We consider chiral symmetry breaking at nonzero chemical potential and discuss the relation with the spectrum of the Dirac operator. We solve the so called Silver Blaze Problem that the chiral condensate at zero temperature does not depend on the chemical potential while this is not the case for the Dirac spectrum and the weight of the partition function.

Keywords: Chiral Symmetry, Dirac Spectrum, Nonzero Chemical Potential

1. Introduction

The QCD phase diagram has been explored in great detail during the past decade. New phases have been discovered and a variety of parameters have been considered. In particular, a great deal of progress has been made for QCD at large nonzero baryon chemical potential (denoted by $\mu$). Based on model calculations and asymptotic expansions, it is now generally accepted that the ground state of QCD in this region is in a phase with a diquark condensate in a color-flavor locked state. The QCD phase diagram is best known at $\mu = 0$ where it has been explored extensively by lattice simulations. At a critical temperature of about 160 MeV we expect a cross-over from a phase with a nonzero chiral condensate to phase with restored chiral symmetry. At zero temperature, the QCD partition function is independent of $\mu$ until it has reached the lightest excitation with nonzero baryon number. This means that the chiral condensate remains constant up to $\mu = m_N/3 - \epsilon$ (with $\epsilon$ the nuclear binding energy). The transition to the chirally restored phase is believed to be of first order at low temperatures and ends in a critical endpoint. A schematic QCD phase diagram is given in Fig. 1. In this lecture we will consider the QCD partition function at zero temperature for chemical potentials $\mu \ll m_N/3$, where QCD can be described by a chiral Lagrangian.
### 2. The QCD Partition Function

The QCD partition function at temperature $1/\beta$ is given by

$$Z_{\text{QCD}}(\mu, T) = \sum_k e^{-\beta(E_k - \mu N_k)},$$

where the sum is over all quantum states of QCD with energy $E_k$ and quark number $N_k$. The partition function is a monotonously nondecreasing function of the chemical potential. For $T = 0$ its value remains constant up to $\mu = mN/3 - \epsilon$ (see Fig. 2). This implies that thermodynamic observables, such as for example the chiral condensate, do not depend on the chemical potential up to this point. However, this simple fact becomes a mystery when we consider the Euclidean QCD partition function given by (with quark masses denoted by $m_f$)

$$Z_{\text{QCD}}(\mu, T) = \langle \prod_{f=1}^{N_f} \det(D + \mu \gamma_0 + m_f) \rangle = \langle \prod_{f=1}^{N_f} \prod_{k} (\lambda_k + m_f) \rangle. \tag{2}$$

Here and below, the brackets denote the average over gauge field configurations weighted by the Euclidean Yang-Mills action. The quark determinant has been expressed in terms of the eigenvalues of the Dirac operator given by

$$(D + \mu \gamma_0)\phi_k = \lambda_k \phi_k. \tag{3}$$

The chiral condensate is thus given by (we only consider $m_f = m$ from now on)

$$|\langle \bar{q} q \rangle| = \lim \frac{1}{V} \frac{1}{N_f} \partial_m \log Z_{\text{QCD}}(\mu, T) = \lim \frac{1}{V} \left\langle \sum_k \frac{1}{\lambda_k + m} \right\rangle_{\text{QCD}}. \tag{4}$$

The mystery is that the eigenvalues and the quark determinant strongly depend on the chemical potential but the partition function and the chiral condensate do not. Thomas Cohen coined this problem as the Silver Blaze problem after the title of
Fig. 2. The QCD partition function as a function of the chemical potential for zero and nonzero temperature in the absence of nuclear binding energy. The temperature of the critical endpoint in Fig. 1 is denoted by $T_{\text{end}}$.

3. Solution of the Silver Blaze Problem

The spectral “density” of the Dirac operator is given by

$$\rho(z, z^*, m, \mu) = \left\{ \sum_k \delta^2(z - \lambda_k) \det(D + \mu \gamma_0 + m) \right\}.$$  

The quenched spectral density is obtained for $N_f = 0$. We also use the resolvent defined by (here and below we take the volume $V$ finite but large)

$$G(z, z^*, m, \mu) = \left\{ \frac{1}{V} \sum_k \frac{1}{z + \lambda_k} \det(D + \mu \gamma_0 + m) \right\},$$  

so that (notice that $\rho(-z, -z^*, m, \mu) = \rho(z, z^*, m, \mu)$)

$$\frac{\rho(z, z^*, m, \mu)}{V} = \frac{1}{\pi} \partial_{z^*} G(z, z^*, m, \mu).$$

Since the chiral condensate is given by,

$$|\langle \bar{q} q \rangle| = \int d^2 z \frac{1}{V z + m} \rho(z, z^*, m, \mu) = G(z = m, z^* = m, m, \mu)$$

we have solved the Silver Blaze problem if we can show that the spectral density satisfies the constraint with $G(z = m, z^* = m, m, \mu)$ independent of $\mu$. At $\mu = 0$, the discontinuity of the chiral condensate is due to an accumulation of eigenvalues on the imaginary axis (Banks-Casher relation). At $\mu \neq 0$ the QCD Dirac operator does not have any hermiticity properties. The eigenvalues are scattered in the complex plane and, because of the phase of the fermion determinant, the eigenvalue density is complex. The solution of the Silver Blaze problem will provide us with different relation between the discontinuity of the chiral condensate and the spectral density.
4. The Dirac Spectrum at Nonzero Chemical Potential

The spectrum of the quenched Dirac operator is well understood\textsuperscript{15,16,17,18,22,19,20}.

In the mean field limit limit at fixed $\mu \ll \Lambda_{\text{QCD}}$ the eigenvalues are scattered uniformly in the strip $|\text{Re}(z)| < 2\mu^2 F^2 / \Sigma$. The mean field limit of the quenched resolvent is therefore given by

$$G_Q(z, z^*, \mu) = \frac{\text{Re}(z)\Sigma^2}{2\mu^2 F^2} \quad \text{for} \quad |\text{Re}(z)| < 2\mu^2 F^2 / \Sigma,$$

$$G_Q(z, z^*, \mu) = \Sigma \quad \text{for} \quad |\text{Re}(z)| > 2\mu^2 F^2 / \Sigma. \quad (9)$$

In the unquenched case we do not have a simple expression for the resolvent except that it has to satisfy the constraint (7) at zero temperature.

The spectral density of the Dirac operator can be obtained from \textsuperscript{16,21}

$$\rho(z, z^*, m, \mu) = \lim_{n \to 0} \frac{1}{\pi n} \partial_z \partial_{z^*} \log Z_n(z, z^*, m, \mu) \quad (11)$$

with generating function $Z_n(z, z^*, m, \mu)$ defined by \textsuperscript{16,22}

$$Z_n(z, z^*, m, \mu) = \langle \det(D + \mu \gamma_0 + m) \det^n(D + \mu \gamma_0 + z) \det^n(D + \mu \gamma_0 + z^*) \rangle = \langle \det(D + \mu \gamma_0 + m) \det^n(D + \mu \gamma_0 + z) \det^n(D - \mu \gamma_0 + z^*) \rangle. \quad (12)$$

In general, the replica limit $n \to 0$ is not well defined\textsuperscript{23}. However, for $m, |z| \ll F^2 / (\Sigma \sqrt{V})$ and $\mu \ll \Lambda_{\text{QCD}}$ this family of partition functions satisfies the Toda lattice hierarchy which allows one to obtain the $n \to 0$ result from a recursion relation\textsuperscript{24,25} both in the quenched\textsuperscript{28} and in the unquenched case\textsuperscript{30}. The unquenched spectral density, which was first obtained\textsuperscript{26} from an equivalent random matrix model, is a strongly oscillating complex function (see Fig. 3). The amplitude of the oscillations grows exponentially with $\mu^2 F^2 V$ while the period goes as $1 / \Sigma V$. It can be shown\textsuperscript{27} that the resolvent, obtained by integrating the oscillating spectral density, satisfies the constraint that $G(z = m, z^* = m, m, \mu)$ is independent of $\mu$. The discontinuity of the chiral condensate is $\mu$-independent with contributions from the entire oscillating region\textsuperscript{27}. This solves the Silver Blaze problem.

The spectral density can also be evaluated in the mean field limit. Then the resolvent does not depend on the number of replicas\textsuperscript{16,21} and the replica limit can be taken trivially. Assuming that this procedure gives the correct result we simply consider $n = 1$. The mean field generating function \textsuperscript{12} for $n = 1$ and real $z$ was analyzed before\textsuperscript{21} by means of a chiral Lagrangian. Interpreting $m$ as the strange quark mass and $z = z^*$ as the light quark masses, the valence “pion” and “kaon” masses are given by,

$$m_{\pi}^2 = \frac{2x \Sigma}{F^2} \quad \text{and} \quad m_K^2 = \frac{(x + m)\Sigma}{F^2} \quad \text{with} \quad x = \text{Re}(z) \quad (13)$$

respectively. With this identification, the isospin chemical potential is equal to $\mu_I = 2\mu$ and the strangeness chemical potential $\mu_s = -\mu$. The phase diagram in the $x - \mu^2$ plane for $\text{Im}(z) = 0$ is shown in the left panel of Fig. 3. We observe three different
phases, a normal phase (N), a pion condensation phase (π) and a kaon condensation phase (K) bordered by the dashed curve and the solid curve \( x = 4\mu^2 F^2 / \Sigma - m \). At fixed chemical potential, a transition from the π-phase to the K-phase takes place when \( m_{\nu,\pi} = m_{\nu,K} \). Evaluating the mean field resolvent in each of the phases we find

\[
G_N = \Sigma, \quad G_K = \Sigma, \quad G_\pi = \frac{\text{Re}(z)\Sigma^2}{2\mu^2 F^2}. \tag{14}
\]

The resulting spectral density is zero in the N-phase and a constant in the π-phase in agreement with the thermodynamic limit of the exact spectral density. However, the mean field spectral density in the K-phase is zero rather than oscillatory. The discontinuity of the mean field resolvent across the interface of the π-phase and the K-phase gives a delta function in the spectral density. As the quark mass approaches zero, the interface approaches zero resulting in a discontinuity of the chiral condensate due to an accumulation of eigenvalues. This solves the Silver Blaze problem for the mean field spectral density for \( N_f = 1 \) obtained from \( n = 1 \).

5. Discussion

We have described two different mechanisms to obtain a discontinuity in the chiral condensate at nonzero chemical potential. First, in case of the low energy limit of the full QCD partition function, the discontinuity arises from the oscillating part of the spectral density. Second, using a replica mean field result for the spectral density we find that the discontinuity moves with \( m \) and reaches the imaginary axis for \( m = 0 \). At this moment it is not yet clear whether or how these two pictures
are related. One important observation is that the oscillating region of the spectral density is absent for $\mu < m_\pi/2$. We expect that the sign problem is less severe in this region of the phase diagram which could open the door to realistic lattice simulations.

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