UNIQUE FAMILY OF CONSISTENT HISTORIES IN THE
CALDEIRA-LEGGETT MODEL *

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Based on the standard statistical interpretation of mixed quantum states, a unique family of consistent histories has been constructed for the quantum Brownian motion in the Caldeira-Leggett reservoir. Analytic solutions have been shown in the Markovian regime: they are uniquely defined coherent wave packets travelling near classical trajectories.

I. INTRODUCTION

Since the famous work of von Neumann [1], the statistical operator $\rho$ has been used to represent the general quantum state of a given quantum system. The statistical interpretation of $\rho$ is given in frames of the quantum measurement theory.

For the recent years, serious efforts have been made to derive statistical interpretation from the (slightly modified) quantum dynamics itself [2]. To achieve a similar goal, other authors have proposed a certain history-formulation [3] of the quantum mechanics instead of the ordinary one. We do not intend to discuss any of the preceding proposals. Rather we are going to propose a unique and exact history-interpretation for a particular quantum system. Although we use the terminology of works [3], we would express our results in the conservative (standard) language of the quantum mechanics as well (cf. Ref. [4]). Lessons learned from the works [2] are essential even if not made explicit in the present paper.

If one ignores the measurement theory, it is still possible to infer a certain statistical content from a general state $\rho$. We can always decompose $\rho$ into the weighted sum of pure

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state statistical operators:
\[ \rho = \sum_{\alpha} w_{\alpha} \psi_{\alpha} \psi_{\alpha}^\dagger. \] (1.1)

Accordingly, we can interpret the given (mixed) quantum state \( \rho \) as follows: the state of the system is just a pure state singled out at random from the set \( \{ \psi_{\alpha} \} \), with probabilities \( \{ w_{\alpha} \} \), respectively. Hence, for mixed quantum states a genuine statistical interpretation is possible, even without referring to the concept of quantum measurement.

The pure states \( \psi_{\alpha} \) may be called consistent states because the above statistical interpretation is fully consistent with what is expected of a usual statistical ensemble, cf.Refs.[3].

For the consistent states \( \{ \psi_{\alpha} \} \) we can introduce the notion of decoherence [3]. Instead of strict decoherence, usually we find a weaker one, e.g. an asymptotic decoherence:
\[ \psi_{\alpha_1}^\dagger \psi_{\alpha_2} \to 0 \] (1.2a)
if labels \( \alpha_1 \) and \( \alpha_2 \) become "very" different. To be precise, assume the existence of Euclidean norm on the space of labels. Thus the condition for the limes (1.2a) reads:
\[ \| \alpha_1 - \alpha_2 \| \to \infty. \] (1.2b)

We wish to emphasize that, at least in our work, decoherence is not a logical necessity to assign consistent probabilities to the terms of the decomposition (1.1) of mixed states. No doubt, asymptotic decoherence (1.2ab) shows up as a characteristic feature of our consistent states, as will be seen in Par.III.

The decomposition (1.1) is trivial and unique if the state \( \rho \) is already pure. Pure states has no classical statistical content independent of further assumptions like, e.g., performing measurements on the pure state. In general, a pure state turns to be mixed provided we ignore variables belonging to a certain factor space of the system's Hilbert space. Technically, one has to take a trace of the original statistical operator \( \rho \) over the factor space of the ignored variables. In recent works this has been termed as coarse graining [3]. Coarse graining can thus produce mixed quantum states which, in turn, will possess classical statistical content in terms of consistent states and their probabilities, as is seen from Eq.(1.1).

For mixed states, unfortunately, the decomposition (1.1) is not unique. We shall propose a certain way to obtain unique results in case of a wellknown model of coarse graining. The proposal relies upon the fact that certain decompositions are distinguished by the dynamics of the coarse grained system.
II. CONSISTENT HISTORIES IN COARSE GRAINED SYSTEMS

For a given coarse grained (i.e. reduced) dynamics, the statistical operator satisfies linear evolution equation of the general form

$$\rho(t) = J(t)\rho(0), \quad t > 0,$$

where $J$ is the evolution superoperator. A basic feature of $J$ is that Eq.(2.1) generates mixed states from pure ones permanently. The formal generalization of Eq.(1.1) reads:

$$\rho(t) = \sum_{\alpha} w_{\alpha}(t)\psi_{\alpha}(t)\psi_{\alpha}^\dagger(t).$$

This equation must be conform with the Eq.(2.1), regarding especially the linearity of the superoperator $J$. This condition is easy to meet if the unnormalized states $\sqrt{w_{\alpha}(t)}\psi_{\alpha}(t)$, too, satisfy linear evolution equations. Hence, we shall assume that

$$\psi_{\alpha}(t) = \frac{1}{\sqrt{w_{\alpha}(t)}}C_{\alpha}(t)\psi_{\alpha}(0)$$

where $C_{\alpha}(t)$ is time dependent linear evolution operator for the consistent state $\psi_{\alpha}(t)$. Observe that the normalized state $\psi_{\alpha}(t)$ satisfies nonlinear equation though the nonlinearity is only caused by the normalizing prefactor. It is fixed just by the normalization condition:

$$w_{\alpha}(t) = \|C_{\alpha}(t)\psi_{\alpha}(0)\|^2.$$

Let us substitute Eq.(2.3) into Eq.(2.2) and compare the result with Eq.(2.1). Then, the evolution superoperator can be written in terms of the evolution operators of the consistent states:

$$J(t) = \sum_{\alpha} C_{\alpha}(t) \otimes C_{\alpha}^\dagger(t).$$

This superoperator is, as expected, linear.

Let us summarize our proposal. Assume a coarse grained system is given, with known linear evolution superoperator $J$ in Eq.(2.1). Single out a certain dyadic decomposition (2.5) of $J$ in terms of linear evolution operators $C_{\alpha}$. Once the operators $C_{\alpha}$ have been specified, the coarse grained dynamics can be described in terms of consistent histories $\{\psi_{\alpha}(\tau), \tau \in [0, t]\}$ generated by the history operators $C_{\alpha}$ via the nonlinear evolution equation (2.3). A given history is realized with probability $w_{\alpha}(t)$ as expressed by Eq.(2.4).

It is most important to realize that the operators $C_{\alpha}$ determine the dynamics as well as the statistics of the consistent histories. Still the choice of the $C_{\alpha}$ is not unique since Eq.(2.5) offers little constraint on it.
III. CONSISTENT HISTORIES IN THE CALDEIRA-LEGGETT MODEL

Nontrivial, i.e. nonunitary evolution equations of type (2.1) are usually not easy to derive. A well-known exception, calculable explicitly, is the evolution of the state $\rho$ of a Brownian particle interacting with a given bosonic reservoir being in thermal equilibrium [5]. Coarse graining is meant by tracing out the reservoir variables. Then standard calculations lead to the exact form of the evolution superoperator:

$$J(x_f, x'_f, x_i, x'_i, t)$$ (3.1)

as expressed by Eqs.(A.6) and (A.7) in coordinate representation.

Following the proposal of Par.II, the evolution superoperator (3.1) will be decomposed into a specific dyadic form (2.5). We shall exploit the fact that $J$ is expressed by double Gaussian integrals over the paths $x(\tau)$ and $x'(\tau)$, $\tau \in [0, t]$. Also the label $\alpha$ of the operators $C_\alpha$ will actually be a path $\bar{x}(\tau)$ rather than a number and, consequently, the summation in Eq.(2.5) will be replaced by functional integration over $\bar{x}$. In coordinate representation one writes Eq.(2.5) in the form:

$$J(x_f, x'_f, x_i, x'_i, t) = \int D\bar{x} C_{\bar{x}}(x_f, x_i, t) C^*_{\bar{x}}(x'_f, x'_i, t).$$ (3.2)

If we choose Gaussian form for the operator kernel $C_{\bar{x}}(x_f, x_i, t)$ then the superoperator functional $J$, too, will be Gaussian. By a clever choice, we can just obtain the required form (A.6).

Let us assume the following Gaussian expression for the history operators:

$$C_{\bar{x}}(x_f, x_i, t) = \int Dx \exp \left( \frac{i}{\hbar} S[x] \right) \Phi_{\bar{x}}[x]$$ (3.3a)

where

$$\Phi_{\bar{x}}[x] = \exp \left( -\frac{2i}{\hbar} \int_0^t d\tau \int_0^\tau d\sigma x(\tau)\eta(\tau - \sigma)\bar{x}(\sigma) \right.$$ (3.3b)

$$-\frac{1}{\hbar} \int_0^t d\tau \int_0^t d\sigma [x(\tau) - \bar{x}(\tau)]\tilde{\nu}(\tau - \sigma)[x(\sigma) - \bar{x}(\sigma)] \right)$$

and $\tilde{\nu}$ is a certain modification of the noise kernel (A.9a), specified below. Let us substitute Eqs.(3.3ab) into Eq.(3.2) and perform the Gaussian functional integration over $\bar{x}$. The resulting expression will coincide with the form given by Eqs.(A.6) and (A.7), provided the following constraint fulfills [cf.Eq.(3.11) of Ref.6]:

$$\nu = \tilde{\nu} + \eta^T \tilde{\nu}^{-1} $$ (3.4)
where we applied symbolic notation for the convolution of the kernels on the RHS. The retarded dissipation kernel is defined by $\eta^r(\tau) \equiv \theta(\tau) \eta(\tau)$, and $\eta^a(\tau) \equiv \eta^r(-\tau)$. It can be shown that, in general, the implicit equation (3.4) possesses two solutions for $\tilde{\nu}$.

In order to write Eq.(3.3b) into a compact form, observe that the dissipation term simulates an external potential $V[\bar{x}](x, \tau) = 2x \int_0^\tau \eta(\tau - \sigma) \bar{x}(\sigma) d\sigma$ as a retarded function of the label path $\bar{x}$. This term leads to a (label-)path dependent contribution to the action:

$$S[x][x] \equiv -\int_0^t d\tau V[\bar{x}](x, \tau). \quad (3.5)$$

Furthermore, let us introduce the following norm on the space of paths:

$$\|x\|^2 \equiv \frac{1}{\hbar^2} \int_0^\tau d\tau \int_0^\tau d\sigma x(\tau) \tilde{\nu}(\tau - \sigma) x(\sigma). \quad (3.6)$$

Using Eqs.(3.5) and (3.6), the compact form of $\Phi[x][x]$ will be the following:

$$\Phi[x][x] = \exp \left( \frac{i}{\hbar} S[x][x] - \|x - \bar{x}\|^2 \right). \quad (3.7)$$

As we have shown above, in the Caldeira- Leggett model an exact statistical decomposition of the evolution superoperator $J$ can explicitly be constructed. Given the initial wave function $\psi(x, 0)$, invoke Eqs.(2.3), (3.3a) and (3.7); then introduce path dependent histories as follows:

$$\psi[x](x_f, t) \equiv \frac{1}{\sqrt{w[x]}(t)} \int C[x](x_f, x_i) \psi(x_i, 0) dx_i$$

$$= \frac{1}{\sqrt{w[x]}(t)} \int Dx \exp \left( \frac{i}{\hbar} S[x] + \frac{i}{\hbar} S[x][x] - \|x - \bar{x}\|^2 \right) \psi(x_i, 0). \quad (3.8)$$

Let us observe that this expression differs from usual unitary Feynman integrals by the presence of the factor $\exp(-\|x - \bar{x}\|^2)$. This factor discards all paths from the functional integration except for those which are close to the label path $\bar{x}$. Remind that the role of functional metric specifying a distance between two paths is played by the modified noise kernel $\tilde{\nu}$. We expect that a typical history (3.8) is depicted by a wave packet propagating along a certain label path $\bar{x}$. The probability distribution $w[x]$ will mostly be concentrated on classical trajectories, hence most likely label paths will fluctuate around classically allowed trajectories of the central (damped) oscillator.

We have seen that to each possible path $\bar{x}$ and to each path dependent history $\psi[x]$ a certain probability can be attributed in a consistent way. It is not at all necessary that
these consistent histories be fully decoherent. Nevertheless, two different histories will tend to decohere:

$$\int \psi^*_\bar{x}_1(x_f,t)\psi\bar{x}_2(x_f,t)dx_f \to 0 \quad (3.9)$$

if $\bar{x}_1, \bar{x}_2$ are two distant paths, i.e.: $||\bar{x}_1 - \bar{x}_2|| \to \infty$ [cf.Eqs.(1.2ab)]. It is seen heuristically, that decoherence is remarkable when the distance of the two paths is large enough to exclude overlaps between the relevant Feynman paths concentrated along $\bar{x}_1$ or $\bar{x}_2$, respectively. Explicit calculations are possible for the scalar product (3.9) since the Caldeira-Leggett model possesses exact solutions.

For pedagogical reasons, we shall consider the very high temperature regime where the history equations are relatively simple.

IV. MARKOVIAN CONSISTENT HISTORIES

It is known from, e.g., Refs.[5] that in the high temperature limit the memory kernels $\eta, \nu$ become, in a good approximation, local kernels. To consider the simplest nontrivial case we assume high temperature and small velocities $\dot{x}$. Then the noise term will dominate and we shall ignore the frictional term proportional to $\eta$. In fact we take $\eta = 0$, therefore $\nu = \tilde{\nu}$ holds due to Eq.(3.4).

The norm (3.6) on path space simplifies as follows:

$$||x||^2 = \frac{\gamma}{\lambda^2} \int_0^t d\tau x^2(\tau). \quad (4.1)$$

The path dependent history (3.8) takes the following simple form:

$$\psi_{\bar{x}_i}(x_f,t) = \frac{1}{\sqrt{w_{\bar{x}_i}(t)}} \int Dx exp \left( \frac{i}{\hbar} S_R[x] - ||x - \bar{x}||^2 \right) \psi(x_i,0). \quad (4.2)$$

This expression is well-known from the theory of continuous quantum measurements [7,6]. It is known, first of all, that the path dependent history (4.2) is $\psi$-valued Markovian process. Remind the summary in Par.II, according to which the quantity $w_{\bar{x}_i}(t)$ in the normalizing factor yields the probability of the given path $\bar{x}_i$ and of the corresponding history (4.2). It has been shown in Refs.[8] that this process can be described by the following Ito stochastic differential equations:
\[ \dot{\psi}_{[\bar{x}]}(x, \tau) = -\frac{i}{\hbar}(H_R\psi_{[\bar{x}]}) (x, \tau) - \frac{\gamma}{2\lambda_{dB}^2} (x-<x>)^2 \psi_{[\bar{x}]}(x, \tau) + (x-<x>)\psi_{[\bar{x}]}(x, \tau) f(\tau), \] (4.3a)

\[ \bar{x}(\tau) = <x> - \frac{\lambda_{dB}^2}{2\gamma} f(t) \] (4.3b)

where \( f \) is an auxiliary white noise of correlation

\[ <f(\tau)f(0)> = \frac{\gamma}{\lambda_{dB}^2} \delta(\tau). \] (4.3c)

For the history expectation value of the coordinate

\[ <x>_{[\bar{x}],\tau} \equiv \int dx |\psi_{[\bar{x}]}(x, \tau)|^2 \] (4.3d)

the shorthand notation \(<x>\) has been introduced.

We can see that the wave function of the path dependent history and the path itself satisfy coupled stochastic differential equations. From Eq.(4.3b) follows that the ordinary quantum expectation value \(<x>\) of the coordinate operator and the label path coordinate \(\bar{x}\) will coincide in stochastic mean.

Incidentally, it is perhaps instructive to write Eq.(4.3a) into an equivalent Ito form for the pure state statistical operator \(P_{[\bar{x}]}(x, x', \tau) \equiv \psi_{[\bar{x}]}(x)\psi_{[\bar{x}]}^\dagger(x'):\)

\[ \dot{P}_{[\bar{x}]}(x, x', \tau) = -\frac{i}{\hbar} [H_R, P_{[\bar{x}]}) (x, x', \tau) - \frac{\gamma}{2\lambda_{dB}^2} (x-x')^2 P_{[\bar{x}]}(x, x', \tau) \]

\[ + (x + x' -2 <x>) P_{[\bar{x}]}(x, x', \tau) f(\tau). \] (4.4)

Taking the stochastic average of both sides, the nonlinear term cancels and we obtain the wellknown linear Markovian master equation (A.11).

The Markovian history equations (4.3ab) can be solved exactly in the long time limit. In Refs.[8,9] the following result was obtained for the special case, when the renormalized central oscillator is just a free particle, i.e. \(\Omega_R = 0\). The shape of the wave function becomes stabilized at the Gaussian shape, i.e. \(\psi(x, \tau) \sim (2\pi\sigma^2)^{-1/4} \exp \left( ix <p> - \frac{1-i}{4\sigma^2} (x-<x>)^2 \right) \) (4.5a)

of width \(\sigma = (\hbar/2)^{3/4}(\gamma k_B T)^{-1/4} M^{-1/2}\), while the quantum expectation values \(<x>, <p>\) of the coordinate and momentum, resp., satisfy the following stochastic equations:

\[ \frac{d <x>}{d\tau} = \frac{<p>}{M} + 2\sigma^2 f, \]

\[ \frac{d <p>}{d\tau} = \hbar f. \] (4.5b)
We note that similar quality of results would be obtained from more general Ito differential equations (cf. Ref. [10]), had we retained the dissipation term proportional to $\nu$ (A.9b).

V. SUMMARY

Starting from the conservative statistical interpretation of mixed quantum states, we have proposed a family of quantum histories possessing consistent probability distribution. The proposal has been realized for the Caldeira-Leggett model of Brownian motion. Our history expansion is exact and needs no particular tuning of decoherence and coarse graining. In the Markovian regime, we have obtained analytic localized solutions for the Brownian particle’s wave function.

A. Coarse graining in the Caldeira-Leggett model [5]

Our central system is a harmonic oscillator of mass $M$ and frequency $\Omega$, with action

$$S[x] = \frac{M}{2} \int_0^t d\tau (\dot{x}^2 - \Omega^2 x^2). \quad (A.1)$$

The initial quantum state will be denoted by $\rho(x_i, x'_i, 0)$.

Consider a reservoir modeled by a set of harmonic oscillators with masses $m_n$ and with frequencies $\omega_n$. Its action is:

$$S_{\text{res}}[q] = \sum_n m_n \frac{1}{2} \int_0^t d\tau (\dot{q}_n^2 - \omega_n^2 q_n^2). \quad (A.2)$$

At $t = 0$ the state of the reservoir is thermal equilibrium state at some temperature $T$. Consider a certain linear combination $Q = \sum_n c_n q_n$ of the reservoir coordinates. Introduce the complex correlation function of the Heisenberg operators $Q(\tau)$:

$$\nu(\tau) + i\eta(\tau) \equiv \frac{1}{\hbar} <Q(\tau)Q(0)>_T \quad (A.3)$$

where $< ... >_T$ stands for the expectation value taken in the thermal equilibrium state of the reservoir. The real and imaginary parts $\nu, \eta$ are called the noise (or fluctuation) and the dissipation kernels, respectively.

For $t > 0$, a linear coupling is introduced between the central oscillator and the reservoir, represented by the action [12]

$$S_{\text{int}}[x, q] = -\int_0^t d\tau x Q. \quad (A.4)$$
The usual coarse graining of the above system consists of tracing out the variables of the reservoir. Then the statistical operator of the central oscillator obeys to linear evolution equation:

\[ \rho(x_f, x'_f, t) = \int J(x_f, x'_f, x_i, x'_i, t) \rho(x_i, x'_i, 0) dx_i dx'_i. \]  

(A.5)

The superoperator \( J \) takes the following general form:

\[ J(x_f, x'_f, x_i, x'_i, t) = \int Dx \int Dx' \exp \left( \frac{i}{\hbar} S[x] - \frac{i}{\hbar} S[x'] \right) F[x, x'] \]  

(A.6)

with the decoherence functional \( F \) defined by

\[ F[x, x'] = \exp \left( \frac{i}{\hbar} \int_0^t d\tau \int_0^\tau d\tau' [x(\tau) - x'(\tau)] \eta(\tau - \tau') [x(\tau') + x'(\tau')] \right. 

\[ - \frac{1}{2\hbar} \int_0^t d\tau \int_0^t d\tau' [x(\tau) - x'(\tau)] \nu(\tau - \tau') [x(\tau') - x'(\tau')] \right). \]  

(A.7)

For the Caldeira-Leggett reservoir, the noise and dissipation kernels have the following particular forms, respectively:

\[ \nu(\tau) = \frac{\gamma M}{\pi} \int_0^{\omega_{\text{max}}} d\omega \omega \coth \frac{\hbar \omega}{2k_B T} \cos(\omega \tau), \]  

(A.8a)

\[ \eta(\tau) = -\frac{\gamma M}{\pi} \int_0^{\omega_{\text{max}}} d\omega \omega \sin(\omega \tau). \]  

(A.8b)

For high temperatures, Markovian approximation can be applied to the noise and dissipation kernels:

\[ \nu(\tau) = \frac{\gamma h}{\lambda_{dB}^2} \delta(\tau), \]  

(A.9a)

\[ \eta(\tau) = \gamma M \delta'(\tau) \]  

(A.9b)

where \( \lambda_{dB} = \hbar / \sqrt{2Mk_B T} \) is the thermal deBroglie length. In addition, the frequency \( \Omega \) of the central oscillator must be replaced by its renormalized value, defined by \( \Omega_R^2 = \Omega^2 - 2\gamma \Omega / \pi \).

Consequently, the action \( S \) on the RHS. of Eq.(A.6) will be replaced by the renormalized action:

\[ S_R[x] = \frac{M}{2} \int_0^t d\tau (\dot{x}^2 - \Omega_R^2 x^2). \]  

(A.10)

In Markovian approximation the evolution superoperator (A.6) becomes local in time. Hence the evolution equation (A.5) can equivalently be written in form of a linear differential equation. We do not quote the general result but a simplified version valid for small velocities \( \dot{x} \):

\[ \dot{\rho}(x, x', \tau) = -\frac{i}{\hbar} [H_R, \rho](x, x', \tau) - \frac{\gamma}{2\lambda_{dB}^2} (x - x')^2 \rho(x, x', \tau). \]  

(A.11)
The general master equation contains additionally a certain dissipation term proportional to the momentum $p$, and a further fluctuation term [11] as well.

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