AUTOMORPHISMS OF PARABOLIC INOUE SURFACES

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ABSTRACT. We determine explicitly the structure of the automorphism group of a parabollic Inoue surface. We also describe the quotients of the surface by typical cyclic subgroups of the automorphism group.

1. Statement of Results

In this note we determine the automorphism group Aut \( S \) of a parabolic Inoue surface \( S \). The corresponding result for a hyperbolic Inoue surface (Inoue-Hirzebruch surface) was obtained by Pinkham \[8\] more than twenty years ago. But the far easier case of parabolic Inoue surfaces does not seem explicit in the literature.

Any parabolic Inoue surface \( S \) with second betti number \( m > 0 \) contains a unique smooth elliptic curve \( E \) and a cyclic of rational curves \( C = C_1 + \cdots + C_m \), where the self-intersection number \( E^2 = -m \), and \( C_1 \) are nonsingular rational curves with \( C_i^2 = -2 \) when \( m \geq 2 \), while when \( m = 1, C = C_1 \) is a rational curve with a single node with \( C^2 = 0 \). \( S \) contains no other curves and hence any automorphism of \( S \) leaves \( C \) and \( E \) invariant.

Let Aut\(_0\)\( S \) be the identity component of Aut \( S \), and Aut\(_1\)\( S \) the normal subgroup of Aut \( S \) of elements which leave each \( C_i \) invariant. By \[4\] we know that Aut\(_0\)\( S \cong \mathbb{C}^* \), the multiplicative group of nonzero complex numbers. Our purpose is thus to determine the discrete part of Aut \( S \). We shall summarize our results in Theorem 1.1, Corollary 1.2 and the ensuing Remark below. In this note \( \mu_m \) will denote the cyclic group of order \( m \).

**Theorem 1.1.** Let \( S \) be a parabolic Inoue surface with second betti number \( m > 0 \). Then we have the following:

1) \( \text{Aut}_0 S \cong \mathbb{C}^* \) coincides with the center of Aut \( S \).
2) $\Aut_1 S$ is commutative and is isomorphic to $\mathbb{C}^* \times \mu_m$.

3) $\Aut S$ is isomorphic to the semidirect product $\mu_m \rtimes \Aut_1 S$, where the action of a generator of $\mu_m$ maps $(t, s) \in \Aut_1 S \cong \mathbb{C}^* \times \mu_m$ to $(st, s)$, where $s \in \mu_m$ is considered as an $m$-th root of unity.

From this we get easily the following:

**Corollary 1.2.** The following hold:

1) There exist precisely $m$ cyclic groups of order $m$ in $\Aut_1 S$ which have trivial intersection with $\Aut_0 S$. These are conjugate to one another in $\Aut S$.

2) $\Aut S/\Aut_0 S$ is isomorphic to the abelian group $\mu_m \times \mu_m$.

3) $\Aut S/\Aut_1 S$ is isomorphic to the cyclic group $\mu_m$. There exist precisely $m$ cyclic groups of order $m$ in $\Aut S$ which is mapped isomorphically onto this quotient.

**Remark.**

1) $S$ has the infinite cyclic fundamental group and its universal covering space $W$ is a toric surface with a natural action of the algebraic torus $G := \mathbb{C}^{*2}$. Then $\Aut_1 S$ is naturally identified with the maximal subgroup of $G$ whose action commutes with the covering transformations. Every element of $\Aut S$ can be explicitly described in terms of its lift to $\Aut W$ with respect to the toric coordinates on $W$ up to covering transformations.

2) Let $D_m$ be the dihedral group of order $2m$, considered as the symmetry group of the graph $\Gamma$ of the cycle $C$, which a regular $m$-gon. Then the action of $\Aut S$ on $C$ induces a natural homomorphism $u : \Aut S \to D_m$ with kernel $\Aut_1 S$. This is surjective onto the unique cyclic subgroup of order $m$. In particular, there exists no automorphism of $S$ which induces a reflection of $\Gamma$.

3) Any of the $m$ cyclic groups of 3) of Corollary 1.2 acts on $S$ without fixed points and permutes the irreducible components of $C$ cyclically. Correspondingly, we have $m$ quotient surfaces $S_i, 1 \leq i \leq m$, which are parabolic Inoue surfaces with second betti number 1.
4) Since the homology classes of $E$ and $C_i$ generate $H_2(S, \mathbb{Z})$ over $\mathbb{Q}$ and $H_2(S, \mathbb{Z})$ is torsion free, $\text{Aut}_1 S$ is precisely the subgroup of $\text{Aut} S$ of elements which act trivially on $H_2(S, \mathbb{Z})$.

5) Parabolic Inoue surfaces with fixed second betti number $m$ are naturally parametrized by complex numbers $\alpha$ with $0 < |\alpha| < 1$ as $S = S(m, \alpha)$. The structure of the automorphism group of these surfaces are independent of the parameter $\alpha$. It follows that $\{S(m, \alpha)\}_\alpha$ actually gives the moduli space of parabolic Inoue surfaces.

6) The Zariski open set $U := S - C$ is invariant by the action of $\text{Aut} S$ as remarked above. $U$ has the natural structure of a holomorphic line bundle of degree $-m$ over the elliptic curve $E$ with $E \subseteq U$ the zero section (cf. [1]). It turns out that $\text{Aut} S$ is isomorphic to the group of bundle automorphisms of $U \to E$ which induce translations on $E$, the \textit{theta group} in the sense of [6]. The structure of the latter group is given in [6], which gives in fact most of the structures of $\text{Aut} S$ as stated in Theorem 1.1 and Corollary 1.2.

The proof of the above results will be given in the next section. After some generalities on the behaviour of parabolic Inoue surface under Galois coverings in Section 3, we give in Section 4 some geometric description of the quotients of $S$ by typical subgroups of $\text{Aut} S$ in Theorem 1.1.

Our interest on automorphism groups of parabolic Inoue surfaces comes from the question as to which anti-self-dual bihermitian structures on hyperbolic and parabolic Inoue surfaces constructed in [3] are invariant by the automorphisms of the surfaces (cf. also [2]).

2. Proof of Theorem 1.1

A compact connected complex surface is said to be \textit{of class VII} if its first betti number equals one and its Kodaira dimension equals $-\infty$. It is called \textit{of class VII}_0 (resp. VII$^+_0$) if further it is minimal (resp. minimal and with positive second betti number).

A parabolic Inoue surface is a surface of class VII$^+_0$ discovered by Inoue in [5] (cf. also [7]). A parabolic Inoue surface $S$ is written uniquely as

$$S = S(m, \alpha), 0 < |\alpha| < 1,$$
where \( m \) is the second betti number of \( S \) (cf. \[3 \] Lemma 3.6). We explain its structure according to \[7 \] (cf. also \[1 \]). (Note however the slight deviation of the notations from those of \[7 \].) \( S \) has an infinite cyclic fundamental group, and its universal covering space \( W \) is common to all \( S \).

\( W \) is covered by coordinate open subsets \( U_k = C^2(x_k, y_k), k \in \mathbb{Z} \), and \( V = C^*(w) \times C(x) \) with the transition relations

\[
\begin{align*}
(1) & \quad x_{k+1} = y_k^{-1}, \quad y_{k+1} = x_k y_k^2 \quad \text{on} \quad U_k \cap U_{k+1} \\
(2) & \quad w = x_k y_k, \quad x = x_k^{k+1} y_k^k, \quad \text{or} \quad x_k = w^{-k} x, \quad y_k = w^{k+1} x^{-1} \quad \text{on} \quad U_k \cap V
\end{align*}
\]

\( W \) is a toric surface on which the algebraic two-torus \( G := C^*(s) \times C^*(t) \) acts by

\[
\begin{align*}
(3) & \quad (x_k, y_k) \to (s^{-k} t x_k, s^{k+1} t^{-1} y_k) \quad \text{on} \quad U_k \\
(4) & \quad \text{and} \\
(5) & \quad (w, x) \to (sw, tw) \quad \text{on} \quad V.
\end{align*}
\]

The \( C^*(t) \) action fixes pointwise the curve \( \tilde{E} := \{ x = 0 \} \cong C^* \) in \( V \). The equations \( x_k = 0 \) on \( U_k \) and \( y_{k+1} = 0 \) on \( U_{k+1} \) define a compact smooth rational curve \( \tilde{C}_{k+1} \) on \( W \):

\[
\tilde{C}_{k+1} : \ x_k = 0 \quad \text{on} \quad U_k, \quad y_{k+1} = 0 \quad \text{on} \quad U_{k+1}.
\]

Now for any complex number \( \alpha, 0 < |\alpha| < 1 \), and for any positive integer \( m \) we define an automorphism \( \gamma = \gamma_{m, \alpha} \) of \( W \) which maps \( U_{k-m} \) to \( U_k \) and preserves \( V \) by

\[
\begin{align*}
(7) & \quad (x_{k-m}, y_{k-m}) \to (x_k, y_k) = (\alpha^{-k} x_{k-m}, \alpha^{k+1} y_{k-m}) \quad \text{on} \quad U_{k-m} \\
(8) & \quad (w, x) \to (\alpha w, w^m x) \quad \text{on} \quad V.
\end{align*}
\]

Note that for \( m = 0 \) \[7 \] and \[8 \] coincide with the \( C^*(s) \)-action given in \[3 \] and \[5 \] for \( s = \alpha \). Then \( \gamma_{m, \alpha} \) generates a properly discontinuous, cocompact and fixed-point free infinite cyclic group of transformations of \( W \). The quotient \( W/\langle \gamma_{m, \alpha} \rangle \) is by definition a parabolic Inoue surface denoted by

\[
S = S(m, \alpha)
\]
with second betti number $m$. The universal covering of $S$ is $W$, independently of $\alpha$ and $m$. By (8) and (5) $\gamma_{m,\alpha}$ normalizes $G$ in $\text{Aut}W$ by the formula

$$\gamma_{m,\alpha}(s, t)\gamma_{m,\alpha}^{-1} = (s, s^mt).$$

In particular $\gamma_{m,\alpha}$ commutes with $C^*(t)$-action, and for the $C^*(s)$-action we have

$$s^{-1}\gamma_{m,\alpha}s = (1, s^m)\gamma_{m,\alpha}.$$

Thus an element $(s, t) \in G$ commutes with $\gamma_{m,\alpha}$ if and only if $s$ is an $m$-th root of unity.

Fix an $m$-th root $\beta$ of $\alpha$. We set $\gamma_\beta := \gamma_{1,\beta}$. Then by (10) applied to the case $m = 1$ and $\alpha = \beta$ we get:

$$s\gamma^m_\beta s^{-1} = (1, s^{-m})\gamma^m_\beta.$$

Since the action of $\gamma^m_\beta$ on $V$ takes the form

$$(w, x) \rightarrow (\beta^mw, \beta^m(m-1)/2w^mx),$$

if we take $s = \beta^{(m-1)/2}$ (with any one of the two values fixed when $m$ is even), by (8) we obtain

$$\nu^m = s\gamma^m_\beta s^{-1} = \gamma_{m,\alpha},$$

where

$$\nu = \nu_\beta := s\gamma^m_\beta s^{-1}$$

which acts in the following form on $V$:

$$\nu : (w, x) \rightarrow (\beta w, \beta^{-(m-1)/2}wx).$$

We summarize the implications of the above computations in the following:

**Lemma 2.1.** $\text{Aut}S$ contains a subgroup $H$ isomorphic to a semidirect product $\mu_m \ltimes (\mu_m \times C^*)$ as in 3) of Theorem 1.1 realized in the form $H = \langle \nu \rangle \ltimes (\langle \rho \rangle \times C^*(t))$, where $\nu$ is the automorphism of $S$ induced by $\nu$, and $(\rho, t) \in G, \rho = \exp(2\pi \sqrt{-1}/m)$, admits a naturally induced action on $S$. 

Proof. By (12) \( \nu \) induces on \( S \) a fixed point free automorphism \( \bar{\nu} \) of order \( m \). If \( C_i \) is the natural image of the curve \( \tilde{C}_i, 1 \leq i \leq m \), in \( S \), \( C = C_1 + \cdots + C_m \) is the unique cycle of rational curves on \( S \). Since the action of \( \nu \) maps \( \tilde{C}_k \) to \( \tilde{C}_{k+1} \), \( \bar{\nu} \) transforms cyclically the curves \( C_i \). Thus we have \( \langle \bar{\nu} \rangle \cap \text{Aut}_1 S = \{ e \} \), where \( \langle \bar{\nu} \rangle \) is the cyclic group of order \( m \) generated by \( \bar{\nu} \) and \( e \) is the identity of \( \text{Aut} S \). On the other hand, by (9) the action of \((\rho, t) \in G\) commutes with \( \gamma_{m, \alpha} \); hence we may consider \( \langle \rho \rangle \times C^* (t) \) as a subgroup of \( \text{Aut} S \), which clearly is contained in \( \text{Aut}_{1} S \). In view of (9) for \( m = 1 \) these two groups form a semi-direct product \( H := \langle \bar{\nu} \rangle \ltimes (\langle \rho \rangle \times C^* (t)) \) in \( \text{Aut} S \) as in 3) of Theorem 1.1. The identity component of \( H \) is clearly isomorphic to \( C^* (t) \) and the quotient group \( H / C^* (t) \) is isomorphic to \( \mu_m \times \mu_m \), again by (9).

It remains to show that \( H \) coincides with \( \text{Aut} S \). First we prove the following:

**Lemma 2.2.** Let \( h \) be any automorphism of finite order on \( S \). Then the restriction \( h_E \) of \( h \) to \( E \) is a translation.

Proof. First we note that the induced action of \( h \) on \( H^1 (S, O_S) \cong C \) is trivial. Indeed, let \( \hat{S} \) be the quotient of \( S \) by the cyclic group \( H \) generated by \( h \) and \( \tilde{S} \) be a resolution of \( \hat{S} \). We have the natural isomorphisms \( H^1 (S, O_S)^H \cong H^1 (\hat{S}, O_{\hat{S}}) \cong H^1 (\tilde{S}, O_{\tilde{S}}) \), where \( (\ )^H \) denotes the subspace of \( H \)-fixed elements. If the action on \( h \) on \( H^1 (S, O_S) \) is non-trivial, \( H^1 (\hat{S}, O_{\hat{S}}) \cong H^1 (S, O_S)^H = \{ 0 \} \). Then by Kodaira’s classification of surfaces, \( \tilde{S} \) must be a Kähler surface. But then \( S \) also is Kähler, as it is bimeromorphic to a finite covering of a Kähler surface \( \tilde{S} \). This is a contradiction. Thus \( h \) acts trivially on \( H^1 (S, O_S) \). Since the restriction map \( H^1 (S, O_S) \to H^1 (E, O_E) \) is known to be isomorphic and \( h \)-equivariant, this implies that the action of \( h_E \) on \( H^1 (E, O_E) \) also is trivial. This implies that \( h_E \) is a translation.

In passing we ask the following question suggested by the above lemma. Let \( S \) be any compact complex surface in class VII. Does any automorphism \( h \) of \( S \) act trivially on \( H^1 (S, O_S) \cong C \)? By the same proof as above this is true when \( h \) is of finite order. Does \( S \) admit any automorphism of infinite order if \( \dim \text{Aut} S = 0 \)?
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Now $U := S - C = V/\langle \gamma_{m,\alpha} \rangle$ has a natural structure of a holomorphic line bundle $u : U \to E$ over the elliptic curve $E$ such that $E \subseteq U$ is the zero section of $u$ and the induced $C^*$-action on $U$ is the natural fiber action on this line bundle. Now let $g$ be any automorphism of $S$, which necessarily preserves $U$ and $g$ normalizes $C^* = \text{Aut}_0 S$.

**Lemma 2.3.** The above $g$ actually centralizes $C^*$ so that $g$ acts on $U$ as a bundle automorphism of $u$.

**Proof.** Suppose that $g$ does not centralize $C^*$. Then for any $t \in C^*$ and $p \in S$ we have $g^{-1}tg(p) = t^{-1}(p)$. Take $p$ with $p \in U - E$ and let $t$ tend to zero. Then the left hand side tends to a point on $E$, while the right hand side has no limit in $U$ since $t^{-1}(p)$ goes to infinity on the fiber over $u(p)$, which is a contradiction. $\square$

Since the degree of $U$ equals $-m = E^2 < 0$, the group $H(U)$ of bundle automorphisms of $u$ is an extension by $C^*$ of a finite group of automorphisms of $E$ (cf. Lemma 2.5 below).

**Lemma 2.4.** Let $g$ be as above. The induced automorphism $g_E$ of $E$ is a translation.

**Proof.** If $g$ is of finite order, this follows from Lemma 2.2. Thus it suffices to show that the composition $tg$ becomes of finite order for some $t \in C^* = \text{Aut}_0 S$ since $(tg)_E = g_E$. Indeed, since $g_E^k = 1_E$, the identity of $E$, for some $k > 0$, and $u$ is equivariant, $g^k$ is a bundle automorphism of $U$ over the identity of $E$, and hence $g^k$ belongs to $C^*$. Take any element $t$ of $C^*$ with $t^k = (g^k)^{-1}$. Then $tg$ clearly is of order $k$ as desired. $\square$

Let $G(U)$ be the subgroup of $H(U)$ of automorphisms of $U$ which induce on $E$ a translation. We summarize the known structure of $G(U)$ from [6] in the following:

**Lemma 2.5.** $G(U)$ fits into the exact sequence

$$0 \to C^* \to G(U) \to K(U) \to 0$$

of algebraic groups in the sense that $G(U)$ is a central extension of $K(U)$ by $C^*$, where $K(U)$ is the finite subgroup of the group of translations $b$ of $E$ consisting of those $b$ with $b^*U \cong U$, while $C^*$ corresponds to the natural fiberwise $C^*$-action on $U$. Moreover, $K(U)$ is isomorphic to $\mu_m \times \mu$, and $C^*$ coincides with the center of $G(U)$. 
These are all found in [6] where in general line bundles over an abelian variety of arbitrary dimension are treated. Specialized to one-dimensional case we easily deduce the results stated above: See [6, p.225,Th.1] for the sequence (14); [6, p.60,Appl.1] for the finiteness of $K(U)$; [6, p.84,(v)], [6, p.154 Remarks] and [6, p.150, R-R, p.155 (*))] for its computation; and finally, [6, p.155,Cor.2] (cf. [6, p.223,Cor.]) for the last assertion. □

Proof of Theorem 1.1 So far we have shown that $H := \langle \bar{\nu}_\beta \rangle \rtimes (\langle \rho \rangle \times \mathbb{C}^* (t)) \subseteq \text{Aut} S \subseteq K(U)$ and $(\langle \rho \rangle \times \mathbb{C}^* (t)) \subseteq \text{Aut}_1 S$, the second inclusion being due to Lemmas 2.3 and 2.4. From the first inclusion we get the natural inclusion $H/\mathbb{C}^* (t) \subseteq G(U)$, while both groups are isomorphic to $\mu_m \times \mu_m$ by Lemma 2.5. Thus we have the equalities $H = \text{Aut} S = G(U)$. All the assertions of Theorem 1.1 then follows from the structure of $H$ already mentioned or of the structure of $G(U)$ given in Lemma 2.5. □

The statements in Remark after Corollary 1.2 are immediate from our arguments above. Instead of using the structure of the theta group as in the above proof one can also obtain the same results by direct computations, which however we shall omit here.

3. Galois Coverings and parabolic Inoue surfaces

It seems that parabolic Inoue surfaces are closed under finite coverings. Namely we could ask if the following is true: Let $S_1 \to S_2$ be a generically surjective meromorphic map of compact complex surfaces of class VII$_0^+$, $+$ Then $S_1$ is a parabolic Inoue surface if and only if so is $S_2$.

In this section, however, we consider just the simplest cases where $f$ is a (holomorphic) Galois covering. For this purpose the following characterization of a parabolic Inoue surface due to Hausen [4] is useful.

Lemma 3.1. Let $S$ be a surface of class VII$_0^+$. Then $S$ is a parabolic Inoue surface if and only if $S$ admits an effective $\mathbb{C}^*$-action.

One could have also used the characterization of by [7]: a parabolic Inoue surface is precisely a surface of class VII$_0^+$ which carries a smooth elliptic curve and a cycle of rational curves. But the argument seems easier with the above characterization.
**Lemma 3.2.** Let $S$ be a parabolic Inoue surface and $G$ a finite subgroup of $\text{Aut} S$. Let $S' := S/G$ be the quotient surface, and $\bar{S}$ the (unique smooth) minimal model which is bimeromorphic to $S'$. Then $\bar{S}$ is again a parabolic Inoue surface.

*Proof.* Since $G$ centralize $\text{Aut}_0 S \cong C^*$ by Theorem 1.1, the $C^*$-action on $S$ descends to $S'$, then lifts to its minimal resolution $\bar{S}'$, and then descends to $\bar{S}$. On the other hand, since $G$ preserve the unique elliptic curve $E$ on $S$ and acts by translations on $E$, $S'$, and then also $\bar{S}'$ and $\bar{S}$, contain a nonsingular elliptic curve on it. Moreover, if $\bar{E}$ is the elliptic curve on $\bar{S}$, its self-intersection number is negative as well as that of $E$. Hence the second betti number of $\bar{S}$ is positive and $S$ belongs to class $\text{VII}_0^+$. Therefore by Lemma 3.1, $\bar{S}$ is a parabolic Inoue surface. $\square$

The next lemma gives also the propagation of parabolic Inoue property but in the converse direction.

**Lemma 3.3.** Let $\bar{S}$ be a parabolic Inoue surface. Let $u : \hat{S} \rightarrow \bar{S}$ be a finite Galois covering with Galois group $H$ where $\hat{S}$ is a normal complex surface. Let $S$ be the (unique smooth) minimal model which is bimeromorphic to of $\hat{S}$. Then $S$ is again a parabolic Inoue surface.

*Proof.* Let $\mathring{C}$ be the discriminant locus of $u$ on $\bar{S}$ and $\hat{C} = u^{-1}(\mathring{C})$ with reduced structure. Since $\mathring{C}$ has only normal crossings, $\hat{S}$ has only cyclic quotient singularities. Thus we have the naturally defined notion of the sheaf $\Theta_{\hat{S}}(-\log \hat{C})$ of logarithmic tangent vector fields on $\hat{S}$ along $\hat{C}$ (cf. [9]). On the smooth part $\hat{S}_0$ of $\hat{S}$ the natural sheaf homomorphism $\Theta_{\hat{S}}(-\log \hat{C}) \rightarrow u^*\Theta_{\bar{S}}(-\log \bar{C})$ is $H$-equivariantly isomorphic and hence the subspaces of $H$-fixed elements are also isomorphic. Thus any holomorphic vector field which comes from the $C^*$ action gives rise to a section of $u^*\Theta_{\bar{S}}(-\log \bar{C})$ on $\hat{S}_0$; then considered as a section of $\Theta_{\hat{S}}(-\log \hat{C})$ by the above isomorphism, it extends by the normality of $\hat{S}$ to a section of $\Theta_{\hat{S}}(-\log \hat{C})$ on the whole $\hat{S}$. Then it lifts to its minimal resolution $S'$ and then descends to the minimal model $S$. Thus $S$ admits the induced $C^*$-action. Moreover, it is easy to show that the proper transform of the inverse image of the unique elliptic curve $\bar{E}$ on $\bar{S}$ has a negative self-intersection on $S$, and hence that $S$ is in class $\text{VII}_0^+$. Thus by Lemma 3.1, it is a parabolic Inoue surface. $\square$
4. Typical cyclic subgroups

In this section we shall identify the minimal model of the quotients of a parabolic Inoue surface $S$ by the typical finite cyclic subgroups of $\text{Aut } S$ given in Theorem 1.1. In deciding the isomorphism classes of the above minimal models, the following lemma is useful.

**Lemma 4.1.** Let $S$ be a parabolic Inoue surface. Let $u : W \to S$ be the universal covering map of $S$ and $\tilde{E} = u^{-1}(E)$, where $E$ is the unique elliptic curve on $S$. Suppose that $E^2 = -m$ and $\tilde{E} \to E$ is isomorphic to the natural quotient map $C^* \to C^*/\langle \alpha \rangle$, then $S$ is isomorphic to $S(m, \alpha)$.

**Proof.** The $m$-part is clear from the description of Section one. For the $\alpha$-part see Lemma 3.6 of [3].

In Examples below we assume that $S = S(m, \alpha), m \geq 1, 0 < |\alpha| < 1$.

**Example 1.** The $m$ cyclic groups $H_\beta$ of order $m$ with trivial intersection with $\text{Aut}_1 S$ mentioned in 3) of Corollary 1.2 are parametrized by the set $B = B(\alpha)$ of $m$-th roots of $\alpha$ as

$$H_\beta := \langle \bar{\nu}_\beta \rangle, \ \beta \in B$$

with $\nu_\beta$ as in (13). The quotient $S_\beta := S/H_\beta$ fits in an unramified covering $S \to S_\beta \cong S(1, \beta)$. Thus in all we have $m$ such unramified quotients of $S$.

**Example 2.** Let $\triangle_l$ be the unique cyclic subgroup of order $m$ in $C^*(t) = \text{Aut}_0 S$. Let $S' := S/\triangle_l$ be the quotient of $S$ by $\triangle_l$ and $\bar{S}$ the smooth surface obtained by taking the minimal resolution of the singularities of $S'$. Then $\bar{S}$ is a parabolic Inoue surface isomorphic to $S(ml, \alpha)$.

**Proof.** The action of $\triangle_l = \langle \rho \rangle, \rho = \exp(2\pi \sqrt{-1}/l)$, fixes $E$ pointwise and its action on $U_k(\subseteq W)$ takes the form

$$(x_k, y_k) \to (\rho x_k, \rho^{-1} y_k).$$

Thus the quotient $S'$ has $m$ rational double points $p_i'$ of type $A_{l-1}$, namely the images of the nodes $p_i := C_i \cap C_{i+1}, 0 \leq i \leq m - 1$, of $C$, and is smooth otherwise. The minimal
resolutions at each \( p'_i \) give rise to a chain of \( (l - 1)(-2) \)-curves. Together with the proper transforms \( \bar{C}'_i \) in \( \bar{S} \) of images of irreducible components \( C_i \) of \( C \) in \( S' \) these can be shown to form a cycle \( \bar{C} \) of rational curves consisting of \( m(l - 1) + m = ml \) irreducible components. Other than these \( ml \) curves the only curve on the resolved surface \( \bar{S} \) is the inverse image \( \bar{E} \) in \( \bar{S} \) of the image of \( E \) in \( S' \). Since \( \triangle_l \) pointwise fixes the elliptic curve \( E \), we conclude easily that \( \bar{E} \cong E \) and that the self-intersection number of \( \bar{E} \) equals \(-lm\).

Now if \( \bar{S} \) is not minimal, we obtain its minimal model \( T \) contracting some of the irreducible components of \( \bar{C} \) to points. By Lemma 3.2 \( T \) is a parabolic Inoue surface. This, however, implies that the image of \( \bar{C} \) in \( T \) must be a cycle of rational curves with precisely \( ml \) irreducible components, since the contraction is isomorphic at the neighborhood of \( \bar{E} \). This is impossible as \( \bar{C} \) contains only \( ml \) irreducible components. Hence \( \bar{S} \) itself is minimal and is a parabolic Inoue surface. Since it contains an elliptic curve with self-intersection number \(-lm\), it is of the form \( S(ml, \alpha') \) for some \( \alpha' \).

It remains to show that \( \alpha' = \alpha \). For this purpose we apply to \( W \) the construction of \( S' \) and \( \bar{S} \) starting from \( S \). Namely we take the quotient \( W' = W/\triangle_l \) and take its minimal resolution \( \bar{W} \). Since the action of \( C^* \) commutes with \( \gamma = \gamma_{m, \alpha} \), the action of \( \gamma \) descends to \( W' \) and then lifts to \( \bar{W} \). The resulting action of \( \bar{W} \) is again properly discontinuous and fixed point free and the quotient \( \bar{W}/(\gamma) \) is isomorphic to \( \bar{S} \). In particular, \( \bar{W} \) is isomorphic to \( W \). By construction, the inverse image of \( \bar{E} \) in \( \bar{W} \) is naturally and \( \gamma \)-equivariantly identified with \( \{x = 0\} \cong C^*(w) \) in (the original) \( W \). Since the action of \( \gamma \) on \( C^*(w) \) is by multiplication by \( \alpha \), the assertion follows by Lemma 4.1. \( \square \)

**Example 3.** Recall that \( S = S(m, \alpha) \). The \( m \) cyclic groups in \( \text{Aut}_1 S \) which have trivial intersections in \( \text{Aut}_0 S \) are identified with the subgroups, say \( M_j \), of \( G = C^{*2} \) generated by \( g_j := (\rho, \rho^j), 0 \leq j \leq m - 1, \rho = \exp(2\pi\sqrt{-1}/m) \). The actions of \( g_j \) on \( U_k \) and \( V \) take respectively the forms:

\[
g_j : (x_k, y_k) \rightarrow (\rho^{-k+j}x_k, \rho^{k-j+1}y_k), \quad (w, x) \rightarrow (\rho w, \rho^j x).
\]

Let \( H \) be any of these subgroups and \( l \) any integer \( > 1 \) which divides \( m \). Let \( H_l \) be the unique cyclic subgroup of \( H \) of order \( l \). Since the above \( m \) groups are conjugate to each other in \( \text{Aut} S \), the quotient \( S \rightarrow S' := S/H_l \) is up to isomorphisms independent of \( H \). By
the action of $H_l$ on $S$ is fixed point free outside the curve $C$; the points $p_k := C_k \cap C_{k+1}$ are all fixed, but for instance the irreducible components $C_{li+1}$, $0 \leq i < n$, $n := m/l$, are also pointwise fixed. So the singularities of the quotient $S/H_l$ is not easily identified. In what follows, by describing this quotient in another way we identify the (unique smooth) minimal model $\bar{S}$ which is bimeromorphic to $S'$. Namely we show the following:

**Proposition 4.2.** We have $\bar{S} \cong S(n, \alpha^l)$. Moreover, the natural bimeromorphic map $S' \to \bar{S}$ is holomorphic and obtained by contracting the image of each connected component of $C - \bigcup_{0 \leq i < n} C_{li+1}$ to a point.

For the proof we start in general a parabolic Inoue surface $\bar{S} = S(n, \beta)$ with a cycle of rational curves $\bar{C} = \bar{C}_1 + \cdots + \bar{C}_n$ and with a smooth elliptic curve $\bar{E}$ with $\bar{E}^2 = -n$. The line bundle $[\bar{C}]$ defined by $\bar{C}$ belongs to $\text{Pic}_0 \bar{S} \cong \mathbb{C}^*$ corresponding to the number $\beta \in \mathbb{C}^*$ (cf. [7, p.425]). Indeed, by (6) the defining equation $f = 0$ of $\tilde{C} := \bigcup_k \tilde{C}_k$ on $W$ is given by $x_ky_k = 0$ on $U_k$ and $w = 0$ on $V$, and by (5) $\nu_\beta$ acts on $f$ by $\nu_\beta^*f = \beta f$. This shows the assertion.

Hence $[\bar{C}]$ is divisible by $l$ in $\text{Pic}_0 \bar{S}$ (exactly $l$ roots) and thus we obtain $l$ ramified cyclic coverings of degree $l$ which are totally ramified along $\bar{C}$ and unramified otherwise. Let $w : \hat{S} \to \bar{S}$ be one of them and $L$ the corresponding holomorphic line bundle with $L^l = [\bar{C}]$. Then $L^j, 1 \leq j < l$, are never trivial when restricted to the elliptic curve $\bar{E}$; indeed, the restriction map $\text{Pic}_0 \bar{S} \to \text{Pic}_0 \bar{E}$ is identified with the quotient $\mathbb{C}^* \to \bar{E}$ with its kernel generated by $\beta$ and hence by $[\bar{C}]$. This implies in particular that $\bar{E} := w^{-1}(\bar{E})$ is connected and with self-intersection number $-ln$.

$\hat{S}$ has $n$ rational double points of type $A_{l-1}$, and by taking the minimal resolutions of these singularities we obtain from each of them $(l-1)$ rational curves with self-intersection number $-2$. Let $v : S \to \hat{S}$ be the minimal resolution. For $1 \leq i \leq n$ let $\hat{C}_i$ be the inverse images of $\bar{C}_i$ in $\hat{S}$ (with reduced structure), and $C_i$ their proper transforms in $S$. Then we can show that altogether we get a cycle of $n + n(l-1) = nl$ rational curves on $S$. Now we conclude that $S$ is again a parabolic Inoue surface with second betti number $nl$ as follows.

**Lemma 4.3.** Let $\bar{S} = S(n, \beta)$ and $S$ be as above. Then $S$ is a parabolic Inoue surface of the form $S(nl, \alpha)$ for some $\alpha$ with $\alpha^l = \beta$. 
Proof. We first show that $S$ is a parabolic Inoue surface of the form $S(nl, \alpha)$ for some $\alpha$.

By Lemma 3.3 the minimal model $S'$ of $S$ is a parabolic Inoue surface. Since the blowing-down map $u : S \to S'$ is isomorphic in a neighborhood of the proper transform $E$ of $\hat{E}$, the self-intersection number of the image of $E$ in $S'$ is again equal to $-nl$. Thus $S'$ should contain a cycle of $nl$ rational curves. But by construction $S$ contains no curves other than $E$ and the $nl$ rational curves mentioned above. Thus no curve on $S$ cannot be blowing down to a point of $S'$; namely $S$ already is minimal and is a parabolic Inoue surface of the form $S(nl, \alpha)$.

It remains to show that $\alpha^l = \beta$. We first construct the universal covering $b : \hat{W} \to S$ of $S$. Let $a : \hat{W} \to \tilde{S}$ be the universal covering of $\tilde{S} = S(n, \beta)$ so that $a^{-1}(\hat{E}) \to \hat{E}$ is isomorphic to the quotient $C^* \to C^*/(\beta)$. Then the pull-back $b : \hat{W} := \hat{W} \times_S S \to S$ of $a$ to $S$ via $wu : S \to \tilde{S}$ is an infinite cyclic unramified covering of $S$. Since $S$ is a parabolic Inoue surface and hence has an infinite cyclic fundamental group, $b$ must be the universal covering of $S$ and $b^{-1}(E)$ is isomorphic to $C^*$. Moreover, the projection $c : \hat{W} \to \hat{W}$ induces an unramified cyclic covering $c' : b^{-1}(E) \to a^{-1}(\hat{E})$ of degree $l$. Thus $c'$ is isomorphic to the map $C^* \to C^*, s \to s^l$, and the multiplication by $\alpha$ is sent to that by $\alpha^l$ on the image $C^*$. Thus by Lemma 4.1 we have $\alpha^l = \beta$. \hfill $\square$

Proof of Proposition 4.2. By Lemma 4.3 we may assume that $S$ and $\tilde{S}$ are the surfaces in that lemma with $m = ln$. Indeed, $n$ and $\beta$ in $\tilde{S}(n, \beta)$ of the lemma can be chosen arbitrarily and the same is true for $\alpha$ with $\alpha^l = \beta$ by a suitable choice of $\hat{S}$. Let $S \xrightarrow{q} S' \xrightarrow{r} \tilde{S}$ be the Stein factorization of $wu : S \to \tilde{S}$ so that we have the commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{v} & \hat{S} \\
q \downarrow & & w \\
S' & \xrightarrow{r} & \tilde{S},
\end{array}
\]  

(17)

where $v$ and $w$ are the natural maps defined above. The action of the Galois group $K$ of $w : \hat{S} \to \tilde{S}$ lifts to $S$ and $q$ is identified with the quotient map $S \to S/K$. The action of $K$ preserves each irreducible components of the curves on $S$ and therefore $K$ is contained in $\text{Aut}_1 S$. Moreover, since $K$ acts freely on $E(\cong \hat{E})$ in $S$, the intersection of $K$ and $\text{Aut}_0 S$
is trivial. Thus $K$ must be contained in one of the maximal cyclic subgroups $M_j$. Thus the first assertion is proved. The second one follows immediately from the construction of $S$ from $\bar{S}$. □

Example 4: We consider the special case of Example 3 for $l = 2$. In this case we can describe the quotient directly. Let $S = S(m, \alpha)$ as before with $m = 2n$ even. Then we obtain an involution $\iota := g^n_1$ (cf. (16)). Explicitly, this is induced by the involution $\tilde{\iota}$ on $W$ defined by

\[
\tilde{\iota} : (x_k, y_k) \rightarrow ((-1)^k x_k, (-1)^{k+1} y_k) \quad \text{on } U_k, \quad (w, x) \rightarrow (-w, x) \quad \text{on } V.
\]

We have a cycle of rational curves $C = C_1 + \cdots + C_n$ on $S$, where $C_i$ is the natural image of $\tilde{C}_i$. Then $C_1 + C_3 + \cdots + C_{n-1}$ is precisely the fixed point set of $\iota$.

Therefore the quotient $S' := S/\langle \iota \rangle$ is a smooth surface, whose structure is described as follows. Let $C'_i$ be the images of $C_i$. Then we have $(C'_{2k-1})^2 = -4$ and $(C'_{2k})^2 = -1$. Thus we may contract $C'_{2k}$ to points and obtain another smooth surface $\tilde{S}$. Let $\tilde{C}_i$ be the images of $C'_i$ and $\tilde{C}$ that of $C' = \sum_i C'_i$ in $\tilde{S}$. Then $\tilde{C}$ is of the form $\tilde{C} = \tilde{C}_1 + \tilde{C}_3 + \cdots + \tilde{C}_{2l-1}$ with $(\tilde{C}_{2k-1})^2 = -2$. The image $\tilde{E} = E/\langle \iota \rangle$ of the elliptic curve $E$ on $\tilde{S}$ is again a smooth elliptic curve. Clearly $\tilde{E}$ and $\tilde{C}$ are the unique curves on $\tilde{S}$, and $\tilde{S}$ is again a parabolic Inoue surface and with second betti number $n$. Thus the quotient is isomorphic to $S(n, \alpha^2)$.

Let $S \xrightarrow{v} \tilde{S} \xrightarrow{w} \bar{S}$ be the Stein factorization of $S \to \bar{S}$. It turns out that $v$ is nothing but the contraction of the $l$ (-2)-curves $C_{2k}$ to $l$ ordinary double points on the normal surface $\hat{S}$, and $w$ is the branched double covering with branch locus $\tilde{C}$. We thus recover the diagram (17) from the other direction. □

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