Frame dragging and bending of light in Kerr and Kerr–(anti) de Sitter spacetimes

G V Kraniotis

George P and Cynthia W Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, TX 77843, USA
E-mail: kraniotis@physics.tamu.edu

Received 8 August 2005
Published 11 October 2005
Online at stacks.iop.org/CQG/22/4391

Abstract
The equations of general relativity in the form of timelike and null geodesics that describe motion of test particles and photons in Kerr spacetime are solved exactly including the contribution from the cosmological constant. We then perform a systematic application of the exact solutions obtained, to the following cases. The exact solutions derived for null, spherical, polar and non-polar orbits are applied for the calculation of frame dragging (Lense–Thirring effect) for the orbit of a photon around the galactic centre, assuming that the latter is a Kerr black hole for various values of the Kerr parameter including those supported by recent observations. Unbound null polar orbits are investigated, and an analytical expression for the deviation angle of a polar photon orbit from the gravitational Kerr field is derived. In addition, we present the exact solution for timelike and null equatorial orbits. In the former case, we derive an analytical expression for the precession of the point of closest approach (perihelion, periastron) for the orbit of a test particle around a rotating mass whose surrounding curved spacetime geometry is described by the Kerr field. In the latter case, we calculate an exact expression for the deflection angle for a light ray in the gravitational field of a rotating mass (the Kerr field). We apply this calculation for the bending of light from the gravitational field of the galactic centre, for various values of the Kerr parameter, and the impact factor.

PACS numbers: 04.25.g, 04.70.s, 04.80.Cc, 98.35.Jk, 95.10.Eg
1. Introduction

1.1. Motivation

Most of the celestial bodies deviate very little from spherical symmetry, and the Schwarzschild spacetime is an appropriate approximation for their gravitational field [2]. However, for some astrophysical bodies the rotation of the mass distribution cannot be neglected. A more general spacetime solution of the gravitational field equations should take this property into account. In this respect, the Kerr solution [3] represents the curved spacetime geometry surrounding a rotating mass [4]. Moreover, the above solution is also important for probing the strong field regime of general relativity [5]. This is significant, since general relativity has triumphed in large-scale cosmology [7–10], and in predicting solar system effects on planetary orbits like the perihelion precession of Mercury with a very high precision [1, 6].

As was discussed in [11], the investigation of spacetime structures near strong gravitational sources, like neutron stars or candidate black hole (BH) systems, is of paramount importance for testing the predictions of the theory in the strong field regime. The study of geodesics is crucial in this respect, in providing information on the structure of spacetime in the strong field limit.

The study of the geodesics from the Kerr metric is additionally motivated by recent observational evidence of stellar orbits around the galactic centre, which indicates that the spacetime surrounding the Sgr A* radio source, which is believed to be a supermassive black hole of 3.6 million solar masses, is described by the Kerr solution rather than the Schwarzschild solution, with the Kerr parameter [12]

\[ \frac{J}{GM_{BH}/c} = 0.52(\pm 0.1, \pm 0.08, \pm 0.08) \]  

where the reported high-resolution infrared observations of Sgr A* revealed ‘quiescent’ emission and several flares. This is half the maximum value for a Kerr black hole [13]. In the above equation \( J \) denotes the angular momentum of the black hole. (The error estimates here the uncertainties in the period, black hole mass \( M_{BH} \) and distance to the galactic centre, respectively; \( G \) is the gravitational constant and \( c \) the velocity of light.)

Taking into account the cosmological constant \( \Lambda \) contribution, the generalization of the Kerr solution is described by the Kerr–de Sitter metric element which in Boyer–Lindquist (BL) coordinates is given by [18, 19]

\[ ds^2 = \frac{\Delta_r}{\Xi^2 \rho^2} (c \, dt - a \sin^2 \theta \, d\phi)^2 - \frac{\rho^2}{\Delta_r} \, dr^2 - \frac{\rho^2}{\Delta_\theta} \, d\theta^2 - \frac{\Delta_\phi \sin^2 \theta}{\Xi^2 \rho^2} (ac \, dt - (r^2 + a^2) \, d\phi)^2 \]  

where

\[ \Delta_r := \left( 1 - \frac{\Lambda}{3} r^2 \right) \left( r^2 + a^2 \right) - \frac{2GMr}{c^2}; \]

\[ \Delta_\theta := 1 + \frac{a^2 \Lambda}{3} \cos^2 \theta; \]

\[ \Xi := 1 + \frac{a^2 \Lambda}{3}, \quad \rho^2 := r^2 + a^2 \cos^2 \theta. \]

---

1 See also the BepiColombo science mission on Mercury http://sci.esa.int/home/bepicolombo/.
2 \( J = ca \) where \( a \) is the Kerr parameter. The interpretation of \( ca \) as the angular momentum per unit mass was first given by Boyer and Price [15]. In fact, by comparing with the Lense–Thirring calculations [16] they determined the Kerr parameter to be \( a = -\frac{2GM}{c^2} \), where \( \Omega \) and \( l \) denote the angular velocity and radius of the rotating sphere.
3 These coordinates have the advantage that they reduce to the Schwarzschild solution with a cosmological constant in the limit \( a \to 0 \), see [14].
In a recent paper [11], we derived the timelike geodesic equations in Kerr spacetime with a cosmological constant by solving the Hamilton–Jacobi partial differential equation by separation of variables. Subsequently, we solved exactly the corresponding differential equations for an interesting class of possible types of motion for a test particle in Kerr and Kerr-(anti) de Sitter spacetimes. The exact solution of non-spherical geodesics was obtained by using the transformation theory of elliptic functions.

The exact solutions of the timelike geodesic equations obtained in [11] were applied to the following situations.

*Frame dragging from rotating gravitational mass.* An essential property of the geodesics in Schwarzschild spacetime is that although the orbit precesses relativistically it remains in the same plane; the Kerr rotation adds longitudinal dragging to this precession. For instance, in the spherical polar orbits we discussed in [11] (where the particle traverses all latitudes, passes through the symmetry axis $z$, infinitely many times) the angle of longitude increases after a complete oscillation in latitude. This phenomenon is in accordance with *Mach’s principle*.

More specifically, in [11] we calculated the dragging of inertial frames in the following situations. (a) Dragging of a satellite’s spherical polar orbit in the gravitational field of the Earth assuming the Kerr geometry, using as radii the semi-major axis of the polar orbit of the GP-B mission\(^4\) launched in April 2004. (b) Dragging of a stellar, spherical polar orbit, in the gravitational field of a rotating galactic black hole.

It is the purpose of this paper to extend the analysis and applications of the exact solutions obtained in [11] to an interesting class of possible types of motion for a test particle in Kerr and Kerr-(anti) de Sitter spacetimes, as well as to derive the exact solutions of null geodesics in the same spacetimes and explore their physical implications. In the latter case, we apply the exact solutions obtained, to the following situations:

(a) Dragging of a photon’s spherical polar and non-polar orbit in the gravitational field of a rotating galactic centre black hole.

(b) The deflection angle of a light ray from the gravitational field of a rotating black hole, for various values of the Kerr parameter and the impact factor.

The material of this paper is organized as follows. In section 2 we review the derivation of the relevant geodesic equations. In section 2.1 we discuss the definition of Lauricella’s hypergeometric function of many variables, as well as the integral representations that it admits, which are important in our exact treatment of geodesic equations that describe motion of a test particle and photon in Kerr-(anti) de Sitter spacetime. In sections 3.2, 3.1 and 6, we solve exactly spherical polar or non-polar null geodesics with and without the cosmological constant. In the case of spherical polar photonic geodesics and for a vanishing cosmological constant the exact solution for the orbit is given by the Weierstraß elliptic function. The exact expression for frame dragging is proportional to the real half-period of the Weierstraß modular Jacobi form. In the case of non-polar spherical photonic orbits, the exact expression for the Lense–Thirring precession of the photon is given in terms of a hypergeometric function of one variable and Appell’s first generalized hypergeometric function of two variables $F_1$ [20]. Assuming a vanishing cosmological constant and that the galactic centre is a supermassive rotating black hole, we apply the exact solutions obtained for the determination of Lense–Thirring precession for a photon in spherical polar and non-polar orbits around the galactic centre. The corresponding exact expressions in the presence of the cosmological constant are also derived and discussed in sections 3.2 and 7.

\(^4\) http://einstein.stanford. See also [17].
In section 4 we perform a precise calculation for the deflection angle of a photon’s non-spherical polar orbit from the gravitational Kerr field. In this novel case, the exact expressions obtained were written in terms of Lauricella’s hypergeometric function \( F_D \).

Timelike spherical polar (with \( \Lambda_1 \neq 0 \)) and non-polar (with \( \Lambda_1 = 0 \)) orbits are treated in sections 5 and 8 respectively.

In sections 8.4 and 8.5 we study the exact solution of non-spherical timelike and null equatorial orbits respectively. The amount of relativistic precession for a test particle in a timelike orbit, confined to the equatorial plane, in the presence of rotation of the central mass is given in terms of Appell’s first hypergeometric function of two variables \( F_1 \). On the other hand, the exact expression for the deflection angle of a photonic orbit from the Kerr gravitational field surrounding a rotating central mass is given in terms of Appell’s \( F_1 \) hypergeometric function and Lauricella’s fourth hypergeometric function \( F_D \) of three variables [21]. We use section 9 for our conclusions. In the appendices, we collect some of our formal calculations, as well as some useful properties of Appell’s hypergeometric function and definitions of genus-2 theta-functions.

2. Separability of Hamilton–Jacobi’s differential equation in Kerr–(anti) de Sitter metric and derivation of geodesics

In the presence of the cosmological constant it was proved in [11] the important result that the Hamilton–Jacobi differential equation can be solved by separation of variables. Thus in this case, the characteristic function separates and takes the form [11]

\[
W = -Ect + L\phi + \int \frac{\sqrt{[Q + (L - aE)^2 \Xi^2 - \mu^2a^2 \cos^2 \theta]\Delta_\theta - \frac{\Xi^2(aE \sin^2 \theta - L)^2}{\sin^2 \theta}}}{\Delta_\theta} d\theta
+ \int \frac{\sqrt{\Xi^2[(r^2 + a^2)E - aL]^2 - \Delta_r(\mu^2r^2 + Q + \Xi^2(L - aE)^2)}}{\Delta_r} dr.
\]

By differentiating now with respect to constants of integration, \( Q, L, E, \mu \), we obtain the following set of geodesic differential equations,

\[
\begin{align*}
\int \frac{dr}{\sqrt{R'}} = & \int \frac{d\theta}{\sqrt{\Theta'}} \\
\rho^2 \frac{d\phi}{d\lambda} = & \frac{\Xi^2}{\Delta_\theta \sin^2 \theta} (aE \sin^2 \theta - L) + \frac{a\Xi^2}{\Delta_r} [(r^2 + a^2)E - aL] \\
c\rho^2 \frac{dt}{d\lambda} = & \frac{\Xi^2(r^2 + a^2)[(r^2 + a^2)E - aL] - a\Xi^2(aE \sin^2 \theta - L)}{\Delta_\theta} \\
\rho \frac{dr}{d\lambda} = & \pm \sqrt{R'} \\
\rho \frac{d\theta}{d\lambda} = & \pm \sqrt{\Theta'}
\end{align*}
\]

where

\[
R' := \Xi^2[(r^2 + a^2)E - aL]^2 - \Delta_r(\mu^2r^2 + Q + \Xi^2(L - aE)^2)
\]

\[
\Theta' := [Q + (L - aE)^2 \Xi^2 - \mu^2a^2 \cos^2 \theta]\Delta_\theta - \Xi^2(aE \sin^2 \theta - L)^2 \frac{\sin^2 \theta}{\sin^2 \theta}
\]

The first line of equation (4) is a differential equation that relates a hyperelliptic Abelian integral to an elliptic integral which is the generalization of the theory of transformation of
elliptic functions discussed in [11], in the case of a nonzero cosmological constant. The mathematical treatment of such a relationship was first discussed by Abel in [34].

Assuming a zero cosmological constant, as was shown by Carter, one gets
\[ W = -Ec t + \int \frac{\sqrt{R}}{\Delta} \, dt + \int \sqrt{\Theta} \, d\theta + L\phi \] (6)

where
\[ \Theta := Q - \left[ a^2(\mu^2 - E^2) + \frac{L^2}{\sin^2 \theta} \right] \cos^2 \theta \] (7)

and
\[ R := [(r^2 + a^2)E - aL]^2 - \Delta[\mu^2r^2 + (L - aE)^2 + Q] \] (8)

with \( \Delta := r^2 + a^2 - \frac{2GM}{c^2} \). Also \( E, L \) are the constants of integration associated with the isometries of the Kerr metric. Carter’s constant of integration is denoted by \( Q \).

Differentiation of (6), with respect to the integration constants \( E, L, Q, \mu \), leads to the following set of first-order equations of motion [37],
\[ \rho^2 \frac{c \, dt}{d\lambda} = \frac{r^2 + a^2}{\Delta} P - a(aE \sin^2 \theta - L) \]
\[ \rho^2 \frac{dr}{d\lambda} = \pm \sqrt{R} \]
\[ \rho^2 \frac{d\theta}{d\lambda} = \pm \sqrt{\Theta} \]
\[ \rho^2 \frac{d\phi}{d\lambda} = \frac{a}{\Delta} P - aE + \frac{L}{\sin^2 \theta} \]

where
\[ P := E(r^2 + a^2) - aL. \] (9)

Null geodesics are derived by setting \( \mu = 0 \).

2.1. Lauricella’s multivariable hypergeometric functions

Giuseppe Lauricella, building on the work of Appell who had developed hypergeometric functions of two variables, investigated in a systematic way, multiple hypergeometric functions at the end of the nineteenth century [21]. He defined four functions which are named after him and have both multiple series and integral representations. In particular, the fourth of these functions, denoted by \( F_D \), admits integral representations of importance in our exact treatment of geodesic equations, which describe motion of a test particle in Kerr–(anti) de Sitter spacetime.

The fourth Lauricella function of \( m \)-variables is given by
\[ F_D(\alpha, \beta, \gamma; z) = \sum_{n_1, n_2, \ldots, n_m=0}^{\infty} \frac{(\alpha)_{n_1+\cdots+n_m}(\beta_1)_{n_1} \cdots (\beta_m)_{n_m} z_1^{n_1} \cdots z_m^{n_m}}{(\gamma)_{n_1+\cdots+n_m}(1)_{n_1} \cdots (1)_{n_m}} \]
\[ = \sum_{n_1, n_2, \ldots, n_m=0}^{\infty} \frac{\sum(\alpha, n_1 + \cdots + n_m)(\beta_1, n_1) \cdots (\beta_m, n_m) z_1^{n_1} \cdots z_m^{n_m}}{(\gamma, n_1 + \cdots + n_m)n_1! \cdots n_m!} \]

where
\[ z = (z_1, \ldots, z_m) \quad \beta = (\beta_1, \ldots, \beta_m). \] (11)
The Pochhammer symbol \((\alpha)_m\) is defined by
\[
(\alpha)_m = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} = \begin{cases} 
1, & \text{if } m = 0, \\
\alpha(\alpha + 1) \cdots (\alpha + m - 1), & \text{if } m = 1, 2, 3, \ldots.
\end{cases}
\]

The series admits the following integral representation,
\[
F_D(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-z t)^{-\beta} \cdots (1-z_m t)^{-\beta_m} \, dt \tag{12}
\]
which is valid for \(\Re(\alpha) > 0, \Re(\gamma - \alpha) > 0\). It converges absolutely inside the \(m\)-dimensional cuboid:
\[
|z_j| < 1 \quad (j = 1, \ldots, m). \tag{13}
\]

3. Spherical polar null geodesics

Depending on whether or not the coordinate radius \(r\) is constant along a given geodesic, the corresponding particle orbit is characterized as spherical or non-spherical respectively. In this section, we will concentrate on spherical polar photon orbits with a vanishing cosmological constant. We should mention at this point the extreme black hole solutions \(a = 1\) of spherical non-polar photon geodesics obtained in [22] in terms of formal integrals.

The exact solution of the corresponding timelike orbits and their physical applications have been derived and investigated in [11].

Assuming a zero cosmological constant, \(r = r_f\), where \(r_f\) is a constant value setting \(\mu = 0\) and using the last two equations of (9) we obtain
\[
\frac{d\phi}{d\theta} = \frac{a P}{\Delta} - a E + L / \sin^2 \theta \sqrt{\Theta} \tag{14}
\]
where \(\Theta\) now is given by
\[
\Theta = Q - \left[ -a^2 E^2 + \frac{L^2}{\sin^2 \theta} \right] \cos^2 \theta. \tag{15}
\]

It is convenient to introduce the parameters
\[
\Phi := L/E, \quad Q := Q/E^2. \tag{16}
\]

Now by defining \(z := \cos^2 \theta\), the previous equation can be written as follows,
\[
d\phi = \frac{1}{2} \frac{dz}{\sqrt{z^2 - z^2(\alpha + \beta) + \Phi z}} \times \left\{ \frac{a P}{\Delta} - a + \frac{\Phi}{1 - z} \right\} \tag{17}
\]
where
\[
\alpha := -a^2, \quad \beta := Q + \Phi^2. \tag{18}
\]

It has been shown [23] that a necessary condition for an orbit to be polar (meaning to intersect the symmetry axis of the Kerr gravitational field) is the vanishing of the parameter \(L\), i.e., \(L = 0\). Assuming \(\Phi = 0\), in equation (17), we can transform it into the Weierstraß form of an elliptic curve by the following substitution:
\[
z := \frac{-\xi + \alpha \beta}{-\alpha / 4}. \tag{19}
\]

Thus, we obtain the integral equation
\[
\int d\phi = \int \frac{d\xi}{2} \frac{1}{\sqrt{4\xi^3 - 8\xi + 1}} \times \left\{ \frac{a P'}{\Delta} - a \right\} \tag{20}
\]
and this orbit integral can be inverted by the Weierstraß modular Jacobi form\(^5\)

\[ \xi = \wp(\phi/A) \]

(21)

where \( A := -\frac{1}{2} \left( \frac{aP'}{A} - a \right) \), \( P' = (r^2 + a^2) \) and the Weierstraß invariants take the form

\[ g_2 = \frac{1}{12}(\alpha + \beta)^2 - \frac{Q\alpha}{4} \]
\[ g_3 = \frac{1}{216}(\alpha + \beta)^3 - \frac{Q\alpha^2}{48} - \frac{Q\alpha\beta}{48} \]

(22)

3.1. Exact solution for spherical polar null geodesics with a vanishing cosmological constant

In terms of the original variables, the exact solution for the polar orbit of the photon (\( \Phi = 0 \)) takes the form

\[ \wp(\phi + \epsilon) = \alpha'' \cos^2 \theta - \frac{1}{12}(\alpha'' + \beta'') \]

where \( \alpha'' := \alpha'/A^2 = -\frac{1}{A^2} \), \( \beta'' := \beta'/A^2 = \frac{\beta}{A^2} \). Also, \( A' \) is given by the expression

\[ A' := -\frac{GMr}{c^2a^2r^2} \]

(24)

Equation (23) represents the first exact solution of a spherical polar photonic orbit assuming a zero cosmological constant, in closed analytic form, in terms of the Weierstraß Jacobi modular form of weight 2.

The Weierstraß invariants are given by

\[ g_2'' = \frac{(\alpha'' + \beta'')^2}{12} - \frac{Q\alpha''}{4} \]
\[ g_3'' = \frac{(\alpha'' + \beta'')^3}{216} - \frac{Q\alpha'^2}{48} - \frac{Q\alpha''\beta''}{48} \]

\[ = \frac{1}{432\alpha^6A^6}[-2a^6 - 3a^4Q + 3a^2Q^2 + 2Q^3] \]

The sign of the discriminant \( \Delta' \) (\( \Delta' = g_2^3 - 27g_3^2 \)) determines the roots of the elliptic curve: \( \Delta' > 0 \) corresponds to three real roots while for \( \Delta' < 0 \) two roots are complex conjugates and the third is real. In the degenerate case \( \Delta' = 0 \) (where at least two roots coincide) the elliptic curve becomes singular and the solution is not given by modular functions.

The analytic expressions for the three roots of the cubic, which can be obtained by applying the algorithm of Tartaglia and Cardano [26], are given by

\[ e_1 = \frac{(a^2 + 2Q)\Delta^2}{12a^2r^4(GM/c^2)^2} \]
\[ e_2 = \frac{(a^2 - Q)\Delta^2}{12a^2r^4(GM/c^2)^2} \]
\[ e_3 = -\frac{(2a^2 + Q)\Delta^2}{12a^2r^4(GM/c^2)^2} \]

(25)

Since we are assuming spherical orbits, there are two conditions from the vanishing of the polynomial \( R(r) \) and its first derivative\(^6\). Implementing these two conditions, expressions

\(^5\) For more information on the properties of the Weierstraß function, the reader is referred to the monographs [24, 25], and the appendix of [10].

\(^6\) These orbits are unstable since \( \frac{d^2}{dr^2} R_{r} > 0 \). However, they represent interesting new possible types of motion in the Kerr spacetime. They represent a non-trivial generalization of the unstable circular closed orbit (photon sphere) in the Schwarzschild black hole.
for the parameter $\Phi$ and Carter’s constant $Q$ are obtained:
\[
\Phi = \frac{a^2 + r^2}{a}, \quad Q = -\frac{r^4}{a^2}
\]
\[
\Phi = \frac{a^2 GM + a^2 r - 3\frac{GM}{c^2}r^2 + r^3}{a\left(\frac{GM}{c^2} - r\right)}, \quad Q = -\frac{r^3\left(-4a^2 \frac{GM}{c^2} + r\left(-\frac{3GM}{c^2} + r\right)^2\right)}{a^2\left(\frac{GM}{c^2} - r\right)^2}.
\]

However, only the second solution is physical [22].

The two half-periods $\omega$ and $\omega'$ are given by the following Abelian integrals (for $\Delta^c > 0$) [27]:
\[
\omega = \int_{\Gamma_1}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}, \quad \omega' = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{-4t^3 + g_2t + g_3}}.
\]

The values of the Weierstraß function at the half-periods are the three roots of the cubic. For positive discriminant $\Delta^c$, one half-period is real while the second is imaginary\(^7\). The period ratio is defined as $\tau = \frac{\omega}{\omega'}$.

An alternative expression for the real half-period $\omega$ of the Weierstraß function is
\[
\omega = \frac{2}{\sqrt{(a^2 + Q)(a^2 + Q + 2r^2)}} \frac{\pi}{2} \frac{1}{\Gamma\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{a^2}{Q}\right)} \frac{a^2}{\sqrt{Q}},
\]

Thus\(^8\)
\[
\omega = \frac{2}{\sqrt{(a^2 + Q)(a^2 + Q + 2r^2)^2}} \frac{\pi}{2} \frac{1}{\Gamma\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{a^2}{Q}\right)} \frac{a^2}{\sqrt{Q}}.
\]

After a complete oscillation in latitude, the angle of longitude, which determines the amount of dragging for the spherical photon polar orbit in the general theory of relativity (GTR), increases by
\[
\Delta \phi^{GTR} = 4\omega.
\]

We can also integrate the first and the third equation in (9). Then we get
\[
\int c\ dt = -4 \left[ \int_0^1 \frac{1}{2} a^2 \left( \frac{z}{a^2} + 1 \right)^2 \frac{dz}{\sqrt{Q}\sqrt{(1-z)\sqrt{1 - \left(\frac{a^2}{Q}\right)z}}} + \int_0^1 \frac{a^2(1-z)}{2\sqrt{Q}\sqrt{(1-z)\sqrt{1 - \left(\frac{a^2}{Q}\right)z}}} \right].
\]

Thus we obtain the following exact expression for $t$
\[
ct = -4 \left[ -\frac{1}{2} a^2 \frac{\left(\frac{z}{a^2} + 1\right)^2}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{a^2}{Q}\right)} + \frac{a^2}{2\sqrt{Q}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} F\left(\frac{1}{2}, \frac{1}{2}, 2, \frac{-a^2}{Q}\right) \right].
\]

\(^7\) We organize the roots as $e_1 > e_2 > e_3$.

\(^8\) Yet another equivalent representation for $\omega$ is
\[
\omega = \frac{2}{\sqrt{(a^2 + Q)(a^2 + Q + 2r^2)}} \frac{\pi}{2} \frac{1}{\Gamma\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{a^2}{Q}\right)} \frac{a^2}{\sqrt{Q}}.
\]
Weierstraß zeta function. Also where we used equation (23) and the fact that observations, equation (1), we determined the precise frame dragging (Lense–Thirring effect) of the black hole have been recently reported from x-ray flare analysis of the galactic centre [28].

The results are displayed in table 1. Let us also mention that in [29] it has been argued that an upper bound of $\Delta\phi_{\text{GTR}} = 1.32447$ arcsec/revolution.

Similarly, if we integrate the differential equations for $t$ and $\phi$ we obtain

$$\frac{c}{\Delta} \frac{dt}{d\phi} = \frac{r^2 c^2 P'' - a^2 \sin^2 \theta}{\Delta} = a + \frac{r^2 P''}{\Delta} + \frac{a^2 \cos^2 \theta}{\Delta} = a + \frac{r^2 P''}{\Delta} - a$$

(31)

or

$$ct + E = a\phi + \frac{r^2 P'/\Delta}{(-2aA')} \phi - \frac{4a^2/a''}{(-2aA')} \left( \xi(\phi) - \frac{1}{12} (a'' + \beta') \phi \right)$$

(32)

where we used equation (23) and the fact that $\int \phi \, d\phi = -\xi(\phi)$, where $\xi(z)$ denotes the Weierstraß zeta function. Also $E$ denotes a constant of integration.

Assuming that the centre of the Milky Way is a black hole and that the structure of spacetime near the region Sgr A*, is described by the Kerr geometry as is indicated by observations, equation (1), we determined the precise frame dragging (Lense–Thirring effect) of a null orbit with a spherical polar geometry. We repeated the analysis for a value of the Kerr parameter as high as $a_{\text{Galactic}} = 0.9939$. Such high values for the angular momentum of the black hole have been recently reported from x-ray flare analysis of the galactic centre [28]. The results are displayed in table 1. Let us also mention that in [29] it has been argued that an upper bound of $a$ is given by $a = 0.99616$.

**3.2. Null spherical polar geodesics with the cosmological constant, Lense–Thirring effect and Appell hypergeometric functions**

We now derive an exact expression for the amount of dragging for a photonic spherical polar orbit in the presence of the cosmological constant, thus generalizing the results of the previous section. After a complete oscillation in latitude, the angle of longitude $\Delta\phi$, which determines the amount of dragging for the spherical polar orbit, is given by

$$\Delta\phi_{\text{GTR}} = -4 \left[ a_1 \int_0^1 \frac{dz}{\sqrt{f(z)}} + b_1 \int_0^1 z \frac{dz}{\sqrt{f(z)}} \right]$$

$$= -4 \left[ a_1 \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} \times F_1 \left( \frac{1}{2}, \frac{1}{2}, 1, 1, \frac{x^2 a^2 + \Xi^2 a^2}{Q}, -a^2 \Lambda \right) \right]$$

$$+ b_1 \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} \times F_1 \left( \frac{3}{2}, \frac{1}{2}, 1, 2, \frac{x^2 a^2 + \Xi^2 a^2}{Q}, -a^2 \Lambda \right)$$

(33)
where \( f(z) = z(1 - z)(Q + z(Qa^2 + \Xi z)) \left( 1 + \frac{a^2}{\Lambda} z \right)^2 \) and \( \alpha_1 = \frac{2a^2}{\sqrt{Q}} - \frac{1}{2} \frac{a^2(1 + i\kappa)}{\Lambda}, \beta_1 = -\frac{1}{2} \frac{a^2(1 + i\kappa)}{\Lambda} \). The function \( F_1(\alpha, \beta, \gamma, x, y) \) is the first of the four Appell’s hypergeometric functions of two variables \( x, y \) \[20\]

\[
F_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n
\]

which admits the following integral representation:

\[
\int_0^1 u^{\alpha-1} (1-u)^{\gamma-a-1} (1-ux)^{-\beta'} (1-uy)^{-\beta} \, du = \frac{\Gamma(\alpha)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} F_1(\alpha, \beta, \beta', \gamma, x, y).
\]

The double series converges when \(|x| < 1\) and \(|y| < 1\). The above Euler integral representation is valid for \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\gamma - \alpha) > 0 \). Also \( \Gamma(p) = \int_0^\infty x^{p-1} e^{-x} \, dx \) denotes the gamma function.

For a zero cosmological constant \( (\Lambda = 0, \beta_1 = 0) \) we obtain the correct limit,

\[
\Delta \phi_{\text{GTR}} = -\frac{4}{\sqrt{Q}} \frac{\alpha_1 \pi}{\Gamma(\gamma - \alpha)} F_1 \left( \frac{1}{2}, \frac{1}{2}, 1, 1, -\frac{a^2}{\Lambda} \right)
\]

where \( \alpha_1 = -\frac{aGMc}{\sqrt{\Delta}} \), in the limit of a vanishing cosmological constant.

We can also obtain an exact expression for time. After a quarter of an oscillation in latitude the time elapses as

\[
ct = \int_0^{\frac{1}{4} \pi} \frac{(\gamma_1 + \delta_1 z)}{\sqrt{f(z)}} \, dz
\]

\[
= \frac{\gamma_1}{\sqrt{\Xi}} \frac{\pi}{\sqrt{\Xi}} F_1 \left( \frac{1}{2}, \frac{1}{2}, 1, 1, -\frac{Qa^2 + \Xi a^2}{\Xi}, -\frac{a^2}{\Lambda} \right)
\]

\[
+ \frac{\delta_1}{\sqrt{\Xi}} \frac{\pi}{\sqrt{\Xi}} F_1 \left( \frac{3}{2}, \frac{1}{2}, 1, 2, -\frac{Qa^2 + \Xi a^2}{\Xi}, -\frac{a^2}{\Lambda} \right).
\]

In the limit \( \Lambda = 0, \gamma_1 = -\frac{1}{2} (\lambda^2 + \kappa^2), \delta_1 = -\frac{1}{2}, \) and equation (30) is recovered.

The conditions from the vanishing of the polynomial \( R \) and its first derivative result in equations which generalize (26) and are provided in appendix C.

4. Non-spherical polar null geodesics

In this case, the relevant differential equation for the calculation of deviation angle \( \Delta \phi \) of light from the rotating black hole (or rotating central mass) is as follows,

\[
\frac{d\phi}{dr} = \frac{2aGMr}{c^2 \Delta} \frac{1}{\sqrt{R}}
\]

where the quartic polynomial \( R(r) \) is given by the expression

\[
R = r^4 + r^2(a^2 - Q) + 2GMr \frac{2GMr}{c^2} - a^2 Q.
\]

9 The expression \( (\lambda, \kappa) = \lambda(\lambda + 1) \cdots (\lambda + \kappa - 1) \), and the symbol \( (\lambda, 0) \) represents 1.
Expressing the roots of \( \Delta \) as \( r_+, r_- \), which are the locations of the event horizons of the black hole, and using partial fractions we derive the expression
\[
\frac{d\phi}{dr} = \frac{A_+^p}{(r-r_+)^{3/2}} + \frac{A_-^p}{(r-r_-)^{3/2}}
\]
\[
= \frac{A_+^p}{(r-r_+)^{3/2}(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}
\]
\[
+ \frac{A_-^p}{(r-r_-)^{3/2}(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}
\]
where \( A_{\pm}^p \) are given by the equations
\[
A_{\pm}^p = \pm \frac{2aGM \pm c^2(r_+ - r_-)}{c^2(r_+ - r_-)}.
\]
(40)

Also the radii of the event horizons are located at
\[
r_{\pm} = \frac{GM}{c^2} \pm \sqrt{\left( \frac{GM}{c^2} \right)^2 - a^2}.
\]
(41)

In order to calculate the angle of deflection we need to integrate the above equation from the distance of closest approach (e.g., from the maximum positive root of the quartic) to infinity. We denote the roots of the quartic by \( \alpha, \beta, \gamma, \delta, \alpha > \beta > \gamma > \delta \).

\[\text{We organize all roots in ascending order of magnitude as follows:} \]
\[
\alpha_{\mu} > \alpha_{\nu} > \alpha_i > \alpha_{\rho} \]
(42)

where \( \alpha_{\mu} = \alpha_{\mu+1}, \alpha_{\nu} = \alpha_{\mu+2}, \alpha_{\rho} = \alpha_{\mu} \) and \( \alpha_i = \alpha_{\mu-i}, i = 1, 2, 3 \) and we have that \( \alpha_{\mu-1} \geq \alpha_{\mu-2} > \alpha_{\mu-3} \).

By applying the transformation
\[
r = \frac{\alpha_{\mu+2} - \alpha_{\nu+1}}{\alpha_{\mu} - \alpha_{\mu+1}}
\]
or equivalently
\[
z = \frac{\alpha_{\mu} - \alpha_{\mu+1}}{\alpha_{\mu} - \alpha_{\mu+2}} \left( \frac{r - \alpha_{\mu+1}}{r - \alpha_{\mu+2}} \right)
\]
(44)

we can bring our integrals into the familiar integral representation of Lauricella’s \( F_1 \) and Appell’s hypergeometric function \( F \) of three and two variables respectively. Indeed, we derive
\[
\Delta \phi = 2 \int_0^{1/\omega} \frac{-A_+^p \omega (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} \frac{dz}{\sqrt{z(1-z)(1-\kappa_z^2 z)}} \sqrt{1-\mu_z^2 z}
\]
\[
+ \int_0^{1/\omega} \frac{A_+^p \omega (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} \frac{z \, dz}{\sqrt{z(1-z)(1-\kappa_z^2 z)}} \sqrt{1-\mu_z^2 z}
\]
\[
+ \int_0^{1/\omega} \frac{-A_-^p \omega (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-} \frac{dz}{\sqrt{z(1-z)(1-\kappa_z^2 z)}} \sqrt{1-\mu_-^2 z}
\]
\[
+ \int_0^{1/\omega} \frac{A_-^p \omega (\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-} \frac{z \, dz}{\sqrt{z(1-z)(1-\kappa_z^2 z)}} \sqrt{1-\mu_-^2 z}
\]
(46)

\[\text{We have the correspondence } a_{\mu+1} = a, a_{\mu+2} = \beta, a_{\mu-1} = r_+ = a_{\mu-2}, a_{\mu-3} = \gamma, a_{\mu} = \delta.\]
where the moduli $\kappa^2_\pm$, $\mu^2$ are
\[
\kappa^2_\pm = \left( \frac{\alpha_\mu - \alpha_{\mu+1}}{\alpha_\mu - \alpha_{\mu+2}} \right) \left( \frac{\alpha_{\mu+2} - \alpha_{\mu-1}}{\alpha_{\mu+1} - \alpha_{\mu-1}} \right) \quad \mu^2 = \left( \frac{\alpha_\mu - \alpha_{\mu+1}}{\alpha_\mu - \alpha_{\mu+2}} \right) \left( \frac{\alpha_{\mu+2} - \alpha_{\mu-1}}{\alpha_{\mu+1} - \alpha_{\mu-3}} \right). \tag{47}
\]

Also
\[
H^\pm = \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})(\alpha_{\mu+1} - \alpha_{\mu-1})/\sqrt{\alpha_{\mu+1} - \alpha_{\mu-1} - \alpha_\mu} \tag{48}
\]

and $\alpha^\pm_{\mu-1} = r_\pm$. By defining a new variable $\omega' := \omega\zeta$ we can express the angle $\Delta \phi$ in terms of Lauricella’s hypergeometric function $F_D$,
\[
\Delta \phi_{\text{GTR}} = 2 \left[ -2A^p \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2}) \frac{H^+}{F_D} \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa^2_+, \mu^2 \right) \right. \tag{49}
\]
\[+ \frac{A^p \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} \left( \frac{3}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa^2_+, \mu^2 \right) \left( \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \right) \]
\[+ \frac{A^p \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^-} \left( \frac{3}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa^2_-, \mu^2 \right) \left( \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \right) \cleft]\]

where the variables of the function $F_D$ are given in terms of the roots of the quartic and the radii of the event horizons by the expressions
\[
\frac{1}{\omega} = \frac{\alpha_\mu - \alpha_{\mu+2}}{\alpha_\mu - \alpha_{\mu+1}} = \delta - \beta \quad \frac{\alpha_\mu - \alpha_{\mu+1}}{\alpha_{\mu+1} - \alpha_{\mu-1}} = \delta - \alpha \quad \kappa^2_\pm = \frac{\alpha_{\mu+2} - \alpha_{\mu-1}^\pm}{\alpha_{\mu+1} - \alpha_{\mu-1}} = \beta - r_\pm \quad \mu^2 = \frac{\alpha_{\mu+2} - \alpha_{\mu-3}}{\alpha_{\mu+1} - \alpha_{\mu-3}} = \beta - \gamma. \tag{50}
\]

An equivalent expression is as follows:
\[
\Delta \phi_{\text{GTR}} = 2 \left[ -2A^p \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2}) \frac{H^+}{F_D} \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa^2_+, \mu^2 \right) \right. \tag{51}
\]
\[+ \frac{A^p \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2})}{H^+} \left( - \frac{1}{\kappa^2_+} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \mu^2 \right) \right) \left( \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \right) \]
\[+ \frac{1}{\kappa^2_+} F_D \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa^2_+, \mu^2 \right) \left( \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \right) \cleft]\]
\[+ \frac{1}{\kappa^2_-} F_D \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa^2_-, \mu^2 \right) \left( \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \right) \cleft]\]
\[+ \frac{1}{\kappa^2_+} F_D \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{\omega}, \kappa^2_+, \mu^2 \right) \left( \frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} \right) \cleft]\]
In going from equations (49) to (51) we used the identity which is proven in appendix C,

$$F_D \left( \frac{3}{2}, \frac{1}{2}, 1, \frac{5}{2} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)} = -\frac{1}{\kappa_c^2} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{\kappa^2}{\omega^2}, \mu^2 \right) \frac{\Gamma(1/2)}{\Gamma(3/2)}$$

$$+ \frac{1}{\kappa_c^2} F_D \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{5}{2} \right) \frac{\Gamma(3/2)}{\Gamma(5/2)}$$

(52)

The phenomenological applications of equation (49) for gravitational bending and lensing studies from a galactic black hole, as well as its generalization in the presence of the cosmological constant will be the subject of detailed investigation in a future publication.

### 4.1. Exact solution of timelike spherical orbits with a cosmological constant

Using the second and the fifth lines of equation (4), for $L = 0$ and assuming a constant value for $r$, we obtain

$$\frac{d\phi}{d\theta} = \frac{\Xi(\omega^2 + a^2)E}{\Delta_c \sqrt{\Theta}}$$

$$= -\frac{\Xi^2 E}{\Delta_c \sqrt{\Theta}}$$

(53)

where $B := \frac{\Xi(\omega^2 + a^2)E}{\Delta_c \sqrt{\Theta}}$.

Similarly, using the third and fifth line we obtain

$$\frac{c \, dt}{d\theta} = \frac{\Xi^2 (r^2 + a^2)^2 E}{\Delta_c \sqrt{\Theta}} - \frac{\alpha^2 \Xi^2 E \sin^2 \theta}{\Delta_c \sqrt{\Theta}}$$

$$= \frac{\Gamma \Delta_c - \alpha^2 \Xi^2 E \sin^2 \theta}{\Delta_c \sqrt{\Theta}}$$

(54)

and $\Gamma := \frac{\Xi^2 (r^2 + a^2)^2 E}{\Delta_c \sqrt{\Theta}}$.

Now using the variable $z = \cos^2 \theta$, we obtain the following system of integral equations:

$$\phi = \int \frac{\Xi^2 a^2 E}{\sqrt{f}(z)} \, dz + \int \frac{B(1 + \frac{a^2}{\Delta_c} z)/( -2 )}{\sqrt{f}(z)} \, dz$$

$$ct = \int \frac{r}{f} \left( 1 + \frac{\alpha^2}{\Delta_c} z \right) dz - \int \frac{a^2 \Xi^2 E (1 - z)}{\sqrt{f}(z)} \, dz$$

(55)

or

$$\phi = \int z^\gamma (\alpha_1 + \beta_1 z) \, dz$$

$$ct = \int z^\gamma (\gamma_1 + \delta_1 z) \, dz$$

(56)

where $f(z) = z(1 - z)(Q + z(Qa^2 + \Xi \Xi^2 a^2 E^2 - \mu^2 a^2) + z^2(-\mu^2 a^2 \frac{\omega}{\lambda^2})) (1 + a^2 \frac{\omega}{\lambda^2} z)^2$. Also we have defined

$$\alpha_1 = \frac{\Xi^2 a^2 E}{2} - \frac{1}{2} \frac{a^2 \Xi^2 (r^2 + a^2) E}{\Delta_c} \frac{a^2 \Xi^2 (r^2 + a^2) E}{3 \Delta_c}$$

$$\beta_1 = -\frac{1}{2} \frac{a^2 \Xi^2 (r^2 + a^2) E}{\Delta_c}$$

$$\gamma_1 = -\frac{1}{2} \frac{a^2 \Xi^2 (r^2 + a^2) E}{\Delta_c} + \frac{a^2 \Xi^2 E}{2}$$

$$\delta_1 = -\frac{a^2 \Xi^2 (r^2 + a^2) E}{6 \Delta_c}$$

(57)

Equation (55) is a system of equations of Abelian integrals, whose inversion, in principle, involves genus-2 Abelian–Siegel'sche modular functions. Indeed, this system is a particular
case of Jacobi’s inversion problem of hyperelliptic Abelian integrals of genus 2 [43–49] (see appendix A for details). Then, one can express \( z \) as a single valued genus-2 Abelian theta-function with arguments \( t, \phi \). However, since the polynomial \( f(z) \) of sixth degree posses a double root it may well be that the Abelian genus-2 theta-function degenerates and the final result can be expressed in terms of genus-1 modular functions.

5. Frame dragging in spherical polar timelike geodesics with a cosmological constant

After a quarter of oscillation in latitude the change of longitude is

\[
\Delta \phi = \int_0^1 \frac{(\alpha + \beta z)}{\sqrt{f(z)}} \, dz
\]

\[
= \frac{\alpha}{\sqrt{Q}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} F_D \left( 1, 1, \frac{1}{2}, \frac{1}{2}, 1, 1; z_1', z_2', -a^2 \Lambda \right)
\]

\[
+ \frac{\beta}{\sqrt{Q}} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} F_D \left( 3, 1, 1, \frac{1}{2}, 1, 2; z_1', z_2', -a^2 \Lambda \right)
\]

where \( z'_i = 1/z_i \) and

\[
z_1 = -\frac{3a^2 E^2 \Xi^3 + 3a^2 - a^2 Q \Lambda - \sqrt{12 Q \Lambda a^4 + (3a^4 E^2 \Xi^3 - 3a^2 + a^2 Q \Lambda)^2}}{2a^4 \Lambda},
\]

\[
z_2 = -\frac{3a^2 E^2 \Xi^3 + 3a^2 - a^2 Q \Lambda + \sqrt{12 Q \Lambda a^4 + (3a^4 E^2 \Xi^3 - 3a^2 + a^2 Q \Lambda)^2}}{2a^4 \Lambda}.
\]

6. Spherical non-polar null geodesics

In this section, we investigate spherical null geodesics with a nonzero value of the parameter \( \Phi \). Now one has to calculate the integral

\[
\Delta \phi = -\frac{1}{2} \int \frac{dz}{\sqrt{\alpha z^3 - z^2(\alpha + \beta) + \Phi \zeta}} - \frac{1}{2} \int \frac{dz \Phi}{(1-z)\sqrt{\alpha z^3 - z^2(\alpha + \beta) + \Phi \zeta}}
\]

where

\[
\alpha := -a^2, \quad \beta := Q + \Phi^2, \quad P = r^2 + a^2 - a\Phi.
\]

The first integral can be brought into the Weierstraß form with the invariants \( g_2, g_3 \)

\[
g_2'' = \frac{1}{12}(\alpha' + \beta'\beta')^2 - \frac{\Phi'' \alpha''}{4}
\]

\[
= \frac{1}{12a^4 A^d} (-a^2 + \Phi - \Phi^2)^2 + \frac{Q}{4a^4 A^d}
\]

\[
g_3'' = \frac{1}{432a^6 A^6} [2(-a^2 + \Phi + \Phi^2)^3 - 9Q a^4 + 9Q a^2 (Q + \Phi^2)]
\]

and

\[
A' = \frac{\Phi}{2a^2} - \frac{GMr}{c^2a^2}
\]

\[
+ 1 - \frac{2GMr}{c^2a^2}.
\]

Now let us discuss the second integral in equation (58). We define a new variable

\[
u_1' z = v
\]
then
\[
- \frac{1}{2} \int_0^1 \frac{dz}{(1 - z)\sqrt{\alpha z^3 - z^2(\alpha + \beta) + \Omega z^2}} = - \frac{\Phi}{2\sqrt{\Omega}u'_i} \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) F_1 \left( \frac{1}{2}, 1, 1, \chi_1, \chi_2 \right)
\]

where \( u'_i = \frac{1}{u_i}, i = 1, 2 \) and

\[
\begin{align*}
    u_1 &= -\frac{a^2 + Q + \Phi^2 - \sqrt{4a^2Q + (a^2 - Q - \Phi^2)^2}}{2a^2} \\
    u_2 &= -\frac{a^2 + Q + \Phi^2 + \sqrt{4a^2Q + (a^2 - Q - \Phi^2)^2}}{2a^2}
\end{align*}
\]

Also
\[
\chi_1 := \frac{1}{u'_1}, \quad \chi_2 := \frac{u'_2}{u'_1}
\] (62)

Thus, we expressed the above integral in terms of Appell’s first hypergeometric function of two variables, \( F_1(\alpha, \beta, \beta', y, x, y) \).

After a complete oscillation in latitude, the angle of longitude, which determines the amount of dragging for the spherical non-polar photonic orbit in the general theory of relativity (GTR), is given by

\[
\Delta \phi^{\text{GTR}} = 4 \pi \frac{\sqrt{\alpha} - a}{2\sqrt{\Omega}} F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, \frac{u'_i}{u'_j}, 1, \frac{u'_j}{u'_i}, 1, \chi_1, \chi_2 \right) + 4 \frac{\Phi}{2\sqrt{\Omega}u'_i} \pi F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, 1, \chi_1, \chi_2 \right)
\]

Orbits with \( \Delta \phi^{\text{GTR}} > 0 \) are called prograde and those with \( \Delta \phi^{\text{GTR}} < 0 \) are called retrograde.

As before we can obtain an exact expression for time. After a quarter of an oscillation in latitude,

\[
\begin{align*}
    \int c \, dt &= - \frac{1}{2} \int_0^1 \frac{dz}{\sqrt{\alpha z^2 - z(\alpha + \beta) + \Omega z^2}} + \frac{dz}{\sqrt{\alpha z^2 - z(\alpha + \beta) + \Omega}} \\
    &= \left[ - \frac{r^2 + a^2}{\Delta} [(r^2 + a^2) - a \Phi] - \frac{\Phi}{2} \frac{1}{\sqrt{\Omega}u'_i} \pi F_1 \left( \frac{1}{2}, 1, 1, 1, \frac{u'_j}{u'_i}, 1, \frac{u'_j}{u'_i}, \frac{1}{2} \right) \right. \\
    &\left. + \frac{a^2}{2\sqrt{\Omega}} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} F_1 \left( \frac{1}{2}, 1, 1, 1, \frac{u'_j}{u'_i}, \frac{1}{2} \right) \right]
\] (64)
The above equation has the correct limit for $\Phi = 0$ and reproduces the corresponding exact expression (30) for spherical null polar geodesics. Indeed, for $\Phi = 0$, $\frac{\phi}{\lambda} = \frac{-a^2}{c^2}$, $u_1' = 1$, and the Appell function has the following limit:

$$
\frac{\Gamma(\frac{1}{2}) \Gamma(2)}{\Gamma(\frac{3}{2})} F_1 \left( \begin{array}{c} 1, 1, 1, 5, 1, -a^2 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \end{array} \right) = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{2}{3})}{\Gamma(2)} F \left( \begin{array}{c} 1, 1, 2, 2, -a^2 \\ \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2} \end{array} \right). 
$$

Assuming that the centre of the Milky Way is a rotating black hole and that the structure of spacetime near the region Sgr A* is described by the Kerr geometry, we determined the precise frame dragging of a null orbit with a spherical non-polar geometry. The results are displayed in tables 2 and 3.

### Table 2. Predictions for frame dragging from a galactic black hole, with Kerr parameter $a_{\text{Galactic}} = 0.52 \frac{G M_{\text{BH}}}{c^2}$, for different values of photon angular momentum and Carter’s constant.

| Parameters | Predicted dragging |
| --- | --- |
| $\Phi = 1$, $Q = 22.693$, $r = 2.7452$ | $\Delta \phi_{\text{GTR}} = 7.72736 = 442.7^\circ$ per revolution $= 1.59 \times 10^6 \text{ arcsec per revolution}$ |
| $\Phi = -1$, $Q = 26.9984$, $r = 2.99523$ | $\Delta \phi_{\text{GTR}} = -5.0551 = -289.6^\circ$ per revolution $= -1.04 \times 10^6 \text{ arcsec per revolution}$ |
| $\Phi = -3$, $Q = 23.0508$, $r = 3.2239$ | $\Delta \phi_{\text{GTR}} = -5.2014 = -298^\circ$ per revolution $= -1.07 \times 10^6 \text{ arcsec per revolution}$ |

### Table 3. Predictions for frame dragging from a galactic black hole, with Kerr parameter $a_{\text{Galactic}} = 0.9939 \frac{G M_{\text{BH}}}{c^2}$, for different values of photon angular momentum and Carter’s constant.

| Parameters | Predicted dragging |
| --- | --- |
| $\Phi = 1$, $Q = 16.1443$, $r = 2.02083$ | $\Delta \phi_{\text{GTR}} = 10.7355 = 615^\circ$ per revolution $= 2.2 \times 10^6 \text{ arcsec per revolution}$ |
| $\Phi = -1$, $Q = 25.8865$, $r = 2.73783$ | $\Delta \phi_{\text{GTR}} = -3.73503 = -214^\circ$ per revolution $= -770.405 \text{ arcsec per revolution}$ |
| $\Phi = -3$, $Q = 25.8628$, $r = 3.23713$ | $\Delta \phi_{\text{GTR}} = -4.32779 = -247.9^\circ$ per revolution $= -892.671 \text{ arcsec per revolution}$ |

7. **Frame dragging in spherical non-polar null geodesics with a cosmological constant**

Including the contribution from the cosmological constant, the relevant differential equation is

$$
\frac{d \phi}{d \theta} = \frac{-a \Sigma^2}{\Delta_{\theta}} + \frac{\Phi \Sigma^2}{\Delta_{\phi} \sin \theta} + \frac{B'}{\Delta_{r} \sqrt{\theta}} 
$$

(66)

where $B' \coloneqq \Sigma^2 \alpha P$.

Introducing the variable $z$, we find that the angle of longitude that measures the frame dragging of a spherical non-polar photon orbit, after a complete oscillation in latitude is given by the exact expression

$$
\Delta \Phi_{\text{GTR}} = -4 \left[ -\frac{1}{2} \phi \Sigma^2 \frac{1}{\sqrt{Q}} \frac{\pi}{u_1'} F_D \left( \frac{1}{2}, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right] 
$$

$$
+ \frac{1}{2} \frac{a \Sigma^2}{\sqrt{Q} u_1'} \pi \left( \frac{1}{2} \phi \Sigma^2 \frac{1}{\sqrt{Q}} \frac{\pi}{u_1'} F_1 \left( \frac{1}{2}, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right) 
$$

$$
- \frac{B'}{\Delta_{r} \sqrt{\theta}} \frac{\pi}{2} F \left( \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right] 
$$

(67)
where \( u' = 1/u_i \), \( i = 1, 2 \) and the roots \( u_i \) are

\[
\begin{align*}
  u_1 &= \frac{\Lambda \Sigma^2 a^4 - 2\Lambda \Sigma^2 \Phi a^3 + 3\Sigma^2 a^2 + \Lambda \Sigma^2 \Phi^2 a^2 + Q \Lambda a^2 - 3\Sigma^2 \Phi^2 - 3Q + \sqrt{H_\Lambda}}{2a^2((\Lambda a^2 - 2\Lambda \Phi a + \Lambda \Phi^2 + 3)\Sigma^2 + Q \Lambda)} \\
  u_2 &= \frac{\Lambda \Sigma^2 a^4 - 2\Lambda \Sigma^2 \Phi a^3 + 3\Sigma^2 a^2 + \Lambda \Sigma^2 \Phi^2 a^2 + Q \Lambda a^2 - 3\Sigma^2 \Phi^2 - 3Q - \sqrt{H_\Lambda}}{2a^2((\Lambda a^2 - 2\Lambda \Phi a + \Lambda \Phi^2 + 3)\Sigma^2 + Q \Lambda)}
\end{align*}
\]  

(68)

and

\[
H_\Lambda := 12Q((\Lambda a^2 - 2\Lambda \Phi a + \Lambda \Phi^2 + 3)\Sigma^2 + Q \Lambda)a^2 + ((a - \Phi)(\Lambda a^2 - \Lambda \Phi a^2 + 3\alpha + 3\Phi)\Sigma^2 + Q(a^2 \Lambda - 3))^2.
\]  

(69)

Equation (67) involves three hypergeometric functions, Gauß’s \( F \), Appell’s \( F_1 \) and Lauricella’s \( F_D \). For a vanishing cosmological constant the above expression reduces exactly to (63).

We also derived an exact expression for time. After a complete oscillation in latitude the time elapsed is

\[
ct = -4 \left[ -\frac{1}{2} \frac{\Sigma^2(r^2 + a^2)[r^2 + a^2 - a \Phi]}{\Delta_r} \frac{1}{\sqrt{Q u'_1}} F \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{u'_1}{u'_1} \right) \pi \\
- \frac{a \Phi \Sigma^2}{2\sqrt{Q u'_1}} F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{a^2 \Lambda}{3u'_1}, \frac{u'_1}{u'_1} \right) \pi \\
+ \frac{a^2 \Sigma^2}{2\sqrt{Q u'_1}} F_D \left( \frac{1}{2}, -1, 1, \frac{1}{2}, 1, \frac{a^2 \Lambda}{3u'_1}, \frac{u'_1}{u'_1} \right) \pi \right].
\]  

(70)

8. Spherical timelike geodesics with \( L \neq 0 \)

The relevant equation for integration is [11]

\[
d\phi = -\frac{1}{2} \frac{dz}{\sqrt{\Sigma^2 \alpha - z^2(\alpha + \beta) + Qz}} \times \left\{ \frac{aP}{\Lambda} - aE + \frac{L}{1 - z} \right\}
\]  

(71)

where

\[
\alpha = a^2(1 - E^2), \quad \beta = Q + L^2
\]  

(72)

and \( P \) is provided from equation (10).\(^{11}\)

Let us define

\[
\Pi := \int \frac{d\xi}{\sqrt{4\xi^3 - g_\xi^2 - g_3}}
\]  

(73)

thus \( \xi = \varphi(\Pi + \epsilon) \), and \( A'' \int \frac{dz}{\sqrt{\Sigma^2 \alpha - z^2(\alpha + \beta) + Qz}} = A'' \int \frac{d\xi}{\sqrt{4\xi^2 - g_\xi^2 - g_3}} = A'' \Pi, A'' := -\frac{1}{2} (\frac{aP}{\Lambda} - aE) \) and \( \epsilon \) is a constant of integration. Now

\[
-\frac{1}{2} L \int \frac{dz}{(1 - z)\sqrt{\Sigma^2 \alpha - z^2(\alpha + \beta) + Qz}}
\]  

(74)

\(^{11}\) The extreme black hole \( a = 1 \) solutions for \( L \neq 0 \) have been investigated in [38].
the equation for the orbit is given by
\[ \frac{u}{4408} G V \chi \]
and
\[ u \]
where \( \chi \)
where
\[ \wp \]
by the expressions
\[ F = \frac{ct}{\Delta_1 \phi} \]
spherical orbits is as follows,
\[ 2 = \frac{1}{2} (1 - \frac{2GM}{\alpha^2 c^2}) \]
An alternative exact expression for time \( t \) the expression
\[ ct = \frac{r^2 + a^2}{L} + \frac{a E}{2} (aE - L) + \frac{a^2 E}{2} \left( -\frac{1}{3} \alpha + \beta \right) + \frac{a^2 E}{2} \frac{4}{\alpha} (\Pi). \]
Similarly using the first and third line of equation (9), we obtain for \( t \) the expression
\[ \frac{u}{4408} G V \chi \]
where \( w := \frac{u}{4408} G V \chi \phi (v_0) \). Also \( \sigma (z) \) denotes the Weierstraß sigma function. Thus
\[ \phi = \int d\phi = A^\alpha \Pi - \frac{L \alpha}{8} \left[ \log \frac{\sigma (\Pi + \epsilon - v_0)}{\sigma (\Pi + \epsilon + v_0)} + 2 \Pi \zeta (v_0) \right] \times \frac{1}{\phi (v_0)} \]
and \( \phi z^2 (v_0) = 4 \phi^3 (v_0) - 2 g \phi (v_0) - g_3 = 4 w^3 - 2 g w - g_3 \). In terms of the integration constants, \( w \) is given by the expression
\[ w = \frac{a^2 (1 - E^2)}{4} - \frac{a^2 (1 - E^2) + Q + L^2}{12}. \]
Similarly using the first and third line of equation (9), we obtain for \( t \) the expression
\[ \frac{u}{4408} G V \chi \]
An alternative exact expression for time \( t \) in terms of the ordinary hypergeometric of one-variable \( F \) and Appell’s first hypergeometric function of two variables is as follows:
\[ ct = \frac{r^2 + a^2}{\Delta} \frac{P}{P - 2} + \frac{a E}{2} \left( aE - L \right) + \frac{a^2 E}{2} \left( -\frac{1}{3} \alpha + \beta \right) + \frac{a^2 E}{2} \frac{4}{\alpha} (\Pi). \]
The variables \( u_i \) are given in terms of the constants of integration and the Kerr parameter \( a \) by the expressions
\[ u_1 = \frac{a^2 (1 - E^2) + L^2 + Q - \sqrt{-a^2 (1 - E^2) - L^2 - Q}^2 - 4 a^2 (1 - E^2) Q}{2 a^2 (1 - E^2)} \]
\[ u_2 = \frac{a^2 (1 - E^2) + L^2 + Q + \sqrt{-a^2 (1 - E^2) - L^2 - Q}^2 - 4 a^2 (1 - E^2) Q}{2 a^2 (1 - E^2)} \]
and \( u_i' = 1/u_i, i = 1, 2 \).
Similarly, an alternative expression for the amount of dragging for timelike non-polar spherical orbits is as follows,
\[ \Delta \phi^{GTR} = 4 \frac{1}{2 \sqrt{Q}} \sqrt{\frac{a}{\alpha}} \left( a \left( \frac{L}{\alpha} + \frac{2GM}{c^2 \alpha^2} \right) \right) \frac{\pi}{\sqrt{u_1}} F \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{u_2}{u_1}, \frac{u_2'}{u_1'} \right) + \frac{4}{2 \sqrt{Q} \sqrt{\frac{a}{\alpha}}} \pi \chi_1 F \left( \frac{1}{2}, \frac{1}{2}, 1, \chi_1, \chi_2 \right) \]
where \( \chi_1 = \frac{1}{\alpha}, \chi_2 = \frac{u_2}{\alpha}, \chi_3 = \frac{u_2'}{\alpha} \).
Table 4. Predictions for frame dragging from a galactic black hole, with the indicated values of Kerr parameter \(a\) for different values of the test particle’s angular momentum, a particular value for Carter’s constant and for a fixed radius \(r = 10\). The values of the radii \(a, L\) are in units of \(GM_{BH}/c^2\), while Carter’s constant \(Q\) is in units of \((GM_{BH}/c^2)^2\). The period ratios, \(\tau\), are \(0.58i, 2.69743i, 2.69625i\) respectively.

| Parameters | Predicted dragging |
|------------|-------------------|
| \(a = 0.52, L = -2.03566, E = 0.957665, Q = 11\) | \(\Delta \phi^{\text{GTR}} = -6.06933 \times 10^9 \text{ arcsec revolution}^{-1}\) |
| \(a = 0.9939, L = -2.25773, E = 0.959284, Q = 11\) | \(\Delta \phi^{\text{GTR}} = -5.86166 \times 10^9 \text{ arcsec revolution}^{-1}\) |
| \(a = 0.99616, L = -2.25883, E = 0.959292, Q = 11\) | \(\Delta \phi^{\text{GTR}} = -5.86161 \times 10^9 \text{ arcsec revolution}^{-1}\) |

We have calculated the amount of frame dragging for galactic black holes for the values of the Kerr parameter given in equation (1) and for fixed values for radii and Carter’s constant \(Q\). The invariant parameters \(L, E\) are determined by the two conditions for spherical orbits. We repeated the analysis for different values of the Kerr parameter [28]. The results are displayed in table 4.

8.1. General solution for non-spherical geodesics in the Kerr metric

In the general case with a nonzero cosmological constant of non-spherical orbits one has to solve the differential equations

\[
\int_0^\theta \frac{d\theta}{\sqrt{\Theta'}} = \int_r^r \frac{dr}{\sqrt{R'}}
\]

where \(R'(r)\) is a quartic polynomial for null geodesics (\(\mu = 0\) in equation (5)) which is given by

\[
R' = E^2(\Sigma^2[(r^2 + a^2) - a\Phi]^2 - \Delta r[\Sigma^2(\Phi - a)^2 + Q])
\]

and \(\Theta'(\theta)\) is given by

\[
\Theta' = E^2 \left\{[Q + (\Phi - a)^2\Sigma^2]\Delta_\theta - \Sigma^2 \frac{(a \sin^2 \theta - \Phi)^2}{\sin^2 \theta} \right\}
\]

Note that this equation is an equation between two elliptic integrals, in the general case when all roots are distinct. The left-hand side can be transformed into an elliptic integral with variable \(z\) or \(\xi\) in the Weierstrass normal form, see equation (19). In order to solve this differential equation and determine \(r\) as a function of \(\theta\), one can employ the theory of Abel for the transformation of elliptic functions [32], which was first applied in [11] for the case of timelike orbits. A detailed exposition of the theory of transformation of elliptic functions based on [32] can also be found in [11]. The interesting connection with modular equations [11] is outlined in appendix B.

We note at this point that the corresponding relationship for non-spherical timelike orbits in the presence of the cosmological constant relates a hyperelliptic integral (the polynomial \(R'\) is a sectic in this case) to an elliptic integral, and therefore involves the transformation theory of hyperelliptic functions. This fact was first observed in [11] and will be the subject of another publication.

8.2. Transforming the geodesic elliptic integrals into Abel’s form

The transformation

\[
x \to e_3 + \frac{(e_1 - e_3)}{x^2}
\]
transforms the elliptic integral in the Weierstraß form into Abel’s and Jacobi’s form,
\[
\int \frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}} \to -\frac{1}{\sqrt{e_1-e_3}} \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},
\]
with \(k^2 = \frac{\alpha - \beta}{\alpha - \gamma}\).

Similarly, the quartic can be brought to Jacobi’s form \(y^2 = (1-x^2)(1-k_1^2x^2)\).

Indeed, as we saw in section 3, we have
\[
\frac{d\theta}{\sqrt{\Theta}} = -\frac{1}{2} \sqrt{z^2 - z^2(\alpha + \beta) + Qz} = -\frac{1}{2} \sqrt{4\xi^3 - g_2\xi - g_3}
\]
where
\[
g_2 = \frac{1}{12} (-a^2 + Q + \Phi^2)^2 - \frac{Q}{4} (-a^2)
\]
\[
g_3 = \frac{1}{216} (-a^2 + Q + \Phi^2)^3 - \frac{Q}{48} a^4 - \frac{Q}{48} (-a^2 (Q + \Phi^2)).
\]

In this case the three roots of the cubic \(4\xi^3 - g_2\xi - g_3\) are given by the expressions
\[
e_1 = \frac{1}{24} (-a^2 + \Phi^2 + Q + 3\sqrt{a^4 + 2a^2Q + Q^2 - 2a^2\Phi^2 + 2Q\Phi^2 + \Phi^4})
\]
\[
e_2 = \frac{1}{12} (a^2 - Q - \Phi^2)
\]
\[
e_3 = \frac{1}{24} (-a^2 + \Phi^2 + Q - 3\sqrt{a^4 + 2a^2Q + Q^2 - 2a^2\Phi^2 + 2Q\Phi^2 + \Phi^4}).
\]

Thus the Jacobi modulus \(k^2 = \frac{\alpha - \gamma}{\alpha - \beta}\) is given by
\[
k^2 = \frac{a^2 - Q - \Phi^2 + \sqrt{a^4 + 2a^2(Q - \Phi^2) + (Q + \Phi^2)^2}}{2\sqrt{a^4 + 2a^2(Q - \Phi^2) + (Q + \Phi^2)^2}}.
\]

It has the correct limit for photon spherical geodesics with \(\Phi = 0\)
\[
k^2(\Phi = 0) = \frac{a^2}{a^2 + Q}.
\]

Also
\[
\frac{1}{\sqrt{e_1-e_3}} = \frac{2}{(a^4 + 2a^2(Q - \Phi^2) + (Q + \Phi^2)^2)^{1/4}}
\]
and it also has the correct limit for spherical photon orbits with zero \(\Phi\),
\[
\frac{1}{\sqrt{e_1-e_3}} = \frac{2}{\sqrt{a^4 + Q}}
\]

Thus, we get
\[
\int \frac{d\theta}{\sqrt{\Theta}} = -\frac{1}{2} \frac{1}{\sqrt{e_1-e_3}} \int \frac{d\xi}{\sqrt{(1-x^2)(1-k^2x^2)}}
\]

Applying Luchterhand’s transformation formula [36] on the radial integral, Jacobi’s form can be recovered,
\[
\frac{\partial x}{M\sqrt{(1-x^2)(1-k_1^2x^2)}} = \frac{\partial y}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}
\]
where the Jacobi modulus \( k_1 \) and the coefficient \( M \) are given in terms of the roots \( \alpha, \beta, \gamma, \delta \) of the quartic, by the following expressions,

\[
k_1 = \sqrt{(\alpha - \delta)(\beta - \gamma)} \quad \frac{1}{\sqrt{(\alpha - \gamma)(\beta - \delta)}}
\]

and the integration variables are related by

\[
1 - x = \frac{(y - \delta)(y - \alpha)(y - \beta)}{(\alpha - \beta)(y - \gamma)(y - \delta)}.
\]

Also we assume that the roots \( \alpha, \beta, \gamma, \delta \) of the quartic are real and are organized in the following ascending order of magnitude:

\( \alpha > \beta > \gamma > \delta \).

We can also provide a nice formula for \( r \) in terms of Jacobi’s sinus amplitudinus function\(^12\):

\[
\frac{(\gamma - \delta)(r - \alpha)(r - \beta)}{(\alpha - \beta)(r - \gamma)(r - \delta)} = \frac{1 - \text{sn}(M \int \frac{d\theta}{\sqrt{\Theta_{1,k}^1}})}{1 + \text{sn}(M \int \frac{d\theta}{\sqrt{\Theta_{1,k}^1}})}.
\]

8.3. Equatorial geodesics including the contribution of the cosmological constant

The equatorial geodesics (i.e., \( \theta = \pi/2, \eta = 0 \)), with a nonzero cosmological constant, may be obtained by equation (4) for the particular values of \( Q, \theta \). The characteristic function in this case has the form\(^11\)

\[
W = -Ect + \int \frac{\sqrt{R}}{\Delta r} \, dr + L\phi
\]

and the geodesics are given by the expressions

\[
\frac{dr}{\sqrt{R}} = \frac{d\lambda}{r^2},
\]

\[
r^2 \frac{d\phi}{d\lambda} = \frac{a(1 + \frac{1}{3}a^2\Lambda)^2}{(1 - \frac{\Lambda}{3}r^2)}(E(r^2 + a^2) - La) + (L - aE) \left( 1 + \frac{1}{3}a^2\Lambda \right)^2,
\]

\[
r^2 \frac{dr}{d\lambda} = \frac{c}{\Delta r} \left( 1 + \frac{1}{3}a^2\Lambda \right)^2 \left( (r^2 + a^2)E - aL \right) + \left( 1 + \frac{1}{3}a^2\Lambda \right)^2 a(L - aE)
\]

where

\[
R' = \left( 1 + \frac{1}{3}a^2\Lambda \right)^2 \left( (r^2 + a^2)E - aL \right)^2 - \Delta_r ((L - aE)^2)
\]

for null geodesics and

\[
R' = \left( 1 + \frac{1}{3}a^2\Lambda \right)^2 \left( (r^2 + a^2)E - aL \right)^2 - \Delta_r ((L - aE)^2) - \Delta_r (\mu^2r^2)
\]

for timelike geodesics.

Equatorial orbits are of particular interest for various astrophysics applications. The exact solution of circular equatorial orbits with a cosmological constant was presented in [11]. In what follows, we shall concentrate on the cases of non-circular equatorial timelike and null orbits that describe the motion of test particles and photons under the assumption of a vanishing cosmological constant. We shall derive new exact expressions for the precession of perihelion or periapsis for the orbit of a test particle in the gravitational field of Kerr, as well as for the deflection angle of a light ray from the Kerr gravitational field, in terms of multivariable hypergeometric functions. In the latter case, we apply the exact calculation

\[\int \frac{d\theta}{\sqrt{\Theta_{1,k}^1}} = \int \frac{d\theta}{\sqrt{(r^2 + a^2)(r^2 - \alpha)(r^2 - \beta)(r^2 - \gamma)(r^2 - \delta)}} = \int \frac{d\theta}{\sqrt{\Theta_{1,k}^1}}.\]
obtained for determining the bending angle of a light ray from the gravitational field of the
galactic centre of Milky Way assuming that the Sgr A$^*$ region is a supermassive rotating black
hole for various values of the Kerr parameter, which are supported by recent observations, and
of the impact factor. The more general case, in the presence of the cosmological constant, is
a task for a future publication.

8.4. Exact solution of timelike equatorial geodesics

We now proceed to determine the exact expression for the precession of equatorial timelike
orbits in Kerr spacetime. We have $r^2(\dot{r}) = \sqrt{R}$. This can be rewritten as

$$
(r^2)^2 = E^2 + \frac{a^2 E^2}{r^2} - \frac{L^2}{r^2} + \frac{2GM}{c^2 r^3} (L - aE)^2 - \frac{\Delta}{r^2}.
$$

(103)

By defining a new variable $u = 1/r$ we obtain the following expression:

$$
u^{-4} u^2 = E^2 + a^2 E^2 u^2 - L^2 u^2 + \frac{2GM}{c^2} (L - aE)^2 u^3 - \left(1 + a^2 u^2 - \frac{2GM}{c^2} u\right) \equiv B' (u).
$$

(104)

Similarly $\phi^2 = u^4 \frac{d^2 \phi}{u' d\phi}$ where

$$
A(u) = L + u \alpha_S (aE - L), \quad D(u) = 1 + u^2 - \alpha_S u, \quad \alpha_S := \frac{2GM}{c^2},
$$

(105)

Thus, we obtain the differential equation

$$
\frac{d\phi}{du} = \frac{A(u)}{D(u)} \frac{1}{\sqrt{B'(u)}}.
$$

(106)

We now write

$$
\frac{A(u)}{D(u)} = \frac{A_+}{u_+ - u} + \frac{A_-}{u_- - u}
$$

(107)

where $u_+ = \frac{r_+}{\alpha^2}$, $u_- = \frac{r_-}{\alpha^2}$ and

$$
r_\pm = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}
$$

(108)

where the quantities $A_+$, $A_-$ are given by

$$
A_+ = \frac{L + \frac{\alpha_S}{a^2} (aE - L) u_+}{u_- - u_+}, \quad A_- = \frac{- \frac{L}{a^2} - \frac{\alpha_S}{a^2} (aE - L) u_-}{u_- - u_+}.
$$

(109)

For the calculation of the perihelion (periapsis) precession of a test particle in the orbit around
a rotating mass we need to calculate the integral $\Delta \phi^{\text{GTR}} = 2 \int_{u'_+}^{u'_-} d\phi$. Then

$$
\int d\phi = \int \frac{du'}{(\frac{Ma'}{u'} - u)} \frac{1}{\sqrt{\frac{\alpha_S (L - aE)^2}{(\frac{c^2}{2})}} \sqrt{(u' - u'_1)(u'_1 - u')(u'_2 - u')}}
$$

$$
+ \int \frac{du'}{(\frac{Ma'}{u'} - u')} \frac{1}{\sqrt{\frac{\alpha_S (L - aE)^2}{(\frac{c^2}{2})}} \sqrt{(u' - u'_1)(u'_1 - u')(u'_2 - u')}}
$$

(110)

and we have defined $u' := u \frac{GM}{c^2}$. Using new variables

$$
u' \rightarrow u'_\xi + \xi^2 (u'_2 - u'_1)
$$
we can bring our expression to the integral representation of Appell’s first hypergeometric function

\[
\Delta \phi^{\text{GTR}} = \frac{2}{\sqrt{u'_1 - u'_3}} \frac{1}{\sqrt{\frac{a_{\infty} (a - \Phi)}{u''_1 - u''_3}}} \left\{ \frac{A_+}{u'_1 - u'_3} F_1 \left( \frac{1}{2}, 1, 1, \frac{u'_2 - u'_3}{u'_1 - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \frac{\Gamma(1/2)^2}{\Gamma(1)} \right. \\
+ \frac{A_-}{u'_1 - u'_3} F_1 \left( \frac{1}{2}, 1, 1, \frac{u'_2 - u'_3}{u'_1 - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \frac{\Gamma(1/2)^2}{\Gamma(1)} \left. \right\}. \tag{111}
\]

We can now provide an exact expression for time \( t \). The first differential equation in (9) for equatorial orbits can be written in terms of the variable \( u \) as follows:

\[
c t = E (1 + a^2 u^2) + a \alpha_S u^3 (a E - \Phi) \\

\frac{c t}{D u} = \frac{E}{u^2 D(u) \sqrt{B'(u)}} + \frac{E a^2}{D(u) \sqrt{B'(u)}} + \frac{a \alpha_S (a E - \Phi)}{D(u) \sqrt{B'(u)}}. \tag{112}
\]

By integrating we can express \( t \) in terms of Appell’s and Lauricella’s generalized hypergeometric functions

\[
ct = \frac{2}{\sqrt{u'_1 - u'_3}} \frac{1}{\sqrt{\frac{a_{\infty} (a - \Phi)}{u''_1 - u''_3}}} \left\{ \frac{A'_+}{u'_1 - u'_3} F_1 \left( \frac{1}{2}, 1, 1, \frac{u'_2 - u'_3}{u'_1 - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \pi \\
+ \frac{A'_-}{u'_1 - u'_3} F_1 \left( \frac{1}{2}, 1, 1, \frac{u'_2 - u'_3}{u'_1 - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \pi \right. \\
+ 2 \frac{A'_+}{u'_1 - u'_3} \frac{1}{\sqrt{\frac{a_{\infty} (a - \Phi)}{u''_1 - u''_3}}} \frac{1}{u''_1 - u''_3} F_D \left( \frac{1}{2}, 2, 1, 1, \frac{u'_2 - u'_3}{u'_1 - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \pi \\
- \frac{1}{u''_1 - u''_3} F_D \left( \frac{1}{2}, 2, 1, 1, \frac{u'_2 - u'_3}{u'_1 - u'_3}, \frac{u'_2 - u'_3}{u'_1 - u'_3} \right) \pi \right\}. \tag{113}
\]

8.5. Exact solution of null equatorial geodesics

In this case, we arrive at the differential equation first derived in [14],

\[
\frac{d \phi}{d u} = \frac{\Phi + u \alpha_S (a - \Phi)}{D(u)} \frac{1}{\sqrt{B^N(u)}} \tag{115}
\]

where the cubic polynomial \( B^N(u) \) is given by the expression

\[
B^N(u) = \alpha_S (\Phi - a)^2 u^3 + u^2 (a^2 - \Phi^2) + 1. \tag{116}
\]

In order to calculate the angle of deflection it is necessary to calculate the integral:

\[
\Delta \phi^{\text{GTR}} = 2 \int_0^{\phi'} \frac{d \phi}{\sqrt{B^N(u)}}. \tag{117}
\]
Using partial fractions as in the timelike case we obtain an elegant exact expression for $\Delta \phi^{GTR}$ in terms of Lauricella’s fourth hypergeometric function of three variables $F_D$, and Appell’s first hypergeometric function of two variables $F_1$.

\[
\Delta \phi^{GTR} = \frac{2}{\sqrt{u_1 - u_3}} \frac{1}{\sqrt{|u_2(\Phi - a\phi)|}} \left( A_+ \left( \frac{G M_c}{e c^2} - u_3 \right) F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{u_2' - u_3'}{u_2' - u_3}, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3} \right) \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}) \Gamma(1) \right)

\]  

\[
+ \frac{2}{\sqrt{u_1 - u_3}} \frac{1}{\sqrt{|u_2(\Phi - a\phi)|}} \left( -A_+ \left( \frac{G M_c}{e c^2} - u_3 \right) \right)

\]  

\[
\times F_D \left( \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3} \right)

\]  

\[
\times 2F_D \left( \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3} \right) \right)

\]  

or

\[
\Delta \phi^{GTR} = \frac{2}{\sqrt{u_1 - u_3}} \frac{1}{\sqrt{|u_2(\Phi - a\phi)|}} \left( A_+ \left( \frac{G M_c}{e c^2} - u_3 \right) F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{u_2' - u_3'}{u_2' - u_3}, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3} \right) \pi \right)

\]  

\[
- 2 \sqrt{F_D} \left( \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3} \right)

\]  

\[
+ \frac{2}{\sqrt{u_1 - u_3}} \frac{1}{\sqrt{|u_2(\Phi - a\phi)|}} \left( -A_+ \left( \frac{G M_c}{e c^2} - u_3 \right) \right)

\]  

\[
\times F_D \left( \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3} \right) \right)

\]  

\[
- 2 \sqrt{F_D} \left( \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3}, \frac{u_3' - u_3}{u_3' - u_3} \right) \right).

\]  

During the derivation of equation (118) we have used at an intermediate step of the calculation the identity

\[
\frac{1}{\sqrt{u_1 u_2}} F_D \left( \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{u_3'}{u_1}, \frac{u_3'}{u_2} \right) = \frac{1}{\sqrt{u_1 u_2}} F_D \left( \frac{1}{2}, 1, \frac{3}{2}, 2, -u_3' \sqrt{u_1 - u_3' \sqrt{u_2' - u_3'}} \right)

\]  

\[
\times F_D \left( \frac{1}{2}, 1, \frac{3}{2}, 2, -u_3' \sqrt{u_1 - u_3' \sqrt{u_2' - u_3'}} \right).

\]  

The quantities $A_{\pm}$ are now given in terms of the impact parameter $\Phi$ and the Kerr parameter $a$ by the expressions

\[
A_{\pm} = \frac{\pm \Phi \pm a \sqrt{(\Phi - a \phi)^2}}{2a^2}.

\]  

(120)
Frame dragging and bending of light in Kerr and Kerr–(anti) de Sitter spacetimes

null

The angle of deflection $\delta$ of light rays from the gravitational field of a galactic rotating black hole or a massive star is defined to be the deviation of $\Delta \phi$ from the transcendental number $\pi$

$$\delta = \Delta \phi_{\text{GTR}} - \pi.$$ (121)

We have calculated the deflection angle $\delta$ of light rays from the gravitational field of a galactic rotating black hole for different values of the Kerr parameter $a$ and the impact parameter $\Phi$. The results are displayed in figure 1 and tables 5 and 6. It is clear from figure 1 that especially for smaller values of the impact parameter $\Phi$, there is a strong dependence of the deflection angle on the spin of the black hole. This has implications for gravitational lensing studies and can lead, in principle, to an independent measurement of the Kerr parameter at the strong field regime.

The roots of the cubic are organized as $u'_1 > u'_2 > 0 > u'_3$.

### 8.5.1. Null equatorial geodesics with a cosmological constant

In this case, the generalization of equation (115) is given by

$$\frac{d\phi}{du} = \frac{\Phi u^2 - \alpha_s u^3 (\Phi - a)(\Phi - a)(a^2 u^2)}{\left[\left(u^2 - \frac{a^2}{\Phi^2}\right)(1 + a^2 u^2) - \alpha_s u^3 \right] \sqrt{B^4(u)}}.$$ (122)
where $B^\Lambda(u)$ is still a cubic polynomial

$$B^\Lambda(u) = 1 + a^2 u^2 - \Phi^2 u^2 + \alpha_3 u^3 (\Phi - a)^2 + \frac{\Lambda}{3} (1 + a^2 u^2)(\Phi - a)^2.$$  \hspace{1cm} (123)

For a vanishing cosmological constant the above cubic polynomial reduces to equation (116).

9. Conclusions

In this work, we have investigated the motion of a test particle and light in the gravitational field of Kerr spacetime with and without the cosmological constant. We have derived a number of useful analytical expressions for measurable physical quantities.

In the case of null orbits, we solved exactly the geodesic equations for spherical polar and non-polar photon orbits. The exact solution for the orbit of a photon with zero angular momentum $\Phi$ and a vanishing cosmological constant was provided by the Weierstraß elliptic function $\wp(z)$.

The exact expressions that determine the amount of frame dragging (Lense–Thirring effect) for the corresponding photon orbits, assuming a vanishing cosmological constant, were written in terms of the Weierstraß function real half-period or equivalently in terms of the Gauß hypergeometric function $F$ for a photon spherical orbit with $\Phi = 0$ and the Gauß hypergeometric function $F$ and Appell’s generalized hypergeometric function of two variables $F_1$ for photon orbits with $\Phi \neq 0$ and a constant radius. The corresponding expressions in the presence of the cosmological constant were given in terms of Appell’s hypergeometric function $F_1$ for the case of spherical null polar orbits, and in terms of Lauricella’s hypergeometric function of three variables $F_D$, Appell’s $F_1$ and Gauß ordinary hypergeometric function $F$ in the case of spherical photonic orbits with a non-vanishing value for the invariant parameter $\Phi$.

We subsequently applied our exact solutions for the determination of the Lense–Thirring effect that a photon experiences in a spherical polar and non-polar orbit around and close to our galactic centre, assuming the latter is a rotating black hole whose surrounding spacetime structure is described by the Kerr geometry as supported by recent observations. We repeated the analysis for various values of the Kerr parameter.

We also solved exactly non-spherical polar null unbound orbits. We derived analytical results for the deflection angle of a light ray from the gravitational field of a rotating black hole’s pole. The resulting expression for the deflection angle was written elegantly in terms of Lauricella’s hypergeometric function $F_D$.

We then investigated, non-circular orbits confined to the equatorial plane (timelike and null-like equatorial geodesics) for which the value of Carter’s constant invariant parameter vanishes. For the case of a vanishing cosmological constant, we derived an exact expression for the amount of relativistic precession for a test particle in a timelike orbit, around a rotating central mass. The corresponding novel expression was given in terms of Appell’s first hypergeometric function of two variables $F_1$. The application of this exact solution as well as of those that describe non-spherical orbits not necessarily confined to the equatorial plane, for the determination of the effect of rotation of central mass on the perihelion precession of a test particle (Mercury around Sun) or periapsis precession for a star such as S2 in a high eccentricity orbit around the galactic centre [42] is beyond the scope of this work and will be the subject of a future publication [51].

We have also derived an exact expression for the deflection angle of a light ray from the gravitational field of a rotating mass (the Kerr field). The corresponding expression was given in terms of Lauricella’s $F_D$ and Appell’s $F_1$ generalized hypergeometric functions. We applied this calculation for the bending of light from the gravitational field of the galactic centre of the Milky Way, assuming the latter is a supermassive Kerr black hole for various
values of the Kerr parameter and the impact factor. We emphasized in the main text the strong
dependence of the bending angle on the Kerr parameter for small values of the impact factor.
These results should be useful for gravitational lensing studies, where one treats the black hole
as a gravitational ‘lens’ [4], especially in the strong field region, when the bending angle can
be very large.

The synergy between theory and experiment for probing and measuring relativistic effects
is going to be one of the most exciting and fruitful scientific endeavours.

Acknowledgments

This work is supported by DOE grant no DE-FG03-95-Er-40917.

Appendix A. Definitions of genus-2 theta functions that solve Jacobi’s inversion problem

Riemann’s theta function [48] for genus $g$ is defined as follows,

$$
\Theta(u) := \sum_{n_1, \ldots, n_g} \exp(2\pi i u n + i\pi \Omega n^2)
$$

(A.1)

where $\Omega n^2 := \Omega_{11}n_1^2 + \cdots + 2\Omega_{12}n_1n_2 + \cdots$ and $u_n := u_1n_1 + \cdots u_gn_g$. The symmetric $g \times g$
complex matrix $\Omega$ whose imaginary part is positive definite is a member of the set called Siegel
upper-half-space denoted as $S_g$. It is clearly the generalization of the ratio of half-periods $\tau$
in the genus $g = 1$ case. For genus $g = 2$ the Riemann theta function can be written in matrix
form:

$$
\Theta(u, \Omega) = \sum_{n \in \mathbb{Z}^2} \exp(\pi i n \Omega u + 2\pi i nu)
$$

$$
= \sum_{n_1, n_2} \exp(\pi i (n_1 n_2) (\Omega_{11,12} n_1 + \Omega_{12,12} n_2) + 2\pi i (n_1 u_1 + n_2 u_2))
$$

(A.2)

Riemann’s theta function with characteristics is defined by

$$
\Theta(u; q, q') := \sum_{n_1, \ldots, n_g} \exp(2\pi i u(n + q') + i\pi \Omega(n + q')^2 + 2\pi i q(n + q')).
$$

(A.3)

Here $q$ denotes the set of $g$-quantities $q_1, \ldots, q_g$ and $q'$ denotes the set of $g$-quantities
$q_1', \ldots, q_g'$. Equation (A.3) can be rewritten in a suggestive matrix form:

$$
\begin{bmatrix} q' \\ q \end{bmatrix}(u, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i' (n + q')\Omega(n + q') + 2\pi i' (n + q')(u + q)), \quad q, q' \in Q^g.
$$

(A.4)

The theta functions whose quotients provide a solution to Abel–Jacobi’s inversion problem
are defined as follows [49],

$$
\theta(u; q, q') := \sum \exp(au^2 + 2hu(n + q') + b(n + q')^2 + 2i\pi q(n + q'))
$$

(A.5)

where the summation extends to all positive and negative integer values of $g, n_1, \ldots, n_g$, $a$
is any symmetrical matrix whatever of $g$ rows and columns, $h$ is any matrix whatever of $g$
rows and columns, in general not symmetrical, $b$ is any symmetrical matrix whatever of $g$
rows and columns, such that the real part of the quadratic form $bm^2$ is necessarily negative for all
real values of the quantities $m_1, \ldots, m_g$, other than zero, and $q, q'$ constitute the characteristics
of the function. The matrix $b$ depends on $\frac{1}{g}(g + 1)$ independent constants; if we put $i\pi \Omega = b$
and denote the $g$-quantities $hu$ by $i\pi U$, we obtain the relation with Riemann’s theta function:

$$
\theta(u; q, q') = e^{au^2} \Theta(U; q, q').
$$

(A.6)
The dependence of genus-2 theta functions on two complex variables is denoted by \( \theta(u; q, q') = \theta(u_1, u_2; q, q') \), the dependence on the Siegel moduli matrix \( \Omega \) by \( \theta(u_1, u_2; \Omega, q, q') \). With every half-period one can associate a set of characteristics. For instance, the period \( w^{a, b} = \frac{1}{2}(1, 0) \) while \( \theta(a) \) is a theta function of two variables with zero characteristics, i.e., \( \theta(a) = \theta(u; 0, 0) = \theta^{1, 0}(u, \Omega) \). Also, Weierstrass had associated a symbol for each of the six odd theta functions with characteristics and the ten even theta functions of genus 2. For example, \( \theta(a) \) is associated with the Weierstrass symbol 5 or occasionally the number appears as a subscript, i.e., \( \theta(a) \).

Let the genus \( g \) Riemann hyperelliptic surface be described by the equation

\[
y^2 = 4(x - a_1) \cdots (x - a_g)(x - c_1) \cdots (x - c_g). \tag{A.7}
\]

For \( g = 2 \) the above hyperelliptic Riemann algebraic equation reduces to

\[
y^2 = 4(x - a_1)(x - a_2)(x - c_1)(x - c_2) \tag{A.8}
\]

where \( a_1, a_2, c_1, c_2 \) denote the finite branch points of the surface.

Jacobi’s inversion problem involves finding the solutions for \( x_i \) in terms of \( u_i \), for the following system of equations of Abelian integrals [49],

\[
u_1^{x_{1,i_1}} + \cdots + u_1^{x_{1,d_1}} \equiv u_1
\]

\[
\vdots + \cdots + \vdots
\]

\[
u_g^{x_{g,i_1}} + \cdots + u_g^{x_{g,d_1}} \equiv u_g
\]

where \( u_i^{x_{i,\mu}} = \int_{a_1}^{x_i} \frac{dx}{y}, u_i^{x_{i,\nu}} = \int_{a_2}^{x_i} \frac{dx}{y} \) and \( u_i^{x_{i,\rho}} = \int_{a_3}^{x_i} \frac{dx}{y} \cdot \)

For \( g = 2 \) the above system of equations takes the form

\[
\int_{a_1}^{x_i} \frac{dx}{y} + \int_{a_2}^{x_i} \frac{dx}{y} \equiv u_1
\]

\[
\int_{a_1}^{x_i} \frac{dx}{y} \equiv u_2
\]

where \( u_1, u_2 \) are arbitrary. The solution is given by the five equations [49]

\[
\frac{\theta^2(u|u^{b,a})}{\theta^2(u)} = A(b - x_1)(b - x_2) \cdots (b - x_g)
\]

\[
= A(b - x_1)(b - x_2)
\]

where \( f(x) = (x - a_1)(x - a_2)(x - c_1)(x - c_2) \), and \( e^{\pi i PP'} = \pm 1 \) accordingly as \( u^{b,a} \) is an odd or even half-period. Also \( a \) denotes a finite branch point and the branch place \( a \) being at infinity [49]. The symbol \( \theta(u|u^{b,a}) \) denotes a genus-2 theta function with characteristics, \( \theta(u; q, q') \) [49], where \( u, \) \( \theta(u_1, u_2) \) denotes two independent variables. From any two of these equations (A.11) the upper integration bounds \( x_1, x_2 \) of the system of differential equations (A.10) can be expressed as single valued functions of the arbitrary arguments \( u_1, u_2 \). For instance, \n
\[
x_1 = a_1 + \frac{1}{A_1(x_2 - a_1)} \frac{\theta^2(u|u^{a_1,a})}{\theta^2(u)} \tag{A.12}
\]

and

\[
x_2 = -\frac{[\theta^2(a_2 - a_1)(a_2 + a_1) + \frac{1}{A_1} \theta^2(a_1|a_2^{a_1,a}) - \frac{1}{A_1} \theta^2(a_2|a_1^{a_1,a})]}{2(a_1 - a_2)} \pm \sqrt{\left[\frac{\theta^2(a_2 - a_1)(a_2 + a_1) + \frac{1}{A_1} \theta^2(a_1|a_2^{a_1,a}) - \frac{1}{A_1} \theta^2(a_2|a_1^{a_1,a})}{2(a_1 - a_2)}\right]^2 - 4(a_1 - a_2)\eta} \tag{A.13}
\]
where
\[ \eta := a_2a_1(a_1 - a_2) - \frac{a_2 \theta^2(u|\mu_1,\nu_1)}{A_1} + \frac{a_1 \theta^2(u|\mu_2,\nu_2)}{A_2}. \] (A.14)

Also, \( A_i = \pm \frac{1}{\sqrt{a_i \mu_0 f(u_0)}}. \)

The solution can be re-expressed in terms of generalized Weierstraß functions,
\[ x_k^{(1,2)} = \frac{\varphi_{2,2}(u) \pm \sqrt{\varphi_{2,2}^2(u) + 4\varphi_{2,1}(u)}}{2}, \quad k = 1, 2 \] (A.15)

where
\[ \varphi_{2,2}(u) = \frac{(a_1 - a_2)(a_2 + a_1) - \frac{1}{A_1} \theta^2(u|\mu_1,\nu_1) + \frac{1}{A_2} \theta^2(u|\mu_2,\nu_2)}{a_1 - a_2}, \] (A.16)

and
\[ \varphi_{2,1}(u) = \frac{-a_1a_2(a_1 - a_2) - \frac{a_1}{A_2} \theta^2(u|\mu_2,\nu_2) + \frac{a_2}{A_1} \theta^2(u|\mu_1,\nu_1)}{a_1 - a_2}. \] (A.17)

Thus, \( x_1, x_2 \), which solve Jacobi’s inversion problem (A.10), are solutions of a quadratic equation [44, 49]
\[ U^2 - U'x + U'' = 0 \] (A.18)

where \( U, U', U'' \) are functions of \( u_1, u_2 \). In the particular case when the coefficient of \( x^5 \) in the quintic polynomial is equal to 4, \( U = 1, U' = \varphi_{2,2}(u), U'' = \varphi_{2,1}(u) \).

The matrix elements \( h_{ij}, \Omega_{ij} \) can be explicitly written in terms of the half-periods \( U_{i}^{(\mu,\nu)} \). For clarity, \( U_{2}^{(\mu,\nu)} = \int_{\epsilon_1}^{\epsilon_2} x dy / \sqrt{\varphi_i}, U_{1}^{(\mu,\nu)} = \int_{\epsilon_1}^{\epsilon_2} dx / \sqrt{\varphi_i} \) etc. The roots have been arranged in ascending order of magnitude and are denoted by \( \epsilon_{2g}, \epsilon_{2g-1}, \ldots, \epsilon_0, g = 2 \), so that \( \epsilon_{2i}, \epsilon_{2i-1} \) are, respectively, \( c_{x-i-1}, c_{x-i} \) and \( c_0 \) is \( c \). For instance, the matrix element
\[ h_{11} = \frac{U_{2}^{(\mu,\nu)}}{2(U_{1}^{(\mu,\nu)} + U_{2}^{(\mu,\nu)} - U_{1}^{(\mu,\nu)}U_{2}^{(\mu,\nu)})} \times \pi, \quad \text{while} \quad \Omega_{11} = \frac{U_{1}^{(\mu,\nu)}U_{2}^{(\mu,\nu)}}{U_{2}^{(\mu,\nu)}U_{1}^{(\mu,\nu)} - U_{1}^{(\mu,\nu)}U_{2}^{(\mu,\nu)}}. \]

A.1. A particular inversion problem of Jacobi

An indefinite elliptic integral of the third kind can be regarded as a special form of a hyperelliptic integral
\[ \int_{0}^{x} \frac{(\alpha + \beta x) dx}{\sqrt{x(1 - x)(1 - \lambda^2 x)(1 - \mu^2 x)}} \] (A.19)

in the special case when two of the moduli are equal, e.g., \( \lambda = \mu \), and therefore one can consider the following Jacobi’s inversion problem:
\[ u = \int_{0}^{x_1} \frac{(\alpha + \beta x) dx}{(1 - \lambda^2 x)^{1/2}(1 - x)(1 - \mu^2 x)} + \int_{0}^{x_2} \frac{(\alpha + \beta x) dx}{(1 - \lambda^2 x)^{1/2}(1 - x)(1 - \mu^2 x)} \] \[ v = \int_{0}^{x_1} \frac{(\alpha' + \beta' x) dx}{(1 - \lambda^2 x)^{1/2}(1 - x)(1 - \mu^2 x)} + \int_{0}^{x_2} \frac{(\alpha' + \beta' x) dx}{(1 - \lambda^2 x)^{1/2}(1 - x)(1 - \mu^2 x)} \] (A.20)

For a convenient choice of the constants \( \alpha, \beta, \alpha', \beta' \) the solution of the above Jacobi’s inversion problem can be expressed in terms of genus-1 Jacobi theta functions.
\[ \pm \kappa \sqrt{x_1 x_2} = \frac{\theta_1(a) e^{-v} \theta_1(a - u + a)}{\theta_1(a) e^{-v} \theta_1(a - u) + e^v \theta_1(a + u)} \]

\[ \kappa \sqrt{(1 - x_1)(1 - x_2)} = \frac{\theta_2(a) e^{-v} \theta_2(a - u) - e^v \theta_2(u + a)}{\theta_2(a) e^{-v} \theta_2(a - u) + e^v \theta_2(u + a)} \]  

\[ \frac{1}{\kappa} \sqrt{(1 - \kappa^2 x_1)(1 - \kappa^2 x_2)} = \frac{\theta_3(a) e^{-v} \theta_3(a - u) + e^v \theta_3(u + a)}{\theta_1(a) e^{-v} \theta_1(a - u) + e^v \theta_1(a + u)} \]  

(A.21)

### Appendix B. Transformation theory of elliptic functions and modular equations

One of the applications supplied by the transformation theory of elliptic functions, which is of great importance in number theory, is the modular equations described below [32, 33, 35].

For a rational solution of the differential equation

\[ \frac{dy}{\sqrt{(1 - y^2)(1 - e_1^2 y^2)}} = C \frac{dx}{\sqrt{(1 - x^2)(1 - e_2^2 x^2)}} \]  

(B.1)

the necessary conditions among the periods

\[ K(e_1) = a_0 CK(e) + a_1 C K'(e) \quad \text{and} \quad i K'(e_1) = b_0 CK(e) + b_1 C K'(e) \]  

(B.2)

with the period ratios (moduli) of the associated modular theta functions being given by

\[ \tau = \frac{b_0 + b_1 \tau'}{a_0 + a_1 \tau'} \]  

(B.3)

are also sufficient, when

\[ a_0 b_1 - a_1 b_0 = n \]  

(B.4)

is a positive integer number. The integer \( n \) is called the degree of transformation.

Equation (B.4) for \( a_0, b_1, a_1, b_0 \in \mathbb{Z} \) when viewed as the determinant of a matrix \( \in GL(2, \mathbb{Z}) \), sometimes is called a modular correspondence of level \( n \).

It can be shown that the *inequivalent reduced forms of modular correspondences*

\[ \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix}, \]

are of the form

\[ \begin{pmatrix} q & 0 \\ 16\xi & q' \end{pmatrix} \]

where \( q \), a positive part of \( n \), represents \( q' := \frac{n}{q} \), and \( 0 \leq \xi < q' - 1 \). For instance, for \( n = p \) a prime number, there are \( p + 1 \) inequivalent reduced forms of the form13

\[ \begin{pmatrix} 1 & 0 \\ 16 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 16 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 16 & p \end{pmatrix}, \cdots, \begin{pmatrix} 1 & 0 \\ 16(p - 1) & p \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}. \]

Also the multiplication factor \( C \) in equation (B.1) is given by

\[ C = \frac{1}{q} \frac{K(e_1)}{K(e)} \]  

(B.5)

13 In a more familiar notation these classes of inequivalent reduced forms are

\[ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad ad = n, \quad (a, b, d) = 1, \quad 0 \leq b < d \text{ and } a, b, d \in \mathbb{Z}. \]
which for a degree of transformation that is a prime number \((n = p)\) is equal to \(\frac{K(e_1)}{K(e)}\) or \((1/n)\frac{K(e_1)}{K(e)}\). Equation (B.5) can be re-expressed in terms of Jacobi theta functions as follows:

\[
C = \frac{1}{q} \frac{\vartheta_2^2(0, \tau)}{\vartheta_2^2(0, \frac{q^{-1} + \xi}{q})}. \tag{B.6}
\]

The modular equations are the equations relating the Jacobi modulus \(e(p \tau)\) to \(e(\tau)\) which are of the form

\[
F_p \left[ \sqrt{e(\tau)}, \sqrt{e \left( \frac{\tau - 16 \xi}{p} \right)} \right] = 0 \tag{B.7}
\]

where \(\left( \frac{2}{p} \right)\) denotes the Legendre symbol\(^{14}\). Equivalently, modular equations can be written in terms of the absolute modular invariant function \(j(\tau)\), and relate the reduced absolute modular invariant \(j^*\) to \(j\) by polynomial equations of the form

\[
\Phi_p(j^*, j) = 0 \tag{B.8}
\]

where \(j^* := j \alpha_i = j \left( \frac{24 a^3}{d^2} \right)\). The explicit form of \(\Phi_2(j^*, j) = 0\) has been given in [41].

Appendix C. Conditions for radii for spherical null geodesics with a cosmological constant and differential equations of Appell’s function

The conditions from the vanishing of the polynomial \(R\) and its first derivative result in the equations which generalize (26)

\[
\Phi = \frac{r^3 \Lambda^2 a^5 + Ya^3 + X \alpha \pm \sqrt{3} \sqrt{f_1}}{r^3 \Lambda^2 a^4 + (2 \Lambda^2 r^5 + 6 \frac{GM}{c^2} \Lambda r^2 - 9 r + 9 \frac{GM}{c^2}) a^2 + r^6 \Lambda \left( \Lambda r^3 - 3 r + 6 \frac{GM}{c^2} \right)} \tag{C.1}
\]

where

\[
f_1 := r^2 \left( 3a^6 + r \left( \frac{GM}{c^2} K + r(2 \Lambda^2 r^6 - 15 \Lambda^2 r^4 + 9 \Lambda r^2 + 54) \right) + r^2 \left( 54(\Lambda r^2 + 2) \left( \frac{GM}{c^2} \right)^2 + 12 r K_2 \frac{GM}{c^2} + r^2 K_1 \right) a^2 + 3 \left( \frac{GM}{c^2} - r \right) r^6 \Lambda \left( \Lambda r^3 - 3 r + 6 \frac{GM}{c^2} \right) \right).
\]

Indeed, for a vanishing cosmological constant it has the correct limit \(\Phi \to \frac{a^3 r^2}{r^2}\) derived in the previous subsection.

Then the parameter \(Q\) is given by the expression

\[
Q = \frac{\Xi^2 r^4 + \Xi^2 r^2 (a^2 - \Phi^2) + \frac{20 M}{c^2} (\Phi - a)^2 + \frac{4}{3} r^2 (r^2 + a^2) \Xi^2 (\Phi - a)^2}{\Delta_r} \tag{C.2}
\]

\(^{14}\left( \frac{2}{q} \right) = e^{\frac{q^{-1}-1}{4}}\).
where
\[ X := \Lambda^2 r^7 + 6 \frac{GM}{c^2} \Lambda r^4 - 9 \frac{GM}{c^2} r^2 \]
\[ Y := 2 \Lambda^2 r^5 + \frac{GM}{c^2} (6 \Lambda r^2 + 9) \]
\[ Z := \Lambda^3 r^6 - 6 \Lambda^2 r^4 + 27 \]
\[ K := 5 \Lambda^2 r^4 - 9 \Lambda r^2 - 36 \]
\[ K_1 := \Lambda^3 r^6 - 12 \Lambda^2 r^4 + 18 \Lambda r^2 + 27 \]
\[ K_2 = 2 \Lambda^2 r^4 - 6 \Lambda r^2 - 9. \]

(C.3)

C.1. Proof of identity equation (52)
\[
\int_0^1 \frac{z'}{(1 - \frac{z'}{\kappa^2 z'})} \left( \frac{1}{1 - \mu^2 z'} \right) \left( \frac{1}{1 - 1 - \kappa^2 z'} \right) \left( \frac{1}{1 - 1 - \kappa^2 z'} \right)
= \int_0^1 \frac{dz'}{\sqrt{z'(1 - \frac{z'}{\kappa^2 z'})}} \left( \frac{1}{1 - \mu^2 z'} \right)
= \int_0^1 \frac{dz'}{\sqrt{z'(1 - \frac{z'}{\kappa^2 z'})}} \left( \frac{1}{1 - \mu^2 z'} \right)
= \frac{1}{\kappa^2} \int_0^1 \frac{dz'}{\sqrt{z'(1 - \frac{z'}{\kappa^2 z'})}} \left( \frac{1}{1 - \mu^2 z'} \right)
+ \frac{1}{\kappa^2} \int_0^1 \frac{dz'}{\sqrt{z'(1 - \frac{z'}{\kappa^2 z'})}} \left( \frac{1}{1 - \mu^2 z'} \right).
\]

(C.4)

Picard had developed a theory for finding solutions of the system of differential equations that the Appell hypergeometric function obeys. More precisely he showed, by direct substitution, that solutions are provided by definite integrals of the form [30]
\[
\int_a^b \frac{u^{b-1}(u - 1)^{b-1}(u - x)^{b-1}(u - y)^{b-1}}{du}
\]
where \( g, h \) denote two of the quantities 0, 1, \( x, y, \infty \) and we have the correspondence
\[ b_1 = 1 + \beta + \beta' - \gamma, \quad b_2 = \gamma - \alpha, \quad \mu = 1 - \beta, \quad \lambda = 1 - \beta'. \]

(C.6)

This is the generalization of Kummer’s work who found that the standard hypergeometric equation has 24 solutions [31]. The system of linear differential equations of Appell’s function \( F_1 \) is
\[
x(1 - x)(x - y) \frac{\partial^2 F_1}{\partial x^2} + [\gamma(y - x) - (\alpha + \beta + 1)x^2 + (\alpha + \beta - \beta' + 1)xy + \beta'y] \frac{\partial F_1}{\partial x}
- \beta y(1 - y) \frac{\partial F_1}{\partial y} - \alpha \beta(x - y) F_1 = 0,
\]
\[
y(1 - y)(y - x) \frac{\partial^2 F_1}{\partial y^2} + [\gamma(y - x) - (\alpha + \beta' + 1)y^2 + (\alpha + \beta' - \beta + 1)xy + \beta'x] \frac{\partial F_1}{\partial y}
- \beta' x(1 - x) \frac{\partial F_1}{\partial x} - \alpha \beta'(y - x) F_1 = 0,
\]
\[
(x - y) \frac{\partial^2 F_1}{\partial x \partial y} = \beta' \frac{\partial F_1}{\partial x} - \beta \frac{\partial F_1}{\partial y}.
\]
For instance, the integral is represented as follows,
\[
\int_1^\infty u^{a-\beta-\gamma} (u - 1)^{\gamma-\alpha-1} (u - x)^{-\beta} (u - y)^{-\beta} \, du
= B(1 + \beta + \beta' - \gamma, \gamma - \alpha) x^{-\beta} y^{-\beta} F_1
\times \left( 1 + \beta + \beta' - \gamma, \beta, \beta', 1 + \beta + \beta' - \alpha, \frac{1}{x}, \frac{1}{y} \right)
\] (C.8)
and \( B(p, q) = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p + q)} \) denotes the beta function.

References

[1] Einstein A 1915 Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie Sitz. Preuss. Akad. Wissenschaften p 831
[2] Schwarzschild K 1916 Über das Gravitationfeld eines Massenpunktes nach der Einsteinschen Theorie Sitz. Königlichen Preuss. Akad. Wissenschaften (Berlin) pp 189–96
[3] Kerr R P 1963 Gravitational field of a spinning mass as an example of algebraically special metrics Phys. Rev. Lett. 11 237
[4] Ohanian H and Ruffini R 1994 Gravitation and Spacetime (New York: Norton)
[5] Rees M J 2004 Black holes in the real universe and their prospects as probes of relativistic gravity Future of Theoretical Physics and Cosmology ed G W Gibbons et al (Cambridge: Cambridge University Press) pp 217–35 (Preprint astro-ph/0401365)
[6] Kraniotis G V and Whitehouse S B 2003 Compact calculation of the perihelion precession of Mercury in general relativity, the cosmological constant and Jacobi’s inversion problem Class. Quantum Grav. 20 4817–35
[7] Bahcall N A, Ostriker J P, Perlmutter S and Steinhardt P J 1999 Science 284 1481
[8] Perlmutter S et al 1999 Astrophys. J. 517 565
[9] Filippenko A V et al 1998 Astron. J. 116 1009
[10] deBernardis P et al 2000 Nature 404 955
[11] Jaffe A H et al 2000 Phys. Rev. Lett. 86 3475
[12] Kraniotis G V and Whitehouse S B 2002 General relativity, the cosmological constant and modular forms Class. Quantum Grav. 19 5073–100 (Preprint gr-qc/0105022)
[13] Kraniotis G V 2004 Precise relativistic orbits in Kerr and Kerr–(anti) de Sitter spacetimes Class. Quantum Grav. 21 4743–69 (Preprint gr-qc/0405095)
[14] Genzel R et al 2003 Near-infrared flares from accreting gas around the supermassive black hole at the galactic centre Nature 425 934–7
[15] Bardeen J M, Press W M and Teukolsky S A 1972 Rotating black holes: locally, non-rotating frames, energy extraction and scalar synchrotron radiation Astrophys. J. 178 347–69
[16] Melia F, Bromley C, Liu S and Walker C K 2001 Measuring the black hole spin in Sgr A* Astrophys. J. 554 L37–40
[17] Boyer R H and Lindquist R W 1967 Maximal analytic extension of the Kerr metric J. Math. Phys. 8 265–81
[18] Boyer R H and Price T G 1965 An interpretation of the Kerr metric in general relativity Proc. Camb. Phil. Soc. 61 531
[19] Lense J and Thirring H 1918 Über den Einfluß der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie Phys. Z. 19 156
[20] Lammerzahl C, Everitt C W F and Hehl F W (ed) 2001 Gyros, Clocks, Interferometers… Testing Relativistic Gravity in Space (Lecture Notes in Physics vol 562) (Berlin: Springer) pp 52–82
[21] Appell P 1882 Sur les fonctions hypergéométriques de deux variables J. Math. Pure Appl. 8 173–216
[22] Lauricella G 1893 Sulle funzioni ipergeometriche a più variabili Rend. Circ. Mat. Palermo 7 111–58
[22] Teo E 2003 Spherical photon orbits around a Kerr black hole Gen. Rel. Grav. 35 1909
[23] Stogianides E and Tsoubelis D 1987 Polar orbits in the Kerr space-time Gen. Rel. Grav. 19 1235–49
[24] Eichler M and Zagier D 1985 The Theory of Jacobi Forms (Progress in Mathematics vol. 55) (Basel: Birkhäuser)
[25] Silverman J H and Tate J 1992 Rational Points on Elliptic Curves (Berlin: Springer)
[26] Cardano G 1545/1968 Artis Magnae sive de regulis algebraicis/The Great Art or the Rules of Algebra transl. ed T R Witmer (Cambridge, MA: MIT Press) (Engl. Transl.)
[27] Whittaker E J 1947 A Treatise on the Analytical Dynamics of Particles and Rigid Bodies (Cambridge: Cambridge University Press)
[28] Aschenbach B, Grosso N, Porquet D and Predehl P 2004 X-ray flares reveal mass and angular momentum of the galactic centre black hole Astron. Astrophys. 417 71–8 (Preprint astro-ph/0401589)
[29] Aschenbach B 2004 Measuring mass and angular momentum of black holes with high-frequency quasi-periodic oscillations Astron. Astrophys. 425 1075–82 (Preprint astro-ph/0406545)
[30] Picard E 1881 Sur une extension aux fonctions de deux variables du problème de Riemann relatif aux fonctions hypergéométriques Ann. Sci. Ecole Norm. Sup. 10 305–22
[31] Kummer E E 1836 Über die hypergeometrische Reihe 1 + \frac{a}{17}x + \frac{a(a+1)}{17 \cdot 37}x^2 + \cdots Crelle’s J. Math. 15 39–172
[32] Abel N H 1928 Solution d’un problème général concernant la transformation des fonctions elliptiques Astronomische Nachrichten, herausgegeben von Schumacher (Altona) Bd 6, Nr 138
[33] Abel N H 1828 Recherches sur les fonctions elliptiques Crelle’s J. Math. 3 160–90
[34] Abel N H 1828 Précis d’une théorie des fonctions elliptiques Crelle’s J. Math. 3 236–77
[35] Abel N H 1829 Crelle’s J. Math. 4 309–48
[36] Jacobi C G J 1827 Extraits de deux lettres de M Jacobi de universite de Königsberg A M Schumacher Astronomische Nachrichten, herausgegeben von Schumacher (Altona) Nr 127
[37] Luchterhandt R A 1837 De transformatione expressionis \sqrt{\frac{a}{x} - \sqrt{a x}} in formam simpliciorem \frac{\sqrt{\frac{a}{x} - \sqrt{a x}}}{x} adhibita substitutione x = \frac{\sin x}{\sin^2 x} Crelle’s J. Math. 17 248–56
[38] Carter B 1968 Global structure of the Kerr family of gravitational fields Phys. Rev. 174 1559–71
[39] Wilkins D C 1972 Bound geodesics in the Kerr metric Phys. Rev. D 5 814–22 see also Johnston M and Ruffini R 1974 Phys. Rev. D 10 2324–9
[40] Merloni A et al 1999 Mon. Not. R. Astron. Soc. 304 135–9
[41] Stäckel P 1893 Über die Integration der Hamilton–Jacobi’schen Differentialgleichung mittels Separation der Variablen Habilitationsschrift Halle (1891) Math. Ann. 42 546–9
[42] Stäckel P 1895 Sur l’integration de l’équation différentielle de Hamilton C. R. Acad. Sci. 121 489–93
[43] Ciufolini I 1986 Measurement of the Lense–Thirring drag on high-altitude, laser-ranged artificial satellites Phys. Rev. Lett. 56 278–81
[44] Ciufolini I 2002 Preprint gr-qc/0209109
[45] Ciufolini I and Pavlis E C 2004 Nature 431 958–60
[46] Yui N 1978 Explicit form of the modular equation Crelle’s J. Math. 299/300 185–200
[47] Schödel R et al 2002 Nature 419 694–6
[48] Gebhardt K 2002 Nature 419 675–6
[49] Abel N H 1828 Remarques sur quelques propriétés générales d’une certaine sorte de fonctions transcendantes Crelle’s J. Math. 3 313–23
[50] Jacobi C G J 1832 Considerationes generales de transcendentibus Abelianis Crelle’s J. Math. 9 394
[51] Jacobi C G J 1835 Crelle’s J. Math. 13 55
[52] Jacobi C G J 1834 Über die vierfach periodischen Functionen zweier variablen (Ostwald’s klassiker der exakten Wissenschaften Nr 64) (Leipzig: Engelmann) pp 1–41
[53] Göpel A 1847 Entwurf einer Theorie der Abel’schen Transcendenten erster Ordnung Crelle’s J. Math. 35 277–312
[54] Rosenhain G 1850 Auszug mehrerer Schreiben des Herrn Dr. Rosenhain an Herrn Prof. C G J Jacobi über die Hyperelliptischen Transcendenten Crelle’s J. Math. 40 319
[55] Weierstraß K 1854 Zur Theorie der Abel’schen Functionen Crelle’s J. Math. 47 289
[56] Riemann B 1857 Theorie der Abelschen Functionen Crelle’s J. Math. 54 115
[57] Baker H F 1995 Abelian Functions: Abel’s Theorem and the Allied Theory of Theta Functions (Cambridge: Cambridge University Press)
[58] Ohanian H C 1987 The black hole as a gravitational lens Ann. J. Phys. 55 428–32
[59] Kranoti G V in preparation