Ricci solitons and Einstein-scalar field theory

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Abstract
B List has recently studied a geometric flow whose fixed points correspond to static Ricci flat spacetimes. It is now known that this flow is in fact Ricci flow modulo pullback by a certain diffeomorphism. We use this observation to associate with each static Ricci flat spacetime a local Ricci soliton in one higher dimension. Also, solutions of Euclidean-signature Einstein gravity coupled to a free massless scalar field with nonzero cosmological constant are associated with shrinking or expanding Ricci solitons. We exhibit examples, including an explicit family of complete expanding solitons. These solitons can also be thought of as a Ricci flow for a complete Lorentzian metric. The possible generalization to Ricci-flat stationary metrics leads us to consider an alternative to Ricci flow.

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1. Introduction

Geometric flows have become important tools in Riemannian geometry and general relativity. Quite early on, Geroch [10] introduced the inverse mean curvature flow in an argument in support of the conjecture that later became the positive mass theorem. The method eventually led to the Huisken–Ilmanen proof of the Riemannian Penrose conjecture [13]. More recently, the powerful tool of Ricci flow has been used to prove the Poincaré conjecture [20] and, it appears, the Thurston conjecture as well [4, 19]. This flow is given as

$$\frac{\partial g_{\mu\nu}}{\partial \lambda} = -2R_{\mu\nu}. \tag{1.1}$$

The flow is often generalized by pulling back along an evolving (i.e., $\lambda$-dependent) diffeomorphism. This yields the Hamilton–DeTurck flow

$$\frac{\partial g_{\mu\nu}}{\partial \lambda} = -2R_{\mu\nu} + \xi X g_{\mu\nu}, \tag{1.2}$$

where $X$ is the vector field generating the diffeomorphism.
The power of geometric flows derives in part from the (quasi-)parabolic character of the flow equations on Riemannian geometries. This property is typically not present for flows on pseudo-Riemannian geometries (spacetimes), unless the problem can be phrased in Riemannian terms, as is the case for the Riemannian Penrose conjecture. Following this reasoning, one may expect quasi-parabolicity of the Ricci flow on a spacetime if the metric has a static or perhaps even a stationary Killing vector field (recall that the Ricci flow preserves isometries).  

A geometric flow of static Lorentzian metrics was studied by List [18]. He did not begin with the Ricci flow of a static metric (i.e., a metric with timelike, hypersurface-orthogonal Killing field). Instead he presented a system of flow equations whose fixed points solve the static vacuum Einstein equations, but which seemed better suited to obtaining estimates than Ricci flow. List’s system is

$$\frac{\partial g_{ij}}{\partial \lambda} = -2(\nabla_i u \nabla_j u),$$

$$\frac{\partial u}{\partial \lambda} = \Delta u,$$

where, for each value of the flow parameter $\lambda$, $g_{ij}(\lambda; x)$ is a Riemannian metric on an $n$-manifold, $u(\lambda; x)$ is a function, $\Delta u := g^{ij} \nabla_i \nabla_j u$ is the Laplacian of $u$, $k_n$ is an arbitrary constant (which we will set to $\sqrt{n-2}$ when $n > 2$), and $-e^{2u} dt^2 + e^{2u} g_{ij}$ is then the (flowing) spacetime static metric.

Static metrics have always yielded important examples of exact solutions in general relativity. They also arise, for example, as the result of limiting processes along a family of metrics. This occurs in two contexts that we know of: Anderson’s approach to geometrization of 3-manifolds [1, 2], and Bartnik’s approach to quasi-local mass [3]. In fact, the motivation behind List’s flow was to have a method of generating families of metrics that might be well suited to studying questions arising from Bartnik’s mass definition [12]. In the special case of a bounded region $B$ in a time-symmetric slice of spacetime, Bartnik defines the quasilocal mass of $B$ by considering all asymptotically flat Riemannian manifolds having non-negative scalar curvature into which $B$ can be embedded (smoothly up to $\partial B$), without there being a stable minimal sphere outside $B$. Each such manifold has a non-negative ADM mass. Bartnik’s quasilocal mass is the infimum of these ADM masses, and is therefore non-negative as well. What is not clear a priori is whether it is positive. However, Bartnik has conjectured that the infimum in the definition is realized by an extension of $B$ which solves the static vacuum Einstein equations: by the positive mass theorem, $m_B$ will be nonzero unless $B$ embeds in flat space. The problem is then to show that such a mass-minimizing extension exists.

Huisken and Ilmanen [13] have since proved that $m_B > 0$ unless $B$ is isometric to a domain in flat space, but Bartnik’s conjecture remains open. One method of attacking it would be to try to construct the conjectured static metric as a geometric limit of a suitable flow starting from some initial data, say a positive scalar curvature metric on the extension, and subject to boundary conditions at $\partial B$. List’s flow is a promising candidate because, as well as having static metrics as fixed points, it preserves asymptotic flatness, has compactness properties to

1 Parabolic flows sometimes arise even in nonstationary spacetimes. Consider the interesting example of Robinson–Trautman 4-metrics, for which the Einstein equation becomes the fourth-order Calabi flow in two dimensions [8, 24].

2 At $\partial B$, ‘geometric boundary conditions’ are imposed. The induced metric and mean curvature must match across $\partial B$, but the full extrinsic curvature of $\partial B$ need not.

3 Indeed, the reason for the condition banning stable minimal spheres outside $B$ is that manifolds which such spheres can have arbitrarily small ADM masses, which would cause that the Bartnik mass to be driven to zero; see [3].
make sense of limits [18], and ‘almost’ preserves positive scalar curvature (along the flow, the scalar curvature is bounded below by $-\text{const}/(1 + \lambda)$, which tends to zero as $\lambda \to \infty$).

It is now realized that List’s system of flow equations is in fact the pullback by a certain diffeomorphism of a class of Ricci flows in one higher dimension; i.e., List’s system is really a certain Hamilton–DeTurck flow. This does not make List’s flow any less interesting however. An advantage of this is that Ricci flow results can be employed to study List’s flow. But as well, certain Ricci flow results may be more evident from the perspective of List’s flow; i.e., by employing the diffeomorphism gauge that leads to List’s flow equations. The point of this paper is to document one example of this interplay, arising from a search for simple solutions of List’s equations.

Applications of geometric flow problems often require quite detailed analytical arguments, for which it is best first to have intuition developed from exact solutions. In the case of Ricci flow, the simplest solutions are the fixed points, i.e., Ricci-flat metrics. The next simplest are the Ricci solitons.

**Definition 1.1.** A Ricci soliton is a manifold-with-metric $(M^{n+1}, g_{\mu\nu})$ and vector field $X$ on it such that

$$R_{\mu\nu} - \frac{1}{2} \mathcal{L}_X g_{\mu\nu} = \kappa g_{\mu\nu}$$

(1.5)

for some constant $\kappa$.

Here $R_{\mu\nu}$ is the Ricci curvature of $g_{\mu\nu}$. The soliton is gradient if $X = \nabla \varphi$ for some function $\varphi$ and steady if $\kappa = 0$. If $\kappa < 0$ the soliton is called an expander; if $\kappa > 0$ it is a shrinker. Finally, a local soliton is one that solves (1.5) on an open region which might not admit an extension to a complete manifold with the soliton metric.

Given a pair $(X, g)$ solving (1.5), the metric $\kappa \lambda \varphi^*(\lambda) g_{\mu\nu}$, obtained by pulling back $g_{\mu\nu}$ along $\frac{1}{\kappa \lambda} X$ and rescaling by $\kappa \lambda$, solves (1.1) [7]. Solitons are not fixed points of (1.1) but evolve only by diffeomorphism and scaling, and in this sense are the simplest nontrivial solutions.

As we said earlier, List’s flow admits fixed point solutions corresponding to static spacetime metrics. The next simplest solutions of List’s flow are solitons of List’s flow, which are flows constructed from pairs $(u, g_{ij})$ that obey

$$R_{ij} - k^2 \nabla_i u \nabla_j u - \frac{1}{2} \mathcal{L}_X g_{ij} = \kappa g_{ij},$$

(1.6)

$$\Delta u + \mathcal{L}_X u = 0,$$

(1.7)

for some constant $\kappa$ and vector field $X$, where now $R_{ij}$ is the Ricci curvature of $g_{ij}$. When $X$ vanishes, equations (1.6) and (1.7) are well known in general relativity:

**Definition 1.2.** We will call the equations

$$R_{ij} - k^2 \nabla_i u \nabla_j u = \kappa g_{ij},$$

(1.8)

$$\Delta u = 0,$$

(1.9)

the Einstein free-scalar-field system. When $\kappa = 0$ as well, equations (1.8) and (1.9) are called the static vacuum Einstein equations.

The terminology Einstein free-scalar-field arises because, for Lorentzian signature $g_{ij}$, equations (1.8) and (1.9) describe Einstein gravity with cosmological constant, coupled to a free scalar field. But note that we will use the terminology without regard to the signature of
The static vacuum Einstein terminology arises because, if \( g_{ij} \) has Euclidean signature and when \( \kappa = 0 \) and the conventional choice \( k_n = \sqrt{\frac{\kappa}{n-2}} \) is made, equations (1.8) and (1.9) imply that the metric on \( \mathbb{R} \times M^n, n > 2 \), given by

\[
dS^2 = G_{\mu\nu} \, dx^\mu \, dx^\nu = -e^{2u} \, dt^2 + e^{-2u} \, g_{ij} \, dx^i \, dx^j
\]

is Ricci-flat and \( \frac{\partial}{\partial t} \) is a hypersurface-orthogonal Killing vector field.

In fact, equation (1.9) is redundant for any \( \kappa \in \mathbb{R} \) because it can be derived from (1.8) using the contracted second Bianchi identity. Thus it plays only the role of an integrability condition for (1.8).

The main result of this paper is the observation that List’s flow is a Hamilton–DeTurck flow leads directly to a nice connection between Ricci solitons and the Einstein free-scalar-field system:

**Observation 1.3.** Solutions of the Einstein free-scalar-field system correspond to Ricci solitons.

Indeed, any soliton of List’s flow (i.e., any solution of (1.6) and (1.7)) corresponds to a Ricci soliton, but we will focus on the relation to relativity theory expressed in the observation.

We give the precise correspondence in section 2. Many of the solitons that arise in this manner are only local solitons, in the sense that the metric is not complete. This is not unexpected in view of various theorems in physics for static vacuum Einstein and Einstein-scalar systems, such as the theorem of Lichnerowicz [17], ‘no hair’ theorems and singularity theorems (see, e.g., Chase [5]).

In section 3, we identify the local solitons that arise from several familiar Einstein-scalar solutions. We also construct an example of a complete soliton arising from the Einstein-scalar system with negative cosmological constant. This may also be thought of as a nontrivial Ricci flow on a complete Lorentzian manifold.

In section 4, we generalize from static to stationary metrics. Stationary Lorentzian metrics have a timelike Killing vector field, but are more general than static metrics because this vector field is no longer necessarily hypersurface orthogonal; when it is not, we say the metric is rotating. We ask whether there is a Ricci flow adapted to stationary metrics in the way that List’s flow is adapted to static metrics; namely, are there metric-diffeomorphism pairs satisfying (1.2) such that the fixed points yield Ricci-flat stationary rotating metrics? If so, then our soliton construction would extend to that case. However, we find that for the most obvious choice of diffeomorphism at least, the fixed points of the resulting Hamilton–DeTurck flow equations do not coincide with rotating Ricci-flat metrics. This leads us to propose an alternative flow, for which fixed points are Ricci-flat stationary metrics, but which we do not derive from Hamilton–DeTurck flow.

**2. Solitons and free scalar fields**

**2.1. The precise relationship**

The precise form of observation 1.3 is the following result:

**Proposition 2.1.** If the pair \((u, g_{ij})\) solves (1.8) (and thus (1.9) as well), the metric

\[
dx^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = e^{2k_0 \, dt^2} + g_{ij} \, dx^i \, dx^j
\]

is a local Ricci soliton on \( \mathbb{R} \times M^n \) solving (1.5).
Proof. The Ricci curvature of the metric (2.1) is
\[
R_{\mu\nu} \, dx^\mu \wedge dx^\nu = -e^{2ku}(k_n \Delta u + k_n^2 |\nabla u|^2) \, dt^2 + \left( R_{ij}^k - k_n^2 \nabla_i u \nabla_j u - k_n \nabla_i \nabla_j u \right) dx^i \wedge dx^j,
\]
where we used (1.8) and (1.9) and the shorthand $|\nabla u|^2 := g^{ij} \nabla_i u \nabla_j u$. On the other hand, define the vector field
\[
X := -\kappa t \frac{\partial}{\partial t} - k_n g^{ij} \nabla_i u \frac{\partial}{\partial x^j}.
\]
Using the metric (2.1), it is straightforward to compute that
\[
\nabla_\mu X_\nu = \begin{bmatrix}
-2e^{2ku}(k_n^2 |\nabla u|^2 + \kappa) & \kappa k_n e^{2ku} \nabla_i u \\
-\kappa k_n e^{2ku} \nabla_i u & -k_n \nabla_i \nabla_j u
\end{bmatrix}.
\]
We note in passing that since this is not symmetric, $X$ is not a gradient vector field, but $X \wedge dX = 0$ so $X$ is hypersurface orthogonal (here we have used $X$ to denote both the vector field and its metric-dual 1-form). From (2.4) we obtain that
\[
\nabla_X g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu = \begin{bmatrix}
-2e^{2ku}(k_n^2 |\nabla u|^2 + \kappa) & 0 \\
0 & -2k_n \nabla_i \nabla_j u
\end{bmatrix}.
\]
Combining (2.2) and (2.5) yields (1.5). □

Remark 2.2. From (1.5), for each sign of $\kappa$, there are constant $u$ solitons $dr^2 + g_{ij} \, dx^i \wedge dx^j$ with $g_{ij}$ an Einstein metric. If $\kappa = 0$, these exist for $t \in S^1$ as well as for $t \in \mathbb{R}$, but for $\kappa \neq 0$, the vector field $X$ would not be single valued if $t$ were periodic.

Remark 2.3. The invariance $u \rightarrow -u$ of equation (1.8) yields a second, distinct soliton by replacing $u$ by $-u$ in (2.4). This is an example of the Buscher duality described in [7].

Remark 2.4. We can replace (2.1) by
\[
ds^2 = -e^{2ku} \, dt^2 + g_{ij} \, dx^i \wedge dx^j,
\]
thereby obtaining a Ricci soliton of Lorentzian signature.

Remark 2.5. It is common to represent scaling solitons in the $\lambda$-dependent form. In the present case, two such forms are
\[
ds^2 = G_{\mu\nu} \, dx^\mu \wedge dx^\nu = -2\kappa \lambda \left( \pm e^{2ku} \, dt^2 + g_{ij} \, dx^i \wedge dx^j \right),
\]
and
\[
ds^2 = G_{\mu\nu} \, dx^\mu \wedge dx^\nu = \pm e^{2ku} \, dt^2 - 2\kappa \lambda g_{ij} \, dx^i \wedge dx^j,
\]
where we take $\lambda \in (-\infty, 0)$ if $\kappa > 0$ for the shrinker and $\lambda \in (0, \infty)$ if $\kappa < 0$ for the expander. These metrics solve the Hamilton–DeTurck flow equation
\[
\frac{\partial G_{\mu\nu}}{\partial \lambda} = -2R_{\mu\nu}^G + \nabla_Y G_{\mu\nu}
\]
with vector fields $Y = -\frac{1}{2\kappa} X$ and $Y = \frac{k_n}{2\kappa} g^{ij} \nabla_i u \frac{\partial}{\partial x^j}$, respectively.
2.2. Completeness

The question arises as to when the solitons (2.1) are complete. Some conclusions can be drawn from simple properties of global solutions of (1.8) and (1.9):

**Lemma 2.6.** If \((M^n, g)\) is a closed manifold, then it is Einstein, \(u = \text{const}\), and the soliton \(dt^2 + g_{ij}\) is complete. If \((M^n, g)\) is noncompact and complete, then \(\kappa \leq 0\). If, further, \(\kappa = 0\) and if \(|\nabla u| \to 0\) at infinity, then \(u = \text{const}\) and \((M, g)\) and the soliton are Ricci flat.

**Remark 2.7.** If \(n = 3\) and \(\kappa = 0\), Anderson’s generalization [2] of Lichnerowicz’s theorem [17] shows that the soliton will be flat space, even without any assumption on \(|\nabla u|\).

**Proof of 2.6.** The first statement of lemma 2.6 follows from (1.9) and the strong maximum principle. The second statement follows from (1.8) and Myers’ diameter estimate. To prove the third statement, set \(\kappa = 0\) in (1.8) and use it and (1.9) to compute that

\[
\Delta(|\nabla u|^2) = 2k_2^2(|\nabla u|^2)^2 + 2|\nabla u|^2 \geq 0. \tag{2.10}
\]

Since \(M\) is complete and \(u \in C^2(M)\) is globally defined and \(|\nabla u| \to 0\) at infinity, it then follows from the strong maximum principle that \(u\) is constant.\(^4\)

Incompleteness can arise because of a singularity—i.e., inextenbility of (2.1) in any coordinates—or it can also arise merely because the coordinate system breaks down at fixed points of the Killing field \(\frac{\partial}{\partial t}\). The latter case is analogous to the incompleteness in the Schwarzschild exterior black hole metric after Wick rotation to Riemannian signature, which is cured by adding in the fixed point at \(r = 2m\). Lemma 2.6 does not distinguish between these sources of incompleteness. The following lemma shows that the soliton will have a singularity when the Einstein-scalar system does. This is useful because, for \(g_{ij}\) a Lorentzian metric, the Einstein-scalar system has been studied in the context of ‘no hair’ results, so we can Wick rotate the conclusions to our case. These results show that, in some circumstances at least, extension of (2.1) to a fixed point of the Killing field is not possible. We will return to this point later (see footnote 7).

**Lemma 2.8.** The soliton is inextendible wherever the norm of either \(\sec[g]\), \(|\nabla u|\), or \(|\nabla\nabla u|\) diverges.

Here \(\sec[g]\) refers to the sectional curvatures of \(g\); i.e., certain orthonormal components of \(\text{Riem}[g]\) that fully determine \(\text{Riem}[g]\).

**Proof.** We compute scalar invariants of the soliton metric (2.1) to obtain

\[
\begin{align*}
R &= -k_2^2|\nabla u|^2 + nk = -R^2 + 2nk, \tag{2.11} \\
R_{\mu\nu}R^{\mu\nu} &= k_2^4(|\nabla u|^2)^2 + nk_2^2|\nabla u|^2 \\
R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} &= 4k_2^4|\nabla\nabla u + k_n\nabla u\nabla u|^2 + |\text{Riem}[g]|^2. \tag{2.13}
\end{align*}
\]

Curvatures on the left-hand side refer to the metric (2.1). On the right-hand side, \(|\cdot|\) is the norm with respect to \(g^{ij}\) and \(\text{Riem}[g]\) is the Riemann tensor of \(g_{ij}\). Then the result follows by inspection of these equations. \(\square\)

\(^4\) That is, since \(|\nabla u|^2\) tends to zero at infinity, it must achieve a maximum in \(M\). At a maximum point, the left-hand side of (2.10) would be \(\leq 0\) but the right-hand side would be \(\geq 0\), so both sides must be zero. This forces \(|\nabla u|^2 = 0\) to be the maximum, so \(|\nabla u|^2\) vanishes pointwise. \(u = \text{const}\).
In the remainder of this section, we briefly discuss a case left open by lemma 2.6. Consider \( \kappa = 0 \) and \( |\nabla u| \to c > 0 \) or \( |\nabla u| \to \infty \). Assume further that there is a unit speed, geodesic ray \( \gamma : [0, \infty) \to M \) along which
\[
\dot{\gamma} \cdot \nabla u = |\nabla u| \cos \theta_0 \geq \epsilon \cos \theta_0 =: C > 0.
\]
(2.14)
This condition always holds for \( (M, g_{ij}) \) asymptotically flat with \( u \sim \text{const}/r \) at large \( r \). In such a case, we can foliate the asymptotic region by convex level sets of \( u \) and choose \( \gamma \) such that \( \dot{\gamma} \) makes an angle with \( \nabla u \) that is never greater than some \( \theta_0 < \pi/2 \).

Lemma 2.9. There are no solutions of (1.8) for which (2.14) holds along a ray \( \gamma : [0, \infty) \to M \), and thus no asymptotically flat solitons (2.1) with \( |\nabla u| \to c \in (0, \infty) \).

The argument relies on properties of (1.8) but not on the soliton interpretation, and is a simple variation on the textbook proof of Myers’ diameter estimate using Synge’s formula. Contrary to Myers’ estimate, (1.8) implies that only one eigenvalue of Ricci is nonzero, but this is sufficient in the given circumstances.

Proof. In the standard way construct an orthonormal basis \( \{\dot{\gamma}(t), e_{(i)}(t) \mid i = 2, \ldots, n\} \) along \( \gamma \) and use it to define \( (n-1) \) orthogonal variation vector fields that vanish at \( \gamma \)’s endpoints:
\[
v_{(i)} := f(t)e_{(i)} = \sin \left( \frac{\pi t}{b-a} \right) e_{(i)}.
\]
(2.15)
Synge’s formula [22] for the second variation of \( \gamma \)’s energy along \( v_{(i)} \) reads
\[
E_{(i)}'' = \int_a^b [g(\nabla_{\dot{\gamma}} v_{(i)}, \nabla_{\dot{\gamma}} v_{(i)}) - g(R(v_{(i)}, \dot{\gamma})\dot{\gamma}, v_{(i)})] \, dt.
\]
(2.16)
In (2.16), \( i \) labels distinct vector fields and so there is no sum over it, but if we do sum over \( i \in [2, \ldots, n] \) we obtain
\[
\sum_{i=2}^n E_{(i)}'' = \int_a^b [(n-1)f^2(t) - f^2(t)R_{ij}\dot{\gamma}^i\dot{\gamma}^j] \, dt
\]
\[
= \int_a^b [(n-1)f^2(t) - k^2_n f^2(t)(\dot{\gamma}^i\nabla_i u)^2] \, dt
\]
\[
\leq \int_a^b \left[ (n-1)\pi^2 \cos^2 \left( \frac{\pi t}{b-a} \right) - C^2 k^2_n \sin^2 \left( \frac{\pi t}{b-a} \right) \right] \, dt
\]
\[
= \frac{(n-1)\pi^2}{2(b-a)} - \frac{1}{2} C^2 k^2_n (b-a),
\]
(2.17)
where we have used (1.8), (2.15), and the condition on \( \gamma \cdot \nabla u \). For \( (b-a) \) large enough, the right-hand side of (2.17) is less than zero, and then there is an \( i \) such that \( E_{(i)}'' < 0 \). By standard results, there is then a shorter geodesic than \( \gamma \); i.e., \( \gamma : [a, b] \to M \) cannot be a ray. This is a contradiction. \( \square \)

5 A ray minimizes the arc length between any two of its points.
3. Examples

3.1. Introduction

In this section, we give examples of solitons constructed from solutions of the Einstein-free scalar field system.

Perhaps one’s first thought is to ask what soliton (or solitons) arises in this manner from the Schwarzschild metric. Therefore we develop in this subsection the specific form of equation (1.8) as it applies to metrics \( g_{ij} \) of the form

\[
d_s^2 = g_{ij} \, dx^i \, dx^j = dr^2 + f^2(r) \, g_{ab} \, dx^a \, dx^b = \, dr^2 + f^2(r) \, d\Omega^2_k,
\]

(3.1)

where \( d\Omega^2_k \) is now an Einstein metric with scalar curvature \( k \) normalized to \(-1, 0, \) or \(1\). This form includes the Schwarzschild \( SO(n-1) \)-symmetric case but is somewhat more general, allowing us to apply the equations developed here to most other, but not entirely all, solitons that we discuss in subsequent subsections.

Taking \( u = u(r) \), then the integrability condition (1.9) shows that

\[
u'(r) = \frac{A}{(f(r))^{n-1}}, \quad A = \text{const}.
\]

(3.2)

We take \( A \neq 0 \) since the \( u = \text{const} \) case was discussed in remark 2.2. Note that \( u(r) \) has no critical point in the interior of the domain of \( r \). This is consistent with the maximum principle.

The Ricci curvature of the metric \( g_{ij} \) is

\[
R_{ij} \, dx^i \, dx^j = -(n-1) \frac{f''(r)}{f(r)} \, dr^2 + [(n-2)(k-f^2(r)) - f(r)f''(r)] \, d\Omega^2_k.
\]

(3.3)

Then equation (1.8) leads to the two equations

\[
\frac{f''(r)}{f(r)} = -\frac{1}{(n-1)} \left[ k^2 \frac{u^2}{(r)^n} + \kappa \right] = -\frac{A^2}{(n-2)f^{(n-1)}(r)} - \frac{\kappa}{n-1}, \quad (3.4)
\]

\[
\frac{f''(r)}{f(r)} = (n-2) \left( k - f^2(r) \right) - \kappa, \quad (3.5)
\]

where we have used (3.2). Eliminating \( f''(r) \), we obtain

\[
f^2(r) + \frac{\kappa}{(n-1)} f^2(r) = \frac{k^2A^2}{(n-1)(n-2)f^{(n-4)}(r)} = k. \quad (3.6)
\]

Since (3.6) may also be obtained by multiplying (3.4) by \( f(r)f'(r) \) and integrating, the problem reduces to solving (3.6) and discarding any \( f'(r) = 0 \) solutions that do not solve (3.4) and (3.5).

Now (3.6) has the familiar form of a unit mass with total energy \( k/2 \) moving in a central potential

\[
\frac{1}{2} \rho^2 + V(\rho) = k/2, \quad (3.7)
\]

\[
V(\rho) = \frac{1}{2} \left( \kappa \rho^2 - \frac{A^2}{(n-2)^2 \rho^{2n-4}} \right), \quad (3.8)
\]

where \( \rho := f(r), \dot{\rho} := f'(r) \), and we’ve used \( k^2 = \frac{n-1}{n-2} \).
Figure 1. Plot of representative cases of the potential of equation (3.8) in $n = 3$ dimensions.

3.2. $n = 3$ dimensions

For simplicity of the presentation, we will fix the dimension and consider in this subsection only the $n = 3$ case. Then

$$V(\rho) = \frac{1}{2} \left( \frac{\kappa \rho^2}{2} - \frac{A^2}{\rho^2} \right). \quad (3.9)$$

There are $\rho = \rho_0 \equiv \text{const} > 0$ solutions of (3.7) and (3.9) but some of these are spurious and do not solve (3.4) and (3.5). There is, however, a genuine solution with $\rho = \rho_0 = \sqrt{k/\kappa}$ and $\kappa < 0, k < 0$. If we take $\kappa = k = -1$ the soliton metric is then $e^{2\rho} \, dt^2 + dr^2 + d\Omega_1^2$ and is an expander.

The $\dot{\rho} \neq 0$ solutions climb the $V(\rho)$ potential well until they reach one of the horizontal lines $k = 0, \pm 1$, then turn around and go to $V(\rho) \to -\infty$ as either $\rho \to \infty$ or $\rho \to 0$. In the $\kappa < 0$ case, it is possible to pass between $\rho \to 0$ and $\rho \to \infty$ without encountering a turning point (see the bottom curve of the figure).

Integrating (3.9) for small $\rho$, we see that the distance coordinate $r$ is bounded as $\rho \to 0$, and $\dot{\rho} = f'(r)$ blows up there. The sectional curvature in planes perpendicular to $\frac{\partial}{\partial r}$ contains a $(f'(r)/f(r))^2$ term and so also blows up. Thus, these metrics are incomplete unless $\rho$ is bounded away from zero. The only solutions with this property are those that move along curves such as the higher of the two $\kappa < 0$ curves in the figure 1. If a solution starts out on this curve to the right of the rightmost intersection point of this curve with the $k = -1$ line, then the solution will move inward to the intersection point, then turn and role back down the $\kappa < 0$ curve, escaping to infinity. Such solutions are complete. A concrete example is provided in section 3.2.2, but first we address the question posed at the beginning of the section.
3.2.1. Schwarzschild soliton. The four-dimensional (thus $n = 3$) Schwarzschild metric gives rise to a $\kappa = 0$ local soliton. Writing the Schwarzschild metric in the form of (3.1)

$$ds^2 = - \left( 1 - \frac{2m}{r} \right) dt^2 + \frac{dr^2}{\left( 1 - \frac{2m}{r} \right)} + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2)$$

$$ds^2 = -e^{2u} dr^2 + e^{-2u} ds^2,$$  \hspace{1cm} (3.10)

$$ds^2 = dr^2 + r^2 \left( 1 - \frac{2m}{r} \right)(d\theta^2 + \sin^2 \theta \, d\phi^2),$$  \hspace{1cm} (3.11)

we can read off $f$ and $u$. In particular,

$$u = \frac{1}{2} \log \left( 1 - \frac{2m}{r} \right)$$  \hspace{1cm} (3.12)

$$f(r) = \sqrt{r(r - 2m)}.$$  \hspace{1cm} (3.13)

From (2.1), the corresponding soliton is

$$ds^2 = \left( 1 - \frac{2m}{r} \right)^{\sqrt{2}} dr^2 + dr^2 + r(r - 2m)(d\theta^2 + \sin^2 \theta \, d\phi^2).$$  \hspace{1cm} (3.14)

Since $\kappa = 0$, we know that the soliton metric will be incomplete (e.g., by remark 2.7). Indeed, from (3.12) and (2.11), the soliton has scalar curvature $-\frac{2m^2}{r^2(r - 2m)}$, which diverges as $r \to 2m$.

Ivey has given examples of complete steady solitons in his PhD thesis [14]. Our solitons are less general than Ivey’s, since in our case the same function $u$ appears in both the potential for the vector field $X$ in the soliton equation (1.5) and the norm of the Killing field $\partial/\partial t$. The only simultaneous solution of Ivey’s soliton equations and ours in dimension 4 is the local soliton (3.14).\(^6\)

If we Wick rotate both $t \mapsto it$ and $\phi \mapsto i\phi$ in (3.10), we are led to a different choice of $u$ and $f(r)$ and ultimately to the soliton

$$(r \sin \theta)^2 \left[ (r \sin \theta)^{\sqrt{2}} d\phi^2 + \left( 1 - \frac{2m}{r} \right) dr^2 + \frac{dr^2}{\left( 1 - \frac{2m}{r} \right)} + r^2 d\theta^2 \right],$$  \hspace{1cm} (3.15)

which of course is also incomplete.

3.2.2. A complete soliton. To obtain a complete soliton we must choose $\kappa < 0$ and $k = -1$. For illustrative purposes, we also choose the definite values $\kappa = -1$ and $A = 1/\sqrt{2}$. Then we can solve (3.6) to obtain

$$f^2(r) = 1 + C \, e^{\frac{1}{\sqrt{2}}}.$$  \hspace{1cm} (3.16)

Choose the plus sign and $C = 1$ for definiteness. Then we get (using (3.2))

$$ds^2 = g_{ij} \, dx^i \, dx^j = dr^2 + (1 + e^{\sqrt{2}r}) \, d\Omega_{k=1}^2,$$  \hspace{1cm} (3.17)

$$\sqrt{2} u(r) = r - \frac{1}{\sqrt{2}} \log(1 + e^{\sqrt{2}r}).$$  \hspace{1cm} (3.18)

The corresponding soliton is obtained by inserting these expressions into (2.1) to obtain

$$ds^2 = \frac{e^{\sqrt{2}}}{(1 + e^{\sqrt{2}r})^{\sqrt{2}}} dr^2 + dr^2 + (1 + e^{\sqrt{2}r}) \, d\Omega_{k=1}^2.$$  \hspace{1cm} (3.19)

\(^6\) One must take the $k = 1$ case of Ivey’s equations (cf chapter 5 of his thesis [14]) and the $\kappa = 0$ case of ours.
Note that \( r \) takes all real values. The sectional curvatures are all negative except the sectional curvature in \( t-r \) planes, which is positive iff \( r > -\frac{\log 2}{\sqrt{2}} \).

Since there is a compact hyperbolic 2-manifold for each integer \( g \geq 2 \) and 3g – 3 distinct choices for the hyperbolic metric \( d\Omega_{n-1} \) on each such manifold, (3.19) represents a countably infinite family of solitons. Also, we can, if we wish, choose Lorentzian signature by Wick rotating \( t \rightarrow it \), thereby obtaining a Lorentzian Ricci soliton.

### 3.3. \( n = 4 \) Einstein-scalar solutions

Above we regarded static vacuum metrics in \( (n + 1) \) dimensions as giving rise to a pair \((u, g_{ij})\) solving the static Einstein equations an \( n \)-dimensional metric. A concrete example was given by \( n \)-dimensional Schwarzschild metrics, which produced a three-dimensional \( g_{ij} \), leading back to a four-dimensional (local) soliton metric. But we could also consider the trivial five-dimensional solitons arising from the pair \((u, g) = (0, g^{\text{Sch}})\) comprised of the zero function (or any constant function) and a four-dimensional Schwarzschild metric \( g^{\text{Sch}} \). Now this pair belongs within a family of solutions of the static Einstein equations

\[
\begin{align*}
\text{ds}^2 &= \left(1 - \frac{2m}{r}\right)^\delta \, \text{dr}^2 + \left(1 - \frac{2m}{r}\right)^{-\delta} \, \text{dr}^2 + r^2 \left(1 - \frac{2m}{r}\right)^{1-\delta} \, \text{d}\Omega^2, \\
u &= \frac{1}{2} \sqrt{1-\delta} \log \left(1 - \frac{2m}{r}\right),
\end{align*}
\tag{3.20}
\]

where \( \text{d}\Omega^2 \) is the standard round metric on \( S^{n-2} \) and \( \delta \in [0, 1] \). These metrics are Wick rotated solutions of the four-dimensional Einstein-free scalar field equations \( [9] \). The higher dimensional generalization was obtained in \( [25] \).

The local soliton corresponding to this solution is

\[
\begin{align*}
\left(1 - \frac{2m}{r}\right)^{3\sqrt{\tau - \gamma}/2} \, \text{dr}^2 + \text{ds}^2,
\end{align*}
\tag{3.22}
\]

with \( \text{ds}^2 \) given by (3.20). The volumes (i.e., surface areas) of the \((n - 2)\)-spheres of constant-\( r \) collapse to zero as \( r \searrow 2m \). Simultaneously, sectional curvatures of \( g_{ij} \) diverge, as does the scalar field. By (2.13), the five-dimensional soliton constructed from this metric inherits the divergence and is incomplete (as it must be, since \( \kappa = 0 \)).

As well, there are known conformally flat solutions in four dimensions. One class is due to Penney \( [21] \). Rotated to Euclidean signature, they are given by

\[
\begin{align*}
g_{ij} &= (1 + a_i x^i) \delta_{ij} \tag{3.23} \\
u &= \sqrt{\frac{3}{2}} \log(1 + a_i x^i), \tag{3.24}
\end{align*}
\]

where \( C \) and \( a_i \)'s are arbitrary constants. Sectional curvatures of \( g_{ij} \) diverge on approach to the hyperplane \( a_i x^i = -1 \). G"urses \( [11] \) found another class of such solutions, again in four dimensions. These are

\[\text{Chase} [5] \text{ proved a no hair theorem showing that such singularities occur even in the absence of the SO(3) rotational symmetry. Chase used Lorentzian signature, prior to Wick rotation of } \tau, \text{ and assumed four dimensions, asymptotic flatness, and const/r + O(1/r^2) fall-off for the radial part of the scalar field (and one more power for the gradient). He found that, in the presence of a nonzero scalar field, the Kretschmann scalar cannot remain bounded on the domain of outer communications to the future of an initial surface which may or may not contain a 2-sphere apparent horizon.}\]
Here \( r^2 = \delta_{ij} x^i x^j \). Sectional curvatures diverge on approach to the sphere \( r = \sqrt{k} \). For both these examples, the solitons constructed from them using (2.1) also have divergent sectional curvature (by (2.13)) and are incomplete.

4. Stationary metrics

To close, we consider the more general class of stationary metrics

\[
ds^2 = \pm e^{2u} \left( dt + A_i \, dx^i \right)^2 + g_{ij} \, dx^i \, dx^j ,
\]

where \( \frac{\partial u}{\partial t} = 0 \), \( \frac{\partial A_i}{\partial t} = 0 \), and \( \frac{\partial g_{ij}}{\partial t} = 0 \). The \( \pm \) sign will allow us to consider both signatures simultaneously. The Ricci flow of this metric can be read off from equation (4.9) of [15].

A reasonable approach is to mimic the procedure in the static case, eliminating certain unwanted second derivative terms from the Ricci flow by adding Lie derivative terms arising from the pullback via an evolving diffeomorphism generated by vector field \( -\sqrt{\frac{n-1}{n-2}} \nabla u \) on the ‘base manifold’ \( (M, g) \). As well, to eliminate an unwanted term in the evolution equation for \( A_i \), we perform the ‘gauge transformation’

\[
B_i := A_i - \nabla_i \Lambda,
\]

\[
\frac{\partial \Lambda}{\partial \lambda} = -\sqrt{\frac{n-1}{n-2}} \left( B + \nabla \Lambda \right) \cdot \nabla u ,
\]

where we define

\[
F_{ij}[A] := \nabla_i A_j - \nabla_j A_i ,
\]

with \( \nabla \) being the Levi-Cevită connection of \( g_{ij} \). Of course \( F_{ij}[A] \equiv F_{ij}[B] \equiv F_{ij} \) is gauge invariant. Then we obtain

\[
\frac{\partial u}{\partial \lambda} = \Delta u + \frac{1}{4} \left( n - 2 \right) e^{2u} \sqrt{\frac{n-1}{n-2}} |F|^2 ,
\]

\[
\frac{\partial B_i}{\partial \lambda} = -\nabla^j F_{ij} - 2 \sqrt{\frac{n-1}{n-2}} F_{ij} \nabla^j u ,
\]

\[
\frac{\partial g_{ij}}{\partial \lambda} = -2 R_{ij} + 2 \left( \frac{n-1}{n-2} \right) \nabla_i u \nabla_j u \pm e^{2u} \sqrt{\frac{n-1}{n-2}} g^{kl} F_{ik} F_{lj} .
\]

This system couples a scalar heat flow, a vector Yang–Mills flow, and a generalization of Ricci flow, and can be made parabolic by adding further gauge and diffeomorphism terms. Setting \( F = 0 \), we recover the static Einstein equations (1.8) and (1.9).

By way of comparison, the condition for the stationary metric

\[
ds^2 = -e^{2u} (dt + B_i \, dx^i)^2 + e^{-\frac{2u}{n-2}} g_{ij} \, dx^i \, dx^j
\]

to be Ricci flat is

\[
S_j := \left( \begin{array}{c} S_0 \\ S_i \\ S_{ij} \end{array} \right) = 0 ,
\]

(4.9)
where

\[ S_0 = \Delta u + \frac{1}{4} e^{2\sqrt{\frac{n-1}{n-2}}u} |F|^2, \]  

\[ S_i = -\nabla^j F_{ij} - 2 \left( \frac{n-1}{n-2} \right) F_{ij} \nabla^j u, \]  

\[ S_{ij} = -2 R_{ij} + 2 \left( \frac{n-1}{n-2} \right) \nabla_i u \nabla_j u e^{2\sqrt{\frac{n-1}{n-2}}u} \left[ F_{ik} F_{jk} - \frac{1}{2(n-2)} g_{ij} |F|^2 \right]. \]

Fixed points of the system (4.5)–(4.7) are obtained by setting the time derivatives to zero so that the right-hand sides vanish as well. The resulting equations differ from the system (4.9)–(4.12) in the coefficients preceding some of the \(F\)-terms. One consequence is that the fixed point condition for the system (4.5)–(4.7) leads to an integrability condition

\[ e^{2\sqrt{\frac{n-1}{n-2}}u} |F|^2 = \text{const} \]  

which does not arise for the system (4.9)–(4.12).

If \( F_{ij} \neq 0 \), the fixed point condition for (4.5)–(4.7) does not coincide with the condition that the stationary metric (4.8) be Ricci flat.

There are two alternative strategies. One is to apply a different diffeomorphism and gauge choice than that used to obtain (4.5)–(4.7).\(^8\) Perhaps such a technique may produce fixed points for which \( S = 0 \). However, a second technique is to study the flow

\[ \frac{\partial}{\partial \lambda} \begin{pmatrix} u \\ B_i \\ g_{ij} \end{pmatrix} = S. \]  

(4.13)

This flow has the same well-posedness properties as (4.5)–(4.7) and its fixed points \((u, B_i, g_{ij})\) make the stationary metric (4.8) Ricci flat. It is an open question whether, as we believe, (4.13) is a genuinely new flow, or whether a diffeomorphism can be found to bring (4.13) to the Hamilton–DeTurck form (1.2).

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References

[1] Anderson M T 1997 Comparison Geometry (MSRI Publications vol 30) (Cambridge: Cambridge University Press) p 49
[2] Anderson M T 1999 Geom. Funct. Anal. 9 855
[3] Bartnik R 2002 Proc Int Congress Math (Beijing), vol 2, p 231 (arXiv:math/0304259)
  Yau S-T (ed) 1995 Tsing Hua Lectures on Geometry and Analysis (Cambridge, MA: International Press) pp 5–27
[4] Cao H-D and Zhu X-P 2006 Asian J. Math. 10 169
[5] Chase J E 1970 Commun. Math. Phys. 19 276

\(^8\) Presumably this would be combined with ‘field redefinitions’ in (4.1); e.g., rewriting \( A \) as \( f(u)A \) with some suitably chosen function \( f \).
[6] Chow B and Knopf D 2004 *The Ricci Flow: An Introduction* (Mathematical Surveys and Monographs vol 110) (Providence, RI: American Mathematical Society)

[7] Chow B et al 2007 *The Ricci Flow: Techniques and Applications: Geometric Aspects* (Mathematical Surveys and Monographs vol 135) (Providence, RI: American Mathematical Society)

[8] Chruściel P T 1991 *Commun. Math. Phys.* 137 289

[9] Fisher I Z 1948 Zh. Eksp. Teor. Fiz. 18 636 (arXiv:gr-qc/9911008)

[10] Buchdahl H A 1959 *Phys. Rev.* 115 1325

[11] Janis A I, Newman E T and Winicour J 1968 *Phys. Rev. Lett.* 20 878

[12] Wyman M 1981 *Phys. Rev.* D 24 839

[13] Virbhadra K S 1997 *Int. J. Mod. Phys.* A 12 4831 (arXiv:gr-qc/9701021)

[14] Chruściel P T 1991 *Commun. Math. Phys.* 137 289

[16] Huisken G Private communication

[17] Huisken G and Ilmanen T 2001 *J. Diff. Geom.* 59 353

[18] Ivey T 1992 On solitons of the Ricci flow PhD Thesis Duke University

[19] Ivey T 1994 *Proc. Am. Math. Soc.* 122 241

[20] J-F Li Unpublished notes

[21] J-F Li Unpublished notes

[22] Lichnerowicz A 1955 *Théories relativistes de la gravitation et de l’électromagnétisme* (Masson, Paris) p 138

[23] Morgan J and Tian G 2008 arXiv:0809.4040

[24] Morgan J and Tian G 2007 *Ricci Flow and the Poincaré Conjecture* (Providence, RI: American Mathematical Society) and references therein

[25] Morgan J and Tian G 2007 *Ricci Flow and the Poincaré Conjecture* (Providence, RI: American Mathematical Society) and references therein

[26] Penney R V 1976 *Phys. Rev.* D 14 910

[27] Petersen P 2006 *Riemannian Geometry* 2nd edn (Berlin: Springer) p 159

[28] Sullivan D 1983 *J. Diff. Geom.* 18 723

[29] Tod P 1989 *Class. Quantum Grav.* 6 1159

[30] Xanthopoulos B C and Zannias T 1989 *Phys. Rev.* D 40 2564