GEODESIC LINES ON NEARLY KÄHLER $S^3 \times S^3$

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Abstract. A Nearly Kähler manifold is an almost Hermitian manifold with the weakened Kähler condition, that is, instead of being zero, the covariant derivative of the almost complex structure is skew-symmetric. We give the explicit parametrization of geodesic lines on nearly Kähler $S^3 \times S^3$.

1. Introduction

A nearly Kähler manifold is an almost Hermitian manifold $(M, g, J)$ with the property that the tensor field $\nabla J$ is skew-symmetric: $(\nabla_X J)Y + (\nabla_Y J)X = 0$, for all $X, Y \in TM$, where $\nabla$ is the Levi-Civita connection of the metric $g$. The first example of such manifolds was introduced on $S^6$ by Fukami and Ishihara in [8] and, later, these manifolds have been intensively studied by A. Gray in [9], who generalized the classical holonomy concept by introducing a classification principle for non-integrable special Riemannian geometries and studied the defining differential equations of each class. The structure theorem of Nagy [11, 12] asserts that a strict and complete nearly Kähler manifold (of arbitrary dimension) writes as a Riemannian product of homogeneous nearly Kähler spaces, twistor spaces over quaternionic Kähler manifolds and 6-dimensional nearly Kähler manifolds. Moreover, Butruille has shown in [4] that the only nearly Kähler homogeneous manifolds of dimension 6 are the compact spaces $S^6$, $S^3 \times S^3$, $CP^3$ and the flag manifold $SU(3)/U(1) \times U(1)$ (where the last three are not endowed with the standard metric). Furthermore, Foscolo et Haskins found in [6] the first two complete non-homogeneous nearly Kähler structures on $S^6$ and on $S^3 \times S^3$. This way they addressed an important problem in the field, namely the absence of any complete non-homogeneous examples.

More recently interest in nearly Kähler manifolds increased because these manifolds are examples of geometries with torsion and therefore they have applications in mathematical physics [1]. Moreover, 6-dimensional nearly Kähler manifolds are Einstein and are related to the existence of a Killing spinor [2], which inspires their further investigation. In this paper we continue the research of the nearly Kähler manifold $S^3 \times S^3$ (see the previous work in [3], [5], [10], for instance) by studying its geodesics. In Section 2 we recall the basic properties of the nearly Kähler $S^3 \times S^3$ and in Section 3 our main results are stated and proved. Theorem 3.1. presents the parametrization of geodesic lines on $S^3 \times S^3$ and Proposition 3.1. presents their features.

2. Preliminaries

In this section we recall the homogeneous nearly Kähler structure of $S^3 \times S^3$. For more details we refer the reader to [2]. By the natural identification $T_{(p, q)}(S^3 \times S^3) \cong T_pS^3 \oplus T_qS^3$, we write a tangent vector at $(p, q)$ in $S^3 \times S^3$ as $Z(p, q) = (U(p, q), V(p, q))$ or simply $Z = (U, V)$. Regarding the 3-sphere in $\mathbb{R}^4$ as the set of all unit quaternions in $\mathbb{H}$ and using the notations $i, j, k$ to denote
the imaginary units of \( \mathbb{H} \), the vector fields defined by
\[
\begin{align*}
\tilde{E}_1(p, q) &= (pi, 0), & \tilde{F}_1(p, q) &= (0, qi), \\
\tilde{E}_2(p, q) &= (pj, 0), & \tilde{F}_2(p, q) &= (0, qj), \\
\tilde{E}_3(p, q) &= -(pk, 0), & \tilde{F}_3(p, q) &= (0, qk),
\end{align*}
\]
(2.1)
are mutually orthogonal with respect to the usual Euclidean product metric on \( S^3 \times S^3 \). The Lie brackets are \([\tilde{E}_i, \tilde{E}_j] = -2\varepsilon_{ijk} \tilde{E}_k, [\tilde{F}_i, \tilde{F}_j] = -2\varepsilon_{ijk} \tilde{F}_k\) and \([\tilde{E}_i, \tilde{F}_j] = 0\), where
\[
\varepsilon_{ijk} = \begin{cases} 
1, & \text{if } (ijk) \text{ is an even permutation of } (123), \\
-1, & \text{if } (ijk) \text{ is an odd permutation of } (123), \\
0, & \text{otherwise.}
\end{cases}
\]

The almost complex structure \( J \) on the nearly Kähler \( S^3 \times S^3 \) is defined by
\[
J(U, V)_{(p, q)} = \frac{1}{\sqrt{3}}(2pq^{-1}V - U, -2qp^{-1}U + V)_{(p, q)},
\]
for \((U, V) \in T_{(p, q)}(S^3 \times S^3)\) and the Hermitian metric associated with the usual Euclidean product metric on \( S^3 \times S^3 \) on \( S^3 \times S^3 \) is given by
\[
g(Z, Z') = \frac{1}{2}\left((Z, Z') + (JZ, JZ')\right) = \frac{2}{3}\left((U, U') + (V, V')\right) - \frac{2}{3}(p^{-1}U, q^{-1}V) - (p^{-1}U', q^{-1}V'),
\]
(2.3)
where \( Z = (U, V), Z' = (U', V') \), in the first line \( \langle \cdot, \cdot \rangle \) stands for the usual Euclidean product metric on \( S^3 \times S^3 \), while in the second line \( \langle \cdot, \cdot \rangle \) stands for the usual Euclidean metric on \( S^3 \). By definition, the almost complex structure \( J \) is compatible with the metric \( g \).

Using the Koszul formula (see Lemma 2.1, in [2] for more details), one finds that the Levi-Civita connection \( \nabla \) on \( S^3 \times S^3 \) with respect to the metric \( g \) is given by
\[
\nabla_{\tilde{E}_i} \tilde{E}_j = -\varepsilon_{ijk} \tilde{E}_k, \quad \nabla_{\tilde{E}_i} \tilde{F}_j = \frac{\varepsilon_{ijk}}{3}(\tilde{E}_k - \tilde{F}_k),
\]
\[
\nabla_{\tilde{F}_i} \tilde{E}_j = \varepsilon_{ijk}(\tilde{E}_k - \tilde{F}_k), \quad \nabla_{\tilde{F}_i} \tilde{F}_j = -\varepsilon_{ijk} \tilde{F}_k.
\]
Consequently, computing
\[
(\nabla_{\tilde{E}_i} J) \tilde{E}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(\tilde{E}_k + 2\tilde{F}_k), \quad (\nabla_{\tilde{E}_i} J) \tilde{F}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(\tilde{E}_k - \tilde{F}_k),
\]
\[
(\nabla_{\tilde{F}_i} J) \tilde{E}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(\tilde{E}_k - \tilde{F}_k), \quad (\nabla_{\tilde{F}_i} J) \tilde{F}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(2\tilde{E}_k + \tilde{F}_k),
\]
(2.4)
we conclude that the (1,2)-tensor field \( G = \nabla J \) is skew-symmetric and therefore \( S^3 \times S^3 \), equipped with \( g \) and \( J \), becomes a nearly Kähler manifold. Moreover \( G \) satisfies
\[
G(X, JY) = -JG(X, Y), \quad g(G(X, Y), Z) + g(G(X, Z), Y) = 0,
\]
(2.5)
for any vector fields \( X, Y, Z \) tangent to \( S^3 \times S^3 \).

The almost product structure \( P \) introduced in [2] is defined as
\[
PZ = (pq^{-1}V, qp^{-1}U), \quad Z = (U, V) \in T_{(p, q)}(S^3 \times S^3)
\]
(2.6)
and it has the following properties:
\[
P^2 = Id \quad (P \text{ is involutive}),
\]
\[
P J = -JP \quad (P \text{ and } J \text{ anti-commute}),
\]
\[
g(PZ, PZ') = g(Z, Z') \quad (P \text{ is compatible with } g),
\]
\[
g(PZ, Z') = g(Z, PZ') \quad (P \text{ is symmetric}).
\]
Moreover, the almost product structure \( P \) can be expressed in terms of the usual product structure \( QZ = Q(U, V) = (-U, V) \) and vice versa:
\[
QZ = \frac{1}{\sqrt{3}}(2PQJZ - JZ), \quad PZ = \frac{1}{2}(Z - \sqrt{3}QJZ).
\]
Next, we recall the relation between the Levi-Civita connections \( \tilde{\nabla} \) of \( g \) and \( \nabla^E \) of the Euclidean product metric \( \langle \cdot, \cdot \rangle \).
Lemma 2.1. \[5\] The relation between the nearly Kähler connection $\hat{\nabla}$ and the Euclidean connection $\nabla^E$ is

\begin{equation}
\nabla^E_X Y = \hat{\nabla}_X Y + \frac{1}{2}(JG(X, PY) + JG(Y, PX)).
\end{equation}

We also recall here a useful formula from \([5]\), decomposing $DX Y$ along the tangent and the normal directions:

\begin{equation}
DX Y = \nabla^E_X Y + \frac{1}{2}(DX Y, (p, q))(p, q) + \frac{1}{2}(DX Y, (-p, q))(-p, q),
\end{equation}

where $D$ is the Euclidean connection on $\mathbb{R}^8$ and $X, Y$ are tangent vector fields on $S^3 \times S^3$.

3. Geodesic lines on $S^3 \times S^3$

In order to study and classify certain types of submanifolds of $S^3 \times S^3$, it is interesting and useful to know how its geodesic lines look like. Since for unitary quaternions $a, b$ and $c$, the map $f : S^3 \times S^3 \to S^3 \times S^3$ given by $(p, q) \mapsto (ap^{-1}, bqc^{-1})$ preserves the usual metric $\langle \cdot, \cdot \rangle$ and the almost complex structure $J$, it is an isometry of $(S^3 \times S^3, g)$ (cf. \([5]\) and remark after Lemma 2.2 in \([13]\)). Therefore, it is enough to obtain the parametrization of geodesic lines through the point $(1, 1)$, as the geodesic lines through the point $(a, b)$ are given by $\tilde{\gamma}(t) = (ax(t), by(t))$. We prove the following.

Theorem 3.1. The geodesic lines on $S^3 \times S^3$ through the point $(1, 1)$ have the following parametrization:

1. $\gamma(t) = (\cos(at) + \sin(at)i, \cos(at) - \sin(at)i)$, $a \in \mathbb{R} \setminus \{0\}$;
2. $\gamma(t) = (\cos(at) + \sin(at)i, \cos(\alpha t) + \sin(\alpha t)i)$, where $c_1 \in \text{Im}H \setminus \{0\}, d_1 \in \mathbb{R}, a = \frac{1 + d_1}{2}||c_1||$,
3. $\tilde{\alpha} = \frac{1 - d_1}{2}||c_1||$;

\begin{align*}
\gamma(t) &= \left(\left(\frac{1}{1 + \varphi^2} \cos(At) + \frac{\varphi^2}{1 + \varphi^2} \cos(Bt)\right) i + \left(\frac{\varphi}{1 + \varphi^2} \sin(At) - \frac{\varphi}{1 + \varphi^2} \sin(Bt)\right) j - \left(\frac{\varphi}{1 + \varphi^2} \cos(\tilde{A}t) + \frac{\varphi^2}{1 + \varphi^2} \cos(\tilde{B}t)\right) k\right) + \\
&\quad \left(\left(\frac{1}{1 + \varphi^2} \sin(\tilde{A}t) - \frac{\varphi}{1 + \varphi^2} \sin(\tilde{B}t)\right) j - \left(\frac{\varphi}{1 + \varphi^2} \cos(\tilde{A}t) + \frac{\varphi^2}{1 + \varphi^2} \cos(\tilde{B}t)\right) k\right),
\end{align*}

where $c_1, c_2 \in \text{Im}H \setminus \{0\}, d_1 \in \mathbb{R}, a = \frac{1 + d_1}{2}||c_1||, b = \frac{1}{2}||c_2||, c = \frac{2}{3}||c_1||$, $\tilde{a} = \frac{1 - d_1}{2}||c_1||$, $\tilde{b} = -\frac{1}{2}||c_2||$, $\tilde{c} = c$,

\begin{align*}
A &= c + \sqrt{(2a - c)^2 + 4b^2}, & B &= c - \sqrt{(2a - c)^2 + 4b^2}, \\
\tilde{A} &= c + \sqrt{(2\tilde{a} - \tilde{c})^2 + 4\tilde{b}^2}, & \tilde{B} &= c - \sqrt{(2\tilde{a} - \tilde{c})^2 + 4\tilde{b}^2}, \\
\varphi &= \frac{c - 2a + \sqrt{(c - 2a)^2 + 4b^2}}{2b}, & \tilde{\varphi} &= \frac{c - 2\tilde{a} + \sqrt{(c - 2\tilde{a})^2 + 4\tilde{b}^2}}{2b}.
\end{align*}
Remark 1. With the substitution $\varphi = \tan \theta$, $\theta \in \mathbb{R}$, (analogously for $\tilde{\theta}$) $\frac{1}{1+\varphi^2} = \cos^2 \theta$, $\frac{\varphi}{1+\varphi^2} = \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$, the parametrization in the case (3) becomes

$$\gamma(t) = \left( \cos^2 \theta \cos(At) + \sin^2 \theta \cos(Bt) \right) i + \sin \theta \cos \theta (\sin(At) - \sin(Bt)) j - \sin \theta \cos \theta (- \cos(At) + \cos(Bt)) k,$$

from (3.7) it follows \( (\alpha', \alpha'' + \beta' + y\beta') \).

Recall that for imaginary quaternions $\alpha_1, \alpha_2$ one has $\alpha_1 \cdot \alpha_2 = -<\alpha_1, \alpha_2> + \alpha_1 \times \alpha_2$. Thus, $\alpha \cdot \alpha = -\|\alpha\|^2$ and we may write

$$\gamma'' = (x(\alpha' - \|\alpha\|^2), y(\beta' - \|\beta\|^2)) = (x\alpha', y\beta') - \|\alpha\|^2(x, 0) - \|\beta\|^2(0, 0).$$

Identifying the tangent and normal parts in the previous relation, using Lemma 2.1, it follows

$$\nabla E_\gamma \gamma' = (x\alpha', y\beta').$$

and Lemma 2.1 yields

$$\nabla E_{\gamma'} = \nabla \gamma' + JG(\gamma', P\gamma').$$

An easy computation establishes the additional formula which allows us to evaluate $G$ for any tangent vector fields:

$$G(X, Y) = \frac{2}{3\sqrt{3}} \left( p(\beta \times \gamma + \alpha \times \delta + \alpha \times \gamma - 2\beta \times \delta), q(-\alpha \times \delta - \beta \times \gamma + 2\alpha \times \gamma - \beta \times \delta) \right),$$

for $X = (p\alpha, q\beta), Y = (p\gamma, q\delta) \in T_{(p, \gamma)}\mathbb{S}^3 \times \mathbb{S}^3, \alpha, \beta, \gamma, \delta \in \text{Im} \mathbb{H}$. As $\gamma' = (x\alpha, y\beta), P\gamma' = (x\beta, y\alpha)$, relation (3.3) implies $G(\gamma', P\gamma') = \frac{2}{\sqrt{3}}(x(\alpha \times \beta), y(\alpha \times \beta))$, which then gives

$$JG(\gamma', P\gamma') = \frac{2}{3}(x(\alpha \times \beta), -y(\alpha \times \beta)).$$

Finally, as $\gamma$ is a geodesic, using (3.4) and (3.6), we compute

$$\gamma'(t) = \frac{2}{3} \alpha(t) \times \beta(t), \quad \gamma''(t) = \frac{2}{3} \alpha(t) \times \beta(t).$$

From (3.7) it follows $(\alpha(t) + \beta(t))' = 0$ and therefore there exists $c_1 \in \text{Im} \mathbb{H}$ such that

$$\alpha(t) + \beta(t) = c_1.$$

Using (3.7) and (3.8) it follows

$$(\alpha(t) - \beta(t))' = -\frac{2}{3} c_1 \times (\alpha(t) - \beta(t)).$$

If $c_1 = 0$, using (3.8) and (3.9), then $\alpha$ and $\beta$ are constants. Without loss of generality, we can assume that they are imaginary quaternions collinear with $i$:

$$\alpha = -\beta = ai, \quad a \in \mathbb{R} \setminus \{0\}.$$
If $c_1 \neq 0$, since $< \alpha(t) - \beta(t), c_1 > = < \alpha(t) - \beta(t), \alpha(t) + \beta(t) >$ and $c_1$ is a constant vector, we have $< \alpha(t) - \beta(t), c_1 > = -\frac{2}{7}c_1 \times (\alpha(t) - \beta(t)), c_1 > = 0$, namely, $< \alpha(t) - \beta(t), c_1 > = d_1 \in \mathbb{R}$. Let us denote by $\varepsilon$ the part of $\alpha - \beta$ which is orthogonal to $c_1$:

\begin{equation}
\varepsilon(t) = \alpha(t) - \beta(t) - \frac{< \alpha(t) - \beta(t), c_1 >}{\|c_1\|^2} c_1
\end{equation}

and compute

\begin{equation}
\varepsilon'(t) = -\frac{2}{7}c_1 \times \varepsilon(t),
\end{equation}

\begin{equation}
\varepsilon''(t) = -\frac{2}{7}\|c_1\|^2 c_1 \varepsilon(t).
\end{equation}

Solving (3.13) gives

\begin{equation}
\varepsilon(t) = \cos \left(\frac{2}{3}\|c_1\|t\right) c_2 + \sin \left(\frac{2}{3}\|c_1\|t\right) c_3,
\end{equation}

for $c_2, c_3 \in Im\mathbb{H}$, $c_2, c_3 \perp c_1$. Having in mind (3.12), we conclude that

$$\|c_1\|c_2 = c_1 \times c_3, \quad \|c_1\|c_3 = -c_1 \times c_2.$$ 

Therefore

\begin{equation}
c_3 = -\frac{c_1}{\|c_1\|} \times c_2
\end{equation}

and, consequently, $c_3 \perp c_1, c_2$.

For $c_2 = 0$ it follows that $c_3 = 0, \varepsilon = 0, \alpha - \beta = d_1 c_1$ $(d_1 = \frac{d_1}{\|c_1\|^2})$, $\alpha + \beta = c_1$ and therefore

$$\alpha = \frac{1 + d_1}{2} c_1, \quad \beta = \frac{1 - d_1}{2} c_1.$$ 

We can assume that $c_1$ is collinear with $i$ and get

\begin{equation}
\alpha = ai, \quad \beta = \bar{a}i,
\end{equation}

where $a = \frac{\|c_1\|}{\|d_1\|}, \bar{a} = \frac{-\|d_1\|}{\|c_1\|}$.

For $c_2 \neq 0$, the vectors $\frac{c_1}{\|c_1\|}, \frac{c_2}{\|c_2\|}, -\frac{c_3}{\|c_3\|}$ form an orthonormal basis of $Im\mathbb{H}$. Since it is always possible to find a unit quaternion $h \in S^3$ such that

$$h^{-1} \frac{c_1}{\|c_1\|} h = i, \quad h^{-1} \frac{c_2}{\|c_2\|} h = j, \quad h^{-1} \frac{c_3}{\|c_3\|} h = -k,$$

we may assume to be working with the basis $\{i, j, -k\}$. Using (3.11) and (3.14) we obtain

\begin{equation}
\alpha(t) + \beta(t) = c_1 = \|c_1\| h i h^{-1},
\end{equation}

\begin{equation}
\alpha(t) - \beta(t) = d_1 \|c_1\| h i h^{-1} + \|c_2\| \cos \left(\frac{2}{3}\|c_1\|t\right) h j h^{-1} - \|c_2\| \sin \left(\frac{2}{3}\|c_1\|t\right) h k h^{-1}.
\end{equation}

Since $\|c_3\| = \|c_2\|$ from (3.15) and $d_1 = \frac{d_1}{\|c_1\|^2} \in \mathbb{R}$, it gives

\begin{equation}
\alpha(t) = \frac{1 + d_1}{2} \|c_1\| hi h^{-1} + \frac{1}{2} \|c_2\| \cos \left(\frac{2}{3}\|c_1\|t\right) hj h^{-1}
\end{equation}

\begin{equation}
- \frac{1}{2} \|c_2\| \sin \left(\frac{2}{3}\|c_1\|t\right) hk h^{-1},
\end{equation}

\begin{equation}
\beta(t) = \frac{1 - d_1}{2} \|c_1\| hi h^{-1} - \frac{1}{2} \|c_2\| \cos \left(\frac{2}{3}\|c_1\|t\right) hj h^{-1} + \frac{1}{2} \|c_2\| \sin \left(\frac{2}{3}\|c_1\|t\right) hk h^{-1},
\end{equation}

where $h \in S^3, c_1, c_2 \in Im\mathbb{H} \setminus \{0\}, d_1 \in \mathbb{R}$.
Remark 2. Notice that we had the freedom to make a rotation of the basis, in the following sense. The map \((x, y) \to (hx^{-1}, hy^{-1})\) is an isometry (rotation) of \(S^3 \times S^3\) and if \((1, 1) \in \gamma = (x, y)\) then also \((1, 1) \in \tilde{\gamma} = (\tilde{x}, \tilde{y}) = (hx^{-1}, hy^{-1})\). Moreover, \(\tilde{\gamma}\) stays tangent to \(S^3 \times S^3\) since, for \(\alpha, \beta : \mathbb{R} \to Im\mathbb{H}\) and using \(3.21\), we have
\[
\beta' = (hx'h^{-1}, hy'h^{-1}) = (hxah^{-1}, hy\beta h^{-1})
\]
for \(\tilde{\alpha} = hah^{-1}, \tilde{\beta} = h\beta h^{-1}, \tilde{\alpha}, \tilde{\beta} : \mathbb{R} \to Im\mathbb{H}\). Therefore, it is enough to solve
\[
(x'(t), y'(t)) = (x(t)\alpha(t), y(t)\beta(t))
\]
for
\[
\alpha(t) = \frac{1 + di}{2}||c_1||i + \frac{1}{2}||c_2|| \cos \left(\frac{\pi}{2}||c_1||t\right) j - \frac{1}{2}||c_2|| \sin \left(\frac{\pi}{2}||c_1||t\right) k, \\
\beta(t) = \frac{1 - di}{2}||c_1||i + \frac{1}{2}||c_2|| \cos \left(\frac{\pi}{2}||c_1||t\right) j + \frac{1}{2}||c_2|| \sin \left(\frac{\pi}{2}||c_1||t\right) k.
\]
In all the cases considered, we have to solve the system of differential equations
\[
\alpha(t) = ai + bj \cos(ct)j - b \sin(ct)k, \\
\beta(t) = \tilde{\alpha}i + \tilde{\beta}j \cos(\tilde{c}t)j - \tilde{b} \sin(\tilde{c}t)k,
\]
which reduces to solving an equation of the form
\[
f'(t) = f(t)(ai + bj \cos(ct) - bk \sin(ct)),
\]
with the following constants:
1. if \(c_1 = 0\), then \(\tilde{a} = -a \neq 0, b = \tilde{b} = c = \tilde{c} = 0\);
2. if \(c_1 \neq 0, c_2 = 0\), then \(a = \frac{1 + di}{2}||c_1||, \tilde{a} = \frac{1 - di}{2}||c_1||, b = \tilde{b} = c = \tilde{c} = 0\);
3. if \(c_1 \neq 0, c_2 \neq 0\), then \(a = \frac{1 + di}{2}||c_1||, b = \frac{1}{2}||c_2||, c = \frac{2}{3}||c_1||, \tilde{a} = \frac{1 - di}{2}||c_1||, \tilde{b} = -\frac{1}{2}||c_2||, \tilde{c} = \tilde{c} = c\).

Cases (1) and (2) lead to the same differential equation of the form \(f'(t) = f(t) \cdot ai\), which has an obvious solution which satisfies \(f(0) = 1\):
\[
f(t) = \cos(at) + i \sin(at).
\]
The geodesics in these two cases have the parametrization
\[
\gamma(t) = (\cos(at) + i \sin(at))i, \cos(at) + i \sin(at))i,
\]
where in case (1) we put \(a = -a \neq 0\) (so \(a + \tilde{a} = 0\)), and in case (2) \(a + \tilde{a} = ||c_1|| \neq 0\). In the third case, we write explicitly
\[
f(t) = f_1(t) + if_2(t) + jjf_3(t) + kf_4(t) =: g_1(t) + jg_2(t),
\]
for \(f_i\) real functions, where \(l = 1, 4\). Then the equation \(3.24\) becomes
\[
g_1' + jg_2' = (g_1 + jg_2)(ai + bj e^{ict})
\]
and we get the following system of differential equations
\[
g_1'(t) = g_1 ai - \overline{g_2} be^{ict}, \\
g_2'(t) = g_2 ai + \overline{g_1} be^{ict}.
\]
When we differentiate one of these equations and combine it with the other, we get the second order linear equations
\[
g_1'' - cig_1 + (a^2 + b^2 - ac)g_1 = 0, \\
g_2'' - cig_2 + (a^2 + b^2 - ac)g_2 = 0.
\]
The characteristic equation is
\[
\lambda^2 - c\lambda + (a^2 + b^2 - ac) = 0,
\]
which has the solutions \(\lambda_1 = Ai, \lambda_2 = Bi\), where
\[
A = \frac{c + \sqrt{(2a - c)^2 + 4b^2}}{2}, \quad B = \frac{c - \sqrt{(2a - c)^2 + 4b^2}}{2}.
\]
We then find the general solutions for (3.26) as

\[ (3.29) \]

\[
g_1(t) = (\mu_1 + i\nu_1)e^{i\gamma t} + (\nu_1 + i\mu_1)e^{iBt},
\]

\[
g_2(t) = (\eta_1 + i\xi_1)e^{i\gamma t} + (\xi_1 + i\eta_1)e^{iBt},
\]

where \( \mu_1, \nu_1, \eta_1, \xi_1 \in \mathbb{R}, l = 1, 2. \) Since \( A \neq B, \) the functions \( \sin(At), \cos(At), \sin(Bt), \cos(Bt) \) are linearly independent. After substitution of (3.29) in (3.25) we get that the following hold

\[
(A - a)\mu_1 = b\xi_2, \\
(B - a)\nu_1 = b\eta_2,
\]

\[
(A - a)\mu_2 = b\xi_1, \\
(B - a)\nu_2 = b\eta_1,
\]

\[
(A - a)\eta_1 = -b\nu_2, \\
(B - a)\xi_1 = -b\mu_2,
\]

\[
(A - a)\eta_2 = -b\nu_1, \\
(B - a)\xi_2 = -b\mu_1.
\]

Since

\[
\frac{A - a}{b} = \frac{-b}{B - a} = \frac{c - 2a + \sqrt{(c - 2a)^2 + 4b^2}}{2b} =: \varphi,
\]

we get four relations among coefficients

\[ (3.30) \]

\[
\varphi = \frac{\xi_2}{\mu_1} = \frac{\xi_1}{\mu_2} = -\frac{\nu_1}{\eta_2} = -\frac{\nu_2}{\eta_1}.
\]

Case \( b = 0 \) leads to \( c_2 = 0, \) which has already been taken into consideration.

Hence, the solution for the function \( f \) is

\[ (3.31) \]

\[
f(t) = (\mu_1 \cos(At) - \mu_2 \sin(At) - \varphi \eta_2 \cos(Bt) + \varphi \eta_1 \sin(Bt)) +
\]

\[
(\mu_2 \cos(At) + \mu_1 \sin(At) - \varphi \eta_1 \cos(Bt) - \varphi \eta_2 \sin(Bt))i +
\]

\[
(\eta_1 \cos(At) - \eta_2 \sin(At) + \varphi \mu_2 \cos(Bt) - \varphi \mu_1 \sin(Bt))j -
\]

\[
(\eta_2 \cos(At) + \eta_1 \sin(At) + \varphi \mu_1 \cos(Bt) + \varphi \mu_2 \sin(Bt))k.
\]

Since \( f(t) \) is a curve on the sphere \( \mathbb{S}^3 \) through the point 1, we get the following relations

\[
\mu_2 = \eta_1 = 0, \quad \mu_1 = \frac{1}{1 + \varphi^2}, \quad \eta_2 = -\frac{\varphi}{1 + \varphi^2}.
\]

Finally, the solution of the equation is

\[
f(t) = \left( \frac{1}{1 + \varphi^2} \cos(At) + \frac{\varphi^2}{1 + \varphi^2} \sin(Bt) \right) +
\]

\[
\left( \frac{\varphi}{1 + \varphi^2} \sin(At) - \frac{\varphi^2}{1 + \varphi^2} \sin(Bt) \right)i +
\]

\[
\left( \frac{-\varphi}{1 + \varphi^2} \cos(At) + \frac{\varphi^2}{1 + \varphi^2} \cos(Bt) \right)j -
\]

\[
\left( \frac{-\varphi}{1 + \varphi^2} \cos(At) + \frac{\varphi^2}{1 + \varphi^2} \cos(Bt) \right)k.
\]

Let us notice that this is indeed a curve on \( \mathbb{S}^3 \) since \( \mu_1^2 + \mu_2^2 + \eta_1^2 + \eta_2^2 = \frac{1}{1 + \varphi^2} = \cos^2 \theta. \) We get the parametrization (3.1) of the geodesics when we substitute (3.31), with the proper constants \( A, B, A, B \) into \( \gamma(t) = (x(t), y(t)). \)

\[\square\]

**Proposition 3.1.**

1. The geodesic lines on \( \mathbb{S}^3 \times \mathbb{S}^3 \) with respect to nearly Kähler metric coincide with geodesic lines with respect to usual Euclidean product metric if and only if \( c_1 = 0 \) or \( c_2 = 0. \)

2. \( \bullet \) The tangent vector of the geodesic line is the eigenvector of the product structure \( P \) with eigenvalue \(-1\) if and only if \( c_1 = 0. \)

3. \( \bullet \) The tangent vector of the geodesic line is the eigenvector of the product structure \( P \) with eigenvalue \(1\) if and only if \( c_2 = 0, d_1 = 0. \)

3. The geodesic line on \( \mathbb{S}^3 \times \mathbb{S}^3 \) is closed if and only if it has the parametrization from the case Theorem 3.1 (1), or the fractions \( \frac{A}{A}, \frac{B}{A} \) and \( \frac{A}{A} \) are rational numbers in the cases (2) and (3) of Theorem 3.1, respectively.

**Proof.**

1. From relation (3.33) it is evident that \( \nabla \gamma' = 0 \) is equivalent to \( \alpha' = \beta' = 0, \) namely \( \alpha = \beta = \text{const}. \) We have already seen that this is true if and only if \( c_1 = 0 \) or \( c_2 = 0. \)

2. As \( \gamma' = (x\alpha, y\beta), \) \( P\gamma' = (x\beta, y\alpha), \) condition \( P\gamma' = \pm \gamma' \) reduces to \( \alpha = \pm \beta \) and the conclusion follows from (3.10) and (3.14).
(3) It is apparent that $\gamma(t)$ is a periodic function in the case Theorem 3.1 (1). Also, in the case Theorem 3.1 (2) both coordinate functions of $\gamma(t)$ are periodic with the same period if and only if the fraction $\frac{a}{\tilde{a}}$ is a rational number (if one of the constants $a, \tilde{a}$ is 0, then this condition stands for the second one which is nonzero). In the case Theorem 3.1 (3) the first coordinate function of $\gamma(t)$ is periodic if and only if the functions $\sin(A t)$, $\cos(A t)$, $\sin(B t)$, $\cos(B t)$ are periodic with the same period, which is again true if and only if the fraction $\frac{B}{A}$ is a rational number (if one of the constants $A, B$ is 0, then this condition stands for the second one which is nonzero). A similar conclusion is valid for the second coordinate function.

□

Remark 3. Using the formula (2.3) we can calculate the length of the tangent vector

$$\|\gamma'\| = \sqrt{g(\gamma', \gamma')} = \sqrt{\frac{1}{3} + d_1^2}\|c_1\|^2 + \|c_2\|^2.$$

Since this is constant, it is easy to get the arclength parametrization where $\gamma'$ is a unit length vector field along the geodesics, only with the change of parameter $s = t\sqrt{(\frac{1}{3} + d_1^2)}\|c_1\|^2 + \|c_2\|^2$.

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