Uniform Sampling of Undirected and Directed Graphs with a Fixed Degree Sequence*

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Abstract

Many applications in network analysis require algorithms to sample uniformly at random from the set of all graphs with a prescribed degree sequence. We present a Markov chain based approach which converges to the uniform distribution of all realizations for both the directed and undirected case. It remains an open challenge whether these Markov chains are rapidly mixing.

For the case of directed graphs, we also explain in this paper that a popular switching algorithm fails in general to sample uniformly at random because the state graph of the Markov chain decomposes into different isomorphic components. We call degree sequences for which the state graph is strongly connected arc swap sequences. To handle arbitrary degree sequences, we develop two different solutions. The first uses an additional operation (a reorientation of induced directed 3-cycles) which makes the state graph strongly connected, the second selects randomly one of the isomorphic components and samples inside it. Our main contribution is a precise characterization of arc swap sequences, leading to an efficient recognition algorithm. Finally, we point out some interesting consequences for network analysis.

1 Introduction

We consider the problem of sampling uniformly at random from the set of all realizations of a prescribed degree sequence as simple, labeled graphs or digraphs, respectively, without loops.

Motivation. In complex network analysis, one is interested in studying certain network properties of some observed real graph in comparison with an ensemble of graphs with the same degree sequence to detect deviations from randomness. For example, this is used to study the motif content of classes of networks. To perform such an analysis, a uniform sampling from the set of all realizations is required. A general method to sample random elements from some set of objects is via rapidly mixing Markov chains. Every Markov chain can be viewed as a random walk on a directed graph, the so-called state graph. In our context, its vertices (the states) correspond one-to-one to the set of all realizations of prescribed degree sequences. For a survey on random walks, we refer to Lovász.

A popular variant of the Markov chain approach to sample among such realizations is the so-called switching-algorithm. It starts with a given realization, and then performs a sequence of 2-swaps.

In the undirected case, a 2-swap replaces two non-adjacent edges \(\{a, b\}, \{c, d\}\) either by \(\{a, c\}, \{b, d\}\) or by \(\{a, d\}, \{b, c\}\), provided that both new edges have not been contained in the graph before the swap operation. Likewise, in the directed case, given two arcs \((a, b), (c, d)\) with all vertices \(a, b, c, d\) being distinct, a 2-swap replaces these two arcs by \((a, d), (c, b)\) which are currently not included in the realization (the latter is crucial to avoid parallel arcs). The switching algorithm is usually stopped heuristically after a certain number of iterations, and then outputs the resulting realization as a “random element”.

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For undirected graphs, one can prove that this switching algorithm converges to a random stage. The directed case, however, turns out to be much more difficult. The following example demonstrates that the switching algorithm does not even converge to a random stage.

**Example 1.1.** Consider the following class of digraphs $D = (V, A)$ with $3n$ vertices $V = \{v_1, v_2, \ldots, v_{3n}\}$, see Figure 1. Roughly speaking, this class consists of induced directed 3-cycles $C_i$ formed by triples $V_i = \{v_{3i}, v_{3i+1}, v_{3i+2}\}$ of vertices, and arcs $A_i = \{(v_{3i}, v_{3i+1}), (v_{3i+1}, v_{3i+2}), (v_{3i+2}, v_{3i})\}$ for $i \in \{0, \ldots, n-1\}$. All vertices of cycle $C_i$ are connected to all other vertices of cycles with larger index than $i$. More formally, let $A' := \{(v, w)|v \in V_i, w \in V_j, i < j\}$. We set $A := A' \cup (\cup_{i=1}^{n} A_i)$.

It is easy to check that no 2-swap can be applied to this digraph. However, we can independently reorient each of the $n$ induced 3-cycles, leading to $2^{n/3}$ many (isomorphic) realizations of the same degree sequence. Thus, if we use a random walk on the state graph of all realizations of this degree sequence and use only 2-swaps to define the possible transitions between realizations, this state graph consists exactly of $2^{n/3}$ many singleton components. Hence, a “random walk” on this graph will be stuck in a single realization although exponentially many realizations exist.

More examples of graph classes of this type will be given in the Appendix. It is interesting to note that 2-swap operations suffice to sample directed graphs with loops as has been proven by Ryser [Rys57] in the context of square matrices with \{0,1\}-entries which can be interpreted as node-node adjacency matrices of digraphs with loops.

**Realizability of degree sequences.** In order to use a Markov chain approach one needs at least one feasible realization. In applications from complex network analysis, one can usually take the degree sequence of some observed real world graph. Otherwise, one has to construct a realization.

The realizability problem, i.e., characterizing the existence and finding at least one realization, has quite a long history. First results go back to the seminal work by Tutte who solved the more general $f$-factor problem [Tut52]. Given a simple graph $G = (V, E)$ and a function $f : V(G) \rightarrow \mathbb{N}_0$, an $f$-factor is a subgraph $H$ of $G$ such that every vertex $v \in V$ in this subgraph $H$ has exactly degree $d_G(v) = f(v)$. Tutte gave a polynomial time transformation of the $f$-factor problem to the perfect matching problem. This implies the first polynomial time algorithm for finding some $f$-factor [Tut54]. For a survey on efficient algorithms for the $f$-factor problem by matching or network flow techniques, we refer to Chapter 21 of Schrijver [Sch03]. Clearly, if the given graph $G$ is complete, then every $f$-factor is a solution of the degree sequence problem. Erdős and Gallai [EG60] proved a simpler Tutte-type result for the degree sequence problem. Already in 1955, Havel [Hav55] developed a simple greedy-like algorithm to construct a realization of a given degree sequence as a simple undirected graph without loops. A few years later, Hakimi [Hak62] studied the simpler case of undirected graphs with multiple edges.

It is also well-known how to test whether a prescribed degree sequence can be realized as a digraph. Chen [Che66] presented necessary and sufficient conditions for the realizability of degree sequences which can be checked in linear time. Again, the construction of a concrete realization is equivalent to an $f$-factor problem on a corresponding undirected bipartite graph. Kleitman and Wang [KW73] found a greedy-type algorithm generalizing previous work by Havel [Hav55] and Hakimi [Hak62, Hak65]. This approach has recently been rediscovered by Erdős et al. [EMT09].

**Related work.** Kannan et al. [KTV99] showed how to sample bipartite undirected graphs via Markov chains. They proved polynomial mixing time for regular and near-regular graphs. Cooper et al. [CDG07] extended this work to non-bipartite undirected, $d$-regular graphs and proved a polynomial mixing time.
Our contribution. In this paper, we prove the following results.

Random walks and Markov chains. Let us briefly review the basic notions of random walks and their relation to Markov chains. See [Lov96] for more details. A random walk (Markov chain) on a digraph $D = (V, A)$ is a sequence of vertices $v_0, v_1, \ldots, v_t$ where $(v_i, v_{i+1}) \in A$. Vertex $v_0$ represents the initial state. Denote by $d^+_D(v)$ the out-degree of vertex $v \in V$. At the $t$th step we move to an arbitrary neighbor of $v_t$ with probability $1/d^+_D(v_t)$ or stay at $v_t$ with probability $(1 - 1/d^+_D(v_t))$. Furthermore, we define the distribution of $V$ at time $t \in \mathbb{Z}^+$ as the function $P_t \in [0, 1]^{\#V}$ with $P_t(i) := \text{Prob}(v_t = i)$. A well-known result [Lov96] is that $P_t$ tends to the uniform stationary distribution for $t \to \infty$, if the digraph is (1) non-bipartite (that means aperiodic), (2) strongly connected (i.e., irreducible), (3) symmetric, and (4) regular. A digraph $D$ is $d_D$-regular if all vertices have the same in- and out-degrees $d_D$.

In this paper, we will view all Markov chains as random walks on symmetric $d_D$-regular digraphs $D = (V, A)$ whose vertices correspond to the state space $V$. The transition probability on each arc $(v, w) \in A$ will be the constant $1/d_D$.

Our contribution. In this paper, we prove the following results.

- For undirected graphs we analyze the well-known switching algorithm. It is straight-forward to translate the switching algorithm into a random walk on an appropriately defined Markov chain. This Markov chain corresponds to a symmetric, regular, strongly connected, non-bipartite simple digraph with directed loops allowed. Thus, it converges to the uniform distribution of all realizations. Each realization of the degree sequence is a vertex of this digraph, and two realizations are mutually connected by arcs if and only if their symmetric difference is an alternating 4-cycle (i.e., corresponds to a 2-swap). This graph becomes regular by adding additional loops, see Section 2.

- Cooper et al. [CDG07] already showed in the context of regular graphs that the underlying digraph of this Markov chain is strongly connected, but we give a much simpler proof of this property. Its diameter is bounded by the number $m$ of edges in the prescribed degree sequence.

- Carefully looking at our Example 1.1, we observe that in the directed case the state graph becomes strongly connected if we add a second type of operation to transform one realization into another: Simply reorient the arcs of an induced directed 3-cycle. We call this operation 3-cycle reorientation. We give a graph-theoretical proof that 2-swaps and 3-cycle reorientations suffice not only here, but also in general for arbitrary prescribed degree sequences. These observations allow us to define a Markov chain, very similar to the undirected case. The difference is that two realizations are mutually connected by arcs if and only if their symmetric difference is either an alternating directed 4-cycle or 6-cycle with exactly three different vertices. Again, this digraph becomes regular by adding additional loops, see Section 3. The transition probabilities are of order $O(1/m^2)$, and the diameter can be bounded by $O(m)$, where $m$ denotes the number of arcs in the prescribed degree sequence.
In the context of \((0,1)\)-matrices with given marginals (i.e., prescribed degree sequences in our terminology), Rao et al. \cite{RaoJ09} similarly observed that switching operations on so-called “compact alternating hexagons” are necessary. A compact alternating hexagon is a \(3 \times 3\)-submatrix, which can be interpreted as the adjacency matrix of a directed 3-cycle subgraph. They define a random walk on a series of digraphs, starting with a non-regular state graph which is iteratively updated towards regularity, i.e. their Markov chain converges asymptotically to the uniform distribution. However, it is unclear how fast this process converges and whether this is more efficient than starting directly with a single regular state graph. Since Rao et al. work directly on matrices, their transition probabilities are of order \(O(1/n^6)\), i.e., by several orders smaller than in our version.

Very recently, Erdős et al. \cite{Erd09} proposed a similar Markov chain approach using 2-swaps and 3-swaps. The latter type of operation exchanges a simple directed 3-path or 3-cycle \((v_1, v_2, v_3, v_4)\) (the first and last vertex may be identical) by \((v_1, v_3), (v_3, v_2), (v_2, v_4)\), but is a much larger set of operations than ours.

- Although in directed graphs 2-swaps alone do not suffice to sample uniformly in general, the corresponding approach is still frequently used in network analysis. One reason for the popularity of this approach — in addition to its simplicity — might be that it empirically worked in many cases quite well \cite{MKI04}. In this paper, we study under which conditions this approach can be applied and provably leads to correct uniform sampling. We call such degree sequences arc-swap sequences, and give a graph-theoretical characterization which can be checked in polynomial time. More specifically, we can recognize arc-swap sequences in \(O(n^2)\) time using matching techniques. Using a parallel Havel-Hakimi algorithm by LaMar \cite{LaM09}, originally developed to realize Euler sequences with an odd number of arcs, the recognition problem can even be solved in linear time. This algorithm also allows us to determine the number of induced directed 3-cycles which appear in every realization.

However, the simpler approach comes with a price: our bound on the diameter of the state graph becomes \(mn\) and so is by one order of \(n\) worse in comparison with using 2-swaps and 3-cycle reorientations. Since half of the diameter is a trivial lower bound on the mixing time and the diameter also appears as a factor in known upper bounds, we conjecture that the classical switching algorithm requires a mixing time \(\tau_x\) with an order of \(n\) more steps as the variant with 3-cycle reorientation.

In those cases where 2-swaps do not suffice to sample uniformly, the state graph decomposes into \(2^k\) strongly connected components, where \(k\) is the number of induced directed 3-cycles which appear in every realization. We can also efficiently determine the number of strongly connected components of the state graph (of course, without explicitly constructing this exponentially sized graph). However, all these components are isomorphic. This can be exploited as follows: For a non-arc-swap sequence, we first determine all those induced directed 3-cycles which appear in every realization. By reducing the \(\text{in-}\) and \(\text{out-}\)degrees for all vertices of these 3-cycles by one, we then obtain a new sequence, now guaranteed to be an arc-swap sequence. On the latter we can either use the switching algorithm or our variant with additional 3-cycle reorientations on a smaller state graph with a reduced diameter \(n(m - 3k)\) or \(m - 3k\), respectively, yielding an important practical advantage.

Our results give a theoretical foundation to compute certain network characteristics on unlabeled digraphs in a single component using 2-swaps only. For example, this includes the analysis of the motif content \cite{MSO02}. Likewise we can still compute the average diameter among all realizations if we work in a single component. However, for other network characteristics, for example betweenness centrality on edges \cite{KLP05}, this leads in general to incorrect estimations.

**Overview.** The remainder of the paper is structured as follows. In Section 2 we start with the undirected case. We introduce appropriately defined state graphs underlying our Markov chains, and show for these graphs crucial properties like regularity and strong connectivity. We also upper bound their diameter. The more difficult directed case is presented in Section 3. Afterwards, in Section 4 we characterize those degree sequences for which a simpler Markov chain based on 2-swaps provably leads to uniform sampling in the directed case. We also describe a few consequences and applications. Finally, we conclude with a short summary and remarks on future work.
2 Sampling Undirected Graphs

In this section we show how to sample undirected graphs with a prescribed degree sequence uniformly at random with a random walk. This section is structured as follows. We first give a formal problem definition and introduce some notation. Then we introduce an appropriately defined Markov chain and prove that it has all desired properties.

**Formal problem definition.** In the undirected case, a degree sequence $S$ of order $n$ is the ordered set $(a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{Z}^+, a_i > 0$. Let $G = (V, E)$ be an undirected labeled graph $G = (V, E)$ without loops and parallel edges and $|V| = n$. We define the degree-function $d : V \rightarrow \mathbb{Z}^+$ which assigns to each vertex $v_i \in V$ the number of incident edges. We call $S$ a graphical sequence if and only if there exists at least one undirected labeled graph $G = (V, E)$ without any loops or parallel edges which satisfies $d(v_i) = a_i$ for all $v_i \in V$ and $i \in \{1, \ldots, |V|\}$. Any such undirected graph $G$ is called realization of $S$.

We define an alternating walk $P$ for a graph $G = (V, E)$ as a sequence $P := (v_1, v_2, \ldots, v_k)$ of vertices $v_i \in V$ where either $\{v_i, v_{i+1}\} \in E(G)$ and $\{v_{i-1}, v_i\} \notin E(G)$ or $\{v_i, v_{i+1}\} \notin E(G)$ and $\{v_{i-1}, v_i\} \in E(G)$ for $i \mod 2 = 1$. The length of a walk (or path, cycle, respectively) is the number of its edges. We call an alternating walk $C$ of even length alternating cycle if $v_1 = v_k$ is fulfilled. For two realizations $G, G'$, the symmetric difference of their edge sets $E(G)$ and $E(G')$ is denoted as $G \Delta G' := (E(G) \setminus E(G')) \cup (E(G') \setminus E(G))$. A graph is called Eulerian if every vertex has even degree. Note that the symmetric difference $G \Delta G'$ of two realizations $G, G'$ is Eulerian and hence always decomposes into a number of alternating cycles.

**The Markov chain.** We denote by $\Psi = (V_\Psi, A_\Psi)$ the digraph for our random walk, the state graph, for short. Its underlying vertex set $V_\Psi$ is the set of all realizations of a given degree sequence $S$. For a realization $G$, we denote by $V_G$ the corresponding vertex in $V_\Psi$. The arc set $A_\Psi$ is defined as follows.

a) We connect two vertices $V_G, V_{G'} \in V_\Psi, G \neq G'$ with arcs $(V_G, V_{G'})$ and $(V_{G'}, V_G)$ if and only if $|G \Delta G'| = 4$ is fulfilled.

b) We set for each pair of non-adjacent edges $\{v_{i_1}, v_{i_2}\}, \{v_{i_3}, v_{i_4}\} \in E(G), i_j \in \{1, \ldots, n\}$ a directed loop $(V_G, V_G)$ if and only if $\{v_{i_1}, v_{i_2}\} \in E(G) \cup \{v_{i_3}, v_{i_4}\} \in E(G)$.

c) We set for each pair of non-adjacent edges $\{v_{i_1}, v_{i_2}\}, \{v_{i_3}, v_{i_4}\} \in E(G), i_j \in \{1, \ldots, n\}$ a directed loop $(V_G, V_G)$ if and only if $\{v_{i_1}, v_{i_3}\} \in E(G) \cup \{v_{i_2}, v_{i_4}\} \in E(G)$.

d) We set one directed loop $(V_G, V_G)$ for all $V_G \in V_\Psi$.

**Lemma 2.1.** The state graph $\Psi = (V_\Psi, A_\Psi)$ is non-bipartite, symmetric, and regular.

**Proof.** Non-bipartiteness follows from the insertion of directed loops. Likewise, symmetry is obvious since we always introduce arcs in both directions in case a). For each pair of non-adjacent edges of a realization $G$, we introduce exactly two arcs in $\Psi$. These arcs either connect two neighboring states or are directed loops. Thus each vertex $V_G \in V_\Psi$ has an out-degree of twice the number of non-adjacent edges in $G$ plus one (for the loop in step d)). Due to symmetry, the out-degree equals the in-degree. For each realization $G$, the number of pairs of non-adjacent edges is exactly $\binom{n}{2} - \sum_{v_i \in V(G)} (d_-(v_i)) = \binom{n}{2} - \sum_{i=1}^{n} \binom{2}{2}$, that is a constant independent of $G$.

The next step is to show that the state graph is strongly connected. We first prove the following auxiliary proposition which asserts that the symmetric difference of two different realizations always contains a vertex-disjoint path of length three.

**Proposition 2.2.** Let $S$ be a graphical sequence and $G$ and $G'$ be two different realizations, i.e., $G \Delta G' \neq \emptyset$. Then there exists a vertex-disjoint alternating walk $P = (v_1, v_2, v_3, v_4)$ in $G \Delta G'$ with $\{v_1, v_2\}, \{v_3, v_4\} \in E(G)$ and $\{v_2, v_3\} \in E(G')$.

**Proof.** In the proof of this proposition, we argue only about edges in the symmetric difference $G \Delta G'$ which is assumed to be non-empty. Therefore, there are edges $\{v_1, v_2\}, \{v_2, v_3\}$ with $\{v_1, v_2\} \in E(G)$ and $\{v_2, v_3\} \in E(G')$ and $v_1 \neq v_3$. If there is also an edge $\{v_3, v_4\} \in E(G)$ with $v_4 \neq v_1$, we are done with
the vertex-disjoint alternating walk \( P = (v_3, v_2, v_3, v_4) \) as desired. Otherwise, the symmetric difference must contain the edge \( \{v_3, v_1\} \in E(G) \) and also some edge \( \{v_1, v_4\} \in E(G') \). Note that \( v_4 \neq v_2 \) and \( v_4 \neq v_3 \). This implies the existence of another edge \( \{v_4, v_5\} \in E(G) \). Note also that \( v_5 \neq v_3 \), since we are in the case that \( \{v_3, v_4\} \) does not exist. Either \( v_5 = v_2 \) or \( v_5 \) is a new vertex disjoint from \( \{v_1, \ldots, v_4\} \).

Therefore, in both cases \( P = (v_3, v_1, v_4, v_5) \) is a vertex-disjoint alternating walk composed of edges from the symmetric difference.

\[ \square \]

**Lemma 2.3.** Let \( S \) be a graphical sequence and let \( G \neq G' \) be two realizations. Then there exist realizations \( G_0, G_1, \ldots, G_k \) with \( G_0 := G, G_k := G' \) and \( |G_i \Delta G_{i+1}| = 4 \) where \( k \leq \frac{1}{2}|G \Delta G'| - 1 \).

**Proof.** We prove the lemma by induction according to the cardinality of the symmetric difference \(|G \Delta G'| = 2k\). For \( k := 2 \) we get \(|G \Delta G'| = 4\). The correctness of our claim follows with \( G_1 := G' \). We assume the correctness of our claim for all \( k \leq \ell \). Consider \(|G \Delta G'| = 2\ell + 2\). According to Proposition 2.2 there exists in \( G \Delta G' \) an alternating vertex-disjoint alternating walk \( P = (v_1, v_2, v_3, v_4) \) with \( \{v_1, v_2\}, \{v_3, v_4\} \in E(G) \) and \( \{v_3, v_2\} \in E(G') \).

**case 1:** Assume \( \{v_1, v_3\} \in E(G') \setminus E(G) \).

This implies \( \{v_1, v_3\} \notin G \Delta G' \). An alternating path of an alternating cycle \( C = (v_4, v_5, \ldots, v_j, v_1, v_2, v_3, v_4) \) with \( \{v_1, v_4\}, \{v_1, v_3\} \in A(G) \) of length \( |C| \geq 6 \). We construct a new alternating cycle \( C^* := (C \setminus P) \cup \{v_1, v_4\} \) with length \( |C^*| = |C| - 2 \geq 4 \). We swap the arcs in \( C^* \) and get a realization \( G^* \) of \( S \) with \( |G_0 \Delta G^*| = |C^*| \leq 2\ell \) and \( |G^* \Delta G'| = |G \Delta G'| - (|C| - 1) + 1 \leq 2\ell \). The symmetric difference \( G \Delta G' \) may consist of several connected components, but each of them has strictly smaller cardinality. Thus, we obtain realizations \( G_1, G_2, \ldots, G_k \) with \( G_k := G' \) and \( |G_i \Delta G_{i+1}| = 4 \) where \( k - 1 \leq \frac{1}{2}|G \Delta G'| - 1 \). Hence, we get the sequence \( G_0, G_1, \ldots, G_k \) with \( k = 1 + \frac{1}{2}|G \Delta G'| - 1 = \frac{1}{4}|(G \Delta G')| - 1 \).

**case 2:** Assume \( \{v_1, v_3\} \notin E(G) \cap E(G') \).

This implies \( \{v_1, v_3\} \notin G \Delta G' \). An alternating path of an alternating cycle \( C = (v_4, v_5, \ldots, v_j, v_1, v_2, v_3, v_4) \) with \( \{v_1, v_4\}, \{v_1, v_3\} \in A(G) \) of length \( |C| \geq 6 \). We construct a new alternating cycle \( C^* := (C \setminus P) \cup \{v_1, v_4\} \) with length \( |C^*| = |C| - 2 \geq 4 \). We swap the arcs in \( C^* \) and get a realization \( G^* \) of \( S \) with \( |G_0 \Delta G^*| = |C^*| \leq 2\ell \) and \( |G^* \Delta G'| = |G \Delta G'| - (|C^*| - 1) + 1 \leq 2\ell \). The symmetric difference \( G \Delta G' \) may consist of several connected components, but each of them has strictly smaller cardinality. Thus, we obtain realizations \( G_1, G_2, \ldots, G_k \) with \( G_k := G' \) and \( |G_i \Delta G_{i+1}| = 4 \) where \( k - 1 \leq \frac{1}{2}|G \Delta G'| - 1 \). Hence, we get the sequence \( G_0, G_1, \ldots, G_k \) with \( k = 1 + \frac{1}{2}|G \Delta G'| - 1 = \frac{1}{4}|(G \Delta G')| - 1 \).

**case 3:** Assume \( \{v_1, v_3\} \in E(G) \setminus E(G') \).

This implies \( \{v_1, v_3\} \notin G \Delta G' \). Assume first that the symmetric difference \( G \Delta G' \) contains an alternating cycle \( C \) which avoids \( P \). Then, we can apply the induction hypothesis to \( C \). Swapping the edges of \( C \), we get a realization \( G^* \) of sequence \( S \) with \( |G_0 \Delta G^*| = |C^*| \leq 2\ell \) and \( |G^* \Delta G'| = |G \Delta G'| - |C| \leq 2\ell \). According to the induction hypothesis there exist sequences \( G_0^* := G, G_1^*, \ldots, G_k^* := G' \) and \( G_0^* := G, G_1^*, \ldots, G_k^* := G' \) with \( k \leq \frac{1}{2}|G \Delta G'| - 1 \) and \( k \leq \frac{1}{2}|G \Delta G'| - 1 \). We arrange these sequences one after another and get a sequence which fulfills \( k = 1 + k_2 = \frac{1}{4}|G \Delta G'| - 1 + \frac{1}{4}|G \Delta G'| - (|C| - 1) + 1 - 1 = \frac{1}{2}|(G \Delta G')| - 1 \).

It remains to consider the case that such a cycle \( C \) does not exist. In other words, every alternating cycle in \( G \Delta G' \) includes edges from \( P \).

The alternating walk \( P \) can be extended to an alternating cycle \( C^* = (v_1, v_2, v_3, v_4, v_5, \ldots, v_t, v_1) \), \( t \geq 3 \) using only arcs from \( G \Delta G' \). To construct \( C^* \), start with \( P \), and keep adding alternating edges until you reach the start vertex \( v_1 \) for the first time with an edge \( \{v_1, v_4\} \in E(G') \). Since the symmetric difference is Eulerian, you will not get stuck before reaching \( v_1 \) with such an edge. Note that \( C^* \) must contain the edge \( \{v_1, v_4\} \), as otherwise an alternating cycle of type \( C \) would exist. This also implies the existence of an alternating sub-walk \( P_1 = (v_1, v_5, \ldots, v_9, v_4) \) of \( C^* \) of odd length (at least of length 3), starting and ending with edges in \( E(C^*) \). Likewise, there must be another alternating sub-walk \( P_2 = (v_1, v_7, \ldots, v_8, v_1) \), also of odd length (at least of length 3), starting and ending with edges in \( E(C^*) \). The situation is visualized in Figure 2.

In this scenario, we have \( v_5 \neq v_7 \), as otherwise \( (E(P_1) \setminus \{v_7, v_1\}) \cup \{v_7 = v_5, v_4\} \) would be an alternating cycle of the form we have excluded above. We have four subcases with respect to the existence of edges between \( v_5 \) and \( v_7 \).
case a) \( \{v_5, v_7\} \in E(G) \setminus E(G') \):
This would imply the existence of the alternating cycle \( C = (v_5, v_4, v_1, v_7, v_5) \), excluded above.

case b) \( \{v_5, v_7\} \in E(G') \setminus E(G) \):
This would imply the existence of the alternating cycle \( C = (v_7, v_5, \ldots, v_6, v_4, v_1, v_8, \ldots, v_7) \), also excluded above.

case c) \( \{v_5, v_7\} \in E(G) \cap E(G') \):
Then there is an alternating cycle \( C' = (v_7, v_5, v_4, v_1, v_7) \) on which we can swap the edges in a single step. This leads to a realization \( G^* = G_1 \) with \( |G_0 \Delta G^*| = 4 \) and \( |G^* \Delta G'| = 2\ell \).

case d) \( \{v_5, v_7\} \not\in E(G) \) and \( \{v_5, v_7\} \not\in E(G') \):
As in case c), we consider the alternating cycle \( C' = (v_7, v_5, P_1 \backslash \{v_4, v_5\}, \{v_1, v_4\}, P_2 \backslash \{v_7, v_1\}, v_7) \). Swapping edges on \( C' \), we get a realization \( G^* = G_{k-1} \), but this time, \( |G_0 \Delta G^*| = |C'| \leq 2\ell \) and \( |G^* \Delta G'| = |G_0 \Delta G'| - (|C'| - 1) + 1 \leq 2\ell \). According to the induction hypothesis there exist sequences \( G_0^k := G, G_1^k, \ldots, G_{k-1}^k := G^* \) and \( G_0^\ell := G^*, G_1^\ell, \ldots, G_k^\ell = G' \) with \( k_1 \leq \frac{1}{2}|G_0 \Delta G^*| - 1 = \frac{1}{2}(|G_0 \Delta G'| - |C'| + 2) - 1 \) and \( k_2 \leq \frac{1}{2}|G^* \Delta G'| - 1 = \frac{1}{2}|C'| - 1 \). We arrange these sequences one after another and get a sequence which fulfills \( k = k_1 + k_2 \leq \frac{1}{2}(|G_0 \Delta G'| - |C'| + 2) - 1 + \frac{1}{2}|C'| - 1 = \frac{1}{2}(|G_0 \Delta G'| - 1).

case 4: Assume \( \{v_1, v_4\} \not\in E(G) \cup E(G') \). This implies \( \{v_1, v_4\} \not\in G_0 \Delta G^* \). It exists the alternating cycle \( C := (v_1, v_2, v_3, v_4, v_1) \) with \( \{v_1, v_4\} \not\in E(G) \). \( G_1 := (G_0 \backslash \{v_1, v_2, \{v_3, v_4\}\}) \cup \{\{v_3, v_2\}, \{v_1, v_4\}\} \) is a realization of \( S \) and it follows \( |G_0 \Delta G_1| = 4 \) and \( |G_1 \Delta G'| = 2\ell + 2 - 2 = 2\ell \). According to the induction hypothesis there exist realizations \( G_1, G_2, \ldots, G_k \) with \( G_k := G' \) where and \( k \leq \frac{1}{2}(|G_0 \Delta G'| - 2) \). Hence, we get the sequence \( G_0, G_1, \ldots, G_k \) with \( k \leq \frac{1}{2}(|G_0 \Delta G'| - 1) \).

\[ \blacksquare \]

We have shown that the state graph \( \Psi = (V_0, A_0) \) is a d-regular, symmetric, non-bipartite, and strongly connected digraph. Hence, the corresponding Markov chain has the uniform distribution as its stationary distribution. A random walk on \( \Psi = (V_0, A_0) \) can be described by Algorithm \( \square \). This algorithm requires a data structure \( DS \) containing all pairs of non-adjacent edges in \( G \).

## 3 Sampling Digraphs

We now turn the directed case. As before, we start with the formal problem definition and some additional notation. Then, we introduce our Markov chain and analyze its properties.

**Formal problem definition** In the directed case, we define a degree sequence \( S \) as a sequence of 2-tuples \( \left( \binom{a_1}{b_1}, \binom{a_2}{b_2}, \ldots, \binom{a_n}{b_n} \right) \) with \( a_i, b_i \in \mathbb{Z}_+^n, i \in \{1, \ldots, n\} \) where \( a_i > 0 \) or \( b_i > 0 \).

Let \( G = (V, A) \) be a directed labeled graph \( G = (V, A) \) without loops and parallel arcs and \( |V| = n \). We define the in-degree-function \( d_+^G : V \to \mathbb{Z}_+^n \) which assigns to each vertex \( v_i \in V \) the number of incoming arcs and the out-degree-function \( d_-^G : V \to \mathbb{Z}_0^+ \) which assigns to each vertex \( v_i \in V \) the number of outgoing arcs. We denote \( S \) as graphical sequence if and only if there exists at least one directed labeled graph \( G = (V, A) \) without any loops or parallel arcs which satisfies \( d_+^G(v_i) = b_i \) and \( d_-^G(v_i) = a_i \) for all \( v_i \in V \) and \( i \in \{1, \ldots, n\} \). Any such graph \( G \) is called realization of \( S \). Let \( H \) be a subdigraph of \( G \). We say that \( H = (V_H, A_H) \) is an induced subdigraph of \( G \) if every arc of \( A \) with both end vertices in \( V_H \) is also in \( A_H \). We write \( H = G \langle V_H \rangle \).
The symmetric difference $G\Delta G'$ of two realizations $G \neq G'$ is defined analogously to the undirected case. Consider for example the realizations $G$ and $G'$ with $A(G) := \{(v_1, v_2), (v_3, v_4)\}$ and $A(G') := \{(v_1, v_3), (v_2, v_4)\}$ consisting of exactly two arcs. Then the symmetric difference is the alternating directed 4-cycle $C := (v_1, v_2, v_3, v_4, v_1)$ where $(v_i, v_{i+1}) \in A(G)$ for $i \in \{1, 3\}$ and $(v_{i+1}, v_i) \in A(G')$ taking indices $i \mod 4$. We define an alternating directed walk $P$ for a directed graph $G = (V, A)$ as a sequence $P := (v_1, v_2, \ldots, v_t)$ of vertices $v_i \in V$ where either $(v_i, v_{i+1}) \in A(G)$ and $(v_{i+1}, v_i) \notin A(G)$ or $(v_i, v_{i+1}) \notin A(G)$ and $(v_{i+1}, v_i) \in A(G)$ for $i \mod 2 = 1$. We call an even alternating directed walk $C$ alternating directed cycle if $v_1 = v_t$ is fulfilled. The symmetric difference of two realizations always decomposes into a number of alternating directed cycles, see Figs. 3 and 4.

The Markov chain. In the directed case, we denote the state graph for our random walk by $\Phi = (V_\phi, A_\phi)$. Its underlying vertex set $V_\phi$ is the set of all realizations of a given degree sequence $S$. For a realization $G$, we denote by $V_G$ the corresponding vertex in $V_\phi$. The arc set $A_\phi$ is defined as follows.

- a) We connect two vertices $V_G, V_G' \in V_\phi, G \neq G'$ with arcs $(V_G, V_G')$ and $(V_G', V_G)$ if and only if one of the two following constraints is fulfilled:
  1. $|G\Delta G'| = 4$
  2. $|G\Delta G'| = 6$ and $G\Delta G'$ contains exactly three different vertices.

- b) We set a directed loop $(V_G, V_G)$
  1. for each pair of non-adjacent arcs $(v_{i_1}, v_{i_2}), (v_{i_3}, v_{i_4}) \in A(G), i_j \in \{1, \ldots, n\}$ if and only if $(v_{i_1}, v_{i_2}) \notin A(G) \lor (v_{i_3}, v_{i_4}) \notin A(G)$ in a realization $G$. 

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**Algorithm 1 Switching Algorithm**

**Input:** sequence $S$, an undirected graph $G = (V, E)$ with $d_G(v_i) = a_i$ for all $i \in \{1, \ldots, n\}$ and $v_i \in V$, a mixing time $\tau$.

**Output:** A sampled undirected graph $G' = (V, E')$ with $d_G(v_i) = a_i$ for all $i \in \{1, \ldots, n\}$ and $v_i \in V$.

1: $t := 0$, $G' := G$ // initialization
2: while $t < \tau$ do
3: Choose an element $p$ from $DS$ uniformly at random. // $p$ is a pair of nonAdjacent edges.
4: Let $p$ be the pair of edges $\{v_{i_1}, v_{i_2}\}, \{v_{i_3}, v_{i_4}\}$.
5: Choose with probability $\frac{1}{2}$ between case a) and case b).
6: if case a) then
7: // Either walk on to an adjacent realization
8: Delete $\{v_{i_1}, v_{i_2}\}, \{v_{i_3}, v_{i_4}\}$ in $E(G')$.
9: Add $\{v_{i_1}, v_{i_4}\}, \{v_{i_2}, v_{i_3}\}$ to $E(G')$.
10: else
11: // or walk a loop: ‘Do nothing’
12: end if
13: else
14: // case b)
15: if $\{v_{i_1}, v_{i_3}\}, \{v_{i_2}, v_{i_4}\} \notin E(G')$ then
16: // Either walk on to an adjacent realization
17: Delete $\{v_{i_1}, v_{i_3}\}, \{v_{i_2}, v_{i_4}\}$ in $E(G')$.
18: Add $\{v_{i_1}, v_{i_2}\}, \{v_{i_3}, v_{i_4}\}$ to $E(G')$.
19: end if
20: else
21: // or walk a loop: ‘Do nothing’
22: end if
23: end if
24: update data structure $DS$
25: $t \leftarrow t + 1$
26: end while
exists a realization of $G$ with an incoming and an outgoing arc at each vertex. $\sum_{\text{adjacent arcs}}$ of $G$ is not bipartite. For the proof of regularity, note, that we consider at each vertex $V_i$ a directed $b_i$-cycle then we get condition $b)2.i$. Hence, $\Phi$ is not bipartite. For the proof of regularity, note, that we consider at each vertex $V_i$ the number of pairs of non-adjacent arcs in a realization $G$. This is the number of all possible arc pairs minus the number of adjacency arcs $\left(\frac{|A(G)|}{2} - \sum_{i=1}^{n} a_i \frac{n}{2} + \sum_{i=1}^{n} a_i b_i\right)$ where $\sum_{i=1}^{n} \binom{n}{2}$ is the number of all incoming arc pairs at each vertex, $\sum_{i=1}^{n} b_i$ is the number of all outgoing arc pairs at each vertex and $\sum_{i=1}^{n} a_i b_i$ is the number of directed 2-paths in a realization $G$. Hence, the number of non-adjacent arcs is a constant value for each realization $G$. For each of these arc pairs we either set a directed loop or an incoming and an outgoing arc at each vertex $V_i \in V_G$. For each 2-path in $G$ we set a loop if it is not part of a directed 3-cycle $C = (v_{i1}, v_{i2}, v_{i3}, v_{i4})$ which is an induced subgraph $C = G\left(\{v_{i1}, v_{i2}, v_{i3}\}\right)$. If it is the case it exists a realization $G'$ with $|G\Delta G'| = 6$ and $G\Delta G'$ contains exactly 3 different vertices. Hence, we set for the 2-path in $C$ with $i < j$ and $i < j''$ with $j, j' \in \{1, 2, 3\}$ the directed arcs $(V_{G'}, V_{G''})$ and $(V_{G''}, V_{G})$ for both other 2-paths in $C$ a directed loop. Generally, we set for all 2-paths in a realization an incoming and an outgoing arc at each $V_i$. The number of 2-paths in each realization is the constant value $\sum_{i=1}^{n} a_i b_i$. Hence, the vertex degree at each vertex is $d_{\Phi} := d_{\Phi}^{+} = d_{\Phi}^{-} = \left(\frac{|A(G)|}{2}\right) - 2 \sum_{i=1}^{n} a_i$. 

In the next section we have to prove that our constructed graphs are strongly connected. This is sufficient to prove the reachability of each realization independent of the starting realization. Fig. 5 shows an example how the realization $G$ from Fig. 3 can be transformed to the realization $G'$ by a sequence of swap operations.
3.1 Symmetric differences of two different realizations

**Proposition 3.2.** Let $S$ be a graphical sequence and $G$ and $G'$ be two different realizations. If $G \Delta G'$ is exactly one weak component and $|G \Delta G'| \neq 6$ then there exists in $G \Delta G'$ a vertex-disjoint alternating 3-walk of type $P$ or $Q$, where $P = (v_1, v_2, v_3, v_4)$ with $(v_1, v_2), (v_3, v_4) \in A(G)$ and $(v_3, v_2) \in A(G')$ and $Q = (w_1, w_2, w_3, w_4)$ with $(w_1, w_2), (w_3, w_4) \in A(G')$ and $(w_3, w_2) \in A(G)$.

**Proof.** Note that in $G \Delta G'$ an alternating cycle of length two is not possible. Otherwise, there exists an arc $(u, v) \in A(G) \cap A(G')$ in contradiction to our assumption that $(u, v) \in G \Delta G'$. The symmetric difference $G \Delta G'$ may decompose into a number of alternating cycles $(G \Delta G')_i$. We consider a decomposition into the minimum number of such cycles. If one of these alternating cycles $(G \Delta G')_i$ contains a vertex-disjoint alternating 3-walk $P$ or $Q$ as claimed, we are done. Otherwise, each vertex is repeated at each third step in $(G \Delta G')_i$. Hence, we get the alternating cycles $(G \Delta G')_i := (v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}, v_{i_6})$ where $(v_{i_1}, v_{i_2}), (v_{i_3}, v_{i_4}) \in A(G)$ and $(v_{i_5}, v_{i_1}), (v_{i_6}, v_{i_2}), (v_{i_3}, v_{i_4}) \in A(G')$. The cycle cannot be longer, as the graph induced by $G \Delta G' \{v_1, v_2, v_3\}$ is already complete. Since $|(G \Delta G')_i| = 6$, there must be $(G \Delta G')_j$ with $i \neq j$. $(G \Delta G')_j$ shares at least one vertex with $(G \Delta G')_i$, because $G \Delta G'$ is weakly connected. There must be exactly one $v_{i_1} = v_{j_1}$, since otherwise these two cycles were not arc-disjoint. The union of these two cycles is an alternating cycle, in contradiction to the minimality of the decomposition. \hfill \Box

Note that the above proposition does not assert that the symmetric difference contains $P$ and $Q$. The smallest counter-example are the realizations $G = (V, A)$ and $G' = (V, A')$ with $V = \{v_1, v_2, v_3, v_4\}$ and $A = \{(v_1, v_3), (v_3, v_2), (v_2, v_4), (v_4, v_1)\}$ and $A' = \{(v_1, v_2), (v_2, v_1), (v_3, v_4), (v_4, v_3)\}$.

**Proposition 3.3.** Let $S$ be a graphical sequence and $G$ and $G'$ be two different realizations. If $|G \Delta G'| = 6$, then there exist

a) realizations $G_0, G_1, G_2$ with $G_0 := G$, $G_2 := G'$ and $|G_i \Delta G_{i+1}| = 4$ for $i \in \{0, 1\}$ or

b) $G$ and $G'$ are different in the orientation of exactly one directed 3-cycle.

**Proof.** First observe that the symmetric difference is weakly connected whenever $|G \Delta G'| = 6$. We consider the alternating 6-cycle $C := G \Delta G'$.

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Figure 5: Transforming $G$ from Fig. into $G'$ by a sequence of swap operations.
case 2: $C$ contains at least four different vertices. Assume first that $C$ contains four different vertices. The only possibility to realize this scenario is $C = (v_1, v_2, v_3, v_1, v_4, v_3, v_1)$ with $(v_1, v_2), (v_3, v_1), (v_4, v_3) \in A(G)$ and $(v_3, v_2), (v_4, v_1), (v_1, v_3) \in A(G')$. (A permutation of $\{1, 2, 3\}$ does not influence the result.) We get the alternating vertex-disjoint walk $P = (v_3, v_1, v_2)$.

(i): Assume $(v_4, v_2) \notin A(G)$. It follows $(v_4, v_2) \notin A(G')$. Otherwise, we would get $(v_4, v_2) \in G \Delta G'$ in contradiction to our assumption. We set

$$G_1 := (G_0 \setminus \{(v_4, v_3), (v_1, v_2)\}) \cup \{(v_4, v_2), (v_1, v_3)\}$$

and

$$G_2 := (G_1 \setminus \{(v_4, v_2), (v_3, v_1)\}) \cup \{(v_4, v_1), (v_3, v_2)\}.$$ 

We get $G_2 = G'$ and realizations $G_0, G_1, G_2$ with $|G_i \Delta G_{i+1}| = 4$.

(ii): Assume $(v_4, v_2) \in A(G)$. It follows $(v_4, v_2) \in A(G')$. Otherwise, we would get $(v_4, v_2) \in G \Delta G'$ in contradiction to our assumption. We set

$$G_1 := (G_0 \setminus \{(v_4, v_2), (v_3, v_1)\}) \cup \{(v_4, v_1), (v_3, v_2)\}$$

and

$$G_2 := (G_1 \setminus \{(v_4, v_3), (v_1, v_2)\}) \cup \{(v_4, v_2), (v_1, v_3)\}.$$ 

We get $G_2 = G'$ and realizations $G_0, G_1, G_2$ with $|G_i \Delta G_{i+1}| = 4$.

We can argue analogously if $C$ contains five or six different vertices.

case 2: $C$ contains exactly three different vertices. Then $C$ is the alternating cycle $C = (v_1, v_2, v_3, v_1, v_2, v_3, v_1)$ with $(v_1, v_2), (v_2, v_3), (v_3, v_1) \in A(G)$ and $(v_1, v_2), (v_2, v_1), (v_1, v_3) \in A(G')$. Hence, $G$ and $G'$ are different in the orientation of exactly one directed 3-cycle.

\[\square\]

**Lemma 3.4.** Let $S$ be a graphical sequence and $G$ and $G'$ be two different realizations. There exist realizations $G_0, G_1, \ldots, G_k$ with $G_0 := G, G_k := G'$ and

1. $|G_i \Delta G_{i+1}| = 4$ or
2. $|G_i \Delta G_{i+1}| = 6$

where $k \leq \frac{1}{2}|G \Delta G'| - 1$. In case (2), $G_i \Delta G_{i+1}$ consists of a directed 3-cycle and its opposite orientation.

**Proof.** We prove the lemma by induction according to the cardinality of the symmetric difference $|G \Delta G'| = 2k$. For $k := 2$ we get $|G \Delta G'| = 4$. The correctness of our claim follows with $G_1 := G'$. For $k := 3$ we get a sequence of realizations $G_0, G_1, G_2$ with case a) of Proposition 3.3. In case b) we get a directed 3-cycle with its opposite orientation. In both cases it follows $k \leq 2$.

We assume the correctness of our claim for all $k \leq \ell$. Let $|G \Delta G'| = 2\ell + 2$. We can assume that $k > 3$. Assume further, that the symmetric difference consists of $k$ weakly connected components $(G \Delta G_i')$, for $i \in \{1, \ldots, k\}$.

Consider first the case that for all these components $|G \Delta G_i'| = 6$ and that each component contains exactly three distinct vertices, then each of them is a directed 3-cycle and its reorientation. We choose $(G \Delta G_i')$, perform a 3-cycle reorientation on it, and obtain realization $G_i$. Thus $|G' \Delta G_i'| = 2\ell - 4$. By the induction hypothesis, there are realizations $G_0 = G, G_1, \ldots, G_k = G'$ such that $k \leq \frac{1}{2}|G' \Delta G_i'| - 1 < \frac{1}{2}|G \Delta G'| - 1$. Combining the first 3-cycle reorientation with this sequence of realizations gives the desired bound. If there is a component $|G \Delta G_i'| = 6$ with at least four distinct vertices, we can apply Proposition 3.3b) case a) to it and handle the remaining components by induction.

Otherwise, there is a component with $|G \Delta G_i'| \geq 8$. Due to Proposition 3.2a) we may assume that there is a vertex-disjoint walk $P = (v_1, v_2, v_3, v_4)$ with $(v_1, v_2), (v_3, v_2) \in A(G)$ and $(v_3, v_2) \in A(G')$. Otherwise, there exists $Q = (w_1, w_2, w_3, w_4)$ with $(w_1, w_2), (w_3, w_4) \in A(G')$ and $(w_3, w_2) \in A(G)$. In that case we can exchange the roles of $G$ and $G'$ and consider $G' \Delta G$. Clearly, a sequence of realizations $G' = G_0, G_1', \ldots, G'_k = G$ can be reversed and then fulfills the conditions of the lemma. So from now on we work with $P$. 

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case 1: Assume \((v_1, v_2) \in A(G') \setminus A(G)\). This implies \((v_1, v_2) \in G \Delta G'\). \(G_1 := (G_0 \setminus \{(v_1, v_2), (v_3, v_4)\}) \cup \{(v_3, v_2), (v_1, v_4)\}\) is a realization of \(S\) and it follows \(|G_0 \Delta G_1| = 4\) and \(|G_1 \Delta G'| = 2\ell + 2 - 4 = 2(\ell - 1)\). Note that after this step, \(G_1 \Delta G'\) may consist of several connected components, but each of them has strictly smaller cardinality. Therefore, we can apply the induction hypothesis on \(|G_1 \Delta G'|\). Thus, we obtain realizations \(G_1, G_2, \ldots, G_k\) with \(G_k := G'\) and \(|G_k \Delta G_{k+1}| = 4\) or \(|G_k \Delta G_{k+1}| = 6\) where \(k - 1 \leq \frac{1}{2}(G_1 \Delta G'| - 1)\). Hence, we get the sequence \(G_0, G_1, \ldots, G_k\) with \(k = 1 + \frac{1}{2}|G_1 \Delta G'| - 1 = \frac{1}{2}|G_1 \Delta G'| - 1\) which fulfills 1. and 2.

case 2: Assume \((v_1, v_2) \in A(G) \cap A(G')\). This implies \((v_1, v_2) \notin G \Delta G'\). Consider an alternating cycle \(C = (v_4, v_i, \ldots, v_j, v_1, v_2, v_3, v_4)\) of \((G \Delta G')\), such that each vertex has in-degree two or out-degree two. Then \(P\) is an alternating subpath of \(C\) with \((v_1, v_2), (v_1, v_1) \in A(G')\). We construct a new alternating cycle \(C^* := (C \setminus P) \cup \{(v_1, v_4)\}\) with length \(|C^*| = |C| - 2\). We swap the arcs in \(C^*\) and get a realization \(G^*\) of \(S\) with \(|G_0 \Delta G^*| = |C^*| \leq 2\ell\) and \(|G^* \Delta G'| = |G \Delta G'| - (|C^*| - 1) + 1 \leq 2\ell\). According to the induction hypothesis there exist sequences \(G_0^1 := G_1^1, G_1^1, \ldots, G_k^1 := G^*\) and \(G_0^2 := G^*, G_1^2, \ldots, G_k^2 := G^*\) with \(k_1 = \frac{1}{2}|G_0 \Delta G^*| - 1\) and \(k_2 = \frac{1}{2}|G^* \Delta G'| - 1\). We arrange these sequences one after another and get a sequence which fulfills 1. and 2. and \(k = k_1 + k_2 = \frac{1}{2}|G_0 \Delta G^*| - 1 + \frac{1}{2}|G^* \Delta G'| - 1 = \frac{1}{2}|G_0 \Delta G^*| - 1\).

case 3: Assume \((v_1, v_4) \in A(G) \setminus A(G')\). This implies \((v_1, v_4) \in G \Delta G'\). The alternating walk \(P\) can be extended to an alternating cycle \(C = \{v_1, v_2, v_3, v_4, v_5, \ldots, v_{2\ell}, v_1\}, \ell \geq 3\) using only arcs from \((G \Delta G')\). To construct \(C\), start with \(P\), and keep adding alternating arcs until you reach the start vertex \(v_1\) for the first time. Obviously, you will not get stuck before reaching \(v_1\). Note that the arc \((v_1, v_4)\) does not belong to \(C\). Therefore, there exists an alternating sub-cycle \(C' := C \cup \{(v_1, v_4)\} \setminus P\) formed by arcs in \((G \Delta G')\). We swap the arcs in \(C'\) and get a realization \(G^*\) of \(S\) with \(|G_0 \Delta G^*| = |C'| \leq 2\ell\) and \(|G^* \Delta G'| = |G \Delta G'| - (|C'| - 1) + 1 \leq 2\ell\). According to the induction hypothesis there exist sequences \(G_0^1 := G_1^1, G_1^1, \ldots, G_{k^1}^1 := G^*\) and \(G_0^2 := G^*, G_1^2, \ldots, G_{k^2}^2 := G^*\) with \(k_1 = \frac{1}{2}|G_0 \Delta G^*| - 1\) and \(k_2 = \frac{1}{2}|G^* \Delta G'| - 1\). We arrange these sequences one after another and get a sequence which fulfills 1. and 2. and \(k = k_1 + k_2 = \frac{1}{2}|G_0 \Delta G^*| - 1 + \frac{1}{2}|G^* \Delta G'| - 1 = \frac{1}{2}|G_0 \Delta G^*| - 2\).

case 4: Assume \((v_1, v_4) \notin A(G) \cup A(G')\). This implies \((v_1, v_4) \notin G \Delta G'\). It exists the alternating cycle \(C := (P_1(v_1, v_4))\) (with \((v_1, v_4) \notin A(G)\), \(G_1 := (G_0 \setminus \{(v_1, v_2), (v_3, v_4)\}) \cup \{(v_3, v_2), (v_1, v_4)\}\) is a realization of \(S\) and it follows \(|G_0 \Delta G_1| = 4\) and \(|G_1 \Delta G'| = 2\ell + 2 - 2\ell = 2\ell\). According to the induction hypothesis there exist realizations \(G_1, G_2, \ldots, G_k\) with \(G_k := \Phi\) which fulfill 1. and 2.) where \(k_1 := 1\) and \(k_2 := \frac{1}{2}|G_1 \Delta G'| - 1\). Hence, we get the sequence \(G_0, G_1, \ldots, G_k\) with \(k = k_1 + k_2 = 1 + \frac{1}{2}|G_1 \Delta G'| - 1 = \frac{1}{2}|G_1 \Delta G'| - 1\) which fulfills 1. and 2.

\[\square\]

**Corollary 3.5.** State graph \(\Phi\) is a strongly connected directed graph.

### 3.2 Random Walks

A random walk on \(\Phi = (V_\Phi, A_\Phi)\) can be described by Algorithm 2. We now require a data structure \(DS\) containing all pairs of non-adjacent arcs and all directed 2-paths in the current realization.

**Theorem 3.6.** Algorithm 2 is a random walk on state graph \(\Phi\) which samples uniformly at random from a directed graph \(G' = (V, A)\) as a realization of sequence \(S\) for \(\tau \rightarrow \infty\).

**Proof.** Algorithm 2 chooses elements in \(DS\) with the same constant probability. For a vertex \(V_G \in V_\Phi\) there exist for all these pairs of arcs in \(A(G')\) either incoming and outgoing arcs on \(V_G\) in \(\Phi\) or a loop. Let \(d_G := \binom{|A(G)|}{2} - 2 \sum_{i=1}^{n} \binom{n_i}{2}\). We get a transition matrix \(M\) for \(\Phi\) with \(p_{ij} = \frac{1}{d_G}\) for \(i, j \in A(\Phi), i \neq j\), \(p_{ij} = 1 - \sum_{i \in (\cup (A(\Phi), i = j)} \frac{1}{d_G}\) for \(i, j \in V_\Phi, i = j\), otherwise we set \(p_{ij} = 0\). Since, \(\Phi\) is a regular, strongly connected, symmetrical and non-bipartite directed graph, the distribution of all realizations in a \(\tau\)th step converges asymptotically to the uniform distribution. \(\square\)
Algorithm 2 Sampling realization digraphs

Input: sequence $S$, a directed graph $G = (V, A)$ with $(d_{G}^{+}(v_i), d_{G}^{-}(v_i)) = (\frac{n}{d}, \frac{n}{d}) \forall i \in \{1, \ldots, n\}$ and $v_i \in V$, a mixing time $\tau$.

Output: A sampled directed graph $G' = (V, A')$ with $(d_{G'}^{+}(v_i), d_{G}^{-}(v_i)) = (\frac{n}{d_{G'}}, \frac{n}{d_{G}^{-}}) \forall i \in \{1, \ldots, n\}$ and $v_i \in V$.

1: $t := 0$, $G' := G$ //initialization
2: while $t < \tau$ do
3:     Choose an element $p$ from $DS$ uniformly at random. // $p$ is a pair of non-adjacent arcs or a directed 2-path.
4:     if $p$ is a pair of non-adjacent arcs $(v_1, v_2), (v_3, v_4)$ then
5:         if $(v_1, v_4), (v_3, v_2) \notin A(G')$ then
6:             // Either walk on $\Phi$ to an adjacent realization $G'$
7:             Delete $(v_1, v_2), (v_3, v_4)$ in $A(G')$.
8:             Add $(v_1, v_3), (v_2, v_4)$ to $A(G')$.
9:         else
10:             // or walk a loop: ‘Do nothing’
11:     end if
12: else
13:     if $p$ is a directed 2-path $P = (v_1, v_2, v_3, v_4)$
14:         if $(v_1, v_2), (v_3, v_4) \notin A(G')$ and $(v_1, v_4), (v_3, v_2) \notin A(G')$ then
15:             // Walk on $\Phi$ to an adjacent realization $G'$ with a reoriented directed 3-cycle
16:             Delete $(v_1, v_2), (v_3, v_4), (v_3, v_2)$ in $A(G')$.
17:             Add $(v_1, v_3), (v_2, v_4)$ to $A(G')$.
18:         else
19:             // Walk a loop: ‘Do nothing’
20:     end if
21: end if
22: update data structure $DS$
23: $t \leftarrow t + 1$
24: end while

4 Arc-Swap Sequences

In this section, we study under which conditions the simple switching algorithm works correctly for digraphs. The Markov chain used in the switching algorithm works on the following simpler state graph $\Phi = (V_\Phi, A_\Phi)$. We define $A_\Phi$ as follows.

a) We connect two vertices $V_G, V_{G'} \subseteq V_\Phi$, $G \neq G'$ with arcs $(V_G, V_{G'})$ and $(V_{G'}, V_G)$ if and only if $|G \Delta G'| = 4$ is fulfilled.

b) We set for each pair of non-adjacent arcs $(v_1, v_2), (v_3, v_4) \in A(G), i_j \in \{1, \ldots, n\}$ a directed loop $(V_G, V_G)$ if and only if $(v_1, v_4) \in A(G) \lor (v_1, v_4) \in A(G)$.

c) We set one directed loop $(V_G, V_G)$ for all $V_G \subseteq V_\Phi$.

Lemma 4.1. The state digraph $\Phi = (V_\Phi, A_\Phi)$ is non-bipartite, symmetric, and regular.

Proof. Since each vertex $V_G \subseteq V_\Phi$ contains a loop, $\Phi$ is not bipartite. At each time we set an arc we also do this for its opposite direction. Hence, $\Phi$ is symmetric. The number of incoming and outgoing arcs at each $V_G$ equals the number of non-adjacent arcs in $G$, which is the constant value $\left(\binom{n}{2}\right) - (\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i)$. Thus, we get the regularity of $\Phi$. \[ \square \]

4.1 Characterization of Arc-Swap Sequences

As shown in Example 1.1 in the Introduction, $\Phi$ decomposes into several components, but we are able to characterize sequences $S$ for which strong connectivity is fulfilled in $\Phi$. In fact, we will show that there
are numerous sequences which only require switching by 2-swaps. In the following we give necessary and sufficient conditions allowing to identify such sequences in polynomial running time.

**Definition 4.1.** Let \( S \) be a graphical sequence and let \( G = (V, A) \) be an arbitrary realization. We denote a vertex subset \( V' \subseteq V \) with \( |V'| = 3 \) as an induced cycle set \( V' \) if and only if for each realization \( G^* = (V, A^*) \) the induced subdigraph \( G^*(V') \) is a directed 3-cycle.

**Definition 4.2.** Let \( S \) be a graphical sequence and \( G = (V, A) \) an arbitrary realization. We call \( S \) an arc-swap-sequence if and only if for each subset \( V' \subseteq V \) of vertices with \( |V'| = 3 \) is not an induced cycle set.

This definition enables us to use a simpler state graph for sampling a realization \( G \) for arc-swap-sequences. In Theorem 4.5 we will show show that in these cases we have only to switch the ends of two non-adjacent arcs.

Before, we study how to recognize arc-swap sequences efficiently. Clearly, we may not determine all realizations to identify a sequence as an arc-swap-sequence. Fortunately, we are able to give a characterization of sequences allowing us to identify an arc-swap-sequence in only considering one realized digraph. We need a further definition for a special case of symmetric differences.

**Definition 4.3.** Let \( S \) be a graphical sequence and \( G = (V, A) \) and \( G^* = (V, A^*) \) arbitrary realizations. We call \( G \Delta G^* \) simple symmetric cycle if and only if each vertex \( v \in V \) \((G \Delta G^*) \) possesses vertex in-degree \( d^+_{G \Delta G^*}(v) \leq 2 \) and vertex out-degree \( d^-_{G \Delta G^*}(v) \leq 2 \), and if \( G \Delta G^* \) is an alternating directed cycle.

Note that the alternating directed cycle \( C_1 \) in Fig. 4 is not a simple symmetric cycle, because \( d^+_{G \Delta G^*}(4) = 4 \). Cycle \( C_1 \) decomposes into two simple symmetric cycles \( C_1^* = \{v_1, v_2, v_3, v_4, v_1\} \) and \( C_2^* = \{v_2, v_3, v_5, v_4, v_2\} \).

**Theorem 4.2.** A graphical sequence \( S \) is an arc-swap-sequence if and only if for any realization \( G = (V, A) \) the following property is true:

For each induced, directed 3-cycle \( G(V') \) of \( G \) there exists a realization \( G^* = (V, A^*) \) so that \( G \Delta G^* \) is a simple symmetric cycle and that the induced subdigraph \( G^*(V') \) is not a directed 3-cycle.

**Proof.** \( \Rightarrow \): Let \( S \) be a graphical arc-swap sequence and \( G = (V, A) \) be an arbitrary realization. With Definition 4.3 it follows that each subset \( V' \subseteq V \) with \( |V'| = 3 \) is not an induced cycle set. Hence, there exists for each induced, directed 3-cycle \( G(V') \) of \( G \) a realization \( G^* = (V, A^*) \) with symmetric difference \( G \Delta G^* \) where the induced subdigraph \( G^*(V') \) is not a directed cycle. If the symmetric difference \( G \Delta G^* \) is not a simple symmetric cycle we delete as long alternating cycles in \( G \Delta G^* \) as we get an alternating directed cycle \( C^* \) where each vertex in \( C^* \) has at most vertex in-degree two and at most vertex out-degree two. Furthermore, \( C^* \) shall contain at least one arc \((v, v') \in V' \times V'\). This is possible, because \( G \Delta G^* \) contains at least one such arc. On the other hand the alternating cycle \( C^* \) does not contain all possible six of such arcs. Otherwise, the induced subdigraph \( G^*(V') \) is a directed cycle. Now, we construct the realization \( G^* = (V, A^*) \) with \( A^* := (A(G) \setminus (A(C^*) \cap A(G))) \cup (A(C^*) \cap A(G')) \). It follows \( G \Delta G^* = C^* \) is a simple symmetric difference.

\( \Leftarrow \): Let \( G \) be any realization of sequence \( S \). We only have to consider 3-tuples of vertices \( V' \) inducing directed 3-cycles in \( G \). With our assumption there exists for each \( V' \) a realization \( G^* \) so that \( G^*(V') \) is not a directed 3-cycle. Hence, we find for each subset \( V' \subseteq V \) of vertices with \( |V'| = 3 \) a realization \( G^* = (V, A) \), so that the induced subdigraph \( G^*(V') \) is not a directed 3-cycle. We conclude that \( S \) is an arc-swap sequence.

This characterization allows us to give a simple polynomial-time algorithm to recognize arc-swap-sequences. All we have to do is to check for each induced 3-cycle of the given realization, if it forms an induced cycle set. Therefore, we check for each arc \((v, w)\) in an induced 3-cycle whether there is an alternating walk from \( v \) to \( w \) (not using arc \((v, w)\)) which does not include all five remaining arcs of the 3-cycle and its reorientation. Moreover, each node on this walk has at most in-degree 2 and at most out-degree 2. Such an alternating walk can be found in linear time by using a reduction to an \( f \)-factor problem in a bipartite graph. In this graph we search for an undirected alternating path by growing alternating trees (similar to matching algorithms in bipartite graphs, no complications with blossoms will occur), see for example Sch03. The trick to ensure that not all five arcs will appear in the alternating cycle is to iterate over these five arcs and exclude exactly one of them from the alternating path search.
between \( v \) and \( w \). Of course, this loop stops as soon as one alternating path is found. Otherwise, no such alternating path exists. As mentioned in the Introduction, a linear-time recognition is possible with a parallel Havel-Hakimi algorithm of LaMar \[\text{[LaM09]}\].

Next, we are going to prove that \( \Phi \) is strongly connected for arc-swap-sequences. The structure of the proof is similar to the case of \( \Phi \), but technically slightly more involved.

**Lemma 4.3.** Let \( S \) be a graphical arc-swap-sequence and \( G \) and \( G^* \) be two different realizations. Assume that \( V' := \{v_1, v_2, v_3\} \subseteq V \) such that \( G(V') \) is an induced directed 3-cycle but \( G^*(V') \) is not an induced directed 3-cycle. Moreover, assume that \( G \Delta G^* \) is a simple symmetric cycle. Then there are realizations \( G_0, G_1, \ldots, G_k \) with \( G_0 := G, G_k := G^* \), \( |G_i \Delta G_{i+1}| = 4 \) and \( k \leq \frac{1}{2}|G \Delta G^*| \).

**Proof.** We prove this lemma by induction on the cardinality of \( G \Delta G^* \). The base case \( |G \Delta G^*| = 4 \) is trivial. Consider next the case \( |G \Delta G^*| = 6 \). We distinguish between two subcases.

case a) \( G \Delta G^* \) consists of at least four different vertices.

By Proposition [3.3] case a), there are realizations \( G = G_0, G_1, G_2 = G^* \) with \( |G_i \Delta G_{i+1}| = 4 \).

case b) \( G \Delta G^* \) consists of exactly three vertices \( v_4, v_5, v_6 \).

Observe that \( G \Delta G^* \) contains at least one arc from \( G(V') \) or its reorientation but not all three vertices \( V' \) as otherwise \( G^*(V') \) would be an induced 3-cycle. In fact, it turns out that \( G \Delta G^* \) contains exactly one arc, say \( (v_2, v_3) \), from \( G(V') \) and its opposite arc \( (v_3, v_2) \), because \( G \Delta G^* \) is the directed alternating cycle \( C := (v_2, v_3, v_4, v_2, v_3, v_4, v_2) \) with \( v_2 \neq v_1 \). We have two subcases. Assume first that \( (v_1, v_4) \notin A(G) \cap A(G^*) \). So we can swap the directed alternating cycle \( (v_1, v_2, v_3, v_4, v_2, v_3) \) in a single step. We then obtain the directed alternating 6-cycle \( (v_2, v_3, v_4, v_2, v_1, v_4, v_2) \) which consists of four different vertices. By Proposition [3.3] case a), we can swap the arcs of this cycle in two steps, thus in total in three steps as claimed. Otherwise, \( (v_1, v_4) \in A(G) \cap A(G^*) \). Then we obtain the directed alternating cycle \( (v_1, v_4, v_2, v_3, v_4, v_2, v_3) \) which can be swapped in a single step. By that, we obtain a new cycle \( (v_1, v_3, v_4, v_2, v_3, v_4, v_1) \) which consists of four different vertices. By Proposition [3.3] case a), we can swap the arcs of this cycle in two steps, thus in total in three steps as claimed.

For the induction step, let us consider \( |G \Delta G^*| = 2\ell + 2 \geq 8 \). Then \( G \Delta G^* \) contains between one and five arcs from \( G(V') \) and its reorientation. By Proposition [3.2] there is a vertex-disjoint alternating directed walk \( P = (w_1, w_2, w_3, w_4) \) in \( G \Delta G^* \) with \( (w_1, w_2) \in A(G) \setminus A(G^*) \) or \( (w_1, w_2) \in A(G^*) \setminus A(G) \).

Suppose that \( P \) contains no arc from \( G(V') \) and its reorientation. We consider the case \( (w_1, w_2) \in A(G) \setminus A(G^*) \). If \( (w_1, w_4) \in A(G) \cap A(G^*) \), then we consider \( C = (G \Delta G^*) \cup \{(w_1, w_4)\} \setminus P \). We swap the arcs of \( C \) and obtain as realization \( G^{**} \). Clearly, \( G \Delta G^{**} \) contains an arc from \( G(V') \) or its reorientation, and is a simple symmetric cycle. As \( |G \Delta G^{**}| = 2\ell \), we can apply the induction hypothesis. We obtain a sequence of realizations \( G = G_0, G_1, \ldots, G_k = G^{**} \) with \( |G_i \Delta G_{i+1}| = 4 \) and \( k \leq \frac{1}{2}|G \Delta G^*| \leq \frac{1}{2}(|G \Delta G^*| - 2) \). Finally, we apply a last swap on the cycle \( (w_1, w_2, w_3, w_4, w_1) \) and thereby transform \( G^{**} \) to \( G^* \). In total, the number of swap operations is \( k \leq \frac{1}{2}|G \Delta G^*| \).

The case \( (w_1, w_4) \notin A(G) \cap A(G^*) \) is similar. This time, we start with a single swap on the cycle \( (w_1, w_2, w_3, w_4, w_1) \) and afterwards apply induction to the remaining cycle. We can treat the case \( (w_1, w_2) \in A(G^*) \setminus A(G) \) analogously. Thus we can exclude the existence of any vertex-disjoint directed alternating 3-walk which does not contain at least one arc from \( G(V') \) and its reorientation.

It remains to consider the case that there is a vertex-disjoint directed alternating 3-walk \( P = (w_1, w_2, w_3, w_4) \) in \( G \Delta G^* \) with \( (w_1, w_2) \in A(G) \setminus A(G^*) \) or \( (w_1, w_2) \in A(G^*) \setminus A(G) \) but at least one arc of \( P \) is from \( G(V') \) and its reorientation, say \( (v_1, v_2) \).

Recall that \( G \Delta G^* \) contains between one and five arcs from \( G(V') \) and its reorientation. We distinguish between three cases:

case I: \( G \Delta G^* \) contains exactly one of these arcs, say \( (v_1, v_2) \in A(G) \setminus A(G^*) \). (The case that \( (v_2, v_1) \in A(G^*) \setminus A(G) \) is this special arc can be treated analogously.)

We claim that the cycle \( G \Delta G^* \) must have the form \( (v_1, v_2, v_4, v_5, v_6, v_4, v_5, v_6, v_1) \). Note that \( v_4, v_5, v_6 \) are repeated every third step, as otherwise we would obtain a vertex-disjoint alternating cycle as excluded above. The cycle cannot be longer than eight, since then we would either obtain a vertex-disjoint 3-walk \( (v_4, v_5, v_6, v_7) \), also excluded above, or if \( v_3 = v_7 \) we would violate
simplicity of the symmetric difference. It might be that \( v_5 = v_3 \), but \( v_4, v_6 \neq v_3 \) as otherwise the symmetric difference would contain more than one arc from \( G(V') \) and its reorientation. If \((v_1, v_4) \in A(G) \cap A(G^*)\) there is the alternating directed 4-cycle \((v_1, v_4, v_5, v_6, v_1)\) which can be swapped. In the remaining 6-cycle the arc \((v_1, v_2)\) is contained, so the induction hypothesis can be applied. Otherwise, if \((v_1, v_4) \notin A(G) \cap A(G^*)\), we first apply the induction hypothesis to the 6-cycle \((v_1, v_2, v_4, v_5, v_6, v_4, v_1)\), and afterwards we swap the remaining 4-cycle \((v_1, v_4, v_5, v_6, v_1)\).

case II: \( G \Delta G^* \) contains exactly two of these arcs.
Suppose first that these two arcs are adjacent, say \((v_1, v_2), (v_3, v_2)\). Consider the following arcs \((v_3, v_4), (v_5, v_4), (v_5, v_6)\) along the symmetric difference. Now \( v_5 = v_2 \) as otherwise there is an alternating directed walk \((v_2, v_3, v_4, v_5)\). Depending whether \((v_5, v_2) \in A(G) \cap A(G^*)\) or not, we can either swap the alternating 4-cycle \((v_3, v_4, v_5, v_2)\) or the remaining part of the symmetric difference together with \((v_5, v_2)\) by the induction hypothesis. Moreover, \( v_6 = v_3 \), so the symmetric difference would be the vertex-disjoint alternating directed 3-walk \((v_3, v_4, v_5, v_6)\) excluded above. But then \((v_3, v_2)\) is also in the symmetric difference, a contradiction. Thus, the two arcs from \( G(V') \) and its reorientation are not adjacent. Then, there are at least two other arcs between them (otherwise the one arc between them would also be from \( G(V') \) and its reorientation). By our assumption, there is a vertex-disjoint alternating directed 3-walk \( P = (w_1, w_2, w_3, w_4)\) with at least one arc from \( G(V') \) and its reorientation. In our scenario it must be exactly one such arc. Depending whether \((w_1, w_4) \in A(G) \cap A(G^*)\) or not, we can either swap the alternating 4-cycle \((w_1, w_2, w_3, w_4, w_1)\) or the remaining part of the symmetric difference together with \((w_1, w_4)\) by the induction hypothesis.

case III: \( G \Delta G^* \) contains between three and five of these arcs.
Suppose first all of them follow consecutively on the alternating directed cycle. Consider the last two of these arcs, and append the next arc which must end in a vertex \( v_4 \notin V' \). Then we have a vertex-disjoint alternating directed 3-walk which contains two arcs from \( G(V') \) and its reorientation, and the remaining part of the symmetric difference has also such an arc. Thus we can apply the induction hypothesis and are done. Otherwise the three to five arcs from \( G(V') \) and its reorientation are separated. So no alternating directed 3-walk may contain all of them, in particular not \( P \). We can proceed as in case II).

\[\]

**Proposition 4.4.** Let \( S \) be a graphical arc-swap-sequence and \( G \) and \( G' \) be two different realizations. If \(|G \Delta G'| = 6\) and \( G \Delta G' \) consists of exactly three vertices \( V' := \{v_1, v_2, v_3\} \), then there exist realizations \( G_0, G_1, \ldots, G_k \) with \( G_0 := G, G_k := G' \), \( |G_i \Delta G_{i+1}| = 4 \) and \( k \leq 2n + 2 \).

**Proof.** Since \( S \) is an arc-swap-sequence, Theorem 4.2 implies the existence of a realization \( G^* \) such that \( G \Delta G^* \) is a simple symmetric cycle and \( G^*(V') \) is not a directed 3-cycle. By Lemma 4.3 there are realizations \( G_0, G_1, \ldots, G_k \) with \( |G_i \Delta G_{i+1}| = 4 \) and \( k' \leq \frac{1}{2}|G \Delta G^*| \leq n \), since \( G \Delta G^* \) is simple. Moreover, we have \(|G^* \Delta G'| \leq |G \Delta G^*| + 4 \leq 2n + 4\) since \( G \) and \( G' \) differ only in their orientation of the 3-cycle induced by \( V' \). The symmetric difference \( G^* \Delta G' \) is not necessarily a simple symmetric cycle, but can be decomposed into simple symmetric cycles, each containing at least one arc from \( G(V') \) and its reorientation. On each of these simple symmetric cycles we apply our auxiliary Lemma 4.3. We obtain a sequence \( G^* := G_0', \ldots, G_k' := G' \) with \(|G_i \Delta G_{i+1}| = 4 \) and \( k'' \leq n + 2 \). Combining both sequences we obtain a sequence with \( k = k' + k'' \leq 2n + 2 \).

**Lemma 4.5.** Let \( S \) be a graphical arc-swap-sequence, and \( G \) and \( G' \) be two different realizations. Then there exist realizations \( G_0, G_1, \ldots, G_k \) with \( G_0 := G, G_k := G' \) and \( |G_i \Delta G_{i+1}| = 4 \), where \( k \leq \frac{1}{2}|G \Delta G'| - 1 \cdot (n + 1) \).

**Proof.** We prove the lemma by induction according to the cardinality of the symmetric difference \(|G \Delta G'| = 2k\). For \( k := 2 \) we get \(|G \Delta G'| = 4\). The correctness of our claim follows with \( G_1 := G' \).

For \( k := 3 \) we distinguish two cases. If \( G \Delta G' \) consists of exactly three vertices, then by Proposition 4.3 we get a sequence of realizations \( G_0, G_1, \ldots, G_k \) and \( k \leq 2n + 2 = 2(n + 1) \), as claimed. Otherwise, the symmetric difference \( G \Delta G' \) consists of more than three vertices. By Proposition 4.3 case a), there are realizations \( G_0, G_1, G_2 := G' \).

We assume the correctness of our induction hypothesis for all \( k \leq \ell \). Let \(|G \Delta G'| = 2\ell + 2\). We can assume that \( k > 3 \). Suppose first that the symmetric difference \( G \Delta G' \) decomposes into \( \ell \) simple symmetric
cycles $|(G \Delta G')| = 6$. Suppose further that all these $(G \Delta G')_i$ consist of exactly three vertices. Clearly, $|G \Delta G'| = 6t$. We apply our Proposition 4.3 to each of these $t$ cycles one after another and get a sequence of realizations $G_0, G_1, \ldots, G_k = G'$ with $k \leq 2t(n + 1) \leq (3t - 1)(n + 1) = \left(\frac{1}{2}G \Delta G'\right)(n + 1)$. 

Otherwise, there is a $(G \Delta G')_1$ which contains at least four vertices. Swapping the arcs in $(G \Delta G')_1$ leads to a realization $G'$. By Proposition 3.3 there are realizations $G = G_0, G_1, G_2 = G'$ with $|G_1 \Delta G_1| = 4$. We can apply the induction hypothesis on the remaining part of the symmetric difference. Obviously, we obtain the desired bound in this case.

It remains the case that there exists a simple symmetric cycle $(G \Delta G')_1$ of $G \Delta G'$ with $|(G \Delta G')_1| \neq 6$. If $|(G \Delta G')_1| = 4$, we use a single swap on $(G \Delta G')$, and obtain a realization $G'$, where $|G' \Delta G'| = |G \Delta G'| - 4$. By the induction hypothesis, there is a sequence of realizations $G' = G_1, G_2, \ldots, G_k = G'$ with $k - 1 \leq \left(\frac{1}{2}G' \Delta G'\right)(n + 1) = \left(\frac{1}{2}G \Delta G'\right)(n + 1)$. Otherwise, $|(G \Delta G')_1| \geq 8$. Using Proposition 4.2, we may assume that there exists a vertex-disjoint directed alternating walk $P = (v_1, v_2, v_3, v_4)$ in $(G \Delta G')_1$ with $(v_1, v_2), (v_3, v_4) \in A(G)$ and $(v_3, v_2) \in A(G')$, for the same reasons as in the proof of Lemma 3.3.

case 1: Assume $(v_1, v_4) \in A(G') \setminus A(G)$.

This implies $(v_1, v_4) \in G \Delta G'$. $G_1 := (G \setminus \{(v_1, v_2), (v_3, v_4)\}) \cup \{(v_3, v_2), (v_1, v_4)\}$ is a realization of $S$ and it follows $|G \Delta G_1| = 4$ and $|G_1 \Delta G'| = 2t - 2 = 2t - 4 = 2(\ell - 1)$. Therefore, we can apply the induction hypothesis on $|G_1 \Delta G'|$. Thus, we obtain realizations $G_1, G_2, G_3, G_k$ with $k = 2G'$ and $|G_1 \Delta G_1| = 4$. where $k = 1 \leq \left(\frac{1}{2}G_1 \Delta G_1\right)(n + 1) = \left(\frac{1}{2}G \Delta G'\right)(n + 1)$. Note that the arc $G_1 = (v_1, v_2, v_3, v_4)$ does not belong to $(G \Delta G')_1$. Therefore, there exists an alternating sub-cycle $C' := (G \Delta G')_1 \setminus P$ formed by arcs in $G \Delta G'$. We swap the arcs in $C'$ and get a realization $G'$ of $S$ with $|G_0 \Delta G'| = |C'| \leq 2t$ and $|G' \Delta G'| = |G \Delta G'| - |C'| = |C'| + 1 \leq 2t$. According to the induction hypothesis there exist sequences $G_1^1 := G_1, G_1^2, \ldots, G_k^1 = G'$ and $G_2^0 := G_1^2, G_2^1, \ldots, G_k^2 = G'$ with $k^1 \leq \left(\frac{1}{2}G_0 \Delta G_1\right)(n + 1)$ and $k^2 \leq \left(\frac{1}{2}G_0 \Delta G_1\right)(n + 1)$. We arrange these sequences one after another and get a sequence with $k = k^1 + k^2 = \left(\frac{1}{2}G_0 \Delta G_1\right)(n + 1) = \left(\frac{1}{2}G \Delta G'\right)(n + 1)$.

case 2: Assume $(v_1, v_4) \in A(G) \setminus A(G')$.

This implies $(v_1, v_4) \notin G \Delta G'$. We construct a new alternating cycle $C' := (G \Delta G')_1 \setminus P \setminus \{(v_1, v_4)\}$ with length $|C'| = |(G \Delta G')_1| - 2$. We swap the arcs in $C'$ and get a realization $G'$ of $S$ with $|G_0 \Delta G'| = |C'| \leq 2t$ and $|G' \Delta G'| = |G \Delta G'| - |C'| = |C'| + 1 \leq 2t$. According to the induction hypothesis there exist sequences $G_0^0 := G, G_1^0, \ldots, G_k^0 = G'$ and $G_0^1 := G^0, G_2^0, \ldots, G_k^1 = G'$ with $k^0 \leq \left(\frac{1}{2}G_0 \Delta G^0\right)(n + 1)$ and $k^1 \leq \left(\frac{1}{2}G_0 \Delta G^0\right)(n + 1) = \left(\frac{1}{2}G_0 \Delta G_1\right)(n + 1)$. We arrange these sequences one after another and get a sequence with $k = k^1 + k^2 = \left(\frac{1}{2}G_0 \Delta G^0\right)(n + 1) = \left(\frac{1}{2}G \Delta G'\right)(n + 1)$.

case 3: Assume $(v_1, v_4) \in A(G) \setminus A(G')$.

This implies $(v_1, v_4) \in G \Delta G'$. Note that the arc $(v_1, v_4)$ does not belong to $(G \Delta G')_1$. Therefore, there exists an alternating sub-cycle $C' := (G \Delta G')_1 \setminus P \setminus \{(v_1, v_4)\}$ with length $|C'| = |(G \Delta G')_1| - 2$. We swap the arcs in $C'$ and get a realization $G'$ of $S$ with $|G_0 \Delta G'| = |C'| \leq 2t$ and $|G' \Delta G'| = |G \Delta G'| - |C'| \leq 2t$. According to the induction hypothesis there exist sequences $G_0^0 := G, G_1^0, \ldots, G_k^0 = G'$ and $G_0^1 := G^0, G_2^0, \ldots, G_k^1 = G'$ with $k^0 \leq \left(\frac{1}{2}G_0 \Delta G^0\right)(n + 1)$ and $k^1 \leq \left(\frac{1}{2}G_0 \Delta G^0\right)(n + 1) = \left(\frac{1}{2}G \Delta G'\right)(n + 1)$. We arrange these sequences one after another and get a sequence with $k = k^1 + k^2 = \left(\frac{1}{2}G_0 \Delta G^0\right)(n + 1) = \left(\frac{1}{2}G \Delta G'\right)(n + 1)$.

case 4: Assume $(v_1, v_4) \notin A(G) \cup A(G')$. This implies $(v_1, v_4) \notin G \Delta G'$. It exists the alternating cycle $C := (P_1(v_1, v_4))$ with $(v_1, v_4) \notin A(G)$. $G_1 := (G \setminus \{(v_1, v_2), (v_3, v_4)\}) \cup \{(v_3, v_2), (v_1, v_4)\}$ is a realization of $S$ and it follows $|G_0 \Delta G_1| = 4$ and $|G_1 \Delta G'| = 2t - 2 = 2t$. According to the induction hypothesis there exist realizations $G_1, G_2, \ldots, G_k$ with $k = k_1 + k_2 \leq \left(\frac{1}{2}G_1 \Delta G_1\right)(n + 1)$. Hence, we get the sequence $G_0, G_1, \ldots, G_k$ with $k = k_1 + k_2 = 1 + \left(\frac{1}{2}G \Delta G'\right)(n + 1) = \left(\frac{1}{2}G \Delta G'\right)(n + 1)$. 

\[ \square \]

**Corollary 4.6.** State graph $\overline{G}$ is a strongly connected directed graph if and only if a given sequence $S$ is an arc-swap-sequence.

An arc-swap-sequence implies the connectedness of the simple realization graph $\overline{G}$. Therefore, for such sequences we are able to make random walks on the simple state graph $\overline{G}$ which can be implemented easily. We simplify the random walk Algorithm 2 for arc-swap-sequences in using realization graph $\overline{G}$. Hence, our data structure $\text{DS}$ only contains pairs of non-adjacent arcs. We can ignore lines 12 to 20 in Algorithm 2. We denote this modified algorithm as the **Arc-Swap-Realization-Sample** Algorithm 3.
Theorem 4.7. Algorithm 3 is a random walk on the state graph \( \overline{\Phi} \) which uniformly samples a directed graph \( G' = (V, A) \) as a realization of an arc-swap-sequence \( S \) for \( \tau \rightarrow \infty \).

Proof. Algorithm 3 chooses all elements in DS with the same constant probability. For a vertex \( V_G \in \overline{\Phi} \) there exist for all these pairs of arcs in \( A(G') \) either incoming and outgoing arcs on \( V_G \) or a loop. We get a transition matrix \( M \) for \( \overline{\Phi} \) with \( p_{ij} = \frac{1}{N} \) for \( i, j \in A(\overline{\Phi}) \). If \( i, j \in V_G \) then \( p_{ij} = 1 \) for \( i, j \in A(\overline{\Phi}) \). Since, \( \overline{\Phi} \) is a regular, strongly connected, symmetric, and non-bipartite directed graph, the distribution of all realizations in a \( t \)th step converges asymptotically to the uniform distribution, see Lovasz [Lov96].

4.2 Practical Insights And Applications

As mentioned in the Introduction, many “practitioners” use the switching algorithm for the purpose of network analysis, regardless whether the corresponding degree sequence is an arc-swap-sequence or not. In this section we would like to discuss under which circumstances this common practice can be well justified and when it may lead to wrong conclusions.

What would happen if we sample using the state graph \( \overline{\Phi} \) for a sequence \( S \) which is not an arc-swap-sequence? Clearly, we get the insight that \( \overline{\Phi} \) has several connected components, but as we will see \( \overline{\Phi} \) consists of at most \( 2^{\lfloor|V|/2\rfloor} \) isomorphic components containing exactly the same realizations up to the orientation of directed 3-cycles each consisting of an induced cycle set \( V' \). Fortunately, we can identify all induced cycle sets using our results in Theorem 4.7 by only considering an arbitrary realization \( G \).

Proposition 4.8. Let \( S \) be a graphical sequence which is not an arc-swap-sequence and has at least two different induced cycle sets \( V' \) and \( V'' \). Then it follows \( V' \cap V'' = \emptyset \).

Proof. Without loss of generality we can label the vertices in \( V' \) with \( v'_1, v'_2, v'_3 \) and in \( V'' \) with \( v''_1, v''_2, v''_3 \). Let \( G \) be a realization where \( \{(v'_1, v'_2), (v'_2, v'_3), (v'_3, v'_1), (v''_1, v''_2), (v''_2, v''_3), (v''_3, v''_1)\} \subset A(G) \). We distinguish between two cases.

a): Assume \( |V' \cap V''| = 1 \) where \( v'_1 = v''_1 \). If it exists arc \( (v''_1, v'_1) \in A(G) \) we find the alternating 4-cycle \( (v''_1, v''_2, v'_2, v'_1) \) which implies a new realization \( G^* \) where \( G^*(V') \) is not a directed cycle in contradiction to our assumption that \( V' \) is an induced cycle set. Hence, it follows \( (v''_1, v'_1) \notin A(G) \).

In this case we find the alternating cycle \( (v''_1, v'_1, v'_2, v'_3) \).

b): Assume \( |V' \cap V''| = |\{v'_1, v'_2\}| = 2 \) where \( v'_1 = v''_1 \) and \( v'_2 = v''_2 \). If arc \( (v''_1, v'_1) \notin A(G) \) exists we find the alternating 4-cycle \( (v''_1, v''_2, v'_1, v'_2) \) which implies a new realization \( G^* \) where \( G^*(V') \) is not a directed cycle in contradiction to our assumption that \( V' \) is an induced cycle set. Hence, it follows \( (v''_1, v'_1) \in A(G) \). In this case we find the alternating cycle \( (v''_1, v'_1, v'_2, v'_3) \).

As the induced 3-cycles which appear in every realization are vertex-disjoint, we can reduce the in- and out-degrees of all vertices in these cycles by one, and obtain a new degree sequence which must be an arc-swap sequence.

Theorem 4.9. Let \( S \) be a sequence. Then the state graph \( \overline{\Phi} \) consists of at most \( 2^{|V|} \) isomorphic components.

Proof. We assume that \( S \) is not an arc-swap-sequence, otherwise we apply Theorem 4.6 and get a strongly connected digraph \( \overline{\Phi} \). With Proposition 4.8 it follows the existence of at most \( 2^{|V|/2} \) induced cycle sets for \( S \). Consider all realizations \( G^* \) possessing a fixed orientation of these induced 3-cycles which implies \( G^*(V_i) = G^*(V_i) \) for all such realizations. We pick out one of these orientation scenarios and consider the symmetric difference \( G^* \Delta G^* \) of two such realizations. Since, all induced 3-cycles are identical in \( G^* \) and \( G^* \), we can delete each arc of these induced cycle sets \( V' \) and get the reduced graphs \( G_1 \) and \( G_2 \). Both are realizations of an arc-swap-sequence \( S' \). Applying Theorem 4.6 we obtain, that there exist realizations \( G_0 := G_1, \ldots, G_k := G_2 \) \( |G_i \Delta G_{i+1}| = 4 \) and \( k \leq |G_1 \Delta G_2| \). Hence, each induced subdigraph \( \overline{\Phi} \{V_G, |V_G| \in V_0 \} \) is a realization for one fixed orientation scenario \( \{V_G, V_G \in V_0 \} \) is strongly connected. On the other hand, we get for each fixed orientation scenario exactly the same realizations \( G^* \). Since, all
induced 3-cycles are isomorphic, it follows that all realizations which are only different in the orientation of such directed 3-cycles are isomorphic. By Theorem 4.2 there does not exist an alternating cycle destroying an induced 3-cycle. Hence, the state graph $\Phi$ consists of exactly $2^k$ strongly connected isomorphic components where $k$ is the number of induced cycle sets $V'$.

**Applications in Network Analysis** Since the switching algorithm samples only in one single component of $\Phi$, one has to be careful to get the correct estimations for certain network statistics. For network statistics on unlabeled graphs, it suffices to sample in a single component which reduces the size of $V_\Phi$ by a factor $2^k$, the number of components in $\Phi$, where $k$ is the number of induced cycle sets of the prescribed degree sequence. Examples where this approach is feasible are network statistics like the average diameter or the motif content over all realizations.

For labelled graphs, however, the random walk on $V_\Phi$ systematically over- and under-samples the probability that an arc is present. Suppose that the random walk starts with a realization $G = (V,A)$. If an arc $(v_1, v_2) \in A(G)$ belongs to an induced cycle set, it appears with probability 1 in all realizations of the random walk. The opposite arc $(v_2, v_1) \notin A(G)$, will never occur. In an unbiased sampling over all realizations, each of these arcs, however, occurs with probability $1/2$. All other arcs occur with the same probability in a single component of $V_\Phi$ as in the whole state graph. This observation can be used to compute correct probabilities for all arcs.

5 Concluding Remarks

In this paper, we have presented Markov chains for sampling uniformly at random undirected and directed graphs with a prescribed degree sequence. The key open problem remains to analyze whether these Markov chains are rapidly mixing or not.

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Appendix

A Further Examples Where Switching Fails

As we have seen in Example 1.1, the switching algorithm will fail in general. Here we give further non-trivial classes of graphs where it also fails. All problematic instances are realizations which are different in at least one directed 3-cycle but not all of them are not changeable with alternating 4-cycles.

Consider the following Figures 6 and 7. Both examples cannot be changed to a realization which is only different in the orientation of the directed 3-cycle by a sequence of alternating 4-cycles.

Figure 6: All vertices in a 3-cycle are incident in one direction with vertices in an arbitrary subdigraph.

Figure 7: All vertices in a 3-cycle are incident in both directions with a directed clique. An independent set of vertices is arbitrarily incident with the directed clique.