SMOOTHNESS OF HILL’S POTENTIAL AND LENGTHS OF SPECTRAL GAPS

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Abstract. Let \( \{ \gamma_0(n) \}_{n \in \mathbb{N}} \) be the lengths of spectral gaps in a continuous spectrum of the Hill-Schrödinger operators

\[
S(q)u = -u'' + q(x)u, \quad x \in \mathbb{R},
\]

with 1-periodic real-valued potentials \( q \in L^2(\mathbb{T}) \). Let weight function \( \omega : [1, \infty) \to (0, \infty) \). We prove that under the condition

\[
\exists s \in [0, \infty) : \quad k^s \ll \omega(k) \ll k^{s+1}, \quad k \in \mathbb{N},
\]

the map \( \gamma : q \mapsto \{ \gamma_0(n) \}_{n \in \mathbb{N}} \) satisfies the equalities:

1) \( \gamma(H^\omega) = h^\omega_{\gamma}, \quad 11) \gamma^{-1}(h^\omega_{\gamma}) = H^\omega, \)

where the real function space

\[
H^\omega = \left\{ f = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ik2\pi x} \in L^2(\mathbb{T}) \left| \sum_{k \in \mathbb{N}} \omega^2(k)|\hat{f}(k)|^2 < \infty, \quad \hat{f}(k) = \overline{\hat{f}(-k)}, \quad k \in \mathbb{Z} \right. \}
\]

and

\[
h^\omega = \left\{ a = \{a(k)\}_{k \in \mathbb{N}} \left| \sum_{k \in \mathbb{N}} \omega^2(k)|a(k)|^2 < \infty \right. \}, \quad h^\omega_{\gamma} = \{a = \{a(k)\}_{k \in \mathbb{N}} \in h^\omega \left| a(k) \geq 0 \right\} \}
\]

If the weight \( \omega \) is such that

\[
\exists a > 1, c > 1 : \quad c^{-1} \leq \frac{\omega(M)}{\omega(t)} \leq c \quad \forall t \geq 1, \quad \lambda \in [1, a]
\]

then the function class \( H^\omega \) is a real Hörmander space \( H^\omega_2(\mathbb{T}, \mathbb{R}) \) with the weight \( \omega(\sqrt{1 + t^2}) \).

1. Introduction

Let consider on the complex Hilbert space \( L^2(\mathbb{R}) \) the Hill-Schrödinger operators

\[
S(q)u := -u'' + q(x)u, \quad x \in \mathbb{R},
\]

with 1-periodic real-valued potentials

\[
q(x) = \sum_{k \in \mathbb{Z}} \hat{q}(k)e^{ik2\pi x} \in L^2(\mathbb{T}, \mathbb{R}), \quad \mathbb{T} := \mathbb{R}/\mathbb{Z}.
\]

Last condition means that

\[
\sum_{k \in \mathbb{Z}} |\hat{q}(k)|^2 < \infty \quad \text{and} \quad \hat{q}(k) = \overline{\hat{q}(-k)}, \quad k \in \mathbb{Z}.
\]

It is well known that the operators \( S(q) \) are lower semibounded and self-adjoint. Their spectra are absolutely continuous and have a zone structure [ReSi]. Spectra of the operators \( S(q) \) are completely defined by the location of the endpoints of spectral gaps \( \{ \lambda_0(q), \lambda_\pm(n)(q) \}_{n=1}^\infty \), which satisfy the inequalities:

\[
-\infty < \lambda_0(q) < \lambda_1^-(q) \leq \lambda_1^+(q) < \lambda_2^-(q) \leq \lambda_2^+(q) < \cdots.
\]

2010 Mathematics Subject Classification. Primary 34L40; Secondary 47A10, 47A75.
Key words and phrases. Hill-Schrödinger operators, spectral gaps, Hörmander spaces.

*The investigation is partially supported by DFFD of Ukraine under grant 28.1/017.
For even/odd numbers \( n \in \mathbb{Z}_+ \) the endpoints of spectral gaps \( \{\lambda_0(q), \lambda_n^+(q), \lambda_n^-(q)\}_{n=1}^{\infty} \) are eigenvalues of the periodic/semiperiodic problems on the interval \([0, 1] \):

\[
S_{\pm}(q)u := -u'' + q(x)u = \lambda u,
\]

\[
\text{Dom}(S_{\pm}(q)) := \left\{ u \in H^2[0, 1] \mid u^{(j)}(0) = \pm u^{(j)}(1), \ j = 0, 1 \right\}.
\]

Interiors of spectral bands (stability or tied zones)

\[ B_0(q) := (\lambda_0(q), \lambda^+_n(q)), \quad B_n(q) := (\lambda^+_n(q), \lambda^-_{n+1}(q)), \quad n \in \mathbb{N}, \]

together with the collapsed gaps

\[ \lambda = \lambda^+_n = \lambda^-_n, \quad n \in \mathbb{N} \]

are characterized as a locus of those real \( \lambda \in \mathbb{R} \) for which all solutions of the equation \( S(q)u = \lambda u \) are bounded. Open spectral gaps (instability or forbidden zones)

\[ G_0(q) := (-\infty, \lambda_0(q)), \quad G_n(q) := (\lambda^-_{n+1}(q), \lambda^+_n(q)) \neq \emptyset, \quad n \in \mathbb{N} \]

are a locus of those real \( \lambda \in \mathbb{R} \) for which any nontrivial solution of the equation \( S(q)u = \lambda u \) is unbounded.

We study the behaviour of the lengths of spectral gaps

\[ \gamma_q(n) := \lambda^+_n(q) - \lambda^-_n(q), \quad n \in \mathbb{N} \]

of the Hill-Schrödinger operators \( S(q) \) in terms of the behaviour of the Fourier coefficients \( \{\tilde{q}(n)\}_{n \in \mathbb{N}} \) of the potentials \( q \) with respect to appropriate weight spaces, that is in terms of potential regularity.

Hochstadt [Hech1, Hech2], Marchenko and Ostrovskii [MrOs], McKeans and Trubowitz [McKT] proved that the potential \( q(x) \) is an infinitely differentiable function if and only if the lengths of spectral gaps \( \{\gamma_q(n)\}_{n=1}^{\infty} \) decrease faster than arbitrary power of \( 1/n \):

\[ q(x) \in C^\infty(T, \mathbb{R}) \Leftrightarrow \gamma_q(n) = O(n^{-k}), \ n \to \infty \quad \forall k \in \mathbb{Z}_+. \]

Marchenko and Ostrovskii [MrOs] (see also [Mrch]) discovered that:

\[ q \in H^s(T, \mathbb{R}) \Leftrightarrow \sum_{n \in \mathbb{N}} (1 + 2n)^{2s} \gamma^2_q(n), \quad s \in \mathbb{Z}_+, \]

where \( H^s(T, \mathbb{R}), \ s \in \mathbb{Z}_+, \) denotes the Sobolev space of 1-periodic real-valued functions on the circle \( T \).

To characterize regularity of potentials in the finer way we shall use the real Hörmander spaces \( H^\omega(T, \mathbb{R}) \) where \( \omega(\cdot) \) is a positive weight (see Appendix). In the case of the Sobolev spaces it is a power one.

Djakov, Mityagin [DjMt2], Pöschel [Psch] extended the Marchenko-Ostrovskii Theorem (3) to the general class of weights \( \Omega = \{\Omega(k)\}_{k \in \mathbb{N}} \) satisfying the following conditions:

i) \( \Omega(k) \nearrow \infty, \ k \in \mathbb{N}; \) (monotonicity)

\[ \Omega(k + m) \leq \Omega(k)\Omega(m), \quad k, m \in \mathbb{N}; \] (submultiplicity)

\[ \log \Omega(k) \underset{k \to \infty}{\sim} 0. \] (subexponentiality).

For such weights they proved that

\[ q \in H^\Omega(T, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^\Omega(\mathbb{N}). \]

Here \( h^\Omega(\mathbb{N}) \) is the Hilbert space of weighted sequences generated by the weight \( \Omega(\cdot) \).

Earlier Kappeler, Mityagin [KpMt2] proved the direct implication in (4) under the only assumption of submultiplicity. In the special cases of the Abel-Sobolev weights, the Gevrey weights and the slowly increasing weights the relationship (4) was established by Kappeler, Mityagin [KpMt1] (\( \Rightarrow \)) and Djakov, Mityagin [DjMt, DjMt1] (\( \Leftarrow \)). Detailed exposition of these results is given in the survey [DjMt2]. It should be noted that Kappeler, Mityagin [KpMt1, KpMt2], Djakov, Mityagin [DjMt1, DjMt2] and Pöschel [Psch] studied also the more general case of complex-valued potentials.
2. Main result

The main purpose of this paper is to prove the following result.

**Theorem 1.** Let \( q \in L^2(\mathbb{T}, \mathbb{R}) \) and the weight \( \omega = \{\omega(k)\}_{k \in \mathbb{N}} \) satisfy conditions:
\[
k^s \ll \omega(k) \ll k^{1+s}, \quad s \in [0, \infty).
\]

Then the map \( \gamma : q \mapsto \{\gamma_q(n)\}_{n \in \mathbb{N}} \) satisfies the equalities:

i) \( \gamma(H^\omega(\mathbb{T}, \mathbb{R})) = h^\omega_+(\mathbb{N}) \),

ii) \( \gamma^{-1}(h^\omega_+(\mathbb{N})) = H^\omega(\mathbb{T}, \mathbb{R}) \).

**Corollary 1.1.** Let for the weight \( \omega = \{\omega(k)\}_{k \in \mathbb{N}} \) exist the order
\[
\lim_{k \to \infty} \frac{\log \omega(k)}{\log k} = s \in [0, \infty),
\]

and let for \( s = 0 \) the values of the weight \( \omega = \{\omega(k)\}_{k \in \mathbb{N}} \) be separated from zero. Then
\[
q \in H^\omega(\mathbb{T}, \mathbb{R}) \iff \{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N}).
\]

From Corollary 1.1 we receive the following result.

**Corollary 1.2** ([MiMl]). Let the weight \( \omega = \{\omega(k)\}_{k \in \mathbb{N}} \) be a regular varying sequence in the Karamata sense with the index \( s \in [0, \infty) \), and let for \( s = 0 \) its values be separated from zero. Then
\[
q \in H^\omega(\mathbb{T}, \mathbb{R}) \iff \{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N}).
\]

Note that the assumption of Corollary 1.2 holds, for instance, for the weight
\[
\omega(k) = (1 + 2k)^{s \cdot (\log(1 + k))^{r_1} \cdot (\log \log(1 + k))^{r_2} \cdot \ldots} \cdot \log \log \ldots \log(1 + k)),
\]
\[
s \in (0, \infty), \quad \{r_1, \ldots, r_p\} \subset \mathbb{R}, \quad p \in \mathbb{N},
\]
see [BnGITz].

The following Example A shows that statement (4) does not cover Corollary 1.1 and moreover Theorem 1.

**Example A.** Let \( s \in [0, \infty) \). Set
\[
w(k) := \begin{cases} k^s \log(1 + k) & \text{if } k \in 2\mathbb{N}, \\ k^s & \text{if } k \in (2\mathbb{N} - 1). \end{cases}
\]

Then the weight \( \omega = \{\omega(k)\}_{k \in \mathbb{N}} \) satisfies the conditions of Corollary 1.1. But one can prove that it is not equivalent to any monotonic weight.

**Remark 1.1.** Theorem 1 shows that if the sequence \( \{\gamma_q(n_k)\}_{k=1}^\infty \) decreases particularly fast on a certain subsequence \( \{n_k\}_{k=1}^\infty \subset \mathbb{N} \), then so does sequence \( \{\gamma_q(n_k)\}_{k=1}^\infty \) on the same subsequence. Inverse statement is also true.

3. Preliminaries

Here, for convenience, we define Hilbert spaces of weighted two-sided sequences and formulate the Convolution Lemma 2.

For every positive sequence \( \omega = \{\omega(k)\}_{k \in \mathbb{N}} \) there exists its unique extension on \( \mathbb{Z} \) which is a two-sided sequence satisfying the conditions:

i) \( \omega(0) = 1 \);

ii) \( \omega(-k) = \omega(k) \quad \forall k \in \mathbb{N} \);

iii) \( \omega(k) > 0 \quad \forall k \in \mathbb{Z} \).
Let $h^\omega(Z) \equiv h^\omega(Z, \mathbb{C})$ be the Hilbert space of two-sided sequences:

$$h^\omega(Z) := \left\{ a = \{a(k)\}_{k \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \omega^2(k)|a(k)|^2 < \infty \right. \right\},$$

$$(a, b)_{h^\omega(Z)} := \sum_{k \in \mathbb{Z}} \omega^2(k)a(k)b(k), \quad a, b \in h^\omega(Z),$$

$$\|a\|_{h^\omega(Z)} := (a, a)^{1/2}_{h^\omega(Z)}, \quad a \in h^\omega(Z).$$

By $h^\omega(n)$ for a convenience we will denote the n-th element of a sequence $a = \{a(k)\}_{k \in \mathbb{Z}}$ in $h^\omega(Z)$.

Basic weights which we use are the power ones:

$$w_s = \{w_s(k)\}_{k \in \mathbb{Z}} : \quad w_s(k) = (1 + 2|k|)^s, \quad s \in \mathbb{R}.$$  

In this case it is convenient to use shorter notations:

$$h^\omega_s(Z) \equiv h^s(Z), \quad s \in \mathbb{R}.$$  

Operation of convolution for two-sided sequences

$$a = \{a(k)\}_{k \in \mathbb{Z}} \quad \text{and} \quad b = \{b(k)\}_{k \in \mathbb{Z}}$$

is formally defined as follows:

$$(a, b) \mapsto a \ast b,$$

$$(a \ast b)(k) := \sum_{j \in \mathbb{Z}} a(k - j)b(j), \quad k \in \mathbb{Z}.$$  

Sufficient conditions for the convolution to exist as a continuous map are given by the following known lemma, see for example [KpMh, Mhr].

**Lemma 2** (The Convolution Lemma). Let $s, r \geq 0$, and $t \leq \min(s, r)$, $t \in \mathbb{R}$. If $s + r - t > 1/2$ then the convolution $(a, b) \mapsto a \ast b$ is well defined as a continuous map acting in the spaces:

(a) $h^s(Z) \times h^r(Z) \to h^{t}(Z)$,

(b) $h^{-t}(Z) \times h^s(Z) \to h^{-r}(Z)$.

In the case $s + r - t < 1/2$ this statement fails to hold.

4. The Proofs

Basic point of our proof of Theorem 1 is sharp asymptotic formulae for the lengths of spectral gaps $\{\gamma_q(n)\}_{n \in \mathbb{N}}$ of the Hill-Schrödinger operators $S(q)$ and fundamental result of [GrTr, Theorem 1].

**Lemma 3.** The lengths of spectral gaps $\{\gamma_q(n)\}_{n \in \mathbb{N}}$ of the Hill-Schrödinger operators $S(q)$ with $q \in H^s(\mathbb{T}, \mathbb{R})$, $s \in [0, \infty)$, uniformly on the bounded sets of potentials $q$ in the corresponding Sobolev spaces $H^s(\mathbb{T})$ for $n \geq n_0$, $n_0 = n_0(\|q\|_{H^r(\mathbb{T})})$, satisfy the following asymptotic formulae:

$$\gamma_q(n) = 2|\bar{q}(n)| + h^{1+s}(n).$$

**Proof of Lemma 3.** To prove asymptotic formulae (5) we use [KpMt2, Theorem 1.2] and the Convolution Lemma 2 (see also [KpMt2, Appendix]). Indeed, applying [KpMt2, Theorem 1.2] with $q \in H^s(\mathbb{T}, \mathbb{R})$, $s \in [0, \infty)$, we get

$$\sum_{n \in \mathbb{N}} (1 + 2n)^{2(1+s)} \left( \min_{\pm} |\gamma_q(n)| \pm 2\sqrt{(\bar{q} + \frac{q}{(n+1)^2})}\sqrt{(\bar{q} + \frac{q}{n})} \right) \leq C (\|q\|_{H^r(\mathbb{T})}),$$

where

$$\gamma_q(n) := \frac{1}{\pi^2} \sum_{j \in \mathbb{Z} \setminus \{\pm n\}} \frac{\omega(n - j)\bar{q}(n + j)}{(n - j)(n + j)}.$$
Without losing generality we assume that
\begin{equation}
\hat{q}(0) := 0.
\end{equation}

Taking into account that the potentials $q$ are real-valued we have
\[ \hat{q}(k) = \overline{q(-k)}, \quad q(k) = q(-k), \quad k \in \mathbb{Z}. \]

Then from (6) we get the estimates
\begin{equation}
\{ \gamma_n(q) - 2|\hat{q}(n) + q(n)| \}_{n \in \mathbb{N}} \in h^{1+s}(\mathbb{N}).
\end{equation}

Further, as by assumption $q \in H^s(\mathbb{T}, \mathbb{R})$, that is $\{\hat{q}(k)\}_{k \in \mathbb{Z}} \in h^s(\mathbb{Z})$, then taking into account
\begin{equation}
\left\{ \frac{\hat{q}(k)}{k} \right\}_{k \in \mathbb{Z}} \in h^{1+s}(\mathbb{Z}), \quad s \in [0, \infty).
\end{equation}

Applying the Convolution Lemma 2 we obtain
\begin{equation}
q(n) = \frac{1}{\pi^2} \sum_{j \in \mathbb{Z}} \frac{\hat{q}(n-j) \hat{q}(n+j)}{(n-j)(n+j)} = \frac{1}{\pi^2} \sum_{j \in \mathbb{Z}} \frac{\hat{q}(2n-j)}{2n-j} \frac{\hat{q}(j)}{j} = \left( \left\{ \frac{\hat{q}(k)}{k} \right\}_{k \in \mathbb{Z}} \ast \left\{ \frac{\hat{q}(k)}{k} \right\}_{k \in \mathbb{Z}} \right)(2n) \in h^{1+s}(\mathbb{N}).
\end{equation}

Finally, from (8) and (9) we get the necessary estimates (5).

The proof of Lemma 3 is complete. □

**Proof of Theorem 1.** Let $q \in L^2(\mathbb{T}, \mathbb{R})$ and $\omega = \{\omega(k)\}_{k \in \mathbb{N}}$ be a given weight satisfying the conditions of Theorem 1:
\begin{equation}
k^s \ll \omega(k) \ll k^{1+s}, \quad s \in [0, \infty).
\end{equation}

At first, we need to prove the statement
\begin{equation}
q \in H^s(\mathbb{T}, \mathbb{R}) \Leftrightarrow \{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N}).
\end{equation}

Due to formulae (10) the continuous embeddings
\begin{align}
H^{1+s}(\mathbb{T}) & \hookrightarrow H^s(\mathbb{T}), \\
h^{1+s}(\mathbb{N}) & \hookrightarrow h^s(\mathbb{N}), \quad s \in [0, \infty),
\end{align}

are valid because
\begin{equation}
H^{1+s}(\mathbb{T}) \hookrightarrow H^{2s}(\mathbb{T}), \quad h^{1+s}(\mathbb{N}) \hookrightarrow h^{2s}(\mathbb{N}) \quad \text{if only} \quad \omega_1 \gg \omega_2.
\end{equation}

Let $q \in H^s(\mathbb{T}, \mathbb{R})$, then from (12) we get $q \in H^s(\mathbb{T}, \mathbb{R})$. Due to Lemma 3 we find that
\[ \gamma_q(n) = 2|\hat{q}(n)| + h^{1+s}(n). \]

Applying (13) from the latter we derive
\[ \gamma_q(n) = 2|\hat{q}(n)| + h^\omega(n). \]

And, as a consequence, we obtain that $\{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N})$.

Direct implication in (11) has been proved.

Let $\{\gamma_q(\cdot)\} \in h^\omega(\mathbb{N})$. Applying (13) we get $\{\gamma_q(\cdot)\} \in h^s(\mathbb{N})$. Further, from (4) with $\Omega(k) = (1 + 2k)^s$, $s \in [0, \infty)$, we obtain $q \in H^s(\mathbb{T}, \mathbb{R})$.

We have already proved the implication
\[ q \in H^s(\mathbb{T}, \mathbb{R}) \Rightarrow \gamma_q(n) = 2|\hat{q}(n)| + h^\omega(n). \]

Hence $\{\hat{q}(\cdot)\} \in h^\omega(\mathbb{N})$, i.e., $q \in H^s(\mathbb{T}, \mathbb{R})$.

Inverse implication in (11) has been proved.

Now we are ready to prove the statement of Theorem 1.

From relationship (11) we get
\begin{equation}
\gamma(H^\omega(\mathbb{T}, \mathbb{R})) \subset h^\omega_s(\mathbb{N}).
\end{equation}
To establish the equality i) of Theorem 1 it is necessary to prove the inverse inclusion in latter formula (15). So, let \( \{ \gamma(n) \}_{n \in \mathbb{N}} \) be an arbitrary sequence from the space \( h_{\omega}^{+}(\mathbb{N}) \). Then \( \{ \gamma(n) \}_{n \in \mathbb{N}} \in h_{\omega}^{+}(\mathbb{N}) \). Due to [GrTr, Theorem 1] potential \( q \in L^{2}(\mathbb{T}, \mathbb{R}) \) exists for which the sequence \( \{ \gamma(n) \}_{n \in \mathbb{N}} \in h_{\omega}^{+}(\mathbb{N}) \) is a corresponding sequence of the lengths of spectral gaps. As by assumption \( \{ \gamma(n) \}_{n \in \mathbb{N}} \in h_{\omega}^{+}(\mathbb{N}) \) due to (11) we conclude that \( q \in H^{\omega}(\mathbb{T}, \mathbb{R}) \). I.e., the inclusion

\[
gamma \left( H^{\omega}(\mathbb{T}, \mathbb{R}) \right) \supset h_{\omega}^{+}(\mathbb{N}) \tag{16}
\]

holds.

Finally, the inclusions (15) and (16) give the necessary equality i).

Now, let prove the equality ii) of Theorem 1. Let \( \{ \gamma(n) \}_{n \in \mathbb{N}} \) be an arbitrary sequence from the space \( h_{\omega}^{+}(\mathbb{N}) \). Similarly as above we prove that potential \( q \in H^{\omega}(\mathbb{T}, \mathbb{R}) \) exists for which the sequence \( \{ \gamma(n) \}_{n \in \mathbb{N}} \in h_{\omega}^{+}(\mathbb{N}) \) is a corresponding sequence of the lengths of spectral gaps. That is

\[
\gamma^{-1} \left( h_{\omega}^{+}(\mathbb{N}) \right) \subset H^{\omega}(\mathbb{T}, \mathbb{R}).
\tag{17}
\]

Inversely. Let \( q \) be an arbitrary function from the Hörmander space \( H^{\omega}(\mathbb{T}, \mathbb{R}) \). Then due to (11) we have \( \gamma_{q} = \{ \gamma_{q}(n) \}_{n \in \mathbb{N}} \in h_{\omega}^{+}(\mathbb{N}) \). I.e.,

\[
\gamma^{-1} \left( h_{\omega}^{+}(\mathbb{N}) \right) \supset H^{\omega}(\mathbb{T}, \mathbb{R}).
\tag{18}
\]

The inclusions (17) and (18) give the equality ii) of Theorem 1.

The proof of Theorem 1 is complete. \( \square \)

**Appendix A. Hörmander spaces on the circle**

Let \( \text{OR} \) be a class of all measurable by Borel functions \( \omega : (0, \infty) \to (0, \infty) \) for which real numbers \( a, c > 1 \) exist such that

\[
c^{-1} \leq \frac{\omega(\lambda t)}{\omega(t)} \leq c \quad \forall t \geq 1, \quad \lambda \in [1, a].
\]

The space \( H^{2}_{\omega}(\mathbb{R}^{n}) \), \( n \in \mathbb{N} \), consists of all complex-valued distributions \( u \in S'(\mathbb{R}^{n}) \) such that their Fourier transformations \( \hat{u} \) are locally integrable by Lebesgue on \( \mathbb{R}^{n} \) and \( \omega((\xi))|\hat{u}(\xi)| \in L^{2}(\mathbb{R}^{n}) \) with \( \langle \xi \rangle := (1 + \xi^{2})^{1/2} \). This space is a Hilbert space with respect to the inner product

\[
\langle u_{1}, u_{2} \rangle_{H^{2}_{\omega}(\mathbb{R}^{n})} := \int_{\mathbb{R}^{n}} \omega^{2}(\langle \xi \rangle) \bar{\hat{u}_{1}}(\xi) \hat{u}_{2}(\xi) \, d\xi.
\]

It is a particular case of the isotropic Hilbert spaces of Hörmander [Hor]. If \( \Omega \) is a domain in \( \mathbb{R}^{n} \) with smooth boundary, then the spaces \( H^{2}_{\omega}(\Omega) \) are defined in a standard way.

Let \( \Gamma \) be an infinitely smooth, closed and oriented manifold of dimension \( n \geq 1 \) with density \( dx \) given on it. Let \( \mathcal{D}'(\Gamma) \) be a topological vector space of distributions on \( \Gamma \) which is dual to \( C^{\infty}(\Gamma) \) with respect to the extension by continuity of the inner product in the space \( L^{2}(\Gamma) := L^{2}(\Gamma, dx) \).

Now, let define the Hörmander spaces on the manifold \( \Gamma \). Choose a finite atlas from the \( C^{\infty} \)-structure on \( \Gamma \) formed by the local charts \( \alpha_{j} : \mathbb{R}^{n} \leftrightarrow U_{j}, \quad j = 1, \ldots, r, \) where the open sets \( U_{j} \) form a finite covering of the manifold \( \Gamma \). Let functions \( \chi_{j} \in C^{\infty}(\Gamma), \quad j = 1, \ldots, r, \) form a partition of unity on \( \Gamma \) satisfying the condition \( \text{supp} \chi_{j} \subset U_{j} \). By definition, the linear space \( H^{2}_{\omega}(\Gamma) \) consists of all distributions \( f \in \mathcal{D}'(\Gamma) \) such that \( (\chi_{j} f) \circ \alpha_{j} \in H^{2}_{\omega}(\mathbb{R}^{n}) \) for every \( j \), where \( (\chi_{j} f) \circ \alpha_{j} \) is a representation of the distribution \( \chi_{j} f \) in the local chart \( \alpha_{j} \). In the space \( H^{2}_{\omega}(\Gamma) \) the inner product is defined by the formula

\[
\langle f_{1}, f_{2} \rangle_{H^{2}_{\omega}(\Gamma)} := \sum_{j=1}^{r} \langle (\chi_{j} f_{1}) \circ \alpha_{j}, (\chi_{j} f_{2}) \circ \alpha_{j} \rangle_{H^{2}_{\omega}(\mathbb{R}^{n})},
\]

and induces the norm \( \| f \|_{H^{2}_{\omega}(\Gamma)} := \langle f, f \rangle_{H^{2}_{\omega}(\Gamma)}^{1/2} \).

There exists an alternative definition of the space \( H^{2}_{\omega}(\Gamma) \) which shows that this space does not depend (up to equivalence of norms) on the choice of the local charts, the partition of unity and that it is a Hilbert space.
Let a $\Psi$DO $A$ of order $m > 0$ be elliptic on $\Gamma$, and let it be a positive unbounded operator on the space $L^2(\Gamma)$. For instance, we can set $A := (1 - \triangle_{\Gamma})^{1/2}$, where $\triangle_{\Gamma}$ is the Beltrami-Laplace operator on the Riemannian manifold $\Gamma$. Redefine the function $\omega \in OR$ on the interval $0 < t < 1$ by the equality $\omega(t) := \omega(1)$ and introduce the norm
\begin{equation}
(\text{A.1}) \quad f \mapsto \|\omega(A^{1/m})f\|_{L^2(\Gamma)}, \quad f \in C^\infty(\Gamma).
\end{equation}

**Theorem A.1.** If $\omega \in OR$, then the space $H^2_\omega(\Gamma)$ coincides up to equivalence of norms with the completion of the linear space $C^\infty(\Gamma)$ by the norm (A.1).

As the operator $A$ has a discrete spectrum, therefore the space $H^2_\omega(\Gamma)$ can be described by means of the Fourier series. Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be a monotonically non-decreasing, positive sequence of all eigenvalues of the operator $A$, enumerated with regard to their multiplicity. Let $\{h_k\}_{k \in \mathbb{N}}$ be an orthonormal basis in the space $L^2(\Gamma)$ formed by the correspondent eigenfunctions of the operator $A$: $Ah_k = \lambda_k h_k$. Then for any distribution the following expansion into the Fourier series converging in the linear space $\mathcal{D}'(\Gamma)$:
\begin{equation}
(\text{A.2}) \quad f = \sum_{k=1}^{\infty} c_k(f)h_k, \quad f \in \mathcal{D}'(\Gamma), \quad c_k(f) := \langle f, h_k \rangle,
\end{equation}
holds.

**Theorem A.2.** The following formulae are fulfilled:
\begin{equation}
(\text{A.3}) \quad H^2_\omega(\Gamma) = \left\{ f = \sum_{k=1}^{\infty} c_k(f)h_k \in \mathcal{D}'(\Gamma) \mid \sum_{k=1}^{\infty} \omega^2(k^{1/n})|c_k(f)|^2 < \infty \right\},
\end{equation}
\begin{equation}
(\text{A.4}) \quad \|f\|_{H^2_\omega(\Gamma)}^2 = \sum_{k=1}^{\infty} \omega^2(k^{1/n})|c_k(f)|^2.
\end{equation}

Note, that for every distribution $f \in H^2_\omega(\Gamma)$ series (A.2) converges by the norm of the space $H^2_\omega(\Gamma)$. If values of the function $\omega$ are separated from zero, then $H^2_\omega(\Gamma) \subseteq L^2(\Gamma)$, and everywhere above we may change the space $\mathcal{D}'(\Gamma)$ by the space $L^2(\Gamma)$. For more details, see [MiMr1, MiMr2].

**Example.** Let $\Gamma = \mathbb{T}$. Then $n = 1$, and we can choose $A = (1 - d^2/dx^2)^{1/2}$, where $x$ defines the natural parametrization on $\mathbb{T}$. The eigenfunctions $h_k = e^{ik2\pi x}$, $k \in \mathbb{Z}$, of the operator $A$ form an orthonormal basis in the space $L^2(\mathbb{T})$. For $\omega \in OR$ we have
\begin{equation}
f \in H^2_\omega(\mathbb{T}) \iff f = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ik2\pi x}, \quad \sum_{k \in \mathbb{Z}\setminus\{0\}} |\hat{f}(k)|^2 \omega^2(|k|) < \infty.
\end{equation}

In this case the function $f$ is real-valued if and only if $\hat{f}(k) = \overline{\hat{f}(-k)}$, $k \in \mathbb{Z}$. Therefore the class $H^\omega$ coincides with the Hörmander space $H^2(\mathbb{T}, \mathbb{R})$ with the weight function $\omega(\sqrt{1 + \xi^2})$ if $\omega \in OR$. In details the class OR is described, for example, in [BuGiTg, p. 74].

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