Global optimal solution on blending problem

Yingchun Zheng
School of Science, Xi’an University of Science and Technology, Xi’an 710054, China
E-mail: zhychun1979@163.com

Abstract. In order to study the algorithm of the global optimal solution of the blending problem. The general optimization model for blending problems is given in this paper, at the same time, the blending problem was transformed into the concave minimization problem and linear programming problem, the relevant proof was also given. Research shows that the blending problem is formulated as a concave minimization problem and linear programming problem, can reduce the number of variables blending problem. Blending problem is transformed into a linear programming enable blending problem solving becomes simple and easy.

Keywords: blending problem; concave minimization problems; linear programming

1. Introduction
The general concave minimization problem is to minimize the general concave function \( f(x) \) in polyhedroid \( D \) of \( \mathbb{R}^n \), that is the concave programming problem. Polyhedroid \( D \) is a compact convex set that is usually defined by inequalities like \( g_i(x) \leq 0, (i = 1, 2, \ldots, m) \). The function \( g_i(x) \) is convex under an appropriate collection \( A \) where \( D \subseteq A \subseteq \mathbb{R}^n \). Many practical problems can be classified as concave programming problems [1-2], such as fixed costs and economies of scale, bid evaluation, the new material blending problems. At the same time, many optimization problems can be transformed into concave minimization problems [3-4], including integer programming, dual linear programming, the complementary problem and some product planning problem. Pardalos and Rosengave a comprehensive review on application of concave minimization problem in references [5-7] in 1987. Concave programming problem is a difficult global optimization problem[8-9] that has a plurality of extreme values. For example, it is easy to construct a concave function \( f(x) \) and the polytope \( D \), making each vertex of \( D \) be the minimum points of \( f(x) \) in \( D \). However, on the other hand, some basic properties of the concave programming problem make it easier to handle than the general extreme value global optimization problems. Of course, it is true to apply \( D \) polyhedron in the design of the algorithm to solve concave programming. that is, defined by a finite number of linear inequalities of convex sets. In addition, the global minimum of \( f(x) \) in the polytope \( D \) is at the vertex of \( D \). The optimal conditions of pole holds when \( D \) is a compact convex set. This article will focus on the research on concave minimization problem-blending problems [10-11], which is widely applied in the practice. First, general problems blending optimization model is given, belonging to a class product planning. Second, the general concave minimization problem is converted into the general minimization problem, making it possible to solve the blending problems by taking advantage of the mixed algorithm of concave minimization problem. Then the blending problems can be further...
converted into the linear programming problems through an identity. On one hand, the number of variables the blending problems can be reduced. It can be seen that the optimal solution of the problem has nothing to do with variable $t$ from the subsequent analysis. On the other hand, the blending problem solving becomes simpler and easier since it is converted into linear programming.

2. Basic results
General concave minimization problem is usually given as following:

$$\min_{x \in D} f(x)$$

where $f(x)$ is concave function in polytope $D$; polytope $D$ is a compact convex set that is usually defined by inequalities like $g_i(x) \leq 0, (i = 1, 2, L, m)$, $g_i(x)$ is convex in the collection including $D$.

There are many good results on theory of general concave minimization problem, the theorems below refer to the literature [12].

**Theorem 1** The function $f : S \rightarrow \mathbb{R}$ on the tight convex set $S \subset \mathbb{R}^n$, its global minimization can be obtained in a pole $S$. When $f$ is not strictly concave, global minimum point may also occur in the pole. In particular, when looking for the global minimum point of a concave function in polytope $S$, we can focus on finding the global minimum point in finite number of vertices $S$. if $f$ is strictly concave, global (local) minimum point can only appear in the poles of $S$, easy to construct examples of the polytope and concave function, make each vertex is a local minimum point. By the theorem I, the global minimum of concave function $f$ in the polytope $S$ can be get in vertex of $S$. We can solve the polytope concave minimization problem. Avis and Fukuda found by linear inequalities system defined the polytope vertex, said later, people put forward many effective algorithms, such as, cutting plane algorithm, outer approximation algorithm, the approximation algorithm, branch and bound algorithm [13-17].

By planning is usually into a concave minimization problem to deal with, the general multiplicative programming is given the following form:

$$\min_{x \in D} f(x) = \prod_{i=1}^{p} f_i(x)$$

where $f_i(x) = a_i^T x + b_i, a_i \in \mathbb{R}^n \setminus \{0\}, b_i \in \mathbb{R}$. $f(x) \geq e > 0, (i = 1, 2, L, p)$. By planning objective function is not concave, but because of the logarithmic function $\ln t$ is the increasing function in $[e, +\infty)$, so (2) can be converted into

$$\min_{x \in D} g(x) = \ln f(x) = \sum_{i=1}^{p} \ln(f_i(x)), \ g(x) = \sum_{i=1}^{p} \ln(a_i^T x + b_i)$$

The blending problem has extensive practical application background, the following use an example to describe the process of mixed problem model. On the basis of a model is set up

$$\max f(x, t) = \sum_{i=1}^{p} x_i f_i(t) \quad s.t. \begin{cases} Ax \leq b, \\ \sum_{i=1}^{p} x_i = 1, x \geq 0, t \in [\alpha, \beta] \end{cases}$$

where convex function $(i = 1, 2, L, p)$.

The model is the general form of the admix problem, this article will mainly discuss blending problem solving method. Because in the mixed problem about constraint of $t$ don’t depend on $x$, the fixed $x \geq 0, f(x, t)$ is the convex function about $t$. This can get the following theorem:

**Theorem 2.1** $\max f(x, t) = \max g(x), \ g(x) = \max \left\{ \sum_{i=1}^{p} x_i f_i(\alpha), \sum_{i=1}^{p} x_i f_i(\beta) \right\}.$

Proof: Because $f_i(t)$ is a convex function in $[\alpha, \beta], \forall t \in [\alpha, \beta], \exists \lambda \in [0, 1]$ make $t = \lambda \alpha + (1-\lambda) \beta,$
\[ f_i(t) = f_i(\lambda \alpha + (1-\lambda) \beta) \leq \lambda f_i(\alpha) + (1-\lambda) f_i(\beta) \leq \max \{ f_i(\alpha), f_i(\beta) \} \]

then because the negative of \( x_i \) and \( f_i(t) \) \[ \sum_{i=1}^{p} x_i f_i(t) \leq \sum_{i=1}^{p} \lambda x_i f_i(\alpha) + (1-\lambda) x_i f_i(\beta) \]. When \( t = \alpha \) or \( t = \beta \), type option on the equal sign, so \( \max_{x_i} f(x,t) = \max_{x_i} g(x) \).

**Corollary 2.1** The blending problem of the optimal solution has nothing to do with temperature \( t \).

As \( g(x) \) is the maximum number of linear functions that relate to \( x \), And is the protruding, the blending problem can be converted into the following problem

\[ \min_{x} -g(x) \quad s.t. \quad \sum_{i=1}^{p} A x_i \leq b \quad (5) \]

Formula (5) not only transforms the blending problem into a concave minimization problem, but also reduces the number of decision variables. The following discussion how to dig into the general linear programming problems.

**Theorem 2.2** suppose \( g(x) \) is the same as theorem 2.1, then

\[ g(x) = \max \left\{ \sum_{i=1}^{p} x_i f_i(\alpha), \sum_{i=1}^{p} x_i f_i(\beta) \right\} = \frac{1}{2} \left( \sum_{i=1}^{p} x_i [f_i(\alpha) + f_i(\beta)] + \sum_{i=1}^{p} x_i [f_i(\alpha) - f_i(\beta)] \right) . \]

Proof: We validate \( \sum_{i=1}^{p} x_i f_i(\alpha) \geq \sum_{i=1}^{p} x_i f_i(\beta) \) and \( \sum_{i=1}^{p} x_i f_i(\alpha) \leq \sum_{i=1}^{p} x_i f_i(\beta) \), the type is established. So, the problem (2.4) can be converted into.

\[ \max \frac{1}{2} \left( \sum_{i=1}^{p} x_i f_i(\alpha) + f_i(\beta) \right) + \left| \sum_{i=1}^{p} x_i f_i(\alpha) - f_i(\beta) \right| \] \[ \text{s.t.} \quad \sum_{i=1}^{p} x_i = 1, x \geq 0 \quad (6) \]

Convenience for writing, count \( c_i = f_i(\alpha) + f_i(\beta), d_i = f_i(\alpha) - f_i(\beta) \). Due to the objective function contains absolute value (2.6), so it is not a linear programming.

Suppose \( u = \frac{1}{2} \left( \sum_{i=1}^{p} d_i x_i + \sum_{i=1}^{p} d_i x_i \right), v = \frac{1}{2} \left( \sum_{i=1}^{p} d_i x_i - \sum_{i=1}^{p} d_i x_i \right), \) (6) can be transformed into:

\[ \max \frac{1}{2} \left( \sum_{i=1}^{p} c_i x_i + u + v \right) \quad \text{s.t.} \quad \begin{cases} A x_i \leq b \quad \\
\sum_{i=1}^{p} x_i = 1, x \geq 0, u \geq 0, v \geq 0 \quad \\
u = 0 \quad \text{or} \quad v = 0 \end{cases} \quad (7) \]

We can make use of the linear programming algorithm to solve the problem of blending.

**3. Numerical examples**

The following three examples, we were transformed into concave minimization problem.

**Case 1** \( f_i(t) = e^{\frac{t}{200}}, t \in [0,100], i = 1,2,3 \quad A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 2 \\ 3 & 6 & 7 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix} \).

**Case 2** \( f_i(t) = it^2, t \in [-10,30], i = 1,2,3 \quad A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 3 & 1 \\ 2 & 4 & 5 \\ 0 & 2 & 1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 6 \\ 8 \\ 4 \end{bmatrix} \).

**Case 3** \( f_i(t) = it^2 - i^2t, t \in [-10,20], i = 1,2,3 \quad A = \begin{bmatrix} 2 & 0 & 5 \\ 4 & 7 & 8 \\ 1 & 3 & 1 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix} \).
3.1 Using concave minimization algorithm to solve the blending problem

Case 1 can be transformed into a concave minimal planning as follows:

\[
\begin{align*}
\min_x -g(x) \quad & s.t. \quad Ax \leq b \\
& \sum_{i=1}^{3} x_i = 1, x \geq 0,
\end{align*}
\]

where \( g(x) = \max \left\{ \sum_{i=1}^{3} x_i, \sum_{i=1}^{3} e_i^2 x_i \right\} = \max \left\{ 1, \sum_{i=1}^{3} e_i^2 x_i \right\} . \)

By using Lingo 11.0, the global optimal solution is \((x_1, x_2, x_3) = (0, 0, 1)\), the optimal objective function value is 4.48169, the number of iterations is 10.

Cases 2 can be transformed into a concave minimal planning as follows:

\[
\begin{align*}
\min_x -g(x) \quad & s,t \quad Ax \leq b \\
& \sum_{i=1}^{3} x_i = 1, x \geq 0, \quad \text{where} \quad g(x) = \max \left\{ \sum_{i=1}^{3} 100ix_i, \sum_{i=1}^{3} 900ix_i \right\} .
\end{align*}
\]

The global optimal solution is \((x_1, x_2, x_3) = (0, 0.25, 1, 0.75)\), the optimal objective function value is 3150, the number of iterations is 9.

Case 3 can be transformed into a concave minimal planning as follows:

\[
\begin{align*}
\min_x -g(x) \quad & s, t \quad Ax \leq b \\
& \sum_{i=1}^{3} x_i = 1, x \geq 0, \quad \text{where} \quad g(x) = \max \left\{ \sum_{i=1}^{3} (100 + 10^2)x_i, \sum_{i=1}^{3} (400 + 20^2)x_i \right\} .
\end{align*}
\]

The global optimal solution is \((x_1, x_2, x_3) = (0.25, 0, 0.75)\), the optimal objective function value is 860, the number of iterations is 4.

| Table 1. The numerical results of blending problem by concave minimization algorithm |
|-----------------------------------------------|----------------|
| case1 | optimal solution | optimal value | iterations |
| case2 | (0,0.25,1,0.75) | 3150 | 9 |
| case3 | (0.25,0,0.75) | 860 | 4 |

3.2 Using of linear programming algorithm to solve the blending problems

Case 1 can be converted into linear programming as following:

\[
\begin{align*}
\max \frac{1}{2} \left[ \sum_{i=1}^{3} \left( 1 + e_i^2 \right) x_i + u + v \right] \quad & s.t. \quad Ax \leq b \\
& \sum_{i=1}^{3} \left( 1 - e_i^2 \right) x_i + v + u = 0 \\
& \sum_{i=1}^{3} x_i = 1, x \geq 0, u \geq 0, v \geq 0 \\
& u = 0 \quad \text{or} \quad v = 0
\end{align*}
\]

The global optimal solution is \((x_1, x_2, x_3) = (0, 0, 1)\), the optimal objective function value is 4.48169, the number of iterations is 2.

Case 2 can be converted into linear programming as following:

\[
\begin{align*}
\max \frac{1}{2} \left[ \sum_{i=1}^{3} 1000ix_i + u + v \right] \quad & s.t. \quad Ax \leq b \\
& -\sum_{i=1}^{3} 800ix_i + v + u = 0 \\
& \sum_{i=1}^{3} x_i = 1, x \geq 0, u \geq 0, v \geq 0 \\
& u = 0 \quad \text{or} \quad v = 0
\end{align*}
\]

The global optimal solution is \((x_1, x_2, x_3, x_4) = (0, 0.25, 1, 0.75)\), the optimal objective function value is 3150, the number of iterations is 3.
Case 3 can be converted into linear programming as following:

\[
\max \frac{1}{2} \sum_{i=1}^{3} (500i - 10i^2)x_i + u + v \]

\[
s.t \begin{cases} 
Ax \leq b \\
\sum_{i=1}^{3} (-300i + 30i^2)x_i + v - u = 0 \\
\sum_{i=1}^{3} x_i = 1, x \geq 0, u \geq 0, v \geq 0 \\
u = 0 \quad \text{or} \quad v = 0.
\end{cases}
\]

The global optimal solution is \((x_1, x_2, x_3) = (0.25, 0, 0.75)\), the optimal objective function value is 860, the number of iterations is 2.

The numerical comparison of table 1 and table 2 shows that the linear programming algorithm is more effective than the concave minimization algorithm to solve the blending problems.

**Table 2.** The numerical results of blending problem by linear programming algorithm

| case   | optimal solution | optimal value | iterations |
|--------|------------------|---------------|------------|
| case1  | (0,0,1)          | 4.48169       | 2          |
| case2  | (0.0,25,1,0.75)  | 3150          | 3          |
| Case3  | (0.25,0,0.75)    | 860           | 2          |

4. **Concluding remarks**

It is quite difficult to solve general global optimal solution of the concave minimize problem. This paper studied the problem of mixed type in concave minimization problem. On one hand, the problem can be converted into concave minimization problem according to its characteristics. This kind of concave minimization problem is essentially a minimax problem, which is difficult to solve the global optimal solution of the minimax problem. On the other hand, the blending problem can be converted into general linear programming problems, making it easy to solve the global optimal solution of the linear program problem. Numerical experiments show that the method is feasible and effective to solve the problem.

**References**

[1] Altannar, C. Panos,M.P and Rentsen,E.Global minimization algorithms for concave quadratic programming problems,A Journal of Mathematical Programming and Operations Research,54(6),2005,627-639.

[2] Yuelin G, Zihui R and Chengxian X.A Branch and Bound-PSO Hybrid Algorithm for Solving Integer Separable Concave Programming Problems,Applied Mathematical Sciences,11(1),2007, 517 - 525.

[3] Gene A. B. Extended reverse-convex programming: an approximate enumeration approach to global optimization,Journal of Global Optimization,65(2),2016, 191–229.

[4] Robert,H. Timm.O. and Robert,W. Note on the complexity of the mixed-integer hull of a polyhedron, Operations Research Letters, 43(3),2015, 279-282.

[5] Floudas,C. A and Pardalos,P.M.,A Collection of Test Problems for Constrained Global Optimization Algorithms, Lecture Notes in Computer Science ,Springer Verlag,Berlin,1990.

[6] Horst,R.and Tuy,H.,Global Optimization:Deterministic Approaches,3rd ed. Springer Verlag, Heidelberg,1996.

[7] Pardalos,P.M. and Rosen,J.B.,Constrained Global Optimization: Algorithms and Applications, Springer Verlag, Berlin,1987.

[8] Floudas, C. A. and C. E. Gounaris. A review of recent advances in global optimization. Journal of Global Optimization, 45(1),2009, 3 - 38.

[9] Yuelin Gao and Siqiao Jin, A Global Optimization Algorithm for Sum of Linear Ratios Problem, Journal of Applied Mathematics, Volume 2013 (2013), Article ID 276245.
[10] Juan F. R. Herrera, Leocadio G. Casado, Eligius M. T. Hendrix and Inmaculada García, Pareto optimality and robustness in bi-blending problems, An Official Journal of the Spanish Society of Statistics and Operations Research, 22(1), 2014, 254–273.

[11] Umit, S. S. and Burak, B. A spreadsheet-based decision support tool for blending problems in brass casting industry, Computers & Industrial Engineering, 56(2), 2009, 724-735.

[12] R. Horst, P. M. Pardalos, N. V. Thoai, Global Optimization Introduction, Tsinghua University Press, 2003.

[13] Horst, R., and Thoai, N. V., DC Programming: Overview, Journal of Optimization Theory and Applications, 103(1), 1999, 1-43.

[14] Kamath, A. Pand Karmaker, N. K., An Iteration Algorithm for Computer Bounds in Quadratic Optimization Problems in Complexity in Numerical Optimization, World Scientific, Singapore, 1993, 254-268.

[15] Murty, K. G., Linear Complementarity, linear, and nonlinear Programming, Heldermann, Berlin, 1988.

[16] Pardalos, P. M., editor, Complexity in Numerical Optimization, World Scientific, Singapore, 1993.

[17] Klinz, B., and Tuy, H., Minimum Concave-Cost Network Flow Problem with a Single Nonlinear Arc Cost, in Network Optimization Problems: Algorithms, Applications and Complexity, World Scientific, Singapore, 1993, 125-146.