Measurements and majorization

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Abstract

Majorization is an outstanding tool to compare the purity of mixed states or the amount of information they contain and also the degrees of entanglement presented by such states in tensor products. States are compared by their spectra and majorization defines a partial order on those. This paper studies the effect of measurements on the majorization relation among states. It, then, proceeds to study the effect of local measurements on the agents sharing an entangled global state. If the result of the measurement is recorded, Nielsen and Vidal [7] showed that the expected spectrum after any P.O.V.M. measurement majorizes the initial spectrum, i.e., a P.O.V.M. measurement cannot, in expectation, reduce the information of the observer. A new proof of this result is presented and, as a consequence, the only if part of Nielsen’s [5] characterization of LOCC transformations is generalized to n-party entanglement. If the result of a bi-stochastic measurement is not recorded, the initial state majorizes the final state, i.e., no information may be gained by such a measurement. This strengthens a result of A. Peres [8]. In the n-party setting, no local trace preserving measurement by Alice can change the local state of another agent.

1 Introduction

In this paper measurement means P.O.V.M. measurement, i.e., a family \( \{f_k\}_{k=1}^m \) of operators \( f_k : H \rightarrow H \) such that

\[
\sum_{k=1}^m f_k^* \circ f_k = id_H. \tag{1}
\]

A bi-stochastic measurement is a measurement that also satisfies

\[
\sum_{k=1}^m f_k \circ f_k^* = id_H. \tag{2}
\]
The term *state* means mixed state, i.e., a self-adjoint, weakly positive operator \( \rho : H \to H \) of trace 1. All Hilbert spaces are finite-dimensional. The majorization relation is written \( \succeq \) both for vectors of non-negative real numbers and for self-adjoint operators. We shall only use the majorization relation to compare two vectors that sum up to the same value (often 1 but not always) or two self-adjoint operators of equal traces (often 1 but not always). The entropy of a state \( \sigma \) is denoted \( E(\sigma) \). The spectrum of a state \( \sigma \) is denoted \( \text{Sp}(\sigma) \). The projection on \( x \) is denoted \( |x\rangle\langle x| \). The conjugate of a complex number \( c \) is denoted \( \overline{c} \).

2 The effect of measurements

2.1 Spectra

If one decides to measure some observable defined by a family \( \{f_k\}_{k=1}^m \) one will obtain some result, i.e., some \( k \), for one’s measurement. The initial state of the
system defines a probability distribution on the possible states that can be the result of the measurement, not a single state. If the initial state is $\sigma$ then the probability of obtaining result $k$ is given by:

$$p_k = Tr(f_k \circ \sigma \circ f_k^*).$$ \hfill (3)

If $p_k = 0$, the result $k$ is never obtained. If $p_k > 0$, the state that results from the measurement is given by:

$$\sigma'_k = \frac{1}{p_k} f_k \circ \sigma \circ f_k^*.$$ \hfill (4)

Our purpose is to compare the spectrum of the state before the measurement and the spectrum after the measurement. There are two spectra that come to mind as candidates for the spectrum after the measurement. The first one is the spectrum of the state $\sum_{k=1}^m p_k \sigma'_k = \sum_{k=1}^m f_k \circ \sigma \circ f_k^*$. In the literature, when this is the result considered, one calls the measurement a trace-preserving measurement. But there is another spectrum that can be considered to be the result of the measurement: the expected spectrum defined as a convex combination of the spectra of the states $\sigma'_k$ where the spectrum of $\sigma'_k$ is weighted by the probability $p_k$ of obtaining the state $\sigma'_k$. For defining this combination, we consider a spectrum to be a decreasing vector of non-negative real numbers and add componentwise. In other term, the $k$'th largest value of the expected spectrum is the expected value of the $k$th largest values obtained in the different possible outcomes and the spectrum considered is:

$$\sum_{k=1}^m p_k Sp(\sigma'_k) = \sum_{k=1}^m Sp(f_k \circ \sigma \circ f_k^*).$$

When such a result is considered, the literature calls the measurement an efficient measurement.

In summary we want to compare $Sp(\sigma)$, $\sum_{k=1}^m Sp(f_k \circ \sigma \circ f_k^*)$ and $Sp(\sum_{k=1}^m f_k \circ \sigma \circ f_k^*)$. We shall show that, for any measurement, the final expected spectrum majorizes the initial spectrum: $\sum_{k=1}^m Sp(f_k \circ \sigma \circ f_k^*) \succeq Sp(\sigma)$ and that, for any bi-stochastic measurement: $Sp(\sigma) \succeq Sp(\sum_{k=1}^m f_k \circ \sigma \circ f_k^*)$.

If we consider entropy, a possible measure of information, one notes that there are two natural ways of measuring the entropy resulting from an efficient measurement. One may consider the entropy of the expected spectrum defined above, but one may also consider the expected entropy. The former quantity is defined by $S_1 = S(\sum_{k=1}^m p_k Sp(\sigma'_k))$ and the latter by $S_2 = \sum_{k=1}^m p_k S(\sigma'_k)$. The result below implies that $S_1 \leq S(\sigma)$. The concavity of entropy implies $S_2 \leq S_1$. The result above therefore implies a definite strengthening of the fact that, in expectation, the entropy cannot be increased by an efficient measurement.

### 3 Efficient measurements

#### 3.1 Past work

Theorem 3 below is Theorem 12 of [7]. The authors present the result as a corollary of the characterization of LOCC transformations in 2-party systems
in a pure state obtained in [5]. The 1-party result is proved by reduction to the 
2-party result by purification. The proof presented below is a direct proof.

We shall rely on two results. The first is a corollary of a theorem of Y. Fan
(Theorem 1 of [2]). It may be found in [4] p. 241.

**Theorem 1** For any self-adjoint matrices $A$ and $B$, $A, B : \mathcal{H} \to \mathcal{H}$, one has $Sp(A) + Sp(B) \succeq Sp(A + B)$.

The second one is most probably well-known, but no reference for it has
been found.

**Theorem 2** Let $A$ be a finite dimensional Hilbert space and $f : A \to A$ a linear operator. Then, $Sp(f^* \circ f) = Sp(f \circ f^*)$.

**Proof:** Note, first, that both $f^* \circ f$ and $f \circ f^*$ are self-adjoint and therefore have $\dim(A)$ real eigenvalues. We shall show that every eigenvalue $\lambda$ of $f^* \circ f$, different from zero, is an eigenvalue of $f \circ f^*$ with the same multiplicity. To this effect we note that if $x \in A$ is an eigenvector of $f^* \circ f$ for some eigenvalue $\lambda \neq 0$, then $f(x)$ is an eigenvector of $f \circ f^*$ for eigenvalue $\lambda$. Suppose indeed $x$ and $\lambda$ are as assumed, then $f^*(f(x)) = \lambda x \neq 0$ and therefore $f(x) \neq 0$. But $(f \circ f^*)(f(x)) = f((f^* \circ f)(x)) = f(\lambda x) = \lambda f(x)$. We are left to show that the multiplicity of $\lambda$ for $f \circ f^*$ is at least its multiplicity for $f^* \circ f$. For this, we note that if $y \in A$ is orthogonal to $x$, then $f(y)$ is orthogonal to $f(x)$. Indeed, $\langle f(y) \mid f(x) \rangle = \langle y \mid (f^* \circ f)(x) \rangle = \langle y \mid \lambda x \rangle = \lambda \langle y \mid x \rangle = 0$.

### 3.2 Result

**Theorem 3** Let $\sigma$ be a state and $\{f_k\}_{k=1}^m$ a measurement. Then,

\[
\sum_{k=1}^m Sp(f_k \circ \sigma \circ f_k^*) \succeq Sp(\sigma). \tag{5}
\]

**Proof:** The operator $\sigma$ is self-adjoint and weakly positive, it has therefore a square root, i.e., a self-adjoint, weakly positive operator $\alpha : \mathcal{H} \to \mathcal{H}$ such that $\sigma = \alpha \circ \alpha^*$. Let $\beta_k = f_k \circ \alpha$. We have $f_k \circ \sigma \circ f_k^* = \beta_k \circ \beta_k^*$. By Theorem 2 we have: $Sp(\beta_k \circ \beta_k^*) = Sp(\beta_k^* \circ \beta_k)$. We conclude that

\[
Sp(f_k \circ \sigma \circ f_k^*) = Sp(\alpha^* \circ f_k^* \circ f_k \circ \alpha).
\]

By Theorem 1 and Equation (1) one has:

\[
\sum_{k=1}^m Sp(\alpha^* \circ f_k^* \circ f_k \circ \alpha) \succeq Sp(\sum_{k=1}^m \alpha^* \circ f_k^* \circ f_k \circ \alpha) = Sp(\alpha^* \circ \alpha) = Sp(\sigma). \tag{6}
\]
4 Trace preserving measurements

4.1 Past work

On p. 262, A. Peres [8] shows that a trace preserving von Neumann measurement cannot decrease entropy, or any concave function of the state for that matter. The meaning of such a result is that a measurement whose result is not recorded can never increase the observer's information, but it can, in fact, decrease this information. We strengthen Peres' result on two counts. First, by showing that the initial state majorizes the final state. This indeed is a strengthening of Peres' result since Schur's characterization of real functions that preserve majorization (Theorem 3.A.4 in [4]) implies that entropy and any concave function considered by Peres, anti-preserve majorization: $\sigma \succeq \rho$ implies $E(\sigma) \leq E(\rho)$; see, for example Proposition 4.2.1 in [6]. Then, the result is proved for any bi-stochastic measurement, not only von Neumann measurements.

4.2 Result

The following is due to A. Uhlmann [10]. The proof given here for completeness' sake is streamlined from the proof of Theorem 5.1.3 in [6].

Theorem 4 Let $\sigma, \tau : H \to H$ be states. Then

$$\sigma \succeq \tau \iff \tau = \sum_{k=1}^{m} f_k \circ \sigma \circ f_k^*$$  \hspace{1cm} (7)

for some bi-stochastic measurement $\{f_k\}_{k=1}^{m}$.

Proof: We first deal with the if direction. Assume $\{f_k\}_{k=1}^{m}$ is a bi-stochastic measurement. Let us make, at first, the facilitating assumption that there is a basis $\{x_i\}_{i=1}^{n}$ of eigenvectors of both $\sigma$ and $\tau \triangleq \sum_{k=1}^{m} f_k \circ \sigma \circ f_k^*$. Let $\sigma(x_i) = \lambda_i x_i$ and $\tau(x_i) = \mu_i x_i$ for any $i$ and let $\lambda$ (resp. $\mu$) be the real column vector $[\lambda_i]$ (resp. $[\mu_i]$).

Define $c_{i,j}^{k} = \langle x_i \mid f_k \mid x_j \rangle$. We have $f_k(x_j) = \sum_{l=1}^{n} c_{i,j}^{k} x_j$. We have, for any $i, j$: $\langle x_j \mid f_k^* \mid x_i \rangle = c_{j,i}^{k}$ and $f_k^*(x_i) = \sum_{j=1}^{n} c_{j,i}^{k} x_j$. Therefore

$$\langle x_j \mid f_k \circ \sigma \circ f_k^* \mid x_i \rangle = (f_k^*(x_j) \mid \sigma \mid f_k^*(x_i)) = \sum_{s=1}^{n} c_{j,s}^{k} \sum_{l=1}^{n} c_{s,l}^{k} \lambda_l c_{i,l}^{k}.$$  

Similarly

$$\langle x_j \mid f_k \circ f_k^* \mid x_i \rangle = \sum_{l=1}^{n} c_{j,l}^{k} c_{l,i}^{k}$$

and

$$\langle x_j \mid f_k^* \circ f_k \mid x_i \rangle = \sum_{l=1}^{n} c_{j,l}^{k} c_{l,i}^{k}.$$
Therefore:

\[ \mu_i = \langle x_i \mid \tau \mid x_i \rangle = \langle x_i \mid \sum_{k=1}^{m} (f_k \circ \sigma \circ f_k^*) \mid x_i \rangle = \sum_{l=1}^{n} \lambda_l \left( \sum_{k=1}^{m} c_{i,l}^k c_{i,j}^k \right). \]

Let the \( n \times n \) matrix \( B \) be defined by: \( b_{i,j} = \sum_{k=1}^{m} c_{i,j}^k \). We see that

\[ \mu = B \lambda. \tag{8} \]

The summation of the elements of the \( i \)th row of \( B \) is:

\[ \sum_{j=1}^{n} b_{i,j} = \sum_{k=1}^{m} \sum_{j=1}^{n} c_{i,j}^k c_{i,j}^k = \sum_{k=1}^{m} \langle x_i \mid f_k \circ f_k^* \mid x_i \rangle = 1. \]

Similarly for the summation of the elements of the \( j \)th column:

\[ \sum_{i=1}^{n} b_{i,j} = \sum_{k=1}^{m} \sum_{i=1}^{n} c_{i,j}^k c_{i,j}^k = \sum_{k=1}^{m} \langle x_j \mid f_k \circ f_k^* \mid x_j \rangle = 1. \]

We note that the matrix \( B \) is bi-stochastic and conclude by Theorem 2.A.4 of [1] that \( \lambda \geq \mu \).

We have proved our claim under the assumption that \( \sigma \) and \( \tau \) commute. Let us now treat the general case. There is a unitary transformation \( U \) such that \( U \circ \sigma \circ U^* \) and \( \tau \) commute. Consider the family \( g_k = f_k \circ U^* \). The \( g_k \) form a bi-stochastic measurement. We have just proven that

\[ U \circ \sigma \circ U^* \succeq \sum_{k=1}^{m} g_k \circ \sigma \circ U^* \circ g_k^* = \sum_{k=1}^{m} f_k \circ \sigma \circ f_k^*. \]

We conclude that \( \sigma \succeq \sum_{k=1}^{m} f_k \circ \sigma \circ f_k^* \).

The proof of the only if direction is easier. Notations are as above. Assume \( \sigma \succeq \tau \). Assume, at first, that \( \sigma \) and \( \tau \) commute. By results of Hardy, Littlewood and Pólya [3] and Birkhoff [1] (see Theorem 3.1.2 in [6]) there is a vector \( \{p_i\}_{k=1}^{m} \) of non-negative real numbers that sum up to 1 and permutation matrices \( \{P_k\}_{k=1}^{m} \) such that \( \mu = \sum_{k=1}^{m} p_k P_k \lambda \). Considering the diagonal matrices representing \( \tau \) and \( \sigma \) in the basis of their joint eigenvectors. One sees that \( \tau = \sum_{k=1}^{m} p_k P_k \circ \sigma \circ P_k^* \). The family \( f_k = \sqrt{p_k} P_k \) forms a bi-stochastic measurement with the desired properties. Now we want to get rid of the assumption that \( \sigma \) and \( \tau \) commute. There is a unitary transformation \( U \) such that \( \sigma \) and \( \rho = U \circ \tau \circ U^* \) commute. We have \( \sigma \succeq \rho \) and we just proved there is a bi-stochastic measurement \( \{f_k\}_{k=1}^{m} \) such that \( \rho = \sum_{k=1}^{m} f_k \circ \sigma \circ f_k^* \). We have \( \tau = \sum_{k=1}^{m} U^* \circ f_k \circ \sigma \circ f_k^* \circ U \) and the family \( \{U^* \circ f_k\} \) is a suitable bi-stochastic measurement.

It is known that trace preserving measurements may decrease the entropy and therefore the bi-stochastic assumption in Theorem [4] cannot be dispensed with.
5 Entangled systems

We now wish to study the effect of local measurements on local and global states in entangled systems. We assume each of $n$ parties, i.e., agents, has some piece of a quantic system. The pieces do not have to be similar. Let $H = H_1 \otimes H_2 \otimes \ldots \otimes H_n$ be a tensor product of $n$ finite-dimensional Hilbert spaces. We shall denote by $G_i$ the tensor product of all spaces $H_j$ for $j \neq i$. We consider that the global system represented by $H$ is made of $n$ different parts, represented by $H_i$, for $i = 1, \ldots, n$, the $i$'s part being controlled by agent $i$. In accordance with tradition, we assume agent 1 is Alice. Bob will be used as a generic name for any agent other than Alice. If the global state of the system is described by state $\sigma : H \rightarrow H$, the local state of agent $i$ is described by the partial trace of $\sigma$ on $G_i$: $\text{Tr}_{G_i}(\sigma) : H_i \rightarrow H_i$. We focus here on the effect of a measurement performed by Alice on her own state, Bob's state and the global state.

The effect of a local measurement $\{f_k\}_{k=1}^m$, $f_k : H_1 \rightarrow H_1$ on the global system is that of the global measurement $\{g_k = (f_k \otimes \text{id}_{G_1})\}_{k=1}^m$, $g_k : H \rightarrow H$. Note that, indeed, the latter is a measurement and that it is bi-stochastic iff the local measurement is. Note also that the effect of the local measurement on Alice's state is as expected: the effect of the local measurement on the local state.

$$\text{Tr}_{G_i}(g_k \circ \sigma \circ g_k^*) = f_k \circ \text{Tr}_{G_i}(\sigma) \circ f_k^*.$$ 

6 Local efficient measurement

6.1 Majorization of local states

As remarked in Section 5, a local measurement of Alice acts on the global state as a global measurement would do and on Alice’s state as it would do if Alice were alone. We conclude, by Theorem 5 that the expected spectrum of the global state majorizes the spectrum of the initial global state and the expected spectrum of Alice’s local state majorizes the spectrum of her initial state. The following describes what happens to the spectrum of Bob’s (or any agent different from Alice) state.

**Theorem 5** Let $\sigma$ be a state of $H$ and $\{f_k\}_{k=1}^m$, $f_k : H_1 \rightarrow H_1$ be a local measurement of Alice. Let $\{g_k = (f_k \otimes \text{id}_{G_1})\}_{k=1}^m$. For any agent $i > 1$, one has:

$$\sum_{k=1}^m \text{Sp}(\text{Tr}_{G_i}(g_k \circ \sigma \circ g_k^*)) \geq \text{Sp}(\text{Tr}_{G_i}(\sigma)).$$ (9)

**Proof:** Since $i > 1$, $\text{Tr}_{G_i}(g_k \circ \sigma \circ g_k^*) = \text{Tr}_{G_i}(\sigma \circ g_k^* \circ g_k)$. By Theorem 1

$$\sum_{k=1}^m \text{Sp}(\text{Tr}_{G_i}(\sigma \circ g_k^* \circ g_k)) \geq \text{Sp}(\sum_{k=1}^m \text{Tr}_{G_i}(\sigma \circ g_k^* \circ g_k)) = \text{Sp}(\text{Tr}_{G_i}(\sigma \circ g_k^* \circ g_k)).$$
6.2 LOCC operations weakly increase the spectra of all local states in the majorization order

**Theorem 6** In any LOCC protocol, the spectrum of any initial local state is majorized by its expected final local spectrum in the majorization order.

**Proof:** We have shown in Section 6 that any measurement operation brings about, for any agent, a situation in which the expected spectrum majorizes the initial one. A local unitary operation of Alice does not change the local state of Bob and does not change the spectrum of her own local state. Classical communication does not change the global quantum state. We see that no step in a LOCC protocol can decrease any local spectrum in the majorization order.

6.3 Derivation of a generalization of one-half of Nielsen’s characterization

We can now derive a generalization of one half (the only if part) of Nielsen’s Theorem 1 in [5].

**Corollary 1** If there is an $n$-party protocol consisting of local unitary operations, local generalized measurements and classical communication that, starting in a mixed global state $\sigma$ terminates for sure, i.e., with probability one, in mixed global state $\sigma'$, then, for every agent $i$, $\text{Tr}_{G_i}(\sigma') \succeq \text{Tr}_{G_i}(\sigma)$.

**Proof:** At each step of the protocol, we have shown that, for any agent, the initial mixed local state is majorized by the expected spectrum of the final mixed local state. If the final global state is, for sure, $\sigma'$, the final mixed local states are $\rho_i'$ and the expected spectra are $\text{Sp}(\rho_i')$. We conclude that, for every $i$, $1 \leq i \leq n$ one has: $\rho_i' \succeq \rho_i$, proving our claim.

One may note that our results do not use Schmidt’s decomposition, which is used heavily in [5].

7 Local trace preserving measurements

We can now show that the result of Section 4.2 can be extended and strengthened. A trace preserving bi-stochastic local measurement of Alice cannot bring about any additional information concerning the global state, Alice’s own state or Bob’s state. In fact, it leaves Bob’s state unchanged. Our claim concerning the global state and Alice’s state follows directly from Theorem 4 since the
transformations of those states are bi-stochastic measurements. Let us deal with Bob’s case and show that his state is not affected by Alice’s measurement.

**Theorem 7** Let $\sigma$ be a state of $H$ and $\{f_k\}_{k=1}^m$, $f_k : H_1 \rightarrow H_1$ be a local measurement of Alice. Let $\{g_k = (f_k \otimes id_{G_1})\}_{k=1}^m$. For any agent $i > 1$, one has:

$$Tr_{G_i} \left( \sum_{k=1}^m g_k \circ \sigma \circ g_k^* \right) = Tr_{G_i} (\sigma).$$

**Proof:** Since $g_k$ is the identity on $H_i$, one has:

$$Tr_{G_i} (g_k \circ \sigma \circ g_k^*) = Tr_{G_i} (\sigma \circ g_k^* \circ g_k)$$

and

$$Tr_{G_i} \left( \sum_{k=1}^m g_k \circ \sigma \circ g_k^* \right) = Tr_{G_i} \left( \sum_{k=1}^m \sigma \circ g_k^* \circ g_k \right) = Tr_{G_i} (\sigma \circ \sum_{k=1}^m g_k \circ g_k^*) = Tr_{G_i} (\sigma).$$

We conclude that trace preserving measurements by Alice cannot change the local states of any other agent. A bi-stochastic trace preserving measurement by Alice can only degrade the information contained in Alice’s local state or in the global state.

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