Hyperbolicity constraints in extended gravity theories

Yotam Sherf

Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel
E-mail: sherfyo@post.bgu.ac.il

Received 4 March 2019, revised 21 March 2019
Accepted for publication 26 March 2019
Published 4 June 2019

Abstract
We study the characteristic structure of the Einstein–Hilbert (EH) action when modifications of the form of $R^2$, $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$, and $C_{\mu\nu\rho\sigma}$ are included. We show that when these quadratic terms are significant, the initial value problem is generically ill-posed. We do so by demanding the hyperbolicity of the effective metric for propagation of perturbations. Here, we find a general expression for the effective metric in field space and calculate it explicitly about the cosmological Friedman–Robertson–Walker spacetime, and the spherically symmetric Schwarzschild solution. We find that when these quadratic contributions are non-negligible, the signature of the effective metric becomes non-Lorentzian and hence non-hyperbolic. As a consequence, we conclude that theories suggesting the inclusion of these terms can only be considered as a perturbative extension of the EH action and therefore cannot constitute a true alternative to general relativity.

Keywords: general relativity, alternative gravity theories, relativistic wave equation, causality and hyperbolicity, gravitation

1. Introduction

Large-distance scale phenomena are well-described in the framework of general relativity (GR). While at short distances, the classical GR description is incomplete, indicating the need for modifications of the Einstein–Hilbert (EH) action. Among them, the modifications most likely expected to appear are higher derivative terms in the form of higher powers of the curvature tensors [1–3].

Higher derivative theories are relevant in several contexts. First, in string theory where higher derivative terms naturally appear in the low energy limit [4–8] and in the study of black hole (BH) solutions [9, 10]. Moreover, renormalization of the stress–energy tensor in one-loop quantum gravity suggests that the presence of quadratic terms in the gravitational Lagrangian is a priori expected [11–14]. Here we are mainly interested in their relevancy within the context of low curvature backgrounds such as the cosmological Friedman–Robertson–Walker (FRW) spacetime [1, 3, 15–21], where in these models, modifications to GR are studied in order to explain dark energy, early-time inflation with late-time cosmic acceleration, and cosmological perturbations.

Here we would like to examine the possibility of viewing higher derivative theories as an alternative to the Einstein theory. This implies that the additional terms in the form of higher derivative induce significant modification to the classical GR also in large-distance scales. Otherwise, they can be viewed as a perturbative correction to the GR [22]. We then study the violation of causality by investigating the hyperbolicity of the modified EOMs of perturbations. We assume that the contribution of higher derivative terms may be comparable or larger than that of the Einstein term, as explained in detail below.

We consider modifications to the EH action in four dimensions (4D) of the form

$$S = \frac{1}{16\pi l_p^2} \int d^4x \sqrt{-g} \left( R + \lambda_1 R^2 + \lambda_2 R_{\mu\nu}R_{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \right),$$

(1)

1 The case of the quadratic Weyl tensor will be discussed separately.
where the $\lambda_i$ are dimensionful coupling constants of a typical length scale $\lambda_i \sim l_i^2$. These coefficients determine the length scale at which corrections are important. Obviously, when these coefficients are significantly small $l_i^2/l_0^2 \ll 1$ or at low curvature backgrounds, the contribution of the additional curvature terms is negligible and can be considered as a perturbative extension of the EH action. On the other hand, when approaching the ultraviolet length scale they become dominant over the Einstein term. To act as a truly alternative theory of gravity, the higher order terms have to induce modifications to the solution of the EOM in the non-perturbative regime. This can happen in two scenarios. First, the coefficients are anomalously larger then the cutoff length scale. Second, the length scale is anomalously large so that curvature terms are large. In general, we claim that when $\sum \lambda_i R^2 \gtrsim R$ or alternatively $\lambda_i R \gtrsim 1$, significant modifications are induced in the EOM, thus the theory can be viewed as a modified gravity. We later show explicitly how these relations are the conditions for the EOM to be substantially modified.

Here, we identify the effective metric—the metric that determines the hyperbolicity of the EOM for propagation of perturbations [23]. A more general related method for identifying the characteristic structure in higher derivative theories is the method of characteristics [24–26]. This method identifies the regions where the Cauchy surface evolves non-uniquely. In this method, the determining criterion for the hyperbolicity of the EOM is the positivity of the principle symbol, which in our case will be strongly related to the demand of maintaining a Lorentzian structure of the effective metric section 2.1. Then, since a necessary condition for a hyperbolic EOM is the Lorentzian structure of the effective metric, its non-Lorentzian structure implies a non-hyperbolic EOM.

Before examining the hyperbolic structure in these extended theories, we direct the reader to some simple higher derivative scalar field models. In [27] and later in [23, 28, 29] it is shown that when the contribution of additional terms in the form $\nabla_\mu \phi \nabla_\nu \phi$ induces significant modification to the EOM, the hyperbolicity effective metric is no longer guaranteed.

To proceed, we review the ‘effective metric method,’ manifested in [23], which enables us to identify the effective metric in field space for propagation of perturbations. This method is implemented for higher derivative theories expanded around the cosmological FRW background and for the spherically symmetric Schwarzschild solution. From the demand for the Lorentzian signature of the effective metric we conclude that the EOM of perturbations are not always hyperbolic when the higher order contributions are significant.

2. Effective metric for quadratic curvature models

In this section we discuss the model that is given in equation (1) along the same lines of the discussion of the scalar field models. In principle, the main complication in replacing the higher derivative terms of the scalar fields $\nabla_\mu \phi \nabla_\nu \phi$ with their gravitational analog, namely the $R^2$, $R_{\mu
u} R^{\mu\nu}$, terms stems from the gravitational index structure and the gauge-redundancy. We overcome these complications by studying gauge-invariant tensor perturbations about FRW spacetime and the Schwarzschild solution. To begin, we consider the action in equation (1) with the Lagrangian density

$$L = \sqrt{-g} (R + \lambda_1 R^2 + \lambda_2 R_{\mu
u} R^{\mu\nu} + \lambda_3 R_{\mu
u\rho\sigma} R^{\mu\nu\rho\sigma}).$$

The vacuum EOM with respect to the background metric are given by,

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R + (\lambda_2 + 4 \lambda_3) \Box R_{\mu\nu}$$

$$+ \frac{1}{2} (\lambda_2 + 4 \lambda_3) g_{\mu\nu} \Box R$$

$$- (2 \lambda_1 + \lambda_2 + 2 \lambda_3) \nabla_\mu \nabla_\nu R + 2 \lambda_3 R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} + 2 \lambda_2$$

$$+ 2 \lambda_1 R_{\mu\nu\rho\sigma} R^{\rho\sigma}$$

$$- 4 \lambda_3 R_{\mu\nu\rho\sigma} R^{\rho\sigma} + 2 \lambda_1 R_{\mu\nu} R_{\rho\sigma}$$

$$- \frac{1}{2} g_{\mu\nu} (\lambda_1 R^2 + \lambda_2 R^{\alpha\beta} R_{\alpha\beta} + \lambda_3 R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma}) = 0.$$  

Equation (3) contains fourth order derivative terms in the metric field like $\nabla_\mu \nabla_\nu \Box R_{\rho\sigma}$, and $R_{\rho\sigma}$. These terms dominate the characteristic structure of propagation of perturbations and determine the well-possedness of the initial value problem, which eventually can lead to a non-hyperbolic EOMs [24, 26]. In addition, EOM of order higher than two, suffer severe instabilities, such as Ostrogradsky type instabilities [30]. Their Hamiltonian formulation is multi-valued in the canonical momentum, which implies that the initial value problem time evolution is non-injective [31]. We argue that if one wishes to construct an alternative and consistent theory of gravity, the terms containing derivatives that are higher than two in the metric have to be dismissed. As a consequence, the modified EOMs would take the differential form as do the Einstein equations take—second order in the metric tensor.

We briefly mention that the higher order derivative terms can be eliminated by applying the method that is reviewed in [32]. There it is shown that a second order field equations are obtained by an appropriate redefinition of the tensor perturbations, in addition to a proper choice of boundary conditions. Then, by specifying the initial value data on a Cauchy surface the resulted EOMs are second order. It is also important to mention that the transverse-traceless (TT) gauge is still maintained after the application of the method that is given in [32]. Since as explained in [33] the ability to choose tensor perturbations that are TT relies solely on the geometrical properties of the backgrounds and not on their gravitational action. Therefore it is possible to apply the TT gauge for tensor perturbations about maximally symmetric subspace.

Now, in accordance with the above-mentioned and without further details we ignore the fourth order derivative terms We will be satisfied with the recognition that the
possibility of eliminating these terms is exists. The detailed description of this process is out of the scope of this paper. From now and on, our interest would be solely subjected to the examination of the hyperbolicity of the second order field equations.

Next, we expand the metric $g_{\mu\nu}$ about the background $g_{\mu\nu}$, \( g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \). Then, in accordance, we expand equation (3) to first order in $h_{\mu\nu}$. Again, we emphasize that our aim is to derive an expression for the effective metric, we do so by isolating the kinetic terms—terms of the form $\nabla \nabla h$. Then the effective metric $G_{\alpha\beta}$ is identified in the form $G_{\alpha\beta} \nabla^\alpha \nabla^\beta h_{\mu\nu}$.

For our purpose, we evaluate the effective metric for tensor perturbations around 4D maximally symmetric subspace with an Euclidean line element. The generalization for dimensions higher than four is immediate and yields similar results.

Then, for the isotropic FRW spacetime, tensor perturbations around maximally symmetric spaces are TT,

$$h_i^j h_i^j, \nabla h_i^j = 0, \quad (4)$$

where $i = 1, 2, 3$ denote the spatial components. We point out that the choice of tensor modes that are TT with respect to a maximally symmetric subspaces is always possible, as explained in detail at [33]. There it is shown that this choice relies on the geometric properties of the space, and not on the gravitational action.

The expansion of equation (3) is carried out by applying the following relations. The first order expansion of the Riemann tensor is given by,

$$\delta R^{(1)}_{\mu\nu\rho\sigma} = \frac{1}{2} (\nabla_i \nabla_j h_{i\sigma} + \nabla_i \nabla_j h_{j\sigma} - \nabla_i \nabla_j h_{i\rho} - \nabla_i \nabla_j h_{j\rho}). \quad (5)$$

According to the mode choice equation (4), the expansion of the Ricci tensor to first order is given by

$$\delta R^{(1)}_{\mu\nu} = \frac{1}{2} (g_{\sigma\rho} \nabla^\sigma \nabla^\rho h_{\mu\nu} - \nabla^\sigma \nabla^\rho h_{\mu\nu}), \quad (6)$$

where the term $\nabla^\sigma \nabla^\rho h_{\mu\nu}$ is not a kinetic term and has no contribution to the dynamics of perturbations, rather it is a mass term. To see this we apply the commutation relations of covariant derivatives in terms of the Riemann tensor as follows

$$[\nabla_i, \nabla^j] h_{k\ell} = -R^\alpha_{\mu\sigma} h_{i\alpha \ell} - R^\alpha_{\mu\sigma} h_{i\ell \alpha}, \quad (7)$$

$$\nabla^\sigma \nabla^\rho h_{\mu\nu} = -R^\alpha_{\rho \sigma} h_{i\alpha \mu} - R^\alpha_{\sigma \mu} h_{i\ell \alpha}. \quad (8)$$

Therefore, in this process we inverted a kinetic term in disguise into a mass terms Then the kinetic contribution of equation (6) takes the form

$$\delta R^{(1)}_{\mu\nu} = \frac{1}{2} g_{\sigma\rho} \nabla^\sigma \nabla^\rho h_{\mu\nu} + \text{mass terms}. \quad (9)$$

The contribution of the Ricci scalar to the effective metric vanishes due to the traceless condition equation (4). In general, for tensor perturbations equation (4), the only contribution to the effective metric from equation (3) comes from the Riemann and Ricci tensors.

Then, the effective metric of the EH action is given as the coefficient of the kinetic term

$$\delta (2R_{\mu\nu} - g_{\mu\nu} \nabla^\alpha \nabla^\beta h_{\mu\nu}) = g_{\alpha\beta} \nabla^\alpha \nabla^\beta h_{\mu\nu} + \text{mass terms}. \quad (10)$$

As expected, $G_{\alpha\beta} = g_{\alpha\beta}$. The effective metric in GR is the background metric and it has a hyperbolic EOM and a well-defined causal structure.

Then, following the above relations, in addition to the Riemann symmetry properties, we obtain a general expression for the perturbed EOM

$$\delta G_{\mu\nu} = \Box h_{\mu\nu} + 4 \lambda_3 R^{abc} (\nabla_a \nabla_b h_{\mu\nu} + \nabla_b \nabla_a h_{\mu\nu}) \quad (11)$$

Now, we would like to isolate the kinetic term, so we would have an expression of the form $G_{\alpha\beta} \nabla^\alpha \nabla^\beta h_{\mu\nu}$. We argue that because of the low symmetry demonstrated by such models, an implicit expression of the effective metric in terms of the curvature tensors is very complicated and is also irrelevant for our purpose. In contrast to other theories, such as Lovelock theories, where it is shown [23] that due to the high degree of symmetry of the fully anti-symmetric Lovelock Lagrangians, an implicit expression of the effective metric in terms of the curvature tensors takes a simple form and its derivation is significantly simpler. Here, we will be satisfied with an explicit calculation of the curvature tensors about a specific background solutions. The calculations are carried out by considering the non-vanishing components of the Riemann tensor in field space as we explain in detail below.

### 2.1. The effective metric in FRW spacetime

The homogeneous and isotropic 4D FRW spacetime with the line element

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j, \quad (12)$$

where $i, j = 1, 2, \ldots, D - 1$ denote spatial components, and $\gamma_{ij} dx^i dx^j$ is the Euclidean line element. This type of spacetime solves action equation (1) in the presence of matter. A detailed description of the solutions is irrelevant for our purpose, and can be found at [1, 15, 16]. Here, we use the solutions that describes a universe undergoing a decelerated or accelerated expansion.

As previously mentioned, we evaluate the effective metric for tensor perturbations, which are defined about FRW backgrounds as

$$h_{\hat{a} \hat{b}} = 0, \quad h_{\hat{a} t} = 0, \quad \nabla_t h_{\hat{a} \hat{b}} = 0, \quad (13)$$

where barred indices $\hat{a}, \hat{b} = 1, 2, 3$, denote spatial components. The non-vanishing components of the Riemann tensor
are the following

\[ R_{\alpha\beta}^{ab} = H^2 \delta_{\alpha\beta}^{ab}, \quad R_{\alpha\beta}^{\mu\nu} = \delta_{\alpha\beta}^{\mu\nu}, \]  

(14)

where \((\dot{a}/a)^2 = H^2\) and \(\ddot{a}/a = H^2 + \dot{H}\). Now, by taking the Riemann tensor in terms of the background metric we are able to derive an explicit expression of the effective metric.

We now wish to explain how to evaluate the effective metric for tensor perturbations about the FRW background. For simplicity, we first consider the term \(R_{cc}^{ab}(\nabla_c \nabla^b h_{bd} + \nabla_d \nabla^c h_{bd})\) (taken from equation (11)). The evaluation is carried out in four steps.

1. Express the Riemann tensor in terms of its different background values metric equation (14)

\[
R_{cc}^{ab}(\nabla_c \nabla^b h_{bd} + \nabla_d \nabla^c h_{bd}) = R_{cc}^{ab}(\nabla_c \nabla^b h_{bd} + \nabla_d \nabla^c h_{bd}) + R_{cc}^{ab}(\nabla_c \nabla^c h_{bd} + \nabla_d \nabla^c h_{bd}) = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a} + H^2 \nabla_c \nabla^c h_{bd} + \nabla_d \nabla^c h_{bd}. \]

2. Perform contraction and apply mode choice equation (13)

\[
\overline{\frac{a}{a}} \nabla_c \nabla^c h_{bd} + H^2 (\nabla_c \nabla^c h_{bd} + \nabla_d \nabla^c h_{bd}) + \nabla_d \nabla^c h_{bd} = \overline{\frac{a}{a}} \nabla_c \nabla^c h_{bd} + H^2 (\nabla_c \nabla^c h_{bd} + \nabla_d \nabla^c h_{bd}). \]

3. Dismiss mass terms by using the commutation relations of covariant derivatives equation (8)

\[
\overline{\frac{a}{a}} \nabla_c \nabla^c h_{bd} + H^2 \nabla_c \nabla^c h_{bd} + \text{mass terms}. \]

4. Gather the different time–time and space–space derivative terms separately, then move into covariant representation

\[
G^{ab} \nabla_a \nabla_b h_{c\delta} = \frac{\ddot{a}}{a} g^{ab} \nabla_a \nabla_b h_{c\delta} + H^2 g^{ac} \nabla_a \nabla_b h_{d\delta}. \]

(18)

In this specific example each derivative term has one contribution, so step 4 is trivial. In general, this is not the case and more terms are involved as seen in equation (11).

Following the above instructions, we apply them for equation (3) and obtain the expression for effective metric about FRW spacetime.

The time–time component

\[
G^{ac} \nabla_a \nabla_c h_{d\delta} = g^{ac} (1 + \dot{H}(2\lambda_3 + 3\lambda_2 + 12\lambda_1) + H^2(-10\lambda_3 + 5\lambda_2 + 24\lambda_1)) \nabla_a \nabla_c h_{d\delta}. \]

(19)

The space–space component

\[
G^{ac} \nabla_a \nabla_c h_{d\delta} = g^{ac} (1 + H(-6\lambda_3 + \lambda_2 + 12\lambda_1)) \nabla_a \nabla_c h_{d\delta}. \]

(20)

First, it is clear the effective metric has no dependence on the different graviton polarization components \(h_{c\delta}\), the reason stems from the fact that the tensor modes equation (13) are defined about the maximally symmetric subspace of the FRW spacetime. In contrast, if the perturbation modes are defined about different subspaces, the effective metric becomes polarization dependent. To illustrate it, we consider a specific graviton \(h_{c\delta}\) with polarization components \((\epsilon, \delta)\). Then if the effective metric equation (11) contains a term of the form \(R^{abcde}\) it becomes polarization dependent, since the term \(R^{abcde}\) varies for different values of graviton components \((\epsilon, \delta)\).

It is important to note that the effective metric time–time and the space–space components have different proportionality factors. Such that, eventually the effective metric is not proportional to the background metric \(g^{ab}\), thus, enabling the breakdown of the Lorentzian structure. In particular, since the multiplying factors in parenthesis of the respective time equation (19) and space equation (20) components can have a different sign, then in general the effective metric is not Lorentzian. Alternatively, if these factors are of the same sign, then the effective metric is Lorentzian.

In addition, we note that the factors that multiplies \(H^2\) are identical in both the spatial and temporal components. In contrast, the multiplying factors of \(H\) are different. Thus, deviations from the Lorentzian pattern can only stems from the \(H\) term and its multiplying factors.

We now turn to discuss the results and to examine the conditions under which the EOM are non-hyperbolic. We do so by considering the different combinations of the higher order terms in action equation (1). We show that the conditions for the well-posedness of the perturbed EOM are in general depends on the specific values of the metric parameters; \(a(t)\) in FRW spacetime and the BH mass the Schwarzschild spacetime. The classification of the different combinations will allow us to determine which theories yield an ill-posed initial value problem and therefore are inapplicable as an alternative theories of gravity.

It is important to mention that by using the Gauss–Bonnet (GB) invariant, the Riemann squared term in action equation (1), can be expressed in terms of the Ricci tensor and the scalar curvature. This is due to the fact that the GB term, namely \(L_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\) is a total derivative in 4D, and therefore has no influence on the solutions of the EOM. However, though the EOM with respect to the first variation of action equation (1) yields identical solutions, the EOM for perturbation with respect to the second variation are essentially different. This can easily be seen in equations (11), (15), (16), where the contribution of the Riemann tensor to the perturbed EOM has an additional terms, involving the contribution from the different degrees of freedom. In general, these extra terms would not be present if one applies the GB identity. Therefore, we consider the action equation (1) in its explicit form.
We recall again that our purpose is to show that when the higher derivative terms are large, the hyperbolicity effective metric is no longer guaranteed and therefore, the extended theory cannot be viewed as a consistent theory of gravity.

2.1.1. \( R^2 \) theories. We consider the Lagrangian term of the form \( \mathcal{L} = R + \lambda R^2 \). In this form, the effective metric is immediately found by taking \( \lambda_1 \neq 0, \lambda_2 = \lambda_3 = 0 \) in equations (19), (20). Thus obtaining,

\[
\mathcal{G}^a = g^a(1 + \lambda_1(12H + 24H^2)), \quad \mathcal{G}^{\alpha\varepsilon} = g^{\alpha\varepsilon}(1 + \lambda_1(12H + 24H^2)).
\]

It is clear that since the factors multiplying \( g^a \) and \( g^{\alpha\varepsilon} \) are identical, the effective metric is proportional to the background metric and therefore its signature is always Lorentzian, no matter the magnitude of \( \lambda_1 \).

This result is not surprising at all, since as already demonstrated by Starobinsky at [17] (for a detailed review [18]), the extensions of the EH action by terms that are powers of Ricci scalar is equivalent to the insertion of scalar potential in the action or alternatively as the addition of mass terms, which in both cases has no influence on the hyperbolicity of the EOM for tensor perturbations as shown at [19]. We therefore conclude that extensions of the form \( R^a \) has no influence on the dynamics of the perturbed EOM.

2.1.2. \( R_{\mu
u}R^{\mu
u} \) extensions. The Lagrangian term we are considering is \( \mathcal{L} = R + \lambda_3 R_{\mu
u}R^{\mu
u} \). Replacing \( \lambda_3 = 0, \lambda_1 = \lambda_2 = 0 \) in equations (19), (20). The corresponding effective metric is,

\[
\mathcal{G}^a = g^a \left( 1 + \lambda_3 H^2 \left( \frac{2H}{H^2} - 10 \right) \right), \quad \mathcal{G}^{\alpha\varepsilon} = g^{\alpha\varepsilon} \left( 1 - \lambda_3 H^2 \left( \frac{6H}{H^2} + 10 \right) \right).
\]

First, it is clear that for \( \lambda_3 = 0 \) we recover the Einstein result, so the effective metric becomes the background metric. For \( \lambda_3 \neq 0 \) the factors multiplying \( g^a \) and \( g^{\alpha\varepsilon} \) are different and in some cases can have opposite signs. In accordance with the assumption mentioned at section 1, we assume that the correction terms are non-negligible; \( \lambda_3 H^2 \gtrless 1 \). Then, the hyperbolicity is governed by the \( H/H^2 \) term. So, for \( \lambda_3 > 0 \) the signature becomes non-Lorentzian in the range \( H/H^2 > 5 \), \( H/H^2 < -5/3 \).

To understand the significance of the conditions, let us consider a solution of decelerated expansion [15, 16] of the form \( \alpha(t) \sim t^\alpha \) with \( 0 < \alpha < 1 \). In this case, \( H = -H^2/\alpha \).

Then, the signature becomes non-Lorentzian for \( \alpha < 3/5 \). This condition (when \( \lambda_3 H^2 \gtrless 1 \)) rules out a large range of the parameter space of decelerated expanding isotropic solutions.

Another way of interpreting these results is by examining the influence of these higher derivative terms on the propagation speed of perturbations. For simplicity, we consider a radially propagating perturbation. The EOM are given by

\[
\mathcal{G}^{\alpha\varepsilon} \nabla_\alpha \nabla_\varepsilon h_{\mu\nu} = \mathcal{G}^a \nabla_a^2 h_{\mu\nu} + \mathcal{G}^{a\varepsilon} \nabla_a h_{\mu\varepsilon}. \tag{25}
\]

The null geodesics define the effective speed of propagation \( c_f^2 = -\mathcal{G}_a/\mathcal{G}^{\alpha\varepsilon} \). Then \( c_f^2 \) has been substantially modified when the higher derivative terms are significant. An interesting observation is that one can still have a well-defined causal structure, even though the propagation speed exceeds the speed of light figure 1. Particularly interesting are the conditions where \( c_f \) becomes imaginary, that are identical to the non-Lorentzian conditions mentioned above. The imaginary regime of \( c_f \) indicates on a causal structure that is badly defined and also the degeneracy of the Cauchy surface.

We now wish to explain the condition \( \lambda H^2 \gtrsim 1 \). Recalling that significant modifications to the EH action are induced when the higher order terms satisfies \( \lambda \mathcal{R} \gtrsim 1 \) section 1. Then, since in FRW spacetime \( \mathcal{R} \sim H^2 \), the condition \( \lambda H^2 \gtrsim 1 \) coincides with the above relations. The interpretation in terms of the \( \alpha \) parameter implying that for for \( \alpha(t) \sim t^{\alpha} \) we have \( \lambda \alpha^2 \gtrsim 1 \) which always valid at some early times. Alternatively, at late times, one can require the coefficient \( \lambda \) to be large such that \( \lambda H^2 \gtrsim 1 \) is valid.

In general, we claim that the breakdown of the causal structure can be avoided if one constrains the correction terms, such that they are subdominant. For example, by

\[3\] The case of \( \lambda < 0 \) is identical to the \( \lambda > 0 \) case.
and the quadratic Weyl tensor is

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R_{\mu\nu}R_{\mu\nu} - \frac{1}{6}R^2. \quad (32)$$

The effective metric is easily obtained from equations (19), (20) by replacing $$\lambda_1 = \lambda, \lambda_2 = -2\lambda, \lambda_3 = -\frac{1}{2}\lambda$$, thus

$$G^{\mu\nu} = g^{\mu\nu} \left( 1 - \lambda H^2 \left( \frac{H}{H^2} + 24 \right) \right). \quad (33)$$

$$G^{\alpha\epsilon} = \left( 1 - \lambda H^2 \left( 10 \frac{H}{H^2} + 24 \right) \right). \quad (34)$$

The, for $$\lambda H^2 > \frac{1}{12}$$ We conclude that the signature is non-Lorentzian when $$-4 < H/H^2 < -2.4$$. The interpretation of this result in terms of scale factor in the form $$a(t) \sim t^\alpha$$ with $$\alpha > 0$$ shows that when $$\frac{1}{4} < \alpha < \frac{5}{12}$$ the EOM for perturbations are non-hyperbolic. Which again, rules out partially the range describing decelerated expansion solutions.

### 2.3. Effective metric in Schwarzschild spacetime

Here, we apply the effective metric method for the static spherically symmetric Schwarzschild BH solution [9, 10]. Particularly interesting is the behavior of the EOM for perturbation in large curvature background, and especially near the horizon.

The static spherically symmetric Schwarzschild spacetime is a solution of the modified action equation (1). This observation is immediately seen by exploiting the properties of the GB invariant equation (32). Indicating that the Lagrangian equation (1) can be decomposed into two independent contributions. In particularly by $$\alpha R^2 + \beta R_{\mu\nu}R_{\mu\nu}$$ [10]. As a consequence, any solution that satisfies the vacuum Einstein equation will also satisfy the modified EOM.

To proceed, the static spherically symmetric Schwarzschild BH solutions are

$$ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2, \quad (35)$$

where $$d\Omega^2$$ is the 2D unit sphere. The non-vanishing components of the Riemann components are the following,

$$R_{tt}^{\mu\nu} = -\frac{f''(r)}{2}, \quad R_{tt}^{ij} = -\frac{f(r)}{r^2} g_{ij}^{\mu\nu}, \quad R_{ij}^{\alpha\beta} = -\frac{f'(r)}{2r} g_{ij}^{\alpha\beta}. \quad (36)$$

where $$f(r) = (1 - 2M/r)$$, the indices $$i, j, k, l = 1, 2$$ denote angular coordinates and $$\alpha = i, r$$. We are interested in the tensor modes about the maximally symmetric subspace. Recalling that such choice is always possible when expanding about the Euclidean line element equation (35). They are given by

$$h_{\alpha\beta}^i = 0, \quad h^{\alpha i} = 0, \quad h_i^i = 0, \quad \nabla_i h^i = 0. \quad (37)$$
The calculation of the effective metric is carried out in a similar way to the one performed in the previous FRW examples. The main difference compared to the FRW case is that the Ricci tensor and the scalar curvature are vanishing, enabling us to perform the calculation with more ease. The effective metric component are given by

\[
G_{\alpha\alpha} = g_{\alpha\alpha} \left( 1 - \lambda \left( \frac{f(r)}{r} - 6 \frac{f(r)}{r^2} \right) \right),
\]

(38)

\[
G_{\beta\kappa} = g_{\beta\kappa} \left( 1 - 2 \phi \frac{f(r)}{r^2} \right)
\]

(39)

As in the FRW case, symmetry results in an equal effective metric for all polarizations. First, it is clear that the multiplying factors in parenthesis are different, implying the possibility for a deviation from the standard Lorentzian pattern. Second, since both \( f(r), f'(r) > 0 \), the issue is whether the factor multiplying \( \lambda \) in equation (38) can change it sign. We derive an explicit conditions for the non-Lorentzian signature by substituting the metric function \( f(r) \) in the above equations. The corresponding effective metric

\[
G_{\alpha\alpha} = g_{\alpha\alpha} \left( 1 - \frac{\lambda M}{r^2} (6\alpha - 16) \right).
\]

(40)

\[
G_{\beta\kappa} = g_{\beta\kappa} \left( 1 - \frac{\lambda M}{r^2} (2\alpha - 1) \right).
\]

(41)

where \( \alpha = \frac{r}{2M} \) and \( \lambda > 0 \). The unitless coefficient \( \frac{\lambda M}{r^2} \) was extracted, since when it satisfies \( \frac{\lambda M}{r^2} \gtrsim 1 \) the higher derivative can induce significant modifications to the EH action. Then, noting that \( \mathcal{R} \sim \frac{M^2}{r^7} \), the above relations coincides with \( \lambda \mathcal{R} \gtrsim 1 \).

To proceed, it is clear that for any value of \( \lambda \) and at large enough distances the effective metric reduces to the background Schwarzschild metric, so it is Lorentzian in this region of large \( r \). Therefore, the effective metric can be non-Lorentzian only for a smaller values of \( r \), where the corrections due to the higher derivative terms are large. Here we find that the initial value problem is ill-posed when \( 2 < \alpha < 2.75 \). The conclusion is that there exist a physical region exterior to the BH horizon \( 2M < r < 2.75M \) where perturbation are not propagate in a causal way.

3. Summary and conclusions

In this paper we examine whether modifications to the EH action in the form of quadratic curvature terms can provide an alternative theory of gravity. We analyzed the causal structure of the EOM for perturbation about the FRW and Schwarzschild spacetimes. We found that when the higher derivative terms are comparable or larger than the Einstein term, such that \( \lambda \mathcal{R} \gtrsim 1 \) the dynamics is ill-defined. Therefore, we conclude that these theories cannot constitute a modified theory of gravity. On the other hand, when these terms are small and therefore act as a perturbative correction of the Einstein term, the EOM are hyperbolic because they are hyperbolic for GR.

First, we review the effective metric method. Then, following the formalism [23], we derive the effective metric for the gravitational case when quadratic terms are included in the EH action. We emphasize that our interest is mainly subjected to EOM that are in the form of Einstein equations—second order in the background field.

Then, we performed explicit calculations of the effective metric about the cosmological FRW spacetime and the spherically symmetric Schwarzschild solution. We considered different combination of extensions in the EH action section 2.1. We found that in general, the Lorentzian pattern of the effective metric in FRW spacetime equations (19), (20) is governed by the magnitude of \( \lambda \). Then, we analyzed the results by assuming \( \lambda H^2 > 1 \), then the non-Lorentzian conditions were expressed in terms of the ratio \( H/H^2 \). The results were also interpreted in terms of the specific values of \( \alpha \) parameters. We find that due to the presence of the higher order terms the EOM for perturbations are no longer hyperbolic, which indicated that causality is violated for a large range of \( \alpha \) parameters that describes a universe undergoing an inflation or deceleration.

We stress again that the hyperbolicity of the EOM can be ensured when the cutoff scale of the theory is set by the correction terms, thus making them subdominant. For example, if one impose a cutoff on the theory such that \( \lambda H^2 < 1 \), or alternatively if one constraints \( H \) (allowing \( \lambda H^2 \) to be large), then the correction terms become subdominant and can be considered as a perturbative extension of GR. Thus, making the EOM for perturbations hyperbolic.

Finally, we demonstrated the method for the Schwarzschild solution. The analysis was carried out along the same lines of the FRW spacetime. We found that when the correction term are large \( \frac{\lambda M}{r^2} \gtrsim 1 \), then, outside the horizon, the effective metric becomes non-Lorentzian for \( 2M < r < 2.75M \). The conclusion, in similar to the FRW case, indicating the possibility of causality violation when the higher derivative terms are treated in the non-perturbative regime.

To finish, the conclusion is that the breakdown of predictability in higher derivative theories indicates that these theories can only be considered as a perturbative extension of the EH action, and not as a true alternative to GR.

Acknowledgments

I would like to Ramy Brustein for the useful comments and the valuable discussions, Yoav Zigdon and Dror Sherf for the comments. The research was supported by the Negev Hi-Tech scholarship, and by the Israel Science Foundation Grant No. 1294/16.

ORCID iDs

Yotam Sherf © https://orcid.org/0000-0003-4687-5454

---

4 The case of \( \lambda < 0 \) is identical to the \( \lambda > 0 \) case.

5 In general, we consider the largest contribution from \( \mathcal{R} \), originates in \( R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{\mathcal{R}}{r^4} \).
References

[1] Clifton T, Ferreira P G, Padilla A and Skordis C 2012 Modified gravity and cosmology Phys. Rep. 513 1
[2] Salvio A 2018 Quadratic gravity Front. Phys. 6 77
[3] Salvio A 2017 Inflationary perturbations in no-scale theories Eur. Phys. J. C 77 267
[4] Candelas P, Horowitz G T, Strominger A and Witten E 1985 Nucl. Phys. B 258 46
[5] Myers R C 1987 Higher derivative gravity, surface terms and string theory Phys. Rev. D 36 392
[6] Green M B, Schwarz J H and Witten E 1982 Superstring Theory, Vol. 1: Introduction (Cambridge: Cambridge University Press) p 469 (Cambridge Monographs On Mathematical Physics)
[7] Polchinski J 1998 String Theory Vol. 1: An introduction to the Bosonic String (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press) (https://doi.org/10.1017/CBO9780511816079)
[8] Polchinski J 1998 String Theory Vol. 2: Superstring Theory and Beyond (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press) (https://doi.org/10.1017/CBO9780511618123)
[9] Lu H, Perkins A, Pope C N and Stelle K S 2015 Black holes in higher-derivative gravity Phys. Rev. Lett. 114 171601
[10] Stelle K S 1978 Classical gravity with higher derivatives Gen. Relativ. Gravit. 9 353
[11] Stelle K S 1977 Renormalization of higher derivative quantum gravity Phys. Rev. D 16 953
[12] Birrell N and Davies P 1982 Quantum Fields in Curved Space (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press) (https://doi.org/10.1017/CBO9780511622632)
[13] Mazzitelli F D 1992 Higher derivatives and renormalization in quantum cosmology Phys. Rev. D 45 2814
[14] Kleidis K, Kairouz-Pires A and Papadopoulos D B 2002 Phys. Lett. B 546 112
[15] Carroll S M, De Felice A, Duvvuri V, Easson D A, Trodden M and Turner M S 2005 The Cosmology of generalized modified gravity models Phys. Rev. D 71 063513
[16] Nojiri S and Odintsov S D 2011 Unified cosmic history in modified gravity: from f(R) theory to Lorentz non-invariant models Phys. Rep. 505 59
[17] Starobinsky A A 1980 A new type of isotropic cosmological models without singularity Phys. Lett. B 91 99
[18] De Felice A and Tsujikawa S 2010 f(R) theories Living Rev. Relativ. 13 3
[19] Alvarez-Gaume L, Keahas A, Kounnas C, Läst D and Riotto A 2016 Aspects of quadratic gravity Fortsch. Phys. 64 176
[20] Nojiri S and Odintsov S D 2006 eConf C 0602061 06
[21] Nojiri S and Odintsov S D 2007 Int. J. Geom. Methods Mod. Phys. 4 115
[22] Nojiri S, Odintsov S D and Ogushi S 2002 Phys. Rev. D 65 023521
[23] Camanho X O, Edelstein J D, Maldacena J and Zhiboedov A 2016 Causality constraints on corrections to the graviton three-point coupling J. High Energy Phys. JHEP02 (2016)020
[24] Brustein R and Sherf Y 2018 Causality violations in lovelock theories Phys. Rev. D 97 084019
[25] Benakli K, chman S, Darmř L and Oz Y 2016 Phys. Rev. D 94 084026
[26] Izumi K 2014 Causal structures in Gauss–Bonnet gravity Phys. Rev. D 90 044037
[27] Papallo G and Reall H S 2017 On the local well-posedness of Lovelock and Horndeski theories Phys. Rev. D 96 044019
[28] Aharonov Y, Komar A and Susskind L 1969 Superluminal behavior, causality, and instability Phys. Rev. 182 1400
[29] Armendariz-Picon C and Lim E A 2005 Haloos of k-essence J. Cosmol. Astropart. Phys. JCAP08(2005)007
[30] Bruneton J P 2007 On causality and superluminal behavior in classical field theories: applications to k-essence theories and MOND-like theories of gravity Phys. Rev. D 75 085013
[31] Chen T J, Fasiello M, Lim E A and Tolley A J 2013 Higher derivative theories with constraints: exercising ostrogradski’s ghost J. Cosmol. Astropart. Phys. JCAP02(2013)042
[32] Woodard R P 2015 Ostrogradsky’s theorem on Hamiltonian instability Scholarpedia 10 32243
[33] Brustein R and Medved A J M 2012 Graviton n-point functions for UV-complete theories in Anti-de Sitter space Phys. Rev. D 85 084028
[34] Higuchi A 1987 Symmetric tensor spherical harmonics on the N sphere and their application to the de sitter group SO(N,1) J. Math. Phys. 28 1553
[35] Higuchi A 2002 Symmetric tensor spherical harmonics on the N sphere and their application to the de sitter group SO(N,1) J. Math. Phys. 43 6385 (erratum)
[36] Donoghue J F and Menezes G 2018 Gauge assisted quadratic gravity: a framework for UV complete quantum gravity Phys. Rev. D 97 126005