Non-equilibrium quantum relaxation across a localization-delocalization transition

Gergő Roósz,1,2 Uma Divakaran,3,4 Heiko Rieger,4 and Ferenc Iglói1,2

1 Wigner Research Centre, Institute for Solid State Physics and Optics, H-1525 Budapest, P.O.Box 49, Hungary
2 Institute of Theoretical Physics, Szeged University, H-6720 Szeged, Hungary
3 Department of Physics, Indian Institute of Technology Kanpur- 208016, India
4 Theoretische Physik, Universität des Saarlandes, 66041 Saarbrücken, Germany

(Dated: July 30, 2014)

We consider the one-dimensional XX-model in a quasi-periodic transverse-field described by the Harper potential, which is equivalent to a tight-binding model of spinless fermions with a quasi-periodic chemical potential. For weak transverse field (chemical potential), $h < h_c$, the excitations (fermions) are delocalized, but become localized for $h > h_c$. We study the non-equilibrium relaxation of the system by applying two protocols: a sudden change of $h$ (quench dynamics) and a slow change of $h$ in time (adiabatic dynamics). For a quench into the delocalized (localized) phase, the entanglement entropy grows linearly (saturates) and the order parameter decreases exponentially (has a finite limiting value). For a critical quench the entropy increases algebraically with time, whereas the order parameter decreases with a stretched-exponential. The density of defects after an adiabatic field change through the critical point is shown to scale with a power of the rate of field change and a scaling relation for the exponent is derived.

I. INTRODUCTION

Non-equilibrium relaxation in a closed quantum system following a change of some parameter(s) in the Hamiltonian is of recent interest, both experimentally and theoretically. Considering the speed of variation of the parameter, we generally discriminate between two limiting processes. For the quench dynamics, the parameter is modified instantaneously, which experimentally can be realized in ultra cold atomic gases using the phenomenon of Feshbach resonance. In this process the evolution of different observables after the quench is of interest, as well as the possible existence and properties of the stationary state, in particular in integrable and non-integrable systems. In the other limiting relaxation process, in the so called adiabatic dynamics the parameter is varied very slowly, usually linearly in time with a rate $1/\tau$ across a phase-transition point. In this case one is interested in the density of defects, which are produced when the system falls out of equilibrium close to the critical point.

Most of the results for non-equilibrium quantum relaxation are obtained for homogeneous systems, for which the eigenstates are generally extended. As a consequence after a quench the general (time- and space-dependent) correlation functions decay exponentially, which can be explained (even quantitatively) within a semi-classical theory. In the stationary state thermalization is expected to hold for non-integrable models whereas for integrable models the stationary state is generally described by a so called generalized Gibbs ensemble (GGE).

Concerning adiabatic dynamics, variants of the Kibble-Zurek scaling theory are found to hold: the density of defects scales as $\tau^{-\kappa}$ and $\kappa$ is related to the static critical exponents $z$ and $\nu$, as well as to the dimension of the system.

Among inhomogeneous quantum systems, random quantum spin chains have most frequently been studied in the context of non-equilibrium relaxation. In these disordered one-dimensional systems, the eigenstates are localized even in the presence of interactions, which prevents thermalization after a quench. Consequently an unusual relaxation can be observed: after a quench both the average entanglement entropy and the magnetization approach a non-vanishing stationary value. After a critical quench (i.e. a quench to the critical point), the dynamics is ultra-slow: the entanglement entropy grows in time as $\ln \ln (\tau)$, whereas the magnetization behaves as $[\ln (\tau)]^{-A}$ with a disorder dependent exponent, $A$. For the adiabatic dynamics the defect density is found to scale as $1/\ln^2 (\tau)$, which is a consequence of the equilibrium dynamical scaling relation: $\xi \sim \ln^2 (\tau)$, $\xi$ being the correlation length.

Localization of eigenstates can exist in non-disordered systems, too, as for instance in quasi-periodic systems. A well known example is the Aubry-André model, which is a one-dimensional hopping model with a specific quasi-periodic potential denoted as Harper’s potential. This model could be experimentally realized by ultra cold atomic gases in optical lattices having two periodic optical waves with different incommensurate wavelengths. For weak quasi-periodic potential the eigenstates are extended, but they become localized for a sufficiently strong potential. A similar scenario has been predicted for interacting particles: sufficiently strong quasi-periodic potential leads to many-body localization. The quench dynamics in the Aubry-André model for hard-core bosons has been studied recently, where the GGE scenario was shown to be valid in the extended phase, but fails in the localized phase.

In the present paper we revisit the non-equilibrium relaxation properties of the Aubry-André model. New features of our study are the following. We consider a magnetic model, the $S = 1/2$ XX-chain in a quasi-periodic transverse field, which after a Jordan-Wigner transformation - is equivalent to a tight-binding model of spinless
fermions in a quasi-periodic chemical potential. We study the non-equilibrium dynamics after a sudden change of the amplitude of the transverse field and compute the dynamical evolution of the entanglement entropy, as well as the relaxation of the magnetization. We investigate separately, when the quench is performed to the extended or to the localized phase, as well as to the transition point. We also study adiabatic dynamics, which has not been considered before, and calculate the density of defects which are created during the process, when the amplitude of the transverse field is passed linearly through the localization-delocalization transition point.

The paper is organized as follows: The model and the observables of interest are introduced in Sec. II. Results for the quench dynamics and the adiabatic dynamics are shown in Secs. III and IV, respectively. Our paper is closed by a discussion in the last section.

II. MODEL AND OBSERVABLES

A. Quasi-periodic XX-chain

We consider the spin-1/2 XX-chain in the presence of a position dependent transverse field, which is defined by the Hamiltonian:

$$\mathcal{H} = -\frac{J}{4} \sum_{n=1}^{L} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) - \sum_{n=1}^{L} h_n \sigma_n^z ,$$

(1)

in terms of the $\sigma_n^{x,y,z} \!$ Pauli-matrices at site $n$. Generally we apply periodic boundary conditions, thus $\sigma_{L+1}^x \equiv \sigma_1^x$ and $\sigma_{L+1}^y \equiv \sigma_1^y$. In the following we fix $J = 1$ and use a quasi-periodic potential:

$$h_n = h \cos(2\pi \beta n)$$

(2)

where $\beta$ is an irrational number: typically we use $\beta = \frac{\sqrt{5} - 1}{2}$ the inverse of the golden mean, which is the “most” irrational number. Using the Jordan-Wigner transformation the Hamiltonian is expressed in terms of fermion creation ($c_n^\dagger$) and annihilation ($c_n$) operators:

$$\mathcal{H} = -\frac{1}{2} \sum_{n=1}^{L} (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) - h \sum_{n=1}^{L} \cos(2\pi \beta n)c_n^\dagger c_n ,$$

(3)

thus in Eq. (3) we have a tight-binding model of spinless fermions in a quasi-periodic chemical potential.

This type of potential appears first in Harper’s paper [2], in which he showed that Hamiltonian in Eq. (3) for $h = 1$ describes an electron on a square lattice in a perpendicular magnetic field. Introducing a new set of fermion operators $\eta_q$ through the canonical transformation:

$$\eta_q = \sum_{n=1}^{L} \phi_{q,n} c_n ,$$

(4)

with $\sum_{q=1}^{L} \phi_{q,n} \phi_{q,n'} = \delta_{n,n'}$ the Hamiltonian in Eq. (3) is transformed to a diagonal form:

$$\mathcal{H} = \sum_{q} \varepsilon_q (\eta_q^\dagger \eta_q - 1/2) .$$

(5)

Here the energy of modes, $\varepsilon_q$, and the components of vectors, $\phi_{q,n}$ satisfy the almost Mathieu equation [9].

$$\frac{1}{2} \phi_{q,n-1} + h_n \phi_{q,n} + \frac{1}{2} \phi_{q,n+1} = -\varepsilon_q \phi_{q,n} .$$

(6)

There is a vast literature about properties of the almost Mathieu equation, as well as on the properties quasi-periodic Hamiltonians both in mathematica [21] and physical points of view.

B. Aubry-André duality

Following Aubry and André [3] a new set of fermion operators are introduced:

$$c_k = \frac{1}{\sqrt{L}} \sum_{n} \exp(i 2\pi k \beta n)c_n$$

(7)

which are eigenstates of the momentum operator with eigenvalue: $k = kF_{n-1}\text{mod}F_n$, where $F_n$ is the $n$-th Fibonacci number and $L = F_n$. In terms of these the Hamiltonian is given by:

$$\mathcal{H} = -\frac{\hbar^2}{2} \left( \sum_{k=-\infty}^{\infty} (c_k^\dagger c_{k+1} + c_{k+1}^\dagger c_k) - \frac{\hbar}{2 \beta} \sum_{k=1}^{L} \cos(2\pi \beta k)c_k^\dagger c_k \right) .$$

(8)

Note that Eq. (8) is in the same form as that in Eq. (3), thus the Hamiltonian satisfies the duality relation:

$$\mathcal{H}(\hbar) = \hbar \mathcal{H}(1/\hbar) .$$

(9)

Through Eq. (9) the small $\hbar$ regime of the Hamiltonian, in which the eigenstates are extended in the real space are connected with the large $\hbar$ regime, in which the eigenstates have extended properties in the Fourier space, thus these are in the real space localized. The localization transition takes place at the self-duality point, thus the critical amplitude of the field is $h_c = 1$. For $h > 1$ the localized states have a finite correlation length, $\xi$, which is given by [31]:

$$\xi = \frac{1}{\ln(h)} , \quad h > 1 ,$$

(10)

for all eigenstates of $\mathcal{H}$. Similar conclusion holds for the eigenvectors, $\phi_{q,n}$ in Eq. (5) which are used to diagonalize the Hamiltonian in Eq. (3). The $\phi_{q,n}$-s are localized in the $h > 1$ regime with the same correlation length given in Eq. (10) and for large $|\hbar|$ these are given by:

$$\phi_{q,n} = \delta_{n,n_q}, \quad \varepsilon_q = -h \cos(2\pi \beta n_q) , \quad |\hbar| \gg 1 .$$

(11)
In the quench process the amplitude of the transverse field is suddenly changed from a value of $h_0$ for $t < 0$ to another value, say $h$ for $t > 0$ and the Hamiltonians are denoted by $H_0$ and $H$, respectively. For $t < 0$ the system is in the ground state of the initial Hamiltonian, $|\Psi_0^{(0)}\rangle$, while for $t > 0$ its time-evolution involves the new Hamiltonian, $H$, and given by $|\Psi_0(t)\rangle = \exp(-iHt)|\Psi_0^{(0)}\rangle$, thus generally $|\Psi_0(t)\rangle$ is not an eigenstate of $H$. We set $h$ to unity through out this paper. The expectation value $\langle A(t) \rangle$ of an observable, $A$, is given by $\langle \Psi_0^{(0)}|\hat{A}(t)|\Psi_0^{(0)}\rangle$, where $\hat{A}(t) = \exp(iHt)\hat{A}\exp(-iHt)$ is $\hat{A}$ in the Heisenberg picture. One can calculate time-dependent correlation functions in similar way.

In the actual problem we calculate the entanglement entropy $S_\ell(t)$ of the first $\ell$ spins of the chain and the rest of the system, which is defined as: $S_\ell(t) = \text{Tr}_\ell \left[ \rho_\ell(t) \ln \rho_\ell(t) \right]$. Here $\rho_\ell(t) = \text{Tr}_{\ell+1} \left[ |\Psi_0(t)\rangle \langle \Psi_0(t)| \right]$ is the reduced density matrix with $|\Psi_0(t)\rangle$ being the state of the complete system at time $t$ obtained after solving the Schrödinger equation. In a homogeneous chain for $L \to \infty$ and $\ell \gg 1$ the entanglement entropy has two different regions:\footnote{D. Density of defects in the adiabatic dynamics}

For $t < \ell/v_{\text{max}}$, where $v_{\text{max}}$ is some maximal velocity of quasi-particles, the entanglement entropy increases linearly: $S_\ell(t) \sim t$; while for $t > \ell/v_{\text{max}}$, its saturates as $S_\ell(t) \sim \ell$. For random quantum spin chains, due to localized excitations the entanglement entropy saturates at a finite value, except at the critical point, where there is an ultra-slow increase of the form:\footnote{D. Density of defects in the adiabatic dynamics} $S_\ell(t) \sim \ln \ln t$.

In the one-dimensional Fibonacci quasi-crystal, where the spectrum of excitations is singular continuous:\footnote{D. Density of defects in the adiabatic dynamics}, the entropy grows in a power-law form: $S_\ell(t) \sim t^\sigma$, with $0 < \sigma < 1$ being a function of the quench parameters:\footnote{D. Density of defects in the adiabatic dynamics}

Another observable we calculate is the local order-parameter (magnetization), $m_\ell(t)$, at a position $\ell$ in an open chain. Here we follow the method of Yang\footnote{D. Density of defects in the adiabatic dynamics} and define $m_\ell(t)$ for large $L$ by the off-diagonal element: $m_\ell(t) = \langle \Psi_0^{(0)} | \hat{P}_{\ell,\ell+1}^{(0)} | \Psi_0^{(0)} \rangle$, where $| \Psi_0^{(0)} \rangle$ is the first excited state of $H_0$. In a homogeneous chain of infinite length ($L \to \infty$), the magnetization for a bulk site $\ell \gg 1$ has an exponential decay:\footnote{D. Density of defects in the adiabatic dynamics} both in time: $m_\ell(t) \sim \exp(-t/\tau)$ for $t < \ell/v_{\text{max}}$ and in space: $m_\ell(t) \sim \exp(-\xi/\ell)$ for $\ell \gg \ell/v_{\text{max}}$. Here the non-equilibrium relaxation time, $\tau$, and the non-equilibrium correlation length, $\xi$, are given functions of the quench parameters, $h_0$ and $h$. For random quantum spin chains the local magnetization relaxes to a finite limiting value, except at the critical point, where the decay is logarithmically slow:\footnote{D. Density of defects in the adiabatic dynamics} $m_\ell(t) \sim (\ln t)^{-A}$ and $A$ depends on the form of the disorder. In the one-dimensional Fibonacci quasi-crystal the relaxation of the bulk magnetization is given in a stretched-exponential form:\footnote{D. Density of defects in the adiabatic dynamics} $m_\ell(t) \sim \exp(-C/t^\mu)$. Here the exponent $\mu$ and the exponent of the entanglement entropy, $\sigma$, are found to be close to each other, at least in the so called non-oscillatory phase.
Note that $P_t$ is normalized in the sense that $0 \leq P_t \leq 1$.

In the actual calculation we have taken two limiting final states: i) $t = 0$, when the quench is performed at the middle of the extended phase and ii) $t = \infty$, when the quench goes across the extended phase and ends at the other limiting side of the localized phase. In the first case the localization-delocalization transition point is crossed once at $h = -1$, while in the second protocol it is crossed twice, at $h = \pm 1$.

III. QUENCH DYNAMICS

In the (sudden) quench dynamics we have used $\beta = (\sqrt{5} - 1)/2$, the inverse golden-mean ratio for the parameter of the Harper potential and the length of the finite chains were fixed to a Fibonacci number $F_n$. We have calculated the entanglement entropy and the local magnetization up to $L = F_{17} = 1597$.

A. Entanglement entropy

The entanglement entropy, $S_t$, is calculated between a block of length, $\ell = F_{n-2}$ and its environment of length $F_{n-1}$ with periodic boundary conditions. (For details of the calculation of the entanglement entropy in the free-fermion basis see the Appendix of Ref. [95].) We used the ground state corresponding to the initial field $h_0 = 0$ as the initial state and then made quenches to the extended $(0 < h < 1)$ and to the localized phases $(h > 1)$, as well as to the critical point $(h = 1)$. Numerical results for $S_t(t)$ are shown in Figure 1. The dynamics of the entanglement entropy has two different regimes (as for homogeneous chain): for short times it is an increasing function of time and for long times it saturates to some value. For quenches to the extended phase the time-dependence in the initial period is linear, $S_t(t) \approx \alpha(h)t$ and the saturation value is $S_t \sim \ell$. This behavior is qualitatively similar to homogeneous system. Estimates of the prefactor of the linear term, $\alpha$ are shown in Fig. 2. Starting from $h = h_0 = 0 \alpha$ is first increasing, has a maximum around $h = 0.5$ and then decreasing to 0 at $h = 1$.

After a quench into the localized phase the entropy saturates quickly to an $\ell$ independent value: $S_t = S(h)$, $h > 1$. We have checked that close to the transition point $S(h)$ diverges:

$$S(h) \sim |\ln(h)|^{-\sigma'},$$

with an exponent: $\sigma' = 0.50(4)$, see in the lower panel of Fig. 1.

Finally, if the quench is performed to the transition point the growth of the entropy is given in a power-low form:

$$S(t) \sim t^{\sigma},$$

with an exponent $\sigma = 0.43(5)$. Using phenomenological scaling theory a relation between the exponents $\sigma'$ and $\sigma$ can be derived in the following way. Under uniform scaling transformation, when lengths are rescaled by a factor $b > 1$ the entanglement entropy behaves as: $S(\ln h, t) = b^sS(b/\ln h, t/b^z)$ for $h \geq 1$, where we have used the form of the correlation length in Eq. [10] and $z = 1$ is the dynamical exponent. Now taking the scale factor $b = t^{1/z}$ we obtain $S(\ln h, t) = t^{s/z}S(t^{1/z} / \ln h)$. At the critical point, $h = 1$, the scaling function has the limiting value $\lim_{u \to -\infty} S(u) = \text{cst}$, thus $\sigma = s/z = s$.

Similarly, taking $b = 1/\ln(h)$ we can show that $\sigma' = s$, thus $\sigma = \sigma'$ in agreement with the numerical results.

The properties of the dynamical entropy can be explained in terms of anomalously diffusing quasiparticles, see in Sec. III C.
for the characteristic time, $\tilde{\tau} \sim h^2 \ln(h)$ time-dependent bulk magnetizations after a quench from $h_0 = 0$ to different values of $h$.

FIG. 2: (Color online) Prefactor of the linear part of the dynamical entanglement entropy (left axis) and the relaxation time (right axis) after a quench from $h_0 = 0$ to different values of $h$.

The local magnetization, $m_b(t)$ is measured in a chain of length $L = F_n$ at a position $l = F_{n-2}$, for technical details see the Appendix of Ref. 93. In this region of the chain the local magnetization is practically independent of $l$ and we consider it as the bulk magnetization and will be denoted by $m_b(t)$. The numerically calculated time-dependent bulk magnetizations after a quench from $h_0 = 0$ to different values of $h$ are shown in Fig. 3. If the quench is performed to the extended phase ($0 < h < 1$) the decay of magnetization is exponential: $m_b(t) \sim \exp(-t/\tilde{\tau})$, as in the homogeneous system. Estimates for the characteristic time, $\tilde{\tau}(h)$ are given in Fig. 2 with varying $h$ it has similar characteristic as the prefactor of the linear part of the entanglement entropy. If the quench is performed to the localized phase $h > 1$ the magnetization approaches a finite limiting value. Finally, for the critical quench ($h = 1$) the decay is stretched exponential:

$$m_b(t) \sim A(t) \exp(-C t^\mu) ,$$

where $A(t)$ is some oscillatory function and $\mu = 0.47(5)$. This is illustrated in the inset of Fig. 2. This behavior is interpreted in terms of quasiparticles in the following section.

C. Quasiparticle interpretation

Non-equilibrium quench dynamics is well described within the framework of a semiclassical theory. It is based on the concept of quasiparticle that are produced uniformly in the system during the quench and which move classically after production. We regard these quasiparticles as wave packets, which are localized at some site at $t = 0$ and which perform afterwards a diffusive motion. Following previous studies in quasicrystals, we construct the wave packet connecting sites $n$ and $n'$ at time $t$ in the form:

$$W_{n,n'}(t) = \sum_q \cos(\epsilon_q t) \phi_{q,n} \phi_{q,n'} ,$$

in terms of the eigenvectors and eigenvalues of Eq. (17) calculated with the amplitude $h$, i.e., after the quench. Due to normalization of the eigenvectors $W_{n,n'}(0) = \delta_{n,n'}$. The width of the wave-packet created at site $n$ after time $t$ is given by:

$$d(n,t) = \left[ \sum_{n'} (n - n')^2 |W_{n,n'}(t)|^2 \right]^{1/2} ,$$

which is than averaged over the starting positions, thus $d(t) = \langle d(n,t) \rangle$.

We have calculated $d(t)$ for different values of the amplitude of the transverse field and these are shown in Fig. 4. In agreement with previous studies, $d(t)$ grows linearly in the extended phase ($0 < h < 1$) thus the quasiparticles move ballistically. From this follows - repeating the arguments of the semiclassical theory - thus the dynamical entropy grows linearly and the bulk magnetization has an exponential decay. In the localized phase ($h > 1$) the width of the wave-packet stays finite, $d(t) \rightarrow \tilde{d}$. We have checked that close to the transition point this limiting value scales as the localization length in the system: $\tilde{d} \sim \xi$, see in the lower panel of Fig. 4.

Finally, at the transition point ($h = 1$) the width of the wave packet grows algebraically with time: $d(t) \sim t^D$, where the diffusion exponent is estimated as $D = 0.477(10)$. In the semiclassical theory the anomalous diffusion of quasiparticles manifests itself in the modified

FIG. 3: (Color online) Bulk magnetization after a quench from $h_0 = 0$ to different values of $h$. In the inset quench to the critical region is shown in agreement with the stretched-exponential form in Eq.(17) (the straight lines have a slope $\mu = 0.47$).
form of the dynamical entanglement entropy in Eq. (16) and of the bulk magnetization in Eq. (17). The corresponding exponents, $\sigma$, $\mu$ and $D$ should be equal, which is indeed satisfied within the error bars of the numerical estimates.

**IV. ADIABATIC DYNAMICS**

The adiabatic dynamics is calculated numerically in systems of finite size $L = 2F_n$ with $\beta = F_{n-1}/F_n$ as an approximant of the inverse golden mean ratio. We set $|h_{\text{max}}| = 10$ for the largest amplitude of the transverse field and checked that the numerical results are stable: they do not change if we used instead $|h_{\text{max}}| = 20$. The differential equation in Eq. (12) is integrated numerically using a Runge-Kutta method with adaptive stepsize in time to keep the relative error less than $10^{-6}$.

Numerical results of the excitation probability as a function of the time-scale, $\tau$, are shown in Fig. 5 for the two types of final states, with $t = 0$ and $t = \infty$, respectively. In the first case, $t = 0$ the largest Fibonacci parameter in the calculation was $n = 18$, while for $t = \infty$ it was $n = 17$. In both cases the excitation probability has an asymptotic power-law dependence:

$$P(\tau) \sim A_0(\tau)\tau^{-\kappa},$$

but the prefactors, $A_0(\tau)$ have different functional forms. In the first case with $t = 0$ when the localization-delocalization transition is crossed once (at $h = -1$) the prefactor has a weak, approximately log-periodic oscillating form: $A_0(\tau) \sim \sin^2(\log(\tau/\tau_0))$. This type of log-periodic oscillations are due to discrete scale invariance and these are often present in quasi-periodic and aperiodic systems. Due to this correction the decay exponent, $\kappa$ can only be estimated with some uncertainty:

$$\kappa = 0.45(5).$$

In the second protocol with $t = \infty$ when the localization-delocalization transition is crossed twice (at $h = -1$ and $h = 1$) the prefactor has oscillations in $\tau$, $A_\infty(\tau) \sim \sin^2(\tau/\tau_\infty + \text{cst.})$ with $\tau_\infty \approx 0.15$, to which also a log-periodic correction is supplemented. This oscillatory phase is analogous to the St"uckelberg oscillations of a periodically driven two-level system which arises due to the interference of probability amplitude between the ground and the excited state, when the region of avoided level crossing is passed twice. In the second case due to the oscillations the estimate of $\kappa$ is somewhat less accurate. We checked, however, that the numerical data in Fig. 5 are compatible with the estimate for $\kappa$ in Eq. (21).

In the following we explain the numerical value of the decay exponent in Eq. (21) and relate it to the combination of other exponents. First, let us recapitulate the reasoning of traditional scaling theory. The amplitude of the transverse field at time $t$ is given by $h(t) = 1 + t/\tau$, and therefore the distance from the critical point $\delta(t) = t/\tau$. This implies that the equilibrium relaxation time

**FIG. 4:** Time-dependent width of the wave packet at different amplitudes of the transverse field. (Color online)

**FIG. 5:** Excitation probability as a function of the time-scale, $\tau$, after an adiabatic process from $h = -\infty$ to $h = 0$ (upper panel) and to $h = \infty$ (lower panel) calculated in finite systems of sizes $L = 2F_n$ with $n = 13, 14, \ldots, 18$. (Color online)
of the system at time $\tilde{t}$ is $\tilde{t}^* \sim \xi^z \sim \delta^{-\nu z} = (\tilde{t}/\tau)^{-\nu z}$, where $\xi$ is the equilibrium correlation length. When the relaxation time $\tilde{t}^*$ is of the same order as the time $\tilde{t}$ the system falls out of equilibrium, i.e. the ground state cannot follow adiabatically the field change any more. The condition $\tilde{t} = \tilde{t}^*$ implies

$$\tilde{t} \sim \tau^{z/\nu z}. \quad (22)$$

For $|\tilde{t}| < \tilde{t}$ defects are produced and transitions to excited states occur. The typical distance between neighbouring defects is then given by $\xi (\sim \tilde{t}^{1/2})$, thus the phase-space of excitations in a $d$-dimensional system is $\Omega \sim \xi^d \sim \tau^{-z/\nu z}$. Then, it is usually expected that the elementary transition probabilities, such as $p_{q,q'}$ in Eq. (13) are independent of the scale thus $P(\tau) \sim \Omega$ and we arrive at the scaling relation:

$$P_{sc}(\tau) \sim \tau^{-z/\nu z}. \quad (23)$$

For the Aubry-André model with $d = 1$ and $\nu = z = 1$ the prediction of traditional scaling theory is $\kappa_{sc} = 0.5$, which is somewhat larger than (although at the border of) the numerical estimate in Eq. (21). However, the assumptions used in the derivation of $P_{sc}(\tau)$ are not valid for the Aubry-André model since the ground state of the Hamiltonian in Eq. (13) is not a continuous function of the amplitude of the transverse field at $h = \pm 1$. Therefore we study numerically the scaling behavior of the elementary transition probabilities, $p_{q,q'}$, calculated at $t = 0$, i.e. for the first protocol. First we notice that $p_{q,q'} = p_{q',q}$ and arrange the $p_{q,q'}$-s in decreasing order. Then in Eq. (13) we sum up the contribution of the largest $N$ terms:

$$P(\tau) = \frac{2}{L} \sum_{q \in Q^*} \sum_{q' \in Q^-} N' p_{q,q'}, \quad (24)$$

which is denoted by the prime at the summation and this quantity is called the partial excitation probability. For large $N$ we can rearrange the parameters $q$ (and also $q'$), such that in Eq. (13) by restricting the summations to $q, q' \leq \sqrt{N}$ we get (asymptotically) $P(\tau) \sim \pi(\tau^{z/\nu z}/N^2) \pi$. Generally, for $q_1 < q_2$ ($q'_1 < q'_2$) the free-fermionic energies in Eq. (10) satisfy $\epsilon_{q_2} < \epsilon_{q_1} < 0$ ($\epsilon_{q'_2} < \epsilon_{q'_1}$).

We have calculated the partial excitation probability, $P(\tau)$, normalized with its limiting value $P_0(\tau)$ for different sizes and for different decay parameters. For large $N$ and $L$ the partial excitation probability is found to be a function $N^2/L^2$, thus $P(N, L, \tau) \sim P(\tau)$. As illustrated in the upper panel of Fig. 4 for different values of $L$ at a fixed value of $\tau$. The $\tau$-dependence of $P(\tau)$ is shown in the lower panel of Fig. 4 at a fixed (large) $L$ and for different values of $\tau < L$. With increasing $\tau$ the scaling functions appear to approach the same limiting curve, thus $P(N, L, \tau)$ is factorized as

$$P(N, L, \tau) = \pi(N^2/L^2)P_0(\tau) \quad (25)$$

for large enough $\tau$.

As seen in the lower panel of Fig. 4 in the log-log plot $\pi(N^2/L^2)$ has a linear section over several decades and then it saturates for large arguments, say for $N < N_{\text{eff}}$. Thus we can approximate

$$P(N, L, \tau) \approx \left\{ \begin{array}{ll} P(N, L, \tau) & \text{if} \quad N \leq N_{\text{eff}} \\ P_0(\tau) \sim (N/N_{\text{eff}})^{\omega} & \text{if} \quad N > N_{\text{eff}} \end{array} \right. \quad (26)$$

From the data in the lower panel of Fig. 4 we estimate $\omega = 0.90(2)$. Now let us consider the scaling behavior of $P(N, L, \tau) = P(N, L^2, \tau)$, when lengths are rescaled be a factor $b > 1$. Keeping in mind that $N_{\text{eff}}/L^2 \sim \Omega$ is the phase-space of excitations given by $\Omega \sim \xi^{-1}$ we obtain:

$$P(N, L^2, \tau) = b^{-\omega} P(b N/L^2, b^{-1/\nu z} \tau). \quad (27)$$

Here the prefactor, $b^{-\omega}$ follows from Eq. (26) and the scaling dimension of $\tau$ can be read from Eq. (22). Now taking $b = \tau^{-1/\nu z}$ we get:

$$P(N, L^2, \tau) = \tau^{-z/\nu z} \pi(\tau^{z/\nu z}/N^2). \quad (28)$$

thus at $N \approx N_{\text{eff}}$:

$$P_0(\tau) \sim \tau^{-z/\nu z}, \quad (29)$$

and $\kappa = \kappa_{sc}\omega$ (also $N_{\text{eff}}/L^2 \sim \tau^{-z/\nu z}$). With the measured value of $\omega$ we get $\kappa \simeq 0.45$ in agreement with the direct estimate in Eq. (21).

V. DISCUSSION

In this paper we have studied the non-equilibrium dynamics of the Aubry-André model for the $S = 1/2$-spin XX-chain in the presence of a quasi-periodically modulated transverse field, which is equivalent to a tight-binding model of spinless fermions in a quasi-periodic chemical potential. In this model there is a localization-delocalization quantum phase-transition separating the extended and the localized phases. By varying the amplitude of the transverse field in time, $h(t)$, we have studied the properties of non-equilibrium quantum relaxation at zero temperature. We considered in details two limiting cases of the dynamics.

First we studied quench dynamics, in which $h(t)$ is changed suddenly at $t = 0$ and focussed on the dynamics of the entanglement entropy, as well as on the relaxation of the local order-parameter. For quenches to the extended phase the non-equilibrium dynamics turns out to be qualitatively similar as in the homogeneous model: the entanglement entropy increases linearly, while the local order-parameter decays exponentially. The characteristic parameters, the prefactor of the linear part of the entanglement entropy, as well as the relaxation time are found to depend on the details of the quench process. This type of non-equilibrium behavior is consistent with
the GGE scenario. In contrast to this, after a quench into the localized phase there is no thermalization in the stationary state: both the entanglement entropy and the local order-parameter approaches a finite limiting value. Finally, for a critical quench the entanglement entropy increases as a power law, whereas the local order-parameter decays with a stretched exponential. This type of behavior is related to the singular continuous form of the spectrum of the critical Hamiltonian, as already noticed in the quench dynamics of quantum Fibonacci quasicrystals. The properties of the critical quench have been explained in the frame of a semi-classical theory in terms of anomalously diffusing quasi-particles, which are created uniformly in space during the quench.

In the second type of non-equilibrium process we have varied $h(t)$ linearly in time with a rate $1/\tau$ and studied the density of defects in the ground state created during this process. If the localization-delocalization transition point is passed once the density of defects follows a power-law dependence, $\sim \tau^{-\kappa}$, while if two symmetrically placed transition points are passed then the density of defects has a multiplicative oscillating correction, similar to the St"uckelberg phase of periodically driven two-level systems. Using scaling arguments we have related $\kappa$ to another critical exponent as given in Eq. (29).

Finally, we discuss the question of the non-equilibrium dynamics of the Hamiltonian in Eq. (11) for different values of the quasi-periodicity parameter $\beta$ in Eq. (8). If $\beta$ is a rational number of the form $\beta = 1/(2q)$ with $q$ being an integer, then in the adiabatic process the decay exponent is given by $\kappa = q/(q + 1)$. The same result holds for $\beta = p/(2q)$, when $p$ is an odd integer and $p$ and $q$ are relative primes, at least for not too large values of $q$. Thus these results can not be analytically continued to the case, when $\beta$ is an irrational number. If $\beta$ is an irrational number and different from the inverse of the golden mean ratio studied in this paper, than the critical exponents of the non-equilibrium dynamics are expected to be $\beta$ dependent. Some hint in favor of this assumption can be found in the diffusion properties of the quasiparticles, see in Sec III C. Indeed the diffusion exponent, $D$, is measured to be $\beta$ dependent, and the same is expected to hold for the non-equilibrium exponents $\sigma$ and $\mu$.

Acknowledgments

This work has been supported by the Hungarian National Research Fund under grant No. OTKA K109577. G. R. and F. I. thanks to the Institute of Theoretical Physics, Saarland University for hospitality and G. R. thanks to the Campus Hungary for a travelling grant. U. D. acknowledges financial support from Alexander von Humboldt foundation with which this work was carried out at Saarland University, and the hospitality of Wigner Research Center, Institute of Solid State Physics, Budapest during her visit.
A. Polkovnikov, Phys. Rev. B 72, 161201(R) (2005); A. Polkovnikov and V. Gritsev, Nature Phys. 4, 477 (2008).

P. Calabrese and J. Cardy, J. Stat. Mech. P04010 (2005), and Phys. Rev. Lett. 96, 136801 (2006).

T. Caneva, R. Fazio and G. E. Santoro, Phys. Rev. B 76, 144427 (2007).

R. W. Cherng and L. S. Levitov, Phys. Rev. A 73, 043614 (2006).

V. Mukherjee, U. Divakaran, A. Dutta, and D. Sen, Phys. Rev. B 76, 174303 (2007); U. Divakaran, A. Dutta, and D. Sen, Phys. Rev. B 78, 144301 (2008); S. Deng, G. Ortiz, and L. Viola, EPL 84, 67008 (2008); U. Divakaran, V. Mukherjee, A. Dutta, and D. Sen, J. Stat. Mech: Theory Exp. P02007 (2009); V. Mukherjee and A. Dutta, EPL 92, 37004 (2010).

A. Dutta, R. R. P. Singh, and U. Divakaran, EPL 89, 67001 (2010); T. Hikichi, S. Suzuki, and K. Sengupta, Phys. Rev. B 82, 174305 (2010).

K. Sengupta, D. Sen, and S. Mondal, Phys. Rev. Lett. 100, 077204 (2008); S. Mondal, D. Sen, and K. Sengupta, Phys. Rev. B 78, 045101 (2008).

C. de Grandi, V. Gritsev, and A. Polkovnikov, Phys. Rev. E 86, 012303 (2010).

D. Patané, A. Silva, L. Amico, R. Fazio, and G. E. Santoro, Phys. Rev. Lett. 101, 175701 (2008).

C. de Grandi, R. Barankov, and A. Polkovnikov, Phys. Rev. Lett. 101, 230402 (2008).

A. Bermudez, D. Patanè, L. Amico, and M. A. Martin-Delgado, Phys. Rev. Lett. 102, 135702 (2009).

S. Mondal, S. Vishveshwara, and P. G. Harper, Proc. Phys. Soc. A 166, 972 (2009).

J. Dziarmaga, Advances in Physics 59, 1063 (2010); A. Dutta, U. Divakaran, D. Sen, B. K Chakrabarti, T. F. Rosenbaum, and G. Aeppli, arXiv:1012.0653.

M. Thakurathi, W. DeGottardi, D. Sen, S. Vishveshwara, Phys. Rev. B 85, 165425 (2012).

G. De Chiara, S. Montangero, P. Calabrese, R. Fazio, J. Stat. Mech., L03001 (2006).

F. Iglói, Zs. Szatmári, and Y.-C. Lin, Phys. Rev. B 85, 094417 (2012).

G. C. Levine, M. J. Bantegui, J. A. Burg, Phys. Rev. B 86, 174202 (2012).

J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. 109, 017202 (2012).

R. Vosk, E. Altman, Phys. Rev. Lett. 110, 067204 (2013).

F. Iglói (unpublished).

For a review, see: F. Iglói and C. Monthus, Physics Reports 412, 277, (2005).

S. Aubry and G. André, Ann. Israel Phys. Soc. 3 133 (1980).

P. G. Harper, Proc. Phys. Soc. A 68, 874 (1955).

G. Roati, C. D. Errico, L. Fallani, M. Fattori, C Fort, M. Zaccanti, G. Modugno, M. Modugno and M. Inguscio, Nature 453, 895 (2008); M. Modugno, New J. Phys. 11, 033023 (2009); B. Deissler, E. Lucioni, M. Modugno, G. Roati, L. Tanci, M. Zaccanti, M. Inguscio and M. Modugno, New J. Phys. 13, 023020 (2011).

S. Iyer, V. Oganesyan, G. Refael and D. A. Huse, Phys. Rev. B 87, 134202 (2013).

V.P. Michal, B.L. Altshuler, G.V. Shlyapnikov, arXiv:1402.4796.

Ch. Gramsch, M. Rigol, Phys. Rev. A 86, 053615 (2012).

E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) 16, 407 (1961).

H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, Schrödinger Operators (Berlin: Springer) (1987).

A. Sütő, Beyond Quasicrystals, ed F. Axel and D. Gratias (Springer-Verlag & Les Editions de Physique) p. 481 (1995).

M. Kohmoto, Phys. Rev. Lett. 51, 1198 (1983); D. J. Thouless, 1983 Phys. Rev. B 28, 4272 (1983), G.-L. Ingold, A. Wobst, C. Aulbach and P. Hänggi, Eur. Phys. J. B 30, 175 (2002).

F. Iglói, G. Roósz and Y.-C. Lin, New J. Phys. 15, 023036 (2013).

C. N. Yang, Phys. Rev. 85, 808 (1952).

F. Iglói, Zs. Szatmári, and Y.-C. Lin, Phys. Rev. B 80, 024405 (2009).

S. J. Poon, Adv. Phys. 41, 303 (1992); H. Q. Yuan, U. Grimm, P. Repetowicz, and M. Schreiber, Phys. Rev. B 62, 15569 (2000); H. Schulz-Baldes and J. Bellissard, Rev. Math. Phys. 10, 1 (1998); B. Huckestein and L. Schweitzer, Phys. Rev. Lett. 72, 713 (1994); S. Thiem and M. Schreiber, Phys. Rev. B 85, 224205 (2012).

M. Wilkinson and J. Austin, Phys. Rev. B 50, 1420 (1994).

D. Karevski and L. Turban, J. Phys. A29, 3461 (1996).

E. C. G. Stueckelberg, Helv. Phys. Acta 10, 1 (1937).