On conjectures of Chern concerning parity bias in partitions

Damanvir Singh Binner
Department of Mathematics
Sant Longowal Institute of Engineering and Technology, India
damanvirbinnar@iisermohali.ac.in

Abstract

We prove recent conjectures of Chern concerning nonnegativity of a certain $q$-series related to parity bias in integer partitions.

1 Introduction

Recently, Kim, Kim, and Lovejoy [2] studied the phenomenon of parity bias in integer partitions. Let $p_o(n)$ denote the number of partitions of $n$ that have more odd parts than even parts and let $p_e(n)$ denote the number of partitions of $n$ that have more even parts than odd parts.

**Theorem 1** (B. Kim, E. Kim and Lovejoy (2020)). For any $n \neq 2$, we have

$$p_o(n) \geq p_e(n).$$

This result was generalized by Kim and Kim [3] in a subsequent work. Let $p_{a,b,m}(n)$ denote the number of partitions of $n$ that have more parts congruent to $a$ modulo $m$ than parts congruent to $b$ modulo $m$.

**Theorem 2** (B. Kim and E. Kim (2021)). Let $m \geq 2$ be an integer. Then,

$$p_{1,m}(n) \geq p_{m,1}(n),$$
$$p_{1,m-1}(n) \geq p_{m-1,1}(n).$$

In a very recent work, Chern [1] obtained a strong generalization of these results.

**Theorem 3** (Chern (2022)). Let $m \geq 2$ be an integer. Then, for any integers $a$ and $b$ with $1 \leq a < b \leq m$,

$$p_{a,b,m}(n) \geq p_{b,a,m}(n).$$

Note that Theorem 1 can be obtained by setting $(a, b, m) = (1, 2, 2)$ in Theorem 3 while Theorem 2 can be obtained by setting $(a, b) = (1, m)$ and $(a, b) = (1, m-1)$ in Theorem 3.

Chern [1, Section 2] proves Theorem 3 using $q$-series techniques for any $(a, b) \neq (1, 2)$. The author [1, Section 3] follows a completely different approach involving some $q$-series analysis followed by some lengthy combinatorial arguments to handle the case $(a, b) = (1, 2)$. 
Finally, the author [1, Section 4] states that to handle all the cases uniformly using the approach described in [1, Section 2], one needs to prove that the following $q$-series

$$
\frac{(q, q^2; q^m)_\infty}{(q; q)_\infty} \sum_{j \geq 0} \sum_{k \geq 1} \frac{q^{3j+k}(1-q^k)}{(q^m; q^m)_j(q^m; q^m)_{j+k}}.
$$

has nonnegative coefficients in its expansion. To prove this, it is clearly sufficient to prove the following statement.

**Conjecture 4** (Chern (2022)). For $m \geq 2$, the double series

$$
\sum_{j \geq 0} \sum_{k \geq 1} \frac{q^{3j+k}(1-q^k)}{(q^m; q^m)_j(q^m; q^m)_{j+k}}
$$

has nonnegative coefficients in its expansion.

Further, the author notes that we can rearrange the terms of this double sum as follows.

$$
\sum_{j \geq 0} \sum_{k \geq 1} \frac{q^{3j+k}(1-q^k)}{(q^m; q^m)_j(q^m; q^m)_{j+k}} = \sum_{j \geq 0} \frac{q^{3j}}{(q^m; q^m)_j(q^m; q^m)_{j}} \sum_{k \geq 1} \frac{q^k(1-q^k)}{(q^{(j+1)m}; q^m)_k}.
$$

From here, it is clearly sufficient to prove the nonnegativity of the inner series.

**Conjecture 5** (Chern (2022)). For $m, s \geq 1$, the $q$-series

$$
\sum_{k \geq 0} \frac{q^k(1-q^k)}{(q^s; q^m)_k}
$$

has nonnegative coefficients in its expansion.

In this note, we obtain a very short and simple counting proof of Conjecture 5. As described above, this leads to a proof of Conjecture 4 as well, and also provides a uniform proof for the cases $(a, b) \neq (1, 2)$ and $(a, b) = (1, 2)$ in the proof of Theorem 3. This greatly simplifies the proof of Theorem 3 provided in [1].

## 2 Proof of Conjecture 5

In this section, we prove Conjecture 5.

**Proof of Conjecture 5** Let $P_{s,m,k}(n)$ denote the number of partitions of $n$ with all parts lying in \( \{ s, s + m, s + 2m, \ldots, s + m(k-1) \} \). Then, the coefficient of $q^n$ in the series

$$
\sum_{k \geq 0} \frac{q^k(1-q^k)}{(q^s; q^m)_k}
$$

...
is given by

\[ \sum_{k \geq 1} P_{s,m,k}(n - k) - P_{s,m,k}(n - 2k) = \sum_{k \geq 1, \, k \text{ odd}} P_{s,m,k}(n - k) + \sum_{k \geq 2, \, k \text{ even}} \left( P_{s,m,k}(n - k) - P_{s,m,k}(n - k) \right), \]

which is easily seen to be nonnegative using the fact that for given \( s, m \) and \( n \), \( P_{s,m,k}(n) \) is an increasing function of \( k \).

\[ \square \]

References

[1] S. Chern, Further results on biases in integer partitions, Bull. Korean Math. Soc., 59(1):111–117, 2022.

[2] B. Kim, E. Kim and J. Lovejoy, Parity bias in partitions, European J. Combin., 89:103159, 19, 2020.

[3] B. Kim and E. Kim, Biases in integer partitions, Bull. Aust. Math. Soc., 1–10, 2021.