Mass and first law for static asymptotically Randall-Sundrum black holes

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We give a new, intrinsic, mass definition for spacetimes asymptotic to the Randall-Sundrum braneworld models, RS1 and RS2. For this mass, we prove a first law for static black holes, including variations of the bulk cosmological constant, brane tensions, and RS1 interbrane distance. Our first law defines a thermodynamic volume and a gravitational tension that are braneworld analogs of the corresponding quantities in asymptotically AdS black hole spacetimes and asymptotically flat compactifications, respectively. This paper is the first in a series on asymptotically RS black holes.

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I. INTRODUCTION

Static and stationary black holes should obey four classical laws [11]. The first law expresses conservation of energy, and has been proven in spacetimes with four dimensions [12, 2] and higher dimensions [3–5], including a compact extra dimension [6], but not in spacetimes asymptotic to the Randall-Sundrum (RS) braneworld models [7, 8]. In this paper, we close this gap by proving a general first law for static asymptotically RS black holes. For this purpose, we provide a new, intrinsic, mass definition. Our first law also defines a thermodynamic volume and a gravitational tension that are braneworld analogs of quantities found previously in [1, 2] and [3–5], respectively.

The RS models are phenomenologically interesting and have holographic interpretations [9] in the AdS/CFT correspondence. In the RS models, our observed universe is a brane surrounded by a bulk spacetime with a negative cosmological constant. The RS1 model [7] has two branes of opposite tension, with our universe on the negative-tension brane. Tuning the interbrane distance appropriately predicts the production of small black holes at TeV-scale collider energies [10], and LHC experiments already predicts the production of small black holes at any point in the parameter space.

In [10], we will use this first law to develop a variational principle: for an asymptotically RS black hole initially at rest, initial data that extremizes the mass (with other physical quantities fixed) yields a static black hole. In [11], we will apply this principle in RS2 to find solutions for small static black holes, both on and off the brane.

This paper is organized as follows. After reviewing the RS spacetimes in section II, we define the mass for an asymptotically RS spacetime in section III and evaluate the mass for a static asymptotic solution in section IV. We prove the first law for static black holes in section V and conclude in section VI.

Throughout this paper, we use two branes, so our results apply to spacetimes asymptotic to either RS1 or RS2 in the appropriate limit. We work on the orbifold region (between the branes). The D-dimensional spacetime has metric $g_{ab}$. A timelike surface has metric $\gamma_{ab}$, extrinsic curvature $K_{ab} = \gamma^{ac} \nabla_c n_b$, and outward unit normal $n_a$. A spatial hypersurface $\Sigma$ has unit normal $u_a$, metric $h_{ab}$, and covariant derivative $D_a$. Each boundary $B$ of $\Sigma$ has metric $\sigma_{ab}$, extrinsic curvature $k_{ab} = h_{a^c}D_{b^c}n_b$, and outward unit normal $n_a$. The boundaries $B$ of $\Sigma$ are: $B_1$, $B_2$ (the branes), $B_\infty$ (spatial infinity), and $B_H$ (the black hole horizon in $\Sigma$).

II. THE RANDALL-SUNDRUM SPACETIMES

The RS spacetimes [7, 8] are portions of an anti-de Sitter (AdS) spacetime, with metric

$$ds^2_{RS} = \Omega(Z)^2 \left(-dt^2 + dp^2 + r^2 d\omega^2_{D-3} + dZ^2\right).$$

Here $d\omega^2_{D-3}$ denotes the unit $(D - 3)$-sphere. The warp factor is $\Omega(Z) = \ell/Z$, where $Z$ is the extra dimension and $\ell$ is the AdS curvature length, related to the bulk cosmological constant $\Lambda < 0$ given below. The RS1 model [7] contains two branes, which are the surfaces $Z = Z_i$ with brane tensions $\lambda_i$, where $i = 1, 2$. The brane tensions $\lambda_i$...
and bulk cosmological constant $\Lambda$ are
\[ \lambda_1 = -\lambda_2 = \frac{2(D-2)}{8\pi G_D \ell^2}, \quad \Lambda = \frac{(D-1)(D-2)}{2\ell^2}. \] (2)

The dimension $Z$ is compactified on an orbifold $(S^1/Z_2)$ and the branes have orbifold mirror symmetry: in the covering space, symmetric points across a brane are identified. There is a discontinuity in the extrinsic curvature $K_{ab}$ across each brane given by the Israel condition \[ [18]. Using orbifold symmetry, the Israel condition requires the extrinsic curvature at each brane to satisfy
\[ 2K_{ab} = \frac{8\pi G_D \lambda}{D-2} \gamma_{ab}, \quad 2k_{ab} = \frac{8\pi G_D \lambda}{D-2} \sigma_{ab}. \] (3)

The RS2 spacetime \[ [8] \] is obtained from RS1 by removing the negative-tension brane (now a regulator) to infinite distance ($Z_2 \rightarrow \infty$) and the orbifold region has $Z \geq Z_1$.

**III. MASS DEFINITION**

For an asymptotically RS spacetime, we will define the mass $M$ using a counterterm method \[ [19]. This is an intrinsic approach, and is well suited to the variations we will perform in section \[ IV. For comparison, other generally useful definitions, such as the Abbott-Deser mass \[ [20], or the Deser-Soldate mass \[ [21] with a compact dimension, use an auxiliary reference spacetime.

In the counterterm approach for a spacetime with metric $g_{ab}$, one first evaluates the bare action $S$. If this diverges, one constructs an action counterterm $S_\alpha$ to render the sum $S = \bar{S} + S_{ct}$ finite, as follows. Let the metric $g_{ab}$ asymptote to $g_{ab}^{(0)}$ whose bare action $\bar{S}_0$ also diverges. We express $\bar{S}_0$ in terms of its intrinsic boundary invariants, and define the action counterterm $S_{ct} = -\bar{S}$ where $\bar{S}$ is the same functional of its boundary geometry that $\bar{S}_0$ is of its boundary geometry. This gives $S_0 = 0$ for $g_{ab}^{(0)}$ and a finite action $S$ for $g_{ab}$.

To define the mass $M$, one proceeds from the action to the Hamiltonian (defined on an arbitrary initial value spatial hypersurface $\Sigma$), which is given by a bulk term involving initial value constraints, and surface terms. For a solution to the constraints, the bulk term vanishes and the bare mass at spatial infinity is \[ [22]
\[ \bar{M} = -\frac{1}{8\pi G_5} \int_{B_\infty} d^{D-2}x N \sqrt{\sigma} k. \] (4)

Here $k$ is the extrinsic curvature of the boundary $B_\infty$ and for a static spacetime, the lapse function is $N = \sqrt{-g_{tt}}$.

If $\bar{M}$ diverges, one constructs a mass counterterm
\[ M_{ct} = \int_{B_\infty} d^{D-2}x N \sqrt{\sigma} u_{a\bar{b}} \left[ \frac{2}{\sqrt{-\gamma}} \delta S_{ct} / \delta \gamma_{a\bar{b}} \right] \] (5)

such that the mass $M$ is finite, defined by
\[ M = \bar{M} + M_{ct}. \] (6)

Following the above procedure, we begin with the RS spacetime. The bare action $\bar{S}_{RS}$ consists of a bulk term $\bar{S}_\Sigma$ and a Gibbons-Hawking term at each boundary,
\[ \bar{S}_{RS} = \bar{S}_\Sigma + S_1 + S_2 + S_\infty. \] (7)

Here
\[ S_\Sigma = \frac{1}{16\pi G_D} \int d\tau \int_{\Sigma} d^{D-1}x \sqrt{-g} (R - 2\Lambda), \]
\[ S_i = \frac{1}{8\pi G_D} \int d\tau \int_{B_i} d^{D-2}x \sqrt{-\gamma} \left( K - \frac{8\pi G_D \lambda_i}{2} \right), \]
\[ S_\infty = \frac{1}{8\pi G_D} \int d\tau \int_{B_\infty} d^{D-2}x \sqrt{-\gamma} K. \] (8)

For the RS solution \[ [1], the Ricci scalar is $R = 2\Lambda D/(D-2)$ and after integrating $\bar{S}_\Sigma$ in $Z$, we find
\[ \bar{S}_\Sigma + S_1 + S_2 = 0. \] (9)

We now specialize to the case $D = 5$, for which
\[ \bar{S}_{RS} = S_\infty = \frac{1}{8\pi G_5} \int d^4x \sqrt{-\gamma} \frac{2}{\Omega \rho}. \] (10)

This diverges as $\rho \rightarrow \infty$. The metric $\gamma_{a\bar{b}}$ on this boundary is \[ [1] with $\rho = \text{constant}$. Let $\tilde{\sigma}_{a\bar{b}}$ be the submetric with $Z = \text{constant}$, and let $\delta_{a\bar{b}}$ be the submetric on a 2-sphere (constant $\rho$, $Z$, $t$). Their Ricci scalars are
\[ \tilde{\mathcal{R}}(\tilde{\sigma}) = \mathcal{R}(\delta) = \frac{2}{(\Omega \rho)^2}. \] (11)

If we express $\bar{S}_{RS}$ in terms of $\tilde{\mathcal{R}}$, then an asymptotically RS spacetime has action counterterm $S_{ct} = -\bar{S}$ where $\bar{S}$ is the same functional of its boundary geometry that $\bar{S}_{RS}$ is of its geometry. Thus
\[ S_{ct} = -\frac{\sqrt{2}}{8\pi G_5} \int d^4x \sqrt{-\gamma} \sqrt{\mathcal{R}}. \] (12)

All quantities in \[ [12] refer to the boundary geometry of a general metric $g_{a\bar{b}}$, not the RS metric \[ [1]. As shown in Appendix \[ A, the mass counterterm \[ (5) then yields our mass definition for an asymptotically RS spacetime,
\[ M = \frac{1}{8\pi G_5} \int_{B_\infty} d^4x N \sqrt{\sigma} \left( -k + \sqrt{2\mathcal{R}} \right). \] (13)

**IV. STATIC ASYMPTOTIC SOLUTION**

Here we evaluate our mass definition \[ [13] for a static asymptotic solution, which we will use in section \[ IV below. We define the branes as the surfaces $Z = Z_1$ and $Z = Z_2$. We consider a static asymptotically RS metric with functions $F_\nu$ that fall off at large $\rho$ as follows,
\[ ds^2 = \Omega^2 \left( -e^{2F_0} dt^2 + e^{2F_3} d\rho^2 + e^{2F_2} \rho^2 d\omega_2^2 + e^{2F_2} dZ^2 \right) \]
\[ F_\nu = \frac{a_0(\rho)}{\rho^2} + \frac{b_0(\rho)}{\rho^2} + c_0(\rho) \rho^2 + O(1/\rho^4). \] (14)
For these asymptotics, the mass \([13]\) evaluates to

\[
M = \frac{1}{G_5} \int_{Z_1}^{Z_2} dZ \Omega^3 \left( a_\rho + \frac{a_Z}{2} \right). \tag{15}
\]

The value of \(M\) also appears in the solution to \([14]\). To see this, we find it is necessary to solve the Einstein equations through third order. The first order solutions are

\[
a_i(Z) = a_\rho(Z) + \mu_0, \tag{16a}
\]

\[
a_\omega(Z) = a_\rho(Z) + \mu_1, \tag{16b}
\]

\[
a_Z(Z) = -Z a_\rho'. \tag{16c}
\]

Here \(\mu_0\) and \(\mu_1\) are integration constants and \(t = d/dZ\). The constant \(\mu_1\) can be removed by a gauge transformation \(\rho \to \rho + \mu_1/2\). Two identities we will need are

\[
\int_{Z_1}^{Z_2} dZ \Omega^3(a_i + a_\rho + a_Z) = 0, \tag{17a}
\]

\[
\int_{Z_1}^{Z_2} dZ \Omega^{n+1}(na_\rho + a_Z) = -\ell (\Omega^n a_\rho)\big|_{Z_1}^{Z_2}. \tag{17b}
\]

The identity \([17b]\) also holds with \(a_\rho\) replaced by \(a_i\) or \(a_\omega\). Using \([17b]\), the mass can be written as

\[
M = -\frac{\ell}{2G_5} (\Omega^2 a_\rho)\big|_{Z_1}^{Z_2}. \tag{18}
\]

The third order solutions \(c_\rho\) involve an integration constant \(q_0\). The Israel condition provide one equation at each brane, which can be solved for \(\mu_0\) and \(q_0\) as

\[
\mu_0(\Omega_2^2 - \Omega_1^2) = -2(\Omega^2 a_\rho)\big|_{Z_1}^{Z_2}, \tag{19}
\]

\[
q_0(\Omega_2^2 - \Omega_1^2) = 2a_\rho\big|_{Z_1}^{Z_2}. \tag{20}
\]

Here \(\Omega_i = \ell/Z_i\) the warp factor at each brane. We see from \([17b]\) and \([19]\) that \(M\) is proportional to \(\mu_0\),

\[
M = \frac{\ell\mu_0}{4G_5} (\Omega_2^2 - \Omega_1^2). \tag{21}
\]

We see from \([20]\) that \(q_0\) is proportional to a quantity \(Q\) parametrizing the interbrane distance near \(\rho \to \infty\),

\[
L_{\text{branes}} = L + \frac{Q}{\rho} + O(1/\rho^2), \tag{22}
\]

where the distance \(L\) at infinity and the constant \(Q\) are

\[
L = \ell \ln \left( \frac{\Omega_1}{\Omega_2} \right), \quad Q = -\ell a_\rho\big|_{Z_1}^{Z_2}. \tag{23}
\]

We will refer to \(L\) and \(Q\) in the next section. We will also use the fact that \(M\) and \(Q\) appear in the values of \(a_\rho\) at each brane, which we find by solving \([18]\) and \([23]\),

\[
a_\rho(Z_i) = \frac{2G_5 M - \Omega_j^2 Q}{\ell(\Omega_1^2 - \Omega_2^2)}, \quad j \neq i. \tag{24}
\]

In the RS1 case, \(M\) and \(Q\) have lower-dimensional interpretations on each brane. This follows since there is an effective Brans-Dicke gravity on each brane \([23]\), and Brans-Dicke gravity contains two asymptotic quantities, a tensor mass and a scalar mass \([24]\). On each brane, one can verify that \(M\) and \(Q\) are proportional to the effective tensor mass and scalar mass, respectively.

V. FIRST LAW FOR STATIC BLACK HOLES

A. Preliminary form

For a static or stationary black hole, the first law relates the variations of mass, black hole horizon area, and other physical quantities. We will include variations of the bulk cosmological constant \(\Lambda\) and brane tensions \(\lambda_i\) that preserve the RS conditions \([2]\),

\[
-\frac{\delta M}{\ell} = \frac{\delta \lambda_1}{\lambda_1} = \frac{\delta \lambda_2}{\lambda_2} = \frac{\delta \Lambda}{2\Lambda}. \tag{25}
\]

We will also include the variation \(\delta L\) of the interbrane separation. From \([23]\),

\[
\delta L = \delta \ell \ln \left( \frac{\Omega_1}{\Omega_2} \right) + \ell \left( \frac{\delta \Omega_1}{\Omega_1} - \frac{\delta \Omega_2}{\Omega_2} \right). \tag{26}
\]

Our setup is general: it applies to a black hole localized on a brane, or isolated in the bulk (away from either brane), and also applies to the asymptotically RS black string \([25]\). Our method is based on the Hamiltonian approach of \([2]\), with the additional variations \([25]–[26]\). The full Hamiltonian contains a bulk term and surface terms. The bulk term on an initial data surface \(\Sigma\) is

\[
H_\Sigma = \int_\Sigma d^{D-1}x \left( NC_0 + N^a C_a \right). \tag{27}
\]

Here \(N\) and \(N^a\) are the lapse and shift in the standard decomposition of the spacetime metric. Our focus is the initial data \((h_{ab}, p^{ab})\) where \(h_{ab}\) is the spatial metric and \(p^{ab}\) is its canonically conjugate momentum,

\[
16\pi G_D p^{ab} = \sqrt{h} K^{ab} - K h^{ab}, \quad K_{ab} = h_a^{\ \rho} \nabla_c u_b. \tag{28}
\]

Initial data must satisfy constraints, \(C_0 = 0\) and \(C_a = 0\), which we henceforth assume, where

\[
C_0 = \frac{\sqrt{h}}{16\pi G_D} \left( 2\Lambda - \mathcal{R} \right) + \frac{16\pi G_D}{\sqrt{h}} \left( p^{ab} p_{ab} - \frac{p^2}{D-2} \right),
\]

\[
C_a = -2D_b p_a^\ b. \tag{29}
\]

Here \(\mathcal{R}\) and \(D_a\) are the Ricci scalar and covariant derivative associated with \(h_{ab}\). We now consider the change \(\delta H_\Sigma\) under variations \((\delta h_{ab}, \delta p^{ab})\). One finds \(\delta C_0\) and \(\delta C_a\) involve derivatives \((D_i \delta h_{ab}, D_a \delta p^{ab})\). Integrating by parts to remove these derivatives yields surface terms \(I_B\),

\[
\delta H_\Sigma = \int_\Sigma d^{D-1}x \left[ p^{ab} \delta h_{ab} + h_{ab} \delta p^{ab} \right]
+ \frac{\delta \Lambda}{8\pi G_D} \int_\Sigma d^{D-1}x N\sqrt{h} + \sum_B I_B. \tag{30}
\]
We will not need the explicit forms of $P_{ab}$ and $\mathcal{H}_{ab}$. They appear in the evolution equations,
\[
\dot{h}_{ab} = \mathcal{H}_{ab}, \quad \dot{p}^{ab} = -p^{ab},
\]  
(31)
where the dot denotes the Lie derivative along the time evolution vector field $\xi^a = Nu^a + N^a$ with $u^a$ the unit normal to $\Sigma$. The sum in $\mathcal{H}_{ab}$ is over all boundaries of $\Sigma$, which are $B_1$, $B_2$ (the branes), $B_{\infty}$ (spatial infinity), and $B_H$ (the black hole horizon in $\Sigma$).

For variations from one solution of the constraints to another solution of the constraints, we have $\delta H_{ab} = 0$. We now assume a static black hole with timelike Killing field $\xi^a$ and choose $t^a = \xi^a$. For a static solution, the following vanish: $\rho^{ab}, N^a, P^{ab}, H_{ab}$. Then $\delta p_{ab}$ gives
\[
0 = \frac{\delta A}{8\pi G_D} \int_\Sigma d^{D-1}x N \sqrt{h} + \sum_B I_B.
\]
(32)
This equation is our preliminary form of the first law. It simply remains to express $I_B$ in terms of physical quantities. Each surface term $I_B$ can be written $\delta \Phi$
\[
I_B = \int_{B^a} \sqrt{-g} \left( \delta \mathcal{L}_{\text{matter}} - \frac{\delta \mathcal{L}}{\delta \phi} \phi + \delta \mathcal{L}_{\text{geometry}} \right) + J_B
\]
(33)
where
\[
8\pi G_D s^{ab} = -k^{ab} + [k + n^c (D_c N)/N] \sigma^{ab},
\]
(34)
\[
16\pi G_D J_B = \sum_{B^a \neq B^b} \int_{B \cap B'} d^{D-3}x N \sqrt{\sigma} n'_a \sigma^{ab} \delta n^b.
\]
(35)
Here $\delta$ denotes the metric on $B \cap B'$. In what follows, we will have $J_B = 0$. This is due to $N = 0$ on $B_H$, and due to orthogonality ($n^a n_a = 0$) at the other boundaries $B$. We now evaluate the boundary terms $\delta \Phi$ for $D = 5$.

The results are, with $A$ the black hole horizon area,
\[
I_{B_H} = \frac{\kappa}{8\pi G_5} \delta A, \quad I_B = \frac{\delta \gamma}{2} \int_{B_i} d^3x N \sqrt{\sigma}.
\]
(36)
These results are straightforward to derive. At the black hole horizon, we have $N = 0$ and $D_a N = -\kappa n_a$ where $\kappa$ is the constant surface gravity $\kappa$. This gives $8\pi G_5 N s^{ab} = -\kappa \sigma^{ab}$ and the result in $\delta \Phi$ follows. At each brane, we use $n_c (D_c N)/N = K - k$, which is a general result $[22]$ valid when $u^a n_a = 0$. Using $\delta \Phi$ gives $s^{ab} = (\Lambda_s/2)\sigma^{ab}$ which yields the result in $\delta \Phi$. In Appendix $[13]$ we show the boundary term $I_{B_{\infty}}$ is
\[
I_{B_{\infty}} = -\delta M + \mathcal{F}_{\infty} \delta \ell + \mathcal{U}_1 \frac{\delta \Omega_1}{\Omega_1} - \mathcal{U}_2 \frac{\delta \Omega_2}{\Omega_2},
\]
(37)
where the boundary quantity $\mathcal{F}_{\infty}$ at infinity is
\[
\mathcal{F}_{\infty} = -\frac{1}{2G_5} \int_{Z_2} dZ \Omega_3 (a_t + 2a z),
\]
(38)
and the coefficients $\mathcal{U}_i$ are
\[
\mathcal{U}_i = M \left( \frac{\Omega_i^2}{\Omega_1^2 - \Omega_2^2} \right) - \frac{3Q}{2G_5} \left( \frac{\Omega_1^2 \Omega_2^2}{\Omega_1^2 - \Omega_2^2} \right).
\]
(39)

Here $Q$ parametrizes the asymptotic interbrane separation $\delta = 2$. We also define $\mathcal{F}$ by the following sum,
\[
\mathcal{F} \delta \ell = \frac{\delta A}{8\pi G_5} \int_\Sigma d^3x N \sqrt{h} + \int_{B_1} I_{B_1} + \int_{B_{\infty}} \mathcal{F}_{\infty} \delta \ell.
\]
(40)
Here the two brane terms $I_{B_i}$ render the volume integral finite, as one can verify. The term $\mathcal{F}_{\infty}$ renders $\mathcal{F}$ gauge invariant, as shown in Appendix $[C]$. We also define $V$ by
\[
V = \left( \frac{\ell}{2P} \right) \mathcal{F}, \quad -V \delta P = \mathcal{F} \delta \ell,
\]
(41)
where $P = -\Lambda/(8\pi G_5)$ is the pressure due to the cosmological constant. We have now evaluated all the terms needed to rewrite the preliminary first law $[22]$.

**B. The first law**

We will give four versions of the first law, corresponding to different choices of variations. Substituting $\delta M, \delta P, \delta \Omega_1, \delta \Omega_2$ into (32) gives the first law in the form
\[
\delta M = \kappa \delta A - V \delta P + U_1 \frac{\delta \Omega_1}{\Omega_1} - U_2 \frac{\delta \Omega_2}{\Omega_2}.
\]
(42)
The area term is standard. The last two terms are changes in mass due to changes in the branes' gravitational field, since $\delta \Omega_i$ are variations of gravitational redshift factors on each brane. The last term is absent in RS2, which removes the negative-tension brane to $Z_2 \to \infty$, for which $\Omega_2 \to 0$ and $U_2/\Omega_2 \to 0$ by $[22]$.

For discussion purposes, we will take $V > 0$. This is easily verified for the static asymptotically RS black string $[25]$, which is the only known exact solution for an asymptotically RS black object in 5-dimensional spacetime. We will also see in $[13]$ below that $V > 0$ if and only if a gravitational tension $T_0$ is positive.

The coefficient of $\delta P$ defines a thermodynamic volume $V_{\text{eff}}$ in a black hole first law $[15, 27]$. For a static asymptotically AdS black hole, it was found in $[H]$ that $V_{\text{eff}} > 0$ is the volume removed by the black hole (the volume of pure AdS space minus the volume outside the black hole). In our first law, $V_{\text{eff}} = -V < 0$ suggests that net volume is added outside the black hole (compared to the case with no black hole). Added volume makes sense physically: in RS2 the black hole repels the positive-tension brane, and in RS1 we would expect a version of the black hole Archimedes effect $[28, 29]$, where the black hole increases the size of the compact dimension (here the interbrane distance). We also note that $V_{\text{eff}} < 0$ occurs, with a natural interpretation as an added volume, in AdS-Taub-NUT spacetime $[30]$. In RS1, there are three ways the variation $\delta L$ of the interbrane distance can be introduced into the first law, using $[26]$. In each case, the coefficient of $\delta L$ defines a gravitational tension $T$ that depends on which quantities are held fixed. The three values we refer to below are
\[
T_0 = \frac{2P V}{L}, \quad T_1 = \frac{U_1}{\ell}, \quad T_2 = \frac{U_2}{\ell}.
\]
(43)
Using (26) in (42) to change variables from $\delta t$ to $\delta L$ gives

$$\delta M = \frac{\kappa \delta A}{8\pi G_5} + \mathcal{T}_0 \delta L + \sum_{i=1,2} \pm \left( U_i - \mathcal{T}_0 \ell \right) \frac{\delta \Omega_i}{\Omega_i}. \tag{44}$$

Here $\pm$ is the sign of each brane tension $\lambda_i$ and $\mathcal{T}_0$ is a gravitational tension at fixed values of $(A, \Omega_1, \Omega_2)$. Using (26) in (42) to eliminate $\delta \Omega_i$ or $\delta \Omega_2$ gives the first law as

$$\delta M = \frac{\kappa \delta A}{8\pi G_5} + \mathcal{T}_1 \delta L + \left( -V + \frac{\mathcal{T}_1 L}{2P} \right) \delta P + M \frac{\delta \Omega_2}{\Omega_2} \tag{45}$$

and

$$\delta M = \frac{\kappa \delta A}{8\pi G_5} + \mathcal{T}_2 \delta L + \left( -V + \frac{\mathcal{T}_2 L}{2P} \right) \delta P + M \frac{\delta \Omega_1}{\Omega_1} \tag{46}$$

Each term $\mathcal{T} \delta L$ in (44)–(46) is the work needed to vary the RS1 interbrane distance (with different quantities held fixed), analogous to the work terms in the first law in the case of a compact dimension without branes [6]. Our gravitational tensions (43) are easily verified to be positive for the asymptotically RS1 black string [25]. We would also expect our gravitational tensions to be positive due to the black hole’s attraction to its images in the covering space, which has been shown [29] in the case of a compact dimension without branes.

Since each version of the first law reparametrizes the geometry, in (45) and (46) each thermodynamic volume $V_{eff} = -V + \mathcal{T}_i L / P$ differs from $-V$. For $-V < 0$, this indicates that positive gravitational tension $\mathcal{T}_1$ opposes the black hole Archimedes effect, and the sign of $V_{eff}$ depends on their relative strengths.

Reparametrizing the geometry also transforms the brane terms in the first law, but with the interesting property that the coefficients of $\delta \Omega_i / \Omega_i$, always add to $M$, in each version of the first law. A brane term in (42) or (44) shifts into both $V_{eff} \delta P$ and $\mathcal{T}_i \delta L$ in (45) and (46), and this shift incorporates the brane’s orbifold symmetry into the gravitational tension, since $\mathcal{T}_i$ is due to the black hole’s attraction to its orbifold mirror images.

VI. CONCLUSION

We have derived the first law for a static asymptotically RS black hole whose mass $M$ is defined in (13) and (15). Four versions of this law are given in (42)–(46) for different choices of variations. In both RS1 and RS2, the general first law contains brane terms and a thermodynamic volume. In RS1, we can define both a thermodynamic volume and a gravitational tension, due to the presence of both a cosmological constant and a compact interbrane distance. This differs from the first law in previously studied spacetimes (with either a cosmological constant or a compact dimension), where the analogs of our thermodynamic volume and gravitational tension are isolated from each other, appearing in the separate first laws of separate spacetimes.

The general set of variations considered in this paper ($\delta L, \delta \Omega_i, \delta t, \delta \lambda_i, \delta \lambda_i$) will be set to zero in the two next papers in this series [16] [17]. In [16], we will use the first law in this paper to develop the following variational principle: for an asymptotically RS black hole initially at rest, initial data that extremizes the mass (holding other physical quantities fixed) yields a static black hole. In [17], we will apply this variational principle and show that a black hole on an orbifold-symmetric brane in RS2 is stable against leaving the brane, which generalizes to other models with an orbifold-symmetric brane. On such a brane, small black holes produced in high energy experiments could be studied directly (instead of leaving behind a signature of missing energy), which is an important result for future collider experiments.

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Appendix A: Mass counterterm $M_{ct}$

In this appendix, we derive the mass counterterm $M_{ct}$ from [12] and [13], and thereby prove the mass formula [13]. We begin with the variation of [12],

$$\delta S_{ct} = - \int d^4x \sqrt{-\gamma} \left( \sqrt{\mathcal{R}} \gamma^{ab} \delta \gamma_{ab} + \delta \mathcal{R} \right). \tag{A1}$$

The standard variation of the Ricci scalar is

$$\delta \mathcal{R} = - \delta \gamma^{ab} \delta \gamma_{ab} + \delta a v^a \tag{A2}$$

where $v^a = 2 \delta b [a] \delta d [b] \delta \sigma_{cd}$ and $\delta a$ is the covariant derivative associated with $\delta a$. This gives

$$\delta S_{ct} = \int d^4x S^{ab} \delta \gamma_{ab} - (8\sqrt{2} \pi G_5) I, \tag{A3}$$

where $S^{ab}$ is given below and, with $J = \sqrt{-\gamma_{tt} \gamma_{zz} / \mathcal{R}}$,

$$I = \int dt \int dZ \int d^2x \sqrt{\sigma} \left[ \delta_a (J v^a) - v^a \delta_a J \right]. \tag{A4}$$

We conclude that $I = 0$, as follows. The first term is a total divergence, but the 2-sphere has no boundary. Also, $\delta_a J = 0$ since we take $J$ independent of the angular coordinates. Then (A3) gives

$$S^{ab} = \delta S_{ct} \delta \gamma_{ab} \tag{A5}$$

In (A1) we now use

$$\gamma^{ab} \delta \gamma_{ab} = \gamma^{tt} \delta \gamma_{tt} + \delta \gamma^{ab} \delta \sigma_{ab} + \sigma_{ZZ} \sigma_{ZZ} \tag{A6}$$
which gives
\[ \frac{\delta S_{ct}}{\delta \gamma_{tt}} = -\frac{1}{8\sqrt{2}\pi G_5} \sqrt{-\gamma R} \gamma_{tt}, \quad (A7a) \]
\[ \frac{\delta S_{ct}}{\delta \sigma_{ab}} = -\frac{1}{8\sqrt{2}\pi G_5} \sqrt{-\gamma R} \left( \sigma_{ab} - \frac{\hat{R}_{ab}}{\hat{R}} \right), \quad (A7b) \]
\[ \frac{\delta S_{ct}}{\delta \sigma_{\bar{Z}\bar{Z}}} = -\frac{1}{8\sqrt{2}\pi G_5} \sqrt{-\gamma R} \sigma_{\bar{Z}\bar{Z}}. \quad (A7c) \]
The mass counterterm from (5) and (A7a) is then
\[ M_{ct} = \frac{\sqrt{2}}{8\pi G_5} \int d^3 x \sqrt{-\gamma R}. \quad (A8) \]
Combining this with (4) now gives the mass formula (13).

**Appendix B: Boundary term \(I_B\) at infinity**

Here we evaluate the term \(I_{B\infty}\) given by (33) at the boundary \(\rho \to \infty\). Throughout this appendix, \(\simeq\) denotes evaluating at leading order and neglecting terms of higher order in \(1/\rho\). We will relate \(I_{B\infty}\) to the mass variation \(\delta M\) and additional terms. The mass is a sum of two terms, \(M = M + M_{ct}\), whose individual variations are
\[ \delta \tilde{M} = -\frac{1}{8\pi G_5} \int d^3 x \left[ N \delta \left( \sqrt{\sigma} k + \sqrt{\sigma} k \delta N \right) \right] \quad (B1) \]
and
\[ \delta M_{ct} = \int_{B\infty} d^3 x \left[ \frac{\delta M_{ct}}{\delta \sigma_{ab}} \delta \sigma_{ab} + \frac{\sqrt{2} \delta R}{8\pi G_5} \delta N \right]. \quad (B2) \]
Note \(\delta M_{ct}/\delta \sigma_{ab} = -\delta S_{ct}/\delta \sigma_{ab}\) since \(\int dt M_{ct} = -S_{ct}\) by (A8) and (12). From (A7b) and (A7c), we find
\[ \frac{\delta M_{ct}}{\delta \sigma_{ab}} = \frac{2}{\sqrt{2}} \sqrt{\sigma} \delta_{ab} = X^{ab} \quad (B3) \]
at large \(\rho\), where the quantities \(X^{ab}\) are given below. Using (B1)–(B3), we rewrite the boundary term (33) as
\[ I_{B\infty} = -\delta M + \int_{B\infty} d^3 x X^{ab} \delta \sigma_{ab} \]
\[ -\frac{1}{8\pi G_5} \int_{B\infty} d^3 x \sqrt{\left( k - \sqrt{2} R \right)} \delta N. \quad (B4) \]
For the metric asymptotics (14), the quantities \(X^{ab}\) are
\[ 16\pi G_5 X^{\chi \chi} \simeq \frac{\Omega \sin \chi}{\rho^2} (a_t + a_\rho + a_Z), \quad (B5a) \]
\[ 16\pi G_5 X^{\phi \phi} \simeq \frac{\Omega}{\rho^2 \sin \chi} (a_t + a_\rho + a_Z), \quad (B5b) \]
\[ 16\pi G_5 X^{\bar{Z} \bar{Z}} \simeq \Omega \sin \chi (a_t + 2a_\rho). \quad (B5c) \]
Here \(\chi\) is the polar angle on the 2-sphere with radius \(\rho\). We now proceed to evaluate (B4). We begin with three convenient variables \((\ell, Z_1, Z_2)\) and then express results in terms of three physical variables \((\ell, \Omega_1, \Omega_2)\). We first consider the variation \(\delta \ell\) at fixed \((Z_1, Z_2)\). At large \(\rho\), we have \(\delta g_{ab} \simeq 2(\delta \ell/\ell) g_{ab}\). Then
\[ \int_{B\infty} d^3 x X^{ab} \delta \sigma_{ab} = F_\infty \delta \ell \quad (B6) \]
where
\[ F_\infty = \frac{1}{2G_5} \int_{Z_1} dZ \Omega^3 (a_t + 2a_\rho). \quad (B7) \]
This is entirely due to \(\delta \sigma_{\bar{Z}\bar{Z}}\) since the integral contributions from \(\delta \sigma_{\phi\phi}\) and \(\delta \sigma_{\chi}\) vanish by the identity (17a). This identity can also be used to rewrite \(F_\infty\) in the form given in (38). The last line in (B4) yields \(\delta M/\ell\). Hence (B4) yields, at fixed \((Z_1, Z_2)\),
\[ (I_{B\infty})_{Z_1 Z_2} = -\delta M + \left( \frac{F_\infty + M_\ell}{\ell} \right) \delta \ell. \quad (B8) \]
We now consider variations \((\delta Z_1, \delta Z_2)\) at fixed \(\ell\). We then perform a coordinate transformation
\[ Z \to \tilde{Z} = (1 - \epsilon) Z - \zeta \quad (B9) \]
such that the branes again reside at \(Z = Z_1\). The required transformation is
\[ \epsilon = \frac{\delta Z_2 - \delta Z_1}{Z_2 - Z_1}, \quad \zeta = \frac{Z_2 \delta Z_1 - Z_1 \delta Z_2}{Z_2 - Z_1}. \quad (B10) \]
At large \(\rho\), the resulting metric perturbation is
\[ \delta g_{ab} \simeq 2\Omega^2 \left[ \epsilon (\delta \sigma^a_b - \eta_{ab}) - \frac{\zeta}{\ell} \Omega \eta_{ab} \right], \quad (B11) \]
where \(\eta_{ab}\) is the 5-dimensional Minkowski metric. Then (B4) becomes, at fixed \(\ell\),
\[ (I_{B\infty})_\epsilon = -\delta M - \epsilon M - \zeta \mathcal{I}, \quad (B12) \]
where the integral \(\mathcal{I}\) is
\[ \mathcal{I} = \frac{3}{2G_5} \int_{Z_1}^{Z_2} dZ \Omega^4 (a_t + 2a_\rho + a_Z). \quad (B13) \]
For the case when all three quantities \((\ell, Z_1, Z_2)\) are varied, we combine (B8) and (B12) to obtain
\[ I_{B\infty} = -\delta M + \left( \frac{F_\infty + M_\ell}{\ell} \right) \delta \ell - \epsilon M - \zeta \mathcal{I}. \quad (B14) \]
We can evaluate the integral \(\mathcal{I}\) using (10a) and the identity (17b). We then express the result in terms of \(M\) and \(Q\) using (21) and (24). This gives
\[ \mathcal{I} \ell = M \left( \frac{\Omega_1^3 - \Omega_2^3}{\Omega_1^2 - \Omega_2^2} \right) - 3Q \left( \frac{\Omega_1^2 \Omega_2^2}{\Omega_1^2 - \Omega_2^2} \right) \Omega_1. \quad (B15) \]
We also express \(\epsilon\) and \(\zeta\) in terms of three physical variables \((\ell, \Omega_1, \Omega_2)\) using
\[ \delta Z_i = \frac{1}{\Omega_i} \left( \delta \ell - \frac{\ell}{\Omega_i} \delta \Omega_i \right). \quad (B16) \]
Using (B15) and (B16) in (B14) then yields the result for \(I_{B\infty}\) given in (37).
Appendix C: Gauge invariance

It is important to confirm that our quantities \( (M, Q, \nu, L, T_0, T_1, U_4) \) are gauge invariant at infinity. As one can verify, these quantities are invariant under the following metric transformation that leaves the branes fixed,

\[
a_{\nu} \rightarrow a_{\nu} - \frac{\Omega'}{\Omega} w - \delta^Z \frac{\rho}{\rho^2} w', \quad w(Z_1) = w(Z_2) = 0, \tag{C1}
\]

with \( \delta = d/dZ \). This is generated by the coordinate transformation \( x^a \rightarrow x^a + \epsilon^a \), where to leading order in \( 1/\rho \),

\[
\epsilon^Z = \frac{w}{\rho}, \quad \epsilon^\rho = \frac{W}{\rho^2}, \quad w = W'. \tag{C2}
\]

In particular, we consider the quantity \( F \), and write

\[
F = F_{\Sigma} + F_\infty \tag{C3}
\]

where \( F_{\Sigma} \) is the sum of bulk and brane terms in \( \mathcal{L} \),

\[
F_{\Sigma} \delta \ell \equiv \frac{\delta \mathcal{L}}{8\pi G_5 \int d^4x \sqrt{\mathcal{H}} + I_{B_1} + I_{B_2}}. \tag{C4}
\]

We note that \( F \) is gauge invariant, but neither \( F_{\Sigma} \) nor \( F_\infty \) is separately invariant, since they transform as

\[
F_{\Sigma} \rightarrow F_{\Sigma} - \varphi, \quad F_\infty \rightarrow F_\infty + \varphi, \tag{C5}
\]

where

\[
\varphi = \frac{3}{2G_5 \ell^2} \int Z_1 dZ \Omega^3 w. \tag{C6}
\]