DIRAC COHOMOLOGY FOR THE DEGENERATE AFFINE HEcke
CLIFFORD ALGEBRA

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Abstract. We define an analogue of the Dirac operator for the degenerate affine Hecke-
Clifford algebra. A main result is to relate the central characters of the degenerate affine
Hecke-Clifford algebra with the central characters of the Sergeev algebra via Dirac
cohomology. The action of the Dirac operator on certain modules is also computed.
Results in this paper could be viewed as a projective version of the Dirac cohomology
of the degenerate affine Hecke algebra.

1. Introduction
Throughout this paper, we work over the ground field $\mathbb{C}$. Let $S_n$ be the symmetric
group. It is well-known that $S_n$ admits a non-trivial central extension
$$1 \to \mathbb{Z}_2 \to \tilde{S}_n \to S_n \to 1,$$
where $\tilde{S}_n$ is a double cover of $S_n$ defined in Section 2.6. The projective representations,
or so-called spin representations of $S_n$ are linear representations of $\tilde{S}_n$ which do not factor
through $S_n$. Those representations over $\mathbb{C}$ have been well understood for a long time by
Schur [9] (1911). A more modern approach to study the projective representations of $S_n$
is via the Sergeev algebra $\text{Seg}_n$ (see Definition 2.10) introduced in [10]. It is known that the
projective representations of $S_n$ can be obtained from $\text{Seg}_n$-modules.

The degenerate affine Hecke-Clifford algebra $H_n^{Cl}$ (see Definition 2.15) is introduced by
Nazarov [8] to study the Young’s symmetrizers of the projective representations of $S_n$. This
algebra contains $\text{Seg}_n$ as a subalgebra and could be viewed as the projective counterpart
of the degenerate affine Hecke algebra $H_n$ of Lusztig.

The purpose of this paper is to establish Dirac cohomology theory for $H_n^{Cl}$ as a projective
analogue of the one recently developed for $H_n$ by Barbasch, Ciubotaru and Trapa [1].
Motivated by the Dirac element in [1], the Dirac type element in $H_n^{Cl}$ is defined as
$$\sum_{i=1}^{n} x'_i c_i,$$
where $x'_i$ is defined in Section 4.1. This Dirac type element can be viewed as the square root
of a certain Casmir type element in the center of $H_n^{Cl}$ (Theorem 4.7). For an $H_n^{Cl}$-module
$(\pi, X)$, the Dirac cohomology is defined as $H_D(X) = \ker \pi(D)/(\ker \pi(D) \cap \text{im} \pi(D))$, which
is a $\text{Seg}_n$-module.

One of our main results (Theorem 4.8) says that if $X$ is irreducible and $H_D(X)$ is nonzero,
then any irreducible $\text{Seg}_n$-module in $H_D(X)$ determines the central character of $X$. This
is an analogue to a statement for Harish-Chandra modules called Vogan’s conjecture [5].
A key step in the proof of Theorem 4.8 is to establish a canonical algebra homomorphism from the center of $\mathbb{H}^\mathbb{C} l_n$ to the center of $\text{Seg}_n$ (Theorem 3.4). This homomorphism is shown to map onto the even elements of the center of $\text{Seg}_n$ via the study of the Dirac cohomology on some modules (Corollary 5.19).

We outline the structure of this paper. In Section 2, we review some properties of superalgebras and definitions of several superalgebras studied in this paper. In Section 3, we define a certain algebraic structure which leads to a more general result than Theorem 4.8. Such algebraic structure includes the consideration of other real reflection groups and has potential to cover some other algebras such as the degenerate affine Hecke-Clifford algebras of other classical types defined by Wang and Khongsap [3]. We prove in Section 4 that $\mathbb{H}^\mathbb{C} l_n$ satisfies such algebraic structure and so complete the proof of Theorem 4.8 for $\mathbb{H}^\mathbb{C} l_n$.

In Section 5, we compute the action of the Dirac type element $D$ on some interesting modules. On one hand, the element $D$ has some similar behavior to the one in $\mathbb{H}_n$ defined in [1]. This suggests our $D$ is a nice projective analogue. On another hand, those computations also show that certain modules have non-zero Dirac cohomologies (Theorem 5.16) and thus Theorem 4.8 for $\mathbb{H}^\mathbb{C} l_n$ covers several interesting examples. More precisely, there exists a certain kind of induced modules playing the role of tempered modules of $\mathbb{H}_n$. Those modules can be parametrized by the partitions of $n$. When such module corresponds to a partition of distinct parts, we compute that they have a non-zero Dirac cohomology. This coincides with the expectation from the case of $\mathbb{H}_n$ in [1].

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2. Preliminaries

2.1. Notation for modules. In this paper, all the algebras are associative with an unit over $\mathbb{C}$. Let $\mathcal{A}$ be an algebra. An $\mathcal{A}$-module is denoted $(\pi, X)$ or simply $X$, where $X$ is a vector space and $\pi$ is the map defining the action of $\mathcal{A}$ on $X$. For $a \in \mathcal{A}$ and $x \in X$, the action of $a$ on $x$ is written by $\pi(a)x$ or $a.x$.

Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Define $\text{Ind}_{\mathcal{B}}^\mathcal{A} Y = \mathcal{A} \otimes_{\mathcal{B}} Y$, where $Y$ is a $\mathcal{B}$-module. The left adjoint functor of $\text{Ind}_{\mathcal{B}}^\mathcal{A}$ is the restriction functor denoted $\text{Res}_{\mathcal{B}}^\mathcal{A}$.

2.2. Superalgebras and supermodules. A super vector space $V$ is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. A super vector subspace $W$ of $V$ is a subspace of $V$ such that $W = (W \cap V_0) \oplus (W \cap V_1)$. We say an element $a$ in $V_0$ (resp. $V_1$) has even (resp. odd) degree, denoted $\text{deg}(v) = 0$ (resp. $\text{deg}(v) = 1$).

A superalgebra $\mathcal{A}$ is an algebra with a super vector space structure $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for $i, j \in \mathbb{Z}_2$. A subalgebra $\mathcal{C}$ of a superalgebra $\mathcal{A}$ is said to be a super
subalgebra of $A$ if $C = (A_0 \cap C) \oplus (A_1 \cap C)$. A super ideal $\mathcal{I}$ of a superalgebra $A$ is an ideal of $A$ such that $\mathcal{I} = (A_0 \cap \mathcal{I}) \oplus (A_1 \cap \mathcal{I})$.

For superalgebras $A$ and $B$, a superalgebra homomorphism from $A$ to $B$ is an algebra homomorphism with $f(A_i) \subset B_i$ for $i \in \mathbb{Z}_2$.

For superalgebras $A$ and $B$, the super tensor product of $A$ and $B$, denoted $A \otimes B$ is a superalgebra isomorphic to $A \otimes B$ as vector spaces with the multiplication determined by:

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')} (aa' \otimes bb'),$$

where $a, a' \in A$ and $b, b' \in B$ are homogeneous elements.

Let $A$ be a superalgebra. An $A$-supermodule $X$ is an $A$-module with a super vector space structure $X = X_0 \oplus X_1$ and the property that $A \cdot X_j \subseteq X_{i+j}$, where $i, j \in \mathbb{Z}_2$. A supersubmodule $Y$ of an $A$-supermodule $X$ is a submodule of $X$ such that $Y = (X_0 \cap Y) \oplus (X_1 \cap Y)$. An $A$-supermodule $X$ is irreducible if there is no proper non-zero supersubmodule of $X$.

For an $A$-supermodule $X = X_0 \oplus X_1$, define a map $\delta : X \to X$ such that $\delta(v) = v$ if $v \in X_0$ and $\delta(v) = -v$ if $v \in X_1$.

Let $\text{Mod}_{\text{sup}}(A)$ be the category of $A$-supermodules. Let $\Pi : \text{Mod}_{\text{sup}}(A) \to \text{Mod}_{\text{sup}}(A)$ be a parity change functor. That means for an $A$-supermodule, $\Pi(M)$ and $M$ are isomorphic as $A$-modules, but have opposite $\mathbb{Z}_2$-grading.

2.3. Relations between irreducible supermodules and irreducible modules. Let $A = A_0 \oplus A_1$ be a superalgebra. Given an irreducible $A$-module $(\pi, Y)$, we construct a supermodule as follows. Let $(\pi, \overline{Y})$ be an irreducible $A$-module such that $\overline{Y}$ is identified with $Y$ as vector spaces and the $A$-action on $\overline{Y}$ is determined for any homogenous element $a \in A$ and for $v \in Y$ by

$$\overline{\pi}(a)v = (-1)^{\deg(a)} \pi(a)v.$$  

Let $(\pi_{X_Y}, X_Y)$ be an $A$-supermodule such that $X_Y = Y \oplus \overline{Y}$ as vector spaces and the action of $A$ on $X_Y = Y \oplus \overline{Y}$ is as: $\pi_{X_Y}(a)(v, \overline{v}) = (\pi(a)v, \overline{\pi(a)}\overline{v})$. Let $(X_Y)_0 = \{(v, \overline{v}) \in X_Y : v = \overline{v}\}$ and let $(X_Y)_1 = \{(v, \overline{v}) \in X_Y : v = -\overline{v}\}$. It is elementary to check $X_Y = (X_Y)_0 \oplus (X_Y)_1$ is an $A$-supermodule.

**Lemma 2.1.** Let $Y$ be an irreducible $A$-module. Let $X_Y = Y \oplus \overline{Y}$ be an $A$-supermodule with the supermodule structure described above. Then

1. $Y$ is an irreducible $A$-supermodule if and only if $Y$ and $\overline{Y}$ are non-isomorphic as $A$-modules.
2. If $Y$ and $\overline{Y}$ are isomorphic as $A$-modules, then there is a supermodule structure on $Y$.

**Proof.** For (1), we first prove that if $X_Y$ is an irreducible $A$-supermodule, then $Y$ and $\overline{Y}$ are not isomorphic as $A$-modules. Suppose instead there exists an $A$-module isomorphism $f : Y \to \overline{Y}$ and we will derive a contradiction. Recall that $\overline{Y}$ is identified with $Y$ as vector spaces and thus there exists a natural vector space isomorphism $\theta : \overline{Y} \to Y$ such that
\((-1)^{\deg(a)}\pi(a)\theta = \theta \pi(a)\) for any homogenous \(a \in \mathcal{A}\). Then \(\theta \circ f\) satisfies the property that for any homogenous element \(a \in \mathcal{A}\),
\[
\pi(a)(\theta \circ f)(x) = (-1)^{\deg(a)}(\theta \circ f)(\pi(a)x)
\]
Then the map \((\theta \circ f)^2\) is an \(\mathcal{A}\)-module automorphism of \(Y\). Thus, by Schur’s lemma and a suitable normalization, we may assume \((\theta \circ f)^2\) is an identity map. Then as vector spaces
\[
Y = \ker(\theta \circ f - \mathrm{Id}) \oplus \ker(\theta \circ f + \mathrm{Id})
\]
For \(\epsilon = 0, 1\), let
\[
\ker_{\epsilon} = \left\{(v, (-1)^{\epsilon}v) \in X_Y : v \in \ker(\theta \circ f - (-1)^{\epsilon}\mathrm{Id})\right\}
\]
Then it is straightforward to verify \(\ker_0 \oplus \ker_1 \subset X_Y\) gives a proper super submodule of \(X_Y\).

We now prove if \(Y\) and \(\overline{Y}\) are not isomorphic as \(\mathcal{A}\)-modules, \(X_Y\) is an irreducible \(\mathcal{A}\)-supermodule. Suppose instead that there exists a proper super submodule \(M\) of \(X_Y\) and we will get a contradiction. Let \(M^i = \{v \in Y : (v, (-1)^iv) \in M \cap (X_Y)_i\}\) for \(i \in \mathbb{Z}_2\), which are regarded as vector subspaces of \(Y\). We first see that \(M^0 \cap M^1 = 0\). Otherwise, there exists some nonzero \(v \in Y\) such that \((v, v) \in M\) and \((v, -v) \in M\), and so \((v, 0), (0, v) \in M\). The irreducibility of \(Y\) and \(\overline{Y}\) implies \(M = X_Y\), contradicting \(X_Y\) is proper. Furthermore the irreducibility of \(Y\) implies \(Y = M^0 \oplus M^1\) (as vector spaces). Define a map \(f : (\pi, Y) \to (\overline{\pi}, \overline{Y})\) determined by \(f(v) = (-1)^iv\) for \(v \in M^i\) \((i \in \mathbb{Z}_2)\). One can check \(f\) is an \(\mathcal{A}\)-module isomorphism and so this gives a contradiction.

We now consider (2). By (1), \(X_Y\) is not an irreducible \(\mathcal{A}\)-supermodule. Let \(X'_Y\) be an irreducible super submodule of \(X_Y\). Then by the construction of \(X_Y\), \(X'_Y\) is isomorphic to \(Y = \overline{Y}\) as \(\mathcal{A}\)-modules. Then this gives a supermodule structure to \(Y\).

Q.E.D.

We can also start with an irreducible \(\mathcal{A}\)-supermodule and decompose it into irreducible \(\mathcal{A}\)-module(s).

**Lemma 2.2.** Let \(X\) be an irreducible \(\mathcal{A}\)-supermodule. Let \(\delta\) be a linear automorphism on \(X\) such that \(\delta(v) = (-1)^iv\) for \(v \in X_i\) \((i = 0, 1)\). If \(X\) is not an irreducible \(\mathcal{A}\)-module, then there exists an irreducible \(\mathcal{A}\)-submodule \(Y\) of \(X\) such that
(1) \(\delta(Y)\) is also an \(\mathcal{A}\)-submodule of \(X\) and \(\delta(Y) = \overline{Y}\); and
(2) \(Y\) and \(\delta(Y)\) are non-isomorphic \(\mathcal{A}\)-modules; and
(3) \(X = Y \oplus \delta(Y)\) as \(\mathcal{A}\)-modules.

**Proof.** (1) follows from \(a.\delta(v) = (-1)^{\deg(a)}\delta(a.v)\) for any homogenous element \(a \in \mathcal{A}\) and \(v \in Y\). (2) and (3) are (a reformulation of) [2] Lemma 2.3.

Q.E.D.

**Lemma 2.3.** Let \(X\) and \(X'\) be irreducible \(\mathcal{A}\)-supermodules. If \(X\) and \(X'\) are isomorphic as \(\mathcal{A}\)-modules, then \(X\) and \(X'\) are isomorphic, up to applying the functor \(\Pi\), as \(\mathcal{A}\)-supermodules.
Proof. Suppose $X$ and $X'$ are also irreducible $\mathcal{A}$-modules. Then $X_0, X_1, X'_0, X'_1$ are irreducible $\mathcal{A}_0$-modules. Then either $X_0 = X'_0$ or $X_0 = X'_1$ as $\mathcal{A}_0$-modules. Then either $X = X'$ or $X = \Pi(X')$ as $\mathcal{A}$-supermodules.

Suppose $X$ is not an irreducible $\mathcal{A}$-module. Let $X = Y \oplus \delta(Y)$ and $X' = Y' \oplus \delta(Y')$ be the decomposition of $X$ into $\mathcal{A}$-modules as in Lemma 2.2. Without loss of generality, we may assume $Y = Y'$ as $\mathcal{A}$-modules. Let $f : Y \to Y'$ be an $\mathcal{A}$-module isomorphism. Then $f$ also induces an $\mathcal{A}$-module isomorphism $\overline{f} : \delta(Y) \to \delta(Y')$ such that $\overline{f} = \delta \circ f \circ \delta$. Then one can show the map $f \oplus \overline{f}$ is an $\mathcal{A}$-supermodule isomorphism by checking the map preserves grading. In particular, we also have $\Pi(X) = X$ as $\mathcal{A}$-supermodules in this case.

Q.E.D.

Let $\text{Irr}(\mathcal{A})$ (resp. $\text{Irr}_{\sup}(\mathcal{A})$) be the set of irreducible $\mathcal{A}$-modules (resp. irreducible $\mathcal{A}$-supermodules). Let $\sim$ be the equivalence relation on $\text{Irr}(\mathcal{A})$: $Y \sim Y'$ if and only if $Y = Y'$ or $Y = \overline{Y}'$. Let $\sim_{\Pi}$ be the equivalence relation on $\text{Irr}_{\sup}(\mathcal{A})$: $X \sim_{\Pi} X'$ if and only if $X = X'$ or $X = \Pi(X')$.

**Proposition 2.4.** There is a natural bijection

$$\text{Irr}_{\sup}(\mathcal{A})/ \sim_{\Pi} \longleftrightarrow \text{Irr}(\mathcal{A})/ \sim.$$

*Proof.* Lemmas 2.1 and 2.3 define a map from $\text{Irr}(\mathcal{A})/ \sim$ to $\text{Irr}_{\sup}(\mathcal{A})/ \sim_{\Pi}$. Lemma 2.2 defines a map in the opposite direction. The two maps are inverse to each other by Lemma 2.3.

Q.E.D.

2.4. **Central characters of supermodules.** For a superalgebra $\mathcal{A}$, let $Z(\mathcal{A})$ be the center of $\mathcal{A}$. Note that $Z(\mathcal{A})$ is a super subalgebra of $\mathcal{A}$. Recall that $Z(\mathcal{A})_0$ is the set of even elements in $Z(\mathcal{A})$.

**Proposition 2.5.** Let $X$ be an irreducible $\mathcal{A}$-supermodule. For $z \in Z(\mathcal{A})_0$, $z$ acts on $X$ by the multiplication of a scalar.

*Proof.* If $X$ is an irreducible $\mathcal{A}$-module, then the statement follows from (ordinary) Schur’s lemma (for this case). If $X$ is not an irreducible $\mathcal{A}$-module, then we could decompose $X = Y \oplus \delta(Y)$ as $\mathcal{A}$-modules as in Lemma 2.2. Then $z$ acts on the two modules $Y$ and $\delta(Y)$ by scalars, denoted $\lambda$ and $\lambda'$ respectively. Then for $v \in Y$,

$$z(v + \delta(v)) = \frac{\lambda + \lambda'}{2} (v + \delta(v)) + \frac{\lambda - \lambda'}{2} (v - \delta(v))$$

Note that $\delta(v + \delta(v)) = v + \delta(v)$ and so $v + \delta(v) \in X_0$, and similarly $v - \delta(v) \in X_1$. Then since $z$ is of even degree, $\lambda = \lambda'$.

Q.E.D.

By Proposition 2.5, we can define the following:
Definition 2.6. Let \( A \) be a superalgebra. Let \((\pi, X)\) be a supermodule of \( A \). Define the central character \( \chi_{\pi} \) to be the map from \( Z(A)_0 \) to \( \mathbb{C} \) such that \( \chi_{\pi}(z) \) is the value of \( z \) acting on \( X \).

The central character defined above is only for even elements in the center of a superalgebra. However, the central character indeed determines the action of odd elements in the center in the following sense:

Proposition 2.7. Let \( z \in Z(A)_1 \). Let \( X \) be an irreducible \( A \)-supermodule. If \( X \) is also an irreducible \( A \)-module, then \( z \) acts by zero on \( X \). If \( X \) is not an irreducible \( A \)-module, then \( z \) acts on the two irreducible \( A \)-submodules of \( X \) by two distinct scalars \( \sqrt{\lambda} \) and \( -\sqrt{\lambda} \), where \( \lambda \) is the value that \( z^2 \in Z(A)_0 \) acts on \( X \).

Proof. For (1), suppose \( X \) is an irreducible \( A \)-module. Then by Schur’s Lemma, \( z \) acts on \( X \) by a value denoted by \( \lambda \). Then by Lemmas 2.1 and 2.3, \( X = X \) as \( A \)-modules. This implies \( z \) also acts by \( -\lambda \) on \( X \) as \( z \) is an odd element. Hence \( \lambda = 0 \).

Now suppose \( X \) is not an irreducible \( A \)-module. Then \( z^2 \) is an even element in the center and hence acts by a scalar, denoted \( \lambda \). Then \( z \) acts on the irreducible \( A \)-submodules of \( X \) by scalars \( \sqrt{\lambda} \) and \( -\sqrt{\lambda} \).

Q.E.D.

2.5. Notation for the root system of type \( A_{n-1} \). Let \( e_1, \ldots, e_n \) be the standard basis of \( \mathbb{R}^n \). The root system \( R \) of type \( A_{n-1} \) is the set

\[
\{ e_i - e_j \in \mathbb{R}^n : i \neq j \}.
\]

Fix a set \( R^+ \) of positive roots

\[
R^+ = \{ e_i - e_j \in \mathbb{R}^n : i < j \}.
\]

The set of simple roots \( \Delta \) is

\[
\{ e_i - e_{i+1} : i = 1, \ldots, n-1 \}.
\]

A partition of \( n \) is a sequence of positive integers \( (n_1, \ldots, n_r) \) such that \( n_1 \geq n_2 \geq \ldots \geq n_r \) and \( n_1 + \ldots + n_r = n \).

For a partition \( \lambda = (n_1, \ldots, n_r) \) of \( n \), let \( I_\lambda = \{1, \ldots, n\} \setminus \{n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_r\} \) and let

\[
\Delta_\lambda = \{ e_i - e_{i+1} : i \in I_\lambda \}.
\]

Let \( V_\lambda \) be the real span of \( \Delta_\lambda \) in \( \mathbb{R}^n \) and let \( R^+_\lambda = V_\lambda \cap R^+ \).

For \( i \neq j \), let

\[
\alpha_{ij} = \begin{cases} 
  e_i - e_j & \text{if } i < j \\
  e_j - e_i & \text{if } i > j 
\end{cases}.
\]

Thus \( \alpha_{ij} \) is always a positive root. For simplicity, we also set \( \alpha_i = \alpha_{i,i+1} \) for \( i = 1, \ldots, n-1 \).
2.6. **The superalgebra** $\mathbb{C}[\tilde{S}_n]^-$. Let $S_n$ be the symmetric group. For $i = 1, \ldots, n-1$, let $s_{\alpha_i} = (i, i+1) \in S_n$ be the transposition of $i$ and $i+1$.

Let $\tilde{S}_n$ be the group generated by the elements $\psi, \tilde{t}_{\alpha_1}, \ldots, \tilde{t}_{\alpha_{n-1}}$ subject to the following relations:

$$\overline{\tilde{t}}_{\alpha} = 1$$
$$\overline{\tilde{t}}_{\alpha_i} \overline{\tilde{t}}_{\alpha_{i+1}} = 1 \quad \text{for } i = 1, \ldots, n-1$$
$$\overline{\tilde{t}}_{\alpha_i} \overline{\tilde{t}}_{\alpha_j} = \psi \overline{\tilde{t}}_{\alpha_j} \overline{\tilde{t}}_{\alpha_i} \quad \text{for } |i - j| > 1,$$
$$\psi \overline{\tilde{t}}_{\alpha_i} = \overline{\tilde{t}}_{\alpha_i} \psi \quad \text{for } i = 1, \ldots, n-1,$$
$$\psi^2 = 1.$$ 

Then $\tilde{S}_n$ is a double cover of $S_n$ via the map determined by $\overline{\tilde{t}}_{\alpha_i} \mapsto s_{\alpha_i}$ and $\psi \mapsto 1$. Denote by $\mathbb{C}[\tilde{S}_n]$ the group algebra of $\tilde{S}_n$ with a basis labeled as $\{e_\overline{w} : \overline{w} \in \tilde{S}_n\}$. Define $\mathbb{C}[\tilde{S}_n]^-=: \mathbb{C}[\tilde{S}_n]/(e_\overline{w} + 1)$. We shall simply write $\overline{w}$ for the image of $e_\overline{w}$ in $\mathbb{C}[\tilde{S}_n]^-$.

There is a superalgebra structure on $\mathbb{C}[\tilde{S}_n]^-$ with $\deg(\overline{\tilde{t}}_{\alpha}) = 1$ for all $\alpha \in \Delta$.

**Lemma 2.8.** Given a $S_n$ representation $U$ and a $\mathbb{C}[\tilde{S}_n]^-$-supermodule $U'$, there exists a natural $\mathbb{C}[\tilde{S}_n]^-$-supermodule structure on $U \otimes U'$ characterized by

$$\overline{\tilde{t}}_{\alpha}(u \otimes u') = (s_{\alpha} u) \otimes (\overline{\tilde{t}}_{\alpha} u'),$$

where $\alpha \in \Delta$, $u \in U$ and $u' \in U'$.

Define an equivalence relation on $\text{Irr}(\mathbb{C}[\tilde{S}_n]^-)$: $U \sim_{\text{sgn}} U'$ if and only if $U = U'$ or $U = \text{sgn} \otimes U'$ as $\mathbb{C}[\tilde{S}_n]^-$-supermodules, where $\text{sgn}$ is the sign representation of $S_n$ and the $\mathbb{C}[\tilde{S}_n]^-$-supermodule structure of $\text{sgn} \otimes U'$ is defined in Lemma 2.8.

**Proposition 2.9.** There is a natural bijection

$$\text{Irr}_{\text{sup}}(\mathbb{C}[\tilde{S}_n]^-) / \sim_{\Pi} \leftrightarrow \text{Irr}(\mathbb{C}[\tilde{S}_n]^-) / \sim_{\text{sgn}}.$$ 

**Proof.** It suffices to see that the equivalence relation $\sim$ in Proposition 2.8 is the same as $\sim_{\text{sgn}}$. This follows from $\deg(\overline{\tilde{t}}_{\alpha}) = 1$ for all $\alpha \in \Delta$ and definitions.

Q.E.D.

2.7. **Sergeev algebra.**

**Definition 2.10.** The Sergeev algebra, denoted $\text{Seg}_n$, is an associative algebra with an unit generated by symbols $\{f_w : w \in S_n\}$ and $\{c_i\}_{i=1}^n$ such that

1. the map from the group algebra $\mathbb{C}[S_n] = \bigoplus_{w \in S_n} \mathbb{C}w$ to $\text{Seg}_n$ given by $w \mapsto f_w$ is an algebra injection;
2. $c_i c_j = -c_j c_i$ for $i \neq j$ and $c_i^2 = -1$ for all $i$;
3. $f_w c_i = c_{w(i)} f_w$ for all $w \in S_n$ and for all $i$. 
We may simply write $w$ for $f_w$ as an element in $\text{Seg}_n$. For example, $s_\alpha \in \text{Seg}_n$ denotes $f_{s_\alpha}$. The algebra $\text{Seg}_n$ has a superalgebra structure with $\text{deg}(c_i) = 1$ and $\text{deg}(w) = 0$ for $w \in S_n$.

Let $\text{Cl}_n$ be the super subalgebra of $\text{Seg}_n$ generated by $c_i$ ($i = 1, \ldots, n$). There exists a unique, up to applying the functor $\Pi$, irreducible supermodule of $\text{Cl}_n$. Let $U(n)$ be a fixed choice of an irreducible supermodule of $\text{Cl}_n$. The dimension of $U(n)$ is $2^{n/2}$ for $n$ even and $2^{(n+1)/2}$ for $n$ odd.

For $i \neq j$, define $c_{\alpha_{ij}}$ as

$$c_{\alpha_{ij}} = \begin{cases} \frac{1}{\sqrt{2}}(c_i - c_j) & \text{if } i < j \\ \frac{1}{\sqrt{2}}(c_j - c_i) & \text{if } j < i \end{cases}. $$

The relation between subalgebras $\text{Seg}_n$ and $\mathbb{C}\tilde{S}_n^-$ is the following.

**Lemma 2.11.** $\text{Seg}_n$ is isomorphic to $\mathbb{C}\tilde{S}_n^- \otimes \text{Cl}_n$ as superalgebras.

**Proof.** Define a map:

$$s_\alpha \mapsto \tilde{t}_\alpha \otimes c_{\alpha}, \quad (i = 1, \ldots, n - 1), \quad c_i \mapsto c_i \quad (i = 1, \ldots, n).$$

It is straightforward to check the map is an isomorphism.

Q.E.D.

Here is an analogue of Lemma 2.8.

**Lemma 2.12.** Given a $S_n$ representation $U$ and a $\text{Seg}_n$-module $U'$, there exists a natural $\text{Seg}_n$-module structure on $U \otimes U'$ characterized by

$$s_\alpha.(u \otimes u') = (s_\alpha.u) \otimes (s_\alpha.u'),$$

and

$$c_i.(u \otimes u') = u \otimes (c_i.u'),$$

where $\alpha \in \Delta$, $i = 1, \ldots, n$, $u \in U$ and $u' \in U'$.

2.8. **Relation between supermodules of $\mathbb{C}\tilde{S}_n^-$ and $\text{Seg}_n$.** Recall from [2] (our formulation here is a bit different) a natural functor $F$:

$$F : \text{Mod}_{\text{sup}}(\mathbb{C}\tilde{S}_n^-) \to \text{Mod}_{\text{sup}}(\text{Seg}_n),$$

$$X \mapsto X \otimes U(n).$$

The $\text{Seg}_n$-supermodule structure of $X \otimes U(n)$ is given by

$$s_\alpha.(x \otimes u) = -(-1)^{\text{deg}(x)}(\tilde{t}_\alpha.x) \otimes (c_\alpha.u),$$

$$c_i.(x \otimes u) = (-1)^{\text{deg}(x)}x \otimes (c_i.u).$$

It is straightforward to check the above equations define a $\text{Seg}_n$-module. Next, define

$$G : \text{Mod}_{\text{sup}}(\text{Seg}_n) \to \text{Mod}_{\text{sup}}(\mathbb{C}\tilde{S}_n^-),$$

$$Y \mapsto \text{Hom}_{\text{Cl}_n}(U(n), Y).$$
The $\mathbb{C}[\tilde{S}_n]$-$\mathfrak{m}$-module structure is given by for $\theta \in \text{Hom}_{\mathfrak{m}}(U(n), Y)$,

\[(\tilde{t}_\alpha, \theta)(u) = (s_\alpha c_\alpha) \cdot \theta(u).\]

**Proposition 2.13.** [2, Theorem 3.4] The functors $F$ and $G$ form an adjoint pair i.e. there is a natural isomorphism

\[\text{Hom}_{\text{Seg}_n}(F(U), U') = \text{Hom}_{\mathbb{C}[\tilde{S}_n]}(U, G(U')).\]

Furthermore if $n$ is even, $G \circ F = \text{Id}$ and $F \circ G = \text{Id}$. If $n$ is odd, $G \circ F = \text{Id} \oplus \Pi$ and $F \circ G = \text{Id} \oplus \Pi$, where $\Pi$ is defined in Section 2.2.

Let $U_{\text{Cl}_n}$ be a Seg$_n$-module defined by

\[U_{\text{Cl}_n} = \text{Ind}_C^{\text{Seg}_n} \text{triv} = \text{Seg}_n \otimes \mathbb{C}[\tilde{S}_n] \text{triv},\]

where $\mathbb{C}[S_n]$ is regarded as the subalgebra of Seg$_n$ generated by the elements $f_{s_n}$ for all $\alpha \in \Delta$ and triv is the trivial representation of $\mathbb{C}[S_n]$. In particular, $\dim_{\mathbb{C}} U_{\text{Cl}_n} = 2^n$.

Let $\text{Cl}'_n$ be the super subalgebra of $\text{Cl}_n$ generated by $c_\alpha$ for $\alpha \in \Delta$. There exists a superalgebra surjection from $\mathbb{C}[\tilde{S}_n]$ to $\text{Cl}'_n$ determined by

\[\bar{t}_\alpha \mapsto \sqrt{-1} c_\alpha\]

for any $\alpha \in \Delta$. Let $U(n)'$ be a fixed choice of irreducible supermodules of $\text{Cl}'_n$ such that $\text{Hom}_{\text{Cl}'_n}(U(n)', \text{Res}_{\text{Cl}_n}^{\text{Seg}_n} U_{\text{Cl}_n}) \neq 0$. Let $U_{\text{spin}}$ be the pull-back to a $\mathbb{C}[\tilde{S}_n]$-supermodule from $\text{Cl}'_n$-supermodule $U(n)'$ under the surjection. The dimension of $U_{\text{spin}}$ is $2^{n/2}$ for $n$ even and $2^{(n-1)/2}$ for $n$ odd.

The relation of $U_{\text{spin}}$ and $U_{\text{Cl}_n}$ is given below.

**Lemma 2.14.** $F(U_{\text{spin}}) = U_{\text{Cl}_n}$.

*Proof.* Let $\mathcal{A}$ be the super subalgebra of Seg$_n$ generated by $s_\alpha c_\alpha$ for all $\alpha \in \Delta$. Note that $\mathcal{A}$ is isomorphic to $\mathbb{C}[\tilde{S}_n]$ via the map $s_\alpha c_\alpha \mapsto \tilde{t}_\alpha$. Let $\tau : U_{\text{Cl}_n} \rightarrow U_{\text{Cl}_n}$ be a linear map such that $\tau(c_{i_1} \ldots c_{i_k}) = (-\sqrt{-1})^{k} (c_{i_1} \ldots c_{i_k})$ for $1 \leq i_1 < \ldots < i_k \leq n$. Note that $\tau(s_\alpha c_\alpha) = \sqrt{-1} c_\alpha \cdot \tau(u)$. Thus by the definition of $U_{\text{spin}}$ and the fact that $\text{Hom}_{\text{Cl}'_n}(U(n)', \text{Res}_{\text{Cl}_n}^{\text{Seg}_n} U_{\text{Cl}_n}) \neq 0$, we have

\[\text{Hom}_{\mathbb{C}[\tilde{S}_n]}(U_{\text{spin}}, \text{Res}_{\mathcal{A}}^{\text{Seg}_n} U_{\text{Cl}_n}) \neq 0,\]

where we regard the $\mathcal{A}$-module as a $\mathbb{C}[\tilde{S}_n]$-module.

Assume $n$ is even. By Proposition 2.13 there exists an irreducible $\mathbb{C}[\tilde{S}_n]$-supermodule $U$ such that $F(U) = U_{\text{Cl}_n}$. Note that $\text{Res}_{\mathcal{A}}^{\text{Seg}_n}(F(U)) = \oplus_{j=1}^{\dim U(n)} U$ as $\mathbb{C}[\tilde{S}_n]$-modules. Thus

\[0 \neq \text{Hom}_{\mathbb{C}[\tilde{S}_n]}(U_{\text{spin}}, \text{Res}_{\mathcal{A}}^{\text{Seg}_n} F(U)) = \oplus_{j=1}^{\dim U(n)} \text{Hom}_{\mathbb{C}[\tilde{S}_n]}(U_{\text{spin}}, U)\]

Thus $U = U_{\text{spin}}$ and $F(U_{\text{spin}}) = U_{\text{Cl}_n}$. For $n$ odd, we can similarly prove $F(U_{\text{spin}}) = U_{\text{Cl}_n}$.

Q.E.D.
2.9. The degenerate affine Hecke-Clifford algebra $\mathbb{H}_n^{CI}$.

**Definition 2.15.** The degenerate affine Hecke-Clifford algebra for type $A_{n-1}$, denoted $\mathbb{H}_n^{CI}$, is the associative algebra with an unit generated by the symbols $\{x_i\}_{i=1}^n$, $\{c_i\}_{i=1}^n$, and $\{f_w : w \in S_n\}$ determined by the following properties:

1. the map from the group algebra $\mathbb{C}[S_n] = \oplus_{w \in S_n} \mathbb{C}w$ to $\mathbb{H}_n^{CI}$ given by $w \mapsto f_w$ is an algebra injection;
2. $x_i x_j = x_j x_i$ for all $i, j$;
3. $x_i c_j = c_j x_i$ for $i \neq j$ and $x_i c_i = -c_i x_i$ for all $i$;
4. $c_i c_j = -c_j c_i$ for $i \neq j$ and $c_i^2 = -1$ for all $i$;
5. $f_w c_i = c_w(i) f_w$ for $w \in S_n$ and for all $i$;
6. $f_{s_{i,j}} x_i - x_{i+1} f_{s_{i,j}} = -1 + c_i c_{i+1}$ for all $i = 1, \ldots, n - 1$ and $f_{s_{i,j}} x_j = x_j f_{s_{i,j}}$ for all $i, j$ with $|i - j| > 1$.

Again, we may later simply write $w$ for $f_w$. The algebra has a superalgebra structure with $\deg(c_i) = 1$, $\deg(w) = 0$ for $w \in S_n$, and $\deg(x_i) = 0$.

The superalgebra of $\mathbb{H}_n^{CI}$ generated by all $f_w$ for $w \in S_n$ and $c_i$ ($i = 1, \ldots, n$) is isomorphic to $\text{Seg}_n$. We shall still denote this superalgebra by $\text{Seg}_n$.

The superalgebra $\mathbb{H}_n^{CI}$ has a PBW type basis:

**Proposition 2.16.** [7, Theorem 14.2.2] The set

$$\{x_1^{m_1} \cdots x_n^{m_n} c_1^{\epsilon_1} \cdots c_n^{\epsilon_n} w : m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}, \epsilon_1, \ldots, \epsilon_n \in \{0, 1\}, w \in S_n\}$$

forms a basis for $\mathbb{H}_n^{CI}$.

The center of $\mathbb{H}_n^{CI}$ plays an important role in Theorem 4.3 and computations in Section 5.

**Proposition 2.17.** [7, Theorem 14.3.1] The center $Z(\mathbb{H}_n^{CI})$ of $\mathbb{H}_n^{CI}$ is the set of all symmetric polynomials in $\mathbb{C}[x_1^2, x_2^2, \ldots, x_n^2]$. In particular, any element in $Z(\mathbb{H}_n^{CI})$ is of even degree.

**Definition 2.18.** Recall that the central character $\chi_\pi : Z(\mathbb{H}_n^{CI})_0 \to \mathbb{C}$ of an irreducible supermodule $(\pi, X)$ is defined in Definition 2.6. By Proposition 2.17 we can also write $\chi_\pi : Z(\mathbb{H}_n^{CI}) \to \mathbb{C}$.

For an element $\gamma = (a_1, \ldots, a_n) \in \mathbb{C}^n$, define $\chi'_\gamma : \mathbb{C}[x_1^2, \ldots, x_n^2] \to \mathbb{C}$ such that $\chi'_\gamma(x_2^2) = a_1$. Define $\chi_\gamma$ to be the restriction of $\chi'_\gamma$ to $Z(\mathbb{H}_n^{CI})$. For the central character $\chi_\pi$ of $X$, there exists a unique $\gamma \in \mathbb{C}^n$, up to permutations of coordinates, such that $\chi_\pi = \chi_\gamma$. We may also say $\gamma$ is the central character of $X$.

A module $(\pi, X)$ is said to be quasisimple if any element in $Z(\mathbb{H}_n^{CI})$ acts by a scalar. In this case, a map $\chi_\pi : Z(\mathbb{H}_n^{CI}) \to \mathbb{C}$ defined as above is still called the central character of $X$. 
3. Dirac cohomology in $\mathcal{H}_W$

In this section, we consider a specific algebraic structure before specializing to $\mathbb{H}^C_n$ in the next section.

3.1. $\mathcal{H}_W$ and a Dirac type element in $\mathcal{H}_W$. Fix a real reflection group $W$. Let $V$ be a representation of $W$. Fix a $W$-invariant inner product on $V$. Let $\{x_1, \ldots, x_n\}$ be an orthogonal basis for $V$.

**Definition 3.1.** An associative algebra $\mathcal{H}_W = \mathcal{H}_W(V)$ is said to have property (*) if it satisfies the following properties. First $\mathcal{H}_W$ is an algebra generated by symbols $f_w$ ($w \in W$), $c_i$ ($i = 1, \ldots, n$) and $x_i$ ($i = 1, \ldots, n$) such that the map from $\mathbb{C}[W]$ to $\mathcal{H}_W$ sending $w$ to $f_w$ is an injection and the algebra has a natural basis of elements having the form $x_1^{k_1} \cdots x_n^{k_n} c_1^{\varepsilon_1} \cdots c_n^{\varepsilon_n} f_w$ ($k_1, \ldots, k_n$ non-negative integers, $w \in W$, $\varepsilon_i = 0$ or $1$). Again we shall write $w$ for $f_w$ for simplicity. Let $\text{Seg}(W)$ be the subalgebra of $\mathcal{H}_W$ generated by all $x_i$ and $c_i$ ($i = 1, \ldots, n$). Furthermore, the generators of $\mathcal{H}_W$ satisfy the following relations:

\begin{align*}
  (3.1) & \quad w x_i w^{-1} = w(x_i) \\
  (3.2) & \quad [x_i, x_j] \in \text{Seg}(W) \\
  (3.3) & \quad c_j x_i = x_i c_j \quad \text{for } i \neq j \\
  (3.4) & \quad c_i x_i = -c_i x_i \\
  (3.5) & \quad c_i c_j = -c_j c_i \quad \text{for } i \neq j \quad \text{and } c_i^2 = -1 \\
  (3.6) & \quad w c_i = w(c_i) w.
\end{align*}

Here $w(x_i)$ is the action of $w$ on $V$. There is a natural action of $W$ on the vector space spanned by all $c_i$ from the action of $W$ on $V$, and $w(c_i)$ represents such action of $w$ on $c_i$. $\mathcal{H}_W$ has a superalgebra structure with $\deg(c_i) = 1$, $\deg(x_i) = \deg(w) = 0$ ($i = 1, \ldots, n$ and $w \in W$).

It is not obvious that $\mathbb{H}^C_n$ satisfies the property (*) at this moment since the generators in the definition of $\mathbb{H}^C_n$ (Definition 2.15) do not directly satisfy the relation (3.1). We will replace those symbols $x_i$ in $\mathbb{H}^C_n$ by some other elements and then show that those new generators satisfy the property (*) in Section 4. We hope this more general setting can cover some other interesting examples such as the Hecke-Clifford algebra for other classical types defined in [4].

In the rest of this section, $\mathcal{H}_W$ denotes an algebra satisfying the property (*). Define a Dirac type element $D$ in $\mathcal{H}_W$:

\begin{equation}
  (3.7) \quad D = \sum_{i=1}^{n} x_i c_i.
\end{equation}

An analogue of a Casimir element in $\mathcal{H}_W$ is the following element:

\begin{equation}
  \Omega_{\mathcal{H}_W} = \sum_{i=1}^{n} x_i^2.
\end{equation}
Note that $\Omega_{\mathcal{H}_W}$ is in the center of $\mathcal{H}_W$ by the orthogonality of the basis $\{x_i\}$.

The following two properties will be used several times:

**Lemma 3.2.**
1. $wD = Dw$ for any $w \in W$;
2. $c_iD = -Dc_i$ for any $i$.

**Proof.**
(1) follows from the fact that $\{x_i\}$ forms an orthogonal basis and property (3.1).
(2) follows from the properties (3.3), (3.4) and (3.5).

Q.E.D.

The following statement says that $D$ can be viewed as the square root of $\Omega_{\mathcal{H}_W}$ and explains why $D$ is called a Dirac type element.

**Lemma 3.3.** $D^2 = \Omega_{\mathcal{H}_W} - \Omega_{\text{Seg}(W)}$ for some element $\Omega_{\text{Seg}(W)}$ in the center of $\text{Seg}(W)$.

**Proof.** Let $\Omega = D^2 - \Omega_{\mathcal{H}_W}$. It suffices to show $\Omega \in Z(\text{Seg}(W))$. By Lemma 3.2, $D^2$ commutes with any element in $\text{Seg}(W)$. Since $\Omega_{\mathcal{H}_W}$ is in the center of $\mathcal{H}_W$, $\Omega_{\mathcal{H}_W}$ also commutes with any element in $\text{Seg}(W)$ and so is $\Omega$. It remains to show that $\Omega \in \text{Seg}(W)$. This follows from the following equation

$$\Omega = D^2 - \Omega_{\mathcal{H}_W} = \sum_{i < j} [x_i, x_j]c_ic_j \in \text{Seg}(W)$$

Q.E.D.

### 3.2. Relation between central characters for $\mathcal{H}_W$ and $\text{Seg}(W)$.

Let $d: \mathcal{H}_W \to \mathcal{H}_W$, $d(h) = Dh - (-1)^{\deg(h)}hD$.

A relation between $Z(\mathcal{H}_W)_0$ and $Z(\text{Seg}(W))_0$ is the following:

**Theorem 3.4.** For any $z \in Z(\mathcal{H}_W)_0$, there exists a unique element $\tilde{z} \in Z(\text{Seg}(W))_0$ such that

$$z - \tilde{z} \in \text{im } d.$$

Let $\zeta: Z(\mathcal{H}_W)_0 \to Z(\text{Seg}(W))_0$ be the map that $\zeta(z)$ is such unique element $\tilde{z}$ in $Z(\text{Seg}(W))_0$. Then $\zeta$ is a superalgebra homomorphism.

Our main result in this paper is the following which says the central character of a $\mathbb{H}_n^{CL}$-supermodule $X$ is determined by the central characters of irreducible $\text{Seg}_n$-supermodules in the Dirac cohomology $H_D(X)$. Here $H_D(X)$ is defined in the theorem.

**Theorem 3.5.** Let $\mathcal{H}_W$ be an algebra satisfying property (*) (Definition 3.1). Let $(\pi, X)$ be an irreducible supermodule of $\mathcal{H}_W$ with the central character $\chi_\pi$ (Definition 2.6). Let the Dirac cohomology $H_D(X)$ of $X$ be

$$H_D(X) = \ker \pi(D)/(\ker \pi(D) \cap \text{im } \pi(D)).$$

Then $H_D(X)$ has a natural $\text{Seg}(W)$-module structure. Let $(\sigma, U)$ be an irreducible $\text{Seg}(W)$-module with the central character $\chi_\sigma$ (Definition 2.6) such that

$$\text{Hom}_{\text{Seg}(W)}(U, H_D(X)) \neq 0.$$
Let $\zeta : \mathcal{H}(\mathcal{H}_W)_0 \to Z(\text{Seg}(\mathcal{W}))_0$ be the map in Theorem 3.4. Let $\chi^\sigma : \mathcal{H}(\mathcal{H}_W)_0 \to \mathbb{C}$, (3.8) \[ \chi^\sigma(z) = \chi_\sigma(\zeta(z)). \]

Then $\chi_\pi = \chi^\sigma$.

Since $wD = Dw$ and $c_i D = -Dc_i$ by Lemma 3.2, $\ker \pi(D)$ and $\ker \pi(D) \cap \text{im} \pi(D)$ are invariant under the action of $\text{Seg}(\mathcal{W})$. Thus $H_D(X)$ has a natural $\text{Seg}(\mathcal{W})$-module structure from the $\mathcal{H}_W$-module structure. The proofs of Theorems 3.4 and 3.5 are given at the end of the next subsection. Theorem 3.5 directly follows from Theorem 3.4. Readers who only want to know how Theorem 3.4 implies Theorem 3.5 may jump to the end of the next subsection.

3.3. Proof of Theorems 3.4 and 3.5. The proofs of the theorems basically follow from the ideas of proofs in [5] Chapter 3 and [1] Section 4. We provide some technical details for this specific case.

Let $S^j(V)$ be the vector space of polynomials of $x_1, \ldots, x_n$ with degree less than or equal to $j$. Let $\mathcal{H}_W^0$ be the vector space spanned by elements of the form \[ \{pw : w \in \text{Seg}(\mathcal{W}), p \in S^j(V)\}. \]

Note that $\mathcal{H}_W^0 \subseteq \mathcal{H}_W^1 \subseteq \ldots$ gives a filtration for $\mathcal{H}_W$. Define \[ \overline{\mathcal{H}}_W = \mathcal{H}_W/\mathcal{H}_W^{-1}, \]

for $r = 0, 1, \ldots$ and $\mathcal{H}_W^{-1} = 0$. Let $\overline{\mathcal{H}}_W = \oplus_{j=0}^\infty \overline{\mathcal{H}}_W^j$. Note that $\overline{\mathcal{H}}_W$ has a natural superalgebra structure from $\mathcal{H}_W$.

Let $d_j : \overline{\mathcal{H}}_W^j \to \overline{\mathcal{H}}_W^{j+1}$ be the map induced from $d$ and let $\overline{d} = \oplus_{j=0}^\infty d_j$. For any element $h \in \mathcal{H}_W$, we still write $h$ for its corresponding element in $\overline{\mathcal{H}}_W$. Let $y_i = x_i c_i$ $(i = 1, \ldots, n)$. Let $\mathcal{Y}$ be the supersubalgebra of $\overline{\mathcal{H}}_W$ generated by all $y_i$. Note that $\overline{d}(\mathcal{Y}) \subseteq \mathcal{Y}$. Let $\overline{d}$ be the restriction of $\overline{d}$ to $\mathcal{Y}$.

In the following lemmas, one can see $\ker \overline{d}$, $\text{im} \overline{d}$, $\ker d$, $(\ker d \cap \text{im} d)_{\text{Seg}(\mathcal{W})}$ and so on are supersubspaces by using the fact that $D$ is an homogenous element.

**Lemma 3.6.** As supersubspaces of $\mathcal{Y}$, \[ \ker \overline{d} = \text{im} \overline{d} \oplus \mathbb{C}. \]

Here $\mathbb{C}$ is regarded as the $\mathbb{C}$-subalgebra of $\mathcal{Y}$ generated by 1.

**Proof.** Note that any element in $\mathcal{Y}$ can be uniquely written as a linear combination of elements of the form $p y_{i_1} y_{i_2} \ldots y_{i_r}$ for $0 < i_1 < \ldots < i_r \leq n$ and $p \in \mathbb{C}[y_1^2, \ldots, y_n^2]$. Note that $D = \sum_{i=1}^n y_i^2$. Using the relation $y_i y_j = -y_j y_i$ (in $\mathcal{Y}$) for $i \neq j$ and $y_i^2 y_j = y_j y_i^2$ (in $\mathcal{Y}$) for any $i, j$, one can see the action of $\overline{d}$ is determined by \[ \overline{d}(p y_{i_1} y_{i_2} \ldots y_{i_r}) = 2 \sum_{k=1}^r (-1)^{k-1} y_k^2 p y_{i_1} \ldots \hat{y}_k \ldots y_{i_r}, \]

where $p \in \mathbb{C}[y_1^2, \ldots, y_n^2]$. 


In order to apply the known cohomology of the Koszul complex, we identify $\mathcal{Y}$ with $\mathbb{C}[x_1, \ldots, x_n] \otimes \wedge^\bullet \mathbb{C}^n$ as vector spaces, where $\wedge^\bullet \mathbb{C}^n$ is the exterior algebra, via the linear isomorphism $\eta$ from $\mathbb{C}[x_1, \ldots, x_n] \otimes \wedge^\bullet \mathbb{C}^n$ to $\mathcal{Y}$ determined by

$$\eta: p(x_1, \ldots, x_n) \otimes e_i \wedge \ldots \wedge e_k \mapsto p(y_1^i, \ldots, y_n^i)y_1 \ldots y_k,$$

where \{e_1, \ldots, e_n\} is the standard basis of $\mathbb{C}^n$. Then, by the above description of the action of $\overrightarrow{\partial}$, the map $\eta^{-1} \circ \overrightarrow{\partial} \circ \eta$ is a multiple of the differential map in the standard Koszul resolution. Then the result follows from the well-known cohomology of the Koszul resolution.

Q.E.D.

**Proposition 3.7.** As supersubspaces of $\overline{H}_W$,

$$\ker \overrightarrow{\partial} = \text{im} \overrightarrow{\partial} \oplus \text{Seg}(W).$$

*Proof.* By the property (*) of $\mathcal{H}_W$, $x_1^{m_1}x_2^{m_2} \ldots x_n^{m_n}c_1^{\epsilon_1} \ldots c_n^{\epsilon_n}w$ ($m_i \in \mathbb{Z}_{\geq 0}$, $\epsilon_i \in \{0, 1\}$ and $w \in S_n$) form a basis for $\mathcal{H}_W$. Then $y_1^{m_1}y_2^{m_2} \ldots y_n^{m_n}c_1^{\epsilon_1} \ldots c_n^{\epsilon_n}w$ ($m_i \in \mathbb{Z}_{\geq 0}$, $\epsilon_i \in \{0, 1\}$ and $w \in S_n$) also form a basis for $\overline{H}_W$. Then as linear vector spaces, we may identify $\overline{H}_W$ with $\mathcal{Y} \otimes \text{Seg}(W)$ via the following map:

$$y_1^{m_1}y_2^{m_2} \ldots y_n^{m_n}c_1^{\epsilon_1} \ldots c_n^{\epsilon_n}w \mapsto y_1^{m_1} \ldots y_n^{m_n} \otimes c_1^{\epsilon_1} \ldots c_n^{\epsilon_n}w.$$

For any $h \in \overline{H}_W$, $\overrightarrow{\partial}(hw) = \overrightarrow{\partial}(h)w$ for $w \in W$ and $\overrightarrow{\partial}(hc_i) = \overrightarrow{\partial}(h)c_i$. Then the map $\overrightarrow{\partial}$ in $\mathcal{H}_W$ is the same as $\overrightarrow{\partial} \otimes \text{Id}$ in $\mathcal{Y} \otimes \text{Seg}(W)$ under the above identification. Then by Lemma 3.6, one has

$$\ker \overrightarrow{\partial} = (\ker \overrightarrow{\partial}) \otimes \text{Seg}(W) = (\text{im} \overrightarrow{\partial} \oplus \mathbb{C}) \otimes \text{Seg}(W) = \text{im} \overrightarrow{\partial} \oplus \text{Seg}(W).$$

Q.E.D.

For any subspace $H$ of $\mathcal{H}_W$, define $H^{\text{Seg}(W)}$ to be the set of all element commuting with elements in $\text{Seg}(W)$. If we view $\text{Seg}(W)$ as a subalgebra of $\overline{H}_W$, we could similarly define $\overline{H}^{\text{Seg}(W)}$ for any subspace $\overline{H}$ of $\overline{H}_W$. Proposition 3.7 implies the following:

**Corollary 3.8.** As supersubspaces of $\overline{H}_W$,

$$(\ker \overrightarrow{\partial})^{\text{Seg}(W)} = (\text{im} \overrightarrow{\partial})^{\text{Seg}(W)} \oplus Z(\text{Seg}(W)).$$

We say that an element $h \in \mathcal{H}_W$ supercommutes with elements in $\text{Seg}(W)$ if $wh = (-1)^{\deg(w)}hw$ for any homogenous $w \in \text{Seg}(W)$.

**Lemma 3.9.** For any element $h \in (\mathcal{H}_W)^{\text{Seg}(W)}$, if $h$ supercommutes with elements in $\text{Seg}(W)$, then $d^2(h) = 0$.

*Proof.* By Lemma 3.3, $D^2 = \Omega_{\mathcal{H}_W} - \Omega_{\text{Seg}(W)}$ for some element in $\Omega_{\text{Seg}(W)} \in Z(\text{Seg}(W))$. Then $d^2(h) = D^2h - hD^2 = (\Omega_{\mathcal{H}_W} - \Omega_{\text{Seg}(W)})h - h(\Omega_{\mathcal{H}_W} - \Omega_{\text{Seg}(W)})$. Then as $h$ supercommutes with any element in $\text{Seg}(W)$ and $\deg(\Omega_{\text{Seg}(W)}) = 2$, we have $\Omega_{\text{Seg}(W)}h = h\Omega_{\text{Seg}(W)}$. As $\Omega_{\mathcal{H}_W} \in Z(\mathcal{H}_W)$, $\Omega_{\mathcal{H}_W}h = h\Omega_{\mathcal{H}_W}$. Thus we have $d^2(h) = 0$.

Q.E.D.
Lemma 3.10. As supersubspaces of $\mathcal{H}_W$, 

$$(\ker d)^{\text{Seg}(W)} = (\ker d \cap \im d)^{\text{Seg}(W)} \oplus Z(\text{Seg}(W)).$$

Proof. It is clear that $Z(\text{Seg}(W))$ and $(\ker d \cap \im d)^{\text{Seg}(W)}$ are subspaces of $(\ker d)^{\text{Seg}(W)}$ and thus $(\ker d \cap \im d)^{\text{Seg}(W)} \oplus Z(\text{Seg}(W)) \subset (\ker d)^{\text{Seg}(W)}$. We will prove another inclusion by induction on the degree of filtration of an element in $(\ker d)^{\text{Seg}(W)}$.

Let $h$ be an element in $(\ker d)^{\text{Seg}(W)}$ such that $h \in \mathcal{H}_W^i$ and $h \notin \mathcal{H}_W^{i-1}$ for some $i$. When $i = 0$, $\mathcal{H}_W^0 = \text{Seg}(W)$ and so the statement is clearly true. Now assume $i > 0$. Let $\overline{h}$ be the image of $h$ in $\overline{\mathcal{H}}_W$. Then by Corollary 3.3, $\overline{h} = \overline{d(h_0)}$ for some unique $h_0$ in $\overline{\mathcal{H}}_W^{i-1}$ such that $\overline{d(h_0)} \in (\overline{\mathcal{H}}_W)^{\text{Seg}(W)}$. For any representative $h_0' \in \mathcal{H}_W^{i-1}$ of $\overline{h}_0$, let

$$h_0 = \frac{1}{2^n|W|} \sum_{k=1}^n \sum_{i_1 < \ldots < i_k \in W} (-1)^k (c_{i_1} \ldots c_{i_k}) w_{h_0}^{-1} (c_{i_1} \ldots c_{i_k})^{-1}.$$

Since $\overline{h}_0$ supercommutes with any element in $\text{Seg}(W)$, $h_0$ is also a representative of $\overline{h}_0$. Furthermore, $h_0$ supercommutes with elements in $\text{Seg}(W)$ and $d(h_0) \in (\mathcal{H}_W^i)^{\text{Seg}(W)}$. By Lemma 3.9, $d^2(h_0) = 0$ and so $d(h - d(h_0)) = 0$. By the induction hypothesis, $h - d(h_0) \in (\im d)^{\text{Seg}(W)} \oplus Z(\text{Seg}(W))$. Hence, we also have $h \in (\im d)^{\text{Seg}(W)} \oplus Z(\text{Seg}(W))$ since $d(h_0) \in (\im d)^{\text{Seg}(W)}$. This completes the proof.

Q.E.D.

Lemma 3.11. $(\ker d)^{\text{Seg}(W)}$ is a super subalgebra of $\mathcal{H}_W$ and $(\ker d \cap \im d)^{\text{Seg}(W)}$ is a two sided super ideal of $(\ker d)^{\text{Seg}(W)}$.

Proof. Let $z_1, z_2 \in (\ker d)^{\text{Seg}(W)}$. Then $d(z_i) = 0$ and so $Dz_i = \epsilon(z_i)D$. Now $d(z_1 z_2) = D(z_1 z_2) - \epsilon(z_1 z_2)D = \epsilon(z_1 z_2)D - \epsilon(z_1 z_2)D = 0$. Hence $z_1 z_2 \in (\ker d)^{\text{Seg}(W)}$. Hence $(\ker d)^{\text{Seg}(W)}$ is a subalgebra of $\mathcal{H}_W$.

We next show that $(\ker d \cap \im d)^{\text{Seg}(W)}$ is a two sided ideal of $(\ker d)^{\text{Seg}(W)}$. Let $z \in \ker d \cap \im d)^{\text{Seg}(W)}$ and $z' \in (\ker d \cap \im d)^{\text{Seg}(W)}$. We have to show $zz', z'z \in (\ker d \cap \im d)^{\text{Seg}(W)}$. Write $z' = Dh - \epsilon(h)D$ for some $h \in \mathcal{H}_W$. Since $d(z) = Dz - \epsilon(z)D = 0$,

$$zz' = zDh - ze(h)D = Dz(h) - ze(h)D = Dz(h) - \epsilon(ze(h))D \in \im d.$$

We also proved in the beginning that $zz' \in \ker d$ and thus $zz' \in (\ker d \cap \im d)^{\text{Seg}(W)}$. The proof for $z'z \in (\ker d \cap \im d)^{\text{Seg}(W)}$ is similar.

Q.E.D.

Proof of Theorem 3.4. Since $z \in Z(\mathcal{H}_W)$, by Lemma 3.10 there exists a unique $\overline{z} \in Z(\text{Seg}(W))$ such that $z - \overline{z} \in (\ker d \cap \im d)^{\text{Seg}(W)} \subset \im d$.

It remains to prove that $\zeta$ is a superalgebra homomorphism. First $\zeta$ preserves the $\mathbb{Z}_2$-grading since the decomposition in Lemma 3.10 is between super vector spaces. To see $\zeta$ is an algebra map, let $z_1 \in Z(\mathcal{H}_W)$ and $h_1 \in (\ker d \cap \im d)^{\text{Seg}(W)}$. Then $z_1 h_1 = \zeta(z_1)\zeta(h_1) + \zeta(z_1)h_1 + \zeta(h_1)z_1$. By Lemma 3.11 $z_1 h_1 - \zeta(z_1)\zeta(h_1) \in (\ker d \cap \im d)^{\text{Seg}(W)}$. Thus $\zeta(z_1 h_1) = \zeta(z_1)\zeta(h_1)$. This completes the proof.
Proof of Theorem 3.3. By our hypothesis, there exists a non-zero element \( v \in H_D(X) \) such that \( v \) is in the isotypic component \( U \) of \( H_D(X) \). Let \( \tilde{v} \) be a representative of \( v \) in \( \ker \pi(D) \). Now by Theorem 3.4 for any \( z \in Z(H_W)_0 \), \( z - \zeta(z) = Da - c(a)D \) for some \( a \in H_W \). Then \( \pi(z - \zeta(z))\tilde{v} = \pi(Da - c(a)D)\tilde{v} = \pi(Da)\tilde{v} \in \operatorname{im} \pi(D) \). On another hand, \( \pi(z - \zeta(z))\tilde{v} = \chi_\sigma(z)\tilde{v} - (\chi_\sigma(\zeta(z))\tilde{v} + \tilde{v}') \) for some \( \tilde{v}' \in \ker \pi(D) \cap \operatorname{im} \pi(D) \) and so \( (\chi_\pi(z) - \chi_\sigma(\zeta(z)))\tilde{v} \in \ker \pi(D) \). We also have \( (\chi_\pi(z) - \chi_\sigma(\zeta(z)))\tilde{v} \in \ker \pi(D) \) as \( \tilde{v} \in \ker \pi(D) \). Thus \( \chi_\pi(z) = \chi_\sigma(\zeta(z)) = \chi_\sigma'(z) \). This completes the proof.

4. Dirac cohomology in \( \mathbb{H}_n^C \)

We specialize the setting to the case of \( \mathbb{H}_n^C \) in this section. We use the notation in Section 2.9. Readers should not mix up with the notation in Section 3.

4.1. Some commutation relations. The main statement of this subsection is Proposition 4.3, which says \( \mathbb{H}_n^C \) satisfies property (*) defined in Definition 3.1.

Let \( \tilde{s}_\alpha = s_\alpha c_{\alpha} \). For later convenience, we also set \( \tilde{s}_{ij} = \tilde{s}_{\alpha_{ij}} = s_{\alpha_{ij}} c_{\alpha_{ij}}, \ y_i = x_i c_i, \ y'_i = y_i + \sqrt{2} \sum_{i \neq j} \tilde{s}_{ij} \) and \( x'_i = y'_i c_i \). Note that \( \mathbb{C}[S_n]^- \) embeds into \( \mathbb{H}_n^C \) via the map \( \ell_\alpha \mapsto \tilde{s}_\alpha \).

Lemma 4.1. \begin{enumerate}
\item[(1)] \( c_i y_j = -y_j c_i \) for any \( i, j \);  
\item[(2)] \( \tilde{s}_{ij} c_k = -c_k \tilde{s}_{ij} \) for any \( i, j, k \);  
\item[(3)] \( c_i y'_j = -y'_j c_i \) for any \( i, j \);  
\item[(4)] For \( \alpha \in \mathbb{R}^+ \) and \( w \in S_n \), \( w \tilde{s}_\alpha w^{-1} = \tilde{s}_{w(\alpha)} \) if \( w(\alpha) > 0 \), and \( w \tilde{s}_\alpha w^{-1} = -\tilde{s}_{-w(\alpha)} \) if \( w(\alpha) < 0 \).
\end{enumerate}

The above lemma is elementary. We skip the proof.

For \( w \in S_n \), define \( l(w) = | \{ e_i - e_j \in \mathbb{R}^+ : w(e_i - e_j) < 0 \} | \).

Lemma 4.2. Let \( w \in S_n \). Then
\[
wy_i w^{-1} - y_w(i) = \sqrt{2} \sum_{\beta > 0, w^{-1}(\beta) < 0, (\beta, w(e_i)) \neq 0} \tilde{s}_\beta.
\]
In particular, for \( \alpha \in \mathbb{R}^+ \),
\[
\tilde{s}_\alpha y_i \tilde{s}_\alpha^{-1} + y_{s_\alpha(i)} = -\sqrt{2} \sum_{\beta > 0, s_\alpha^{-1}(\beta) < 0, (\beta, w(e_i)) \neq 0} \tilde{s}_\beta.
\]

Proof. When \( l(w) = 1 \), \( w = s_\alpha \) for some \( \alpha \in \Delta \). We consider three cases. When \( \langle e_i, \alpha \rangle = 0 \), it is easy to see \( s_\alpha y_i s_\alpha - y_i = 0 \). Now consider the case \( \langle e_i, \alpha \rangle = 1 \). In this case, we have
\[
s_\alpha y_i s_\alpha = s_\alpha x_i c_i s_\alpha = x_i c_i + (-1 + c_i c_i) c_i s_\alpha = x_i + c_i c_i s_\alpha = x_i + s_\alpha(c_i - c_i + 1) = y_i + \sqrt{2} \tilde{s}_\alpha.
\]
For \( \langle e_i, \alpha \rangle = -1 \), by using \( s_\alpha s_\alpha = -s_\alpha \) and the computation in the case \( \langle e_i, \alpha \rangle = 1 \), we have
\[
s_\alpha y_{i+1} s_\alpha = y_i + \sqrt{2}s_\alpha.
\]
We now use an induction on \( l(w) \). Assume \( l(w) = k \) for some \( k > 1 \). Write \( w = s_\alpha w' \) for some simple reflection \( s_\alpha \) and \( w' \in W \) with \( l(w') = k - 1 \). Set \( \epsilon = 1 \) if \( \langle \alpha, w(e_i) \rangle \neq 0 \) and \( \epsilon = 0 \) otherwise. Then
\[
wy_{i}^{-1} = s_\alpha w' y_{i} w'^{-1} s_\alpha
\]
\[
= s_\alpha y_{w'(i)} s_\alpha + \sqrt{2} \sum_{\beta > 0, w'^{-1}(\beta) < 0, \langle \beta, w'(e_i) \rangle \neq 0} s_\alpha s_\beta s_\alpha \quad \text{(induction hypothesis)}
\]
\[
= y_{s_\alpha w'(i)} + \epsilon \sqrt{2}s_\alpha + \sqrt{2} \sum_{\beta > 0, w'^{-1}(\beta) < 0, \langle \beta, w'(e_i) \rangle \neq 0} s_\beta
\]
\[
= y_{s_\alpha w'(i)} + \epsilon \sqrt{2}s_\alpha + \sqrt{2} \sum_{\beta > 0, w'^{-1}(\beta) < 0, \langle \beta, s_\alpha w'(e_i) \rangle \neq 0} s_\beta
\]
This proves the first assertion. The second assertion follows from the first one with the equation that
\[
s_\alpha y_{i} s_\alpha^{-1} = s_\alpha c_\alpha y_{i} (-c_\alpha s_\alpha) = s_\alpha (c_\alpha^2) y_{i} s_\alpha = -s_\alpha y_{i} s_\alpha.
\]
Q.E.D.

Lemma 4.3.

\[
[x'_i, x'_j] c_i c_j = y'_i y'_j + y'_j y'_i \in \text{Seg}_n.
\]

Proof.
\[
y'_i y'_j + y'_j y'_i
\]
\[
= (y_i + \sqrt{2} \sum_{k \neq i} \bar{s}_{i,k})(y_j + \sqrt{2} \sum_{l \neq j} \bar{s}_{l,j}) + (y_j + \sqrt{2} \sum_{l \neq j} \bar{s}_{l,j})(y_i + \sqrt{2} \sum_{k \neq i} \bar{s}_{i,k})
\]
\[
= y_i y_j + y_j y_i + \sqrt{2} \left( \sum_{k \neq i} \bar{s}_{i,k} y_j + y_j \sum_{l \neq j} \bar{s}_{l,j} + y_i \sum_{l \neq j} \bar{s}_{l,j} + \sum_{l \neq j} \bar{s}_{l,j} y_j \right)
\]
\[
+ \frac{1}{2} \left( \sum_{k \neq i} \bar{s}_{i,k} \sum_{l \neq j} \bar{s}_{l,j} + \sum_{l \neq j} \bar{s}_{l,j} \sum_{i \neq k} \bar{s}_{i,k} \right)
\]
\[
= \frac{\sqrt{2}}{2} \left( \sum_{k \neq i} \bar{s}_{i,k} y_j + y_j \sum_{l \neq j} \bar{s}_{l,j} + y_i \sum_{l \neq j} \bar{s}_{l,j} + \sum_{l \neq j} \bar{s}_{l,j} y_j \right) + \frac{1}{2} \left( \sum_{l \neq j} \sum_{k \neq i} \bar{s}_{l,j} \bar{s}_{i,k} + \sum_{l \neq j} \sum_{i \neq k} \bar{s}_{i,k} \bar{s}_{l,j} \right)
\]
By Lemma 4.2 the term \( \frac{\sqrt{2}}{2} \left( \sum_{k \neq i} \bar{s}_{i,k} y_j + y_j \sum_{k \neq i} \bar{s}_{i,k} + y_i \sum_{l \neq j} \bar{s}_{l,j} + \sum_{l \neq j} \bar{s}_{l,j} y_j \right) \) is in \( \text{Seg}_n \). This completes the proof.

Q.E.D.
Lemma 4.4.  

(1) \(wx'_iw^{-1} = x'_{w(i)}\);

(2) \(c_ix'_i = -c_ix'_i\) and \(c_jx'_i = x'_ic_j\) for \(i \neq j\).

Proof. For (1), it suffices to show when \(w = s_\alpha\) for some \(\alpha \in \Delta\). By the definition of \(x'_i\), it suffices to show \(s_\alpha y'_i s_\alpha = y'_i s_{s_\alpha(i)}\). We consider two cases. In the case that \(\langle e_i, \alpha \rangle = 0\), \(s_\alpha(s_{i,j} s_\alpha) > 0\) for any \(j \neq i\). Then \(s_\alpha s_{i,j} s_\alpha = s_{i,s_{\alpha}(j)}\) for any \(j \neq i\). Thus, the last equality becomes

\[
s_\alpha y'_i s_\alpha^{-1} = y_i + \frac{\sqrt{2}}{2} \sum_{j \neq i} s_{i,s_{\alpha}(j)} = y'_i
\]

In the case that \(\langle e_i, \alpha \rangle \neq 0\), let \(k = i - 1\) or \(i + 1\) such that \(\alpha = \alpha_{i,k}\). Then, by Lemmas 4.3 and 4.4,

\[
(4.9) \quad s_\alpha y'_i s_\alpha^{-1} = y_k + \frac{\sqrt{2}}{2} \sum_{j \neq k} s_{k,j}
\]

(4.10) \[= y'_k\]

(4.11)

For (2), it is straightforward from Lemma 4.1 and \(y'_i = x'_ic_i\).

Q.E.D.

Proposition 4.5. The degenerate affine Hecke-Clifford algebra \(HCl_n\) satisfies the property (*) in Definition 3.1.

Proof. By setting \(W\) in Definition 3.1 equal to \(S_n\) and replacing \(x_i\) in Definition 3.1 with \(x'_i\) defined in the beginning of Section 4.1, the proposition follows from Lemmas 4.3 and 4.4.

Q.E.D.

4.2. Dirac element \(D\). With the notation in Section 4.1, the Dirac element \(D\) for \(HCl_n\) is defined as

\[
(4.12) \quad D = \sum_{i=1}^{n} x'_ic_i.
\]

Using the expressions in Section 4.1, we also have

\[
D = \sum_{i=1}^{n} x_ic_i + \sqrt{2} \sum_{\alpha > 0} s_\alpha c_\alpha = \sum_{i=1}^{n} y_i + \sqrt{2} \sum_{\alpha > 0} \tilde{s}_\alpha.
\]

Lemma 4.6.

\[
\left( \sum_{\alpha > 0} \tilde{s}_\alpha \right)^2 = \sum_{\alpha > 0, \beta > 0, s_\alpha(\beta) < 0} \tilde{s}_\alpha \tilde{s}_\beta.
\]

Proof. It suffices to show that

\[
\sum_{\alpha > 0, \beta > 0, s_\alpha(\beta) > 0} \tilde{s}_\alpha \tilde{s}_\beta = 0.
\]
Set \( \tilde{R} = \{ (\alpha, \beta) \in R^+ \times R^+ : s_{\alpha}(\beta) > 0 \} \). Note that for any \((\alpha, \beta) \in \tilde{R}\), either \(s_\beta(\alpha) < 0\) or \(s_{s_\alpha(\beta)}(\alpha) < 0\). We define a map \( \iota : \tilde{R} \to \tilde{R} \) such that
\[
\iota(\alpha, \beta) = \begin{cases} 
(\beta, s_\beta(\alpha)) & \text{if } s_\beta(\alpha) > 0 \\
(s_\alpha(\beta), \alpha) & \text{if } s_{s_\alpha(\beta)}(\alpha) > 0
\end{cases}
\]
It is not hard to verify \( \iota \) is well-defined and is an involution. For \( \iota(\alpha, \beta) = (\alpha', \beta') \), one can also check that \( \tilde{s}_\alpha \tilde{s}_\beta + \tilde{s}_{\alpha'} \tilde{s}_{\beta'} = 0 \). Thus each term \( \tilde{s}_\alpha \tilde{s}_\beta \) in the expression
\[
\sum_{\alpha>0, \beta>0, s_\alpha(\beta)>0} \tilde{s}_\alpha \tilde{s}_\beta
\]
can be paired with another one and gets canceled. This proves the expression is zero.

Q.E.D.

We restate Lemma 3.3 for the case of \( H_{\text{Cl}}^n \) in an explicit form below. This is an analogue of \([1, \text{Theorem 2.11}]\).

Theorem 4.7.
\[
D^2 = \Omega_{H_{\text{Cl}}^n} - \Omega_{\text{Seg}^n},
\]
where
\[
\Omega_{H_{\text{Cl}}^n} = \sum_{i=1}^{n} x_i^2,
\]
\[
\Omega_{\text{Seg}^n} = 2 \sum_{\alpha>0, \beta>0, s_\alpha(\beta)<0} \tilde{s}_\alpha \tilde{s}_\beta.
\]

Proof. By Lemma 4.2, for any \( \alpha \in R^+ \),
\[
\sum_{i=1}^{n} y_i \tilde{s}_\alpha + \tilde{s}_\alpha \sum_{i=1}^{n} y_i = -2 \sqrt{2} \sum_{\beta>0, s_\alpha(\beta)<0} \tilde{s}_\alpha \tilde{s}_\beta
\]
Now, by (4.13) and Lemma 4.6,
\[
D^2 = \left( \sum_{i=1}^{n} y_i + \sqrt{2} \sum_{\alpha} \tilde{s}_\alpha \right)^2
\]
\[
= \left( \sum_{i=1}^{n} y_i \right)^2 + 2 \sqrt{2} \sum_{i=1}^{n} y_i \sum_{\alpha} \tilde{s}_\alpha + \sqrt{2} \sum_{\alpha} \tilde{s}_\alpha \sum_{i=1}^{n} y_i + 2 \left( \sum_{\alpha>0} \tilde{s}_\alpha \right)^2
\]
\[
= \sum_{i=1}^{n} x_i^2 - 2 \sum_{\alpha>0, \beta>0, s_\alpha(\beta)<0} \tilde{s}_\alpha \tilde{s}_\beta
\]
Q.E.D.

The conclusion of this section is a version of Theorem 3.5 in the specific case of \( H_{\text{Cl}}^n \).

Theorem 4.8. Let \((\pi, X)\) be an irreducible supermodule of \( H_{\text{Cl}}^n \) with the central character \( \chi_{\pi} \) (Definition 3.18). Let \( D \) be the Dirac element in \( H_{\text{Cl}}^n \) in (4.12). Define the Dirac cohomology \( H_D(X) \) as in Theorem 3.5. Then \( H_D(X) \) has a natural \( \text{Seg}^n \)-module structure.

Suppose
\[
\text{Hom}_{\text{Seg}^n}(U, H_D(X)) \neq 0,
\]
for some \( \text{Seg}^n \)-module \((\sigma, U)\). Then \( \chi_{\pi} = \chi^\sigma \), where \( \chi^\sigma \) is defined as in (3.8).
Proof. This immediately follows from Theorem 3.8 and Proposition 4.5.

Q.E.D.

5. Spectrum of the Dirac operator

In this section, we compute the action of $D$ on some interesting $H_n^{CI}$-modules. Thus, we shall see Theorem 4.8 for $H_n^{CI}$ has interesting consequences. We keep using the notations in Section 4.

5.1. Steinberg type module and induced modules. Let us recall the construction of some $H_n^{CI}$-modules in [6]. Fix a partition $\lambda = (n_1, n_2, \ldots, n_r)$ of $n$. Let $S_{\lambda}$ be the subgroup of $S_n$ generated by $s_{i,i+1}$ for $i = \{1, \ldots, n\} \setminus \{n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_r\}$. It is easy to see that $S_{\lambda}$ is isomorphic to $S_{n_1} \times \ldots \times S_{n_r}$. Let $H_n^{CI}_{\lambda}$ be the subalgebra of $H_n^{CI}$ generated by all $w \in S_{\lambda}$, $x_i$ ($i = 1, \ldots, n$) and $c_i$ ($i = 1, \ldots, n$). Let $\text{Seg}_{\lambda}$ be the subalgebra of $H_n^{CI}_{\lambda}$ generated by all $w \in S_{\lambda}$ and $c_i$ ($i = 1, \ldots, n$).

We define an $H_n^{CI}_{\lambda}$-module $\tilde{\text{St}}_{\lambda}$, which is identified with $\text{Cl}_n$ as vector spaces. The action of $H_n^{CI}_{\lambda}$ is determined by the following:

\begin{align*}
  c_i.1 &= c_i, \\
  s_{\alpha}.1 &= 1,
\end{align*}

where $1$ is the identity in $\text{Cl}_n$ and for $i \in \{n_{k-1} + 1, \ldots, n_k\}$ ($k = 1, \ldots, r$ and $n_0 = 0$)

\[ x_i.v = \sum_{n_{k-1} + 1 \leq j < i \leq n_k} s_{i,j}(1 - c_ic_j).v, \]

where $v$ is any vector in $\text{Cl}_n$ and the actions of $s_{i,j}$ and $c_i, c_j$ are the ones defined in (5.14) and (5.15). When $\lambda = (n)$, we will simply write $\tilde{\text{St}}$ for $\tilde{\text{St}}_{(n)}$. The notation $\tilde{\text{St}}$ stands for a Steinberg type module as it performs the role of Steinberg module in the degenerate affine Hecke algebra. It is straightforward to check the above actions defines an $H_n^{CI}_{\lambda}$-module by verifying the defining relations of $H_n^{CI}_{\lambda}$. Some details can be found in [6 Proposition 4.1.1].

Lemma 5.1. The element $x_i^2$ acts on $\tilde{\text{St}}_{\lambda}$ by a scalar $(i - n_{k-1})(i - n_k)$ where $k = 0, \ldots, r - 1$ and $i = n_k + 1, \ldots, n_{k+1}$.

Proof. Direct computation or see [6 Proposition 4.1.1].

Q.E.D.

Define the Dirac type element $D_{\lambda}$ in $H_n^{CI}_{\lambda}$ as:

\[ D_{\lambda} = \sum_{i=1}^{n} y_i + \sqrt{2} \sum_{\alpha \in R^+_n} \tilde{s}_{\alpha}. \]

Proposition 5.2. The element $D_{\lambda}$ acts as zero on the $H_n^{CI}_{\lambda}$-module $\tilde{\text{St}}_{\lambda}$. In particular, $D$ acts as zero on $\tilde{\text{St}}$. 

The central character of the information discussed in the next subsections.

To compute the Dirac cohomology of the above induced modules, we need some more

Proof. The assertion follows from the following computation:

\[
\pi(D_\lambda)v = \sum_{k=1}^{r-1} \sum_{n_k+1 \leq i < n_{k+1}} s_{ij}(1 - c_i c_j) c_i v + \sqrt{2} \sum_{\alpha \in R_\lambda^+} \tilde{s}_\alpha v
\]

\[
= \sum_{k=1}^{r-1} \sum_{n_k+1 \leq i < n_{k+1}} s_{ij}(c_i - c_j)v + \sqrt{2} \sum_{\alpha \in R_\lambda^+} \tilde{s}_\alpha v
\]

\[
= \left( -\sqrt{2} \sum_{k=1}^{r-1} \sum_{n_k+1 < i < n_{k+1}} \tilde{s}_{ji} + \sqrt{2} \sum_{\alpha \in R_\lambda^+} \tilde{s}_\alpha \right) v
\]

\[
= 0
\]

Q.E.D.

Define

(5.16) \[ X_\lambda = \text{Ind}_{St_\lambda}^{\mathbb{H}^C_n} \tilde{St}_\lambda = \mathbb{H}^C_n \otimes_{\tilde{S}_\lambda} \tilde{St}_\lambda \]

with the map \( \pi_\lambda \) defining the action of \( \mathbb{H}^C_n \) on \( X_\lambda \). Note that \( X_\lambda \) is quasisimple (Definition 2.15). The central character of \( X_\lambda \) can be represented by

\[(1(1-1), \ldots, n_1(n_1-1), \ldots, 1(1-1), \ldots, n_r(n_r-1)) \in \mathbb{R}^n.\]

To compute the Dirac cohomology of the above induced modules, we need some more information discussed in the next subsections.

5.2. \( S_n \)-structure and \( \text{Seg}_n \)-structure of \( (\pi_\lambda, X_\lambda) \). We continue to fix a partition \( \lambda \) of \( n \). Recall that in Definition 2.15(1) \( \mathbb{H}^C_n \) contains \( \mathbb{C}[S_n] \) as a subalgebra. Let \( (\pi_V, V = \mathbb{C}^n) \) be the \( S_n \)-representation such that elements in \( S_n \) permute the coordinates.

Lemma 5.3. The restriction of \( X_\lambda \) to \( \mathbb{C}[S_n] \) is isomorphic to

\[ \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{Res}_{\mathbb{C}[S_\lambda]}^{\mathbb{C}[S_n]} \left( \bigoplus_{i=0}^{n} \wedge^i V \right), \]

as \( \mathbb{C}[S_n] \)-modules.

Proof. Note that the restriction of \( \tilde{St}_\lambda \) to \( \mathbb{C}[S_n] \) is isomorphic to \( \text{Res}_{\mathbb{C}[S_\lambda]}^{\mathbb{C}[S_n]}(\bigoplus_{i=0}^{n} \wedge^i V) \). Then \( \mathbb{H}^C_n \otimes_{\tilde{S}_\lambda} \tilde{St}_\lambda \) and \( \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{Res}_{\mathbb{C}[S_\lambda]}^{\mathbb{C}[S_n]} \left( \bigoplus_{i=0}^{n} \wedge^i V \right) \) are isomorphic as \( \mathbb{C}[S_n] \)-modules.

Q.E.D.

It is well-known that we have the following \( \mathbb{C}[S_n] \)-isomorphism:

\[ \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{Res}_{S_\lambda}^{S_n} \left( \bigoplus_{i=0}^{n} \wedge^i V \right) \cong \left( \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_\lambda]} \text{triv} \right) \otimes \bigoplus_{i=0}^{n} \wedge^i V \]

Here the module in the right hand side is viewed as the tensor product of two \( S_n \)-representations. The isomorphism is given by

\[ w \otimes (v_1 \wedge \ldots \wedge v_i) \mapsto (w \otimes 1) \otimes (\pi_V(w)v_1 \wedge \ldots \wedge \pi_V(w)v_i). \]
Note that the space $\oplus_{i=1}^n \Lambda^i V$ can be identified with $\text{Cl}_n$ via the map determined by 
$$e_1 \wedge \ldots \wedge e_i \mapsto c_1 \ldots c_i,$$
where $\{e_1, \ldots, e_n\}$ is the standard basis of $V = \mathbb{C}^n$. Thus $X_\lambda = \text{Ind}_{\text{Cl}_n}^{\mathbb{H}^{\text{CI}}_{\tilde{\text{St}}}} \tilde{\text{St}}_\lambda$ can be identified with, as vector spaces, $(\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_n]} \text{triv}) \otimes \text{U}_{\text{Cl}_n}$ via the identification in Lemma 5.3 and the above identification between $\oplus_{i=1}^n \Lambda^i V$ and $\text{Cl}_n$. Then if we translate the action of the subalgebra $\text{Seg}_n$ under the above identifications, then we have:
$$\pi_\lambda(w)(u' \otimes 1 \otimes c_{i_1} \ldots c_{i_r}) = uw' \otimes 1 \otimes c_{w(i_1)} \ldots c_{w(i_r)},$$
$$\pi_\lambda(c_i)(u' \otimes 1 \otimes c_{i_1} \ldots c_{i_r}) = u' \otimes 1 \otimes c_i c_{i_1} \ldots c_{i_r}.$$ 

We have just proven that:

**Lemma 5.4.** As $\text{Seg}_n$-supermodules,

$$\text{Res}^{\mathbb{H}^{\text{CI}}_{\text{Seg}_n}}_{\text{Seg}_n} X_\lambda = (\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_n]} \text{triv}) \otimes \text{U}_{\text{Cl}_n},$$

where the supermodule in the right hand side has the $\text{Seg}_n$-supermodule structure described in Lemma 2.13.

Recall that $F$ is the functor defined in Section 2.8.

**Proposition 5.5.** As $\text{Seg}_n$-supermodules,

$$\text{Res}^{\mathbb{H}^{\text{CI}}_{\text{Seg}_n}}_{\text{Seg}_n} X_\lambda = F((\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_n]} \text{triv}) \otimes \text{U}_{\text{spin}}),$$

where $(\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_n]} \text{triv}) \otimes \text{U}_{\text{spin}}$ has $\mathbb{C}[\tilde{S}_n]$-supermodule described in Lemma 2.18.

**Proof.** By Lemma 2.14 it suffices to show

$$(\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_n]} \text{triv}) \otimes \text{U}_{\text{Cl}_n} = (\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_n]} \text{triv}) \otimes \text{U}_{\text{spin}} \otimes \text{U}(n).$$

By Lemma 2.14 there is a $\text{Seg}_n$-module isomorphism $f$ from $\text{U}_{\text{Cl}_n}$ to $F(\text{U}_{\text{spin}}) = \text{U}_{\text{spin}} \otimes \text{U}(n)$. Then define a vector space isomorphism from $\text{Seg}_n$-module $(\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_n]} \text{triv}) \otimes \text{U}_{\text{Cl}_n} \rightarrow (\mathbb{C}[S_n] \otimes_{\mathbb{C}[S_n]} \text{triv}) \otimes \text{U}_{\text{spin}} \otimes \text{U}(n)$ determined by

$$(w \otimes 1) \otimes (c_{i_1} \ldots c_{i_r} \otimes 1) \mapsto (w \otimes 1) \otimes f(c_{i_1} \ldots c_{i_r} \otimes 1).$$

Using the module structure described before Lemma 5.4, one can check the linear isomorphism is $\text{Seg}_n$-equivariant.

Q.E.D.

5.3. **Hermitian form on** $(\pi_\lambda, X_\lambda)$. We continue to fix a partition $\lambda$ of $n$. In this subsection, we shall construct a Hermitian form on the $\mathbb{H}^{\text{CI}}_{\tilde{\text{St}}}$-module $(\pi_\lambda, X_\lambda)$ such that the adjoint operator of $\pi_\lambda(D)$ with respect to such form is $-\pi_\lambda(D)$. We will see this makes the computation for the Dirac cohomology $H_D(X)$ of those modules $X$ much easier.

Recall that $\text{Seg}_\lambda$ is a subalgebra of $\mathbb{H}^{\text{CI}}_{\lambda}$. 

**Lemma 5.6.** There exists a $\text{Seg}_\lambda$-invariant positive definite Hermitian form on $\tilde{\text{St}}_\lambda$. 


Proof. Since \( \text{Res}_{\text{Seg}_\lambda}^{\text{Seg}_n} \tilde{\text{St}}_\lambda = \text{Res}_{\text{Seg}_n}^{\text{Seg}_\lambda} U_{\text{Cl}_n} \) as Seg\(\lambda\)-modules, it suffices to consider the case when \( \lambda = (n) \). Recall that \( U_{\text{Cl}_n} = \text{Seg}_n \otimes \mathbb{C}[S_n] \text{triv} \) in Section 2.3. Define \( \langle ., . \rangle : U_{\text{Cl}_n} \times U_{\text{Cl}_n} \to \mathbb{C} \) such that for \( 1 \leq i_1 < \cdots < i_r \leq n \) and \( 1 \leq j_1 < \cdots < j_s \leq n \),

\[
\langle c_{i_1} c_{i_2} \ldots c_{i_r} \otimes 1, c_{j_1} c_{j_2} \ldots c_{j_s} \otimes 1 \rangle = \begin{cases} 1 & \text{if } \{i_1, \ldots, i_r\} = \{j_1, \ldots, j_s\} \\ 0 & \text{otherwise} \end{cases}
\]

It is straightforward to check \( \langle ., . \rangle \) satisfies the desired properties. 

Q.E.D.

We denote the Seg\(\lambda\)-invariant Hermitian form on \( \tilde{\text{St}}_\lambda \) in the above lemma by \( \langle ., . \rangle_{\lambda} \). Recall that \( X_\lambda = \mathbb{H}_n^{CI} \otimes_{\mathbb{R}_\lambda^{CI}} \tilde{\text{St}}_\lambda \). We define a bilinear form \( \langle ., . \rangle \) on \( X_\lambda \) characterized by:

\[
\langle w_1 \otimes v_1, w_2 \otimes v_2 \rangle = \delta_{w_1 S_\lambda w_2 S_\lambda} (\lambda(w_2^{-1}w_1)v_1, v_2)_{\lambda}
\]

where \( w_1, w_2 \in S_n \) and \( \delta_{w_1 S_\lambda w_2 S_\lambda} = 1 \) if \( w_1 S_\lambda = w_2 S_\lambda \) and \( \delta_{w_1 S_\lambda w_2 S_\lambda} = 0 \) otherwise.

Lemma 5.7. \( \langle ., . \rangle \) defined above is a positive definite Hermitian form.

Proof. This follows from the property that \( \langle ., . \rangle_{\lambda} \) is positive definite and Hermitian.

Q.E.D.

We next compute the adjoint operator of \( \pi_\lambda(D_\lambda) \) with respect to \( \langle ., . \rangle \). We begin with some lemmas.

Lemma 5.8. For \( v_1, v_2 \in \tilde{\text{St}}_\lambda \), \( \langle D \otimes v_1, 1 \otimes v_2 \rangle = \langle 1 \otimes v_1, D \otimes v_2 \rangle = 0 \).

Proof. Since for \( \alpha \in R^+ \setminus R^+_\lambda \),

\[
\langle \tilde{s}_\alpha \otimes v_1, 1 \otimes v_2 \rangle = 0,
\]

Then since

\[
D = D_\lambda + \sqrt{2} \sum_{\alpha > 0, \alpha \in R^+ \setminus R^+_\lambda} \tilde{s}_\alpha,
\]

\( \langle D \otimes v_1, 1 \otimes v_2 \rangle = \langle D_\lambda, v_1, v_2 \rangle_{\lambda} \). Then we have \( \langle D \otimes v_1, 1 \otimes v_2 \rangle = 0 \) by Proposition 5.2. The proof for \( \langle 1 \otimes v_1, D \otimes v_2 \rangle = 0 \) is similar.

Q.E.D.

Lemma 5.9. Suppose \( \beta_1 \neq \beta_2 \) and \( \beta_1, \beta_2 \in R^+ \setminus R^+_\lambda \). Then \( s_{\beta_1} s_{\beta_2} \notin S_\lambda \).

Proof. In the following, we implicitly use several facts that the theory of element in \( S_\lambda \) cannot send a positive root not in \( R_\lambda \) to a negative root. If \( \langle \beta_1, \beta_2 \rangle = 0 \), then \( s_{\beta_1} s_{\beta_2}(\beta_2) = -\beta_2 < 0 \). Since \( \beta_2 \notin R_\lambda \), \( s_{\beta_1} s_{\beta_2} \notin S_\lambda \). If \( \langle \beta_1, \beta_2 \rangle = -1 \), then \( s_{\beta_2}(\beta_1) = \beta_1 + \beta_2 > 0 \). Moreover, \( s_{\beta_1} s_{\beta_2} s_{\beta_1}(\beta_1) = -\beta_1 < 0 \). Since \( \beta_1 + \beta_2 \notin R_\lambda \), \( s_{\beta_1} s_{\beta_2} \notin S_\lambda \). If \( \langle \beta_1, \beta_2 \rangle = 1 \), then either \( s_{\beta_1}(\beta_2) > 0 \) or \( s_{\beta_2}(\beta_1) > 0 \). In the case that \( s_{\beta_1}(\beta_2) > 0 \), \( s_{\beta_1} s_{\beta_2}(\beta_2) = -s_{\beta_1}(\beta_2) < 0 \). Then since \( \beta_2 \notin R_\lambda \), \( s_{\beta_1} s_{\beta_2} \notin S_\lambda \). Similar argument by considering \( (s_{\beta_1} s_{\beta_2})^{-1} \) can prove another case.
Proposition 5.10. The adjoint operator of $\pi_\lambda(D)$ with respect to $\langle \cdot, \cdot \rangle$ is $-\pi_\lambda(D)$.

Proof. It suffices to show that

$$\langle Dw_1 \otimes v_1, w_2 \otimes v_2 \rangle = \langle w_1 \otimes v_1, -Dw_2 \otimes v_2 \rangle$$

for any $w_1, w_2 \in S_\lambda$ and $v_1, v_2 \in X_\lambda$. To this end, we consider two cases. Suppose $w_1S_\lambda = w_2S_\lambda$. Then,

$$\langle Dw_1 \otimes v_1, w_2 \otimes v_2 \rangle = \langle w_2^{-1}Dw_1 \otimes v_1, 1 \otimes v_2 \rangle = \langle Dw_2^{-1}w_1 \otimes v_1, 1 \otimes v_2 \rangle = \langle D \otimes (w_2^{-1}w_1), v_1, 1 \otimes v_2 \rangle = 0 \quad \text{(by Lemma 5.8)}$$

Similarly, we also have

$$\langle w_1 \otimes v_1, Dw_2 \otimes v_2 \rangle = 0.$$

and so $\langle Dw_1 \otimes v_1, w_2 \otimes v_2 \rangle = \langle w_1 \otimes v_1, -Dw_2 \otimes v_2 \rangle$.

Now we suppose that $w_1S_\lambda \neq w_2S_\lambda$. Without loss of generality, assume that $w_2^{-1}w_1$ is a minimal representative in $w_2^{-1}w_1S_\lambda$.

$$\langle w_2^{-1}w_1D \otimes v_1, 1 \otimes v_2 \rangle = \langle w_2^{-1}w_1, \sqrt{2} \sum_{\alpha > 0} \bar{s}_\alpha \otimes v_1, 1 \otimes v_2 \rangle = \langle 1 \otimes v_1, \sqrt{2} \sum_{\alpha > 0} \bar{s}_\alpha w_1^{-1}w_2 \otimes v_2 \rangle = -(1 \otimes v_1, Dw_1^{-1}w_2 \otimes v_2) + \langle 1 \otimes v_1, w_1^{-1}w_2 D \otimes v_2 \rangle + \langle 1 \otimes v_1, \sqrt{2} \sum_{\alpha > 0} \bar{s}_\alpha w_1^{-1}w_2 \otimes v_2 \rangle$$

It remains to show

$$\langle 1 \otimes v_1, w_1^{-1}w_2 D \otimes v_2 \rangle + \sqrt{2}(1 \otimes v_1, \sum_{\alpha > 0} \bar{s}_\alpha w_1^{-1}w_2 \otimes v_2) = 0.$$

By Lemma 5.5, there exists at most one $\beta \in R^+ \setminus R_\lambda$ such that $w_1^{-1}w_2s_\beta \in S_\lambda$. If such $\beta$ does not exist, then the two terms in the left hand side of the above equation are both zero and so the equation holds. If such unique $\beta$ exists, let $\beta' = -w_1^{-1}w_2(\beta)$. Note that $\beta' > 0$ otherwise $w_1^{-1}w_2s_\beta \notin S_\lambda$. Then

$$\langle 1 \otimes v_1, w_1^{-1}w_2 D \otimes v_2 \rangle + \sqrt{2}(1 \otimes v_1, \sum_{\alpha > 0} \bar{s}_\alpha w_1^{-1}w_2 \otimes v_2) = \sqrt{2}(1 \otimes v_1, \bar{s}_\beta w_1^{-1}w_2 \otimes v_2) + \sqrt{2}(1 \otimes v_1, \bar{s}_\beta' w_1^{-1}w_2 \otimes v_2) \quad \text{(by definition of $D$ and $\langle \cdot, \cdot \rangle$)}$$

$$= -\sqrt{2}(1 \otimes v_1, \bar{s}_\beta w_1^{-1}w_2 \otimes v_2) + \sqrt{2}(1 \otimes v_1, \bar{s}_\beta' w_1^{-1}w_2 \otimes v_2) \quad \text{(by Lemma 5.3.4)}$$

$$= 0$$

This completes the proof.

Q.E.D.
Q.E.D.

**Proposition 5.11.** Let \((\pi_\lambda, X_\lambda)\) be the \(\mathbb{H}_{n}^{cl}\)-module as in (5.10). Then
\[
\ker \pi_\lambda(D) = \ker \pi_\lambda(D^2)
\]
and
\[
\ker \pi_\lambda(D) \cap \text{im} \pi_\lambda(D) = 0.
\]
In particular, \(H_D(X_\lambda) = \ker \pi_\lambda(D^2)\).

**Proof.** It is clear that \(\ker \pi_\lambda(D) \subset \ker \pi_\lambda(D^2)\). For \(v \in \ker \pi_\lambda(D^2)\), \(\langle \pi_\lambda(D)v, -\pi_\lambda(D)v \rangle = \langle \pi_\lambda(D^2)v, v \rangle = 0\) by Proposition 5.10. Since \(\langle \cdot, \cdot \rangle\) is positive definite by Lemma 5.7, \(\pi_\lambda(D)v = 0\). This proves the first equation \(\ker \pi_\lambda(D) = \ker \pi_\lambda(D^2)\). The equation \(\ker \pi_\lambda(D) \cap \text{im} \pi_\lambda(D) = 0\) follows from the first one.

Q.E.D.

5.4. **Dirac cohomology of** \(X_\lambda\). Let \(\mathcal{P}_n\) be the set of partitions of \(n\). One can attach an element in \(\mathcal{P}_n\) to a point in \(\mathbb{R}^n\) via the Jacobson-Morozov triple. The map, denoted \(\Phi_1 : \mathcal{P}_n \to \mathbb{R}^n\) can be explicitly described as:
\[
(n_1, n_2, \ldots, n_r) \mapsto (-n_1 + 1, -n_1 + 3, \ldots, n_1 - 1, \ldots, -n_r + 1, -n_r + 3, \ldots, n_r - 1).
\]
There is another way to attach an element in \(\mathcal{P}_n\) to a point in \(\mathbb{R}^n\) via the central characters of the modules \(X_\lambda\). This map, denoted \(\Phi_2 : \mathcal{P}_n \to \mathbb{R}^n\) is:
\[
(n_1, n_2, \ldots, n_r) \mapsto (\sqrt{(1 - 1)}1, \ldots, \sqrt{(n_1 - 1)n_1}, \ldots, \sqrt{(1 - 1)}1, \ldots, \sqrt{n_r(n_r - 1)}).
\]

The first interesting computational fact is the following:

**Lemma 5.12.** For a partition \(\lambda\) of \(n\), \(|\Phi_1(\lambda)| = |\Phi_2(\lambda)|\), where \(|\cdot|\) denotes the standard Euclidean norm in \(\mathbb{R}^n\).

**Proof.** This follows from the computation that
\[
\sum_{k=1}^{n_i} (-n_i + 2k - 1)^2 = \sum_{k=1}^{n_i} k(k - 1) = \frac{1}{3} (n_i - 1)n_i(n_i + 1).
\]
Q.E.D.

For each \(\lambda \in \mathcal{P}_n\), define a \(S_n\)-representation:
\[
W_\lambda = (\text{Ind}_{\mathbb{C}[S_{\lambda'}]}^{\mathbb{C}[S_n]} \text{sgn}) \cap (\text{Ind}_{\mathbb{C}[S_{\lambda'}]}^{\mathbb{C}[S_n]} \text{triv}),
\]
where \(\text{sgn}\) and \(\text{triv}\) are respectively the sign and trivial representations of \(S_\lambda\), and \(\lambda'\) is the conjugate of \(\lambda\). It is well-known that \(W_\lambda\) exhausts the list of irreducible representations of \(S_n\).

Define
\[
\Omega_{\mathbb{C}[S_n]}^- = 2 \sum_{\alpha > 0, \beta > 0, s_{\alpha}(\beta) < 0} \bar{t}_{\alpha}\beta \in \mathbb{C}[\tilde{S}_n]^-.
\]
Recall that \( \text{Irr}_{\text{sup}} \mathbb{C}[\widetilde{S}_n]^\sim \) (resp. \( \text{Irr}_{\text{sup}} \text{Seg}_n \)) is the set of irreducible supermodules of \( \mathbb{C}[\widetilde{S}_n]^\sim \) (resp. \( \text{Seg}_n \)). Recall that the equivalence relation \( \sim_\Pi \) on \( \text{Irr}_{\text{sup}} \mathbb{C}[\widetilde{S}_n]^\sim \) or \( \text{Irr}_{\text{sup}} \text{Seg}_n \) is defined in Section 2.8.

**Proposition 5.13.** \( \text{[3] Part of Theorem 1.0.1} \) (also see [1]) There exists a bijection \( \Psi_1 : \mathcal{P}_n^{\text{dist}} \to \text{Irr}_{\text{sup}} \mathbb{C}[\widetilde{S}_n]^\sim / \sim_\Pi \) such that for each partition \( \lambda \) of \( n \), there exists a representative \( (\sigma, U) \in \Psi_1(\lambda) \) with the properties that

\[
|\Phi_1(\lambda)|^2 = \chi_\sigma(\Omega_{\mathbb{C}[\widetilde{S}_n]}^\sim)
\]

and

\[
\text{Hom}_{\mathbb{C}[\widetilde{S}_n]^\sim}(U, W_\lambda \otimes \text{U}_{\text{spin}}) \neq 0.
\]

**Proof.** In [3] Theorem 1.0.1, the set \( \text{Irr} \mathbb{C}[\widetilde{S}_n]^\sim / \sim_{\text{sgn}} \) is considered instead of \( \text{Irr}_{\text{sup}} \mathbb{C}[\widetilde{S}_n]^\sim / \sim_\Pi \). By Proposition 2.9 there is a natural bijection between \( \text{Irr} \mathbb{C}[\widetilde{S}_n]^\sim / \sim_{\text{sgn}} \) and \( \text{Irr}_{\text{sup}} \mathbb{C}[\widetilde{S}_n]^\sim / \sim_\Pi \). Then one can now apply [3] Theorem 1.0.1.

Q.E.D.

Here is an analogue of Proposition 5.13

**Proposition 5.14.** There exists a bijection \( \Phi_2 : \mathcal{P}_n^{\text{dist}} \to \text{Irr}_{\text{sup}} \text{Seg}_n / \sim_\Pi \) such that there exists a representative \( (\sigma, U) \in \Phi_2(\lambda) \) with the properties that

\[
|\Phi_2(\lambda)|^2 = \chi_\sigma(\Omega_{\text{Seg}_n})
\]

and

\[
\text{Hom}_{\text{Seg}_n}(U, F(W_\lambda \otimes \text{U}_{\text{spin}})) \neq 0.
\]

**Proof.** Note that for an irreducible \( \mathbb{C}[\widetilde{S}_n]^\sim \)-supermodule \( U \), \( F(U) \) is either an irreducible supermodule or the direct sum of two irreducible supermodules of opposite grading. Thus we could define \( \Psi_2(\lambda) \) to be the unique equivalence class in \( \text{Irr}_{\text{sup}} \text{Seg}_n / \sim_\Pi \) containing the irreducible supermodule(s) in \( F(U) \) for a representative \( U \in \Phi_1(\lambda) \), where \( \Phi_1 \) is defined in Proposition 5.13.

It remains to check those two properties. Recall \( F(U) = U \otimes U(n) \) and the action of \( \text{Seg}_n \) on \( F(U) \) is defined in Section 2.8. Then for \( u \otimes u' \in U \otimes U(n) \),

\[
\Omega_{\text{Seg}_n}(u \otimes u') = 2 \sum_{\alpha, \beta > 0, \alpha(\beta) < 0} \widetilde{s}_\alpha \widetilde{s}_\beta (u \otimes u')
\]

\[
= 2 \left( \sum_{\alpha, \beta > 0, \alpha(\beta) < 0} \tilde{t}_\alpha \tilde{t}_\beta u \otimes u' \right)
\]

\[
= \chi_\sigma(\Omega_{\mathbb{C}[\widetilde{S}_n]^\sim}) u \otimes u'
\]

Thus for any irreducible supermodule \( (\sigma', U') \) in \( F(U) \), \( \chi_\sigma'(\Omega_{\text{Seg}_n}) = \chi_\sigma(\Omega_{\mathbb{C}[\widetilde{S}_n]^\sim}) \). Then combining with Lemma 5.12 and Proposition 5.13 we have shown the first property.

The second property follows from

\[
\text{Hom}_{\text{Seg}_n}(F(U'), F(W_\lambda \otimes \text{U}_{\text{spin}})) = \text{Hom}_{\mathbb{C}[\widetilde{S}_n]^\sim}(U', G \circ F(W_\lambda \otimes \text{U}_{\text{spin}})) \neq 0,
\]

where the last equality follows from Propositions 2.13 and 5.13.
Lemma 5.15. For a partition $\lambda$ of $n$ with distinct parts, there exists a representative $U \in \Phi_2(\lambda)$ such that
\[ \text{Hom}_{\text{Seg}_n}(U, \text{Res}_{\text{Seg}_n} X_\lambda) \neq 0. \]

Proof. This follows from
\[ \text{Hom}_{\text{Seg}_n}(U, \text{Res}_{\text{Seg}_n} X_\lambda) = \text{Hom}_{\text{Seg}_n}(U, F(\mathbb{C}[S_n] \otimes \mathbb{C}[S_\lambda]^{\text{triv}} \otimes U_{\text{spin}})) \quad (\text{by Lemma 5.15}) \]
\[ \supset \text{Hom}_{\text{Seg}_n}(U, F(\sigma \otimes U_{\text{spin}})) \quad (\text{by definition of } \sigma) \]
\[ \supset \text{Hom}_{\text{Seg}_n}(U, (\mathbb{C}[S_n] \otimes \mathbb{C}[S_\lambda]^{\text{triv}} \otimes U_{\text{spin}})) \quad (\text{by Proposition 2.13}) \]
The statement now follows from Proposition 5.14.

Q.E.D.

The following theorem states that the induced modules $(\pi_\lambda, X_\lambda)$ with $\lambda$ of distinct parts have non-zero Dirac cohomologies.

Theorem 5.16. Let $\lambda$ be a partition of $n$ with distinct parts. Let $(\pi_\lambda, X_\lambda)$ be the $\mathbb{H}^n_{\text{CI}}$-module defined in (5.16). Let $\Psi_2$ be the map defined in Proposition 5.14. Then there exists a representative $U$ in $\Phi_2(\lambda)$ such that
\[ \text{Hom}_{\text{Seg}_n}(U, \text{HD}(X_\lambda)) \neq 0. \]
In particular, $\text{HD}(X_\lambda)$ is non-zero.

Proof. For a fixed $\lambda \in \mathcal{P}^\text{dist}_n$, let $U$ be a $\text{Seg}_n$-module with the property in Lemma 5.15. Then there exists a non-zero vector $v$ in the isotypical component $U$ of $X_\lambda$. By Theorem 4.7, Lemma 5.1 and Proposition 5.14, $\pi_\lambda(D^2)v = (\chi_{\pi_\lambda}(\Omega_{\mathbb{H}^n_{\text{CI}}}) - \chi_{\Psi_2(\lambda)}(\Omega_{\text{Seg}_n}))v = (|\Phi_2(\lambda)|^2 - \chi_{\Psi_2(\lambda)}(\Omega_{\text{Seg}_n}))v = 0$. Hence, $v \in \ker(\pi_\lambda(D^2))$. By Proposition 5.11 $v \in \text{HD}(X_\lambda) = \ker \pi_\lambda(D^2)$. This proves the theorem.

Q.E.D.

The Dirac cohomology $\text{HD}(X)$ also provides a way to realize irreducible $\text{Seg}_n$-supermodules.

Corollary 5.17. For each $\lambda \in \mathcal{P}^\text{dist}_n$, there exists a unique irreducible $\text{Seg}_n$-supermodule $U$, up to the equivalence of $\sim_\Pi$, such that $\text{Hom}_{\text{Seg}_n}(U, \text{HD}(X_\lambda)) \neq 0$. Let $[\text{HD}(X_\lambda)]$ be an irreducible submodule of $\text{HD}(X_\lambda)$. Then
\[ \text{ Irr}_{\text{sup}} \text{Seg}_n = \bigsqcup_{\lambda \in \mathcal{P}^\text{dist}_n} \{ [\text{HD}(X_\lambda)], \Pi([\text{HD}(X_\lambda)]) \}, \]
where $\bigsqcup$ means the disjoint union.

Proof. For the first assertion, the existence has been proved in Theorem 5.16 and we only have to prove the uniqueness. Let $(\sigma', U')$ be an irreducible $\text{Seg}_n$-module such that
\[ \text{Hom}_{\text{Seg}_n}(U', \text{HD}(X_\lambda)) \neq 0. \]
Then $\chi_{\pi_\lambda} = \chi_{\sigma'}$ by Theorem 4.8 and Theorem 5.16. On another hand, by Proposition 5.14, $(\sigma', U')$ is in $F_2(\lambda')$ for some $\lambda' \in P_{\text{dist}}^n$. Then

$$\text{Hom}_{\text{Seg}_n}(U', H_D(X_{\lambda'})) \neq 0$$

and by Theorem 4.8 again, $\chi_{\pi_{\lambda'}} = \chi_{\sigma'}$. Thus $\chi_{\pi_{\lambda'}} = \chi_{\pi_\lambda}$ and so $\lambda = \lambda'$. This implies the uniqueness.

The second assertion follows from the first assertion and the bijectivity of $\Phi_2$ in Proposition 5.14.

Q.E.D.

Let $K(\mathbb{H}_n^{CI})$ (resp. $K(\text{Seg}_n)$) be the Grothendieck group of finite-dimensional $\mathbb{H}_n^{CI}$-supermodules (resp. finite-dimensional Seg$_n$-supermodules). Then the Dirac cohomology $H_D$ induces a map, still denoted $H_D$, from $K(\mathbb{H}_n^{CI})$ to $K(\text{Seg}_n)$. Corollary 5.17 implies the following:

**Corollary 5.18.** The image of $H_D : K(\mathbb{H}_n^{CI}) \to K(\text{Seg}_n)$ has a finite index of $K(\text{Seg}_n)$.

Recall that the superalgebra homomorphism $\zeta : Z(\mathbb{H}_n^{CI}) \to Z(\text{Seg}_n)_0$ is defined in Theorem 3.14. We also have:

**Corollary 5.19.** The map $\zeta : Z(\mathbb{H}_n^{CI}) \to Z(\text{Seg}_n)_0$ is surjective.

**Proof.** It suffices to show $\dim(\text{im} \zeta) \geq \dim Z(\text{Seg}_n)_0$. By Theorem 4.8 and Theorem 5.16 for any partition $\lambda \in P_{\text{dist}}^n$, there exists $(\sigma_\lambda, U_\lambda) \in \text{Irr}_{\text{sup}} \text{Seg}_n$, such that $\chi_{\pi_\lambda} = \chi_{\sigma_\lambda}$. Since the central characters $\{\chi_{\pi_\lambda}\}_{\lambda \in P_{\text{dist}}^n}$ are linearly independent over $\mathbb{C}$, $\{\chi_{\sigma_\lambda}\}_{\lambda \in P_{\text{dist}}^n}$ are also linearly independent. Then we have $\dim(\text{im} \zeta)$ is not less than the cardinality of $P_{\text{dist}}^n$. Now the statement follows from the fact that $\dim Z(\text{Seg}_n)_0$ is equal to the cardinality of $\text{Irr}_{\text{sup}}(\text{Seg}_n)/\sim_{\Pi}$, which is the same as the cardinality of $P_{\text{dist}}^n$.

Q.E.D.

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