Quality of local equilibria in discrete exchange economies

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Abstract

This paper defines the notion of a local equilibrium of quality $q$, $0 \leq q \leq 1$, in a discrete exchange economy: a partial allocation and item prices that guarantee certain stability properties. The quality $q$ measures the fit between the allocation and the prices: the closer to 1 the better. This notion provides a graceful degradation for the conditional equilibria of [10]: local equilibria of quality 1 are exactly the conditional equilibria. Any local equilibrium of quality $q$ provides, without any assumption on the type of the agents' valuations, an allocation whose value is at least $\frac{q^2}{1+q}$ the optimal fractional allocation.

In an economy in which all agents' valuations are $a$-submodular, i.e., exhibit complementarity bounded by $a \geq 1$, there is always a local equilibrium of quality at least $\frac{1}{a}$. In such an economy any greedy allocation provides a local equilibrium that is at least $\frac{1}{1+a}$ the optimal fractional allocation.

Keywords: discrete exchange economies; local equilibria; item prices

1 Background

Economic theory, since Adam Smith [18], has a Leibnizian flavor: some "invisible hand" provides an efficient situation. It also has a distributed flavor: there is no central command. The means by which such a feat is achieved are prices: publicly posted prices are accepted by the agents and this enables
them to sell and buy to enhance their welfare without negotiating with other agents and such sales and purchases lead to a situation that is favorable to everybody. Léon Walras [21] proposed a formal description of the situation and the prices obtained, now called a competitive or Walrasian equilibrium. Walrasian prices equate supply and demand. Once such prices are publicly known and no one can influence them, if each agent pursues his or her own individual interest without any consideration for others, each agent will obtain exactly all he or she wants. This stands in sharp contrast with other social situations in which, for the benefit of all, each one has to give up on some of his or her wishes.

It was only Abraham Wald [20] who brought to the attention of economists the problem of proving rigorously the existence of such equilibria. Intuitively, the existence of competitive equilibria, or their enforcement in a market, depends on each trader renouncing the idea of influencing the prices and on the prices equating supply and demand. This typically happens in markets with a large number of traders.

For economies of divisible objects, the existence of a competitive equilibrium has been proved under two very different types of assumptions. Arrow, Debreu [1] and many others assume that the valuation of every trader is concave, a very restrictive and quite unintuitive assumption: one may well hesitate between a beef roast for \( x \) shekels and a lamb shoulder for \( y \) shekels but not be interested in buying half the roast and half the shoulder for \( \frac{x+y}{2} \) shekels. Their result holds for any number of traders. Aumann [2] assumes nothing about the form of the valuations of the individual traders, but assumes that there is a continuum of traders, also quite a restrictive assumption.

For economies of indivisible objects, Kelso and Crawford [14] proved, for any number of agents, the existence of a competitive equilibrium under a different assumption: they assume that the valuation of each trader is (gross) substitutes. In [15] it was shown that substitutes valuations have zero measure among all valuations and therefore, given any substitutes valuation, some arbitrarily small modification will define a valuation that is not substitutes. The substitutes assumption is therefore very restrictive. Larger families of valuations for which a Walrasian equilibrium always exists are described in [19, 13, 12] but they assume some additional structure on the set of items.

Walrasian equilibria can be considered a suitable justification of Adam Smith’s general perspective only if a typical exchange economy has a Wal-
asian equilibrium. The author does not know of any result evaluating the probability that a typical exchange economy has a Walrasian equilibrium, but the results mentioned above, and others, show that, in a typical economy, no Walrasian equilibrium is guaranteed.

Markets cannot, in general, be assumed to possess a Walrasian equilibrium. But item prices are ubiquitous in markets and we should ask what drives such prices and what is their function?

2 This paper

This paper proposes a perspective change: given an exchange economy, don’t ask whether it possesses a Walrasian equilibrium, ask how good is the best equilibrium. To this purpose the notion of an equilibrium of quality \( q \), for \( 0 \leq q \leq 1 \) will be defined in Definition 2. An equilibrium consists of a partial allocation and a price vector. Any partial allocation and any price vector form an equilibrium of quality \( q \) for some \( 0 \leq q \leq 1 \). The notion of an equilibrium developed in this work, termed a local equilibrium, generalizes the notion of a conditional equilibrium defined in [10]. A local equilibrium of quality 1, is exactly a conditional equilibrium.

One may have preferred to consider a notion of equilibrium that generalizes that of a Walrasian equilibrium, not that of a conditional (local) equilibrium. No such notion has been shown, so far, to have interesting properties. Appendix D defines such a possible notion of equilibrium and shows that, in even simple exchange economies, only equilibria of quality 0 may exist, in stark contradiction with Theorems 6 and 7 below.

Many recent works, e.g., [10, 6, 9, 3] have studied the problem of approximating the social optimum, the role that prices can play in doing so and the properties of different notions of equilibria. The general impression that one gathers is that as long as the agents’ valuations are submodular, things are reasonably well understood and one can justify the general perspective portrayed by Adam Smith, by generalizing Walras’ model. But, on the whole, such approaches have not been convincing so far when the agents’ valuations exhibit complementarities, as is often the case for real life agents. Section 7 of [15] introduced the notion of an \( a \)-submodular valuation, i.e., a valuation exhibiting complementarity bounded by \( a \), \( 1 \leq a \). The parameter \( a \) measures how far a valuation is from being submodular. This paper shows that most results obtained about exchange economies of submodular agents can
be extended to such economies of $a$-submodular agents: the strength of the result obtained depends gracefully on the parameter $a$. Since essentially any valuation is $a$-submodular for some $a$, this paper generalizes what is known for exchanges of submodular agents to almost arbitrary agents.

This paper proposes an original justification for Adam Smith’s general perspective. Each agent has an initial endowment. Agents perform simple profitable bilateral trades: two agents agree that a single item will be transferred from one agent to the other for a certain amount of money. Such trades have a double effect: first the social welfare is increased and secondly item prices become more and more publicly known. After a certain time we expect to find the economy in a state where the social welfare cannot be improved upon by transfers of a single item from an agent to another (we shall call such a situation a local optimum) and where a price is publicly known for each item. Such prices support the allocation obtained in a way to be described below that we shall call a local equilibrium. The quality $q$ of a local equilibrium measures the fit between the allocation and the prices. We shall show that any local optimum, i.e., any allocation in which no simple bilateral trade can be profitable to both the seller and the buyer, provides a local equilibrium whose quality depends on the amount of complementarity exhibited by the agents’ valuations. The less complementarity, the better the quality. We shall also show that any local equilibrium of high quality has a high social value, i.e., its value is close to the socially optimal value. The general perspective becomes: the agents trade bilaterally and in doing so item prices crystallize and all this leads to a situation in which no simple bilateral trade can be profitable for both agents and to prices making the situation a local equilibrium. In typical situations, we claim, the amount of complementarity is limited, the quality of the local equilibrium obtained is high and therefore its social value is close to optimal, but not optimal. At the end the agents are not allocated their preferred bundle at the posted prices but a bundle that cannot be improved upon by any simple bilateral trade at the posted prices. If one wants to design a market in which an increased social welfare is attained, one should design means to support trades more complex than transfers of single items, e.g., bilateral trading of bundles or multilateral trades.
3 Plan of this paper

Section 4 defines discrete exchange economies, fractional allocations and the fractional optimal allocation. Section 5 briefly recalls Walrasian equilibria and points to an original presentation of the main results about them in an Appendix. In Section 6 we define a notion of equilibrium, weaker than a Walrasian equilibrium, \( q \)-local equilibrium, indexed by a quality parameter \( q, 0 \leq q \leq 1 \). In Section 7 we show that \( q \)-local equilibria are \( \frac{q}{1+q} \)-efficient. The remainder of this paper is devoted to the defense of the following thesis: typical economies possess high quality local equilibria and bilateral trading can discover them or move the economy towards them. In Section 8 we describe and discuss examples of economies illustrating the absence of high quality local equilibria. Section 9 recalls the bounded complementarity introduced in [15]: \( a \)-submodular valuations are valuations whose degree of complementarity is bounded by the parameter \( a, 1 \leq a \). Almost all valuations are \( a \)-submodular for some \( a \). It deepens their study. Section 10 provides a characterization of \( q \)-local equilibria in \( a \)-submodular economies. Section 11 recalls the notion of a local optimum from [3]. It proves a second welfare theorem: in an \( a \)-submodular economy, any local optimum can be associated with a price vector to provide a \( \frac{1}{a} \)-local equilibrium. Therefore, in an \( a \)-submodular economy every local optimum provides a high-quality local equilibrium. In particular, if all valuations are submodular any local optimum can be associated with a 1-local equilibrium. Section 12 sharpens the results of [15] about greedy allocations in the presence of bounded complementarity. In such situations any greedy allocation method provides a local equilibrium and a good approximation of the social optimum. Section 13 deals with the case all agents’ valuations are substitutes. In this case Walrasian equilibria are exactly those 1-local equilibria that satisfy an additional condition: no agent is interested in exchanging one of his items for an item he does not possess at the given prices. Section 14 presents a list of open questions. Section 15 concludes with a reflexion on the role item prices play in exchange economies.
4 Exchange economies, allocations, fractional allocations and prices

We consider exchange economies of indivisible objects, with private values, quasi-linear utilities, no externalities, free disposal and normalization.

**Definition 1** An exchange economy is defined by a finite set of indivisible objects $X$, of size $m$, a finite set $N$, of size $n$, of agents and by $n$ valuations: the valuation $v_i : 2^X \rightarrow \mathbb{R}$, $i \in N$ describes the preferences of agent $i$. A bundle $D \subseteq X$ possesses the value $v_i(D)$ for agent $i \in N$. We shall assume that those valuations satisfy:

- **Free disposal:** $v_i(A) \leq v_i(B)$, whenever $A \subseteq B$.
- **Normalization:** $v_i(\emptyset) = 0$.

In an exchange economy a partial allocation (hereafter called allocation) is a function $f : X \rightarrow N \cup \{\text{unallocated}\}$. Item $j$ of $X$ is allocated to agent $f(j)$ or left unallocated. An allocation is *total* if no item is left unallocated. The set of items allocated to agent $i$, $f^{-1}(i)$ in allocation $f$ will be denoted by $S_i^f$. For an allocation $f$, we define its social value by: $val(f) = \sum_{i \in N} v_i(S_i^f)$. In a given exchange economy, the social value of the allocation of maximal social value is denoted $M$. We shall follow the convention that is now well established in complexity theory and say that an allocation $f$ is an $a$-approximation ($a \geq 1$) of the social optimum iff $M \leq a \cdot val(f)$.

In [5] Bikhchandani and Mamer considered fractional allocations, a generalization of the notion of an allocation. A fractional allocation consists of a nonnegative number $x_i^D$ for every $i \in N$ and every $D \subseteq X$ satisfying the constraints:

1. for any $i \in N$, 
   $$ \sum_{D \subseteq X} x_i^D \leq 1, $$  
   and  
2. for any $j \in X$, 
   $$ \sum_{i \in N} \sum_{D \subseteq X, j \in D} x_i^D \leq 1. $$
The value of a fractional allocation $x$ is defined by:

$$\text{val}(x) = \sum_{i \in N} \sum_{D \subseteq X} x_i^D v_i(D).$$

The value of the fractional allocation of maximal value will be denoted $M_F$. It is the solution of the linear program $\text{LP}$.

**Linear Programming (LP):**

Maximize

$$\sum_{i \in N} \sum_{D \subseteq X} x_i^D v_i(D) \quad (3)$$

under the constraints

$$\sum_{D \subseteq X, j \in D} x_i^D \leq 1, \text{ for all } j \in X, \quad (4)$$

$$\sum_{D \subseteq X} x_i^D \leq 1 \text{ for all } i \in N, \text{ and } \quad (5)$$

$$x_i^D \in [0, 1], \text{ for all } i \in N, D \subseteq X. \quad (6)$$

The optimal solution set to $\text{LP}$ can be viewed as the set of efficient fractional allocations. Agents may be allocated fractional bundles of the form: $\alpha \in [0, 1]^2$, as long as $\sum_{D \subseteq X} \alpha(D) \leq 1$ (this is constraint (5) ) and, for each item, the sum of the fractions of it that are allocated does not exceed one (this is constraint (4) ). Agent $i$ values the fractional bundle $\alpha$ at:

$$\sum_{D \subseteq X} \alpha(D) v_i(D).$$

Any allocation $f$ is the fractional allocation for which $x_i^D = 1$ iff $D = S_i^f$ and $x_i^D = 0$ otherwise. Clearly, in any exchange economy, $M$ is the solution of the integer version of $\text{LP}$ where $x_i^D \in \{0, 1\}$. Therefore we have $M \leq M_F$.

A price vector is a function $p : X \rightarrow \mathcal{R}_+$ that assigns a nonnegative real number (its price) to each item. The price of item $j$, $p(j)$ will be denoted $p_j$.

### 5 Walrasian equilibria

In preparation for Section 6 where another notion of equilibrium will be defined, the reader can find in Appendix A the definition and the properties of the classical notion of a Walrasian equilibrium. The presentation there is original.
In a Walrasian equilibrium every agent is allocated the bundle he prefers amongst all possible bundles, if only he considers he cannot have any influence on the prices. A Walrasian equilibrium is the best of all possible situations for each and every agent, at the publicly posted prices.

The existence of a Walrasian equilibrium is not guaranteed in general. It is only in exchange economies in which every agent has a (gross) substitutes valuation that such an equilibrium is guaranteed to exist, by a result of [14]. By [11], to any valuation that is not substitutes one may add unit-demand valuations to define an exchange economy without a Walrasian equilibrium. The family of substitutes valuations has zero measure, as shown in [15], which implies that any substitutes valuation can be approached as close as one wants by valuations that are not substitutes. Therefore one can say that Walrasian equilibria are quite rare.

This paper’s goal is to propose a less optimistic but more realistic view of the states into which economies can evolve. Agents will not find themselves in the best of all possible worlds but in a relatively good situation, a situation that cannot be improved upon easily.

6 Local equilibria

We shall now define the central notion of this paper. A local equilibrium comprises a partial allocation, a price vector and a quality parameter. The quality parameter is a real number \( q \in [0, 1] \). A local equilibrium of the highest quality, 1, is exactly a conditional equilibrium as defined in [10].

In a 1-local equilibrium, i.e., a conditional equilibrium, the allocation gives every agent a bundle with a nonnegative utility, at the given prices. In other terms no agent is willing to give his whole bundle back, given the prices, but he could, for example, prefer selling a subset of his bundle or exchanging an item \( k \) he has been allocated for an item \( l \) allocated to some other agent and pay \( p_l - p_k \). Also, no agent wishes to buy any set of items he does not own, at the given prices.

We generalize this definition by adding a quality parameter \( q \). The number \( q \) is a discount factor for the prices.

**Definition 2** Suppose an economy \( E = (N, X, v_i, i \in N) \) is given and let \( 0 \leq q \leq 1 \). A \( q \)-local equilibrium \((f, p)\) is a pair where \( f \) is a partial allocation of the items to the agents and \( p \) is a price vector that satisfy the following three conditions:
1. for any $j \in X$ such that $f(j) = \text{unallocated}$ one has $p_j = 0$,

2. **Individual Rationality** for any $i \in N$ one has

   $$v_i(S_i) \geq q \sum_{j \in S_i} p_j,$$

   (7)

3. **Outward Stability** for any $i \in N$ and any $A \subseteq X$ such that $A \cap S_i = \emptyset$ one has

   $$v_i(A | S_i) \leq \frac{1}{q} \sum_{j \in A} p_j.$$

   (8)

We are using the same parameter $q$ in Equations (7) and (8) for expediency reasons, but we could have chosen to define local equilibria by two different quality parameters since Lemma 2 below uses two different parameters. Note that **Outward Stability** is formulated for a bundle $A$, not for a single item.

It is easy to see that

1. any allocation, with any prices, provides a 0-local equilibrium,

2. if $q \leq q'$ any $q'$-local equilibrium is a $q$-local equilibrium,

3. any conditional equilibrium, and in particular any Walrasian equilibrium is a 1-local equilibrium, and

4. any 1-local equilibrium is a conditional equilibrium.

An example will show that a 1-local equilibrium need not be Walrasian.

**Example 1** Let $X = \{a, b\}$ and $N = \{1, 2\}$. Both agents are unit-value: $v_1(a) = v_2(b) = 4$, $v_1(b) = v_2(a) = 3$ and $v_i(ab) = 4$ for any $i$.

Let $f$ be the sub-optimal allocation which allocates $a$ to agent 2 and $b$ to agent 1 and let $p_a = p_b = 2$. The pair $(f, p)$ is a 1-local equilibrium that is not a Walrasian equilibrium. With the price vector $p_a = p_b = \frac{1}{2}$ the allocation $f$ provides a $\frac{1}{2}$-local equilibrium.

The question of the existence of high quality local equilibria is postponed to Section 8 and we shall now show that any $q$-local equilibrium provides a $(1 + \frac{1}{q^2})$-approximation of the optimal fractional allocation.
7 First local social welfare theorem

The first local social welfare theorem says that the (partial) allocation $f$ of any Walrasian equilibrium has maximal social value: $\text{val}(f) = M$. We shall show a similar result for $q$-local equilibria. Any $q$-local equilibrium is a $1 + \frac{1}{q^2}$-approximation, i.e., is at least $\frac{q^2}{1+q^2}$ efficient. The strength of this result is that no assumption on the valuations of the agents is necessary.

**Theorem 1 (First local social welfare theorem)** In any exchange economy, if $(f, p)$ is a $q$-local equilibrium, then $\text{val}(x) \leq (1 + \frac{1}{q^2}) \text{val}(f)$ for any fractional allocation $x$ and therefore $\text{val}(g) \leq (1 + \frac{1}{q^2}) \text{val}(f)$ for any partial allocation $g$.

**Proof:** By Lemma 2 in Appendix B, taking $a = b = \frac{1}{q}$. One sees that any 1-local equilibrium provides a 2-approximation of the fractional optimum. Theorem 1 therefore improves on Proposition 1 in [10]: a conditional equilibrium always provide a 2-approximation of the fractional optimum, not only of the integral optimum. Two examples will now suggest that Theorem 1 cannot be significantly improved. Our first example shows that, when $q = 1$ the number $2 = 1 + \frac{1}{q^2}$ cannot be improved upon.

**Example 2** Suppose $X = \{a, b\}$, $N = \{1, 2\}$, $v_1(a) = 2$, $v_1(b) = 1$, $v_2(a) = 1$, $v_2(b) = 2$ and $v_1(ab) = v_2(ab) = 2$. Both agents are additive with a budget constraint. The allocation of the item that gives $a$ to 1 and $b$ to 2 has value 4, and with the price vector $(1.5, 1.5)$ provides a Walrasian equilibrium. The allocation $g$ is therefore a fractional optimum, by Theorem 1. The allocation $f$ that gives $b$ to 1 and $a$ to 2 with price vector $(1, 1)$ is a 1-local equilibrium of value 2, a 2-approximation.

Our second example will show that Theorem 1 cannot be significantly improved upon for arbitrarily small values of $q$.

**Example 3** Consider a single item and two agents. Agent 1 values the item at 1 and agent 2 values it at $\epsilon > 0$. The allocation of the item to agent 2 with a price of $\sqrt{\epsilon}$ is a $\sqrt{\epsilon}$-local equilibrium. Theorem 1 claims the allocation is a $1 + \frac{1}{\epsilon}$-approximation of the optimal fractional allocation. The optimal fractional allocation has a value of 1 and therefore the allocation is, truly, a $\frac{1}{\epsilon}$-approximation. For $\epsilon$ close to 0 Theorem 1 cannot be significantly improved.
8 Existence

Does every exchange economy possess a 1-local equilibrium? We shall discuss two examples. Our first example is presented in [11] as an example of an economy without a Walrasian equilibrium.

**Example 4** Let \( X = \{a, b, c\} \) and \( N = \{1, 2\} \). The two agents have the same valuation: \( v_1 = v_2 \). This valuation is symmetric: its gives a zero value to any bundle of less than 2 items, a value of 3 to any bundle of two elements and a value of 4 to the set \( X \).

The optimal fractional allocation gives 1/4 of each of the three bundles of two elements to each agent: every agent gets, on the whole, 3/4 of a bundle and each item is part of four bundles, in equal parts. Its value is 4.5. The values given to the dual variables by the Dual Linear Program (see Section [C] of the Appendix) are \( \pi_1 = \pi_2 = 0 \) and \( p_a = p_b = p_c = 1.5 \). An optimal allocation gives all three items to any one of the agents and has value 4. Let \( (f, p) \) be a \( q \)-local equilibrium. It must be of one of the three following types:

1. all three items are allocated to a single agent,
2. two items are allocated to an agent and one item to the other agent, or
3. two items are allocated to an agent and the third item is unallocated.

In the first case we must have

\[
p_a + p_b + p_c \leq 4/q, \quad p_a + p_b + p_c \geq 4q, \quad p_a + p_b \geq 3q, \quad p_b + p_c \geq 3q, \quad p_a + p_c \geq 3q
\]

which implies \( q \leq \frac{2\sqrt{7}}{3} \). In the second and third case, suppose \( a \) and \( b \) are allocated to a single agent. Then, in both cases, we must have \( p_c = 0 \) and

\[
p_a + p_b \leq 3/q, \quad p_a \geq 3q, \quad p_b \geq 3q
\]

which implies \( q \leq \frac{\sqrt{3}}{2} \leq \frac{2\sqrt{3}}{3} \). But fixing

\[
p_a = p_b = p_c = \frac{9}{4\sqrt{2}}
\]

and allocating all three items to agent 1 is a \( \frac{2\sqrt{7}}{3} \)-local equilibrium. We conclude that the best quality attainable is \( q = \frac{2\sqrt{3}}{3} \). Such quality may be
obtained for the socially optimal allocation, with suitable prices, but there is no local equilibrium of quality 1. Note also that Theorem 1 guarantees for the local equilibrium just described a social value of at least $\frac{81}{34}$ which is lower than its actual social value: 4.

An exchange economy has many different $q$-local equilibria. Our next example enables us to consider the question: which of those will be attained? or which of those is the best? One can think of two general answers. First, since the prices are driving the market, one can expect the price structure to determine the allocation that fits the prices. But one could also expect the market activity to generate an allocation of high social value and the prices be determined by the allocation. The question needs further research.

**Example 5** Let $X = \{a, b, c\}$ and $N = \{1, 2, 3\}$. Let $v_1(ab), v_2(bc), v_3(ca)$ and $v_i(abc)$ for any $i$ be equal to 1 and let the values of all other bundles, for any $i$, be equal to 0.

We are interested in exploring the $q$-local equilibria of this economy. Let us, first, consider prices that can be said to be reasonable, or natural, and study the quality of local equilibria under such prices. Since the economy in Example 5 is unchanged under a permutation of the items one could perhaps expect that all item prices, at equilibrium, will be equal: $p = p_a = p_b = p_c$. One may notice that in the fractional optimum, of value $\frac{3}{2}$, the prices are equal with $p = \frac{1}{2}$. In any local equilibrium of strictly positive quality, $p > 0$, no item is unallocated, and no agent is allocated a single item or a pair of items that he or she values at 0. We conclude that in such an equilibrium all items are allocated to a single agent, which, by the way, provides a social optimum. Without loss of generality, let us assume all items are allocated to agent 1. The constraints on the price $p$ and the quality $q$ are:

$$1 \geq 3qp, \quad 1 \leq \frac{2p}{q}.$$ 

We conclude that the highest quality that can be attained by a local equilibrium of this type is $\frac{\sqrt{3}}{\sqrt{2}}$. Such quality is attained with $p = \frac{1}{\sqrt{3}}$. Note that such a price is less than $\frac{1}{2}$, the price suggested by the fractional optimum. For any inferior quality $q \leq \frac{\sqrt{3}}{\sqrt{2}}$ any price $p$ in the interval $[\frac{1}{2}, \frac{1}{\sqrt{3}}]$ will provide a $q$-local equilibrium. For $p = \frac{1}{2}$ the quality obtained is $\frac{2}{3}$.

But, aren’t there local equilibria of higher quality? Any 1-local equilibrium has, by Theorem 1, a value of at least $\frac{3}{4}$. Therefore it has value 1 and
is an optimal allocation. Without loss of generality, we shall assume that agent 1 receives \( a \) and \( b \). If item \( c \) is unallocated or is allocated to one of agents 2 or 3 we must have \( p_c = 0 \), \( 1 \geq p_a = p_c \), \( 1 \leq p_b \) and \( 1 \leq p_a \), which is impossible. Any 1-local equilibrium must allocate all items to the same agent. The constraints are:

\[
1 \geq p_a + p_b + p_c , \quad 1 \geq p_b + p_c , \quad 1 \geq p_a + p_c .
\]

There is a unique solution: \( p_a = p_b = 0 \) and \( p_c = 1 \). If we consider that the economy should attain the optimal allocation in which all items are allocated to agent 1 and ask what are the item prices that, with such an allocation, provide a local equilibrium of the highest quality we find that we can obtain the highest quality, 1, with surprising prices:

- the items \( a \) and \( b \) have a zero price notwithstanding the fact they are valued by agent 1, and
- agent 1 is ready to pay a high price for an item, \( c \), that is useless to him.

The explanation may be that item \( c \) is of interest to both agents 2 and 3 whereas items \( a \) and \( b \) are each of interest to one other agent only.

In the exchange economy of Example 5 should we expect an invisible hand to drive the market to an optimal integral solution and to prices forming a high quality local equilibrium, or should we expect this invisible hand to drive the prices of the different items to be equal?

Note that the allocation that allocates item \( a \) to agent 2, \( b \) to agent 3 and \( c \) to agent 1 has value 0 but is a local optimum (see Section 11). Theorem 1 then implies that all local equilibria based on this allocation have quality 0.

Note that, with equal prices and \( p = \frac{1}{2} \) an optimal integral solution such as giving \( \{ab\} \) to agent 1 and letting \( c \) be unallocated satisfies all but one condition to be a Walrasian equilibrium: every agent gets one of its preferred bundles at the posted prices, but \( c \) stays unallocated while its price is not zero, and therefore this is not even a local equilibrium. Should we expect to see unallocated items with positive prices?

9 Bounded complementarity

In this section we shall recall some definitions and results from [15] and prove some more. I wish to propose the thesis that most real life valuations have low
complementarity. It will be shown that most of the properties of exchange economies of submodular agents degrade gracefully with the parameter $a$ when $a$-submodular economies are considered. We shall use $v_W$ to denote the marginal valuation $v_W(A) = v(A \cup W) - v(W)$ for any disjoint bundles $W, A$. The following definition appears in [15].

**Definition 3** Let $a \geq 1$. A valuation $v$ is said to be $a$-submodular iff for any $W, A \subseteq X$, $W \cap A = \emptyset$, and for any $x \in X - W - A$

$$v_W(A \cup \{x\}) \leq v_W(A) + a v_W(x).$$

An obvious example of a valuation that is $a$-submodular for no $a$ is a valuation that values at 0 each of two items separately but values them at a strictly positive value together. Such valuations that exhibit unbounded complementarity have been considered in the literature (see, e.g., [17] ) but, in real economies, it seems that complementarity is bounded. Note that if each of the two items above have value 1 and not 0 and the pair has value 4, a considerable complementarity, the valuation is still a 3-submodular valuation.

**Theorem 2** Let $v$ be $a$-submodular. For any $S, T, A \subseteq X$, such that $S \subseteq T$ and $A \cap T = \emptyset$ we have

$$v(A \mid T) \leq a v(A \mid S).$$

**Proof:** Let $v$, $S$, $T$ and $A$ be as in the assumptions. We shall prove our claim by induction on the size of $A$. If $A = \emptyset$ the claim is obvious. For the induction step, let $x \in X - A - T$.

$$v(A \cup \{x\} \mid T) = v(A \mid T) + v(x \mid A \cup T).$$

By the induction hypothesis $v(A \mid T) \leq a v(A \mid S)$. Let $B = A \cup T$ and $C = A \cup S$. We have $C \subseteq B$ and

$$v(x \mid B) = v(C+(B-C)+x)-v(C+(B-C)) = v_C((B-C)+x)-v_C(B-C) \leq v_C(B-C) + a v_C(x) - v_C(B-C) = a v(x \mid C).$$

We conclude that

$$v(A \cup \{x\} \mid T) \leq a v(A \mid S) + a v(x \mid A \cup S) = a v(A + x \mid S).$$
Theorem 3 Let $v$ be $a$-submodular. Let $A, B$ be disjoint bundles and for each $x \in A$ let $B_x \subseteq B$. Then one has:

$$v(A \mid B) \leq a \sum_{x \in A} v(x \mid B_x).$$

Proof: By induction on the size of $A$. For $A = \emptyset$ the claim is obvious. For the induction step, let $y \in X - A - B$ and $B_y \subseteq B$. By the induction hypothesis and Theorem 2,

$$v(A + y \mid B) = v(A \mid B) + v(y \mid A \cup B) \leq a \sum_{x \in A} v(x \mid B_x) + a v(y \mid B_y) = a \sum_{x \in A + y} v(x \mid B_x).$$

The following result will be instrumental in Section 10.

Theorem 4 Let the valuation $v$ be $a$-submodular.

1. For any $A \subseteq S \subseteq X$,

$$a \sum_{j \in A} v(j \mid S - \{j\}) \geq \sum_{j \in A} v(j \mid S - A)$$

and for any $A, S \subseteq X$ such that $A \cap S = \emptyset$,

$$v(A \mid S) \leq a \sum_{j \in A} v(j \mid S).$$

Proof: Let $A = \{j_1, j_2, \ldots, j_k\}$ and let $A_i = \{j_1, \ldots, j_{i-1}\}$. We have

$$v(A \mid S - A) = \sum_{i=0}^{k-1} v(j_{i+1} \mid S - A + A_i).$$

By Theorem 2 we have $v(j_{i+1} \mid S - \{j_{i+1}\}) \leq a v(j_{i+1} \mid S - A + A_i)$ and this proves our first claim. Now, by Definition 3

$$v(A \mid S) \leq v(A_k \mid S) + a v(j_k \mid S) \leq$$

$$v(A_{k-1} \mid S) + a v(j_{k-1} \mid S) + a v(j_k \mid S) \leq \ldots \leq a \sum_{i=1}^k v(j_i \mid S).$$
10 Local equilibria in $\alpha$-submodular economies

One of the main results of this paper is: in an exchange economy in which every agent’s valuation is $\alpha$-submodular, one may weaken the requirements in Definition 2 to requirements dealing with a single item.

**Theorem 5** In any $\alpha$-submodular exchange economy, if $f$ is a partial allocation and $p$ is a price vector that satisfy

1. for any $j \in X$ such that $f(j) = \text{unallocated}$ one has $p(j) = 0$,
2. for any $i, k \in N$, $i \neq k$ and any $j \in S_i^f$ one has

$$v_k(j \mid S_k^f) \leq p_j \leq v_i(j \mid S_i^f - \{j\})$$

then the pair $(f, p)$ is a $\frac{1}{\alpha}$-local equilibrium and moreover, for any $A \subseteq S_i^f$

$$\alpha v_i(A \mid S_i^f) \geq \sum_{j \in A} p_j. \tag{9}$$

Note that Theorem 5 guarantees more than just a $\frac{1}{\alpha}$-local equilibrium. The local equilibrium obtained also satisfies Equation (9), meaning that an agent is unwilling to sell at the posted prices suitably discounted (upwards), any subset of the items allocated to him. This property reinforces the stability of the equilibrium and probably its social value.

**Proof:** Let $i \in N$ and $A \subseteq S_i^f$. By Theorem 4

$$\sum_{j \in A} v_i(j \mid S_i^f - \{j\}) \leq \alpha v_i(A \mid S_i^f - A).$$

Therefore, by our assumption, Equation (9) is satisfied and we prove the Individual Rationality property of Definition 2 by taking $A = S_i^f$. For the Outward Stability property, let $i \in N$ and $A \subseteq X$, $A \cap S_i^f = \emptyset$. By Theorem 4 and our assumption

$$v_i(A \mid S_i^f) \leq \alpha \sum_{j \in A} v_i(j \mid S_i^f) \leq \alpha \sum_{j \in A} p_j.$$

\[ \square \]

**Definition 4** Any $q$-local equilibrium satisfying Equation (4) will be called a special $q$-local equilibrium.
An example will show that Theorem 5 cannot be improved significantly.

Example 6 Suppose two items and two agents. Agent 1 values any of the items to 1 and both items to $1 + a$ ($a \geq 1$). His valuation is $a$-submodular. Agent 2 has an additive valuation: each item is valued at 1 and the whole set of two items at 2.

The allocation that gives both items to agent 2 with prices 1 to each item satisfies the assumptions of Theorem 5. It is a $\frac{2}{1+a}$-local equilibrium since $2 \leq \frac{1+a}{2}$, $1 \geq \frac{2}{1+a}$, and $1 + a \leq \frac{1+a}{2}$. When $a$ is large, $\frac{2}{1+a}$ is of the same order as $\frac{1}{a}$.

11 Local optima

We shall now recall the definition of a local optimum as presented in [3]. It formalizes the notion of an allocation that is Pareto-optimal under simple transfers of single items. We shall then show that, in an $a$-submodular exchange economy, any local optimum can be associated with a price vector to form a special $\frac{1}{a}$-local equilibrium. In [3] the authors show that, in a submodular economy, every local optimum is a 2-approximation of the fractional optimum. Theorem 6 generalizes this result.

In an exchange economy, agents trade items and they can trade in many different, sometimes complex, patterns involving a number of agents. But bilateral trades, i.e., trades between two agents seem to be most prevalent. It even seems that, typically, bilateral trades consist of one agent selling a bundle to another agent: one agent delivers a bundle and receives money, the other agent gives money and receives a bundle. Most prevalent seems to be the transfer of a single item, in exchange for money, from an agent to another one. If we limit ourselves to the consideration of such simple bilateral actions, we expect, at the long end, to find the economy in a situation in which no such bilateral trade can be profitable to both the seller and the buyer. Such situations are natural candidates for allocations that are part of some kind of equilibrium. Such a situation has been termed a local optimum in [3]. Note that no prices are involved here.

Definition 5 An total allocation $f : X \rightarrow N$ is said to be a local optimum iff for any distinct agents $i, k \in N$, $i \neq k$ and for any item $j \in S_i$ allocated
to agent $i$, one has:

$$v_i(S_i^f - \{j\}) + v_k(S_k^f \cup \{j\}) \leq v_i(S_i^f) + v_k(S_k^f),$$  \hspace{1cm} (10)

equivalently $v_i(j | S_i^f - \{j\}) \geq v_k(j | S_k^f)$.

If the allocation of items is a local optimum, in a secondary market only complex trades will be performed: transfers of bundles, exchanges, or trades involving more than two agents.

Note that any allocation that maximizes social value, i.e., any global optimum, is a local optimum. Therefore any exchange economy possesses a local optimum.

Any local optimum defines in a natural way, for each item, a set of prices: prices that support the allocation of the item to the agent it is allocated to in a second price auction.

**Definition 6** Let $f$ be a local optimum and let $j \in X$. The agent $f(j)$ is the agent to whom $j$ is allocated and therefore $v_{f(j)}(j | S_{f(j)}^f - \{j\}) \geq v_i(j | S_i^f)$ for any agent $i \neq f(j)$. We say that any number $\alpha$ such that, for any agent $i \neq f(j)$

$$v_i(j | S_i^f) \leq \alpha \leq v_{f(j)}(j | S_{f(j)}^f - \{j\})$$

is a suitable price for item $j$ given the local optimum $f$ and that any price vector $p$ such that $p_j$ is a suitable price for every item $j$ given $f$ is a supporting price vector for $f$.

Caution: the term supporting has a different meaning in [7, 10]. The following is obvious.

**Lemma 1** Every local optimum admits a supporting price vector.

In any exchange economy one can obtain a local optimum by starting from any allocation and executing a sequence of moves in which a single item is transferred from an agent to another one, if this move strictly benefits the social value. The procedure must terminate in a local optimum. The complexity of finding a local optimum has been studied in [3] and its communication complexity has been studied in [4]. It follows from results there that the sequence of moves above can be of an exponential length even for submodular economies.
Theorem 6 In an $a$-submodular exchange economy, if $f$ is a local optimum and $p$ is a supporting price vector then the pair $(f, p)$ is a special $\frac{1}{a}$-local equilibrium and $\text{val}(x) \leq (1 + a^2) \text{val}(f)$ for any fractional allocation $x$. Therefore any local optimum is a $1 + a^2$-approximation of the fractional optimum and the integrality gap of an $a$-submodular economy is less or equal to $1 + a^2$.

Proof: By Theorems 5 and 1.

The following corollary is a second welfare theorem for 1-local equilibria, i.e., conditional equilibria. It strengthens Proposition 3 and Corollary 1 of [10] very significantly: it applies to any local optimum, not only to a welfare maximizing allocation, and to $a$-submodular economies (for any $a$) not only to submodular economies.

Corollary 1 In an $a$-submodular economy, if $f$ is a local optimum, then there is a price vector $p$ such that $(f, p)$ is a $1$-local equilibrium.

Proof: By Lemma 1.

Note that, in Example 5, the valuations are not $a$-submodular for any $a$. Theorem 6 cannot be applied to the local optimum of value 0 described there.

Note that, in Example 6, the allocation of both items to agent 2 is a local optimum that is a $\frac{1+a}{2}$-approximation of the fractional optimum. It is easy to see that there is no worse local optimum and therefore every local optimum is a $\frac{1+a}{2}$-approximation of the fractional optimum. Theorem 6 claims only that every local optimum is a $1 + a^2$-approximation. I do not know of an economy for which the bound in Theorem 6 is sharp.

Note also that the $\frac{1}{a}$ bound on the quality of the local equilibrium holds for any set of supporting prices, but some vectors of supporting prices may provide local equilibria of better quality than others.

12 Greedy allocation in economies with bounded complementarity

In Section 11 we described how, in an $a$-submodular exchange economy, a local optimum defines a local equilibrium, we shall now describe a different way to obtain a local equilibrium. The family of greedy allocation algorithms introduced in [15] was claimed there to provide a $1 + a$-approximation
of the integral optimum when all the agents’ valuations are \(a\)-submodular. Such algorithms require only polynomial time. An improved result has been presented orally at [16], to the effect that, for \(a = 1\), they provide a 2-approximation of the fractional optimum. We shall now show that greedy algorithms provide a \(\frac{1}{a}\)-local equilibrium, and a \(1 + a\)-approximation of the fractional optimum. This is better than the \(1 + a^2\)-approximation guaranteed by Theorem 6. A greedy allocation can be implemented by a sequence of single-item auctions, auctioning the items separately.

A greedy allocation consists in the choice of a total ordering of the items of \(X\): \(j_1, \ldots, j_m\). An iterative process then allocates the items one by one in the order chosen: an item is allocated to the agent for which it has the highest marginal value. At stage 0 we set \(S_0^i = \emptyset\) for any \(i \in N\). At stage \(k\), for \(k = 1, \ldots, m\) we choose an agent \(i_k\) such that
\[
    v_{i_k}(j_k | S_{i_k}^{k-1}) \geq v_l(j_k | S_l^{k-1})
\]
for any agent \(l\) and set \(S_{i_k}^{k+1} = S_{i_k}^k \cup \{j_k\}\) and \(S_l^{k+1} = S_l^k\) for any agent \(l\), \(l \neq i_k\). The resulting allocation \(f\) is defined by \(f(j_k) = i_k\) for any \(k\). The procedure may be used to define a price for each of the items. The price \(p_k\) of item \(k\) is fixed, at the time \(k\) is allocated, at any value less or equal to its marginal value for the agent it is allocated to and larger or equal to its marginal value for any of the other agents. The price of item \(k\), once fixed, is never modified.

**Theorem 7** In an \(a\)-submodular exchange economy, any greedy allocation algorithm results in an allocation that, with the prices defined just above, provides a \(\frac{1}{a}\)-local equilibrium in which the discount factor in Equation (7) is 1 (not \(\frac{1}{a}\)). The allocation obtained, \(f\), is a \(1 + a\)-approximation, i.e., \(\text{val}(x) \leq (1 + a) \text{val}(f)\) for any fractional allocation \(x\).

**Proof:** The allocation provided is a total allocation, therefore condition 1 of Definition 2 is satisfied.

We show, by induction on \(k\), that, for any \(k\), \(0 \leq k \leq m\) and for any agent \(i\):
\[
    v_i(S_i^k) \geq \sum_{l \in S_i^k} p_l. \tag{11}
\]

First, for any agent \(i\):
\[
    v_i(S_i^0) = v_i(\emptyset) = 0 \geq \sum_{l \in \emptyset} p_l.
\]

Let \(i\) be the agent to which item \(k + 1\) is allocated. For any agent \(d\) different from \(i\), \(S_{d}^{k+1} = S_{d}^k\) and Equation (11) holds for \(d\) and \(k + 1\) by the induction
hypothesis. Since the price of $k + 1$, $p_{k+1}$ is less or equal to item $k + 1$’s marginal value for $i$

$$
v_i(S_i^{k+1}) = v_i(S_i^{k}) + v_i(k + 1 | S_i^{k}) \geq \sum_{l \in S_i^{k}} p_l + p_{k+1} = \sum_{l \in S_i^{k+1}} p_l.
$$

We conclude that the final allocation satisfies Individual Rationality with $q = 1$.

We now want to show that for any agent $i$, any stage $k$ and any bundle $A$ of already allocated items, $A \subseteq \bigcup_{l \in N} S_l^k$, $A \cap S_i^k = \emptyset$ we have

$$v_i(A | S_i^k) \leq a \sum_{l \in A} p_l.
$$

(12)

After the allocation of item $k$, we only need to check the two cases below.

- For the agent $i$ to whom $k$ has been allocated. A bundle $A$ such that $A \cap S_i^{k+1} = \emptyset$ does not include $k$. We have, by Theorem $3$

$$v_i(A | S_i^k \cup \{k\}) \leq a \sum_{x \in A} v_i(x | S_i^{r(x)}) \leq a \sum_{x \in A} p_x
$$

where $r(x)$ is the stage at which item $x$ has been allocated.

- For any other agent $j$ for any $A$ that includes $k$. By the induction hypothesis, Theorem $2$ and the choice of $p_k$

$$v_j(A' \cup \{k\} | S_j) = v_j(A' | S_j) + v_j(k | S_j \cup A') \leq
$$

$$a \sum_{l \in A'} p_l + a v_j(k | S_j) \leq a \sum_{l \in A'} p_l + a p_k = a \sum_{l \in A} p_l.
$$

We have shown that the greedy allocation, together with the prices defined by the greedy process satisfy the conditions of Theorem $2$ with $b = 1$. Our claims now follow from the theorem.

Note that the local equilibrium obtained is not, in general, a special local equilibrium (with discount factor equal to 1) since the marginal value of item $k$ for the agent to whom it has been allocated is different in the final allocation from what it was at the time $k$ was allocated and $p_k$ was set. This marginal value may have decreased and may now be smaller than $p_k$.

The following follows immediately from Theorem $7$ and generalizes a result of $8$ for submodular economies.
**Corollary 2** In an $a$-submodular economy the integral gap is at most $1 + a$.

The following example shows that a greedy allocation does not always provide a special local optimum even in submodular economies.

**Example 7** Consider two items $a$, $b$ and two agents 1, 2. Let $v_1(a) = v_1(b) = 5$, $v_1(ab) = 7$ and $v_2(a) = 4$, $v_2(b) = 1$ and $v_2(ab) = 5$.

Both valuations are submodular. If, in a greedy allocation, $a$ is allocated before $b$, agent 1 is allocated $a$ and $b$. But, then, $v_1(a | b) = 2 < 4 = v_2(a)$.

## 13 Substitutes economies

In an exchange economy in which all agents have a substitutes valuation, one may pinpoint exactly which of the 1-local equilibria are Walrasian: if no agent is interested in exchanging, at the posted prices, an item allocated to him for an item not in his possession.

**Theorem 8** In an exchange economy in which all agents have substitutes valuations, $(f, p)$ is a 1-local equilibrium such that for any $i \in N$, any $j \in S^f_i$ and any $k \in X - S^f_i$ one has $v_i(S^f_i) - v_i(S^f_i - j + k) \geq p_j - p_k$, iff $(f, p)$ is a Walrasian equilibrium.

**Proof:** The if part is obvious and does not need the substitutes assumption. For the only if part assume $v_i$ is substitutes for any agent $i$ and that $p$ is a price vector. The valuation $u_i(A) = v_i(A) - \sum_{j \in A} p_j$ is also substitutes. The assumptions ensure that, for every $i \in N$, $u_i(S^f_i) \geq u_i(A)$ for any $A \subseteq X$ such that the size of the symmetric difference $S^f_i \Delta A$ is less or equal to 2. The single improvement condition shown to be equivalent to the substitutes property in Theorem 1 of [11] implies that $S^f_i$ maximizes $u_i$ over all subsets of $X$. We conclude that $(f, p)$ is a Walrasian equilibrium.

## 14 Summary and open questions

This paper proposes the notion of a $q$-local equilibrium to understand the role of item prices in discrete exchange economies. It focuses on such economies in which all agents have an $a$-submodular valuation. Two different processes that build local equilibria have been put in evidence. The first one is based on
simple bilateral trades and seems close to the way real markets function. It provides a local optimum with a price vector that is defined by the allocation. This allocation is a $1 + a^2$-approximation of the fractional optimum. The second one is greedy allocation with historical prices, prices corresponding to the moment the item has been allocated. It may resemble the birth of a market accommodating more and more items. This allocation is a $1 + a$-approximation of the fractional optimum. The discrepancy in the quality of approximations requires further study. Do random greedy allocations really provide higher social value than the local optima obtained from random initial allocations by sequences of simple bilateral trades? How do simple bilateral trades perform on initial allocations that are already the result of a greedy process?

The following questions require for further research. Can the $1 + a^2$-approximation for any local optimum be improved? At the moment no $a$-submodular economy with a local optimum that is only a $1 + a^2$-approximation is known, for $a > 1$. Can one prove a better approximation result for a restricted class of local optima, e.g., special local optima or Pareto optimal allocations under all bilateral trades? How prevalent can the absence of a 1-local equilibrium be? Can the optimal allocation always support prices that exhibit the highest quality local equilibrium? Could it be that most typical economies have a 1-local equilibrium? Are local equilibria typically stable, i.e., does a small change in valuations or in prices bring only a small change in the local equilibrium? Some high quality local equilibria, as in Example 5, seem surprising. Do all 1-local equilibria have economic significance? What is the dynamics of the revelation of such equilibrium prices?

The results presented in this paper do not depend on the number of agents or items. There is, I think, a general feeling that a better equilibrium can be reached in an a large economy, i.e., an economy in which a large number of agents actively participate. Could it be that the approximation obtained by any local optimum in which a large number of agents are allocated a non-empty bundle is better than the one promised in Theorem 6?

As noticed in [15] maximizing social welfare in a discrete economy is a problem of maximizing a function over a matroid. One should consider our results from this point of view too. Is the notion of a local maximum interesting there? The notion of a function close to submodular?

Can the notions of local optimum and local equilibrium be of use in the study of markets of divisible goods?
15 Conclusion: the role of prices

The view presented in this paper is that markets attain a local equilibrium through the advent of suitable item prices. The role of prices in this process is significantly different from their role according to the view that markets attain a Walrasian equilibrium. According to this last view, prices, through a *tatonnement* process, converge towards equilibrium prices that are the best possible: if agents accept those prices and trade, at those prices, to improve their individual welfare, every agent will find himself in the best possible situation. No profitable trade is prevented by the equilibrium prices. Apart from the convergence towards equilibrium prices, we do not expect prices to vary, and any divergence from the equilibrium prices can only hamper progress towards equilibrium. If an invisible hand would reveal equilibrium prices from the start, convergence towards equilibrium would only be sped up.

In the local equilibrium view of prices, prices have a different role. They play the traditional role of guiding the market towards a (local) equilibrium, but, once such a local equilibrium is attained, those prices can prevent trades that would improve the social welfare. Consider a $q$-local equilibrium ($q < 1$) in which agent 1 holds an item he values at $x$, but that agent 2 values at $y > x$. Note that the allocation is not a local optimum and that a trade would improve the social welfare. If the price $p$ of the item is greater than $y$ or less than $x$, no trade at price $p$ can take place, even though it could happen at another price. In such a situation local equilibrium prices may have a negative effect: they can prevent an increase in social welfare resulting from trade. There, a change in prices may enable profitable trades that were impossible previously. We expect that changes in the revealed prices can help the market to move from a local equilibrium to another local equilibrium of higher social value. Such price modifications may also change the quality of a local equilibrium, but the forces behind such process are still unclear.

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A Walrasian equilibria

In a Walrasian equilibrium \((f, p)\), every agent \(i \in N\), at the given prices \(p\), prefers his allocated bundle \(S^f_i\) to any other bundle. For an agent \(i \in N\), its utility for bundle \(A \subseteq X\) is defined as \(u_i(A) = v_i(A) - \sum_{j \in A} p_j\).

Definition 7 Suppose an economy \(E = (N, X, v_i, i \in N)\) is given. A Walrasian equilibrium \((f, p)\) is a pair where \(f\) is a partial allocation of the items to the agents and \(p\) is a price vector that satisfy the following conditions:

1. for any \(j \in X\) such that \(f(j) = \text{unallocated}\) one has \(p_j = 0\),
2. for any \(i \in N\) and any \(D \subseteq X\) one has
\[
    u_i(S^f_i) \geq u_i(D). \tag{13}
\]

In a Walrasian equilibrium every agent is allocated the bundle he prefers amongst all possible bundles, if only he considers he cannot have any influence on the prices. A Walrasian equilibrium is the best of all possible situations for each and every agent.

The basic properties of Walrasian equilibria are described in Theorem 9. They summarize the two theorems of welfare economics and the results of [5] in an original manner.

Theorem 9 • If \((f, p)\) is a Walrasian equilibrium then \(\text{val}(f) = M_F\), i.e., \(f\) is a fractional optimum,

• if \(f\) is an allocation and \(\text{val}(f) = M_F\), then there exists a price vector \(p\) such that \((f, p)\) is a Walrasian equilibrium,

• if \((f, p)\) is a Walrasian equilibrium and \(g\) is an allocation such that \(\text{val}(g) = M_F\), then \((g, p)\) is a Walrasian equilibrium.

Proof: First, under the assumptions, for any fractional allocation \(x^D_i\):
\[
    \text{val}(x) = \sum_{i \in N} \sum_{D \subseteq X} x^D_i v_i(D) \leq \sum_{i \in N} \sum_{D \subseteq X} x^D_i v_i(S^f_i) = \sum_{i \in N} v_i(S^f_i) \sum_{D \subseteq X} x^D_i \leq \sum_{i \in N} v_i(S^f_i) = \text{val}(f).
\]

Secondly, under the assumptions, \(f\) is the solution to the linear program \(\text{LP}\). The dual program \(\text{DLP}\) is described in Appendix C. We shall take the variables \(p_j, j \in X\) of the dual for prices in the equilibrium. It follows from general results on linear programming, that for any \(i \in N\), \(D \subseteq X\)
• $\pi_i \geq v_i(D) - \sum_{j \in D} p_j$ and
• if $x_i^D > 0$ then $\pi_i = v_i(D) - \sum_{j \in D} p_j$.

But $x_i^{S_f^l} = 1$ and therefore

\[
v_i(S_i^f) - \sum_{j \in S_i^f} p_j = \pi_i \geq v_i(D) - \sum_{j \in D} p_j.
\]

For the third part of our claim, we shall show that, for any $i \in N$, one has

\[
v_i(S_i^q) - \sum_{j \in S_i^q} p_j = v_i(S_i^f) - \sum_{j \in S_i^f} p_j.
\]

Since $(f, p)$ is a Walrasian equilibrium, we know that, for any $i \in N$,

\[
v_i(S_i^f) - \sum_{j \in S_i^f} p_j \geq v_i(S_i^q) - \sum_{j \in S_i^q} p_j.
\]

But, since $\text{val}(g) = \text{val}(f)$ and then by the fact that if $f(j) = \text{unalloc}$ one has $p_j = 0$

\[
\sum_{i \in N} (v_i(S_i^q) - \sum_{j \in S_i^q} p_j) = \sum_{i \in N} \sum_{j \in S_i^q} p_j \geq \sum_{i \in N} (v_i(S_i^f) - \sum_{j \in X} p_j =
\]

\[
\sum_{i \in N} \sum_{j \in S_i^f} p_j = \sum_{i \in N} (v_i(S_i^f) - \sum_{j \in S_i^f} p_j).
\]

\]

B A technical lemma

The following shows sufficient conditions for an allocation to be a $1 + a + b$-approximation ($a, b \geq 0$) of the fractional optimal allocation. It is used to prove Theorem 1. Note that the result holds for any $a, b \geq 0$, but is used only for $a, b \geq 1$, and also that in an exchange economy the agents’ valuations are assumed to satisfy both Free disposal and Normalization but that Normalization is not used in Lemma 2. Its strength is that no further assumption is made on the agents’ valuations.
Lemma 2 In any exchange economy, let $a, b \geq 0$ and assume that $f$ is a partial allocation and $p$ is a price vector that satisfy the following two conditions:

1. for any $i \in N$
   \[ bv_i(S_i^f) \geq \sum_{j \in S_i^f} p_j, \]  
   (14)

2. for any $i \in N$ and any $A \subseteq X$ such that $A \cap S_i^f = \emptyset$ one has
   \[ v_i(A \mid S_i^f) \leq a \sum_{j \in A} p_j, \]  
   (15)

then, for any fractional allocation $x$:

\[ \text{val}(x) \leq (1 + ab) \text{val}(f) + a \sum_{j \in X, f(j) = \text{unalloc}} p_j. \]  
(16)

Proof: By definition, \( \text{val}(x) = \sum_{i \in N} \sum_{D \subseteq X} x_i^D v_i(D) \). By the free disposal assumption, then:

\[ \text{val}(x) \leq \sum_{i \in N} \sum_{D \subseteq X} x_i^D v_i(D \cup S_i^f) = \sum_{i \in N} \sum_{D \subseteq X} x_i^D v_i(S_i^f) + \sum_{i \in N} \sum_{D \subseteq X} x_i^D v_i(D - S_i^f \mid S_i^f). \]

First, by Equation (1)

\[ \sum_{i \in N} \sum_{D \subseteq X} x_i^D v_i(S_i^f) = \sum_{i \in N} v_i(S_i^f) \sum_{D \subseteq X} x_i^D \leq \sum_{i \in N} v_i(S_i^f) = \text{val}(f). \]

Then, by Equation (15), then Equation (2) and finally by Equation (14) we have

\[ \sum_{i \in N} \sum_{D \subseteq X} x_i^D v_i(D - S_i^f \mid S_i^f) \leq \sum_{i \in N} \sum_{D \subseteq X} a x_i^D \sum_{j \in D - S_i^f} p_j \leq \sum_{j \in X} \sum_{i \in N} \sum_{D \subseteq X, j \in D} a x_i^D p_j = a \sum_{j \in X} p_j \sum_{i \in N} \sum_{D \subseteq X, j \in D} x_i^D \leq a \sum_{j \in X} p_j = a \sum_{i \in N} \sum_{j \in S_i^f} p_j + a \sum_{j \in X, f(j) = \text{unalloc}} p_j \leq a b \sum_{i \in N} v_i(S_i^f) + a \sum_{j \in X, f(j) = \text{unalloc}} p_j. \]

We conclude that \( \text{val}(x) \leq (1 + ab) \text{val}(f) + a \sum_{j \in X, f(j) = \text{unalloc}} p_j. \)
C Dual linear program

The dual of LP will be described now.

Dual Linear Program (DLP):

Minimize

\[ \sum_{j \in X} p_j + \sum_{i \in N} \pi_i \]  

(17)

under the constraints

\[ p_j \geq 0, \pi_i \geq 0 \text{ for all } j \in X, i \in N, \text{ and } \]

(18)

\[ \sum_{j \in D} p_j + \pi_i \geq v_i(D), \text{ for all } D \subseteq X \text{ and } i \in N. \]  

(19)

D Quasi-Walrasian equilibria

A quasi-Walrasian equilibrium of quality \( q, 0 \leq q \leq 1 \) consists of a partial allocation and a price vector such that every agent gets from his bundle, at the given prices, a utility that is at least the utility he would get from any other bundle discounted by \( q \).

**Definition 8** Suppose an economy \( E = (N, X, v_i, i \in N) \) is given and let \( 0 \leq q \leq 1 \). A \( q \)-quasi-Walrasian equilibrium \((f, p)\) is a pair where \( f \) is a partial allocation of the items to the agents and \( p \) is a price vector that satisfy the following two conditions:

1. for any \( j \in X \) such that \( f(j) = \text{unallocated} \) one has \( p_j = 0 \),

2. for any \( i \in N \) and for any \( A \subseteq X \) one has

\[ v_i(S_i^f) - \sum_{j \in S_i^f} p_j \geq q \left( v_i(A) - \sum_{j \in A} p_j \right). \]  

(20)

Clearly a pair \((f, p)\) is a 1-quasi-Walrasian equilibrium iff it is a Walrasian equilibrium and any allocation together with zero prices provides a 0-quasi-Walrasian equilibrium.

The value of the allocation of a \( q \)-quasi-Walrasian equilibrium is at least \( q \) times the social optimum.
Theorem 10 (First social quasi welfare theorem) If \((f, p)\) is a \(q\)-quasi-Walrasian equilibrium, then \(val(f) \geq q \cdot val(g)\) for any partial allocation \(g\).

Note that the approximation \(q\) here is better than the \(\frac{q^2}{1+q^2}\) of Theorem \([4]\), but that the comparison is, here, with the integral social optimum, not with the fractional, higher, optimum.

**Proof:** Let \(g\) be the social optimum. We have, by condition \([1]\) and then condition \([2]\) of Definition \([8]\):

\[
val(f) = \sum_{i \in N} v_i(S_i^f) = \sum_{i \in N} (v_i(S_i^f) - \sum_{j \in S_i^f} p_j) + \sum_{j \in X} p_j \geq q \sum_{i \in N} (v_i(S_i^g) - \sum_{j \in S_i^g} p_j) + \sum_{j \in X} p_j \geq q \cdot val(g).
\]

Lemma 3 Any \(q\)-quasi-Walrasian equilibrium is a \(q\)-local-equilibrium.

**Proof:** Let \((f, p)\) be a \(q\)-quasi-Walrasian equilibrium. Let us show that the three conditions of Definition \([2]\) are satisfied. Condition \([1]\) is explicitly satisfied by Definition \([8]\). For Individual Rationality, notice that, for any \(i \in N\),

\[
v_i(S_i^f) - \sum_{j \in S_i^f} p_j \geq q (v_i(\emptyset) - 0) = 0.
\]

Equation \([7]\) is satisfied even with a parameter \(q\) equal to 1. For Outward Stability note that

\[
v_i(S_i^f) - \sum_{j \in S_i^f} p_j \geq q (v_i(S_i^f \cup A) - \sum_{j \in S_i^f \cup A} p_j)
\]

and therefore

\[
\sum_{j \in A} p_j \geq v_i(S_i^f \cup A) - v_i(S_i^f) = v_i(A | S_i^f).
\]
in a \( q \)-quasi-Walrasian equilibrium. His utility is 0 and therefore, comparing with receiving two of the three items, we see that

\[
0 \geq q (3 - p_1 - p_2), 0 \geq q (3 - p_1 - p_3), 0 \geq q (3 - p_2 - p_3)
\]

and therefore, if \( q > 0 \), \( p_1 + p_2 + p_3 \geq 4.5 \). Note that the parameter \( q \) has disappeared from the inequality. If agent 2 is allocated the whole bundle of three items, he must prefer this to the empty bundle and we must have:

\[
4 - p_1 - p_2 - p_3 \geq q 0 = 0
\]

which is impossible. But, if an item, say item 3 is unallocated, we must have \( p_3 = 0 \) and \( p_1 + p_2 \geq 4.5 \). Allocating the bundle \((1, 2)\) to agent 2 would imply

\[
3 - p_1 - p_2 \geq q 0 = 0
\]

which is impossible. We conclude that at least two items must be unallocated, but this can be shown similarly to imply that no item is allocated, which is clearly impossible. We have shown that there is no \( q \)-quasi-Walrasian equilibrium for \( q > 0 \) in which some agent receives an empty bundle.

Suppose now that agent 1 is allocated a single item, say item 3. We must have

\[
0 - p_3 \geq q 0 = 0
\]

and therefore \( p_3 = 0 \). Since agent 1 prefers item 3 to all three items we have:

\[
0 - p_3 = 0 \geq q (4 - p_1 - p_2)
\]

and \( p_1 + p_2 \geq 4 \). In such a situation, agent 2 cannot be allocated the pair \((1, 2)\), nor can he be allocated any single item.

We conclude that, if \( q > 0 \), in any \( q \)-quasi-Walrasian equilibrium each of the two agents must be allocated at least two of the three items, which is impossible. The only \( q \)-quasi-Walrasian equilibria have \( q = 0 \).