µ − τ Reflection Symmetry in Lepton Mixing and Neutrino Oscillations

P. F. Harrison
Physics Department, Queen Mary University of London
Mile End Rd. London E1 4NS. UK

and

W. G. Scott
Rutherford Appleton Laboratory
Chilton, Didcot, Oxon OX11 0QX. UK

Abstract

We examine the possibility that the lepton mixing matrix may exhibit a strong form of µ − τ universality, in which corresponding elements of the µ- and τ-neutrino flavour eigenstates have equal moduli, and find that it is consistent with present data on neutrino oscillations. We point out that in the Dirac case, this is equivalent to symmetry under µ − τ reflection, ie. the combined operation of µ − τ flavour exchange in the MNS matrix with a CP transformation on the whole leptonic sector. We give the most general form for such a mixing matrix, examine the lepton mass matrices under such a symmetry, and explore the observable manifestations of the symmetry in neutrino oscillations.

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1 Strong $\mu - \tau$ Universality in Lepton Mixing

The data on atmospheric neutrinos indicate that the muon neutrino couples to the heaviest neutrino mass eigenstate, $\nu_3$, with a strength $|U_{\mu 3}| \approx 1/\sqrt{2}$, where $U$ is the conventional MNS lepton mixing matrix. Independently, the data from the CHOOZ and PALO VERDE reactor neutrino experiments indicate that the electron neutrino has at most a small coupling to the heaviest neutrino, $|U_{e 3}|^2 \lesssim 0.03$.

These two facts taken together with the normalisation condition on the $\nu_3$ eigenstate imply that $|U_{\tau 3}| \approx 1/\sqrt{2}$ and hence, in particular, that $|U_{\mu 3}| \approx |U_{\tau 3}|$. Since $U_{e 3}$ is small and $|U_{\mu 3}| \approx |U_{\tau 3}|$, it follows immediately from the orthogonality of the $\nu_3$ eigenstate with the $\nu_1$ and $\nu_2$ eigenstates respectively that $|U_{\mu i}| \approx |U_{\tau i}|$ for $i = 1, 2$ also. Thus each of the moduli of the three $\mu$-flavour mixing elements is at least approximately equal to that of the corresponding $\tau$-flavour mixing element.

In this paper, we examine the possibility that $|U_{\mu i}| = |U_{\tau i}|$, exactly, for all $i = 1, 2, 3$ (1)

and find that it is consistent with all the present data on neutrino oscillations (excluding the LSND results, which are awaiting independent confirmation). This strong form of $\mu - \tau$ universality (as distinct from the familiar lepton flavour universality implicit in the Standard Model, in which $|U_{\alpha 1}|^2 + |U_{\alpha 2}|^2 + |U_{\alpha 3}|^2 = 1, \forall \alpha$) is interesting, as it removes two of the otherwise free parameters of the lepton mixing matrix.

Several previously suggested hypotheses for the lepton mixing matrix satisfy Eq. (1), in particular, the trimaximal, bimaximal and Altarelli-Ferruglio models, (as well as the more recently proposed tri-bimaximal and tri-\(\chi\)maximal schemes). The generalisation of these hypotheses to a unitary mixing matrix simply satisfying Eq. (1) was already identified as interesting in and such a mixing matrix was implicit in [13]. In this paper, we examine this class of mixing matrices and the associated phenomenology in more detail.

2 Mixing Matrix Parameterisation and $\mu - \tau$ Reflection Symmetry

We develop here a simple parameterisation of mixing matrices satisfying Eq. (1) in terms of only moduli of the mixing matrix elements, as distinct from the standard parameterisation in terms of mixing angles and a phase. A unitary matrix cannot contain two identical rows. Hence, in a unitary mixing matrix satisfying Eq. (1), the elements of the two rows corresponding to the $\nu_\mu$ and $\nu_\tau$ respectively must contain relative phases. For Dirac neutrinos, we are free to rephase independently the rows and columns of the mixing matrix with no observable consequence, and we can exploit this freedom (for columns) to arrange that $U_{\mu i} = U_{\tau i}^*$ for each such pair of elements. We can then write:

$$U = \begin{pmatrix} u \\ v \\ u^* \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ v_1^* & v_2^* & v_3^* \end{pmatrix}$$

(2)
which is sufficient to ensure Eq. (1). The normalisation of any column of $U$ leads to

$$|v_i| = \left(\frac{1 - |u_i|^2}{2}\right)^{1/2}, \forall i$$

while the orthogonality of any pair of columns gives

$$u_i u_j^* = -2 \text{Re}(v_i v_j^*) \quad \forall i \neq j.$$  

Eq. (4) can be satisfied only if all three elements of $u$ have a common phase and we use the freedom to rephase the electron neutrino flavour eigenstate to set $u$ real and positive. Using Eq. (3), we may now re-write Eq. (4) as

$$\cos \alpha_{ij} = \frac{-u_i u_j}{(1 - u_i^2)^{1/2}(1 - u_j^2)^{1/2}} \quad \forall i \neq j$$

where $\alpha_{ij}$ is the relative phase between $v_i$ and $v_j$. The row $v$ (and hence also the row $v^*$) is now completely determined up to an overall phase by the magnitudes of the elements of $u$ (only two of which are independent, $u$ being normalised to unity).

If the MNS matrix is a matrix of the form, Eq. (2), with $u$ real, the standard action is invariant under the simultaneous interchange of the $\nu_\mu$ and $\nu_\tau$ flavour eigenstates ($v$ and $v^*$ respectively) and complex conjugation of the MNS matrix. However, we know [13] that complex conjugation of the MNS matrix is formally equivalent to making a $CP$ transformation on the whole leptonic sector of the Standard Model (modified to include Dirac neutrino masses and standard MNS mixings). We will refer to the combined operation of $\mu - \tau$ flavour exchange in the MNS matrix and $CP$ transformation on the leptonic sector as “$\mu - \tau$ reflection”. Symmetry under $\mu - \tau$ reflection implies an MNS matrix which may be written with the $\nu_e$ flavour eigenstate real and the $\nu_\mu$ and $\nu_\tau$ flavour eigenstates complex conjugates of each other, Eq. (2). This is sufficient to ensure the strong $\mu - \tau$ universality expressed by Eq. (1), the two definitions being equivalent (for Dirac neutrinos). As we show at the end of this section, $\mu - \tau$ reflection symmetry is consistent with all confirmed neutrino oscillation measurements currently available.

For a mixing matrix which is $\mu - \tau$ reflection symmetric, as in Eq. (2), Jarlskog’s $CP$-violating invariant [14] is given by

$$J = \text{Im}(u_i u_j^* v_i^* v_j)$$

$$= \frac{1}{2} u_i u_j (1 - u_i^2)^{1/2}(1 - u_j^2)^{1/2} \sin \alpha_{ij}$$

$$= \frac{1}{2} u_i u_j (1 - u_i^2 - u_j^2)^{1/2}$$

1Unitarity triangles formed from the orthogonality condition of $u$ with $v$ or $v^*$ are always acute-angled triangles. Those formed from pairs of columns are isosceles triangles.

2For Majorana neutrinos, $\mu - \tau$ reflection symmetry is more constraining on the observable parameters of the mixing matrix than the strong $\mu - \tau$ universality of Eq. (1). In the following sections of this paper however, there is no distinction between the Majorana and Dirac cases, and the universality and the symmetry may be taken to be equivalent.
a strikingly simple result (we recall the fact that the $u_i$ have been taken to be real).

We note that quite generally, for a given, fixed pair of magnitudes of elements of the electron flavour eigenstate, $u$, requiring $\mu - \tau$ reflection symmetry, is equivalent to requiring maximal $CP$ violation in the leptonic sector (the proof is not given here but follows readily by differentiation of the complete expression for $\mathcal{J}$ keeping $|u_2|$ and $|u_3|$ fixed; it may also be seen in the standard parameterisation [14], where Eq. (1) corresponds to fixing $\theta_{23} = \pi/4$ and $\delta = \pm \pi/2$).

In order to make mixing matrices of the form Eq. (1) $\mu - \tau$ reflection symmetric, we have defined the phase convention of the first row and all the columns, and maintaining these conventions restricts somewhat our freedom to rephase the remaining rows of the matrix. However, we are still free to make a phase transformation on the matrix $U \to \Phi U$ where $\Phi = \exp\{i \text{ diag}(0, \phi, -\phi)\}$, without any observable consequence, and without disturbing the $\mu - \tau$ reflection symmetry. We may therefore choose $\phi$ such that any chosen neutrino mass eigenstate (any column) is wholly real. With such a phase convention, the matrix $U$ has only two free parameters, which can be taken to be any pair of the (real) elements of $u$.

It is perhaps of the most practical convenience to choose $\phi$ such that the $\nu_3$ eigenstate is real (so that the three parameters presently constrained directly by experiment, $U_{e2}$, $U_{e3}$ and $U_{\mu 3}$ are all real) and to take as the two free parameters, $U_{e2} = u_2$ and $U_{e3} = u_3$. We know from the HOMESTAKE [17], SUPER-K [18] and SNO [19] solar neutrino measurements that $|U_{e2}| \simeq 1/\sqrt{3}$ and from the CHOOZ [4] and PALO VERDE [5] reactor experiments that $|U_{e3}| \lesssim 0.17$. Hence, the hypothesis of $\mu - \tau$ reflection symmetry accommodates all atmospheric, solar, and reactor neutrino data currently available (each of these results is confirmed by more than one experiment). The LSND result [6], cannot be accommodated simultaneously with all these confirmed data using a $3 \times 3$ MNS lepton mixing matrix, and would require a fourth family of neutrinos [20]. We have not considered the question of $\mu - \tau$ reflection symmetry in such an extended picture.

3 $\mu - \tau$ Reflection Symmetry in the Flavour Basis

Any symmetry of the lepton mixing matrix should have its origin in a corresponding symmetry of the lepton mass matrices expressed in a weak basis, namely a basis in which the weak interaction is flavour diagonal and universal. Such a basis is not uniquely defined however, and we choose to consider the special case [7] in which the charged lepton mass matrix is diagonal (the flavour basis), which is defined up to some unobservable phase transformation. We will furthermore consider the Hermitian squares, $M^2 \equiv MM^\dagger$, of the mass matrices, $M$, in this basis, in order to avoid an arbitrariness in the determination of the mass matrices themselves due to the non-observation of right-handed charged currents.

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\[ = \frac{1}{2} u_1 u_2 u_3, \quad \text{(6)} \]
In the case of $\mu - \tau$ reflection symmetry (ie. assuming $U$ has the form Eq. (2), with $\underline{u}$ real), the neutrino mass matrix (squared) is given in the flavour basis by

$$M^2_\nu = U D^2_\nu U^\dagger$$

(7)

where $D^2_\nu = \text{diag}(m^2_1, m^2_2, m^2_3)$, is the diagonal neutrino mass matrix squared. Then

$$M^2_\nu = \begin{pmatrix} z & w & w^* \\ w^* & x & y \\ w & y^* & x \end{pmatrix},$$

(8)
a form [21] which again explicitly respects $\mu - \tau$ reflection symmetry (in this basis, the rows and columns of $M^2_\nu$ are labelled by the flavour indices $(e, \mu, \tau)$):

$$M^2_\nu \rightarrow (M^2_\nu)' \equiv E_{\mu\tau} M^2_\nu E_{\mu\tau}^\dagger = (M^2_\nu)^*,$$

(9)

where $E_{\mu\tau}$ is the $\mu - \tau$ flavour exchange operator:

$$E_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  

(10)

$M^2_\nu$ has 5 observable degrees of freedom: $z, x, |w|, |y|$ and the phase of the combination $w^2y$, which together correspond to the three neutrino masses, and the two mixing degrees of freedom in the matrix $U$. In general, $CP$ violation enters when the combination $w^2y$ has a non-trivial phase: $\text{Im}(w^2y) = \Delta m^2_{21} \Delta m^2_{31} \Delta m^2_{32} \mathcal{J}$ (where $\Delta m^2_{ij} = m^2_i - m^2_j$). The elements of $M^2_\nu$ are given by

$$z = m^2_1 + \Delta m^2_{21} u^2_2 + \Delta m^2_{31} u^2_3,$$
$$x = m^2_1 + \Delta m^2_{21} |v^2_{e2}| + \Delta m^2_{31} |v^2_{e3}|,$$
$$w = \Delta m^2_{21} u_2 v^*_{e2} + \Delta m^2_{31} u_3 v^*_{e3},$$
$$y = \Delta m^2_{21} v^2_{e2} + \Delta m^2_{31} v^2_{e3}.$$  

(11)

These equations may be combined with Eqs. (3) and (5) to find closed expressions in terms of any desired pair of elements of the vector $\underline{w}$. We have left the unphysical phase of $\underline{v}$ (and $\underline{v}^*$) undefined, hence the overall phases of $w$ and $y$ are similarly unphysical and undefined, and may be fixed by the choice of the phase $\phi$ discussed in Section 2.

Tri-$\chi$ maximal mixing [12] is a special case of $\mu - \tau$ reflection symmetry defined by $u_2 = 1/\sqrt{3}$, $u_3 = \sqrt{2/3} \sin \chi$, and (in a particular phase convention) corresponds to

$$z = m^2_1 + (1/3) \Delta m^2_{21} + (2/3) \Delta m^2_{31} \sin^2 \chi,$$
$$x = m^2_1 + (1/3) \Delta m^2_{21} + \Delta m^2_{31}[(1/2) - (1/3) \sin^2 \chi],$$
$$w = (1/3) \Delta m^2_{21} + \Delta m^2_{31}[-(1/3) \sin^2 \chi - 3i \mathcal{J}],$$
$$y = (1/3) \Delta m^2_{21} + \Delta m^2_{31}[-(1/2) \cos^2 \chi + (1/6) \sin^2 \chi - 3i \mathcal{J}]$$

(12)

where $\mathcal{J} = \sin 2\chi/(6\sqrt{3})$. Tri-bimaximal mixing [11] is obtained by setting $\chi = 0$.
Reflection Symmetry in Neutrino Oscillations

The amplitude \( A_{\alpha\beta} \) for a neutrino of flavour \( \alpha \) to be detected as a neutrino of flavour \( \beta \) is given as a function of propagation distance \( L \) by the (matrix) equation:

\[
A = \exp(-iHL) \tag{13}
\]

where \( H \) is the effective neutrino Hamiltonian in the flavour basis (in the rest of this paper, all calculations are assumed to be in this flavour basis). We may take:

\[
H = M^2_\nu/2E \tag{14}
\]

where \( M^2_\nu \) is the Hermitian square of the neutrino mass matrix with \( \mu - \tau \) reflection symmetry given in Eq. (8), and \( E \) is the neutrino energy. Obviously, since \( M^2_\nu \) is \( \mu - \tau \) reflection symmetric, then so is \( H \), and we note further that any power or polynomial (with real coefficients) of \( H \) shares this symmetry.

Eq. (13) may be re-written [22]:

\[
A = \sum_i X^i \exp(-i\lambda_i L) \tag{15}
\]

where the \( \lambda_i \) are the eigenvalues of \( H \) and the \( X^i \) are hermitian (projection) operators given by second-order polynomials (with real coefficients) of \( H \):

\[
X^i = \frac{\prod_{j \neq i}(H - \lambda_j)}{\prod_{j \neq i}(\lambda_i - \lambda_j)}. \tag{16}
\]

Clearly, the \( X^i \) share the \( \mu - \tau \) reflection symmetry of Eq. (8). Comparison with the alternative approach in which the Hamiltonian is diagonalised before exponentiation allows us to identify the elements of the \( X^i \) with the familiar combinations of the lepton mixing matrix elements [8, 22]:

\[
X^i_{\alpha\beta} = U_{\alpha i}U^*_{\beta i} \tag{17}
\]

(no summation over \( i \) is implied).

The squared amplitude \( |A_{\alpha\beta}|^2 \) for \( \alpha \neq \beta \) (\( \alpha = \beta \)) gives the appearance (survival) probabilities as a function of neutrino energy and propagation distance and may be decomposed as a sum of two matrices:

\[
P_{\alpha\beta}(L/E) = |A_{\alpha\beta}|^2 = S_{\alpha\beta}(L/E) + A_{\alpha\beta}(L/E) \tag{18}
\]

where the \( CP \) (and \( T \)) symmetric and anti-symmetric terms are respectively:

\[
S_{\alpha\beta}(L/E) = \delta_{\alpha\beta} - 4 \sum_{i<j} K_{\alpha i}^{\beta j} \sin^2 \left( \frac{\Delta_{ij}L}{2} \right)
\]

and

\[
A_{\alpha\beta}(L/E) = 8J_{\alpha\beta} \sin \left( \frac{\Delta_{12}L}{2} \right) \sin \left( \frac{\Delta_{23}L}{2} \right) \sin \left( \frac{\Delta_{31}L}{2} \right) \tag{19}
\]

(the dependence on neutrino energy is implicit in the factors \( \Delta_{ij} = (m_i^2 - m_j^2)/2E \)).

The respective \( CP \)-even coefficients are:

\[
K_{\alpha\beta}^{i j} = \text{Re}(U_{\alpha i}U^*_{\beta i}U_{\alpha j}U^*_{\beta j}) = \text{Re}(X^i_{\alpha\beta}X^j_{\alpha\beta}) \tag{20}
\]
and the $CP$-odd one is:

$$J_{\alpha\beta} = \text{Im}(U_{\alpha i}^{\dagger} U_{\beta j}^{\dagger} U_{\alpha j} U_{\beta i}) = \text{Im}(X_{\alpha\beta}^{i} X_{\alpha\beta}^{j*}) = \epsilon_{\alpha\beta} J \quad (21)$$

where the flavour-antisymmetric matrix $\epsilon$ is

$$\epsilon = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad (22)$$

and $J$ is Jarlskog’s $CP$- and $T$-odd invariant. For anti-neutrinos, $P_{\alpha\beta}(L/E)$ is given by the same formulation modified by $J \rightarrow -J$.

As already noted, the projection operators $X^{i}$ are symmetric under $\mu - \tau$ reflection:

$$X^{i} \rightarrow (X^{i})' \equiv E_{\mu\tau} X^{i} E_{\mu\tau}^{\dagger} = (X^{i})^* \quad (23)$$

where $E_{\mu\tau}$ was defined in Eq. (11). Hence, the real matrices $K^{ij}$ and $J$ derived from them are respectively symmetric (anti-symmetric) under $\mu - \tau$ flavour exchange (as well as under the $CP$-transformation), and are both therefore symmetric under full $\mu - \tau$ reflection [4]. Hence, the $CP$-even and $CP$-odd components of $P(L/E)$ defined in Eq. (19) are separately $\mu - \tau$ reflection symmetric, $S(L/E)$ having the analogous form to that of $P(L/E)$ with all parameters real, while $A(L/E)$, is proportional to $\epsilon J$ whose factors flip sign under $\mu - \tau$ flavour exchange and under $CP$ respectively.

By unitarity (applied separately for $\nu$ and $\overline{\nu}$), the rows and columns of $S(L/E)$ each sum to unity (at each value of $L/E$), while those of $A(L/E)$ each sum to zero. By $CPT$ symmetry, $S(L/E)$ is a symmetric matrix ($S = S^T$; for brevity, we drop the dependence on $L/E$), while $A(L/E)$ is anti-symmetric ($A = -A^T$). In the general vacuum case, these constraints reduce the number of independent functions of $L/E$ which are needed to specify $P(L/E)$ to three $CP$-even (eg. $S_{\alpha\beta}(L/E), \alpha < \beta$) and one $CP$-odd (eg. $A_{\alpha\mu}(L/E)$). Hence survival probabilities can be written entirely in terms of the $CP$-conserving parts of appearance probabilities ($P_{\alpha\alpha} = 1 - S_{\alpha\beta} - S_{\alpha\gamma}, \alpha \neq \beta \neq \gamma$) and vice versa ($S_{\alpha\beta} = \frac{1}{2}(1 - P_{\alpha\alpha} - P_{\beta\beta} + P_{\gamma\gamma})$).

Adding the constraints of $\mu - \tau$ reflection symmetry further reduces the number of independent $CP$-even functions of $L/E$ to two and we may therefore write:

$$P = \begin{pmatrix} 1 - 2S_{ee} & S_{e\mu} & S_{e\mu} \\ S_{e\mu} & 1 - S_{e\mu} - S_{\mu\tau} & S_{e\mu} \\ S_{e\mu} & S_{\mu\tau} & 1 - S_{e\mu} - S_{\mu\tau} \end{pmatrix} + A \quad (24)$$

where $A = \epsilon A_{e\mu}$ and $\epsilon$ was given in Eq. (22), or equivalently, in terms of survival probabilities ($S_{\alpha\alpha} = P_{\alpha\alpha}$):

$$P = \begin{pmatrix} \frac{1}{2}(1 - S_{ee}) & \frac{1}{2}(1 - S_{ee}) & \frac{1}{2}(1 - S_{ee}) \\ \frac{1}{2}(1 - S_{ee}) & S_{\mu\mu} & S_{\mu\mu} \\ \frac{1}{2}(1 - S_{ee}) & \frac{1}{2}(1 + S_{ee} - 2S_{\mu\mu}) & S_{\mu\mu} \end{pmatrix} + A. \quad (25)$$

The complex matrices (in flavour-space) of plaquettes $\Pi^{ij} = K^{ij} + iJ$ ($i < j$) are symmetric under $\mu - \tau$ reflection, themselves having the form Eq. (8).
Hence there are only three functions to evaluate in order to completely specify the matrix $\mathcal{P}(L/E)$.

For completeness, we provide here the expressions for the various components of $\mathcal{P}$, (noting from above that $S_{ee}$ and $S_{\mu\nu}$ are not independent of $S_{ee}$ and $S_{\mu\mu}$):

\[
S_{ee}(L/E) = 1 - 4 \sum_{i<j} u_i^2 u_j^2 \sin^2 (\Delta_{ij}L/2) \tag{26}
\]

\[
S_{\mu\mu}(L/E) = 1 - \sum_{i<j} (1 - u_i^2)(1 - u_j^2) \sin^2 (\Delta_{ij}L/2) \tag{27}
\]

\[
S_{e\mu}(L/E) = 2 \sum_{i<j} u_i^2 u_j^2 \sin^2 (\Delta_{ij}L/2) \tag{28}
\]

\[
S_{\mu\tau}(L/E) = \sum_{i\neq j\neq k} (u_k^2 - u_i^2 u_j^2) \sin^2 (\Delta_{ij}L/2) \tag{29}
\]

\[
A_{e\mu}(L/E) = 4(u_1 u_2 u_3) \sin (\Delta_{12}L/2) \sin (\Delta_{23}L/2) \sin (\Delta_{31}L/2). \tag{30}
\]

Of course, only two elements of $u$ are independent, and the third may be simply determined from a given pair using the normalisation of $u$.

5 Matter Effects

Matter effects are of considerable phenomenological importance. In this section, we discuss how they affect the predictions of $\mu - \tau$ reflection symmetry under quite general circumstances. When (anti-)neutrinos propagate in matter, the Hamiltonian in the flavour basis is modified: $H \rightarrow \tilde{H}$, with

\[
\tilde{H} = H \pm \text{diag}(a(L), 0, 0) \tag{31}
\]

where $a(L) = \sqrt{2}G_FN_e(L)$, $N_e(L)$ is the (in general position-dependent) electron number density of the matter, and the “−” sign applies in the case of anti-neutrinos. This modifies both the eigenvalues and eigenstates of the Hamiltonian in a non-trivial way, in general as a function of $L/E$. However, in the case of a $\mu - \tau$ reflection symmetric Hamiltonian, Eq. (8), its symmetry is unaffected, as only the parameter $z$ is changed ($z \rightarrow \tilde{z} = z \pm 2Ea(L)$). Hence, matter effects respect $\mu - \tau$ reflection symmetry and any result which is a consequence of the symmetry, is unaltered when neutrinos pass through matter.

We first consider the most general case of arbitrary matter profile, and no $\mu - \tau$ reflection symmetry. It is still convenient to decompose the matrix $\mathcal{P}(L/E)$ of survival and appearance probabilities into $CP$-even and $CP$-odd parts, $\mathcal{S}(L/E)$ and $\mathcal{A}(L/E)$ respectively. Without solving the Schroedinger Equation for the particular $L$-dependent Hamiltonian of Eq. (11) the functional dependences of $\mathcal{S}$ and $\mathcal{A}$ on $L/E$ are unknown. Moreover, in arbitrary matter, several of the familiar constraints of the vacuum case are lost, in particular because $CPT$ symmetry does not in general hold. For example, the matrix $\mathcal{S}(L/E)$ is no longer symmetric, and $\mathcal{A}(L/E)$ is no longer anti-symmetric; the survival probabilities $\mathcal{P}_{\mu\mu}$ and $\mathcal{P}_{\tau\tau}$ acquire $CP$-violating contributions [24]. However, we can still learn a considerable amount from general
principles: appealing to unitarity, as in the previous section, we see that in general, there are now four independent CP-even functions of $L/E$, eg. we may write:

$$S = \begin{pmatrix}
1 - S_{e\mu} - S_{e\tau} & S_{e\mu} & S_{e\tau} \\
S_{\mu e} & 1 - S_{\mu e} - S_{\mu\tau} & S_{\mu\tau} \\
S_{e\tau} - (S_{\mu e} - S_{e\mu}) & S_{\mu\tau} + (S_{\mu e} - S_{e\mu}) & 1 - S_{e\mu} - S_{e\tau}
\end{pmatrix}. \quad (32)$$

Similarly, there are three CP-odd ones (we use the fact that $A_{ee} = 0$)

$$A = \begin{pmatrix}
0 & A_{e\mu} & -A_{e\mu} \\
A_{\mu e} & A_{\mu\mu} & -A_{\mu e} - A_{\mu\mu} \\
-A_{\mu e} - A_{\mu\mu} & A_{e\mu} + A_{\mu e} + A_{\mu\mu}
\end{pmatrix}. \quad (33)$$

The $\mu - \tau$ reflection symmetry makes the following predictions:

$$S_{e\mu}(L/E) = S_{e\tau}(L/E) \quad ; \quad S_{\mu e}(L/E) = S_{\tau e}(L/E) \quad (34)$$

$$S_{\mu\tau}(L/E) = S_{\tau\mu}(L/E) \quad ; \quad S_{\mu\mu}(L/E) = S_{\tau\tau}(L/E) \quad (35)$$

$$A_{e\mu}(L/E) = -A_{e\tau}(L/E) \quad ; \quad A_{\mu e}(L/E) = -A_{\tau e}(L/E) \quad (36)$$

$$A_{\mu\tau}(L/E) = -A_{\tau\mu}(L/E) \quad ; \quad A_{\mu\mu}(L/E) = -A_{\tau\tau}(L/E) \quad (37)$$

(clearly, some though not all of them are redundant relative to the constraints of unitarity already invoked, eg. Eq. (36)). Although following simply from $\mu - \tau$ reflection symmetry, we have verified these predictions in detail using the elegant formalism for arbitrary matter profiles.

It is easy to see that the constraints from $\mu - \tau$ reflection symmetry given in Eqs. (34) and (35) reduce $S(L/E)$ to the form already given in Eq. (24), so that now it has only two independent components, as before. Similarly, constraints (36) and (37) simplify the form of $A(L/E)$ so that its complete specification now requires two functions to be determined, eg. $A_{\mu\tau}(L/E)$ and $A_{\mu\mu}(L/E)$ (or whichever are most convenient). Now, (using unitarity) we may write:

$$A = \begin{pmatrix}
0 & A_{\mu\tau} - A_{\mu\mu} & A_{\mu\mu} - A_{\mu\tau} \\
-A_{\mu\tau} - A_{\mu\mu} & A_{\mu\mu} & -A_{\mu\tau}
\end{pmatrix}. \quad (38)$$

For $T$-symmetric matter profiles, $A_{\mu\mu} = 0$, and the symmetry of $A(L/E)$ reduces to that of the vacuum case, Eq. (22).

6 Application to Atmospheric Neutrinos

The observation from atmospheric neutrinos that $|U_{\mu 3}| \approx 1/\sqrt{2}$, was important.

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5It is interesting to note that in the case of the element $P_{\mu\tau} = S_{\mu\tau} + A_{\mu\tau}$, the $\mu - \tau$ exchange operator, $E_{\mu\tau}$, has the same effect as the time-reversal operator, $T$, and that the resulting constraints on these matrix elements due to $\mu - \tau$ reflection symmetry are identical to those of $CPT$ symmetry. This accidental mimicking of $CPT$ symmetry is, however, restricted to the $P_{\mu\tau}$ and $P_{\tau\mu}$ elements, and $\mu - \tau$ reflection symmetry generally leads to constraints distinct from those of $CPT$ symmetry.
to the argument in Section 1 that Nature respects strong $\mu - \tau$ universality, at least approximately. However, $|U_{\mu 3}| = 1/\sqrt{2}$ is not a prediction of $\mu - \tau$ reflection symmetry in general. On the contrary, from Eq. (3), if the symmetry is good, we have that $|U_{\mu 3}| = (1-|U_{e3}|^2)^{1/2}$, which gives $|U_{\mu 3}| = 1/\sqrt{2}$ only if $U_{e3} = 0$, a possibility which is not excluded, but which is by no means essential. In fact, if $0 \leq |U_{e3}| \lesssim 0.17$ as is currently thought to be the case [4, 3], then $\mu - \tau$ reflection symmetry predicts
\[
0.696 \lesssim |U_{\mu 3}| = |U_{\tau 3}| \leq 1/\sqrt{2},
\]
a rather precise prediction, and certainly in agreement with the data.

One piece of experimental data which we have not used in the present paper, is the non-observation of electron neutrino disappearance (or appearance) in the atmospheric neutrino oscillation data [3]. It is tempting to think that the closeness of the measured atmospheric electron neutrino rate to the expected value, independent of zenith angle, is evidence for the fact that $|U_{e3}| << 1$. We will show here that under $\mu - \tau$ reflection symmetry, and in the (realistic) approximation of an initial flux ratio of muon neutrinos to electron neutrinos, $\phi_{\nu_e}/\phi_{\nu_\mu} = 2$, the measured electron neutrino survival probability is close to unity, independent of the value of $|U_{e3}|$, and hence that there is essentially no sensitivity to $|U_{e3}|$ in the rate of atmospheric electron neutrino events.

From Eq. (24), it can be seen that a prediction of $\mu - \tau$ reflection symmetry is that the $CP$-conserving part of the appearance probability for electron neutrinos in a muon neutrino beam, $S_{\mu e}$, is exactly half of the disappearance probability from an electron neutrino beam, $1 - P_{ee}$, for all $L/E$ (it is even true for arbitrary matter profiles, Eq. (34)). In the limit $\phi_{\nu_e}/\phi_{\nu_\mu} = 2$, these two terms compensate each other exactly in the measured rate of atmospheric electron neutrinos. Defining the cross-section-weighted neutrino flavour fractions $r = \sigma_{\nu_e}/(\sigma_{\nu_\mu} + \sigma_{\nu_e} + \sigma_{\tau_e}/\sigma_{\tau_e})$, $s = \sigma_{\nu_\mu}/(\sigma_{\nu_\mu} + \sigma_{\nu_e} + \sigma_{\tau_e}/\sigma_{\tau_e})$, and $\bar{s} = \sigma_{\tau_e}/(\sigma_{\nu_\mu} + \sigma_{\nu_e} + \sigma_{\tau_e}/\sigma_{\tau_e})$, the ratio of the observed electron neutrino rate to that expected with no oscillations is given by:
\[
R_e(L/E) = \frac{r[P_{ee}(L/E) + (\phi_{\nu_e}/\phi_{\nu_\mu})P_{\mu e}(L/E)] + (1 - r)[\nu \leftrightarrow \tau]}{r + (1 - r) + sA_{\mu e} + \bar{s}A_{\tau e}} \quad \text{for } \phi_{\nu_\mu}/\phi_{\nu_e} = \phi_{\tau_\mu}/\phi_{\tau_e} = 2
\]
\[
\simeq 1
\]
(40)
where the $CP$-violating effects are suppressed by the smallness of $U_{e3}$ and $\Delta m_{12}^2$, which would explain why present data are consistent with unity. While such compensation theorems have been noted previously by several authors for specific models [8, 13], we emphasise here the perfect compensation of the $CP$-even terms for the generality of models with $\mu - \tau$ reflection symmetry, and for arbitrary matter profiles. We note furthermore, that although naively there is a partial cancellation in Eq. (40) between the terms involving $A_{\mu e}$ and $A_{\tau e}$ (owing to a relative minus sign in their dependences on $J$), matter effects have different influences on their respective $L/E$ dependences, and the cancellation may be incomplete, even if $s = \bar{s}$.

Any observed deviation from $R_e(L/E) = 1$ is therefore evidence for one of three things: i) a deviation from $(\phi_{\nu_\mu}/\phi_{\nu_e}) = 2$ and/or from $(\phi_{\tau_\mu}/\phi_{\tau_e}) = 2$, either of which would appear in the leading oscillations, given sufficient $L/E$ resolution; ii) a deviation
from $\mu - \tau$ reflection symmetry, which would also appear in the leading oscillation; 
iii) $CP$ violation, which is potentially distinguishable from i) and ii) as it would be 
manifest only on the longer distance scale of $\Delta m_{12}^2 L / 4E \ll 1$.

7 Conclusions

Current data from atmospheric, solar and reactor neutrino oscillation experiments are 
all consistent with the hypothesis of $\mu - \tau$ reflection symmetry for the leptonic sector 
of the Standard Model (augmented with neutrino masses and standard MNS mixings). 
The $\mu - \tau$ reflection transformation is the product of the $\mu - \tau$ exchange operator, 
$E_{\mu\tau}$, applied to the neutrino flavour eigenstates, and the $CP$ operator. $CPE_{\mu\tau}$ is 
clearly a unitary transformation and is also Hermitian with $(CPE_{\mu\tau})^2 = I$, so that 
its eigenvalues are $\pm 1$. If $\mu - \tau$ reflection symmetry is a good symmetry of the neu-
trino oscillation Hamiltonian, one would expect the $CPE_{\mu\tau}$ quantum number to be 
conserved in such processes. One might call this quantum number “$\mu - \tau$ reflectivity” 
or simply “mutativity”. The eigenstates of mutativity are clearly:

\begin{align}
\text{mutativity} + 1 & : \frac{1}{\sqrt{2}}(\nu_\mu + \bar{\nu}_\tau) \text{ and } \frac{1}{\sqrt{2}}(\bar{\nu}_\mu + \nu_\tau) \\
\text{mutativity} - 1 & : \frac{1}{\sqrt{2}}(\nu_\mu - \bar{\nu}_\tau) \text{ and } \frac{1}{\sqrt{2}}(\bar{\nu}_\mu - \nu_\tau)
\end{align}

although it is not easy to see how one might prepare a beam of neutrinos in such an 
eigenstate to test directly the conservation law in a non-trivial way.

As we have explored in the main body of this paper, in addition to the above 
postulated conservation law, the hypothesis of $\mu - \tau$ reflection symmetry makes a set 
of specific predictions about the $CP$-even and $CP$-odd parts of neutrino appearance 
and survival probabilities. Further detailed measurements are required to falsify this 
hypothesis, by finding deviations from the predictions, Eqs. (34)-(37), valid even in 
the presence of terrestrial matter, and Eqs. (1), (3), (5), (6), (39) and (40).

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Appendix A: $\mu - \tau$ Reflection Symmetry in the Neutrino Mass Basis

It is also interesting to consider the manifestations of $\mu - \tau$ reflection symmetry in another special weak basis, namely that in which the neutrino mass matrix is diagonal (the neutrino mass basis). In this basis, the lepton mixing degrees of freedom are all contained in the charged lepton mass matrix-squared (the weak interaction, and the neutrino mass matrix are both diagonal by definition in this basis), which is then given by

$$M^2_\ell = U^\dagger D^2_\ell U$$

(43)

where $D^2_\ell = \text{diag}(m^2_e, m^2_\mu, m^2_\tau)$, from which we obtain

$$(M^2_\ell)_{ij} = \left(\frac{m^2_\mu + m^2_\tau}{2}\right) \delta_{ij} + \left[m^2_e - \left(\frac{m^2_\mu + m^2_\tau}{2}\right)\right] u_i u_j + i\epsilon_{ijk} \left(\frac{m^2_\tau - m^2_\mu}{2}\right) u_k,$$

(44)

whose real part is symmetric under the interchange $\mu \leftrightarrow \tau$ and whose imaginary part is anti-symmetric (in this basis, the rows and columns of $M^2_\ell$ are labelled by the neutrino mass eigenstate indices, the flavour indices labelling only the charged lepton mass eigenvalues). This matrix explicitly respects $\mu - \tau$ reflection symmetry, as the effects of $\mu \leftrightarrow \tau$ flavour exchange are clearly reversed by a $CP$ transformation, which is again equivalent to complex conjugation. The fact that the symmetry is manifest also in this basis demonstrates that the symmetry is not simply an accident in the flavour basis considered in the main text. $M^2_\ell$ encodes 5 degrees of freedom - the three charged lepton masses, and any pair of elements of the $\nu_e$ flavour eigenstate, $u$. 
References

[1] Y. Fukuda et al. Phys. Let. B 436 (1998) 33 (hep-ex/9805006).
    Y. Fukuda et al. Phys. Rev. Lett. 85 (2000) 3999 (hep-ex/0009001).
    W. W. M. Allison et al. Phys. Lett. B (1999) 137 (hep-ex/9901024).
[2] Y. Fukuda et al. Phys. Rev. Lett. 81 (1998) 1562 (hep-ex/9807003).
[3] Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. 28 (1962) 870.
    B. W. Lee, S. Pakvasa, R. Shrock and H. Sugawara, Phys. Rev. Lett. 38 (1977)
    937.
    B. W. Lee and R. Shrock, Phys. Rev. D16 (1977) 1444.
[4] M. Apollonio et al. Phys. Lett. B 420 (1998) 397 (hep-ex/9711002);
    Phys. Lett. B. 466 (1999) 415 (hep-ex/9907037).
[5] F. Boehm et al. Phys. Rev. D64 (2001) 112001 (hep-ex/0107009).
[6] C. Athanassopoulos et al. Phys. Rev. Lett. 75 (1995) 2650 (hep-ex/9504002);
    Phys. Rev. Lett. 77 (1996) 3082 (hep-ex/9605003);
    Phys. Rev. Lett. 81 (1998) 1774 (hep-ex/9709006).
[7] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 349 (1995) 357;
    Phys. Lett. B 396 (1997) 186 (hep-ph/9702243).
[8] P. F. Harrison, D. H. Perkins, W. G. Scott, Phys. Lett. B458 (1999), 79 (hep-
    ph/9904297).
[9] F. Vissani. hep-ph/9708483 (unpublished).
    V. Barger et al. Phys. Lett. B437 (1998) 107 (hep-ph/9806387).
    A. J. Bahz, A. S. Goldhaber and M. Goldhaber. Phys. Rev. Lett. 81 (1998) 5730
    (hep-ph/9806540).
    D. V. Ahluwalla. Mod. Phys. Lett. A 18 (1998) 2249.
    H. Giorgi and S. L. Glashow. hep-ph/9808293.
[10] G. Altarelli and F. Feruglio. Phys. Lett. B439 (1998) 112 (hep-ph/9807333);
    G. Altarelli and F. Feruglio. Phys. Lett. B451 (1999) 388 (hep-ph/9912475);
    see also A. Ghosal and D. Majundar, hep-ph/0209280, for a special case of this
    scenario.
[11] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530 (2002) 167
    (hep-ph/0202074).
[12] P. F. Harrison and W. G. Scott, Phys. Lett. B 535 (2002) 163 (hep-ph/0203203).
[13] D.V. Ahluwalia, Y. Liu and I. Stancu, Mod. Phys. Lett. A17 (2002), 13 (hep-
    ph/0008303).
[14] D.E. Groom et al. Eur. Phys. J. C15 (2000) 1.
[15] C. Jarlskog ”CP Violation”, World Scientific (1989), 3. In particular, we refer to Eq. (7.1) on p. 22.

[16] C. Jarlskog, Z. Phys. C29 (1985) 491; Phys. Rev. Lett. 55 (1985) 1039.

[17] B. T. Cleveland et al. AstroPart. Phys. 496 (1998) 505.

[18] S. Fukuda et al. Phys. Lett. B539 (2002) 179 (hep-ex/0205073).

[19] Q. R. Ahmad et al. Phys. Rev. Lett. 87 (2001) 071301 (nucl-ex/0106013).
    Q. R. Ahmad et al. Phys. Rev. Lett. 89 (2002) 011302 (nucl-ex/0204008).

[20] M. Maltoni, T. Schwetz, J.W.F. Valle, Phys. Rev. D65 (2002) 093004 (hep-ph/0112103);
    but see also M. Maltoni, M.A. Tortola, T. Schwetz and J.W.F. Valle, hep-ph/0207154.

[21] A matrix with similar symmetry, but with all elements real appeared in W. Grimus
    and L. Lavoura, Acta Phys. Polon. B32 (2001) 3719 (hep-ph/0110041).

[22] P.F. Harrison, W.G. Scott, Phys. Lett. B535 (2002) 229 (hep-ph/0203021).

[23] These quantities (for \( \alpha \neq \beta \)) were dubbed “plaquettes” in the context of quark
    mixing phenomenology, see eg. J.D. Bjorken and I. Dunietz, Phys. Rev. D36
    (1987) 2109;
    and “boxes” in the context of neutrino oscillations, D.J. Wagner and T.J. Weiler,
    Phys. Rev. D59 (1999) 113007 (hep-ph/9801327).

[24] H. Minakata and S. Watanabe, Phys. Lett. B468 (1999) 256 (hep-ph/9906530).

[25] T.K. Kuo and J. Pantaleone, Phys. Lett. B198 (1987) 406.

[26] H. Yokomakura, K. Kimura and A. Takamura, Phys. Lett. B544 (2002) 286 (hep-ph/0207174). In particular, we use the equations in Appendix A of the reference,
    setting \( \theta_{23} = \pi/4 \) and \( \delta = \pm \pi/2 \) therein.