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A numerical simulation of fractional order mathematical modeling of COVID-19 disease in case of Wuhan China

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A B S T R A C T

The novel Covid-19 was identified in Wuhan China in December, 2019 and has created medical emergency world wise and distorted many life in the couple of month, it is being burned challenging situation for the medical scientist and virologists. Fractional order derivative based modeling is quite important to understand the real world problems and to analyse realistic situation of the proposed model. In the present investigation a fractional model based on Caputo-Fabrizio fractional derivative has been developed for the transmission of CORONA VIRUS (COVID-19) in Wuhan China. The existence and uniqueness solutions of the fractional order derivative has been investigated with the help of fixed point theory. Adams-Bashforth numerical scheme has been used in the numerical simulation of the Caputo-Fabrizio fractional order derivative. The analysis of susceptible population, exposed population, infected population, recovered population and concentration of the virus of COVID-19 in the surrounding environment with respect to time for different values of fractional order derivative has been shown by means of graph. The comparative analysis has also been performed from classical model and fractional model along with the certified experimental data.

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1. Introduction

In the past December 2019, an outbreak of a mysterious pneumonia symbolized by dry cough along with fever, fatigue and occasional gastrointestinal symptoms identified in a seafood wholesale wet market in Hubei, Wuhan, China. This disease is officially announced as Coronavirus Disease (COVID-19, by WHO on February 11, 2020). Nowadays this pandemic is spreading across the globe and causes a severe outbreak of viral pneumonia, catching the eyes of the world. Due to high rate of spreading of COVID-19 is rapidly at an unprecedented scale across continents and has emerged as the single biggest risk in the world has faced in the present time. It is an infectious disease due to acute respiratory syndrome coronavirus 2 (SARS-CoV-2). Researchers and scientists from US, Europe, China, Japan, WHO show there is no certain conclusion to origin of COVID-19. China currently has more 81,600 cases of the novel coronavirus that first broke out in December last year in Wuhan city.

In the current century, Covid-19 is the third emerging human to transmission disease after the outbreak acute respiratory syndrome coronavirus (SARS-CoV) in 2002 that spread to 37 countries and the Middle East respiratory syndrome corona virus (MERS-CoV) in 2012 that spread to 27 countries. The symptoms of COVID-19 is serious bilateral lung infiltration including dry cough, fever, difficulty in breathing, fatigue and similar to those symptom caused by SARS-CoV and MERS-CoV infections [8]. The virus of this disease spread from one to another person through the respiratory droplets produced during coughing. It is also found that this is also spread from touching contaminated surfaces and then touching one’s face. There is no vaccine or specific antiviral treatment for COVID-19 therefore it is a challenging problem for medical scientist and virologist. However, there are many ongoing clinical trials evaluating potential treatments. The best way to prevent and slow down transmission is be well informed about the COVID-19 virus, the disease it causes and how it spreads. Protect yourself and others from infection by washing your hands or using an alcohol based rub frequently and not touching your face.

A number of fractional order modeling studied [1–4] and recently A. Atangana et al., [1,5,6] developed a fractional order model for the COVID-19 pandemic. Wu et al. [7] developed SEIR model to study the transmission of the Covid-19 and reported the basic reproductive number for validated data recoded from December 31, 2019 to January 28, 2020. The SEIR model based on Poisson-distributed daily time increments have been investigated by Read.
et al., [9] and estimated the basic reproductive number. Tang et al., [10] developed mathematical model based on deterministic compartmental and analysed the clinical progression of the disease. Imai et al., [11] investigated computational modeling of potential epidemic trajectories to evaluate the disease outbreak in Wuhan. Gao et al., [12] investigated an algorithm to study novel corona virus and predict its potential hosts. They observed that bats and minks may be two animal hosts of this virus.

New definition of fractional order derivative given by Caputo and Fabrizio [13]. León [14] proved lemma to Caputo fractional derivatives of Lyapunov functions and shown the uniform asymptotic stability of some epidemic systems. Later on several mathematical modeling based on fractional order derivative developed by scientist and researchers to analyse to real world’s scenario of spreading outbreak of epidemic. It is well known fact that the virus causing of many epidemic as dengue, influenza, Ebola and COVID-19. Influenza is an infectious disease caused by an influenza virus. Parra et al., [15] was developed a fractional order derivative based mathematical model of outbreaks of influenza. They examined that in the case of fractional order model observed that the next state depends not only upon its current state but also upon all of its historical states. Therefore they found that the fractional model is more general mathematical model compare to classical epidemic modeling. The fractional order derivative based mathematical modeling for the Ebola epidemic had been provided by Area et al., [16]. They studied classical model (susceptible, exposed, infections, removed) and fractional order SEIR Ebola epidemic model. Then they showed comparison study with real data examined by World Health Organization (WHO). Latha et al., [17] investigated the fractional-order model with time-delay to report the transition of Ebola virus infection. They examined that the fractional-order derivative based model with time-delay can more realistic for analysis of stability condition of fractional-order infection model. Area et al., [18] studied mathematical model based on new Caputo-fractional and performed numerical simulation for the extended model by using Owerola and Atangana new numerical technique through Adam-Basford method for the Caputo-Fabrizio fractional based derivative. Dokuyucu and Dutta [19] modeled fractional order based mathematical model for Ebola Virus spreading in certain parts of Africa. They provided numerical solution for the generalised model by using Atangana and Owerola numerical method. Mathematical model based on fractional order derivative for HIV infection was modeled by Ding and Ye [20]. They showed that model has non-negative solutions, as preferred in any population dynamics and also point out analysis on the stability of equilibrium in a detailed.

Arshad et al., [21] was also investigated a fractional order derivative model and obtained numerical simulation for immunogenic tumours. They studied the model based on fractional derivative growing tumour cell population and also observed that growth rate in death of immune cells has significant role in tumour dynamical and system consisting saddle-node and transcritical bifurcation analysis. A mathematical model based on fractional order derivative for the HIV/AIDS epidemic had been investigated by Zafar et al. [22]. They performed numerical simulations to study the influence of the parameter involving mathematical model for outbreak of the disease. A non linear dynamics and chaos analysis in mathematical model based on fractional order derivative for the HIV was obtained by Ye and Ding [23]. Later on several mathematician focused on to developing the mathematical modeling based on fractional order derivative for the pandemic. Ucar et al., [24] modeled fractional order derivative based model for immune cells influenced by cancer cells. Sardar et al., [25] provided a mathematical model for dengue transmission along with memory. They proposed new compartmental mathematical model of dengue transmission with memory between human to mosquito and mosquito to human. Kiliçman [26] also investigated mathematical model based on fractional order for dengue dengue transmission. He showed that his proposed model is well validated by published weekly dengue cases in Malaysia and found that the proposed model provides more understanding analysis to study the dynamic of dengue disease. Later on, Qureshi and Atangana [27] was also modeled mathematical modeling for dengue fever outbreak based on fractional order derivative operators. They showed that efficacy rate of obtained results in the case of fractional order modeling is more high compare to the case of classical model. Chittnis et al., [28] was studied the bifurcation analysis of a mathematical model for malaria transmission. A novel mathematical model based on fractional derivative TB investigated by Khan et al., [29]. They showed that fractional order derivative bestow more realistic situation and deeper knowledge about the insolvability of the dynamics of TB model with relapse. Recently Baleanu et al. [30] modeled fractional order derivative based model for human liver. They showed existence of a unique solution of the proposed model and a comparative study is made between the predicted values by the model and the certified clinical data.

Mathematical models in epidemiology are used widely in order to understand better the dynamics of an infectious disease. Fractional order models are more reliable and helpful in the real phenomena than the classical models due to hereditary properties and the description of memory [31,32] therefore Fractional order derivative based modeling is used to a important tool for analyzed real world problem to find better ways of understanding and providing more realistic situation and deeper information about the complexity of the proposed model. Also, in the explanation of real world scenario, the dynamics between two different points is not able to explored by classical derivative. To analyze such types of failure in the classical dynamical system, fractional order derivative concept have been investigated. Fractional order models render a better fit to the real data for different diseases and other experimental work in the field of modeling and simulation.

Nowadays, fractional order derivative is widely used in the mathematical modeling and have noticeable importance [33]. Several fractional differential operators like Riemann-Liouville, Hilfer, Caputo, etc. are mostly used in the modeling of physical problems. However, these fractional derivative possess a power law kernel and have own limitations, and reduce the field of application of fractional derivative. To deal with such type of difficulty, Caputo and Fabrizio [13] have developed an alternate fractional differential operator having a non-singular kernel with exponential decay. The Caputo-Fabrizio (C-F) operator has attracted many research scholars due to the fact that it has a non-singular kernel and to be found most appropriate for modeling some class of real world problem. Some researcher [34–39] have been used in the modeling and have received tremendous success.

Keeping this idea to mind Qureshi and Yusef [40] modeled a fractional order mathematical model for a chickenpox disease and Öztürk and Özkoca [41] examine stability analysis of fractional order mathematical model of tumor immune system interaction. After this study, recently Fanelli and Piazza [42] analyzed transmission of coronavirus disease - 2019 which is spreading in China, France and Italy. The transmission of the COVID-19 virus from human to human has been a real worry within the community of modelers as this virus has destroyed many life in the last past years. Only science and technology in the hands of capable scientists and innovators can come to our rescue in developing innovative but effective solutions as we prepare for a future, which in the very short term looks increasingly uncertain. As per the authors knowledge no attempt has been made till date to develop a fractional order modeling for the COVID-19. The present investigation focused to develop a fraction order based mathematical model for Covid-19 disease in Wuhan China. The Caputo-Fabrizio fractional
derivative concept has been used in the development of this fractional order mathematical model. Two step Adams-Bashforth numerical scheme has been used in the numerical simulation of Caputo-Fabrizio fractional order derivative. The analysis of susceptible population, exposed population, infected population, recovered and concentration of virus with respect to time for different value of fractional order has been shown by means of graph. The comparative analysis has also been performed from classical model and different order fractional along with the certified experimental data.

2. Definitions and basic concepts

In this section, we present some basic definitions, theorems and results related to Riemann-Liouville and Caputo-Fabrizio fractional order derivatives which are commonly used in the formulation of fractional order mathematical model.

**Definition 2.1.** [31,43] The integrability of a function \( f(\theta) \) for any arbitrary real order \( \Theta > 0 \) in the Riemann-Liouville sense is defined by the integral in the following form:

\[
D_{0+}^{\Theta} \left[f(\theta)\right] = \frac{1}{\Gamma(\Theta)} \int_{0}^{\theta} (\theta - s)^{\Theta-1} f(s) ds, \quad \Theta > 0.
\]  

**Definition 2.2.** [31,43] The Caputo-fabrizio fractional order derivative of any absolutely continuous function \( f(\theta) \), where \( f(\theta) \in C^{n}[0, T] \) with \( \Theta > 0 \) is represented in the form of following integral:

\[
CD_{0+}^{\Theta} \left[f(\theta)\right] = \frac{1}{\Gamma(n-\Theta)} \int_{0}^{\theta} (\theta - s)^{n-\Theta-1} f^n(s) ds, \quad n-1 < \Theta < n, n \in \mathbb{N}.\]

where \( n-1 < \Theta < n, n \in \mathbb{N} \). If \( \Theta \rightarrow 1 \) then \( CD_{0+}^{\Theta} f(\theta) \rightarrow f'(\theta) \).

**Theorem 2.1.** [50] Mean Value Theorem for a fractional order derivative For any function \( g(y) \in C[0, T] \) and \( CD_{0+}^{\Theta} g(\theta) \in (0, T] \), then

\[
g(\Theta) = g(0) + \frac{1}{\Gamma(\Theta)} CD_{0+}^{\Theta} g(\theta)\]

with \( 0 \leq s \leq \Theta, \forall \theta \in [0, T] \).

**Definition 2.3.** [13] The Caputo-Fabrizio derivative of a function \( f(\theta) \) of order \( \Theta > 0 \) is defined as:

\[
CD_{0+}^{\Theta} \left[f(\theta)\right] = \frac{\phi(\Theta)}{1-\Theta} \int_{0}^{\theta} f(s) \exp \left[-\Theta \frac{(\theta - s)}{1-\Theta}\right] ds, \quad \Theta > 0,
\]

where \( \phi(\Theta) \) is noted as normalization function such that \( \phi(0) = \phi(1) = 1 \) and \( f \in H^{1}(0, T), T > 0.\)

**Definition 2.4.** For a given function \( f \) the fractional integral is given as

\[
D_{0+}^{\Theta} f(t) = \frac{2(1-\Theta)}{(2-\Theta)\phi(\Theta)} f(t) + \frac{2-\Theta}{(2-\Theta)\phi(\Theta)} \int_{0}^{t} f(\tau) d\tau, \quad t \geq 0.
\]

where, \( \Theta \) is the order of fractional integral, such that \( 0 < \Theta < 1.\)

3. Formulation of the mathematical model

3.1. Classical mathematical model (non-fractional order)

In the present part of this work, we are considered to investigate a classical mathematical model developed by Yang and Wang [44], for pandemic COVID-19, which was firstly reported in Wuhan, China in December 2019 and then spread out quickly across the globe.

In the considered transmission model total human population in Wuhan China is divided into five sub compartments, where \( S \) denotes susceptible population, which represent the section of human population who are susceptible to contact the virus and become infectious if exposed. \( E \) denotes exposed population, which represent human population who are infected but not infectious so far. \( I, R, V \) denotes infected population, who have fully developed the symptoms of COVID-19 and can spread the virus through contact with the susceptible population. The populations section who have fully recovered after the treatment and they have no symptoms of disease (free from the disease) is denoted by \( R \) and \( V \) represents the concentration of the virus of COVID-19 in the surrounding environment.

The transmission model for COVID-19 pandemic is given by following system of ordinary differential equations:

\[
\begin{align*}
\frac{dS}{dt} &= \Pi - \beta_{E} S E - \beta_{I} S I - \beta_{V} S V - \mu S, \\
\frac{dE}{dt} &= \beta_{E} S E + \beta_{I} S I + \beta_{V} S V - (\alpha_{c} + \mu_{c}) E, \\
\frac{dI}{dt} &= \alpha_{c} E - (\omega + \gamma + \mu_{c}) I, \\
\frac{dR}{dt} &= \gamma_{c} I - \mu_{c} R, \\
\frac{dV}{dt} &= \psi_{1} E + \psi_{2} I - \tau_{c} V,
\end{align*}
\]

where the initial conditions for the considered transmission model of COVID-19 is given as

\[
S(0) = a_{1}, \quad E(0) = a_{2}, \quad I(0) = a_{3}, \quad R(0) = a_{4}, \quad V(0) = a_{5}.
\]

The parameter \( \Pi \) define as influx of population, \( \mu_{c} \) denotes the natural death rate of population. The quarantine period of the infected population is denoted by parameter \( (\omega_{c})^{-1} \), whereas \( \gamma_{c} \) denote the rate of recovery and \( \psi_{1}, \psi_{2} \) denotes the exposed and infected population which contributing the coronavirus in the surrounding environment, \( \omega_{c} \) and \( \tau_{c} \) denotes the disease-induced death rate and removal rate of the COVID-19 virus from surrounding environment respectively.

In the dynamical system of equations the rate of human to human transmission of disease between the exposed and susceptible individuals are given by the function \( \beta_{E} \) and the rate of human to human transmission of disease between the infected and susceptible individuals are denoted by the function \( \beta_{I} \). The rate of transmission disease due to environmental contact to human individuals is denoted by the function \( \beta_{V} \).

In the serve mathematical model we are considered all functions \( \beta_{E}, \beta_{I}, \beta_{V} \) are non-negative and non-increasing.

3.2. Mathematical model of COVID-19 in Wuhan, China, based on Caputo-Fabrizio fractional derivative

Although sometime integer derivative based mathematical models can not replicate accurately the real world problem, of course due to that fact that the translation from real world observed problem to mathematical formula is not really accurate due to lack of information, or lack of accuracy to converting reality to mathematical formula. However, these models have been used in the last past decades with great success, thus their use is important to mankind for prediction. These prediction help human you have an idea of what could happen in near future, such that they can take some control measures to avoid worst case scenario. Therefore in the present section, we devolved a Caputo-Fabrizio fractional derivative based mathematical model that could be used to predict the spread of covid-19 for case of Wuhan china.

For the purpose of above, Now applying the Caputo-Fabrizio fractional derivative [13,34–39] in the classical mathematical model (3.1). The fractional order transmission model for COVID-19 dynamics is given by the following system of non-linear fractional differential equation;
\[ CF_0 D_0^\alpha S_i = \Pi_c - \beta E_i S_i E_i - \beta I_i S_i I_i - \beta V_i S_i V_i - \mu_c S_i, \]
\[ CF_0 D_0^\alpha E_i = \beta E_i S_i E_i + \beta I_i S_i I_i + \beta V_i S_i V_i - (\alpha_c + \mu_c)E_i, \]
\[ CF_0 D_0^\alpha I_i = \alpha_c E_i - (\omega_c + \gamma_c + \mu_c)I_i, \]
\[ CF_0 D_0^\alpha R_i = \gamma_c I_i - \mu_c R_i, \]
\[ CF_0 D_0^\alpha V_i = \psi_1 E_i + \psi_2 I_i - \tau V_i. \]

Now, Eq. (3.2) is the complete fractional order mathematical model for the transmission of covid-19 disease in the case of Wuhan China, which can portray the real world problem to a desired level of accuracy, offering valuable predictions. Such a prediction can lead to assess the forthcoming situations thereby, making one to adopt controlling measures well before time in order to avoid worst case scenario.

We present below the solution analysis of the mathematical model, that includes the first testing of positivity of the solution, existence of the equilibrium points, reproductive number, and local stability analysis.

Since the propose mathematical model (3.2), predict the characteristic of real world problem of covid-19 disease. Therefore for analysis of positivity of the solution of the model (3.2), we used mean value theorem given as in [50], and we consider
\[ R^*_p = \{ \xi \in \mathbb{R}^5 : \xi \geq 0 \}, \quad \xi(t) = [S_i(t), E_i(t), I_i(t), R_i(t), V_i(t)]^T. \] (3.3)

**Lemma 3.1.** The solution \( \xi(t) \) of the proposed model (3.2) is positive, unique and lies in \( R^*_p \).

**Proof.** The analysis for the positivity of the solution, we show that all the components are bounded in the positive quadrant due to fact that we are dealing with population model. Since the vector field tends to \( R^*_p \), then
\[ CF_0 D_0^\alpha S_i = 0, \]
\[ CF_0 D_0^\alpha E_i = 0, \]
\[ CF_0 D_0^\alpha I_i = 0, \]
\[ CF_0 D_0^\alpha R_i = 0, \]
\[ CF_0 D_0^\alpha V_i = 0. \]

**Corollary 3.1.** Assume that \( u(\xi) \in \Theta(a, b) \) and \( CF_0 D_0^\alpha u(\xi) \in \Theta(0, 1) \). Then, it follows that \( CF_0 D_0^\alpha u(\xi) \geq 0 \), for all \( \xi \in (a, b) \) and \( CF_0 D_0^\alpha u(\xi) \leq 0 \), for all \( \xi \in (a, b) \), for increasing and decreasing function \( u(\xi) \), respectively.

The solution lies in \( R^*_p \), which is associated with a biological meaningful and defined by
\[ \Xi = \{ (S_i, E_i, I_i, R_i, V_i) \in R^*_p : \xi \geq 0 \}. \] (3.5)

Now, we can modify the condition in model (3.2) with \( \alpha = 1 \) and can be write in the matrix form as
\[ \frac{d \xi}{d t} = \partial(\xi(t)), \] (3.6)
where, \( \xi(t) = [S_i(0), E_i(0), I_i(0), R_i(0), V_i(0)]^T \) and
\[ \partial(\xi(t)) = \begin{pmatrix} \Pi_c - \beta E_i S_i E_i - \beta I_i S_i I_i - \beta V_i S_i V_i - \mu_c S_i \\ \beta E_i S_i E_i + \beta I_i S_i I_i + \beta V_i S_i V_i - (\alpha_c + \mu_c)E_i \\ \alpha_c E_i - (\omega_c + \gamma_c + \mu_c)I_i \\ \gamma_c I_i - \mu_c R_i \\ \psi_1 E_i + \psi_2 I_i - \tau V_i \end{pmatrix}, \]
with \( \partial : R^5 \rightarrow R^5 \) and \( \partial \in C^\infty(R^5) \), where
\[ R^*_p = \{ (S_i, E_i, I_i, R_i, V_i) \in R^*_p : S_i \geq 0, E_i \geq 0, I_i \geq 0, R_i \geq 0, V_i \geq 0 \} \] (3.7)
and \( \partial \in C^\infty(R^5) \) denotes infinitely differentiable functions.

The equilibrium states of fractional model (3.2) are acquired from solving \( CF_0 D_0^\alpha (\cdot) = 0 \) or \( \xi(S_i, E_i, I_i, R_i) = 0 \), \( i = 1, 2, 3, 4, 5 \), i.e.
\[ 0 = \Pi_c - \beta E_i S_i E_i - \beta I_i S_i I_i - \beta V_i S_i V_i - \mu_c S_i, \]
\[ 0 = \beta E_i S_i E_i + \beta I_i S_i I_i + \beta V_i S_i V_i - (\alpha_c + \mu_c)E_i, \]
\[ 0 = \alpha_c E_i - (\omega_c + \gamma_c + \mu_c)I_i, \]
\[ 0 = \gamma_c I_i - \mu_c R_i, \]
\[ 0 = \psi_1 E_i + \psi_2 I_i - \tau V_i. \] (3.8)

To explore the infected state of the disease, investigation of disease free and endemic equilibrium of an epidemiological models are prominent. The exploration of steady state of the fractional order model system of Eq. (3.2) can be written as:
The disease free equilibrium condition of Eq. (3.8) is given as,
\[ Q_D = (S_c(0), E_c(0), L_c(0), R_c(0), V_c(0)) = \left( \frac{P_0}{R_0}, 0, 0, 0, 0 \right). \]

For the analysis of endemic equilibrium point, first we need to have to the expression for a threshold quality called the basic reproduction number \( R_0 \).

The infection components in the model systems are \( E_c, L_c \) and \( V_c \). The new infection matrix \( \mathcal{S} \) and the transition matrix \( \nu \) are given by
\[
\mathcal{S} = \begin{bmatrix}
\beta_c S_c(0) & \beta_c S_c(0) & \beta_c S_c(0) \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix},
\]
and
\[
\nu = \begin{bmatrix}
\alpha_c + \mu_c & 0 & 0 \\
-\alpha_c & \omega_c + \gamma_c + \mu_c & 0 \\
-\psi_{1c} & -\psi_{2c} & \tau_c 
\end{bmatrix}.
\]

The basic reproduction number of fractional mathematical model (3.2) is defined as in the term of spectral radius of the next generation matrix \( \mathcal{S} \nu^{-1} \) and calculated as
\[
R_0 = \rho(\mathcal{S} \nu^{-1}) = \left[ \frac{\beta_c S_c(0)}{(\alpha_c + \mu_c)} + \frac{\alpha_c \beta_c S_c(0)}{(\omega_c + \gamma_c + \mu_c)(\alpha_c + \mu_c)} + \frac{(\omega_c + \gamma_c + \mu_c)(\alpha_c + \mu_c)}{\tau_c(\omega_c + \gamma_c + \mu_c)(\alpha_c + \mu_c)} \right].
\]

In the fractal mathematical model Eq. (3.8), the second equilibrium point are endemic equilibrium (EE) is given as
\[
S^*_c = \frac{1}{\mu_c} (\Pi_c - (\alpha_c + \mu_c)E^*_c), \quad E^*_c = \frac{\omega_c + \gamma_c + \mu_c}{\alpha_c} R^*_c, \quad R^*_c = \frac{\gamma_c I_c}{\mu_c}.
\]
\[
V^*_c = \frac{\psi_{1c}(\omega_c + \gamma_c + \mu_c) + \alpha_c \psi_{2c} I_c}{\tau_c \alpha_c}.
\]

It view of the first two equations of (3.8) that \( S_c \) can be represented by a function of \( I_c \), namely,
\[
S^*_c = \Sigma(I_c) = \frac{1}{\mu_c} \left( \Pi_c - (\omega_c + \gamma_c + \mu_c)(\alpha_c + \mu_c) I_c \right).
\]

With the help of second equation of (3.8) and Eq. (3.9), we have
\[
S^*_c = \Lambda(I_c) = (\alpha_c + \mu_c) \left( \frac{\beta_c (\omega_c + \gamma_c + \mu_c) I_c}{\alpha_c} + \frac{\alpha_c}{(\omega_c + \gamma_c + \mu_c)} \beta_c \right)
+ \frac{\psi_{1c}(\omega_c + \gamma_c + \mu_c) + \alpha_c \psi_{2c} I_c}{\tau_c \alpha_c} \frac{\psi_{1c}(\omega_c + \gamma_c + \mu_c) + \alpha_c \psi_{2c} I_c}{\tau_c \alpha_c}^{-1}.
\]

Now, we assume that the curves \( S^*_c = \Sigma(I_c), \quad I_c \geq 0 \) and \( S^*_c = \Lambda(I_c), \quad I_c \geq 0 \). In particular, the intersections of these two curves in \( R^2 \) examine the non-DFE equilibrium. Clearly, \( \Sigma(I_c) \) is strictly decreasing, whereas \( \Lambda(I_c) \) is increasing since \( \beta_c = \frac{\omega_c + \gamma_c + \mu_c}{\alpha_c} \), \( \beta_c \) and \( \beta_c = \frac{\psi_{1c}(\omega_c + \gamma_c + \mu_c) + \alpha_c \psi_{2c} I_c}{\tau_c \alpha_c} \) are positive and decreasing functions of \( I_c \). Furthermore, we can easily examine that \( \Sigma(0) = S_c(0) \), \( \Sigma(I_c) = 0 \), where \( I_c = \frac{\alpha c}{\omega_c + \mu_c} \).

Now, \( \Lambda(0) = (\alpha_c + \mu_c) \left( \beta_c (0) + \frac{\alpha_c}{(\omega_c + \gamma_c + \mu_c)} \beta_c (0) + \frac{\psi_{1c}(\omega_c + \gamma_c + \mu_c) + \alpha_c \psi_{2c} I_c}{\tau_c \alpha_c} \beta_c (0) \right)^{-1} \)
\[
= \frac{S_c(0)}{R_0}.
\]

Thus we have
(i) If the basic reproduction number \( R_0 > 1 \) then these two curves have a unique intersection lying in the interior of \( R^2 \), since \( \Lambda(0) < \Sigma(0) \) and \( \Lambda(I_c) > \Lambda(0) > 0 = \Sigma(I_c) \). Moreover, at this intersection point, Eq. (3.9) gives a unique endemic equilibrium (EE) \( (Q^*_1, Q^*_2, Q^*_3, Q^*_4, Q^*_5) \).

(ii) If the basic reproduction number \( R_0 \leq 1 \) then the two curves have no intersection in the interior of \( R^2 \) as \( \Lambda(0) \geq \Sigma(0) \).

Therefore, by Eq. (3.8) we find that the model (3.2) admits a unique equilibrium, the DFE \( Q_D \), if \( R_0 \leq 1 \) and it admits two equilibria, the DFE \( Q_D \) and the EE \( Q^*_1 \) if \( R_0 > 1 \).

4. Solutions for the Caputo-fabrizio fractional order mathematical model

In the present section of this study, we will investigate uniqueness and existence of the solution of fractional order differential equation with the help of fixed point theory and fraction derivative [35, 36, 47] on the fractional order differential Eq. (3.2), we obtain as
\[
S_c(t) - S_c(0) = \lim_{t \to 0} [\Pi_c - \beta_c S_c E_c - \beta_c S_c L_c - \beta_c S_c V_c - \mu_c S_c],
\]
\[
E_c(t) - E_c(0) = \lim_{t \to 0} [\beta_c S_c E_c + \beta_c S_c L_c + \beta_c S_c V_c - (\alpha_c + \mu_c) E_c],
\]
\[
L_c(t) - L_c(0) = \lim_{t \to 0} [\alpha_c E_c - (\omega_c + \gamma_c + \mu_c) L_c],
\]
\[
V_c(t) - V_c(0) = \lim_{t \to 0} [\psi_{1c}(\omega_c + \gamma_c + \mu_c) + \alpha_c \psi_{2c} I_c].
\]
\[ R_c(t) - R_c(0) = \int_0^t \int_0^t \{ \gamma \dot{k} - \mu_c R_c \} \, dy. \]

\[ E_c(t) - E_c(0) = \int_0^t \int_0^t \{ \psi \dot{k} + \psi_2 k - \tau V_c \} \, dy. \]

Now using the fractional order derivative concept given in [47], we have

\[ S_c(t) - S_c(0) = \int_0^t \int_0^t \{ \Pi_c - \beta_k S_c E_c \} \, dy. \]

\[ E_c(t) - E_c(0) = \int_0^t \int_0^t \{ \beta_k S_c E_c \} \, dy. \]

\[ I_c(t) - I_c(0) = \int_0^t \int_0^t \{ \alpha_{\mu k} E_c - (\omega_c + \gamma \mu_k) k \} \, dy. \]

\[ V_c(t) - V_c(0) = \int_0^t \int_0^t \{ \psi \dot{k} + \psi_2 k - \tau V_c \} \, dy. \]

After simplifying the above equation, we can be written as

\[ \alpha_1(t, S_c) = \Pi_c - \beta_k S_c E_c - \beta_k S_c k - \beta_k S_c V_c - \mu_c S_c. \]

\[ \alpha_2(t, E_c) = \beta_k S_c E_c + \beta_k S_c k + \beta_k S_c V_c - (\alpha_c + \mu_k) E_c. \]

\[ \alpha_3(t, k) = \alpha_{\mu k} E_c - (\omega_c + \gamma_\mu k) k. \]

\[ \alpha_4(t, R_c) = \gamma \dot{k} - \mu_c R_c. \]

\[ \alpha_5(t, V_c) = \psi \dot{k} + \psi_2 k - \tau V_c. \]

**Theorem 4.1.** In the proposed fractional order mathematical model systems the kernels \( \alpha_1, \alpha_2, \alpha_3, \) and \( \alpha_5 \) in Eq. (3.2) will be holds good for Lipschitz and contraction condition, if satisfy the following condition;

\[ 0 \leq ((\beta_k + \beta_k k + \mu_c) \Phi + \mu_c) < 1. \]

**Proof:** Let us consider functions \( S_c \) and \( S_c(t) \) and proceed from \( \alpha_1 \), then we applying the following process;

\[ \alpha_1(t, S_c) - \alpha_1(t, S_c(t)) = -\beta_k E_c(S_c(t) - S_c(t)) - \beta_k V_c(S_c(t) - S_c(t)) - \mu_c(S_c(t) - S_c(t)). \]

Now, applying norm on Eq. (4.2) and simplifying. We obtain

\[ \| \alpha_1(t, S_c) - \alpha_1(t, S_c(t)) \| \leq \| \beta_k E_c(S_c(t) - S_c(t)) \| + \| \beta_k V_c(S_c(t) - S_c(t)) \| + \| \mu_c(S_c(t) - S_c(t)) \|. \]

\[ \leq ((\beta_k + \beta_k k + \mu_c) \Phi + \mu_c) \| (S_c(t) - S_c(t)) \|. \]

Now, suppose \( \xi_1 = ((\beta_k + \beta_k k + \mu_c) \Phi + \mu_c) \), where \( \| E_c \| \leq \Phi, \| I_c \| \leq \Phi \) and \( \| V_c \| \leq \Phi \) are bounded, we attain

\[ \| \alpha_1(t, S_c) - \alpha_1(t, S_c(0)) \| \leq \xi_1 \| (S_c(t) - S_c(t)) \|. \]

Therefore, the Eq. (4.3) implies that \( \alpha_1 \) is satisfied Lipschitz condition, however the condition \( 0 \leq ((\beta_k + \beta_k k + \mu_c) \Phi + \mu_c) < 1 \) provides the condition of contraction. Similarly, we can show that the Lipschitz condition for other cases as;

\[ \| \alpha_2(t, E_c) - \alpha_2(t, E_c(t)) \| \leq \xi_2 \| (E_c(t) - E_c(t)) \|. \]

\[ \| \alpha_3(t, k) - \alpha_3(t, k(t)) \| \leq \xi_3 \| (k(t) - k(t)) \|. \]

\[ \| \alpha_4(t, R_c) - \alpha_4(t, R_c(t)) \| \leq \xi_4 \| (R_c(t) - R_c(t)) \|. \]

\[ \| \alpha_5(t, V_c) - \alpha_5(t, V_c(t)) \| \leq \xi_5 \| (V_c(t) - V_c(t)) \|. \]

Now, Eq. (4.2) can be written as;

\[ S_c(t) = S_c(0) + \int_0^t \int_0^t \{ \Pi_c - \beta_k S_c E_c \} \, dy. \]

\[ E_c(t) = E_c(0) + \int_0^t \int_0^t \{ \beta_k S_c E_c \} \, dy. \]

\[ I_c(t) = I_c(0) + \int_0^t \int_0^t \{ \alpha_{\mu k} E_c - (\omega_c + \gamma_\mu k) k \} \, dy. \]

\[ V_c(t) = V_c(0) + \int_0^t \int_0^t \{ \psi \dot{k} + \psi_2 k - \tau V_c \} \, dy. \]

Now, applying Recursive relation, we have

\[ S_{cn}(t) = \int_0^t \int_0^t \{ \Pi_c - \beta_k S_c E_c \} \, dy. \]
\[ E_{cn}(t) = \frac{2(1 - \Theta)}{(2 - \Theta)} \left( \omega_2(x, E_{cn(t-1)}) + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_2(y, E_{cn(t-1)}) \right) dy. \]

\[ I_{cn}(t) = \frac{2(1 - \Theta)}{(2 - \Theta)} \left( \omega_3(x, I_{cn(t-1)}) + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_3(y, I_{cn(t-1)}) \right) dy. \]

\[ R_{cn}(t) = \frac{2(1 - \Theta)}{(2 - \Theta)} \left( \omega_4(x, R_{cn(t-1)}) + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_4(y, R_{cn(t-1)}) \right) dy. \]

\[ V_{cn}(t) = \frac{2(1 - \Theta)}{(2 - \Theta)} \left( \omega_5(x, V_{cn(t-1)}) + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_5(y, V_{cn(t-1)}) \right) dy. \]

with the help of following conditions
\[ S_{0}(t) = S_{0}(0), \quad E_{0}(t) = E_{0}(0), \quad I_{0}(t) = I_{0}(0), \quad V_{0}(t) = V_{0}(0). \]

Furthermore, applying the difference of successive terms, we find out as
\[ Q_{1n}(t) = S_{cn}(t) - S_{cn(t-1)} = \frac{2(1 - \Theta)}{(2 - \Theta)} \left( \omega_1(x, S_{cn(t-1)}) - \omega_1(x, S_{cn(t-2)}) \right) + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_1(y, S_{cn(t-1)}) - \omega_1(y, S_{cn(t-2)}) \right) dy. \]
\[ Q_{2n}(t) = E_{cn}(t) - E_{cn(t-1)} = \frac{2(1 - \Theta)}{(2 - \Theta)} \left( \omega_2(x, E_{cn(t-1)}) - \omega_2(x, E_{cn(t-2)}) \right) + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_2(y, E_{cn(t-1)}) - \omega_2(y, E_{cn(t-2)}) \right) dy. \]
\[ Q_{3n}(t) = I_{cn}(t) - I_{cn(t-1)} = \frac{2(1 - \Theta)}{(2 - \Theta)} \left( \omega_3(x, I_{cn(t-1)}) - \omega_3(x, I_{cn(t-2)}) \right) + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_3(y, I_{cn(t-1)}) - \omega_3(y, I_{cn(t-2)}) \right) dy. \]
\[ Q_{4n}(t) = R_{cn}(t) - R_{cn(t-1)} = \frac{2(1 - \Theta)}{(2 - \Theta)} \left( \omega_4(x, R_{cn(t-1)}) - \omega_4(x, R_{cn(t-2)}) \right) + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_4(y, R_{cn(t-1)}) - \omega_4(y, R_{cn(t-2)}) \right) dy. \]
\[ Q_{5n}(t) = V_{cn}(t) - V_{cn(t-1)} = \frac{2(1 - \Theta)}{(2 - \Theta)} \left( \omega_5(x, V_{cn(t-1)}) - \omega_5(x, V_{cn(t-2)}) \right) + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_5(y, V_{cn(t-1)}) - \omega_5(y, V_{cn(t-2)}) \right) dy. \]

It is worth to be noted that
\[ S_{cn}(t) = \sum_{i=1}^{n} Q_{1i}(t), \quad E_{cn}(t) = \sum_{i=1}^{n} Q_{2i}(t), \quad I_{cn}(t) = \sum_{i=1}^{n} Q_{3i}(t), \quad R_{cn}(t) = \sum_{i=1}^{n} Q_{4i}(t), \quad V_{cn}(t) = \sum_{i=1}^{n} Q_{5i}(t). \]

Estimating with the same processore, we have
\[ \|Q_{1n}(t)\| = \left\|S_{cn}(t) - S_{cn(t-1)}(t)\right\| = \left\| \frac{2(1 - \Theta)}{(2 - \Theta)} \left( \omega_1(x, S_{cn(t-1)}) - \omega_1(x, S_{cn(t-2)}) \right) + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_1(y, S_{cn(t-1)}) - \omega_1(y, S_{cn(t-2)}) \right) dy. \right\|. \]

With the help of triangle inequality, Eq. (4.7) converted as
\[ \left\|S_{cn}(t) - S_{cn(t-1)}(t)\right\| \leq \frac{2(1 - \Theta)}{(2 - \Theta)} \left\| \left( \omega_1(x, S_{cn(t-1)}) - \omega_1(x, S_{cn(t-2)}) \right) \right\| + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_1(y, S_{cn(t-1)}) - \omega_1(y, S_{cn(t-2)}) \right) dy. \]

As Lipschitz condition satisfied by Kernel, we have
\[ \left\|S_{cn}(t) - S_{cn(t-1)}(t)\right\| \leq \frac{2(1 - \Theta)}{(2 - \Theta)} \left\| \omega_1(x, S_{cn(t-1)}) - \omega_1(x, S_{cn(t-2)}) \right\| + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_1(y, S_{cn(t-1)}) - \omega_1(y, S_{cn(t-2)}) \right) dy. \]

Now, we get
\[ \left\|Q_{1n}(t)\right\| \leq \frac{2(1 - \Theta)}{(2 - \Theta)} \left\| \omega_1(x, S_{cn(t-1)}) \right\| + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_1(y, S_{cn(t-1)}) \right) dy. \]

Similarly,
\[ \left\|Q_{2n}(t)\right\| \leq \frac{2(1 - \Theta)}{(2 - \Theta)} \left\| \omega_2(x, S_{cn(t-1)}) \right\| + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_2(y, S_{cn(t-1)}) \right) dy. \]
\[ \left\|Q_{3n}(t)\right\| \leq \frac{2(1 - \Theta)}{(2 - \Theta)} \left\| \omega_3(x, S_{cn(t-1)}) \right\| + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_3(y, S_{cn(t-1)}) \right) dy. \]
\[ \left\|Q_{4n}(t)\right\| \leq \frac{2(1 - \Theta)}{(2 - \Theta)} \left\| \omega_4(x, S_{cn(t-1)}) \right\| + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_4(y, S_{cn(t-1)}) \right) dy. \]
\[ \left\|Q_{5n}(t)\right\| \leq \frac{2(1 - \Theta)}{(2 - \Theta)} \left\| \omega_5(x, S_{cn(t-1)}) \right\| + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_5(y, S_{cn(t-1)}) \right) dy. \]

The existence of the solution is showed by operating the results stated in Eq. (4.8).

**Theorem 4.2.** The proposed fractional order mathematical model system has exact coupled-solutions of COVID-19 transmission if satisfying the following condition. That is we can find \( t_0 \) such that
\[ \frac{2(1 - \Theta)}{(2 - \Theta)} \left\| \omega_1(x, S_{cn(t-1)}) \right\| + \frac{2\Theta}{(2 - \Theta)} \int_0^t \left( \omega_1(y, S_{cn(t-1)}) \right) dy < 1 \]
where $S_c(t)$, $E_c(t)$, $I_c(t)$, $R_c(t)$ and $V_c(t)$ are bounded functions. Therefore, the Lipschitz condition is satisfied by Kernels as,

$$
\|Q_{1n}(t)\| \leq \|L_{cn}(0)\| \left[ \frac{2(1-\theta)}{(2-\theta)} \right] (\zeta_1^2 + \frac{2\theta}{(2-\theta)} \zeta_3 t^n) ,
$$

$$
\|Q_{2n}(t)\| \leq \|E_{cn}(0)\| \left[ \frac{2(1-\theta)}{(2-\theta)} \right] (\zeta_2^2 + \frac{2\theta}{(2-\theta)} \zeta_3 t^n) ,
$$

$$
\|Q_{3n}(t)\| \leq \|I_{cn}(0)\| \left[ \frac{2(1-\theta)}{(2-\theta)} \right] (\zeta_3^2 + \frac{2\theta}{(2-\theta)} \zeta_3 t^n) ,
$$

$$
\|Q_{4n}(t)\| \leq \|R_{cn}(0)\| \left[ \frac{2(1-\theta)}{(2-\theta)} \right] (\zeta_4^2 + \frac{2\theta}{(2-\theta)} \zeta_4 t^n) ,
$$

$$
\|Q_{5n}(t)\| \leq \|V_{cn}(0)\| \left[ \frac{2(1-\theta)}{(2-\theta)} \right] (\zeta_5^2 + \frac{2\theta}{(2-\theta)} \zeta_5 t^n) .
$$

(4.10)

Now, follow the following processure

$$
S_c(t) - S_c(0) = S_{cn}(t) - X_{1n}(t),
$$

$$
E_c(t) - E_c(0) = E_{cn}(t) - X_{2n}(t),
$$

$$
I_c(t) - I_c(0) = I_{cn}(t) - X_{3n}(t),
$$

$$
R_c(t) - R_c(0) = R_{cn}(t) - X_{4n}(t),
$$

$$
V_c(t) - V_c(0) = V_{cn}(t) - X_{5n}(t) .
$$

(4.11)

Therefore, we have

$$
\|X_{1n}(t)\| = \left\| \frac{2(1-\theta)}{(2-\theta)} \zeta_1 \left( \varphi_1(t, S_{cn}) - \varphi_1(t, S_{c(n-1)}) \right) + \frac{2\theta}{(2-\theta)} \zeta_3 \int_0^t \left( \varphi_1(y, S_{cn}) - \varphi_1(y, S_{c(n-1)}) \right) dy \right\| .
$$

$$
\leq \frac{2(1-\theta)}{(2-\theta)} \zeta_1 \left\| \varphi_1(t, S_{cn}) - \varphi_1(t, S_{c(n-1)}) \right\| + \frac{2\theta}{(2-\theta)} \zeta_3 \int_0^t \left\| \varphi_1(y, S_{cn}) - \varphi_1(y, S_{c(n-1)}) \right\| dy .
$$

$$
\leq \frac{2(1-\theta)}{(2-\theta)} \zeta_1 t \left\| S_c - S_{c(n-1)} \right\| + \frac{2\theta}{(2-\theta)} \zeta_3 t \left\| S_c - S_{c(n-1)} \right\| t .
$$

We followed as

$$
\|X_{1n}(t)\| \leq \left( \frac{2(1-\theta)}{(2-\theta)} \zeta_1 + \frac{2\theta}{(2-\theta)} \zeta_3 t \right) \zeta_1^{n+1} c .
$$

(4.13)

Then at $t_0$, we have

$$
\|X_{1n}(t)\| \leq \left( \frac{2(1-\theta)}{(2-\theta)} \zeta_1 + \frac{2\theta}{(2-\theta)} \zeta_3 t_0 \right) \zeta_1^{n+1} c .
$$

(4.14)

We can obtain from Eq. (4.14) as,

$$
\|X_{1n}(t)\| \to 0, \quad n \to \infty .
$$

(4.15)

Similarly, we have,

$$
\|X_{2n}(t)\| \to 0, \quad n \to \infty .
$$

$$
\|X_{3n}(t)\| \to 0, \quad n \to \infty .
$$

$$
\|X_{4n}(t)\| \to 0, \quad n \to \infty .
$$

$$
\|X_{5n}(t)\| \to 0, \quad n \to \infty .
$$

This state that the proposed fractional order mathematical system (3.2) has a solution.

For the uniqueness of the solution of system (3.2), on the contrary, we suppose that $S_{c1}(t)$, $E_{c1}(t)$, $I_{c1}(t)$, $R_{c1}(t)$, $V_{c1}(t)$ is another solution of system (3.2), then

$$
S_c(t) - S_{c1}(t) = \frac{2(1-\theta)}{(2-\theta)} \zeta_1 \left( \varphi_1(t, S_{c1}) - \varphi_1(t, S_{c1}) \right) + \frac{2\theta}{(2-\theta)} \zeta_3 \int_0^t \left( \varphi_1(y, S_{c1}) - \varphi_1(y, S_{c1}) \right) dy .
$$

(4.16)

With the help of norm Eq. (4.16) takes the form

$$
\|S_c(t) - S_{c1}(t)\| \leq \frac{2(1-\theta)}{(2-\theta)} \zeta_1 \left\| \varphi_1(t, S_{c1}) - \varphi_1(t, S_{c1}) \right\| + \frac{2\theta}{(2-\theta)} \zeta_3 \int_0^t \left\| \varphi_1(y, S_{c1}) - \varphi_1(y, S_{c1}) \right\| dy .
$$

(4.17)

Further, Lipschitz condition of kernel gives

$$
\|S_c(t) - S_{c1}(t)\| \leq \frac{2(1-\theta)}{(2-\theta)} \zeta_1 \left\| S_c(t) - S_{c1}(t) \right\| + \frac{2\theta}{(2-\theta)} \zeta_3 \int_0^t \zeta_1 t \left\| S_c(t) - S_{c1}(t) \right\| dy .
$$

(4.18)
Thus
\[
\|S_c(t) - S_{c1}(t)\| \left( 1 - \frac{2(1 - \Theta)}{(2 - \Theta) \, \frac{\partial}{\partial t}} \right) \left( \frac{2\Theta}{(2 - \Theta) \, \frac{\partial}{\partial t}} \right) \leq 0. \quad (4.19)
\]
Therefore, we have
\[
\|E_c(t) - E_{c1}(t)\| \left( 1 - \frac{2(1 - \Theta)}{(2 - \Theta) \, \frac{\partial}{\partial t}} \right) \leq 0.
\]
\[
\|I_c(t) - I_{c1}(t)\| \left( 1 - \frac{2(1 - \Theta)}{(2 - \Theta) \, \frac{\partial}{\partial t}} \right) \leq 0.
\]
\[
\|R_c(t) - R_{c1}(t)\| \left( 1 - \frac{2(1 - \Theta)}{(2 - \Theta) \, \frac{\partial}{\partial t}} \right) \leq 0.
\]
\[
\|V_c(t) - V_{c1}(t)\| \left( 1 - \frac{2(1 - \Theta)}{(2 - \Theta) \, \frac{\partial}{\partial t}} \right) \leq 0.
\]

**Theorem 4.3.** The proposed fractional order mathematical model system (3.2) will be a unique solution if
\[
\left( 1 - \frac{2(1 - \Theta)}{(2 - \Theta) \, \frac{\partial}{\partial t}} \right) \leq 0. \quad (4.20)
\]
**Proof:** If condition (4.20) is true then in view of condition (4.19) implies that
\[
\|S_c(t) - S_{c1}(t)\| = 0. \quad \text{Therefore, we get}
\]
\[
S_c(t) = S_{c1}(t). \quad (4.21)
\]
Furthermore, we have
\[
E_c(t) = E_{c1}(t), \quad I_c(t) = I_{c1}(t). \quad (4.22)
\]
\[
R_c(t) = R_{c1}(t), \quad V_c(t) = V_{c1}(t). \quad (4.23)
\]
Hence, the proposed fractional order mathematical model (3.2) is unique.

5. Numerical Scheme

In the present section of this investigation, we have described a new numerical technique (Agangana and Owolabi [47]) by using the new Caputo-Fabrizio fractional order derivative for the discretization of fractional differential equation. Agangana and Owolabi [47] assumed the given fractional differential equation.
\[
\frac{\partial}{\partial t}^{\alpha} g(t) = \frac{\partial}{\partial t} \int_0^t \left( t - \xi \right) \exp \left( - \frac{\Theta}{1 - \Theta} (t - \xi) \right) \frac{\partial}{\partial \xi} g(\xi, z(\xi)) \, d\xi. \quad (5.1)
\]
In the view of Equation (5.1) and from the fundamental theorem of analysis, we have
\[
z(t) - z(0) = \frac{(1 - \Theta)}{\frac{\partial}{\partial t}} g(t, z(t)) + \frac{\Theta}{\frac{\partial}{\partial t}} \int_0^t g(\xi, z(\xi)) \, d\xi. \quad (5.2)
\]
According to above,
\[
z(t_{n+1}) - z(0) = \frac{(1 - \Theta)}{\frac{\partial}{\partial t}} g(t_n, z(t_n)) + \frac{\Theta}{\frac{\partial}{\partial t}} \int_{t_n}^{t_{n+1}} g(t, z(t)) \, dt, \quad (5.3)
\]
and
\[
z(t_n) - z(0) = \frac{(1 - \Theta)}{\frac{\partial}{\partial t}} g(t_{n-1}, z(t_{n-1})) + \frac{\Theta}{\frac{\partial}{\partial t}} \int_0^{t_n} g(t, z(t)) \, dt. \quad (5.4)
\]
In view of Eqs. (5.3) and (5.4), we have obtained the following system of equations.
\[
z(t_{n+1}) - z(t_n) = \frac{(1 - \Theta)}{\frac{\partial}{\partial t}} [g(t_n, z(t_n)) - g(t_{n-1}, z(t_{n-1}))] + \frac{\Theta}{\frac{\partial}{\partial t}} \int_{t_n}^{t_{n+1}} g(t, z(t)) \, dt, \quad (5.5)
\]
where
\[
\int_{t_n}^{t_{n+1}} g(t, z(t)) \, dt = \int_{t_n}^{t_{n+1}} \left\{ \frac{g(t_n, z_n)}{h} (t - t_{n-1}) - \frac{g(t_{n-1}, z_{n-1})}{h} (t - t_n) \right\} \, dt,
\]
and
\[
\frac{3h}{2} g(t_n, z_n) - \frac{h}{2} g(t_{n-1}, z_{n-1}).
\]
Therefore, from Eq. (5.5), we get
\[
z(t_{n+1}) - z(t_n) = \frac{(1 - \Theta)}{\frac{\partial}{\partial t}} [g(t_n, z_n) - g(t_{n-1}, z_{n-1})] + \frac{3h \Theta}{2 \frac{\partial}{\partial t}} g(t_n, z_n) - \frac{\Theta h}{2 \frac{\partial}{\partial t}} g(t_{n-1}, z_{n-1}), \quad (5.6)
\]

which indicates that
\[ z(t_{n+1}) - z(t_n) = \frac{(1 - \Theta)}{\frac{d}{d\Theta} W(\Theta)} + \frac{3\Theta h}{2W(\Theta)} \right) g(t_n, z_n) + \left( \frac{(1 - \Theta)}{\frac{d}{d\Theta} W(\Theta)} + \frac{\Theta h}{2W(\Theta)} \right) g(t_{n-1}, z_{n-1}) \] 
(5.7)

Hence,
\[ z_{n+1} = z_n + \frac{(1 - \Theta)}{\frac{d}{d\Theta} W(\Theta)} + \frac{3\Theta h}{2W(\Theta)} \right) g(t_n, z_n) + \left( \frac{(1 - \Theta)}{\frac{d}{d\Theta} W(\Theta)} + \frac{\Theta h}{2W(\Theta)} \right) g(t_{n-1}, z_{n-1}) \] 
(5.8)

The Eq. (5.8) is to correspond two-step Adams-Bashforth numerical scheme for the Caputo-Fabrizio fractional order derivative.

**Theorem 5.1.** Let us suppose that \( z(t) \) be a solution of fractional order differential equation \( 0^C D^\alpha \gamma z(t) = g(t, z(t)) \) and \( g \) is a continuous bounded function for the Caputo-Fabrizio fractional order derivative [48], then
\[ z_{n+1} = z_n + \frac{(1 - \Theta)}{\frac{d}{d\Theta} W(\Theta)} + \frac{3\Theta h}{2W(\Theta)} \right) g(t_n, z_n) + \left( \frac{(1 - \Theta)}{\frac{d}{d\Theta} W(\Theta)} + \frac{\Theta h}{2W(\Theta)} \right) g(t_{n-1}, z_{n-1}) + W_n^\alpha, \]

where \( \| W_n^\alpha \| \leq M \).

### 6. Numerical scheme for the transmission of fractional order mathematical model of COVID-19

In the present analysis, we utilize the newly developed numerical scheme invented by [35,36,47] for simulation of new Caputo-Fabrizio fractional derivative in the proposed model system (3.2) for fractional order COVID-19 disease. For approximate solution of the proposed model system with the help of numerical iteration of this scheme, firstly we use the fundamental theorem of calculus to rearrange the model system (3.2) in the following fractional equation.

\[ S_c(t) - S_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^t \frac{d}{d\Theta} S_c(t, S_c(t)) \, dt \]
\[ E_c(t) - E_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^t \frac{d}{d\Theta} E_c(t, E_c(t)) \, dt \]
\[ I_c(t) - I_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^t \frac{d}{d\Theta} I_c(t, I_c(t)) \, dt \]
\[ R_c(t) - R_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^t \frac{d}{d\Theta} R_c(t, R_c(t)) \, dt \]
\[ V_c(t) - V_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^t \frac{d}{d\Theta} V_c(t, V_c(t)) \, dt \]

Therefore,
\[ S_c(t_{n+1}) - S_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^{t_{n+1}} \frac{d}{d\Theta} S_c(t, S_c(t)) \, dt \]
\[ E_c(t_{n+1}) - E_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^{t_{n+1}} \frac{d}{d\Theta} E_c(t, E_c(t)) \, dt \]
\[ I_c(t_{n+1}) - I_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^{t_{n+1}} \frac{d}{d\Theta} I_c(t, I_c(t)) \, dt \]
\[ R_c(t_{n+1}) - R_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^{t_{n+1}} \frac{d}{d\Theta} R_c(t, R_c(t)) \, dt \]
\[ V_c(t_{n+1}) - V_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^{t_{n+1}} \frac{d}{d\Theta} V_c(t, V_c(t)) \, dt \]

and
\[ S_c(t_n) - S_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^{t_n} \frac{d}{d\Theta} S_c(t, S_c(t)) \, dt \]
\[ E_c(t_n) - E_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^{t_n} \frac{d}{d\Theta} E_c(t, E_c(t)) \, dt \]
\[ I_c(t_n) - I_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^{t_n} \frac{d}{d\Theta} I_c(t, I_c(t)) \, dt \]
\[ R_c(t_n) - R_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^{t_n} \frac{d}{d\Theta} R_c(t, R_c(t)) \, dt \]
\[ V_c(t_n) - V_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_0^{t_n} \frac{d}{d\Theta} V_c(t, V_c(t)) \, dt \]

In view of Eqs. (6.1) and (6.2), the following equation system is obtained.
\[ S_c(t_{n+1}) - S_c(0) = \frac{1}{\frac{d}{d\Theta} W(\Theta)} \{ \frac{d}{d\Theta} S_c(t_n, S_c(t_n)) - \frac{d}{d\Theta} S_c(t_{n-1}, S_c(t_{n-1})) \} + \frac{1}{\frac{d}{d\Theta} W(\Theta)} \int_{t_n}^{t_{n+1}} \frac{d}{d\Theta} S_c(t, S_c(t)) \, dt \]
\[ E_C(t_{n+1}) - E_C(t_0) = \frac{1 - \Theta}{\Theta} \{ \phi_2(t_n, E_c(t_n)) - \phi_2(t_{n-1}, E_c(t_{n-1})) \} + \frac{\Theta}{\Theta} \int_{t_n}^{t_{n+1}} \phi_2(t, E_c(t)) \, dt, \]
\[ I_c(t_{n+1}) - I_c(t_0) = \frac{1 - \Theta}{\Theta} \{ \phi_3(t_n, I_c(t_n)) - \phi_3(t_{n-1}, I_c(t_{n-1})) \} + \frac{\Theta}{\Theta} \int_{t_n}^{t_{n+1}} \phi_3(t, I_c(t)) \, dt, \]
\[ R_C(t_{n+1}) - R_C(t_0) = \frac{1 - \Theta}{\Theta} \{ \phi_4(t_n, R_c(t_n)) - \phi_4(t_{n-1}, R_c(t_{n-1})) \} + \frac{\Theta}{\Theta} \int_{t_n}^{t_{n+1}} \phi_4(t, R_c(t)) \, dt, \]
\[ V_c(t_{n+1}) - V_c(t_0) = \frac{1 - \Theta}{\Theta} \{ \phi_5(t_n, V_c(t_n)) - \phi_5(t_{n-1}, V_c(t_{n-1})) \} + \frac{\Theta}{\Theta} \int_{t_n}^{t_{n+1}} \phi_5(t, V_c(t)) \, dt, \]

where

\[ \int_{t_n}^{t_{n+1}} \phi_1(t, S_c(t)) \, dt = \int_{t_n}^{t_{n+1}} \left\{ \frac{\phi_1(t_n, S_c(t_n)) - \phi_1(t_{n-1}, S_c(t_{n-1}))}{h} \right\} \, dt, \]
\[ = \frac{3h}{2} \phi_1(t_n, S_c(t_n)) - \frac{h}{2} \phi_1(t_{n-1}, S_c(t_{n-1})). \]

\[ \int_{t_n}^{t_{n+1}} \phi_2(t, E_c(t)) \, dt = \int_{t_n}^{t_{n+1}} \left\{ \frac{\phi_2(t_n, E_c(t_n)) - \phi_2(t_{n-1}, E_c(t_{n-1}))}{h} \right\} \, dt, \]
\[ = \frac{3h}{2} \phi_2(t_n, E_c(t_n)) - \frac{h}{2} \phi_2(t_{n-1}, E_c(t_{n-1})). \]

\[ \int_{t_n}^{t_{n+1}} \phi_3(t, I_c(t)) \, dt = \int_{t_n}^{t_{n+1}} \left\{ \frac{\phi_3(t_n, I_c(t_n)) - \phi_3(t_{n-1}, I_c(t_{n-1}))}{h} \right\} \, dt, \]
\[ = \frac{3h}{2} \phi_3(t_n, I_c(t_n)) - \frac{h}{2} \phi_3(t_{n-1}, I_c(t_{n-1})). \]

\[ \int_{t_n}^{t_{n+1}} \phi_4(t, R_c(t)) \, dt = \int_{t_n}^{t_{n+1}} \left\{ \frac{\phi_4(t_n, R_c(t_n)) - \phi_4(t_{n-1}, R_c(t_{n-1}))}{h} \right\} \, dt, \]
\[ = \frac{3h}{2} \phi_4(t_n, R_c(t_n)) - \frac{h}{2} \phi_4(t_{n-1}, R_c(t_{n-1})). \]

\[ \int_{t_n}^{t_{n+1}} \phi_5(t, V_c(t)) \, dt = \int_{t_n}^{t_{n+1}} \left\{ \frac{\phi_5(t_n, V_c(t_n)) - \phi_5(t_{n-1}, V_c(t_{n-1}))}{h} \right\} \, dt, \]
\[ = \frac{3h}{2} \phi_5(t_n, V_c(t_n)) - \frac{h}{2} \phi_5(t_{n-1}, V_c(t_{n-1})). \]

From Eq. (6.5),

\[ S_c(t_{n+1}) = S_c(t_0) + \left( 1 - \frac{\Theta}{\Theta} \right) \{ \phi_1(t_n, S_c(t_n)) \} + \left( \frac{1 - \Theta}{\Theta} + \frac{\Theta}{2 \Theta} \right) \{ \phi_1(t_{n-1}, S_c(t_{n-1})) \}, \]
\[ E_C(t_{n+1}) = E_C(t_0) + \left( 1 - \frac{\Theta}{\Theta} \right) \{ \phi_2(t_n, E_c(t_n)) \} + \left( \frac{1 - \Theta}{\Theta} + \frac{\Theta}{2 \Theta} \right) \{ \phi_2(t_{n-1}, E_c(t_{n-1})) \}, \]
\[ I_c(t_{n+1}) = I_c(t_0) + \left( 1 - \frac{\Theta}{\Theta} \right) \{ \phi_3(t_n, I_c(t_n)) \} + \left( \frac{1 - \Theta}{\Theta} + \frac{\Theta}{2 \Theta} \right) \{ \phi_3(t_{n-1}, I_c(t_{n-1})) \}, \]
\[ R_C(t_{n+1}) = R_C(t_0) + \left( 1 - \frac{\Theta}{\Theta} \right) \{ \phi_4(t_n, R_c(t_n)) \} + \left( \frac{1 - \Theta}{\Theta} + \frac{\Theta}{2 \Theta} \right) \{ \phi_4(t_{n-1}, R_c(t_{n-1})) \}, \]
\[ V_c(t_{n+1}) = V_c(t_0) + \left( 1 - \frac{\Theta}{\Theta} \right) \{ \phi_5(t_n, V_c(t_n)) \} + \left( \frac{1 - \Theta}{\Theta} + \frac{\Theta}{2 \Theta} \right) \{ \phi_5(t_{n-1}, V_c(t_{n-1})) \}. \]

In the view of a Theorem (5.1), we obtain

\[ S_c(t_{n+1}) = S_c(t_0) + \left( 1 - \frac{\Theta}{\Theta} \right) \{ \phi_1(t_n, S_c(t_n)) \} + \left( \frac{1 - \Theta}{\Theta} + \frac{\Theta}{2 \Theta} \right) \{ \phi_1(t_{n-1}, S_c(t_{n-1})) \} + \frac{\Theta}{\Theta} \{ \phi_1(t_{n-2}, S_c(t_{n-2})) \}. \]
Fig. 1. Variation of population density against time for fixed value of fractional order parameter.

Fig. 2. Variation of exposed population against time for different value of fractional order parameter.

Fig. 3. Variation of infected population against time for different value of fractional order parameter.
Fig. 4. Variation of recovered population against time for different value of fractional order parameter.

$$E_c(t_{n+1}) = E_c(t_n) + \left( \frac{1 - \Theta}{W(\Theta)} + \frac{3\Theta h}{2W(\Theta)} \right) \{ \omega_2(t_n, E_c(t_n)) \} + \left( \frac{1 - \Theta}{W(\Theta)} + \frac{\Theta h}{2W(\Theta)} \right) \{ \omega_2(t_{n-1}, E_c(t_{n-1})) \} + 2 \phi_i^m,$$

$$I_c(t_{n+1}) = I_c(t_n) + \left( \frac{1 - \Theta}{W(\Theta)} + \frac{3\Theta h}{2W(\Theta)} \right) \{ \omega_3(t_n, I_c(t_n)) \} + \left( \frac{1 - \Theta}{W(\Theta)} + \frac{\Theta h}{2W(\Theta)} \right) \{ \omega_3(t_{n-1}, I_c(t_{n-1})) \} + 3 \phi_i^m,$$

$$R_c(t_{n+1}) = R_c(t_n) + \left( \frac{1 - \Theta}{W(\Theta)} + \frac{3\Theta h}{2W(\Theta)} \right) \{ \omega_4(t_n, R_c(t_n)) \} + \left( \frac{1 - \Theta}{W(\Theta)} + \frac{\Theta h}{2W(\Theta)} \right) \{ \omega_4(t_{n-1}, R_c(t_{n-1})) \} + 4 \phi_i^m,$$

$$V_c(t_{n+1}) = V_c(t_n) + \left( \frac{1 - \Theta}{W(\Theta)} + \frac{3\Theta h}{2W(\Theta)} \right) \{ \omega_5(t_n, V_c(t_n)) \} + \left( \frac{1 - \Theta}{W(\Theta)} + \frac{\Theta h}{2W(\Theta)} \right) \{ \omega_5(t_{n-1}, V_c(t_{n-1})) \} + 5 \phi_i^m,$$

where

$$\| \phi_i^m \|_\infty < \frac{\Theta}{W(\Theta)} (n + 1)! h^{n+1} M, \quad i = 1, 2, 3, 4, 5.$$
Fig. 6. Comparative study of exposed population against time for fixed value of fractional order parameter.

Fig. 7. Comparative study of infected population against time for fixed value of fractional order parameter.
7. Numerical simulation

For numerical simulation and graphical illustration of susceptible, exposed, infected, recovered and concentration of virus of COVID-19 in the surrounding environment, following values are considered for the chosen parameters:

\[ \Pi_e = 8859.23 \times 10^4, \quad \beta_{1e} = 6.11 \times 10^{-8}, \quad \beta_{2e} = 2.62 \times 10^{-8}, \quad \beta_{3e} = 3.03 \times 10^{-9}, \quad \mu_c = 3.01 \times 10^{-2}, \quad \alpha_c = 0.143, \quad \omega_c = 0.01, \quad \gamma_c = 0.67, \quad \psi_{2e} = 1.30, \quad \psi_{2e} = 0.06, \quad \tau_c = 2.0. \]  

All parameter values are given in Table 1. The variation of population density in fractional order mathematical model of covid-19 disease in Wuhan China with respect to time for fixed value of faction order parameter has been portrayed in Fig. 1. It is to be noted that the value of infected population is very high and lowest value is the recovered population. It is also to be emphasized that initially high rate of infected population observed but after certain time rate of infected population density become slower and same result also obtained for the case of recovered population.

The impact of fractional order parameter in fractional order mathematical model of covid-19 disease in Wuhan China for exposed population density with respect to time has been depicted in Fig. 2. From the examination of Fig. 2, it is revealed that, the value of fractional order parameter in the favour of exposed population density. It is also remarkable that when the value of fractional order parameter tending to unity the variation of exposed population found very smooth and apparent.

Fig. 3 shows the variation of infected population against time for different value of fraction order parameter in fractional order mathematical model for Covid-19 disease in Wuhan China. The observed variation of infected population we have found that fractional order parameter in the favor of infected population and growth rate is observed very smooth when the value of fractional order parameter closed to unity, therefore the fractional order mathematical model reflect the more realistic situation of this disease in Wuhan China. The effect fractional order parameter on the mathematical model for recovered population and concentration of virus in environments exhibited in Fig. 4 and 5 respectively.

It has been investigated from Fig. 4 and 5 that recovered population and concentration of virus in environments increased with increment in the value of fractional order parameter. The comparative analysis between the classical model and fractional order model for the Covid-19 disease in Wuhan China has been portrayed in Fig. 6-10. Fig. 6 shows the comparative analysis between the classical model and fractional order model for the exposed population. In Fig. 6 solid line correspond to fractional model system for the Covid-19 disease in Wuhan China and dotted line for the case of classical model. Further, it is reported from all the figures (Fig. 6, 7, 8, 9) that the growth of exposed population, infected population and recovered population in the case of fractional order mathematical model for Covid-19 disease in Wuhan China more apparent in to the compression of classical model.

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**Table 1**
Parameters, definitions and numerical values.

| Parameters | Interpretation                                                                 | Values               | Reference |
|------------|--------------------------------------------------------------------------------|----------------------|-----------|
| \( \Pi_e \) | influx of population                                                           | \( 8859.23 \times 10^4 \) | [46]      |
| \( \beta_{1e} \) | the rate of human to human transmission of disease between the exposed and susceptible individuals | \( 6.11 \times 10^{-8} \) | [10]      |
| \( \beta_{2e} \) | the rate of transmission disease due to environmental contact to human individuals | \( 2.62 \times 10^{-8} \) | [10]      |
| \( \beta_{3e} \) | the natural death rate of population                                           | \( 3.03 \times 10^{-8} \) | fitted    |
| \( \mu_c \) | the disease-induced death rate                                                 | 0.01                 | [46]      |
| \( \alpha_c \) | the disease-induced removal rate                                               | 0.67                 | [48]      |
| \( \tau_c \) | the exposed population which contributing the coronavirus in the surrounding environment | 1.30                 | fitted    |
| \( \psi_{2e} \) | the infected population which contributing the coronavirus in the surrounding environment | 0.06                 | fitted    |
| \( \Pi_e \) | the disease-induced removal rate                                               | 2.0                  | [45]      |
To comparatively analyze population density in classical model and fractional order model for the Covid-19 disease in Wuhan China has been depicted in Fig. 10. It has been found from the observation of Fig. 10 that infected population in the classical model is very high compared to other population (exposed, infected, recovered) and the same results also obtained in the case of fractional order mathematical model. In addition of this it is also obtained that difference in population density in the fractional order mathematical model show the realistic situation of Covid-19 disease according to recorded data analysis by different medical and health care agency of the China.

8. Concluding remarks

The present investigation focused to develop a fractional order based mathematical model for Covid-19 disease in Wuhan China. The Caputo-Fabrizio fractional derivative concept has been used in the development of this fractional order mathematical model. Two step Adams- Bashforth numerical scheme has been used in the numerical simulation of Caputo-Fabrizio fractional derivative. The analysis of susceptible population, exposed population, infected population, recovered population and concentration of virus with respect to time for different value of fractional order has been shown by means of graph. The comparative analysis has also been performed from classical model and different order fractional along with the certified experimental data. The major outcomes of the present study can be encapsulated as follows:

1. In the graphical analysis it is reported that at initially the infected population is very high after the certain time the growth of infected population become slow, therefore this study reflect that at initially social distancing and lockdown is very effective on controlling of Covid-19.

2. Through a comparative examination of population density in the case of fractional order mathematical model, the growth of infected population is high compare to recovered population and the same results was also obtained in the classical mathematical model.

3. In the fractional order mathematical model growth of recovered population is less than exposed population for Covid-19 disease in Wuhan China.

The detailed examination of growth of exposed population, infected population and recovered population against time in the fractional order mathematical model for Covid-19 disease in Wuhan China may help to future in detail deep study of the dangerous pandemic Covid-19 disease. The present analysis of the graphical illustration may contribute to help in the analysis of control and data analysis for medical science.

Credit author statement

Dr. Ram Prasad Yadav and Dr. Renu Verma have equal contribution in the development of Mathematical Model and performing numerical simulations. Dr. Renu carried out discussion and conclusion of the manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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