§1. Introduction

The aim of this paper is to give an account of the birational point of view on nilpotent orbits in a complex simple Lie algebra. Let $\mathfrak{g}$ be a complex simple Lie algebra and $G$ the adjoint group. An adjoint orbit $O$ in $\mathfrak{g}$ is called a nilpotent orbit if $O$ consists of nilpotent elements of $\mathfrak{g}$. The closure $\bar{O}$ of $O$ is then an affine variety with singularities. In general, $\bar{O}$ is not necessarily normal (see for example [K-P] in this direction). In this paper we shall take its normalization $\bar{\mathfrak{O}}$ and consider the birational geometry on its (partial) resolutions. Each variety $\bar{O}$ has symplectic singularities. More precisely, the smooth locus $\bar{O}_{\text{reg}}$ admits the Kostant-Kirillov 2-form $\omega$, which is $d$-closed and non-degenerate. Moreover, if we take a resolution $\mu : Y \to \bar{O}$, then $\omega$ extends to a regular 2-form on $Y$. A resolution $\mu : Y \to \bar{O}$ is called a crepant resolution if $K_Y = \mu^* K_{\bar{O}}$. The nilpotent cone $N$ is defined to be the subset of $\mathfrak{g}$ which consists of all nilpotent elements of $\mathfrak{g}$. By definition $N$ is a disjoint union of all nilpotent orbits of $\mathfrak{g}$. There is a largest nilpotent orbit $O_r$ and $N$ coincides with its closure. Moreover, $N$ is a normal variety. Let $B$ be a Borel subgroup of $G$ and let $T^*(G/B)$ be the cotangent bundle of the flag variety $G/B$. By using the Killing form of $\mathfrak{g}$, one can identify $T^*(G/B)$ with a vector bundle $G \times^B [\mathfrak{b}, \mathfrak{b}]$ over $G/B$. Then there is a natural map

$$\nu : G \times^B [\mathfrak{b}, \mathfrak{b}] \to \mathfrak{g}$$

defined by $[g, x] \to Ad_g(x)$. The image of $\nu$ coincides with $N$ and $\nu$ gives a resolution of $N$ ([Sp]). We call $\nu$ the Springer resolution of $N$. Since $T^*(G/B)$ admits a canonical symplectic 2-form and it coincides with the pull-back of the Kostant-Kirillov 2-form on $O_r$, the Springer resolution is a crepant resolution. One can generalize this construction to a parabolic subgroup $Q$ of $G$. Let us start with the cotangent bundle $T^*(G/Q)$. Note that $T^*(G/Q)$ is identified with $G \times^Q \mathfrak{n}(q)$ where $\mathfrak{n}(q)$ is the nil-radical of $\mathfrak{q}$. 
In a similar way to the above, we have a map
\[ \nu : T^*(G/Q) \to g, \]
whose image is the closure of a nilpotent orbit \( O \). In general, \( \nu \) is not birational onto its image, but a generically finite projective morphism (see Example (2.5.4) for a non-birational Springer map). When \( \nu \) gives a resolution of \( \bar{O} \), we call \( \nu \) the Springer resolution of \( \bar{O} \). In this case, the Stein factorization
\[ T^*(G/Q) \xrightarrow{\nu^n} \tilde{O} \to \bar{O} \]
gives a crepant resolution of \( \tilde{O} \). B. Fu [Fu 1] proved the following.

**Theorem.** Let \( O \) be a nilpotent orbit of \( g \) and assume that \( \tilde{O} \) admits a crepant resolution. Then it coincides with a Springer resolution. More exactly, there is a parabolic subgroup \( Q \) of \( G \) such that \( \nu^n \) is the given crepant resolution.

However there still remained interesting problems. At first, there actually exists a nilpotent orbit which has no crepant resolutions. Secondly, even if \( \tilde{O} \) has a crepant resolution, it is not unique, that is, the choice of \( Q \) is not unique even up to conjugacy class. Our purpose is to survey complete answers (cf. [Na 1], [Na 2], [Na 3] and [Fu 2]) to these problems.

A substitute for a crepant resolution is a \( Q \)-factorial terminalization. A birational projective morphism \( \mu : Y \to \tilde{O} \) is a \( Q \)-factorial terminalization if \( Y \) has only \( Q \)-factorial terminal singularities and \( K_Y = \mu^*K_{\tilde{O}} \). The existence of a \( Q \)-factorial terminalization is established by Birkar, Cascini, Hacon and McKernan [BCHM]. But, we shall give here more concrete forms of \( Q \)-factorial terminalization. A hint is already in the work of Lustzig and Spaltenstein [LT]. They introduced the notion of an induced orbit. Let us start with a parabolic subgroup \( Q \) of \( G \) and its Levi factor \( L(Q) \). Let \( O' \subset I(q) \) be a nilpotent orbit with respect to the adjoint \( L(Q) \)-action. Then one can make an associated bundle \( G \times^Q (n(q) + O') \) and define a map
\[ \nu : G \times^Q (n(q) + O') \to g \]
by \( \nu([g, x]) = Ad_g(x) \). Since this is a \( G \)-equivariant closed map, its image is the closure of a nilpotent orbit \( O \) of \( g \). Then we say that \( O \) is induced from \( O' \) and write \( O = Ind_{I(q)}^G(O') \). The map \( \nu \) is called the generalized Springer map. The generalized Springer map \( \nu \) is a generically finite projective morphism. But if \( \nu \) is birational onto its image, then the Stein factorization
\[ G \times^Q (n(q) + O') \xrightarrow{\nu^n} \tilde{O} \to \bar{O} \]
gives a partial resolution of $\tilde{O}$. Now one can prove:

**Theorem (2.6.2).** Let $O$ be a nilpotent orbit of a complex simple Lie algebra $\mathfrak{g}$. Then there are a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ and a nilpotent orbit $O'$ of $\mathfrak{l}(\mathfrak{q})$ such that the following holds:

1. $O = \text{Ind}_{\mathfrak{l}(\mathfrak{q})}^\mathfrak{g}(O')$.
2. $\nu^n$ gives a $\mathbb{Q}$-factorial terminalization of $\tilde{O}$.

In order to look for other $\mathbb{Q}$-factorial terminalizations of $\tilde{O}$, we introduce a flat deformation of $\mathfrak{g} \times \mathfrak{q} (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$. For simplicity we put $\mathfrak{l} := \mathfrak{l}(\mathfrak{q})$ and let $L$ be the corresponding Levi subgroup. Let $\mathfrak{r}(\mathfrak{q})$ be the solvable radical of $\mathfrak{q}$ and consider the variety $\mathfrak{g} \times \mathfrak{q} (\mathfrak{r}(\mathfrak{q}) + \tilde{O}')$. Its normalization $X_{\mathfrak{q},O'}$ is isomorphic to $\mathfrak{g} \times \mathfrak{q} (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$. Let $\mathfrak{k}$ be the center of $\mathfrak{q}$. In (3.3) we shall define a map

$$X_{\mathfrak{q},O'} \to \mathfrak{k}$$

whose central fiber $X_{\mathfrak{q},O',0}$ is $\mathfrak{g} \times \mathfrak{q} (\mathfrak{n}(\mathfrak{q}) + \tilde{O}')$. This map factorizes as

$$X_{\mathfrak{q},O'} \xrightarrow{\mu} \text{Spec } \Gamma(X_{\mathfrak{q},O'}, \mathcal{O}_{X_{\mathfrak{q},O'}}) \to \mathfrak{k}.$$

Put

$$Y_{\mathfrak{l},O'} := \text{Spec } \Gamma(X_{\mathfrak{q},O'}, \mathcal{O}_{X_{\mathfrak{q},O'}}).$$

An important fact is that $Y_{\mathfrak{l},O'}$ depends only on $\mathfrak{l}$ and $O'$. Moreover its central fiber $Y_{\mathfrak{l},O',0}$ is isomorphic to $\tilde{O}$. Define

$$S(\mathfrak{l}) := \{\text{parabolic subalgebras } \mathfrak{q}' \text{ of } \mathfrak{g}; \mathfrak{l}(\mathfrak{q}') = \mathfrak{l}\}.$$

We can define $X_{\mathfrak{q}',O'}$ for each $\mathfrak{q}' \in S(\mathfrak{l})$. We also have a map

$$\mu_{\mathfrak{q}'} : X_{\mathfrak{q}',O'} \to Y_{\mathfrak{l},O'}.$$

The map $\mu_{\mathfrak{q}'}$ is a crepant birational morphism. Moreover, $\mu_{\mathfrak{q}',t}$ is an isomorphism for $t \in \mathfrak{t}^{\text{reg}}$, hence $\mu_{\mathfrak{q}'}$ is an isomorphism in codimension one. Define

$$M(L) := \text{Hom}_{\text{alg.gp}}(L, \mathbb{C}^*)$$

and put $M(L)_\mathbb{R} := M(L) \otimes \mathbb{R}$. Then 2-nd cohomology groups $H^2(X_{\mathfrak{q}',O'}, \mathbb{R})$ are naturally identified with $M(L)_\mathbb{R}$. By these identifications the nef cones $\overline{\text{Amp}(\mu_{\mathfrak{q}',O'})}$ are regarded as the cones in $M(L)_\mathbb{R}$.

**Theorem (3.5.1)** For $\mathfrak{q}' \in S(\mathfrak{l})$, the birational map $\mu_{\mathfrak{q}'} : X_{\mathfrak{q}',O'} \to Y_{\mathfrak{l},O'}$ is a $\mathbb{Q}$-factorial terminalization and is an isomorphism in codimension one.
Any \( \mathbb{Q} \)-factorial terminalization of \( Y_{l,O'} \) is obtained in this way. If \( q_1 \neq q_2 \), then \( \mu_{q_1} \) and \( \mu_{q_2} \) give different \( \mathbb{Q} \)-factorial terminalizations. Moreover,

\[
M(L)_R = \bigcup_{q'\in S(l)} \overline{\text{Amp}}(\mu_{q'}).
\]

Two elements of \( S(l) \) are connected by a sequence of the operations called \textit{twists} (cf. (3.1)). Corresponding to a twist \( q_1 \leadsto q_2 \), we have a flop

\[
X_{q_1,O'} \rightarrow Z \leftarrow X_{q_2,O'}. 
\]

So any two \( \mathbb{Q} \)-factorial terminalizations of \( \tilde{O} \) are connected by a sequence of certain flops. Now let us look at the central fibers \( X_{q',O',0} \) of \( X_{q',O'} \rightarrow \mathfrak{k} \). The diagram

\[
X_{q_1,O',0} \rightarrow Z_0 \leftarrow X_{q_2,O',0}
\]

is not necessarily a flop. Twists are divided into those of the first kind and those of the second kind. If the twist \( q_1 \leadsto q_2 \) is of the first kind, then it induces a flop between \( X_{q_1,O',0} \) and \( X_{q_2,O',0} \). These flops are completely classified and we call them Mukai flops (3.2.1). If it is of the second kind, the maps \( X_{q_i,O',0} \rightarrow Z_0 \) \((i = 1, 2)\) are both divisorial birational maps. Define \( S^1(l) \) to be the subset of \( S(l) \) consisting of the parabolic subalgebras \( q' \) obtained from \( q \) by a finite succession of the twists of the first kind. Note that the restriction map \( H^2(X_{q',O',0}, \mathbb{R}) \rightarrow H^2(X_{q',O',0}, \mathbb{R}) \) is an isomorphism and \( \text{Amp}(\mu_{q'}) \) is mapped onto \( \overline{\text{Amp}}(\mu_{q',0}) \).

\textbf{Theorem (3.5.4).} There is a one-to-one correspondence between the set of \( \mathbb{Q} \)-factorial terminalizations of \( \tilde{O} \) and \( S^1(l) \). In other words, every \( \mathbb{Q} \)-factorial terminalization of \( \tilde{O} \) is obtained as \( \mu_{q',0} : X_{q',O',0} \rightarrow \tilde{O} \) for \( q' \in S^1(l) \). Two different \( \mathbb{Q} \)-factorial terminalizations of \( \tilde{O} \) are connected by a sequence of Mukai flops. Moreover

\[
\overline{\text{Mov}}(\mu_{q,0}) = \bigcup_{q'\in S^1(l)} \overline{\text{Amp}}(\mu_{q',0}),
\]

where \( \overline{\text{Mov}}(\mu_{q,0}) \) is the movable cone for \( \mu_{q,0} \) (cf. (3.4.2)).

A direct approach to Theorem (3.5.4) usually needs the classification of the generalized Springer maps which are isomorphisms in codimension one. But our approach using Theorem (3.5.1) does not need this and Mukai flops appear in a very natural way.

Let \( W \) be the Weyl group of \( \mathfrak{g} \) and let \( N_W(L) \) be the subgroup of \( W \) which normalizes \( L \). Then the quotient group

\[
W' := N_W(L)/W(L)
\]
naturally acts on $M(L)$. The interior $\text{Mov}(\mu,0)$ of the movable cone can be characterized as a fundamental domain for this action (Theorem (3.6.1)). The group $W'$ was extensively studied in [Ho]. As explained above, the deformation $X_{q,\Omega'} \to \mathfrak{k}$ of $G \times Q(n(q) + 0')$ played an important role to study the birational geometry for $\tilde{O}$. But this is not merely a flat deformation of $G \times Q(n(q) + 0')$. In fact, $G \times Q(n(q) + 0')$ admits a symplectic 2-form on its regular locus. This symplectic 2-form induces a Poisson structure of the regular part; moreover, it uniquely extends to a Poisson structure of $G \times Q(n(q) + 0')$. One can introduce the notion of a Poisson deformation (cf. §4), and $X_{q,\Omega'} \to \mathfrak{k}$ turns out to be a Poisson deformation of $G \times Q(n(q) + 0')$.

On the other hand, since $\tilde{O}$ has symplectic singularities, $\tilde{O}$ also admits a natural Poisson structure. One can construct a flat deformation of $\tilde{O}$ as follows. Let $G \cdot (\mathfrak{r}(q) + \tilde{O}) \subset \mathfrak{g}$ denote the $G$-orbit of $\mathfrak{r}(q) + \tilde{O}$ by the adjoint $G$-action. By using the adjoint quotient map $\mathfrak{g} \to \mathfrak{h}/W$, we get a map $\chi : G \cdot (\mathfrak{r}(q) + \tilde{O}) \to \mathfrak{h}/W$. The image of $\chi$ is not necessarily normal, but its normalization coincides with $\mathfrak{r}(q)/W'$. Let $G \cdot (\mathfrak{r}(q) + \tilde{O})^n$ be the normalization of $G \cdot (\mathfrak{r}(q) + \tilde{O})$. Then $\chi$ induces a map

$$G \cdot (\mathfrak{r}(q) + \tilde{O})^n \to \mathfrak{r}(q)/W'. $$

One can check that this is a flat map and its central fiber is $\tilde{O}$. Moreover, this is a Poisson deformation of $\tilde{O}$. The two Poisson deformations are combined together by the Brieskorn-Slodowy diagram

$$
\begin{array}{ccc}
X_{q,\Omega'} & \longrightarrow & G \cdot (\mathfrak{r}(q) + \tilde{O})^n \\
\downarrow & & \downarrow \\
\mathfrak{r}(q) & \longrightarrow & \mathfrak{r}(q)/W'
\end{array}
$$

Our last theorem (= Theorem (4.5)) claims that it gives formally universal Poisson deformations of $G \times Q(n(q) + \tilde{0}')$ and $\tilde{O}$.

Finally we shall explain the contents of this paper. The first part of §2 is an introduction to nilpotent orbits and related resolutions. Many concrete examples are described in terms of flags; I believe that they would motivate the following abstract arguments. In the final part of §2, we give a rough sketch of the proof of Theorem (2.6.2) in the classical cases. The readers can find a proof in [Fu 2] when $\mathfrak{g}$ is exceptional. The idea of most arguments in §3 comes from [Na 2]. But all statements are generalized so that one can apply them to generalized Springer maps. §4 is concerned with a Poisson
deformation. We quickly review the notions of Poisson structures and Poisson deformations. After that, we will give a rough sketch of Theorem (4.5) mentioned above. The results of §4 have been already treated in [Na 5] when \( \tilde{O} \) has a crepant resolution.

**Notations.** Let \( G \) be an algebraic group over \( \mathbb{C} \) and \( P \) a closed subgroup of \( G \). If \( V \) is a variety with a \( P \)-action, then we denote by \( G \times^P V \) the associated fiber bundle over \( G/P \) with a typical fiber \( V \). More exactly, \( G \times^P V \) is defined as the quotient of \( G \times P \) by an equivalence relation \( \sim \), where \((g, x) \sim (g', x')\) if there is an element \( p \in P \) such that \( g' = gp \) and \( x' = p^{-1} \cdot x \).

§2. Nilpotent orbits and symplectic singularities

(2.1) Let \( G \) be a semi-simple algebraic group over the complex number field \( \mathbb{C} \) and let \( \mathfrak{g} \) be its Lie algebra. An orbit \( O \) of the adjoint action \( \text{Ad} : G \to \text{Aut}(\mathfrak{g}) \) is called an adjoint orbit. Moreover, if \( O \) consists of nilpotent elements (resp. semi-simple elements), then \( O \) is called a nilpotent orbit (resp. semi-simple orbit). The tangent space \( T_\alpha O \) of an adjoint orbit \( O \) at \( \alpha \) is identified with

\[ [\alpha, \mathfrak{g}] := \{[\alpha, x] ; x \in \mathfrak{g} \}. \]

Since \( \mathfrak{g} \) is semi-simple, the Kostant-Kirillov form

\[ k : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \]

is a non-degenerate symmetric form. We define a skew-symmetric form

\[ \omega_\alpha : T_\alpha O \times T_\alpha O \to \mathbb{C} \]

by

\[ \omega_\alpha([\alpha, x], [\alpha, y]) := k(\alpha, [x, y]). \]

This is well-defined and non-degenerate because if \( [\alpha, x] = 0 \), then \( k(\alpha, [x, y]) = k([\alpha, x], y) = 0 \). If \( \alpha \) runs through all elements of \( O \), the 2-form \( \omega := \{\omega_\alpha\} \) is a \( d \)-closed form on \( O \). In particular, \( O \) is a smooth algebraic variety of even dimension. The symplectic form \( \omega \) is called the Kostant-Kirillov 2-form. A semi-simple orbit is a closed subvariety of \( \mathfrak{g} \). But, a nilpotent orbit \( O \) is not closed in \( \mathfrak{g} \) except when \( O = \{0\} \). If we take the closure \( \tilde{O} \) of \( O \), it is an affine variety with singularities. Note that \( \tilde{O} \) is not necessarily normal. We denote by \( \tilde{O} \) its normalization.
Nilpotent orbits in a classical Lie algebra: Let $\mathfrak{sl}(n)$ be the Lie algebra consisting of $n \times n$ matrices $A$ with $\text{tr}(A) = 0$. Define

$$ so(n) := \{ A \in \mathfrak{sl}(n); A^t J + JA = 0 \}, $$

where

$$ J = \begin{pmatrix} \ldots & 1 \\ \ldots \\ 1 \end{pmatrix}, $$

and $A^t$ is the transposed matrix of $A$. Similarly, define

$$ sp(2n) := \{ A \in \mathfrak{sl}(2n); A^t J' + J' A = 0 \}, $$

where

$$ J' = \begin{pmatrix} \ldots & 1 \\ \ldots \\ -1 \end{pmatrix}. $$

If $\mathfrak{g}$ is of type $A_{n-1}$, then $\mathfrak{g} = \mathfrak{sl}(n)$. If $\mathfrak{g}$ is of type $B_n$, then $\mathfrak{g} = so(2n+1)$. If $\mathfrak{g}$ is of type $C_n$, then $\mathfrak{g} = sp(2n)$. Finally, if $\mathfrak{g}$ is of type $D_n$, then $\mathfrak{g} = so(2n)$. One can associate a Jordan type $d$ to each nilpotent orbit of $\mathfrak{g}$. Here $d := [d_1, d_2, ..., d_k]$ is a partition of $n := \dim V$. Namely, $d_i$ are positive integers such that $d_1 \geq d_2 \geq ... \geq d_k$ and $\Sigma d_i = n$. Another way of writing $d$ is $[d_1^{s_1}, ..., d_k^{s_k}]$ with $d_1 > d_2 > ... > d_k > 0$. Here $d_i^{s_i}$ is an $s_i$ times $d_i$’s: $d_i, d_i, ..., d_i$. The partition $d$ corresponds to a Young diagram. For example, $[5, 4^2, 1]$ corresponds to
When an integer \( e \) appears in the partition \( d \), we say that \( e \) is a member of \( d \). We call \( d \) very even when \( d \) consists with only even members, each having even multiplicity.

The following result can be found, for example, in [C-M, §5].

**Proposition (2.2.1)** Let \( \mathcal{N}o(\mathfrak{g}) \) be the set of nilpotent orbits of \( \mathfrak{g} \).

1. (\( A_{n-1} \)): When \( \mathfrak{g} = \mathfrak{sl}(n) \), there is a bijection between \( \mathcal{N}o(\mathfrak{g}) \) and the set of partitions \( d \) of \( n \).

2. (\( B_n \)): When \( \mathfrak{g} = \mathfrak{so}(2n+1) \), there is a bijection between \( \mathcal{N}o(\mathfrak{g}) \) and the set of the partitions \( d \) of \( 2n+1 \) for which all even members have even multiplicities.

3. (\( C_n \)): When \( \mathfrak{g} = \mathfrak{sp}(2n) \), there is a bijection between \( \mathcal{N}o(\mathfrak{g}) \) and the set of the partitions \( d \) of \( 2n \) for which all odd members have even multiplicities.

4. (\( D_n \)): When \( \mathfrak{g} = \mathfrak{so}(2n) \), there is a surjection \( f \) from \( \mathcal{N}o(\mathfrak{g}) \) to the set of the partitions \( d \) of \( 2n \) for which all even members have even multiplicities. For a partition \( d \) which is not very even, \( f^{-1}(d) \) consists of exactly one orbit, but, for very even \( d \), \( f^{-1}(d) \) consists of exactly two different orbits.

We introduce a partial order in the set of the partitions of (the same number): for two partitions \( d \) and \( f \), \( d \geq f \) if \( \Sigma_{i \leq k} d_i \geq \Sigma_{i \leq k} f_i \) for all \( k \geq 1 \). On the other hand, for two nilpotent orbits \( O \) and \( O' \) in \( \mathfrak{g} \), we write \( O \geq O' \) if \( O' \subset \overline{O} \). Then, \( O_d \geq O_f \) if and only if \( d \geq f \).

**Remark (2.2.2)** In order to classify nilpotent orbits in simple Lie algebras including those of exceptional type, we need a different method. Dynkin [D] associates a weighted Dynkin diagram to each nilpotent orbit. This correspondence is an injection, but is not surjective. Bala and Carter [B-C] determined which weighted Dynkin diagrams come from nilpotent orbits and completed the classification of nilpotent orbits in all simple Lie algebras. For details, see [B-C] and [C-M].

(2.3) **Jacobson-Morozov resolution of \( \overline{O} \):** Let \( O \) be a nilpotent orbit of a complex semi-simple Lie algebra \( \mathfrak{g} \). Fix an element \( x \in O \). By the Jacobson-Morozov theorem (cf. [C-M, 3.2]) one can find a semi-simple element \( h \in \mathfrak{g} \), and a nilpotent element \( y \in \mathfrak{g} \) in such a way that \( [h, x] = 2x \), \( [h, y] = -2y \) and \( [x, y] = h \). For \( i \in \mathbb{Z} \), let \( \mathfrak{g}_i := \{ z \in \mathfrak{g} \mid [h, z] = iz \} \).
Then one can write
\[ g = \bigoplus_{i \in \mathbb{Z}} g_i. \]

Let \( h \) be a Cartan subalgebra of \( g \) with \( h \in h \). Let \( \Phi \) be the corresponding root system and let \( \Delta \) be a base of simple roots such that \( h \) is \( \Delta \)-dominant, i.e. \( \alpha(h) \geq 0 \) for all \( \alpha \in \Delta \). In this situation,
\[ \alpha(h) \in \{0, 1, 2\}. \]

The weighted Dynkin diagram of \( O_x \) is the Dynkin diagram of \( g \) where each vertex \( \alpha \) is labeled with \( \alpha(h) \). A nilpotent orbit \( O_x \) is completely determined by its weighted Dynkin diagram. A Jacobson-Morozov parabolic subalgebra for \( x \) is the parabolic subalgebra \( p \) defined by
\[ p := \bigoplus_{i \geq 0} g_i. \]

Let \( P \) be the parabolic subgroup of \( G \) determined by \( p \). We put
\[ n_2 := \bigoplus_{i \geq 2} g_i. \]

Then \( n_2 \) is an ideal of \( p \); hence, \( P \) has the adjoint action on \( n_2 \). Let us consider the vector bundle \( G \times^P n_2 \) over \( G/P \) and the map
\[ \mu : G \times^P n_2 \to g \]
defined by \( \mu([g, z]) := Ad_g(z) \). Then the image of \( \mu \) coincides with the closure \( \bar{O} \) of \( O \) and \( \mu \) gives a resolution of \( \bar{O} \). We call \( \mu \) the Jacobson-Morozov resolution of \( O \). The construction of \( \mu \) depends on the choices of \( x \in O \) and the \( sl(2) \)-triple \( \{x, y, h\} \). But, for any nilpotent elements \( x \) and \( x' \) of \( O \), two \( sl(2) \)-triple \( \{x, y, h\} \) and \( \{x', y', h'\} \) are conjugate to each other by an element of \( G \) (cf. [C-M, 3.2]). In this sense, the Jacobson-Morozov resolution is unique. But, the Jacobson-Morozov resolution is not a crepant resolution in general.

**Definition (2.4).** Let \( X \) be a normal variety defined over \( \mathbb{C} \). Assume that the regular locus \( X_{reg} \) admits a symplectic 2-form \( \omega \). Then \( (X, \omega) \) (or \( X \)) has symplectic singularities if there is a resolution \( \mu : Y \to X \) such that the 2-form \( \omega \) on \( \mu^{-1}(X_{reg}) \) extends to a regular 2-form on \( Y \).

Remark that if a particular resolution \( \mu : Y \to X \) has the extension property explained above, then all resolutions of \( X \) actually have the property. The following proposition is due to Hinich and Panyushev [Hi],[Pa]:

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Remark that if a particular resolution \( \mu : Y \to X \) has the extension property explained above, then all resolutions of \( X \) actually have the property. The following proposition is due to Hinich and Panyushev [Hi],[Pa]:
Proposition (2.4.1). For the Jacobson-Morozov resolution $\mu : G \times P \to \bar{O}$, the Kostant-Kirillov 2-form on $O$ extends to a regular 2-form on $G \times P \to \bar{O}$. In particular, the normalization $\bar{O}$ of $\bar{O}$ has symplectic singularities.

(2.5) (Induced orbits):

(2.5.1) Let $G$ and $g$ be the same as in (2.1). Let $Q$ be a parabolic subgroup of $G$ and let $q$ be its Lie algebra with Levi decomposition $q = l \oplus n$. Here $n$ is the nil-radical of $q$ and $l$ is a Levi-part of $q$. Fix a nilpotent orbit $O'$ in $l$. Then there is a unique nilpotent orbit $O$ in $q$ meeting $n + O'$ in an open dense subset ([L-S]). Such an orbit $O$ is called the nilpotent orbit induced from $O'$ and we write $O = \text{Ind}^l_l(O')$.

Note that when $O' = 0$, $O$ is called the Richardson orbit for $Q$. Since the adjoint action of $Q$ on $q$ stabilizes $n + O'$, one can consider the variety $G \times Q (n + O')$. There is a map $\nu : G \times Q (n + O') \to \bar{O}$ defined by $\nu([g, z]) := \text{Ad}_g(z)$. Since $\text{Codim}(O') = \text{Codim}_g(O)$ (cf. [C-M], Prop. 7.1.4), $\nu$ is a generically finite dominating map. Moreover, $\nu$ is factorized as $G \times Q (n + O') \to G/Q \times \bar{O} \to \bar{O}$ where the first map is a closed embedding and the second map is the 2-nd projection; this implies that $\nu$ is a projective map. In the remainder, we call $\nu$ the generalized Springer map for $(Q, O')$. When $O' = \{0\}$, we often call $\mu$ the Springer map. Let $\bar{O}'$ be the normalization of $\bar{O}$. Then the normalization of $G \times Q (n + \bar{O}')$ coincides with $G \times Q (n + O')$. The generalized Springer map $\nu$ induces a map $\nu^\#: G \times Q (n + \bar{O}') \to \bar{O}$.

We call $\nu^\#$ the normalized map of $\nu$.

(2.5.2) Assume that there are a parabolic subgroup $Q_L$ of $L$ and a nilpotent orbit $O''$ in the Levi subalgebra $l(Q_L)$ such that $O'$ is the nilpotent orbit induced from $(Q_L, O'')$. Then there is a parabolic subgroup $Q'$ of $G$ such that $Q' \subset Q$, $l(Q') = l(Q_L)$ and $O$ is the nilpotent orbit induced from $(Q', O'')$. If we put $l' := l(Q')$, then this can be written as $\text{Ind}_l^g(\text{Ind}_l^l(O'')) = \text{Ind}_l^g(O'')$. 
The generalized Springer map $\nu'$ for $(Q', O')$ is factorized as

$$G \times Q' (n' + \bar{O}') \rightarrow G \times Q (n + \bar{O}') \rightarrow \bar{O}.$$ 

**Example (2.5.3).** Let $G := SL(n, \mathbb{C})$ and $\mathfrak{g} := sl(n)$. Then $G$ acts naturally on $\mathbb{C}^n$ and any parabolic subgroup $Q$ of $G$ is given as the subgroup of stabilizers of a flag (= a sequence of vector subspaces):

$$V_1 \subset V_2 \subset \ldots \subset V_l = \mathbb{C}^n.$$ 

Put $q_i := \dim V_i - \dim V_{i-1}$ and $(q_1, q_2, \ldots, q_l)$ is called the type of $Q$. If two parabolic subgroups of $G$ have the same type, then they are conjugate to each other. The set of all diagonal matrices in $\mathfrak{g}$ forms a Cartan subalgebra $\mathfrak{h}$. There is a unique Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ such that $\mathfrak{h} \subset \mathfrak{l}$. In our case,

$$\mathfrak{l} = \left\{ \begin{pmatrix} A_{q_1} & 0 & \cdots & 0 \\ 0 & A_{q_2} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{q_l} \end{pmatrix} \in \mathfrak{g} \mid A_{q_i} : q_i \times q_i \text{ matrix} \right\}.$$ 

We take $O' := \{0\}$ as a nilpotent orbit in $\mathfrak{l}$. Rearrange $q_i's$ in such a way that $q_{\sigma(1)} \geq q_{\sigma(2)} \geq \ldots \geq q_{\sigma(l)}$ by using a suitable permutation $\sigma \in S_l$. Then $\mathfrak{q} := (q_{\sigma(1)}, q_{\sigma(2)}, \ldots, q_{\sigma(l)})$ is a partition of $n$. As in (2.4), we associate a Young diagram to $\mathfrak{q}$. Let $d_i$ be the length of the $i$-th column of the Young diagram. The dual partition $\mathfrak{q}'$ of $\mathfrak{q}$ is defined as $\mathfrak{q}' := [d_1, d_2, \ldots, d_s]$. We shall prove that $\text{Ind}_G^H(O') \subset \mathfrak{g}$ is the nilpotent orbit with Jordan type $\mathfrak{q}'$. Take a basis $e_1, e_2, \ldots, e_n$ of $\mathbb{C}^n$ in such a way that

$$V_i = \mathbb{C} < e_1, e_2, \ldots, e_{\sum_{j \leq i} q_j} >$$ 

for all $i$. Define

$$W_1 := \{e_1, \ldots, e_{q_1}\},$$

$$W_2 := \{e_{q_1+1}, \ldots, e_{q_1+q_2}\},$$

$$\ldots$$

$$W_l := \{e_{\sum_{i \leq k \leq \sum_{-1} q_k} + 1}, \ldots, e_{\sum_{i \leq k \leq q_k} } \}.$$ 

Then $V_i$ is spanned by $W_1 \cup \ldots \cup W_l$. Take the 1-st vectors from $W_i$’s and form a set $E_1$ consisting of them. Namely

$$E_1 = \{e_1, e_{q_1+1}, \ldots, e_{q_1+\ldots+q_{l-1}+1}\}.$$
Next take the 2-nd vectors from $W_i$'s. If $q_i = 1$, then there is no 2-nd vector in $W_i$. In this case, we take no vectors from this $W_i$. Let $E_2$ be the set consisting of such vectors. Similarly we define $E_3, \ldots$. Note that $E_i$ has exactly $d_i$ elements. Let us consider the nilpotent endomorphism $x_i$ of $C < E_i >$ corresponding to the Jordan matrix $J_{d_i}$ of size $d_i$:

\[
J_{d_i} := \begin{pmatrix}
0 & 1 & 0 & \cdots & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

We define a nilpotent endomorphism $x$ of $C^n$ by $x := \oplus x_i$. By the construction, $x$ has Jordan type $[d_1, d_2, \ldots, d_s]$ and $x(V_i) \subset V_{i-1}$ for each $i$; hence $x \in \mathfrak{n}$. Let us consider the Springer map

\[
\nu : G \times \mathbb{Q} \mathfrak{n} \to \mathfrak{g}.
\]

Since $x \in \mathfrak{n}$, the image of $\mu$ contains $x$. Let $O_x$ be the nilpotent orbit containing $x$. In order to prove that $\text{Im}(\mu) = \bar{O}_x$, it suffices to prove that $\dim O_x = \dim(G \times \mathbb{Q} \mathfrak{n})$. We put

\[
\mathfrak{g}^x := \{ z \in \mathfrak{g} ; [x, z] = 0 \}.
\]

Note that $\dim O_x = \dim \mathfrak{g} - \dim \mathfrak{g}^x$. By using this fact, one can check that

\[
\dim O_x = n(n + 1) - 2 \sum_{1 \leq i \leq s} id_i.
\]

On the other hand, one has

\[
\dim(G/Q) = n(n + 1)/2 - \sum_{1 \leq i \leq s} id_i.
\]

Since $\dim(G \times \mathbb{Q} \mathfrak{n}) = 2 \dim(G/Q)$, we have the desired result. Finally we shall check that $\nu$ is a birational map. Since $\nu$ is a $G$-equivariant map, we only have to show that $\nu^{-1}(y)$ consists of exactly one point for a particular nilpotent element $y$ with Jordan type $[d_1, d_2, \ldots, d_s]$. We put

\[
y = \begin{pmatrix}
J_{d_1} & 0 & \cdots & 0 \\
0 & J_{d_2} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & J_{d_s}
\end{pmatrix}.
\]
Assume that $\nu([g, z]) = y$. Then $z$ is uniquely determined by $[g]$ as $z := Ad_{g^{-1}}(y)$. Therefore,

$$
\nu^{-1}(y) = \{[g] \in G/Q; y \in Ad_g(n)\}.
$$

Note that $G/Q$ is naturally identified with the set of parabolic subgroups of $G$ which are conjugate to $Q$ by $[g] \to gQg^{-1}$. Moreover, $Ad_g(n) = n(gQg^{-1})$.

As a consequence, $\nu^{-1}(y)$ is identified with the set of parabolic subgroups $Q'$ of $G$ such that $Q'$ is conjugate to $Q$ and $y \in n(Q')$. In terms of flags, $Q'$ corresponds to a flag $V_1 \subset V_2 \subset ... \subset V_l = C^n$ of type $(q_1, q_2, ..., q_l)$ so that $y(V_i) \subset V_{i-1}$ for all $i$. Let us consider the Young diagram corresponding to $[d_1, ..., d_s]$. We fill up each box on the 1-st row by $e_1, ..., e_{d_1}$ from left to right. Next fill up each box on the 2-nd row by $e_{d_1+1}, ..., e_{d_2}$ from left to right, and so on. For example, when $(q_1, q_2, q_3, q_4, q_5) = (3, 4, 3, 3, 1)$ and $[d_1, d_2, d_3, d_4] = [5, 4, 4, 1]$, we have the following tablau:

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|-------|-------|-------|-------|-------|
|       | $e_6$ | $e_7$ | $e_8$ | $e_9$ |
| $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ |
|       |       |       | $e_{14}$ |

Let us consider the $q_1$ boxes on the 1-st column from the top. Then take all vectors in these boxes and form a vector subspace $V_1$ generated by them. In the above example, $q_1 = 3$; hence $V_1 = C < e_1, e_6, e_{10}>$. We next delete these $q_1$ boxes from the original Young tablau to get a new one. The new tablau has exatly $d_i - 1$ boxes on the $i$-th row for $1 \leq i \leq q_1$, and has exactly $d_i$ boxes on the $i$-th row for $i > q_1$:

| $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|-------|-------|-------|-------|
|       | $e_7$ | $e_8$ | $e_9$ |
| $e_{11}$ | $e_{12}$ | $e_{13}$ |
|       |       |       | $e_{14}$ |

Consider the $q_2$ boxes on the 1-st column of the new diagram from the top and take all vectors in these boxes. They and $V_1$ together generate a vector subspace $V_2$. In the above example, $V_2 = C < e_1, e_6, e_{10}, e_2, e_7, e_{11}, e_{14}>.
Deleting the $q_2$ boxes, we get again a new Young tableau. Repeat the similar process and we finally get a desired flag $V_1 \subset V_2 \subset ... \subset V_l = \mathbb{C}^n$ of type $(q_1, ..., q_l)$. One can check that this is a unique flag of type $(q_1, ..., q_l)$ such that $y(V_i) \subset V_{i-1}$. Therefore, $\nu^{-1}(y)$ consists of one point.

Example (2.5.4). By $J'$ in (2.2) we introduce a non-degenerate skew symmetric form $< , >$ on $\mathbb{C}^4$. Define

$$SP(4) := \{ A \in GL(4, \mathbb{C}); A^t J'A = J' \}.$$ 

Its Lie algebra is $sp(4)$. By an easy calculation, we see that

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is an element of $so(4)$. Note that $x$ has Jordan type $[2, 2]$. In general, a parabolic subgroup of $SP(2n)$ is obtained as the group of stabilizers of an isotropic flag $\{ V_i \}_{1 \leq i \leq s}$ of $\mathbb{C}^4$. An isotropic flag is a flag such that $V_i^\perp = V_{s-i}$ for all $i$. The (flag) type of an isotropic flag can be written as $(p_1, ..., p_k, q, p_k, ..., p_1)$ with some positive integers $p_i$ and a non-negative integer $q$. Here we put $q = 0$ if the length of the flag is even. Let $e_1, ..., e_4$ be the standard base of $\mathbb{C}^4$. Since $V := \mathbb{C} < e_1, e_3 >$ is a Lagrangian subspace (i.e. $V^\perp = V$), the flag $V \subset \mathbb{C}^4$ is isotropic of type $(2, 2)$. Let $Q_{2,2}$ be the stabilizer group of this isotropic flag. Since $x : \mathbb{C}^4 \subset V$, $x \cdot V = 0$, we have $x \in n(q_{2,2})$. Since $\dim O_{[2,2]} = 2 \dim SP(4)/Q_{2,2}$, we know that $O_{[2,2]}$ is the Richardson orbit for $Q_{2,2}$. The Springer map

$$\nu_{2,2} : SP(4) \times Q_{2,2} n(q_{2,2}) \to \bar{O}_{[2,2]}$$

is a birational map.

On the other hand, let us consider the isotropic flag:

$$V_1 := \mathbb{C} < e_1 >, \quad V_2 := \mathbb{C} < e_1, e_2, e_3 >.$$ 

Let $Q_{1,2,1} \subset SP(4)$ be the stabilizer group of this flag. Since $\dim O_{[2,2]} = 2 \dim SO(4)/Q_{1,2,1}$ and $x \in n(q_{1,2,1})$, we see that $O_{[2,2]}$ is the Richardson orbit for $Q_{1,2,1}$. The Springer map

$$\nu_{1,2,1} : SP(4) \times Q_{1,2,1} n(q_{1,2,1}) \to \bar{O}_{[2,2]}$$
is not birational. In fact, let us consider the isotropic flag
\[ V_1' := C < e_3 >, \quad V_2' := C < e_1, e_3, e_4 >, \]
and its stabilizer group \( Q_{1,2,1}' \). Then \( x \in n(q_{1,2,1}') \) and \( Q_{1,2,1} \) and \( Q_{1,2,1}' \) are conjugate to each other. This means that \( \nu_{1,2,1}(x) \) contains at least two points. One can prove that \( \deg \nu_{1,2,1} = 2 \) (cf. [He]).

\textbf{Example (2.5.5).} Assume that \( g = sp(m) \) or \( so(m) \). Let \( z \in g \) be a nilpotent element of Jordan type \( d := [d_1^{r_1}, d_2^{r_2}, \ldots, d_k^{r_k}] \). Let \( O_z \) be the nilpotent orbit containing \( z \). Assume that \( d_p \geq d_{p+1} + 2 \) for some \( p \). Put \( r := \sum_{1 \leq j \leq p} s_j \).

We shall show that there are a parabolic subgroup \( Q \) of \( G \) with flag type \( (r, m - 2r, r) \), a Levi decomposition \( q = l \oplus n \), and a nilpotent orbit \( O' \) of \( l \) such that \( O_z = \text{Ind}_{l}^{g}(O') \). Here
\[ l = gl(r) \oplus g', \]
where \( g' = sp(m) \) (resp. \( so(m) \)) if \( g = sp(m) \) (resp. \( so(m) \)). The orbit \( O' \) is a nilpotent orbit of \( g' \) with Jordan type \( d' := [(d_1 - 2)s_1, \ldots, (d_p - 2)s_p, d_{p+1}^{s_{p+1}}, \ldots, d_k^{s_k}] \).

Let us consider the case \( g = sp(m) \). We prepare two skew-symmetric vector space \( V_d \) (d: even), and \( W_{2d} \) (d: odd) as follows. The vector space \( V_d \) is a \( d \)-dimensional vector space with a skew-symmetric form determined by the \( d \times d \) matrix \( J' \) in (2.2). Let \( Z_d \) be the \( d \times d \) matrix such that \( Z_d(i, i + 1) = 1 \) (\( 1 \leq i \leq d/2 \)), \( Z_d(i, i + 1) = -1 \) (\( d/2 + 1 \leq i \leq d - 1 \)) and otherwise \( Z_d(i, j) = 0 \). We denote by \( z_d \) the endomorphism of \( V_d \) determined by \( Z_d \). The vector space \( W_{2d} \) is a \( 2d \) dimensional vector space with a skew-symmetric form determined by the \( 2d \times 2d \) matrix \( J' \) in (2.2). By using the Jordan matrix \( J_d \) we define
\[ Z_{2d} := \begin{pmatrix} J_d & 0 \\ 0 & -J_d \end{pmatrix} \]
and let \( z_{2d} \) be the corresponding endomorphism of \( W_{2d} \).

Note that, in the partition \( d, s_i \) is even if \( d_i \) is odd. When \( d_i \) is even, we put \( U_i := V_{d_i}^{s_i} \) and define \( z_i \in \text{End}(U_i) \) by \( z_i = z_{d_i}^{s_i} \). When \( d_i \) is odd, we put \( U_i := W_{2d_i}^{s_i/2} \) and define \( z_i \in \text{End}(U_i) \) by \( z_i = z_{2d_i}^{s_i/2} \). We may assume that
\[ (C^m, <, >) := \bigoplus_{1 \leq i \leq k} U_i, \]
and \( z = \oplus z_i \). Each \( U_i \) has a filtration \( 0 \subset U_{i,1} \subset U_{i,2} \subset \ldots \subset U_{i,d_i} = U_i \) defined by \( U_{i,j} := \text{Im}(z_i^{d_i-j}) \). We put

\[
F := \oplus_{1 \leq i \leq p} U_{i,1}.
\]

By definition \( \dim F = r \) and \( F \subset F^\perp \). Moreover, one can check that \( z|_{F^\perp/F} \) is a nilpotent endomorphism of Jordan type \( d' = [(d_1 - 2)s_1, \ldots, (d_p - 2)s_p, d_{p+1}, \ldots, d_{d_k}] \). Conversely, \( F \) is the unique \( r \)-dimensional isotropic subspace such that \( z(F) = 0 \) and \( z|_{F^\perp/F} \) has Jordan type \( d' \). Let \( Q \subset SP(m) \) be the stabilizer group of the isotropic flag

\[
0 \subset F \subset F^\perp \subset \mathbb{C}^m,
\]

and let \( O' \) be the nilpotent orbit of \( sp(F^\perp/F) \) which contains \( z|_{F^\perp/F} \). Then \( O_z = \text{Ind}_{O'}^Q (O') \). Moreover, the generalized Springer map

\[
\nu_Q : SP(m) \times Q (n + O') \to O_z
\]

is birational. The case \( g = so(m) \) is similar.

**Example (2.5.6).** Let us consider a nilpotent element \( x \in so(4n + 2) \) with Jordan type \( [2^{2n}, 1^2] \). Denote by \( O_x \) the nilpotent orbit containing \( x \). Let \( V_2 \) be a 2-dimesional vector space with a symmetric form determined by the matrix

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

Let \( W_4 \) be a 4-dimensional vector space with a symmetric form determined by

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

Define \( z \in \text{End}(W_4) \) by the matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
One may assume that
\[ x = z^\oplus n \oplus 0 \in W_4^\oplus n \oplus V_2. \]
Let \( e^{(i)}_1, \ldots, e^{(i)}_4 \) be the (standard) basis of the \( i \)-th direct summand \( W_4 \) of \( W_4^\oplus n \), and let \( f_1 \) and \( f_2 \) be the basis of \( V_2 \). Define a \( 2n + 1 \)-dimensional isotropic subspace \( F \) by
\[ F := C < e^{(1)}_1, e^{(1)}_3, e^{(2)}_1, e^{(2)}_3, \ldots, e^{(n)}_1, e^{(n)}_3, f_1 > \]
and consider the isotropic flag \( \{ F_i \} \) defined by \( F_0 = 0, F_1 := F, F_2 := F^\perp \) and \( F_3 = C^{4n+2} \). One can check that \( x(F_i) \subset F_{i-1} \) for all \( i \). Let \( Q \subset SO(4n+2) \) be the parabolic subgroup stabilizing this flag. One can check that \( x \in \mathfrak{n}(q) \) and \( \dim O_x = 2 \dim SO(4n+2)/Q \). Therefore, \( O_x \) is the Richardson orbit for \( Q \). But \( \{ F_i \} \) is not the unique isotropic flag (of type \( (2n+1, 2n+1) \)) with this property. We put
\[ F' := C < e^{(1)}_1, e^{(1)}_3, e^{(2)}_1, e^{(2)}_3, \ldots, e^{(n)}_1, e^{(n)}_3, f_2 >, \]
and define \( \{ F'_i \} \) by \( F'_0 = 0, F'_1 := F', F'_2 := (F')^\perp \) and \( F'_3 := C^{4n+2} \). Then \( \{ F'_i \} \) has the same property. Let \( Q' \) be the corresponding parabolic subgroup of \( SO(4n+2) \). Then \( O_x \) is the Richardson orbit for \( Q' \). Although \( Q \) and \( Q' \) have the same flag types, \( Q \) and \( Q' \) are not conjugate as the subgroups of \( SO(4n+2) \). As a consequence, we have two Springer map
\[ SO(4n+2) \times^Q \mathfrak{n}(q) \to \bar{O}_x \leftarrow SO(4n+2) \times^{Q'} \mathfrak{n}(q') \]
and both of them are birational. @

(2.6) (Crepant resolutions and \( \mathbb{Q} \)-factorial terminalizations): As before, let \( O \) be a nilpotent orbit of \( \mathfrak{g} \) and let \( \bar{O} \) be the normalization of \( \bar{O} \). It is an important problem to find a crepant resolution or its substitute for \( \bar{O} \). Let us recall

**Definition (2.6.1).** Let \( X \) and \( Y \) be normal variety with rational Gorenstein singularities and let \( \mu : Y \to X \) be a birational projective morphism. Then \( \mu \) is called a crepant resolution (resp. \( \mathbb{Q} \)-factorial terminalization) of \( X \) if \( Y \) is smooth (resp. \( Y \) has only \( \mathbb{Q} \)-factorial terminal singularities) and \( K_Y = \mu^* K_X \).

**Theorem (2.6.2).** Let \( O \) be a nilpotent orbit of a complex simple Lie algebra \( \mathfrak{g} \). Then there are a parabolic subalgebra \( \mathfrak{g} \) of \( \mathfrak{g} \) and a nilpotent orbit \( O' \) of \( \mathfrak{l}(\mathfrak{q}) \) such that the following holds:
(1) \( O = \text{Ind}_{\mathfrak{n}}^\mathfrak{g}(O') \).

(2) Let \( \nu : G \times Q (\mathfrak{n} + \mathfrak{O}') \to \mathfrak{O} \) be the generalized Springer map. Then its normalized map \( \nu' \) (cf. (2.5.1)) gives a \( Q \)-factorial terminalization of \( \mathfrak{O} \).

Theorem (2.6.2) is due to [Na 3] when \( \mathfrak{g} \) is of classical type and is due to [Fu 2] when \( \mathfrak{g} \) is of exceptional type. Here we shall give a rough sketch of the proof when \( \mathfrak{g} \) is classical. First let us consider the case \( \mathfrak{g} = \mathfrak{sl}(n) \). Assume that \( \mathfrak{O} \) has Jordan type \( d \). Let \( d' = \left[ q_1, q_2, \ldots, q_l \right] \) be the dual partition of \( d \). By Example (2.5.1), \( \mathfrak{O} \) is the Richardson orbit for a parabolic subgroup \( Q \subset SL(n) \) of flag type \( (q_1, q_2, \ldots, q_l) \). Moreover, the Springer map \( \nu : G \times Q \mathfrak{n} \to \mathfrak{O} \) is birational. Note that \( G \times Q \mathfrak{n} \) is isomorphic to the cotangent bundle \( T^* (G/Q) \) of the homogeneous space \( G/Q \). The pull-back of the Kostant-Kirillov 2-form \( \omega \) (cf. (2.1)) coincides with the canonical 2-form of \( T^* (G/Q) \); hence \( \nu \) is a crepant resolution. Next let us consider the cases \( \mathfrak{g} = \mathfrak{sp}(m) \) and \( \mathfrak{g} = \mathfrak{so}(m) \). We say that a partition \( d := [d_1, d_2, \ldots, d_k] \) of \( m \) has full members if \( d_i = k + 1 - i \) for all \( i \).

**Proposition (2.6.3).** Assume that \( \mathfrak{g} = \mathfrak{sp}(m) \) or \( \mathfrak{so}(m) \). Then \( \mathfrak{O}_d \) has terminal singularities if and only if \( d \) has full members. If \( d \) has full members, then \( \mathfrak{O}_d \) is \( Q \)-factorial except when \( \mathfrak{g} = \mathfrak{so}(4n + 2), n \geq 1 \) and \( d = [2^{2n}, 1^2] \).

For the proof of Proposition (2.6.3), see [Na 3].

Assume that \( \mathfrak{O} \) does not have \( Q \)-factorial terminal singularities. By (2.6.3), \( \mathfrak{O} \) does not have full members or \( \mathfrak{O} = \mathfrak{O}_{[2^{2n}, 1^2]} \subset \mathfrak{so}(4n + 2) \). In the second case, \( \mathfrak{O} \) is a Richardson orbit and has a crepant resolution by (2.5.6). In the first case, let \( d := [d_1, \ldots, d_k] \) be the Jordan type of \( \mathfrak{O} \). Then \( d_p \geq d_{p+1} + 2 \) for some \( p \). The situation is now the same as (2.5.5). The orbit \( \mathfrak{O} \) is induced, and as in (2.5.5) one can find a generalized Springer map

\[
\nu : G \times Q (\mathfrak{n} + \mathfrak{O}') \to \mathfrak{O},
\]

which is birational. Then \( \mathfrak{O}' \) is again a nilpotent orbit of a smaller Lie algebra of the same type. If \( \mathfrak{O}' \) already has \( Q \)-factorial terminal singularities, then the normalization of \( G \times Q (\mathfrak{n} + \mathfrak{O}') \) gives a \( Q \)-factorial terminalization (cf. Proposition (4.2) below, see also [Na 3, Lemma (1.2.4) \cite{remark}]). If not, then we repeat the same process. By (2.5.2) we have a birational map

\[
\nu' : G \times Q' (\mathfrak{n} + \mathfrak{O}') \to G \times Q (\mathfrak{n} + \mathfrak{O}').
\]

\footnote{The proof of [Na 3], Lemma (1.2.4) contains an error. In fact, \( < y, [v_1, w_1] > \) is claimed to be zero there, but it is not correct. The equality starting from line 9, p 552 should contain the additional term \( < y, [v_1, w_1] > \). The equality on line -2, p 552 should also contain \( < y, [v_1, w_1] > \). But the claim itself is correct.}
Finally we get a $\mathbb{Q}$-factorial terminalization of $\tilde{O}$.

§3. Birational geometry of $\mathbb{Q}$-factorial terminalizations

(3.1) Parabolic subgroups and root systems. Let $G$ be a simple algebraic group over $\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. We fix a maximal torus $T$ of $G$ and denote by $\mathfrak{h}$ its Lie algebra. Let $\Phi$ be the root system for $\mathfrak{g}$ determined by $\mathfrak{h}$ (cf. [Hu 1]). The root system $\Phi$ has a natural involution $-1$. There is a (unique) involution $\varphi_{\mathfrak{g}}$ of $\mathfrak{g}$ which stabilizes $\mathfrak{h}$ and which acts on $\Phi$ via $-1$ (cf. [Hu 1], 14.3). Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the root space decomposition. Let us choose a base $\Delta$ of $\Phi$ and denote by $\Phi^+$ the set of positive roots. Then $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is a Borel subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$. Let $B$ be the corresponding Borel subgroup of $G$. Take a subset $I$ of $\Delta$. Let $\Phi_I$ be the root subsystem of $\Phi$ generated by $I$ and put $\Phi_I^- := \Phi_I \cap \Phi^-$, where $\Phi^-$ is the set of negative roots. Then

$$\mathfrak{q}_I := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_I^-} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

is a parabolic subalgebra containing $\mathfrak{b}$. This parabolic subalgebra $\mathfrak{q}_I$ is called a standard parabolic subalgebra with respect to $I$. Let $Q_I$ be the corresponding parabolic subgroup of $G$. By definition, $B \subset Q_I$. Any parabolic subgroup $Q$ of $G$ is conjugate to a standard parabolic subgroup $Q_I$ for some $I$. Moreover, if two subsets $I, I'$ of $\Delta$ are different, $Q_I$ and $Q_{I'}$ are not conjugate. Thus, a conjugacy class of parabolic subgroups of $G$ is completely determined by $I \subset \Delta$. In this paper, to specify the subset $I$ of $\Delta$, we shall use the marked Dynkin diagram. Recall that $\Delta \subset \Phi$ defines a Dynkin diagram; each vertex corresponds to a simple root (an element of $\Delta$). Now, if a subset $I$ of $\Delta$ is given, we indicate the vertices corresponding to $I$ by white vertices, and other vertices by black vertices. A black vertex is called a marked vertex. A Dynkin diagram with such a marking is called a marked Dynkin diagram, and a marked Dynkin diagram with only one marked vertex is called a single marked Dynkin diagram. Note that the standard parabolic subgroup corresponding to a single marked Dynkin diagram (resp. full marked Dynkin diagram) is a maximal parabolic subgroup (resp. a Borel subgroup). Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$. Let $\mathfrak{r}(\mathfrak{q})$ (resp. $\mathfrak{n}(\mathfrak{p})$) be the solvable radical (resp. nilpotent radical) of $\mathfrak{q}$. We put $\mathfrak{k}(\mathfrak{q}) := \mathfrak{r}(\mathfrak{q}) \cap \mathfrak{h}$. Then

$$\mathfrak{r}(\mathfrak{q}) = \mathfrak{r}(\mathfrak{q}) \oplus \mathfrak{n}(\mathfrak{q}).$$
On the other hand, the Levi factor $l(q)$ of $q$ is defined as $l(q) := g^{t(q)}$. Here, $g^{t(q)} := \{x \in g; [x, y] = 0, \forall y \in t(q)\}$. Note that $t(q)$ is the center of $l(q)$.

Then

$$ q = l(q) \oplus n(q) $$

and

$$ l(q) = t(q) \oplus [t(q), l(q)]. $$

If $q = q_I$, we have

$$ t(q_I) = \{h \in h; \alpha(h) = 0, \forall \alpha \in I\}. $$

Moreover, we define

$$ t(q_I)^{reg} = \{h \in t(q_I); \alpha(h) \neq 0, \forall \alpha \in \Phi \setminus \Phi_I\}. $$

Note that $t(q_I)^{reg}$ is an open subset of $t(q_I)$.

(3.2) (*Parabolic subalgebras with a fixed Levi part*) A subalgebra $l$ of $g$ is called a Levi subalgebra if it is the Levi part of some parabolic subalgebra $q$ of $g$. We fix a Levi subalgebra $l$ and put

$$ \mathcal{S}(l) := \{\text{parabolic subalgebras } q \text{ of } g; l(q) = l\}. $$

Fix a Cartan subalgebra $h$ of $g$ so that $h \subset l$. Let $b$ be a Borel subalgebra of $g$ so that $h \subset b \subset q$. Then $q$ corresponds to a marked Dynkin diagram $D$. Take a marked vertex $v$ of the Dynkin diagram $D$ and consider the maximal connected single marked Dynkin subdiagram $D_v$ of $D$ containing $v$. We call $D_v$ the single marked diagram associated with $v$. When $D_v$ is one of the following, we say that $D_v$ (or $v$) is of the first kind, and when $D_v$ does not coincide with any of them, we say that $D_v$ (or $v$) is of the second kind.

\[ A_{n-1} \quad (k < n/2) \]

\[ D_n \quad (n : \text{odd} \geq 5) \]
In the single marked Dynkin diagrams above, two diagrams in each type (i.e. $A_{n-1}$, $D_n$, $E_{6,I}$, $E_{6,II}$) are called duals. The Weyl group $W$ of $\mathfrak{g}$ does not contain $-1$ exactly when $\mathfrak{g} = A_n (n \geq 2)$, $D_n (n: \text{odd})$ or $E_6$ (cf. [Hu 1], p.71, Exercise 5). This property characterizes the Dynkin diagrams in the list. Moreover, the single marked Dynkin diagrams $D_v$ in the list are characterized by the following property.

(*) Let $\mathfrak{q}_v$ be the parabolic subalgebra of $\mathfrak{g}$ corresponding to $D_v$, and let $\varphi_\mathfrak{g}$ be the automorphism of $\mathfrak{g}$ determined by $-1$ (cf.(3.1)). Then $\varphi_\mathfrak{g}(\mathfrak{q}_v)$ is not conjugate to $\mathfrak{q}_v$. If $\varphi_\mathfrak{g}(\mathfrak{q}_v)$ corresponds to a single marked Dynkin diagram $D_v^*$, then $D_v$ and $D_v^*$ are mutually duals.

Let $\bar{D}$ be the marked Dynkin diagram obtained from $D$ by making $v$ unmarked. Let $\bar{\mathfrak{q}}$ be the parabolic subalgebra containing $\mathfrak{q}$ corresponding to $\bar{D}$. Now let us define a new marked Dynkin diagram $D'$ as follows. If $D_v$ is of the first kind, we replace $D_v \subset D$ by its dual diagram $D_v^*$ to get a new marked Dynkin diagram $D'$. If $D_v$ is of the second kind, we define $D' := D$. The new diagram $D'$ obtained in this way is called an adjacent diagram to $D$. As in (3.1), the set of unmarked vertices of $D$ (resp. $\bar{D}$) defines a subset $I \subset \Delta$ (resp. $\bar{I} \subset \Delta$). By definition, $v \in \bar{I}$. The unmarked vertices of $\bar{D}$ define a Dynkin subdiagram, which is decomposed into the disjoint sum of the connected component containing $v$ and the union of other components. Correspondingly, we have a decomposition $\bar{I} = I_v \cup I'_v$ with $v \in I_v$. The parabolic subalgebra $\mathfrak{q}$ (resp. $\bar{\mathfrak{q}}$) coincides with the standard parabolic subalgebra $\mathfrak{q}_I$ (resp. $\mathfrak{q}_{\bar{I}}$). Let $I_f$ be the (standard) Levi factor of $\mathfrak{q}_I$. Let $\mathfrak{z}(l_f)$ be the center of $l_f$. Then $l_f/\mathfrak{z}(l_f)$ is decomposed into the direct
sum of simple factors. Now let \( I_v \) be the simple factor corresponding to \( I_v \) and let \( I_v' \) be the direct sum of other simple factors. Then

\[
I_I/3(I_I) = I_v \oplus I_v'.
\]

The marked Dynkin diagram \( D_v \) defines a standard parabolic subalgebra \( q_v \) of \( l_{I_v} \). Here let us consider the involution \( \varphi_{l_{I_v}} \in \text{Aut}(l_{I_v}) \) (cf. (3.1)). When \( D_v \) is of the first kind, \( \varphi_{l_{I_v}}(q_v) \) is conjugate to a standard parabolic subalgebra of \( I_v \) with the dual marked Dynkin diagram \( D_v^* \) of \( D_v \). When \( D_v \) is of the second kind, \( \varphi_{l_{I_v}}(q_v) \) is conjugate to \( q_v \) in \( I_v \). Let \( \pi : I_I \to I_I/3(I_I) \) be the quotient homomorphism. Note that

\[
\bar{q} = I_I \oplus n(q),
\]

\[
q = \pi^{-1}(q_v \oplus I_v') \oplus n(q).
\]

Here we define

\[
q' = \pi^{-1}(\varphi_{l_{I_v}}(q_v) \oplus I_v') \oplus n(q).
\]

Then \( q' \in S(I) \) and \( q' \) is conjugate to a standard parabolic subalgebra with the marked Dynkin diagram \( D' \). This \( p' \) is said to be the parabolic subalgebra twisted by \( v \). Two marked diagrams \( D_1 \) and \( D_2 \) are called equivalent if there is a finite chain of adjacent diagrams connecting \( D_1 \) and \( D_2 \).

**Definition (3.2.1) (Mukai flops and primitive pairs):** Let \( q \) and \( q' \) be two parabolic subalgebras of \( g \) corresponding to the dual diagrams in the list above. Assume that \( q \) and \( q' \) have a common Levi factor \( I \). When the diagram is of type \( A_{n-1,k}, D_n \) or \( E_{6,11} \), define \( O' \) to be the 0-orbit in \( I \). When the diagram is of type \( E_{6,11} \), define \( O' \) to be the 0-orbit, \( O'_{[3,2^2,1^3]} \) or \( O'_{[2^2,1^6]} \). Such a pair \( (q, O') \) is called a primitive pair. We put \( O := \text{Ind}^g_i(O') \). Then we have a diagram of normalized maps (cf. (2.5.1)) of (generalized) Springer maps:

\[
G \times Q (n(q) + \tilde{O}') \overset{\nu^n}{\rightarrow} \tilde{O}' \overset{(\nu')^n}{\leftarrow} G \times Q' (n(q')) + \tilde{O}'.
\]

For every primitive pair, \( \nu^n \) and \( (\nu')^n \) are both isomorphisms in codimension one. This diagram is called a Mukai flop.

(3.3) (Brieskorn-Slodowy diagram): Let \( O \subset g \) be the induced orbit from \( O' \subset l(q) \) for a parabolic subalgebra \( q \). Assume that \( \tilde{O}' \) has only \( Q \)-factorial terminal singularities. Let us consider the subvariety

\[
\tau(q) + \tilde{O}' \subset \tau(q) \oplus [l(q), l(q)].
\]
The $Q$-adjoint action stabilizes $\mathfrak{r}(q) + \bar{O}'$ as a set; hence one has an associated fiber bundle
\[ X'_{q, O'} := G \times^Q (\mathfrak{r}(q) + \bar{O}') \]
over $G/Q$. Write $x \in \mathfrak{r}(q) + \bar{O}'$ as $x = x_1 + x_2 + x_3$, where $x_1 \in \mathfrak{r}(q)$, $x_2 \in \mathfrak{n}(q)$ and $x_3 \in [l(q), l(q)]$. Define a map
\[ \eta : X'_{q, O'} \to \mathfrak{t}(q) \]
by $[g, x] \in X_q \to x_1$. This is a well-defined map; in fact, for $q \in Q$, we have $(Ad_q(x_1))_1 = x_1$, $Ad_q(x_2) \in \mathfrak{n}(q)$ and $Ad_q(x_3) \in \mathfrak{n}(q) \oplus [l(q), l(q)]$.

**Lemma (3.3.1).** For $t \in \mathfrak{t}(q)^{reg}$, any orbit of the $Q$-variety $t + \mathfrak{n}(q) + \bar{O}'$ is of the form $Q(t + y)$ with $y \in \bar{O}'$.

**Proof.** We shall prove that $\bigcup_{y \in \bar{O}'Q} \cdot (t + y) = t + \mathfrak{n}(q) + \bar{O}'$. Define $Z_Q(t + y) := \{ q \in Q ; Ad_q(t + y) = t + y \}$. Then $Q \cdot (t + y) \cong Q/Z_Q(t + y)$. Note that $t$ (resp. $y$) is the semi-simple part (resp. nilpotent part) of $t + y$ in the Jordan-Chevalley decomposition because $[t, y] = 0$. Hence $Z_Q(t + y) = Z_Q(t) \cap Z_Q(y)$. Since $t \in \mathfrak{t}(q)^{reg}$, we have $Z_Q(t) = L(Q)$, and $Z_Q(t) \cap Z_Q(y) = Z_{L(Q)}(y)$. Let $O_y \subset l(q)$ be the $L(Q)$-adjoint orbit containing $y$. Then one has
\[ \dim Q/Z_Q(t + y) = \dim \mathfrak{n}(q) + \dim O_y. \]

Let us write $Q = U(Q) \cdot L(Q)$ with the unipotent radical $U(Q)$. Since $L(Q) \cdot (t + y) = t + O_y$ and $U(Q) \cdot (t + y) \subset t + y + \mathfrak{n}(q)$, we see that $U(Q) \cdot (t + y)$ is dense in $t + y + \mathfrak{n}(q)$. But any $U(Q)$-orbit in $t + y + \mathfrak{n}(q)$ is closed (cf. [Hu 2], §17, Exercise 8), $U(Q) \cdot (t + y) = t + y + \mathfrak{n}(q)$. Q.E.D.

We shall prove that the following diagram commutes:

\[
\begin{array}{ccc}
X'_{q, O'} & \longrightarrow & G \cdot (\mathfrak{r}(q) + \bar{O}') \\
\downarrow \eta & & \downarrow \chi \\
\mathfrak{t}(q) & \longrightarrow & \mathfrak{h}/W.
\end{array}
\]

(2)

Here $\chi$ is the composite of the inclusion map $G(\mathfrak{r}(q) + \bar{O}') \to \mathfrak{g}$ and the adjoint quotient map $\mathfrak{g} \to \mathfrak{h}/W$. The horizontal map on the first row is given by $[g, x] \to Ad_g(x)$ and the horizontal map $\iota$ on the second row is the composite of two maps $\mathfrak{t}(q) \to \mathfrak{h}$ and $\mathfrak{h} \to \mathfrak{h}/W$. Define
\[ W' := N_{W}(L(Q))/W(L(Q)), \]
where $N_W(L(Q))$ is the normalizer subgroup of $W$ for $L(Q)$. Then $W'$ acts on $\mathfrak{t}(q)$ and the normalization of $\text{Im}(\iota)$ is $\mathfrak{t}(q)/W'$. Let us check the commutativity of the diagram. Choose $t \in \mathfrak{t}(q)^{reg}$ and $y \in O'$. Then $G \times Q \cdot (t+y)$ is an open dense subset of $\eta^{-1}(t) = G \times Q \cdot (t + n(q) + \bar{O'})$ by Lemma (3.3.1). We only have to check the commutativity of the diagram for an element $[g, t+y] \in G \times Q \cdot (t+y)$:

$$
\begin{array}{ccc}
[g, t+y] & \longrightarrow & Ad_g(t+y) \\
\downarrow & & \downarrow \\
t & \longrightarrow & [t].
\end{array}
$$

The commutativity now follows from the fact that $Ad_g(t+y) = Ad_g(t) + Ad_g(y)$ coincides with the Jordan-Chevalley decomposition of $Ad_g(t+y)$. We put $Y_{l(q), O'} := (\mathfrak{t}(q) \times_{h/W} G \cdot (\mathfrak{r}(q) + \bar{O}'))_{red}$. Note that $\mathfrak{t}(q)$ only depends on the Levi part $l(q)$ of $q$. Moreover, since $G \cdot (\mathfrak{r}(q) + \bar{O}') = G \cdot (\mathfrak{r}(q)^{reg} + \bar{O}')$,

$Y_{l(q), O'}$ only depends on $l(q)$ and $O'$ as the index indicates. The commutative diagram induces a map

$$
\mu_{l(q), O'}^g : X_{l(q), O'}^t \rightarrow Y_{l(q), O'}^t.
$$

A nilpotent orbit of $l(q)$ is not necessarily stable under the $W'$-action. But, in our case, we have:

**Lemma (3.3.2):** All elements $w \in W'$ stabilizes $O'$.

**Proof.** If $O' = 0$, then the statement is obvious. Assume that $O' \neq 0$. Let us consider the decomposition of $l(q)$ into simple factors (up to centers). Then $O'$ is contained in a simple factor $g'$ whose type in not $A$. Then each element $w \in W'$ induces an automorphism of Lie algebra $g'$. If $\text{Aut}(g') = \text{Aut}(g')^0$, then $w$ acts on $g$ as an adjoint action $Ad_g$ for some $g \in G'$. In this case, $O'$ is stable by $W'$. So we may assume that $g$ is of type $D$ or $E_6$. Suppose that $O'$ is sent to a different nilpotent orbit $O'' \subset g'$ by some $\phi \in \text{Aut}(g')$. Then $\phi$ acts on the Dynkin diagram as a graph automorphism. The weighted Dynkin diagram of $O'$ should be sent to that of $O''$ by this graph automorphism. If $g'$ is of type $D$, such things happen only when the orbit is very even or it is $O_{[5,13]} \subset so(8)$. Our $O'$ does not coincide with any of them. If $g'$ is of
type $E_6$, then one can check that there are no such orbits by using the list of [C-M], page 129. Q.E.D.

**Proposition (3.3.3).** The map $\mu'_q$ is a birational projective morphism. In particular, $Y'_{(q),O'}$ is irreducible. Moreover, for $t \in \mathfrak{r}(q)^{\text{reg}}$, the induced map $\eta^{-1}(t) \to \{t\} \times_y \chi^{-1}([t])$ is a bijection.

**Proof.** The map $\mu'_q$ is written as the composite of a closed immersion and a projective morphism: $G \times Q (\mathfrak{r}(q) + O') \to G/Q \times g \to g$. Hence $\mu'_q$ is a projective morphism. By (3.3.1), for $t \in \mathfrak{r}(q)^{\text{reg}}$, the fiber $\chi^{-1}([t])$ coincides with

$$
\bigcup_{y \in \bar{O'}} \bigcup_{s \in \mathfrak{r}(q), [s]=[t]} G \cdot (s + y).
$$

But, if $[s] = [t]$, then $s = Ad_w(t)$ with some $w \in W'$. By (3.3.2), $Ad_w(y) \in \bar{O'}$. Therefore

$$
\chi^{-1}([t]) = \bigcup_{y \in \bar{O'}} G \cdot (t + y).
$$

In particular, $\chi^{-1}([t])$ is irreducible for $t \in \mathfrak{r}(q)^{\text{reg}}$. By the argument above, any point of $\chi^{-1}([t])$ is $G$-equivariant in $t + y$ with $y \in \bar{O'}$. Since $\mu'_q$ is $G$-equivariant, it is sufficient to prove that $\mu'^{-1}_q(t, t + y)$ consists of exactly one point, where $(t, t + y) \in \{t\} \times_y \chi^{-1}([t])$. Assume that $[g, x] \in G \times Q (\mathfrak{r}(q) + O')$ is contained in $\mu'^{-1}_q(t, t + y)$. Then $Ad_g(x) = t + y$. Moreover, by (3.1.1), one can write $x = Ad_q(t + y')$ with some $q \in Q$ and some $y' \in \bar{O'}$. This means that $Ad_q(t + y') = t + y$. Since $t + y$ and $t + y'$ are both Jordan-Chevalley decompositions, we see that $t = Ad_q(t)$ and $y = Ad_g(y')$. By the first equality, we have $gg \in L(Q)$, and hence $g \in Q$. By the second equality, we have

$$
x = Ad_q(t + y') = Ad_q(t + Ad_q^{-1}g^{-1}(y)) = Ad_q(Ad_q^{-1}g^{-1}(t+y)) = Ad_{g^{-1}}(t+y).
$$

As a consequence,

$$
[g, x] = [g, Ad_{g^{-1}}(t + y)] = [1, t + y].
$$

The rest of the argument is the same as [Na 2], Lemma 1.1. Q.E.D.

Let $X_{q,O'}$ be the normalization of $X'_{(q),O'}$ and let $Y'_{(q),O'}$ be the normalization of $Y'_{(q),O'}$. Then $\mu'_q$ induces a commutative diagram

$$
X_{q,O'} \xrightarrow{\mu'_q} Y'_{(q),O'} \quad \text{(4)}
$$

\[\text{Diagram (4)}\]
Let $X_{q,O',0}$ (resp. $Y_{l(q),O',0}$) be the fiber of the map $X_{q,O'} \to \mathfrak{k}(q)$ (resp. $Y_{l(q),O'} \to \mathfrak{k}(q)$) over $0 \in \mathfrak{k}(q)$.

**Lemma (3.3.4).** Assume that

$$\nu^n : G \times^\mathfrak{g} (\mathfrak{n}(q) + \tilde{O}') \to \tilde{O}$$

is birational (cf. (2.5.1). Then one has

$$X_{q,O',0} = G \times^\mathfrak{g} (\mathfrak{n}(q) + \tilde{O}'),$$

$$Y_{l(q),O',0} = \tilde{O}.$$

The map

$$\mu_{q,0} : X_{q,O',0} \to Y_{l(q),O',0}$$

coincides with the normalized map $\nu^n$ of the generalized Springer map.

**Proof.** The first statement is obvious. Since $Y_{q,O'}$ is Cohen-Macaulay and $\mathfrak{k}(q)$ is smooth, $Y_{q,O',0}$ is also Cohen-Macaulay. The map $\mu_{q,0} : X_{q,O',0} \to Y_{q,O',0}^{\text{red}}$ is a birational morphism with connected fibers. It factorizes the generalized Springer map $X_{q,O',0} = G \times^\mathfrak{g} (\mathfrak{n}(q) + \tilde{O}') \to \tilde{O}$. By the assumption, the generalized Springer map is an isomorphism over $O$. This means that $\mu_{q,0}$ is an isomorphism outside a certain codimension 2 subset $Z$ of $Y_{q,O',0}^{\text{red}}$ and $Y_{q,O',0}^{\text{red}} - Z$ is smooth. Take a point $x \in Y_{q,O',0}^{\text{red}} - Z$. Then we have a surjection

$$\mathcal{O}_{Y_{q,O',0}} \to \mathcal{O}_{Y_{q,O',0}^{\text{red}}} \cong \mathcal{O}_{X_{q,O',0},\mu_{q,0}^{-1}(x)}.$$

By Nakayama’s lemma, this implies that $\mathcal{O}_{Y_{q,O',0}} \cong \mathcal{O}_{X_{q,O',0},\mu_{q,0}^{-1}(x)}$. Therefore, $Y_{q,O',0}$ is reduced at $x$, and moreover, $Y_{q,O',0}$ is smooth at $x$. Since $Y_{q,O',0}$ is Cohen-Macaulay and regular in codimension one, $Y_{q,O',0}$ is normal. This means that $Y_{q,O',0} = \tilde{O}$.

**Proposition (3.3.5).** The map $\mu_q$ is crepant and is an isomorphism in codimension one.

**Proof.** Since $X_{q,O'}$ has only terminal singularities and its canonical line bundle is trivial, $K_{X_{q,O'}}$ is $\mu_q$-numerically trivial. By Kawamata-Viehweg vanishing theorem, $R^j(\mu_q)_* \mathcal{O}_{X_{q,O'}} = 0$ for $j > 0$. Therefore, $K_{X_{q,O'}}$ is the pull-back of a line bundle $M$ on $Y_{l(q),O'}$. Since $Y_{l(q),O'}$ has a $\mathbb{C}^*$-action with positive weights, its Picard group is trivial: $\text{Pic}(Y_{l(q),O'}) = 0$. This means that $K_{X_{q,O'}}$ is a trivial line bundle. Then $K_{Y_{l(q),O'}} = (\mu_q)_* K_{X_{q,O'}}$ is also trivial.
and $K_{X_{q,O'}} = (\mu_q)^* K_{Y_{l(q),O'}}$. The second assertion follows from Proposition (3.3.3).

**Corollary (3.3.6).** Assume that

$$\nu^n : G \times^Q (n(q) + \tilde{O}') \to \tilde{O}$$

is birational. Then, for $q' \in S(l(q))$, the normalized map

$$(\nu')^n : G \times^{Q'} (n(q') + \tilde{O}') \to \tilde{O}$$

is birational.

**Proof.** The map

$$\mu_{q',0} : X_{q',O',0} \to Y_{l(q),O'}$$

coinsides with $(\nu')^n : G \times^{Q'} (n(q') + \tilde{O}') \to \tilde{O}$ by (3.3.4). Moreover, $(\nu')^n$ is a generically finite morphism. On the other hand, $\mu_{q'}$ is birational by (3.3.5). Since $Y_{l(q),O'}$ is normal, $\mu_{q',0}$ has connected fibers. This means that $(\nu')^n$ is birational.

(3.4) **Nef cones and flops.** Let $(q, O')$ be the same as in (3.3). Furthermore we assume that

$$\nu^n : G \times^Q (n(q) + \tilde{O}') \to \tilde{O}$$

is birational.

(3.4.1) **Nef cone of $G/Q$:** Let $Q \subset G$ be a parabolic subgroup of $G$. Note that $\text{Pic}(G/Q) \cong H^2(G/Q, \mathbb{Z})$. Define $M(L(Q)) := \text{Hom}_{\text{alg.gp}}(L(Q), \mathbb{C}^*)$. Let $\chi : L(Q) \to \mathbb{C}^*$ be an element of $M(L(Q))$. By the exact sequence

$$1 \to U(Q) \to Q \to L(Q) \to 1,$$

$\chi$ defines a group homomorphism $Q \to \mathbb{C}^*$, which gives rise to a line bundle $L_\chi := G \times^Q \mathbb{C}$ on $G/Q$. The correspondence $\chi \to L_\chi$ gives a map

$$\phi : M(L(Q)) \to \text{Pic}(G/Q)$$

and it turns out be an isomorphism after tensorized with $\mathbb{R}$: $M(L(Q))_\mathbb{R} \cong \text{Pic}(G/Q)_\mathbb{R}$. The nef cone $\text{Amp}(G/Q)$ is a closed convex cone in $H^2(G/Q, \mathbb{R})$ generated by nef line bundles on $G/Q$. Let us describe $\text{Amp}(G/Q)$ as a cone in $M(L(Q))_\mathbb{R}$ in terms of roots. Assume that $Q$ is a standard parabolic subgroups $Q_I$ containing a Borel subgroup $B$ (cf. (3.1)). Recall that $\Delta \setminus I$
corresponds to the set of marked vertices \( \{v_1, \ldots, v_\rho \} \) of the Dynkin diagram. Then \( \rho = b_2(G/Q_I) \). The nef cone \( \overline{\text{Amp}}(G/Q_I) \) is then generated by dominant characters \( \chi \) of \( L(Q_I) \) (i.e. \( \langle \chi, \alpha^\vee \rangle \geq 0, \forall \alpha \in \Delta \)), where \( \alpha^\vee \) is the coroot corresponding to \( \alpha \), and \( \chi \) is regarded as an element of \( \mathfrak{h}^* \)). Moreover, it is a simplicial cone and its codimension 1 face consists of the dominant characters with \( \langle \chi, v_i^\vee \rangle = 0 \) for some \( i \). We denote by \( F_{v_i} \) this face.

(3.4.2) \( \text{(Nef cones of } X_{q,0} \text{ and } X_{q,O',0} ) \): Note that each fiber of the natural projection \( \pi : X_{q,0} \to G/Q \) is isomorphic to the normalization of \( \mathfrak{q} + \tilde{O}' \), which coincides with \( \mathfrak{q} \times \tilde{O}' \). Here \( \tilde{O}' \) is the normalization of \( O' \). Since \( \mathfrak{q} \times \tilde{O}' \) is topologically contractible to the origin (by the natural \( \mathbb{C}^* \)-action), we see that \( \pi^*(\text{Amp}(G/Q)) \cong \text{Amp}(X_{q,O',0}) \). Define \( \overline{\text{Amp}}(\mu_q) \) to be the closed convex cone in \( \text{Amp}(X_{q,O',0}) \) generated by \( \mu_q \)-nef line bundles on \( X_{q,O'} \). As in [Na 2], (P.3), one can prove that

\[
\pi^*(\text{Amp}(G/Q)) = \overline{\text{Amp}}(\mu_q).
\]

Next let us consider \( X_{q,O',0} \). Let \( p : X_{q,O',0} \to G/Q \) be the natural projection. The situation is quite similar to the case of \( X_{q,O'} \). One can prove that

\[
p^*(\text{Amp}(G/Q)) = \overline{\text{Amp}}(\mu_{q,0}).
\]

In particular, \( \overline{\text{Amp}}(\mu_q) \) and \( \overline{\text{Amp}}(\mu_{q,0}) \) are both rational simplicial cones. Finally we shall define the \textit{movable cones} \( \overline{\text{Mov}}(\mu_q) \) for \( \mu_q \) to be the closed convex cone of \( H^2(X_{q,O'}, \mathbb{R}) \) generated by the \( \mu_q \)-movable line bundles. Here a line bundle \( L \in \text{Pic}(X_{q,O'}) \) is called \( \mu_q \)-movable if the support of

\[
\text{Coker}[(\mu_q)^* (\mu_q)_* L \to L]
\]

has codimension \( \geq 2 \). Denote by \( \overline{\text{Mov}}(\mu_q) \) the interior of \( \overline{\text{Mov}}(\mu_q) \). Similarly we define \( \overline{\text{Mov}}(\mu_{q,0}) \) and \( \overline{\text{Mov}}(\mu_{q,0}) \).

(3.4.3) \( \text{Twists and flops} \): We may assume that \( Q \) is a standard parabolic subgroup \( Q_I \) defined in (3.1). Take \( v \in \Delta - I \) and put \( \tilde{I} := I \cup \{v\} \). We shall use the same notation as in (3.2). Then \( \tilde{I} \) is decomposed into two sets

\[
\tilde{I} = I_v \cup I'_v.
\]

Let \( \mathfrak{l}(I) \) be the standard Levi factor of \( \mathfrak{q}_I \) and let \( \mathfrak{z}(\mathfrak{l}(I)) \) be its center. Then \( \mathfrak{l}(\tilde{I})/\mathfrak{z}(\mathfrak{l}(\tilde{I})) \) is decomposed into a simple factor \( \mathfrak{l}_{I_v} \) and other parts \( \mathfrak{l}_{I'_v} \):

\[
\mathfrak{l}(\tilde{I})/\mathfrak{z}(\mathfrak{l}(\tilde{I})) = \mathfrak{l}_{I_v} \oplus \mathfrak{l}_{I'_v}.
\]
Note that \( O' \subset I_{l_c} \) or \( O' \subset I_{l'_c} \). For simplicity, we write \( g_v \) for \( I_{l_c} \) and \( Q \) for \( Q_f \).

Remark that \( q_v := q \cap g_v \) is a parabolic subalgebra of \( g_v \). Identify \( \text{Amp}(\mu_q) \) with \( \text{Amp}(G/Q) \) as in (3.4.2). As in (3.4.1), \( v \) determines a codimension 1 face \( F_v \) of \( \text{Amp}(\mu_q) \). We are now going to construct the birational contraction map of \( X_{g,v}^{O'} \) corresponding to \( F_v \). First look at the \( Q \)-orbit \( Q \cdot (r(q) + O') \) of \( r(q) + O' \). Then we can write

\[
\bar{Q} \cdot (r(q) + O') = \begin{cases} 
q \times G_v \cdot (r(q_v) + O') & (O' \subset g_v) \\
q \times O' \times G_v \cdot r(q_v) & (O' \subset I_{l'_c}) 
\end{cases}
\]

Put \( W_v := W(\mathfrak{g}_v) \) and let \( h_v \) be a Cartan subalgebra of \( g_v \). By the adjoint quotient map \( G_v \cdot (r(q_v) + O') \to h_v/W_v \) (or \( G_v \cdot r(q_v) \to h_v/W_v \)), we have a map \( Q \cdot (r(q) + O') \to h_v/W_v \). As in (3.3), we define a map \( \eta : Q \times Q (r(q) + O') \to \mathfrak{t}(q_v) \). Since there is a natural map \( Q \times Q (r(q) + O') \to Q \cdot (r(q) + O') \), we get a map

\[
\alpha : Q \times Q (r(q) + O') \to Q \cdot (r(q) + O') \times h_v/W_v \mathfrak{t}(q_v).
\]

Since \( \alpha \) is a \( Q \)-equivariant map, we obtain a map:

\[
f : X_{g,v}^{O'} \to G \times Q \ Q \cdot (r(q) + O') \times h_v/W_v \mathfrak{t}(q_v).
\]

Note that \( f \) is a morphism over \( \mathfrak{t}(q) \). By restricting \( f \) to the fibers over \( 0 \in \mathfrak{t}(q) \), we get a map

\[
f_0 : G \times Q (n(q) + O') \to G \times Q \ Q \cdot (n(q) + O').
\]

Define a map \( f_v \) by

\[
f_v : \begin{cases} 
G_v \times Q_v (r(q_v) + O') \to G_v \cdot (r(q_v) + O') \times h_v/W_v \mathfrak{t}(q_v) & (O' \subset g_v) \\
G_v \times Q_v r(q_v) \to G_v \cdot r(q_v) \times h_v/W_v \mathfrak{t}(q_v) & (O' \subset I_{l'_c}) 
\end{cases}
\]

Moreover define \( \nu_v \) to be the generalized Springer map:

\[
\nu_v : \begin{cases} 
G_v \times Q_v (n(q_v) + O') \to G_v \cdot (n(q_v) + O') & (O' \subset g_v) \\
G_v \times Q_v n(q_v) \to G_v \cdot n(q_v) & (O' \subset I_{l'_c}) 
\end{cases}
\]

Note that

\[
Q \times Q (r(q) + O') = \begin{cases} 
q \times G_v \times Q_v (r(q_v) + O') & (O' \subset g_v) \\
q \times O' \times G_v \times Q_v r(q_v) & (O' \subset I_{l'_c}) 
\end{cases}
\]
\[ Q \times Q (n(q) + \bar{O}') = \left\{ \begin{array}{ll} n(q) \times G_v \times \hat{Q_v} \cdot (n(q_v) + \bar{O}') & (O' \subset g_v) \\ n(q) \times \bar{O}' \times G_v \times \hat{Q_v} \cdot n(q_v) & (O' \subset I_{\nu}') \end{array} \right., \]

and

\[ \bar{Q} \cdot (n(q) + \bar{O}') = \left\{ \begin{array}{ll} n(q) \times G_v \cdot (n(q_v) + \bar{O}') & (O' \subset g_v) \\ n(q) \times \bar{O}' \times G_v \cdot n(q_v) & (O' \subset I_{\nu}') \end{array} \right..\]

Therefore, we have the following lemma.

**Lemma (3.4.4).** (1)

\[ X'_{q,O'} = \left\{ \begin{array}{ll} G \times \hat{Q} (r(q) \times G_v \times \hat{Q_v} \cdot (r(q_v) + \bar{O}')) & (O' \subset g_v) \\ G \times \hat{Q} (r(q) \times \bar{O}' \times G_v \times \hat{Q_v} \cdot r(q_v)) & (O' \subset I_{\nu}') \end{array} \right. \]

and

\[ G \times \hat{Q} \cdot (r(q) + \bar{O}') \times h_v/W_v \cdot \hat{v}(q_v) = \left\{ \begin{array}{ll} G \times \hat{Q} (r(q) \times G_v \cdot (r(q_v) + \bar{O}') \times h_v/W_v \cdot \hat{v}(q_v)) & (O' \subset g_v) \\ G \times \hat{Q} \cdot (r(q) \times \bar{O}' \times G_v \cdot r(q_v) \times h_v/W_v \cdot \hat{v}(q_v)) & (O' \subset I_{\nu'}). \end{array} \right. \]

When \( O' \subset g_v \), one has \( f = id_G \times \hat{Q} (id_{r(q)} \times f_v) \). When \( O' \subset I_{\nu}' \), one has \( f = id_G \times \hat{Q} (id_{r(q)} \times O' \times f_v) \).

(2)

\[ G \times Q (n(q) + \bar{O}') = \left\{ \begin{array}{ll} G \times Q (n(q) \times G_v \times \hat{Q_v} \cdot (n(q_v) + \bar{O}')) & (O' \subset g_v) \\ G \times Q (n(q) \times \bar{O}' \times G_v \times \hat{Q_v} \cdot n(q_v)) & (O' \subset I_{\nu}') \end{array} \right. \]

and

\[ G \times \hat{Q} \cdot (n(q) + \bar{O}') = \left\{ \begin{array}{ll} G \times \hat{Q} (n(q) \times G_v \cdot (n(q_v) + \bar{O}')) & (O' \subset g_v) \\ G \times \hat{Q} (n(q) \times \bar{O}' \times G_v \cdot n(q_v)) & (O' \subset I_{\nu'}). \end{array} \right. \]

When \( O' \subset g_v \), one has \( f_0 = id_G \times \hat{Q} (id_{n(q)} \times O' \times \hat{v}_v) \). When \( O' \subset I_{\nu}' \), one has \( f_0 = id_G \times \hat{Q} (id_{n(q)} \times O' \times \nu_v) \).

**Lemma (3.4.5).** (1) \( f_0 \) is birational.

(2) \( \nu_v \) is birational.

(3) \( f_v \) is birational.

(4) \( f \) is birational.

**Proof.** (1): By the assumption, the generalized Springer map \( G \times Q (n(q) + \bar{O}') \rightarrow \bar{O} \) is birational. Since \( f_0 \) factorizes this map, \( f_0 \) is birational.
(2): Since $f_0$ is birational by (1), we see that $\nu_v$ is birational by (3.4.4), (2).

(3): If $\nu_v$ is birational, then $f_v$ is birational by (3.3.3).

(4): Since $f_v$ is birational by (3), $f$ is also birational by (3.4.4), (1).

Q.E.D.

Let $Z_v$ be the normalization of $G \times \bar{Q} \cdot (r(q) + \bar{O}') \times_{\theta_v/W_v} \mathfrak{f}(q_v)$. By Lemma (3.4.5), (4), the map $f$ induces a birational morphism

$$f^n : X_{q,O'} \to Z_v.$$ 

This map $f^n$ is the desired birational contraction map corresponding to $F_v$.

We next let $q'$ be the parabolic subalgebra obtained from $q$ by the twist of $v$. Then we have

$$\mathfrak{f}(q_v) = \mathfrak{f}(q'_v).$$

Thus, there is a diagram of birational morphisms

$$X_{q,O'} \xrightarrow{f^n} Z_v \xleftarrow{(f^n)^{-1}} X_{q',O'},$$

and we have

$$\overline{\text{Amp}(\mu_q)} \cap \overline{\text{Amp}(\mu_{q'})} = F_v.$$

(3.5). Let $O \subset g$ be a nilpotent orbit. Assume that a parabolic subalgebra $q_0$ of $g$ and a nilpotent orbit $O' \subset l(q_0)$ give a $\mathbb{Q}$-factorialization $\nu^n : G \times Q_0 (n(q_0) + \bar{O}') \to \bar{O}$. We put $l := l(q_0)$ and denote by $L$ the corresponding Levi subgroup of $G$. Let us consider $Y_{l,O'}$ defined in (3.3). Note that, for $q \in S(l)$, the nef cone $\overline{\text{Amp}(\mu_q)}$ is regarded as a cone in $M(L)_{\mathbb{R}}$.

**Theorem (3.5.1).** For $q \in S(l)$, the birational map $\mu_q : X_{q,O'} \to Y_{l,O'}$ is a $\mathbb{Q}$-factorial terminalization and is an isomorphism in codimension one. Any $\mathbb{Q}$-factorial terminalization of $Y_{l,O'}$ is obtained in this way. If $q \neq q'$, then $\mu_q$ and $\mu_{q'}$ give different $\mathbb{Q}$-factorial terminalizations. Moreover,

$$M(L)_{\mathbb{R}} = \bigcup_{q \in S(l)} \overline{\text{Amp}(\mu_q)}.$$

**Proof.** We shall first prove that $\mu_{q'}^{-1} \circ \mu_q : X_{q,O'} \to X_{q',O'}$ is not an isomorphism for $q \neq q'$. By Lemma (3.3.1), for $t \in (t_0)^{\text{reg}}$ there is an isomorphism

$$\rho_t : G \times^L (t + \bar{O}') \cong G \times^Q (t + n(q) + \bar{O}').$$
defined by $\rho_t([g, t + y']) = [g, t + y']$. In a similar way, we have an isomorphism

$$\rho'_t : G \times^L (t + \bar{O}') \cong G \times^{Q'} (t + n(q') + \bar{O}') .$$

Note that $\rho'_t \circ (\rho_t)^{-1}$ coincides with $\mu^{-1}_{q,t} \circ \mu_{q,t}$. Assume that $\mu^{-1}_{q,t} \circ \mu_{q}$ is an isomorphism. For $g \in G$ and $q \in Q - Q'$, let us consider two curves in $X_{q,0}$:

$$C_t := \{ \rho_t([g, 0]) \}$$

and

$$X$$

Then

$$\lim_{t \to 0} C_t = \lim_{t \to 0} D_t .$$

Define $C'_t := \mu^{-1}_{q,t} \circ \mu_q(C_t)$ and $D'_t := \mu^{-1}_{q,t} \circ \mu_q(D_t)$. Note that, for $t \in (\mathfrak{g}_0)^{\text{reg}}$, we have $C'_t = \{ \rho'_t([g, 0]) \}$ and $D'_t = \{ \rho'_t([g, q]) \}$. Then

$$\lim_{t \to 0} C'_t = [g, 0] \in G \times^{Q'} (n(q') + \bar{O}') ,$$

and

$$\lim_{t \to 0} D'_t = [gq, 0] \in G \times^{Q'} (n(q') + \bar{O}') .$$

These two points should coincide. But, since $Q \neq Q'$, this is a contradiction. Therefore, $\mu^{-1}_{q,t} \circ \mu_q$ is not an isomorphism. We next prove that any $Q$-factorial terminalization $\mu : X \to Y_{l,0}$ is of the form $\mu_q$. Fix a $\mu$-ample line bundle $L$ on $X$. Let $L^{(0)} \in \text{Pic}(X_{q_0,0})$ be its proper transform. If $L^{(0)}$ is $\mu_{q_0}$-nef, then $X = X_{q_0,0}$. Assume that $L^{(0)}$ is not $\mu_{q_0}$-nef. There is an extremal ray $\mathbb{R}_+[z] \subset \overline{\text{NE}}(\mu_{q_0})$ such that $(L^{(0)}, z) < 0$. Let $F \subset \overline{\text{Amp}}(\mu_{q_0})$ be the corresponding codimension one face. By (3.4) one can find $q_1 \in S(l)$ such that

$$\overline{\text{Amp}}(\mu_{q_0}) \cap \overline{\text{Amp}}(\mu_{q_1}) = F .$$

As constructed in (3.4), we then have a flop $X_{q_0,0} - - - - - - X_{q_1,0}$. We let $L^{(1)} \in X_{q_1,0}$ be the proper transform of $L^{(0)}$ and repeat the same procedure. Thus, we get a sequence of flops

$$X_{q_0,0} - - - - - - X_{q_1,0} - - - - - - X_{q_2,0} - - - - - - ...$$

But, since $S(l)$ is a finite set, this sequence must terminate by the discrepancy argument (cf. [Na 1, Theorem 6.1], [KMM, Proposition 5-1-11]). As a consequence, $X = X_{q_k,0}$ for some $k$.

**Proposition (3.5.2).** Let $Q \subset G$ be a maximal parabolic subgroup (i.e. $b_2(G/Q) = 1$) and let $O' \subset \mathfrak{l}(q)$ be a nilpotent orbit. Assume that $\nu^n : G \times^Q$
(n(q) + 0') \rightarrow 0 is a $\mathbb{Q}$-factorial terminalization which is an isomorphism in codimension one. Then (q, O') is a primitive pair (cf. (3.2.1)).

**Proof.** As in (3.4.3), we may assume that $Q = Q_I$. Since $Q$ is maximal, $\Delta - I = \{v\}$. Let $Q'$ be the parabolic subgroup twisted by $v$. As in (3.4), we have a birational map (over $k$):

$$\gamma : X_{q,O'} \longrightarrow X_{q',O'}.$$ 

By restricting this diagram to the fibers over $0 \in \mathfrak{k}(q)$, we have a birational map

$$\gamma_0 : G \times^Q (n(q) + 0') \longrightarrow G \times^{Q'} (n(q') + 0').$$

Let $p \in G \times^Q (n(q) + 0')$ be a point such that $\nu^n$ is an isomorphism at $p$. Then $\gamma$ is an isomorphism at $p \in X_{q,O'}$. Let $L \in \text{Pic}(X_{q,O'})$ be a $\mu_q$-ample line bundle and denote by $\gamma_*(L) \in \text{Pic}(X_{q',O'})$ the proper transform of $L$ by $\gamma$. Then we have

$$\gamma_*(L)|_{G \times^Q (n(q) + 0')} = (\gamma_0)_*(L|_{G \times^{Q} (n(q) + 0')}).$$

Since $\mu_q$ and $\mu_{q'}$ are different by (3.5.1), the left hand side is not $(\nu')^n$-ample; hence the right hand side is not so. This means that $\gamma_0$ is not an isomorphism.

Suppose $v$ is of the second kind; then $Q$ and $Q'$ are conjugate by an element $w \in W$. Since $\mathfrak{r}(q)$ and $\mathfrak{r}(q')$ are conjugate by $w$, $\mathfrak{r}(q) \cap \mathfrak{h}$ and $\mathfrak{r}(q') \cap \mathfrak{h}$ are also conjugate by $w$. Note that $\mathfrak{l}(q) := \mathfrak{r}(q) \cap \mathfrak{h} = \mathfrak{r}(q') \cap \mathfrak{h}$ and then $\mathfrak{l}(q) = g^{r(q)}$ (cf. (3.1)). This means that $w$ sends $\mathfrak{l}(q)$ to $\mathfrak{l}(q)$; hence $w \in W'$. Since $W'$ stabilizes $O'$ by Lemma (3.3.2), two $\mathbb{Q}$-factorial terminalizations $\nu^n : G \times^Q (n(q) + 0') \rightarrow 0$ and $(\nu')^n : G \times^{Q'} (n(q') + 0') \rightarrow 0$ are the same one. In other words, $\gamma_0$ is an isomorphism; hence $v$ should be of the first kind. Let $D$ be the single marked Dynkin diagram corresponding to $Q$.

Then $\mu(q)$ has only simple factors of type $A$ except when $D$ is of type $E_{6,1}$. If all simple factors of $\mu(q)$ are of type $A$, then $O'$ has $\mathbb{Q}$-factorial terminal singularities only when $O' = 0$. On the other hand, if $D$ is of type $E_{6,1}$, then $\mu(q) = D_5$. In $D_5$, we only have three nilpotent orbits $O'$ for which $O'$ has $\mathbb{Q}$-factorial terminal singularities: $O' = 0$, $O' = O_{3,2^2,1^3}$ and $O' = O_{2^2,1^6}$.

**Corollary (3.5.3).** Assume that $\nu^n : G \times^Q (n(q) + 0') \rightarrow 0$ is a $\mathbb{Q}$-factorial terminalization. Let $(f_0)^n : G \times^Q (n(q) + 0') \rightarrow Z_{v,0}$ be the birational contraction map corresponding to a codimension one face $F_v$ of $\text{Amp}(\mu_q)$ (cf. (3.4)). Then $(f_0)^n$ is an isomorphism in codimension one if and only if $(q_v, O')$ is a primitive pair.
Proof. This follows from (3.4.4), (2) and (3.5.2). Q.E.D.

Let us return to the original situation of (3.5). Define $S^1(l)$ to be the subset of $S(l)$ consisting of the parabolic subalgebras $q$ obtained from $q_0$ by a finite succession of the twists of the first kind.

**Theorem (3.5.4).** There is a one-to-one correspondence between the set of $\mathbb{Q}$-factorial terminalizations of $\tilde{O}$ and $S^1(l)$. In other words, every $\mathbb{Q}$-factorial terminalization of $\tilde{O}$ is obtained as $\mu_{q,0} : X_{q,O',0} \to \tilde{O}$ for $q \in S^1(l)$. Two different $\mathbb{Q}$-factorial terminalizations of $\tilde{O}$ are connected by a sequence of Mukai flops (cf. (3.2.1)). Moreover

$$\overline{\text{Mov}}(\mu_{q_0,0}) = \cup_{q \in S^1(l)} \overline{\text{Amp}}(\mu_{q,0}).$$

Proof. Let $\nu : Z \to \tilde{O}$ be a $\mathbb{Q}$-factorial terminalization of $\tilde{O}$. Note that the birational map $X_{q_0,0} \to Z$ is an isomorphism in codimension one. Fix a $\nu$-ample line bundle $M \in \text{Pic}(Z)$. Let $M^{(0)} \in \text{Pic}(X_{q_0,O',0})$ be its proper transform. Assume that $M^{(0)}$ is not $\mu_{q_0,0}$-nef. There is an extremal ray $R_+[z] \subset \overline{NE}(\mu_{q_0,0})$ such that $(M^{(0)}, z) < 0$. Let $F \subset \overline{\text{Amp}}(\mu_{q_0,0})$ be the corresponding codimension one face. As explained in (3.4.3), $F$ corresponds to a marked vertex $v$ of the marked Dynkin diagram $D$ determined by $q_0$. By Corollary (3.5.3) we see that $v$ is of the first kind (see (3.2)). By (3.4) one can find $q_1 \in S^1(l)$ such that

$$\overline{\text{Amp}}(\mu_{q_0,0}) \cap \overline{\text{Amp}}(\mu_{q_1,0}) = F.$$

As constructed in (3.4), we then have a flop $X_{q_0,O',0} \to X_{q_1,O',0}$. We let $M^{(1)} \in X_{q_1,O',0}$ be the proper transform of $M^{(0)}$ and repeat the same procedure. Thus, we get a sequence of flops

$$X_{q_0,O',0} \to X_{q_1,O',0} \to X_{q_2,O',0} \to \ldots$$

But, since $S^1(l)$ is a finite set, this sequence must terminate by the discrepancy argument (cf. [Na 1, Theorem 6.1], [KMM, Proposition 5-1-11]). As a consequence, $Z = X_{q_k,O',0}$ for some $k$.

(3.6) **Movable cones and the $W'$-action:** Start with the situation in Theorem (3.5.4). Recall that

$$W' := N_W(L_0)/W(L_0).$$
For \( w \in N_W(L_0) \) and \( \chi \in M(L_0) \), we define \( w\chi \in M(L_0) \) by \( w\chi(g) = \chi(w^{-1}gw) \) with \( g \in L_0 \). In this way \( N_W(L_0) \) acts on \( M(L_0)_R \). Note that \( W(L_0) \) coincides with the subgroup of \( N_W(L_0) \) which consists of the elements acting trivially on \( M(L)_R \). Hence, \( W' \) acts on \( M(L_0)_R \) effectively.

**Theorem (3.6.1).**

(i) The set \( S(l) \) contains exactly \( N \cdot |\bar{\pi}(W')| \) elements, where \( N \) is the number of the conjugacy classes of parabolic subalgebras contained in \( S(l) \).

(ii) For any \( q \in S(l) \), there is an element \( w \in N_W(L_0) \) such that \( w(q) \in S^1(l) \).

(iii) For any non-zero element \( w \in W' \), we have
\[
\left(w(\text{Mov} (\mu_{q_0,0})) \cap \overline{\text{Mov}(\mu_{q_0,0})}\right) = \emptyset.
\]

(iv) The set \( S^1(l) \) contains exactly \( N \) elements.

**Proof.** (i): Take two conjugate elements \( q, q' \in S(l) \). Then, there is an element \( w \in W \) such that \( q = w(q') \). Since \( r(q) \) and \( r(q') \) are conjugate by \( w \), \( r(q) \cap h \) and \( r(q') \cap h \) are also conjugate by \( w \). If we put \( l := r(q) \cap h = r(q') \cap h \), then \( l = \mathfrak{g}^l \) (cf. (3.1)). This means that \( w \) sends \( l \) to \( l' \); hence \( w \in N_W(L) \).

We next show that, if \( w(q) = q \) for an element \( q \in S(l) \), then \( w \in W(L) \). Let \( U \) be the unipotent radical of \( Q \). Then one can write \( Q = U \cdot L \). Now we suppose that \( w \) is represented by an element of the normalizer group \( N_G(T) \). Since \( w(Q) = Q \) and \( N_G(Q) = Q \), \( w \in Q \). Let us write \( w = u \cdot l \) with \( u \in U \) and \( l \in L \). By assumption, \( w(L) = L \). This means that \( u(L) = L \). Since any two Levi subgroups of \( Q \) are conjugate by a unique element of \( U \) (cf. [Bo], 14.19), we have \( u = 1 \), which implies that \( w \in W(L) \).

(ii): Let us assume that \( q_0 \) is a standard parabolic subalgebra determined by a marked Dynkin diagram \( D \). By the definition of twists, any \( q \in S(l) \) is conjugate to a standard parabolic subalgebra determined by a marked Dynkin diagram \( D' \) which is equivalent to \( D \) (cf. (3.2)). In order to get \( D' \) from \( D \), we only need the twists of the first kind. This means that there is an element \( q' \in S^1(l) \) such that \( q' \) is conjugate to the standard parabolic subalgebra determined by \( D' \).

(iii), (iv): We shall prove that any two distinct elements of \( S^1(l) \) are not conjugate to each other. Suppose that \( q, q' \in S^1(l) \) are conjugate to each other. Let us consider the diagram
\[
X_{q,0'} \overset{\mu_3}{\rightarrow} Y_{l,0'} \overset{\mu_{q'}}{\leftarrow} X_{q',0'}.
\]
Restrict the diagram over $0 \in \mathfrak{f}$ to get

$$X_{q,O',0} \xrightarrow{\mu_{q',0}} Y_{l,0} \xleftarrow{\mu_{q,0}} X_{q',O',0}. $$

By (3.3.4), this diagram coincides with

$$G \times^Q (n(q) + \tilde{O}) \xrightarrow{\nu^n} \tilde{O} \xleftarrow{\nu' n} G \times^{Q'} (n(q') + \tilde{O'}). $$

Since $Q$ and $Q'$ are conjugate, we see that $\nu^n$ and $(\nu')^n$ give the same $Q$-factorial terminalization of $\tilde{O}$ by (3.3.2). In other words, the birational map $\mu_{q^{-1},0} \circ \mu_{q,0}$ is an isomorphism. We shall prove that $\mu_{q^{-1},0} \circ \mu_{q}$ is an isomorphism. Let $L$ be a $\mu_q$-ample line bundle on $X_{q,O'}$ and let $L' \in \text{Pic}(X_{q',O'})$ be the proper transform of $L$ by $\mu_{q}^{-1} \circ \mu_{q}$. By Theorem (3.5.1), $X_{q,O'}$ and $X_{q',O'}$ are connected by a sequence of birational transformations which are isomorphisms in codimension one. Since $q, q' \in S^1(l)$, these birational transformations all come from twists of the first kind. This means that, there is a closed subset $F$ of $X_{q,O',0}$ with codimension $\geq 2$ such that $\mu_{q^{-1},0} \circ \mu_{q}$ is an isomorphism at each $x \in X_{q,O',0} \setminus F$. Hence we have

$$L'|_{X_{q',O',0}} \cong (\mu_{q^{-1},0} \circ \mu_{q,0})_*(L|_{X_{q,O',0}}). $$

But the right hand side is a $\mu_{q',0}$-ample line bundle. Hence $L'|_{X_{q',O',0}}$ is $\mu_{q',0}$-ample. This shows that $L'$ is $\mu_q$-ample. Indeed, by the $C^*$-action of $X_{q',O'}$, every proper curve $C$ in a fiber of $\mu_{q'}$ is deformed to a curve inside $X_{q',O',0}$; hence $(L', C) > 0$ follows from the ampleness of $L'|_{X_{q',O',0}}$. Therefore, $\mu_{q^{-1},0} \circ \mu_{q}$ is an isomorphism. Then, by Theorem (3.5.1), $q = q'$.

§4. Poisson deformations of nilpotent orbits.

(4.1) Let $X$ be a normal variety with symplectic singularities (cf. (2.4)). We shall define a Poisson structure on $X$ by using the symplectic 2-form $\omega$ on $X_{\text{reg}}$. By $\omega$ the sheaf of 1-forms $\Omega^1_{X_{\text{reg}}}$ is identified with the sheaf of vector fields $\Theta_{X_{\text{reg}}}$. Since $\Omega^2_{X_{\text{reg}}} \cong \wedge^2 \Theta_{X_{\text{reg}}}$, $\omega$ determines a bivector $\Theta \in \wedge^2 \Theta_{X_{\text{reg}}}$. We then define a bracket

$$\{ , \} : \wedge^2 \mathcal{O}_{X_{\text{reg}}} \to \mathcal{O}_{X_{\text{reg}}}$$

by $\{ f, g \} := \Theta(df \wedge dg)$. By definition this bracket is bi-derivation. Moreover, it satisfies the Jacobi identity

$$\{ \{ f, g \}, h \} + \{ \{ g, h \}, f \} + \{ \{ h, f \}, g \} = 0.$$
(cf. [C-G], Theorem 1-2-7). In other words, we have a Poisson structure on $X_{\text{reg}}$. Since $X$ is normal, this bracket uniquely extends to the bracket

$$\{ , \} : \wedge^2 \mathcal{O}_X \to \mathcal{O}_X.$$  

This bracket is also a bi-derivation and satisfies the Jacobi-identity. In this way, $(X, \{ , \})$ is a variety with a Poisson structure. We shall introduce the notion of a Poisson deformation of $(X, \{ , \})$. First recall

**Definition (4.1.1).** Let $T$ be a scheme (resp. complex space). Let $X$ be a scheme (resp. complex space) over $T$. Then $(X, \{ , \})$ is a Poisson scheme (resp. a Poisson space) over $T$ if $\{ , \}$ is an $\mathcal{O}_T$-linear map:

$$\{ , \} : \wedge^2 \mathcal{O}_T \to \mathcal{O}_X$$

such that, for $a, b, c \in \mathcal{O}_X$,

1. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$
2. $\{a, bc\} = \{a, b\}c + \{a, c\}b$.

Let $0 \in T$ be a punctured $\mathbf{C}$-scheme. A Poisson deformation of $(X, \{ , \})$ over $T$ is a pair of a Poisson scheme $(X', \{ , \}_{T'})$ over $T$ and an isomorphism $\phi : X \times_T \text{Spec}(\mathbf{C}) \cong X'$ such that $X'$ is flat over $T$, and the Poisson structure $\{ , \}_{T'}$ induces the original Poisson structure $\{ , \}$ over the closed fiber $X$ by $\phi$. Let $S$ be a local Artin $\mathbf{C}$-algebra with residue field $\mathbf{C}$. Two Poisson deformations $(X, \phi)$ and $(X', \phi')$ over $S$ are equivalent if there is a Poisson isomorphism $\varphi : X \cong X'$ over $\text{Spec}(S)$ which induces the identity map of $X$ over $\text{Spec}(\mathbf{C})$ via $\phi$ and $\phi'$.

The Poisson deformation $X \xrightarrow{f} T$ is called formally universal at $0 \in T$ if, for any Poisson deformation $X' \to T'$ of $X$ with a local Artinian base $T'$, there is a unique map $T' \to T$ such that $X' \cong X \times_T T'$ as a Poisson deformation of $X$ over $T'$. In this case, for a small open neighborhood $V$ of $0 \in T$, the family $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is called the Kuranishi family for the Poisson deformations of $X$, and $V$ is called the Kuranishi space for the Poisson deformations of $X$.

**Proposition (4.2).** Let $O$ be a nilpotent orbit of a complex simple Lie algebra $\mathfrak{g}$ and let $\nu^n : G \times^Q (\mathfrak{n}(q) + \tilde{O}') \to \tilde{O}$ be a $\mathbf{Q}$-factorial terminalization. Then $G \times^Q (\mathfrak{n} + \tilde{O}')$ has symplectic singularities. Moreover,

$$X_{q,O} := G \times^Q (\mathfrak{r}(q) + \tilde{O}') \to \mathfrak{t}(q).$$
is a Poisson deformation of $G \times^Q (n(q) + \tilde{O'})$.

Proof. For $t \in \mathfrak{t}(q)$, the fiber $X_{q,O',t}$ is isomorphic to $G \times^Q (t + n(q) + \tilde{O'})$, whose regular locus is $G \times^Q (t + n(q) + O')$. We have a natural $G$-equivariant map

$$\mu_t : G \times^Q (t + n(q) + O') \to g$$

defined by $[g, t + y + y'] \to Ad_g(t + y + y')$. The image of this map coincides with the closure of an adjoint orbit, say $O_{\mu_t}$. We shall prove that the pull-back $\omega_t$ of the Kostant-Kirillov 2-form on $O_{\mu_t}$ give a symplectic 2-form on $G \times^Q (t + n(q) + O')$. Let $I$ be the Levi part of $\mathfrak{p}$ and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \subset I$. In the remainder of the proof, we shall simply write $n$ for $n(q)$. There is an involution $\phi_g$ of $\mathfrak{g}$ which stabilizes $\mathfrak{h}$ and which acts on the root system $\Phi$ via $-1$. Put $n_+ := \phi_g(n)$. Take a point $[1, t + y + y'] \in G \times^Q (t + n + O')$ so that $y \in n(q)$, $y' \in O'$ and $t + y + y' \in O_{\mu_t}$. The tangent space of $G \times^Q (t + n + \tilde{O'})$ at $[1, t + y + y']$ is decomposed as

$$T_{[1, t+y+y']} = n_- \oplus T_{y+y'}(t+n+\tilde{O'}).$$

Since $Q \cdot (t + y + y')$ coincides with the Zariski open dense subset $O_{\mu_t} \cap (t + n + O') \subset t + n + O'$, an element $v \in T_{[1, t+y+y']}$ can be written as

$$v = v_1 + [v_2, t + y + y'], \quad v_1 \in n_-, \quad v_2 \in q.$$

Let $d\nu_* : T_{[1, y+y']'} \to T_{\nu([1, t+y+y'])}O_{\mu_t}$ be the tangential map for $\mu_t$. Then

$$d(\mu_t)_*(v) = [v_1 + v_2, t + y + y'].$$

Take one more element $w \in T_{[1, t+y+y']}$ in such a way that

$$w = w_1 + [w_2, t + y + y'], \quad w_1 \in n_-, \quad w_2 \in q.$$

Denote by $\langle \cdot, \cdot \rangle$ the Killing form of $\mathfrak{g}$. By the definition of the Kostant-Kirillov form, one has

$$\omega(d(\mu_t)_*(v), d(\mu_t)_*(w)) := \langle t + y + y', [v_1 + v_2, w_1 + w_2] \rangle.$$

Note that $\langle t + y + y', [v_1, w_1] \rangle = \langle y, [v_1, w_1] \rangle$, and $\langle t + y + y', [v_2, w_2] \rangle = \langle y', [v_2, w_2] \rangle$. Therefore,

$$\omega(d(\mu_t)_*(v), d(\mu_t)_*(w)) =$$

$$\langle y, [v_1, w_1] \rangle + \langle t + y + y', [v_1, w_2] \rangle + \langle t + y + y', [v_2, w_1] \rangle + \langle y', [v_2, w_2] \rangle =$$
\(\langle y, [v_1, w_1]\rangle + \langle v_1, [w_2, t + y + y']_n\rangle - \langle [v_2, t + y + y']_n, w_1\rangle + \omega([v_2, y'], [w_2, y'])\),

where \([w_2, t + y + y']_n\) (resp. \([v_2, t + y + y']_n\)) is the nil-radical part of \([w_2, t + y + y']\) in the decomposition \(T_{y+y}'(t+n+\tilde{O}') = n + T_y'O'\). Let \(O_r \subset \mathfrak{g}\) be the Richardson orbit for \(Q\), and let \(\pi : T^*(G/Q) \to \tilde{O}_r\) be the Springer map. The first part \(\langle y, [v_1, w_1]\rangle + \langle v_1, [w_2, t + y + y']_n\rangle - \langle [v_2, t + y + y']_n, w_1\rangle\) corresponds to the 2-form on \(T^*(G/Q)\) obtained by the pull-back of the Kostant-Kirillov 2-form on \(O_r\) by \(\pi\) (cf. [Pa]), which is non-degenerate on \(T^*(G/Q)\). Let us consider the second part \(\omega([v_2, y'], [w_2, y'])\). Denote by \([v_2, t + y + y']_l\) (resp. \([w_2, t + y + y']_l\)) the \(T_y'O'\)-part of \([v_2, t + y + y']\) (resp. \([w_2, t + y + y']\)) in the decomposition \(T_{y+y'}(t+n+\tilde{O}') = n + T_y'O'\). Then \([v_2, y'] = [v_2, t + y + y']_l\) and \([w_2, y'] = [w_2, t + y + y']_l\); hence, the second part is the Kostant-Kirillov form on \(O'\).

Now assume that \(y \in n\) and \(y' \in O'\) (not necessarily \(t + y + y' \in O_{\mu_t}\)). Write \(v \in T_{[1,t+y+y']}\) as

\[v = v_1 + (v_3)_n + (v_3)_l,\]

where \((v_1) \in n_-, (v_3)_n \in n\), and \((v_3)_l \in T_{y'O'}\). Similarly, write \(w \in T_{[1,y+y']}\) as

\[w = w_1 + (w_3)_n + (w_3)_l.\]

The arguments above show that

\[(\mu_t)^*\omega(v, w) = \langle y, [v_1, w_1]\rangle + \langle v_1, (w_3)_n\rangle - \langle (v_3)_n, w_1\rangle + \omega((v_3)_l, (w_3)_l).\]

It is easily checked that \((\mu_t)^*\omega\) is non-degenerate at \([1, t + y + y']\). By the \(G\)-equivariance of \(\mu_t\), we see that \(\omega_t := (\mu_t)^*\omega\) is a symplectic 2-form on \(G \times Q (t + n(q) + O')\).

By this description we can also observe that the family of symplectic 2-forms \(\{\omega_t\}\) defines a relative symplectic 2-form \(\omega\) of \(G \times Q (\mathfrak{r}(q) + O') \to \mathfrak{k}(q)\). This relative symplectic 2-form makes \(G \times Q (\mathfrak{r}(q) + \tilde{O}')\) into a Poisson scheme over \(\mathfrak{k}(q)\). Its central fiber is clearly the original Poisson scheme \(G \times Q (n+\tilde{O}')\).

Q.E.D.

(4.3) We have constructed in (3.3) a map \(\chi : G \cdot (\mathfrak{r}(q) + \tilde{O}') \to \mathfrak{h}/W'\) and have remarked that the normalization of \(\text{Im} (\chi)\) coincides with \(\mathfrak{k}(q)/W'\). Let \(G \cdot (\mathfrak{r}(q) + \tilde{O}')^n\) be the normalization of \(G \cdot (\mathfrak{r}(q) + \tilde{O}')\). Then \(\chi\) induces a map

\[\chi^n : G \cdot (\mathfrak{r}(q) + \tilde{O}')^n \to \mathfrak{k}(q)/W'.\]
**Proposition (4.3.1).** $\chi^n$ is a flat morphism whose central fiber is isomorphic to $\tilde{O}$. Moreover, $\chi^n : G \cdot (\mathfrak{r}(q) + \tilde{O})^n \to \mathfrak{k}(q)$ is a Poisson deformation of $\tilde{O}$.

**Proof.** When $O' = 0$, the statements are exactly Corollary (2.3) and Proposition (2.6) of [Na 5]. The proof in a general case is the same.

**4.4 The period map:** We put $X := G \times^Q (n(q) + \tilde{O}')$ and consider the $C^*$-equivariant Poisson deformation of $X$:

$$X_{q,O'} \to \mathfrak{k}(q).$$

In (4.2) we have defined a relative symplectic 2-form $\omega$ on the regular locus $(X_{q,O'})_{reg} := G \times^Q (\mathfrak{r}(q) + O')$ over $\mathfrak{k}(p)$. We shall construct a period map

$$p : \mathfrak{k}(q) \to H^2(X, C)$$

by using $\omega$. Since the fibers of $X_{q,O'} \to \mathfrak{k}(q)$ are not smooth, we need some technical arguments to define $p$. First of all, note that $X$ is a $Q$-factorial terminalization of $\tilde{O}$, where $\tilde{O}$ has a $C^*$-action with positive weights. The $C^*$-action on $X$ is the lifting of this $C^*$-action. Then $X$ is also $Q$-factorial as a complex analytic space by [Na 4], Proposition (A.9). By [ibid, Theorem 17] we see that $X_{q,O'} \to \mathfrak{k}(q)$ is a locally trivial flat deformation of $X$. In the proof of [ibid], Proposition 24, we have constructed a simultaneous $C^*$-equivariant resolution of $X_{q,O'} \to \mathfrak{k}(q)$:

$$\beta : Z \to X_{q,O'}.$$ 

We now have a commutative diagram

$$\begin{array}{ccc}
Z & \longrightarrow & G \cdot (\mathfrak{r}(q) + \tilde{O})^n \\
\alpha \downarrow & & \downarrow \\
\mathfrak{k}(q) & \longrightarrow & \mathfrak{k}(q)/W'.
\end{array}$$

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Note that $G \cdot (\mathfrak{r}(q) + \tilde{O})^n$ has a $C^*$-action with a unique fixed point and with positive weights. Moreover, the diagram is $C^*$-equivariant and $\alpha : Z \to \mathfrak{k}(q)$ is a simultaneous resolution of $G \cdot (\mathfrak{r}(q) + \tilde{O})^n \to \mathfrak{k}(q)/W'$. Then we see that $Z$ is a $C^\infty$-trivial fiber bundle over $\mathfrak{k}(q)$ by [Slo], Remark at the end of section 4.2. Let $\Omega^2_{Zan/\mathfrak{k}(q)}$ be the relative complex-analytic de Rham complex. Let $\mathcal{K}$ be the subsheaf of $\Omega^2_{Zan/\mathfrak{k}(q)}$ which consists of d-closed
relative 2-forms. By the natural map $K[-2] \to \Omega_{\mathcal{Z}^{an}/P}$, we can define a sequence of maps:

$$\alpha_* K \to R^2 \alpha_* \Omega_{\mathcal{Z}^{an}/P} \cong R^2 \alpha_* \alpha^{-1} \mathcal{O}_{\mathcal{Z}^{an}/P}.$$ 

Since $R^2 \alpha_* \alpha^{-1} \mathcal{O}_{\mathcal{Z}^{an}/P} \cong R^2 \alpha_* \mathcal{O}_{\mathcal{Z}^{an}/P}$ (cf. [Lo], Lemma (8.2)), we have an isomorphism

$$R^2 \alpha_* \alpha^{-1} \mathcal{O}_{\mathcal{Z}^{an}/P} \cong H^2(\mathcal{Z}_0, \mathcal{C}) \otimes \mathcal{O}_{\mathcal{Z}^{an}/P}.$$ 

By pulling back the relative symplectic 2-form $\omega$ of $X_{q, O'}/\mathfrak{k}(q)$ defined in (4.2), we get a relative d-closed 2-form $\omega_Z$ of $\mathcal{Z}/\mathfrak{k}(q)$. Then $\omega_Z$ gives a section $s$ of the sheaf $H^2(\mathcal{Z}_0, \mathcal{C}) \otimes \mathcal{O}_{\mathcal{Z}^{an}/P}$. Let

$$ev_t : H^2(\mathcal{Z}_0, \mathcal{C}) \otimes \mathcal{O}_{\mathcal{Z}^{an}/P} \to H^2(\mathcal{Z}_0, \mathcal{C})$$

be the evaluation map at $t \in \mathfrak{k}(q)$. We define a period map

$$p : \mathfrak{k}(q) \to H^2(\mathcal{Z}_0, \mathcal{C})$$

by $p(t) = ev_t(s)$. By the construction, $p$ is a holomorphic map. The birational morphism $\mathcal{Z} \to X_{q, O'}$ induces a birational morphism $\mathcal{Z}_0 \to X_{q, O', 0}$ of central fibers. Since $X_{q, O', 0}$ has only rational singularities, there is an injection $H^2(X_{q, O', 0}, \mathcal{C}) \to H^2(\mathcal{Z}_0, \mathcal{C})$. We shall prove that $p$ factors through $H^2(X_{q, O', 0}, \mathcal{C})$:

$$p : \mathfrak{k}(q) \to H^2(X_{q, O', 0}, \mathcal{C}) \to H^2(\mathcal{Z}_0, \mathcal{C}).$$

Take a point $t \in \mathfrak{k}(q)$. In order to prove that $p(t) \in H^2(X_{q, O', 0}, \mathcal{C})$, it is enough to show that $(p(t), C) = 0$ for all proper curves $C$ which are mapped to points by the map $\beta_0 : \mathcal{Z}_0 \to X_{q, O', 0}$. Put $p := \beta_0(C) \in X_{q, O', 0}$ and take a small open neighborhood $V$ of $p \in X_{q, O'}$. Set $\bar{V} := \beta^{-1}(V)$. Let us consider the map $V \to \mathfrak{k}(q)$ obtained as the composite $V \subset X_{q, O'} \to \mathfrak{k}(q)$. We regard this map as the flat deformation of the central fiber $V_0$. Let $L$ be the line of $\mathfrak{k}(q)$ passing through 0 and $t$. Restrict the map $V \times_{\mathfrak{k}(q)} L \to L$ to the $n$-th infinitesimal neighborhoods $L_n$ of 0 $\in L$. Then we have a formal deformation $\{V_n\}$ of $V_0$. The map $\beta$ induces a resolution $\bar{V}_n \to V_n$ for each $n \geq 1$. Since $X_{q, O'} \to \mathfrak{k}(q)$ is a locally trivial flat deformation of $X_{q, O', 0}$, we see that $V_n = V_0 \times L_n$. By the construction of $\beta$ (cf. [Na 4], Proposition 24), we have $\bar{V}_n = V_0 \times L_n$. This means that the proper curve $C \subset \bar{V}_0$ deforms
sideways in the flat deformation $\tilde{V}_n \to L_n$ for each $n$. Here let us consider the relative Hilbert scheme

$$H := \text{Hilb}(\mathcal{Z} \times_{t(q)} L/X_{q,O'} \times_{t(q)} L).$$

The argument above shows that there is an irreducible component $H'$ of $H$ such that $[C] \in H'$ and $H'$ dominates $L$ by the composite $H' \to X_{q,O'} \times_{t(q)} L \to L$. Note that $\mathcal{Z} \to \mathfrak{k}(q)$ is $\mathbb{C}^*$-equivariant and $L - \{0\}$ is a $\mathbb{C}^*$-orbit of $\mathfrak{k}(q)$. Since $H'$ dominates $L$, one can find $t' \in L - \{0\}$ such that there is a proper curve $C_{t'} \subset \mathcal{Z}_{t'}$ which is deformation equivalent to $C$. By using the $\mathbb{C}^*$-action, one can also find a proper curve $C_t \subset \mathcal{Z}_t$ which is deformation equivalent to $C$. By the definition of $p$, we have

$$(p(t), C)_{Z_0} = ([\omega_{Z_t}], C_t)_{Z_t}.$$ 

Since the restriction of a holomorphic 2-form to a curve is always zero, $([\omega_{Z_t}], C_t)_{Z_t} = 0$.

Once we have made the period map precise, we can generalize Proposition (2.7) and Theorem (2.8) of [Na 5] as follows. The proof is almost the same as [Na 5].

**Theorem (4.5).** Let $O$ be a nilpotent orbit of a complex simple Lie algebra $\mathfrak{g}$, and let $\nu^n : G \times^Q (\mathfrak{n}(q) + \bar{O}') \to \bar{O}$ be a $\mathbb{Q}$-factorial terminaization. Then the following $\mathbb{C}^*$-equivariant commutative diagram

$$
\begin{array}{ccc}
X_{q,O'} & \longrightarrow & G \cdot (\mathfrak{r}(q) + \bar{O}')^n \\
\downarrow & & \downarrow \\
\mathfrak{k}(q) & \longrightarrow & \mathfrak{k}(q)/W'
\end{array}
$$

(6)

gives formally universal deformations of $G \times^Q (\mathfrak{n}(q) + \bar{O}')$ and $\bar{O}$.

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