Dynamics of spatially distributed delay logistic equation

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Abstract. The local dynamics of spatially distributed logistic equation with delay is studied. It is shown, that the critical cases in problem of equilibrium's stability have infinite dimension. For each of the critical case special replacement is built, which reduce the original problem to a system of parabolic equations — quasi-normal form, the which solutions’ behavior defines local dynamics.

1. Introduction
Logistic equation with delay

\[ \frac{dN}{dt} = r[1 - N(t-h)]N \quad (r, h > 0) \]

appears in a number of applications. This equation plays an especial role in modelling of mathematical ecology problems [1, 2]. Consider more complex model for description of quantity \( N = N(t,x) \), which depends also on spatial variable \( x \in (-\infty, \infty) \):

\[ \frac{\partial N}{\partial t} = r \left[ 1 - \int_{-\infty}^{\infty} F(s)N(t - h, x + s) \, ds \right] N + \delta \frac{\partial^2 N}{\partial x^2}. \]  \hspace{1cm} (1)

Here as above \( r, h > 0 \). We will consider that \( N(t, x) \) is periodic in second variable

\[ N(t, x + 2\pi) \equiv N(t, x). \]  \hspace{1cm} (2)

We will focus our attention below on one type of kernel functions \( F(s) \), which have a specified biological meaning [2]

\[ F(x) = \frac{\sigma}{\sqrt{\pi}} e^{-\sigma^2 x^2}, \quad (\sigma > 0). \]

The graph of the \( F(x) \) is shown on Fig. 1. Note that

\[ \int_{-\infty}^{\infty} F(x) \, dx = 1, \]
so problem (1), (2) have equilibrium state $N \equiv 1$. Let study the local dynamics of (1), (2) in the neighborhood of this equilibrium state.

Main assumption that allows us using asymptotic methods of analysis is that the diffusion coefficient in (1) is a small parameter, and the values of $F(x)$ are small if $x$ is not in the small neighborhood zero

$$\delta = \varepsilon d, \quad \sigma = \varepsilon^{-1} a, \quad 0 < \varepsilon \ll 1.$$  

This means that (1), (2) is singular perturbed problem.

The method of research based on special method of quasynormal forms [3, 4]. The main idea of this method is to build special nonlinear equation, which dynamics describes behavior of initial equation’s solutions in small neighbourhood of steady state.

2. Nonlinear analysis

The linearisation of the boundary value problem (1), (2) at the equilibrium $N \equiv 1$ gives a linear boundary value problem

$$\frac{\partial u}{\partial t} = -r \int_{-\infty}^{\infty} F(s)u(t - h, x + s) \, ds + \varepsilon^2 d^2 \frac{\partial^2 u}{\partial x^2},$$

$$u(t, x + 2\pi) \equiv u(t, x).$$

Using the eigenfunctions $e^{ikx}$ of a periodic boundary value problem we obtain the following characteristic equation

$$\lambda = -r \exp(-\lambda h) \exp(-\frac{\varepsilon^2 k^2}{4a^2}) - d^2 \varepsilon^2 k^2, \quad k = 0, \pm 1, \pm 2, \ldots$$  \hspace{1cm} (3)

The following lemma describes the root location for the equation (3).

**Lemma.** Let $rh < \frac{\pi}{2}$, then all roots of the equation (3) have negative real parts and are bounded away from the imaginary axis for small values of $\varepsilon$. If $rh > \frac{\pi}{2}$, then the equation (3) has at least one root with a positive real part (and separated from 0 for $\varepsilon \to 0$).

Let $rh = \frac{\pi}{2}$, then infinitely many roots of the equation (3) tend to $i\frac{\pi}{2h}$ for $\varepsilon \to 0$.

Thus (see [5]) for $rh < \frac{\pi}{2}$ the dynamics in a neighborhood of the equilibrium state $N \equiv 1$ is trivial: all solutions from its certain neighborhood tend to $N = 1$. For $rh > \frac{\pi}{2}$, the dynamics become nonlocal: there is no stable regime in the some small (but independent of $\varepsilon$ neighborhood of equilibrium.

\hspace{1cm} Figure 1. The graph of the function $F(x)$.
Let \( r_0 \) and \( h_0 \) be positive and such that \( r_0 h_0 = \frac{\pi}{2} \). So the \( r = r_0, h = h_0 \) is the point of bifurcation. Consider

\[
r = r_0 + \varepsilon^p r_1, \quad h = h_0 + \varepsilon^p h_1, \quad 0 < p \leq 2.
\]

We consider only \( 0 < p \leq 2 \), since for \( p > 2 \) all constructions are equivalent to the case \( p = 2 \) and \( r_1 = h_1 = 0 \). Note, that the case \( p = 2 \) is studied in [6].

From lemma it follows that under the condition (4) and for small enough \( \varepsilon \) we have the critical case of infinite dimension. Roots of (3), which tend to the imaginary axis, admit an asymptotic representation

\[
\lambda_n = \frac{\pi}{2h_0} i + \frac{\varepsilon^p}{1 + i\pi/2} \left( \frac{r_0h_1\pi}{2h_0} + r_1i - d^2 z^2 n^2 - \frac{r_0z^2p^2}{4a^2} \right) + o(\varepsilon^p).
\]

The critical spatial modes have the numbers \((z\varepsilon^{p/2-1} + \theta_z)n\), where \( n \in \mathbb{Z}, z \) is arbitrarily fixed, and \( \theta_z = \theta(z(\varepsilon) \in [0,1]) \) such that \( z\varepsilon^{p/2-1} + \theta_z \) takes integer values. However, if \( p = 2 \) then select \( z = 1 \) and \( \theta_z = 0 \).

Let us assume that in (1), (2)

\[
N = 1 + \varepsilon^{p/2} \left( \exp(i \frac{\pi}{2h_0} t) u(\tau, y) + \exp(-i \frac{\pi}{2h_0} t) \pi(\tau, y) \right) + \varepsilon^p u_2(t, \tau, y) + \varepsilon^{3p/2} u_3(t, \tau, y) + \ldots, \quad (5)
\]

where \( \tau = \varepsilon^p t \) is a slow time, \( y = (z\varepsilon^{p/2-1} + \theta_z) x \), the functions \( u_j(t, \tau, x) \) are \( 4h_0 \)-periodic with respect to \( t \). Put the formal series (5) into (1),(2) and collect coefficients at equal powers of \( \varepsilon \). First, define \( u_2(t, \tau, x) \), from the equation of coefficients at \( \varepsilon^p \):

\[
u_2 = \frac{2 - i}{5} \exp(i \frac{\pi}{h_0} t) u^2(\tau, y) + \frac{2 + i}{5} \exp(-i \frac{\pi}{h_0} t) \pi^2(\tau, y).
\]

Then, from the solvability condition of the relevant equation with respect to \( u_3(t, \tau, x) \), we obtain a parabolic boundary value problem (without a small parameter) of the Ginzburg-Landau type

\[
\frac{\partial u}{\partial \tau} (1 + \frac{\pi}{2}) = z^2(d^2 + \frac{r_0}{4a^2}) \frac{\partial^2 u}{\partial x^2} + (r_0 \pi h_1 + r_1i) u + \frac{r_0(1 - 3i)}{5} u |u|^2, \quad (6)
\]

\[
|u(\tau, x) = u(\tau, x + 2\pi). \quad (7)
\]

The main result is the following. The problem (6), (7) is a quasinormal form of (1), (2).

**Theorem.** To each periodic solution \( u(\tau, y) \) of the problem (6), (7) for some \( z \) there corresponds solution (1), (2) with discrepancy up to \( O(\varepsilon^{3p/2}) \) uniformly with respect to \( t \geq 0, x \in [0, 2\pi] \)

\[
N(t, x) = 1 + \varepsilon^{p/2} \left( u(\varepsilon^p t, y) \exp(i \frac{\pi}{2h_0} t) + \varepsilon \pi(\varepsilon^p t, y) \right) + \varepsilon^p \left( \frac{2 - i}{5} \exp(i \frac{\pi}{h_0} t) u^2(\varepsilon^p t, y) + \frac{2 + i}{5} \exp(-i \frac{\pi}{h_0} t) \pi^2(\varepsilon^p t, y) \right), \quad (8)
\]

where \( y = (z\varepsilon^{p/2-1} + \theta_z) x \).

Note, that if \( p < 2 \) then (8) is a family (depending on \( z \)) of solutions which fast oscillates in spatial variable around \( N \equiv 1 \).
3. Conclusion
We built special nonlinear parabolic Ginzburg-Landau type equations (that does not contain asymptotically small or large parameters) to find the main terms of asymptotic of initial equation’s solutions. Existence of arbitrary parameter $z$ in these equations leads to multistability. Constructing the main terms of asymptotic of solutions (1) can help to find solutions with any degree of accuracy.

Obtained results can be generalized to more complex equations — for example, to situation when spatial variable is two-dimensional [7].

Acknowledgments
This study is supported by the Russian Foundation for Basic Research (project no. 18-01-00672).

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