DEFORMATION EQUIVALENCE OF AFFINE RULED SURFACES

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Abstract. A smooth family $\varphi : V \to S$ of surfaces will be called completable if there is a logarithmic deformation $(\bar{V}, \bar{D})$ over $S$ so that $V = \bar{V} \backslash \bar{D}$. Two smooth surfaces $V$ and $V'$ are said to be deformations of each other if there is a completable flat family $V \to S$ of smooth surfaces over a connected base so that $V$ and $V'$ are fibers over suitable points $s, s' \in S$. This relation generates an equivalence relation called deformation equivalence. In this paper we give a complete combinatorial description of this relation in the case of affine ruled surfaces, which by definition are surfaces that admit an affine ruling $V \to B$ over an affine base with possibly degenerate fibers. In particular we construct complete families of such affine ruled surfaces. In a few particular cases we can also deduce the existence of a coarse moduli space.

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Date: May 24, 2013.

Acknowledgements: This research was done during a visit of the first and the second authors at the Fourier Institute, Grenoble, and of all three authors at the Max-Planck-Institute of Mathematics, Bonn. They thank these institutions for the generous support and excellent working conditions.

2010 Mathematics Subject Classification: 14R05, 14R25, 14J10, 14D22.
Key words: affine ruled surface, $A^1$-fibration, moduli space, deformation equivalence.
**Introduction**

In classifying algebraic objects like varieties or vector bundles of a certain class one usually tries to find discrete invariants such that the objects sharing a fixed set of these invariants, form a moduli space. The model case is here the moduli space of smooth complete curves, which is known to be a $3g-3$-dimensional irreducible variety, see [DM]. Another classical case is the Hilbert scheme $H_P$ parameterizing subschemes of the projective space $\mathbb{P}^n$ with fixed Hilbert polynomial $P$. By a classical result of Hartshorne [Ha] $H_P$ is connected. On the other hand it is already an unsolved problem to determine the connected components of the Hilbert scheme of locally Cohen-Macaulay curves of degree $d$ and genus $g$, see [Ha] and the references therein for partial results.

In general it may happen that for a class of varieties one cannot expect a reasonable moduli space. A typical obstruction for its existence are usually $\mathbb{P}^1$-fibrations or, in the case of open varieties, $\mathbb{A}^1$-fibrations. Instead of studying in such cases the connected components of the moduli space it is convenient to consider the relation of deformation equivalence as introduced in [Ca]. Given a deformation $f : X \to S$ with all fibers in a class of varieties $\mathcal{C}$ we consider the relations $f^{-1}(s) \sim f^{-1}(t)$ if and only if $s, t \in S$ belong to the same connected component of $S$. This generates an equivalence relation on $\mathcal{C}$ called deformation equivalence.

In this paper we study this equivalence relation for the class of normal affine surfaces $V$ that can be equipped with an affine ruling. By this we mean a morphism $\pi : V \to B$ onto a smooth affine curve with the general fiber isomorphic to $\mathbb{A}^1$. In this case there is no moduli space available except for a few special cases; see sect. 5. Our main result is that nevertheless one can characterize deformation equivalence completely. The deformations which we consider here are deformations of the open surface $X$ which can be extended to a logarithmic deformation of a completion of $X$ in the sense of [Ka].

The main tool for this characterization is the so called normalized extended graph. For smooth surfaces it can be described as follows. Given an affine ruling $\pi : V \to B$ we can extend $\pi$ to a $\mathbb{P}^1$-fibration $\bar{\pi} : \bar{X} \to \bar{B}$ over the normal completion $\bar{B}$ of $B$. Performing suitable blowups and blowdowns along the boundary $D := \bar{X} \setminus X$ of $X$ one can transform $D$ or, equivalently, its weighted dual graph $\Gamma_D$ into standard form, see [DG] or [FKZ]. The extended divisor $D_{\text{ext}}$ is then the union of $D$ with all singular fibers of $\pi$. This extended divisor is not in general an invariant of the surface alone, but depends on the choice of the affine ruling. The components of $D_{\text{ext}} - D$ are called the feathers. For such feathers one has the notion of a mother component (see Definition 2.16 for details). A feather is not necessarily linked to its mother component. Attaching the feathers to their mother components we obtain from the dual graph $\Gamma_{D_{\text{ext}}}$ the normalized extended graph $N(\Gamma_{D_{\text{ext}}})$. With this terminology our main result is as follows.

**Theorem 0.1.** Two affine ruled surfaces are deformation equivalent if and only if they share the same normalized extended graph unless $D$ is a zigzag, i.e. a linear chain of rational curves. In the latter case two surfaces are deformation equivalent if and only if the normalized extended graphs are equal, possibly after reversing one of them (see sect. 3 for details).
This theorem enables us to construct in section 5 a coarse moduli space for the class of special Gizatullin surface that was studied in [FKZ4]. Let us mention a few special cases. First of all, the Ramanujam theorem [Ra] states that every contractible, smooth surface over $\mathbb{C}$ with a trivial fundamental group at infinity (in particular, every smooth surface over $\mathbb{C}$ homeomorphic to $\mathbb{R}^4$) is isomorphic to the affine plane $\mathbb{A}^2 = \mathbb{A}^2_\mathbb{C}$. Thus the deformation equivalence class for such surfaces is reduced to a single point. The normalized extended graph consists in this case just of two 0-vertices joined by an edge.

A generalization of this theorem due to Gurjar and Shastri [GS] says that every normal contractible surface with a finite fundamental group at infinity is isomorphic to $\mathbb{A}^2/G$, where $G$ is a finite subgroup of $GL_2(\mathbb{C})$ acting freely on $\mathbb{A}^2 \setminus \{0\}$. Fixing $G$, the deformation equivalence class for such surfaces again consists of a single point.

Yet another result of this type is provided by the Danilov-Gizatullin Isomorphism Theorem [DG] (see also [CNR, FKZ5]). Recall that a Danilov-Gizatullin surface is the complement to an ample divisor $H$ in some Hirzebruch surface. Such a surface can be completed by the zigzag $[[0,0,(2)_n]]$, where $n + 1 = H^2$. The Danilov-Gizatullin theorem proves that the deformation equivalence class for such a surface again consists of a single point, once we fix the length of the boundary zigzag.

The paper is organized as follows. In section 1 we recall the notion of standard completion and standard dual graph of an affine ruled surface, and in subsection 2.2 that of an extended divisor. The rest of section 2 is devoted to developing such notions in the relative case. More specifically, in Theorem 2.4 we show the existence of standard completions in completable families, and in the factorization Lemma 2.24 we shed some light onto the structure of extended divisors in the relative case. Central are here the notions of completable families and resolvable families, see Definitions 2.3 and 2.20.

The normalized extended graph is introduced in section 3. Our main result Theorem 3.6 characterizes the deformation equivalence of affine ruled surfaces in terms of these graphs. The proof of this theorem starts in section 3 and is completed in the next section 4, where we construct a versal deformation space of affine ruled surfaces. In section 5 we apply our previous results in order to construct the moduli space of special Gizatullin surfaces, a subclass of the class of all Gizatullin surfaces studied in detail in our previous paper [FKZ4].

All varieties in this paper are assumed to be defined over an algebraically closed field $k$. By a surface we mean a connected algebraic scheme over $k$ such that all its irreducible components are of dimension 2.

1. Preliminaries on standard completions

Let us recall the notions of a (semi-)standard zigzag and (semi-)standard graph.

1.1. Let $X$ be a complete normal algebraic surface, and let $D$ be an SNC (i.e. a simple normal crossing) divisor $D$ contained in the smooth part $X_{\text{reg}}$ of $X$. We say that $D$ is a zigzag if all irreducible components of $D$ are rational and the dual graph $\Gamma_D$ of $D$ is linear.\footnote{By abuse of notation, we often denote an SNC divisor and its dual graph by the same letter.} We abbreviate a chain of curves $C_0, C_1, \ldots, C_n$ of weights $w_0, \ldots, w_n$ by $[[w_0, \ldots, w_n]]$. We also write $[[\ldots,(w)_k,\ldots]]$ if a weight $w$ occurs at $k$ consecutive places.
1.2. A zigzag $D$ is called standard if it is one of the chains

(1) $[(0),i], i \leq 3,$ or $[(0)_{2i}, w_2, \ldots, w_n]$, where $i \in \{0, 1\}$, $n \geq 2$ and $w_j \leq -2 \ \forall j$.

A linear chain $\Gamma$ is said to be semi-standard or $w_1$-standard if it is either standard or one of

(2) $[0, w_1, w_2, \ldots, w_n], \quad [(0, w_1, 0)]$, where $n \geq 1$, $w_1 \in \mathbb{Z}$, and $w_j \leq -2 \ \forall j \geq 2$.

A circular graph is a connected graph with all vertices of degree 2. Such a weighted graph will be denoted by $((w_0, \ldots, w_n))$. A circular graph is called standard if it is one of

(3) $((w_1, \ldots, w_n))$, $((0, 0, w_1, \ldots, w_n))$, $((0, w))$, or $((0, 0, -1, -1))$,

where $w_1, \ldots, w_n \leq -2$, $n > 0$, $0 \leq l \leq 3$ and $w \leq 0$.

By an inner elementary transformation of a weighted graph we mean blowing up at an edge incident to a 0-vertex of degree 2 and blowing down the image of this vertex. By a sequence of inner elementary transformations we can successively move the pair of zeros in the standard zigzag $[[0, 0, w_2, \ldots, w_n]]$ to the right:

$[[0, 0, w_2, \ldots, w_n]] \rightsquigarrow [[w_2, 0, 0, w_3, \ldots, w_n]] \rightsquigarrow \cdots \rightsquigarrow [[w_2, \ldots, w_n, 0, 0]].$

This yields the reversion

(4) $D = [[0, 0, w_2, \ldots, w_n]] \rightsquigarrow [[0, 0, w_n, \ldots, w_2]] =: D^\vee$

(see 1.4 in [FKZ$_3$]).

An outer elementary transformation consists in blowing up at a 0-vertex of degree $\leq 1$ and blowing down the image of this vertex. A birational inner elementary transformation on a surface is rigid i.e., uniquely determined by the associated combinatorial transformation of the dual graph, whereas an outer one depends on the choice of the center of blowup.

1.3. In the sequel there is also the need to consider NC divisors $D$ on a surface $X$. By this we mean that $D \subseteq X_{reg}$ and that the singularities of $D$ are ordinary double points; these are given in the local ring $\mathcal{O}_{X,p}$ by an equation $xy = 0$, where $x, y \in \mathfrak{m} \setminus \mathfrak{m}^2$. In particular, two different components meet in smooth points transversally, while the intersection of three different components is empty. Thus $D$ is an SNC divisor if and only if it is an NC divisor and all its irreducible components are smooth. The dual graph $\Gamma_D$ of an NC divisor $D$ has loops which correspond to the singular points of the components. Vice versa, if the dual graph $\Gamma_D$ of an NC divisor $D$ has no loop then $D$ is SNC. In particular, $D$ is SNC if $\Gamma_D$ is a tree.

An NC completion $(\bar{V}, D)$ of a surface $V$ consists of a complete surface $\bar{V}$ and an NC divisor $D$ on $\bar{V}$ such that $V = \bar{V} \setminus D$.

For instance, a plane nodal cubic is an NC divisor, which is not SNC. Its dual graph consists of one vertex of weight 3 and a loop.

**Definition 1.4.** Let $\Gamma$ be the dual graph of an NC divisor $D$. We use the following notations.

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$^2$We list only those graphs for which the intersection matrix has at most one positive eigenvalue. Actually we do not use circular graphs in this paper, however without these graphs the general notion of a standard graph in 1.4 below would be incomplete.
(1) $B = B(\Gamma)$ is the set of branching points of $\Gamma$.
(2) $S = S(\Gamma)$ is the set of vertices corresponding to non-rational components of $D$.
(3) Following [FKZ 1] a connected component of $\Gamma - (B \cup S)$ is called a segment of $\Gamma$.
(4) A segment will be called outer if it contains an extremal (or end) vertex of $\Gamma$ i.e.,
a vertex of degree 1.

Thus an outer segment is either the whole graph $\Gamma$ and $\Gamma$ is linear, or it is connected
to exactly one vertex of $B \cup S$. The dual graph $\Gamma$ of an NC divisor $D$ on an algebraic
surface $\tilde{V}$ (and also $D$ itself) will be called (semi-)standard if the following hold.

(i) All segments of $\Gamma$ are (semi-)standard;
(ii) If a segment is outer and contains a vertex $v$ of weight 0 then it also has an
extremal vertex of weight 0. For a standard graph we require additionally that
the neighbor in $\Gamma$ of every extremal vertex of weight 0 is as well a zero vertex.

An NC completion $(\tilde{V}, D)$ of an open surface $V$ is called (semi-)standard if so is $D$.

These notions differ from those in [FKZ 1], Definition 2.13], where condition (ii) is
absent.

Remark 1.5. (1) Every normal surface $V$ has a standard NC completion $(\tilde{V}, D)$. For
applying [FKZ 1], Theorem 2.15(b)] every normal surface has an NC completion $(\tilde{V}, D)$
such that $\Gamma_D$ satisfies (i) in [L.A]. By further elementary transformations we can achieve
that also (ii) holds.

(2) The dual graph $\Gamma_D$ of the boundary divisor of a standard NC-completion $(V, D)$
is unique up to elementary transformations as follows from [FKZ 1], Theorem 3.1].

(3) A Gizatullin surface is a normal affine surface $V$ with a completion $(\tilde{V}, D)$, where
$D$ is a standard zigzag. Reversing $D$ by a sequence of inner elementary transformations
performed on $(\tilde{V}, D)$ we obtain a new completion $(\tilde{V}'', D'')$, which is called the reversed
standard completion. It is uniquely determined by $(\tilde{V}, D)$.

In the presence of an affine ruling we have more precise informations.

Lemma 1.6. Let $V$ be a normal affine surface. Given an affine ruling $\pi : V \to B$ over
a smooth affine curve $B$ the following hold.

(a) There is a standard SNC completion $(\tilde{V}, D)$ such that $\pi$ extends to a $\mathbb{P}^1$-fibration
$\bar{\pi} : \tilde{X} \to \bar{B}$ over the smooth completion $\bar{B}$ of $B$. There is a unique curve, say, $C_1$
in $D$, which is a section of $\bar{\pi}$.
(b) With $(\tilde{V}, D)$ as in (a), $\Gamma_D$ is a tree, and $D - C_1$ has only rational components.
(c) A completion as in (a) is unique up to elementary transformations in extremal zero
vertices of $\Gamma_D$. These extremal zero vertices correspond to components of $D - C_1$
that are full fibers of $\bar{\pi}$.

Proof. (a) and (b): Let $(\tilde{V}, D)$ be an SNC completion of $V$. Blowing up $(\tilde{V}, D)$ suitably
we may assume that $\pi$ extends to a regular map $\bar{\pi} : \tilde{V} \to \bar{B}$. There is a unique
horizontal component, say, $C_1$ of $D$ which is a section of $\bar{\pi}$. The surface $V$ being affine
$D$ is connected. Every fiber of $\bar{\pi}$ is a tree of rational curves and so $D$ is as well a tree
with rational components except possibly for $C_1$.

Any branch $\mathcal{B}$ of $D$ at $C_1$ (i.e. connected component of $D - C_1$) is contained in a
fiber of $\bar{\pi}$. If the intersection form of $\mathcal{B}$ is not negatively definite, then by Zariski’s
lemma [FKZ 1] 4.3] $\mathcal{B}$ coincides with the entire fiber of $\bar{\pi}$ and so can be contracted to a
single 0-curve. Thus after contracting at most linear \((-1)\)-curves of \(D\) contained in the fibers we may suppose that every branch of \(\Gamma_D\) at \(C_1\) either is minimal and negative definite or is a single extremal zero vertex. Since \(B\) is affine, extremal zero vertices do exist. Performing outer elementary transformations at such a vertex we can achieve that \((C_1)^2 = 0\). Thus conditions (i) and (ii) of Definition 1.4 are fulfilled and so \(\Gamma_D\) is standard.

(c) Let \((\bar{V}', D')\) be a second SNC completion satisfying (a). We can find a common domination of \((\bar{V}, D)\) and \((\bar{V}', D')\) by a SNC completion \((\bar{V}'', D'')\) of \(V\). We may suppose that there is no \((-1)\)-curve in \(\bar{V}''\) that is contracted in both \(\bar{V}\) and \(\bar{V}'\). If \(C \neq C_1\) is a curve in \(D\), which is not a full fiber, then its proper transform in \(\bar{V}''\) has self-intersection \(\leq -2\). Hence if a \((-1)\)-curve \(E\) in \(\bar{V}''\) is contracted in \(\bar{V}'\), then it is necessarily the proper transform of a component of \(D\) which is a full fibers of \(\bar{\pi}\). Thus the indeterminacy locus of the map \(\bar{V} \dashrightarrow \bar{V}'\) is contained in the union of full fibers, say, \(F_1 \cup \ldots \cup F_a\) of \(\bar{\pi}\) that are contained in \(D\). Now it is a standard fact that \((\bar{V}', D)\) is obtained via a sequence of elementary transformations along \(F_1 \cup \ldots \cup F_a\). \(\square\)

Remarks 1.7. (1) Given a normal affine surface \(V\) with a semi-standard NC completion \((\bar{V}, D)\), the following hold.

(a) If \(\Gamma_D\) contains an extremal 0-vertex, then \(V\) is affine ruled. Indeed, every extremal 0-vertex \(C_0\) of \(\Gamma_D\) induces a \(\mathbb{P}^1\)-fibration \(\bar{\pi} : \bar{V} \rightarrow B\) onto a smooth complete curve \(\bar{B}\) so that \(C_0\) is one of the fibers, see e.g. [BHPV, Chapt. V, Proposition 4.3]. In particular \(\bar{\pi}\) restricts to an affine ruling on \(V\).

(b) Conversely, if \(V\) carries an affine ruling then \(\Gamma_D\) has extremal 0-vertices. For by Remark 1.5(1) the given semi-standard completion is obtained by elementary transformations from a standard one with extremal 0-vertices as in Lemma 1.6. Applying [FKZ1, Corollary 3.33'], (b) follows.

(2) Recall that a normal affine surface different from \(A^1 \times A^1_*\) is Gizatullin if and only if it admits two distinct affine rulings [Gi, Du]. Thus the affine ruling of a non-Gizatullin surface is unique up to an isomorphism of the base. Actually the base of the canonical affine ruling \(V \rightarrow B\) is equal to \(B : = \text{Spec} \text{ML}(V)\), where \(\text{ML}(V) \subseteq \mathcal{O}(V)\) is the Makar-Limanov invariant of \(V\), i.e. the common kernel of nonzero locally nilpotent derivations on \(\mathcal{O}(V)\). Clearly, the ruling \(V \rightarrow B\) is induced by the embedding \(\text{ML}(V) \rightarrow \mathcal{O}(V)\).

(3) We note that by Remark (1) any standard completion \((\bar{V}, D)\) of an affine ruled surface \(V\) yields an affine ruling \(\pi : V \rightarrow B\) so that \((\bar{V}, D)\) is a standard completion associated to \(\pi\) as in Lemma 1.6.

(4) In the situation of Lemma 1.6, performing elementary transformations in extremal 0-vertices one can replace a standard completion by a semi-standard one where \((C_1)^2\) is any given number. This will be useful in the sequel.

Lemma 1.8. Given an affine ruled surface \(V\) with standard completion \((\bar{V}, D)\) the following hold.

(a) If \(V\) is Gizatullin then up to reversion [4] the dual graph \(\Gamma_D\) of \(D\) does not depend on the choice of such a completion.

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3 As before here \(A^1_* = A^1 \setminus \{0\} \).
(b) If \( V \) is non-Gizatullin then the divisor \( D \) and its dual graph \( \Gamma_D \) are unique up to elementary transformations at extremal 0-vertices of \( \Gamma_D \).

Proof. (a) is shown in [DG, Theorem 2.1], see also [FKZ1, Corollary 3.32].
(b) With \( \pi : V \to B \) the affine ruling of \( V \) we let \( (\overline{V},D) \) be a standard completion of \( V \) as constructed in Lemma 1.6(a). By Remark 1.7(1) the dual graph \( \Gamma_D' \) of any other standard completion \( (\overline{V}',D') \) has again an extremal 0-vertex that induces a \( \mathbb{P}^1 \)-fibration \( \overline{\pi}' : \overline{V}' \to \overline{B}' \). Since \( V \) is non-Gizatullin, up to a suitable isomorphism \( \overline{B} \cong \overline{B}' \) the map \( \overline{\pi}' \) restricts again to \( \pi \) on the open part \( V \). Thus \( (\overline{V}',D') \) is also a standard completion as in Lemma 1.6(a). Now the assertion follows form part (c) of that Lemma. \( \square \)

2. Families of affine surfaces

2.1. Families of standard completions. By a family (sometimes called a flat family) we mean a flat morphism \( \varphi : \mathcal{V} \to S \) of algebraic \( k \)-schemes. We call it a family of normal or affine surfaces if every fiber \( \mathcal{V}(s) = \varphi^{-1}(s), s \in S \), has this property.

Definition 2.1. Let \( \varphi : \mathcal{V} \to S \) be a proper family of surfaces over an algebraic \( k \)-scheme \( S \). A Cartier divisor \( D \) in \( \mathcal{V} \) will be called a relative NC divisor or a family of NC-divisors if \( D \) is proper over \( S \) and for every point \( p \in D \) either \( \varphi|_D \) is smooth at \( p \), or in a suitable neighborhood of \( p \) we have \( D = D' \cup D'' \), where \( D', D'' \) and the scheme theoretic intersection \( D' \cap D'' \) are smooth over \( S \).

Every fiber \( D(s), s \in S \), of a relative NC-divisor is an NC divisor. Its dual graph \( \Gamma(s) \) is locally constant in \( S \) with respect to the étale topology. We say that a relative NC divisor \( D \) has constant dual graph \( \Gamma \) if for every irreducible component \( S' \) of \( S \) and all \( s \in S' \) the irreducible components of both \( D'|S' \) and \( D(s) \) are in one-to-one correspondence with the vertices of \( \Gamma \).

Remark 2.2. (1) A relative NC-divisor is contained in the set \( \text{Reg}(\mathcal{V}/S) \) of points in \( \mathcal{V} \) in which \( \varphi \) is smooth.

(2) If \( \text{Sing}(D/S) \) is non-empty then \( \text{Sing}(D/S) \to S \) is an unramified covering. It may happen that the irreducible components of \( D \) and \( D(s) \) are not in one-to-one correspondence even if the base \( S \) is connected (see Example 2.27 below). In such cases the family \( (\Gamma(s))_{s \in S} \) has non-trivial monodromy, and \( \text{Sing}(D/S) \to S \) is a non-trivial covering of \( S \).

(3) If \( \varphi \) is smooth and all irreducible components of \( D \) are smooth then a relative NC-divisor \( D \) amounts locally in \( S \) to a logarithmic deformation of the fiber \( (\mathcal{V}(s), D(s)) \) in the sense of [Ka].

The following definition is central in our considerations.

Definition 2.3. A flat family \( \varphi : \mathcal{V} \to S \) of normal surfaces will be called completable if there is a proper smooth family of surfaces \( \overline{\varphi} : \overline{\mathcal{V}} \to S \) together with an open embedding \( \mathcal{V} \hookrightarrow \overline{\mathcal{V}} \) such that \( D = \mathcal{V}|\mathcal{V} \to S \) is a relative NC divisor with constant dual graph \( \Gamma \). We call such a pair \( (\overline{\mathcal{V}}, D) \) an NC completion of \( \varphi \). This NC completion is called (semi-)standard if so is \( \Gamma \).

The main result in this subsection is the following relative version of the existence of standard NC completions, see Remark 1.5(a).
Theorem 2.4. Let \( \varphi : \mathcal{V} \to S \) be a family of normal surfaces admitting an NC completion \((\bar{\mathcal{V}}, \mathcal{D})\). Then locally with respect to the étale topology there exists a standard NC completion \((\bar{\mathcal{V}}, \mathcal{D})\) of \( \varphi \).

To deduce this result we need some preparations. The following well known lemma is an easy consequence of a result due to Grothendieck; in the analytic setup this is a special case of Kodaira’s theorem on the stability of certain submanifolds in deformations [Ko].

Lemma 2.5. Let \( \varphi : \mathcal{V} \to S \) be a flat family of normal surfaces over \( S \). If the regular part of a special fiber \( \mathcal{V} = \mathcal{V}(s) \) contains a \((-1)\)-curve \( E \) then over some étale neighbourhood of \( s \) there is a unique family of \((-1)\)-curves \( E \subseteq \mathcal{V} \) with \( E = E(s) \).

Proof. The curve \( E \) can be considered as a point \([E]\) in the relative Hilbert scheme \( \mathbb{H}_{X/S} \). Applying the smoothness criterion of Grothendieck [Gro1, Corollaire 5.4] the morphism \( \mathbb{H}_{X/S} \to S \) is étale at \([E]\). Hence on a sufficiently small open neighbourhood \( U \) of \([E]\) in \( \mathbb{H}_{X/S} \) the map \( U \to S \) is étale. Now the universal subspace of \( X \times_S U \) yields the desired family of \((-1)\)-curves. \( \square \)

In the next result we show that in the relative case one can perform blowups and blowdowns just as in the absolute case. In order to keep the formulation short, let us say that a divisor \( E \subseteq \bar{\mathcal{V}} \) is a relative \((-1)\)-curve over \( S \) if it is smooth over \( S \) and restricts to a \((-1)\)-curve in every fiber.

Lemma 2.6. Let \( \varphi : \bar{\mathcal{V}} \to S \) be a proper flat family of surfaces, and let \( \mathcal{D} \) be a relative NC divisor on \( \bar{\mathcal{V}} \) over \( S \) with constant dual graph \( \Gamma \). Then the following hold.

(a) If \( D_i \subseteq \mathcal{D} \) is a relative \((-1)\)-curve, which corresponds to a vertex of degree \( \leq 2 \) in \( \Gamma \), then contracting \( D_i \) fiberwise yields a proper flat family of surfaces \( \varphi : \bar{V}_{\text{cont}} \to S \) and a relative NC divisor \( \mathcal{D}_{\text{cont}} \) on \( \bar{V}_{\text{cont}} \) with a constant dual graph.

(b) Assume that \( D_i, D_j \) are irreducible components of \( \mathcal{D} \) such that \( D_i \cap D_j \cong S \) is a section of \( \varphi \). Blowing up \( \bar{V} \) along this section yields a flat family \( \varphi' : \mathcal{V}' \to S \) together with a relative NC divisor \( \mathcal{D}' = \varphi'^{-1}(\mathcal{D}) \) on \( \mathcal{V}' \) with constant dual graph.

Proof. A proof of (a) can be found e.g. in [FKZ3, Lemma 1.15]. To deduce (b) it is enough to observe that locally near \( D_i \cap D_j \) our family is a product family. \( \square \)

We frequently use the following well known fact on the local triviality of \( \mathbb{P}^1 \)-fibrations.

Lemma 2.7. Let \( \varphi : \mathcal{X} \to S \) be a smooth morphism such that every fiber of \( \varphi \) is isomorphic to \( \mathbb{P}^1 \). Then the following hold.

(a) \( \varphi \) is a locally trivial \( \mathbb{P}^1 \)-bundle in the étale topology.

(b) If \( \varphi \) admits a section \( \sigma : S \to \mathcal{X} \) then \( \varphi \) is locally trivial in the Zariski-topology.

Proof. (b) is well known. It can be shown e.g. along the same lines as Lemma 1.16 in [FKZ3]. The assertion (a) follows from (b) since in a suitable étale neighborhood of a given point \( s \in S \) the map \( \varphi \) has sections. \( \square \)

Lemma 2.6 yields the following result.

Lemma 2.8. Let \( \varphi : \mathcal{V} \to S \) be a family of normal surfaces, which admits an NC completion \((\bar{\mathcal{V}}, \mathcal{D})\) with constant dual graph \( \Gamma \). If \( \Gamma' \dashrightarrow \Gamma \) is a birational transformation
then locally in the étale topology there exists an NC completion \((\tilde{V}', \mathcal{D}')\) of \(\varphi\) such that \(\mathcal{D}'\) has constant dual graph \(\Gamma'\).

**Proof.** It is sufficient to treat the case where \(\Gamma'\) is obtained from \(\Gamma\) by a single blowup or blowdown. The case of a single blowdown follows from Lemma 2.6(a), while in the case of an inner blowup part (b) of that Lemma is applicable. It remains to treat the case of an outer blowup in an irreducible component, say \(\mathcal{D}_i\) of \(\mathcal{D}\). Locally in the étale topology around a point \(s \in S\) there are sections \(S \to \mathcal{D}_i\) not meeting the other components \(\mathcal{D}_j\) with \(\mathcal{D}_j \neq \mathcal{D}_i\). Blowing up such a section yields the desired result. \(\square\)

We are now ready to give the proof of Theorem 2.4.

**Proof of Theorem 2.4.** By Theorem 2.15 in [FKZ1] the dual graph \(\Gamma\) of \((\tilde{V}, \mathcal{D})\) can be transformed into a standard graph by a sequence of blowdowns and blowups. Applying Lemma 2.8 the result follows. \(\square\)

Next we show that for completable families \(\varphi : V \to S\) of affine ruled surfaces the morphism \(\varphi\) is affine.

**Proposition 2.9.** Let \(V \to S\) be a completable family of normal surfaces admitting an NC completion \((\tilde{V}, \mathcal{D})\) with constant connected dual graph \(\Gamma\).

(a) If \(\Gamma\) has an extremal 0-vertex then there exists a relative semi-ample divisor \(A\) on \(\tilde{V}\) supported by \(\mathcal{D}\).

(b) The morphism \(V \to S\) can be factorized into a proper relative simultaneous contraction \(V \to V'\) (the Remmert reduction) and an affine morphism \(V' \to S\).

**Proof.** (b) follows immediately from (a).

(a) If \(\Gamma\) has extremal 0-vertices then it is a tree. Letting \(\mathcal{D} = \sum_{i=0}^{n} \mathcal{D}_i\), where \(\mathcal{D}_0 = \mathcal{D}_{01}\) corresponds to an extremal zero vertex of \(\Gamma\), we consider a \(\mathbb{Q}\)-divisor \(A = \sum_i a_i \mathcal{D}_i\) supported by \(\mathcal{D}\), where \(a_i > 0\) \(\forall i\). We let \(a_0 = 1\), and then we choose the coefficients \(a_i\) rapidly decreasing when the distance in \(\Gamma\) of \(\mathcal{D}_i\) to \(\mathcal{D}_0\) increases. Then in each fiber, \(A(s) \cdot \mathcal{D}_i(s) > 0\) for \(i = 0, \ldots, n\). \(\square\)

**Corollary 2.10.** If \(\varphi : V \to S\) is a completable family of affine ruled surfaces then \(\varphi\) is an affine morphism. In particular if the base \(S\) of the family is affine so is its total space \(V\).

**Remark 2.11.** (1) Let \(\varphi : V \to S\) be a family of normal affine surfaces which admits an NC completion \((\tilde{V}, \mathcal{D})\). If some fiber of \(\varphi\) is an affine ruled surface then so is every fiber of \(\varphi\). Indeed, the dual graph \(\Gamma\) of an NC completion of an affine surface \(V\) can be transformed into a standard one \(\Gamma_{\text{std}}\). Due to Remark 1.7(1) the surface \(V\) is affine ruled if and only if \(\Gamma_{\text{std}}\) has extremal 0-vertices. Since the dual graphs \(\Gamma(s)\) of the fiber over \(s \in S\) and then also \(\Gamma_{\text{std}}(s)\) are constant the result follows.

(2) Later on we will show that, under the assumptions of (1), locally over \(S\) the total space \(V\) of the family carries a relative affine ruling. One can show that then it also carries locally a relative \(\mathbb{G}_a\)-action.

2.2. Extended divisors. In this subsection \(V\) denotes an affine ruled surface. In the sequel we use the following notation.
Notation 2.12. (a) Let $(V, D)$ be a semi-standard completion of $V$. Such a completion induces a $\mathbb{P}^1$-fibration $\tilde{\pi} : \tilde{V} \to \tilde{B}$, see Remark 1.7(1). In the case that $(V, D)$ is standard $\tilde{\pi}$ was called in [FKZ3] the standard fibration associated to $(V, D)$. Let $\pi : V \to B$ be the induced affine ruling. We note that by Lemma 1.6 any affine ruling on $V$ appears in this way.

(b) The singularities of $\tilde{V}$ are all lying in the open part $V$. Thus, if $\varrho : \tilde{V} \to \tilde{V}$ is the minimal resolution of singularities then $D$ can be considered as a divisor in $\tilde{V}$ denoted by the same letter $D$. We call $(\tilde{V}, D)$ a resolved semi-standard completion of $V$ and the induced $\mathbb{P}^1$-fibration $\tilde{\pi} := \bar{\pi} \circ \varrho : \tilde{V} \to B$ again the associated standard ruling.

(c) The horizontal section $C_1 \cong \tilde{B}$ of $\bar{\pi}$ (see Lemma 1.6), considered as a curve in $\tilde{V}$, yields a horizontal section of $\bar{\pi}$. Let $C_{0i} = \bar{\pi}^{-1}(c_{0i})$, $i = 1, \ldots, a$, denote the full fibers of $\bar{\pi}$ contained in $D$, where $c_{0i} \in \tilde{B} \setminus B$ are distinct points. They correspond to the extremal 0-vertices in $\Gamma_D$ and are all adjacent to $C_1$. The vertex $C_1$ is connected to further vertices, say $C_{21}, \ldots, C_{2b}$. The latter represent fiber components of the degenerate fibers, say, $\bar{\pi}^{-1}(c_{21}), \ldots, \bar{\pi}^{-1}(c_{2b})$ of $\bar{\pi}$. We let in the sequel

$$C_0 = \sum_{i=1}^{a} C_{0i} \quad \text{and} \quad C_2 = \sum_{j=1}^{b} C_{2j}.$$ 

Since the component $C_{2i}$ meets the section $C_1$ transversally, it has multiplicity 1 in the fiber $\bar{\pi}^{-1}(c_{2i})$. Hence for every $i = 1, \ldots, b$ the rest of the fiber $\bar{\pi}^{-1}(c_{2i}) - C_{2i}$ can be blown down to a smooth point. In this way we obtain a $\mathbb{P}^1$-ruling (i.e. a locally trivial $\mathbb{P}^1$-fibration) $\psi : X \to \tilde{B}$. Thus $\tilde{V}$ is obtained from $X$ by a sequence of blowups with centers on $C_{2i}\setminus C_1$, $i = 1, \ldots, b$, and at infinitesimally near points. To simplify notation, we keep the same letters for the curves $C_{0i}, C_1$, and $C_{2i}$ on $\tilde{V}, \tilde{V}$ and for their respective images in $X$.

In analogy with the case of Gizatullin surfaces [FKZ3], given a resolved standard completion $(V, D)$ of an affine ruled surface $V$, we associate to it the extended divisor $D_{\text{ext}}$ and its extended dual graph $\Gamma_{\text{ext}}$ of $(\tilde{V}, D)$ as follows.

Definition 2.13. The reduced divisor

$$D_{\text{ext}} = D \cup \bar{\pi}^{-1}(\{c_{21}, \ldots, c_{2b}\})$$

is called the extended divisor of $(\tilde{V}, D)$ and its dual graph $\Gamma_{\text{ext}}$ the extended graph. The connected components of $D_{\text{ext}} - D$ are called feathers, see [FKZ3] 2.3.

As shown there each feather of $D_{\text{ext}}$ is a linear chain of smooth rational curves on $\tilde{V}$

$$\tilde{\mathfrak{c}} : F_0 \to F_1 \to \cdots \to F_k,$$

where the subchain $\mathfrak{R} = \tilde{\mathfrak{c}} - F_0 = F_1 + \cdots + F_k$ (if non-empty) contracts to a cyclic quotient singularity of $V$ and $F_0$ is attached to some component $C$ of $D - C_0 - C_1$. The curve $F_0$ is called the bridge curve of $\tilde{\mathfrak{c}}$. For instance, an $A_k$-singularity on $V$ leads to an $A_k$-feather, where $\mathfrak{R}$ is a chain of $(-2)$-curves of length $k$. In particular, $V$ can have at most cyclic quotient singularities. We note that this also follows from Miyanishi’s Theorem [Mi, Lemma 1.4.4].
In the case of a smooth surface $V$ one has $k = 0$ that is, every feather is an irreducible curve. Otherwise the dual graphs of $R$ correspond to the minimal resolutions of the singularities of $V$ and are independent of the choice of a resolved completion. In contrast, the bridge curve $F_0$, its neighbor in $D$, and its self-intersection may depend on this choice.

We have the following uniqueness results for extended divisors.

**Proposition 2.14.** (a) If the standard fibrations of two standard completions $(\tilde{V}, D)$ and $(\tilde{V}', D')$ of a Gizatullin surface $V$ induce the same affine ruling $V \to \mathbb{A}^1$ then there is an isomorphism of the associated extended divisors $D_{\text{ext}} \cong D'_{\text{ext}}$ and of their extended dual graphs $\Gamma_f : \Gamma_{\text{ext}} \cong \Gamma'_{\text{ext}}$ sending $D$ to $D'$ and $\Gamma_D$ to $\Gamma_D'$.

(b) For a non-Gizatullin affine ruled surface $V$, up to an isomorphism the extended divisor $D_{\text{ext}}$ and the extended graph $\Gamma_{\text{ext}}$ do not depend on the choice of a standard completion $(\bar{V}, D)$. 

**Proof.** Performing an elementary transformation at an extremal zero vertex $C_{bi}$ of $\Gamma_D$ does not affect $D_{\text{ext}} - C_{bi}$ while $C_{bi}$ is replaced by another smooth rational curve $C'_{bi}$. Thus under such an operation $D_{\text{ext}}$ and $\Gamma_{\text{ext}}$ remain unchanged up to isomorphism. Now (a) and (b) follow from Lemma 1.6(c). We note that (a) is also contained in Lemma 5.12 in [FKZ3].

**2.15.** In the setting of Notation 2.12, the irreducible components of $D_{\text{ext}} - C_0 - C_1 - C_2$ in $\tilde{V}$ are obtained via a sequence of blowups starting from the projective ruled surface $X$ as in 2.12,

$$\tilde{V} = X_m \to X_{m-1} \to \ldots \to X_2 \to X_1 = X,$$

with centers lying over $C_2 \setminus C_1$. Every component $F$ of $D_{\text{ext}} - C_0 - C_1 - C_2$ is created by one of the blowups $X_{k+1} \to X_k$ in (5). Since feathers do not contain $(-1)$-curves besides the bridge curves, the center of this blowup is necessarily a point lying on the image of $D$ in $X_k$.

This justifies the following definition.

**Definition 2.16.** (See [FKZ3, 2.3, 2.5] or [FKZ4, 3.2.1].) Suppose that a component $C$ of $D$ is mapped onto a curve $C$ in $X_{k-1}$. If under the blowup $X_k \to X_{k-1}$ the component $F$ of $D_{\text{ext}}$ is created by a blowup on $C$ then $C$ is called a *mother component* of $F$. If the blowup takes place in the image of the point $p_F \in C$ then $p_F$ is called the *base point* of $F$.

The mother component $C$ of a component $F$ of a feather $\mathfrak{F}$ and its base point $p_F$ are uniquely determined by $F$ (see [FKZ3, Lemma 2.4(a)]). Indeed, otherwise $F$ would appear as the exceptional curve of a blowup of an intersection point $C \cap C'$ in some $X_{k}$ with $C, C' \subseteq D$, so that $D$ looses connectedness, which is impossible.

In contrast, a component $F = C$ of $D$ can have two mother components (at most). For, if $C \subseteq D$ is the result of an inner blowup of an intersection point of two curves $C'$ and $C''$ of the image of $D$ in $X_k$, then the proper transforms of $C'$ and $C''$ in $D$ are both mother components of $C$. The curves $C_{bi}$, $C_{2j}$, and $C_1$ are orphans in that they have no mother component. This distinction leads to the following definition.
Definition 2.17. Consider as before a resolved completion \((\bar{V}, D)\) of an affine ruled surface \(V\). We say that \(C \subseteq D\) is a \(-\)-component if it has two mother components in \(D\), and a \(+\)-component otherwise. In particular, the curves \(C_1, C_0,\) and \(C_2\) are \(+\)-components.

Remark 2.18. Although the sequence (5) is not unique, in general, we can replace it by a canonical one by blowing down on each step \(X_k \to X_{k-1}\) simultaneously all \((-1)\)-curves in the fibers of the induced \(\mathbb{P}^1\)-fibration \(\pi_k : X_k \to B\) different from the components of the curve \(C_2\). The latter contractions are defined correctly since, on each step, no two \((-1)\)-components in a degenerate fiber \(\pi_k^{-1}(\pi_k(C_{2j}))\) different from \(C_{2j}\) are neighbors. This new sequence now only depends on the completion \((\bar{V}, D)\).

2.3. Extended divisors in families. Let now \(\varphi : V \to S\) be a family of normal affine surfaces. In order to generalize the preceding construction to the relative case, we will assume that \(\varphi\) is completable and admits a simultaneous resolution of singularities i.e., a proper morphism \(\alpha : V' \to V\) such that the induced map \(\varphi' : V' \to S\) is smooth and \(\alpha(s) : V'(s) \to V(s)\) is a resolution of singularities for every \(s \in S\). Such a resolution will be called minimal if \(\alpha(s) : V'(s) \to V(s)\) is a minimal resolution of singularities for every \(s \in S\).

In many cases, a simultaneous resolution does not exist, see Remark 2.28 for a more thorough discussion. In the case where a simultaneous resolution does exist, it can be chosen to be minimal due to the following result.

Proposition 2.19. Let \((\bar{V}, D)\) be an NC-completion of the family \(\varphi : V \to S\). Assume that \(V\) admits a simultaneous resolution of singularities. Then locally in the \(\acute{e}tale\) topology of \(S\) there is a minimal simultaneous resolution of singularities \(\alpha : \bar{V} \to \bar{V}\).

Proof. Given a simultaneous resolution of singularities \(V' \to V\) there is an NC completion \((\bar{V}, D)\) of \(V' \to S\) by this same relative NC divisor \(\mathcal{D}\) so that \(\bar{V} \to \bar{V}\) is again a simultaneous resolution of singularities.

To make this resolution minimal let us fix a point \(s_0 \in S\). If \(\bar{V}(s_0) \to \bar{V}(s_0)\) is not a minimal resolution then there is a \((-1)\)-curve \(E\) in the fiber \(\bar{V}(s_0)\). By Lemma 2.5 near \(s_0\) there is a unique family of \((-1)\)-curves \(\mathcal{E} \to S\) in \(\mathcal{V}\) with \(\mathcal{E}(s_0) = E\). Since \(\mathcal{V}\) is a family of affine surfaces, the fibers of \(\mathcal{E}\) are contracted in \(\mathcal{V}\) and then also in \(\bar{V}\). According to Lemma 2.6(a) contracting \(\mathcal{E}\) fiberwise leads to a flat family \(\bar{V}'\) over \(S\). Replacing \(\bar{V}\) by \(\bar{V}'\) and repeating the argument, after a finite number of steps we arrive at a simultaneous resolution near \(s_0\) denoted again \(\bar{V} \to \bar{V}\), which is minimal in the fiber over \(s_0\).

We claim that near \(s_0\), the map \(\bar{V}(s) \to \bar{V}(s)\) is a minimal resolution for every \(s \in S\). Indeed, consider the relative Hilbert scheme \(\mathbb{H}_{\bar{V}/S}\). Every of its irreducible components is proper over \(S\) (see Grob2), and the \((-1)\)-curves in \(\mathbb{H}_{\bar{V}/S}\) form a constructible subset, say, \(A\). Then also the image of \(A\) in \(S\) is constructible.

Hence, if in every neighborhood of \(s_0\) there are points for which the resolution \(\bar{V}(s) \to \bar{V}(s)\) is not minimal, then there is a smooth curve \(T\), a morphism \(\gamma : T \to S\), and a point \(t_0 \in T\) with \(\gamma(t_0) = s_0\) together with a \(T\)-flat family of curves \(C \subseteq \bar{V} \times_S T\) such

\footnote{In the \(\acute{e}tale\) topology.}
that over \( t \neq t_0 \) the fiber \( C_t := C(t) \) is a \((-1)\)-curve in \( \tilde{\mathcal{V}}(\gamma(t)) \). Over \( t_0 \) we obtain a curve \( C = C(t_0) \) in \( V = \tilde{\mathcal{V}}(s_0) \).

We claim that \( C \) is contracted in \( V \). To show this, let \( p : C \to \mathcal{V} \) be the induced map. As \( C \) is flat over \( T \), the complement \( C \setminus \mathcal{C} \) is dense in \( C \) and so by continuity \( p(C \setminus \mathcal{C}) \) is dense in \( p(\mathcal{C}) \). Since for \( t \neq t_0 \) the curve \( C_t \) is contracted in \( V \) to a point, the image \( p(C \setminus \mathcal{C}) \) and then also \( p(\mathcal{C}) \) are curves in \( V \). Thus \( C \) is contracted to the point in \( p(\mathcal{C}) \) lying over \( s_0 \) in \( V \), proving the claim.

Because of \( C.D = C_t.D(t) = 0 \) for \( t \neq t_0 \) the curve \( C \) does not meet \( D \) and so is contained in \( \tilde{\mathcal{V}} \setminus D \). In particular, since \( \tilde{\mathcal{V}} \to \mathcal{V} \) is a minimal resolution of singularities every component \( C_i \) of \( C \) is smooth and \( C_i^2 \leq -2 \).

Letting \( C = \sum n_i C_i \), \( n_i > 0 \), and \( K_i \) be the canonical divisor on \( \mathcal{V}(\gamma(t)) \), we get

\[
-1 = C_t.K_t = C.K = \sum n_i C_i.K_i,
\]

where \( K = K_{t_0} \). Hence \( C_i.K \leq -1 \) for some \( i \). This is only possible if \( C_i^2 \geq -1 \), which contradicts the minimality of resolution in the fiber over \( s_0 \). Now the proof is completed.

We need the following definition.

**Definition 2.20.** Let \( \varphi : \mathcal{V} \to S \) be a flat family of normal surfaces, \( \varphi' : \tilde{\mathcal{V}} \to S \) a flat proper family of smooth surfaces and \( \mathcal{D} \subseteq \tilde{\mathcal{V}} \) a divisor. We call \((\tilde{\mathcal{V}}, \mathcal{D})\) a **resolved completion** of \( \varphi : \mathcal{V} \to S \) if the following conditions are satisfied.

(a) \( \varphi' : \mathcal{V}' := \mathcal{V} \setminus \mathcal{D} \to S \) is a minimal simultaneous resolution of singularities of \( \varphi : \mathcal{V} \to S \);

(b) \((\tilde{\mathcal{V}}, \mathcal{D})\) is a NC completion of \( \varphi' : \mathcal{V}' \to S \) as in Definition 2.3, in particular it has a constant dual graph \( \Gamma \).

We call the family \((\tilde{\mathcal{V}}, \mathcal{D})\) **standard or semi-standard** if so is \( \Gamma \).

In the next Proposition we show that one can organize the standard morphisms of Notation 2.12 into a family.

**Proposition 2.21.** Let \( \varphi : \mathcal{V} \to S \) be a family of affine ruled surfaces, which admits a resolved semi-standard completion \((\tilde{\mathcal{V}}, \mathcal{D})\) with associated map \( \tilde{\varphi} : \tilde{\mathcal{V}} \to S \). Then there exists a factorization of \( \tilde{\varphi} \) as

\[
\tilde{\mathcal{V}} \xrightarrow{\tilde{\Pi}} \mathcal{B} \xrightarrow{\pi} S,
\]

where \( \mathcal{B} \to S \) is a family of smooth complete curves and \( \tilde{\Pi} \) induces in every fiber over \( s \in S \) the standard morphism \( \tilde{\pi}_s \) from Notation 2.12.

**Proof.** The extremal zero vertices of \( \Gamma \) correspond to families of curves \( C_{0i} \subseteq \mathcal{D}, i = 1, \ldots, a \). Their common neighbor \( C_1 \subseteq \mathcal{D} \) is a family of curves over \( S \). Given a point \( s \in S \) we let \( C_s \) and \( \tilde{V}_s \) be the fibers of \( \mathcal{C} = C_{0i} \) and \( \tilde{\mathcal{V}} \) over \( s \), respectively. We consider a semi-standard \( \mathbb{P}^1 \)-fibration \( \tilde{\pi}_s : \tilde{V}_s \to \tilde{B}_s \) on \( \tilde{V}_s \) as in 2.12. The curve \( C_s \) is the full fiber \( \tilde{\pi}^{-1}(c_s) \) of \( \tilde{\pi}_s \) over some point \( c_s \in \tilde{B}_s \). The direct image sheaf \( R^1\tilde{\pi}_{ss}(\mathcal{O}_{\tilde{V}_s}(mC_s)) \) vanishes for \( m \geq 0 \), while \( \tilde{\pi}_{ss}(\mathcal{O}_{\tilde{V}_s}(mC_s)) \cong \mathcal{O}_{\tilde{B}_s}(mc_s) \). Applying the Leray spectral sequence we obtain that \( H^1(\tilde{V}_s, \mathcal{O}_{\tilde{V}_s}(mC_s)) \cong H^1(\tilde{B}_s, \mathcal{O}_{\tilde{B}_s}(mc_s)) = 0 \) for \( m \gg 0 \) and
all \( s \in S \). Hence \( R^1\tilde{\varphi}_s(\mathcal{O}_{\tilde{V}}(m\mathcal{C})) \) vanishes and is in particular locally free. By [Ha_2, Chapt. III, Theorem 12.11] the map
\[
\tilde{\varphi}_s(\mathcal{O}_{\tilde{V}}(m\mathcal{C}))_s \rightarrow H^0(\tilde{V}_s, \mathcal{O}_{\tilde{V}_s}(mC_s))
\]
is surjective for all \( s \in S \). Shrinking \( S \) we may assume that there are sections
\[
\beta_0, \ldots, \beta_N \in H^0(S, \tilde{\varphi}_s(\mathcal{O}_{\tilde{V}}(m\mathcal{C})))
\]
whose images in \( H^0(\tilde{V}_s, \mathcal{O}_{\tilde{V}_s}(m\mathcal{C})) \) generate the latter vector space for every \( s \in S \). Consider now the morphism
\[
f = ((\beta_0 : \ldots : \beta_N), \tilde{\varphi}) : \tilde{V} \rightarrow \mathbb{P}^N \times S.
\]

By construction its restriction \( f_s : \tilde{V}_s \rightarrow \mathbb{P}^N \) to the fiber over \( s \) factors through the standard morphism
\[
\tilde{V}_s \xrightarrow{\#s} \tilde{B}_s \xrightarrow{\gamma_s} \mathbb{P}^N.
\]

Here \( \gamma_s \) denotes the map given by the linear system \(|m \cdot c_s|\) on \( B_s \).

We claim that the sheaf \( \mathcal{O}_B = f_*(\mathcal{O}_{\tilde{V}}) \) is flat over \( S \). As this is a local problem, we may suppose in the proof of this claim that \( S = \text{Spec} A \) is affine. Using the vanishing of \( H^1(\tilde{V}_s, \mathcal{O}_{\tilde{V}_s}) \) for all \( s \in S \) by [Ha_2, Chapt. III, Proposition 2.10] the left exact functor
\[
M \mapsto T(M) = H^0(\tilde{V}, \mathcal{O}_\tilde{V} \otimes_A M)
\]
on finite \( A \)-modules is also right exact. Applying Proposition 12.5 in [Ha_2, Chapt. III] the natural map \( T(A) \otimes_A M \rightarrow T(M) \) is an isomorphism. Thus the functor \( M \mapsto T(A) \otimes_A M \) is exact and so \( T(A) \) is a flat \( A \)-module. Consequently
\[
\mathcal{O}_B := f_*(\mathcal{O}_{\tilde{V}})
\]
is a \( S \)-flat sheaf on \( \mathbb{P}^N \times S \), proving the claim.

The sheaf \( \mathcal{O}_B \) gives rise to a flat family of smooth curves \( B \rightarrow S \) with fibers \( \tilde{B}_s \). By construction the morphism \( \tilde{\Pi} = f : \tilde{V} \rightarrow B \) induces over each point \( s \in S \) the standard fibration from Notation 2.12, proving the Lemma. \( \square \)

**Definition 2.22.** Let \( \varphi : \mathcal{V} \rightarrow S \) be as before a family of affine ruled surfaces with a resolved semi-standard completion \((\tilde{V}, \mathcal{D})\). The morphism \( \tilde{\Pi} : \tilde{V} \rightarrow B \) onto a family of curves \( B \rightarrow S \) constructed in the proof of Proposition 2.21 by means of the linear system \(|m\mathcal{C}|\), \( m \gg 0 \), will be called the **standard morphism** associated to \((\tilde{V}, \mathcal{D})\). This is a \( \mathbb{P}^1 \)-fibration over \( B \) which induces in each fiber over \( s \in S \) the standard \( \mathbb{P}^1 \)-fibration.

2.23. The vertex of \( \Gamma \) which corresponds to \( \mathcal{C}_1 \) has neighbors given by \( \mathcal{C}_{01}, \ldots, \mathcal{C}_{0a} \) and further ones given (locally in \( S \)) by smooth families of rational curves \( \mathcal{C}_{21}, \ldots, \mathcal{C}_{2b} \subseteq \mathcal{V} \) over \( S \). The families \( \mathcal{C}_{01}, \ldots, \mathcal{C}_{0a} \) arise as preimages under \( \tilde{\Pi} \) of sections, say, \( \gamma_{0i} : S \rightarrow \mathcal{B}, i = 1, \ldots, a \), while \( \mathcal{C}_{2j} \) are projected in \( B \) to sections, say, \( \gamma_{2j} : S \rightarrow \mathcal{B}, j = 1, \ldots, b \). Moreover \( \mathcal{C}_1 \cong \mathcal{B} \) is a section of the standard morphism \( \tilde{\Pi} \). On the curve \( \tilde{B}_s = \mathcal{B}(s) \) the sections \( \gamma_{0i} \) and \( \gamma_{2j} \) yield the points \( c_{0i} = \gamma_{0i}(s) \) and \( c_{2j} = \gamma_{2j}(s) \) (cf. Definition 2.12).

The following Factorization Lemma will be useful in the sequel.
Lemma 2.24. (Factorization Lemma) Let $\mathcal{V} \to S$ be a family of affine ruled surfaces, which admits a resolved standard completion $(\tilde{\mathcal{V}}, \mathcal{D})$ (see Definition 2.20). Then locally in the étale topology of $S$ there is a factorization of the associated standard morphism $\tilde{\Pi} : \tilde{\mathcal{V}} \to \tilde{B}$ as

$$\tilde{\mathcal{V}} = \mathcal{X}_m \to \mathcal{X}_{m-1} \to \ldots \to \mathcal{X}_1 \to \mathcal{B},$$

where

(a) every morphism $\mathcal{X}_i \to \mathcal{X}_{i-1}$ is a blowup of a section $\gamma_i : S \to \mathcal{X}_{i-1}$ contained in the image of $\mathcal{D}$ in $\mathcal{X}_{i-1}$ with the exceptional divisor $\mathcal{E}_i \subseteq \mathcal{X}_i$ being a relative (-1)-curve over $S$,

(b) $\mathcal{B} \to S$ is a smooth family of complete curves over $S$, and

(c) $\mathcal{X}_1 \to \mathcal{B}$ is a locally trivial $\mathbb{P}^1$-bundle.

Proof. On the fiber over some point $s$ we consider the standard fibration

$$\tilde{\pi}_s =: \tilde{\Pi}|_{\tilde{V}_s} : \tilde{V}_s = \tilde{V}(s) \to \tilde{B}_s.$$ 

We let $C_{0i,s} = C_{0i}(s)$, $C_{1,s} = C_1(s)$, and $C_{2j,s} = C_{2j}(s)$ be the respective fibers over $s$ so that $C_{2j,s}$ is contained in a fiber $\tilde{\pi}_s^{-1}(c_{2j,s})$ over some point $c_{2j,s}$ of $\tilde{B}_s$. Blowing down the divisors $\tilde{\pi}_s^{-1}(c_{2j,s}) - C_{2j,s}$, $j = 1, \ldots, b$, we arrive at a locally trivial $\mathbb{P}^1$-bundle, say, $\Psi : \tilde{V} \to \tilde{B}$ such that $\tilde{V}$ is obtained by a sequence of blowups of $X$ as in (5) with centers on $\bigcup_{j=1}^b \Psi^{-1}(c_{2j}) \setminus C_1$ and infinitesimally near points.

In particular, in the last blowup $X_m = \tilde{V} \to X_{m-1}$ in (5) there is a (-1)-curve, say, $E_m \subseteq \tilde{V}$ which is blown down in $X_{m-1}$. By Lemma 2.25(a) near $s_0$ there is a family of (-1)-curves $\mathcal{E}_m \subseteq \tilde{V}$ inducing $E_m$ over $s_0$. Applying Lemma 2.6(a) we can blow down $\mathcal{E}_m$ and obtain a morphism $\mathcal{X}_m = \tilde{V} \to \mathcal{X}_{m-1}$ inducing $X_m \to X_{m-1}$ in the fiber over $s_0$. Repeating this procedure we arrive at a factorization (6).

It remains to prove that $\mathcal{X}_1 \to \mathcal{B}$ is a locally trivial $\mathbb{P}^1$-fibration. Let us show first that the morphism $\mathcal{X}_1 \to \mathcal{B}$ is flat. Indeed, using in every step of our construction Lemma 2.20(a) $\mathcal{X}_i \to S$ is a flat morphism. In particular $\mathcal{X}_1 \to S$ is flat. Since for every $s \in S$ also $\mathcal{X}_1(s) \to \mathcal{B}(s)$ is flat, the flatness of $\mathcal{X}_1 \to \mathcal{B}$ follows from [Ei, Corollary 6.9].

Now the fact that $\mathcal{X}_1 \to \mathcal{B}$ is a locally trivial $\mathbb{P}^1$-fibration is a consequence of Lemma 2.7. For $C_1 \to \mathcal{B}$ is an isomorphism and so the inclusion $C_1 \hookrightarrow \mathcal{X}_1$ yields a section of $\mathcal{X}_1 \to \mathcal{B}$. \[\square\]

By construction all blowups in the sequence (6) take place over disjoint sections $\gamma_{21}(S), \ldots, \gamma_{2b}(S)$ of $C_2 \setminus C_1 = \bigcup_{j=1}^b C_{2j} \setminus C_1$ over $S$ in $\mathcal{X}_1$. In analogy with the absolute case we introduce the extended divisor of $(\tilde{\mathcal{V}}, \mathcal{D})$ as

$$\mathcal{D}_{\text{ext}} = \mathcal{D} \cup \tilde{\Pi}^{-1}(\gamma_{21}(S) \cup \ldots \cup \gamma_{2b}(S))$$

(see Definition 2.13). We emphasize that this is not, in general, a relative SNC divisor in the sense of Definition 2.1 (see Example 2.27 below). However, we have the following result.

Corollary 2.25. Under the assumptions of Lemma 2.24 the extended divisor $\mathcal{D}_{\text{ext}}$ is flat over $S$, and each fiber $\mathcal{D}_{\text{ext}}(s)$ over $s \in S$ is just the extended divisor of $(\tilde{V}_s, D_s)$.

\[\text{As in Notation 2.12 we use the same letters for } C_{0i}, C_1 \text{ and for their images in } X.\]
Proof. Locally $\mathcal{D}_{ext}$ is the set of zeros of a non-zero divisor on $\tilde{V}$ that is also a non-zero divisor in each fiber. Thus the first part follows e.g. from [E3, Corollary 6.9]. The second part follows from the fact that blowing up a section of $\tilde{V} \to S$ commutes with taking the fiber over $s$. 

Remark 2.26. In analogy with the absolute case we call a component $\mathcal{A}$ of the extended divisor $\mathcal{D}_{ext}$ from (7) a boundary component if it is in $\tilde{V}$ a component of $\mathcal{D}$, and otherwise a feather component. Since the dual graph of $\mathcal{D}(s)$ is constant, $\mathcal{A}$ is a boundary or a feather component if and only if its fiber $\mathcal{A}(s)$ over some point $s \in S$ is. However the neighbouring components of $\mathcal{A}(s)$ may change in nearby fibers. We call this phenomenon “jumping” of feathers.

Let us give an example where this actually happens.

Example 2.27. Letting $V$ be the Danilov-Gizatullin surface with dual zigzag $\Gamma = [0, 0, -2, -2]$ we consider the trivial family $V = V \times S$ over $S = \mathbb{A}^1$. We construct a family of completions $(\tilde{V}, D)$ over $S$ with constant dual zigzag $\Gamma$, where the dual graph $\mathcal{D}_{ext}(s)$ of the extended divisor is not constant on $S$. Actually we will see that the dual graphs of $\mathcal{D}_{ext}(s)$, $s \neq 0$, and $\mathcal{D}_{ext}(0)$ are as follows:

\begin{equation}
\mathcal{D}_{ext}(s) : \begin{array}{cccc}
F_1 & -1 \\
0 & 0 & C_0 & C_1 \\
-2 & -2 & C_2 & C_3 \\
F_2 \\
\end{array}, \quad \mathcal{D}_{ext}(0) : \begin{array}{cccc}
F_1 & -2 \\
0 & 0 & C_0 & C_1 \\
-2 & -2 & C_2 & C_3 \\
F_2 \\
\end{array}
\end{equation}

The construction starts from the quadric $X_1 = Q = \mathbb{P}^1 \times \mathbb{P}^1$ with the $\mathbb{P}^1$-fibration given by the first projection $X_1 \to B = \mathbb{P}^1$ and with the curves $C_0 = \{\infty\} \times \mathbb{P}^1$, $C_1 = \mathbb{P}^1 \times \{\infty\}$, and $C_2 = \{0\} \times \mathbb{P}^1$.

Blowing up $X_1$ at the point $(0, 0)$ creates a feather $F_1$ on the blown up surface $X_2$. Letting $X_2 = X_2 \times \mathbb{A}^1$, $C_i = C_i \times \mathbb{A}^1$, $i = 0, 1, 2$, and $F_1 = F_1 \times \mathbb{A}^1$, we blow up $X_2$ along a “diagonal” section $\gamma_3 : S \to C_2$, which meets $F_1$ over $s = 0$ only. By this blowup in every fiber over $s$ the boundary component $C_3$ is created. Blowing up a section $\gamma_4(S) \subseteq C_3$, which does not meet $C_2 \cup F_1$, we obtain a second feather $F_2$ of $\mathcal{D}_{ext}$ on the new threefold $\tilde{V} = X_4$. Thus the feather $F_1(s)$, $s \neq 0$, jumps from $C_2$ to $C_3$ over the point $s = 0$ of the base as indicated by the dual graphs.

We end this section with a remark on the existence of a resolved completion in our setup.

Remark 2.28. (1) Let $V$ be a normal affine ruled surface with a standard completion $(\tilde{V}, D)$ and associated standard fibration $V \to \bar{B}$. Then every deformation $V$ over an Artinian germ $S$ can be extended to a deformation $(\tilde{V}, D) \to S$ of the completion $(\tilde{V}, D)$. Indeed, the infinitesimal deformations and obstructions of $(\tilde{V}, D)$ are given by

\[ T^1 = H^1(\tilde{V}, R\text{Hom}_{\tilde{V}}(\Omega^1_{\tilde{V}}(D), O_{\tilde{V}})) \quad \text{and} \quad T^2 = H^2(\tilde{V}, R\text{Hom}_{\tilde{V}}(\Omega^1_{\tilde{V}}(D), O_{\tilde{V}})) \]

respectively, see [Se]. Consider the triangle in the derived category

\[ 0 \to \Theta_{\tilde{V}}(D) \to R\text{Hom}_{\tilde{V}}(\Omega^1_{\tilde{V}}(D), O_{\tilde{V}}) \to G^* \to 0. \]
The first and the second cohomology of $G^\bullet$ are just
\[ T^1_{\text{loc}} = \text{Ext}^1_{\bar{V}}(\Omega^1_{\bar{V}}, \mathcal{O}_{\bar{V}}) \quad \text{and} \quad T^2_{\text{loc}} = \text{Ext}^2_{\bar{V}}(\Omega^1_{\bar{V}}, \mathcal{O}_{\bar{V}}), \]
which control the deformations and obstructions of the affine part, respectively (cf. \cite{Se}). Using the long exact cohomology sequence of (9) we get an exact sequence
\[ T^1 \to T^1_{\text{loc}} \to H^2(\bar{V}, \Theta_{\bar{V}}(D)) \to T^2 \to T^2_{\text{loc}}. \]
The term in the middle is dual to $\text{Hom}_{\bar{V}}(\Theta_{\bar{V}}(D), \omega_{\bar{V}})$ and so vanishes as the restriction of an element of the latter group to a generic fiber of $\bar{V} \to \bar{B}$ vanishes.

Thus by standard deformation theory (see \cite{Se}) the functor assigning to a deformation of $(\bar{V}, D)$ the deformations of the open part is formally smooth.

2) Consider, for instance, a surface $V$ which has a cyclic quotient singularity with several components in the versal deformation. Its versal family $V \to S$ admits at least formally a simultaneous completion by (1) but does not admit a simultaneous resolution. See \cite{KS} for examples of cyclic quotients for which the versal deformation space is not irreducible and does not coincide with the Artin component.

3. Deformation equivalence

3.1. Normalized extended graph and deformation equivalence. The purpose of this and the next sections is to characterize in combinatorial terms the deformation equivalence for affine ruled surfaces, which we introduce as follows.

**Definition 3.1.** We say that two normal surfaces $V$ and $V'$ are **deformations of each other** if there exists a flat family of surfaces $\varphi : V \to S$ over a connected base $S$ with the following properties.

(a) $\varphi$ admits a resolved completion (see Definition 2.20), and
(b) $V \cong V(s)$ and $V' \cong V(s')$ for some points $s, s' \in S$.

This generates an equivalence relation called **deformation equivalence**: $V \sim V'$ if and only if there exists a chain $V = V_1, \ldots, V_n = V'$ such that $V_i$ and $V_{i+1}$ are deformations of each other for every $i = 1, \ldots, n - 1$.

In the sequel $V$ will be a normal affine ruled surface. Let $(\bar{V}, D)$ be a standard completion of $V$, and let $\tilde{V} \to \bar{V}$ be the minimal resolution of singularities. The extended divisor $D_{\text{ext}}$ as in Definition 2.13 and the extended graph $\Gamma_{\text{ext}} = \Gamma(D_{\text{ext}})$ depend in general on the choice of a completion $(\bar{V}, D)$. We associate to $\Gamma_{\text{ext}}$ now another graph $N(\Gamma_{\text{ext}})$ called the **normalized extended graph** which will turn out to be independent of the choice of completion and thus is an invariant of the affine surface.

**Definition 3.2.** Given a component $C$ of $\Gamma$ we let $\delta_C$ denote the number of feather components $F$ of $\Gamma_{\text{ext}}$ with mother component $C$ (see Definitions 2.13 and 2.16). The **normalized extended graph** $\Delta = N(\Gamma_{\text{ext}})$ of $(\tilde{V}, D)$ is the weighted graph obtained from $\Gamma = \Gamma_D$ by attaching to every component $C$ of $\Gamma$ exactly $\delta_C$ extremal $(-1)$-vertices called the **feathers** of $N(\Gamma_{\text{ext}})$.

Thus $\Gamma_{\text{ext}}$ and $N(\Gamma_{\text{ext}})$ contain both $\Gamma$ as a distinguished subgraph and have the same number of feather components, and even the same number of them with a given mother component. Furthermore, every feather of $N(\Gamma_{\text{ext}})$ consists of a single extremal $(-1)$-vertex, and these vertices are in one-to-one correspondence with the feather components.
of $\Gamma_{\text{ext}}$. For instance, for the family $\left( \mathcal{X}(s), D(s) \right)$ as in Example 2.27 we have $\delta_{C_2} = \delta_{C_3} = 1$ at any point $s \in S$, so the normalized graph $\Delta$ is in both cases the dual graph on the left in $[3]$.

**Remark 3.3.** The normalized extended graph $N(\Gamma_{\text{ext}})$ can be uniquely recovered from the extended graph $\Gamma_{\text{ext}}$ via the simultaneous contraction procedure described in 2.15.

In the case of a Gizatullin surface $V$ with a standard completion $(\bar{V}, D)$ the boundary zigzag $\Gamma$ can be reversed by moving the two zeros to the other end as in (1). Since $\Gamma$ is contained in the extended divisor $\Gamma_{\text{ext}}$ and also in $\Delta = N(\Gamma_{\text{ext}})$, we can perform the same operation in $\Gamma_{\text{ext}}$ and in $\Delta$. In this way we obtain a new normalized extended graph $\Delta^\vee = N(\Gamma_{\text{ext}}^\vee)$ called the reversion of $\Delta$.

**Example 3.4.** Recall that every Danilov-Gizatullin surface $V = V_n$ (see the Introduction) can be realized as the complement of an ample section $\sigma$ in a Hirzebruch surface $\Sigma_d$ with self-intersection $\sigma^2 = n > d$. By a theorem of Danilov-Gizatullin [DG, Theorem 5.8.1] the isomorphism class of $V_n$ depends only on $n$ and not on the choice of $\sigma$ or of the concrete Hirzebruch surface $\Sigma_d$. According to Example 1.22 in [FKZ3], for every $r$ in the range $1, \ldots, n-1$ the surface $V_n$ admits a standard completion $(\bar{V}_{n,r}, D)$ with extended graph

$$
\delta_0 C_2 C_1 C_0 C_{r+1} C_{r+2} \ldots C_n \delta_1 -r \delta_0 -1
$$

where the bottom line $\Gamma_D$ corresponds to the boundary zigzag $D$, the feather $\delta_1$ consists of a single $(-r)$-component $F_1$, and $\delta_0$ of a single $(-1)$-component (for $r + 1 = n$ both feathers $\delta_0$ and $\delta_1$ are attached to $C_n$). For every $r = 1, \ldots, n$ the mother component of $F_1$ is $C_2$. Thus all these completions have the same normalized extended graph $N(\Gamma_{\text{ext},r}) = \Gamma_{\text{ext},1}$. In particular, $\Gamma_{\text{ext},1}$ is as well the normalized extended graph of the reversed standard completion $(\bar{V}_{n,r}^\vee, D^\vee)$ with the extended graph $\Gamma_{\text{ext},n-r+2}$.

In the next result we show that $N(\Gamma_{\text{ext}})$ is essentially independent of the choice of the completion $(\bar{V}, D)$ of an affine ruled surface $V$ and thus is an invariant of $V$.

**Theorem 3.5.** Let $V$ be a normal affine ruled surface.

(a) If $V$ is not Gizatullin then up to an isomorphism its normalized extended graph $N(\Gamma_{\text{ext}})$ is independent of the choice of a resolved standard completion $(\bar{V}, D)$ of $V$.

(b) If $V$ is Gizatullin then its normalized extended graph $N(\Gamma_{\text{ext}})$ is uniquely determined by $V$ up to reversion.

**Proof.** Assertion (a) follows from Corollary 2.14(b). Indeed, due to this corollary for a non-Gizatullin affine ruled surface $V$ the extended graph $\Gamma_{\text{ext}}$, and hence also the normalized extended graph $N(\Gamma_{\text{ext}})$, does not depend on the choice of a standard completion.

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6See also [CNR, FKZ5].

7Hence this graph $N(\Gamma_{\text{ext}})$ is a combinatorial invariant of the affine ruled surface $V$. 

---
In the smooth case the proof of (b) is a consequence of [FKZ 3 Corollary 3.4.3]. For the normal case we provide a proof in Theorem 3.11 below.

Now we come to the main theorem of our paper. In terms of the normalized extended graphs deformation equivalence can be characterized as follows.

**Theorem 3.6.** Two affine ruled surfaces $V$ and $V'$ with resolved standard completions $(\bar{V}, D)$ and $(\bar{V}', D')$ are deformation equivalent if and only if the following two conditions hold:

(i) The associated normalized extended graphs are isomorphic or, in the case of Gizatullin surfaces, are isomorphic up to reversion.

(ii) The horizontal curves $C_1 \subseteq D$ and $C_1' \subseteq D'$ have the same genus.

We establish the ‘only if’-part in subsection 3.3 and the ‘if’-part in Corollary 4.8 in section 4.

### 3.2. A general matching principle.

Part (b) of Theorem 3.5 follows immediately from Theorem 3.11(b) below. To formulate this theorem we introduce the following notation.

**Notation 3.7.** Let $C$ be a component of $D - C_0 - C_1$. Then $C_1$ is sitting on a unique branch $\Gamma_D$ of the tree $\Gamma_D$ at the vertex $C$; note that $D'$ contains a mother component of $C$. We let $C \cap D' = \{\infty_C\}$. For a $+$-component $C$ (see Definition 2.17) we let $C^* = C \setminus \{\infty_C\} = C \setminus D'$.

A $*$-component $C$ has two mother components, say, $C' \subseteq D'$ and $C'' \subseteq D - D'$. For such a component $C$ we let $C^* = C \setminus (D' \cup D'') = C \setminus \{\infty_C, 0_C\}$, where $D'' \neq D'$ is the branch of $D$ at $C$ containing $C''$ and $\{0_C\} = C \cap D''$.

Next we define the configuration invariant of $V$. In the particular case of smooth Gizatullin surfaces, this invariant was introduced in [FKZ §3.2].

**Definition 3.8.** (cf. [FKZ 3.2.3]) Given a component $C$ of $D - C_0 - C_1$ we consider the cycle on $C$

$$Q_C = \sum_F p_F,$$

where $F$ runs over all feather components with mother component $C$, and $p_F \in C$ is the base point of $F$ (see Definition 2.16). From the fact that every feather has just one mother component it follows that the cycle $Q_C$ is contained in $C^*$. Letting $\delta_C = \text{deg } Q_C$ be the number of feathers with mother component $C$, we consider the Hilbert scheme $\mathbb{H}_{\delta_C}(C^*)$ of subschemes of length $\delta_C$ in $C^*$, or, equivalently, of effective zero cycles of degree $\delta_C$ on $C^*$. This space is the quotient of the Cartesian power $(C^*)^{\delta_C}$ modulo the symmetric group $S_{\delta_C}$. Let $\text{Aut}(C^*, \infty_C)$ denote the group of automorphisms of the curve $C \cong \mathbb{P}^1$ fixing the point $\infty_C = C \cap D'$ and, if $C$ is a $*$-component, also the point $0_C = C \cap D''$ (see 3.7). This group acts diagonally on $(C^*)^{\delta_C}$ commuting with the $S_{\delta_C}$-action. Hence it also acts on the Hilbert scheme $\mathbb{H}_{\delta_C}(C^*)$. Let us consider the quotient

$$\mathcal{E}_{\delta_C}(C^*) = \mathbb{H}_{\delta_C}(C^*)/\text{Aut}(C^*, \infty_C).$$
The cycle $Q_C$ defines a point denoted by the same letter in the space $\mathfrak{C}_{δ_C}(C^*)$. Letting

$$\mathfrak{C}(\bar{V}, D) = \prod_{C \subseteq D-C_0-C_1} \mathfrak{C}_{δ_C}(C^*)$$

we obtain a point

$$Q(\bar{V}, D) = \{Q_C\}_{C \subseteq D-C_0-C_1} \in \mathfrak{C}(\bar{V}, D)$$
called the configuration invariant of $V$.

**Remark 3.9.** Performing in $(\bar{V}, D)$ elementary transformations with centers at the components $C_0i$ of $C_0$ we neither change $\Pi$ nor the extended divisor (except for the self-intersection index $C_i^2$) and thus leave the data $δ_C$ and $Q(\bar{V}, D)$ invariant. Hence we can define these invariants also for any semi-standard completion $(\bar{V}, D)$ of $V$ by sending the latter via elementary transformations with centers on $C_0$ into a standard completion.

In order to show that the configuration invariant is independent of the choice of a standard completion we need the following definition.

**Definition 3.10.** Let $V$ be an affine ruled surface. Given two standard completions $(\bar{V}, D)$ and $(\bar{V}', D')$ of $V$, we consider the birational map $f : \bar{V} \to \bar{V}'$, which extends the identity map of $V$. We distinguish between the following two cases.

1. If $V$ is non-Gizatullin then according to Proposition 2.14(b) the extended divisor of a standard completion is uniquely determined. In other words, $f$ induces a canonical isomorphism of extended divisors $D_{ext}$ and $D'_{ext}$. The component in $D'$ corresponding under this isomorphism to the component $C \subseteq D$ will be denoted $C^f$.

2. Assume further that $V$ is a Gizatullin surface, and let $D = C_0 \cup \ldots \cup C_n$ and $D' = C'_0 \cup \ldots \cup C'_n$ be the standard zigzags of the corresponding completions. Then $(\bar{V}', D')$ is symmetrically linked either to the completion $(\bar{V}, D)$, or to its reversion $(\bar{V}^\vee, D^\vee)$, see [FKZ4 2.2.1-2.2.2]. In the first case we set $C^f_i = C'_i$ and in the second one $C^f_i = C'^{i'}_i$, where

$$i' := n - i + 2.$$ 

If $(\bar{V}', D')$ is symmetrically linked to both completions $(\bar{V}, D)$ and $(\bar{V}^\vee, D^\vee)$, then we define $C^f_i$ to be either $C'_i$ or $C'^{i'}_i$.

The following theorem in the particular case of smooth Gizatullin surfaces is proven in [FKZ4 Proposition 3.3.1].

**Theorem 3.11. (General Matching Principle)** Let $V$ be an affine ruled surface with two standard completions $(\bar{V}, D)$ and $(\bar{V}', D')$, and let $C$ be a component of $D$ with the corresponding component $C^f$ of $D'$. Then the following hold.

1. $C$ is a $*$-component if and only if $C^f$ is;
2. there is a canonical isomorphism $C \cong C^f$ mapping $C^*$ to $(C^f)^*$ and $\infty_C$ to $\infty_{C^f}$.

Under this isomorphism the cycle $Q_C$ on $C$ is mapped onto the cycle $Q_{C^f}$ on $C^f$;
3. the induced isomorphism $\mathfrak{C}(\bar{V}, D) \cong \mathfrak{C}(\bar{V}', D')$ sends the configuration invariant $Q(\bar{V}, D)$ to $Q(\bar{V}', D')$.

---

The latter means that there is a sequence of blowups and blowdowns transforming $(\bar{V}', D')$ into one of $(\bar{V}, D)$ or $(\bar{V}^\vee, D^\vee)$ and inducing an isomorphism of the corresponding dual graphs, which can be written symmetrically as $(\gamma, \gamma^{-1})$, where $\Gamma D' \to \Gamma'$ is a birational transformation.
Proof. Clearly (c) is a consequence of (a) and (b). If $V$ is not a Gizatullin surface, then the extended divisors $D_{ext}$ and $D'_{ext}$ are isomorphic, see Proposition 2.1.14(b). Hence (a) follows in this case. Since the cycles $Q_C$ can be read off from this extended divisor, also (b) follows.

In the case of a Gizatullin surface $V$, the birational map $(\bar{V}, D) \rightsquigarrow (\bar{V}', D')$ induced by $id_V$ can be uniquely decomposed into a sequence

$$D = Z_1 \leadsto Z_2 \leadsto \cdots \leadsto Z_n = D',$$

where each $Z_i$ is a semi-standard zigzag and each step is either

(i) the reversion of a standard zigzag, or

(ii) an elementary transformation at an extremal 0-vertex,

see [DG] Theorem 1 or [FKZ4] Proposition 2.3.3] for the existence part. A transformation of type (ii) does not alter the extended divisor except for the weight $C^*_i$, so it preserves the configuration invariant. Hence we are done in this case as before. It remains to deduce (a) and (b) in the case, where $(\bar{V}', D')$ is the reversion of $(\bar{V}, D)$. This is the content of the following proposition.

Proposition 3.12. Let $V$ be a Gizatullin surface with standard completion $(\bar{V}, D)$ and standard zigzag $D = C_0 \cup \cdots \cup C_n$. Let further $(\bar{V}', D')$ be the reversed completion with the reversed standard zigzag $D' = C'_0 \cup \cdots \cup C'_n$. Then the following hold.

(a) $C_i$ is a $*$-component if and only if $C'_i$ is;

(b) there is a natural isomorphism $C_i \cong C'_i$ mapping $C^*$ to $(C')^*$ and $\infty_C$ to $\infty_{C'_i}$.

Under this isomorphism the cycle $Q_{C_i}$ is mapped onto the cycle $Q_{C'_i}$.

The proof is given in 3.19 below. Let us recall first the following notation.

Notation 3.13. Given a Gizatullin surface $V$ with boundary zigzag $D = C_0 + C_1 + \cdots + C_n$, for $t \in \{1, \ldots, n\}$ we let $D_{ext}^t$ denote the branch of $D_{ext}$ at $t^{-1}$ containing $C_t$. Moreover we let $D_{\geq t} = D \cap D_{ext}^t$ and $D_{\geq t}^t = D_{ext}^t - C_t$ (see [FKZ4] 3.2.1)).

The proof of Proposition 3.12 is similar to the proof for smooth Gizatullin surfaces given in [FKZ4] Prop. 3.3.1). As in loc. cit. the main tool is the correspondence fibration. Let us recall this notion from [FKZ4] 3.3.2-3.3.3).

Definition 3.14. Using inner elementary transformations we can move the pair of zeros in the zigzag $D = [[0, 0, w_2, \ldots, w_n]]$ several places to the right. In this way we obtain a new resolved completion $(W, D_W)$ of $V$ with boundary zigzag

$$D_W = [[w_2, \ldots, w_{t-1}, 0, 0, w_t, \ldots, w_n]]$$

for some $t \in \{2, \ldots, n+1\}$. For $t = 2$, $D_W = D = [[0, 0, w_2, \ldots, w_n]]$ is the original zigzag, while for $t = n + 1$, $D_W = D' = [[w_2, \ldots, w_n, 0, 0]]$ is the reversed one. The new zigzag $D_W$ can also be written as

$$D_W = C'_n \cup \cdots \cup C'_2 \cup C_{t-1} \cup C_t \cup \cdots \cup C_n = D_{\geq t-1} \cup D_{\geq t}^t,$$

where for all $t - 1 \leq i \leq n$ and $t' \leq j \leq n$ we identify $C_i \subseteq \bar{V}$ and $C'_j \subseteq \bar{V}'$ with their proper transforms in $W$. In $W$ we have $C^2_{t-1} = C'^2_t = 0$. Likewise in [FKZ4] 3.3.3] the map

$$\psi : W \rightarrow \mathbb{P}^1.$$
defined by the linear system $|C_{t-1}|$ on $W$ will be called the \textit{correspondence fibration} for the pair of curves $(C_t, C'_{t})$. The components $C_t$ and $C'_{t}$ represent sections of $\psi$. Their projections to the base $\mathbb{P}^1$ yield isomorphisms $C_t \cong C'_{t} \cong \mathbb{P}^1$; in what follows we identify points of all three curves under these isomorphisms.

Since the feathers of $D_{\text{ext}}$ and $D'_{\text{ext}}$ are not contained in the boundary zigzags they are not contracted in $W$. We denote their proper transforms in $W$ by the same letters. It will be clear from the context where they are considered. We observe that every such feather $\mathfrak{F}$ can be written as $\mathfrak{F} = F + \mathfrak{R}$, where $F$ is a unique component of $\mathfrak{F}$ with $F \cdot D_W \geq 1$ and $\mathfrak{R}$ is the exceptional divisor of the minimal resolution in $W$ of a cyclic quotient singularity of $V$ with $\mathfrak{R} \cdot D_W = 0$.

The following lemma is proven in [FKZ4, Lemma 3.3.4] for smooth Gizatullin surfaces; the proof in the general case is similar and can be left to the reader.

\textbf{Lemma 3.15.} Under the assumptions of Proposition 3.12 with the notation as in 3.14 the following hold.

(a) The divisor $D_{\text{ext}}^{\geq t+1}$ is contained in some fiber $\psi^{-1}(q)$, $q \in \mathbb{P}^1$. Similarly, $D'_{\text{ext}}^{\geq t'+1}$ is contained in some fiber $\psi^{-1}(q')$. The points $q$ and $q'$ are uniquely determined unless $D_{\text{ext}}^{\geq t+1}$ and $D'_{\text{ext}}^{\geq t'+1}$ are empty, respectively.

(b) A fiber $\psi^{-1}(p)$ contains at most one component $C$ not belonging to $D_{\text{ext}}^{\geq t} \cup D'_{\text{ext}}^{\geq t'}$. Such a component $C$ meets both $D_{\text{ext}}^t$ and $D'_{\text{ext}}^{t'}$.

The next lemma is crucial in the proof of Proposition 3.11 see [FKZ4, 3.3.6 and 3.3.9] for the case of a smooth Gizatullin surface. The proof in the general case is far more involved.

\textbf{Lemma 3.16.} Given a point $q \in \mathbb{P}^1$, we let $F_0, \ldots, F_k$ denote the feather components of the extended divisor $D_{\text{ext}}$ with mother component $C_t$ contained in the fiber $\psi^{-1}(q)$. Assuming that there exists at least one such component, i.e. $k \geq 0$, the following hold.

(a) With a suitable enumeration, the components $F_0, \ldots, F_k$ form a chain in $D_{\text{ext}}^{\geq t}$ contained in some feather $\mathfrak{F}$ of $D_{\text{ext}}^{\geq t}$.

(b) Let $F_0$ have the smallest distance to $C_t$ in the chain above. Then the fiber $\psi^{-1}(q)$ contains a further component $F_{k+1}$ such that $F_1, \ldots, F_{k+1}$ are all components of $D'_{\text{ext}}$ with mother component $C'_{t'}$ contained in the fiber $\psi^{-1}(q)$. The components $F_1, \ldots, F_{k+1}$ form a chain in some feather $\mathfrak{F}'$ of $D'_{\text{ext}}^{\geq t'}$.

\textbf{Proof.} All base points $p_{F_i}$ are equal since by assumption the curves $F_i$ are contained in the same fiber over $q$. Let us denote this common base point by $p \in C_t$.

To deduce (a) we perform contractions in $D_{\text{ext}}^{\geq t}$ until one of the components $F_0, \ldots, F_k$, say $F_0$, meets the first time the component $C_t$. On the contracted surface, say, $W'$, the images of $F_0, \ldots, F_k$ necessarily form a chain $[[-1,(-2)k]]$. Any curve in $W$ between $F_0$ and $F_j$ has also mother component $C_t$ and the same base point $p$ whence $F_0, \ldots, F_k$ form as well a chain in $D_{\text{ext}}$. Being connected this chain has to be contained in some feather $\mathfrak{F}$.

(b) After renumbering we may suppose that any two consecutive curves in the chain $F_0, \ldots, F_j$ meet. The dual graph of $\mathfrak{F}_0 = F_0 + \ldots + F_k$ in $W'$ is $[[-1,(-2)k]]$, where $W'$ is as in (a). Since under the map $W \to W'$ only components of the fiber $\psi^{-1}(q)$ are contracted, the $\mathbb{P}^1$-fibration $\psi : W \to \mathbb{P}^1$ factors through a $\mathbb{P}^1$-fibration $\psi' : W' \to \mathbb{P}^1$. 
The component $F_0$ meets $C_t$ in $W'$ and so it disconnects the rest of the fiber $\psi^{t-1}(q)$ from $C_t$. Hence this fiber cannot contain any component of the zigzag $D^{\geq t}$.

Being contractible in $W'$ to a smooth point $p \in C_t$ the chain $F_0$ cannot exhaust the full fiber $\psi^{t-1}(q)$. After contracting $W_0$ in $W'$ the section $C_t$ still meets the resulting fiber in one point transversally. Hence there is an extra component $F_{k+1}$ of the fiber $\psi^{t-1}(q)$ with $F_{k+1} \cdot W_0 = F_{k+1} \cdot F_k = 1$. Clearly, $F_{k+1}$ has multiplicity 1 in the fiber. Therefore the divisor $\psi^{t-1}(q) - F_k - F_{k+1}$ can be contracted on $W''$ to a smooth point. Thus we arrive at a smooth surface $W''$ still fibered over $\mathbb{P}^1$ with two $(-1)$-curves $F_k$ and $F_{k+1}$ in the fiber over $q$.

The fiber component $F_{k+1}$ when considered as a curve on $W$, cannot belong to $D^{\geq t'}$ since otherwise the feather $\mathcal{F}$ containing $\mathcal{F}_0$ would meet the boundary twice. If on $W$ the fiber $\psi^{-1}(q)$ contains an extra component $C$ as in Lemma 3.15(b), then $C$ must be contracted on $W''$. Indeed, $C$ has to meet both $D^{\geq t}$ and $D^{\geq t'}$, whereas $F_{k+1}$, $F_k$ when considered in $W$ belong neither to $D^{\geq t'}$ nor to $D^{\geq t}$.

If $F_{k+1}$ were a component of $D^{\geq t}$ on $W$ then it would also be a feather component with mother component $C_t$ and base point $p$, contradicting the maximality of the collection $\{F_0, \ldots, F_k\}$.

According to Lemma 3.15(b) $F_{k+1}$ is a component of $D^{\geq t'}_{\text{ext}}$ with mother component $C_i$ and base point $p' \in C^\vee_i$ over $q$; this can be seen from the fiber structure on the surface $W''$. Thus $F_{k+1}$ must be a component of a feather, say, $\mathcal{F}^\vee$ of $D^{\geq t'}_{\text{ext}}$.

Assume that $k > 0$. Then the feather $\mathcal{F}$ containing $\mathcal{F}_0$ can be written as $\mathcal{F} = F + \mathcal{R}$, where $F$ is the bridge curve of $\mathcal{F}$ and $\mathcal{R}$ contracts to a singular point $x$ on $V$. The image of $F_{k+1}$ on $V$, and then also that of $\mathcal{F}^\vee$, contains $x$. It follows that $\mathcal{F}^\vee = F^{\vee} + \mathcal{R}$ with $F^{\vee}$ being the bridge curve of $\mathcal{F}^\vee$. In particular, the chain $\mathcal{F}_0 - F_0 + F_{k+1} = F_1 + \ldots + F_{k+1}$ is contained in $\mathcal{F}^\vee$.

On the surface $W'$ we can contract all components of the fiber $\psi^{t-1}(q)$ except for $F_0, \ldots, F_{k+1}$. The remaining fiber $F_0 + \ldots + F_k + F_{k+1}$ has then dual graph $[-1, (-2)k, -1]$. The subchain $F_1 + \ldots + F_{k+1}$ of this fiber can be contracted on the resulting surface to the point $p' \in C^\vee_i$. Hence this is a subchain of a maximal chain of feather components of $D^{\geq t'}_{\text{ext}}$ with the same mother component $C^\vee_i$ and the same base point $p'$ as $F_{k+1}$.

If there were a further component $F^{\vee}$ of $D^{\geq t'}_{\text{ext}}$ with the same mother component and the same base point as $F_{k+1}$, then interchanging the roles of $V$ and $V^\vee$ the above reasoning would give at least $k + 2$ feather components in $D^{\geq t}$ with mother component $C_t$ and base point $q = p_{F_0}$, contradicting our assumption. Now the proof is completed.

\begin{definition}
Following [FKZ4, 3.3.7] a pair of feathers $(\mathcal{F}, \mathcal{F}^\vee)$ as in Lemma 3.16 will be called a matching pair.
\end{definition}

The next lemma is shown in [FKZ4, Lemma 3.3.10] for a smooth Gizatullin surface; the proof applies as well to an arbitrary Gizatullin surface.

\begin{lemma}
$C_t$ is a $*$-component if and only if $C^\vee_i$ is. Furthermore, in the latter case the base points $q$ and $q'$ as in Lemma 3.15(a) coincide.
\end{lemma}

Now we are ready to deduce Proposition 3.12.
3.19. **Proof of Proposition 3.12.** (a) is just Lemma 3.18. By Lemma 3.16(b), the identification \( C_t \cong C'_t \) given by the correspondence fibration \( \psi \) provides a one-to-one correspondence between the set of feathers of \( D_{\text{ext}} \) with mother component \( C_t \) and base point \( p \in C_t \) and the set of feathers of \( D'_{\text{ext}} \) with mother component \( C'_t \) and base point \( p' \in C'_t \). Moreover, the points

\[
\{ \infty_{C_t} \} = C_t \cap C_{t-1} \quad \text{and} \quad C'_t \cap C_{t-1} = C'_t \cap C'_{t-1} = \{ \infty_{C'_t} \}
\]

are identified under \( \psi \) and as well the points \( p \) and \( p' \). This leads to the equality \( 0_{C_t} = 0_{C'_t} \) and shows (b). \( \square \)

3.3. **Deformation invariance of the normalized extended graph.** The ‘only if’ part of Theorem 3.6 is an immediate consequence of the following proposition.

**Proposition 3.20.** If a family \( \pi : \mathcal{V} \to S \) of affine ruled surfaces over a connected base \( S \) admits a resolved standard completion \( (\tilde{\mathcal{V}}, \mathcal{D}) \), then the normalized extended graph \( N(\Gamma_{\text{ext}}(s)) \) of \( (\tilde{\mathcal{V}}(s), \tilde{\mathcal{D}}(s)) \) does not depend on \( s \in S \).

**Proof.** The proof is based on the Factorization Lemma 2.24. Given a point \( s_0 \in S \) the fiber \( (\tilde{\mathcal{V}}, \mathcal{D}) = (\tilde{\mathcal{V}}(s_0), \mathcal{D}(s_0)) \) of the pair \( (\mathcal{V}, \mathcal{D}) \) over \( s_0 \) is a resolved standard completion of \( \mathcal{V} \) with dual graph \( \Gamma \) independent of \( s \). According to Lemma 2.24 shrinking \( S \) to a suitable \( \acute{\text{e}} \text{tale} \) neighborhood of \( s_0 \) we can decompose \( \tilde{\mathcal{V}} \to S \) into a sequence

\[
\tilde{\mathcal{V}} = \mathcal{X}_m \to \mathcal{X}_{m-1} \to \ldots \to \mathcal{X}_1 \to \mathcal{B} \to S, \tag{11}
\]

where \( \mathcal{B} \to S \) is a smooth family of curves, \( \mathcal{X}_1 \to \mathcal{B} \) is a locally trivial \( \mathbb{P}^1 \)-fibration, and \( \mathcal{X}_i \to \mathcal{X}_{i-1} \) is a blowup of a section \( \gamma_i : S \to \mathcal{X}_{i-1} \) with exceptional set \( \mathcal{E}_i \subseteq \mathcal{X}_i \), \( i = 2, \ldots, m \). Restricting (11) to the fiber over \( s_0 \) we obtain a decomposition

\[
\tilde{\mathcal{V}} = \mathcal{X}_m \to \mathcal{X}_{m-1} \to \ldots \to \mathcal{X}_1 \to \tilde{\mathcal{B}}. \tag{12}
\]

Let now \( F \) be a component of a feather \( \mathcal{F} \) of \( D_{\text{ext}}(s_0) \) created in the blowup \( \mathcal{X}_i \to \mathcal{X}_{i-1} \) with center on the mother component \( C \subseteq D \) of \( F \). Then \( \gamma_i(s_0) \in \mathcal{D} \) is a smooth point of \( \mathcal{D} \) and so \( \gamma_i(S) \subseteq \mathcal{C} \), where \( \mathcal{C} \) is a component of \( \mathcal{D} \) near \( s_0 \) with \( \mathcal{C}(s_0) = C \). For \( s \in S \) we let \( F(s) \) be the proper transform of \( \mathcal{E}_i(s) \) in \( \tilde{\mathcal{V}}(s) \) so that \( F(s_0) = F \). As follows from the definition of a constant dual graph of a family of completions (see Definition 2.1) \( F(s) \) cannot be a component of \( \mathcal{D}(s) \). Hence this is a feather component in \( \tilde{\mathcal{V}}(s) \) with mother component \( \mathcal{C}(s) \). Since every feather of \( \tilde{\mathcal{V}}(s) \) appears in this way and the total number of components of the extended graph \( D_{\text{ext}}(s) \) stays constant, the number \( \delta_C \) of feather components in \( \tilde{\mathcal{V}}(s) \) with mother component \( \mathcal{C}(s) \) does not depend on \( s \in S \). Hence the normalized extended graph \( N(\Gamma_{\text{ext}}(s)) \) is indeed independent of \( s \in S \). \( \square \)

The above proof provides in fact a little piece more of information.

**Corollary 3.21.** The identification \( N(\Gamma_{\text{ext}}(s_0)) \cong N(\Gamma_{\text{ext}}(s)) \) as in the proof of Proposition 3.20 is independent of the factorization (11). Thus locally in \( S \) with respect to the \( \acute{\text{e}} \text{tale} \) topology there is a natural identification \( N(\Gamma_{\text{ext}}(s_0)) \cong N(\Gamma_{\text{ext}}(s)) \) for \( s \in S \).

**Proof.** Given another factorization

\[
\tilde{\mathcal{V}} = \mathcal{X}_m \to \mathcal{X}_{m-1} \to \ldots \to \mathcal{X}_1' \to \mathcal{B} \to S, \tag{13}
\]
in the first step $X_m \rightarrow X_{m-1}$ and $X'_m \rightarrow X'_{m-1}$ two families of $(-1)$-curves $\mathcal{E}$ and $\mathcal{E}'$, respectively, are contracted. These relative $(-1)$-curves are either equal, or they are disjoint in all fibers by the uniqueness part in Lemma 2.5. In the latter case it is possible to contract $\mathcal{E}$ and $\mathcal{E}'$ simultaneously. In this way we obtain a new family $\mathcal{V}$ with again two different factorizations. Using induction, these two factorizations provide the same identification of extended divisors. □

The monodromy of the base points of the feather components can be non-trivial even in a family with constant dual graph. To exclude this possibility we give the following definition.

**Definition 3.22.** Let $\mathcal{V} \to S$ be a family of affine ruled surfaces with resolved completion $(\check{\mathcal{V}}, \mathcal{D})$ and associated dual graph $\Gamma$. We call $(\check{\mathcal{V}}, \mathcal{D})$ a family of constant type $(\Delta, \Gamma)$ if there is a family of isomorphisms $\Delta \cong \Gamma_{\text{ext}}(s)$ such that for every point $s_0 \in S$ the isomorphism $N(\Gamma_{\text{ext}}(s)) \cong \Delta \cong N(\Gamma_{\text{ext}}(s_0))$ is the natural identification of Corollary 3.21 for $s$ near $s_0$.

4. Versal families of affine ruled surfaces

4.1. **Complete deformation families of surfaces.** In this subsection we construct a sufficiently big family of affine ruled surfaces admitting a resolved completion with a given normalized extended graph $\Delta$. This family is complete in the sense that every other such family can be obtained, at least locally, from this one by a base change. In particular, every individual affine ruled surface admitting a resolved completion with normalized extended graph $\Delta$ appears as a member of our family.

4.1. We let $\Gamma$ be a semi-standard tree with an extremal 0-vertex and a fixed embedding $\Gamma \hookrightarrow \Delta$ into a normalized extended weighted tree. All extremal 0-vertices say, $v_{01}, \ldots, v_{0a}$ of $\Gamma$ are joined to the same vertex $v_1$, which is uniquely determined. The subgraph $\Delta^1$ of $\Delta$ consisting of $v_1$ and all its neighbors in $\Delta$ will be called the *one-skeleton* of $\Delta$. We consider it as a weighted graph by assigning to $v_1$ the same weight as in $\Delta$, while all other vertices get the weight zero. Thus $\Delta^1$ consists of $v_1$, $v_{0i}$, $1 \leq i \leq a$, and the remaining neighbors say, $v_{21}, \ldots, v_{2b}$ of $v_1$. We assume that $\Delta$ is obtained from its 1-skeleton by a sequence of blow ups

\[(14) \quad \Delta = \Delta^m \to \Delta^{m-1} \to \ldots \to \Delta^1.\]

We suppose also that the following conditions are satisfied.

(a) If in a blowup $\Delta^i \to \Delta^{i-1}$ a vertex $v$ of $\Gamma$ is created then in the subsequent blowups we first create all the feathers of $v$, and only after that the next vertices of $\Gamma$. In other words, if $v$ and $w$ are vertices of $\Gamma$ created in the blowups $\Delta^i \to \Delta^{i-1}$ and $\Delta^j \to \Delta^{j-1}$, respectively, where $j > i$, then the vertices created in the further blowups $\Delta^k \to \Delta^{k-1}$, $k > j$, are not feathers of $v$.

(b) There is a genus $g$ assigned to $v_1$, and $v_1$ has weight $-2g$.

The construction of a versal family of surfaces associated to $\Delta$ proceeds in several steps as follows.

4.2. We let $\mathcal{M}$ denote the moduli space of marked curves $(C, p)$ of genus $g$ with $p \in C$ and with a level $l$ structure, where $l \geq 3$. The latter means that we fix a symplectic
basis of the group $H^1(C; \mathbb{Z}/l\mathbb{Z})$. It is known that $\mathcal{M}$ is irreducible, quasi-projective \cite{DM} and smooth \cite{Po}. According to Theorem 10.9 in \cite{Po} Lect. 10 there exists a universal family of curves $\mathcal{Z} \to \mathcal{M}$, which constitutes a smooth projective morphism.

**Step 0:** By \cite{Gro} Théorème 3.2] the relative Picard functor $\text{Pic}_{\mathcal{Z}/\mathcal{M}}$ is representable over $\mathcal{M}$. In view of the fact that there is a section provided by the marking, the representability of the Picard functor means that there exists a scheme $\mathcal{P}$ locally of finite type over $\mathcal{M}$ and a universal line bundle $\mathcal{L}$ over $\mathcal{Z} \times_\mathcal{M} \mathcal{P}$, see \cite{Gro} Corollaire 2.4. Letting $\mathcal{P}' \subseteq \mathcal{P}$ be the connected component corresponding to the line bundles of degree $-2g$ and $B^0 = \mathcal{Z} \times_\mathcal{M} \mathcal{P}'$, there are morphisms$^{9}$

$$X^0 = \mathbb{P}(\mathcal{O}_{\mathcal{B}^0} \otimes \mathcal{L}) \to B^0 \to S^0 = \mathcal{P'},$$

where the first one is a locally trivial $\mathbb{P}^1$-fibration and $B^0 \to S^0$ is a complete family of marked curves of genus $g$ with a level $l$ structure over $S^0$. The composition $X^0 \to S^0$ is a complete family of projective ruled surfaces over $S^0$. The marking gives rise to a section $\sigma : S^0 \to B^0$. Taking the fiber product yields a family of curves $C_{01}^0 = X^0 \times_\sigma S^0$ in $X^0$ over $S^0$. The projection $\mathcal{O}_{\mathcal{B}^0} \otimes \mathcal{L} \to \mathcal{O}_{\mathcal{B}^0}$ defines a section $C_0^1 \cong B^0$, where

$$C_0^0 = \mathbb{P}(\mathcal{O}_{\mathcal{B}^0}) \hookrightarrow X^0 = \mathbb{P}(\mathcal{O}_{\mathcal{B}^0} \otimes \mathcal{L}).$$

Thus the fiber $X_t = X^0(t)$ over a point $t \in S^0$ contains the curves $C_{01} = C_{01}^0(1)$ and $C_1 = C_0^1(t)$. Clearly $X_1$ is a $\mathbb{P}^1$-bundle over the genus $g$ curve $B = B^0(t)$, the curve $C_{01}$ is a full fiber of $X_1 \to B$ and $C_1 \cong B$ is a section. By construction $\mathcal{O}_{C_1}(C_1) \cong \mathcal{L}|B$ so $C_1$ has in $X_1$ self-intersection $C_1^2 = \deg(\mathcal{L}|B) = -2g$.

**Step 1:** Let now

$$(B^0)^{a+b-1} = B^0 \times_{S^0} \ldots \times_{S^0} B^0$$

be the $(a + b - 1)$-fold fiber product and $S^1 \subseteq (B^0)^{a+b-1}$ be the subset in the fiber over $s \in S^0$ consisting of all points

$$(p_{02}, \ldots, p_{0a}, p_{21}, \ldots, p_{2b})$$

with pairwise distinct coordinates different from $p_{01} = \sigma(s)$. By base change $S^1 \to S^0$ we obtain from (15) morphisms

$$X^1_1 \to B^1 \to S^1.$$

On $X_1^1$ we have again the families of curves $C_{01}^i = C_{01}^0 \times_{S^0} S^1$ (corresponding to the section $\sigma$) and $C_i^1 = C_i^0 \times_{S^0} S^1$. With $\pi_i : S^1 \to B^0$ being the $i$th projection the morphisms

$$\pi_i \times \text{id} : S^1 \to B^1 \subseteq B^0 \times_{S^0} S^1, \quad i = 1, \ldots, a + b - 1,$$

yield sections $\sigma_{02}, \ldots, \sigma_{0a}$ and $\sigma_{21}, \ldots, \sigma_{2b}$. The preimages of $\sigma_{ij}(S^1) \subseteq B^1$ under $X^1 \to B^1$ give rise to families of curves $C_{02}^1, \ldots, C_{0a}^1$ and $C_{21}^1, \ldots, C_{2b}^1$ in $X^1$.

**Further steps:** We construct in 1.3.4.4 below a sequence of morphisms

$$S^m \to S^{m-1} \to \ldots \to S^1,$$

which corresponds to the sequence (14). Furthermore, for each $i = 1, \ldots, m$ we define

---

$^{9}$By abuse of notation, we denote the restriction $\mathcal{L}|B^0$ still by $\mathcal{L}$. 
(a) a sequence of morphisms

\[ \mathcal{X}_i^i \to \mathcal{X}_{i-1}^i \to \ldots \to \mathcal{X}_1^i \to \mathcal{B}^i \to \mathcal{S}^i, \]

where \( \mathcal{B}^i = \mathcal{B}^{i-1} \times_{\mathcal{S}^{i-1}} \mathcal{S}^i \), and

(b) families of curves over \( \mathcal{S}^i \)

\[ \mathcal{E}_j^i \subseteq \mathcal{X}^i, \quad j \leq i, \quad \mathcal{C}_{\alpha\alpha}^i \subseteq \mathcal{X}^i, \quad 1 \leq \alpha \leq a, \quad \mathcal{C}_{1}^i \subseteq \mathcal{X}^i \quad \text{and} \quad \mathcal{C}_{2\beta}^i \subseteq \mathcal{X}^i, \quad 1 \leq \beta \leq b, \]

with the following properties.

(i) Except for the morphism \( \mathcal{X}_i^i \to \mathcal{X}_{i-1}^i \), (18) is obtained from the corresponding sequence of morphisms in level \( i - 1 \) by base change \( \mathcal{S}^i \to \mathcal{S}^{i-1} \).

(ii) \( \mathcal{E}_j^i \) is a family of \((-1)\)-curves obtained by blowing up a section of \( \mathcal{X}_{i-1}^i \to \mathcal{S}^i \).

Moreover we have \( \mathcal{E}_j^i = \mathcal{E}_j^i \times_{\mathcal{S}^i} \mathcal{S}^i \) for \( j < i \) and similarly

\[ \mathcal{C}_{\alpha\alpha}^i = \mathcal{C}_{\alpha\alpha}^i \times_{\mathcal{S}^i} \mathcal{S}^i, \quad \mathcal{C}_1^i = \mathcal{C}_1^i \times_{\mathcal{S}^i} \mathcal{S}^i \quad \text{and} \quad \mathcal{C}_{2\beta}^i = \mathcal{C}_{2\beta}^i \times_{\mathcal{S}^i} \mathcal{S}^i. \]

(iii) The divisor \( \mathcal{D}_{\text{ext}}^i \) on \( \mathcal{X}^i \) formed of the families of curves \( \mathcal{E}_j^i, \mathcal{C}_{\alpha\alpha}^i, \mathcal{C}_1^i \) and \( \mathcal{C}_{2\beta}^i \), restricts in each fiber over \( \mathcal{S}^i \) to an SNC divisor with normalized dual graph \( \Delta_i \) as in (14).

(iv) The divisor, say, \( \mathcal{D}^i \) formed of families of curves as in (ii) that correspond to vertices in \( \Gamma \), represents a family of SNC divisors with dual graph \( \Gamma \cap \Delta_i \).

4.3. Suppose that the sequences (17) and (18) are already constructed up to a step \( i < m \). Then at the step \( i + 1 \) we construct these data as follows.

(A) Suppose that in the blowup \( \Delta^{i+1} \to \Delta_i \) in (14) a feather \( v \) is created by an outer blowup of, say, \( v' \in \Gamma \cap \Delta_i \). Let \( \mathcal{C}' \) be the corresponding family of curves on \( \mathcal{X}^i \) so that \( \mathcal{C}' = \mathcal{E}_j^i \) for some \( j \leq i \), or \( \mathcal{C}' \) is one of the families of curves \( \mathcal{C}^i_{\alpha,\beta} \) for some \( \beta \). Letting

\[ S^{i+1} = \mathcal{C}' \setminus \mathcal{D}' \quad \text{with} \quad \mathcal{D}' = \mathcal{D}^i - \mathcal{C}', \]

we define \( \mathcal{X}^i_{i+1} \) to be the blowup of \( \mathcal{X}^i_i = \mathcal{X}^i_i \times_{\mathcal{S}^i} S^{i+1} \) along \( S^{i+1} \), which we consider as a subscheme of \( \mathcal{X}^i_{i+1} \) via the diagonal embedding into \( \mathcal{X}^i_i \times_{\mathcal{S}^i} S^{i+1} \). Now we let \( \mathcal{E}^i_{i+1} \) be the resulting family of \((-1)\)-curves on \( \mathcal{X}^i_{i+1} \), while the remaining curves in (b) are defined as in (ii).

(B) Suppose next that in the blowup \( \Delta^{i+1} \to \Delta_i \) in (14) a vertex \( v \) of \( \Gamma \) is created. Then we proceed as follows according to whether \( v \) has one or two mother components.

(B1) If \( v \) has one mother component, then it is created by an outer blowup of, say, \( v' \in \Gamma \cap \Delta_i \). Let \( \mathcal{C}' \) be the corresponding component of \( \mathcal{D}^i \). We let \( S^{i+1} = \mathcal{C}' \setminus \mathcal{D}' \), where \( \mathcal{D}' = \mathcal{D}^i - \mathcal{C}' \), and define now \( \mathcal{X}^i_{i+1} \) and \( \mathcal{E}^i_{i+1} \) similarly as in (A).

(B2) If \( v \) has two mother components, then it is created by an inner blowup of an edge connecting two vertices \( v', v'' \in \Gamma \cap \Delta_i \). These vertices correspond to families of curves \( \mathcal{C}' \) and \( \mathcal{C}'' \) of \( \mathcal{D}^i \), respectively. In this case we let \( S^{i+1} = S^i \), and we let \( \mathcal{X}^i_{i+1} \) be the blowup of \( \mathcal{X}^i_i \) along the section \( \mathcal{C}' \cap \mathcal{C}'' \).

4.4. On the last step \( m \) we arrive at a family of surfaces \( \mathcal{X}_j = \mathcal{X}_j^m \) over \( S = \mathcal{S}^m \) with morphisms

\[ \mathcal{X}_m \to \mathcal{X}_{m-1} \to \ldots \to \mathcal{X}_1 \to \mathcal{B} = \mathcal{B}^m \to \mathcal{S}. \]

Likewise, omitting the upper index \( m \) we obtain families of curves \( \mathcal{C}_{\alpha\alpha}, \mathcal{C}_1, \mathcal{C}_{2\beta} \) and \( \mathcal{E}_j \). These yield a relative SNC divisor \( \mathcal{D} \) over \( S \) with dual graph \( \Gamma \), and in every fiber of
\[ \mathcal{X}_m \to S \text{ over } s \in S \text{ also an extended divisor } \mathcal{D}_{\text{ext}}(s). \]

The normalized dual graph of \( \mathcal{D}_{\text{ext}}(s) \) is \( \Delta \) for each point \( s \in S \), while the dual graph of \( \mathcal{D}_{\text{ext}}(s) \) might depend on \( s \). We arrive in this way at a family of smooth quasi-projective surfaces \( \mathcal{X}_m \setminus \mathcal{D} \) over \( S \). In a final step using Proposition \[2.9\] we can contract all feather components in \( \mathcal{X}_m \setminus \mathcal{D} \) that are complete curves in \( \mathcal{X}_m \setminus \mathcal{D} \). Thus we obtain a flat family of normal affine surfaces \( \mathcal{V} \to S \) along with a resolved \((-2g)\)-standard completion \((\tilde{\mathcal{V}}, \mathcal{D}) = (\mathcal{X}_m, \mathcal{D})\).

**Definition 4.5.** Given a normalized extended tree \( \Delta \) together with a subtree \( \Gamma \) as in \[4.1\] we call
- \( S = S_{\Delta, \Gamma} \) as in \[4.4\] the presentation space of normal affine surfaces of type \((\Delta, \Gamma)\);
- \( \mathcal{V} = \mathcal{V}_{\Delta, \Gamma} \to S \) the universal family over \( S \) of normal surfaces of type \((\Delta, \Gamma)\);
- \((\tilde{\mathcal{V}}, \mathcal{D}) = (\tilde{\mathcal{V}}_{\Delta, \Gamma}, D_{\Delta, \Gamma})\) the universal resolved \((-2g)\)-standard completion of \( \mathcal{V} \to S \) of type \((\Delta, \Gamma)\).

**Remark 4.6.** 1. The construction of the presentation space \( S_{\Delta, \Gamma} \) and the universal families \((\tilde{\mathcal{V}}_{\Delta, \Gamma}, D_{\Delta, \Gamma})\) depends \( \text{à priori} \) on the order of blowups in the sequence \[1.1\]. However, the reader can easily check that different orders satisfying (a) and (b) in \[4.1\] will result in canonically isomorphic presentation spaces and families over them.

2. The graph \( \Gamma \) does not determine the normalized extended graph \( \Delta \) as in \[4.1\] even if we restrict to smooth Gizatullin surfaces. For instance, in the zigzag \([0, 0, -2, -1, -2]\) the component \( C_3 \) is a \( \ast \)-component, while in \([0, 0, -1, -2, -1]\) all of them are \( \pm \)-components. However, blowing up suitable feathers we can obtain from both zigzags the chain \( \Gamma = [[0, 0, -2, -2, -2]] \). In the case of the zigzag \([0, 0, -2, -1, -2]\) the resulting surface is an affine pseudo-plane, see [MM] or [FZ], while in the other case the result is a Danilov-Gizatullin surface.

3. In the same way, given a normalized tree, we can construct an associated universal resolved \( s \)-standard completion provided that \( s \leq -2g \). This is easily seen from the proof. If however \( s > -2g \), \( B \) is a family of complete curve of genus \( g \) and \( L \) is a line bundle on \( B \) of degree \( s \) in each fiber, then fiberwise there are non-trivial extensions \( 0 \to \mathcal{O}_{B_s} \to \mathcal{G} \to L_s \to 0 \). Even worse, such extensions cannot be organized into a reasonable moduli space, in general.

In the next result we show that the universal family from Definition \[4.5\] is complete in the sense of deformation theory.

**Proposition 4.7.** Let \( \mathcal{V}' \to S' \) be a family of affine ruled surfaces admitting a resolved \((-2g)\)-standard completion \((\tilde{\mathcal{V}}', \mathcal{D}')\). Then locally in the étale topology of \( S' \) there is a morphism \( S' \to S = S_{\Delta, \Gamma} \) such that \((\tilde{\mathcal{V}}', \mathcal{D}')\) can be obtained from the universal family \((\tilde{\mathcal{V}}, \mathcal{D})\) via a base change \( S' \to S = S_{\Delta, \Gamma} \).

**Proof.** By Proposition \[2.21\] there is a family \( B' \to S' \) of curves of genus \( g \) and the morphism \( \tilde{\Pi}' : \tilde{\mathcal{V}}' \to B \) such that for every \( s \in S \) the restriction of \( \tilde{\Pi}' \) to the fiber over \( s \) is the standard morphism. By the Factorization Lemma \[2.24\] locally there is a factorization
\[
\tilde{\mathcal{V}}' = \mathcal{X}'_n \to \mathcal{X}'_{n-1} \to \ldots \to \mathcal{X}'_1 \to B' \to S',
\]

\[\text{So } \mathcal{D}_{\text{ext}} \text{ is not, in general, a relative SNC divisor over } S.\]
where $B' \to S'$ is a smooth family of curves of genus $g$. In each step we blow up a section $S \to X'_i$, with the order of blowups corresponding to that in (14).

Locally the family $q' : B' \to S'$ admits a level $l$ structure. The position of the family of curves $C'_{0i}$ in $V'$ gives rise to a section $\sigma' : S' \to B'$ and thus to a marking. Hence it is obtained from the universal family $\mathcal{Z} \to \mathcal{M}$ in [4.2] by a base change $S' \to \mathcal{M}$. The $\mathbb{P}^1$-bundle $\varphi' : X'_1 \to B'$ possesses a section $C'_1$ and so $X'_1 \cong \mathbb{P}^1(B')$ is the projective bundle associated to the 2-bundle $G' = \varphi'_*(\mathcal{O}_{X'_1}(C'_1))$. The exact sequence

$$0 \to \mathcal{O}_{X'_1} \to \mathcal{O}_{X'_1}(C'_1) \to \mathcal{L}' := \mathcal{O}_{C'_1}(C'_1) \to 0$$

yields a sequence $0 \to \mathcal{O}_{B'} \to \mathcal{G}' \to \mathcal{L}' \to 0$. In every fiber over $s' \in S'$ the curve $C'_1 = C_1(s')$ satisfies $C'_1^2 = -2g$, see [4.1]. Hence $\mathcal{L}'$ is a family of line bundles of degree $-2g$ that is obtained from the universal family $\mathcal{L}$ from [4.2] by a base change $S' \to \mathcal{P}' = S^0$. Furthermore the sequence $0 \to \mathcal{O}_{B'} \to \mathcal{G}' \to \mathcal{L}' \to 0$ can be regarded as an element in

$$\text{Ext}^1_{\mathcal{O}_{B'}}(\mathcal{L}', \mathcal{O}_{B'}) \cong H^1(B', \mathcal{L}'^\vee).$$

Since $\deg \mathcal{L}'^\vee = 2g$ the sheaf $\mathcal{R}^1q'_*\mathcal{L}'$ vanishes. Hence locally in $S'$ the group $\text{Ext}^1$ vanishes as well. In other words, there is a splitting $\mathcal{G}' \cong \mathcal{O}_{B'} \oplus \mathcal{L}'$. It follows that $X'_i \cong X_i \times_{S^0} S'$ locally in $S'$.

Assume now that, for all $j \leq i$, $X'_j$ is already obtained from $X'_i$ by a base change $f_j : S' \to S^i$. The morphism $X'_{i+1} \to X'_i$ is then the blowup of a section, say, $\sigma : S' \to X'_i$ with exceptional set $E'_{i+1} \subseteq X'_{i+1}$, see Lemma [2.24]. Assume that in the blowup $\Delta^{i+1} \to \Delta^i$ a vertex $v$ is created. As in [1.3] we distinguish between the following cases (A) and (B).

In case (A) the vertex $v$ corresponds to a feather component of $\Delta$ and is created by an outer blowup at a unique vertex $v' \in \Gamma$. The section $\sigma$ must be contained in the corresponding family of curves, say, $C' \subseteq X'_i$, which is necessarily a component of the image, say, $D'_i$ of $D'$ in $X'_i$. The section $\sigma$ cannot meet any component of $D'_i$ different from $C'$ since otherwise in a fiber over some point $s' \in S'$ a feather is created by an inner blowup of $D'_i = D'_i(s')$. Thus this newly created feather will divide the zigzag, which is impossible.

Using the construction of $S^{i+1}$ in (A), the section $\sigma$ induces a morphism $f_{i+1} = (f_i, \sigma) : S' \to S^{i+1}$ such that $X'_{i+1} \cong X_i \times_{S^i} S'$, as desired. The argument in case (B) of [1.3] (where the newly created vertex $v$ is a vertex of $\Gamma$) is similar and so we leave it to the reader.

\begin{corollary}
Let $V'$ and $V''$ be two normal affine ruled surfaces admitting resolved standard completions $(\tilde{V}', D')$ and $(\tilde{V}'', D'')$, respectively, with the same dual graph $\Gamma$, the same normalized extended tree $\Delta$, and the same genus assigned to the vertex $v_1$ of $\Gamma$. Then $V'$ and $V''$ are deformation equivalent.
\end{corollary}

\begin{proof}
Performing elementary transformations in one of the extremal 0-vertices $C'_{0i}$ and $C''_{0i}$, we may suppose that the curves $C'_i$ and $C''_i$ have the same self-intersection index $-2g$. Then the induced completions $(\tilde{V}', D')$ and $(\tilde{V}'', D'')$ of $V'$ and $V''$, respectively, arise as fibers over certain points $s', s'' \in S$ of the corresponding complete family $(\mathcal{X}_n, D)$ over $S$. Since the moduli space of curves $\mathcal{M}$ is irreducible, also the base $S$ is. In particular, $V'$ and $V''$ are deformation equivalent, as required.
\end{proof}
4.2. The map into the configuration space. We let as before $\varphi : V \rightarrow S$ be a family of affine ruled surfaces, which admits a minimal resolved completion $(\tilde{V}, D)$ of constant type $(\Delta, \Gamma)$.

Given a point $s \in S$, we consider the completion $(\tilde{V}, D) = (\tilde{V}(s), D(s))$ of the affine surface $V = V(s)$ and the configuration invariant

$$Q(\tilde{V}, D) \in \mathcal{C} = \prod_{C \subseteq D - C_0 - C_1} \mathcal{C}_{\delta_C}(C^n),$$

see Definition 3.8. The configuration space $\mathcal{C}$ does not depend on the choice of $s \in S$ and only depends on the tree $\Delta = N(\Gamma_{\text{ext}})$ and the subtree $\Gamma = \Gamma_D$. Note that $\delta_C$ is just the number of feathers of $\Delta$ at the vertex $C$.

**Definition 4.9.** $\mathcal{C} = \mathcal{C}(\Delta, \Gamma)$ is called the configuration space associated to $(\Delta, \Gamma)$.

With these notation and assumptions we have the following result.

**Proposition 4.10.** The map $S \rightarrow \mathcal{C}$ assigning to $s \in S$ the configuration invariant of the fiber $V(s)$ is a regular morphism.

**Proof.** This follows immediately from the Factorization Lemma 2.24. Indeed, let

$$\tilde{V} = \mathcal{X}_n \rightarrow \mathcal{X}_{n-1} \rightarrow \ldots \mathcal{X}_1 \rightarrow \mathcal{B} \rightarrow S$$

be the factorization as in Lemma 2.24. In each step $\mathcal{X}_{i+1} \rightarrow \mathcal{X}_i$ the center of the blowup is a section $S \hookrightarrow \mathcal{X}_i$ of $\mathcal{X}_i \rightarrow S$. Thus restricting to the fiber over $s$ the center of blowup varies regularly with $s$. This readily implies the result. \[ \square \]

5. Moduli spaces of special Gizatullin surfaces

5.1. Special Gizatullin surfaces. In this section we discuss the existence of a coarse moduli space of affine ruled surfaces. This does not follow immediately from the construction in section 4. For instance, the moduli space of the Danilov-Gizatullin surfaces $V_n$ consists of one point, while even in simple cases as in Example 2.27, where $n = 3$, our construction leads to a multi-dimensional versal family. However, we show that in certain cases the moduli space does exist and can be cooked out using the deformation family.

Fixing a tree $\Gamma$ with an extremal 0-vertex as in 4.1 and a normalized extended tree $\Delta$, we consider families $\mathcal{V} \rightarrow S$ of normal affine surfaces over an algebraic $k$-scheme $S$ with a resolved completion $(\tilde{V}, D)$ of constant type $(\Delta, \Gamma)$. Two such families $\mathcal{X} \rightarrow S$ and $\mathcal{X}' \rightarrow S$ are isomorphic if there is an $S$-isomorphism $\mathcal{X} \xrightarrow{\sim} \mathcal{X}'$.

**Definition 5.1.** Given an algebraic variety $S$ we let $F(S)$ be the set of isomorphism classes of families $\mathcal{X} \rightarrow S$ which admit a resolved completion of constant type $(\Delta, \Gamma)$. This yields a functor

$$F = F_{\Delta, \Gamma} : \text{ASch}_k \longrightarrow \text{sets}$$

from the category of algebraic $k$-schemes into sets. For a morphism $S' \rightarrow S$ the corresponding map $F(S) \rightarrow F(S')$ is given by the fiber product.

Restricting to families of smooth surfaces $\mathcal{X} \rightarrow S$ we obtain a functor $F_s : \text{ASch}_k \longrightarrow \text{sets}$. 
In general \( F \) is not a sheaf. This means that two families \( X \to S \) and \( X' \to S \) that are locally in \( S \) isomorphic, do not need to be \( S \)-isomorphic globally. On the other hand, a representable functor is always a sheaf. Thus in order to study representability it is necessary to consider the sheaf \( \tilde{F} \) associated to \( F \). We present below concrete classes of surfaces for which \( \tilde{F} \) has a fine moduli space, which we denote by \( \mathcal{M}(\Delta, \Gamma) \).

Clearly then \( \mathcal{M}(\Delta, \Gamma) \) will be as well a coarse moduli space for \( F \).

**Example 5.2.** Consider a Danilov–Gizatullin surface \( V_n = \Sigma \) of type \( n \), where \( C \subseteq \Sigma \) is an ample section with \( C^2 = n \) in the Hirzebruch surface \( \Sigma \). According to [DG] (see also [CNR, FKZ]) \( V_n \) only depends on \( n \) and neither on \( d \) nor on the choice of the section \( C \). The normalized extended graph \( \Delta_n \) of \( V_n \) is

\[
\Delta_n : \begin{array}{ccccccc}
0 & 0 & -2 & -2 & \cdots & -2 & C_n \\
F_1 & F_0 & -1 & -1 & & & \\
C_0 & C_1 & C_2 & C_3 & \cdots & C_n
\end{array}
\]

By the Isomorphism Theorem of Danilov and Gizatullin cited in the Introduction, the coarse moduli space for such surfaces consists of a single reduced point.

**Example 5.3.** According to [FKZ] Definition 1.0.4 a *special Gizatullin surface* with invariants \((n, r, t)\) is a smooth Gizatullin surface with normalized extended graph \( \Delta = \Delta(n, r, t) \):

\[
\Delta : \begin{array}{ccccccccccc}
0 & 0 & -2 & -2 & \cdots & -2 & -2 & \{F_i\}_{i=1}^r & -2 & -2 & \cdots & -2 & -2 & C_n \\
C_0 & C_1 & C_2 & C_3 & \cdots & C_t & C_t & C_{t+1} & C_{t+1} & C_{t+1} & \cdots & C_n
\end{array}
\]

where \( n \geq 3 \) and \( \{F_i\}_{i=1}^r \) is a family of \( r \) feathers joined to \( C_t \) and consisting each one of a single \((-1)\)-curve. In the case where \( t = 2 \) or \( t = n \) the number of \((-1)\)-feathers attached to \( C_t \) is \( r + 1 \). Thus there are \( \delta_t \) feathers with mother component \( C_t \), where \( \delta_t = r + 1 \) for \( t \in \{2, n\} \) and \( \delta_t = r \) otherwise. The associated configuration space is \( \mathcal{C} = \mathcal{C}_{\delta_t}(C^*_t) \).

Assigning to \( S \) the isomorphism classes of completable families over \( S \) of special Gizatullin surfaces with invariants \((n, r, t)\) we obtain as before a moduli functor \( F \). With this notation we can reformulate Corollary 6.1.4 in [FKZ] as follows.

**Theorem 5.4.** \( \mathcal{C} := \mathcal{C}_{\delta_t}(C^*_t) \) is a coarse moduli space for \( F \).

**Proof.** By Proposition [110] there is a functorial morphism

\[
\alpha_S : F(S) \to \text{Hom}(S, \mathcal{C})
\]

As follows from Corollary 6.1.4 in [FKZ], the elements of \( F(0) \) are in one-to-one correspondence with the elements of \( \mathcal{C} \), where \( 0 \) denotes the reduced point. It remains to show that for every other space \( \mathcal{M} \) together with a functorial morphism \( \beta : F(S) \to \)
Hom(S, M) there is a unique morphism $\varphi : C \to M$ such that the diagram

\[
\begin{array}{ccc}
F(S) & \xrightarrow{\alpha} & \text{Hom}(S, C) \\
\downarrow & & \downarrow \varphi \circ \downarrow & & \downarrow \\
\text{Hom}(S, C) & \xrightarrow{\varphi^*} & \text{Hom}(S, M)
\end{array}
\]

(23)

commutes. Let $S(\Gamma)$ be the space of presentations as in 4.5 and $V(\Gamma) \to S(\Gamma)$ be the universal family. Using $\beta$ this family induces a morphism $\psi = \beta([V(\Gamma) \to S(\Gamma)]) : S(\Gamma) \to M$.

By Corollary 6.1.4 in [FKZ 4] this morphism is constant on the fibers of $S(\Gamma) \to C$. For any two elements in the fiber define the same element in $F(0)$. Using the fact that $C$ is normal and $S(\Gamma) \to C$ is surjective $\psi$ induces a morphism $\varphi : C \to M$ making the diagram (23) commutative, as required. □

5.2. Appendix: An example. Let us recall from Corollary 2.10 that for a completable family of affine ruled surfaces $V \to S$ the total space $V$ is affine if so is $S$.

We emphasize that this does not remain true if we allow degenerations of $D$ in our completable families. Namely, allowing such degenerations we show below the existence of a smooth family of affine surfaces $V \to S$ over $S = \mathbb{A}^1$ with a completion $(\overline{V}, D)$ of $V$ such that the total space $V$ is not affine and even not quasi-affine. We write in the sequel $\mathbb{A}^n_{x_1, \ldots, x_n} = \text{Spec} \mathbb{k}[x_1, \ldots, x_n]$.

Example 5.5. Similarly as in Example 2.27 we consider the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ with the zigzag $C_0 + C_1 + C_2$, where

\[
C_0 = \{\infty\} \times \mathbb{P}^1, \quad C_1 = \mathbb{P}^1 \times \{\infty\}, \quad \text{and} \quad C_2 = \{0\} \times \mathbb{P}^1.
\]

We let $V_1 \to Q$ be the blowup in $(0,0)$ to create a feather $F$, and $V_2 \to V_1$ the blowup of $F \cap C_2$ to create a boundary curve $C_3$ on $V_2$. After choosing an isomorphism $j : S \cong C_2 \setminus C_1$ we can blow up the image of the diagonal embedding $(1, j) : S \hookrightarrow S \times C_2$ on $S \times V_2$. Thus we obtain a surface $\overline{V}$ with a relative $(-1)$-curve $F'$ on it and a divisor $D = \sum_{i=0}^3 C_i$, where $C_i$ is the proper transform of $S \times C_i$ in $\overline{V}$. We let $V$ be the complement $\overline{V} \setminus D$. The extended graphs of $(\overline{V}(s), D(s))$ over $s \neq 0$ and $s = 0$ are

\[
\begin{array}{cccc}
0 & 0 & -1 & F \\
C_0 & C_1 & C_2 & C_3
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
0 & 0 & -2 & -1 \\
C_0 & C_1 & C_2 & F'
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
-2 & -1 & -2 \quad \text{and} \quad -3 \quad \text{and} \quad -2 \\
C_0 & C_1 & C_2 & C_3
\end{array}
\]

respectively, where $F'$ stands for the fiber of $F'$ over $s$.

Proposition 5.6. (a) Letting $z = y/x$ we obtain

\[H^0(V, \mathcal{O}_V) \cong \mathbb{k}[s, zx, z^2x - sz, x] \subseteq \mathbb{k}[s, x, z],\]

while $H^0(V(0), \mathcal{O}_{V(0)}) = \mathbb{k}[x, z]$.

(b) The variety $V$ is not quasi-affine.
Proof. We only give a rough sketch of the argument; details will be given elsewhere.

(a) implies (b), since the global functions on $\mathcal{V}$ all vanish on the line $x = 0$ of $\mathcal{V}(0)$.

To deduce (a) let us simplify the setup by considering $V_i':= V_i \setminus (C_0 \cup C_1)$, $i = 1, 2$, so that the restricted maps $V_i' \to V_i' \to \mathbb{A}^2$ are blowups. In coordinates $V_i' \to \mathbb{A}^2$ can be described by

$$(x_1, y_1) = \left(\frac{x}{y}, y\right),$$

where $C_2$ corresponds to the $y_1$-axis and $F$ to the $x_1$-axis.

Clearly $V_i' \setminus C_2 \cong \mathbb{A}^2_{x,z}$ with $z = y/x$ since the other chart of the blowup $V_i'$ is given by $(x, z = y/x)$. Blowing up $C_2 \cap F$ with coordinates $(0, 0)$ we obtain $V_2'$ with coordinates

$$(x_2, y_2) = \left(\frac{x_1}{y_1}, y_1\right) = \left(\frac{x}{y^2}, y\right),$$

where

$C_2$ corresponds to the $y_2$-axis and the exceptional set $C_3$ to the $x_2$-axis.

The affine surfaces $V_2 = V_i' \setminus (C_2 \cup C_3)$ and $V_1 = \mathbb{A}^2_{x,z}$ are clearly isomorphic. Next we have to blow up the product $S \times V_2'$ along the curve $\{(s, 0, s) \mid s \in S\}$, where the last two coordinates are the $(x_2, y_2)'$-coordinates as above. The ideal $I$ of this curve is given by $I = (x_2, y_2 - s)$. Blowing it up yields a 3-fold $\mathcal{V}$ with a coordinate chart $U_3 \cong \mathbb{A}^3$ and coordinates

$$(s, x_3, y_3) = \left(s, x_2, \left(\frac{y_2 - s}{x_2}\right)\right) = \left(s, x/y^2, (y^3 - sy^2)/x\right),$$

where the new exceptional set $\mathcal{F}'$ corresponds to $\{x_3 = 0\}$. By construction $\mathcal{V} = \mathcal{V}' \setminus (C_2 \cup C_3)$. Moreover $C_3$ is given on $U_3$ by $y_2 = 0$, or in $(x_3, y_3)$ coordinates as $x_3y_3 + s = 0$. The threefold $\mathcal{V}$ is equal to the union of the two coordinate charts

$$S \times \mathbb{A}^2_{x,z} \quad \text{and} \quad D(x_3y_3 + s) \subseteq S \times \mathbb{A}^2_{x_3, y_3}.$$ 

Accordingly

$$A := H^0(\mathcal{V}, \mathcal{O}_\mathcal{V}) = \mathbb{k}[s, x, z] \cap \mathbb{k}[s, x_3, y_3, (x_3y_3 + s)^{-1}].$$

Using (24) it is easy to see that $A$ contains the functions

$$s, \quad zx = x_3y_3 + s, \quad x = \frac{(zx)^2}{z^2x}, \quad z^2x - sz = \frac{z^3x^2 - sz^2x}{zx}.$$ 

It can be shown that $A$ is actually generated by them over $\mathbb{k}$. Thus the result follows. 

\[ \square \]

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