The Boltzmann Equation in Scalar Field Theory

F. T. Brandt, J. Frenkel, A. Guerra

Instituto de Física, Universidade de São Paulo, CP 66318, 05315-970, São Paulo, SP, Brasil

(March 28, 2022)

We derive the classical transport equation, in scalar field theory with a $g^2V(\phi)$ interaction, from the equation of motion for the quantum field. We obtain a very simple, but iterative, expression for the effective action $\Gamma$ which generates all the $n$-point Green functions in the high-temperature limit. An explicit and closed form is given for $\Gamma$ in the static case.

1. INTRODUCTION

In thermal field theory, one is often interested in the behavior of Green functions at high temperatures. These hard thermal functions have been much studied in QCD [1–5] where they have a fundamental role in the resummation procedure of Braaten and Pisarski [6]. This program is usually based on quantum field theory, which involves in general rather complicated calculations. On the other hand, it was shown that the hard thermal effects can be described in a simpler and more transparent way using classical transport equations [7–9].

One may ask about the relevance of considering this approach in the framework of a scalar field theory, given that it has been already applied to more realistic theories like QED and QCD. Our motivation is twofold. First, we recall that an essential ingredient in the derivation of classical transport equations is the equation of motion for a particle propagating in a external background field. In QED, the equation of motion for the electron is governed by the well known Lorentz force. The natural generalization to a non-Abelian gauge theory like QCD, is the Wong equation [10]. The geodesic equation in gravity [11] is another example where the particle has a well defined equation of motion. The common feature of these
theories is that they are based on a *gauge invariance principle* which prescribes in a well defined way how the particle couples to the external field. Nonlinear scalar theories, on the other hand, have no such a simple prescription and require a rather different approach [for an alternative treatment, see also reference [12]]. Our second purpose is to obtain an exact non-perturbative expression for the effective thermal action, which contains the dominant static contributions arising in each order of perturbation theory. (Such a closed form result for the corresponding action in gauge theories is generally not known.)

In section 2 we derive the Boltzmann equation, by considering the expectation values of the relevant quantum-mechanical quantities. The analysis shows that the condition for the classical limit requires that the distances over which the background field varies must be large compared with the particle de Broglie wavelength, i.e:

$$\frac{1}{k} \gg \frac{\hbar}{p},$$

where $k$ is a typical wavenumber of the background field and $p$ is the kinetic momentum of the particle.

The behavior of Green functions at high temperatures is governed by the region involving large values of momenta of the particle propagating in the thermal loops, a domain which is consistent with the condition given by Eq. (1). Using the properties of the classical transport equation, we show in section 3 that the effective action which generates the high-temperature contributions of the Green functions in this domain has a form such that:

$$-\frac{\delta \Gamma}{\delta V^{(2)}(\phi)} = C_{(n)} \sum_{l=0}^{\infty} g^{2(l+1)} \int d^m p \left[ \frac{1}{2p \cdot \partial} \partial^\mu V^{(2)}(\phi) \frac{\partial}{\partial p^\mu} \right]^l f_0(p),$$

where $C_{(n)}$ is an appropriate normalization constant which depends on the space-time dimension $n$, and $f_0(p)$ is a function proportional to the Bose-Einstein probability distribution. This result is not obvious, so far as we know, since in scalar field theory the $n$-point Green functions are sub-leading for $n > 2$.

The above action has two interesting features:

- a non-locality in configuration space in the form of powers of the operator $(p \cdot \partial)^{-1}$. 


• in the static case the effective action can be expressed in closed form in terms of modified Bessel functions.

II. DERIVATION OF THE BOLTZMANN EQUATION

Consider the scalar field theory described by the Lagrangian density:
\[ \mathcal{L} = -\frac{1}{2} (\partial_{\mu}\Phi)^2 - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \Phi^2 - \frac{g^2}{\hbar^2} V(\Phi), \]  \hspace{1cm} (3)

where \( m \) is the mass of the field and \( V(\Phi) \) describes the interactions between the scalar fields. Examples of particular interest are the \( \Phi^4 \) theory in 4 space-time dimensions and the \( \Phi^3 \) theory in 6 dimensions. In what follows we shall work in units where \( c = 1 \), but will keep explicit the dependence on the Planck constant.

We now divide \( \Phi \) into a sum of a quantum field \( \psi \), associated with the particle, and a classical background field \( \phi \):
\[ \Phi = \psi + \phi. \]  \hspace{1cm} (4)

For our purpose, it is sufficient to consider only the terms in \( \mathcal{L} \) which are quadratic in \( \psi \):
\[ \mathcal{L}_2 = -\frac{1}{2} (\partial_{\mu}\psi)^2 - \frac{m^2 c^2}{2\hbar^2} \psi^2 - \frac{g^2}{2\hbar^2} V^{(2)}(\phi)\psi^2. \]  \hspace{1cm} (5)

This leads to the linearized Euler-Lagrange equations:
\[ \hbar^2 \partial^2 \psi = [m^2 + g^2 V^{(2)}(\phi)]\psi. \]  \hspace{1cm} (6)

It is convenient to write this equation in Hamiltonian form. To this end we introduce two wave functions \( \psi_1 \) and \( \psi_2 \) [13]
\[ \psi = \psi_1 + \psi_2, \quad m (\psi_1 - \psi_2) = i\hbar \frac{\partial \psi}{\partial t}, \]  \hspace{1cm} (7)

together with the single-column matrix \( \Psi \) and the Pauli matrices given by:
\[ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}; \quad (8) \]

\[ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9) \]

It is then straightforward to verify that (8) is completely equivalent to the equation of motion:

\[ i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi, \quad (10) \]

where the Hamiltonian operator \( \hat{H} \) is given by:

\[ \hat{H} = \frac{1}{2m} (\sigma_3 + i \sigma_2) [\hat{P}^2 + g^2 V^{(2)}(\phi)] + m \sigma_3, \quad (11) \]

and \( \hat{P} \) is the canonical momentum operator \( -i\hbar \nabla \).

In the Heisenberg representation, we have the operator equation:

\[ i\hbar \frac{d\hat{P}}{dt} = [\hat{P}, \hat{H}]. \quad (12) \]

Using the form of the Hamiltonian given by (11), this leads to the equation of motion for the canonical momentum:

\[ \frac{d\hat{P}}{dt} = -\frac{g^2}{2m} (\sigma_3 + i \sigma_2) \nabla V^{(2)}(\phi). \quad (13) \]

We now consider the expectation value of this quantity:

\[ < \frac{d\hat{P}}{dt} > = \int d^{n-1} x \Psi^\dagger \sigma_3 \frac{d\hat{P}}{dt} \Psi. \quad (14) \]

To evaluate the space integral, we make the assumption that the background field is roughly constant over a distance of the order of the particle de Broglie wavelength. Taking into account the spatial Lorentz contraction, we then obtain:

\[ < \frac{d\hat{P}}{d\tau} > = -\frac{g^2}{2m} \nabla V^{(2)}(\phi), \quad (15) \]
where τ is the proper time of the particle.

Let us compare (15) with the classical equation of motion of a scalar particle moving in an external field, which is obtained varying the action:

\[ S = - \int_{\tau_i}^{\tau_f} d\tau \left[ m + \frac{g^2}{2m} V^{(2)}(\phi) \right]. \]  

(16)

Proceeding as usual, we get the following equation of motion:

\[ \left[ 1 + \frac{g^2}{2m^2} V^{(2)}(\phi) \right] \frac{dp^\mu}{d\tau} = -\frac{g^2}{2m} \left[ \partial^\mu V^{(2)}(\phi) - \frac{p^\mu}{p^2} p \cdot \partial V^{(2)}(\phi) \right], \]  

(17)

where \( p^\mu \) denotes the kinetic momentum of the particle. Note that (17) is consistent with the on-shell condition \( p^2 = -m^2 \) which should hold for a physical particle.

In order to compare this relation with equation (15), we replace \( p \cdot \partial V^{(2)} \) by \( m dV^{(2)}/d\tau \), which follows by the use of the chain rule. Then, we can write (17) in the form:

\[ \frac{d}{d\tau} \left\{ \left[ 1 + \frac{g^2}{2m^2} V^{(2)}(\phi) \right] p^\mu \right\} = -\frac{g^2}{2m} \partial^\mu V^{(2)}(\phi). \]  

(18)

This expression is in agreement with (15), provided the canonical and kinetic momentum are related by:

\[ \mathbf{P} = \left[ 1 + \frac{g^2}{2m^2} V^{(2)}(\phi) \right] \mathbf{p}. \]  

(19)

Furthermore, the classical limit requires the condition:

\[ p \gg \hbar k, \]  

(20)

where \( k \) is a typical wavenumber of the background field.

Equation (17) can be used to determine the evolution of the distribution function \( f(x, p) \), which represents the probability of finding a particle in the state \((x, p)\). From the property:

\[ \frac{df(x, p)}{d\tau} = 0, \]  

(21)

which holds in the collisionless case, and using the constitutive relation \( p_\mu = m dx_\mu/d\tau \), we obtain the Boltzmann equation:
\[ p \cdot \partial f(x, p) = \frac{g^2}{2} \frac{1}{1 - g^2 V^{(2)}/2p^2} \left[ \partial^\mu V^{(2)} - \frac{p^\mu}{p^2} p \cdot \partial V^{(2)} \right] \frac{\partial f(x, p)}{\partial p^\mu} = 0. \] (22)

We remark that the Planck constant does not appear in this equation, as one would have expected for a classical transport equation.

Let us expand \( f(x, p) \) in powers of the coupling constant:

\[ f(x, p) = f_0(p) + g^2 f_1(x, p) + g^4 f_2(x, p) + \cdots, \] (23)

where:

\[ f_0(p) = \theta(p_0) \delta(p^2 + m^2) N(p_0). \] (24)

Here, the theta and delta functions guarantee positivity of the energy and on-shell evolution, and \( N(p_0) \) is the Bose probability distribution:

\[ N(p_0) = \frac{1}{e^{\beta p_0} - 1}. \] (25)

Inserting the expansion of \( f(x, p) \) into the transport equation (22) and identifying the coefficients of the powers of the coupling constant, we can determine recursively the components \( f_i(x, p) \). We find, for example:

\[ f_1(x, p) = \frac{1}{2} \left[ \frac{1}{p \cdot \partial} \partial^\mu V^{(2)} - \frac{p^\mu}{p^2} V^{(2)} \right] \frac{\partial f_0}{\partial p^\mu}, \] (26)

\[ f_2(x, p) = \frac{1}{4} \left[ \frac{1}{p \cdot \partial} \partial^\mu V^{(2)} \frac{\partial}{\partial p^\mu} \right]^2 f_0(p) - \frac{1}{4p^2} \frac{1}{p \cdot \partial} \left[ \partial^\mu V^{(2)} \frac{\partial}{\partial p^\mu} (p^\nu V^{(2)}) \right] \frac{\partial f_0(p)}{\partial p^\nu} + \cdots, \] (27)

where \( \cdots \) denote similar terms involving powers of \( 1/p^2 \). It turns out that such terms are not connected with the contributions obtained, in the high-temperature limit, from the 1-particle irreducible Green functions.
III. THE EFFECTIVE ACTION

In order to determine the form of the effective action, we will consider only those terms in the expansion of $f(x, p)$ which involve powers of the operator:

$$D(\phi) = \frac{1}{2} \cdot \frac{1}{p \cdot \partial} \partial^\mu V^{(2)}(\phi) \frac{\partial}{\partial p^\mu},$$

(28)

which has in configuration space a non-locality of the form $(p \cdot \partial)^{-1}$. Note that in momentum space, $D$ is a function of degree zero in $k$.

The above terms are generated by the series:

$$F(x, p) = \sum_{l=0}^{\infty} g^{2l} D^l(\phi) f_0(p),$$

(29)

which satisfies the differential equation:

$$p \cdot \partial F(x, p) - \frac{g^2}{2} \partial^\mu V^{(2)}(\phi) \frac{\partial F(x, p)}{\partial p^\mu} = 0.$$  

(30)

It is next shown that this function is relevant in connection with the generating functional $\Gamma$ defined by:

$$-\frac{\delta \Gamma}{\delta V^{(2)}(\phi)} = C_{(n)} g^2 \int d^n p \; F(x, p)$$

$$= C_{(n)} g^2 \sum_{l=0}^{\infty} g^{2l} \int d^n p \; D^l(\phi) f_0(p),$$

(31)

where $C_{(n)}$ is a normalization factor.

To illustrate the content of the above quantity, let us consider for definiteness the $\phi^4$ theory in 4 space-time dimensions. For $l = 0$, we then get in momentum space the expression:

$$-\frac{\delta \Gamma}{\delta \phi^2} = \frac{C_{(4)} g^2}{2} \int d^3 p \theta(p_0) \delta(p^2 + m^2) N(p_0),$$

(32)

which represents the scalar self-energy function $\Pi_2$ shown in Fig. 1, provided we choose $C_{(4)} = 1/(2\pi)^3$. As is well known [14] this function has a leading behavior proportional to $T^2$.

For $l = 1$, a simple calculation which makes use of integration by parts, gives in momentum space a contribution like:
\[
\frac{\delta^2 \Gamma}{\delta \phi^2 \delta \phi^2} = -\frac{g^4}{4(2\pi)^3} \int d^4p \theta(p_0) \delta(p^2 + m^2)N(p_0) \frac{k^2}{2(p \cdot k)^2}, \tag{33}
\]
where $\hbar k$ denotes the total momentum of a pair of fields $\phi^2$.

We shall now compare this result with the one obtained from the 1-particle irreducible Green function shown in Fig. 2. This contribution can be expressed in terms of a forward scattering amplitude of an on-shell particle in an external field \[5\] as indicated by the two graphs in Fig. 3. Also, an integration over the particle’s momenta $p^\mu$ must be performed with a weight factor given by the Bose probability distribution $N(p_0)$. We then obtain a contribution of the form:

\[
\Pi_4 = \frac{g^4}{4} \int \frac{d^3p}{(2\pi)^3} \frac{N(\omega)}{2\omega} \left[ \frac{1}{\hbar^2 k^2 + 2\hbar k \cdot p} + \frac{1}{\hbar^2 k^2 - 2\hbar k \cdot p} \right]. \tag{34}
\]

We now consider the high-temperature behavior, which is governed by the region involving high values of $p$. Expanding the Feynman propagators in this region, we get:

\[
\frac{1}{\hbar^2 k^2 + 2\hbar k \cdot p} + \frac{1}{\hbar^2 k^2 - 2\hbar k \cdot p} = -\frac{k^2}{2(p \cdot k)^2} + O \left( \frac{\hbar^2 k^2}{p^4} \right). \tag{35}
\]

Note that the leading term, which is of degree zero in $k$, is classical since the $\hbar$ dependence cancels out. Therefore, as far as the high-temperature domain is concerned, (34) effectively reduces to the classical amplitude (33), yielding a $\ln T$ contribution.

These calculations show that the operator $D$ generates iteratively, in the high-temperature limit, the dominant contributions associated with the 1-particle irreducible Green functions. It is clear that this mechanism generalizes to more general forms of interactions between the scalar fields. Hence, the generating functional $\Gamma$ given by equation (31) can be interpreted as the high-temperature effective action in scalar field theory.

We have not been able to evaluate $\Gamma$ in closed form for arbitrary external fields. However, this is possible when the fields are static, in which case the function $F(x, p)$ can be determined exactly. It is easy to verify that the static solution of the equation (30), which reduces to $f_0$ in the absence of interactions, is given by:

\[
F(x, p) = \theta(p_0) \delta(p^2 + m^2 + g^2 V^{(2)}(\phi))N(p_0). \tag{36}
\]
With this form, \( \Gamma \) may be integrated to give the following local expression:

\[
\Gamma = C_n \int d^n x \int d^n p \theta(p_0) \theta[-(p^2 + m^2 + g^2 V(\phi))] N(p_0). \tag{37}
\]

Using the constraints imposed by the \( \theta \) functions, the \( p \) integrations can be done, with the result:

\[
\Gamma = \frac{2C_n}{n-1} \Gamma[(n-1)/2] \int d^n x \int_M^\infty dp_0 \frac{(p_0^2 - M^2)^{(n-1)/2}}{2^{n/2} \Gamma[(n-1)/2]} N(p_0). \tag{38}
\]

where \( M^2 = m^2 + g^2 V(\phi) \).

After expanding \( N(p_0) \) as a power series in exponentials, the \( p_0 \) integrations can be expressed in terms of modified Bessel functions \[15\]. We then obtain, in the static case, the following expression for \( \Gamma \):

\[
\Gamma = \pi^{n/2-1} C_n \sum_{l=1}^\infty \left( \frac{2T}{l} \right)^{n/2} \int d^n x M^{n/2} K_{n/2} \left( \frac{lM}{T} \right). \tag{39}
\]

The properties of this result may be illustrated in the case of the \( \phi^4 \) theory in 4 space-time dimensions. Making a high-temperature expansion of the Bessel function \( K_2 \) in powers of \( lM/T \), and dropping an irrelevant constant term, we obtain a series of the form:

\[
\Gamma = -\frac{T^2}{8\pi^2} \int d^4 x M^2 \left[ \frac{\pi^2}{3} + \frac{1}{8} \frac{M^2}{T^2} \ln \frac{M^2}{T^2} + \frac{\xi^{(1)}(-2)M^4}{24T^4} + \cdots \right], \tag{40}
\]

where \( \xi^{(1)} \) is the derivative of the Riemann zeta function and \( M^2 = m^2(1 + g^2 \phi^2/2m^2) \).

The first term in equation \( \Gamma \) corresponds to the leading \( T^2 \) contribution associated with the scalar self-energy function. The second term contains, in addition to sub-leading self-energy function, the \( \ln T \) contribution of the 4-point function in the high-temperature limit. The contributions associated with the scalar 6-point function first appear in the third term.

It is interesting to remark that the expression \( \Gamma \) also contains a non-perturbative term involving \( \ln[1 + g^2 \phi^2/2m^2] \). This contribution may be expanded in a power series of the coupling constant when \( g^2 \phi^2 < 2m^2 \), but in very strong external fields the perturbative expansion is no longer meaningful. Hence, aside from the static condition, we expect that the result \( \Gamma \) may be applicable in perturbation theory for not too strong external fields.
REFERENCES

[1] A. H. Weldon, Phys. Rev. 26, 1394 (1982).

[2] K. Kajantie and J. Kapusta, Ann. Phys. (NY). 160, 477 (1985).

[3] U. Heinz, K. Kajantie and T. Toimela, Ann. Phys. (NY). 176, 215 (1987).

[4] R. D. Pisarski, Nucl. Phys. B 309, 476 (1988).

[5] J. Frenkel and J. C. Taylor, Nucl. Phys. B 334, 199 (1990); Nucl. Phys. B 374, 156 (1992).

[6] E. Braaten and R. D. Pisarski, Nucl. Phys. B 337, 569 (1990); Nucl. Phys. B 339, 310 (1990).

[7] H. -Th.Elze and U. Heinz, Phys. Rep. 183, 81 (1989).

[8] D. Bak, R. Jackiw and S. Y. Pi, Phys. Rev. D 49, 6778 (1994).

[9] P. F. Kelly, Q. Liu, C. Lucchesi and C. Manuel, Phys. Rev. D 50, 4209 (1994).

[10] S. Wong, Nuovo Cim. A 65, 689 (1970).

[11] F. T. Brandt, J. Frenkel and J. C. Taylor, Nucl. Phys. B 437, 433 (1995).

[12] S. Mrówczyński, Phys. Res D 56, 2265 (1997).

[13] A. S. Davydov, Quantum Mechanics (Addison-Wesley, 1965).

[14] J. Kapusta, Finite Temperature Field Theory (Cambridge University Press, 1989).

[15] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products (Academic Press, 1980).
FIGURE CAPTIONS

Fig. 1 The scalar self-energy function $\Pi_2$.

Fig. 2 The irreducible 4-point function $\Pi_4$.

Fig. 3 The forward scattering amplitudes associated with the 4-point function.