Strong Error Estimates for a Space-Time Discretization of the Linear-Quadratic Control Problem with the Stochastic Heat Equation with Linear Noise

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Abstract

We propose a time-implicit, finite-element based space-time discretization of the necessary and sufficient optimality conditions for the stochastic linear-quadratic optimal control problem with the stochastic heat equation driven by linear noise of type \[ X(t) + \sigma(t) dW(t), \] and prove optimal convergence w.r.t. both, space and time discretization parameters. In particular, we employ the stochastic Riccati equation as a proper analytical tool to handle the linear noise, and thus extend the applicability of the earlier work [16], where the error analysis was restricted to additive noise.

Keywords: Error estimate, stochastic linear quadratic problem, stochastic heat equation, Pontryagin’s maximum principle, stochastic Riccati equation

AMS 2010 subject classification: 49J20, 65M60, 93E20

1 Introduction

Let \( D \subset \mathbb{R}^d \) be a bounded domain with \( C^2 \) boundary and \( T > 0 \) be given. Our goal is to numerically approximate the \( \mathbb{F} \)-adapted control process \( U^* = \{ U^*(t); t \in [0, T] \} \) on the filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) that minimizes the quadratic functional \((\alpha \geq 0)\)

\[
J(X, U) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \|X(t)\|_{L^2}^2 + \|U(t)\|_{L^2}^2 \, dt + \alpha \|X(T)\|_{L^2}^2 \right]
\]

subject to the (controlled forward) stochastic PDE (SPDE, for short) of the form

\[
\begin{cases}
    dX(t) = [\Delta X(t) + U(t)] \, dt + [X(t) + \sigma(t)] \, dW(t) & \forall t \in [0, T], \\
    X(0) = X_0,
\end{cases}
\]

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for a proper function $\sigma : [0, T] \to L^2$, a suitable initial datum $X_0$, homogeneous Dirichlet boundary conditions, and a Wiener process $W := \{ W(t) : t \in [0, T] \}$, which here is $\mathbb{R}$-valued for the sake of simplicity. A unique (strong) minimizer $(X^*, U^*)$ of the stochastic optimal control problem: ‘minimize (1.1) subject to (1.2)’ may then be deduced — which we below refer to as problem SLQ; see e.g. [2, 13], and Section 2 for a further specification of the data.

SLQ is a prototypic stochastic optimization problem on (infinite-dimensional) Hilbert spaces, for which the numerical analysis so-far is rare in the literature; cf. the references in [16]. In the foregoing work [16], optimal strong error estimates were shown for a space-time discretization of a corresponding problem where the equation (1.2) was driven by additive noise. On the level of Pontryagin’s maximum principle that we apply below to develop numerical methods for SLQ this difference, in particular, simplified a lot the (numerical analysis of the) backward stochastic PDE (BSPDE, for short), which for the current problem SLQ with solution tuple $(Y, Z)$ reads:

\[
\begin{align*}
    dY(t) &= [-\Delta Y(t) - Z(t) + X(t)] \, dt + Z(t) \, dW(t) \quad \forall t \in [0, T], \\
    Y(T) &= -\alpha X(T).
\end{align*}
\] (1.3)

Note that in the case of additive noise in (1.2), $Z(t)$ in (1.3) does not appear in the drift term, which is the reason why the tools which were developed for the corresponding numerical analysis in [16] do to cover the case of linear noise as present in (1.2). For SLQ, the optimality system consists of (1.2), (1.3) and Pontryagin’s maximum condition

\[
0 = U(t) - Y(t) \quad \forall t \in (0, T),
\] (1.4)

which then uniquely determines the optimal process tuple denoted by $(X^*, U^*)$ of problem SLQ.

Based on Pontryagin’s maximum principle, problem SLQ may be numerically accessed by solving the coupled system (1.2), (1.3) and (1.4), which is a forward-backward stochastic PDEs (FBSPDE, for short). In order to derive strong error estimates in [16] for the related stochastic control problem with additive noise in (1.2), a spatial semi-discretization via finite elements (with step size parameter $h$ for the mesh) with solution $(X_h^*, Y_h, Z_h, U_h^*)$ was considered in a first step, for which optimal convergence rates were obtained; the key link to show optimal strong error estimates for the full space-time discretization (with additional time step parameter $\tau$) in a second step then depended on the $h$-independent stability results for the above solution quadruple $(X_h^*, Y_h, Z_h, U_h^*)$, whose derivation via Malliavin calculus rested on the fact that $Z_h$ did not enter the drift part in (the finite element version of) the corresponding modification of BSPDE (1.3) within (the finite element version of) the coupled optimality system FBSPDE. In this work, we use the stochastic Riccati equation (3.22) as proper representation tool for the solution of the semi-discretization SLQ$_h$ (see (3.3)–(3.4)) to deduce the relevant $h$-independent stability results for the solution quadruple $(X_h^*, Y_h, Z_h, U_h^*)$ of the related optimality system (FBSPDE)$_h$, which is (3.4)–(3.6); these results may then be used to prove optimal convergence rates for optimal tuple of the space-time discretization SLQ$_{h\tau}$, which is (3.9)–(3.10) in equivalent form. Specifically, a relevant result is the following, which bounds the temporal variation of the component $Z_h$ of the solution to (FBSPDE)$_h$ (see Lemmata 3.8, 3.13),

\[
\mathbb{E}[\|Z_h(t) - Z_h(s)\|_{L^2}^2] \leq C |t - s| \quad \forall t, s \in [0, T],
\] (1.5)

where $C > 0$ is independent of $h$. We remark that (3.9)–(3.10) is a modification of the implicit Euler method, which is again due to the role that $Z_h$ plays in the drift part in (FBSPDE)$_h$, and that needs be properly addressed numerically; see Remark 3.4.
To computationally solve this discrete, coupled optimality system requires huge computational resources; instead, we again return to the fully discretized problem SLQ_{h,\tau} (3.7)-(3.8) and exploit its character as a minimization problem to initiate a decoupled gradient descent method to successively determine approximations of the optimal control; this method, which is close to the one in [8] where also computational experiments are provided, is detailed in Section 4, and an optimal convergence rate is shown for this iteration — which is the second goal in this work.

The rest of this paper is organized as follows. In Section 2, we introduce notations, and review relevant rates of convergence for a discretization in space and time of SPDE (1.2) and a semi-discretization in space of BSPDE (1.3), which are both needed in Section 3. In Section 3, we prove rates of convergence for a space-time discretization of a coupled FBSPDE, which is related to problem SLQ. Convergence of the related iterative gradient descent method towards the optimal pair (X^*, U^*) of problem SLQ is shown in Section 4.

2 Preliminaries

2.1 Notations and assumptions — involved processes and the finite element method

Let \((\mathcal{K}, (\cdot, \cdot)_{\mathcal{K}})\) be a separable Hilbert space. By \(\| \cdot \|_{L^2} \) resp. \((\cdot, \cdot)_{L^2}\), we denote the norm resp. the scalar product in Lebesgue space \(L^2 := L^2(D)\). By \(\| \cdot \|_{H^1}, \| \cdot \|_{H^2}, \| \cdot \|_{H^3}\), we denote norms in Sobolev spaces \(H^1 := H^1_0(D), H^2 := H^2(D), H^3 := H^3(D)\) respectively. Let \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})\) be a complete filtered probability space, where \(\mathbb{P} = \{\mathcal{F}_t\}_{t \in [0, T]}\) is the filtration generated by the \(\mathbb{P}\)-valued Wiener process \(W\), which is augmented by all the \(\mathbb{P}\)-null sets. The space of all \(\mathbb{F}\)-adapted processes \(X : \Omega \times [0, T] \to \mathbb{K}\) satisfying \(\mathbb{E}[\int_0^T \|X(t)\|^2_{\mathbb{K}} \, dt] < \infty\) is denoted by \(L^2_\mathbb{F}(0, T; \mathbb{K})\); the space of all \(\mathbb{F}\)-adapted processes \(X : \Omega \times [0, T] \to \mathbb{K}\) with continuous path satisfying \(\mathbb{E}[\sup_{t \in [0, T]} \|X(t)\|^2_{\mathbb{K}}] < \infty\) is denoted by \(L^2_\mathbb{F}(\Omega; C([0, T]; \mathbb{K}))\); for any \(t \in [0, T]\), the space of \(\mathbb{K}\)-valued \(\mathcal{F}_t\)-measurable random variables \(\eta\) satisfying \(\mathbb{E}[\|\eta\|^2_{\mathbb{K}}] < \infty\) is denoted by \(L^2_{\mathbb{F}}(\Omega; \mathbb{K})\).

We partition the bounded domain \(D \subset \mathbb{R}^d\) via a regular triangulation \(T_h\) into elements \(K\) with maximum mesh size \(h := \max\{\text{diam}(K) : K \in T_h\}\), and consider space

\[
\mathcal{V}_h := \{\phi \in H^1_0 : \phi|_K \in \mathcal{P}_1(K) \quad \forall \, K \in T_h\},
\]

where \(\mathcal{P}_1(K)\) denotes the space of polynomials of degree 1; see e.g. [4]. We define the discrete Laplacian \(\Delta_h : \mathcal{V}_h \to \mathcal{V}_h\) by \((-\Delta_h \xi_h, \phi_h)_{L^2} = (\nabla \xi_h, \nabla \phi_h)_{L^2}\) for all \(\xi_h, \phi_h \in \mathcal{V}_h\), the \(L^2\)-projection \(\Pi_h : L^2 \to \mathcal{V}_h\) by \((\Pi_h \xi - \xi, \phi_h)_{L^2} = 0\) for all \(\xi \in L^2, \phi_h \in \mathcal{V}_h\), and the Ritz-projection \(\mathcal{R}_h : H^1_0 \to \mathcal{V}_h\) by \((\nabla \mathcal{R}_h \xi - \xi, \nabla \phi_h)_{L^2} = 0\) for all \(\xi \in H^1_0, \phi_h \in \mathcal{V}_h\). Via definition of \(\mathcal{R}_h\), it is easy to get that

\[
\|\nabla \mathcal{R}_h \xi\|_{L^2} \leq \|\nabla \xi\|_{L^2} \quad \forall \, \xi \in H^1_0. \quad (2.1)
\]

We denote by \(I_\tau = \{t_n\}_{n=0}^N \subset [0, T]\) a time mesh with maximum step size \(\tau := \max\{t_{n+1} - t_n : n = 0, 1, \cdots, N - 1\}\), and \(\Delta_n W := W(t_n) - W(t_{n-1})\) for all \(n = 1, \cdots, N\). For given time mesh \(I_\tau\), we can define \(\mu(\cdot), \nu(\cdot)\) by

\[
\mu(t) := t_{n+1}, \quad \nu(t) := t_n \quad \forall \, t \in [t_n, t_{n+1}), \, n = 0, 1, \cdots, N - 1. \quad (2.2)
\]

Throughout this work, we assume that \(\tau \leq 1\). For simplicity, we choose a uniform partition, i.e. \(\tau = T/N\). The results in this work still hold for quasi-uniform partitions. Throughout this work,
we shall make use of the following

**Assumption (A)**: \( X_0 \in \mathbb{H}_0^1 \cap \mathbb{H}^3 \), and \( \sigma \in C^1(0,T;\mathbb{H}_0^1 \cap \mathbb{H}^3) \).

### 2.2 The stochastic heat equation and rates of strong convergence for its space-time discretization

Under Assumption (A), in particular, and given \( U \in L^2_{\mathbb{F}}(0,T;\mathbb{H}_0^1) \) in SPDE (1.2), there exists a unique strong solution \( X \in L^2_{\mathbb{F}}(0,T;\mathbb{H}_0^1) \) which satisfies the following estimate (see e.g. [5]),

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \|X(t)\|_{\mathbb{H}_0^1}^2 + \int_0^T \|X(t)\|_{\mathbb{H}_0^1}^2 \, dt \right] \leq C \mathbb{E}\left[ \|X_0\|_{\mathbb{H}_0^1}^2 + \int_0^T \|U(t)\|_{\mathbb{H}_0^1}^2 + \|\sigma(t)\|_{\mathbb{H}_0^1}^2 \, dt \right].
\]

(2.3)

Obviously, \( X \) satisfies the following variational form \( \mathbb{P} \)-a.s. for all \( t \in [0,T] \),

\[
(X(t), \phi)_{L^2} - (X_0, \phi)_{L^2} + \int_0^t \left( \nabla X(s), \nabla \phi \right)_{L^2} - \left( U(s), \phi \right)_{L^2} \, ds = \int_0^t \left( X(s) + \sigma(s), \phi \right)_{L^2} \, dW(s), \quad \forall \phi \in \mathbb{H}_0^1.
\]

(2.4)

A finite element discretization of (2.4) which we later refer to as SPDE\(_h\) then reads: For all \( t \in [0,T] \), find \( X_h \in L^2_{\mathbb{F}}(0,T;\mathbb{H}_0^1) \) such that \( \mathbb{P} \)-a.s. and for all times \( t \in [0,T] \)

\[
(X_h(t), \phi_h)_{L^2} - (X_h(0), \phi_h)_{L^2} + \int_0^t \left( \nabla X_h(s), \nabla \phi_h \right)_{L^2} - \left( U(s), \phi_h \right)_{L^2} \, ds = \int_0^t \left( X_h(s) + \mathcal{R}_h \sigma(s), \phi_h \right)_{L^2} \, dW(s), \quad \forall \phi_h \in \mathbb{V}_h.
\]

(2.5)

Equation (2.5) may be recast into the following stochastic differential equation,

\[
\begin{align*}
\frac{dX_h(t)}{dt} &= \left[ \Delta_h X_h(t) + \Pi_h U(t) \right] \, dt + \left[ X_h(t) + \mathcal{R}_h \sigma(t) \right] \, dW(t) \quad \forall t \in [0,T], \\
X_h(0) &= \mathcal{R}_h X_0.
\end{align*}
\]

(2.6)

Thanks to this equivalence, we do not distinguish between SPDE\(_h\) (2.5) and equation (2.6) throughout this paper.

The derivation of an error estimate is well-known (see e.g. [21]), which uses the improved (spatial) regularity properties of the strong solution to deduce

\[
\sup_{t \in [0,T]} \mathbb{E}\left[ \|X_h(t) - X(t)\|_{L^2}^2 \right] + \mathbb{E}\left[ \int_0^T \|\nabla [X_h(t) - X(t)]\|_{L^2}^2 \, dt \right] \leq C h^2.
\]

(2.7)

We now consider a time-implicit discretization of (2.5) on a partition \( I_\tau \) of \([0,T]\). The problem then reads: For every \( 0 \leq n \leq N - 1 \), find a solution \( X_h^{n+1} \in L^2_{\mathbb{F}_{n+1}}(\Omega;\mathbb{V}_h) \) such that \( \mathbb{P} \)-a.s.

\[
(X_h^{n+1} - X_h^n, \phi_h)_{L^2} + \tau \left[ \left( \nabla X_h^{n+1}, \nabla \phi_h \right)_{L^2} - \left( U(t_n), \phi_h \right)_{L^2} \right] = (X_h^n, \phi_h)_{L^2} \Delta \tau W.
\]

(2.8)

The verification of the error estimate (see [21])

\[
\max_{0 \leq n \leq N} \mathbb{E}\left[ \|X_h(t_n) - X_h^n\|_{L^2}^2 \right] + \tau \sum_{n=1}^N \mathbb{E}\left[ \|\nabla [X_h(t_n) - X_h^n]\|_{L^2}^2 \right] \leq C \tau
\]

(2.9)
rests on stability properties of the implicit Euler, Assumption (A), as well as the further assumption that
\[ \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \|U(t) - U(t_n)\|_{L^2}^2 \, dt \right] \leq C_T. \]

When solving problem SLQ, we select the optimal control \( U^* \) for \( U \), in particular, which inherits improved regularity conditions from the solution of the adjoint equation to the optimal control \( U^* \) via the optimality condition.

2.3 The backward stochastic heat equation — a finite element based spatial discretization

Let \( Y_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{H}_0^1) \) and \( f \in L^2_{\mathcal{F}}(0,T; \mathbb{L}^2) \). A strong solution to the backward stochastic heat equation
\[
\begin{aligned}
&\text{d}Y(t) = [\Delta Y(t) - Z(t) + f(t)] \, dt + Z(t) \, dW(t) \quad \forall \, t \in [0,T], \\
&Y(T) = Y_T
\end{aligned}
\]
(2.10)
with homogeneous Dirichlet boundary data is a pair of square integrable \( \mathbb{F} \)-adapted processes \((Y, Z) \in (L^2_{\mathbb{F}}(\Omega; C([0,T]; \mathbb{H}_0^1)) \cap L^2_{\mathcal{F}}(0,T; \mathbb{H}_0^1 \cap \mathbb{H}^2)) \times L^2_{\mathcal{F}}(0,T; \mathbb{H}_0^1)\) that satisfies the following variational form \( \mathbb{P} \)-a.s. for all \( t \in [0,T] \),
\[
(Y(t), \phi)_{L^2} - (Y(t), \phi)_{L^2} - \int_t^T (\nabla Y(s), \nabla \phi)_{L^2} - (Z(s), \phi)_{L^2} + (f(s), \phi)_{L^2} \, ds
= \int_t^T (Z(s), \phi)_{L^2} \, dW(s) \quad \forall \, \phi \in \mathbb{H}_0^1.
\]
(2.11)
The existence of a strong solution to (2.10), as well as its uniqueness are shown in [7]; moreover, there exists a constant \( C \equiv C(D,T) > 0 \) such that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|Y(t)\|_{\mathbb{H}_0^1}^2 \right] + \mathbb{E} \left[ \int_0^T \|Y(t)\|_{\mathbb{H}_0^1}^2 + \|Z(t)\|_{\mathbb{H}_0^1}^2 \, dt \right] \leq C \mathbb{E} \left[ \|Y_T\|_{\mathbb{H}_0^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 \, dt \right].
\]
(2.12)

We may consider a finite element discretization of the BSPDE (2.10). Let \( Y_{T,h} = \mathcal{R}_h Y_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{V}_h) \), an approximation of \( Y_T \). The problem BSPDE\(_h\) then reads: Find a pair \((Y_h, Z_h) \in L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{V}_h)) \times L^2_{\mathcal{F}}(0,T; \mathbb{V}_h)\) such that \( \mathbb{P} \)-a.s. for all \( t \in [0,T] \)
\[
(Y_{T,h}, \phi_h)_{L^2} - (Y_h(t), \phi_h)_{L^2} = \int_t^T (\nabla Y_h(s), \nabla \phi_h)_{L^2} - (Z(s), \phi_h)_{L^2} + (f(s), \phi_h)_{L^2} \, ds
= \int_t^T (Z_h(s), \phi_h)_{L^2} \, dW(s) \quad \forall \, \phi_h \in \mathbb{V}_h.
\]
(2.13)
Actually, equation (2.13) is equivalent to the following BSDE:
\[
\begin{aligned}
&\text{d}Y_h(t) = [-\Delta_h Y_h(t) - Z_h(t) + \Pi_h f(t)] \, dt + Z_h(t) \, dW(t) \quad \forall \, t \in [0,T], \\
&Y_h(T) = Y_{T,h}
\end{aligned}
\]
(2.14)
and we do not distinguish between BSPDE\(_h\) (2.13) and BSDE (2.14) throughout this paper. The existence and uniqueness of a solution tuple \((Y_h, Z_h)\) follows from [9, Theorem 2.1]. Moreover, there exists \( C \equiv C(T) > 0 \) such that
\[
\sup_{t \in [0,T]} \mathbb{E} \left[ \|\nabla Y_h(t)\|_{L^2}^2 \right] + \mathbb{E} \left[ \int_0^T \|\Delta_h Y_h(t)\|_{L^2}^2 + \|\nabla Z_h(t)\|_{L^2}^2 \, dt \right] \leq C \mathbb{E} \left[ \|\nabla Y_{T,h}\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2 \, dt \right];
\]
The numerical analysis of a temporal discretization of problem SLQ — the role of Malliavin derivatives

2.4 Temporal discretization of problem SLQ — the role of Malliavin derivatives

The numerical analysis of a temporal discretization of problem SLQ requires Malliavin calculus to bound temporal increments such as $E[|Z_h(t) - Z_h(s)|^2]$ in terms of $|t - s|$, where $s, t \in [0, T]$, and $Z_h$ is the second component of BSDE (2.14) with $\Pi_h f = X_h^\ast$. We therefore briefly recall the definition and results of the Malliavin derivative of processes, which will be applied below. For further details, we refer to [15, 9].

We define the Itô isometry $\mathbb{W} : L^2(0, T; \mathbb{R}) \rightarrow L_{\mathcal{F}_T}^2(\Omega; \mathbb{R})$ by

$$\mathbb{W}(g) = \int_0^T g(t) \, dW(t).$$

For $\ell \in \mathbb{N}$, we denote by $C_p^\infty(\mathbb{R}^\ell)$ the space of all smooth functions $s : \mathbb{R}^\ell \rightarrow \mathbb{R}$ such that $s$ and all of its partial derivatives have polynomial growth. Let $\mathcal{P}$ be the set of $\mathbb{R}$-valued random variables of the form

$$F = s(\mathbb{W}(g_1), \mathbb{W}(g_2), \ldots, \mathbb{W}(g_\ell))$$

for some $s \in C_p^\infty(\mathbb{R}^\ell)$, $\ell \in \mathbb{N}$, and $g_1, \ldots, g_\ell \in L^2(0, T; \mathbb{R})$. To any $F \in \mathcal{P}$ we define its $\mathbb{R}$-valued Malliavin derivative $DF := \{D_\theta F; 0 \leq \theta \leq T\}$ process via

$$D_\theta F = \sum_{i=1}^\ell \frac{\partial s}{\partial x_i}(\mathbb{W}(g_1), \mathbb{W}(g_2), \ldots, W(g_\ell))g_i(\theta).$$

In general, we can define the $k$-th iterated derivative of $F$ by $D^k F = D(D^{k-1} F)$, for any $k \in \mathbb{N}$. Note that, generally, for any $\theta \in [0, T]$, $D_\theta F$ is $\mathcal{F}_T$-measurable; and if $F$ is $\mathcal{F}_T$-measurable, then $D_\theta F = 0$ for any $\theta \in (t, T)$.

Now we can extend the derivative operator to $\mathbb{K}$-valued variables. For any $k \in \mathbb{N}$, and $u$ in the set of $\mathbb{K}$-valued variables:

$$\mathcal{P}_K = \left\{ u = \sum_{j=1}^n F_j \phi_j : F_j \in \mathcal{P}, \phi_j \in \mathbb{K}, n \in \mathbb{N} \right\},$$

we can define the $k$-th iterated derivative of $u$ by $D^k u = \sum_{j=1}^n D^k F_j \otimes \phi_j$. For $p \geq 1$, we define the norm $\| \cdot \|_{k,p}$ via

$$\|u\|_{k,p} := \left( E[\|u\|_K^p] + \sum_{j=1}^k E[\|D^j u\|_{(L^2(0,T;\mathbb{R}))^\otimes j \otimes \mathbb{K}}^p] \right)^{1/p}.$$
3 Rates of convergence for a spatio-temporal discretization of problem SLQ

3.1 The discrete maximum principle and main results

In this part, we discretize the original problem SLQ within two steps, starting with its semi-discretization in space (which is referred to as SLQ$_h$), which is then followed by a discretization in space and time (which is referred to as SLQ$_{h,r}$). Our goal in this section is to prove rates of convergence in both cases. By [13], problem SLQ is uniquely solvable, and its optimal pair $(X^*, U^*)$ may be characterized by the following coupled FBSPDE (supplemented by homogeneous Dirichlet data for $X^*, Y$) with a unique solution $(X^*, Y, Z, U^*)$,

$$
\begin{cases}
\left\{
\begin{array}{l}
\d X^*(t) = \left[\Delta X^*(t) + U^*(t)\right]dt + \left[X^*(t) + \sigma(t)\right]dW(t) \quad \forall t \in (0, T), \\
\d Y(t) = \left[ -\Delta Y(t) - Z(t) + X^*(t)\right]dt + Z(t)dW(t) \quad \forall t \in (0, T), \\
X^*(0) = X_0, \quad Y(T) = -\alpha X^*(T),
\end{array}
\right.
\end{cases}
$$

(3.1)

with the condition

$$U^*(t) - Y(t) = 0 \quad \forall t \in (0, T).$$

(3.2)

By (3.2) and the fact that $Y \in L_2^2(\Omega; C([0, T]; \mathbb{H}_0^1))$, we find that the optimal control $U^*$ has continuous paths, taking zero values on the boundary $\partial D$.

The spatial semi-discretization SLQ$_h$ of problem SLQ reads as follows: Find an optimal pair $(X^*_h, U^*_h) \in L_2^2(\Omega; \mathcal{C}([0, T]; \mathbb{H}_h)) \times L_2^2(0, T; \mathbb{V}_h)$ that minimizes the functional

$$\mathcal{J}(X_h, U_h) = \frac{1}{2} \mathbb{E} \int_0^T \|X_h(t)\|^2_{L^2} + \|U_h(t)\|^2_{L^2} dt$$

subject to the equation

$$\begin{cases}
\left\{
\begin{array}{l}
\d X_h(t) = \left[\Delta_h X_h(t) + U_h(t)\right]dt + \left[X_h(t) + \mathcal{R}_h \sigma(t)\right]dW(t) \quad \forall t \in [0, T], \\
X_h(0) = \mathcal{R}_h X_0,
\end{array}
\right.
\end{cases}
$$

(3.4)

The existence of a unique optimal pair $(X^*_h, U^*_h)$ follows from [22], thanks to its characterization via Pontryagin’s maximum principle, i.e.,

$$0 = U^*_h(t) - Y_h(t) \quad \forall t \in (0, T),$$

(3.5)

where the adjoint $(Y_h, Z_h) \in L_2^2(\Omega; \mathcal{C}([0, T]; \mathbb{V}_h)) \times L_2^2(0, T; \mathbb{V}_h)$ solves the BSPDE$_h$

$$\begin{cases}
\left\{
\begin{array}{l}
\d Y_h(t) = \left[ -\Delta_h Y_h(t) - Z_h(t) + X^*_h(t)\right]dt + Z_h(t)dW(t) \quad \forall t \in [0, T], \\
Y_h(T) = -\alpha X^*_h(T).
\end{array}
\right.
\end{cases}
$$

(3.6)

In [8], error estimates have been obtained for $(X^*_h, Y_h, Z_h)$ with the help of a fixed point argument — which crucially exploits $T > 0$ to be sufficiently small. One main goal in this work is to derive corresponding estimates for $(X^*_h, Y_h, Z_h, U^*_h)$ for arbitrary $T > 0$ via a variational argument which exploits properties of the cost functional $\mathcal{J}$ in (3.3): once an estimate for $\mathbb{E} \left[ \int_0^T \|U^*(s) - U^*_h(s)\|^2_{L^2} ds \right]$ stands, we use the results from Sections 2.2 and 2.3 to derive estimates for the remaining processes in $(X^*_h, Y_h, Z_h, U^*_h)$. 


7
Theorem 3.1. Under Assumption (A), let \((X^*, U^*)\) be the solution to problem \(\text{SLQ}\), and 
\((X^*_h, U^*_h)\) solve problem \(\text{SLQ}_h\). Then, there exists \(C \equiv C(X_0, \sigma, T) > 0\) independent of \(h > 0\) such that

\[
\begin{align*}
(i) \quad & \mathbb{E} \left[ \int_0^T \|U^*(t) - U^*_h(t)\|_{L^2}^2 \, dt \right] \leq C h^2, \\
(ii) \quad & \sup_{0 \leq t \leq T} \mathbb{E} \left[ \|X^*(t) - X^*_h(t)\|_{L^2}^2 \right] + \mathbb{E} \left[ \int_0^T \|X^*(t) - X^*_h(t)\|_{\mathbb{H}^1_0}^2 \, dt \right] \leq C h^2, \\
(iii) \quad & \sup_{0 \leq t \leq T} \mathbb{E} \left[ \|Y(t) - Y^*_h(t)\|_{L^2}^2 \right] + \int_0^T \mathbb{E} \left[ \|Y(t) - Y^*_h(t)\|_{\mathbb{H}^1_0}^2 + \|Z(t) - Z^*_h(t)\|_{L^2}^2 \right] \, dt \leq C h^2.
\end{align*}
\]

We postpone its proof to Section 3.2. — In a second step, we propose a temporal discretization of problem \(\text{SLQ}_h\) which will be analyzed in Section 3.4. For this purpose, we use a mesh \(I_r\) covering \([0, T]\), and consider step size processes \((X_{t_n}, U_{t_n}) \in \mathbb{X}_{t_n} \times \mathbb{U}_{t_n} \subset L^2_2(0, T; \mathbb{V}_h) \times L^2_2(0, T; \mathbb{V}_h)\), where

\[
\mathbb{X}_{t_n} := \{ X \in L^2_2(0, T; \mathbb{V}_h) : X(t) = X(t_n), \forall t \in [t_n, t_{n+1}], \, n = 0, 1, \ldots, N-1 \},
\]

\[
\mathbb{U}_{t_n} := \{ U \in L^2_2(0, T; \mathbb{V}_h) : U(t) = U(t_n), \forall t \in [t_n, t_{n+1}], \, n = 0, 1, \ldots, N-1 \},
\]

and define for any \(X \in \mathbb{X}_{t_n}\) and \(U \in \mathbb{U}_{t_n}\),

\[
\|X\|_{\mathbb{X}_{t_n}} := \left( \tau \sum_{n=1}^{N} \mathbb{E} \left[ \|X(t_n)\|_{L^2}^2 \right] \right)^{1/2}, \quad \text{and} \quad \|U\|_{\mathbb{U}_{t_n}} := \left( \tau \sum_{n=0}^{N-1} \mathbb{E} \left[ \|U(t_n)\|_{L^2}^2 \right] \right)^{1/2}.
\]

Note that the norms of \(\mathbb{X}_{t_n}\), \(\mathbb{U}_{t_n}\) differ: the reason is that for a control system, we should control ‘from now on’ — when the current state is known, and the future state is what we care about.

Problem \(\text{SLQ}_{t_n}\) then reads as follows: Find an optimal pair \((X^*_h, U^*_h) \in \mathbb{X}_{t_n} \times \mathbb{U}_{t_n}\) which minimizes the quadratic cost functional

\[
\mathcal{J}_\tau(X_{t_n}, U_{t_n}) = \frac{1}{2} \|X_{t_n}\|_{\mathbb{X}_{t_n}}^2 + \|U_{t_n}\|_{\mathbb{U}_{t_n}}^2 + \frac{\alpha}{2} \mathbb{E} \left[ \|X_{t_n}(T)\|_{L^2}^2 \right],
\]

subject to a forward difference equation

\[
\begin{align*}
X_{t_n}(t_{n+1}) - X_{t_n}(t_n) = & \tau \left[ \Delta_h X_{t_n}(t_{n+1}) + U_{t_n}(t_n) \right] + \left[ X_{t_n}(t_n) + \mathcal{R}_h \sigma(t_n) \right] \Delta_n W, \\
X_{t_n}(0) = & \mathcal{R}_h X_0.
\end{align*}
\]

The following result states a (discrete) Pontryagin-type maximum principle for the uniquely solvable problem \(\text{SLQ}_{t_n}\). We mention the appearing mapping \(K_{t_n}\) in (3.9), which is used to give the discrete optimality condition (3.10); see also Remark 3.4. In the sequel, this principle is used to verify rates of convergence for the solution to problem \(\text{SLQ}_{t_n}\) towards the solution to \(\text{SLQ}_h\).

Theorem 3.2. Let \(A_0 := (I - \tau \Delta_h)^{-1}\). The unique optimal pair \((X^*_h, U^*_h) \in \mathbb{X}_{t_n} \times \mathbb{U}_{t_n}\) of
problem $\text{SLQ}_{ht}$ solves the following equalities for $n = 0, 1, \cdots, N - 1$:

$$
\begin{cases}
X_{ht}^*(t_{n+1}) = A_0^{n+1} \prod_{j=1}^{n+1} (1 + \Delta_j W) X_{ht}(0) + \tau \sum_{j=0}^{n} A_0^{n+1-j} \prod_{k=j+2}^{n+1} (1 + \Delta_k W) U_{ht}^*(t_j) \\
\quad + \sum_{j=0}^{n} A_0^{n+1-j} \prod_{k=j+2}^{n+1} (1 + \Delta_k W) \mathcal{R}_h \sigma(t_j) \Delta_{j+1} W, \\
(K_{ht} X_{ht}^*)(t_n) := -\tau \mathbb{E} \left[ \sum_{j=n+1}^{N} A_0^{j-n} \prod_{k=j+2}^{N} (1 + \Delta_k W) X_{ht}^*(t_j) \big| \mathcal{F}_{t_n} \right] \\
\quad - \alpha \mathbb{E} \left[ A_0^{N-n} \prod_{k=n+2}^{N} (1 + \Delta_k W) X_{ht}^*(T) \big| \mathcal{F}_{t_n} \right], \\
X_{ht}^*(0) = \mathcal{R}_h X_0,
\end{cases}
$$

(3.9)

To obtain the results in $[16]$, we verify this characterization of the unique minimizer of problem $\text{SLQ}_{ht}$ in Section 3.4, and then use it to prove our second main result.

**Theorem 3.3.** Under Assumption (A), let $(X_{ht}^*, U_{ht}^*)$ be the solution to problem $\text{SLQ}_{ht}$, and $(X_{ht}^*, U_{ht}^*)$ solves problem $\text{SLQ}_{ht}$. Then, there exists $C = C(X_0, \sigma, T) > 0$ independent of $h, \tau > 0$ such that

(i) $\sum_{k=0}^{N-1} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \|U_{ht}^*(t) - U_{ht}^*(t_k)\|_{L^2}^2 \, dt \right] \leq C \tau$;

(ii) $\max_{0 \leq k \leq N} \mathbb{E} \left[ \|X_{ht}^*(t_k) - X_{ht}^*(t_k)\|_{L^2}^2 \right] + \tau \sum_{k=1}^{N} \mathbb{E} \left[ \|X_{ht}^*(t_k) - X_{ht}^*(t_k)\|_{L^2}^2 \right] \leq C \tau$.

**Remark 3.4.** 1. The strategy of proof in Section 3.4 for Theorems 3.2 and 3.3 can also be used to obtain the results in $[16]$, where the noise in the SPDE $\text{SLQ}_{ht}$ (2.6) — as part of the optimality conditions for $\text{SLQ}_{ht}$ — is additive instead; but not vice versa. In the setting of $[16]$, the coupled FBSPDE $\text{SLQ}_{ht}$ (3.4)–(3.6) is replaced by

$$
\begin{align*}
\text{(i)} \quad & \frac{dX_h(t)}{dt} = \left[ \Delta_h X_h(t) + Y_h(t) \right] dt + \mathcal{R}_h \sigma(t) dW(t) \quad \forall t \in [0, T], \\
\text{(ii)} \quad & \frac{dY_h(t)}{dt} = -\Delta_h Y_h(t) + X_h(t) dt + Z_h(t) dW(t) \quad \forall t \in [0, T], \\
& X_h(0) = \Pi_h X_0, \quad Y_h(T) = -\alpha X_h(T),
\end{align*}
$$

(3.11)

and to solve the corresponding problem to $\text{SLQ}_{ht}$ is equivalent to solving the following forward-backward stochastic difference equation for $n = 0, 1, \cdots, N - 1$,

$$
\begin{align*}
& [1 - \tau \Delta_h] X_{ht}(t_{n+1}) = X_{ht}(t_n) + \tau Y_{ht}(t_n) + \mathcal{R}_h \sigma(t_n) \Delta_{n+1} W, \\
& [1 - \tau \Delta_h] Y_{ht}(t_n) = \mathbb{E} \left[ Y_{ht}(t_{n+1}) - \tau X_{ht}(t_{n+1}) \big| \mathcal{F}_{t_n} \right], \\
& X_{ht}(0) = \Pi_h X_0, \quad Y_{ht}(T) = -\alpha X_{ht}(T).
\end{align*}
$$

(3.12)

based on which the optimal strong error estimates in $[16, \text{Theorem 4.2}]$ are shown.
2. Equation (3.9)\(_1\) is the time-implicit approximation of SPDE\(_h\) (3.1)\(_1\), while \(K_{h\tau}X_{h\tau}^*\) is an approximation of \(Y_h\) to BSPDE\(_h\) (3.1)\(_2\). In fact, (3.9)\(_2\) is different from the temporal discretization of (3.1)\(_2\) via the implicit Euler method; see also Lemma 3.10 for \(Y_h\)’s approximation based on implicit Euler method. In Lemma 3.11, we estimate the difference of these two approximations. When SPDE (1.2) is driven by additive noise (i.e., \(\sigma(t)\mathrm{d}W(t)\)), \(K_{h\tau}X_{h\tau}^*\) defined in (3.9)\(_2\) is just \(Y_{h\tau}\) in (3.12) by changing \(\Pi_hX_0\) to \(\mathcal{R}_hX_0\), and Theorem 3.2 turns to [16, Theorem 4.2]; see also item 1.

3. Compared to BSPDE\(_h\) (3.12)\(_2\), the adjoint equation (3.6)\(_2\) for problem SLQ\(_h\) contains \(Z_h\) in the drift term. This induces extra difficulties (such as to estimate \(\mathbb{E}[\|Z_h(t) - Z_h(s)\|_{L^2}^2]\), \(s, t \in [0, T]\)) when deducing a convergence rate for the temporal discretization (3.6)\(_2\). In this work, without any extra assumptions on data, we adopt the stochastic Riccati equation to overcome these difficulties; see Section 3.3 for further details.

The optimality system (3.9)\(-\)(3.10) is still not amenable to an actual implementation, but serves as a key step towards the practical Algorithm 4.1, which approximately solves SLQ\(_{h\tau}\); its convergence will be shown in Section 4.

3.2 Spatial semi-discretization SLQ\(_h\): Proof of Theorem 3.1

We remark that by (3.1)\(_1\), \(X^*\) may be written as \(X^* = S(U^*)\), where

\[S : L^2_0(0, T; L^2) \to L^2_0(\Omega; C([0, T]; \mathbb{H}^1_0)) \cap L^2(0, T; \mathbb{H}^1_0 \cap \mathbb{H}^2)\]

is the bounded ‘control-to-state’ map. If \(X_0 \equiv 0\) and \(\sigma \equiv 0\), we denote this solution map by \(S^0\). Moreover, we introduce the reduced functional

\[\hat{J} : L^2_0(0, T; L^2) \to \mathbb{R} \quad \text{via} \quad \hat{J}(U) = J(S(U), U),\]

where \(J\) is defined in (1.1). The solution to equation (3.1)\(_2\) may be written in the form \((Y, Z) = (T^1(X^*), T^2(X^*))\), where

\[T^1 : L^2_0(\Omega; C([0, T]; \mathbb{L}^2)) \to L^2_0(\Omega; C([0, T]; \mathbb{H}^1_0)) \cap L^2(0, T; \mathbb{H}^1_0 \cap \mathbb{H}^2),\]

\[T^2 : L^2_0(\Omega; C([0, T]; \mathbb{L}^2)) \to L^2_0(0, T; \mathbb{H}^1_0),\]

which are both bounded.

**Lemma 3.5.** For every \(U \in L^2_0(0, T; \mathbb{L}^2)\), the Fréchet derivative \(D\hat{J}(U)\) is a bounded operator on \(L^2_0(0, T; \mathbb{L}^2)\) which takes the form

\[D\hat{J}(U) = U - T^1(S(U)).\]  

**Proof.** By (2.3), the stability of SPDE (3.1)\(_1\), we have

\[\mathbb{E}[\|S^0(V)(t)\|_{L^2}^2] \leq C\|V\|_{L^2_0(0, T; L^2)}^2 \quad \forall \ t \in [0, T], \ V \in L^2_0(0, T; \mathbb{L}^2).\]

Hence, for any \(U, V \in L^2_0(0, T; \mathbb{L}^2)\), applying the fact \(S(U + V) = S(U) + S^0(V)\), we can get

\[\hat{J}(U + V) - \hat{J}(U) - \left[\langle S(U), S^0(V) \rangle_{L^2_0(0, T; L^2)} + \left(\langle \alpha S(U)(T), S^0(V)(T) \rangle_{L^2_{\mathbb{R}}(\Omega; \mathbb{L}^2)} + \langle U, V \rangle_{L^2_0(0, T; \mathbb{L}^2)}\right)\right] = \frac{1}{2}\left[\|S^0(V)\|_{L^2_0(0, T; L^2)}^2 + \|V\|_{L^2_0(0, T; L^2)}^2 + \alpha \|S^0(V)(T)\|_{L^2_{\mathbb{R}}(\Omega; \mathbb{L}^2)}^2\right].\]
On the other side, Itô’s formula to \((S^0(V), T^1(S(U)))_{L^2}\) yields to
\[
(S(U), S^0(V))_{L^2(0,T;L^2)} + (\alpha S(U)(T), S^0(V)(T))_{L^2(\Omega;L^2)} = -(T^1(S(U)), V)_{L^2(0,T;L^2)}.
\]
Subsequently, by the definition of Fréchet derivative and above three inequalities, we prove the desired result.

By the unique solvability property of (3.4), we associate to this equation the bounded solution operator
\[
S_h : L^2_F(0,T;\mathbb{V}_h) \rightarrow L^2_F(\Omega;C([0,T];\mathbb{V}_h)),
\]
which allows to introduce the reduced functional
\[
\hat{J}_h : L^2_F(0,T;\mathbb{V}_h) \rightarrow \mathbb{R}, \quad \text{via} \quad \hat{J}_h(U_h) = J(S_h(U_h), U_h),
\]
where \(J\) is defined in (1.1). The solution pair to equation (3.6) may be written as \((Y_h, Z_h) = (T^1_h(X^h), T^2_h(X^h))\), where
\[
T^1_h : L^2_F(\Omega;C([0,T];\mathbb{V}_h)) \rightarrow L^2_F(\Omega;C([0,T];\mathbb{V}_h)), \quad T^2_h : L^2_F(\Omega;C([0,T];\mathbb{V}_h)) \rightarrow L^2_F(0,T;\mathbb{V}_h).
\]

We are now in a position to prove Theorem 3.1.

**Proof of Theorem 3.1.** For every \(U_h \in L^2_F(0,T;\mathbb{V}_h)\), the Fréchet derivative \(D\hat{J}_h(U_h)\) is a bounded operator (uniformly in \(h\)) on \(L^2_F(0,T;\mathbb{V}_h)\), and has the form
\[
D\hat{J}_h(U_h) = U_h - T_h(S_h(U_h)),
\]
which can be deduced by the similar procedure as that in Lemma 3.5. Let \(U_h \in L^2_F(0,T;\mathbb{V}_h)\) be arbitrary; it is due to the quadratic structure of the reduced functional (3.14) that
\[
(D^2\hat{J}_h(U_h)R_h, R_h)_{L^2_F(0,T;L^2)} \geq \|R_h\|^2_{L^2_F(0,T;L^2)} \quad \forall R_h \in L^2_F(0,T;\mathbb{V}_h).
\]
As a consequence, on putting \(R_h = U^*_h - \Pi_h U^*\),
\[
\|U^*_h - \Pi_h U^*\|^2_{L^2_F(0,T;L^2)} \leq (D^2\hat{J}_h(U_h)(U^*_h - \Pi_h U^*), U^*_h - \Pi_h U^*)_{L^2_F(0,T;L^2)} = (D\hat{J}_h(U^*_h), U^*_h - \Pi_h U^*)_{L^2_F(0,T;L^2)} - (D\hat{J}_h(\Pi_h U^*), U^*_h - \Pi_h U^*)_{L^2_F(0,T;L^2)}.
\]
Note that \(D\hat{J}_h(U^*_h) = 0\) by (3.5), as well as \(D\hat{J}(U^*) = 0\) by (3.2), such that the last line equals
\[
= \left[ (D\hat{J}(U^*), U^*_h - \Pi_h U^*)_{L^2_F(0,T;L^2)} - (D\hat{J}(\Pi_h U^*), U^*_h - \Pi_h U^*)_{L^2_F(0,T;L^2)} \right] + \left[ (D\hat{J}(\Pi_h U^*), U^*_h - \Pi_h U^*)_{L^2_F(0,T;L^2)} - (D\hat{J}(\Pi_h U^*), U^*_h - \Pi_h U^*)_{L^2_F(0,T;L^2)} \right].
\]
Hence,
\[
\|U^*_h - \Pi_h U^*\|^2_{L^2_F(0,T;L^2)} \leq 2 \left( \|D\hat{J}(U^*) - D\hat{J}(\Pi_h U^*)\|^2_{L^2_F(0,T;L^2)} + \|D\hat{J}(\Pi_h U^*) - D\hat{J}(\Pi_h U^*)\|^2_{L^2_F(0,T;L^2)} \right) \quad (3.16)
\]
\[= 2(I + II).\]
We use (3.13) to bound $I$ as follows,

$$I \leq 2\left(\|U^* - \Pi_h U^*\|^2_{L^2_tC(0,T;\mathbb{H}_0^1)} + \|T^1(S(\Pi_h U^*)) - T^1(S(U^*))\|^2_{L^2_tC(0,T;\mathbb{H}_0^1)}\right).$$

By stability properties (see also (2.12)) for BSPDE (3.1)$_2$, as well as SPDE (3.1)$_1$ (see also (2.3)), the last term in the above inequality reads

$$\leq C\left(\|(S(U^*) - S(\Pi_h U^*))(T)\|^2_{L^2_Tc(\Omega;\mathbb{H}_0^2)} + \|S(U^*) - S(\Pi_h U^*)\|^2_{L^2_Tc(0,T;\mathbb{H}_0^2)}\right) \leq C\|U^* - \Pi_h U^*\|^2_{L^2_Tc(0,T;\mathbb{H}_0^2)}.$$

By optimality condition (3.2), and the regularity properties of the solution of FBSPDE (3.1), we know that already $U^* \in L^2_Tc(0,T;\mathbb{H}_0^1)$; as a consequence, the right-hand side of (3.17) is bounded by $Ch^2$.

We use the representation (3.15) to bound $II$ via

$$II \leq \|T^1(S(\Pi_h U^*)) - T^1_h(S_h(\Pi_h U^*))\|^2_{L^2_Tc(0,T;\mathbb{H}_0^2)} \leq 2\left[\|T^1(S(\Pi_h U^*)) - T^1(S_h(\Pi_h U^*))\|^2_{L^2_Tc(0,T;\mathbb{H}_0^2)} + \|T^1(S_h(\Pi_h U^*)) - T^1_h(S_h(\Pi_h U^*))\|^2_{L^2_Tc(0,T;\mathbb{H}_0^2)}\right]$$

$$= 2(II_1 + II_2).$$

In order to bound $II_1$, we use stability properties for BSPDE (3.1)$_2$, in combination with the error estimate (2.7) for (2.6) to conclude

$$II_1 \leq C\left(\|S(\Pi_h U^*) - S_h(\Pi_h U^*)\|^2_{L^2_Tc(0,T;\mathbb{H}_0^2)} + \|S(\Pi_h U^*)(T) - S_h(\Pi_h U^*)(T)\|^2_{L^2_Tc(\Omega;\mathbb{H}_0^2)}\right) \leq Ch^2.$$

In order to bound $II_2$, we use the error estimate for BSPDE (3.1)$_2$, and estimate for SPDE$_h$ (1.2) with $U_h = \Pi_h U^*$ to find

$$II_2 \leq Ch^2\left(\|T^1(S_h(\Pi_h U^*))\|^2_{L^2_Tc(0,T;\mathbb{H}_0^2)} + \|T^2(S_h(\Pi_h U^*))\|^2_{L^2_Tc(0,T;\mathbb{H}_0^1)}\right) \leq Ch^2\left(\|X_0\|^2_{\mathbb{H}_0^1} + \|Y\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\sigma\|^2_{L^2(0,T;\mathbb{H}_0^1)}\right) \leq Ch^2.$$

We now insert these estimates into (3.16), and utilize the optimal condition (3.2) to obtain the bound

$$\|U^* - U_h^*\|^2_{L^2_Tc(0,T;\mathbb{H}_0^1)} \leq 2\left(\|U^* - \Pi_h U^*\|^2_{L^2_Tc(0,T;\mathbb{H}_0^1)} + \|U_h^* - \Pi_h U^*\|^2_{L^2_Tc(0,T;\mathbb{H}_0^1)}\right) \leq Ch^2.$$

This just part (i) of the theorem.

Since $U^* \in L^2_Tc(0,T;\mathbb{H}_0^1)$, and (i), the estimates (ii) and (iii) can be deduced as (2.7) and Theorem 2.1 respectively.
3.3 Some $h$-independent stability bounds and time regularity properties for the solutions to SPDE$_h$ and BSPDE$_h$

The following lemmata validate $h$-independent bounds and time regularity in relevant norms for the solution $X^*_h$ of SPDE$_h$ (3.4) where $U_h \equiv U^*_h$, and of $(Y_h, Z_h)$ which solves BSPDE$_h$ (3.6), which are both crucial to derive the convergence rate for proposed discretization. To obtain time regularity, two types of assumptions are usually made on terminal conditions: (1) Malliavin differentiability (see e.g. [11, 19]), or (2) Markovianity (see e.g. [23, 20]). In this work, due to the optimal control framework, we adopt the state feedback strategy to show $h$-independent stability bounds of the optimal pair $(X^*_h, U^*_h)$ to problem SLQ$_h$ and $(Y_h, Z_h)$ as well as time regularity of $Z_h$. Specifically, we introduce the stochastic Riccati equation (3.22) in combination with a backward ODE (3.23) and, in particular, prove that the (stochastic) Riccati operator $P_h$ that solves (3.22) may be bounded uniformly in $h$; see Lemma 3.6. Then, we apply the state feedback control $U^*_h = -P_h X^*_h - \varphi_h$ to deduce $h$-independent stability bounds for the tuple $X^*_h$, then for $U^*_h$ and $(Y_h, Z_h)$. Also by the aforementioned feedback control, we further conclude the Malliavin differentiability of $X^*_h, U^*_h, Y_h$, obtain $h$-independent bounds for it, and thus conclude the needed time regularity of $Z_h$; see (1.5).

To begin with, we introduce a family of SLQ$_h$ problems, parametrized by $t \in [0,T]$; for this purpose, we consider the controlled SPDE$_h$

$$
\begin{cases}
\text{d}X_h(s) = \left[ \Delta_h X_h(s) + U_h(s) \right] \text{d}s + \left[ X_h(s) + R_h \sigma(s) \right] \text{d}W(s) & \forall s \in [t,T],
\end{cases}
$$

with $X_t \in \mathbb{H}^1_0$, and the (parametrized) cost functional

$$
\mathcal{J}_h(t, X_t; U_h) := \frac{1}{2} \mathbb{E} \left[ \int_t^T \|X_h(s)\|_{L_2}^2 + \|U_h(s)\|_{L_2}^2 \text{d}s \right] + \frac{\alpha}{2} \mathbb{E} \left[ \|X_h(T)\|_{L_2}^2 \right].
$$

We define the value function as follows:

$$
V_h(t, X_t) := \inf_{U_h \in L^2_2(t; \mathcal{X}^2)} \mathcal{J}_h(t, X_t; U_h).
$$

Obviously $\mathcal{J}(X_h, U_h) = \mathcal{J}_h(0, X_0; U_h)$, and $\mathcal{J}(X^*_h, U^*_h) = V_h(0, X_0)$. The stochastic Riccati equation related to SLQ$_h$ then reads:

$$
\begin{cases}
P^*_h(t) + P_h(t) \Delta_h + \Delta_h P_h(t) + P_h(t) + I_h - P_h(t) P_h(t) &= 0 & \forall \ t \in [0,T],
\end{cases}
$$

and we consider a backward ODE,

$$
\begin{cases}
\varphi'_h(t) + \left[ \Delta_h - P_h(t) \right] \varphi_h(t) + P_h(t) R_h \sigma(t) &= 0 & \forall \ t \in [0,T],
\varphi_h(T) &= 0.
\end{cases}
$$

Here $I_h$ denotes the identity operator on $\mathcal{V}_h$. By [22, Chapter 6, Theorems 6.1 & 7.2], we know that the stochastic Riccati equation (3.22) admits a unique solution $P_h \in C([0,T]; \mathcal{L}(\mathcal{V}_h; \mathcal{V}_h))$ which is nonnegative, symmetric, subsequently (3.23) has a unique solution $\varphi \in C([0,T]; \mathcal{V}_h)$, and that

$$
V_h(t, X_t) = \frac{1}{2} \left( P_h(t) R_h X_t, R_h X_t \right)_{L^2} + \left( \varphi_h(t), R_h X_t \right)_{L^2} + \frac{1}{2} \int_t^T \left( (P_h(s) R_h \sigma(s), R_h \sigma(s))_{L^2} + \|\varphi_h(s)\|_{L^2}^2 \right) \text{d}s.
$$

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Lemma 3.6. Let \( P_h \) be the solution of (3.22), and \( \varphi_h \) solves (3.23). Then there exists a constant \( C > 0 \) independent of \( h > 0 \) such that

\[
\begin{align*}
(\text{i}) \quad & \sup_{t \in [0,T]} \| P_h(t) \|_{L^\infty(\mathbb{L}^2 | \mathcal{F}_t \cup \mathcal{F}_h)} \leq C, \\
(\text{ii}) \quad & \sup_{t \in [0,T]} \| \varphi_h(t) \|_{\mathbb{L}^2}^2 + \int_0^T \| \nabla \varphi_h(s) \|_{\mathbb{L}^2}^2 + (P_h(s) \varphi_h(s), \varphi_h(s))_{L^2} \, ds \leq C \| \sigma \|_{L^\infty(0,T;\mathbb{L}^2)}^2.
\end{align*}
\]

Proof. 1) We consider problem \( \text{SLQ}_h \) for (3.19) with \( \sigma \equiv 0 \). In this case, we denote the solution to (3.19) by \( X_h^0 = X_h^0(\cdot; \mathcal{R}_h X_t, U_h) \). For \( U_h \equiv 0 \), Itô’s formula then yields

\[
\sup_{s \in [t,T]} \mathbb{E}[\| X_h^0(s; \mathcal{R}_h X_t, 0) \|_{\mathbb{L}^2}^2] \leq C^T \mathbb{E}[\| \mathcal{R}_h X_t \|_{\mathbb{L}^2}^2].
\]

Hence, (3.21) and (3.24) lead to

\[
(P_h(t) \mathcal{R}_h X_t, \mathcal{R}_h X_t)_{\mathbb{L}^2} = 2V_h(t, X_t) \leq C \mathbb{E}[\| \mathcal{R}_h X_t \|_{\mathbb{L}^2}^2],
\]

which, together with the facts that \( P_h \) is nonnegative and \( \mathcal{R}_h \) is surjective, implies assertion (i).

2) To verify (ii), we infer from (3.23) that

\[
\int_0^T \| \varphi_h(s) \|_{\mathbb{L}^2}^2 \, ds + \int_0^T \| P_h(s) \mathcal{R}_h \sigma(s) \|_{\mathbb{L}^2}^2 \, ds.
\]

Then Gronwall’s inequality and (i) settle the assertion.

The following lemma collects bounds for the solution \( X_h^* \) of \( \text{SPDE}_h \) (3.4) with \( U_h \equiv U_h^* \), by exploiting the state feedback representation (3.26) of the optimal control \( U_h^* \), and the bounds for the stochastic Riccati operator in Lemma 3.6, in particular.

Lemma 3.7. Let \( X_h^* \) solve \( \text{SPDE}_h \) (3.4) with \( U_h \equiv U_h^* \). Then for any \( t \in [0,T] \), \( X_h^*(t) \in \mathbb{D}^{2,2}(\mathbb{L}^2) \), and there exists an \( h \)-independent constant \( C > 0 \) such that

\[
\begin{align*}
\sup_{\theta \in [0,T]} \mathbb{E}[\| X_h^*(t) \|_{\mathbb{L}^2}^2] & + \sup_{\theta \in [0,T]} \mathbb{E}[\| D_h X_h^*(t) \|_{\mathbb{L}^2}^2] + \sup_{\theta \in [0,T]} \mathbb{E}[\| D_h D_h X_h^*(t) \|_{\mathbb{L}^2}^2] \\
& \leq C \left( \| X_0 \|_{\mathbb{D}^{2,2}}^2 + \| \sigma \|_{C([0,T];\mathbb{L}^2)}^2 \right), \\
\sup_{\theta \in [0,T]} \mathbb{E}[\| \Delta_h X_h^*(t) \|_{\mathbb{L}^2}^2] & + \mathbb{E}[\int_0^T \| \Delta_h X_h^*(t) \|_{\mathbb{L}^2}^2 \, dt] \leq C \left( \| X_0 \|_{\mathbb{D}^{2,2}}^2 + \| \sigma \|_{C([0,T];\mathbb{L}^2)}^2 \right), \\
\sup_{\theta \in [0,T]} \mathbb{E}[\| \nabla D_h X_h^*(t) \|_{\mathbb{L}^2}^2] & + \mathbb{E}[\int_0^T \| \nabla D_h X_h^*(t) \|_{\mathbb{L}^2}^2 \, dt] \\
& \leq C \left( \| X_0 \|_{\mathbb{D}^{2,2}}^2 + \| \sigma \|_{C([0,T];\mathbb{L}^2)}^2 \right), \\
\mathbb{E}[\| X_h^*(t) - X_h^*(s) \|_{\mathbb{L}^2}^2] & \leq C|t-s| \left( \| X_0 \|_{\mathbb{D}^{2,2}}^2 + \| \sigma \|_{C([0,T];\mathbb{L}^2)}^2 \right) \quad t, s \in [0,T], \\
\sup_{\theta \in [0,T]} \mathbb{E}[\| (D_{\theta_1} - D_{\theta_2}) X_h^*(t) \|_{\mathbb{L}^2}^2] & \leq C|\theta_1 - \theta_2| \left( \| X_0 \|_{\mathbb{D}^{2,2}}^2 + \| \sigma \|_{C([0,T];\mathbb{L}^2)}^2 \right) \quad \theta_2 \leq \theta_1, \\
\sup_{\theta \in [0,T]} \mathbb{E}[\| U_h^*(t) \|_{\mathbb{L}^2}^2] & + \left( \sup_{\theta \in [0,T]} \mathbb{E}[\| U_h^*(t) \|_{\mathbb{L}^2}^2] \right)^{1/2} \leq C \left( \| X_0 \|_{\mathbb{D}^{2,2}}^2 + \| \sigma \|_{C([0,T];\mathbb{L}^2)}^2 \right).
\end{align*}
\]
Proof. 1) By [22, Chapter 6, Theorem 6.1, 7.2], the optimal control \( U_h^* \) has the following state feedback form

\[
U_h^* = -P_h X_h^* - \varphi_h, \tag{3.26}
\]

where \( P_h, \varphi_h \) are solutions to (3.22), (3.23). Itô’s formula leads to

\[
\mathbb{E}\left[ \|X_h^*(t)\|^2_{L^2} \right] + 2\mathbb{E}\left[ \int_0^t \|\nabla X_h^*(s)\|^2_{L^2} + (P_h(s)X_h^*(s), X_h^*(s))_{L^2} \right] ds \\
\leq \|R_h X_0\|^2_{L^2} + \int_0^T \left[ \|\varphi_h(t)\|^2_{L^2} + 2\|R_h \sigma(t)\|^2_{L^2} \right] dt + 3\mathbb{E}\left[ \int_0^t \|X_h^*(t)\|^2_{L^2} ds \right].
\]

By Gronwall’s inequality, Lemma 3.6, (ii), and stability properties of \( R_h \) we conclude that

\[
\sup_{t \in [0,T]} \mathbb{E}\left[ \|X_h^*(t)\|^2_{L^2} \right] \leq C \left( \|R_h X_0\|^2_{L^2} + \int_0^T \left[ \|\varphi_h(t)\|^2_{L^2} + \|R_h \sigma(t)\|^2_{L^2} \right] dt \right) \\
\leq C \left( \|X_0\|^2_{\mathbb{H}_0^2} + \int_0^T \|\sigma(t)\|^2_{\mathbb{H}_0^2} dt \right).
\]

2) To estimate the second term on the left-hand side of (3.25)_1, by noting that \( P_h \) and \( \varphi_h \) are deterministic and [15, Theorem 2.2.1], the first Malliavin derivative of \( X_h^* \) exists. We take the Malliavin derivative on both sides of (3.4) with the state feedback control (3.26), and then

\[
\begin{cases}
  \mathrm{d}D_0 X_h^*(t) = [\Delta_h - P_h(t)] D_0 X_h^*(t) \, dt + D_0 X_h^*(t) \, dW(t) & \forall t \in [0,T], \\
  D_0 X_h^*(\theta) = X_h^*(\theta) + R_h \sigma(\theta), \\
  D_0 X_h^*(t) = 0 & \forall t \in [0,\theta].
\end{cases}
\tag{3.27}
\]

Itô’s formula leads to

\[
\sup_{\theta \in [0,T]} \sup_{t \in [\theta,T]} \mathbb{E}\left[ \|D_0 X_h^*(t)\|^2_{L^2} \right] \leq C \sup_{\theta \in [0,T]} \mathbb{E}\left[ \|X_h^*(\theta) + R_h \sigma(\theta)\|^2_{L^2} \right] \leq C \left( \|X_0\|^2_{\mathbb{H}_0^2} + \|\sigma\|^2_{L^2([0,T];\mathbb{H}_0^2)} \right).
\]

Hence, \( X_h^* \in \mathbb{D}^{1,2}(\mathbb{L}^2) \). In a similar vein, we can deduce \( X_h^* \in \mathbb{D}^{2,2}(\mathbb{L}^2) \) and the remaining part of (3.25)_1.

3) For (3.25)_2, similarly to 1), we estimate

\[
\sup_{t \in [0,T]} \mathbb{E}\left[ \|\nabla X_h^*(t)\|^2_{L^2} \right] + \mathbb{E}\left[ \int_0^T \|\Delta_h X_h^*(t)\|^2_{L^2} dt \right] \\
\leq C\left( \|R_h X_0\|^2_{L^2} + \int_0^T \left( \|P_h(t)X_h^*(t)\|^2_{L^2} + \|\varphi_h(t)\|^2_{L^2} + \|\nabla R_h \sigma(t)\|^2_{L^2} \right) dt \right) \\
\leq C \left( \|X_0\|^2_{\mathbb{H}_0^2} + \|\sigma\|^2_{L^2([0,T];\mathbb{H}_0^2)} \right),
\]

where Lemma 3.6, assertion (3.25)_1, and stability properties of \( R_h \) are applied.
4) We use Itô’s formula for the solution of (3.27) to find via Lemma 3.6, (i) that

\[
E[\|\nabla D_{\theta}X^h(t)\|^2_{L^2}] + E\left[\int_\theta^t \|\Delta_{\delta}D_{\theta}X^h(s)\|^2_{L^2} \, ds\right]
\leq \left( E[\|\nabla D_{\theta}X^\star(\theta)\|^2_{L^2}] + E\left[\int_\theta^T \|P_{h}(s)D_{\theta}X^h(s)\|^2_{L^2} \, ds\right]\right) + E\left[\int_\theta^t \|\nabla D_{\theta}X^h(s)\|^2_{L^2} \, ds\right]
\leq C \left(E[\|\nabla X^\star(\theta)\|^2_{L^2}] + \|\nabla \mathcal{R}_{h}\sigma(\theta)\|_{L^2}^2 + E\left[\int_\theta^T \|D_{\theta}X^h(s)\|^2_{L^2} \, ds\right]\right) + E\left[\int_\theta^t \|\nabla D_{\theta}X^h(s)\|^2_{L^2} \, ds\right].
\]

Then, by Gronwall’s inequality as well as (3.25)_1, (3.25)_2, this shows (3.25)_3.

5) The verification of estimate (3.25)_4 can now be deduced via SPDE\_h (3.4) and (3.25)_1, (3.25)_2.

6) We estimate (3.25)_5. Using equation (3.27), Itô’s formula, and (3.25)_1–(3.25)_3, we find that

\[
sup_{t \in [\theta, T]} E\left[\| (D_{\theta_1} - D_{\theta_2})X^h(t) \|^2_{L^2}\right]
\leq C \left(E[\|D_{\theta_1}X^h(\theta_1) - D_{\theta_2}X^h(\theta_2)\|^2_{L^2}] + E\left[\|D_{\theta_2}X^h(\theta_1) - D_{\theta_2}X^h(\theta_2)\|^2_{L^2}\right]\right)
\leq C \left(E\left[\| (X^h(\theta_1) + \mathcal{R}_{h}\sigma(\theta_1)) - (X^h(\theta_2)) - \mathcal{R}_{h}\sigma(\theta_2))\|^2_{L^2}\right] + (\theta_1 - \theta_2)E\left[\int_0^{\theta_1} \|\Delta_{\delta}D_{\theta}X^h(s)\|^2_{L^2} \, ds\right]
\right) + E\left[\int_{\theta_2}^{\theta_1} \|D_{\theta_2}X^h(s)\|^2_{L^2} \, ds\right] \right)
\leq C|\theta_1 - \theta_2| \left(\|X_0\|^2_{H^1} + (\theta_1 - \theta_2) \int_{\theta_2}^{\theta_1} \|\sigma'(t)\|^2_{H^1} dt + \|\sigma\|^2_{L^2[C([0,T];H^1)]}\right),
\]

which settles the assertion.

7) By (3.26), Lemma 3.6, and (3.25)_1, we easily find the first estimate in (3.25)_6. Then, the application of Itô’s formula to $\|X^h\|^2_{L^2}$ leads to the remaining estimate in (3.25)_6.

We now may use these estimates for $X^h$ to bound the solution $(Y_h, Z_h)$ of BSPDE\_h (3.6) — in which $X^h$ appears as well.

**Lemma 3.8.** Suppose that $(Y_h, Z_h)$ solves BSPDE\_h (3.6) and $I_{\tau}$ is a uniform time mesh of $[0,T]$. Then there exists a constant $C > 0$ independent of $h > 0$ such that

\[
\begin{align*}
\sup_{t \in [0,T]} E[\|Y_h(t)\|^2_{L^2}] + \sup_{t \in [0,T]} E\left[\int_0^T \|\nabla Y_h(t)\|^2_{L^2} + \|Z_h(t)\|^2_{L^2} dt\right] &\leq C \left(\|X_0\|^2_{H^1} + \|\sigma\|^2_{L^2[C([0,T];H^1)]}\right),
\sup_{t \in [0,T]} E[\|Y_h(t)\|^2_{L^2}] + \sup_{t \in [0,T]} E\left[\int_0^T \|\Delta_{\delta}Y_h(t)\|^2_{L^2} + \|\nabla Z_h(t)\|^2_{L^2} dt\right] &\leq C \left(\|X_0\|^2_{H^1} + \|\sigma\|^2_{L^2[C([0,T];H^1)]}\right),
\|Y_h - Y_h\|^2_{L^2([0,T];L^2)} &\leq C \left(\|X_0\|^2_{H^1} + \|\sigma\|^2_{L^2[C([0,T];H^1)]}\right),
\sup_{t \in [0,T]} \sup_{\theta \in [0,T]} E[\|D_{\theta}Y_h(t)\|^2_{L^2}] + \sup_{t \in [0,T]} \sup_{\theta \in [0,T]} E\left[\int_\theta^T \|D_{\theta}D_{\theta}Y_h(t)\|^2_{L^2} + \|\nabla D_{\theta}Z_h(t)\|^2_{L^2} dt\right] &\leq C \left(\|X_0\|^2_{H^1} + \|\sigma\|^2_{L^2[C([0,T];H^1)]}\right),
\sup_{t \in [0,T]} \sup_{\theta \in [0,T]} E[\|D_{\theta}Y_h(t)\|^2_{L^2}] + \sup_{t \in [0,T]} \sup_{\theta \in [0,T]} E\left[\int_\theta^T \|\Delta_{\delta}D_{\theta}Y_h(t)\|^2_{L^2} + \|\nabla D_{\theta}Z_h(t)\|^2_{L^2} dt\right] &\leq C \left(\|X_0\|^2_{H^1} + \|\sigma\|^2_{L^2[C([0,T];H^1)]}\right),
E[\|Z_h(t) - Z_h(s)\|^2_{L^2}] &\leq C|t - s| \left(\|X_0\|^2_{H^1} + \|\sigma\|^2_{L^2[C([0,T];H^1)]}\right),
\end{align*}
\]
where the (piecewise constant) operator $\Pi_\tau : L^2_\mathcal{F}(\Omega; C([0,T]; \mathbb{V}_h))) \to \mathbb{U}_{\Pi\tau}$ is defined by
\[
\Pi_\tau U_h(t) := U_h(t_n) \quad \forall t \in [t_n, t_{n+1}) \quad n = 0, 1, \ldots, N - 1. \tag{3.29}
\]

**Proof.**

1) Applying Itô’s formula for $\|\nabla Y_h\|_{L^2_{\mathcal{F}}}$ in BSPDEh (3.6), we see that
\[
\sup_{t \in [0,T]} \mathbb{E}[\|\nabla Y_h(t)\|_{L^2_{\mathcal{F}}}^2] + \mathbb{E}\left[\int_0^T \|\Delta_h Y_h(t)\|_{L^2_{\mathcal{F}}}^2 + \|\nabla Z_h(t)\|_{L^2_{\mathcal{F}}}^2 \, dt\right] 
\leq C \left(\mathbb{E}[\|\nabla X_h^\mu(T)\|_{L^2_{\mathcal{F}}}^2] + \mathbb{E}\left[\int_0^T \|X_h^\mu(t)\|_{L^2_{\mathcal{F}}}^2 \, dt\right]\right). \tag{3.30}
\]
By (3.25)$_1$, (3.25)$_2$ and above inequality, we can get (3.28)$_2$. Estimate (3.28)$_1$ can be obtained in the same vein.

2) BSPDEh (3.6) and (3.25)$_1$, (3.28)$_1$, (3.28)$_2$ lead to
\[
\|Y_h - \Pi_\tau Y_h\|_{L^2_\mathcal{F}([0,T]; L^2_{\mathcal{F}})}^2 
= \sum_{k=0}^{N-1} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \|\int_{t_k}^t -\Delta_h Y_h(s) - Z_h(s) + X_h^\mu(s) \, ds \|_{L^2_{\mathcal{F}}}^2 \, dt\right] 
\leq C \tau \mathbb{E}\left[\|\int_0^T \|\Delta_h Y_h(t)\|_{L^2_{\mathcal{F}}}^2 \, dt\right] + \mathbb{E}\left[\|\int_0^T \|Z_h(t)\|_{L^2_{\mathcal{F}}}^2 \, dt\right] 
\leq C \left(\|X_0\|_{H_0^1}^2 + \|\sigma\|_{L^2([0,T]; H_0^1)}^2\right),
\]
which is (3.28)$_3$.

3) Noting that $X_h^\mu(t) \in \mathbb{D}_{2,2}(L^2)$, by [9, Proposition 5.3] we can get
\[
\begin{cases}
    dD_\theta Y_h(t) = \left[-\Delta_h D_\theta Y_h(t) - D_\theta Z_h(t) + D_\theta X_h^\mu(t)\right] \, dt + D_\theta Z_h(t) \, dW(t) & t \in [\theta, T], \\
    D_\theta Y_h(T) = -\alpha D_\theta X_h^\mu(T),
\end{cases}
\]
and
\[
\begin{cases}
    dD_\mu D_\theta Y_h(t) = \left[-\Delta_h D_\mu D_\theta Y_h(t) - D_\mu D_\theta Z_h(t) + D_\mu D_\theta X_h^\mu(t)\right] \, dt \\
    + D_\mu D_\theta Z_h(t) \, dW(t) & t \in [\mu \lor \theta, T], \\
    D_\mu D_\theta Y_h(T) = -\alpha D_\mu D_\theta X_h^\mu(T).
\end{cases}
\]
Then, Itô’s formula and (3.25)$_1$, (3.25)$_3$ lead to (3.28)$_4$ and (3.28)$_5$.

4) To estimate (3.28)$_6$, by applying the fact that $Z_h(\cdot) = D Y_h(\cdot)$, a.e., we arrive at
\[
\mathbb{E}\left[\|D_t - D_s\|_{L^2_{\mathcal{F}}}^2\right] 
= \mathbb{E}\left[\|D_t Y_h(t) - D_s Y_h(s)\|_{L^2_{\mathcal{F}}}^2\right] 
\leq 2 \mathbb{E}\left[\|D_t - D_s\|_{L^2_{\mathcal{F}}}^2\right] + 2 \mathbb{E}\left[\|D_s (Y_h(t) - Y_h(s))\|_{L^2_{\mathcal{F}}}^2\right]. \tag{3.33}
\]
For the first term on the right side of (3.33), Itô’s formula and (3.25)$_3$ lead to
\[
\begin{align*}
\mathbb{E}\left[\|D_t - D_s Y_h(t)\|_{L^2_{\mathcal{F}}}^2\right] 
& \leq C \left(\mathbb{E}\left[\|D_t - D_s\|_{L^2_{\mathcal{F}}}^2\right] + \mathbb{E}\left[\int_t^T \|D_t - D_s\|_{L^2_{\mathcal{F}}}^2 \, dt\right]\right) 
\leq C |t-s| \left(\|X_0\|_{H_0^1}^2 + \|\sigma\|_{C^1([0,T]; H_0^1)}^2\right). \tag{3.34}
\end{align*}
\]
Similar to (3.30), by virtue of (3.25)1, (3.28)4, (3.28)5,
\[
\mathbb{E}\left[\|D_s(Y_h(t) - Y_h(s))\|^2_{L^2_h}\right] \\
\leq C\left(\mathbb{E}\left[\int_s^t \|D_sD_0Y_h(\theta)\|^2_{L^2_h} d\theta\right] + (t-s)\mathbb{E}\left[\int_s^t \|\Delta h Y_h(\theta)\|^2_{L^2_h} + \|D_sZ_h(\theta)\|^2_{L^2_h} + \|D_sX_h^*(\theta)\|^2_{L^2_h} d\theta\right]\right) \\
\leq C|t-s|\left(\|X_0\|^2_{H^1_0} + \|\sigma\|^2_{C^1([0,T];\mathbb{H}_1)}\right).
\] (3.35)

Now (3.28)6 can be derived by (3.33)–(3.35).

Based on Lemma 3.7 and Lemma 3.8, we can now sharpen our results to the following one.

**Lemma 3.9.** Let \(X^*_h\) solve SPDE\(_h\) (3.4) with \(U_h = U^*_h\), and \((Y_h, Z_h)\) be the solution to BSPDE\(_h\) (3.6). Then
\[
\begin{align*}
\sup_{t \in [0,T]} \mathbb{E}\left[\|\Delta h X^*_h(t)\|^2_{L^2_h}\right] & + \mathbb{E}\left[\int_0^T \|\nabla \Delta h X^*_h(t)\|^2_{L^2_h} dt\right] \\
\leq &\ C\left(\|X_0\|^2_{H^1_0} + \|\sigma\|^2_{C^1([0,T];\mathbb{H}^1)}\right), \\
\sup_{t \in [0,T]} \mathbb{E}\left[\|\nabla \Delta h X^*_h(t)\|^2_{L^2_h}\right] & + \mathbb{E}\left[\int_0^T \|\Delta h X^*_h(t)\|^2_{L^2_h} dt\right] \\
\leq &\ C\left(\|X_0\|^2_{H^1_0} + \|\sigma\|^2_{C^1([0,T];\mathbb{H}^1)}\right), \\
\sup_{t \in [0,T]} \mathbb{E}\left[\|\Delta h Y_h(t)\|^2_{L^2_h}\right] & + \mathbb{E}\left[\int_0^T \|\nabla \Delta h Y_h(t)\|^2_{L^2_h} + \|\Delta h Z_h(t)\|^2_{L^2_h} dt\right] \\
\leq &\ C\left(\|X_0\|^2_{H^1_0} + \|\sigma\|^2_{C^1([0,T];\mathbb{H}^1)}\right), \\
\mathbb{E}\left[\|\nabla (X^*_h(t) - X^*_h(s))\|^2_{L^2_h}\right] & \leq C|t-s|\left(\|X_0\|^2_{H^1_0} + \|\sigma\|^2_{C^1([0,T];\mathbb{H}^1)}\right),
\end{align*}
\] (3.36)
for all \(t, s \in [0, T]\).

**Proof.** Firstly, by definitions of \(\Delta h, R_h\), and the fact that \(X_0 \in \mathbb{H}^1_0\), we have
\[
(\Delta h R_h X_0, \phi_h)_{L^2} = - (\nabla X_0, \nabla \phi_h)_{L^2} = (\Delta X_0, \phi_h)_{L^2} = (\Pi_h \Delta X_0, \phi_h)_{L^2} \quad \forall \phi_h \in V_h.
\]
Hence, we deduce that
\[
\Delta h R_h X_0 = \Pi_h \Delta X_0. \quad (3.37)
\]
Based on the state feedback control (3.26), Itô’s formula to \(\|\Delta h X^*_h\|^2_{L^2_h}\) and (3.5), lead to
\[
\begin{align*}
\mathbb{E}\left[\|\Delta h X^*_h(t)\|^2_{L^2_h}\right] & + \mathbb{E}\left[\int_0^t \|\nabla \Delta h X^*_h(s)\|^2_{L^2_h} ds\right] \\
\leq &\ \|\Delta h R_h X_0\|^2_{L^2_h} + C\mathbb{E}\left[\int_0^T \|\nabla U_h(t)\|^2_{L^2_h} + \|\Delta h R_h \sigma(t)\|^2_{L^2_h} dt\right] + \mathbb{E}\left[\int_0^t \|\Delta h X^*_h(s)\|^2_{L^2_h} ds\right] \\
= &\ \|\Delta h R_h X_0\|^2_{L^2_h} + C\mathbb{E}\left[\int_0^T \|\nabla Y_h(t)\|^2_{L^2_h} + \|\Delta h R_h \sigma(t)\|^2_{L^2_h} dt\right] + \mathbb{E}\left[\int_0^t \|\Delta h X^*_h(s)\|^2_{L^2_h} ds\right],
\end{align*}
\]
which, together with (3.37), (3.28)1 and Gronwall’s inequality, leads to (3.36)1. Applying Itô’s formula to \(\|\Delta h Y_h\|^2_{L^2_h}\) and (3.36)1, we can derive (3.36)3. In the same vein, on using (3.36)3, (3.37), and the \(\mathbb{H}^1\)-stability of the \(L^2\)-projection \(\Pi_h\) (see e.g. [6]), as well as the fact
\[
\|\nabla \Delta h X^*_h(0)\|^2_{L^2_h} = \|\nabla \Delta h R_h X_0\|^2_{L^2_h} = \|\nabla \Pi_h \Delta X_0\|^2_{L^2_h} \leq C\|\Delta X_0\|^2_{\mathbb{H}^1} \leq C\|X_0\|^2_{\mathbb{H}^1},
\]

thanks to Assumption (A), and
\[ \mathbb{E}[\int_0^T \|\nabla \Delta_h \mathcal{R}_h \sigma(t)\|_{L^2}^2 \, dt] \leq C \mathbb{E}[\int_0^T \|\sigma(t)\|_{H^3}^2 \, dt], \]
we can derive that
\[ \sup_{t \in [0,T]} \mathbb{E}[\|\nabla \Delta_h X_h^*(t)\|_{L^2}^2] + \mathbb{E}\left[ \int_0^T \|\Delta_h^2 X_h^*(t)\|_{L^2}^2 \, dt \right] \]
\[ \leq C \left( \|\nabla \Delta_h \mathcal{R}_h X_0\|_{L^2}^2 + \mathbb{E}\left[ \int_0^T \|\Delta_h Y_h(t)\|_{L^2}^2 + \|\nabla \mathcal{R}_h \sigma(t)\|_{L^2}^2 \, dt \right] \right) \]
\[ \leq C \left( \|X_0\|_{H^3}^2 + \|\sigma\|_{L^2(0,T;H^3)}^2 \right). \]
That is (3.36)2.

By SPDE$_h$ (3.4), and (3.36)$_1$, (3.25)$_2$ in Lemma 3.7, (3.28)$_2$ in Lemma 3.8 as well as 2.1, for $s, t \in [0,T], s \leq t$, we can derive that
\[ \mathbb{E}[\|\nabla(X_h^*(s) - X_h^*(t))\|_{L^2}^2] \]
\[ \leq C \int_s^t \mathbb{E}[\|\nabla \Delta_h X_h^*(\tau)\|_{L^2}^2 + \|\nabla X_h^*(\tau)\|_{L^2}^2 + \|\nabla Y_h(\tau)\|_{L^2}^2 + \|\nabla \mathcal{R}_h \sigma(\tau)\|_{L^2}^2] \, d\tau \]
\[ \leq C|t-s| \left( \|X_0\|_{H^3}^2 + \|\sigma\|_{C([0,T];H^3)}^2 + \|\sigma\|_{L^2(0,T;H^3)}^2 \right). \]
That is (3.36)$_4$. \hfill \qed

### 3.4 Spatio-temporal discretization SLQ$_{h\tau}$: Proof of Theorems 3.2 and 3.3

#### Proof of Theorem 3.2
The proof is similar to that of [16, Theorem 4.2], and for the completeness, we provide it here; it consists of two steps.

1) Recall the definition of $A_0$ in Theorem 3.2. We define the bounded operators $\Gamma : \mathbb{V}_h \to \mathcal{X}_{h\tau}$ and $L : \mathbb{U}_{h\tau} \to \mathcal{X}_{h\tau}$ as follows,
\[ (\Gamma X_{h\tau}(0))(t_n) = A_0^n \prod_{j=1}^n (1 + \Delta_j W) X_{h\tau}(0), \]
\[ (LU_{h\tau})(t_n) = \tau \sum_{j=0}^{n-1} A_0^{n-j} \prod_{k=j+2}^n (1 + \Delta_k W) U_{h\tau}(t_j) \quad \forall \, n = 1, 2, \ldots, N, \] respectively, where $(X_{h\tau}, U_{h\tau})$ is an admissible pair in problem SLQ$_{h\tau}$; see (3.7)–(3.8). We also need $f(\cdot)$, which we define as
\[ f(t_n) = \sum_{j=0}^{n-1} A_0^{n-j} \prod_{k=j+2}^n (1 + \Delta_k W) \mathcal{R}_h \sigma(t_j) \Delta_{j+1} W \quad \forall \, n = 1, 2, \ldots, N, \]
and use the abbreviations
\[ \hat{\mathcal{R}}_h X_0 := (\Gamma \mathcal{R}_h X_0)(T), \quad \hat{L} U_{h\tau} := (LU_{h\tau})(T), \quad \hat{f} := f(T). \]
By (3.8), we can find that
\[ X_{h^\tau}(t_n) = (\Gamma X_{h^\tau}(0))(t_n) + (LU_{h^\tau})(t_n) + f(t_n) \quad n = 1, 2, \ldots, N. \] (3.40)

**Claim:** For any \( \xi \in \mathcal{X}_{h^\tau} \), and any \( \eta \in L^2_{\mathcal{F}_{h^\tau}}(\Omega; \mathbb{V}_h) \),
\[
\begin{align*}
(L^*\xi)(t_j) &= \tau \mathbb{E}\left[ \sum_{n=j+1}^{N} A_0^{n-j} \prod_{k=j+2}^{n} (1 + \Delta_k W)\xi(t_n) | \mathcal{F}_{t_j} \right], \\
(\hat{L}^*\eta)(t_j) &= \mathbb{E}\left[ A_0^{N-j} \prod_{k=j+2}^{N} (1 + \Delta_k W)\eta | \mathcal{F}_{t_j} \right] \quad j = 0, 1, \ldots, N - 1.
\end{align*}
\] (3.41)

**Proof of Claim:** Let \( U_{h^\tau} \in \mathbb{U}_{h^\tau} \) be arbitrary. By the definition of \( L \) and the fact \( A_0 = A_0^* \), we can calculate that
\[
\tau \sum_{n=1}^{N} \mathbb{E}\left[ ((LU_{h^\tau})(t_n), \xi(t_n))_{L^2} \right] = \tau \sum_{j=0}^{N-1} \mathbb{E}\left[ (U_{h^\tau}(t_j), \tau \mathbb{E}\left[ \sum_{n=j+1}^{N} A_0^{n-j} \prod_{k=j+2}^{n} (1 + \Delta_k W)\xi(t_n) | \mathcal{F}_{t_j} \right] )_{L^2} \right],
\]
which is the first part of the claim. The remaining part can be deduced similarly.

2) By (3.40), and (3.38) together with (3.39), we can rewrite \( \mathcal{J}_\tau(X_{h^\tau}, U_{h^\tau}) \) in (3.7) as follows:
\[
\mathcal{J}_\tau(X_{h^\tau}, U_{h^\tau}) = \frac{1}{2} \left\{ [\Gamma \mathcal{R}_h X_0 + LU_{h^\tau} + f, \Gamma \mathcal{R}_h X_0 + LU_{h^\tau} + f]_{L^2_{\mathcal{F}}(0, T; L^2)} + (U_{h^\tau}, U_{h^\tau})_{L^2_{\mathcal{F}}(0, T; L^2)} \right. \\
+ \alpha \left( \hat{\Gamma} \mathcal{R}_h X_0 + \hat{L} U_{h^\tau} + \hat{f}, \hat{\Gamma} \mathcal{R}_h X_0 + \hat{L} U_{h^\tau} + \hat{f} \right)_{L^2_{\mathcal{F}}(0, T; L^2)} \right\},
\]
Rearranging terms then further leads to
\[
= \frac{1}{2} \left\{ [\Gamma \mathcal{R}_h X_0 + f, \Gamma \mathcal{R}_h X_0 + f]_{L^2_{\mathcal{F}}(0, T; L^2)} + (U_{h^\tau}, U_{h^\tau})_{L^2_{\mathcal{F}}(0, T; L^2)} \right. \\
+ \alpha \left( \hat{\Gamma} \mathcal{R}_h X_0 + \hat{f}, \hat{\Gamma} \mathcal{R}_h X_0 + \hat{f} \right)_{L^2_{\mathcal{F}}(0, T; L^2)} \right\}.
\]

We use this re-writing of \( \mathcal{J}_\tau(X_{h^\tau}, U_{h^\tau}) \) which involves mappings \( N \) and \( H \), and where the last term does not depend on \( U_{h^\tau} \), to now involve the optimality condition for \( U^*_{h^\tau} \): since \( N = 1 + L^* L + \alpha \hat{L}^* \hat{L} \) is positive definite, there exists a unique \( U^*_{h^\tau} \in \mathbb{U}_{h^\tau} \) such that
\[
NU^*_{h^\tau} + H(\mathcal{R}_h X_0, f) = 0.
\]

Therefore, for any \( U_{h^\tau} \in \mathbb{U}_{h^\tau} \) such that \( U_{h^\tau} \neq U^*_{h^\tau} \),
\[
\mathcal{J}_\tau(X_{h^\tau}, U_{h^\tau}) - \mathcal{J}_\tau(X^*_{h^\tau}, U^*_{h^\tau}) = \frac{1}{2} (N(U_{h^\tau} - U^*_{h^\tau}), U_{h^\tau} - U^*_{h^\tau})_{L^2_{\mathcal{F}}(0, T; L^2)} > 0,
\]
which means that \( U^*_{h^\tau} \) is the unique optimal control, and \( (X^*_{h^\tau}, U^*_{h^\tau}) \) is the unique optimal pair.

Finally, by the definition of \( N, H, L^*, \hat{L}^* \), and properties (3.41) and (3.38), we can get
\[
0 = NU^*_{h^\tau} + H(\mathcal{R}_h X_0) = U^*_{h^\tau} + [L^* X^*_{h^\tau} + \alpha \hat{L}^* \hat{L} X^*_{h^\tau}(T)].
\]
By the definition of $K_{h\tau}$ in (3.9)\textsubscript{2}, we can arrive at
\[ L^*X_{h\tau}^* + \alpha \hat{L}^*X_{h\tau}^*(T) = -K_{h\tau}X_{h\tau}^*. \]

Then (3.10) can be deduced by these two equalities. That completes the proof. \[ \]

In Remark 3.4 we already mentioned that $K_{h\tau}X_{h\tau}^*$ in (3.9)\textsubscript{2} neither solves the temporal discretization of \textsc{BSPDE}_h (3.6) by the explicit Euler, nor by the implicit Euler method. The following two lemmas study the difference between $K_{h\tau}X_{h\tau}^*$ and the temporal discretization of \textsc{BSPDE}_h (3.6) by the implicit Euler method.

**Lemma 3.10.** Suppose that $(Y_0, Z_0)$ solves the following \textsc{BSDE}_h:
\[
\begin{cases}
  dY_0(t) = [-\Delta_h Y_0(\nu(t)) - \check{Z}_0(\nu(t)) + X_{h\tau}^*(\mu(t))] \, dt + Z_0(t) \, dW(t) & \forall \, t \in [0, T], \\
  Y_0(t_N) = -\alpha X_{h\tau}^*(T),
\end{cases}
\]
\[
(3.42)
\]
where $X_{h\tau}^*$ is the optimal state of problem \textsc{SLQ}_{h\tau}, $\mu(\cdot), \nu(\cdot)$ are defined in (2.2), and $\check{Z}_0$ is a piecewise constant process which is defined by
\[
\check{Z}_0(t) = \frac{1}{\tau} \mathbb{E}\left[ \int_{t_n}^{t_{n+1}} Z_0(s) \, ds \bigg| \mathcal{F}_{t_n} \right] \quad \forall \, t \in [t_n, t_{n+1}), \quad n = 0, 1, \cdots, N - 1.
\]

Then,
\[
Y_0(t_j) = -\alpha \mathbb{E}\left[ A_0^{N-j} \prod_{k=j+1}^{N} (1 + \Delta_k W) \, X_{h\tau}^*(T) \bigg| \mathcal{F}_{t_j} \right]
\]
\[ - \tau \mathbb{E}\left[ \sum_{k=j+1}^{N} A_0^{k-j} \prod_{i=j+1}^{k} (1 + \Delta_i W) \, X_{h\tau}^*(t_k) \bigg| \mathcal{F}_{t_j} \right] \quad j = 0, 1, \cdots, N - 1.
\]

Note that $(Y_0, \check{Z}_0)$ is the numerical solution to \textsc{BSPDE}_h (3.6) by the implicit Euler method; see e.g. [1].

**Proof.** By (3.42), for any $n = 0, 1, \cdots, N - 1,$
\[
\tau \check{Z}_0(t_n) = \mathbb{E}\left[ (Y_0(t_{n+1}) - \tau X_{h\tau}^*(t_{n+1})) \Delta_{n+1} W \bigg| \mathcal{F}_{t_n} \right],
\]
which yields the desired results. \[ \]

The solution $(Y_0, Z_0)$ in (3.42) depends on $X_{h\tau}^*$. Hence, in what follows, we may write it in the form $(Y_0(\cdot; X_{h\tau}^*), Z_0(\cdot; X_{h\tau}^*))$.

Similar to $\mathcal{S}$, $\mathcal{S}_h$, we can define difference equation (3.8)’s solution operator by
\[
\mathcal{S}_{h\tau} : \mathcal{U}_{h\tau} \to \mathcal{X}_{h\tau}.
\]

In the next lemma, we estimate the difference between $(K_{h\tau} \mathcal{S}_{h\tau}(\Pi, U_{h\tau}^*))(\cdot)$, which was introduced in (3.9)\textsubscript{2}, and $Y_0(\cdot; \mathcal{S}_{h\tau}(\Pi, U_{h\tau}^*))$, which is crucial in proving rates of convergence for the temporal discretization.
Lemma 3.11. Suppose that \((Y_0, Z_0)\) solves (3.42). Then it holds that
\[
\max_{0 \leq t \leq N-1} \mathbb{E} \left[ \left\| Y_0(t_j; S_{hT}(\Pi_T U^*_h)) - (K_{hT} S_{hT}(\Pi_T U^*_h))(t_j) \right\|_{L^2}^2 \right] \leq C \tau ,
\]
where \(\Pi_T\) is defined in (3.29), where \(K_{hT}\) is given by (3.9)_2, and \(C\) is independent of \(h\) and \(\tau\).

Proof. By triangular inequality, it holds that, for any \(j = 0, 1, \cdots, N-1,\)
\[
\mathbb{E} \left[ \left\| Y_0(t_j; S_{hT}(\Pi_T U^*_h)) - (K_{hT} S_{hT}(\Pi_T U^*_h))(t_j) \right\|_{L^2}^2 \right] \\
\leq 2 \alpha^2 \mathbb{E} \left[ \left\| A_{0}^{N-j} \left( \prod_{k=j+1}^{N} - \prod_{k=j+2}^{N} \right) (1 + \Delta_k W) S_{hT}(T; \Pi_T U^*_h) \right\|_{L^2}^2 \right] \\
+ \mathbb{E} \left[ \left\| \tau \sum_{k=j+1}^{N} A_{0}^{k-j} \left( \prod_{i=j+1}^{k} - \prod_{i=j+2}^{k} \right) (1 + \Delta_i W) S_{hT}(t_i; \Pi_T U^*_h) \right\|_{L^2}^2 \right] \\
=: 2 (I_{j1} + I_{j2}) .
\]

We only present the estimate of \(I_{j2}\), since \(I_{j1}\) can be bounded in a similar vein. Applying (3.9)_1, we can get
\[
I_{j2} \leq \tau^2 N \sum_{k=j+1}^{N} \mathbb{E} \left[ \left\| \prod_{i=j+1}^{k} (1 + \Delta_i W) \Delta_{j+1} W S_{hT}(t_k; \Pi_T U^*_h) \right\|_{L^2}^2 \right] \\
\leq 3 \tau \sum_{k=j+1}^{N} \mathbb{E} \left[ \left\| \prod_{i=j+1}^{k} (1 + \Delta_i W) \Delta_{j+1} W \prod_{l=1}^{k} (1 + \Delta_l W) X_{hT}(0) \right\|_{L^2}^2 \right] \\
+ \tau^2 N \sum_{l=0}^{k-1} \mathbb{E} \left[ \left\| \prod_{i=j+1}^{k} (1 + \Delta_i W) \Delta_{j+1} W \prod_{m=l+1}^{k} (1 + \Delta_m W) U^*_h(t_l) \right\|_{L^2}^2 \right] \\
+ \sum_{l=0}^{k-1} \mathbb{E} \left[ \left\| \prod_{i=j+1}^{k} (1 + \Delta_i W) \Delta_{j+1} W \prod_{m=l+1}^{k} (1 + \Delta_m W) R_h \sigma(t_l) \Delta_{l+1} W \right\|_{L^2}^2 \right] .
\]

For the last term in (3.44), we use the fact that \(\{\Delta_j W\}_{j=1}^{N}\) is a sequence of mutually independent random variables, such that for \(0 \leq l_1, l_2 \leq k - 1, \) and \(l_1 \neq l_2,\)
\[
\mathbb{E} \left[ \left( \prod_{m=l_1+1}^{k} (1 + \Delta_m W) \Delta_{l_1+1} W \right) \times \left( \prod_{m=l_2+1}^{k} (1 + \Delta_m W) \Delta_{l_2+1} W \right) \right] = 0 .
\]
Also by the mutual independence of \(\{\Delta_j W\}_{j=1}^{N}\), and since \(X_{hT}(0)\) is deterministic, we can arrive at
\[
\mathbb{E} \left[ \left\| \prod_{i=j+1}^{k} (1 + \Delta_i W) \Delta_{j+1} W \prod_{i=1}^{k} (1 + \Delta_i W) X_{hT}(0) \right\|_{L^2}^2 \right] \\
= \prod_{i=j+1}^{k} \mathbb{E} \left[ (1 + \Delta_i W)^4 \mathbb{E} \left[ (\Delta_{j+1} W + \Delta_{j+1} W^2)^2 \right] \prod_{i=1}^{j} \mathbb{E} \left[ (1 + \Delta_i W)^2 \right] \right] \left\| X_{hT}(0) \right\|_{L^2}^2 \\
= (\tau + 3 \tau^2) \prod_{i=j+1}^{k} (1 + 4 \tau + 3 \tau^2) \prod_{i=1}^{j} \left\| X_{hT}(0) \right\|_{L^2}^2 \\
\leq C \tau \left\| X(0) \right\|_{H^1}^2 ,
\]
where $\Delta^2_{j+1}W := (\Delta_{j+1}W)^2$.

In the sequel, we tend to estimate $\mathbb{E}\left[\left\| \prod_{i=j+2}^{k} (1 + \Delta_i W) \Delta_{j+1}W \prod_{m=l+2}^{k} (1 + \Delta_m W) U_h^*(t_l) \right\|_{L^2}^2 \right]$ and $\mathbb{E}\left[\left\| \prod_{i=j+2}^{k} (1 + \Delta_i W) \Delta_{j+1}W \prod_{m=l+2}^{k} (1 + \Delta_m W) \mathcal{R}_h \sigma(t_l) \Delta_{l+1}W \right\|_{L^2}^2 \right]$ under the following two cases.

**Case I.** $l \leq j$. In this case, using the same trick as that in (3.45), we can see that

$$
\mathbb{E}\left[\left\| \prod_{i=j+2}^{k} (1 + \Delta_i W) \Delta_{j+1}W \prod_{m=l+2}^{k} (1 + \Delta_m W) U_h^*(t_l) \right\|_{L^2}^2 \right] \\
\leq \prod_{i=j+2}^{k} \mathbb{E}\left[ (1 + \Delta_i W)^4 \right] \mathbb{E}\left[ (\Delta_{j+1}W + \Delta^2_{j+1}W)^2 \right] \prod_{m=l+2}^{k} \mathbb{E}\left[ (1 + \Delta_m W)^2 \right] \mathbb{E}\left[ ||U_h^*(t_l)||_{L^2}^2 \right] \\
\leq C\tau \sup_{t \in [0,T]} \mathbb{E}\left[ ||U_h^*(t)||_{L^2}^2 \right]
$$

and

$$
\mathbb{E}\left[\left\| \prod_{i=j+2}^{k} (1 + \Delta_i W) \Delta_{j+1}W \prod_{m=l+2}^{k} (1 + \Delta_m W) \mathcal{R}_h \sigma(t_l) \Delta_{l+1}W \right\|_{L^2}^2 \right] \\
\leq \prod_{i=j+2}^{k} \mathbb{E}\left[ (1 + \Delta_i W)^4 \right] \mathbb{E}\left[ (\Delta_{j+1}W + \Delta^2_{j+1}W)^2 \right] \prod_{m=l+2}^{k} \mathbb{E}\left[ (1 + \Delta_m W)^2 \right] \mathbb{E}\left[ ||\sigma(t_l)||_{H^4}^2 \right] \mathbb{E}\left[ ||U_h^*(t_l)||_{L^2}^2 \right] \\
\leq C\tau^2 ||\sigma||_{C([0,T];H^4)}^2.
$$

**Case II.** $l > j$. Still applying the mutual independence of $\{\Delta_j W\}_{j=1}^N$, we can deduce that

$$
\mathbb{E}\left[\left\| \prod_{i=j+2}^{k} (1 + \Delta_i W) \Delta_{j+1}W \prod_{m=l+2}^{k} (1 + \Delta_m W) U_h^*(t_l) \right\|_{L^2}^2 \right] \\
\leq \prod_{i=l+2}^{k} \mathbb{E}\left[ (1 + \Delta_i W)^4 \right] \mathbb{E}\left[ (1 + \Delta_{l+1}W)^2 \right] \\
\times \left\{ \prod_{m=j+2}^{l} \mathbb{E}\left[ (1 + \Delta_m W)^4 \right] \mathbb{E}\left[ (\Delta_{j+1}W)^4 \right] \mathbb{E}\left[ ||U_h^*(t_l)||_{L^2}^4 \right] \right\}^{1/2} \\
\leq C\tau \left( \sup_{t \in [0,T]} \mathbb{E}\left[ ||U_h^*(t)||_{L^2}^4 \right] \right)^{1/2},
$$

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Proof. We use a standard argument for the implicit method to solve BSDEs (see e.g. Lemma 3.12). For simplicity, we write \(Y_h\) instead of \(Y\) and \(h\) instead of \(t\). In the above both cases, utilizing (3.50), we get

\[
\begin{align*}
&\left| I_{j+2} \right| \leq \sum_{n=1}^{N} \mathbb{E}\left[ \left| \int_{t_n}^{t_{n+1}} \Delta_h \tilde{Y}_h(s) \right| \right] + \int_{t_n}^{t_{n+1}} \mathbb{E}[|\tilde{Z}_h(s)|^2] ds \leq C \tau \|\sigma\|_{C([0,T];\mathcal{M}_h)}.
\end{align*}
\]

In the above both cases, utilizing (3.25) of Lemma 3.7, and combining with (3.44), (3.45), we can bound \(I_{j+2}\) by \(C \tau\), and then prove the desired result.

**Lemma 3.12.** Suppose that \((Y_h, Z_h), (Y_0, Z_0)\) solve (3.6), (3.42) respectively. Then it holds that

\[
\max_{0 \leq n \leq N} \mathbb{E}\left[ \left| Y_h(t_n; S_{h^*}(\Pi_{t^*} U_{h^*}^*)) - Y_0(t_n; S_{h^*}(\Pi_{t^*} U_{h^*}^*)) \right|^2 \right] \leq C \tau.
\]

**Proof.** We use a standard argument for the implicit method to solve BSDEs (see e.g. [23, 19]); since we need to deal with a family of BSDEs, which is obtained via finite element discretization of BSDE and thus depends on the parameter \(h\), for the completeness, we provide the proof.

For simplicity, we write

\[
\left( Y_h(\cdot; S_{h^*}(\Pi_{t^*} U_{h^*}^*)), Z_h(\cdot; S_{h^*}(\Pi_{t^*} U_{h^*}^*)) \right) \text{ resp. } \left( Y_0(\cdot; S_{h^*}(\Pi_{t^*} U_{h^*}^*)), Z_0(\cdot; S_{h^*}(\Pi_{t^*} U_{h^*}^*)) \right)
\]

as \((\tilde{Y}_h(\cdot), \tilde{Z}_h(\cdot))\) resp. \((\tilde{Y}_0(\cdot), \tilde{Z}_0(\cdot))\) in the proof. Set \(e_V^0 = \tilde{Y}_h(t_n) - \tilde{Y}_0(t_n)\). Subtracting (3.42) from (3.6) by changing \(X^*_h\) to \(S_{h^*}(\Pi_{t^*} U_{h^*}^*)\), we can get

\[
\begin{align*}
(I - \tau \Delta_h) e_V^0 &= \int_{t_n}^{t_{n+1}} (\tilde{Z}_h(s) - \tilde{Z}_0(s)) dW(s) \\
= e_V^{n+1} + \int_{t_n}^{t_{n+1}} \Delta_h(\tilde{Y}_h(t_n) - \tilde{Y}_h(s)) + \frac{1}{\tau} \mathbb{E}\left[ \int_{t_n}^{t_{n+1}} \tilde{Z}_0(t) d|\mathcal{F}_t| - \tilde{Z}_h(s) \right] + S_{h^*}(t_n, \Pi_{t^*} U_{h^*}^*) - S_{h^*}(t_{n+1}, \Pi_{t^*} U_{h^*}^*) ds.
\end{align*}
\]

Following the same procedure as in the proof of [19, Theorem 4.1], we get for all \(\varepsilon > 0\),

\[
\begin{align*}
&\mathbb{E}\left[ \left\| (I - \tau \Delta_h) e_V^0 \right\|_{L^2}^2 \right] + \mathbb{E}\left[ \int_{t_n}^{t_{n+1}} \left\| \tilde{Z}_h(s) - \tilde{Z}_0(s) \right\|_{L^2}^2 ds \right] \\
&\quad \leq (1 + 5\varepsilon)\mathbb{E}\left[ \left\| e_V^{n+1} \right\|_{L^2}^2 \right] + \mathbb{E}\left[ \int_{t_n}^{t_{n+1}} \left\| \Delta_h(\tilde{Y}_h(s) - \tilde{Y}_h(t_n)) \right\|_{L^2}^2 ds \right] \\
&\quad \quad + 2\tau \mathbb{E}\left[ \int_{t_n}^{t_{n+1}} \left\| \tilde{Z}_h(s) - \tilde{Z}_h(t_n) \right\|_{L^2}^2 ds \right] + \mathbb{E}\left[ \int_{t_n}^{t_{n+1}} \left\| \tilde{Z}_0(s) - \tilde{Z}_0(s) \right\|_{L^2}^2 ds \right] \\
&\quad \quad + \tau^2 \mathbb{E}\left[ \left\| S_{h^*}(t_n, \Pi_{t^*} U_{h^*}^*) - S_{h^*}(t_{n+1}, \Pi_{t^*} U_{h^*}^*) \right\|_{L^2}^2 \right].
\end{align*}
\]
By taking \( \varepsilon \geq \frac{\tau}{1-5\tau} \) (for example we may choose \( \tau \) small enough such that \( 5\tau \leq 1/2 \) and take \( \varepsilon = 2\tau \), we will see that
\[
\mathbb{E} \left[ \| e_0^\varepsilon \|^2_{L^2} \right] \leq (1 + 10\tau) \mathbb{E} \left[ \| e_{n+1}^\varepsilon \|^2_{L^2} \right] + \left( 5 + \frac{1}{2\tau} \right) \left( \tau \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \| \Delta_h (\hat{Y}_h(s) - \hat{Y}_h(t_n)) \|^2_{L^2} ds \right] + \tau \mathbb{E} \left[ \| S_{h\tau}(t_n, \Pi_r U_n^\varepsilon) - S_{h\tau}(t_{n+1}, \Pi_r U_n^\varepsilon) \|^2_{L^2} \right] \right).
\]

Then the discrete Gronwall’s inequality leads to
\[
\max_{0 \leq n \leq N-1} \mathbb{E} \left[ \| e_n^\varepsilon \|^2_{L^2} \right] \leq C \sum_{n=0}^{N-1} \left\{ \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \| \Delta_h (\hat{Y}_h(s) - \hat{Y}_h(t_n)) \|^2_{L^2} ds \right] + \tau \mathbb{E} \left[ \| S_{h\tau}(t_n, \Pi_r U_n^\varepsilon) - S_{h\tau}(t_{n+1}, \Pi_r U_n^\varepsilon) \|^2_{L^2} \right] \right\} =: C (I_1 + I_2 + I_3).
\]

In what follows, we estimate the three terms on the right side of (3.51). For \( I_1 \), by \textbf{BSPDE}_h (3.6) (changing \( X_h^* \) to \( S_{h\tau}(\Pi_r U_n^\varepsilon) \)) and Itô’s formula to \( \| \nabla \hat{Y} \|^2_{L^2} \), we estimate
\[
I_1 \leq C \tau \mathbb{E} \left[ \int_0^T \| \Delta_h \hat{Y}(t) \|^2_{L^2} dt + \sup_{0 \leq n \leq N} \mathbb{E} \left[ \| \Delta_h S_{h\tau}(t_n; \Pi_r U_n^\varepsilon) \|^2_{L^2} \right] \right] + C \tau \sup_{0 \leq n \leq N} \mathbb{E} \left[ \| \Delta_h S_{h\tau}(t_n; \Pi_r U_n^\varepsilon) \|^2_{L^2} \right].
\]

In the below, we tend to estimate two terms on the right side of (3.52). By Lemma 3.9 and the following calculation:
\[
\mathbb{E} \left[ \| \Delta_h S_{h\tau}(t_n; \Pi_r U_n^\varepsilon) \|^2_{L^2} \right] \leq 3 \prod_{j=1}^{n} (1 + \tau) \| \Delta_h X_{h\tau}(0) \|^2_{L^2} + 3\tau^2 N \sum_{j=0}^{n-1} \prod_{k=j+2}^{n} (1 + \tau) \mathbb{E} \left[ \| \Delta_h U_n^\varepsilon(t_j) \|^2_{L^2} \right] \]
\[
+ 3\tau \sum_{j=0}^{n-1} \prod_{k=j+2}^{n} (1 + \tau) \| \Delta_h R_h \sigma(t_j) \|^2_{L^2} \leq C \left[ \| X_0 \|^2_{H^2} + \| \sigma \|^2_{C^0(0,T);H^2} \right],
\]
we can see that the second term on the right side of (3.52) is bounded. In the below, we estimate the first term. By (3.9) (with \( U_{h\tau} = \Pi_r U_n^\varepsilon \)), for any \( k = 0, 1, \cdots, N \),
\[
\mathbb{E} \left[ \| \nabla \Delta_h A_0^N \prod_{j=1}^{N} (1 + \Delta_j W) X_{h\tau}(0) \|^2_{L^2} \right] = (1 + \tau)^N \| \nabla \Delta_h \hat{X}_{h\tau}(T) \|^2_{L^2},
\]
where \( \hat{X}_{h\tau}(t_k) = A_0^k X_{h\tau}(0), \) for \( k = 0, 1, \cdots, N \) solves
\[
\begin{cases}
\hat{X}_{h\tau}(t_{k+1}) - \hat{X}_{h\tau}(t_k) = \tau \Delta_h \hat{X}_{h\tau}(t_{k+1}) & k = 0, 1, \cdots, N - 1, \\
\hat{X}_{h\tau}(0) = X_{h\tau}(0).
\end{cases}
\]
Multiplying (3.55) by $\Delta_h^2 \hat{X}_{hT}(t_{k+1})$ then leads to
\[
\|\nabla \Delta_h \hat{X}_{hT}(t_{k+1})\|_{L^2}^2 = \|\nabla \Delta_h A_0 \hat{X}_{hT}(t_k)\|_{L^2}^2
\]
\[
= \|\nabla \Delta_h \hat{X}_{hT}(t_k)\|_{L^2}^2 - \|\nabla \Delta_h (\hat{X}_{hT}(t_{k+1}) - \hat{X}_{hT}(t_k))\|_{L^2}^2 - \tau \|\Delta_h^2 \hat{X}_{hT}(t_{k+1})\|_{L^2}^2
\]
\[
\leq \|\nabla \Delta_h \hat{X}_{hT}(t_k)\|_{L^2}^2
\]
which, together with (3.54), leads to
\[
E\left[\|\nabla \Delta_h A_0^N \prod_{j=1}^N (1 + \Delta_j W) X_{hT}(0)\|_{L^2}^2\right] \leq C\|\nabla \Delta_h X_{hT}(0)\|_{L^2}^2.
\]
(3.57)

With a similar trick, we can show that
\[
E\left[\left\|\nabla \Delta_h A_0^{l-j} \prod_{k=j+1}^n (1 + \Delta_k W) R_h \sigma(t_j) \Delta_{j+1} W\right\|_{L^2}^2\right] \leq C\|\nabla \Delta_h \sigma(t_j)\|_{L^2}^2 \leq C\|\sigma(t_j)\|_{H^3}^2.
\]
(3.58)

On the other side, by applying the maximum condition (3.5), estimate (3.56), Lemma 3.9 and Itô’s formula to $\|\nabla \Delta_h A_0^N Y_h\|_{L^2}^2$, we can find that
\[
\sup_{t \in [0,T]} E\left[\|\nabla \Delta_h A_0^k U_h^k(t)\|_{L^2}^2\right] + E\left[\int_0^T \|\Delta_h^2 A_0^k U_h^k(t)\|_{L^2}^2 + \|\nabla \Delta_h A_0^k Z_h(t)\|_{L^2}^2 dt\right]
\]
\[
\leq C\left[E\left[\left\|\nabla \Delta_h A_0^k X_h^k(T)\right\|_{L^2}^2\right] + \|\nabla \Delta_h X_h^k(0)\|_{L^2}^2\right] \leq C\left[E\left[\left\|\nabla \Delta_h X_h^k(T)\right\|_{L^2}^2\right] + \|\nabla \Delta_h X_h^k(0)\|_{L^2}^2\right] \leq C \left[\|X_0\|_{H^3}^2 + \|\sigma\|_{L^2([0,T];H^3)}^2\right].
\]
(3.59)

Combining with (3.57), (3.58) and (3.59), we can deduce that
\[
E\left[\left\|\nabla \Delta_h S_{hT}(T; \Pi_u U_h^*)\right\|_{L^2}^2\right] \leq C \left[\|X_0\|_{H^3}^2 + \|\sigma\|_{C([0,T];H^3)}^2\right].
\]
Therefore, by (3.52), we arrive at
\[
I_1 \leq C \left[\|X_0\|_{H^3}^2 + \|\sigma\|_{C([0,T];H^3)}^2\right] \tau.
\]

For $I_2$, we still can prove that
\[
I_2 \leq C \left[\|X_0\|_{H^3}^2 + \|\sigma\|_{C([0,T];H^3)}^2\right] \tau.
\]
(3.60)

We postpone the proof to Lemma 3.13.
For $I_3$, (3.8) (with $U_{hT} = \Pi_u U_h^*$) yields to
\[
E\left[\left\|S_{hT}(t_n; \Pi_u U_h^*) - S_{hT}(t_{n+1}; \Pi_u U_h^*)\right\|_{L^2}^2\right]
\]
\[
= E\left[\|\tau [\Delta_h S_{hT}(t_{n+1}; \Pi_u U_h^*) + U_h^*(t_n)] + [S_{hT}(t_n; \Pi_u U_h^*) + R_h \sigma(t_n)] \Delta_{n+1} W\|_{L^2}^2\right]
\]
\[
\leq C \tau \max_{0 \leq n \leq N} E\left[\left\|S_{hT}(t_n; \Pi_u U_h^*)\right\|_{L^2}^2\right] \leq C \left[\|X_0\|_{H^3}^2 + \|\sigma\|_{C([0,T];H^3)}^2\right] \tau.
\]
(3.61)

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(3.53), together with (3.25)_5 and (3.61) then leads to
\[
I_3 \leq C \max_{0 \leq n \leq N - 1} \mathbb{E}\left[\|S_{hT}(t_n; \Pi_s U_h^*) - S_{hT}(t_{n+1}; \Pi_s U_h^*)\|_{L^2}^2\right] \\
\leq C \left[\|X_0\|_{H^1}^2 + \|\sigma\|_{C([0,T]; H^1)}^2\right] \tau. 
\]
(3.62)

That completes the proof.

Lemma 3.13. Suppose that \((Y_h, Z_h)\) solves (3.6). Then for all \(t \in [0,T]\),
\[
\mathbb{E}\left[\|Z_h(t; S_{hT}(\Pi_s U_h^*)) - Z_h(\nu(t); S_{hT}(\Pi_s U_h^*))\|_{L^2}^2\right] \leq C|t - \nu(t)| \left(\|X_0\|_{H^1}^2 + \|\sigma\|_{C([0,T]; H^1)}^2\right),
\]
where \(\nu(\cdot)\) is defined in (2.2)

Proof. The proof is long. Hence we divide it into three steps.

1) We claim that, for any \(n = 0, 1, \cdots, N\), \(S_{hT}(t_n; \Pi_s U_h^*) \in \mathbb{D}^{2,2}(L^2)\).

Indeed, by (3.40), we know that
\[
S_{hT}(t_n; \Pi_s U_h^*) = A_0^n \prod_{j=1}^n (1 + \Delta_j W) X_{hT}(0) + \tau \sum_{j=0}^{n-1} A_0^{n-j} \prod_{k=j+2}^n (1 + \Delta_k W) U_h^*(t_j) \\
+ \sum_{j=0}^{n-1} A_0^{n-j} \prod_{k=j+2}^n (1 + \Delta_k W) \mathcal{R}_h \sigma(t_j) \Delta_{j+1} W.
\]

In the following, we only prove that the second term on the right-hand side of the above representation is in \(\mathbb{D}^{2,2}(L^2)\). The other two terms can be proved in a similar vein.

By maximum condition (3.5) and Lemma 3.8, we know that for any \(j = 0, 1, \cdots, N - 1\), \(U_h^*(t_j) \in \mathbb{D}^{2,2}(L^2)\). By the chain rule, for any \(\theta, \mu \in [0,T]\) without loss of generality, suppose that \(\theta \in [t_l, t_{l+1})\), \(\mu \in [t_m, t_{m+1})\). Then, by the fact that \(D_0(1 + \Delta_k W) = \delta_{lk}\), \(D_\mu(1 + \Delta_k W) = \delta_{mk}\), where \(\delta_{lk}, \delta_{mk}\) are Kronecker delta functions, we have
\[
D_\theta \left(\tau \sum_{j=0}^{n-1} A_0^{n-j} \prod_{k=j+2}^n (1 + \Delta_k W) U_h^*(t_j)\right)
= \tau \sum_{j=0}^{n-1} A_0^{n-j} \prod_{k=j+2}^n (1 + \Delta_k W) U_h^*(t_j) + \tau \sum_{j=0}^{n-1} A_0^{n-j} \prod_{k=j+2}^n (1 + \Delta_k W) D_\theta U_h^*(t_j),
\]
\[
D_\mu D_\theta \left(\tau \sum_{j=0}^{n-1} A_0^{n-j} \prod_{k=j+2}^n (1 + \Delta_k W) U_h^*(t_j)\right)
= \tau \sum_{j=0}^{n-1} A_0^{n-j} \delta_{ml} \prod_{k=j+2}^n (1 + \Delta_k W) U_h^*(t_j) + \tau \sum_{j=0}^{n-1} A_0^{n-j} \prod_{k=j+2}^n (1 + \Delta_k W) D_\mu U_h^*(t_j)
+ \tau \sum_{j=0}^{n-1} A_0^{n-j} \prod_{k=j+2}^n (1 + \Delta_k W) D_\theta D_\mu U_h^*(t_j),
\]

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and subsequently, by (3.28)₃ and (3.28)₄ in Lemma 3.8,

\[
\mathbb{E} \left[ \left\| D_\mu D_\nu \left( \tau \sum_{j=0}^{n-1} A_{n-j} \prod_{k=j+2}^{n} (1 + \Delta_k W) U_h^*(t_j) \right) \right\|_{L^2}^2 \right] \\
\leq C \left[ \sup_{0 \leq j \leq N-1} \mathbb{E} \left[ \left\| U_h^*(t_j) \right\|_{L^2}^2 \right] + \sup_{0 \leq j \leq N-1} \mathbb{E} \left[ \left\| D_\mu D_\nu U_h^*(t_j) \right\|_{L^2}^2 \right] \right] \\
\leq C \left[ \left\| X_0 \right\|_{H^1_0}^2 + \left\| \sigma \right\|_{C([0,T];L^2)}^2 + \left\| \sigma \right\|_{L^2(0,T;H^1_0)}^2 \right],
\]

which leads to \( \tau \sum_{j=0}^{n-1} A_{n-j} \Pi_{k=j+2} (1 + \Delta_k W) U_h^*(t_j) \in \mathbb{D}^{2,2}(L^2) \). Hence, for any \( t \in [0, T] \), \( S_{h\tau}(\nu(t); \Pi_U h^*) \in \mathbb{D}^{2,2}(L^2) \), and

\[
\sup_{\mu, \theta \in [0,T]} \sup_{\nu(t) \in [\mu \vee \theta, T]} \mathbb{E} \left[ \left\| D_\mu D_\nu S_{h\tau}(\nu(t); \Pi_U h^*) \right\|_{L^2}^2 \right] \leq C \left[ \left\| X_0 \right\|_{H^1_0}^2 + \left\| \sigma \right\|_{C([0,T];L^2)}^2 + \left\| \sigma \right\|_{L^2(0,T;H^1_0)}^2 \right].
\]

(3.64)

2) Applying the same trick to estimate \( \mathbb{E} \left[ \left\| \nabla \Delta_h S_{h\tau}(T; \Pi_U h^*) \right\|_{L^2}^2 \right] \) in the proof of Lemma 3.12, we can deduce that for any \( \theta \in [0, T] \),

\[
\mathbb{E} \left[ \left\| D_\mu D_\nu S_{h\tau}(T; \Pi_U h^*) \right\|_{L^2}^2 \right] \leq C \left[ \left\| X_0 \right\|_{H^1_0}^2 + \left\| \sigma \right\|_{C([0,T];H^1_0)}^2 \right].
\]

(3.65)

Also, by the same procedure as that in the proof of (3.28)₄, (3.28)₅ in Lemma 3.8, thanks to (3.64), we can obtain

\[
\sup_{\theta \in [0,T]} \mathbb{E} \left[ \int_\theta^T \left\| D_\mu Z_h(t; S_{h\tau}(\Pi_U h^*)) \right\|_{L^2}^2 dt \right] \\
\leq C \sup_{\theta \in [0,T]} \mathbb{E} \left[ \left\| D_\mu S_{h\tau}(\nu(t); \Pi_U h^*) \right\|_{L^2}^2 \right] \\
\leq C \left[ \left\| X_0 \right\|_{H^1_0}^2 + \left\| \sigma \right\|_{C([0,T];L^2)}^2 + \left\| \sigma \right\|_{L^2(0,T;H^1_0)}^2 \right],
\]

(3.66)

\[
\sup_{\mu, \theta \in [0,T]} \sup_{\nu(t) \in [\mu \vee \theta, T]} \mathbb{E} \left[ \left\| D_\mu D_\nu Y_h(t; S_{h\tau}(\Pi_U h^*)) \right\|_{L^2}^2 \right] \\
\leq C \sup_{\mu, \theta \in [0,T]} \mathbb{E} \left[ \left\| D_\mu D_\nu S_{h\tau}(\nu(t); \Pi_U h^*) \right\|_{L^2}^2 \right] \\
\leq C \left[ \left\| X_0 \right\|_{H^1_0}^2 + \left\| \sigma \right\|_{C([0,T];L^2)}^2 + \left\| \sigma \right\|_{L^2(0,T;H^1_0)}^2 \right],
\]

(3.67)

and

\[
\sup_{\theta \in [0,T]} \mathbb{E} \left[ \int_\theta^T \left\| \Delta_h D_\nu Y_h(t; S_{h\tau}(\Pi_U h^*)) \right\|_{L^2}^2 dt \right] \\
\leq C \sup_{\theta \in [0,T]} \mathbb{E} \left[ \left\| \nabla D_\nu S_{h\tau}(\nu(t); \Pi_U h^*) \right\|_{L^2}^2 \right] \\
\leq C \left[ \left\| X_0 \right\|_{H^1_0}^2 + \left\| \sigma \right\|_{C([0,T];H^1_0)}^2 \right].
\]

(3.68)

3) Applying the fact that \( Z_h(\cdot; S_{h\tau}(\Pi_U h^*)) = D.Y_h(\cdot; S_{h\tau}(\Pi_U h^*)) \) a.e., for any \( t \in [t_n, t_{n+1}] \), \( n = 0, 1, \cdots, N - 1 \), we arrive at

\[
\mathbb{E} \left[ \left\| Z_h(t; S_{h\tau}(\Pi_U h^*)) - Z_h(\nu(t); S_{h\tau}(\Pi_U h^*)) \right\|_{L^2}^2 \right] \\
= \mathbb{E} \left[ \left\| D_\nu Y_h(t; S_{h\tau}(\Pi_U h^*)) - D_t X_h(t_n; S_{h\tau}(\Pi_U h^*)) \right\|_{L^2}^2 \right] \\
\leq 2 \mathbb{E} \left[ \left\| (D_t - D_t) X_h(t_n; S_{h\tau}(\Pi_U h^*)) \right\|_{L^2}^2 \\
+ \left\| D_t X_h(t_n; S_{h\tau}(\Pi_U h^*)) - Y_h(t_n; S_{h\tau}(\Pi_U h^*)) \right\|_{L^2}^2 \right].
\]

(3.69)
For the first term on the right side of (3.69), the fact \( t, t_n \in [t_n, t_{n+1}) \) leads to
\[
(D_t - D_{t_n})S_{h^\tau}(\cdot; \Pi_t U_h^*) \equiv 0 \quad \text{a.e.},
\]
and then
\[
(D_t - D_{t_n})Y_h(\cdot; S_{h^\tau}(\Pi_t U_h^*)) \equiv 0 \quad \text{a.e.}. \tag{3.70}
\]
Similar to (3.30), by virtue of (3.64), (3.66)–(3.64), we can get
\[
\mathbb{E}[\|D_{t_n}(Y_h(t; S_{h^\tau}(\Pi_t U_h^*)) - Y_h(t_n; S_{h^\tau}(\Pi_t U_h^*)))\|_{L^2}^2] 
\leq C \left( \mathbb{E} \left[ \int_{t_n}^t \|D_{t_n}D\theta Y_h(\theta; S_{h^\tau}(\Pi_t U_h^*))\|_{L^2}^2 \, d\theta \right] 
+ (t - t_n) \mathbb{E} \left[ \int_{t_n}^T \|\Delta_h D_{t_n}Y_h(\theta; S_{h^\tau}(\Pi_t U_h^*))\|_{L^2}^2 + \|D_{t_n}Z_h(\theta; S_{h^\tau}(\Pi_t U_h^*))\|_{L^2}^2 \, d\theta \right] \right)
\leq C|t - t_n| \left[ \|X_0\|_{H_0^0}^2 + \|\sigma\|_{C([0,T];H_0^0)}^2 \right].
\tag{3.71}
\]
Now desired result (3.63) can be derived by (3.69)–(3.71).

**Remark 3.14.** For SPDE (1.2) driven by additive noise (i.e., \( \sigma(t) \, dW(t) \)), \( Z_h \) does not appear in the drift of BSPDE\(_h^\tau\) (3.6). For this case, in order to prove Lemma 3.12, we simply multiply by \( \epsilon_{n}\) both sides of (3.50) and then take expectations to settle Lemma 3.12. For our problem, however, this approach fails due to the presence of \( Z_h \) in the drift.

We are now ready to verify rates of convergence for the solution to problem SLQ\(_h^\tau\); it is as in Section 3.1 that the reduced cost functional \( \tilde{J}_{h^\tau}: \mathbb{U}_{h^\tau} \to \mathbb{R} \) is used, which is defined via
\[
\tilde{J}_{h^\tau}(U_{h^\tau}) = J_{h^\tau}(S_{h^\tau}(U_{h^\tau}), U_{h^\tau}),
\]
where \( S_{h^\tau}: \mathbb{U}_{h^\tau} \to \mathbb{X}_{h^\tau} \) is the solution operator to the forward equation (3.9)\(_1\).

**Proof of Theorem 3.3.** We divide the proof into two steps.

1) We follow the argumentation in the proof of Theorem 3.1. For every \( U_{h^\tau}, R_{h^\tau} \in \mathbb{U}_{h^\tau} \), the first Fréchet derivative \( D\tilde{J}_{h^\tau}(U_{h^\tau}) \), and the second Fréchet derivative \( D^2\tilde{J}_{h^\tau}(U_{h^\tau}) \) satisfy
\[
D\tilde{J}_{h^\tau}(U_{h^\tau}) = U_{h^\tau} - K_{h^\tau}S_{h^\tau}(U_{h^\tau}) \quad \text{(3.72)}
\]
By putting \( R_{h^\tau} = U_{h^\tau}^* - \Pi_t U_{h^\tau}^* \) in (3.72), and applying the fact \( D\tilde{J}_{h^\tau}(U_{h^\tau}^*) = D\tilde{J}_{h^\tau}(U_{h^\tau}^*) = 0 \), we see that
\[
\|U_{h^\tau}^* - \Pi_t U_{h^\tau}^*\|_{L^2(0,T;L^2)}^2 \leq \left[ \|D\tilde{J}_{h^\tau}(U_{h^\tau}^*) - D\tilde{J}_{h^\tau}(\Pi_t U_{h^\tau}^*), U_{h^\tau}^* - \Pi_t U_{h^\tau}^*\|_{L^2(0,T;L^2)}^2 \right]
+ \|D\tilde{J}_{h^\tau}(\Pi_t U_{h^\tau}^*) - D\tilde{J}_{h^\tau}(\Pi_t U_{h^\tau}^*), U_{h^\tau}^* - \Pi_t U_{h^\tau}^*\|_{L^2(0,T;L^2)}^2. \tag{3.73}
\]
Therefore,
\[
\|U_{h^\tau}^* - \Pi_t U_{h^\tau}^*\|_{L^2(0,T;L^2)}^2 
\leq 3\left[\|D\tilde{J}_{h^\tau}(U_{h^\tau}^*) - D\tilde{J}_{h^\tau}(\Pi_t U_{h^\tau}^*)\|_{L^2(0,T;L^2)}^2 + \|\tilde{J}_{h^\tau}(S_{h^\tau}(\Pi_t U_{h^\tau}^*)) - Y_0(S_{h^\tau}(\Pi_t U_{h^\tau}^*))\|_{L^2(0,T;L^2)}^2 \right]
+ \|Y_0(S_{h^\tau}(\Pi_t U_{h^\tau}^*)) - K_{h^\tau}S_{h^\tau}(\Pi_t U_{h^\tau}^*)\|_{L^2(0,T;L^2)}^2 \tag{3.74}
\]
\[= 3(I' + II' + III').\]
We use (3.15) and (3.5) to bound $I'$ as follows,
\[
I' = \|U_h^* - \Pi_r U_h^* + C \tau \left( S_h(\Pi_r U_h^*) - T_h^1(\Pi_r U_h^*) \right) \|_{L^2(0,T;L^2)}^2 \\
\leq 2 \left[ \|U_h^* - \Pi_r U_h^*\|_{L^2(0,T;L^2)}^2 + \|T_h^1(S_h(\Pi_r U_h^*)) - T_h^1(S_h(U_h^*))\|_{L^2(0,T;L^2)}^2 \right].
\]  
(3.75)

By stability properties of solutions to BSPDE $h$ (3.6) with $X_h^* = S_h(\Pi_r U_h^*)$ and SPDE $h$ (3.4) with $U_h = \Pi_r U_h^*$, we obtain
\[
\|T_h^1(S_h(\Pi_r U_h^*)) - T_h^1(S_h(U_h^*))\|_{L^2(0,T;L^2)}^2 \\
\leq C \left[ \|S_h(U_h^*) - S_h(\Pi_r U_h^*)\|_{L^2(0,T;L^2)}^2 + \|S_h(\Pi_r U_h^*) - S_h(\Pi_r U_h^*)\|_{L^2(0,T;L^2)}^2 \right].
\]  
(3.76)

By the optimality condition (3.5), estimates (3.28)$_3$, (3.28)$_5$ of Lemma 3.8, we have
\[
\|U_h^* - \Pi_r U_h^*\|_{L^2(0,T;L^2)}^2 \leq C \|Y_h - \Pi_r Y_h\|_{L^2((0,T;L^2)}^2 \leq C \tau \|X_0\|_{H^1}^2.
\]  
(3.77)

Next, we turn to $II'$. Triangular inequality leads to
\[
II' \leq 2 \left( \|T_h^1(S_h(\Pi_r U_h^*)) - T_h^1(S_h(\Pi_r U_h^*))\|_{L^2(0,T;L^2)}^2 \\
+ \|T_h^1(S_h(\Pi_r U_h^*)) - Y_0(\Pi_r U_h^*)\|_{L^2(0,T;L^2)}^2 \right).
\]

In order to bound $II'_1$, we use stability properties for SPDE $h$ (2.6), BSPDE $h$ (3.6), in combination with the error estimate (2.9) for (2.6) to conclude
\[
II'_1 \leq C \left[ \|S_h(\Pi_r U_h^*) - S_{h_r^*}(\Pi_r U_h^*)\|_{L^2(0,T;L^2)}^2 + \|S_h(\Pi_r U_h^*) - S_{h_r^*}(\Pi_r U_h^*)\|_{L^2(0,T;L^2)}^2 \right] \leq C \tau.
\]

To bound $II'_2$, it is easy to see
\[
II'_2 \leq 2 \sum_{n=0}^{N-1} E \left[ \int_{t_n}^{t_{n+1}} \|Y_h(t; S_{h_r}(\Pi_r U_h^*)) - Y_h(t_n; S_{h_r}(\Pi_r U_h^*))\|_{L^2}^2 dt \right] \\
+ 2T \max_{0 \leq t \leq N} E \left[ \|Y_h(t_n; S_{h_r^*}(\Pi_r U_h^*)) - Y_0(t_n; S_{h_r^*}(\Pi_r U_h^*))\|_{L^2}^2 \right].
\]

Like (3.30), we can get
\[
\sum_{n=0}^{N-1} E \left[ \int_{t_n}^{t_{n+1}} \|Y_h(t; S_{h_r}(\Pi_r U_h^*)) - Y_h(t_n; S_{h_r}(\Pi_r U_h^*))\|_{L^2}^2 dt \right] \\
\leq C \tau E \left[ \int_{0}^{T} \|\Delta_t Y_h(t; S_{h_r}(\Pi_r U_h^*))\|_{L^2}^2 + \|Z_h(t; S_{h_r}(\Pi_r U_h^*))\|_{L^2}^2 + \|S_h(t; \Pi_r U_h^*)\|_{L^2}^2 dt \right].
\]  
(3.78)

Therefore, by Lemma 3.12 and (3.78), we can get
\[
II'_2 \leq C \tau.
\]
Finally, Lemma 3.11 leads to

$$III' \leq C\tau.$$  

Now we insert above estimates into (3.74) to obtain assertion (i).

2) For all \( k = 0, 1, \ldots, N \), we define \( e^k_X = X^*_h(t_k) - X^*_h(t_k) \). Subtracting (3.8) from (3.4) leads to

\[
e^{k+1}_X = e^k_X \tau + e^k_X \Delta X_{k+1} + \tau [U^*_h(t_k) - U^*_h(t_k)]\]

\[+ \int_{t_k}^{t_{k+1}} \left([X^*_h(s) - X^*_h(t_k)] + [\mathcal{R}_h \sigma(s) - \mathcal{R}_h \sigma(t_k)]\right) dW(s)\]

\[+ \int_{t_k}^{t_{k+1}} \left([U^*_h(s) - U^*_h(t_k)]\right) ds.

Testing with \( e^{k+1}_X \), and using binomial formula, Poincaré’s inequality, independence, and absorption lead to

\[
\frac{1}{2} \mathbb{E}[\|e^{k+1}_X\|^2_{L^2} - \|e^k_X\|^2_{L^2}] + \frac{1}{2} \mathbb{E}[\|e^{k+1}_X - e^k_X\|^2_{L^2}] + \frac{\tau}{2} \mathbb{E}[\|
abla e^{k+1}_X\|^2_{L^2}]
\]

\[\leq \tau \mathbb{E}[\|e^{k+1}_X\|^2_{L^2}] + 2\tau \mathbb{E}[\|e^k_X\|^2_{L^2}] + \frac{\tau}{2} \mathbb{E}[\|U^*_h(t_k) - U^*_h(t_k)\|^2_{L^2}]\]

\[+ C \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \left[\|
abla [X^*_h(s) - X^*_h(t_{k+1})]\|^2_{L^2} + \|X^*_h(s) - X^*_h(t_k)\|^2_{L^2}\right.\]

\[\left. + \|\sigma(s) - \sigma(t_k)\|^2_{H^1} + \|U^*_h(s) - U^*_h(t_k)\|^2_{L^2} ds\right].
\]

By the discrete Gronwall’s inequality, and then taking the sum over all \( 0 \leq k \leq N - 1 \), and noting that \( e^0_X = 0 \), we find that

\[
\max_{0 \leq n \leq N} \mathbb{E}[\|e^n_X\|^2_{L^2}] + \sum_{n=1}^{N} \tau \mathbb{E}[\|
abla e^n_X\|^2_{L^2}]\]

\[\leq C\tau \sum_{k=0}^{N-1} \mathbb{E}[\|U^*_h(t_k) - U^*_h(t_k)\|^2_{L^2}]\]

\[+ C \sum_{k=0}^{N-1} \mathbb{E}\left[\int_{t_k}^{t_{k+1}} \left[\|
abla [X^*_h(s) - X^*_h(t_{k+1})]\|^2_{L^2} + \|X^*_h(s) - X^*_h(t_k)\|^2_{L^2}\right.\]

\[\left. + \|\sigma(s) - \sigma(t_k)\|^2_{H^1} + \|U^*_h(s) - U^*_h(t_k)\|^2_{L^2} ds\right].
\]

By (3.74), the first term on the right-hand side is bounded by \( C\tau \). By (3.25) and (3.36), we can bound the second and third terms by \( C\tau \). The fourth term can be bounded by \( C\tau \|\sigma\|^2_{C^1([0,T];H^1)} \).

By the optimal condition (3.5) and (3.28) in Lemma 3.8, the last term is bounded by \( C\tau \). That is assertion (ii).

\[\square\]

4 The gradient descent method to solve problem SLQ

By Theorem 3.2, solving problem SLQ is equivalent to solving the system of the coupled forward-backward difference equations (3.9) and (3.10). We may exploit the variational character of problem SLQ to construct a gradient descent method where approximate iterates of the optimal control \( U^*_h \) in the Hilbert space \( U_h \) are obtained; see also [14, 12] for more details.
Algorithm 4.1. Let $U^{(0)}_{h^2} \in U_{h^2}$, and fix $\kappa > 0$. For any $\ell \in \mathbb{N}_0$, update $U^{(\ell)}_{h^2} \in U_{h^2}$ as follows:

1. Compute $X^{(\ell)}_{h^2} \in X_{h^2}$ by

$$
\begin{align*}
&\left\{
(1 - \tau \Delta h)X^{(\ell)}_{h^2}(t_{n+1}) = X^{(\ell)}_{h^2}(t_n) + \tau U^{(\ell)}_{h^2}(t_n) + \left[X^{(\ell)}_{h^2}(t_n) + R_h \sigma(t_n)\right] \Delta_{n+1} W \\
&\quad n = 0, 1, \ldots, N - 1,
\end{align*}
$$

(4.1)

2. Use $X^{(\ell)}_{h^2} \in X_{h^2}$ to compute $Y^{(\ell)}_{h^2} \in X_{h^2}$ via

$$
Y^{(\ell)}_{h^2}(t_n) = -\tau \mathbb{E}\left[\sum_{j=n+1}^N A_{0,j-n}^2 \prod_{k=n+2}^j (1 + \Delta k W) X^{(\ell)}_{h^2}(t_j) \left| \mathcal{F}_{t_n} \right\} \right] - \alpha \mathbb{E}\left[A_{0,n-n}^2 \prod_{k=n+2}^N (1 + \Delta k W) X^{(\ell)}_{h^2}(T) \left| \mathcal{F}_{t_n} \right\} \right] n = 0, 1, \ldots, N - 1.
$$

3. Update $U^{(\ell+1)}_{h^2} \in U_{h^2}$ via

$$
U^{(\ell+1)}_{h^2} = U^{(\ell)}_{h^2} - \frac{1}{\kappa} (U^{(\ell)}_{h^2} - Y^{(\ell)}_{h^2}).
$$

If compared with (3.9)-(3.10), steps 1 and 2 are now decoupled: the first step requires to solve a space-time discretization of $\text{SPDE}_h$ (3.1), while the second requires to solve a space-time discretization of the $\text{BSPDE}$ (3.1) which is not the numerical solution by the implicit Euler method (see Lemmas 3.10 and 3.11 for the difference). A similar method to solve problem $\text{SLQ}_{h^2}$ has been proposed in [8, 16]. We refer to related works on how to approximate conditional expectations (e.g. [3, 10, 1, 18, 17]).

In the below, we tend to present a lower bound for $\kappa$ and show convergence rate of Algorithm 4.1. For this purpose, we first recall Lipschitz continuity of $D_2 \widehat{\mathcal{J}}_{h^2}$. By the definition of Fréchet derivative,

$$
D^2 \widehat{\mathcal{J}}_{h^2}(U_{h^2}) = \left[1 + L^* L + \alpha \widehat{L}^* \widehat{L}\right] U_{h^2},
$$

where operators $L$, $\widehat{L}$ are defined in (3.38)-(3.39). Next, we can find $K := \|1 + L^* L + \alpha \widehat{L}^* \widehat{L}\|_{\mathcal{L}(U_{h^2}, \bar{U}_{h^2})}$, such that

$$
\|D_2 \widehat{\mathcal{J}}_{h^2}(U^1_{h^2}) - D_2 \widehat{\mathcal{J}}_{h^2}(U^2_{h^2})\|_{\mathcal{L}(U_{h^2}, \bar{U}_{h^2})} \leq K \|U^1_{h^2} - U^2_{h^2}\|_{\bar{U}_{h^2}}.
$$

Indeed, noting that $||(1 - \tau \Delta h)^{-1}||_{\mathcal{L}(U_{h^2}, \bar{U}_{h^2})} \leq 1$, we find that

$$
\|LU_{h^2}\|_{\bar{U}_{h^2}}^2 = \sum_{n=1}^N \tau \mathbb{E}\|\tau \sum_{j=0}^{n-1} \left[(1 - \tau \Delta h)^{-1}\right]_{n-j} \prod_{k=j+2}^n (1 + \Delta k W) U_{h^2}(t_j)\|_{L^2}^2 \leq T^2 e^T \|U_{h^2}\|_{\bar{U}_{h^2}}^2.
$$

In a similar vein, we can prove that

$$
\|\widehat{L}U_{h^2}\|_{L^2(\Omega; L^2)}^2 \leq T e^T \|U_{h^2}\|_{L^2}^2.
$$

Hence

$$
K = \|1 + L^* L + \alpha \widehat{L}^* \widehat{L}\|_{\mathcal{L}(U_{h^2}, \bar{U}_{h^2})} \leq 1 + \alpha Te^T + T^2 e^T.
$$

Since Algorithm 4.1 is the gradient descent method for $\text{SLQ}_{h^2}$, we have the following estimates.
Theorem 4.2. Suppose that $\kappa \geq K$. Let $\{U_{h}^{(\ell)}\}_{\ell \in \mathbb{N}_0} \subset \mathbb{U}_{h_r}$ be generated by Algorithm 4.1, and $U_{h_r}^{*}$ solve $\text{SLQ}_{h_r}$. Then for $\ell = 1, 2, \cdots$,

\begin{enumerate}[(i)]
    \item \[\|U_{h}^{(\ell)} - U_{h_r}^{*}\|_{\mathbb{U}_{h_r}}^2 \leq \left(1 - \frac{1}{\kappa}\right)^\ell \|U_{h}^{(0)} - U_{h_r}^{*}\|_{\mathbb{U}_{h_r}}^2;\]
    \item \[\mathcal{F}_{h_r}(U_{h}^{(\ell)}) - \mathcal{F}_{h_r}(U_{h_r}^{*}) \leq \frac{2\kappa \|U_{h}^{(0)} - U_{h_r}^{*}\|_{\mathbb{U}_{h_r}}^2}{\ell};\]
    \item \[\max_{0 \leq n \leq N} \mathbb{E}[\|X_{h_r}^{*}(t_n) - X_{h_r}^{(\ell)}(t_n)\|_{L_2}^2] \leq C\left(1 - \frac{1}{\kappa}\right)^\ell \|U_{h}^{(0)} - U_{h_r}^{*}\|_{\mathbb{U}_{h_r}}^2,\]
\end{enumerate}

where $C$ is independent of $h, \tau, \ell$.

Proof. The estimates (i) and (ii) are standard for the gradient descent method (see e.g. [14, Theorem 1.2.4]); more details can also be found in [16, Section 5]. In the following, we restrict to assertion (iii).

For all $n = 0, 1, \cdots, N$, define $\hat{e}_{X}^{n,\ell} = X_{h_r}^{*}(t_n) - X_{h_r}^{(\ell)}(t_n)$. Subtracting (4.1) from (3.8) (where $U_{h_r} = U_{h_r}^{*}$) leads to

\[\hat{e}_{X}^{n+1,\ell} - \hat{e}_{X}^{n,\ell} = \Delta_{h}\hat{e}_{X}^{n+1,\ell} + \hat{e}_{X}^{n,\ell}\Delta_{n+1}W + \tau\left[U_{h_r}^{*}(t_n) - U_{h_r}^{(\ell)}(t_n)\right].\]

Multiplication with $\hat{e}_{X}^{n+1,\ell}$, and then taking expectations, as well as applying Cauchy-Schwartz inequality, we arrive at

\[(1 - \tau)\mathbb{E}[\|\hat{e}_{X}^{n+1,\ell}\|_{L_2}^2] + 2\tau\mathbb{E}[\|\nabla \hat{e}_{X}^{n+1,\ell}\|_{L_2}^2] \leq (1 + \tau)\mathbb{E}[\|\hat{e}_{X}^{n,\ell}\|_{L_2}^2] + \tau\mathbb{E}[\|U_{h_r}^{*}(t_n) - U_{h_r}^{(\ell)}(t_n)\|_{L_2}^2].\]

By the discrete Gronwall’s inequality, then taking the sum over all $0 \leq n \leq N - 1$, and utilizing the fact $\hat{e}_{X}^{0,\ell} = 0$, we find that

\[\max_{0 \leq n \leq N} \mathbb{E}[\|\hat{e}_{X}^{n,\ell}\|_{L_2}^2] + \sum_{n=1}^{N} \tau\mathbb{E}[\|\nabla \hat{e}_{X}^{n,\ell}\|_{L_2}^2] \leq C\tau \sum_{n=0}^{N-1} \mathbb{E}[\|U_{h_r}^{*}(t_n) - U_{h_r}^{(\ell)}(t_n)\|_{L_2}^2],\]

which, together with (i), imply assertion (iii).

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