Critical $\overline{\partial}$ problems in one complex dimension and some remarks on conformally invariant variational problems in two real dimensions.

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Abstract

We will study a linear first order system, a connection $\overline{\partial}$ problem, on a vector bundle equipped with a connection, over a Riemann surface. We show optimal conditions on the connection forms which allow one to find a holomorphic frame, or in other words to prove the optimal regularity of our solution. The underlying geometric principle, discovered by Koszul-Malgrange, is classical and well known; it gives necessary and sufficient conditions for a connection to induce a holomorphic structure on a vector bundle over a complex manifold. Here we explore the limits of this statement when the connection is not smooth and our findings lead to a very short proof of the regularity of harmonic maps in two dimensions as well as re-proving a recent estimate of Lamm and Lin concerning conformally invariant variational problems in two dimensions.

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1 Introduction

We will consider a square integrable connection on a smooth vector bundle $E^m$ over a Riemann surface $\Sigma$. Our vector bundle may be real or complex, however the problems we wish to consider will largely require us to complexify $E$ when it is real (unless the connection happens to be flat). Since we are working over a Riemann surface we may consider the related $\overline{\partial}$-problem associated to sections of $\wedge^{(1,0)} T^* \Sigma \otimes E$. We ask: Under what circumstances can we locally find a holomorphic frame? Or, can we find a cover of $\Sigma$ with a collection of bundle trivialisations such that the transition charts are holomorphic? The latter question is equivalent to being able to find a holomorphic frame over each trivialisation.

Since the question is of a local nature, we work with a small piece of $\Sigma$ over which $E$ is trivial, therefore we may simply consider the case $\Sigma = D \subset \mathbb{C}$ the unit disc and $E = D \times \mathbb{F}^m$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Now the connection is defined entirely by one-forms which we denote by $\omega \in L^2(D, \text{gl}(m, \mathbb{C}) \otimes \wedge^1 \mathbb{F}^2)$. The frame $S : D \to GL(m, \mathbb{C})$ that we are going to look for will solve

$$\overline{\partial} S = -\omega S.$$  

1Initially we consider $\omega$ as a $\text{gl}(m, \mathbb{C})$-valued real one form, i.e. $\omega = \omega^x dx + \omega^y dy$ with $\omega^x, \omega^y : D \to \text{gl}(m, \mathbb{C})$; but we can just as easily express $\omega$ with respect to $dz$ and $d\overline{z}$, at which point $\omega^z := \frac{1}{2} (\omega^x + i \omega^y) dz = \omega^{(0,1)}$ is the $(0,1)$ part of $\omega$. 

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or, to put it another way
\[(S^{-1}dS + S^{-1}\omega S)^{(0,1)} = 0\]  

In words, we could say that re-writing the connection forms with respect to this new frame (or trivialisation) \(S\) forces the \((0,1)\)-part of the new connection forms to be zero. A classical theorem of Koszul-Malgrange \([7,\text{Theorem 1}]\) c.f. Theorem \(8.2\) tells us that when \(\omega\) is smooth then we can find \(S\) if and only if \(F_\omega^{(0,2)} \equiv 0\) where
\[F = d\omega + [\omega, \omega]\]
is the curvature of our connection\(^2\). Thus in our setting, under the assumption that \(\omega\) is smooth, one can \textit{always} find such a frame \(S\). However this stops being true when we only assume \(\omega \in L^2\) (or even \(\omega \in L^{2,q}\) for all \(q > 1\), see section \(4\)). The problem is that we do not get \textit{a-priori} \(L^\infty\) estimates for \(S\) when \(\omega \in L^2\), which makes it impossible to guarantee that we can find an invertible matrix \(S\) solving \((\text{I})\). However if we assume that \(\|\omega\|_{L^2}^2\) is sufficiently small then we can find \(S\) via a fixed point argument and ensure that it stays a bounded distance from the identity, Theorem \(8.3\). It follows that \(\omega \in L^p\) for \(p > 2\) also works and we can consider \(\omega \in L^2\) as being a borderline case that fails to hold. Therefore we will assume a further structural condition on \(\omega \in L^2\) that will ensure the existence of \(S\), Theorem \(2.1\).

A corollary of Theorem \(2.1\) allows one to prove regularity for maps (or sections) \(\alpha \in L^2(D, C^m \otimes \wedge^{(1,0)} T^* C)\) solving
\[
\overline{\partial}_\omega(\alpha) = \overline{\partial}_\alpha + \omega \wedge \alpha = 0,
\]  
in particular one can show that with \(S\) solving \((\text{I})\), we have
\[
\overline{\partial}(S^{-1}\alpha) = 0
\]
and the highest regularity of \(\alpha\) we can expect is the same as that of \(S\). We remark that the PDE \((\text{II})\) is critical in the sense that we have \(\overline{\partial}\alpha \in L^1\) so with standard Calderon-Zygmund estimates we can conclude that \(\nabla \alpha \in L^{1,\infty}\) (a space with \(L^1\) as a strict subset) i.e. that
\[
|\{z \in D : |\nabla \alpha(z)| > s\}| \leq Cs^{-1}.
\]

However we will show that we can find such an \(S \in L^\infty \cap W^{1,2}\) at which point these estimates pass locally onto \(\alpha\). In fact we also end up with an estimate for \(|\alpha|^2\) in the local Hardy space \(h^1\) on the whole disc. The regularity for \(\alpha\) is therefore much higher than we would expect, due to the geometric nature of the problem. Essentially \(\alpha\) is geometrically holomorphic and when the geometry is ‘sufficiently nice’ we can understand it to be locally genuinely holomorphic.

This theory is closely related to Hélein’s \([6]\) regularity theory for harmonic maps from a Riemann surface to a closed Riemannian manifold \(\mathcal{N}\); indeed it provides a short proof for the full regularity theory in two dimensions using \textit{only Wente-type estimates and Coulomb gauge methods} without requiring that \(T\mathcal{N}\) be trivial. The assumption that \(T\mathcal{N}\) be trivial can be made without loss of generality if \(\mathcal{N}\) is sufficiently regular: When \(\mathcal{N}\) is \(C^4\) Hélein proved that there is a totally geodesic embedding of \(\mathcal{N}\) into a torus, thus harmonic maps into \(\mathcal{N}\) lift to harmonic maps into a torus (and we may therefore consider only targets with trivial tangent bundle). Theorem \(2.1\) allows us to side-step this technicality and we require the minimal regularity assumptions on \(\mathcal{N}\), that it is a \(C^2\) submanifold of \(\mathbb{R}^m\) with bounded\(^4\) \(\omega\) is a two form given by \([\omega, \omega](X,Y) = \omega([X,Y])\) where \([,]\) is the Lie bracket of \(gl(m, \mathbb{C})\).

\(^3\)Perhaps the reader should compare with the analogous statement in the real setting; that one can find a parallel frame solving
\[S^{-1}dS + S^{-1}\omega S = 0\]
if and only if \(F \equiv 0\), i.e. when the connection is flat.

\(^4\)The notation \((p,q)\) will refer either to \(p+q\) forms of type \(dz^1 \wedge \cdots \wedge dz^p \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^q\) or if \(G\) is a \(p+q\) form then \(G^{(p,q)}\) is the projection of \(G\) onto forms of type \((p,q)\).
second fundamental form (which follows trivially by the first assumption if $N$ is closed). We mention here that Hélein’s theory simplifies if $N$ is $C^2$ with trivial tangent bundle. Under the assumption that the tangent bundle is trivial, one can employ Coulomb gauge methods and write the harmonic map equation as (2) with the assumption that the tangent bundle is trivial, one can employ Coulomb gauge methods mention here that Hélein’s theory simplifies if $N$ is $C^2$ with trivial tangent bundle. Under the assumption that the tangent bundle is trivial, one can employ Coulomb gauge methods mention here that Hélein’s theory simplifies if $N$ is $C^2$ with trivial tangent bundle. Under the assumption that the tangent bundle is trivial, one can employ Coulomb gauge methods mention here that Hélein’s theory simplifies if $N$ is $C^2$ with trivial tangent bundle. Under the assumption that the tangent bundle is trivial, one can employ Coulomb gauge methods.

The theory here is also related to the work of Rivière [11], who generalised the regularity theory of Hélein, where he proves the full regularity for critical points of conformally invariant elliptic Lagrangians in two dimensions by considering a geometric divergence problem (vs a geometric $\mathcal{J}$ problem). Specifically he considered maps $u \in W^{1,2}(B_1, \mathbb{R}^m)$ weakly solving

$$0 = d^*(du) = *d\omega \wedge *du = -\Delta u - \omega.\nabla u$$

(3)

for $\omega \in L^2(B_1, so(m) \otimes \Lambda^1 T^* \mathbb{R}^2)$ on the unit ball $B_1 \subset \mathbb{R}^2$. This PDE is critical in the sense that the best one can do with straightforward elliptic estimates is to get estimates on $\nabla u$ in $L^{2,\infty}$ (i.e. $|\nabla u|^2 \leq L^{1,\infty}$). However Rivière proved the existence of a frame $A \in L^\infty \cap W^{1,2}(B_1, gl(m, \mathbb{R}))$ (a perturbed Coulomb gauge) such that

$$d^*(dA - A\omega) = 0$$

(4)

when $\|\omega\|_{L^2}$ is sufficiently small. This enables one to re-write (5) and uncovers hidden Jacobian determinant terms. By using classical Wente estimates one can show that $\nabla u \in L^p$ for every $p < \infty$ (see [14] or [12]). Finally he observed that critical points of conformally invariant elliptic Lagrangians in two dimensions solve a PDE of the form (3) to conclude the full regularity of solutions under the weakest regularity assumptions on the Lagrangian. In particular one can conclude regularity of harmonic maps into $C^2$ targets, or the regularity of conformal immersions of the disc in $\mathbb{R}^3$ with bounded mean curvature and finite area.

Going back to (4) we could conclude that there exists a matrix $B \in W^{1,2}(B_1, gl(m, \mathbb{R}))$ such that

$$dA - *dB = A\omega$$

or, to write it another way

$$\mathcal{J}(A - iB) = A\omega^\ast$$

(compare this with (1) and remember that we cannot control $\|B\|_{L^\infty}$ and therefore there is no reason that $A - iB$ be invertible).

In section 3 we show that critical points of conformally invariant elliptic Lagrangians solve (2) and, under an added regularity assumption, we can find the frame $S$ solving (1). Unlike the case for harmonic maps we require the theory of Rivière, namely the existence of the frame $A$ solving (4), in order to find $S$. However this can still be used to re-prove a recent estimate of Lamm and Lin [8], Theorem 3.3. We remark that it might be possible to drop the added regularity and still be able to find $S$ in this setting; either a positive or a negative answer to this question would provide further insight into these regularity problems.

Another interesting problem would be to extend this theory to higher dimensional complex domains, but of course one would have to impose the condition that $F_{\omega}^{(0,2)} = 0$ in a weak sense (which is given for free in one dimension), and find the right borderline spaces for $\omega$ to lie in. The author does not know of any geometric situation where the higher dimensional theory would apply.

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2 Results

Since the PDE we are trying to solve only involves \( \omega \) we may consider local connection forms \( \omega \in L^2(D, u(m) \otimes \Lambda^1 T^* \mathbb{R}^2) \) without loss of generality (see Remark 2.3). The assumptions we want to impose are that such \( \omega \) admit the following Hodge decomposition:

\[
\omega = da + d^*b
\]

with \( a \in W^{1,2}, \ b \in W^{1,(2,1)}_0 \). We say that such an \( \omega \) satisfies condition \( \dag \). If for some \( \varepsilon \) we know that

\[
\| \omega \|_{L^2(D)} + \| \nabla b \|_{L^2,1(D)} \leq \varepsilon
\]

then we say that \( \omega \) satisfies condition \( \dag \varepsilon \). Another way of writing this condition is that \( \omega \in L^2(D, u(m) \otimes \Lambda^1 T^* \mathbb{R}^2) \) satisfies

\[
\nabla \Delta^{-1}(d\omega) \in L^{2,1}
\]

with

\[
\| \omega \|_{L^2(D)} + \| \nabla \Delta^{-1}(d\omega) \|_{L^2,1(D)} \leq \varepsilon.
\]

**Theorem 2.1.** There exists \( \varepsilon > 0 \) such that whenever \( \omega \in L^2(D, u(m) \otimes \Lambda^1 T^* \mathbb{R}^2) \) satisfies condition \( \dag \varepsilon \), there exists a change of frame \( S \in L^\infty \cap W^{1,loc}(D, Gl(m, \mathbb{C})) \) such that

\[
\overline{\partial} S = -\omega \overline{\partial} S
\]

with

\[
\| \text{dist}(S, U(m)) \|_{L^\infty(D)} \leq \frac{1}{3}
\]

and for any \( U \subset \subset D \) there exists some \( C = C(U) < \infty \) such that

\[
\| \nabla S \|_{L^2(U)} \leq C \| \omega \|_{L^2}.
\]

Such an \( S \) is called a holomorphic frame. Going back to the general case of a smooth vector bundle \((E, \pi)\) over a Riemann surface \( \Sigma \) with an \( L^2 \) connection \( D_E \), if we could find a cover of \( \Sigma \), \( \{U_i\} \) such that the connection forms over each \( U_i \) satisfy condition \( \dag \varepsilon \), then we can skew our trivialisations by the non-smooth changes of frame \( S_i \). In other words where we had a smooth diffeomorphisms \( \phi_i = (\pi, \varphi_i) : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^m \), we replace them by non-smooth \( \tilde{\phi}_i(e) = (\pi(e), S_i(\pi(e))\varphi_i(e)) \). The reader can check that the new transition charts \( \tilde{\phi}_{ij} = S_i S_j^{-1} \) will be holomorphic\(^5\) (at the expense of the trivialisations being non-smooth).

**Corollary 2.2.** Let \( \alpha \in L^2(D, \mathbb{C}^m \otimes \Lambda^{(1,0)} T^*_C \mathbb{R}^2) \) and \( \omega \in L^2(D, u(m) \otimes \Lambda^1 T^* \mathbb{R}^2) \). Suppose that \( \omega \) satisfies condition \( \dag \) and that \( \overline{\partial}_m \alpha = 0 \), i.e.

\[
\overline{\partial}_m \alpha = -\omega \overline{\partial} \wedge \alpha
\]

then \( \alpha \in L^\infty \cap W^{1,2}(U) \) for all \( U \subset \subset D \). There exists an \( \varepsilon > 0 \) such that if \( \omega \) satisfies \( \dag \varepsilon \) then there exists a \( C = C(U) < \infty \) such that

\[
\| \alpha \|_{L^\infty(U)} \leq C \| \alpha \|_{L^1}
\]

and

\[
\| \nabla \alpha \|_{L^2(U)} \leq C \| \alpha \|_{L^1}(1 + \| \omega \|_{L^2}).
\]

Moreover, under these assumptions we have \( |\alpha|^2 \in h^1(D) \) with

\[
\| |\alpha|^2 |_{h^1(D)} \| \leq C \| \alpha \|_{L^2(D)}^2.
\]

\(^5\) Compare with the flat scenario, where by skewing our trivialisations by parallel frames yields locally constant (rather than holomorphic) transition functions.
As per the introduction, re-writing everything in terms of $S$ means the connection $\nabla$ problem is a 'genuine' $\nabla$ problem. The space $h^1$ is the local Hardy space, see for instance [14, Appendix A.2] for a brief introduction or [4].

Remark 2.3. We remark here that given any $\hat{\omega} \in L^2(D, gl(m, \mathbb{C}) \otimes \wedge^1 T^* \mathbb{R}^2)$ we can always find a unique $\omega \in L^2(D, u(m) \otimes \wedge^1 T^* \mathbb{R}^2)$ such that $\omega = \hat{\omega}$. Indeed if we write

$$\hat{\omega} = (\hat{\omega}_1 + i \hat{\omega}_2) dz$$

with $\hat{\omega}_j : D \to gl(m, \mathbb{R})$. Then we can decompose each $\hat{\omega}_j$ into its symmetric and antisymmetric part,

$$\hat{\omega}_j = \hat{\omega}_j^S + \hat{\omega}_j^A$$

thus letting $\omega^x = 2(\hat{\omega}_1^S + i \hat{\omega}_2^S) : D \to u(m)$ and $\omega^y = 2(\hat{\omega}_2^A - i \hat{\omega}_1^A) : D \to u(m)$ and

$$\omega = \omega^x dx + \omega^y dy \in L^2(D, u(m) \otimes \wedge^1 T^* \mathbb{R}^2)$$

we have

$$\omega = \frac{1}{2}(\omega^x + i \omega^y) dz = \hat{\omega}.$$ 

Therefore for any such $\hat{\omega}$ we can apply Theorem 2.1 and Corollary 2.2 if $\omega$ satisfies condition $\dagger$.

3 Applications to Harmonic maps and conformally invariant Lagrangians

3.1 Harmonic maps

When one considers a harmonic function $u : U \subset \mathbb{R}^n \to \mathbb{R}^m$ there are a few equivalent viewpoints that can be used to understand the PDE that is solved. Harmonic functions $u$, are critical points of the Dirichlet energy

$$E(v) := \frac{1}{2} \int_U |\nabla v|^2 \ dx,$$

which is equivalent to $u$ being a solution to

$$-\Delta u = -\text{div}(\nabla u) = d^* du = 0.$$

One might also consider the PDE not in terms of $u$, but $du$, and rather pedantically write the coupled system

$$d^*(du) = 0 \quad \text{and} \quad d(du) = 0$$

at which point we could say that $du$ is a harmonic one form, or equivalently that each of the $m$ functions $H^i = (u^i_1, \ldots, u^i_n) : U \to \mathbb{R}^n$ solve the Cauchy-Riemann equations. When $n = 2$ we can more succinctly write this as

$$\bar{\nabla}(du) = 0$$

where we are now considering $U \subset \mathbb{C}$ and $d = \partial + \bar{\partial}$ is the usual splitting i.e. $\partial v = \frac{\partial v}{\partial z} dz$ and similarly for $\partial$.

From now on we will restrict to considering two dimensional domains and the target a Riemannian manifold $(\mathcal{N}, h)$. Due to the conformal invariance of the problems we are looking at, we take the unit disc $B_1 \subset \mathbb{R}^2$ with the Euclidean metric as our domain. In order to be able
to write down the PDE appearing below we will be implicitly using coordinates on $\mathcal{N}$ and therefore we are assuming that $u$ is at least continuous so that we may always assume that it remains in a single coordinate chart. Under this assumption we can consider the pull back bundle $u^*\mathcal{T}\mathcal{N}$ to be trivial and the pulled back Levi-Civita connection is defined entirely by the form

$$\omega_j^i := \Gamma^i_{jk}(u) du^k \in L^2(B_1, gl(m, \mathbb{R}) \otimes \Lambda^1 \mathbb{T}^* \mathbb{R}^2).$$

One has that $u$ is harmonic if

$$d^u_{\mathcal{N}}(du) = -\Delta u - \Gamma^i_{jk}(u) \left( \frac{\partial u^j}{\partial x} \frac{\partial u^k}{\partial x} + \frac{\partial u^j}{\partial y} \frac{\partial u^k}{\partial y} \right) = 0.$$

Here we have considered the connection as a covariant exterior derivative

$$d_{u^*\mathcal{N}} : \Gamma(\mathcal{N}) \to \Gamma(\mathcal{N})$$

and $d^u_{\mathcal{N}}$ is the formal adjoint for $k = 0$.

In this case we also have

$$d_{u^*\mathcal{N}}(du) = d(du^i) + \Gamma^i_{jk}(u) du^k \wedge du^j = 0.$$  

It is thus unsurprising that by considering the domain as being complex we actually have

$$\overline{\partial}_{u^*\mathcal{N}}(\overline{\partial}u) = \overline{\partial}\partial u^i + \Gamma^i_{jk}(u) \overline{\partial}u^j \wedge \partial u^k = 0.$$  

Now we make another rash assumption: suppose that the connection is flat i.e. there exists a frame $S : B_1 \to GL(m, \mathbb{R})$ solving

$$S^{-1} dS + S^{-1} \omega S = 0.$$  

(5)

In this case we can re-write our PDE with respect to $S$ and we actually have the coupled system

$$d^*(S^{-1} du) = 0 \quad \text{and} \quad d(S^{-1} du) = 0,$$

or equivalently

$$\overline{\partial}(S^{-1} \partial u) = 0.$$  

The assumption that the connection is flat is of course too strong (unless $\mathcal{N}$ is flat), therefore being able to find $S$ solving (5) is impossible. However we may utilise the complex domain here and consider the $\mathcal{F}$ problem by considering $\partial u$ as a section of $u^*\mathcal{T}\mathcal{N} \otimes \mathbb{C} \otimes \Lambda(1, 0) \mathbb{T}^* \mathbb{R}^2$ and finding a holomorphic frame $S$ for the pulled back connection for the complexified bundle.

Unfortunately we have presupposed that $u$ is continuous in order to have these observations, however the potential lack of continuity of $u$ can be overcome by considering $(\mathcal{N}, \overline{\partial})$ to be

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6Given a vector bundle with a connection and a trivialisation the covariant exterior derivative is given simply by (where $e_i$ is our local frame, $\omega$ are our connection forms and $v = v^i dx^i \otimes e_i$)

$$d_E(v) = dv + \omega \wedge v = (dv^i + \omega^i_j \wedge dx^j \otimes e_i) \otimes e_i.$$  

If we have a metric on our vector bundle and we impose that the connection be compatible then when we choose an orthonormal (unitary) frame for our vector bundle, the connection forms are skew-symmetric (skew-hermitian) and the metric is trivial with respect to our trivialisation. From here the formal adjoint $d^*_E$ on sections $z \in \Gamma(E \otimes \Lambda^1 \mathbb{T}^* \mathbb{R}^2)$ ($z = (z^i dx^i + z^i_j dy^j) \otimes e_i$) is locally given by

$$d^*_E(z) = d^*(z) - \ast(\omega \wedge z) = (d^*(z^i dx^i + z^i_j dy^j) - \ast(z^i_j \wedge \ast(z^i_k dx^k + z^i_k dy^k)) \otimes e_i$$

as can be checked directly. However we are not using an orthonormal frame for our trivialisation above so one must be more careful when computing the adjoint. It turns out that we do indeed have

$$d^*_{u^*\mathcal{N}}(du) = -\Delta u^i - \Gamma^i_{jk}(u) \left( \frac{\partial u^j}{\partial x} \frac{\partial u^k}{\partial x} + \frac{\partial u^j}{\partial y} \frac{\partial u^k}{\partial y} \right)$$

as can be checked directly.
isometrically embedded in some Euclidean space $\mathbb{R}^m$ at which point we can consider the

critical points of the Dirichlet energy amongst maps in

$$W^{1,2}(B_1, \mathcal{N}) := \{ v \in W^{1,2}(B_1, \mathbb{R}^m) : v(z) \in \mathcal{N} \text{ for almost every } z \in B_1 \}$$

and we can write the harmonic map equation as $(\Delta u)^T = 0$, or the projection of $\Delta u$ onto $T_u \mathcal{N}$ is zero. Therefore we have

$$\Delta u + A(u)(u_x, u_z) + A(u)(u_y, u_y)$$

or in coordinates

$$\Delta u^i + A^i_{jk}(u)(\partial_{x^j} u^k + \partial_{x^k} u^j) = 0$$

where $A^i_{jk}$ are the components of the second fundamental form of $\mathcal{N} \hookrightarrow \mathbb{R}^m$ (actually we have first extended the second fundamental form to a neighbourhood of $\mathcal{N}$, $\mathcal{N}'$ so that $A : \mathcal{N}' \rightarrow T^*\mathbb{R}^m \otimes T\mathbb{R}^m \otimes T\mathbb{R}^m$ with $A^i_{jk}(u) = A^i_{jk}(u)$ and the vector $\{A^i_{jk}(u)\}_{i,j}^m$ is normal to $\mathcal{N}$ for each $j$ and $k$. – see [3] Chapter 1) for details. The observation of Rivi`ere was to let $\omega^i_j := (A^i_{jk}(u) - A^i_{jk}(u))du^k \in L^2(B_1, so(m) \otimes \wedge^1 T^*\mathbb{R}^2)$ be our connection forms, and using the properties of $A$ it can be checked that we have

$$d_\omega^i(du) = d^*(du) - * (\omega \wedge du) = 0$$

from which the higher regularity can be obtained by using the perturbed Coulomb gauge $A$ (the anti-symmetry of $\omega$ and the $L^\infty$ bound on $A$ are essential here). As is our wont, we can also write

$$d_\omega(du) = d(du) + \omega \wedge du = 0$$

– this can be checked directly by using the properties of $A$ and implies that the Hopf differential is holomorphic.

Therefore we also have

$$\overline{\omega}^i_j = (A^i_{jk}(u) - A^i_{jk}(u))\overline{du}^k.$$ 

Under the added assumption that the normal bundle is trivial (for instance when $\mathcal{N}$ is diffeomorphic to a sphere, or an orientable hypersurface in $\mathbb{R}^m$ etc), we can have a global normal frame $\{\nu_K\}_{K \in \mathbb{N}+1}$ for $\mathcal{N}$ that is $C^1$, and we can express $A$ with respect to this frame via the Weingarten equation (note that this is independent of the choice of orthonormal normal frame):

$$A^i_{jk}(z) := - \sum_K \partial_{z^k} \nu^j_K$$

where $z$ is the standard coordinate of $\mathbb{R}^m$ and $\nu$ is extended arbitrarily off $\mathcal{N}$. Thus we can write

$$\omega^i_j = (du^i_K(u))\nu^j_K(u) - (du^i_K(u))\nu^j_K(u)$$

where we are summing over repeated indices (i.e. over $K$), thus $(\omega^i_j) = \overline{\partial}u^i_K(u)\nu^j_K(u) - \overline{\partial}u^i_K(u)\nu^j_K(u)$. Thus for a Hodge decomposition $\omega = da + d^*b$ with $b \in W^{1,2}_0$ we have

$$\Delta^i_j = 2du^i_K(u) \wedge \nu^j_K(u)$$

\footnote{We have $\omega^x u_y = \omega^y u_x$ thus we can conclude (an exercise using the anti-symmetry of $\omega$) that $\Delta u$ is perpendicular to both $u_x$ and $u_y$ (of course this is obvious for harmonic maps but works more generally when we assume $\omega^x u_y = \omega^y u_x$). The Hopf differential

$$\psi := (u^x h)^{(2,0)} := (u_x, u_x)dz \wedge dz = \phi dz \wedge dz$$

where $(,)$ is the Euclidean inner product extended complex linearly. Therefore

$$\overline{\partial} \phi = 2(u_x, u_y) = 0.$$}
and therefore \( \nabla b \in L^{2,1} \) by Theorem 3.1 with
\[
\| \nabla b \|^2_{L^{2,1}} \leq C \| \nabla u \|^2_{L^2}.
\]
Notice that here we have \( C = C(\sup |A|) \). As mentioned earlier if \( T\mathcal{N} \) is trivial we also have an easy proof using a \( \overline{\mathcal{D}} \) problem see [9].

Following [10] we can utilise this idea for general \( C^2 \) target manifolds \( \mathcal{N} \) by considering a smooth partition of unity \( \{ \chi_\alpha \} \) over \( \mathcal{N} \) such that over the support of each \( \chi_\alpha \) we know that the normal bundle of \( \mathcal{N} \) is trivialised by \( \{ \nu_{\alpha,K} \}_K^{m=|N|+1} \). Setting \( \nu_{\alpha,K} \) to be zero outside of the support of \( \chi_l \) and defining
\[
\hat{\mu}_{\alpha,K}(z) := \chi_\alpha(z)\nu_{\alpha,K}(z)
\]
we see that \( \hat{\mu} \) is smooth over \( \mathcal{N} \) with
\[
|\nabla \hat{\mu}_{\alpha,K}(z)| \leq \sup_\alpha |A| + \sup |\nabla \chi_\alpha| \leq C(\sup |A|)
\]
where the second inequality follows since \( \sup |A| \) uniformly controls the diameter \( R \) of intrinsic balls over which the normal bundle can be trivialised. Thus our partition of unity can be constructed by smoothing out characteristic functions over balls of a fixed radius. Now define \( \mu_{\alpha,K} := \hat{\mu}_{\alpha,K}(u) \) and note that we can write
\[
\omega_{i}^j := (d\mu_{\alpha,K}^i)\nu_{\alpha,K}^j(u) - (d\nu_{\alpha,K}^j)\mu_{\alpha,K}^i(u)
\]
where we are summing over both \( \alpha \) and \( K \).

Again for a Hodge decomposition as above we have
\[
\Delta b_j^i = (d\mu_{\alpha,K}^i) \wedge (d\nu_{\alpha,K}^j(u)) - (d\nu_{\alpha,K}^i) \wedge (d\nu_{\alpha,K}^j(u))
\]
and therefore \( \nabla b \in L^{2,1} \) with
\[
\| \nabla b \|^2_{L^{2,1}} \leq C \| \nabla u \|^2_{L^2}
\]
(again \( C = C(\sup |A|) \)) and \( \omega \) satisfies condition \( \dagger \) with
\[
\| \omega \|^2_{L^2} + \| \nabla b \|^2_{L^{2,1}} \leq C \| \nabla u \|^2_{L^2}
\]
whenever \( \| \nabla u \|^2_{L^2} \leq 1 \).

Thus, Corollary 2.2 immediately gives Lipschitz estimates on \( u \). The full regularity (along with smooth estimates) for harmonic maps follows from an easy boot-strapping argument using standard Calderon-Zygmund and Schauder estimates.

**Theorem 3.1 (Hélein).** Suppose \( u : B_1 \to \mathcal{N} \) is a weakly harmonic map where \( \mathcal{N} \) is a \( C^l \) submanifold of \( \mathbb{R}^m \) such that the second fundamental form is bounded with respect to the induced metric and \( l \geq 2 \). Then for all \( \alpha \in (0,1) \) there exist \( \varepsilon = \varepsilon(\sup |A|) \) and \( C = C(\mathcal{N},\alpha) \) such that if
\[
\| \nabla u \|^2_{L^2(B_1)} \leq \varepsilon
\]
then
\[
[\nabla^i u]_{BMO(B_{\frac{1}{2}})} + \| u \|^2_{C^{l-1,\alpha}(B_{\frac{1}{2}})} \leq C \| \nabla u \|^2_{L^2(B_1)}.
\]

We also recover the following Energy convexity theorem in [2], from which local uniqueness of harmonic maps follows easily in two dimensions. The proof can be found in [2, Appendix C] however now we can assume that \( \mathcal{N} \) is \( C^2 \) with bounded second fundamental form and we do not need to make any assumptions on the tangent bundle.
Theorem 3.2 (Colding-Minicozzi). Let \( u, v \in W^{1,2}(B_1, \mathcal{N}) \) and suppose that \( u \) is weakly harmonic map where \( \mathcal{N} \) is a \( C^2 \) submanifold of \( \mathbb{R}^m \) with bounded second fundamental form. Then there exists some \( \varepsilon = \varepsilon(\sup |A|) \) such that if \( u - v \in W^{1,2}_0 \) and
\[
\|\nabla u\|_{L^2(B_1)} \leq \varepsilon,
\]
then
\[
\int_{B_1} |\nabla v|^2 - |\nabla u|^2 \geq \frac{1}{2} \int_{B_1} |\nabla (v - u)|^2.
\]

3.2 Conformally invariant Lagrangians and an estimate of Lamm and Lin

Here we will recover (and marginally improve) the following result of Lamm and Lin (stated as a Corollary in [8]).

Theorem 3.3. Let \( \mathcal{N} \hookrightarrow \mathbb{R}^m \) be an isometrically embedded, closed Riemannian manifold which is \( C^2 \) with bounded second fundamental form. Let \( \gamma \in C^{1,1}(\mathcal{N}, \Lambda^2 T^*\mathcal{N}) \) then every critical point in \( W^{1,2}(B_1, \mathcal{N}) \) of the Lagrangian
\[
F(u) = \hat{\Omega}_{B_1} \left( \frac{1}{2} |\nabla u|^2 dx \wedge dy + u^* \gamma \right)
\]
solves
\[
\mathcal{D}_{\omega}(\partial u) = 0
\]
where
\[
\omega^j_i := (d\pi^j_a, K) \nu^j_a, K(u) - (d\mu^j_a, K) \nu^j_a, K(u) + *\lambda^j_k(u) du^k
\]
(see the previous section for definitions if necessary). Here \( \lambda^j_k(u) = (d\pi^j_a(\gamma))(e_i, e_j, e_k) \) where \( \pi^j_a \) is the orthogonal projection onto \( \mathcal{N} \) defined in some small tubular neighbourhood. Moreover \( \omega \) satisfies condition \( \dagger \varepsilon \) with
\[
\varepsilon \leq C\|\nabla u\|_{L^2(B_1)}.
\]
Then whenever \( E(u) \) is sufficiently small we have
\[
\| |\nabla u|^2 |_{H^1(B_1)} \leq C\|\nabla u\|_{L^2(B_1)}^2
\]
and locally smooth estimates for \( u \) in terms of \( E(u) \).

Proof of theorem \[7,3\] First of all notice that \( \omega \in L^2(B_1, u(m) \otimes \Lambda^1 T^* \mathbb{R}^2) \) (since it is real and antisymmetric). We already know that (see been or \[8\])
\[
d^*_\omega(du) = 0
\]
and it follows from the symmetries of \( \mathcal{A} \) and \( \lambda \) that
\[
d_\omega(du) = d(du) + \omega \wedge du = 0.
\]
Therefore we can conclude that
\[
\mathcal{D}_{\omega}(\partial u) = \mathcal{D}(\partial u) + \omega^* \wedge \partial u = 0.
\]

\[^{8}\text{It is known that all Lagrangians of this form are conformally invariant, moreover any Lagrangian that is conformally invariant and quadratic in the gradient takes this form when it is } C^2 \text{ regular – see } [8].\]
We now use Rivièrè’s decomposition to find $A$ and $B$ solving
\[ dA - A\omega = *dB. \]

Inspecting the proof of [8, Proposition 4.1] we have that
\[ \|\nabla B\|_{L^2(B_1)} \leq C\|\nabla u\|_{L^2(B_1)}. \]

Therefore
\[ d\omega = dA^{-1} \wedge dA - d(A^{-1} \ast dB) \]
from which we can conclude that
\[ \|\nabla A^{-1}(d\omega)\|_{L^2(B_1)} \leq C\|\nabla u\|_{L^2(B_1)}. \]

The rest of the proof follows from applying Corollary 2.2.

\[ \square \]

4 Optimality of condition \dagger

Here we present an example to show that the condition on the Hodge decomposition is sharp.

Consider $\alpha : D \to C^2$ given by $\alpha(z) = \frac{1}{z \log(\frac{e}{|z|})}(1, -i)dz \in L^2(D, C^2 \otimes (1,0)T^\star D)$ and we define $\omega \in L^2(D, so(2) \otimes \wedge^1 T^\star R^2)$ by
\[ \omega = \frac{1}{z^2 \log(\frac{e}{|z|})} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (ydx - xdy) \]
so that
\[ \omega^\ast = \frac{i/2}{z \log(\frac{e}{|z|})} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} d\bar{z}. \]

A short calculation yields that
\[ \partial_\bar{z} \omega \wedge d\bar{z} = \frac{1/2}{(|z| \log(\frac{e}{|z|}))^2} (1, -i) d\bar{z} \wedge d\bar{z} = -\omega \wedge \alpha. \]

Therefore $\nabla_\omega \alpha = 0$ but $\alpha \notin L^\infty_{loc}(D)$

and
\[ \partial_\bar{z} \omega = \left( \frac{1}{z \log(\frac{e}{|z|})} + \frac{1/2}{(|z| \log(\frac{e}{|z|}))^2} \right) (1, -i) \not\in L^1_{loc}(D) \]

thus Theorem 2.1 cannot hold in this case.

It is easy to see that
\[ \omega = *du \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
where $u = \log \log(\frac{e}{|z|})$ is the Frehse example, thus setting $b = *u$ one can easily check that $d^\ast b = *du \in L^{2,q}$ for all $q > 1$ but $d^\ast b \notin L^{2,1}$, hence condition \dagger is sharp in this sense.
5 Proof of the regularity result, Corollary 2.2

Suppose that \( \omega \) satisfies \( \epsilon \) given by Theorem 2.1. We check that
\[
\partial (S^{-1}\alpha) = -S^{-1}\partial SS^{-1} \wedge \alpha - S^{-1}\omega^z \wedge \alpha = 0.
\]
Therefore \( \alpha = Sh \) for some holomorphic \( h \) and the estimates follow by standard theory. A simple covering argument completes the first part of the proof.

The proof of the final assertion (that \( |\alpha|^2 \in h^1 \)) follows from the following fact that is easily verified: Given a holomorphic function \( h \in L^2(D) \), then \( |h|^2 \in h^1(D) \) with
\[
\|\|h|^2\|_{h^1(D)} \leq C\|h\|_{L^2(D)}.
\]

To see this first notice that \( h = f_z \) for some holomorphic \( f \in W^{1,2}(D; u(m) \otimes \wedge^2 T^*D) \) (this follows from the Poincaré lemma, for instance). Thus we have (since \( f \) is holomorphic)
\[
|h|^2 = |f_z|^2 = -d(df_1 \wedge df_2) \in h^1(D)
\]
by the main result in \( \Pi \) (coupled with an extension argument).

In our case we have \( \alpha = Sh \) so that there exists some \( C \) with
\[
C^{-1}|h|^2 \leq |\alpha|^2 \leq C|h|^2.
\]
Thus we have
\[
\|\|\alpha|^2\|_{h^1(B_1)} \leq C\|\|h|^2\|_{h^1(B_1)} \leq C\|\|h\|^2_{L^2(D)} \leq C\|\alpha\|^2_{L^2(D)}.
\]

6 Proof of the existence of a holomorphic gauge, Theorem 2.1

We start by finding the Coulomb frame associated to \( da \); using Theorem 8.1 we can find \( P \in W^{1,2}(D; U(m)) \) and \( \eta \in W^{1,2}_0(D, u(m) \otimes \wedge^2 T^*D) \) such that
\[
P^{-1}daP + P^{-1}daP = d^*\eta \tag{7}
\]
and
\[
\|\nabla P\|_{L^2(D)} + \|\eta\|_{W^{1,2}(D)} \leq C\|da\|_{L^2(D)} \leq C\|\omega\|_{L^2(D)}.
\]

Thus on \( D \) we have a solution to
\[
\Delta \eta = d\bar{P}^T \wedge dP + d(P\bar{P}^T daP).
\]
Or, in coordinates we have
\[
\Delta \eta_j = d\bar{P}^k \wedge dP^k + d(P_i^l P^k_j) \wedge da_k^l.
\]
Again we sum over repeated indices here so that we sum over \( k \) in the first term, and both \( k \) and \( l \) in the second.

The estimates from Theorem 8.1 give
\[
\|\nabla \eta\|_{L^2(Z^1(D))} \leq C\|\omega\|^2_{L^2(D)}.
\]
Now we check how $P$ transforms $\omega$, by (7) we have
\[ P^{-1}dP + P^{-1}\omega = \omega P = d^*\eta + \beta P - d^*\beta P \in L^{(2,1)}(D) \] (8)
and
\[ \|d^*\eta + \beta P - d^*\beta P\|_{L^{2,1}(D)} \leq C\varepsilon. \]
We can see here the significance of condition †, essentially it allows us to change the connection forms so that the whole of the transformed connection lies in $L^{2,1}$.

We can now take the $(0,1)$-part of (8) to give
\[ P^{-1}\partial P + P^{-1}\omega = d^*\eta + \beta P - d^*\beta P \in L^{(2,1)}(D) \] (9)
which after applying Theorem 8.3 (by setting $\varepsilon$ small enough) gives us the existence of some $Q \in C^0 \cap W^{1,2,1}(D, GL(k, \mathbb{C}))$ satisfying
\[ \partial Q = -d^*\eta + \beta P - d^*\beta P \]
\[ \|\text{dist}(Q, \text{Id})\|_{L^\infty(D)} \leq \frac{1}{3} \]
and for any $U \subset D$ there exists $C = C(U) < \infty$ such that
\[ \|\nabla Q\|_{L^{2,1}(U)} \leq C\|\omega\|_{L^2(D)}. \]
Thus we have
\[ P^{-1}\partial P + P^{-1}\omega P = -\theta Q Q^{-1} \]
and therefore setting $S = P Q \in L^\infty(D, GL(k, \mathbb{C})) \cap W^{1,2}(D, GL(k, \mathbb{C}))$ we have
\[ \partial S = -\omega S \]
with the desired estimates.

7 A few remarks

We could generalise this, and simply consider maps $v \in L^2(B_1, \mathbb{C}^m \otimes \Lambda^1 T^* \mathbb{R}^2)$ solving
\[ d^*\omega(v) = 0 \]
and
\[ d\omega(v) = 0 \]
for some connection $\omega \in L^2(B_1, u(m) \otimes \Lambda^1 T^* \mathbb{R}^2)$. As above we can check that $v^{(1,0)} \in L^2(B_1, \mathbb{C}^m \otimes \Lambda^{(1,0)} T^* \mathbb{C})$ solves
\[ \overline{\partial}\omega(v^{(1,0)}) = 0. \]
Now we can ask, under what conditions can we find a holomorphic change of frame $S$ as in Theorem 2.1 in order to conclude $v \in (L^\infty \cap W^{1,2})_{\text{loc}}$. In general we cannot do this unless $\omega$ satisfies condition † because of the counter-example presented in section 4. However we are still free to change our frame via a map $P \in W^{1,2}(B_1, U(m))$, and writing $v^{(1,0)} := P^{-1} v^{(1,0)}$ we have
\[ \overline{\partial}\omega_P(v^{(1,0)}) = 0 \]
where
\[ P^{-1}dP + P^{-1}\omega P = \omega P. \]
Now we can ask whether \( \omega_P \) satisfies condition \( \dagger \)? In particular this is the case if \( d(\omega_P) = 0 \) (the ‘opposite’ of what is achieved in considering a Coulomb frame) or \( d(\omega_P) \in H^1 \). More generally this is true if

\[
\nabla \Delta^{-1}(d(\omega_P)) = \nabla \Delta^{-1}(dP^{-1} \wedge dP + d(P^{-1} \omega_P)) \in L^{2,1}.
\]

Therefore because of Theorem \([8,1]\) we can reduce this condition to being able to find a frame \( P \) such that

\[
\nabla \Delta^{-1}(d(P^{-1} \omega_P)) \in L^{2,1}.
\]

The bottom line here is the following:

**Theorem 7.1.** Let \( \omega \in L^2(D, u(m) \otimes \wedge^1 T^* \mathbb{R}^2) \), and suppose there exists a change of frame \( P \in W^{1,2}(B_1, U(m)) \) such that

\[
\omega_P = P^{-1} dP + P^{-1} \omega P
\]

satisfies condition \( \dagger \). Then there exists \( \varepsilon > 0 \) such that whenever \( \omega_P \) satisfies condition \( \dagger \varepsilon \) there exists a change of frame \( S \in L^\infty \cap W^{1,2}(D_2, Gl(k, \mathbb{C})) \) such that

\[
\partial S = -\omega \gamma S
\]

with

\[
\|\text{dist}(S, U(m))\|_{L^\infty(D)} \leq \frac{1}{3}
\]

and

\[
\|\nabla S\|_{L^2(D_2)} \leq C\|\omega\|_{L^2}.
\]

### 7.1 Conformal immersions of surfaces into Riemannian manifolds

In this section we consider a conformal immersion \( u : B_1 \to N \hookrightarrow \mathbb{R}^m \) with bounded area and mean curvature \( H : B_1 \to \mathbb{R}^m \). It is well known that \( u \) solves

\[
\tau(u) = H(|u_x|^2 + |u_y|^2)
\]

(10)

where \( \tau(u)^i := -\Delta u^i - \mathcal{A}_{jk} \nabla u^k \cdot \nabla u^j \) is the tension field of \( u \). Since \( u \) is conformal we also have

\[
|u_x|^2 - |u_y|^2 = (u_x, u_y) = 0
\]

and we consider the surface \( \Sigma = (B_1, \rho^2(dx^2 + dy^2)) \) so that \( u \) is an isometry \( u : \Sigma \to u(B_1) \) and \( \rho = |u_x| \). When \( H \in L^2(\Sigma) \) we can find \( \omega \in L^2(\Sigma, so(m) \otimes T^* \mathbb{R}^2) \) such that (10) can be written

\[
\quad d_\omega(du) = \omega(du) = \partial_\omega(du) = 0.
\]

To see this let

\[
\omega_j := (\omega_N)^j + (\omega_H)^j
\]

\[
:= (d\mu^i_{\alpha,K})\nu^i_{\alpha,K}(u) - (d\mu^j_{\alpha,K})\nu^j_{\alpha,K}(u) + (H^i du^j - H^j du^i)
\]

so that

\[
d_\omega(du) = d(du^i) + \omega_j^i \wedge du^j
\]

\[
= (\mathcal{A}_{jk}(u) - \mathcal{A}_{jk}(u)) du^k \wedge du^j + (H^i du^j - H^j du^i) \wedge du^j
\]

\[
= 0,
\]
and
\[
\delta^*(du) = d^*(du^i) - \ast(\omega^i \wedge du^i)
\]
\[
= -\Delta u^i - (A^i_{jk}(u) - A^i_{jk}(u)) \ast (du^k \wedge \ast du^j) - \ast((H^i du^j - H^j du^i) \wedge \ast du^j)
\]
\[
= \tau(u) - H(|u_x|^2 + |u_y|^2) = 0,
\]
which together imply that
\[
\partial_\omega(\partial u) = \partial_\omega(u) + \omega^z \wedge \partial u = \frac{1}{4} \Delta u \partial_\omega d\bar{z} \wedge dz + (A^i_{jk}(u) - A^i_{jk}(u)) \partial u^k \wedge \partial u^j + (H^i \partial u^j - H^j \partial u^i) \wedge \partial u^j
\]
\[
= -\frac{1}{4} \delta^*(du) d\bar{z} \wedge dz + \frac{1}{2} \delta_\omega(du) = 0.
\]

We therefore see that for \( H \in W^{1,2}(B_1, \mathbb{R}^m) \) with \( H \in L^2(\Sigma) \) there exists \( \eta = \eta(\mathcal{N}, m, \varepsilon) \) such that if
\[
\text{Area}(u(B_1)) + \| \nabla H \|_{L^2(B_1)} + \| H \|_{L^2(\Sigma)} \leq \eta
\]
then \( \omega \) satisfies \( \dagger \).

**Problem 7.2.** The requirement that \( H \in W^{1,2} \) does not seem to be natural, therefore one could ask whether only considering \( H \in L^2(\Sigma) \) and
\[
\text{Area}(u(B_1)) + \| H \|_{L^2(\Sigma)} \leq \eta
\]
is enough to find a holomorphic gauge? Using the previous section this would amount to finding a change of frame \( P \in W^{1,2}(B_1, SO(m)) \) such that
\[
P^{-1} \omega H \quad \text{satisfies condition } \dagger \quad \text{with}
\]
\[
\| \nabla \Delta^{-1}(d(P^{-1} \omega H P)) \|_{L^2(B_1)} \leq \| H \|_{L^2(\Sigma)}.
\]

In the next section we essentially show that this is possible with \( P = Id \) under the (strong) assumptions that the mean curvature is parallel, and \( \mathcal{N} = \mathbb{R}^m \).

### 7.1.1 Parallel mean curvature

Here we consider the situation where \( \mathcal{N} = \mathbb{R}^m \) and \( u(B_1) \) has parallel mean curvature (PMC). This condition means that \( \nabla^\perp H = 0 \) where \( \nabla^\perp \) is the induced connection on the normal bundle of \( \Sigma \). This is equivalent to the condition that \( \frac{\partial H}{\partial x} \) and \( \frac{\partial H}{\partial y} \) are tangent to \( \Sigma \) and therefore we may conclude that \( |H|^2 \) is a constant (since \( H \) is normal to \( \Sigma \)). Moreover we will use the expressions
\[
\frac{\partial H}{\partial x} = H_x = \langle H_x, u_x \rangle \frac{u_x}{\rho^2} + \langle H_x, u_y \rangle \frac{u_y}{\rho^2},
\]
\[
\frac{\partial H}{\partial y} = H_y = \langle H_y, u_x \rangle \frac{u_x}{\rho^2} + \langle H_y, u_y \rangle \frac{u_y}{\rho^2},
\]
which hold since we have PMC. Also
\[
\langle H_x, u_x \rangle = -\langle H, u_{xx} \rangle,
\]
\[
\langle H_y, u_y \rangle = -\langle H, u_{yy} \rangle.
\]
which hold simply because $H$ is normal.

Now, we still have
$$d^*\omega_H(du) = d\omega_H(du) = 0$$
and a computation using the fact that we have PMC gives
$$\frac{(d\omega_H)^i_j}{\rho^2} = dH^i \wedge du^j - dH^j \wedge du^i = \frac{(H_x u_y^i - H_y u_x^i + H^t u_y^t - H^t u_x^t)dx \wedge dy}{\rho^2} + \frac{(H_y, u_y) u_x^i u_y^t - u_x^t u_y^i}{\rho^2} dx \wedge dy$$
$$= -\frac{(H, \Delta u)}{\rho^2} du^i \wedge du^j = 2|H|^2 du^i \wedge du^j.$$

Moreover we also have
$$d^*\omega_H = 0$$
weakly since
$$\frac{(d \omega_H)^i_j}{\rho^2} = H^i d \ast du^j - H^j d \ast du^i + dH^i \wedge \ast du^j - dH^j \wedge \ast du^i = \frac{(H_x u_y^i - H_y u_x^i + H^t u_y^t - H^t u_x^t)dx \wedge dy}{\rho^2} + \frac{(H_y, u_y) u_x^i u_y^t - u_x^t u_y^i}{\rho^2} dx \wedge dy$$
$$= 0.$$ 

This tells us two things: Firstly that $\omega$ satisfies condition $\dagger$ when
$$|H|^2 \text{Area}(u(B_1))$$
is sufficiently small, and secondly we also have that $\Delta u$ is a sum of Wente terms by writing
$$\omega_H = \ast d\eta$$
and
$$-\Delta u = \ast (d\eta \wedge du).$$
The latter fact is obvious when $m = 3.$

8 Wente estimates and changes of frame

We will use the following well known estimate, which follows from the results of [11] and [3] but in a simpler form is due to [10]. We also use implicitly here the continuous embedding $W^{1,1}(B_1) \hookrightarrow L^{2,1}(B_1)$ when $B_1 \subset \mathbb{R}^2$. A proof of this fact can be found in [6].

**Theorem 8.1.** Suppose that $\phi \in W^{1,2}_0(B_1)$ is a solution to
$$\Delta \phi = \ast (da \wedge db)$$
with $a, b \in W^{1,2}(B_1, \mathbb{R})$ on the unit disc $B_1 \subset \mathbb{R}^2$. Then $\phi \in W^{2,1}$ with
$$\|\nabla^2 \phi\|_{L^1(B_1)} + \|\nabla \phi\|_{L^2(B_1)} + \|\phi\|_{C^0(B_1)} \leq C\|\nabla a\|_{L^2(B_1)} \|\nabla b\|_{L^2(B_1)}.$$
Critical problems

We state here the following classical theorem of Koszul-Malgrange [7, Theorem 1], below we use \( G \) to denote a complex Lie group and \( g \) its Lie algebra.

**Theorem 8.2.** Let \( U \subset \mathbb{C}^n \) be open and \( \alpha \in C^\infty(U, g \otimes \wedge^{(0,1)} T^*_C \mathbb{C}^n) \). Then, for any open \( V \subset U \) there exists \( f \in C^\infty(V, G) \) solving

\[
f^{-1} \bar{\partial} f = \alpha
\]

if and only if

\[
\bar{\partial} \alpha + [\alpha, \alpha] = 0 \quad (11)
\]
on \( V \).

In the case that \( \alpha \) is the \((0,1)-part\) of a connection form, the expression \((11)\) is precisely the \((0,2)\)-part of the curvature. As we have mentioned previously, in the case that \( n = 1 \) it is clear that such an \( f \) always exists since the condition \((11)\) is vacuously true. The following is a non-smooth version of Theorem 8.2 for \( n = 1 \) and \( G = GL(m, \mathbb{C}) \), the proof of which can be found in [6].

**Theorem 8.3.** Let \( \omega \in L^{2,1}(D, gl(k, \mathbb{C}) \otimes \wedge^{(0,1)} T^*_C \mathbb{R}^2) \) then there exists \( \varepsilon > 0 \) such that whenever \( \| \omega \|_{L^{2,1}(D)} \leq \varepsilon \) there is a \( \eta \in W^{1,2}(D, GL(k, \mathbb{C})) \) such that

\[
\bar{\partial} \eta = -\omega \eta
\]

and

\[
\| \text{dist}(\eta, \text{Id}) \|_{L^\infty(D)} \leq \frac{1}{3}.
\]

It follows from standard estimates for harmonic maps that the smallness condition on \( \| \omega \|_{L^{2,1}} \) cannot be dropped in order that we keep the \( L^\infty \) estimate on \( \eta \). For instance if one considers a sequence of harmonic maps \( \{ u_n \} : B_1 \to S^2 \) with uniformly bounded energy, that undergoes bubbling, then one has \( \| \nabla u_n \|_{L^{2,1}} \) is uniformly bounded.\footnote{In this instance we know there exist functions \( B_j \) such that

\[
\Delta u = \varepsilon (dB_j^* \wedge du^*)
\]

and

\[
\| \nabla B_j^* \|_{L^2} \leq C \| \nabla u \|_{L^2}.
\]See [6].} We also know that \( \alpha_n = \partial u_n \) solves

\[
\bar{\partial} \alpha_n, \alpha_n = 0
\]

with \( \| \alpha_n \|_{L^{2,1}} \leq C \| \nabla u_n \|_{L^{2,1}} \) so that if one could find such maps \( Q_n \), bounded in \( L^\infty \) then we could conclude that \( \| \nabla u_n \|_{L^\infty} \) is uniformly bounded, contradicting the assumption that the maps undergo bubbling.

We would also like to recall some results about existence of Coulomb (or Uhlenbeck see [15]) gauges. We provide a proof of the following, communicated to us by Ernst Kuwert, as we have not seen it elsewhere, although similar Theorems are proved in [11] and [13].

**Theorem 8.4.** Let \( \omega \in L^2(B_1, u(m) \otimes \wedge^1 T^* \mathbb{R}^2) \) then we can find maps \( P \in W^{1,2}(B_1, U(m)) \) and \( \eta \in W^{1,2}(B_1, u(m) \otimes \wedge^2 T^* \mathbb{R}^2) \) such that

\[
P^{-1} dP + P^{-1} \omega P = d^* \eta.
\]

Moreover

\[
\| \nabla \eta \|_{L^2(B_1)} \leq \| \omega \|_{L^2(B_1)}
\]

and

\[
\| \nabla P \|_{L^2(B_1)} \leq 2 \| \omega \|_{L^2(B_1)}.
\]
Proof. The Coulomb gauge $P$ is found by minimising the following energy

$$E(P) := \int_{B_1} |P^{-1} dP + P^{-1} \omega P|^2 = \int_{B_1} |dP + \omega P|^2,$$

which effectively is trying to minimise the $L^2$ distance of our connection to the exterior derivative.

Clearly $P \equiv Id$ is admissible so that we choose a minimising sequence $\{P_n\}$ with

$$E(P_n) \leq E(Id) = \|\omega\|^2_{L^2}.$$

We also have that

$$\|\nabla P_n\|^2_{L^2(B_1)} = E(P_n) - \|\omega\|^2_{L^2} - \int_{B_1} 2\langle dP_n, \omega P_n \rangle$$

$$\leq \|\omega\|^2_{L^2} - \|\omega\|^2_{L^2} + 2\varepsilon \|\nabla P_n\|^2_{L^2(B_1)} + \frac{1}{2\varepsilon}\|\omega\|^2_{L^2}$$

by Young’s inequality. Therefore

$$\|\nabla P_n\|^2_{L^2(B_1)} \leq 4\|\omega\|^2_{L^2}$$

and we can find $P \in W^{1,2}(B_1, gl(m, \mathbb{C}))$ such that $P_n \rightharpoonup P$ in $W^{1,2}$. This tells us that in particular, $dP_n \rightharpoonup dP$ in $L^2$ and $P_n \to P$ pointwise almost everywhere. It follows that $P \in W^{1,2}(B_1, U(m))$ and that

$$E(P_n) - E(P) = \int_{B_1} |\nabla P_n|^2 - |\nabla P|^2 + |\omega|^2 - |\omega|^2 + 2\langle dP_n, \omega P_n \rangle - 2\langle dP, \omega P \rangle$$

$$= \int_{B_1} |\nabla P_n|^2 - |\nabla P|^2 + 2\langle dP_n, \omega(P_n - P) \rangle + 2\langle dP_n - dP, \omega P \rangle.$$

Now,

$$\int_{B_1} |\omega(P_n - P)|^2 = \int_{B_1} \langle \omega P_n, \omega P_n \rangle + \langle \omega P, \omega P \rangle - 2\langle \omega P_n, \omega P \rangle \to 0$$

as $n \to \infty$ by Lebesgue’s dominated convergence theorem.

We also have

$$\int_{B_1} \langle dP_n - dP, \omega P \rangle \to 0$$

and

$$\liminf \|\nabla P_n\|^2_{L^2} \geq \|\nabla P\|^2_{L^2}.$$

Putting these things together yields the lower semi-continuity of $E$ and

$$E(P) \leq \liminf E(P_n) \leq \|\omega\|^2_{L^2}.$$ 

Let $\omega_P := P^{-1} dP + P^{-1} \omega P$ and using the fact that $P$ is a critical point of $E$ we consider variations $\phi \in C^1(\bar{B}_1, u(m))$ and we get

$$\frac{d}{dt} \int_{B_1} |d(Pe^{t\phi}) + \omega Pe^{t\phi}|^2 \bigg|_{t=0} = 0.$$

Using

$$|d(Pe^{t\phi}) + \omega Pe^{t\phi}|^2 = |d(e^{t\phi}) + P\omega Pe^{t\phi}|^2 = |d(e^{t\phi}) + \omega Pe^{t\phi}|^2$$

it can be easily checked that

$$0 = \frac{d}{dt} \int_{B_1} |d(Pe^{t\phi}) + \omega Pe^{t\phi}|^2 \bigg|_{t=0} = 2 \int_{B_1} \langle d\phi, \omega_P \rangle$$
for any $\phi \in W^{1,2}(B_1)$ by approximation. In fact we can allow $\phi \in L^2_{\text{loc}}(B_1)$ by approximating with $\phi_r(x) = \phi(rx)$ as $r \uparrow 1$.

By a linear Hodge decomposition we know there exists $\alpha \in W^{1,2}_0(B_1, u(m))$, $\eta \in W^{1,2}_0(B_1, u(m) \otimes \wedge^2 T^* \mathbb{R}^2)$ and a harmonic one form $h \in L^2(B_1, u(m) \otimes \wedge^1 T^* \mathbb{R}^2)$ - see for instance [9] - such that

$$\omega_P = d\alpha + d^*\eta + h.$$

Let $\gamma \in L^2_{\text{loc}}(B_1)$ be the harmonic function given by

$$d\gamma = h$$

giving $\nabla \gamma \in L^2(B_1)$. Thus we have that $\alpha + \gamma$ is an admissible test function. This fact, coupled with

$$\int_{B_1} \langle d\phi, d^*\eta \rangle = 0$$

for all such $\phi$ gives

$$\int_{B_1} \langle d\phi, d\alpha + h \rangle = 0,$$

and therefore

$$0 = \int_{B_1} \langle d\alpha + h, d\alpha + h \rangle = \int_{B_1} |d\alpha + h|^2 = \int_{B_1} |d\alpha|^2 + |h|^2.$$

Therefore

$$dP + \omega P = Pd^*\eta$$

giving the final estimate and completing the proof.

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