Gravitating Isovector Solitons

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We formulate the nonlinear isovector model in a curved background, and calculate the spherically symmetric solutions for weak and strong coupling regimes. The usual belief that gravity does not have appreciable effects on the structure of solitons will be examined, in the framework of the calculated solutions, by comparing the flat-space and curved-space solutions. It turns out that in the strong coupling regime, gravity has essential effects on the solutions. Masses of the self-gravitating solitons are calculated numerically using the Tolman expression, and its behavior as a function of the coupling constant of the model is studied.

I. INTRODUCTION

There has been considerable interest in the localized solutions of the Einstein’s equations with nonlinear field sources in recent years ([1], [2], [3], [4], [5]). Gravitating non-abelian solitons and black holes with Yang-Mills fields is investigated in [6]. Such problems were not investigated earlier in the history of GR, mainly because of two reasons:
1. It was widely accepted that the gravitational effects are too weak to affect -in an essential way- the properties of soliton solutions of nonlinear field theories.
2. The resulting equations are usually formidable such that the ordinary analytical approaches become idle. More recently, however, the availability of high speed computers and advanced numerical methods have changed the case, and extensive numerical attempts have been made in this direction (see e.g. 387N Term Project in [7]). It has emerged from recent studies that the effects due to the inclusion of gravity are not always negligible. Consider, for example, the Einstein-Yang-Mills (EYM) system. It has been shown that the EYM equations have both soliton and black hole solutions ([1], [2] and [8]). This is in contrast to the fact that vacuum Einstein and pure Yang-Mills equations do not have by themselves soliton solutions. We can therefore conclude that gravity may have dramatic effects on the existence or non-existence of soliton solutions of nonlinear field equations.

Another interesting example is the discovery that black hole solutions may have Skyrmion hair [11]. It was previously believed that stationary black holes can only have global charges given by surface integrals at spatial infinity (the so-called no-hair theorem).

In the ordinary O(3) model, spherically symmetric solutions have an energy density which behave like \(1/r^2\) at large distances ([9]). When formulated in a curved background, this model leads to a spacetime which is not asymptotically flat, and the ADM mass is not well defined ([5]). A nonlinear O(3) model (thereafter referred to as the isovector model) was introduced in ([10]), which possesses spherical, soliton-like solutions with a \(1/r^4\) energy density at large distances. Such a model, is therefore expected to be well behaved in an asymptotically flat background. In the present paper, we examine this model, and discuss its self-gravitating solutions. These new solutions are compared with those obtained previously in a flat spacetime.

The present manuscript is organized in the following way. In section II, we will review the isovector model of [10]. In section III, flat-space solitons of the isovector model and their resemblance to charged particles are introduced. In section IV, the isovector model will be reformulated in a curved background. The resulting differential equations for a spherically symmetric ansatz will be introduced in this section, together with the necessary boundary conditions. These equations will be solved numerically, for several choices of the coupling constant. We will compare the self gravitating solutions with those obtained for a flat spacetime. Soliton masses using the Tolman formalism will be discussed in section V, together with the behavior as a function of the model parameter. Section VI will contain the summary and conclusion.
II. ISOVECTOR MODEL

Consider an isovector field \( \phi_a \) \((a = 1, 2, 3)\) with a \( S^2 \) vacuum at

\[
\phi_a \phi_a = \phi_o^2.
\]  

(1)

Each component \( \phi_a \) is a pseudo-scalar under spacetime transformations, and \( \phi_o \) is a constant. A topological current can be defined for such a field according to (10)

\[
J^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\alpha\beta} \epsilon_{abc} \partial_\nu \phi_a \partial_\alpha \phi_b \partial_\beta \phi_c.
\]  

(2)

For the time being, spacetime is assumed to be the flat Minkowski spacetime and \( \mu, \nu, ... \) \( = 0, 1, 2, 3 \) with \( x^o = t \) \((c = 1 \) is assumed through this paper). \( \epsilon^{\mu\nu\alpha\beta} \) and \( \epsilon_{abc} \) are the totally anti-symmetric tensor densities in 4 and 3 dimensions, respectively. It can be easily shown that the current (2) is identically conserved \((\partial_\mu J^\mu = 0)\), and the total charge is quantized

\[
Q = \int J^o d^3x = \frac{1}{2\pi} \oint dS_\phi dS_x = ne,
\]  

(3)

where \( e \equiv 2\phi_o^3 \). In this equation, \( dS_x \) and \( dS_\phi \) are area elements of \( S^2 \) surfaces in the \( x \)-space \((r \rightarrow \infty)\) and \( \phi \)-space \((\phi \rightarrow \phi_o)\), respectively. The current (2) can identically be written as the covariant divergence of an anti-symmetric, second-rank tensor

\[
\partial_\mu F^{\mu\nu} = J^\nu,
\]  

(4)

in which \( B_\alpha \) is an auxiliary vector field. The dual field \( *F \) with the tensorial components

\[
*F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \frac{1}{4\pi} (2\epsilon_{abc} \phi_a \partial^\mu \phi_b \partial^\nu \phi_c + \partial^\nu C^\mu - \partial^\mu C^\nu),
\]  

(5)

satisfies the equation

\[
\partial_\mu *F^{\mu\nu} = 0,
\]  

(6)

provided that the vector field \( C^\mu \) is a solution of the following wave equation

\[
\square C^\mu - \partial^\mu (\partial_\alpha C^\alpha) = 2\epsilon_{abc} \partial_\nu (\phi_a \partial^\mu \phi_b \partial^\nu \phi_c).
\]  

(7)

It can be easily shown that the right hand side of this equation defines another conserved current

\[
K^\mu = 2\epsilon_{abc} \partial_\alpha (\phi_a \partial^\mu \phi_b \partial^\alpha \phi_c),
\]  

(8)

with

\[
\partial_\mu K^\mu = 0.
\]  

(9)

Using the language of differential forms, (2) can be written in the following form

\[
F = G + H,
\]  

(10)

where the components of the 2-forms \( G \) and \( H \) are given by

\[
G_{\mu\nu} = \frac{1}{2\pi} \epsilon^{\alpha\beta}_{\mu\nu} \epsilon_{abc} \phi_a \partial_\alpha \phi_b \partial_\beta \phi_c,
\]  

(11)

and

\[
H_{\mu\nu} = \frac{1}{2\pi} \epsilon^{\alpha\beta}_{\mu\nu} \partial_\beta B_\alpha.
\]  

(12)
We now have

\[ dF = 0, \tag{14} \]

and

\[ d^*H = 0. \tag{15} \]

The 2-form \( F \) is therefore Hodge-decomposable, and cohomologous with \( G \) (i.e. they belong to the same cohomology class, since they differ only by an exact form). The resemblance of equations (4) and (7) to the Maxwell’s equations and the capability of this model to provide non-singular solutions behaving like charged particles were discussed in [10]. In the next section, we will only outline the main results valid in a flat spacetime.

III. FLAT SPACE SOLITONS

The requirement of having non-singular, finite energy and stable solitons, severely restrict the possible choices of the lagrangian density of the isovector field. Let us follow [10], and adopt the following lagrangian density which satisfies the above requirements:

\[ \mathcal{L} = \lambda (\partial_\mu \phi_a \partial^\mu \phi_a)^2 - b_o (1 - \phi/\phi_o)^2, \tag{16} \]

with \( \lambda < 0 \), and \( b_o \) real constants. The potential \( V(\phi) = b_o (1 - \phi/\phi_o)^2 \) satisfies the following conditions

\[ V(\phi_o) = 0, \quad (\frac{\partial V}{\partial \phi})_{\phi_o} = 0, \quad \text{and} \quad (\frac{\partial^2 V}{\partial \phi^2})_{\phi_o} > 0, \tag{17} \]

which leads to the spontaneous breaking of the (global) \( SO(3) \) symmetry of the system. The dynamical equation for the isovector field is easily obtained, using the variational principle \( \delta \int \mathcal{L} d^4x = 0 \), which leads to

\[ \partial_\mu \partial_\nu \phi_b \partial^\mu \phi_a \partial^\nu \phi_a + \partial_\nu \phi_b \partial_\mu \phi_b \partial^\mu \phi_a + \partial_\nu \phi_b \partial^\nu \phi_b \Box \phi_a = -\frac{1}{4\lambda} \frac{\partial V}{\partial \phi_a}. \tag{18} \]

Similar to the ansatz used in the Skyrme model ([12]), we start with the so-called hedgehog ansatz

\[ \phi_a = \phi(r) \frac{x^a}{r}, \tag{19} \]

where \( x^a, a = 1, 2, 3 \) represent the Euclidean coordinates \( x, y, \) and \( z, \) respectively. This ansatz immediately leads to

\[ K^\mu = 0, \quad B^\mu = 0, \quad \vec{B} = 0, \tag{20} \]

and

\[ \vec{E} = \frac{2\phi^3}{r^2} \hat{r} = \frac{\epsilon}{r^2} \left( \frac{\phi}{\phi_o} \right)^3 \hat{r}, \tag{21} \]

where \( \hat{r} \) is the unit vector in the radial direction, and \( \epsilon \) is the elementary topological charge defined in (3). Note that \( F^{oi} = E_i \) and \( F^{ij} = -\epsilon_{ijk} B_k \). The charge density corresponding to this ansatz is easily obtained to be

\[ \rho = J^\mu = \frac{3}{2\pi} \frac{\phi^2}{r} \frac{d\phi}{dr}, \tag{22} \]

which leads to

\[ Q = \int \rho 4\pi r^2 dr = 2\phi_o^3 = \epsilon, \tag{23} \]

showing that the ansatz bears unit topological charge. Note that in deriving this result, we have used the boundary conditions

\[ \phi_a(r = 0) = 0, \tag{24} \]
\[
\phi \rightarrow \phi_0, \quad \text{as} \quad r \rightarrow \infty. \quad (25)
\]

It can be shown that the following asymptotic solutions are valid:
\[
\phi(r) \simeq \phi_0 \left[ \frac{\alpha_0}{r_0} - \frac{1}{10\alpha_0^2} \left( \frac{r}{r_0} \right)^2 + \ldots \right], \quad (26)
\]
close to the center of the soliton \((r \rightarrow 0)\), where \(\alpha_1\) is a dimensionless constant, and
\[
\phi(r) \simeq \phi_0 \quad (27)
\]
and
\[
\vec{E} \simeq \frac{e}{r^2} \hat{r}, \quad (28)
\]
far from the soliton (i.e. \(r \rightarrow \infty\)).

It can be seen that the total soliton energy
\[
M = \int T_0^0 d^3x = \int_0^\infty \left[ -\lambda \left( \frac{\partial \phi}{\partial r} \right)^4 + 4 \frac{\phi^4}{r^2} + 4 \frac{\phi^2}{r} \left( \frac{d \phi}{dr} \right)^2 + V(\phi) \right] 4\pi r^2 dr = 5.21 \phi_0^3 [b_0(-4\lambda)]^{1/4}. \quad (29)
\]

Using the scale transformation \(r \rightarrow \alpha r\) (while keeping \(\phi\) unchanged), it can be shown that \(M(\alpha)\) has a minimum at \(\alpha = 1\), which is a signature of the stability of the soliton under radial perturbations.

### IV. SELF-GRAVITATING ISOVECTOR SOLITONS

By self-gravitating isovector solitons, we mean static solutions of the coupled isovector-gravitational equations which are everywhere regular and represent localized lumps of energy. By numerically integrating the coupled nonlinear equations, we will show that such solutions do arise depending on the value of the model parameters. Based on our results, we will also criticize the widely expressed view that gravity has only a minute effect on the structure and properties of extended solitons.

Let us start with the action
\[
A = \int \left( -\frac{R}{16\pi G} + L_M \right) \sqrt{-g} d^3x, \quad (30)
\]
in which \(G\) is the gravitational constant, \(R\) is the curvature scalar, \(L_M\) is the lagrangian density of the matter source, and \(g\) is the determinant of the metric tensor. As the source of gravity, we consider the isovector field \((16)\). By varying the action \((30)\) with respect to \(g_{\mu\nu}\) and \(\phi_a\), we obtain the corresponding field equations:
\[
R_{\mu\nu} = 8\pi G \lambda \left[ 4(\partial^\beta \phi_b \partial^\beta \phi_b)(\partial_\mu \phi_a \partial_\nu \phi_a) - g_{\mu\nu}(\partial^\beta \phi_b \partial^\beta \phi_b)^2 - g_{\mu\nu} \frac{V(\phi)}{\lambda} \right], \quad (31)
\]
and
\[
(\partial^\beta \phi_b \partial^\beta \phi_b) \Box \phi_a + g^{\mu\nu} \partial_\mu \phi_a \partial_\nu (\partial^\beta \phi_b \partial^\beta \phi_b) = -\frac{1}{4\lambda} \frac{\phi_a}{\phi} \frac{\partial V(\phi)}{\partial \phi}. \quad (32)
\]

By contracting equation \((31)\), we obtain the following equation for the curvature scalar:
\[
R = -32\pi GV(\phi). \quad (33)
\]

This equation is useful, since it expresses a simple relation between the curvature scalar and the self-interaction potential of the isovector field, and provides a means to check some of the calculations. We employ the coordinates \(x^\mu = (t, r, \theta, \phi)\), and the general, spherically symmetric, static metric \(g_{\mu\nu} = \text{diag}(-B(r), A(r), r^2, r^2 \sin \theta)\). For the hedgehog ansatz \((19)\), the independent field equations become
\[
\frac{B'}{B} + \frac{A'}{A} = -64\pi G \lambda \left[ \frac{r\phi^4}{2A} + \frac{\phi^2 \phi'^2}{r} \right]^2, \quad (34)
\]
\[
\frac{B'}{B} - \frac{A'}{A} = -64\pi G\lambda \left( \frac{r\phi'^4}{4A} - \frac{A\phi'}{r^3} + \frac{rAV(\phi)}{4\lambda} \right) + \frac{2A}{r} - \frac{2}{r},
\]  
\text{(35)}

and

\[
\frac{3\phi'^2\phi''}{A^2} + \frac{2\phi'^2\phi''}{r^2A} - \frac{3A'\phi'^3}{2A^3} + \frac{B'\phi'^3}{2A^2B} + \frac{2\phi\phi'^2}{rA^2} + \frac{A'\phi'^2\phi'}{r^2A^2} + \frac{B'\phi'^2\phi'}{r^2AB} - \frac{4\phi'^3}{r^4} = \frac{1}{4\lambda} \frac{\partial V}{\partial \phi}.
\]  
\text{(36)}

Note that equation (36) reduces to the flat-space equation (18), by putting \(A = B = 1\). Equations (34) and (35), however, become inconsistent for obvious reasons (the spacetime cannot be flat in the presence of matter sources).

Let us introduce the following three length scales

\[
r_o \equiv \left( \frac{-4\lambda}{b_o} \right)^{1/4},
\]  
\text{(37)}

\[
r_{g1} \equiv \frac{1}{4\sqrt{\pi Gb_o}},
\]  
\text{(38)}

and

\[
r_{g2} \equiv \sqrt{-\pi G\lambda b_o^2},
\]  
\text{(39)}

which appear naturally in equations (34) to (36). It can be seen that \(r_o\) is proportional to the geometric mean of \(r_{g1}\) and \(r_{g2}\):

\[
r_o = \sqrt{\frac{r_{g1}r_{g2}}{2}}.
\]  
\text{(40)}

Emergence of new length scales is similar to what happens in non-abelian gauge fields ([14]), and leads to the appearance of new branches of spherical, static solitons. Having additional length scales, one expects departures from the Einstein-Maxwell solutions and the appearance of more subtle features. Using these two length scales, equations (34) to (36) can be made dimensionless and suitable for numerical integration;

\[
\frac{B'}{B} + \frac{A'}{A} = \frac{1}{\epsilon} \left( \frac{xu'^4}{2A} + \frac{u'^2u'^2}{x} \right)^2,
\]  
\text{(41)}

\[
\frac{B'}{B} - \frac{A'}{A} = \frac{1}{\epsilon} \left( \frac{xu'^4}{4A} - \frac{Au'^4}{x^3} - xAW(u) \right) + \frac{2A}{x} - \frac{2}{x},
\]  
\text{(42)}

and

\[
\frac{3u'^2u''}{A^2} + \frac{2u'^2u''}{x^2A} - \frac{3A'u'^3}{2A^3} + \frac{B'u'^3}{2A^2B} + \frac{2u'^3}{xA^2} + \frac{2uu'^2}{x^2A} - \frac{4u'^3}{x^2AB} + \frac{4u'^3}{x^4} = \frac{\partial W(u)}{\partial u}.
\]  
\text{(43)}

In these equations, \(u = \phi/\phi_o, x = r/r_o\), and \(\epsilon \equiv \alpha^2\) which is the dimensionless parameter of the system:

\[
\epsilon = \left( \frac{r_{g1}}{r_o} \right)^2 = \frac{1}{32\pi G\phi_o^2\sqrt{-\lambda b_o}},
\]  
\text{(44)}

and

\[
W(u) = (1 - u)^2.
\]  
\text{(45)}

The asymptotic behavior of the non-singular solutions can be found by using the following series ansatze:

\[
u(x) = \sum_{i=0}^{\infty} c_i x^i, \quad A(x) = \sum_{i=0}^{\infty} a_i x^i, \quad B(x) = \sum_{i=0}^{\infty} b_i x^i,
\]  
\text{(46)}
for $x \to 0$, and

$$u(x) = 1 + \sum_{i=1}^{\infty} \frac{c_i'}{x^i}, \quad A(x) = 1 + \sum_{i=1}^{\infty} \frac{a_i'}{x^i}, \quad B(x) = 1 + \sum_{i=1}^{\infty} \frac{b_i'}{x^i},$$

for $x \to \infty$. The unknown coefficients are calculated by inserting these ansätze into equations (41) to (43) and balancing the terms of the same order in $x$. Table I shows the leading coefficients for the above asymptotics.

In order to solve the coupled nonlinear DEs numerically, we used the Gerald’s shooting method, which is based on two guesses for the initial slope of the unknown functions. Using these initial guesses, the equations are integrated via the Runge-Kutta-Fehlberg method, reaching the end point of the independent variable. A better guess for the initial slope is then found by comparing the end point values with the boundary conditions, and interpolating for the

### Table I: Leading, non-vanishing coefficients of the asymptotic solutions.

| $r \to 0$ | $r \to \infty$ |
|-----------|----------------|
| $a_0 = 1$ | $a_0' = -\frac{1}{2}$ |
| $a_2 = \frac{1}{24} \frac{9 \omega^4}{e^4}$ | $a_2' = \frac{1}{4 \pi}$ |
| $a_3 = \frac{1}{10} \frac{\omega}{e}$ | $a_3' = -\frac{1}{40} + \frac{2}{15} \omega^2$ |
| $a_4 = \frac{1}{10} \frac{12555\omega^{10} - 12240 \omega^6 - 2800 \omega^4 + 2000 \omega^2 - 936}{e^4}$ | $a_4' = \frac{5}{32 \pi} - \frac{\omega^2}{60}$ |
| $b_0 = 1$ | $b_0' = \frac{1}{2 \pi}$ |
| $b_2 = \frac{1}{24} \frac{9 \omega^4}{e^4}$ | $b_2' = -\frac{2}{\pi}$ |
| $b_3 = -\frac{13 \omega}{50}$ | $b_{10}' = -\frac{656}{96 \pi}$ |
| $b_4 = \frac{1}{100} \frac{101 \omega^{10} + 230 \omega^6 + 144 \omega^4 + 268}{e^4}$ | $b_{12}' = -\frac{5 \omega^2}{96 \pi}$ |
| $c_1 = \omega$ | $c_4' = -2$ |
| $c_2 = -\frac{1}{100} \frac{105 \omega^{10} - 35 \omega^6 + 30 \omega^4 - 96}{e^4}$ | $c_8' = -28$ |
| $c_3 = \frac{1}{7200} \frac{15505 \omega^{10} - 6620 \omega^6 + 10920 \omega^4 - 1356}{e^4}$ | $c_{10}' = -\frac{28}{\pi}$ |
| $c_4 = \frac{1}{100} \frac{10 \omega^{10} - 35 \omega^6 + 30 \omega^4 - 96}{e^4}$ | $c_{12}' = -1648$ |

FIG. 1: Variations of the isovector field amplitude for several values of the parameter $\epsilon$. Downward: flat space, $\epsilon=0.2, 0.1$, and 0.05.
initial slope (14). The boundary conditions for asymptotically flat spacetime read

\[ u \to 0 \quad \text{as} \quad x \to 0, \quad (48) \]

and

\[ u \to 1, \quad A \to 1, \quad \text{and} \quad B \to 1, \quad \text{as} \quad x \to \infty. \quad (49) \]

The procedure described above is iterated until the correct boundary values are reached with a reasonable accuracy.

In order to test the method, flat space solutions were first computed and compared to the solutions obtained via energy minimization algorithms (10). Figure 1 shows numerical variations of the \( u(x) \) function for several values of the parameter \( \epsilon \). The corresponding results for the metric coefficients \( A(x) \) and \( B(x) \) are shown in Figures 2 and 3, respectively.

It is seen that for \( \epsilon \) of the order of unity, the self-gravitating solutions differ only slightly from the flat-space solution. The difference vanishes completely as \( \epsilon \to \infty \). As \( \epsilon \) becomes much smaller than 1, significant differences with the flat-space solution emerge. For example, the metric signature changes at some \( r \) for sufficiently small \( \epsilon \).

\section{V. SOLITON MASSES}

Although there are still controversies about an exact definition for the total mass of a self-gravitating system (15, 16, 17), we adopt the Tolman formalism for computing the total mass of the gravitating solitons:

\[ M_T = \int I_T \sqrt{-g} \, d^3x, \quad (50) \]

where

\[ I_T = T^0_0 - T^1_1 - T^2_2 - T^3_3, \quad (51) \]
FIG. 3: Variations of the metric coefficient $B$ for several values of the parameter $\epsilon$ (0.2, 0.1, 0.5, and 0.05 downward).

FIG. 4: Variations of the integral $S_T$ as a function of the dimensionless parameter $\epsilon$.

\[
T^{\mu\nu} = \frac{2}{\sqrt{-g}} \left[ \frac{\partial(\sqrt{-g}L_M)}{\partial g_{\mu\nu}} - \frac{d}{dx^\sigma} \frac{\partial(\sqrt{-g}L_m)}{\partial g_{\mu\nu,\sigma}} \right].
\]

(52)

In most cases (including the model we are discussing here), the second term is absent since the matter lagrangian...
does not contain derivatives of the metric tensor. For the isovector model, we have

$$I_T = 4\lambda \left( \frac{\phi'^2}{A} + \frac{2\dot{\phi}^2}{r^2} \right) \left( \frac{2r\phi'\phi''}{A} - \frac{rA'\phi'^2}{A} + \frac{4\phi\phi'}{r} - \frac{4\phi^2}{r^2} \right) - 2r\phi\frac{\partial V}{\partial \phi} - \left( \frac{rA'}{A} + \frac{rB'}{B} + 6 \right) \left[ -\lambda \left( \frac{\phi'}{A} + \frac{2\phi^3}{r^2} \right) + V(\phi) \right].$$

(53)

Using the transformations (44) and (45), the total mass can be written in terms of a dimensionless integral:

$$M_T = \pi \phi_0^3 [b_0(4\lambda)^3]^{1/4} S_T,$$

(54)

where

$$S_T = -4 \int_a^\infty \left( \frac{u^2}{A} + \frac{2u^2}{A} + \frac{2u^2}{x^2} \right) \left( \frac{2xu'\phi''}{A} - \frac{xu'\phi'^2}{A^2} - \frac{u^2}{A} + \frac{4u'\phi}{x} - \frac{4u^2}{x^2} \right) +$$

$$\left( 6 + \frac{xA'}{A} + \frac{xB'}{B} \right) \left( W(u) + \frac{1}{4} \left( \frac{u^2}{A} + \frac{2u^2}{r^2} \right)^2 \right) + \frac{2xu}{\partial W(u)/\partial u} \sqrt{ABx^2} dx.$$  

(55)

and \( \eta = -4\lambda \). The dimensionless integral \( S_T \) were computed numerically for several values of the parameter \( \epsilon \). The results are shown in Figure 5.

It is seen that (1) \( S_T \rightarrow 1.66 \) as \( \epsilon \rightarrow \infty \), leading to the asymptotic (flat space) mass, (2) There is a maximum mass around \( \epsilon \simeq 1 \), and (3) The total mass decreases as \( \epsilon \rightarrow 0 \), with \( M_T \simeq 0 \) at \( \epsilon \simeq 0.055 \). It is also interesting to note that in the asymptotic series solutions summarized in Table 1, the coefficients \( a_0 \) and \( b_0 \) vanish, which imply vanishing total mass of the soliton as deduced from the asymptotic form of the metric (19).

VI. SUMMARY AND CONCLUSION

We extended the isovector model to incorporate the effects of gravity. The resulting equations were integrated numerically for spherically symmetric ansatz, using the Gerald’s shooting method. It was found that for large values of the dimensionless parameter of the system the effect of gravity is negligible. For small values of \( \epsilon \), however, gravity has a considerable effect on the qualitative and quantitative behavior of the solutions. Such dramatic changes in the behavior of the spherical solutions in the presence of gravity were also reported in the framework of EYM equations (1, 2, and 3). Gravitating solitons of the isovector model in an asymptotically flat background bear quantized topological charges, exactly similar to the flat-space solitons. The quantization is due to a $\pi^2$ homotopy between the boundary of the curved space ($S^2$ at \( r \rightarrow \infty \)), and the vacuum $S^2$ of the isovector field ($\phi_a\phi_a = \phi_0^2$). This is in analogy with the quantization of the magnetic pole intensity in the ’t Hooft Polyakov monopoles (18). Solutions presented in this paper (Figures 5 to 9) do not exhibit horizons. Using the well-known result from general relativity (19), \( \tau_2/\tau_1 = \sqrt{B(r_2)/B(r_1)} \), where \( \tau_1 \) and \( \tau_2 \) are the emitted (at \( r_1 \) ) and detected (at \( r_2 \) ) periods of photons, event horizons correspond to \( B(r_1) = 0 \), which are not fulfilled by the present solutions. However, the appearance of horizons and signature changes is not ruled out and will be addressed elsewhere. In particular, it should be interesting to know whether isovector black holes have hair. As pointed out in (13), solitons can make bound states with ordinary black holes to form hairy black holes. In such a case, the total ADM mass of the hairy black hole is the sum of the mass of the bare black hole, the mass of the soliton, and the gravitational binding energy between the two (13).

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