A NOTE ON THE C-NUMERICAL RADIUS AND THE \( \lambda \)-ALUTHGE TRANSFORM IN FINITE FACTORS

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Abstract. We prove that for any two elements \( A, B \) in a factor \( \mathcal{M} \), if \( B \) commutes with all the unitary conjugates of \( A \), then either \( A \) or \( B \) is in \( CI \). Then we obtain an equivalent condition for the situation that the \( C \)-numerical radius \( \omega_C(\cdot) \) is a weakly unitarily invariant norm on finite factors and we also prove some inequalities on the \( C \)-numerical radius on finite factors. As an application, we show that for an invertible operator \( T \) in a finite factor \( \mathcal{M} \), \( f(\Delta_\lambda(T)) \) is in the weak operator closure of the set \( \{ \sum_{i=1}^{n} z_i f(T)U_i^*U_i \mid n \in \mathbb{N}, U_i \in \mathcal{M}, \sum_{i=1}^{n} |z_i| \leq 1 \} \), where \( f \) is a polynomial, \( \Delta_\lambda(T) \) is the \( \lambda \)-Aluthge transform of \( T \) and \( 0 \leq \lambda \leq 1 \).

1. NOTATION AND INTRODUCTION

Denote by \( B(\mathcal{H}) \) the set of bounded linear operators on a Hilbert space \( \mathcal{H} \) and \( M_n(\mathbb{C}) \) the self-adjoint algebra of the \( n \times n \) matrices. A von Neumann algebra \( \mathcal{M} \) on \( \mathcal{H} \) is a unital weak operator closed \( * \)-algebra. A von Neumann algebra \( \mathcal{M} \) is said to be a factor if \( \mathcal{M} \cap \mathcal{M}' = CI \), where \( I \) is the identity of \( \mathcal{M} \). A von Neumann algebra \( \mathcal{M} \) is finite if it has a faithful normal tracial state. If \( \mathcal{M} \) is a finite factor with a faithful normal trace \( \tau \), denote by \( \| \cdot \|_1 \) the norm on \( \mathcal{M} \) to be \( \tau(\cdot | \cdot) \). Then denote by \( L^1(\mathcal{M}, \tau) \) the completion of \( \mathcal{M} \) with respect to \( \| \cdot \|_1 \) norm. Also to each normal linear functional \( f \) on \( \mathcal{M} \) corresponds a unique element \( X \in L^1(\mathcal{M}, \tau) \) such that \( f(\cdot) = \tau(X \cdot) \). Denote by \( \mathcal{U}(\mathcal{M}) \) the set of all the unitary operators in a von Neumann algebra \( \mathcal{M} \).

Let \( \text{tr} \) be the normalized trace of \( M_n(\mathbb{C}) \). Given a matrix \( C \in M_n(\mathbb{C}) \) and set

\[
\omega_C(A) = \max_{U \in \mathcal{U}(M_n(\mathbb{C}))} |\text{tr}(CUA^*)|.
\]

Then \( \omega_C(A) \) is called the \( C \)-numerical radius of \( A \). We say a norm \( \| \cdot \| \) on \( M_n(\mathbb{C}) \) weakly unitarily invariant if \( \| A \| = \| UAU^* \| \) for all \( A \in M_n(\mathbb{C}), U \in \mathcal{U}(M_n(\mathbb{C})) \).

Note that for every \( C \in M_n(\mathbb{C}) \), the \( C \)-numerical radius \( \omega_C \) is a weakly unitarily invariant seminorm on \( M_n(\mathbb{C}) \). It is a norm on \( M_n(\mathbb{C}) \) if and only if \( C \) is not a scalar and has nonzero trace. The family \( \omega_C \) of \( C \)-numerical radius, where \( C \) is not a scalar and has nonzero trace, plays a role analogous to that of Ky Fan norms in the family of unitarily invariant norm [3] Theorem IV.4.7. A norm \( \| \cdot \| \) on \( M_n(\mathbb{C}) \) is called a unitarily invariant norm if \( \| A \| = \| UAU^* \| \) for all \( A \in M_n(\mathbb{C}), U, V \in \mathcal{U}(M_n(\mathbb{C})) \). The concept of unitarily invariant norms was introduced by von Neumann [17] for the purpose of metrizing matrix spaces. Von Neumann and his associates established that the class of unitarily invariant norms

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of \( n \times n \) complex matrices coincides with the class of symmetric gauge function of their \( s \)-numbers. These norms have now been variously generalized and utilized in many contexts. For historical perspectives and surveys, we refer the reader to \((3, 6, 8, 11, 13, 15)\) and etc.

Let \( T \in B(H) \) and let \( T = U|T| \) be its polar decomposition. The Aluthge transform of \( T \) is the operator \( \Delta(T) = |T|^2 U|T|^2 \). This was first studied in \(11\) and has received much attention in recent years. One reason the Aluthge transform is interesting is in relation to the invariant subspace problem. Jung, Ko and Percy in \(10\) proved that other spectral data are also preserved by \( \Delta \)-Aluthge transform. Dykema and Schultz in \(12\) proved the Brown measures are unchanged by the Aluthge transform. Another reason is related with iterated Aluthge transform. Let \( \Delta^n(T) = \Delta(\Delta^{n-1}(T)) \) for every \( n \in \mathbb{N} \). It was conjectured in \(10\) that the sequence \( \{\Delta^n(T)\}_{n \in \mathbb{N}} \) converges in the norm topology. For more surveys, we refer the reader to \((1, 2, 5, 10, 12, 13)\) and etc.

The \( \lambda \)-Aluthge transform of \( T \) is defined in \(12\) by \( \Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda} \), \( 0 \leq \lambda \leq 1 \). In particular, for \( \lambda = \frac{1}{2} \), \( \Delta_{\frac{1}{2}}(T) \) is just the Aluthge transform \( \Delta(T) \). Okubo in \(12\) proved that for an invertible operator \( T \in B(H) \), \( \|f(\Delta_\lambda(T))\| \leq \|f(T)\| \) for any polynomial \( f \) and \( \| \cdot \| \) a weakly unitarily invariant norm. Fore more results on the \( \lambda \)-Aluthge transform, we refer the reader to \((1, 12, 13)\) and etc.

This paper is organized as follows.

The key motivation for studying the C-numerical radius \( \omega_C \) on finite factors stems from the fact that for the finite dimensional case, i.e., \( M_n(\mathbb{C}) \), it has a relation with weakly unitarily invariant norms on \( M_n(\mathbb{C}) \). So in section 2, we use some knowledge on dual norms to show that relation.

In section 3, We first prove that if \( \mathcal{M} \) is a factor, then for any non-trivial projection \( P \) in \( \mathcal{M} \), all the unitary conjugates of \( P \) generate the whole von Neumann algebra \( \mathcal{M} \) (see Lemma \(3.4\)). Then using this lemma we prove a technical result in this paper.

**Theorem 1.1** (see Theorem \(3.2\)). Let \( \mathcal{M} \) be a factor and \( A, B \in \mathcal{M} \). If \( UAU^*B = BUU^*A \) holds for every \( U \in \mathcal{U}(\mathcal{M}) \), then either \( A \) or \( B \) is in \( \mathcal{C}I \).

We define the C-numerical radius on finite factors.

**Definition 1.2.** Let \( \mathcal{M} \) be a finite factor with a faithful normal trace \( \tau \) and for \( A, C \in \mathcal{M} \), the C-numerical radius of \( A \) is defined as

\[
\omega_C(A) = \sup_{U \in \mathcal{U}(\mathcal{M})} |\tau(CUAU^*)|.
\]

Observe that the C-numerical radius of \( A \) is a weakly unitarily invariant seminorm on \( \mathcal{M} \).

In section 4, as one application of Theorem 1.1, we prove the following corollary.

**Corollary 1.3** (see Corollary \(3.1\)). Let \( \mathcal{M} \) be a finite factor with a faithful normal trace \( \tau \). The C-numerical radius \( \omega_C \) is a norm on \( \mathcal{M} \) if and only if...
(1) \( C \) is not a scalar multiple of \( I \) and;
(2) \( \tau(C) \neq 0 \).

We also prove some inequalities for the \( C \)-numerical radius \( \omega_C \) on finite factors (see Theorem 4.2).

In section 5, we discuss some properties of the \( \lambda \)-Aluthge transform of an invertible operator in a finite factor. Using three line theorem and some results in section 4, we obtain the following result.

**Proposition 1.4** (see Proposition 5.3). Let \( M \) be a finite factor with a faithful normal trace \( \tau \). Assume \( T \in M \) is an invertible operator with polar decomposition \( T = U|T| \) and \( f \) is a polynomial, then for \( 0 \leq \lambda \leq 1 \), \( f(|T|^\lambda U|T|^{1-\lambda}) \) is in the weak operator closure of the set \( \{ \sum_{i=1}^n z_i U_i f(T) U_i^* | n \in \mathbb{N}, (U_i)_{1 \leq i \leq n} \in \mathcal{U}(M), \sum_{i=1}^n |z_i| \leq 1 \} \).

In this paper, we assume all the factors have separable predual.

### 2. Relation between weakly unitarily invariant norms and the \( C \)-numerical radius \( \omega_C \) on \( M_n(\mathbb{C}) \)

In this section, a finite von Neumann algebra \((M, \tau)\) means a finite von Neumann algebra \( M \) with a faithful normal tracial state \( \tau \). Recall the definition and some properties of dual norms in [7].

Let \( ||| \cdot ||| \) be a norm on a finite von Neumann algebra \((M, \tau)\). For \( T \in M \), define
\[
|||T|||_M^\sharp = \sup \{ |\tau(TX)| : X \in M, |||X||| \leq 1 \}.
\]
When no confusion arises, we write \( ||| \cdot ||| \) instead of \( ||| \cdot |||_M^\sharp \).

**Lemma 2.1** ([7]). \( ||| \cdot ||| \) is a norm on \((M, \tau)\).

**Definition 2.2** ([7]). \( ||| \cdot ||| \) is called the dual norm of \( ||| \cdot ||| \) on \( M \) with respect to \( \tau \).

**Definition 2.3.** A norm \( ||| \cdot ||| \) on \((M, \tau)\) is weakly unitarily invariant if \( |||UTU^*||| = |||T||| \) for all \( T \in M \) and \( U \in \mathcal{U}(M) \).

**Lemma 2.4** ([7]). If \( ||| \cdot ||| \) is a norm on \((M_n(\mathbb{C}), tr)\) and \( ||| \cdot ||| \) is the dual norm of \( ||| \cdot ||| \) on \( M \) with respect to \( tr \), then \( ||| \cdot ||| = ||| \cdot |||\). \( ||| \cdot ||| \) is also a weakly unitarily invariant norm on \((M, \tau)\).

**Lemma 2.5.** If \( ||| \cdot ||| \) is a weakly unitarily invariant norm on a finite von Neumann algebra \((M, \tau)\), then \( ||| \cdot ||| \) is also a weakly unitarily invariant norm on \((M_n(\mathbb{C}), tr)\).

**Proof.** Let \( U \in \mathcal{U}(M) \). Then \( |||UTU^*||| = \sup \{ |\tau(UTU^*X)| : X \in M, |||X||| \leq 1 \} = \sup \{ |\tau(UTU^*X)| : X \in M, |||U^*XU||| \leq 1 \} = |||T|||^2 \).

We now proceed to the relation between weakly unitarily invariant norms and the \( C \)-numerical radius on \((M_n(\mathbb{C}), tr)\).

**Proposition 2.6.** If \( ||| \cdot ||| \) is a weakly unitarily invariant norm on \((M_n(\mathbb{C}), tr)\), then \( |||T||| = \sup_{|||X||| \leq 1} \omega_X(T) \).
Proof. For $T \in (M_n(\mathbb{C}), tr)$, by Lemma 2.6, Lemma 2.3 and the definition of dual norm, we have
\[ |||T||| = |||T|||_{\mathbb{F}} = \sup_{U \in \mathcal{U}(\mathcal{M})} \|T(U)\|_{\mathbb{F}} \]
\[ = \sup_{U \in \mathcal{U}(\mathcal{M})} \sup_{||X||_{1} \leq 1} \{ |\tau(TUXU^*)|, X \in M_n(\mathbb{C}) \} \]
\[ = \sup_{||X||_{1} \leq 1} \sup_{U \in \mathcal{U}(\mathcal{M})} \{ |\tau(TUXU^*)|, X \in M_n(\mathbb{C}) \} \]
\[ = \sup_{||X||_{1} \leq 1} \omega_X(T). \]

Note that when proving Proposition 2.6, we use Lemma 2.3 [7, Lemma 6.18], so we may ask whether this result can be generalized to finite factors.

3. A result on factors

In this section, we show a technical result (Theorem 3.2), which is the most difficult part of this paper. To prove that result, we first need the following lemma.

Lemma 3.1. Let $\mathcal{M}$ be a factor and $P$ be a non-trivial projection in $\mathcal{M}$. Then the von Neumann algebra generated by $\{ UPU^* : U \in \mathcal{U}(\mathcal{M}) \}$ is $\mathcal{M}$.

Proof. We divide the proof into four cases according to the type of $\mathcal{M}$.

(i) For the case $\mathcal{M} = B(H)$, where $\dim(H) \leq 1$.

Take two projections $P_1 \leq P$ and $P_1 \leq 1 - P$ with $\dim(P_i(H)) = 1$ for $i = 0, 1$ and write $Q = P - P_0 + P_1$, then $P_0 = P(1 - Q)$ and we can find some unitary operator $V \in \mathcal{U}(\mathcal{M})$ such that $VPV^* = Q$, since $P$ and $Q$ are equivalent. Then we have $\{ UPU^* : U \in \mathcal{U}(\mathcal{M}) \} \subseteq \{ UPU^* : U \in \mathcal{U}(\mathcal{M}) \}$. Note that the von Neumann algebra generated by $\{ UPU^* : U \in \mathcal{U}(\mathcal{M}) \}$ is $\mathcal{M}$. Hence we prove our result.

(ii) For the case $\mathcal{M}$ is a II$_1$ factor with a faithful normal tracial state $\tau$.

Write $\tau(P) = \lambda \in (0, 1)$ and we may assume $\lambda \leq \frac{1}{2}$. Then for any $0 < t \leq \lambda$, we can find two projections $P_t \leq P$ and $F_t \leq 1 - P$ with $\tau(P_t) = \tau(F_t) = t$. Write $Q_t = P - P_t + F_t$, then $P_t = P(1 - Q_t)$. Again we can find some unitary operator $V \in \mathcal{U}(\mathcal{M})$ such that $VPV^* = Q_t$. Hence $\{ UPU^* : \tau(P_t) = t \in (0, \lambda], P_t \leq P, U \in \mathcal{U}(\mathcal{M}) \} \subseteq \{ UPU^* : U \in \mathcal{U}(\mathcal{M}) \}$. Note that the von Neumann algebra generated by $\{ UPU^* : \tau(P_t) = t \in (0, \lambda], P_t \leq P, U \in \mathcal{U}(\mathcal{M}) \}$ is the whole $\mathcal{M}$. Then we have our result.

(iii) For the case $\mathcal{M}$ is a II$_\infty$ factor with a faithful normal tracial weight $Tr$.

Write $Tr(P) = \lambda \in (0, \infty]$ and we may assume $Tr(1 - P) \geq Tr(P)$. Then using the same trick in case (ii), we prove our result.

(iv) For the case $\mathcal{M}$ is a type III factor.

This case is trivial, since all the non-trivial projections in a type III factor are equivalent. 

Our main theorem is the following.

Theorem 3.2. Let $\mathcal{M}$ be a factor and $A, B \in \mathcal{M}$. If $UAU^* B = BUAU^*$ holds for any $U \in \mathcal{U}(\mathcal{M})$, then either $A$ or $B$ is in $\mathbb{C}I$. 

Proof. Let $P$ be a projection in $\mathcal{M}$, then we can write $A$ and $B$ in the matrix form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where $A_{11}, B_{11} \in PMP$, $A_{12}, B_{12} \in PMP^\perp$, $A_{21}, B_{21} \in P^\perp MP$, $A_{22}, B_{22} \in P^\perp MP^\perp$.

Let $\theta \in [0, 2\pi]$, $U = \begin{pmatrix} e^{i\theta} P_n & 0 \\ 0 & P_n^\perp \end{pmatrix}$, it is clear that $U$ is a unitary operator. Then we have $UAU^* = \begin{pmatrix} A_{11} & e^{i\theta} A_{12} \\ e^{-i\theta} A_{21} & A_{22} \end{pmatrix}$.

$$UAU^* B = \begin{pmatrix} A_{11} & e^{i\theta} A_{12} \\ e^{-i\theta} A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11} B_{11} + e^{i\theta} A_{12} B_{21} \\ * \\ e^{-i\theta} A_{21} B_{22} + A_{22} \end{pmatrix}.$$

and

$$B A U^* = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} A_{11} & e^{i\theta} A_{12} \\ e^{-i\theta} A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} B_{11} A_{11} + e^{-i\theta} B_{12} A_{21} \\ * \\ e^{i\theta} B_{21} A_{22} + A_{22} \end{pmatrix}.$$

It follows that

\[(3.1) \quad A_{11} B_{11} - B_{11} A_{11} + e^{i\theta} A_{12} B_{21} - e^{-i\theta} B_{12} A_{21} = 0 \]

since $UAU^* B = B A U^*$. Note that (3.1) holds for any $\theta \in [0, 2\pi]$, a not difficult calculation implies

\[(3.2) \quad A_{11} B_{11} = B_{11} A_{11}, A_{12} B_{21} = B_{12} A_{21} = 0. \]

Observe that for any $U, V \in \mathcal{U}(\mathcal{M})$, $U V A V^* U^* B = B U V A V^* U^*$ still holds, in particular, we can choose $V = \begin{pmatrix} V_1 & 0 \\ 0 & P_n^\perp \end{pmatrix}$, where $V_1 \in \mathcal{U}(PMP)$, then

\[(3.3) \quad V_1 A_{11} V_1^* B_{11} = B_{11} V_1 A_{11} V_1^*. \]

(i) For the case $\mathcal{M} = B(\mathcal{H})$, where $\dim(\mathcal{H}) = \infty$.

For $n \in \mathbb{N}$, let $P_n$ be a projection of dimension $n$ and $P_n \leq P_{n+1}$.

By a result of finite dimension case, i.e., if $A, B \in M_n(\mathbb{C})$ and $UAU^* B = B A U^*$ holds for any $U \in \mathcal{U}(M_n(\mathbb{C}))$, then either $A$ or $B$ is in $CI_n$, where $I_n$ is the identity of $M_n(\mathbb{C})$(cf. proof of [3 Proposition IV.4.4]). Then by (3.3), we have either $A_{11}$ or $B_{11}$ is in $CI_n$, i.e., $P_n A P_n$ or $P_n B P_n$ is in $CI_n$, for any $n \in \mathbb{N}$. Assume $P_n A P_n$ is in $CI_n$, while $P_n B P_n$ not. For $m > n$, if $P_m A P_m$ isn’t in $CI_m$, while $P_m B P_m$ is in $CI_m$, that would contradict the assumption $P_n B P_n$ isn’t in $CI_n$. Hence we have for all $n \in \mathbb{N}$, $P_n A P_n$ is in $CI_n$, which implies $A$ is in $CI$.

(ii) For the case $\mathcal{M}$ is a II$_1$ factor with trace $\tau$ or a type III factor.

If $\mathcal{M}$ is a II$_1$ factor, then assume $\tau(P) = \frac{1}{2}$. Otherwise if $\mathcal{M}$ is a type III factor, then assume $P \neq 0$ or $P \neq 1$. Then we have $\mathcal{M} \cong M_2(\mathbb{C}) \otimes PMP$ and we can write $A, B$ in the matrix form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad A_{ij}, B_{ij} \in PMP, \text{ for } 1 \leq i, j \leq 2.$$
operator $V_2 \in \mathcal{U}(PMP)$, which implies $B_{21} = 0$. Moreover, put $V' = \begin{pmatrix} 0 & V_1 \\ V_2 & 0 \end{pmatrix}$, then

$$V'AV'^* = \begin{pmatrix} V_1A_{22}V_1^* & V_1A_{21}V_2^* \\ V_2A_{12}V_1^* & V_2A_{11}V_2^* \end{pmatrix}.$$ 

Using the same trick as above, we obtain that if $A_{12} \neq 0$, then $B_{12} = 0$. Thus we have if $A_{12} \neq 0$, then $B_{21} = B_{12} = 0$. Similarly, we would have if $A_{21} \neq 0$, then $B_{21} = B_{12} = 0$.

Observe that if we replace $A$ with $UAU^*$ for every $U \in \mathcal{U}(M)$ and replace $B$ with $VBV^*$ for every $V \in \mathcal{U}(M)$, then the above fact still holds.

Then we can argue as follows.

Assume that $A \notin CI$, we try to show $B \in CI$.

Case 1: If there exists $U \in \mathcal{U}(M)$ such that $(UAU^*)_{12}$ or $(UAU^*)_{21}$ is non-zero, then from above, we know that $(VBV^*)_{12} = (VBV^*)_{21} = 0$ for every $V \in \mathcal{U}(M)$. Hence $VBV^*P = PVBV^*$ for every $V \in \mathcal{U}(M)$. Then apply Lemma 3.1 to get $B \in CI$.

Case 2: If for every $U \in \mathcal{U}(M)$, $(UAU^*)_{12} = (UAU^*)_{21} = 0$. Then $UAU^*P = PUAU^*$ for every $U \in \mathcal{U}(M)$. Again using Lemma 3.1 we have $A \in CI$, which is a contradiction. Hence this case actually does not appear under the assumption that $A \notin CI$.

(iii) For the case $M$ is a $II_\infty$ factor.

Note that $M = B(H) \otimes N$, where $N$ is a $II_1$ factor. For any $n \in \mathbb{N}$, let $P'_n$ be a projection of dimension $n$ in $B(H)$, $I'$ be the identity of $N$ and $P_n = P'_n \otimes I'$, then $P_nMP_n$ is a type $II_1$ factor. Hence using the same trick in case (i) and the result in case (ii), our result follows.

\section{The C-numerical radius $\omega_C$ on finite factors}

In this section, we show some applications of Theorem 3.2 and discuss some properties of the C-numerical radius $\omega_C$ on finite factors.

We use Theorem 3.2 and the same technique in [3, Proposition IV.4.4], to prove our next corollary, for reader’s convenience, we write the proof below.

**Corollary 4.1.** Let $M$ be a finite factor with trace $\tau$. The C-numerical radius $\omega_C$ is a weakly unitarily invariant norm on $M$ if and only if

1. $C$ is not a scalar multiple of $I$ and;
2. $\tau(C) \neq 0$.

**Proof.** If $C = \lambda I$ for any $\lambda \in \mathbb{C}$, then $\omega_C(A) = ||\lambda||\tau(A)$, and this is zero if $\tau(A) = 0$, which means $\omega_C$ can’t be a norm on $M$. If $\tau(C) = 0$, then $\omega_C(I) = 0$. Again $\omega_C$ is not a norm.

Conversely, suppose $\omega_C$ is not a norm on $M$ and $\omega_C(A) = 0$. If $A = \lambda I$ for any $\lambda \in \mathbb{C}$, this would mean that $\tau(C) = 0$. So, if $\tau(C) \neq 0$, then $A \notin CI$. We claim that $C \in CI$. Since $e^{itK}$ is in $\mathcal{U}(M)$ for all $t \in \mathbb{R}$ and $K = K^* \in M$, the condition $\omega_C(A) = 0$ implies in particular that $\tau(Ce^{itK}Ae^{-itK}) = 0$ if $t \in \mathbb{R}$ and $K = K^* \in M$. Differentiating this relation at $t = 0$, one gets $\tau((AC - CA)K) = 0$ for all $K = K^* \in M$. Hence we obtain that $\tau((AC - CA)T) = 0$ for all $T \in M$. Hence $\tau(CA) = 0$. Note that $\omega_C(A) = \omega_C(UAU^*)$ for all $U \in \mathcal{U}(M)$, so that $UAU^*C = CUAU^*$ for all $U \in \mathcal{U}(M)$. Hence the result $C$ is in $CI$ follows from Theorem 3.2. \qed
Observe that for $A, C \in M$, by the definition of the $C$-numerical radius $\omega_C$, we have $\omega_C(A) = \omega_A(C)$ and $\omega_C(\cdot)$ is normal on $M$.

**Theorem 4.2.** Let $M$ be a finite factor with a faithful normal trace $\tau$. For $A, B \in M$, the following conditions are equivalent.

1. $\omega_C(A) \leq \omega_C(B)$ for all operators $C \in M$ that are not scalars and have nonzero trace;
2. $\omega_C(A) \leq \omega_C(B)$ for all operators $C \in M$;
3. Let $K = \{\sum_{i=1}^{n} z_i U_i B U_i^{*} | n \in \mathbb{N}, (U_i)_{1 \leq i \leq n} \in \mathcal{U}(M), \sum_{i=1}^{n} |z_i| \leq 1\}$ and $\Gamma$ be the weak operator closure of $K$. Then $A \in \Gamma$.

**Proof.** (1) $\Rightarrow$ (2). Assume $C \in M$ and $\tau(C) = 0$. Put $C_n = C + \frac{1}{n}$, then $\tau(C_n) = \frac{1}{n}$ and $\|C_n - C\| \to 0$. Moreover, we have

$$|\omega_A(C_n) - \omega_A(C)| \leq \sup_{U \in \mathcal{U}(M)} |\tau(U(C_n - C)U^{*})|$$

$$= \sup_{U \in \mathcal{U}(M)} \frac{1}{n}|\tau(A)|$$

$$\to 0.$$ 

Similarly, we would have $\omega_B(C_n) \to \omega_B(C)$. Note that $\omega_A(C_n) \leq \omega_B(C_n)$, then we have $\omega_A(C) \leq \omega_B(C)$.

Let $P \in M$ be a projection with trace not equal to 0 or 1. Let $C_n = P + (1 - \frac{1}{n})(1 - P)$, then $C_n$ is not a scalar, $\tau(C_n) \neq 0$ and $\|C_n - 1\| \to 0$. Hence we have $\omega_A(C_n) \leq \omega_B(C_n)$ and for any operator $T \in M$,

$$|\omega_T(C_n) - \omega_T(I)| \leq |\omega_T(C_n - I)|$$

$$= \sup_{U \in \mathcal{U}(M)} |\tau(T(U(C_n - I)U^{*})|$$

$$\leq \|C_n - 1\||T||_1$$

$$\to 0.$$ 

It follows that $\omega_A(I) \leq \omega_B(I)$.

(2) $\Rightarrow$ (3). Assume $A \notin \Gamma$, then there exists a linear normal functional $f$ on $M$ and $a > b$, such that $\text{Re } f(A) \geq a > b \geq \text{Re } f(D), \forall D \in \Gamma$. Since $f$ is a normal linear functional on $M$, there exists a $C \in L^1(M, \tau)$ such that $f(T) = \tau(CT)$ for all $T \in M$.

Observe that $\omega_C(A) = \sup_{U \in \mathcal{U}(M)} |\tau(C U A U^{*})| \geq |\tau(CA)| = |f(A)|$ and

$$\text{Re } f(A) > \sup_{D \in \Gamma} \text{Re } f(D) \geq \sup_{\theta, U} \text{Re } f(e^{i\theta} U B U^{*}) = \sup_{U \in \mathcal{U}(M)} |f(U B U^{*})| = \omega_C(B).$$

Let $C = V |C|$ be the polar decomposition of $C$ in $L^1(M, \tau)$ and $H_n = \chi_{[0,n]}(|C|)|C|$, then $\|H_n - |C||_1 \to 0$. Put $C_n = V H_n$. Then we have

$$|\omega_{C_n}(A) - \omega_C(A)| \leq |\omega_A(C_n) - \omega_A(C)|$$

$$\leq \sup_{U \in \mathcal{U}(M)} |\tau((C_n - C) U A U^{*})|$$

$$\leq \|C_n - C||_1 \|A\|$$

$$\to 0.$$
Similarly, $|ω_{C_m}(B) − ω_C(B)| → 0$. Hence there exists $m ∈ N$ such that $ω_{C_m}(A) > ω_{C_m}(B)$, which contradicts to (3) since $C_m ∈ M$.

(3) ⇒ (1).

For all operators $C ∈ M$ that are not scalars and have nonzero trace, by Corollary 4.1, we obtain that $ω_C$ is a norm, hence $ω_C(T) ≤ ω_C(B)$ for all $T ∈ K$. Hence our result follows since $ω_C$ is normal.

Remark 4.3. If $||| · |||$ is a weakly unitarily invariant norm on $(M_n(ℂ), tr)$. By Theorem 12 and Proposition 5.6, we have [3] Theorem IV.4.7.

5. λ-Aluthge transform of an invertible operator in a finite factor

Let $T ∈ B(ℋ)$ and let $T = U|T|$ be its polar decomposition. The Aluthge transform of $T$ is the operator $△(T) = |T|^½ U|T|^½$. The λ-Aluthge transform of $T$ is defined by $△_λ(T) = |T|^λ U|T|^{1−λ}$, $0 ≤ λ ≤ 1$.

In this section, we show some results on the λ-Aluthge transform of an invertible operator in a finite factor.

For the infinite factor $B(ℋ)$, Okubo in [12] proved that if $T ∈ B(ℋ)$ is an invertible operator, then for any polynomial $f$, $0 ≤ λ ≤ 1$ and $||| · |||$ a weakly unitarily invariant norm, we have $|||f(△_λ(T))||| ≤ |||f(T)|||$. Note that the $C$-numerical radius is a weakly unitarily invariant seminorm on a finite factor $M$ and we have already given an equivalent condition for the situation that when this seminorm is a norm in section 4.

The idea of proving the following theorem comes from [12].

Theorem 5.1. Let $M$ be a finite factor with a faithful normal trace $τ$, $T ∈ M$ be an invertible operator with polar decomposition $T = U|T|$ and $B ∈ M$ commute with $T$. Let $ω_C(·)$ be the $C$-numerical radius on $M$. Then

$ω_C( |T|^λ BU|T|^{1−λ}) ≤ ω_C(BT)$, for $0 ≤ λ ≤ 1$.

Proof. On the strip $\{ z : −\frac{1}{2} ≤ Re(z) ≤ \frac{1}{2} \}$, consider the operator-valued function $φ(z)$ defined by

$φ(z) = |T|^\frac{1}{2} \tau^z BU|T|^{\frac{1}{2} + z}$.

It is clear that $φ(z)$ is analytic in the interior of the strip.

For any $U ∈ ℋ(M)$, define $f_U(z) = τ(CUφ(z)U^*)$. Then $f_U(z)$ is uniformly bounded on the strip and analytic since $τ$ is linear and $φ(z)$ is analytic. Applying three line theorem (see [3] pp. 136-137)) to $f_U(z)$ we would obtain that the function

$x → Log sup_{y ∈ ℝ} |f_U(x + iy)|$ is a convex function on $[-\frac{1}{2}, \frac{1}{2}]$.

Put $F_U(x) = Log sup_{y ∈ ℝ} |f_U(x + iy)|$, then for $−\frac{1}{2} ≤ x ≤ \frac{1}{2}$,

$F_U(x) ≤ F_U(−\frac{1}{2})(x + \frac{1}{2}) + F_U(\frac{1}{2})(\frac{1}{2} − x)$,

so that

$\sup_{U ∈ ℋ(M)} F_U(x) ≤ \sup_{U ∈ ℋ(M)} F_U(−\frac{1}{2})(x + \frac{1}{2}) + \sup_{U ∈ ℋ(M)} F_U(\frac{1}{2})(\frac{1}{2} − x)$.
For $-\infty < y < \infty$, since $|T|^{\frac{1}{2}+iy}$ is a unitary operator and $\phi(\frac{1}{2}+iy) = |T|^{-iy}BU|T||T|^{iy}$ and $\omega_C(\cdot)$ is a weakly unitarily invariant seminorm on $M$, we have $\omega_C(\phi(\frac{1}{2}+iy)) = \omega_C(BU|T|)$. Note that

$$\phi(-\frac{1}{2}+iy) = |T|^{-iy}|T|BU|T|^{iy} = |T|^{-iy}U^*|T|BU|T|^{iy},$$

by using the commutativity of $T$ and $B$, we have $\omega_C(\phi(-\frac{1}{2}+iy)) = \omega_C(BU|T|)$.

Note that

$$\sup_{U \in \mathcal{U}(M)} F_U(-\frac{1}{2}) = \sup_{U \in \mathcal{U}(M)} \log \sup_{y \in \mathbb{R}} |f_U(-\frac{1}{2}+iy)|$$

$$= \log \sup_{y \in \mathbb{R}} \sup_{U \in \mathcal{U}(M)} |f_U(-\frac{1}{2}+iy)|$$

$$= \log \sup_{y \in \mathbb{R}} |\tau(CU\phi(-\frac{1}{2}+iy)U^*)|$$

$$= \log \omega_C(\phi(-\frac{1}{2}+iy))$$

$$= \log \omega_C(BU|T|).$$

Similarly,

$$\sup_{U \in \mathcal{U}(M)} F_U(\frac{1}{2}) = \sup_{U \in \mathcal{U}(M)} \log \sup_{y \in \mathbb{R}} |f_U(\frac{1}{2}+iy)|$$

$$= \log \sup_{y \in \mathbb{R}} \sup_{U \in \mathcal{U}(M)} |f_U(\frac{1}{2}+iy)|$$

$$= \log \sup_{y \in \mathbb{R}} |\tau(CU\phi(\frac{1}{2}+iy)U^*)|$$

$$= \log \omega_C(\phi(\frac{1}{2}+iy))$$

$$= \log \omega_C(BU|T|).$$

Then inequality (5.2) implies that for $-\frac{1}{2} \leq x \leq \frac{1}{2}$,

$$\sup_{U \in \mathcal{U}(M)} F_U(x) = \sup_{U \in \mathcal{U}(M)} \log \sup_{y \in \mathbb{R}} |f_U(x+iy)|$$

$$= \log \sup_{y \in \mathbb{R}} \omega_C(\phi(x+iy))$$

$$\leq \log \omega_C(BT),$$

which means that $\omega_C(\phi(x+iy)) \leq \omega_C(BT)$, $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $-\infty < y < \infty$, hence

$$\omega_C(|T|^{\lambda}BU|T|^{1-\lambda}) \leq \omega_C(BT),$$

for $0 \leq \lambda \leq 1$.

□

The proof of the following proposition is exactly the same as [12, Proposition 4], so we state it as follows without a proof.

**Proposition 5.2.** Let $\mathcal{M}$ be a finite factor with a faithful normal trace $\tau$, $T \in \mathcal{M}$ be an invertible operator with polar decomposition $T = U|T|$. Let $\omega_C(\cdot)$ be the
$C$-numerical radius on $\mathcal{M}$ and $f(x)$ be a polynomial. Then
\[
\omega_C(f(|T|^\lambda U|T|^{1-\lambda})) \leq \omega_C(f(T)), \text{ for } 0 \leq \lambda \leq 1.
\]
Applying Theorem 4.2 and Proposition 5.2, we can obtain that

**Proposition 5.3.** Let $\mathcal{M}$ be a finite factor with a faithful normal trace $\tau$. Assume $T \in \mathcal{M}$ is an invertible operator with polar decomposition $T = U|T|$ and $f$ is a polynomial, then for $0 \leq \lambda \leq 1$, $f(|T|^\lambda U|T|)|T|^{1-\lambda}$ is in the weak operator closure of the set $\{\sum_{i=1}^n z_i U_i f(T) U_i^* | n \in \mathbb{N}, (U_i)_{1 \leq i \leq n} \in \mathcal{W} (\mathcal{M}), \sum_{i=1}^n |z_i| \leq 1\}$.

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