FREE JUMP DYNAMICS IN CONTINUUM

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Abstract. The evolution is described of an infinite system of hopping point particles in $\mathbb{R}^d$. The states of the system are probability measures on the space of configurations of particles. Under the condition that the initial state $\mu_0$ has correlation functions of all orders which are:

(a) $k^{(n)}_0 \in L^\infty((\mathbb{R}^d)^n)$ (essentially bounded); (b) $\|k^{(n)}_0\|_{L^\infty((\mathbb{R}^d)^n)} \leq Cn$, $n \in \mathbb{N}$ (sub-Poissonian), the evolution $\mu_0 \mapsto \mu_t$, $t > 0$, is obtained as a continuously differentiable map $k_{\mu_0} \mapsto k_t$, $k_t = (k^{(n)}_t)_{n \in \mathbb{N}}$, in the space of essentially bounded sub-Poissonian functions. In particular, it is proved that $k_t$ solves the corresponding evolution equation, and that for each $t > 0$ it is the correlation function of a unique state $\mu_t$.

1. INTRODUCTION

In this paper, we study the dynamics of an infinite system of point particles $x \in \mathbb{R}^d$, $d \geq 1$. States of the system are discrete subsets of $\mathbb{R}^d$ – configurations, which constitute the set

$$\Gamma = \{ \gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \text{ for any compact } \Lambda \subset \mathbb{R}^d \},$$

where $|\cdot|$ stands for cardinality. Note that $\Gamma$ contains also finite configurations, including the empty one. The set $\Gamma$ can be completely and separably metrized, see [1, 7], and thus equipped with the corresponding Borel $\sigma$-field $\mathcal{B}(\Gamma)$. The elements of $\Gamma$ are considered as point states of the system in the sense that, for a suitable function $F : \Gamma \to \mathbb{R}$, the number $F(\gamma)$ is treated as the value of observable $F$ in state $\gamma$. Along with point states $\gamma \in \Gamma$ one employs states determined by probability measures on $\mathcal{B}(\Gamma)$. In this case, the corresponding value is the integral

$$\langle F, \mu \rangle := \int_{\Gamma} Fd\mu,$$

and the system’s dynamics are described as maps $\mu_0 \mapsto \mu_t$, $t > 0$.

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In the Markov approach, the map \( \mu_0 \mapsto \mu_t \) is obtained from the Fokker-Planck equation

\[
\frac{d}{dt} \mu_t = L^\mu \mu_t, \quad \mu_t|_{t=0} = \mu_0, \quad t > 0,
\]

in which ‘generator’ \( L^\mu \) specifies the model. In order to solve \((1.2)\) in the set of probability measures one has to introduce an appropriate mathematical setting, e.g., a Banach space of signed measures, and then to define \( L^\mu \) as a linear operator in this space. However, for infinite systems such a direct way is rather impossible. By the duality

\[
\langle F_0, \mu_t \rangle = \langle F_t, \mu_0 \rangle, \quad t > 0,
\]

the observed evolution \( \langle F, \mu_0 \rangle \mapsto \langle F, \mu_t \rangle \) can also be considered as the evolution \( \langle F_0, \mu \rangle \mapsto \langle F_t, \mu \rangle \) obtained from the Kolmogorov equation

\[
\frac{d}{dt} F_t = L F_t, \quad F_t|_{t=0} = F_0, \quad t > 0,
\]

where \( L \) and \( L^\mu \) are dual in the sense of \((1.3)\). Thus, also ‘generator’ \( L \) specifies the model. Various types of such generators are discussed in [6]. In this paper, we consider the model specified by

\[
(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x-y) [F(\gamma \setminus x \cup y) - F(\gamma)] \, dy.
\]

Here and in the sequel in the corresponding context, \( x \in \mathbb{R}^d \) is also treated as a single-point configuration \( \{x\} \). The jump kernel \( a(x) = a(-x) \geq 0 \) is supposed to satisfy the condition

\[
\int_{\mathbb{R}^d} a(x)dx =: \alpha < +\infty.
\]

The ‘generator’ in \((1.5)\) describes free jumps of the elements of configurations. In models where jumps are not free, the kernel \( a \) depends also on \( \gamma \), see [2, 6].

Similarly as above, to solve \((1.4)\) one should define \( L \) as a linear operator in an appropriate Banach space of functions, which can also be problematic as the sum in \((1.5)\) typically runs over an infinite set. One of the possibilities here is to construct a Markov process \( \gamma_t \) with state space \( \Gamma \), which starts from a fixed configuration \( \gamma_0 \in \Gamma \). Then \( \mu_t^0 \) – the law of \( \gamma_t \), solves \((1.2)\) with \( \mu_0 = \delta_{\gamma_0} \) (the Dirac measure). However, since the evolution of the model \((1.5)\) includes simultaneous jumps of an infinite number of points, there can exist \( \gamma_0 \) such that with probability one at some \( t > 0 \) infinitely many points appear in a bounded \( \Lambda \), see the corresponding discussion in [8]. The reason for this is that the configuration space \((1.1)\) appears to be too big and cannot serve as a state-space for the corresponding process. In [8], under a more restrictive condition than that in \((1.6)\) (see eq. (39) in that paper), the Markov process corresponding to \((1.5)\) was constructed for \( \gamma_0 \), and hence all \( \gamma_t, t > 0 \), lying in a certain proper measurable subset \( \Theta \subset \Gamma \).
By this result, the evolution $\mu_0 \mapsto \mu_t$ corresponding to (1.2) with $\mu_0(\Theta) = 1$
can be obtained by the formula
$$
\mu_t(\cdot) = \int_\Gamma \mu_t^\gamma(\cdot) \mu_0(d\gamma),
$$
which guarantees also that $\mu_t(\Theta) = 1$ for all $t > 0$.

There exists another approach to solving (1.2) in which instead of restricting
the set of configurations where the process takes its values one restricts
the set of initial measures $\mu_0$. This restriction amounts to imposing a con-
dition, formulated in terms of the so called correlation measures. For $n \in \mathbb{N}$
and a probability measure $\mu$ on $\mathcal{B}(\Gamma)$, the $n$-th order correlation measure
$\chi^{(n)}_{\mu}$ is related to $\mu$ by the following formula
$$
\int_\Gamma \left( \sum_{\{x_1, \ldots, x_n\} \subseteq \gamma} \chi^{(n)}_{\mu}(x_1, \ldots, x_n) \right) \mu(d\gamma)
= \frac{1}{n!} \int_{(\mathbb{R}^d)^n} \chi^{(n)}_{\mu}(dx_1, \ldots, dx_n),
$$
which ought to hold for all bounded, compactly supported measurable functions
$G^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$, see, e.g., [6, 7]. Now the mentioned condition is that,
for each $n \in \mathbb{N}$, $\chi^{(n)}_{\mu}$ is absolutely continuous with respect to the correspond-
ing Lebesgue measure with Radon-Nikodym derivative $\frac{d\mu}{d\lambda} \in L^\infty((\mathbb{R}^d)^n)$. Clearly,
this condition excludes Dirac measures, and hence the possibility
to solve (1.2) by means of stochastic processes. Instead one can drop
the mentioned above restrictions on the jump kernel $a$ and on the support of $\mu_t$.
In the present paper, we follow this way.

Until this time, there have been published only two papers [2, 8] dealing
with jump models on $\Gamma$. In [2], the approach based on essentially bounded
correlation functions is applied to the model in which the jump kernel depend
also on the configuration $\gamma \setminus x$ in a specific way. In that paper, by means of
the so called Ovchinnikov method [11], the evolution $\mu_0 \mapsto \mu_t$ is constructed
for $t \in [0, T)$, with $T < \infty$ dependent on the kernel $a$. In the present paper,
we consider the free case (no dependence of $a$ on $\gamma$), which allows us to
employ semigroup methods and obtain the evolution $\mu_0 \mapsto \mu_t$ for all $t \geq 0$.

2. Basic Notions and the Result

The main idea of the approach used in this paper is to obtain the evolution
of states from the evolution of their correlation functions. This includes
the following steps: (a) passing from problem (1.2) to the corresponding
problem for correlation functions; (b) obtaining $k^{(n)}_{\mu_0} \Rightarrow k^{(n)}_{\mu_t}$; (c)
proving that, for each $t > 0$, there exists a unique $\mu_t$ such that $k^{(n)}_{\mu_t}$ is its
correlation function for all $n \in \mathbb{N}$. We perform this steps in subsection
2.2. In subsection 2.1, we present some details of the method. Further
information on the methods used in this work can be found in [2, 4, 5, 6, 7].
2.1. Configuration spaces and correlation functions. By $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_b(\mathbb{R}^d)$ we denote the set of all Borel and all bounded Borel subsets of $\mathbb{R}^d$, respectively. The set of configurations $\Gamma$ defined in (1.1) is equipped with the vague topology – the weakest topology which makes the maps

$$\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x) \in \mathbb{R}$$

continuous for all compactly supported continuous functions $f : \mathbb{R} \to \mathbb{R}$. This topology can be completely and separably metrized, that turns $\Gamma$ into a Polish spaces, see [1, 7]. By $\mathcal{B}(\Gamma)$ and $\mathcal{P}(\Gamma)$ we denote the Borel $\sigma$-field of subsets of $\Gamma$ and the set of all probability measures on $\mathcal{B}(\Gamma)$, respectively.

The set of finite configurations

$$\Gamma_0 = \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}, \quad (2.1)$$

is the disjoint union of the sets of $n$-particle configurations:

$$\Gamma^{(0)} = \{\emptyset\}, \quad \Gamma^{(n)} = \{\gamma \in \Gamma : |\gamma| = n\}, \quad n \in \mathbb{N}.$$  

For $n \geq 2$, $\Gamma^{(n)}$ can be identified with the symmetrization of the set

$$\left\{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j, \text{ for } i \neq j \right\}, \quad (2.2)$$

which allows one to introduce the corresponding (Euclidean) topology on $\Gamma^{(n)}$. Then by (2.1) one defines also the topology on the whole $\Gamma_0$: $A \subset \Gamma_0$ is said to be open if its intersection with each $\Gamma^{(n)}$ is open. This topology differs from that induced on $\Gamma_0$ by the vague topology of $\Gamma$. At the same time, as a set $\Gamma_0$ is in $\mathcal{B}(\Gamma)$. Thus, a function $G : \Gamma_0 \to \mathbb{R}$ is measurable as a function on $\Gamma$ if and only if its restrictions to each $\Gamma^{(n)}$ are Borel functions. Clearly, these restrictions fully determine $G$. In view of (2.2), the restriction of $G$ to $\Gamma^{(n)}$ can be extended to a symmetric function $G^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$, $n \in \mathbb{N}$, such that

$$G(\gamma) = G^{(n)}(x_1, \ldots, x_n), \quad \text{ for } \gamma = \{x_1, \ldots, x_n\}. \quad (2.3)$$

It is convenient to complement (2.3) by putting $G(\emptyset) = G^{(0)} \in \mathbb{R}$.

**Definition 2.1.** A measurable function $G : \Gamma_0 \to \mathbb{R}$ is said to have *bounded support* if the following holds: (a) there exists $N \in \mathbb{N}_0$ such that $G^{(n)} \equiv 0$ for all $n > N$; (b) there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that, for all $n \in \mathbb{N}$,

$$G^{(n)}(x_1, \ldots, x_n) = 0 \text{ whenever } x_j \in \mathbb{R}^d \setminus \Lambda \text{ for some } j = 1, \ldots, n.$$  

By $\mathcal{B}_{bs}(\Gamma_0)$ we denote the set of all such functions.

Let all $G^{(n)}$, $n \in \mathbb{N}$, be bounded Borel functions and $G$ be related to $G^{(n)}$ by (2.3). For such $G$, we then write

$$\int_{\Gamma_0} G(\gamma) \lambda(d\gamma) = G^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \ldots, x_n) dx_1 \cdots dx_n. \quad (2.4)$$
This expression determines a $\sigma$-finite measure $\lambda$ on $\Gamma_0$, called the Lebesgue-Poisson measure. Then the formula in (1.7) can be written in the following way

$$\int_{\Gamma} \left( \sum_{\eta \in \gamma} G(\eta) \right) \mu(d\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta) \lambda(d\eta), \quad (2.5)$$

where the sum on the left-hand side runs over all finite sub-configurations of $\gamma$. Like in (2.3), $k_\mu : \Gamma_0 \to \mathbb{R}$ is determined by its restrictions $k_\mu^{(n)}$. Note that $k_\mu^{(0)} \equiv 1$ for all $\mu \in \mathcal{P}(\Gamma)$.

Prototype examples of measures with property $k_\mu^{(n)} \in L^\infty((\mathbb{R}^d)^n)$ are the Poisson measures $\pi_\rho$ for which

$$k_\mu^{(n)}(x_1, \ldots, x_n) = \prod_{j=1}^n \rho(x_j), \quad n \in \mathbb{N}. \quad (2.6)$$

Here $\rho \in L^\infty(\mathbb{R}^d)$, and the case of constant $\rho \equiv \infty > 0$ corresponds to the homogeneous Poisson measure. Along with the spatial properties of correlation functions, it is important to know how do they depend on $n$. Having in mind (2.6) we say that a given $\mu \in \mathcal{P}(\Gamma)$ is sub-Poissonian if its correlation functions are such that

$$k_\mu^{(n)}(x_1, \ldots, x_n) \leq C^n, \quad (2.7)$$

holding for some $C > 0$, all $n \in \mathbb{N}$, and Lebesgue-almost all $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$. Then a state with property (2.7) is similar to the Poisson states $\pi_\rho$ in which the particles are independently scattered over $\mathbb{R}^d$. At the same time, the increase of $k_\mu^{(n)}$ as $n!$ corresponds to the appearance of clusters in state $\mu$. For the so called continuum contact model, it is known [5] that, for any $t > 0$,

$$\text{const} \cdot n! k_\mu^{(n)}(x_1, \ldots, x_n) \leq \text{const} \cdot n! C_n,$$

where the left-hand inequality holds if all $x_i$ belong to a ball of small enough radius.

Recall that by $B_{bs}(\Gamma_0)$ we denote the set of all $G : \Gamma_0 \to \mathbb{R}$ which have bounded support, see Definition 2.1. For each such $G$ and $\gamma \in \Gamma$, the expression

$$(KG)(\gamma) := \sum_{\eta \in \gamma} G(\eta) \quad (2.8)$$

is well-defined as the sum has finitely many terms only. Note that $KG$ is $\mathcal{B}(\Gamma_0)$-measurable for each $G \in B_{bs}(\Gamma_0)$. Indeed, given $G \in B_{bs}(\Gamma_0)$, let $N$ and $\Lambda$ be as in Definition 2.1. Then

$$(KG)(\gamma) = G^{(0)} + \sum_{x \in \gamma \cap \Lambda} G^{(1)}(x) + \cdots + \sum_{\{x_1,\ldots,x_N\} \subset \gamma \cap \Lambda} G^{(N)}(x_1, \ldots, x_N).$$
The inverse of (2.8) has the form
\[(K^{-1}F)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta|/|\xi|} F(\xi). \tag{2.9}\]

As was shown in [7], \(K\) and \(K^{-1}\) are linear isomorphisms between \(B_{bs}(\Gamma_0)\) and the set of cylinder functions \(F : \gamma \to \mathbb{R}\).

Let us now turn to the following analog of the classical moment problem: given a function \(k : \Gamma_0 \to \mathbb{R}\), which properties of \(k\) could guarantee that there exists \(\mu \in P(\Gamma)\) such that \(k = k_\mu\)? The answer to this question is given by the following statement, see Theorems 6.1, 6.2 and Remark 6.3 in [7], in which
\[B_{bs}^+(\Gamma_0) := \{ G \in B_{bs}(\Gamma_0) : (KG)(\gamma) \geq 0, \ \gamma \in \Gamma \}. \tag{2.10}\]

**Proposition 2.2.** Let \(k : \Gamma_0 \to \mathbb{R}\) be such that: (a) \(k^{(0)} \equiv 1\) and for each \(G \in B_{bs}^+(\Gamma_0)\) the following holds
\[\langle G, k \rangle := \int_{\Gamma_0} G(\eta)k(\eta)\lambda(\eta) d\eta \geq 0; \tag{2.11}\]
(b) there exists \(C > 0\) such that each \(k^{(n)}, n \in \mathbb{N}\), satisfies (2.7). Then there exists a unique \(\mu \in P(\Gamma)\) such that \(k\) is its correlation function, i.e., it is the Radon-Nikodym derivative of the corresponding correlation measure \(\chi_\mu\).

Note that \(B_{bs}^+(\Gamma_0)\) contains not only positive functions, cf. (2.9). That is, the positivity of \(k\) as in (2.11), which readily follows from (2.5), (2.8), and (2.10), is a stronger property than the usual positivity.

### 2.2. The evolution equation

If we rewrite (2.5) in the form, cf. (2.8),
\[\int_{\Gamma} (KG)(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)k_\mu(\eta)\lambda(\eta) d\eta,\]
then the action of \(L\) on \(F\) in (1.5) can be transferred to \(G\), and then to \(k_\mu\), as follows
\[\int_{\Gamma} [L(KG)](\gamma)\mu(d\gamma) = \int_{\Gamma_0} (\hat{L}G)(\eta)k_\mu(\eta)\lambda(\eta) d\eta = \int_{\Gamma_0} G(\eta)(L^\Delta)k_\mu(\eta)\lambda(\eta) d\eta.\]

Thus, by (2.8) and (2.9), we see that
\[\hat{L} = K^{-1}LK,\]
and that \(L^\Delta\) is the adjoint of \(\hat{L}\) with respect to the pairing in (2.11). Then the problem in (1.4) is being transformed into the following one
\[\frac{d}{dt}k_t = L^\Delta k_t, \quad k_t|_{t=0} = k_0 = k_{\mu_0}. \tag{2.12}\]
For a more general model of jumps in $\mathbb{R}^d$, the calculations of $L$ and $L^\Delta$ were performed in [6, Section 4]. The peculiarity of our simple case is that the action of both these ‘operators’ is the same, and, in fact, the same as that in (1.5). That is,

$$L^\Delta = A^\Delta + B^\Delta,$$

(2.13)

$$(A^\Delta k)(\eta) = -\alpha |\eta| k(\eta),$$

$$(B^\Delta k)(\eta) = \sum_{x \in \eta} \int_{\mathbb{R}^d} a(x - y) k(\eta \setminus x \cup y) dy, \quad \eta \in \Gamma_0.$$
which holds for all appropriate functions $f : \mathbb{R}^d \times \Gamma_0 \to \mathbb{R}$ and $g : \Gamma_0 \to \mathbb{R}$.

A more general form of this identity is

$$
\int_{\Gamma_0} \left( \sum_{\xi \in \eta} f(\xi) \right) g(\eta) \lambda(d\eta) = \int_{\Gamma_0} \int_{\Gamma_0} f(\xi) g(\eta \cup \xi) \lambda(d\xi) \lambda(d\eta). \quad (2.18)
$$

Consider

$$
\widehat{G}_\theta = \{ v \in \mathcal{G}_\theta : A^\Delta v \in \mathcal{G}_\theta \}, \quad (2.19)
$$

which is a dense linear subset of $\mathcal{G}_\theta$. Now let us define in $\mathcal{G}_\theta$ linear operators $\widehat{A}_\theta$ and $\widehat{B}_\theta$ by the expressions for $A^\Delta$ and $B^\Delta$, respectively, given in (2.13). Namely, the action of $A_\theta$ and $B_\theta$ on the elements of $\widehat{G}_\theta$ is given by the right-hand sides of the second and third expressions in (2.13), respectively. Clearly $\widehat{A}_\theta$ with domain $\widehat{G}_\theta$ is a closed linear operator in $\widehat{G}_\theta$. Moreover, by (2.17) we get

$$
\|B^\Delta v\|_{\mathcal{G}_\theta,1} \leq \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{x \in \eta} a(x - y)|v(\eta \setminus x \cup y)| \exp(-\vartheta|\eta|) \lambda(d\eta) dy
$$

$$
= \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{x \in \eta} a(x - y)|v(\eta)| \exp(-\vartheta|\eta|) \lambda(d\eta) dy \quad (2.20)
$$

$$
= \alpha \int_{\Gamma_0} |\eta||v(\eta)| \exp(-\vartheta|\eta|) \lambda(d\eta) = \|A^\Delta v\|_{\mathcal{G}_\theta,1},
$$

where we have taken into account (1.6) and twice used (2.17). Thus, $(\widehat{B}_\theta, \widehat{G}_\theta)$ is also well-defined, which allows us to define the sum $\widehat{L}_\theta = \widehat{A}_\theta + \widehat{B}_\theta$ with domain $\widehat{G}_\theta$. It can readily be shown that $(\widehat{L}_\theta, \widehat{G}_\theta)$ is closed, and its action on the the elements of $\widehat{G}_\theta$ is

$$
(\widehat{L}_\theta v)(\eta) = -\alpha|\eta|v(\eta) + \int_{\mathbb{R}^d} \sum_{x \in \eta} a(x - y)v(\eta \setminus x \cup y) dy. \quad (2.21)
$$

By (2.17), for $v \in \widehat{G}_\theta$ and $u \in B_{\text{ba}}(\Gamma_0)$, we get

$$
\langle \widehat{A}_\theta v, u \rangle = \langle v, A^\Delta u \rangle, \quad \langle \widehat{B}_\theta v, u \rangle = \langle v, B^\Delta u \rangle, \quad (2.22)
$$

that is, the action of the dual operators is again given by the same expressions (2.13). We use this fact to define the adjoint operator $\widehat{L}_\theta^*$. Its domain is

$$
\mathcal{D}_\theta = \{ u \in \mathcal{K}_\theta : \forall v \in \mathcal{G}_\theta \exists w \in \mathcal{K}_\theta \langle \widehat{L}_\theta v, u \rangle = \langle v, w \rangle \}. \quad (2.23)
$$

Then the action of $\widehat{L}_\theta^*$ on the elements of $\mathcal{D}_\theta \subset \mathcal{K}_\theta$ is again given by (2.13) or by the right-hand side of (2.22). By construction, the operator $(\widehat{L}_\theta^*, \mathcal{D}_\theta)$ is closed. Let $Q_\theta$ be the closure of $\mathcal{D}_\theta$ in $\mathcal{K}_\theta$. Note that $Q_\theta$ is a proper subset of $\mathcal{K}_\theta$. By $L_\theta^\circ$ we define the part of $\widehat{L}_\theta^*$ in $Q_\theta$. That is, it is the restriction of $\widehat{L}_\theta^*$ to the set

$$
\mathcal{D}_\theta^\circ := \{ u \in \mathcal{D}_\theta : \widehat{L}_\theta^* u \in Q_\theta \}. \quad (2.24)
$$
Lemma 2.3. The operator $(L_0^\vartheta, D_0^\vartheta)$ is the generator of a $C_0$-semigroup of bounded linear operators $S_0^\vartheta(t) : Q_\vartheta \to Q_\vartheta$, $t \geq 0$. Furthermore, for each $\vartheta' > \vartheta$, it follows that $K_{\vartheta'} \subset D_0^\vartheta$.

The proof of the lemma is given in the next section. Now we turn to the problem in (2.12). By the very definition of $L_0^\vartheta$, its action on the elements of $D_0^\vartheta$ is given by the right-hand side of (2.21). Then the version of (2.12) in $Q_\vartheta \subset G_\vartheta$ is

$$\frac{d}{dt} k_t = L_0^\vartheta k_t, \quad k_t|_{t=0} = k_0 \in D_0^\vartheta. \quad (2.25)$$

Definition 2.4. By the classical global solution of the problem in (2.25) we mean a function $[0, +\infty) \ni t \mapsto k_t \in Q_\vartheta$ which is continuously differentiable on $[0, +\infty)$, lies in $D_0^\vartheta$, and satisfies (2.25).

Theorem 2.5. For each $\vartheta \in \mathbb{R}$ and $k_0 \in D_0^\vartheta$, the problem in (2.12) has a unique global classical solution $k_t \in Q_\vartheta \subset K_\vartheta$ given by the formula

$$k_t = S_0^\vartheta(t)k_0, \quad t > 0,$$

where $S_0^\vartheta$ is as in Lemma 2.3. This, in particular, holds if $k_0 \in K_{\vartheta'}$ for some $\vartheta' > \vartheta$. Furthermore, if $k_0$ is the correlation function of some $\mu_0 \in \mathcal{P}(\Gamma)$, then, for each $t > 0$, there exists a unique $\mu_t \in \mathcal{P}(\Gamma)$ such that $k_t$ is the the correlation function of this $\mu_t$.

Note that in [2] where the jumps were not free, the evolution $K_{\vartheta'} \ni k_0 \mapsto k_t \in K_\vartheta$ was obtained on a bounded time interval $[0, T)$ with $T$ dependent on the difference $\vartheta' - \vartheta$.

3. The Proofs

We first prove Lemma 2.3 by means of a statement, which we present here. Let $X$ be a Banach space with a cone of positive elements, $X^+$, which is convex, generating $(X = X^+ - X^+)$, and proper $(X^+ \cap (-X^+) = \{0\})$. Let also the norm of $X$ be additive on $X^+$, that is, $\|x + x'\|_X = \|x\|_X + \|x'\|_X$ for $x, x' \in X^+$. Then there exists a positive (hence bounded) linear functional $\varphi$ on $X$ such that $\varphi(x) = \|x\|_X$ for each $x \in X^+$. Let now $X_1 \subset X$ be a dense linear subset equipped with its own norm in which it is also a Banach space, and let the embedding $X_1 \hookrightarrow X$ be continuous. Assume also that the norm of $X_1$ is additive on the cone $X_1^+ := X_1 \cap X^+$, and $\varphi_1$ is the functional with property $\varphi_1(x) = \|x\|_{X_1}$ for each $x \in X_1^+$. Let $S := \{S(t)\}_{t \geq 0}$ be a $C_0$-semigroup of bounded linear operators $S(t) : X \to X$. It is called substochastic (resp. stochastic) if $S(t) : X^+ \to X^+$ and $\|S(t)\|_X \leq 1$ (resp. $\|S(t)\|_X = 1$) for all $t > 0$. Suppose now that $(A_0, D(A_0))$ be the generator of a substochastic semigroup $S_0$ on $X$. Set $\hat{S}_0(t) = S_0(t)|_{X_1}$, $t > 0$, and assume that the following holds:

(a) for each $t > 0$, $S_0(t)X_1 \to X_1$;

(b) $\hat{S}_0 := \{\hat{S}_0(t)\}_{t \geq 0}$ is a $C_0$-semigroup on $X_1$. 

Under these conditions, $\tilde{A}_0$ – the generator of $\tilde{S}_0$, is the part of $A_0$ in $X_1$, see [3, Proposition II.2.3]. That is, $\tilde{A}_0$ is the restriction of $A_0$ to, cf. (2.24),

$$D(\tilde{A}_0) := \{x \in D(A_0) \cap X_1 : A_0x \in X_1\}.$$  

The next statement is an adaptation of Proposition 2.6 and Theorem 2.7 of [10].

**Proposition 3.1.** Let conditions (a) and (b) given above hold, and $-A_0$ be a positive linear operator in $X$. Let also $B$ be positive and such that its domain in $X$ contains $D(A_0)$ and

$$\varphi((A_0 + B)x) = 0, \quad x \in D(A_0) \cap X^+.$$  \hspace{1cm} (3.1)

Additionally, suppose that $B : D(\tilde{A}_0) \to X_1$ and the following holds

$$\varphi_1((A_0 + B)x) \leq C\varphi_1(x) - \varepsilon\|A_0x\|_X, \quad x \in D(\tilde{A}_0) \cap X^+,$$  \hspace{1cm} (3.2)

for some positive constants $C$ and $\varepsilon$. Then $(A, D(A))$ – the closure of $(A_0 + B, D(A_0))$, is the generator of a stochastic semigroup $S$ on $X$, which leaves $X_1$ invariant. That is, for each $t$, $S(t) : X_1 \to X_1$.

3.1. **Proof of Lemma 2.3.**

**Proof.** The space $\mathcal{G}_\vartheta$ possesses all the properties of the space $X$ assumed above. The corresponding functional is, cf. (2.15),

$$\varphi(v) = \int_{\Gamma_0} v(\eta) \exp(-\vartheta|\eta|)\lambda(d\eta).$$  \hspace{1cm} (3.3)

Then $(\tilde{A}_\vartheta, \tilde{G}_\vartheta)$, see (2.19) and (2.22), generates the sub-stochastic $C_0$-semigroup $S_0$ of multiplication operators defined by the formula

$$(S_0(t)v)(\eta) = \exp(-\alpha|\eta|)v(\eta).$$

Now let $\beta : N_0 \to [0, +\infty)$ be such that $\beta(n) \to +\infty$ as $n \to +\infty$. Set

$$\|v\|_{\vartheta, \beta} = \int_{\Gamma_0} |v(\theta)|\beta(|\eta|)\exp(-\vartheta|\eta|)\lambda(d\eta)$$

$$\varphi_{\beta}(v) = \int_{\Gamma_0} v(\theta)\beta(|\eta|)\exp(-\vartheta|\eta|)\lambda(d\eta),$$

$$\mathcal{G}_{\vartheta, \beta} = \{v \in \mathcal{G}_\vartheta, 1 : \|v\|_{\vartheta, \beta} < +\infty\}.$$  

Then clearly $S_0 : \mathcal{G}_{\vartheta, \beta} \to \mathcal{G}_{\vartheta, \beta}$, and $\|S_0(t)v - v\|_{\vartheta, \beta} \to 0$ as $t \downarrow 0$ by the dominated convergence theorem. Thus, both conditions (a) and (b) above are satisfied. Next, let $\tilde{B}_\vartheta$ be as in (2.22), (2.13). Then $(\tilde{L}_\vartheta, \tilde{G}_\vartheta)$, see (2.21),
is closed. As in (2.20), by (2.17) and (3.3) we obtain

\[
\varphi(\hat{L}_\vartheta v) = -\alpha \int_{\Gamma_0} |\eta| v(\eta) \exp(-\vartheta|\eta|) \lambda(d\eta) \\
+ \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{x \in \eta} a(x - y) v(\eta \setminus x \cup y) \exp(-\vartheta|\eta|) \lambda(d\eta) dy \\
= 0.
\]

That is, the condition in (3.1) holds in our case. Moreover, again by (2.17) we obtain that

\[
\varphi(\hat{L}_\vartheta v) = 0,
\]

hence, in our case (3.2) is satisfied if, for some \(C > 0\), the following holds

\[
\beta(n) \geq Cn, \quad n \in \mathbb{N}.
\]

Then we choose, e.g., \(\beta(n) = n\) and obtain by Proposition 3.1 that \((\hat{L}_\vartheta, \hat{G}_\vartheta)\) generates a stochastic \(C_0\)-semigroup on \(\mathcal{G}_\vartheta\), which we denote by \(\hat{S}_\vartheta\). By the construction performed in (2.23) and (2.24) and by [9, Theorem 10.4, page 39], the semigroup in question is obtained as the restriction of the adjoint semigroup \(\hat{S}_\vartheta^*\) to \(\mathcal{Q}_\vartheta\). □

3.2. Proof of Theorem 2.5

Proof. The proof of the first part Theorem 2.5 readily follows by Lemma 2.3 and [9, Theorem 1.3, page 102]. So, it remains to prove that, for each \(t > 0\), the solution \(k_t\) is the correlation function for a unique \(\mu_t \in \mathcal{P}(\Gamma)\). According to Proposition 2.2, to this end we have to show that, for each \(G \in B_{\text{bs}}(\Gamma_0)\) and all \(t > 0\),

\[
\langle \langle G, k_t \rangle \rangle = \langle \langle G, \hat{S}_\vartheta^* (t)k_0 \rangle \rangle \geq 0,
\]

whenever this property holds for \(t = 0\). The proof of (3.4) follows along the following line of arguments. For a \(\Lambda \in B_b(\mathbb{R}^d)\), we let

\[
\Gamma_\Lambda = \{\gamma \in \Gamma : \gamma \subset \Lambda\},
\]

which is a Borel subset of \(\Gamma\) such that \(\Gamma_\Lambda \subset \Gamma_0\), see (1.1) and (2.1). Hence, \(\Gamma_\Lambda \in \mathcal{B}(\Gamma_0)\). Next, by \(\mu_0^\Lambda\) we denote the projection of \(\mu_0\) on \(\Gamma_\Lambda\), i.e., \(\mu_0^\Lambda := \mu_0 \circ p^{-1}_\Lambda\), where \(p_\Lambda(\gamma) = \gamma \cap \Lambda\) is the projection of \(\Gamma\) onto \(\Gamma_\Lambda\). It can be shown that \(\mu_0^\Lambda\) is absolutely continuous with respect to the Lebesgue-Poisson measure \(\lambda\). Let \(R_0^\Lambda\) be its Radon-Nikodym derivative. For \(N \in \mathbb{N}\), we also set \(R_0^{N,\Lambda}(\eta) = R_0^\Lambda(\eta) I_N(\eta)\), where \(I_N(\eta) = 1\) if \(|\eta| \leq N\) and \(I_N(\eta) = 0\) otherwise. By this construction, \(R_0^{N,\Lambda} \in \mathcal{G}_\vartheta\) for each \(\theta \in \mathbb{R}\), see (2.16). Thus, we can get

\[
R_t^{N,\Lambda} = \hat{S}_\vartheta(t) R_0^{N,\Lambda}, \quad t > 0,
\]

(3.5)
where $\tilde{S}_\vartheta$ is the semigroup constructed in the proof of Lemma 2.3. Then $R_t^{N,\Lambda}(\eta) \geq 0$ and $\|R_t^{N,\Lambda}\|_{\vartheta,1} \leq 1$. Set
\[ q_t^{N,\Lambda}(\eta) = \int_{\Gamma_0} R_t^{N,\Lambda}(\eta \cup \xi) \lambda(d\xi). \tag{3.6} \]
By (2.8), (2.9), and (2.18) for $G \in B_{bs}^+(\Gamma_0)$, we then get
\[ \langle \langle G, q_t^{N,\Lambda} \rangle \rangle = \langle \langle KG, R_t^{N,\Lambda} \rangle \rangle \geq 0. \tag{3.7} \]
On the other hand, by (3.6) for $t = 0$ we have
\[ 0 \leq q_0^{N,\Lambda}(\eta) \leq \int_{\Gamma_0} R_0^\Lambda(\eta \cup \xi) \lambda(d\xi) = k_{\mu_0}(\eta) \mathbb{I}_\Lambda(\eta) \leq k_{\mu_0}(\eta), \]
where $\mathbb{I}_\Lambda$ is the corresponding indicator function. Hence, $q_0^{N,\Lambda} \in D^{\vartheta}_0$ and we get
\[ k_t^{N,\Lambda} = S_\vartheta^0(t)q_0^{N,\Lambda}, \quad t > 0. \tag{3.8} \]
As in [2, Appendix], one can show that
\[ \lim_{l \to +\infty} \lim_{n \to +\infty} \langle \langle G, k_l^{N_i,\Lambda_n} \rangle \rangle = \langle \langle G, k \rangle \rangle, \tag{3.9} \]
for certain increasing sequences $\{N_i\}_{i \in \mathbb{N}}$ and $\{\Lambda_n\}_{n \in \mathbb{N}}$ such that $N_i \to +\infty$ and $\Lambda_n \to \mathbb{R}^d$. In (3.7), $k_t$ is the same as in (3.4). Thus, by (3.7) and (3.9), we can obtain (3.4) by proving that
\[ \langle \langle G, q_t^{N,\Lambda} \rangle \rangle = \langle \langle G, k_t^{N,\Lambda} \rangle \rangle, \quad t > 0. \tag{3.10} \]
Set
\[ \phi(t) = \langle \langle G, q_t^{N,\Lambda} \rangle \rangle, \quad \psi(t) = \langle \langle G, k_t^{N,\Lambda} \rangle \rangle. \]
By (3.8) as well as by (2.25) and (2.23) we then get
\[ \psi'(t) = \langle \langle \tilde{L}_\vartheta G, k_t^{N,\Lambda} \rangle \rangle, \tag{3.11} \]
which makes sense since elements of $B_{bs}(\Gamma_0)$ clearly belong to the domain of $\tilde{L}_\vartheta$ for each $\vartheta \in \mathbb{R}$. Moreover, the operator $\tilde{L}_\vartheta$, see (2.21), can be defined as a bounded operators acting from $G_{\vartheta'}$ to $G_\vartheta$, for each $\vartheta' < \vartheta$. Hence, one can define also its powers $\tilde{L}_\vartheta^n : G_{\vartheta'} \to G_\vartheta$ such that, see [2] eq. (3.21), page 1039,
\[ \| \tilde{L}_\vartheta^n \|_{\vartheta' \vartheta} \leq \left( \frac{2an}{e(\vartheta - \vartheta')} \right)^n, \tag{3.12} \]
where $\| \cdot \|_{\vartheta' \vartheta}$ is the corresponding operator norm. Since $B_{bs}(\Gamma_0) \subset G_{\vartheta'}$ for each $\vartheta'$, by the latter estimate we conclude that $\psi$ can be continued to a function analytic in some neighborhood of the point $t = 0$. Its derivatives are
\[ \psi^{(n)}(0) = \langle \langle \tilde{L}_\vartheta^n G, q_0^{N,\Lambda} \rangle \rangle, \quad n \in \mathbb{N}. \tag{3.13} \]
On the other hand, by (3.5) and (3.6), as well as by the fact that the action of all operators like $\tilde{L}_\vartheta$ on their domains is the same as that in (2.13), we obtain
\[ \phi'(t) = \langle \langle \tilde{L}_\vartheta G, q_t^{N,\Lambda} \rangle \rangle, \]
cf. (3.11) and (3.12), which yields that also φ can be continued to a function analytic in some neighborhood of the point $t = 0$, with the derivatives, cf (3.13), $φ^{(n)}(0) = ψ^{(n)}(0)$. These facts readily yield (3.10), which completes the proof.

□

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