EFFECT OF INFORMATION ON THE STRATEGIC BEHAVIOR OF CUSTOMERS IN A DISCRETE-TIME BULK SERVICE QUEUE

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(Communicated by Shoji Kasahara)

Abstract. We consider the equilibrium and socially optimal behavior of strategic customers in a discrete-time queue with bulk service. The service batch size varies from a single customer to a maximum of ‘b’ customers. We study the equilibrium and socially optimal balking strategies under two information policies: observable and unobservable. In the former policy, a service provider discloses the queue length information to arriving customers and conceals it in the latter policy. The effect of service batch size and other queueing parameters on the equilibrium strategies under both information policies are compared and illustrated with numerical experiments.

1. Introduction. The game theoretic study of queueing systems has emerged as an important area of study and is attracting several investigators since its inception in the late sixties by Naor [18] who studied an observable M/M/1 queue (Naor’s model). In Naor’s model, the queue length information is disclosed to the customers at their arrival epochs. Based on this information, customers make their decision whether to join or to balk depending on their accumulated net benefit. Few years later, Edelson and Hilderbrand [6] complemented Naor’s study by considering the corresponding unobservable M/M/1 system, where the arriving customers make their joining/balking decision without having the queue length information. In both the studies, a linear cost-reward structure is imposed on the queuing system under consideration. Customers act strategically and enter the queue only when their expected waiting cost is less than the reward received on service completion. This strategic behavior is studied using a game theoretic approach among the potential customers, and it helps to determine the corresponding equilibrium strategies in each case. The monographs of Hassin and Haviv [10] and Hassin [9] report various results in the economic analysis of several Markovian queueing systems. There is a vast literature on equilibrium balking strategies in continuous-time Markovian queueing systems and is increasing steadily in different directions such as server

2010 Mathematics Subject Classification. Primary: 60K25, 68M20; Secondary: 90B22.

Key words and phrases. Discrete-time queue, bulk service, strategic customers, equilibrium strategies, social optimization.
vacations [3, 22], Bernoulli schedule [13], customer impatience [19, 20, 26], catastrophes [1], breakdown and repair [5], group arrivals, different service policies and so on. Similar works in the continuous-time non-Markovian set up can be found in [4, 12, 27, 21] and the references therein.

Discrete-time queueing systems have been extensively studied in the past and are more challenging than their continuous-time counter parts due to their wide use in telecommunication networks, computer networks and digital communications ([24]). A complete study on discrete-time queueing models has been presented in [11]. Although the literature on equilibrium analysis of discrete-time queues with strategic customers is very less in compared to their continuous-time counterparts, there are some well known studies in several directions such as vacations [15, 23, 14, 7], breakdowns and repairs [25], pricing [16], comparison of information policies [8] etc.

Queuing systems with batch service deal with situations where the server serves a group of customers at a time. Such scenarios are frequently observed in real life applications, for example in transportation systems (shuttle service, cargo ships, airport trips, elevators), mass evacuation systems in medical emergency or natural calamities, manufacturing industry (bulk production, packaging, logistics services), digital communication (information-transmission systems handling messages, web browsing), inventory management, etc. There are several works on performance evaluation of bulk service queues, but rarely any work is done in the direction of economic analysis. Recently, one such work is presented in the continuous-time queues in [2]. To the best of our knowledge, there is no work on the discrete-time queueing model with batch service from a game theoretic viewpoint. In this work, the aim is to study the strategic behavior of customers in the framework of a single server discrete-time queue with batch service.

In this work, we analyze the individual and socially optimal equilibrium balking strategies in the discrete-time queue with batch services. Arriving customers are either informed or uninformed about the queue length and decide whether to join or to balk the queue based on a reward-cost structure which incorporates their desire for service as well as their unwillingness to wait. Two decision policies with respect to the available queue length information are explored in the following way: in one policy known as the observable, arriving customers obtain the queue length information on arrival whereas in the other policy called unobservable, they do not have the queue length information. The steady-state analysis of the corresponding systems are carried out and the Nash equilibrium solutions are derived. Some performance measures like the average queue length, average system length, mean waiting time and mean sojourn time, balking probability are computed. The equilibrium balking strategies for individual customers and the society as a whole is analyzed in each policy. The behavior of individual and social benefits under equilibrium with respect to different system parameters are compared against both information policies. Furthermore, we present several numerical experiments to explore the effect of several parameters on the equilibrium customer behavior.

The bulk service queueing model with strategic customers can be used effectively to study decentralized decision-making problems arising in the areas of digital communication like cognitive radio networks, wireless sensor networks and transportation services like ridesharing platforms, elevators or escalators at shopping malls and others. An example of the present model can be seen in the functioning of wireless sensor networks, which are used for remote monitoring of environmental conditions.
Wireless sensor networks usually consist of several sensor nodes (customers) that connect to a base station through another relay node or gateways (server). A sensor node consists of a radio transmitter, battery, micro-controller, analog circuit, and a sensor interface. Sensor nodes collect local information (data packets) and send them to the gateway. A gateway is assigned to several sensor nodes, and several gateways connect the base station. Data packets are stored in a single queue at the gateway in a first-come, first-served way. Further, the queue capacity may be small (finite) or large (infinite) depending on the implementation scenario. The gateway inspects the queue periodically (only in active mode) and sends the received data packet(s) to the base station for data aggregation and analysis. When sensor nodes collect and transmit data, they consume battery power and each transmission creates interference with other nodes in their gateway cluster. Each node as a decision maker tries to minimize the interference effect and conserve battery. Since the decisions are made by these players independently, there is a non-cooperative symmetric game among the players.

The main contributions of our study can be summarized as follows:

- We observe that the socially optimal strategy is unique in both information policies, but there are multiple equilibrium strategies in the unobservable policy.
- The informed customers’ benefit under equilibrium is a constant function of the arrivals whereas it alternates between zero and a positive value for different service rates. Also, the benefit is inversely proportional to the service rate.
- Social welfare always dominates the selfish benefits of customers when they are fully informed, and the dominance is reversed when they are uninformed. Thus, the observable policy is socially preferable by the service provider. In both of the information policies, the social benefit under equilibrium strategy increases with lower values of service batch sizes but remains fixed for higher values.
- In the observable policy, the system congestion increases with an increase in service batch size and attains optimum and then decreases with the further increase, whereas it is monotonically decreasing in the unobservable policy.
- Social benefit increases strictly for lower values of service rate and remains stagnant for higher values of service rate in both policies. But, the equilibrium threshold strategy is a linear function of the service rate.

This paper is organized as follows. Section 2 describes the system model. Sections 3 and 4 analyze the model and find the equilibrium and socially optimal strategies in observable and unobservable policies, respectively. Section 5 presents the equilibrium behavior of customers through numerical results. Section 6 concludes the paper.

2. Model description. We consider a discrete-time Markovian queue with infinite buffer capacity wherein the arrivals follow a Bernoulli process and a single server provide service to a group of customers. Let the time axis be divided into fixed size slots of equal intervals as 0, 1, ..., n, .... A potential arrival occurs in the slot \((n^-, n)\) and a potential batch departure takes place in the slot \((n, n^+)\). Even if a customer arrives in the interval \((n^-, n)\) and encounters an empty bulk service system, he has to wait until the time interval \((n, n^+)\) for his service, that is, the bulk service queueing model is considered as a late arrival system with delayed access (LAS-DA), see Hunter [11] for a complete information on the LAS-DA type
models. The various time epochs at which events take place are described in Fig. 1. The state of the bulk service system occurs only around the slot boundaries, that is, both individual arrivals and bulk departures are possible only at slot boundaries. Further, we assume that arrival within \((0^-,0)\) and departure in \((0,0^+)\) are not possible. At any time instant, the server may either be busy (serving a batch of size 1 to \(b\)) or idle (empty system). Upon arrival, if a customer encounters the server in an idle state, then the customer’s service starts immediately. On the other hand, if a customer encounters the server in a busy state, then the customer will wait in the queue. The queue discipline is the classical first-come first-served (FCFS). Customers arriving during the service of an incomplete batch are not allowed to join service even if there is space for them in the service batch. This is known as the non-accessible policy in the batch service system. The inter-arrival times are independent and geometrically distributed with probability mass function (p.m.f.)

\[
P(A = k) = \bar{\lambda}^{k-1} \lambda, \quad k \geq 1, 0 < \lambda < 1,
\]

where the random variable \(A\) is the generic of inter-arrival times and \(\bar{x} = 1 - x\) for any real number \(x \in [0, 1]\). Here, \(\lambda (\bar{\lambda})\) is the probability of an arrival (no arrival) in every slot. The service times \((S)\) are also independent and geometrically distributed with p.m.f.

\[
P(S = k) = \bar{\mu}^{k-1} \mu, \quad k \geq 1, 0 < \mu < 1,
\]

where \(\mu\) is the probability that a batch of customers depart in a slot and \(\bar{\mu}\) is the complementary probability with no customer departure in a slot. We consider that the server serves waiting customers in batches according to the general bulk service rule, that is, the maximum capacity of the server is \(b\). On completion of a batch service, if the server finds less than \(b\) customers waiting in the queue, then the server takes all of them in a batch for service. If the server finds more than \(b\) customers in the queue, then it takes a batch of a size \(b\) for service, while the remaining customers wait in the queue for their term. We assume that the service times can be initiated and completed only at the slot boundaries and the service time durations are integral multiples of slot lengths. Further, the service time of a batch is independent of the number of customers in that batch. Define \(\rho = \lambda \bar{\mu}\) as the maximum load of the queue, because the server can serve up to \(b\) customers simultaneously in an average service time duration \(\mu^{-1}\). The inter-arrival and bulk service times are considered to be mutually independent. The above bulk service discrete-time queueing system may be represented as a Geo/Geo\((1,b)\)/1 queue under the LAS-DA policy.

\[
\begin{array}{c}
A (\text{Potential arrival}) \\
D (\text{Potential departure}) \\
{(n+1)^-} \\
{(n+1)^+} \\
(n+, \ldots) \\
n^-: \text{Epoch prior to a potential arrival} \\
n^+: \text{Epoch after a potential departure} \\
(n+1)^+: \text{Outside observer’s interval} \\
\end{array}
\]

**Figure 1.** Various time epochs in a late-arrival system with delayed access (LAS-DA)

Let the random variables \(L_n\) and \(\zeta_n\) denote the number of customers in the queue and the number of customers in the service batch, respectively at time \(n^+\).
According to the assumptions in the LAS-DA model, a customer who finishes service and leaves at \( n^+ \) does not counted in \( L_n \) while one arrives at \( n^- \) is counted in \( L_n \). The derivation of the queue length process and the service batch size \( \{(L_n, \zeta_n)\} \) builds a discrete-time Markov chain (DTMC) with state space \( \Omega = \{(0,0), (i,j) : i \geq 0, j = 1,2,\ldots,b\} \). The state \((0,0)\) represents an empty system, that is, the queue is empty \( (L_n = 0) \) and the server is idle \( (\zeta_n = 0) \). The state \((i,j)\) represents the server is active with a service batch of size \( j = 1,2,\ldots,b \) and \( i \geq 0 \) number of customers waiting in the queue. The conditional probabilities of the state transitions from state \((i,j)\) to state \((k,l)\) are defined as \( P_{(i,j)(k,l)} = P(L_{n+1} = k, \zeta_{n+1} = l | L_n = i, \zeta_n = j) \) and the non-zero transition probabilities are

\[
\begin{align*}
P_{(0,0)(0,0)} &= \lambda, \\
P_{(0,0)(0,1)} &= \lambda, \\
P_{(0,1)(0,1)} &= \lambda \mu, \\
P_{(0,j)(0,j)} &= \lambda \mu, j = 2,\ldots,b, \\
P_{(i,j)(i,j)} &= \lambda \mu, i = 1,\ldots,j = 1,2,\ldots,b, \\
P_{(i,j)(0,1)} &= \lambda \mu, i = 0,1,\ldots,j = 1,2,\ldots,b, \\
P_{(i+b,j)(i,b)} &= \lambda \mu, i = 1,2,\ldots,j = 1,2,\ldots,b, \\
P_{(i,j)(i+1,j)} &= \lambda \mu, i = 0,1,\ldots,j = 1,2,\ldots,b, \\
P_{i,j)(0,i+1)} &= \lambda \mu, i = 0,1,\ldots,j = 1,2,\ldots,b, \\
P_{i,j)(i+1,j)} &= \lambda \mu, i = 0,1,\ldots,j = 1,2,\ldots,b,
\end{align*}
\]

The corresponding state transition diagram is illustrated in figure 2. Let \( P_{k,j}(n) \) denote the probability that the system is in state \((k,j)\) at time \( n^+ \) and is defined by \( P_{k,j}(n) = P(L_n = k, \zeta_n = j) \). Define the corresponding probabilities in steady-state as \( P_{k,j} = \lim_{n \to \infty} P_{k,j}(n), \) \( k \geq 0, \) \( 1 \leq j \leq b, \) and \( P_{0,0} = \lim_{n \to \infty} P_{0,0}(n). \)

\[\text{Figure 2. State transition diagram for the original model with maximum batch size } b\]
The problem associated in queueing systems with strategic customers is that an arriving customer decides whether to join or to balk depending on the information/knowledge about the queue length at the instant of arrival. We assume that a fixed reward of $R$ unit is gained by each customer after their service completion and at the same time they are charged a waiting cost of $C$ units per time unit to remain in the system. Using a linear cost-reward function, we study a customer’s expected net benefit, defined as $R - CE(W)$, where $E(W)$ represents the expected mean sojourn time of an arriving customer. Customers desire to maximize their expected net benefit by taking decisions only at their arrival instants. The decisions of customers are irrevocable in the sense that retrials of balking and reneging of entering customers are not allowed. We are interested in the equilibrium behavior of the strategic customers under different levels of information. We discuss the equilibrium balking strategies for individual customers and the optimal joining strategy for the society under two different policies such as observable and unobservable.

3. Observable Geo/Geo$^{(1,b)}/1$ queue. In this section, we consider an observable discrete-time bulk service queueing model in which arriving customers are informed about the number of customers in the queue. Based on this information, customers will decide only at their arrival instant whether to join or to balk the system. If a tagged customer encounters $L_n$ (> $b$) number of customers in the queue upon arrival in slot $n$ and decides to join, then his service will start after the completion of all complete service batches of size ‘$b$’. The possible positions of the tagged customer can be obtained from the functional relation $L_n = M_n b + J_n$, where $J_n \in \{0, 1, ..., b-1\}$ is the number of customers in the incomplete batch in slot $n$ and $M_n = \lfloor L_n/b \rfloor$ is the number of complete batch services possible in slot $n$ ($\lfloor \cdot \rfloor$ denotes the greatest integer function).

We are interested in the equilibrium solution of the model under individual and social optimization. Here, the (Nash) equilibrium solution is a pure strategy (join or balk) of threshold type, i.e., there exist a positive integer $n_e$ such that an arriving customer will join the system if and only if the queue length upon arrival is smaller than $n_e$. This pure threshold strategy is a dominant one in the sense that it is the best response against any other strategies. Alternatively it maximizes the customer’s net benefit irrespective of the strategies adopted by other customers. The socially optimal solution is also a pure threshold type strategy. Before the equilibrium threshold analysis, we need the steady-state solution of the underlying observable discrete-time bulk service queue.

The steady-state probabilities for the DTMC $\{(L_n, \zeta_n)\}$ can be defined similarly as in the original model, but with a changed state space $\Omega_o = \{(0,0), (i,j) : i = 0, 1, \ldots, n_e; j = 1, 2, \ldots, b\}$. The state transition diagram of the Markov chain is illustrated in Fig.3.

The steady-state queue length probabilities can be computed from the balance equations (1a) - (1b).

\[
P_{0,0} = \bar{\lambda} P_{0,0} + \bar{\lambda} \mu \sum_{j=1}^{b} P_{0,j}, \quad (1a)
\]

\[
P_{0,1} = \lambda P_{0,0} + \bar{\lambda} \mu P_{0,1} + \bar{\lambda} \mu \sum_{j=1}^{b} P_{1,j} + \lambda \mu \sum_{j=1}^{b} P_{0,j}, \quad (1b)
\]
In particular, taking different values of \( n \) from \( \{n - 1, n - 2, \ldots, n - b + 1\} \) and substituting the values obtained in the previous iteration and continuing the process, we will have

\[
P_{t,b} = (1 + x)^{n - i - 1} x P_{n,e,b}, \quad i = n_e - 1, n_e - 2, \ldots, n_e - b,
\]

where \( x = \frac{\mu}{\lambda} \). Separating the terms in (1g), we have

\[
P_{t-1,b} = (1 + x) P_{t,b}, \quad i = n_e - 1, n_e - 2, \ldots, n_e - b + 1.
\]
Rearranging the terms of (1f) and substituting the expressions evaluated before, we get
\[ P_{n_e-b-1,b} = \left( (1 + x)^b - \frac{1}{\mu} \right) x P_{n_e,b} - \frac{x}{\mu} \sum_{j=1}^{b-1} P_{n_e,j}. \]

From equation (1d), we have
\[ P_{i,j} = (1 + x)^{n_e-i-1} P_{n_e,j}, \quad 0 \leq i \leq n_e - 1, \quad 1 \leq j \leq b - 1. \]
Similarly for \( n = n_e - b - 2, n_e - b - 3, \ldots, 1, 0 \), equation (1e) reduces to
\[ P_{i-1,b} = (1 + x) P_{i,b} - x \left( \lambda P_{i+b,b} + \lambda P_{i-1+b,b} \right) - \frac{x^2}{\mu} (1 + x)^{n_e-b-1} \sum_{j=1}^{b-1} P_{n_e,j}. \]
From (1a), \( P_{0,0} \) can be evaluated as
\[ P_{0,0} = \frac{\bar{\lambda} \mu}{\lambda} \sum_{j=1}^{b-1} P_{0,j}. \]
Using (1b) and (1c), we have
\[ \sum_{j=1}^{b-1} P_{0,j} = \frac{\mu}{\mu + \lambda \bar{\mu}} \left[ ((1 + x)^{n_e} - (1 + x)^{n_e-b}(1 + \lambda x)) \sum_{k=1}^{b-1} P_{n_e,k} + \sum_{j=0}^{b-2} P_{j,b} + \bar{\lambda} P_{0-1,b} \right]. \]
The remaining probability \( P_{n_e,b} \) can be found from the normalization condition.

**Computational Algorithm:**
**Step 1:** For \( i = n_e - 1, \ldots, 1, 0 \), calculate \( P_{i,j} \) in terms of \( P_{n_e,j} \), \( 1 \leq j \leq b - 1 \) as follows
\[ P_{i,j} = x(1 + x)^{n_e-i-1} P_{n_e,j}. \]
**Step 2:** For \( i = n_e, n_e - 1, \ldots, 1, 0 \), calculate \( P_{i,b} \) in terms of \( P_{n_e,b} \) as follows
\[ P_{i,b} = \psi_i P_{n_e,b} + \xi_i \sum_{j=1}^{b-1} P_{n_e,j}, \]
where \( \psi_i \) and \( \xi_i \) are
\[ \psi_i = \begin{cases} 1, & \text{for } i = n_e, \\ x(1 + x)^{n_e-i-1}, & \text{for } i = n_e - 1, \ldots, n_e - b, \\ x(1 + x)^b - \frac{x}{\mu}, & \text{for } i = n_e - b - 1, \\ (1 + x)\psi_{i+1} - x \left( \lambda \psi_{i+1+b} + \lambda \psi_{i+b} \right), & \text{for } i = n_e - b - 2, \ldots, 1, 0. \end{cases} \]
\[ \xi_i = \begin{cases} 0, & \text{for } i = n_e, \ldots, n_e - b, \\ -\frac{x}{\mu}, & \text{for } i = n_e - b - 1, \\ (1 + x)\xi_{i+1} - x \left( \lambda \xi_{i+1+b} + \lambda \xi_{i+b} \right) - \frac{x^2}{\mu} (1 + x)^{n_e-b-i}, & \text{for } i = n_e - b - 2, \ldots, 0. \end{cases} \]
**Step 3:** Compute \( P_{0,0} \) in terms of \( P_{n_e,b} \) as
\[ P_{0,0} = \frac{\bar{\lambda} \mu}{\lambda} \left( x(1 + x)^{n_e-1} + \xi_0 \right) \sum_{j=1}^{b-1} P_{n_e,j} + \frac{\bar{\lambda} \mu}{\lambda} \psi_0 P_{n_e,b}. \]
Step 4: Compute \( \sum_{j=1}^{b-1} P_{n_e,j} \) in terms of \( P_{n_e,b} \) as follows

\[
\sum_{j=1}^{b-1} P_{n_e,j} = TP_{n_e,b},
\]

where \( T = \frac{\mu (\psi_n + (1+x)^n (1+x)\sum_{k=1}^{n-1} \xi_k - T\xi_{b,k-1})}{\mu + \mu (1-(1+x)^n (1+x)\sum_{k=1}^{n-1} \xi_k - T\xi_{b,k-1})} \).

Step 5: For \( i = 0, 1, \ldots, n_e \), calculate \( P_{i,b} \) in terms of \( P_{n_e,b} \) as

\[
P_{i,b} = (\psi_i + T\xi_i)P_{n_e,b}.
\]

Step 6: Using normalization condition, determine \( P_{n_e,b} \) as

\[
P_{n_e,b} = \left[ \frac{\lambda\mu}{\lambda}((x(1+x)^{n_e-1} + \xi_0)T + \psi_0) + (1+x)^{n_e}T + (1+x)^b \right]^{n_e-b-1}
+ \sum_{i=0}^{n_e-b-1} P_{0,j}(\psi_i + T\xi_i).\]

Thus, all the stationary probabilities \( P_{0,0}, P_{i,j}, i = 0, \ldots, n_e, \) and \( j = 1, \ldots, b \) are computed. When an arrival finds \( n_e \) customers in the queue, it will balk as the equilibrium threshold is reached. Thus, the probability of balking is \( \sum_{j=1}^{b} P_{n_e,j} \) and the effective arrival rate to the system in equilibrium is \( \lambda (1 - \sum_{j=1}^{b} P_{n_e,j}) \). The probability that an arrival encounters the busy server is given by \( \sum_{i=0}^{n_e} \sum_{j=1}^{b} P_{i,j} \).

The average number of customers in the queue, \( E(L_q) \), and in the system, \( E(L) \) are

\[
E(L_q) = \sum_{i=1}^{n_e} \sum_{j=1}^{b} iP_{i,j}, 
E(L) = \sum_{i=0}^{n_e} \sum_{j=1}^{b} (i+j)P_{i,j}.
\]

3.1. Equilibrium joining strategy. In the observable queue, customers are informed about the number of waiting customers upon arrivals and their decision whether to join or to balk will depend on their net benefit which is the difference between the reward to be received after service completion and the total cost of waiting in the system to be paid to the service provider. If \( R \) is the service reward and \( C \) is the waiting cost per unit time for a tagged customer, then his net benefit will be \( \Delta_n = R - CT(n) \), where \( T(n) \) is the conditional mean sojourn time of the tagged customer who finds \( n \) customers in queue. There are two possibilities depending on the value of \( n \). If \( n < b \), then the customer has to wait for the completion of the remaining batch service and his own batch service time. As the geometric distribution has the memoryless property, the remaining batch service time is also geometrically distributed with same parameter \( \mu \). Therefore, \( T(n) = 1/\mu + 1/\mu \).

As the service is rendered in batches of maximum size \( b \), then the above situation is equivalent to have an incomplete batch in the queue. On the other hand if the tagged one finds \( n \geq b \) customers in the queue upon arrival and decides to join, then his sojourn time is the sum of the remaining batch service time, all complete batch service times and the service time of his own batch. If we consider \( n = kb + j \) with \( k = \lfloor n/b \rfloor > 0 \) being the number of complete batches in queue and \( 0 \leq j \leq b-1 \) is the incomplete batch, then \( T(n) = 1/\mu + k/\mu + 1/\mu \). It is observed from both policies that the conditional mean sojourn time is a function of the number of complete batches, say \( k \), in queue. Hence

\[
T(n) = T_b(k) = \frac{k + 2}{\mu}, \ k = 0, 1, 2, \ldots \text{ and } k = \left\lfloor \frac{n}{b} \right\rfloor.
\]
is a more convenient method to study the equilibrium threshold strategies as compared to the number in queue, \( T(n) \). Here, \( T_b(k) \) denotes the conditional mean sojourn time of the tagged customer who finds \( k \) complete batches of customers in the queue. If a tagged customer encounters an empty system upon arrival, then his sojourn time is equal to his service time, \( 1/\mu \). Therefore the conditional mean sojourn time \( T_0(0) \) is given by

\[
T_0(0) = \frac{1}{\mu}.
\]

We consider the net benefit of a customer who finds an empty system to be positive, otherwise no one will join an empty system. Hence, the assumption \( R\mu > C \) is made for equilibrium analysis. In the observable model, the equilibrium strategies are of threshold type with respect to \( k \), that is customers observing a smaller \( k \) are more willing to join compare to the customers finding higher \( k \) in queue. Customers who wish to maximize their net benefit will follow a pure threshold strategy, that is, there exists a positive integer \( k_e \) such that the future arriving customers will join the queue if the number of complete batches in the queue is smaller than \( k_e \). When arriving customers observe at least \( k_e \) complete batches in queue, they prefer to balk as their net benefit becomes negative. The unique pure equilibrium strategy \( k_e \) can be obtained by solving the following inequalities.

\[
\Delta(k - 1) = R - C \frac{k + 1}{\mu} \geq 0
\]
\[
\Delta(k) = R - C \frac{k + 2}{\mu} < 0
\]

Solving the above inequalities, we get

\[
k + 1 \leq \frac{R\mu}{C} < k + 2,
\]

or equivalently,

\[
k_e = \left\lfloor \frac{R\mu}{C} - 1 \right\rfloor. \tag{2}
\]

Under this equilibrium strategy, arriving customers balk the queue if they observe \( k_e \) or more number of complete batches in the queue otherwise they will prefer to join the queue. By the assumption \( R\mu \geq C \), the threshold \( k_e \) is non-negative. Here, the equilibrium balking threshold \( k_e \) depends on the batch service rate, waiting cost and service completion reward, but not on the arrival rate. In terms of the number of customers in the queue, the equilibrium balking threshold is \( n_e = \left\lfloor \frac{n_e}{b} \right\rfloor \) or \( n_e = \min_{j \in J_n} \{k_e b + j\} = k_e b \). Hence, arriving customers decide to balk if they find \( n_e \) or more in the queue and will join if the queue length is at most \( n_e - 1 \).

**Remark 1.** In Naor’s model, arriving customers will join the queue if they find at most \( n_s - 1 \) customers in the system and balk if they observe at least \( n_s \) customers in the system, where \( n_s = \left\lfloor \frac{R\mu}{C} \right\rfloor \) is the equilibrium balking threshold. But, in the batch service model, the equilibrium balking threshold is \( n_e = \left\lfloor \frac{R\mu}{C} - 1 \right\rfloor b \). In Naor’s model, the expected net benefit of a tagged customer who enters the system is a linearly decreasing function of his position in the queue while joining. In our model, the expected net benefit is a non-increasing function because of the batch service rule. That is, regardless of their position within a service batch, the customers that have the same waiting time will experience equal expected net benefit. In [2], customers upon arrival receive information about the number of complete batches and the
number of customers in an incomplete batch (in the observable policy). There exist multiple equilibrium strategies of threshold type that depend on the number of customers in the incomplete batch. The thresholds concerning the incomplete batch \( m_{K-1} = \left\lfloor \frac{R}{C} - 1 \right\rfloor \) gradually decrease with a decrease in the number of customers in the incomplete batch. In the present study, the equilibrium threshold strategy depends on the number of complete batches and not on the number of waiting customers in any incomplete batch.

3.2. Socially optimal joining strategy. In the case of social optimization, we are interested to find a threshold which maximizes the overall social benefit of the system. Due to the BASTA property, the probability that an arrival encounters the system on state \((n_e, j)\) and balks is \( \sum_{j=1}^{b} P_{n_e, j} \). Hence the social benefit per time unit when all present customers follow the equilibrium threshold policy “While arriving at time \( n \), enter if \( k < k_e \) and balk otherwise” is given by

\[
\Delta_s(n_e) = \lambda(1 - \sum_{j=1}^{b} P_{n_e, j}) \cdot R - C \cdot E(L).
\]

The expected total profit under some strategy \( n \) is given by

\[
\Delta_s(n) = \lambda(1 - \sum_{j=1}^{b} P_{n, j}) R - C \cdot E(L).
\]

The analytical expression of the socially optimal strategy is more complex and difficult to write in closed form, but it can be computed numerically.

4. Unobservable Geo/Geo\(^{(1,b)}\)/1 queue. In the unobservable queues, customers don’t know the system state upon arrival and follow a mixed strategy as the probability of joining \( f \). If all the customers follow the same joining strategy \( f \), the evolution of the queue length, \( L_n \) and the service batch size, \( \zeta_n \) at an arbitrary slot \( n \) forms a discrete-time Markov chain \( \{ (L_n, \zeta_n) \} \) with state space \( \Omega_u = \{(0,0),(n,j) : n = 0,1,... ; j = 1,2,...,b \} \). The state transition diagram is illustrated in Fig. 4. the underlying model is similar to the original model with customer arrival rate \( \lambda f \). For the steady-state analysis, we assume the traffic intensity \( \rho = \lambda f / b \mu \) to be smaller than unity otherwise the system will be overcrowded leading to infinite wait. The stationary queue length distributions can be obtained by solving the following set of balance equations.

\[
P_{0,0} = \lambda_f P_{0,0} + \lambda_f \mu \sum_{j=1}^{b} P_{0,j}, \quad (3a)
\]

\[
P_{0,1} = \lambda f P_{0,0} + \lambda_f \mu P_{0,1} + \lambda_f \mu \sum_{j=1}^{b} P_{1,j} + \lambda_f \mu \sum_{j=1}^{b} P_{0,j}, \quad (3b)
\]

\[
P_{0,j} = \lambda_f \mu P_{0,j} + \lambda_f \mu \sum_{k=1}^{b} P_{j,k} + \lambda_f \mu \sum_{k=1}^{b} P_{j-1,k}, \quad j = 2,\ldots,b, \quad (3c)
\]

\[
P_{i,j} = \lambda f \mu P_{i-1,j} + \lambda_f \mu P_{i,j}, \quad i \geq 1, \quad j = 1,2,\ldots,b-1, \quad (3d)
\]

\[
P_{i,b} = \lambda f \mu P_{i-1,b} + \lambda_f \mu P_{i,b} + \lambda_f \mu \sum_{j=1}^{b} P_{i+b,j} + \lambda_f \mu \sum_{j=1}^{b} P_{i+b-1,j}, \quad i \geq 1, \quad (3e)
\]
The system of homogeneous linear difference equations with constant coefficients (3d), can be rearranged to

\[ P_{i,j} = \frac{\lambda f \bar{\mu}}{1 - \lambda f \bar{\mu}} P_{i-1,j}, \text{ for } i \geq 1; j = 1, 2, \ldots, b - 1, \]

which on successive recursion for \( i \) reduces to

\[ P_{i,j} = y^i P_{0,j}, \text{ for } i \geq 1; j = 1, 2, \ldots, b - 1, \]

where \( y = \frac{\lambda f \bar{\mu}}{\mu + \lambda f \bar{\mu}} \). To find the solution to the system of non-homogeneous linear difference equation with constant coefficients (3e), we rewrite the system of equations using the displacement operator \( D \) defined by \( DP_{i,b} = P_{i+1,b} \), as the following

\[ P_{i-1,b} \left[ D - (\lambda f + \lambda f D)(\bar{\mu} + \mu D^b) \right] = \lambda \lambda f \mu \sum_{j=1}^{b-1} y^{i+b} P_{0,j} + \lambda f \mu \sum_{j=1}^{b-1} y^{i+b-1} P_{0,j}, \text{ for } i \geq 1, \]

\[ P_{i,b} \left[ D - (\lambda f + \lambda f D)(\bar{\mu} + \mu D^b) \right] = \mu(\lambda f y + \lambda f) y^{i+b} \sum_{j=1}^{b-1} P_{0,j}, \text{ for } i \geq 0. \]

The solution to the corresponding homogeneous systems

\[ P_{i,b} \left[ D - (\lambda f + \lambda f D)(\bar{\mu} + \mu D^b) \right] = 0, \text{ for } i \geq 0, \]

can be obtained in terms of the roots of the characteristic equation

\[ K(z) = z - (\lambda f + \lambda f z)(\bar{\mu} + \mu z^b) = 0. \]

There are \( b + 1 \) characteristic roots including \( z = 1 \). Using Rouche’s theorem, it can be proved that the characteristic equation has a unique root inside the unit circle and the remaining \( b - 1 \) roots outside the unit circle. Further the unique root is real if and only if \( \rho < 1 \) (Medhi [17]). Denoting the unique root inside the unit circle by

Figure 4. State transition diagram for the unobservable batch service queueing model with maximum batch size \( b \)
r and the outside roots by \( r_i, \ i = 1, 2, \ldots, b - 1 \), the solution of the homogeneous system can be written as

\[
P_{i,b}^{(h)} = A_0 r^i + \sum_{n=1}^{b-1} A_n r_n^i, \ i \geq 0.
\]  
(6)

Since, \( P_{i,b}^{(h)} \)'s are probabilities, hence the constants \( A_n = 0, \ n = 1, 2, \ldots, b - 1 \). Therefore, equation (6) becomes

\[
P_{i,b}^{(h)} = A_0 r^i, \ i \geq 0.
\]  
(7)

The solution to the non-homogeneous systems (4) can be obtained using the method of undetermined coefficients

\[
P_{i,b}^{(p)} = \frac{\mu y^{b+1} \sum_{j=1}^{b-1} P_{0,j} y^j}{\mu K(y)} = -y^j \sum_{j=1}^{b-1} P_{0,j}, \ i \geq 0.
\]  
(8)

The general solution to the system (3e) is

\[
P_{i,b} = A_0 r^i - y^j \sum_{j=1}^{b-1} P_{0,j}, \ i \geq 0.
\]  
(9)

Summing the equations (3b) and (3c) and using (3a), we get

\[
B_0 = \frac{\mu (1 - r^{b-1}(\lambda f r + \lambda f))}{(1 - r)(\mu + \lambda f \mu)}.
\]  
(10)

Using the above expression in (9), we get

\[
P_{i,b} = A_0 (r^i - B_0 y^j), \ i \geq 0,
\]  
(11)

where \( B_0 = \frac{\mu (1 - r^{b-1}(\lambda f r + \lambda f))}{(1 - r)(\mu + \lambda f \mu)} \). Substituting the values of \( P_{i,j} \) and \( P_{i,b} \) in (3a), we obtain

\[
P_{0,0} = \frac{\lambda f \mu}{\lambda f} A_0
\]  
(12)

Again, substituting the values of \( P_{0,0} \), \( P_{1,j} \), \( P_{i,b} \) in (3b) and (3c), we obtain

\[
P_{0,1} = A_0 \frac{\mu (1 + \lambda f r)}{\mu + \lambda f \mu},
\]  
(13)

\[
P_{0,j} = A_0 \frac{\mu (\lambda f + \lambda f r) r^{j-1}}{\mu + \lambda f \mu}, \ j = 2, \ldots, b - 1.
\]  
(14)

Now all the stationary probabilities are expressed in terms of the only unknown \( A_0 \). The expression of \( A_0 \) can be evaluated using the normalization condition \( P_{0,0} + \sum_{i=1}^{\infty} \sum_{j=1}^{b} P_{i,j} = 1 \) as

\[
A_0 = \frac{\lambda f (1 - r)}{\lambda f + \lambda f \mu (1 - r)}.
\]

The average number of customers waiting in the queue is

\[
E(L_q) = \sum_{i=1}^{\infty} \sum_{j=1}^{b} i P_{i,j} = \frac{A_0 r}{(1 - r)^2} = \frac{\lambda f r}{(1 - r)(\lambda f + \lambda f \mu (1 - r))}.
\]
The average number of customers in the system is
\[
E(L) = \sum_{i=0}^{\infty} \sum_{j=1}^{b} (i + j) P_{i,j} = \frac{\lambda f r}{(1 - r)(\lambda f \mu (1 - r) + \lambda f) + \lambda f m}.
\]

Hence, the mean sojourn time of a customer who makes up mind to join upon his/her arrival can be incurred by using Littles law:
\[
E(W) = \frac{r}{(1 - r)(\lambda f \mu (1 - r) + \lambda f) + \frac{1}{\mu}}.
\]

And if \( \lambda f < h \mu \), \( E(W) \) is strictly increasing for \( f \in [0, 1] \). The first and second derivatives of \( r \) with respect to \( f \) are calculated from (5) after replacing \( z \) by \( r \) and are given as
\[
r' = \frac{\lambda (1 - r)(\mu + \mu r^b)}{\lambda f (\mu + \mu r^b) + \mu (1 - r^b) - \mu br^{b-1}(r + \lambda f (1 - r))}.
\]

It can be easily checked that the first and second derivatives of \( E(W) \) with respect to \( f \) are positive. Hence, \( E(W) \) is a strictly convex function of \( f \).

4.1. **Equilibrium joining strategy.** A tagged customer upon arrival will join the system with probability \( f \) if his reward for service completion is greater than his waiting cost, that is
\[
\Delta_u(f) = R - CE(W) = R - C \left( \frac{r}{(1 - r)(\lambda f \mu (1 - r) + \lambda f) + \frac{1}{\mu}} \right) \geq 0. \tag{15}
\]

As the sojourn time is a strictly convex function, it attains a unique global minimum at a point say, \( f^* \) in its domain and the minimum value will be \( E(W^*) \). The evaluation of equilibrium strategies will depend on the value of the net benefit evaluated at this global minimum. If it is negative then all customers will balk otherwise they will join the system following a mixed strategy, that is, the arriving customers will choose the pure strategies (joining or balking) with a fixed probability. Let \( f_e \) denote the equilibrium joining probability and \( \lambda f_e \) the corresponding effective arrival rate.

**Case 1.** \( \frac{R}{C} < E(W^*) \).
In this case, if all customers balk, then the tagged customer who decides to enter has negative benefit, \( \Delta_{au}(f) \leq 0 \). Hence, \( f_e = 1 \) can not be an equilibrium strategy. If all other customers balk, then the tagged customer is benefited by his balking decision. Hence, \( f_e = 0 \) is an equilibrium strategy.

**Case 2.** \( \frac{R}{C} = E(W^*) \).
In this case, if all customers decide to balk, then the tagged customer is benefited by joining the system as he will get immediate service. Hence, \( f_e = 0 \) is not an equilibrium strategy. If all customers join, then the tagged customer who decides to enter has positive benefit, \( \Delta_{au}(f) \geq 0 \). Hence, \( f_e = 1 \) is an equilibrium strategy. Here, joining is a dominant strategy. Alternatively, if all customers join with probability \( f^* \), then the tagged customer is benefited by joining with the same probability and \( f_e = f^* \) is another equilibrium strategy if \( f^* < 1 \). Thus, multiple mixed strategies occur for \( f^* \leq 1 \).

**Case 3.** \( \frac{R}{C} > E(W^*) \).
In this case, if all customers join, then the tagged customer who decides to enter has a positive benefit. Hence, \( f_e = 1 \) is an equilibrium strategy. On the other hand, if all customers balk, then the tagged customer by entering the
system has a positive benefit. Hence, \( f_c = 0 \) is not an equilibrium strategy. A proper mixed strategy \( f_c \in (0, 1) \) is an equilibrium strategy if and only if \( \Delta_u(f) = 0 \). Due to the strict convexity of \( E(W) \), there exists exactly two solutions to \( \Delta_u(f) = 0 \). Let the two solutions be \( f_1 \) and \( f_2 \) with \( f_1 < f_c < f_2 \). If \( f_2 < 1 \), then there exists multiple proper equilibrium strategies \( f_1 \) and \( f_2 \). If \( f_1 > 1 \), then there is no proper equilibrium strategy.

4.2. Socially optimal joining strategy. In this case, we are interested to find a joining probability which maximizes the social welfare of the system. When all customers follow the same equilibrium mixed strategy “While arriving in slot \( n \), enter with probability \( f \) and balk with probability \( 1 - f \)”, the social welfare per time unit is given by

\[
\Delta_{us}(f) = \lambda f R - C \left( \frac{\lambda f r}{(1-r)(\lambda f \mu (1-r) + \lambda f)} + \frac{\lambda f}{\mu} \right),
\]

with social welfare becoming zero for an empty system, that is, all customers prefer to balk. The first and second derivatives of the social benefit can be calculated as

\[
\Delta'_{us} = \lambda R - \lambda C E(W) - \lambda f C E(W)',
\]
\[
\Delta''_{us} = -2 C E(W)' - \lambda f C E(W)''.
\]

Since \( E(W)' \) and \( E(W)'' \) are always positive, \( \Delta''_{us} \) is negative for \( 0 \leq f \leq 1 \). Hence, \( \Delta_{us}(f) \) is strictly concave in \([0, 1]\) and has a unique global maximum, say \( f^* \in [0, 1] \). When all customers follow the same strategy, the socially optimal joining strategy, \( f_s \) can be calculated from (16) under the following policies.

Case 1. If \( \frac{R}{C} < E(W_s) \), then \( \Delta_{us}(f) \leq 0, \forall f \in [0, 1] \). In particular, the maximum value is attained for \( f = 0 \), that is, \( \Delta_{us}(f) = 0 \) and this is the unique point of global maximum. Hence, there exists a unique socially optimal strategy \( f_s = 0 \).

Case 2. If \( \frac{R}{C} = E(W_s) \), then the roots of \( \Delta_{us}(f) = 0 \) are \( f = 0 \) and \( f = f_s \). Using the second derivative test (17), these two points are found to be the global maximum points. Hence, there exists two socially optimal strategies \( f_s = 0 \) and \( f_s = f_s \). Thus, multiple socially optimal strategies occur in this case.

Case 3. If \( \frac{R}{C} > E(W_s) \), then \( \Delta_{us}(f) > 0 \). The global maximum of \( \Delta_{us}(f) \) will be attained for some \( f \in (0, 1] \). Since, \( \Delta_{us}(f) \) is a strictly concave function of the joining probability \( f \), it has unique maximum point. Therefore, \( \Delta_{us}(f) \) attains its maximum at \( f = f^* \) and is strictly decreasing in \([f^*, 1]\) and strictly
increasing for $[0, f^*]$. Hence, there exists a unique socially optimal strategy $f_s = f^*$.

**Remark 2.** In [2], the mixed strategy $f = 0$ of balking is always an equilibrium strategy whereas in our case it is not. This is due to the general bulk service rule $(1, b)$. But the social optimal strategies remains unchanged. Under the restriction $R\mu - C > R\mu r^b$, the socially optimal strategy is unique, but there are multiple equilibrium strategies.

5. **Numerical results.** In this section, some numerical experiments are discussed to study the behavioral change of strategic customers on varying parameters of the batch service queueing model under two different information policies explained above. The effect of service batch size, average arrival rate and mean service rate on the equilibrium and social benefit is studied under both information policies. A comparison study between the equilibrium benefits and social benefits for the observable as well as unobservable policies is presented for different queueing parameters. Further, the variation of the threshold policy in the observable policy is discussed for different parameters. The interest lies in the selection of appropriate model parameters that will help the system administrator to optimize the benefit by controlling the information shared with the arriving customers.

![Figure 6](image_url) **Figure 6.** Effect of customer arrivals on the benefit function under different information policies with parameters $\mu = 0.15, b = 10, R = 30, C = 1$.

![Figure 7](image_url) **Figure 7.** Effect of service rate on the benefit function under different information policies with parameters $\lambda = 0.75, b = 10, R = 30, C = 1$.

The main concern of the system administrator in strategic queueing games is to control the level of information (either to reveal or to conceal information from the customers) for optimal use of the system components and at the same time the system revenue. In the first set of numerical experiments, we have investigated the dependence of equilibrium benefit and social benefit per time unit on the service rate, arrival rate and service batch size. The results obtained are compared for both information levels and illustrated in the following figures. In Fig. 6, we plot the equilibrium benefit and social benefit per time unit under the equilibrium threshold strategy as functions of the mean arrival rate, $\lambda$. We consider batch service queueing models with finite buffer Geo/Geo$^{(1,10)}/1/n_e$ and infinite buffer...
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Geo/Geo\(^{(1,10)}\)/1 with same model parameters \(\mu = 0.15, R = 30, C = 1, b = 10\) and \(\lambda\). The uninformed customers’ benefits under equilibrium always dominate the individual equilibrium benefits of their counterparts. The service provider will reveal more information to the arriving customers to lower their benefit out of the system. By considering the overall benefit of the system as a whole, an opposite behavior is observed. At both the information levels, the social benefit under equilibrium begins with a minimum difference for lower values of \(\lambda\). The difference increases with more arrivals. This behavior is quite significant in compared with the Markovian single server system with fixed batch service capacity in [2].

**Fig. 7** depicts threshold strategies under both information policies with parameters \(\lambda = 0.75, b = 10, R = 30, C = 1\) for various service rates. We observe similar behavior as in the previous experiment with a difference that the informed customers’ benefits under equilibrium is a constant function of the arrivals whereas it alternates between zero and a positive value for different service rates. Most importantly, the benefit is inversely proportional to the service rate. The overall benefit of the system is higher for an informed system. So to increase the social benefit, the system administrator has to reveal more information to the customers. In the uninformed system, the social benefit is an increasing function of service rate, but it is much smaller than the observable counterpart. In both information levels, the social benefit increases strictly for lower values of \(\mu\) and remains stagnant for higher service rates. We observe that the equilibrium threshold strategy is a linear function of the service rate, whereas it remains constant for various values of \(\lambda\). Hence, from the above two results, it is observed that with the above set of model parameters the service provider will maximize the revenue if he employs the full information policy.

In the second set of numerical experiments, we find the behavior of strategic customers on the variation of service batch size. In the observable policy, the equilibrium joining threshold increases linearly with service batch size, see Fig. 8. The customers with the batch size information make decisions to join even if the system is congested, but customers without batch size information decide to balk the system when it is congested. In Fig. 9, uninformed customers equilibrium joining rate
decreases as the service batch size increase to 13 and remains unchanged with further increase in batch size. Fig. 10 illustrates the dependence of equilibrium benefit and social benefit under equilibrium for both information policies for various batch sizes. When the customers are well informed, the overall benefit of the system dominates the individual benefits of customers. But the reverse scenario prevails in the unobservable policy, where the selfish benefit of uninformed customers dominates the social welfare of the system. Hence, the observable model is socially preferable by the service provider. We observe that the rate of increase in a social benefit under equilibrium is higher for informed customers. In both of the information policies the social benefit under equilibrium strategy increases with lower service batch sizes but remains fixed for higher values of $b$, and in this case, it is 25. The equilibrium benefit is non-decreasing in service batch size for the observable policy and remains unchanged for the uninformd customers. The social benefit is also a non-decreasing function in the batch size whereas in [2] it is a decreasing function.
Thus, the batch service rule plays an important role in the control of social benefit in both policies. Fig. 11 shows several performance measures in the informed policy. We notice that the average system length and server busy are increasing functions with the increase of arrival rate $\lambda$. But the arrival rate does not have an impact on the equilibrium threshold. Fig. 12 and Fig. 13 compare the behavior of average system length with service batch size and arrival rate, respectively. The arrivals make the system more crowded in the observable policy in compare with the unobservable policy. Similarly, in the observable policy, the system congestion increases with an increase in $b$, attains optimum and then decreases with further increase in $b$, whereas it is monotonically decreasing in the unobservable policy. These experiments will help the service provider to select the information policy as well as the service batch size to control system congestion effectively.

6. Conclusions. In this paper, we have studied the strategic behavior of customers in discrete-time batch service queueing systems. Two different policies, observable and unobservable, distinguished by the level of information given to arriving customers were examined and, the equilibrium and the socially optimal strategies for each case was determined. We found that customers get higher benefits when they follow the crowd whereas the overall benefit of the system is higher in the observable policy. Service batch information has an adverse effect on the individual behavior, but has a positive effect on the society as a whole. We numerically related the equilibrium strategy with the socially optimal strategy and noticed that customers are inclined to overuse the system and make individual decisions maximizing their profit. It is observed that the optimal social benefits always lie in between the equilibrium benefits for suitable system parameters of arrival and service rates. Hence, the equilibrium benefit and the social benefit under equilibrium can be effectively controlled by choosing the ideal system parameters and disclosing the suitable information policy. It is seen that the social benefit under equilibrium strategy increases with lower service batch sizes but remains fixed for higher values of $b$ in both information policies. A possible extension to the present model may be to integrate a profit-maximizing framework where the system manager enforces an entrance fee. Another direction for future studies is to include the non-Markovian batch service queues with several vacation policies.

Acknowledgments. We would like to thank the anonymous referees, whose constructive comments and suggestions have helped us significantly improve the presentation of this paper.

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Received February 2018; 1st revised August 2018; 2nd revised October 2018.

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