On a two-player transversal game on a square grid

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Abstract

We give a short analysis of the transversal achievement game on a square grid due to M. Erickson (2010).

Keywords: 2-player game, integer grid.

1 Introduction

The following problem can be found in the Open Problem Garden [7]:

Two players alternately write 0’s (first player) and X’s (second player) in the unoccupied cells of an \(n \times n\) grid, \(n \geq 2\). The first player (if any) to occupy a set of \(n\) cells having no two cells in the same row or column is the winner. What is the outcome of the game given optimal play?

Recently, Krishna [5] proved that the game is a draw. His proof uses mathematical induction and case analysis. Here we offer a much shorter direct proof.

Theorem 1. The outcome of the game given optimal play is a draw for all \(n\).

Related work. One of the most common two-player games played on a square grid is tic-tac-toe, where the goal of each player is to get many consecutive points (cells) of the same color. Several variants and interesting results can be found in [1, Ch. 9]. The \(n^d\) tic-tac-toe is the \(d\)-dimensional version where the winning sets are the \(n\)-in-a-line. One can prove for instance that the \(3^3\) game is a first player win whereas the \(8^3\) game is a draw; the status of the \(5^3\) game appears to be unknown. See also [2, 3, 4, 6] for other related problems and results.

Notation. A configuration of a player is the set of cells occupied by that player in the \(n \times n\) grid (at a specified time/move in the game). Let \([n]\) denote the set \(\{1, 2, \ldots, n\}\). For a configuration \(\Xi\), let \(R(\Xi)\) and \(C(\Xi)\) denote the sets of row indexes and respectively, column indexes appearing in \(\Xi\) (\(R(\Xi)\) and \(C(\Xi)\) are subsets of \([n]\)).

2 A simple drawing strategy

In this section we prove Theorem 1. We show that drawing the game is as easy as counting up to \(n\). Let \(A\) and \(B\) (Alice and Bob) be the two players, where \(A\) moves first. By the rules of the game, the game is decided by the end of the \(n\)th move at the latest: either a row/column repetition occurs or the goal is achieved in the \(n\)th move. Note that \(2n \leq n^2\) for \(n \geq 2\).

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Proof. We show that either player can block the other one from winning. A key observation is: if a player’s configuration $\Xi$ of $n - 1$ cells can be completed to a winning configuration, such a configuration is unique. Indeed, both $R(\Xi)$ and $C(\Xi)$ consist (each) of distinct elements, hence $|\{n\} \setminus R(\Xi)| = |\{n\} \setminus C(\Xi)| = 1$.

A simple strategy for $B$ is: fill in arbitrarily chosen unoccupied cells until $B$’s turn in move $n - 1$. If $A$’s configuration of $n - 1$ cells can be completed to a winning configuration, $B$ fills (blocks) the unique completion cell as desired. (Otherwise any move will do.) $A$’s strategy is similar: fill in arbitrarily chosen unoccupied cells until $A$’s turn in move $n$. If $B$’s configuration of $n - 1$ cells can be completed to a winning configuration, $A$ fills (blocks) the unique completion cell as desired. □

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