On the global regularity for the dissipative surface quasi-geostrophic equation

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Abstract: In this paper, we consider the two-dimensional surface quasi-geostrophic equation with fractional horizontal dissipation and fractional vertical thermal diffusion. Global existence of classical solutions is established when the dissipation powers are restricted to a suitable range. Due to the nonlocality of these 1D fractional operators, some of the standard energy estimate techniques such as integration by parts no longer apply, to overcome this difficulty, we establish several anisotropic embedding and interpolation inequalities involving fractional derivatives. In addition, in order to bypass the unavailability of the classical Gronwall inequality, we establish a new logarithmic type Gronwall inequality, which may be of independent interest and potential applications.

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1. Introduction

This paper concerns itself with the initial-value problem for the two-dimensional (2D) surface quasi-geostrophic (abbr. SQG) equation with fractional horizontal dissipation and fractional vertical thermal diffusion, which can be written as

\[\begin{aligned}
\partial_t \theta + (u \cdot \nabla) \theta + \mu \Lambda_{x_1}^{2\alpha} \theta + \nu \Lambda_{x_2}^{2\beta} \theta &= 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \\
\theta(x, 0) &= \theta_0(x),
\end{aligned}\]

where \(\theta\) is a scalar real-valued function, \(\mu \geq 0, \nu \geq 0, \alpha \in (0, 1), \beta \in (0, 1)\) are real constants, and the velocity \(u \equiv (u_1, u_2)\) is determined by the Riesz transforms of the potential temperature \(\theta\) via the formula

\[u = (u_1, u_2) = \left( -\frac{\partial_{x_2}}{\sqrt{-\Delta}} \theta, \frac{\partial_{x_1}}{\sqrt{-\Delta}} \theta \right) = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) := \mathcal{R}^\perp \theta,\]

where \(\mathcal{R}_1, \mathcal{R}_2\) are the standard 2D Riesz transforms. Clearly, the velocity \(u = (u_1, u_2)\) is divergence free, namely \(\partial_{x_1} u_1 + \partial_{x_2} u_2 = 0\). The fractional operators \(\Lambda_{x_1} := \sqrt{-\partial_{x_1}^2}\) and \(\Lambda_{x_2} := \sqrt{-\partial_{x_2}^2}\) are defined through the Fourier transform, namely

\[\mathcal{F}_{x_1}^{2\alpha} f(\xi) = |\xi_1|^{2\alpha} \hat{f}(\xi), \quad \mathcal{F}_{x_2}^{2\beta} f(\xi) = |\xi_2|^{2\beta} \hat{f}(\xi),\]

where

\[\hat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) \, dx.\]
The SQG equation arises from the geostrophic study of the highly rotating flow (see for instance [30]). In particular, it is the special case of the general quasi-geostrophic approximations for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers, see [10, 30] and the references cited there. Mathematically, as pointed out by Constantin, Majda and Tabak [10], the inviscid SQG equation (i.e., (1.1) with \( \mu = \nu = 0 \)) shares many parallel properties with those of the 3D Euler equations such as the vortex-stretching mechanism and thus serves as a lower-dimensional model of the 3D Euler equations. We remark that the inviscid SQG equation is probably among the simplest scalar partial differential equations, however, the global regularity problem still remains open.

The system (1.1) is deeply related to the classical fractional dissipative SQG equation, with its form as follows

\[
\begin{aligned}
\partial_t \theta + (u \cdot \nabla) \theta + \mu \Lambda^{2\alpha} \theta &= 0, \\
\theta(x,0) &= \theta_0(x),
\end{aligned}
\]

where the classical fractional Laplacian operator \( \Lambda^{2\alpha} := (-\Delta)^\alpha \) is defined through the Fourier transform, namely

\[
\hat{\Lambda}^{2\alpha} f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi).
\]

Obviously, the above system (1.2) can be deduced from the system (1.1) with \( \alpha = \beta \) and \( \mu = \nu \). Because of its important physical background and profound mathematical significance, the SQG equation attracts interest of scientists and mathematicians. The first mathematical studies of the SQG equation was carried out in 1994s by Constantin, Majda and Tabak [10], where they considered the inviscid SQG case, and established the local well-posedness and blow-up criterion in the Sobolev spaces. Since then, the global regularity issue concerning the SQG has recently been studied very extensively and important progress has been made (one can see [5] for a long list of references). Let us briefly recall some related works on the system (1.2). Due to its analogy with 3D incompressible Navier-Stokes equations, the cases \( \alpha > \frac{1}{2} \), \( \alpha = \frac{1}{2} \) and \( \alpha < \frac{1}{2} \) are called sub-critical, critical and super-critical, respectively. The global regularity of the SQG equation seems to be in a satisfactory situation in the subcritical and critical cases. The subcritical case has been essentially resolved in [12, 31] (see also [19, 22, 32] and references therein). Constantin, Córdoba and Wu in [9] first address the global regularity issue for the critical case and obtained a small data global existence result. More precisely, they showed that there is a unique global solution when \( \theta_0 \) is in the critical space \( H^1 \) under a smallness assumption on \( \|\theta_0\|_{L^\infty} \). In fact, due to the balance of the nonlinear term and the dissipative term in (1.2), the global existence of the critical case is a very challenge issue, whose global regularity without small condition has been successfully established by two elegant papers with totally different approach, namely Caffarelli-Vasseur [2] via the De Giorgi iteration method and Kiselev, Nazarov-Volberg [26] relying on a new nonlocal maximum principle. We also refer to Kiselev-Nazarov [25] and Constantin-Vicol [11] for another two delicate and still quite different proofs of the same issue. See also the works [1, 18, 20, 29] where same type of results have been obtained. However, in terms of the supercritical case whether solutions (for large data) remain globally regular or not is a remarkable open problem. Although the global well-posedness for arbitrary initial data is still open for the supercritical SQG equation, some interesting regularity criteria (see for example [12, 6, 20, 21]) and small data global existence results (see for
instance \([8, 14, 22, 35, 37]\) have been established. Moreover, the global existence of weak solutions and the eventual regularity of the corresponding weak solutions to supercritical SQG equation have been established (see, e.g. \([31, 16, 24, 33, 15]\)). For many other interesting results on the SQG equation, we refer to \([13, 18, 34, 36]\), just to mention a few.

As stated in the previous paragraph, on the one hand, it is not hard to establish the global regularity for the SQG equation \((1.2)\) with \(\alpha > \frac{1}{2}\). However, on the other hand, the global regularity problem of the inviscid SQG equation is still an open problem. Comparing these two extreme cases, it is natural for us to consider the intermediate cases. Note that in all the papers mentioned above, the equation is assumed to have the standard fractional dissipation. In fact, compared with the SQG equation with the standard fractional dissipation, little has been done for the system \((1.1)\) as many techniques such as integration by parts no longer apply. Very recently, the author with collaborators in \([38]\) proved the global regularity result of the system \((1.1)\) with \(\mu > 0, \nu = 0, \alpha = 1\) or \(\mu = 0, \nu > 0, \beta = 1\). In this paper, we consider the intermediate case to explore how fractional horizontal dissipation and fractional vertical thermal diffusion would affect the regularity of solutions to the SQG equation. To the best of our knowledge, such system of equation as in \((1.1)\) has never been studied before. The main purpose of this paper is to establish the global regularity when the dissipation powers are restricted to a suitable range. More specifically, the main result of this paper is the following global regularity result.

**Theorem 1.1.** Let \(\theta_0 \in H^s(\mathbb{R}^2)\) for \(s \geq 2\). If \(\alpha \in (0, 1)\) and \(\beta \in (0, 1)\) satisfy

\[
\beta > \begin{cases} 
\frac{1}{2\alpha + 1}, & 0 < \alpha \leq \frac{1}{2}, \\
\frac{1 - \alpha}{2\alpha}, & \frac{1}{2} < \alpha < 1,
\end{cases}
\tag{1.3}
\]

then the system \((L.1)\) admits a unique global solution \(\theta\) such that for any given \(T > 0\),

\[
\theta \in L^\infty([0, T]; H^s(\mathbb{R}^2)), \quad \Lambda_{x_1}^\alpha \theta, \ \Lambda_{x_2}^\beta \theta \in L^2([0, T]; H^s(\mathbb{R}^2)).
\]

We outline the main ideas and difficulties in the proof of this theorem. A large portion of the efforts are devoted to obtaining global \textit{a priori} bounds for \(\theta\) on the interval \([0, T]\) for any given \(T > 0\). The proof is largely divided into two steps, namely, the global \(H^1\)-estimate and the global \(H^2\)-estimate. The first difficulty comes from the presence of the general 1D fractional Laplacian dissipation which is a nonlocal operator, and thus some of the standard energy estimate techniques such as integration by parts no longer apply. Concerning the difficulty caused by the presence of the 1D nonlocal operator, we need to establish the anisotropic embedding and the interpolation inequalities involving fractional derivatives. The second major difficulty lies in the unboundedness of the Riesz transform between the space \(L^\infty\). More precisely, if one tries to establish the global \(H^1\)-estimate, then one needs to control the quantity \(\|u(t)\|_{L^\infty}\). However, due to the relation \(u = R^\perp \theta\), the boundedness of \(\|u(t)\|_{L^\infty}\) is obviously not guaranteed even if we have \(\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}\). To overcome this kind of difficulty, we first resort to following
logarithmic Sobolev interpolation inequality

\[ \|f\|_{L^\infty} \leq C(1 + \|f\|_{L^2} + \|f\|_{\dot{B}^0_{\infty, \infty}} \ln (e + \|\Lambda^\sigma f\|_{L^2}), \quad \forall \sigma > 1. \]

In what follows, the resulting corresponding \(H^1\)-estimate is of the logarithmic type

\[ \frac{d}{dt} A(t) + B(t) \leq \tilde{C}_1 (A(t) + e) + \tilde{C}_2 \left( \ln (A(t) + B(t) + e) \right)^\rho (A(t) + e) \tag{1.4} \]

for some absolute constants \(\tilde{C}_1 > 0\) and \(\tilde{C}_2 > 0\). The natural next step would be to make use of the logarithmic Gronwall inequality, but the power \(\rho > 1\) leads to the unavailability of the known Gronwall inequality including the very recent result (Lemma 2.3 of [28]). This motives us to consider the relationship between \(A(t)\) and \(B(t)\). As a matter of fact, by fully exploiting of the dissipation of the SQG equation (1.1), we obtain the key estimate

\[ B(t) \geq C_1 A^\gamma(t), \quad \gamma > 1 \tag{1.5} \]

for some absolute constant \(C_1 > 0\). Fortunately, if the relationship (1.5) holds, then it indeed implies the boundedness of the quantity \(A(t)\) (see Lemma 2.1 for details), which is nothing but the desired global \(H^1\)-estimate. Next, we are able to obtain the global \(H^2\)-estimate by combining the anisotropic Sobolev inequality (see Lemma 2.5) and the obtained global \(H^1\)-estimate. Finally, the global existence of \(H^s\)-estimate follows easily.

The method adopted in proving Theorem 1.1 may also be adapted with almost no change to the study of the following 2D incompressible porous medium equation with partial dissipation:

\[
\begin{align*}
\partial_t \theta + (u \cdot \nabla) \theta + \mu \Lambda_{x_1}^{2\alpha} \theta + \nu \Lambda_{x_2}^{2\beta} \theta &= 0, \\
u &= -\nabla p - \theta e_2, \\
\nabla \cdot u &= 0, \\
\theta(x, 0) &= \theta_0(x).
\end{align*}
\tag{1.6}
\]

More precisely, the result can be stated as follows.

**Theorem 1.2.** Let \(\theta_0 \in H^s(\mathbb{R}^2)\) for \(s \geq 2\). If \(\alpha \in (0, 1)\) and \(\beta \in (0, 1)\) satisfy (1.3), then the system (1.6) admits a unique global solution \(\theta\) such that for any given \(T > 0\),

\[ \theta \in L^\infty([0, T]; H^s(\mathbb{R}^2)), \quad \Lambda_{x_1}^{\alpha} \theta, \, \Lambda_{x_2}^{\beta} \theta \in L^2([0, T]; H^s(\mathbb{R}^2)). \]

**Remark 1.3.** As a matter of fact, the equation \(u = -\nabla p - \theta e_2\) and the incompressible condition \(\nabla \cdot u = 0\) allow us to conclude

\[ u = (-\mathcal{R}_1 \mathcal{R}_2 \theta, \, \mathcal{R}_1 \mathcal{R}_1 \theta). \]

Whence, performing the same manner as adopted in proving Theorem 1.1 one may complete the proof of Theorem 1.2 immediately. To avoid redundancy, we omit the details.

The present paper is organized as follows. In Section 2, we provide several useful lemmas which play a key role in the main proof. Then we dedicate to the proof of Theorem 1.1 in Section 3.
2. Preliminaries

In this section, we collect some preliminary results, including a logarithmic type Gronwall inequality, an anisotropic Sobolev inequality and several interpolation inequalities involving fractional derivatives, which will be used in the rest of this paper. In this paper, all constants will be denoted by $C$ that is a generic constant depending only on the quantities specified in the context. If we need $C$ to depend on a parameter, we shall indicate this by subscripts.

We first establish the following logarithmic type Gronwall inequality which will play an important role in the proof of Theorem 1.1.

Lemma 2.1. Assume that $l(t), m(t), n(t)$ and $f(t)$ are all nonnegative and integrable functions on $(0, T)$ for any given $T > 0$. Let $A \geq 0$ and $B \geq 0$ be two absolutely continuous functions on $(0, T)$, satisfying for any $t \in (0, T)$

$$A'(t) + B(t) \leq \left[ l(t) + m(t) \ln(A(t) + e) + n(t) \left( \ln(A(t) + B(t) + e) \right)^{\alpha} \right] (A(t) + e) + f(t), \quad (2.1)$$

where $\alpha > 1$. Assume further that for some positive constant $C_1 > 0$

$$B(t) \geq C_1 A^{\gamma}(t), \quad \gamma > 1, \quad (2.2)$$

and for constants $K \in [0, \infty), \beta \in [0, \frac{2-1}{\gamma})$ such that for any $t \in (0, T)$

$$n(t) \leq K \left( A(t) + B(t) + e \right)^{\beta}. \quad (2.3)$$

Then the following estimate holds true

$$A(t) + \int_0^t B(s) \, ds \leq \tilde{C}(C_1, l, m, n, f, \alpha, \gamma, K, t), \quad (2.4)$$

for any $t \in (0, T)$. In particular, for the case $\beta = 0$, namely,

$$n(t) \leq K, \quad (2.5)$$

the estimate $2.4$ still holds true.

Remark 2.2. It is worthwhile to mention that Li-Titi [28] established a logarithmic type Gronwall inequality with $\alpha \leq 1$ and $\beta = 0$, but without the restriction (2.2), we also refer to Cao-Li-Titi [3] for more general result. We remark that the restriction $\alpha \leq 1$ is a crucial condition in the previous works. Noticing that the PDEs of the fluid mechanic with some certain dissipation, the condition (2.2) may be true, and corresponding with some $\alpha > 1$. This motives us to establish a logarithmic type Gronwall inequality like Lemma 2.1.

Proof of Lemma 2.1. First, denoting

$$A_1 := A + e + \sigma, \quad B_1 := A + B + e + \sigma,$$

where $\sigma > 0$ to be fixed hereafter, we thus obtain

$$A_1' + B_1 = A' + A + B + e + \sigma$$

$$\leq \left[ l(t) + m(t) \ln(A + e) + n(t) \left( \ln(A + B + e) \right)^{\alpha} \right] (A + e) + A + e + \sigma + f(t)$$

$$= \left[ l(t) + m(t) \ln(A_1 - \sigma) + n(t) \left( \ln(B_1 - \sigma) \right)^{\alpha} \right] (A_1 - \sigma) + A_1 + f(t)$$
≤ \left[ 1 + l(t) + m(t) \ln A_1 + n(t) \left( \ln B_1 \right)^\alpha \right] A_1 + f(t). \quad \text{(2.6)}

Dividing both sides of the above differential inequality (2.6) by $A_1$ and using the fact $A_1 \geq 1$, we further have
\[(\ln A_1)' + \frac{B_1}{A_1} \leq 1 + l(t) + m(t) \ln A_1 + n(t) \left( \ln B_1 \right)^\alpha + f(t). \quad \text{(2.7)}\]

It follows from (2.2) that
\[B_1(t) \geq C_1 \frac{A_1^\gamma}{2^{\gamma-1} A_1^\gamma}, \quad \gamma > 1. \quad \text{(2.8)}\]

As a matter of fact, one has
\[B_1 = A + B + e + \sigma \]
\[= A_1 + B \]
\[\geq A_1 + C_1 A^\gamma \]
\[= A_1 + C_1 (A_1 - e - \sigma)^\gamma \]
\[= \left( \frac{1}{A_1^{\gamma-1}} + C_1 \left( 1 - \frac{e + \sigma}{A_1} \right)^\gamma \right) A_1^\gamma \]
\[\geq \max \left\{ \frac{1}{2^{\gamma-1}(\sigma + e)^{\gamma-1}}, \frac{C_1}{2^\gamma} \right\} A_1^\gamma \]
\[\geq \frac{C_1}{2^{\gamma-1}} A_1^\gamma, \]

where in the sixth line we have used
\[\frac{1}{A_1^{\gamma-1}} + C_1 (1 - \frac{\sigma + e}{A_1})^\gamma \geq \begin{cases} \frac{1}{2^{\gamma-1}(\sigma + e)^{\gamma-1}}, & \sigma + e \leq A_1 \leq 2(\sigma + e), \\ \frac{C_1}{2^\gamma}, & A_1 \geq 2(\sigma + e), \end{cases} \]

and in the last line we have taken $\sigma$ satisfying
\[\sigma \geq \left( \frac{2}{C_1} \right)^\gamma - e \Rightarrow \frac{C_1}{2^\gamma} \geq \frac{1}{2^{\gamma-1}(\sigma + e)^{\gamma-1}}. \]

Now under the assumption of (2.8), we will show the key bound
\[(\ln B_1)^\alpha \leq C_2 \frac{B_1^{\theta_1}}{A_1^{\theta_2}} + C_3 \ln A_1, \quad \text{(2.9)}\]

where $C_3, C_4, \theta_1, \theta_2$ are positive constants satisfying $\theta_2 < \gamma \theta_1$. To this end, we define a function
\[F(B_1) = C_2 \frac{B_1^{\theta_1}}{A_1^{\theta_2}} + C_3 \ln A_1 - \left( \ln B_1 \right)^\alpha. \]

Next we will find some conditions to guarantee that $F(B_1)$ is a nondecreasing function for $B_1 \geq \frac{C_1}{2^{\gamma-1}} A_1^\gamma$. As a result, if (2.9) holds, then it suffices
\[F(B_1) \geq F\left( \frac{C_1}{2^{\gamma-1}} A_1^\gamma \right) = \frac{C_2 C_1^{\theta_1}}{2^{\gamma-1} \theta_1} A_1^{\gamma \theta_1 - \theta_2} + C_3 \ln A_1 - \left( \ln C_1 - (\gamma - 1) \ln 2 + \gamma \ln A_1 \right)^\alpha. \]
Thanks to $\theta_2 < \gamma \theta_1$, it is not hard to check that there exists a suitable large $\sigma_1 = \sigma_1(C_1, C_2, C_3, \alpha, \gamma, \theta_1, \theta_2) > 0$ such that for all $\sigma \geq \sigma_1$, we have

$$\frac{C_2 C_1^{\theta_1}}{2(\gamma - 1) \theta_1} A_1^{\theta_1 - \theta_2} + C_3 \ln A_1 - \left( \ln C_1 - (\gamma - 1) \ln 2 + \gamma \ln A_1 \right) \alpha \geq 0.$$  

In order to show the non decreasing property of $F(B_1)$, we differentiate it to get

$$F'(B_1) = \left( C_2 \theta_1 \frac{B_1^{\theta_1}}{A_1^{\theta_2}} - \alpha \left( \ln B_1 \right)^{\alpha - 1} \right) \frac{1}{B_1}.$$  

By the fact $B_1 \geq \frac{C_1}{2^{\gamma - 1}} A_1^\gamma$, one has

$$C_2 \theta_1 \frac{B_1^{\theta_1}}{A_1^{\theta_2}} - \alpha \left( \ln B_1 \right)^{\alpha - 1} \geq C_2 \theta_1 \left( \frac{C_1}{2^{\gamma - 1}} \right)^{\frac{\theta_2}{\gamma}} B_1^{\theta_1 - \frac{\theta_2}{\gamma}} - \alpha \left( \ln B_1 \right)^{\alpha - 1}.$$  

Similarly, one can show that there exists a suitable large $\sigma_2 = \sigma_2(C_1, C_2, \alpha, \gamma, \theta_1, \theta_2) > 0$ such that for all $\sigma \geq \sigma_2$, we obtain

$$C_2 \theta_1 \left( \frac{C_1}{2^{\gamma - 1}} \right)^{\frac{\theta_2}{\gamma}} B_1^{\theta_1 - \frac{\theta_2}{\gamma}} - \alpha \left( \ln B_1 \right)^{\alpha - 1} \geq 0.$$  

Now the above bound yields that $F'(B_1) \geq 0$ for $B_1 \geq \frac{C_1}{2^{\gamma - 1}} A_1^\gamma$. Combining the above analysis, if we take $\sigma \geq \max\{\sigma_1, \sigma_2\}$, then the desired (2.9) indeed holds. Notice that

$$n(t) \leq K \left( A(t) + B(t) + e \right)^{\beta} \leq KB_1^\beta.$$  

and using (2.9), it is not hard to check

$$n(t) \left( \ln B_1 \right)^{\alpha} \leq n(t) \left( C_2 \frac{B_1^{\theta_1}}{A_1^{\theta_2}} + C_3 \ln A_1 \right)$$

$$= C_2 n(t) \frac{B_1^{\theta_1}}{A_1^{\theta_2}} + C_3 n(t) \ln A_1$$

$$\leq C_2 KB_1^\beta \frac{B_1^{\theta_1}}{A_1^{\theta_2}} + C_3 n(t) \ln A_1$$

$$= C_2 K \left( \frac{B_1}{A_1} \right)^{\beta + \theta_1} + C_3 n(t) \ln A_1$$

$$\leq \frac{B_1}{2 A_1} + C(C_2, \theta_1, \theta_2, \alpha, \beta, K) + C_3 n(t) \ln A_1,$$  

(2.10)

where we have used the following condition

$$\theta_2 = \beta + \theta_1 < 1.$$  

This along with $\theta_2 < \gamma \theta_1$ implies

$$\frac{\beta}{\gamma - 1} < \theta_1 < 1 - \beta,$$

which leads to the restriction

$$\beta < \frac{\gamma - 1}{\gamma}.$$
Therefore, we first fix $C_2$, $C_3$, $\theta_1$ and $\theta_2$, then we choose
\[ \sigma \geq \max \left\{ \left( \frac{2}{C_1} \right)^{\frac{1}{\gamma - 1}} - e, \sigma_1, \sigma_2 \right\}, \]
where $\sigma_1 = \sigma_1(C_1, \alpha, \beta, \gamma)$ and $\sigma_2 = \sigma_2(C_1, \alpha, \beta, \gamma) > 0$. Summing up \((2.7)\) and \((2.10)\), we conclude
\[ (\ln A_1)' + \frac{B_1}{2A_1} \leq (m(t) + C_3 n(t)) \ln A_1 + C(C_1, \alpha, \beta, \gamma) + l(t) + f(t). \quad (2.11) \]
For the sake of simplicity, we denote
\[ X(t) := \ln A_1(t) + \int_0^t \frac{B_1(s)}{2A_1(s)} \, ds, \]
then it follows from \((2.11)\) that
\[ X'(t) \leq (C_1, \alpha, \beta, \gamma) + l(t) + f(t) + (m(t) + C_3 n(t)) X(t). \]
Whereas by using a standard Gronwall inequality, we obtain
\[ X(t) \leq e^{\int_0^t (m(s) + C_3 n(s)) \, ds} \left( X(0) + \int_0^t ((C_1, \alpha, \beta, \gamma) + l(s) + f(s)) \, ds \right) \]
\[ = C(C_1, l, m, n, f, \alpha, \beta, \gamma, K, t). \quad (2.12) \]
According to the definition of $X$, we infer
\[ A_1(t) \leq e^{X(t)} \leq e^{C(C_1, l, m, n, f, \alpha, \beta, \gamma, K, t)}. \]
Moreover, it is also easy to see that
\[ \int_0^t B_1(s) \, ds = \int_0^t 2A_1(s) \frac{B_1(s)}{2A_1(s)} \, ds \]
\[ \leq \int_0^t 2 \left( \max_{0 \leq \tau \leq t} A_1(\tau) \right) \frac{B_1(s)}{2A_1(s)} \, ds \]
\[ \leq 2e^{C(C_1, l, m, n, f, \alpha, \beta, \gamma, K, t)} \int_0^t \frac{B_1(s)}{2A_1(s)} \, ds \]
\[ \leq 2C(C_1, l, m, n, f, \alpha, \beta, \gamma, K, t) e^{C(C_1, l, m, n, f, \alpha, \beta, \gamma, K, t)}. \quad (2.13) \]
This concludes the proof of Lemma \textit{2.1}. □

The following anisotropic Sobolev inequalities will be frequently used later.

\textbf{Lemma 2.3.} \textit{The following anisotropic interpolation inequalities hold true for $i = 1, 2$}
\[ \| \Lambda_{x_i}^s f \|_{L^2} \leq C \| f \|_{L^2}^{\frac{1-\gamma}{\gamma}} \| \Lambda_{x_i}^\delta \partial_{x_i} f \|_{L^2}^{\frac{\gamma}{\gamma - \delta}}, \quad (2.14) \]
where $0 \leq s \leq \delta + 1$. In particular, we have
\[ \| \Lambda_{x_i}^s f \|_{L^2} \leq C \| f \|_{L^2}^{1-\gamma} \| \Lambda_{x_i}^\delta f \|_{L^2}^{\gamma}, \quad 0 \leq \gamma \leq \varrho. \quad (2.15) \]

\textit{Proof of Lemma 2.3.} It suffices to show \((2.14)\) for $i = 1$ as $i = 2$ can be performed as the same manner. By the interpolation inequality and the Young inequality, it is obvious to check that
\[ \| \Lambda_{x_1}^s f \|^2_{L^2} = \int_{\mathbb{R}} \int_{\mathbb{R}} |\Lambda_{x_1}^s f(x_1, x_2)|^2 \, dx_1 \, dx_2 \]
one may conclude which is nothing but the desired result (2.14). Following the proof of (2.14), the estimate (2.15) immediately holds true. This completes the proof of the lemma.

The following anisotropic interpolation inequalities hold true for Lemma 2.4.

\[ \| \partial_{x_1} f \|_{L^2(\gamma + 1)} \leq C \| f \|_{L^\infty}^{\gamma} \| \Lambda_{x_1} \partial_{x_1} f \|_{L^2}^{1-\gamma}, \quad \gamma \geq 0, \quad (2.16) \]

\[ \| \Lambda_{x_1}^\delta f \|_{L^{2(\rho + 1)}(\delta)} \leq C \| f \|_{L^\infty}^{1-\rho} \| \Lambda_{x_1}^\delta \partial_{x_1} f \|_{L^2}^{\rho}, \quad 0 \leq \rho \leq \rho + 1. \quad (2.17) \]

**Proof of Lemma 2.4.** It is sufficient to prove (2.16) and (2.17) for \( i = 1 \). We first recall the following one-dimensional Sobolev inequality

\[ \| \partial_{x_1} g \|_{L^{2(\gamma + 1)}(2\gamma)}(x_1) \leq C \| g \|_{L^\infty(2\gamma)}(x_1) \| \Lambda_{x_1}^{\gamma} \partial_{x_1} g \|_{L^2(2\gamma)}(x_1), \]

where we have used the sub-index \( x_1 \) with the Lebesgue spaces to emphasize that the norms are taken in one-dimensional Lebesgue spaces with respect to \( x_1 \). Thanks to the above interpolation inequality and the Young inequality, we have

\[ \| \partial_{x_1} f \|_{L^{2(\gamma + 1)}(2\gamma)}(x_1) = \int_{x_2} \| \partial_{x_1} f(x_1, x_2) \|_{L^{2(\gamma + 1)}(2\gamma)}(x_1) dx_2 \]

\[ \leq C \int_{x_2} \| f(x_1, x_2) \|_{L^\infty(2\gamma)}(x_1) \| \Lambda_{x_1}^{\gamma} \partial_{x_1} f(x_1, x_2) \|_{L^2(2\gamma)}(x_1) dx_2 \]

\[ \leq C \| f(x_1, x_2) \|_{L^\infty(2\gamma)}(x_1) \int_{x_2} \| \Lambda_{x_1}^{\gamma} \partial_{x_1} f(x_1, x_2) \|_{L^2(2\gamma)}(x_1) dx_2 \]

\[ = C \| f \|_{L^\infty(\gamma)}(x_1) \| \Lambda_{x_1}^{\gamma} \partial_{x_1} f \|_{L^2(\gamma)}(x_1), \]

which implies that

\[ \| \partial_{x_1} f \|_{L^{2(\gamma + 1)}(2\gamma)}(x_1) \leq C \| f \|_{L^\infty(\gamma)}(x_1) \| \Lambda_{x_1}^{\gamma} \partial_{x_1} f \|_{L^2(\gamma)}(x_1). \]

Similarly, using the following one-dimensional Sobolev inequality

\[ \| \Lambda_{x_1}^\delta g \|_{L^{2(\rho + 1)}(\rho + 1)}(\delta) \leq C \| g \|_{L^\infty(\rho + 1)}(\rho + 1) \| \Lambda_{x_1}^\delta \partial_{x_1} g \|_{L^2(\rho + 1)}(\rho + 1), \]

one may conclude

\[ \| \Lambda_{x_1}^\delta f \|_{L^{2(\rho + 1)}(\rho + 1)}(\delta) = \int_{x_2} \| \Lambda_{x_1}^\delta f(x_1, x_2) \|_{L^{2(\rho + 1)}(\rho + 1)}(\delta) dx_2 \]

\[ \leq C \int_{x_2} \| f(x_1, x_2) \|_{L^\infty(\rho + 1)}(\rho + 1) \| \Lambda_{x_1}^\delta \partial_{x_1} f(x_1, x_2) \|_{L^2(\rho + 1)}(\rho + 1) dx_2 \]
\[ \leq C\|f(x_1, x_2)\|_{L_\infty}^{2(\theta+1-\delta)} \int_{\mathbb{R}} \|\Lambda_{x_1}^\delta \partial_{x_1} f(x_1, x_2)\|_{L_2}^2 \, dx_2 \]

\[ = C\|f\|_{L_\infty}^{2(\theta+1-\delta)} \|\Lambda_{x_1}^\delta \partial_{x_1} f\|_{L_2}^2, \]

which leads to the following desired estimate

\[ \|\Lambda_{x_1}^\delta f\|_{L_2}^{2(\theta+1)} \leq C\|f\|_{L_\infty}^{1-\frac{\delta}{\theta+1}} \|\Lambda_{x_1}^\delta \partial_{x_1} f\|_{L_2}^{\frac{\delta}{\theta+1}}. \]

We therefore conclude the proof of Lemma 2.4. \( \square \)

In order to obtain the higher regularity, we need to establish the following anisotropic Sobolev inequality.

**Lemma 2.5.** Let \( f \in L^p_{x_2} L^q_{x_1}(\mathbb{R}^2) \) for \( p, q \in [2, \infty] \). If \( g, h \in L^2(\mathbb{R}^2), \Lambda_{x_1}^\gamma_1 g, \Lambda_{x_2}^\gamma_2 h \in L^2(\mathbb{R}^2) \) for any \( \gamma_1 \in \left(\frac{1}{p}, 1\right) \) and \( \gamma_2 \in \left(\frac{1}{q}, 1\right) \), then it holds true

\[ \int \int |f g h| \, dx_1 dx_2 \leq C \|f\|_{L^p_{x_2} L^q_{x_1}} \|g\|_{L^2}^{1-\frac{1}{\gamma_1 p}} \|\Lambda_{x_1}^\gamma_1 g\|_{L^2}^{\frac{1}{\gamma_1 p}} \|h\|_{L^2}^{1-\frac{1}{\gamma_2 q}} \|\Lambda_{x_2}^\gamma_2 h\|_{L^2}^{\frac{1}{\gamma_2 q}}, \quad (2.18) \]

where here and in sequel, we use the notation

\[ \|h\|_{L^p_{x_2} L^q_{x_1}} := \left( \int \int |h(\cdot, x_2)|^q_{L^q_{x_1}} \, dx_2 \right)^{\frac{1}{q}}. \]

In particular, let \( f, g, h \in L^2(\mathbb{R}^2) \) and \( \Lambda_{x_1}^\gamma_1 g, \Lambda_{x_2}^\gamma_2 h \in L^2(\mathbb{R}^2) \) for any \( \gamma_1, \gamma_2 \in \left(\frac{1}{2}, 1\right) \), then it holds true

\[ \int \int |f g h| \, dx_1 dx_2 \leq C \|f\|_{L^2} \|g\|_{L^2}^{\frac{1-\frac{1}{\gamma_1 p}}{\gamma_1 p}} \|\Lambda_{x_1}^\gamma_1 g\|_{L^2}^{\frac{1}{\gamma_1 p}} \|h\|_{L^2}^{\frac{1-\frac{1}{\gamma_2 q}}{\gamma_2 q}} \|\Lambda_{x_2}^\gamma_2 h\|_{L^2}^{\frac{1}{\gamma_2 q}}, \quad (2.19) \]

where \( C \) is a constant depending on \( \gamma_1 \) and \( \gamma_2 \) only.

**Proof of Lemma 2.5.** The proof of this lemma can be found in [39]. For the convenience of the reader, we provide the details. Now we recall the one-dimensional Sobolev inequality

\[ \|g\|_{L^2_{x_1}(\mathbb{R})} \leq C \|g\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{\gamma_1 p}} \|\Lambda_{x_1}^\gamma_1 g\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{\gamma_1 p}}, \quad \gamma_1 \in \left(\frac{1}{p}, 1\right), \quad (2.20) \]

where here and in what follows, we adopt the convention \( \frac{2p}{p-2} = \infty \) for \( p = 2 \). By means of (2.20) and the Hölder inequality, one deduces

\[ \int \int |f g h| \, dx_1 dx_2 \leq C \int |f|_{L^p_{x_2}} \|g\|_{L^2_{x_1}}^{\frac{2p}{p-2}} \|h\|_{L^2_{x_1}} \, dx_2 \]

\[ \leq C \int \|f\|_{L^p_{x_2}} \|g\|_{L^2_{x_1}}^{\frac{1}{\gamma_1 p}} \|\Lambda_{x_1}^\gamma_1 g\|_{L^2_{x_1}}^{\frac{1}{\gamma_1 p}} \|h\|_{L^2_{x_1}} \, dx_2 \]

\[ \leq C \left( \int \|f\|_{L^p_{x_2}}^q \, dx_2 \right)^{\frac{1}{q}} \left( \int \|g\|_{L^2_{x_1}}^q \, dx_2 \right)^{\frac{1}{q}} \left( \int \|\Lambda_{x_1}^\gamma_1 g\|_{L^2_{x_1}}^q \, dx_2 \right)^{\frac{1}{q}} \left( \int \|h\|_{L^2_{x_1}}^q \, dx_2 \right)^{\frac{1}{q}} \]

\[ \leq C \|f\|_{L^p_{x_2} L^q_{x_1}} \|g\|_{L^2}^{\frac{1-\frac{1}{\gamma_1 p}}{\gamma_1 p}} \|\Lambda_{x_1}^\gamma_1 g\|_{L^2}^{\frac{1}{\gamma_1 p}} \|h\|_{L^2}^{\frac{1-\frac{1}{\gamma_2 q}}{\gamma_2 q}} \|\Lambda_{x_2}^\gamma_2 h\|_{L^2}^{\frac{1}{\gamma_2 q}}, \quad (2.21) \]
According to the Minkowski inequality and (2.20), we have
\[
\|h\|_{L^2_t L^{\frac{2q}{2q-1}}_x} \leq C \left( \int_{\mathbb{R}} \|h(x_1, x_2)\|^2_{L^2_t L^{\frac{2q}{2q-1}}_x} \, dx_1 \right)^{\frac{1}{2}} \\
\leq C \left( \int_{\mathbb{R}} \|h(x_1, x_2)\|^2_{L^2_t L^{\frac{2q}{2q-1}}_x} \|\Lambda_{x_2}^q h(x_1, x_2)\|^2_{L^{\frac{2q}{2q-1}}_x} \, dx_1 \right)^{\frac{1}{2}} \\
\leq C \left( \int_{\mathbb{R}} \|h(x_1, x_2)\|^2_{L^2_t L^{\frac{2q}{2q-1}}_x} \left( \int_{\mathbb{R}} \|\Lambda_{x_2}^q h(x_1, x_2)\|^2_{L^{\frac{2q}{2q-1}}_x} \, dx_1 \right)^{\frac{1}{2}} \right) \\
= C \|h\|_{L^2_t L^{\frac{2q}{2q-1}}_x} \|\Lambda_{x_2}^q h\|_{L^{\frac{2q}{2q-1}}_x}.
\tag{2.22}
\]
Inserting (2.22) into (2.21) gives
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} |f \, g \, h| \, dx_1 dx_2 \leq C \|f\|_{L^q_t L^p_x} \|g\|_{L^q_t L^p_x} \|\Lambda_{x_2}^q g\|_{L^p_x} \|h\|_{L^2_t L^{\frac{2q}{2q-1}}_x} \|\Lambda_{x_2}^q h\|_{L^{\frac{2q}{2q-1}}_x},
\]
which is the desired inequality (2.19). This completes the proof of the lemma. \(\square\)

Finally, the following standard commutator estimate will also be used as well, which can be found in [23, p.614].

**Lemma 2.6.** Let \(s \in (0, 1)\) and \(p \in (1, \infty)\). Then
\[
\|\Lambda^s (f \, g) - f \Lambda^s g - g \Lambda^s f\|_{L^p(\mathbb{R}^d)} \leq C \|g\|_{L^\infty(\mathbb{R}^d)} \|\Lambda^s f\|_{L^p(\mathbb{R}^d)},
\tag{2.23}
\]
where \(d \geq 1\) denotes the spatial dimension and \(C = C(d, s, p)\) is a constant. In particular, it holds true
\[
\|\Lambda^s (f \, g) - f \Lambda^s g\|_{L^p(\mathbb{R}^d)} \leq C \|g\|_{L^\infty(\mathbb{R}^d)} \|\Lambda^s f\|_{L^p(\mathbb{R}^d)}.
\]

3. **The proof of Theorem 1.1**

It is worthwhile pointing out that the existence and uniqueness of local smooth solutions can be established without difficulty. Thus, in order to complete the proof of Theorem 1.1, it is sufficient to establish *a priori* estimates that hold for any fixed \(T > 0\).

The following proposition states the basic bounds.

**Proposition 3.1.** Assume \(\theta_0\) satisfies the assumptions stated in Theorem 1.1 and let \((u, \theta)\) be the corresponding solution. Then, for any \(t > 0\),
\[
\|\theta(t)\|^2_{L^2} + 2 \int_0^t (\|\Lambda_{x_1}^\alpha \theta(\tau)\|^2_{L^2} + \|\Lambda_{x_2}^\beta \theta(\tau)\|^2_{L^2}) \, d\tau \leq \|\theta_0\|^2_{L^2},
\]
\[
\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \quad 2 \leq p \leq \infty.
\]

**Proof of Proposition 3.1.** Multiplying the first equation of (1.1) by \(\theta\), using the divergence-free condition and integrating with respect to the space variable, we have
\[
\frac{1}{2} \frac{d}{dt} \|\theta(t)\|^2_{L^2} + \|\Lambda_{x_1}^\alpha \theta\|^2_{L^2} + \|\Lambda_{x_2}^\beta \theta\|^2_{L^2} = 0.
\]
Integrating with respect to time yields
\[ \|\theta(t)\|^2_{L^2} + 2 \int_0^t (\|\Lambda_{x_1}^\alpha \theta(\tau)\|^2_{L^2} + \|\Lambda_{x_2}^\beta \theta(\tau)\|^2_{L^2}) d\tau \leq \|\theta_0\|^2_{L^2}. \] (3.1)

We multiply the first equation of (1.1) by \(|\theta|^{p-2}\theta\) and use the divergence-free condition to derive
\[ \frac{1}{p} \frac{d}{dt} \|\theta(t)\|^p_{L^p} + \int_{\mathbb{R}^2} \Lambda_{x_1}^{2\alpha} \theta(|\theta|^{p-2}\theta) dx + \int_{\mathbb{R}^2} \Lambda_{x_2}^{2\beta} \theta(|\theta|^{p-2}\theta) dx = 0. \]

Invoking the lower bounds
\[ \int_{\mathbb{R}^2} \Lambda_{x_1}^{2\alpha} \theta(|\theta|^{p-2}\theta) dx = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Lambda_{x_1}^{2\alpha} \theta(x_1, x_2) (|\theta(x_1, x_2)\|^2 \theta(x_1, x_2)) dx_1 dx_2 \]
\[ \geq C \int_{\mathbb{R}^2} \Lambda_{x_1}^{\alpha} \theta(x_1, x_2) \left( \frac{|\theta(x_1, x_2)|}{2} \right)^2 dx_1 dx_2 \]

and
\[ \int_{\mathbb{R}^2} \Lambda_{x_1}^{2\alpha} \theta(|\theta|^{p-2}\theta) dx = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Lambda_{x_2}^{2\beta} \theta(x_1, x_2) (|\theta(x_1, x_2)\|^2 \theta(x_1, x_2)) dx_1 dx_2 \]
\[ \geq C \int_{\mathbb{R}^2} \Lambda_{x_2}^{\beta} \theta(x_1, x_2) \left( \frac{|\theta(x_1, x_2)|}{2} \right)^2 dx_1 dx_2, \]

it follows that
\[ \|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \quad 2 \leq p \leq \infty. \]

This ends the proof of the proposition.

We now prove the following global \(H^1\)-bound for \(\beta > \max \left\{ \alpha, \frac{1}{2 \alpha + 1} \right\}\).

**Proposition 3.2.** Assume \(\theta_0\) satisfies the assumptions stated in Theorem 1.1 and let \((u, \theta)\) be the corresponding solution. If \(\alpha\) and \(\beta\) satisfy
\[ \beta > \max \left\{ \alpha, \frac{1}{2 \alpha + 1} \right\}, \]
then, for any \(t > 0\),
\[ \|\nabla \theta(t)\|^2_{L^2} + \int_0^t (\|\Lambda_{x_1}^\alpha \nabla \theta(\tau)\|^2_{L^2} + \|\Lambda_{x_2}^\beta \nabla \theta(\tau)\|^2_{L^2}) d\tau \leq C(t, \theta_0), \] (3.2)

where \(C(t, \theta_0)\) is a constant depending on \(t\) and the initial data \(\theta_0\).

**Proof of Proposition 3.2.** Taking the inner product of (1.1) with \(\Delta \theta\) and using the divergence-free condition \(\partial_{x_1} u_1 + \partial_{x_2} u_2 = 0\), we infer that
\[ \frac{1}{2} \frac{d}{dt} \|\nabla \theta(t)\|^2_{L^2} + \|\Lambda_{x_1}^\alpha \nabla \theta\|^2_{L^2} + \|\Lambda_{x_2}^\beta \nabla \theta\|^2_{L^2} = \int_{\mathbb{R}^2} (u \cdot \nabla) \theta \Delta \theta \, dx \]
\[ = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4, \] (3.3)

where
\[ \mathcal{H}_1 = -\int_{\mathbb{R}^2} \partial_{x_1} u_1 \partial_{x_1} \theta \partial_{x_1} \theta \, dx, \quad \mathcal{H}_2 = -\int_{\mathbb{R}^2} \partial_{x_1} u_2 \partial_{x_2} \theta \partial_{x_1} \theta \, dx, \]
\[ \mathcal{H}_3 = -\int_{\mathbb{R}^2} \partial_{x_2} u_1 \partial_{x_1} \theta \partial_{x_2} \theta \, dx, \quad \mathcal{H}_4 = -\int_{\mathbb{R}^2} \partial_{x_2} u_2 \partial_{x_2} \theta \partial_{x_2} \theta \, dx. \]
In what follows, we shall estimate the terms at the right hand side of (3.3) one by one. To estimate the first term, we use \( \partial_{x_1} u_1 + \partial_{x_2} u_2 = 0 \) and the commutator (2.23) to conclude

\[
\mathcal{H}_1 = \int_{\mathbb{R}^2} \partial_{x_2} u_2 \partial_{x_1} \theta \partial_{x_1} \theta \, dx
\]

\[
= -2 \int_{\mathbb{R}^2} u_2 \partial_{x_1} \theta \partial_{x_2} x_1 \theta \, dx
\]

\[
= -2 \int_{\mathbb{R}^2} \Lambda_{x_2}^{1-\delta} (u_2 \partial_{x_1} \theta) \Lambda_{x_2}^{\delta} \Lambda_{x_2}^{-1} \partial_{x_2} \partial_{x_1} \theta \, dx
\]

\[
\leq C \| \Lambda_{x_2}^{\delta} \partial_{x_1} \theta \|_{L^2} \| \Lambda_{x_2}^{1-\delta} (u_2 \partial_{x_1} \theta) \|_{L^2}
\]

\[
\leq C \| \Lambda_{x_2}^{\delta} \partial_{x_1} \theta \|_{L^2} \| (\Lambda_{x_2}^{1-\delta} (u_2 \partial_{x_1} \theta) - \Lambda_{x_2}^{1-\delta} u_2 \partial_{x_1} \theta) \|_{L^2} + \| \Lambda_{x_2}^{1-\delta} u_2 \partial_{x_1} \theta \|_{L^2}
\]

\[
\leq C \| \Lambda_{x_2}^{\delta} \partial_{x_1} \theta \|_{L^2} (\| u_2 \|_{L^\infty} \| \Lambda_{x_2}^{1-\delta} \partial_{x_1} \theta \|_{L^2} + \| \Lambda_{x_2}^{1-\delta} u_2 \partial_{x_1} \theta \|_{L^2})
\]

\[
:= \mathcal{H}_{11} + \mathcal{H}_{12},
\]

where \( \mathcal{H}_{11} \) and \( \mathcal{H}_{12} \) are given by

\[
\mathcal{H}_{11} = C \| \Lambda_{x_2}^{\delta} \partial_{x_1} \theta \|_{L^2} \| u_2 \|_{L^\infty} \| \Lambda_{x_2}^{1-\delta} \partial_{x_1} \theta \|_{L^2}, \quad \mathcal{H}_{12} = C \| \Lambda_{x_2}^{\delta} \partial_{x_1} \theta \|_{L^2} \| \Lambda_{x_2}^{1-\delta} u_2 \partial_{x_1} \theta \|_{L^2}.
\]

In light of the interpolation inequality (2.15), one obtains for \( 1 - \beta \leq \delta < \beta \)

\[
\mathcal{H}_{11} \leq C \| \partial_{x_1} \theta \|_{L^2}^{1-\delta} \| \Lambda_{x_2}^{\beta} \partial_{x_1} \theta \|_{L^2} \| u_2 \|_{L^\infty} \| \partial_{x_1} \theta \|_{L^2}^{1-\delta} \| \Lambda_{x_2}^{\beta} \partial_{x_1} \theta \|_{L^2}^{1-\delta}
\]

\[
\leq C \| \Lambda_{x_2}^{\beta} \nabla \theta \|_{L^2} \| u_2 \|_{L^\infty} \| \nabla \theta \|_{L^2}^{2-\delta}
\]

\[
\leq \epsilon \| \Lambda_{x_2}^{\beta} \nabla \theta \|_{L^2}^{2} + C(\epsilon) \| u_2 \|_{L^\infty} \| \nabla \theta \|_{L^2}^{2}.
\]

Observing the interpolation inequality (see Lemma 2.3), it follows that

\[
\mathcal{H}_{12} \leq C \| \partial_{x_1} \theta \|_{L^2}^{1-\delta} \| \Lambda_{x_2}^{\beta} \partial_{x_1} \theta \|_{L^2} \| \Lambda_{x_2}^{1-\delta} u_2 \|_{L^2} \| \partial_{x_1} \theta \|_{L^2}^{2(1+\beta)}\| \partial_{x_1} \theta \|_{L^2(\alpha+1)}
\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2}^{1-\delta} \| \Lambda_{x_2}^{\beta} \partial_{x_1} \theta \|_{L^2} \| u_2 \|_{L^\infty} \| \Lambda_{x_2}^{1-\delta} \partial_{x_2} u_2 \|_{L^2} \| \theta \|_{L^\infty} \| \Lambda_{x_2}^{1-\delta} \partial_{x_1} \theta \|_{L^2}^{1-\delta}
\]

\[
\leq C \| \nabla \theta \|_{L^2}^{1-\delta} \| \Lambda_{x_2}^{\beta} \nabla \theta \|_{L^2} \| u_2 \|_{L^\infty} \| \Lambda_{x_2}^{1-\delta} \nabla \theta \|_{L^2} \| \theta \|_{L^\infty} \| \Lambda_{x_2}^{1-\delta} \nabla \theta \|_{L^2}^{1-\delta}
\]

\[
\leq \epsilon \| \Lambda_{x_2}^{\beta} \nabla \theta \|_{L^2}^{2} + \epsilon \| \Lambda_{x_2}^{\beta} \nabla \theta \|_{L^2}^{2} + C(\epsilon) \| u_2 \|_{L^\infty} \| \theta \|_{L^\infty} \| \theta \|_{L^\infty}^{1-\delta} \| \nabla \theta \|_{L^2}^{2}
\]

\[
\leq \epsilon \| \Lambda_{x_2}^{\beta} \nabla \theta \|_{L^2}^{2} + \epsilon \| \Lambda_{x_2}^{\beta} \nabla \theta \|_{L^2}^{2} + C(\epsilon) \| u_2 \|_{L^\infty} \| \nabla \theta \|_{L^2}^{2}
\]

where in the first line \( \delta \) should further satisfy

\[
\frac{1-\delta}{1+\beta} + \frac{1}{\alpha+1} = 1 \quad \text{or} \quad \delta = \frac{1-\alpha\beta}{1+\alpha}.
\]

As a result, the above estimates \( \mathcal{H}_{11} \) and \( \mathcal{H}_{12} \) would work as long as \( \delta \) satisfies

\[
1 - \beta \leq 1 - \frac{\alpha(\beta+1)}{\alpha+1} < \beta.
\]

The above constraint is in particular satisfied

\[
\beta > \max \left\{ \alpha, \frac{1}{2\alpha+1} \right\}.
\]
A simple computation shows that
\[
\max \left\{ \alpha, \frac{1}{2\alpha + 1} \right\} \geq \frac{1}{2}
\]
Substituting the above estimates into (3.4) yields
\[
\mathcal{H}_1 \leq \epsilon \| \Lambda_{x_1}^\alpha \nabla \theta \|_{L^2}^2 + 2\epsilon \| \Lambda_{x_2}^\beta \nabla \theta \|_{L^2}^2 + C(\epsilon) \left( \| u_2 \|_{L^\infty}^{\frac{2\beta}{2\beta + 1}} + \| u_2 \|_{L^\infty}^{\frac{2\beta}{2\beta + 1}} \right) \| \nabla \theta \|_{L^2}^2. \tag{3.5}
\]
Similarly, arguing as the estimates of \( \mathcal{H}_{11} \) and \( \mathcal{H}_{12} \), we thus have
\[
\mathcal{H}_2 = \int_{\mathbb{R}^2} \theta \partial_{x_2 x_1} u_2 \partial_{x_1} \theta \, dx + \int_{\mathbb{R}^2} \theta \partial_{x_1} u_2 \partial_{x_2 x_1} \theta \, dx
\leq C \| \Lambda_{x_2}^\delta \partial_{x_1} u_2 \|_{L^2} \| \Lambda_{x_2}^{1-\delta} \theta \partial_{x_1} \theta \|_{L^2} + C \| \Lambda_{x_2}^\delta \partial_{x_1} \theta \|_{L^2} \| \Lambda_{x_2}^{1-\delta} \theta \partial_{x_1} u_2 \|_{L^2}
\leq C \| \Lambda_{x_2}^\delta \partial_{x_1} \theta \|_{L^2} \left( \| \theta \|_{L^\infty} \| \Lambda_{x_2}^{1-\delta} \partial_{x_1} \theta \|_{L^2} + \| \Lambda_{x_2}^{1-\delta} \partial_{x_1} u_2 \|_{L^2} \right)
+ C \| \Lambda_{x_2}^\delta \partial_{x_1} \theta \|_{L^2} \left( \| \theta \|_{L^\infty} \| \Lambda_{x_2}^{1-\delta} \partial_{x_1} u_2 \|_{L^2} + \| \Lambda_{x_2}^\delta \partial_{x_1} u_2 \|_{L^2} \right)
\leq \epsilon \| \Lambda_{x_2}^\beta \nabla \theta \|_{L^2}^2 + C(\epsilon) \| \nabla \theta \|_{L^2}^2 + C(\epsilon) \| u_2 \|_{L^\infty \frac{2\beta}{2\beta + 1}} \| \nabla \theta \|_{L^2}^2.
\] (3.6)
For the term \( \mathcal{H}_3 \), one directly obtains
\[
\mathcal{H}_3 = \int_{\mathbb{R}^2} u_1 \partial_{x_2 x_1} \theta \partial_{x_2} \theta \, dx + \int_{\mathbb{R}^2} u_1 \partial_{x_1} \theta \partial_{x_2 x_2} \theta \, dx
\leq C \| \Lambda_{x_2}^{1-\beta} \partial_{x_1} \theta \|_{L^2} \| \Lambda_{x_2}^\beta (u_1 \partial_{x_2} \theta) \|_{L^2} + C \| \Lambda_{x_2}^\delta \partial_{x_2} \theta \|_{L^2} \| \Lambda_{x_2}^{1-\delta} (u_1 \partial_{x_1} \theta) \|_{L^2}
:= \mathcal{H}_{31} + \mathcal{H}_{32}. \tag{3.7}
\]
Applying the same manner dealing with \( \mathcal{H}_{11} \) and \( \mathcal{H}_{12} \), we immediately get
\[
\mathcal{H}_{32} \leq C \| \Lambda_{x_2}^\delta \partial_{x_2} \theta \|_{L^2} \| \Lambda_{x_2}^{1-\delta} (u_1 \partial_{x_1} \theta) \|_{L^2}
\leq C \| \Lambda_{x_2}^\delta \partial_{x_2} \theta \|_{L^2} \left( \| u_1 \|_{L^\infty} \| \Lambda_{x_2}^{1-\delta} \partial_{x_1} \theta \|_{L^2} + \| \Lambda_{x_2}^{1-\delta} u_1 \partial_{x_1} \theta \|_{L^2} \right)
\leq \epsilon \| \Lambda_{x_2}^\alpha \nabla \theta \|_{L^2}^2 + 2\epsilon \| \Lambda_{x_2}^\beta \nabla \theta \|_{L^2}^2 + C(\epsilon) \left( \| u_1 \|_{L^\infty}^{\frac{2\beta}{2\beta + 1}} + \| u_2 \|_{L^\infty \frac{2\beta}{2\beta + 1}} \right) \| \nabla \theta \|_{L^2}^2 \tag{3.8}
\]
For \( \beta > \frac{1}{2} \), it follows from the interpolation inequalities (see Lemma 2.4) and Lemma 2.3) and the commutator (2.23) that
\[
\mathcal{H}_{31} \leq C \| \Lambda_{x_2}^{1-\beta} \partial_{x_1} \theta \|_{L^2} (\| u_1 \|_{L^\infty} \| \Lambda_{x_2}^\beta \partial_{x_2} \theta \|_{L^2} + \| \Lambda_{x_2}^\beta u_1 \partial_{x_2} \theta \|_{L^2})
\leq C \| \partial_{x_1} \theta \|_{L^2}^{1-\frac{\beta}{\beta + 1}} \| \Lambda_{x_2}^\beta \partial_{x_1} \theta \|_{L^2} \left( \| u_1 \|_{L^\infty} \| \Lambda_{x_2}^\beta \partial_{x_2} \theta \|_{L^2} \right)
+ C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda_{x_2}^\beta \partial_{x_1} \theta \|_{L^2} \left( \| u_1 \|_{L^\infty} \| \Lambda_{x_2}^\beta \partial_{x_2} \theta \|_{L^2} \right)
\leq C \| \partial_{x_1} \theta \|_{L^2}^{1-\frac{\beta}{\beta + 1}} \| \Lambda_{x_2}^\beta \partial_{x_1} \theta \|_{L^2} \left( \| u_1 \|_{L^\infty} \| \Lambda_{x_2}^\beta \partial_{x_2} \theta \|_{L^2} \right)
+ C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda_{x_2}^\beta \partial_{x_1} \theta \|_{L^2} \left( \| u_1 \|_{L^\infty} \| \Lambda_{x_2}^\beta \partial_{x_2} \theta \|_{L^2} \right)
\leq C \| u_1 \|_{L^\infty} \| \nabla \theta \|_{L^2}^{\frac{2-\frac{\beta}{\beta + 1}}{2}} \| \Lambda_{x_2}^\beta \nabla \theta \|_{L^2} + C \| \nabla \theta \|_{L^2}^{\frac{2-\frac{\beta}{\beta + 1}}{2}} \| \Lambda_{x_2}^\beta \nabla \theta \|_{L^2} \| u_1 \|_{L^\infty \| \theta \|_{L^\infty}^{\frac{2\beta}{2\beta + 1}}}
\[
\begin{align*}
\|\Lambda^\beta_{x_2} \nabla \theta\|_{L^2}^2 &\leq \epsilon \|\Lambda^\beta_{x_2} \nabla \theta\|_{L^2}^2 + C(\epsilon) \|u_1\|_{L^\infty}^{\frac{2}{3+\beta}} \|\nabla \theta\|_{L^2}^2 + C(\epsilon) \left( \|u_1\|_{L^\infty}^{\frac{2}{3+\beta}} + \|u_1\|_{L^\infty}^{\frac{2\beta}{3(3+\beta)-1}} \right) \|\nabla \theta\|_{L^2}^2, \\
\|\Lambda^\beta_{x_2} \nabla \theta\|_{L^2}^2 &\leq \epsilon \|\Lambda^\beta_{x_2} \nabla \theta\|_{L^2}^2 + C(\epsilon) \left( \|u_1\|_{L^\infty}^{\frac{2}{3+\beta}} + \|u_1\|_{L^\infty}^{\frac{2\beta}{3(3+\beta)-1}} \right) \|\nabla \theta\|_{L^2}^2.
\end{align*}
\]

Inserting the above two estimates (3.9) and (3.9) into (3.7) yields

\[
\mathcal{H}_3 \leq \epsilon \|\Lambda^\alpha_{x_1} \nabla \theta\|_{L^2}^2 + 3\epsilon \|\Lambda^\beta_{x_2} \nabla \theta\|_{L^2}^2 + C(\epsilon) \left( \|u_2\|_{L^\infty}^{\frac{2\beta}{3(3+\beta)-1}} + \|u_1\|_{L^\infty}^{\frac{2\beta}{3(3+\beta)-1}} \right) \|\nabla \theta\|_{L^2}^2,
\]

Finally, following the estimate of \(\mathcal{H}_{31}\), one directly gets for \(\beta > \frac{1}{2}\)

\[
\begin{align*}
\mathcal{H}_4 &\leq 2 \int_{\mathbb{R}^2} u_2 \partial_{x_2 x_2} \theta \partial_{x_2} \theta \, dx \\
&\leq C \|\Lambda^1_{x_2} \partial_{x_2} \theta\|_{L^2} \|\Lambda^\beta_{x_2} (u_2 \partial_{x_2} \theta)\|_{L^2} \\
&\leq \epsilon \|\Lambda^\beta_{x_2} \nabla \theta\|_{L^2}^2 + C(\epsilon) \left( \|u_2\|_{L^\infty}^{\frac{2\beta}{3(3+\beta)-1}} + \|u_2\|_{L^\infty}^{\frac{2\beta}{3(3+\beta)-1}} \right) \|\nabla \theta\|_{L^2}^2.
\end{align*}
\]

Collecting the estimates (3.3), (3.5), (3.6), (3.10) and (3.11), and selecting \(\epsilon\) suitable small, it follows that

\[
\frac{d}{dt} \|\nabla \theta(t)\|_{L^2}^2 + \|\Lambda^\alpha_{x_1} \nabla \theta\|_{L^2}^2 + \|\Lambda^\beta_{x_2} \nabla \theta\|_{L^2}^2 \leq H(t) \|\nabla \theta\|_{L^2}^2,
\]

where

\[
H(t) = C \left( \|u\|_{L^\infty}^{\frac{2\beta}{3(3+\beta)-1}} + \|u_2\|_{L^\infty}^{\frac{2\beta}{3(3+\beta)-1}} + \|u\|_{L^\infty}^{\frac{2(\alpha+1)\beta}{(\alpha+1)\beta-1}} + \|u\|_{L^\infty}^{\frac{2\beta}{3(3+\beta)-1}} \right).
\]

Obviously, it is easy to show

\[
H(t) \leq C \left( 1 + \|u\|_{L^\infty}^\rho \right)
\]

where

\[
\rho = \max \left\{ \frac{2\beta}{2\beta-1}, \frac{2\beta}{(\alpha+1)\beta-1} \right\} > 1.
\]

By denoting

\[
A(t) := \|\nabla \theta(t)\|_{L^2}^2, \quad B(t) := \|\Lambda^\alpha_{x_1} \nabla \theta(t)\|_{L^2}^2 + \|\Lambda^\beta_{x_2} \nabla \theta(t)\|_{L^2}^2,
\]

we therefore obtain

\[
\frac{d}{dt} A(t) + B(t) \leq CA(t) + C\|u\|_{L^\infty}^\rho A(t).
\]

We deduce by Lemma 2.3 that

\[
\begin{align*}
\|\partial_{x_1} \theta(t)\|_{L^2} &\leq C(\|\theta(t)\|_{L^2}^{\frac{1}{\alpha+1}} \|\Lambda^\alpha_{x_1} \partial_{x_1} \theta(t)\|_{L^2}^{\frac{1}{\alpha+1}} \leq C\|\theta(t)\|_{L^2}^{\frac{1}{\alpha+1}} \|\Lambda^\alpha_{x_1} \nabla \theta(t)\|_{L^2}^{\frac{1}{\alpha+1}}, \n\|\partial_{x_2} \theta(t)\|_{L^2} &\leq C(\|\theta(t)\|_{L^2}^{\frac{1}{\alpha+1}} \|\Lambda^\beta_{x_2} \partial_{x_2} \theta(t)\|_{L^2}^{\frac{1}{\alpha+1}} \leq C\|\theta(t)\|_{L^2}^{\frac{1}{\alpha+1}} \|\Lambda^\beta_{x_2} \nabla \theta(t)\|_{L^2}^{\frac{1}{\alpha+1}}.
\end{align*}
\]

This further allows us to deduce

\[
C^{-1} A^\gamma(t) \leq B(t), \quad \gamma = \min\{\alpha, \beta\} + 1 > 1.
\]

Thanks to Lemma 2.3 again, we have

\[
\|\partial_{x_1}^\sigma \theta(t)\|_{L^2} \leq C(\|\theta(t)\|_{L^2}^{\frac{\sigma}{\alpha+1}} \|\Lambda^\alpha_{x_1} \partial_{x_1} \theta(t)\|_{L^2}^{\frac{\sigma}{\alpha+1}} \leq C\|\theta(t)\|_{L^2}^{\frac{\sigma}{\alpha+1}} \|\Lambda^\alpha_{x_1} \nabla \theta(t)\|_{L^2}^{\frac{\sigma}{\alpha+1}},
\]

\[
\|\partial_{x_2}^\sigma \theta(t)\|_{L^2} \leq C(\|\theta(t)\|_{L^2}^{\frac{\sigma}{\alpha+1}} \|\Lambda^\beta_{x_2} \partial_{x_2} \theta(t)\|_{L^2}^{\frac{\sigma}{\alpha+1}} \leq C\|\theta(t)\|_{L^2}^{\frac{\sigma}{\alpha+1}} \|\Lambda^\beta_{x_2} \nabla \theta(t)\|_{L^2}^{\frac{\sigma}{\alpha+1}}.
\]
\[ \| \partial_{x_2}^{\sigma} \theta(t) \|_{L^2} \leq C \| \theta(t) \|_{L^2}^{\frac{\alpha}{2\alpha}} \| \Lambda_{x_2}^{\beta} \partial_{x_2} \theta(t) \|_{L^2}^{\frac{\beta}{\alpha}} \leq C \| \theta_0 \|_{L^2}^{\frac{\alpha}{2\alpha}} \| \Lambda_{x_2}^{\beta} \nabla \theta(t) \|_{L^2}^{\frac{\beta}{\alpha}}, \]

where \( 0 \leq \sigma \leq \min\{\alpha, \beta\} + 1 \). Now taking some \( 1 < \sigma \leq \min\{\alpha, \beta\} + 1 \), we obtain

\[ \| \Lambda^\sigma \theta(t) \|_{L^2} \leq \| \partial_{x_2}^{\sigma} \theta(t) \|_{L^2} + \| \partial_{x_2}^{\sigma} \theta(t) \|_{L^2} \]

\[ \leq C (\| \Lambda_{x_1}^{\alpha} \nabla \theta(t) \|_{L^2} + \| \Lambda_{x_2}^{\beta} \nabla \theta(t) \|_{L^2}), \] (3.15)

which leads to

\[ \| \Lambda^\sigma \theta(t) \|_{L^2} \leq e + B(t). \]

By the following logarithmic Sobolev interpolation inequality (see for instance [27])

\[ \| f \|_{L^\infty} \leq C (1 + \| f \|_{L^2} + \| f \|_{H^\infty_{\beta_{c, \infty}}} \ln \left( e + \| \Lambda^\sigma f \|_{L^2} \right), \quad \forall \sigma > 1, \]

we deduce from (3.13) and (3.15) that

\[ \frac{d}{dt} A(t) + B(t) \leq CA(t) + C \| u(t) \|_{B^\infty_{0, \infty}}^{\sigma} \left( \ln \left( e + \| \Lambda^\sigma u(t) \|_{L^2} \right) \right)^{\sigma} A(t) \]

\[ \leq CA(t) + C \| \mathcal{R}^{-\theta} (t) \|_{B^\infty_{0, \infty}}^{\sigma} \left( \ln \left( e + \| \Lambda^\sigma \mathcal{R}^{-\theta} (t) \|_{L^2} \right) \right)^{\sigma} A(t) \]

\[ \leq CA(t) + C \| \theta(t) \|_{L^\infty}^{\sigma} \left( \ln \left( e + \| \Lambda^\sigma \theta(t) \|_{L^2} \right) \right)^{\sigma} A(t) \]

\[ \leq CA(t) + C \| \theta_0 \|_{L^\infty}^{\sigma} \left( \ln \left( e + B(t) \right) \right)^{\sigma} A(t) \]

where we have used the boundness of the Riesz transform \( \mathcal{R} \) between the homogenous Besov space and the embedding \( L^\infty \hookrightarrow B^0_{\infty, \infty} \). We finally get

\[ \frac{d}{dt} A(t) + B(t) \leq C \left( A(t) + e \right) + C \| \theta_0 \|_{L^\infty} \left( \ln \left( A(t) + B(t) + e \right) \right)^{\sigma} \left( A(t) + e \right). \] (3.16)

Applying the logarithmic type Gronwall inequality (see Lemma 2.1) to (3.16), we therefore obtain

\[ A(t) + \int_0^t B(s) \, ds \leq C, \]

which is nothing but the desired estimate (3.2). Consequently, we complete the proof of Proposition 3.2. \( \square \)

Next we will prove the global \( H^1 \)-bound for \( \beta > \frac{1-\alpha}{2\alpha} \) and \( \alpha > \frac{1}{2} \).

**Proposition 3.3.** Assume \( \theta_0 \) satisfies the assumptions stated in Theorem 7.1 and let \((u, \theta)\) be the corresponding solution. If \( \alpha \) and \( \beta \) satisfy

\[ \alpha > \beta > \frac{1-\alpha}{2\alpha}, \] (3.17)

then, for any \( t > 0 \),

\[ \| \nabla \theta(t) \|_{L^2}^2 + \int_0^t (\| \Lambda_{x_1}^{\alpha} \nabla \theta(\tau) \|_{L^2}^2 + \| \Lambda_{x_2}^{\beta} \nabla \theta(\tau) \|_{L^2}^2) \, d\tau \leq C(t, \theta_0), \] (3.18)

where \( C(t, \theta_0) \) is a constant depending on \( t \) and the initial data \( \theta_0 \).
**Remark 3.4.** In the above proposition, we need \( \beta \) smaller than \( \alpha \). It is a technical assumption. In common sense, it is commonly believed that the diffusion term is always a good term and the larger the power \( \beta \) is, the better effects it produces. In this sense, we can ignore the up bound of \( \beta \).

**Proof of Proposition 3.2.** It follows from (3.3) that

\[
\frac{1}{2} \frac{d}{dt} \| \nabla \theta(t) \|^2_{L^2} + \| \Lambda^\alpha_{x_1} \nabla \theta \|^2_{L^2} + \| \Lambda^\beta_{x_2} \nabla \theta \|^2_{L^2} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4. \tag{3.19}
\]

By means of the commutator (2.23), it ensures for \( \alpha > \frac{1}{2} \)

\[
\mathcal{H}_1 = - \int_{\mathbb{R}^2} \partial_{x_1} u_1 \partial_{x_1} \theta \partial_{x_1} \theta \, dx
\]

\[
= 2 \int_{\mathbb{R}^2} u_1 \partial_{x_1} \theta \partial_{x_1} \theta \, dx
\]

\[
\leq C \| \Lambda^\alpha_{x_1} \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} (u_1 \partial_{x_1} \theta) \|_{L^2}
\]

\[
\leq C \| \Lambda^\alpha_{x_1} \partial_{x_1} \theta \|_{L^2} (\| u_1 \|_{L^\infty} \| \Lambda^\alpha_{x_1} \partial_{x_1} \theta \|_{L^2} + \| \Lambda^\alpha_{x_1} u_1 \partial_{x_1} \theta \|_{L^2})
\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_1} \theta \|_{L^2}\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_1} \theta \|_{L^2}\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_1} \theta \|_{L^2}\]

\[
\leq \epsilon \| \Lambda^\alpha_{x_1} \nabla \theta \|^2_{L^2} + C(\epsilon) \| u \|_{L^\infty} \| \nabla \theta \|^2_{L^2}
\]

\[
\leq C \left( \| u \|_{L^\infty} \| \nabla \theta \|^2_{L^2} + C(\epsilon) \| u \|_{L^\infty} \| \nabla \theta \|^2_{L^2} \right)
\]

\[
\leq \epsilon \| \Lambda^\alpha_{x_1} \nabla \theta \|^2_{L^2} + C(\epsilon) \| u \|_{L^\infty} \| \nabla \theta \|^2_{L^2} + C(\epsilon) \| u \|_{L^\infty} \| \nabla \theta \|^2_{L^2}.
\tag{3.20}
\]

We rewrite \( \mathcal{H}_2 \) as

\[
\mathcal{H}_2 = \int_{\mathbb{R}^2} u_2 \partial_{x_1} x_2 \partial_{x_1} \theta \, dx + \int_{\mathbb{R}^2} u_2 \partial_{x_2} \partial_{x_1} x_1 \theta \, dx = \mathcal{H}_{21} + \mathcal{H}_{22}.
\]

According to the estimate of (3.20), we infer that

\[
\mathcal{H}_{21} \leq C \| \Lambda^\alpha_{x_1} \partial_{x_2} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} (u_2 \partial_{x_2} \theta) \|_{L^2}
\]

\[
\leq \epsilon \| \Lambda^\alpha_{x_1} \nabla \theta \|^2_{L^2} + C(\epsilon) \| u \|_{L^\infty} \| \nabla \theta \|^2_{L^2} + C(\epsilon) \| u \|_{L^\infty} \| \nabla \theta \|^2_{L^2}.
\]

As \( \alpha \) and \( \beta \) satisfy the condition (3.17), we may choose \( \tilde{\delta} \in (1 - \alpha, \alpha) \) as

\[
\tilde{\delta} = \frac{\alpha + 1}{\beta + 1} \beta.
\]

Now the term \( \mathcal{H}_{22} \) can be estimated as follows

\[
\mathcal{H}_{22} \leq C \| \Lambda^\beta_{x_1} \partial_{x_1} \theta \|_{L^2} \| \Lambda^\beta_{x_1} (u_2 \partial_{x_2} \partial_{x_1} x_1) \|_{L^2}
\]

\[
\leq C \| \Lambda^\beta_{x_1} \partial_{x_1} \theta \|_{L^2} (\| u_2 \|_{L^\infty} \| \Lambda^\alpha_{x_1} \partial_{x_2} \theta \|_{L^2} + \| \Lambda^\alpha_{x_1} u_2 \partial_{x_2} \theta \|_{L^2})
\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_2} \theta \|_{L^2}\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_2} \theta \|_{L^2}\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_2} \theta \|_{L^2}\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_2} \theta \|_{L^2}\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_2} \theta \|_{L^2}\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_2} \theta \|_{L^2}\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_2} \theta \|_{L^2}\]

\[
\leq C \| \partial_{x_1} \theta \|_{L^2} \| \Lambda^\alpha_{x_1} \partial_{x_2} \theta \|_{L^2}\]
Proposition 3.5. Assume \( \theta_0 \) satisfies the assumptions stated in Theorem 1.1 and let \( (u, \theta) \) be the corresponding solution. If \( \alpha \) and \( \beta \) satisfy (1.3), then, for any \( t > 0 \),

\[
\| \Delta \theta(t) \|_{L^2} + \int_0^t \left( \| \Lambda_{x_1}^\alpha \Delta \theta(\tau) \|_{L^2}^2 + \| \Lambda_{x_2}^\beta \Delta \theta(\tau) \|_{L^2}^2 \right) d\tau \leq C(t, \theta_0),
\]

(3.22)
where $C(t, \theta_0)$ is a constant depending on $t$ and the initial data $\theta_0$.

Proof of Proposition 3.3: Applying $\Delta$ to the first equation of (1.21), multiplying the resulting identity by $\Delta \theta$ and integrating over $\mathbb{R}^2$ by parts, we immediately deduce that

$$\frac{1}{2} \frac{d}{dt} \parallel \Delta \theta(t) \parallel_{L^2}^2 + \parallel \Lambda_x^\alpha \Delta \theta \parallel_{L^2}^2 + \parallel \Lambda_x^\beta \Delta \theta \parallel_{L^2}^2 = - \int_{\mathbb{R}^2} \Delta \{ (u \cdot \nabla) \theta \} \Delta \theta \, dx. \quad (3.23)$$

Using the divergence free condition, the term at the right hand side of (3.23) can be rewritten as

$$- \int_{\mathbb{R}^2} \Delta \{ (u \cdot \nabla) \theta \} \Delta \theta \, dx$$

$$= \int_{\mathbb{R}^2} \Delta (u_1 \partial_{x_1} \theta + u_2 \partial_{x_2} \theta) \Delta \theta \, dx$$

$$= \int_{\mathbb{R}^2} \Delta u_1 \partial_{x_1} \theta \Delta \theta \, dx + \int_{\mathbb{R}^2} \Delta u_2 \partial_{x_2} \theta \Delta \theta \, dx + 2 \int_{\mathbb{R}^2} \partial_{x_1} u_1 \partial_{x_1} \theta \Delta \theta \, dx$$

$$+ 2 \int_{\mathbb{R}^2} \partial_{x_2} u_1 \partial_{x_1} \theta \Delta \theta \, dx + 2 \int_{\mathbb{R}^2} \partial_{x_1} u_2 \partial_{x_2} \theta \Delta \theta \, dx + 2 \int_{\mathbb{R}^2} \partial_{x_2} u_2 \partial_{x_2} \theta \Delta \theta \, dx$$

$$:= \mathcal{T}_1 + \mathcal{T}_2 + \cdots + \mathcal{T}_6. \quad (3.24)$$

Our next goal is to handle the six terms at the right hand side of (3.24). Let us first notice some basic estimates. Due to Plancherel’s Theorem and the following simple inequality

$$|\xi_2|^{2\alpha} |\xi_1|^2 \leq |\xi_1|^{2\alpha} |\xi_1|^2,$$

we arrive at

$$\parallel \Lambda_x^\alpha \partial_{x_1} \theta \parallel_{L^2} \leq \parallel \Lambda_x^\alpha \nabla \theta \parallel_{L^2}. \quad (3.25)$$

Keeping in mind the fact $u = (-R_2, R_1)$ and using the same argument adopted in proving (3.25), one may conclude the following estimates which will be needed to estimate the terms $\mathcal{T}_1 - \mathcal{T}_6$

$$\parallel \Lambda_x^\alpha \Delta u_1 \parallel_{L^2} = \parallel \Lambda_x^\alpha \Delta R_2 \theta \parallel_{L^2} \leq \parallel \Lambda_x^\beta \Delta \theta \parallel_{L^2}, \quad (3.26)$$

$$\parallel \Lambda_x^\beta \Delta u_2 \parallel_{L^2} \leq \parallel \Lambda_x^\beta \Delta R_1 \theta \parallel_{L^2} \leq \parallel \Lambda_x^\alpha \Delta \theta \parallel_{L^2}, \quad (3.27)$$

$$\parallel \Lambda_x^\beta \partial_{x_1} \theta \parallel_{L^2} \leq \parallel \Lambda_x^\alpha \Delta \theta \parallel_{L^2}, \quad (3.28)$$

$$\parallel \Lambda_x^\alpha \partial_{x_1} u_1 \parallel_{L^2} \leq \parallel \Lambda_x^\alpha \partial_{x_1} R_2 \theta \parallel_{L^2} \leq \parallel \Lambda_x^\beta \nabla \theta \parallel_{L^2}, \quad (3.29)$$

$$\parallel \Lambda_x^\alpha \partial_{x_1} \theta \parallel_{L^2} \leq \parallel \Lambda_x^\alpha \Delta \theta \parallel_{L^2}, \quad (3.30)$$

$$\parallel \Lambda_x^\beta \partial_{x_2} u_1 \parallel_{L^2} \leq \parallel \Lambda_x^\beta \partial_{x_2} R_2 \theta \parallel_{L^2} \leq \parallel \Lambda_x^\beta \nabla \theta \parallel_{L^2}, \quad (3.31)$$

$$\parallel \Lambda_x^\alpha \partial_{x_1} \theta \parallel_{L^2} \leq \parallel \Lambda_x^\alpha \Delta \theta \parallel_{L^2}, \quad (3.32)$$

$$\parallel \Lambda_x^\beta \partial_{x_1} \theta \parallel_{L^2} \leq \parallel \Lambda_x^\beta \Delta \theta \parallel_{L^2}, \quad (3.33)$$

$$\parallel \Lambda_x^\alpha \partial_{x_2} u_2 \parallel_{L^2} \leq \parallel \Lambda_x^\alpha \partial_{x_2} R_1 \theta \parallel_{L^2} \leq \parallel \Lambda_x^\alpha \nabla \theta \parallel_{L^2}, \quad (3.34)$$

$$\parallel \Lambda_x^\beta \partial_{x_2} \theta \parallel_{L^2} \leq \parallel \Lambda_x^\beta \Delta \theta \parallel_{L^2}. \quad (3.35)$$
It should be mentioned that if $\alpha$ and $\beta$ satisfy (1.3), then $\alpha > \frac{1}{2}$ or $\beta > \frac{1}{2}$ holds true. Therefore, we split the proof into two cases, namely,

**Case 1:** $\alpha > \frac{1}{2}$; \hspace{1cm} **Case 2:** $\beta > \frac{1}{2}$.

For the **Case 1**, the inequality (2.19) implies the following bounds

\[
\mathcal{T}_1 = \int_{\mathbb{R}^2} \Delta u_1 \partial_{x_1} \theta \Delta \theta \, dx
\]

\[
\leq C \|\Delta \theta\|_{L^2} \|\partial_{x_1} \theta\|_{L^2} \|\Lambda_{x_1}^\alpha \partial_{x_1} \theta\|_{L^2} \|\Lambda_{x_2}^\alpha \partial_{x_2} \theta\|_{L^2} \|\Delta u_1\|_{L^2} \|\Lambda_{x_1}^\alpha \Delta u_1\|_{L^2} \left(\text{using (3.25)}\right)
\]

\[
\leq \epsilon \|\Lambda_{x_1}^\alpha \Delta \theta\|_{L^2}^2 + C(\epsilon) \|\nabla \theta\|_{L^2}^{\frac{2(2\alpha-1)}{4\alpha-1}} \|\Lambda_{x_1}^\alpha \nabla \theta\|_{L^2} \|\Delta \theta\|_{L^2}^2
\]

\[
\leq \epsilon \|\Lambda_{x_1}^\alpha \Delta \theta\|_{L^2}^2 + C(\epsilon) \|\nabla \theta\|_{L^2}^{\frac{2(2\alpha-1)}{4\alpha-1}} (1 + \|\Lambda_{x_1}^\alpha \nabla \theta\|_{L^2}^2) \|\Delta \theta\|_{L^2}^2, \hspace{1cm} (3.36)
\]

\[
\mathcal{T}_2 = \int_{\mathbb{R}^2} \Delta u_2 \partial_{x_2} \theta \Delta \theta \, dx
\]

\[
\leq C \|\Delta \theta\|_{L^2} \|\partial_{x_2} \theta\|_{L^2} \|\Lambda_{x_1}^\alpha \partial_{x_2} \theta\|_{L^2} \|\Lambda_{x_1}^\alpha \partial_{x_2} \theta\|_{L^2} \|\Delta u_2\|_{L^2} \|\Lambda_{x_1}^\alpha \Delta u_2\|_{L^2} \left(\text{using (3.27)}\right)
\]

\[
\leq \epsilon \|\Lambda_{x_1}^\alpha \Delta \theta\|_{L^2}^2 + C(\epsilon) \|\nabla \theta\|_{L^2}^{\frac{2(2\alpha-1)}{4\alpha-1}} (1 + \|\Lambda_{x_1}^\alpha \nabla \theta\|_{L^2}^2) \|\Delta \theta\|_{L^2}^2, \hspace{1cm} (3.37)
\]

\[
\mathcal{T}_3 = 2 \int_{\mathbb{R}^2} \partial_{x_1} u_1 \partial_{x_1} x_1 \theta \Delta \theta \, dx
\]

\[
\leq C \|\Delta \theta\|_{L^2} \|\partial_{x_1} u_1\|_{L^2} \|\Lambda_{x_1}^\alpha \partial_{x_1} \theta\|_{L^2} \|\Lambda_{x_2}^\alpha \partial_{x_1} \theta\|_{L^2} \|\Delta \theta\|_{L^2} \left(\text{using (3.28)}\right)
\]

\[
\leq \epsilon \|\Lambda_{x_1}^\alpha \Delta \theta\|_{L^2}^2 + C(\epsilon) \|\nabla \theta\|_{L^2}^{\frac{2(2\alpha-1)}{4\alpha-1}} (1 + \|\Lambda_{x_1}^\alpha \nabla \theta\|_{L^2}^2) \|\Delta \theta\|_{L^2}^2, \hspace{1cm} (3.38)
\]

\[
\mathcal{T}_4 = 2 \int_{\mathbb{R}^2} \partial_{x_2} u_1 \partial_{x_2} x_1 \theta \Delta \theta \, dx
\]

\[
\leq C \|\Delta \theta\|_{L^2} \|\partial_{x_2} u_1\|_{L^2} \|\Lambda_{x_1}^\alpha \partial_{x_2} \theta\|_{L^2} \|\Lambda_{x_2}^\alpha \partial_{x_2} \theta\|_{L^2} \|\Delta \theta\|_{L^2} \left(\text{using (3.30)}\right)
\]

\[
\leq \epsilon \|\Lambda_{x_1}^\alpha \Delta \theta\|_{L^2}^2 + C(\epsilon) \|\nabla \theta\|_{L^2}^{\frac{2(2\alpha-1)}{4\alpha-1}} (1 + \|\Lambda_{x_1}^\alpha \nabla \theta\|_{L^2}^2) \|\Delta \theta\|_{L^2}^2. \hspace{1cm} (3.39)
\]

\[
\mathcal{T}_5 = 2 \int_{\mathbb{R}^2} \partial_{x_1} u_2 \partial_{x_2} x_2 \theta \Delta \theta \, dx
\]

\[
\leq C \|\Delta \theta\|_{L^2} \|\partial_{x_1} u_2\|_{L^2} \|\Lambda_{x_1}^\alpha \partial_{x_1} \theta\|_{L^2} \|\Lambda_{x_2}^\alpha \partial_{x_2} \theta\|_{L^2} \|\Delta \theta\|_{L^2} \left(\text{using (3.32)}\right)
\]

\[
\leq C \|\Delta \theta\|_{L^2} \|\nabla \theta\|_{L^2} \|\Lambda_{x_1}^\alpha \nabla \theta\|_{L^2} \|\Delta \theta\|_{L^2} \left(\text{using (3.27)}\right)
\]
For the Case 2, one may conclude by using the inequality (2.19) that

\[ \mathcal{T}_1 = \int_{\mathbb{R}^2} \Delta u_1 \partial_{x_1} \theta \Delta \theta \, dx \]
\[ \leq C \| \Delta \theta \|_{L^2} \| \partial_{x_1} \theta \|_{L^2} \| \Lambda_{x_1}^\alpha \|_{L^2} \| \Delta u_1 \|_{L^2} \| \Lambda_{x_1}^\beta \|_{L^2} \]
\[ \leq C \| \Delta \theta \|_{L^2} \| \nabla \theta \|_{L^2} \| \Lambda_{x_2}^\beta \|_{L^2} \| \Delta \theta \|_{L^2} \]
\[ \leq \epsilon \| \Lambda_{x_1}^\alpha \Delta \theta \|_{L^2}^2 + C(\epsilon) \| \nabla \theta \|_{L^2}^2 \] (3.42)

\[ \mathcal{T}_2 = \int_{\mathbb{R}^2} \Delta u_2 \partial_{x_2} \theta \Delta \theta \, dx \]
\[ \leq C \| \Delta \theta \|_{L^2} \| \partial_{x_2} \theta \|_{L^2} \| \Lambda_{x_2}^\beta \|_{L^2} \| \Delta u_2 \|_{L^2} \]
\[ \leq C \| \Delta \theta \|_{L^2} \| \nabla \theta \|_{L^2} \| \Lambda_{x_2}^\beta \|_{L^2} \| \Delta \theta \|_{L^2} \]
\[ \leq \epsilon \| \Lambda_{x_2}^\beta \Delta \theta \|_{L^2}^2 + C(\epsilon) \| \nabla \theta \|_{L^2}^2 \] (3.43)

\[ \mathcal{T}_3 = 2 \int_{\mathbb{R}^2} \partial_{x_1} u_1 \partial_{x_1} x_1 \theta \Delta \theta \, dx \]
\[ \leq C \| \Delta \theta \|_{L^2} \| \partial_{x_1} u_1 \|_{L^2} \| \Lambda_{x_1}^\beta \|_{L^2} \| \partial_{x_1} \theta \|_{L^2} \]
\[ \leq C \| \Delta \theta \|_{L^2} \| \nabla \theta \|_{L^2} \| \Lambda_{x_2}^\beta \|_{L^2} \| \Delta \theta \|_{L^2} \]
\[ \leq \epsilon \| \Lambda_{x_1}^\alpha \|_{L^2}^2 + C(\epsilon) \| \nabla \theta \|_{L^2}^2 \] (3.44)

\[ \mathcal{T}_4 = 2 \int_{\mathbb{R}^2} \partial_{x_2} u_1 \partial_{x_2} x_1 \theta \Delta \theta \, dx \]
\[ \leq C \| \Delta \theta \|_{L^2} \| \partial_{x_2} u_1 \|_{L^2} \| \Lambda_{x_2}^\beta \|_{L^2} \| \partial_{x_2} \theta \|_{L^2} \]
\[ \leq C \| \Delta \theta \|_{L^2} \| \nabla \theta \|_{L^2} \| \Lambda_{x_2}^\beta \|_{L^2} \| \Delta \theta \|_{L^2} \]
\[ \leq \epsilon \| \Lambda_{x_2}^\beta \|_{L^2}^2 + C(\epsilon) \| \nabla \theta \|_{L^2}^2 \] (3.45)
\[ T_5 = 2 \int_{\mathbb{R}^2} \partial_{x_1} u_2 \partial_{x_1} \theta \Delta \theta \, dx \]
\[ \leq C \| \Delta \theta \|_{L^2} \| \partial_{x_1} u_2 \|^{1 - \frac{2 \alpha}{\delta_1}}_{L^2} \| \Lambda_{x_2} \partial_{x_2} \|^{\frac{1}{\delta_1}}_{L^2} \| \partial_{x_2} \theta \|^{1 - \frac{2 \alpha}{\delta_2}}_{L^2} \| \Lambda_{x_2} \partial_{x_2} \|^{\frac{1}{\delta_2}}_{L^2} \] 
\[ \leq C \| \Delta \theta \|_{L^2} \| \nabla \theta \|^{\frac{2(2\alpha - 1)}{4\beta - 1}}_{L^2} \| \Lambda_{x_2} \partial_{x_2} \|^{1 + \frac{1}{2\beta}}_{L^2} \] 
\[ \leq \epsilon \| \Lambda_{x_2} \partial_{x_2} \|_{L^2}^2 + C(\epsilon) \| \nabla \theta \|_{L^2}^{\frac{2(2\alpha - 1)}{4\beta - 1}} (1 + \| \Lambda_{x_2} \partial_{x_2} \|_{L^2}^{2}) \| \Delta \theta \|_{L^2}^2 \quad (3.46) \]

\[ T_6 = 2 \int_{\mathbb{R}^2} \partial_{x_2} u_2 \partial_{x_2} \theta \Delta \theta \, dx \]
\[ \leq C \| \Delta \theta \|_{L^2} \| \partial_{x_2} u_2 \|^{1 - \frac{2 \alpha}{\delta_1}}_{L^2} \| \Lambda_{x_2} \partial_{x_2} \|^{\frac{1}{\delta_1}}_{L^2} \| \partial_{x_2} \theta \|^{1 - \frac{2 \alpha}{\delta_2}}_{L^2} \| \Lambda_{x_2} \partial_{x_2} \|^{\frac{1}{\delta_2}}_{L^2} \] 
\[ \leq C \| \Delta \theta \|_{L^2} \| \nabla \theta \|^{\frac{2(2\alpha - 1)}{4\beta - 1}}_{L^2} \| \Lambda_{x_2} \partial_{x_2} \|^{1 + \frac{1}{2\beta}}_{L^2} \] 
\[ \leq \epsilon \| \Lambda_{x_2} \partial_{x_2} \|_{L^2}^2 + C(\epsilon) \| \nabla \theta \|_{L^2}^{\frac{2(2\alpha - 1)}{4\beta - 1}} (1 + \| \Lambda_{x_2} \partial_{x_2} \|_{L^2}^{2}) \| \Delta \theta \|_{L^2}^2 \quad (3.47) \]

Combining the above estimates and taking \( \epsilon \) suitable small, it allows us to get
\[
\frac{d}{dt} \| \Delta \theta(t) \|_{L^2}^2 + \| \Lambda_{x_1} \Delta \theta \|_{L^2}^2 + \| \Lambda_{x_2} \Delta \theta \|_{L^2}^2 \leq C \| \nabla \theta \|_{L^2}^{\frac{2(2\alpha - 1)}{4\beta - 1}} (1 + \| \Lambda_{x_2} \partial_{x_2} \|_{L^2}^2) \| \Delta \theta \|_{L^2}^2
\]
for \( \alpha > \frac{1}{\delta_1} \), while for \( \beta > \frac{1}{\delta_2} \), one deduces
\[
\frac{d}{dt} \| \Delta \theta(t) \|_{L^2}^2 + \| \Lambda_{x_1} \Delta \theta \|_{L^2}^2 + \| \Lambda_{x_2} \Delta \theta \|_{L^2}^2 \leq C \| \nabla \theta \|_{L^2}^{\frac{2(2\alpha - 1)}{4\beta - 1}} (1 + \| \Lambda_{x_2} \partial_{x_2} \|_{L^2}^2) \| \Delta \theta \|_{L^2}^2
\]

Applying the classical Gronwall inequality and noticing the key bounds (3.2) as well as (3.18), we immediately conclude
\[
\| \Delta \theta(t) \|_{L^2}^2 + \int_0^t (\| \Lambda_{x_1} \Delta \theta(\tau) \|_{L^2}^2 + \| \Lambda_{x_2} \Delta \theta(\tau) \|_{L^2}^2) \, d\tau \leq C(t, \theta_0).
\]

Therefore, the proof of Proposition 3.5 is concluded. \( \square \)

With the global \( H^2 \)-bound of \( \theta \) in hand, we are now ready to establish the global \( H^\alpha \)-estimate of \( \theta \) to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1** First, we need the following anisotropic interpolation inequality, whose proof will be provided in the Appendix
\[
\| h \|_{L^\infty} \leq C \| h \|_{L^2}^{1 - \frac{\delta_1}{2\delta_1}} \| \Lambda_{x_1} \partial_{x_1} h \|_{L^2}^{\frac{1}{\delta_1}} \| \Lambda_{x_2} \partial_{x_2} h \|_{L^2}^{\frac{1}{2\delta_2}},
\]
where \( \delta_1 > 0 \) and \( \delta_2 > 0 \) satisfy \( \frac{1}{\delta_1} + \frac{1}{\delta_2} < 2 \). The above inequality further allows us to show that
\[
\| \nabla \theta \|_{L^\infty} \leq C \| \nabla \theta \|_{L^2}^{1 - \frac{1}{2(1 + \alpha)}} \| \Lambda_{x_1} \nabla \theta \|_{L^2}^{\frac{1}{2(1 + \alpha)}} \| \Lambda_{x_2} \nabla \theta \|_{L^2}^{\frac{1}{2(1 + \beta)}}
\]
\[
\leq C \| \nabla \theta \|_{L^2}^{1 - \frac{1}{2(1 + \alpha)}} \| \Lambda_{x_1} \nabla \theta \|_{L^2}^{\frac{1}{2(1 + \alpha)}} \| \Lambda_{x_2} \Delta \theta \|_{L^2}^{\frac{1}{2(1 + \beta)}}
\]
and
\[
\| \nabla u \|_{L^\infty} \leq C \| \nabla u \|_{L^2}^{1 - \frac{1}{2(1 + \alpha)}} \| \Lambda_{x_1} \nabla u \|_{L^2}^{\frac{1}{2(1 + \alpha)}} \| \Lambda_{x_2} \nabla u \|_{L^2}^{\frac{1}{2(1 + \beta)}}
\]
\[
\leq C \| \nabla \theta \|_{L^2}^{1 - \frac{1}{2(1 + \alpha)}} \| \Lambda_{x_1} \nabla \theta \|_{L^2}^{\frac{1}{2(1 + \alpha)}} \| \Lambda_{x_2} \nabla \theta \|_{L^2}^{\frac{1}{2(1 + \beta)}}
\]
≤ C∥∇θ∥1\(L^2\)\(_{2+α}θ\)∥Λ_1^α ∆θ∥2\(L^2\)\(\frac{1}{2+α}\)∥Λ_2^β ∆θ∥2\(L^2\)\(\frac{1}{2+α}\).

The obtained estimates in (3.2), (3.18) and (3.22) yield
\[\int_0^t (∥\nabla θ(s)∥^{\frac{4(1+α)(1+β)}{2+α+β}}_L^{2+α+β} + ∥\nabla u(s)∥^{\frac{4(1+α)(1+β)}{2+α+β}}_L^{2+α+β}) ds ≤ C(t, θ_0).\]

The basic \(H^s\)-estimate of the system (1.1) reads
\[\frac{d}{dt}∥θ(t)∥^2_Hs + ∥Λ_1^α θ∥^2_Hs + ∥Λ_2^β θ∥^2_Hs ≤ C(1 + ∥∇ u∥_L^∞ + ∥∇ θ∥_L^∞)∥θ∥^2_Hs.\]

It is then clear that
\[∥θ(t)∥^2_Hs + \int_0^t (∥Λ_1^α θ(τ)∥^2_Hs + ∥Λ_2^β θ(τ)∥^2_Hs) dτ ≤ C(t, θ_0).\]

This completes the proof of Theorem 1.1 \(\Box\)

APPENDIX A. AN ALTERNATIVE PROOF OF (3.18)

Here we give the proof of the anisotropic interpolation inequality (3.18). Before proving this inequality, we point out that the anisotropic interpolation inequality established in [17, Lemma A.2] is an easy consequence of the inequality (3.18). By means of the following one-dimensional Sobolev inequality
\[∥g∥_{L^2_1(\mathbb{R})} ≤ C∥g∥_{L^1_1(\mathbb{R})}^{\frac{2γ-1}{2}}∥Λ_1^γ g∥_{L^2_1(\mathbb{R})}^{\frac{1}{2}}, \quad γ > \frac{1}{2},\]

it is clear that by choosing the intermediate variables \(ε_1, ε_2 > \frac{1}{2}\) and noticing \(δ_2 > \frac{1}{2}\)
\[∥h(x_1, x_2)∥_L^∞ = ∥h(x_1, x_2)∥_{L^∞_x^1 L^∞_x^2} ≤ C∥h(x_1, x_2)∥_{L^{2ε_1}_x^1 L^{2_2}_x^2}^{\frac{1}{2_2}}∥Λ_1^{δ_1} h(x_1, x_2)∥_{L^{2_2}_x^1 L^{2_2}_x^2}^{\frac{1}{2_2}} \leq C∥h(x_1, x_2)∥_{L^{2ε_2}_x^1 L^{2_2}_x^2}^{\frac{1}{2}}∥Λ_2^{δ_2} h(x_1, x_2)∥_{L^{2_2}_x^1 L^{2_2}_x^2}^{\frac{1}{2}},\]

\[= C∥h(x_1, x_2)∥_{L^{2_2}_x^1 L^{2_2}_x^2}^{\frac{1}{2}}∥Λ_2^{δ_2} h(x_1, x_2)∥_{L^{2_2}_x^1 L^{2_2}_x^2}^{\frac{1}{2}} \times ∥Λ_1^{δ_1} h(x_1, x_2)∥_{L^{2_2}_x^1 L^{2_2}_x^2}^{\frac{1}{2}}∥Λ_2^{δ_2} Λ_1^{δ_1} h(x_1, x_2)∥_{L^{2_2}_x^1 L^{2_2}_x^2}^{\frac{1}{2}}.\]

(A.1)

Now if we further assume \(δ_1, ε_2 ≤ δ_2\) and \(\frac{δ_1}{δ_1} + \frac{ε_2}{δ_2} ≤ 1\), then we obtain
\[∥Λ_1^{δ_1} h(x_1, x_2)∥_{L^2} ≤ ∥h(x_1, x_2)∥_{L^∞_x^1 L^∞_x^2}^{\frac{δ_1 - ε_1}{δ_1}}∥Λ_1^{δ_1} h(x_1, x_2)∥_{L^∞_x^1 L^∞_x^2}^{\frac{ε_1}{δ_1}},\]

(A.2)

and
\[∥Λ_2^{δ_2} Λ_1^{δ_1} h(x_1, x_2)∥_{L^2} = \left( \int_{\mathbb{R}^2} |ξ_2|^{2_{ε_2}} |ξ_1|^{2_{δ_1}} |\hat{h}(ξ)|^2 dξ \right)^{\frac{1}{2}}.\]
\[
\begin{align*}
= \left( \int_{\mathbb{R}^2} \left( |\xi_2|^{2\delta_2} \left| \hat{h}(\xi) \right|^2 \right) \left( |\xi_1|^{2\delta_1} \left| \hat{h}(\xi) \right|^2 \right) \left| \hat{h}(\xi) \right|^{2-\frac{2\delta_2}{\delta_1} - \frac{2\delta_1}{\delta_2}} \, d\xi \right)^{\frac{1}{2}} \\
\leq C \left( \int_{\mathbb{R}^2} |\xi_2|^{2\delta_2} \left| \hat{h}(\xi) \right|^2 \, d\xi \right)^{\frac{\delta_2}{\delta_1}} \left( \int_{\mathbb{R}^2} |\xi_1|^{2\delta_1} \left| \hat{h}(\xi) \right|^2 \, d\xi \right)^{\frac{\delta_1}{\delta_2}} \\
\times \left( \int_{\mathbb{R}^2} \left| \hat{h}(\xi) \right|^2 \, d\xi \right)^{\frac{1}{2} - \frac{\delta_1}{\delta_2} - \frac{\delta_2}{\delta_1}} \\
= C \left\| \Lambda_{x_2}^{\delta_2} h(x_1, x_2) \right\|_{L^2}^{\frac{\delta_2}{\delta_1}} \left\| \Lambda_{x_1}^{\delta_1} h(x_1, x_2) \right\|_{L^2}^{\frac{\delta_1}{\delta_2}} \left\| \hat{h}(x_1, x_2) \right\|_{L^2}^{1 - \frac{\delta_1}{\delta_2} - \frac{\delta_2}{\delta_1}}. \quad (A.3)
\end{align*}
\]

Combining the above estimates, it yields
\[
\left\| h(x_1, x_2) \right\|_{L^\infty} \leq C \left\| h(x_1, x_2) \right\|_{L^2}^{\frac{2\delta_1}{\delta_1} - \frac{2\delta_2}{\delta_2} + \frac{\delta_1}{\delta_1} - \frac{\delta_2}{\delta_2} + \frac{1}{2} \left( \frac{\delta_1}{\delta_1} + \frac{\delta_2}{\delta_2} \right)} \times \left( \frac{\delta_1}{\delta_1} + \frac{\delta_2}{\delta_2} \right) \left\| \Lambda_{x_1}^{\delta_1} h(x_1, x_2) \right\|_{L^2} \left\| \Lambda_{x_2}^{\delta_2} h(x_1, x_2) \right\|_{L^2} \left\| \hat{h}(x_1, x_2) \right\|_{L^2},
\]

where the intermediate variables \( \varepsilon_1 \) and \( \varepsilon_2 \) should be satisfied \( \frac{1}{2} < \varepsilon_1 \leq \delta_1, \frac{1}{2} < \varepsilon_2 \leq \delta_2 \) and \( \frac{\delta_1}{\delta_1} + \frac{\delta_2}{\delta_2} \leq 1 \). Thus, it leads to \( \frac{\delta_1}{\delta_1} > \frac{1}{2\delta_1} \) and \( \frac{\delta_2}{\delta_2} > \frac{1}{2\delta_2} \), which together with the condition \( \frac{\delta_1}{\delta_1} + \frac{\delta_2}{\delta_2} \leq 1 \) implies
\[
\frac{1}{2\delta_1} + \frac{1}{2\delta_2} < 1 \quad \text{or} \quad \frac{1}{\delta_1} + \frac{1}{\delta_2} < 2. \quad (A.4)
\]

The above argument implies that the intermediate variables \( \varepsilon_1 \) and \( \varepsilon_2 \) do exist as long as \( (A.4) \) holds true. This completes the proof of the inequality \( (3.48) \).

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