A note on commensurability induced by oblateness at L₃

Arantza Jency A¹, Ram Krishan Sharma*²
¹Department of Aerospace Engineering, Karunya Institute of Technology and Sciences, Coimbatore 641114, Tamil Nadu, India, ramkrishan@karunya.edu; ramkrishansharma@gmail.com

Introduction

The locations of the Lagrangian points in the restricted three-body problem (CRTBP) by assuming both the primaries as oblate spheroids with their equatorial planes coincident with the plane of motion was calculated in [6]. In [7] the location of the collinear points in the same problem was studied numerically for some systems of astronomical interest. These equilibria were shown to be unstable in general, though the existence of conditional infinitesimal (linearized) periodic orbits around them was established. However, the secular effect of oblateness of the primaries on the motion of the primaries was not included. Later the oblateness of only the more massive primary was considered and the secular effect of oblateness [8] on the mean motion of the primaries was included in [9, 10, 11]. In [9, 10] the critical mass value μc was found to decrease with oblateness. In [10, 11] a numerical investigation of the locations of the five equilibrium points was made for some systems of astronomical interest. Periodic solutions of the linearized equations of motion around the five equilibrium points were studied. The angular frequency in the z-direction (s₃) was found to be more than the mean motion n. In [12] it was established that the oblateness induces a one-to-one commensurability at the exterior point L₃ to the right of the more massive primary, when the more massive primary is considered as an oblate spheroid with its equatorial plane coincident with the plane of motion. This study will be useful in the generation of halo orbits at L₃. For Saturn-Titan system, the values of the μ and oblateness coefficient (A₁) are 0.000236695 and 0.00003965396, respectively [10]. It is interesting to note that the value of A₁ obtained for one-to-one commensurability is 0.0000344978 for this system, which is very close to the actual value. Thus, halo orbits of small size can be generated in Saturn-Titan system at L₃.

Abstract

It is known that there exists a near one-to-one commensurability ratio between the planar angular frequencies (s₁, s₂) and the corresponding angular frequency (s₃) in the z-direction at the three collinear points (L₁, L₂, L₃) in the three-dimensional restricted three-body problem. It is significant for small and practically important values of the mass parameter (μ). In this note we have used a new mean motion expression which includes the secular effects of oblateness on argument of perigee, right ascension of ascending node and mean anomaly, which was used in the studies [2, 3, 4, 5]. We establish that there is one-to-one commensurability at the external points L₃ (to the right of the more massive primary), when the more massive primary is considered as an oblate spheroid with its equatorial plane coincident with the plane of motion. This study will be useful in generating the halo orbits at L₃. For Saturn-Titan system, the values of the μ and oblateness coefficient (A₁) are 0.000236695 and 0.00003965396, respectively [10]. It is interesting to note that the value of A₁ obtained for one-to-one commensurability is 0.0000344978 for this system, which is very close to the actual value. Thus, halo orbits of small size can be generated in Saturn-Titan system at L₃.

Keywords: Circular Restricted Three-Body Problem (CRTBP), Oblateness, Mean Motion, Lagrangian Points, One-to-one commensurability.

Equations of Motion

The problem is defined in the non-dimensional pulsating synodic coordinate frame as given by Figure 1. The barycentre of the primaries mark the origin of the system which rotates about the z-axis (perpendicular to the plane of motion of primaries. The mass ratio is the ratio of the mass of less massive primary m₂ to the sum of the masses of the primaries m₁ + m₂ which is unity in the non-dimensional system. Point represents the point mass (with infinitesimal mass).

The equations of motion in terms of the dimensionless frame is given by Equations (1) [9, 11]. The force function Ω [9, 11] in the equations of motion is given by Equation (2). The oblateness of the more massive primary A₁ = (AE²-AP²)/5R², AE and AP are equatorial and polar radii, respectively, and R is the distance between them, affects the force function of the system.
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Referring Figure 1, the distances \( r_1 \) and \( r_2 \) of \( P \) from the more massive and the smaller primaries are related and to the distances \( x \) and \( y \) from the origin by

\[
x'' - 2ny' - \Omega_x = 0, \\
y'' + 2nx' = \Omega_y,
\]

\[
\Omega = \frac{n^2}{2}((1-\mu)r_1^2 + \mu r_2^2) + \left[ \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right] \frac{(1-\mu)A_1}{2r_1^2}.
\]

Mean Motion

The mean motion equation (Equation 5) for this study is derived using the effect of \( J_2 \), given by Equations (4)

\[
\frac{dM_s}{dt} = \pi \left[ 1 + \frac{3r_1}{2(1-\mu^2)} \right] \frac{d\Omega_s}{dt} = \pi \left[ -\frac{3r_1}{2\sqrt{1-\mu^2}} \right] \frac{d\Omega_s}{dt} = \pi \left[ \frac{-3r_1}{2\sqrt{1-\mu^2}} \right]
\]

The mean motion \( n \) is the summation of the changes in \( M_s, \Omega_s \) and \( \Omega_x \) after one revolution [1], given by

\[
n = 1 + \frac{3A_1}{2\sqrt{1-\mu^2}}(1 + \sqrt{1-\mu^2})
\]

Location of Collinear Equilibrium Points

The equations of motion (1) are found to have singular solutions at five points [13] called the Lagrange points, liberation points or equilibrium points. Three of these equilibrium points (collinear equilibrium points - \( L_1, L_2 \) and \( L_3 \)) lie in the line connecting the primaries and the other two (triangular equilibrium points - \( L_4 \) and \( L_5 \)) form nearly equilateral triangles [11] with the primaries. These equilibrium points satisfy the conditions that the first derivatives of the force function equation equate to zero i.e., \( \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \) [11, 13].

As the collinear equilibrium points lie on the \( x \)-axis, in addition to the conditions \( \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \), they also satisfy \( y = 0 \). Therefore, by equating \( \Omega_x \) and \( y \) to zero and making the corresponding substitutions from Equations (6), we get the seventh degree polynomials given by Equations (7), (8) and (9) for the locations of \( L_1, L_2 \) and \( L_3 \), respectively, which upon solving with the help of MATLAB for different values of \( \mu \) and \( A_1 \) gives the locations of the collinear equilibrium [5]. It is interesting to note that all the three collinear points move towards the more massive primary with oblateness with the new mean motion [5]. Earlier in [11] with the mean motion \( n = L_1 + 3A_1/2 \), it was noticed that only moves towards the bary center.

\[
\begin{align*}
L_1 & = x_1 = 1 - \mu \\
x_2 & = \mu - 1 + \mu \\
x_3 & = \mu + 1 - \mu \\
y_1 & = 0 \\
y_2 & = 0 \\
y_3 & = 0.
\end{align*}
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Polynomial approximations using Taylor’s series expansions for (9), we get

From (10) and (11), we notice that the equilibrium point \( L_3 \) moves towards the more massive primary with its oblateness.

\[
\rho_3 = \left( \frac{7}{12} \mu - \frac{1377}{1000} \mu^2 + \ldots \right) - A_1 \left( \frac{77}{24} \mu - \frac{567}{2500} \mu^2 + \frac{7385}{25000} \mu^4 + \ldots \right).
\]

and

\[
x = x + \rho_3.
\]

As in \([12]\), the angular frequency \( s_j \) and \( s_z(L_3) \) are given by:

\[
\begin{align*}
\{s_j, s_z(L_3)\} & = \{x, y, z, \theta, \phi\}, \\
\{s_j, s_z(L_3)\} & = \{x, y, z, \theta, \phi\},
\end{align*}
\]

\[
\begin{align*}
\eta & = \{x, y, z, \theta, \phi\}, \\
\beta & = 3A_1(1-1/2\mu).
\end{align*}
\]

Commensurability at \( L_3 \)

As can be seen from (14) and (15), at \( L_3 \) the in-plane frequencies \( s_j \) are greater than \( s_z \) without oblateness, however, both \( s_j \) and \( s_z \) increase with oblateness effect. The increase in \( s_z \) is more than in \( s_j \) with oblateness. So there is a possibility for one-to-one commensurability between the two frequencies, for a suitable choice of \( A_1 \), for \( 0 \leq \mu \leq \frac{1}{2} \).

The values of \( A_1 \) has been obtained using MATHEMATICA with an initial estimate:

\[
A_1 = \frac{7 \cdot 248 \cdot 784 - 195 \cdot 103 \cdot 1234 \cdot 7 \cdot 10^3}{7 \cdot 8 \cdot 9 \cdot 10^3} \times 10^{-12}
\]

obtained by setting \( s_j = s_z(L_3) \) from (14) and (15). It may be noted that the values of \( A_1 \) are small for small \( \mu \). For Saturn-Titan system, the values of \( \mu \) and \( A_1 \) are 0.000236695 and 0.000039653936, respectively \([1]\). It is interesting to note that the value of \( A_1 \) obtained for one-to-one commensurability is 0.0000344978 for this system, which is very close to the actual value of 0.000039653936. Thus, halo orbits of small size can be generated in Saturn-Titan system at \( L_3 \). Most satellites in halo orbit serve scientific purposes, such as space telescopes.

Conclusions

With the secular perturbations effects of oblateness on argument of perigee, right ascension of ascending node and mean anomaly on the mean motion \([1]\), it is found that the mean motion increases further. The CRTBP with the more massive primary as an oblate spheroid with its equatorial plane coincident with the plane of motion is studied with the new mean motion. The locations of the three Lagrangian points of the CRTBP are computed. Series expansion for the location of \( L_3 \) is found. It is noticed that \( L_3 \) moves towards the more massive primary with its inclusion of the oblateness.



| S. No | \( \mu \) | \( A_1 \) | \( A_1 \) (From Eq.) |
|-------|-----------|-----------|----------------------|
| 1     | \( 10^{-6} \) | 0.1458329725x10^{-6} | 0.1458329725x10^{-6} |
| 2     | \( 10^{-5} \) | 0.1458297254x10^{-5} | 0.1458297255x10^{-5} |
| 3     | \( 10^{-4} \) | 0.1457972546x10^{-4} | 0.1457972546x10^{-4} |
| 4     | \( 10^{-3} \) | 0.1454725453x10^{-3} | 0.1454725454x10^{-3} |
| 5     | \( 10^{-2} \) | 0.1422377438x10^{-2} | 0.1422377438x10^{-2} |

Table 1: Values of \( A_1 \) when \( s_j = s_z \) for small values of \( \mu \).

References

1. Sharma RK, Sellamuthu, H and Jency AA. Perturbed Trojan dynamics in the solar system, AAS AIAA Astrodynamics Specialist Conference
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2. Johnson A, Sharma RK. Locations of Lagrangian points and periodic orbits around triangular points in the photo gravitational elliptic restricted three-body problem with oblateness. International Journal of Advanced Astronomy. 2019;7(2):25-38. doi:10.14419/ijaa.v7i2.29377.

3. Jency AA, Sharma RK. Location and stability of the triangular Lagrange points in photo-gravitational elliptic restricted three body problem with the more massive primary as an oblate spheroid. International Journal of Advanced Astronomy. 2019;7(2):57-62. DOI:10.14419/ijaa.v7i2.29814

4. Arohan R, Sharma RK. Periodic orbits in the planar restricted photo-gravitational problem when the smaller primary is an oblate spheroid. Indian Journal of Science and Technology. 2020; 13(16):1630-1640. https://doi.org/10.17485/IJST/v13i16.401

5. Jency AA, Sharma RK, Gaagandeep Singh. Stationary solutions, critical mass, Tadpole orbits in the circular restricted three-body problem with the more massive primary as an oblate spheroid. Indian Journal of Science and Technology (Accepted for publication, October 2020).

6. Sharma RK. Perturbations of Lagrangian points in the restricted three-body problem. Indian Journal of Pure and Applied Mathematics, 1975; 6:1099-1102.

7. Sharma RK, Subba Rao PV. Collinear equilibria and their characteristic exponents in the restricted three-body problem when the primaries are oblate spheroids. Celestial Mechanics, 1975; 12,189-201. https://doi.org/10.1007/BF01230211.

8. McCuskey SW. Introduction to Celestial Mechanics. Addison-Wesley Publishing Company. 1963.

9. Subba Rao PV, Sharma RK. A note on the stability of the triangular points of equilibrium in the restricted three-body problem. Astronomy and Astrophysics. 1975; 43:381-383. Available from: http://adsabs.harvard.edu/abs/1975A%26A43..381S.

10. Sharma R. K. Study of Periodic Orbits in Restricted Problem taking the Bigger Primary an Oblate Spheroid. Ph.D. Thesis. University of Delhi. 1976

11. Sharma RK, Subba Rao PV. Stationary solutions and their characteristic exponents in the restricted three-body problem when the more massive primary is an oblate spheroid. Celestial Mechanics. 1976; 13:137–149. https://doi.org/10.1007/BF01232721.

12. Sharma RK, Subba Rao PV. A case of commensurability induced by oblateness. Celestial Mechanics. 1978; 18,185-194. Available from: adsabs.harvard.edu › full › 1978CeMec..18..185S

13. Szebehely V, and. Theory of Orbits. Academic Press: New York, 1967. eBook ISBN: 9780323143462.

14. Curtis HD. Orbital Mechanics for Engineering Students. Elsevier: Waltham, 2014. ISBN-13: 978-0080977478.