Article
The Pauli Problem for Gaussian Quantum States:
Geometric Interpretation

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Abstract: We solve the Pauli tomography problem for Gaussian signals using the notion of Schur complement. We relate our results and method to a notion from convex geometry, polar duality. In our context polar duality can be seen as a sort of geometric Fourier transform and allows a geometric interpretation of the uncertainty principle and allows to apprehend the Pauli problem in a rather simple way.

Keywords: covariance matrix; polar duality; uncertainty principle; reconstruction problem

1. The Pauli Problem and Quantum Tomography

The problem goes back to Pauli’s question [1]:

The mathematical problem as to whether, for given probability densities \( W(p) \) and \( W(x) \), wave function \( \psi(...) \) is always uniquely determined, has still not been investigated in its generality.

The answer to Pauli’s question is negative [2]; there is a general nonuniqueness of the solution (for a detailed discussion of the Pauli problem and its applications, see [3]). The problem can actually be formulated as from statistical quantum mechanics as follows: can we estimate the density matrix of the said state using repeated measurements on identical quantum systems? After having obtained measurements on these identical systems, can we make a statistical inference about their probability distributions (e.g., [4])? Such a procedure is an instance of quantum state tomography, and is practically implemented using a set of measurements of a so-called quorum of observables. It can be performed using various mathematical techniques, for instance the Radon–Wigner transform that we discussed in [5]; the latter has important applications in medical imaging [6]. For details and explicit constructions, see [7–14], and [15] by Man’ko and Man’ko.

Remark 1. Everything in this paper extends mutatis mutandis to time-frequency analysis, replacing the notion of wave function by that of a signal. In this case, one takes \( \hbar = 1/2\pi \) and replaces phase-space variables \((x, p)\) with time-frequency variables \((x, \omega)\).

2. A Simple Example

Let us discuss the Pauli problem on the simplest possible example, that of a Gaussian wave function in one spatial dimension. Assuming for simplicity, it is centered at the origin and is given by formula

\[
\psi(x) = \left( \frac{1}{2\pi\sigma_{xx}} \right)^{1/4} e^{-\frac{x^2}{4\sigma_{xx}}} e^{i\sigma_{xp}\frac{\hbar}{2\sigma_{xx}}x^2}
\]

where \( \sigma_{xx} \) is the variance in the position variable, and \( \sigma_{xp} \) the covariance in the position and momentum variables. Fourier transform

\[
\hat{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}px} \psi(x) dx
\]
of the \( \psi \) is explicitly given by

\[
\hat{\psi}(p) = \left( \frac{1}{2\pi \hbar} \right)^{1/4} e^{-\frac{p^2}{2\hbar^2}} e^{-\frac{i}{\hbar} \sum_{pp} \sigma_{pp} p^2}
\]

(2)

hence, the knowledge of \( \sigma_{xx} \) and of \( \sigma_{pp} \), that is, of moduli \( |\psi(x)|^2 \) and \( |\hat{\psi}(p)|^2 \), determines covariance \( \sigma_{xp} \) up to a sign because state \( \psi \) saturates the Robertson–Schrödinger inequality; so, we have

\[
\sigma_{xx} \sigma_{pp} - \sigma_{xp}^2 = \frac{1}{2} \hbar^2
\]

(3)

This identity can be solved in \( \sigma_{xp} \) yielding \( \sigma_{xp} = \pm (\sigma_{xx} \sigma_{pp} - \frac{1}{2} \hbar^2)^{1/2} \). The state and its Fourier transform are given by formulas

\[
\psi_\pm(x) = \left( \frac{1}{2\pi \sigma_{xx}} \right)^{1/4} e^{-\frac{1}{2\sigma_{xx}}} e^{\pm i \sigma_{xx} x^2}
\]

(4)

and

\[
\hat{\psi}_\pm(p) = \left( \frac{1}{2\pi \sigma_{pp}} \right)^{1/4} e^{-\frac{1}{2\sigma_{pp}}} e^{\pm i \sigma_{pp} p^2}.
\]

(5)

Both functions \( \psi_+ \) and \( \psi_- \) and their Fourier transforms \( \hat{\psi}_+ \) and \( \hat{\psi}_- \) satisfy conditions \( |\psi_+(x)|^2 = |\psi_-(x)|^2 \) and \( |\hat{\psi}_+(p)|^2 = |\hat{\psi}_-(p)|^2 \) showing that the Pauli problem does not have a unique solution. In Corbett’s [16] terminology \( \psi_+ \) and \( \psi_- \) are “Pauli partners”. Let us now have a look at these things from the perspective of the Wigner transform

\[
W\psi(x, p) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} e^{-\frac{1}{2\pi} p y} \psi(x + \frac{1}{2} y) \psi^*(x - \frac{1}{2} y) dy
\]

of Gaussian \( \psi \). A straightforward calculation involving Gaussian integrals [17] yields, setting \( z = (x, p) \), normal distribution

\[
W\psi_\pm(z) = \frac{1}{2\pi \sqrt{\det \Sigma_{\pm}}} e^{-\frac{1}{2} \Sigma_{\pm}^{-1} z \cdot z}
\]

(6)

where covariance matrix

\[
\Sigma_{\pm} = \begin{pmatrix}
\sigma_{xx} & \pm \sigma_{xp} \\
\pm \sigma_{px} & \sigma_{pp}
\end{pmatrix}
\]

has determinant \( \det \Sigma_{\pm} = \frac{1}{2} \hbar^2 \) in view of equality (3); hence,

\[
W\psi_\pm(z) = \frac{1}{\pi \hbar} e^{-\frac{1}{2} \Sigma_{\pm}^{-1} z \cdot z}.
\]

(7)

Associated covariance matrices are thus

\[
\Omega_{\pm} = \left\{ z : \frac{1}{2} \Sigma_{\pm}^{-1} z \cdot z \leq 1 \right\}.
\]

3. Multivariate Case: Asking the Right Questions

We generalize the discussion to the multivariate case where the real variables \( x \) and \( p \) are replaced with real vectors \( x = (x_1, ..., x_n) \), \( p = (p_1, ..., p_n) \).

The Wigner function cannot be directly measured, but its marginal distributions can (they are classical probability densities). In analogy with Formula (6) we determine a (centered) Gaussian, \( \psi \) such that

\[
W\psi(z) = \left( \frac{1}{2\pi} \right)^n \frac{1}{\sqrt{\det \Sigma}} e^{-\frac{1}{2} \Sigma^{-1} z \cdot z}
\]

(8)
where \( z = (x, p) \), and the covariance matrix is
\[
\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix}, \quad \Sigma_{PX} = \Sigma_{XP}^T.
\]  

(9)

Here, the \( n \)-dimensional Wigner transform \( W\psi \) is defined by
\[
W\psi(x, p) = \left( \frac{1}{2\pi\hbar} \right)^n \int e^{-\frac{i}{\hbar} p \cdot y} \psi(x + \frac{1}{2} y) \psi^*(x - \frac{1}{2} y) d^n y.
\]

The most straightforward way to determine this state is to use the properties of the Wigner transform itself. Let us start with the marginal properties [17]:
\[
\int W\psi(x, p) d^n p = |\psi(x)|^2
\]
\[
\int W\psi(x, p) d^n x = |\hat{\psi}(p)|^2
\]

(10)

(11)

where the \( n \)-dimensional Fourier transform \( \hat{\psi} \) is given by
\[
\hat{\psi}(p) = \left( \frac{1}{2\pi\hbar} \right)^{n/2} \int e^{-\frac{i}{\hbar} p \cdot x} |\psi(x)|^2 d^n x.
\]

These formulas hold as soon as both \( \psi \) and \( \hat{\psi} \) are in \( L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) [17]. These quantities allow for determining matrices
\[
\Sigma_{XX} = (\sigma_{x_jx_k})_{1 \leq j,k \leq n} \quad \text{and} \quad \Sigma_{PP} = (\sigma_{p_jp_k})_{1 \leq j,k \leq n}
\]

by usual formulas
\[
\sigma_{x_jx_k} = \int x_j x_k |\psi(x)|^2 d^n x, \quad \sigma_{p_jp_k} = \int p_j p_k |\hat{\psi}(p)|^2 d^n p
\]

and an elementary calculation of Gaussian integrals yields the values
\[
|\psi(x)| = \left( \frac{1}{2\pi} \right)^{n/4} (\det \Sigma_{XX})^{-1/4} e^{-\frac{1}{4} \Sigma_{XX}^{-1} x \cdot x}
\]
\[
|\hat{\psi}(p)| = \left( \frac{1}{2\pi} \right)^{n/4} (\det \Sigma_{PP})^{-1/4} e^{-\frac{1}{4} \Sigma_{PP}^{-1} p \cdot p}
\]

(12)

(13)

Here, we are exactly in the situation discussed by Pauli: \( |\psi(x)| \) and \( |\hat{\psi}(p)| \) are what we can measure, so we can determine covariance blocks \( \Sigma_{XX} \) and \( \Sigma_{PP} \), but not covariance \( \Sigma_{XP} \); knowledge of the latter (and hence of \( \Sigma_{PX} = \Sigma_{XP}^T \)) is necessary to entirely determine state \( \psi \). In the previous section, the problem was solved: in case \( n = 1 \), blocks \( \Sigma_{XX}, \Sigma_{PP}, \) and \( \Sigma_{XP} \) were scalars \( \sigma_{xx}, \sigma_{pp}, \) and \( \sigma_{xp}, \) and these are related by the uncertainty principle in the form of \( \sigma_{xx} \sigma_{pp} - \sigma_{xp}^2 = \frac{1}{4} \hbar^2 \) yielding two possible values \( \sigma_{xp} = \pm (\sigma_{xx} \sigma_{pp} - \frac{1}{4} \hbar^2)^{1/2} \), and hence the two states (5). In the multidimensional, case we also have a simple (but not immediately obvious) formula connecting the blocks of the covariance matrix. The way out of this problem consists in using a general formula [17–19], which was initially proved by Bastiaans [20] in connection with first-order optics. Let \( X \) and \( Y \) be real \( n \times n \) matrices, such that \( X = X^T > 0 \) and \( Y = Y^T \), and set
\[
\psi_{X,Y}(x) = \left( \frac{1}{2\pi} \right)^{n/4} (\det X)^{1/4} e^{-\frac{1}{2} X^{-1/2} (X + iY) x \cdot x}.
\]

(14)

This function is normalized to unity: \( \|\psi_{X,Y}\|_{L^2} = 1 \), and its Wigner transform is given by
\[
W\psi_{X,Y}(z) = \left( \frac{1}{2\pi} \right)^n e^{-\frac{1}{2} \hbar^2 z \cdot z}
\]

(15)
where $G$ is the symmetric matrix

$$G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix}. \quad (16)$$

A fundamental fact, which is related to the uncertainty principle, is that $G$ is a symplectic matrix, i.e., it belongs to symplectic group $\text{Sp}(n)$. Equivalently, since $G = G^T$, $G^T J G = G J G = J$ where

$$J = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{pmatrix}$$

is the standard symplectic matrix. We have $G = S^T S$, where

$$S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2} & X^{-1/2} \end{pmatrix} \quad (17)$$

is clearly symplectic. Assuming that function $\psi$ for which we are looking is a Gaussian, comparing Formulas (8) and (15) leads to identification

$$\Sigma = \hbar^2 G^{-1}. \quad (18)$$

Since $G^T J G = J$ the inverse $G^{-1}$ is $-JG$, explicit formula

$$G^{-1} = \begin{pmatrix} X^{-1} & -X^{-1}Y \\ -YX^{-1} & X + YX^{-1}Y \end{pmatrix}$$

so that there remains to solve matrix equation

$$\begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} X^{-1} & -X^{-1}Y \\ -YX^{-1} & X + YX^{-1}Y \end{pmatrix}. \quad (19)$$

It immediately follows that we have $X = \frac{\hbar}{2} \Sigma_{XX}^{-1}$ and $Y = -\frac{\hbar}{2} \Sigma_{XP} \Sigma_{XX}^{-1}$, so the unknown Gaussian for which we were looking is

$$\psi(x) = \left( \frac{1}{2\pi} \right)^{n/4} \left( \det \Sigma_{XX} \right)^{-1/4} \exp \left[ -\frac{1}{2} \Sigma_{XX}^{-1} x \cdot x \right], \quad (20)$$

which is the $n$-dimensional variant of (1), replacing $\sigma_{xx}$ with $\Sigma_{XX}$ and $\sigma_{xp}$ with $\Sigma_{XP}$. This does not solve completely our problem, however, because we do not know matrix $\Sigma_{XP}$. The crucial step is to notice that, as a bonus, we obtained from (18) the matrix form of the saturated Robertson–Schrödinger equality, namely,

$$\Sigma_{PP} \Sigma_{XX} - \Sigma_{XP}^2 = \frac{1}{4} \hbar^2 I_{n \times n}. \quad (21)$$

From this formula we can deduce $\Sigma_{XP}^2$, and one finds two Pauli partners

$$\psi_{\pm}(x) = \left( \frac{1}{2\pi} \right)^{n/4} \left( \det \Sigma_{XX} \right)^{-1/4} \exp \left[ -\frac{1}{2} \Sigma_{XX}^{-1} x \cdot x \pm \frac{\hbar}{2} \Sigma_{XP} \Sigma_{XX}^{-1} x \cdot x \right], \quad (22)$$

once a value of $\Sigma_{XP}$ is determined (even if $\Sigma_{XP}^2 = 0$, we can have $\Sigma_{XP} \neq 0$). Here, we solved a so-called “phase retrieval problem” (see Klibanov et al. [21] for a good review of the topic): in view of Formula (12), we know that

$$\psi(x) = e^{i\Phi(x)} \left( \frac{1}{2\pi} \right)^{n/4} \left( \det \Sigma_{XX} \right)^{-1/4} e^{-\frac{1}{2} \Sigma_{XX}^{-1} x \cdot x}$$
where $\Phi$ is an unknown real function of the position variable. We identified this phase here as being function

$$\Phi(x) = -\left(\frac{1}{2\hbar} \sum_{X \in P} \sum_{x \in X} x \cdot x\right).$$

4. Geometric Interlude

We introduce the notion of $h$-polarity and duality; we see in the next section that this notion from convex geometry is quite unexpectedly related to the Pauli problem, of which it gives a limpid geometric interpretation. For a very detailed study of polarity, see Charalambos and Aliprantis [22]. In both sources, alternative competing definitions are also described; the one we use here is the most common and the best fitted to our needs.

Let $X$ be a nonempty subset of $n$-dimensional configuration space $\mathbb{R}_n$; this may be, for instance, a set of position measurements performed on some physical system with $n$ degrees of freedom. One defines the polar set of $X$ as the set $X^o$ of all points $p = (p_1, ..., p_n)$ in the momentum space $\mathbb{R}_n^p$, such that

$$px = p_1 x_1 + \cdots + p_n x_n \leq 1$$

for all points $x = (x_1, ..., x_n)$ in $X$. Similarly, if $P$ is a subset of $\mathbb{R}_n^p$, one defines its polar $P^o$ as the set of all $x$ in $\mathbb{R}_n^p$, such that $px \leq 1$ for all $p$ in $P$. We use a rescaled variant of the notion of polarity here, which we call $\hbar$ polarity. By definition, the $h$-polar $X^h$ of $X$ is the set of all $p$, such that

$$px = p_1 x_1 + \cdots + p_n x_n \leq \hbar$$

for all points $x$ in $X$. We have $X^h = hX^o$ and $P^h = hP^o$ likewise.

From now on, we assume for simplicity that $X$ and $P$ are convex bodies, i.e., they are convex, compact, and with a nonempty interior; we also assume that they are symmetric (i.e., $X = -X$), which implies, by convexity, that they contain 0 in their interior. Simple examples of such sets are balls and ellipsoids centered at the origin. Polar duals have the following remarkable properties:

- **Biduality:** $(X^h)^h = X$
- **Antimonotonicity:** $X \subseteq Y =\Rightarrow Y^h \subseteq X^h$
- **Scaling property:** $L \in GL(n, \mathbb{R}) =\Rightarrow (LX)^h = (L^T)^{-1} X^h$.

Let $B_X^h(R)$ (resp. $B_Y^h(R)$) be the ball $\{x : |x| \leq R\}$ in $\mathbb{R}_n^p$ (resp. $\{p : |p| \leq R\}$ in $\mathbb{R}_n^p$). We have

$$B_X^h(\sqrt{\hbar}) = B_Y^h(\sqrt{\hbar})$$

(23)

and one can show that $B_X^h(\sqrt{\hbar})$ is the only self $h$-dual set in $\mathbb{R}_n^p$. Let us extend this to the case of ellipsoids. An ellipsoid in $\mathbb{R}_n^p$ centered at the origin (which is just an ordinary plane ellipse when $n = 1$) can always be viewed as the image of ball $B_X^h(\sqrt{\hbar})$ by some invertible linear transformation $L$, in which case, it is given by inequality

$$L^{-1}x \cdot L^{-1}x = (LL^T)^{-1}x \cdot x \leq h.$$

Conversely, if $A$ is a positive definite symmetric matrix, inequality $Ax \cdot x \leq h$ always defines an ellipsoid, since it is equivalent to the above inequality, taking for $L$ inverse square root $A^{-1/2}$ of $A$. It immediately follows from the scaling property that the $h$-polar of the ellipsoid is obtained by inverting the matrix of the ellipsoid:

$$X : Ax^2 \leq h \iff X^h : A^{-1}p \cdot p \leq h$$

(24)

(that we have an equivalence follows from biduality property $(X^h)^h = X$).
5. The Pauli Problem and Polar Duality

Let us return to the Wigner transform of Gaussian states; using Formula (15), we can explicitly calculate \( W\psi_\pm \), and one finds

\[
W\psi_\pm (z) = (\pi \hbar)^{-n} e^{-\frac{1}{2} \Sigma_\pm^T z \Sigma_\pm z}
\]

where covariance matrices \( \Sigma_\pm \) are given by

\[
\Sigma_\pm = \begin{pmatrix} \Sigma_{XX} \pm \Sigma_{XP} \\
\pm \Sigma_{PX} \Sigma_{PP} \end{pmatrix}
\]

with \( \Sigma_{PX} = \Sigma_{XP}^T \). Two ellipsoids \( \Omega_\pm \) centered at the origin correspond to \( \Sigma_\pm \). Let us determine orthogonal projections \( \Omega_X, \Omega_P \) of \( \Omega_\pm \) on the position and momentum spaces \( \mathbb{R}_n^x \) and \( \mathbb{R}_n^p \).

5.1. Case \( n = 1 \)

We begin with case \( n = 1 \), and projections are line segments. Here, \( \Sigma_{XX} = \sigma_{xx} \), \( \Sigma_{PP} = \sigma_{pp} \), and \( \Sigma_{XP} = \sigma_{xp} \) and covariance ellipses \( \Omega_\pm \) are defined by

\[
\sigma_{pp} x^2 + \frac{\sigma_{xp}}{D} px + \sigma_{xx} p^2 \leq 1
\]

where \( D = \sigma_{xx} \sigma_{pp} - \sigma_{xp}^2 = \frac{1}{2} \hbar^2 \) (cf. Formula (3)). Orthogonal projections \( \Omega_X, \Omega_P \) of \( \Omega_\pm \) on the \( x \) and \( p \) axes are the same:

\[
\Omega_X = [-\sqrt{2\sigma_{xx}}, \sqrt{2\sigma_{xx}}], \quad \Omega_P = [-\sqrt{2\sigma_{pp}}, \sqrt{2\sigma_{pp}}].
\]

Let \( \Omega_X^\perp \) be the polar dual of \( \Omega_X \); it is the set of all numbers \( p \), such that \( px \leq \hbar \) for \(-\sqrt{2\sigma_{xx}} \leq x \leq \sqrt{2\sigma_{xx}} \) and is thus the interval

\[
\Omega_X^\perp = [-\hbar/\sqrt{2\sigma_{xx}}, \hbar/\sqrt{2\sigma_{xx}}].
\]

Since \( \sigma_{xx} \sigma_{pp} \geq \frac{1}{2} \hbar \), we have inclusion

\[
\Omega_X^\perp \subset \Omega_P \tag{27}
\]

and this inclusion reduces to equality \( \Omega_X^\perp = \Omega_P \) if and only if the Heisenberg inequality is saturated, i.e., \( \sigma_{xx} \sigma_{pp} = \frac{1}{4} \hbar^2 \), which is equivalent to \( \sigma_{xp} = 0 \).

5.2. General Case

We have similar properties in arbitrary dimension \( n \). To study this case, we first must find the orthogonal projections of covariance ellipsoid \( \Omega \) on the position and momentum spaces. Ellipsoid \( \Omega \) is given by equation \( Mz \cdot z \leq \hbar \) where \( M = \frac{1}{2} \Sigma^{-1} \) is symmetric and positive definite (\( M > 0 \)). Writing \( M \) in block form

\[
M = \begin{pmatrix} M_{XX} & M_{XP} \\
M_{PX} & M_{PP} \end{pmatrix}
\]

where \( M_{XX} = M_{XX}^T, M_{PP} = M_{PP}^T \), and \( M_{XP} = M_{XP}^T \) are \( n \times n \) matrices; since \( M > 0 \), we also have \( M_{XX} > 0 \) and \( M_{PP} > 0 \). Then, the projections of \( \Omega \) on \( \mathbb{R}_n^x \) and \( \mathbb{R}_n^p \) are ellipsoids given by, respectively [23],

\[
\Omega_X : (M/M_{PP})z \cdot z \leq \hbar \quad \text{and} \quad \Omega_P : (M/M_{XX})p \cdot p \leq \hbar.
\]
where symmetric matrices
\[ \frac{M}{M_{PP}} = M_{XX} - M_{XP}M_{PP}^{-1}M_{PX} \] (29)
\[ \frac{M}{M_{XX}} = M_{PP} - M_{PX}M_{XX}^{-1}M_{XP} \] (30)
are Schur complements in \( M \) of \( M_{PP} \) and \( M_{XX} \); we have \( \frac{M}{M_{PP}} \succ 0 \) and \( \frac{M}{M_{XX}} \succ 0 \) so that \( \Omega_X \) and \( \Omega_P \) are nondegenerate (see Zhang’s treatise [24] for a detailed study of the Schur complement). To prove that inclusion \( \Omega_X^0 \subset \Omega_P \) holds, we must show that cf. implication (24) that
\[ \left( \frac{M}{M_{PP}} \right) \left( \frac{M}{M_{XX}} \right) \leq I_{n \times n}, \] (31)
that is, that the eigenvalues of \( \left( \frac{M}{M_{PP}} \right) \left( \frac{M}{M_{XX}} \right) \) must be smaller than 1. To prove this, we use the following essential remark: we showed above that matrix \( M = \frac{1}{2} \Sigma^{-1} \) is symplectic; therefore, its entries obey some constraints. Considering that \( M \) is also symmetric, these constraints are
\[ M_{XX}M_{PP} - M_{XP}^2 = I_{n \times n} \] (32)
\[ M_{XX}M_{PX} = M_{XP}M_{XX} \] (33)
\[ M_{PX}M_{PP} = M_{PP}M_{XP}. \] (34)
Using Identities (33) and (34), it follows that Schur complements (29) and (30) can be rewritten as
\[ \frac{M}{M_{PP}} = M_{XX} - M_{PP}^{-1}M_{PX} \]
\[ = M_{PP}^{-1}(M_{PP}M_{XX} - M_{PX}^2) \]
\[ = M_{PP}^{-1} \]
the last equality by using the transpose of Identity (32). Similarly,
\[ \frac{M}{M_{XX}} = M_{PP} - M_{XX}^{-1}M_{XP}^2 = M_{XX}^{-1} \]
So, summarizing, Schur complements are given by
\[ \frac{M}{M_{PP}} = M_{PP}^{-1}, \frac{M}{M_{XX}} = M_{XX}^{-1}. \] (35)
It follows that
\[ \left( \frac{M}{M_{PP}} \right) \left( \frac{M}{M_{XX}} \right) = M_{PP}^{-1}M_{XX}^{-1} = (M_{XX}M_{PP})^{-1}. \]
We show that \( \left( \frac{M}{M_{PP}} \right) \left( \frac{M}{M_{XX}} \right) \leq I_{n \times n} \); equivalently, \( M_{XX}M_{PP} \succeq I_{n \times n} \). Now, since \( M = \frac{1}{2} \Sigma^{-1} \) is symplectic, so is matrix
\[ M^{-1} = \frac{2}{\hbar} \Sigma = \left( \begin{array}{cc} \frac{2}{\hbar} \Sigma_{XX} & \frac{2}{\hbar} \Sigma_{XP} \\ \frac{2}{\hbar} \Sigma_{PX} & \frac{2}{\hbar} \Sigma_{PP} \end{array} \right) \]

hence, reinverting,
\[ M = \left( \begin{array}{cc} \frac{2}{\hbar} \Sigma_{PP} & -\frac{2}{\hbar} \Sigma_{PX} \\ -\frac{2}{\hbar} \Sigma_{XP} & \frac{2}{\hbar} \Sigma_{XX} \end{array} \right) \] (36)
so that \( M_{XX}M_{PP} = \frac{4}{\hbar^2} \Sigma_{PP} \Sigma_{XX} \). In view of the generalized RSUP (20), we have
\[ \Sigma_{PP} \Sigma_{XX} - \Sigma_{XP}^2 = \frac{1}{4} \hbar^2 I_{n \times n} \] (37)

hence
\[ M_{XX}M_{PP} = I_{n \times n} + \frac{4}{\hbar^2} \Sigma_{XP}^2 \] (38)
and we are finished, provided that we can prove that \( \Sigma^2_{XP} \geq 0 \) (which is obvious if \( n = 1 \)), or, which amounts to the same \( M_{XX}^2 \geq 0 \). For this, since \( M_{XX}M_{PX} = M_{XP}M_{XX} \) (Formula (33)), we have

\[
M_{XP} = M_{XX}M_{PX}M_{XX}^{-1}
\]

hence, \( M_{XP} \) and \( M_{PX} \) have the same eigenvalues; since \( M_{PX} = M_{XP}^T \), these eigenvalues must be real, and those of \( M_{XX}^2 \) must be \( \geq 0 \).

For completeness, we still need to discuss what happens when \( \Omega_X^L = \Omega_P \). In view of Formulas (28) and Equivalence (24), this means that (31) reduces to equality

\[
(M/M_{PP})(M/M_{XX}) = I_{n \times n}
\]

that is, by (35), \( M_{XX}M_{PP} = I_{n \times n} \). Taking (38) into account, we must thus have \( M_{XP}^2 = 0 \), which does not imply that \( M_{XP} = 0 \). We are in the presence of states (21) in this case, saturating the Heisenberg inequalities.

6. Discussion and Outlook

Our discussion of polar duality suggests that a quantum system localized in the position representation in a set \( X \) cannot be localized in the momentum representation in a set smaller than that of its polar dual \( X^\perp \). The notion of polar duality thus appears informally as a generalization of the uncertainty principle of quantum mechanics, as expressed in terms of variances and covariances (see [23]). The idea of such generalizations is not new, and can already be found in the work of Uffink and Hilgevoord [25,26]; see Butterfield’s discussion in [27]. It would certainly be interesting to explore the connection between convex geometry and quantum mechanics, but very little work has been conducted so far.

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