p} \) pointwise pure of weight 0. We denote by \( t_a \) the trace function attached to \( \mathcal{F}_a \).

ii) The \( \ell \)-adic Fourier transform \( \text{FT}(\mathcal{F}_1) \) is an \( r \)-bountiful sheaf,

iii) for all \( y \in \mathbb{F}_p \), there exists \( \tau_y \in \text{PGL}_2(\mathbb{F}_p) \), such that \( \tau_i \neq \tau_j \) if \( i \neq j \) and
\[
K_a(y) = K_1(\tau_y, a),
\]
for any \( a \in \mathbb{F}_p^X \), where \( K_a(\cdot) \) denote the trace functions attached to \( \text{FT}(\mathcal{F}_a) \).
Definition 1.4. A family of $r$-acceptable families $\mathcal{F} := ((\mathcal{F}(a,p))_{a \in \mathbb{F}_p^\times})_p$ is $r$-coherent if there exists $C \geq 1$ such that

$$c(\mathcal{F}(a,p)) \leq C,$$

for any $p$ prime and $a \in \mathbb{F}_p^\times$. We call the smallest $C$ with this property the conductor of the family and we denote it by $C_{\mathcal{F}}$.

Definition 1.5. Let $\mathcal{F}$ be a $r$-coherent family and for any $A > 0$ we define

$$D_{\mathcal{F}}(A) := \liminf_{p \to \infty} \frac{1}{p-1}|\{a \in \mathbb{F}_p^\times : M(t,a,p) > A\}|.$$

Example 1.1. The following families are $2$-coherent:

i) The family of Artin-Schreier sheaves $\left(\left(\mathcal{L}(\frac{ax + x}{p})\right)_{a \in \mathbb{F}_p^\times}\right)_p$. Indeed for any $a \in \mathbb{F}_p^\times$, $\mathcal{L}(\frac{ax + x}{p})$ is a middle-extension $\ell$-adic Fourier sheaf pointwise pure of weight $0$ with $\text{cond}(\mathcal{L}(\frac{ax + x}{p})) = 2$. Moreover $\text{FT}(\mathcal{L}(\frac{ax + x}{p})) = K\ell_2$, the Kloosterman sheaf of rank $2$ which is $2$-bountiful ([FKM15b, Paragraph 3.2]) and

$$\text{FT}\left(\frac{ax + x}{p}\right)(y) = Kl(a + y, 1; p),$$

so we can take $\tau_y := \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$.

ii) The family of Artin-Schreier sheaves $\left(\left(\mathcal{L}(\frac{x + bx^3}{p})\right)_{b \in \mathbb{F}_p^\times}\right)_p$. It is enough to argue as above and to observe that

$$\text{FT}\left(\frac{x + bx^3}{p}\right)(y) = Kl(by, 1; p),$$

so we can take $\tau_y := \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$.

iii) Fix $m \in \mathbb{Z}$. The family of Artin-Schreier sheaves $\left(\left(\mathcal{L}(\frac{ax + m}{p})\right)_{a \in \mathbb{F}_p^\times}\right)_p$ is $2$-coherent. Also in this case one argues as above and observes that

$$\text{FT}\left(\frac{ax + m}{p}\right)(y) = Kl(my + m, 1; p),$$

so we can take $\tau_y := \begin{pmatrix} my & m \\ 1 & 0 \end{pmatrix}$.

iv) With similar arguments one shows that the families $\left(\left(\mathcal{L}(\frac{ax + x}{p})\right)_{a \in \mathbb{F}_p^\times}\right)_p$, $\left(\left(\mathcal{L}(\frac{ax + x^3}{p})\right)_{a \in \mathbb{F}_p^\times}\right)_p$ and $\left(\left(\mathcal{L}(\frac{ax + m(x^3)}{p})\right)_{a \in \mathbb{F}_p^\times}\right)_p$ are $2$-coherent families.

Then Theorem 1.2 and 1.3 are consequences of the following

Theorem 1.9. Let $0 < \varepsilon < 1$. Let $\mathcal{F} := ((\mathcal{F}(a,p))_{a \in \mathbb{F}_p^\times})_p$ be a $2$-coherent. For all $p$ there exists $S_p \subset \mathbb{F}_p^\times$ such that
i) for any \( a \in S_p \) one has 
\[
M(t_{a,p}) \geq \left( \frac{1-\varepsilon}{\sqrt{2\pi}} + o(1) \right) \log \log p.
\]

ii) \(|S_p| \gg_{\varepsilon,c} p^{1-\frac{\log(4)}{\log p}}\).

Similarly, Theorem 1.5 and 1.6 are consequence of:

**Theorem 1.10.** Let \( \mathcal{F} = (\mathcal{F}_{a,p})_{a \in \mathbb{F}_p^*} \) be a \( r \)-coherent family. There exist two positive constant \( C > 1 \) and \( c<1 \) depending only on \( c_F \) such that for any fixed \( k \geq 1 \)
\[
(c^{2k} + o(1))(\log k)^{2k} \leq \frac{1}{p-1} \sum_{a \in \mathbb{F}_p^*} M(t_{a,p})^{2k} \leq ((Ck)^{2k} + o(1))(\log \log p)^{2k}.
\]

If moreover one has that there exists an \( \varepsilon > 0 \) such that
\[\sum_{N \leq x \leq N+H} t_{a,p}(x) \ll_{c,F} H^{1-\varepsilon}\] uniformly for any \( 1 < N < p, p^{1/2-\varepsilon/2} < H < p^{1/2+\varepsilon/2} \) and \( a \in \mathbb{F}_p^* \), then for any fixed \( k \geq 1 \) one has
\[
(c^{2k} + o(1))(\log k)^{2k} \leq \frac{1}{p-1} \sum_{a \in \mathbb{F}_p^*} M(t_{a,p})^{2k} \leq (C^{2k} + o(1))P(k).
\]

We then get the following

**Corollary 1.11.** Same notation as in Theorem 1.10. Then:

i) for any \( A > 0 \) one has
\[
D_{\mathcal{F}}(A) \geq \exp(-\exp(bA)),
\]
where \( b > 0 \) depends only on \( c_{\mathcal{F}} \),

ii) if the condition (6) holds, there exists a \( B > 0 \) depending only on \( c_{\mathcal{F}} \) such that for \( A \to \infty \) one has
\[
D_{\mathcal{F}}(A) \leq \exp\left(-\exp\left(BA^{1/2-o(1)}\right)\right).
\]

**2 Proof of Theorem 1.9**

**First step: Fourier expansion and Féjer Kernel**

The first step for both Theorems is to get a quantitative version of the Fourier expansion for \( \frac{1}{\sqrt{p}} \sum_{x \leq N} t(x) \):

**Lemma 2.1.** Let \( t : \mathbb{F}_p \to \mathbb{C} \) be a complex valued function on \( \mathbb{F}_p \), then for any for any \( 0 < a < 1 \) we have
\[
\frac{1}{\sqrt{p}} \sum_{x \leq \alpha p} t(x) = -\frac{1}{2\pi i} \sum_{1 \leq |n| \leq N} K(n) \frac{1}{n} (1 - e(-an)) + oK(0) + O\left( \frac{\|t\|_{\infty} \sqrt{p \log p}}{N} \right),
\]
for any \( 1 \leq N \leq p \), where the implied constant is absolute.
Proof. We use the same strategy used in [Pol18]. Let us introduce the function

$$\Phi(s) = \begin{cases} 
1 & \text{if } 0 < s < 2\pi\alpha, \\
\frac{1}{2} & \text{if } s = 0 \text{ or } s = 2\pi\alpha, \\
0 & \text{if } 2\pi\alpha < s < 2\pi.
\end{cases}$$

Then the Fourier series of $\Phi$ is

$$\Phi(s) = \alpha + \sum_{n>0} \frac{\sin 2\pi\alpha n}{\pi n} \cos(ns) - \frac{\cos 2\pi\alpha n - 1}{n\pi} \sin(ns)$$

$$= \alpha + \frac{1}{\pi} T(s) - \frac{1}{\pi} T(s - 2\pi\alpha),$$

where

$$T(x) := \sum_{n>0} \frac{\sin nx}{n}.$$ 

Observe that for any $N > 1$ one has

$$T(x) = \sum_{0<n\leq N} \frac{\sin nx}{n} + R_N(x),$$

with $R_N(0) = R_N(\pi)$, $R_N(2\pi - x) = -R_N(x)$ and $|R_N(x)| = O(1/Nx)$ for any $x \in (0, \pi]$ [Pol18 eq. 10]. Then we have

$$\frac{1}{\sqrt{p}} \sum_{x \leq \alpha p} t(x) = \frac{1}{\sqrt{p}} \sum_{x < p} t(x) \Phi\left(\frac{2\pi x}{p}\right) + O\left(\|t\|_\infty / \sqrt{p}\right)$$

$$= \frac{1}{\sqrt{p}} \sum_{x < p} t(x) \left(\alpha + \frac{1}{\pi} T\left(\frac{2\pi x}{p}\right) - \frac{1}{\pi} T\left(\frac{2\pi x}{p} - 2\pi\alpha\right)\right) + O\left(\|t\|_\infty \sqrt{p}\right)$$

$$= \frac{1}{\sqrt{p}} \sum_{x < p} t(x) \left(\alpha + \frac{1}{\pi} \sum_{0<n\leq N} \frac{\sin \left(\frac{2\pi n x}{p}\right)}{n} + \frac{1}{\pi} R_N\left(\frac{2\pi x}{p}\right)\right)$$

$$= \frac{1}{\sqrt{p}} \sum_{0<n\leq N} \sin\left(\frac{2\pi n x}{p} - 2\pi\alpha n\right) + \frac{1}{\pi} R_N\left(\frac{2\pi x}{p} - 2\pi\alpha\right) + O\left(\|t\|_\infty \sqrt{p}\right)$$

$$= \frac{1}{\sqrt{p}} \sum_{0<n\leq N} \sin\left(\frac{2\pi n x}{p}\right) - \frac{1}{\pi} \sum_{0<n\leq N} \sin\left(\frac{2\pi n x}{p} - 2\pi\alpha n\right) + \alpha K(0) + O\left(\|t\|_\infty \sqrt{p} \log p N\right).$$

On the other hand we have

$$\sin\left(\frac{2\pi n x}{p}\right) = \frac{e\left(\frac{n x}{p}\right) - e\left(-\frac{n x}{p}\right)}{2i}$$

and

$$\sin\left(\frac{2\pi n x}{p} - 2\pi\alpha n\right) = \frac{e\left(\frac{n x}{p} - \alpha n\right) - e\left(-\frac{n x}{p} - \alpha n\right)}{2i}. $$
Then one has
\[ \frac{1}{\sqrt{p}} \sum_{x \in \mathbb{F}_p} t(x) \left( e\left(\frac{nx}{p}\right) - e\left(\frac{-nx}{p}\right) \right) = -\frac{1}{2i} (K(n) - K(-n)) \]
and similarly
\[ \frac{1}{\sqrt{p}} \sum_{x \in \mathbb{F}_p} t(x) \left( e\left(\frac{nx}{p} - \alpha n\right) - e\left(\frac{-nx}{p} - \alpha n\right) \right) = -\frac{1}{2i} (e(-\alpha n)K(n) - e(\alpha n)K(-n)). \]

Now we use the same strategy of \[\text{Pal32}\] introducing the Féjér’s kernel:

**Lemma 2.2.** For any \( t : \mathbb{F}_p \to \mathbb{C} \) one has
\[ M(t) \geq \max_{\alpha \in [0,1]} \left| \frac{1}{4\pi} \sum_{1 \leq |n| < N} K(n) \left(1 - e(-\alpha n)\right) \right| + O(\|K\|_\infty) \]

**Proof.** The quantitative version of the Fourier transform leads to
\[ \frac{1}{\sqrt{p}} \sum_{x \leq \alpha p} t(x) = -\frac{1}{2\pi i} \sum_{1 \leq |n| \leq p} K(n) \left(1 - e(-\alpha n)\right) + O(1) \]
\[ = -\frac{1}{2\pi} \sum_{1 \leq |n| < p} K(n) \left(1 - e(-\alpha n)\right) + O(\|K\|_\infty), \]
at this point we extend the outer sum to all values modulo \( p \) using the Féjér’s kernel: for any \( 1 < N < p \) we have
\[ \frac{1}{2\pi i} \sum_{1 \leq |n| \leq N} K(n) \left(1 - e(-\alpha n)\right) = \frac{1}{2\pi i} \sum_{1 \leq |n| \leq p} K(n) \left(1 - e(-\alpha n)\right) \times \]
\[ \times \sum_{1 \leq |a| \leq N} \phi(a) \int_0^1 e((a - n)d) d\theta + O(\|K\|_\infty) \]
\[ = \int_0^1 A_\theta \Phi_N(\theta) d\theta + O(\|K\|_\infty), \quad (7) \]

where \( \phi(a) := 1 - \frac{|a|}{N}, \quad \Phi_N(\theta) := \sum_{|a| \leq N} \phi(a)e(a\theta) = \frac{1}{N} \left( \frac{\sin \frac{N\theta}{2}}{\sin \frac{\theta}{2}} \right)^2, \quad (8) \)
is the Féjér Kernel, and
\[ A_\theta := \frac{1}{2\pi i} \sum_{1 \leq |n| \leq p} K(n) \left(1 - e(-\alpha n)\right)e(-\vartheta n). \]
On the other hand the triangular inequality leads to
\[
 \max_{\vartheta \in [0,1]} |A_{\vartheta}| \leq 2 \max_{\alpha \in (0,1)} \left| \frac{1}{2 \pi} \sum_{1 \leq |n| \leq p} \frac{K(n)}{n} (1-e(-\alpha n)) \right|
\]
\[
\leq 2 \max_{\alpha \in (0,1)} \left| \frac{1}{\sqrt{p}} \sum_{x \leq \alpha p} t(x) \right| + O(\|K\|_{\infty})
\]
\[
\leq 2M(t) + O(\|K\|_{\infty})
\]
So we obtain the bound
\[
\left| \frac{1}{4 \pi i} \sum_{1 \leq |n| \leq N} \frac{K(n)}{n} (1-e(-\alpha n)) \right| \leq \left( M(t) + O(\|K\|_{\infty}) \right) \cdot \int_{\vartheta}^{1} \Phi_N(\vartheta) d\vartheta.
\] (9)

On the other hand using the fact
\[
\int_{0}^{1} \Phi_N(\vartheta) d\vartheta = 1
\]
we conclude the proof.

To conclude, it is enough to prove the following

**Proposition 2.3.** Same assumption as in Theorem 1.9. Let 0 < \(\varepsilon\) < 1. Then for all \(p\) there exists \(S_p \subset \mathbb{F}_p^\times\) such that for any \(a \in S_p\)
\[
K_{1,p}(\tau_n \cdot a) \geq \sqrt{2},
\]
for any \(1 \leq n \leq (\log p)^{1-\varepsilon}\) odd, and:
\[
K_{1,p}(\tau_n \cdot a) \leq -\sqrt{2},
\]
for any \(- (\log p)^{1-\varepsilon} \leq n \leq -1\) odd. Moreover \(|S_p| \gg_{\varepsilon,c} p^{1 - \log(4) - \log p}\).

Assuming this Proposition, which we prove in the next section, let us prove Theorem 1.9. We have that
\[
M(t_{a,p}) = \max_{\alpha \in [0,1]} \left| \frac{1}{\sqrt{p}} \sum_{x \leq \alpha p} t_{a,p}(x) \right|
\]
\[
\geq \frac{1}{4 \pi} \max_{\alpha \in [0,1], 1 \leq N < p} \left| \sum_{1 \leq |n| \leq N} \frac{K_{a,p}(n)}{n} (1-e(-\alpha n)) \right| + O(1)
\]
\[
= \frac{1}{4 \pi} \max_{\alpha \in [0,1], 1 \leq N < p} \left| \sum_{1 \leq |n| \leq N} \frac{K_{1,p}(\tau_n \cdot a)}{n} (1-e(-\alpha n)) \right| + O(1)
\]
\[
\geq \frac{1}{4 \pi} \left| \sum_{1 \leq |n| \leq (\log p)^{1-\varepsilon}} \frac{K_{1,p}(\tau_n \cdot a)}{n} (1 + (-1)^{n+1}) \right| + O(1)
\]
\[
\geq \frac{2\sqrt{2}}{4 \pi} \sum_{1 \leq |n| \leq (\log p)^{1-\varepsilon}} \frac{1}{n} + O(1)
\]
\[
\geq \left( \frac{1 - \varepsilon}{\sqrt{2} \pi} + o(1) \right) \log \log p.
\]
for any \(a \in S_p\), where in the second step uses the fact that \(K_{a,p}(n) = K_{1,p}(\tau_n \cdot a)\) (the family is 2-bountiful).
Proof of Lemma 2.3 via Chebyshev Polynomials

From now on $p$ is a fixed prime number. We consider an irreducible 2-bountiful sheaf $K$ on $\mathbb{P}^1_{\mathbb{F}_p}$ and we will denote the trace function attached to it by $K(\cdot)$. The 2-bountiful condition on the sheaf $K$ implies that for any $a \in \mathbb{F}_p$, one has

$$K(a) = 2 \cos(\theta(a)).$$

with $\theta(a) \in [0, \pi]$. We call $\theta(a)$ the angle associated to $K(a)$. We recall that there exist polynomials $U_n$ for $n \geq 0$ such that

$$U_n(2 \cos \theta) = \sin((n + 1) \theta) \sin \theta,$$

for all $\theta \in [0, \pi]$. In terms of Representation Theory, these are related to the characters of the symmetric power of the standard representation of $\text{SU}_2$. In particular by Peter-Weyl Theorem, these form an orthonormal basis of $L^2([0, \pi], \mu_{\text{ST}})$. Note that we can see $U_n(K(\cdot))$ as the trace function attached to the sheaf $\text{Sym}^n(K)$. Moreover we call trigonometric polynomial of degree $s \geq 0$ any $Y \in L^2([0, \pi], \mu_{\text{ST}})$ written in the form

$$Y = \sum_{i=0}^{s} y(i)U_i,$$

with $y(s) \neq 0$. Let us start by proving some property of the sheaf $\text{Sym}^n(K)$:

**Lemma 2.4.** Let $K$ as above. For any $n > 0$:

i) The geometric monodromy group of $\text{Sym}^n(K)$ is given by

$$G_{\text{geom}}^{\text{Sym}^n(K)} \cong \begin{cases} \text{SU}_2 & \text{if } n \text{ is odd}, \\ \text{SU}_2/\{\pm 1\} & \text{if } n \text{ is even}. \end{cases}$$

ii) The projective automorphism group

$$\text{Aut}_0(\text{Sym}^n(K)) := \{ \gamma \in \text{PGL}_2(\mathbb{F}_p) : \gamma^* \text{Sym}^n(K) \cong \text{Sym}^n(K) \otimes L \}$$

for some rank 1 sheaf $L$ is trivial.

iii) The conductor of $\text{Sym}^n(K)$ is bounded by

$$c(\text{Sym}^n(K)) \leq n \cdot c(K).$$

**Proof.** Let us start with part (i): by the definition of the geometric monodromy one has that $G_{\text{geom}}^{\text{Sym}^n(K)} = \text{Sym}^n(G_{\text{geom}}^K)$. Then the result follows because $G_{\text{geom}}^K = \text{SU}_2$ by hypothesis. Let us prove now part (ii). Let $\gamma \in \text{PGL}_2(\mathbb{F}_p)$. First observe that

$$t_{\text{Sym}^n(K)}(x) = \frac{\sin((n + 1)\theta(x))}{\sin(\theta(x))},$$

$$t_{\gamma^* \text{Sym}^n(K)}(x) = \frac{\sin((n + 1)\theta(\gamma \cdot x))}{\sin(\theta(\gamma \cdot x))},$$

where $t_{K}(x) = 2 \cos \theta(x)$. Thanks to the fact that $K$ is a bountiful sheaf we know that the angles $\{((\theta(x), \theta(\gamma \cdot x)) : x \in \mathbb{F}_p^r \}$ become equidistributed in $([0, \pi] \times [0, \pi], \mu_{\text{ST}} \otimes \mu_{\text{ST}})$ when
$r \to \infty$ (Goursat-Kolchin-Ribet criterion). By contradiction, assume that $\gamma^* \text{Sym}^n(K) \cong \text{Sym}^n(K) \otimes L$ for some rank 1 sheaf. We may assume that $\mathcal{L}$ is of weights 0. Let $U$ be a dense open set where $\gamma^* \text{Sym}^n(K), \text{Sym}^n(K)$ and $\mathcal{L}$ are lisse. Using the equidistribution we can find $x \in U$ such that

$$t_{\text{Sym}^n(K)}(x) < 1/4, \quad t_{\gamma^* \text{Sym}^n(K)}(x) > 3/4.$$ 

On the other hand in $U$ one would have

$$|t_{\text{Sym}^n(K)}(x)| = |t_{\gamma^* \text{Sym}^n(K)}(x)|.$$

and this is absurd. Part (iii) is just a consequence of Deligne’s Equidistribution Theorem (see for example [Kat88, Paragraph 3.6]).

Lemma 2.5. Let $(Y_i)_{i=0}^n$ be a family of trigonometric polynomials as above such that for any $i$, $\deg Y_i \leq d$, and let $(\tau_i)_{i=1}^n \in \text{PGL}_2(\mathbb{F}_p)$ such that $\tau_i \neq \tau_j$ if $i \neq j$ then

$$\left| \sum_{a \in \mathbb{F}_p^*} \prod_{i=0}^n Y_i(\theta(\tau_i \cdot a)) - p \prod_{i=0}^n y_i(0) \right| \leq nC^n c(K)^2 d^{2n+2} y^n \sqrt{p},$$

where $y = \max_{i,j} |y_i(j)|$ and the constant $C$ is absolute.

Proof. To prove the lemma it is enough to bound

$$S = \sum_{a \in \mathbb{F}_p^*} \prod_{i=0}^n U_{n_i}(K(\tau_i \cdot a)),$$

when at least one of the $n_i \neq 0$. Thanks to Lemma 2.4 and [FKM15b Paragraph 3.1] we can apply [FKM15b Theorem 2.7] getting

$$|S - p \prod_{i=0}^n M_{n_i}| \leq L \sqrt{p},$$

where

$$L := C'n \cdot (\max_{n_i} \text{Rank}(\text{Sym}^{n_i}(K)))^n \cdot (\max_{n_i} c(\text{Sym}^{n_i}(K)))^2,$$

with $C'$ absolute constant (see [PG16, Proposition 4.4]), and for any $n_i$

$$M_{n_i} := \text{Mult}(1, \text{Sym}^{n_i} \text{ Std}) = \begin{cases} 1 & \text{if } n_i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and Std denotes the standard representation on $\text{SU}_2$. Then if at least one of the $n_i \neq 0$ we have

$$|S| \leq L \sqrt{p}.$$ 

The result then follows from the fact that

$$\text{Rank Sym}^{n_i}(K) = n_i + 1 \leq 2d, \quad c(\text{Sym}^{n_i}(K)) \leq n_i c(K) \leq dc(K),$$

because $n_i \leq d$ for all $i$ by assumption. \qed
Proof of Proposition \ref{prop:main}

We can now prove Proposition \ref{prop:main}. Let \( z \in \mathbb{N} \) be an odd positive number and let \( \gamma \in \mathbb{N} \), we denote by \( \theta := (\theta_{2j} - z)^2_j = 0 \in [0, \pi]^{z+1} \) and by \( \chi_{+}(\cdot) \) (resp. \( \chi_{-}(\cdot) \)) the characteristic function of \([0, \gamma - \frac{\pi}{2}] \) (resp. \([\gamma + \frac{\pi}{2}, \pi] \)). To prove Proposition \ref{prop:main} we start approximating the function

\[
\prod_{i=1}^{z} \chi_{+}(\theta(\tau_i \cdot a)) \prod_{i=1}^{z} \chi_{-}(\theta(\tau_{-1} \cdot a))
\]

using Chebyshev polynomials. We use the same method adopted in \cite[Section 3]{KLSW10}, for any \( z \), we find an integer \( L \equiv -1 \mod 2\gamma \) and two families of trigonometric polynomials \( \{\alpha_i\} \) and \( \{\beta_i\} \) such that if we define

\[
A_L(\theta) := \prod_{1 \leq i \leq z} \alpha_{L,i}(\frac{\theta}{\pi}) \quad B_L(\theta) = \sum_{1 \leq i \leq z} \beta_{L,i}(\frac{\theta}{\pi}) \prod_{j \neq i} \alpha_{L,j}(\frac{\theta}{\pi}),
\]

the following inequality holds

\[
A_L(\theta) - B_L(\theta) \leq \prod_{i=1}^{z} \chi_{+}(\theta_i) \prod_{i=1}^{z} \chi_{-}(\theta_{-i}),
\]

for any \( \theta \in [0, \pi]^{z+1} \). Moreover we will prove

Lemma \ref{lem:main}. With the notation as above, we have:

i) There exist two constant \( L_0 \geq 1 \) and \( c > 0 \) depending only on \( \gamma \), such that the contribution \( \Delta \) of the constant term in the Chebyshev expansions of \( A_L(\theta) \) – \( B_L(\theta) \) satisfies:

\[
\Delta \geq \frac{1}{2}(\frac{1}{2} - \frac{1}{\gamma})^{z+1},
\]

if \( L \) is the smallest integer such that \( L \equiv -1 \mod 2\gamma \) satisfying \( L \geq \max(cz, L_0) \).

ii) All coefficients in the Chebyshev expansion of the factors in \( A_L(\theta) \) and the terms in \( B_L(\theta) \) are bounded by 1.

iii) The degrees, in terms of the Chebyshev expansion, of the factors of \( A_L(\theta) \) and \( B_L(\theta) \) are \( \leq 2L \).

Once we have this Lemma we can easily get Proposition \ref{prop:main}. Fix \( \gamma = \frac{1}{4} \) in Lemma \ref{lem:main} and denote \( S_p \) the set of \( a \in \mathbb{F}_p^* \) which satisfy the property in the Proposition \ref{prop:main}. Let \( L \) be as in part (i) of Lemma \ref{lem:main} then we have

\[
|S_p| = \sum_{a \in \mathbb{F}_p^*} \prod_{i=1}^{z} \chi_{+}(\theta(\tau_i \cdot a)) \prod_{i=1}^{z} \chi_{-}(\theta(\tau_{-1} \cdot a))
\geq \sum_{a \in \mathbb{F}_p^*} A_L(\theta(\tau_{-2} \cdot a), \ldots, \theta(\tau_{2} \cdot a)) - B_L(\theta(\tau_{-2} \cdot a), \ldots, \theta(\tau_{2} \cdot a))
\geq p\Delta + O(zC^2 c_4 L^{2n+2} \sqrt{p})
\geq \frac{1}{2}(\frac{1}{4})^{z+1} p + O(zC^2 c_4 L^{2n+2} \sqrt{p}),
\]

as in part (i) of Lemma \ref{lem:main}.
where in the second step we are using Lemma 2.5 notice that

i) The condition \( \tau_i \neq \tau_j \) if \( i \neq j \) is satisfied by definition of acceptable family.

ii) thanks to part (ii) of Lemma 2.6 we have that \( y \) in Lemma 2.5 is equal to 1.

Let us denote \( \delta = 1 - \varepsilon \) and consider \( z = [(\log p)^{\delta}] \). By part (i) of Lemma 2.6 we know that \( \max(cz, L_0) \leq L \leq \max(2\gamma cz, L_0) \), moreover we may assume \( cz \leq L \leq 2\gamma cz \) because \( L_0 \) is an absolute constant (it depends only on \( \gamma = \frac{1}{4} \)). Then

\[
zC^2 e^{\frac{1}{\delta}} L^{2n+2} \sqrt{p} \leq (\log p)^{\delta} C^{(\log p)^{\delta}} e^\frac{1}{\delta} (2\gamma c(\log p)^{\delta})^2 (\log p)^{\delta} \sqrt{p} = o((\log p)^{4\delta} (\log p)^{\delta} \sqrt{p}) = o(p^{\frac{1}{2} + \eta}),
\]

for any \( \eta > 0 \). On the other hands, we have

\[
\left(\frac{1}{4}\right)^{z+1} \gg \left(\frac{1}{4}\right)^{\log p} = e^{-\log(4)(\log p)} = e^{-\log(4) \cdot \frac{\log p}{\log(4)^{\delta}}} = p^{-\frac{\log(4)}{\log(4)^{\delta}}}.
\]

Thus we obtain

\[
|S_p| \gg p^{1-\log(4) \cdot \frac{\log p}{\log(4)^{\delta}}},
\]

as we wanted.

**Proof of Lemma 2.6** The main references for this proof are [KLSW10, Lemma 3.2] and [BMV01]. We define

\[
A_L(x) := \prod_{i=1}^{z} \alpha_{L,+}(x_i) \prod_{i=1}^{1} \alpha_{L,-}(x_{-i}),
\]

where:

i) \( \alpha_{L,+} \) is a trigonometric polynomial in one variable of the form

\[
\sum_{|l| \leq L} \alpha_{L,+}(l)e(lx)
\]

defined as in [BMV01, (2.2) Lemma 5, (2.17)] with \( u = 0, v = \frac{1}{2} - \frac{1}{4} \).

ii) \( \alpha_{L,-} \) is a trigonometric polynomial in one variable of the form

\[
\sum_{|l| \leq L} \alpha_{L,-}(l)e(lx)
\]

defined as in [BMV01, (2.2) Lemma 5, (2.17)] with \( u = \frac{1}{2} + \frac{1}{4}, v = 1 \).

Instead we define

\[
B_L(x) := \sum_{i=1}^{z} \beta_{L,+}(x_i) \prod_{j=1}^{z} \alpha_{L,+}(x_j) \prod_{j=1}^{z} \alpha_{L,-}(x_{-j})
\] + \[
\sum_{i=1}^{z} \beta_{L,-}(x_{-i}) \prod_{j=1}^{z} \alpha_{L,+}(x_j) \prod_{j=1}^{z} \alpha_{L,-}(x_{-j}),
\]

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We can rewrite the above trigonometric polynomials as

\[
\beta_{L,+}(x) = \frac{1}{2^{L+1}} \left( \sum_{|l| \leq L} \left( 1 - \frac{|l|}{L+1} \right) e(lx) + \sum_{|l| \leq L} \left( 1 - \frac{|l|}{L+1} \right) e(l(x - \frac{1}{2} + \frac{1}{4})) \right)
\]

\[
\beta_{L,-}(x) = \frac{1}{2^{L+1}} \left( \sum_{|l| \leq L} \left( 1 - \frac{|l|}{L+1} \right) e(l(x - \frac{1}{2} - \frac{1}{4})) + \sum_{|l| \leq L} \left( 1 - \frac{|l|}{L+1} \right) e(l(x - 1)) \right).
\]

We can rewrite the above trigonometric polynomials as

\[
\beta_{L,+}(x) = \frac{1}{2^{L+1}} \left( 2 + \sum_{1 \leq l \leq L} \left( 1 - \frac{1}{l+1} \right) (\cos(\pi l - \frac{2\pi l}{\pi}) + \sin(\pi l + \frac{2\pi l}{\pi} + 1) \cos(2\pi lx) \right)
\]

\[
\beta_{L,-}(x) = \frac{1}{2^{L+1}} \left( 2 + \sum_{1 \leq l \leq L} \left( 1 - \frac{1}{l+1} \right) (\cos(-\pi l - \frac{2\pi l}{\pi}) + \sin(-\pi l - \frac{2\pi l}{\pi}) + 1) \cos(2\pi lx) \right).
\]

Remember that the \( n \)-th coefficient in the Chebychev expansion of \( \alpha_{L,\pm} \) and \( \beta_{L,\pm} \) are given by

\[
\int_0^\pi \alpha_{L,\pm}(\frac{\theta}{\pi}) U_n(\theta) d\mu_{st}, \quad \int_0^\pi \beta_{L,\pm}(\frac{\theta}{\pi}) U_n(\theta) d\mu_{st},
\]

then part (iii) immediately follows because the above integrals vanishes if \( n > 2L \). Moreover in [BMV01] Lemma 5 it is shown that \( 0 \leq \alpha_{L,\pm}(x) \leq 1 \) for \( x \in [0,1] \) and the same holds for the \( |\beta_{L,\pm}| \) by definition. Using Cauchy-Schwarz inequality we get

\[
\left| \int_0^\pi \alpha_{L,\pm} U_n(\theta) d\mu_{st} \right|^2 \leq \int_0^\pi |\alpha_{L,\pm}(\frac{\theta}{\pi})|^2 d\mu_{st} \cdot \int_0^\pi |U_n(\theta)|^2 d\mu_{st} \leq 1,
\]

the same argument can be used for \( \beta_{L,\pm} \) and this proof part (ii). It remains to prove only part (i), as we have just observed for any trigonometric polynomial \( Y \) the constant term of its Chebyshev expansion is given by

\[
\int_0^\pi Y(\theta) d\mu_{st},
\]

so we have that \( \Delta \) in part (i) is given by

\[
\Delta = \left( \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st} \right) \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st}
\]

\[
- \frac{z+1}{2} \int_0^\pi \beta_{L,+}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st}
\]

\[
- \frac{z+1}{2} \int_0^\pi \beta_{L,-}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st}
\]

Using the definition of \( \beta_{L,\pm} \) we get

\[
\int_0^\pi \beta_{L,\pm}(\frac{\theta}{\pi}) d\mu_{st} = \frac{1}{L+1},
\]

so we can write \( \Delta \) as

\[
\Delta = \left( \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st} \right) \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st}
\]

\[
- \frac{z}{2L+2} \left( \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st} \right) \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st}
\]

\[
- \frac{z}{2L+2} \left( \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st} \right) \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,+}(\frac{\theta}{\pi}) d\mu_{st} \int_0^\pi \alpha_{L,-}(\frac{\theta}{\pi}) d\mu_{st}
\]

\[
.\]
When \( L \to \infty \), \( \alpha_{L, \pm} \to \chi_{\pm} \) in \( L^2([0, 1]) \), moreover from [BMV01, (2.6)] one has

\[
|\chi_{\pm} - \alpha_{L, \pm}| \leq |\beta_{L, \pm}|
\]

and from the Fourier expansion of \( \beta_{L, \pm}(x) \) we have

\[
||\beta_{L, \pm}||^2_{L^2} \leq \frac{8 + 3L}{(2L + 2)^2} \to 0,
\]

thus we have

\[
\int_{0}^{\pi} \alpha_{L, \pm}(\frac{\theta}{\pi}) d\mu_{st} \to \int_{0}^{\pi} \chi_{\pm}(\frac{\theta}{\pi}) d\mu_{st} = \frac{1}{2} - \frac{1}{\gamma} + \frac{\sin(\gamma)}{\gamma} \cos(\frac{\pi}{\gamma}).
\]

This implies that there exist \( L_0 \), such that the integral in the left hand side of the equation above is \( \geq \frac{1}{2} - \frac{1}{\gamma} \) so we get:

\[
\Delta \geq \left( \frac{1}{2} - \frac{1}{\gamma} \right)^{z-1} \left( \left( \frac{1}{2} - \frac{1}{\gamma} \right)^2 - \frac{3z}{2L + 2} \right), \tag{14}
\]

If we assume \( 2L + 2 \geq 6z \left( \frac{1}{2} - \frac{1}{\gamma} \right)^{-2} \) we get:

\[
\Delta \geq \frac{1}{2} \left( \frac{1}{2} - \frac{1}{\gamma} \right)^{z+1}, \tag{15}
\]

as we wanted.

3 Moments

The Auxiliary Lemma

Let us start with the following Lemma

**Lemma 3.1.** With the same notation as in Theorem [10], let \( 0 \leq \alpha < \beta \leq 1 \), then for any \( k \geq 2 \) there exist two constant \( \gamma, \delta \geq 1 \) depending only on \( c_8 \), such that

\[
\frac{1}{p - 1} \sum_{a \in \mathbb{Z}_p} \sum_{a_p < x \leq \beta_p} t_{a, p}(x)^{2k} \leq \gamma^{2k} (\log k)^{2k} \left( \frac{\pi}{\beta - \alpha} \right)^{-\frac{2k}{\beta - \alpha}} + \delta^{2k} p^{-\frac{1}{2}} (\log p)^{2k}.
\]

**Proof.** Let’s start with the quantitative form of the Fourier expansion:

\[
\frac{1}{\sqrt{p}} \sum_{a_p < x \leq \beta_p} t_{a, p} = \frac{1}{2\pi i} \sum_{1 \leq |n| \leq p/2} \frac{K_{a, p}(n)}{n} (1 - e((\beta - \alpha)n)) e(an) + (\beta - \alpha)K_{a, p}(0) + O(1)
\]

To simplify the notation, for any \(-p/2 \leq n \leq p/2\) we define

\[
c_n := \frac{(1 - e((\beta - \alpha)n)) e(an)}{n}.
\]
so we can write the equation above as

$$\frac{1}{\sqrt{p}} \sum_{\alpha \leq x \leq \delta p} t_{a,p}(x) = \frac{1}{2\pi i} \sum_{1 < |n| < p/2} K_{a,p}(n) c_n + (\beta - \alpha)K_{a,p}(0) + O(1).$$

By the triangular inequality one gets

$$\frac{1}{p-1} \sum_{a \in \mathbb{F}_p^*} \left| \frac{1}{\sqrt{p}} \sum_{\alpha \leq x \leq \delta p} t_{a,p}(x) \right|^{2k} \leq \frac{1}{(p-1)p^{2k}} \sum_{a \in \mathbb{F}_p^*} \left| \sum_{1 < |n| < p/2} K_{a,p}(n) c_n \right|^{2k}
+ O(2^{4k} + 2^{4k}(\beta - \alpha)^{2k} \delta^2 \frac{1}{p^2})
= \left( \frac{1}{(p-1)p^{2k}} \sum_{a \in \mathbb{F}_p^*} \left| \sum_{1 < |n| < p/2} K_{a,p}(\tau_n \cdot a) c_n \right|^{2k}
+ O(2^{4k} + 2^{4k}(\beta - \alpha)^{2k} \delta^2 \frac{1}{p^2}) \right),$$

where in the first inequality we use the fact that $K_{a,p}(0) \leq c(F_{a,p}) \leq c_3$ by hypothesis. To conclude the proof of the Lemma it is enough to provide a bound for the first term in the right hand side. Extending the $2k$-power we get

$$\sum_{a \in \mathbb{F}_p^*} \sum_{\gamma_1} \cdots \sum_{\gamma_k} \sum_{t_1} \cdots \sum_{t_k} \sum_{\gamma_1} \cdots \sum_{\gamma_k} K_{1,p}(\tau_{\gamma_1} \cdot a) \cdots K_{1,p}(\tau_{\gamma_k} \cdot a) \cdot K_{1,p}(\tau_{t_1} \cdot a) \cdots K_{1,p}(\tau_{t_k} \cdot a) c_{\gamma_1} \cdots c_{\gamma_k} c_{t_1} \cdots c_{t_k}.$$ 

An application of [FKM15b, Corollary 1.7] implies

$$\left| \sum_{a \in \mathbb{F}_p^*} K_{1,p}(\tau_{\gamma_1} \cdot a) \cdots K_{1,p}(\tau_{\gamma_k} \cdot a) K_{1,p}(\tau_{t_1} \cdot a) \cdots K_{1,p}(\tau_{t_k} \cdot a) - m(n,1) p \right| \leq \delta_1^{2k} \sqrt{p},$$

where the constant $\delta_1$ depends only on $c_3$ and moreover $m(n,1) \neq 0$ if and only if every entries of the array $(n,1)$ have even multiplicity (see [FKM15b Corollary 1.7] for a precise definition of $m(n,1)$). Thanks to this we get

$$\sum_{a \in \mathbb{F}_p^*} \left| \sum_{1 < |n| < p/2} K_{a,p}(n) c_n \right|^{2k} \leq A + B,$$

where

$$A := \left| \sum_{n,1} c_{n_1} \cdots c_{n_k} c_{t_1} \cdots c_{t_k} m(n,1) \right|,$$

and

$$B := \left| \sqrt{p\delta_1^{2k}} \sum_{n,1} c_{n_1} \cdots c_{n_k} c_{t_1} \cdots c_{t_k} \right|.$$ 

Let us bound $B$ first.

$$B \leq \sqrt{p\delta_1^{2k}} \sum_{n,1} |c_{n_1}| \cdots |c_{n_k}| |c_{t_1}| \cdots |c_{t_k}| = \sqrt{p\delta_1^{2k}} \left( \sum_n |c_n| \right)^2.$$ 

On the other hand

$$|c_n| \leq 2 \min \left( \frac{1}{n} \frac{\beta - \alpha}{\pi} \right) \leq \frac{2}{n},$$

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hence we get $B \leq \sqrt{p_{2k}^2(\log p)^{2k}}$ for some $\delta_2 > 0$ depending only on $c_2$.

To bound $A$ we can proceed as follows: first observe that if $m(n, 1) \neq 0$ then there exists a constant dependent only on $c_2$, let’s say $\gamma_1$, such that $m(n, 1) \leq \gamma_1^{2k}$ (again the reference here is [FKM15b Corollary 1.7]). Thus

$$A \leq \gamma_1^{2k} p \sum_n \sum_{(n_1, \ldots, n_{2k}) \in m(n)} c_{n_1} \cdots c_{n_{2k}},$$

where $m(n) := \{(n_1, \ldots, n_{2k}) : n_1 \cdots n_{2k} = n \text{ any } n_i \text{ appears an even number of times}\}$.

On the other hand we have that

$$c_{n_1} \cdots c_{n_{2k}} \leq 2^{2k} \min \left( \frac{1}{n}, \left( \frac{\beta - \alpha}{\pi} \right)^{2k} \right) =: b(n).$$

Let us focus our attention on the size of $|m(n)|$. First observe that by definition, $|m(n)| = 0$ is not a square. Moreover for any $(n_1, \ldots, n_{2k}) \in m(n^2)$ we can find two set $S_1, S_2 \subset \{1, \ldots, 2k\}$ such that

$$|S_1| = |S_2| = k \quad S_1 \cap S_2 = \emptyset \quad n = \prod_{i \in S_1} n_i = \prod_{i \in S_2} n_i,$$

thus

$$|m(n^2)| \leq \left( \frac{2k}{k} \right) d_k(n^2).$$

Inserting this in equation (16) we get

$$A \leq \gamma_1^{2k} \left( \frac{2k}{k} \right) p \sum_n d_k(n^2) b(n^2) \leq \gamma_1^{2k} \left( \frac{2k}{k} \right) p \sum_{n \leq p} d_k(n^2) b(n^2) + O_{k, \epsilon}(p^\epsilon).$$

On the other hand is shown in [BGI13 Lemma 4.1] that

$$\sum_{n \leq p} d_k(n^2) b(n^2) \leq 2^k (\log k)^{2k} \left( \frac{\pi}{\beta - \alpha} \right)^{2k}$$

and this conclude the proof.

\[\square\]

**Proof of Theorem 1.10**

We are finally ready to prove our result on moments

**Proof of Theorem 1.10** Let us start with the lower bound. By Lemma 2.2 we have that

$$M(t_{a, p}) \geq \frac{1}{4\pi} \left| \sum_{1 \leq n \leq k} \frac{K_{1, p}(\tau_n \cdot a)}{n} (1 - e(an)) \right| + O_{\epsilon}(3)(1)$$

for any $p$ large enough and any $a \in \mathbb{F}_p^\times$. On the other hand $K_{1, p}$ is a bountiful sheaf, so the sheaves $\mathcal{K}_{1, p}, \tau_{-1} \mathcal{K}_{1, p}, \ldots, \tau_{-k} \mathcal{K}_{1, p}, \tau_k \mathcal{K}_{1, p}$ satisfy the Goursat-Kolchin-Ribet criterion and so

$$a \mapsto (\mathcal{K}_{1, p}(\tau_1 \cdot a), \mathcal{K}_{1, p}(\tau_{-1} \cdot a), \ldots, \mathcal{K}_{1, p}(\tau_{-k} \cdot a), \mathcal{K}_{1, p}(\tau_k \cdot a))$$
become equidistributed in \((\prod_{i=1}^{2k} G_{\text{geom}}, \mu_{\text{geom}}^{2k})\) when \(p \to \infty\). Now if we define
\[
S_{k,p} := \{ a \in F_p^\times : K_{1,p}(\tau_i \cdot a) > \sqrt{2} \forall 0 < i < k, i \equiv 1(2), \quad K_{1,p}(\tau_{i-1} \cdot a) < -\sqrt{2} \forall 0 < i < k, i \equiv 1(2) \},
\]
we have
\[
M(t_{a,p}) \geq \left( \frac{1}{\sqrt{2\pi}} + o(1) \right) \log k
\]
for any \(a \in S_{k,p}\). Hence
\[
\frac{1}{p-1} \sum_{a \in F_p^\times} M(t_{a,p})^{2k} \geq \frac{1}{p-1} \sum_{a \in S_{k,p}} M(t_{a,p}) \geq \left( \frac{c}{\sqrt{2\pi}} + o(1) \right)^{2k} (\log k)^{2k},
\]
where \(c = \{ a \in F_p^\times : K_{1,p}(a) > \sqrt{2} \}\) which depends only on \(c_3\). Let us prove now the upper bound. For any \(a \in F_p^\times\) let \(N_{a,p}\) be the smallest integer such that
\[
M(t_{a,p}) = \frac{1}{\sqrt{p}} \sum_{x \leq N_{a,p}} t_{a,p}(x).
\]
At this point we would like to apply Lemma 3.1 but the issue here is that the \(N_{a,p}\)s could be very different each other. To turn around to this problem, following the same strategy as in [MV79] and [BG13], we will use the Rademacher-Menchov trick: first of all write \(\frac{N_{a,p}}{p}\) in base two
\[
\frac{N_{a,p}}{p} = \sum_{j=1}^{\infty} a_j 2^{-j} \quad a_j \in \{0, 1\},
\]
and let \(N_{a,p}(L_p)/p\) be the truncation of this series to the factor of power \(L_p\). Then we have
\[
M(t_{a,p}) \leq \left| \frac{1}{\sqrt{p}} \sum_{x \leq N_{a,p}(L_p)} t_{a,p}(x) \right| + \left| \frac{1}{\sqrt{p}} \sum_{N_{a,p}(L_p) < x \leq N_{a,p}} t_{a,p}(x) \right|,
\]
notice that the length of the second summation is \(\leq \frac{1}{2^{L_p}}\), let’s denote \(E(a, p, L_p)\) this sum. An application of the Hölder inequality leads to
\[
M(t_{a,p})^{2k} \leq 2^{2k} \left( \sum_{l \leq L_p} \frac{1}{l^{2k+1}} \right)^{2k-1} \left( \sum_{l \leq L_p} l^{2ka} \right) \left( \sum_{N_{a,p}(L_p) < x \leq N_{a,p}(l+1)} t_{a,p}(x) \right)^{2k} + 2^{2k} E(a, p, L_p)^{2k},
\]
at this point we observe at first that \(N_{a,p}(l+1) \leq N_{a,p}(l) + p^{2^{-l+1}}\), moreover for the value of \(N_{a,p}(l)\) one has \(2^{l-1}\) possibility so
\[
M(t_{a,p})^{2k} \leq 2^{2k} \left( \sum_{l \leq L_p} \frac{1}{l^{2k+1}} \right)^{2k-1} \left( \sum_{l \leq L_p} l^{2ka} \sum_{0 \leq m \leq 2^{l-1}} \left| \frac{1}{\sqrt{p}} \sum_{\frac{m}{2^l} < x < \frac{m+1}{2^l}} t_{a,p}(x) \right|^2 \right) + 2^{2k} E(a, p, L_p)^{2k}.
\]

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We can now apply the Lemma 3.1 and choose \( \alpha = 3/2 \) getting
\[
\frac{1}{p-1} \sum_{a \in \mathbb{F}_p^*} M(t_{a,p})^{2k} \leq 2^{2k} \left( \sum_{l \leq L_p} \frac{1}{l^{2k+\varepsilon}} \right)^{2k-1} \left( \sum_{l \leq L_p} l^{3k}2^l (\gamma)^{2k} (\log k)^{2k} 2^{-\frac{4k}{\log p}} + \varepsilon \right) + \frac{2^{2k}}{p-1} \sum_{a \in \mathbb{F}_p^*} E(a, p, L_p)^{2k}.
\]

Let \( L_p := \log_2 \left( \frac{p^2}{(\log p)^2} \right) \), then
\[
(2\delta)^{2k} \left( \sum_{l \leq L_p} \frac{1}{l^{2k+\varepsilon}} \right)^{2k-1} \sum_{l \leq L_p} l^{3k}2^l p^{-\frac{1}{4}} (\log p)^{2k} \ll_k (\log p)^{8k} 2^L p^{-\frac{1}{4}} \ll_k 1,
\]
where in the first step we are using that
\[
\sum_{l} \frac{1}{l^{2k+\varepsilon}} \ll 1.
\]

Moreover using the inequality
\[
\sum_{l \leq L_p} l^{3k}2^l p^{-\frac{1}{4}} (\log p)^{2k} \leq \exp(3k \log k + O(k))
\]
proven in \cite{BG13} Theorem 1.1 we get
\[
(2\gamma)^{2k} (\log k)^{2k} \left( \sum_{l \leq L_p} \frac{1}{l^{2k+\varepsilon}} \right)^{2k-1} \sum_{l \leq L_p} l^{2k}2^l 2^{-\frac{k}{\log p}} \ll_k 1.
\]

On the other hand we have already pointed out that the length of \( E(a, p, L_p) \) is at most \( \frac{p^2}{(\log p)^2} = p^4 (\log p)^{4k} \), so thanks to \cite{FKM+17} Theorem 1.1] we have
\[
|E(a, p, L_p)| \leq 16 \log(4e^5 (\log p)^3)
\]
then
\[
\frac{1}{p-1} \sum_{a \in \mathbb{F}_p^*} E(a, p, L_p)^{2k} \leq (Ck)^k (\log \log p)^{2k}.
\]

Now let us assume that
\[
\left| \sum_{N \leq x \leq N+H} t_{a,p}(x) \right| \ll \alpha H^{1-\varepsilon} \tag{17}
\]
holds uniformly for any \( 1 < N < p, p^{1/2-\varepsilon/2} < H < p^{1/2+\varepsilon/2} \) and \( a \in \mathbb{F}_p^* \). Starting again from
\[
\frac{1}{p-1} \sum_{a \in \mathbb{F}_p^*} M(t_{a,p})^{2k} \leq 2^{2k} \left( \sum_{l \leq L_p} \frac{1}{l^{2k+\varepsilon}} \right)^{2k-1} \left( \sum_{l \leq L_p} l^{2k\alpha} (\gamma)^{2k} (\log k)^{2k} 2^{-\frac{4k}{\log p}} + \varepsilon \right) + \frac{2^{2k}}{p-1} \sum_{a \in \mathbb{F}_p^*} E(a, p, L_p)^{2k}.
\]
We can now choose $L_p := \frac{1}{2} \log_2 p$, then (17) implies

$$E(a, p, L_p) = \left| \frac{1}{\sqrt{p}} \sum_{N_{a,p}(x) < x \leq N_{a,p}} t_{a,p}(x) \right| \ll_{c_3} p^{-\varepsilon}.$$  

Then

$$\frac{1}{p-1} \sum_{a \in F_p^*} E(a, p, L_p) \ll_{c_3} p^{-\varepsilon'},$$

for some $\varepsilon' > 0$. Moreover observe that

$$p^{-\frac{1}{2}} (\delta \log p)^{2k} \left( \sum_{l \leq L_p} l^{2k \alpha} l^l \right) \leq p^{-\frac{1}{2}} (\delta \log p)^{2k} \left( \sum_{l \leq L_p} l^{4k \alpha} l^l \right) \ll_{k,c_3} p^{-\frac{1}{2}} (\log p)^{2k L_p} \ll p^{-\varepsilon/2} (\log p)^{2k}.$$

So we get

$$\frac{1}{p-1} \sum_{a \in F_p^*} M(t_{a,p})^{2k} \leq (2 \gamma \log k)^{2k} \left( \sum_{l \leq L_p} \frac{1}{l^{2k \alpha}} \right)^{2k-1} \left( \sum_{l \leq L_p} l^{2k \alpha} l^l 2^{-\frac{1}{2} \log p} \right) + O_{k,c_3} (p^{-\varepsilon''}),$$

where $\varepsilon'' > 0$. On the other hand we have (BG13, Theorem 1.1)

$$\sum_{l \leq L_p} \frac{1}{l^{2k \alpha}} \leq (\alpha - 1)^{1-2k}, \quad \sum_{l \leq L_p} l^{2k \alpha} l^l 2^{-\frac{1}{2} \log p} \leq \exp(2k \alpha \log \log k + O(k)),$$

so choosing $\alpha = 1 + 1/\log \log k$ we get the result.

We conclude with

Proof of Corollary 1.11 For (i) observe that from the proof of lower bond of Theorem 1.10 it follows that any element, $a$, in the set

$$S_{h,p} := \{ a \in F_p^* : K_{1,p}(\tau_i \cdot a) > \sqrt{2} \forall 0 < i < h, i \equiv 1(2), \quad K_{1,p}(\tau_{i-1} \cdot a) < -\sqrt{2} \forall 0 < i < h, i \equiv 1(2) \}$$

is such that $M(t_{a,p}) > \text{const} \cdot \log h$. Moreover we have that $|S_{h,p}| > c^{2h}$ for some constant $0 < c < 1$ depending on $c_3$. Choosing $h = \exp((\text{const})^{-1} \cdot A)$ we get

$$D_{g}(A) \geq |S_{\exp((\text{const})^{-1} \cdot A)}|.$$  

To conclude, the proof of (ii) is exactly the same as in [BG13, Theorem 1.3].

References

[BG13] J. W. Bober and L. Goldmakher. The distribution of the maximum of character sums. Mathematika, 59(2):427–442, 2013.
[BMV01] J. T. Barton, H. L. Montgomery, and J. D. Vaaler. Note on a Diophantine inequality in several variables. *Proc. Amer. Math. Soc.*, 129(2):337–345, 2001.

[FKM14] Étienne Fouvry, Emmanuel Kowalski, and Philippe Michel. Trace functions over finite fields and their applications. In *Colloquium De Giorgi 2013 and 2014*, volume 5 of *Colloquia*, pages 7–35. Ed. Norm., Pisa, 2014.

[FKM15a] É. Fouvry, E. Kowalski, and P. Michel. Algebraic twists of modular forms and Hecke orbits. *Geom. Funct. Anal.*, 25(2):580–657, 2015.

[FKM15b] É. Fouvry, E. Kowalski, and P. Michel. A study in sums of products. *Philos. Trans. Roy. Soc. A*, 373(2040):20140309, 26, 2015.

[FKM+17] É. Fouvry, E. Kowalski, P. Michel, C. S. Raju, J. Rivat, and K. Soundararajan. On short sums of trace functions. *Ann. Inst. Fourier (Grenoble)*, 67(1):423–449, 2017.

[GL12] L. Goldmakher and Y. Lamzouri. Lower bounds on odd order character sums. *Int. Math. Res. Not. IMRN*, (21):5006–5013, 2012.

[GL14] L. Goldmakher and Y. Lamzouri. Large even order character sums. *Proc. Amer. Math. Soc.*, 142(8):2609–2614, 2014.

[GS07] A. Granville and K. Soundararajan. Large character sums: pretentious characters and the Pólya-Vinogradov theorem. *J. Amer. Math. Soc.*, 20(2):357–384, 2007.

[Hoo78] C. Hooley. On the greatest prime factor of a cubic polynomial. *J. Reine Angew. Math.*, 303/304:21–50, 1978.

[Kat88] Nicholas M. Katz. *Gauss sums, Kloosterman sums, and monodromy groups*, volume 116 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1988.

[KLSW10] E. Kowalski, Y.-K. Lau, K. Soundararajan, and J. Wu. On modular signs. *Math. Proc. Cambridge Philos. Soc.*, 149(3):389–411, 2010.

[KS16] E. Kowalski and W. F. Sawin. Kloosterman paths and the shape of exponential sums. *Compos. Math.*, 152(7):1489–1516, 2016.

[Lam18] Y. Lamzouri. On the distribution of the maximum of cubic exponential sums. *Journal of the Institute of Mathematics of Jussieu*, 2018.

[MV77] H. L. Montgomery and R. C. Vaughan. Exponential sums with multiplicative coefficients. *Invent. Math.*, 43(1):69–82, 1977.

[MV79] H. L. Montgomery and R. C. Vaughan. Mean values of character sums. *Canad. J. Math.*, 31(3):476–487, 1979.

[Pal32] R. E. A. C. Paley. A Theorem on Characters. *J. London Math. Soc.*, 7(1):28–32, 1932.

[PG16] C. Perret-Gentil. Probabilistic aspects of short sums of trace functions over finite fields. *Ph.D. Thesis*, 2016.
[Pol18] G. Polya. Über die verteilung der quadratischen reste und nichtreste. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, pages 21–29, 1918.

[Vin18] I. M. Vinogradov. Sur la distribution des residus and nonresidus des puissances. J. Soc. Phys. Math. Univ. Permi, pages 18–29, 1918.