AN OVERTWISTED CONVEX HYPERSURFACE IN HIGHER DIMENSIONS

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Abstract. We show that the germ of the contact structure surrounding a certain kind of convex hypersurfaces is overtwisted. We then find such hypersurfaces close to any plastikstufe with toric core so that these imply overtwistedness. All proofs in this article are explicit, and we hope that the methods used here might hint at a deeper understanding of the size of neighborhoods in contact manifolds.

In the appendix we reprove in a concise way that the Legendrian unknot is loose if the ambient manifold contains a large enough neighborhood of a 2-dimensional overtwisted disk. Additionally we prove the folklore result that the singular distribution induced on a hypersurface \( \Sigma \) of a contact manifold \((M, \xi)\) determines the germ of the contact structure around \( \Sigma \).

The fundamental distinction between tight and overtwisted contact structures was discovered first by Eliashberg in dimension 3 [Eli89], and then generalized to arbitrary dimensions by Borman–Eliashberg–Murphy [BEM15]. The main feature is that an overtwisted contact structure is flexible. However, the original high dimensional definition makes it practically unverifiable if a given contact structure is overtwisted. Thanksto Casals–Murphy–Presas [CMP19] we know that most previously existing conjectural definitions for overtwistedness are actually equivalent to the one given in [BEM15].

Still many open questions persist. One of them was lifted by Huang [Hua17]: The second author of this article proposed a definition of overtwistedness called plastikstufe [Nie06]. The results in [MNPS13] combined with those of [CMP19] showed that certain very special plastikstufes imply overtwistedness, while Huang explains in his article that this result extends to any plastikstufe. Nevertheless, several points in his proof are unclear to us and more detailed arguments may be desired.

In this article, we reprove Huang’s result for the more restrictive case of plastikstufes with toric core (generalizing results by Adachi [Ada16]).

Our strategy is based on the following observation about a certain kind of convex hypersurfaces. Consider for any \( C > 0 \) the manifold
\[
\Sigma_C = D^2_{\leq \tau} \times (-C, C)^{2n}
\]
carrying a singular distribution \( \mathcal{D}_C \) given as the kernel of the 1-form \( \beta = r \sin r \, d\theta - \sum_{j=1}^{n} t_j \, ds_j \), where \((r, \theta)\) are polar coordinates on the disk, and \((s_j, t_j)\) are the natural coordinates on the cube \((-C, C)^n \times (-C, C)^n\).

**Theorem A.** There exists a constant \( C_{OT} > 0 \) such that every contact manifold \((M, \xi)\) of dimension \( \geq 5 \) is overtwisted, if it is possible to embed a hypersurface \((\Sigma_C, \mathcal{D}_C)\) with \( C > C_{OT} \) such that \( \xi \) induces the singular distribution \( \mathcal{D}_C \) on \( \Sigma_C \).

Consequently, we call any embedded hypersurface \((\Sigma_C, \mathcal{D}_C)\) with \( C > C_{OT} \) an overtwisted convex disk.

**Remark 1.**
(a) This definition is closely related to the characterization of overtwisted contact structures in terms of “large neighborhoods” given in [NP10, CMP19]. Our result states that instead of considering large embedded balls, it is already sufficient to find a large hypersurface.

(b) In the model used in Section 3, one sees directly that there is an obvious contact vector field \( Z (= \partial_z \text{ in fact}) \) that is transverse to the overtwisted convex disk. It is tempting to try to characterize the constant \( C_{OT} \) more explicitly by trying to recognize some sort of higher dimensional Giroux criterion for convex hypersurfaces. Unfortunately, the dividing
set for $Z$ is non-compact, and we have not succeeded in finding a more suitable contact vector field.

(c) Even though we are unable to give a specific value for the size parameter that appears in the definition of overtwistedness in [BEM15] (and the equivalent formulation via large neighborhoods in [CMP19]), it follows from our argument that the size parameter can be chosen uniformly for all dimensions.

Proof of Theorem A. We show in Section 2 and 3 that every neighborhood of $D^{2} \leq \pi \times (\pi, C, C)$ contains the embedding of a certain type of open subset $B(h) \times (-\frac{5}{6}C, \frac{5}{6}C)^{2n}$. See Corollary 3.3 for the details. As explained first in [MNPS13, p. 1813], the Legendrian unknot is loose, if $C$ is chosen sufficiently large. We give in Appendix A a streamlined proof of this statement.

It was proved in [CMP19] that any contact manifold in which the unknot is loose is overtwisted. □

Question 1. Is it possible to explicitly show that in any neighborhood of a hypersurface $\Sigma_{C}$ with $C > C_{OT}$ one can embed a hypersurface $\Sigma_{C'}$ with $C' > 2C$ as in the analogous claim for loose charts [Mur12, Proposition 4.4]?

It is likely that the contact germs around the overtwisted convex disk $(\Sigma_{C}, D_{C})$ correspond to the thick neighborhoods which were shown in [CMP19] to be overtwisted. Unfortunately the formulation of thick neighborhoods in [CMP19] and the one used here are not directly comparable, and the verification of the equivalence would force us to dig through the proofs of the corresponding statements in [CMP19]. We have preferred to take instead the detour over the loose unknots which allows us to split the argument cleanly into separate steps.

Even if it might be unclear at the moment if Theorem A is more than a curious observation, it allows us to prove in an extremely elementary way that every contact manifold containing a plastikstufe with toric core is overtwisted, as claimed in [Hua17].

In Appendix A we show that the Legendrian unknot is loose in a large neighborhood of an overtwisted 2-disk; in Appendix B we prove the folklore result that a hypersurface in a contact manifold determines together with the induced singular distribution the germ of the contact structure.

Acknowledgments. We thank Sylvain Courte and Emmanuel Giroux and Patrick Massot for useful and interesting discussions. During a short conversation with Emmy Murphy, we learned that she had come to similar conclusions regarding the “height” of the overtwisted model.

1. A plastikstufe with toric core implies overtwistedness

Denote the cylindrical coordinates on $\mathbb{R}^{3}$ by $(r, \vartheta, z)$. The 1-form

$$\alpha_{OT} = \cos(r)\, dz + r \sin(r)\, d\vartheta$$

is then a well-defined contact form. The disk $D_{OT} := \{ (r, \vartheta, z) \mid r \leq \pi, z = 0 \}$ is overtwisted, and we will call it the standard overtwisted disk.

We choose a small cylindrical box of height $h$ around $D_{OT}$ of the form

$$B(h) := D^{2}_{\pi + h} \times (-h, h).$$

Let $(M, \xi)$ be a $(2n + 3)$-dimensional contact manifold that contains a plastikstufe of the form $D_{OT} \times \mathbb{T}^{n}$. We will show that we can find an arbitrarily “large” hypersurface $D_{2\pi}^{2} \times [-C, C]^{2n}$ in any neighborhood of the plastikstufe by successively unwinding each of the $S^{1}$-factors of the torus. This way we obtain the following corollary.

Corollary 1.1. Every contact manifold that contains a plastikstufe $D_{OT} \times \mathbb{T}^{n}$ with toric core, also admits an embedding of an overtwisted convex disk.

Proof. There is a neighborhood of the plastikstufe that is contactomorphic to an open neighborhood of $D_{OT} \times \mathbb{T}^{n}$ in

$$(\mathbb{R}^{3} \times T^{n}, \alpha_{OT} + \lambda_{can})$$;
Lemma 1.3. Any neighborhood is contactomorphic to a product of the form $B(\varepsilon) \times D_{< \delta}(T^*\mathbb{T}^n)$, where $B(\varepsilon) \subset \mathbb{R}^3$ is a cylindrical box as defined in [1,1] and $D_{< \delta}(T^*\mathbb{T}^n)$ is the disk bundle of radius $\delta$ in $T^*\mathbb{T}^n$.

We can now apply to this neighborhood Lemma [1,2] in dimension 5, or Lemma [1,3] in the general case to find a hypersurface of the form $D_{< \pi}(\mathbb{R}^3 \times (-C,C)^n) \times (-a,a)^n$ for $a = \frac{\delta}{2\sqrt{n}}$, and where $C > 0$ can be chosen to be arbitrarily large. The singular distribution induced by the contact structure agrees with the kernel of $r \sin r d\vartheta - \sum_{j=1}^{n} t_j ds_j$, where $(r, \vartheta)$ are polar coordinates on the disk, and $(s_j, t_j)$ are the natural coordinates on the rectangle $(-C,C) \times (-a,a)$.

If $C > 0$ is chosen larger than $2C^2_{OT}/a$, then it suffices to apply to each coordinate pair $(s_j, t_j)$ the diffeomorphism $(s_j, t_j) \mapsto (\mu^{-1}s_j, \mu t_j)$ with $\mu = \frac{2C_{OT}}{a}$ to contain the desired overtwisted convex disk $(\Sigma_{\mathcal{C}}, \mathcal{D}_{\mathcal{C}})$ for some appropriate $\bar{C} > C_{OT}$.

We will now show how to “unwrap” the toric plastikstufe. Consider for simplicity first a contact manifold of dimension 5 so that $\mathbb{T}^n = \mathbb{S}^1$.

Lemma 1.2. Choose any $\varepsilon > 0$ and $\delta > 0$, and let $B(\varepsilon) \times D_{< \delta}(T^*\mathbb{S}^1)$ be a neighborhood of a plastikstufe $D_{OT} \times \mathbb{S}^1$ in $(\mathbb{R}^3 \times T^*\mathbb{S}^1, \alpha_{OT} + \lambda_{can})$.

For any arbitrarily large $C > 0$ it is possible to embed the hypersurface

$$S_C := D_{< \pi}^2 \times (-C,C) \times (-\frac{\delta}{2}, \frac{\delta}{2})$$

into $B(\varepsilon) \times D_{< \delta}(T^*\mathbb{S}^1)$ such that the contact structure induces the singular distribution $D := \ker (r \sin r d\vartheta - t ds)$ on $S_C$. Here $(r, \vartheta)$ are polar coordinates on the disk, and $(s,t)$ are the natural coordinates on the rectangle $(-C,C) \times (-\frac{\delta}{2}, \frac{\delta}{2})$.

Proof. Define for any $h > 0$ an embedding

$$\Psi_h : D_{< \pi}^2 \times \mathbb{R}^2 \to \mathbb{R}^3 \times T^*\mathbb{S}^1$$

$$(r, \vartheta; s,t) \mapsto (r, \vartheta, hs; q = e^{is}, p = t + h \cos r)$$

“reeling up” the hypersurface along the $\mathbb{S}^1$-core of the plastikstufe. One easily verifies that

$$\Psi_h(\alpha_{OT} + \lambda_{can}) = r \sin r d\vartheta - t ds$$

so that the singular distribution induced by $\ker (\alpha_{OT} + \lambda_{can})$ is indeed equal to $D$.

Choose $h = \varepsilon/C$ and suppose that $h < \frac{\delta}{2}$, we see that $\Psi_h(S_C)$ lies in the given neighborhood $B(\varepsilon) \times D_{< \delta}(T^*\mathbb{S}^1)$.

For general dimensions, the embedding is only slightly more complicated. Denote the standard coordinates on $\mathbb{T}^n$ by $q = (q_1, \ldots, q_n)$ and those on $T^*\mathbb{T}^n$ by $(q, p) = (q_1, \ldots, q_n; p_1, \ldots, p_n)$.

Lemma 1.3. Any neighborhood $B(\varepsilon) \times D_{< \delta}(T^*\mathbb{T}^n)$ of a plastikstufe $D_{OT} \times \mathbb{T}^n$ with $\varepsilon > 0$ and $\delta > 0$ arbitrarily small contains an embedded hypersurface of the form

$$S_C := D_{< \pi}^2 \times \left\{(s_1, \ldots, s_n; t_1, \ldots, t_n) \in \mathbb{R}^{2n} \mid |s_j| < C, |t_j| < \frac{\delta}{2\sqrt{n}} \right\}$$

with $C > 0$ arbitrarily large such that the contact structure induces the singular distribution $D = \ker (r \sin r d\vartheta - \sum_{j=1}^{n} t_j ds_j)$ on $S_C$.

Proof. Choose constants $h_1, \ldots, h_n > 0$ that are linearly independent over $\mathbb{Q}$, and define a map $\Psi : D_{< \pi}^2 \times \mathbb{R}^{2n} \to \mathbb{R}^3 \times T^*\mathbb{T}^n$ by

$$(r, \vartheta; s_1, \ldots, s_n; t_1, \ldots, t_n) \mapsto (r, \vartheta, z = \sum_{j=1}^{n} h_j s_j; \quad q_1 = e^{is_1}, \ldots, q_n = e^{is_n}; \quad p_1 = t_1 + h_1 \cos r, \ldots, p_n = t_n + h_n \cos r)$$

It is easy to verify that $\Psi^*(\alpha_{OT} + \lambda_{can}) = r \sin r d\vartheta - (t_1 ds_1 + \cdots + t_n ds_n)$ induces the distribution $D$ on $S_C$. It is also immediately clear that $\Psi$ is an immersion.
To see that Ψ is injective, use first that the images of two points \((r, \vartheta; s_1, \ldots; s_n; t_1, \ldots; t_n)\) and \((r', \vartheta'; s'_1, \ldots; s'_n; t'_1, \ldots; t'_n)\) by Ψ can only agree if \(r = r'\), \(\vartheta = \vartheta'\), and \(t_j = t'_j\) for all \(j = 1, \ldots, n\), and if \(s_j - s'_j\) is for every \(j = 1, \ldots, n\) an integer multiple of \(2\pi\). The equation \(h_1 s_1 + \cdots + h_n s_n = h'_1 s'_1 + \cdots + h'_n s'_n\) implies that \(h_1 (s_1 - s'_1) + \cdots + h_n (s_n - s'_n) = 0\), but by our assumption that the \(h_j\) are linearly independent over \(\mathbb{Q}\) it follows that all coefficients \(s_1 - s'_1\) need to vanish so that Ψ is injective.

We still need to verify that the image of Ψ lies in the neighborhood \(B(\varepsilon) \times \mathbb{D}_{<\delta}(T^*\mathbb{T}^n)\). To respect the \(z\)-height, it suffices to choose \(h_1 + \cdots + h_n < \varepsilon/C\), so that the \(z\)-coordinate of Ψ is bounded by \(\varepsilon\). For the radius of the fibers in \(\mathbb{D}_{<\delta}(T^*\mathbb{T}^n)\) choose \(h_j < \frac{\delta}{2\sqrt{n}}\) so that Ψ also stays inside the \(\delta\)-disk bundle of \(T^*\mathbb{T}^n\).

\[\square\]

2. The standard overtwisted contact structure on \(\mathbb{R}^3\)

For a cylindrical box of height \(h\) around the standard overtwisted disk \(\mathbb{D}_{\text{OT}}\) in \((\mathbb{R}^3, \alpha_{\text{OT}})\) of the form

\[B(h) := \mathbb{D}_2 \times (-h, h),\]

give a contact vector field that can be used to prove this fact by hand, but instead one can also easily convince oneself that all \(B(h)\) are overtwisted at infinity which uniquely characterizes by a contact structure on \(\mathbb{R}^3\).

The main technical problem that we will deal with in this article is to show that the choice of the \(h\)-parameter also remains largely irrelevant for the contactomorphism type when we take the product with a Liouville domain.

We will now discuss a contact vector field \(X\) whose flow compresses any large box \(B(h)\) into an arbitrarily small neighborhood of the standard overtwisted disk. Ideally we would like \(X\) to be a strict contact vector field or at least to have a constant scaling factor \(c\) such that \(\mathcal{L}_X \alpha_{\text{OT}} = c \cdot \alpha_{\text{OT}}\).

Unfortunately such a vector field cannot exist: firstly, \(X\) should be contracting and thus it needs to reduce the total volume. This implies that \(c\) would have to be strictly negative on a predominant part of its domain. On the other hand, \(c\) cannot be everywhere strictly negative as this would allow us to squeeze with the strategy of Section 3 a high-dimensional overtwisted chart into an arbitrarily “thin” set, thus contradicting the existence of tight contact manifolds.

The following vector field arose in discussions with Patrick Massot around 2010,

\[X := -z \partial_z - \frac{r \cos(r) \sin(r)}{r + \cos(r) \sin(r)} \partial_r\]

is well-defined and induces a contact flow on \((\mathbb{R}^3, \alpha_{\text{OT}})\).

Even though this vector field might at first appear overly complicated, note that all coordinates in \(X\) are uncoupled. This allows us to see that its time \(T\) flow preserves the \(\vartheta\)-coordinate, and it contracts the \(z\)-coordinate by the factor \(e^{-T}\). The radial coordinate is fixed on every cylinder of radius \(r \in \frac{\pi}{2} \mathbb{N}\). The cylinders of radius \(r \in \frac{\pi}{2} + \pi \mathbb{N}\) are repelling; the cylinders of radius \(r \in \pi \mathbb{N}\) are attracting, in the sense that all points between these cylinders are pushed away from the repelling cylinder towards the attracting one, see Fig 1.

The height of the box \(B(h)\) is squeezed by the flow of \(X\) by an exponential factor, while the radial direction also shrinks, without becoming ever smaller than \(\pi\) though. This way an arbitrarily tall box \(B(h)\) can be squeezed by a contactomorphism into an arbitrarily small neighborhood of the standard overtwisted disk. In fact, one can convince oneself that the image of a box \(B(h)\) for a certain choice of \(\delta > 0\) will be of the form \(\mathbb{D}_{<\pi+\delta} \times (-h', h')\) for some smaller \(h' > 0\) and \(\delta' > 0\).

Finally note that \(\mathcal{L}_X \alpha_{\text{OT}} = g \cdot \alpha_{\text{OT}}\) where

\[g(r, \vartheta, z) = -\frac{\cos r (r \cos r + \sin r)}{r + \cos r \sin r}.\]

As we claimed above, \(g\) takes both positive and negative values. More precisely:
Figure 1. The flow of $X$ preserves the cylinders of radius $r \in \frac{\pi}{2} + \pi \mathbb{N}$. Note that the boundary of the standard overtwisted disk sits on an attracting cylinder.

**Lemma 2.1.** The function $g: B(h) \to \mathbb{R}$ is everywhere negative on $B(h)$ except for the domain lying between $r_m = \pi/2$ and $r_M \approx 2.03$ such that $r_M = -\tan r_M$. See also the graph in Fig. 2.

**Proof.** The function $g$ only depends on the radial coordinate $r$. Its denominator is everywhere positive while the numerator changes once its sign at $r = \pi/2$, where $\cos r$ vanishes, and then again at $r \approx 2.03$, where $r \cos r + \sin r$ vanishes.

We can also read off from the graph, Fig. 2, that $g$ is everywhere smaller than $0.1$; i.e., even though $g$ becomes positive it only becomes very slightly so.

![Graph of $g$](image.png)

**Figure 2.** The graph of $g = -\frac{\cos r (r \cos r + \sin r)}{r \cos r \sin r}$

3. **Contactomorphism on Product Structure**

Let $(M, \xi)$ be a contact manifold with contact form $\alpha$. Assume that $X$ is a contact vector field such that $\mathcal{L}_X \alpha = g \cdot \alpha$ for some function $g: M \to \mathbb{R}$.

Choose an exact symplectic manifold $(W, d\lambda)$ that has a Liouville vector field $Y$, then we easily check that the contact form of $(M \times W, \alpha + \lambda)$ is preserved by the vector field

$$\tilde{X} = X + g \cdot Y.$$ 

Note that even if $Y$ points outwards and is expanding on $(W, d\lambda)$, the behavior of $\tilde{X}$ on the product manifold $M \times W$ is controlled in $W$-direction by the sign of the function $g$ that might take positive or negative values.
We consider now the main example we will be interested in: Let $B(h) \subset (\mathbb{R}^3, \alpha_{\text{OT}})$ be a box of height $h$ as defined in [1.1], and let $(L, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold that does not need to be closed or geodesically complete. We denote the disk bundle of radius $c$ in $(T^*L, d\lambda_{\text{can}})$ by

$$D_{<c}(T^*L) := \{ \nu \in T^*L \mid \|\nu\| < c \}.$$ 

**Proposition 3.1.** There exists a positive constant $\mu_0 < 7/6$ such that every contact domain

$$\left( B(h) \times D_{<\mu_0}(T^*L), \ker(\alpha_{\text{OT}} + \lambda_{\text{can}}) \right)$$

with $c > 0$ can be embedded by a contactomorphism into

$$\left( B(h') \times D_{<\mu_0}(T^*L), \ker(\alpha_{\text{OT}} + \lambda_{\text{can}}) \right)$$

independently of the choices of $h, h' > 0$.

**Proof.** For $h' \geq h$ the claim is obvious; for $h' < h$ the strategy is to use the flow of the vector field $\tilde{X} := X + g \cdot Y$ with $X$ and $g$ introduced in the previous section, and $Y$ the Liouville vector field on $T^*L$ that is defined by $e_Y d\lambda_{\text{can}} = \lambda_{\text{can}}$.

By writing $Y$ in a coordinate chart, one easily convinces oneself that the time $t$ flow of $Y$ simply consists of multiplying the fiber of $T^*L$ by $e^t$. In particular, even if $L$ is open or has boundary there is no danger that the flow of $\tilde{X}$ escapes transversely through the fibers of $T^*L$. Now let us study the behavior of the flow $\Phi^{\tilde{X}}_t$ in more details.

Recall that the coordinates are uncoupled by the flow of $X$ given in (2.1). We can thus write

$$\Phi^{\tilde{X}}_t (r, \vartheta, z) = (F(r,T), \vartheta, e^{-T} z),$$

where $F(r,T)$ is the solution of the ODE

$$y'(t) = - \frac{y(t) \cos y(t) \sin y(t)}{y(t) + \cos y(t) \sin y(t)}, \quad \text{and} \quad y(0) = r.$$ 

The flow $\Phi^{\tilde{X}}_t$ is therefore of the form

$$\Phi^{\tilde{X}}_t (r, \vartheta, z; \mathbf{q}, \mathbf{p}) = \left( \Phi^X_t (r, \vartheta, z); \mathbf{q}, e^{G(r,t)} \cdot \mathbf{p} \right).$$

That is, the flow on the $\mathbb{R}^3$-factor simply reduces to the corresponding flow of $X$ and can be evaluated independently of the $T^*L$-part; the flow on the cotangent bundle factor is obtained by multiplying the fiber direction by a positive function $e^{G}$ that can be computed via

$$G(r,t) = \int_0^t g(\Phi^X_s (r, \vartheta, z)) \, ds = \int_0^t g(F(r,s)) \, ds.$$ 

If $T$ is chosen to be $T = \ln \frac{h}{\mu_0}$, it follows that $\Phi^{\tilde{X}}_T$ squeezes the first factor of $B(h) \times D_{<\mu_0}(T^*L)$ into $B(h')$. By Lemma 3.2 below, $G(r,t) < \ln(7/6)$ for any point in $B(h)$ and any $t \geq 0$. This implies as desired that the initial domain is squeezed into $B(h') \times D_{<\frac{7}{6}

\mu_0}(T^*L)$.

**Lemma 3.2.** The function $G(r,t)$ given in (3.1) is bounded from above by $\ln(7/6)$ for all $r \in [0, \pi + \delta]$ and all $t \in [0, \infty)$.

The sharp upper bound in the lemma is $\ln\left( \frac{2r_M \sin(r_M)}{\pi} \right)$ with $r_M$ specified in Lemma 2.1.

**Proof.** Denote the $r$-coefficient of the vector field $X$ by

$$f(r) = - \frac{r \cos(r) \sin(r)}{r + \cos(r) \sin(r)}.$$ 

Then $F(r,t)$ is the flow of the field $X_f(r) := f(r) \partial_r$, on $[0, \pi + \delta]$, that is, $F$ is the solution of the ordinary differential equation $\partial_t F(r,t) = f(F(r,t))$ with initial condition $F(r,0) = r$.

The only critical points of $X_f$ are the points $r \in \frac{\pi}{2} \mathbb{N}$; see also Fig. 1. Furthermore, recall that $r = \frac{\pi}{2}$ and $r = \frac{3\pi}{2}$ are repelling, and that $r = \pi$ is an attracting critical point.

According to Lemma 2.1, the function $g$ is everywhere on $[0, \pi + \delta]$ negative except for the interval $[r_m, r_M]$ with $r_m = \frac{\pi}{2}$, and $r_M \approx 2.03$ given by $r_M = - \tan r_M$. 

Since all trajectories of $X_r$ starting in $[0, \pi/2]$ are trapped inside this interval, the function $G(r, t)$ will be negative for all $r \in [0, \pi/2]$ and all $t \geq 0$. Similarly, the points in $[\pi, \pi + \delta]$ are pulled by the flow towards $r = \pi$ without ever crossing this point. Thus $G(r, t)$ will also be negative for all $r \in [\pi, \pi + \delta]$ and all $t \geq 0$.

It only remains to understand the behavior of $G(r, t)$ for $r \in (\pi/2, \pi)$. Since $f$ is strictly positive on this interval, it follows that, for every initial value $r \in (\pi/2, \pi)$,

$$F(r, \cdot) : \mathbb{R} \to (\pi/2, \pi)$$

is an orientation preserving diffeomorphism, and in particular there is a unique time $T_r \in \mathbb{R}$ such that $F(r, T_r) = r_M$.

For every fixed $r \in (\pi/2, r_M]$ and all positive $t \geq 0$, the upper bound of $G(r, t)$ in (3.1) is then given by

$$G(r, T_r) = \int_0^{T_r} g(F(r, s)) \, ds,$$

because $g$ is strictly positive up to $t = T_r$ so that $G(r, \cdot)$ increases up to that moment; for all later times $t > T_r$, the trajectory $F(r, t)$ lies in the zone $[r_M, \pi)$ where $g$ is negative so that $G(r, t) \leq G(r, T_r)$ for all $t \geq T_r$.

To compute $G(r, T_r)$ use that $F(r, \cdot) : \mathbb{R} \to (\pi/2, \pi)$ is for every choice of $r \in (\pi/2, \pi)$ a diffeomorphism, so that we can substitute $u = F(r, s)$ in the integral using that $\frac{du}{ds} = f(F(r, s)) = f(u)$, and obtain

$$G(r, T_r) = \int_0^{T_r} g(F(r, s)) \, ds = \int_{F(r, 0)}^{F(r, T_r)} \frac{g(u)}{f(u)} \, du = \int_r^{r_M} \frac{u \cos u + \sin u}{u \sin u} \, du = \ln(u \sin u) \bigg|_r^{r_M} = \ln \left( \frac{r_M \sin r_M}{r \sin r} \right).$$

The denominator $r \sin r$ is increasing on $[\pi/2, r_M]$ so that its smallest value on this interval is attained at $r = \pi/2$. We obtain the estimate

$$G(r, t) \leq \ln \left( \frac{r_M \sin r_M}{r \sin r} \right) < \ln \left( \frac{r_M \sin r_M}{\pi/2 \sin \pi/2} \right) = \ln \left( \frac{2r_M \sin r_M}{\pi} \right) < \ln(7/6)$$

finishing the proof. \hfill \Box

In particular, we obtain the following result.

**Corollary 3.3.** Let $L$ be a manifold that does not need to be closed, and let

$$\Sigma := \mathbb{D}_{\leq \pi}^2 \times \mathbb{D}_{< c_0} (T^* L)$$

be a hypersurface in a contact manifold $(M, \xi)$ such that the singular distribution induced by $\xi$ on the hypersurface agrees with the kernel of the 1-form $\beta = r \sin r \, d\vartheta + \lambda_{can}$, where $(r, \vartheta)$ denotes the polar coordinates on $\mathbb{D}_{\leq \pi}^2$.

Let $c > 0$ be such that $\mu_0 c < c_0$ with the constant $\mu_0 < 7/6$ in Proposition 3.1. Then we can embed the contact domain

$$\left( B(h) \times \mathbb{D}_{< c} (T^* L), \ker(\alpha_{OT} + \lambda_{can}) \right)$$

of “width” $c > 0$ and of any chosen “height” $h > 0$ into an arbitrarily small neighborhood of the hypersurface $\mathbb{D}_{\leq \pi}^2 \times \mathbb{D}_{< c_0} (T^* L)$.

**Proof.** The induced singular distribution of a hypersurface determines by Proposition 3.1 the germ of the contact structure on a neighborhood of the hypersurface. We can thus assume that $\Sigma$ has a neighborhood $U$ that is contactomorphic to $(B(\varepsilon) \times \mathbb{D}_{< c_0} (T^* L), \ker(\alpha_{OT} + \lambda_{can}))$ for small $\varepsilon > 0$. By Proposition 3.1 we can thus embed $B(h) \times \mathbb{D}_{< c} (T^* L)$ into $U$. \hfill \Box
APPENDIX A. THE LEGENDRIANUNKNOT IS LOOSE IN A SUFFICIENTLY LARGE OVERTWISTED CHART

By now, it is well-known that Legendrian unknots are loose in ambient manifolds containing a large neighborhood of an overtwisted chart. The argument was first sketched in [MNPS13], and Huang gave later a more detailed proof in [Hua17]. Unfortunately though he uses piecewise smooth Legendrians so that a lot of the potential clarity was lost. In this appendix, we take a new attempt to write down the proof. Once a certain 3-dimensional result is accepted as a black-box, we show that the main step to pass from dimension 3 to higher dimensions essentially boils down to a careful inspection of the original definition of looseness given by Murphy [Mur12].

Let \( \mathbb{R}^{2n+1}, \xi_0 = \ker(dz - \sum_{j=1}^{n} y_j dx_j) \) with coordinates \((x, y, z) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z)\) be the standard contact space. The Legendrian unknot \( \Lambda_0 \) in \( \mathbb{R}^{2n+1} \) can be given by the embedding

\[
S^n \hookrightarrow (\mathbb{R}^{2n+1}, \xi_0), \quad (x, s) \mapsto (x, -sx, s^3) .
\]

By extension, any Legendrian \( \Lambda \) in a contact manifold is called a Legendrian unknot if there exists a Darboux chart containing \( \Lambda \) in its interior such that \( \Lambda \) agrees in the chart with \( \Lambda_0 \).

Let \((M, \xi)\) be a contact manifold. We want to study Legendrians in \( M \) that look locally like product submanifolds in the following sense: Suppose that there is an open subset \( U \subset M \) that is diffeomorphic to \( U_M \times U_W \) where \( U_M \) is a manifold that carries a contact form \( \alpha \), and \( U_W \) is an open Liouville domain with Liouville form \( \lambda \) such that \( \xi|_U = \ker(\alpha + \lambda) \). Then assume that a Legendrian \( \Lambda \) satisfies

\[\Lambda \cap (U_M \times U_W) = L \times N ,\]

where \( L \) is a Legendrian in \((U_M, \alpha)\) and \( N \) is an exact Lagrangian in \((U_W, \lambda)\) with \( \lambda|_{TN} = 0 \).

The key notion we want to study in this appendix is due to Murphy [Mur12].

**Definition 1.** Let \( \Lambda \) be a Legendrian in \((M, \xi)\) that is locally in product form \( L \times Z_\rho \) in a chart \((C \times V_\rho, \ker(\alpha_0 + \lambda_{\text{can}}))\), where

- \( C = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \in (-1, 1)\}\) is a cube with side lengths 2 and standard contact form \( \alpha_0 = dz - dy dx \),
- \( V_\rho = \{(q, p) \in \mathbb{R}^{2n-2} | \|q\| < \rho, \|p\| < \rho\}\) with Liouville form \( \lambda_{\text{can}} = -\sum_j p_j dq_j \),
- \( L \) is a properly embedded Legendrian arc whose front is a zig-zag and which is equal to the set \{\(y = z = 0\)\} near the boundary,
- \( Z_\rho = \{(q, p) \in V_\rho | p = 0\}\).

We say that \( \Lambda \) is loose if \( \rho > 1 \).

**Remark A.1.** Note that if we replace the cube \( C \) in the definition above with any cube of side lengths smaller than 2, then it can be seen with the argument in [Mur12, Proposition 4.4] that the corresponding Legendrian is still loose.

The result we want to show in this appendix is the following proposition.

**Proposition A.2.** There exists a \( \rho_0 > 0 \) (that is independent of the dimension of \( V_\rho \)) such that the Legendrian unknot \( \Lambda_0 \) is loose in every contact manifold

\[(B(1) \times V_\rho, \ker(\alpha_{\text{OT}} + \lambda_{\text{can}}))\]

for which \( \rho > \rho_0 \). Here, \( \alpha_{\text{OT}} = \cos(r) dz + r \sin(r) d\theta \) denotes the standard overtwisted contact form on \( B(1) \), see also Section 2

**Proof.** By [Dym01] or [Etn12], see also the details in [MNPS13], the Legendrian unknot is in any overtwisted 3-manifold the stabilization of another Legendrian knot \( L_1 \). More precisely, let \((B(1), \alpha_{\text{OT}})\) be the cylindrical box surrounding an overtwisted disk described in Section 2 We can assume that there is a Darboux chart \( U_1 \) centered around a point of \( L_1 \) such that
• the restriction of \( \alpha_{\Omega} \) agrees in the coordinates of the Darboux chart with the standard form \( dz - y \, dx \),
• \( U_1 \) is a cube of size \( \varepsilon_1 < 1 \),
• \( L_1 \cap U_1 \) is the Legendrian arc \{ \( y = z = 0 \) \}.

We stabilize \( L_1 \) inside \( U_1 \) by adding a zig-zag. The resulting knot is then a Legendrian unknot \( L_0 \).

In particular, there exists a Darboux chart \( U_0 \) in \( B(1) \) such that \( L_0 \) lies in standard position \([A.1]\) inside \( U_0 \). The restriction \( \alpha_{\Omega} |_{U_0} \) with respect to the coordinates of \( U_0 \) will be of the form \( e^{f(x,y,z)} (dz - y \, dx) \) for some smooth function \( f \) that probably cannot be chosen to be equal to 0. Nonetheless, we can assume that there are constants \( c_0 > 0 \), and \( C_0 > 1 \) such that \( c_0 \leq e^f \leq C_0 \).

Using the Darboux chart \( U_0 \) in \( B(1) \), we can find in the product contact manifold

\[
(B(1) \times V_{\rho'}, \ker(\alpha_{\Omega} + \lambda_{\text{can}}))
\]

a higher dimensional Darboux chart \( U_0 \times V_{\rho'} \) for \( \rho' = \frac{e^f}{c_0} \) embedded by

\[
U_0 \times V_{\rho'} \to U_0 \times V_{\rho}, \,(x,y,z; q, p) \mapsto (x, y, z; q, e^{f(x,y,z)} \, p).
\]

If \( \rho' > 1 \), we can embed the Legendrian unknot \( \Lambda_0 \) into this Darboux chart using the standard map \([A.1]\)

\[
S^{n+1} \to (U_0 \times V_{\rho'}, \xi_0), \,(x_0, x, s) \mapsto (x_0, -sx_0, \frac{\varepsilon_3}{2}; x, -sx).
\]

Note that the intersection of \( \Lambda_0 \) with the slice \( B(1) \times \{0\} \) is precisely the unknot \( L_0 \) in \( U_0 \) that we had singled out in \( B(1) \).

By slightly deforming \( \Lambda_0 \) close to \( L_0 \times \{0\} \) we can assume that \( \Lambda_0 \) is locally in product form \( L_0 \times \mathbb{R}^\varepsilon \) for some small constant \( \varepsilon > 0 \). This can be easily seen using the front projection (even tough “seeing” the front projection requires from dimension 7 on some experience). More explicitly, let \( g: [0, 1] \to [0, 1] \) be a monotonous smooth function such that \( g \) is equal to 0 in a neighborhood of 0 and 1 in a neighborhood of 1. We can then consider the deformed sphere \( S^{n+1}_g \subset \mathbb{R}^{n+2} \) given by the equation

\[
x_0^2 + s^2 + g(x^2) \cdot x^2 = 1.
\]

Interpolating linearly between the equation of the round sphere and the deformed one, one sees that both are isotopic to each other.

We define a Legendrian embedding of \( S^{n+1}_g \) by

\[
S^{n+1}_g \to (U_0 \times V_{\rho'}, \xi_0), \,(x_0, x, s) \mapsto \left(x_0, -sx_0, \frac{\varepsilon_3}{2}; x, -sx \left(g(x^2) + x^2 g'(x^2)\right)\right).
\]

We denote this deformed sphere by \( \Lambda'_0 \). It is Legendrian isotopic to the initial Legendrian unknot, and it is composed of a cylindrical part \( L_0 \times \mathbb{R}^\varepsilon \) for small values of \( x \) in \( U_0 \times V_{\rho'} \).

Recall that the Darboux chart \( U_1 \times V_{\rho'} \) had been embedded into \( B(1) \times V_{\rho} \) via the map

\[
(x, y, z; q, p) \mapsto (x, y, z; q, e^{f(x,y,z)} \, p).
\]

By looking at the preimage of this embedding, it follows that \( \Lambda'_0 \) also has a cylindrical segment in \( B(1) \times V_{\rho} \). More explicitly, we have shown that after an isotopy the Legendrian unknot \( \Lambda_0 \) in \( B(1) \times V_{\rho} \) has a cylindrical part of the form \( L_0 \times \mathbb{R}^\varepsilon \) in the open ball \( B(1) \times V_{\rho} \) with \( \delta' = \delta/c_0 \).

We stretch out \( \Lambda'_0 \) in the \( V_{\rho} \)-direction of \( B(1) \times V_{\rho} \) using an isotopy of the form \( (x, y, z; q, p) \mapsto (x, y, z; e^t \, q, e^{-t} \, p) \) where the maximal size of \( t \geq 0 \) depends on the width of \( V_{\rho} \). If there is enough space, then we can suppose that the cylindrical part found above expands to be of the form \( L_0 \times \mathbb{R}^\eta \) in the open ball \( B(1) \times V_{\rho} \) with \( \eta > 1 \).

If we now consider the Darboux chart \( U_1 \subset B(1) \), we see that the intersection of the deformed Legendrian sphere with \( U_1 \times V_{\rho} \) is \( L \times \mathbb{R}^\eta \), where \( L \) is a Legendrian arc in \( U_1 \) whose front is a zig-zag, just as in the definition of looseness. If \( \eta > 1 \) and if \( U_1 \) is a cube of size smaller than 1 (see remark above following the definition of looseness), then the deformed unknot and thus also \( \Lambda_0 \) are loose.
Appendix B. Contact germ along a hypersurface

A folklore result states that a hypersurface in a contact manifold determines the germ of the contact structure surrounding it. Not having found a proof for dimension $> 3$ in the literature we have decided to add it here.

**Proposition B.1.** Let $(M_0, \xi_0)$ and $(M_1, \xi_1)$ be two $(2n + 1)$-dimensional contact manifolds, and let $\Sigma$ be a (not necessarily closed) $2n$-dimensional manifold. Assume that there are two embeddings
\[
i_0 : \Sigma \hookrightarrow M_0 \quad \text{and} \quad \iota_1 : \Sigma \hookrightarrow M_1
\]
such that the singular distributions $D_0 = (D_{i_0})^{-1}(\xi_0)$ and $D_1 = (D_{i_1})^{-1}(\xi_1)$ agree.

Then there exist a neighborhood $U_0 \subset M_0$ of $\iota_0(\Sigma)$, a neighborhood $U_1 \subset M_1$ of $\iota_1(\Sigma)$, and a contactomorphism
\[
\Phi : (U_0, \xi_0) \rightarrow (U_1, \xi_1)
\]
such that $\Phi \circ \iota_0 = \iota_1$.

**Remark B.2.** To be able to apply Proposition B.1 to a hypersurface $\Sigma$ with non-empty boundary, one needs to attach a small collar along $\partial \Sigma$, and extend the embeddings $\iota_0$ and $\iota_1$ in such a way that the singular distributions $D_0 = (D_{i_0})^{-1}(\xi_0)$ and $D_1 = (D_{i_1})^{-1}(\xi_1)$ agree.

We split the proof of Proposition B.1 into several lemmas. The first one is due to Giroux, but we learned about it from Mas08.

**Lemma B.3.** Let $\Sigma$ be a (not necessarily closed) manifold carrying a (cooriented) singular distributions $D$ that is given as the kernel of a 1-form $\beta$ such that $d\beta$ does not vanish at the singular points of $\Sigma$; i.e., at the points where $\beta = 0$.

If $\beta'$ is any other 1-form such that $D = \ker \beta'$ inducing the same coorientation, and such that $d\beta'$ does not vanish either at the singular points of $D$, then there exists a smooth positive function $f : \Sigma \rightarrow [0, \infty)$ such that
\[
\beta = f \cdot \beta'.
\]

**Proof.** Denote the set of all regular points of the distributions $D$ by $U_{\text{reg}} = \{ p \in \Sigma \mid D_p \neq T_p \Sigma \}$. On $U_{\text{reg}}$, we can simply define $f$ to be the quotient $\beta(X)/\beta'(X)$, where $X$ is any vector field on $U_{\text{reg}}$ that is transverse to $D$. We are thus left with studying the singular points $p \notin U_{\text{reg}}$ of $D$, where $\beta$ and $\beta'$ both vanish, and proving that $f$ extends to a non-vanishing smooth function.

Use a coordinate chart for $\Sigma$ centered at $p \in \Sigma \setminus U_{\text{reg}}$ with coordinates $x = (x_1, \ldots, x_n)$. We can then write
\[
\begin{align*}
\beta &= g_1 \, dx_1 + \cdots + g_n \, dx_n \\
\beta' &= g'_1 \, dx_1 + \cdots + g'_n \, dx_n
\end{align*}
\]
with functions $g_1, \ldots, g_n$ and $g'_1, \ldots, g'_n$ such that all $g_j$ and all $g'_j$ vanish at the origin. In fact for each $j$, the two functions $g_j$ and $g'_j$ vanish precisely on the same subset. By our assumption $d\beta' \neq 0$ at $p$, so that we can assume after possibly permuting the coordinates that $\frac{\partial g'_i}{\partial x_2}(0) \neq 0$.

We will now show that $f$ extends in the chart smoothly to a neighborhood of the origin such that $g_1(x) = f(x) \cdot g'_1(x)$. Note that $\{ x \mid g'_1(x) \neq 0 \}$ is a subset of $U_{\text{reg}}$ so that $f$ is a well-defined function on this subset. The condition $\frac{\partial g'_i}{\partial x_2}(0) \neq 0$ allows us to apply the implicit function theorem to find a new set of coordinates $y = (y_1, \ldots, y_n)$ for which $g'_1$ simplifies to $g'_1(y) = y_2$. In this new chart, we obtain that $f$ is defined in particular for all points $\{ y_2 \neq 0 \} \subset U_{\text{reg}}$.

Consider now the function $g_1$ represented with respect to the $y$-coordinates. It also vanishes precisely along the hyperplane $\{ y_2 = 0 \}$ so that there exists a smooth functions $\tilde{g}_1$ allowing us to write $g_1$ as
\[
g_1(y) = y_2 \, \tilde{g}_1(y),
\]
see for example Mil63 Lemma 2.1. Using this representation we see that $f(y) = \tilde{g}_1(y)$ extends to a smooth function on the whole chart so that it obviously satisfies the equation $g_1 = f \cdot g'_1$.

In particular, since $U_{\text{reg}}$ is dense in $\Sigma$ the continuous extension of $f$ is unique and does not depend on our choice of charts. This way, $f$ can be defined smoothly on all of $\Sigma$, and it satisfies $\beta = f \cdot \beta'$ on $\Sigma$. 

It remains to prove that $f$ does not vanish anywhere, but this is clear because if $f$ ever vanished at a singular point $p$ of $\mathcal{D}$, we would find from $\beta = f \cdot \beta'$ that $d\beta = 0$ at $p$ — contrary to our assumption that $d\beta \neq 0$ along the singular set of $\mathcal{D}$. □

Lemma B.4. Let $(M_0, \xi_0)$ and $(M_1, \xi_1)$ be two $(2n+1)$-dimensional contact manifolds with contact forms $\alpha_0$ and $\alpha_1$ respectively, and let $\Sigma$ be a (not necessarily closed) manifold of dimension $2n$.

Suppose that there are two embeddings

$$\iota_0: \Sigma \hookrightarrow (M_0, \xi_0) \text{ and } \iota_1: \Sigma \hookrightarrow (M_1, \xi_1)$$

of $\Sigma$ into $M_0$ and $M_1$ such that $\iota_0^*\alpha_0 = \iota_1^*\alpha_1$.

Then, there is a bundle isomorphism

$$\Phi: TM_0|_{\Sigma} \to TM_1|_{\Sigma}$$

such that:

(i) $\Phi|_{T\Sigma} = \text{id}_{T\Sigma}$ (We identify here, and in the rest of the proof, $T\Sigma$ with the tangent spaces of $\iota_0(\Sigma)$ and $\iota_1(\Sigma)$.)

(ii) $\alpha_0 \circ \Phi = \alpha_0$ on $TM_0|_{\Sigma}$.

(iii) The linear interpolation $(1 - \tau) \, d\alpha_0 + \tau \, (d\alpha_1 \circ \Phi)$ is for every $\tau \in [0,1]$ a symplectic form on $\xi_0|_{\Sigma}$.

Proof. Denote the 1-form $\iota_0^*\alpha_0 = \iota_1^*\alpha_1$ by $\beta$. To construct the desired bundle isomorphism $\Phi$, we distinguish two types of subsets of $\Sigma$: Define

$$U_{\text{reg}} = \{ p \in \Sigma \mid \beta_p \neq 0 \} \quad \text{and} \quad U_{\text{symp}} = \{ p \in \Sigma \mid d\beta^n_p \neq 0 \}.$$ 

Both sets are open and their union covers all of $\Sigma$, because $d\beta = \iota_0^*d\alpha_0$ is at every point $p \in \Sigma$ where $T_p\Sigma = \xi_j(p)$ a maximally non-degenerate form on $T_p\Sigma$; that is, $d\beta$ is a symplectic form on $T_p\Sigma$ at every point $p$ where $\beta$ vanishes.

We construct now separate bundle isomorphisms over $U_{\text{reg}}$ and over $U_{\text{symp}}$ that we then glue together using a partition of unity.

Over $U_{\text{symp}}$, we can decompose $TM_j$ as $T\Sigma \oplus \text{span}(R_j)$, where $R_j$ is the Reeb vector field of $\alpha_j$. This allows us to define a first bundle isomorphism

$$\Phi_{\text{symp}}: TM_0|_{U_{\text{symp}}} \to TM_1|_{U_{\text{symp}}}$$

by $\Phi_{\text{symp}}(v + cR_0) = v + cR_1$ for every $v \in T\Sigma|_{U_{\text{symp}}}$ and every $c \in \mathbb{R}$. It is easy to verify that $\alpha_0$ and $\alpha_1 \circ \Phi_{\text{symp}}$ agree on $TM_0|_{U_{\text{symp}}}$. Furthermore, $d\alpha_0$ and $d\alpha_1 \circ \Phi_{\text{symp}}$ also agree, because for any pair of vectors $v, v' \in T\Sigma$, we have $d\alpha_j(v, v') = d\beta(v, v')$ on one hand, and $d\alpha_j(R_j, v) = 0$ on the other, for both $j = 0, 1$.

To construct a bundle isomorphism over $U_{\text{reg}}$, we define the characteristic foliation $\mathcal{F}$ on $\Sigma$. It is characterized over $U_{\text{reg}}$ as the (singular) subdistribution of $\ker \beta = T\Sigma \cap \xi$ on which $d\beta|_{\ker \beta}$ vanishes. A dimension count shows that $\mathcal{F}$ is of dimension 1.

Choose a volume form $d\text{vol}_{\Sigma}$ on $\Sigma$, and let $X$ be the vector field determined by the equation

$$\iota_X d\text{vol}_{\Sigma} = \beta \wedge (d\beta)^{n-1}.$$ 

Since $X$ only vanishes at points where $\beta$ vanishes, it follows that $X$ is everywhere on $U_{\text{reg}}$ non-singular, and it is easy to convince oneself that the characteristic foliation is generated by $X$.

Choosing compatible complex structures $J_0$ on $(\xi_0, d\alpha_0)$ and $J_1$ on $(\xi_1, d\alpha_1)$, we define two vector fields $Y_0 = J_0 \cdot X$ and $Y_1 = J_1 \cdot X$ along $\iota_0(\Sigma)$ and $\iota_1(\Sigma)$ respectively. These vector fields are everywhere over $U_{\text{reg}}$ transverse to $\Sigma$ and they lie in the kernel of $\alpha_j$. This way, we can split the tangent bundles as

$$TM_j|_{U_{\text{reg}}} = T\Sigma|_{U_{\text{reg}}} \oplus \text{span}(Y_j)|_{U_{\text{reg}}}.$$ 

and use these decompositions to define the bundle isomorphism

$$\Phi_{\text{reg}}: TM_0|_{U_{\text{reg}}} \to TM_1|_{U_{\text{reg}}}.$$
by $\Phi_{\text{reg}} \left(v + c \gamma_0 \right) = v + c Y_1$ for every $v \in T \Sigma |_{U_{\text{reg}}}$ and every $c \in \mathbb{R}$. Again, we easily check that $\alpha_1 \circ \Phi_{\text{reg}}$ agrees on $TM_{\Sigma}|_{U_{\text{reg}}}$ with $\alpha_0$ so that $\Phi_{\text{reg}}(\xi_0|_{U_{\text{reg}}}) = \xi_1|_{U_{\text{reg}}}$. To understand the interpolation between $d \alpha_0$ and $d \alpha_1 \circ \Phi_{\text{reg}}$, choose at a point $p \in U_{\text{reg}}$ a basis of $\gamma_0(p)$ of the form $v_1, \ldots, v_{2n-2}, X(p), Y_0(p)$, where the $v_j$ all lie in $\ker \beta$ and are complementary to $X$. Assume $\beta$ is ordered in such a way that $d \alpha_0^{-1}(v_1, \ldots, v_{2n-2}) = d \beta^{-1}(v_1, \ldots, v_{2n-2}) = d \alpha_1^{-1}(v_1, \ldots, v_{2n-2}) > 0$. Note that $d \alpha_0(X, \cdot)$ and $d \alpha_1(X, \cdot)$ vanish on all the vectors $v_1, \ldots, v_2n-2, X$.

Define $d \alpha_\tau = (1 - \tau) d \alpha_0 + \tau (d \alpha_1 \circ \Phi_{\text{reg}})$ for any $\tau \in [0, 1]$. Then we compute for all $\tau \in [0, 1]$ that

$$d \alpha_\tau (v_1, \ldots, v_{2n-2}, X, Y_0) = n d \alpha_\tau (X, Y_0) \cdot d \alpha_\tau^{-1}(v_1, \ldots, v_{2n-2}) > 0,$$

because $d \alpha_\tau (X, Y_0) = (1 - \tau) d \alpha_0 (X, J_0 X) + \tau d \alpha_1 (X, J_1 X) > 0$.

We glue now $\Phi_{\text{reg}}$ and $\Phi_{\text{sym}}$ to produce a global bundle isomorphism. Choose a smooth function $\rho : \Sigma \to [0, 1]$ with support in $U_{\text{reg}}$ such that $1 - \rho$ has support in $U_{\text{sym}}$ so that $\rho$ and $1 - \rho$ form a partition of unity subordinate to $\{U_{\text{reg}}, U_{\text{sym}}\}$. Define a bundle homomorphism

$$\Phi : TM_{\Sigma} |_{\Sigma} \to TM_{\Sigma}$$

to produce a global bundle isomorphism. Choose a smooth

$$by\ mapping\ a\ vector\ v \in T_p M_{\Sigma}\ at\ a\ point\ p \in \Sigma\ to\ \Phi(v) = \rho(p) \cdot \Phi_{\text{reg}}(v) + (1 - \rho(p)) \cdot \Phi_{\text{sym}}(v).$$

It is obvious that $\Phi$ is a bundle homomorphism such that $\Phi|_{T \Sigma} = \text{id}_{T \Sigma}$ and such that $\alpha_0 = \alpha_1 \circ \Phi$ on $TM_{\Sigma}$ proving properties (i) and (ii) in the lemma.

It remains to verify property (iii). Define the interpolation $d \alpha_\tau := (1 - \tau) d \alpha_0 + \tau (d \alpha_1 \circ \Phi)$ for $\tau \in [0, 1]$. Since $\Phi$ agrees with $\Phi_{\text{sym}}$ at the points where $\beta = 0$, we obtain that $d \alpha_\tau = d \alpha_0$ is non-degenerate at any such point. We study now the desired property at points at which $\beta \neq 0$ and thus $X \neq 0$.

Since $d \alpha_\tau|_{T \Sigma} = d \beta$ is independent of $\tau$, we see that $d \alpha_\tau(X, \cdot)$ vanishes on every vector that lies in $\ker \beta$. Using the same basis chosen above with $Y_0 = J_0 X$ and $Y_1 = J_1 X$, it follows that the sign of $d \alpha_\tau^2(v_1, \ldots, v_{2n-2}, X, Y_0) = n d \alpha_\tau(X, Y_0) \cdot d \beta^{-1}(v_1, \ldots, v_{2n-2})$ only depends on the sign of the term $d \alpha_\tau(X, Y_0)$.

For this term we obtain $d \alpha_\tau(X, Y_0) = (1 - \tau) d \alpha_0(X, J_0 X) + \tau d \alpha_1(X, \Phi(Y_0))$. The first term is clearly positive, and for the second one write $d \alpha_1(X, \Phi(Y_0)) = \rho d \alpha_1(X, J_1 X) + (1 - \rho) d \alpha_1(X, \Phi_{\text{sym}}(Y_0))$, where the first term is again positive. Recall that $d \alpha_0 = d \alpha_1 \circ \Phi_{\text{sym}}$, so that we can simplify the second term as $d \alpha_1(X, \Phi_{\text{sym}}(Y_0)) = d \alpha_1(\Phi_{\text{sym}}(X), \Phi_{\text{sym}}(Y_0)) = (d \alpha_1 \circ \Phi_{\text{sym}})(X, Y_0) = d \alpha_0(X, Y_0)$. Thus $d \alpha_\tau(X, Y_0) = (1 - \tau) d \alpha_0(X, Y_0) + \tau (\rho d \alpha_1(X, Y_1) + (1 - \rho) d \alpha_0(X, Y_0))$ is positive as a convex combination of positive terms, and we have shown property (iii).

**Proof of Proposition 3.1** Let $\alpha_0$ be a contact form for $\xi_0$, and let $\alpha_1$ be a contact form for $\xi_1$. By Lemma [B.3] there is a smooth function $f : \Sigma \to \mathbb{R}_{>0}$ such that $\iota_0^* \alpha_0 = f \cdot \iota_1^* \alpha_1$. We would like to extend $f \circ \iota_1^{-1}$ to all of $M_1$ to normalize $\alpha_1$ globally; in general though, if $\Sigma$ is not closed, this might be impossible.

Denote the normal bundle of $\iota_1(\Sigma)$ in $M_1$ by $\nu_1 \Sigma \cong \Sigma$, and recall that there is a tubular neighborhood $U_1$ of $\iota_1(\Sigma)$ that is diffeomorphic to a neighborhood $V_1$ of the 0-section in $\nu_1 \Sigma$ (of course $V_1$ will generally not have uniform radius in the fiber directions, when $\Sigma$ is not closed). The function $f \circ \pi$ is a smooth positive function on $\nu_1 \Sigma$. We will replace $M_1$ by the open subset $U_1$, and use $f \circ \pi$ to rescale $\alpha_1$ on $U_1$ so that we can assume that $\iota_0^* \alpha_0 = \iota_1^* \alpha_1$. This allows us to apply Lemma [B.4] to obtain a bundle isomorphism $\Phi$ between $TM_0|_{\iota_0(\Sigma)}$ and $TM_1|_{\iota_1(\Sigma)}$.

Let $U_0$ be a tubular neighborhood of $\iota_0(\Sigma)$ in $M_0$ such that the exponential map $\exp_0$ (with respect to some Riemannian metric) defines a diffeomorphism $\exp_0 : V_0 \to U_0$, where $V_0$ is a neighborhood of the 0-section of the normal bundle of $\Sigma$ in $M_0$. Similarly, let $\exp_1$ be the exponential map on $M_1$. By suitably reducing the size of $U_0$ and $U_1$, we can assume that

$$\Psi := \exp_1 \circ \Phi \circ \exp_0^{-1} : U_0 \to U_1$$

is a diffeomorphism. To simplify our setup, pull-back $\alpha_1$ to $U_0$, and work in the fixed ambient manifold $U_0$. For simplicity we also write $\alpha_1$ for its pull-back. Then it follows that $U_0$ contains

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the submanifold $\Sigma$, and carries two contact structures given by contact forms $\alpha_0$ and $\alpha_1$ such that $\alpha_0$ and $\alpha_1$ agree at all points of $\Sigma$, and such that the linear interpolation of $d\alpha_0$ and $d\alpha_1$ is a path of symplectic structures on $\xi_0|_{\Sigma} = \xi_1|_{\Sigma}$.

The rest of the proof is a simple application of the Moser trick: Clearly the interpolation $\alpha_{\tau} := (1 - \tau) \alpha_0 + \tau \alpha_1$ satisfies along $\Sigma$ for every $\tau \in [0, 1]$ the contact condition. There is thus a small neighborhood of $\Sigma$ in $U_0$ on which all $\alpha_\tau$ are contact forms.

As in the standard proof of Gray stability, we define a vector field $X_{\tau}$ on this neighborhood by the equations

$$\alpha_{\tau}(X_{\tau}) = 0 \quad \text{and} \quad d\alpha_{\tau}(X_{\tau}, \cdot) = f_{\tau} \alpha_{\tau} - \dot{\alpha}_{\tau}$$

with $f_{\tau} := \alpha_{\tau}(R_{\tau})$, where $R_{\tau}$ is the Reeb field of $\alpha_{\tau}$. Note that the right hand side of the second equation vanishes along $\Sigma$, thus it follows that $X_{\tau}(p) = 0$ at every $p \in \Sigma$. By reducing to a smaller neighborhood of $\Sigma$ in $U_0$, the flow of $X_{\tau}$ will be defined up to time 1 giving a contact isotopy between $\xi_0$ and $\xi_1$.

Composing this isotopy with $\Psi$, we find the desired contactomorphism. □

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