A New Scheme To Solve The Two-Fluid Cosmic-Ray Magnetohydrodynamic Equations

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Abstract. We examine the two-fluid cosmic-ray magnetohydrodynamic (CR MHD) equations to take account of the dynamical effects of cosmic rays (CRs) in the simplest form. For simplicity we assume that the pressure of the CRs is proportional to the energy density, \(P_{\text{cr}} = (\gamma_{\text{cr}} - 1)E_{\text{cr}}\), where \(\gamma_{\text{cr}}\) denotes the adiabatic index of CRs. We find the fully conservative form of the CR MHD equations and derive the Rankine-Hugoniot relation for a shock. One component of the CR MHD equations describes conservation of total number of CRs where the number density is defined as \(n_{\text{cr}} = P_{\text{cr}}^{1/(\gamma_{\text{cr}} - 1)}\). We also find the Riemann solution, which provides us approximate Riemann fluxes for numerical solutions. We have also identified origin of spurious oscillation originating from the pressure balance mode across which the CR pressure has a jump while the total pressure is continuous. We propose the two step method to solve the CR MHD equations. We use 1D shock tube problems to compare our method with others.

1. Introduction

Cosmic rays, referred to CRs in the following, are important constituents in many astronomical objects such as the galactic disks, solar winds and supernova remnants. The energy density of CRs are comparable to those of magnetic field, turbulence and thermal gas. Thus CRs are expected to influence the dynamics of the interstellar gas. As pointed out by Parker [1,2], CRs inflate magnetic flux tubes to rise from the galactic plane. They also contribute to expansion of supernova remnants [3] and interaction of the solar system with the interstellar medium flow in the heliosphere [4].

The dynamical effects of CRs have been incorporated into the hydrodynamic (HD) equations under the fluid approximation in which CRs are assumed to have only pressure and energy density. The pressure of CRs is often assumed to be proportional to the energy density, \(P_{\text{cr}} = (\gamma_{\text{cr}} - 1)E_{\text{cr}}\), where \(E_{\text{cr}}\) and \(\gamma_{\text{cr}}\) denote the energy density and adiabatic index, respectively. The time evolution of CRs is described by the advection-diffusion equation. They are extended to the CR magnetohydrodynamic (MHD) equations by including the magnetic force and the induction equation. These CR MHD equations are a powerful tool in analyzing the dynamics of interstellar medium, though anisotropy and energy spectrum of CRs cannot be taken into account.

Although it has passed more than thirty years since the first formulation by Drury and Völk [5], some basic properties of the CR HD equations remain still unknown. When the diffusion
term is omitted, the CR HD equations are hyperbolic and hence should have the Riemann solution, which are still unknown. Even the Rankine-Hugoniot relation at a shock front has not been obtained yet.

The deficit in our knowledge on the CR MHD equations are serious in solving them numerically since modern numerical technics are based on the Riemann solution. First the fully conservative form of the CR HD equations have not been known in the literature, although we solve the ordinary MHD equations in the conservative form. The work by the CR pressure has been treated as a source term in the literature, though it contains a spatial derivative. Second, the pressure balance mode, which appears as a new wave in the CR HD equations, is known to show a suspicious oscillation in the numerical solution [6]. We need the fully conservative form and the Riemann solution to find a smart way to solve them.

In this paper we derive the fully conservative form of the CR evolution. The newly derived equation describes the conservation of the number of CRs. We define the number density of CRs to be the $\gamma_{cr}$-th power of $P_{cr}$. Then we derive the Rankine-Hugniot relation at the shock front from the conservative form. We find that the ratio of the number density of CRs to the gas density is continuous at the shock front. We also find that numerical diffusion of CRs generates the oscillation from the pressure balance mode. Based on these new findings we propose a new scheme to solve the CR HD equations.

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This paper is organized as follows. We analyze the CR MHD equations and derive the conservation form in §2. The extension to MHD is tedious but straightforward. The Rankine-Hugniot relation and approximate Riemann solvers for CR HD equations are also derived in §2. The numerical results of 1D shock tube problems are shown in §3.

2. The two-fluid CR-MHD system

2.1. Basic equations

When the pressure of CRs, $P_{cr}$, is taken into account, the MHD equations [7] are expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left[ \rho \mathbf{v} \mathbf{v} + \left( P_{g} + \frac{B^2}{2} \right) \mathbf{I} - \mathbf{B} \mathbf{B} \right] = -\nabla P_{cr},$$

$$\frac{\partial}{\partial t} \left( \frac{P_{g}}{\gamma_{g} - 1} + \frac{1}{2} \rho \mathbf{v}^2 + \frac{B^2}{2} \right) + \nabla \cdot \left[ \left( \frac{\gamma_{g} - 1}{\gamma_{g} - 5/3} \right) P_{g} + \frac{1}{2} \rho \mathbf{v}^2 \right] \mathbf{v} - \left( \mathbf{v} \times \mathbf{B} \right) \times \mathbf{B} \right] = -\mathbf{v} \cdot \nabla P_{cr},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$

Here, $\rho$ denotes the gas density, $\mathbf{v}$ does the velocity, $\mathbf{B}$ does the magnetic field, and $P_{g}$ does the gas pressure. The gas is assumed to an ideal gas with the specific heat ratio, $\gamma_{g} = 5/3$. The pressure force of CRs appear in equations (2) and (3), i.e., in the momentum conservation and energy conservation. The equation of continuity and induction equations are unchanged.

CRs are assumed to have no mass and to have the constant specific ratio, $\gamma_{cr} = 4/3$. Then the energy conservation of CRs is expressed as

$$\frac{\partial}{\partial t} \left( \frac{P_{cr}}{\gamma_{cr} - 1} \right) + \nabla \cdot \left( \frac{\gamma_{cr}}{\gamma_{cr} - 1} \right) P_{cr} \mathbf{v} = \mathbf{v} \cdot \nabla P_{cr} + \nabla \cdot \mathbf{F}_{\text{diff}},$$

$$\mathbf{F}_{\text{diff}} = \kappa_{\perp} \nabla P_{cr} + \frac{\left( \kappa_{\parallel} - \kappa_{\perp} \right)}{|\mathbf{B}|^2} (\mathbf{B} \cdot \nabla P_{cr}) \mathbf{B},$$
by taking account of anisotropic diffusion. Here the symbols, $\kappa_\parallel$ and $\kappa_\perp$, denote the diffusion coefficients in the direction parallel and perpendicular to the magnetic field, respectively. For simplicity, we omit the diffusion in the following analysis, since it is evaluated separately in a numerical analysis, i.e., by use of an operator splitting method.

While equations (1), (2), and (4) are written in the conservation form, equations (3) and (5) are not. From the sum of equations (3) and (5), we obtain the total energy conservation,

$$\frac{\partial}{\partial t} \left( \rho H - P_T + \frac{B^2}{2} \right) + \nabla \cdot [\rho H \mathbf{v} - (\mathbf{v} \times \mathbf{B}) \times \mathbf{B}] = 0,$$

in the conservation form, where

$$H = \frac{v^2}{2} + \frac{\gamma_k}{\gamma_k - 1} \frac{P_g}{\rho} + \frac{\gamma_{cr}}{\gamma_{cr} - 1} \frac{P_{cr}}{\rho},$$

$$P_T = P_g + P_{cr}.$$  

Equation (5) is equivalent to

$$\frac{\partial}{\partial t} \rho_{cr} + \nabla \cdot (\rho_{cr} \mathbf{v}) = 0,$$

where

$$\rho_{cr} = P_{cr}^{1/\gamma_{cr}}.$$  

Equation (10) denotes the conservation of CR particle numbers. Equation (11) implies that energy of each CR particle increases in proportion to $\rho_{cr}^{\gamma_{cr}-1} = \rho_{cr}^{1/3}$ when compressed. It is consistent with the assumption that CRs consist of ultra-relativistic particles.

By replacing equations (3) and (5) with (7) and (10), we can express all the components of the CR MHD equations in the conservation form.

### 2.2. Rankine-Hugoniot Relation

The Rankine-Hugoniot relation, the jump condition at a discontinuity, is derived easily from the CR MHD equations in the conservation form; the fluxes should be constant across the discontinuity when seen in the comoving frame. In the following we consider a standing shock of which wave plane is normal to the $x$-axis. Then the Rankine-Hugoniot relation is expressed as

$$\rho v_x = \text{const.},$$

$$\rho v_x^2 + P_T \frac{B_y^2 + B_z^2 - \frac{B_x^2}{2}}{2} = \text{const.},$$

$$\rho v_x v_y - B_x B_y = \text{const.},$$

$$\rho v_x v_z - B_x B_z = \text{const.},$$

$$\rho H v_x + (B_y^2 + B_z^2) v_x - (v_y B_y + v_z B_z) B_x = \text{const.},$$

$$v_x B_y - v_y B_x = \text{const.},$$

$$v_x B_z - v_z B_x = \text{const.},$$

$$\rho_{cr} v_x = \text{const.}.$$  

Combining equations (12) and (19) we obtain

$$\frac{\rho_{cr}}{\rho} = \text{const.}.$$
across the shock. Here, the subscripts, $x$, $y$, and $z$ specify the $x$-, $y$-, and $z$-components, respectively. We can derive also

$$\frac{D}{Dt} \left( \frac{\rho \varepsilon_x}{\rho} \right) = 0,$$

(21)

and hence

$$\frac{D}{Dt} \left( \frac{P_{ex}}{\rho^{\gamma_{ex}}} \right) = 0,$$

(22)

from equations (1) and (10). The ratio of the CR number density to the gas remains constant in a gas element.

The effects of CRs on the shock are summarized as follows. First CRs contribute to the normal component of the momentum flux and the energy flux. Thus the fast and slow modes are modified, while the Alfvén waves are not. The propagation speeds of the fast and slow modes are modified according to the change in the sound speed defined as

$$c_s = \left( \frac{dP_T}{d\rho} \right)^{1/2} = \left( \frac{\gamma_k P_k + \gamma_{ex} P_{ex}}{\rho} \right)^{1/2}.$$

(23)

Second, the contact discontinuity is degenerate; both $\rho$ and $\rho_{ex}$ can be discontinuous while $P_T$, $\mathbf{v}$ and $\mathbf{B}$ should be continuous. We define the entropy wave as the discontinuity in $\rho$ and the pressure balance mode as that in $\rho_{ex}/\rho$. The entropy is defined as

$$s = \ln P_k - \gamma_k \ln \rho,$$

(24)

in this paper.

2.3. Approximate Riemann Solvers

We can apply the Godunov method, the current standard in the grid based simulations, to the CR MHD equations since they have the conservation form as shown in the previous section. The HLL Riemann solver [8], the simplest one among the Godunov type, can be easily derived from the conservation form. However, the HLL Riemann solver provides a diffusive solution as shown later. Thus we have developed a Roe-type solver [9] to reduce diffusion around the contact discontinuity. Although a solution of the latter is less diffusive, it suffers from an artificial oscillation originating from the pressure balance mode. The artificial oscillation can be suppressed when the advection of the pressure mode is evaluated separately. We describe the Roe-type Riemann solver and its modification to avoid the artificial oscillation.

In this subsection we show the Roe-type solver not for the CR MHD equations but for the CR HD in order to avoid complexity due to the existence of magnetic field. Inclusion of CRs changes nothing but the equation of state. Extension to the CR MHD equations is straightforward.

The 1D CR HD equations are expressed in the vector form,

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0, \quad \mathbf{U} = \begin{bmatrix} \rho \\ \rho \mathbf{v}_x \\ \rho H - P_T \\ \rho_{ex} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho \mathbf{v}_x \\ \rho \mathbf{v}_x^2 + P_T \\ \rho H \mathbf{v}_x \\ \rho_{ex} \mathbf{v}_x \end{bmatrix},$$

(25)

where $\mathbf{U}$ and $\mathbf{F}$ denote the state and flux vectors, respectively. The Jacobi-matrix, $\mathbf{A} \equiv \partial \mathbf{F}/\partial \mathbf{U}$, has four eigenvalues,

$$\lambda_1 = v_x + c_s, \quad \lambda_2 = v_x, \quad \lambda_3 = v_x, \quad \lambda_4 = v_x - c_s,$$

(26)
and corresponding right eigenvectors,

$$
    r_1 = \begin{bmatrix}
        1 \\
        v_x + c_s \\
        H + c_sv_x \\
        \rho \\
        \rho_{cr} \\
        \rho 
    \end{bmatrix},
    r_2 = \begin{bmatrix}
        1 \\
        v_x \\
        \frac{v_x^2}{2} + \frac{\rho_{cr}}{\rho} \alpha \\
        \rho \\
        \rho_{cr} \\
        \rho 
    \end{bmatrix},
    r_3 = \begin{bmatrix}
        0 \\
        0 \\
        1 \\
        \alpha \\
        1 \\
        1
    \end{bmatrix},
    r_4 = \begin{bmatrix}
        1 \\
        v_x - c_s \\
        H - c_sv_x \\
        \rho \\
        \rho_{cr} \\
        \rho 
    \end{bmatrix},
$$

(27)

where the subscript, $k$, specifies the waves; $k = 1$ and 4 for the sound waves, $k = 2$ for the entropy wave, and $k = 3$ for the pressure balance mode [6]. The entropy wave and pressure balance mode denote the advection of $s$ and $\rho_{cr}/\rho$, respectively. We use these eigenvectors to decompose the spatial change in the state vector, $U$.

The Roe-type numerical flux is expressed as

$$
    F_{j+1/2}^{Roe} = \frac{1}{2} \left( F_{j+1} + F_j - \sum_k w_k |\lambda_k| r_k \right),
$$

(28)

where

$$
    w_{1,4} = \frac{1}{2c_s^2} (\Delta P_{1} \pm \tilde{\rho}_s \Delta v_x),
    w_2 = \Delta \rho - \frac{\Delta P_1}{c_s^2},
    w_3 = \Delta \rho_{cr} - \frac{\tilde{\rho}_{cr}}{\rho} \Delta \rho.
$$

(29)

Here the symbols with subscript $j$ denote the values at the center of the $j$-th cell. The symbol with $\Delta$ denote the differences between the adjacent cells, while those with bar denote the Roe average,

$$
    \tilde{\rho} = \sqrt{\frac{\rho_{j+1} + \rho_j}{2}},
    \tilde{v}_x = \frac{\sqrt{P_{j+1}v_{x,j+1} + \sqrt{P_j}v_{x,j}}}{\sqrt{P_{j+1} + \sqrt{P_j}}},
    \tilde{H} = \frac{\sqrt{P_{j+1}H_{j+1} + \sqrt{P_j}H_j}}{\sqrt{P_{j+1} + \sqrt{P_j}}},
    \tilde{\rho}_{cr} = \frac{\sqrt{P_{j+1}\rho_{cr,j+1} + \sqrt{P_j}\rho_{cr,j}}}{\sqrt{P_{j+1} + \sqrt{P_j}}},
    \tilde{P}_T = \frac{\sqrt{P_{j+1}P_{T,j+1} + \sqrt{P_j}P_{T,j}}}{\sqrt{P_{j+1} + \sqrt{P_j}}}.
$$

The sound speed is defined as

$$
    c_s^2 = (\gamma_k - 1) \left( \tilde{H} - \frac{\tilde{v}_x^2}{2} - \frac{\tilde{\rho}_{cr}}{\rho} \alpha \right),
    \alpha = \frac{\gamma_k - \gamma_{cr}}{(\gamma_k - 1)(\gamma_{cr} - 1)} \frac{\Delta P_{cr}}{\Delta \rho_{cr}}.
$$

(30)

The numerical flux given by equation (28) satisfies the property $U$,

$$
    U_{j+1} - U_j = \sum_{k=1}^{4} w_k r_k,
    F_{j+1} - F_j = \sum_{k=1}^{4} \lambda_k w_k r_k.
$$

(31)

We obtain the solution of the first order accuracy,

$$
    U_{j+1}^n = U_j^n - \frac{\Delta t}{\Delta x} \left( F_{j+1/2}^{Roe} - F_{j-1/2}^{Roe} \right),
$$

(32)

where $U_{j+1}^n$ and $U_j^n$ denote the state vector at the $n + 1$-th and $n$-th time steps, respectively. From $U_j^n$ we obtain the primitive variables, $\rho, v_x, P_x$, and $P_{cr}$ at the $n$-th time step.

As mentioned earlier and shown later, a small oscillation of numerical origin arises from the pressure balance mode in a solution given by equation (32). In order to remove the oscillation we
propose a two step approach in which the advection of CRs is taken into account in the second step. In the first step we use the modified numerical flux,

\[ F^*_{j+1/2} = \frac{1}{2} \left( F_{j+1} + F_j - \sum_k w_k |\lambda_k| r_k^j \right), \]  

(33)

\[ r_2' = \begin{bmatrix} \frac{1}{\bar{v}} \\ \frac{\bar{v}^2}{2} + \frac{\bar{\rho}_c}{\bar{\rho}} \alpha \\ 0 \end{bmatrix}, \quad r_3' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad r_k' = r_k \text{ (otherwise)}. \]  

(34)

Using this modified numerical flux, we obtain the intermediate state,

\[ U^*_j = U^n_j - \frac{\Delta t}{\Delta x} \left( F^*_{j+1/2} - F^*_{j-1/2} \right) = \begin{bmatrix} \rho_{n+1} \\ \rho_{n+1}^{n+1} \bar{v}_{x,j+1/2}^2 + \frac{\rho_{n+1}^{n+1}}{\gamma_{s,j+1/2} - 1} \\ \rho_{n+1}^{n+1} \gamma_{cr,j} \end{bmatrix}. \]  

(35)

In the first step we obtain \( \rho_{n+1}^{n+1}, v_{x,j+1/2}^{n+1} \) from \( U^n_j \).

In the second step we evaluate \( P_{n+1}^{n+1} \) and then \( \rho_{n+1}^{n+1}, v_{x,j+1/2}^{n+1} \) by the following procedures. First we obtain

\[ \rho_{cr,j} = \rho_{cr,j} - \frac{\Delta t}{2\Delta x} \left( \bar{v}_{x,j+1/2} q_{j+1/2} + \bar{v}_{x,j-1/2} q_{j-1/2} \right), \]  

(36)

\[ q_{j+1/2} = \rho_{cr,j+1} - \rho_{cr,j} - \frac{\rho_{cr,j+1/2}}{\bar{\rho}_{j+1/2}} \frac{P_{T,j+1} - P_{T,j}}{c_{s,j+1/2}^2}. \]  

(37)

to evaluate the change in the CR density by advection. Note that equation (37) denotes the spatial change in \( \rho_{cr,j} \) to be advected. We evaluate the advection of CRs by using

\[ P_{cr,j}^{n+1} = \left( \rho_{cr,j} \right)^{\gamma_{cr,j}} - \frac{\Delta t}{\Delta x} \left[ \min(\bar{v}_{x,j+1/2}, 0) q_{j+1/2}^{max} + \max(\bar{v}_{x,j-1/2}, 0) q_{j-1/2}^{max} \right], \]  

(38)

\[ q_{j+1/2}^{max} = \frac{P_{cr,j+1} - P_{cr,j}}{\rho_{cr,j+1} - \rho_{cr,j}}. \]  

(39)

In other words we respect the conservation of CR energy rather than that of CR numbers about the advection. We evaluate \( \rho_{cr,j}^{n+1} \) from \( P_{cr,j}^{n+1} \) and \( P_{g,j}^{n+1} \) from the third component of \( U^*_j \). This new scheme is referred to the two step method in the following.

3. Numerical Tests

We use the 1D shock tube problem to examine the validity of the numerical methods shown in the previous section. We use the explicit scheme of the first order accuracy for time integration to show the dependence on the Riemann solver directly. All the test problems are solved on a uniform grid and with a constant time step. Subsections 3.1 and 3.2 are dedicated to CR HD and CR MHD problems, respectively.
3.1. CR HD Shock Tube Problems

First we examine the case when the left side has a higher pressure and a higher density. Table 1 summarizes the gas pressure, the CR pressure and the density in the initial state. This problem is similar to the shock tube problem in pure hydrodynamics but the CR pressure is taken into account. The Riemann solution is illustrated in Figure 1. The shock wave propagates with the phase speed, \( \lambda_1 = 2.36684 \) and the head of the rarefaction wave with that of \( \lambda_4 = -2.16025 \). The contact discontinuity and pressure balance mode move at \( \lambda_2 = \lambda_3 = 1.56214 \).

**Table 1.** The initial state in the CR HD shock tube problem.

|       | \( \rho \) | \( v_x \) | \( P_g \) | \( P_{cr} \) |
|-------|----------|----------|--------|------------|
| left (\( x < 0 \)) | 1.0      | 0.0      | 2.0    | 1.0        |
| right (\( x \geq 0 \)) | 0.2      | 0.0      | 0.02   | 0.1        |

![Diagram](image)

**Figure 1.** Illustration of the Riemann solution for the CR HD shock tube problem.

Figure 2 shows the numerical solution obtained with the two step method on the grid of \( \Delta x = 1/128 \). The time step is constant at \( \Delta t = 2.0 \times 10^{-3} \) so that the CFL number is 0.61 at the shock front. The thick curves and dots denote the numerical solution while the thin lines do the analytic Riemann solution. The upper panels denote the density (\( \rho \)), velocity (\( v_x \)), and entropy (\( s \)) at \( t = 0.1 \), i.e. 50 time steps from the initial, as a function of \( x \) from left to right. The lower left panel denote the gas pressure (\( P_g \)) and the total pressure (\( P_T \)). The lower middle and lower right panels denote the CR pressure (\( P_{cr} \)) and the ratio of the CR number density to gas density (\( \rho_{cr}/\rho \)), respectively. The numerical solution reproduces the analytic Riemann solution well with the first order accuracy in space and in time. The velocity at the contact discontinuity is \( v_x = 1.565 \) in the numerical solution and \( v_x = 1.562 \) in the analytic solution. The total pressure at the contact discontinuity is \( P_T = 0.8609 \) in the numerical solution and \( P_T = 0.8595 \) in the analytic solution. The relative errors are about 0.2%. Both \( v_x \) and \( P_T \) are fairly flat between the the shock front and the tail of the rarefaction. Note that the CR number density to the gas density, \( \rho_{cr}/\rho \) is constant except for the jump at the pressure balance mode. Also the gas entropy is constant except for the jump at the shock front and pressure balance mode.

We have solved the same problem with various methods. The results are shown in Figure 3. All the solutions are obtained on the same grid with the same time step for fair comparison. The
Figure 2. The numerical solution of the CR HD shock tube problem compared with the analytic Riemann solution (thin black solid curves). The numerical solution is obtained with the two step method. The upper panels show $\rho$, $v_x$ and $s$, while the lower do $P_g$, $P_T$, $P_{cr}$ and $\rho_{cr}/\rho$. The dashed lines denote the initial values. The red and green curves denote the solutions obtained with the CR HD equations in the conserved form while the blue and purple do those obtained with the CR HD equations with some source terms, i.e., in the original form. The source term is $v \cdot \nabla P_{cr}$ in the solution denoted by the blue curves while it is $P_{cr} \cdot \nabla v$ in that denoted by the purple ones. The former scheme is similar to Kuwabara et al. [10], while the latter is similar to Hanasz et al. [11], Rasera and Chandran [12], and Yang et al [13]. The red curves denote the solution obtained with the two step method, while the other solutions are obtained with the HLL flux. All the panels show only the interval of $0 < x < 0.30$ and the rarefaction waves are omitted in the panels.

The deviation from the Riemann solution is most serious in the solution denoted by the blue curves. The propagation of the shock wave is slowest mainly due to numerical diffusion of CR particles from the pressure balance mode. The CR HD equations denotes the adiabatic compression of CRs while they do not take account of diffusive shock acceleration (DSA). DSA term should be taken into account separately. The numerical diffusion of CRs is also serious in the solution denoted by the purple curves.

The HLL solution of the CR HD equations in the conserved form (green) provides a better estimate on the shock propagation. However it shows a spurious hump in $P_g$ around the contact discontinuity. The hump is smaller in the solution obtained with the two step method (red).

Roe type flux and the two step method give us almost the same result for the test problem shown in Figure 2. This is because the pressure balance mode has only a small amplitude in the test problem. The second step used in the two step method is important to solve the pressure balance mode. In order to demonstrate the importance we have solved the advection of the pressure balance mode. Both the left and right side has the same total pressure in this advection test while the CR pressure is higher in the left. The velocity is taken to be uniform at $v_x = 1.0$ as summarized in Table 2.

Figure 4 shows the numerical solutions at the first three time steps in the advection test. The
Figure 3. Comparison of the numerical solutions of the CR HD shock tube problem. The green and red curves denote the solutions obtained with the CR HD equations in the conservation form while the blue and purple curves do those obtained with the equations with source terms. The spatial grid and time step are the same as those used in the solution shown in Fig. 2.

Table 2. The initial state in the test problem of the advection of pressure balance mode.

|   | $\rho$ | $v_x$ | $P_g$ | $P_{cr}$ |
|---|---|---|---|---|
| left ($x < 0$) | 1.0 | 1.0 | 0.1 | 1.0 |
| right ($x \geq 0$) | 1.0 | 1.0 | 1.0 | 0.1 |

upper, middle and lower panels denote the solutions at the initial, the first, and the second time steps, respectively. The left panels denote $P_g$ and $P_{cr}$. The middle and right panels denote $P_T$ and $v_x$, respectively. The green, purple and red lines denote the solutions obtained with HLL, Roe and the two step method, respectively. HLL and Roe schemes produce spurious increase in $P_T$ at the first step and spurious change in $v_x$ at the second step.

The spurious increase in $P_T$ is due to numerical diffusion in $P_{cr}$. The inevitable numerical diffusion modifies equation (10) into

$$\frac{\partial}{\partial t} \rho_{cr} + \nabla \cdot (\rho_{cr} v) = \nabla \cdot (\eta \nabla \rho_{cr}),$$

where $\eta$ denotes the effective diffusion coefficient. We can obtain

$$\frac{\partial}{\partial t} \left( \frac{P_{cr}}{\gamma_{cr} - 1} \right) + \nabla \cdot \left[ \frac{\gamma_{cr}}{\gamma_{cr} - 1} P_{cr} v - \eta \nabla \left( \frac{P_{cr}}{\gamma_{cr} - 1} \right) \right] = v \cdot \nabla P_{cr} - \frac{\eta}{\gamma_{cr} P_{cr}} |\nabla P_{cr}|^2,$$

from equation (40). The second term in the right hand side denotes decrease in $P_{cr}$ and hence that in the CR energy density in equation (41). Since the total energy is conserved, the decrease
in the CR energy is supplemented by increase in the gas energy. Hence the gas pressure increases to increase the total pressure. Figure 5 illustrates this mechanism. Equation (10) gives a good solution for a shock but not for the pressure balance mode. Thus we need the two step method.

3.2. MHD Shock Tube Test

We can extend the two step method for CR MHD equations. In this subsection we demonstrate that our two step method works well in a MHD shock tube problem. The initial state in our MHD shock tube problem is summarized in Table 3. It is similar to the problem of Brio & Wu [14], while the CR pressure is taken into account in ours.

Figure 6 shows the solution obtained with the two step method. The solution is obtained on the uniform grid of $\Delta x = 1/512$ with the constant time step of $\Delta t = 4 \times 10^{-4}$. The solid
Table 3. Initial left- and right-state in the shock tube problem of modified Brio & Wu [14].

|            | $\rho$ | $v_x$ | $v_y$ | $v_z$ | $B_x$ | $B_y$ | $B_z$ | $P_g$ | $P_{cr}$ |
|------------|--------|-------|-------|-------|-------|-------|-------|-------|----------|
| Left ($x < 0$) | 1.0    | 0.0   | 0.0   | 0.0   | 1.0   | 1.0   | 0.0   | 1.0   | 0.4      |
| Right ($x \geq 0$) | 0.125  | 0.0   | 0.0   | 0.0   | 1.0   | -1.0  | 0.0   | 0.1   | 0.04     |

curves denote the solution at $t = 0.08$, while the dashed lines do the initial state ($t = 0$). We find the fast rarefaction propagating rightward in the range of $0.26 \leq x \leq 0.35$ and that propagating leftward in $-0.18 \leq x \leq -0.05$. The slow shock located at $x = 0.14$ is obtained without numerical oscillation. The gas entropy ($s$) remains constant except at the slow shock, contact discontinuity ($x = 0.07$) and the edge of the slow compound ($x = -0.04$). The ratio of the CR density to gas density ($\rho_{cr}/\rho$) changes only at the contact discontinuity. Also the transverse velocity ($v_y$) and magnetic field ($B_y$) are obtained successfully without any numerical oscillations.

Figure 6. An example of CR MHD shock tube problem. The upper panels denote $\rho$, $P_g$, $P_{cr}$, $P_T$, and $s$. The lower panels denote $v_x$, $v_y$, $B_y$, and $\rho_{cr}/\rho$.

4. Conclusion

We have examined the CR MHD equations and found the Rankine-Hugoniot relation for a shock. The CR MHD equations have the fully conservative form which includes the conservation of CR numbers. It means that they describe adiabatic compression of CRs at the shock. We need to another term in the CR MHD equations if we take account of the diffusive shock acceleration. The Riemann solution and Roe-type Riemann solver are obtained for the CR MHD equations. We find that numerical diffusion of CRs generates spurious oscillation from the pressure balance mode. The oscillation can be removed by our two step method.
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