HADAMARD GAP SERIES IN WEIGHTED-TYPE SPACES ON THE UNIT BALL

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Abstract. We give a sufficient and necessary condition for an analytic function \( f(z) \) on the unit ball \( \mathbb{B} \) in \( \mathbb{C}^n \) with Hadamard gaps, that is, for \( f(z) = \sum_{k=1}^{\infty} P_{n_k}(z) \) where \( P_{n_k}(z) \) is a homogeneous polynomial of degree \( n_k \) and \( n_k+1/n_k \geq c > 1 \) for all \( k \in \mathbb{N} \), to belong to the weighted-type space \( H^\infty_\mu \) and the corresponding little weighted-type space \( H^\infty_{\mu,0} \), under some condition posed on the weighted function \( \mu \). We also study the growth rate of those functions in \( H^\infty_\mu \). Finally, we characterize the boundedness and compactness of weighted composition operator from weighted-type space \( H^\infty_\mu \) to mixed norm spaces.

1. Introduction

Let \( \mathbb{B} \) be the open unit ball in \( \mathbb{C}^n \) with \( \mathbb{S} \) as its boundary and \( H(\mathbb{B}) \) the collection of all holomorphic functions in \( \mathbb{B} \). \( H^\infty(\mathbb{B}) \) denotes the Banach space consisting of all bounded holomorphic functions in \( \mathbb{B} \) with the norm \( \|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)| \).

A positive continuous function \( \mu \) on \([0,1)\) is called normal if there exists positive numbers \( \alpha \) and \( \beta \), \( 0 < \alpha < \beta \), and \( \delta \in (0,1) \) such that (see, e.g., [9])

\[
\frac{\mu(r)}{(1-r)^\alpha} \text{ is decreasing on } [\delta, 1), \quad \lim_{r \to 1^-} \frac{\mu(r)}{(1-r)^\alpha} = 0,
\]

\[
\frac{\mu(r)}{(1-r)^\beta} \text{ is increasing on } [\delta, 1), \quad \lim_{r \to 1^-} \frac{\mu(r)}{(1-r)^\beta} = \infty.
\]

Note that a normal function \( \mu : [0,1) \to [0,\infty) \) is decreasing in a neighborhood of 1 and satisfies \( \lim_{r \to 1^-} \mu(r) = 0 \).

Date: September 1, 2015.

2010 Mathematics Subject Classification. 32A05, 32A37, 47B33.

Key words and phrases. Weighted-type space, Hadamard gaps, weighted composition operator, mixed norm space.
An \( f \in H(B) \) is said to belong to the weighted-type space, denoted by \( H^\infty_\mu = H^\infty_\mu(B) \) if
\[
\| f \| = \sup_{z \in B} \mu(|z|)|f(z)| < \infty,
\]
where \( \mu \) is normal on \([0, 1)\) (see, e.g. [11]). It is well-known that \( H^\infty_\mu \) is a Banach space with the norm \( \| \cdot \| \).

The little weighted-type space, denoted by \( H^{\infty,0}_\mu \), is the closed subspace of \( H^\infty_\mu \) consisting of those \( f \in H^\infty_\mu \) such that
\[
\lim_{|z| \to 1^-} \mu(|z|)|f(z)| = 0.
\]
When \( \mu(|z|) = (1 - |z|^2)^\alpha, \alpha > 0 \), the induced spaces \( H^\infty_\mu \) and \( H^{\infty,0}_\mu \) become the Bers-type space and little Bers-type space respectively.

Let \( \phi \) be a normal function on \([0, 1)\). For \( 0 < p, q < \infty \), the mixed-norm space \( H^{(p,q,\phi)}(B) = H^{(p,q,\phi)}(B) \) is the space consisting of all \( f \in H(B) \) such that
\[
\| f \|_{H^{(p,q,\phi)}} = \left( \int_0^1 M_q^p(f, r) \frac{\phi^p(r)}{1 - r} dr \right)^{1/p} < \infty,
\]
where
\[
M_q(f, r) = \left( \int_S |f(r\zeta)|^q d\sigma(\zeta) \right)^{1/q},
\]
and \( \sigma \) is the normalized area measure on \( S \).

Let \( \varphi \) be a holomorphic self-map of \( B \) and \( u \in H(B) \). For \( f \in H(B) \), the \text{weighted composition operator} \( uC_{\varphi} \) is defined by
\[
(uC_{\varphi}f)(z) = u(z)f(\varphi(z)), \quad z \in B.
\]
The weighted composition operator can be regarded as a generalization of the \text{multiplication operator} and the \text{composition operator}, which are defined by \( M_u(f) = (uf)(z) \) and \( (C_{\varphi}f)(z) = f(\varphi(z)) \), respectively. See [4] for more information on this topic.

We say that an \( f \in H(B) \) has the \text{Hadamard gaps} if
\[
f(z) = \sum_{k=0}^{\infty} P_{n_k}(z),
\]
where \( P_{n_k} \) is a homogeneous polynomial of degree \( n_k \) and there exists some \( c > 1 \) (see, e.g., [10]),
\[
\frac{n_{k+1}}{n_k} \geq c, \; \forall k \geq 0.
\]

Hadamard gap series on spaces of holomorphic functions in the unit disc \( D \) or in the unit ball \( B \) has been studied quite well. We refer the
readers to the related results in [2, 5, 6, 10, 11, 13, 14, 15, 16, 17] and the reference therein.

In [15], the authors studied the Hadamard gap series and the growth rate of the functions in \( H^\infty_\mu \) in the unit disk. Motivated by [15], the aim of this paper is to study the Hadamard gap series in \( H^\infty_\mu \), as well as its little space \( H^\infty_{\mu,0} \) on the unit ball. Moreover, as an application of our main result, we characterize the growth rate of those functions in \( H^\infty_\mu \). Finally, we give some sufficient and necessary conditions for the boundedness and compactness of weighted composition operators from weighted-type space \( H^\infty_\mu \) to mixed norm spaces.

Throughout this paper, for \( a, b \in \mathbb{R} \), \( a \lesssim b \) (\( a \gtrsim b \), respectively) means there exists a positive number \( C \), which is independent of \( a \) and \( b \), such that \( a \leq Cb \) (\( a \geq Cb \), respectively). Moreover, if both \( a \lesssim b \) and \( a \gtrsim b \) hold, then we say \( a \simeq b \).

2. Hadamard gap series in \( H^\infty_\mu \) and \( H^\infty_{\mu,0} \)

Let \( f(z) = \sum_{k=0}^{\infty} P_k(z) \) be a holomorphic function in \( \mathbb{B} \), where \( P_k(z) \) is a homogeneous polynomial with degree \( k \). For \( k \geq 0 \), we denote

\[
M_k = \sup_{\xi \in \mathbb{S}} |P_k(\xi)|.
\]

We have the following estimations on \( M_k \) of a holomorphic function \( f \in H^\infty_\mu \) (or \( f \in H^\infty_{\mu,0} \), respectively).

**Theorem 2.1.** Let \( \mu \) be a normal function on \([0,1)\). Let \( f(z) = \sum_{k=0}^{\infty} P_k(z), z \in \mathbb{B} \). Then the following statements hold.

1. If \( f \in H^\infty_\mu \), then \( \sup_{k \geq 0} M_k \mu (1 - \frac{1}{k}) < \infty \).
2. If \( f \in H^\infty_{\mu,0} \), then \( \lim_{k \to \infty} M_k \mu (1 - \frac{1}{k}) = 0 \).

**Proof.** (1). Suppose \( f \in H^\infty_\mu \). Fix a \( \xi \in \mathbb{S} \) and denote

\[
f_\xi(w) = \sum_{k=0}^{\infty} P_k(\xi) w^k = \sum_{k=0}^{\infty} P_k(\xi w), \ w \in \mathbb{D}.
\]

Since \( f \in H(\mathbb{B}) \), it is known that for a fixed \( \xi \in \mathbb{S} \), \( f_\xi(w) \) is holomorphic in \( \mathbb{D} \) (see, e.g., [7]). Hence, for any \( r \in (0,1) \), we have
\begin{align}
M_k &= \sup_{\xi \in \mathbb{S}} |P_k(\xi)| = \sup_{\xi \in \mathbb{S}} \left| \frac{1}{2\pi i} \int_{|w|=r} \frac{f(\xi w)}{w^{k+1}} \, dw \right| \\
&= \frac{1}{2\pi} \sup_{\xi \in \mathbb{S}} \left| \int_{|w|=r} \frac{f(\xi w)}{r^{k+1}} \, dw \right| \\
&\leq \frac{1}{2\pi} \sup_{\xi \in \mathbb{S}} \int_{|w|=r} \frac{|f(\xi w)|}{r^{k+1}} \, dw \\
&= \frac{1}{2\pi} \sup_{\xi \in \mathbb{S}} \int_{|w|=r} \frac{|f(\xi w)| \mu(|\xi w|)}{r^{k+1} \mu(r)} \, dw \\
&\leq \frac{\|f\|}{r^k \mu(r)}.
\end{align}

In (2.1), letting \( r = 1 - \frac{1}{k}, k \geq 2, k \in \mathbb{N} \), we have

\[ M_k \leq \frac{\|f\|}{(1 - \frac{1}{k})^k \mu(1 - \frac{1}{k})}. \]

Thus, for each \( k \geq 2 \),

\[ M_k \mu \left(1 - \frac{1}{k}\right) \leq \frac{\|f\|}{(1 - \frac{1}{k})^k} \leq 4\|f\|, \]

which implies that

\[ \sup_{k \geq 1} M_k \mu \left(1 - \frac{1}{k}\right) \leq \max \{\mu(0), 4\|f\|\} < \infty. \]

(2). Suppose \( f \in H^\infty_{\mu,0} \), that is, for any \( \varepsilon > 0 \), there exists a \( \delta \in (0, 1) \), when \( \delta < |z| < 1 \),

\[ \mu(|z|)|f(z)| < \varepsilon. \]

Take \( N_0 \in \mathbb{N} \) satisfying \( \delta < 1 - \frac{1}{k} < 1 \) when \( k > N_0 \). Then for any \( k > N_0 \) and \( r = 1 - \frac{1}{k} \), as the proof in the previous part, we have

\[ M_k \leq \frac{1}{(1 - \frac{1}{k})^k \mu(1 - \frac{1}{k})} \cdot \sup_{\delta < |z| < 1} \mu(|z|)|f(z)| < \frac{\varepsilon}{(1 - \frac{1}{k})^k \mu(1 - \frac{1}{k})}, \]

which implies

\[ M_k \mu \left(1 - \frac{1}{k}\right) \leq \frac{\varepsilon}{(1 - \frac{1}{k})^k} \leq 4\varepsilon, \quad k > N_0. \]

Hence we have \( \lim_{k \to \infty} M_k \mu \left(1 - \frac{1}{k}\right) = 0. \) \qed
Theorem 2.2. Let $\mu$ be a normal function on $[0,1)$. Let $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$ with Hadamard gaps, where $P_{n_k}$ is a homogeneous polynomial of degree $n_k$. Then the following assertions hold.

1. $f \in H^\infty_\mu$ if and only if $\sup_{k \geq 1} \mu \left( 1 - \frac{1}{n_k} \right) M_{n_k} < \infty$.
2. $f \in H^\infty_{\mu,0}$ if and only if $\lim_{k \to \infty} \mu \left( 1 - \frac{1}{n_k} \right) M_{n_k} = 0$.

Proof. By Theorem 2.1, it suffices to show the sufficiency of both statements.

1. Noting that $|f(z)| = \left| \sum_{k=0}^{\infty} P_{n_k} \left( \frac{z}{|z|} \right) \right| \leq \sum_{k=0}^{\infty} M_{n_k} |z|^{n_k} \leq \sum_{k=0}^{\infty} \mu \left( 1 - \frac{1}{n_k} \right) |z|^{n_k}$, from the proof of [15, Theorem 2.3], we have

$$\frac{|f(z)|}{1 - |z|} \lesssim \sum_{m=1}^{\infty} \left( \sum_{n_k \leq m} \frac{1}{\mu \left( 1 - \frac{1}{n_k} \right)} \right) |z|^m \lesssim \sum_{m=1}^{\infty} \frac{|z|^m}{\mu \left( 1 - \frac{1}{m} \right)} \lesssim \frac{1}{(1 - |z|)\mu(|z|)}$$

which implies $f \in H^\infty_\mu$, as desired.

2. Since $\lim_{k \to \infty} \mu \left( 1 - \frac{1}{n_k} \right) M_{n_k} = 0$, we have $\sup_{k \geq 1} \mu \left( 1 - \frac{1}{n_k} \right) M_{n_k} < \infty$. Hence by part (1), we have $f \in H^\infty_\mu$. For any $\varepsilon > 0$, there exists a $N_0 \in \mathbb{N}$ satisfying when $m > N_0$,

$$M_{n_m} \mu \left( 1 - \frac{1}{n_m} \right) < \varepsilon.$$

For each $m \in \mathbb{N}$, put $f_m(z) = \sum_{k=0}^{m} P_{n_k}(z)$. Note that

$$\mu(|z| |f_m(z)|) \leq \mu(|z|) \left( \sum_{k=0}^{m} |P_{n_k}(z)| \right) \leq \mu(|z|) \sum_{k=0}^{m} \left| P_{n_k} \left( \frac{z}{|z|} \right) |z|^{n_k} \right| \leq K_m \mu(|z|) \sum_{k=0}^{m} |z|^{n_k} \leq mK_m \mu(|z|),$$
where \( K_m = \max\{M_{n_0}, M_{n_1}, M_{n_2}, \ldots, M_{n_m}\} \). Noting that \( \lim_{|z| \to 1^-} \mu(|z|) = 0 \), we have \( \lim_{|z| \to 1^-} \mu(|z|)|f_m(z)| = 0 \), which implies for each \( m \in \mathbb{N} \), \( f_m \in H^\infty_{\mu, 0} \). Hence it suffices to show that \( \| f_m - f \| \to 0 \) as \( m \to \infty \).

Indeed, for \( m > N_0 \), we have

\[
|f_m(z) - f(z)| = \left| \sum_{k=m+1}^{\infty} P_{n_k}(z) \right| \leq \sum_{k=m+1}^{\infty} M_{n_k} |z|^{n_k} \leq \sum_{k=m+1}^{\infty} \frac{|z|^{n_k}}{\mu \left( 1 - \frac{1}{n_k} \right)}.
\]

From this, the result easily follows from the proof of part (1).

\[\Box\]

3. Growth rate

As an application of Theorem 2.2 in this section, we show the following result.

**Theorem 3.1.** Let \( \mu \) be a normal function on \([0, 1)\). Then there exists a positive integer \( M = M(n) \) with the following property: there exists \( f_i \in H^\infty_{\mu}, 1 \leq i \leq M \), such that

\[
\sum_{i=1}^{M} |f_i(z)| \gtrsim \frac{1}{\mu(|z|)}, \quad z \in \mathbb{B}.
\]

Note that the result in [15, Theorem 2.5] in the unit disc is a particular case of Theorem 3.1 when \( n = 1 \).

**Remark 3.2.** We observe that \( M \) cannot be 1. Indeed, assume that there exists a \( f \in H^\infty_{\mu} \), such that

\[
|f(z)| \gtrsim \frac{1}{\mu(|z|)}, \quad z \in \mathbb{B}.
\]

It implies that \( f(z) \) has no zero in \( \mathbb{B} \), and it follows that there exists \( g \in H(\mathbb{B}) \), such that \( f = e^g \). Thus,

\[
|f(z)| = |e^{g(z)}| = e^{\text{Re} g(z)},
\]

which implies that \( e^{\text{Re} g(z)} \gtrsim \frac{1}{\mu(|z|)} \) and hence \( \text{Re} g(z) \gtrsim \log \frac{1}{\mu(|z|)} \). For each \( r \in (0, 1) \), integrating on both sides of the above inequality on \( r\mathbb{S} = \{z \in \mathbb{B}, |z| = r\} \), we have

\[
\int_{r\mathbb{S}} \text{Re} g(z) d\sigma \gtrsim \int_{r\mathbb{S}} \log \left( \frac{1}{\mu(|z|)} \right) d\sigma = \log \left( \frac{1}{\mu(r)} \right) \cdot \sigma(r\mathbb{S}).
\]

By the mean value property, we have \( \text{Re} g(0) \gtrsim \log \left( \frac{1}{\mu(r)} \right), \forall r \in (0, 1) \), which is impossible.
Before we formulate the proof of our main result, we need some preliminary results. In the sequel, for $\xi, \zeta \in S$, denote
\[
d(\xi, \zeta) = (1 - |\langle \xi, \zeta \rangle|^2)^{1/2}.
\]
Then $d$ satisfies the triangle inequality (see, e.g., [1]). Moreover, we write $E_\delta(\zeta)$ for the $d$-ball with radius $\delta \in (0, 1)$ and center at $\zeta \in S$:
\[
E_\delta(\zeta) = \{ \xi \in S : d(\xi, \zeta) < \delta \}.
\]
We say that a subset $\Gamma$ of $S$ is $d$-separated by $\delta > 0$, if $d$-balls with radius $\delta$ and center at points of $\Gamma$ are pairwise disjoint.

We begin with several lemmas, which play important role in the proof of our main result.

**Lemma 3.3.** [3, 12] For each $a > 0$, there exists a positive integer $M = M_n(a)$ with the following property: if $\delta > 0$, and if $\Gamma \subset S$ is $d$-separated by $a\delta$, then $\Gamma$ can be decomposed into $\Gamma = \bigcup_{j=1}^M \Gamma_j$ in such a way that each $\Gamma_j$ is $d$-separated by $\delta$.

**Lemma 3.4.** [3, Lemma 2.3] Suppose that $\Gamma \subset S$ is $d$-separated by $\delta$ and let $k$ be a positive integer. If
\[
P(z) = \sum_{\zeta \in \Gamma} \langle z, \zeta \rangle^k, \quad z \in \mathbb{B},
\]
then
\[
|P(z)| \leq 1 + \sum_{m=1}^\infty (m + 2)^{2n-2} e^{-m^2 \delta^2 k/2}.
\]

**Proof of Theorem 3.1.** We will prove the theorem by constructing $f_i \in H^\infty_\mu$ satisfying the given property only near the boundary (then, by adding a proper constant, one obtains the given property on all of the unit ball). Since $\mu$ is normal, by the definition of normal function, there exists positive numbers $\alpha, \beta$ with $0 < \alpha < \beta$, and $\delta \in (0, 1)$ satisfy (1.1). Take and fix some small positive number $A < 1$ such that
\[
\sum_{m=0}^\infty (m + 2)^{2n-2} e^{-m^2 \frac{\alpha^2}{2A^2}} \leq \frac{1}{27}.
\]
Let $M = M_n \left( \frac{A}{2} \right)$ be a positive integer provided by Lemma 3.3 with $A/2$ in place of $a$. Let $p$ be a sufficiently large positive integer so that
\[
1 - \frac{1}{p} \geq \delta,
\]
For each positive integer $j \leq M$, set $\delta_{j,0}$ such that
\[
(3.6) \quad A^2 p^j \delta^2_{j,0} = 1
\]
and inductively choose $\delta_{j,v}$ such that
\[
(3.7) \quad p^M \delta^2_{j,v} = \delta^2_{j,v-1}, \quad v = 1, 2, \ldots
\]
From (3.6) and (3.7), we get
\[
(3.8) \quad A^2 p^{vM+j} \delta^2_{j,v} = 1.
\]
For each fixed $j$ and $v$, let $\Gamma_{j,v}$ be a maximal subset of $S$ subject to the condition that $\Gamma_{j,v}$ is $d$-separated by $A\delta_{j,v}/2$. Then by Lemma 3.3
write
\[
(3.9) \quad \Gamma_{j,v} = \bigcup_{l=1}^{M} \Gamma_{j,vM+l}
\]
in such a way that each $\Gamma_{j,vM+l}$ is $d$-separated by $\delta_{j,v}$.
For each $i, j = 1, 2, \ldots, M$ and $v \geq 0$, set
\[
P_{i,vM+j}(z) = \sum_{\xi \in \Gamma_{j,vM+\tau^i(j)}} \langle z, \xi \rangle p^{vM+j},
\]
where $\tau^i$ is the $i$th iteration of the permutation $\tau$ on $\{1, 2, \ldots, M\}$ defined by
\[
\tau(j) = \begin{cases} j+1, & j < M; \\ 1, & j = M. \end{cases}
\]
By (3.8), Lemma 3.4 and (3.1), we get that
\[
(3.10) \quad |P_{i,vM+j}(z)| \leq 1 + \sum_{m=1}^{\infty} (m + 2)^{2n-2} e^{-m^2 \delta^2_{j,v} p^{vM+j}/2} \leq 1 + \sum_{m=1}^{\infty} (m + 2)^{2n-2} e^{-\frac{m^2}{2\lambda^2}} \leq 2, \quad z \in \mathbb{B},
\]
for all $i, j = 1, 2, \ldots, M$ and $v \geq 0$. 

Define

\[ g_{i,j}(z) = \sum_{v=0}^{\infty} \frac{P_{i,vM+j}(z)}{\mu \left( 1 - \frac{1}{p^{vM+j}} \right)}, \quad z \in \mathbb{B}. \]

By Theorem 2.2, it is clear that for each \( i, j \in \{1, 2, \ldots, M\} \), \( g_{i,j} \in H^\infty \).

We will show that for every \( v \geq 0, 1 \leq j \leq M \) and \( z \in \mathbb{B} \) with that

(3.11) \[ 1 - \frac{1}{p^{vM+j}} \leq |z| \leq 1 - \frac{1}{p^{vM+j} + \frac{1}{2}}, \]

there exists an \( i \in \{1, 2, \ldots, M\} \) such that

\[ |g_{i,j}(z)| \geq \frac{C}{\mu(|z|)}, \]

where \( C \) is some constant independent of the choice of \( i, j \) and \( z \).

Fix \( v, j \) and \( z \) for which (3.11) holds. Let \( z = |z|\eta \) where \( \eta \in \mathbb{S} \).

Since \( d \)-balls with radius \( A\delta_{j,v} \) and centers at points of \( \Gamma^{j,v} \) cover \( \mathbb{S} \) by maximality, there exists some \( \zeta \in \Gamma^{j,v} \) such that

\[ \eta \in E_{A\delta_{j,v}}(\zeta). \]

Note that \( \zeta \in \Gamma^{j,vM+l} \) for some \( 1 \leq l \leq M \) by (3.9) and hence \( \zeta \in \Gamma^{j,vM+j+l}(j) \) for some \( 1 \leq i \leq M \).

We now estimate \( |g_{i,j}(z)| \). By (3.10),

\[ |g_{i,j}(z)| = \sum_{k=0}^{\infty} \frac{P_{i,kM+j}(z)}{\mu \left( 1 - \frac{1}{p^{kM+j}} \right)} \geq \frac{P_{i,vM+j}(z)}{\mu \left( 1 - \frac{1}{p^{vM+j}} \right)} - \sum_{k \neq v} \frac{P_{i,kM+j}(z)}{\mu \left( 1 - \frac{1}{p^{kM+j}} \right)} \]

\[ = \frac{|z|p^{vM+j}|P_{i,vM+j}(\eta)|}{\mu \left( 1 - \frac{1}{p^{vM+j}} \right)} - \sum_{k \neq v} \frac{|z|p^{kM+j}P_{i,kM+j}(\eta)|}{\mu \left( 1 - \frac{1}{p^{kM+j}} \right)} \geq \frac{|z|p^{vM+j}|P_{i,vM+j}(\eta)|}{\mu \left( 1 - \frac{1}{p^{vM+j}} \right)} - 2 \sum_{k=0}^{v-1} \frac{|z|p^{kM+j}}{\mu \left( 1 - \frac{1}{p^{kM+j}} \right)} \]

\[ - 2 \sum_{k=v+1}^{\infty} \frac{|z|p^{kM+j}}{\mu \left( 1 - \frac{1}{p^{kM+j}} \right)} = I_1 - I_2 - I_3, \]
where
\[ I_1 = \frac{|z|^{p_{v+J}} |P_{i,v+J}(\eta)|}{\mu \left( 1 - \frac{1}{p_{v+J}} \right)}, \quad I_2 = 2 \sum_{k=0}^{v-1} \frac{|z|^{p_{k+J}}}{\mu \left( 1 - \frac{1}{p_{k+J}} \right)} \]
and
\[ I_3 = 2 \sum_{k=v+1}^{\infty} \frac{|z|^{p_{k+J}}}{\mu \left( 1 - \frac{1}{p_{k+J}} \right)} \].

Now we estimate \( I_1, I_2 \) and \( I_3 \) respectively.

- **Estimation of \( I_1 \).**

  By (3.3) and (3.11), we obtain
  \[ |z|^{p_{v+J}} \geq \left( 1 - \frac{1}{p_{v+J}} \right)^{p_{v+J}} \geq \frac{1}{3} \],
  and therefore
  \[ I_1 \geq \frac{|P_{i,v+J}(\eta)|}{3\mu \left( 1 - \frac{1}{p_{v+J}} \right)} \geq \frac{2}{27\mu \left( 1 - \frac{1}{p_{v+J}} \right)}. \]

- **Estimation of \( I_2 \).**

  By the definition of normal function and (3.2), we have for each \( s \in \mathbb{N} \),
  \[ \left( 1 - \left( 1 - \frac{1}{p_{s+J}} \right) \right)^{\alpha} \leq \frac{\mu \left( 1 - \frac{1}{p_{s+J}} \right)}{\mu \left( 1 - \frac{1}{p_{s+1}s+J} \right)} \leq \left( 1 - \left( 1 - \frac{1}{p_{s+1}s+J} \right) \right)^{\beta}, \]
  that is,
  \[ \frac{\mu \left( 1 - \frac{1}{p_{s+J}} \right)}{\mu \left( 1 - \frac{1}{p_{s+1}s+J} \right)} \leq p_{s+J}^{M_{s+J}}. \]
Combining this with (3.4), we have

\[
I_2 \leq 2 \sum_{k=0}^{v-1} \frac{1}{\mu \left(1 - \frac{1}{p^{k+1}M^j} \right)}
\]

\[
= \frac{2}{\mu \left(1 - \frac{1}{p^{v}M^j} \right)} \sum_{k=0}^{v-1} \left[ \mu \left(1 - \frac{1}{p^{k+1}M^j} \right) \mu \left(1 - \frac{1}{p^{(v-k)+1}M^j} \right) \right]
\]

\[
\times \mu \left(1 - \frac{1}{p^{(k+1)+1}M^j} \right)
\]

\[
\leq \frac{2}{\mu \left(1 - \frac{1}{p^{v}M^j} \right)} \sum_{k=0}^{v-1} \frac{1}{p^{vM(v-k)}
\]

\[
\leq \frac{2}{\mu \left(1 - \frac{1}{p^{v}M^j} \right)} \cdot \frac{1}{p^{vM} - 1}
\]

\[
\leq \frac{1}{100\mu \left(1 - \frac{1}{p^{v}M^j} \right)}.
\]

**Estimation of \( I_3 \).**

Noting that by (3.3) and (3.11), we have

\[
(3.13) \quad |z|^p_{M+j} \leq \left(1 - \frac{1}{p^{vM+j+\frac{1}{2}}} \right)^{p^{vM+j+\frac{1}{2}}p^{-\frac{1}{2}}} \leq \left(\frac{1}{2}\right)^{p^{-\frac{1}{2}}}.
\]

Hence, by (3.5), (3.12) and (3.13), we have

\[
I_3 = \frac{2|z|^p_{M+j}}{\mu \left(1 - \frac{1}{p^{v}M^j} \right)} \cdot \sum_{k=v+1}^{\infty} \left[ \mu \left(1 - \frac{1}{p^{kM+j}} \right) \mu \left(1 - \frac{1}{p^{vM+j}} \right) \mu \left(1 - \frac{1}{p^{(v-k)+1}M^j} \right) \mu \left(1 - \frac{1}{p^{(k+1)+1}M^j} \right) \right]
\]

\[
\times \mu \left(1 - \frac{1}{p^{(k+1)+1}M^j} \right)
\]

\[
\leq \frac{2|z|^p_{M+j}}{\mu \left(1 - \frac{1}{p^{v}M^j} \right)} \cdot \sum_{k=v+1}^{\infty} \left[ p^{(3M)(k-v)} \left(p^{kM+j} - p^{(v+1)M+j} \right) \right]
\]
\[ |g_{i,j}(z)| \geq I_1 - I_2 - I_3 \geq \frac{1}{\mu \left(1 - \frac{1}{p^{v+1}M+1}\right)} \left( \frac{2}{27} - \frac{1}{100} - \frac{1}{100} \right) \]

\[
\geq \frac{1}{20\mu \left(1 - \frac{1}{p^{v+1}M+1}\right)} = \frac{1}{20\mu \left(1 - \frac{1}{p^{v+1}M+1}\right)} \cdot \frac{\mu \left(1 - \frac{1}{p^{v+1}M+1}\right)}{\mu \left(1 - \frac{1}{p^{v+1}M+1}\right)} \cdot \frac{1}{20\mu \left(1 - \frac{1}{p^{v+1}M+1}\right)} \\
\geq \frac{1}{20p^{\frac{v}{2}} \mu \left(1 - \frac{1}{p^{v+1}M+1}\right)} \geq \frac{1}{20p^{\frac{v}{2}} \mu (|z|)}.
\]

In summary, we have

\[ \sum_{i=1}^{M} \sum_{j=1}^{M} |g_{i,j}(z)| \geq \frac{1}{20p^{\frac{v}{2}} \mu (|z|)}, \quad (3.14) \]
for all $z$ such that $1 - \frac{1}{p^k} \leq |z| \leq 1 - \frac{1}{p^{k+\frac{1}{2}}}, k = 1, 2, \ldots$.

Next, pick a sequence of positive integers $q_k$ such that $0 \leq q_k - p^{k+\frac{1}{2}} < 1$ and for each $1 \leq j \leq M$, a sequence of positive numbers $\varepsilon_{j,v}$ such that $A^2 q_{vM+j} \varepsilon_{j,v}^2 = 1$.

Choose a sequence of subsets $\Psi_{j,v}$ of $S$ with the following property: for each nonnegative integer $v$, the set $\bigcup_{l=1}^M \Psi_{j,vM+l}$ is a maximal subset of $S$ which is $d$-separated by $A\varepsilon_{j,v}/2$, and each $\Psi_{j,vM+l}$ is $d$-separated by $\varepsilon_{j,v}$.

For each $i, j = 1, 2, \ldots, M$ and $v \geq 0$, set

$$Q_{i,vM+j}(z) = \sum_{\xi \in \Psi_{j,vM+i(j)}} \langle z, \xi \rangle^{q_{vM+j}}$$

and define

$$h_{i,j}(z) = \sum_{v=0}^{\infty} \frac{Q_{i,vM+j}(z)}{\mu \left( 1 - \frac{1}{q_{vM+j}} \right)}.$$ 

Then $h_{i,j}$ is in the Hadamard gap since for each $v \geq 0$,

$$\frac{q_{vM+j}}{q_{(v-1)M+j}} \geq \frac{p^{vM+j+\frac{1}{2}}}{p^{(v-1)M+j+\frac{1}{2}} + 1} \geq \frac{p^{M}}{2} > 1.$$ 

Moreover, the homogeneous polynomials $Q_{i,vM+j}$ are uniformly bounded by 2 as before. Hence each $h_{i,j}$ belongs to $H^\infty_{\mu}$ by Theorem $2.2$ and an easy modification of the previous arguments yields for each $v \geq 0, 1 \leq j \leq M$ and $z \in \mathbb{B}$ satisfying

$$1 - \frac{1}{p^{vM+j+\frac{1}{2}}} \leq |z| \leq 1 - \frac{1}{p^{vM+j+1}}$$

there exists an index $i \in \{1, 2, \ldots, M\}$, such that

$$|h_{i,j}(z)| \geq \frac{C_p}{\mu(|z|)},$$

where $C_p > 0$.

Hence

$$\sum_{i=1}^M \sum_{j=1}^M |h_{i,j}(z)| \geq \frac{C_p}{\mu(|z|)},$$

for all $z$ such that $1 - \frac{1}{p^{k+\frac{1}{2}}} \leq |z| \leq 1 - \frac{1}{p^{k+1}}, k = 1, 2, \ldots.$
Consequently, we finally have
\[ \sum_{i=1}^{M} \sum_{j=1}^{M} (|g_{i,j}(z)| + |h_{i,j}(z)|) \geq \frac{C}{\mu(|z|)} \]
for all \( z \in \mathbb{B} \) sufficiently close to the boundary and for some constant \( C \). Therefore the proof is complete. \( \square \)

As a corollary, we get the following description of the growth rate on Bers-type space \( H^{\infty}_\alpha (\alpha > 0) \), by taking \( \mu(|z|) = (1 - |z|^2)^{\alpha} \) in Theorem 3.1.

**Corollary 3.5.** There exists some positive integer \( M \) and a sequence of functions \( f_i \in H^{\infty}_\alpha, 1 \leq i \leq M \), such that
\[ \sum_{i=1}^{M} |f_i(z)| \gtrsim \frac{1}{(1 - |z|^2)^{\alpha}}, \quad z \in \mathbb{B}. \]

4. **Weighted composition operator** \( uC_\varphi : H^{\infty}_\mu \to H(p, q, \phi) \)

In this section, we will use Theorem 3.1 to characterize the boundedness and compactness of the operator \( uC_\varphi : H^{\infty}_\mu \to H(p, q, \phi) \). Our main result is the following.

**Theorem 4.1.** Let \( \varphi \) be a holomorphic self-map of \( \mathbb{B} \) and \( u \in H(\mathbb{B}) \). Suppose that \( 0 < p, q < \infty \) and \( \mu, \phi \) are normal on \([0, 1)\). Then the following statements are equivalent:
(i) The operator \( uC_\varphi : H^{\infty}_\mu \to H(p, q, \phi) \) is bounded;
(ii) The operator \( uC_\varphi : H^{\infty}_\mu \to H(p, q, \phi) \) is compact;
(iii)
\[ \int_{0}^{1} \left( \int_{\mathbb{S}} \frac{|u(r\xi)|^q}{\mu^q(|\varphi(r\xi)|)} d\sigma(\xi) \right)^{p/q} \phi^p(r) \frac{1}{1-r} dr < \infty; \]
(iv)
\[ \lim_{t \to 1} \int_{0}^{1} \left( \int_{|\varphi(r\xi)| > t} \frac{|u(r\xi)|^q}{\mu^q(|\varphi(r\xi)|)} d\sigma(\xi) \right)^{p/q} \phi^p(r) \frac{1}{1-r} dr = 0. \]

Before proving Theorem 4.1, we need the following auxiliary result, which can be proved by standard arguments (see, e.g., Proposition 3.11 of [4]).

**Lemma 4.1.** Let \( \varphi \) be a holomorphic self-map of \( \mathbb{B} \) and \( u \in H(\mathbb{B}) \). Suppose that \( 0 < p, q < \infty \) and \( \mu, \phi \) are normal on \([0, 1)\). Then \( uC_\varphi : \)
$H^\infty_\mu \rightarrow H(p,q,\phi)$ is compact if and only if $uC_\phi : H^\infty_\mu \rightarrow H(p,q,\phi)$ is bounded and for any bounded sequence $\{f_k\}_{k \in \mathbb{N}}$ in $H^\infty_\mu$ which converges to zero uniformly on compact subsets of $\mathbb{B}$ as $k \rightarrow \infty$, we have $\|uC_\phi f_k\|_{H(p,q,\phi)} \rightarrow 0$ as $k \rightarrow \infty$.

**Lemma 4.2.** [8] If $a > 0, b > 0$, then the following elementary inequality holds.

$$ (a + b)^p \leq \begin{cases} a^p + b^p, & p \in (0, 1) \\ 2^p(a^p + b^p), & p \geq 1 \end{cases} $$

It is obvious that Lemma 4.2 holds for the sum of finite terms, that is,

$$ (a_1 + \cdots + a_j)^p \leq C(a_1^p + \cdots + a_j^p) $$

where $a_1, \ldots, a_j$ are nonnegative numbers, and $C$ is a positive constant.

**Proof of Theorem 4.1.** (ii) $\Rightarrow$ (i). It is obvious.

(i) $\Rightarrow$ (iii). Suppose that $uC_\phi : H^\infty_\mu \rightarrow H(p,q,\phi)$ is bounded. From Theorem 3.1, we pick functions $f_1, \cdots, f_M \in H^\infty_\mu$ such that

$$ \sum_{j=1}^M |f_j(z)| \geq \frac{1}{\mu(|z|)} \quad z \in \mathbb{B}. \quad (4.1) $$

The assumption implies that

$$ \int_0^1 \left( \int_S |(uC_\phi f_j)(r \xi)|^q d\sigma(\xi) \right)^{p/q} \phi^p(r) \frac{1}{1-r} \, dr < \infty, \quad j = 1, \cdots, M, $$

which together with (4.1) and Lemma 4.2 imply

$$ \sum_{j=1}^M \int_0^1 \left( \int_S |u(r \xi)|^q \left( \sum_{j=1}^M |f_j(\varphi(r \xi))| \right)^q d\sigma(\xi) \right)^{p/q} \phi^p(r) \frac{1}{1-r} \, dr $$

$$ \leq \sum_{j=1}^M \int_0^1 \left( \int_S \left| u(r \xi) \varphi(\xi) \right|^q d\sigma(\xi) \right)^{p/q} \phi^p(r) \frac{1}{1-r} \, dr $$

$$ \leq \sum_{j=1}^M \int_0^1 \left( \int_S |(f_j \circ \varphi)(r \xi)|^q d\sigma(\xi) \right)^{p/q} \phi^p(r) \frac{1}{1-r} \, dr $$

$$ \leq \sum_{j=1}^M \int_0^1 \left( \int_S |(uC_\phi f_j)(r \xi)|^q d\sigma(\xi) \right)^{p/q} \phi^p(r) \frac{1}{1-r} \, dr $$

$$ < \infty, $$

as desired.
(iii) \Rightarrow (iv). This implication follows from the dominated convergence theorem.

(iv) \Rightarrow (ii). Assume that (iv) holds. To prove that \( uC_\varphi : H^\infty_\mu \rightarrow H(p, q, \phi) \) is compact, it suffices to prove that if \( \{ f_k \}_{k \in \mathbb{N}} \) is a bounded sequence in \( H^\infty_\mu \) such that \( \{ f_k \}_{k \in \mathbb{N}} \) converges to zero uniformly on compact subsets of \( \mathbb{B} \), then \( \| uC_\varphi f_k \|_{H(p, q, \phi)} \rightarrow 0 \), as \( k \rightarrow \infty \). Take such a sequence \( \{ f_k \} \subset H^\infty_\mu \). We have

\[
\int_0^1 \left( \int_{|\varphi(r\xi)| > \xi} |u(r\xi)|^q |(f_k \circ \varphi)(r\xi)|^q d\sigma(\xi) \right)^{\frac{p}{q} \phi^p(r)} \frac{1}{1-r} dr \\
\leq \| f_k \|^p \int_0^1 \left( \int_{|\varphi(r\xi)| > \xi} \frac{|u(r\xi)|^q}{\mu^q(|\varphi(r\xi)|)} d\sigma(\xi) \right)^{\frac{p}{q} \phi^p(r)} \frac{1}{1-r} dr
\]

(4.2)

\[
\lesssim \int_0^1 \left( \int_{|\varphi(r\xi)| > \xi} \frac{|u(r\xi)|^q}{\mu^q(|\varphi(r\xi)|)} d\sigma(\xi) \right)^{\frac{p}{q} \phi^p(r)} \frac{1}{1-r} dr,
\]

for all \( k \). Take \( \varepsilon > 0 \). (iv) and (4.2) imply that there exists \( t_0 \in (0, 1) \) such that

\[
\int_0^1 \left( \int_{|\varphi(r\xi)| > \frac{t_0}{r}} |u(r\xi)|^q |(f_k \circ \varphi)(r\xi)|^q d\sigma(\xi) \right)^{\frac{p}{q} \phi^p(r)} \frac{1}{1-r} dr < \varepsilon,
\]

(4.3)

for all \( k \). For the above \( \varepsilon \), since \( \{ f_k \} \) converges to 0 on any compact subset of \( \mathbb{B} \), there exists a \( k_0 \) such that

\[
\int_0^1 \left( \int_{|\varphi(r\xi)| \leq \frac{t_0}{r}} |u(r\xi)|^q |(f_k \circ \varphi)(r\xi)|^q d\sigma(\xi) \right)^{\frac{p}{q} \phi^p(r)} \frac{1}{1-r} dr < \varepsilon,
\]

(4.4)

for all \( k > k_0 \). Hence by (4.3) and (4.4), we have

\[
\| uC_\varphi f_k \|_{H(p, q, \phi)}
\]

\[
= \int_0^1 \left( \int_{\mathbb{B}} |u(r\xi)|^q |(f_k \circ \varphi)(r\xi)|^q d\sigma(\xi) \right)^{\frac{p}{q} \phi^p(r)} \frac{1}{1-r} dr
\]

\[
\lesssim \int_0^1 \left( \int_{|\varphi(r\xi)| > \frac{t_0}{r}} |u(r\xi)|^q |(f_k \circ \varphi)(r\xi)|^q d\sigma(\xi) \right)^{\frac{p}{q} \phi^p(r)} \frac{1}{1-r} dr
\]

\[
+ \int_0^1 \left( \int_{|\varphi(r\xi)| \leq \frac{t_0}{r}} |u(r\xi)|^q |(f_k \circ \varphi)(r\xi)|^q d\sigma(\xi) \right)^{\frac{p}{q} \phi^p(r)} \frac{1}{1-r} dr \\
\lesssim \varepsilon, \text{ as } k > k_0,
\]

from which we obtain \( \lim_{k \rightarrow \infty} \| uC_\varphi f_k \|_{H(p, q, \phi)} = 0 \). Thus \( uC_\varphi : H^\infty_\mu \rightarrow H(p, q, \phi) \) is compact by Lemma 4.1. This completes the proof of this theorem.

**Acknowledgement.** This project was partially supported by the Macao Science and Technology Development Fund (No.098/2013/A3),
NSF of Guangdong Province(No.S2013010011978) and NNSF of China(No.11471143).

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