Models of Finsler Geometry on Lie algebroids

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Abstract

Horizontal endomorphisms, almost complex structures, vertical, horizontal and complete lifts on prolongation of a Lie algebroid are considered. Then using exact sequences, semisprays are constructed. Moreover, important geometrical objects such as classical distinguished connections, torsions and partial curvatures are studied on prolongation of Lie algebroids. Considering pullback bundle, covariant derivatives are scrutinized based on anchor map. Several Finsler geometry models on Lie algebroid structures, will organized via recent arguments. Finally, it will be overhauled some special Finsler Lie algebroid spaces.

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1 Introduction

The notion of Lie algebroids was first introduced by Pradines [26]. Researching on this field, is continuous by mathematicians with various purposes up to now. Lie algebroids are studied pure or in relation with other subjects [1, 3, 5, 6, 10, 13, 19, 32, 34]. Precisely, a Lie algebroid is a vector bundle with the property that its sections involve a real Lie algebra. Each section is anchored on a vector field, by means of a linear bundle map named as anchor map, which is further supposed to induce a Lie algebra homomorphism. Specially when the base manifold \( \mathcal{M} \) is a point, a Lie algebroid reduces to a Lie algebra. The most simple examples of Lie algebroids are the zero bundle over \( \mathcal{M} \) which is denoted by \( \mathcal{M} \) and tangent bundle over \( \mathcal{M} \) with identity as anchor map which is denoted by \( T \mathcal{M} \). Then the tangent bundle is a special case of Lie algebroid structure. Therefore a Lie algebroid is a generalization of a Lie algebra and vector bundle.

The Lie algebroid is a good extension of tangent bundle, since the homomorphism property of the anchor map grants the basic notions of tangent bundle to the vector bundle (for example the torsion notion that has not a good meaning in vector bundles). However, some theorems on tangent bundle do not work here as we shall see. Indeed, any generalization need to using anchor map to reconstruct the most useful objects to compare geometric structures at different points of a manifold on the vector bundle, namely; covariant derivation and connection concepts. In [14], one can see how they have generalized on Lie algebroids by a worthwhile work as the first step. Another acute extension is curvature. In this way, a good conceptualization is [9]. In [11], Riemannian Lie algebroids have introduced and basic facts like Levi-Civita connection, Riemannian metric and curvature, sectional curvature, geodesics and integrability have studied.

Recently, Lie algebroids are important issues in physics and mechanics since the extension of Lagrangian and Hamiltonian systems to their entity [4, 17, 18, 20, 37] and catching the poisson structure [24]. They are wrested with nonsmooth optimization [25] and studied on Banach vector bundles [2]. They have such a flexibility that holonomy of orbit foliation carried on them [14]. Thus Lie algebroids are strong assorted structures to assemble the Physics and mechanics notions on them. For a good details about penetration of Lie algebroids, see [35].

The aim of this paper is rebuild the Finsler geometry concepts on Lie algebroid structures. For instance, this matter is discussed in [36, 25] of course. Finsler geometry is a generalization of Riemannian geometry such that interfering of direction and position duplicates the degree of freedom in view of configuration. Variety of tensors in Finsler geometry is more than Riemannian case. One can study with more details about Finsler geometry, say in [7, 8, 33].

In Finsler geometry there are two approaches. The first is global make up, and the second is localization. Indeed, non of them have any advantage to the other until one would like use them as gadgets to derive the conclusion or to receive the target. For example, when we want to see the anchor role precisely, it prefered the locally approach shall be applied and when we want to have a boxed and index-free formula to categorize the results, we choose the global viewpoint. Accordingly, we tried to designate the approaches in the sense of the case.

The paper is organized as follows. In Section 2, we recall differential, contrac-
tion and Lie differential operators on Lie algebroids and we study the relation between these operators. We also mention the generalized Frölicher-Nijenhuis bracket on Lie algebroids. Then vertical and complete lifts on a Lie algebroid is considered and some important properties of these objects is studied. In Section 3, building on the notion of prolongation $\mathcal{L}^E \to E$ of the Lie algebroid $E \to M$, we present vertical and complete lifts of a section of $E$ to $\mathcal{L}^E$. But the major concept of this section is to construct the vertical endomorphism on a special exact sequence. Then Liouville section and semispray on $\mathcal{L}^E$ will be introduced. Moreover, some interesting results on them will be gathered. The aim of Section 4 is to make up pre-curvature concepts like torsion and tension by decomposing $\mathcal{L}^E$ using horizontal endomorphisms. The almost complex structure on $\mathcal{L}^E$ and Berwald endomorphism will be introduced in this section also. Finally, it will be shown that how sections on $\mathcal{L}^E$ lift into the horizontal space of $\mathcal{L}^E$ using the horizontal endomorphism. In Section 5, distinguished connections on $\mathcal{L}^E$ are introduced and torsion and curvature tensor fields of these connections are considered. In particular, we introduce Berwald-type and Yano-type connections on $\mathcal{L}^E$ as two important classes of distinguished connections. In Section 6, we construct $\rho$-covariant derivatives in $\pi^*\pi$ as generalization of covariant derivative in $\pi^*\pi$ to $\mathcal{L}^E$. Moreover, Berwald and Yano derivatives as two important classes of $\rho$-covariant derivatives in $\pi^*\pi$ are introduced in this section. Section 7 is busy with Finsler algebroids and related materials. Important endomorphisms like Conservative and Barthel, Cartan tensor and some distinguished connections like Berwald, Cartan, Chern-Rund and Hashiguchi are studied by Szilasi and his collaborators from a special point view based on pullback bundle \cite{27, 28, 29, 30, 31}. In this section we construct them on Finsler algebroids and obtain some results on these concepts. In section 8, $h$-basic distinguished connections are introduced on Finsler algebroids. Specially, Ichijyō connection that is a special $h$-basic distinguished connection is more studied. Generalized Berwald Lie algebroids are presented next. The section will ended by Wagner-Ichijyō connection that is a special case of Ichijyō’s one.

2 Basic concepts on Lie algebroids

Let $E$ be a vector bundle of rank $n$ over a manifold $M$ of dimension $m$ and $\pi : E \to M$ be the vector bundle projection. Denote by $\Gamma(E)$ the $C^\infty(M)$-module of sections of $\pi : E \to M$. A Lie algebroid structure $([.,.]_E, \rho)$ on $E$ is a Lie bracket $[.,.]_E$ on the space $\Gamma(E)$ and a bundle map $\rho : E \to TM$, called the anchor map, such that if we also denote by $\rho : \Gamma(E) \to \chi(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map then

$$[X, fY]_E = f[X, Y]_E + \rho(X)(f)Y, \quad \forall X, Y \in \Gamma(E), \forall f \in C^\infty(M).$$

Moreover, we have the relations

$$[\rho(X), \rho(Y)] = \rho([X, Y]_E), \quad (1)$$

and

$$[X, [Y, Z]_E]_E + [Y, [Z, X]_E]_E + [Z, [X, Y]_E]_E = 0. \quad (2)$$

Then triple $(E, [.,.]_E, \rho)$ is called a Lie algebroid over $M$. 

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Trivial examples of Lie algebroids are real Lie algebras of finite dimension, the tangent bundle $TM$ of an arbitrary manifold $M$ and an integrable distribution of $TM$.

If $(E, [\cdot, \cdot]_E, \rho)$ is a Lie algebroid over $M$, then the anchor map $\rho : \Gamma(E) \rightarrow \chi(M)$ is a homomorphism between the Lie algebras $(\Gamma(E), [\cdot, \cdot]_E)$ and $(\chi(M), [\cdot, \cdot])$.

On Lie algebroids $(E, [\cdot, \cdot]_E, \rho)$, we define the differential of $E$, $d^E : \Gamma(\wedge^k E^*) \rightarrow \Gamma(\wedge^{k+1} E^*)$, as follows

$$d^E \mu(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \rho(X_i)(\mu(X_0, \ldots, \hat{X}_i, \ldots, X_k))$$

$$+ \sum_{i<j} (-1)^{i+j} \mu([X_i, X_j]_E, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k),$$

for $\mu \in \Gamma(\wedge^k E^*)$ and $X_0, \ldots, X_k \in \Gamma(E)$. In particular, if $f \in \Gamma(\wedge^0 E^*) = C^\infty(M)$ we have $d^E f(X) = \rho(X) f$. Using the above equation it follows that $(d^E)^2 = 0$.

If we take local coordinates $(x^i)$ on $M$ and a local basis $\{e_\alpha\}$ of sections of $E$, then we have the corresponding local coordinates $(x^i, \gamma^\alpha)$ on $E$, where $x^i = x^i \circ \pi$ and $\gamma^\alpha(u)$ is the $\alpha$-th coordinate of $u \in E$ in the given basis. Such coordinates determine local functions $\rho_\alpha^i$, $L^\alpha_{\beta\gamma}$ on $M$ which contain the local information of the Lie algebroid structure, and accordingly they are called the structure functions of the Lie algebroid. They are given by

$$\rho(e_\alpha) = \rho_\alpha^i \frac{\partial}{\partial x^i}, \quad [e_\alpha, e_\beta]_E = L^\gamma_{\alpha\beta} e_\gamma.$$  

Using (1) and (2), these functions should satisfy the following relations

$$(i) \quad \rho_\alpha^i \frac{\partial \rho_\beta^j}{\partial x^i} - \rho_\beta^j \frac{\partial \rho_\alpha^i}{\partial x^j} = \rho_\gamma^i L^\gamma_{\alpha\beta}, \quad (ii) \quad \sum_{(\alpha, \beta, \gamma)} [\rho_\alpha^i \frac{\partial L^\gamma_{\beta\gamma}}{\partial x^i} + L^\gamma_{\alpha\mu} L^\mu_{\beta\gamma}] = 0, \quad (3)$$

which are usually called the structure equations. We have also,

$$d^Ef = \frac{\partial f}{\partial x^i} \rho_\alpha^i e_\alpha, \quad \forall f \in C^\infty(M), \quad (4)$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$. On the other hand, if $\omega \in \Gamma(E^*)$ and $\omega = \omega_\gamma e^\gamma$ it follows

$$d^E \omega = (\frac{\partial \omega_\gamma}{\partial x^i} \rho_\gamma^i - \frac{1}{2} \omega_\gamma L^\gamma_{\alpha\beta} \omega^\beta \wedge e_\alpha).$$

In particular,

$$d^E x^i = \rho_\alpha^i e_\alpha, \quad d^E \omega^\alpha = -\frac{1}{2} L^\gamma_{\alpha\beta} e^\beta \wedge e^\gamma.$$

A section $\omega$ of $E^*$ also defines a function $\hat{\omega}$ on $E$ by means of

$$\hat{\omega}(u) = \left< \omega_m, u \right>, \quad \forall u \in E_m.$$  

If $\omega = \omega_\alpha e^\alpha$, then the linear function $\hat{\omega}$ is

$$\hat{\omega}(x, y) = \omega_\alpha y^\alpha.$$  

In particular, using (1) we have

$$\overline{d^E f} = \frac{\partial f}{\partial x^i} \rho_\alpha^i y^\alpha.$$  

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2.1 Generalized Frölicher-Nijenhuis bracket

For $X \in \Gamma(\wedge^k E)$, the contraction $i_X : \Gamma(\wedge^p E^*) \to \Gamma(\wedge^{p-k} E^*)$ is defined in standard way and the Lie differential operator $\mathcal{L}_X^E : \Gamma(\wedge^p E^*) \to \Gamma(\wedge^{p-k+1} E^*)$ is defined by

$$\mathcal{L}_X^E = i_X \circ d^E - (-1)^k d^E \circ i_X.$$ 

Note that if $E = TM$ and $X \in \Gamma(E) = \chi(M)$, then $d^TM$ and $\mathcal{L}_X^TM$ are the usual differential and the usual Lie derivative with respect to $X$, respectively.

Let $K \in \Gamma(\wedge^k E^* \otimes E)$, then the contraction

$$i_K : \Gamma(\wedge^n E^*) \to \Gamma(\wedge^{n+k-1} E^*),$$

is defined in the natural way. In particular, for simple tensor $K = \mu \otimes X$, where $\mu \in \Gamma(\wedge^k E^*)$, $X \in \Gamma(E)$, we set

$$i_K \nu = \mu \wedge i_X \nu.$$ 

The corresponding Lie differential is defined by the formula

$$\mathcal{L}_K^E = i_K \circ d^E + (-1)^k d^E \circ i_K,$$

and, in particular,

$$\mathcal{L}_{\mu \otimes X}^E = \mu \wedge \mathcal{L}_X^E + (-1)^k d^E \mu \wedge i_X.$$

The contraction $i_K$ can be extended to an operator

$$i_K : \Gamma(\wedge^n E^* \otimes E) \to \Gamma(\wedge^{n+k-1} E^* \otimes E),$$

by the formula $i_K(\mu \otimes X) = i_K(\mu) \otimes X$. The following theorem contains a list of well-known formulas [15]:

**Theorem 2.1.** Let $\mu \in \Gamma(\wedge^k E^*)$, $\nu \in \Gamma(E^*)$ and $X, Y \in \Gamma(E)$. Then we have

1. $d^E \circ d^E = 0$,
2. $d^E(\mu \wedge \nu) = d^E \mu \wedge \nu + (-1)^k \mu \wedge d^E \nu$,
3. $i_X(\mu \wedge \nu) = i_X \mu \wedge \nu + (-1)^k \mu \wedge i_X \nu$,
4. $\mathcal{L}_X^E(\mu \wedge \nu) = \mathcal{L}_X^E \mu \wedge \nu + (-1)^k \mu \wedge \mathcal{L}_X^E \nu$,
5. $\mathcal{L}_X^E \circ \mathcal{L}_Y^E - \mathcal{L}_Y^E \circ \mathcal{L}_X^E = [X,Y]_E$, 
6. $\mathcal{L}_X^E \circ \mathcal{L}_Y^E - \mathcal{L}_Y^E \circ \mathcal{L}_X^E = [X,Y]_E$.

The generalized Frölicher-Nijenhuis bracket is defined for simple tensors $\mu \otimes X \in \Gamma(\wedge^k E^* \otimes E)$ and $\nu \otimes Y \in \Gamma(\wedge^l E^* \otimes E)$ by the formula

$$[\mu \otimes X, \nu \otimes Y]^{F-N} = (\mathcal{L}_{\mu \otimes X} \nu) \otimes Y - (-1)^{kl}(\mathcal{L}_{\nu \otimes Y} \mu) \otimes X + \mu \wedge \nu \otimes [X,Y]_E. \quad (5)$$

Moreover, for $K \in \Gamma(\wedge^k E^* \otimes E)$, $L \in \Gamma(\wedge^l E^* \otimes E)$ and $N \in \Gamma(\wedge^\alpha E^* \otimes E)$ we have

$$\mathcal{L}_K^L \circ \mathcal{L}_N^L = \mathcal{L}_K^L \circ \mathcal{L}_N^L - (-1)^{kl} \mathcal{L}_L^K \circ \mathcal{L}_N^L, \quad (6)$$

$$(-1)^{kn}[K, [L, N]]^{F-N} + (-1)^{jk}[L, [N, K]]^{F-N} + (-1)^{nl}[N, [K, L]]^{F-N} = 0. \quad (7)$$
From (5) and (6) we get

\[ [K, Y]_{E^{-N}}(X) = [K(X), Y]_E - K[X, Y]_E, \]
\[ [K, L]_{E^{-N}}(X, Y) = [K(X), L(Y)]_E + [L(X), K(Y)]_E + (K \circ L + L \circ K)[X, Y]_E \]
\[ = K[X, L(Y)]_E - K[L(X), Y]_E - L[K(X), Y]_E, \]

where \( K \in \Gamma(E^* \otimes E) \), \( L \in \Gamma(E^* \otimes E) \) and \( X, Y \in \Gamma(E) \). (see [15]).

### 2.2 Vertical and complete lifts on Lie algebroids

For a function \( f \) on \( M \) one defines its vertical lift \( f^\gamma \) on \( E \) by \( f^\gamma(u) = f(\pi(u)) \)

for \( u \in E \). Now, let \( X \) be a section of \( E \). Then, we can consider the vertical lift of \( X \) as the vector field on \( E \) given by \( X^\gamma(u) = X(\pi(u))_u, u \in E \), where \( \gamma : E_{\pi(u)} \to T_u(E_{\pi(u)}) \) is the canonical isomorphism between the vector spaces \( E_{\pi(u)} \) and \( T_u(E_{\pi(u)}) \).

**Lemma 2.2.** Let \( \{e_\alpha\} \) be a basis of sections of \( E \). Then we have

\[ e_\alpha^\gamma = \frac{\partial}{\partial y^\alpha}. \]

**Proof.** We have

\[ dy^\alpha (e_\beta^\gamma(u)) = dy^\alpha \left( \frac{d}{dt} \bigg|_{t=0}(u + te_\beta) \right) = \frac{d}{dt} \bigg|_{t=0}(y^\alpha(u + te_\beta)) \]
\[ = \frac{d}{dt} \bigg|_{t=0}(y^\alpha(u) + t\delta^\alpha_\beta) = \delta^\alpha_\beta. \]

From the above lemma we result that if \( X = X^\alpha e_\alpha \in \Gamma(E) \), then the vertical lift \( X^\gamma \) has the locally expression

\[ X^\gamma = (X^\alpha \circ \pi) \frac{\partial}{\partial y^\alpha}. \]

Using the locally expression of \( X^\gamma \) we can deduce

**Lemma 2.3.** If \( X, Y \) are sections of \( E \) and \( f \in C^\infty(M) \), then

\( (X + Y)^\gamma = X^\gamma + Y^\gamma \), \( (fX)^\gamma = f^\gamma X^\gamma \), \( X^\gamma f^\gamma = 0 \).

The complete lift of a smooth function \( f \in C^\infty(M) \) into \( C^\infty(E) \) is the smooth function

\[ f^c : E \longrightarrow \mathbb{R}, \quad f^c(u) = d^E f(u) = \rho(u)f. \]

In the local basis we have

\[ f^c(u) = f^c(u^\alpha e_\alpha) = \rho(u^\alpha e_\alpha)(f) = u^\alpha \rho(e_\alpha)(f) = u^\alpha \rho^\alpha_i \frac{\partial f}{\partial x^i} \]
\[ = (y^\alpha((\rho^\alpha_i \frac{\partial f}{\partial x^i}) \circ \pi))(u), \]

i.e.,

\[ f^c|_{\pi^{-1}(U)} = y^\alpha((\rho^\alpha_i \frac{\partial f}{\partial x^i}) \circ \pi). \]

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Lemma 2.4. If \( X \) is a section on \( E \) and \( f, g \in C^\infty(M) \), then

(i) \((f + g)^c = f^c + g^c\),  
(ii) \((fg)^c = f^c g^c + f^c g^c\),  
(iii) \(X^c f^c \circ (\rho(X)f)^c\).

Proof. The proof of (i) is obvious. Thus we only prove (ii) and (iii). Using the definition of \( f^c \) we get

\[
(fg)^c(u) = \rho(u)(fg) = (\rho(u)f)(g \circ \pi)(u) + (f \circ \pi)(u)(\rho(u)g) = f^c(u)g^c(u) + f^c(u)g^c(u).
\]

So we have (ii). Using the locally expressions, we obtain

\[
(X^c f^c)(u) = [(X^c \circ \pi) \frac{\partial}{\partial y^\alpha}((\rho^\beta \frac{\partial f}{\partial x^i}) \circ \pi)](u) = [(X^c \circ \pi) \frac{\partial}{\partial y^\alpha}]((\rho^\beta \frac{\partial f}{\partial x^i} - \nabla_\gamma L^\alpha_{\gamma \beta}) \circ \pi)(u)
\]

\[
= ((\rho(X)f) \circ \pi)(u) = (\rho(X)f)^c(u).
\]

\[
\square
\]

Let \( X \) be a section on \( E \). Then there exist a unique vector field \( X^c \) on \( E \), the complete lift of \( X \), satisfying the following conditions:

i) \( X^c \) is \( \pi \)-projectable on \( \rho(X) \),

ii) \( X^c(\hat{X}) = \hat{X} \).

where \( \alpha \in \Gamma(E^*) \). It is known that \( X^c \) has the following coordinate expression,[15], [16]:

\[
X^c = \{(X^c \circ \pi) \frac{\partial}{\partial y^\alpha} + y^\alpha \{(\rho^\beta \frac{\partial X^c}{\partial x^i} - \nabla_\gamma L^\alpha_{\gamma \beta}) \circ \pi\} \frac{\partial}{\partial y^\alpha}\}.
\]

(11)

Lemma 2.5. Let \( X \) be a section of \( E \). Then

\[
X^c f^c = (\rho(X)f)^c, \quad \forall f \in C^\infty(M).
\]

Proof. Using (10) we get

\[
(\rho(X)f)^c = (X^c \circ \pi) \frac{\partial}{\partial x^i} \circ \pi\} = y^\alpha \{\{(\rho^\beta \frac{\partial X^c}{\partial x^i} - \nabla_\gamma L^\alpha_{\gamma \beta}) \circ \pi\} \frac{\partial}{\partial y^\alpha}\} \frac{\partial}{\partial x^i} (X^c \circ \pi) \frac{\partial}{\partial y^\alpha}\} \frac{\partial}{\partial x^i} \circ \pi\}
\]

(12)

Again (10) and (11) give us

\[
X^c f^c = \{(X^c \circ \pi) \frac{\partial}{\partial x^i} \circ \pi\} \frac{\partial}{\partial y^\alpha} \{y^\alpha \{(\rho^\beta \frac{\partial X^c}{\partial x^i} - \nabla_\gamma L^\alpha_{\gamma \beta}) \circ \pi\} \frac{\partial}{\partial y^\alpha}\} \frac{\partial}{\partial x^i} \circ \pi\}
\]

It is easy to see that \( \pi_*(\frac{\partial}{\partial y^\alpha}) = \frac{\partial}{\partial x^i} \) which gives us \( \frac{\partial}{\partial x^i} (f \circ \pi) = \frac{\partial}{\partial x^i} \circ \pi = f \circ \pi \) for all \( f \in C^\infty(M) \). Thus from the above equation one can deduce the following

\[
X^c f^c = y^\alpha \{(X^c \circ \pi) \frac{\partial}{\partial x^i} \circ \pi\} \frac{\partial}{\partial y^\alpha} \{y^\alpha \{(\rho^\beta \frac{\partial X^c}{\partial x^i} - \nabla_\gamma L^\alpha_{\gamma \beta}) \circ \pi\} \frac{\partial}{\partial y^\alpha}\} \frac{\partial}{\partial x^i} \circ \pi\}
\]

Using (i) of (3), the above relation and (12) yields

\[
X^c f^c = y^\alpha \{(\rho^\beta \frac{\partial}{\partial x^i} (X^c \circ \pi) \frac{\partial}{\partial y^\alpha}\} \circ \pi\} = (\rho(X)f)^c.
\]

\[
\square
\]
Corollary 2.6. Let $X$ be a section of $E$. Then
\[ X^c f^\gamma = (\rho(X)f)^\gamma, \quad \forall f \in C^\infty(M). \]

**Proof.** Using the above lemma, we obtain
\[ \frac{1}{2} X^c (f^2)^\gamma = X^c (f^c f^\gamma) = (X^c f^c) f^\gamma + f^c (X^c f^\gamma) = (\rho(X)f)^c f^\gamma + f^c (X^c f^\gamma). \]
In other hand, we deduce
\[ \frac{1}{2} X^c (f^2)^c = \frac{1}{2} (\rho(X)f)^c = (f \rho(X)f)^c = f^c (\rho(X)f)^\gamma + f^\gamma (\rho(X)f)^c. \]
The above equations give us $X^c f^\gamma = (\rho(X)f)^\gamma$. \( \square \)

Using the locally expressions of vertical and complete lifts we have

**Lemma 2.7.** If $X$ and $Y$ are sections of $E$, then
\[ [X^c, Y^c] = [X, Y]^c_E, \quad [X^c, Y^\gamma] = [X, Y]^\gamma_E, \quad [X^\gamma, Y^\gamma] = 0. \]

### 3 The Prolongation of a Lie algebroid

In this section we will recall the notion of the prolongation of a Lie algebroid and we will consider a Lie algebroid structure on it. We also study the vertical and complete lifts on the prolongation of a Lie algebroid.

Let $\mathcal{L}^\pi E$ be the subset of $E \times TE$ defined by $\mathcal{L}^\pi E = \{(u, z) \in E \times TE | \rho(u) = \pi_*(z)\}$ and denote by $\pi_\mathcal{L} : \mathcal{L}^\pi E \to E$ the mapping given by $\pi_\mathcal{L}(u, z) = \pi_E(z)$, where $\pi_E : TE \to E$ is the natural projection. Then $(\mathcal{L}^\pi E, \pi_\mathcal{L}, E)$ is a vector bundle over $E$ of rank $2n$. Indeed, the total space of the prolongation is the total space of the pull-back of $\pi_* : TE \to TM$ by the anchor map $\rho$.

We introduce the vertical subbundle
\[ v\mathcal{L}^\pi E = \ker \tau_\mathcal{L} = \{(u, z) \in \mathcal{L}^\pi E | \pi_\mathcal{L}(u, z) = 0\}, \]
where $\tau_\mathcal{L} : \mathcal{L}^\pi E \to E$ is the projection onto the first factor, i.e., $\tau_\mathcal{L}(u, z) = u$. Therefore an element of $\mathcal{L}^\pi E$ is of the form $(0, z) \in E \times TE$ such that $\pi_*(z) = 0$ which is called vertical. Since $\pi_*(z) = 0$ and $\ker \pi_* = vE = (\pi_* : TE \to TM)$, then we deduce that if $(0, z)$ is vertical then $z$ is a vertical vector on $E$.

For local basis $\{e_\alpha\}$ of sections of $E$ and coordinates $(x^i, y^\alpha)$ on $E$, we have local coordinates $(x^i, y^\alpha, k^\alpha, z^\alpha)$ on $\mathcal{L}^\pi E$ given as follows. If $(u, z)$ is an element of $\mathcal{L}^\pi E$, then by using $\rho(u) = \pi_*(z)$, $z$ has the form
\[ z = (\rho^\alpha u^\alpha \circ \pi) \frac{\partial}{\partial x^i}|_v + z^\alpha \frac{\partial}{\partial y^\alpha}|_v, \quad z \in T_v E. \]
The local basis $\{X_\alpha, Y_\alpha\}$ of sections of $\mathcal{L}^\pi E$ associated to the coordinate system is given by
\[ X_\alpha(v) = (e_\alpha(\pi(v)), (\rho^i \circ \pi) \frac{\partial}{\partial x^i}|_v), \quad Y_\alpha(v) = (0, \frac{\partial}{\partial y^\alpha}|_v). \quad (14) \]
If \( V \) is a section of \( \pi^p E \) which in coordinates writes
\[
V(x, y) = (x^i, y^\alpha, Z^\alpha(x, y), V^\alpha(x, y)),
\]
then the expression of \( V \) in terms of base \( \{ X_\alpha, V_\alpha \} \) is [18]
\[
V = Z^\alpha X_\alpha + V^\alpha V_\alpha.
\]
We may introduce the vertical lift \( X^V \) and the complete lift \( X^C \) of a section \( X \in \Gamma(E) \) as the sections of \( \pi^p E \to E \) given by
\[
X^V(u) = (0, X^V(u)), \quad X^C(u) = (X(\pi(u)), X^C(u)), \quad u \in E.
\]
Using the coordinate expressions of \( X^V \) and \( X^C \), the coordinate expressions of \( X^V \) and \( X^C \) as follows:
\[
X^V = (X^\alpha \circ \pi)V_\alpha, \quad X^C = (X^\alpha \circ \pi)X_\alpha + \frac{\partial X^\alpha}{\partial \xi^\beta} - X^\gamma L^\alpha_{\gamma\beta} \circ \pi)V_\alpha, \quad (15)
\]
where \( X = X^\alpha e_\alpha \in \Gamma(E) \). In particular we have
\[
e^V_\alpha = V_\alpha. \quad (16)
\]
Here, we consider the anchor map \( \rho_X : \pi^p E \to TE \) defined by \( \rho_X(u, z) = z \) and the bracket \([.,.],_E \) satisfying the relations
\[
[X^V, Y^V]_E = 0, \quad [X^V, Y^C]_E = [X, Y]^V_E, \quad [X^C, Y^C]_E = [X, Y]^C_E, \quad (17)
\]
for \( X, Y \in \Gamma(E) \). Then this vector bundle \((\pi^p E, \pi_E, E)\) is a Lie algebroid with structure \([.,.]_E, \rho_E\).

Using (17) we can deduce the following

**Lemma 3.1.** The Lie brackets of basis \( \{ X_\alpha, V_\alpha \} \) are
\[
[X_\alpha, X_\beta]_E = (L^\gamma_{\alpha\beta} \circ \pi)X_\gamma, \quad [X_\alpha, V_\beta]_E = 0, \quad [V_\alpha, V_\beta]_E = 0.
\]

### 3.1 A setting for semispray on \( \pi^p E \)

A section of \( \pi \) along smooth map \( f : N \to M \) is a smooth map \( \sigma : N \to E \) such that \( \pi \circ \sigma = f \). The set of sections of \( \pi \) along \( f \) will be denoted by \( \Gamma_f(\pi) \). Then there is a canonical isomorphism between \( \Gamma(f^\ast \pi) \) and \( \Gamma_f(\pi) \) (see [29]). Now we consider pullback bundle \( \pi^\ast \pi = (\pi^\ast E, pr_1, E) \) of vector bundle \((E, \pi, M)\), where
\[
\pi^\ast E := E \times_M E := \{(u, v) \in E \times E | \pi(u) = \pi(v)\},
\]
and \( pr_1 \) is the projection map onto the first component. The fibres of \( \pi^\ast \pi \) are the \( n \)-dimensional real vector spaces
\[
\{u\} \times E_{\pi(u)} \cong E_{\pi(u)}.
\]
Therefore any section in \( \Gamma(\pi^\ast \pi) \) is of the form
\[
\bar{X} : u \in E \to \bar{X}(u) = (u, X(u)),
\]
where $X : E \to E$ is a smooth map such that $\pi \circ X = \pi$. In these terms, the map

$$\tilde{X} \in \Gamma(\pi^*\pi) \to X \in \Gamma_\pi(\pi),$$

is an isomorphism of $C^\infty(E)$-modules. Therefore we have

$$\Gamma(\pi^*\pi) \cong \Gamma_\pi(\pi).$$

In $\Gamma(\pi^*\pi)$, there is a distinguished section

$$\delta : u \in E \to \delta(u) = (u, u) \in \pi^*E,$$

that called the canonical section along $\pi$. This section corresponds to the identity map $1_E$ under the isomorphism $\Gamma_\pi(\pi) \cong \Gamma(\pi^*\pi)$.

For any section $X$ on $E$, the map

$$\hat{X} : E \to \pi^*E,$$

defined by

$$\hat{X}(u) = (u, X \circ \pi(u))$$

called the lift of $X$ into $\Gamma(\pi^*\pi)$. We consider the following sequence

$$0 \to \pi^*(E) \to \mathcal{L}_E \to \pi^*(E) \to 0,$$

with $j(u, z) = (\pi_E(z), Id(u)) = (v, u), z \in T_v E$, and $i(u, v) = (0, v^\vee)$ where $v^\vee : C^\infty(E) \to \mathbb{R}$ is defined by $v^\vee(F) = \frac{d}{dt}|_{t=0} F(u + tv)$. Indeed we have $v^\vee = \frac{d}{dt}|_{t=0}(u + tv)$. Function $J = i \circ j : \mathcal{L}_E \to \mathcal{L}_E$ is called the vertical endomorphism (almost tangent structure) of $\mathcal{L}_E$.

From the definitions of $i, j$ and $J$ we get

$$\text{Im} J = \text{Im} i = v\mathcal{L}_E, \quad \ker J = \ker j = v\mathcal{L}_E, \quad J \circ J = 0.$$

Moreover, $i$ is injective and $j$ is surjective. Therefore the sequence given by (19) is exact sequence.

**Lemma 3.2.** Let $J$ be the vertical endomorphism of $\mathcal{L}_E$. Then

$$J(\mathcal{V}_\alpha) = \mathcal{V}_\alpha, \quad J(\mathcal{V}_\alpha) = 0.$$  

**Proof.** The definition of $J$ implies

$$J(\mathcal{V}_\alpha(v)) = i \circ j(e_\alpha(\pi(v)), (\rho_\alpha \circ \pi) \frac{\partial}{\partial \tilde{X}_v} |_v) = i(v, e_\alpha(\pi(v))) = (0, e_{\alpha}(\pi(v))^\vee)$$

$$= (0, \frac{\partial}{\partial Y_\alpha} |_v) = \mathcal{V}_\alpha(v).$$

We also deduce

$$J(\mathcal{V}_\alpha(v)) = i \circ j(0, \frac{\partial}{\partial Y_\alpha} |_v) = i(v, 0) = (0, 0).$$  

$\blacksquare$
Corollary 3.3. Let \( \{X^\alpha, Y^\alpha\} \) be the corresponding dual basis of \( \{X_\alpha, Y_\alpha\} \). Then
\[
J = V_\alpha \otimes X^\alpha.
\] (21)

Using the above corollary and (15) we obtain
\[
J(X^\alpha V_\alpha) = 0, \quad J(X^\alpha C) = (X^\alpha \circ \pi) V_\alpha = X^\alpha.
\]

Definition 3.4. Let \( \delta \) be the canonical section along \( \pi \) given by (249). Then the section \( C \) given by
\[
C := i \circ \delta,
\]
is called Liouville or Euler section.

Using the definition of Liouville section we have
\[
C(u) = (i \circ \delta)(u) = i(u, u) = (0, u^\alpha \circ \pi) \frac{\partial}{\partial y^\alpha}
\]
where \( u = u^\alpha e_\alpha \in \Gamma(E) \). Therefore, the Liouville section \( C \) has the coordinate expression
\[
C = y^\alpha V_\alpha,
\] (22)
with respect to \( \{X_\alpha, V_\alpha\} \). It is easy to prove the following

Lemma 3.5. Let \( X \) be a section of \( E \). Then we have
\[
(i) \quad [J, C]_E^F = J, \quad (ii) \quad [X^\alpha V_\alpha, C]_E = X^\alpha, \quad (iii) \quad JC = 0.
\] (23)

Definition 3.6. Section \( \tilde{X} \) of vector bundle \( (E^x, \pi^x, E) \) is said to be homogeneous of degree \( r \), where \( r \) is an integer, if \( [C, \tilde{X}]_E = (r-1)\tilde{X} \). Moreover, \( \tilde{f} \in C^\infty(E) \) is said to be homogeneous of degree \( r \) if \( E^x \tilde{f} = \rho_E(C)(\tilde{f}) = r \tilde{f} \).

Now, let \( \tilde{X} = \tilde{X}^\alpha X_\alpha + \tilde{Y}^\alpha V_\alpha \). Then we obtain
\[
[C, \tilde{X}]_E = y^\alpha \frac{\partial \tilde{X}^\beta}{\partial y^\alpha} X_\beta + (y^\alpha \frac{\partial \tilde{Y}^\beta}{\partial y^\alpha} - \tilde{Y}^\beta)V_\beta.
\]
Thus \( [C, \tilde{X}]_E = (r-1)\tilde{X} \) if and only if
\[
y^\alpha \frac{\partial \tilde{X}^\beta}{\partial y^\alpha} = (r-1)\tilde{X}^\beta, \quad y^\alpha \frac{\partial \tilde{Y}^\beta}{\partial y^\alpha} = r\tilde{Y}^\beta.
\] (24)

Therefore we have

Lemma 3.7. Section \( \tilde{X} = \tilde{X}^\alpha X_\alpha + \tilde{Y}^\alpha V_\alpha \) of \( E^x \) is homogeneous of degree \( r \) if and only if (24) holds.

Now, let \( \tilde{f} \in C^\infty(E) \) be homogeneous of degree 1. Then we have
\[
E^x \tilde{f} = \rho_E(C)(\tilde{f}) = r \tilde{f}.
\]
The above equation gives us
\[
y^\alpha \rho_E(V_\alpha) \tilde{f} = y^\alpha \frac{\partial \tilde{f}}{\partial y^\alpha} = r \tilde{f}.
\]
Therefore we have
Lemma 3.8. Real valued smooth function \( \tilde{f} \) on \( E \) is homogenous of degree \( r \) if and only if \( y^\alpha \frac{\partial \tilde{f}}{\partial y^\alpha} = r \tilde{f} \).

Definition 3.9. A section \( S \) of the vector bundle \((\mathcal{L}^\pi E, \pi_\mathcal{L}, E)\) is said to be a semispray if it satisfies the condition \( J(S) = C \). Moreover if \( S \) is homogenous of degree 2, i.e., \([C, S]_\mathcal{L} = S\), then we call it spray.

Let \( S = A^\alpha \mathcal{X}_\alpha + S^\alpha \mathcal{V}_\alpha \) be a semispray on \( \mathcal{L}^\pi E \). Then by using (20) and (22) we deduce \( A^\alpha = y^\alpha \). Therefore semispray \( S \) has the following coordinate expression:

\[ S = y^\alpha \mathcal{X}_\alpha + S^\alpha \mathcal{V}_\alpha. \tag{25} \]

Moreover, from the above lemma we deduce that \( S \) is a spray if and only if

\[ 2S^\beta = y^\alpha \frac{\partial S^\beta}{\partial y^\alpha}. \tag{26} \]

Using (10) and (26), it is easy to see that

\[ \rho_\mathcal{L}(S)(f^\circ) = f^\circ. \tag{27} \]

Lemma 3.10. Let \( S_1 \) be a spray on \( \mathcal{L}^\pi E \) and \( \tilde{f} : E \to \mathbb{R} \) be a smooth function on \( E = \{0\} \). Then \( S_2 = S_1 + \tilde{f} C \) is a spray on \( \mathcal{L}^\pi E \) if and only if \( \tilde{f} \) is homogenous of degree 1.

Proof. Let \( \tilde{f} \) be a homogenous function of of degree 1. Then we have \( \rho_\mathcal{L}(C)\tilde{f} = \tilde{f} \). In other hand, since \( S_1 \) is a spray on \( \mathcal{L}^\pi E \) then we have \( JS_1 = C \) and \([C, S_1]_\mathcal{L} = S_1 \). Therefore

\[ JS_2 = JS_1 + \tilde{f}JC = C, \]

and

\[ [C, S_2]_\mathcal{L} = [C, S_1 + \tilde{f} C]_\mathcal{L} = [C, S_1]_\mathcal{L} + [C, \tilde{f} C]_\mathcal{L} = S_1 + (\rho_\mathcal{L}(C)\tilde{f})C = S_1 + \tilde{f}C = S_2. \]

Thus \( S_2 \) is a spray on \( \mathcal{L}^\pi E \). Conversely, let \( S_2 \) be a spray on \( \mathcal{L}^\pi E \). Then we have

\[ S_1 + \tilde{f}C = S_2 = [C, S_2]_\mathcal{L} = S_1 + (\rho_\mathcal{L}(C)\tilde{f})C. \]

Thus we get \( \rho_\mathcal{L}(C)\tilde{f} = \tilde{f} \), i.e., \( C \) is homogenous of degree 1.

The spray \( S_2 \) given in the above lemma is said to be projective change of \( S_1 \) by \( \tilde{f} \).

Definition 3.11. A Lie symmetry of semispray \( S \) is a section \( X \) of \( E \) such that \([S, X^\mathcal{C}]_\mathcal{L} = 0 \). Moreover a dynamical symmetry of semispray \( S \) is a section \( \tilde{X} \) of \( \mathcal{L}^\pi E \) such that \([S, \tilde{X}]_\mathcal{L} = 0 \).

Proposition 3.12. A section \( X = X^\alpha e_\alpha \) of \( E \) is a Lie symmetry of \( S \) if and only if

\[ y^\beta y^\gamma (\rho_\lambda (\circ \pi)) \frac{\partial (X^\alpha \circ \pi)}{\partial x^\beta} - ((X^\lambda \rho_\lambda (\circ \pi)) \frac{\partial S^\alpha}{\partial x^\gamma} + S^\lambda (X^\alpha \circ \pi) - y^\beta (X^\lambda \circ \pi) \frac{\partial S^\lambda}{\partial y^\beta} = 0, \]

where \( X^\alpha_{|\lambda} := \rho_\lambda \frac{\partial X^\alpha}{\partial x^\lambda} - X^\gamma L^\alpha_{\gamma \beta}. \)
Proof. Using (15) and (25) we obtain

\[ [S, X^C]_E = [y^\alpha X_\alpha + S^\alpha V_\alpha, (X^\lambda \circ \pi)X_\lambda + y^\beta (X^\lambda |_{\beta} \circ \pi)V_\lambda]_E \]

\[ = \{ y^\lambda \rho^\alpha_\lambda \frac{\partial (X^\alpha \circ \pi)}{\partial x^\lambda} + y^\beta ((X^\lambda L^\alpha_{\gamma}) \circ \pi) - y^\beta (X^\lambda |_{\beta} \circ \pi)\}X_\alpha \]

\[ + \{ y^\beta y^\lambda (\rho^\beta_\lambda \circ \pi) \frac{\partial (X^\alpha \circ \pi)}{\partial x^\lambda} - ((X^\lambda \rho^\beta_\lambda) \circ \pi) \frac{\partial S^\alpha}{\partial x^\lambda} + S^\lambda (X^\lambda |_{\beta} \circ \pi) \]

\[ - y^\beta (X^\lambda |_{\beta} \circ \pi) \frac{\partial S^\alpha}{\partial x^\lambda} \}V_\alpha. \]

Using direct calculation we deduce that the coefficient of \( X_\alpha \) vanishes. Therefore \([S, X^C]_E = 0\) if and only if the coefficient of \( V_\alpha \) is zero. \( \square \)

**Proposition 3.13.** A section \( \tilde{X} = \tilde{X}^\alpha X_\alpha + \tilde{Y}^\alpha V_\alpha \) of \( E \) is dynamical symmetry of \( S \) if and only if

\[ \tilde{X}^\alpha = \tilde{Y}^\alpha, \quad y^\beta \rho^\beta_\alpha \frac{\partial \tilde{Y}^\alpha}{\partial x^\beta} - \tilde{X}^\beta y^\gamma L^\alpha_{\gamma \beta} = 0, \]

where \( \tilde{X}^\alpha := \rho_E(S)\tilde{X}^\alpha + \tilde{X}^\alpha y^\beta L^\alpha_{\gamma \beta} = y^\beta \rho^\beta_\alpha \frac{\partial \tilde{Y}^\alpha}{\partial y^\beta} + S^\lambda (S) \frac{\partial \tilde{Y}^\alpha}{\partial y^\beta} \).

**Proof.** Using (25) we obtain

\[ [S, \tilde{X}]_E = [y^\alpha X_\alpha + S^\alpha V_\alpha, \tilde{X}^\beta X_\beta + \tilde{Y}^\beta V_\beta]_E \]

\[ = \{ y^\beta \rho^\beta_\alpha \frac{\partial \tilde{X}^\alpha}{\partial x^\beta} + S^\lambda (S) \frac{\partial \tilde{X}^\alpha}{\partial y^\beta} - \tilde{X}^\beta y^\gamma L^\alpha_{\gamma \beta} \}X_\alpha \]

\[ + \{ y^\beta \rho^\beta_\alpha \frac{\partial \tilde{Y}^\alpha}{\partial x^\beta} - \tilde{X}^\beta y^\gamma L^\alpha_{\gamma \beta} \}V_\alpha. \]

Therefore \([S, \tilde{X}]_E = 0\) if and only if the coefficients of \( X_\alpha \) and \( V_\alpha \) are zero. \( \square \)

### 4 Horizontal lift on \( E \)

In this section we introduce horizontal endomorphisms to decompose \( E \) to horizontal and vertical subbundles. Then we consider torsions, tension and curvature of a horizontal endomorphism. Moreover, some new results are obtained on horizontal endomorphisms and using a horizontal endomorphism, the horizontal lift of a section of \( E \) to \( E \) is constructed.

#### 4.1 Horizontal endomorphism

**Definition 4.1.** A function \( h : E \rightarrow E \) is called a horizontal endomorphism if \( h \circ h = h \), \( \ker h = v E \) and \( h \) is smooth on \( E = E - \{ 0 \} \). Also, \( v := Id - h \) is called vertical projector associated to \( h \).

Setting \( h E := Im h \) we have the following splitting for \( E \):

\[ \epsilon E = v E \oplus h E. \quad (28) \]
From the above relation we deduce $\text{Im} v = v \mathcal{L}^\pi E$. Thus, using $\ker J = v \mathcal{L}^\pi E$, we obtain

$$0 = Jv\tilde{X} = J(\tilde{X} - h\tilde{X}) = J\tilde{X} - Jh\tilde{X}, \quad \tilde{X} \in \mathcal{L}^\pi E.$$ 

Also, from the definition of the horizontal endomorphism we have

$$\ker h = \text{Im} J = \ker J = \text{Im} v = v \mathcal{L}^\pi E.$$ 

(i) $hJ = hv = Jv = 0$, (ii) $v \circ v = v$, (iii) $vh = 0$, (iv) $Jh = Jv = vJ$. (29) 

Coordinate expression of $h$. Let $h = (A^\alpha_\beta X^\alpha + B^\alpha_\beta V^\alpha \otimes X^\beta) + (C^\alpha_\beta X^\alpha + D^\alpha_\beta V^\alpha \otimes V^\beta)$. Then using $\ker h = v \mathcal{L}^\pi E$, we have

$$0 = h(V^\gamma) = (C^\alpha_\gamma X^\alpha + D^\alpha_\gamma V^\alpha).$$ 

Therefore $C^\alpha_\gamma = D^\alpha_\gamma = 0$. Also (iv) of (29) yields

$$V^\gamma = J(X^\gamma) = Jh(X^\gamma) = J(A^\alpha_\gamma X^\alpha + B^\alpha_\gamma V^\alpha) = A^\alpha_\gamma V^\alpha.$$ 

Therefore we have $A^\alpha_\beta = \delta^\alpha_\beta$. Hence $h$ has the following locally expression:

$$h = (X^\beta + B^\alpha_\beta V^\alpha \otimes X^\beta).$$ (30) 

Definition 4.2. For $k \in \mathbb{N}$, $K \in \Gamma(\wedge^k E^* \otimes E)$ is called semibasic if

$$J_0 K = 0, \quad i_{JX} K = 0, \quad \forall X \in \Gamma(E).$$

Definition 4.3. Let $h$ be a horizontal endomorphism on $\mathcal{L}^\pi E$. Then $H = [h, C]_{\mathcal{L}^\pi E} : \mathcal{L}^\pi E \to \mathcal{L}^\pi E$ is called the tension of $h$, where $[h, C]_{\mathcal{L}^\pi E}$ is the generalized Frölicher-Nijenhuis bracket on $\mathcal{L}^\pi E$. If $H = 0$, then $h$ is called homogeneous.

Using (8), (22) and (30), we obtain

$$H(X^\lambda) = [h, C]_{\mathcal{L}^\pi E}(X^\lambda) = [h(X^\lambda), C]_{\mathcal{L}^\pi E} - h[X^\lambda, C]_{\mathcal{L}^\pi E}$$

$$= B^\alpha_\lambda \rho E(V^\alpha)(\bar{y}^\gamma) V^\gamma - \bar{y}^\gamma \rho E(V^\gamma)(B^\alpha_\lambda) V^\alpha$$

$$= (B^\alpha_\lambda - \bar{y}^\gamma \frac{\partial B^\alpha_\lambda}{\partial y^\gamma}) V^\alpha,$$

$$H(V^\lambda) = [h, C]_{\mathcal{L}^\pi E}(V^\lambda) = [h(V^\lambda), C]_{\mathcal{L}^\pi E} - h[V^\lambda, C]_{\mathcal{L}^\pi E} = 0.$$ 

Using the above equations $H$ has the coordinate expression

$$H = (B^\alpha_\beta - \bar{y}^\gamma \frac{\partial B^\alpha_\beta}{\partial y^\gamma}) V^\alpha \otimes X^\beta.$$ (31) 

Since $J(V^\alpha) = 0$, then we obtain $J \circ H = 0$ and $i_{JX} H = 0$, where $\tilde{X} \in \Gamma(\mathcal{L}^\pi E)$. Therefore $H$ is semibasic.

From (31) we have
Lemma 4.4. The horizontal endomorphism $h$ is homogeneous if and if 

$$B'_\beta = y^\gamma \frac{\partial B'_\gamma}{\partial y^\beta}.$$  

Definition 4.5. Let $h$ be a horizontal endomorphism on $\mathcal{L}^\pi E$. Then $t = [J, h]_{\mathcal{L}^\pi E} \in \Gamma(\mathcal{L}^\pi E)$ is called the weak torsion of $h$.

Using (31) we get

$$t(X_\alpha, X_\beta) = \frac{1}{2} t^\gamma_{\alpha \beta} X^\alpha \wedge X^\beta \otimes V_\gamma,$$  

where

$$t^\gamma_{\alpha \beta} := \frac{\partial B^\gamma_\alpha}{\partial y^\alpha} - \frac{\partial B^\gamma_\beta}{\partial y^\beta} - (L^\gamma_{\alpha \beta} \circ \pi).$$  

Lemma 4.6. The weak torsion $t$ is semibasic.

Proof. Since $J(V_\alpha) = 0$, then we deduce $J \circ t = 0$. Also we have $i_{\tilde{X}}(X^\alpha) = 0$, for each $\tilde{X} \in \Gamma(\mathcal{L}^\pi E)$. Therefore we obtain

$$i_{\tilde{X}} t = \frac{1}{2} t^\gamma_{\alpha \beta} i_{\tilde{X}} (X^\alpha \wedge X^\beta) \otimes V_\gamma = \frac{1}{2} t^\gamma_{\alpha \beta} (i_{\tilde{X}} (X^\alpha) \wedge X^\beta - X^\alpha \wedge i_{\tilde{X}} (X^\beta)) \otimes V_\gamma = 0.$$  

Therefore $t$ is semibasic.

Definition 4.7. The strong torsion of $h$ is defined by $T = i_{st} + H$.

Lemma 4.8. The strong torsion $T$ has the following coordinate expression:

$$T = (B'_\beta - y^\gamma \frac{\partial B'_\gamma}{\partial y^\beta} - y^\gamma (L^\gamma_{\alpha \beta} \circ \pi)) V_\alpha \otimes X^\beta.$$  

Proof. Using (31) we get 

$$T(X_\alpha) = (i_{st} t)(X_\alpha) + H(X_\alpha) = (i_{st} t)(X_\alpha) + (B'_\alpha - y^\gamma \frac{\partial B'_\gamma}{\partial y^\alpha}) V_\gamma.$$  

But using (29) and (30) we obtain

$$i_{st} = \frac{1}{2} t^\gamma_{\alpha \beta} (y^\alpha X^\beta - y^\beta X^\alpha) \otimes V_\gamma.$$  

Thus

$$(i_{st} t)(X_\alpha) = y^\alpha t^\gamma_{\alpha \beta} V_\gamma = y^\alpha (\frac{\partial B'_\gamma}{\partial y^\alpha} - \frac{\partial B'_\gamma}{\partial y^\alpha} - (L^\gamma_{\alpha \beta} \circ \pi)) V_\gamma.$$  

Setting the above equation in (35) we obtain (34).
It is easy to see that \(J \circ T = 0\) and \(i_{J}T = 0\), for each \(\tilde{X} \in \Gamma(L^\sigma E)\). Thus \(T\) is semibasic.

**Definition 4.9.** The curvature of a horizontal endomorphism \(h\) is defined by 
\[
\Omega = -N_h,
\]
where \(N_h\) is the Nijenhuis tensor of \(h\) given by 
\[
N_h(\tilde{X}, \tilde{Y}) = [h\tilde{X}, h\tilde{Y}] - h[h\tilde{X}, \tilde{Y}] - h[\tilde{X}, h\tilde{Y}] + h[\tilde{X}, \tilde{Y}], \quad \forall \tilde{X}, \tilde{Y} \in \Gamma(L^\sigma E).
\]

**Lemma 4.10.** For sections \(\tilde{X}\) and \(\tilde{Y}\) of \(L^\sigma E\) we have
\[
\Omega(\tilde{X}, \tilde{Y}) = \Omega(h\tilde{X}, h\tilde{Y}) = -v[h\tilde{X}, h\tilde{Y}]_E. \tag{37}
\]

**Proof.** At first it is easy to check that \([v\tilde{X}, v\tilde{Y}]_E \in vL^\sigma E\). Thus using \(\ker h = vL^\sigma E\) and \(hv = 0\) we get
\[
\Omega(v\tilde{X}, v\tilde{Y}) = -N_h(v\tilde{X}, v\tilde{Y}) = -h[v\tilde{X}, v\tilde{Y}]_E = 0.
\]
Also it is easy to that \(N_h(h\tilde{X}, v\tilde{Y}) = 0\) and consequently \(\Omega(h\tilde{X}, v\tilde{Y}) = 0\). Therefore we obtain
\[
\Omega(\tilde{X}, \tilde{Y}) = \Omega(h\tilde{X} + v\tilde{X}, h\tilde{Y} + v\tilde{Y}) = \Omega(\tilde{X}, \tilde{Y})
\]
\[
= -h[\tilde{X}, h\tilde{Y}]_E + h[h\tilde{X}, h\tilde{Y}]_E + h[h\tilde{X}, h\tilde{Y}]_E - h[h\tilde{X}, h\tilde{Y}]_E
\]
\[
= -v[h\tilde{X}, h\tilde{Y}]_E.
\]

**Proposition 4.11.** The curvature \(\Omega\) has the following coordinate expression:
\[
\Omega = -\frac{1}{2} R^\gamma_{\alpha\beta} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma, \tag{38}
\]
where
\[
R^\gamma_{\alpha\beta} = (\rho^\gamma_{\alpha} \circ \pi) \frac{\partial B^\gamma_{\beta}}{\partial x^\gamma} - (\rho^\gamma_{\beta} \circ \pi) \frac{\partial B^\gamma_{\alpha}}{\partial x^\gamma} + B^\lambda_{\alpha} \frac{\partial B^\gamma_{\beta}}{\partial y^\gamma} - B^\lambda_{\beta} \frac{\partial B^\gamma_{\alpha}}{\partial y^\gamma} + (L^\lambda_{\alpha} \circ \pi) B^\gamma_{\lambda}. \tag{39}
\]

**Proof.** Using \([37]\) we have
\[
\Omega(\mathcal{X}_\alpha, \mathcal{X}_\beta) = -v[h\mathcal{X}_\alpha, h\mathcal{X}_\beta]_E = -v[\mathcal{X}_\alpha \otimes B^\beta_{\lambda} \mathcal{V}_\lambda, \mathcal{X}_\beta \otimes B^\gamma_{\beta} \mathcal{V}_\gamma]_E
\]
\[
= -v \left( (L^\gamma_{\alpha\beta} \circ \pi) \mathcal{X}_\gamma + \rho_L(\mathcal{X}_\alpha)(B^\gamma_{\beta}) \mathcal{V}_\gamma - \rho_L(\mathcal{X}_\beta)(B^\gamma_{\alpha}) \mathcal{V}_\lambda \right)
\]
\[
+ B^\lambda_{\alpha} \rho_L(\mathcal{V}_\lambda)(B^\gamma_{\beta}) \mathcal{V}_\gamma - B^\lambda_{\beta} \rho_L(\mathcal{V}_\gamma)(B^\gamma_{\alpha}) \mathcal{V}_\lambda
\]
\[
= - \left( (\rho^\gamma_{\alpha} \circ \pi) \frac{\partial B^\gamma_{\beta}}{\partial x^\gamma} - (\rho^\gamma_{\beta} \circ \pi) \frac{\partial B^\gamma_{\alpha}}{\partial x^\gamma} + B^\lambda_{\alpha} \frac{\partial B^\gamma_{\beta}}{\partial y^\gamma} - B^\lambda_{\beta} \frac{\partial B^\gamma_{\alpha}}{\partial y^\gamma} \right) v \mathcal{V}_\gamma,
\]
\[
- (L^\gamma_{\alpha\beta} \circ \pi) v \mathcal{X}_\gamma. \tag{40}
\]

Using \(v = Id - h\) and \([50]\) we deduce that
\[
v \mathcal{V}_\alpha = \mathcal{V}_\alpha, \quad v \mathcal{X}_\alpha = -B^\beta_{\alpha} \mathcal{V}_\beta.
\]
Plugging the above equation into (40) yields
\[
\Omega(\mathcal{X}_\alpha, \mathcal{X}_\beta) = \left( (L_{\alpha \beta}^\lambda \circ \pi) B_\gamma^\lambda - (\rho^\lambda_\alpha \circ \pi) \frac{\partial B_\gamma^\lambda}{\partial x^\alpha} + (\rho^\lambda_\beta \circ \pi) \frac{\partial B_\gamma^\lambda}{\partial x^\beta} - B_\alpha^\lambda \frac{\partial B_\beta^\gamma}{\partial y^\lambda} + B_\beta^\lambda \frac{\partial B_\alpha^\gamma}{\partial y^\beta} \right) V_\gamma = -R^\gamma_{\alpha \beta} V_\gamma.
\]

Similarly, we have
\[
\Omega(\mathcal{X}_\alpha, \mathcal{V}_\beta) = \Omega(\mathcal{V}_\alpha, \mathcal{V}_\beta) = 0.
\]

Similar to the proof of Lemma 4.6, we can prove the following

**Lemma 4.12.** The curvature \( \Omega \) of horizontal endomorphism \( h \) is semibasic.

**Proposition 4.13.** Let the horizontal endomorphism \( h \) be given on \( \mathcal{E}^\pi \). If \( S \) is an arbitrary semispray of \( \mathcal{E}^\pi \), then \( \bar{S} = hS \) is also a semispray of \( \mathcal{E}^\pi \) which does not depend on the choice of \( S \). \( \bar{S} \) is called the semispray associated to \( h \).

**Proof.** Since \( Jh = J \) then we have
\[
J\bar{S} = J(hS) = Jh(S) = JS = C.
\]
Thus \( S' \) is a semispray. Now let \( S' \) be another semispray of \( \mathcal{E}^\pi \). Then we have
\[
J(S - S') = JS - JS' = C - C = 0.
\]
Thus \( S - S' \in \ker J = v\mathcal{E}^\pi \), which gives us \( 0 = h(S - S') = hS - hS' \), i.e., \( hS = hS' \).

**Proposition 4.14.** If the horizontal endomorphism \( h \) is homogeneous, then the semispray associated to \( h \) is spray.

**Proof.** Let \( S \) be a semispray of \( \mathcal{E}^\pi \). Since \( h \) is homogeneous, then we have
\[
0 = H(S) = [h, C]_{\mathcal{E}^\pi}F^{-N}(S) = [hS, C]_{\mathcal{E}^\pi} - h[S, C]_{\mathcal{E}^\pi}
= [hS, C]_{\mathcal{E}^\pi} - h([S, C]_{\mathcal{E}^\pi} + S) + hS. \tag{41}
\]
But we can obtain
\[
[S, C]_{\mathcal{E}^\pi} + S = (2 S^\alpha - y^\alpha \frac{\partial S^\alpha}{\partial y^\beta}) V_\alpha,
\]
and consequently \( h([S, C]_{\mathcal{E}^\pi} + S) = 0 \). Plugging this equation into (41) implies \([C, hS]_{\mathcal{E}^\pi} = hS \), i.e., \( hS \) is a spray of \( \mathcal{E}^\pi \).

**Lemma 4.15.** If \( h_1 \) and \( h_2 \) are horizontal endomorphisms on \( \mathcal{E}^\pi \), then \( h_1 - h_2 \in v\mathcal{E}^\pi \). Moreover
\[
J((h_1 - h_2)(\tilde{X}), S)_{\mathcal{E}^\pi} = (h_1 - h_2)(\tilde{X}), \quad \forall \tilde{X} \in \Gamma(\mathcal{E}^\pi). \tag{42}
\]
Proof. From (39) we have
\[ h_1 = (\mathcal{X}_\beta + h_1 \mathcal{B}^\alpha_\beta) \otimes \mathcal{X}^\beta, \quad h_2 = (\mathcal{X}_\beta + h_2 \mathcal{B}^\alpha_\beta) \otimes \mathcal{X}^\beta. \]

Thus
\[ h_1 - h_2 = (h_1 \mathcal{B}^\alpha_\beta - h_2 \mathcal{B}^\alpha_\beta) \mathcal{V}_\alpha \otimes \mathcal{X}^\beta. \]

Now let \( \tilde{X} = \tilde{X} \mathcal{X}_\beta + \tilde{Y} \mathcal{X}_\beta \in \Gamma(\mathcal{E}^* \mathcal{E}). \) Then we obtain
\[ (h_1 - h_2)(\tilde{X}) = \tilde{X} (h_1 \mathcal{B}^\alpha_\beta - h_2 \mathcal{B}^\alpha_\beta) \mathcal{V}_\alpha \otimes \mathcal{X}^\beta \in v \mathcal{E}^* \mathcal{E}. \]

Now, we prove the second part of the lemma. The above equation implies that
\[ J [(h_1 - h_2)(\tilde{X}), \mathcal{S}]_{\mathcal{E}} = J [\tilde{X} (h_1 \mathcal{B}^\alpha_\beta - h_2 \mathcal{B}^\alpha_\beta) \mathcal{V}_\alpha, y^* \mathcal{X}_\gamma + S^* \mathcal{V}_\gamma]_{\mathcal{E}} \]
\[ = \tilde{X} (h_1 \mathcal{B}^\alpha_\beta - h_2 \mathcal{B}^\alpha_\beta) J (\mathcal{X}_\alpha) = \tilde{X} (h_1 \mathcal{B}^\alpha_\beta - h_2 \mathcal{B}^\alpha_\beta) \mathcal{V}_\alpha \]
\[ = (h_1 - h_2)(\tilde{X}). \]

\[ \blacksquare \]

**Theorem 4.16.** If \( h_1 \) and \( h_2 \) are horizontal endomorphisms with same associated semisprays and strong torsion, then \( h_1 = h_2. \)

Proof. Let \( K = h_1 - h_2. \) Since \( J h_1 = J h_2 = J \) and \( h_1 J = h_2 J = 0, \) then we obtain \( J \circ K = 0 \) and \( i_{\tilde{X}} K = K(\tilde{X}) = 0, \) for each \( \tilde{X} \in \Gamma(\mathcal{E}^* \mathcal{E}). \) Thus \( K \) is a semibasic. Since \( h_1 \) and \( h_2 \) have the same associated semisprays, then \( h_1 S = h_2 S, \) and consequently \( K S = 0. \) But we have
\[ t_2 = [J, h_1]_{\mathcal{E}}^{F-N} = [J, h_2]_{\mathcal{E}}^{F-N} + [J, K]_{\mathcal{E}}^{F-N} = t_1 + [J, K]_{\mathcal{E}}^{F-N}. \]

Similarly we obtain
\[ H_2 = H_1 + [K, C]_{\mathcal{E}}^{F-N}. \]

The above equations give us
\[ T_2 = i_{\tilde{X}} t_2 + H_2 = T_1 + i_{\tilde{X}} [J, K]_{\mathcal{E}}^{F-N} + [K, C]_{\mathcal{E}}^{F-N}. \]

Since \( T_1 = T_2, \) then from the above equation we deduce
\[ i_{\tilde{X}} [J, K]_{\mathcal{E}}^{F-N}(\tilde{X}) = -[K, C]_{\mathcal{E}}^{F-N}(\tilde{X}), \quad \forall \tilde{X} \in \Gamma(\mathcal{E}^* \mathcal{E}). \quad (43) \]

Since \( J \circ K = K \circ J = K S = 0 \) and \( JS = C, \) then using (39) we get
\[ i_{\tilde{X}} [J, K]_{\mathcal{E}}^{F-N}(\tilde{X}) = [J, K]_{\mathcal{E}}^{F-N}(S, \tilde{X}) \]
\[ = [C, K \tilde{X}]_{\mathcal{E}} - J[S, K \tilde{X}]_{\mathcal{E}} \]
\[ - K[S, J \tilde{X}]_{\mathcal{E}} - K[JS, \tilde{X}]_{\mathcal{E}}. \]

Setting the above equation in (43) and using (8) imply that
\[ J[S, K \tilde{X}]_{\mathcal{E}} = K[J \tilde{X}, S]_{\mathcal{E}}. \]

Using (42) and the above equation we obtain
\[ -K \tilde{X} = J[S, K \tilde{X}]_{\mathcal{E}} = K[J \tilde{X}, S]_{\mathcal{E}} = K([J \tilde{X}, S]_{\mathcal{E}} - \tilde{X}) + K \tilde{X}. \]

It is easy to see that \([J \tilde{X}, S]_{\mathcal{E}} - \tilde{X} \in v \mathcal{E}^* \mathcal{E} \) and \([v \mathcal{E}^* \mathcal{E}] \subset \ker K. \) Thus \( K([J \tilde{X}, S]_{\mathcal{E}} - \tilde{X}) = 0. \) Therefore the above equation gives us \( K \tilde{X} = 0 \) and consequently \( h_1 = h_2. \)

\[ \blacksquare \]
4.2 Almost complex structure on $\mathcal{L}^\pi E$

Let $S$ be the semispray associated to $h$. We consider the map $F : \mathcal{L}^\pi E \to \mathcal{L}^\pi E$ given by $F := h[S, h]^{F - N}_{\mathcal{L}} - J$. Since $J^2 = 0$ and $J h = J$, then we have

$$F^2 = (h[S, h]^{F - N}_{\mathcal{L}} - J)^2 = (h[S, h]^{F - N}_{\mathcal{L}})^2 - J[h[S, h]^{F - N}_{\mathcal{L}} - h[S, h]^{F - N}_{\mathcal{L}}]J. \quad (44)$$

But we have

$$h[S, h]^{F - N}_{\mathcal{L}} \tilde{X} = h[h[\tilde{X}, S]_{\mathcal{L}} - [h \tilde{X}, S]_{\mathcal{L}}] = h[\tilde{X} - h \tilde{X}, S]_{\mathcal{L}} = h[v \tilde{X}, S]_{\mathcal{L}}.$$  

Therefore

$$(h[S, h]^{F - N}_{\mathcal{L}})^2 \tilde{X} = h[v(h[v \tilde{X}, S]_{\mathcal{L}}), S]_{\mathcal{L}} = 0, \quad (45)$$

because $v h = 0$. In other hand, by a direct computation, we get

$$J[h[S, h]^{F - N}_{\mathcal{L}} \tilde{X}] = (J[v \tilde{X}, S]_{\mathcal{L}} - v \tilde{X}) + (h[J \tilde{X}, S]_{\mathcal{L}} - h \tilde{X}), \quad (46)$$

But we have

$$J[v \tilde{X}, S]_{\mathcal{L}} = J[(\tilde{Y}^\alpha - \tilde{X}^\gamma B^\gamma_{\alpha\beta}) V_\alpha, \beta \gamma \chi_\beta + S^\beta \chi_\beta]_{\mathcal{L}} = (\tilde{Y}^\alpha - \tilde{X}^\gamma B^\gamma_{\alpha\beta}) V_\alpha (\beta \gamma \chi_\beta) J(\chi_\beta) = (\tilde{Y}^\alpha - \tilde{X}^\gamma B^\gamma_{\alpha\beta}) V_\alpha = v \tilde{X}, \quad (47)$$

where $\tilde{X} = \tilde{X}^\alpha \chi_\alpha + \tilde{Y}^\alpha V_\alpha$. Also, we can obtain $[J \tilde{X}, S]_{\mathcal{L}} - \tilde{X} \in v \mathcal{L}^\pi E$. Thus

$$h[J \tilde{X}, S]_{\mathcal{L}} - h \tilde{X} = 0. \quad (48)$$

Setting (47) and (48) in (46) give us

$$J[h[S, h]^{F - N}_{\mathcal{L}} \tilde{X}] + h[h[S, h]^{F - N}_{\mathcal{L}} J \tilde{X}] = \tilde{X}. \quad (49)$$

Plugging (45) and (49) into (44) yield $F^2 = -1_{\mathcal{L}^\pi E}$. Thus $F$ is an almost complex structure on $\mathcal{L}^\pi E$ which is called the almost complex structure induced by $h$.

Lemma 4.17. The following relations hold

(i) $F \circ J = h$, (ii) $F \circ h = -J$, (iii) $J \circ F = v$, (iv) $F \circ v = h \circ F$. \hspace{1cm} (50)

Proof. Since $J^2 = 0$, then we have

$$F \circ J = (h[S, h]^{F - N}_{\mathcal{L}} - J) \circ J = h[S, h]^{F - N}_{\mathcal{L}} J.$$  

But we have

$$h[S, h]^{F - N}_{\mathcal{L}} J \tilde{X} - h \tilde{X} = h(h[J \tilde{X}, S]_{\mathcal{L}} - [h J \tilde{X}, S]_{\mathcal{L}}) - h \tilde{X} = h([J \tilde{X}, S]_{\mathcal{L}} - J) = 0,$$

because $[J \tilde{X}, S]_{\mathcal{L}} - \tilde{X} \in v \mathcal{L}^\pi E$. Therefore $F \circ J = h$. Now we prove the second equation. Since $J h = J$, then we have

$$F \circ h = (h[S, h]^{F - N}_{\mathcal{L}} - J) \circ h = h[S, h]^{F - N}_{\mathcal{L}} h - J.$$  

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But we have
\[ h[S, h]_{E}^{F-N} h \tilde{X} = h(h[h, X], S) - [h^2 \tilde{X}, S] = h^2[h, \tilde{X}] - h[h, \tilde{X}] = 0. \] (51)

Therefore \( F \circ h = -J \). Using the definition of \( F \) we deduce
\[ J \circ F = J(h[S, h]_{E}^{F-N} - J) = Jh[S, h]_{E}^{F-N} = Jh[S, h]_{E}^{F-N}. \]

But using (47) we get
\[ J(h[S, h]_{E}^{F-N} - J) = Jh[S, h]_{E}^{F-N} \tilde{X} = (h[S, h]_{E}^{F-N} - J) \tilde{X} = (h[S, h]_{E}^{F-N} - J) v \tilde{X} = (F \circ v) \tilde{X}. \]

Let \( F = h[S, h]_{E}^{F-N} - J \) be the almost complex structure induced by \( h \).
Since \( S \) is the semispray associated to \( h \), then we have \( S = hS' \), where \( S' \) is a semispray of \( L^\pi E \). Using (20), (25) and (30) we obtain
\[ F(X_\alpha) = -B_\alpha^\gamma(X_\beta + B_\beta^\gamma \psi_\beta) - \psi_\alpha, \quad F(\psi_\alpha) = X_\alpha + B_\alpha^\beta \psi_\beta. \]

Therefore \( F \) has the following coordinate expression
\[ F = -(B_\alpha^\gamma(X_\beta + B_\beta^\gamma \psi_\beta) + \psi_\alpha) \otimes X^\alpha + (X_\alpha + B_\alpha^\beta \psi_\beta) \otimes \psi_\alpha. \] (52)

**Proposition 4.18.** Let \( h \) be a horizontal endomorphism on \( L^\pi E \) and \( j : L^\pi E \to E \times_M E \) be the map introduced in (17). Then we have
\[ j \circ h = j. \] (53)

**Proof.** Since \( \text{Im} v = \ker j = v L^\pi E \), then
\[ j \circ h = j \circ (\text{Id} - v) = j - jv = j. \]

Let \( H := F \circ i : E \times_M E \to L^\pi E \). Then using (i) of Lemma 11 and (53)
\[ j \circ H \circ j = j \circ F \circ i \circ j = j \circ F \circ J = j \circ h = j. \]

Since \( j \) is surjective, then the above equation gives us \( j \circ H = 1_{E \times_M E} \). Therefore \( H \) is a right splitting of (exacts). We call \( H \) the horizontal map for \( L^\pi E \) associated to \( h \).

Now we consider
\[ V := j \circ F : L^\pi E \to E \times_M E. \]

Then we have
\[ V \circ i = j \circ F \circ i = j \circ H = 1_{E \times_M E}. \]

Therefore \( V \) is a left splitting of (exacts), which is called the vertical map for \( L^\pi E \) associated to \( h \).
Corollary 4.19. The following sequence is a double short exact sequence

\[ 0 \to \pi^* E \xrightarrow{j} \xi \to \pi^* E \to 0 \]

Proof. We obtain

\[ \mathcal{V} \circ \mathcal{H} = (j \circ F) \circ (F \circ i) = j \circ (-1_{\xi \to E}) \circ i = -j \circ i = 0. \]

Thus \( \text{Im} \mathcal{H} = \ker \mathcal{V} \). Moreover \( \mathcal{V} \) is surjective, because \( j \) is surjective. Similarly, since \( i \) is injective, then \( \mathcal{H} \) is injective. These complete the proof. \( \square \)

Using (i) and (iii) of Lemma 4.17 we can get

(i) \( h = \mathcal{H} \circ j \), \quad (ii) \( v = i \circ \mathcal{V} \). \((54)\)

### 4.3 Berwald endomorphism

Let \( S \) be a semispray on \( \mathcal{L}^\infty E \). We consider the map \( h_S : \mathcal{L}^\infty E \to \mathcal{L}^\infty E \) given by \( h_S := \frac{1}{2}(1_{\mathcal{L}^\infty E} + [J, S]_{\mathcal{L}^\infty E}^{F-N}) \). Using \((20)\) and \((25)\) we can obtain

\[ h_S(X_\alpha) = X_\alpha + \frac{1}{2}(\partial S^\gamma_{\nu\alpha} - \gamma^\beta (L^\gamma_{\alpha\beta} \circ \pi))\gamma, \quad h_S(V_\gamma) = 0. \]

Therefore \( h_S \) has the coordinate expression

\[ h_S = (X_\alpha + E^\gamma_\alpha \gamma) \otimes X^\alpha, \]

where

\[ E^\gamma_\alpha = \frac{1}{2}(\partial S^\gamma_{\nu\alpha} - \gamma^\beta (L^\gamma_{\alpha\beta} \circ \pi)). \]

Now one can easily check that \( h_S \circ h_S = h_S, J \circ h_S = J, h_S \circ J = 0 \) and consequently \( \ker h_S = \ker J = v.\mathcal{L}^\infty E \). Therefore \( h_S \) is a horizontal endomorphism on \( \mathcal{L}^\infty E \) called horizontal endomorphism generated by semispray \( S \).

**Theorem 4.20.** The horizontal endomorphism generated by semispray \( S \) is torsion free. Moreover, we have \( H_S = \frac{1}{2}[[C, S]_{\mathcal{L}^\infty} - S, J]_{\mathcal{L}^\infty E}^{F-N} \), where \( H_S \) is the tension of \( h_S \).

Proof. Let \( t_S \) be the weak torsion of \( h_S \). Then using \((7)\) we have

\[ t_S = [J, h_S]_{\mathcal{L}^\infty E}^{F-N} = \frac{1}{2}[J, [J, S]_{\mathcal{L}^\infty E}^{F-N}]_{\mathcal{L}^\infty E}^{F-N} = \frac{1}{2}[J, [S, J]_{\mathcal{L}^\infty E}^{F-N}]_{\mathcal{L}^\infty E}^{F-N} = \frac{1}{2}[J, [S, J]_{\mathcal{L}^\infty E}^{F-N}]_{\mathcal{L}^\infty E}^{F-N} = -t_S. \]

Therefore \( t_B = 0. \) \((7)\) and (i) of \((23)\) give us

\[ \frac{1}{2}[[C, S]_{\mathcal{L}^\infty} - S, J]_{\mathcal{L}^\infty E}^{F-N} = \frac{1}{2}([[[C, S]_{\mathcal{L}^\infty} - S, J]_{\mathcal{L}^\infty E}^{F-N} - [S, J]_{\mathcal{L}^\infty E}^{F-N}]) = \frac{1}{2}([[J, S]_{\mathcal{L}^\infty E}^{F-N}, C]_{\mathcal{L}^\infty E}^{F-N} - [[J, C]_{\mathcal{L}^\infty E}^{F-N}, S]_{\mathcal{L}^\infty E}^{F-N} - [S, J]_{\mathcal{L}^\infty E}^{F-N}) = \frac{1}{2}([[J, S]_{\mathcal{L}^\infty E}^{F-N}, C]_{\mathcal{L}^\infty E}^{F-N} - [J, S]_{\mathcal{L}^\infty E}^{F-N} - [S, J]_{\mathcal{L}^\infty E}^{F-N}) = \frac{1}{2}[[J, S]_{\mathcal{L}^\infty E}^{F-N}, C]_{\mathcal{L}^\infty E}^{F-N} = [h_S, C]_{\mathcal{L}^\infty E}^{F-N} = H_S. \]

\( \square \)
Using (31) and (56) we deduce that $H_B$ has the following coordinate expression:

$$H_S = \frac{1}{2} \left( \frac{\partial S^\alpha}{\partial y^\alpha} - y^\gamma \frac{\partial^2 S^\alpha}{\partial y^\gamma \partial y^\beta} \right) \mathcal{V}_\alpha \otimes \mathcal{X}^\beta. $$

**Lemma 4.21.** Let $h_S$ be the horizontal endomorphism generated by semispray $S$. Then the semispray associated by $h_S$ is $\frac{1}{2}(S + [C, S]_E)$.

**Proof.** We have

$$h_S S = \frac{1}{2} (1_E + [J, S]_E) = \frac{1}{2} (S + [J, S]_E) - J[S, S]_E = \frac{1}{2} (S + [C, S]_E).$$

**Corollary 4.22.** Let $h_S$ be the horizontal endomorphism generated by spray $S$. Then the spray associated by $h_S$ is $S$. Moreover $h_S$ is homogenous.

**Definition 4.23.** The horizontal endomorphism generated by an spray is called Berwald endomorphism.

**Theorem 4.24.** Let $h$ be a homogenous horizontal endomorphism on $E$ and $S$ be the semispray associated to $h$. Then we have

$$h_S = h - \frac{1}{2} i_S t,$$

where $t$ is the weak torsion of $h$ and $h_S$ is the horizontal endomorphism generated by $S$.

**Proof.** Since $h$ is homogenous, then $S$ is spray. Therefore $h_S$ is the Berwald endomorphism and consequently from Lemma 4.21 and Corollary 4.22 we deduce $h_S$ is homogenous and $h_S S = S$. Also since $h = (X_\beta + B^\alpha_\beta \mathcal{V}_\alpha) \otimes \mathcal{X}^\beta$ is homogenous and $hS = S$, we obtain

$$(i) \ B^\alpha_\beta = y^\gamma \frac{\partial B^\alpha_\beta}{\partial y^\gamma}, \quad (ii) \ S^\alpha = y^\beta B^\alpha_\beta. \quad (57)$$

From $(ii)$ of (57) we get

$$y^\alpha \frac{\partial B^\gamma_\beta}{\partial y^\alpha} = \frac{\partial S^\alpha}{\partial y^\gamma} - B^\alpha_\beta. \quad (58)$$

Using (56) and (58) we obtain

$$h(X_\beta) = \frac{1}{2} (i_S t)(X_\beta) = \mathcal{X}_\beta + \{B^\gamma_\beta - \frac{1}{2} y^\alpha \frac{\partial B^\gamma_\beta}{\partial y^\alpha} + \frac{1}{2} y^\alpha \frac{\partial B^\gamma_\alpha}{\partial y^\beta} + \frac{1}{2} y^\alpha (L_{\gamma \alpha} \circ \pi) \} \mathcal{V}_\gamma.$$ 

Setting $(i)$ of (57) in the above equation gives us

$$h(X_\beta) = \frac{1}{2} (i_S t)(X_\beta) = \mathcal{X}_\beta + \{\frac{1}{2} B^\gamma_\beta + \frac{1}{2} y^\alpha \frac{\partial B^\gamma_\alpha}{\partial y^\beta} + \frac{1}{2} y^\alpha (L_{\gamma \alpha} \circ \pi) \} \mathcal{V}_\gamma.$$
Plugging (59) into the above equation implies that
\[ h(X_\beta) - \frac{1}{2}(i st)(X_\beta) = X_\beta + \frac{1}{2} \partial S^h + y^\alpha (L^\gamma_{\alpha \beta} \circ \pi) \nu_\gamma = h_S(X_\beta). \]

Similarly, we obtain
\[ h(V_\beta) - \frac{1}{2}(i st)(V_\beta) = h_S(V_\beta). \]

\[ \square \]

4.4 Horizontal lift

Let \( h \) be a horizontal endomorphism on \( \mathcal{L} \pi E \). We consider the map
\[ X \in \Gamma(E) \to X^h := hX^C \in h\mathcal{L} \pi E, \]
and we call it horizontal lift by \( h \). If \( X = X^\alpha e_\alpha \), then we have
\[ X^h = (X^\alpha \circ \pi)(X_\alpha + B^\beta_{\alpha \beta} V_\beta). \] (59)

**Lemma 4.25.** Let \( h \) be a horizontal endomorphism on \( \mathcal{L} \pi E \) and \( X, Y \in \Gamma(E) \).

Then
\( i \) \( JX^h = X^V \), \( ii \) \( h[X^h, Y^h]_E = [X, Y]_E^h \), \( iii \) \( [X, Y]_E^V = J[X^h, Y^h]_E \).

**Proof.** We have
\[ JX^h = JhX^C = JX^C = X^V. \]

Thus (i) is proved. Now let \( X = X^\alpha e_\alpha \) and \( Y = Y^\beta e_\beta \). Then by a direct calculation we get
\[ [X, Y]_E = (X^\alpha \rho_\alpha \frac{\partial Y_\gamma}{\partial x^\gamma} - Y^\beta \rho_\beta \frac{\partial X_\gamma}{\partial x^\gamma} + X^\alpha Y^\beta L^\gamma_{\alpha \beta} e_\gamma, \] (61)
and
\[ [X^h, Y^h]_E = \left( (X^\alpha \rho_\alpha \frac{\partial Y_\gamma}{\partial x^\gamma} - Y^\beta \rho_\beta \frac{\partial X_\gamma}{\partial x^\gamma} + X^\alpha Y^\beta L^\gamma_{\alpha \beta}) \circ \pi \right) X_\gamma \\
+ \left( ((X^\alpha \rho_\alpha) \circ \pi) \frac{\partial}{\partial x^\gamma} \left( (Y^\beta \circ \pi) B^\gamma_{\beta} \right) - ((Y^\alpha \rho_\beta \circ \pi) \frac{\partial}{\partial x^\gamma} ((X^\beta \circ \pi) B^\gamma_{\beta}) + ((X^\alpha Y^\beta) \circ \pi) B^\gamma_{\beta} \frac{\partial \gamma}{\gamma^\gamma} \right) V_\gamma. \] (62)

Therefore
\[ [X, Y]_E^h = \left( (X^\alpha \rho_\alpha \frac{\partial Y_\gamma}{\partial x^\gamma} - Y^\beta \rho_\beta \frac{\partial X_\gamma}{\partial x^\gamma} + X^\alpha Y^\beta L^\gamma_{\alpha \beta}) \circ \pi \right) (X_\gamma + B^\beta_{\alpha \beta} V_\beta) \\
= h[X^h, Y^h]_E. \]

Thus we have (ii). Also, using (61) and (62) we obtain (iii) as follows
\[ [X, Y]_E^V = \left( (X^\alpha \rho_\alpha \frac{\partial Y_\gamma}{\partial x^\gamma} - Y^\beta \rho_\beta \frac{\partial X_\gamma}{\partial x^\gamma} + X^\alpha Y^\beta L^\gamma_{\alpha \beta}) \circ \pi \right) V_\gamma = J[X^h, Y^h]_E. \]

\[ \square \]
Lemma 4.26. Let \( h \) be a horizontal endomorphism on \( L^\pi E \) and \( X, Y \in \Gamma(E) \). Then

\[
\tau(X^h, Y^h) = [X^h, Y^V]_E - [Y^h, X^V]_E - [X, Y]^V.
\] (63)

Proof. Using the definition of the weak torsion we have

\[
\tau(X^h, Y^h) = [J, h]E^{-N}(X^h, Y^h) = [JX^h, hY^h]_E + [hX^h, JY^h]_E
+ Jh[X^h, Y^h]_E + hJ[X^h, Y^h]_E - J[X^h, hY^h]_E
- J[hX^h, Y^h]_E - h[X^h, JY^h]_E - h[JX^h, Y^h]_E.
\]

Using \( JX^h = X^h \), (i) of (60) and (ii), (iv) of (29) in the above equation, we get

\[
\tau(X^h, Y^h) = [X^h, Y^h]_E + [X^h, Y^V]_E - J[X^h, hY^h]_E
- h[X^h, Y^h]_E - h[X^V, Y^h]_E. \] (64)

But we can obtain

\[
[X^h, Y^V]_E = \left( (X^o \rho^i_\alpha \frac{\partial Y^\beta}{\partial x^i}) \circ \pi - ((X^o \gamma^i) \circ \pi) \frac{\partial B^\beta}{\partial y^i} \right) V^\beta.
\]

Therefore \( h[X^h, Y^V]_E = 0 \). Similarly we have \( h[X^V, Y^h]_E = 0 \). Setting this equation and (iii) of (65) in (64) we obtain (63).

\[
\text{Lemma 4.26.}
\]

Proposition 4.27. If \( h \) and \( \tilde{h} \) are homogenous horizontal endomorphisms on \( L^\pi E \) such that

\[
[X^h, Y^V]_E = [X^h, Y^V]_E, \quad \forall X, Y \in \Gamma(E),
\] (65)

then \( h = \tilde{h} \).

Proof. Let \( h = (X\alpha + B^\beta_\alpha V^\beta) \otimes X^\beta \) and \( \tilde{h} = (X\alpha + B^\beta_\alpha V^\beta) \otimes X^\beta \). Since \( h \) and \( \tilde{h} \) are homogenous, then we have

\[
B^\beta_\alpha = y^\gamma \frac{\partial B^\beta_\alpha}{\partial y^\gamma}, \quad B^\beta_\alpha = y^\gamma \frac{\partial B^\beta_\alpha}{\partial y^\gamma}\] (66)

Setting \( X = e_\alpha \) and \( Y = e_\beta \) in (65), we have \([e^h_\alpha, e^V_\beta]_E = [e^\tilde{h}_\alpha, e^V_\beta]_E\). This equation gives us

\[
\frac{\partial B^\beta_\alpha}{\partial y^\gamma} = \frac{\partial B^\beta_\alpha}{\partial y^\gamma}.
\]

Contracting the above equation by \( y^\gamma \) and using (66) we deduce \( B^\beta_\alpha = B^\beta_\alpha \) and consequently \( h = \tilde{h} \).

We set \( \delta_\alpha = e^h_\alpha \). Then we have \( \delta_\alpha = X\alpha + B^\beta_\alpha V^\beta = h(X\alpha) \). It is easy to see that \( h\delta_\alpha = \delta_\alpha \), \( v\delta_\alpha = 0 \) and

\[
\rho\xi(h\delta_\alpha) = \left( \rho^i_\alpha \circ \pi \right) \frac{\partial}{\partial x^i} + B^\beta_\alpha \frac{\partial}{\partial y^\gamma}.
\] (67)

Moreover, \( \{\delta_\alpha\} \) generate a basis of \( hL^\pi E \) and the frame \( \{\delta_\alpha, V_\alpha\} \) is a local basis of \( L^\pi E \) adapted to splitting (29) which is called adapted basis. The dual adapted basis is \( \{X^\alpha, \delta^V_\alpha\} \), where

\[
\delta^V_\alpha = \gamma^\alpha - B^\beta_\alpha X^\beta.
\]
Proposition 4.28. The Lie brackets of the adapted basis \( \{ \delta_\alpha, V_\alpha \} \) are

\[
[\delta_\alpha, \delta_\beta]_\mathcal{L} = (L^\gamma_{\alpha\beta} \circ \pi) \delta_\gamma + R^\gamma_{\alpha\beta} V_\gamma, \quad [\delta_\alpha, V_\beta]_\mathcal{L} = \frac{\partial B^\gamma_{\beta\alpha}}{\partial y^\gamma} V_\gamma, \quad [V_\alpha, V_\beta]_\mathcal{L} = 0,
\]

where \( R^\gamma_{\alpha\beta} \) is given by (69).

Using (30) and (52), \( h \) and \( F \) have the following coordinate expressions with respect to adapted basis

(i) \( h = \delta_\alpha \otimes X^\alpha \), \quad F = -V_\alpha \otimes X^\alpha + \delta_\alpha \otimes \delta V^\alpha.

5 Distinguished connections on Lie algebroids

This section is appertained to constructing distinguished connections on Lie algebroids. Intrinsic \( v \)-connections and Berwald-type and Yano-type connections are also studied. Ultimately, The Douglas tensor of a Berwald endomorphism based on Yano connection is introduced.

A linear connection on a Lie algebroid \( (\mathcal{E}, [\cdot, \cdot]_\mathcal{E}, \rho) \) is a map

\[
D : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})
\]

which satisfies the rules

\[
D_{fX}Y + D_XZ = fD_XY + D_YZ, \\
D_X(fY + Z) = (\rho(X)f)Y + fD_XY + D_XZ,
\]

for any function \( f \in C^\infty(M) \) and \( X, Y, Z \in \Gamma(\mathcal{E}) \).

**Definition 5.1.** Let \( D \) be a linear connection on \( \mathcal{E} \) and \( h \) be a horizontal endomorphism on \( \mathcal{E} \). Then \( (D, h) \) is called a distinguished connection (or \( d \)-connection) on \( \mathcal{E} \), if

i) \( D \) is reducible, i.e., \( Dh = 0 \),

ii) \( D \) is almost complex, i.e., \( DF = 0 \),

where \( F \) is the almost complex structure associated by \( h \).

**Lemma 5.2.** If \( D \) is reducible respect to \( h \), then we have

(i) \( D_XhY = hD_XY \in h\mathcal{E} \), \quad (ii) \( D_XvY = vD_XY \in v\mathcal{E} \),

where \( X \) and \( Y \) are sections of \( \mathcal{E} \).

**Proof.** Since \( Dh = 0 \), then we have

\[
0 = Dh(X, Y) = D_XhY - hD_XY,
\]

which gives us (i). Similarly we can prove (ii).

Since \( \text{Im}h = h\mathcal{E} \) and \( \text{Im}v = v\mathcal{E} \), then we have

**Corollary 5.3.** If \( \tilde{Y} \) and \( \tilde{Z} \) are sections of \( v\mathcal{E} \) and \( h\mathcal{E} \), respectively, then we have \( D_X\tilde{Y} \in v\mathcal{E} \) and \( D_X\tilde{Z} \in h\mathcal{E} \).
Lemma 5.4. If the linear connection $D$ is almost complex on $\mathcal{L}^\pi E$, then $D$ is determined on $\mathcal{L}^\pi E \times v\mathcal{L}^\pi E$, completely.

Proof. From $DF = 0$, we deduce $D_{\tilde{X}}F\tilde{Y} = FD_{\tilde{X}}\tilde{Y}$, for all $\tilde{X}, \tilde{Y} \in \Gamma(\mathcal{L}^\pi E)$. Thus we have

$$D_{v\tilde{X}}h\tilde{Y} = D_{v\tilde{X}}FJ\tilde{Y} = FD_{v\tilde{X}}J\tilde{Y},$$

(71)

$$D_{h\tilde{X}}h\tilde{Y} = D_{h\tilde{X}}FJ\tilde{Y} = FD_{h\tilde{X}}J\tilde{Y}.$$  

(72)

Lemma 5.5. If $(D, h)$ is a d-connection, then $DJ = 0$.

Proof. Let $\tilde{X}$ and $\tilde{Y}$ be sections of $\mathcal{L}^\pi E$ and $v\mathcal{L}^\pi E$, respectively. Then from the above lemma we have $D_{\tilde{X}}\tilde{Y} \in \Gamma(v\mathcal{L}^\pi E)$. Thus, since $ImJ = v\mathcal{L}^\pi E$, then we have $J\tilde{Y} = 0$ and $JD_{\tilde{X}}\tilde{Y} = 0$. Therefore we obtain

$$DJ(\tilde{Y}, \tilde{X}) = D_{\tilde{X}}J\tilde{Y} - JD_{\tilde{X}}\tilde{Y} = 0.$$  

$$D_{\tilde{X}}J\tilde{Y} = JD_{\tilde{X}}\tilde{Y} = 0.$$  

$$D_{\tilde{X}}J\tilde{Y} = JD_{\tilde{X}}\tilde{Y} = 0.$$  

Using (ii) of Lemma 5.2 we have $D_{\delta_{\tilde{X}}}V_{\beta} \in v\mathcal{L}^\pi E$ and $D_{V_{\alpha}}V_{\beta} \in v\mathcal{L}^\pi E$. Thus these have the following coordinate expressions

$$D_{\delta_{\tilde{X}}}V_{\beta} = F_{\alpha\beta}^\gamma V_{\gamma}, \quad D_{V_{\alpha}}V_{\beta} = C_{\alpha\beta}^\gamma V_{\gamma}.$$  

(73)

From (69), (72) and the above equation we obtain

$$D_{h\delta_{\tilde{X}}}h_{\beta} = D_{h\delta_{\tilde{X}}}h_{\beta} = F_{\alpha\beta}J_{\gamma} = FD_{h\delta_{\tilde{X}}}V_{\beta} = F_{\alpha\beta}^\gamma h_{\gamma}. $$

(74)

Similarly (69), (71) and (73) imply that

$$D_{v\delta_{\tilde{X}}}h_{\beta} = C_{\alpha\beta}^\gamma h_{\gamma}. $$

(75)

Definition 5.6. Let $(D, h)$ be a d-connection. Then

$$\begin{align*}
&D^h : \Gamma(\mathcal{L}^\pi E) \times \Gamma(\mathcal{L}^\pi E) \to \Gamma(\mathcal{L}^\pi E) \\
&(\tilde{X}, \tilde{Y}) \mapsto D_{h\tilde{X}}\tilde{Y} := D_{h\tilde{X}}\tilde{Y} \\
\end{align*}$$

and

$$\begin{align*}
&D^v : \Gamma(\mathcal{L}^\pi E) \times \Gamma(\mathcal{L}^\pi E) \to \Gamma(\mathcal{L}^\pi E) \\
&(\tilde{X}, \tilde{Y}) \mapsto D_{v\tilde{X}}\tilde{Y} := D_{v\tilde{X}}\tilde{Y} \\
\end{align*}$$

are called $h$-covariant derivative and $v$-covariant derivative, respectively. Moreover,

$$\begin{align*}
&h^*(DC) : \Gamma(\mathcal{L}^\pi E) \to \Gamma(\mathcal{L}^\pi E) \\
&\tilde{X} \mapsto DC(h\tilde{X}) := D_{h\tilde{X}}C \\
\end{align*}$$

(76)

and

$$\begin{align*}
&v^*(DC) : \Gamma(\mathcal{L}^\pi E) \to \Gamma(\mathcal{L}^\pi E) \\
&\tilde{X} \mapsto DC(v\tilde{X}) := D_{v\tilde{X}}C \\
\end{align*}$$

are called $h$-deflection and $v$-deflection of $(D, h)$, respectively.
Similarly we obtain
\[ D^h_{\delta_\alpha} \delta_\beta = F^\gamma_{\alpha\beta} \delta_\gamma, \quad D^h_{\delta_a} \mathcal{V}_\beta = F^\gamma_{\alpha\beta} \mathcal{V}_\gamma, \quad D^h_{\delta_a} \delta_\beta = D^h_{\delta_a} \mathcal{V}_\beta = 0. \]  
(77)

Similarly we obtain
\[ D^\nu_{\delta_\alpha} \delta_\beta = C^\gamma_{\alpha\beta} \delta_\gamma, \quad D^\nu_{\delta_a} \mathcal{V}_\beta = C^\gamma_{\alpha\beta} \mathcal{V}_\gamma, \quad D^\nu_{\delta_a} \delta_\beta = D^\nu_{\delta_a} \mathcal{V}_\beta = 0. \]  
(78)

Using (73), (74) and (75) we get
\[ h^*(DC)(\delta_\alpha) = D_{\delta_\alpha} C = D_{\delta_a} (\nu^\gamma \mathcal{V}_\beta) = \rho(\delta_\alpha) (\nu^\gamma) \mathcal{V}_\beta + \nu^\gamma D_{\delta_a} \mathcal{V}_\beta = (B^\alpha_{\gamma} + \nu^\gamma F^\gamma_{\alpha\beta}) \mathcal{V}_\gamma, \]
and \( h^*(DC)(\mathcal{V}_\alpha) = 0. \) Therefore \( h^*(DC) \) has the following coordinate expression:
\[ h^*(DC) = (B^\alpha_{\gamma} + \nu^\gamma F^\gamma_{\alpha\beta}) \mathcal{V}_\gamma \otimes \mathcal{X}^\alpha. \]  
(79)

Similarly, we can see that \( \nu^*(DC) \) has the following coordinate expression:
\[ \nu^*(DC) = (\delta^\alpha_{\gamma} + \nu^\gamma C^\gamma_{\alpha\beta}) \mathcal{V}_\gamma \otimes \delta \mathcal{X}^\alpha, \]  
(80)

where \( \delta^\alpha_{\gamma} \) is the Kronicher symbole.

**Theorem 5.7.** Let \( (D, h) \) be a d-connection on \( \mathcal{L}^\infty E \). Then the torsion tensor field \( T \) of \( D \) determined by the following, completely:

\[ A(\tilde{X}, \tilde{Y}) : = hT(h\tilde{X}, h\tilde{Y}) = D_{h\tilde{X}} h\tilde{Y} - D_{h\tilde{Y}} h\tilde{X} - \Pi_{h\tilde{X}, h\tilde{Y}]}, \]  
(81)
\[ B(\tilde{X}, \tilde{Y}) : = hT(h\tilde{X}, h\tilde{Y}) = -D_{h\tilde{Y}} h\tilde{X} - h[h\tilde{X}, h\tilde{Y}], \]  
(82)
\[ R^1(\tilde{X}, \tilde{Y}) : = \nu T(h\tilde{X}, h\tilde{Y}) = -\nu h[h\tilde{X}, h\tilde{Y}], \]  
(83)
\[ P^1(\tilde{X}, \tilde{Y}) : = \nu T(h\tilde{X}, h\tilde{Y}) = D_{h\tilde{X}} \mathcal{Y} - v[h\tilde{X}, h\tilde{Y}], \]  
(84)
\[ S^1(\tilde{X}, \tilde{Y}) : = \nu T(h\tilde{X}, h\tilde{Y}) = D_{h\tilde{X}} \mathcal{Y} - \Pi_{h\tilde{X}, h\tilde{Y}], \]  
(85)

where \( A, B, R^1, P^1 \) and \( R^1 \) are called h- horizontal, h- mixed, v- horizontal, v- mixed and v- vertical torsion, respectively.

**Proof.** We have
\[ hT(\tilde{X}, \tilde{Y}) = hT(h\tilde{X}, h\tilde{Y}) + hT(h\tilde{X}, v\tilde{Y}) + hT(v\tilde{X}, h\tilde{Y}) + hT(v\tilde{X}, v\tilde{Y}) = A(\tilde{X}, \tilde{Y}) + B(\tilde{X}, \tilde{Y}) - B(\tilde{X}, \tilde{Y}) + hT(v\tilde{X}, v\tilde{Y}). \]
It is easy to check that \( T(v\tilde{X}, v\tilde{Y}) \in v\mathcal{L}^\infty E \) and consequently \( hT(v\tilde{X}, v\tilde{Y}) = 0. \)

Therefore we obtain
\[ hT(\tilde{X}, \tilde{Y}) = A(\tilde{X}, \tilde{Y}) + B(\tilde{X}, \tilde{Y}) - B(\tilde{Y}, \tilde{X}). \]  
(86)

Similarly we get
\[ \nu T(\tilde{X}, \tilde{Y}) = P^1(\tilde{X}, \tilde{Y}) + R^1(\tilde{X}, \tilde{Y}) + S^1(\tilde{Y}, \tilde{X}) - P^1(\tilde{Y}, \tilde{X}). \]  
(87)

Summing (86) and (87) we conclude that the torsion \( T \) of \( D \) completely determined by (81)–(85).
Similarly, using (90) we deduce that

\begin{align*}
A &= T^\gamma_{\alpha\beta} \delta \otimes X^\alpha \otimes X^\beta, \\
B &= -C^\gamma_{\alpha\beta} \delta \otimes X^\alpha \otimes X^\beta,
\end{align*}

where

\begin{align*}
(i) \ T^\gamma_{\alpha\beta} &= F^\gamma_{\alpha\beta} - F^\gamma_{\beta\alpha} - (L^\gamma_{\alpha\beta} \circ \pi), \\
(ii) \ P^\gamma_{\alpha\beta} &= F^\gamma_{\alpha\beta} + \frac{\partial B^\gamma_{\alpha\beta}}{\partial y^\lambda}, \\
(iii) \ S^\gamma_{\alpha\beta} &= C^\gamma_{\alpha\beta} - C^\gamma_{\beta\alpha}.
\end{align*}

(88)

**Theorem 5.8.** Let \((D, h)\) be a d-connection on \(E^\pi E\). Then the curvature tensor field \(K\) of \(D\) completely determined by the following

\begin{align*}
(i) \ R(\tilde{\chi}, \tilde{\gamma})\tilde{Z} &= \tilde{K}(h\tilde{\chi}, h\tilde{\gamma})\tilde{J}\tilde{Z}, \\
(ii) \ P(\tilde{\chi}, \tilde{\gamma})\tilde{Z} &= \tilde{K}(h\tilde{\chi}, J\tilde{\gamma})\tilde{J}\tilde{Z}, \\
(iii) \ Q(\tilde{\chi}, \tilde{\gamma})\tilde{Z} &= \tilde{K}(J\tilde{\chi}, J\tilde{\gamma})\tilde{J}\tilde{Z}.
\end{align*}

\(R, P\) and \(Q\) are called horizontal, mixed and vertical curvature, respectively.

**Proof.** Since \(D\) is a d-connection, then we have

\[ D_{\chi} J\tilde{Y} = JD_{\chi}\tilde{Y}, \quad D_{\chi} F\tilde{Y} = FD_{\chi}\tilde{Y}. \]

From the above relation we get

\begin{align*}
JK(\tilde{\chi}, \tilde{\gamma})\tilde{Z} &= \tilde{K}(h\tilde{\chi}, \tilde{\gamma})\tilde{J}\tilde{Z}, \\
FK(\tilde{\chi}, \tilde{\gamma})\tilde{Z} &= \tilde{K}(h\tilde{\chi}, J\tilde{\gamma})\tilde{J}\tilde{Z}, \\
JFK(\tilde{\chi}, \tilde{\gamma})\tilde{Z} &= \tilde{K}(J\tilde{\chi}, J\tilde{\gamma})\tilde{J}\tilde{Z}.
\end{align*}

(90)

Therefore using (i), (iii) of (51) we obtain

\begin{align*}
hK(\tilde{\chi}, \tilde{\gamma})\tilde{Z} &= FJK(\tilde{\chi}, \tilde{\gamma})\tilde{J}\tilde{Z} = FK(\tilde{\chi}, \tilde{\gamma})\tilde{J}\tilde{Z} = \tilde{K}(h\tilde{\chi}, h\tilde{\gamma})\tilde{J}\tilde{Z} \\
&\quad + FK(h\tilde{\chi}, v\tilde{\gamma})\tilde{J}\tilde{Z} + FK(v\tilde{\chi}, v\tilde{\gamma})\tilde{J}\tilde{Z} + FK(v\tilde{\chi}, h\tilde{\gamma})\tilde{J}\tilde{Z} \\
&= FR(\tilde{\chi}, \tilde{\gamma})\tilde{Z} + FP(\tilde{\chi}, F\tilde{\gamma})\tilde{Z} + FQ(\tilde{\chi}, F\tilde{\gamma})\tilde{Z} = FP(\tilde{\gamma}, F\tilde{\chi})\tilde{Z}.
\end{align*}

Similarly, using (89) we deduce

\[ vK(\tilde{\chi}, \tilde{\gamma})\tilde{Z} = R(\tilde{\chi}, \tilde{\gamma})F\tilde{Z} + P(\tilde{\chi}, F\tilde{\gamma})F\tilde{Z} + Q(F\tilde{\chi}, F\tilde{\gamma})F\tilde{Z} = P(\tilde{\gamma}, F\tilde{\chi})F\tilde{Z}. \]

Summing two above equation we derive that \(K\) completely determined by \(R, P\) and \(Q\).

By a direct calculation, we can see that the horizontal, mixed and vertical curvature, have the following coordinate expressions:

\begin{align*}
R &= R^\lambda_{\alpha\beta\gamma} \lambda \ V_\lambda \otimes X^\alpha \otimes X^\beta \otimes X^\gamma, \\
P &= P^\lambda_{\alpha\beta\gamma} \lambda \ V_\lambda \otimes X^\alpha \otimes X^\beta \otimes X^\gamma, \\
Q &= Q^\lambda_{\alpha\beta\gamma} \lambda \ V_\lambda \otimes X^\alpha \otimes X^\beta \otimes X^\gamma.
\end{align*}
where

\[
R_{\alpha\beta\gamma}^\lambda = (\rho^i_\alpha \circ \pi) \frac{\partial F_{\beta\gamma}^\lambda}{\partial x^i} + B^\delta_{\beta\gamma} \frac{\partial F_{\delta\alpha}^\lambda}{\partial y^\mu} - (\rho^j_\beta \circ \pi) \frac{\partial F_{\alpha\gamma}^\lambda}{\partial x^j} + B^\mu_{\beta\gamma} \frac{\partial F_{\alpha\mu}^\lambda}{\partial y^\nu} - F^\mu_{\alpha\gamma} F_{\beta\mu}^\lambda - (L^\mu_{\alpha\beta} \circ \pi) F_{\mu\gamma}^\lambda - R_{\alpha\beta\gamma}^\lambda C_{\mu\gamma}^\lambda
\]  

\eqref{91}

\[
P_{\alpha\beta\gamma}^\lambda = (\rho^i_\alpha \circ \pi) \frac{\partial C_{\beta\gamma}^\lambda}{\partial x^i} + B^\delta_{\beta\gamma} \frac{\partial C_{\delta\alpha}^\lambda}{\partial y^\mu} + C^\mu_{\beta\gamma} F_{\alpha\mu}^\lambda - \frac{\partial F_{\delta\gamma}^\lambda}{\partial y^\delta} - F^\mu_{\alpha\gamma} C_{\beta\mu}^\lambda + \frac{\partial B^\mu_{\alpha\gamma}}{\partial y^\mu} C_{\beta\mu}^\lambda.
\]  

\eqref{92}

\[
S_{\alpha\beta\gamma}^\lambda = \frac{\partial C_{\beta\gamma}^\lambda}{\partial y^\alpha} + C^\mu_{\beta\gamma} C_{\alpha\mu}^\lambda - \frac{\partial C_{\delta\gamma}^\lambda}{\partial y^\delta} - C^\mu_{\alpha\gamma} C_{\beta\mu}^\lambda.
\]  

\eqref{93}

**Definition 5.9.** Let \((D, h)\) be a d-connection on \(\mathcal{L}^\pi E\). Then the tensor field

\[
\begin{align*}
P_{\text{ric}} : \Gamma(\mathcal{L}^\pi E) \times \Gamma(\mathcal{L}^\pi E) &\rightarrow C^\infty(E), \\
(\tilde{X}, \tilde{Y}) &\rightarrow \text{tr}(F \circ (\tilde{Z} \rightarrow P(\tilde{Y}, \tilde{Z})\tilde{X})),
\end{align*}
\]

is called mixed Ricci tensor of d-connection \((D, h)\), where \(F\) is the almost complex structure associated to \(h\).

By a direct calculation we can see that the mixed Ricci tensor of \((D, h)\) has the following coordinate expression

\[
P_{\text{ric}} = P_{\alpha\beta} \mathcal{X}^\alpha \otimes \mathcal{X}^\beta,
\]

where \(P_{\alpha\beta} = P_{\alpha\beta\gamma}^\gamma\).

### 5.1 Intrinsic v-connections

**Definition 5.10.** The canonical map

\[
\begin{align*}
\hat{\delta} : \Gamma(\mathcal{L}^\pi E) \times \Gamma(\mathcal{L}^\pi E) &\rightarrow \Gamma(\mathcal{L}^\pi E), \\
(\tilde{J}\tilde{X}, \tilde{J}\tilde{Y}) &\rightarrow D_{\tilde{J}\tilde{X}}^\hat{\delta} \tilde{J}\tilde{Y} := [J, J\tilde{Y}]_{\mathcal{L}} - J[J, \tilde{Y}]_{\mathcal{L}} - J[J, \tilde{X}]_{\mathcal{L}} - J[J, \tilde{X}]_{\mathcal{L}} = 0.
\end{align*}
\]

is called intrinsic or the flat v-connection in \(v\mathcal{L}^\pi E\).

**Lemma 5.11.** Let \(\tilde{X}\) and \(\tilde{Y}\) be two section of \(\mathcal{L}^\pi E\). Then we have

\[
\hat{\delta}_{\tilde{J}\tilde{X}} \tilde{J}\tilde{Y} := J[\tilde{J}\tilde{X}, \tilde{J}\tilde{Y}]_{\mathcal{L}}, \quad \hat{\delta}_{\tilde{v}\tilde{X}} \tilde{J}\tilde{Y} := J[\tilde{v}\tilde{X}, \tilde{J}\tilde{Y}]_{\mathcal{L}}.
\]

**Proof.** From \(N_J = 0\), we obtain

\[
[J\tilde{X}, J\tilde{Y}]_{\mathcal{L}} - J[\tilde{X}, J\tilde{Y}]_{\mathcal{L}} - J[J, \tilde{X}]_{\mathcal{L}} - J[J, \tilde{Y}]_{\mathcal{L}} = 0.
\]

Therefore we get

\[
\hat{\delta}_{\tilde{J}\tilde{X}} \tilde{J}\tilde{Y} := [J, J\tilde{Y}]_{\mathcal{L}} - [J, \tilde{Y}]_{\mathcal{L}} - J[J, \tilde{X}]_{\mathcal{L}} = J[J, \tilde{Y}]_{\mathcal{L}}.
\]

Also since \(v = J \circ F\), then the above equation gives us

\[
\hat{\delta}_{\tilde{v}\tilde{X}} \tilde{J}\tilde{Y} = \hat{\delta}_{JF\tilde{X}} \tilde{J}\tilde{Y} = J[\tilde{J}F\tilde{X}, \tilde{J}\tilde{Y}]_{\mathcal{L}} = J[\tilde{v}\tilde{X}, \tilde{J}\tilde{Y}]_{\mathcal{L}}.
\]

\[\square\]
Let $D^v$ be the intrinsic $v$-connection. We consider the map
\[\tilde{D}^i: \Gamma(v\mathcal{L}^\pi E) \times \Gamma(\mathcal{L}^\pi E) \to \Gamma(v\mathcal{L}^\pi E)\]
defined by
\[\tilde{D}^i_j X\tilde{Y} = D^i_j X\tilde{Y}, \quad \tilde{D}^i_j h\tilde{Y} = F^i D^i_j X\tilde{Y}.\]

It is easy to see that
\[\tilde{D}^i_j h\tilde{Y} = h[Jh\tilde{X}, \tilde{Y}]_E, \quad \tilde{D}^i_j h\tilde{Y} = h[Jh\tilde{X}, \tilde{Y}]_E.\]  

(94)

**Theorem 5.12.** Let $(D, h)$ be a $d$-connection on $\mathcal{L}^\pi E$ and $\tilde{D}$ be given by [94]. If $\tilde{D}$ is the map
\[
\begin{cases}
\tilde{D}: \Gamma(\mathcal{L}^\pi E) \times \Gamma(\mathcal{L}^\pi E) \to \Gamma(\mathcal{L}^\pi E), \\
\tilde{\pi}(X, Y) \to \tilde{D}\tilde{X}\tilde{Y} := D^i h\tilde{X} + D^c h\tilde{Y},
\end{cases}
\]
then $(\tilde{D}, h)$ is a $d$-connection on $\mathcal{L}^\pi E$, which is called the $d$-connection associated to $(D, h)$.

**Proof.** At first we show that $\tilde{D}$ is a linear connection on $\mathcal{L}^\pi E$. Let $f \in C^\infty(M)$. Then we have $D^i h\tilde{X} f\tilde{Y} = \rho_L(h\tilde{X})(f)\tilde{Y} + fD^c h\tilde{Y}$, because $D$ is a linear connection. Direct calculations give us
\[\tilde{D}^{vc} f\tilde{Y} = D^{vc} v\tilde{Y} + D^{vc} f\tilde{Y} = D^{vc} h\tilde{Y} + D^{vc} fJF\tilde{Y}
= h[v\tilde{X}, f\tilde{Y}]_E + J[v\tilde{X}, f\tilde{Y}]_E = h[\rho_L(v\tilde{X})(f)\tilde{Y} + f[v\tilde{X}, \tilde{Y}]_E]
+ J[\rho_L(v\tilde{X})(f)\tilde{Y} + f[v\tilde{X}, \tilde{Y}]_E] = \rho_L(v\tilde{X})(f)\tilde{Y} + fh[v\tilde{X}, \tilde{Y}]_E
+ \rho_L(v\tilde{X})(f)v\tilde{Y} + fJ[v\tilde{X}, F\tilde{Y}]_E = \rho_L(v\tilde{X})(f)(\tilde{Y}) + fD^{vc} \tilde{Y}.
\]

Therefore we have
\[\tilde{D}^i f\tilde{Y} = \rho_L(h\tilde{X})(f)\tilde{Y} + \rho_L(v\tilde{X})(f)\tilde{Y} + fD^c h\tilde{Y} + fD^{vc} \tilde{Y} = \rho_L(h\tilde{X})(f)(\tilde{Y}) + fD^{vc} \tilde{Y}.
\]

Similarly we can prove
\[\tilde{D}^i(\tilde{Y} + \tilde{Z}) = \tilde{D}^i \tilde{Y} + \tilde{D}^i \tilde{Z}, \quad \tilde{D}_{f\tilde{X} + \tilde{Y}} \tilde{Z} = f\tilde{D}^i \tilde{Z} + \tilde{D}^i \tilde{Z}.
\]

Thus $\tilde{D}$ is a linear connection on $\mathcal{L}^\pi E$. Now, we show that $\tilde{D}$ is reducible. Since $D$ is reducible, then we have $Dh = 0$. So,
\begin{align*}
(\tilde{D}^i h)(\tilde{Y}) &= \tilde{D}^i \tilde{h}\tilde{Y} - h\tilde{D}^i \tilde{Y} = D^i h\tilde{Y} + D^{vc} h\tilde{Y} - hD^c h\tilde{Y} - hD^{vc} h\tilde{Y} \\
&= (D^i h)(\tilde{Y}) + D^{vc} h\tilde{Y} - hD^{vc} h\tilde{Y} - hD^{vc} h\tilde{Y} \\
&= vD^{vc} h\tilde{Y} - hD^{vc} h\tilde{Y} \\
&= vh[v\tilde{X}, \tilde{Y}]_E + hj[v\tilde{X}, F\tilde{Y}]_E = 0.
\end{align*}

Similarly, we can show that $\tilde{D}F = 0$, i.e., $\tilde{D}$ is an almost complex connection. Therefore $(\tilde{D}, h)$ is a $d$-connection on $\mathcal{L}^\pi E$. 

\[31\]
Let \( \bar{X} = \bar{X}^\alpha \delta_\alpha + \bar{X}^\alpha V_\alpha \) and \( Y = \bar{Y}^\beta \delta_\beta + \bar{Y}^\beta V_\beta \) are sections of \( L^\pi E \) and \( (F^\alpha_{\alpha \beta}, C^\gamma_{\alpha \beta}) \) are the local coefficients of d-connection \( D \). Using (67), (94) and (95) we deduce the following coordinate expression for \( \bar{D} \):

\[
\bar{D}_\bar{X} \bar{Y} = \left( \bar{X}^\alpha (\rho^\alpha_\alpha \circ \pi) \frac{\partial \bar{Y}^\beta}{\partial \bar{x}^j} + \bar{X}^\alpha B^\alpha_\alpha \frac{\partial \bar{Y}^\beta}{\partial \bar{y}^\gamma} + \bar{X}^\alpha \bar{Y}^\gamma F^\beta_{\alpha \gamma} + \bar{X}^\alpha \frac{\partial \bar{Y}^\beta}{\partial \bar{y}^\alpha} \right) \delta_\beta \\
+ \left( \bar{X}^\alpha (\rho^\lambda_\alpha \circ \pi) \frac{\partial F^\lambda_{\alpha \gamma}}{\partial \bar{x}^j} + \bar{X}^\alpha B^\alpha_\alpha \frac{\partial F^\lambda_{\alpha \gamma}}{\partial \bar{y}^\gamma} + \bar{X}^\alpha \bar{Y}^\gamma F^\beta_{\alpha \gamma} + \bar{X}^\alpha \frac{\partial F^\lambda_{\alpha \gamma}}{\partial \bar{y}^\alpha} \right) Y_\beta. \tag{96}
\]

If we denote the local coefficients of d-connection \( \bar{D} \) by \( (\bar{F}^\gamma_{\alpha \beta}, \bar{C}^\gamma_{\alpha \beta}) \), then from the above equation we conclude \( \bar{F}^\gamma_{\alpha \beta} = F^\gamma_{\alpha \beta} \) and \( \bar{C}^\gamma_{\alpha \beta} = 0 \). Therefore using (91), (92) and (93) we derive that

\[
\bar{R}^\lambda_{\alpha \beta \gamma} = (\rho^\lambda_\alpha \circ \pi) \frac{\partial F^\lambda_{\alpha \gamma}}{\partial \bar{x}^j} + B^\alpha_\alpha \frac{\partial F^\lambda_{\beta \gamma}}{\partial \bar{y}^\mu} - (\rho^\lambda_\beta \circ \pi) \frac{\partial F^\lambda_{\alpha \gamma}}{\partial \bar{x}^j} - B^\lambda_\beta \frac{\partial F^\lambda_{\alpha \gamma}}{\partial \bar{y}^\gamma} + F^\mu_{\beta \gamma} F^\lambda_{\alpha \mu} \\
- F^\mu_{\beta \gamma} F^\lambda_{\alpha \mu} - (L^\mu_{\alpha \beta} \circ \pi) F^\lambda_{\gamma \iota}, \\
\bar{P}^\lambda_{\alpha \beta \gamma} = - \frac{\partial F^\lambda_{\alpha \gamma}}{\partial \bar{y}^j}, \quad \bar{S}^\lambda_{\alpha \beta \gamma} = 0, \tag{97}
\]

where \( \bar{R}^\lambda_{\alpha \beta \gamma}, \bar{P}^\lambda_{\alpha \beta \gamma} \) and \( \bar{S}^\lambda_{\alpha \beta \gamma} \) are the coefficients of the horizontal, mixed and vertical curvatures of d-connection \( (\bar{D}, \bar{h}) \), respectively. Therefore the vertical curvature of d-connection \( \bar{D} \) is vanished. Also, it is easy to see that

**Proposition 5.13.** The mixed curvature \( \bar{P} \) of \( \bar{D} \) satisfies

\[
\bar{P} (X^C, Y^C) Z^C = - |J, D_X^C Z^C| Z^C - N Y^C.
\]

### 5.2 Berwald-type connection

Let \( h \) be a horizontal endomorphism on \( L^\pi E \). Then the map

\[
\begin{aligned}
\bar{D} &\colon \Gamma(L^\pi E) \times \Gamma(L^\pi E) \to \Gamma(L^\pi E), \\
(\bar{X}, \bar{Y}) &\to \bar{D}_{\bar{X}} \bar{Y},
\end{aligned}
\]

defined by

\[
\bar{D}_{\bar{X}} \bar{Y} := hF[h\bar{X}, J\bar{Y}]_E + v[h\bar{X}, v\bar{Y}]_E + h[v\bar{X}, \bar{Y}]_E + J[v\bar{X}, F\bar{Y}]_E
\]

is a linear connection on \( L^\pi E \). Similar to \( \bar{D} \), we can prove that \( \bar{D} h = \bar{D} F = 0 \). Therefore \( (\bar{D}, h) \) is a d-connection, which is called the **Berwald-type connection**.

If, in particular, \( h \) is a Berwald endomorphism, then we call \( (\bar{D}, h) \) a **Berwald connection**.

It is easy to see that

\[
\begin{aligned}
\bar{D}_{h_{\alpha}} \delta_\beta &= - \frac{\partial \bar{h}^\gamma_{\beta}}{\partial \bar{y}^\gamma} \delta_\alpha, & \bar{D}_{v_{\alpha}} v_\beta &= 0, \\
\bar{D}_{v_{\alpha}} \delta_\beta &= - \frac{\partial \bar{h}^\gamma_{\alpha}}{\partial \bar{y}^\gamma} v_\beta, & \bar{D}_{v_{\alpha}} v_\beta &= 0. \tag{98}
\end{aligned}
\]
If we denote the local coefficients of Berwald connection \( \hat{D} \) by \( (\hat{F}_{\alpha\beta}^\gamma, \hat{C}_{\alpha\beta}) \), then from the above equation we conclude \( \hat{F}_{\alpha\beta}^\gamma = -\frac{\partial \alpha \beta \gamma}{\partial y_i} \) and \( \hat{C}_{\alpha\beta} = 0 \). Therefore using (91), (92) and (93) we derive that

\[
\hat{R}_{\alpha\beta\gamma}^\lambda = -\left( \hat{\rho}_{\beta}^\lambda \circ \pi \right) \frac{\partial^2 \hat{B}_{\alpha}^\lambda}{\partial x^i \partial y_i} - \hat{B}_{\beta}^\rho \frac{\partial^2 \hat{B}_{\alpha}^\lambda}{\partial y^\rho \partial y_i} + \left( \hat{\rho}_{\beta}^\lambda \circ \pi \right) \frac{\partial^2 \hat{B}_{\alpha}^\lambda}{\partial x^i \partial y^\rho} + \hat{B}_{\beta}^\rho \frac{\partial^2 \hat{B}_{\alpha}^\lambda}{\partial y^\rho \partial y_i} + \hat{B}_{\beta}^\rho \frac{\partial^2 \hat{B}_{\alpha}^\lambda}{\partial y^\rho \partial y_i} + (L_{\alpha\beta}^\mu \circ \pi) \frac{\partial \hat{B}_{\alpha}^\lambda}{\partial y^\rho}, \quad (99)
\]

\[
\hat{P}_{\alpha\beta\gamma}^\lambda \equiv \frac{\partial^2 \hat{B}_{\alpha}^\lambda}{\partial y^\beta \partial y^\gamma}, \quad (100)
\]

\[
\hat{S}_{\alpha\beta\gamma}^\lambda = 0, \quad (101)
\]

where \( \hat{R}_{\alpha\beta\gamma}^\lambda \), \( \hat{P}_{\alpha\beta\gamma}^\lambda \) and \( \hat{S}_{\alpha\beta\gamma}^\lambda \) are the coefficients of the horizontal, mixed and vertical curvatures of d-connection \( (\hat{D}, \hat{h}) \), respectively. Therefore the vertical curvature of d-connection \( \hat{D} \) vanishes.

**Proposition 5.14.** Let \( (\hat{D}, \hat{h}) \) be the Berwald-type connection. Then

(i) The \( h \)-deflection of \( (\hat{D}, \hat{h}) \) coincides with the tension of \( h \).

(ii) The torsion tensor field \( \hat{T} \) of \( \hat{D} \) can be represented in the form

\[
\hat{T} = F \circ t + \Omega, \quad (102)
\]

where \( t \) and \( \Omega \) are the weak torsion and the curvature of \( h \).

**Proof.** (i) Let \( \hat{X} = \hat{X}^\alpha \delta_\alpha + \hat{X}^\alpha \hat{V}_\alpha \) be a section of \( \hat{E}^* \hat{E} \). Then using (22), (31), (67) and (76) we have

\[
h^* (DC)(\hat{X}) = D_{\hat{X}^\alpha \delta_\alpha + \hat{X}^\alpha \hat{V}_\alpha} (y^\beta \hat{V}_\beta) = \hat{X}^\alpha (\hat{B}^\gamma_\alpha - y^\beta \frac{\partial \hat{B}^\gamma_\beta}{\partial y^\beta}) \hat{V}_\gamma = H(\hat{X}).
\]

(ii) Using (82), (84), (85), (87) and (88) we obtain

\[
\hat{T} (\delta_\alpha, \delta_\beta) = \left( \frac{\partial \hat{B}^\gamma_\alpha}{\partial y^\beta} - (L_{\alpha\beta}^\mu \circ \pi) \right) \delta_\gamma - \hat{R}_{\alpha\beta\gamma}^\lambda \hat{V}_\gamma = F t(\delta_\alpha, \delta_\beta) + \Omega(\delta_\alpha, \delta_\beta),
\]

\[
\hat{T} (\delta_\alpha, \hat{V}_\beta) = 0 = F t(\delta_\alpha, \hat{V}_\beta) + \Omega(\delta_\alpha, \hat{V}_\beta),
\]

\[
\hat{T} (\hat{V}_\alpha, \hat{V}_\beta) = 0 = F t(\hat{V}_\alpha, \hat{V}_\beta) + \Omega(\hat{V}_\alpha, \hat{V}_\beta).
\]

\( \square \)

Similar to Lemma 3.7 we can prove

**Lemma 5.15.** A section \( \tilde{X} = \tilde{X}^\alpha \delta_\alpha + \tilde{X}^\alpha \hat{V}_\alpha \) is homogenous of degree \( v \) if and only if

\[
y^\gamma \frac{\partial \tilde{X}^\beta}{\partial y^\gamma} = (r - 1) \tilde{X}^\beta, \quad y^\gamma \frac{\partial \tilde{X}^\beta}{\partial y^\gamma} + \tilde{X}^\gamma (y^\rho \frac{\partial \hat{B}^\beta_\gamma}{\partial y^\rho} - \hat{B}^\beta_\gamma) = r \tilde{X}^\beta.
\]
**Proposition 5.16.** The mixed curvature \( \bar{P} \) of Berwald-type connection \( \bar{D} \) is symmetric with respect to last two variables. Moreover, if \( h \) is torsion free, then \( \bar{P} \) is symmetric with respect to all variables.

**Proof.** Equation (100) told us that \( P_{\alpha\beta\gamma}^\lambda \) is symmetric with respect to last two indices. Therefore \( \bar{P} \) is symmetric with respect to last two variables. Now, let \( h \) be torsion free. Then using (33) we obtain

\[
\bar{P}_{\alpha\beta\gamma}^\lambda = \frac{\partial^2 B_{\alpha}^\lambda}{\partial y^\alpha \partial y^\gamma} = \frac{\partial}{\partial y^\gamma} \left( \frac{\partial B_{\alpha}^\lambda}{\partial y^\gamma} + \left( L_{\alpha}^\beta \circ \pi \right) \right) = \frac{\partial^2 B_{\beta}^\lambda}{\partial y^\alpha \partial y^\gamma} = \bar{P}_{\beta\alpha\gamma}^\lambda.
\]

Similarly, we can prove \( \bar{P}_{\alpha\beta\gamma}^\lambda = \bar{P}_{\gamma\beta\alpha}^\lambda \).

**Proposition 5.17.** Let \( h \) be a homogenous horizontal endomorphism on \( \pi^* E \). Then the mixed curvature \( \bar{P} \) of \( (\bar{D}, h) \) is homogenous of degree \(-1\). Moreover if the weak torsion of \( h \) is zero, then for any semispray \( S \) we have \( i_S \bar{P} = 0 \).

**Proof.** To proof the first part of proposition, using the above lemma, we must show \( y^\mu \frac{\partial \bar{P}_{\alpha\beta\gamma}^\lambda}{\partial y^\mu} = - \bar{P}_{\alpha\beta\gamma}^\lambda \). Since \( h \) is homogenous, then we have \( y^\beta \frac{\partial B_{\alpha}^\lambda}{\partial y^\beta} = B_{\alpha}^\lambda \). Differentiating with respect to \( y^\gamma \) we obtain

\[
y^\beta \frac{\partial^2 B_{\alpha}^\lambda}{\partial y^\mu \partial y^\gamma} = 0.
\]

Differentiating (103) with respect to \( y^\mu \) gives us

\[
y^\beta \frac{\partial^3 B_{\alpha}^\lambda}{\partial y^\mu \partial y^\gamma^2} = - \frac{\partial^2 B_{\alpha}^\lambda}{\partial y^\mu \partial y^\gamma}.
\]

Therefore we have

\[
y^\mu \frac{\partial \bar{P}_{\alpha\beta\gamma}^\lambda}{\partial y^\mu} = y^\beta \frac{\partial^3 B_{\alpha}^\lambda}{\partial y^\mu \partial y^\gamma^2} = - \frac{\partial^2 B_{\alpha}^\lambda}{\partial y^\mu \partial y^\gamma} = - \bar{P}_{\alpha\beta\gamma}^\lambda.
\]

Now, we proof the second part of assertion. From the above Proposition, we deduce that \( \bar{P} \) is symmetric with respect to all variables. Thus we have \( (i_S \bar{P})(\bar{X}, \bar{Y}) = \bar{P}(\bar{X}, S)\bar{Y} \). Thus using (100) and (103) we get

\[
(i_S \bar{P})(\bar{X}, \bar{Y}) = \bar{X}^\alpha \gamma^\beta \gamma^\gamma \bar{P}_{\alpha\beta\gamma}^\lambda \mathcal{V}_\lambda = \bar{X}^\alpha \gamma^\beta \gamma^\gamma \frac{\partial^2 B_{\alpha}^\lambda}{\partial y^\gamma^2} = 0,
\]

where \( \bar{X} = \bar{X}^\alpha \Delta_\alpha + \bar{X}^\alpha \mathcal{V}_\alpha, \bar{Y} = \bar{Y}^\beta \Delta_\beta + \bar{Y}^\beta \mathcal{V}_\beta \).

**Proposition 5.18.** The mixed curvature \( \bar{P} \) of Berwald-type connection \( D^0 \) satisfies

\[
\bar{P}(X^C, Y^C)Z^C = [X^h, Y^V]_E, Z^V_\|E.
\]
Proof. Let $X = X^\alpha e_\alpha$, $Y = Y^\beta e_\beta$ and $Z = Z^\gamma e_\gamma$ are sections of $E$. Then we can obtain
\[
\bar{P}(X^C,Y^C)Z^C = ((X^\alpha Y^\beta Z^\gamma) \circ \pi) \frac{\partial^2 B^\lambda}{\partial y^\beta \partial y^\gamma} V_\lambda = [[X^h,Y^V],Z^V]_E.
\]

Proposition 5.19. Let $h$ be a homogenous horizontal endomorphism on $L^\pi E$. The mixed Ricci tensor $\bar{P}_{ric}$ of Berwald-type connection $(D,h)$ is homogenous of degree $-1$. Moreover, we have
\[
\frac{n}{\bar{L}_C} \bar{P}_{ric}=\frac{n}{D_C} \bar{P}_{ric}=-\bar{P}_{ric}.
\]

Proof. Using (100) and (104) we have
\[
y^\lambda \frac{\partial P_{\alpha\gamma\beta}}{\partial y^\gamma} = y^\lambda \frac{\partial^2 B^\gamma}{\partial y^\beta \partial y^\gamma} = - \frac{\partial^2 B^\gamma}{\partial y^\beta \partial y^\gamma} = - \frac{n}{P_{\alpha\gamma\beta}}.
\]
Thus from Lemma 5.15 we deduce $-\frac{n}{\bar{P}_{ric}}$ is homogenous of degree $-1$. Also, using (105) we get $D_C \bar{P}_{ric}(\delta_\alpha,\nu_\beta) = D_C \bar{P}_{ric}(\nu_\alpha,\delta_\beta) = 0$ and
\[
(D_C \bar{P}_{ric})(\delta_\alpha,\delta_\beta) = \frac{n}{\bar{P}_{ric}} \bar{D}_C \bar{P}_{ric}(\delta_\alpha,\delta_\beta) = y^\lambda \frac{\partial P_{\alpha\beta}}{\partial y^\gamma} = - \frac{n}{P_{\alpha\beta}}.
\]
Therefore we deduce $D_C \bar{P}_{ric} = - \bar{P}_{ric}$. Similarly we have $(\frac{n}{L_C} \bar{P}_{ric})(\delta_\alpha,\nu_\beta) = (\frac{n}{L_C} \bar{P}_{ric})(\nu_\alpha,\delta_\beta) = 0$ and
\[
(\frac{n}{L_C} \bar{P}_{ric})(\delta_\alpha,\delta_\beta) = \bar{C}(\bar{P}_{ric}(\delta_\alpha,\delta_\beta)) = y^\lambda \frac{\partial P_{\alpha\beta}}{\partial y^\gamma} = - \bar{P}_{ric}.
\]
Therefore $(\frac{n}{L_C} \bar{P}_{ric}) = - \bar{P}_{ric}$.

Proposition 5.20. Let $(D,h)$ be a Berwald-type $d$-connection and $K \in \Gamma(\wedge^k E^* \otimes E)$ be a semibasic. Then
\[
D_X K = \frac{n}{L_X} K, \quad \forall X \in \Gamma(E).
\]

Proof. 
\[
(L_X K)(\delta_{a_1}, \ldots, \delta_{a_k}) = (\bar{L}_X K)(\delta_{a_1}, \ldots, \delta_{a_k}) = [X^V,K(e^h_{a_1}, \ldots, e^h_{a_k})]_E - \sum_{i} K(e^h_{a_1}, \ldots, [X^V,e^h_{a_i}], \ldots, e^h_{a_k})
\]
\[
= [X^V,K(e^h_{a_1}, \ldots, e^h_{a_k})]_E - [JFK(e^h_{a_1}, \ldots, e^h_{a_k}),X^V]_E = [JX^V,F^{-N}(FK(X^h_{a_1}, \ldots, X^h_{a_k})) - J[FK(X^h_{a_1}, \ldots, X^h_{a_k}),X^V]_E
\]
\[
= J[X^C,K(e^h_{a_1}, \ldots, e^h_{a_k})]_E = \frac{n}{D_X} K(e^h_{a_1}, \ldots, e^h_{a_k})
\]
\[
= (D_X K)(e^h_{a_1}, \ldots, e^h_{a_k}) + \sum_{i} K(e^h_{a_1}, \ldots, D_X e^h_{a_i}, \ldots, e^h_{a_k})
\]
\[
= (D_X K)(e^h_{a_1}, \ldots, e^h_{a_k})
\]
\[
= (D_X K)(\delta_{a_1}, \ldots, \delta_{a_k}).
\]

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5.3 Yano-type connection

Let \( h \) be a horizontal endomorphism on \( \mathcal{L}^\pi E \) with associated almost complex structure \( F \) and \( \omega \in \Gamma(\wedge^2(\mathcal{L}^\pi E)^*) \) be a symmetric tensor, satisfying the condition

\[
\iota_S \omega = 0,
\]

where \( S \) is an arbitrary semispray on \( \mathcal{L}^\pi E \). We define the mapping

\[
D : \Gamma(\mathcal{L}^\pi E) \times \Gamma(\mathcal{L}^\pi E) \to \Gamma(\mathcal{L}^\pi E),
\]

by the following rules:

\[
\begin{align*}
D_{v\tilde{X}}v\tilde{Y} &= J[v\tilde{X}, F\tilde{Y}]_E = D_{v\tilde{X}}v\tilde{Y}, \\
D_{h\tilde{X}}v\tilde{Y} &= v[h\tilde{X}, v\tilde{Y}]_E + \omega(\tilde{X}, F\tilde{Y})\tilde{U} = D_{h\tilde{X}}v\tilde{Y} + \omega(\tilde{X}, F\tilde{Y})\tilde{U}, \\
D_{v\tilde{X}}h\tilde{Y} &= h[v\tilde{X}, \tilde{Y}]_E = D_{v\tilde{X}}h\tilde{Y}, \\
D_{h\tilde{X}}h\tilde{Y} &= hF[h\tilde{X}, J\tilde{Y}]_E + \omega(\tilde{X}, \tilde{Y})F\tilde{U} = D_{h\tilde{X}}h\tilde{Y} + \omega(\tilde{X}, \tilde{Y})F\tilde{U},
\end{align*}
\]

where \( \tilde{U} \) is a nonzero section of \( v\mathcal{L}^\pi E \).

\[
D_{\tilde{X}}\tilde{Y} = hF[h\tilde{X}, J\tilde{Y}]_E + v[h\tilde{X}, v\tilde{Y}]_E + h[v\tilde{X}, \tilde{Y}]_E + J[v\tilde{X}, F\tilde{Y}]_E + \omega(\tilde{X}, \tilde{Y})F\tilde{U} = D_{h\tilde{X}}h\tilde{Y} + \omega(\tilde{X}, \tilde{Y})F\tilde{U}.
\]

It is easy to see that \( (D, h) \) is a d-connection on \( \mathcal{L}^\pi E \). In the coordinate expression we have

\[
\begin{align*}
D_{\delta} \delta &= (\omega_{\alpha\beta}\tilde{U}^\gamma - \frac{\partial\gamma}{\partial\eta})\delta, & D_{\gamma} V_\gamma &= 0, \\
D_{\beta} V_\beta &= (\omega_{\alpha\beta}\tilde{U}^\gamma - \frac{\partial\gamma}{\partial\eta})V_\gamma, & D_{\gamma} \delta &= 0,
\end{align*}
\]

and consequently

\[
D_{\tilde{X}}\tilde{Y} = \left( \tilde{X}^\alpha \{(\rho_\alpha^0 \circ \pi) \frac{\partial\tilde{Y}^\gamma}{\partial x^\alpha} + B^\alpha_\beta \frac{\partial\tilde{Y}^\gamma}{\partial y^\beta} \} + \tilde{X}^\alpha \tilde{Y}^\beta \{- \frac{\partial B^\gamma_\alpha}{\partial y^\beta} + \omega_\alpha\beta\tilde{U}^\gamma \} + \tilde{X}^\alpha \frac{\partial\tilde{Y}^\gamma}{\partial y^\alpha} \right) \delta, \\
&+ \left( \tilde{X}^\alpha \{(\rho_\alpha^0 \circ \pi) \frac{\partial\tilde{Y}^\gamma}{\partial x^\alpha} + B^\alpha_\beta \frac{\partial\tilde{Y}^\gamma}{\partial y^\beta} \} + \tilde{X}^\alpha \tilde{Y}^\beta \{- \frac{\partial B^\gamma_\alpha}{\partial y^\beta} + \omega_\alpha\beta\tilde{U}^\gamma \} + \tilde{X}^\alpha \frac{\partial\tilde{Y}^\gamma}{\partial y^\alpha} \right) V_\gamma,
\]

where \( \tilde{X} = \tilde{X}^\alpha \delta_\alpha + \tilde{X}^\alpha V_\alpha, \tilde{Y} = \tilde{y}^\beta \delta_\beta + \tilde{Y}^\beta V_\beta, \tilde{U} = \tilde{U}^\gamma V_\gamma \) and \( \omega_\alpha\beta = \omega(\delta_\alpha, \delta_\beta) \).

**Remark 5.21.** From [108] and [109] we deduce that \( \omega(h\tilde{X}, v\tilde{Y})\tilde{U} = \omega(v\tilde{X}, v\tilde{Y})\tilde{U} = 0 \). Therefore we have \( \omega(\delta_\alpha, V_\beta)\tilde{U} = \omega(V_\alpha, V_\beta)\tilde{U} = 0 \).

**Theorem 5.22.** Let \( (D, h) \) be the d-connection given by [107] – [109]. Then

(i) the v-mixed torsion of \( D \) is \( P^1 = \omega \otimes \tilde{U} \),

(ii) the h-mixed torsion \( B \) of \( D \) vanishes.

Moreover, if

(iii) the h-deflection of \( (D, h) \) vanishes,

(iv) the h-horizontal torsion of \( D \) vanishes, then the horizontal endomorphism \( h \) is homogeneous and torsion free.
Proof. Using Remark 5.21 and (112) we get

\[ P^1(\delta_\alpha, \delta_\beta) = \omega_{\alpha\beta} \tilde{U} \tilde{V} = \omega(\delta_\alpha, \delta_\beta) \tilde{U}, \]
\[ P^1(\delta_\alpha, \nu_\beta) = 0 = \omega(\delta_\alpha, \nu_\beta) \tilde{U}, \]
\[ P^1(\nu_\alpha, \nu_\beta) = 0 = \omega(\nu_\alpha, \nu_\beta) \tilde{U}. \]

Therefore we have (i). Also, using (112) we deduce \( B(\delta_\alpha, \delta_\beta) = 0 \) that gives us (ii). Now let (iii) and (iv) hold. (iii) gives us

\[ B^\gamma_\alpha + y^\beta \omega_{\alpha\beta} \tilde{U} \tilde{V} - y^\beta \partial B^\gamma_\alpha \partial y^\beta = 0. \]

But from the condition \( i_S \omega = 0 \) we derive that \( y^\beta \omega_{\alpha\beta} = 0 \). Setting this in the above equation we have

\[ B^\gamma_\alpha = y^\beta \partial B^\gamma_\alpha \partial y^\beta, \]

i.e., \( h \) is homogenous. (iv) gives us

\[ 0 = \partial B^\gamma_\beta \partial y^\alpha - \partial B^\gamma_\alpha \partial y^\beta - (L^\gamma_\alpha \circ \pi) = t^\gamma_\alpha. \]

Therefore \( h \) is torsion free.

Here, let \( h \) be a homogenous and torsion free horizontal endomorphism on \( L^*E \) and \( \hat{P} \) be the mixed Ricci tensor of the Berwald-type connection \((D, h)\). From Proposition 5.17 we can deduce that \( i_S P^\text{ric} = 0 \).

Replacing \( \omega \) and \( \tilde{U} \) in (107)-(110) by \( \frac{1}{n+1} \hat{P} \) and Liouville section \( C \), respectively, where \( n = \text{rank}E \), we have the following d-connection

\[ \gamma \}
\]
\( C_{\alpha\beta\gamma} = 0 \). Therefore using (111), (112) and (113) we derive that

\[
{\gamma} R_{\alpha\beta\gamma} = (\rho_\alpha \circ \pi) \left( \frac{1}{n+1} \frac{\partial^3 B_\beta^\mu}{\partial x^\alpha \partial y^\nu \partial y^\gamma} y^\lambda - \frac{\partial^2 B_\beta^\lambda}{\partial x^\alpha \partial y^\gamma} - \frac{1}{n+1} \frac{\partial^2 B_\beta^\lambda}{\partial y^\nu \partial y^\gamma} \frac{\partial B_\beta^\mu}{\partial y^\gamma} \right) + \frac{1}{n+1} \frac{\partial^2 B_\beta^\mu}{\partial y^\nu \partial y^\gamma} \frac{\partial B_\beta^\mu}{\partial y^\gamma} y^\lambda
\]

\[
+ B_\alpha^\mu \left( \frac{1}{n+1} \frac{\partial^3 B_\beta^\nu}{\partial x^\alpha \partial y^\mu \partial y^\gamma} y^\lambda - \frac{\partial^2 B_\beta^\nu}{\partial x^\alpha \partial y^\gamma} + \frac{1}{n+1} \frac{\partial^2 B_\beta^\nu}{\partial y^\mu \partial y^\gamma} \frac{\partial B_\beta^\nu}{\partial y^\gamma} y^\lambda \right) + \frac{1}{n+1} \frac{\partial^2 B_\beta^\nu}{\partial y^\mu \partial y^\gamma} \frac{\partial B_\beta^\nu}{\partial y^\gamma} y^\lambda
\]

\[
= (\rho_\beta \circ \pi) \left( \frac{\partial^2 B_\alpha^\lambda}{\partial x^\beta \partial y^\gamma} - \frac{1}{n+1} \frac{\partial^2 B_\alpha^\mu}{\partial x^\beta \partial y^\nu \partial y^\gamma} y^\lambda - \frac{1}{n+1} \frac{\partial^2 B_\alpha^\mu}{\partial y^\nu \partial y^\gamma} \frac{\partial B_\alpha^\mu}{\partial y^\gamma} y^\lambda \right)
\]

\[
+ B_\alpha^\nu \left( \frac{\partial^2 B_\beta^\nu}{\partial x^\alpha \partial y^\mu \partial y^\gamma} y^\lambda - \frac{1}{n+1} \frac{\partial^2 B_\beta^\nu}{\partial x^\alpha \partial y^\gamma} + \frac{1}{n+1} \frac{\partial^2 B_\beta^\nu}{\partial y^\mu \partial y^\gamma} \frac{\partial B_\beta^\nu}{\partial y^\gamma} y^\lambda \right) + \frac{1}{n+1} \frac{\partial^2 B_\beta^\nu}{\partial y^\mu \partial y^\gamma} \frac{\partial B_\beta^\nu}{\partial y^\gamma} y^\lambda
\]

\[
= (I_{\alpha\beta}^\mu \circ \pi) \left( \frac{\partial^2 B_\alpha^\mu}{\partial x^\beta \partial y^\gamma} - \frac{1}{n+1} \frac{\partial^2 B_\alpha^\mu}{\partial y^\nu \partial y^\gamma} y^\lambda \right), \tag{118}
\]

\[
P_{\alpha\beta\gamma} = \frac{\partial^2 B_\alpha^\lambda}{\partial y^\beta \partial y^\gamma} - \frac{1}{n+1} \left( \frac{\partial^2 B_\alpha^\mu}{\partial y^\nu \partial y^\gamma} \frac{\partial^2 B_\alpha^\nu}{\partial y^\gamma} + \frac{\partial^2 B_\alpha^\mu}{\partial y^\nu \partial y^\gamma} \frac{\partial^2 B_\alpha^\nu}{\partial y^\gamma} \right) y^\lambda, \tag{119}
\]

\[
\tilde{S}_{\alpha\beta\gamma} = 0, \tag{120}
\]

where \( \tilde{R}_{\alpha\beta\gamma} \), \( \tilde{P}_{\alpha\beta\gamma} \), and \( \tilde{S}_{\alpha\beta\gamma} \) are the coefficients of the horizontal, mixed and vertical curvatures of Yano-type connection \( \tilde{D} \), respectively. Therefore the vertical curvature of \( d \)-connection \( \tilde{D} \) is vanished.

From theorem 5.22 we have

**Corollary 5.23.** Let \( (\tilde{D}, h) \) be the Yano-type \( d \)-connection. Then

(i) the \( v \)-mixed torsion of \( \tilde{D} \) is \( P^\lambda = \frac{1}{n+1} (P_{\text{ric}} \otimes C) \),

(ii) the \( h \)-mixed torsion \( \tilde{P} \) of \( \tilde{D} \) vanishes.

**Proposition 5.24.** Let \( h \) be a torsion free, homogeneous horizontal endomorphism. If \( (\tilde{D}, h) \) and \( (D, h) \) are the induced Berwald-type and Yano-type connections with mixed curvature and mixed Ricci tensors \( \tilde{P}, P \) and \( \tilde{P}_{\text{ric}}, P_{\text{ric}} \), respectively, then we have

\[
P = \tilde{P} = \frac{n}{n+1} D_{\text{J}} P_{\text{ric}} \otimes C - \frac{1}{n+1} P_{\text{ric}} \otimes J. \tag{121}
\]

\[
P_{\text{ric}} = \frac{2}{n+1} \tilde{P}_{\text{ric}}, \tag{122}
\]

where \( n = \text{rank} E \).

**Proof.** Using (111) we can obtain

\[
(P_{\text{ric}} = \frac{1}{n+1} D_{\text{J}} P_{\text{ric}} \otimes C - \frac{1}{n+1} P_{\text{ric}} \otimes J)(\delta_\alpha, \delta_\beta, \delta_\gamma)
\]

\[
= \left( \frac{\partial^2 B_\alpha^\lambda}{\partial y^\nu \partial y^\gamma} - \frac{1}{n+1} \frac{\partial^2 B_{\alpha}^\mu}{\partial y^\nu \partial y^\gamma} \delta_\beta^\mu - \frac{1}{n+1} \frac{\partial^2 B_{\alpha}^\mu}{\partial y^\nu \partial y^\gamma} \delta_\gamma^\mu \right) \nu_\lambda. \tag{123}
\]
Since $h$ is torsion free, then we get
\[ \frac{\partial^3 B^\mu}{\partial y^\alpha \partial y^\mu \partial y^\gamma} = \frac{\partial^3 B^\mu}{\partial y^\beta \partial y^\mu \partial y^\gamma}. \]

Setting the above equation in (100) and using (119) we deduce (121). To prove the (122) we let $\tilde{P}_{\alpha\beta}$ and $\bar{P}_{\alpha\gamma}$ be the coefficients of mixed Ricci tensors of Berwald-type and Yano-type connections, respectively. Then using (119) we obtain
\[ \tilde{P}_{\alpha\gamma} = \frac{\partial^3 B^\lambda}{\partial y^\alpha \partial y^\mu \partial y^\gamma} = \frac{1}{n+1} \left( \frac{\partial^2 B^\lambda}{\partial y^\alpha \partial y^\gamma} - \frac{\partial^2 B^\mu}{\partial y^\beta \partial y^\gamma} y^\lambda \right), \]
where $n = \text{rank} E$. Since $h$ is homogenous, then we have (104). Setting (104) in the above equation and using (100) we get
\[ \bar{P}_{\alpha\gamma} = \frac{2}{n+1} \frac{\partial^2 B^\lambda}{\partial y^\alpha \partial y^\gamma} = \frac{2}{n+1} P_{\alpha\gamma}. \]

5.3.1 The Douglas tensor of a Berwald endomorphism

Let $h$ be a Berwald endomorphism on the manifold $\mathcal{E}$. If $(\tilde{D}, h)$ is the Yano connection induced by $h$ and $\bar{D}$ is the mixed curvature of $\tilde{D}$, then the tensor
\[ \mathbf{D} = \bar{D} - \frac{1}{2}(\tilde{P}_{ric} \otimes J + \bar{P}_{ric}), \]
is said to be the Douglas tensor of the Berwald endomorphism. Using (21) and (119), the Douglas tensor $\mathbf{D}$ has the following coordinate expression:
\[ \mathbf{D} = D^\lambda_{\alpha\beta\gamma} X^\alpha \otimes X^\beta \otimes X^\gamma, \quad (124) \]
where
\[ D^\lambda_{\alpha\beta\gamma} = \frac{\partial^2 B^\lambda}{\partial y^\beta \partial y^\gamma} - \frac{1}{n+1} \left( \frac{\partial^2 B^\mu}{\partial y^\alpha \partial y^\gamma} \delta^\lambda_\mu - \frac{\partial^2 B^\mu}{\partial y^\alpha \partial y^\beta} y^\lambda + \frac{\partial^2 B^\mu}{\partial y^\beta \partial y^\gamma} \delta^\lambda_\mu + \frac{\partial^2 B^\mu}{\partial y^\gamma \partial y^\beta} \delta^\lambda_\mu \right). \quad (125) \]

Theorem 5.25. Let $\mathbf{D}$ be the Douglas tensor of a Berwald endomorphism. Then $i_s \mathbf{D} = 0$ and $\mathbf{D}_{ric} = 0$.

Proof. Let $\tilde{X} = \tilde{X}^\alpha \delta_\alpha + \tilde{X}^\alpha \nu_\alpha$ and $\tilde{Y} = \tilde{Y}^\gamma \delta_\gamma + \tilde{Y}^\gamma \nu_\gamma$. Since $\mathbf{D}$ is symmetric then using (124) we get
\[ (i_s \mathbf{D})(\tilde{X}, \tilde{Y}) = \mathbf{D}(\tilde{X}, S) \tilde{Y} = y^\beta \tilde{X}^\alpha \tilde{Y}^\gamma D^\lambda_{\alpha\beta\gamma}. \]
But using (103) and (104) we deduce $y^\beta D^\lambda_{\alpha\beta\gamma} = 0$. Therefore we have $i_s \mathbf{D} = 0$. Now we prove the second part of assertion. It is easy to see that
\[ \mathbf{D}_{ric} = D_{\alpha\gamma} X^\alpha \otimes X^\gamma, \]
where $D_{\alpha\gamma} = D^\lambda_{\alpha\gamma}$. But using (104) and (125) we deduce $D_{\alpha\gamma} = 0$ and consequently $\mathbf{D}_{ric} = 0$. \[ \square \]
Theorem 5.26. The Douglas tensor of a Berwald endomorphism is invariant under the projective changes of the associated spray.

Proof. Let $h$ be a Berwald endomorphism on $\mathcal{L}^n E$ with associated spray $S$ and $D$ be the Douglas tensor of $h$. Also, let $\tilde{S}$ be the projective change of $S$ by $\tilde{f}$. Then $\tilde{S}$ generates a Berwald endomorphism $\bar{h}$. Denote by $\bar{D}$ the Douglas tensor of $\bar{h}$. If $S = y^\alpha X_\alpha + S^\alpha V_\alpha$ and $\tilde{S} = y^\alpha X_\alpha + \tilde{S}^\alpha V_\alpha$, then $\tilde{S} = S + \tilde{f}C$ gives us

$$S^\alpha = S^\alpha + y^\alpha \tilde{f}.$$ (126)

From (55) and (56), $h$ and $\bar{h}$ have the following coordinate expressions:

$$h = (\lambda_\alpha + \mathcal{B}_\alpha^\gamma V_\gamma) \otimes \lambda^\alpha,$$ (127)

where

$$\mathcal{B}_\alpha^\gamma = \frac{1}{2}(\frac{\partial S^\gamma}{\partial y^\alpha} - y^\beta (L^\gamma_{\alpha\beta} \circ \pi)), \quad \bar{\mathcal{B}}_\alpha^\gamma = \frac{1}{2}(\frac{\partial \tilde{S}^\gamma}{\partial y^\alpha} - y^\beta (L^{\gamma}_{\alpha\beta} \circ \pi)).$$ (128)

Using (126) and (128) we get

$$\bar{\mathcal{B}}_\alpha^\gamma = \mathcal{B}_\alpha^\gamma + \tilde{f}_{\alpha}^\gamma,$$ (129)

where

$$\tilde{f}_{\alpha}^\gamma = \frac{1}{2}(\tilde{f} \delta_{\alpha}^\gamma + y^\beta \frac{\partial \tilde{f}}{\partial y^\alpha}).$$

If we denote by $D^\alpha_{\beta\gamma}$ and $\bar{D}^\alpha_{\beta\gamma}$ the coefficients of $D$ and $\bar{D}$, respectively, then using (125) and (129) we get

$$D^\alpha_{\beta\gamma} = \bar{D}^\alpha_{\beta\gamma} + \frac{\partial^2 \tilde{f}_{\alpha}}{\partial y^\beta \partial y^\gamma} - \frac{1}{n+1}(\frac{\partial^2 \tilde{f}_{\alpha}}{\partial y^\beta \partial y^\gamma} \delta_{\beta}^\gamma + \frac{\partial^2 \tilde{f}_{\beta}}{\partial y^\alpha \partial y^\gamma} \delta_{\alpha}^\gamma + \frac{\partial^3 \tilde{f}_{\beta}}{\partial y^\alpha \partial y^\gamma \partial y^\delta} \delta_{\alpha}^\delta).$$ (130)

Since $\tilde{f}$ is homogenous of degree 1, then we can obtain

$$\frac{\partial^3 \tilde{f}}{\partial y^\beta \partial y^\gamma \partial y^\alpha} y^\alpha = - \frac{\partial^2 \tilde{f}}{\partial y^\gamma \partial y^\alpha}.$$

The above equation and direct calculation give us

$$\frac{\partial^2 \tilde{f}_{\alpha}}{\partial y^\beta \partial y^\gamma} = \frac{1}{2}(\frac{\partial^2 \tilde{f}}{\partial y^\beta \partial y^\gamma} \delta_{\alpha}^\gamma + \frac{\partial^2 \tilde{f}}{\partial y^\gamma \partial y^\alpha} \delta_{\beta}^\gamma + \frac{\partial^2 \tilde{f}}{\partial y^\beta \partial y^\gamma} \delta_{\gamma}^\alpha + \frac{\partial^3 \tilde{f}}{\partial y^\beta \partial y^\gamma \partial y^\delta} \delta_{\gamma}^\delta),$$ (131)

$$\frac{\partial^2 \tilde{f}_{\alpha}}{\partial y^\beta \partial y^\gamma} = \frac{1}{2}(n+1) \frac{\partial^2 \tilde{f}}{\partial y^\gamma \partial y^\alpha},$$ (132)

$$\frac{\partial^2 \tilde{f}_{\alpha}}{\partial y^\beta \partial y^\gamma} = \frac{1}{2}(n+1) \frac{\partial^2 \tilde{f}}{\partial y^\gamma \partial y^\alpha},$$ (133)

$$\frac{\partial^2 \tilde{f}_{\beta}}{\partial y^\alpha \partial y^\gamma} = \frac{1}{2}(n+1) \frac{\partial^2 \tilde{f}}{\partial y^\gamma \partial y^\alpha},$$ (134)

$$\frac{\partial^3 \tilde{f}_{\beta}}{\partial y^\alpha \partial y^\gamma \partial y^\delta} = \frac{1}{2}(n+1) \frac{\partial^3 \tilde{f}}{\partial y^\gamma \partial y^\alpha \partial y^\beta}. $$ (135)

Setting (131)–(135) in (130) we obtain $\bar{D}^\alpha_{\beta\gamma} = D^\alpha_{\beta\gamma}$, i.e., $\bar{D} = D$. \[\square\]
6 $\rho_\pi$-covariant derivatives in $\pi^*\pi$

In this section, we investigate geometric properties of $\rho_\pi$-covariant derivatives in $\pi^*\pi$ like torsion and partial curvature. Results are in a deep relation with Berwald derivative.

We can deduce the following double-exact short sequence from the double-exact short sequence (19)

$$0 \longrightarrow \Gamma(\pi^*\pi) \xrightarrow{\bar{i}} \Gamma(\mathcal{E}E) \xrightarrow{\bar{j}} \Gamma(\pi^*\pi) \longrightarrow 0,$$

such that for every $\bar{X} \in \Gamma(\pi^*\pi)$ and $\xi \in \Gamma(\mathcal{E}E)$ the followings hold

$$\bar{i}(\bar{X}) := i \circ \bar{X}, \quad \bar{j}(\xi) := j \circ \xi, \quad \bar{\mathcal{H}}(\bar{X}) := \mathcal{H} \circ \bar{X}, \quad \bar{\nabla}(\xi) := \nabla \circ \xi. \quad (136)$$

**Proposition 6.1.** Let $X$ belongs to $\Gamma(E)$. Then we have the followings

(i) $\bar{i}(\bar{X}) = X^V$,
(ii) $\bar{j}(X^V) = 0$;
(iii) $\bar{j}(X^C) = \bar{X}$,
(iv) $\bar{\mathcal{H}}(\bar{X}) = X^h$;
(v) $\bar{\nabla}(X^V) = \bar{X}$,
(vi) $\bar{\nabla}(X^h) = 0$.

**Proof.** Let $u \in E$. Then we have

$$\bar{i}(\bar{X})(u) = i \circ \bar{X} = i(u, X(\pi(u))) = (0, X(\pi(u))^\pi)$$

$$= (0, X^V(u)) = X^V(u),$$

that gives us the first one. The second one is obvious. For the third, since $J(X^C) = X^V$, then we have $i \circ j(X^C) = X^V = i(\bar{X})$. Because $i$ is injective, $j(X^C) = \bar{X}$ and consequently $\bar{j}(X^C) = \bar{X}$. For the forth, we can deduce

$$\bar{\mathcal{H}}(\bar{X}) = \mathcal{H} \circ \bar{X} = F \circ i \circ \bar{X} = F \circ X^V = F \circ J(X^C) = h(X^C) = X^h.$$  

Using (133), the fifth equation proves as follows

$$\bar{\nabla}(X^V) = j \circ F \circ X^V = j \circ F \circ J(X^C) = j \circ h(X^C) = j \circ X^C = \bar{X}.$$  

The last one obvious.

**Remark 6.2.** The mapping $\bar{i}$ is an isomorphism between $\Gamma(\pi^*\pi)$ and $\Gamma(\mathcal{E}E)$. Thus every section of $\mathcal{E}E$ can be shown like $i\bar{X}$ where $\bar{X} \in \Gamma(\pi^*\pi)$. Moreover, since $\bar{j}$ is surjective, then every member of $\Gamma(\pi^*\pi)$ has the format $\bar{j}(\xi)$, where $\xi \in \Gamma(\mathcal{E}E)$.  

**Definition 6.3.** Operator $\nabla^v$ with properties

(i) $\nabla^v_h \tilde{f} := \rho_h(i\tilde{X})\tilde{f}$,

(ii) $\nabla^v_\bar{X} \bar{Y} := \bar{j}[(\bar{i} \bar{X}, \bar{\mathcal{H}} \bar{Y})_L]$,

(iii) $(\nabla^v_\bar{X} \bar{\alpha})(\bar{Y}) := \rho_h(i\tilde{X})(\bar{\alpha}(\bar{Y})) - \bar{\alpha}(\nabla^v_\bar{X} \bar{Y})$,

is called the canonical $v$-covariant differential, where $\tilde{f} \in \mathcal{C}^\infty(E)$, $\bar{X}, \bar{Y} \in \Gamma(\pi^*\pi)$, $\bar{\alpha} \in \Omega^1(\pi)$.
Remark 6.4. The second condition of the above definition is independent of choosing $\mathcal{H}$. Indeed since $j$ is surjective, there is some $\tilde{Y} \in \Gamma(\mathcal{L}^n E)$, such that $\bar{Y} = j\tilde{Y}$. Thus

$$\nabla_X^v j\tilde{Y} = j[\bar{i}\dot{X}, \mathcal{H} \circ j\tilde{Y}]_E = j[\bar{i}\dot{X}, h\tilde{Y}]_E.$$ 

But $[\bar{i}\dot{X}, h\tilde{Y}]_E$ is vertical. Therefore

$$\nabla_X^v j\tilde{Y} = j[\bar{i}\dot{X}, \tilde{Y}]_E.$$

Let $\bar{A} \in T^k_\pi$. Then we define

$$(\nabla^v_X \bar{A})(\alpha_1, \alpha_2, ..., \alpha_k, \bar{X}_1, \bar{X}_2, ..., \bar{X}_i) := \rho_E(\bar{i}\dot{X})(\alpha_1, \alpha_2, ..., \alpha_k, \bar{X}_1, \bar{X}_2, ..., \bar{X}_i)$$

$$- \sum_{i=1}^k \bar{A}(\alpha_1, \alpha_2, ..., \alpha_k, \bar{X}_1, \bar{X}_2, ..., \bar{X}_i)$$

$$- \sum_{i=1}^l \bar{A}(\alpha_1, \alpha_2, ..., \alpha_k, X_1, ..., \nabla^v_i X, ..., \bar{X}_i).$$

Moreover, for $\bar{A} \in T^k_{\pi} \pi$ tensor field $\nabla^v \bar{A} \in T^k_{\pi + 1}(\pi)$ is defined by the following rule

$$(\nabla^v_X \bar{A})(\bar{X}, \alpha_1, \alpha_2, ..., \alpha_k, \bar{X}_1, \bar{X}_2, ..., \bar{X}_i) := (\nabla^v_X \bar{A})(\alpha_1, \alpha_2, ..., \alpha_k, \bar{X}_1, \bar{X}_2, ..., \bar{X}_i).$$

Definition 6.5. Let $\tilde{f}$ be a smooth function on $E$. Then tensor field

$$\nabla^v \nabla^v \tilde{f} := \nabla^v (\nabla^v \tilde{f}) \in T^v_\pi(\pi),$$

is said to be hession of $\tilde{f}$.

Proposition 6.6. Function $\tilde{f} \in C^\infty(E)$ is homogenous of degree 1 if and only if $\nabla^v \tilde{f} = \tilde{f}$.

Proof. Let $\tilde{f}$ be a homogenous function of degree 1 on $E$. Then we have $\rho_E(C)\tilde{f} = \tilde{f}$. Thus

$$\nabla^v \tilde{f} = \rho_E(\bar{i}\dot{X})\tilde{f} = \rho_E(\bar{i} \circ \bar{\delta})\tilde{f} = \rho_E(C)\tilde{f} = \tilde{f}.$$ 

From the above equation, also we can deduce the convers of assertion. $\square$

Proposition 6.7. Let $X$ and $Y$ be sections of $E$ and $\tilde{f} \in C^\infty(E)$. Then

$$\nabla^v \nabla^v \tilde{f}(\dot{X}, \dot{Y}) = \rho_E(X^v)(\rho_E(Y^v)\tilde{f}).$$

Moreover, the hessian of $\tilde{f}$ is symmetric.

Proof. Using the definition of hessian of $\tilde{f}$, (i) of proposition 6.1 and (iii) of definition 6.3 we get

$$\nabla^v \nabla^v \tilde{f}(\dot{X}, \dot{Y}) = (\nabla^v_X (\nabla^v \tilde{f}))(\dot{Y}) = \rho_E(i\dot{X})(\rho_E(i\dot{Y})\tilde{f} - \nabla^v \tilde{f}(\nabla_X \dot{Y}))$$

$$= \rho_E(i\dot{X})(\rho_E(i\dot{Y})\tilde{f} - \rho_E(i\nabla_X \dot{Y})\tilde{f}$$

$$= \rho_E(X^v)(\rho_E(Y^v)\tilde{f}) - \rho_E(i\nabla_X \dot{Y})\tilde{f}.$$ (138)
But using (i), (ii) and (iv) of proposition 6.1 we deduce

$$\nabla^v_X Y = j[iX, \hat{Y}]_E = j[X^V, Y^h]_E = 0,$$

because $[X^V, Y^h]_E \in \Gamma(v L^x E)$. Plugging the above equation into (138) implies the first part of assertion. Now, we prove the second part of assertion. Since $[X^V, Y^V]_E = 0$, then using the first part of assertion we get

$$\nabla^v \nabla^v \tilde{f}(\hat{X}, \hat{Y}) = \rho_L(X^V)(\rho_L(Y^V)\tilde{f}) = \rho_L([X^V, Y^V]_E)(\tilde{f})$$

$$= \rho_L(Y^V)(\rho_L(X^V)\tilde{f})$$

$$= \nabla^v \nabla^v \tilde{f}(\hat{Y}, \hat{X}).$$

**Proposition 6.8.** Let $\tilde{f} \in C^\infty(E)$ be a homogenous function of degree 1. Then

$$\nabla^v_X (\nabla^v \nabla^v \tilde{f}) = -\nabla^v \nabla^v \tilde{f}.$$

**Proof.** Setting $\tilde{A} = \nabla^v \nabla^v \tilde{f}$, we must show $\nabla^v_X \tilde{A} = -\tilde{A}$. Let $X$ and $Y$ be sections of $E$. Then we have

$$(\nabla^v_X \tilde{A})(\hat{X}, \hat{Y}) = \rho_L(i \delta)A(\hat{X}, \hat{Y}) - A(\nabla^v_X \hat{X}, \hat{Y}) - \tilde{A}(\nabla^v_X \hat{Y}).$$

But using (ii) of definition 6.3 we deduce

$$\nabla^v_X \hat{X} = j[i \delta, \hat{H} \hat{Y}]_E = j[C, Y^h]_E = 0,$$

because $[C, Y^h]_E \in \Gamma(v L^x E)$. Similarly we have $\nabla^v_Y \hat{Y} = 0$. Therefore reduce to the following

$$(\nabla^v_X \tilde{A})(\hat{X}, \hat{Y}) = \rho_L(C)\tilde{A}(\hat{X}, \hat{Y}) = \rho_L(C)\left(\rho_L(X^V)(\rho_L(Y^V)\tilde{f})\right).$$

In other hand, using (ii) of (23) we get

$$\tilde{A}(\hat{X}, \hat{Y}) = \rho_L(X^V)(\rho_L(Y^V)\tilde{f}) = \rho_L([X^V, C]_E)(\rho_L(Y^V)\tilde{f})$$

$$= [\rho_L(X^V), \rho_L(C)](\rho_L(Y^V)\tilde{f}) = \rho_L(X^V)\left(\rho_L(C)(\rho_L(Y^V)\tilde{f})\right)$$

$$- \rho_L(C)\left(\rho_L(X^V)(\rho_L(Y^V)\tilde{f})\right) = \rho_L(X^V)\left(\rho_L(C)(\rho_L(Y^V)\tilde{f})\right)$$

$$+ \rho_L(Y^V)(\rho_L(C)\tilde{f}) - \rho_L(C)\left(\rho_L(X^V)(\rho_L(Y^V)\tilde{f})\right)$$

$$= \rho_L(X^V)\left(\rho_L[C, Y^V]_E \tilde{f} + \rho_L(Y^V)(\rho_L(C)\tilde{f})\right)$$

$$- \rho_L(C)\left(\rho_L(X^V)(\rho_L(Y^V)\tilde{f})\right).$$

Since $\tilde{f}$ is homogenous of degree 1, then we have $\rho_L(C)\tilde{f} = \tilde{f}$. Setting this in the above equation and using (ii) of (23) we get

$$\tilde{A}(\hat{X}, \hat{Y}) = -\rho_L(C)\left(\rho_L(X^V)(\rho_L(Y^V)\tilde{f})\right).$$

From (140) and (141) we have the assertion. □

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Definition 6.9. Let $h$ be a horizontal endomorphism and $\mathcal{H}$ be a horizontal map of $\pi$ associated to $h$. Operator $\nabla^h$ with properties

(i) $\nabla^h_X \tilde{f} = \rho_{\mathcal{E}}(\mathcal{H} \tilde{X}) \tilde{f}$,

(ii) $\nabla^h_X \tilde{Y} := \nabla[\mathcal{H} \tilde{X}, \tilde{Y}]_{\mathcal{E}}$,

(iii) $(\nabla^h_X \tilde{h})(Y) := \rho_{\mathcal{E}}(\mathcal{H} \tilde{X})(\tilde{h}(Y)) - \tilde{h}(\nabla^h_X Y)$,

is called the canonical $h$-covariant differential, where $\tilde{f} \in \mathcal{C}^\infty(E)$, $\tilde{X}, \tilde{Y} \in \Gamma(\pi^* \pi)$, $\tilde{h} \in \Omega^1(\pi)$.

Lemma 6.10. Let $H$ be the tension of $h$ and $\tilde{X}$ be a section of $\mathcal{L}^* E$. Then

$$(\nabla^h \delta)(j \tilde{X}) = \nabla H(\tilde{X}). \tag{142}$$

Proof. Using (ii) of the above definition we get

$$(\nabla^h \delta)(j \tilde{X}) = \nabla^h_{\tilde{X}} \delta = \nabla[\mathcal{H} j \tilde{X}, \delta]_{\mathcal{E}} = \rho_{\mathcal{E}}(\mathcal{H} j \tilde{X}, \tilde{X}) = \rho_{\mathcal{E}}(h, C)_{\mathcal{E}} = \rho_{\mathcal{E}}(\rho_{\mathcal{E}}(h, C)_{\mathcal{E}}) = \nabla H(\tilde{X}).$$

Since $\tilde{v} \delta = v$, then \((142)\) gives us

$$\tilde{v}(\nabla^h \delta)(j \tilde{X}) = v H(\tilde{X}) = H(\tilde{X}). \tag{143}$$

By reason of the above relation, the $(1, 1)$ tensor field $\mathcal{H} = \nabla^h \delta$ is called the tension of the horizontal map $\mathcal{H}$. Indeed, we have

$$\mathcal{H}(\tilde{X}) = \nabla[\mathcal{H} \tilde{X}, C]_{\mathcal{E}}, \quad \forall \tilde{X} \in \Gamma(\pi^* \pi). \tag{144}$$

Let $\bar{A} \in \mathcal{T}^k(\pi)$. Then we define

$$(\nabla^h_X \bar{A})(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_l) := \rho_{\mathcal{E}}(\mathcal{H} \tilde{X})(\tilde{h}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_l)$$

$$- \sum_{i=1}^k \bar{A}(\tilde{\alpha}_1, \ldots, \nabla^h_{\tilde{X}_1} \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k, \tilde{X}_2, \ldots, \tilde{X}_l)$$

$$- \sum_{i=1}^l \bar{A}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k, \tilde{X}_1, \ldots, \nabla^h_{\tilde{X}_2} \tilde{X}_1, \ldots, \tilde{X}_l).$$

Moreover, For $A \in \mathcal{T}^k(\pi)$ tensor field $\nabla^h \bar{A} \in \mathcal{T}^k(\pi)$ is defined by the following rule

$$(\nabla^h \bar{A})(\tilde{X}, \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_l) := (\nabla^h_X \bar{A})(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k, \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_l).$$

Now, we consider map

$$\begin{cases} D : \Gamma(\mathcal{L}^* E) \times \Gamma(\pi^* \pi) &\longrightarrow \Gamma(\pi^* \pi), \\
(\tilde{X}, \tilde{Y}) &\longmapsto D_{\tilde{X}} \tilde{Y}, \end{cases} \tag{145}$$

satisfies

(i) $D_{f \tilde{X} + \tilde{Y}} \tilde{Z} = fD_{\tilde{X}} \tilde{Z} + D_{\tilde{Y}} \tilde{Z}$. 

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(ii) \(D\tilde{f}Z = \tilde{f}D\tilde{X}Z + \rho_{\tilde{X}}(\tilde{X})(\tilde{f})Z,\)

(iii) \(D\tilde{X}(Z + W) = D\tilde{X}Z + D\tilde{X}W.\)

We call this map a \(\rho_{\tilde{X}}\)-covariant derivative in \(\Gamma(\pi^*\pi).\)

**Theorem 6.11.** Let \(h\) be a horizontal endomorphism and \(\mathcal{H}\) be a horizontal map of \(\pi\) associated to \(h.\) Then

\[
\nabla : \Gamma(L^\pi E) \times \Gamma(\pi^*\pi) \longrightarrow \Gamma(\pi^*\pi),
\]

given by

\[
\nabla_{\tilde{X}}\tilde{Y} := \nabla^u_{\tilde{X}}\tilde{Y} + \nabla^h_{\tilde{X}}\tilde{Y}, \tag{146}
\]

is a \(\rho_{\tilde{X}}\)-covariant derivative in \(\Gamma(\pi^*\pi),\) where \(\tilde{X} \in \Gamma(L^\pi E)\) and \(\tilde{Y} \in \Gamma(\pi^*\pi).\)

**Proof.** Let \(\tilde{f} \in C^\infty(E).\) Then we have

\[
\nabla_{\tilde{X}}\tilde{f}\tilde{Y} = \nabla^u_{\tilde{X}}\tilde{f}\tilde{Y} + \nabla^h_{\tilde{X}}\tilde{f}\tilde{Y} = \rho_{\tilde{X}}(\tilde{f})\tilde{Y} + \rho_{\tilde{X}}(\mathcal{H}\tilde{X})\tilde{f} + \tilde{f}\nabla^u_{\tilde{X}}\tilde{Y} + \tilde{f}\nabla^h_{\tilde{X}}\tilde{Y} = \rho_{\tilde{X}}(\tilde{f})\tilde{Y} + \rho_{\tilde{X}}(\mathcal{H}\tilde{X})\tilde{f} + \tilde{f}\nabla_{\tilde{X}}\tilde{Y}.
\]

It is easy to show that \(\tilde{f}\tilde{X} = v\tilde{X}\) and \(\mathcal{H}\tilde{X} = h\tilde{X}.)\) Therefore the above equation gives us

\[
\nabla_{\tilde{X}}\tilde{f}\tilde{Y} = \rho_{\tilde{X}}(v)\tilde{f}\tilde{Y} + \rho_{\tilde{X}}(h)\tilde{f}\tilde{Y} + \tilde{f}\nabla_{\tilde{X}}\tilde{Y} = \rho_{\tilde{X}}(\tilde{X})\tilde{f} + \tilde{f}\nabla_{\tilde{X}}\tilde{Y}.
\]

Similarly we can show \(\nabla_{\tilde{X}}(\tilde{Y} + Z) = \nabla_{\tilde{X}}\tilde{Y} + \nabla_{\tilde{X}}\tilde{Z}\) and \(\nabla_{\tilde{f}\tilde{X} + \tilde{Y}}\tilde{Z}\) \(= \tilde{f}\nabla_{\tilde{X}}\tilde{Z} + \nabla_{\tilde{Y}}\tilde{Z}\) Therefore \(\nabla\) is a \(\rho_{\tilde{X}}\)-covariant derivative in \(\Gamma(\pi^*\pi).\)

The \(\rho_{\tilde{X}}\)-covariant derivative \(\nabla\) introduced by the above theorem is called Berwald derivative generated by \(h.\) Indeed the Berwald derivative is as follows:

\[
\nabla_{\tilde{X}}\tilde{Y} = \tilde{j}[v\tilde{X}, \mathcal{H}\tilde{Y}]_E + \tilde{V}[h\tilde{X}, \tilde{Y}]_E, \quad \forall \tilde{X} \in \Gamma(L^\pi E), \quad \forall \tilde{Y} \in \Gamma(\pi^*\pi). \tag{147}
\]

Using the above equation we can obtain

\[
\nabla_{\tilde{X}}v\tilde{Y} = 0, \quad \nabla_{\tilde{X}}h\tilde{Y} = \tilde{V}[X^h, Y^V]_E, \tag{148}
\]

\[
\nabla_{\tilde{X}}\tilde{Y} = \tilde{j}[v\tilde{X}, \mathcal{H}\tilde{Y}]_E, \quad \nabla_{\mathcal{H}\tilde{X}}\tilde{Y} = \tilde{V}[\mathcal{H}\tilde{X}, \tilde{Y}]_E. \tag{149}
\]

where \(X\) and \(Y\) are sections of \(E\) and \(\tilde{X}, \tilde{Y} \in \Gamma(\pi^*\pi).\)

Now consider the local basis \(\{e_\alpha\} \) of \(\Gamma(E).\) Then \(\{e_\alpha\} \) is a basis of \(\Gamma(\pi^*\pi),\) where \(e_\alpha(u) = (u, e_\alpha(\pi(u))),\) for all \(u \in E.\) Using \([136],\) proposition \([137]\) and the definition of \(j,\) it is easy to check that

\[
\mathcal{H}e_\alpha = \delta_\alpha, \quad \tilde{V}e_\alpha = V_\alpha, \quad \tilde{j}(\delta_\alpha) = e_\alpha, \quad \tilde{V}(V_\alpha) = e_\alpha. \tag{150}
\]

Also we can deduce \(\tilde{V}(\delta_\alpha) = 0.\) Therefore using the above equation, \([136]\) and \([138]\) we obtain

\[
\nabla_{\delta_\alpha}e_\beta = \tilde{V}[\delta_\alpha, e^V_\beta]_E = \tilde{V}[\delta_\alpha, V_\beta]_E = -\frac{\partial B^\gamma_{\alpha \beta}}{\partial y^\gamma}e_\gamma, \tag{140}
\]

\[
\nabla_{V_\alpha}e_\beta = \tilde{j}[V_\alpha, e^h_\beta]_E = \tilde{j}[V_\alpha, \delta_\beta]_E = 0, \tag{141}
\]

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and consequently
\[\nabla_{\tilde{X}} \tilde{Y} = \left( \tilde{X}^\alpha ((p_\alpha^j \circ \pi) \frac{\partial \tilde{Y}^\beta}{\partial x^i} + B^\alpha_\beta \frac{\partial \tilde{Y}^\beta}{\partial y^\gamma}) - \tilde{X}^\alpha \tilde{Y}^\gamma \frac{\partial B^\alpha_\beta}{\partial y^\gamma} + \tilde{X}^\alpha \tilde{B}^\beta_\gamma \right) e^i, \tag{151}\]
where \(\tilde{X} = \tilde{X}^\alpha \delta_\alpha + \tilde{X}^\alpha \nu_\alpha \in \Gamma(\mathcal{L}^E)\) and \(\tilde{Y} = \tilde{Y}^\beta \epsilon_\beta \in \Gamma(\pi^* \pi)\).

**Definition 6.12.** A \(\rho_\mathcal{L}\)-covariant derivative operator \(D\) in \(\Gamma(\pi^* \pi)\) is said to be associated to the horizontal map \(\mathcal{H}\) if \(D\delta = \nu\).

**Lemma 6.13.** Let \(\nabla\) be the Berwald derivative induced by \(h\). Then
\[\nabla \delta = \check{H} \circ \check{j} + \check{\nu}. \tag{152}\]

**Proof.** Using (ii) of definition 6.3, (ii) of definition 6.9 and (144) we get
\[\langle \nabla \delta \rangle (\tilde{X}) = \nabla_{\nu \tilde{X}} \delta + \nu^h \frac{j}{j} \check{\nu} \frac{j}{j} \check{\delta} = \check{j} \left[ \nu \check{\tilde{X}}, \check{H} \delta \right]_{\mathcal{L}} + \check{H} (j \check{X}) = \check{j} [v \check{\tilde{X}}, \check{H} \delta]_{\mathcal{L}} + \check{H} (j \check{X}). \tag{153}\]

Now let \(\tilde{X} = \tilde{X}^\alpha \delta_\alpha + \tilde{X}^\alpha \nu_\alpha\). It is easy to see that \(\delta = y^\alpha \epsilon_\alpha\). Then using (150) we obtain
\[\check{j} [v \check{\tilde{X}}, \check{H} \delta]_{\mathcal{L}} = \check{j} [\tilde{X}^\alpha \nu_\alpha, y^\beta \delta_\beta]_{\mathcal{L}} = \tilde{X}^\alpha \check{j} (\delta_\alpha) = \tilde{X}^\alpha \epsilon_\alpha = \check{\nu} (\check{X}).\]

Setting the above equation in (153) implies (152). \(\square\)

**Proposition 6.14.** Let \(S\) be a spray on \(\mathcal{L}^E\) and \(h\) be the horizontal endomorphism generated by it. If \(\mathcal{H}\) be the horizontal map generated by \(h\) and \(\nabla\) be the Berwald derivative induced by \(h\), then \(\nabla_S \delta = 0\).

**Proof.** From the above lemma we have
\[\nabla_S \delta = \check{H} \check{j} (S) + \check{\nu} (S).\]

Using (150) it is easy to see that \(\check{j} S = y^\alpha \epsilon_\alpha = \delta\). Thus we have
\[\nabla_S \delta = \check{H} \delta + \check{\nu} (S). \tag{154}\]

But (144) gives us \(\check{H} \delta = \check{\nu} \left[ H \delta, C \right]_{\mathcal{L}}\). In other hand, from Corollary 4.22 we have \(hS = S\). Therefore we get
\[S = hS = H \check{j} S = H \delta,\]
and consequently \(H \delta = \check{\nu} [S, C]_{\mathcal{L}}\). Since \(S\) is a spray then \([S, C]_{\mathcal{L}} = -S\). Therefore \(H \delta = -\check{\nu} (S)\). Setting this equation in (154) we obtain \(\nabla_S \delta = 0\). \(\square\)

### 6.1 Torsions and partial curvatures

Let \(D\) be a \(\rho_\mathcal{L}\)-covariant derivative in \(\Gamma(\pi^* \pi)\). The \(\pi^* \pi\)-valued two-forms
\[
T^h(D)(\tilde{X}, \tilde{Y}) := D_{\tilde{X}} \check{j} \tilde{Y} - D_{\check{j} \tilde{X}} \tilde{Y} - \check{j} \tilde{[X, Y]}_{\mathcal{L}},
\[
T^\nu(D)(\tilde{X}, \tilde{Y}) := D_{\tilde{X}} \check{\nu} \tilde{Y} - D_{\check{\nu} \tilde{X}} \tilde{Y} - \check{\nu} \tilde{[X, Y]}_{\mathcal{L}},
\]

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are said to be the horizontal and the vertical torsion of \( D \), respectively, where \( \bar{X} \) and \( \bar{Y} \) belong to \( \Gamma(\mathcal{L}^\pi E) \).

Let \( \bar{X}, \bar{Y} \in \Gamma(\pi^*\pi) \). The maps \( A \) and \( B \) given by

\[
A(\bar{X}, \bar{Y}) := T^h(D)(\bar{H}\bar{X}, \bar{H}\bar{Y}), \quad B(\bar{X}, \bar{Y}) := T^h(D)(\bar{H}\bar{X}, \bar{i}\bar{Y}),
\]

are called the \( h \)-horizontal and the \( h \)-mixed torsion of \( D \) (with respect to \( \bar{H} \)), respectively. \( A \) will also be mentioned as the torsion of \( D \), while for \( B \) we use the term Finsler torsion as well. \( D \) is said to be symmetric if \( A = 0 \) and \( B \) is symmetric. The maps \( R^1, P^1 \) and \( Q^1 \) given by

\[
R^1(\bar{X}, \bar{Y}) := T^v(D)(\bar{H}\bar{X}, \bar{H}\bar{Y}), \quad P^1(\bar{X}, \bar{Y}) := T^v(D)(\bar{H}\bar{X}, \bar{i}\bar{Y}), \quad Q^1(\bar{X}, \bar{Y}) := T^v(D)(\bar{i}\bar{X}, \bar{i}\bar{Y}), \quad \forall \bar{X}, \bar{Y} \in \Gamma(\pi^*\pi),
\]

are called the \( v \)-horizontal, the \( v \)-mixed and the \( v \)-vertical torsion of \( D \), respectively. Using \( (149), (155), (156) \) and \( (157) \) we can obtain

**Lemma 6.15.** Let \( D \) be a \( \rho_E \)-covariant derivative in \( \Gamma(\pi^*\pi) \). Then all of the partial torsions of the \( \rho_E \)-covariant derivative operator \( D \) are tensor fields of type \((1, 2)\) on \( \Gamma(\pi^*\pi) \). Moreover, for any vector fields \( \bar{X}, \bar{Y} \) belong to \( \Gamma(\pi^*\pi) \) we have

\[
A(\bar{X}, \bar{Y}) = D_{\bar{H}\bar{X}}\bar{Y} - D_{\bar{H}\bar{Y}}\bar{X} - \bar{j}[\bar{H}\bar{X}, \bar{H}\bar{Y}]_E,
B(\bar{X}, \bar{Y}) = -D_{\bar{H}\bar{X}}\bar{Y} - \bar{j}[\bar{H}\bar{X}, \bar{i}\bar{Y}]_E = -D_{\bar{H}\bar{Y}}\bar{X} + \bar{\nabla}_H\bar{X},
R^1(\bar{X}, \bar{Y}) = -\bar{\nabla}[\bar{H}\bar{X}, \bar{H}\bar{Y}]_E,
P^1(\bar{X}, \bar{Y}) = D_{\bar{H}\bar{X}}\bar{Y} - \bar{\nabla}[\bar{H}\bar{X}, \bar{i}\bar{Y}]_E = D_{\bar{H}\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{H}\bar{X}}\bar{Y},
Q^1(\bar{X}, \bar{Y}) = D_{\bar{i}\bar{X}}\bar{Y} - D_{\bar{i}\bar{Y}}\bar{X} - \bar{\nabla}[\bar{i}\bar{X}, \bar{i}\bar{Y}]_E,
\]

where \( \nabla \) is the Berwald derivative given by \((146)\).

**Corollary 6.16.** A \( \rho_E \)-covariant derivative in \( \Gamma(\pi^*E) \) is the Berwald derivative induced by a given horizontal endomorphism if and only if, its Finsler torsion and \( v \)-mixed torsion vanish.

Using the above lemma we get

\[
A(\bar{j}\bar{X}, \bar{j}\bar{Y}) = D_{\bar{h}\bar{X}}\bar{j}\bar{Y} - D_{\bar{h}\bar{Y}}\bar{j}\bar{X} - \bar{j}[\bar{h}\bar{X}, \bar{h}\bar{Y}]_E,
B(\bar{j}\bar{X}, \bar{v}\bar{Y}) = -D_{\bar{v}\bar{Y}}\bar{j}\bar{X} - \bar{j}[\bar{h}\bar{X}, \bar{v}\bar{Y}]_E,
-B(\bar{j}\bar{Y}, \bar{v}\bar{X}) = D_{\bar{v}\bar{X}}\bar{j}\bar{Y} + \bar{j}[\bar{h}\bar{Y}, \bar{v}\bar{X}]_E.
\]

Since \([\bar{v}\bar{X}, \bar{v}\bar{Y}] \in \Gamma(v\mathcal{L}^\pi E)\), then \( \bar{j}[\bar{v}\bar{X}, \bar{v}\bar{Y}] = 0 \). Therefore summing the above equations give us

\[
A(\bar{j}\bar{X}, \bar{j}\bar{Y}) + B(\bar{j}\bar{X}, \bar{v}\bar{Y}) - B(\bar{j}\bar{Y}, \bar{v}\bar{X}) = D_{\bar{X}}\bar{j}\bar{Y} - D_{\bar{Y}}\bar{j}\bar{X} - \bar{j}[\bar{X}, \bar{Y}]_E
= T^h(D)(\bar{X}, \bar{Y}).
\]

Thus we have

**Lemma 6.17.** The horizontal torsion \( T^h(D) \) is completely determined by the torsion \( A \) and the Finsler torsion \( B \). Indeed, we have

\[
T^h(D)(\bar{X}, \bar{Y}) = A(\bar{j}\bar{X}, \bar{j}\bar{Y}) + B(\bar{j}\bar{X}, \bar{v}\bar{Y}) - B(\bar{j}\bar{Y}, \bar{v}\bar{X}), \quad \forall \bar{X}, \bar{Y} \in \Gamma(\mathcal{L}^\pi E).
\]
Lemma 6.18. Let \( D \) be a \( \rho_L \)-covariant derivative in \( \Gamma(\pi^*\pi) \). If \( D \) is associated to the horizontal map \( \mathcal{H} \), then for every section \( \bar{X} \) of \( \pi^*\pi \) we have
\[
B(\delta, \bar{X}) = 0, \quad P^1(\bar{X}, \delta) = -\bar{H}(\bar{X}).
\]

Proof. Since \( D \) is associated to the horizontal map \( \mathcal{H} \), then \( D\delta = \bar{\nabla} \). Therefore using lemma 6.15 we get
\[
B(\delta, \bar{X}) = -D_{\bar{X}}\delta - \bar{j}[\bar{H}\delta, \bar{X}]_L = -\bar{\nabla}(\bar{X}) - \bar{j}[\bar{H}\delta, \bar{X}]_L = -\bar{X} - \bar{j}[\bar{H}\delta, \bar{X}]_L.
\]
Now let \( \bar{X} = \bar{X}^\alpha\bar{e}_\alpha \). Then we deduce \( \bar{j}\bar{X} = \bar{X}^\alpha\bar{\nabla}_\alpha \) and consequently
\[
\bar{j}[\bar{H}\delta, \bar{X}]_L = \bar{j}[\nabla^\alpha\delta, \bar{X}]_L = -\bar{j}(\bar{X}^\alpha\delta) = -\bar{X}^\alpha\bar{e}_\alpha = -\bar{X}.
\]
Setting the above equation in (158) we derive that \( B(\delta, \bar{X}) = 0 \). Using (152) and lemma 6.15 we get
\[
P^1(\bar{X}, \delta) = D_{\bar{H}\bar{X}}\delta - \bar{\nabla}_{\bar{H}\bar{X}}\delta = \bar{\nabla}\bar{H}\bar{X} - (\bar{H} \circ \bar{j} + \bar{\nabla})(\bar{H}\bar{X}) = -\bar{H} \circ \bar{j}(\bar{H}\bar{X}).
\]
But we have \( \bar{j}\bar{H} = 1_{\Gamma(\pi^*\pi)} \). Therefore the above equation gives us the second part of the assertion. \( \square \)

Definition 6.19. Let \( D \) be a \( \rho_L \)-covariant derivative in \( \Gamma(\pi^*\pi) \). Then the maps \( R, P \) and \( Q \) given by
\[
R(\bar{X}, \bar{Y})\bar{Z} := K^D(\bar{H}\bar{X}, \bar{H}\bar{Y})\bar{Z},
\]
\[
P(\bar{X}, \bar{Y})\bar{Z} := K^D(\bar{H}\bar{X}, \bar{i}\bar{Y})\bar{Z},
\]
\[
Q(\bar{X}, \bar{Y})\bar{Z} := K^D(\bar{i}\bar{X}, \bar{i}\bar{Y})\bar{Z},
\]
are said to be the horizontal or Riemann curvature, the mixed or Berwald curvature and the vertical or Berwald-Cartan curvature of \( D \) (with respect to \( \mathcal{H} \)), respectively.

Lemma 6.20. Let \( D \) be a \( \rho_L \)-covariant derivative in \( \Gamma(\pi^*\pi) \). If \( D \) is associated to the horizontal map \( \mathcal{H} \), then we have
\[
R(\bar{X}, \bar{Y})\delta = R^1(\bar{X}, \bar{Y}), \quad P(\bar{X}, \bar{Y})\delta = P^1(\bar{X}, \bar{Y}), \quad Q(\bar{X}, \bar{Y})\delta = Q^1(\bar{X}, \bar{Y}),
\]
where \( \bar{X}, \bar{Y} \in \Gamma(\pi^*\pi) \). Moreover, if the Finsler torsion is symmetric, then \( Q(\cdot, \cdot)\delta = Q^1 \equiv 0 \).

Proof. Since \( D \) is associated to the horizontal map \( \mathcal{H} \), then \( D\delta = \bar{\nabla} \) and therefore
\[
D_{\bar{H}\bar{X}}\delta = 0, \quad D_{\bar{X}}\delta = \bar{X}, \quad \forall \bar{X} \in \Gamma(\pi^*\pi).
\]
Using the above equations, the proof of the first part of the assertion is obvious. Now we prove the second part. From the first part we have
\[
Q(\bar{X}, \bar{Y})\delta = Q^1(\bar{X}, \bar{Y}) = D_{\bar{X}}\bar{Y} - D_{\bar{Y}}\bar{X} - \bar{\nabla}[\bar{X}, \bar{Y}]_L.
\]
Since the Finsler torsion \( B \) is symmetric, then
\[
0 = B(\bar{X}, \bar{Y}) - B(\bar{Y}, \bar{X}) = D_{\bar{X}}\bar{Y} - D_{\bar{Y}}\bar{X} - \bar{j}[\bar{H}\bar{X}, \bar{i}\bar{Y}]_L + \bar{j}[\bar{H}\bar{Y}, \bar{i}\bar{X}]_L.
\]
Therefore from (32) we deduce

\[ Q(\bar{X}, \bar{Y})\delta = j[H\bar{X}, i\bar{Y}]_E - j[H\bar{Y}, i\bar{X}]_E - \bar{V}[i\bar{X}, i\bar{Y}]_E. \]

Since \( \tilde{j} \) is surjective, then there exist \( \bar{X}, \bar{Y} \in \Gamma(\mathcal{L}^E) \) such that \( \bar{X} = j\bar{X} \) and \( \bar{Y} = \tilde{j}\bar{Y} \). Setting these equations in the above equation imply

\[ Q(j\bar{X}, j\bar{Y})\delta = j[h\bar{X}, J\bar{Y}]_E - j[h\bar{Y}, J\bar{X}]_E - \bar{V}[J\bar{X}, J\bar{Y}]_E, \]

and consequently

\[ i(Q(j\bar{X}, j\bar{Y})\delta) = J[h\bar{X}, J\bar{Y}]_E - J[h\bar{Y}, J\bar{X}]_E - v[J\bar{X}, J\bar{Y}]_E \]

\[ = J[\bar{X}, J\bar{Y}]_E + J[J\bar{X}, J\bar{Y}]_E = [J\bar{X}, J\bar{Y}]_E \]

\[ = -N_j(\bar{X}, \bar{Y}) = 0. \]

Since \( \tilde{i} \) is injective, then the above equation gives us \( Q(j\bar{X}, j\bar{Y})\delta = 0 \) and therefore \( Q(X, Y)\delta = 0. \)

Now we denote the torsions and the curvatures of the Berwald derivative \( \nabla \)
by \( \hat{A}, \hat{B}, \hat{R}^1, \hat{P}^1, \hat{Q}^1 \) and \( \hat{R}, \hat{P}, \hat{Q}, \) respectively. Using (151) and Lemma 6.15 it is easy to prove the following

**Lemma 6.21.** Let \( \nabla \) be the Berwald derivative induced by \( h \) and \( \{e_\alpha\} \) be a basis of \( E \). Then

\[ \hat{A} = \frac{1}{2} R^{\alpha\beta} e_\alpha \wedge e_\beta \otimes \bar{e}_\gamma, \]

\( \hat{R}^1 = -\frac{1}{2} R^{\alpha\beta\gamma} e_\alpha \wedge e_\beta \otimes \bar{e}_\gamma, \]

\( \hat{B} = 0, \hat{P}^1 = 0, \hat{Q}^1 = 0, \)

where \( \{\bar{e}_\alpha\} \) is a dual basis of \( \{e_\alpha\} \) and \( t^\gamma_{\alpha\beta} \) and \( R^{\alpha\beta}_{\gamma}\) are given by (33) and (37).

Using \( \hat{A} \) and \( \hat{R}^1 \) we introduce the following tensor fields:

\[ \hat{A}_\alpha: \Gamma(\mathcal{L}^E) \times \Gamma(\mathcal{L}^E) \rightarrow \Gamma(\mathcal{L}^E), \]

\[ \hat{A}_\alpha (\bar{X}, \bar{Y}) = i \hat{A} (j\bar{X}, j\bar{Y}) \]

\[ \hat{R}^1: \Gamma(\mathcal{L}^E) \times \Gamma(\mathcal{L}^E) \rightarrow \Gamma(\mathcal{L}^E), \]

\[ \hat{R}^1 (\bar{X}, \bar{Y}) = i \hat{R}^1 (j\bar{X}, j\bar{Y}) \]

Using (159) and (162) we can obtain

\[ \hat{A}_\alpha (\delta_\alpha, \delta_\beta) = t^\gamma_{\alpha\beta} \gamma, \]

\[ \hat{A}_\gamma (\nu_\alpha, \nu_\beta) = \hat{A}_\gamma (\nu_\alpha, \nu_\beta) = 0. \]

Therefore from (32) we deduce

\[ \hat{A}_\gamma = \frac{1}{2} t^\gamma_{\alpha\beta} \chi^\alpha \wedge \chi^\beta \otimes \nu_\gamma = t, \]
where \( t \) is the weak torsion of \( h \). Similarly using (160) and (163) we obtain
\[
\overset{\circ}{R}^i_0 = -\frac{1}{2} R^\gamma_{\alpha\beta} \Lambda^\alpha \land \Lambda^\beta \otimes \nu_\gamma = \Omega,
\]
where \( \Omega \) is the curvature of \( h \) given in (38).

**Proposition 6.22.** Let \( \nabla \) be the Berwald derivative induced by \( h \) in \( \Gamma(\pi^* \pi) \).

Then \( \overset{\circ}{A}_0 = t \) and \( \overset{\circ}{R}^1_0 = \Omega \), where \( t \) and \( \Omega \) are weak torsion and curvature of \( h \), respectively.

Using (150), (151) and definition 6.19 we can deduce

**Theorem 6.23.** Let \( \nabla \) be the Berwald derivative induced by \( h \) in \( \Gamma(\pi^* \pi) \) and \( \{ e_\alpha \} \) be a basis of \( E \).

Then
\[
\overset{\circ}{R} = R_{\alpha\beta\gamma}^\lambda e^\alpha \otimes e^\beta \otimes e^\gamma,
\]
\[
\overset{\circ}{P} = P_{\alpha\beta\gamma}^\lambda e^\alpha \otimes e^\beta \otimes e^\gamma,
\]
\[
\overset{\circ}{Q} = Q_{\alpha\beta\gamma}^\lambda e^\alpha \otimes e^\beta \otimes e^\gamma,
\]
where
\[
R_{\alpha\beta\gamma}^\lambda = -(\rho^i_\alpha \circ \pi) \frac{\partial^2 B^\lambda_\alpha}{\partial x^i \partial y^\gamma} - B^\mu_\alpha \frac{\partial^2 B^\lambda_\alpha}{\partial y^\mu \partial y^\gamma} + (\rho^j_\beta \circ \pi) \frac{\partial^2 B^\lambda_\beta}{\partial x^j \partial y^\gamma} + B^\nu_\beta \frac{\partial^2 B^\lambda_\beta}{\partial y^\nu \partial y^\gamma} + \frac{\partial B^\mu_\alpha}{\partial y^\nu} \frac{\partial B^\lambda_\alpha}{\partial y^\nu} - \frac{\partial B^\mu_\beta}{\partial y^\nu} \frac{\partial B^\lambda_\beta}{\partial y^\nu} + (L^\mu_{\alpha\beta} \circ \pi) \frac{\partial B^\lambda_\beta}{\partial y^\nu} + B^\nu_{\alpha\beta} \frac{\partial B^\lambda_\beta}{\partial y^\nu},
\]
(164)
\[
P_{\alpha\beta\gamma}^\lambda = \frac{\partial^2 B^\lambda_\alpha}{\partial y^\mu \partial y^\gamma},
\]
(165)
\[
Q_{\alpha\beta\gamma}^\lambda = 0.
\]
(166)

Using \( \overset{\circ}{R} \), \( \overset{\circ}{P} \) and \( \overset{\circ}{Q} \) we introduce the following tensor fields:

\[
\begin{align*}
\overset{\circ}{R}_0 &\colon \Gamma(\mathcal{L}X E) \times \Gamma(\mathcal{L}X E) \to \Gamma(\mathcal{L}X E), \\
\overset{\circ}{R}_0 (\tilde{X}, \tilde{Y}) &\equiv \overset{\circ}{R} (\tilde{X}, j\tilde{Y}), \\
\overset{\circ}{P}_0 &\colon \Gamma(\mathcal{L}X E) \times \Gamma(\mathcal{L}X E) \to \Gamma(\mathcal{L}X E), \\
\overset{\circ}{P}_0 (\tilde{X}, \tilde{Y}) &\equiv \overset{\circ}{P} (\tilde{X}, j\tilde{Y}), \\
\overset{\circ}{Q}_0 &\colon \Gamma(\mathcal{L}X E) \times \Gamma(\mathcal{L}X E) \to \Gamma(\mathcal{L}X E), \\
\overset{\circ}{Q}_0 (\tilde{X}, \tilde{Y}) &\equiv \overset{\circ}{Q} (\tilde{X}, j\tilde{Y}).
\end{align*}
\]
(167)
(168)
(169)

Using the above theorem, (99)-(101) and (167)-(169) we derive that

**Proposition 6.24.** Let \( \nabla \) be the Berwald derivative induced by \( h \). Then
\[
\overset{\circ}{\tilde{R}} = \overset{\circ}{R}, \quad \overset{\circ}{\tilde{P}} = \overset{\circ}{P}, \quad \overset{\circ}{\tilde{Q}} = \overset{\circ}{Q},
\]
where \( \overset{\circ}{R}, \overset{\circ}{P} \) and \( \overset{\circ}{Q} \) are the horizontal, mixed and vertical curvatures of Berwald connection \( (\tilde{D}, h) \), respectively.

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Proposition 6.25. Let $\nabla$ be the Berwald derivative induced by $h$. Then for sections $X$, $Y$ and $Z$ of $E$ we have

$$\bar{\partial}(\hat{X}, \hat{Y})\hat{Z} = \bar{\nabla}[[X^h, Y^V]|_E, Z^V]|_E.$$ 

Proof. Let $X = X^\alpha e_\alpha$, $Y = Y^\beta e_\beta$ and $Z = Z^\gamma e_\gamma$ be sections of $E$. Then we have $\hat{X} = (X^\alpha \circ \pi)e_\alpha$, $\hat{Y} = (Y^\beta \circ \pi)e_\beta$ and $\hat{Z} = (Z^\gamma \circ \pi)e_\gamma$. Therefore, \[105\] implies

$$\bar{\partial}(\hat{X}, \hat{Y})\hat{Z} = ((X^\alpha Y^\beta Z^\gamma) \circ \pi) \frac{\partial B_\alpha^\lambda}{\partial y^\alpha \partial y^\gamma} \hat{e}_\lambda.$$ 

Similarly we can obtain

$$\bar{\nabla}[[X^h, Y^V]|_E, Z^V]|_E = \bar{\nabla}[[X^\alpha \circ \pi]e_\alpha, (Y^\beta \circ \pi)\hat{V}_\beta]|_E, (Z^\gamma \circ \pi)\hat{V}_\gamma]|_E$$

$$= ((X^\alpha Y^\beta Z^\gamma) \circ \pi) \frac{\partial B_\alpha^\lambda}{\partial y^\alpha \partial y^\gamma} \hat{e}_\lambda.$$ 

Two above equations gives us the assertion. □

With the help of the mixed curvature $\bar{\partial}$, we define an important change of the Berwald derivative $\nabla$ by the formula

$$D \tilde{X} \tilde{Y} := \nabla \tilde{X} \tilde{Y} + \frac{1}{n+1}(\text{tr} \bar{\partial}(\tilde{X}, \tilde{Y}))*\delta. \quad (170)$$

The covariant derivative operator $D$ so obtained is called the Yano derivative induced by $\bar{\partial}$. Using \[155\] and the above equation we get

$$D \tilde{X} \tilde{Y} = \bar{X} ^\alpha ((\rho^\alpha_\beta \circ \pi) \frac{\partial Y^\beta}{\partial x^\alpha} + \bar{B}_\alpha^\beta \frac{\partial Y^\beta}{\partial y^\gamma}) + \bar{X} ^\gamma \frac{\partial B_\alpha^\beta}{\partial y^\gamma} + \bar{X} ^\alpha \frac{\partial B_\alpha^\beta}{\partial y^\alpha}$$

$$+ \frac{1}{n+1} \bar{X} ^\alpha y^\gamma \frac{\partial B_\alpha^\beta}{\partial y^\gamma} \hat{e}_\beta,$$

where $\hat{X} = \bar{X} ^\alpha \hat{e}_\alpha + \bar{X} ^\alpha \hat{V}_\alpha \in \Gamma(L^\pi E)$ and $\hat{Y} = \bar{Y} ^\beta \hat{e}_\beta \in \Gamma(\pi^* \pi)$. In particular case we have

$$D \tilde{X} \tilde{e}_\beta = \left( \frac{1}{n+1} y^\gamma \frac{\partial B_\alpha^\lambda}{\partial y^\gamma \partial y^\lambda} - \frac{\partial B_\alpha^\lambda}{\partial y^\gamma} \hat{e}_\gamma, \right.$$

$$D \tilde{V}_\alpha \hat{e}_\beta = 0.$$

7 Finsler algebroids

This section is devoted to Finsler algebroids and their outputs. We will derive a pseudo-Riemannian metric from Finsler algebroid. Gradient of smooth functions on Lie algebroid bundle and their lifts is studied. Special case of horizontal endomorphism named conservative, are visited. Barthel endomorphism on Finsler algebroids is proceeded too. Cartan tensor and some distinguished connections on Finsler algebroids are studied finally.
Definition 7.1. Finsler algebroid \((E,F)\) is a Lie algebroid \(\mathcal{L}^\pi E\) provided with a fundamental Finsler function \(F : E \to \mathbb{R}\) satisfying the conditions:

(i) \(F\) is a scalar differentiable function on the manifold \(\hat{E} = E - \{0\}\) and continuous on the null section of \(\pi : E \to M\).

(ii) \(F\) is a positive function and homogeneous of degree 2, i.e., \(\mathcal{L}^\pi F = 2F\).

(iii) The fundamental form \(\omega = d^\xi d^\eta F\) is nondegenerate, where

\[
d^\xi F = i_\xi d^\eta F = d^\eta F \circ J.
\]

For the basis \(\{X_\alpha, V_\alpha\}\) of \(\Gamma(\mathcal{L}^\pi E)\) and the dual basis \(\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}\) of it, we get

\[
d^\xi F(V_\alpha) = 0\quad \text{and}\quad d^\eta F(X_\alpha) = \frac{\partial F}{\partial y^\alpha}.
\]

Therefore \(d^\eta F\) has the following coordinate expression:

\[
d^\eta F = \frac{\partial F}{\partial y^\alpha} \mathcal{X}^\alpha. \quad (171)
\]

Lemma 7.2. The fundamental form \(\omega\) of a Finsler algebroid has the following coordinate expression:

\[
\omega = \left((\rho^\alpha_\beta \circ \pi) \frac{\partial^2 F}{\partial x^\alpha \partial y^\beta} - \frac{1}{2} \frac{\partial F}{\partial y^\gamma} (L^\gamma_{\alpha\beta} \circ \pi)\right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta = \frac{\partial^2 F}{\partial y^\gamma \partial y^\beta} \mathcal{X}^\alpha \wedge \mathcal{V}^\beta. \quad (172)
\]

Proof. Using \(171\) we have

\[
\omega = d^\xi d^\eta F = d^\xi (\frac{\partial F}{\partial y^\alpha}) \wedge \mathcal{X}^\alpha + \frac{\partial F}{\partial y^\gamma} d^\xi \mathcal{X}^\gamma. \quad (173)
\]

It is easy to see that \((d^\xi \mathcal{X}^\gamma)(X_\alpha, X_\beta) = -(L^\gamma_{\alpha\beta} \circ \pi)\) and \((d^\xi \mathcal{X}^\gamma)(X_\alpha, V_\beta) = (d^\xi \mathcal{X}^\gamma)(V_\alpha, V_\beta) = 0\). Thus we have

\[
d^\xi \mathcal{X}^\gamma = -\frac{1}{2} (L^\gamma_{\alpha\beta} \circ \pi) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta.
\]

Also it is easy to check that \((d^\xi (\frac{\partial F}{\partial y^\gamma}))(X_\beta) = \rho^\beta_\gamma \frac{\partial^2 F}{\partial x^\gamma \partial y^\beta}\) and \((d^\xi (\frac{\partial F}{\partial y^\gamma}))(V_\beta) = \frac{\partial^2 F}{\partial y^\gamma \partial y^\beta}\). Thus we have

\[
d^\xi (\frac{\partial F}{\partial y^\gamma}) = (\rho^\beta_\gamma \circ \pi) \frac{\partial^2 F}{\partial x^\gamma \partial y^\beta} \mathcal{X}^\beta + \frac{\partial^2 F}{\partial y^\gamma \partial y^\beta} \mathcal{V}^\beta.
\]

Setting the above two equations in \(173\) imply \(172\). \(\square\)

From \(172\) we deduce that the fundamental form \(\omega\) is nondegenerate if and only if the symmetric matrix \(\left(\frac{\partial^2 F}{\partial y^\alpha \partial y^\beta}\right)\) is regular.

Proposition 7.3. For the fundamental form \(\omega\) we have the following identities:

(i) \(i_\omega = 0\),

(ii) \(L^\xi_\omega = \omega\),

(iii) \(i_\omega = d^\xi F\).

Proof. We have

\[
i_\omega = i_{\mathcal{X}^\alpha \circ \mathcal{V}^\gamma} \mathcal{X}^\gamma + i_{\mathcal{V}^\alpha} \mathcal{V}^\gamma.
\]

It is easy to check that \(i_{\mathcal{V}^\gamma} \mathcal{X}^\alpha = 0\) and \(i_{\mathcal{V}^\gamma} \mathcal{V}^\alpha = \delta^\alpha_\gamma\). Therefore from \(172\) we get

\[
i_{\mathcal{V}^\gamma} \mathcal{X}^\alpha = \frac{\partial^2 F}{\partial y^\alpha \partial y^\gamma} \mathcal{X}^\alpha, \quad \text{and consequently}
\]

\[
i_{\mathcal{V}^\gamma} \mathcal{V}^\alpha = \frac{\partial^2 F}{\partial y^\alpha \partial y^\gamma} \mathcal{X}^\gamma \wedge \mathcal{X}^\alpha.
\]

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It is easy to see that \( \frac{\partial^2 F}{\partial y^\gamma \partial y^\beta} X^\gamma \wedge X^\alpha = - \frac{\partial^2 F}{\partial y^\alpha \partial y^\gamma} X^\gamma \wedge X^\alpha \). Thus we deduce \( i_C \omega = 0 \). Now we prove (ii). Since \( [C, X_\alpha] = 0 \), then using (172) we derive that
\[
(L_C^E \omega)(X_\alpha, X_\beta) = \rho_C(C) \left( (\rho^i \rho^j \pi) \frac{\partial^2 F}{\partial x^i \partial y^j} - (\rho^i \rho^j \pi) \frac{\partial^2 F}{\partial x^i \partial y^j} \right)
= \gamma^j \left( (\rho^i \rho^j \pi) \frac{\partial F}{\partial y^j} (L^\gamma_{\alpha j} \circ \pi) \right)
- \frac{\partial^2 F}{\partial y^\gamma \partial y^\beta} (L^\gamma_{\alpha j} \circ \pi).
\]
Since \( F \) is homogeneous of degree 2, then we can obtain
\[
\frac{\partial F}{\partial y^\gamma} = \gamma^j \frac{\partial^2 F}{\partial y^j \partial y^\beta}.
\]
Using this equation in the above equation we get
\[
(L_C^E \omega)(X_\alpha, X_\beta) = (\rho^i \rho^j \pi) \frac{\partial^2 F}{\partial x^i \partial y^j} - (\rho^i \rho^j \pi) \frac{\partial^2 F}{\partial x^i \partial y^j} \gamma^j = \omega(X_\alpha, X_\beta).
\]
Similarly, we can obtain
\[
(L_C^E \omega)(X_\alpha, V_\beta) = - \frac{\partial^2 F}{\partial y^\alpha \partial y^\beta} = \omega(X_\alpha, V_\beta), \quad (L_C^E \omega)(V_\alpha, V_\beta) = 0 = \omega(V_\alpha, V_\beta).
\]
Thus we have (ii). It is easy to check that \( i_C X^\gamma = 0 \) and \( i_C V^\gamma = y^\gamma \). Thus using (172) and (174) we get
\[
i_C \omega = \gamma^\alpha \frac{\partial^2 F}{\partial y^\alpha \partial y^\beta} X^\beta = \frac{\partial F}{\partial y^\beta} X^\beta = df^E.
\]

\[\square\]

**Definition 7.4.** Let \((E, F)\) be a Finsler algebroid with fundamental form \( \omega \). Map
\[
G : \Gamma(vE) \times \Gamma(vE) \to C^\infty (E),
\]
defined by \( G(JX, JY) := \omega(JX, JY) \) is called the vertical metric of Finsler algebroid \((E, F)\).

**Remark 7.5.** It is easy to check that \( G \) is bilinear, symmetric and nondegenerate on \( vE \).

From remark 7.5 we have

**Proposition 7.6.** Let \( h \) be a horizontal endomorphism and \( G \) be the vertical metric of Finsler manifold \((E, F)\). Then the function \( \tilde{G} : \Gamma(E) \times \Gamma(E) \to C^\infty (E) \) given by
\[
\tilde{G}(X, Y) := G(JX, JY) + G(vX, vY), \quad \forall X, Y \in \Gamma(E).
\]
is a pseudo-Riemannian metric on \( E \).
The pseudo-Riemannian metric $\tilde{G}$ introduced in the above proposition, is called the prolongation of $G$ along $h$.

In the coordinate expression, using (172) we obtain
\[
G_{\alpha\beta} := G(V_\alpha, V_\beta) = \omega(V_\alpha, X_\beta) = \partial^2 F / \partial y^\alpha \partial y^\beta.
\] (176)

Also, using (175) we can obtain
\[
\tilde{G}(\delta_\alpha, \delta_\beta) = G_{\alpha\beta}, \quad \tilde{G}(\delta_\alpha, V_\beta) = 0, \quad \tilde{G}(V_\alpha, V_\beta) = G_{\alpha\beta},
\]
and consequently
\[
\tilde{G} = G_{\alpha\beta} X^\alpha \otimes X^\beta + G_{\alpha\beta} \delta V^\alpha \otimes \delta V^\beta.
\] (177)

**Proposition 7.7.** For metrics $G, \tilde{G}$ and sections $X, Y$ of $\tilde{E}$, we have
\[
\tilde{G}(X^V, Y^V) = G(X^V, Y^V) = \rho_L(X^V)(\rho_L(Y^V)F),
\] (178)
\[
\tilde{G}(C, C) = G(C, C) = 2F.
\] (179)

**Proof.** Using (178) we get
\[
G(C, C) = y^\alpha y^\beta \partial^2 F / \partial y^\alpha \partial y^\beta.
\]
Since $F$ is homogenous of degree 2, then we can obtain $y^\alpha y^\beta \partial^2 F / \partial y^\alpha \partial y^\beta = 2F$. Thus we deduce $G(C, C) = 2F$. Using (iii) of (23) and (175) we deduce
\[
\tilde{G}(C, C) = G(JC, JC) + G(vC, vC) = G(C, C) = 2F.
\]

Now let $X = X^\alpha e_\alpha$ and $Y = Y^\beta e_\beta$ be sections of $\tilde{E}$. Then we have
\[
G(X^V, Y^V) = G((X^\alpha \circ \pi)\nu_\alpha, (Y^\beta \circ \pi)\nu_\beta) = (X^\alpha \circ \pi)(Y^\beta \circ \pi) \partial^2 F / \partial y^\alpha \partial y^\beta
\]
\[= \rho_L(X^V)(\rho_L(Y^V)F).
\]

Using (175) and the above equation we can obtain
\[
\tilde{G}(X^V, Y^V) = \rho_L(X^V)(\rho_L(Y^V)F).
\]

Let $h$ be a horizontal endomorphism on $L^\pi E$ and $\tilde{G}$ be a pseudo-Riemannian metric given by (175). We consider
\[
K_h(\tilde{X}, \tilde{Y}) = \tilde{G}(\tilde{X}, J\tilde{Y}) - \tilde{G}(J\tilde{X}, \tilde{Y}), \quad \forall \tilde{X}, \tilde{Y} \in \Gamma(L^\pi E),
\]
and we call it the Kähler form with respect to $\tilde{G}$.

**Proposition 7.8.** We have $K_h = i_v \omega$. 

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Proof. Let \( \tilde{X}, \tilde{Y} \in \Gamma(\mathcal{L}^n E) \). Then we have

\[
(i_\omega)(\tilde{X}, \tilde{Y}) = \omega(v \tilde{X}, \tilde{Y}) + \omega(\tilde{X}, v \tilde{Y}) = \omega(v \tilde{X}, \tilde{Y}) - \omega(v \tilde{Y}, \tilde{X})
\]

\[
= \tilde{G}(\tilde{X}, J\tilde{Y}) - \tilde{G}(J\tilde{X}, \tilde{Y}) = K_h(\tilde{X}, \tilde{Y}).
\]

Using \([177]\), Kähler form \( K_h \) has the following coordinate expression with respect to \( \{\delta_\alpha, V_\alpha\} \):

\[
K_h = G_\alpha^\beta \delta V_\alpha \wedge X^\beta.
\]

Definition 7.9. Let \((E, F)\) be a Finsler algebroid with fundamental form \(\omega\). If \(\phi : E \to \mathbb{R}\) is a smooth function, then the section \(\text{grad} \phi \in \Gamma(\mathcal{L}^n E)\) characterized by

\[
d\mathcal{L}_\phi = i_{\text{grad} \phi} \omega,
\]

is called the gradient of \(\phi\).

Remark 7.10. In the above definition, the nondegeneracy of \(\omega\) guarantees the existence and unicity of the gradient section.

If \(\beta\) is a nonzero 1-form on \(\mathcal{L}^n E\), we denote by \(\beta^\sharp\) the section corresponding to \(\omega\), i.e., \(i_{\beta^\sharp} \omega = \beta\). Thus we can introduce the gradient of \(\phi\) by \(\text{grad} \phi = (d^E \phi)^\sharp\).

Since \(\text{grad} \phi \in \Gamma(\mathcal{L}^n E)\), then we can write it as follow

\[
\text{grad} \phi = (\text{grad} \phi)^\alpha X_\alpha + (\text{grad} \phi)^\bar{\alpha} V_\bar{\alpha}.
\]

Thus using \([172]\) and \([180]\) we get

\[
\frac{\partial \phi}{\partial y^\beta} = (d^E \phi)(V_\beta) = (i_{\text{grad} \phi} \omega)(V_\beta) = -(\text{grad} \phi)^\alpha \frac{\partial^2 F}{\partial y^\alpha \partial y^\beta} = -\text{grad} \phi^\alpha G_{\alpha \beta},
\]

which yields

\[
(\text{grad} \phi)^\alpha = -G^{\alpha \beta} \frac{\partial \phi}{\partial y^\beta},
\]

where \((G^{\alpha \beta})\) is the inverse matrix of \((G_{\alpha \beta})\). Similarly, using \([172]\), \([180]\) and the above equation we obtain

\[
(p_\beta^\lambda \circ \pi) \frac{\partial \phi}{\partial x^\gamma} = (d^E \phi)(X_\beta) = (i_{\text{grad} \phi} \omega)(X_\beta) = -G^{\alpha \beta} \frac{\partial \phi}{\partial y^\gamma} \left( (p_\alpha^\lambda \circ \pi) \frac{\partial^2 F}{\partial x^\lambda \partial y^\beta} ight.
\]

\[
- \left. (p_\beta^\gamma \circ \pi) \frac{\partial^2 F}{\partial x^\beta \partial y^\gamma} \right) + (\text{grad} \phi)^\alpha G_{\alpha \beta},
\]

which gives us

\[
(\text{grad} \phi)^\alpha \equiv G^{\alpha \beta} \left\{ (p_\beta^\lambda \circ \pi) \frac{\partial \phi}{\partial x^\gamma} + G^{\lambda \gamma} \frac{\partial \phi}{\partial y^\gamma} \left( (p_\lambda^\alpha \circ \pi) \frac{\partial^2 F}{\partial x^\alpha \partial y^\beta} - (p_\beta^\alpha \circ \pi) \frac{\partial^2 F}{\partial x^\beta \partial y^\alpha} \right) 
\]

\[
- \frac{\partial F}{\partial y^\gamma} (L_{\bar{\lambda} \bar{\beta}} \circ \pi) \right\}. \quad (183)
\]
Thus from (184), we deduce that gradient

\[ \text{grad}\phi = -G^{\alpha\beta} \frac{\partial \phi}{\partial y^\beta} X_\alpha + G^{\alpha\beta} \left\{ (\rho^i_\beta \circ \pi) \frac{\partial \phi}{\partial x^i} + G^{\lambda\gamma} (\rho^i_\lambda \circ \pi) \left( \frac{\partial^2 F}{\partial x^i \partial y^\beta} - (\rho^i_\beta \circ \pi) \frac{\partial^2 F}{\partial x^i \partial y^\beta} \right) \right\} V_\alpha. \]  

(184)

Proposition 7.11. Let \((E, F)\) be a Finsler algebroid and \(f \in C^\infty(M)\). Then we have

(i) \(\text{grad}\ f^\vee \in \Gamma(\mathcal{L}^E E)\), (ii) \(\lbrack C, \text{grad}\ f^\vee \rbrack_E = -\text{grad}\ f^\wedge\), (iii) \(\rho_E(\text{grad}\ f^\vee)(F) = f^c\).

Proof. Since \(f^\vee = f \circ \pi\) is a function with respect to \((x^i)\), then we have \(\frac{\partial f^\vee}{\partial x^i} = 0\). Thus from (184), we deduce that gradient \(f^\vee\) has the following coordinate expression

\[ \text{grad}\ f^\vee = G^{\alpha\beta}(\rho^i_\beta \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} V_\alpha. \]  

(185)

Thus we have (i). The above equation and (22) give us

\[ \lbrack C, \text{grad}\ f^\vee \rbrack_E = \left( y^\alpha \frac{\partial G^{\alpha\gamma}}{\partial y^\gamma} (\rho^i_\gamma \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} - G^{\alpha\gamma}(\rho^i_\gamma \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \right) V_\beta. \]

But using (174) we can deduce \(\frac{\partial G^{\alpha\gamma}}{\partial y^\gamma} = 0\) and consequently \(\frac{\partial G^{\alpha\gamma}}{\partial y^\gamma} = 0\). Setting this equation in the above equation implies

\[ \lbrack C, \text{grad}\ f^\vee \rbrack_E = -G^{\alpha\gamma}(\rho^i_\gamma \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} V_\beta = -\text{grad}\ f^\wedge. \]

Thus we have (ii). To prove (iii), we use (173) and (185) as follows

\[ \rho_E(\text{grad}\ f^\vee)(F) = G^{\alpha\beta}(\rho^i_\beta \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \rho_E(V_\alpha)(F) = G^{\alpha\beta}(\rho^i_\beta \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial F}{\partial y^\alpha} = G^{\alpha\beta}(\rho^i_\beta \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} y^\beta V_\alpha = y^\beta(\rho^i_\beta \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} = f^c. \]

\[
\square
\]

7.1 Conservative endomorphism on Finsler algebroids

Definition 7.12. Horizontal endomorphism \(h\) on Finsler algebroid \((E, F)\) is called conservative if \(d_H^E F = 0\).

Using (63), it is easy to check that \(h\) is conservative if and only if

\[ (\rho^i_\alpha \circ \pi) \frac{\partial F}{\partial x^i} + B^i_\alpha \frac{\partial F}{\partial y^\beta} = 0. \]  

(186)

Proposition 7.13. Let \(h\) be a conservative horizontal endomorphism on Finsler algebroid \((E, F)\). Then we have \(d_H^E F = 0\), where \(H\) is the tension of \(h\).
Proof. Using (31) we can obtain $d_H^F(X_\alpha) = 0$ and

$$d_H^F(X_\alpha) = (B^3_\alpha - y^\gamma \frac{\partial B^3_\alpha}{\partial y^\gamma}) \frac{\partial F}{\partial y^3}. \tag{187}$$

Since $h$ is conservative, then differentiating (186) with respect to $y^\gamma$ we obtain

$$(\rho_\alpha^\gamma \circ \pi) \frac{\partial^2 F}{\partial x^i \partial y^\gamma} + \frac{\partial B^3_\alpha}{\partial y^\gamma} \frac{\partial F}{\partial y^3} + B^3_\alpha \frac{\partial^2 F}{\partial y^3 \partial y^\gamma} = 0. \tag{188}$$

Contracting the above equation by $y^\gamma$ and using homogeneity of $F$ we get

$$(\rho_\alpha^\gamma \circ \pi) \frac{\partial F}{\partial x^i} + \frac{\partial B^3_\alpha}{\partial y^\gamma} \frac{\partial F}{\partial y^3} = 0. \tag{189}$$

Setting the above equation in (187) and using (186) we deduce $d_H^F(X_\alpha) = 0$. Therefore $d_H^F = 0$.

Lemma 7.14. If $\omega$ is the fundamental two-form of Finsler algebroid $(E,F)$ and $h$ is a conservative horizontal endomorphism on $\pi^*E$, then

$$i_h \omega = \omega + i_t d_H^F. \tag{190}$$

Proof. Since $h$ is conservative, then we have (188). Thus using (172) and (188) we get

$$(i_h \omega)(X_\alpha, X_\beta) = (\rho_\alpha^\gamma \circ \pi) \frac{\partial^2 F}{\partial x^i \partial y^\gamma} - (\rho_\beta^\gamma \circ \pi) \frac{\partial^2 F}{\partial x^i \partial y^\gamma} - 2 \frac{\partial F}{\partial y^\gamma} (L_\gamma^\alpha \circ \pi)$$

$$- \frac{\partial B^3_\alpha}{\partial y^\gamma} \frac{\partial F}{\partial y^3} + \frac{\partial B^3_\beta}{\partial y^\gamma} \frac{\partial F}{\partial y^3} + \frac{\partial F}{\partial y^\gamma} (L_\gamma^\beta \circ \pi).$$

Also, (32) and (33) give us

$$(i_t d_H^F)(X_\alpha, X_\beta) = \frac{\partial F}{\partial y^\gamma} \left( \frac{\partial B^3_\alpha}{\partial y^\gamma} - \frac{\partial B^3_\beta}{\partial y^\gamma} \right).$$

Two above equations yield

$$(i_h \omega - i_t d_H^F)(X_\alpha, X_\beta) = (\rho_\alpha \circ \pi) \frac{\partial^2 F}{\partial x^i \partial y^\beta} - (\rho_\beta \circ \pi) \frac{\partial^2 F}{\partial x^i \partial y^\beta} - \frac{\partial F}{\partial y^\gamma} (L_\gamma^\beta \circ \pi)$$

$$= \omega(X_\alpha, X_\beta).$$

Similarly we get

$$(i_h \omega - i_t d_H^F)(X_\alpha, V_\beta) = (i_h \omega)(X_\alpha, X_\beta) = \omega(hX_\alpha, V_\beta) = \omega(X_\alpha, V_\beta),$$

and

$$(i_h \omega - i_t d_H^F)(V_\alpha, V_\beta) = 0 = \omega(V_\alpha, V_\beta).$$

Corollary 7.15. If $\omega$ is the fundamental two-form of Finsler algebroid $(E,F)$ and $h$ is a torsion free conservative horizontal endomorphism on $\pi^*E$, then

$$i_h \omega = \omega.$$
On any Finsler algebroid there is a spray \( S_o : E \to \mathcal{E} E \), which is uniquely determined on \( \mathcal{E} E \) by the formula

\[
i_{S_o} \omega = -d^F, \quad (190)
\]

This spray is called the canonical spray of the Finsler algebroid.

Using (177) and the above equation, the canonical spray \( S_o \) has the coordinate expression \( S_o = y^\alpha \mathcal{X}_\alpha + S_o^\gamma \mathcal{V}_\gamma \), where

\[
S_o^\gamma = G^{\alpha \beta} \left( (\rho^i_\beta \circ \pi) \frac{\partial F}{\partial x^i} + y^\gamma (\frac{\partial F}{\partial y^\alpha} (L^\lambda_{\gamma \beta} \circ \pi) - (\rho^i_\gamma \circ \pi) \frac{\partial^2 F}{\partial x^i \partial y^\beta}) \right), \quad (191)
\]

and \((G^{\alpha \beta})\) is the inverse matrix of \((G_{\alpha \beta})\).

**Proposition 7.16.** Let \( S_o \) be the canonical spray and \( h \) be a conservative horizontal endomorphism on Finsler algebroid \((E, F)\) with the associated semispray \( S \). Then we have

\[
S - S_o = (d^F)_{i \mapsto t} \omega,
\]

where \( i_{(d^F)_{i \mapsto t}} \omega = d^F_{i \mapsto t}. \)

**Proof.** Let \( h = (X_\alpha + B^\alpha_\gamma \mathcal{V}_\beta) \otimes \mathcal{X}_\alpha \), \( S = y^\alpha \mathcal{X}_\alpha + S^\gamma \mathcal{V}_\gamma \) and \( S_o = y^\alpha \mathcal{X}_\alpha + S_o^\gamma \mathcal{V}_\gamma \), where \( S^\gamma \) are given by (191). Since \((i_{V_\gamma} \omega)(X_\beta) = \frac{\partial F}{\partial y^\gamma} = G_{\alpha \beta} \) and \((i_{V_\gamma} \omega)(V_\beta) = 0\), then we have \( i_{V_\gamma} \omega = G_{\alpha \beta} \mathcal{X}_\beta \). Therefore, using (191) we get

\[
i_{S - S_o} \omega = (S - S_o) i_{V_\gamma} \omega = (S^\alpha \frac{\partial^2 F}{\partial y^\alpha \partial y^\beta} - (\rho^i_\beta \circ \pi) \frac{\partial F}{\partial x^i} - y^\gamma \frac{\partial F}{\partial y^\alpha} (L^\lambda_{\gamma \beta} \circ \pi)
+ (\rho^i_\gamma \circ \pi) y^\gamma \frac{\partial^2 F}{\partial x^i \partial y^\beta}) \mathcal{X}_\beta.
\]

From \( S = hS_o \) we deduce \( S^\gamma = y^\gamma B^\gamma_\gamma \). Setting this in the above equation gives us

\[
i_{S - S_o} \omega = (y^\gamma B^\gamma_\gamma \frac{\partial^2 F}{\partial y^\alpha \partial y^\beta} - (\rho^i_\beta \circ \pi) \frac{\partial F}{\partial x^i} - y^\gamma \frac{\partial F}{\partial y^\alpha} (L^\lambda_{\gamma \beta} \circ \pi)
+ (\rho^i_\gamma \circ \pi) y^\gamma \frac{\partial^2 F}{\partial x^i \partial y^\beta}) \mathcal{X}_\beta.
\]

Since \( h \) is conservative, then we have (180), (188), and (189). Using these equations in the above equation and using (33) we get

\[
i_{S - S_o} \omega = y^\alpha \frac{\partial B^\gamma_\gamma}{\partial y^\alpha} - \frac{\partial B^\gamma_\gamma}{\partial y^\beta} - (L^\lambda_{\alpha \beta} \circ \pi)) \frac{\partial F}{\partial y^\gamma} \mathcal{X}_\beta = y^\alpha \frac{\partial B^\gamma_\alpha}{\partial y^\beta} \mathcal{X}_\beta
= \frac{1}{2} L^\gamma_{\alpha \beta} \frac{\partial F}{\partial y^\gamma} (y^\alpha \mathcal{X}_\beta - y^\beta \mathcal{X}_\alpha) = \frac{1}{2} B^\gamma_{\alpha \beta} (y^\alpha \mathcal{X}_\beta - y^\beta \mathcal{X}_\alpha)_{|\gamma} \mathcal{V}_\gamma (F)
= \frac{1}{2} L^\gamma_{\alpha \beta} (y^\alpha \mathcal{X}_\beta - y^\beta \mathcal{X}_\alpha) i_{V_\gamma} d^F = i_{i \mapsto t} d^F = d^F_{i \mapsto t} = i_{(d^F)_{i \mapsto t}} \omega.
\]
7.1.1 Barthel endomorphism on Finsler algebroids

Let $S_0$ be the canonical spray on Finsler algebroid $(E, F)$. We consider
\[ h = \frac{1}{2} (1_{\Gamma(E)} + [J, S_0]_E^N). \]

In the coordinate expression, we can obtain
\[ h_\circ = \left( X_\alpha + \frac{1}{2} \left( \frac{\partial S_\circ^\beta}{\partial y^\alpha} - y^\gamma (L_\circ^\beta \circ \pi) V_\beta \right) \right) \otimes X_\alpha. \quad (192) \]

From the above equation we deduce $h_\circ^2 = h_\circ$ and $\ker h_\circ = v_\pi E$. Thus $h_\circ$ is a horizontal endomorphism on $\pi E$ which is called Barthel endomorphism. Since $S_0$ is a spray on $(E, F)$, then we can deduce that the Barthel endomorphism is homogenous.

**Proposition 7.17.** Let $h$ be a conservative and homogenous horizontal endomorphism and $h_\circ$ be the Barthel endomorphism on Finsler algebroid $(E, F)$. Then we have
\[ h = h_\circ + \frac{1}{2} i_{\mathcal{S}^t} - \frac{1}{2} [J, (d_{\mathcal{S}^t}^t)^2]_E. \]

**Proof.** Let $S$ be the semispray associated to $h$ and $h'$ be the horizontal endomorphism generated by $S$. Then using theorem 4.24 we get
\begin{align*}
    h_\circ &= \frac{1}{2} (1_{\Gamma(E)} + [J, S_\circ]_E) = \frac{1}{2} (1_{\Gamma(E)} + [J, S]_E - [J, (d_{\mathcal{S}^t}^t)^2]_E) \\
    &= h' - \frac{1}{2} [J, (d_{\mathcal{S}^t}^t)^2]_E = h - \frac{1}{2} i_{\mathcal{S}^t} - \frac{1}{2} [J, (d_{\mathcal{S}^t}^t)^2]_E.
\end{align*}

**Theorem 7.18.** Barthel endomorphism of Finsler algebroid $(E, F)$ is conservative.

**Proof.** Using (190) it is sufficient to show that
\[ (\rho_\alpha \circ \pi) \frac{\partial F}{\partial x^\alpha} + B_\alpha^\beta \frac{\partial F}{\partial y^\beta} = 0, \quad (193) \]

where $B_\alpha^\beta = \frac{1}{2} (\frac{\partial S_\circ^\beta}{\partial y^\alpha} - y^\gamma (L_\circ^\beta \circ \pi))$ and $S_\circ^\beta$ are given by (191). Using (174) we deduce
\begin{align*}
    (i) \quad \frac{\partial F}{\partial y^\gamma} = y^\lambda G_{\gamma \lambda}, & \quad (ii) \quad y^\mu \frac{\partial^2 F}{\partial y^\gamma \partial y^\lambda \partial y^\mu} = 0. \quad (194)
\end{align*}

Thus using (i) of (194) we derive that
\[ B_\alpha^\beta \frac{\partial F}{\partial y^\beta} = \frac{1}{2} \left( \frac{\partial S_\circ^\beta}{\partial y^\alpha} - y^\gamma (L_\circ^\beta \circ \pi) y^\mu G_{\mu \beta}. \right) \quad (195) \]
Using (191) we obtain
\[
\frac{\partial S_3}{\partial y^\alpha} y^\mu G_{\mu \beta} = y^\mu G_{\mu \beta} \left( \frac{\partial G^{\alpha \sigma}}{\partial y^\alpha} \right) \left( (\rho^i_\sigma \circ \pi) \frac{\partial F}{\partial x^i} + y^\gamma \frac{\partial F}{\partial y^\gamma} (L^\lambda_{\sigma \gamma} \circ \pi) \right)
\]
\[\quad - (\rho^i_\sigma \circ \pi) \frac{\partial^2 F}{\partial x^i \partial y^\alpha} \right) + y^\alpha \left( (\rho^i_\sigma \circ \pi) \frac{\partial^2 F}{\partial x^i \partial y^\alpha} \right)
\]
\[\quad + \frac{\partial F}{\partial y^\alpha} (L^\lambda_{\sigma \gamma} \circ \pi) - (\rho^i_\sigma \circ \pi) \frac{\partial^2 F}{\partial x^i \partial y^\alpha} + y^\gamma \frac{\partial^2 F}{\partial y^\gamma} (L^\lambda_{\sigma \gamma} \circ \pi)
\]
\[\quad - y^i (\rho^i_\sigma \circ \pi) \frac{\partial^3 F}{\partial x^i \partial y^\alpha} \partial y^\alpha \right) .
\] (196)

But (ii) of (194) implies
\[
y^\mu G_{\mu \beta} \frac{\partial G^{\alpha \sigma}}{\partial y^\alpha} = -y^\mu G_{\mu \beta} \frac{\partial G^{\alpha \sigma}}{\partial y^\alpha} = -y^\mu G_{\mu \beta} \frac{\partial^3 F}{\partial y^\alpha \partial y^\alpha \partial y^\beta} = 0.
\]

Moreover, we have \(y^\alpha \frac{\partial^2 F}{\partial x^i \partial y^\alpha} = \frac{\partial^2 F}{\partial x^i \partial y^\alpha} \) and \(y^\alpha \frac{\partial^2 F}{\partial y^\gamma \partial y^\alpha} = \frac{\partial^2 F}{\partial y^\gamma \partial y^\alpha} \). Therefore (196) reduce to
\[
\frac{\partial S_3}{\partial y^\alpha} y^\mu G_{\mu \beta} = y^\mu \frac{\partial F}{\partial y^\alpha} (L^\lambda_{\sigma \gamma} \circ \pi) - 2 (\rho^i_\sigma \circ \pi) \frac{\partial F}{\partial x^i}.
\]

Setting the above equation in (195) we deduce \(B^\alpha_\beta \frac{\partial F}{\partial y^\alpha} = -(\rho^i_\sigma \circ \pi) \frac{\partial F}{\partial x^i} \). Therefore we have (193).

**Theorem 7.19.** Let \(h_1 \) and \(h_2 \) be conservative horizontal endomorphisms on Finsler algebroid \((E, F)\). If \(h_1 \) and \(h_2 \) have common strong torsions, then \(h_1 = h_2 \).

**Proof.** We denote by \(S_1 \) and \(S_2 \) the associated semispray of \(h_1 \) and \(h_2 \), respectively and we let \(T_1 \) and \(T_2 \) be the strong torsions of \(h_1 \) and \(h_2 \), respectively. Then from hypothesis we have \(d^E_{i_1} F = d^E_{i_2} F = 0 \) and \(T_1 = T_2 \). Also, from the last equation in the proof of proposition 7.16 we deduce \(i_{S_1 - S_2} \omega = d^E_{i_3} \omega = d^E_{i_{S_1} t_1} \omega \) and consequently
\[
\quad i_{S_1 - S_2} \omega = d^E_{i_{S_1} t_1} \omega - d^E_{i_{S_2} t_2} \omega .
\] (197)

where \(t_1 \) and \(t_2 \) are weak torsions of \(h_1 \) and \(h_2 \), respectively. From the definition of strong torsion we have
\[
d^E_{i_{S_1} t_1} \omega = d^E_{i_{S_1} t_1} \omega - d^E_{i_{S_2} t_2} \omega ,
\]

because \(d^E_{H_1} \omega = 0 \), where \(H_1 \) is the tension of \(h_1 \). Similarly we obtain \(d^E_{i_{S_1} t_1} \omega = d^E_{i_{S_2} t_1} \omega \). Setting this equation together the above equation in (197) we deduce \(i_{S_1 - S_2} \omega = d^E_{i_3} \omega = d^E_{i_{S_1} t_1} \omega \). Since \(\omega \) is nondegenerate, then this equation gives us \(S_1 = S_2 \) and consequently using theorem 4.16 we deduce \(h_1 = h_2 \).

From the above results we understand that Barthel endomorphism is homogenous, conservative and torsion free. Moreover, since Barthel endomorphism
is homogenous and torsion free, then we deduce that the it’s strong torsion is zero. Also, from the above theorem we derive that if $h$ is a homogenous, conservative and torsion free horizontal endomorphism then it is coincide with Barthel endomorphism. Thus we have the following

**Theorem 7.20.** There exists a unique horizontal endomorphism on Finsler algebroid $(E, F)$ such that it is homogenous, conservative and torsion free.

### 7.2 Cartan tensor on Finsler algebroids

Here, we consider the tensor

$$
\{ C : \Gamma(\mathcal{E}^o E) \times \Gamma(\mathcal{E}^o E) \to \Gamma(\mathcal{E}^o E), (\tilde{X}, \tilde{Y}) \to C(\tilde{X}, \tilde{Y}), \}
$$

on Finsler algebroid $(E, F)$ which satisfies in

$$
J \circ C = 0,
$$

(199)

$$
\mathcal{G}(C(\tilde{X}, \tilde{Y}), J\tilde{Z}) = \frac{1}{2}(\mathcal{E}_{\tilde{X} \tilde{Y}} J^* \mathcal{G})(\tilde{Y}, \tilde{Z}),
$$

(200)

where $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(\mathcal{E}^o E)$ and we call it the first Cartan tensor. Also, the lowered tensor $C_\flat$ of $C$ is defined by

$$
C_\flat(\tilde{X}, \tilde{Y}, \tilde{Z}) = \mathcal{G}(C(\tilde{X}, \tilde{Y}), J\tilde{Z}), \forall \tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(\mathcal{E}^o E).
$$

(201)

(199) told us that $C(\tilde{X}, \tilde{Y})$ belongs to $\Gamma(v \mathcal{E}^o E)$. Also, from (200) we deduce that $C(\mathcal{X}_\alpha, \mathcal{V}_\beta) = C(\mathcal{V}_\alpha, \mathcal{V}_\beta) = 0$ and

$$
C(\mathcal{X}_\alpha, \mathcal{X}_\beta) = \frac{1}{2} \frac{\partial^3 \mathcal{F}}{\partial y^\alpha \partial y^\beta \partial y^\gamma} G^{\gamma\lambda} \mathcal{V}_\lambda.
$$

Therefore the first Cartan tensor has the following coordinate expression:

$$
C = C^\gamma_{\alpha\beta} \mathcal{X}^\alpha \otimes \mathcal{X}^\beta \otimes \mathcal{V}_\gamma,
$$

(202)

where

$$
C^\gamma_{\alpha\beta} = \frac{1}{2} \frac{\partial^3 \mathcal{F}}{\partial y^\alpha \partial y^\beta} G^{\gamma\lambda} = \frac{1}{2} \frac{\partial^3 \mathcal{F}}{\partial y^\alpha \partial y^\beta \partial y^\gamma} G^{\gamma\lambda}.
$$

From (202) and the above equation, we can deduce

**Proposition 7.21.** The first Cartan tensor is semibasic. Moreover, it and the lowered tensor of it, are symmetric tensors.

Using (201) and (202) we can obtain the following coordinate expression for the lowered tensor:

$$
C_\flat = C_{\alpha\beta\gamma} \mathcal{X}^\alpha \otimes \mathcal{X}^\beta \otimes \mathcal{X}^\gamma,
$$

where

$$
C_{\alpha\beta\gamma} = C^\lambda_{\alpha\beta\gamma} G_{\gamma\lambda} = \frac{1}{2} \frac{\partial^3 \mathcal{F}}{\partial y^\alpha \partial y^\beta \partial y^\gamma}.
$$
Proposition 7.22. If $S$ is a semispray on $\mathcal{L}^\gamma E$, then we have $i_\Sigma C = i_\Sigma C_s = 0$.

Proof. Let $\tilde{Y} = \tilde{Y}^\alpha X_\alpha + \tilde{Y}^\beta V_\beta$ and $\tilde{Z} = \tilde{Z}^\gamma \chi_\gamma$ be sections of $\mathcal{L}^\gamma E$. Then using (194), we have

\[(i_\Sigma C_s)(\tilde{Y}, \tilde{Z}) = C_s(S(\tilde{Y}), \tilde{Z}) = \frac{1}{2} \frac{\partial^2 F}{\partial y^\alpha \partial y^\beta} \tilde{y}^\gamma = 0.\]

Similarly we can prove $i_\Sigma C = 0$.

Now we consider a horizontal endomorphism $h$ on $\mathcal{L}^\gamma E$, and the prolongation $\tilde{G}$ of the vertical metric $G$ along $h$. The second Cartan tensor

\[
\tilde{C} : \Gamma(\mathcal{L}^\gamma E) \times \Gamma(\mathcal{L}^\gamma E) \to \Gamma(\mathcal{L}^\gamma E),
\]

(belonging to $h$) is defined by the rules

\[
J \circ \tilde{C} = 0,
\]

\[
\tilde{G}(\tilde{C}(\tilde{X}, \tilde{Y}), J\tilde{Z}) = \frac{1}{2} (\mathcal{L}_h \tilde{G})(J\tilde{Y}, J\tilde{Z}),
\]

where $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(\mathcal{L}^\gamma E)$. Also, the lowered tensor $\underline{C}$ of $\tilde{C}$ is defined by

\[
\underline{C}_s(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{G}(\tilde{C}(\tilde{X}, \tilde{Y}), J\tilde{Z}), \quad \forall \tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(\mathcal{L}^\gamma E).
\]

Similar to the first Cartan tensor, using (203) and (204), we can deduce that the second Cartan tensor has the following coordinate expression:

\[
\tilde{C} = \tilde{C}_\alpha^\gamma X^\alpha \otimes X^\beta \otimes V_\gamma,
\]

where

\[
\tilde{C}_\alpha^\gamma = \frac{1}{2} \left( (\rho^\lambda \circ \pi)^{\alpha} \frac{\partial G_{\beta \mu}}{\partial x^\mu} G^{\gamma \mu} + B^\lambda_{\alpha} \frac{\partial G_{\beta \mu}}{\partial y^\mu} G^{\gamma \mu} + \frac{\partial B^\lambda_{\alpha}}{\partial y^\beta} G^{\gamma \mu} G_{\beta \lambda} \right).
\]

From (207), it is easy to see that the second Cartan tensor is semibasic. Moreover, (207) and (208) give us

\[
\tilde{C}_s = \tilde{C}_\alpha^\gamma X^\alpha \otimes X^\beta \otimes X^\gamma,
\]

where

\[
\tilde{C}_\alpha^\beta = \tilde{C}_\alpha^\beta_{\lambda \gamma} = \frac{1}{2} \left( (\rho^\lambda \circ \pi)^{\alpha} \frac{\partial G_{\beta \gamma}}{\partial x^\lambda} + B^\lambda_{\alpha} \frac{\partial G_{\beta \gamma}}{\partial y^\lambda} + \frac{\partial B^\lambda_{\alpha}}{\partial y^\beta} G_{\lambda \gamma} + \frac{\partial B^\lambda_{\alpha}}{\partial y^\beta} G_{\beta \lambda} \right).
\]

Proposition 7.23. Let $(E, F)$ be a Finsler algebroid. Then we have

\[
2C_s(X^C, Y^C, Z^C) = \rho_L(X^C)(\rho_L(Y^C)(\rho_L(Z^C)),
\]

\[
2\tilde{C}_s(X^C, Y^C, Z^C) = [Y^C, [X^C, Z^C]] + \rho_L(Y^C)(\rho_L(X^C)(\rho_L(Z^C)).
\]
Proof. Let $X, Y$ and $Z$ be sections of $E$. Using the second part of (15) we get
\[2\mathcal{C}_\gamma(X^C, Y^C, Z^C) = 2(Y^\alpha \circ \pi)(Y^\beta \circ \pi)(Z^\gamma \circ \pi)\mathcal{C}_\nu(X_{\alpha}, X_{\beta}, X_{\gamma}),\]
\[= (X^\alpha \circ \pi)(Y^\beta \circ \pi)(Z^\gamma \circ \pi)\frac{\partial^3 F}{\partial y^\alpha \partial y^\beta \partial y^\gamma},\]
\[= (X^\alpha \circ \pi)(Y^\beta \circ \pi)(Z^\gamma \circ \pi)\rho_E(V_\alpha)(\rho_E(V_\beta)(\rho_E(V_\gamma)F)).\]
\[= \rho_E(X^V)(\rho_E(Y^V)(\rho_E(Z^V)F)).\]

Now we prove (212). Direct calculation gives us
\[ [Y^V, [X^h, Z^V]_E] + \rho_E(Y^V)(\rho_E(Z^V)(\rho_E(X^h)F)) = (X^\alpha \circ \pi)(Y^\beta \circ \pi)(Z^\gamma \circ \pi)(\rho_\mu \circ \pi)\frac{\partial^3 F}{\partial y^\alpha \partial y^\beta \partial y^\gamma} + \frac{\partial^2 F}{\partial y^\beta \partial y^\gamma \partial y^\lambda}\]
\[+ \frac{\partial^3 F}{\partial y^\alpha \partial y^\gamma \partial y^\lambda}.\]

But using (213), we can see that the above equation is equal to $2\mathcal{C}_\gamma(X^C, Y^C, Z^C)$. Thus we have (212).

**Proposition 7.24.** Let $(E, F)$ be a Finsler algebroid. If $h$ is a torsion free and conservative horizontal endomorphism on $E$, then the lowered second Cartan tensor is symmetric.

**Proof.** (210) told us that $\tilde{C}_{\alpha\beta\gamma}$ is symmetric with respect to last two variables. Thus it is sufficient to prove that $\tilde{C}_{\alpha\beta\gamma}$ is symmetric with respect to first two variables. Since $h$ is conservative, then using (193) and (i) of (194) in (210) we obtain
\[ \tilde{C}_{\alpha\beta\gamma} = \frac{1}{2} Y^\mu \frac{\partial^2 B^\alpha_{\beta\gamma}}{\partial y^\mu \partial y^\nu} G_{\lambda\nu}.\]

Since $h$ is torsion free, then using (33) we have $\frac{\partial^2 B^{\lambda}_{\mu\nu}}{\partial y^\mu \partial y^\nu} = \frac{\partial^2 B^{\lambda}_{\mu\nu}}{\partial y^\mu \partial y^\nu}$. Setting this equation in the above equation implies $\tilde{C}_{\alpha\beta\gamma} = \tilde{C}_{\beta\alpha\gamma}$.

### 7.3 Distinguished connections on Finsler algebroids

**Theorem 7.25.** Let $(E, F)$ be a Finsler algebroid and $h$ be a conservative horizontal endomorphism on $E$. Then there exists a unique $d$-connection $\nabla^h$ on $(E, F)$ such that the $v$-mixed and $h$-mixed torsions of $\nabla^h$ are zero.

**Proof.** Let there exist a $d$-connection $\nabla^h$ on $(E, F)$ such that the $v$-mixed and $h$-mixed torsions of are zero. If we denote by $P^h$, the $v$-mixed torsion of $\nabla^h$, then we have
\[ 0 = P^h(\tilde{X}, F\tilde{Y}) = v T(h\tilde{X}, v\tilde{Y}) = v(D_{h\tilde{X}} v\tilde{Y} - D_{v\tilde{Y}} h\tilde{X} - [h\tilde{X}, v\tilde{Y}]_E) = D_{h\tilde{X}} v\tilde{Y} - v[h\tilde{X}, v\tilde{Y}]_E, \]
where $T$ is the torsion of $\nabla^h$. The above equation gives us
\[ D_{h\tilde{X}} v\tilde{Y} = v[h\tilde{X}, v\tilde{Y}]_E. \]
Since the \( h \)-mixed torsion of \( \mathbf{D} \) is zero, then we have
\[
0 = \mathbf{D} \left( \tilde{Y}, F \tilde{X} \right) = h^T (h \tilde{Y}, v \tilde{X}) = h(\mathbf{D}_{h\tilde{Y}} v \tilde{X} - \mathbf{D}_{v \tilde{X}} h \tilde{Y} - [h \tilde{Y}, v \tilde{X}]_\ell)
\]
\[
= - \mathbf{D}_{v \tilde{X}} h \tilde{Y} - h[h \tilde{Y}, v \tilde{X}]_\ell = - \mathbf{D}_{v \tilde{X}} h \tilde{Y} - h[\tilde{Y}, v \tilde{X}]_\ell,
\]
where \( \mathbf{D} \) is the \( h \)-mixed torsion of \( \mathbf{D} \). The above equation gives us
\[
\mathbf{D}_{v \tilde{X}} h \tilde{Y} = h[v \tilde{X}, \tilde{Y}]_\ell. \tag{214}
\]
Since \( \mathbf{D} \) is d-connection, then using (213), (iv) of (29) and (i), (iv) of (50) we get
\[
\mathbf{D}_{h \tilde{X}} h \tilde{Y} = F \mathbf{D}_{h \tilde{X}} J \tilde{Y} = F \mathbf{D}_{h \tilde{X}} v J \tilde{Y} = F v[h \tilde{X}, v J \tilde{Y}]_\ell
\]
\[
= h F[h \tilde{X}, J \tilde{Y}]_\ell. \tag{215}
\]
Since \( \mathbf{D} \) is d-connection, then (iii), (iv) of (51), (ii), (iv) of (29) and (214) give us
\[
\mathbf{D}_{v \tilde{X}} v \tilde{Y} = \mathbf{D}_{v \tilde{X}} (v \tilde{Y}) = \mathbf{D}_{v \tilde{X}} J (F v \tilde{Y}) = J \mathbf{D}_{v \tilde{X}} h F \tilde{Y} = J h[v \tilde{X}, F \tilde{Y}]_\ell
\]
\[
= J[v \tilde{X}, F \tilde{Y}]_\ell. \tag{216}
\]
Relations (213)-(216) prove the existence and uniqueness of \( \mathbf{D} \).

Using (213)-(216), the d-connection \( \mathbf{D} \) has the following coordinate expression:
\[
\begin{array}{ll}
\mathbf{D}_{\delta \alpha} \delta \beta &= - \frac{\partial \mathbf{B}^\gamma_\beta}{\partial \lambda} \delta \gamma, & \mathbf{D}_{v \alpha} \mathbf{V}_\beta = 0, \\
\mathbf{D}_{\delta \alpha} \mathbf{V}_\beta &= - \frac{\partial \mathbf{B}^\gamma_\beta}{\partial y^\gamma} \mathbf{V}_\gamma, & \mathbf{D}_{v \alpha} \delta \beta = 0.
\end{array} \tag{217}
\]

**Proposition 7.26.** Let \((E, \mathcal{F})\) be a Finsler algebroid, \( h \) be a conservative horizontal endomorphism on \( \mathcal{L}^n E \) and \( \mathbf{D} \) be the d-connection given by (217). If \( h \)-deflection and \( h \)-horizontal torsion of \( \mathbf{D} \) are zero, then \( h \) is the Barthel endomorphism.

**Proof.** It is sufficient to show that \( h \) is homogenous and torsion free. Since \( h \)-deflection of \( (\mathbf{D}, h) \) is zero, then we have
\[
0 = h^*(DF(C))\delta \alpha = \mathbf{D}_{h \delta \alpha}(C) = \mathbf{D}_{h \delta \alpha}(y^\beta \mathbf{V}_\beta) = (B_{\alpha}^{\beta} - y^\gamma \frac{\partial B_{\alpha}^{\gamma}}{\partial y^\nu}) \mathbf{V}_\beta.
\]
The above equation shows that \( h \) is homogenous. Also, since the \( h \)-horizontal torsion of \( \mathbf{D} \) is zero, then we get
\[
0 = hT(\delta \alpha, \delta \beta) = h(\mathbf{D}_{\delta \alpha} \delta \beta - \mathbf{D}_{\delta \beta} \delta \alpha - [\delta \alpha, \delta \beta]_\ell)
\]
\[
= h \left( \frac{\partial B_{\alpha}^{\gamma}}{\partial y^\nu} - \frac{\partial B_{\beta}^{\gamma}}{\partial y^\mu} - (L^\alpha_{\alpha \beta} \circ \pi) \delta \gamma - R^\gamma_{\alpha \beta} \mathbf{V}_\gamma \right)
\]
\[
= \ell_{\alpha \beta} \delta \gamma.
\]
From the above equation we deduce that the weak torsion of \( h \) is zero. \( \square \)
If \( h \) is the Barthel endomorphism of Finsler algebroid \((E, F)\), then the d-connection \( \tilde{\nabla} \) given in (217) is called the Berwald connection of \((E, F)\).

Theorem 7.27. Let \((E, F)\) be a Finsler algebroid, \( h \) be a torsion free and conservative horizontal endomorphism on \( E \), \( \tilde{G} \) be the prolongation of \( G \) along \( h \). Then there exists a unique d-connection \( \tilde{\nabla} \) on \((E, F)\) such that \( \tilde{\nabla} \) is metrical, i.e., \( \tilde{\nabla} \tilde{G} = 0 \) and the v-vertical and h-horizontal torsions of \( \tilde{\nabla} \) are zero.

Proof. Let there exist a d-connection \( \tilde{\nabla} \) such that \( \tilde{\nabla} \) is metrical and the v-vertical and h-horizontal torsions of \( \tilde{\nabla} \) are zero. Since \( \tilde{\nabla} \) is metrical, then we have

\[
\rho_x(\delta_\alpha)\tilde{\nabla}(\delta_\beta, \delta_\gamma) = \tilde{\nabla}(\tilde{D}_\delta_\alpha, \delta_\beta, \delta_\gamma) + \tilde{\nabla}(\delta_\beta, \tilde{D}_\delta_\alpha, \delta_\gamma),
\]

(218)

\[
\rho_x(\delta_\beta)\tilde{\nabla}(\delta_\beta, \delta_\alpha) = \tilde{\nabla}(\tilde{D}_\delta_\beta, \delta_\alpha) + \tilde{\nabla}(\delta_\gamma, \tilde{D}_\delta_\alpha, \delta_\gamma),
\]

(219)

\[-\rho_x(\delta_\gamma)\tilde{\nabla}(\delta_\alpha, \delta_\beta) = -\tilde{\nabla}(\tilde{D}_\delta_\alpha, \delta_\beta) - \tilde{\nabla}(\delta_\alpha, \tilde{D}_\delta_\beta).
\]

(220)

Since the h-horizontal torsion of \( \tilde{\nabla} \) is zero, then we have

\[
\tilde{D}_\delta_\beta \delta_\alpha - \tilde{D}_\delta_\alpha \delta_\beta = [\delta_\alpha, \delta_\beta]_L = (L^\gamma_{\alpha\beta} \circ \pi)\delta_\gamma + R^\gamma_{\alpha\beta} \gamma.
\]

Summing (218)-(220) and using the above equation give us

\[
\tilde{\nabla}(\tilde{D}_\delta_\beta, \delta_\gamma) = \frac{1}{2} (\rho_x^\prime \circ \pi) \frac{\partial \tilde{G}}{\partial x^\gamma} + \frac{\partial B^\lambda}{\partial y^\lambda} \frac{\partial \tilde{G}}{\partial y^\lambda} + (\rho_\beta^\prime \circ \pi) \frac{\partial \tilde{G}}{\partial x^\gamma} + \frac{\partial B^\lambda}{\partial y^\lambda} \frac{\partial \tilde{G}}{\partial y^\lambda} \frac{\partial \tilde{G}}{\partial y^\lambda} - (\rho_\gamma^\prime \circ \pi) \frac{\partial \tilde{G}}{\partial y^\lambda} - (L^\lambda_{\alpha\beta} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\beta\alpha} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\gamma\lambda} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\gamma\lambda} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\gamma\lambda} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\gamma\lambda} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\gamma\lambda} \circ \pi) \tilde{G}_\lambda^\beta.
\]

(221)

Since \( h \) is torsion free, then using (33) in the above equation we get

\[
\tilde{D}_\delta_\beta \delta_\alpha = \frac{1}{2} \tilde{G}^\gamma_{\alpha\beta} \frac{\partial \tilde{G}}{\partial x^\gamma} + \frac{\partial B^\lambda}{\partial y^\lambda} \frac{\partial \tilde{G}}{\partial y^\lambda} + (\rho_\beta^\prime \circ \pi) \frac{\partial \tilde{G}}{\partial x^\gamma} + \frac{\partial B^\lambda}{\partial y^\lambda} \frac{\partial \tilde{G}}{\partial y^\lambda} - (\rho_\gamma^\prime \circ \pi) \frac{\partial \tilde{G}}{\partial y^\lambda} - (L^\lambda_{\alpha\beta} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\beta\alpha} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\gamma\lambda} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\gamma\lambda} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\gamma\lambda} \circ \pi) \tilde{G}_\lambda^\beta - (L^\lambda_{\gamma\lambda} \circ \pi) \tilde{G}_\lambda^\beta.
\]

(222)

Since \( h \) is conservative, then we have (188). Differentiation of this equation with respect to \( y \) give us

\[
(\rho_\beta^\prime \circ \pi) \frac{\partial \tilde{G}}{\partial y^\lambda} + \tilde{D}_\delta_\beta \frac{\partial \tilde{G}}{\partial y^\lambda} + \tilde{D}_\beta \frac{\partial \tilde{G}}{\partial y^\lambda} + \tilde{D}_\lambda \frac{\partial \tilde{G}}{\partial y^\lambda} + \tilde{D}_\gamma \frac{\partial \tilde{G}}{\partial y^\lambda} + \tilde{D}_\lambda \frac{\partial \tilde{G}}{\partial y^\lambda} + \tilde{D}_\gamma \frac{\partial \tilde{G}}{\partial y^\lambda} + \tilde{D}_\lambda \frac{\partial \tilde{G}}{\partial y^\lambda} = 0.
\]

(223)

Setting two above equation in (221) we obtain

\[
\tilde{D}_\delta_\beta \delta_\beta = \frac{1}{2} \tilde{G}^\gamma_{\alpha\beta} \frac{\partial \tilde{G}}{\partial x^\gamma} + \frac{\partial B^\lambda}{\partial y^\lambda} \frac{\partial \tilde{G}}{\partial y^\lambda} + \frac{\partial B^\lambda}{\partial y^\lambda} \frac{\partial \tilde{G}}{\partial y^\lambda} - \tilde{D}_\beta \frac{\partial \tilde{G}}{\partial y^\lambda} + \tilde{D}_\lambda \frac{\partial \tilde{G}}{\partial y^\lambda} + \tilde{D}_\gamma \frac{\partial \tilde{G}}{\partial y^\lambda}.
\]

(224)
Since the \( v \)-horizontal torsion of \( \overset{c}{D} \) is zero, then we have
\[
\overset{c}{D}_{V_\alpha}V_\beta - \overset{c}{D}_{V_\beta}V_\alpha = [V_\alpha, V_\beta]_E = 0.
\]
If we replace \( \delta_\alpha, \delta_\beta, \delta_\gamma \) by \( V_\alpha, V_\beta, V_\gamma \) in \((218)-(220)\), then summing these equations and using the above equation we get
\[
\overset{c}{D}_{V_\alpha}V_\beta = \frac{1}{2} \left( \frac{\partial G_{\alpha\beta}}{\partial y^\alpha} + \frac{\partial G_{\alpha\beta}}{\partial y^\beta} - \frac{\partial G_{\alpha\beta}}{\partial y^\gamma} \right) = \frac{1}{2} \frac{\partial G_{\alpha\beta}}{\partial y^\alpha},
\]
which gives us
\[
\overset{c}{D}_{V_\alpha}V_\beta = \frac{1}{2} \frac{\partial G_{\alpha\beta}}{\partial y^\alpha} G^{\mu\nu} V_\mu.
\]
(225)

Since \( \overset{c}{D} \) is d-connection, then using the above equation we obtain
\[
\overset{c}{D}_{V_\alpha} \delta_\beta = \overset{c}{D}_{V_\alpha} F V_\beta = F \overset{c}{D}_{V_\alpha} V_\beta = \frac{1}{2} \frac{\partial G_{\alpha\beta}}{\partial y^\gamma} G^{\mu\nu} F(V_\mu),
\]
which gives us
\[
\overset{c}{D}_{V_\alpha} \delta_\beta = \frac{1}{2} \frac{\partial G_{\alpha\beta}}{\partial y^\gamma} G^{\mu\nu} \delta_\mu.
\]
(226)

Similarly, using \((224)\) we get
\[
\overset{c}{D}_{\delta_\alpha} V_\beta = \overset{c}{D}_{\delta_\alpha} J \delta_\beta = J \overset{c}{D}_{\delta_\alpha} \delta_\beta
\]
\[
= \frac{1}{2} G^{\gamma\mu}\left( \left( \rho^\alpha \circ \pi \right) \frac{\partial G_{\alpha\gamma}}{\partial x^\alpha} + B^\lambda_B \frac{\partial G_{\beta\gamma}}{\partial y^\lambda} - \frac{\partial B^\lambda_B}{\partial y^\gamma} G_{\lambda\gamma} + \frac{\partial B^\lambda_B}{\partial y^\gamma} G_{\lambda\beta} \right) J(\delta_\mu),
\]
which gives us
\[
\overset{c}{D}_{\delta_\alpha} V_\beta = \frac{1}{2} G^{\gamma\mu}\left( \left( \rho^\alpha \circ \pi \right) \frac{\partial G_{\alpha\gamma}}{\partial x^\alpha} + B^\lambda_B \frac{\partial G_{\beta\gamma}}{\partial y^\lambda} - \frac{\partial B^\lambda_B}{\partial y^\gamma} G_{\lambda\gamma} + \frac{\partial B^\lambda_B}{\partial y^\gamma} G_{\lambda\beta} \right) \delta_\mu + B_\mu^\nu V_\mu.
\]
(227)

Relations \((224)-(227)\) prove the existence and uniqueness of \( \overset{c}{D} \). \( \square \)

**Proposition 7.28.** Let \((E, F)\) be a Finsler algebroid, \( h \) be a torsion free and conservative horizontal endomorphism on \( \overset{c}{E} \) and \( \overset{c}{D} \) be the d-connection given by the above theorem. If \( h \)-deflection of \( \overset{c}{D} \) is zero, then \( h \) is the Barthel endomorphism.

**Proof.** It is sufficient to show that \( h \) is homogenous. Since \( h \)-deflection of \( \overset{c}{D}, h \) is zero, then using \((184)\) and \((224)\) we obtain
\[
0 = h^\ast(\overset{c}{D} C)(\delta_\alpha) = \overset{c}{D}_{h \delta_\alpha}(C) = \overset{c}{D}_{h \delta}(y^\beta V_\beta)
\]
\[
= \frac{1}{2} G^{\mu\gamma}\left( \left( \rho^\alpha \circ \pi \right) \frac{\partial^2 F}{\partial x^\alpha \partial y^\gamma} - y^\beta \frac{\partial B^\lambda_B}{\partial y^\beta} G_{\lambda\gamma} + \frac{\partial B^\lambda_B}{\partial y^\gamma} y^\beta G_{\lambda\beta} \right) V_\mu + B_\mu^\nu V_\mu.
\]
Since \( h \) is conservative, then we have \((185)\). Using this equation in the above equation we deduce
\[
0 = \frac{1}{2} G^{\mu\gamma}\left( - B_\alpha^\beta \frac{\partial F}{\partial y^\beta} - y^\beta \frac{\partial B^\lambda_B}{\partial y^\beta} G_{\lambda\gamma} \right) V_\mu + B_\mu^\nu V_\mu = \frac{1}{2} \left( B_\alpha^\mu - y^\beta \frac{\partial B_\alpha^\nu}{\partial y^\beta} \right) V_\mu.
\]
The above equation shows that \( h \) is homogenous. \( \square \)
If $h$ is the Barthel endomorphism of Finsler algebroid $(E, F)$, then the connection $\tilde{\nabla}$ given by (224)-(227) is called the Cartan connection of $(E, F)$. Using (221), (223), (225) and (224)-(227) we can obtain

$$
\tilde{\nabla}^\lambda_{\alpha\beta\gamma} = - (\rho^\alpha_\beta \circ \pi) \frac{\partial}{\partial x^\gamma} \left( \frac{1}{2} \frac{\partial^2 B^\nu_\beta}{\partial y^\gamma \partial y^\nu} \frac{\partial F}{\partial y^\rho} G^{\lambda \rho} + \frac{\partial B^\lambda_\beta}{\partial y^\gamma} \right) - B^\mu_\alpha \frac{\partial}{\partial y^\mu} \left( \frac{\partial B^\lambda_\beta}{\partial y^\gamma} + \frac{1}{2} \frac{\partial^2 B^\nu_\rho}{\partial y^\gamma \partial y^\nu} G^{\lambda \rho} \right) + (\rho^\nu_\beta \circ \pi) \frac{\partial}{\partial x^\nu} \left( \frac{1}{2} \frac{\partial^2 B^\rho_\gamma}{\partial y^\nu \partial y^\rho} \frac{\partial F}{\partial y^\sigma} G^{\lambda \sigma} + \frac{\partial B^\lambda_\rho}{\partial y^\gamma} \right)
$$

$$(231) $$

$$(230) $$

$$(229) $$

$$(228) $$

$$(227) $$

Let $\tilde{X}$ and $\tilde{Y}$ are sections of $L^\circ E$. Then using (224)-(227) we can obtain the following formula for Cartan connection:

$$
\tilde{\nabla} \tilde{X} \tilde{Y} = \tilde{D}_{\tilde{X}} \tilde{v} \tilde{Y} + \tilde{D}_{\tilde{v} \tilde{X}} \tilde{h} \tilde{Y} + \tilde{D}_h \tilde{X} \tilde{v} \tilde{Y} + \tilde{D}_v \tilde{X} h \tilde{Y},
$$

where

$$
\tilde{D}_h \tilde{X} h \tilde{Y} = hF[h, \tilde{X}, J\tilde{Y}]_E + F\tilde{C}(\tilde{X}, \tilde{Y}),
$$

$$
\tilde{D}_v \tilde{X} \tilde{v} \tilde{Y} = J[v, \tilde{X}, F\tilde{Y}]_E + \tilde{C}(F\tilde{X}, F\tilde{Y}),
$$

$$
\tilde{D}_{\tilde{v} \tilde{X}} \tilde{h} \tilde{Y} = h[v, \tilde{X}, J\tilde{Y}]_E + \tilde{F}(\tilde{X}, \tilde{Y}),
$$

$$
\tilde{D}_h \tilde{X} \tilde{v} \tilde{Y} = v[h, \tilde{X}, J\tilde{Y}]_E + \tilde{C}(\tilde{X}, F\tilde{Y}).
$$
Theorem 7.29. Let \((E, F)\) be a Finsler algebroid, \(h\) be a torsion free and conservative horizontal endomorphism on \(E\). \(\mathcal{G}\) be the prolongation of \(\mathcal{G}\) along \(h\). Then there exists a unique d-connection \(\overline{D}\) on \((E, F)\) such that \(\overline{D}\) is h-metrical, (i.e., \( \forall X \in \Gamma(\mathcal{E}\mathcal{E}E), \overline{D}_hX = 0\)). \(J^h\overline{D} = J^h\overline{D}\) and the h-horizontal torsion of \(\overline{D}\) is zero. Moreover, if the h-deflection of \(\overline{D}\) is zero, then \(h\) is the Barthel endomorphism.

Proof. Let there exists a d-connection \(\overline{D}\) on \((E, F)\) such that \(\overline{D}\) is h-metrical, \(J^h\overline{D} = J^h\overline{D}\) and the h-horizontal torsions of \(\overline{D}\) is zero. Since \(\overline{D}\) is h-metrical and the h-horizontal torsion of \(\overline{D}\) is zero, then similar to the proof of theorem 7.27, we can deduce

\[
\overline{D}_{\delta\epsilon,\beta} = \frac{1}{2} G^{\mu\gamma} \left( (\rho^i_{\alpha} \circ \pi) \frac{\partial G_{\lambda\gamma}}{\partial x^i} + B_{\alpha}^{\lambda} \frac{\partial G_{\beta\gamma}}{\partial y^\alpha} - \frac{\partial B_{\lambda}^{\beta}}{\partial y^\gamma} G_{\lambda\gamma} + \frac{\partial B_{\lambda}^{\beta}}{\partial y^\gamma} G_{\lambda\beta} \right) \delta_\mu. \tag{232}
\]

Also, since \(\overline{D}\) is d-connection, then above equation gives us

\[
\overline{D}_{\delta\epsilon,\beta} \gamma = \frac{1}{2} G^{\mu\gamma} \left( (\rho^i_{\alpha} \circ \pi) \frac{\partial G_{\lambda\gamma}}{\partial x^i} + B_{\alpha}^{\lambda} \frac{\partial G_{\beta\gamma}}{\partial y^\alpha} - \frac{\partial B_{\lambda}^{\beta}}{\partial y^\gamma} G_{\lambda\gamma} + \frac{\partial B_{\lambda}^{\beta}}{\partial y^\gamma} G_{\lambda\beta} \right) \gamma. \tag{233}
\]

The condition \(J^h\overline{D} = J^h\overline{D}\) and \(217\) gives us

\[
\overline{D}_V \gamma = D_{J\delta\epsilon,\beta} \gamma = D_{J\delta\epsilon,\beta} \gamma = D_{V\gamma} \gamma = 0, \tag{234}
\]

and consequently

\[
\overline{D}_V \gamma = 0. \tag{235}
\]

Relations (232), (235) prove the existence and uniqueness of \(\overline{D}\). The proof of the second part of assertion is similar to proposition 7.28.

If \(h\) is the Barthel endomorphism of Finsler algebroid \((E, F)\), then the d-connection \(\overline{D}\) given by (232) - (235) is called the Chern-Rand connection of \((E, F)\).

Using (21), (22), (23) and (232) - (235) we can get

\[
\overline{R}^{\lambda}_{\alpha\beta\gamma} = -\left( \rho^i_{\alpha} \circ \pi \right) \frac{\partial}{\partial x^i} \left( \frac{1}{2} \frac{\partial^2 B_{\lambda}^{\beta}}{\partial y^\gamma \partial y^\mu} \frac{\partial F}{\partial y^\lambda} + \frac{\partial B_{\lambda}^{\beta}}{\partial y^\mu} \right) + \frac{\partial B_{\lambda}^{\beta}}{\partial y^\mu} \frac{\partial}{\partial y^\gamma} \left( \frac{1}{2} \frac{\partial^2 B_{\lambda}^{\beta}}{\partial y^\gamma \partial y^\mu} \frac{\partial F}{\partial y^\lambda} + \frac{\partial B_{\lambda}^{\beta}}{\partial y^\mu} \right)
\]

\[
+ \left( \rho^i_{\beta} \circ \pi \right) \frac{\partial}{\partial y^i} \left( \frac{1}{2} \frac{\partial^2 B_{\lambda}^{\beta}}{\partial y^\gamma \partial y^\mu} \frac{\partial F}{\partial y^\lambda} + \frac{\partial B_{\lambda}^{\beta}}{\partial y^\mu} \right)
\]

\[
+ \left( \rho^i_{\gamma} \circ \pi \right) \frac{\partial}{\partial y^i} \left( \frac{1}{2} \frac{\partial^2 B_{\lambda}^{\beta}}{\partial y^\gamma \partial y^\mu} \frac{\partial F}{\partial y^\lambda} + \frac{\partial B_{\lambda}^{\beta}}{\partial y^\mu} \right)
\]

\[
+ \left( L^\mu_{\alpha\beta} \circ \pi \right) \left( \frac{1}{2} \frac{\partial^2 B_{\lambda}^{\beta}}{\partial y^\gamma \partial y^\mu} \frac{\partial F}{\partial y^\lambda} + \frac{\partial B_{\lambda}^{\beta}}{\partial y^\mu} \right). \tag{236}
\]

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Let  and  are sections of . Then using (232)-(235) we can obtain the following formula for Chern-Rand connection:

\[ D_X Y = D_{eX} vY + D_{eX} hY + D_{hX} vY + D_{hX} hY, \]

where

\[ D_{hX} hY = hF[hX, JY], \]

\[ D_{eX} vY = J[vX, FY], \]

\[ D_{eX} hY = h[vX, Y], \]

\[ D_{hX} vY = v[hX, vY] + \tilde{C}(X, FY). \]

**Theorem 7.30.** Let be a Finsler algebroid, be a conservative horizontal endomorphism on , be the prolongation of along . Then there exists a unique d-connection such that is v-metrical, (i.e., ), the v-vertical and v-mixed torsions of are zero.

**Proof.** Let there exists a d-connection on such that is v-metrical and the v-vertical and v-mixed torsions of are zero. Since is v-metrical and the v-vertical torsion of is zero, then similar to the proof of theorem we can deduce

\[ D_{\nu \sigma} = \frac{1}{2} \frac{\partial G_{\nu \sigma}}{\partial y} \hat{G} \nu \sigma \nu \mu, \]

(240)

Also, since is d-connection, then using the above equation we obtain

\[ D_{\nu \sigma} = \frac{1}{2} \frac{\partial G_{\nu \sigma}}{\partial y} \hat{G} \nu \sigma \nu \mu. \]

(241)

Moreover, since the v-mixed torsion of is zero, then we can obtain

\[ D_{hX} vY = v[\delta_{\alpha}, Y] = -\frac{\partial B^\mu_{\delta \beta}}{\partial y} \nu \mu, \]

(242)

and consequently

\[ D_{hX} \delta_{\beta} = -\frac{\partial B^\mu_{\delta \beta}}{\partial y} \nu \mu, \]

(243)

because is d-connection. Relations (240)-(243) prove the existence and uniqueness of .

**Proposition 7.31.** Let be a Finsler algebroid, be a conservative horizontal endomorphism on and be the d-connection given by the above theorem. If h-horizontal torsion and h-deflection of are zero, then is the Barthel endomorphism.
Proof. The proof is similar to proof of proposition \[ \square \]

If \( h \) is the Barthel endomorphism of Finsler algebroid \((E,F)\), then the d-connection \( \tilde{D} \) given by (240)-(243) is called the Hashiguchi connection of \((E,F)\). Using (91), (92), (93) and (240)-(243) we can obtain
\[
\tilde{R}^\lambda_{\alpha\beta\gamma} = - (\rho^\alpha_\beta \circ \pi) \frac{\partial^2 B^\lambda_\alpha}{\partial x^\beta \partial y^\gamma} - B^\mu_\alpha \frac{\partial^2 B^\lambda_\beta}{\partial y^\mu \partial y^\gamma} + (\rho^\beta_\beta \circ \pi) \frac{\partial^2 B^\lambda_\beta}{\partial x^\mu \partial y^\gamma} + B^\mu_\beta \frac{\partial^2 B^\lambda_\beta}{\partial y^\mu \partial y^\gamma} + \frac{1}{2} R^\mu_{\alpha\beta\gamma} \frac{\partial g_{\gamma\xi}}{\partial y^\mu} g^{\lambda\xi},
\]
\[
\tilde{P}^\lambda_{\alpha\beta\gamma} = (\rho^\alpha_\alpha \circ \pi) \frac{\partial G_{\gamma\xi}}{\partial y^\alpha} + \frac{1}{2} B^\mu_\alpha \frac{\partial G_{\gamma\xi}}{\partial y^\mu} - \frac{1}{2} B^\mu_\beta \frac{\partial G_{\gamma\xi}}{\partial y^\mu} + \frac{1}{2} G_{\gamma\xi} \frac{\partial B^\mu_\beta}{\partial y^\mu} + \frac{1}{2} \frac{\partial^2 B^0_\beta}{\partial y^\gamma} + \frac{1}{2} \frac{\partial^2 B^0_\beta}{\partial y^\gamma} + \frac{1}{4} \frac{G_{\gamma\rho} \frac{\partial G_{\xi\sigma}}{\partial y^\rho}}{\partial y^\gamma} + \frac{1}{4} \frac{G_{\gamma\rho} \frac{\partial G_{\xi\sigma}}{\partial y^\rho}}{\partial y^\gamma}.
\]

Let \( \tilde{X} \) and \( \tilde{Y} \) are sections of \( \mathcal{L}^\circ E \). Then using (240)-(243) we can obtain the following formula for Hashiguchi connection:
\[
\tilde{D}_{\tilde{X}} \tilde{Y} = D_{\tilde{X}} \tilde{Y} + \frac{1}{2} \left( \frac{\partial G_{\gamma\xi}}{\partial y^\alpha} - \frac{\partial G_{\gamma\xi}}{\partial y^\alpha} \right) + \frac{1}{4} \left( G_{\gamma\rho} \frac{\partial G_{\xi\sigma}}{\partial y^\rho} \right) + \frac{1}{4} \left( G_{\gamma\rho} \frac{\partial G_{\xi\sigma}}{\partial y^\rho} \right),
\]
where
\[
\tilde{D}_{h_{\tilde{X}}} h_{\tilde{Y}} = hF[h_{\tilde{X}}, J\tilde{Y}]_E, \tag{244}
\]
\[
\tilde{D}_{v_{\tilde{X}}} v_{\tilde{Y}} = J[v_{\tilde{X}}, F\tilde{Y}]_E + C(F\tilde{X}, F\tilde{Y}), \tag{245}
\]
\[
\tilde{D}_{h_{\tilde{X}}} h_{\tilde{Y}} = h[v_{\tilde{X}}, \tilde{Y}]_E + FC(F\tilde{X}, \tilde{Y}), \tag{246}
\]
\[
\tilde{D}_{v_{\tilde{X}}} v_{\tilde{Y}} = v[h_{\tilde{X}}, v\tilde{Y}]_E. \tag{247}
\]

**Theorem 7.32.** Let \( h \) be the Barthel endomorphism on Finsler algebroid \((E,F)\). Then the Cartan connection \( \tilde{D} \)

(i) is Chern-Rund connection if \( J^* \tilde{D} = J^* \tilde{D} \),

(ii) is Hashiguchi connection if \( ^{\circ h} \tilde{D} = \tilde{D} \),

(iii) is Berwald connection if it is the Chern-Rund connection and the Hashiguchi connection at the same time.

8 **Generalized Berwald Lie algebroids**

In this section, \( h \)-basic distinguished connections are introduced on Finsler algebroids. We have more attention to Ichijyō connection. Dealing with conservative endomorphisms, generalized Berwald Lie algebroid is introduced and Wagner-Ichijyō connection as a special case is studied notably.
Definition 8.1. Let \((D,h)\) be a d-connection on \(\mathcal{E} E\). We call it a \(h\)-basic d-connection if there is a linear connection \(\nabla\) on \(E\) such that

\[
D_{X^h}Y^V = (\nabla_X Y)^V, \quad \forall X, Y \in \Gamma(E).
\] (248)

Linear connection \(\nabla\) in the above definition is called the basic connection

belongs to \((D,h)\). Note that the base connection of a \(h\)-basic d-connection is unique.

Proposition 8.2. Let \((D,h)\) be a d-connection on \(\mathcal{E} E\) and \((\tilde{D},h)\) be the d-connection associated to \((D,h)\) given by (77). Then \((D,h)\) is \(h\)-basic if and only if the mixed curvature of \((\tilde{D},h)\) is zero.

Proof. Let \((D,h)\) be a \(h\)-basic d-connection on \(\mathcal{E} E\) and \(\{e_\alpha\}\) be a basis of \(\Gamma(E)\). Since \(\nabla_{e_\alpha} e_\beta\) belongs to \(\Gamma(E)\), then we can write it as \(\nabla_{e_\alpha} e_\beta = \Gamma_\alpha^\beta e_\gamma\), where \(\Gamma_\alpha^\beta\) are local functions on \(M\). From (248) we can deduce

\[
D_{\delta_\alpha} V_\beta = D_{\alpha_\delta} e_\beta = (\nabla_{e_\alpha} e_\beta)^V = (\Gamma_\alpha^\beta \circ \pi) V_\gamma.
\]

Thus we have \(F^\gamma_{\alpha \beta} = (\Gamma_\alpha^\beta \circ \pi),\) where \(\Gamma_\alpha^\beta\) are the local coefficients of \(D_{\delta_\alpha} V_\beta\). Since \(F^\gamma_{\alpha \beta}\) are functions with respect to \((x^h),\) only, then using the first part of (77) we get \(P^\lambda_{\alpha \beta \gamma} = 0\), i.e., the mixed curvature of \((\tilde{D},h)\) is zero.

Conversely, let the mixed curvature of \((\tilde{D},h)\) be zero. Then from (77) we derive that \(F^\gamma_{\alpha \beta}\) are functions with respect to \((x^h),\) only. Now we define \(\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)\) by \((\nabla_X Y)^V := D_X Y^V\). Since the vertical lift of a section of \(E\) is unique, then \(\nabla\) is well defined. Also, we have

\[
(\nabla_X (fY))^V = D_X (fY)^V = D_X ((f^v)Y^V) = \rho_L(X^h)(f^v)Y^V + f^v D_X Y^V,
\]

where \(X,Y \in \Gamma(E)\) and \(f \in C^\infty(M)\). It is easy to check that \(\rho_L(X^h)(f^v) = (\rho(X)f)^v\). Setting this in the above equation we get

\[
(\nabla_X (fY))^V = (\rho(X)f)^v Y^V + f^v D_X Y^V = (\rho(X)f)^v Y^V + f^v (\nabla_X Y)^V
\]

which gives us \(\nabla_X (fY) = \rho(X)(fY) + f\nabla_X Y\), because the vertical lift is unique.

Similarly we can obtain \(\nabla_X (gY) = f\nabla_X Y + g\nabla_X Z\) and \(\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z\), for all \(X,Y,Z \in \Gamma(E)\) and \(f,g \in C^\infty(M)\). Thus \(\nabla\) is a linear connection on \(E\) and consequently \((D,h)\) is \(h\)-basic. \(\square\)

Let \(\nabla\) be a linear connection on \(E\), \(\{e_\alpha\}\) be a basis of \(\Gamma(E)\) and \(\nabla_{e_\alpha} e_\beta = \Gamma_\alpha^\beta e_\gamma\). Then

\[
h_{\nabla} = (X_\alpha - g^\lambda(\Gamma_\alpha^\beta \circ \pi) V_\gamma) \otimes X^\alpha,
\] (249)

is a horizontal endomorphism on \(\mathcal{E} E\). Indeed we have

\[
(\nabla_X Y)^V = [X^{h_{\nabla}}, Y^V]_E, \quad \forall X,Y \in \Gamma(E).
\]

We call \(h_{\nabla}\) given by (249) the horizontal endomorphism generated by \(\nabla\). It is easy to see that \(h_{\nabla}\) is homogenous and it is smooth on the whole \(\mathcal{E} E\).
Lemma 8.3. Let $\nabla$ be a linear connection on $E$ and $h_{\nabla}$ be the horizontal endomorphism generated by $\nabla$. If $K_{\alpha\beta\gamma}^{\lambda}$ and $R_{\alpha\beta}^{\lambda}$ are the local coefficients of the curvature tensors of $\nabla$ and $h_{\nabla}$, respectively, then we have $\hat{y}(K_{\alpha\beta\gamma}^{\lambda} \circ \pi) = -R_{\alpha\beta}^{\lambda}$.

Proof. Setting $\nabla$ endomorphism generated by $h$, we have

$$R_{\alpha\beta}^{\lambda} = y\gamma((\Gamma_{\alpha\gamma}^{\lambda} \circ \pi)) = \frac{\partial (\Gamma_{\alpha\gamma}^{\lambda} \circ \pi)}{\partial x^\lambda} + (\Gamma_{\alpha\gamma}^{\lambda} \circ \pi)\frac{\partial (\Gamma_{\alpha\gamma}^{\lambda} \circ \pi)}{\partial x^\lambda} - (\Gamma_{\alpha\gamma}^{\lambda} \circ \pi)(\Gamma_{\beta\gamma}^{\lambda} \circ \pi) = -y\gamma(K_{\alpha\beta\gamma}^{\lambda} \circ \pi).$$

\[ \Box \]

Corollary 8.4. Let $\nabla$ be a linear connection on $E$ and $h_{\nabla}$ be the horizontal endomorphism generated by $\nabla$. Then the curvature of $\nabla$ is zero if and only if the curvature of $h_{\nabla}$ vanishes.

Proposition 8.5. Let $(D, h)$ be a $h$-basic d-connection with base connection $\nabla$ and $h_{\nabla}$ be the horizontal endomorphism generated by $\nabla$. Then

$$D_{\alpha\beta} C = X^h - X^{h_{\nabla}}.$$

Proof. Let $F_{\alpha\beta}^\gamma$ be the local coefficients of $D_{\delta\alpha} V_{\beta}$ and $\Gamma_{\alpha\beta\gamma}^{\lambda}$ be the local coefficients of $\nabla_{\alpha\beta\gamma}$. In the above proposition we show that $F_{\alpha\beta}^\gamma = (\Gamma_{\alpha\beta}^{\gamma} \circ \pi)$, because $(D, h)$ be a $h$-basic d-connection with base connection $\nabla$. Thus we can obtain

$$D_{\alpha\beta} C = (X^\alpha \circ \pi)(B_{\alpha\beta}^\gamma + y\gamma F_{\alpha\gamma}^\beta) V_{\beta} = (X^\alpha \circ \pi)(B_{\alpha\beta}^\gamma + y\gamma(\Gamma_{\alpha\gamma}^{\beta} \circ \pi)) V_{\beta}, \quad (250)$$

where $X = X^\alpha e_\alpha$, $X^h = (X^\alpha \circ \pi) \delta_\alpha$ and $h$ is given by (250), (250), (249) and the above equation give us

$$X^h - X^{h_{\nabla}} = (X^\alpha \circ \pi)(X_\alpha + B_{\alpha\beta}^\gamma V_{\beta}) - (X^\alpha \circ \pi)(X_\alpha - y\gamma(\Gamma_{\alpha\gamma}^{\beta} \circ \pi)V_{\beta}) = D_{\alpha\beta} C.$$

\[ \Box \]

Corollary 8.6. Let $(D, h)$ be a $h$-basic d-connection with base connection $\nabla$ and $h_{\nabla}$ be the horizontal endomorphism generated by $\nabla$. Then $h_{\nabla}$ coincides with $h$ if and only if the $h$-deflection of $(D, h)$ is zero.

Proof. If $h_{\nabla} = h$, then from the above proposition we have $D_{\alpha\beta} C = 0$ and in particular $D_{\alpha\beta} C = D_{\delta\alpha} C = 0$. Therefore we deduce $h^\ast(DC)(\delta_\alpha) = D_{\delta\alpha} C = 0$, i.e., the $h$-deflection of $(D, h)$ vanishes. Conversely, if the $h$-deflection of $(D, h)$ is zero, then we deduce $D_{\delta\alpha} C = 0$ and consequently $D_{\alpha\beta} C = 0$. Thus from the above proposition we derive that $X^h = X^{h_{\nabla}}$ and consequently $h = h_{\nabla}$. \[ \Box \]

Corollary 8.7. Let $(D, h)$ be a $h$-basic d-connection with base connection $\nabla$ and $h_{\nabla}$ be the horizontal endomorphism generated by $\nabla$. If the $h$-deflection of $(D, h)$ is zero, then we have

\[ (i) \quad D_{h\bar{X}} v \bar{Y} = v[h\bar{X}, v \bar{Y}]_E, \quad (ii) \quad D_{h\bar{X}} h \bar{Y} = h F[h \bar{X}, J \bar{Y}]_E, \]

where $\bar{X}, \bar{Y} \in \Gamma(\mathcal{L}^E)$. 72
Proof. Let $\tilde{X} = \tilde{X}^\alpha \delta_\alpha + \tilde{X}^\alpha V_\alpha$ and $\tilde{Y} = \tilde{Y}^\beta \delta_\beta + \tilde{Y}^\beta V_\beta$ be sections of $\mathcal{L}^E$. Since the $h$-deflection of $(D, h)$ is zero, then using the above corollary we have $h = h_\nabla$ and consequently $B^\alpha_\beta = -y^\lambda (\Gamma^\lambda_{\alpha\beta} \circ \pi)$. Thus we can obtain

$$v[h\tilde{X}, v\tilde{Y}]_E = \tilde{X}^\alpha \left( (\rho^\alpha_\beta \circ \pi) \frac{\partial \tilde{Y}^\beta}{\partial \tilde{X}^\alpha} - y^\lambda (\Gamma^\lambda_{\alpha\beta} \circ \pi) \frac{\partial \tilde{Y}^\beta}{\partial y^\lambda} \right) V_\beta + X^\alpha y^\beta (\Gamma^\gamma_{\alpha\beta} \circ \pi) V_\gamma$$

$$= D_h \tilde{X} v\tilde{Y},$$

because $F^\gamma_{\alpha\beta} = (\Gamma^\gamma_{\alpha\beta} \circ \pi)$, where $F^\gamma_{\alpha\beta}$ are the local coefficients of $D_h V_\beta$. Therefore we have (i). Similar to (215), using (i) we can deduce (ii).

Proposition 8.8. Let $(D, h)$ be a $h$-basic $d$-connection with base connection $\nabla$ and $h$ be a homogeneous horizontal endomorphism. Then $h$-deflection of $(D, h)$ is zero if and only if the $v$-mixed torsion of $D$ is zero.

Proof. Using (33) we have

$$P^1(\delta_\alpha, \delta_\beta) = D_\delta_\alpha V_\beta - v[\delta_\alpha, V_\beta]_E = ((\Gamma^\gamma_{\alpha\beta} \circ \pi) + \frac{\partial B^\gamma_{\alpha\beta}}{\partial y^\gamma}) V_\gamma. \quad (251)$$

Thus $P^1 = 0$ if and only if $\frac{\partial B^\gamma_{\alpha\beta}}{\partial y^\gamma} = - (\Gamma^\gamma_{\alpha\beta} \circ \pi)$. But since $h$ is homogeneous, then we have $y^\gamma \frac{\partial B^\gamma_{\alpha\beta}}{\partial y^\gamma} = B^\gamma_\alpha$. Thus we can deduce $P^1 = 0$ if and only if $B^\gamma_\alpha = -y^\beta (\Gamma^\gamma_{\alpha\beta} \circ \pi)$ (this equation gives us $h = h_\nabla$). Therefore the vanishing of $P^1$ is equivalent to vanishing of the $h$-deflection of $(D, h)$.

Remark 8.9. Since in corollaries 8.6, 8.7 and proposition 8.8 we work on the vanishing of $h$-deflection of $(D, h)$, then we have $h = h_\nabla$. But $h_\nabla$ is smooth on the whole $\mathcal{L}^E$. Therefore the horizontal endomorphism $h$ should be smooth on the whole $\mathcal{L}^E$.

Proposition 8.10. Let $(D, h)$ be a $h$-basic $d$-connection with base connection $\nabla$ and the horizontal endomorphism $h$ be smooth on whole $\mathcal{L}^E$. Then the $h$-deflection of $(D, h)$ coincides with the tension of $h$ if and only if the $v$-mixed torsion of $D$ is zero.

Proof. Let the $v$-mixed torsion of $D$ be zero. Then from (251) we can deduce $(\Gamma^\gamma_{\alpha\beta} \circ \pi) = -\frac{\partial B^\gamma_{\alpha\beta}}{\partial y^\gamma}$. But from (250) we have

$$D_\delta_\alpha C = (B^\gamma_\alpha + y^\gamma (\Gamma^\gamma_{\alpha\beta} \circ \pi)) V_\gamma.$$

Setting $(\Gamma^\gamma_{\alpha\beta} \circ \pi) = -\frac{\partial B^\gamma_\alpha}{\partial y^\gamma}$ in the above equation and using (31) we obtain

$$h^*(DC)(\delta_\alpha) = D_\delta_\alpha C = (B^\gamma_\alpha + y^\gamma \frac{\partial B^\gamma_\alpha}{\partial y^\gamma}) V_\gamma = H(\delta_\alpha).$$

Conversely, if $h^*(DC) = H$ and $h$ is smooth on whole $\mathcal{L}^E$ then using (31) and (250) we obtain $\frac{\partial B^\gamma_\alpha}{\partial y^\gamma} = - (\Gamma^\gamma_{\alpha\beta} \circ \pi)$. Setting this equation in (251) we deduce $P^1 = 0$.

Theorem 8.11. Let $(D, h)$ be a $h$-basic $d$-connection on Finsler algebroid $(E, F)$ and the first Cartan tensor be nonzero on $(E, F)$. Then $(D, h)$ is $h$-metrical if and only if $h$ is conservative and the $h$-deflection of $(D, h)$ is zero.
Proof. Let \((D, h)\) be \(h\)-metrical. Then we get
\[
X^h \mathcal{F} = \frac{1}{2} X^h (\tilde{\mathcal{G}}(C, C)) = \tilde{\mathcal{G}}(C, D_X h C) = (X^\alpha \circ \pi)(B_\alpha^\beta + y^\gamma (\Gamma^\alpha_{\beta\gamma} \circ \pi)) \frac{\partial F}{\partial y^\beta} \\
= \langle D_X h C \rangle \mathcal{F}.
\]
But from proposition \(8.6\) we have
\[
(D_X h C) \mathcal{F} = X^h \mathcal{F} - X^{h^c} \mathcal{F}.
\]
Two above equations gives us \(X^{h^c} \mathcal{F} = 0\) and consequently \(d_{h^c} \mathcal{F} = 0\). Thus \(h^c\) is conservative. Direct calculation we obtain
\[
X^{h^c} \tilde{\mathcal{G}}(\nu_\beta, \nu_\lambda) = \tilde{\mathcal{G}}(D_X \nu_\beta, \nu_\lambda) - \tilde{\mathcal{G}}(\nu_\beta, D_X \nu_\lambda) = (X^\alpha \circ \pi) \left( (\rho_\alpha^\beta \circ \pi) \frac{\partial \tilde{\mathcal{G}}}{\partial y^\alpha} \right) \\
- y^\gamma (\Gamma^\mu_{\alpha\gamma} \circ \pi) \frac{\partial \tilde{\mathcal{G}}}{\partial y^\mu} - (\Gamma^\gamma_{\alpha\lambda} \circ \pi) \tilde{\mathcal{G}} - (\Gamma^\gamma_{\alpha\lambda} \circ \pi) \tilde{\mathcal{G}} \beta \right). \tag{252}
\]
Since \(h^c\) is conservative, then we have \(B_\lambda^\alpha = -y^\mu (\Gamma^\alpha_{\mu\beta} \circ \pi)\). Setting this equation in \(252\) we can see that the right side of the above equation vanishes. Therefore we have
\[
X^{h^c} \tilde{\mathcal{G}}(\nu_\beta, \nu_\lambda) = \tilde{\mathcal{G}}(D_X \nu_\beta, \nu_\lambda) + \tilde{\mathcal{G}}(\nu_\beta, D_X \nu_\lambda). \tag{253}
\]
In other hand, since \((D, h)\) is \(h\)-metrical, then we have
\[
X^h \tilde{\mathcal{G}}(\nu_\beta, \nu_\lambda) = \tilde{\mathcal{G}}(D_X \nu_\beta, \nu_\lambda) + \tilde{\mathcal{G}}(\nu_\beta, D_X \nu_\lambda).
\]
Two above equations give us
\[
(X^{h^c} - X^h) \tilde{\mathcal{G}}(\nu_\beta, \nu_\lambda) = 0. \tag{254}
\]
For vertical metric \(\mathcal{G}\), using \(201\) we can obtain
\[
\mathcal{G}(C(\delta_\alpha, \delta_\beta), X^h - X^{h^c}) = (X^\sigma \circ \pi)(B_\sigma^\lambda + y^\gamma (\Gamma^\lambda_{\sigma\gamma} \circ \pi)) \mathcal{G}(C(\delta_\alpha, \delta_\beta), \nu_\lambda) \\
= \frac{1}{2} (X^\sigma \circ \pi)(B_\sigma^\lambda + y^\gamma (\Gamma^\lambda_{\sigma\gamma} \circ \pi)) (J \nu_\gamma \mathcal{G})(\delta_\beta, \delta_\lambda) \\
= \frac{1}{2} (X^\sigma \circ \pi)(B_\sigma^\lambda + y^\gamma (\Gamma^\lambda_{\sigma\gamma} \circ \pi)) (\nu_\alpha \mathcal{G}(\nu_\beta, \nu_\lambda)).
\]
Since \(\nu_\alpha \mathcal{G}(\nu_\beta, \nu_\lambda) = \nu_\lambda \mathcal{G}(\nu_\alpha, \nu_\beta)\), then using this equation in the above equation and using \(254\) we deduce
\[
\mathcal{G}(C(\delta_\alpha, \delta_\beta), X^h - X^{h^c}) = \frac{1}{2} (X^\sigma \circ \pi)(B_\sigma^\lambda + y^\gamma (\Gamma^\lambda_{\sigma\gamma} \circ \pi)) (\nu_\lambda \mathcal{G}(\nu_\alpha, \nu_\beta)) \\
= (X^{h^c} - X^h) \mathcal{G}(\nu_\alpha, \nu_\beta) \\
= (X^{h^c} - X^h) \tilde{\mathcal{G}}(\nu_\alpha, \nu_\beta) = 0.
\]
From the above equation we can derive that \(\mathcal{G}(C(\tilde{Y}, \tilde{Z}), X^h - X^{h^c}) = 0\), for all \(\tilde{Y}, \tilde{Z} \in \Gamma(\mathcal{E}^* E)\). Since \(\mathcal{G}\) is non-degenerated, then this equation gives us \((X^h - X^{h^c} = 0) \) or \(X^h = X^{h^c}\) and consequently \(h = h^c\). Thus \(h\) is conservative and using corollary \(8.6\) the \(h\)-deflection of \((D, h)\) vanishes. Conversely, let \(h\)
be the conservative horizontal endomorphism and the $h$-deflection of $(D, h)$ be zero. Then from corollary 8.6, $h$ coincides with $h_{\nabla}$ and so $h_{\nabla}$ is conservative. Therefore we have (253) which gives us

$$
(D X, \tilde{G})(V_\alpha, V_\beta) = (X^{h} - X^{h_{\nabla}})\tilde{G}(V_\alpha, V_\beta) = 0.
$$

Also, since $h = h_{\nabla}$ and $h$ is conservative, then using (ii) of corollary 8.6 and (222) we obtain

$$
X^{h}\tilde{G}(V_\beta, V_\lambda) - \tilde{G}(D X^{h} V_\beta, V_\lambda) - \tilde{G}(V_\beta, D X^{h} V_\lambda) = 0,
$$

which gives us $(D X^{h}\tilde{G})(\delta_\alpha, \delta_\beta) = 0$. Therefore we can deduce $D_{h}X = 0$, for all $\tilde{X} \in \mathcal{E}^\pi E$.

8.1 Ichijyō connection

**Theorem 8.12.** Let $(E, F)$ be a Finsler algebroid, $\nabla$ be a linear connection on $E$, $h_{\nabla}$ be the horizontal endomorphism generated by $\nabla$ and $G$ be the prolongation of vertical metric along $h_{\nabla}$. Then there is a unique $d$-connection $(\tilde{D}, h_{\nabla})$ on $(E, F)$ such that

(i) $\tilde{D}$ is $v$-metrical,

(ii) The $v$-vertical torsion of $\tilde{D}$ is zero,

(iii) The $h$-deflection of $(\tilde{D}, h_{\nabla})$ is zero,

(iv) The mixed curvature of $(\tilde{D}, h_{\nabla})$ is zero,

where $(\tilde{D}, h_{\nabla})$ the $d$-connection associated to $(\tilde{D}, h_{\nabla})$ given by (95).

**Proof.** Let there exists a $d$-connection $\tilde{D}$ on $(E, F)$ such that $\tilde{D}$ satisfies in (i)-(iv). Since $\tilde{D}$ is $v$-metrical and the $v$-vertical of $\tilde{D}$ is zero, then similar to the proof of theorem 7.27 we can deduce

$$
\tilde{D}_{V_\alpha} V_\beta = \frac{1}{2} \frac{\partial G_{\beta\gamma}}{\partial y_\alpha} G^{\gamma\mu} V_\mu = C_{\gamma\beta\alpha} V_\mu.
$$

(255)

Also, since $\tilde{D}$ is $d$-connection, then using the above equation we obtain

$$
\tilde{D}_{V_\alpha} \delta_\beta = \frac{1}{2} \frac{\partial G_{\beta\gamma}}{\partial y_\alpha} G^{\gamma\mu} \delta_\mu = C_{\gamma\beta\alpha} \delta_\mu.
$$

(256)

Condition (iv) together proposition 8.2 told us that $(\tilde{D}, h_{\nabla})$ is $h$-basic. Thus there exists a unique linear connection $\tilde{\nabla}$ on $E$ such that $(\tilde{\nabla} X Y)^V = D_{X^{h_{\nabla}}} Y^V$. But using (iii) and corollary 8.6, we deduce that $\tilde{\nabla}$ coincides with $\nabla$. Thus we have

$$
\tilde{\nabla} D_{X^{h_{\nabla}}} Y^V = (\nabla X Y)^V, \quad \forall X, Y \in \Gamma(E).
$$

From the above equation we obtain

$$
\tilde{D}_{h_\alpha} V_\beta = (\Gamma^{\gamma}_{\alpha\beta} \circ \pi) V_\gamma,
$$

(257)
Using the above equations we can obtain the following formula for Ichijyō connection:

\[ \tilde{\nabla} \] on Finsler algebroid \((E, F)\).

Let \( \tilde{X} \) and \( \tilde{Y} \) are sections of \( L^0 E \). Then using \( 255 \) we can obtain the following formula for Ichijyō connection:

\[
\tilde{\nabla} \tilde{X} \tilde{Y} = \tilde{\nabla}_{\tilde{v} \tilde{X}} \tilde{v} \tilde{Y} + \tilde{\nabla}_{\tilde{v} \tilde{X}} \tilde{v} \tilde{Y} + \tilde{\nabla}_{\tilde{h} \tilde{X}} \tilde{v} \tilde{Y} + \tilde{\nabla}_{\tilde{h} \tilde{X}} \tilde{v} \tilde{Y},
\]

where

\[
\tilde{\nabla}_{\tilde{v} \tilde{X}} \tilde{v} \tilde{Y} = h^v [\nabla \tilde{X}, \tilde{Y}] + \tilde{\nabla}_{\tilde{v} \tilde{X}} \tilde{v} \tilde{Y} + \tilde{\nabla}_{\tilde{h} \tilde{X}} \tilde{v} \tilde{Y} + \tilde{\nabla}_{\tilde{h} \tilde{X}} \tilde{v} \tilde{Y},
\]

Using the above equations we can obtain

\[
\tilde{\nabla}_{X^\alpha} Y_{hv} = \left( (X^\alpha \circ \pi) (\rho^\alpha \circ \pi) \frac{\partial (Y^\gamma \circ \pi)}{\partial x^\alpha} + (X^\alpha \circ \pi) (Y^\beta \circ \pi) (\Gamma_{\alpha \beta} \circ \pi) \right) \delta_{hv} = (\nabla_X Y^\gamma)_{hv},
\]

\[
\tilde{\nabla}_{X^\alpha} Y^\gamma = (X^\alpha \circ \pi) (Y^\beta \circ \pi) \Gamma^{\gamma}_{\alpha \beta} = C(X^\alpha, Y^h),
\]

\[
\tilde{\nabla}_{X^\alpha} Y_{hv} = (X^\alpha \circ \pi) (Y^\beta \circ \pi) \Gamma^h_{\alpha \beta} \delta_{hv} = FC(X^\alpha, Y^h),
\]

\[
\tilde{\nabla}_{X^\alpha} Y^\gamma = \left( (X^\alpha \circ \pi) (\rho^\alpha \circ \pi) \frac{\partial (Y^\gamma \circ \pi)}{\partial x^\alpha} + (X^\alpha \circ \pi) (Y^\beta \circ \pi) (\Gamma_{\alpha \beta} \circ \pi) \right) \delta_{hv} = (\nabla_X Y^\gamma),
\]

where \( X, Y \in \Gamma(E) \).

**Proposition 8.13.** Let \((E, F)\) be a Finsler algebroid, \( \nabla \) a linear connection on \( E \) and \((\tilde{\nabla}, h^v)\) be the d-connection induced by \( \nabla \). Then

\[
(\tilde{\nabla}_{\tilde{V}^\alpha} C)(\tilde{Y}, \tilde{Z}) = (\tilde{\nabla}_{\tilde{V}^\alpha} C)(\tilde{X}, \tilde{Z}), \quad \forall \tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma( L^0 E).
\]

**Proof.** It is sufficient to show that \((\tilde{\nabla}_{\tilde{V}^\alpha} C)(\delta_{\beta}, \delta_{\gamma}) = (\tilde{\nabla}_{\tilde{V}^\alpha} C)(\delta_{\alpha}, \delta_{\gamma})\). Using the local expression of the first Cartan tensor and \( 255 \) we get

\[
(\tilde{\nabla}_{\tilde{V}^\alpha} C)(\delta_{\beta}, \delta_{\gamma}) = \frac{1}{4} \left( 2 \frac{\partial^2 \Gamma^{\gamma}_{\alpha \lambda}}{\partial y^\alpha \partial y^\beta} \rho^\gamma_{\lambda \mu} + 2 \frac{\partial \Gamma^{\gamma}_{\alpha \lambda}}{\partial y^\gamma} \rho^\lambda_{\mu \nu} + \frac{\partial \Gamma^{\gamma}_{\alpha \lambda}}{\partial y^\gamma} \rho^\lambda_{\nu \sigma} \rho^\sigma_{\lambda \mu} \right) \delta_{hv}.
\]
Since $\mathcal{G}^{\lambda\sigma} \frac{\partial \mathcal{G}^{\sigma\mu}}{\partial y^\alpha} = -\mathcal{G}_{\nu\lambda} \frac{\partial \mathcal{G}^{\lambda\mu}}{\partial y^\alpha}$, then we get
\[
\frac{\partial \mathcal{G}_{\nu\lambda} \mathcal{G}^{\lambda\sigma} \frac{\partial \mathcal{G}^{\sigma\mu}}{\partial y^\alpha}}{\partial y^\beta} = -\frac{\partial \mathcal{G}_{\nu\lambda} \mathcal{G}^{\lambda\mu}}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^\gamma}.
\]

Similarly we obtain
\[
\frac{\partial \mathcal{G}_{\nu\lambda} \mathcal{G}^{\lambda\sigma} \frac{\partial \mathcal{G}^{\sigma\mu}}{\partial y^\alpha}}{\partial y^\beta} = \frac{\partial \mathcal{G}_{\nu\lambda} \mathcal{G}^{\lambda\mu}}{\partial y^\beta} \frac{\partial y^\alpha}{\partial y^\gamma}.
\]

Setting two above equation in (267) give us
\[
\nabla (D_{\nu\alpha} C)(\delta_\beta, \delta_\gamma) = \frac{1}{4} \left( \frac{\partial^2 \mathcal{G}_{\nu\lambda} \mathcal{G}^{\lambda\mu}}{\partial y^\alpha \partial y^\beta} + \frac{\partial \mathcal{G}_{\nu\lambda} \mathcal{G}^{\lambda\mu}}{\partial y^\alpha} \frac{\partial \mathcal{G}^{\lambda\mu}}{\partial y^\beta} - \frac{\partial \mathcal{G}_{\nu\lambda} \mathcal{G}^{\lambda\mu}}{\partial y^\alpha} \frac{\partial \mathcal{G}^{\lambda\mu}}{\partial y^\beta} \right) \forall y,
\]

Similarly we can obtain
\[
\nabla (D_{\nu\alpha} C)(\delta_\alpha, \delta_\gamma) = \frac{1}{4} \left( \frac{\partial^2 \mathcal{G}_{\nu\lambda} \mathcal{G}^{\lambda\mu}}{\partial y^\alpha \partial y^\beta} + \frac{\partial \mathcal{G}_{\nu\lambda} \mathcal{G}^{\lambda\mu}}{\partial y^\alpha} \frac{\partial \mathcal{G}^{\lambda\mu}}{\partial y^\beta} - \frac{\partial \mathcal{G}_{\nu\lambda} \mathcal{G}^{\lambda\mu}}{\partial y^\alpha} \frac{\partial \mathcal{G}^{\lambda\mu}}{\partial y^\beta} \right) \forall y.
\]

Two above equation show that $(\nabla (D_{\nu\alpha} C))(\delta_\beta, \delta_\gamma) = (\nabla (D_{\nu\beta} C))(\delta_\alpha, \delta_\gamma)$. 

Let $t_\nu$ be the weak torsion of $h_\nu$ and $T_\nu$ be the torsion of $\nabla$. Then using (33) and (240) we deduce
\[
\left( \Gamma^\gamma_{\beta\alpha} - \Gamma^\gamma_{\beta\alpha} - e_\beta \right) \circ \pi = (T_\nu(e_\alpha, e_\beta))^{\nu\gamma},
\]
where $t^\gamma_{\alpha\beta}$ are the coefficient of $t_\nu$. If we denote by $\nabla T$, the torsion of Ichijyô connection $(D, h_\nu)$ then we get
\[
\nabla T(\delta_\alpha, \delta_\beta) = \left( \left( \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} - e_\beta \right) \circ \pi \right) \delta_\gamma - R^\gamma_{\alpha\beta} \nu_\gamma,
\]
\[
= t^\gamma_{\alpha\beta} \delta_\gamma + \Omega(\delta_\alpha, \delta_\beta) = F_\nu t_\nu(\delta_\alpha, \delta_\beta) + \Omega_\nu(\delta_\alpha, \delta_\beta),
\]
\[
= (T_\nu(e_\alpha, e_\beta))^{\nu\gamma} + \Omega_\nu(\delta_\alpha, \delta_\beta),
\]
\[
\nabla T(\delta_\alpha, \nu_\gamma) = -\frac{1}{2} \frac{\partial \mathcal{G}^{\lambda\mu}}{\partial y^\beta} \mathcal{G}^{\lambda\mu} \mathcal{C}(\delta_\alpha, \delta_\beta) = -F_\nu \mathcal{C}(\delta_\alpha, \delta_\beta),
\]
\[
\nabla T(\nu_\alpha, \nu_\beta) = 0,
\]
where $\Omega_\nu$ is the curvature tensor of $h_\nu$. From the above equations we can conclude the following

**Proposition 8.14.** Let $(\nabla, h_\nu)$ be the Ichijyô connection on Finsler algebroid $(E, \mathcal{F})$ with base connection $\nabla$. Then the torsion tensor of $\nabla$ satisfies
\[
\nabla T(\delta X, \delta Y) = F_\nu t_\nu(h_\nu X, h_\nu Y) + \Omega(h_\nu X, h_\nu Y) = F_\nu \mathcal{C}(h_\nu X, h_\nu Y)
\]
\[
+ F_\nu \mathcal{C}(F_\nu v_\nu X, h_\nu Y), \quad \forall X, Y \in \Gamma(\mathbb{E}, E).
\]
Corollary 8.15. Let \( \overline{D}, h_{\nabla} \) be the Ichijyô connection on Finsler algebroid \((E, \mathcal{F})\) with base connection \( \nabla \). Then for all \( X, Y \in \Gamma(E) \) we have

\[
\begin{align*}
\overline{\nabla}^T (X^{h_{\nabla}}, Y^{h_{\nabla}}) &= (T_{\nabla}(X, Y))^{h_{\nabla}} + \Omega_{\nabla}(X^{h_{\nabla}}, Y^{h_{\nabla}}), \\
\overline{\nabla}^T (X^{h_{\nabla}}, Y^{V}) &= -F_{\nabla}C(X^{h_{\nabla}}, F_{\nabla}V^{Y}), \\
\overline{\nabla}^T (X^{V}, Y^{V}) &= 0.
\end{align*}
\]

Let \( \overline{\nabla}_{\alpha\beta\gamma}^\lambda, \overline{P}_{\alpha\beta\gamma}^\lambda \) and \( \overline{S}_{\alpha\beta\gamma}^\lambda \) be the coefficients of the horizontal, mixed and vertical curvatures of Ichijyô connection \( \overline{D}, h_{\nabla} \), respectively. Then using (91)-(92) and (255)-(258) we get

\[
\begin{align*}
\overline{\nabla}_{\alpha\beta\gamma}^\lambda &= (\rho^i_\alpha \circ \pi) \frac{\partial (\Gamma^\lambda_{\beta\gamma} \circ \pi)}{\partial x^i} - (\rho^i_\beta \circ \pi) \frac{\partial (\Gamma^\lambda_{\alpha\gamma} \circ \pi)}{\partial x^i} + (\Gamma^\mu_{\beta\gamma} \circ \pi)(\Gamma^\lambda_{\alpha\mu} \circ \pi) \\
&\quad - (\Gamma^\mu_{\alpha\gamma} \circ \pi)(\Gamma^\lambda_{\beta\mu} \circ \pi) - (L^\mu_{\alpha\beta} \circ \pi)(\Gamma^\lambda_{\mu\gamma} \circ \pi) - R^{\mu}_{\alpha\beta} C^\lambda_{\mu\gamma} \\
&= -\frac{\partial R^{\mu}_{\alpha\beta}}{\partial y^\gamma} - R^{\mu}_{\alpha\beta} C^\lambda_{\mu\gamma}, \\
\overline{P}_{\alpha\beta\gamma}^\lambda &= (\rho^i_\alpha \circ \pi) \frac{\partial C^\lambda_{\beta\gamma}}{\partial x^i} - y^\nu (\Gamma^\mu_{\alpha\nu} \circ \pi) \frac{\partial C^\lambda_{\beta\gamma}}{\partial y^\mu} + C^\mu_{\beta\gamma}(\Gamma^\lambda_{\alpha\mu} \circ \pi) - (\Gamma^\mu_{\alpha\gamma} \circ \pi)C^\lambda_{\beta\mu} \\
&\quad - (\Gamma^\mu_{\alpha\beta} \circ \pi)C^\lambda_{\mu\gamma}, \\
\overline{S}_{\alpha\beta\gamma}^\lambda &= \frac{\partial C^\lambda_{\beta\gamma}}{\partial y^\alpha} + C^\mu_{\beta\gamma} C^\lambda_{\alpha\mu} - \frac{\partial C^\lambda_{\alpha\gamma}}{\partial y^\beta} - C^\mu_{\alpha\gamma} C^\lambda_{\beta\mu}.
\end{align*}
\]

Using the above equations we conclude the following proposition which gives us the global expressions of horizontal, mixed and vertical curvatures of Ichijyô connection.

Proposition 8.16. Let \( \overline{D}, h_{\nabla} \) be the Ichijyô connection on Finsler algebroid \((E, \mathcal{F})\) with base connection \( \nabla \). Then we have

\[
\begin{align*}
\overline{\nabla}^R (\tilde{X}, \tilde{Y}, \tilde{Z}) &= [J, \Omega_{\nabla}(\tilde{X}, \tilde{Y})]_{\mathcal{L}^N}(h_{\nabla}\tilde{Z}) + C(F_{\nabla} \Omega_{\nabla}(\tilde{X}, \tilde{Y}), \tilde{Z}), \\
\overline{\nabla}^P (\tilde{X}, \tilde{Y}, \tilde{Z}) &= (\overline{\nabla}_{h_{\nabla}}\tilde{X} \circ C)(h_{\nabla}\tilde{Y}, h_{\nabla}\tilde{Z}), \\
\overline{\nabla}^Q (\tilde{X}, \tilde{Y}, \tilde{Z}) &= C(F_{\nabla}C(\tilde{X}, \tilde{Z}), \tilde{Y}) - C(\tilde{X}, F_{\nabla}C(\tilde{Y}, \tilde{Z})),
\end{align*}
\]

where \( \tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma(\mathcal{L}^{\circ}E) \).

Corollary 8.17. The horizontal curvature of Ichijyô connection is zero if and only if the curvature of \( h_{\nabla} \) (or the curvature of base connection \( \nabla \)) is zero.

Proof. If the curvature of \( h_{\nabla} \) vanishes, then we have \( R^{\lambda}_{\alpha\beta} = 0 \). Therefore from (268) we deduce \( \overline{R}^{\lambda}_{\alpha\beta\gamma} = 0 \), i.e., the horizontal curvature of Ichijyô connection is zero. Conversely, if \( R^{\lambda}_{\alpha\beta\gamma} = 0 \), then from (268) we derive that

\[
\frac{\partial R^{\lambda}_{\alpha\beta}}{\partial y^\gamma} + R^{\mu}_{\alpha\beta} C^\lambda_{\mu\gamma} = 0.
\]
Multiplying $y^\gamma$ in the above equation and using $y^\gamma c^\lambda_{\mu\gamma} = 0$, give us $y^\gamma \frac{\partial R^\lambda_{\alpha\beta}}{\partial y^\gamma} = 0$.

But it is easy to see that $y^\gamma \frac{\partial R^\lambda_{\alpha\beta}}{\partial y^\gamma} = R^\lambda_{\alpha\beta}$. Thus we deduce $R^\lambda_{\alpha\beta} = 0$, i.e., the curvature of $h_\nabla$ is zero. Note that from corollary 8.4 we deduce that the vanishing of the horizontal curvature of Ichijyō connection is equivalent to the vanishing of the curvature of base connection $\nabla$.

From the second relation of proposition 8.16 we conclude

**Corollary 8.18.** The mixed curvature of Ichijyō connection is zero if and only if the $h$-covariant derivative of the first Cartan tensor with respect to $\hat{D}$ (i.e., $\nabla_{D_{h\nabla}} C$) vanishes.

If we denote by $\nabla_A$, $\nabla_B$, $\nabla_{R^1}$, $\nabla_{P^1}$, $\nabla_{Q^1}$ the components of torsion of Ichijyō connection, then using (88), (89) and (255)-(258) we obtain

\[
\nabla_A (\delta_\alpha, \delta_\beta) = \left( (\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} - L^\gamma_{\alpha\beta}) \circ \pi \right) \delta_\gamma = t^\gamma_{\alpha\beta} \delta_\gamma = F_{\nabla t} \nabla (\delta_\alpha, \delta_\beta),
\]

(271)

\[
\nabla_B (\delta_\alpha, \delta_\beta) = -C^\alpha_{\alpha\beta} \delta_\gamma = -F_{\nabla C} (\delta_\alpha, \delta_\beta),
\]

(272)

\[
\nabla_{R^1} (\delta_\alpha, \delta_\beta) = -R^\gamma_{\alpha\beta} \nabla_\gamma = \Omega_{\nabla} (\delta_\alpha, \delta_\beta),
\]

(273)

\[
\nabla_{P^1} = 0, \quad \nabla_{Q^1} = 0.
\]

(274)

From the above equation we conclude the following

**Proposition 8.19.** Let $(\hat{D}, h_\nabla)$ be the Ichijyō connection on Finsler algebroid $(E, F)$ with base connection $\nabla$. Then for all sections $X$ and $Y$ of $E$ we have

\[
\nabla_A (X^{h_\nabla}, Y^{h_\nabla}) = \left( T_{\nabla} (X, Y) \right)^{h_\nabla} = F_{\nabla t} \nabla (X^{h_\nabla}, Y^{h_\nabla}),
\]

\[
\nabla_B (X^{h_\nabla}, Y^{h_\nabla}) = -F_{\nabla C} (X^{h_\nabla}, Y^{h_\nabla}),
\]

\[
\nabla_{R^1} (X^{h_\nabla}, Y^{h_\nabla}) = \Omega_{\nabla} (X^{h_\nabla}, Y^{h_\nabla}),
\]

\[
\nabla_{P^1} = 0, \quad \nabla_{Q^1} = 0.
\]

From the first equation of the above proposition we have

**Corollary 8.20.** The $h$-horizontal torsion of the Ichijyō connection is zero if and only if the torsion tensor of $\nabla$ (or the weak torsion of $h_\nabla$) vanishes.

8.2 Generalized Berwald Lie algebroid

**Definition 8.21.** Let $(E, F)$ be a Finsler algebroid and $\nabla$ be a linear connection on $E$. Then $(E, F, \nabla)$ is called generalized Berwald Lie algebroid, if the horizontal endomorphism $h_\nabla$ is conservative.
Proposition 8.22. Let \((E, F, \nabla)\) be a Finsler algebroid and \(\nabla\) be a linear connection on \(E\). Then the following items are equivalent:

(i) \((E, F, \nabla)\) is a generalized Berwald Lie algebroid.

(ii) Second Cartan tensor \(\tilde{C}_\nabla\) belonging to \(\nabla\) is zero.

(iii) Ichijō connection \((\bar{D}, h\nabla)\) is \(h\nabla\)-metrical.

Proof. (i) \(\Rightarrow\) (ii). Since \(h\nabla\) is conservative, then we have (186). Setting \(B^\lambda_\alpha = -y^\sigma (\Gamma^\lambda_{\alpha\sigma} \circ \pi)\) in this equation we have

\[
(\rho^\alpha_\lambda \circ \pi) \frac{\partial F}{\partial x^\lambda} - y^\sigma (\Gamma^\lambda_{\alpha\sigma} \circ \pi) \frac{\partial F}{\partial y^\lambda} = 0. \tag{275}
\]

Differentiating the above equation with respect to \(y^\beta\) and \(y^\mu\) gives us

\[
(\rho^\alpha_\lambda \circ \pi) \frac{\partial^3 F}{\partial x^\lambda \partial y^\beta \partial y^\mu} - (\Gamma^\lambda_{\alpha\beta} \circ \pi) \frac{\partial^2 F}{\partial y^\beta \partial y^\lambda} - (\Gamma^\lambda_{\alpha\mu} \circ \pi) \frac{\partial^2 F}{\partial y^\beta \partial y^\lambda} - y^\sigma (\Gamma^\lambda_{\alpha\sigma} \circ \pi) \frac{\partial^3 F}{\partial y^\beta \partial y^\sigma \partial y^\lambda} = 0. \tag{276}
\]

If we multiply \(g^{\mu\nu}\) in the above equation, then we obtain \(\tilde{C}^\alpha_{\alpha\beta} = 0\), where \(\tilde{C}^\alpha_{\alpha\beta}\) are the coefficients of second Cartan tensor \(\tilde{C}_\nabla\) given by (208).

(ii) \(\Rightarrow\) (i). Since Second Cartan tensor \(\tilde{C}_\nabla\) belonging to \(\nabla\) is zero, then we have \(\tilde{C}^\alpha_{\alpha\beta} = 0\). Thus setting \(B^\lambda_\alpha = -y^\sigma (\Gamma^\lambda_{\alpha\sigma} \circ \pi)\) in (205) and multiply \(g_{\gamma\mu}\) in it, we deduce (276). Multiplying \(y^\beta y^\mu\) in (276) and using (ii) of (194) and (174) we obtain (275). Thus \(h\nabla\) is conservative.

(iii) \(\Rightarrow\) (ii). Since \(\bar{D}\) is \(h\)-metrical, then we have \(\bar{D}_{h\nabla} \tilde{G} = 0\). Thus we get

\[
0 = (\bar{D}_{h\nabla} \delta_\beta)(\delta_\beta, \delta_\gamma) = (\rho^\alpha_\lambda \circ \pi) \frac{\partial g^{\delta_\gamma}}{\partial x^\lambda} - (\Gamma^\lambda_{\alpha\beta} \circ \pi)g^{\delta_\gamma}_{\lambda} - (\Gamma^\lambda_{\alpha\gamma} \circ \pi)g^{\delta_\beta}_{\lambda} - y^\sigma (\Gamma^\lambda_{\alpha\sigma} \circ \pi) \frac{\partial^2 g^{\delta_\gamma}}{\partial y^\beta \partial y^\lambda}.
\]

Therefore we have (276), i.e., the second Cartan tensor \(\tilde{C}_\nabla\) belonging to \(\nabla\) is zero.

(ii) \(\Rightarrow\) (iii). If (ii) holds, then we have (276). Using this equation it is easy to check that \(\nabla \bar{D}_{h\nabla} \delta_\beta)(\delta_\beta, \delta_\gamma) = (\bar{D}_{h\nabla} \delta_\beta)(\delta_\beta, \delta_\gamma) = 0\). Also, we have \(\nabla \bar{D}_{h\nabla} \delta_\beta)(\delta_\beta, \delta_\gamma) = 0\). Thus Ichijō connection \((\bar{D}, h\nabla)\) is \(h\nabla\)-metrical. \(\square\)

Proposition 8.23. Let \((E, F, \nabla)\) be a generalized Berwald Lie algebroid. Then the mixed curvature of Ichijō connection \((\bar{D}, h\nabla)\) is zero.

Proof. It is sufficient to show that \(\nabla \tilde{C}^\lambda_{\alpha\beta\gamma} = 0\). Using (209) we have

\[
\tilde{P}^\lambda_{\alpha\beta\gamma} = \frac{1}{2} \left( \rho^\alpha_\lambda \circ \pi \right) \left( \frac{\partial^2 G^\lambda_{\beta\sigma}}{\partial x^\lambda \partial y^\gamma} G^{\sigma \lambda} + \frac{\partial G^\lambda_{\beta\sigma}}{\partial x^\lambda} \frac{\partial G^{\sigma \lambda}}{\partial y^\gamma} \right) - \frac{1}{2} y^\sigma (\Gamma^\mu_{\alpha\nu} \circ \pi) \left( \frac{\partial^2 G^\lambda_{\beta\sigma}}{\partial y^\nu \partial y^\gamma} G^{\sigma \lambda} \right)
\]

\[
+ \frac{\partial G^\lambda_{\beta\sigma}}{\partial y^\gamma} \frac{\partial G^{\sigma \lambda}}{\partial y^\mu} + \frac{1}{2} \frac{\partial G^\lambda_{\beta\sigma}}{\partial y^\gamma} \frac{\partial G^{\sigma \lambda}}{\partial y^\mu} \left( \Gamma^\lambda_{\alpha\mu} \circ \pi \right) - \frac{1}{2} \frac{\partial G^\lambda_{\beta\sigma}}{\partial y^\gamma} \frac{\partial G^{\sigma \lambda}}{\partial y^\mu} \left( \Gamma^\mu_{\alpha\gamma} \circ \pi \right)
\]

\[
- \frac{1}{2} \frac{\partial G^\lambda_{\beta\sigma}}{\partial y^\gamma} G^{\sigma \lambda} (\Gamma^\mu_{\alpha\beta} \circ \pi). \tag{277}
\]
Since Ichijyō connection is $h$-metrical, then we have

$$0 = \nabla_{h \nabla, \delta_u} G^\sigma_\lambda = (\rho^i_\alpha \circ \pi) \frac{\partial G^\sigma_\lambda}{\partial x^i} - \psi^\nu (\Gamma^\mu_\alpha \circ \pi) \frac{\partial G^\sigma_\lambda}{\partial y^\mu} + G^\sigma_\mu (\Gamma^\lambda_\alpha \circ \pi) + G^\lambda_\mu (\Gamma^\sigma_\alpha \circ \pi),$$

which gives us

$$(\rho^i_\alpha \circ \pi) \frac{\partial G^\sigma_\lambda}{\partial x^i} - \psi^\nu (\Gamma^\mu_\alpha \circ \pi) \frac{\partial G^\sigma_\lambda}{\partial y^\mu} + G^\sigma_\mu (\Gamma^\lambda_\alpha \circ \pi) = -G^\lambda_\mu (\Gamma^\sigma_\alpha \circ \pi).$$

Setting the above equation in (277) we get

$$\nabla^\lambda P_{\alpha \beta \gamma} = \frac{1}{2} (\rho^i_\alpha \circ \pi) \frac{\partial^2 G^\sigma_\lambda}{\partial x^i \partial y^\gamma} - \frac{1}{2} \psi^\nu (\Gamma^\mu_\alpha \circ \pi) \frac{\partial^2 G^\sigma_\lambda}{\partial y^\mu \partial y^\gamma} G^\sigma_\lambda$$

$$- \frac{1}{2} \frac{\partial G^\sigma_\lambda}{\partial y^\mu} \psi^\nu (\Gamma^\mu_\alpha \circ \pi) - \frac{1}{2} \frac{\partial G^\sigma_\mu}{\partial y^\gamma} G^\lambda_\mu (\Gamma^\sigma_\alpha \circ \pi).$$

Since $h_\nabla$ is conservative, then using (276) the right side of the above equation vanishes. Thus we have $\nabla^\lambda P_{\alpha \beta \gamma} = 0$. □

Let $(E, F, \nabla)$ be a generalized Berwald Lie algebroid and $f$ be a non-constant smooth function on $E$. We define $h_\nabla := h_\nabla - df^\nu \otimes C$. Since $df^\nu = (\rho^i_\alpha \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \lambda^\alpha$, then using (249) we can see that $h_\nabla$ has the local expression

$$h_\nabla = (X_\alpha + B^\beta_\alpha Y_\beta) \otimes X^\alpha, \tag{278}$$

where

$$B^\beta_\alpha = -\psi^\beta (\rho^i_\alpha \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} + \psi^\lambda (\Gamma^\beta_\alpha \circ \pi). \tag{279}$$

Using two above equation it is easy to check that $h_\nabla$ is an everywhere smooth function and $h_\nabla\vert E = h_\nabla$, $\ker h_\nabla = \Gamma (v \cdot L^e E)$. Thus $h_\nabla$ is an everywhere smooth, horizontal endomorphism on $L^e E$. Moreover we can obtain $\psi^\lambda \frac{\partial G^\sigma_\lambda}{\partial y^\nu} \sigma^\nu_\beta = B^\beta_\alpha$, i.e., $h_\nabla$ is a homogenous horizontal endomorphism.

Lemma 8.24. Let $(E, F, \nabla)$ be a generalized Berwald Lie algebroid and $\{e_\alpha\}$ be a basis of sections of $E$. Then $h_\nabla$ is conservative if and only if $\rho (e_\alpha) f = 0$.

Proof. Using (186), $h_\nabla$ is conservative, if and only if

$$(\rho^i_\alpha \circ \pi) \frac{\partial F}{\partial x^i} + B^\beta_\alpha \frac{\partial F}{\partial y^\nu} = 0, \tag{280}$$

where $B^\beta_\alpha$ are given by (249). Setting (249) in the above equation give us

$$(\rho^i_\alpha \circ \pi) \frac{\partial F}{\partial x^i} = \psi^\beta (\rho^i_\alpha \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial F}{\partial y^\beta} - \psi^\alpha (\Gamma^\beta_\alpha \circ \pi) \frac{\partial F}{\partial y^\beta} = 0.$$
Two above equations gives us
\[ y^\beta (\rho_\alpha^i \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial F}{\partial y^\beta} = 0, \]
and consequently
\[ (\rho_\alpha^i \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} F = 0, \]
because \( F \) is homogenous of degree 2. But since \( F \) is non-zero, then from the above equation we deduce \( (\rho_\alpha^i \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} = 0 \) or \( (\rho(e_\alpha)) f)^{\gamma} = 0 \). Thus \( h_{\nabla} \) is conservative if and only if \( \rho(e_\alpha) f = 0 \).

**Corollary 8.25.** Let \( (E, F, \nabla) \) be a generalized Berwald Lie algebroid and the anchor map \( \rho \) be injective. Then \( h_{\nabla} \) is not conservative.

Now we consider the linear connection \( \nabla_{e_\alpha} e_\beta = \bar{\Gamma}^{\gamma}_{\alpha \beta} e_\gamma \), where
\[
(\bar{\Gamma}^{\gamma}_{\alpha \beta} \circ \pi) = -\frac{\partial B^{\gamma}_{\alpha}}{\partial y^\beta} - \delta^\gamma_{\beta} (\rho_\alpha \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} + (\bar{\Gamma}^{\gamma}_{\alpha \beta} \circ \pi),
\]
or
\[
\bar{\Gamma}^{\gamma}_{\alpha \beta} = \delta^\gamma_{\beta} (\rho_\alpha \frac{\partial f}{\partial x^i}) + (\Gamma^{\gamma}_{\alpha \beta}) \tag{281}
\]
and we call it the linear connection generated by \( h_{\nabla} \).

**Proposition 8.26.** Let \( (E, F, \nabla) \) be a generalized Berwald Lie algebroid and \( \nabla \) be the linear connection generated by \( h_{\nabla} \). Then the mixed curvature of Ichijyo connection \( (D, h_{\nabla}) \) vanishes.

**Proof.** Using (269) and (281) we get
\[
\bar{P}^{\lambda}_{\alpha \beta \gamma} = \nabla^\lambda \bar{P}^{\lambda}_{\alpha \beta \gamma} - \frac{1}{2} y^\mu (\rho_\alpha \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial^2 G^{\beta \sigma \lambda}}{\partial y^\mu \partial y^\gamma} \frac{\partial f}{\partial y^\gamma} = 0.
\]
Since \( (E, F, \nabla) \) is a generalized Berwald Lie algebroid, then \( h_{\nabla} \) is conservative.

Thus according to proposition 8.23 \( \bar{P}^{\lambda}_{\alpha \beta \gamma} = 0 \). Moreover, we have
\[
y^\mu \frac{\partial^2 G^{\beta \sigma \lambda}}{\partial y^\gamma \partial y^\gamma} = -\frac{\partial G^{\beta \sigma \lambda}}{\partial y^\gamma}, \quad y^\mu \frac{\partial G^{\sigma \lambda}}{\partial y^\mu} = 0,
\]
because \( \frac{\partial G^{\beta \sigma \lambda}}{\partial y^\gamma} \) and \( G^{\sigma \lambda} \) are homogenous functions of degree -1 and 0, respectively.

Therefore, (282) reduce to the following
\[
\bar{P}^{\lambda}_{\alpha \beta \gamma} = \frac{1}{2} (\rho_\alpha \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial^2 G^{\beta \sigma \lambda}}{\partial y^\gamma \partial y^\gamma} G^{\beta \sigma \lambda} + \frac{1}{2} \frac{\partial G^{\beta \sigma \lambda}}{\partial y^\gamma} (\rho_\alpha \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} = 0.
\]
Definition 8.27. Generalized Berwald Lie algebroid \((E, F, \nabla)\) is called Berwald Lie algebroid, if \(\nabla\) be a torsion free linear connection on \(E\).

Proposition 8.28. Let \((E, F)\) be a Finsler Lie algebroid and \(h_\circ\) be a Barthel endomorphism of it. Then \((E, F)\) is Berwald Lie algebroid if and only if there is a linear connection on \(E\) such that
\[
(\nabla_X Y)^V = [X^{h_\circ}, Y^V]_\mathcal{L}, \quad \forall X, Y \in \Gamma(E).
\]

Proof. Let \((E, F)\) be a Finsler Lie algebroid. Then there is a torsion free linear connection \(\nabla\) on \(E\) such that \(h_\nabla\) is conservative. From torsion freeness of \(\nabla\) we conclude that \(\tau_\nabla\) is zero and consequently \(h_\nabla\) is homogenous. Thus \(h_\nabla\) is the Barthel horizontal endomorphism and consequently \(h_\nabla = h_\circ\), because the Barthel connection is unique. Therefore we have
\[
(\nabla_X Y)^V = [X^{h_\circ}, Y^V]_\mathcal{L}, \quad \forall X, Y \in \Gamma(E).
\]

Theorem 8.29. A Finsler Lie algebroid is a Berwald Lie algebroid if and only if the Hashiguchi connection of it, is a Ichijyö connection.

Proof. Let \((E, F)\) be a Berwald Lie algebroid. Then from the above proposition, \(h_\nabla = h_\circ\), where \(h_\nabla\) is a horizontal endomorphism generated by \(\nabla\) and \(h_\circ\) is the Barthel endomorphism. Thus we have \(\mathcal{B}^\circ_m = -\gamma^\gamma(\Gamma^\alpha_{\gamma\mu} \circ \pi).\) Setting this equation in \(242\) and \(243\) we obtain
\[
\nabla \delta_\alpha V_\beta = (\Gamma^\mu_{\alpha\beta} \circ \pi)V_\mu = D\delta_\alpha V_\beta,
\]
\[
\nabla \delta_\alpha \delta_\beta = (\Gamma^\mu_{\alpha\beta} \circ \pi)\delta_\mu = D\delta_\alpha \delta_\beta.
\]

Also, from \(240\), \(241\), \(255\) and \(256\) we have
\[
\nabla \delta_\alpha V_\beta = D\delta_\alpha V_\beta, \quad \nabla \delta_\alpha \delta_\beta = D\delta_\alpha \delta_\beta.
\]

Thus \(D = \nabla\). Conversely, if the Hashiguchi connection of a Finsler algebroid \((E, F)\) is a Ichijyö connection, then it is easy to see that \(h_\nabla = h_\circ\). Thus according to the above proposition we conclude that \((E, F)\) is a Berwald Lie algebroid.

Let \((E, F, \nabla)\) be a Berwald Lie algebroid. If \(\nabla\) is a flat connection then we call \((E, F, \nabla)\), the locally Minkowski Lie algebroid.

Theorem 8.30. A Finsler Lie algebroid \((E, F)\) is a locally Minkowski Lie algebroid if and only if there is a torsion free and flat linear connection on \(E\) such that Ichijyö connection \((\nabla, h_\nabla)\) is \(h_\nabla\)-metrical.

Proof. Let \((E, F)\) be a locally Minkowski Lie algebroid. Then there exist torsion free and flat linear connection \(\nabla\) on \(E\) such that \((E, F, \nabla)\) is a generalized Berwald Lie algebroid. Therefore, from proposition \(8.22\) we deduce that Ichijyö connection \((\nabla, h_\nabla)\) is \(h_\nabla\)-metrical. Using proposition \(8.22\) the proof of the converse of the theorem is obvious.
Proposition 8.31. Let \((E, F, \nabla)\) be a generalized Berwald Lie algebroid. Then we have
\[
S_\nabla = S_\circ + (d_{i_t \nabla}^E F)^\sharp,  
\]
\[
h_\nabla = h_\circ + \frac{1}{2} i_{S_\nabla} \nabla + \frac{1}{2} J. (d_{i_t \nabla}^E F)^\sharp t.  
\]  
Proof. Since \((E, F, \nabla)\) be a generalized Berwald Lie algebroid, then \(h_\nabla\) is conservative. Thus from propositions 7.16 and 7.17 the proof is obvious.

Theorem 8.32. Let \((E, F, \nabla_1)\) and \((E, F, \nabla_2)\) be generalized Berwald Lie algebroids. Then \(\nabla_1\) is equal to \(\nabla_2\) if and only if the torsion tensor fields of these are equal.

Proof. If \(\nabla_1 = \nabla_2\), then \(T_\nabla_1 = T_\nabla_2\). Conversely, if \(T_\nabla_1 = T_\nabla_2\) then the horizontal endomorphisms \(h_\nabla_1\) and \(h_\nabla_2\) have the same weak torsion and since these horizontal endomorphisms are homogenous, then they have the same strong torsion. Therefore using theorem 7.19 we deduce that \(h_\nabla_1 = h_\nabla_2\) and consequently \(\nabla_1 = \nabla_2\).

Proposition 8.33. Let \((E, F, \nabla)\) be generalized Berwald Lie algebroids. If spray \(S_\nabla\) generated by \(\nabla\) is the projective change of spray \(S_\circ\), then \(S_\nabla = S_\circ\) and consequently \((E, F)\) is a Berwald manifold.

Proof. Since \(S_\nabla\) is the projective change of \(S_\circ\), then the exist a function \(\tilde{f} : E \to \mathbb{R}\) that is smooth on \(E - \{0\}\) such that \(S_\nabla = S_\circ + \tilde{f} C\). Then using (283) we have \((d_{i_t \nabla}^E F)^\sharp = \tilde{f} C\). Thus using (iii) of proposition 7.3 we obtain
\[
i_{S_\nabla - S_\circ} \omega = i_{(d_{i_t \nabla}^E F)^\sharp} \omega = i_{\tilde{f} C} \omega = \tilde{f} d_{i_t \nabla}^E F.  
\]

Also, we have
\[
i_{S_\nabla - S_\circ} \omega = d_{i_t \nabla}^E \nabla F.  
\]

Two above equation give us
\[
d_{i_t \nabla}^E \nabla F = \tilde{f} d_{i_t \nabla}^E F.  
\]  
Thus we have
\[
d_{i_t \nabla}^E \nabla F(S) = d^E F(i_{S_\nabla} t_\nabla(S)) = d^E F(t_\nabla(S_\nabla, S)) 
= d^E F(t_\nabla(S, S)) = d^E F(0) = 0.  
\]

Also from (171) we have \(d_{i_t \nabla}^E F(S) = y^a \frac{\partial F}{\partial y^a} = 2F\). Setting this equation and the above equation in (285) we deduce \(\tilde{f} F = 0\) and consequently \(\tilde{f} = 0\). Therefore we have \(S_\nabla = S_\circ\).

8.3 Wagner-Ichijyō connection

Let \(\nabla\) be a linear connection on \(E\) and \(f\) be a smooth function on \(M\). If \((\nabla, h_\nabla)\) is a Ichijyō connection such that the \(h\)-horizontal torsion of \(\nabla\) satisfies in
\[
\nabla A = d^E f^\nabla \wedge h_\nabla = d^E f^\nabla \otimes h_\nabla - h_\nabla \otimes d^E f^\nabla,  
\]

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then we call \( \overline{D}, h_\nabla, f \) the Wagner-Ichiyô connection generated by \( \nabla \).

From (286) we deduce that \( A(V_\alpha, V_\beta) = A(\delta_\alpha, \delta_\beta) = 0 \) and

\[
\overline{A}(\delta_\alpha, \delta_\beta) = d^E f^\gamma(\delta_\alpha) h_\nabla(\delta_\beta) - h_\nabla(\delta_\alpha) d^E f^\gamma(\delta_\beta) \\
= \rho_\alpha(\delta_\alpha)(f \circ \pi) \delta_\beta - \rho_\beta(\delta_\beta)(f \circ \pi) \delta_\alpha \\
= \left( \left( \rho_\alpha^i \circ \pi \right) \frac{\partial(f \circ \pi)}{\partial x^i} \delta_\beta^i - \left( \rho_\beta^i \circ \pi \right) \frac{\partial(f \circ \pi)}{\partial x^i} \delta_\alpha^i \right) \delta_\gamma. \tag{287}
\]

**Lemma 8.34.** Let \( \overline{D}, h_\nabla, f \) be a Wagner-Ichiyô connection on Finsler algebroid \( E, F \). Then we have

\[
T_\nabla(X, Y) = d^E f(X)Y - d^E f(Y)X, \quad \forall X, Y \in \Gamma(E),
\]

\[
t_\nabla = d^E f^\gamma \wedge J = d^E f^\gamma \otimes J - J \otimes d^E f^\gamma,
\]

\[
i_{S_\nabla} t_\nabla = f^\gamma J - d^E f^\gamma \otimes C.
\]

**Proof.** Using (287) we obtain

\[
\overline{A}(\delta_\alpha, \delta_\beta) = \left( \left( \rho_\alpha^i \frac{\partial f}{\partial x^i} \right) \delta_\beta^i - \left( \rho_\beta^i \frac{\partial f}{\partial x^i} \right) \delta_\alpha^i \right) \delta_\gamma = \left( \rho(e_\alpha)(f)e_\beta - \rho(e_\beta)(f)e_\alpha \right) \delta_\gamma.
\]

Also, from (271) we have \( \overline{A}(\delta_\alpha, \delta_\beta) = (T_\nabla(e_\alpha, e_\beta))^h \). Therefore we obtain

\[
T_\nabla(e_\alpha, e_\beta) = d^E f(e_\alpha)e_\beta - d^E f(e_\beta)e_\alpha,
\]

that gives us the first equation of the lemma. Also, from (271) and (287) we obtain

\[
F_\nabla t_\nabla(\delta_\alpha, \delta_\beta) = \overline{A}(\delta_\alpha, \delta_\beta) = d^E f^\gamma(\delta_\alpha) h_\nabla(\delta_\beta) - h_\nabla(\delta_\alpha) d^E f^\gamma(\delta_\alpha).
\]

Applying \( F_\nabla \) to the above equation and using \( F_\nabla h_\nabla = -J \) and \( F_\nabla F_\nabla = -1 \) give us

\[
t_\nabla(\delta_\alpha, \delta_\beta) = d^E f^\gamma(\delta_\alpha) J(\delta_\beta) - J(\delta_\alpha) d^E f^\gamma(\delta_\beta),
\]

which gives us the second equation of the lemma. Using the above equation and (10) we get

\[
i_{S_\nabla} t_\nabla(\delta_\beta) = t_\nabla(S_\nabla, \delta_\beta) = y^\alpha t_\nabla(\delta_\alpha, \delta_\beta) = y^\alpha d^E f^\gamma(\delta_\alpha) V_\beta - y^\alpha V_\alpha d^E f^\gamma(\delta_\beta) \\
= \left( \left( \rho_\alpha^i \circ \pi \right) \frac{\partial f}{\partial x^i} \right) \delta_\beta^i = \left( \rho_\alpha^i \circ \pi \right) \frac{\partial f}{\partial x^i} \delta_\beta^i - C d^E f^\gamma(\delta_\beta)
\]

\[
= f^\gamma J(\delta_\beta) - d^E f^\gamma(\delta_\beta) C,
\]

which gives us the third equation of the lemma. \( \square \)

**Definition 8.35.** Let \( (E, F, \nabla) \) be a generalized Berwald Lie algebroid and \( f \) be a smooth function on \( E \). Then \( (E, F, \nabla, f) \) is called Wagner Lie algebroid if the torsion of linear connection \( \nabla \) satisfies in the following relation

\[
T_\nabla(X, Y) = d^E f(X)Y - d^E f(Y)X, \quad \forall X, Y \in \Gamma(E). \tag{288}
\]
Theorem 8.36. Let \((E, F)\) be a Lie algebroid, \(f\) be a smooth function on \(M\) and \(\nabla\) be a linear connection on \(E\). Then the following items are equivalent:

(i) \((E, F, \nabla, f)\) is a Wagner Lie algebroid.

(ii) Wagner-Ichiyô connection \((D, h\nabla, f)\) generated by \(\nabla\), is h-metrical.

(iii) Horizontal endomorphism \(h\nabla\) satisfies in the following

\[
h\nabla = h_\circ + f^c J - F[J, \text{grad} f^\vee]_E - d_f^c F \otimes \text{grad} f^\vee. \tag{289}
\]

Proof. From proposition \(8.22\) the equivalence of (i) and (ii) is obvious. Thus it is sufficient to prove that (i) is equivalent to (iii). Let (i) holds. Since \((E, F, \nabla, f)\) is a Wagner Lie algebroid, then \((E, F, \nabla)\) is a generalized Berwald Lie algebroid and consequently from proposition \(8.31\) we have the formula \(284\) for \(h\nabla\). Using the third equation of lemma \(8.34\) and \(180\) we obtain

\[
(d_{i\nabla}^c t\nabla F)(\delta_\beta) = (d^c F \circ i_{\nabla t\nabla}) = d^c F(t\nabla(S\nabla, \delta_\beta))
\]

\[
= d^c F(f^c J(\delta_\beta) - d^c f^\vee(\delta_\beta) C)
\]

\[
= f^c d^c F(J(\delta_\beta)) - d^c f^\vee(\delta_\beta) d^c F(C)
\]

\[
= f^c d^c F(J(\delta_\beta)) - (i_{\text{grad} f^\vee \omega})(\delta_\beta) d^c F(C). \tag{290}
\]

Since \(F\) is homogenous of degree 2, then we deduce

\[
d^c F(C) = \rho_F(C)(F) = \nu^a \frac{\partial F}{\partial \nu^a} = 2F.
\]

Also, from (iii) of proposition \(7.3\) we get

\[
d^c F(J(\delta_\beta)) = (d_f^c F)(\delta_\beta) = (i_C \omega)(\delta_\beta).
\]

Setting two above equations in \(290\), we obtain

\[
d_{i\nabla}^c t\nabla F = i_f C - 2F \text{grad} f^\vee,
\]

which gives us

\[
(d_{i\nabla}^c t\nabla F)^2 = f^c C - 2F \text{grad} f^\vee. \tag{291}
\]

Setting the third equation of lemma \(8.34\) and the above equation in \(284\), we get

\[
h\nabla = h_\circ + \frac{1}{2}(f^c J - d^c f^\vee \otimes C) + \frac{1}{2}[J, f^c C]_E - [J, F \text{grad} f^\vee]_E. \tag{292}
\]

Direct calculation we can obtain the following equations

\[
[J, f^c C]_E = f^c J + d_f^c f^c \otimes C,
\]

\[
[J, F \text{grad} f^\vee]_E = F[J, \text{grad} f^\vee]_E + d_f^c F \otimes \text{grad} f^\vee.
\]

Setting two above equations in \(292\), we deduce

\[
h\nabla = h_\circ + \frac{1}{2}(f^c J - d^c f^\vee \otimes C) + \frac{1}{2}f^c J + \frac{1}{2}d_f^c f^c \otimes C
\]

\[
- \frac{F[J, \text{grad} f^\vee]_E - d_f^c F \otimes \text{grad} f^\vee. \tag{293}
\]

But we have

\[
(d_f f^c)(\delta_\alpha) = df^c(\nu_\alpha) = \frac{\partial f^c}{\partial \nu^a} = (\rho^a_\circ \pi \frac{\partial (f \circ \pi)}{\partial x^a} = (d^c f^\vee)(\delta_\alpha),
\]

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and \((df^c)(\mathcal{V}_a) = 0 = (df^d)(\mathcal{V}_a)\). Thus we have \(df^c = df^d\). Setting this equation in (293) we obtain (289), i.e., (iii) holds. Now we let (iii) holds and we prove (i). Let

\[ h_\alpha = (\mathcal{X}_a + B^a_\beta \mathcal{V}_\beta) \otimes \mathcal{X}^\alpha, \quad h_\mathcal{V} = (\mathcal{X}_a + B^a_\beta \mathcal{V}_\beta) \otimes \mathcal{X}^\alpha. \]

Then using (183) and (289) we can obtain

\[
\tilde{B}^\alpha_\beta = B^\alpha_\beta + y^\gamma (\rho^i_\gamma \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \delta^\alpha_\beta - \mathcal{F} \frac{\partial G^\beta_\gamma}{\partial y^\alpha} (\rho^i_\gamma \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial F}{\partial y^\beta} - \frac{\partial F}{\partial y^\beta} \tilde{G}^{\beta \gamma} (\rho^i_\gamma \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial F}{\partial y^\gamma}.
\]

Since \(h_\alpha\) is conservative, then using (180) we have \((\rho^i_\alpha \circ \pi) \frac{\partial F}{\partial x^i} + B^\alpha_\beta \frac{\partial F}{\partial y^\beta} = 0\). Thus using the above equation we get

\[
(\rho^i_\alpha \circ \pi) \frac{\partial F}{\partial x^i} + \tilde{B}^\alpha_\beta \frac{\partial F}{\partial y^\beta} = -\mathcal{F} \frac{\partial G^\beta_\gamma}{\partial y^\alpha} (\rho^i_\gamma \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial F}{\partial y^\beta}.
\]

Using (i) of (194) in the above equation, the sum of the first and third sentences of the right side of the above equation vanishes. Thus the above equation reduce to

\[
(\rho^i_\alpha \circ \pi) \frac{\partial F}{\partial x^i} + \tilde{B}^\alpha_\beta \frac{\partial F}{\partial y^\beta} = -\mathcal{F} \frac{\partial G^\beta_\gamma}{\partial y^\alpha} (\rho^i_\gamma \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial F}{\partial y^\beta}.
\]

But from (194) we deduce

\[
\frac{\partial G^\beta_\gamma}{\partial y^\alpha} \frac{\partial F}{\partial y^\beta} = y^\gamma \frac{\partial G^\beta_\gamma}{\partial y^\alpha} \tilde{G}_{\lambda \beta} = -y^\gamma \frac{\partial G^\beta_\gamma}{\partial y^\alpha} \tilde{G}_{\beta \gamma} = 0.
\]

Two above equations give \((\rho^i_\alpha \circ \pi) \frac{\partial F}{\partial x^i} + \tilde{B}^\alpha_\beta \frac{\partial F}{\partial y^\beta} = 0\). Thus \(h_\mathcal{V}\) is conservative and consequently \(E, \mathcal{F}, \nabla\) is a generalized Berwald Lie algebroid. Now we show that the torsion of \(\nabla\) satisfies in (289). Differentiating of (294) with respect to \(y^\mu\) we obtain

\[
\frac{\partial \tilde{B}^\alpha_\beta}{\partial y^\mu} = \frac{\partial B^\alpha_\beta}{\partial y^\mu} + (\rho^i_\alpha \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \delta^\alpha_\beta - \mathcal{F} \frac{\partial G^\beta_\gamma}{\partial y^\alpha} (\rho^i_\gamma \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial F}{\partial y^\beta} - \frac{\partial F}{\partial y^\beta} \frac{\partial G^\beta_\gamma}{\partial y^\alpha} (\rho^i_\gamma \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \frac{\partial F}{\partial y^\gamma}.
\]

Rechanging \(\alpha\) and \(\mu\) in the above equation we can obtain \(\frac{\partial \tilde{B}^\alpha_\beta}{\partial y^\mu}\). Therefore we can obtain

\[
\tilde{G}^\beta_{\mu \alpha} = \frac{\partial \tilde{B}^\beta_\alpha}{\partial y^\mu} - \frac{\partial \tilde{B}^\beta_\mu}{\partial y^\alpha} - (L^\beta_\alpha \circ \pi) = \frac{\partial B^\beta_\alpha}{\partial y^\mu} - \frac{\partial B^\beta_\mu}{\partial y^\alpha} = - (L^\beta_\mu \circ \pi) + (\rho^i_\mu \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \delta^\beta_\alpha - (\rho^i_\alpha \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \delta^\beta_\mu.
\]
where \( \tilde{t}_{\mu\alpha} \) are the coefficients of weak torsion \( t_{\nabla} \) of \( h_{\nabla} \) and \( t^\beta_{\mu\alpha} \) are the coefficients of weak torsion \( t_{\circ} \) of Barthel endomorphism \( h_{\circ} \) given by (33). But the Barthel endomorphism is torsion free. So \( t^\beta_{\mu\alpha} = 0 \). Therefore from the above equation we obtain

\[
T_{\nabla}(\delta_\mu, \delta_\alpha) = \tilde{t}_{\mu\alpha} \nabla_\beta = (\rho_\mu^i \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \nabla_\alpha - (\rho_\alpha^i \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \nabla_\mu.
\]

But from (271) and the above equation we deduce

\[
(T_{\nabla}(e_\mu, e_\alpha))^{h_{\nabla}} = F_{\nabla} t_{\nabla}(\delta_\mu, \delta_\alpha) = (\rho_\mu^i \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \delta_\alpha - (\rho_\alpha^i \circ \pi) \frac{\partial (f \circ \pi)}{\partial x^i} \delta_\mu
\]

\[
= (\rho(e_\mu)(f)e_\alpha - \rho(e_\alpha)(f)e_\mu)^{h_{\nabla}} = (dE f(e_\mu)e_\alpha - dE f(e_\alpha)e_\mu)^{h_{\nabla}}
\]

which gives us \( T_{\nabla}(e_\mu, e_\alpha) = dE f(e_\mu)e_\alpha - dE f(e_\alpha)e_\mu \). Therefore (288) holds and consequently \( (E, F, \nabla, f) \) is a Wagner Lie algebroid.

**Corollary 8.37.** If \( (E, F, \nabla, f) \) is a Wagner Lie algebroid, then spray \( S_{\nabla} \) generated by \( h_{\nabla} \) satisfies in the following relation

\[
S_{\nabla} = S_0 + f^C - 2F \text{grad} f^C.
\]

**Proof.** Since \( (E, F, \nabla, f) \) is a Wagner Lie algebroid, then we have (291). Setting (291) in (283) the proof completes.

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