HAUSDORFF DIMENSION IN R-ANALYTIC PROFINITE GROUPS

GUSTAVO A. FERNÁNDEZ-ALCOBER, EUGENIO GIANNELLI, AND JON GONZÁLEZ-SÁNCHEZ

Abstract. In this paper, we study the Hausdorff dimension of R-analytic subgroups in an R-analytic profinite group, where R is an arbitrary discrete valuation pro-p ring. In particular, we prove that the set of such Hausdorff dimensions is a finite subset of the rational numbers.

1. Introduction

The study of Hausdorff dimension in profinite groups was initiated by Abercrombie in [1], and has attracted special attention in recent times (see, for example, [2, 3, 4, 5, 6, 10]).

If G is a countably based profinite group, we can consider a descending series \{G_n\}_{n \in \mathbb{N}} of open subgroups of G which is a base of neighborhoods of 1. Any such a series is called a filtration of G. Observe that we do not require the subgroups G_n to be normal in G; if they are normal, then we speak of a normal filtration. Given a filtration \{G_n\}_{n \in \mathbb{N}}, then

\[d(x,y) = \inf \{|G : G_n|^{-1} \mid xy^{-1} \in G_n\}\]

defines a metric d on G, since the intersection of all the subgroups G_n is 1, and the topology induced by d coincides with the original topology in G.

The metric d defines a Hausdorff dimension function on all subsets of G, which we denote by \(h\text{Dim}_{\{G_n\}}\). Clearly different filtrations may define different metrics on G, and also different Hausdorff dimension functions. As shown in [4, Theorem 2.4], if the filtration \{G_n\}_{n \in \mathbb{N}} is normal and H is a closed subgroup of G, then

\[h\text{Dim}_{\{G_n\}}(H) = \liminf_{n \to \infty} \frac{\log |H : H \cap G_n|}{\log |G : G_n|}.
\]

Obviously, the last expression is independent of the base to which we take the logarithm. If G is a pro-p group (throughout the paper p will denote a prime number), it is customary to use p as the base for logarithms.

2010 Mathematics Subject Classification. Primary 20E18; Secondary 28A78.

Key words and phrases. Profinite groups, analytic groups, Hausdorff dimension.

The first and third authors are supported by the Spanish Government, grant MTM2011-28229-C02, and by the Basque Government, grant IT753-13.
In [4], Barnea and Shalev focused their attention on the study of Hausdorff dimension for closed subgroups of $p$-adic analytic pro-$p$ groups, with respect to the natural filtration given by $G_n = G^{p^n}$ for all $n \in \mathbb{N}$. They proved that
\[ h\text{Dim}_{(G^{p^n})}(H) = \frac{\text{dim}(H)}{\text{dim}(G)} \]
for every closed subgroup $H$ of $G$. Here and throughout the paper, if $G$ is an analytic group, $\text{dim}(G)$ denotes the dimension of $G$ as an analytic manifold.

In this paper we will study Hausdorff dimension in $R$-analytic profinite groups, where $R$ is an arbitrary discrete valuation pro-$p$ ring. If $G$ is such a group, then $G$ is virtually a pro-$p$ group, but the subgroups $G^{p^n}$ need not be open, and so they need not form a filtration. We will consider instead the natural filtration induced by any standard open subgroup of $G$. As we will see, given two different standard open subgroups $S$ and $T$ of $G$, the Hausdorff dimension of a closed subgroup with respect to the natural filtration induced by $S$ is the same as the one calculated with respect to the natural filtration induced by $T$. This remark will allow us to define the concept of standard Hausdorff dimension of a closed subgroup $H$, denoted by $h\text{Dim}_{\text{Std}}(H)$, and will lead us to prove the following result.

**Main Theorem.** Let $G$ be an $R$-analytic profinite group and let $H$ be an $R$-analytic subgroup of $G$. Then
\[ h\text{Dim}_{\text{Std}}(H) = \frac{\text{dim}(H)}{\text{dim}(G)}. \]

By an $R$-analytic subgroup of $G$ we mean a subgroup $H$ that is also an $R$-analytic submanifold of $G$; $H$ is then closed in $G$. A crucial remark is that while the converse is true in a $p$-adic analytic group, i.e. every closed subgroup is analytic, that property is not necessarily true for an arbitrary pro-$p$ ring $R$. For example, $\mathbb{F}_p[[t^d]]$ is a closed subgroup of $\mathbb{F}_p[[t]]$ for every positive integer $d$, but it is not an analytic subgroup for $d > 1$, even if it is an $\mathbb{F}_p[[t]]$-analytic group in its own right.

Observe that (1) implies that the $R$-analytic spectrum of $G$, defined by
\[ \text{Spec}_R(G) = \{ h\text{Dim}_{\text{Std}}(H) \mid H \text{ is an } R\text{-analytic subgroup of } G \}, \]
is finite and consists only of rational values for every $R$-analytic profinite group $G$. This is contrast with the result, also proved by Barnea and Shalev [4, Lemma 4.1], that the spectrum of $\mathbb{F}_p[[t]]$ corresponding to all closed subgroups is the full interval $[0, 1]$. Thus our main theorem is pointing to the fact that most closed subgroups of $\mathbb{F}_p[[t]]$ are non-analytic.

The paper is structured as follows. In Section 2 we give some basic background on analytic groups over pro-$p$ rings and we recall the main facts about Hausdorff dimension in countably based profinite groups. In Section
3 we firstly define the concept of standard filtration and then we show that the Hausdorff dimension of a closed subgroup of an \( R \)-analytic profinite group is independent of the choice of the standard filtration. We conclude the section with the proof of our main theorem.

Notation. For every set \( X \) and every integer \( d \geq 1 \), we represent by \( X^{(d)} \) the cartesian product of \( d \) copies of \( X \). Also, we use the symbols \( \subseteq_o \) and \( \subseteq_c \) to indicate that a subset of a given topological space is open or closed, respectively.

2. Preliminaries

Analytic groups. Throughout this paper we will let \( R \) be a pro-\( p \) ring, that is, a Noetherian local commutative ring with maximal ideal \( m \neq 0 \) such that \( R/m \) is a finite field of characteristic \( p \), and \( R \) is complete in the \( m \)-adic topology. We furthermore require \( R \) to be a principal ideal domain, which in this setting is equivalent to \( R \) being a discrete valuation ring. For example, the ring \( \mathbb{F}[[t]] \) of formal power series in one indeterminate over a finite field \( \mathbb{F} \) of characteristic \( p \) is a discrete valuation pro-\( p \) ring. If \( \pi \in R \) is an element of valuation 1, then \( m = (\pi) \). Also, if the residue field \( R/m \) is of size \( q \), we have \( |m^n : m^{n+1}| = q \) for every \( n \geq 1 \).

A topological group \( G \) with an \( R \)-analytic manifold structure is an \( R \)-analytic group if multiplication and inversion are analytic functions. We will always assume, without loss of generality, that the manifold structure is given by a full atlas. The basics of the theory of \( R \)-analytic groups can be found in [5, Chapter 13]; see also [9, Part II, Chapters 3 and 4] for a comprehensive description of manifold structures on topological groups.

A subgroup \( H \) of \( G \) is said to be an analytic subgroup if it is also an analytic submanifold of \( G \). This latter property means [9, Section 3.11] that for every \( h \in H \) there exist an open neighbourhood \( U_h \) of \( h \) in \( H \), a chart \((V_h, \phi_h)\) of \( G \), and a linear subspace \( E_h \) of \( K^{(d)} \) (where \( K \) is the field of fractions of \( R \)) such that:

(i) \( U_h \subseteq V_h \).

(ii) \( \phi_h(U_h) = \phi_h(V_h) \cap E_h \).

Analytic submanifolds are locally closed in the ambient manifold, that is, they are the intersection of an open subset and a closed subset. Equivalently, an analytic submanifold is open in its closure. Since open subgroups are closed in any topological group, it follows that an analytic subgroup of an \( R \)-analytic group is always closed. As already mentioned in the introduction, the converse is not true in general. However, one can readily check that an open subgroup \( H \) (actually, any open subset) is a submanifold when considered with the restrictions of the charts of \( G \), and \( \dim H = \dim G \).
A fundamental class of analytic groups is that of standard groups. An $R$-analytic group $S$ is called $R$-standard of dimension $d$ if it admits a global atlas consisting of a chart $(S, \phi, d)$ such that $\phi = (\phi_1, \ldots, \phi_d)$ defines a homeomorphism between $S$ and $m^{(d)}$. It is also required that $\phi(1) = 0$ and that for $j \in \{1, \ldots, d\}$, there exist formal power series $F_j(X, Y) \in R[[X, Y]]$ (where $X$ and $Y$ are $d$-tuples of indeterminates), without constant term, such that

$$\phi(xy) = F(\phi(x), \phi(y)), \quad \text{for every } x, y \in S. \tag{2}$$

Here $F = (F_1, \ldots, F_d)$ is called the formal group law of dimension $d$ associated to $S$. Observe that $F$ defines an operation on $m^{(d)}$ which, according to [2], provides $m^{(d)}$ with a new group structure, other than its natural additive structure. Obviously, $\phi$ is then an isomorphism between the multiplicative group $(S, \cdot)$ and the group $(m^{(d)}, F)$.

One of the cornerstones in the theory of $R$-analytic groups is that every such group $G$ contains an open (and so analytic) $R$-standard subgroup $S$ [5, Theorem 13.20]. Then the global chart $\phi$ of $S$ belongs to the full atlas of $G$. Notice also that if a profinite group is $R$-analytic, then it is countably based, since an $R$-standard subgroup is obviously countably based.

Before moving forward we need some more information about the formal group law $F$ in a standard group. We refer the reader to [5, pp. 331-334] for a detailed account.

**Lemma 2.1.** Let $F$ be a formal group law of dimension $d$ associated to a standard group $S$. Then

$$F(X, Y) = X + Y + G(X, Y),$$

where every monomial involved in $G$ has total degree at least 2, and contains a non-zero power of $X_r$ and of $Y_s$ for some $r, s \in \{1, \ldots, d\}$. Moreover, there exists a formal inverse $I(X) \in R[[X]]$ such that

$$F(X, I(X)) = 0 = F(I(X), X),$$

and $I(X) = -X + H(X)$, where every monomial involved in $H$ has total degree at least 2.

It readily follows that $(m^n)^{(d)}$ is a subgroup of $(m^{(d)}, F)$. This allows us to introduce a special type of filtrations in an $R$-analytic group.

**Definition 2.2.** Let $G$ be an $R$-analytic group and let $S$ be a standard open subgroup of $G$, with global chart $(S, \phi, d)$. Then for every positive integer $n \geq 0$, and for every subset $A \subseteq S$, we define

$$\pi^n_\phi A = \phi^{-1}(\pi^n \phi(A)).$$
We say that \( \{ \pi_n S \}_{n \in \mathbb{N}} \) is the natural filtration of \( G \) induced by the standard subgroup \( S \).

Observe that
\[
\pi_n S = \phi^{-1}(\langle m^{n+1} \rangle),
\]
which implies that \( \pi_n S \) is an open subgroup of \( G \), and also that \( \{ \pi_n S \}_{n \in \mathbb{N}} \) is a filtration of \( G \). Actually, it is not difficult to prove that \( \pi_n S \trianglelefteq S \) (see [3, Proposition 13.22]). The following lemma will be a key step in the proof of our main theorem.

**Lemma 2.3.** Let \( (S, \phi, d) \) be an \( R \)-standard group. Then:

(i) For every \( x, y \in S \), we have \( \phi(xy^{-1}) \in \langle m^{n+1} \rangle \) if and only if \( \phi(x) - \phi(y) \in \langle m^{n+1} \rangle \).

(ii) \( |S : \pi_n S| = |m(d) : \langle m^{n+1} \rangle| = q^{dn} \) for every \( n \geq 0 \).

(iii) \( \phi \) is an isometry between the multiplicative group \( S \) with the metric induced by the filtration \( \{ \pi_n S \}_{n \in \mathbb{N}} \) and the additive group \( \langle m(d) \rangle + \) with the metric induced by the filtration \( \{ \langle m^n \rangle \}_{n \in \mathbb{N}} \).

**Proof.** (i) Since \( \phi(1) = 0 \), we may assume that \( x \neq y \). Let \( k \in \mathbb{N} \) be such that \( \phi(xy^{-1}) \in \langle m^k \rangle - \langle m^{k+1} \rangle \). Then since
\[
\phi(x) = \phi(xy^{-1}) + \phi(y) = \phi(xy^{-1}) + \phi(y) + G(\phi(xy^{-1}), \phi(y)),
\]
by Lemma 2.1 we have
\[
\phi(x) - \phi(y) \equiv \phi(xy^{-1}) \pmod{\langle m^{k+1} \rangle}.
\]
Hence also \( \phi(x) - \phi(y) \in \langle m^k \rangle - \langle m^{k+1} \rangle \), and (i) follows.

(ii) Observe that (i) can be rewritten as
\[
xy^{-1} \in \pi_n S \iff \phi(x) - \phi(y) \in \langle m^{n+1} \rangle,
\]
or what is the same,
\[
x \cdot \pi_n S = y \cdot \pi_n S \iff \phi(x) + \langle m^{n+1} \rangle = \phi(y) + \langle m^{n+1} \rangle.
\]
This proves (ii), since \( \phi \) is a bijection.

(iii) According to the definition of the metric associated to a fitration, it is clear that (ii) and [3] together imply that \( \phi \) is an isometry. \( \square \)

**Hausdorff dimension.** Now we fix the notation and state some of the main results about Hausdorff dimension in metric spaces (for which we refer to Sections 2.1 and 2.2 of Falconer’s book [7]), and more specifically in countably based profinite groups.

Let \( (M, d) \) be a metric space, let \( X \subseteq M \), and let \( z, r \) be positive numbers. Define
\[
H^z_r(X) = \inf \sum_{n=1}^{\infty} (\text{diam}(U_n))^z,
\]
where \( \{U_n\}_{n \in \mathbb{N}} \) is a cover of \( X \) by sets of diameter at most \( r \), and the infimum is calculated over all such covers. We observe that the limit

\[
H^z(X) = \lim_{r \to 0} H^z_r(X)
\]

exists, since \( H^z_r(X) \) is non-decreasing as \( r \to 0 \). It turns out that \( H^z \) is an outer measure on \( M \); it is usually called the \( z \)-dimensional Hausdorff measure on \( M \). The following result holds.

**Lemma 2.4.** If \( H^z(X) < \infty \) and \( z < w \), then \( H^w(X) = 0 \).

This allows us to define the *Hausdorff dimension* of \( X \) as follows:

\[
\text{hDim}(X) = \sup \{ z \mid H^z(X) = \infty \} = \inf \{ z \mid H^z(X) = 0 \}.
\]

Clearly if \( X \subseteq Y \) then \( \text{hDim}(X) \leq \text{hDim}(Y) \), and it can also be shown that

\[
\text{hDim}\left( \bigcup_{n \in \mathbb{N}} X_n \right) = \sup_{n \in \mathbb{N}} \text{hDim}(X_n),
\]

for subsets \( X_n \subseteq M \). If \( f \) is an isometry of \( M \), then \( \text{hDim}(f(X)) = \text{hDim}(X) \), and more generally, we have the following result (see [7, Corollary 2.4]).

**Lemma 2.5.** Let \((M_1, d_1)\) and \((M_2, d_2)\) be two metric spaces, and suppose that \( f : M_1 \to M_2 \) is a bi-Lipschitz map, i.e. that there exist \( \lambda, \mu \in \mathbb{R}^+ \) such that

\[
\lambda d_1(x, y) \leq d_2(f(x), f(y)) \leq \mu d_1(x, y),
\]

for all \( x, y \in M_1 \). Then \( \text{hDim}(f(X)) = \text{hDim}(X) \) for every \( X \subseteq M_1 \).

**Corollary 2.6.** Let \( d_1 \) and \( d_2 \) be two distances on the same set \( M \). Suppose that there exist \( \lambda, \mu \in \mathbb{R}^+ \) such that

\[
\lambda d_1(x, y) \leq d_2(x, y) \leq \mu d_1(x, y),
\]

for all \( x, y \in M \). Then \( d_1 \) and \( d_2 \) induce the same Hausdorff dimension function on \( M \).

In the following two lemmas we collect some properties about Hausdorff dimension in countably based profinite groups (so valid, in particular, in \( R \)-analytic profinite groups) that will be needed later in the paper.

**Lemma 2.7.** Let \( G \) be a countably based profinite group with filtration \( \{G_n\}_{n \in \mathbb{N}} \). Let \( H \) be a closed subgroup of \( G \), and let \( U \) be a non-empty open subset of \( H \). Then

\[
\text{hDim}_{\{G_n\}}(H) = \text{hDim}_{\{G_n\}}(U).
\]
Proof. Since $U$ is non-empty and contained in $H$, we have $H = \bigcup_{h \in H} Uh$. Now $H$ is compact, being closed in the profinite group $G$. Hence there exist $h_1, \ldots, h_k \in H$ such that $H = \bigcup_{i=1}^k Uh_i$. By (4), we have

$$h\dim_{(G_n)}(H) = \sup_{i=1,\ldots,k} h\dim_{(G_n)}(U h_i).$$

Now, according to the definition of the metric associated to the filtration $\{G_n\}_{n \in \mathbb{N}}$, right multiplication by an element of $G$ is an isometry of $G$. Hence

$$h\dim_{(G_n)}(U h_i) = h\dim_{(G_n)}(U) \quad \text{for every } i = 1, \ldots, k,$$

and consequently

$$h\dim_{(G_n)}(H) = h\dim_{(G_n)}(U),$$

as desired. □

If $G$ is a countably based profinite group and $S$ in an open subgroup of $G$, then any filtration $\{S_n\}_{n \in \mathbb{N}}$ of $S$ is also a filtration of $G$. Hence we can calculate the Hausdorff dimension of a subset $X$ of $S$ with respect to the metric induced by $\{S_n\}_{n \in \mathbb{N}}$ in $S$ or in $G$, which we denote by $h\dim_{(S_n)}(X)$ and $h\dim_{(G_n)}(X)$, respectively. Our next lemma shows that there is actually no need to introduce this new notation.

**Lemma 2.8.** Let $G$ be a countably based profinite group and let $S$ be an open subgroup of $G$. If $\{S_n\}_{n \in \mathbb{N}}$ is a filtration of $S$, then

$$h\dim_{(G_n)}(X) = h\dim_{(S_n)}(X)$$

for every $X \subseteq S$.

**Proof.** Let $d_S$ and $d_G$ be the distances induced by the filtration $\{S_n\}_{n \in \mathbb{N}}$ in $S$ and $G$, respectively. Given $x, y \in S$, we have

$$d_S(x, y) = \inf \{|S : S_n|^{-1} | xy^{-1} \in S_n\} = |G : S| \inf \{|G : S_n|^{-1} | xy^{-1} \in S_n\} = |G : S| d_G(x, y),$$

and $|G : S|$ is finite, since $S$ is open in $G$. Thus the inclusion map from $S$ to $G$ is bi-Lipschitz, and the result follows from Lemma 2.5. □

As already mentioned in the introduction, the theory of the Hausdorff dimension in profinite groups developed in [1] implies the following result, which will be important in proving our main theorem.

**Theorem 2.9.** Let $G$ be a countably based profinite group and let $\{G_n\}_{n \in \mathbb{N}}$ be a normal filtration of $G$. Then

$$h\dim_{(G_n)}(H) = \liminf_{n \to \infty} \frac{\log |H : H \cap G_n|}{\log |G : G_n|},$$

for every closed subgroup $H$ of $G$.

Thus if $G$ is infinite, then open subgroups have Hausdorff dimension equal to 1.
3. Standard Hausdorff dimension

In this section we will prove our main results. In order to do this we start by giving sufficient conditions to get the same Hausdorff dimension with respect to different filtrations.

**Proposition 3.1.** Let \( G \) be a countably based profinite group. Let \( \{S_n\}_{n \in \mathbb{N}} \) and \( \{T_n\}_{n \in \mathbb{N}} \) be two filtrations of \( G \) satisfying the following conditions:

(i) There exist natural numbers \( k_1, c, \) and \( d, \) such that

\[
S_{n+c} \leq T_n \leq S_{n-d} \quad \text{for all} \quad n \geq k_1.
\]

(ii) There exist natural numbers \( k_2 \) and \( M \) such that

\[
|S_n : S_{n+1}| \leq M \quad \text{for} \quad n \geq k_2.
\]

Then \( h\text{Dim}\{S_n\}(X) = h\text{Dim}\{T_n\}(X) \) for every \( X \subseteq G. \)

**Proof.** Let \( d_S \) and \( d_T \) be the distances induced in \( G \) by the two filtrations \( \{S_n\}_{n \in \mathbb{N}} \) and \( \{T_n\}_{n \in \mathbb{N}}, \) respectively. According to Corollary 2.6, it suffices to find \( \lambda, \mu \in \mathbb{R}^+ \) such that

\[
\lambda d_S(x, y) \leq d_T(x, y) \leq \mu d_S(x, y)
\]

for all \( x, y \in G. \)

Let us then consider two elements \( x, y \in G. \) Put \( k = \max\{k_1, k_2\}. \) If \( xy^{-1} \in T_k \) then

\[
d_T(x, y) = \inf \{|G : T_n|^{-1} \mid n \geq k, \ xy^{-1} \in T_n\}.
\]

If \( xy^{-1} \in T_n \) with \( n \geq k, \) then by \( (5) \) we also have \( xy^{-1} \in S_{n-d}, \) and

\[
|G : T_n|^{-1} \geq |G : S_{n+c}|^{-1} \geq M^{-(c+d)}|G : S_{n-d}|^{-1},
\]

where the last inequality follows from \( (6). \) It follows that

\[
\inf \{|G : T_n|^{-1} \mid n \geq k, \ xy^{-1} \in T_n\} \geq M^{-(c+d)} \inf \{|G : S_n|^{-1} \mid xy^{-1} \in S_n\},
\]

and consequently,

\[
d_T(x, y) \geq M^{-(c+d)}d_S(x, y).
\]

On the other hand, if \( xy^{-1} \notin T_k \) then

\[
d_T(x, y) \geq |G : T_k|^{-1} \geq |G : T_k|^{-1}d_S(x, y),
\]

since \( d_S \) takes values in the interval \([0, 1]\). Thus the first inequality in \( (7) \) holds by taking \( \lambda = \min\{M^{-(c+d)}, |G : T_k|^{-1}\}. \)

Now in order to get the second inequality in \( (7), \) it suffices to observe that \( (6) \) implies that

\[
T_{n+d} \leq S_n \leq T_{n-c} \quad \text{for all} \quad n \geq k_1 + c,
\]
and that, by \([5]\) and \([6]\), we have

\[
|T_n : T_{n+1}| \leq |S_{n-d} : S_{n+c+1}| \leq M^{c+d+1} \quad \text{for } n \geq k + d.
\]

Hence conditions (i) and (ii) in the statement of the theorem equally hold with the role of the two filtrations \(\{S_n\}_{n \in \mathbb{N}}\) and \(\{T_n\}_{n \in \mathbb{N}}\) reversed. Thus the argument in the previous paragraph yields the existence of \(\mu \in \mathbb{R}^+\) such that \(d_S(x,y) \geq \mu^{-1}d_T(x,y)\). This completes the proof. 

□

We note that we are only going to need Proposition 3.1 to calculate the Hausdorff dimension of closed subgroups of \(G\), and not of arbitrary subsets of \(G\). In that case, an alternative proof could be given by using the formula in Theorem 2.9. Since that proof is not significantly shorter than the proof given above, we have preferred to state the result in all generality.

**Remark 3.2.** Let \(G\) be an \(R\)-analytic profinite group of dimension \(d\), and let \(S\) be an open standard subgroup of \(G\), with corresponding chart \(\phi\). Then, according to Lemma 2.3, the natural filtration \(\{\pi^n_\phi S\}_{n \in \mathbb{N}}\) always satisfies condition (ii) in Proposition 3.1, with \(M = q^d\).

Actually, as we prove below, given two open standard subgroups of an \(R\)-analytic profinite group, the corresponding natural filtrations also satisfy condition (i) in Proposition 3.1, and we have the following consequence.

**Proposition 3.3.** Let \(G\) be a profinite \(R\)-analytic group, and let \(S\) and \(T\) be two open standard subgroups of \(G\). If \(\{\pi^n_\phi S\}\) and \(\{\pi^n_\psi T\}\) are the natural filtrations of \(G\) induced by \(S\) and \(T\), respectively, then

\[
hDim(\pi^n_\phi S)(X) = hDim(\pi^n_\psi T)(X)
\]

for every \(X \subseteq G\).

**Proof.** It suffices to see that the natural filtrations \(\{\pi^n_\phi S\}\) and \(\{\pi^n_\psi T\}\) satisfy condition (i) of Proposition 3.1.

As indicated in Section 2, the charts \(\phi\) and \(\psi\) belong to the full atlas of \(G\), and so the two functions \(\psi\phi^{-1}|_{\phi(S \cap T)}\) and \(\phi\psi^{-1}|_{\phi(S \cap T)}\) are analytic. Since \(\phi(S \cap T)\) is open in \(S\), it follows that \(\psi\phi^{-1}\) can be evaluated in \((m^k)^{d}\) for some \(k\). By [4] Lemma 6.45, there exists a natural number \(c\) such that

\[
\psi\phi^{-1}((m^c)^{(d)}) \subseteq R^{(d)}.
\]

Now, since \(\psi\phi^{-1}(0) = 0\), the previous inclusion implies that

\[
\psi\phi^{-1}((m^{n+c+1})^{(d)}) \subseteq (m^{n+1})^{(d)}
\]

for every non-negative integer \(n\). This implies that \(\pi^{n+c}_\phi S \leq \pi^n_\psi T\). Arguing similarly with \(\phi\psi^{-1}\), we obtain that \(\pi^{n+d}_\psi T \leq \pi^n_\phi S\), for some natural number \(d\), and for all \(n \geq 0\). Thus

\[
\pi^{n+c}_\phi S \leq \pi^n_\psi T \leq \pi^{n-d}_\phi S
\]
for every \( n \geq d \), and the result follows. \( \square \)

Thus all open standard subgroups of an \( R \)-analytic profinite group define the same Hausdorff dimension function. This allows us to give the following definition.

**Definition 3.4.** Let \( G \) be an \( R \)-analytic profinite group and let \( X \) be a subset of \( G \). Then the **standard Hausdorff dimension** of \( X \), \( \text{hDim}_{\text{Std}}(X) \), is the Hausdorff dimension of \( X \) calculated with respect to the natural filtration induced by any given standard open subgroup of \( G \).

For the proof of our main theorem, we need one last lemma regarding Hausdorff dimension of \( R \)-submodules in an \( R \)-module.

**Lemma 3.5.** Let \( R \) be a discrete valuation pro-\( p \) ring with maximal ideal \( m \), and let \( M \) be a free \( R \)-module. Then for every submodule \( N \) of \( M \), we have

\[
\text{hDim}_{\{m^n M\}_{n \in \mathbb{N}}} N = \frac{\text{rank}_R N}{\text{rank}_R M}.
\]

**Proof.** Since \( R \) is a principal ideal domain, we can find a basis \( \{v_1, \ldots, v_d\} \) of \( M \) such that \( \{r_1 v_1, \ldots, r_e v_e\} \) is a basis of \( N \) for some \( r_1, \ldots, r_e \in R \) and a non-negative integer \( e \leq d \). Put \( \tilde{N} = \langle v_1, \ldots, v_e \rangle_R \). Then \( N \) is open in \( \tilde{N} \), and then by Lemma 2.7

\[
\text{hDim}_{\{m^n M\}_{n \in \mathbb{N}}} N = \text{hDim}_{\{m^n M\}_{n \in \mathbb{N}}} \tilde{N}.
\]

Since \( \tilde{N} \cap m^n M = m^n \tilde{N} \), it follows from Theorem 2.9 that

\[
\text{hDim}_{\{m^n M\}_{n \in \mathbb{N}}} \tilde{N} = \liminf_{n \to \infty} \frac{\log |\tilde{N} : m^n \tilde{N}|}{\log |M : m^n M|} = \liminf_{n \to \infty} \frac{\log(q^{ne})}{\log(q^{nd})} = \frac{e}{d},
\]

where \( q \) is the size of the residue field \( R/m \). This proves the result. \( \square \)

We are now ready to prove our main theorem.

**Proof of the Main Theorem.** Let \( \dim(G) = d \) and \( \dim(H) = e \). Since \( H \) is an \( R \)-analytic submanifold of \( G \), there exist an open subset \( U \) of \( H \) containing 1, and a chart \((V, \phi_0, d)\) of \( G \) such that \( U \subseteq V \) and \( \phi_0(U) = E \cap \phi_0(V) \), for some vector subspace \( E \) of \( K^{(d)} \) of dimension \( e \) (see [9, Page 89]). Here \( K \) is the field of fractions of \( R \).

From the proof of [5, Theorem 13.20] we see that there exist a natural number \( N \) and an open subgroup \( S \) of \( G \) such that \( S \subseteq V \) and \( (S, \pi^{-N} \phi_0, d) \) is a standard subgroup of \( G \). Let \( \phi \) be the map \( \pi^{-N} \phi_0 \). Since \( U \cap S \subseteq_o U \subseteq_o H \), we have

\[
\text{hDim}_{\text{Std}}(H) = \text{hDim}_{\text{Std}}(U \cap S),
\]
by Lemma 2.7. On the other hand, since $\phi$ is an isometry between the multiplicative group $S$ and the additive group $(m^{(d)}, +)$ by Lemma 2.3 we get

$$h\text{Dim}_{\text{Std}}(U \cap S) = h\text{Dim}_{\{m^n\}_{n \in \mathbb{N}}}(\phi(U \cap S)).$$

Now it is easy to check that $\phi(U \cap S) = E \cap m^{(d)}$, and this is an $R$-submodule of $m^{(d)}$ of rank $e$. By applying Lemma 3.5 it follows that

$$h\text{Dim}_{\{m^n\}_{n \in \mathbb{N}}}(\phi(U \cap S)) = \frac{e}{d},$$

which completes the proof. □

The following corollary is a straightforward consequence of our main theorem.

**Corollary 3.6.** Let $G$ be an $R$-analytic group of dimension $d$. Then

$$\text{Spec}_R(G) \subseteq \{0, \frac{1}{d}, \ldots, \frac{d-1}{d}, 1\}.$$

In particular, $G$ has finite $R$-analytic spectrum.

We conclude with a final observation.

**Remark 3.7.** Let $\text{Spec}(G)$ be the set of the Hausdorff dimensions of all the closed subgroups of $G$. If $R = \mathbb{Z}_p$ then we have $\text{Spec}(G) = \text{Spec}_R(G)$, since all closed subgroups are then analytic. This is definitely false in general. For example, let $R = \mathbb{F}_p[[t]]$ and let $G$ be the additive abelian group $\mathbb{F}_p[[t]]$, with its natural 1-dimensional $\mathbb{F}_p[[t]]$-analytic structure. Consider the metric on $G$ naturally defined by the filtration $\{t^n\mathbb{F}_p[[t]]\}_{n \in \mathbb{N}}$. Then, by Theorem [4] and [11] Lemma 4.1, we have

$$\{0, 1\} = \text{Spec}_R(G) \subsetneq \text{Spec}(G) = [0, 1].$$

The reason for such a diversity is, of course, that most closed subgroups of $(\mathbb{F}_p[[t]], +)$ are not $\mathbb{F}_p[[t]]$-analytic. For instance, $\mathbb{F}_p[[t^2]]$ is closed but not analytic, and has Hausdorff dimension equal to $\frac{1}{2}$.

**Acknowledgements**

The results of this article have been achieved during the internship of the second author with the GRECA (Groups, Representations, and Algebraic Combinatorics) research group at the University of the Basque Country. He gratefully acknowledges the members of this group for their support.
References

[1] J.L. Abercrombie, Subgroups and subrings of profinite rings, Math. Proc. Cambr. Phil. Soc. 116 (1994), 209–222.
[2] M. Abért, B. Virág, Dimension and randomness in groups acting on rooted trees, J. Amer. Math. Soc. 18 (2005), 157–192.
[3] Y. Barnea, B. Klopsch, Index-subgroups of the Nottingham group, Adv. Math. 180 (2003), 187–221.
[4] Y. Barnea, A. Shalev, Hausdorff dimension, pro-$p$ groups, and Kac-Moody algebras, Trans. Amer. Math. Soc. 349 (1997), no. 12, 5073–5091.
[5] J. Dixon, M. Du Sautoy, A. Mann, D. Segal, Analytic pro-$p$ groups, 2nd ed., Cambridge University Press, Cambridge, 1999.
[6] M. Ershov, On subgroups of the Nottingham group of positive Hausdorff dimension, Comm. Algebra 35 (2006), 193–206.
[7] K. Falconer, Fractal geometry: mathematical foundations and applications, John Wiley and Sons, New York, 1990.
[8] G.A. Fernández-Alcober, A. Zugadi-Reizabal, GGS-groups: order of congruence quotients and Hausdorff dimension, Trans. Amer. Math. Soc. 366 (2014), 1993–2017.
[9] J.-P. Serre, Lie algebras and Lie groups, Lecture Notes in Mathematics 1500, Springer-Verlag, Berlin-Heidelberg-New York, 1992.
[10] O. Siegenthaler, Hausdorff dimension of some groups acting on the binary tree, J. Group Theory 11 (2008), 555–567.

Department of Mathematics, University of the Basque Country UPV/EHU, 48080 Bilbao, Spain
E-mail address: gustavo.fernandez@ehu.es

Department of Mathematics, Royal Holloway University of London, Egham TW20 0EX, United Kingdom
E-mail address: Eugenio.Giannelli.2011@live.rhul.ac.uk

Department of Mathematics, University of the Basque Country UPV/EHU, 48080 Bilbao, Spain
E-mail address: jon.gonzalez@ehu.es