On small matrix subalgebras with a trivial centralizer *

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Abstract

Given an integer $n \geq 3$, we investigate the minimal dimension of a subalgebra of $M_n(\mathbb{K})$ with a trivial centralizer. It is shown that this dimension is 5 when $n$ is even and 4 when it is odd. In the latter case, we also determine all 4-dimensional subalgebras with a trivial centralizer.

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1 Introduction

Here, we set an integer $n \geq 2$ and a field $\mathbb{K}$. Using the French convention, we let $\mathbb{N}$ denote the set of non-negative integers and $\mathbb{N}^*$ the one of positive integers. We let $M_n(\mathbb{K})$ denote the algebra of square matrices of order $n$ with entries in $\mathbb{K}$ and $T_n(\mathbb{K})$ its subalgebra of upper triangular matrices. We let $M_{p,q}(\mathbb{K})$ denote the set of matrices with $p$ rows, $q$ columns and entries in $\mathbb{K}$.

All subalgebras of $M_n(\mathbb{K})$ are required to contain the unit matrix $I_n$: the subalgebra $\text{Span}(I_n)$ will be called trivial.

For $(i, j) \in [1, n]^2$, we let $E_{i,j}$ denote the elementary matrix of $M_n(\mathbb{K})$ with a zero entry in every position except for $(i, j)$ where the entry is 1.

Given a subset $V$ of $M_n(\mathbb{K})$, we let

$$\mathcal{C}(V) := \{ A \in M_n(\mathbb{K}) : \forall M \in V, AM = MA \}$$

denote its centralizer, and we simply write $\mathcal{C}(A) = \mathcal{C}(\{A\})$ when $A$ is a matrix of $M_n(\mathbb{K})$. Recall that $\mathcal{C}(V)$ is always a subalgebra of $M_n(\mathbb{K})$ and that $\mathcal{C}(V)$ is

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also the centralizer of the subalgebra generated by $V$.
The Jordan matrix of order $n$ will be written $J_n = (\delta_{i+1,j})_{1 \leq i,j \leq n}$, where $\delta_{a,b}$ equals 1 if $a = b$, and 0 otherwise.

In this paper, we will focus on subalgebras of $M_n(\mathbb{K})$ with a small dimension and a trivial centralizer. The basic motivation for studying subalgebras with a trivial centralizer comes from the theory of representations of algebras. Let $A$ be a subalgebra of $M_n(\mathbb{K})$, which we identify with the algebra of linear endomorphisms of the vector space $\mathbb{K}^n$. This yields a structure of $A$-module on $\mathbb{K}^n$ for which the endomorphisms naturally correspond to the matrices in the centralizer $C(A)$. When $\mathbb{K}$ is algebraically closed and $\mathbb{K}^n$ is a simple $A$-module (i.e. when it has no non-trivial submodule), then $C(A)$ is trivial, however the converse may not hold. In the case $A$ is generated by a finite subgroup of $\text{GL}_n(\mathbb{K})$ and $\mathbb{K}$ has characteristic 0, then the converse is classically true because $A$ is then semi-simple (see the theory of linear representations of finite groups). In the general case of an arbitrary field and an arbitrary subalgebra of $M_n(\mathbb{K})$, the condition $C(A) = \text{Span}(I_n)$ may thus be seen as an alternative notion of simplicity or semi-simplicity, which provides motivation enough for studying it systematically.

Non-trivial subalgebras of $M_n(\mathbb{K})$ with a trivial centralizer are actually commonplace. A classical example is that of $T_n(\mathbb{K})$. Indeed, let $A \in C(T_n(\mathbb{K}))$.
Then $A$ commutes with $E_{i,i}$ for every $i \in [1, n]$, which shows that $A$ is diagonal. The commutation of $A$ with $E_{1,i}$ for every $i \in [2, n]$ then shows that all diagonal entries of $A$ are equal.

It is somewhat harder to produce such subalgebras with a small dimension. Our main goal is to find the smallest dimension for such a subalgebra and to classify the subalgebras of minimal dimension.

**Definition 1.** We let $t_n(\mathbb{K})$ (or simply $t_n$ when the field is obvious) denote the smallest dimension of a subalgebra of $M_n(\mathbb{K})$ with a trivial centralizer.

Notice first that a subalgebra of dimension 2 is always of the form $\mathbb{K}[A]$ for some matrix $A$ which is not a scalar multiple of $I_n$, so its centralizer contains $A$ and is therefore non-trivial. This essentially solves the case $n = 2$.

**Proposition 1.** One has $t_2 = 3$. More precisely, $T_2(\mathbb{K})$ has a trivial centralizer and dimension 3.

We will assume $n \geq 3$ from now on. Our main results are stated below:
Theorem 2. If \( n \geq 3 \) is even, then \( t_n(\mathbb{K}) = 5 \).
If \( n \geq 3 \) is odd, then \( t_n(\mathbb{K}) = 4 \).

Proposition 3. Let \( p \in \mathbb{N} \setminus \{0,1\} \) and consider the linear subspace \( \mathcal{F}_{2p} \) of \( M_{2p}(\mathbb{K}) \) generated by the matrices
\[
\begin{bmatrix}
I_p & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & I_p
\end{bmatrix}, \begin{bmatrix}
0 & I_p \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & J_p \\
0 & 0
\end{bmatrix} \text{ and } \begin{bmatrix}
0 & J_p \\
0 & 0
\end{bmatrix}.
\]
Then \( \mathcal{F}_{2p} \) is a 5-dimensional subalgebra of \( M_{2p}(\mathbb{K}) \) with a trivial centralizer.

Proposition 4. Let \( p \in \mathbb{N}^* \). Consider the matrices \( C_p = \begin{bmatrix} I_p & 0 \end{bmatrix} \) and \( D_p = \begin{bmatrix} 0 & I_p \end{bmatrix} \) in \( M_{p,p+1}(\mathbb{K}) \), and define \( \mathcal{H}_{2p+1} \) as the linear subspace of \( M_{2p+1}(\mathbb{K}) \) generated by the matrices
\[
\begin{bmatrix}
I_p & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & I_{p+1}
\end{bmatrix}, \begin{bmatrix}
0 & C_p \\
0 & 0
\end{bmatrix} \text{ and } \begin{bmatrix}
0 & D_p \\
0 & 0
\end{bmatrix}.
\]
Then \( \mathcal{H}_{2p+1} \) is a 4-dimensional subalgebra of \( M_{2p+1}(\mathbb{K}) \) with a trivial centralizer.

Finally, we will prove that the latter example is essentially unique:

Proposition 5. Let \( p \in \mathbb{N}^* \) and \( A \) be a subalgebra of \( M_{2p+1}(\mathbb{K}) \) of dimension 4 with a trivial centralizer. Then \( A \) is conjugate to either \( \mathcal{H}_{2p+1} \) or its transposed subalgebra \( \mathcal{H}^t_{2p+1} \), i.e. there is a non-singular matrix \( P \in GL_{2p+1}(\mathbb{K}) \) such that
\[
A = PH_{2p+1}P^{-1} \quad \text{or} \quad A = PH^t_{2p+1}P^{-1}.
\]

Remark 1. It is an easy exercise to prove that \( \mathcal{H}_{2p+1} \) and \( \mathcal{H}^t_{2p+1} \) are not conjugate one to the other.

2 Checking the examples
We will start with a little lemma.

Lemma 6. Let \( n \geq 2 \) be an integer. Then \( \text{Span}(J_n, J^t_n) \) has a trivial centralizer.

Proof. Since \( J_n \) is cyclic, its centralizer is \( \mathbb{K}[J_n] \) (see [5] Theorem 5 p.23), and it thus contains only upper triangular matrices with equal diagonal entries. Similarly, every matrix of \( \mathbb{K}[J^t_n] \) is lower triangular. It follows that every matrix in the centralizer of \( \text{Span}(J_n, J^t_n) \) must be scalar. \( \square \)
The examples featured in Propositions 3 and 4 are based upon the same idea. Consider a decomposition \( n = p + q \) and a linear subspace \( V \) of \( \mathbb{M}_{p,q}(\mathbb{K}) \). It is then easily checked that

\[
\mathcal{H} := \left\{ \begin{bmatrix} aI_p & K \\ 0 & bI_q \end{bmatrix} \mid (a, b) \in \mathbb{K}^2, K \in V \right\}
\]

is always a subalgebra of \( \mathbb{M}_n(\mathbb{K}) \) with dimension \( \dim V + 2 \). Straightforward computation also shows that the centralizer of the matrix \( P = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \) is

\[
\mathcal{C}(P) = \left\{ \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \mid (X, Y) \in \mathbb{M}_p(\mathbb{K}) \times \mathbb{M}_q(\mathbb{K}) \right\}.
\]

Let \( M = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{C}(P) \). Then \( M \) belongs to \( \mathcal{C}(\mathcal{H}) \) if and only if \( \forall K \in V, X K = K Y \). Of course, this last relation need only be tested on a basis of \( V \).

From there, our claims may easily be proven.

The example in Proposition 3.
Here, \( q = p \) and \( V = \text{Span}(I_p, J_p, J^t_p) \).

Let \( (X, Y) \in \mathbb{M}_p(\mathbb{K})^2 \) such that \( X I_p = I_p Y, X J_p = J_p Y \) and \( X J^t_p = J^t_p Y \). Then \( Y = X \) and \( X \) commutes with both \( J_p \) and \( J^t_p \). By Lemma 6, \( X \) is a scalar multiple of \( I_p \) hence \( \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \) is a scalar multiple of \( I_n \). This proves Proposition 3.

The example in Proposition 4.
Here \( q = p + 1 \) and \( V = \text{Span}(C_p, D_p) \).

Let \( (X, Y) \in \mathbb{M}_p(\mathbb{K}) \times \mathbb{M}_{p+1}(\mathbb{K}) \) such that \( \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{C}(\mathcal{H}) \).

The identity \( X C_p = C_p Y \) entails that \( Y = \begin{bmatrix} X & 0 \\ L & \alpha \end{bmatrix} \) for some \((\alpha, L) \in \mathbb{K} \times \mathbb{M}_{1,p}(\mathbb{K})\), whilst identity \( X D_p = D_p Y \) shows that \( Y = \begin{bmatrix} \beta & L' \\ 0 & X \end{bmatrix} \) for some \((\beta, L') \in \mathbb{K} \times \mathbb{M}_{1,p}(\mathbb{K})\). We thus have

\[
\begin{bmatrix} X & 0 \\ L & \alpha \end{bmatrix} = \begin{bmatrix} \beta & L' \\ 0 & X \end{bmatrix}.
\]
Starting from the first column of $X$, an easy induction shows that $X$ is upper triangular with all diagonal entries equal to $\beta$. Also, starting from the last column of $X$, an easy induction shows that $L' = 0$ and $X$ is lower triangular. This yields $X = \beta I_p$ and $Y = \beta I_{p+1}$ hence \[
\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} = \beta I_{2p+1}, \]
which proves Proposition 4.

3 A lower bound for $t_n(\mathbb{K})$

3.1 Introduction

Here, we will prove that $t_n \geq 4$ when $n$ is odd, and $t_n \geq 5$ when $n$ is even. In other words, we will prove that every subalgebra of $M_n(\mathbb{K})$ has a non-trivial centralizer provided it has dimension $p \leq 3$ when $n$ is odd, and dimension $p \leq 4$ when $p$ is even. The proof is essentially laid out as follows:

- by extending the ground field, we reduce the study to the case of an algebraically closed field;
- in this case, we discard all unispectral subalgebras (i.e. subalgebras in which every operator has a sole eigenvalue);
- in the remaining cases, the considered subalgebra contains a non-trivial idempotent which we use to split the algebra $\mathcal{A}$ into several remarkable subspaces; we then use that splitting to find a non-scalar matrix in the centralizer of $\mathcal{A}$ when $\dim \mathcal{A}$ is small enough.

From now on, we set an integer $n \geq 3$ and a subalgebra $\mathcal{A}$ of $M_n(\mathbb{K})$. The following elementary facts will be used repeatedly:

- for every $P \in \text{GL}_n(\mathbb{K})$, the conjugate subalgebra $PAP^{-1}$ has the same dimension as $\mathcal{A}$ and its centralizer $P \mathcal{C}(\mathcal{A}) P^{-1}$ is trivial if and only if $\mathcal{C}(\mathcal{A})$ is trivial.
- the transposed subalgebra $\mathcal{A}^t := \{ M^t \mid M \in \mathcal{A} \}$ has the same dimension as $\mathcal{A}$ and its centralizer $\mathcal{C}(\mathcal{A})^t$ is trivial if and only if $\mathcal{C}(\mathcal{A})$ is trivial.

3.2 Reduction to the case of an algebraically closed field

Let $\mathbb{L}$ be a field extension of $\mathbb{K}$. Recall that when $\mathcal{A}$ is a subalgebra of $M_n(\mathbb{K})$ and we let $\mathcal{A}_\mathbb{L}$ denote the linear $\mathbb{L}$-subspace of $M_n(\mathbb{L})$ generated by $\mathcal{A}$, then
\(A_L\) is a subalgebra of \(M_n(L)\). The natural isomorphism of \(L\)-algebras \(M_n(L) \simeq M_n(K) \otimes_K L\) maps \(A_L\) to \(A \otimes_K L\) hence \(\dim_L A_L = \dim_K A\). Also, the centralizer of \(A \otimes_K L\) in \(M_n(K) \otimes_K L\) is clearly \(C(A) \otimes_K L\), therefore \(A\) has a trivial centralizer in \(M_n(K)\) if and only if \(A_L\) has a trivial centralizer in \(M_n(L)\). We deduce that 
\[t_n(K) \geq t_n(L)\.

Therefore, by Steinitz’s theorem and the examples discussed earlier, it will suffice to prove theorem 2 when \(K\) is algebraically closed.

In the rest of this section, we assume \(K\) is algebraically closed.

### 3.3 The case \(A\) is unispectral

**Definition 2.** We call a matrix \(A \in M_n(K)\) unispectral when it has a sole eigenvalue.

A subalgebra of \(M_n(K)\) is called unispectral when all its elements are unispectral.

The standard example is the subalgebra of matrices of the form \(\lambda I_n + T\) for some strictly upper triangular matrix \(T\). Conversely, every unispectral subalgebra is conjugate to a subalgebra of the preceding one:

**Proposition 7.** Let \(A\) be a unispectral subalgebra of \(M_n(K)\). Then there is a non-singular \(P \in GL_n(K)\) such that \(PAP^{-1} \subset T_n(K)\).

**Proof.** We use Burnside’s theorem (see [3] p.213) by induction on \(n\). The case \(n = 1\) is trivial. Assume \(n \geq 2\) and our claim holds for every non-negative integer \(p < n\) and every unispectral subalgebra of \(M_p(K)\). Let \(A\) be a unispectral subalgebra of \(M_n(K)\). Clearly, \(A \subseteq M_n(K)\) hence Burnside’s theorem shows there is a non-singular \(P \in GL_n(K)\) and an integer \(p \in [1, n - 1]\) such that every \(M \in A\) splits as

\[M = P^{-1} \begin{bmatrix} A(M) & * \\ 0 & B(M) \end{bmatrix} P \quad \text{where } A(M) \in M_p(K) \text{ and } B(M) \in M_{n-p}(K).\]

Then \(A_1 := \{A(M) \mid M \in A\}\) (resp. \(A_2 := \{B(M) \mid M \in A\}\)) is a unispectral subalgebra of \(M_p(K)\) (resp. of \(M_{n-p}(K)\)). Using the induction hypothesis, there are non-singular matrices \(P_1 \in GL_p(K)\) and \(P_2 \in GL_{n-p}(K)\) such that \(P_1A_1P_1^{-1} \subset T_p(K)\) and \(P_2A_2P_2^{-1} \subset T_{n-p}(K)\). Setting \(Q := \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\), we have \(Q \in GL_n(K)\) and \((QP)A(QP)^{-1} \subset T_n(K)\). \[\square\]
Corollary 8. Let \( p \geq 2 \) be an integer. Then every unispectral subalgebra of \( M_p(\mathbb{K}) \) has a non-trivial centralizer.

Proof. It suffices to prove the statement for any unispectral subalgebra \( \mathcal{A} \) of \( T_p(\mathbb{K}) \). However any matrix \( M \) of \( \mathcal{A} \) must have identical diagonal entries and must be upper triangular: it easily follows that the elementary matrix \( E_{1,n} \) lies in \( C(\mathcal{A}) \).

In what follows, we will assume \( \mathcal{A} \) is not unispectral.

3.4 The basic splitting

Let us choose a non-unispectral matrix \( M \in \mathcal{A} \). Choose then a spectral projection \( P \) associated to \( M \): hence \( P \in \mathbb{K}[M] \subset \mathcal{A} \) (see [1] chapter 8, 4 corollary 3 p.271) and we have therefore found a non-trivial idempotent in \( \mathcal{A} \). By conjugating \( \mathcal{A} \) with an appropriate non-singular matrix, we are reduced to the case \( \mathcal{A} \) contains, for some \( p \in [1, n-1] \), the idempotent matrix

\[
P := \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \in M_n(\mathbb{K}).
\]

Remark 2. The rest of the proof will only rely on this assumption and the condition on the dimension of \( \mathcal{A} \). In particular, the reader will check that it does not use the fact that \( \mathbb{K} \) is algebraically closed.

In what follows, we set \( q := n - p \). Notice then that \( \mathcal{A} \) also contains \( Q := I_n - P = \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix} \). Since \( \mathcal{A} \) is a subalgebra containing both \( P \) and \( Q \), one clearly has:

\[
P \mathcal{A} P = \left\{ \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \mid M \in M_p(\mathbb{K}) \right\} \cap \mathcal{A}, \quad P \mathcal{A} Q = \left\{ \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \mid M \in M_{p,q}(\mathbb{K}) \right\} \cap \mathcal{A},
\]

\[
Q \mathcal{A} P = \left\{ \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \mid M \in M_{q,p}(\mathbb{K}) \right\} \cap \mathcal{A}, \quad Q \mathcal{A} Q = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \mid M \in M_q(\mathbb{K}) \right\} \cap \mathcal{A}.
\]

Those four linear subspaces of \( \mathcal{A} \) will respectively be denoted by \( \mathcal{A}_{1,1}, \mathcal{A}_{1,2}, \mathcal{A}_{2,1} \) and \( \mathcal{A}_{2,2} \). For every \( M \in M_n(\mathbb{K}) \), write:

\[
M = I_n M I_n = (P + Q) M (P + Q) = P M P + P M Q + Q M P + Q M Q.
\]
It follows that
\[ \mathcal{A} = \mathcal{A}_{1,1} \oplus \mathcal{A}_{1,2} \oplus \mathcal{A}_{2,1} \oplus \mathcal{A}_{2,2} \]
hence
\[ \dim \mathcal{A} = \dim \mathcal{A}_{1,1} + \dim \mathcal{A}_{1,2} + \dim \mathcal{A}_{2,1} + \dim \mathcal{A}_{2,2}. \]
Notice also that \( P \in \mathcal{A}_{1,1} \) and \( Q \in \mathcal{A}_{2,2} \), so that
\[ \dim \mathcal{A}_{1,1} \geq 1 \quad \text{and} \quad \dim \mathcal{A}_{2,2} \geq 1. \]
In what follows, we will discuss various cases depending on the respective dimensions of the \( \mathcal{A}_{i,j} \)'s. One case can readily be done away: if \( \mathcal{A}_{1,2} = \{0\} \) and \( \mathcal{A}_{2,1} = \{0\} \), then \( P \) belongs to \( \mathcal{C}(\mathcal{A}) \setminus \text{Span}(I_n) \).

We will now assume \( \mathcal{A}_{1,2} \neq \{0\} \) or \( \mathcal{A}_{2,1} \neq \{0\} \).

### 3.5 The case \( \dim \mathcal{A} = 3 \)

It only remains to investigate the case one of \( \mathcal{A}_{2,1} \) and \( \mathcal{A}_{1,2} \) has dimension 1 and the other 0. Transposition of \( \mathcal{A} \) only leaves us with the case \( \mathcal{A}_{2,1} = \{0\} \) and \( \mathcal{A}_{1,2} \) is generated by some \( C \in \mathbb{M}_{p,q}(\mathbb{K}) \setminus \{0\} \). Hence \( \mathcal{A}_{1,1} = \text{Span}(I_p) \) and \( \mathcal{A}_{2,2} = \text{Span}(I_q) \), so the considerations of Section 2 show that, in order to prove that \( \mathcal{C}(\mathcal{A}) \) is non-trivial, it will suffice to prove that, for some pair \( (X,Y) \in \mathbb{M}_p(\mathbb{K}) \times \mathbb{M}_q(\mathbb{K}) \), one has \( XC = CY \) and \( \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \) is not a scalar multiple of \( I_n \). Consider then the linear map:
\[
\begin{aligned}
f : \mathbb{M}_p(\mathbb{K}) \times \mathbb{M}_q(\mathbb{K}) & \to \mathbb{M}_{p,q}(\mathbb{K}) \\
(X,Y) & \mapsto XC - CY.
\end{aligned}
\]
By the rank theorem:
\[ \dim \text{Ker} f \geq \dim \left( \mathbb{M}_p(\mathbb{K}) \times \mathbb{M}_q(\mathbb{K}) \right) - \dim \mathbb{M}_{p,q}(\mathbb{K}) = p^2 + q^2 - pq = (p-q)^2 + pq \geq pq. \]
Since \( n \geq 3 \) and \( p \in [1, n-1] \), one has \( pq \geq 2 \), which shows that \( \text{Ker} f \neq \text{Span}((I_p, I_q)) \). Therefore, \( \mathcal{C}(\mathcal{A}) \) is non-trivial.
3.6 The case $\dim \mathcal{A} = 4$

We now assume $\dim \mathcal{A} = 4$ and set

$$\nu(\mathcal{A}) = (\dim \mathcal{A}_{1,1}, \dim \mathcal{A}_{1,2}, \dim \mathcal{A}_{2,1}, \dim \mathcal{A}_{2,2}).$$

Transposition and conjugation by a permutation matrix help us reduce the situation to only three cases:

- $\nu(\mathcal{A}) = (2, 1, 0, 1);$  
- $\nu(\mathcal{A}) = (1, 1, 1, 1);$  
- $\nu(\mathcal{A}) = (1, 2, 0, 1).$

In the first two cases, we will show that $\mathcal{C}(\mathcal{A})$ is non-trivial, even when $n$ is odd. In the last case, we will show that $\mathcal{C}(\mathcal{A})$ is non-trivial when $n$ is even.

3.6.1 The case $\nu(\mathcal{A}) = (2, 1, 0, 1)$.

In this case, there is some $C \in M_p(\mathbb{K}) \setminus \text{Span}(I_p)$ and some $V \in M_{p,q}(\mathbb{K}) \setminus \{0\}$ such that $\mathcal{A}$ is generated by the matrices

$$\begin{bmatrix} I_p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}.$$

Since the product of the last two matrices belongs to $\mathcal{A}$, one must have $C V = \lambda V$ for some $\lambda \in \mathbb{K}$. Replacing $C$ with $C - \lambda I_p$, we may assume $C V = 0$, in which case the matrix $\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}$ clearly belongs to $\mathcal{C}(\mathcal{A}) \setminus \text{Span}(I_n)$.

3.6.2 The case $\nu(\mathcal{A}) = (1, 1, 1, 1)$.

In this case, there are matrices $U \in M_{q,p}(\mathbb{K}) \setminus \{0\}$ and $V \in M_{p,q}(\mathbb{K}) \setminus \{0\}$ such that $\mathcal{A}$ is generated by the four matrices

$$\begin{bmatrix} I_p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ U & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}.$$

A matrix belongs to $\mathcal{C}(\mathcal{A})$ if and only if it has the form $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$, where $(X, Y) \in M_p(\mathbb{K}) \times M_q(\mathbb{K})$ satisfies $UX = YU$ and $XV = VY$.
One obvious element in this centralizer is $\begin{bmatrix} VU & 0 \\ 0 & UV \end{bmatrix}$. Assume now that $C(A)$ is trivial. There would then exist some $\lambda \in \mathbb{K}$ such that $VU = \lambda I_p$ and $UV = \lambda I_q$.

Two situations may arise:

- The case $\lambda \neq 0$. Replacing $V$ by $\frac{1}{\lambda} V$, we may assume $\lambda = 1$. Then standard rank considerations show that $p = q$ and $V = U^{-1}$. Equality $XV = VY$ thus implies $UX = YU$, hence $A$ has the same centralizer as the subalgebra generated by $P$, $Q$ and $\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$, which has been shown to be non-trivial in Section 3.5. There lies a contradiction.

- The case $\lambda = 0$. Then $UV = 0$ and $VU = 0$. Since neither $U$ nor $V$ is zero, we deduce that $\text{Ker} U$ and $\text{Im} V$ are non-trivial subspaces of $\mathbb{K}^p$ and $\text{Ker} V$ and $\text{Im} U$ are non-trivial subspaces of $\mathbb{K}^q$. We can therefore construct rank 1 matrices $X \in M_p(\mathbb{K})$ and $Y \in M_q(\mathbb{K})$ such that $A$ is generated by $\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$. The considerations of Section 2 show that the kernel of $g : M_p(\mathbb{K}) \times M_q(\mathbb{K}) \rightarrow M_{p,q}(\mathbb{K})^2$ \((X, Y) \mapsto (XU - UY, XV - VY)\) has a dimension greater than 1 if and only if $C(A)$ is non-trivial.

**Lemma 9.** With the preceding notations, for to have $\dim \text{Ker} g = 1$, it is necessary that:

\section*{3.6.3 The case $\nu(A) = (1, 2, 0, 1)$}

We actually lose no generality assuming $p \leq q$ (if not, we simply replace $A$ with $A^t$ and conjugate it by an appropriate permutation matrix). We then find linearly independent matrices $U$ and $V$ in $M_{p,q}(\mathbb{K})$ such that $A$ is generated by $\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$.
• either $q = p$ and at least one of the matrices $U$ or $V$ has rank $p$;
• or $q = p + 1$ and both matrices $U$ and $V$ have rank $p$.

In order to prove this lemma, we will start with another one:

**Lemma 10.** Let $W \in M_{p,q}(\mathbb{K})$ be such that $\text{rk} W < p \leq q$. Then the linear map

$$f : \begin{cases} M_p(\mathbb{K}) \times M_q(\mathbb{K}) \rightarrow M_{p,q}(\mathbb{K}) \\ (X,Y) \mapsto XW - WY \end{cases}$$

is not onto.

**Proof.** Notice that $\text{rk} f$ is unchanged by replacing $W$ with an equivalent matrix: for every $(P_1,Q_1) \in \text{GL}_p(\mathbb{K}) \times \text{GL}_q(\mathbb{K})$ and $(X,Y) \in M_p(\mathbb{K}) \times M_q(\mathbb{K})$, one can indeed write:

$$XP_1 WQ_1 - P_1 WQ_1 Y = P_1 \left[ (P_1^{-1} XP_1) W - W(Q_1 Y Q_1^{-1}) \right] Q_1.$$

Letting $r := \text{rk} W$, we then lose no generality assuming that $W = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. In this case however, every matrix of $\text{Im} f$ has a zero entry in position $(p,q)$, hence $f$ is not onto.

**Proof of Lemma 9.** Setting $f_1 : (X,Y) \mapsto XU - UY$ and $f_2 : (X,Y) \mapsto XV - VY$, we find that $\text{rk} g \leq \text{rk} f_1 + \text{rk} f_2$ whilst Lemma 10 shows that $\text{rk} f_1 + \text{rk} f_2 \leq 2 M_{p,q}(\mathbb{K}) - m$ where $m$ is the number of matrices in $\{U,V\}$ which have rank lesser than $p$. We deduce that $\text{rk} g \leq 2pq - m$. The rank theorem then shows that

$$\dim \ker g \geq p^2 + q^2 - 2pq + m = (q - p)^2 + m.$$

If $q \geq p + 2$, then $\dim \ker g \geq 4$. If $q = p + 1$ and $m \geq 1$, then $\dim \ker g \geq 2$. If $q = p$ and $m = 2$, then $\dim \ker g \geq 2$. This proves the claimed results.

We may now complete the proof of Theorem 2. Assume $n$ is even. Then Lemma 9 shows $\mathcal{C}(A)$ is non-trivial unless $p = q = \frac{n}{2}$ and one of the matrices $U$ and $V$ is non-singular.

Assume then $p = q = \frac{n}{2}$ and $U$ is non-singular (for example). Conjugating $A$
by \( \begin{bmatrix} I_p & 0 \\ 0 & U \end{bmatrix} \), we are then reduced to the case \( U = I_p \): in this case \( V \notin \text{Span}(I_p) \) and the matrix \( \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} \) clearly belongs to \( \mathcal{C}(A) \setminus \text{Span}(I_n) \).

Let us finish this section by summing up the results in the case \( \dim A = 4 \). Recall that we have assumed that \( K \) is algebraically closed or that \( A \) contains a non-trivial idempotent.

(i) If \( n \) is even, then \( \mathcal{C}(A) \) is non-trivial.

(ii) If \( n \) is odd and \( \mathcal{C}(A) \) is trivial, then, setting \( p := \frac{n+1}{2} \), there are linearly independent matrices \( U \) and \( V \) in \( M_{p,p+1}(K) \) with rank \( p \) and a non-singular matrix \( P \in \text{GL}_n(K) \) such that either

\[
PAP^{-1} = \left\{ \begin{bmatrix} a.I_p & c.U + d.V \\ 0 & b.I_{p+1} \end{bmatrix} \mid (a, b, c, d) \in \mathbb{K}^4 \right\}
\]

or

\[
PAP^{-1} = \left\{ \begin{bmatrix} a.I_p & c.U + d.V \\ 0 & b.I_{p+1} \end{bmatrix} \mid (a, b, c, d) \in \mathbb{K}^4 \right\}.
\]

This of course completes the proof of Theorem 2.

4 On subalgebras of dimension 4 of \( M_{2n+1}(K) \) with a trivial centralizer

Here, we establish Proposition 5 by prolonging the proof of Section 3.6 in the case \( \dim A = 4 \). We must first return to the situation where \( K \) is an arbitrary field.

**Lemma 11.** Let \( n \in \mathbb{N}^* \) and \( A \) be a 4-dimensional subalgebra of \( M_{2n+1}(K) \) with a trivial centralizer. Then \( A \) contains a rank \( n \) idempotent.

**Proof.** Choose an algebraically closed extension \( L \) of \( K \). Then \( A_L \) has a trivial centralizer in \( M_{2n+1}(L) \). The proof in Sections 3.2, 3.3, 3.4 and 3.6 then shows that there is a 2-dimensional subspace \( P \subset M_{n,n+1}(K) \) such that \( A_L \) is conjugate to either the subalgebra

\[
\mathcal{H} := \left\{ \begin{bmatrix} a.I_n & M \\ 0 & b.I_{n+1} \end{bmatrix} \mid (a, b) \in \mathbb{L}^2, \ M \in P \right\}
\]
or its transpose $\mathcal{H}^t$. In any case, the set of unispectral elements in $\mathcal{A}_L$ is a linear hyperplane of $\mathcal{A}_L$: this is the case indeed when $\mathcal{A}_L = \mathcal{H}$ since this subset is then

$$\left\{ \begin{bmatrix} a.I_n & M \\ 0 & a.I_{n+1} \end{bmatrix} \mid a \in \mathbb{L}, M \in P \right\}.$$  

Also, every non-unispectral element of $\mathcal{A}_L$ is clearly diagonalisable with exactly two eigenvalues of respective orders $n$ and $n+1$.

Every basis of the $\mathbb{K}$-vector space $\mathcal{A}$ is also a basis of the $\mathbb{L}$-vector space $\mathcal{A}_L$ and therefore must contain a matrix which is not unispectral in $M_n(\mathbb{L})$. Let us choose such a matrix $M \in \mathcal{A}$, with eigenvalues $\lambda$ and $\mu$ of respective orders $n$ and $n+1$. Notice then that $\lambda$ and $\mu$ belong to $\mathbb{K}$. Indeed:

- the minimal polynomial of $M$ is $X^2 - (\lambda + \mu)X + \lambda \mu$, so $\lambda + \mu \in \mathbb{K}$ (implicit here is the fact that the minimal polynomial of a matrix is unchanged by extending the field of scalars);
- also $\text{tr}(M) = n(\lambda + \mu) + \mu$, which entails $\mu \in \mathbb{K}$ and therefore $\lambda \in \mathbb{K}$.

We deduce that $\frac{1}{\lambda - \mu} (M - \mu.I_{2n+1})$ is a rank $n$ idempotent in $\mathcal{A}$. \hfill \Box

Proposition 5 can now be proven. Since $\mathcal{A}$ contains an idempotent of rank $n$, the proof from Sections 3.4 and 3.6 shows that we can reduce the study to the situation where there is a 2-dimensional subspace $\mathcal{P} \subset M_{n,n+1}(\mathbb{K})$ such that

$$\mathcal{A} = \left\{ \begin{bmatrix} a.I_n & M \\ 0 & b.I_{n+1} \end{bmatrix} \mid (a, b) \in \mathbb{K}^2, M \in \mathcal{P} \right\}.$$  

Let $U \in \mathcal{P} \setminus \{0\}$ and choose $V$ such that $(U, V)$ is a basis of $\mathcal{P}$. Then Lemma 11 shows that $U$ (and $V$) must have rank $n$. For every extension $\mathbb{L}$ of $\mathbb{K}$, the subalgebra $\mathcal{A}_L$ has a trivial centralizer, which shows $\text{rk} U = n$ for every $U \in P_\mathbb{L} \setminus \{0\}$. We will then use the following result to see that $\mathcal{P}$ is equivalent to the 2-dimensional subspace $\text{Span}(C_n, D_n)$ i.e. $\mathcal{P} = P \text{ Span}(C_n, D_n) Q$ for some pair $(P, Q) \in \text{GL}_n(\mathbb{K}) \times \text{GL}_{n+1}(\mathbb{K})$:

**Proposition 12.** Let $A$ and $B$ in $M_{n,n+1}(\mathbb{K})$. Assume that every non-trivial linear combination of $A$ and $B$ over any field extension of $\mathbb{K}$ has rank $n$. Then there is a pair $(P, Q) \in \text{GL}_n(\mathbb{K}) \times \text{GL}_{n+1}(\mathbb{K})$ such that $A = P C_n Q^{-1}$ and $B = P D_n Q^{-1}$.

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Assuming this to be true, consider a basis \((A, B)\) of \(P\) and a pair \((P, Q)\) associated to it as in Proposition 12. Then a straightforward computation shows that 
\[
R \mathcal{H}_{2n+1}(K) R^{-1} = A \quad \text{for} \quad R = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix},
\]
which completes the proof of Theorem 5.

**Proof of Proposition 12.** We let \(x\) denote an indeterminate. We use the Kronecker-Weierstrass reduction for pencils of matrices (see chapter XII of [2] and the appendix of [4] for the case of an arbitrary field). Since \(A\) and \(B\) have rank \(n\) and belong to \(M_{n,n+1}(K)\), the canonical form of the pencil \(A + xB\) cannot contain any block of the forms
\[
\begin{bmatrix}
x & 0 \\
1 & x & 0 \\
0 & 1 & x \\
\vdots & \ddots & \ddots \\
& & & x
\end{bmatrix} \in M_{p+1,p}(K[x]) ;
\]
or
\[
\begin{bmatrix}
x & 1 & 0 \\
0 & x & 1 \\
\vdots & \ddots & \ddots \\
& & & 1 & x
\end{bmatrix} \in M_{p}(K[x])
\]
and contains at most one block of the form
\[
L_p := \begin{bmatrix}
1 & x & 0 \\
0 & 1 & x \\
\vdots & \ddots & \ddots \\
1 & \cdots & x
\end{bmatrix} \in M_{p,p+1}(K[x]).
\]

It follows that there exists some \(p \in \{0, n-1\}\), some non-singular \(C \in GL_p(K)\) and some pair \((P, Q) \in GL_n(K) \times GL_{n+1}(K)\) such that
\[
P^{-1} (A + xB) Q = \begin{bmatrix} C + xI_p & 0 \\ 0 & L_{n-p} \end{bmatrix}.
\]
However, if \( p > 0 \), then \( \operatorname{rk}(A - \lambda B) < n \) for any eigenvalue \( \lambda \) of \( C \), which contradicts the assumptions. It follows that \( p = 0 \), hence \( P^{-1} (A + x B) Q = L_n = C_n + x D_n \), which shows \( A = P C_n Q^{-1} \) and \( B = P D_n Q^{-1} \).

\[ \square \]

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