Dynamic programming principle for a controlled FBSDE system and associated extended HJB equation

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Abstract: This paper investigates the dynamic programming principle for a general stochastic control problem in which the state processes are described by a forward-backward stochastic differential equation (FBSDE). Using the method of $S$-topology, we show that there exists an optimal control for the value function. Then a dynamic programming principle is established. As a consequence, an extended Hamilton-Jacobi-Bellman (HJB) equation is derived. The existence and uniqueness of both smooth solution and a new type of viscosity solution are investigated for this extended HJB equation. Compared with the extant researches on stochastic maximum principle, this paper is the first normal work on partial differential equation (PDE) method for a controlled FBSDE system.

Keywords: Dynamic programming principle; Forward-backward stochastic differential equation; Hamilton-Jacobi-Bellman equation; Viscosity solution; $S$-topology.

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1 Introduction

In the present paper, we study the following value function of a general cost functional,

\[ v(t, x) = \inf_{u(t) \in \mathcal{U}[t, T]} \mathbb{E} \left[ \int_t^T f(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s) \, ds + G(X_T^{t,x,u}) \right], \tag{1.1} \]

where the controlled states \((X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u})_{t \leq s \leq T}\) are the solutions of stochastic differential equation (SDE)

\[ X_s^{t,x,u} = x + \int_t^s \mu(r, X_r^{t,x,u}, u_r) \, dr + \int_t^s \sigma(r, X_r^{t,x,u}, u_r) \, dB_r, \tag{1.2} \]

and backward stochastic differential equation (BSDE)

\[ Y_s^{t,x,u} = \Phi(X_T^{t,x,u}) + \int_s^T h(r, X_r^{t,x,u}, Y_r^{t,x,u}, Z_r^{t,x,u}, u_r) \, dr - \int_s^T Z_r^{t,x,u} \, dB_r, \tag{1.3} \]

with \(\mu, \sigma, h, f\) and \(\Phi, G\) being deterministic functions, and \((u_r)_{t \leq r \leq T}\) being a given control process.

In the traditional optimal control problem, one considers a particular case of cost functional (1.1), where \(f\) does not depend on states \((Y_s^{t,x,u}, Z_s^{t,x,u})_{t \leq s \leq T}\). It is well-known that there are two key methods, the stochastic maximum principle and the dynamic programming principle, used to study the optimal control problem. For the traditional optimal control problem, under mild conditions, one can prove that the value function \(v\) is the unique viscosity solution of an HJB equation. We refer readers to [22, 23] and the monographs [9] and [31] for the basic theory of maximum principle and viscosity solution of HJB equation for the optimal control problem. Subsequent developments for recursive optimal control are referred to [25, 27, 3, 29, 15, 26, 18].

For the stochastic maximum principle of the value function (1.1) when \(f\) depends on the states \((Y_s^{t,x,u}, Z_s^{t,x,u})\), the first result was established by [24]. Then subsequent results were developed by [5], [17], [28], [32], [13], and [14], among many others. To the best of the authors’ knowledge, there are no work on either dynamic programming principle or PDE method for the value function (1.1). The difficulty comes from that the cost function not only depends on the forward diffusion \((X_s^{t,x,u})\) but also depends on the solution \((Y_s^{t,x,u}, Z_s^{t,x,u})\) of a backward equation. In spirit of [1] and [2], we predict that this control system should be associated to a vector valued PDE which is called the extended HJB equation. When the BSDE (1.3) in the state degenerates to the classical
linear expectation, the control system coincides with the one investigated by [2], who mainly studies the general mean-variance problem. See [7] for further developments. The main contribution of this paper is to establish a dynamic programming principle for the value function (1.1) and to derive the associated extended HJB equation. Both smooth solution and viscosity solution are investigated for the this extended HJB equation. Hence our paper is the first normal work who provides the PDE method for a controlled FBSDE system.

As we know that the dynamic programming principle plays a vital role on deriving the related PDE. Usually, it is difficult to obtain the dynamic programming principle for a non-Markovian state $Y_{t,x,u}(\cdot)$. To solve the problem, we introduce a Lipschitz-continuous control set. When the control $u(\cdot)$ is a function of $(t, x)$, following the results given in [20] and [16], we obtain that $Y_{t,x,u}^t$ is a deterministic function of $(t, x)$. Therefore, we can investigate the related PDE for $Y_{t,x,u}^t$ with a given control $u(t, x)$. Moreover, under some conditions for the coefficients $\mu, \sigma$ in (1.2) and assuming $f$ does not depend on states $(Y_{t,x,u}^t, Z_{t,x,u}^t)$, Theorem 6.4 in Chapter VI of [10] showed that the optimal control of the value function is Lipschitz-continuous in variable $x$. This observation motivates us to introduce a control set $U[t, T]$ which consists of all functions $u(t, x)$ satisfying Lipschitz-continuous in $x$ and right-continuous and left-limit on $t$. In fact, when considering a feedback control of an optimal control problem, one needs an admissible feedback control set which is considered in Chapter VI of [10], including the above Lipschitz control set $U[t, T]$ as an important case.

Thus, in this study, we focus on the Lipschitz-continuous control set $U[t, T]$, and show the existence of an optimal control for the value function (1.1). Applying the method of $S$-topology developed in [11] and [12], combining the properties of the Lipschitz control set $U[t, T]$, we show that there exists an optimal control $u^*(t, x), (t, x) \in [0, T] \times \mathbb{R}^n$. See [4] for more details on the application of $S$-topology to the existence of relaxed optimal control in a particular case where $\sigma$ does not depend on control $u(\cdot)$ and $h, f$ do not depend on the state $(Z_{s,x,u}^t)_{t \leq s \leq T}$. To derive the dynamic programming principle and develop the related extended HJB equation for (1.1) and (1.3), we define an auxiliary value function

$$g^u(t, x) = Y_{t,x,u}^t, \ u(\cdot) \in U[t, T]$$

to represent the term $Y_{t,x,u}^t$ in the cost functional of value function (1.1), and denote
\[ g(t, x) = g^u(t, x) \] where \( u^*(\cdot) \) is an optimal control of value function (1.1). The main contribution of this study is threefold:

(i) Under mild conditions, we obtain the existence of an optimal control to the value function (1.1). Then, the dynamic programming principle for \( v(t, x) \) is established.

(ii) When \( v(t, x), g(t, x) \in C^{1,2}([0, T] \times \mathbb{R}^n) \), we verify that \( v(t, x) \) and \( g(t, x) \) satisfy the following extended HJB equation:

\[
\begin{aligned}
\inf_{u(t,x) \in U_t} \{ Dv(t, x) + f(t, x, g(t, x), \sigma^{*\top} \partial_x g(t, x), u(t, x)) \} &= 0, \\
D^* g(t, x) + h(t, x, g(t, x), \sigma^{*\top} \partial_x g(t, x), u^*(t, x)) &= 0,
\end{aligned}
\]  

(1.4)

where \( U_t = \lim_{s \to t} U[t, s] \), \( v(T, x) = G(x) \), and \( g(T, x) = \Phi(x) \). The infinitesimal operator \( D \) is \( D(\cdot) = [\partial_t + \mu \partial_x + \frac{1}{2} \sigma \sigma^{\top} \partial_x x](\cdot) \), \( D^*(\cdot) = [\partial_t + \mu^* \partial_x + \frac{1}{2} \sigma^* \sigma^{*\top} \partial_x x](\cdot) \), \( \mu^*(t, x) = \mu(t, x, u^*(t, x)) \) and \( \sigma^*(t, x) = \sigma(t, x, u^*(t, x)) \); Furthermore, the classical verification theorem is given.

(iii) We show that \( v(t, x) \) and \( g(t, x) \) are the unique viscosity solution of the extended HJB equation.

Note that the extended HJB equation (1.4) is a vector valued PDE. The theory of viscosity solution for vector valued PDEs is not well-developed due to the lack of a general comparison theorem. If \( f(t, x, y, z, u) \) does not depend on \( z \), then the comparison theorem holds for the equation (1.4). Hence one can define the viscosity solution in the traditional way and further prove the uniqueness. However, when \( f(t, x, y, z, u) \) depends on \( z \), the comparison theorem does not hold in general. See [30] for a detailed description. To solve the problem, we present a new version of viscosity solution by imposing additional first-order smoothness to the state coefficients. In fact, there is little difference between differential functions of first-order and Lipschitz-continuous functions in that Lipschitz-continuity implies differentiability almost everywhere.

The remainder of this paper is organized as follows: In Section 2, we formulate a general optimal control problem, in which the cost functional is driven by a coupled FBSDE. Based on the \( S \)-topology and a Lipschitz control set, we show that the optimal control system admits an optimal control in Section 3. In Section 4, the dynamic programming principle for the optimal control problem is investigated, and a classical verification theorem is established. In Section 5, we introduce a new type of viscosity solution and prove that the value function is the unique viscosity solution of the extended HJB equation. As an application, we consider the utility maximization problem.
in Section 6. Section 7 concludes the paper.

2 General Framework

Let $B$ be an $m$-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_s\}_{s \geq t})$, where $\{\mathcal{F}_s\}_{s \geq t}$ is the $P$-augmentation of the natural filtration generated by the Brownian motion $B$. Let $T > 0$ be given. The controlled state processes $(X_{t,x,u}^s)_{s \in [t,T]}$ and $(Y_{t,x,u}^s, Z_{t,x,u}^s)_{s \in [t,T]}$ are given by

$$
\text{d}X_{t,x,u}^s = \mu(s, X_{t,x,u}^s, u(s, X_{t,x,u}^s)) \text{d}s + \sigma(s, X_{t,x,u}^s, u(s, X_{t,x,u}^s)) \text{d}B_s, \quad X_{t,x,u}^0 = x,
$$

and

$$
\text{d}Y_{t,x,u}^s = -h(s, X_{t,x,u}^s, Y_{t,x,u}^s, Z_{t,x,u}^s, u(s, X_{t,x,u}^s)) \text{d}s + Z_{t,x,u}^s \text{d}B_s, \quad Y_{T,x,u}^T = \Phi(X_{T,x,u}^T),
$$

where the coefficients are functions $\mu : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times m}$, $h : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \times U \to \mathbb{R}$, $\Phi : \mathbb{R}^n \to \mathbb{R}$.

We restrict ourself to the control set $\mathcal{U}[t,T] = \{u(\cdot) : [t,T] \to U, |u(s, x) - u(s, x')| \leq K |x - x'|, \text{ and } u(\cdot, x) \text{ is right-continuous and left-limit function on } [t,T] \text{ for } x, x' \in \mathbb{R}^n \text{ and } s \in [t,T]\}$, where $K$ is a nonnegative constant, and $U$ is a subset of $\mathbb{R}^k$ with a given positive integer $k$.

For given deterministic functions $f$ and $G$, we introduce the following cost functional:

$$
J(t, x; u(\cdot)) = \mathbb{E} \left[ \int_t^T f(s, X_{s,x,u}^t, Y_{s,x,u}^t, Z_{s,x,u}^t, u(s, X_{s,x,u}^t)) \text{d}s + G(X_{T,x,u}^T) \right].
$$

Then, the value function is defined as

$$
v(t, x) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x, u(\cdot)).
$$

For the simplicity of notations, we omit the time variable in $\mu, \sigma, h, f$. We assume that $f, G$ are uniformly continuous and of polynomial growth on independent variables, and $\mu, \sigma, h, \Phi$ satisfy the following Lipschitz and linear growth conditions.

**Assumption 2.1** There exists a constant $c > 0$ such that

$$
|\mu(x_1, u_1) - \mu(x_2, u_2)| + |\sigma(x_1, u_1) - \sigma(x_2, u_2)|
$$

$$
+ |h(x_1, y_1, z_1, u_1) - h(x_2, y_2, z_2, u_2)| + |\Phi(x_1) - \Phi(x_2)|
$$

$$
\leq c (|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| + |u_1 - u_2|),
$$
∀(x₁, y₁, z₁, u₁), (x₂, y₂, z₂, u₂) ∈ ℜⁿ × ℜ × ℜᵐ × U.

**Assumption 2.2** There exists a constant c > 0 such that

\[ |\mu(x, u)| + |\sigma(x, u)| + |h(x, y, z, u)| + |\Phi(x)| \leq c(1 + |x| + |u|), \quad \forall (x, y, z, u) ∈ ℜⁿ × ℜ × ℜᵐ × U. \]

The following results come from [31].

**Theorem 2.1** Let Assumptions 2.1 and 2.2 hold. Then for any given \((t, x) ∈ [0, T) × ℜⁿ\), FBSDE (2.1)–(2.2) admits a unique solution \((X^{t,x,u}_s, Y^{t,x,u}_s, Z^{t,x,u}_s)_{s ∈ [t, T]} \) under norm,

\[ \mathbb{E} \left[ \sup_{t ≤ s ≤ T} |X^{t,x,u}_s|^2 + \sup_{t ≤ s ≤ T} |Y^{t,x,u}_s|^2 + \int_{t}^{T} |Z^{t,x,u}_s|^2 ds \right] < ∞. \]

**Remark 2.1** Peng [24] introduced a nonlinear term \(γ(Y^{t,x,u}_t)\) in the cost functional. Namely,

\[ \tilde{J}(t, x; u(·)) = \mathbb{E} \left[ \int_{t}^{T} f(t, X^{t,x,u}_s, Y^{t,x,u}_s, Z^{t,x,u}_s, u(s)) ds + γ(Y^{t,x,u}_t) + G(X^{t,x,u}_T) \right], \]

and the related maximum principle is established. In fact, the above cost functional \( \tilde{J} \) is equivalent to the one \( J \) defined in (2.3).

Assuming \( γ(y) \) has continuous second-order derivatives on \( ℜ \) and applying Itô’s formula to \( γ(Y^{t,x,u}_t) \), it follows that,

\[ γ(Y^{t,x,u}_t) = \mathbb{E} \left[ \int_{t}^{T} \left( h(s)γ_y(Y^{t,x,u}_s) - \frac{1}{2}Z^{t,x,u}_s(Z^{t,x,u}_s)^\top γ_{yy}(Y^{t,x,u}_s) \right) ds + γ(Φ(X^{t,x,u}_T)) \right], \]

where \( h(s) := h(s, X^{t,x,u}_s, Y^{t,x,u}_s, Z^{t,x,u}_s, u(s)) \). Denote

\[ \tilde{f}(t, x, y, z, u) = f(t, x, y, z, u) + h(t)γ_y(y) - \frac{1}{2}Z^{t,x,u}_s^\top γ_{yy}(y), \quad \tilde{G}(x) = G(x) + γ(Φ(x)). \]

Therefore, the cost functional \( J \) includes \( \tilde{J} \) as a special case while the functions \( f, G \) are replaced by \( \tilde{f}, \tilde{G} \). This observation implies that the optimal control problem in this paper is equivalent to the one studied in [24].

As an example of \( \tilde{J}(t, x; u(·)) \), we take

\[ f = h = 0, \quad Φ(x) = x, \quad G(x) = x^2, \quad γ(y) = -y^2. \]
Then the cost functional is

\[ \tilde{J}(t, x; u(\cdot)) = \mathbb{E}[\langle X_t^{x,u} - \mathbb{E}[X_T^{x,u}] \rangle^2], \]

which is the well-known dynamic mean-variance problem ([1], [2], [7]).

Replacing the functions \( f, G \) by \( \tilde{f}, \tilde{G} \), we obtain

\[ \tilde{J}(t, x; u(\cdot)) = \mathbb{E} \left[ \int_t^T Z_s^{t,x,u}(Z_s^{t,x,u})^\top ds \right], \]

which is within the scope of our model. For notational simplification, we present all results in the rest of the paper under the cost functional \( J \) defined in (2.3).

Since we aim to establish the dynamic programming principle of the value function \( v(t, x) \), we need to consider a weak control framework \( (\Omega, \mathcal{F}, P; \{\mathcal{F}(s)\}_{s \geq t}, B, u) \). For convenience, we still use the notation \( U[t, T] \) to denote \( (\Omega, \mathcal{F}, P; \{\mathcal{F}(s)\}_{s \geq t}, B, u) \).

### 3 Existence of optimal controls

In this section, we investigate the existence of an optimal control for the value function \( v(t, x) \). First, we introduce a linear growth condition for the control set \( U[t, T] \).

**Assumption 3.1** The total variation \( V[u(\cdot, x)] \) is linear growth on \( x \), for all \( x \in \mathbb{R}^n \) and \( u(\cdot, x) \in U[t, T] \). Namely, there exists a constant \( c \) such that

\[ V[u(\cdot, x)] + \sup_{t \leq s \leq T} u(s, 0) < c(1 + |x|), \]

where \( V[u(\cdot, x)] = \sup\{\sum_{i=1}^p |u(t_{i-1}, x) - u(t_i, x)| : t = t_0 < t_1 \cdots < t_p = T\} \).

The main result is given as follows:

**Theorem 3.1** Let Assumptions 2.1, 2.2, and 3.1 hold. Then there exists an optimal control for the value function (2.4).

Before proving Theorem 3.1, we first investigate the tightness of the related processes. The main idea is to introduce the \( S \)-topology developed in [11] for the control set \( U[t, T] \). Let \( \{u^n(\cdot)\}_{n=1}^\infty \) be a sequence such that

\[ \lim_{n \to \infty} J(t, x; u^n(\cdot)) = \inf_{u(\cdot) \in U[t,T]} J(t, x; u(\cdot)). \]
Let $\Theta^n = (X^n, Y^n, Z^n)$ be the solution of the following FBSDE,

\[
\begin{align*}
\begin{cases}
    dX^n_r &= \mu(r, X^n_r, u^n(r, X^n_r))dr + \sigma(r, X^n_r, u^n(r, X^n_r))dB_r, \\
          X^n_0 &= x, \\
    dY^n_r &= -h(r, \Theta^n_r, u^n(r, X^n_r))dr + Z^n_r dB_r, \\
          Y^n_T &= \Phi(X^n_T).
\end{cases}
\end{align*}
\] (3.1)

**Lemma 3.1** Let Assumptions 2.1, 2.2 hold. Then, $\Theta^n$ is the unique solution of FB-SDE (3.1) and there exists a constant $c$ such that

\[
\sup_{n \geq 1} E \left[ \sup_{t \leq s \leq T} |X^n_s|^2 + \sup_{t \leq s \leq T} |Y^n_s|^2 + \int_t^T |Z^n_s|^2 ds \right] < c.
\]

**Proof.** Note that $u^n(\cdot) \in U[t, T]$ satisfies the uniformly Lipschitz condition for any given $x \in \mathbb{R}^n$. Thus, based on Itô’s formula and classical inequalities, we can obtain a uniform bound for the sequence $\Theta^n$. □

Based on a standard argument and Lemma 3.1, we directly have the following results.

**Lemma 3.2** Let Assumptions 2.1, 2.2 hold. Then the sequence $(X^n, Y^n, \int_t^T Z^n_s dB_s)$ is tight on the space $C([t, T], \mathbb{R}^n) \times C([t, T], \mathbb{R}) \times C([t, T], \mathbb{R})$, endowed with the topology of uniform convergence.

**Lemma 3.3** Let Assumptions 2.1, 2.2, 3.1 hold. Then, there exists a constant $c$ such that the control sequence $u^n(\cdot, X^n)$ satisfies

\[
\sup_{n \geq 1} E \left[ \sup_{t \leq s \leq T} |u^n(s, X^n_s)|^2 + CV[u^n(\cdot, X^n(\cdot))] \right] < c,
\]

and $u^n(\cdot, X^n)$ is tight on the right-continuous and left-limit space $D([t, T], \mathbb{R}^n)$, where $CV[u^n(\cdot, X^n(\cdot))]$ is the conditional variation of $u^n(\cdot, X^n(\cdot))$, namely,

\[
CV[u^n(\cdot, X^n(\cdot))] = \sup \left\{ \mathbb{E} \left[ \sum_{i=1}^p \mathbb{E} \left[ u(t_{i-1}, X^n_{t_{i-1}}) - u(t_i, X^n_{t_i}) \mid \mathcal{F}_{t_{i-1}} \right] \right] : t_0 < t_1 \cdots < t_p = T \right\}.
\]

**Proof.** Note that $u^n(\cdot) \in U[t, T]$, we have that

\[
|u^n(s, X^n_s)|^2 \leq 2|u^n(s, 0)|^2 + 2|u^n(s, X^n_s) - u^n(s, 0)|^2 \leq 2|u^n(s, 0)|^2 + 2K|X^n_s|^2,
\]
where $K$ is the Lipschitz constant in control set $\mathcal{U}[t, T]$. Furthermore,

$$
\mathbb{E}[\sup_{t \leq s \leq T} |u^n(s, X^n_s)|^2] \leq 2 \sup_{t \leq s \leq T} |u^n(s, 0)|^2 + 2K \mathbb{E}[\sup_{t \leq s \leq T} |X^n_s|^2].
$$

By Assumption 3.1 and Lemma 3.1, there exists a constant $c_1$ such that

$$
\sup_{n \geq 1} \mathbb{E}[\sup_{t \leq s \leq T} |u^n(s, X^n_s)|^2] \leq c_1. \tag{3.2}
$$

Now, we consider the term $CV[u^n(\cdot, X^n(\cdot))]$. For a given partition $t = t_0 < t_1 \cdots < t_p = T$, it follows that,

$$
\mathbb{E} \left[ \sum_{i=1}^p \mathbb{E}[u(t_{i-1}, X^n_{t_{i-1}}) - u(t_i, X^n_{t_i}) \mid \mathcal{F}_{t_{i-1}}] \right] \\
\leq \sum_{i=1}^p \mathbb{E}[u(t_{i-1}, X^n_{t_{i-1}}) - u(t_i, X^n_{t_i})] \\
\leq \sum_{i=1}^p \mathbb{E} \left[ |u(t_{i-1}, X^n_{t_{i-1}}) - u(t_i, X^n_{t_i})| + |u(t_i, X^n_{t_i}) - u(t_i, X^n_{t_{i-1}})| \right] \\
\leq \mathbb{E} \left[ \sum_{i=1}^p |u(t_{i-1}, X^n_{t_{i-1}}) - u(t_i, X^n_{t_i})| \right] + K \mathbb{E} \left[ \sum_{i=1}^p |X^n_{t_{i-1}} - X^n_{t_i}| \right].
$$

Denote

$$
I_1(p, n) := \mathbb{E} \left[ \sum_{i=1}^p |u(t_{i-1}, X^n_{t_{i-1}}) - u(t_i, X^n_{t_i})| \right],
$$
and

$$
I_2(p, n) := \mathbb{E} \left[ \sum_{i=1}^p |X^n_{t_{i-1}} - X^n_{t_i}| \right].
$$

By Assumption 3.1 and Lemma 3.1, there exists a constant $c_2$ which does not depend on $p$ and $n$ such that $I_1(p, n) \leq c_2$. Since $X^n$ satisfies FBSDE (3.1), we have that

$$
I_2(p, n) \leq \mathbb{E} \left[ \int_t^T |\mu(r, X^n_r, u^n(r, X^n_r))| \, dr + \left( \int_t^T |\sigma(r, X^n_r, u^n(r, X^n_r))|^2 \, dr \right)^{\frac{1}{2}} \right].
$$

Then from Assumption 2.1, 2.2, Lemma 3.1, and combining inequality (3.2), we deduce that there exists a constant $c_3 > 0$ such that $I_2(p, n) \leq c_3$, where $c_3$ does not depend on $p$ and $n$.

Combining the inequalities for $I_1(p, n)$ and $I_2(p, n)$, we obtain that

$$
\sup_{n \geq 1} \mathbb{E} \left[ \sum_{i=1}^p \mathbb{E}[u(t_{i-1}, X^n_{t_{i-1}}) - u(t_i, X^n_{t_i}) \mid \mathcal{F}_{t_{i-1}}] \right] \leq c_2 + c_3,
$$
which implies that $\sup_{n \geq 1} CV[u^n(\cdot, X^n(\cdot)) < c_2 + c_3$. The proof is complete. \hfill \Box

Now, we come back to prove the main result.
Proof of Theorem 3.1. Let \( \{u^n(\cdot)\}_{n=1}^\infty \) be a sequence such that
\[
\lim_{n \to \infty} J(t, x; u^n(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x; u(\cdot)),
\]
and \( \Theta^n = (X^n, Y^n, Z^n) \) be the solution of FBSDE (3.1).

By Lemmas 3.2 and 3.3, we deduce that the sequence of processes
\[
Q^n(\cdot) = (X^n, Y^n, \int_t^T Z^n_s dB_s, u^n(\cdot, X^n))
\]
is tight on the space \( \mathcal{V} = C([t, T], \mathbb{R}^n) \times C([t, T], \mathbb{R}) \times C([t, T], \mathbb{R}^m) \times D([t, T], \mathbb{R}^n) \), equipped with product topology of the uniform convergence on the first, second and third factors and the \( \mathcal{S} \)-topology on the fourth factor. Following the results of [11, 12] or [4], there exists a probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{B}) \) and a sequence \( \tilde{Q}^n(\cdot) = (\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n, u^n(\cdot, \tilde{X}^n)) \)
and \( \hat{Q}(\cdot) = (\hat{X}, \hat{Y}, \hat{Z}, u(\cdot, \hat{X})) \) such that
1. For any given \( n \geq 1 \), \( Q^n(\cdot) \) and \( \hat{Q}^n(\cdot) \) are identically distributed;
2. There exists a subsequence of \( \tilde{Q}^n(\cdot) \) converges to \( \tilde{Q}(\cdot) \), \( \hat{P} - a.s. \), the subsequence
is still denoted as \( \tilde{Q}^n(\cdot) \) if without abuse of notations;
3. Let \( n \to \infty \), then
\[
(\sup_{t \leq s \leq T} |\tilde{X}^n_s - \tilde{X}_s|, \sup_{t \leq s \leq T} |\tilde{Y}^n_s - \tilde{Y}_s|, \int_t^T |\tilde{Z}^n_s - \tilde{Z}_s|^2 ds)
\]
converges to \((0, 0, 0)\), \( \tilde{P} - a.s. \);
4. For any given \( x \in \mathbb{R}^n \), \( u^n(\cdot, x) \) converges to \( u(\cdot, x) \), \( dt - a.s. \), and \( u(\cdot, x) \in \mathcal{U}[t, T] \);
5. \( u^n(\cdot, \tilde{X}^n) \) converges to \( u(\cdot, \hat{X}) \), \( d\delta \times \hat{P} - a.s. \).

Let \( \tilde{Q}^n(\cdot) \) be the solution of the following FBSDE,
\[
\begin{cases}
d\tilde{X}^n_r = \mu(r, \tilde{X}^n_r, u^n(r, \tilde{X}^n_r))dr + \sigma(r, \tilde{X}^n_r, u^n(r, \tilde{X}^n_r))d\tilde{B}_r, & \tilde{X}^n_t = x, \\
\quad d\tilde{Y}^n_r = -h(r, \tilde{\Theta}^n_r, u^n(r, \tilde{X}^n_r))dr + \tilde{Z}^n_r d\tilde{B}_r, & \tilde{Y}^n_T = \Phi(\tilde{X}^n_T),
\end{cases}
\]
where \( \tilde{\Theta}^n = (\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n). \) Letting \( n \to \infty \), from the above 2) and 4), we can show that \( \tilde{Q}(\cdot) \) satisfies the following FBSDEs,
\[
\begin{cases}
d\hat{X}_r = \mu(r, \hat{X}_r, u(r, \hat{X}_r))dr + \sigma(r, \hat{X}_r, u(r, \hat{X}_r))d\hat{B}_r, & \hat{X}_t = x, \\
\quad d\hat{Y}_r = -h(r, \hat{\Theta}_r, u(r, \hat{X}_r))dr + \hat{Z}_r d\hat{B}_r, & \hat{Y}_T = \Phi(\hat{X}_T),
\end{cases}
\]
where \( \hat{\Theta} = (\hat{X}, \hat{Y}, \hat{Z}). \)
Now we show the following limit holds in probability,
\[
\lim_{n \to \infty} \int_t^T \mu(r, \tilde{X}_r^n, u^n(r, \tilde{X}_r^n))dr = \int_t^T \mu(r, \tilde{X}_r, u(r, \tilde{X}_r))dr.
\]
Other terms can be proved with similar arguments.

It follows Chebyshev inequality and Assumption 2.1 that
\[
P\left(\left|\int_t^T \mu(r, \tilde{X}_r^n, u^n(r, \tilde{X}_r^n))dr - \int_t^T \mu(r, \tilde{X}_r, u(r, \tilde{X}_r))dr\right| > \varepsilon\right) \\
\leq \frac{1}{\varepsilon} E \int_t^T |\mu(r, \tilde{X}_r^n, u^n(r, \tilde{X}_r^n))dr - \mu(r, \tilde{X}_r, u(r, \tilde{X}_r))|dr \\
\leq \frac{1}{\varepsilon} E \int_t^T |\tilde{X}_r^n - \tilde{X}_r|dr.
\]

From 3) and the Dominated Convergence Theorem, we have
\[
\lim_{n \to \infty} P\left(\left|\int_t^T \mu(r, \tilde{X}_r^n, u^n(r, \tilde{X}_r^n))dr - \int_t^T \mu(r, \tilde{X}_r, u(r, \tilde{X}_r))dr\right| > \varepsilon\right) = 0.
\]
The proof is complete. \(\square\)

4 Dynamic programming principle and extended HJB equation

In this section, we first establish a dynamic programming principle for \(v\) and \(g\). Then the associated extended HJB equation is derived. A verification theorem is given at last.

4.1 Dynamic programming principle

Denote by \(u^*(\cdot)\) an optimal control of the value function (2.4). To prove the main results, using the same method given in Lemma 3.2 of [31], we introduce the following results.

Lemma 4.1 Let \((t, x) \in [0, T] \times \mathbb{R}^n\). Then, for any given \(s \in [t, T]\) and \((\mathcal{F}_s)_{s \in [t, T]}\)-progressively measurable process \(\xi(\cdot)\),
\[
J(s, \xi_s; u(\cdot)) = E \left[ \int_s^T f(r, \Theta^s_{r, \xi_s, u}, u(r, X^{s, \xi_s, u}_r))dr + G(X^s_T, \xi_s, u) \bigg| \mathcal{F}_s \right], \tag{4.1}
\]
where \(\Theta^s_{r, \xi_s, u} = (X^s_{r, \xi_s, u}, Y^s_{r, \xi_s, u}, Z^s_{r, \xi_s, u}), s \leq r \leq T\).
Now, we investigate the dynamic programming principle for the value function \( v(t, x) \).

**Theorem 4.1** Let Assumptions 2.1 and 2.2 hold. Then, for any given \((t, x) \in [0, T) \times \mathbb{R}^n\), and \(s \in [t, T]\),

\[
v(t, x) = \inf_{u(\cdot) \in U[t, s]} \mathbb{E} \left[ \int_t^s f(r, \Theta_t^{t,x,u}, u(r, X_t^{t,x,u}))dr + v(s, X_s^{t,x,u}) \right],
\]

where \( \Theta_t^{t,x,u} = (X_t^{t,x,u}, Y_t^{t,x,u}, Z_t^{t,x,u}) \), \( t \leq r \leq s \), is the solution of the following FBSDE:

\[
\begin{cases}
\quad dX_t^{t,x,u} = \mu(r, X_t^{t,x,u}, u(r, X_t^{t,x,u}))dr + \sigma(r, X_t^{t,x,u}, u(r, X_t^{t,x,u}))dB_r, \quad X_t^{t,x,u} = x, \\
\quad dY_t^{t,x,u} = -h(r, X_t^{t,x,u}, Y_t^{t,x,u}, Z_t^{t,x,u}, u(r, X_t^{t,x,u}))dr + Z_t^{t,x,u}dB_r, \quad Y_s^{t,x,u} = Y_s^{t,x,u},
\end{cases}
\]

and \( \Theta_t^{t,x,u^*} = (X_t^{t,x,u^*}, Y_t^{t,x,u^*}, Z_t^{t,x,u^*}) \), \( s \leq r \leq T \) is the solution of the following FBSDE:

\[
\begin{cases}
\quad dX_t^{t,x,u^*} = \mu(r, X_t^{t,x,u^*}, u^*(r, X_t^{t,x,u^*}))dr + \sigma(r, X_t^{t,x,u^*}, u^*(r, X_t^{t,x,u^*}))dB_r, \quad X_t^{t,x,u^*} = X_t^{t,x,u}, \\
\quad dY_t^{t,x,u^*} = -h(r, X_t^{t,x,u^*}, Y_t^{t,x,u^*}, Z_t^{t,x,u^*}, u^*(r, X_t^{t,x,u^*}))dr + Z_t^{t,x,u}dB_r, \quad Y_T^{t,x,u^*} = \Phi(X_T^{t,x,u^*}),
\end{cases}
\]

with \( u^*(\cdot) \) being an optimal control of the value function (2.4).

**Proof.** We denote the right side of equation (4.2) by \( \tilde{v}(t, x) \). For any given sufficiently small \( \varepsilon > 0 \), based on the definition of \( v(t, x) \), there exists

\[
u^{s-t}(r, x) = \begin{cases} u_1(r, x), & t \leq r < s \\
u^*(r, x), & s \leq r \leq T,\end{cases}
\]

which belongs to \( U[t, T] \) such that
\[ v(t, x) + \varepsilon > J(t, x; u^{s-t}(\cdot)) \]
\[ = \mathbb{E} \left[ \int_{t}^{T} f(r, \Theta \tilde{t}^{r,x,u^{s-t}}, u^{s-t}(r, X_{T}^{t,x,u^{s-t}}))dr + G(X_{T}^{t,x,u^{s-t}}) \right] \]
\[ = \mathbb{E} \left[ \int_{t}^{T} f(r, \Theta \tilde{t}^{r,x,u^{s-t}}, u^{s-t}(r, X_{T}^{t,x,u^{s-t}}))dr \right. \]
\[ + \mathbb{E} \left[ \int_{t}^{T} f(r, \Theta \tilde{t}^{r,x,u^{s-t}}, u^{s-t}(r, X_{T}^{t,x,u^{s-t}}))dr + G(X_{T}^{t,x,u^{s-t}}) \mid F_{s} \right] \]
\[ = \mathbb{E} \left[ \int_{t}^{T} f(r, \Theta \tilde{t}^{r,x,u^{s-t}}, u_{1}(r, X_{T}^{t,x,u^{s-t}}))dr \right. \]
\[ + \mathbb{E} \left[ \int_{t}^{T} f(r, \Theta \tilde{t}^{r,x,u^{s-t}}, u^{s-t}(r, X_{T}^{t,x,u^{s-t}}))dr + G(X_{T}^{t,x,u^{s-t}}) \mid F_{s} \right] \]
\[ = \mathbb{E} \left[ \int_{t}^{s} f(r, \Theta \tilde{t}^{r,x,u^{s-t}}, u_{1}(r, X_{T}^{t,x,u^{s-t}}))dr + J(s, X_{T}^{t,x,u^{s-t}}; u^{s}(\cdot)) \right] \]
\[ \geq \mathbb{E} \left[ \int_{t}^{s} f(r, \Theta \tilde{t}^{r,x,u^{s-t}}, u_{1}(r, X_{T}^{t,x,u^{s-t}}))dr + v(s, X_{T}^{t,x,u^{s-t}}) \right] \]
\[ \geq \tilde{v}(t, x). \]

The second equality is derived from Lemma 4.1. Conversely, by the definition of the value function \( v(t, x) \), for a given

\[ u^{s-t}(r) = \begin{cases} 
    u_{1}(r, x), & t \leq r < s \\
    u^{s}(r, x), & s \leq r \leq T,
\end{cases} \]

we have

\[ v(s, X_{T}^{t,x,u^{s-t}}) = J(s, X_{T}^{t,x,u^{s-t}}; u^{s}(\cdot)). \]

Note that

\[ v(t, x) \leq J(t, x; u^{s-t}(\cdot)), \]

which implies that

\[ v(t, x) \leq \mathbb{E} \left[ \int_{t}^{s} f(r, \Theta \tilde{t}^{r,x,u^{s-t}}, u_{1}(r, X_{T}^{t,x,u^{s-t}}))dr + J(s, X_{T}^{t,x,u^{s-t}}; u^{s}(\cdot)) \right]. \]

Therefore,

\[ v(t, x) \leq \tilde{v}(t, x). \]

This completes the proof. \( \square \)
4.2 Extended HJB equation

To investigate the related HJB equation for the value function \( v(t, x) \) in (2.4). We introduce another value function for the solution of BSDE (2.2),

\[
g^u(t, x) = Y^t_{t,x,u}, \quad u \in \mathcal{U}[t, T].
\]  

(4.3)

For notational simplicity, we denote \( g^u \) by \( g \). Then the related extended HJB equation is given as follows:

\[
\inf_{u(t,x) \in \mathcal{U}_t} \{ Dv(t, x) + f(t, x, g(t, x), \sigma^* \partial_x g(t, x), u(t, x)) \} = 0,
\]

(4.4)

\[
\mathcal{D}^* g(t, x) + h(t, x, g(t, x), \sigma^* \partial_x g(t, x), u^*(t, x)) = 0,
\]

(4.5)

subject to \( v(T, x) = G(x) \) and \( g(T, x) = \Phi(x) \), where \( \mathcal{U}_t = \lim_{s \to t} \mathcal{U}[t, s], u^*(\cdot) \) is an optimal control of value function (2.4). The infinitesimal operator \( \mathcal{D} \) is \( \mathcal{D}(\cdot) = \{ \partial_t + \mu \partial_x + \frac{1}{2} \sigma \sigma^\top \partial_{xx} \}(\cdot) \), \( \mathcal{D}^*(\cdot) = \{ \partial_t + \mu^* \partial_x + \frac{1}{2} \sigma^* \sigma^{*\top} \partial_{xx} \}(\cdot) \) and \( \mu^*(t, x) = \mu(t, x, u^*(t, x)) \), \( \sigma^*(t, x) = \sigma(t, x, u^*(t, x)) \).

Let \( C^{1,2}([0, T] \times \mathbb{R}^n) \) be the space of functions with continuous first-order derivatives on \( t \in [0, T] \) and continuous second-order derivatives on \( x \in \mathbb{R}^n \). We first establish the following result.

**Theorem 4.2** Suppose Assumptions 2.1, 2.2 hold, and value functions \( v, g \in C^{1,2}([0, T] \times \mathbb{R}^n) \). Then \((v, g)\) is a solution of the extended HJB equation (4.4)–(4.5).

**Proof.** For any given \((t, x) \in [0, T] \times \mathbb{R}^n\) and a control \(u(\cdot) \in \mathcal{U}[t, s]\), define

\[
u_1(r, x) = \begin{cases} 
  u(t, x), & t \leq r < s \\
  u^*(r, x), & s \leq r \leq T,
\end{cases}
\]

where \( u^*(r, x) \) is an optimal control of value function \( v \) on \((s, T]\). Let \( X^{t,x,u_1}(\cdot) \) be the corresponding solution of equation (2.1), and \((Y^{t,x,u_1}(\cdot), Z^{t,x,u_1}(\cdot))\) be the solution of equation (2.2). According to Theorem 4.1, we have

\[
v(t, x) \leq \mathbb{E} \left[ \int_t^s f(r, \Theta^{t,x,u}_r, u(r, X^{t,x,u}_r))dr + v(s, X^{t,x,u}_s) \right],
\]

which implies that

\[
0 \leq \frac{\mathbb{E}[v(s, X^{t,x,u}_s) - v(t, x)]}{s - t} + \frac{1}{s - t} \mathbb{E} \int_t^s f(r, \Theta^{t,x,u}_r, u(t, X^{t,x,u}_r))dr
\]

\[
= \frac{1}{s - t} \mathbb{E} \int_t^s \left[ \mathcal{D}v(r, X^{t,x,u}_r) + f(r, \Theta^{t,x,u}_r, u(t, X^{t,x,u}_r)) \right]dr.
\]
Letting \( s \downarrow t \), we have
\[
0 \leq Dv(t, x) + f(t, x, g(t, x), \sigma^* \partial_x g(t, x), u(t, x)).
\]

Thus, it follows that
\[
0 \leq \inf_{u(t, x) \in U_t} \{ Dv(t, x) + f(t, x, g(t, x), \sigma^* \partial_x g(t, x), u(t, x)) \}. \tag{4.6}
\]

For any given \( \varepsilon > 0 \), there exists a control
\[
u(r, x) = \begin{cases} u^\varepsilon(r, x), & t \leq r < s \\ u^*(r, x), & s \leq r \leq T, \end{cases}
\]
such that
\[
v(t, x) + \varepsilon(s - t) \geq \mathbb{E} \left[ \int_t^s f(r, \Theta_r^{t,x,u^\varepsilon}, u^\varepsilon(r, X_r^{t,x,u^\varepsilon})) dr + v(s, X_s^{t,x,u^\varepsilon}) \right].
\]

Thus, it follows from Itô’s formula that,
\[
\varepsilon \geq \mathbb{E}[v(s, X_s^{t,x,u^\varepsilon}) - v(t, x)] + \frac{1}{s - t} \mathbb{E} \int_t^s f(r, \Theta_r^{t,x,u^\varepsilon}, u^\varepsilon(r, X_r^{t,x,u^\varepsilon})) dr
\]
\[
= \frac{1}{s - t} \mathbb{E} \int_t^s \left[ Dv(r, X_r^{t,x,u^\varepsilon}) + f(r, \Theta_r^{t,x,u^\varepsilon}, u^\varepsilon(r, X_r^{t,x,u^\varepsilon})) \right] dr.
\]

Therefore,
\[
0 \geq \inf_{u(t, x) \in U_t} \{ Dv(t, x) + f(t, x, g(t, x), \sigma^* \partial_x g(t, x), u(t, x)) \}. \tag{4.7}
\]

Then, combining equations (4.6) and (4.7), the value function \( v \) is the solution of equation (4.4).

Based on the nonlinear Feynman-Kac formula (see [20]), the function \( g(t, x) = Y_t^{t,x,u^*} \in C^{1,2}([0, T] \times \mathbb{R}^n) \) is the classical solution of
\[
D^* g(t, x) + h(t, x, g(t, x), \sigma^* \partial_x g(t, x), u^*(t, x)) = 0, \quad g(T, x) = \Phi(x).
\]

The proof is complete. \( \square \)

Now, we show how to construct an optimal control \( u^*(\cdot) \) of value function (2.4).
For a given control \( u_1(\cdot) \in \mathcal{U}[t, T] \), we assume that \( v_1(t, x) \) is defined as
\[
v_1(t, x) = \sup_{u \in \mathcal{U}[t, T]} \mathbb{E} \left[ \int_t^T f \left( s, X_s^{t,x,u}, Y_s^{t,x,u_1}, Z_s^{t,x,u_1}, u(s, X_s^{t,x,u}) \right) ds + G(X_T^{t,x,u}) \right],
\]
where \( X_{t,x,u} \) satisfies
\[
\frac{dX_{t,x,u}}{dt} = \mu(s, X_{t,x,u}, u(s, X_{t,x,u})) + \sigma(s, X_{t,x,u}, u(s, X_{t,x,u}))dB_s, \quad X_{t,x,u} = x,
\]
and \((Y_{t,x,u}^1, Z_{t,x,u}^1)\) satisfies
\[
\begin{cases}
\frac{dY_{t,x,u}^1}{dt} = \mu(s, X_{t,x,u}^1, u_1(s, X_{t,x,u}^1)) + \sigma(s, X_{t,x,u}^1, u_1(s, X_{t,x,u}^1))dB_s, \quad Y_{t,x,u}^1 = x, \\
\frac{dY_{t,x,u}^1}{dt} = -h(s, X_{t,x,u}^1, Y_{t,x,u}^1, Z_{t,x,u}^1, u_1(s, X_{t,x,u}^1))dt + Z_{t,x,u}^1dB_s, \quad Y_{T,x,u}^1 = \Phi(X_{T,x,u}^1).
\end{cases}
\]

Assume \( v_1(t, x), g^{u_1}(t, x) = Y_{t,x,u}^1 \in C^{1,2}([0, T] \times \mathbb{R}^n) \). Applying Theorem 4.2, we get that \((v_1(t, x), g^{u_1}(t, x))\) is the classical solution of the following extended HJB equation:
\[
\begin{align}
\inf_{u(t, x) \in U} \{ &Dv_1(t, x) + f(t, x, g^{u_1}(t, x), \sigma^1 \partial_x g^{u_1}(t, x), u(t, x)) \} = 0, \\
D^1 g^{u_1}(t, x) + &h(t, x, g^{u_1}(t, x), \sigma^1 \partial_x g^{u_1}(t, x), u_1(t, x)) = 0,
\end{align}
\]
where \( D^1(\cdot) = [\partial_t + \mu^1 \partial_x + \frac{1}{2} \sigma^1 \sigma^1 \partial_{xx}] (\cdot) \) and \( \mu^1(t, x) = \mu(t, x, u_1(t, x)), \quad \sigma^1(t, x) = \sigma(t, x, u_1(t, x)) \). From Theorem 3.1, the value function \( v_1(t, x) \) admits an optimal control, denoted by \( u_2(\cdot) \), which minimizes the first equation of (4.8). Thus, we can define a sequence \((v_n(t, x), g^{u_n}(t, x), u_n(t, x))_{n=1}^{\infty}\) which satisfies
\[
v_n(t, x) = \sup_{u \in U[t, T]} \mathbb{E} \left[ \int_t^T f(s, X_{t,x,u}^n, Y_{t,x,u}^n, Z_{t,x,u}^n, u(s, X_{t,x,u}^n)) ds + G(X_{T,x,u}^n) \right].
\]
Furthermore, when \((v_n(t, x), g^{u_n}(t, x)) \in C^{1,2}([0, T] \times \mathbb{R}^n)\), we have that \((v_n(t, x), g^{u_n}(t, x))\) is the classical solution of the following extended HJB equation:
\[
\begin{align}
\inf_{u(t, x) \in U} \{ &Dv_n(t, x) + f(t, x, g^{u_n}(t, x), \sigma^n \partial_x g^{u_n}(t, x), u(t, x)) \} = 0, \\
D^n g^{u_n}(t, x) + &h(t, x, g^{u_n}(t, x), \sigma^n \partial_x g^{u_n}(t, x), u_n(t, x)) = 0,
\end{align}
\]
where \( D^n(\cdot) = [\partial_t + \mu^n \partial_x + \frac{1}{2} \sigma^n \sigma^n \partial_{xx}] (\cdot) \) and \( \mu^n(t, x) = \mu(t, x, u_n(t, x)), \quad \sigma^n(t, x) = \sigma(t, x, u_n(t, x)) \).

Based on Assumptions 2.1, 2.2, and 3.1, using a similar argument as in Section 3, we can show that a subsequence of \((v_n(t, x), g^{u_n}(t, x), u_n(t, x))_{n=1}^{\infty}\) converge in space \( C([0, T], \mathbb{R}^n) \times C([0, T], \mathbb{R}^n) \times D([0, T], \mathbb{R}^n) \). We denote the limits by \( v(t, x), g(t, x), u^*(t, x) \).

Supposing \( v(t, x), g(t, x) \in C^{1,2}([0, T], \mathbb{R}^n)\), from Theorem 4.2, we can show that \( v(t, x), g(t, x) \) are the classical solution of the extended HJB equation (4.4)–(4.5).

We conclude the above results as follows:
Proposition 4.1 Let Assumptions 2.1, 2.2, and 3.1 hold and assume \( \{v_n(t, x), g^{un}(t, x)\}_{n=1}^{\infty} \), \( v(t, x), g(t, x) \in C^{1,2}([0, T], \mathbb{R}^n) \). Then, we can construct an optimal control \( u^*(\cdot) \) for value function (2.4).

4.3 Verification theorem

Now, we establish the well-known verification theorem for value function (2.4).

Theorem 4.3 Let Assumptions 2.1, 2.2 hold, and \( (v, g) \) be the classical solution of equations (4.4) and (4.5). Then, we have

(i) \( v(t, x) \leq J(t, x; u(\cdot)) \) for any given \( u(\cdot) \in \mathcal{U}[t, T] \) and \( (t, x) \in [0, T] \times \mathbb{R}^n \);

(ii) For any \( (t, x) \in [0, T] \times \mathbb{R}^n \), \( v(t, x) = J(t, x; u^*(\cdot)) \) and therefore \( u^*(\cdot) \) is an optimal control of value function (2.4).

Proof. (i) Note that \( v, g \) are the classical solution of equations (4.4) and (4.5). Thus for an optimal control \( u^*(t, x) \in \mathcal{U}_t \), and \( (t, x) \in [0, T] \times \mathbb{R}^n \),

\[
0 = D^*v(t, x) + f(t, x, g(t, x), \sigma^*\partial_x g(t, x), u^*(t, x)),
\]

which deduces that

\[
0 = D^*v(s, X^{t,x,u^*}_s) + f(s, X^{t,x,u^*}_s, g(s, X^{t,x,u^*}_s), \sigma^*\partial_x g(s, X^{t,x,u^*}_s), u^*(s, X^{t,x,u^*}_s)).
\]

Then, applying Itô’s formula to \( v(s, X^{t,x,u^*}_s) \), we have

\[
v(t, x) = \mathbb{E}\left[ -\int_t^T D^*v(r, X^{t,x,u^*}_r)dr + G(X^{t,x,u^*}_T) \right],
\]

and therefore

\[
v(t, x) = \mathbb{E}\left[ \int_t^T f(r, X^{t,x,u^*}_r, g(r, X^{t,x,u^*}_r), \sigma^*\partial_x g(r, X^{t,x,u^*}_r), u^*(r, X^{t,x,u^*}_r)) dr + G(X^{t,x,u^*}_T) \right].
\]

Furthermore, we can verify that

\[
(X^{t,x,u^*}_r, g(r, X^{t,x,u^*}_r), \sigma^*\partial_x g(r, X^{t,x,u^*}_r)) = (X^{t,x,u^*}_r, Y^{t,x,u^*}_r, Z^{t,x,u^*}_r), \quad t \leq r \leq T
\]
is the unique solution of the following FBSDE:

\[
\begin{align*}
dX^{t,x,u^*}_r &= \mu(r, X^{t,x,u^*}_r, u^*(r, X^{t,x,u^*}_r))dr + \sigma(r, X^{t,x,u^*}_r, u^*(r, X^{t,x,u^*}_r))dB_r, \quad X^{t,x,u^*}_t = x, \\
dY^{t,x,u^*}_r &= -h(r, X^{t,x,u}_r, Y^{t,x,u^*}_r, Z^{t,x,u^*}_r, u^*(r, X^{t,x,u^*}_r))dr + Z^{t,x,u^*}_r dB_r, \quad Y^{t,x,u^*}_T = \Phi(X^{t,x,u}_T).
\end{align*}
\]
Thus, equation (4.9) can be rewritten as

\[
v(t, x) = E \left[ \int_t^T f \left( r, X_r^{t,x,u^*}, Y_r^{t,x,u^*}, Z_r^{t,x,u^*}, u^*(r, X_r^{t,x,u^*}) \right) dr + G(X_T^{t,x,u^*}) \right].
\]

Note that \(u^*(\cdot)\) is an optimal control of value function (2.4). We obtain that,

\[
v(t, x) \leq J(t, x; u(\cdot)),
\]

for any given control \(u(\cdot) \in U[t, T]\).

(ii) From the proof given in (i), we have that for any \((t, x) \in [0, T] \times \mathbb{R}^n\) and \(u(\cdot) \in U[t, T]\), \(v(t, x) \leq J(t, x; u(\cdot))\). Thus, if \(u^*(\cdot) \in U[t, T]\) such that \(v(t, x) = J(t, x; u^*(\cdot))\), then \(u^*(\cdot)\) is an optimal control for value function (2.4). □

5 Viscosity solution

In this section, we discuss the viscosity solution of the extended HJB equation (4.4)–(4.5) in two situations: \(f\) is independent of \(Z^{t,x,u}\), and \(f\) depends on \(Z^{t,x,u}\). In the first case, based on the method given in [21] and following the classical definition of viscosity solution, we show that the value function \((v, g)\) is the unique viscosity solution of (4.4) and (4.5). While \(f\) depends on \(Z^{t,x,u}\), to the best of our knowledge, there is no result about the viscosity solution of HJB equations (4.4) and (4.5). One of the reasons is that the comparison theorem for the vector valued value function \((v, g)\) does not hold in general. To overcome the difficulty, we add a first-order smoothness condition on the coefficients of value function \(g\). See subsection 5.2 for more details. As we know that, compared with the Lipschitz continuous condition, the first-order smoothness condition is trivial since the uniformly Lipschitz continuity implies absolute continuity, and absolute continuity implies almost everywhere differentiability.

5.1 When \(f\) does not depend on \(Z^{t,x,u}\)

Insipired by [21], we first give a definition of a viscosity solution of the vector valued PDE (4.4) and (4.5) according to the classical type:

\begin{definition}
Let \((w_1, w_2) \in C([0, T] \times \mathbb{R}^n)\). \((w_1, w_2)\) is a viscosity sub-solution of (4.4) and (4.5), if \(\forall (\Gamma_1, \Gamma_2) \in C^{1,2}([0, T] \times \mathbb{R}^n)\), for any \((t, x) \in [0, T] \times \mathbb{R}^n\) such that
\end{definition}
\[ \Gamma_i \geq w_i \text{ and } \Gamma_i(t, x) = w_i(t, x), \ i = 1, 2, \text{ we have} \]

\[ \inf_{u(t, x) \in \mathcal{U}} \{D\Gamma_1(t, x) + f(t, x, \Gamma_2(t, x), u(t, x))\} \geq 0, \quad (5.1) \]

and

\[ D^*\Gamma_2(t, x) + h(t, x, \Gamma_2(t, x), \sigma^* D_x \Gamma_2(t, x), u^*(t, x)) \geq 0. \quad (5.2) \]

\((w_1, w_2)\) is a viscosity super-solution of \((4.4)\) and \((4.5)\), if \(\forall (\Gamma_1, \Gamma_2) \in C^{1,2}([0, T] \times \mathbb{R}^n)\), for any \((t, x) \in [0, T] \times \mathbb{R}^n\) such that \(\Gamma_i \leq w_i\) and \(\Gamma_i(t, x) = w_i(t, x), \ i = 1, 2, \) we have

\[ \inf_{u(t, x) \in \mathcal{U}} \{D\Gamma_1(t, x) + f(t, x, \Gamma_2(t, x), u(t, x))\} \leq 0, \quad (5.3) \]

and

\[ D^*\Gamma_2(t, x) + h(t, x, \Gamma_2(t, x), \sigma^* D_x \Gamma_2(t, x), u^*(t, x)) \leq 0, \quad (5.4) \]

\((w_1, w_2)\) is a viscosity solution of \((4.4)\) and \((4.5)\), if it is both a viscosity super-solution and a viscosity sub-solution of \((4.4)\) and \((4.5)\) over \([0, T] \times \mathbb{R}^n\).

Before checking the viscosity solution, we first show that the value function \(v\) has the following continuous properties. Here, the function \(f\) can depend on \(Z^{t-x,u}\).

**Lemma 5.1** Let Assumptions 2.1 and 2.2 hold. The value functions \(v\) and \(g\) defined in (2.4) and (4.3) are continuous and of linear growth. Namely, there exists a constant \(c > 0\) such that, for every \((t, x) \in [0, T] \times \mathbb{R}^n\), we have

\[ |v(t, x)| + |g(t, x)| \leq c(1 + |x|), \]

and \(\forall \varepsilon > 0\), there exists a constant \(\delta > 0\), such that \(\sqrt{|t-s| + |x-y|} < \delta\), we have

\[ |v(t, x) - v(s, y)| + |g(t, x) - g(s, y)| < \varepsilon. \]

**Proof.** Based on Assumption 2.2, we obtain that \(v\) is of linear growth. Next, we prove the continuity of \(v\) at \((t, x)\): For a given \(u(\cdot) \in \mathcal{U}[t, T]\) and the initial data \((t, x) \in [0, T] \times \mathbb{R}^n\) and \((s, y) \in [0, T] \times \mathbb{R}^n\) with \(t \leq s \leq T\), we have

\[
J(t, x; u(\cdot)) - J(s, y; u(\cdot)) = \mathbb{E} \left[ \int_{t}^{T} f(r, \Theta_r^{t,x,u}, u(r, X_r^{t,x,u}))dr - \int_{s}^{T} f(r, \Theta_r^{t,x,u}, u(r, X_r^{s,y,u}))dr + G(X_T^{t,x,u}) - G(X_T^{s,y,u}) \right] \\
= \mathbb{E} \left[ \int_{t}^{s} f(r, \Theta_r^{t,x,u}, u(r, X_r^{t,x,u}))dr + \int_{s}^{T} \left[ f(r, \Theta_r^{t,x,u}, u(r, X_r^{t,x,u})) - f(r, \Theta_r^{t,x,u}, u(r, X_r^{s,y,u})) \right]dr \right] \\
+ \mathbb{E} \left[ G(X_T^{t,x,u}) - G(X_T^{s,y,u}) \right].
\]
Denote
\[ I_1 = E \left[ \int_T^s f(r, \Theta_r^{t,x,u}, u(r, X_r^{t,x,u}))dr \right. \]
\[ + \left. \int_s^T \left[ f(r, \Theta_r^{t,x,u}, u(r, X_r^{t,x,u})) - f(r, \Theta_r^{s,y,u}, u(r, X_r^{s,y,u})) \right] dr \right], \]
\[ I_2 = E \left[ G(X_T^{t,x,u}) - G(X_T^{s,y,u}) \right]. \]

For any given \( \varepsilon > 0 \), by Assumption 2.1, there exists \( \delta > 0 \) and \( \sqrt{|t - s| + |x - y|} < \delta \) such that \(|I_1| + |I_2| < \frac{\varepsilon}{2}\). Therefore, we have
\[ |J(t, x; u(\cdot)) - J(s, y; u(\cdot))| \leq \frac{\varepsilon}{2}, \]
which further implies that \(|v(t, x) - v(s, y)| \leq \frac{\varepsilon}{2}\). Similarly, we can prove that \(|g(t, x) - g(s, y)| \leq \frac{\varepsilon}{2}\).

\[ \square \]

**Theorem 5.1** Let Assumptions 2.1, 2.2 hold. Then the value function \((v, g)\) is a viscosity solution of the extended HJB equation (4.4)–(4.5).

**Proof:** For a given optimal control \(u^*(\cdot)\) of value function (2.4), following the results of [25], we deduce that \(g(t, x)\) is a viscosity solution of equation (4.5). Then, based on the results of [21], we have that \(v(t, x)\) is the viscosity solution of (4.4). \[ \square \]

**Theorem 5.2** Let Assumptions 2.1, 2.2. The value function \((v, g)\) is the unique viscosity solution of the extended HJB equation (4.4)–(4.5). Namely,

(i) For a given optimal control \(u^*(\cdot)\) of value function (2.4), the value function \(g(t, x)\) of (4.3) is the unique viscosity solution of (4.5);

(ii) The value function \(v(t, x)\) of (2.4) is the unique viscosity solution of (4.4).

**Proof:** For a given optimal control \(u^*(\cdot)\) of value function (2.4), the uniqueness of viscosity solution of (4.5), subject to \(g(T, x) = \Phi (x)\), is obtained directly by the results of [6].

Now we consider equation (4.4) for given \(g(t, x) \in C([0, T] \times \mathbb{R}^n)\). Based on the results of [6], we can obtain the uniqueness of the viscosity solution of (4.4). Then, from Theorem 5.1, the value function \(v(t, x)\) of (2.4) is the unique viscosity solution of (4.4).
For another $u^*(\cdot)$ and the related $g(t, x)$, repeat the above procedure, we can still prove that the value function $v(t, x)$ of (2.4) is the unique viscosity solution of (4.4). Hence the uniqueness of viscosity solution to (2.4) does not depend on either the choice of an optimal control or the related $g(t, x)$. □

**Remark 5.1** Theorem 5.2 tells us that the value function (2.4) is the unique viscosity solution of (4.4), while there may be several functions $g(t, x)$ for (4.5) depending given optimal controls $u^*(\cdot)$. In fact, equation (4.5) is an auxiliary equation. What we are concerned about is the the first equation (4.4).

### 5.2 When $f$ depends on $Z^{t,x,u}$

We first give an example to show that Definition 5.1 is not suitable for equations (4.4) and (4.5) when $f$ depends on $Z^{t,x,u}$. Thus we need to redefine the viscosity solution.

**Example 5.1** We consider a simple two dimensional ordinal differential equations which are similar with equations (4.4) and (4.5):

\[
\begin{align*}
2 - \frac{du(x)}{dx} - \frac{dv(x)}{dx} &= 0, \\
1 - \frac{dv(x)}{dx} &= 0, \quad 0 \leq x \leq 2, \\
u(0) = u(2) = v(0) = v(2) = 0.
\end{align*}
\]

(5.5)

Applying Definition 5.1, we can verify that \(\tilde{v}(x)\) is the unique viscosity solution of the second equation in (5.5):

\[
1 - \left| \frac{dv(x)}{dx} \right| = 0, \quad v(0) = v(2) = 0.
\]

Now, we consider the first equation in (5.5),

\[
2 - \left| \frac{du(x)}{dx} \right| - \frac{dv(x)}{dx} = 0, \quad 0 \leq x \leq 2, \quad u(0) = u(2) = 0.
\]

(5.6)

when $x \neq 1$, we can show that \(\tilde{u}(x)\) is given by:

\[
\tilde{u}(x) = \begin{cases} 
  x + 2, & \text{if } 0 \leq x < 1, \\
  6 - 3x, & \text{if } 1 < x \leq 2,
\end{cases}
\]
is the viscosity solution of equation (5.6). Furthermore, let $\tilde{u}(1) = 3$. Thus $\tilde{u}(\cdot)$ is continuous on interval $[0, 2]$.

Applying Definition 5.1, we consider the point $x = 1$, and functions $\phi_1(\cdot), \phi_2(\cdot) \in C^1(\mathbb{R})$, which satisfy

$$\phi_1(x) \geq \tilde{u}(x), \quad \phi_1(1) = \tilde{u}(1), \quad \phi_2(x) \geq \tilde{v}(x), \quad \phi_2(1) = \tilde{v}(1).$$

It follows that

$$\frac{d\phi_1(1)}{dx} \in [-3, 1], \quad \frac{d\phi_2(1)}{dx} \in [-1, 1].$$

Thus, when $\frac{d\phi_1(1)}{dx} = -3$, $\frac{d\phi_2(1)}{dx} = 1$,

$$2 - \left| \frac{d\phi_1(1)}{dx} \right| - \frac{d\phi_2(1)}{dx} = -2 < 0;$$

When $\frac{d\phi_1(1)}{dx} = 0$, $\frac{d\phi_2(1)}{dx} = 0$,

$$2 - \left| \frac{d\phi_1(1)}{dx} \right| - \frac{d\phi_2(1)}{dx} = 2 > 0.$$

Hence, $\tilde{u}(\cdot)$ is not a viscosity solution of equation (5.6).

In fact, we cannot find a viscosity solution for equation (5.6). Thus, Definition 5.1 is inapposite to equations (4.4) and (4.5). The problem is that the term $\sigma^T \partial_x \Gamma_2(t, x)$ in Definition 5.1 may cause the ill-posedness of the viscosity solution of equation (4.4).

To overcome the problem in the definition of viscosity solution in Definition 5.1, we introduce the following first-order smoothness condition for the coefficients of FB-SDE (2.1) and (2.2), which is used to obtain the continuity of solution $Z_t^{t,x,u^*} = \sigma^T(t, x) \partial_x g(t, x)$ at $(t, x) \in [0, T] \times \mathbb{R}^n$. The following assumption and lemma come from [16].

**Assumption 5.1** For any given optimal control $u^*(\cdot)$ of value function (2.4), $\mu^* \in C_b^{0,1}([0, T] \times \mathbb{R}^n)$, $\sigma^* \in C_b^{0,1}([0, T] \times \mathbb{R}^n)$, $h^* \in C_b^{0,1}([0, T] \times \mathbb{R}^n)$, $\Phi(x) \in C_b^1(\mathbb{R})$, i.e., the partial derivatives of $\mu^*, \sigma^*, h^*, \Phi$ to $x \in \mathbb{R}^n$ are continuous and uniformly bounded, where

(5.7) $$
\mu^*(t, x) = \mu(t, x, u^*(t, x)), \quad \sigma^*(t, x) = \sigma(t, x, u^*(t, x)), \quad h^*(t, x, y, z) = h(t, x, y, z, u^*(t, x)), \quad (t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m.
$$
Remark 5.2 Note that uniformly Lipschitz continuity implies absolute continuity, and absolute continuity implies almost everywhere differentiability. Thus, comparing with Lipschitz continuity, Assumption 5.1 is not a too strong condition.

Lemma 5.2 Suppose Assumption 5.1 holds. Then, we have that

(i) \( Z_{s}^{t,x,u^*} = \sigma^* \partial_x Y_{s}^{t,x,u^*} \), where \( (Y_{s}^{t,x,u^*}, Z_{s}^{t,x,u^*})_{t \leq s \leq T} \) is the solution of BSDE (2.2) with optimal control \( u^*(\cdot) \);

(ii) \( (Y_{t}^{t,x,u}, Z_{t}^{t,x,u}) = (g(t,x), \sigma^* \partial_x g(t,x)) \) is continuous on \([0,T] \times \mathbb{R}^n\).

Based on Lemma 5.2, we introduce a new definition for the viscosity solution of equations (4.4) and (4.5):

Definition 5.2 Suppose \((w_1, w_2) \in C([0,T] \times \mathbb{R}^n) \times C^{0,1}([0,T] \times \mathbb{R}^n)\). \((w_1, w_2)\) is a viscosity sub-solution of (4.4) and (4.5), if \( \forall (\Gamma_1, \Gamma_2) \in C^{1,2}([0,T] \times \mathbb{R}^n) \), for any \((t, x) \in [0,T] \times \mathbb{R}^n\) such that \( \Gamma_i \geq w_i \) and \( \Gamma_i(t, x) = w_i(t, x), \ i = 1, 2\), we have

\[
\inf_{u(t,x) \in U_t} \{ D \Gamma_1(t, x) + f(t, x, \Gamma_2(t, x), \sigma^* \partial_x \Gamma_2(t, x), u(t, x)) \} \geq 0, \tag{5.8}
\]

and

\[
D^* \Gamma_2(t, x) + h(t, x, \Gamma_2(t, x), \sigma^* \partial_x \Gamma_2(t, x), u^*(t, x)) \geq 0. \tag{5.9}
\]

\((w_1, w_2)\) is a viscosity super-solution of (4.4) and (4.5), if \( \forall (\Gamma_1, \Gamma_2) \in C^{1,2}([0,T] \times \mathbb{R}^n) \), for any \((t, x) \in [0,T] \times \mathbb{R}^n\) such that \( \Gamma_i \leq w_i \) and \( \Gamma_i(t, x) = w_i(t, x), \ i = 1, 2\), we have

\[
\inf_{u(t,x) \in U_t} \{ D \Gamma_1(t, x) + f(t, x, \Gamma_2(t, x), \sigma^* \partial_x \Gamma_2(t, x), u(t, x)) \} \leq 0, \tag{5.10}
\]

and

\[
D^* \Gamma_2(t, x) + h(t, x, \Gamma_2(t, x), \sigma^* \partial_x \Gamma_2(t, x), u^*(t, x)) \leq 0. \tag{5.11}
\]

\((w_1, w_2)\) is a viscosity solution of (4.4) and (4.5), if it is both a viscosity super-solution and a viscosity sub-solution of (4.4) and (4.5) over \([0,T] \times \mathbb{R}^n\).

Remark 5.3 Obviously, the Definition 5.2 of viscosity solution is more strong than that given in Definition 5.1, since we require the viscosity solution of the auxiliary equation to be differential in \(x\). However, we did not add any additional condition to \(w_1\). Hence Definition 5.2 could degenerate to the classical case when \(f \equiv 0\).

Since \(w_2 \in C^{0,1}([0,T] \times \mathbb{R}^n)\), \(\Gamma_2 \geq w_2\) and \(\Gamma_2(t, x) = w_2(t, x)\), we get that \(\partial_x \Gamma_2(t, x) = \partial_x w_2(t, x)\), which is an important observation in the proof of Theorem 5.3. This result is also true for the viscosity super-solution in Definition 5.2.
In the following, we present the main result of this section.

**Theorem 5.3** Let Assumptions 2.1, 2.2, and 5.1 hold. Then the value function \((v, g)\) is a viscosity solution of the extended HJB equation (4.4)–(4.5).

**Proof:** Let \(\Gamma_1, \Gamma_2 \in C^{1,2}([0,T] \times \mathbb{R}^n)\). For a given \((t, x) \in [0,T] \times \mathbb{R}^n\) such that \(\Gamma_1 \geq v, \Gamma_2 \geq g\) and \(\Gamma_1(t, x) = v(t, x), \Gamma_2(t, x) = g(t, x)\), from Remark 5.3, we have \(\partial_x \Gamma_2(t, x) = \partial_x g(t, x)\). Now, we prove the viscosity sub-solution inequalities (5.8) and (5.9).

**Step 1:** First we prove that \(\Gamma_2(t, x)\) satisfies inequality (5.9). Applying Itô’s formula to \(\Gamma_2(t, X_t^r,x,u^*)\), for \(t < s < T\), we have that
\[
\mathbb{E}[\Gamma_2(s, X_t^r,x,u^*)] - \Gamma_2(t, x) = \mathbb{E}[\Gamma_2(s, X_t^r,x,u^*)] - g(t, x) = \mathbb{E} \int_t^s D^* \Gamma_2(r, X_t^r,x,u^*) dr. \tag{5.12}
\]

From Lemma 5.2, it follows that
\[
\mathbb{E}[g(s, X_t^r,x,u^*)] - g(t, x) = \mathbb{E} \left[ \int_t^s -h(r, X_t^r,x,u^*, Y_t^r,x,u^*, Z_t^r,x,u^*, u^*(s, X_t^r,x,u^*)) dr \right].
\]
and \(Z_t^r,x,u^* = \sigma^*\top g(s, X_t^r,x,u^*)\). Combining equation (5.12), we obtain
\[
\mathbb{E} \int_t^s D^* \Gamma_2(r, X_t^r,x,u^*) dr \geq \mathbb{E} \left[ \int_t^s -h(r, X_t^r,x,u^*, Y_t^r,x,u^*, Z_t^r,x,u^*, u^*(r, X_t^r,x,u^*)) dr \right].
\]
That is,
\[
\mathbb{E} \int_t^s \left[ D^* \Gamma_2(r, X_t^r,x,u^*) + h(r, X_t^r,x,u^*, Y_t^r,x,u^*, Z_t^r,x,u^*, u^*(r, X_t^r,x,u^*)) dr \right] \geq 0.
\]
Letting \(s \to t\), it follows that
\[
D^* \Gamma_2(t, x) + h(t, x, \Gamma_2(t, x), \sigma^*\top \partial_x g(t, x), u^*(t, x)) \geq 0.
\]
Since \(\partial_x \Gamma_2(t, x) = \partial_x g(t, x)\), we have
\[
D^* \Gamma_2(t, x) + h(t, x, \Gamma_2(t, x), \sigma^*\top \partial_x \Gamma_2(t, x), u^*(t, x)) \geq 0.
\]

**Step 2:** We now prove that \(v\) is the sub-viscosity solution of (4.4). Define
\[
u_1(r, x) = \begin{cases} u(t, x), & t \leq r < s \\ u^*(r, x), & s \leq r \leq T, \end{cases}
\]
where \( u^*(\cdot) \) is an optimal control of value function \( v \) on \((s,T)\). By Itô’s formula, for \( t < s < T \), we have that

\[
\mathbb{E}[\Gamma_1(s, X_{s}^{t,x,u})] - v(t, x) = \mathbb{E}[\Gamma_1(s, X_{s}^{t,x,u})] - \Gamma_1(t, x) = \mathbb{E} \int_{t}^{s} \mathcal{D}\Gamma_1(r, X_{r}^{t,x,u})dr.
\]

(5.13)

Based on the dynamic programming principle established in Theorem 4.1, we have

\[
v(t, x) = \inf_{u(\cdot) \in \mathcal{U}[t,s]} \mathbb{E}\left[ \int_{t}^{s} f(\Theta_{r}^{t,x,u}, u(t, X_{r}^{t,x,u}))dr + v(s, X_{s}^{t,x,u}) \right],
\]

which implies that

\[
v(t, x) \leq \mathbb{E}\left[ \int_{t}^{s} f(\Theta_{r}^{t,x,u}, u(t, X_{r}^{t,x,u}))dr + v(s, X_{s}^{t,x,u}) \right].
\]

By equation (5.13) and observing that for \((t, x)\) \(\in [0,T] \times \mathbb{R}^n\), \( \Gamma_1 \geq v \) and \( \Gamma_1(t, x) = v(t, x) \), we deduce that

\[
0 \leq \mathbb{E}\left[ \int_{t}^{s} f(\Theta_{r}^{t,x,u}, u(t, X_{r}^{t,x,u}))dr + v(s, X_{s}^{t,x,u}) \right] - v(t, x)
\]

\[
\leq \mathbb{E}\left[ \int_{t}^{s} f(\Theta_{r}^{t,x,u}, u(t, X_{r}^{t,x,u}))dr + \Gamma_1(s, X_{s}^{t,x,u}) \right] - \Gamma_1(t, x)
\]

\[
\leq \mathbb{E}\int_{t}^{s} \left[ \mathcal{D}\Gamma_1(r, X_{r}^{t,x,u}) + f(\Theta_{r}^{t,x,u}, u(t, X_{r}^{t,x,u})) \right]dr.
\]

Dividing on both sides of the above inequality by \( s - t \), we get that

\[
0 \leq \frac{1}{s - t} \mathbb{E}\int_{t}^{s} \left[ \mathcal{D}\Gamma_1(r, X_{r}^{t,x,u}) + f(\Theta_{r}^{t,x,u}, u(t, X_{r}^{t,x,u})) \right]dr.
\]

Now, letting \( s \downarrow t \), it follows that

\[
\mathcal{D}\Gamma_1(t, x) + f\left(t, x, g(t, x), Z_{t}^{t,x,u^*}, u(t, x)\right) \geq 0.
\]

Since, by Lemma 5.2, \( Z_{t}^{t,x,u^*} = \sigma^{*\top} \partial_x g(t, x) = \sigma^{*\top} \partial_x \Gamma_2(t, x) \), thus we have

\[
\inf_{u(t,x) \in \mathcal{U}_t} \{ \mathcal{D}\Gamma_1(t, x) + f\left(t, x, \Gamma_2(t, x), \sigma^{*\top} \partial_x \Gamma_2(t, x), u(t, x)\right) \} \geq 0.
\]

**Step 3:** In the following, we prove the viscosity super-solution. Let \( \Gamma_1, \Gamma_2 \in C^{1,2}([0,T] \times \mathbb{R}^n) \). For given \((t, x) \in [0,T] \times \mathbb{R}^n\) such that \( \Gamma_1 \leq v \), \( \Gamma_2 \leq g \) and \( \Gamma_1(t, x) = v(t, x) \), \( \Gamma_2(t, x) = g(t, x) \), from Remark 5.3, we have \( \partial_x \Gamma_2(t, x) = \partial_x g(t, x) \).
The proof of $\Gamma_2$ satisfying inequality (5.11) is similar with that in Step 1. Thus, we omit it.

We now use the method of “proof by contradiction” to verify that $\Gamma_1$ satisfies inequality (5.10). Let us assume that, there exist $(t, x) \in [0, T] \times \mathbb{R}^n$, with $\Gamma_1 \leq v$ on $[0, T] \times \mathbb{R}^n$ and $\Gamma_1(t, x) = v(t, x)$, and $\delta_1 > 0$ such that for any control $u(\cdot) \in \mathcal{U}[t, T]$,

$$D\Gamma_1(t, x) + f(t, x, \Gamma_2(t, x), \sigma^* \partial_x \Gamma_2(t, x), u(t, x)) \geq 2\delta_1.$$  

(5.12)

Let $u(\cdot) \in \mathcal{U}[t, T]$, and there exists $t < s$ such that $\mathbb{E}[(X^{t,x,u}_s - x)] \leq \rho(\delta_1)$, where $\rho(\cdot)$ is a continuous function on $\mathbb{R}$ and is independent of $u$ with $\rho(0) = 0$. Thus, for any $r \in [t, s]$, we have

$$\mathbb{E} \left[ D\Gamma_1(r, X^{t,x,u}_r) + f(r, X^{t,x,u}_r, g(r, X^{t,x,u}_r), \sigma^* \partial_x \Gamma_2(r, X^{t,x,u}_r), u(r, X^{t,x,u}_r)) \right] \geq \delta_1.$$  

(5.14)

On the other hand, since $\Gamma_1 \leq v$ on $[0, T] \times \mathbb{R}^n$, by the definition of $J$ and $v$, we have that

$$J(t, x; u_1(\cdot)) = \mathbb{E} \left[ \int_t^s f(r, \Theta^{t,x,u}_r, u(r, X^{t,x,u}_r)) \, dr + v(s, X^{t,x,u}_s) \right]$$

$$\geq \mathbb{E} \left[ \int_t^s f(r, \Theta^{t,x,u}_r, u(r, X^{t,x,u}_r)) \, dr + \Gamma_1(s, X^{t,x,u}_s) \right],$$

where

$$u_1(r, x) = \begin{cases} u(t, x), & t \leq r < s \\ u^*(r, x), & s \leq r \leq T. \end{cases}$$

It follows equation (5.14) that

$$J(t, x; u_1(\cdot)) \geq \mathbb{E} \int_t^s \left[ \delta_1 - D\Gamma_1(r, X^{t,x,u}_r) \right] \, ds + \mathbb{E} \left[ \Gamma_1(s, X^{t,x,u}_s) \right].$$

(5.15)

In addition, similar to (5.13), it follows that

$$\mathbb{E}[\Gamma_1(s, X^{t,x,u}_s)] - v(t, x) = \mathbb{E}[\Gamma_1(s, X^{t,x,u}_s)] - \Gamma_1(t, x) = \mathbb{E} \int_t^s D\Gamma_1(r, X^{t,x,u}_r) \, dr.$$  

(5.16)

Therefore, from equations (5.15) and (5.16), we have

$$J(t, x; u_1(\cdot)) \geq (s - t)\delta_1 + \Gamma_1(t, x) = (s - t)\delta_1 + v(t, x).$$
Taking the minimum over $u(t, x) \in \mathcal{U}_t$, we have that

$$v(t, x) \geq (s - t)\delta_1 + v(t, x).$$

This contradicts to the fact that $\delta_1 > 0$. Therefore, $v(t, x)$ is a viscosity super-solution of equation (4.4). □

**Theorem 5.4** Let Assumptions 2.1, 2.2 and 5.1 hold. Then the value function $(v, g)$ is the unique viscosity solution of the HJB equations (4.4) and (4.5). Namely,

(i) For a given optimal control $u^*(\cdot)$ of value function (2.4), the function (4.3) is the unique viscosity solution of (4.5);

(ii) The value function (2.4) is the unique viscosity solution of (4.4).

**Proof:** For a given optimal control $u^*(\cdot)$ of value function (2.4), we first consider equation (4.5) with $g(T, x) = \Phi(x)$. Note that the new Definition 5.2 of viscosity solution is more strong than Definition 5.1. Following the classical Definition 5.1 of viscosity solution, we can obtain the uniqueness of viscosity solution of (4.5) based on the results of [6], which implies the uniqueness of viscosity solution in Definition 5.2.

Now, from Lemma 5.2 and Theorem 5.3, the value function $g(t, x) \in C^{0,1}([0, T] \times \mathbb{R}^n)$ of (4.3) is the unique viscosity solution of (4.5) under a given optimal control $u^*(\cdot)$. We now consider equation (4.4) for any given $(t, x) \in [0, T] \times \mathbb{R}^n$. Note that $g(t, x) \in C^{0,1}([0, T] \times \mathbb{R}^n)$. Hence the second item in equation (4.4) are continuous in $(t, x)$. Therefore, the uniqueness of the viscosity solution in Definition 5.2 is obtained directly by [6]. Then, from Theorem 5.3, the value function $v(t, x)$ of (2.4) is the unique viscosity solution of (4.4).

For another optimal control $u^*(\cdot)$ and the related $g(t, x)$, repeat the above procedure, we can still prove that the value function $v(t, x)$ of (2.4) is the unique viscosity solution of (4.4). Hence the uniqueness of viscosity solution to (2.4) does not depend on either the choice of an optimal control or the related $g(t, x)$. □

6 Utility maximization of expected terminal wealth

We now consider a maximization problem in which an investor concerns about the utility of the expected terminal wealth. We start with a market in which two assets are available for investment: a riskless asset (bond) with constant interest rate $r > 0$, ...
and a risky asset (stock). The stock price evolves according to the following geometric Brownian motion:

\[ dS_t = \mu S_t dt + \sigma S_t dB_t, \quad (6.1) \]

where \( B_t \) is a standard 1-dimensional Brownian motion, the constants \( \mu, \sigma \) represent the expected return rate and the volatility respectively.

A self-financing wealth process \( (W_t) \) can be described by

\[ dW_t = [rW_t + (\mu - r) \pi(s, W_t)] dt + \sigma \pi(s, W_t) dB_t, \quad (6.2) \]

where \( W_t = x \) and \( \pi(\cdot) \) is the dollar amount invested in the stock, standing for a trading strategy which is the function of \( (s, W_t) \).

Let \( T > 0 \) be the investment horizon. We adopt an exponential utility as the reward function:

\[ J(t, x; \pi(\cdot)) = \mathbb{E} \left[ \int_t^T f(Y^t, x, \pi) ds + U(W_T) \right], \quad (6.3) \]

where the functions \( f = U := -\frac{1}{2} e^{-\gamma x}, \gamma > 0 \) is the risk aversion of the investor; \( (Y^t, x, \pi) \) is the solution of the following BSDE,

\[ Y^t, x, \pi = W^t, x, \pi - \int_t^T Z^t, x, \pi dB_s. \quad (6.4) \]

Though \( f, U \) are not of polynomial growth, one can check that our main results still hold for this case without much technically modification.

Let \( \pi^*(t, x) \) be an optimal control of value function \( v(t, x) = \inf_{u(\cdot) \in U[t,T]} J(t, x, u(\cdot)) \) and define \( g(t, x) = Y^t, x, \pi^* \). By Theorem 4.2, we can verify that \( v(t, x) \) and \( g(t, x) \) satisfy the following extended HJB equation:

\[ \begin{cases} 
\inf_{\pi(t,x) \in U_t} \{ \mathcal{D} v(t, x) + f(g(t, x)) \} = 0, & v(T, x) = U(x) \\
\mathcal{D}^* g(t, x) = 0, & g(T, x) = x.
\end{cases} \quad (6.5) \]

The infinitesimal operator is \( \mathcal{D} = [\partial_t + [rx + (\mu - r) \pi] \partial_x + \frac{1}{2} \sigma^2 \pi^2 \partial_{xx}] \) and \( \mathcal{D}^* = [\partial_t + [rx + (\mu - r) \pi^*] \partial_x + \frac{1}{2} \sigma^2 \pi^* \partial_{xx}] \).

From the first equation of (6.5), we obtain the optimal formula of \( \pi^* \) which satisfies

\[ \pi^*(t, x) = -\frac{\mu - r}{\sigma^2} \frac{\partial_x v}{\partial_{xx} v}. \quad (6.6) \]
Now we assume that $v$ and $g$ have the following structures:

$$v(t, x) = -\frac{1}{\gamma} e^{A(t)x} B(t), \quad (6.7)$$

$$g(t, x) = C(t)x + D(t), \quad (6.8)$$

where

$$A(T) = -\gamma, \quad B(T) = 1, \quad C(T) = 1, \quad D(T) = 0.$$ Substituting (6.6), (6.7), and (6.8) into the second equation of (6.5), it follows that

$$\begin{cases}
C'(t) + rC(t) = 0, & C(T) = 1 \\
D^2 \frac{\beta^2 C(t)}{A(t)} = 0, & D(T) = 0,
\end{cases} \quad (6.9)$$

where

$$\beta = \frac{\mu - r}{\sigma}.$$ Thus, we have that

$$C(t) = e^{-r(t-T)}, \quad D(t) = \int_t^T \frac{\beta^2 C(s)}{A(s)} ds.$$ Substituting (6.6) and (6.7) into the first equation of (6.5), we get

$$\begin{cases}
A(t) = -\gamma C(t), \\
B'(t) - \frac{1}{2} \beta^2 B(t) + e^{-\gamma D(t)} = 0, & B(T) = 1.
\end{cases} \quad (6.10)$$ The solutions of (6.9) and (6.10) are given as follows:

$$A(t) = -\gamma e^{-r(t-T)}, \quad B(t) = e^{\frac{\beta^2}{2} (T-t)} + \int_t^T e^{\frac{\beta^2}{2} (s+T-t-2T)} ds, \quad C(t) = e^{-r(t-T)}, \quad D(t) = \frac{\beta^2}{\gamma} (T-t).$$ Thus, we obtain the solutions for $v(t, x)$, $g(t, x)$, and an optimal control

$$\pi^*(t, x) = \frac{(\mu - r)e^{r(t-T)}}{\gamma \sigma^2}.$$ 7 Conclusion

In this paper, we first established a dynamic programming principle for a general cost functional in which the states is a coupled FBSDE. Then we deduce an extended HJB
equation for the controlled system. Both smooth solution and viscosity solution were investigated for the extended HJB equation.

Since the value function (2.4) depends on the solution of a controlled FBSDE, it is difficult to construct an optimal control $u^*(\cdot)$. To the best of our knowledge, there are no literature on the dynamic programming principle of the value function (2.4) and its related PDE method. To overcome the difficulties, we introduce a feedback control structure where the control set $\mathcal{U}[t,T]$ consists of functions $u(t,x)$ satisfying Lipschitz continuity in $x$ and right-continuous and left-limit on $t$. Applying the method of $\mathcal{S}$-topology, we showed the existence of an optimal control $u^*(t,x)$ on $[0,T] \times \mathbb{R}^n$. Since there is a BSDE states in the cost functional, the controlled system corresponds to an extended HJB equation which is in fact a vector valued PDE. Due to the lack of a good comparison theorem for multi-dimensional PDEs, the viscosity method usually does not apply to multi-dimensional PDEs. A new definition of viscosity solution is given with additional first-order smoothness on the auxiliary equation. Therefore we obtained the existence and uniqueness of a viscosity solution for the extended HJB system.

In summary, compared with the extant method of maximum principle, see for instance [24], this paper provides a PDE method for a general control problem in which the states are described by a FBSDE system. Of course, there are several other related improvements should be considered. For instance, more general feedback control structure should be further investigated; When the FBSDE are fully coupled, i.e., the forward diffusion also depends on the solution of the BSDE, especially, the initial value of the SDE depends on $(x+Y_t)$, it is open on how to derive the extended HJB equation and how to define and prove the existence and uniqueness of the viscosity solutions.

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