Low-rank matrix estimation in multi-response regression with measurement errors: Statistical and computational guarantees

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Abstract

In this paper, we investigate the matrix estimation problem in multi-response regression with measurement errors. A nonconvex error-corrected estimator is proposed to estimate the matrix parameter via a combination of the loss function and the nuclear norm regularization. Then under the low-rank constraint, we analyse the statistical and computational theoretical properties of global solution of the nonconvex regularized estimator from a general point. In the statistical aspect, we establish the recovery bound for the global solution of the nonconvex estimator, under restricted strong convexity on the loss function. In the computational aspect, we solve the nonconvex optimization problem via the proximal gradient method. The algorithm is proved to converge to a near-global solution and achieve a linear convergence rate. In addition, we also establish sufficient conditions for the general results to be held for specific types of corruptions, including the additive noise and missing data. Probabilistic consequences are obtained by applying the general results. Finally, we demonstrate our theoretical consequences by several numerical experiments on the corrupted errors-in-variables multi-response regression models. Simulation results show remarkable consistency with our theory under high-dimensional scaling.

Keywords: Low-rank matrix; Measurement error; Nonconvex estimation; Recovery bound; Statistical consistency; Proximal gradient method; Convergence rate

1 Introduction

Massive data sets have posed a variety of challenges to the field of statistics and machine learning in recent decades. In view of these challenges, researchers have developed different classes of statistical models to deal with the complexities of modern data, such as sparse linear regression models, matrix regression models with rank constraints, graphical models and various nonparametric models. Effective estimation methods based on convex or nonconvex optimization problems have also been proposed to analyse these models from both statistical
and computational aspects. By now, high-dimensional statistics have gained fruitful results and have been successfully used in a wide range of application fields; see the books [9, 30] for an overall review.

In standard formulations of statistical inference problems, it is assumed that the collected data are clean enough, meaning that there exists no measurement error. However, this hypothesis is neither realistic nor reasonable, since in many real-world problems, due to the instrumental constraint or the lack of observation, the collected data, such as human genetic data, may always be corrupted and tend to be noisy or partially missing, indicating that measurement error cannot be avoided in general. There have been a variety of researches focusing on models with corrupted data for regression problems; see, e.g., [6, 10, 27] and references therein. However, much of the previous theoretical work is for classical low-dimensional statistics, that is, the sample size $n$ diverges while the problem dimension $d$ is fixed. As for the high-dimensional scenario (i.e., $n \ll d$), authors in [25] pointed out that misleading inference results can still only be obtained if the method for clean data is applied naively to the corrupted data. Thus it is necessary to take measurement errors into consideration and develop new error-corrected methods for high-dimensional models.

Recently, some regularization methods have been proposed to deal with high-dimensional errors-in-variables regression. For example, in [17], Loh and Wainwright proposed a nonconvex Lasso-type estimator via substituting the corrupted Gram matrix with unbiased surrogates for additive and missing data cases, and established the statistical errors for global solutions. Later in [19], the authors further extended the $\ell_1$-norm regularizer to the nonconvex regularizers, namely SCAD and MCP, and investigate liner and logistic regression models. Then in order to avoid the nonconvexity, Datta and Zou developed the convex conditional Lasso (CoCoLasso) by defining the nearest positive semi-definite matrix in [12]. The CoCoLasso thus enjoys the benefits of convex optimization and is shown to possess nice estimation accuracy in linear regression. Rosenbaum and Tsybakov [23] worked on the sparse linear model with corrupted covariates, and proposed a modified form of the Dantzig selector, called matrix uncertainty selector (MUS). The MUS is proved to be variable selection consistency under strong conditions while the consistency for parameter estimation is not guaranteed. Further development of the method MUS included modification to obtain statistical consistency, and generalization to handle the cases of unbounded and dependent measurement errors as well as the generalized linear models [4, 5, 24, 26].

There are some methods fall out the category of regularization methods. For instance, authors in [14] first performed model selection and then made use of the corrected least squares on the reduced model for estimation; Chen and Caramanis [11] modified the orthogonal matching pursuit algorithm for variable selection in errors-in-variables linear regression; the measurement error boosting (MEBoost) algorithm [8] was based on the idea of classical estimation equation and implemented measurement error corrected variable selection at every iterative path.

However, the aforementioned researches mainly focused on the high-dimensional errors-in-variables regression with univariate responses, and relatively little attention is paid for the case of multivariate responses, a model which is also an special and important instance of matrix regression [29]. Though a simple and natural idea is to vectorize both the response matrix and the coefficient matrix so that methods for univariate response case can be directly applied, it may ignore the particular low-dimensional structures of the coefficient matrix such as low rankness and row/column sparsity as well as the multivariate nature of the responses.
Moreover, the multi-response linear regression has a substantial wider application than that of the univariate case in modern large-scale association analysis, such as genome-wide association studies and social network analyses. Therefore, in this work, we shall deal with the multi-response case in the high-dimensional errors-in-variables regression. More precisely, we shall consider the multi-response regression model $Y = X\Theta^* + \epsilon$, where $\Theta^* \in \mathbb{R}^{d_1 \times d_2}$ is the unknown underlying matrix parameter. The covariate matrix $X$ does not need to be fully observed, instead, two types of corruption will be discussed, including the additive noise and missing data cases.

The major contributions of this paper areas follows. First, we develop a new error-corrected method for estimating the parameter $\Theta^*$. Then in the statistical aspect, under the low-rank assumption on the true parameter $\Theta^*$, we provide the squared $\ell_2$ recovery bound for any global solution $\hat{\Theta}$ of the optimization problem as $\|\hat{\Theta} - \Theta^*\|_2^2 = O(\lambda^2 - qRq)$; see Theorem 1. In the aspect of computation, we apply the proximal gradient method to solve a modified version of the optimization problem. The algorithm is proved to converge linearly to a near-global solution of the nonconvex regularized problem; see Theorem 2. Apart from the general and deterministic results, i.e., Theorems 1 and 2 in addition, we also discuss probabilistic consequences for the additive noise and missing data cases by establishing sufficient conditions for the general results to be held; see Corollaries 1–4. Numerical experiments are also performed to demonstrate the consequences.

The remainder of the article is organized as follows. In Section 2 we provide background on the high-dimensional multi-response regression model with measurement errors and then propose a novel error-corrected estimator. Some regularity conditions are also imposed to facilitate the following analysis. In Section 3 we establish our main results on statistical consistency and algorithmic rate of convergence. In Section 4 probabilistic consequences for specific corruption models are obtained by verifying the required regularity conditions. In Section 5 we perform several numerical experiments to demonstrate statistical and computational results. Conclusions and future work are discussed in Section ??.

Technical lemmas are presented in Appendix.

We end this section by introducing some useful notations. For $d \geq 1$, let $I_d$ stand for the $d \times d$ identity matrix. For a matrix $X \in \mathbb{R}^{n \times d}$, let $X_{ij}$ ($i = 1, \ldots, n, j = 1, 2, \ldots, d$) denote its $ij$-th entry, $X_i$ ($i = 1, \ldots, n$) denote its $i$-th row, $X_j$ ($j = 1, 2, \ldots, d$) denote its $j$-th column, and $\text{diag}(X)$ stand for the diagonal matrix with its diagonal entries equal to $X_{11}, X_{22}, \ldots, X_{dd}$. We write $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ to denote the minimal and maximum eigenvalues of a matrix $X$, respectively. For a matrix $\Theta \in \mathbb{R}^{d_1 \times d_2}$, we define $d = \min\{d_1, d_2\}$, and denote its ordered singular values by $\sigma_1(\Theta) \geq \sigma_2(\Theta) \geq \cdots \geq \sigma_d(\Theta) \geq 0$. We use $\|\cdot\|$ to denote different types of matrix norms based on singular values, including the nuclear norm $\|\Theta\|_* = \sum_{j=1}^d \sigma_j(\Theta)$, the spectral or operator norm $\|\Theta\|_{op} = \sigma_1(\Theta)$, and the Frobenius norm $\|\Theta\|_F = \sqrt{\text{trace}(\Theta^\top \Theta)} = \sqrt{\sum_{j=1}^d \sigma_j^2(\Theta)}$. For a pair of matrices $\Theta$ and $\Gamma$ with equal dimensions, we let $\langle (\Theta, \Gamma) \rangle = \text{trace}(\Theta^\top \Gamma)$ denote the trace inner product on matrix space. For a function $f : \mathbb{R}^d \to \mathbb{R}$, $\nabla f$ is used to denote a gradient or subgradient depending on whether $f$ is differentiable or nondifferentiable but convex, respectively.
2 Problem setup

In this section, we begin with background on the high-dimensional errors-in-variables multi-response regression, and then a precise description on the proposed error-corrected estimation methods. Finally, we introduce some regularity conditions that will facilitate the following analysis.

2.1 Model setting

Consider the high-dimensional multi-response regression model which links the response vector $Y_i \in \mathbb{R}^{d_2}$ to a covariate vector $X_i \in \mathbb{R}^{d_1}$

$$Y_i = \Theta^* X_i + \epsilon_i, \quad \text{for } i = 1, 2, \cdots, n,$$

where $\Theta^* \in \mathbb{R}^{d_1 \times d_2}$ is the unknown parameter matrix, $\epsilon_i \in \mathbb{R}^{d_2}$ is the observation noise independent of $X_i$. Model (1) can be written in a more compact form using matrix notation. Particularly, define the multi-response matrix $Y = (Y_1, Y_2, \cdots, Y_n)^\top \in \mathbb{R}^{n \times d_2}$ with similar definitions for the covariate matrix $X \in \mathbb{R}^{n \times d_1}$ ($n \ll d_1$) and the noise matrix $\epsilon \in \mathbb{R}^{n \times d_2}$ in terms of $\{X_i\}_{i=1}^n$ and $\{\epsilon_i\}_{i=1}^n$, respectively. Then model (1) is re-written as

$$Y = X\Theta^* + \epsilon.$$  

We work within a high-dimensional framework which allows the number of covariates $d_2$ to grow and possibly more than the sample size $n$ in this paper. It is already known that consistent estimation cannot be obtained when $n \ll d_1$ unless the model is imposed with additional structure, such as low-rankness in the matrix estimation problems. In the following, unless otherwise specified, $\Theta^* \in \mathbb{R}^{d_1 \times d_2}$ is assumed to be of either exactly low-rank, i.e., it has rank far less than $\min\{d_1, d_2\}$, or near low-rank, meaning that it can be approximated by a matrix of low-rank perfectly. One popular way to measure the degree of low-rank is to use the matrix $\ell_q$-ball (Accurately speaking, when $q \in [0, 1)$, these sets are not real “ball”s, as they fail to be convex), which is defined as, for $q \in [0, 1]$, and a radius $R_q > 0$,

$$\mathbb{B}_q(R_q) := \{\Theta \in \mathbb{R}^{d_1 \times d_2} | \sum_{i=1}^d |\sigma_i(\Theta)|^q \leq R_q\},$$

where $d = \min\{d_1, d_2\}$. Note that the $\ell_0$ matrix ball corresponds to the case of exact low-rank, meaning rank at most $R_0$; while the $\ell_q$ matrix ball for $q \in (0, 1]$ corresponds to the case of near low-rank, which enforces a certain decay rate on the ordered of the singular values of the matrix $\Theta \in \mathbb{B}_q(R_q)$. In this paper, we fix $q \in [0, 1]$, and assume that the true parameter $\Theta^* \in \mathbb{B}_q(R_q)$ unless otherwise specified.

In standard formulations, the true covariates $X_i$’s are assumed to be fully-observed. However, this assumption is not realistic for many real-world applications, in which the covariates may be observed with measurement errors, and we use $Z_i$’s to denote corrupted versions of the corresponding $X_i$’s. $Z_i$ is usually assumed to be linked to $X_i$ via some conditional distribution as follows

$$Z_i \sim Q(\cdot | X_i) \quad \text{for } i = 1, 2, \cdots, n.$$  

As has been discussed in previous literatures (e.g., [17, 19]), there are mainly two types of corruption:
(a) Additive errors: For each \( i = 1, 2, \ldots, n \), we observe \( Z_i = X_i + W_i \), where \( W_i \in \mathbb{R}^{d_1} \) is a random vector independent of \( X_i \) with mean 0 and known covariance matrix \( \Sigma_w \).

(b) Missing data: For each \( i = 1, 2, \ldots, n \), we observe a random vector \( Z_i \in \mathbb{R}^{d_1} \), such that for each \( j = 1, 2, \ldots, d_1 \), we independently observe \( Z_{ij} = X_{ij} \) with probability \( 1 - \rho \), and \( Z_{ij} = 0 \) with probability \( \rho \), where \( \rho \in [0, 1) \).

Throughout this paper, we assume a Gaussian model on the covariate and error matrices. Specifically, the matrices \( X \), \( W \) and \( \epsilon \) are assumed to be random matrices with independent and identically distributed (i.i.d.) rows as sampled from Gaussian distributions \( \mathcal{N}(0, \Sigma_x) \), \( \mathcal{N}(0, \sigma^2_w \mathbb{I}_{d_1}) \) and \( \mathcal{N}(0, \sigma^2_\epsilon \mathbb{I}_{d_2}) \), respectively. Then one has that \( \Sigma_w = \sigma^2_w \mathbb{I}_{d_1} \).

Define \( Z = (Z_1, Z_2, \ldots, Z_n)^\top \). Then \( Z \) is the observed covariate matrix, which involves certain types of measurement error, and will be specified according to the context.

2.2 Error-corrected M-estimators

When there exists no measurement error, meaning that the covariate matrix \( X \) is correctly obtained, previous literatures have proposed various methods for the rank-constrained problems; see, e.g., \([20, 22, 31]\) and references therein. Most of the estimators are formulated as solutions to certain semidefinite programs (SDPs) based on the nuclear norm regularization. Recall that for a matrix \( \Theta \in \mathbb{R}^{d_1 \times d_2} \), the nuclear or trace norm is defined by \( \|\Theta\|_* = \sum_{j=1}^d \sigma_j(\Theta) \) (\( d = \min(d_1, d_2) \)), corresponding to the sum of its singular values. For instance, given model \([2]\), define \( N = nd_2 \), a commonly-used estimator is based on solving the following SDP:

\[
\hat{\Theta} \in \arg\min_{\Theta \in \mathcal{S}} \left\{ \frac{1}{2N} \|Y - X\Theta\|_F^2 + \lambda_N \|\Theta\|_* \right\},
\]

where \( \mathcal{S} \) is a subset of \( \mathbb{R}^{d_1 \times d_2} \), and \( \lambda_N > 0 \) is a regularization parameter. The nuclear norm of a matrix offers a natural convex relaxation of the rank constraint, analogous to the \( \ell_1 \)-norm as a convex relaxation of the cardinality of a vector. The statistical and computational property of the nuclear norm as a regularizer has been studied deeply and applied widely in various fields, such as matrix completion, matrix decomposition and so on. Apart from the nuclear norm, other regularizers including the elastic net, SCAD and MCP, which were first proposed to solve sparse recovery problem in linear regression, have also been used in low-rank matrix estimation problem; see \([31]\) for a detailed investigation.

Note that \((5)\) can be re-written as follows

\[
\hat{\Theta} \in \arg\min_{\Theta \in \mathcal{S}} \left\{ \frac{1}{2N} \langle X^\top X\Theta, \Theta \rangle - \frac{1}{N} \langle X^\top Y, \Theta \rangle + \lambda_N \|\Theta\|_* \right\}.
\]

Recall the relation \( N = nd_2 \), the SDP optimization problem \((6)\) is transformed to

\[
\hat{\Theta} \in \arg\min_{\Theta \in \mathcal{S}} \frac{1}{d_2} \left\{ \frac{1}{2n} \langle X^\top X\Theta, \Theta \rangle - \frac{1}{n} \langle X^\top Y, \Theta \rangle + \lambda_n \|\Theta\|_* \right\},
\]

where we have defined \( \lambda_n = d_2 \lambda_N \). In the case of measurement errors, the quantities \( \frac{X^\top X}{n} \) and \( \frac{X^\top Y}{n} \) are both unknown, which means that this estimator does not work. However, this transformation still provides some useful intuition for estimation via the plug-in principle.
Specifically, given a set of samples, one way is to find suitable estimates of the quantities $X_n^\top X_n$ and $X_n^\top Y_n$ that are adapted to the cases of additive noise and/or missing data.

Let $(\hat{\Gamma}, \hat{\Upsilon})$ denote estimates of $(X_n^\top X_n, X_n^\top Y_n)$. Inspired by (7) and ignoring the constant, we propose the following estimator to solve the low-rank estimation problem in the measurement error case:

$$\hat{\Theta} \in \arg \min_{\Theta \in \mathcal{S}} \left\{ \frac{1}{2} \langle \langle \hat{\Gamma} \Theta, \Theta \rangle \rangle - \langle \langle \hat{\Upsilon}, \Theta \rangle \rangle + \lambda \|\Theta\|_* \right\},$$

(8)

where $\lambda$ is the regularization parameter. The feasible region is specialized as $\mathcal{S} = \{\Theta \in \mathbb{R}^{d_1 \times d_2} : \|\Theta\|_* \leq \omega \}$, and the parameter $\omega > 0$ must be chosen carefully to guarantee $\Theta^* \in \mathcal{S}$. We include this side constraint is because of the nonconvex nature of the estimator, which will be explained in detail in the following. Then any matrix $\Theta \in \mathcal{S}$ will also satisfy $\|\Theta\|_* \leq \omega$, and since the optimization subject function is continuous, it is guaranteed by Weierstrass extreme value theorem that a global solution $\hat{\Theta}$ always exists.

Note that the estimator (8) is just a general expression, and concrete values for $(\hat{\Gamma}, \hat{\Upsilon})$ still need to be determined. For the specific additive noise and missing data cases, as discussed in [17], an unbiased choice of the pair $(\hat{\Gamma}, \hat{\Upsilon})$ is given respectively by

$$\hat{\Gamma}_{\text{add}} := Z_n^\top Z_n - \Sigma_w$$

and

$$\hat{\Upsilon}_{\text{add}} := Z_n^\top Y_n,$$

(9)

$$\hat{\Gamma}_{\text{mis}} := \frac{\hat{Z}_n^\top \hat{Z}_n}{n} - \rho \cdot \text{diag} \left( \frac{\hat{Z}_n^\top \hat{Z}_n}{n} \right)$$

and

$$\hat{\Upsilon}_{\text{mis}} := \frac{\hat{Z}_n^\top Y_n}{n} \left( \hat{Z} = \frac{Z}{1-\rho} \right).$$

(10)

In the high-dimensional regime ($n \ll d_1$), the matrices $\hat{\Gamma}_{\text{add}}$ and $\hat{\Gamma}_{\text{mis}}$ in (9) and (10) are always negative definite; indeed, both the matrices $Z_n^\top Z_n$ and $\hat{Z}_n^\top \hat{Z}_n$ are with rank at most $n$, and then the positive definite matrices $\Sigma_w$ and $\rho \cdot \text{diag} \left( \frac{\hat{Z}_n^\top \hat{Z}_n}{n} \right)$ are subtracted to arrive at the estimates $\hat{\Gamma}_{\text{add}}$ and $\hat{\Gamma}_{\text{mis}}$, respectively. Therefore, the above estimator (8) involves solving a nonconvex optimization problem. Due to the nonconvexity, it is generally impossible to obtain a global solution through a polynomial-time algorithm. Nevertheless, this issue is not significant in our setting, and we shall establish that a simple proximal gradient algorithm converges linearly to a matrix extremely close to any global optimum of the problem (8) with high probability.

### 2.3 Regularity conditions

Now we impose some regularity conditions on the surrogate matrices $\hat{\Gamma}$ and $\hat{\Upsilon}$, which will be beneficial to the statistical and computational analysis for the estimator (8).

In high-dimensional linear regression with univariate responses (i.e., $d_2 = 1$), when the true covariate matrix $X$ is correctly obtained, it is well understood that a type of restricted eigenvalue condition (REC) is sufficient to guarantee nice $\ell_2$ recovery for the Lasso; see, e.g., [7, 28]. The REC then was utilized to establish $\ell_2$ recovery bound in the errors-in-variables linear regression [17]. More general conditions called restricted strong convexity/smoothness (RSC/RSM) have also been adopted in the analysis of matrix regression, and is applicable when the loss function is nonquadratic or nonconvex; see, e.g., [16, 19, 20]. In this paper, the following RSC/RSM conditions are required.
Definition 1. [Restricted strong convexity] The matrix $\hat{\Gamma}$ is said to satisfy a restricted strong convexity with curvature $\alpha_1 > 0$ and tolerance $\tau(n,d_1,d_2) > 0$ if
\[
\langle \langle \hat{\Gamma} \Delta, \Delta \rangle \rangle \geq \alpha_1 \|\Delta\|_F^2 - \tau \|\Delta\|_*^2 \quad \text{for all} \ \Delta \in R^{d_1 \times d_2}.
\] (11)

Definition 2. [Restricted strong smoothness] The matrix $\hat{\Gamma}$ is said to satisfy a restricted strong smoothness with smoothness $\alpha_2 > 0$ and tolerance $\tau(n,d_1,d_2) > 0$ if
\[
\langle \langle \hat{\Gamma} \Delta, \Delta \rangle \rangle \leq \alpha_2 \|\Delta\|_F^2 - \tau \|\Delta\|_*^2 \quad \text{for all} \ \Delta \in R^{d_1 \times d_2}.
\] (12)

It has been shown that for linear/matrix regression without measurement errors, the RSC/RSM conditions are satisfied by various types of random matrices with high probability [1, 20]. Similar results will be established for our choice of $\hat{\Gamma}$ in the cases of additive noise and missing data in the following.

Recall that $(\hat{\Gamma}, \hat{\Theta})$ are estimates for the unknown quantities $(X^TX/n, X^TY/n)$. Then a deviation bound is needed to measure the approximate degree. Particularly, we assume that there is some function $\phi(Q, \sigma)$, depending on the two sources of corruptions in our setting: the conditional distribution $Q$ (cf. (4)) that links the true covariates $X_t$ to the corrupted versions $Z_i$ and the standard deviation $\sigma_e$ of the observation noise $\epsilon_i$. With this notation, we consider the following deviation condition:
\[
\|\hat{\Theta} - \hat{\Theta}^*\|_o \leq \phi(Q, \sigma_e) \sqrt{\frac{\max d_1, d_2}{n}}.
\] (13)

Similar to the RSC/RSM conditions, the deviation condition (13) will also be verified for various forms of corruptions, with the quantity $\phi(Q, \sigma_e)$ changing according to the specific model.

3 Main results

In this section, we establish our main results including general statistical guarantee for the nonconvex estimator [3] and convergence rate for the proximal gradient algorithm to solve it. Consequences for the additive noise and missing data cases will be discussed in the next section.

Before we proceed, some additional notations are required to facilitate the analysis of exact/near low-rank matrices. First let $\Psi(\Theta) = \frac{1}{2} \langle \langle \hat{\Gamma} \Theta, \Theta \rangle \rangle - \langle \langle \hat{\Theta}, \Theta \rangle \rangle + \lambda \|\Theta\|_*$ denote the objective function to be minimized, and $L(\Theta) = \frac{1}{2} \langle \langle \hat{\Gamma} \Theta, \Theta \rangle \rangle - \langle \langle \hat{\Theta}, \Theta \rangle \rangle$ denote the quadratic loss function. Then one has that $\Psi(\Theta) = L(\Theta) + \lambda \|\Theta\|_*$.

Note that the parameter matrix $\Theta^*$ has a singular value decomposition of the form $\Theta^* = UDV^T$, where $U \in R^{d_1 \times d}$ and $V \in R^{d_2 \times d}$ are orthonormal matrices with $d = \min\{d_1, d_2\}$ and without loss of generality, assume that $D$ is diagonal with the singular values in nonincreasing order, i.e., $\sigma_1(\Theta^*) \geq \sigma_2(\Theta) \geq \cdots \sigma_d(\Theta) \geq 0$. For each integer $r \in \{1, 2, \cdots, d\}$, let $U^r \in R^{d_1 \times r}$ and $V^r \in R^{d_2 \times r}$ be the sub-matrices consist of singular vectors associated with the top $r$ singular values of $\Theta^*$. Then two subspaces of $R^{d_1 \times d_2}$ associated with $\Theta^*$ are defined as follows:

\[
A(U^r, V^r) := \{\Delta \in R^{d_1 \times d_2} | \text{row}(\Delta) \subseteq \text{col}(V^r), \text{col}(\Delta) \subseteq \text{col}(U^r)\} \quad \text{and} \quad \text{(14a)}
\]
\[
B(U^r, V^r) := \{\Delta \in R^{d_1 \times d_2} | \text{row}(\Delta) \subseteq (\text{col}(V^r))^T, \text{col}(\Delta) \subseteq \text{col}(U^r))^T\} \quad \text{and} \quad \text{(14b)}
\]
where \( \text{row}(\Delta) \in \mathbb{R}^d \) and \( \text{col}(\Delta) \in \mathbb{R}^d \) denote the row space and column space of the matrix \( \Theta^* \), respectively. When the sub-matrices \((U^r, V^r)\) are clear from the context, the shorthand notation \( \mathcal{A}^r \) and \( \mathcal{B}^r \) are adopted instead. Then for any pair of matrices \( \Theta \in \mathcal{A}(U^r, V^r) \) and \( \Theta' \in \mathcal{B}(U^r, V^r) \), it holds that \( \|\Theta + \Theta'\|_* = \|\Theta\|_* + \|\Theta'\|_* \), i.e., the nuclear norm is decomposable with respect to the subspaces \( \mathcal{A}^r \) and \( \mathcal{B}^r \).

Still consider the singular decomposition \( \Theta^* = UDV^\top \). For any positive number \( \eta > 0 \) to be chosen, we define the set corresponding to \( \Theta^* \):

\[
K_\eta := \{ j \in \{1, 2, \ldots, d\} \mid |\sigma_j(\Theta^*)| > \eta \}. 
\]

According to the above notations, the matrix \( U^{\lfloor K_\eta \rfloor} \) (resp., \( V^{\lfloor K_\eta \rfloor} \)) represents the \( d_1 \times |K_\eta| \) (resp., \( d_2 \times |K_\eta| \)) orthogonal matrix consisting of the first \( |K_\eta| \) columns of \( U \) (resp., \( V \)). With this choice, the matrix \( \Theta_{K_\eta}^* := \Pi_{B^{\lfloor K_\eta \rfloor}}(\Theta^*) \) has rank at most \( d - |K_\eta| \), with singular values \( \{\sigma_j(\Theta^*), j \in K_\eta^c\} \). Moreover, recall the true parameter \( \Theta^* \in B(q(R_q)) \). Then the cardinality of \( K_\eta \) and the approximation error \( \|\Theta_{K_\eta}^*\|_* \) can both be bounded from above. Indeed, by a standard argument (see, e.g., [20]), one checks that

\[
|K_\eta| \leq \eta^{-q} R_q \quad \text{and} \quad \|\Theta_{K_\eta}^*\|_* \leq \eta^{1-q} R_q. 
\]

Using the notations, we now state a useful technical lemma that shows, for the true parameter matrix \( \Theta^* \) and any matrix \( \Theta \), we can decompose \( \Delta := \Theta - \Theta^* \) as the sum of two matrices \( \Delta' \) and \( \Delta'' \) such that the rank of \( \Delta' \) is not too large. This lemma will serve as the key decomposition for proving out main results.

**Lemma 1.** For a positive integer \( r \in \{1, 2, \ldots, d\} \), let \( U^r \in \mathbb{R}^{d_1 \times r} \) and \( V^r \in \mathbb{R}^{d_2 \times r} \) be matrices consisting of the top \( r \) left and right singular vectors of \( \Theta^* \), respectively. Let \( \Theta \in \mathbb{R}^{d_1 \times d_2} \) be an arbitrary matrix. Then for the error matrix \( \Delta := \Theta - \Theta^* \), there exists a decomposition \( \Delta = \Delta' + \Delta'' \) such that:

(i) the matrix \( \Delta' \) satisfies the constraint \( \text{rank}(\Delta') \leq 2r \);

(ii) moreover, suppose that \( \Theta = \hat{\Theta} \) is a global optimum of the optimization problem \( \mathcal{S} \). Then if \( \lambda \geq 2\phi(Q, \sigma_t)\sqrt{\frac{\max(d_1, d_2)}{n}} \), the nuclear norm of \( \Delta'' \) is bounded as

\[
\|\Delta''\|_* \leq 3\|\Delta'\|_* + 4 \sum_{j=r+1}^{d} \sigma_j(\Theta^*). 
\]

**Proof.** (i) Write the SVD as \( \Theta^* = UDV^\top \), where \( U \in \mathbb{R}^{d_1 \times d_1} \) and \( V \in \mathbb{R}^{d_2 \times d_2} \) are orthogonal matrices, and \( D \) is the matrix consisting of the singular values of \( \Theta^* \). Then it is obvious to see that \( U^r \) and \( V^r \) are formed by the first \( r \) columns of \( U \) and \( V \), respectively. Define the matrix \( \Xi = U^\top \Delta V \in \mathbb{R}^{d_1 \times d_2} \). Partition \( \Xi \) in block form as follows

\[
\Xi := \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{pmatrix}, \quad \text{where} \quad \Xi_{11} \in \mathbb{R}^{r \times r} \text{, and} \quad \Xi_{22} \in \mathbb{R}^{(m_1 - r) \times (m_2 - r)}. 
\]

Set the matrices as

\[
\Delta'' := U \begin{pmatrix} 0 & 0 \\ 0 & \Xi_{22} \end{pmatrix} V^\top, \quad \text{and} \quad \Delta' := \Delta - \Delta''.
\]
Then the rank of $\hat{\Delta}'$ is upper bounded as
\[
\text{rank}(\Delta') = \text{rank}( \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & 0 \end{pmatrix} ) \leq \text{rank}( \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ 0 & 0 \end{pmatrix} ) + \text{rank}( \begin{pmatrix} \Xi_{11} & 0 \\ \Xi_{21} & 0 \end{pmatrix} ) \leq 2r,
\]
which established Lemma 1(i). Moreover, by the construction of $\Delta''$, it follows that
\[
\|\Pi_{A'}(\Theta^*) + \Delta''\|_* = \|\Pi_{A'}(\Theta^*)\|_* + \|\Delta''\|_*
\]
(18)

(ii) We now turn to the proof of Lemma 1(ii). By the feasibility of $\Theta^*$ and optimality of $\hat{\Theta}$, one has that $\Psi(\hat{\Theta}) \leq \Psi(\Theta^*)$. Then it follows from some elementary algebra and Hölder’s inequality that
\[
\frac{1}{2} \langle (\hat{\Gamma}, \Delta) \rangle \leq \langle (\hat{\Gamma} - \hat{\Gamma}^* \Theta^*, \Delta) \rangle + \lambda \|\Theta^*\|_* - \lambda \|\hat{\Theta}\|_* \leq \|\hat{\Gamma} - \hat{\Gamma}^* \Theta^*\|_{\text{op}} \|\Delta\|_* + \lambda \|\Theta^*\|_* - \lambda \|\hat{\Theta}\|_* \tag{19}
\]
which implies that
\[
0 \leq \|\hat{\Gamma} - \hat{\Gamma}^* \Theta^*\|_{\text{op}} \|\Delta\|_* + \lambda \|\Theta^*\|_* - \lambda \|\hat{\Theta}\|_* \tag{19}
\]
Note that the decomposition $\Theta^* = \Pi_{A'}(\Theta^*) + \Pi_{B'}(\Theta^*)$ holds. This equality, together with the triangle inequality and (18), implies that
\[
\|\hat{\Theta}\|_* = \|\Pi_{A'}(\Theta^*) + \Delta''\|_* + \|\Pi_{B'}(\Theta^*) + \Delta'\|_* \geq \|\Pi_{A'}(\Theta^*) + \Delta''\|_* - \|\Pi_{B'}(\Theta^*) + \Delta'\|_* \geq \|\Pi_{A'}(\Theta^*)\|_* + \|\Delta''\|_* - \{\|\Pi_{B'}(\Theta^*)\|_* + \|\Delta'\|_*\}
\]
Consequently, we have
\[
\|\Theta^*\|_* - \|\hat{\Theta}\|_* \leq \|\Theta^*\|_* - \|\Pi_{A'}(\Theta^*)\|_* - \|\Delta''\|_* + \{\|\Pi_{B'}(\Theta^*)\|_* + \|\Delta'\|_*\} \leq 2\|\Pi_{B'}(\Theta^*)\|_* + \|\Delta'\|_* - \|\Delta''\|_*.
\]
Substituting this inequality into (19), we obtain that
\[
0 \leq \|\hat{\Gamma} - \hat{\Gamma}^* \Theta^*\|_{\text{op}} \|\Delta\|_* + \lambda \{2\|\Pi_{B'}(\Theta^*)\|_* + \|\Delta'\|_* - \|\Delta''\|_*\}.
\]
Finally, by the assumption that $\lambda \geq 2\phi(Q, \sigma) \sqrt{\frac{\max(d_1, d_2)}{n}}$ and the deviation condition (13), we conclude that
\[
0 \leq \lambda \{2\|\Pi_{B'}(\Theta^*)\|_* + \frac{3}{2} \|\Delta'\|_* - \frac{1}{2} \|\Delta''\|_*\}.
\]
Then (17) follows from the fact that $\|\Pi_{B'}(\Theta^*)\|_* = \sum_{j=r+1}^d \sigma_j(\Theta^*)$. \qed
3.1 Statistical results

**Theorem 1.** Let $R_q > 0$ and $\omega > 0$ be positive numbers such that $\Theta^* \in \mathcal{B}_q(R_q) \cap \mathcal{S}$. Let $\hat{\Theta}$ be a global optimum of the optimization problem (8). Suppose that the surrogate matrices $(\hat{\Gamma}, \hat{\Upsilon})$ satisfies the deviation condition (13), and that $\hat{\Gamma}$ satisfies the RSC condition (11) with

$$\tau \leq \frac{\phi(Q, \sigma_e)}{\omega} \sqrt{\frac{\max(d_1, d_2)}{n}}. \quad (21)$$

Assume that $\lambda$ is chosen to satisfy

$$\lambda \geq 2\phi(Q, \sigma_e) \sqrt{\frac{\max(d_1, d_2)}{n}}. \quad (22)$$

Then we have that

$$\|\hat{\Theta} - \Theta^*\|_F^2 \leq 544 R_q \left( \frac{\lambda}{\alpha_1} \right)^{2-q}, \quad (23)$$

$$\|\hat{\Theta} - \Theta^*\|_* \leq (4 + 32\sqrt{17}) R_q \left( \frac{\lambda}{\alpha_1} \right)^{1-q}. \quad (24)$$

**Proof.** Set $\hat{\Delta} := \hat{\Theta} - \Theta^*$. By the feasibility of $\Theta^*$ and optimality of $\hat{\Theta}$, one has that $\Psi(\hat{\Theta}) \leq \Psi(\Theta^*)$. Then it follows from some elementary algebra and the triangle inequality that

$$\frac{1}{2} \langle \langle \hat{\Gamma} \hat{\Delta}, \hat{\Delta} \rangle \rangle \leq \langle \langle \hat{\Gamma} - \hat{\Gamma} \Theta^*, \hat{\Delta} \rangle \rangle + \lambda \|\Theta^*\|_* - \lambda \|\Theta^* + \hat{\Delta}\|_* \leq \langle \langle \hat{\Gamma} - \hat{\Gamma} \Theta^*, \hat{\Delta} \rangle \rangle + \lambda \|\hat{\Delta}\|_* \quad (25).$$

Applying Hölder’s inequality and by the deviation condition (13), one has that

$$\langle \langle \hat{\Gamma} - \hat{\Gamma} \Theta^*, \hat{\Delta} \rangle \rangle \leq \phi(Q, \sigma_e) \sqrt{\frac{\max(d_1, d_2)}{n}} \|\hat{\Delta}\|_* \quad (26).$$

Combining the above two inequalities, and noting (22) we obtain that

$$\langle \langle \hat{\Gamma} \hat{\Delta}, \hat{\Delta} \rangle \rangle \leq 3\lambda \|\hat{\Delta}\|_* \quad (26).$$

Applying the RSC condition (11) to the left-hand side of (26), yields that

$$\alpha_1 \|\hat{\Delta}\|_F^2 - \tau \|\hat{\Delta}\|_*^2 \leq 3\lambda \|\hat{\Delta}\|_* \quad (27).$$

On the other hand, by assumptions (21) and (22) and noting the fact that $\|\hat{\Delta}\|_* \leq \|\Theta^*\|_* + \|\hat{\Theta}\|_* \leq 2\omega$, the left-hand side of (27) is lower bounded as

$$\alpha_1 \|\hat{\Delta}\|_F^2 - \tau \|\hat{\Delta}\|_*^2 \geq \alpha_1 \|\hat{\Delta}\|_F^2 - 2\tau \omega \|\hat{\Delta}\|_* \geq \alpha_1 \|\hat{\Delta}\|_F^2 - \lambda \|\hat{\Delta}\|_* \quad (28).$$

Combining this inequality with (27), one has that

$$\alpha_1 \|\hat{\Delta}\|_F^2 \leq 4\lambda \|\hat{\Delta}\|_* \quad (29).$$
Then it follows from Lemma 1 that there exists a matrix \( \hat{\Delta}' \) such that

\[
\| \hat{\Delta} \|_* \leq 4 \| \hat{\Delta}' \|_* + 4 \sum_{j=r+1}^d \sigma_j(\Theta^*) \leq 4 \sqrt{2r} \| \hat{\Delta}' \|_F + 4 \sum_{j=r+1}^d \sigma_j(\Theta^*), \tag{30}
\]

where \( \text{rank}(\hat{\Delta}') \leq 2r \) with \( r \) to be chosen later, and the second inequality is due to the fact that \( \| \hat{\Delta}' \|_* \leq \sqrt{2r} \| \hat{\Delta}' \|_F \). Combining (29) and (30), we obtain that

\[
\alpha_1 \| \hat{\Delta} \|_F^2 \leq 16\lambda \left( \sqrt{2r} \| \hat{\Delta}' \|_F + \sum_{j=r+1}^d \sigma_j(\Theta^*) \right) 
\leq 16\lambda \left( \sqrt{2r} \| \hat{\Delta} \|_F + \sum_{j=r+1}^d \sigma_j(\Theta^*) \right). \tag{31}
\]

Then it follows that

\[
\| \hat{\Delta} \|_F^2 \leq \frac{512r\lambda^2 + 32\alpha_1\lambda \sum_{j=r+1}^d \sigma_j(\Theta^*)}{\alpha_1^2}. \tag{32}
\]

Recall the set \( K_\eta \) defined in (15) and set \( r = |K_\eta| \). Combining (32) with (16) and setting \( \eta = \frac{\lambda}{\alpha_1} \), we arrive at (23). Moreover, it follows from (30) that (24) holds. The proof is complete.

3.2 Computational results

We now apply the proximal gradient method [21] to solve the proposed nonconvex optimization problem (8) and then establish the linear convergence result to the global solution. Recall the quadratic loss function \( L(\Theta) = \frac{1}{2} \langle \hat{\Gamma}\Theta, \Theta \rangle - \langle \hat{\Upsilon}, \Theta \rangle \), and the optimization objective function \( \Psi(\Theta) = L(\Theta) + \lambda \| \Theta \|_* \). The gradient of the loss function takes the form \( \nabla L(\Theta) = \hat{\Gamma}\Theta - \hat{\Upsilon} \).

Then it is easy to see that the optimization objective function consists of a differentiable but nonconvex function and a nonsmooth but convex function (i.e., the nuclear norm). The proximal gradient method proposed in [21] is applied to (8) to obtain a sequence of iterates \( \{\Theta^t\}_{t=0}^\infty \) as

\[
\Theta^{t+1} \in \arg\min_{\Theta \in S} \left\{ \frac{1}{2} \| \Theta - \left( \Theta^t - \nabla L(\Theta^t) \right) \|_F^2 + \frac{\lambda}{v} \| \Theta \|_* \right\}, \tag{33}
\]

where \( v \) is the step size.

Recall the feasible region \( S = \{ \Theta \in \mathbb{R}^{d_1 \times d_2} : \| \Theta \|_* \leq \omega \} \). Given \( \Theta^t \), one can follow [19] to generate the next iterate \( \Theta^{t+1} \) via the following three steps; see [19, Appendix C.1] for details.

1. First optimize the unconstrained optimization problem

   \[
   \hat{\Theta}^t \in \arg\min_{\Theta \in \mathbb{R}^{d_1 \times d_2}} \left\{ \frac{1}{2} \| \Theta - \left( \Theta^t - \nabla L(\Theta^t) \right) \|_F^2 + \frac{\lambda}{v} \| \Theta \|_* \right\},
   \]

2. If \( \| \Theta^t \|_* \leq \omega \), define \( \Theta^{t+1} = \hat{\Theta}^t \).
(3) Otherwise, if \( \| \Theta^t \|_r > \omega \), optimize the constrained optimization problem

\[
\Theta^{t+1} \in \arg \min_{\| \Theta \|_r \leq \omega} \left\{ \frac{1}{2} \left\| \Theta - \left( \Theta - \frac{\nabla L(\Theta)}{v} \right) \right\|_F^2 \right\}.
\]

Before we state our main result that the algorithm defined by (33) converges linearly to a small neighbourhood of the global solution \( \hat{\Theta} \), we shall need some notations to simplify our expositions.

Recall the RSC and RSM conditions in (11) and (12), respectively. Recall that the true underlying parameter \( \Theta^* \in B_q(R_q) \) (cf. (3)). Let \( \hat{\Theta} \) be a global solution of the optimization problem (8). Then unless otherwise specified, we define

\[
\bar{\epsilon}_{\text{stat}} := 8 R_q^{\frac{1}{2}} \lambda^{\frac{1}{2}} \left( \sqrt{2} \| \hat{\Theta} - \Theta^* \|_F + R_q^{\frac{1}{2}} \lambda^{\frac{1}{2}} \right),
\]

\[
\kappa := \left\{ 1 - \frac{\alpha_1}{8v} + \frac{256 R_q \tau \lambda^{-q}}{\alpha_1} \right\} \left\{ 1 - \frac{256 R_q \tau \lambda^{-\frac{2}{q}}}{\alpha_1} \right\}^{-1},
\]

\[
\xi := \tau \left\{ \alpha_1 \frac{512 R_q \tau \lambda^{\frac{2}{2}}}{8v} + 5 \right\} \left\{ 1 - \frac{256 R_q \tau \lambda^{-\frac{2}{q}}}{\alpha_1} \right\}^{-1}.
\]

For a given number \( \delta > 0 \) and an integer \( T > 0 \) such that

\[
\Psi(\Theta^t) - \Psi(\hat{\Theta}) \leq \delta, \quad \forall \ t \geq T,
\]

define

\[
\epsilon(\delta) := 2 \min \left( \frac{\delta}{\lambda ; \omega} \right).
\]

With this setup, we now state our main algorithmic result.

**Theorem 2.** Let \( R_q > 0 \) and \( \omega > 0 \) be positive numbers such that \( \Theta^* \in B_q(R_q) \cap S \). Let \( \hat{\Theta} \) be a global solution of the optimization problem (8). Suppose that the RSC/RSM conditions \([11]\) and \([12]\) are satisfied with

\[
\tau \leq \frac{\phi(Q, \sigma_e)}{\omega} \sqrt{\frac{\max(d_1, d_2)}{n}}.
\]

Let \( \{ \Theta^t \}_{t=0}^{\infty} \) be a sequence of iterates generated via (33) with an initial point \( \Theta^0 \) and step size \( v \geq \max\{4\alpha_1, \alpha_2\} \). Assume that \( \lambda \) is chosen to satisfy

\[
\lambda \geq \max \left\{ \left( \frac{128 R_q \tau}{\alpha_1} \right)^{1/q} \sqrt{\frac{\max(d_1, d_2)}{n}}, 4 \phi(Q, \sigma_e) \sqrt{\frac{\max(d_1, d_2)}{n}} \right\}.
\]

Then for any tolerance \( \delta^* \geq \frac{8 \kappa}{1 - \kappa} \bar{\epsilon}_{\text{stat}}^2 \) and any iteration \( t \geq T(\delta^*) \), we have that

\[
\left\| \Theta^t - \hat{\Theta} \right\|_F^2 \leq \frac{4}{\alpha_1} \left( \delta^* + \frac{\delta^*}{2 \tau \omega^2} + 2 \tau \bar{\epsilon}_{\text{stat}}^2 \right),
\]

where

\[
T(\delta^*) := \log_2 \log_2 \left( \frac{\omega \lambda}{\delta^*} \right) \left( 1 + \frac{\log 2}{\log (1/\kappa)} \right) + \frac{\log((\Psi(\Theta^0) - \Psi(\hat{\Theta}))/\delta^*)}{\log(1/\kappa)},
\]

and \( \bar{\epsilon}_{\text{stat}}, \kappa, \xi \) are defined in (34)-(73), respectively.
Before providing the proof of Theorem 2, we need several useful lemmas first.

**Lemma 2.** Suppose that the conditions of Theorem 2 are satisfied, and that there exists a pair \((\delta, T)\) such that Theorem 2 holds. Then for any iteration \(t \geq T\), it holds that

\[
\left\| \Theta^t - \Theta \right\|_* \leq 4\sqrt{2R_q^2 \lambda^{-\frac{q}{2}}} \left\| \Theta^t - \Theta \right\|_F + \epsilon_{\text{stat}} + \epsilon(\delta).
\]

**Proof.** We first show that if \(\lambda \geq 4\phi(Q, \sigma)\sqrt{\frac{\max(d_1, d_2)}{n}}\), then for any \(\Theta \in \mathcal{S}\) satisfying

\[
\Psi(\Theta) - \Psi(\Theta^*) \leq \delta,
\]

it holds that

\[
\left\| \Theta - \Theta^* \right\|_* \leq 4\sqrt{2R_q^2 \lambda^{-\frac{q}{2}}} \left\| \Theta - \Theta^* \right\|_F + 4R_q \lambda^{1-q} + 2 \min \left( \frac{\delta}{\lambda^{-q}}, \omega \right).
\]

Set \(\Delta := \Theta - \Theta^*\). From (41), we obtain that

\[
\mathcal{L}(\Theta^* + \Delta) + \lambda \left\| \Theta^* + \Delta \right\|_* - \lambda \left\| \Theta^* \right\|_* \leq -\langle \hat{\Gamma} \Theta^* - \hat{\Upsilon}, \Delta \rangle + \delta.
\]

We now claim that

\[
\lambda \left\| \Theta^* + \Delta \right\|_* - \lambda \left\| \Theta^* \right\|_* \leq \frac{\lambda}{2} \left\| \Delta \right\|_* + \delta.
\]

In fact, combining (42) with the RSC condition (11) and the H"older’s inequality, one has that

\[
\frac{1}{2} \left\{ \alpha_1 \left\| \Delta \right\|_F^2 - \tau \left\| \Delta \right\|_*^2 \right\} + \lambda \left\| \Theta^* + \Delta \right\|_* - \lambda \left\| \Theta^* \right\|_* \leq \left\| \hat{\Upsilon} - \hat{\Gamma} \Theta^* \right\|_{op} \left\| \Delta \right\|_* + \delta.
\]

This inequality, together with the deviation condition (13) and the assumption that \(\lambda \geq 4\phi(Q, \sigma)\sqrt{\frac{\max(d_1, d_2)}{n}}\), implies that

\[
\frac{1}{2} \left\{ \alpha_1 \left\| \Delta \right\|_F^2 - \tau \left\| \Delta \right\|_*^2 \right\} + \lambda \left\| \Theta^* + \Delta \right\|_* - \lambda \left\| \Theta^* \right\|_* \leq \frac{\lambda}{4} \left\| \Delta \right\|_* + \delta.
\]

Noting the facts that \(\alpha_1 > 0\) and that \(\left\| \Delta \right\|_* \leq \left\| \Theta^* \right\|_* + \left\| \Theta \right\|_* \leq 2\omega\), one arrives at (44) by the assumption that \(\lambda \geq 4\tau\omega\). On the other hand, it follows from Lemma (i) that there exists two matrices \(\Delta'\) and \(\Delta''\) such that \(\Delta = \Delta' + \Delta''\), where \(\text{rank}(\Delta') \leq 2r\) with \(r\) to be chosen later. Recall the definitions of \(A^*\) and \(B^*\). Then the decomposition \(\Theta^* = \Pi_{A^*}(\Theta^*) + \Pi_{B^*}(\Theta^*)\) holds. This equality, together with the triangle inequality and Lemma (i) as well as (18), implies that

\[
\left\| \Theta \right\|_* = \left\| (\Pi_{A^*}(\Theta^*) + \Delta'') + (\Pi_{B^*}(\Theta^*) + \Delta') \right\|_* \geq \left\| (\Pi_{A^*}(\Theta^*) + \Delta'') \right\|_* - \left\| (\Pi_{B^*}(\Theta^*) + \Delta') \right\|_* \\
\geq \left\| (\Pi_{A^*}(\Theta^*)) \right\|_* + ||\Delta''||_* - \left\{ \left\| \Pi_{B^*}(\Theta^*) \right\|_* + ||\Delta'||_* \right\}.
\]
Consequently, we have

\[
\|\Theta^*\|_s - \|\Theta\|_s \leq \|\Theta^*\|_s - \|\Pi_{B^*}(\Theta^*)\|_s - \|\Delta''\|_s + \{\|\Pi_{B^*}(\Theta^*)\|_s + \|\Delta'\|_s \}
\leq 2\|\Pi_{B^*}(\Theta^*)\|_s + \|\Delta'\|_s - \|\Delta''\|_s.
\]  

(45)

Combining (45) and (44) and noting the fact that \(\|\Pi_{B^*}(\Theta^*)\|_s = \sum_{j=r+1}^{\lambda} \sigma_j(\Theta^*)\), one has that \(0 \leq \frac{3}{2}\|\Delta'\|_s - \frac{\lambda}{2}\|\Delta''\|_s + 2\lambda\sum_{j=r+1}^{\lambda} \sigma_j(\Theta^*) + \delta\), and consequently, \(\|\Delta''\|_s \leq 3\|\Delta'\|_s + 4\sum_{j=r+1}^{\lambda} \sigma_j(\Theta^*) + \frac{3\delta}{\lambda}\). Using the trivial bound \(\|\delta\|_s \leq 2\omega\), one has that

\[
\|\Delta\|_s \leq 4\sqrt{2}\|\Delta\|_F + 4\sum_{j=r+1}^{\lambda} \sigma_j(\Theta^*) + 2\min\left(\frac{\delta}{\lambda}, \omega\right).
\]  

(46)

Recall the set \(K_{\eta}\) defined in (15) and set \(r = |K_{\eta}|\). Combining (46) with (16) and setting \(\eta = \lambda\), we arrive at (42). We now verify that (41) is held by the vector \(\hat{\Theta}\) and \(\Theta^\star\), respectively. Since \(\hat{\Theta}\) is the optimal solution, it holds that \(\Psi(\hat{\Theta}) \leq \Psi(\Theta^\star)\), and by assumption (37), it holds that \(\Psi(\Theta^\star) \leq \Psi(\Theta) + \delta \leq \Psi(\Theta^\star) + \delta\). Consequently, it follows from (42) that

\[
\|\hat{\Theta} - \Theta^\star\|_s \leq 4\sqrt{2}\|\hat{\Theta} - \Theta^\star\|_F + 4\|\Delta\|_F + 4\sum_{j=r+1}^{\lambda} \sigma_j(\Theta^\star) + 2\min\left(\frac{\delta}{\lambda}, \omega\right).
\]

(47)

By the triangle inequality, we then conclude that

\[
\|\Theta^\star - \Theta^\star\|_s \leq \|\Theta^\star - \Theta^\star\|_s + \|\Theta^\star - \Theta^\star\|_s
\leq 4\sqrt{2}\|\Theta^\star - \Theta^\star\|_F + \|\Theta^\star - \Theta^\star\|_F + 4\|\Delta\|_F + 4\sum_{j=r+1}^{\lambda} \sigma_j(\Theta^\star) + 2\min\left(\frac{\delta}{\lambda}, \omega\right)
\leq 4\sqrt{2}\|\Theta^\star - \Theta^\star\|_F + \epsilon_{\text{stat}} + \epsilon(\delta).
\]

The proof is complete.

\[\square\]

**Lemma 3.** Suppose that the conditions of Theorem 2 are satisfied and that there exists a pair \((\Delta, T)\) such that (37) holds. Then for any iteration \(t \geq T\), we have that

\[
\mathcal{L}(\hat{\Theta}) - \mathcal{L}(\Theta^\star) - \langle \nabla \mathcal{L}(\Theta^\star), \hat{\Theta} - \Theta^\star \rangle \geq -\tau(\epsilon_{\text{stat}} + \epsilon(\delta))^2,
\]  

(47)

\[
\Psi(\Theta^\star) - \Psi(\hat{\Theta}) \geq \frac{\alpha_1}{4}\|\Theta^\star - \hat{\Theta}\|_F^2 - \tau(\epsilon_{\text{stat}} + \epsilon(\delta))^2,
\]  

(48)

\[
\Psi(\Theta^\star) - \Psi(\hat{\Theta}) \leq \kappa^{-T}(\Psi(\Theta^\star) - \Psi(\hat{\Theta})) + \frac{2\xi}{1 - \kappa}(\epsilon_{\text{stat}}^2 + \epsilon^2(\delta)).
\]  

(49)

**Proof.** By the RSC condition (11), one has that

\[
\mathcal{L}(\Theta^\star) - \mathcal{L}(\hat{\Theta}) - \langle \nabla \mathcal{L}(\Theta^\star), \Theta^\star - \hat{\Theta} \rangle \geq \frac{1}{2} \left\{\alpha_1\|\Theta^\star - \hat{\Theta}\|_F^2 - \tau\|\Theta^\star - \hat{\Theta}\|_s^2\right\}.
\]  

(50)

It then follows from Lemma 2 and the assumption that \(\lambda \geq \left(\frac{128R\xi}{\alpha_1}\right)^{1/q}\) that

\[
\mathcal{L}(\hat{\Theta}) - \mathcal{L}(\Theta^\star) - \langle \nabla \mathcal{L}(\hat{\Theta}), \Theta^\star - \hat{\Theta} \rangle \geq \frac{1}{2} \left\{\alpha_1\|\Theta^\star - \hat{\Theta}\|_F^2 - \tau\|\Theta^\star - \hat{\Theta}\|_s^2\right\} \geq -\tau(\epsilon_{\text{stat}} + \epsilon(\delta))^2,
\]  

14
which establishes \((\ref{eq:47})\). Furthermore, it follows from the convexity of \(\|\cdot\|_*\) that
\[
\lambda \|\Theta^t\|_* - \lambda \|\hat{\Theta}\|_* - \langle \nabla \{ \lambda \|\hat{\Theta}\|_* \} , \Theta^t - \hat{\Theta} \rangle \geq 0,
\]
and the first-order optimality condition for \(\hat{\Theta}\) that
\[
\langle \nabla \Psi(\hat{\Theta}), \Theta^t - \hat{\Theta} \rangle \geq 0.
\]
Combining \((\ref{eq:50})\), \((\ref{eq:51})\) and \((\ref{eq:52})\), one has that
\[
\Psi(\Theta^t) - \Psi(\hat{\Theta}) \geq \frac{1}{2} \left\{ \alpha_1 \|\Theta^t - \hat{\Theta}\|_F^2 - \tau \|\Theta^t - \hat{\Theta}\|_*^2 \right\}.
\]
Then using Lemma \((\ref{lem:2})\) to bound the term \(\|\Theta^t - \hat{\Theta}\|_*^2\) and noting the assumption that \(\lambda \geq \left(\frac{128R_{\alpha}^2}{\alpha_1}\right)^{1/q}\), we arrive at \((\ref{eq:48})\). Now we turn to prove \((\ref{eq:49})\). Define
\[
\Psi_t(\Theta^t) := \mathcal{L}(\Theta^t) + \langle \nabla \mathcal{L}(\Theta^t), \Theta - \Theta^t \rangle + \frac{v}{2} \|\Theta - \Theta^t\|_F^2 + \lambda \|\Theta\|_*^2,
\]
which is the optimization objective function minimized over the feasible region \(\mathcal{S} = \{ \Theta : \|\Theta\|_* \leq \omega \}\) at iteration count \(t\). For any \(a \in [0, 1]\), it is easy to see that the matrix \(\Theta_a = a\hat{\Theta} + (1-a)\Theta^t\) belongs to \(\mathcal{S}\) by the convexity of \(\mathcal{S}\). Since \(\Theta^{t+1}\) is the optimal solution of the optimization problem \((\ref{eq:33})\), we have that
\[
\Psi_t(\Theta^{t+1}) \leq \Psi_t(\Theta_a) = \mathcal{L}(\Theta^t) + \langle \nabla \mathcal{L}(\Theta^t), \Theta_a - \Theta^t \rangle + \frac{v}{2} \|\Theta_a - \Theta^t\|_F^2 + \lambda \|\Theta_a\|_*^2
\]
\[
\leq \mathcal{L}(\Theta^t) + \langle \nabla \mathcal{L}(\Theta^t), a\hat{\Theta} - a\Theta^t \rangle + \frac{va^2}{2} \|\hat{\Theta} - \Theta^t\|_F^2 + a\lambda \|\hat{\Theta}\|_*^2 + (1-a)\lambda \|\Theta^t\|_*^2,
\]
where the last inequality is from the convexity of \(\|\cdot\|_*\). Then by \((\ref{eq:17})\), one has that
\[
\Psi_t(\Theta^{t+1}) \leq (1-a)\mathcal{L}(\Theta^t) + a\mathcal{L}(\hat{\Theta}) + a\tau(\epsilon_{stat} + \epsilon(\delta))^2 + \frac{va^2}{2} \|\hat{\Theta} - \Theta^t\|_F^2 + a\lambda \|\hat{\Theta}\|_*^2 + (1-a)\lambda \|\Theta^t\|_*^2
\]
\[
\leq \Psi(\Theta^t) - a(\Psi(\Theta^t) - \Psi(\hat{\Theta})) + \tau(\epsilon_{stat} + \epsilon(\delta))^2 + \frac{va^2}{2} \|\hat{\Theta} - \Theta^t\|_*^2.
\]
Applying the RSM condition \((\ref{eq:12})\) on the pair \((\Theta^{t+1}, \Theta^t)\), on has by assumption \(v \geq \alpha_2\) that
\[
\mathcal{L}(\Theta^{t+1}) - \mathcal{L}(\Theta^t) - \langle \nabla \mathcal{L}(\Theta^t), \Theta^{t+1} - \Theta^t \rangle \leq \frac{1}{2} \left\{ \alpha_2 \|\Theta^{t+1} - \Theta^t\|_F^2 + \tau \|\Theta^{t+1} - \Theta^t\|_*^2 \right\}
\]
\[
\leq \frac{v}{2} \|\Theta^{t+1} - \Theta^t\|_F^2 + \frac{\tau}{2} \|\Theta^{t+1} - \Theta^t\|_*^2.
\]
Adding \(\lambda \|\Theta^{t+1}\|_*\) to both sides of the former inequality yields that
\[
\Psi(\Theta^{t+1}) \leq \mathcal{L}(\Theta^t) + \langle \nabla \mathcal{L}(\Theta^t), \Theta^{t+1} - \Theta^t \rangle + \lambda \|\Theta^{t+1}\|_*^2 + \frac{v}{2} \|\Theta^{t+1} - \Theta^t\|_F^2 + \frac{\tau}{2} \|\Theta^{t+1} - \Theta^t\|_*^2
\]
\[
= \Psi_t(\Theta^{t+1}) + \frac{\tau}{2} \|\Theta^{t+1} - \Theta^t\|_*^2.
\]
This, together with (53), implies that
\[
\Psi(\Theta^{t+1}) \leq \Psi(\Theta^t) - a(\Psi(\Theta^t) - \Psi(\hat{\Theta})) + \frac{va^2}{2} \| \hat{\Theta} - \Theta \|^2_F + \frac{\tau}{2} \| \Theta^{t+1} - \Theta^t \|^2_s + \tau(\bar{\epsilon}_{\text{stat}} + \epsilon(\delta))^2.
\]
Define \( \Delta^t := \Theta^t - \hat{\Theta} \). Then it follows that \( \| \Theta^{t+1} - \Theta^t \|^2_s \leq (\| \Delta^{t+1} \|^2_s + \| \Delta^t \|^2_s) \leq 2 \| \Delta^{t+1} \|^2_s + 2 \| \Delta^t \|^2_s \). Combining this inequality with (54), one has that
\[
\Psi(\Theta^{t+1}) \leq \Psi(\Theta^t) - a(\Psi(\Theta^t) - \Psi(\hat{\Theta})) + \frac{va^2}{2} \| \hat{\Theta} - \Theta \|^2_F + \frac{\tau}{2} (\| \Delta^{t+1} \|^2_s + \| \Delta^t \|^2_s) + \tau(\bar{\epsilon}_{\text{stat}} + \epsilon(\delta))^2.
\]
To simplify the notations, we define \( \psi := \tau(\bar{\epsilon}_{\text{stat}} + \epsilon(\delta))^2, \zeta := R_q \tau \lambda^{-q} \) and \( \delta_t := \Psi(\Theta^t) - \Psi(\hat{\Theta}) \). Applying Lemma 2 to bound the term \( \| \Delta^{t+1} \|^2_s \) and \( \| \Delta^t \|^2_s \), we obtain that
\[
\Psi(\Theta^{t+1}) \leq \Psi(\Theta^t) - a(\Psi(\Theta^t) - \Psi(\hat{\Theta})) + \frac{va^2}{2} \| \Delta^t \|^2_F + 64R_q \tau \lambda^{-q}(\| \Delta^{t+1} \|^2_F + \| \Delta^t \|^2_F) + 5\psi
= \Psi(\Theta^t) - a(\Psi(\Theta^t) - \Psi(\hat{\Theta})) + \left( \frac{va^2}{2} + 64\zeta \right) \| \Delta^t \|^2_F + 64\zeta \| \Delta^{t+1} \|^2_F + 5\psi.
\]
Subtracting \( \Psi(\hat{\Theta}) \) from both sides of (55), we have by (48) that
\[
\delta_{t+1} \leq (1 - a)\delta_t + \frac{2va^2 + 256\zeta}{\alpha_1}(\delta_t + \psi) + \frac{256\zeta}{\alpha_1}(\delta_{t+1} + \psi) + 5\psi.
\]
Setting \( a = \frac{\alpha_1}{2v} \in (0, 1) \), one has by the former inequality that
\[
\left( 1 - \frac{256\zeta}{\alpha_1} \right) \delta_{t+1} \leq \left( 1 - \frac{\alpha_1}{8v} + \frac{256\zeta}{\alpha_1} \right) \delta_t + \left( \frac{\alpha_1}{8v} + \frac{512\zeta}{\alpha_1} + 5 \right) \psi,
\]
or equivalently, \( \delta_{t+1} \leq \kappa \delta_t + \xi(\bar{\epsilon}_{\text{stat}} + \epsilon(\delta))^2 \), where \( \kappa \) and \( \xi \) were previously defined in (72) and (73), respectively. Finally, we obtain that
\[
\Delta_t \leq \kappa^{t-T} \Delta_T + \xi(\bar{\epsilon}_{\text{stat}} + \epsilon(\delta))^2(1 + \kappa + \kappa^2 + \cdots + \kappa^{t-T-1})
\leq \kappa^{t-T} \Delta_T + \frac{\xi}{1 - \kappa}(\bar{\epsilon}_{\text{stat}} + \epsilon(\delta))^2 \leq \kappa^{t-T} \Delta_T + \frac{2\xi}{1 - \kappa}(\bar{\epsilon}_{\text{stat}} + \epsilon^2(\delta)).
\]
The proof is complete. \( \square \)

By virtue of the above lemmas, we are now ready to prove Theorem 2.

**Proof of Theorem 2.** We first prove the inequality as follows:
\[
\Psi(\Theta^t) - \Psi(\hat{\Theta}) \leq \delta^*, \quad \forall t \geq T(\delta^*).
\]
Divide the iterations \( t = 0, 1, \cdots \) into a series of disjoint epochs \( [T_k, T_{k+1}] \) and define an associated sequence of tolerances \( \delta_0 \geq \delta_1 \geq \cdots \) such that
\[
\Psi(\Theta^t) - \Psi(\hat{\Theta}) \leq \delta_k, \quad \forall t \geq T_k,
\]
as well as the associated error term \( \epsilon_k := 2 \min \{ \frac{k}{k^2}, \omega \} \). The values of \( \{ (\delta_k, T_k) \}_{k \geq 1} \) will be chosen later. Then at the first iteration, we apply Lemma 3 (cf. (49)) with \( \epsilon_0 = 2 \omega \) and \( T_0 = 0 \) to conclude that

\[
\Psi(\Theta^t) - \Psi(\hat{\Theta}) \leq \kappa^t (\Psi(\Theta^0) - \Psi(\hat{\Theta})) + \frac{2\kappa}{1 - \kappa} (\epsilon_{stat}^2 + 4\omega^2), \quad \forall t \geq T_0.
\]  

(57)

Set \( \delta_1 := \frac{4\kappa}{1 - \kappa} (\epsilon_{stat}^2 + 4\omega^2) \). Noting that \( \kappa \in (0, 1) \) by assumption, one has by (57) that for \( T_1 := \left\lceil \frac{\log(2\delta_0/\delta_1)}{\log(1/\kappa)} \right\rceil \),

\[
\Psi(\Theta^t) - \Psi(\hat{\Theta}) \leq \frac{\delta_1}{2} + \frac{2\kappa}{1 - \kappa} (\epsilon_{stat}^2 + 4\omega^2) = \delta_1 \leq \frac{8\kappa}{1 - \kappa} \max \{ \epsilon_{stat}^2, \epsilon_k^2 \}, \quad \forall t \geq T_1.
\]

For \( k \geq 1 \), we define

\[
\delta_{k+1} := \frac{4\kappa}{1 - \kappa} (\epsilon_{stat}^2 + \epsilon_k^2) \quad \text{and} \quad T_{k+1} := \left\lceil \frac{\log(2\delta_k/\delta_{k+1})}{\log(1/\kappa)} + T_k \right\rceil.
\]

(58)

Then Lemma 3 (cf. (49)) is applicable to concluding that for all \( t \geq T_k \),

\[
\Psi(\Theta^t) - \Psi(\hat{\Theta}) \leq \delta_{k+1} \leq \frac{8\kappa}{1 - \kappa} \max \{ \epsilon_{stat}^2, \epsilon_k^2 \}, \quad \forall t \geq T_{k+1}.
\]

From (58), we obtain the following recursion for \( \{ (\delta_k, T_k) \}_{k=0}^\infty \)

\[
\delta_{k+1} \leq \frac{8\kappa}{1 - \kappa} \max \{ \epsilon_k^2, \epsilon_{stat}^2 \}, \quad T_k \leq k + \frac{\log(2\delta_k/\delta_{k+1})}{\log(1/\kappa)}.
\]

(59a)

(59b)

Then by [2, Section 7.2], one sees that (59a) implies that

\[
\delta_{k+1} \leq \frac{\delta_k}{4^{2^k+1}} \quad \text{and} \quad \frac{\delta_{k+1}}{\lambda} \leq \frac{\omega}{4^{2^k}}, \quad \forall k \geq 1.
\]

(60)

Now let us show how to decide the smallest \( k \) such that \( \delta_k \leq \delta^* \) by applying (60). If we are in the first epoch, \( (56) \) is clearly from (59a). Otherwise, by (59b), we see that \( \delta_k \leq \delta^* \) holds after at most

\[
k(\delta^*) \geq \frac{\log(\log(\omega\lambda/\delta^*)/\log 4)}{\log(2)} + 1 = \log_2 \log_2 (\omega\lambda/\delta^*)
\]

epochs. Combining the above bound on \( k(\delta^*) \) with (59b), we conclude that \( \Psi(\Theta^t) - \Psi(\hat{\Theta}) \leq \delta^* \) holds for all iterations

\[
t \geq \log_2 \log_2 \left( \frac{\omega\lambda}{\delta^*} \right) \left( 1 + \frac{\log 2}{\log(1/\kappa)} \right) + \frac{\log(\delta_0/\delta^*)}{\log(1/\kappa)}.
\]

17
which proves (56). Finally, as (56) is proved, it follows from (48) in Lemma 3 and the assumption that \( \lambda \geq \left( \frac{128 R \tau^2}{a_1} \right)^{1/q} \) that, for any \( t \geq T(\delta^*) \),

\[
\frac{\alpha_1}{4} \left\| \Theta^t - \hat{\Theta} \right\|_F^2 \leq \Psi(\Theta^t) - \Psi(\hat{\Theta}) + \tau \left( \epsilon(\delta^*) + \tilde{\epsilon}_{stat} \right)^2 \leq \delta^* + \tau \left( \frac{2\delta^*}{\lambda} + \tilde{\epsilon}_{stat} \right)^2.
\]

Consequently, by assumptions (38) and (39), we conclude that for any \( t \geq T(\delta^*) \),

\[
\left\| \Theta^t - \hat{\Theta} \right\|_F^2 \leq \frac{4}{\alpha_1} \left( \delta^* + \frac{\delta^*^2}{2\tau \omega^2} + 2\tau \tilde{\epsilon}_{stat}^2 \right). 
\]

The proof is complete. \( \square \)

4 Consequences for the measurement error cases

As have been mentioned previously, both Theorems 1 and 2 are deterministic results. Consequences for specific statistical models requires some probabilistic discussions in order to verify that the stated conditions are satisfied, namely, the RSC/RSM conditions (11)/(12) and the deviation condition (13). We now turn to the statements of probabilistic consequences of these theorems for different cases of additive noise and missing data. The following propositions are needed first to facilitate the discussions.

4.1 Additive noise case

In the additive noise case, let us first set \( \Sigma_z = \Sigma_x + \Sigma_w \), \( \sigma_z^2 = \|\Sigma_x\|_{op}^2 + \sigma_w^2 \) for notational simplicity. Then define

\[
\tau_{add} = \lambda_{\min}(\Sigma_x) \max \left( \frac{d_2(\|\Sigma_x\|_{op}^2 + \sigma_w^2)^2}{\lambda_{\min}^2(\Sigma_x)}, \frac{d_2(\|\Sigma_x\|_{op}^2 + \sigma_w^2)}{\lambda_{\min}(\Sigma_x)} \right) \frac{2 \max(d_1, d_2) + \log(\min(d_1, d_2))}{n}, \tag{61a}
\]

\[
\phi_{add} = \sqrt{\lambda_{\max}(\Sigma_x)}(\sigma_x + \omega \sigma_w). \tag{61b}
\]

**Proposition 1** (RSC/RSM conditions, additive noise case). **In the additive noise case, there exist universal positive constants \( c_0, c_1 \) such that the matrix \( \Gamma_{add} \) satisfies the RSC and RSM conditions (cf. (11) and (12)) with parameters \( \alpha_1 = \frac{\lambda_{\min}(\Sigma_x)}{2}, \alpha_2 = \frac{3\lambda_{\max}(\Sigma_x)}{2}, \) and \( \tau = c_0\tau_{add} \), with probability at least \( 1 - 2 \exp\left( -c_1 n \min \left( \frac{\lambda_{\min}(\Sigma_x)}{d_1^2(\|\Sigma_x\|_{op}^2 + \sigma_w^2)^2}, \frac{\lambda_{\min}(\Sigma_x)}{d_2(\|\Sigma_x\|_{op}^2 + \sigma_w^2)} \right) \right) \).**

**Proof.** Set

\[
r = \frac{1}{c'} \min \left( \frac{\lambda_{\min}(\Sigma_x)}{d^2\sigma_z^4}, \frac{\lambda_{\min}(\Sigma_x)}{d_2^2\sigma_z^2} \right) \frac{n}{2 \max(d_1, d_2) + \log(\min(d_1, d_2))}, \tag{62}
\]

with \( c' > 0 \) being chosen sufficiently small so that \( r \geq 1 \). Then noting that \( \hat{\Gamma}_{add} - \Sigma_x = \frac{Z^T Z}{n} - \Sigma_z \) and by Lemma A.3 one sees that it suffices to show that

\[
\sup_{\Delta \in K(2r)} \left| \langle \langle \frac{Z^T Z}{n} - \Sigma_z, \Delta \rangle \rangle \Delta \right| \leq \frac{\lambda_{\min}(\Sigma_x)}{24}.
\]

18
holds with high probability. Let \( D(r) = \sup_{\Delta \in \mathbb{K}(2r)} \| (\frac{Z^T Z}{n} - \Sigma_z) \Delta, \Delta \| \) for simplicity. Note that the matrix \( Z \) is sub-Gaussian with parameters \((\Sigma_z, \sigma_z^2)\). Then it follows from Lemma A.5 that there exists a universal positive constant \( c' \) such that

\[
P \left[ D(r) \geq \frac{\lambda_{\min}(\Sigma_x)}{24} \right] \leq 2 \exp \left( -c' n \min \left( \frac{\lambda_{\min}^2(\Sigma_x)}{376d_1^2\sigma_z^4}, \frac{\lambda_{\min}(\Sigma_x)}{24d_2\sigma_z^2} \right) \right).
\]

This inequality, together with (62), implies that there exists universal positive constant \((c_0, c_1)\) such that \( \tau = c_0 \lambda_{\min}(\Sigma_x) \max \left( \frac{d_2(\| \Sigma_w \|_{\text{op}}^2 + \sigma_z^2)^2}{\lambda_{\min}^2(\Sigma_x)}, \frac{d_2(\| \Sigma_x \|_{\text{op}}^2 + \sigma_z^2)}{\lambda_{\min}(\Sigma_x)} \right) \frac{2 \max(d_1, d_2) + \log(\min(d_1, d_2))}{n}, \) and

\[
P \left[ D(r) \geq \frac{\lambda_{\min}(\Sigma_x)}{24} \right] \leq 2 \exp \left( -c_1 n \min \left( \frac{\lambda_{\min}^2(\Sigma_x)}{d_2^2\sigma_z^4}, \frac{\lambda_{\min}(\Sigma_x)}{d_2\sigma_z^2} \right) + \log d_2 \right),
\]

which completes the proof.

**Proposition 2** (Deviation condition, additive noise case). In the additive noise case, there exist universal positive constants \((c_0, c_1, c_2)\) such that the deviation condition (cf. (13)) holds with parameter \( \phi(Q, \sigma_z) = c_0 \phi_{\text{add}}, \) with probability at least \( 1 - c_1 \exp(-c_2 \log(\max(d_1, d_2)))).

**Proof.** By the definition of \( \hat{\Gamma}_{\text{add}} \) and \( \hat{\Gamma}_{\text{add}}(\text{cf. (7)}) \), one has that

\[
\left\| \hat{\Gamma}_{\text{add}} - \hat{\Gamma}_{\text{add}}\Theta^* \right\|_{\text{op}} = \left\| \frac{Z^TY}{n} - \left( \frac{Z^TZ}{n} - \Sigma_w \right) \Theta^* \right\|_{\text{op}}
\]

\[
= \left\| \frac{Z^T(X\Theta^* + \epsilon)}{n} - \left( \frac{Z^TZ}{n} - \Sigma_w \right) \Theta^* \right\|_{\text{op}}
\]

\[
\leq \left\| \frac{Z^T\epsilon}{n} \right\|_{\text{op}} + \left\| (\Sigma_w - \frac{Z^TW}{n}) \Theta^* \right\|_{\text{op}}
\]

\[
\leq \left\| \frac{Z^T\epsilon}{n} \right\|_{\text{op}} + \left( \left\| \Sigma_w \right\|_{\text{op}} + \left\| \frac{Z^TW}{n} \right\|_{\text{op}} \right) \left\| \Theta^* \right\|_\infty,
\]

where the second inequality is from the fact that \( Y = X\Theta^* + \epsilon, \) and the third inequality is due to the triangle inequality. Recall the assumption that the matrices \( X, W \) and \( \epsilon \) are assumed to be with i.i.d. rows sampled from Gaussian distributions \( \mathcal{N}(0, \Sigma_x) \), \( \mathcal{N}(0, \sigma_w^2 I_{d_1}) \) and \( \mathcal{N}(0, \sigma_z^2 I_{d_2}) \), respectively. Then one has that \( \Sigma_w = \sigma_w^2 I_{d_1} \) and \( \left\| \Sigma_w \right\|_{\text{op}} = \sigma_w. \) It follows from (20) Lemma 3] that there exists universal positive constants \((c_3, c_4, c_5)\) such that

\[
\left\| \hat{\Gamma}_{\text{add}} - \hat{\Gamma}_{\text{add}}\Theta^* \right\|_{\text{op}} \leq c_3 \sigma_x \sqrt{\lambda_{\max}(\Sigma_z)} \sqrt{\frac{d_1 + d_2}{n}} + (\sigma_w + c_3 \sigma_w \sqrt{\lambda_{\max}(\Sigma_z)} \sqrt{\frac{2d_1}{n}}) \left\| \Theta^* \right\|_\infty,
\]

with probability at least \( 1 - c_4 \exp(-c_5 \log(\max(d_1, d_2))). \) Recall that the nuclear norm of \( \Theta \) is assumed to be bounded by \( \left\| \Theta \right\|_\infty \leq \omega. \) Then up to constant factors, we conclude that there exists universal positive constants \((c_0, c_1, c_2)\) such that

\[
\left\| \hat{\Gamma}_{\text{add}} - \hat{\Gamma}_{\text{add}}\Theta^* \right\|_{\text{op}} \leq c_0 \sqrt{\lambda_{\max}(\Sigma_z)} (\sigma_x + \omega \sigma_w) \sqrt{\frac{\max(d_1, d_2)}{n}},
\]

with probability at least \( 1 - c_1 \exp(-c_2 \log(\max(d_1, d_2))). \) The proof is complete. \qed
Now we are ready to state statistical and computational consequences for the multiresponse regression with additive noise. The conclusion follow by applying Propositions 1 and 2 on Theorems 1 and 2 respectively, and so the proofs are omitted.

**Corollary 1.** Let $R_q > 0$ and $\omega > 0$ be positive numbers such that $\Theta^* \in \mathbb{B}_q(R_q) \cap \mathcal{S}$. Let $\hat{\Theta}$ be a global optimum of the optimization problem (8) with $(\hat{\Gamma}_{\text{add}}, \hat{\Phi}_{\text{add}})$ given by (9) in place of $(\hat{\Gamma}, \hat{\Phi})$. Then there exist universal positive constants $c_i$ ($i = 0, 1, 2, 3, 4$) such that if $T_{\text{add}} \leq 20 \phi_{\text{add}}^2 \sqrt{\max(d_1,d_2)/\nu}$, and $\lambda$ is chosen to satisfy $\lambda \geq c_1 \phi_{\text{add}} \sqrt{\max(d_1,d_2)/n}$, then it holds with probability at least $1 - 2 \exp \left( -c_2 n \min \left( \frac{\lambda_{\min}(\Sigma_x)}{d_2 \| \Sigma_x \|_{\text{op}}^2 + \sigma_x^2}, \frac{\lambda_{\min}(\Sigma_x)}{d_2 \| \Sigma_x \|_{\text{op}}^2 + \sigma_x^2} \right) + \log d_2 \right) - c_3 \exp( -c_4 \log(\max(d_1,d_2)) )$

$$\left\| \hat{\Theta} - \Theta^* \right\|_F^2 \leq 1644 R_q \left( \frac{2 \lambda}{\lambda_{\min}(\Sigma_x)} \right)^{2-q},$$

$$\left\| \hat{\Theta} - \Theta^* \right\|_F \leq (4 + 32\sqrt{17}) R_q \left( \frac{2 \lambda}{\lambda_{\min}(\Sigma_x)} \right)^{1-q}.$$ Define

$$\kappa_{\text{add}} := \left\{ \frac{1 - \lambda_{\min}(\Sigma_x)}{16 \nu} + \frac{512 R_q \tau_{\text{add}} \lambda^{-q}}{\lambda_{\min}(\Sigma_x)} \right\} \left\{ 1 - \frac{512 R_q \tau_{\text{add}} \lambda^{-q}}{\lambda_{\min}(\Sigma_x)} \right\}^{-1}, \quad (63)$$

$$\xi_{\text{add}} := \tau_{\text{add}} \left\{ \frac{\lambda_{\min}(\Sigma_x)}{16 \nu} + \frac{1024 R_q \tau_{\text{add}} \lambda^{-q}}{\lambda_{\min}(\Sigma_x)} + 5 \right\} \left\{ 1 - \frac{512 R_q \tau_{\text{add}} \lambda^{-q}}{\lambda_{\min}(\Sigma_x)} \right\}^{-1}. \quad (64)$$

**Corollary 2.** Let $R_q > 0$ and $\omega > 0$ be positive numbers such that $\Theta^* \in \mathbb{B}_q(R_q) \cap \mathcal{S}$. Let $\hat{\Theta}$ be a global solution of the optimization problem (8) with $(\hat{\Gamma}_{\text{add}}, \hat{\Phi}_{\text{add}})$ given by (9) in place of $(\hat{\Gamma}, \hat{\Phi})$. Let $\{\Theta^t_{\text{add}}\}_{t=0}^{\infty}$ be a sequence of iterates generated via (33) with an initial point $\Theta^0_{\text{add}}$ and step size $\nu \geq \max(2\lambda_{\min}(\Sigma_x), \frac{3}{2} \lambda_{\max}(\Sigma_x))$. Then there exist universal positive constants $c_i$ ($i = 0, 1, 2, 3, 4, 5$) such that, if $T_{\text{add}} \leq \frac{c_0 \phi_{\text{add}}^2}{\omega} \sqrt{\frac{\lambda_{\min}(\Sigma_x)}{d_2 \| \Sigma_x \|_{\text{op}}^2 + \sigma_x^2}}$, and $\lambda$ is chosen to satisfy

$$\lambda \geq \max \left\{ \left( \frac{c_1 R_q \tau_{\text{add}}}{\lambda_{\min}(\Sigma_x)} \right)^{1/q}, \frac{c_2 \phi_{\text{add}} \sqrt{\max(d_1,d_2)}}{n} \right\},$$

then for any tolerance $\delta^* \geq \frac{8 \xi_{\text{add}} \lambda_{\min}(\Sigma_x)}{1 - \kappa_{\text{add}}} \sigma_{\text{stat}}^2$ and any iteration $t \geq T(\delta^*)$, it holds with probability at least $1 - 2 \exp \left( -c_3 n \min \left( \frac{\lambda_{\min}(\Sigma_x)}{d_2 \| \Sigma_x \|_{\text{op}}^2 + \sigma_x^2}, \frac{\lambda_{\min}(\Sigma_x)}{d_2 \| \Sigma_x \|_{\text{op}}^2 + \sigma_x^2} \right) + \log d_2 \right) - c_4 \exp( -c_5 \log(\max(d_1,d_2)) )$

$$\left\| \Theta^t_{\text{add}} - \Theta^* \right\|_F^2 \leq \frac{8}{\lambda_{\min}(\Sigma_x)} \left( \delta^* + \frac{\delta^2}{2 \tau_{\text{add}} \omega} + 2 \tau_{\text{add}} \epsilon_{\text{stat}}^2 \right),$$

where

$$T(\delta^*) := \log(2) \log \left( \frac{\omega \lambda}{\delta^*} \right) \left( 1 + \frac{\log(2)}{\log(1/\kappa_{\text{add}})} \right) + \log \left( \frac{\lambda_{\min}(\Sigma_x)}{d_2 \| \Sigma_x \|_{\text{op}}^2 + \sigma_x^2} \right) \sigma_{\text{stat}}^2 + \frac{\log(2 \tau_{\text{add}} \omega \delta^*)}{\log(1/\kappa_{\text{add}})} + \frac{\log(\| \Psi(\Theta^0_{\text{add}}) - \Psi(\hat{\Theta}) \|/\delta^*)}{\log(1/\kappa_{\text{add}})},$$

and $\epsilon_{\text{stat}}$ is given in (34).
4.2 Missing data case

In the missing data case, we first define a matrix \( M \in \mathbb{R}^{d_1 \times d_1} \) satisfying \( M_{ij} = (1 - \rho)^2 \) for \( i \neq j \) and \( M_{ij} = 1 - \rho \) for \( i = j \). Let \( \otimes \) and \( \odot \) denote element-wise multiplication and division, respectively, and set \( \Sigma_z = \Sigma_x \otimes M \). Then define

\[
\tau_{\text{mis}} = \min(\Sigma_x) \max\left( \frac{1}{(1 - \rho)^4}, \frac{d_2^2 ||\Sigma_x||_p^2}{\lambda_{\min}(\Sigma_x)^2} \right) \frac{2 \max(d_1, d_2) + \log(\min(d_1, d_2))}{n},
\]

\[
\phi_{\text{mis}} = \frac{\max(\Sigma_z)}{1 - \rho} \left( \frac{\omega}{1 - \rho} ||\Sigma_x||_{\text{op}} + \sigma_x \right).
\]

**Proposition 3** (RSC/RSM conditions, missing data case). *In the missing data case, there exist universal positive constants \((c_0, c_1)\) such that the matrix \( \tilde{\Gamma}_{\text{mis}} \) satisfies the RSC and RSM conditions (cf. [11] and [12]) with parameters \( \alpha_1 = \frac{\lambda_{\min}(\Sigma_x)}{2}, \alpha_2 = \frac{3\lambda_{\max}(\Sigma_x)}{2}, \) and \( \tau = c_0 \tau_{\text{mis}}, \)

with probability at least \( 1 - 2 \exp\left( -c_1 n \min\left( (1 - \rho)^4 d_2^2 ||\Sigma_x||_p^2, (1 - \rho)^2 \lambda_{\min}(\Sigma_x) \right) + \log d_2 \right) \).

**Proof.** This proof is similar to that of Proposition 1 in the additive noise case. Set \( \sigma^2 = \frac{||\Sigma_x||_p^2}{(1 - \rho)^2}, \)

and

\[
r = \frac{1}{c'} \min\left( \frac{\lambda_{\min}(\Sigma_x)}{d_2^2 \sigma^2}, \frac{\lambda_{\min}(\Sigma_x)}{d_2 \sigma^2} \right) \frac{n}{2 \max(d_1, d_2) + \log(\min(d_1, d_2))},
\]

with \( c' > 0 \) being chosen sufficiently small so that \( r \geq 1 \). Note that

\[
\tilde{\Gamma}_{\text{mis}} = \frac{1}{(1 - \rho)^2} \frac{Z^T Z}{n} - \rho \cdot \text{diag}\left( \frac{1}{(1 - \rho)^2} \frac{Z^T Z}{n} \right) = \frac{Z^T Z}{n} \odot M,
\]

and thus

\[
\tilde{\Gamma}_{\text{mis}} - \Sigma_x = \frac{Z^T Z}{n} \odot M - \Sigma_x = \left( \frac{Z^T Z}{n} - \Sigma_z \right) \odot M.
\]

By Lemma A.3, one sees that it suffices to show that

\[
\sup_{\Delta \in \mathbb{K}(2r)} \left| \left| \left\langle \left( \frac{Z^T Z}{n} - \Sigma_z \right) \odot M, \Delta \right\rangle \right| \right| \leq \frac{\lambda_{\min}(\Sigma_x)}{24}
\]

holds with high probability. Let \( D(r) = \sup_{\Delta \in \mathbb{K}(2r)} \left| \left| \left\langle \left( \frac{Z^T Z}{n} - \Sigma_z \right) \odot M, \Delta \right\rangle \right| \right| \) for simplicity. On the other hand, one has that

\[
\left| \left| \left\langle \left( \frac{Z^T Z}{n} - \Sigma_z \right) \odot M, \Delta \right\rangle \right| \right| \leq \frac{1}{(1 - \rho)^2} \left| \left| \left\langle \left( \frac{Z^T Z}{n} - \Sigma_z \right) \Delta, \Delta \right\rangle \right| \right|
\]

Note that the matrix \( Z \) is sub-Gaussian with parameters \((\Sigma_z, ||\Sigma_x||_{\text{op}}^2)\) [18]. Then it follows from Lemma A.5 that there exists a universal positive constant \( c'' \) such that

\[
P \left[ \left| \left| \left\langle \left( \frac{Z^T Z}{n} - \Sigma_z \right) \Delta, \Delta \right\rangle \right| \right| \geq (1 - \rho)^2 \frac{\lambda_{\min}(\Sigma_x)}{24} \right] \leq 2 \exp\left( -c'' n \min\left( (1 - \rho)^4 \frac{\lambda_{\min}(\Sigma_x)^2}{576 d_2^2 ||\Sigma_x||_{\text{op}}^4}, (1 - \rho)^2 \frac{\lambda_{\min}(\Sigma_x)}{24 \rho \cdot d_2 ||\Sigma_x||_{\text{op}}^2} \right) + \log d_2 + 2r(2 \max(d_1, d_2) + \log(\min(d_1, d_2)) \right) \leq 2 \exp\left( -c'' n \min\left( (1 - \rho)^4 \frac{\lambda_{\min}(\Sigma_x)^2}{576 d_2^2 ||\Sigma_x||_{\text{op}}^4}, (1 - \rho)^2 \frac{\lambda_{\min}(\Sigma_x)}{24 \rho \cdot d_2 ||\Sigma_x||_{\text{op}}^2} \right) + \log d_2 + 2r(2 \max(d_1, d_2) + \log(\min(d_1, d_2)) \right) \]
This inequality, together with (62), implies that there exists universal positive constants \((c_0, c_1)\) such that
\[
\tau = c_0 \lambda_{\min}(\Sigma_x) \max \left( \frac{1}{(1-\rho)^2 \Lambda_{\min}(\Sigma_x)}, \frac{1}{(1-\rho)^2 \Lambda_{\min}(\Sigma_x)} \right) \geq \tau \geq \frac{\lambda_{\min}(\Sigma_x)}{24}
\]
\[
\Pr \left[ D(\tau) \geq \frac{\lambda_{\min}(\Sigma_x)}{24} \right] \leq 2 \exp \left( -c_1 n \min \left( (1-\rho)^4 \lambda_{\min}(\Sigma_x), (1-\rho)^2 \lambda_{\min}(\Sigma_x) \right) + \log \left( 1 + d_2 \right) \right),
\]
which completes the proof.

**Proposition 4** (Derivation condition, missing data case). In the missing data case, there exist universal positive constants \((c_0, c_1, c_2)\) such that the deviation condition (cf. (13)) holds with parameter \(\phi(Q, \sigma) = c_0 \phi_{\text{mis}}\), with probability at least \(1 - c_1 \exp(-c_2 \max(d_1, d_2))\).

**Proof.** Note that the matrix \(Z\) is sub-Gaussian with parameters \((\Sigma_z, \|\Sigma_x\|_{\text{op}})^{\infty}\). The following discussion is divided into two parts. First consider the quantity \(\|\hat{\Gamma}_{\text{mis}} - \Sigma_x \Theta^*\|_{\text{op}}\). By the definition of \(\hat{\Gamma}_{\text{mis}}\) (cf. (10)) and the fact that \(Y = X \Theta^* + \epsilon\), one has that
\[
\left\| \hat{\Gamma}_{\text{mis}} - \Sigma_x \Theta^* \right\|_{\text{op}} = \frac{1}{1-\rho} \left\| \frac{1}{n} Z^{\top} Y - (1-\rho) \Sigma_x \Theta^* \right\|_{\text{op}}
\]
\[
= \frac{1}{1-\rho} \left\| \frac{1}{n} Z^{\top} (X \Theta^* + \epsilon) - (1-\rho) \Sigma_x \Theta^* \right\|_{\text{op}}
\]
\[
\leq \frac{1}{1-\rho} \left( \left\| \frac{1}{n} Z^{\top} X - (1-\rho) \Sigma_x \right\|_{\text{op}} + \left\| \frac{1}{n} Z^{\top} \epsilon \right\|_{\text{op}} \right). \tag{68}
\]
It then follows from the assumption that \(\|\Theta^*\|_* \leq \omega\) that
\[
\left\| \hat{\Gamma}_{\text{mis}} - \Sigma_x \Theta^* \right\|_{\text{op}} \leq \frac{1}{1-\rho} \left( \left\| \frac{1}{n} Z^{\top} X \right\|_{\text{op}} + (1-\rho)\|\Sigma_x\|_{\text{op}} \right) \|\Theta^*\|_* + \left\| \frac{1}{n} Z^{\top} \epsilon \right\|_{\text{op}} \tag{69}
\]
\[
\leq \frac{1}{1-\rho} \left( \left\| \frac{1}{n} Z^{\top} X \right\|_{\text{op}} + (1-\rho)\|\Sigma_x\|_{\text{op}} \right) \omega + \left\| \frac{1}{n} Z^{\top} \epsilon \right\|_{\text{op}} \right). \tag{70}
\]
Recall the assumption that the matrices \(X, W\) and \(\epsilon\) are assumed to be with i.i.d. rows sampled from Gaussian distributions \(\mathcal{N}(0, \Sigma_x), \mathcal{N}(0, \sigma^2 z_1, \sigma^2 z_2)\) and \(\mathcal{N}(0, \sigma^2 z_2, \sigma^2 z_2)\), respectively. Then it follows from [20, Lemma 3] that there exists universal positive constants \((c_3, c_4, c_5)\) such that
\[
\left\| \hat{\Gamma}_{\text{mis}} - \Sigma_x \Theta^* \right\|_{\text{op}} \leq c_3 \frac{\sigma}{1-\rho} \sqrt{\lambda_{\max}(\Sigma_x)} \sqrt{\frac{d_1 + d_2}{n}} + \left( c_3 \frac{\|\Sigma_x\|_{\text{op}}}{1-\rho} \sqrt{\lambda_{\max}(\Sigma_x)} \sqrt{\frac{2d_1}{n}} + \|\Sigma_x\|_{\text{op}} \right) \omega \tag{70}
\]
with probability at least \(1 - c_4 \exp(-c_5 \log(\max(d_1, d_2))\). Now let us consider the quantity \(\|\hat{\Gamma}_{\text{mis}} - \Sigma_x \Theta^*\|_{\text{op}}\). By the definition of \(\hat{\Gamma}_{\text{mis}}\) (cf. (10)), one has that
\[
\left\| \hat{\Gamma}_{\text{mis}} - \Sigma_x \Theta^* \right\|_{\text{op}} = \left\| ((\frac{Z^{\top} Z}{n} - \Sigma_x) \odot M) \Theta^* \right\|_{\text{op}} \leq \frac{1}{(1-\rho)^2} \left\| \frac{Z^{\top} Z}{n} - \Sigma_x \right\|_{\text{op}} \omega \leq \frac{1}{(1-\rho)^2} \left( \left\| \frac{Z^{\top} Z}{n} \right\|_{\text{op}} + \|\Sigma_x\|_{\text{op}} \right) \omega
\]

22
This inequality, together with [20 Lemma 3], implies that there exists universal positive constants \((c_6, c_7, c_8)\) such that

\[
\left\| (\hat{\Gamma}_{\text{mis}} - \Sigma_x) \Theta^* \right\|_{\text{op}} \leq c_0 \frac{1}{(1 - \rho)^2} \left\| \Sigma_x \right\|_{\text{op}} \sqrt{\lambda_{\text{max}}(\Sigma_x)} \sqrt{\frac{2d_1}{n}} \omega + \frac{1}{(1 - \rho)^2} \left\| \Sigma_x \right\|_{\text{op}} \omega
\]  

(71)

with probability at least \(1 - c_7 \exp(-c_8 \log(\max(d_1, d_2)))\). Combining (70) and (71), up to constant factors, we conclude that there exists universal positive constants \((c_0, c_1, c_2)\) such that

\[
\left\| \hat{\Upsilon}_{\text{mis}} - \hat{\Upsilon}_{\text{mis}} \Theta^* \right\|_{\text{op}} \leq c_0 \frac{\lambda_{\text{max}}(\Sigma_x)}{1 - \rho} \left( \frac{\omega}{1 - \rho} \left\| \Sigma_x \right\|_{\text{op}} + \sigma_c \right) \sqrt{\frac{\max(d_1, d_2)}{n}},
\]

with probability at least \(1 - c_1 \exp(-c_2 \max(d_1, d_2))\). The proof is complete. 

\[\square\]

Now we are ready to state statistical and computational consequences for the multi-response regression with missing data. The conclusion follow by applying Propositions 3 and 4 on Theorems 1 and 2 respectively, and so the proofs are omitted.

**Corollary 3.** Let \(R_q > 0\) and \(\omega > 0\) be positive numbers such that \(\Theta^* \in \mathbb{B}_q(R_q) \cap S\). Let \(\hat{\Theta}\) be a global optimum of the optimization problem \([8]\) with \((\hat{\Gamma}_{\text{mis}}, \hat{\Upsilon}_{\text{mis}})\) given by \((10)\) in place of \((\hat{\Gamma}, \hat{\Upsilon})\). Then there exist universal positive constants \(c_i\) \((i = 0, 1, 2, 3, 4)\) such that if \(\tau_{\text{mis}} \leq c_0 \frac{\phi_{\text{mis}}}{\omega} \sqrt{\frac{\max(d_1, d_2)}{n}}\), and \(\lambda\) is chosen to satisfy \(\lambda \geq c_1 \phi_{\text{mis}} \sqrt{\frac{\max(d_1, d_2)}{n}}\), then it holds with probability at least \(1 - 2 \exp\left(-c_2 n \min\left((1 - \rho)^{\frac{\lambda_{\text{min}}(\Sigma_x)}{d_2 ||\Sigma_x||_{\text{op}}}}, (1 - \rho)^{\frac{2 \lambda_{\text{min}}(\Sigma_x)}{d_2 ||\Sigma_x||_{\text{op}}}} + \log d_2\right) + c_3 \exp(-c_4 \max(d_1, d_2))\) that

\[
\left\| \hat{\Theta} - \Theta^* \right\|_F^2 \leq 544 R_q \left( \frac{2 \lambda}{\lambda_{\text{min}}(\Sigma_x)} \right)^{2 - q},
\]

\[
\left\| \hat{\Theta} - \Theta^* \right\|_F \leq (4 + 32 \sqrt{17}) R_q \left( \frac{2 \lambda}{\lambda_{\text{min}}(\Sigma_x)} \right)^{1 - q}.
\]

Define

\[
\kappa_{\text{mis}} := \left\{ 1 - \frac{\lambda_{\text{min}}(\Sigma_x)}{16v} + \frac{512 R_q \tau_{\text{mis}} \lambda^{-q}}{\lambda_{\text{min}}(\Sigma_x)} \right\} \left\{ 1 - \frac{512 R_q \tau_{\text{mis}} \lambda^{-q}}{\lambda_{\text{min}}(\Sigma_x)} \right\}^{-1},
\]

(72)

\[
\xi_{\text{mis}} := \tau_{\text{mis}} \left\{ \frac{\lambda_{\text{min}}(\Sigma_x)}{16v} + \frac{1024 R_q \tau_{\text{mis}} \lambda^{\frac{2}{3}}}{\lambda_{\text{min}}(\Sigma_x)} + 5 \right\} \left\{ 1 - \frac{512 R_q \tau_{\text{mis}} \lambda^{-q}}{\lambda_{\text{min}}(\Sigma_x)} \right\}^{-1}.
\]

(73)

**Corollary 4.** Let \(R_q > 0\) and \(\omega > 0\) be positive numbers such that \(\Theta^* \in \mathbb{B}_q(R_q) \cap S\). Let \(\hat{\Theta}\) be a global solution of the optimization problem \([8]\) with \((\hat{\Gamma}_{\text{mis}}, \hat{\Upsilon}_{\text{mis}})\) given by \((10)\) in place of \((\hat{\Gamma}, \hat{\Upsilon})\). Let \((\Theta_{\text{mis}}^t)_{t=0}^\infty\) be a sequence of iterates generated via \((33)\) with an initial point \(\Theta_{\text{mis}}^0\) and step size \(v \geq \max(2 \lambda_{\text{min}}(\Sigma_x), \frac{3}{2} \lambda_{\text{max}}(\Sigma_x))\). Then there exist universal positive constants \(c_i\) \((i = 0, 1, 2, 3, 4, 5)\) such that, if \(\tau_{\text{mis}} \leq c_0 \frac{\phi_{\text{mis}}}{\omega} \sqrt{\frac{\max(d_1, d_2)}{n}}\), and \(\lambda\) is chosen to satisfy

\[
\lambda \geq \max\left\{ \left( \frac{c_1 R_q \tau_{\text{mis}}}{\lambda_{\text{min}}(\Sigma_x)} \right)^{1/q}, c_2 \phi_{\text{mis}} \sqrt{\frac{\max(d_1, d_2)}{n}} \right\},
\]

23
then for any tolerance $\delta^* \geq \frac{8\rho_{\text{mis}}^2\bar{\epsilon}_{\text{stat}}^2}{1 - \rho_{\text{mis}}}$ and any iteration $t \geq T(\delta^*)$, it holds with probability at least $1 - 2 \exp\left(-c_3 n \min\left((1 - \rho)^2 \lambda_{\text{min}}^2(\Sigma_x) d_\Sigma \|x\|_{op}, (1 - \rho)^2 \lambda_{\text{min}}^2(\Sigma_x) d_\Sigma \|x\|_{op} + \log d_2\right) - c_4 \exp\left(-c_5 \max(d_1, d_2)\right)$ that

$$
\left\|\Theta_t^\text{add} - \Theta\right\|_F^2 \leq \frac{8}{\lambda_{\text{min}}(\Sigma_x)} \left(\delta^* + \frac{\delta^3}{2 \rho_{\text{mis}}^2 \omega^2} + 2 \bar{\epsilon}_{\text{stat}}^2\right),
$$

where

$$
T(\delta^*) := \log_2 \log_2 \left(\frac{\omega \lambda}{\delta^*}\right) \left(1 + \frac{\log 2}{\log(1/\rho_{\text{mis}})}\right) + \frac{\log((\Psi(\Theta_0^0) - \Psi(\Theta))/(\delta^*))}{\log(1/\rho_{\text{mis}})},
$$

and $\bar{\epsilon}_{\text{stat}}$ is given in (34).

### 5 Simulations

In this section, we implement several numerical experiments on the multi-response regression model with measurement errors to illustrate our main theoretical results. The following simulations will be performed with the loss function $L_n$ corresponding to the additive noise and missing data cases, respectively, and the nuclear norm regularizer. All numerical experiments are performed in MATLAB R2013a and executed on a personal desktop (Intel Core i7-6700, 2.80 GHz, 16.00 GB of RAM).

The simulated data are generated as follows. Specifically, the true parameter is generated as a square matrix $\Theta^* \in \mathbb{R}^{d \times d}$, and we consider the exact low rank case as an instance with $\text{rank}(\Theta^*) = r = 10$. Explicitly, let $\Theta^* = AB^\top$, where $A, B \in \mathbb{R}^{d \times r}$ consist of i.i.d. $\mathcal{N}(0, 1)$ entries. Then we generate i.i.d. true covariates $X_i \sim \mathcal{N}(0, I_d)$ and the noise term $e \sim \mathcal{N}(0, (0.1)^2 I_n)$. The data $y$ are generated according to (2). The corrupted term is set to $W_i \sim \mathcal{N}(0, (0.2)^2 I_d)$ and $\rho = 0.2$ for the additive noise and missing data cases, respectively. The problem sizes $d$ and $n$ will be specified based on the experiments. The data are then generated at random for 100 times.

For all simulations, the regularization parameter is set as $\lambda = \sqrt{\frac{d}{n}}$, and $r = 1.1\|\Theta^*\|_*$ to ensure the feasibility of $\Theta^*$. Iteration (33) is then implemented with the step size $v = 2\lambda_{\text{max}}(\Sigma_x)$ and the initial point $\Theta^0$ being a zero matrix. Performance of the estimator $\Theta$ is characterized by the relative error $\left\|\hat{\Theta} - \Theta^*\right\|_F / \left\|\Theta^*\right\|_F$ and is illustrated by averaging across the 100 numerical results.

The first experiment is performed to demonstrate the statistical guarantee for multi-response linear regression in additive noise and missing data cases, respectively. Fig. 1(a) plots the relative error on a logarithmic scale versus the sample size $n$ for three different matrix dimensions $d \in \{64, 128, 256\}$ in the additive noise case. For each matrix dimension, as the sample size increases, the relative error decreases to zero, implying the statistical consistency of the estimators. However, larger matrices need larger sample sizes, which is reflected by the rightward shift of the curves as the dimension $d$ is increased. Fig. 1(b) shows the same set of simulation results as in Fig. 1(a), but now the relative error is plotted versus the rescaled sample size $n/m$. We can see from Fig. 1(b) that the three curves nearly match with one another under different matrix dimensions $d$, coinciding with Corollary 1. Hence, Fig. 1 shows that $n/d$ actually acts as the effective sample size in this high-dimensional setting. Similar results on the statistical consistency for the missing data case are displayed in Fig. 2.
Figure 1: Statistical consistency for multi-response regression with additive error.

Figure 2: Statistical consistency for multi-response regression with missing data.
The second experiment is designed to illustrate the algorithmic linear convergence rate in additive noise and missing data cases, respectively. We have investigated the performance for a broad range of dimensions $d$ and $n$, and the results are comparatively consistent across these choices. Hence we here report results for $p = 128$ and a range of the sample sizes $n = \lceil \alpha d \rceil$ with $\alpha \in \{15, 30, 50\}$. In the additive noise case, we can see from Fig. 3(a) that for the three sample sizes, the algorithm reveal exact linear convergence rate. As the sample size becomes larger, the convergence speed turns faster and achieves a more accurate estimation level. Fig. 3(b) depicts analogous results to Fig. 3(a) in the case of missing data.

Figure 3: Algorithmic convergence rate for multi-response regression with measurement error.

Appendix A  Technical lemmas

In this appendix, several technical lemmas are provided, which are used to establish propositions on RSC/RSM conditions and deviation conditions for different observation models (cf. Propositions 1–4). The first three lemmas are in preparation for the RSC/RSM conditions, while the next two lemmas are for the deviation conditions. The following lemma tells us that the intersection of the matrix $\ell_1$-ball with the matrix $\ell_2$-ball can be bounded by virtue of a simpler set. For a symbol $x \in \{0, *, F\}$ and a positive real number $r \in \mathbb{R}^+$, define $M_x(r) = \{ A \in \mathbb{R}^{d_1 \times d_2} \| A \|_x \leq r \}$, where $\| A \|_0$ denotes the rank of matrix $A$.

Lemma A.1. For any constant $r \geq 1$, it holds that

$$M_*(\sqrt{r}) \cap M_F(1) \subseteq 2\text{cl}\{\text{conv}\{M_0(r) \cap M_F(1)\}\},$$

(A.1)

where $\text{cl}\{\cdot\}$ and $\text{conv}\{\cdot\}$ denote the topological closure and convex hull, respectively.

Proof. Note that when $r > \min\{d_1, d_2\}$, this containment is trivial, since the right-hand set is equal to $M_F(2)$, and the left-hand set is contained in $M_F(1)$. Thus, we will assume $1 \leq r \leq \min\{d_1, d_2\}$.

Let $A \in M_*(\sqrt{r}) \cap M_F(1)$. Then it follows that $\| A \|_* \leq \sqrt{r}$ and $\| A \|_F \leq 1$. Consider a singular value decomposition of $A$:

$$A = UDV^\top,$$

(A.2)
where \( U \in \mathbb{R}^{d_1 \times d_1} \) and \( V \in \mathbb{R}^{d_2 \times d_2} \) are orthogonal matrices, and \( D \in \mathbb{R}^{d_1 \times d_2} \) consists of \( \sigma_1(D), \sigma_2(D), \ldots, \sigma_k(D) \) on the “diagonal” and 0 elsewhere with \( k = \text{rank}(A) \). Write \( D = \text{diag}(\sigma_1(D), \sigma_2(D), \ldots, \sigma_k(D)) \), and use \( \text{vec}(D) \) to denote the vectorized form of the matrix \( D \). Then it follows that \( \|\text{vec}(D)\|_1 \leq \sqrt{r} \) and \( \|\text{vec}(D)\|_2 \leq 1 \). Partition the support of \( \text{vec}(D) \) into disjoint subsets \( T_1, T_2, \ldots \), such that \( T_1 \) is the index set corresponding to the first \( r \) largest elements in absolute value of \( \text{vec}(D) \), \( T_2 \) indexes the next \( r \) largest elements, and so on. Write \( D_i = \text{diag}(\text{vec}(D)_{T_1}) \), and \( A_i = UD_iV^T \). Then one has that \( \|A_i\|_0 = \text{rank}(A_i) \leq r \) and \( \|A_i\|_F \leq 1 \). Write \( B_i = 2A_i/\|A_i\|_F \) and \( t_i = \|A_i\|_F/2 \). Then \( B_i \in 2\{M_0(r) \cap M_F(1)\} \) and \( t_i \geq 0 \). Now we check that \( A \) can be expressed as a convex combination of matrices in \( 2\{\text{conv}\{M_0(r) \cap M_F(1)\}\} \), namely \( A = \sum_{i \geq 1} t_iB_i \). Since the zero matrix contains in \( 2\{M_0(r) \cap M_F(1)\} \), it suffices to show that \( \sum_i t_i \leq 1 \), which is equivalent to \( \sum_{i \geq 1} \|\text{vec}(D)_{T_i}\|_2 \leq 2 \). To prove this, first note that \( \|\text{vec}(D)_{T_1}\|_2 \leq \|\text{vec}(D)\|_2 \). Second, note that for \( i \geq 2 \), each elements of \( \text{vec}(D)_{T_i} \) is bounded in magnitude by \( \|\text{vec}(D)_{T_{i-1}}\|_1/r \), and thus \( \|\text{vec}(D)_{T_i}\|_2 \leq \|\text{vec}(D)_{T_{i-1}}\|_1/r \). Combining these two facts, one has that

\[
\sum_{i \geq 1} \|\text{vec}(D)_{T_i}\|_2 \leq 1 + \sum_{i \geq 2} \|\text{vec}(D)_{T_i}\|_2 \leq 1 + \sum_{i \geq 2} \|\text{vec}(D)_{T_{i-1}}\|_1/r \leq 1 + \|\text{vec}(D)\|_1/r \leq 2.
\]

The proof is complete. \( \square \)

For ease of notation, define the sparse set \( \mathbb{K}(r) := M_0(r) \cap M_F(1) \) and the cone set \( \mathbb{C}(r) := \{A \in \mathbb{R}^{d_1 \times d_2} | \|A\|_* \leq \sqrt{r} \|A\|_F \} \).

Lemma A.2. Let \( \Gamma \in \mathbb{R}^{d_1 \times d_1} \) be a fixed matrix, \( r \geq 1 \), and \( \delta > 0 \) be a tolerance. Suppose that the following condition holds

\[
\|\langle \Gamma \Delta, \Delta \rangle \| \leq \delta, \quad \forall \Delta \in \mathbb{K}(2r).
\]

(A.3)

Then we have that

\[
\|\langle \Gamma \Delta, \Delta \rangle \| \leq 12\delta (\|\Delta\|_F^2 + \frac{1}{r} \|\Delta\|_2^2), \quad \forall \Delta \in \mathbb{R}^{d_1 \times d_2}.
\]

(A.4)

Proof. We begin by establishing the inequalities

\[
\|\langle \Gamma \Delta, \Delta \rangle \| \leq 12\delta \|\Delta\|_F^2, \quad \forall \Delta \in \mathbb{C}(r), \quad \text{(A.5a)}
\]

\[
\|\langle \Gamma \Delta, \Delta \rangle \| \leq \frac{12\delta}{r} \|\Delta\|_2^2, \quad \forall \Delta \notin \mathbb{C}(r), \quad \text{(A.5b)}
\]

then (A.4) then follows directly.

Now we turn to prove (A.5). By rescaling, (A.5a) holds if we can show that

\[
\|\langle \Gamma \Delta, \Delta \rangle \| \leq 12\delta, \quad \text{for all } \Delta \text{ satisfying } \|\Delta\|_F^2 = 1 \text{ and } \|\Delta\|_* \leq \sqrt{r}. \quad \text{(A.6)}
\]

It then follows from Lemma A.1 and continuity that (A.6) can be reduced to the problem of proving that

\[
\|\langle \Gamma \Delta, \Delta \rangle \| \leq 12\delta, \quad \forall \Delta \in 2\text{conv}\{\mathbb{K}(r)\} = \text{conv}\{M_0(r) \cap M_F(2)\}. \quad \text{(A.7)}
\]

For this purpose, consider a weighted linear combination of the form \( \Delta = \sum_i t_i \Delta_i \), with weights \( t_i \geq 0 \) such that \( \sum_i t_i = 1 \), and \( \|\Delta_i\|_0 \leq r \) and \( \|\Delta_i\|_F \leq 2 \) for each \( i \). Then one has that

\[
\langle \Gamma \Delta, \Delta \rangle = \langle \Gamma (\sum_i t_i \Delta_i), (\sum_i t_i \Delta_i) \rangle = \sum_{i,j} t_i t_j \langle \langle \Gamma \Delta_i, \Delta_j \rangle \rangle.
\]
On the other hand, it holds that for all $i, j$

$$|\langle\langle \Gamma \Delta_i, \Delta_j \rangle \rangle| = \frac{1}{2} |\langle\langle \Gamma (\Delta_i + \Delta_j), (\Delta_i + \Delta_j) \rangle \rangle - \langle\langle \Gamma \Delta_i, \Delta_i \rangle \rangle - \langle\langle \Gamma \Delta_j, \Delta_j \rangle \rangle|.$$  \hspace{1cm} (A.8)

Noting that $\frac{1}{2} \Delta_i$, $\frac{1}{2} \Delta_j$, $\frac{1}{4} (\Delta_i + \Delta_j)$ all belong to $\mathbb{K}(2r)$, and then combining (A.8) with (A.3), we have that

$$|\langle\langle \Gamma \Delta_i, \Delta_j \rangle \rangle| \leq \frac{1}{2}(16\delta + 4\delta + 4\delta) = 12\delta,$$

for all $i, j$, and thus $\langle\langle \Gamma \Delta, \Delta \rangle \rangle \leq \sum_{i,j} t_i t_j (12\delta) = 12\delta (\sum_i t_i)^2 = 12\delta$, which establishes (A.5a).

As for inequality (A.5b), note that for $\Delta \notin \mathbb{C}(r)$, one has that

$$\frac{1}{\sqrt{r}} \sup_{\|U\|_* \leq \sqrt{r}, \|U\|_F \leq 1} |\langle\langle \Gamma U, \Delta \rangle \rangle| \leq \frac{12\delta}{r},$$  \hspace{1cm} (A.9)

where the first inequality follows by the substitution $U = \sqrt{r} \frac{\Delta}{\|\Delta\|_F}$, and the second inequality is due to the same argument used to establish (A.5a) as $U \in \mathbb{C}(r)$. Rearranging (A.9) yields (A.5b). The proof is complete.

**Lemma A.3.** Let $r \geq 1$ be a positive integer. Suppose that $\hat{\Gamma}$ is an estimator of $\Sigma_x$ satisfying

$$|\langle\langle \hat{\Gamma} - \Sigma_x \rangle \rangle| \leq \frac{\lambda_{\min}(\Sigma_x)}{24}, \quad \forall \Delta \in \mathbb{K}(2r).$$  \hspace{1cm} (A.10)

Then we have that

$$\langle\langle \hat{\Gamma} \Delta, \Delta \rangle \rangle \geq \frac{\lambda_{\min}(\Sigma_x)}{2} \|\Delta\|_F^2 - \frac{\lambda_{\min}(\Sigma_x)}{2r} \|\Delta\|_*^2,$$  \hspace{1cm} (A.11)

$$\langle\langle \hat{\Gamma} \Delta, \Delta \rangle \rangle \leq \frac{3\lambda_{\max}(\Sigma_x)}{2} \|\Delta\|_F^2 + \frac{\lambda_{\min}(\Sigma_x)}{2r} \|\Delta\|_*^2.$$  \hspace{1cm} (A.12)

**Proof.** Setting $\Gamma = \hat{\Gamma} - \Sigma_x$ and $\delta = \frac{\lambda_{\min}(\Sigma_x)}{24}$. It then follows directly from Lemma A.2 that

$$|\langle\langle \hat{\Gamma} - \Sigma_x \rangle \rangle| \leq \frac{\lambda_{\min}(\Sigma_x)}{2} (\|\Delta\|_F^2 + \frac{1}{r} \|\Delta\|_*^2),$$

which implies that

$$\langle\langle \hat{\Gamma} \Delta, \Delta \rangle \rangle \geq \langle\langle \Sigma_x \Delta, \Delta \rangle \rangle - \frac{\lambda_{\min}(\Sigma_x)}{2} (\|\Delta\|_F^2 + \frac{1}{r} \|\Delta\|_*^2),$$

$$\langle\langle \hat{\Gamma} \Delta, \Delta \rangle \rangle \leq \langle\langle \Sigma_x \Delta, \Delta \rangle \rangle + \frac{\lambda_{\min}(\Sigma_x)}{2} (\|\Delta\|_F^2 + \frac{1}{r} \|\Delta\|_*^2).$$

Then the conclusion follows from the fact that $\lambda_{\min}(\Sigma_x) \|\Delta\|_F^2 \leq \langle\langle \Sigma_x \Delta, \Delta \rangle \rangle \leq \lambda_{\max}(\Sigma_x) \|\Delta\|_F^2$. The proof is complete.

**Lemma A.4.** Let $X \in \mathbb{R}^{n \times d_1}$ be a zero-mean sub-Gaussian matrix with parameters $(\Sigma_x, \sigma_x^2)$. Then for any fixed matrix $\Delta \in \mathbb{R}^{d_1 \times d_2}$, there exists a universal positive constant $c$ such that

$$P \left[ \frac{\|X\Delta\|_F^2}{n} - \mathbb{E} \left( \frac{\|X\Delta\|_F^2}{n} \right) \geq t \right] \leq 2 \exp \left( -cn \min \left( \frac{t^2}{d_2^2 \sigma_x^2}, \frac{t}{d_2 \sigma_x} \right) + \log d_2 \right).$$  \hspace{1cm} (A.13)
Proof. By the definition of matrix Frobenius norm, one has that
\[
\frac{\|X\Delta\|_F^2}{n} - \mathbb{E}\left(\frac{\|X\Delta\|_F^2}{n}\right) = \sum_{j=1}^{d_2} \left[ \frac{\|X\Delta_j\|_2^2}{n} - \mathbb{E}\left(\frac{\|X\Delta_j\|_2^2}{n}\right) \right].
\] (A.14)

Then it follows from elementary probability theory that
\[
\mathbb{P}\left[ \left| \frac{\|X\Delta\|_F^2}{n} - \mathbb{E}\left(\frac{\|X\Delta\|_F^2}{n}\right) \right| \leq t \right] = \mathbb{P}\left\{ \left| \sum_{j=1}^{d_2} \left[ \frac{\|X\Delta_j\|_2^2}{n} - \mathbb{E}\left(\frac{\|X\Delta_j\|_2^2}{n}\right) \right] \right| \leq t \right\}
\geq \mathbb{P}\left\{ \left| \sum_{j=1}^{d_2} \left[ \frac{\|X\Delta_j\|_2^2}{n} - \mathbb{E}\left(\frac{\|X\Delta_j\|_2^2}{n}\right) \right] \right| \leq \frac{t}{d_2} \right\}
\geq \sum_{j=1}^{d_2} \mathbb{P}\left[ \left| \frac{\|X\Delta_j\|_2^2}{n} - \mathbb{E}\left(\frac{\|X\Delta_j\|_2^2}{n}\right) \right| \leq \frac{t}{d_2} \right] - (d_2 - 1)
\] (A.15)

On the other hand, note the assumption that \(X\) is a sub-Gaussian matrix with parameters \((\Sigma_x, \sigma^2_x)\). Then [18, Lemma 14] is applicable to conclude that there exists a universal positive constant \(c\) such that
\[
\mathbb{P}\left[ \left| \frac{\|X\Delta\|_F^2}{n} - \mathbb{E}\left(\frac{\|X\Delta\|_F^2}{n}\right) \right| \leq t \right] \geq d_2 \left(1 - 2 \exp\left(-cn \min\left(\frac{t^2}{d_2^2 \sigma_x^4}, \frac{t}{d_2^2 \sigma_x^2}\right)\right)\right) - (d_2 - 1)
\geq 1 - 2 \exp\left(-cn \min\left(\frac{t^2}{d_2^2 \sigma_x^4}, \frac{t}{d_2^2 \sigma_x^2}\right) + \log d_2\right),
\] (A.16)

which completes the proof. \(\square\)

Recall the set \(\mathcal{K}(r)\) for a positive integer \(r\).

Lemma A.5. Let \(X \in \mathbb{R}^{n \times d_1}\) be a zero-mean sub-Gaussian matrix with parameters \((\Sigma_x, \sigma^2_x)\). Then there exists a universal positive constant \(c\) such that
\[
\mathbb{P}\left[ \sup_{\Delta \in \mathcal{K}(2r)} \left| \frac{\|X\Delta\|_F^2}{n} - \mathbb{E}\left(\frac{\|X\Delta\|_F^2}{n}\right) \right| \geq t \right] \leq 2 \exp\left(-cn \min\left(\frac{t^2}{d_2^2 \sigma_x^4}, \frac{t}{d_2^2 \sigma_x^2}\right) + \log d_2 + 2r(2\max(d_1, d_2) + \log(\min(d_1, d_2)))\right)
\] (A.17)

Proof. For an index set \(J \subseteq \{1, 2, \cdots, \min\{d_1, d_2\}\}\), we define the set \(S_J = \{\Delta \in \mathbb{R}^{d_1 \times d_2} \mid \|\Delta\|_F \leq 1, \text{supp}(\sigma(\Delta)) \subseteq J\}\), where \(\sigma(\Delta)\) refers to the singular vector of the matrix \(\Delta\). Then it is easy to see that \(\mathcal{K}(2r) = \bigcup_{|J| \leq 2r} S_J\). Let \(G = \{U_1, U_2, \cdots, U_m\}\) be a 1/3-cover of \(S_J\), then for every \(\Delta \in S_J\), there exists some \(U_i\) such that \(\|\hat{\Delta}\|_F \leq 1/3\), where \(\hat{\Delta} = \Delta - U_i\). It then follows from [3, Section 7.2] that one can construct \(G\) with \(|G| \leq 2^{4r \max(d_1, d_2)}\). Define \(\Psi(\Delta_1, \Delta_2) = \langle (X^n - \Sigma_x)\Delta, \hat{\Delta} \rangle\). Then we obtain that
\[
\sup_{\Delta \in S_J} |\Psi(\Delta, \Delta)| \leq \max_i |\Psi(U_i, U_i)| + 2 \sup_{\Delta \in S_J} |\max_i \Psi(\hat{\Delta}, U_i)| + \sup_{\Delta \in S_J} |\Psi(\hat{\Delta}, \hat{\Delta})|.
\]

29
It then follows from the fact that
\[
\sup_{\Delta \in \mathcal{S}_J} |\Psi(\Delta, \Delta)| \leq \max_i |\Psi(U_i, U_i)| + \sup_{\Delta \in \mathcal{S}_J} \left( \frac{2}{3} |\Psi(\Delta, \Delta)| + \frac{1}{9} |\Psi(\Delta, \Delta)| \right),
\]
and hence, \(\sup_{\Delta \in \mathcal{S}_J} |\Psi(\Delta, \Delta)| \leq \frac{9}{2} \max_i |\Psi(U_i, U_i)|\). It follows from Lemma A.4 and a union bound that there exists a universal positive constant \(c'\) such that
\[
P\left[ \sup_{\Delta \in \mathcal{S}_J} \left| \frac{\|X\Delta\|_F^2}{n} - \mathbb{E} \left( \frac{\|X\Delta\|_F^2}{n} \right) \right| \geq t \right] \leq 27^{4r \max(d_1,d_2)} \cdot 2 \exp \left( -c' n \min \left( \frac{t^2}{d_2^2 \sigma_x^4}, \frac{t}{d_2 \sigma_x^2} \right) + \log d_2 \right).
\]
Finally, taking a union bound over the \(\min\left(\frac{d_1}{2r}, \frac{d_2}{2r}\right)\) choice of \(J\) yields that there exists a universal positive constant \(c\) such that
\[
P\left[ \sup_{\Delta \in \mathcal{S}(2r)} \left| \frac{\|X\Delta\|_F^2}{n} - \mathbb{E} \left( \frac{\|X\Delta\|_F^2}{n} \right) \right| \geq t \right] \leq 2 \exp \left( -cn \min \left( \frac{t^2}{d_2^2 \sigma_x^4}, \frac{t}{d_2 \sigma_x^2} \right) + \log d_2 + 2r \left(2 \max(d_1,d_2) + \log(\min(d_1,d_2)) \right) \right).
\]
The proof is complete.

References

[1] A. Agarwal, S. Negahban, and M. J. Wainwright. Fast global convergence of gradient methods for high-dimensional statistical recovery. *Annals of Statistics*, 40(5):2452–2482, 2012.

[2] A. Agarwal, S. N. Negahban, and M. J. Wainwright. Supplementary material: Fast global convergence of gradient methods for high-dimensional statistical recovery. *Ann. Statist.*, 2012.

[3] P. Alquier, K. Bertin, P. Doukhan, and R. Garnier. High-dimensional var with low-rank transition. *Statistics and Computing*, pages 1–15, 2020.

[4] A. Belloni, M. Rosenbaum, and A. B. Tsybakov. An \(\ell_1, \ell_2, \ell_\infty\)-regularization approach to high-dimensional errors-in-variables models. *Electron. J. Stat.*, 10(2):1729–1750, 2016.

[5] A. Belloni, M. Rosenbaum, and A. B. Tsybakov. Linear and conic programming estimators in high dimensional errors-in-variables models. *J. Royal Stat. Soc. B*, 79(3):939–956, 2017.

[6] P. J. Bickel and Y. Ritov. Efficient estimation in the errors in variables model. *Annals of Statistics*, pages 513–540, 1987.

[7] P. J Bickel, Y. Ritov, and A. B. Tsybakov. Simultaneous analysis of Lasso and Dantzig selector. *Annals of statistics*, 37(4):1705–1732, 2009.

[8] B. Brown, T. Weaver, and J. Wolfson. MEBoost: Variable selection in the presence of measurement error. *Stat. Med.*, 38(15):2705–2718, 2019.
[9] P. Bühlmann and S. Van De Geer. *Statistics for high-dimensional data: methods, theory and applications*. Springer Science & Business Media, 2011.

[10] R. J. Carroll, D. Ruppert, L. A. Stefanski, and C. M. Crainiceanu. *Measurement error in nonlinear models: a modern perspective*. CRC press, 2006.

[11] Y. D. Chen and C. Caramanis. Noisy and missing data regression: Distribution-oblivious support recovery. In *International Conference on Machine Learning*, pages 383–391, 2013.

[12] A. Datta and H. Zou. Cocolasso for high-dimensional error-in-variables regression. *Ann. Stat.*, 45(6):2400–2426, 2017.

[13] A. J. Izenman. Modern multivariate statistical techniques: regression, classification and manifold learning.

[14] A. Kaul, H. L. Koul, A. Chawla, and S. N. Lahiri. Two stage non-penalized corrected least squares for high dimensional linear models with measurement error or missing covariates. *arXiv preprint arXiv:1605.03154*, 2016.

[15] X. Li, D. Y. Wu, Y. Cui, B. Liu, H. Walter, G. Schumann, C. Li, and T. Z. Jiang. Reliable heritability estimation using sparse regularization in ultrahigh dimensional genome-wide association studies. *BMC Bioinformatics*, 20(1):219, 2019.

[16] X. Li, D. Y. Wu, C. Li, J. H. Wang, and J.-C. Yao. Sparse recovery via nonconvex regularized M-estimators over $\ell_q$-balls. *Computational Statistics & Data Analysis*, 152:107047, 2020.

[17] P.-L. Loh and M. J. Wainwright. High-dimensional regression with noisy and missing data: Provable guarantees with nonconvexity. *Ann. Stat.*, 40(3):1637–1664, 2012.

[18] P.-L. Loh and M. J. Wainwright. Supplementary material: High-dimensional regression with noisy and missing data: Provable guarantees with nonconvexity. *Ann. Stat.*, 2012.

[19] P.-L. Loh and M. J. Wainwright. Regularized M-estimators with nonconvexity: statistical and algorithmic theory for local optima. *J. Mach. Learn. Res.*, 16(1):559–616, 2015.

[20] S. Negahban and M. J. Wainwright. Estimation of (near) low-rank matrices with noise and high-dimensional scaling. *Annals of Statistics*, pages 1069–1097, 2011.

[21] Y. Nesterov. Gradient methods for minimizing composite objective function. Technical report, Université catholique de Louvain, Center for Operations Research and Econometrics (CORE), 2007.

[22] B. Recht, M. Fazel, and P. A. Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM review*, 52(3):471–501, 2010.

[23] M. Rosenbaum and A. B. Tsybakov. Sparse recovery under matrix uncertainty. *Ann. Stat.*, 38(5):2620–2651, 2010.

[24] M. Rosenbaum and A. B. Tsybakov. Improved matrix uncertainty selector. In *From Probability to Statistics and Back: High-Dimensional Models and Processes–A Festschrift in Honor of Jon A. Wellner*, pages 276–290. Institute of Mathematical Statistics, 2013.
[25] Ø. Sørensen, A. Frigessi, and M. Thoresen. Measurement error in LASSO: Impact and likelihood bias correction. *Statistica sinica*, pages 809–829, 2015.

[26] Ø. Sørensen, K. H. Hellton, A. Frigessi, and M. Thoresen. Covariate selection in high-dimensional generalized linear models with measurement error. *J. Comput. Graph. Stat.*, 27(4):739–749, 2018.

[27] L. A. Stefanski and R. J. Carroll. Conditional scores and optimal scores for generalized linear measurement-error models. *Biometrika*, 74(4):703–716, 1987.

[28] S. A. Van De Geer and P. Bühlmann. On the conditions used to prove oracle results for the Lasso. *Electronic Journal of Statistics*, 3:1360–1392, 2009.

[29] M. J. Wainwright. Structured regularizers for high-dimensional problems: Statistical and computational issues. *Annu. Rev. Stat. Appl.*, 1:233–253, 2014.

[30] M. J. Wainwright. *High-dimensional statistics: A non-asymptotic viewpoint*, volume 48. Cambridge University Press, 2019.

[31] H. Zhou and L. X. Li. Regularized matrix regression. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 76(2):463–483, 2014.