On strongly starlike and convex functions of order $\alpha$ and type $\beta$

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Abstract

In this note we investigate the inclusion relationship between the class of strongly starlike functions of order $\alpha$ and type $\beta$, $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, which satisfy

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} - \beta \right\} \right| < \frac{\pi}{2\alpha}$$

and the class of strongly convex functions of order $\alpha$ and type $\beta$ which satisfy

$$\left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} - \beta \right\} \right| < \frac{\pi}{2\alpha}$$

in the unit disk, where $f$ is an analytic function defined on the unit disk and satisfies $f(0) = f'(0) - 1 = 1$. Some applications of our main result are also presented which contains various classical results for the typical subclasses of starlike and convex functions.

Key words: univalent function, strongly starlike function, strongly convex function

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1. Introduction

Let $\mathcal{A}$ denote the set of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$.

Let $\alpha$ be a real number with $\alpha \in (0, 1]$. A function $f \in \mathcal{A}$ is called strongly starlike of order $\alpha$ if it satisfies

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2\alpha}$$
for all \( z \in \mathbb{D} \) and strongly convex of order \( \alpha \) if
\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2\alpha}
\]
for all \( z \in \mathbb{D} \). Let us denote by \( S^*(\alpha) \) the class of functions strongly starlike of order \( \alpha \), and by \( \mathcal{K}(\alpha) \) the class of functions strongly convex of order \( \alpha \). The class \( S^*(\alpha) \) was introduced first by Stankiewicz [13] and by Brannan and Kirwan [2], independently. It is clear from the definitions that \( S^*(\alpha_1) \subset S^*(\alpha_2) \) and \( \mathcal{K}(\alpha_1) \subset \mathcal{K}(\alpha_2) \) for \( 0 < \alpha_1 < \alpha_2 \leq 1 \). The case when \( \alpha = 1 \), i.e., \( S^*(1) \) and \( \mathcal{K}(1) \) correspond to well known classes of starlike and convex functions respectively, and therefore all the functions which belong to \( S^*(\alpha) \) or \( \mathcal{K}(\alpha) \) are univalent in \( \mathbb{D} \). We denote by \( S^* \) and \( \mathcal{K} \) the classes of starlike and convex functions. For the general reference of classes of starlike and convex functions, see, for instance [3].

Mocanu [9] obtained the following result (see also [11]). Here, set
\[
\rho(\alpha) = \tan^{-1}\left( \frac{\alpha}{1-\alpha} \left( \frac{1-\alpha}{1+\alpha} \right)^{\frac{\pi}{2\alpha}} \sin \left( \frac{\pi}{2} (1-\alpha) \right) \right) - \frac{1}{1+\left( \frac{\alpha}{1-\alpha} \left( \frac{1-\alpha}{1+\alpha} \right)^{\frac{\pi}{2\alpha}} \cos \left( \frac{\pi}{2} (1-\alpha) \right) \right) - \frac{\alpha}{1-\alpha} \left( \frac{1-\alpha}{1+\alpha} \right)^{\frac{\pi}{2\alpha}}}
\]

and
\[
\gamma(\alpha) = \alpha + \frac{2}{\pi} \rho(\alpha).
\]

**Theorem A.** \( \mathcal{K}(\gamma(\alpha)) \subset S^*(\alpha) \) for each \( \alpha \in (0, 1] \).

We remark that the function \( \gamma(\alpha) \) is continuous and strictly increases from 0 to 1 when \( \alpha \) moves from 0 to 1. Further investigations for the above theorem can be found in [3].

Now we shall introduce the class of functions \( S^*(\alpha, \beta) \) and \( \mathcal{K}(\alpha, \beta) \), \( \alpha \in (0, 1] \) and \( \beta \in [0, 1) \), whose members satisfy the conditions
\[
\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} - \beta \right) \right| < \frac{\pi}{2\alpha}
\]
and
\[
\left| \arg \left( \frac{zf''(z)}{f'(z)} - \beta \right) \right| < \frac{\pi}{2\alpha}
\]
for all \( z \in \mathbb{D} \), respectively. We call a function \( f \in S^*(\alpha, \beta) \) strongly starlike of order \( \alpha \) and type \( \beta \). In the same way, a function \( f \in \mathcal{K}(\alpha, \beta) \) is strongly convex of order \( \alpha \) and type \( \beta \). It is obvious that \( S^*(\alpha, 0) = S^*(\alpha) \) and \( \mathcal{K}(\alpha, 0) = \mathcal{K}(\alpha) \). Also the following relations are true from the definitions;

i) \( S^*(\alpha_1, \beta) \subset S^*(\alpha_2, \beta) \),
ii) \( \mathcal{K}(\alpha_1, \beta) \subset \mathcal{K}(\alpha_2, \beta) \),
iii) \( S^*(\alpha, \beta_1) \supset S^*(\alpha, \beta_2) \),
iv) \( \mathcal{K}(\alpha, \beta_1) \supset \mathcal{K}(\alpha, \beta_2) \).
for $0 < \alpha_1 < \alpha_2 \leq 1$ and $0 \leq \beta_1 < \beta_2 < 1$. That is why all functions belong to $S^*(\alpha, \beta)$ or $K(\alpha, \beta)$ are univalent on $\mathbb{D}$.

A sufficient condition for which $f \in A$ lies in $S^*(\alpha, \beta)$ was proved by the second author et al. [12]. The authors also proposed in [12] the open problem about an inclusion relationship between $K(\alpha, \beta)$ and $S^*(\alpha, \beta)$. However, it seems that no results concerning this question have been known.

Our main result is the following;

**Theorem 1.** $K(\gamma(\alpha), \beta) \subset S^*(\alpha, \beta)$ for each $\alpha \in (0, 1]$ and $\beta \in [0, 1)$.

The above theorem includes Theorem A as the case when $\beta = 0$.

We should notice the reader that this estimation is not sharp for each $\alpha \in (0, 1]$ and $\beta \in [0, 1)$ (see also [5]). We will discuss about this problem in section 2 with the proof of Theorem 1. Our main theorem yields several applications which will be shown in the last section.

### 2. Proof of Theorem 1

Our proof relies on the following lemma which was obtained by the second author [10, 11];

**Lemma B.** Let $p(z)$ be analytic and satisfies $p(0) = 1$, $p(z) \neq 0$ in $\mathbb{D}$. Let us assume that there exists a point $z_0 \in \mathbb{D}$ such that $|\arg p(z)| < \pi\alpha/2$ for $|z| < |z_0|$ and $|\arg p(z_0)| = \pi\alpha/2$ where $\alpha > 0$. Then we have

$$\frac{z_0p'(z_0)}{p(z_0)} = i\alpha k$$

where $k \geq \frac{1}{2}(a + \frac{1}{a})$ when $\arg p(z_0) = \pi\alpha/2$ and $k \leq -\frac{1}{2}(a + \frac{1}{a})$ when $\arg p(z_0) = -\pi\alpha/2$, where $p(z_0)^{1/\alpha} = \pm i\alpha$ and $a > 0$.

The next result will be used later;

**Lemma 2.** $\tan^{-1}\alpha \geq \rho(\alpha)$ for all $\alpha \in (0, 1)$, where $\rho$ is defined by (1).

**Proof.** Put $\phi(\alpha) = (1/(1 - \alpha))/((1 - \alpha)/(1 + \alpha))^{1/\alpha}$. It is enough to prove that

$$\alpha \geq \frac{\alpha\phi(\alpha) \sin[\pi(1 - \alpha)/2]}{1 + \alpha\phi(\alpha) \cos[\pi(1 - \alpha)/2]}$$

for all $\alpha \in (0, 1]$. Since $\phi(\alpha) < 1$ because of $\phi(0) = 1$ and $\phi'(\alpha) < 0$, we obtain $\alpha > \phi(\alpha)$ and therefore

$$\frac{\alpha \sin[\pi(1 - \alpha)/2]}{1 + \alpha \cos[\pi(1 - \alpha)/2]} > \frac{\alpha\phi(\alpha) \sin[\pi(1 - \alpha)/2]}{1 + \alpha\phi(\alpha) \cos[\pi(1 - \alpha)/2]}.$$

It remains to show that

$$\alpha \geq \frac{\alpha \sin[\pi(1 - \alpha)/2]}{1 + \alpha \cos[\pi(1 - \alpha)/2]}$$

for all $\alpha \in (0, 1]$ and this is clear. \qed
Proof of Theorem[4] Let us suppose that \( f \) satisfies the assumption of the theorem and let

\[
p(z) = \frac{1}{1 - \beta} \left( \frac{zf'(z)}{f(z)} - \beta \right).
\]

Then \( p(0) = 1 \), and calculations show that

\[
1 + \frac{zf''(z)}{f'(z)} - \beta = (1 - \beta)p(z) \left\{ 1 + \frac{zp'(z)}{p(z)} \right\}.
\]

We note that \( p(z) \neq 0 \) holds for all \( z \in \mathbb{D} \) since \( 1 + zf''(z)/f'(z) - \beta \neq \infty \) on \( \mathbb{D} \) from our assumption.

Now we derive a contradiction by using Lemma[5]. If there exists a point \( z_0 \) such that \( \left| \arg p(z) \right| < \pi \alpha/2 \) for \( |z| < |z_0| \) and \( \left| \arg p(z_0) \right| = \pi \alpha/2 \), where \( \alpha \in (0, 1) \), then by Lemma[5], \( p \) must satisfy \( z_0p'(z_0)/p(z_0) = i\alpha k \) when \( k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \) when \( \arg p(z_0) = \pi \alpha/2 \) and \( k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \) when \( \arg p(z_0) = -\pi \alpha/2 \), where \( p(z_0)^{1/\alpha} = \pm ia \) and \( a > 0 \).

At first we suppose that \( \arg p(z_0) = \pi \alpha/2 \). Then from (3) we have

\[
\arg \left\{ 1 + \frac{z_0f''(z_0)}{f'(z_0)} - \beta \right\} = \arg \left\{ (1 - \beta)p(z_0) \left\{ 1 + \frac{zp'(z_0)}{p(z_0)} \right\} \right\}.
\]

We shall estimate the second term of the second line of above. Geometric observations show that the point \( 1 + [iak/((1 - \beta)p(z_0) + \beta)] \) lies on the subarc \( C \) of the circle which passes through \( 1, 1 + iak \) \( \text{ and } \) \( 1 + [iak/p(z_0)] \), where \( C \) connects \( 1 + iak \) \( \text{ and } \) \( 1 + [iak/p(z_0)] \) and does not pass through \( 1 \). Further, we can find out that the value \( \{\arg z : z \in C\} \) attains its minimum at the end points of \( C \). Therefore we have

\[
\arg \left\{ 1 + \frac{iak}{(1 - \beta)p(z_0) + \beta} \right\} \geq \min \left\{ \arg \{1 + iak\}, \arg \left\{ 1 + \frac{iak}{p(z_0)} \right\} \right\}.
\]

Here, the first value in the above minimum can be evaluated by \( \arg(1 + iak) \geq \tan^{-1} \alpha \) since \( k \geq 1 \). For the second value, we note that \( a^{1-\alpha} + a^{-1-\alpha} \) takes its minimum value at \( a = \sqrt{(1 + \alpha)/(1 - \alpha)} \). Therefore

\[
\arg \left\{ 1 + \frac{iak}{p(z_0)} \right\} = \arg \left\{ 1 + e^{z(1-\alpha)i} \cdot \frac{\alpha}{2} \left[ a^{1-\alpha} + a^{-1-\alpha} \right] \right\} \geq \arg \left\{ 1 + e^{z(1-\alpha)i} \cdot \frac{\alpha}{2} \left[ \left( 1 + \frac{1}{1 - \alpha} \right)^{1/\alpha} + \left( 1 + \frac{1}{1 - \alpha} \right)^{-1/\alpha} \right] \right\} = \rho(\alpha).
\]
By Lemma 2 we conclude that
\[
\arg \left\{ 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right\} - \beta \geq \frac{\pi}{2} \alpha + \min \left\{ \tan^{-1} \alpha, \rho(\alpha) \right\} = \frac{\pi}{2} \gamma(\alpha)
\]
and this contradicts our assumption.

In the same fashion, if \( p(z_0) = -\pi \alpha / 2 \) then a similar argument shows that
\[
\arg \left\{ 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right\} - \beta \leq -\frac{\pi}{2} \alpha + \max \left\{ \tan^{-1}(-\alpha), -\rho(\alpha) \right\} = -\frac{\pi}{2} \gamma(\alpha).
\]
This also contradicts our assumption and our proof is completed. □

We remark that we expect this theorem to be room for improvement in our method because the inequality (4) is a rough estimation except the case when \( \beta = 0 \), whereas it seems to be not easy to give a precise estimation for the left hand side of (4).

3. Applications

We would like to give a further discussion to the relationship between \( S^*(\alpha, \beta) \) and \( \mathcal{K}(\alpha, \beta) \) by using Theorem 1.

3.1.

It is well known that a convex function is a starlike function, that is, \( \mathcal{K} \subset S^* \). Furthermore, Mocanu [8] showed that \( \mathcal{K}(\alpha) \subset S^*(\alpha) \) for all \( \alpha \in (0, 1] \). Now we give the next result which includes these properties as special cases;

**Corollary 3.** \( \mathcal{K}(\alpha, \beta) \subset S^*(\alpha, \beta) \) for each \( \alpha \in (0, 1] \) and \( \beta \in [0, 1) \).

**Proof.** Since \( \alpha \leq \gamma(\alpha) \) for all \( \alpha \in (0, 1] \), \( \mathcal{K}(\alpha, \beta) \subset \mathcal{K}(\gamma(\alpha), \beta) \subset S^*(\alpha, \beta) \) by ii) in (2) and Theorem 1 which is our desired inclusion. □

Corollary 3 yields the following property;

**Corollary 4.** If \( z f'(z) \in S^*(\alpha, \beta) \), then \( f \in S^*(\alpha, \beta) \).

**Proof.** It is obvious that \( g \in \mathcal{K}(\alpha, \beta) \) if and only if \( zg'(z) \in S^*(\alpha, \beta) \). Thus if \( zg'(z) \in S^*(\alpha, \beta) \) then \( g \in \mathcal{K}(\alpha, \beta) \subset S^*(\alpha, \beta) \) from Corollary 3. Hence our assertion follows if we put \( f(z) = zg'(z) \). □

This corollary is equivalent to the following: \( S^*(\alpha, \beta) \) is preserved by the Alexander transformation, where the Alexander transformation \([1]\) is the integral transformation defined by
\[
f(z) \mapsto \int_0^\infty \frac{f(u)}{u} du
\]
for \( f \in \mathcal{A} \).
3.2.
If \( \alpha = 1 \), then the class \( S^*(1, \beta) \) and \( K(1, \beta) \) is called \textit{starlike of order} \( \beta \) and \textit{convex of order} \( \beta \), respectively. It is easy to see that \( f \in S^*(1, \beta) \) satisfies

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta
\]

and \( f \in K(1, \beta) \) satisfies

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta.
\]

Marx [7] and Strohhäcker [14] showed that \( K(1, 0) \subset S^*(1, 1/2) \). Jack [4] proposed the more general problem; What is the largest number \( \beta_0 \) which satisfies \( K(1, \beta) \subset S^*(1, \beta_0) \)? Later MacGregor [6] and Wilken and Feng [15] answered the problem to give the exact value of \( \beta_0 \);

\textbf{Theorem C.} \( K(1, \beta) \subset S^*(1, \delta(\beta)) \) for all \( \beta \in [0, 1) \), where

\[
\delta(\beta) = \begin{cases} 
1 - 2\beta & \text{if } \beta \neq \frac{1}{2}, \\
\frac{1}{2 \log 2} & \text{if } \beta = \frac{1}{2}.
\end{cases}
\]

This estimation is sharp for each \( \beta \in [0, 1) \).

Setting \( \beta = 0 \), we have the result of Marx and Strohhäcker. We can obtain a similar estimation to above that “\( K(\gamma(\alpha), \delta(\beta)) \subset S^*(\alpha, \beta) \) for all \( \alpha \in (0, 1] \) and \( \beta \in [0, 1) \)” by Theorem I since \( \beta < \delta(\beta) \) for all \( \beta \in [0, 1) \). However, the following problem is still open;

\textbf{Open Problem.} \( K(\gamma(\alpha), \beta) \subset S^*(\alpha, \delta(\beta)) \) for each \( \alpha \in (0, 1] \) and \( \beta \in [0, 1) \).

This problem implies Theorem I because \( S^*(\alpha, \delta(\beta)) \subset S^*(\alpha, \beta) \) for all \( \alpha \in (0, 1] \) and \( \beta \in [0, 1) \), and Theorem C as the case when \( \alpha = 1 \).

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