Viscoplastic fluid flow modeling with the "solidification" effect in the straight channel of the annular cross section

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Abstract. On the basis of the viscoplastic fluid rheological model, which takes into account the manifestation of the “solidification” effect, a flow has been simulated in the annular cross section channel. It is shown that, depending on the pressure drop level, three different flow patterns are possible. For each pattern, the expressions that determine the fluid velocity distribution in the channel's cross section have been obtained. It is shown that the dependence of the volume fluid flow rate on the pressure drop is not monotonic. Moreover, the increase in the pressure drop critical level ultimately leads to the flow rate decrease due to the narrowing of the channel's passage section, which is caused by the formation of the “solidified” liquid layers.

1. Introduction
Some suspensions types based on polymer liquids and fine particles at certain predetermined combination of their size and concentration [1-3] can show certain anomalies, such as, viscous behavior.

The main feature of the viscous behavior of such working environments can be outlined as follows. As the shear rate approaches a certain critical level, the liquid begins to exhibit dilatant properties. At the same time, the increase in the viscosity becomes so significant that the liquid behaves like a solid, which allows us to talk about its "hardening" or the manifestation of the "solidification" effect. As the shear rate decreases below a critical level, the fluid "restores" its initial viscosity.

It should be noted that we are not talking about the traditional “liquid - solid” phase transition, but about the change in the material internal structure caused by the formation of the fine particles into associations or clusters of the "solid" structures type.

This type of behavior does not allow us to use the traditional rheological dilatant fluid models such as the viscosity power law model. In this regard, the corresponding rheological model was proposed in [4] and the problem of the rotational fluid flow with this kind of viscous behavior anomaly between the two coaxial cylinders was considered.

In the case when the liquid component of such suspensions is a viscoplastic continuum, this, of course, introduces appropriate adjustments to the final mechanical behavior. Specifically, such liquid can behave like a solid in two limiting cases. Firstly, this applies to the situation when the stresses in the fluid do not exceed the yield strength. The second limiting case will occur when the shear rate approaches the threshold, critical level and the manifestation of the “solidification” effect. Naturally, in each of these two cases we don’t talk about the entire flow area, but only about its individual zones.
Moreover, in different (divorced) zones, at the same time, each of these two limiting cases can manifest itself separately.

Note that it is the plasticity that makes it possible to produce “fixed” form products from such materials. However, the rather complicated continua behavior, as a rule, is discussed from the point of view of technical applications only at the qualitative level [5-7]. However, the viscoplastic behavior itself has not been reflected at the mathematical modeling level. In this regard, this paper proposes a rheological model in the context of solving the problem of the viscoplastic fluid flow in the annular cross-section rectilinear channel with the “solidification” effect and considers the mathematical modeling method of the continuous medium flow of this kind.

2. Rheological model

Let us consider a fluid whose rheological model for the one-dimensional flow case is defined as follows [8]

\[
|\dot{\gamma}| = \begin{cases} 0; & |\dot{\gamma}| < \tau_p; \\
1 - \left( \frac{\tau_s - |\dot{\gamma}|}{\tau_s - \tau_p} \right)^n \cdot \dot{\gamma}; & \tau_p \leq |\dot{\gamma}| \leq \tau_s; \\
0 < n < 1; & |\dot{\gamma}| \leq \dot{\gamma}_s, \end{cases}
\]

where \( \dot{\gamma} \) is the shear rate; \( \tau \) are the shearing stress; \( \dot{\gamma}_s \) is the critical value of the shear rate modulus, upon approaching which the liquid demonstrates the manifestation of the “solidification” effect; \( \tau_p \) is the yield stress; \( \tau_s \) is the shear stress value which is achieved at the shear rate limit value (\( |\dot{\gamma}| = \dot{\gamma}_s \)); \( n \) is a rheological model parameter.

Note that the fluid viscous properties are characterized by the flow curve steepness. This value is the curve slope to the shear stress versus shear rate graph. In particular, for the classical Newtonian fluid, in which the flow curve graph is a straight line, this value is exactly equal to the dynamic viscosity.

From this perspective, the condition for the “solidification” effect manifestation, when the liquid viscosity increases indefinitely, can be represented at the model level as follows

\[
\lim_{\dot{\gamma} \to \dot{\gamma}_s} \left( \frac{d|\dot{\gamma}|}{d|\dot{\gamma}|} \right) = \infty.
\]

(2)

In the shear flow region, the dependence of the shear stress on the shear rate is determined taking into account (1) by the relation of the form

\[
|\tau| = \tau_s - \left( \tau_s - \tau_p \right) \left( 1 - \frac{|\dot{\gamma}|}{\dot{\gamma}_s} \right)^n.
\]

Given that, the “solidification” condition (2) for the considered range \( 0 < n < 1 \) of the rheological model parameter is unconditionally fulfilled.

Note that the rheological model (1) admits the following generalization to the case of shear flow in an arbitrary spatial domain

\[
\tau_{ij} = -P \cdot \delta_{ij} + 2 \cdot \mu( I_2 ) \cdot \varepsilon_{ij}; \quad i, j = 1, 2, 3;
\]
\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right); \quad I_2 = \varepsilon_{11} \cdot \varepsilon_{22} + \varepsilon_{22} \cdot \varepsilon_{33} + \varepsilon_{33} \cdot \varepsilon_{11} - \varepsilon_{12}^2 - \varepsilon_{23}^2 - \varepsilon_{31}^2; \]

\[ \mu(I_2) = \frac{1}{2 \cdot \sqrt{|I_2|}} \left\{ \tau_s - (\tau_s - \tau_p) \cdot \left[ 1 - \frac{|I_2|}{\sqrt{|I_2|}} \right] \right\}; \quad |I_2| \leq I_{2,\text{crit}}. \]

Where \( \tau_{ij} \), \( \varepsilon_{ij} \) are the stress tensors and strain rates components; \( P \) is the hydrostatic pressure; \( u_i \) are the velocity vector components; \( \delta_{ij} \) is the Kronecker symbol; \( \mu(I_2) \) is a function of the second strain rate tensor invariant \( I_2 \), that characterizes the liquid transverse viscosity; \( I_{2,\text{crit}} \) is the second invariant modulus value of the strain rate tensor approaching which \( (I_2) \rightarrow I_{2,\text{crit}} \) the “solidification” effect begins to manifest itself.

In the case of a one-dimensional shear flow, the obvious relation is valid

\[ I_{2,\text{crit}} = -\frac{1}{4} \dot{\gamma}_s^2. \]

The rheological model under consideration suggests that the “solidification” transition is carried out on the corresponding spatial zones surface, where condition (2) is satisfied. In other words, condition (2) defines the boundaries of the zones filled with the “solidified” liquid material. Moreover, in the future, within such zones, the “solidification” state remains unchanged. However, this is a simplifying assumption.

3. Formulation of the problem

Let us consider a one-dimensional, steady-state, laminar, axisymmetric fluid flow with a rheological model (1) under the action of a given pressure drop in the straight channel of the annular cross section.

We introduce a cylindrical coordinate system as shown in figure 1.

Taking into account the assumptions above, the motions equations and the flow continuity condition take the form

\[ \frac{\partial P}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} (r \cdot \tau_{rz}); \quad \frac{\partial P}{\partial r} = 0; \quad \frac{\partial u_z}{\partial z} = 0, \quad (3) \]

Here

\[ u_z = u(r); \quad \tau_{rz} = 2 \cdot \mu(I_2) \cdot \varepsilon_{rz}; \quad I_2 = -\frac{1}{4} \cdot \varepsilon_{rz}^2; \quad \varepsilon_{rz} = \frac{1}{2} \cdot \dot{\gamma} = \frac{1}{2} \cdot \frac{du}{dr}, \]

where \( u_z \) is the only identically non-zero longitudinal velocity which is a yet unknown function of the radial coordinate.

The following expression for the pressure distribution directly follows from (3)

\[ P(z) = P_0 - (P_0 - P_I) \cdot \frac{z}{L}, \]

where \( P_0, P_I \) are the pressure values in the channel sections at its inlet and outlet, respectively; \( L \) is the channel length.
Figure 1. Flow region.

To represent the main quantities in the dimensionless notation (marked with upper strokes below), we use the following relations:

\[ r' = \frac{r}{R_2}; \quad z' = \frac{z}{L}; \quad R_I' = \frac{R_I}{R_2}; \quad P' = \frac{P}{\tau_s}; \quad \Delta P' = \frac{\Delta P}{\tau_s}; \quad \tau_{r z}' = \frac{\tau_{r z}}{\tau_s}; \quad \tau_p' = \frac{\tau_p}{\tau_s}; \quad u' = \frac{u}{V}; \quad \gamma' = \frac{\gamma}{\gamma_s}; \quad Q' = \frac{Q}{\gamma_s' \cdot R_2}; \quad G = \frac{R_2}{L}, \]

where \( R_I, R_2 \) are the inner and outer radii of the cylindrical surfaces bounding the flow region; \( \Delta P \) is the pressure drop across the channel length; \( V \) is the characteristic speed value, taken as large-scale; \( Q \) - is the volume fluid flow rate through the channel; \( G \) is the geometric problem parameter.

A preliminary analysis, taking into account the fluid mechanical behavior peculiarities inherent in model (1), shows that, depending on the pressure drop level across the channel length, there are three possible different flow patterns. Let us consider the flow modeling for each of these three options.

4. The first flow pattern
First of all, note that the yield strength presence assumes the existence of the pressure drop minimum threshold value

\[ \Delta P_{min}' = \frac{2 \cdot \tau_p'}{G \cdot (1 - R_I')}, \]

below the level of which the considered liquid flow through the channel is impossible.

If this threshold level (\( \Delta P' > \Delta P_{min}' \)) is exceeded, a flow begins in the channel. In this case, a priori, it can be expected that in the central part of the channel section a rigid zone of plastic flow will form with a constant velocity value. At the same time, the shear flows that satisfy the system of equations (3) will be realized in the annular gaps between the channel walls and this rigid zone. The structural diagram of such flow is presented in Figure 2.
Figure 2. The first flow pattern structure. 1, 2 are the shear flow zones, \( p \) is the plastic flow rigid zone.

Let us write down the boundary conditions for such flow pattern

\[
\begin{align*}
  r' &= R'_1; & u'(1) &= 0; & r' &= R'_{1p}; & \tau'(1) &= \tau'_p; \\
  r' &= R'_{2p}; & \tau'(2) &= -\tau'_p; & r' &= 1; & u'(2) &= 0 .
\end{align*}
\] (4)

Here it is also necessary to demand the fulfillment of one more obvious condition

\[
u'(1)(R'_{1p}) = u'(2)(R'_{2p}) = u'(p) .
\] (5)

where \( R'_{1p}, R'_{2p} \) are the radii of the plastic flow rigid zone cylindrical boundaries, which are still unknown parameters of the problem.

In (4), (5) and further, superscripts 1, 2 in parentheses indicate that the characteristics under consideration belong to the first or the second shear flow zones, as well as to the rigid plastic flow zone, presented in Figure 2.

Solving then (3), taking into account the boundary conditions (4) within the framework of the first flow pattern, we arrive at the following distribution of the fluid velocity in the channel cross section

\[
u' = \begin{cases} 
  u'(1)(r'); & R'_1 \leq r' \leq R'_{1p}; \\
  u'(p)(r'); & R'_{1p} \leq r' \leq R'_{2p}; \\
  u'(2)(r'); & R'_{2p} \leq r' \leq 1;
\end{cases}
\] (6)

\[
u'(1)(r') = \int_{R'_1}^{r'} f^{(1)}(\xi, R'_{1p}, \Delta P') \cdot d\xi; \quad \nu'(2)(r') = -\int_{r'}^{R'_{2p}} f^{(2)}(\xi, R'_{2p}, \Delta P') \cdot d\xi ;
\]

\[
u'(p) = \int_{R'_1}^{R'_{1p}} f^{(1)}(\xi, R'_{1p}, \Delta P') \cdot d\xi = -\int_{R'_{2p}}^{R'_{2p}} f^{(2)}(\xi, R'_{2p}, \Delta P') \cdot d\xi = \text{const} ;
\]
should be determined by the following system of the three equations:

\[
f^{(1)}(r', R'_{1p}, \Delta P') = \dot{\gamma}'^{(1)} = \frac{du'^{(1)}}{dr'} = 1 - \left\{ \frac{1}{1 - \tau_p'} \left[ I - \Delta P' \cdot G \cdot \left( \frac{R'_{1p}^2 - r'^2}{2 \cdot r'} - \frac{R'_{1p}}{r'} \right) \right] \right\}^{\frac{1}{n}};
\]

\[
f^{(2)}(r', R'_{2p}, \Delta P') = \dot{\gamma}'^{(2)} = \frac{du'^{(2)}}{dr'} = \left\{ \frac{1}{1 - \tau_p'} \left[ I - \Delta P' \cdot G \cdot \left( \frac{r'^2 - R'_{2p}^2}{2 \cdot r'} - \frac{R'_{2p}}{r'} \right) \right] \right\}^{\frac{1}{n}} - 1.
\]

In this case, the shear stress distributions in the shearing flow zones are determined by the following relations

\[
\tau'^{(1)} = 1 - (1 - \tau_p') \cdot (1 - \dot{\gamma}'^{(1)})^n; \quad \tau'^{(2)} = 1 - (1 - \tau_p') \cdot (1 + \dot{\gamma}'^{(2)})^n.
\]

The yet unknown radii values \( R'_{1p} \) и \( R'_{2p} \) which bound the rigid zone of the plastic flow are determined by the following equations system solution

\[
\begin{align*}
(1 + R'_1) - (1 - \tau_p') & \cdot \left\{ R'_1 \left[ I - f^{(1)}(R'_1, R'_{1p}, \Delta P') \right] \right\}^n + \left[ I + f^{(2)}(1, R'_{2p}, \Delta P') \right]^n = \\
&= \frac{\Delta P' \cdot G}{2} \cdot (1 - R'_1^2); \\
\int_{R'_1}^{R'_p} f^{(1)}(r', R'_{1p}, \Delta P') \cdot dr' & = - \int_{R'_2}^{R'_p} f^{(2)}(r', R'_{2p}, \Delta P') \cdot dr'.
\end{align*}
\]

In its meaning, the first equation of this system is an equilibrium condition for a control liquid volume, and the second equation is condition (5).

In view of (6), the volume fluid flow rate through the channel is determined by the following relation

\[
Q' = 2 \cdot \pi \cdot \int_{R'_1}^{R'_p} r' \cdot u''^{(1)}(r') \cdot dr' + \pi \cdot (R'_{2p}^2 - R'_{1p}^2) \cdot u''^{(p)} + 2 \cdot \pi \cdot \int_{R'_2}^{R'_p} r' \cdot u''^{(2)}(r') \cdot dr'.
\]

As the pressure drop increases, the cylindrical surfaces shear rate moduli values on limit the flow region, increase. As our numerical experiments show, the following condition holds:

\[
|\dot{\gamma}'^{(2)}(R'_{2p})| < |\dot{\gamma}'^{(1)}(R'_{1p})|.
\]

Therefore, when the differential pressure reaches certain critical value \( \Delta P' = \Delta P'_{crit,1} \), the shear rate will reach the threshold level \( |\dot{\gamma}'^{(1)}(R'_{1p})| = 1 \) corresponding to the “solidification” effect onset on the inner cylindrical surface \( r' = R'_{1p} \). This differential pressure critical value, together with its corresponding values \( R'_{1p} \) и \( R'_{2p} \) should be determined by the following system of the three equations:

\[
\begin{align*}
(1 + R'_1) - (1 - \tau_p') & \cdot \left\{ R'_1 \left[ I - f^{(1)}(R'_1, R'_{1p}, \Delta P'_{crit,1}) \right] \right\}^n + \left[ I + f^{(2)}(1, R'_{2p}, \Delta P'_{crit,1}) \right]^n = \\
&= \frac{\Delta P' \cdot G}{2} \cdot (1 - R'_1^2); \\
\int_{R'_1}^{R'_p} f^{(1)}(r', R'_{1p}, \Delta P'_{crit,1}) \cdot dr' & = - \int_{R'_2}^{R'_p} f^{(2)}(r', R'_{2p}, \Delta P'_{crit,1}) \cdot dr'.
\end{align*}
\]
\[
\frac{\Delta P'_{\text{crit},l}}{2} \cdot G \cdot (1 - R_j^2);
\]

\[
\int_{R_j}^{R_{1p}} f^{(1)} (r', R_{1p} \cdot \Delta P'_{\text{crit},l}) \cdot dr' = - \int_{R_{2p}}^{1} f^{(2)} (r', R_{2p} \cdot \Delta P'_{\text{crit},l}) \cdot dr';
\]

\[
\Delta P'_{\text{crit},l} \cdot G \cdot (R_{1p}^2 - R_j^2) + 2 \cdot \tau'_{p} \cdot R_{1p} = 2 \cdot R_{1},
\]

From here it follows that the next pressure drop range determines the first flow pattern region of existence

\[
\Delta P'_{\text{min}} \leq \Delta P' \leq \Delta P'_{\text{crit},l}.
\]

5. The second flow pattern

When the pressure drop exceeds the critical level \( \Delta P'_{\text{crit},l} \), a zone filled with the “solidified” fluid material appears on the inner cylindrical surface, and a new (second) flow pattern begins in the channel. Structurally, this flow version is presented in Figure 3.

![Figure 3. The second flow pattern structure. 1, 2 are the shearing flow zones, \( p \) is the plastic flow rigid zone; \( s_1 \) is the zone filled with the “solidified” fluid material.](image)

Let us write down the boundary conditions for the second flow pattern

\[
\begin{align*}
& r' = R_{1s}^1; \quad u^{(1)} = 0; \quad \tau^{(1)} = I; \quad r' = R_{1p}^1; \quad \tau^{(1)} = \tau'_{p}; \\
& r' = R_{2p}^1; \quad \tau^{(2)} = - \tau'_{p}; \quad r' = 1; \quad u^{(2)} = 0,
\end{align*}
\]

where \( R_{1s}^1 \) is the yet unknown outer boundary radius of the zone filled with the “solidified” fluid material and formed on the channel’s inner cylindrical surface.
After solving the equations system (3), taking into account the boundary conditions (9), the expressions for the velocity distribution over the main flow zones take on the following form:

\[
u^{(1)} = \int_{R_{1p}^i}^{r'} f^{(1)}(\xi, R_{1p}, \Delta P') \cdot d\xi; \quad u^{(2)} = -\frac{1}{r'} \int_{R_{2p}^i}^{r} f^{(2)}(\xi, R_{2p}, \Delta P') \cdot d\xi;
\]

\[
u^{(p)} = \int_{R_{1s}^i}^{R_{1p}^i} f^{(1)}(\xi, R_{1p}, \Delta P') \cdot d\xi = -\frac{1}{R_{2p}^i} \int_{R_{2p}^i}^{R_{1p}^i} f^{(2)}(\xi, R_{2p}, \Delta P') \cdot d\xi = \text{const}.
\]

In this case, the shear stress distributions over the shearing flow zones are determined by the relations that coincide with (7). As for the yet unknown set of the parameters \(R_{1p}^i, R_{2p}^i, R_{1s}^i\), they should be determined by the solution of the following equations system

\[
(I + R_{1s}^i) - (1 - \tau'_p) \cdot \left[ I + f^{(2)}(I, R_{2p}, \Delta P') \right]^n = \frac{\Delta P' \cdot G}{2} \cdot (I - R_{2s}^2);
\]

\[
\int_{R_{1s}^i}^{R_{1p}^i} f^{(1)}(r', R_{1p}, \Delta P') \cdot dr' = -\frac{1}{R_{2p}^i} \int_{R_{2p}^i}^{R_{1p}^i} f^{(2)}(r', R_{2p}, \Delta P') \cdot dr';
\]

\[
\Delta P' \cdot G \cdot (R_{1p}^2 - R_{1s}^2) + 2 \cdot \tau'_p \cdot R_{1p}^i = 2 \cdot R_{1p}^i.
\]

The meaning of the first two equations of this system has already been analyzed when we considered the first flow pattern. The third equation follows from the condition that the shear rate is equal to the threshold value at the “solidified” fluid boundary zone at \(r' = R_{1s}^i\).

The volume fluid flow rate through the channel, taking into account the found expressions for the velocity distribution, is determined as follows

\[
Q'_2 = 2 \cdot \pi \cdot \int_{R_{1s}^i}^{R_{1p}^i} r' \cdot u^{(1)}(r') \cdot dr' + \pi \cdot (R_{2p}^2 - R_{1p}^2) \cdot u^{(p)} + 2 \cdot \pi \cdot \int_{R_{1p}^i}^{R_{1p}^i} r' \cdot u^{(2)}(r') \cdot r'.
\]

The second flow pattern will exist until the pressure drop, as it increases, reaches another critical level \(\Delta P' = \Delta P'_{\text{crit}, 2} > \Delta P'_{\text{crit}, 1}\), at which another zone filled with the “solidified” fluid material begins to form on the channel outer wall (\(r' = R'_2\)).

Such critical value of the pressure drop, together with the corresponding values \(R_{1p}^i, R_{2p}^i, R_{1s}^i\) should be determined by the solution of the following system of the four equations

\[
(I + R_{1s}^i) - (1 - \tau'_p) \cdot \left[ I + f^{(2)}(I, R_{2p}^i, \Delta P'_{\text{crit}, 2}) \right]^n = \frac{\Delta P'_{\text{crit}, 2} \cdot G}{2} \cdot (I - R_{2s}^2);
\]

\[
\int_{R_{1s}^i}^{R_{1p}^i} f^{(1)}(r', R_{1p}^i, \Delta P'_{\text{crit}, 2}) \cdot dr' = \frac{1}{R_{2p}^i} \int_{R_{2p}^i}^{R_{1p}^i} f^{(2)}(r', R_{2p}^i, \Delta P'_{\text{crit}, 2}) \cdot dr';
\]

\[
\Delta P'_{\text{crit}, 2} \cdot G \cdot (R_{1p}^2 - R_{1s}^2) + 2 \cdot \tau'_p \cdot R_{1p}^i = 2 \cdot R_{1p}^i, \quad \Delta P'_{\text{crit}, 2} \cdot G \cdot (I - R_{2p}^2) + 2 \cdot \tau'_p \cdot R_{2p}^i = 2.
\]

The last equation in this system is the condition that the shear rate is equal to the threshold value at the flow region external boundary at \(r' = I\).
Thus, the second flow pattern can be implemented in the following pressure drop range:

$$\Delta P'_{\text{crit},1} \leq \Delta P' \leq \Delta P'_{\text{crit},2}.$$ 

6. The third flow pattern

When the pressure drop exceeds the second critical level ($\Delta P' > \Delta P'_{\text{crit},2}$) in the channel, another zone begins to form on the external cylindrical surface filled with the “solidified” fluid material, and the third flow pattern begins to be implemented in the channel. Structurally, this flow version is presented in Figure 4.

![Figure 4. The third flow pattern structure.](image)

[1, 2] are the shearing flow zones; $s_1$, $s_2$ are the zones filled with the “solidified” fluid material.

Let us write down the boundary conditions

$$r' = R'_{1s}; \quad u''(1) = 0; \quad \tau''(1) = I; \quad r' = R'_{1p}; \quad \tau''(1) = \tau'_p;$$

$$r' = R'_{2p}; \quad \tau''(2) = -\tau'_p; \quad r' = R'_{2s}; \quad u''(2) = 0; \quad \tau''(2) = -1,$$

where $R'_{2s}$ is the yet unknown outer boundary radius of the zone filled with the “solidified” fluid material and formed on the channel’s outer cylindrical surface.

After solving the equations system (3) once more, taking into account the boundary conditions (9), we arrive at the expressions of the velocity distribution over the main flow zones:

$$u''(1) = \int_{R'_{1s}}^{r'} f^{(1)}(\xi, R'_{1p}, \Delta P') \cdot d\xi; \quad u''(2) = -\int_{r'}^{R'_{2s}} f^{(2)}(\xi, R'_{2p}, \Delta P') \cdot d\xi;$$

$$u''(p) = \int_{R'_{1s}}^{R'_{1p}} f^{(1)}(\xi, R'_{1p}, \Delta P') \cdot d\xi = -\int_{R'_{2p}}^{R'_{2s}} f^{(2)}(\xi, R'_{2p}, \Delta P') \cdot d\xi = \text{const}.$$ 

In this case, the same as the second pattern, the shear stress distributions over the shearing flow zones are determined by the relations that coincide with (7). As for the yet unknown set of the
parameters \( R'_{1p}, \ R'_{2p}, \ R'_{1s}, \ R'_{2s} \) they should be determined by the solution of the following equations system

\[
\Delta P' \cdot G \cdot ( R'_{2s} - R'_{1s} ) = 2; \quad \int_{R'_{1p}}^{R'_{2s}} f^{(1)}( r', R'_{1p}, \Delta P'_{crit,l} ) \cdot dr' = - \int_{R'_{2p}} f^{(2)}( r', R'_{2p}, \Delta P'_{crit,l} ) \cdot dr';
\]

\( \Delta P' \cdot G \cdot ( R'_{1p} - R'_{1s} ) + 2 \cdot \tau' \cdot R'_{1p} = 2 \cdot R'_{1s}; \quad \Delta P' \cdot G \cdot ( R'_{2s}^2 - R'_{2p}^2 ) + 2 \cdot \tau' \cdot R'_{2p} = 2 \cdot R'_{2s}. \)

The meaning of the first three equations of this system has already been analyzed when we considered the first and the second flow patterns. The fourth equation follows from the condition that the shear rate is equal to the threshold value at the second “solidified” fluid boundary zone at \( r' = R'_{2s}. \)

The volume fluid flow rate through the channel for the third flow pattern is determined as follows

\[
Q'_3 = 2 \cdot \pi \cdot \int_{R'_{1p}}^{R'_{2s}} r' \cdot u'^{(1)}( r' ) \cdot dr' + \pi \cdot ( R'_{2s}^2 - R'_{1p}^2 ) \cdot u'^{(p)} + 2 \cdot \pi \cdot \int_{R'_{1p}}^{R'_{2s}} r' \cdot u'^{(2)}( r' ) \cdot dr' \quad (12).
\]

7. The dependence of the volume fluid flow rate on the pressure drop analysis

Taking into account (8), (10), (12), we find the expression for determining the dependence of the volume fluid flow rate on the pressure drop over its size of changing, as follows

\[
Q' \left( \Delta P' \right) = \begin{cases} 
0; & 0 \leq \Delta P' \leq \Delta P'_{min}; \\
Q'_1( \Delta P' ); & \Delta P'_{min} \leq \Delta P' \leq \Delta P'_{crit,l}; \\
Q'_2( \Delta P' ); & \Delta P'_{crit,l} \leq \Delta P' \leq \Delta P'_{crit,2}; \\
Q'_3( \Delta P' ); & \Delta P' > \Delta P'_{crit,2}.
\end{cases}
\]

An idea of the nature of the behavior of function (13) is given by the example given in figure 5.

**Figure 5.** The dependence of the volumetric flow rate on the pressure drop for \( R'_j = 0.2; \) \( G = 0.05; \) \( \tau'_p = 0.1; \) \( n = 0.2; \) \( \Delta P'_{min} = 5; \) \( \Delta P'_{crit,l} = 28.047; \) \( \Delta P'_{crit,2} = 5432. \)

If we analyze the presented dependence, we can see that the “expected” volume fluid flow rate increase with the growing pressure drop takes place only for the first flow pattern.
Further pressure drop increase (for the second and third flow patterns ranges) leads to the volume flow rate decrease and the manifestation of the channel “blocking” effect.

This behavior is due to the formation of the “solidified” liquid material zones in the flow region, and, as a result, the channel cross section decrease.

8. Conclusion

We have considered the problem of viscoplastic fluid flow with the “solidification” effect in the straight channel of the annular cross section. According to the solution results, the manifestation of the channel “blocking” effect has been shown. The methodology for solving the considered problem, which involves the allocation of the three zones in the flow region (the viscous shearing flow zone, the rigid plastic flow zone and the “solidified” fluid material zone), can be recommended for fluid flows modeling of this kind and of a more complex geometry.

Speaking about further development of the considered rheological model, it should be noted that in a more accurate formulation it should also take into account the reverse process of transformation of the “solidified” fluid into a liquid state not only on the surface of the corresponding zones filled with the “solidified” fluid material, but also inside such zones.

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