Instantons, Twistors, and Emergent Gravity

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Abstract

Motivated by potential applications to holography on space-times of positive curvature, and by the successful twistor description of scattering amplitudes, we propose a new dual matrix formulation of \( \mathcal{N} = 4 \) gauge theory on \( S^4 \). The matrix model is defined by taking the low energy limit of a holomorphic Chern-Simons theory on \( \mathbb{CP}^3/4 \), in the presence of a large instanton flux. The theory comes with a choice of \( S^4 \) radius \( \ell \) and a parameter \( N \) controlling the overall size of the matrices. The flat space variant of the 4D effective theory arises by taking the large \( N \) scaling limit of the matrix model, with \( \ell_{pl}^2 \sim \ell^2/N \) held fixed. Its massless spectrum contains both spin one and spin two excitations, which we identify with gluons and gravitons. As shown in the companion paper [1], the matrix model correlation functions of both these excitations correctly reproduce the corresponding MHV scattering amplitudes. We present evidence that the scaling limit defines a gravitational theory with a finite Planck length. In particular we find that in the \( \ell_{pl} \rightarrow 0 \) limit, the matrix model makes contact with the CSW rules for amplitudes of pure gauge theory, which are uncontaminated by conformal supergravity. We also propose a UV completion for the system by embedding the matrix model in the physical superstring.

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1 Introduction

While there is little doubt that string theory provides a consistent theory of quantum gravity, it has proven rather difficult to specify the physical foundations of the theory. Part of the issue is that in situations where maximal theoretical control is available, space-time is treated as a classical background, rather than as an emergent concept. Related to this, the understanding of holography on space-times of positive curvature remains elusive.

In this paper we propose and develop a new dual matrix formulation of 4D field theory, in which the space-time and field theory degrees of freedom simultaneously emerge from a large $N$ double scaling limit. The basic idea is to view $\mathcal{N} = 4$ gauge theory on $S^4$ as an effective low energy description of an underlying topological large $N$ gauge theory, without any local propagating degrees of freedom. This topological theory takes the form of holomorphic Chern-Simons (hCS) theory defined on projective supertwistor space $\mathbb{CP}^3|4$.

Motivated by the (partial) successes of earlier matrix reformulations of string theory [2–6], and of the twistor string reformulation of gauge theory [7], we consider the hCS theory in the presence of a large background flux [8]. We choose this flux to be a homogeneous $U(N)$ instanton on $S^4$ with the maximal possible instanton number $k_N$. This special flux configuration, known as the $U(N)$ Yang monopole, breaks the $U(Nc)$ gauge symmetry of the UV theory down to $U(Nc)$, the gauge group of the IR effective gauge theory.

As we will show, the low energy physics in the presence of the flux is governed by a large $N$ matrix model, with matrices of size $k_N \times k_N$. The guiding principle that determines the form and content of the matrix model action is the requirement that the matrix equations of motion encapsulate the space of deformations of the Yang monopole, and reduce to the ADHM equations in the appropriate flat space limit. The finite matrices arise because, in the presence of the instanton flux, all charged excitations are forced to move in Landau levels. The lowest level is built up from $k_N$ Planck cells – or fuzzy points – of a non-commutative $\mathbb{CP}^3|4$, where each fuzzy point represents one elemental instanton. The coordinates of the supermanifold become operators, that satisfy a commutator algebra

$$[Z^\alpha, Z^\dagger_\beta] = \delta^\alpha_\beta.$$  \hspace{1cm} (1.1)

The oscillator number is bounded to be less than some integer $N$, which translates into an upper limit on the allowed angular momentum on the $S^4$.

A striking aspect of this type of non-commutative deformation is that, unlike other more
standard versions of space-time non-commutativity, it preserves all space-time isometries. Moreover, because the non-commutative deformation does not directly affect the holomorphic coordinates, the basic geometric link between the twistor variables and 4D physics is kept intact. Nevertheless, the instanton density of the background flux introduces a UV length scale on the $S^4$, which in units of the $S^4$ radius $\ell$ scales as

$$\ell_{pl}^2 \sim \frac{\ell^2}{N}. \quad (1.2)$$

Below this length scale, the physics is described by a holomorphic Chern-Simons theory, expanded around a trivial background and without local excitations. At much larger scales, however, the collective motion of the instanton background produces rich dynamics that, as we will show, takes the form of a quantum field theory with a UV cutoff given by (1.2). The continuum QFT arises by taking the large $N$ limit while keeping the $S^4$ radius $\ell$ fixed. One can instead take a combined large $N$ and flat space limit, by simultaneously sending $N$ and $\ell$ to infinity while keeping $\ell_{pl}$ fixed. As we will argue, this results in a 4D gravitational theory with a finite Planck length.

The emergence of gravity from an a priori non-gravitational dual system is a remarkable phenomenon that deserves further exploration. At a superficial level, our matrix model looks like a regulated version of the twistor string theory introduced by Witten as a dual description of $\mathcal{N} = 4$ gauge theory. Twistor string theory also gives rise to gravitational degrees of freedom in the form of conformal supergravity. However, our set-up differs in several essential aspects from the twistor string. Most importantly, our theory comes with a natural length scale that breaks conformal invariance. Moreover, the emergence of gravity has a clear geometric origin via the non-commutativity of twistor space.

The non-commutativity of the twistor coordinates causes gauge transformations to act like diffeomorphisms. Correspondingly, a non-zero gauge field background on non-commutative twistor space translates into a deformation of the emergent 4D space-time geometry. This relationship can be made precise, via a direct variant of Penrose’s non-linear graviton construction of self-dual solutions to the 4D Einstein equations. In a companion paper [1] we use this insight to evaluate correlation functions of bilinear matrix observables, and show that quite remarkably, the end result reproduces MHV graviton scattering amplitudes of ordinary 4D Einstein (super)gravity.

The matrix model is the low energy approximation to a more complete UV theory. In the first part of the paper, the UV theory is taken to be the hCS theory, or topological B-model open string theory, defined on super twistor space. In the last section, we propose an alternative UV completion in terms of type IIA superstring theory. The brane configuration that we consider is given by a D0-brane bound to a stack of D8-branes and an O8-plane. The background geometry is taken to be 10 non-compact space-time dimensions. In the
Figure 1: The twistor matrix model captures the low energy physics of holomorphic Chern-Simons theory on \( \mathbb{CP}^{3|4} \) in the presence of an instanton background. The ingredients of the matrix model have a natural connection with the ADHM construction of instantons. Correlation functions of the matrix model compute amplitudes of a 4D space-time theory, which in the flat space limit coincide with scattering amplitudes of gluons and gravitons.

presence of a background magnetic flux, the low temperature limit of a gas of fermionic 0-8 strings reproduces the degrees of freedom and action of the twistor matrix model. An intriguing aspect of this construction is that it (potentially) gives rise to a 4D theory coupled to gravity, without first having to choose some Kaluza-Klein compactification manifold.

The rest of this paper is organized as follows. In section 2 we review some basics of twistor geometry. Next, in section 3 we introduce holomorphic Chern-Simons theory with the Yang monopole flux and explain how it naturally gives rise to a non-commutative deformation of twistor space. This is followed in section 4 by the presentation of the twistor matrix model, and the gauge currents of the defect system. Section 5 is devoted to matching a particular limiting case of the matrix model to the ADHM construction of instantons. We study the continuum limit of the matrix model in section 6. Section 7 discusses the sense in which the theory describes a quantum theory of gravity, and in section 8 we provide a UV completion of the matrix model based on thermodynamics of a stringy brane construction. We present our conclusions and avenues of future investigation in section 9.

2 Twistor Preliminaries

In this section we review some preliminary aspects of twistor geometry, as well as the connection between hCS and self-dual Yang Mills. We also discuss quantization of twistor space.
2.1 Twistor Correspondence

Twistors were introduced by Penrose as a natural geometric description of massless field theories [11,12] (see e.g. [13,14] for reviews). The construction starts with the introduction of spinor helicity variables. In complexified Minkowski space, light-like momenta $p^{\dot{a}}_a$ can be represented as a product $p^{\dot{a}}_a = \bar{\pi}_a \pi^\dot{a}$ of a chiral and an anti-chiral spinor. Next, one notices that given the spinor $\pi_a$ and a space-time point $x^{\dot{a}}a$ on complexified Minkowski space, one can define a corresponding two component complex spinor $\omega^{\dot{a}}$ via

$$\omega^{\dot{a}} = ix^{\dot{a}}a \pi_a.$$  (2.1)

This relation is invariant under simultaneous complex rescaling of the two spinors $\pi_a$ and $\omega^{\dot{a}}$. One thus obtains a set of homogeneous coordinates $Z^a = (\omega^{\dot{a}}, \bar{\pi}_a)$ on projective twistor space $\mathbb{P}T^\bullet$. Similarly, one can introduce a dual twistor space, with coordinates $\bar{Z}_{\dot{b}} = (\bar{\omega}_{\dot{a}}, \bar{\pi}^a)$, denoted by $\mathbb{P}T^\bullet$. Geometrically, $\mathbb{P}T^\bullet$ and its dual both define a three-dimensional complex projective space $\mathbb{C}P^3$. We will raise and lower two component spinor indices with the help of the corresponding two component epsilon symbols\(^2\) The associated antisymmetric spinor inner products are denoted via

$$\langle \pi_1 \pi_2 \rangle = \varepsilon^{ab} \pi_{1a} \pi_{2b}, \quad [\omega_1 \omega_2] = \varepsilon_{\dot{a}\dot{b}} \omega_{1\dot{a}} \omega_{2\dot{b}}.$$  (2.2)

Equation (2.1) combines two complex linear relations in $\mathbb{C}P^3$. For a given point $x$ in complexified space-time, it defines a $\mathbb{C}P^1$, which we will call the *twistor line* associated with $x$. The space $\mathbb{C}^4 \times \mathbb{C}P^1$ with coordinates $(x, \pi)$ is known as the “correspondence space”. This admits a map to $\mathbb{C}P^3$ via $\omega^{\dot{a}} = ix^{\dot{a}a} \pi_a$.

A convenient presentation of complexified, conformally compactified Minkowski space is given as the zero locus of the Klein quadric in $\mathbb{C}P^5$:

$$\varepsilon_{\alpha \beta \gamma \delta} X^{\alpha \beta} X^{\gamma \delta} = 0.$$  (2.3)

Here the $X^{\alpha \beta} = -X^{\beta \alpha}$ are the six homogeneous coordinates of $\mathbb{C}P^5$. The constraint (2.3) is automatically solved by introducing a pair of points in twistor space, with homogeneous coordinates $Z^a$ and $W^\beta$, via

$$X^{\alpha \beta} = Z^{[a} W^{\beta]}.$$  (2.4)

These homogeneous coordinates satisfy: $X^{[\alpha \beta} Z^{\gamma]} = 0$. For a given $X^{\alpha \beta}$ these equations are solved by all points on the line through $Z$ and $W$.

Since the $X^{\alpha \beta}$ are homogeneous coordinates, they are only sensitive to the conformal

\(^2\)Our conventions are as follows: $\omega^\dot{a}_b = \omega^{\dot{a}} \varepsilon_{\dot{a}b}$, $\pi^b_a = \pi_a \varepsilon^{ab}$. 

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structure of space-time. Conformal symmetry is broken by designating a choice of two index anti-symmetric bi-twistor, called the \textit{infinity twistor}, which we denote by $I_{\alpha\beta} = -I_{\beta\alpha}$. The ‘inverse’ bi-twistor is denoted by $I^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}I_{\gamma\delta}$. The infinity twistor allows us to raise and lower the index of the twistor coordinates $Z^\alpha$ via $Z_\alpha \equiv I_{\alpha\beta}Z^\beta$. Furthermore, it defines an anti-symmetric pairing $\langle ZW \rangle$ between two twistors $Z$ and $W$ via

$$\langle ZW \rangle = I_{\alpha\beta}Z^\alpha W^\beta$$

(2.5)

As the name suggests, the infinity twistor selects the twistor lines that map to space-time points at infinity: a pair $ZW$ describes an infinite point if $\langle ZW \rangle = 0$. Deleting the locus $\langle ZW \rangle = 0$, we can introduce affine coordinates $x^{\alpha\beta}$, and their duals $\tilde{x}_{\alpha\beta}$, via

$$x^{\alpha\beta} = \frac{Z^{[\alpha}W^{\beta]}}{\langle ZW \rangle}, \quad \tilde{x}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}x^{\gamma\delta}$$

(2.6)

Raising and lowering of a single index is accomplished via the infinity bi-twistor. These coordinates satisfy the relations $x^{\alpha\gamma}x_{\gamma\beta} = -x^{\alpha\beta}$, and $\tilde{x}_{\alpha\gamma}\tilde{x}_{\gamma\beta} = -\tilde{x}_{\alpha\beta}$. The dual coordinates $\tilde{x}_{\alpha\beta}$ act as projection matrices onto the twistor line associated to the space-time point $\hat{X}$:

$$\tilde{x}_{\alpha\beta}U^\beta = 0$$

(2.7)

which is solved by all points $U = aZ + bW$ on the twistor line through $Z$ and $W$.

The form of the infinity twistor depends on the choice of space-time signature and metric. In most of this paper, we will work in Euclidean signature. Euclidean twistor space can be obtained from $\mathbb{CP}^3$ by introducing a map $\sigma$ which acts on the coordinates $Z^\alpha$ as:

$$(Z^1, Z^2, Z^3, Z^4) \xrightarrow{\sigma} (\bar{Z}^2, -\bar{Z}^1, \bar{Z}^4, -\bar{Z}^3)$$

(2.8)

where $\bar{Z}^\alpha$ denotes the complex conjugate of $Z^\alpha$. The point $Z^\alpha$ and its image $W^\alpha = \sigma(Z^\alpha)$ define a corresponding space-time point via equation (2.4). None of the resulting twistor lines intersect. This is why $\mathbb{CP}^3$ can be described as an $S^2$ fibered over $S^4$.

The infinity twistor of the euclidean four sphere of radius $\ell$ can be chosen of the form

$$I_{\alpha\beta} = \frac{1}{\ell} \begin{pmatrix} \gamma\varepsilon_{\dot{a}\dot{b}} & 0 \\ 0 & \varepsilon^{ab} \end{pmatrix}. \quad (2.9)$$

Note that pairs $ZW$ with $\langle ZW \rangle = 0$ indeed map to points in complexified space-time with $x^{\alpha\beta} = \infty$. For $S^4$, this infinity locus does not intersect with the real four sphere. In the flat space limit, $\ell \to \infty$, the infinity twistor degenerates, and selects $\pi_a = 0$ as the twistor line associated with space-like infinity.
When $\gamma = 1$, we have a round $S^4$ with isometry group $SO(5)$. When $\gamma = 0$, we have the infinity twistor for Minkowski space. Decomposing the twistor into two component spinors

$$Z^\alpha = (\omega^\dot{\alpha}, \pi_\alpha), \quad Z_\alpha = \frac{1}{\ell} \left( \frac{\gamma \omega^\dot{\alpha}}{\pi_\alpha} \right),$$

(2.10)

the twistor inner product takes the form

$$\langle Z_1 Z_2 \rangle = \frac{1}{\ell} \langle \pi_1 \pi_2 \rangle + \gamma \ell \left[ \omega_1 \omega_2 \right],$$

(2.11)

with $\langle \ldots \rangle$ and $[\ldots]$ the spinor pairings \[2.2\]. Note that, unlike for Minkowski space, the infinity twistor \[2.9\] for non-zero $\gamma$ is invertible.

The correspondence between the two twistor line equations \[2.1\] and \[2.7\] is made as follows. The dual space-time coordinates $\tilde{x}_{\alpha\beta}$ can be parameterized with the help of five coordinates $y_A$ constrained to live on a round $S^4$ of unit radius, via

$$x_{\alpha\beta} = \left( \frac{1}{2} (1 + y_5) \epsilon^{ab}_{\alpha\beta} - i y^a_b, iy^b_a \frac{1}{2} (1 - y_5) \epsilon_{ab} \right), \quad y^A y_A = 1.$$

(2.12)

Here $y^\dot{a}_b = \frac{1}{2} y_\mu (\sigma^\mu)^{\dot{a}}_b$ with $\sigma^\mu = (\sigma^i, -i \mathbf{1})$ the usual Pauli matrices. With this parametrization, the four component twistor line equation \[2.7\] reduces to the standard two-component twistor line equation \[2.1\]

$$x^\dot{a} = \frac{2y^\dot{a}_b}{1 - y^b}.$$

(2.13)

The flat space limit amounts to zooming in on the south pole region of the $S^4$ near $y_5 \simeq -1$. In this limit, the remaining four $S^4$ coordinates $y^\mu$ become identified with the flat space coordinates $x^\mu$.

The geometry of twistor space also extends to theories with supersymmetry. Here we focus on theories with $\mathcal{N} = 4$ supersymmetry. This is achieved by supplementing the four bosonic coordinates by four fermionic coordinates $\psi^i i = 1, ..., 4$. The resulting supertwistor space $\mathbb{CP}^{3|4}$ is a Calabi-Yau supermanifold. The homogeneous coordinates of $\mathbb{CP}^{3|4}$ will be denoted as $Z^I = (Z^\alpha | \psi^i).$ Chiral Minkowski superspace is then described by coordinates $(x^\dot{a}, \theta^a_+)$.

One can also introduce coordinates on Minkowski superspace given as $(x^\dot{a}, x^\dot{a}_-, \theta^a_+, \theta^a_-)$ subject to the constraint equations:

$$x^\dot{a}_+ = x^\dot{a} \pm \theta^a_+ \theta^a_-$$

(2.14)

so that the independent coordinates on complexified superspactime $\mathbb{M}^{4|16}$ are $(x, \theta^a_+, \theta^a_-)$.
Chiral and anti-chiral superspace are then parameterized by \((x^{a\dot{a}}, \theta^{ia})\) and \((x^{a\dot{a}}, \theta^{\dot{a}i})\), respectively. Let us note that in the complexified setting, \(\theta_{+}\) and \(\theta_{-}\) constitute independent coordinates. A bosonic line in supertwistor space is specified by:

\[
\omega^{\dot{a}} = ix^{a\dot{a}}\pi_{a} \quad ; \quad \psi^{i} = \theta^{ia}\pi_{a}
\]  

which are invariant under re-scalings of the superspace coordinate \(Z^{I}\), and so define homogeneous coordinates on projective supertwistor space \(\mathbb{CP}^{3|4}\). Equations \((2.15)\) define a bosonic \(\mathbb{CP}^{1|0}\) in \(\mathbb{CP}^{3|4}\). The correspondence space is now parameterized by coordinates \((x, \theta_{-}, \pi)\) for \(\mathbb{C}^{4|8} \times \mathbb{C}^{1|0}\), with the map to \(\mathbb{CP}^{3|4}\) specified by equations \((2.15)\).

There is a natural extension of the infinity twistor to a super bi-twistor, \(\mathcal{I}_{IJ}\). In the case of \(S^{4|8}\), the super bi-twistor is:

\[
\mathcal{I}_{IJ} = \frac{1}{\ell} \begin{pmatrix}
\gamma \varepsilon_{\dot{a}b} & 0 & 0 \\
0 & \varepsilon^{ab} & 0 \\
0 & 0 & \eta_{ij}
\end{pmatrix}
\]  

where \(\eta_{ij}\) is a symmetric tensor.

### 2.2 Holomorphic Chern-Simons

Two of the most notable successes of twistor theory are the construction of anti-selfdual Yang-Mills equations, and the application of twistor techniques to the study of perturbative scattering amplitudes in \(\mathcal{N} = 4\) gauge theory. Central to both applications is the link between 4D (self-dual) gauge theory and holomorphic Chern-Simons (hCS) theory on twistor space. Following Witten, we will consider the hCS theory as the field theory of the physical modes of the B-model topological open string theory, attached to suitable twistor space-filling D-branes.

Supertwistor space \(\mathbb{CP}^{3|4}\) is a Calabi-Yau supermanifold, making it possible to introduce a topological B-model. The corresponding open string subsector defines a decoupled theory. The physical degree of freedom is a \((0, 1)\) component \(\mathcal{A}\) of a gauge connection on \(\mathbb{CP}^{3|4}\). The action is given by the holomorphic Chern-Simons form

\[
S_{hCS} = \int_{\mathbb{CP}^{3|4}} \Omega \wedge \text{tr} \left( \mathcal{A} \overline{\mathcal{A}} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right)
\]  

The holomorphic \((3|4)\) form \(\Omega\) for the Calabi-Yau space \(\mathbb{CP}^{3|4}\) is given by:

\[
\Omega = \frac{\varepsilon_{\alpha \beta \gamma \delta}}{4!} Z^{\alpha} dZ^{\beta} dZ^{\gamma} dZ^{\delta} \frac{\varepsilon_{ijkl}}{4!} d\psi^{i} d\psi^{j} d\psi^{k} d\psi^{l}.
\]
where the $Z^\alpha$ and $\psi^i$ for $\alpha = 1, \ldots, 4$ and $i = 1, \ldots, 4$ are respectively bosonic and fermionic homogeneous coordinates for $\mathbb{CP}^{3|4}$. The measure is invariant under $PSL(4|4)$.

The critical points of $S_{hCS}$ correspond to gauge bundles with vanishing $(0,2)$ curvature:

$$\mathcal{F}^{(0,2)} = \overline{\partial}A + A \wedge A = 0. \quad (2.19)$$

In other words, they are holomorphic bundles whose complex structure is compatible with the complex structure of the ambient $\mathbb{CP}^{3|4}$. Via the Ward correspondence [15] (see also [16, 17]), there is a special class of such holomorphic bundles which describe anti-self-dual instantons on $S^4$. The basic equivalence is between the following two types of mathematical objects:

- Anti-self dual Yang-Mills connections of $GL(n, \mathbb{C})$, defined on the four sphere $S^4$
- Holomorphic rank $n$ vector bundles on $\mathbb{CP}^3$, that are trivial on every twistor line

The instanton number of the gauge bundle on $S^4$ corresponds to the second Chern class of the vector bundle on $\mathbb{CP}^{3|4}$. Ward’s construction represents the analog for gauge fields of Penrose’s ‘non-linear graviton’ construction of the anti-selfdual Einstein equations [18]. For more details of the Ward correspondence, we refer to the standard literature [15, 16].

The connection between the variables of twistor space, and super Yang-Mills theory can also be seen through the expansion of $A$ in the fermionic coordinates:

$$A = a + \psi^i \chi_i + \frac{1}{2!} \psi^i \psi^j \phi_{ij} + \frac{1}{3!} \psi^i \psi^j \psi^k \eta_{ijk} \phi_{[ijk]} + \frac{1}{4!} \psi^i \psi^j \psi^k \psi^l \eta_{ijkl} b_{[ijkl]} \quad (2.20)$$

The variable $\psi$ has homogeneity $+1$. From this we conclude that $a$ is a degree zero $(0,1)$ form on bosonic twistor space, and $b$ is a degree $-4$ $(0,1)$ form. Via the correspondence between homogeneity and space-time helicity (reviewed in footnote 11), these components of the expansion are to be identified with the components of the $\mathcal{N} = 4$ vector multiplet.

Expanding out the components and performing the fermionic integrations, the holomorphic Chern-Simons action reduces to the following action defined over the bosonic twistor space $\mathbb{CP}^3$:

$$S_{hCS} = \int_{\mathbb{CP}^3} \Omega' \wedge \text{tr} \left( b \wedge (\overline{\partial}a + a \wedge a) + \phi^{ij} \wedge (\overline{\partial} + a) \phi_{ij} \right) + \text{fermionic terms.} \quad (2.21)$$

The bosonic holomorphic three-form $\Omega'$ is an element of $\Omega^{(3,0)}(\mathcal{O}(4))$, while the integrand is an element of $\Omega^{(0,3)}(\mathcal{O}(-4))$. The equation of motion for $b$ selects a choice of $a$ such that the

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To be slightly more precise, the $d\psi$ entering here is more appropriately viewed as a measure on the Berezinian in the fermionic directions [7].
(0, 2) component of the corresponding field strength vanishes. This defines a holomorphic bundle over \( \mathbb{CP}^3 \), which includes all anti-self-dual instantons on \( S^4 \). Hence the hCS theory on \( \mathbb{CP}^{3|4} \) is physically equivalent to the anti-self-dual sector of \( \mathcal{N} = 4 \) gauge theory \([7]\), provided that \( \mathcal{A} \) is restricted to be flat over each twistor line.

In the influential paper \([7]\), Witten proposed to extend this relationship to the complete \( \mathcal{N} = 4 \) gauge theory. The construction, motivated by Nair’s observation that MHV amplitudes are localized on twistor lines, involved supplementing the hCS theory with D1-instantons, wrapping holomorphic curves inside \( \mathbb{CP}^{3|4} \). In this paper, we will follow a different procedure: we will consider the pure holomorphic Chern-Simons theory, without adding any additional degrees of freedom. Rather than introducing D1-instantons by hand, we will study the perturbative dynamics around a non-trivial instanton background for the gauge field \( \mathcal{A} \) on \( \mathbb{CP}^{3|4} \), with a very large instanton number. The effect of this flux background is quite similar to adding D1-branes\(^5\), in that it leads to additional defect degrees of freedom that localize on the twistor lines. But there are important differences with Witten’s proposal. Most notably, our setup introduces a UV scale, via the size of the constituent instantons that comprise the total flux background.

Two additional comments are in order. First, note that not all holomorphic vector bundles on \( \mathbb{CP}^{3|4} \) correspond to instantons on \( S^4 \): the critical points of the hCS theory also include vector bundles which do not trivialize over twistor lines. Mathematically, such bundles leads to additional data beyond that specified in the usual ADHM construction of instantons on \( S^4 \).\(^6\) Secondly, the reduction from a string field theory to the topological B-model may also allow for certain deformations that can not be captured purely in terms of purely holomorphic data. In particular, like in type II theory, one can choose to turn on a 2-form \( B \)-field background, which deforms the string worldsheet action. The 2-form field acts like a \( U(1) \) magnetic field, under which the two open string end points are oppositely charged. Upon taking the decoupling limit, each end point is forced into lowest Landau level orbits. From the point of view of the hCS theory, turning on this 2-form \( B \)-field implements a non-commutative deformation of twistor space.

### 2.3 Twistor Quantization

Twistor space and its dual space naturally combine into a quantized phase space \([12]\). Indeed, the Heisenberg commutation relation between positions and momenta implies that

\(^5\)Indeed, lower dimensional branes embedded inside of higher dimensional branes dissolve into topologically non-trivial flux configurations of the higher dimensional gauge field.

\(^6\)See for example \([19]\) for some discussion of this extension in the math literature.
the twistors $Z^a = (\omega^a, \pi_a)$ and dual twistors $\tilde{Z}_\beta = (\tilde{\pi}_b, \tilde{\omega}^b)$ satisfy the commutation algebra:

$$\left[ Z^a, \tilde{Z}_\beta \right] = \hbar \delta^a_\beta \quad \Rightarrow \quad \begin{cases} \left[ \omega^a, \tilde{\pi}_b \right] = \hbar \delta^a_b \\ \left[ \pi_a, \tilde{\omega}^b \right] = \hbar \delta^b_a \end{cases} \quad (2.22)$$

The Hilbert space realization of this commutator algebra provides a representation of $sl(4, \mathbb{C})$, the algebra of the complexified conformal group in four dimensions, generated by respectively the translations, conformal boosts, Lorentz rotations and dilatation

$$P_{aa} = \pi_\ld{a} \pi_a \quad , \quad K_{ab} = \tilde{\omega}_a \omega_\ld{a} \quad , \quad D = \frac{1}{2} (\pi_\ld{a} \omega^a - \tilde{\omega}^a \pi_a). \quad (2.23)$$

The passage to a real space-time signature is achieved by a convention for hermitian conjugation of the $Z^a$ variables and the corresponding $\tilde{Z}_\beta$ variable, or equivalently, a choice of realization of $sl(4, \mathbb{C})$. The three signatures ($++--$), ($-+++$) and ($+++$) correspond to the realizations of $sl(4, \mathbb{C})$ respectively by $sl(4, \mathbb{R})$, $su(2, 2)$ and $su(4)$. Hermitian conjugation in the first two cases acts via

$$su(2, 2) : \quad \omega^\dagger_a = \tilde{\omega}_a, \quad \pi^\dagger_a = \pi_\ld{a} \quad \text{and} \quad \hbar \in \mathbb{R} \quad (2.24)$$

$$sl(4, \mathbb{R}) : \quad \begin{cases} \omega^\dagger_a = \omega_\ld{a}, \quad \pi^\dagger_a = \pi_a \\ \tilde{\omega}^\dagger_a = \tilde{\omega}_a, \quad \pi^\dagger_\ld{a} = \pi_\ld{a} \end{cases} \quad \text{and} \quad \hbar \in i\mathbb{R} \quad (2.25)$$

The reality condition on $\hbar$ is enforced by considering how hermitian conjugation acts on the canonical commutator relations. For euclidean signature, hermitian conjugation relates the $\pi$ and $\omega$ spinors via

$$su(4) : \quad \omega^\dagger_\ld{a} = \tilde{\pi}_\ld{a}, \quad \pi^\dagger_a = \tilde{\omega}^a \quad \text{and} \quad \hbar \in \mathbb{R}. \quad (2.26)$$

The resulting commutation relations are:

$$\left[ \omega^\dagger_a, \omega^\dagger_b \right] = \hbar \delta^a_b \quad ; \quad \left[ \pi_a, \pi^\dagger_b \right] = \hbar \delta^b_a. \quad (2.27)$$

Note that daggering an oscillator changes an upper index to a lower one and visa versa. Hermitian conjugation (2.26) acts on the conformal symmetry generators as:

$$(P_{\ld{a}a})^\dagger = K^{\ld{a}a}, \quad (K_{\ld{a}a})^\dagger = P^{\ld{a}a}, \quad (J_{ab})^\dagger = J^{ab}, \quad (\tilde{J}_{\ld{a}\ld{b}})^\dagger = \tilde{J}^{\ld{a}\ld{b}}, \quad D^\dagger = -D \quad (2.28)$$

The above reality conventions may be deformed by including relative complex phases. This provides a means to analytically continue from one space-time signature to another.
The Euclidean theory is therefore most naturally viewed as a radially quantized theory on an $S^4$. Working with respect to this realization, note that in $su(4) \simeq so(6)$ the infinity bitwistor (2.9) transforms as a vector under $so(6)$. The generators which leave it invariant span a $usp(4) \simeq so(5)$ subalgebra, which we identify with the symmetries of the four sphere. The Hermitian $so(5)$ generators corresponding to motion along a great circle are:

$$P_{\dot{a}a} = P_{\dot{a}a} + K_{\dot{a}a}.$$  \hspace{1cm} (2.29)

Starting from $S^4$, we can take the flat space limit by sending the radius of the sphere to infinity and zooming in on a local flat region of the sphere. This procedure is equivalent to performing a Wigner-In"on"u contraction, and amounts to rescaling the dual oscillators as:

$$\omega_{\dot{a}} = \gamma^{-1} \tilde{\pi}_{\dot{a}}, \pi_{a} = \gamma \tilde{\omega}_{a}$$  \hspace{1cm} (2.30)

where the parameter $\gamma$, the same one that appears in the $S^4$ bitwistor (2.9), is being sent to zero. In this limit, the $so(4)$ subalgebra spanned by $J$ and $\tilde{J}$ stays fixed, while the Hermitian translation generators become $P + \gamma^2 K$. The contraction to the Euclidean Poincaré algebra then proceeds via $\gamma \to 0$. See figure 2 for a depiction of the flat space limit.

The extension to space-time supersymmetric theories is achieved by including four fermionic oscillators $\psi_i$ with $\{\psi^i, \tilde{\psi}_j\} = \hbar \delta_{ij}$, which then provides a representation of $psl(4|4)$. In what follows we shall often leave implicit the extension to the fermionic case. The supersymmetric hermitian charges are the symmetries of the supersphere $S^{4|8}$.

Finally, we emphasize that the above discussion follows the standard viewpoint of twistor...
quantization [12]. In particular, up to this point, the introduction of quantized twistor space does not lead to any a priori space-time non-commutativity. A key feature of twistor theory is that space time is a derived notion, obtained via the correspondence at the level of holomorphic geometry. Since (for any of the three possible space-time signatures) the holomorphic twistor coordinates $Z^{\alpha} = (\omega^{a}, \pi_{a})$ all commute among each other, the notions of twistor lines and space-time points are essentially unaffected by the quantum relation with the dual twistors. Hence in most applications of twistor methods – for example, the construction of instantons and the study of scattering amplitudes – one does not need take into account the quantum nature of the twistor coordinates.

3 Instantons and Fuzzy Twistors

In the following sections, we will introduce a twistor matrix model, which in a large $N$ scaling limit reproduces the 4D continuum theory. The matrix model is to be viewed as the low energy limit of a more UV complete theory given by ordinary holomorphic Chern-Simons theory on ordinary supertwistor space. By expanding around a flux background which retains the symmetries of the 4D space-time, we obtain a low energy effective theory. Motivated, in part, by the correspondence with the 4D Quantum Hall effect studied in [9], the gauge field that we will turn on is the twistor lift of the Yang monopole [20,21].

The Yang monopole is a non-abelian generalization of the Dirac monopole to an $SU(2)$ gauge theory defined in five dimensions [20,21]. The monopole sits at the origin of a 5D space, surrounded by a four sphere $S^4$. On the $S^4$, this leads to an $so(5)$ homogeneous instanton density. In addition to the parameters specifying an ordinary instanton, the configuration space of the Yang monopole contains an $S^2$. This $S^2$ is the space of possible identifications between the gauge theory $SU(2)$ and a space-time $SU(2) \subset SO(5)$ of the isometries of the $S^4$. A particle charged under this background flux experiences the local geometry $S^4 \times S^2$ geometry. Globally, this $S^2$ fibers non-trivially over the $S^4$, so that for a single instanton, we obtain a three dimensional complex projective space $\mathbb{C}\mathbb{P}^3$. This construction generalizes to $SU(n)$ gauge theory via the principal embedding of the spin $N/2$ representation of $SU(2)$ in the fundamental of $SU(n)$ where $n = N + 1$. Via the Ward correspondence, the twistor lift of the rank $n$ bundle on $S^4$, combined with the abelian flux, realizes a $U(1) \times SU(n)$ gauge bundle $A_Y$ on $\mathbb{C}\mathbb{P}^{3|4}$.

Proceeding in the other direction, we can start form the hCS theory with gauge group $G = U(nN_c)$. Turning on a $U(1) \times SU(n)$ gauge field $A_Y$ then breaks the gauge group $G$ to

\footnote{See [22] for a discussion of the relation between the 4D QHE and twistor geometry, [23] for an embedding of the 4D QHE in string theory and [24] for a discussion of the connection between the QHE effect on $S^4$ and $\mathbb{C}\mathbb{P}^3$. For the relation between the lowest Landau level and twistor geometry, see also [25].}
U(N_c). The low energy limit is a U(N_c) hCS theory, defined on fuzzy twistor space, coupled to fundamental matter localized on the twistor lines. From the open string perspective, the non-commutativity arises because the open string end points are charged under the background flux. The presence of the flux introduces an explicit choice of scale, and the open string spectrum is gapped with excitations of energy inversely proportional to this UV length scale [8]. The gapped excitations decouple at large N and in the low energy limit, the system is forced to lie in the lowest Landau level.

Our aim in this section will be to study the effective geometry experience by particles moving in the lowest Landau level. We begin by reviewing the topology of the Yang monopole configuration.

### 3.1 Yang Monopole

The Yang monopole is a 5 dimensional non-abelian generalization of the Dirac monopole. The basic example consists of an abelian flux extended over the full twistor space \( \mathbb{CP}^3 \), correlated with a homogeneous \( SU(2) \) instanton configuration along the four sphere \( S^4 \). The bundle data is equivalently characterized in terms of the two Hopf fibrations \( S^7 \to \mathbb{CP}^3 \) and \( S^7 \to S^4 \) with fibers \( S^1 = U(1) \) and \( S^3 = SU(2) \), respectively:

\[
\begin{array}{ccc}
U(1) & \to & S^7 \\
\downarrow & & \downarrow \\
\mathbb{CP}^3 & \to & S^4 \\
\end{array}
\]  

(3.1)

The basic Yang monopole configuration is given by the original Hopf fibration, which carries a single unit of flux. In this case, the abelian part of the monopole is given by the line bundle \( O(1) \to \mathbb{CP}^3 \), while the non-abelian part is given by a single \( SO(5) \) homogeneous instanton of \( SU(2) \) gauge theory. We refer to the fibrations with fiber \( S^1 \) and \( S^3 \) as respectively the abelian and non-abelian ‘parts’ of the Yang monopole.

A concise way to present both contributions is given by working in terms of quaternionic coordinates. Given complex coordinates \( Z^a = \text{Re}(Z_a) + i \text{Im}(Z_a) \) for \( \mathbb{C}^4 \), the quaternionic plane \( \mathbb{H}^2 \) can be parameterized as \( Q_1 = Z^1 + Z^2 j \) and \( Q_2 = Z^3 + Z^4 j \), where \( i, j \) and \( k \) denote the usual quaternionic generators. Modding out by \( \mathbb{C}^* \) on \( \mathbb{C}^4 \) realizes complex projective three-space \( \mathbb{CP}^3 \), while modding out by the left action of the non-zero quaternions on \( \mathbb{H}^2 \) realizes the quaternionic projective line \( \mathbb{HP}^1 \simeq S^4 \). In terms of the \( Q \) coordinates, the abelian and non-abelian parts of the Yang monopole can now be written as [26]:

\[
A_Y = \frac{1}{2} \left( dQ_a^\dagger Q_a - Q_a^\dagger dQ_a \right) .
\]  

(3.2)

This represents a \( U(1) \times SU(2) \) gauge connection via the identification of quaternionic
generators with Lie algebra elements of $SU(2)$. More general choices of flux can be accommodated via the principal embedding of $SU(2) \to SU(n)$, defined by identifying the spin $N/2 = (n - 1)/2$ representation of $SU(2)$ with the fundamental representation of $SU(n)$:

$$1 \to N1_{n \times n}, \ i \to -2i1, \ j \to -2i2, \ k \to -2i3,$$  

(3.3)

where $I_k$ denote the spin $N/2$ representation of the $SU(2)$ algebra. Here and in the following, the integers $N$ and $n$ are related via $N = n + 1$. The abelian contribution to the Yang monopole defines a $U(1)$ gauge connection

$$A_{U(1)} = \frac{N}{2} (dZ_\alpha^1 Z^\alpha - Z_\alpha^1 dZ^\alpha),$$  

(3.4)

with curvature equal to $N$ times the standard Kähler form on $\mathbb{CP}^3$. The non-abelian contribution defines an $SU(n)$ gauge connection over $\mathbb{HP}^1 \simeq S^4$, which via the Ward correspondence lifts to an $so(5)$ symmetric rank $n$ holomorphic vector bundle over $\mathbb{CP}^3$.

In the following, the Yang monopole will refer to a gauge connection on twistor space, given by the sum of the abelian flux (3.4) and the twistor lift of the $SU(n)$ connection. The instanton number of the resulting $SU(n)$ gauge field is maximized by choosing the irreducible representation, giving an $so(5)$ homogeneous instanton with instanton number:

$$k_{\text{inst}} = \frac{n(n^2 - 1)}{6}. $$  

(3.5)

A somewhat more explicit description of the Yang monopole, which also brings out its natural relation to the twistor geometry of the four sphere, is as follows [27]. We can parametrize the seven sphere $S^7$ by means of a four-component complex vector $Z^\alpha$ (which we can view as a spinor of $so(5)$), subject to the constraint

$$Z_\alpha^1 Z^\alpha = \hbar N,$$  

(3.6)

where for later convenience, we chose the radius of the $S^7$ equal to $\sqrt{\hbar N}$. For now, however, $Z$ and $Z^\dagger$ are just classical coordinates on $S^7$. The $U(1)$ action in the first Hopf fibration $S^7 \to \mathbb{CP}^3$ in (3.1) represents the phase rotation

$$(Z, Z^\dagger) \to (e^{i\phi} Z, e^{-i\phi} Z^\dagger).$$  

(3.7)

Similarly, we can parametrize the four sphere $S^4$ with the help of a five-component real vector $y_A$ (which we can view as a vector of $so(5)$), satisfying

$$y^A y_A = \ell^2.$$  

(3.8)
We would like to find a parametrization of $Z_\alpha$ in terms of the geometric data of the second Hopf fibration $S^7 \to S^4$ in (3.1). In fact, we have already found this parametrization: it is given by the twistor correspondence for $S^4$ summarized in section 2.1. Concretely, we can package the $S^4$ coordinates $y^A$ into a $4 \times 4$ matrix $\tilde{x}_{\alpha}^\beta$ as in (2.6), and impose that $\tilde{x}_{\alpha\beta}$ and $Z$ satisfy the twistor line equation (2.7). Indeed, in (5) spinor notation (and setting $\ell = 1$), the parametrization (2.12) amounts to setting $\tilde{x}_{\alpha}^\beta = \frac{1}{2}(1 - \Gamma_A y^A)$, with $\Gamma_A$ the $so(5)$ gamma matrices. The twistor line equation (2.7) then takes the standard form of the Hopf equation: $Z + y^A \Gamma_A Z = 0$. This equation is explicitly solved, up to an overall phase, via

$$Z^\alpha = \frac{1}{\sqrt{\ell (\ell + y_5)}} \begin{pmatrix} -iy_\mu \sigma^\mu (u_1) \\ (\ell + y_5) (u_1) \end{pmatrix} ; \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sqrt{\frac{\hbar N}{1 + n_3}} \begin{pmatrix} 1 + n_3 \\ n_1 + i n_2 \end{pmatrix}$$ (3.9)

where the $n_i$ are normalized unit vectors on $\mathbb{R}^3$, defining a unit two-sphere $n_i n^i = 1$. They parametrize the twistor line above the given point $y^A$ on the four sphere. Combined, the $y^A$ and $n_i$ represent a complete coordinate system on $\mathbb{CP}^3$.

Following [27], we may now characterize the abelian part of the Yang monopole (3.4) as the Berry connection of the spinor $Z^\alpha$ defined over $\mathbb{CP}^3$. Since the associated holonomy is just a phase, the Berry phase Lagrangian for the spinor coordinates is

$$\frac{1}{\hbar} L = \frac{i}{\hbar} Z^\alpha_\alpha = -N \left( \frac{\varepsilon_{\delta i j} n_i \dot{\hat{n}}_j}{1 + n_3} + \eta^i_{\mu \nu} n_i y^\mu \dot{y}^\nu \right)$$ (3.10)

where $\eta^i_{\mu \nu} = \epsilon_{i\mu\nu4} + \delta_{i\mu} \delta_{\nu4} - \delta_{i\nu} \delta_{\mu4}$ denotes the ’t Hooft symbol. Using this, we can reconstruct a gauge field flux on both the $S^4$ direction, and along the $S^2$ fiber:

$$A_\mu = N \frac{\eta^i_{\mu \nu} n_i \dot{y}^\nu}{\ell (\ell + y_5)} ; A_5 = 0, \quad A_i = N \frac{\varepsilon_{\delta i j} n_i}{1 + n_3}$$ (3.11)

which defines the abelian Yang monopole connection on $\mathbb{CP}^3$.

Both terms in (3.11) have a familiar form. The $S^2$ part $A_i$ represents the constant magnetic flux through a unit two sphere, produced by a Dirac monopole of charge $N$ at its center. It arises because the two Hopf fibrations in eqn (3.1) are related to each other via the basic Hopf fibration $S^3 \to S^2$ that embeds $U(1)$ inside $SU(2)$. Similarly, the $S^4$ part $A_\mu$ looks like the abelianized twistor lift of the basic $SU(2)$ instanton – or rather its embedding via the spin $N/2$ representation inside $SU(n)$ – via the identification

$$n_i \leftrightarrow \frac{2i}{N} \mathbf{I}_i$$ (3.12)
Indeed, making this substitution inside of the $S^4$ part $A_\mu$ in (3.11) gives a maximal homogeneous $SU(n)$ instanton on $S^4$. Again we see that the two components of the monopole, the abelian flux (3.4) and the non-abelian homogeneous instanton background on $S^4$, are directly linked via the substitution (3.12) of the $S^2$ coordinate with an $SU(2)$ generator embedded in $SU(n)$. The complete rank $n$ Yang monopole $A_Y$ is the sum of the abelian magnetic field (3.11) with the twistor lift of the homogeneous $SU(n)$ instanton.

3.2 Lowest Landau Level

Let us look a little more closely at the effective geometry experienced charged particles in the abelian flux background. At low energy, the particles are forced into their lowest Landau level (LLL). This system was analyzed in detail in [9, 27]. The LLL states are most succinctly characterized as the Hilbert state obtained by canonical quantization of the abelian Berry holonomy Lagrangian (3.10).

We can do this in two different ways. We can first quantize the $Z_\alpha$ coordinates, and then impose the constraint (3.6) or we can first solve the constraint, as done via the $S^4 \times S^2$ parametrization (3.9). Let us first follow the second route. From the second form of the Berry action in (3.10), we can read off the commutation relations among the $n_i$ and $y_\mu$, by following the usual rules of canonical quantization. One finds coordinates which satisfy the following commutator algebra [9, 27]

$$\begin{align*}
[n_i, n_j] &= \frac{2}{N} \epsilon_{ijk} n_k, \\
[y_\mu, y_\nu] &= \frac{\ell^2}{2N} \eta^{ij}_{\mu\nu} n_i, \\
[n_i, y_\mu] &= \frac{2}{N} \eta_{i\mu\nu} y^\nu.
\end{align*}$$

(3.13)

Rescaling the $n_i$ coordinates that parameterize the two sphere, we see that they turn into generators of an $su(2)$ algebra that rotates the $S^2$. Since the rescaled $S^2$ has radius $N$, we learn that the $su(2)$ acts via a spin $N/2$ representation. The $S^2$ has turned into a non-commutative sphere with $n = N + 1$ fuzzy points. Moreover, we see that the $su(2)$ algebra does not commute with the position coordinates $y^\mu$, but instead act like the generators of chiral space-time rotations. The space-time coordinates $y_\mu$ coordinates also do not commute among each other: their commutator is a generator of a chiral $su(2)$ rotation. We will refer to the non-abelian commutator algebra (3.13) later on, when we begin our study of the low energy physics of the hCS theory with flux.

The above discussion also clarifies the mapping (3.12) between the abelian and non-abelian flux: it identifies the spin $N/2$ representation of chiral $su(2)$ rotations with $su(2)$ gauge generators embedded inside an $SU(n)$ gauge group. The replacement (3.12) results in a homogeneous, $so(5)$ invariant instanton configuration, by virtue of the fact that we can combine the chiral $su(2)$ space-time rotations with global $su(2)$ rotations, that acts on the Lie algebra labels of the $SU(n)$ gauge field. Indeed one can show that the total space
spanned by the Landau level wave functions preserves an overall so(5) symmetry, and that moreover, the lowest Landau level transforms as a representation of su(4) \[9\].

We now turn to a more practical description of the LLL states: we first quantize the \(Z_\alpha\) coordinates, and then afterwards impose the constraint (3.6). This procedure identifies the LLL states with the space of Planck cells, or fuzzy points, on non-commutative \(\mathbb{CP}^3\). The su(4) symmetry of the LLL level is manifest in this description\[9\].

From the first form of the Berry Lagrangian (3.10) we immediately read off that the four coordinates \(Z^\alpha\) act as bosonic oscillators with commutators:

\[
[Z^\alpha, Z^\beta_\dagger] = \hbar \delta^\alpha_\beta. \tag{3.14}
\]

These generate a Fock space obtained by acting with the \(Z^\dagger_\beta\) oscillators on the vacuum state \(|0\rangle\) annihilated by the \(Z^\alpha\) oscillators. Each basis state in the Fock space represents one Planck cell of the non-commutative space \(\mathbb{C}^4\), Since this space is non-compact, the associated Hilbert space \(\mathcal{H}_{\mathbb{C}^4}\) is infinite dimensional.

Now let us impose the constraint (3.6). At the quantum level, this will automatically lead to a projection from \(\mathbb{C}^4\) onto the complex projective projective space \(\mathbb{CP}^3\). Introduce the level operator \(H_0 = Z^\dagger_\alpha Z^\alpha\). This operator has an integer spectrum, given by the sum of the oscillator levels of the \(Z^\dagger\) number eigenstates. The constraint (3.6) is now imposed at the level of the Hilbert states

\[
H_0 |\Psi\rangle = \hbar N |\Psi\rangle \quad ; \quad H_0 = Z^\dagger_\alpha Z^\alpha. \tag{3.15}
\]

We denote by \(\mathcal{H}_{\mathbb{CP}^3}(N)\) the Hilbert space of states that satisfies this condition. At a geometric level, states in \(\mathcal{H}_{\mathbb{CP}^3}(N)\) represent holomorphic sections of the degree \(N\) line bundle \(\mathcal{O}_{\mathbb{CP}^3}(N)\). Note that the level constraint \(H_0 = N\) indeed eliminates one complex dimension: it fixes the absolute value of \(Z^\alpha\) but also implements the \(U(1)\) invariance under phase rotations \(Z^\alpha \rightarrow e^{i\phi} Z^\alpha\). Eqn (3.15) is the non-commutative way of realizing \(\mathbb{CP}^3\) as the Kähler quotient \(\mathbb{C}^4//U(1)\). Observe that the states of \(\mathcal{H}_{\mathbb{CP}^3}(N)\) are created by homogeneous degree \(N\) polynomials in the \(Z^\dagger\)s. We can immediately count that

\[
\dim \mathcal{H}_{\mathbb{CP}^3}(N) = \frac{(N+1)(N+2)(N+3)}{6} \equiv k_N. \tag{3.16}
\]

This formula counts the number of Planck cells that fit inside the compact space \(\mathbb{CP}^3\), or equivalently, the relative inverse volume of a fuzzy twistor point. We will focus on the leading behavior in the limit of large \(N\). In this limit, the size of the Planck cells, \textit{i.e.} the scale of non-commutativity, tends to zero relative to the total size of the projective space.

\[9\]The approach to non-commutative geometry we consider has been developed in [28], and we refer the interested reader there for additional details (see also [29], [30]).
The supersymmetric generalization of the story is straightforward. In addition to the bosonic oscillators, introduce four fermionic oscillators $\psi^i$ satisfying:

$$\{\psi^i, \psi^j\} = \hbar \delta^i_j.$$ (3.17)

The Hilbert space of points for fuzzy $\mathbb{CP}^{3|4}$ is given by the Fock space of states $\mathcal{H}_{\mathbb{C}P^{3|4}}$ generated by $Z^i_j$. The restriction to fuzzy $\mathbb{CP}^{3|4}$ is achieved by introducing the Hamiltonian constraint:

$$H_0 = Z^i_\alpha Z^\alpha_i + \psi^i_\alpha \psi^\alpha_i.$$ (3.18)

### 3.3 Space-Time and Locality

One of the key features of twistor theory is that space-time physics is a derived notion, obtained via the correspondence between complex lines in twistor space and space-time points. Since this is a correspondence at the level of the holomorphic geometry, it is essentially left intact by the non-commutativity.

Let us first discuss holomorphic subspaces in $\mathbb{CP}^3$. In commutative geometry, a holomorphic divisor $S \subset \mathbb{CP}^3$ is specified by the vanishing locus of a degree $d$ holomorphic polynomial $f(Z^\alpha)$. In the fuzzy setting, the Hilbert space of points for $S$ is given by states of $\mathcal{H}_{\mathbb{C}P^3}(N)$ annihilated by $f(Z^\alpha)$. Let us note that generically this space is non-empty, since $f : \mathcal{H}_{\mathbb{C}P^3}(N) \to \mathcal{H}_{\mathbb{C}P^3}(N - d)$ is a linear map to a vector space of smaller dimension. Intersections of divisors proceed in a similar fashion. Given two holomorphic polynomials $f_1$ and $f_2$ of respective degrees $d_1$ and $d_2$, the space of states annihilated by both $f_1$ and $f_2$ defines the Hilbert space of points for a fuzzy curve in $\mathbb{CP}^3$. Finally, given polynomials $f_1$, $f_2$, and $f_3$, we generically obtain a discrete collection of fuzzy points. See [28] for further discussion of intersection theory on fuzzy spaces.

To any point $p$ of commutative twistor space, we can associate a corresponding Hilbert space $\mathcal{H}_p$. To see this, note that $p$ can be viewed as the intersection of three linear divisors, $f^{(i)}_\alpha Z^\alpha = 0$ for $i = 1, 2, 3$. The space of states annihilated by all three polynomials defines a one-dimensional Hilbert space. This also provides a fiber bundle over the commutative $\mathbb{CP}^3$ which we can identify with the abelian part of the Yang monopole.

Especially important for the twistor correspondence is that for any given point $(x^a, \theta^i)$ on the commutative complexified Minkowski space, there is a corresponding fuzzy $\mathbb{C}P^4(x, \theta)$, with an $n = N + 1$ dimensional Hilbert space $\mathcal{H}_{x, \theta}(N)$ spanned by all states $|x, \theta\rangle$ that satisfy the annihilator equations:

$$(\omega^a - i x^a \pi_a) |x, \theta\rangle = 0 ; \quad (\psi^i - \theta^i \pi_a) |x, \theta\rangle = 0$$ (3.19)
These equations implement the commutative twistor line equations (2.15) as a linear projection on the Hilbert space $\mathcal{H}_{x,\theta}(N)$. This will be a key point of our further discussion: although twistor space has become non-commutative, there is still a continuous moduli space of twistor lines. Similarly we can define a Hilbert space of dual bra states $\mathcal{H}^{\vee}_{x,\theta}$ via:

$$\langle x, \theta | (\omega_{\dot{a}} - ix_{\dot{a}a} \pi^{a\dot{a}}) = 0 ; \langle x, \theta | (\psi^i - \theta_{ia} \pi^{a\dot{a}}) = 0 .$$

(3.20)

The index $i$ of $\theta_{ia}$ has been lowered using the infinity twistor $I^I_J$. We can thus speak of a space of ket states $\mathcal{H}_{x,\theta}$ and bra states $\mathcal{H}^{\vee}_{x,\theta}$ located at any given space-time point $(x, \theta)$.

Given a specific twistor location $(\omega_{\dot{a}}, \pi_a, \psi^i) = (x^{\dot{a}b} \pi_b, \pi_a, \theta_{ia} \pi^{a\dot{a}})$ on the twistor line at $(x, \theta)$, we can associate an element of $\mathcal{H}_{x,\theta}$ via the condition:

$$(\pi^2 - \lambda \pi^1) |x, \theta; \lambda \rangle = 0 .$$

(3.21)

Here $\lambda = \pi_2/\pi_1$ denotes the affine coordinate on the $\mathbb{CP}^1$. When combined, the conditions (3.19) and (3.21) select a one dimensional Hilbert space within $\mathcal{H}_{\mathbb{CP}^3|4}(N)$.

On the finite radius $S^4$, there is a closely related way to define position states $|y, \theta; \lambda \rangle$, where $y^A = (y^\mu, y^5)$ denote the $S^4$ coordinates introduced in eqn (2.12), as follows. Begin with a state on the $\mathbb{CP}^1$ at the origin where $y_\mu = 0$ and $y_5 = 1$. This state $|0, 0; \lambda \rangle$ is annihilated by $\omega^\dot{a}$, $\psi^i$ and $\pi_2 - \lambda \pi_1$. Next, we introduce a finite symmetry transformation of the supersphere $S^{4|8}$, which we denote by $\mathcal{R} (y, \theta)$, which maps the origin to the point $(y, \theta)$. Its bosonic part is an $SO(5)$ rotation. We can thus obtain general position eigenstates by applying the symmetry transformation

$$|x, \theta; \lambda \rangle = \mathcal{R} (x, \theta) |0, 0; \lambda \rangle .$$

(3.22)

This defines a unitary parallel transport operation. In other words, each point $(y, \theta)$ on the sphere $S^{4|8}$ now comes equipped with an $n$ dimensional linear space $\mathcal{H}_{y,\theta}$ which via (3.22) turns into an $su(n)$ bundle over $S^{4|8}$, or more specifically, the spin $N/2$ lift of an $su(2)$ bundle over $S^4$. This is the gauge bundle for the non-abelian part of the Yang monopole.

Since the Hilbert space $\mathcal{H}_{\mathbb{CP}^3|4}(N)$ is finite dimensional, it is evident that the states associated with different space-time points can not all be independent. Rather, we should expect that states at nearby space-time points have a non-zero overlap. The short distance scale $\ell_{pl}$, where locality breaks down, is determined by the overlap between two neighboring states $|y, \theta; \lambda \rangle$ and $|y', \theta; \lambda \rangle$. A simple calculation, outlined in [118], shows that for $y$ and $y'$ close to each other (and close to the south pole where $y_\mu \ll y_5$)

$$(y', \theta, \lambda | y, \theta; \lambda \rangle = \exp \left( - \frac{N}{8\ell_{pl}^2} |y - y'|^2 \right) \times (0, \theta, \lambda |0, \theta; \lambda \rangle .$$

(3.23)
In the large \( N \) limit, the prefactor approaches a regulated delta function, smeared out over a small region on \( S^4 \) of linear size

\[
\ell_{pl} \simeq \frac{\ell}{\sqrt{N}}.
\]  

(3.24)

As suggested by our notation, this will play the role of the Planck length.

An equivalent way to see the breakdown of locality is by considering the range of allowed angular momenta of functions on \( S^4 \). To this end, we introduce a holomorphic position operator \( \hat{X}_{\alpha\beta} \) which acts on a matrix \( \Phi \) as:

\[
\hat{X}_{\alpha\beta} \Phi = Z_\alpha \Phi Z_\beta^\dagger - Z_\beta \Phi Z_\alpha^\dagger.
\]  

(3.25)

This definition is motivated by the commutative relation (2.4) between the space-time coordinates \( X_{\alpha\beta} \) and pairs of twistors \( Z_\alpha \) and \( W_\beta \). The operator \( \hat{X}_{\alpha\beta} \) indeed selects a space-time point, as follows. It is not hard to show that the eigen operators, defined via

\[
\hat{X}_{\alpha\beta} \Phi(x) = x_{\alpha\beta} \hat{X}_0 \Phi(x)
\]  

(3.26)

where \( \hat{X}_0 = I^{\alpha\beta} \hat{X}_{\alpha\beta} \) and \( x_{\alpha\beta} \) is a c-number, are annihilated from the left and right by the associated twistor line conditions (3.19) and (3.20). This suggests that we can construct a space of functions associated with the space-time as follows.

Introduce the hermitian conjugate coordinates \( \hat{X}^*_{\alpha\beta} \Phi = Z_{[\alpha} \Phi Z_{\beta]} \). Starting from the vacuum state \( \vert 0 \rangle \langle 0 \vert \), we can repeatedly act via \( \hat{X}^*_{\alpha\beta} \). After acting \( N \) times, we obtain a subset of operators on \( H_{\mathbb{C}P^3|4}(N) \), which we denote by \( H_{S^4}(N) \). These matrices form an irreducible representation of \( su(4) \), given by a Young tableau with two rows of \( N \) boxes. The dimension of this irreducible representation is:

\[
\dim H_{S^4}(N) = \frac{1}{12} (N + 3) (N + 2)^2 (N + 1) = \frac{1}{2} k_N (n + 1)
\]  

(3.27)

which is the number of independent functions on \( S^4 \) up to a maximal angular momentum. We can further decompose this into representations of \( so(5) \), which are labelled by Young tableaux with two rows of lengths \( r_1 \) and \( r_2 \leq r_1 \), with dimension (see e.g. [31])

\[
D(r_1, r_2) = \frac{1}{6} (r_1 + r_2 + 2)(r_1 - r_2 + 1)(3 + 2r_1)(1 + 2r_2).
\]  

The functions on a \( S^4 \) up to a cutoff angular momentum \( N \) are given by the direct sum of the \( r \)-fold symmetric product on the vector representation, i.e. representations with \( r_2 = 0 \):

\[
H_{S^4} = \bigoplus_{r=0}^N D(r, 0).
\]  

(3.28)

\(^{10}\)Note that LLL states transform in the \( so(5) \) irrep with \( r_1 = r_2 = N/2 \), with dimension \( k_N \).
In \cite{31,32} this space of states was interpreted as the space of functions for a fuzzy $S^4$. Here, we are simply considering a theory on an $S^4$ where we truncate the spherical harmonics to a maximal angular momentum. Since the space of functions scales as $N^4$, the number of pixels which can be reconstructed is of order $\sqrt{N^4}$. The minimal length scale which can be resolved by this angular momentum cutoff is $\ell_{pl}$ of equation (3.24).

4 Twistor Matrix Model

After collecting the main geometric ingredients, we are finally ready to introduce the physical system of interest. In this section we wish to extract the effective matrix model that captures the low energy dynamics of holomorphic Chern-Simons theory in the background of the Yang monopole flux. The theory is formulated in terms of finite size matrices, and so in particular describes a theory on space-time with a truncated number of degrees of freedom. See \cite{33} for earlier discussion on a potential matrix model characterization of twistor string theory.

4.1 Chern-Simons with Flux

We wish to study the perturbative dynamics of a $U(nN_c)$ holomorphic Chern-Simons theory in the Yang monopole background. As explained, this monopole has two components: an abelian flux given by (3.4), and a non-abelian flux given by the twistor lift of a homogeneous $SU(n)$ instanton with maximal instanton charge $k_N$. Both components have an important influence on the low energy effective theory. We first discuss the abelian flux.

The proper way to view the abelian flux is through the lens of the topological B-model. In any ordinary Chern-Simons gauge theory, all fields transform in the adjoint of the gauge group and the overall $U(1)$ gauge group factor decouples. Hence turning on an abelian flux would not have any effect on the low energy dynamics. Instead, we will consider the holomorphic Chern-Simons theory as defined via an open topological string theory, or equivalently, by taking the zero slope limit of the open twistor string theory introduced and studied in \cite{34,35}. In the open string theory one can choose to turn on a $B$-field background, which acts like an abelian magnetic field, under which the open string end points are oppositely charged. In the zero slope limit, each end point is forced into a lowest Landau level orbit, or equivalently, is compelled to occupy a state in $\mathcal{H}_{\mathbb{C}P^{3/4}}$.

This gives us a first hint of what the low energy theory should look like: by turning on the abelian flux, we have deformed the original $U(nN_c)$ holomorphic Chern-Simons theory into a non-commutative $U(nN_c)$ hCS theory defined on twistor space $\mathbb{C}P^{3/4}$, deformed via commutation relations (3.14). What does this deformation correspond to in space-time? Via the appropriate generalization of the Penrose-Atiyah-Ward correspondence, classical
solutions to the non-commutative hCS theory, that is, holomorphic $U(nN_c)$ gauge bundles with $F^{(0,2)} = 0$, correspond to instanton backgrounds on a fuzzy $S^4$. This fuzzy four sphere is characterized by the commutator algebra (3.13), or equivalently, by the collection of fuzzy twistor lines introduced in the previous section. The most symmetric instanton background is the non-abelian Yang monopole with rank $n$ and maximal instanton number $k_N$.

What are the low energy consequences of turning on the non-abelian flux? A first obvious consequence is that the unbroken low energy gauge group is reduced from $U(nN_c)$ to $U(N_c)$. So we should expect the low energy theory to contain a non-commutative hCS sector with gauge group $U(N_c)$. But we should also look for other low energy remnants of the instantons. From a D-brane perspective, adding an instanton background amounts to adding $k_N$ co-dimension four D-branes, one brane per constituent instanton. We will refer to the elementary instantons, or co-dimension four D-branes, as defects. Based on our experience with intersecting branes, we should thus anticipate the existence of additional defect degrees of freedom associated to the non-abelian instantons. An alternative and possibly more direct explanation for the existence of defect modes comes from considering the ADHM description of instantons.

One guiding principle for fixing the form of the matrix model is that the low energy theory should be compatible with the symmetries preserved by the Yang monopole. Since this configuration comes with a choice of scale, specified by the instanton density, we should expect conformal symmetry to be broken. However, we do require the bosonic $SO(5)$ symmetry to be manifest. Moreover, because of the close link between hCS theory and instantons on $S^4$, we expect that, in an appropriate limit, the configuration space of the matrix model should reproduce the ADHM construction of instantons.

This ADHM correspondence provides important guidance, so let us make the connection a bit more precise. Since the Yang monopole breaks conformal symmetry, the low energy theory is aware of the radial size $\ell$ of the $S^4$. There is only one scale because the instanton density on $S^4$ is homogeneous. However, if we give up $SO(5)$ symmetry, we can separate the scale defined by the abelian flux from the scale defined by the non-abelian flux.

In fuzzy twistor terms, this is done as follows. The LLL states admit the action of an $su(4)$ algebra, generated by the hermitian charges $J$, $\tilde{J}$, $P + K$, $i(P - K)$ and $iD$. The hermiticity convention $P^\dagger = K$ depends on the strength of the abelian flux. The $su(4)$ symmetry is broken to $so(5)$ via the introduction of the infinity twistor (2.9), provided we make the $so(5)$ invariant choice $\gamma = 1$. As we will see, the appearance infinity twistor is linked to the presence of the non-abelian flux. Taking $\gamma \neq 1$ breaks $so(6)$ further down to $so(4)$. These infinity twistors are associated with $SU(n)$ instanton backgrounds that are homogeneous relative to a four sphere of a different radius, i.e. with non-hermitian symmetry generators $P + \gamma K$ relative to the inner product set by the abelian flux. Taking the limit $\gamma \to 0$ corresponds to non-abelian instanton configurations which are centered.
around the south pole of the $S^4$. In this limit the abelian flux gets diluted, and we expect to make contact with the ADHM construction of $SU(n)$ instantons. This check is performed in section 5.

4.2 The Matrix Model

Let us write the matrix model action. It consists of two terms. The first term is the non-commutative holomorphic Chern-Simons action and the second term is the defect action

\[ S_{\text{MM}} = S_{\text{hCS}}(A) + S_{\text{defect}}(Q, \bar{Q}, A). \tag{4.1} \]

\[ S_{\text{hCS}}(A) = \text{Tr} \left( \Omega_{\alpha\beta\gamma} D^\alpha D^\beta D^\gamma \right) \psi^4 \tag{4.2} \]

\[ S_{\text{defect}}(Q, \bar{Q}, A) = \text{Tr} \left( \mathcal{I}_{IJ} \bar{Q} \mathcal{D}^I Q Z^J \right) \tag{4.3} \]

where $\Omega_{\alpha\beta\gamma}$ is a three index anti-symmetric tensor which is the non-commutative analogue of the holomorphic three-form. Here the symbol $\text{Tr}$ is the trace over the Hilbert space $\mathcal{H}_{CP^{3|4}}(N)$ tensored with the $u(N_c)$ color space. We give the precise definition of each matrix variable in the following subsections. Roughly, each symbol defines a linear operator that acts on $H_{CP^{3|4}}(N) \otimes u(N_c)$. The $D^\alpha = Z^\alpha + A^\alpha$ are non-commutative versions of anti-holomorphic covariant derivatives and $\mathcal{D}^I$ is an extension of this derivative to superspace. The $Q$ and $\bar{Q}$'s are the defect modes, and $Z^I = (Z^\alpha, \psi^i)$ are the non-commutative supertwistor coordinates. Finally, $\mathcal{I}_{IJ}$ is the supersymmetric infinity bi-twistor (2.16), and the subscript $(...)_{\psi^4}$ indicates the projection onto the top superfield component.

All matrix variables in the action (4.1)-(4.3) have direct geometric meaning as functions and sections of bundles over twistor space. Functions correspond to maps from $\mathcal{H}_{CP^{3|4}}(N)$ to $\mathcal{H}_{CP^{3|4}}(N)$. A convenient representation of such maps is in terms of a power series in the oscillators, which we may normal order as $M = M_{mn|ab} (Z^\dagger)^m (\psi^4)^a Z^n \psi^b$. Matrix multiplication then corresponds to multiplying successive normal ordered power series. Similarly, we can describe sections of degree $l$ line bundles by rectangular matrices. These correspond to maps from $\mathcal{H}_{CP^{3|4}}(N + p)$ to $\mathcal{H}_{CP^{3|4}}(N + q)$, where the net degree is $l = q - p$. The corresponding map has degree $N + p$ in the $Z$ oscillators, and $N + q$ in the $Z^\dagger$ oscillators. Hence, it can be identified in the commutative geometry with the corresponding section of $O(l)$. Note, however, that there is some freedom in how to assign $p$ and $q$ to a degree $l$ line bundle. This is because complex conjugation and dualization of a bundle are naturally related by Hermitian conjugation of operators. Finally, integration of a product of such functions or sections proceeds by tracing over the appropriate Hilbert space [28].

In the matrix action, the covariant derivatives $D^\alpha$ and $\mathcal{D}^I$ are (0,1) forms on $CP^{3|4}$ and
like the coordinates $Z_\alpha$ and $Z^I$ (which also act like $\overline{\partial}$ operators) and describe non-square matrices of degree $-1$, that change the rank from $N + p$ to $N + p - 1$. The $Q$’s and $\tilde{Q}$ variables are sections of a degree 1 line bundle, and thus increase the rank from $N + p$ to $N + p + 1$. Modulo such small shifts, all symbols in (4.1)-(4.3) are $k_N$ times $k_N$ matrices.

### 4.3 Matrix Chern Simons

To motivate the form of the holomorphic Chern-Simons action, it is actually helpful to at first enlarge the field content, to cover all $\mathbb{CP}^3|_4$’s up to level $N$. This provides an eight-dimensional formulation, as in integral over a ball of square radius $N$ inside $\mathbb{C}^4|_4$, from which the hCS contribution arises as a boundary term at level $N$. We introduce a fuzzy ball $\mathbb{B}^{4|4}$ with a Hilbert space of points:

$$\mathcal{H}_{\mathbb{B}^{4|4}} = \bigoplus_{M=0}^{N} \mathcal{H}_{\mathbb{CP}^3|_4}(M)$$

(4.4)

The matrices $D^\alpha$ correspond to $(0,1)$-forms on $\mathbb{B}^{4|4}$. We focus on the bosonic part of this differential form content. Each $D^\alpha$ is specified by a power series in the $Z^\dagger$’s, $Z$ and $\psi$ which we schematically write as: $D^\alpha = D^\alpha_{lm}|a}(Z^\dagger)^l Z^m (\psi)^a$ with specified coefficients $D^\alpha_{lm}|a}$. Let us note that this expansion provides a convenient way to deal with matrices which are of different sizes. The field content is specified in terms of the direct sum of $K_M \times K_{M+1}$ matrices for $M = 0$ up to $M = N$, which we denote by:

$$D^\alpha \equiv \bigoplus_{M=0}^{N} D^\alpha_{M \times (M+1)}$$

(4.5)

Each term in this direct sum is controlled by the same expansion coefficients $D^\alpha_{lm}|a}$. The degrees of freedom of the matrix model truncates at finite order set by $N$. The $D^\alpha$ transform in the adjoint representation of the $U(N_c)$ gauge group, and are to be thought of as covariant derivatives. To ensure that the gauge fields project to $(0,1)$ forms on $\mathbb{CP}^3|_4$, they are subject to the constraint [28]:

$$Z^\dagger_{\alpha} D^\alpha = \hbar N.$$  

(4.6)

Expanding around $D^\alpha = Z^\alpha + A^\alpha$, this becomes the condition $Z^\dagger_{\alpha} A^\alpha = 0$. This projects out the direction orthogonal to $\mathbb{CP}^3|_4$ inside of $\mathbb{C}^4|_4$. See [28] for further discussion.

The fuzzy holomorphic Chern-Simons action can now be defined as a trace over the 8-dimensional ball $\mathbb{B}^{4|4}$

$$S_{hCS}(A) = \frac{1}{g^2} \text{Tr}_{\mathbb{B}^{4|4}} \left( \varepsilon_{\alpha\beta\gamma\delta} D^\alpha D^\beta D^\gamma D^\delta \right) \psi^4$$

(4.7)

where we have introduced $g$, the gauge coupling of the matrix model. This is the non-
commutative version of the continuum action $\int_{\mathbb{B}^4_{4|4}} d^4 \psi d^4 Z \text{tr} (\mathcal{F} \wedge \mathcal{F})$ on $\mathbb{B}^{4|4}$ where $\mathcal{F} = \overline{\partial} A + A \wedge A$. This continuum action is a total derivative, and so integrates to just a boundary term, which is the hCS action.\footnote{D. Skinner has also considered an eight-dimensional formulation of hCS theory.} A similar effect occurs in our case: the matrix operator inside the trace in (4.7) naively takes the form of a total commutator:

$$\varepsilon_{\alpha\beta\gamma\delta} \left[ D^\alpha, D^\beta D^\gamma D^\delta \right]$$

and thus should have a vanishing trace. However, the $D^\alpha$’s are non-square matrices that lower the rank of the Hilbert space by one unit: they map $\mathcal{H}_{\mathbb{C}P^3_{|4}}(N+p)$ to $\mathcal{H}_{\mathbb{C}P^3_{|4}}(N+p-1)$. So if we cycle $D^\alpha$ around the trace, it lowers the rank of the Hilbert space over which the trace is taken on one unit. It is easy to see that, via this mechanism, the trace over all states inside the ball $\mathbb{B}^{4|4}$ cancel out, but that one is in fact left with boundary term at level $N$. This boundary term includes the hCS action as written in (4.2).

There is in fact a subtlety in this partial integration argument, that will play an important role later on. The eight-dimensional form (4.7) of the holomorphic Chern-Simons action is manifestly invariant under the gauge transformations:

$$D^\alpha \rightarrow e^{ih} D^\alpha e^{-ih}.$$ \hspace{1cm} (4.9)

where $h = T^A h_A$ is a $u(N_c)$ Lie algebra element and $h_A$ are arbitrary degree zero polynomials in the $Z^\dagger$ and $Z$’s. Hence the result after the partial integration should also be gauge invariant. However, as the attentive reader may already have noticed, the action (4.2) is not automatically gauge invariant, since, even though it is a color singlet, the holomorphic three-form $\Omega_{\alpha\beta\gamma}$ does not commute with general non-commutative gauge parameters $h_A(Z, Z^\dagger)$. This effect is subleading in $1/N$, and thus disappears in the commutative limit, but nonetheless should be taken into account if one insists on writing the hCS action as a trace over $\mathcal{H}_{\mathbb{C}P^3_{|4}}(N)$. There are two ways to deal with this subtlety: (i) stick to the covariant action (4.7), or (ii) promote $\Omega_{\alpha\beta\gamma}$ in (4.2) to a dynamical variable that acts like a compensator field. We will return to this point in section 7.

### 4.3.1 Small Phase Space Description

The twistor matrix model as just given can be viewed as a large phase space description. It makes the $su(4|4)$ symmetry manifest, but at the expense of working with non-square matrices. There is also a small phase description which sometimes gives a practical alternative. Introduce three oscillators $\zeta^i$ for an affine patch of $\mathbb{C}^3$. On this space, we can restrict to $\mathbb{B}^3(N)$, all states of $\mathbb{C}^3$ of level less than or equal to $N$. In this description, a homogeneous polynomial of degree $d$ in the remaining $Z$’s is replaced by a polynomial of degree $d$ in the
Starting from the bulk action of equation (4.2), there is a local direction normal to \( \mathbb{CP}^3 \), which we can identify with \( \mathbb{Z}^4 \). This amounts to the formal replacement of \( D^4 \) and \( Z^4 \) by the identity. Viewing all fields as polynomials in the \( \zeta \)'s and \( \zeta^\dagger \)'s, with the maximal degree fixed by their homogeneous counterparts, the hCS action simplifies to

\[
S_{hCS} = \frac{1}{g^2} \text{Tr} \left( \varepsilon_{stu} D^s D^t D^u \right) \psi_4 .
\]  

(4.10)

where the trace is over the three ball \( \mathbb{B}^{3/4} \) of states in \( C^3 \) with level less or equal to \( N \).

### 4.4 Defect Action

We now turn to describe the properties of the defect action (4.3). We first discuss the free action, with the coupling to the holomorphic Chern-Simons gauge field turned off. So in (4.3) we replace the gauge covariant derivative \( D^I \) by the non-commutative twistor coordinate \( Z^I \). Also, to keep the discussion a bit more transparent, we will for the most part restrict our attention to the bosonic sector. The generalization to the supersymmetric formulas is straightforward. The free bosonic defect action takes the simple form

\[
S(\tilde{Q}, Q) = \text{Tr}(\tilde{Q} D Q),
\]

(4.11)

\[
D Q \equiv I_{\alpha\beta} Z^\alpha Q Z^\beta ,
\]

(4.12)

where the trace is over the Hilbert space of points on \( \mathbb{CP}^3 \) and over the \( u(N_c) \) color space. This action defines a gaussian matrix model, where the integration variables \( \tilde{Q} \) and \( Q \) are slightly non-square matrices of size \( k_{N+1} \times k_N \), which in addition may carry a color index, transforming respectively in the \( N_c \) and \( N_c \) of \( U(N_c) \).

The gaussian matrix model (4.11) turns out to have quite remarkable properties, which are studied in some detail in the companion paper [1]. Here our main task is to motivate why (4.11) is the correct action for the defect modes, that is, for the low energy fluctuations around the homogeneous instanton background. So let us list the main characteristics of (4.11), and compare them with our wish list.

- **Global symmetries.** Our first requirement is that the defect action should respect the \( SO(5) \) symmetry of the homogeneous instanton background. The action (4.11) clearly does – but it is instructive to see how it works out. Let us first look at the global symmetries of the kinetic operator \( \overline{D} \) defined in (4.12). The operator \( \overline{D} \) is in fact invariant under a full set of \( gl(4, \mathbb{C}) \) transformations. Consider a \( gl(4, \mathbb{C}) \) generator \( M_{\alpha\beta} = Z^\dagger_\alpha Z_\beta \), and define its action on \( Q \) via \( M_{\alpha\beta} Q \equiv M_{\alpha\beta} Q - Q M_{\beta\alpha} \). Note that the indices \( \alpha \) and \( \beta \) switch locations between the two terms. A short computation shows that \( [\overline{D}, M_{\alpha\beta}] = 0 \), and so the kinetic operator is invariant under the group generated by all \( M_{\alpha\beta} \) operators. It would not be
correct, however, to conclude that the defect action preserves $gl(4, \mathbb{C})$: the action (4.11) involves a trace, which is defined with respect to a choice of inner product. So the most we could hope for is the symmetry algebra $u(4)$. Additionally, the action contains the infinity bi-twistor $I_{\alpha\beta}$, which transforms as a vector of $so(6) \simeq su(4)$. This breaks the symmetry group to $so(5)$. Concretely, hermitian generators that leave the action invariant are

$$M(v) = v^\alpha_\beta Z^\dagger_\alpha Z^\beta, \quad v^\alpha_\beta = v^{\beta\alpha}, \quad (4.13)$$

where the raising of the second index is accomplished with the $I^{\beta\gamma}$ bi-twistor. This leaves 10 hermitian charges, which are the symmetry generators of $so(5)$. In a similar way, we find that the supersymmetric action (4.3) preserves the symmetries of the supersphere $S^{4|8}$.

- **Ultra-locality.** A second desired property is that the defect action should reflect the quantum Hall intuition, that its excitations are bound to Landau orbits localized along the $S^4$. Alternatively, using the twistor string theory terminology of [7], we wish to see that the defect action shares the properties of an effective action for (the ground states of) open strings that stretch between the instantonic D-branes and the space-filling D-brane. Both arguments indicate that the defect action should be ultra-local in space-time. Eqn (4.11) satisfies this beautifully [1]. Consider the position operator $X_{\alpha\beta}$ introduced in eqn (3.25). An easy computation shows that these operators commute with the kinetic operator:

$$[\overline{D}, X_{\alpha\beta}] = 0. \quad (4.14)$$

The same is true for the hermitian conjugate operators $X^\dagger_{\alpha\beta}$. Hence the kinetic operator acts within the eigen space (3.26) of $X_{\alpha\beta}$. It is the precise sense in which the defect action is ultra-local along the $S^4$. So it has the right to be interpreted as the action of modes that are tied to local defects, that wrap the twistor lines. It is easy to verify that this result extends to the supersymmetric case.

- **$\mathbb{CP}^1$ Propagation.** This ultra-local property indicates that the defect modes propagate along the $\mathbb{CP}^1$ fiber directions. Continuing to take guidance from twistor string theory, we would like to see that, when restricted to a particular fiber, the action reduces to that of a 2-dimensional chiral free field. Via $SO(5)$ symmetry, it is sufficient to verify this for the twistor line at the origin. So let $\Phi$ be a field localized at the origin, i.e. it is a linear map on $H_{\mathbb{CP}^1}$ that is made up only from $\pi_a$ and $\pi^\dagger_a$ oscillators. On this subspace, the kinetic operator $\overline{D}$ reduces to

$$\overline{D}\Phi|_{\mathbb{CP}^1} = \varepsilon^{ab} \pi_a \Phi \pi_b \quad (4.15)$$

Via the commutation relation $[\pi_a, \pi^\dagger_b] = \varepsilon_{ab}$, we immediately see the right-hand side indeed acts on $\Phi$ as a standard Dolbeault operator $\overline{\partial} = \pi^a \frac{\partial}{\partial \pi^a}$ on $\mathbb{CP}^1$. Based on this alone, we
expect that the propagator of the $D$ operator looks like the propagator of a two-dimensional chiral free field. The propagator is studied in some detail the companion paper [1], where this expectation is confirmed. For additional discussion of the Dolbeault operator on fuzzy $\mathbb{CP}^1$, see e.g. [36–38].

- **ADHM correspondence.** The defect modes $Q$ and $\tilde{Q}$ have a familiar analogue the ADHM construction of instantons. From the comparison with ADHM we learn that they naturally transform in the fundamental and anti-fundamental of $u(N_c)$. In the full system described by the action (4.1)-(4.3), they provide a source term to the equation of motion of the holomorphic Chern-Simons gauge field. Looking at (4.3) and the small phase space form (4.11) of the hCS action, we can already see that the (top component of) the $D^\alpha$ equation of motion takes the schematic form of the ADHM equation. We will make this match more precise in the next section.

- **MHV correspondence.** The gauged defect action is invariant under gauge rotations

\[
Q \to e^{ih}Q, \quad \tilde{Q} \to \tilde{Q}e^{-ih}, \quad D^I \to e^{ih}D^Ie^{-ih}. \tag{4.16}
\]

We read off that the bulk gauge field couples to the current:

\[
J^A_\beta = T^A_{ij}Q^iZ^\beta\tilde{Q}^j \tag{4.17}
\]

where $T^A_{ij}$ is a $U(N_c)$ generator in the fundamental representation. This current is conserved on-shell: $Z^\beta J^A_\beta = 0$. Hence, after taking the large $N$ limit, the matrix model has all elements in place to provide a candidate dual description of 4D $\mathcal{N} = 4$ scattering amplitudes. In particular, following [39] and [7], we are led to identify the correlation functions of these current operators with MHV gluon amplitudes. As shown in [1], the two sides indeed match. We will study this relationship in more detail in section 6.

We have verified that the defect action (4.3) passes several non-trivial checks, which support its candidacy as the correct action for low energy fluctuations around the instanton background. In the next sections, we will study the physical properties of the matrix model in more detail. In particular, we will present evidence that, in a suitable large $N$ limit, its correlation functions approach those of $\mathcal{N} = 4$ gauge theory, coupled to gravity. Various technical details will be delegated to [1].

## 5 ADHM Limit

Let us briefly review the realization of ADHM in terms of bound states of branes. The brane construction of ADHM is discussed in [40–42] (see [43] for a concise review). To frame our discussion, consider type IIB string theory on $\mathbb{R}^{9,1}$ with $n$ D7-branes filling $\mathbb{R}^{3,1} \times \mathbb{R}^4$, and $k$
probe D3-branes filling $\mathbb{R}^{3,1}$. We focus on the $\mathbb{R}^4$ directions, relative to which the D3-branes look like instantons.

The theory on the probe D3-branes is given by a $U(k)$ $\mathcal{N} = 2$ supersymmetric field theory. In $\mathcal{N} = 1$ language, the motion of the D3-branes is parametrized by three chiral superfields, $\phi_1, \phi_2$ point parallel and $\varphi$ points normal to the seven-brane. All three fields are in the adjoint of $U(k)$. Additionally, there are bifundamentals $\tilde{q}$ and $q$ which define $n \times k$ and $k \times n$ matrices. These correspond to $3 - 7$ strings. The superpotential reads:

$$W_{\text{probe}} = \text{Tr}_{U(k)} (\varphi [\phi_1, \phi_2]) - \text{Tr}_{U(n)} (\tilde{q} \varphi q)$$

(5.1)

In this system, the trace over $U(n)$ is associated with the flavor symmetry of the D7-branes. There is also a D-term potential. To study the classical vacua, it is enough to consider solutions to the F-term equations, modulo the complexified gauge group $GL(k, \mathbb{C})$:

$$[\phi_1, \phi_2] = q \tilde{q}, \ [\varphi, \phi_2] = [\varphi, \phi_1] = 0, \ \tilde{q} \varphi = \varphi q = 0.$$  

(5.2)

These are the ADHM equations. The moduli space consists of several branches: the Coulomb branch, the Higgs branch, and mixed branches. The Coulomb branch, $\tilde{q} = q = 0$, describes the motion of the D3-branes away from the D7s. On the Higgs branch, $\varphi = 0$, and $q$ and $\tilde{q}$ are non-zero. In this case, the D3-branes have dissolved as instantons inside the D7-branes. On the mixed branches, some combinations of both are non-zero.

The Higgs branch coincides with the $k$-instanton moduli space of the $GL(n, \mathbb{C})$ gauge theory on $\mathbb{R}^4$. This case is characterized by $[\phi_1, \phi_2] = q \tilde{q}$, and the space of solutions modulo $GL(k, \mathbb{C})$ has real dimension $4kn$, which is the dimension of the instanton moduli space. More generally, there can be isolated solutions with all fields switched on. From the perspective of the D3-branes, such mixed branches describe configurations where the D3-brane has partially puffed up in the directions normal to the seven-brane.

Let us now see how this matches up with the behavior of the matrix model. To do this, we will consider a slightly broader notion of the system, where all modes are complexified. We can see roughly how the match works. There are three independent adjoint fields $\phi_1, \phi_2$ and $\varphi$, which are naturally identified as the covariant derivatives $D^a$. In a local patch of the $S^4$, we can pick a preferred origin and a corresponding $\mathbb{R}^4$. Equipping this with a complex structure we have a $\mathbb{C}^2$, and two corresponding vector fields $D_1$ and $D_2$. The third component $D_0$ points along the direction of the twistor line at the origin. In addition, the defect modes $\tilde{Q}$ and $Q$ have their analogues with the D3-D7 bi-fundamentals $\tilde{q}$ and $q$. Note,

---

The counting is as follows. We have $4k^2 + 4kn$ real degrees of freedom from $\phi_1 \oplus \phi_2$ and $\varphi \oplus q$. The F-term constraints impose $2k^2$ conditions, and modding out by $GL(k, \mathbb{C})$ removes a further redundancy of $2k^2$. The resulting moduli space has dimension $4kn$. On $S^4$, the instanton moduli space is $4kn - n^2$, but if one allows gauge transformations at infinity, the dimension is again $4kn$.  

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however, that the size of the matrices do not yet match: $\tilde{Q}$ and $Q$ are $k_{N+1} \times k_N$ matrices, whereas $\bar{q}$ and $q$ are $k \times n$ matrices.

We now provide a more precise match with ADHM. First let us specify the limit in which we should expect that match to become exact. For this we need to be able to turn off the non-commutativity. This is done as follows. Recall the form of the infinity bitwistor:

$$
\mathcal{L}_{IJ} = \frac{1}{\ell} \begin{pmatrix}
\gamma \varepsilon_{ab} & 0 & 0 \\
0 & \varepsilon^{ab} & 0 \\
0 & 0 & \eta_{ij}
\end{pmatrix}
$$

(5.3)

Let us consider the limit $\gamma \to 0$ with $\ell$ held fixed. As explained, the $\gamma$ parameter describes the ratio of two $S^4$ radii, namely (i) the radius of the $S^4$ relative to which the abelian flux of the Yang monopole is homogeneous, and (ii) the radius of the $S^4$ relative to which the $SU(n)$ instanton part is homogeneous. So sending $\gamma \to 0$ amounts to localizing the abelian flux in a small region near the north pole relative to the non-abelian flux, or equivalently, localizing the non-abelian flux in a small region near the south pole relative to the abelian flux. From either perspective, it is clear that in this limit, the effect of the abelian flux on the non-abelian configuration becomes negligible. Let us take the perspective where we zoom in on the south pole region of the $S^4$. In this limit, the four sphere starts to look like flat space time, and moreover, only the non-abelian part of the configuration is retained.

Varying the matrix action (4.3) with respect to $\tilde{Q}$, we obtain the equation of motion

$$
\bar{D}_AQ \equiv \mathcal{L}_{IJ}D^JQZ^I = 0
$$

(5.4)

When $\mathcal{L}_{IJ}$ is invertible, this equation generically possesses no solutions. Indeed, via the rank-nullity theorem, a map $\bar{D}_A : K_{N+1} \times K_N \to K_N \times K_{N+1}$ will in general have a trivial kernel. In the specific limit $\gamma \to 0$, however, $\mathcal{L}_{IJ}$ is no longer invertible and the matrix system will have a moduli space of solutions. Looking at eqns (5.3) and (5.4) we see that derivatives in the $\omega$ direction cost much less energy than derivatives in the other directions of the supertwistor space. The effective geometry experienced by the modes is $\mathbb{C}^{2|4} \times \mathbb{CP}^{1|0}$ where the $\mathbb{CP}^{1|0}$ factor is composed from the $\omega$ directions. (In the presence of a background gauge field, the overall direction of this constraint will change, but this can be absorbed into the definition of the oscillators.) Without loss of generality, the constraint is then:

$$
\pi_a Q = \psi^i Q = 0
$$

(5.5)

These conditions project $Q$ down to a $(n + 1) \times K_N$ supermatrix, which is purely bosonic in the $(n + 1)$-component part. In other words, the left index on $Q$ are states of a bosonic $\mathbb{CP}^{1|0}$ at level $n$. The kinetic term for these modes then fixes the form of $\tilde{Q}$, so that it is given by a $K_{N+1} \times n$ supermatrix.
In this limit, it is also appropriate to take a specific basis for the bulk gauge fields. There are the two derivatives in directions transverse to the $\mathbb{C}P^{10}$, given by $D^1$ and $D^2$. In addition, there is the derivative along the $\mathbb{C}P^{10}$, given by $D^0$. To analyze the remaining equations of motion, it is helpful to pass to a description in terms of an affine patch. This leaves us with three matrices $D^0$, $D^1$, $D^2$ which admit a $\psi$-expansion in terms of $k_N \times k_N$ bosonic matrices, and $Q$ and $\tilde{Q}$ which respectively admit $\psi^\dagger$- and $\psi^\dagger$ expansions in terms of bosonic matrices of respective sizes $(n+1) \times k_N$ and $k_N \times (n+1)$. The resulting action is:

$$S = \text{Tr}_{k_N}(D^0 [D^1, D^2]) \psi^4 + \text{Tr}_n(Q D^0 \tilde{Q})$$ (5.6)

The ADHM equations now naturally appear by varying with respect to the $\psi^4$ component of $D^0 = d^0 + \ldots + \psi^4 b^0$. First, observe that $\psi^4 b^0$ is sandwiched between the supermatrix parts of $Q$ and $\tilde{Q}$. Since $\psi^4$ corresponds to an annihilator operator when acting to the right on $\tilde{Q}$, the only non-zero coupling involves the $\psi^4$ component of $\tilde{Q}$, which we denote by $\tilde{q}$. On the other side, the only term that survives is $q$, the purely bosonic component of $Q$. The terms of the bosonic action involving $b^0$ are then:

$$S = \text{Tr}_{\mathbb{C}P^3}(b^0 [d^1, d^2]) + \text{Tr}_{\mathbb{C}P^{10}}(q b^0 \tilde{q})$$ (5.7)

where $d^i$ is the bosonic component of $D^i$. Setting to zero all other components of the supermatrices, we see that the resulting equations of motion are identical to those of line (5.2). Working modulo $GL(k_N, \mathbb{C})$ transformations, we obtain a moduli space of instantons for $su(n+1)$ gauge theory at instanton number $k_N$. Note that this is the maximal instanton number for which it is possible to define an $so(5)$ homogeneous instanton configuration:

$$k_N = \frac{(N + 1)(N + 2)(N + 3)}{6} = \frac{(n + 1) ((n + 1)^2 - 1)}{6}. \quad (5.8)$$

This is again in accord with the fact that we are considering instanton configurations which are deformations of the non-abelian part of the Yang monopole.

### 6 Continuum Limit

Our discussion so far has focussed on the finite $N$ regime of the matrix model. The finite matrices furnish a basis of functions on $S^4$ but only up to some cutoff angular momentum. To arrive at a continuum theory, it is necessary to take a large $N$ limit.

In addition to $N$, the matrix model comes with a length scale, given by the $S^4$ radius $\ell$, and a parameter $\gamma$ which controls the breaking of the $so(5)$ symmetry. Depending on the scaling of these parameters, we arrive at different limits for 4D physics. The most basic large $N$ limit is taken while keeping $\gamma = 1$, which preserves the $so(5)$ symmetry.
Alternatively, we can take a combined large $N$ and flat space limit, which corresponds to performing a Wigner-Inönü contraction of the original $so(5)$ algebra. This involves rescaling the momentum and conformal boost generators, or equivalently, rescaling the $\pi^\dagger$ oscillator relative to the $\omega^\dagger$ oscillator.

We can consider two types of flat space continuum limits. In the first scenario, we first send $N$ to infinity, and then take the flat space limit $\ell \to \infty$. As we will argue, this continuum limit produces pure $\mathcal{N} = 4$ SYM theory. The correspondence arises in a similar way as in twistor string theory, and the dictionary looks almost identical. But there are some important contrasts, since our treatment of the defect system is quite different from that in [7]. Most significantly, we will present clear evidence that in our set-up, the continuum $\mathcal{N} = 4$ gauge theory arises without any coupling to conformal gravity.

A second possibility is to take a double scaling limit, in which both $N$ and $\ell$ are sent to infinity, but such that the ratio $\ell^2_{pl} = \ell^2 / N$ is held fixed:

$$N \to \infty, \ell \to \infty, \quad \text{with} \quad \ell^2_{pl} = \frac{\ell^2}{N} \text{ fixed}. \quad (6.1)$$

We will argue that in this limit, the system does contain gravity. Moreover, we will present evidence that the emerging gravitational theory is described by the usual Einstein action, with a finite Newton constant $G_N$ of order $\ell^4_{pl}$. The strongest evidence in support of this inference is that the matrix model is able to reproduce the MHV graviton amplitudes. The calculation that leads to this result is described in the companion paper [1]. In the next section we will try to give a more conceptual explanation of how and why the gravitational degrees of freedom can arise from a matrix model.

### 6.1 Emergent Gauge Theory

Let us first begin with a discussion of the purely gauge theory sector. In the matrix model, this is arranged by taking $\ell_{pl} \to 0$. Here we will study this limit directly at the level of the action.

The basic dictionary is relatively simple and standard. Starting from the matrix model action (4.1)-(4.3), the first step is to rescale all twistor coordinates by a factor of $1/\sqrt{N}$, so that the commutation relations take the form $[Z^\alpha, Z^\dagger_\beta] = \delta^\alpha_\beta / N$, etc. Hence, upon taking the large $N$ limit, the twistor coordinates become ordinary commutative variables. Accordingly, the trace of the Hilbert space $\mathcal{H}_{\mathbb{C}P^{3|4}}$ of points, reduces to an integral over the commutative supertwistor space $\mathbb{C}P^{3|4}$. The covariant derivatives $D^\alpha$, instead must be defined with an extra prefactor of $N$, so that $Z^\alpha$ reduces to an anti-holomorphic derivative $\frac{\partial}{\partial Z^\alpha}$. We will
thus write

\[ D^\alpha = N Z^\alpha + A^\alpha \]

(6.2)

where \( A^\alpha \) denotes the gauge field. Applying this dictionary to the matrix version of the holomorphic Chern-Simons action gives the continuum hCS action, in the form explained at the end of subsection 4.3. In particular, the stationary points of the hCS action are holomorphic flat connections which satisfy (2.19).

The appearance of a continuum field theory in the large \( N \) limit, while standard and reasonable, in fact needs a bit more justification. For any finite \( N \), regardless how large, the space of functions on a non-commutative space includes arbitrarily non-local maps. Indeed, the matrix model involves integration over matrices \( D^\alpha, \tilde{Q} \) and \( Q \), which include maps from one point to any other point on the fuzzy \( \mathbb{C}P^3 \). In our context, these non-local maps are exponentially suppressed, due to the specific form of the matrix action. For example, as seen from (2.19), the saddle points of the matrix hCS action are maps that are locally holomorphic in the \( Z^\alpha \)'s. Since the \( Z^\alpha \)'s commute among each other, holomorphic functions multiply locally. In other words, all non-locality is associated with the \( Z^\dagger_\alpha \) dependence of the fields. As we will see more explicitly in the next section, the size of this non-locality is set by \( \ell_{pl} \). So in the \( \ell_{pl} \to 0 \) limit, the theory becomes local.

The correspondence with local 4D physics proceeds by projecting the \( \mathbb{C}P^3 \) gauge field \( A^\alpha \) down onto \( S^4 \). This is done via the Penrose correspondence, which repackages the components of \( A^\alpha \) into those of \( \mathcal{N} = 4 \) SYM theory. Adjusted to our context, the Penrose transform works as follows:

Suppose \( A \) is an on-shell mode. So on a suitable subset of the Hilbert space, it commutes with the \( Z^\alpha \) coordinates. In particular, it preserves the twistor line equations (2.15) that select the \( N + 1 \) dimensional Hilbert space \( \mathcal{H}_x \) associated with a given space-time location \( x \). (Here and below \( x \) is short-hand for a point \((x, \theta)\) on the supersphere \( S^4 \).) We can

13Functions or sections of degree \( p \) line bundles on \( \mathbb{C}P^3 \) can typically not be globally holomorphic, since it should be expandable as a degree zero (or degree \( p \)) polynomial in terms of \( Z^\alpha \) and \( Z^\dagger_\alpha \). A function \( f(Z) \) locally holomorphic is if its commutator with the holomorphic coordinates \( Z^\alpha \) is an operator that annihilates a subspace \( \mathcal{H}(S) \) of the Hilbert space \( \mathcal{H}_{\mathbb{C}P^3}(N) \). This subspace \( \mathcal{H}(S) \) represents the local region within which we can consider \( f(Z) \) as a holomorphic function of the \( Z \) coordinates. For a more precise definition of locally holomorphic functions, see [I].

14In commutative twistor theory, the basic correspondence is between space-time fields of helicity \( h \), and certain cohomology elements on \( \mathbb{P}T' \), that is, \( \mathbb{C}P^3 \) with the \( \mathbb{C}P^1 \) at infinity removed:

\[ \{ \text{helicity } h \text{ space-time fields} \} = H^1(\mathbb{P}T', \mathcal{O}(2h - 2)) \]

(6.3)

where this is defined with respect to a sheaf cohomology. Consider a meromorphic \((0,1)\) form defined over commutative twistor space, which we denote by \( \Psi(\omega, \pi) \). Imposing the constraint \( \omega^\dagger_\dot{a} = i x^{\dot{a} a} \pi_a \), we obtain a space-time dependent \((0,1)\) form \( \Psi(ix\pi, \pi) \). Given this, we obtain a
therefore decompose $\mathcal{A}$ as a sum of operators

$$\mathcal{A}^\alpha = \int d^4|\mathcal{A}^\alpha(x)\rangle$$

where $\mathcal{A}^\alpha(x)$ denotes a linear map from $\mathcal{H}_x$ to $\mathcal{H}_x$. As motivated in more detail in \[1\], an on-shell mode corresponds to taking $\mathcal{A}^\alpha(x)$ of the special form

$$\mathcal{A}^\alpha(x) = \oint \langle \lambda d\lambda \rangle A^\alpha(x, \lambda) |x, \lambda\rangle$$

where, in order for $\mathcal{A}$ to represent a holomorphic function of the twistor coordinates $(\omega^a, \pi_a)$, $A$ must be a function of the form $A(ix^{\alpha a} \lambda_a, \lambda_a)$. Equation (6.7) is the quantum version of the Penrose transform. The classical version is obtained by sandwiching both sides between the two position eigenstates $(x, 0|$ and $|x, 0)$, and using that $(x, 0|x, \lambda) = (x, \lambda|x, 0) = 1$.

Applying this reduction procedure to the pure hCS theory only gives the self-dual gauge theory on $S^4$. The role of the defect modes is to supplement the rest of the $\mathcal{N} = 4$ SYM theory. At the level of perturbation theory, the match proceeds via the CSW rules for constructing scattering amplitudes \[11\]. To see how this emerges from the large $N$ matrix model, let us look more closely at the defect contribution.

As we have seen, the defect action is ultra-local in space-time. The fields $Q$ and $\tilde{Q}$ do not propagate along the space-time directions, and hence do not correspond ordinary space-time fields. The defect modes appear quadratically in (4.3). So it is natural to integrate them out, and collect their contribution to the effective action for the space-time gauge field $\mathcal{A}$ in the form of a functional (or rather, for us, just a regular) determinant.

Let us write the defect action (4.3) in the short hand notation

$$S_{\text{defect}} = \text{Tr}(\tilde{Q}D\mathcal{A}Q)$$

with $D$ the free kinetic operator defined in eqn (4.12). The trace is over the Hilbert space $\mathcal{H}_{\mathbb{C}P^3}(N)$ of point on twistor space. The (lowest superfield components of the) defect modes $\tilde{Q}$ and $Q$ are both $k_{N+1} \times k_N$ matrices. In the following, the space of all such matrices will be denoted by $M_k$. Now consider the operator $D^{-1} \circ D\mathcal{A} = 1 + D^{-1} \mathcal{A}$. This defines a map
from $M_k$ to $M_k$. So we can consider its determinant. Integrating out the defect modes leads to an $\mathcal{A}$-dependent effective action

$$S_{\text{eff}}(\mathcal{A}) = \text{Tr}_{M_k} \log (1 + \bar{D}^{-1}\mathcal{A}), \quad (6.9)$$

where we used the standard identity $\log \det = \text{Tr} \log$. We will now rewrite this in the form of a generating function of MHV gluon amplitudes.

Let us specialize to the case that $\mathcal{A}$ describes a sum of on-shell modes, of the form (6.6)-(6.7). The defect mode kinetic operator $\bar{D}$ and the on-shell field $\mathcal{A}$ then both commute with the position operators $X_{\alpha\beta}$

$$[\bar{D}, X_{\alpha\beta}] = 0 \quad ; \quad [\mathcal{A}, X_{\alpha\beta}] = 0. \quad (6.10)$$

The operator inside the trace in (6.9) also commutes with $X_{\alpha\beta}$, and thus defines an ultra-local operator on space-time. Given this result, we can naturally decompose the trace over $M_k$ into two factors: a trace over the space-time directions, and a trace $\text{Tr}_x$ over the space of maps that act on the $N+1$ dimensional Hilbert space $\mathcal{H}_x$ associated with the space-time point $x$. In the large $N$ limit, this decomposition amounts to making the replacement

$$\text{Tr}_{M_k} \rightarrow \int d^4|x\rangle \text{Tr}_x. \quad (6.11)$$

Here we have made use of the fact that the overlap between states located at different space-time points takes the form of a sharply peaked gaussian (3.23), which in the large $N$ limit turns into a delta-function. In the continuum theory, (6.11) simply amounts to writing an integral over $\mathbb{CP}^3$ as an integral over the $S^4$ base, times an integral over the $\mathbb{CP}^1$ fiber. The $\bar{D}$ operator in (6.13) takes the form (4.15), which in the large $N$ limit becomes the standard Dolbeault operator $\bar{\partial}$ on $\mathbb{CP}^1$. In the companion paper [1], we work out the precise form of the Green’s function $\bar{D}^{-1}$ and show that at large $N$ it coincides with the continuum version. We can view $\bar{D}$ as the kinetic operator of a free chiral field theory in two dimensions. To reflect this, we shall make the replacement $\bar{D} = \bar{\partial}$ when acting on a given twistor line.

We thus conclude that (6.9) in fact describes a local space-time action

$$S_{\text{eff}}(\mathcal{A}) = \int d^4|x\rangle \mathcal{L}_{\text{eff}}(\mathcal{A}(x)) \quad (6.12)$$
The effective Lagrangian takes the form

\[ \mathcal{L}_{\text{eff}}(\mathcal{A}(x)) = \text{Tr}_x \log \left( 1 + \overline{\partial}^{-1} \mathcal{A}(x) \right) \]  

(6.13)

\[ = \log \det_x \left( 1 + \overline{\partial}^{-1} \mathcal{A}(x) \right) \]  

(6.14)

where \( \text{Tr}_x \) and \( \det_x \) respectively denote the trace and determinant over the space of linear maps on \( \mathcal{H}_x \), or in more geometric terms, the space of functions on the commutative limit of a fuzzy \( \mathbb{CP}^3 \) twistor line located at \( x \). Equation (6.13) is then the generating function 1PI correlation functions of bi-linear currents of the chiral free fields, of the form (4.17). In [1] we show that the large \( N \) limit of these current correlation functions reproduces the Parke-Taylor formula.

At this point we have made contact with the works [45, 46], where it is shown that the continuum version of the effective action (6.12)–(6.13) is the generating function of gluon MHV amplitudes. Importantly, as pointed out in these references, the effective action (6.13) evades the troubling appearance of conformal gravity amplitudes, that plague the twistor string proposal of [7, 47]. The main difference between the two proposals is that in the twistor string theory, the effective action that gave the MHV amplitudes arose from integrating out the open string modes attached to defect D1-instantons, which then were integrated over their moduli space of positions. This latter procedure gives an effective lagrangian of the form \( \det(1 + \overline{\partial}^{-1} \mathcal{A}) \), rather than \( \log \det(1 + \overline{\partial}^{-1} \mathcal{A}) \). The \( \det \) expression of twistor string theory has several drawbacks. First, it is not by itself gauge invariant, and to be well-defined, requires additional couplings to a closed string sector. By contrast, the \( \log \det \) term is gauge invariant. In the continuum theory, this is due to a subtle interplay with \( \mathcal{N} = 4 \) superspace\(^{15}\). For us, the gauge invariance of (6.13) is manifest, since it was derived by integrating out a regulated and manifestly gauge invariant action (6.8).

Let us note that when there is reduced supersymmetry, the cancellation of chiral anomalies is a bit more subtle. In this case, the 2D chiral determinant in (6.13) would produce an anomaly outflow, that somehow needs to get cancelled by some anomaly sink somewhere else. The appearance of the anomaly is related to subtleties with the analogue of equation (6.11). With less than \( \mathcal{N} = 4 \) supersymmetry, the \( \mathbb{CP}^3 \) Hilbert space does not neatly factorize as assumed in (6.11). The obstruction is the Berry phase associated with passing from one coherent state \( |x\rangle \) to another \( |x'\rangle \). The associated Berry connection is the Yang monopole background. The Yang monopole background transforms under gauge transformations, and in this way, is capable of transporting the anomaly from one twistor line to another. So we see that the matrix model regulates the \( \log \det \) action proposed in [45, 46],\(^{15}\)

\(^{15}\)Indeed, without the superspace integrations, the MHV action of [45, 46] would not have been gauge invariant, and moreover, would have contained additional divergences due to ultra-local interactions and accompanying factors of \( \delta(0) \).
and should make sense with reduced supersymmetry.

Returning to the $\mathcal{N} = 4$ symmetric case, one could perhaps wonder how it could be possible that in the $\ell_{pl} \to 0$ limit, one can recover a superconformally invariant continuum theory. At any finite $\ell_{pl}$, only the $SO(5)$ symmetry is unitarily realized in the matrix model, so where could the other generators come from? The mechanism is expected to be the same as for any non-conformal theory with a (non-trivial) IR fixed point: conformal symmetry is broken as long as the UV scale is present, and arises as a linearly and unitarily realized symmetry only in the strict IR limit.

Finally let us comment on non-MHV amplitudes. The basic story here is the same as in the continuum theory. The total matrix action combines the holomorphic Chern-Simons action with the effective action (6.12)-(6.13). As in [44,48], general types of amplitudes are obtained by gluing together MHV amplitudes via propagators of the hCS theory. Here we do not have much to add to the continuum discussion, except for the following comment: instead of integrating out the defect modes, we can alternatively integrate out the bulk modes. Specifically, one can choose an axial gauge \footnote{Another natural and potentially useful gauge condition on $A$ is $Z^\alpha A_\alpha = 0$. Note that this condition involves the infinity twistor.} for which the interaction terms of the hCS theory drop out. So we can just perform the gaussian integral over the gauge field. The effective action for the defect field $Q$ and $\tilde{Q}$ will then contain the kinetic term, and a non-trivial quartic interaction term, which can be included iteratively. It is tempting to conjecture that the BCFW recursion relations [49] can be interpreted as the loop equation of this interacting matrix model. We leave the exploration of non-MHV and loop amplitudes for future study.

7 Emergent Gravity

In the previous sections we have provided evidence that the continuum limit of the matrix model specifies a $U(N_c)$ gauge theory on an $S^4$. We have also seen signs that hint at the presence of gravity. This is reflected in the appearance of a minimal length scale $\ell_{pl}$, and the fact that gauge transformations on the non-commutative space look like diffeomorphisms.

In this section we assemble these hints into concrete evidence that the matrix model contains a gravitational subsector. In the companion paper [1] we show that MHV graviton scattering amplitudes can be represented as correlation function of currents of the gaussian matrix model, that describes the defect modes in a fixed $A$ background. In this section, we will try to embed this result inside a more global geometric perspective. The appearance of a gravitational subsector is in some sense anticipated by earlier work on twistor string theory. However, as opposed to there, the matrix model comes with an intrinsic length scale, and therefore does not seem to describe conformal gravity. We comment on this
point further in subsection 7.4. In this section, we will put $N_c = 1$.

### 7.1 Currents and Compensators

In this subsection we show that the matrix model contains gravitational current operators. To some extent, the presence of this sector is necessary in order for the non-commutative Chern-Simons action to even be well-defined. As explained earlier, starting from the hCS action (4.7), written as an integral over the four-ball $B^4$, one could in principle perform a partial integration and obtain an action of the form (4.2). However, the naive action (4.2) is not invariant under the gauge transformations $D^\alpha \to e^{ih} D^\alpha e^{-ih}$, since the $h$’s do not commute with $\Omega_{\alpha\beta\gamma}$. To rectify this problem, we can introduce a compensator field $\Omega^\alpha = \frac{1}{6} \varepsilon_{\alpha\beta\gamma\delta} \Omega_{\beta\gamma\delta}$ which transforms in the adjoint under non-commutative $U(1)$ gauge transformations. The correct action is then:

$$S_{\text{fCS}} = \frac{1}{g^2} \varepsilon_{\alpha\beta\gamma\delta} \text{Tr} \left( \Omega^\alpha D^\beta D^\gamma D^\delta \right).$$  \hspace{1cm} (7.1)

One can view the $\Omega$ as a residual boundary or ‘singlet’ component of the $U(N_c)$ gauge field on the eight dimensional ball $B^4$, which is left over after performing the partial integration.

This gives a clear indication that the adjoint $U(1)$ factor of the fuzzy system is on a different footing from the other gauge symmetries. In commutative terms, maintaining gauge invariance via the compensator can be interpreted as allowing the holomorphic form $\Omega = \varepsilon_{\alpha\beta\gamma\delta} Z^\alpha dZ^\beta dZ^\gamma dZ^\delta$ to be deformed, akin to a Kodaira-Spencer theory of gravity [50]. In so doing, the geometric data of the twistor space become dynamical. This already suggests the appearance of a gravitational subsector.

It is reasonable to assume that the compensator should also enter in the defect action. Indeed, there is a natural place for it. Viewing the defect action as descending from a holomorphic Chern-Simons action in the presence of a flux defect (as is common in the study of intersecting brane configurations), the compensator appears in the action as:

$$S_{\text{defect}} = \text{Tr} \left( I_{\alpha\beta} \tilde{Q} D^\alpha Q \Omega^\beta \right).$$  \hspace{1cm} (7.2)

The $Q$ and $\tilde{Q}$ modes have become bifundamentals of $u(N_c) \times u(1)$. The system is invariant under the enlarged group of gauge transformations:

$$Q \rightarrow e^{ih} Q e^{-ig}, \quad \tilde{Q} \rightarrow e^{ig} Q e^{-ih}, \quad D^\alpha \rightarrow e^{ih} D^\alpha e^{-ih}, \quad \Omega^\alpha \rightarrow e^{ig} \Omega^\alpha e^{-ig}. \hspace{1cm} (7.3)$$

where $h$ is a Lie algebra element of $u(N_c)$ and $g$ is a phase, and both are local functions of $Z$ and $Z^\dagger$. In addition to the usual color gauge rotations, we see that the gauge transformations can also shift the locations of fuzzy points. In other words, the full symmetry is $u(N_c) \times u(N_c) \times u(1)$.
$u(1) \times gl(k_N) \times \tilde{gl}(k_N)$ where the two $gl(k)$ factors act by left and right multiplication on the defect fields. In keeping with the interpretation of the $\Omega^\alpha$ mode as a compensator field, it is natural to focus on the diagonal $gl(k_N)_{diag} \subset gl(k_N) \times \tilde{gl}(k_N)$. Indeed, if we view the matrix $Q$ as a field located at some holomorphic point $x$, then the simultaneous left and right action by the diagonal $gl(k_N)_{diag}$ corresponds to a diffeomorphism on the 4D space-time which is capable of shifting $x$ to another location.

Following the Noether procedure, we obtain the $gl(k_N) \times \tilde{gl}(k_N)$ ‘currents’ (here we switched notation, and explicitly write the $gl(k_N)$ matrix indices)

$$J_\beta(T) = T_{a b} Q^\alpha Z_\beta \tilde{Q}^{\bar{c} \bar{d}}, \quad \tilde{J}_\beta(T) = \tilde{T}_{a b} \tilde{Q}^\alpha Z_\beta Q^{\bar{c} \bar{d}}.$$  

(7.4)

where $T_{a \bar{b}}$ is a generator of $gl(k_N)$. The diagonal currents are the linear combinations

$$J_\beta(T) = J_\beta(T) - \tilde{J}_\beta(T).$$  

(7.5)

In principle, one can consider charges associated with arbitrary $gl(k_N)$ transformations, involving an arbitrary number of $Z^\dagger$ oscillators. However, we can anticipate that the equations of motion of the corresponding bulk gauge field disfavors such anti-holomorphic dependence. So it is reasonable to restrict our attention to currents of the form (7.4) which contain only one single $Z^\dagger$ oscillator. We will write out these currents in the next subsection.

Associated with the diagonal $gl(k_N)$ is a gauge field $A^\beta$, which is a function of $Z$ and $Z^\dagger$. It couples to the defect system via the vertex operator:

$$\mathcal{T}(\mathfrak{A}) = \text{Tr} \left( \mathfrak{A}^\beta J_\beta \right) - \text{Tr} \left( \mathfrak{A}^\beta \tilde{J}_\beta \right)$$  

(7.6)

Based on the index structure of the diagonal $gl(k_N)$ charges (7.4), we see that the symmetries define $(0,1)$-forms valued in the holomorphic cotangent and tangent bundle to twistor space. This is the same mode content expected for a conformal graviton [47], but with a few important differences. First, we see that they couple to a defect action which involves the infinity bitwistor. This means that although these currents are obtained from conformal gravitons, they are subject to the constraint that they leave the defect action, and thus the infinity twistor invariant. This imposes conditions which truncate the physical mode content. The combined presence of a UV cut-off and of the infinity twistor both break conformal invariance, which suggests that the emerging gravitational sector will not be conformal.

We can extend our discussion to the supersymmetric case with defect action:

$$S_{\text{defect}} = \text{Tr} \left( \mathcal{I}_I J_\beta \tilde{Q}^I J Q^\dagger \Omega^I \right).$$  

(7.7)
The bosonic currents (i.e. with $I$ and $J$ bosonic) provide us with the gauged translations and $u(N_c)$ rotations. The currents with $I$ and $J$ both fermionic give rise to $su(4)$ gauge fields. Further, there is a fermionic current as well which couples to the gravitinos. All this is in accord with the gauge symmetries of an $\mathcal{N} = 4$ supergravity theory.\footnote{The bulk gauge field $\mathcal{A}^I$ also can be viewed as a vector on superspace. We are making the simplifying assumption that these additional components of the would be gauge field in the fermionic directions have been set to zero. It would be interesting to study this more general situation, though it is unclear to us that it is a necessary element of the construction.}

### 7.2 MHV Graviton Scattering

To confirm the proposed geometric interpretation of the $gl(k_N)$ symmetry, in the companion paper [1] we show that correlation functions of the currents (7.4) reproduce MHV graviton scattering amplitudes. Here we just briefly comment on how this structure emerges from the matrix model, and on how it relates to existing results obtained from ordinary twistor space.

MHV amplitudes in gravity involve a single ingoing negative helicity graviton and an arbitrary number of ingoing positive helicity gravitons. Collectively, the plus helicity gravitons represent a selfdual background geometry, on which the minus helicity graviton propagates. The MHV amplitudes arise by expanding out the background field in terms of linearized perturbations around flat space-time. The twistor realization of this calculation builds on Penrose’s non-linear graviton [18]. In this construction, (anti-)selfdual space time backgrounds are represented as complex structure deformations of twistor space. Infinitesimal deformations correspond to vector fields, that via their Lie derivative, bend the location of the holomorphic twistor lines. As shown in [51], this geometric description allows a computation of MHV graviton amplitudes, that reproduces the well-known BGK formula [52].

The basic setup for scattering theory in the matrix model is to zoom in on a small and locally flat patch near the south pole of the $S^4$. This allows us to define a flat space limit via the Wigner-Inönü contraction of the $so(5)$ algebra:

$$\mathcal{P} \rightarrow P + \ell^{-2}K$$  \hspace{1cm} (7.8)

so that locally, translations are given by just the $P$. As shown in [1], the asymptotic data of the scattering theory are specified by vertex operators $T$ such that the adjoint $U(1)$ gauge field takes the special form $\mathcal{A}^\beta = Z^\beta T$, where $T(Z, Z^\dagger)$ is some non-commutative function. The $gl(k_N)$ generators of interest for MHV graviton scattering are of the form which locally have a single $Z^\dagger$ oscillator:

$$T_v(p) = (v \cdot \mathcal{P})\Psi(p)$$  \hspace{1cm} (7.9)

where $v^{\dagger a}$ is a polarization tensor for the graviton, $v \cdot \mathcal{P} = v^{\dagger a}P_{aa} + v_{aa}K^{\dagger a}$ is a complexified
so(5) generator, and \( \Psi(p) \) is a “momentum eigenstate” matrix satisfying \( [P_{aa}, \Psi(p)] = p_{a\dot{a}} \Psi(p) \) for complexified momentum \( p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} \). The corresponding currents for the plus helicity gravitons build up a self-dual background, and are given by:

\[
T_+ = \text{Tr} \left( [T, Q] \overline{D} \overline{Q} \right)
\]

(7.10)

the minus helicity gravitons are specified by the related set of currents:

\[
T_- = \text{Tr} \left( T Q \overline{D} \overline{Q} \right)
\]

(7.11)

which are acted upon by the \( T_+ \) generators. In an \( \mathcal{N} = 4 \) supersymmetric theory these modes sit in different supermultiplets, and so can naturally be associated with different types of currents. Quite remarkably, the correlation functions of these currents reproduce the MHV graviton scattering amplitude of Einstein gravity in flat space-time. This correspondence is worked out in detail in [1], and forms the main piece of evidence that our large \( N \) twistor matrix model gives rise to a space-time theory that contains gravity.

### 7.3 Geometric Action

In light of the detailed evidence that the defect sector of the matrix model contains the MHV sector of gravity, it is appropriate to ask whether the result can be extended to include non-MHV dynamics. Following the gauge theory lead, the logical place to look is in the hCS gauge sector and its coupling with the gravitational currents of the defect system. Indeed, we have already seen hints of a gravitational mode in the hCS system, in the form of an obstruction to decoupling the adjoint \( u(1) \) gauge transformations.

To discuss the action of the adjoint \( u(1) \) in the large \( N \) limit, it is helpful to reformulate the matrix multiplication of the matrix fields in terms of a Moyal product on the homogeneous coordinates:

\[
Z^\alpha \ast Z^\beta = Z^\alpha Z^\beta + \alpha^{\alpha\beta}
\]

(7.12)

for \( \alpha^{\alpha\beta} \) a (1, 1) bivector compatible with the symplectic structure induced by the Kähler form. The Moyal product for two functions \( f \) and \( g \) is, to leading order in \( \alpha \):

\[
f \ast g = fg + \alpha^{\sigma\tau} (\partial_\sigma f \partial_\tau g - \partial_\tau f \partial_\sigma g)
\]

(7.13)

where \( \sigma \) and \( \tau \) are holomorphic and anti-holomorphic tangent bundle indices, respectively. Given a non-commutative \( U(1) \) gauge field \( A_\tau \) and a field \( Q \) in the adjoint, the covariant

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\textsuperscript{18} Though the physical context is different, for somewhat related discussion on non-commutative gauge theory and emergent gravity, see [54, 57] and [58] for a review.
derivative is:
\[ \nabla_\tau Q = \partial_\tau Q + A_\tau * Q - Q * A_\tau. \] (7.14)

In terms of the Moyal expansion, we have:
\[ \nabla_\tau Q = \partial_\tau Q + \tilde{a}_\tau^\sigma \partial_\sigma Q - a_\tau^\sigma \partial_\sigma Q \] (7.15)

where we have introduced the modes:
\[ a_\sigma^\tau = 2 \sigma^\tau \partial_\sigma A_\tau, \quad \tilde{a}_\tau^\sigma = 2 \sigma^\tau \partial_\sigma A_\tau \] (7.16)

Note that what actually couples to the mode \( Q \) is the gradient of \( A \) rather than \( A \) itself. This again is an indication that the adjoint \( U(1) \) behaves differently from other types of modes in the fuzzy system.

The modes \( a \) and \( \tilde{a} \) transform as \((0,1)\) forms valued respectively in \( T^{(1,0)} \) and \( T^{(0,1)} \), i.e. the holomorphic and anti-holomorphic parts of the tangent bundle. A zero mode of \( Q \) is specified to zeroth order by the condition \( \partial_\tau Q = 0 \). To first order, this is corrected to the condition \( \partial_\tau Q - a_\sigma^\tau \partial_\sigma Q = 0 \). By inspection, \( a_\sigma^\tau \) defines an almost complex structure, and the derivative \( \nabla_\tau \) corresponds to the gauge field associated with complex structure deformations. This is of course in keeping with the expectation that twistor space is sensitive to complex structure deformations, and so a putative bulk gravitational theory will be of Kodaira-Spencer type.

To further bolster this interpretation, we can also work out the \((0,2)\) component of the field strength \( F_{\tau \bar{\beta}} = [\nabla_\tau, \nabla_\beta] \). A lengthy but straightforward computation shows that the field strength takes the form:
\[ F = \bar{\partial} \mathfrak{A} + \{ \mathfrak{A}, \mathfrak{A} \} \] (7.17)

where \( \mathfrak{A} = \frac{1}{2} \left( a_\sigma^\tau \partial_\sigma + \tilde{a}_\tau^\sigma \partial_\tau \right) \) is a \((0,1)\) form valued in the tangent bundle of twistor space. Here, \( \{ \mathfrak{A}, \mathfrak{A} \} \) denotes the Lie bracket for the \((0,1)\)-form \( \mathfrak{A} \) taking values in the tangent bundle. In other words, the adjoint \( U(1) \) of the fuzzy system is naturally associated with the algebra of vector fields on twistor space.

Given that the effective low energy theory describes physics derived from a holomorphic Chern-Simons field theory, it is natural to expect that the geometric action for the adjoint \( U(1) \) gauge field \( \mathfrak{A} \) can be packaged in the form of a \( BF \) type action that imposes the equation of motion \( F_{(0,2)} = 0 \). So it is reasonable to write an action of the form
\[ S_{u(1)} = \int \Omega \wedge \tilde{\mathfrak{A}} \wedge (\bar{\partial} \mathfrak{A} + \{ \mathfrak{A}, \mathfrak{A} \}) \] (7.18)

where \( \tilde{\mathfrak{A}} \) is a \((0,1)\)-form in the cotangent bundle which forms a pairing with the “color indices” of \( F \). The critical points of this action enforce the condition that the \((0,2)\) component
of the field strength is compatible with the complex structure of the geometry. Returning
to our discussion between equations (7.16)-(7.17), note further that to leading order in the
gauge field fluctuations, only the almost complex structure deformation $a^\alpha_\beta$ appears in a zero
mode equation. Setting $\tilde{a}^\tau_\alpha = 0$, the field strength $\mathcal{F}$ reduces to the Nijenhuis tensor for the
almost complex structure. Vanishing of the field strength enforces the condition that it is
integrable. This is of course quite familiar from earlier work on twistor string theory, and
the appearance of a Kodaira-Spencer theory of gravity, as in [47]. BF type actions for the
self-dual sector of supergravity have been considered in [51, 59] (see also [60]).

But as opposed to the twistor string, the coupling of the mode $A$ to the defect system
comes with additional structure, as reflected in the computation of MHV graviton amplitudes.
Indeed, in the MHV amplitude, the $gl(k_N)$ charges which couple to $a^\alpha_\beta \in \Omega^{(0,1)}(T^{(1,0)})$
are invariant under longitudinal shifts $v_{\dot{a}a} \rightarrow v_{\dot{a}a} + p_{\dot{a}a}$ in the polarization tensor, which allows one to pick the gauge condition $p \cdot v = 0$. The positive helicity gravitons then describe
special modes for which the $(0,1)$-form $a^\alpha_\beta$ is Hamiltonian with respect to the infinity twistor $I^{\alpha\beta}$ [51]:

$$a^\alpha_\beta = I^{\alpha\sigma} \partial_\alpha h_{\sigma\beta}.$$  \hspace{1cm} (7.19)

for $h_\sigma$ a line bundle valued $(0,1)$-form. A similar redundancy condition holds for the dual
mode $\tilde{A}$ [51], and the corresponding physical mode $\tilde{h}$. These have the interpretation as the two helicities of a graviton.

To give a bit more detail, recall that Einstein gravitons are given as elements $h_{--} \in H^1(O(2))$ and $h_{++} \in H^1(O(-6))$, on $\mathbb{P}^T$, twistor space with a line at infinity removed. By
contrast, the conformal graviton is instead represented by $(0,1)$-forms in the holomorphic
tangent and cotangent bundle, i.e. $\mathcal{C} \in H^1(T^{(1,0)})$ and $\tilde{\mathcal{C}} \in H^1(\Omega^{(1,0)})$. With no additional
restrictions, $\mathcal{C}$ is simply the mode $a$ of line (7.16). This is a linearized complex structure
deformation for a Kodaira-Spencer theory of gravity on twistor space. To get back the
usual modes of Einstein gravity, one introduces a redundancy for the $\mathcal{C}$ and $\tilde{\mathcal{C}}$ modes, so
that only $h_{--}$ and $h_{++}$ survive as physical excitations. As explained in [51], the precise
link between the two is given by:

$$\mathcal{C} = I^{\alpha\beta} \partial h_{--} \frac{\partial}{\partial Z^\alpha} \frac{\partial}{\partial Z^\beta}, \quad h_{++} = I^{\alpha\beta} \partial \tilde{\mathcal{C}}_\beta \frac{\partial}{\partial Z^\alpha}$$

where $I^{\alpha\beta}$ is the inverse bitwistor for Minkowski space. The redundancy is manifest for $\mathcal{C}$,
and for $\tilde{\mathcal{C}}$ is obtained through the conditions:

$$\tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}} + \partial m + n (\pi d\pi) \hspace{1cm} (7.20)$$

$$\tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}} + \bar{\chi} \hspace{1cm} (7.21)$$

where $m$ is an element of $\Omega^{(0,1)}(O(-4))$ and $n$ is an element of $\Omega^{(0,1)}(O(-6))$. In addition,
\( \chi \) is an element of \( \Omega^{(1,0)}(O(-4)) \), and the corresponding redundancy equation indicates \( \tilde{C} \) is represented by a cohomology class.

On the 4D space-time, the exchange of the \( \tilde{h}/h \) pair between twistor lines (i.e. space-time points) then generates the usual gravitational potential for a spin two excitation with the strength of Newton’s constant set by the ratio of an overall coefficient multiplying \( (7.18) \) and the bulk defect coupling \( (7.6) \). It would be interesting to refine this discussion further and fix the precise value of Newton’s constant.

### 7.4 Einstein versus Conformal Gravity

What is the low energy effective action for this putative theory of gravity? The leading order terms consistent with the symmetries of the system are:

\[
S_{\text{grav}} = \alpha \Lambda \int d^4x \sqrt{-g} (R - 2\Lambda) + \beta \int d^4x \sqrt{-g} W^2 + \ldots \tag{7.22}
\]

where \( R \) is the scalar curvature, \( \Lambda \) is the cosmological constant, \( W \) is the Weyl tensor, and \( \alpha \) and \( \beta \) are dimensionless couplings. In Einstein gravity, we would set \( \alpha = 1/(16\pi G_N \Lambda) \). The case \( \alpha = 0 \) corresponds to conformal gravity. This latter theory contains a number of pathologies compared to ordinary Einstein gravity. For example, the kinetic term is quartic in the momentum, and the resulting spectrum of modes contains ghost-like excitations. This is also the gravitational sector of the twistor string theories of \cite{7,34,47}.

We will now list a set of arguments that indicate that in our case, rather than getting conformal gravity, the emergent gravity theory is described by the Einstein action with a Newton coupling set by \( \sqrt{G_N} \sim \ell_{pl} \sim \ell/\sqrt{N} \).

- **Graviton scattering**: In principle, we can determine the effective action by computing all possible graviton scattering amplitudes which descend from the matrix model. So far, we have only computed the MHV amplitudes. Are these sufficient to distinguish the two theories? This is in fact a somewhat subtle question, because some features of Einstein gravity in (anti-)de Sitter space can be mimicked by conformal gravity with Neumann boundary conditions at spatial infinity \cite{61}.

MHV graviton scattering amplitudes describe the process of an incoming negative helicity graviton bouncing off a self-dual space-time background. Self-dual backgrounds are solutions to both the Einstein and the conformal theory. So we can set up the calculation in both theories, by starting with a self-dual geometry that asymptotically looks like an (anti-)de Sitter space-time. Conformal gravity contains two propagating graviton modes, one that behaves like the ordinary graviton and a ghost mode. By picking appropriate asymptotic conditions, we can isolate the Einstein graviton. The conformal gravity calculation then looks identical to the Einstein gravity calculation. The difference between the
two theories is that one theory has a dimensionful constant and the other a dimensionless one. The reason that we can compare the two directly is because we are considering the system on de Sitter space with finite curvature radius $\ell^2 \sim 1/\Lambda$.

For the sake of argument, let us suppose that the gravitational theory described by the matrix model is dominated by the conformal gravity term proportional to $\beta$. The effective Newton’s constant would then fixed by the coefficient $\beta$ and the de Sitter radius $\Lambda$: 

$$G_N \sim \frac{1}{\beta \Lambda}.$$ 

If we were to match our space-time theory to this prescription, we would need to set $\Lambda \sim 1/\ell^2$ and $G_N \sim \ell_{pl}^2 \sim \ell^2/N$ which would imply that $\beta \sim N$. In the large $N$/flat space scaling limit, this looks absurd. Indeed, conformal gravity (with any finite coupling $\beta$) does not have a non-zero MHV graviton scattering amplitude in flat space. Further, explicit calculations of graviton scattering in twistor string theory vanish for Einstein gravitons. Our double scaled matrix model does have finite flat space scattering amplitudes.

- **Absence of conformal symmetry at finite $\ell_{pl}$**: Conformal symmetry is explicitly broken in the double scaled matrix model, both due to the presence of a UV cut-off $\ell_{pl}$ and due to the introduction of the infinity bi-twistor. In the non-commutative twistor theory, only the compact $SU(4)$ subgroup of the complexified conformal group $SL(4, \mathbb{C})$ is unitarily realized. Moreover, the defect action, which involves the infinity twistor, further breaks the symmetry to $SO(5)$. The $SO(5)$ generators therefore play a distinguished role among the currents of the defect system. The (MHV manifestation) of the graviton modes are made up from these $SO(5)$ currents – specifically, those associated with the hermitian ‘translation’ generators (7.8). The hermitian generators (7.8) are precisely the ones that are compatible with the boundary conditions that selects the Einstein gravitons from the conformal gravity fluctuations.

- **CSW correspondence**: As explained in section 6.1 and in [1], integrating out the defect modes produces an effective action that, in the continuum limit, matches with the MHV effective action in [45, 46], which (by design in [45, 46], but derived here) evades the troubling appearance of conformal gravity amplitudes, that plague the twistor string proposal of [7, 47]. The appearance of gravity for us indeed has a different geometric origin,

19 Taking $\beta \sim N \to \infty$ amounts to taking a classical limit in conformal gravity. One could suspect that this classical limit amounts to a decoupling limit, in which the ghost mode of conformal gravity can be consistently decoupled from the rest of the theory. If this were possible, the left over graviton mode would propagate and interact like an ordinary graviton in Einstein gravity, with a finite Planck length equal to $\ell_{pl}$.

20 Note also that, by starting from the $S^4$ with finite radius $\ell$, we are able to define hermitian translation generators (7.8), and avoid the non-unitarity that plagued the twistor string theory of [47].
in which the Einstein gravitons immediately play a distinguished role.

Suppose the theory does indeed give rise to Einstein gravity, as we have argued. What kind of theory is it, and does the matrix model – or rather its UV parent theory, the holomorphic Chern-Simons theory with flux – provide a UV complete description? With regard to the first question, indications are that the IR theory will take the form of $\mathcal{N} = 4$ supergravity, possibly coupled to $\mathcal{N} = 4$ gauge theory. We have not found a clear condition that limits the possible rank of the gauge group. This suggests that the answer to the second question should be: No, the matrix model (and even its embedding in the topological B-model string theory) is not UV complete, but should be viewed as the low energy effective description of a more complete theory. To illustrate how such an embedding may arise, we now describe a potential realization of the matrix model in the physical superstring.

8 Embedding in Superstring Theory

In this section we propose an embedding of the twistor matrix model in a brane construction in superstring theory. In particular, this will establish a concrete UV completion for our system. The basic setup we consider is given by a D0-brane bound to a stack of eight-branes in type IIA superstring theory. We work in ten non-compact space-time dimensions, which we write as $\mathbb{R}_{\text{time}} \times \mathbb{R}^8 \times \mathbb{R}_L$. The eight-branes are assumed to be parallel, filling $\mathbb{R}_{\text{time}} \times \mathbb{R}^8$ and sitting at various points of $\mathbb{R}_L$. The D0-brane we shall be considering will be bound to one stack of eight-branes, but will be free to move in the worldvolume $\mathbb{R}_{\text{time}} \times \mathbb{R}^8$. The basic idea is that when a suitable flux is switched on along the worldvolume of the eight-branes, we obtain four copies of the supersymmetric harmonic oscillator. The statistical mechanics of this system is studied by compactifying on a thermal circle of radius $\beta$. We argue that the many body problem for a gas of 0-8 strings at low temperature realizes the twistor matrix model.

8.1 Brane Construction

Our main interest will be in the worldvolume theory of a single D0-brane probing a stack of $N_{D8}$ D8-branes coincident with an O8-plane. However, to frame our discussion, let us first briefly review the worldvolume theory of $N_{D0}$ D0-branes, first in the absence of other branes, and then in the presence of eight-branes.

Consider first the case of $N_{D0}$ D0-branes in ten flat space-time directions. This is a system which preserves sixteen real supercharges, and is described at low velocity by a quantum mechanics with $(8,8)$ supersymmetry. For $N_{D0}$ D0-branes, we have a $U(N_{D0})$ gauge theory with gauge field $A_{D0}$. All of the 0-0 strings transform in the adjoint of $U(N_{D0})$. The bosonic matter content consists of eight real scalars $X^I$ for $I = 1,..,8$ which
describe motion parallel to the D8-brane and an additional real scalar $X_\perp$ describing motion transverse to the D8-brane. In addition there are sixteen fermions, which transform as a Majorana-Weyl spinor of $so(9,1)$. Under the decomposition $so(9,1) \supset so(8) \times so(1,1)$, these fermions transform in the $8_s$ and $8_c$ of $so(8)$. We denote these two sets by $S_A \oplus \tilde{S}_{A'}$ for $A,A'=1,...,8$.

Introducing a stack of D8-branes leads to additional dynamics for the system. Due to their high codimension, adding D8-branes introduces a number of subtleties. For example, at finite distance from the D8-brane, the dilaton blows up. Consistent treatment of the system then requires introducing O-planes to prevent this. When the tadpole is not locally cancelled, there will also be a Romans mass on one side of the D8-brane\cite{63}.

The effective theory of the D0-brane in the presence of $N_{D8}$ D8-branes has been treated previously in the literature, see for example\cite{64}. This system preserves a chiral $(8,0)$ supersymmetry. In addition to the 0-0 strings, the worldvolume theory now contains an additional 0-8 string charged in the $(N_{D8},N_{D0})$, which is a complex fermion we denote by $\chi$. This mode is a singlet under the (on-shell) supersymmetry transformations, which is possible due to the low dimension and chiral nature of the supersymmetry. The resulting D0-D8 system is BPS. In particular, the D0-brane experiences no force in the direction transverse to the D8-brane. See\cite{64} for further discussion.

The system we shall mainly be interested in for our purposes is the case of $N_{D8}$ D8-branes coincident with an O8-plane. Local tadpole cancellation then fixes $N_{D8} = 16$, that is, the maximal gauge group is $SO(16)$. Let us note that $E_8$ can also be realized, though non-perturbatively. The presence of the orientifold plane affects the 0-0 and 0-8 strings as well. The D0-brane gauge group is now $O(N_{D0})$\footnote{As explained for example in\cite{65} the reason the gauge group is $O(N_{D0})$ rather than $SO(N_{D0})$ is because upon compactifying on the transverse circle $S^1_\perp$, there are $\mathbb{Z}_2$ valued Wilson line configurations available in the T-dual description.}. The 0-0 strings, which were initially in the adjoint of $U(N_{D0})$, are now two index representations of $O(N_{D0})$. The modes in the two index symmetric representation descend from $X^I \oplus S_A$, while the modes in the two index anti-symmetric descend from $A_D \oplus \tilde{S}_{A'} \oplus X_\perp$. The 0-8 strings now transform in the $(N_{D0},N_{D8})$, which is a real representation. To maintain a holomorphy convention, we shall often decompose $SO(N_{D8}) \supset U(N_{D8}/2)$. The 0-8 string content then consists of a vector-like pair of complex fermions $\chi \oplus \tilde{\chi}$, which are subject to the reality condition $\chi^\dagger = \tilde{\chi}$.

Consider now the special case of a single D0-brane. When $N_{D0} = 1$, we project out the mode describing motion transverse to the D8 stack, and also eliminate half of the fermionic modes. The gauge group is also in this case reduced to $O(1) = \mathbb{Z}_2$. The effective Lagrangian is (in units where the D0-brane mass is unity):

$$L_{\text{eff}} = \frac{1}{2} \partial_t X^I \partial_t X^I + \frac{i}{2} S_A \partial_t S_A + \frac{i}{2} (\tilde{\chi} D\chi + \chi D\tilde{\chi})$$

\begin{equation}
\end{equation}
where the covariant derivative acts on the fermionic modes as:

\[
D\chi = (\partial_t + igA_D8)\chi \quad (8.2)
\]

\[
\tilde{D}\tilde{\chi} = (\partial_t - igA_D8)\tilde{\chi} \quad (8.3)
\]

where \(A_{D8}\) is the pullback of the D8-brane gauge field to the D0-brane worldvolume, with \(g\) the gauge coupling of the D8-brane theory. In the above, we have specialized to the case of a gauge field activated in the \(U(N_{D8}/2)\) subgroup.

This system describes a D0-brane which is free to move inside of the D8-brane. In the quantum theory, the conjugate momenta to \(X^I\) are \(\Pi^I = \partial_t X^I\). This results in the commutation relations:

\[
[X^I, \Pi^J] = i\delta^{IJ}, \{S_A, S_B\} = \delta_{AB}, \quad \{\chi_k, \chi_l^\dagger\} = \delta_{kl} \quad (8.4)
\]

where \(k\) (resp. \(\bar{l}\)) denotes an index in the fundamental (resp. anti-fundamental) of \(U(N_{D8}/2)\). Observe that there is a degeneracy of ground states for the system. These correspond to the possible positions of the D0-brane in the superspace \(\mathbb{R}^{8|8}\). We will soon switch on a flux which identifies a canonical choice of complex structure, in which case we have \(\mathbb{C}^{4|4}\).

To break this degeneracy in the ground state energies we now switch on a magnetic flux which takes values in the overall \(U(1) \subset U(N_{D8}/2)\). The effective energy scale for the flux is a tunable parameter of the D8-brane worldvolume theory. Since the D0-brane is more accurately thought of as a superparticle, it is appropriate to consider a background gauge field configuration:

\[
A_{D8} = F^{(v)}_{IJ} X^I \partial_t X^J + F^{(s)}_{AB} S_A S_B \quad (8.5)
\]

where here, \(F^{(v)}\) and \(F^{(s)}\) reflect the presentation of the flux in the \(8_v\) and \(8_s\) of \(so(8)\). By triality, we can take an embedding of \(su(4) \times u(1) \subset so(8)\) so that the \(8_v\) and \(8_s\) decompose as:

\[
so(8) \supset su(4) \times u(1) \quad (8.6)
\]

\[
8_v \rightarrow 4_{+1} + \bar{4}_{-1} \quad (8.7)
\]

\[
8_s \rightarrow 4_{-1} + \bar{4}_{+1} \quad (8.8)
\]

The original basis of modes can now be traded for the usual coordinates of supertwistor space, \(Z^a \oplus \psi^i\). In terms of this basis, the pullback of the gauge field is:

\[
A_{D8} = F_{\alpha\beta} \left( Z^a \partial_t \overline{Z^\beta} - \overline{Z^a} \partial_t Z^\beta \right) + F_{\sigma\psi^i} \overline{\psi^j} \quad (8.9)
\]
The profile of this flux is taken to be:

\[ F = \frac{f_1}{2} dz^1 \wedge d\bar{z}^1 + \frac{f_2}{2} dz^2 \wedge d\bar{z}^2 + \frac{f_3}{2} dz^3 \wedge d\bar{z}^3 + \frac{f_4}{2} dz^4 \wedge d\bar{z}^4. \]  

(8.10)

In general, this choice of flux will break supersymmetry on the D8-brane, though the D0-brane may still experience an effective supersymmetric quantum mechanics.

Let us now study the quantum mechanics of the endpoint for the 0-8 string which is attached to the D8-brane. Evaluating in the background where the 0-8 string is present, the Lagrangian is:

\[ L_{\text{eff}} = \frac{1}{2} \partial_t X^I \partial_t X^I + \frac{i}{2} S_A \partial_t S_A + i g F_{I J}^{(v)} X^I \partial_t X^J + i g F_{A B}^{(s)} S_A S_B. \]  

(8.14)

Quantization is now of the standard type for a superparticle moving in a background magnetic flux. This is described by four supersymmetric harmonic oscillators, which by abuse of notation we label as \( Z^\alpha \) and \( \psi^i \). These modes have the same supercommutators as fuzzy supertwistor space, with characteristic frequencies:

\[ \omega_\alpha = \frac{2f_\alpha}{m_D 0}. \]  

(8.15)

Here we have rescaled the coordinates to include the explicit dependence on the D0-brane

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\textsuperscript{22}Following \[66-68\], to study the unbroken supersymmetries in the presence of this flux, introduce the block diagonal matrix \( M \) in the 8c representation (see e.g. \[67, 69-71\]):

\[ M = \bigoplus_{i=1}^4 M_i, \quad M_i = \begin{bmatrix} \cos 2\pi v_i & \sin 2\pi v_i \\ -\sin 2\pi v_i & \cos 2\pi v_i \end{bmatrix}. \]  

(8.11)

where each \( v_j \) is given by:

\[ \exp 2\pi i v_j = \frac{1 + i f_j}{1 - i f_j}. \]  

(8.12)

The condition for unbroken supersymmetry is that in the 8c representation, \( \rho(-M) \) has a unit eigenvalue. The amount of supersymmetry which is preserved is dictated by the maximal subalgebra of \( u(4) \subset so(8) \) which commutes with \( \rho(-M) \). A necessary condition for some supersymmetry to be preserved is:

\[ v_1 + ... + v_4 \in \mathbb{Z}. \]  

(8.13)

For a generic choice which retains some supersymmetry, this leaves a system with only two real supercharges. The case we shall eventually specialize to is given by taking all \( f_j = f \). In this case, we see that generically, all supersymmetry will be broken. However, in the limit \( f \to +\infty \), we see that all eight supercharges are retained. This corresponds to the limit \( v_i \to 1/2 \).
Figure 3: Depiction of the D0-brane bound to a stack of $N_c$ D8-branes and an O8-plane in the presence of a magnetic flux threading the D8-brane. The 0-8 strings are attached to the D0-brane, and the other end can wander inside the D8-brane. Switching on a background flux through the D8-branes, a gas of 0-8 strings descend into Landau levels graded by integers $M \geq 0$. Here, we have depicted a low temperature configuration which realizes fuzzy supertwistor space at level $N = M_{\text{max}}$.

mass. The effective Hamiltonian is:

$$H_{\text{eff}} = \sum_{\alpha=1}^{4} \omega_{\alpha} Z_{\alpha}^\dagger Z_{\alpha} + \sum_{i=1}^{4} \omega_{i} \psi_{i}^\dagger \psi_{i}$$

(8.16)

where the zero point energy cancels out. In what follows we focus on the special case of $\omega = \omega_{\alpha} = \omega_{i}$ for all $\alpha$ and $i$. Indeed, in this case is proportional to the Hamiltonian constraint operator $H_0$ of equation (3.18).

An important feature of this system is that it admits a decoupling limit, where we take $f$ to be large in string units, while keeping the ratio $f/m_{D0}$ small in string units. This is basically the same type of limit studied in [72] for non-commutative Yang-Mills theory. Here, this zero slope limit ensures that the low energy dynamics are well-described by this effective Hamiltonian.

The modes at level $M$ are described by the Hilbert space of fuzzy points for $\mathbb{C}P^{3|4}$, $\mathcal{H}_{\mathbb{C}P^{3|4}}(M)$, so there is a degeneracy of $K_M$ at each $M$. Thus, the phase space available to a 0-8 string is now a direct sum of fuzzy supertwistor spaces. Finally, at each level, there is an additional overall degeneracy, because the fermionic 0-8 string transforms in the fundamental of $U(N_{D8}/2)$. We denote the effective $N_c = N_{D8}/2$. See figure 3 for a depiction of this system.
8.2 Statistical Mechanics

Let us now study the statistical mechanics of a large number of 0-8 strings. We compactify
the temporal direction on a circle of circumference $\beta = 1/T$ and take thermal boundary
conditions for all modes. Though there are some subtleties in such stringy systems at high
temperature, i.e. small thermal radius, we shall mainly confine our discussion to the low
temperature limit.

In the grand canonical ensemble, the chemical potential for the 0-8 strings is controlled
by a flat connection from the D8-brane gauge field. The full gauge field configuration on
the D8-brane is then given by the magnetic flux considered earlier, and a holonomy from
the component which points along the thermal circle direction. The energetics of the 0-8
strings are unaffected by including the contribution from the flat connection, though it
does introduce a chemical potential for the system. Integrating over the thermal circle, we
obtain the fugacity $\zeta$ for the grand canonical ensemble:

$$\zeta = \exp \left( \int_{S^1_\beta} A_{D8} \right). \quad (8.17)$$

The chemical potential is then given by $\mu = T \log \zeta$.

Consider first the zero temperature limit, with a fixed number of 0-8 strings which we
denote by $N_{\text{tot}}$. The phase space for each particle is a non-commutative $\mathbb{C}^{4|4}$. Since the 0-8
strings are fermionic, they will start to fill up a Fermi surface. We will mainly be interested
in the case where the Fermi surface forms a complete shell up to a maximal level $M_{\text{max}} = N$.
This corresponds to taking a fixed number of states:

$$N_{\text{tot}} = \sum_{M=0}^{M_{\text{max}}} K_M. \quad (8.18)$$

Each energy shell corresponds to the Hilbert space of a fuzzy $\mathbb{CP}^{3|4}$ at level $M$. As we
increase $M_{\text{max}}$, we approach a large $N$ limit. The states fill out entries in the direct sum:

$$\mathcal{H}(N_{\text{tot}}) = \mathcal{H}_{\mathbb{CP}^{3|4}}(N) \otimes N_c = \bigoplus_{M=0}^{M_{\text{max}}} \mathcal{H}_{\mathbb{CP}^{3|4}}(M) \otimes N_c \quad (8.19)$$

where the tensor product with the factor of $N_c$ reflects the fact that each particle transforms
in the fundamental of $U(N_c)$. Up to this factor, we can identify this Hilbert space with the
fuzzy four-ball encountered previously in section 4.

The length scale $\ell$ is set by the inverse characteristic frequency of the system $\omega$:

$$\ell \sim \omega^{-1}. \quad (8.20)$$
We can also introduce an anisotropic scaling for the D8-brane magnetic flux, and thereby include the parameter $\gamma$ of the infinity bitwistor.

Returning to the case of uniform energy level spacing in the phase space, we would now like to model the effective dynamics of the many body system at finite temperature. It is helpful to work in terms of fields, which describe the collective excitations. There are two types of excitations which we can identify, corresponding to motion within the Fermi surface, and normal to it.

Excitations above the Fermi surface are 0-8 strings, which are characterized by a field $Q(Z^\dagger, \psi^\dagger, Z, \psi)$ in the fundamental of $U(N_c)$. Holes are 8-0 strings, and are characterized by a field $\tilde{Q}(Z^\dagger, \psi^\dagger, Z, \psi)$ in the anti-fundamental. Moving the location of the D0-brane, we see that there are also four complex scalars. These can be viewed as bound states of the $8-0 \oplus 0-8$ strings. If the particles were free to move inside of the D8-brane, we would have four such complex scalars $A^\alpha(Z^\dagger, \psi^\dagger, Z, \psi)$

These modes define the transformation of the states within the Fermi surface, and are therefore maps $\mathcal{H}(N_{tot}) \rightarrow \mathcal{H}(N_{tot} + 1)$. In other words, they are fuzzy $(0, 1)$-forms acting on $\mathcal{H}_{\mathbb{P}(4)}(N)$ which are also in the adjoint representation of $U(N_c)$. Since we are interested in the low energy dynamics confined to near the Fermi surface, there is an additional constraint:

$$Z^\dagger_\alpha A^\alpha = 0. \quad (8.21)$$

so there are only three independent bosonic degrees of freedom.

We fix the effective action via symmetry considerations. The kinetic operator for the particle/hole pairs corresponds to an excitation in the direction normal to the Fermi surface. In a local patch where we write $\mathbb{C}P^{3|4} \approx \mathbb{C}P^{2|4} \times \mathbb{C}P^{1|0}$, this is in the direction of the bosonic twistor line. From our earlier analysis of the defect kinetic operator, we know that a derivative operator which is local on the “space-time” directions of the twistor fibration is the kinetic operator $\mathcal{D}$. Symmetry consideration then dictate the kinetic operator for the particle/hole pair at low temperatures:

$$S_{\text{defect}} = \text{Tr}_{\mathcal{H}(N_{tot})} \left( I_{IJ} \tilde{Q} D^I Q \Omega^J \right) \quad (8.22)$$

where the form of the derivative operators is fixed by gauge invariance. Here, the defect action is integrated over the entire four-ball of the phase space. Excitations near the top of the Fermi surface cost less energy, and so at low energies, we get back the integral over just the top layer, given by the $\mathbb{C}P^{3|4}$ at level $N = M_{\text{max}}$.

For the bulk modes, there is a single topological term we can write down which is

\[23\text{Although it is tempting to also include four complex fermionic modes } \Psi^I(Z^\dagger, \psi^\dagger, Z, \psi), \text{ the geometric interpretation of these modes is less clear, as we cannot “give a vev” to a fermionic direction.}\]
consistent with the symmetries of the system; it is the holomorphic volume of the phase space in the supermanifold $\mathbb{C}^{4|4}$ occupied by the 0-8 strings:

$$S_{\text{bulk}}(\beta) = \int_{S^1} C_1 \wedge \text{Tr}_{\mathcal{H}(N_{\text{tot}})} \left( \varepsilon_{\alpha\beta\gamma\delta} D^\alpha D^\beta D^\gamma D^\delta \right) \psi^4$$

(8.23)

where $D^\alpha = Z^\alpha + A^\alpha$ and $C_1$ is the RR one-form potential. The coupling constant of the matrix model is obtained by integrating over the Euclidean thermal circle. Taking $C_1$ to be a constant one-form and integrating over the circle yields an overall coupling. In other words, the action becomes:

$$S_{\text{bulk}}(\beta) = \frac{1}{g^2} \text{Tr}_{\mathcal{H}(N_{\text{tot}})} \left( \varepsilon_{\alpha\beta\gamma\delta} D^\alpha D^\beta D^\gamma D^\delta \right) \psi^4$$

(8.24)

Hence, the low temperature dynamics of a large number of 0-8 strings reproduces the basic form of the twistor matrix model.

By embedding the matrix model in the physical superstring, we have equipped the model with a proposed UV completion. Consistency of the UV completion leads to some interesting restrictions on the form of the low energy effective theory. For example, it imposes a sharp upper bound on the rank of the gauge group. Note, however, that this restriction is milder than what is required in conformal $\mathcal{N} = 4$ supergravity for anomaly cancellation (see [73] for a review). A further interesting feature of this construction is that it is formulated with ten non-compact directions. Indeed, the embedding of the twistor matrix model into the superstring does not involve a traditional compactification at all.

Using this perspective, we can ask how various features of the 4D space-time are encoded in the gas of 0-8 strings. First note that the double scaling limit for realizing a flat space theory is quite natural in this setup; we can scale $N \to \infty$, but simultaneously decrease the energy spacing between the Landau levels. The currents of the matrix model are bilinears in the $Q\tilde{Q}$ modes, corresponding to the creation of particle/hole pairs. If we are at low energies, all of these states will be created near the top of the Fermi surface, and at the same Landau level. However, as we increase the available energy, holes can be created further down in the Fermi sea, and can also jump between neighboring Landau levels. Thus, a continuous extra dimension opens up in the direction normal to the Fermi surface. It is tempting to identify this with a holographic RG scale. At even higher excitation energies, the particles are no longer close to the Fermi surface, and simply move throughout the fuzzy $\mathbb{C}^{4|4}$ phase space. Note that even in this case, however, there is an upper bound on

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24In string theory the usual situation is that without a compactification of the “extra dimensions” one obtains a 4D gauge theory which is decoupled from gravity. For example, a D3-brane probing a non-compact Ricci flat manifold will achieve this. Here, the interpretation of the ten dimensions for the 4D theory is different, and gravity emerges along with the space-time.
the available excitation energies.

Though at finite $N$ the continuum theory interpretation is less evident, it is also the case where we have a truncated Hilbert space for Euclidean de Sitter. In this case, the jump in Landau levels is discontinuous. A further curious feature is that the effective ratio $\ell/\ell_{pl} \sim \sqrt{N}$ is modified at high energies so that states would appear to experience distinct values of the cosmological radius. One can also consider the high temperature regime – i.e. high compared to the level spacing – of the gas of 0-8 strings, but below the Hagedorn temperature. This is still a well-defined system, but now the Fermi surface itself will begin to break apart due to thermal fluctuations. Clearly, it would be interesting to study this set of issues further.

8.3 Dual Descriptions

The D0-D8 bound state we have been considering has a number of dual formulations, which provide additional insight into the dynamics of the system. Compactifying on a circle $S^1_\perp$ in the direction transverse to the eight-branes, we obtain a T-dual description in type I string theory, where the D0-brane is now a D1-brane. This is in turn dual to an F1-string of IIB via an S-duality. Using type I/heterotic duality, we can map this to a heterotic string wrapping $\mathbb{R}_{time} \times S^1_\perp$. Observe that the projection on the mode content we have observed is quite natural on the heterotic string side. There is also an intriguing connection to the $\mathcal{N} = 2$ string. Indeed, as advocated for example in [74], the worldvolume theory of the D1-brane is naturally viewed as a four-dimensional theory on $\mathbb{R}^{2,2}$ which has been reduced along a null vector. This suggests a more direct connection between $\mathcal{N} = 2$ strings and the physics of supertwistor space which would be interesting to develop further.

We can also use these dual formulations to realize more intricate gauge theories. The basic idea is to consider the low energy quantum mechanics of a D0-brane, now bound to a more general configuration of eight-branes and lower dimensional branes. Compactifying on a circle, this type of configuration is T-dual to a configuration of intersecting seven-branes in IIB. Going to strong coupling, we can study the resulting dynamics in terms of an effective heterotic string probing an appropriate flux background.

As an example, consider three stacks of branes which we denote as $\mathcal{B}, \mathcal{B}'$ and $\mathcal{B}''$ which wrap a common $\mathbb{R}_{time} \times \mathbb{R}^4$. We assume $\mathcal{B}$ describes a D8-brane, as in our previous discussion. Though it is beyond the scope of our present discussion to provide a full description of this system, we can sketch how additional matter sectors could arise. In addition to the $0 - \mathcal{B}$ strings, there will now be $0 - \mathcal{B}$ and $0 - \mathcal{B}''$ strings. Excitations above the Fermi surface can therefore be of different particle/hole types: They can correspond to pairs of $\mathcal{B} - 0 \oplus 0 - \mathcal{B}$ modes, which we previously identified with the modes of an $\mathcal{N} = 4$ supermultiplet in the adjoint of $U(N_c)$. In addition, we see that there are new effective modes, such
as $\mathcal{B} - 0 \oplus 0 \rightarrow \mathcal{B}'$ which are in the fundamental of the emergent $U(N_c)$. Observe that gauge invariance also dictates the form of the possible interaction terms of these composites.

Note also that the effective dynamics is not limited to unitary gauge groups. Indeed, using the dual description in the heterotic string, there is an effective worldsheet description available in the strong coupling limit of the IIB configurations, so we can also consider E-type gauge theories. This suggests that phenomenologically relevant gauge theories may also be engineered in this type of setup (by embedding in a GUT group). It would be of interest to provide a more explicit constructions along these lines.

### 9 Conclusions

In this paper we have provided evidence that $\mathcal{N} = 4$ SYM theory on an $S^4$ is dual to a large $N$ matrix model defined over non-commutative twistor space. The matrix model is the theory of the lowest Landau level for holomorphic Chern-Simons on $\mathbb{C}P^3$ expanded around the Yang monopole configuration. The matrix model incorporates the symmetries of the original space-time theory, and moreover, contains a natural class of spin 1 and spin 2 symmetry currents. Quite remarkably, in the flat space limit of the matrix model, the correlators of these currents correctly reproduce MHV gluon and graviton scattering [1]. This is a highly non-trivial feature, and provides evidence that the low energy limit is described by Einstein gravity. We have also presented a proposed UV completion of the model based on the low energy dynamics of a D0-D8 bound state. In the remainder of this section we discuss various open directions.

At a pragmatic level, it is important to check that the match to the 4D theory continues to work at loop level in the gauge theory. Indeed, loop level contributions from conformal graviton states is a significant hurdle for the general twistor string program. This would also provide another probe into the nature of the gravitational theory which is coupled to the gauge theory.

The most intriguing feature of the twistor matrix model is that it seems to contain an emergent gravitational sector. Though there are strong constraints on emergent theories of gravity, the basic setup of the matrix model violates some assumptions of the Weinberg-Witten theorem [75]. For example, the graviton and the space-time emerge simultaneously in a double scaling limit. Furthermore, the theory is defined as the infinite radius limit of a theory with positive curvature. Additionally, it is far from evident that our model possesses a local gauge invariant stress energy tensor [25]. Note, however, that although there is a short

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[25] One could ask whether the model comes with additional higher spin currents, which did appear in [9]. Here, note that the interactions involving our spin 2 excitation are suppressed by the small parameter $\ell_{pl}$, and so in the large $N$ limit, it is natural to expect any higher spin excitations to effectively decouple. Even in the finite $N$ setting, these excitations are more appropriately viewed
distance cutoff $\ell_{pl}$, the low energy theory will not contain Lorentz violating operators. This is simply because the model respects $so(5)$ and its Wigner-Inönü contraction in the flat space limit. To further test our conjecture, it would be interesting to see whether other features of quantum gravity such as black hole formation, or a more direct account of holographic entropy bounds are also realized.

We have also seen that fuzzy twistor geometry and the matrix model may naturally arise from a D0-D8 bound state. It is likely that this construction generalizes to situations with less supersymmetry and more realistic matter content. Observe that all ten of the string space-time dimensions are necessary to realize the 4D dual theory. Thus, four space-time dimensions is a rather special part of the construction.

Finally, as the 4D theory is formulated on a four-sphere, it is natural to analytically continue the theory to de Sitter space. In particular, the large but finite $N$ version of the matrix model suggests an intrinsic link between the de Sitter radius and the Planck length via $\ell_{dS}^2/\ell_{pl}^2 \sim N$. In other words, the cosmological constant is always parametrically small in Planck units. The finite number of degrees of freedom at finite $N$ is also suggestive of a holographic theory of de Sitter space \[80\].

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as diffeomorphisms on the fuzzy twistor space, and so are not expected to be 4D higher spins. It would nevertheless be interesting to study this issue further, and in particular, to see whether a truncated version of a higher spin theory along the lines of \[76\] could be implemented in a variant of the matrix model considered in this paper.
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