Non-renormalization Theorem and Cyclic Leibniz Rule in Lattice Supersymmetry

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in collaboration with M. Kato and H. So
based on JHEP 1305(2013)089; arXiv:1311.4962; and in progress
For more than 30 years, no one has succeeded to construct satisfactory lattice models which realize full supersymmetry algebras.
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Are full SUSY algebras necessary to keep crucial features of SUSY on lattice?
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So, I want to know the answer to the question:

Are full SUSY algebras necessary to keep crucial features of SUSY on lattice?

Our results suggest that the answer is possibly negative.
We want to find lattice SUSY transf. $\delta_Q, \delta_{Q'}$ such that

$$\delta_Q S[\phi, \chi, F] = \delta_{Q'} S[\phi, \chi, F] = 0$$

with the SUSY algebra

$$\{ \delta_Q, \delta_{Q'} \} = \delta_P$$
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One might replace $\delta_P$ by a difference operator $\nabla$.

Then, we need to find $\nabla$ which satisfies the **Leibniz rule**.

$$\delta_P (\phi \psi) = (\delta_P \phi) \psi + \phi (\delta_P \psi)$$

$$\delta_P \rightarrow \nabla \quad \nabla (\phi \psi) = (\nabla \psi) \psi + \phi (\nabla \psi)$$
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One might replace $\delta_P$ by a difference operator $\nabla$. Then, we need to find $\nabla$ which satisfies the **Leibniz rule**.

$$\delta_P (\phi \psi) = (\delta_P \phi) \psi + \phi (\delta_P \psi)$$

However, we can show that it is hard to realize the Leibniz rule on lattice!!
To answer the question whether the Leibniz rule can be realized on lattice or not, let us consider generalized difference operators and field products such as

\[
\text{difference operator: } (\nabla \phi)_n \equiv \sum_m \nabla nm \phi_m
\]

\[
\text{field product: } (\phi \ast \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m
\]
No-Go theorem

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\text{field product: } (\phi \ast \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m
\]

No-Go Theorem

There is no difference operator \( \nabla \) satisfying the following three properties:

i) translation invariance
ii) locality
iii) Leibniz rule \( \nabla (\phi \ast \psi) = (\nabla \phi) \ast \psi_n + \phi \ast (\nabla \psi) \)

M.Kato, M.S. & H.So, JHEP 05(2008)057
The No-Go theorem tells us that we cannot realize SUSY algebras with \( \nabla \) equipped with the Leibniz rule.
Our approach to construct lattice SUSY models

The No-Go theorem tells us that we cannot realize SUSY algebras with $\nabla$ equipped with the Leibniz rule.

Our strategy to construct lattice SUSY models is

- full SUSY algebra $\rightarrow$ Nilpotent SUSY algebra
  \[
  (\delta_Q)^2 = (\delta_{Q'})^2 = \{\delta_Q, \delta_{Q'}\} = 0
  \]

- Leibniz rule $\rightarrow$ Cyclic Leibniz rule
Complex SUSY quantum mechanics on lattice
To make our discussions simple, we here put $m=0$.

We can add mass terms as well as *supersymmetric Wilson terms* to prevent the doubling.
N=2 nilpotent SUSYs:

$$(\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0$$

\[\begin{align*}
\delta_+ \phi_+ &= \bar{\chi}_+ \\
\delta_+ \chi_+ &= F_+ \\
\delta_+ \chi_- &= -i \nabla \phi_- \\
\delta_+ F_- &= -i \nabla \bar{\chi}_- \\
\text{others} &= 0
\end{align*}\]

\[\begin{align*}
\delta_- \chi_+ &= i \nabla \phi_+ \\
\delta_- F_+ &= -i \nabla \bar{\chi}_+ \\
\delta_- \phi_- &= -\bar{\chi}_- \\
\delta_- \chi_- &= F_- \\
\text{others} &= 0
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\delta_- \chi_- &= F_- \\
\text{others } &= 0
\end{align*}
\]

\( \delta_\pm S = 0 \)

\[
(\nabla \bar{\chi}_\pm, \phi_\pm \ast \phi_\pm) + (\nabla \phi_\pm, \phi_\pm \ast \bar{\chi}_\pm) + (\nabla \phi_\pm, \bar{\chi}_\pm \ast \phi_\pm) = 0
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N=2 nilpotent SUSYs:

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\[ \delta_\pm S = 0 \]

\[ (\nabla \bar{\chi}_\pm, \phi_\pm \ast \phi_\pm) + (\nabla \phi_\pm, \phi_\pm \ast \bar{\chi}_\pm) + (\nabla \phi_\pm, \bar{\chi}_\pm \ast \phi_\pm) = 0 \]

We call this **Cyclic Leibniz rule.**
We have found that the **Cyclic Leibniz Rule** guarantees the N=2 nilpotent SUSYs.

\[
(\nabla A, \quad B \ast C) + (\nabla B, \quad C \ast A) + (\nabla C, \quad A \ast B) = 0
\]

**Cyclic Leibniz Rule (CLR)**

**Leibniz Rule (LR)**
We have found that the Cyclic Leibniz Rule guarantees the N=2 nilpotent SUSYs.

**Cyclic Leibniz Rule (CLR)**

\[(\nabla A, B \ast C) + (\nabla B, C \ast A) + (\nabla C, A \ast B) = 0\]

**Leibniz Rule (LR)**

\[(\nabla A, B \ast C) + (A, \nabla B \ast C) + (A, B \ast \nabla C) = 0\]
We have found that the **Cyclic Leibniz Rule** guarantees the N=2 nilpotent SUSYs.

**Cyclic Leibniz Rule (CLR)**

\[
(\nabla A, B \ast C) + (\nabla B, C \ast A) + (\nabla C, A \ast B) = 0
\]

**vs.**

**Leibniz Rule (LR)**

\[
(\nabla A, B \ast C) + (A, \nabla B \ast C) + (A, B \ast \nabla C) \neq 0
\]

**No-Go theorem**
We have found that the **Cyclic Leibniz Rule** guarantees the N=2 nilpotent SUSYs.

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**Cyclic Leibniz Rule (CLR)**

vs.

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(\nabla A, B \ast C) + (A, \nabla B \ast C) + (A, B \ast \nabla C) \neq 0
\]

**Leibniz Rule (LR)**

**No-Go theorem**

The cyclic Leibniz rule ensures an **lattice analog of vanishing surface terms**!

\[
(\nabla \phi, \phi \ast \phi) = 0 \quad \text{on lattice}
\]

\[
\int dx \, \partial_x (\phi(x))^3 = 0 \quad \text{in continuum}
\]
An explicit example of the **Cyclic Leibniz Rule**:

\[
(\nabla \phi)_n = \frac{1}{2} (\phi_{n+1} - \phi_{n-1})
\]

\[
(\phi \ast \psi)_n = \frac{1}{6} \left( 2\phi_{n+1}\psi_{n+1} + 2\phi_{n-1}\psi_{n-1} + \phi_{n+1}\psi_{n-1} + \phi_{n-1}\psi_{n+1} \right)
\]

which satisfy i) *translation invariance*, ii) *locality* and iii) *Cyclic Leibniz Rule*.

The field product \((\phi \ast \psi)_n\) should be non-trivial!
**Advantages of our lattice model with CLR are given by**

|                         | CLR | no CLR |
|-------------------------|-----|--------|
| nilpotent SUSYs         |     |        |
| Nicolai maps            |     |        |
| “surface” terms         |     |        |
| non-renormalization     |     |        |
| theorem                 |     |        |
| cohomology              |     |        |
Advantages of our lattice model with **CLR** are given by

|                      | CLR       | no CLR  |
|----------------------|-----------|---------|
| nilpotent SUSYs      | $\delta_+ , \delta_-$ | $\delta = \delta_+ + \delta_-$ |
| Nicolai maps         |           |         |
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|---------------------------------------|------|--------|
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| Nicolai maps                          | 2    | 1      |
| “surface” terms                       |      |        |
| non-renormalization theorem           |      |        |
| cohomology                            |      |        |

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Advantages of our lattice model with \textit{CLR} are given by

|                      | CLR                           | no CLR                        |
|----------------------|-------------------------------|-------------------------------|
| nilpotent SUSYs      | $\delta_+, \delta_-$         | $\delta = \delta_+ + \delta_-$|
| Nicolai maps         | 2                             | 1                             |
| “surface” terms      | $(\nabla \phi, \phi \ast \phi) = 0$ | $(\nabla \phi, \phi \ast \phi) \neq 0$ |
| non-renormalization  |                               |                               |
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| non-renormalization theorem    | ☐         | ☒                     |
| cohomology                     | **non-trivial** | **trivial** |
Advantages of our lattice model with **CLR** are given by

|                     | CLR       | no CLR    |
|---------------------|-----------|-----------|
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| cohomology          | non-trivial | trivial   |
One of the striking features of SUSY theories is the *non-renormalization theorem*. 
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- \textbf{4d N=1 Wess-Zumino model in continuum}

\begin{equation}
S = \int d^4 x \left\{ \int d^2 \theta d^2 \bar{\theta} \, \Phi^\dagger(\bar{\theta}) \Phi(\theta) + \int d^2 \theta \, W(\Phi) + \text{c.c.} \right\}
\end{equation}

- \textit{D term (kinetic terms)}
- \textit{F term (potential terms)}

\textit{chiral superfield}

\textit{superpotential}
One of the striking features of SUSY theories is the **non-renormalization theorem**.

**4d N=1 Wess-Zumino model in continuum**

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S = \int d^4x \left\{ \int d^2\theta d^2\bar{\theta} \Phi^\dagger(\bar{\theta}) \Phi(\theta) + \int d^2\theta W(\Phi) + c.c. \right\}
\]

- **D term** (kinetic terms)
- **F term** (potential terms)

**Non-renormalization Theorem**

There is **no quantum correction to the F-terms** in any order of perturbation theory.
Essence of non-renormalization theorem

\[ S = \int d^4 x \left\{ \int d^2 \theta d^2 \bar{\theta} \, \Phi^\dagger(\bar{\theta})\Phi(\theta) + \int d^2 \theta \, W(\Phi) + c.c. \right\} \]

D term (kinetic terms)  F term (potential terms)
Holomorphy plays an important role in the non-renormalization theorem.

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D term (kinetic terms) \hspace{1cm} F term (potential terms)

\[ \int d^2 \theta \, W_{\text{tree}}(\Phi, \lambda) + \int d^2 \bar{\theta} \, \bar{W}_{\text{tree}}(\Phi^\dagger, \lambda^*) \]

\[ \int d^2 \theta \, W_{\text{eff}}(\Phi, \lambda; \Phi^\dagger, \lambda^*) + \int d^2 \bar{\theta} \, \bar{W}_{\text{eff}}(\Phi^\dagger, \lambda^*; \Phi, \lambda) \]
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- **D term** (kinetic terms)
- **F term** (potential terms)

**Holomorphy**

\[ \int d^2 \theta \ W_{\text{tree}}(\Phi, \lambda) + \int d^2 \bar{\theta} \ W_{\text{tree}}(\Phi^\dagger, \lambda^*) \]

**Effective superpotential**

\[ \int d^2 \theta \ W_{\text{eff}}(\Phi, \lambda; \Phi^\dagger, \lambda^*) + \int d^2 \bar{\theta} \ W_{\text{eff}}(\Phi^\dagger, \lambda^*; \bar{\Phi}, \bar{\lambda}) \]

- **Holomorphy**
- **Anti-holomorphy**

**Quantum corrections**

*Superpotential*
Essence of non-renormalization theorem

\[ S = \int d^4 x \left\{ \int d^2 \theta d^2 \bar{\theta} \, \Phi^\dagger(\bar{\theta}) \Phi(\theta) + \int d^2 \theta \, W(\Phi) + c.c. \right\} \]

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**D term** (kinetic terms)

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\[ \int d^2 \theta \, W_{\text{eff}}(\Phi, \lambda; \chi^\dagger, \chi^*) + \int d^2 \bar{\theta} \, \bar{W}_{\text{eff}}(\Phi^\dagger, \lambda^*; \chi, \chi^*) \]

**Holomorphy**

\[ W_{\text{eff}} = W_{\text{tree}} \]

N. Seiberg, Phys. Lett. B318 (1993) 469

Lattice 2014 at Columbia University, New York, June 23 - 28, 2014
Essence of non-renormalization theorem

\[ S = \int d^4x \left\{ \int d^2\theta d^2\bar{\theta} \, \Phi^\dagger(\bar{\theta})\Phi(\theta) + \int d^2\theta \, W(\Phi) + \text{c.c.} \right\} \]

D term (kinetic terms) \quad F term (potential terms)

Holomorphy plays an important role in the non-renormalization theorem.

\[ \int d^2\theta \, W_{\text{tree}}(\Phi, \lambda) + \int d^2\bar{\theta} \, \bar{W}_{\text{tree}}(\Phi^\dagger, \lambda^*) \]

\[ \int d^2\theta \, W_{\text{eff}}(\Phi, \lambda; \x\; \x^\dagger, \x^*) \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ + \int d^2\bar{\theta} \, \bar{W}_{\text{eff}}(\Phi^\dagger, \lambda^*; \x, \x) \]

No quantum correction!! \quad W_{\text{eff}} = W_{\text{tree}}

N. Seiberg, Phys. Lett. B318 (1993) 469

Lattice 2014 at Columbia University, New York, June 23 – 28, 2014
Difficulty in defining chiral superfield on lattice

The holomorphy requires that the F term $W(\Phi)$ depends only on the _chiral_ superfield $\Phi(x, \theta)$, which is defined by

$$\bar{D}\Phi(x, \theta) \equiv \left( \frac{\partial}{\partial \theta} - i\theta \sigma_\mu \partial_\mu \right) \Phi(x, \theta) = 0 \quad \text{in continuum}$$
The holomorphy requires that the F term $W(\Phi)$ depends only on the **chiral** superfield $\Phi(x, \theta)$, which is defined by

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$$\bar{D}\Phi(\theta)_n \equiv \left( \frac{\partial}{\partial \theta} - i\theta \sigma_\mu \nabla_\mu \right) \Phi(\theta)_n = 0 \quad \text{on lattice}$$
The holomorphy requires that the F term $W(\Phi)$ depends only on the *chiral* superfield $\Phi(x, \theta)$, which is defined by

\[
\bar{D} \Phi(x, \theta) \equiv \left( \frac{\partial}{\partial \theta} - i \theta \sigma_\mu \partial_\mu \right) \Phi(x, \theta) = 0 \quad \text{in continuum}
\]

However, the above definition of the chiral superfield is *ill-defined* because any products of chiral superfields are not chiral due to the *breakdown of LR on lattice!*

\[
\bar{D} \Phi(\theta)_n \equiv \left( \frac{\partial}{\partial \theta} - i \theta \sigma_\mu \nabla_\mu \right) \Phi(\theta)_n = 0 \quad \text{on lattice}
\]

\[
\bar{D} \Phi_1 = \bar{D} \Phi_2 = 0 \quad \implies \quad \bar{D} (\Phi_1 \Phi_2) \neq 0
\]
Superfield formulation in our lattice model

- Lattice superfields

\[ \Psi_{\pm} (\theta_+, \theta_-) \equiv \chi_{\pm} + \theta_{\pm} F_{\pm} + \theta_{\mp} i \nabla \phi_{\pm} + \theta_{\pm} \theta_{\mp} i \nabla \bar{\chi}_{\pm} \]

\[ \Lambda_{\pm} (\theta_{\pm}) \equiv \phi_{\pm} + \theta_{\pm} \chi_{\pm} \]
Superfield formulation in our lattice model

- **Lattice superfields**
  \[
  \Psi_{\pm}(\theta_+, \theta_-) \equiv \chi_{\pm} + \theta_{\pm} F_{\pm} + \theta_+ i \nabla \phi_{\pm} + \theta_- \theta_+ i \nabla \bar{\chi}_{\pm}
  \]
  \[
  \Lambda_{\pm}(\theta_{\pm}) \equiv \phi_{\pm} + \theta_{\pm} \chi_{\pm}
  \]

- **Lattice action in superspace**
  \[
  S = S_{\text{type I}} + S_{\text{type II}}
  \]
  \[
  S_{\text{type I}} = \int d\theta_+ d\theta_- \, \Psi_- \Psi_+ \quad \Longrightarrow \text{kinetic terms (D-term)}
  \]
  \[
  S_{\text{type II}} = \int d\theta_+ d\theta_- \left\{ \theta_- \lambda_+(\Psi_+, \Lambda_+ * \Lambda_+) + \theta_+ \lambda_-(\Psi_-, \Lambda_- * \Lambda_-) \right\}
  \]
  \[
  \Longrightarrow \text{potential terms (F-term)}
  \]
Superfield formulation in our lattice model

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- Lattice action in superspace

\[ S = S_{\text{type I}} + S_{\text{type II}} \]

\[ S_{\text{type I}} = \int d\theta_+ d\theta_- \ K(\Psi_+, \Lambda_+; \Psi_-, \Lambda_-) \]

\[ S_{\text{type II}} = \int d\theta_+ d\theta_- \ \{ \theta_- W(\Psi_+, \Lambda_+) + \theta_+ \bar{W}(\Psi_-, \Lambda_-) \} \]
Superfield formulation in our lattice model

- Lattice superfields

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- Lattice action in superspace

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\[ S_{\text{type II}} = \int d\theta_+ d\theta_- \ \left\{ \theta_- W(\Psi_+, \Lambda_+) + \theta_+ \bar{W}(\Psi_-, \Lambda_-) \right\} \]

\[ S_{\text{type II}} \text{ is SUSY-invariant if and only if } W(\Psi_+, \Lambda_+) \]
depends only on \( \Psi_+, \Lambda_+ \) and is written into the form
\[ W(\Psi_+, \Lambda_+) = \sum_{n} \lambda_+^{(n)}(\Psi_+, \underbrace{\Lambda_+ * \Lambda_+ * \cdots * \Lambda_+}_{n-1}) \]
and \( (\Psi_+, \Lambda_+ * \Lambda_+ * \cdots * \Lambda_+) \) has to obey CLR.

M.Kato, M.S., H.So, in preparation
Lattice 2014 at Columbia University, New York, June 23 – 28, 2014
Non-renormalization theorem in our lattice model

\[ \int d\theta_+ d\theta_- \theta_- W_{\text{tree}}(\Psi_+, \Lambda_+, \lambda_+) \]

\[ W_{\text{tree}} = \lambda_+ (\Psi_+, \Lambda_+ \ast \Lambda_+) \]

quantum corrections
Non-renormalization theorem in our lattice model

\[ \int d\theta_+ d\theta_- \theta_- W_{\text{tree}}(\Psi_+, \Lambda_+, \lambda_+) \]

\[ W_{\text{tree}} = \lambda_+(\Psi_+, \Lambda_+ \ast \Lambda_+) \]

quantum corrections

\[ \int d\theta_+ d\theta_- \theta_- W_{\text{eff}}(\Psi_+, \Lambda_+, \lambda_+; \Psi_-, \Lambda_-, \lambda_-) \]
Non-renormalization theorem in our lattice model

\[ \int d\theta_+ d\theta_- \theta_- W_{\text{tree}}(\Psi_+, \Lambda_+, \lambda_+) \]

\[ W_{\text{tree}} = \lambda_+(\Psi_+, \Lambda_+ \ast \Lambda_+) \]

quantum corrections

\[ \int d\theta_+ d\theta_- \theta_- W_{\text{eff}}(\Psi_+, \Lambda_+, \lambda_+; \color{red} X_- , \color{red} X_- , \color{red} X_- ) \]

SUSY-invariance with CLR forbids them!
The holomorphic property is realized in our lattice model.

\[ \int d\theta_+ d\theta_- \theta_- W_{\text{tree}}(\Psi_+, \Lambda_+, \lambda_+) \]

\[ W_{\text{tree}} = \lambda_+(\Psi_+, \Lambda_+ \ast \Lambda_+) \]

quantum corrections

\[ \int d\theta_+ d\theta_- \theta_- W_{\text{eff}}(\Psi_+, \Lambda_+, \lambda_+; \underline{\underline{X}_-, \underline{X}_-, \underline{X}_-}) \]

SUSY-invariance with CLR forbids them!
Non-renormalization theorem in our lattice model

\[ \int d\theta_+ d\theta_- \theta_- W_{\text{tree}}(\Psi_+, \Lambda_+, \lambda_+) \]

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quantum corrections

\[ \int d\theta_+ d\theta_- \theta_- W_{\text{eff}}(\Psi_+, \Lambda_+, \lambda_+; \bar{\Psi}_-, \bar{\Lambda}_-, \bar{\lambda}_-) \]

SUSY-invariance with \textbf{CLR} forbids them!

The \textit{holomorphic property} is realized in our lattice model.

\textbf{no quantum corrections: } W_{\text{eff}} = W_{\text{tree}}
The non-renormalization theorem holds even for a finite lattice spacing in our lattice model.
Q-exact form and cohomology

\[ S = \delta_+ \delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm) + \delta_+ \tilde{W}(\phi_+, \chi_+) + \delta_- \tilde{\bar{W}}(\phi_-, \chi_-) \]
Q-exact form and cohomology

\[ S = \delta_+ \delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm) + \delta_+ W(\phi_+, \chi_+) + \delta_- \bar{W}(\phi_-, \chi_-) \]

invariant under \( \delta_\pm \) because of the **nilpotency**:

\( (\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0 \)

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Q-exact form and cohomology

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\[ (\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0 \]

invariant under \( \delta_\pm \)
because of the **nilpotency**:

\[ \delta_- W = 0 \]

invariant under \( \delta_\pm \) only if
Q-exact form and cohomology

\[ S = \delta_+ \delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm) + \delta_+ W(\phi_+, \chi_+) + \delta_- \bar{W}(\phi_-, \chi_-) \]

invariant under \(\delta_\pm\) only if \(\delta_- W = 0\)

\(\delta_-\)-exact

because of the **nilpotency**:

\((\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0\)
Q-exact form and cohomology

$$S = \delta_+ \delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm) + \delta_+ W(\phi_+, \chi_+) + \delta_- \bar{W}(\phi_-, \chi_-)$$

- **\(S_{\text{type I}}\)**
  - **\(S_{\text{type II}}\)**

  - Invariant under \(\delta_+\) because of the **nilpotency**: \((\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0\)

  - \(\delta_-\)-exact: \(W = \delta_- K'\)

  - Invariant under \(\delta_+\) only if \(\delta_- W = 0\)
Q-exact form and cohomology

\[ S = \delta_+ \delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm) + \delta_+ W(\phi_+, \chi_+) + \delta_- \bar{W}(\phi_-, \chi_-) \]

\[ \delta_+ \delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm) \]

\[ (\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0 \]

\[ \delta_- \text{-exact} \]

\[ W = \delta_- K' \]

\[ S_{\text{type I}} \]

\[ S_{\text{type II}} \]

\[ \text{invariant under } \delta_\pm \text{ only if } \delta_- W = 0 \]

\[ \text{because of the nilpotency:} \]

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Q-exact form and cohomology

\[ S = \delta_+ \delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm) + \delta_+ W(\phi_+, \chi_+) + \delta_- \bar{W}(\phi_-, \chi_-) \]

- invariant under \( \delta_\pm \) because of the **nilpotency**: 
  \[(\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0\]

- \( \delta_- \)-exact 
  \[ W = \delta_- K' \]

- \( S_{\text{type I}} \)

- \( \delta_- \)-closed but not exact

- invariant under \( \delta_\pm \) only if 
  \[ \delta_- W = 0 \]

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Q-exact form and cohomology

\[ S = S_{\text{type I}} + S_{\text{type II}} \]

\[ S_{\text{type I}} = \delta_+ \delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm) \]

\[ S_{\text{type II}} = \delta_+ W(\phi_+, \chi_+) + \delta_- \bar{W}(\phi_-, \chi_-) \]

\( \delta_+ \delta_- \) invariant under \( \delta_+ \) only if \( \delta_- \) closed but not exact

\( \delta_- \)-exact

\[ W = \delta_- K' \]

\[ \delta_- W \sim (\nabla \phi_+, \phi_+ * \phi_+ * \cdots * \phi_+) \]

\[ W \sim (\chi_+, \phi_+ * \phi_+ * \cdots * \phi_+) \]

because of the nilpotency:

\[ (\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0 \]

\( \delta_- \)-closed but not exact

\[ \delta_- W = 0 \]

\[ W \neq \delta_- K' \]

which has the properties:

\[ \delta_- W \sim (\nabla \phi_+, \phi_+ * \phi_+ * \cdots * \phi_+) = 0 \]

M.Kato, M.S., H.So in preparation

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Q-exact form and cohomology

\[ S = \delta_+ \delta_- K(\phi_\pm, F_\pm, \chi_\pm, \bar{\chi}_\pm) + \delta_+ W(\phi_+, \chi_+) + \delta_- \bar{W}(\phi_-, \chi_-) \]

\( S_{\text{type I}} \)

\( S_{\text{type II}} \)

invariant under \( \delta_\pm \) because of the **nilpotency**:
\[(\delta_+)^2 = (\delta_-)^2 = \{\delta_+, \delta_-\} = 0\]

\( \delta_-\)-exact

\( \delta_-\)-closed but not exact

\( \delta_- W = 0 \)

\( W = \delta_- K' \)

\( W \sim (\chi_+, \phi_+ * \phi_+ * \cdots * \phi_+) \)

which has the properties:

\[ W \neq \delta_- K' \]

\[ \delta_- W \sim (\nabla \phi_+, \phi_+ * \phi_+ * \cdots * \phi_+) = 0 \]

The type II terms are **cohomologically non-trivial**!

M.Kato, M.S., H.So in preparation

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We have proved the **No-Go theorem** that the Leibniz rule cannot be realized on lattice under reasonable assumptions.

We proposed a lattice SUSY model equipped with the **cyclic Leibniz rule** as a modified Leibniz rule.

A striking feature of our lattice SUSY model is that the **non-renormalization theorem** holds for a finite lattice spacing.

Our results suggest that the **cyclic Leibniz rule grasps important properties of SUSY**.
Remaining tasks

- Extension to higher dimensions
  
  We have to extend our analysis to higher dimensions. 
  Especially, we need to find solutions to CLR in more than one dimensions.

- Inclusion of gauge fields

- Nilpotent SUSYs with CLR → full SUSYs
  
  Are nilpotent SUSYs extended by CLR enough to guarantee full SUSYs?
Appendix
SUSY transformations of superfields

\[ \Psi_{\pm}(\theta_+ , \theta_-) \equiv \chi_\pm + \theta_\pm F_\pm + \theta_+ i \nabla \phi_\pm + \theta_\pm \theta_+ i \nabla \bar{\chi}_\pm \]

\[ \Lambda_{\pm}(\theta_\pm) \equiv \phi_\pm + \theta_\pm \chi_\pm \]

transform under SUSY transformations \( \delta_{\pm} \) as

\[ \delta_{\pm} \mathcal{O}(\theta_\pm) = \frac{\partial}{\partial \theta_\pm} \mathcal{O}(\theta_\pm) \]

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Two Nicolai maps:

$$\xi_{\pm} \equiv \nabla \phi_{\pm} \pm \phi_{+\ast} \phi_{+}$$

$$\bar{\xi}_{\pm} \equiv \nabla \phi_{\mp} \pm \phi_{-\ast} \phi_{-}$$

Action: $S = S_B + S_F$

$$S_B = (\bar{\xi}_+, \xi_+) = (\bar{\xi}_-, \xi_-)$$

$$(\nabla \phi_{\pm}, \phi_{\pm \ast} \phi_{\pm}) = 0$$

CLR
Proof of No-Go Theorem

difference operator:  \((\nabla \phi)_n \equiv \sum_{m} \nabla_{nm} \phi_m\)

field product:  \((\phi \star \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m\)

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Proof of No-Go Theorem

\begin{align*}
\text{difference operator: } & \quad (\nabla \phi)_n \equiv \sum_m \nabla_{nm} \phi_m \\
\text{field product: } & \quad (\phi \ast \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m
\end{align*}

i) translation invariance

\begin{align*}
\nabla_{nm} & = \nabla(n - m) \\
M_{nlm} & = M(l - n, m - n)
\end{align*}
Proof of No-Go Theorem

difference operator: \( (\nabla \phi)_{n} \equiv \sum_{m} \nabla_{nm} \phi_{m} \)

field product: \( (\phi \ast \psi)_{n} \equiv \sum_{lm} M_{nlm} \phi_{l} \psi_{m} \)

ii) locality

\[ \nabla(m) \xrightarrow{|m| \to \infty} 0 \quad (\text{exponentially}) \]

\[ M(l, m) \xrightarrow{|l|, |m| \to \infty} 0 \quad (\text{exponentially}) \]

holomorphic representation

\[ \tilde{\nabla}(z) \equiv \sum_{m} \nabla(m) z^{m} \quad \text{on} \quad 1 - \varepsilon < |z|, |\psi| < 1 + \varepsilon \]

\[ \tilde{M}(z, \psi) \equiv \sum_{lm} M(l, m) z^{l} \psi^{m} \]

\( \tilde{\nabla}(z), \tilde{M}(z, \psi) \) have to be holomorphic on \( 1 - \varepsilon < |z|, |\psi| < 1 + \varepsilon \)
Proof of No-Go Theorem

\begin{equation}
(\nabla \phi)_n \equiv \sum_m \nabla nm \phi_m
\end{equation}

\begin{equation}
(\phi \ast \psi)_n \equiv \sum_{lm} M_{nlm} \phi_l \psi_m
\end{equation}

iii) Leibniz rule

\begin{equation}
\nabla (\phi \ast \psi) = (\nabla \phi) \ast \psi + \phi \ast (\nabla \psi)
\end{equation}

\implies M(z, w) (\nabla (zw) - \nabla (z) - \nabla (w)) = 0
\implies \nabla (zw) - \nabla (z) - \nabla (w) = 0
\implies \nabla (z) \propto \log z
\implies \log z \text{ is non-holomorphic on } 1 - \varepsilon < |z| < 1 + \varepsilon.
\implies \text{The Leibniz rule cannot be realized on lattice!}