Homogenization of elliptic boundary value problems in Lipschitz domains

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Abstract  In this paper we study the $L^p$ boundary value problems for $\mathcal{L}(u) = 0$ in $\mathbb{R}^{d+1}_+$, where $\mathcal{L} = -\text{div}(A\nabla)$ is a second order elliptic operator with real and symmetric coefficients. Assume that $A$ is periodic in $x_{d+1}$ and satisfies some minimal smoothness condition in the $x_{d+1}$ variable, we show that the $L^p$ Neumann and regularity problems are uniquely solvable for $1 < p < 2 + \delta$. We also present a new proof of Dahlberg’s theorem on the $L^p$ Dirichlet problem for $2 - \delta < p < \infty$ (Dahlberg’s original unpublished proof is given in the Appendix). As the periodic and smoothness conditions are imposed only on the $x_{d+1}$ variable, these results extend directly from $\mathbb{R}^{d+1}_+$ to regions above Lipschitz graphs. Consequently, by localization techniques, we obtain uniform $L^p$ estimates for the Dirichlet, Neumann and regularity problems on bounded Lipschitz domains for a family of second order elliptic operators arising in the theory of homogenization. The results on the Neumann and regularity problems are new even for smooth domains.

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1 Introduction

Let \( \mathcal{L} = -\operatorname{div}(A \nabla) \) be a second order elliptic operator defined in \( \mathbb{R}^{d+1} = \{ X = (x, t) \in \mathbb{R}^d \times \mathbb{R} \} \), \( d \geq 2 \). We will always assume that the \( (d+1) \times (d+1) \) coefficient matrix

\[
A = A(X) = (a_{i,j}(X)) \text{ is real and symmetric},
\]

and satisfies the ellipticity condition,

\[
\mu |\xi|^2 \leq a_{i,j}(X) \xi_i \xi_j \leq \frac{1}{\mu} |\xi|^2 \text{ for all } X, \xi \in \mathbb{R}^{d+1},
\]

where \( \mu > 0 \). In this paper we shall be interested in boundary value problems for \( \mathcal{L}(u) = 0 \) in the upper half-space \( \mathbb{R}_{+}^{d+1} = \mathbb{R}^d \times (0, \infty) \) with \( L^p \) boundary data, under the assumption that the coefficients are periodic in the \( t \) variable,

\[
A(x, t + 1) = A(x, t) \text{ for } (x, t) \in \mathbb{R}^{d+1}.
\]

More precisely, we will study the solvabilities of the \( L^p \) Dirichlet problem \((D)_p\)

\[
\begin{cases}
\mathcal{L}(u) = 0 \text{ in } \Omega = \mathbb{R}_{+}^{d+1}, \\
u = f \in L^p(\partial \Omega) \text{ n.t. on } \partial \Omega \text{ and } (u)^* \in L^p(\partial \Omega),
\end{cases}
\]

the \( L^p \) Neumann problem \((N)_p\)

\[
\begin{cases}
\mathcal{L}(u) = 0 \text{ in } \Omega = \mathbb{R}_{+}^{d+1}, \\
\frac{\partial u}{\partial \nu} = f \in L^p(\partial \Omega) \text{ on } \partial \Omega \text{ and } N(\nabla u) \in L^p(\partial \Omega),
\end{cases}
\]

where \( \frac{\partial u}{\partial \nu} \) denote the conormal derivative associated with operator \( \mathcal{L} \), and the \( L^p \) regularity problem \((R)_p\)

\[
\begin{cases}
\mathcal{L}(u) = 0 \text{ in } \Omega = \mathbb{R}_{+}^{d+1}, \\
u = f \in \hat{W}^{1,p}(\partial \Omega) \text{ n.t. on } \partial \Omega \text{ and } N(\nabla u) \in L^p(\partial \Omega).
\end{cases}
\]

Here \((u)^*\) denotes the usual nontangential maximal function of \( u \) and \( N(\nabla u) \) a generalized nontangential maximal function of \( \nabla u \). By \( u = f \) n.t. on \( \partial \Omega \) we mean that \( u(X) \) converges to \( f(P) \) as \( X \to P \) nontangentially for a.e. \( P \in \partial \Omega \). Under the periodic condition \((1.3)\) as well as some (necessary) local solvability conditions on \( \mathcal{L} \), we will show that the \( L^p \) Dirichlet problem is uniquely solvable for \( 2 - \delta < p < \infty \), and the \( L^p \) Neumann and regularity problems are uniquely solvable for \( 1 < p < 2 + \delta \). Furthermore, the solution to the Dirichlet problem satisfies the estimate \( \|(u)^*\|_p \leq C \|u\|_p \), while the solutions to the \( L^p \) Neumann and regularity problems satisfy \( \|N(\nabla u)\|_p \leq C \|u\|_p \).