The location of the Fisher zeros and estimates of $y_T = 1/\nu$ are found for the Baxter–Wu model

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Abstract

The Fisher zeros of the Baxter–Wu model are examined for the first time and for two series of finite-sized systems, with ‘spherical’ boundary conditions, their location is found to be extremely simple. They lie on the unit circle in the complex sinh[$2\beta J_3$] plane. This is the same location as the Fisher zeros of the square lattice Ising model with nearest neighbour interactions and Brascamp–Kunz boundary conditions. The Baxter–Wu model is an Ising model with three-site interactions, $J_3$, on the triangle lattice. From the leading Fisher zeros, using finite-size scaling, accurate estimates of the critical exponent $1/\nu$ are obtained and emphasis is placed on using different variables such as $\exp[-2\beta J_3]$, $\exp[-4\beta J_3]$, and $\sinh[2\beta J_3]$ to enhance the accuracy of estimates. Furthermore, using the imaginary parts of the leading zeros versus the real part of the leading zeros, yields different results. This is similar to results of Janke and Kenna for the nearest neighbour, Ising model on the square lattice and extends this behaviour to a multisite interaction system in a different universality class than the pair-interaction cases.

Keywords: partition function zeros, Baxter–Wu model, finite size scaling

1. Introduction

Yang and Lee in the first of two seminal papers [1] established the importance of the zeros of the partition function in statistical mechanics, especially in regards to the occurrence of phase transitions. In the second of their papers [2] they examined the location of zeros in the complex $\exp[-2\beta h]$ plane, where $h$ is the magnetic field and $\beta = 1/kT$, $k$ is the Boltzmann constant and $T$ is the temperature. They proved that for ferromagnetic pair interactions among a set of Ising spins, that is, spins only taking on the two values $\pm 1$, that in the complex $\exp[-2\beta h]$ plane,

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the location of the zeros simply lie on the unit circle. Since then, zeros lying in some plane related to the magnetic field are generally known as Lee–Yang zeros.

Not long afterward Yang and Lee introduced the importance of the zeros of the partition function Fisher [3] began the study of the zeros of the partition function involving a complex plane related to temperature. One such plane is the complex \( \exp[-2\beta J_2] \) plane, where \( J_2 \) is the interaction strength between a pair of Ising spins. These zeros have become known as Fisher zeros.

While the location of the Lee–Yang zeros for the ferromagnetic pair interactions systems involving Ising spins is incredibly simple, this is the exception rather than the rule. This is especially true of Fisher zeros. In their review article, Bena et al [4] give three reasons for the distribution of the Fisher zeros being more complicated than that of the Lee–Yang zeros. They then write ‘these elements explain the scarcity of exact analytical results on the Fisher zeros (as compared to the Yang–Lee zeros) ...’.

In the following I find for certain finite systems of sites and with certain boundary conditions, denoted as ‘spherical’ boundary conditions, that the location of the Fisher zeros for the Baxter–Wu model is both simple and similar to that of the nearest-neighbour interaction (hereafter n.n.), Ising model, on the square lattice. I also present results, based on the finite size scaling (hereafter FSS) behaviour of the ‘leading’ Fisher zeros. These results are of two parts. This first consists of, both estimates for the correlation length critical exponent \( \nu \), and a comparison of accuracy of the estimates using different temperature related variables, such as \( \exp[-2\beta J] \) compared to \( \sinh[2\beta J] \) or \( \exp[-4\beta J] \). In the past authors using ‘leading’ Fisher zeros and FSS simply used a single variable, such as \( \exp[-4\beta J] \). The accuracy does vary and it is best to look at the results of a selection of variables. The second part consists of seeing that the behaviour of the scaling of the imaginary and real parts of the complex zeros differ by a factor of two in the value of a scaling exponent. This later characteristic is similar to what was recently found by Janke and Kenna [5] for the n.n., square lattice, Ising model with Brascamp–Kunz boundary conditions.

The Baxter–Wu model along with notation is presented in the following section. Then there are two sections containing results concerning the Fisher zeros of the model. In the first of these two sections are results regarding the location of the zeros for various finite sized systems and in the second are results regarding critical and shift exponents found using the leading Fisher zeros and FSS. The final section contains conclusions along with some questions raised by these results.

2. Baxter–Wu model and notation

The Baxter–Wu model is a Hamiltonian model on the triangle lattice with Hamiltonian

\[
\mathcal{H}(\sigma) = -J_3 \sum_{\Delta} \sigma_i \sigma_j \sigma_k - h \sum_i \sigma_i,
\]  

(1)

where \( J_3 \) is the interaction strength of the three-site interactions involving Ising spins \( \sigma_i \), \( \sigma_j \), and \( \sigma_k \), and \( h \) is the magnetic field. The first sum is over all elementary triangles forming the lattice and the second is over all sites of the lattice. Here only the case of \( h = 0 \) will be considered. With this restriction this lattice spin system is one of a rather small number of lattice spin systems which have been solved exactly. In particular Baxter and Wu [6–8] showed the critical temperature to be exactly the same as the most famous of the exactly solved lattice spin systems, that of the n.n., square lattice, Ising system solved by Onsager [9]. The critical temperature for both models is \( 2/\log[1 + \sqrt{2}] \). Besides the critical temperature, the value of
several critical exponents were found. Unlike the critical temperature, the critical exponents have values different from those of the n.n., ferromagnetic, Ising model on the square lattice. This is to be expected as the symmetries of both models differ, in particular the Baxter–Wu model lacks the up–down spin reversal symmetry of the pair interaction Ising models (this is one of the reasons for interest in the model) and is in the same universality class as the four-state Potts model.

Knowing exact values for the critical temperature and critical exponents this model has occasionally been used as a test of various approximation methods, e.g., series expansion analysis [10, 11], real-space renormalization group [12], Monte Carlo renormalization group [13], and Wang–Landau sampling methods [14]. However, to the author’s knowledge, there have been no studies of the zeros of the partition function and their implications for this model. For a summary of lattice spin systems for which the Fisher zeros have been studied see section 12 of [4].

For the Baxter–Wu model I calculate the exact partition function for a number of finite site systems as is often done in trying to determine properties of the Fisher zeros. This started back when the Fisher zeros of the square lattice Ising model were first looked at [15, 16], and continues today [17, 18]. The partition function is

$$Z(\beta) = \sum_{\{\sigma\}} e^{-\beta H} = \sum_{n=-N_b}^{N_b} \Omega(n)b^n,$$  \hspace{1cm} (2)

where the first sum is over all configurations, denoted by \{\sigma\}. The second sum indicates the partition function can be written as a generalized polynomial in \(b\) where I have \(b = \exp[\beta J_3]\). The partition function can also be written in terms of \(\exp[\pm \beta J_2]\), \(\exp[\pm 2\beta J_2]\), or \(\exp[\pm 4\beta J_2]\), the latter only in the case of periodic boundary conditions, as well as other quantities, some of which will be presented later. Obviously which variable one chooses to express the partition function clearly impacts the location of the zeros. Fisher’s [3] original results used \(\exp[-2\beta J_2]\) and it is in this complex plane that he conjectured the zeros lie on two intersecting circles in the thermodynamic limit.

3. Location of the Fisher zeros for the Baxter–Wu model

Perhaps the most obvious systems of sites to use in investigating the zeros of the Baxter–Wu model are systems of a square shape where each elementary square is divided by a single diagonal into two triangles creating a triangular lattice. I will denote the size of such clusters as \(L \times L\) with \(L\) being the number of sites along any edge of the original square. This shape is generally studied using free, cylindrical, or toroidal boundary conditions. With any type of periodicity, it is critical to realize that the Baxter–Wu Hamiltonian creates a situation where there are three sublattices of importance and because of this only \(L \times L\) clusters where \(L\) is divisible by 3 should be examined.

The location of the Fisher zeros of the Baxter–Wu model for the \(6 \times 6\) site cluster with toroidal boundary conditions is shown in figure 1. The zeros lie close to, but not on, the two intersecting circles which are the loci of the Fisher zeros of the n.n., square lattice, Ising model with Brascamp–Kunz [19] boundary conditions. In that case the equation of the two interacting circles is given by

$$e^{-2\beta J_2} = 1 \pm \sqrt{2} e^{i\varphi},$$  \hspace{1cm} (3)
where $J_2$ is the n.n. interaction strength. With the Baxter–Wu model I conjecture that with larger and larger square systems of sites and with any of the three commonly used boundary conditions the zeros move toward these two circles, and reach the same two intersecting circles, as given by (3) with $J_2$ replaced by $J_3$, in the thermodynamic limit. This is similar to what happens with finite systems of sites for the n.n., Ising model on the square lattice with these boundary conditions.

For the n.n., square lattice, Ising model Brascamp and Kunz [19] established the fact that with two special boundary conditions of theirs, the zeros lie directly on the intersecting circles for all finite, as well as the infinite, systems. I do something similar for the case of the Baxter–Wu model by presenting two series of systems and a special boundary condition where the zeros for all finite systems investigated lie exactly on the circles given in (3). Unlike the case considered by Brascamp and Kunz [19] there is no expression for the partition function that I have found where it can be shown that for any finite system with these boundary conditions and hence in the thermodynamic limit the zeros lie exactly on the circles. Rather, I present results for small finite systems where the partition function and hence the Fisher zeros can be directly calculated.

The boundary conditions I employ with the Baxter–Wu model might be called ‘spherical’ boundary conditions. They are easiest viewed in the following manner. I begin by considering a seven-site hexagonal system consisting of a single central site and the six surrounding sites, see figure 2(a). One has six three-site interactions with the sites around the corners of each elementary triangle. Now one should imagine this hexagonal system draped over the top half of a sphere thereby forming the upper hemisphere. Then on the lower hemisphere there is a similar hexagon. The two hexagons share sites along the equator of the sphere. Similar boundary conditions have been considered for Ising spin systems by others, e.g. Diego et al [20], and Hoelbling and Lang [21], where finite, n.n. interaction, Ising spin systems on surfaces topologically equivalent to a sphere were studied.

The Fisher zeros of this eight-site system lie precisely on the two intersecting curves described by the above equation (3) where $J_2$ is replaced by the three-site interaction $J_3$ of the Baxter–Wu model.
A series of finite site systems of this form with ‘spherical’ boundary conditions can be constructed. The upper hemisphere of the next larger sized system in the series is shown in figure 2(b). This system with the spherical boundary condition is a 26-site system. Based on the progression from the eight-site system to the 26-site system it is easy to see how to construct a sequence of ever-larger systems. I label this sequence of systems Series A. Of course, as in any study along this line it quickly becomes extremely time consuming and memory demanding to compute the partition function. Using various symmetries and Mathematica I have calculated the exact partition function for the 8, 26, 56, and 98-site systems. The complete set of Fisher zeros for these four finite systems is shown grouped in figure 3, along with the two intersecting circles given by (3). One sees graphically that all the zeros lie on the circles given. More importantly, these Fisher zeros lie precisely on the circles, as can be checked using Mathematica to whatever precision desired.

It is of interest to plot the Fisher zeros in complex planes other than that of \( \exp[\pm 2\beta J_3] \). As the locations are special in many planes. In my initial calculations of the partition function using the variable \( b \) as written above the zeros lie on two Cassini curves, while in the complex
exp[±4βJ3] they lie on a cardioid. Brascamp and Kunz [19] point out that for their system the zeros lie on the unit circle in the complex sinh[2βJ2]. This of course holds for the Fisher zeros of the four systems mentioned above with spherical boundary conditions and the Baxter–Wu Hamiltonian, with J3 replacing J2. In the complex cosh[2βJ3] plane they lie on a lemniscate.

There is a second series of systems, labelled Series B, where the Fisher zeros lie on the same two intersecting circles in the complex exp[±2βJ3]. This is a series of systems best viewed by considering the L × L square lattice with a diagonal along one direction forming a triangle lattice as done at the beginning of this section. The first two finite site systems in this sequence are illustrated in figure 4. Note the square clusters are clipped on the left and right corners as shown. The clusters, similar to what was done in Series A, are thought of as comprising the upper half of a sphere, with an identical cluster on the lower half, and shared sites along the equator of the sphere. In this case the exact partition function of systems of size 8, 18, 32, 50, 72, and 98-sites have been calculated. Their Fisher zeros all lie precisely on the loci as with the previous series, Series A, i.e., circles, Cassini curves, intersecting circles, etc, depending on the complex plane being considered.

4. Critical exponents and finite size scaling

Now with the location of the Fisher zeros for the two series A and B with spherical boundary conditions lying on the unit circle in the complex sinh[2βJ3] plane it would be only reasonable to conjecture, assuming that the critical temperature of the Baxter–Wu model was not already known, that the critical temperature should be given by sinh[2βcJ3] = 1. Having the solution of Baxter and Wu this is no longer a conjecture but clearly, as far as the critical temperature is concerned, the loci of the Fisher zeros shown here reflect exactly what is expected.

I now use the leading Fisher zeros for each finite site system and FSS to obtain estimates of 1/ν, where ν is the correlation length critical exponent. Itzykson et al [22] established a FSS relation between the Fisher zeros, the correlation length critical exponent ν, and the critical temperature writing

$$u_i(L) - u_c = CL^{-1/\nu},$$

where $u = \exp[-4βJ]$, $u_i(L)$ is the ith Fisher zero for a system of size L, $u_c$ is $u$ at its critical value and $C$ is a constant and as Itzykson et al point out, a complex number in general. Several authors, e.g. [23–25], used this relation, or variations of it, see the following, both to examine its ability to estimate the critical temperature and to estimate ν. Most studies in the past have
Table 1. Comparison of estimates of $1/\nu$ for the Baxter–Wu model based on equation (5a).

| Size of systems used | $\text{Im}(\exp(-2\beta J_3))$ | $\text{Im}(\exp(-4\beta J_3))$ | $\text{Im}(\sinh(2\beta J_3))$ |
|---------------------|-------------------------------|-------------------------------|-------------------------------|
| SERIES A            |                               |                               |                               |
| 8 & 26-sites        | 1.647 503                    | 1.526 243                    | 1.265 705                    |
| 26 & 56-sites       | 1.547 942                    | 1.527 331                    | 1.466 931                    |
| 56 & 98-sites       | 1.521 821                    | 1.514 760                    | 1.493 178                    |
| SERIES B            |                               |                               |                               |
| 8 & 18-sites        | 1.617 750                    | 1.466 454                    | 1.155 446                    |
| 18 & 32-sites       | 1.535 870                    | 1.491 876                    | 1.371 021                    |
| 32 & 50-sites       | 1.514 173                    | 1.494 763                    | 1.437 516                    |
| 50 & 72-sites       | 1.506 947                    | 1.496 587                    | 1.465 212                    |
| 72 & 98-sites       | 1.503 884                    | 1.497 688                    | 1.478 694                    |

Table 2. Comparison of estimates of $\lambda$ for the Baxter–Wu model based on equation (6).

| Size of systems used | $\text{Re}(\exp(-2\beta J_3))$ | $\text{Re}(\exp(-4\beta J_3))$ | $\text{Re}(\sinh(2\beta J_3))$ |
|---------------------|-------------------------------|-------------------------------|-------------------------------|
| SERIES A            |                               |                               |                               |
| 8 & 26-sites        | 3.312 064                    | 3.285 579                    | 2.833 919                    |
| 26 & 56-sites       | 3.098 882                    | 3.094 240                    | 3.000 481                    |
| 56 & 98-sites       | 3.044 675                    | 3.043 078                    | 3.010 272                    |
| SERIES B            |                               |                               |                               |
| 8 & 18-sites        | 3.256 787                    | 3.223 863                    | 2.675 126                    |
| 18 & 32-sites       | 3.078 101                    | 3.068 250                    | 2.876 570                    |
| 32 & 50-sites       | 3.031 174                    | 3.026 798                    | 2.938 119                    |
| 50 & 72-sites       | 3.015 407                    | 3.013 066                    | 2.964 877                    |
| 72 & 98-sites       | 3.008 675                    | 3.007 273                    | 2.978 212                    |

dealt with one or more of the following three variations of (4).

\[
\text{Im}[u_i(L)] \sim L^{-\nu_i}[1 + \mathcal{O}(L)] \quad (5a)
\]

\[
|u_i(L) - u_c| \sim L^{-\nu_i}[1 + \mathcal{O}(L)] \quad (5b)
\]

\[
\text{Re}[u_i(L)] - u_c \sim L^{-\nu_i}[1 + \mathcal{O}(L)]. \quad (5c)
\]

But as Janke and Kenna [5, 26] note, (5c) is more properly written

\[
|\text{Re}[z_i(L)] - z_c| \sim L^{-\lambda} \quad (6)
\]

the important point being that $1/\nu$ in (5c) is replaced with $\lambda$, a shift exponent. Here $z_i(L)$ is the $i$th Fisher zero for some temperature variable. While equations (5a)–(5c), and (6) are written for general $i$ it is usually the case, and will be the case here, that only the ‘leading’ or ‘first’ Fisher zeros is used. By ‘leading’ or ‘first’ Fisher zero is meant the zero with the minimum argument measured from the real axis.
Table 3. Comparison of estimates of $1/\nu$ for the Baxter–Wu model based on equation (5b).

| Size of systems used | $u = \text{exp}(-2\beta J_3)$ | $u = \text{exp}(-4\beta J_3)$ | $u = \text{sinh}(2\beta J_3)$ |
|----------------------|--------------------------|--------------------------|--------------------------|
| SERIES A             |                          |                          |                          |
| 8 & 26-sites         | 1.451 184                | 0.284 902                | 0.090 564                |
| 26 & 56-sites        | 1.549 441                | 1.556 637                | 1.500 240                |
| 56 & 98-sites        | 1.522 337                | 1.524 824                | 1.505 014                |
| SERIES B             |                          |                          |                          |
| 8 & 18-sites         | 1.628 394                | 1.677 451                | 1.337 563                |
| 18 & 32-sites        | 1.539 050                | 1.554 211                | 1.438 285                |
| 32 & 50-sites        | 1.515 586                | 1.522 372                | 1.469 059                |
| 50 & 72-sites        | 1.507 703                | 1.511 344                | 1.482 439                |
| 72 & 98-sites        | 1.504 337                | 1.506 520                | 1.489 106                |

Janke and Kenna [5, 26] state the shift exponent in (6) usually equals $1/\nu$ but this is not a consequence of FSS and is not always true. In fact, they illustrate this [5, 26] by showing in the case of finite systems of Ising spins, with n.n. interactions on a square lattice, and Brascamp and Kunz boundary conditions, that $\lambda = 2 = 2/\nu$. The later equality is because for their lattice spin system $\nu = 1$. In the following I show that the same thing occurs with the Baxter–Wu model and Series A or B systems of sites, that is $\lambda = 2/\nu$. With the Baxter–Wu model $\nu = 2/3$ and thus $\lambda = 3$. While in [26] their results are analytical results, covering all sized systems, the results here are only specifically shown for the finite size clusters for which I have been able to explicitly compute the exact partition function.

Using (5a) one can obtain an estimated value for $1/\nu$ given the leading zeros from two cluster sizes using

$$\frac{1}{\nu(L, L')} = \frac{\text{Ln}[\text{Im} z_1(L')/\text{Im} z_1(L)]}{\text{Ln}[L'/L]}.$$ (7)

Two issues must be addressed when using (7). First, unlike many previous analyses, mostly of the square lattice n.n. Ising model, where the values for $L$ and $L'$ are obvious, that is not the case here. Being that the sites can be thought of as on the surface of a sphere, which is two-dimensional, I have used the square root of the number of sites in a particular system for my value of $L$. Second, while in the original derivation of the FSS scaling relation [22], $u = \text{exp}[-4\beta J]$ this is not a requirement. Janke and Kenna stated in regards to (5a)–(5c), and (6) that $u$ need only by some appropriate temperature variable as written in (6). I have used for $u$ the following, $\text{exp}[-2\beta J_3]$, $\text{exp}[-4\beta J_3]$, and $\text{sinh}(2\beta J_3)$. Previous authors, when dealing with n.n. interactions, have generally used either $\text{exp}[-2\beta J_2]$, or $\text{exp}[-4\beta J_2]$ not both. The following shows there are advantages to examining the FSS results using more than a single expression for $u$.

The values of $1/\nu(L, L')$ obtained using (7) and the imaginary part of the leading zeros from both Series A and B are presented in Table 1. Values for the three temperature variables mentioned in the previous paragraph are given. As stated above, Baxter [8] found the value of $\nu$ to be 2/3. Hence one sees from Table 1 that estimates using the Fisher zeros are reasonably accurate and, except for one sequence, monotonically increase or decrease toward the exact value of Baxter. In fact, it is not unreasonable to conclude after examining the results using as the variables $\text{exp}[-2\beta J_3]$ and $\text{exp}[-4\beta J_3]$ that $1/\nu$ must lie between 1.504 and 1.498. In all studies the author is aware of where Fisher are used to estimate critical exponents the authors...
looked at results based on only one variable. The results in table 1 show it can prove beneficial to look at the results from more than one variable. Based on the above, and taking the average of what appears as an upper bound and lower bound on $1/\nu$ this approach gives $1/\nu = 1.501(3)$. Novotny et al [13] get $\gamma_T = 1/\nu = 1.48(3)$ using Monte Carlo renormalization group methods and more recently Velonakis and Martinos [27] obtain $\nu = 0.67(4)$. Estimates based on the Fisher zeros at the level of finite systems used here match or surpass estimates found in the past.

Next using (6) one can obtain estimates for $\lambda$ based on the real part of the leading Fisher zeros from two finite site systems. Again, as when using $\text{Im}[u_i(L)]$, I have presented the results for the cases where $u = \exp[-2\beta J_3]$, $\exp[-4\beta J_3]$, and $\sinh[2\beta J_3]$. These results can be found in table 2 and as stated earlier I get accurate results similar to that found analytically, by Janke and Kenna [5, 26], using Brascamp–Kunz boundary conditions for the n.n., square lattice, Ising model, but here the results apply to a multi-site interaction system. They found the shift exponent to be $\lambda = 2/\nu$. As with the results using the imaginary part of the leading zeros, the series of results are monotonically decreasing or increasing toward what would appear to be a value of 3. Again, considering that one has what would seem to be upper and lower bounds one gets an estimate for the shift exponent of $\lambda = 2.99(2)$.

One final relation should be examined and that is (5b) where both the real and imaginary part of the leading zero comes into play. Once again using the leading Fisher zeros, in some appropriate variable, leads to an estimate of $1/\nu = 1.497(8)$. Results using this approach are presented in table 3.

5. Concluding remarks

After calculating the exact expression of the partition function, in terms of $\exp[\beta J_3]$, for certain finite site systems of Ising spins, denoted Series A and B, governed by the Baxter–Wu Hamiltonian (1), the Fisher zeros were calculated. With the spherical boundary conditions given these clusters, the Fisher zero were shown to lie on very simple loci. Specifically, in the case of the complex $\exp[\pm 2\beta J_3]$-plane the zeros lie on two intersecting circles, just as the Fisher zeros of the n.n., square lattice, Ising model were shown to lie on the same two intersecting circles, when they had Brascamp–Kunz boundary conditions, and in the case of the complex $\exp[\pm 2\beta J_2]$-plane. While it seems, as with the n.n., square lattice, Ising model with Brascamp–Kunz boundary conditions, that this is true for arbitrarily large clusters, this unfortunately has not been established. In proving their result, Brascamp and Kunz relied upon previous expressions [28] for the partition function and properties of their system, which are currently not known for the Baxter–Wu model.

In addition to the location of the Fisher zeros, estimates for both $1/\nu$ and the shift exponent $\lambda$, defined in (6), were obtained. Two series of estimates for $1/\nu$, based on using either the imaginary part or the entire part of the leading Fisher zeros, were presented. Not surprisingly these estimates improved with an increase in the size of the clusters used. More striking is the fact that using variables in one complex plane gave monotonically increasing estimates while those using another variable gave monotonically decreasing estimates. Together they result in what could be conjectured to be upper and lower bounds for the value of $1/\nu$. The estimates based on the Fisher zeros of the Series B clusters gave the more accurate results. This series using (5a) resulted in an estimate of $1/\nu = 1.501(3)$ while using (5b) resulted in an estimate of $1/\nu = 1.497(8)$. These estimates match or better estimates by methods such as Monte Carlo renormalization group methods.

In terms of the shift exponent $\lambda$, it was shown for the systems studied here, that the shift exponent is not equal to $1/\nu$, but rather $\lambda = 2/\nu$, paralleling what was found by Janke and
Kenna for the n.n., square lattice, Ising model with Brascamp–Kunz boundary conditions. This should, as they state with their result, further ‘resolve some hitherto puzzling features of FSS’ [5]. That the Baxter–Wu model is a multi-site interaction, Ising model and not a pair interaction Ising model means these results extend the class of models over which this $\lambda \neq 1/\nu$ behaviour occurs.

The spherical boundary conditions for the Baxter–Wu model are analogous to the boundary conditions of Brascamp and Kunz in that they produce for finite systems simple results for the location of the Fisher zeros. It would be of interest if there is some symmetry or duality relationship that these boundary conditions are associated with that might underlie their producing such simple results. Also, while the n.n., Ising model on the square lattice having the same critical temperature as the Baxter–Wu model may be a coincidence the fact that both systems have the same location for their Fisher zeros raises the question as to what characteristic of their partition functions causes this.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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