Abstract

In this work, we derive, for the first time to our knowledge, the exact closed-form solutions to the $L_1$ and $L_\infty$ optimal triangulation from two views based on the angular reprojection error. Unlike iterative methods (e.g., [13, 17]), the proposed methods guarantee global optimality without any iterations, and unlike polynomial methods (e.g., [10, 22, 30]), they do not involve finding the roots of a higher-degree polynomial. Hence, our methods simultaneously provide the global optimality, speed and simplicity. We also present our own derivation of the $L_2$ optimal solution that is much more compact and geometrically intuitive than the existing one [23]. Since all three methods are based on the angular error, they are not limited to standard perspective cameras and can also be used for fisheye and omnidirectional cameras.

The paper is organized as follows. In the next three sections, we discuss the related work and preliminaries. Section 5, 6 and 7 respectively present the closed-form solutions to the $L_1$, $L_2$ and $L_\infty$ optimal triangulation. To make our paper compact and easily accessible, we separated the proofs from our main findings and put them in the appendix. Section 8 addresses the cheirality constraint. Finally, experimental results are provided in Section 9, followed by the conclusions in Section 10.

2. Related Work

The most widespread approach to triangulation is to find the 3D point that minimizes the $L_2$ norm of image reprojection errors [8]. Assuming that image points are perturbed by Gaussian noise, the $L_2$ optimal solution gives the maximum likelihood estimate (MLE). This can be obtained in closed form by solving a polynomial of degree 6 for two views [10] and degree 47 for three views [30]. Such polynomial methods are, however, computationally expensive and susceptible to ill-conditioning [17]. Besides, an iterative search for the roots may converge to a local minimum [10].

Another two-view method by Kanatani et al. [13] iteratively corrects the 2D projections of the points. Although this method was shown to be faster than the one by Hartley and Sturm [10], it does not satisfy the epipolar constraint [19] in each iteration. Lindstrom [17] solved this problem
with an improved iterative algorithm that is even more stable and faster. However, neither his method nor Kanatani’s guarantees global optimality. Oliensis [23] showed that by formulating the problem as $L_2$ minimization of the sine of angular reprojection errors, an exact closed-form solution can be derived for two-view triangulation.

Instead of minimizing the $L_2$ norm, one may choose to minimize the $L_1$ norm of reprojection errors. The advantage of $L_1$ norm is that it is more robust to outliers as it places less emphasis on larger errors [10, 11]. For two views, Hartley and Sturm [10] showed that the $L_1$ optimal solution can be obtained in closed form by solving a polynomial of degree 8. They also found that the $L_1$ optimization gives slightly more accurate 3D results than the $L_2$ optimization.

In geometric problems, another popular norm is the $L_\infty$ norm. The $L_\infty$ optimal solution corresponds to the MLE under the assumption of uniform noise in the image points [6]. The advantage of the $L_\infty$ cost function over the $L_2$ cost is that it is relatively simpler and has a single minimum [9]. For the case of two views, Nister [22] showed that the optimal solution can be obtained in closed form by keeping the reprojection errors equal in the two views and solving the resulting quartic equation. A main drawback of the $L_\infty$ cost is that it is relatively more sensitive to outliers [9]. This being said, such sensitivity was shown to be useful for outlier removal [28, 25, 16].

While most of the aforementioned works formulate their optimization problem in terms of the image reprojection error, the angular reprojection error is another popular choice. It embodies a better noise model for fisheye or omnidirectional cameras [23, 20]. Even for perspective cameras, the assumption of Gaussian noise is not justified [6], and the angular reprojection error is just as valid as the image reprojection error, if not more so. In the literature, it has been proposed to minimize the sine of angular reprojection errors in $L_2$ norm [23], the tangent in $L_2$ or $L_\infty$ norm [7, 9, 12], and the cosine in negative $L_1$ norm [26, 3]. In contrast to these methods, our $L_1$ and $L_\infty$ optimization do not involve trigonometric functions.

3. Preliminaries on 3D Geometry

Throughout the paper, we adopt the following notation: We use bold letters for vectors and matrices, and light letters for scalars. The Euclidean norm of a vector $v$ is denoted by $\|v\|$, and the unit vector by $\hat{v} = v / \|v\|$. The angle between two lines $L_0$ and $L_1$ is denoted by $\angle(L_0, L_1) \in [0, \pi/2]$. The following vector identities will come in handy later:

1. The distance between a point $p$ and a plane $\Pi_0(x) = m_0 \cdot (x - c_0) = 0$ is given by $\|p - r_0\|$ where $r_0$ is the projection of $p$ onto $\Pi_0$. This is computed as follows:

$$\|p - r_0\| = |\hat{m}_0 \cdot (p - c_0)|. \quad (3)$$

2. The distance between two skew lines $L_0(s_0) = c_0 + s_0m_0$ and $L_1(s_1) = c_1 + s_1m_1$ is given by $\|r_0 - r_1\|$ where $r_0$ and $r_1$ are the points on each line that form the closest pair. Letting $t = c_0 - c_1$ and $q = m_0 \times m_1$, this is computed as follows:

$$\|r_0 - r_1\| = |t \cdot \hat{q}|. \quad (4)$$

The two points can also be obtained individually [14]:

$$r_0 = c_0 + \frac{q \cdot (m_1 \times t)}{\|q\|^2} m_0, \quad (5)$$

$$r_1 = c_1 + \frac{q \cdot (m_0 \times t)}{\|q\|^2} m_1. \quad (6)$$

Equation (4) can be interpreted as the minimum amount of translation required for the two lines to intersect. In this work, it will be also important to know the minimum amount of rotation (or pivot) required for the two lines to intersect. We answer this question in the following lemma:

Lemma 1 (Minimum Pivot Angle for Intersection)

Given two skew lines $L_0(s_0) = c_0 + s_0m_0$ and $L_1(s_1) = c_1 + s_1m_1$, let $L_0'$ be the line that forms the smallest angle $\theta_0 \in [0, \pi/2]$ to $L_0$ among all possible lines that intersect both point $c_0$ and line $L_1$. Then, $L_0'$ is the projection of $L_0$ onto the plane that contains $c_0$ and $L_1$. Furthermore, letting $t = c_0 - c_1$ and $n_1 = m_1 \times t$,

$$\sin(\theta_0) = |\hat{n}_1 \cdot \hat{m}_0|. \quad (7)$$

We call $\theta_0$ the minimum pivot angle for intersection, as it represents the smallest angle required for pivoting line $L_0$ at $c_0$ to make it intersect $L_1$.

Proof. Refer to Appendix A.

4. Preliminaries on Two-View Triangulation

Consider two cameras $C_0$ and $C_1$ observing the same 3D world point $x_w$. Let $c_0$ and $c_1$ be their positions in the world frame, and let $R$ and $t$ be the rotation matrix and translation vector that together transform a point from the camera frame $C_0$ to $C_1$, i.e., $x_1 = Rx_0 + t$, where $x_0 = [x_0, y_0, z_0]^T$ and $x_1 = [x_1, y_1, z_1]^T$ correspond to $x_w$ in camera frame $C_0$ and $C_1$, respectively. Since triangulation is impossible for zero translation, we set $\|t\| = \|c_0 - c_1\| = 1$ without loss of generality. Let $u_0 = (u_0, v_0, 1)^T$ and $u_1 = (u_1, v_1, 1)^T$ be the homoneous pixel coordinates of the estimated correspondence to $x_w$ in each frame. Given the camera calibration matrix $K$, the normalized image coordinates $f_0 = [x_0/z_0, y_0/z_0, 1]^T$ and $f_1 = [x_1/z_1, y_1/z_1, 1]^T$ are related to $u_0$ and $u_1$ by $u_0 = Kf_0$ and $u_1 = Kf_1$. 

Proof. Refer to Appendix A.
The two backprojected rays in frame $C_1$, i.e., $r_1(s_1) = s_1f_1$ and $r_0(s_0) = s_0Rf_0 + t$, do not necessarily intersect due to inaccuracies in the image measurements and camera matrices. For the rays to intersect, $f_0$ and $f_1$ must be corrected to $f'_0$ and $f'_1$ such that the epipolar constraint \([19]\) is satisfied. It ensures the coplanarity of $f'_1$, $Rf'_0$ and $t$, and is given by
\[
f' : (t \times Rf'_0) = 0. \tag{8}
\]

The goal of the optimal triangulation is to minimally correct the feature rays so that they satisfy (8) and intersect at some point $x'_1$ in frame $C_1$. What is meant by “minimal” depends on the chosen cost function and error criterion. Fig. 1 illustrates two most popular error criteria, namely the reprojection error and the angular reprojection error. Formally, they are defined as follows:
\[
d_i := \|u_i - u'_i\| = \|K (f_i - f'_i)\|, \text{ for } i = 0, 1 \tag{9}
\]
\[
\theta_i := \angle (f_i, f'_i) = \angle (K^{-1}u_i, K^{-1}u'_i), \text{ for } i = 0, 1 \tag{10}
\]

In this work, we minimize the latter in $L_1$, $L_2$, and $L_\infty$ norms. Once we have the optimal $f'_0$ and $f'_1$, the point of intersection $x'_1$ can be obtained using either (5) or (6):
\[
x'_1 = t + \frac{z \cdot (t \times f'_1)}{\|z\|^2} Rf'_0 = \frac{z \cdot (t \times Rf'_0)}{\|z\|^2} f'_1 \tag{11}
\]
where $z = f'_1 \times Rf'_0$.

Note that the epipolar constraint (8) is a necessary condition for intersecting the two rays, but not a sufficient one. Fig. 2 illustrates scenarios where the two rays are coplanar, but do not intersect. This happens when the intersection requires negative depth(s), violating the cheirality constraint [7]. In the following analysis (until Section 8), we will temporarily assume that satisfying the epipolar constraint (8) is sufficient for intersecting the rays.

5. Closed-Form $L_1$ Triangulation

The $L_1$ triangulation based on the angular reprojection error (10) finds the feature rays $f'_0$ and $f'_1$ that minimize $\theta_0 + \theta_1$ subject to the epipolar constraint (8). The following lemma reveals a surprising fact that $(\theta_0 + \theta_1)_\text{min}$ is achieved by correcting either one of $f_0$ or $f_1$, but not both:

Lemma 2 ($L_1$ Angle Minimization)

Given two skew lines $L_0(s_0) = c_0 + s_0m_0$ and $L_1(s_1) = c_1 + s_1m_1$, consider any two intersecting lines that also pass $c_0$ and $c_1$, respectively, i.e., $L_0'(s'_0) = c_0 + s'_0m'_0$ and $L_1'(s'_1) = c_1 + s'_1m'_1$. Let $t = c_0 - c_1$, $n_0 = m_0 \times t$, $n_1 = m_1 \times t$, $\theta_0 = \angle (L_0, L'_0)$ and $\theta_1 = \angle (L_1, L'_1)$. Then, $(\theta_0 + \theta_1)$ is minimized for the following $m'_0$ and $m'_1$:
\[
m'_0 = m_0 - (m_0 \cdot \hat{n}_1) \hat{n}_1 \quad \text{and} \quad m'_1 = m_1. \tag{12}
\]

Otherwise,
\[
m'_0 = m_0 - (m_0 \cdot \hat{n}_1) \hat{n}_1 \quad \text{and} \quad m'_1 = m_1. \tag{13}
\]

Proof. Refer to Appendix B.

6. Closed-Form $L_2$ Triangulation

The $L_2$ triangulation based on the angular reprojection error (10) finds the feature rays $f'_0$ and $f'_1$ that minimize $\theta_0^2 + \theta_1^2 \approx \sin^2(\theta_0) + \sin^2(\theta_1)$ subject to the epipolar constraint (8). Note that the small angle approximation by $\sin(\theta)$ is more accurate than the one by $\tan(\theta) \text{ or } 1 - \cos(\theta)$ which have been used in other works [7, 9, 12, 26, 3]. This is easily seen by comparing the Maclaurin expansions. We use the following lemma to obtain the closed-form solution that minimizes (14):

Lemma 3 ($L_2$ Angle Minimization)

Given two skew lines $L_0(s_0) = c_0 + s_0m_0$ and $L_1(s_1) = c_1 + s_1m_1$, consider any two intersecting lines that also pass $c_0$ and $c_1$, respectively, i.e., $L_0'(s'_0) = c_0 + s'_0m'_0$ and $L_1'(s'_1) = c_1 + s'_1m'_1$. Let $t = c_0 - c_1$, $\theta_0 = \angle (L_0, L'_0)$ and $\theta_1 = \angle (L_1, L'_1)$. Then, $(\sin^2 \theta_0 + \sin^2 \theta_1)$ is minimized for
\[
m'_i = m_i - (m_i \cdot \hat{n}) \hat{n} \quad \text{for } i = 0, 1, \tag{15}
\]
where $\hat{n}$ is the second column of the $3 \times 3$ matrix $V$ from
\[
USV^T = \text{SVD} \left( \begin{bmatrix} \hat{m}_0 & \hat{m}_1 \end{bmatrix} \left(I - \tilde{t} \tilde{t}^T \right) \right). \tag{16}
\]
Proof. Refer to Appendix C. ■

Analogously to the $L_1$ method, substituting $Rf_0$ and $f_i$ into $m_0$ and $m_1$ in the above lemma gives $Rf_0 = m_0$ and $f_i' = m_i'$ that satisfy the $L_2$ optimality.

7. Closed-Form $L_\infty$ Triangulation

The $L_\infty$ triangulation based on the angular reprojection error (10) finds the feature rays $f_0'$ and $f_i'$ that minimize $\max(\theta_0, \theta_i)$ subject to the epipolar constraint (8). The following lemma states that this is achieved when $\theta_0 = \theta_1$:

Lemma 4 ($L_\infty$ Angle Minimization)

Given two skew lines $L_0(s_0) = c_0 + s_0 m_0$ and $L_1(s_1) = c_1 + s_1 m_1$, consider any two intersecting lines that also pass $c_0$ and $c_1$, respectively, i.e., $L'_0(s'_0) = c_0 + s'_0 m_0'$ and $L'_1(s'_1) = c_1 + s'_1 m_1'$. Let $t = c_0 - c_1$, $n_a = (\hat{m}_0 + \hat{m}_1) \times t$, $n_b = (\hat{m}_0 - \hat{m}_1) \times t$, $\theta_0 = \angle(L_0, L'_0)$ and $\theta_1 = \angle(L_1, L'_1)$. Then, $\max(\theta_0, \theta_1)$ is minimized when $\theta_0 = \theta_1$. This is achieved for

$$m_i' = m_i - (m_i \cdot \hat{n}') \hat{n}' \quad \text{for} \quad i = 0, 1,$$

where

$$\hat{n}' = \begin{cases} n_a & \text{if} \quad ||n_a|| \geq ||n_b|| \\ n_b & \text{otherwise} \end{cases} \quad (18)$$

Proof. Refer to Appendix D. ■

Analogously to the previous two methods, substituting $Rf_0$ and $f_i'$ into $m_0$ and $m_1$ gives $Rf_0 = m_0$ and $f_i' = m_i'$ that satisfy the $L_\infty$ optimality.

8. Cheirality, Parallax and Outliers

We have used the term lines instead of rays in all lemmas so far, ignoring the cheirality constraint [8]. We argue that if the optimal solution violates the cheirality constraint, the most reasonable choice is to simply discard the result. In the following, we provide the rationale for this choice.

Fig. 4 illustrates five scenarios where the optimal solution violates the cheirality constraint. In case (a), both rays have negative depths at the optimal intersection. Increasing the allowed angular reprojection error, the first intersection with positive depths occurs when the two corrected rays become parallel, resulting in a point at infinity. This point cannot be triangulated, so it should be discarded.

In the remaining cases, the optimal intersection involves only one of the rays having a negative depth. Following the same procedure, the first intersection with positive depths occurs either at infinity (case (b)), at one of the camera centers (case (c)), along the ray parallel to the translation (case (d)), or at a point somewhere else (case (e)).

In case (b), (c) and (d), the newly triangulated point has either infinite, zero or ambiguous depth, so it is reasonable to discard it. In case (e), we found that reattempting the triangulation with positive depths yields either a very large error, a point near the epipole or a low parallax angle. Typically, these are the indicators of low accuracy or an outlier [10, 8], so a reasonable choice is to discard the match. This procedure is outlined in Step 4–6 of Tab. 1.

9. Experimental Results

We evaluate the proposed methods in comparison to the midpoint method [2, 10], Hartley and Sturm’s $L_1$ and $L_2$ method [10], Lindström’s $L_2$ method with five iterations [17], and Nister’s $L_\infty$ method [22]. The evaluation was performed on both synthetic and real datasets. We generated the synthetic datasets as follows: A set of $8 \times 4$ point clouds of 2,500 points each are generated with a Gaussian radial distribution $N(0, d/4)$ where $d$ is the distance from the world origin. Each point cloud is centered at the world origin. Each point cloud is centered at the world origin, (2) “lateral” - the cameras are horizontally offset, (3) “forward” - the cameras are facing the world origin.

Table 1. Summary of the proposed methods.

| Input | Calib. matrix ($K$), relative pose ($R$, $t$), and a match ($u_0$, $u_1$) from two views ($C_0$, $C_1$). |
| Output | Triangulated 3D point ($x'_1$) in ref. frame $C_1$. |
| 1) $f_0 \leftarrow K^{-1}u_0, f_1 \leftarrow K^{-1}u_1, m_0 \leftarrow Rf_0, m_1 \leftarrow f_1$. |
| 2) For $L_1$ triangulation: |
| If $||\hat{m}_0 \times t|| \leq ||\hat{m}_1 \times t||$, use (12) to obtain $m_0'$ and $m_1'$. Otherwise, use (13). |
| 3) $Rf_0' \leftarrow m_0'$ and $f_i' \leftarrow m_i'$. |
| 4) Check cheirality: |
| (i) Obtain $\lambda_0$ and $\lambda_1$ from (11). |
| (ii) Discard the point and terminate if either $\lambda_0 \leq 0$ or $\lambda_1 \leq 0$. |
| 5) Check angular reprojection errors: |
| (i) $\theta_0 \leftarrow \angle(Rf_0, Rf_0')$ and $\theta_1 \leftarrow \angle(f_1, f_1')$. |
| (ii) Discard the point and terminate if $\max(\theta_0, \theta_1) > \epsilon_1$ for some small $\epsilon_1$. |
| 6) Check parallax: |
| (i) $\beta \leftarrow \angle(Rf_0'f_1')$. |
| (ii) Discard the point and terminate if $\beta < \epsilon_2$ for some small $\epsilon_2$. |
| 7) Compute and return $x'_1$ from (11). |
Figure 3. **Top row:** Real dataset images. **Bottom row:** Main segments of the median reconstruction results using the proposed \(L_1\) method.

Figure 4. Five scenarios where the optimal solution violates the cheirality constraint, and the possible reattempts for triangulation.

Figure 5. 3D triangulation errors before (top) and after (bottom) discarding the points with the lowest 5% parallax.

The cameras at \([0, 0, \pm 0.5]^T\) pointing at the point cloud center. The poses are slightly perturbed with uniform noise \(U(0, 0.01)\). For real datasets, we used the Oxford Dinosaur, Model House and Corridor [1], Notre Dame [29] and Fountain [4, 24] dataset. In total, the synthetic and real datasets provide over 5.5 million unique triangulation problems in a wide variety of geometric configurations.

Tab. 2 provides the percentage of the total matches (from both synthetic and real datasets) for which each method yields the lowest error in given criterion. In 100% of the total triangulation problems, all three of our methods yield the lowest errors in their corresponding optimal criterion. We also see that minimizing \(\sin^2(\theta_0) + \sin^2(\theta_1)\) is very close to minimizing \(\theta_0^2 + \theta_1^2\), as discussed in Section 6. Since our \(L_1\) angular method is numerically stable, it sometimes finds better solutions than Hartley-Sturm’s closed-form \(L_1\) method [10] even in the \(L_1\) image error criterion \((d_0 + d_1)\).

In Fig. 5, histograms are given for the 3D reconstruction errors on the synthetic datasets. It shows that 1) all methods exhibit similar 3D accuracy, and 2) discarding low-parallax points (Step 6 of Tab. 1) helps to remove large 3D errors. Qualitatively, we also found that the reconstructions of the real datasets look similar for all methods. Fig. 3 shows the reconstruction results using the proposed \(L_1\) method.

We compare the speed of each algorithm in Tab. 3. The midpoint method is the fastest, as it directly computes the 3D point using (11) without correcting the feature rays or image points. Among the optimal methods, our \(L_1\) and \(L_\infty\) methods are significantly faster than the rest, i.e., at least 1–2 orders of magnitude faster than the state-of-the-art [17].
Table 2. Percentage of the total matches (from all synthetic and real datasets) for which each method yields the lowest error in given criterion. “img/ang”: optimal in the image/angular errors. See the supplementary material for the results from individual datasets.

| Error Criterion | Midpoint | $L_1$ img | $L_2$ img | $L_2^\infty$ img | $L_1$ ang | $L_2$ ang | $L_2^\infty$ ang |
|-----------------|----------|-----------|-----------|-----------------|----------|----------|-----------------|
| $\theta_0 + \theta_1$ | - | - | - | - | 100 % | - | - |
| $\theta_0^2 + \theta_1^2$ | - | - | 7e-5 % | 5e-5 % | - | - | 99.9999 % |
| $\sin^2(\theta_0) + \sin^2(\theta_1)$ | - | - | - | - | - | - | 100 % |
| $\max(\theta_0, \theta_1)$ | - | - | - | - | - | - | 100 % |

Table 3. Speed of computing a 3D point. The relative speed is normalized by that of the midpoint method. Note that this does not take into account Step 4–6 of Tab. 1. All algorithms were implemented in C++ and run on a laptop CPU (Intel i7-4810MQ, 2.8 GHz).

| Points/sec | Relative Speed |
|------------|----------------|
| Midpoint   | L1 img | L2 img | $L_2^\infty$ img | L1 ang | L2 ang | $L_2^\infty$ ang |
| [2, 10]    | [10]    | [10]    | [10]    | [22]    | [17]    | [17]    |
| Points/sec | 42 M     | 65 K     | 92 K     | 270 K    | 1.4 M    | 520 K    | 29 M     | 670 K    | 14 M |
| Relative Speed | 1.0     | 0.0016   | 0.0022   | 0.0064   | 0.033    | 0.013    | 0.71     | 0.016    | 0.33 |

10. Conclusions

In this work, we derived optimal closed-form solutions to the $L_1$, $L_2$ and $L_\infty$ stereo triangulation based on the angular reprojection error. The proposed triangulation methods are extremely simple and fast, and they guarantee global optimality under respective cost functions. We believe that our findings will be particularly useful for large-scale SfM and real-time visual SLAM algorithms.

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Appendix

A. Proof of Lemma 1

Consider a right circular cone with apex $c_0$ and axis $L_0$, lying sideways on a plane $\Pi$ that contains $c_0$ and line $L_1$ (see Fig. 6). The equation of the plane is given by

$$\Pi(x) = n_1 \cdot (x - c_0) = 0 \quad \text{with} \quad n_1 = m_1 \times t.$$  \hfill (19)

The line of intersection between the plane and the cone forms the smallest angle to $L_0$ among all possible lines on the plane that pass $c_0$. That is, it forms the smallest angle to $L_0$ among all possible lines that pass both $c_0$ and $L_1$. Hence, this line of intersection must be $L_0'$. Now, consider a point $a_0 = c_0 + \hat{m}_0$ located one unit away from $c_0$ along $L_0$. Let $r$ be the projection of $a_0$ onto plane $\Pi$. According to lemma 5 in Appendix E, the point $r$ must be located along $L_0'$. Let $d = ||a_0 - r||$, i.e., the distance between $a_0$ and plane $\Pi$. Then, we obtain $\sin(\theta_0)$ as follows:

$$\sin(\theta_0) = d \equiv |\hat{m}_1 \cdot (a_0 - c_0)| = |\hat{m}_1 \cdot \hat{m}_0|.$$  \hfill \blacksquare

B. Proof of Lemma 2

One of the following is true when $(\theta_0 + \theta_1)$ is minimized:

1. $L_0' \neq L_0$ and $L_1' = L_1 \iff \theta_0 > 0$ and $\theta_1 = 0$.
2. $L_0' = L_0$ and $L_1' \neq L_1 \iff \theta_0 = 0$ and $\theta_1 > 0$.
3. $L_0' \neq L_0$ and $L_1' \neq L_1 \iff \theta_0 > 0$ and $\theta_1 > 0$.

Suppose, for the sake of argument, that one of the first two statements is true. In the first case, lemma 1 states that $m_0'$ is obtained by projecting $m_0$ onto the plane with the normal $m_1 \times t$, which leads to (12) and

$$\sin(\theta_0) = \frac{|\hat{m}_0 \cdot (m_1 \times t)|}{||m_1 \times t||} = \frac{|\hat{m}_0 \cdot (\hat{m}_1 \times t)|}{||\hat{m}_1 \times t||}. \hfill (20)$$

Likewise, in the second case, lemma 1 leads to (13), and

$$\sin(\theta_1) = \frac{|\hat{m}_1 \cdot (m_0 \times t)|}{||m_0 \times t||} = \frac{|\hat{m}_0 \cdot (\hat{m}_1 \times t)|}{||m_0 \times t||}. \hfill (21)$$

Figure 6. The angle $\theta_0$ is the smallest angle required for pivoting line $L_0$ at point $c_0$ to make it intersect $L_1$. 

\[\Pi\]

\[c_0\]

\[L_0\]

\[\theta_0\]

\[\hat{m}_0\]

\[L_1\]
Now, the question is how to determine which of the two statements is true. Comparing the right-hand side of (20) and (21), we find that if \( \| \mathbf{m}_0 \times \mathbf{t} \| \leq \| \mathbf{m}_1 \times \mathbf{t} \| \), then

\[
\min_{\theta_0 | \theta_1 = 0} \theta_0 \leq \min_{\theta_1 | \theta_0 = 0} \theta_1, \tag{22}
\]

and \( \min(\theta_0 + \theta_1) \) is equal to the left-hand side of (22), indicating that the first statement is true. Naturally, the second statement is true otherwise. Note that there is an ambiguity if \( \| \mathbf{m}_0 \times \mathbf{t} \| = \| \mathbf{m}_1 \times \mathbf{t} \| \), and the solution is optimal whichever case is considered. This concludes the proof of lemma 2 for the first two cases.

We will now prove that the third case never occurs. Given some angle \( \theta_1 \), minimizing \( (\theta_0 + \theta_1) \) is equivalent to minimizing \( \theta_0 \). This is the identical situation as the first case if we replace \( L_1 \) by \( L'_1 \). We know from the proof of lemma 1 that pivoting a line to intersect another with minimum angle can be modeled by a cone lying sideways on a plane. The top of Fig. 7 illustrates this. Similarly, the bottom of Fig. 7 illustrates the minimization of \( \theta_1 \) with respect to \( L'_0 \) given \( \theta_0 \). Now, since both planes touching each cone contain the same two intersecting lines \( L'_0 \) and \( L'_1 \), they must be the same plane. Let this plane be \( \Pi' \). According to lemma 1, \( L'_0 \) is the projection of \( L_0 \) onto plane \( \Pi' \). Therefore,

\[
\hat{\mathbf{n}}' = \frac{\mathbf{m}_0 - (\mathbf{m}_0 \cdot \hat{\mathbf{n}}') \hat{\mathbf{n}}'}{\| \mathbf{m}_0 - (\mathbf{m}_0 \cdot \hat{\mathbf{n}}') \hat{\mathbf{n}}' \|}, \tag{23}
\]

where \( \hat{\mathbf{n}}' \) is the unit normal of plane \( \Pi' \). Since \( \Pi' \) contains both \( c_0 \) and \( c_1 \), \( \hat{\mathbf{n}}' \) is perpendicular to \( \mathbf{t} = \mathbf{c}_0 - \mathbf{c}_1 \). Hence, computing the dot product with \( \mathbf{t} \) on each side of (23) yields

\[
\mathbf{t} \cdot \hat{\mathbf{m}}_0' = \frac{\mathbf{t} \cdot \mathbf{m}_0}{\| \mathbf{m}_0 - (\mathbf{m}_0 \cdot \hat{\mathbf{n}}') \hat{\mathbf{n}}' \|}. \tag{24}
\]

Note that \( |\mathbf{t} \cdot \hat{\mathbf{m}}_0'| \) corresponds to the magnitude of the projection of \( \mathbf{m}_0 \) onto plane \( \Pi' \) for non-zero \( \theta_0 \), so it must be smaller than \( |\mathbf{t} \cdot \hat{\mathbf{m}}_0| \) = 1. Thus,

\[
|\mathbf{t} \cdot \hat{\mathbf{m}}_0'| = \frac{\| \mathbf{t} \cdot \mathbf{m}_0 \|}{\| \mathbf{m}_0 - (\mathbf{m}_0 \cdot \hat{\mathbf{n}}') \hat{\mathbf{n}}' \|} > |\mathbf{t} \cdot \hat{\mathbf{m}}_0|. \tag{25}
\]

Using (2), this inequality can be written as

\[
|\mathbf{t} \times \hat{\mathbf{m}}_0'| < |\mathbf{t} \times \hat{\mathbf{m}}_0|. \tag{26}
\]

Analogously, we can also derive

\[
|\mathbf{t} \times \hat{\mathbf{m}}_1'| < |\mathbf{t} \times \hat{\mathbf{m}}_1|. \tag{27}
\]

Now, suppose that

\[
\min_{\theta_0 | \theta_1} (\theta_0 + \theta_1) = \theta'_0 + \theta'_1 \text{ with } \theta'_0, \theta'_1 > 0. \tag{28}
\]

Without loss of generality, let us assume that \( |\mathbf{t} \times \hat{\mathbf{m}}_0| \leq |\mathbf{t} \times \hat{\mathbf{m}}_1| \). Then, (26) gives \( |\mathbf{t} \times \hat{\mathbf{m}}_0'| < |\mathbf{t} \times \hat{\mathbf{m}}_1'| \). As we discussed for the first two cases, this means that pivoting \( L'_0 \) to intersect \( L_1 \) takes smaller angle than pivoting \( L_1 \) to intersect \( L'_0 \), i.e., \( \theta'_0 < \theta'_1 \). Thus

\[
\theta'_0 + \theta'_1 < \theta'_0 + \theta'_0. \tag{29}
\]

According to lemma 6 in Appendix E, pivoting a line twice for intersection takes equal or greater angle than the single minimum pivot angle. Therefore,

\[
\min_{\theta_0 | \theta_1 = 0} \theta_0 \leq \theta'_0 + \theta'_1 < \theta'_0 + \theta'_0, \tag{30}
\]

which contradicts (28). Therefore, \( (\theta_0 + \theta_1) \) is minimized when either \( \theta_0 \) or \( \theta_1 \) is zero. ■

C. Proof of Lemma 3

Given some angle \( \theta_1 \), \( (\sin^2(\theta_0) + \sin^2(\theta_1))_{\min} \) is achieved by minimizing \( \theta_0 \) and vice versa. As discussed in the proof of lemma 2, this means that the underlying geometry at \( (\sin^2(\theta_0) + \sin^2(\theta_1))_{\min} \) can be represented by the two cones with apex \( c_0, c_1 \) and skew axes \( L_0, L_1 \), respectively, touching each side of the same plane on their lateral surface. This is visualized in Fig. 7. Let \( \hat{\mathbf{n}}' \) be the normal of plane \( \Pi' \). From Lemma 1, we know that

\[
\sin (\theta_0) = |\hat{\mathbf{n}}' \cdot \hat{\mathbf{m}}_0| \text{ and } \sin (\theta_1) = |\hat{\mathbf{n}}' \cdot \hat{\mathbf{m}}_1|. \tag{31}
\]

Combining these two equations, we get

\[
\sin^2(\theta_0) + \sin^2(\theta_1) = \| M^T \hat{\mathbf{n}}' \|^2 \text{ with } M = [\hat{\mathbf{m}}_0 \ \hat{\mathbf{m}}_1]. \tag{32}
\]

Since plane \( \Pi' \) contains both \( c_0 \) and \( c_1 \), \( \hat{\mathbf{n}}' \) is perpendicular to \( \mathbf{t} = \mathbf{c}_0 - \mathbf{c}_1 \). Therefore, minimizing \( (\sin^2(\theta_0) + \sin^2(\theta_1)) \) is equivalent to solving the following equality-constrained quadratic programming problem:

\[
\arg\min_{\hat{\mathbf{n}}'} \| M^T \hat{\mathbf{n}}' \|^2, \text{ s.t. } \| \hat{\mathbf{n}}' \| = 1 \text{ and } \mathbf{t} \cdot \hat{\mathbf{n}}' = 0. \tag{33}
\]

In [5], it was shown that this problem can be solved using the method of Lagrange multipliers, and \( \| M^T \hat{\mathbf{n}}' \|^2 \) is
minimized when \( \mathbf{n}' \) is the eigenvector corresponding to the smallest nontrivial eigenvalue of \( \mathbf{A} = (I - t \mathbf{t} \mathbf{t}^\top) \mathbf{M} \mathbf{M}^\top \).

Letting \( \mathbf{P} = (I - t \mathbf{t} \mathbf{t}^\top) \), it can be easily shown that \( \mathbf{P} = \mathbf{P}' = \mathbf{P}^\top \mathbf{P} = \mathbf{P}_1 \mathbf{P}_1^\top \). Hence, \( \mathbf{A} = \mathbf{P} \mathbf{M} \mathbf{M}^\top = \mathbf{P}_1 \mathbf{M} \mathbf{M}^\top \).

Therefore, letting \( \mathbf{USV}^\top = \mathbf{SVD} \) with the diagonal entries of \( \mathbf{X} \mathbf{Y} \) the same as that of \( \mathbf{Y} \mathbf{X} \).

This means that the eigenvectors of \( \mathbf{A} = \mathbf{P} \mathbf{SVD} \mathbf{P}^\top \) are the same as those of \( \mathbf{SVD} \mathbf{M} \mathbf{M}^\top \mathbf{P} = (\mathbf{M} \mathbf{P})^\top (\mathbf{M} \mathbf{P}) \), i.e., the right-singular vectors of \( \mathbf{M} \mathbf{P} \). Therefore, letting \( \mathbf{USV}^\top = \mathbf{SVD} (\mathbf{M} \mathbf{P}) \) with the diagonal entries of \( \mathbf{S} \) in descending order, the optimal \( \mathbf{n}' \) is given by the second column of \( \mathbf{V} \). Finally, projecting \( \mathbf{m}_0 \) and \( \mathbf{m}_1 \) onto plane \( \Pi' \) leads to (15).

\[ \mathbf{m}_0 = \mathbf{m}_1 \]

**D. Proof of Lemma 4**

First, we show that \( \theta_0 = \theta_1 \) when \( \max(\theta_0, \theta_1) \) is minimized: Consider two cones with apex \( c_0, c_1 \) and skew axes \( \mathbf{L}_0, \mathbf{L}_1 \). Constrained both their apertures to be \( 2\pi \).

When \( \theta = 0 \), the they are simply two skew lines. As we gradually increase \( \theta \), they will grow at the same rate, and eventually, touch one another. What if \( \theta = \theta' \) at this point. Now, suppose

\[ \theta^* := \min_{\theta_0, \theta_1} \max(\theta_0, \theta_1) < \theta' \quad (34) \]

The definition of \( \theta^* \) implies that setting \( \theta_0 = \theta_1 = \theta^* \) will make the two cones partially overlap in space (at least meet at a point). However, the they do not meet when \( \theta_0 = \theta_1 < \theta' \). This is a contradiction, so the inequality in (34) must be false, and \( \theta^* \) must be equal to \( \theta' \).

That is, \( \theta_0 = \theta_1 = \theta' \) in order for \( \max(\theta_0, \theta_1) \) to be minimized.

We can now represent the underlying geometry at \( (\max(\theta_0, \theta_1))_{\min} \) as two congruent cones with skew axes, touching each side of the same plane \( \Pi' \) on their lateral surface. This is the situation shown in Fig. 7 for \( \theta_0 = \theta_1 \). Let \( \mathbf{n}' \) be the normal of plane \( \Pi' \). Then, from lemma 1, we get

\[ \sin(\theta_0) = \sin(\theta_1) = |\mathbf{n}' \cdot \mathbf{m}_0| = |\mathbf{n}' \cdot \mathbf{m}_1| \quad (35) \]

The last equality in (35) can be written as

\[ (\mathbf{m}_0 + w\mathbf{m}_1) \cdot \mathbf{n}' = 0 \quad (36) \]

where \( w \) is \(-1 \) or \( 1 \), depending on the signs of \( \mathbf{n}' \cdot \mathbf{m}_0 \) and \( \mathbf{n}' \cdot \mathbf{m}_1 \). On the other hand, since plane \( \Pi' \) contains both \( c_0 \) and \( c_1 \), \( \mathbf{n}' \) is perpendicular to \( \mathbf{t} = \mathbf{c}_0 - \mathbf{c}_1: \)

\[ \mathbf{t} \cdot \mathbf{n}' = 0 \quad (37) \]

Combining (36) and (37), \( \mathbf{n}' \) can be expressed as

\[ \mathbf{n}' = \lambda(\mathbf{m}_0 + w\mathbf{m}_1) \times \mathbf{t} \quad (38) \]

where \( \lambda \) is the normalizing factor. Evaluating (38) at \( w = 1 \) and \( w = -1 \) gives two candidates for optimal \( \mathbf{n}' \). The optimal solution is then determined by comparing the values of (35) with each candidate \( \mathbf{n}' \), which amounts to choosing the solution with smaller \( \lambda \). This procedure corresponds to (18). Finally, projecting \( \mathbf{m}_0 \) and \( \mathbf{m}_1 \) onto plane \( \Pi' \) with optimal \( \mathbf{n}' \) leads to (17).

\[ \mathbf{m}_0 = \mathbf{m}_1 \]

**E. Other Geometric Lemmas**

**Lemma 5 (Cone-On-Plane Perpendicularity)**

When a plane is tangent to a right circular cone, the line of intersection is the projection of cone’s axis onto the plane.

**Proof.** Consider a cone with axis \( \mathbf{L}_0 \) and plane \( \Pi \) tangent to this cone. They are both symmetric with respect to the plane that contains \( \mathbf{L}_0 \) and the normal of \( \Pi \). Let this plane be \( \Pi_{\text{sym}} \). For any circular cross-section of the cone, there is a single point touching \( \Pi \). Therefore, this point must lie on \( \Pi_{\text{sym}} \), and must the line of intersection, \( \mathbf{L}_0' \). It follows that \( \Pi_{\text{sym}} \) contains \( \mathbf{L}_0 \) and \( \mathbf{L}_0' \). Since \( \Pi_{\text{sym}} \) is perpendicular to \( \Pi \), \( \mathbf{L}_0' \) is a projection of \( \mathbf{L}_0 \) onto \( \Pi \).

**Lemma 6 (Single vs Multi-Pivot for Intersection)**

Given two skew lines \( \mathbf{L}_0(s_0) = c_0 + s_0\mathbf{m}_0 \) and \( \mathbf{L}_1(s_1) = c_1 + s_1\mathbf{m}_1 \), let \( \mathbf{L}_0' \) be the line that forms the smallest angle \( \theta_0 \in [0, \pi/2] \) to \( \mathbf{L}_0 \) among all possible lines that intersect both point \( c_0 \) and line \( \mathbf{L}_1 \). For any positive integer \( N \), consider the following arbitrary lines passing \( c_0 \) such that

\[ \mathbf{L}_i(s_i) = \begin{cases} \mathbf{L}_0 & \text{for } i = 0 \\ \mathbf{c}_0 + s_i\mathbf{m}_i & \text{for } i = 1, 2, \cdots, N \end{cases} \]

where only \( \mathbf{L}_N \) intersects \( \mathbf{L}_1 \). Then,

\[ \theta_0 \leq \sum_{i=1}^{N} \angle(\mathbf{L}_i', \mathbf{L}_{i-1}') \quad (39) \]

**Proof.** The right-hand side of (39) corresponds to the sum of \( N \) pivot angles that make line \( \mathbf{L}_0' \) to intersect \( \mathbf{L}_1 \). Fig. 8a depicts such operation for \( N = 2 \). Let \( \phi = \angle(\mathbf{L}_0', \mathbf{L}_1') \), \( \psi = \angle(\mathbf{L}_1', \mathbf{L}_2') \) and \( \tau = \angle(\mathbf{L}_0', \mathbf{L}_2') \). Now, consider three arbitrary points \( p, q_1 \) and \( q_2 \) on \( \mathbf{L}_0', \mathbf{L}_1' \) and \( \mathbf{L}_2' \), respectively. A tetrahedron formed by these three points and \( c_0 \) are shown in Fig. 8b. At a vertex of a tetrahedron, the three edges form three angles such that the sum of any two angles is greater than the third one [18, 15]. Thus, \( \tau \leq \phi + \psi \), which proves (39) for \( N = 2 \). Now, for \( N > 2 \), we know that replacing the last two pivots by the corresponding single minimum pivot will produce \( N-1 \) pivots that take equal or smaller angle. Repeating this process until \( N = 1 \) proves (39) for any \( N > 2 \).
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Supplementary Materials

This is the supplementary material to the paper “Closed-Form Optimal Triangulation Based on Angular errors”. Tab. 5–12 present the results of the error criteria comparison from each individual dataset considered in the main paper. See Tab. 4 for the statistics of the datasets. Note that combining the results in Tab. 5–12 leads to Tab. 2 of the main paper.

| Cameras | Orbital | Lateral | Forward | Dinosaur | Model House | Corridor | Fountain | Notre Dame |
|---------|---------|---------|---------|----------|-------------|---------|----------|------------|
| Points  | $10^5$  | $10^5$  | $10^5$  | 4,983    | 672         | 737     | 5,302    | 127,431    |
| Matches | $10^5$  | $10^5$  | $10^5$  | 27,080   | 5,550       | 12,139  | 58,815   | 5,120,077  |

Table 4. Datasets used. “Matches” denotes the number of pairwise instances where a point is visible in two camera views. Generally, the number of matches is larger than the number of points because the points may be visible in more than two views.

| Error Criterion | $\theta_0 + \theta_1$ | $\theta_2^0 + \theta_2^1$ | $\sin^2(\theta_0) + \sin^2(\theta_1)$ | $\max(\theta_0, \theta_1)$ | $d_0 + d_1$ | $d_2^0 + d_2^1$ | $\max(d_0, d_1)$ |
|-----------------|------------------------|-----------------------------|---------------------------------|---------------------------|-------------|----------------|----------------|
| Midpoint L1 img | 2, 10                  | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |
| Midpoint L2 img | 10                     | 10                          | 5 it. [17]                      | 22                        | 100,000     | 18,346         | 100,000        |
| Midpoint L1 ang | 2, 10                  | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |
| Midpoint L2 ang | 10                     | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |
| Midpoint $L_\infty$ ang | 10                     | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |

Table 5. [Orbital] The number of matches for which each method yields the lowest error in given criterion.

| Error Criterion | $\theta_0 + \theta_1$ | $\theta_2^0 + \theta_2^1$ | $\sin^2(\theta_0) + \sin^2(\theta_1)$ | $\max(\theta_0, \theta_1)$ | $d_0 + d_1$ | $d_2^0 + d_2^1$ | $\max(d_0, d_1)$ |
|-----------------|------------------------|-----------------------------|---------------------------------|---------------------------|-------------|----------------|----------------|
| Midpoint L1 img | 2, 10                  | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |
| Midpoint L2 img | 10                     | 10                          | 5 it. [17]                      | 22                        | 100,000     | 183            | 100,000        |
| Midpoint L1 ang | 2, 10                  | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |
| Midpoint L2 ang | 10                     | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |
| Midpoint $L_\infty$ ang | 10                     | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |

Table 6. [Lateral] The number of matches for which each method yields the lowest error in given criterion.

| Error Criterion | $\theta_0 + \theta_1$ | $\theta_2^0 + \theta_2^1$ | $\sin^2(\theta_0) + \sin^2(\theta_1)$ | $\max(\theta_0, \theta_1)$ | $d_0 + d_1$ | $d_2^0 + d_2^1$ | $\max(d_0, d_1)$ |
|-----------------|------------------------|-----------------------------|---------------------------------|---------------------------|-------------|----------------|----------------|
| Midpoint L1 img | 2, 10                  | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |
| Midpoint L2 img | 10                     | 10                          | 5 it. [17]                      | 22                        | 100,000     | 99,993         | -              |
| Midpoint L1 ang | 2, 10                  | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |
| Midpoint L2 ang | 10                     | 10                          | 5 it. [17]                      | 22                        | 100,000     | -              | -              |
| Midpoint $L_\infty$ ang | 10                     | 10                          | 5 it. [17]                      | 22                        | 100,000     | 92,375         | -              |

Table 7. [Forward] The number of matches for which each method yields the lowest error in given criterion.
| Error Criterion       | Midpoint | $L_1$ img | $L_2$ img | $L_\infty$ img | $L_1$ ang | $L_2$ ang | $L_\infty$ ang |
|-----------------------|----------|-----------|-----------|----------------|----------|----------|---------------|
|                       |          | [2, 10]   | [10]      | [10] 5 it. [17] | [22]     |          |               |
| $\theta_0 + \theta_1$ | -        | -         | -         | -               | 27,080   | -        | -             |
| $\theta_0^2 + \theta_1^2$ | -       | -         | -         | -               | -        | 27,080   | -             |
| $\sin^2(\theta_0) + \sin^2(\theta_1)$ | -       | -         | -         | -               | -        | 27,080   | -             |
| max($\theta_0, \theta_1$) | -       | -         | -         | -               | -        | -        | 27,080       |
| $d_0 + d_1$            | -        | 27,080    | -         | -               | -        | -        | -             |
| $d_0^2 + d_1^2$        | -        | -         | 6,598     | 20,482          | -        | -        | -             |
| max($d_0, d_1$)        | -        | -         | -         | 27,080          | -        | -        | -             |

Table 8. [Dinosaur] The number of matches for which each method yields the lowest error in given criterion.

| Error Criterion       | Midpoint | $L_1$ img | $L_2$ img | $L_\infty$ img | $L_1$ ang | $L_2$ ang | $L_\infty$ ang |
|-----------------------|----------|-----------|-----------|----------------|----------|----------|---------------|
|                       |          | [2, 10]   | [10]      | [10] 5 it. [17] | [22]     |          |               |
| $\theta_0 + \theta_1$ | -        | -         | -         | -               | 5,550    | -        | -             |
| $\theta_0^2 + \theta_1^2$ | -       | -         | -         | -               | -        | 5,550    | -             |
| $\sin^2(\theta_0) + \sin^2(\theta_1)$ | -       | -         | -         | -               | -        | 5,550    | -             |
| max($\theta_0, \theta_1$) | -       | -         | -         | -               | -        | -        | 5,550       |
| $d_0 + d_1$            | -        | 3,563     | -         | -               | -        | 1,987    | -             |
| $d_0^2 + d_1^2$        | -        | -         | 2,766     | 2,784           | -        | -        | -             |
| max($d_0, d_1$)        | -        | -         | -         | 5,550           | -        | -        | -             |

Table 9. [Model House] The number of matches for which each method yields the lowest error in given criterion.

| Error Criterion       | Midpoint | $L_1$ img | $L_2$ img | $L_\infty$ img | $L_1$ ang | $L_2$ ang | $L_\infty$ ang |
|-----------------------|----------|-----------|-----------|----------------|----------|----------|---------------|
|                       |          | [2, 10]   | [10]      | [10] 5 it. [17] | [22]     |          |               |
| $\theta_0 + \theta_1$ | -        | -         | -         | -               | 12,139   | -        | -             |
| $\theta_0^2 + \theta_1^2$ | -       | -         | -         | -               | -        | 12,139   | -             |
| $\sin^2(\theta_0) + \sin^2(\theta_1)$ | -       | -         | -         | -               | -        | 12,139   | -             |
| max($\theta_0, \theta_1$) | -       | -         | -         | -               | -        | -        | 12,139       |
| $d_0 + d_1$            | -        | 6,053     | -         | -               | -        | 6,086    | -             |
| $d_0^2 + d_1^2$        | -        | -         | 5,999     | 6,140           | -        | -        | -             |
| max($d_0, d_1$)        | -        | -         | -         | 12,139          | -        | -        | -             |

Table 10. [Corridor] The number of matches for which each method yields the lowest error in given criterion.

| Error Criterion       | Midpoint | $L_1$ img | $L_2$ img | $L_\infty$ img | $L_1$ ang | $L_2$ ang | $L_\infty$ ang |
|-----------------------|----------|-----------|-----------|----------------|----------|----------|---------------|
|                       |          | [2, 10]   | [10]      | [10] 5 it. [17] | [22]     |          |               |
| $\theta_0 + \theta_1$ | -        | -         | -         | -               | 58,815   | -        | -             |
| $\theta_0^2 + \theta_1^2$ | -       | -         | -         | -               | -        | 58,815   | -             |
| $\sin^2(\theta_0) + \sin^2(\theta_1)$ | -       | -         | -         | -               | -        | 58,815   | -             |
| max($\theta_0, \theta_1$) | -       | -         | -         | -               | -        | 58,815   | -             |
| $d_0 + d_1$            | -        | 32,557    | -         | -               | -        | 26,258   | -             |
| $d_0^2 + d_1^2$        | -        | -         | 2,348     | 56,467          | -        | -        | -             |
| max($d_0, d_1$)        | -        | -         | -         | 58,815          | -        | -        | -             |

Table 11. [Fountain] The number of matches for which each method yields the lowest error in given criterion.
| Error Criterion | Midpoint | $L_1$ img | $L_2$ img | $L_2$ img | $L_\infty$ img | $L_1$ ang | $L_2$ ang | $L_\infty$ ang |
|-----------------|----------|-----------|-----------|-----------|----------------|-----------|-----------|----------------|
| $\theta_0 + \theta_1$ | - | - | - | - | - | 5,120,077 | - | - |
| $\theta_0^2 + \theta_1^2$ | - | - | - | - | - | - | 5,120,077 | - |
| $\sin^2(\theta_0) + \sin^2(\theta_1)$ | - | - | - | - | - | - | 5,120,077 | - |
| $\max(\theta_0, \theta_1)$ | - | - | - | - | - | - | - | 5,120,077 |
| $d_0 + d_1$ | - | 3,654,620 | 121 | 123 | - | 1,465,213 | - | - |
| $d_0^2 + d_1^2$ | - | - | 1,190,827 | 3,929,250 | - | - | - | - |
| $\max(d_0, d_1)$ | - | - | - | - | 5,120,077 | - | - | - |

Table 12. [Notre Dame] The number of matches for which each method yields the lowest error in given criterion.