C\textsuperscript{1} MAPPINGS IN \( \mathbb{R}^5 \) WITH DERIVATIVE OF RANK AT MOST 3 CANNOT BE UNIFORMLY APPROXIMATED BY C\textsuperscript{2} MAPPINGS WITH DERIVATIVE OF RANK AT MOST 3

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Abstract. We find a counterexample to a conjecture of Gałęski \cite{G} by constructing for some positive integers \( m < n \) a mapping \( f \in C^1(\mathbb{R}^n, \mathbb{R}^n) \) satisfying \( \text{rank}\ Df \leq m \) that, even locally, cannot be uniformly approximated by \( C^2 \) mappings \( f_\varepsilon \) satisfying the same rank constraint: \( \text{rank}\ Df_\varepsilon \leq m \).

1. Introduction

In the context of geometric measure theory Jacek Gałęski \cite{G} Conjecture 1.1 and Section 3.3] formulated the following conjecture.

**Conjecture 1.** Let \( 1 \leq m < n \) be integers and let \( \Omega \subset \mathbb{R}^n \) be open. If \( f \in C^1(\Omega, \mathbb{R}^n) \) satisfies \( \text{rank}\ Df \leq m \) everywhere in \( \Omega \), then \( f \) can be uniformly approximated by smooth mappings \( g \in C^\infty(\Omega, \mathbb{R}^n) \) such that \( \text{rank}\ Dg \leq m \) everywhere in \( \Omega \).

A weaker form of the conjecture is whether any mapping as in Conjecture 1 can be approximated locally.

**Conjecture 2.** Let \( 1 \leq m < n \) be integers and let \( \Omega \subset \mathbb{R}^n \) be open. If \( f \in C^1(\Omega, \mathbb{R}^n) \) satisfies \( \text{rank}\ Df \leq m \) everywhere in \( \Omega \), then for every point \( x \in \Omega \) there is a neighborhood \( B^n(x, \varepsilon) \subset \Omega \) and a sequence \( f_i \in C^\infty(B^n(x, \varepsilon), \mathbb{R}^n) \) such that \( \text{rank}\ Df_i \leq m \) and \( f_i \) converges to \( f \) uniformly on \( B^n(x, \varepsilon) \).

The following result is easy to prove and it shows that Conjecture 2 is true on an open and dense subset of \( \Omega \).

**Theorem 3.** Let \( 1 \leq m < n \) be integers and let \( \Omega \subset \mathbb{R}^n \) be open. If \( f \in C^1(\Omega, \mathbb{R}^n) \) satisfies \( \text{rank}\ Df \leq m \) everywhere in \( \Omega \), then there is an open and dense set \( G \subset \Omega \) such that for every point \( x \in G \) there is a neighborhood \( B^n(x, \varepsilon) \subset G \) and a sequence \( f_i \in C^\infty(B^n(x, \varepsilon), \mathbb{R}^n) \) such that \( \text{rank}\ Df_i \leq m \) and \( f_i \) converges to \( f \) uniformly on \( B^n(x, \varepsilon) \).

However, in general Conjecture 2 (and hence Conjecture 1) is false and the main result of the paper provides a family counterexamples for certain ranges of \( n \) and \( m \).
Theorem 4. Suppose that \( m + 1 \leq k < 2m - 1, \ell \geq k + 1, r \geq m + 1, \) and the homotopy group \( \pi_k(S^m) \) is non-trivial. Then there is a map \( f \in C^1(\mathbb{R}^\ell, \mathbb{R}^r) \) with rank \( Df \leq m \) in \( \mathbb{R}^\ell \) and a Cantor set \( E \subset \mathbb{R}^\ell \) with the following property:

For every \( x_0 \in E \) and \( \varepsilon > 0 \) there is \( \delta > 0 \) such that

if \( g \in C^{k-m+1}(\mathbb{B}(x_0, \varepsilon), \mathbb{R}^r) \) and \( |f(x) - g(x)| < \delta \) for all \( x \in \mathbb{B}(x_0, \varepsilon) \),

then rank \( Dg \geq m + 1 \) on a non-empty open set in \( \mathbb{B}(x_0, \varepsilon) \).

(Here by a Cantor set we mean a set that is homeomorphic to the ternary Cantor set.)

Therefore the mapping \( f \) cannot be approximated in the supremum norm by \( C^{k-m+1} \) mappings with rank of the derivative \( \leq m \) in any neighborhood of any point of the set \( E \).

Remark 5. In fact, the mapping \( f \) constructed in the proof of Theorem 4 is \( C^\infty \) smooth on \( \mathbb{R}^\ell \setminus E \), so \( G = \mathbb{R}^\ell \setminus E \) is an open and dense set where we can approximate \( f \) smoothly, cf. Theorem 3.

Since the assumptions of the theorem are quite complicated, let us show explicit situations when the approximation cannot hold.

Example 1. If \( n \geq 3, \ell \geq n+2 \) and \( r \geq n+1 \), then there is \( f \in C^1(\mathbb{R}^\ell, \mathbb{R}^r) \) with rank \( Df \leq n \) in \( \mathbb{R}^\ell \) that cannot be locally approximated in the supremum norm by mappings \( g \in C^2(\mathbb{R}^\ell, \mathbb{R}^r) \) satisfying rank \( Dg \leq n \).

Indeed, if \( n \geq 3, k = n+1 \) and \( m = n \), then \( \pi_k(S^m) = \mathbb{Z}_2 \) (see 3) and \( m+1 \leq k < 2m-1 \).

In particular, there is \( f \in C^1(\mathbb{R}^5, \mathbb{R}^5) \) with rank \( Df \leq 3 \) that cannot be locally approximated in the supremum norm by mappings \( g \in C^2(\mathbb{R}^5, \mathbb{R}^5) \) satisfying rank \( Dg \leq 3 \).

Example 2. \( \pi_6(S^4) = \mathbb{Z}_2, k = 6, m = 4, m+1 \leq k < 2m-1 \), so there is \( f \in C^1(\mathbb{R}^7, \mathbb{R}^7) \), rank \( Df \leq 4 \), that cannot be locally approximated by mappings \( g \in C^3(\mathbb{R}^7, \mathbb{R}^7) \) satisfying rank \( Dg \leq 4 \).

Example 3. \( \pi_8(S^5) = \mathbb{Z}_{24}, k = 8, m = 5, m+1 \leq k < 2m-1 \), so there is \( f \in C^1(\mathbb{R}^9, \mathbb{R}^9) \), rank \( Df \leq 5 \), that cannot be locally approximated by mappings \( g \in C^4(\mathbb{R}^9, \mathbb{R}^9) \) satisfying rank \( Dg \leq 5 \).

Infinitely many essentially different situations when the assumptions of Theorem 4 are satisfied can be easily obtained by examining the catalogue of homotopy groups of spheres.

While, in general, Gałęski’s conjecture is not true, Theorem 4 covers only a certain range of dimensions and ranks, leaving other cases unsolved. We believe that the following special case of the conjecture is true.

Conjecture 6. If \( f \in C^1(\mathbb{R}^n, \mathbb{R}^k), n, k \geq 2, \) satisfies rank \( Df \leq 1 \), then \( f \) can be uniformly approximated (at least locally) by mappings \( g \in C^\infty(\mathbb{R}^n, \mathbb{R}^k) \) satisfying rank \( Dg \leq 1 \).

Our belief is based on the fact that in that case the structure of the mapping \( f \) is particularly simple: on the open set where rank \( Df = 1 \), it is a \( C^1 \) curve that branches on the set where rank \( Df = 0 \).
2. Proof of Theorem 3

Let $G \subset \Omega$ be the set of points where the function $x \mapsto \text{rank } Df(x)$ attains a local maximum i.e.,

$$G = \{ x \in \Omega : \exists \varepsilon > 0 \forall y \in \mathbb{B}^n(x, \varepsilon) \text{ rank } Df(y) \leq \text{rank } Df(x) \}.$$ 

We claim that the set $G$ is open, and that rank $Df$ is locally constant in $G$. Indeed, the set \{rank $Df \geq k$\} is open so if $x \in G$ and rank $Df(x) = k$, then rank $Df(y) \geq k$ in a neighborhood $\mathbb{B}^n(x, \varepsilon)$ of $x$, but rank $Df$ attains a local maximum at $x$, so rank $Df(y) = k$ in $\mathbb{B}^n(x, \varepsilon)$. Clearly, $\mathbb{B}^n(x, \varepsilon) \subset G$ and rank $Df$ is constant in the neighborhood $\mathbb{B}^n(x, \varepsilon) \subset G$.

We also claim that the set $G \subset \Omega$ is dense. Let $x \in \Omega$ and $\mathbb{B}^n(x, \varepsilon) \subset \Omega$. Since rank $Df$ can attain only a finite number of values, it attains a local maximum at some point $y \in \mathbb{B}^n(x, \varepsilon)$. Clearly, $y \in G$. That proves density of $G$.

It remains to prove now that $f$ can be locally approximated in $G$. Let $x \in G$. Then rank $Df(x) = k \leq m$. Since rank $Df$ is constant in a neighborhood of $x$, it follows from the Rank Theorem [6, Theorem 8.6.2/2] that there are diffeomorphisms $\Phi$ and $\Psi$ defined in neighborhoods of $x$ and $f(x)$ respectively such that $\Phi(x) = 0$, $\Psi(f(x)) = 0$, and

$$\Psi \circ f \circ \Phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0) \quad \text{in a neighborhood of } 0 \in \mathbb{R}^n.$$ 

Let $\pi_k : \mathbb{R}^n \to \mathbb{R}^n$, $\pi_k(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0)$. Then $\Psi \circ f \circ \Phi^{-1} = \pi_k$, so $f = \Psi^{-1} \circ \pi_k \circ \Phi$ in a neighborhood of $x$. If $\Phi_\varepsilon$ and $(\Psi^{-1})_\varepsilon$ are smooth approximations by mollification, then $f_\varepsilon = (\Psi^{-1})_\varepsilon \circ \pi_k \circ \Phi_\varepsilon$ is $C^\infty$ smooth and it converges uniformly to $f$ in a neighborhood of $x$ as $\varepsilon \to 0$. Clearly, rank $DF_\varepsilon \leq k$ by the chain rule, since rank $D\pi_k = k$. \qed

Remark 7. It is easy to see that in fact rank $f_\varepsilon = k$ in a neighborhood of $x$, provided $\varepsilon$ is sufficiently small. Indeed, $\Phi_\varepsilon = \Phi \circ \varphi_\varepsilon$ (approximation by mollification) so $DF_\varepsilon = (DF) \circ \varphi_\varepsilon$. Since $\det(DF(x)) \neq 0$, for small $\varepsilon > 0$ we also have that $\det(DF_\varepsilon(x)) \neq 0$ and hence $\Phi_\varepsilon$ is a diffeomorphism near $x$. Similarly, $(\Psi^{-1})_\varepsilon$ is a diffeomorphism near $0$.

3. Proof of Theorem 4

In the first step of the proof we shall construct a mapping $F : \mathbb{B}^{k+1} \to \mathbb{R}^{m+1}$ defined on the unit ball $\mathbb{B}^{k+1} = \mathbb{B}^{k+1}(0, 1)$, with the properties announced by Theorem 4.

Lemma 8. Suppose that $m + 1 \leq k < 2m - 1$ and $\pi_k(S^m) \neq 0$. Then there exists a map $F \in C^1(\mathbb{B}^{k+1}, \mathbb{R}^{m+1})$ with rank $DF \leq m$ in $\mathbb{B}^{k+1}$ and a Cantor set $E_F \subset \mathbb{B}^{k+1}$ such that for every $x_0 \in E_F$ and $1 - |x_o| > \varepsilon > 0$ there is $\delta > 0$ with the following property:

if $G \subset C^{k-m+1}(\mathbb{B}^{k+1}(x_o, \varepsilon), \mathbb{R}^{m+1})$ satisfies $|F(x) - G(x)| < \delta$ at all points $x \in \mathbb{B}^{k+1}(x_o, \varepsilon)$, then rank $DG \geq m + 1$ on an open, non-empty set in $\mathbb{B}^{k+1}(x_o, \varepsilon)$.

Before we prove Lemma 8 let us show how Theorem 4 follows from it. To this end, let $\mathbb{B}^{k+1} \subset \mathbb{B}^{k+1}$ be a ball concentric with $\mathbb{B}^{k+1}$, containing the Cantor set $E_F$ and let $\Phi : \mathbb{B}^{k+1} \to \mathbb{R}^{k+1}$ be a diffeomorphism onto $\mathbb{R}^{k+1}$ that is identity on $\mathbb{B}^{k+1}$, so $F \circ \Phi^{-1} : \mathbb{B}^{k+1} \to \mathbb{R}^{m+1}$...
\( \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{m+1} \) coincides with \( F \) on \( \mathbb{B}^{k+1} \) and hence in a neighborhood of the set \( E_F \). Denote the points in \( \mathbb{R}^{k} \) and \( \mathbb{R}^{r} \) by

\[
(x, y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{r-k-1} = \mathbb{R}^{k} \quad \text{and} \quad (z, v) \in \mathbb{R}^{m+1} \times \mathbb{R}^{r-m-1} = \mathbb{R}^{r}
\]

and let \( \pi : \mathbb{R}^{r} \rightarrow \mathbb{R}^{m+1} \), \( \pi(z, v) = z \) be the orthogonal projection.

It easily follows that the mapping

\[
\mathbb{R}^{k} \ni (x, y) \mapsto f(x, y) := (F \circ \Phi^{-1}(x), 0) \in \mathbb{R}^{r}
\]

satisfies the claim of Theorem 4 with \( E \equiv E_F \times \{0\} \subset \mathbb{R}^{k+1} \times \mathbb{R}^{r-k-1} = \mathbb{R}^{k} \).

Indeed, in a neighborhood of \( x_o \in E_F \), \( f(x, y) = (F(x), 0) \).

Suppose that \( g \in C^{k-m+1}(\mathbb{B}^{k+1}(x_o, 0), \mathbb{R}^{r}) \) is such that

\[
|f(x, y) - g(x, y)| < \delta \quad \text{for all} \quad (x, y) \in \mathbb{B}^{k+1}(x_o, 0, \varepsilon).
\]

Then \( G(x) = \pi(g(x, 0)) \in C^{k-m+1}(\mathbb{B}^{k+1}(x_o, \varepsilon), \mathbb{R}^{m+1}) \) satisfies

\[
|F(x) - G(x)| < \delta \quad \text{for all} \quad x \in \mathbb{B}^{k+1}(x_o, \varepsilon)
\]

provided \( \varepsilon > 0 \) is so small that \( f(x, y) = (F(x), 0) \) for all \( x \in \mathbb{B}^{k+1}(x_o, \varepsilon) \).

Hence \( \text{rank } DG \geq m + 1 \) on an open, non-empty set in \( \mathbb{B}^{k+1}(x_o, \varepsilon) \) by Lemma 8. Since \( \text{rank } Dg(x, 0) \geq \text{rank } DG(x) \) and the set \{ \text{rank } Dg \geq m + 1 \} \) is open, rank \( Dg \geq m + 1 \) on an open, non-empty subset of \( \mathbb{B}^{k+1}(x_o, \varepsilon) \), which completes the proof of Theorem 4.

Therefore it remains to prove Lemma 8.

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**Proof of Lemma 8.** Let \( \mathcal{I} \) denote the unit cube \( [-\frac{1}{2}, \frac{1}{2}]^{m+1} \) in \( \mathbb{R}^{m+1} \). Since, by assumption, \( \pi_k(\mathcal{S}^m) \neq 0 \) and \( \partial \mathcal{I} \) is homeomorphic to \( \mathcal{S}^m \), there is a continuous mapping \( \hat{\phi} : \mathcal{S}^k \rightarrow \partial \mathcal{I} \) that is not homotopic to a constant map. Approximating \( \hat{\phi} \) by standard mollification, we obtain a smooth mapping from \( \mathcal{S}^k \) to \( \mathbb{R}^{m+1} \), uniformly close to \( \hat{\phi} \), with the image lying in a small neighborhood of \( \partial \mathcal{I} \). Then, composing it with a \( C^\infty \) smooth mapping \( R \) that is homotopic to the identity and maps a neighborhood of \( \partial \mathcal{I} \) onto \( \partial \mathcal{I} \) we obtain a mapping \( \phi : \mathcal{S}^k \rightarrow \partial \mathcal{I} \) that is not homotopic to a constant map and is \( C^\infty \) smooth as a mapping to \( \mathbb{R}^{m+1} \).

A smooth mapping \( R : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1} \) homotopic to the identity, that maps a neighborhood of \( \partial \mathcal{I} \) onto \( \partial \mathcal{I} \) can be defined by a formula

\[
R(x_1, x_2, \ldots, x_{m+1}) = (\lambda_s(x_1), \lambda_s(x_2), \ldots, \lambda_s(x_{m+1})),
\]

where for \( s \in (0, \frac{1}{2}) \) the function \( \lambda_s : \mathbb{R} \rightarrow \mathbb{R} \) is smooth, odd, non-decreasing and such that \( \lambda_s(t) = t \) when \( ||t - \frac{1}{2}|| > 2s \) and \( \lambda(t) = 1 \) when \( ||t - \frac{1}{2}|| < s \), see the graph on the right. Taking \( s \rightarrow 0 \) gives a homotopy between \( R \) and the identity.

Lemma 8 is a simple consequence of the following result proved in \cite{2} Lemma 5.1. (Note that in the statement of Lemma 5.1 in \cite{2}, \( k \) plays the role of \( m \) and \( m \) plays the role of \( k \).) The self-similarity property of the mapping \( F \) in Lemma 9 is explicitly stated in the proof of Lemma 5.1 in \cite{2}.
Lemma 9. Suppose that $m + 1 \leq k < 2m - 1$ and $\pi_k(S^m) \neq 0$. Then there is a mapping $F \in C^1(\mathbb{R}^{k+1}, \mathbb{R}^k)$ satisfying rank $DF \leq m$ everywhere, such that $F$ maps the boundary $\partial \mathbb{B}^{k+1} = S^k$ to $\partial \mathbb{R}^k$ and $F|_{\partial \mathbb{B}^{k+1}} = \phi$, where $\phi$ has been defined above.

Moreover, $F$ is self-similar in the following sense. There is a Cantor set $E_F \subset \mathbb{R}^{k+1}$ such that for every $x_o \in E_F$ there is a sequence of balls $D_i \subset \mathbb{R}^{k+1}$, $x_o \in D_i$, with radii convergent to zero, and similarity transformations

$$\Sigma_i : \mathbb{B}^{k+1} \to D_i, \quad \Sigma_i(\mathbb{B}^{k+1}) = D_i, \quad T_i : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1},$$

each being a composition of a translation and scaling, such that

$$T_i^{-1} \circ F|_{D_i} \circ \Sigma_i = F.$$

Here the $C^1$ regularity of $F$ means that it is $C^1$ as a mapping into $\mathbb{R}^{m+1}$, with the image being the cube $\mathbb{I}$.

The mappings $T_i$ and $\Sigma_i$ are compositions $T_i = \tau_{j_1} \circ \ldots \circ \tau_{j_l}$ and $\Sigma_i = \sigma_{j_1} \circ \ldots \circ \sigma_{j_l}$ of similarity transformations $\tau_j$ and $\sigma_j$ that are used at the very end of the proof of Lemma 5.1 in [2]. The Cantor set $E_F$ is the same as the Cantor set $C$ in the proof of Lemma 5.1 in [2].

In other words, $F$ restricted to an arbitrarily small ball $D_i$ that contains $x_o$ is a scaled copy of $F : \mathbb{B}^{k+1} \to \mathbb{I}$.

The mapping $F$ is obtained through an iterative construction, described in detail in [2]. We shall present here a sketch of that construction.

Sketch of the construction of the mapping $F$.

By assumption, $\pi_k(S^m) \neq 0$. By Freudenthal’s theorem ([3, Corollary 4.24]), also $\pi_{k-1}(S^{m-1}) \neq 0$; let $h : S^{k-1} \to S^{m-1}$ be a mapping that is not homotopic to a constant.

We begin by choosing in the ball $\mathbb{B}^{k+1}$ disjoint, closed balls $\mathbb{B}_i$, $i = 1, 2, \ldots, N = n^{m+1}$, of radius $\frac{2}{n}$, all inside $\frac{1}{2} \mathbb{B}^{k+1}$. This is possible, if $n$ is chosen sufficiently large, since, for $n$ large, the $(k+1)$-dimensional volume of $\frac{1}{2} \mathbb{B}^{k+1}$ is much larger than the sum of volumes of $\mathbb{B}_i$, $2^{-(k+1)} \gg n^{m+1}2^{k+1}n^{-(k+1)}$.

We define a $C^\infty$-mapping $F$ in $\mathbb{B}^{k+1} \setminus \bigcup_{i=1}^N \mathbb{B}_i$; then, the same mapping is iterated inside each of the balls $\mathbb{B}_i = \mathbb{B}_{i,1}$, which defines $F$ outside a family of $N^2$ second generation balls $\mathbb{B}_{i,2}$, and so on – in this way we obtain a mapping which is $C^\infty$ outside a Cantor set. Finally, we extend $F$ continuously to the Cantor set $C$ defined by the subsequent generations of balls $\mathbb{B}_{i,j}$, as the intersection $C = \bigcap_{j=1}^\infty \bigcup_{i=1}^{N^j} \mathbb{B}_{i,j}$.

The mapping $F$ in $\mathbb{B}^{k+1} \setminus \bigcup_{i=1}^N \mathbb{B}_i$ is (in principle – see comments below) defined as a composition of four steps (see Figure 1):

1. First, we realign all the balls $\mathbb{B}_i$ inside $\mathbb{B}^{k+1}$, by a diffeomorphism $G_1$ equal to the identity near $\partial \mathbb{B}^{k+1}$, so that the images of $\mathbb{B}_i$ are identical, disjoint, closed balls
lying along the vertical axis of $\mathbb{B}^{k+1}$. Obviously, this diffeomorphism has to shrink the balls $\mathbb{B}_i$ somewhat.

(2) The next step, the mapping $H : \mathbb{B}^{k+1} \rightarrow \mathbb{B}^{m+1}$, is defined in the following way: it maps $(k - 1)$-dimensional spheres centered at the vertical axis of $\mathbb{B}^{k+1}$, lying in the hyperplane orthogonal to that axis, to $(m - 1)$-dimensional spheres of the same radius, centered at analogous points on the vertical axis of $\mathbb{B}^{m+1}$. On each such sphere, $H$ is an appropriately scaled copy of the mapping $h$. This way, $H$ restricted to any $k$-sphere centered on the axis (in particular to $\partial \mathbb{B}_{k+1}$ and to $\partial(G_1(\mathbb{B}_i))$) equals (up to scaling) to the suspension of $h$.

(3) Next, we define the diffeomorphism $G_2$: we inflate the ball $\mathbb{B}^{m+1}$ to $\frac{1}{2}\sqrt{m + 1}\mathbb{B}^{m+1}$, so that we can inscribe the unit cube $[-\frac{1}{2}, \frac{1}{2}]^{m+1}$ in it, and inside that ball, we rearrange the $N$ balls $H(G_1(\mathbb{B}_i))$, so that each of them is almost inscribed in one of the cubes of the grid obtained by partitioning the unit cube into $N = n^{m+1}$ cubes of edge length $\frac{1}{n}$.

(4) Finally, we project $\frac{1}{2}\sqrt{m + 1}\mathbb{B}^{m+1} \cup \bigcup_{i=1}^{N} G_2(H(G_1(\mathbb{B}_i)))$ onto the $m$-dimensional skeleton of the grid: first, we project the outside of the unit cube onto the boundary of the cube using the nearest point projection $\pi$, then in each of the $N$ closed cubes of the grid we use the mapping $R$ defined in the proof of Lemma 8. Even though $\pi$ is not smooth, this composition turns out to be smooth (see [2, Lemma 5.3]).

In fact, this construction of $F$ outside $\bigcup_i \mathbb{B}_i$ is almost correct – the resulting mapping is not $C^\infty$, but Lipschitz: it is not differentiable at the points of the vertical axis, and some technical modifications are necessary to make it $C^\infty$. Similarly, some additional work is necessary to glue $F$ with scaled copies of $F$ in each of the balls $\mathbb{B}_i$ in a differentiable way. These are purely technical difficulties, the details are provided in [2].

The third iteration of that construction is depicted in Figure 2.

One easily checks that the derivative of $F$ tends to 0 as we approach the points of the Cantor set $C$, thus the limit mapping, extended to the whole $\mathbb{B}^{k+1}$, is $C^1$. For each point of $\mathbb{B}^{k+1} \setminus C$, the image of its small neighborhood is mapped to the $m$-dimensional skeleton of the grid, thus rank $DF \leq m$ at all these points, and since $DF = 0$ at the points of $C$, the condition rank $DF \leq m$ holds everywhere in $\mathbb{B}^{k+1}$.

\[\square\]

Lemma 9 allows us to complete the proof of Lemma 8 as follows. Let $x_o \in E_F$ and $1 - |x_o| > \varepsilon > 0$ be given. Suppose to the contrary, that there is a sequence $G_j \in C^{k-m+1}(\mathbb{B}^{k+1}(x_o, \varepsilon), \mathbb{R}^{m+1})$ with rank $DG_j \leq m$, that is uniformly convergent to $F$ on $\mathbb{B}^{k+1}(x_o, \varepsilon)$.

Let $D_i$ be a sequence of balls convergent to $x_o$ as in the statement of Lemma 9. If $i$ is sufficiently large, then $D_i \subset \mathbb{B}^{k+1}(x_o, \varepsilon)$ and the sequence $G_j$ converges uniformly to $F$ on $D_i$. Hence

$$\tilde{G}_j := T_i^{-1} \circ G_j \mid_{D_i} \circ \Sigma_i : \mathbb{B}^{k+1} \rightarrow \mathbb{R}^{m+1}$$

converges uniformly to

$$T_i^{-1} \circ F \mid_{D_i} \circ \Sigma_i = F : \mathbb{B}^{k+1} \rightarrow \Pi.$$
Figure 1. The construction of $F$ in $\mathbb{B}^{m+1} \setminus \bigcup_{i=1}^{N} \mathbb{B}_i$.

Figure 2. The third iteration: $F$ outside the third generation of balls $\bigcup_i \mathbb{B}_{3,i}$.
Obviously, \( \text{rank } D\tilde{G}_j \leq m \). Since \( \tilde{G}_j \) is uniformly close to \( F \) on \( \partial B^{k+1} \) and \( F|_{\partial B^{k+1}} : S^k \to \partial I \) is not homotopic to a constant map, it easily follows that for \( j \) sufficiently large the image \( \tilde{G}_j(B^{k+1}) \) contains the cube \( \frac{1}{2}I \) that is concentric with \( I \) and has half the diameter (as otherwise, using a projection onto the boundary of the cube, one could construct a homotopy of \( F|_{\partial B^{k+1}} : S^k \to \partial I \) to a constant map).

Recall that according to Sard’s theorem [4, 5], the map \( \tilde{G}_j \in C^{k-m+1} \) maps the set of its critical points to a set of measure zero. Since \( \text{rank } D\tilde{G}_j \leq m \), all points in \( B^{k+1} \) are critical, so the set \( \tilde{G}_j(B^{k+1}) \) has measure zero, which contradicts the fact that it contains the cube \( \frac{1}{2}I \). The proof is complete. \( \square \)

**References**

[1] Gałęski, J.: Besicovitch–Federer projection theorem for continuously differentiable mappings having constant rank of the Jacobian matrix. *Math. Z.* (2017), https://doi.org/10.1007/s00209-017-1985-x.

[2] Goldstein, P., Hajłasz, P., Pankka, P.: Topologically nontrivial counterexamples to Sard’s theorem. *arXiv:1804.07658*. Submitted.

[3] Hatcher, A.: *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[4] Sard, A.: The measure of the critical values of differentiable maps. *Bull. Amer. Math. Soc.* 48 (1942), 883–890.

[5] Sternberg, S.: *Lectures on differential geometry*. Chelsea Publishing Co., New York, second edition, 1983. With an appendix by Sternberg and Victor W. Guillemin.

[6] Zorich, V. A.: *Mathematical analysis. I.* Second edition. Universitext. Springer-Verlag, Berlin, 2015.

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