Odd/even cube-full numbers

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Abstract: In this paper we use an elementary method to give an asymptotical ratio of odd to even cube-full numbers and show that it is asymptotically $1 : 1 + 2^{-1/3} + 2^{-2/3}$.

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1 Introduction and result

Let $k > 1$ be a fixed integer. A positive integer $n$ is said to be $k$-full if each of its prime factors appears to the power at least $k$. For $k = 2, 3$, these numbers are called square-full and cube-full respectively. Let $N_k(x)$ be the number of $k$-full integers $\leq x$. In 1935, Erdős and Szekeres [1] proved that for $k$ fixed

$$N_k(x) = x^{1/k} \prod_p \left( 1 + \sum_{m=k+1}^{2k-1} p^{-m/k} \right) + O(x^{1/(k+1)}).$$

(1)

For a study of these asymptotic formulae, we refer to [2, Chapter 14.4 ].

In this paper, we study the odd/even dichotomy for the set of cube-full numbers. The motivation follows from work by Scott [4] and Jameson [3], where it was shown that the ratio of odd to even
square-free numbers is asymptotically $2 : 1$. (A positive integer $n$ is called square-free if it is not divisible by the square of any prime). Very recently, Srichan [5] used an elementary method to prove that the ratio of odd to even square-full numbers is asymptotically $1 : 1 + \sqrt{2}/2$. Then, it would be interesting to consider the odd/even dichotomy for the set of cube-full numbers.

Let $G$ be the set of all cube-full numbers. Let $G(x)$, $G_{\text{odd}}(x)$ and $G_{\text{even}}(x)$ be the set of all cube-full numbers, odd cube-full numbers and even cube-full numbers in the interval $[1, x]$, respectively. We denote by $N(x)$, $N_{\text{odd}}(x)$ and $N_{\text{even}}(x)$ the number of members of $G(x)$, $G_{\text{odd}}(x)$ and $G_{\text{even}}(x)$, respectively. We prove the following theorem.

**Theorem 1.1.** As $x \to \infty$, we have

$$\frac{N_{\text{odd}}(x)}{N_{\text{even}}(x)} \sim 2 - 2^{2/3}. \quad (2)$$

**2 Proof of Theorem 1.1**

First, we assume that

$$N_{\text{odd}}(x) \sim a x^{1/3} \quad \text{and} \quad N_{\text{even}}(x) \sim b x^{1/3}, \quad \text{for some } a, b \in \mathbb{R}^+. \quad (3)$$

We wish to show that,

$$\frac{a}{b} = 2 - 2^{2/3}. \quad (4)$$

For an even cube-full number $n$, we have $2 \mid n$, then also $8 \mid n$. Thus, there are no cube-full numbers $n$ such that $n \equiv 2, 4, 6 \pmod{8}$. Then we write $G_{\text{even}}(x) = \{ n \leq x, \ n \in G \text{ and } 8 \nmid n \}$ and $G_{\text{odd}}(x) = \{ n \leq x, \ n \in G \text{ and } n \equiv 1, 3, 5, 7 \pmod{8} \}$. Next, we split $G_{\text{even}}(x)$ into the set $G_{\text{even}1}(x)$ and the set $G_{\text{even}2}(x)$, where $G_{\text{even}1}(x) = \{ n \leq x, \ n \in G_{\text{even}}(x) \text{ and } \frac{n}{8} \in G \}$ and $G_{\text{even}2}(x) = \{ n \leq x, \ n \in G_{\text{even}}(x) \text{ and } \frac{n}{8} \notin G \}$. Let $N_{\text{even}1}(x)$ and $N_{\text{even}2}(x)$ be the number of members of $G_{\text{even}1}(x)$ and $G_{\text{even}2}(x)$, respectively. It is easy to prove that

$$N_{\text{even}1}(x) = N(x/8). \quad (5)$$

Now we will show that

$$N_{\text{even}2}(x) = N_{\text{odd}}(x/16) + N_{\text{odd}}(x/32). \quad (6)$$

A positive integer $n \in G_{\text{even}2}(x)$ has the form as $2^r m$, with $m$ being an odd cube-full number and $r = 4, 5$. Thus, we write $G_{\text{even}2}(x) = G_{\text{even}21}(x) \cup G_{\text{even}22}(x),$ where

$$G_{\text{even}21}(x) = \{ n \leq x, \ n \in G_{\text{even}2}(x) \text{ and } n = 16m \text{ with } m \text{ being odd cube-full } \},$$

and

$$G_{\text{even}22}(x) = \{ n \leq x, \ n \in G_{\text{even}2}(x) \text{ and } n = 32m \text{ with } m \text{ being odd cube-full } \}.$$

Formula (6) follows at once.
In view of (5) and (6), we have

\[ N_{\text{even}}(x) = N(x/8) + N_{\text{odd}}(x/16) + N_{\text{odd}}(x/32). \]  

(7)

Then,

\[ N_{\text{even}}(x) = (N_{\text{even}}(x/8) + N_{\text{odd}}(x/8)) + N_{\text{odd}}(x/16) + N_{\text{odd}}(x/32). \]

In view of (3), we have

\[ bx^{1/3} = \frac{b}{2}x^{1/3} + \frac{a}{2}x^{1/3} + \frac{a}{24/5}x^{1/3} + \frac{a}{25/3}x^{1/3}. \]

This proves (4).

Now it remains to prove the existence of \( a \) and \( b \).

In view of (7), we write

\[ N(x) - N_{\text{odd}}(x) = N(x/8) + N_{\text{odd}}(x/16) + N_{\text{odd}}(x/32) \]

\[ N(x) - N(x/8) = N_{\text{odd}}(x) + N_{\text{odd}}(x/16) + N_{\text{odd}}(x/32). \]

We write \( f(x) = N(x) - N(x/8) \), then we have

\[ f(x) = N_{\text{odd}}(x) + N_{\text{odd}}(x/16) + N_{\text{odd}}(x/32). \]  

(8)

In view of (1), we have

\[ f(x) \sim c x^{1/3}, \]

(9)

for a certain \( c > 0 \). By the mathematical induction on \( m \geq 0 \) and (8), we have

\[ N_{\text{odd}}(x) = \sum_{j=0}^{m} (-1)^j \sum_{i=0}^{j} \binom{j}{i} f\left( \frac{x}{2^{4j+i}} \right) - (-1)^m m+1 \sum_{i=0}^{m+1} \binom{m+1}{i} N_{\text{odd}}\left( \frac{x}{2^{4m+4+i}} \right). \]

For \( m > \log_2 x^{1/4} - 1 \), we have

\[ N_{\text{odd}}(x) = \sum_{j=0}^{\infty} \sum_{i=0}^{j} \binom{j}{i} f\left( \frac{x}{2^{4j+i}} \right) \]

\[ = \sum_{j=0}^{\infty} \sum_{i=0}^{2j+1} \binom{2j+1}{i} f\left( \frac{x}{2^{8j+i}} \right) - \sum_{j=0}^{\infty} \sum_{i=0}^{2j+1} \binom{2j+1}{i} f\left( \frac{x}{2^{8j+i+4}} \right). \]

In view of (9), we know that, for \( \epsilon > 0 \), and for some \( x_0 \),

\[ (c - \epsilon) x^{1/3} \leq f(x) \leq (c + \epsilon) x^{1/3}, \quad \text{for} \quad x > x_0. \]

We note that the inequality \( f(y) \leq (c+\epsilon)y^{1/3} \) only applies to the terms \( y = x/2^{4j+i} \) if \( x/2^{5j} \geq x_0 \).

There exists a positive \( M \) such that \( f(y) \leq M y^{1/3} \) for all \( y \geq 1 \). Suppose that \( k \) and \( x \) are such that \( x \geq 2^{5k}x_0 \). For \( j > k \), we have

\[ \sum_{i=0}^{j} \binom{j}{i} f\left( \frac{x}{2^{4j+i}} \right) \leq M x^{1/3} 2^{-4j/3} \sum_{i=0}^{j} \binom{j}{i} 2^{-i/3} = M \alpha^j x^{1/3}, \]

with \( \alpha = 16^{-1/3} + 32^{-1/3} \). Now we choose \( k \geq \log_{\alpha} \frac{\epsilon(1-\alpha)}{M} - 1 \), we have

\[ M \sum_{j>k} \alpha^j \leq \epsilon. \]  

(10)
In view of (10), and (11) we have

\[
N_{\text{odd}}(x) \geq (c - \epsilon) \sum_{j=0}^{\infty} \sum_{i=0}^{2j} \binom{2j}{i} \frac{x^{1/3}}{2(8j+i+4)/3} - (c + \epsilon) \sum_{j=0}^{k} \sum_{i=0}^{2j+1} \binom{2j+1}{i} \frac{x^{1/3}}{2(8j+i+4)/3} - M \sum_{j>k} \sum_{i=0}^{2j+1} \binom{2j+1}{i} \frac{x^{1/3}}{2(8j+i+4)/3}
\]

\[
= (c - \epsilon)x^{1/3} \sum_{j=0}^{\infty} 2^{-8j/3} \sum_{i=0}^{2j} \binom{2j}{i} 2^{-i/3} - (c + \epsilon)x^{1/3} \sum_{j=0}^{k} 2^{-8j/3} \sum_{i=0}^{2j+1} \binom{2j+1}{i} 2^{-i/3}
\]

\[
- Mx^{1/3} \sum_{j>k} 2^{-8j/3} \sum_{i=0}^{2j+1} \binom{2j+1}{i} 2^{-i/3}
\]

\[
= (c - \epsilon)x^{1/3} \sum_{j=0}^{\infty} 2^{-8j/3} \left(2^{-1/3} + 1\right)^{2j} - (c + \epsilon)x^{1/3} \sum_{j=0}^{k} 2^{-8j/3} \left(2^{-1/3} + 1\right)^{2j+1}
\]

\[
- Mx^{1/3} \sum_{j>k} 2^{-8j/3} \left(2^{-1/3} + 1\right)^{2j+1}
\]

\[
\geq (c - \epsilon)x^{1/3} \sum_{j=0}^{\infty} \left(2^{-5/3} + 2^{-4/3}\right)^{2j} - (c + \epsilon)x^{1/3} \sum_{j=0}^{\infty} \left(2^{-5/3} + 2^{-4/3}\right)^{2j+1}
\]

\[
- Mx^{1/3} \sum_{j>k} \left(2^{-5/3} + 2^{-4/3}\right)^{2j+1}
\]

\[
\geq (c - \epsilon)x^{1/3} \sum_{j=0}^{\infty} \alpha^{2j} - (c + \epsilon)x^{1/3} \sum_{j=0}^{\infty} \alpha^{2j+1} - Mx^{1/3} \sum_{j>k} \alpha^{2j}.
\]  

(11)

In view of (10), and (11) we have

\[
N_{\text{odd}}(x) \geq \left(\frac{c}{1+\alpha} - \frac{\epsilon}{1-\alpha} - \epsilon\right)x^{1/3}.
\]  

(12)

Similary, we have

\[
N_{\text{odd}}(x) \leq \left(\frac{c}{1+\alpha} + \frac{\epsilon}{1-\alpha} + \epsilon\right)x^{1/3}.
\]  

(13)

In view of (12) and (13), we have

\[
\left(\frac{c}{1+\alpha} - \frac{\epsilon}{1-\alpha} - \epsilon\right)x^{1/3} \leq N_{\text{odd}}(x) \leq \left(\frac{c}{1+\alpha} + \frac{\epsilon}{1-\alpha} + \epsilon\right)x^{1/3}.
\]  

(14)

The existence of \(a\) follows from (14) and by the similar proof the existence of \(b\) is obtained.

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