Morse shellability, tilings and triangulations

Nermin Salepci and Jean-Yves Welschinger

October 30, 2019

Abstract

We introduce notions of tilings and shellings on finite simplicial complexes, called Morse tilings and shellings, and relate them to the discrete Morse theory of Robin Forman. Skeletons and barycentric subdivisions of Morse tileable or shellable simplicial complexes are Morse tileable or shellable. Moreover, every closed manifold of dimension less than four has a Morse tiled triangulation, admitting compatible discrete Morse functions, while every triangulated closed surface is even Morse shellable. Morse tilings extend a notion of $h$-tilings that we introduced earlier and which provides a geometric interpretation of $h$-vectors. Morse shellability extends the classical notion of shellability.

Keywords: simplicial complex, shellable complex, tilings, barycentric subdivision, discrete Morse theory, triangulation

Mathematics subject classification 2010: 05E45, 57Q15, 55U10, 52C22.

1 Introduction

We recently [15] introduced a notion of tilings of a finite simplicial complex $K$. It is a partition of $K$, or rather of the underlying topological space, by tiles. A tile is a closed simplex deprived of several facets, that is of codimension one faces. In each dimension $n$, there are thus $n + 2$ different tiles, denoted by $T_0^n, \ldots, T_{n+1}^n$ depending on the number of facets that have been removed, and the closed simplex $\Delta_n$ itself is one of them, namely $T_0^n$, while the open simplex is another one, namely $T_{n+1}^n$. Not all simplicial complexes are tileable, but skeletons and barycentric subdivisions of tileable simplicial complexes are tileable by Theorem 1.9 of [15]. These tilings provide a geometric way to understand the $h$-vectors of finite tileable simplicial complexes, see [9, 17, 19] for a definition. Namely, if $h_k^n$ denotes the number of tiles $T_k^n$ needed to tile a complex $K$, then $(h_0^n, \ldots, h_{n+1}^n)$ coincides with the $h$-vector of $K$ provided that $h_0^n = 1$ and in general, two tilings of $K$ have the same $h$-vector $(h_0^n, \ldots, h_{n+1}^n)$ provided they have the same number of tiles $T_k^n$, see Theorem 1.8 of [15]. These tilings appeared to be useful to produce packings by disjoint simplices of the successive barycentric subdivisions $S^d(K), d > 0$, see § 5 of [15]. They actually also seemed to be closely related to the discrete Morse theory of Robin Forman [8] even though...
this aspect has not been investigated in [15]. The tiles $T_0^n$ behaved as critical points of index zero, the tiles $T_{n+1}^n$ as critical points of index $n$ of a Morse function and the other ones as regular points. No analog though of critical points of intermediate indices. We now fill this gap.

We define a Morse tile to be a closed simplex deprived of several facets together with a unique face of possibly higher codimension. It is critical if and only if this codimension is maximal, see Definition 2.4. A Morse tiling of a finite simplicial complex is a partition by Morse tiles such that for every $j \geq 0$, the union of tiles of dimension greater than $j$ is a simplicial subcomplex, see Definition 2.8. The previous notion of tiling, due to its relation with $h$-vectors, is now called $h$-tiling and slightly generalized to allow for tiles of various dimensions, see Definition 2.11. We moreover define a Morse shellable complex to be a finite simplicial complex $K$ admitting a filtration $\emptyset \subset K_1 \subset \ldots \subset K_N = K$ by simplicial subcomplexes together with a Morse tiling such that for every $i \in \{1, \ldots, N\}$, $K_i \setminus K_{i-1}$ is a single Morse tile, see Definition 2.14. Replacing Morse tiles by basic tiles in this definition, we recover the classical notion of shellability, see Lemma 2.15 and [10] for instance. These definitions actually extend to a larger class of sets, the Morse tileable or shellable sets, see §§2.2 and 2.3. We prove the following tiling theorem, see Corollaries 2.10, 2.20 and Theorem 2.17.

**Theorem 1.1.** Skeletons and barycentric subdivisions of Morse tileable (resp. shellable) sets are Morse tileable (resp. shellable). Moreover, every Morse tiling on such a set induces Morse tilings on its barycentric subdivisions containing the same number of critical tiles with the same indices.

This tiling theorem relies in particular on the fact that the first barycentric subdivision of a Morse tile is itself a disjoint union of Morse tiles, see Theorem 2.18. Given a Morse tiling on a finite simplicial complex $K$, we also deduce packings by disjoint simplices in its successive barycentric subdivisions, see Proposition 2.23. Such packings were used in [15] to improve upper estimates on the expected Betti numbers of random subcomplexes.

We then associate to every Morse tiling a set of discrete vector fields in the sense of Robin Forman [8] which are compatible with the tiling, see Definition 3.11. Their critical points are in one-to-one correspondence with the critical Morse tiles, see §3.2. Moreover, due to Theorem 9.3 of [8], these vector fields are gradient vector fields of discrete Morse functions provided they have no non-stationary closed paths, see §3.1. We provide a criterion for the latter condition to be satisfied, Theorem 3.14, that applies to Morse shellable complexes. Every Morse shelling on a finite simplicial complex is thus a Morse tiling for which any compatible discrete vector field is the gradient vector field of a discrete self-indexing Morse function, see Corollary 3.15. We prove that this result applies to all triangulations of closed manifolds in dimension one and two. Indeed,

**Theorem 1.2.** Every closed triangulated manifold of dimension less than three is Morse shellable.

In dimension three, we are able to prove the existence of Morse tileable triangulations.
Theorem 1.3. Every closed manifold of dimension less than four admits a Morse tiled triangulation such that moreover every discrete vector field compatible with the tiling is the gradient vector field of a discrete self-indexing Morse function.

Not every Morse tiling shares the property of the ones given by Theorem 1.3 and many simplicial complexes are not Morse tileable, see § 3.6. It would be of interest to find a triangulation of a closed manifold which is not Morse tileable. Given a Morse tiling on a closed triangulated manifold, there are many different compatible discrete vector fields and thus many associated discrete Morse functions in the case of Theorems 1.2 and 1.3, but they all have the same number of critical points with same indices. These critical points are in one-to-one correspondence with the critical tiles of the tiling, preserving the index. Such a Morse tiling thus provides an efficient way to bound the topology of the manifold.

Corollary 1.4. Let $X$ be a closed triangulated $n$-manifold equipped with a Morse tiling $T$ admitting a compatible discrete Morse function. Then, each Betti number $b_k(X)$ of $X$ is bounded from above by the number of critical tiles $c_k(T)$ of index $k$ of $T$ and the Morse inequalities hold true, namely $\sum_{i=0}^{k} (-1)^{k-i} b_i(X) \leq \sum_{i=0}^{k} (-1)^{k-i} c_i(T)$ for every $0 \leq k \leq n$ with equality if $k = n$.

It would be of interest to extend Theorem 1.3 to all dimensions. We also actually do not know which are the closed three-manifolds that admit an $h$-tileable triangulation, see § 3.6.

The second section of this paper is devoted to Morse tiles and tilings and the proof of Theorem 1.1 while the third one is devoted to discrete Morse theory and the proofs of Theorems 1.2, 1.3 and Corollary 1.4, given in §§ 3.3 and 3.5 respectively.

Acknowledgement: The second author is partially supported by the ANR project MICROLOCAL (ANR-15CE40-0007-01).

2 Morse tilings

2.1 Morse tiles

Let us recall that an $n$-simplex is the convex hull of $n+1$ points affinely independent in some real affine space and that the standard $n$-simplex $\Delta_n$ is the one spanned by the standard affine basis of $\mathbb{R}^{n+1}$, see [13]. These are all isomorphic one to another by some affine isomorphism. A face of a simplex is the convex hull of a subset of its vertices.

For every $n > 0$ and every $k \in \{0, \ldots, n+1\}$, we set $T^n_k = \Delta_n \setminus (\sigma_1 \cup \ldots \cup \sigma_k)$, where $(\sigma_i)_{i \in \{1, \ldots, n+1\}}$ denote the facets of $\Delta_n$, that is its codimension one faces. In particular, the tile $T^n_{n+1}$ is the open $n$-simplex $\Delta_n$ and $T^n_0 = \Delta_n$ is the closed one. These standard tiles were introduced in [15] and one of their key properties is the following.

Proposition 2.1 (Proposition 4.1 of [15]). For every $n > 0$ and every $k \in \{0, \ldots, n+1\}$, $T^n_{k+1}$ is a cone over $T^n_k$, deprived of its apex if $k \neq 0$. Moreover, $T^n_{k+1}$ is a disjoint union $T^n_{n+2} \sqcup T^n_k \sqcup T^n_{k+1} \sqcup \ldots \sqcup T^n_{n+1}$. In particular, the cone $T^n_{k+1}$ deprived of its base $T^n_k$ is $T^n_{k+1}$.  


Proposition 2.1 provides a partition of the \( \Delta_{n+1} \) into \( k \)-skeletons of \( T^k_n \) for every \( k \in \{0, \ldots, n\} \). Moreover, it contains a unique tile of order \( k \). The result holds true for \( k = 0 \) as well, since by definition \( T^0_n = \Delta_n \). In the case \( n > 0 \), it is uniquely defined by \( c \), the base of \( T^c_{n+1} \). Thus, \( T^0_{n+1} \) is the disjoint union of \( T^k_n \) for all \( k \in \{0, \ldots, n\} \). The result then follows from Proposition 2.1 and a decreasing induction in general.

Definition 2.2. A basic tile is a subset of a simplex isomorphic to a standard tile \( T^k_{n+1} \) via some affine isomorphism. The integer \( n \geq 0 \) is the dimension of the tile while \( k \in \{0, \ldots, n+1\} \) is the order of the tile.

The \( j \)-skeleton of a tile \( T^k_{n+1} \) is by definition the intersection of the \( j \)-skeleton of \( \Delta_n \) with \( T^k_{n+1} \subseteq \Delta_n \). Proposition 2.1 provides a partition of the \( (n-1) \)-skeleton of \( T^k_{n+1} \) by basic tiles and by induction it provides a partition of all its skeletons by basic tiles of the corresponding dimensions.

Proposition 2.3. For every \( n \geq 0 \), every \( 0 \leq k \leq n+1 \) and every \( j \in \{k-1, \ldots, n\} \), any partition of the \( j \)-skeleton of \( T^k_{n+1} \) given by Proposition 2.1 contains only tiles of order \( \geq k \). Moreover, it contains a unique tile of order \( k \) which is the trace of a \( j \)-dimensional face of \( \Delta_n \) on \( T^k_{n+1} \). If \( j < k - 1 \), the \( j \)-skeleton of \( T^k_{n+1} \) is empty.

Proof. This result is given by Proposition 2.1 when \( j = n - 1 \) and \( T^n_{n+1} \) is indeed the trace of a facet of \( \Delta_n \) on \( T^k_{n+1} \), since it is the intersection of the subcomplex \( T^0_{n+1} \sqcup \cdots \sqcup T^{n-1}_{n+1} \) of \( \Delta_n \) with \( T^k_{n+1} \), while \( T^0_{n+1} \sqcup \cdots \sqcup T^{k-1}_{n+1} \) is disjoint from \( T^k_{n+1} \). The result then follows from Proposition 2.1 and a decreasing induction in general.

Let now \( \tau \) be a face of \( \Delta_n \) not contained in \( \sigma_1 \cup \cdots \cup \sigma_k \) and let \( l \) be its dimension which we assume to be less than \( n-1 \), so that \( k \leq l+1 < n \). We set \( T^l_{n+1} = \Delta_n \setminus (\sigma_1 \cup \cdots \cup \sigma_k \cup \tau) \), it is uniquely defined by \( k, l, n \) up to permutation of the vertices of \( \Delta_n \).

Definition 2.4. A Morse tile is a subset of a simplex isomorphic to a standard tile \( T^k_{n+1} \), \( k \in \{0, \ldots, n+1\} \), or a tile \( T^l_{n+1} \), \( 0 < k \leq l+1 < n \), via some affine isomorphism. It is critical when \( l = k - 1 \) and \( k \) is then said to be its index, while \( \Delta_n \) is critical of index zero and \( \Delta_n \) critical of index \( n \). It is regular otherwise.

For every \( n \geq 0 \) and \( k \in \{0, \ldots, n\} \), we also denote the critical Morse tile of index \( k \) by \( C^k_n = T^{n,k-1}_{n+1} \). In the case \( k = 0 \), \( C^0_n = \Delta_n \) is the standard \( n \)-simplex while \( C^0_{n+1} \) is the standard open \( n \)-simplex. The tiles \( T^0_n \setminus T^1_n \) have been excluded in Definition 2.4 they turn out not to be needed to get Theorems 1.1 and 1.3.

The next lemma computes the contribution of each tile to the Euler characteristic of a tiled simplicial complex. Recall that the Euler characteristic is additive and may be computed with respect to the cellular structure of the simplicial complex, given by open simplices.
Lemma 2.25. For every $0 \leq k \leq n$, $\chi(C^n_k) = (-1)^k$. Likewise, for every $0 \leq k \leq l \leq n-1$, $\chi(T^n_k \setminus T^n_l) = 0$.

Proof. If $k = 0$, $T^n_k = C^n_0$ is the standard simplex $\Delta_n$, so that $\chi(T^n_0) = 1$. If $k = n + 1$, $T^n_k = C^n_n$ is the open simplex, so that $\chi(T^n_k) = (-1)^n$.

For every $n \geq 1$, $T^n_1 = T^n_0 \setminus T^{n-1}_0$ has vanishing Euler characteristic. By Proposition 2.1, for every $n \geq 2$, $T^n_2 = T^n_0 \setminus (T^{n-1}_0 \cup T^{n-1}_1)$ satisfies $\chi(T^n_2) = \chi(T^n_0) - \chi(T^{n-1}_0) - \chi(T^{n-1}_1) = 0$. Then, by induction on $k$, for every $k \geq 1$ and every $n \geq k$, $T^n_k = T^n_0 \setminus (T^{n-1}_0 \cup \ldots \cup T^{n-1}_{n-1})$ has vanishing Euler characteristic so that any basic tile has vanishing Euler characteristic unless it is isomorphic to an open or a closed simplex. Then, for every $0 < k < n$, $\chi(C^n_k) = \chi(T^n_k) - \chi(T^{k-1}_k) = 0 - (-1)^{k-1} = (-1)^k$ by definition and the additivity of the Euler characteristic. Likewise for every $0 \leq k \leq l \leq n-1$, $\chi(T^n_{k,l}) = \chi(T^n_k) - \chi(T^n_l) = 0$. \qed

Proposition 2.26. For every $0 \leq k \leq j \leq n$, the $j$-skeleton of $C^n_k$ admits a partition by basic tiles isomorphic to $T^j_l$ with $l > k$ and a unique critical Morse tile isomorphic to $C^j_l$. This skeleton is empty if $j < k$. Likewise, for every $0 < k \leq l < n-1$, the $j$-skeleton of $T^n_{m,l}$ is empty if $j < k$, admits a partition by basic tiles isomorphic to $T^j_m$ with $m > k$ if $k \leq j \leq l$ together with a unique tile isomorphic to $T^j_{k,l}$ if $l < j \leq n$.

Proof. By definition, $C^n_k = T^n_k \setminus T^{k-1}_k$ and by Proposition 2.3, the $j$-skeleton of $T^n_k$ is empty if $j < k-1$ and admits a partition by basic tiles isomorphic to $T^j_l$, $l \geq k$, with a unique tile of order $k$. The latter contains the unique $(k-1)$-dimensional tile of order $k$. The $j$-skeleton of $C^n_k$ thus inherits a partition by tiles isomorphic to $T^j_l$ with $l > k$ and a tile isomorphic to $T^j_k \setminus T^{k-1}_k = C^j_l$. In particular, it is also empty if $j = k-1$ by Proposition 2.3. Likewise, $T^n_{m,l} = T^n_m \setminus T^n_k$ and by Proposition 2.3 the $l$-skeleton of $T^n_m$ admits a partition by basic tiles isomorphic to $T^l_m$ with $m \geq k$, the tile of order $k$ being unique. The $l$-skeleton of $T^n_{m,l}$ thus inherits a partition by basic tiles of order $m > k$. It then follows from Proposition 2.3 that the same holds true for the $j$-skeleton of $T^n_{m,l}$ with $j \leq l$, these skeletons being empty if $j < k$. Finally, if $j > l$, we deduce from Proposition 2.3 that the $j$-skeleton of $T^n_{m,l}$ admits a partition by tiles isomorphic to $T^j_m$ with $m > k$ and a tile isomorphic to $T^j_k \setminus T^j_l = T^j_{k,l}$. Hence the result. \qed

Proposition 2.27. For every $0 < k < n$, $(c \ast C^n_k) \setminus \{c\} = T^{n+1,k}_k$. Moreover, this cone deprived of its base $C^n_k$ is $C^{n+1}_{k+1}$. Similarly, for every $k \leq l \leq n-1$, $(c \ast T^n_{m,l}) \setminus \{c\} = T^{n+1,l+1}_k$ and if this cone is deprived of its base $T^n_{m,l}$, it is $T^{n+1,l+1}_{k+1}$.

Proof. By definition, $C^n_k = T^n_k \setminus T^{k-1}_k$, so that $(c \ast C^n_k) \setminus \{c\} = (c \ast T^n_k) \setminus (c \ast T^{k-1}_k) = T^{n+1}_k \setminus T^{k+1}_k$. If the cone is deprived of its base, it gets $T^{n+1}_k \setminus T^{k+1}_k = C^{n+1}_k$. Similarly, $(c \ast T^n_{m,l}) \setminus \{c\} = (c \ast T^n_{m,l}) \setminus (c \ast T^{k-1}_k) = T^{n+1}_k \setminus T^{k+1}_k = T^{n+1,l+1}_k$. And if the cone is deprived of its base $T^{n}_k$, we get $T^{n+1}_k \setminus T^{l+1}_k = T^{n+1,l+1}_k$. \qed

2.2 Morse tilings

We now introduce Morse tilings of finite simplicial complexes, or more generally of Morse tileable sets. For a definition of simplicial complexes, see for instance [13]. The relation
between Morse tilings and discrete Morse theory is discussed in §3.

**Definition 2.8.** A subset $S$ of a finite simplicial complex $K$ is said to be Morse tileable iff it admits a partition by Morse tiles such that for every $j \geq 0$, the union of tiles of dimension greater than $j$ is the intersection with $S$ of a simplicial subcomplex of $K$. It is Morse tiled iff such a partition, called a Morse tiling, is given. The dimension of $S$ is then the maximal dimension of its tiles.

The dimension of a Morse tileable set does not depend on the tiling, for it is also the maximal dimension of the open simplices contained in $S$. The trivial partition of a simplicial complex by open simplices is not a Morse tiling, though open simplices are Morse tiles. Recall that Proposition 2.1 provides in particular a partition of the boundary $\partial \Delta_{n+1}$ which contains each basic tile $T_k^n$, $k \in \{0, \ldots, n+1\}$, exactly once. This is a Morse tiling of a triangulated sphere, even an $h$-tiling by Definition 2.11 with one critical tile of index 0 and one critical tile of index $n$. Our aim is to prove Theorem 1.3, up to which every closed manifold of dimension less than four has a Morse tileable triangulation.

**Definition 2.9.** Let $S$ be a Morse tiled set. A subset $S'$ of $S$ is said to be a Morse tiled subset iff it is a union of Morse tiles and there exists a subcomplex $L$ of a finite simplicial complex $K$ such that $S \subset K$ and $S' = S \cap L$.

By definition thus, if $S$ is a Morse tiled set, then for every $j \geq 0$, the union of its tiles of dimension greater than $j$ is a Morse tiled subset of $S$.

**Corollary 2.10.** Let $S$ be a Morse tileable set. Then, all its skeletons are Morse tileable. Moreover, given a Morse tiling on $S$, there exist Morse tilings on its skeletons $S^{(i)}$, $i \geq 0$, such that every tile of $S^{(i)}$ is contained in a tile of $S^{(i+1)}$.

**Proof.** By definition, $S$ is a subset of a finite simplicial complex $K$. Let $n$ be the dimension of $S$ and let a Morse tiling be given. By Propositions 2.3 and 2.6 the $(n-1)$-skeleton of every $n$-dimensional tile of $S$ admits a partition by Morse tiles. Then, the union of tiles of $S$ of dimension less than $n$ with the ones given by these partitions induces a partition of $S^{(n-1)}$ with tiles which are either tiles of $S^{(n)} = S$ or contained in such tiles. Moreover, by construction, for every $j \in \{0, \ldots, n-1\}$, the union of tiles of dimension greater or equal to $j$ in this partition is the intersection of $K^{(n-1)}$ with the union of tiles of dimension greater or equal to $j$ in $S$. Since the latter is by definition the intersection with $S$ of a subcomplex $L$ of $K$, the former is the intersection with $S$ of the complex of $L^{(n-1)}$, so that $S^{(n-1)}$ is Morse tileable. The result is then obtained by induction, replacing $S$ with $S^{(n-1)}$.

We prove in §2.4 that the first barycentric subdivision of a Morse tileable set is also Morse tileable, see Corollary 2.20, so that the class of Morse tileable sets is stable under barycentric subdivisions and skeletons, proving Theorem 1.1. We already introduced in 15 a notion of tileable simplicial complexes sharing the same properties. Let us recall and slightly generalize this subclass of Morse tileable simplicial complexes.
Definition 2.11. A subset $S$ of a finite simplicial complex $K$ is said to be $h$-tileable iff it admits a partition by basic tiles such that for every $j \geq 0$, the union of tiles of dimension greater than $j$ is the intersection with $S$ of a simplicial subcomplex of $K$. It is $h$-tiled iff such a partition, called an $h$-tiling, is given.

Definition 2.11 extends the definition given in § 4.2 of [15] where only simplicial complexes and basic tiles of the same dimension are admitted. Such a tiling is now called an $h$-tiling to avoid confusion with Morse tilings and due to its close relation with $h$-vectors discussed in § 4.2 of [15], see § 2.5.

Corollary 2.12. Let $S$ be an $h$-tileable set. Then, all its skeletons are $h$-tileable. Moreover, given an $h$-tiling on $S$, there exist $h$-tilings on its skeletons $S^{(i)}$ such that every tile of $S^{(i)}$ is contained in a tile of $S^{(i+1)}$.

Corollary 2.13. Let $S$ be an $h$-tileable set, then so is its first barycentric subdivision $Sd(S)$.

The proofs of Corollaries 2.12 and 2.13 have already been given in [15] in the case only simplicial complexes and tiles of the same dimension are involved. Since they are similar to the ones of Corollaries 2.10 and 2.20, we do not repeat them.

2.3 Morse shellability

We now introduce another subclass of Morse tileable sets, the Morse shellable ones, which plays a role in [3].

Definition 2.14. A subset $S$ of a finite simplicial complex $K$ is said to be Morse shellable (resp. shellable) iff there exists a Morse tiling on $S$ and a filtration $\emptyset \subset S_1 \subset \ldots \subset S_N = S$ by Morse tiled subsets of $S$ such that for every $i \in \{1, \ldots, N\}$, $S_i \setminus S_{i-1}$ is a single Morse tile (resp. basic tile).

A finite simplicial complex $K$ is classically said to be shellable iff there exists an ordering $\sigma_1, \ldots, \sigma_N$ of its maximal simplices such that for every $i \in \{2, \ldots, N\}$, $\sigma_i \cap (\bigcup_{j=1}^{i-1} \sigma_j)$ is of pure dimension $\dim \sigma_i - 1$, see Definition 12.1 of [10] for instance. This means that each simplex $\sigma_1, \ldots, \sigma_N$ is not a proper face of a simplex in $K$ and every simplex in $\sigma_i \cap (\bigcup_{j=1}^{i-1} \sigma_j)$, $i \in \{2, \ldots, N\}$, is a face of a $(\dim \sigma_i - 1)$-dimensional one in this intersection. Definition 2.14 extends this classical notion of shellability. Indeed.

Lemma 2.15. A finite simplicial complex is shellable in the sense of Definition 2.14 iff it is shellable.

Proof. Let $K$ be a finite simplicial complex which is shellable in the sense of Definition 2.14. There exists then a filtration $\emptyset \subset K_1 \subset \ldots \subset K_N = K$ of $K$ by finite simplicial complexes together with an $h$-tiling on $K$ such that for every $i \in \{1, \ldots, N\}$, $K_i \setminus K_{i-1}$ is a single basic tile $T_i$. Let $\sigma_i$ be the closure of $T_i$ in $K$. Then, $\sigma_i \cap (\bigcup_{j=1}^{i-1} \sigma_j)$ is of pure dimension $\dim \sigma_i - 1$ by Definition 2.2. Moreover, $\sigma_i$ cannot be a proper face of a simplex
in $K$ by Definition 2.11, since otherwise the union of tiles of dimensions greater than $\dim \sigma_i$ would not be a simplicial subcomplex of $K$.

Conversely, let us now assume that $K$ is a shellable simplicial complex, so that there exists an ordering $\sigma_1, \ldots, \sigma_N$ of its maximal simplices such that for every $i \in \{2, \ldots, N\}$, $\sigma_i \cap (\cup_{j=i}^{N-1} \sigma_j)$ is of pure dimension $\dim \sigma_i - 1$. This means that $\sigma_i \cap (\cup_{j=i}^{N-1} \sigma_j)$ is a union of facets of $\sigma_i$ so that $T_i = \sigma_i \setminus (\cup_{j=i}^{N-1} \sigma_j)$ is a basic tile. These tiles provide a partition of $K$ and we have to prove that for every $j \geq 0$, the union of tiles of dimension greater than $j$ is a simplicial subcomplex of $K$. We proceed by induction on $i \in \{1, \ldots, N\}$. For $i = 1$, there is nothing to prove. Assume now that the union $\cup_{j=1}^{i-1} T_j$ is an $h$-tiling of $K_{i-1}$. Then, each open facet of $\sigma_i$ has to be contained in a tile of dimension at least $\dim \sigma_i$ in $K_{i-1}$, since otherwise it would be the interior of one of the tiles $T_j$, $j < i$, and $\sigma_j$ would be a proper face of $\sigma_i$, a contradiction. Since by induction the union of tiles of dimensions at least $\dim \sigma_i$ in $K_{i-1}$ is a simplicial subcomplex of $K_{i-1}$, the same holds true for the union of tiles of dimensions at least $\dim \sigma_i$ in $K_i$, which is a simplicial subcomplex of $K_i$. The result follows.

We may now revisit Propositions 2.3 and 2.6.

Lemma 2.16. The codimension one skeletons of Morse tiles are Morse shellable.

Proof. The $(n-1)$-skeleton of a closed $n$-simplex is well known to be shellable. A shelling is given by Proposition 2.1 where the order of the simplices of $\partial \Delta_n = T_{n-1}^0 \cup \ldots \cup T_n$ is given by the order of the tiles in the sense of Definition 2.2. If the Morse tile is basic, isomorphic to $T_k^n$, then by Proposition 2.1, the $(n-1)$-skeleton of $T_k^n$ is shelled by $T_{k-1}^n \sqcup \ldots \sqcup T_n$, again ordering the tiles by increasing order in the sense of Definition 2.2. Finally, if the Morse tile is not basic, isomorphic to $T_k^{n,l} = T_k^n \setminus T_k^l$ with $k-1 \leq l < n-1$, the $(n-1)$-skeleton of $T_k^{n,l}$ is likewise shelled by $T_{k-1}^{n,l} \sqcup T_{k+1}^{n-1} \sqcup \ldots \sqcup T_{n-1}$.

Theorem 2.17. Let $S$ be a Morse shellable set. Then, all its skeletons are Morse shellable. Moreover, given a Morse shelling on $S$, there exist Morse shellings on its skeletons $S^{(i)}$, $i \geq 0$, such that every tile of $S^{(i)}$ is contained in a tile of $S^{(i+1)}$.

Proof. Let $S$ be equipped with a Morse shelling. By Definition 2.14, there exists a filtration $\emptyset \subset S_1 \subset \ldots \subset S_N = S$ by Morse tiled subsets of $S$ such that for every $i \in \{1, \ldots, N\}$, $S_i \setminus S_{i-1}$ is a single Morse tile. Let $n$ be the dimension of $S$, it is enough to prove this result for the $(n-1)$-skeleton of $S$, since replacing $S$ by $S^{(n-1)}$ we get the result by decreasing induction.

We proceed by induction on $i \in \{1, \ldots, N\}$. If $i = 1$, $S_1$ is a single Morse tile and its $(n-1)$-skeleton is shellable by Lemma 2.16. Let us now assume that this result holds true for $S_{i-1}$ and prove it for $S_i$. By the induction hypothesis, the skeleton $S_{i-1}^{(n-1)}$ is shellable and by Lemma 2.16, the $(n-1)$-skeleton of the Morse tile $S_i \setminus S_{i-1}$ is shellable as well. Then, the concatenation of these shellings provides a shelling of $S_i^{(n-1)}$. Indeed, for every $j \geq 0$, the union of tiles of dimension greater than $j$ in this concatenation is the intersection with $S_i$ of the $(n-1)$-skeleton of $L_j$, where $L_j$ is a subcomplex of a complex $K$ containing $S$ such that the union of tiles of dimension greater than $j$ in $S$ is the trace $L_j \cap S$. □
We likewise prove in §2.4 that barycentric subdivisions of Morse shellable sets are Morse shellable, see Corollary 2.20.

2.4 The tiling theorem

For every Morse tile $T = \Delta_n \setminus (\sigma_1 \cup \ldots \cup \sigma_k \cup \tau)$, we set $\text{Sd}(T) = \text{Sd}(\Delta_n) \setminus \left( \bigcup_{i=1}^{k} \text{Sd}(\sigma_i) \cup \text{Sd}(\tau) \right)$, where $\text{Sd}(\Delta_n)$ denotes the first barycentric subdivision of $\Delta_n$, see [13].

**Theorem 2.18.** The first barycentric subdivision of every Morse tile $T$ is Morse shellable, shelled by tiles of the same dimension as $T$. Moreover, such a Morse shelling can be chosen such that it contains a critical tile iff $T$ is critical and this critical tile is then unique of the same index as $T$.

The fact that the first barycentric subdivision of a basic tile is tileable has already been established in [15] and Theorem 2.18 also recovers the fact that $\text{Sd}(\Delta_n)$ is shellable, see Theorem 5.1 of [2].

**Proof.** Let us first prove the result for basic tiles by induction on their dimension $n > 0$. If $n = 1$, the partitions $\text{Sd}(T_0^1) = T^1_0 \sqcup T^1_1$, $\text{Sd}(T_1^1) = 2T^1_1$ and $\text{Sd}(T_2^1) = T^1_1 \sqcup T^1_2$ are suitable with the filtration $S_1 = T^1_0$ and $S_2 = \text{Sd}(T^1_1)$, see Figure 1.

![Figure 1: Tilings of subdivided one-dimensional tiles.](image)

Now, let us assume that the result holds true for $r \leq n - 1$ and let us prove it for $r = n$. From Proposition 2.1 (see also Corollary 4.2 of [15]), $\partial \Delta_n$ has a partition $\bigcup_{k=0}^{n-1} T^1_k$ which is such that for every $r \in \{0, \ldots, n\}$, $\bigcup_{k=0}^{r} T^1_{k-1}$ is a subcomplex of $\partial \Delta_n$ covered by $r + 1$ tiles. We equip $\text{Sd}(\partial \Delta_n) = \bigcup_{k=0}^{n-1} \text{Sd}(T^1_{k-1})$ with the partition by basic tiles given by the induction hypothesis. There exists a filtration $L_1 \subset \ldots \subset L_{(n+1)!} = \text{Sd}(\partial \Delta_n)$ such that for every $j \in \{1, \ldots, (n+1)\}$, $L_j$ is a subcomplex which is the union of $j$ tiles of the partition. Indeed, if $S^1_k$ is the filtration of $\text{Sd}(T^1_{k-1})$ given by the induction hypothesis, $k \in \{0, \ldots, n\}$, $i \in \{1, \ldots, n!\}$, we set for every $j = kn! + i$, $L_j = \bigcup_{k=0}^{j-1} T^{n-1}_r \sqcup S^1_k$, which is a subcomplex by the induction hypothesis. Then, $\text{Sd}(\Delta_n)$ gets a partition by cones over the tiles of $\text{Sd}(\partial \Delta_n)$ centered at the barycenter of $\Delta_n$ where all the cones except the one over $T^1_0$ are deprived of their apex. From Proposition 2.1 this partition induces a shelling of $\text{Sd}(\Delta_n) = \text{Sd}(T^1_0)$ with a unique tile of order zero and no other critical tile, the cones over the filtration $(L_j)_{j \in \{1, \ldots, (n+1)\}}$ providing the shelling. For every $k \in \{1, \ldots, n+1\}$, we equip $\text{Sd}(T^1_k) = \text{Sd}(\Delta_n) \setminus \bigcup_{j=0}^{k-1} \text{Sd}(T^1_{j-1})$ with the shelling induced by removing the bases of all the cones over the tiles included in $\bigcup_{j=0}^{k-1} \text{Sd}(T^1_{j-1}) \subset \text{Sd}(\partial \Delta_n)$ in the preceding shelling. From Proposition 2.1 these cones deprived of their bases are basic tiles so that we get as
well a shelling of $\text{Sd}(T^n_k)$ which, as in [15], has no more basic tile of order zero as soon as $k > 0$ and gets a unique basic tile of order $n + 1$ when $k = n + 1$. The result is proved in the case of basic tiles.

Let us now prove the result for non-basic tiles $T^{n,l}_k$. The shelling of $\text{Sd}(T^{n,l}_k)$ is again induced by the one of $\text{Sd}(\Delta_n)$. We obtained the shelling of $\text{Sd}(T^n_k)$ by considering the cones deprived of their bases for every tile of the shelling of $\text{Sd}(T^{n-1}_j)$ with $0 \leq j < k$. Among the $(n+1)!$ tiles belonging to the shelling of $\text{Sd}(T^{n,l}_k)$, $n!$ are cones deprived of their apex over the tiles of the shelling of $\text{Sd}(T^{n-1}_j)$ and by induction, for every $k - 1 \leq l \leq n - 1$, $(l+1)!$ of them are iterated cones over the tiles of the shelling of $\text{Sd}(T^n_k)$. If $l = k - 1$, the shelling of $\text{Sd}(T^{k-1}_k)$ contains a unique tile $T^{k-1}_k = \Delta_{k-1}$ together with tiles $T^{k-1}_m$ with $0 < m < k$ by the previous case. Hence, the $k!$ tiles of the shelling of $\text{Sd}(T^n_k)$ which intersect $\text{Sd}(T^{k-1}_k) \subset \text{Sd}(T^n_k)$ consist of a tile $T^n_k$ and tiles $T^{n}_m$ with $0 < m < k$ by Proposition 2.1. The Morse shelling induced on $\text{Sd}(C^n_k) = \text{Sd}(T^n_k) \setminus \text{Sd}(T^{k-1}_k)$ hence consists of a tile $C^n_k = T^n_k \setminus T^{k-1}_k$ together with tiles $T^{n,k-1}_m = T^{n}_m \setminus T^{k-1}_m$ with $0 < m < k$ and tiles $T^{n}_m$ with $0 < m \leq n$. As before, this Morse shelling is a Morse shelling, the Morse shelling being obtained by concatenation. Finally, if $l \geq k$, the shelling of $\text{Sd}(T^n_k)$ contains tiles $T^{n}_m$ with $0 < m \leq l$. The $(l+1)!$ tiles of the shelling of $\text{Sd}(T^n_k)$ which intersect $\text{Sd}(T^{l}_k) \subset \text{Sd}(T^n_k)$ thus consist of tiles $T^{n}_m$ with $0 < m \leq l$ by Proposition 2.1, they are iterated cones of the previous ones. The Morse shelling induced on $\text{Sd}(T^{n,l}_k) = \text{Sd}(T^n_k) \setminus \text{Sd}(T^{l}_k)$ hence consists of tiles $T^{n,l}_m = T^{n}_m \setminus T^{l}_m$ with $0 < m \leq l$ and of tiles $T^{l}_m$ with $0 < j \leq n$.

Remark 2.19.  

1. We actually proved that the regular Morse tiles involved in the partition of $\text{Sd}(C^n_k)$ are either basic, or isomorphic to $T^{n,k-1}_m$ with $0 < m < k$. Likewise, the tiles involved in the partition of $\text{Sd}(T^{n,l}_k)$ are either basic or isomorphic to $T^{n,l}_m$ with $0 < m \leq l$.

2. Theorem 2.18 also extends to subsets $T^{n,l}_m = T^{n}_m \setminus T^{l}_m$ with the same proof, but these have not been declared to be Morse tiles in Definition 2.4 and thus have been excluded.

3. One may check that $\text{Sd}(C^n_3)$ does not admit any partition involving only critical Morse tiles and basic tiles so that non-basic regular Morse tiles are needed to get Theorem 2.18.

Corollary 2.20. Let $S$ be a Morse tileable (resp. shellable) set, then so is its first barycentric subdivision $\text{Sd}(S)$. Moreover, given a Morse tiling (resp. shelling) on $S$, any induced tiling (resp. shelling) on $\text{Sd}(S)$ contains the same number of critical tiles with the same indices.

Proof. Let us first assume that $S$ is a Morse tileable subset of a finite simplicial complex $K$. In order to equip $\text{Sd}(S)$ with a Morse tiling, we first equip $S$ with a Morse tiling and then, for each of its tile $T$, equip $\text{Sd}(T)$ with a Morse tiling given by Theorem 2.18. It is indeed a tiling since for every $j \geq 0$, the union of tiles of dimension greater than $j$ of $\text{Sd}(S)$ is the first barycentric subdivision of the union of tiles of dimension greater than $j$ of $S$, so that if the latter is the intersection with $S$ of a subcomplex $L_j$ of $K$, then the former is the intersection with $\text{Sd}(S)$ of the subcomplex $\text{Sd}(L_j)$ of $\text{Sd}(K)$.
Let us now prove that the barycentric subdivision of $S$ is shellable, provided $S$ is. Let then $S$ be equipped with a Morse shelling. By Definition 2.14 there exists a filtration $\emptyset \subset S_1 \subset \ldots \subset S_N = S$ by Morse tiled subsets of $S$ such that for every $i \in \{1, \ldots, N\}$, $S_i \setminus S_{i-1}$ is a single Morse tile. We proceed by induction on $i \in \{1, \ldots, N\}$. If $i = 1$, $S_1$ is a closed simplex and the result follows from Theorem 2.18. Let now the result be proved up to the rank $i - 1$. Then $S_i \setminus S_{i-1}$ is a Morse tile and we get a shelling of $Sd(S_i)$ by concatenation of the shelling of $Sd(S_{i-1})$ with the shelling of $Sd(S_i \setminus S_{i-1})$ given by Theorem 2.18 as in the proof of Theorem 2.17. By Theorem 2.18 these induced tiling (resp. shelling) on $Sd(S)$ contain the same number of critical tiles of the same indices as the one of $S$. Hence the result. $\square$

2.5 Packings and $h$-vectors

When an $h$-tiling $T$ of a finite $h$-tileable simplicial complex $K$ only involves tiles of the same dimension $n$, we may encode the number of tiles of each order into the $h$-vector $h(T) = (h_0(T), \ldots, h_{n+1}(T))$ of the tiling, see Definition 4.8 of [15]. Then, by Theorem 4.9 of [15], two $h$-tilings $T$ and $T'$ of $K$ have the same $h$-vectors provided $h_0(T) = h_0(T')$ and if moreover $h_0(T) = 1$, this $h$-vector $h(T)$ coincides with the $h$-vector of $K$ by Corollary 4.10 of [15], see also [9, 19] for a definition. In particular, $h$-tilings provide in this situation a geometric interpretation of the $h$-vector as the number of tiles of each order needed to tile the complex. A part of these results remains valid in the case of Morse tilings. Namely, for every Morse tiling $T$ on a Morse tileable set, let us denote by $h_j^0(T)$ (resp. $h_j^1(T)$) the number of basic tiles of dimension $j$ and order zero (resp. order one) contained in $T$, $j \geq 0$.

Proposition 2.21. Let $T$ be a Morse tiling on an $n$-dimensional Morse tileable set $S$. Then, $\sum_{j=0}^n (j+1)h_j^0(T) + h_1(T) = f_0(S)$, where $h_1(T) = \sum_{j=0}^n h_j^1(T)$ and $f_0(S)$ denotes the number of vertices of $S$.

Proof. By Propositions 2.3 and 2.6 the only Morse tiles which contain vertices are basic tiles of order zero and one. The former contain $j + 1$ vertices if they are of dimension $j$ while the latter contain a single vertex, whatever their dimension is. Counting the number of vertices of $S$ by using the partition $T$, we deduce the result. $\square$

Corollary 2.22. Let $T$ and $T'$ be two Morse tilings on a Morse tileable set which contain only tiles of the same dimension. Then, $h_0(T) = h_0(T')$ if and only if $h_1(T) = h_1(T')$. $\square$

As in §5 of [15], Morse tilings can be used to produce packings by disjoint simplices in Morse tileable sets.

Proposition 2.23. Let $T$ be a Morse tiling on a Morse tileable set $S$. Then, it is possible to pack in $Sd(S)$ a disjoint union of simplices containing, for every $j \geq 0$, at least $h_j^0(T) + h_j^1(T)$ $j$-dimensional ones.

Proof. A basic tile $T \subset \Delta_j$ of order zero or one contains at least one vertex $v$ and a $j$-simplex of $Sd(\Delta_j)$ containing $v$ is contained in $Sd(T)$. A choice of such a simplex for each subdivided basic tile of order zero or one provides a suitable packing. $\square$
The packings given by Proposition 2.23 have non-trivial asymptotic under a large number of barycentric subdivisions. Indeed, assume for instance that the Morse tiling $T$ only contains tiles of the same dimension $n$. Then, by Theorem 2.18, $h_0(S_d(S))$ is constant, so that by Proposition 2.21, $h_0(S_d(S)) + h_1(S_d(S)) \sim_{d \to +\infty} f_0(S_d(S))$, while by [3] (see also [6, 16]), $f_0(S_d(S)(n+1)^d)$ converges to a positive limit $q_0$ as $d$ grows to $+\infty$, where $f_n(S)$ denotes the number of $n$-dimensional tiles of $S$. Proposition 2.23 makes it possible to pack at least a number of disjoint $n$-simplices in $S_d(S)$ asymptotic to $q_0 f_n(S)(n+1)^{d-1}$ as $d$ grows to $+\infty$. Such packings were used in [15] to improve upper estimates on the expected Betti numbers of random subcomplexes in a simplicial complex $K$. More general packing results are obtained in § 5 of [15], where simplices are allowed to intersect each other in low dimensions.

3 Morse tileable triangulations and discrete Morse theory

3.1 Discrete Morse theory

Let us recall few notions of the discrete Morse theory introduced by Robin Forman, see [8]. Let $K$ be a finite simplicial complex. For every $p \geq 0$, we denote by $K[p]$ its set of $p$-simplices and for every $\tau, \sigma$ in $K$, $\tau > \sigma$ means that $\sigma$ is a face of $\tau$.

Definition 3.1 (Page 91 of [8]). A function $f: K \to \mathbb{R}$ is a discrete Morse function iff for every $p$-simplex $\sigma$ of $K[p]$, the following two conditions are satisfied:

1. $\#\{\tau \in K[p+1] \mid \tau > \sigma \text{ and } f(\tau) \leq f(\sigma)\} \leq 1$,
2. $\#\{\nu \in K[p-1] \mid \nu < \sigma \text{ and } f(\nu) \geq f(\sigma)\} \leq 1$.

Definition 3.2 (Definition 9.1 of [8]). A discrete vector field on a simplicial complex $K$ is a map $W: K \to K \cup \{0\}$ such that:

1. $\forall p \geq 0, W(K[p]) \subset K[p+1] \cup \{0\}$.
2. For every $\sigma \in K[p]$, either $W(\sigma) = 0$ or $\sigma$ is a face of $W(\sigma)$.
3. If $\sigma \in \text{Im}(W)$, then $W(\sigma) = 0$.
4. For every $\sigma \in K[p]$, $\#\{\nu \in K[p-1] \mid W(\nu) = \sigma\} \leq 1$.

Definition 3.3 (Remark on page 131 of [8]). A critical point of a discrete vector field $W$ on a simplicial complex $K$ is a simplex $\sigma \in K$ such that $W(\sigma) = 0$ and $\sigma \notin \text{Im}(W)$.

We set the index of a critical point $\sigma$ of a discrete vector field $W$ to be the dimension of $\sigma$. 12
Definition 3.4 (Definition 6.1 of [8]). The gradient vector field of a discrete Morse function \( f : K \to \mathbb{R} \) is the discrete vector field \( W_f : K \to K \cup \{0\} \) such that for every \( p \)-simplex \( \sigma \in K \), \( W_f(\sigma) = 0 \) if there is no \((p + 1)\)-simplex \( \tau \) such that \( \tau > \sigma \) and \( f(\tau) \leq f(\sigma) \) while \( W_f(\sigma) = \tau \) otherwise.

Remark 3.5. The gradient vector field is actually defined on oriented simplices in [8] and Definition 3.4 should rather read \( W_f(\sigma) = -\langle \partial \tau, \sigma \rangle \tau \) in case \( \tau > \sigma \) and \( f(\tau) \leq f(\sigma) \). However, orientations do not play any role throughout this paper.

Definition 3.6 (Definition 9.2 of [8]). Let \( W \) be a discrete vector field. A \( W \)-path of dimension \( p \) is a sequence of \( p \)-simplices \( \gamma = \sigma_0, \sigma_1, ..., \sigma_r \) such that:

1. If \( W(\sigma_i) = 0 \), then \( \sigma_{i+1} = \sigma_i \).
2. If \( W(\sigma_i) \neq 0 \), then \( \sigma_{i+1} \neq \sigma_i \) and \( \sigma_{i+1} < W(\sigma_i) \) (i.e. \( \sigma_{i+1} \) is a facet of \( W(\sigma_i) \)).

The path \( \gamma \) is said to be closed iff \( \sigma_r = \sigma_0 \) and to be non-stationary iff \( \sigma_1 \neq \sigma_0 \).

Remark 3.7. These Definitions 3.1 - 3.6 are given in [8] in the more general setting of regular CW-complexes rather than simplicial complexes. They extend to Morse tiled sets in the sense of Definition 2.8 as well, replacing simplices by their interiors.

Theorem 3.8 (Theorem 9.3 of [8]). Let \( W \) be a discrete vector field on a finite simplicial complex. There is a discrete Morse function \( f \) for which \( W \) is the gradient vector field if and only if \( W \) has no non-stationary closed paths. Moreover, for every such \( W \), \( f \) can be chosen to have the property that if a \( p \)-simplex is critical, then \( f(\sigma) = p \).

A Morse function given by Theorem 3.8 is said to be self-indexing. We finally recall that the critical points of a discrete Morse function on a finite simplicial complex span a chain complex which computes its homology, see Theorem 7.3 of [8].

3.2 Compatible discrete vector fields

We are now going to prove that a Morse tiled set carries natural discrete vector fields compatible with the tiling, since every Morse tile carries natural discrete vector fields, see Remark 3.7.

Proposition 3.9. For every \( n \geq 0 \), every \( k \in \{0, ..., n+1\} \) and every decomposition \( T^n_k = T^{n-1}_k \sqcup ... \sqcup T^{n-1}_n \sqcup T^n_{n+1} \) given by Proposition 2.7, the tile \( T^n_k \) has a discrete vector field \( W^n_k \) such that \( W^n_k(T^{n-1}_n) = T^{n+1}_n \) and such that for every \( l \in \{k, ..., n-1\} \) the restriction of \( W^n_k \) to \( T^{l-1}_l \) coincides with one vector field \( W^{n-1}_l \). Such a vector field has no critical point if \( 0 < k < n + 1 \), a unique critical point of index zero if \( k = 0 \) and a unique critical point of index \( n \) if \( k = n + 1 \).
Proof. We proceed by induction on \( n \). If \( n = 0 \), we set \( W^n_k = 0 \) for every \( k \in \{0, 1\} \) and the result holds true. Let us suppose that the result is proved up to the dimension \( n - 1 \) and prove it for the dimension \( n \). Let then \( k \in \{0, \ldots, n + 1\} \) and a decomposition \( T^n_k = T^{n-1}_k \cup \ldots \cup T^n_{n-1} \cup T^n_{n+1} \) be chosen (given by Proposition 3.9). If \( k = n + 1 \), we set \( W^n_{n+1} = 0 \) and the tile \( T^n_{n+1} \) is critical of index \( n \) since it has no facet. Otherwise, we set \( W^n_k(T^n_{n-1}) = T^n_{n+1} \) and for every \( l \in \{k, \ldots, n-1\} \) we set the restriction of \( W^n_k \) to the tile isomorphic to \( T^n_l \) to be \( W^n_{l-1} \) through such an isomorphism. By the induction hypothesis, it has no critical point, unless \( k = 0 \) where it has a unique critical point of index zero.

Proposition 3.9 defines many discrete vector fields on the tile \( T^n_k \), \( n \geq 0 \), \( k \in \{0, \ldots, n+1\} \), which have all been denoted by \( W^n_k \). Indeed, such a vector field depends on the choice of a partition \( T^n_k = T^{n-1}_k \cup \ldots \cup T^n_{n-1} \cup T^n_{n+1} \), but also on a similar choice of a partition of the \( (n-1) \)-dimensional tiles \( T^{n-1}_k, \ldots, T^{n-1}_{n-1} \) and by induction, on such a choice of an \( h \)-tiling on all skeletons of \( T^n_k \), compare [2.1].\footnote{\[}\footnote{\[} In particular, for every face \( \tau \) of \( \Delta_n \) not contained in \( \sigma_1 \cup \cdots \cup \sigma_k \), where \( T^n_k = \Delta_n \setminus (\sigma_1 \cup \cdots \cup \sigma_k) \) and \( \dim \tau = l \in \{k, \ldots, n-2\} \), we may choose these partitions in such a way that \( \tau \setminus (\sigma_1 \cup \cdots \cup \sigma_k) \) is a basic tile of order \( k \) of the \( l \)-skeleton of \( T^n_k \), which is thus preserved by \( W^n_k \). Such a vector field \( W^n_k \) then restricts to a discrete vector field on the complement \( T^n_{k,l} = T^n_k \setminus T^l_k \).

**Corollary 3.10.** For every \( n \geq 0 \) and every \( k \in \{0, \ldots, n\} \), the critical Morse tile \( C^n_k \) inherits from any vector field given by Proposition 3.9 a discrete vector field which has a unique critical point of index \( k \). Moreover, for every \( 0 < k \leq l < n-1 \), the standard regular Morse tile \( T^n_{k,l} \) inherits from any vector field given by Proposition 3.9 which preserves \( T^n_{k,l} \subset T^n_k \) a discrete vector field without any critical point.

Proof. By Proposition 2.3, the \( k \)-skeleton of \( T^n_k \) is tiled by a unique tile \( T^k_k = T^{k-1}_k \cup T^{k+1}_k \) and by Proposition 3.9 \( W^n_k(T^{k-1}_k) = T^{k+1}_k \), for any vector field \( W^n_k \) given by this proposition. Thus, \( W^n_k \) induces a discrete vector field on \( C^n_k = T^n_k \setminus T^{k-1}_k \), just by restriction. The tile \( T^{k+1}_k \subset C^n_k \) is then critical since it is no more in the image of \( W^n_k \), so that this vector field on \( C^n_k \) has a unique critical point of index \( k \). Likewise, the vector field \( W^n_k \) of \( T^n_k \) preserves \( T^n_{k,l} \) and thus restricts to a vector field on \( T^n_{k,l} \) which has no critical point.\footnote{\[}\footnote{\[}
Proof. By Definition 2.8, the Morse tiling on $K$ provides a partition of $K$ by Morse tiles. The vector fields given by Proposition 3.9 and Corollary 3.10 thus induce discrete vector fields on $K$ whose critical points are in one-to-one correspondence with the critical Morse tiles, preserving the index. Now, Theorem 3.8 guarantees that such a vector field is the gradient vector field of some discrete self-indexing Morse function on $K$ provided that it has no non-stationary path.

We finally provide a criterium which ensures that a compatible discrete vector field has no non-stationary closed path. This criterium given by Theorem 3.14 applies to Morse shellings, see Definition 2.14 and Corollary 3.15.

Lemma 3.13. Every discrete vector field given by Proposition 3.9 or Corollary 3.10 has no non-stationary closed path in the corresponding Morse tile.

Proof. It is enough to prove the result for a basic tile $T^k_n$ equipped with a discrete vector field $W^k_n$ given by Proposition 3.9, since vector fields given by Corollary 3.10 on non basic Morse tiles are restriction of the formers, so that every path on a non basic Morse tile is also a path on the corresponding basic tile, with the exception of the stationary path at the critical point in the case of a critical Morse tile. We then prove the result by induction on the dimension $n$ of the tile. If $n = 0$, there is nothing to prove, every path is stationary. Otherwise, let us choose a partition $T^k_n = T^k_{n-1} \cup \ldots \cup T^k_{n-1} \cup T^k_{n+1}$ given by Proposition 2.1 and an associated discrete vector field $W^k_n$. A path of dimension $n$ of $W^k_n$ is stationary, since $W^k_n(T^k_{n+1})$ has to vanish. A path of dimension $n-1$ which begins with $T^k_{n-1}$ continues in one of the tiles $T^k_{n-1}, \ldots, T^k_{n-1}$ and is then stationary as in the previous case. Any other path is contained in one of the tiles $T^k_{n-1}, \ldots, T^k_{n-1}$, so that the result follows from the induction hypothesis.

Theorem 3.14. Let $K_0 \subset K_1 \subset \ldots \subset K_N = K$ be a filtration of Morse tiled finite simplicial complexes such that for every $i \in \{1, \ldots, N\}$, $K_i \setminus K_{i-1}$ is a single Morse tile. Let $W$ be a compatible discrete vector field on $K$ such that its restriction to $K_0$ has no non-stationary closed path. Then, $W$ has no non-stationary closed path and it is the gradient vector field of a discrete self-indexing Morse function on $K$.

Proof. We prove the result by induction on $i \in \{0, \ldots, N\}$. If $i = 0$, the result holds true by hypothesis. Let $i > 0$ and $W$ be a discrete vector field on $K$ compatible with the Morse tiling and whose restriction to $K_0$ has no non-stationary closed path. Then, $K_i \setminus K_{i-1}$ is reduced to a single Morse tile and by Lemma 3.13, it has no non-stationary closed path. Now, a $W$-path on $K_i$ is either contained in $K_i \setminus K_{i-1}$, or it meets $K_{i-1}$ and cannot leave $K_{i-1}$ once it entered in this subcomplex by definition. In both cases, from the induction hypothesis, it cannot have any non-stationary closed path. Hence the result.

Corollary 3.15. Every discrete vector field compatible with a Morse shelling of a finite simplicial complex is the gradient vector field of a discrete self-indexing Morse function.

Proof. From Theorem 3.14, any discrete vector field compatible with any Morse shelling is the gradient vector field of a discrete Morse function, since its restriction to $K_0 = \emptyset$ has no non-stationary closed path. Hence the result.
Some relations between discrete Morse theory and shellability have already been developed in [5].

### 3.3 Proof of Theorem 1.2

**Proof.** Let $K$ be a finite simplicial complex homeomorphic to a closed surface, which we may assume to be connected. We have to prove that there exists a filtration $K_1 \subset \ldots \subset K_N$ of Morse tiled simplicial complexes such that $K_N = K$ and such that for every $i \in \{1, \ldots, N\}$, $K_i$ is the union of $i$ Morse tiles, see Definition 2.14. In order to prove the existence of the filtration, we proceed by induction on $i > 0$. If $i = 1$, we choose any closed simplex in $K$ and declare that $K_1$ is this simplex, tiled by a single critical tile of index 0.

Let us assume by induction that we have constructed a tiled subcomplex $K_i$ with $i$ tiles. If there exists an edge $e$ in $K_i$ which is adjacent to only one triangle of $K_i$, we know from the Dehn-Sommerville relations that $K$ contains a triangle $T$ adjacent to $e$ and not contained in $K_i$. Then $T \setminus K_i$ is isomorphic to a triangle deprived from at least one face of dimension one and thus at most one face of codimension greater than one, so that $T \setminus K_i$ is a Morse tile by Definition 2.4. We set $K_{i+1}$ to be the union of $K_i$ and $T$ (together with its faces) and equip it with the Morse tiling given by the one of $K_i$ completed by $T \setminus K_i$. If now all edges of $K_i$ are adjacent to two triangles of $K_i$, let us prove that $K_i = K$. From the Dehn-Sommerville relations, we know that every edge is adjacent to at most two triangles of $K_i$. We observe that the link of every vertex in $K$ is a triangulated circle, so that the star of a vertex in $K$ is a cone over a polygon, see Figure 2.

![Figure 2: The star of a vertex $v$ in $K$.](image)

Let $v$ be a vertex in $K$. Since the underlying topological space $|K|$ is connected, there exists a path $v_0, v_1, \ldots, v_k$ such that $v_0 \in K_i, v_k = v$ and for every $j \in \{0, \ldots, k - 1\}$, $[v_j, v_{j+1}]$ is an edge of $K$. Then, by construction, $v_0$ is adjacent to a triangle of $K_i$ and since all edges of $K_i$ are adjacent to two triangles, all triangles adjacent to $v_0$ have to be in $K_i$, see Figure 2. Thus $v_1$ belongs to $K_i$ as well and by induction, $v$ belongs to $K_i$. Hence, $K_i$ contains all vertices of $K$ and also all triangles and edges adjacent to them, so that $K_i = K$. The proof is similar in dimension 1. □

### 3.4 Morse tilings on triangulated handles

Recall that in topology, a handle of index $i$ and dimension $n$ is by definition a product of an $i$-dimensional disk with an $(n - i)$-dimensional one, see §6 of [14]. We likewise define a handle of index $i$ in discrete geometry to be the product of simplices $\Delta_i \times \Delta_{n-i}$, or rather
in what follows the product $\hat{\Delta}_i \times \Delta_{n-1}$ of an open simplex of dimension $i$ with a closed $(n-i)$-simplex, suitably triangulated. Our purpose is to define a Morse shelling on such triangulated $i$-handle for $i = 1$ or $n - 1$, the general case being postponed.

**Proposition 3.16.** For every $n \geq 2$, $\Delta_1 \times \Delta_{n-1}$ has a subdivision into $n$ simplices $\sigma_1, \ldots, \sigma_n$ of dimension $n$ turning it into a shellable simplicial complex. Moreover, writing $\partial \Delta_1 = \{0, 1\}$, it can be chosen in such a way that for every $i \in \{1, \ldots, n\}$, $\dim(\sigma_i \cap (\{0\} \times \Delta_{n-1})) = n - i$ and $\dim(\sigma_i \cap (\{1\} \times \Delta_{n-1})) = i - 1$. For every $i \in \{1, \ldots, n\}$, the subcomplex $K_i^n = \sigma_1 \cup \ldots \cup \sigma_i$ inherits the h-tiling made of one basic tile of order zero and $i - 1$ basic tiles of order one.

**Proof.** If $n = 2$, the square $\Delta_1 \times \Delta_1$ is the union of two triangles meeting along a diagonal and the result follows. If $n > 2$, let $c$ be a vertex of $\{0\} \times \Delta_{n-1}$ so that this simplex is the cone $c \ast (\{0\} \times \Delta_{n-2})$ over its facet $\Delta_{n-2}$. Then, the convex domain $\Delta_1 \times \Delta_{n-1}$ is a cone centered at $c$ over the base $(\Delta_1 \times \Delta_{n-2}) \cup (\{1\} \times \Delta_{n-1})$. By induction, the lateral part $\Delta_1 \times \Delta_{n-2}$ has a subdivision $\sigma_1' \cup \ldots \cup \sigma_n'$ such that for every $i \in \{1, \ldots, n - 1\}$, $\dim(\sigma_i' \cap (\{0\} \times \Delta_{n-2})) = n - 1 - i$ and $\dim(\sigma_i' \cap (\{1\} \times \Delta_{n-2})) = i - 1$ and such that $\sigma_1' \cup \ldots \cup \sigma_i' = T_0^{n-1} \sqcup T_1^{n-1} \sqcup \ldots \sqcup T_i^{n-1}$. We then set, for every $i \in \{1, \ldots, n - 1\}$, $\sigma_i = c \ast \sigma_i'$ and $\sigma_n = c \ast (\{1\} \times \Delta_{n-1})$. The result follows, since $(\{1\} \times \Delta_{n-1}) \setminus (\{1\} \times \Delta_{n-2})$ is isomorphic to $T_i^{n-1}$ and the cone over a basic tile of order one remains a basic tile of order one by Proposition 2.1.

**Corollary 3.17.** For every $n \geq 2$, the handles $\hat{\Delta}_1 \times \Delta_{n-1}$, $\Delta_1 \times \Delta_{n-1}$ and the product $T_1^n \times \Delta_{n-1}$ inherit from Proposition 3.16 the structure of Morse shellable sets. Moreover, for every $i \in \{1, \ldots, n\}$, the subset $K_i^n \cap (\Delta_1 \times \Delta_{n-1})$ gets tiled by a disjoint union $\sqcup_{j=0}^{i-1} T_j^{n,i}$, $K_i^n \cap (\Delta_1 \times \hat{\Delta}_{n-1})$ by one critical tile of index $n - 1$ and $i - 1$ basic tiles of order one and $K_i^n \cap (T_1^n \times \Delta_{n-1})$ by basic tiles of order one, where $K_i^n$ is the subcomplex given by Proposition 3.16.

Corollary 3.17 thus provides a Morse shelling on the triangulated one-handle $\hat{\Delta}_1 \times \Delta_{n-1}$ (resp. on the triangulated $(n - 1)$-handle $\Delta_1 \times \hat{\Delta}_{n-1}$) containing a unique critical tile, of index one (resp. of index $n - 1$).

**Proof.** By Proposition 3.16 for every $i \in \{1, \ldots, n\}$, $K_i^n = \sigma_1 \cup \ldots \cup \sigma_i = T_0^n \sqcup T_1^n \sqcup \ldots \sqcup T_i^n$ and $\dim(\sigma_1 \cap (\{0\} \times \Delta_{n-1})) = n - 1$, so that $\{0\} \times \Delta_{n-1}$ is contained in $\sigma_1$ and disjoint from the tiles $T_1^n$. Thus, $K_i^n \cap (T_1^n \times \Delta_{n-1}) = K_i^n \setminus (\{0\} \times \Delta_{n-1})$ inherits the h-tiling $(\sigma_j \setminus (\{0\} \times \Delta_{n-1})) \sqcup_{j=2}^i (\sigma_j \setminus K_j^n)$ made of $i$ basic tiles of order one. The last part of Corollary 3.17 is proved. By Proposition 3.16 now, $\Delta_1 \times \Delta_{n-1} = \sigma_1 \cup \ldots \cup \sigma_n$ with $\dim(\sigma_i \cap (\{1\} \times \Delta_{n-1})) = i - 1$ so that by induction on $i \in \{1, \ldots, n\}$, the intersection of $\{1\} \times \Delta_{n-1}$ with $\sigma_i$ is a face of dimension $i - 1$ not contained in $\sigma_1 \cup \ldots \cup \sigma_{i-1} \cup (\{0\} \times \Delta_{n-1})$.

By Definition 2.4 the one-handle $\hat{\Delta}_1 \times \Delta_{n-1}$ thus inherits the Morse tiling $\sqcup_{j=0}^{i-1} T_{n,j}^i$ made of one critical tile $C_i^n$ of index one and regular Morse tiles $T_{n,l}^i$ with $l \in \{1, \ldots, n - 1\}$ and moreover for every $i \in \{1, \ldots, n\}$, $K_i^n \cap (\hat{\Delta}_1 \times \Delta_{n-1}) = \sqcup_{j=0}^{i-1} (T_{n,j}^i)$.
Let us finally prove the result for the \((n-1)\)-handle \(\Delta_1 \times \hat{\Delta}_{n-1}\) by induction on \(n\). For \(n = 2\), it has already been proved in the first part. In general, as in the proof of Proposition 3.16, let \(c\) be a vertex of \(\{0\} \times \Delta_{n-1}\) so that \(\Delta_1 \times \hat{\Delta}_{n-1}\) is the union of the cone \(c \ast (\Delta_1 \times \Delta_{n-2})\) over the lateral face deprived of its base and apex and the cone \(c \ast \{1\} \times \hat{\Delta}_{n-1}\) over the upper face. The latter is isomorphic to a standard tile \(T_n\) by Proposition 2.1 while by the induction hypothesis, the former is the union of one critical tile \(C_{n-1} = c \ast C_{n-2} \setminus C_{n-2}\) and \(n-2\) basic tiles \(T_n = (c \ast T_{n-1}) \setminus T_{n-1}\), by Propositions 2.7 and 2.1. The same induction provides the result since for every \(i \in \{1, \ldots, n-1\}\), \(K_n^i = c \ast K_{i-1}^n\).

In dimension three, the tiled two-handle \(\Delta_1 \times \hat{\Delta}_2\) given by Corollary 3.17 is obtained from the triangulated three-ball \(\Delta_1 \times \Delta_2\) given by Proposition 3.16 by removing the cylinder \(\Delta_1 \times \partial \Delta_2\). The latter inherits a triangulation with six triangles. Each of these triangles has an edge on the boundary component \(\{0\} \times \partial \Delta_2\) or \(\{1\} \times \partial \Delta_2\) and the opposite vertex on the opposite component, see Figure 3. Denoting by \(d\) the boundary component \(\{0\} \times \partial \Delta_2\) and \(u\) the component \(\{1\} \times \partial \Delta_2\), this triangulation depicted in Figure 3 produces the cyclic word \(w_2 = ududdu\) while this cyclic word encodes in a unique way the triangulation up to homeomorphisms preserving the boundary components and the orientation.

More generally, let us declare that a triangulation of an annulus \(A \cong [0,1] \times \mathbb{R}/\mathbb{Z}\) is simple iff each triangle has one edge on one boundary component of \(A\) and the opposite vertex on the other boundary component.

**Proposition 3.18.** Simple triangulations of an annulus up to homeomorphisms preserving the boundary components and the orientation are in one-to-one correspondence with finite cyclic words in the alphabet \(\{d,u\}\) containing each letter at least once.

**Proof.** Let us encode one boundary component of the annulus by the letter \(d\) and the other one by the letter \(u\). A triangulation of the annulus is a homeomorphism with a two-dimensional simplicial complex and if this triangulation is simple each triangle of this complex has one edge mapped to some boundary component and thus encoded by either \(d\) or \(u\) and the opposite vertex on the other component. We may join the middle points of the two remaining edges by some arc in the triangle. The union of all these arcs then gives a closed curve homotopic to the boundary components and choosing an orientation on this curve, we read on it a finite cyclic word in the alphabet \(\{d,u\}\). Each boundary component has to contain at least one edge so that this cyclic word has to contain each letter at least
once. Conversely, one may reverse the procedure to associate to every such cyclic word a simple triangulation on the annulus, which is uniquely defined up to homeomorphisms preserving the boundary components and the orientation.

Let us finally declare that a compression of two letters in a cyclic word in the alphabet \{d, u\} is the replacement of a sequence \(dd\) (resp. \(uu\)) by the single letter \(d\) (resp. \(u\)), while a transposition is the replacement of \(du\) (resp. \(ud\)) by \(ud\) (resp. \(du\)). We observe the following proposition which will be useful in the proof of Theorem 1.3.

**Proposition 3.19.** It is possible to obtain \(w_2\) from any finite cyclic word in the alphabet \{d, u\} containing at least three times each letter by applying finitely many compressions or transpositions. □

### 3.5 Proofs of Theorem 1.3 and Corollary 1.4

We are now ready to prove Theorem 1.3. We begin by proving it in dimensions one and two since the approach is the same as in dimension three, even though in these dimensions the result follows from the existence of triangulations combined with Theorem 1.2.

**Proof of Theorem 1.3.** If \(n = 1\), \(X\) is homeomorphic to the boundary of a finite union of two-simplices. Such a homeomorphism defines a triangulation with the shelling \(T_0 \sqcup T_1 \sqcup T_2\) given by Proposition 2.1 on each simplex. By Corollary 3.15, any vector field compatible with this shelling is the gradient vector field of a discrete self-indexing Morse function. In general, we know from Morse theory [4, 11, 14] that \(X\) is obtained by successive attachments of handles, that is it decomposes into finitely many sublevels, starting from the empty set and ending with \(X\) in such a way that one passes from a sublevel to the next one by attaching some handle. We can moreover start by attaching all 0-handles and end by attaching all \(n\)-handles, see [11]. We are going to prove the result by having each sublevels being triangulated and equipped with a Morse tiling and by performing each handle attachment by gluing a Morse tiled handle. Moreover, we will check that the conditions of Theorem 3.14 are satisfied to get the result.

Let us now consider the case \(n = 2\). We start with finitely many closed two-simplices corresponding to the 0-handles we have to attach. We then add one after the other basic tiles of order one on the boundary of these two-simplices, in order to get triangulated disks with an increased number of edges on their boundaries so that one may find two edges on each boundary component which are not faces of the same triangle. We then get a finite union of triangulated balls equipped with a Morse shelling. We can now attach a one-handle \(\Delta_1 \times \Delta_1\) equipped with the Morse shelled triangulation given by Corollary 3.17 to these balls and whatever the attachment is, the Morse shelling can be extended through the handle. Namely, the shelled simplicial complex \(\Delta_1 \times \Delta_1\) equipped with the triangulation given by Proposition 3.16 is attached to the boundary of the union of disks along \(\partial \Delta_1 \times \Delta_1\) and we may attach both components of \(\partial \Delta_1 \times \Delta_1\) to the same boundary component of these disks, or not, since these have enough edges. We then get a triangulated two-manifold with boundary equipped with a Morse shelling and may again attach one after the other basic
tiles of order one along the boundary component of this manifold to increase the number of edges in its triangulation. Then we may attach a second one-handle and repeating the procedure, we may attach all one-handles to get a two-manifold with boundary equipped with a Morse shelled triangulation. Each boundary component is then a triangulated circle. If this circle has only three edges, we may directly glue the two-handle $T_3^2 = \Delta_2$ unless all these edges are the faces of a triangle in which case we glue $T_1^2 \cup T_2^2 \cup T_3^2$ to get the shelled triangulation of $\partial \Delta_3$. If this circle has more than three edges, we attach basic tiles of order two to the boundary component to decrease by one the number of edges in this triangulation. At the end it is possible to add all two-handles $\hat{\Delta}_2$ to get a triangulated manifold homeomorphic to $M$ and equipped with a Morse shelling. The result again follows from Corollary 3.15.

If $n = 3$, we may proceed in the same way to attach 0- and 1-handles. Namely, we start by attaching all 0-handles at once, that is we start with finitely many closed three-simplices and get a Morse shelled triangulated three-manifold with boundary. Given such a Morse shelled triangulated three-manifold with boundary, we may attach one after the other basic tiles of order one to its boundary components to get a new triangulation on this three-manifold with boundary with an increased number of triangles on each boundary components, see Lemma 3.25 of [14] (from the Dehn-Sommerville relations, the number of triangles of any triangulated closed surface is even). This makes it possible to attach a one-handle $\hat{\Delta}_1 \times \Delta_2$ with the triangulation given by Corollary 3.17. As before, there is no obstruction to perform any attachment in this way to get a new three-manifold with boundary equipped with a Morse shelled triangulation. We now have to be able to attach a two-handle.

Each boundary component of the three-manifold is homeomorphic to a closed surface equipped with a $(PL)$-triangulation. The two-handle has to be attached along a tubular neighborhood of a two-sided closed curve embedded in such a surface. Let $C'$ be such a closed curve embedded in a boundary component $\Sigma$ of the three-manifold. By Theorem A1 of [7] it can be assumed to be the image of a $PL$-embedding of $S^1$, deforming it by some ambient isotopy if necessary. We may perform a large number of barycentric subdivisions on the triangulation and isotope slightly $C'$ to get a new curve $C$ which does not contain any vertex of the new triangulation, is transverse to the edges of the triangulation and is such that for every triangle $T_3$, either $C$ is disjoint from $T$, or $C$ intersects $T$ along a connected piecewise linear arc joining two different edges, see Figure 4.

![Figure 4: A piecewise-linear arc joining two edges.](image)
The union of all triangles meeting $C$ is then a regular neighborhood homeomorphic to an annulus $\Delta_1 \times C$ equipped with a simple triangulation, that is such that all triangles have an edge on the boundary component of $\partial \Delta_1 \times C$ and have the opposite vertex on the other boundary component, see the end of §3.4. We may as in §3.4 denote by $d$ and $u$ these two boundary components of $\partial \Delta_1 \times C$ and by Proposition 3.18 encode this triangulation of $\Delta_1 \times C$ by a cyclic word in the alphabet $\{d, u\}$. Performing an additional barycentric subdivision and isotopy on $C$ if necessary, we may assume that this cyclic word contains each letter at least once. Then, by attaching basic tiles of order one to $\Sigma$, we may get a new triangulation and a new annulus such that the cyclic word is modified by duplicating letters, namely the ones which encode the triangles on which the basic tiles are attached, see Figure 5.

![Figure 5: Duplication of a triangle.](image)

We may thus assume that this cyclic word contains each letter at least three times. Now, by attaching a basic tile of order two to $\Sigma$ along two triangles encoded by $dd$ or $uu$ (resp. by $du$ or $ud$), we may get a new triangulation and a new annulus such that the cyclic word is modified by the compression $dd \rightarrow d$ of $uu \rightarrow u$ (resp. by the transposition $du \rightarrow ud$ or $ud \rightarrow du$), see Figure 6 and the end of §3.4.

![Figure 6: Transposition of two triangles.](image)

Performing such gluings finitely many times if necessary, we may assume by Proposition 3.19 that this cyclic word is just $w_2 = ududu$. Corollary 3.17 then provides a Morse tiled two-handle that can be attached to the boundary component of $\Sigma$ along such a neighborhood of $C$. From Corollary 2.20, we know that performing barycentric subdivisions to a Morse shellable simplicial complex, we still get a Morse shellable simplicial complex. Corollary 3.17 then ensures that we may attach the two-handle given by Corollary 3.17 to get a new three-manifold with boundary equipped with a Morse shelled triangulation. By induction, we can then perform all attachments of handles of indices 0, 1 and 2 to get a Morse shelled triangulated three-manifold with boundary whose boundary components are homeomorphic to spheres. It remains to attach the three-handles to these boundary components. Each boundary component $\Sigma$ is homeomorphic to a two-sphere equipped
with a triangulation. If this triangulation has just four vertices, we may directly glue the three-handle $\Delta_3$ unless all of these vertices are the vertices of a same three-simplex of the three-manifold, in which case we attach $T^3_1 \sqcup T^3_2 \sqcup T^3_3 \sqcup T^3_4$ to get a three-sphere with the shelled triangulation of $\partial \Delta_4$. If this triangulation has more than four vertices, we are going to prove by induction that we can modify the triangulation to reduce the number of vertices. Namely, if $\Sigma$ contains a vertex $v$ of valence greater than three, then there are two triangles on $\Sigma$ adjacent to $v$ and which are not faces of the same three-simplex. We can then attach a basic tile of order two along these two triangles to get a new Morse shelled triangulation on the three-manifold with the same vertices but the valence of $v$ has decreased by one, see Lemma 3.25 of [14]. The two triangles form a triangulated square and the triangulation has been modified by switching of diagonal in the square without changing the set of vertices, see Figure 7.

![Figure 7: Decreasing the valence of a vertex.](image)

By iterating this process, we can decrease the valence of $v$ to three and moreover by construction, the three triangles adjacent to $v$ are not faces of the same three-simplex. We can then attach a basic tile of order three to get a new triangulation on $\Sigma$ with the same vertices, but $v$, see Figure 8.

![Figure 8: Removal of a vertex.](image)

By induction, we can thus assume that all vertices on all the boundary components of the three-manifold have valence three and moreover that the three triangles adjacent to them are faces of the same three-simplex. The triangulation of the three-manifold with boundary is equipped with a Morse shelling so that every compatible discrete vector field has no non-stationary path by Corollary 3.15. Let now $v$ be a vertex of valence three on $\Sigma$ such that the three triangles adjacent to it are faces of the same three-simplex. Then, the tile covering the interior of this three-simplex has to contain the three triangles in its boundary, so that it is either a basic tile of order one or a basic tile of order zero. In the first case, we may remove the basic tile to get a new triangulation with the same vertices.
on the boundary, but $v$. It is the same modification as in Figure 8. Any discrete vector field compatible with this new Morse tiled triangulation has no non-stationary closed path since such a closed path would be a closed path of the previous one. In the second case, the boundary component of the three-manifold is just $\partial \Delta_3$ and we glue $T^3_1 \sqcup T^3_2 \sqcup T^3_3 \sqcup T^3_4$ to get a three-sphere with the shelled triangulation of $\partial \Delta_4$. The result then follows from Corollary 3.15.

Proof of Corollary 1.4. Let $f$ be a discrete Morse function compatible with $\mathcal{T}$. By Theorem 3.12, the critical points of $f$ are in one-to-one correspondence, preserving the index, with the critical tiles of the tiling, so that for every $k \in \{0, \ldots, n\}$, $f$ has $c_k(\mathcal{T})$ critical points of index $k$. Theorem 7.3 of [8] then provides a chain complex which computes the homology of $X$ and which has dimension $c_k(\mathcal{T})$ in grading $k$, it is the discrete Morse complex. The result then follows from the classical Morse inequalities deduced from this chain complex.

3.6 Final remarks

1. The critical points of the Morse functions given by Theorem 1.3 are in one-to-one correspondence with the critical tiles of the tiling, preserving the index. It would be of interest to prove Theorem 1.3 in any dimension.

2. The $h$-tiling of $\partial \Delta_2$ made of three basic tiles of order one has no critical tile. Example 4.5 of [15] also provides $h$-tiled triangulations on the two-torus having no critical tile, so that not every Morse tiling shares the property given by Theorem 1.3. In these examples, every discrete vector field compatible with the tiling has closed non-stationary path, so that by Theorem 3.8 they cannot be the gradient vector fields of some discrete Morse functions.

3. By Lemma 2.5 and the additivity of the Euler characteristic, an even dimensional closed manifold equipped with an $h$-tiled triangulation has non-negative Euler characteristic. In particular, no triangulation on a closed surface of genus greater than one is $h$-tiled. We do not know which closed three-manifold possess an $h$-tileable triangulation.

4. The existence of triangulations on smooth manifolds is well known, see for example [18], and topological closed manifolds of dimension less than four are known to have a unique smooth structure, see [1, 12].

5. Given a $PL$-triangulation on a closed manifold, one gets a decomposition of the manifold into triangulated handles, see Proposition 6.9 of [14]. However, such a decomposition is far from being optimal and moreover the triangulations on the handles are not standard, they depend on the manifold and the handle. The triangulation of Theorem 1.3 can be obtained using any handle decomposition and the triangulations given on each handle is then standard.
6. The simplicial complex of dimension two made of the three triangles depicted in Figure 9 is not Morse tileable. It would be of interest to exhibit a triangulated closed manifold which is not Morse tileable.

Figure 9: A non Morse tileable simplicial complex.

References

[1] R. H. Bing. An alternative proof that 3-manifolds can be triangulated. *Ann. of Math. (2)*, 69:37–65, 1959.

[2] A. Björner. Shellable and Cohen-Macaulay partially ordered sets. *Trans. Amer. Math. Soc.*, 260(1):159–183, 1980.

[3] F. Brenti and V. Welker. $f$-vectors of barycentric subdivisions. *Math. Z.*, 259(4):849–865, 2008.

[4] J. Cerf and A. Gramain. Le théorème du $h$-cobordisme (Smale). *École Normale Supérieure*, 1968.

[5] M. K. Chari. On discrete Morse functions and combinatorial decompositions. volume 217, pages 101–113. 2000. Formal power series and algebraic combinatorics (Vienna, 1997).

[6] E. Delucchi, A. Pixton, and L. Sabalka. Face vectors of subdivided simplicial complexes. *Discrete Math.*, 312(2):248–257, 2012.

[7] D. B. A. Epstein. Curves on 2-manifolds and isotopies. *Acta Math.*, 115:83–107, 1966.

[8] R. Forman. Morse theory for cell complexes. *Adv. Math.*, 134(1):90–145, 1998.

[9] W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.
[10] D. Kozlov. *Combinatorial algebraic topology*, volume 21 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2008.

[11] J. Milnor. *Lectures on the h-cobordism theorem*. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J., 1965.

[12] E. E. Moise. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung. *Ann. of Math. (2)*, 56:96–114, 1952.

[13] J. R. Munkres. *Elements of algebraic topology*. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[14] C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology*. Springer Study Edition. Springer-Verlag, Berlin-New York, 1982. Reprint.

[15] N. Salepci and J.-Y. Welschinger. Tilings, packings and expected Betti numbers in simplicial complexes. *arXiv:1806.05084*, 2018.

[16] N. Salepci and J.-Y. Welschinger. Asymptotic measures and links in simplicial complexes. *Discrete Comput. Geom.*, 62(1):164–179, 2019.

[17] R. P. Stanley. The upper bound conjecture and Cohen-Macaulay rings. *Studies in Appl. Math.*, 54(2):135–142, 1975.

[18] H. Whitney. *Geometric integration theory*. Princeton University Press, Princeton, N. J., 1957.

[19] G. M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.