Qubit portrait of qudit states and Bell inequalities

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Abstract

A linear map of qudit tomogram onto qubit tomogram (qubit portrait) is proposed as a characteristics of the qudit state. Using the qubit portrait method the Bell inequalities for two qubits and two qutrits are discussed in framework of probability representation of quantum mechanics. Semigroup of stochastic matrices is associated with tomographic probability distributions of qubit and qutrit states. Bell-like inequalities are studied using the semigroup of stochastic matrices. The qudit-qubit map of tomographic probability distributions is discussed as ansatz to provide a necessary condition for separability of quantum states.

1 Introduction

In probability representation of quantum states [1] the states are described by probability distributions. For example, the spin states are described by probability distribution (called spin tomogram) $w(m, \vec{n})$ [2] [3] where $m$ is spin projection on direction determined by unit vector $\vec{n}$. The role of spin tomograms for studying separability and entanglement of quantum states was pointed out in [4]. The aim of our work is to study properties of spin tomograms for one and two spins. In quantum information framework [5] we study qubits and qudits in the context of separable and entangled states. We will obtain that the separable two-qubit states can be associated with 4x4 - stochastic matrices which form a semigroup. This property provides the Bell inequality [6], [7] which serves as a criterion of the separability. The Bell inequalities were considered in context of the probability representation in [8], [9], [10]. The probability representation for spin states was discussed and developed in [11], [12], [13], [14], [15], [16]. The Shanon entropy [17] of spin states was considered in [18], [19]. A linear map of spin tomographic probability distribution (called qudit-tomogram) onto qubit tomogram is constructed. The map provides qubit portrait of qudit states. The qubit portrait is used to get necessary condition of multiqudit state separability. The preliminary remarks on such map were presented in [20]. We will discuss as examples some multiqubit states. The paper is organized as follows: In Sec.2 we review properties of stochastic matrices. In Sec.3 we derive an inequality to be used for studying Bell inequality. In Sec.4 we consider matrices as vectors. In Sec.5 we give a geometrical picture associated with probabilities. In Sec.6 we
give example of 3x3 stochastic matrices. In Sec.7 we present example of qubit states. In Sec.8 we discuss entangled two-qubit states. In Sec.9 we formulate new separability criterion related to semigroup of stochastic matrices. In Sec.10 a new necessary condition of separability is suggested. In Sec.11 example of two-qubit entangled state is considered. In Sec.12 qubit portrait method is applied to qubit-qutrit state. In Sec.13 concrete example is given. In Sec.14 general reduction criterion of separability is formulated. In Sec.15 conclusions and perspectives are discussed.

2 Qubits and stochastic matrices

For one qubit (or for the spin one-half particle state) any state vector \( |\psi\rangle \) has the form

\[
|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \langle \psi | = (a^*, b^*) ,
\]

(1)

where the complex numbers \( a = a_1 + \text{i}a_2 \) and \( b = b_1 + \text{i}b_2 \) satisfy the normalization condition

\[
\langle \psi | \psi \rangle = |a|^2 + |b|^2 = 1.
\]

The 2x2-density matrix of the pure state \( |\psi\rangle \) reads

\[
\rho_\psi = |\psi\rangle \langle \psi | = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}.
\]

(3)

The trace of the density matrix is

\[
\text{Tr} \rho_\psi = |a|^2 + |b|^2 = 1.
\]

(4)

The diagonal elements of the density matrix determine the probabilities for spin projections on z-axis \( m = +1/2 \) and \( m = -1/2 \), i.e.

\[
w(\frac{1}{2}) = |a|^2, \quad w(-\frac{1}{2}) = |b|^2.
\]

(5)

Since the probabilities satisfy condition (2) they can be parameterized as follows

\[
|a|^2 = \cos^2 \Theta, \quad |b|^2 = \sin^2 \Theta.
\]

(6)

Let us introduce the matrix

\[
M = \begin{pmatrix} p & q \\ 1-p & 1-q \end{pmatrix},
\]

(7)

where the real numbers \( p \) and \( q \) satisfy the inequalities

\[
1 \geq p \geq 0, \quad 1 \geq q \geq 0.
\]

(8)

The nonnegative numbers \( p, 1-p \) and \( q, 1-q \) can be considered as probability distributions. Numerical example of such matrix reads

\[
M_N = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{2}{5} \end{pmatrix}.
\]

(9)
There are two probability distributions. Firsts one is \((1/10, 9/10)\). The second one is \((2/5, 3/5)\). Important property of the set of the matrices \(M\) is that the product of two matrices of the form (7) has the same form, i.e.

\[
M_1 M_2 = \begin{pmatrix} p_1 & q_1 \\ 1 - p_1 & 1 - q_1 \end{pmatrix} \begin{pmatrix} p_2 & q_2 \\ 1 - p_2 & 1 - q_2 \end{pmatrix} = \begin{pmatrix} p_3 & q_3 \\ 1 - p_3 & 1 - q_3 \end{pmatrix},
\]

Here

\[
p_3 = p_1 p_2 + q_1 (1 - p_2), \qquad q_3 = p_1 q_2 + q_1 (1 - q_2).
\]

The set of matrices (7) forms the semigroup. The unit matrix belongs to the set. The inverse matrix

\[
M^{-1} = \frac{1}{\det M} \begin{pmatrix} 1 - q & -q \\ p - 1 & p \end{pmatrix}, \quad \det M = p(1 - q) - q(1 - p),
\]

does not satisfy the condition (8) and does not belong to the set of matrices (7). The subset of stochastic matrices of the form

\[
N = \begin{pmatrix} p & 1 - p \\ 1 - p & p \end{pmatrix}
\]

is also the semigroup. In fact

\[
N_1 N_2 = \begin{pmatrix} p_1 & 1 - p_1 \\ 1 - p_1 & p_1 \end{pmatrix} \begin{pmatrix} p_2 & 1 - p_2 \\ 1 - p_2 & p_2 \end{pmatrix} = \begin{pmatrix} p_3 & 1 - p_3 \\ 1 - p_3 & p_3 \end{pmatrix},
\]

where the nonnegative number

\[
p_3 = p_1 p_2 + (1 - p_1)(1 - p_2)
\]

determines the matrix elements of the matrix

\[
N_3 = \begin{pmatrix} p_3 & 1 - p_3 \\ 1 - p_3 & p_3 \end{pmatrix}.
\]

The set of matrices (13) is called semigroup of bistochastic matrices. The sum of numbers both in columns and in rows of bistochastic matrices is equal to one. The bistochastic matrices can be associated with \(n \times n\)-unitary matrices \(u\) with matrix elements \(u_{jk}\) satisfying the condition.

\[
\sum_{k=1}^{n} |u_{jk}|^2 = 1, \quad \sum_{j=1}^{n} |u_{jk}|^2 = 1.
\]

Thus the stochastic matrix \(\varphi\) with matrix elements.

\[
\varphi_{jk} = |u_{jk}|^2
\]

is the bistochastic matrix. It means that the group \(u(n)\) of unitary \(n \times n\) matrices induces the semigroup of bistochastic matrices \(\varphi_{jk} = |u_{jk}|^2\). The tensor product of two bistochastic matrices is a bistochastic matrix. Thus the group of tensor product of unitary matrices \(u(n_1) \otimes u(n_2)\) creates the semigroup which
is tensor product of bistochastic matrices \( \varphi_1 \otimes \varphi_2 \) with matrix elements \( |u(n_1)_{jk}|^2 \) and \( |u(n_2)_{\alpha\beta}|^2 \). Using the property (14) one can introduce the associative product of probability distributions. In fact given two probability distributions \( p_1, 1 - p_1 \) and \( p_2, 1 - p_2 \). One can associate with the probability distributions two vectors

\[
\vec{w}_1 = \begin{pmatrix} p_1 \\ 1 - p_1 \end{pmatrix} \equiv \begin{pmatrix} w_1^{(1)} \\ w_2^{(1)} \end{pmatrix}, \quad (19)
\]

\[
\vec{w}_2 = \begin{pmatrix} p_2 \\ 1 - p_2 \end{pmatrix} \equiv \begin{pmatrix} w_1^{(2)} \\ w_2^{(2)} \end{pmatrix}, \quad (20)
\]

and two matrices

\[
N_1 = \begin{pmatrix} w_1^{(1)} & w_2^{(1)} \\ w_1^{(1)} & w_2^{(1)} \end{pmatrix}, \quad (21)
\]

\[
N_2 = \begin{pmatrix} w_1^{(2)} & w_2^{(2)} \\ w_2^{(2)} & w_1^{(2)} \end{pmatrix}. \quad (22)
\]

We define the associative product \( \vec{w}_3 \) of two vectors (called star-product) \( \vec{w}_1 \ast \vec{w}_2 = \vec{w}_3 \) using the result of multiplication of two matrices \( N_1 \) and \( N_2 \) given by (14) and (15) for finding the component of the vector \( \vec{w}_3 \). We get

\[
w_1^{(3)} = w_1^{(1)} w_1^{(2)} + w_2^{(1)} w_2^{(2)}, \quad (23)
\]

\[
w_2^{(3)} = w_1^{(2)} w_2^{(1)} + w_2^{(1)} w_1^{(2)}. \quad (24)
\]

This result can be generalized to introduce the associative product by means of the same tools for \( N \)-dimensional vectors. The components of the product vector read

\[
p_m = \sum_{k=1}^{N} w_{[k+m-1]N} W_k. \quad (25)
\]

Here \( [k + m - 1]_N \) means the number

\[
\begin{cases} 
  k + m - 1, & \text{if } k + m - 1 < N; \\
  k + m - 1 - N, & \text{if } k + m - 1 > N.
\end{cases} \quad (26)
\]

The eigenvalues of the stochastic matrix (7) are

\[
\lambda_1 = 1, \quad \lambda_2 = p - q. \quad (27)
\]

They satisfy the condition

\[
|\lambda_k| \leq 1, \quad k = 1, 2. \quad (28)
\]

The eigenvectors of the matrix (7) read

\[
|U_1\rangle = \begin{pmatrix} 1 \\ q^{-1}(1 - p) \end{pmatrix};
\]

\[
|U_{p-q}\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \quad (29)
\]
It means that the matrix $M$ can be presented in the form
\[
\begin{pmatrix} p & q \\ 1-p & 1-q \end{pmatrix} = U \begin{pmatrix} 1 & 0 \\ 0 & p-q \end{pmatrix} U^{-1},
\]
where the matrix $U$ reads
\[
U = \begin{pmatrix} 1 & 1 \\ q^{-1}(1-p) & -1 \end{pmatrix}.
\]
In the case $p = q$ the determinant of the stochastic matrix equals zero. The inverse matrix has the form
\[
U^{-1} = \frac{1}{1+q^{-1}(1-p)} \begin{pmatrix} 1 & 1 \\ q^{-1}(1-p) & -1 \end{pmatrix}.
\]
It means that
\[
U^{-1} = U \frac{1}{1+q^{-1}(1-p)},
\]
and
\[
U^2 = (1+q^{-1}(1-p)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
From this property follows
\[
U^{2k} = (1+q^{-1}(1-p))^k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
U^{2k+1} = (1+q^{-1}(1-p))^k U.
\]
From (30) we get
\[
\begin{pmatrix} p & q \\ 1-p & 1-q \end{pmatrix}^n = U \begin{pmatrix} 1 & 0 \\ 0 & (p-q)^n \end{pmatrix} U^{-1}, \quad n = 1, 2, 3, ...
\]
Since $|p - q| \leq 1$, for large $n |(p-q)|^n \ll 1$ In this case the matrix
\[
\begin{pmatrix} 1 & 0 \\ 0 & (p-q)^n \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]
\section{Useful inequality}

We prove now an useful inequality for scalar product of two pairs of real vectors. Let
\[
|(a_1 \overrightarrow{b_1})| < c \quad \text{and} \quad |(a_2 \overrightarrow{b_2})| < c,
\]
where $c$ is a positive number. Then the convex sum $\cos^2 \gamma (a_1 \overrightarrow{b_1}) + \sin^2 \gamma (a_2 \overrightarrow{b_2})$ satisfies the inequality
\[
|\cos^2 \gamma (a_1 \overrightarrow{b_1}) + \sin^2 \gamma (a_2 \overrightarrow{b_2})| < c.
\]
By induction we get the inequality for generic convex sum. If $|\vec{a}_k \vec{b}_k| < c$, then

$$|\sum_k p_k (\vec{a}_k \vec{b}_k)| < c,$$  \hspace{1cm} (41)

where the coefficients

$$1 \geq p_k \geq 0, \quad \sum_k p_k = 1.$$  \hspace{1cm} (42)

In particular, we get the following inequality. If $\vec{b}_1 = \vec{b}_2 = ... = \vec{b}_k = ... = \vec{B}$ the property (41) reads

$$|\sum_k p_k (\vec{a}_k \vec{B})| < c,$$  \hspace{1cm} (43)

i.e

$$|\sum_k ((p_k \vec{a}_k) \vec{B})| < c.$$  \hspace{1cm} (44)

4 Matrices as vectors

We discuss below how matrices can be interpreted as vectors. For example, the real 2x2 matrix

$$\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$  \hspace{1cm} (45)

can be considered as the vector

$$\vec{\mu} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$  \hspace{1cm} (46)

The sum of two matrices $\mu_1$ and $\mu_2$

$$\mu_1 + \mu_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$  \hspace{1cm} (47)

can be interpreted as sum of two vectors with following components

$$\vec{\mu}_1 + \vec{\mu}_2 = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{pmatrix}.$$  \hspace{1cm} (48)

Then the number $Tr(\mu_1^{tr} \mu_2) = a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2$ where $\mu_1^{tr}$ is transposed matrix $\mu_1$, is standard scalar product of two vectors, i.e.

$$Tr(\mu_1^{tr} \mu_2) = (\vec{\mu}_1 \vec{\mu}_2).$$  \hspace{1cm} (49)
Let us make a remark. The stochastic matrix \( M \) \({\text{(7)}}\) becomes new stochastic matrix \( M' \) if one permutes the columns of the matrix \( M \), i.e.

\[
M' = \begin{pmatrix}
q & p \\
1 - q & 1 - p
\end{pmatrix}.
\]  
\( (50) \)

The same property takes place if one permutes rows of the matrix \( M \). In this case we get new stochastic matrix

\[
M'' = \begin{pmatrix}
1 - p & 1 - q \\
p & q
\end{pmatrix}.
\]  
\( (51) \)

5 Geometrical picture

The probabilities \( 1 \geq w_1 \geq 0 \) and \( 1 \geq w_2 \geq 0 \) such that \( w_1 + w_2 = 1 \) can be considered in geometrical terms as points on a simplex which is the line shown in Fig. 1.

As example we show vector with its end posed on the line and it can be given as the column

\[
\vec{w} = \begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}.
\]  
\( (52) \)

The stochastic matrices transform the vector \( \vec{w} \) into another vector \( \vec{W} \), for example

\[
\vec{W} = M \vec{w}'.
\]  
\( (53) \)

One can check that the components of vector

\[
\begin{pmatrix}
W_1 \\
W_1
\end{pmatrix} = \begin{pmatrix}
q & p \\
1 - q & 1 - p
\end{pmatrix} \begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}
\]  
\( (54) \)

satisfy the conditions \( 1 \geq W_1 \geq 0 \), \( 1 \geq W_2 \geq 0 \), \( W_1 + W_2 = 1 \). It means that the stochastic matrices move the initial point on the simplex into another point on the same simplex. The new probability distribution described by the vector \( \vec{W} \) has the components \( W_1 = qw_1 + pw_2 \), \( W_2 = (1 - q)w_1 + (1 - p)w_2 \). For bistochastic matrices, one has the transformation \( W_1 = qw_1 + (1 - q)w_2 \) \( W_2 = (1 - q)w_1 + qw_2 \). The point \( w_1 = 1/2 \), \( w_2 = 1/2 \) is invariant under this action. For distributions with three components, the simplex has the geometrical form of the plane shown in Fig. 2.

All the points on the triangle shown on this figure correspond to all the probability distributions with three outputs. Below we discuss the stochastic matrices which transform point on this simplex into another point of the same simplex.

6 The 3x3-stochastic matrices and linear maps of distributions

Let us discuss now the stochastic matrices of the 3rd order of the form

\[
M = \begin{pmatrix}
p_1 & q_1 & r_1 \\
p_2 & q_2 & r_2 \\
p_3 & q_3 & r_3
\end{pmatrix}.
\]  
\( (55) \)
Here the positive numbers $p_k, q_k, r_k \ (k = 1, 2, 3)$ satisfy the normalization conditions

$$\sum_{k=1}^{3} p_k = \sum_{k=1}^{3} q_k = \sum_{k=1}^{3} r_k = 1.$$  \hfill (56)

It means that the numbers in columns of the matrix $M$ can be interpreted as probability distributions. It is easy to check that the set of all the matrices $M$ (55) form semigroup. Let us give numerical example of such a matrix, i.e.

$$M = \begin{pmatrix}
\frac{1}{10} & \frac{1}{3} & \frac{8}{10} \\
\frac{2}{10} & 0 & \frac{2}{10} \\
\frac{1}{10} & \frac{1}{3} & \frac{1}{10}
\end{pmatrix}. \hfill (57)$$

It is interesting that the eigenvalues of the stochastic matrix $M$ contain $\lambda_1 = 1$. This eigenvalue 1 have stochastic matrices $M_N$ of all dimensions $N \geq 2$. One can see that other eigenvalues of the stochastic matrix $M_N$ can be either real or complex. Also all the eigenvalues of the stochastic matrices $M_N$ satisfy inequality $|\lambda_k| \leq 1, \ k = 1, 2, ..., N$. We point out that the permutations of elements of a chosen column transform the stochastic matrix into another stochastic matrix. The group of all permutations of matrix elements of $M_N$ - stochastic matrix has $(N!)^{N+1}$ symmetry elements. The group elements are independent permutations in each column $(N!)^N$ combined with $N!$ permutation of columns. Trace of stochastic matrix $M_N$ satisfies inequality $\text{Tr}M_N \leq N$. The bistochastic 3x3 - matrices have the form (55) but satisfy extra condition $p_k + q_k + r_k = 1 \ (k = 1, 2, 3)$. The discussed stochastic and bistochastic matrices move the points on the triangle. The point with components $(1/3, 1/3, 1/3)$ is invariant under the action of the bistochastic matrices. Let us consider the first column of 3x3-stochastic matrix (55). The nonnegative matrix elements in this column $p_1, p_2, p_3$ can be mapped onto three pairs of nonnegative numbers:

$$P_1^{(1)} = p_1, \quad P_2^{(1)} = (p_2 + p_3); \hfill (58)$$

$$P_1^{(2)} = p_1 + p_2, \quad P_2^{(2)} = p_3; \hfill (59)$$

$$P_1^{(3)} = p_1 + p_3, \quad P_2^{(3)} = p_2. \hfill (60)$$

Thus we get three probability distributions $(P_1^{(1)}, P_2^{(1)});$ $(P_1^{(2)}, P_2^{(2)});$ $(P_1^{(3)}, P_2^{(3)})$ and distributions obtained by permutations of these numbers. One can see that we constructed the linear map of initial probability distribution with three possible outcomes onto a set of probability distributions with two outcomes. The map is invertible. In fact

$$p_1 = P_1^{(1)}, \quad p_2 = P_1^{(2)} - P_1^{(1)}, \quad p_3 = P_2^{(2)}.$$  \hfill (61)

It means that knowing two probability distributions (58) - (59) we can reconstruct the initial distribution. We call the set of probability distributions (58) and (59) as qubit ”portrait” of initial qutrit distribution. We introduce this terminology because we will apply the constructed map to study necessary conditions of separability for quantum multiqudit states. Using the suggested ansatz one can construct the analogous map for obtaining analogous portraits of joint probability distributions.
7 Qubit

If one takes a convex sum of pure state density matrices we get the density matrix of mixed state of spin $-1/2$ particle (or qubit state). It means that the matrix

$$\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|,$$

where $1 \geq p_k \geq 0$ and $\sum_k p_k = 1$ is hermitian matrix

$$\rho^+ = \rho,$$

and its trace is equal to 1. The density matrix is nonnegative matrix, i.e. its eigenvalues are nonnegative numbers. The tomogram of the qubit state is defined by formula

$$w(m, U) = \begin{pmatrix} w(+\frac{1}{2}, U) \\ w(-\frac{1}{2}, U) \end{pmatrix} = (U^+ \rho U)_{mm}.$$  \hspace{1cm} (64)

Here $U$ is unitary matrix. It has the form

$$U = \begin{pmatrix} \cos \frac{\theta}{2} e^{i(\varphi+\psi)/2} & \sin \frac{\theta}{2} e^{i(\varphi-\psi)/2} \\ -\sin \frac{\theta}{2} e^{-i(\varphi-\psi)/2} & \cos \frac{\theta}{2} e^{-i(\varphi+\psi)/2} \end{pmatrix},$$  \hspace{1cm} (65)

and $\varphi, \theta, \psi$ are Euler angles. In reality the Euler angle $\psi$ is not present in the final expression of the tomogram. The tomogram is probability distribution. In our previous notations we can introduce stochastic matrix using substitutions

$$p = w(+\frac{1}{2}, U_1),$$

$$q = w(+\frac{1}{2}, U_2),$$  \hspace{1cm} (66)

i.e.

$$M = \begin{pmatrix} w(+\frac{1}{2}, U_1) & w(+\frac{1}{2}, U_2) \\ 1 - w(+\frac{1}{2}, U_1) & 1 - w(+\frac{1}{2}, U_2) \end{pmatrix}.$$  \hspace{1cm} (67)

Here $U_1$ is matrix determined by angels $\varphi_1, \theta_1, \psi_1$ and the matrix $U_2$ is determined by angels $\varphi_2, \theta_2, \psi_2$. The constructed stochastic matrix with matrix elements equal to tomographic probabilities has all the properties of stochastic matrices discussed in previous sections.

8 Two qubits, separable and entangled states

Let us introduce a unit vector $\vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ which is normal vector to sphere surface. The tomogram $w(m, U)$ can be considered as function on the sphere

$$w(m, U) \equiv w(m, \vec{n}).$$  \hspace{1cm} (68)
The stochastic matrix $M$ can be rewritten in the form

$$
M = \begin{pmatrix}
    w(+\frac{1}{2}, \vec{n}_1^1) & w(+\frac{1}{2}, \vec{n}_2^1) \\
    1 - w(+\frac{1}{2}, \vec{n}_1^1) & 1 - w(+\frac{1}{2}, \vec{n}_2^1)
\end{pmatrix}.
$$

One has

$$
w(-\frac{1}{2}, \vec{n}_1^1) = 1 - w(+\frac{1}{2}, \vec{n}_1^1),
\quad w(-\frac{1}{2}, \vec{n}_2^1) = 1 - w(+\frac{1}{2}, \vec{n}_2^1).
$$

Let us consider two qubits. It means that we consider 4x4-density matrix $\rho$. The tomogram of two-qubit state reads

$$
w(m_1, m_2, \vec{n}, \vec{N}) = (U^\dagger \rho U)_{m_1 m_2, m_1 m_2}.
$$

Here $U$ is 4x4 unitary matrix which is tensor product of two 2x2 - unitary matrices

$$
U = U_1 \otimes U_2,
$$

where $U_1$ and $U_2$ are given by formula (65) with Euler angles $\varphi_1 \theta_1 \psi_1$, $\varphi_2 \theta_2 \psi_2$, respectively. The vector $\vec{n}$ is determined $\varphi_1 \theta_1 \psi_1$ and vector $\vec{N}$ is determined by Euler angles $\varphi_2 \theta_2 \psi_2$. Simply separable state has the tomogram of the factorized form

$$
w(m_1, m_2, \vec{n}, \vec{N}) = w_1(m_1, m_2, \vec{n}, \vec{N}) \cdot w_2(m_2, \vec{n}, \vec{N}).
$$

Let us construct 4x4- stochastic matrix by following rule. We take 4 vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$. Then we choose 2 vectors $\vec{n}$ to be equal $\vec{a}^1$, $\vec{b}$ and 2 vectors $\vec{N}$ to be equal $\vec{c}$, $\vec{d}^1$. We have two probability distributions for first qubit $w_1(m_1, \vec{a}^1)$, $w_1(m_1, \vec{a})$ and two probability distributions for second qubit $w_2(m_2, \vec{c})$, $w_2(m_2, \vec{d}^1)$. Then our 4x4-stochastic matrix reads

\begin{align*}
(M_4)_{k_1} &= \begin{pmatrix}
    w_1(+\frac{1}{2}, \vec{a})w_2(+\frac{1}{2}, \vec{b}) \\
    w_1(+\frac{1}{2}, \vec{a})w_2(-\frac{1}{2}, \vec{b}) \\
    w_1(-\frac{1}{2}, \vec{a})w_2(+\frac{1}{2}, \vec{b}) \\
    w_1(-\frac{1}{2}, \vec{a})w_2(-\frac{1}{2}, \vec{b})
\end{pmatrix}, \quad k = 1, 2, 3, 4
\end{align*}

\begin{align*}
(M_4)_{k_2} &= \begin{pmatrix}
    w_1(+\frac{1}{2}, \vec{a})w_2(+\frac{1}{2}, \vec{c}) \\
    w_1(+\frac{1}{2}, \vec{a})w_2(-\frac{1}{2}, \vec{c}) \\
    w_1(-\frac{1}{2}, \vec{a})w_2(+\frac{1}{2}, \vec{c}) \\
    w_1(-\frac{1}{2}, \vec{a})w_2(-\frac{1}{2}, \vec{c})
\end{pmatrix}, \quad k = 1, 2, 3, 4
\end{align*}

\begin{align*}
(M_4)_{k_3} &= \begin{pmatrix}
    w_1(+\frac{1}{2}, \vec{d})w_2(+\frac{1}{2}, \vec{b}) \\
    w_1(+\frac{1}{2}, \vec{d})w_2(-\frac{1}{2}, \vec{b}) \\
    w_1(-\frac{1}{2}, \vec{d})w_2(+\frac{1}{2}, \vec{b}) \\
    w_1(-\frac{1}{2}, \vec{d})w_2(-\frac{1}{2}, \vec{b})
\end{pmatrix}, \quad k = 1, 2, 3, 4
\end{align*}

and

\begin{align*}
(M_4)_{k_4} &= \begin{pmatrix}
    w_1(+\frac{1}{2}, \vec{d})w_2(+\frac{1}{2}, \vec{c}) \\
    w_1(+\frac{1}{2}, \vec{d})w_2(-\frac{1}{2}, \vec{c}) \\
    w_1(-\frac{1}{2}, \vec{d})w_2(+\frac{1}{2}, \vec{c}) \\
    w_1(-\frac{1}{2}, \vec{d})w_2(-\frac{1}{2}, \vec{c})
\end{pmatrix}, \quad k = 1, 2, 3, 4.
\end{align*}
This matrix can be presented in the form of tensor product of two stochastic 2x2-matrices, i.e.

$$M_4 = \left( \begin{array}{c} w_1 \left( \frac{1}{2}, \frac{1}{2} \right) \cdot w_1 \left( \frac{1}{2}, \frac{1}{2} \right) \\ w_1 \left( -\frac{1}{2}, -\frac{1}{2} \right) \cdot w_1 \left( -\frac{1}{2}, -\frac{1}{2} \right) \end{array} \right) \otimes \left( \begin{array}{c} w_2 \left( +\frac{1}{2}, -\frac{1}{2} \right) \cdot w_2 \left( +\frac{1}{2}, -\frac{1}{2} \right) \\ w_2 \left( -\frac{1}{2}, +\frac{1}{2} \right) \cdot w_2 \left( -\frac{1}{2}, +\frac{1}{2} \right) \end{array} \right). \quad (77)$$

We call this stochastic matrix as "simply separable stochastic matrix". One can check that the matrix (77) satisfies the inequality ((Bell-CHSH) inequality [7])

$$\left| (M_4)_{11} - (M_4)_{21} - (M_4)_{31} + (M_4)_{41} + (M_4)_{12} - (M_4)_{22} - (M_4)_{32} + (M_4)_{42} \
+ (M_4)_{13} - (M_4)_{23} - (M_4)_{33} + (M_4)_{43} - (M_4)_{14} + (M_4)_{24} + (M_4)_{34} - (M_4)_{44} \right| \leq 2. \quad (78)$$

This inequality can be rewritten in matrix form as $|\text{Tr}(M_4I)| \leq 2$

where

$$I = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}. \quad (79)$$

The inequality has to be preserved if one changes the matrix $I$ by the product matrix $\tilde{I} = IC$, $C = C_1 \otimes C_2$. Here two 2x2-matrices $C_1$ and $C_2$ are arbitrary stochastic matrices. In vector form $M_4 \rightarrow \overrightarrow{M_4}$ and according to rules of Sec.3 $I \rightarrow \overrightarrow{I}$ this inequality reads

$$|\overrightarrow{\tilde{I}M_4}| \leq 2. \quad (80)$$

Due to property of convex sums (41) one can state that if one constructs a convex sum of matrices of the type $M_4$

$$M = \sum_k P_k M_4^{(k)}, \quad P_k \geq 0, \quad \sum_k P_k = 1; \quad (81)$$

we get inequality

$$|\overrightarrow{T \tilde{M}}| \leq 2; \quad (82)$$

or

$$|\text{Tr}(MI)| \leq 2. \quad (83)$$

9 Separable and entangled states

By definition the quantum state of two qubits is separable if the tomogram of the state can be presented in the form of convex sum of simply separable tomograms, i.e.

$$w(m_1m_1\bar{n}_1\bar{n}_2) = \sum_k P_k w_1^{(k)}(m_1\bar{n}_1)w_2^{(k)}(m_2\bar{n}_2); \quad P_k \geq 0; \quad \sum_k P_k = 1. \quad (84)$$
Here the index \( k \) can be understood as a collective index with any number of components including both discrete and continuous ones. One can see that the stochastic matrix corresponding to the tomogram (84) has the form of convex sum of the matrices of type (77), i.e.

\[
M_4 = \sum_k P_k \left( \begin{array}{cc}
\frac{1}{2}, \alpha & \frac{1}{2}, \bar{\alpha} \\
\frac{1}{2}, \beta & \frac{1}{2}, \bar{\beta}
\end{array} \right) \otimes \left( \begin{array}{cc}
\frac{1}{2}, \alpha & \frac{1}{2}, \bar{\alpha} \\
\frac{1}{2}, \beta & \frac{1}{2}, \bar{\beta}
\end{array} \right).
\]  

(85)

We call this stochastic matrix as ”separable stochastic matrix”.

**Lemma**

The product of two stochastic matrices \( M_4^{(1)}, M_4^{(2)} \) corresponding to tomograms of separable states of two qubits is the convex sum of simply separable stochastic matrices.

**Proof**

Let \( F_1 \) be stochastic matrix corresponding to separable two qubit quantum state, i.e. it can be written in the form (85) which we denote as

\[
F_1 = \sum_k P_k w^{(k)}(1).
\]

(86)

Here

\[
w^{(k)}(1) = \left( \begin{array}{cc}
\frac{1}{2}, \alpha_1 & \frac{1}{2}, \bar{\alpha}_1 \\
\frac{1}{2}, \beta_1 & \frac{1}{2}, \bar{\beta}_1
\end{array} \right) \otimes \left( \begin{array}{cc}
\frac{1}{2}, \alpha_1 & \frac{1}{2}, \bar{\alpha}_1 \\
\frac{1}{2}, \beta_1 & \frac{1}{2}, \bar{\beta}_1
\end{array} \right).
\]

(87)

Let \( F_2 \) be another stochastic matrix of the form

\[
F_2 = \sum_s \rho_s w^{(s)}(2).
\]

(88)

Here \( \rho_s \geq 0, \sum_s \rho_s = 1 \) and notation (88) means that we change in (87) \( k \to s, \alpha_1 \to \alpha_2, \beta_1 \to \beta_2, \alpha_1 \to \alpha_2, \beta_1 \to \beta_2, \alpha_1 \to \alpha_2, \beta_1 \to \beta_2, \alpha_1 \to \alpha_2, \beta_1 \to \beta_2 \). Let us calculate the product matrix

\[
F = F_1 F_2 = \sum_{ks} (P_k \rho_s) w^{(k)}(1) w^{(s)}(2).
\]

(89)

Since the rule of multiplication of tensor products of matrices reads

\[
(a \otimes b)(c \otimes d) = (ac) \otimes (bd),
\]

(90)

one has

\[
F = \sum_j Q_j w^j.
\]

(91)

Here \( j \) is collective index \( j = (ks) \), the matrix \( w^{(j)} \) is the 4x4-stochastic matrix of simply separable form. It means that the matrix \( F \) satisfies the Bell-CHSH inequality

\[
|\text{Tr}(FI)| \leq 2.
\]

(92)
10 Necessary condition of separability

We will use this lemma to formulate the necessary condition of the separability of two qubit state. In fact if one has the two qubit separable state with spin tomogram $w(m_1 m_2 n_1 n_2)$ the set of matrices associated with the tomogram using the following rule

$$M(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d}) = \begin{pmatrix}
  w(+\frac{1}{2}, +\frac{1}{2}, n_1, n_2) & w(+\frac{1}{2}, +\frac{1}{2}, c_1, c_2) & w(+\frac{1}{2}, +\frac{1}{2}, d_1, d_2) & w(+\frac{1}{2}, +\frac{1}{2}, b_1, b_2) \\
  w(+\frac{1}{2}, -\frac{1}{2}, n_1, n_2) & w(+\frac{1}{2}, -\frac{1}{2}, c_1, c_2) & w(+\frac{1}{2}, -\frac{1}{2}, d_1, d_2) & w(+\frac{1}{2}, -\frac{1}{2}, b_1, b_2) \\
  w(-\frac{1}{2}, +\frac{1}{2}, n_1, n_2) & w(-\frac{1}{2}, +\frac{1}{2}, c_1, c_2) & w(-\frac{1}{2}, +\frac{1}{2}, d_1, d_2) & w(-\frac{1}{2}, +\frac{1}{2}, b_1, b_2) \\
  w(-\frac{1}{2}, -\frac{1}{2}, n_1, n_2) & w(-\frac{1}{2}, -\frac{1}{2}, c_1, c_2) & w(-\frac{1}{2}, -\frac{1}{2}, d_1, d_2) & w(-\frac{1}{2}, -\frac{1}{2}, b_1, b_2)
\end{pmatrix}$$

(93)

form the semigroup of matrices satisfying the inequality $\leq 1$. This property can be used as criterion of the separability. For example we take the two matrices $M_1(\overrightarrow{a_1} \overrightarrow{b_1} \overrightarrow{c_1} \overrightarrow{d_1})$ and $M_2(\overrightarrow{a_2} \overrightarrow{b_2} \overrightarrow{c_2} \overrightarrow{d_2})$. We check that for both matrices the product $F = M_1 M_2(\overrightarrow{a_1} \overrightarrow{b_1} \overrightarrow{c_1} \overrightarrow{d_1} \overrightarrow{a_2} \overrightarrow{b_2} \overrightarrow{c_2} \overrightarrow{d_2})$ satisfies the inequality $\leq 1$ for arbitrary directions $(\overrightarrow{a_k} \overrightarrow{b_k} \overrightarrow{c_k} \overrightarrow{d_k})$ ($k = 1, 2$). This property can be generalized to any number of directions $k = 1, 2, \ldots$. It is worthy to note that the product of two density matrices of two separable quantum states is not density matrix of quantum state.

11 Example of entangled states

Let us take known example of entangled state of two qubits

$$\rho = \frac{1}{2} \begin{pmatrix}
  1 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  1 & 0 & 0 & 1
\end{pmatrix}.$$  (94)

We construct the tomogram of this state using (71), (72). The result reads

$$w(+\frac{1}{2}, +\frac{1}{2}, n_1, n_2) = \frac{1}{2} (\cos^2 \frac{\Theta_1}{2} \cos^2 \frac{\Theta_2}{2} + \sin^2 \frac{\Theta_1}{2} \sin^2 \frac{\Theta_2}{2}) + \sin \Theta_1 \sin \Theta_2 \cos (\varphi_1 + \varphi_2);$$  

$$w(+\frac{1}{2}, -\frac{1}{2}, n_1, n_2) = \frac{1}{2} (\cos^2 \frac{\Theta_1}{2} \sin^2 \frac{\Theta_2}{2} + \sin^2 \frac{\Theta_1}{2} \cos^2 \frac{\Theta_2}{2}) - \frac{1}{4} \sin \Theta_1 \sin \Theta_2 \cos (\varphi_1 + \varphi_2);$$  

$$w(-\frac{1}{2}, +\frac{1}{2}, n_1, n_2) = \frac{1}{2} (\cos^2 \frac{\Theta_1}{2} \sin^2 \frac{\Theta_2}{2} + \sin^2 \frac{\Theta_1}{2} \cos^2 \frac{\Theta_2}{2}) - \frac{1}{4} \sin \Theta_1 \sin \Theta_2 \cos (\varphi_1 + \varphi_2);$$  

$$w(-\frac{1}{2}, -\frac{1}{2}, n_1, n_2) = \frac{1}{2} (\cos^2 \frac{\Theta_1}{2} \cos^2 \frac{\Theta_2}{2} + \sin^2 \frac{\Theta_1}{2} \sin^2 \frac{\Theta_2}{2}) + \frac{1}{4} \sin \Theta_1 \sin \Theta_2 \cos (\varphi_1 + \varphi_2).$$  (95)

The matrix $M(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d})$ associated with the tomogram has the 16 matrix elements

$$M_{11} = \frac{1}{2} (\cos^2 \frac{\Theta_a}{2} \cos^2 \frac{\Theta_b}{2} + \sin^2 \frac{\Theta_a}{2} \sin^2 \frac{\Theta_b}{2}) + \frac{1}{4} \sin \Theta_a \sin \Theta_b \cos (\varphi_a + \varphi_b);$$  

$$M_{21} = \frac{1}{2} (\cos^2 \frac{\Theta_a}{2} \sin^2 \frac{\Theta_b}{2} + \sin^2 \frac{\Theta_a}{2} \cos^2 \frac{\Theta_b}{2}) - \frac{1}{4} \sin \Theta_a \sin \Theta_b \cos (\varphi_a + \varphi_b);$$  

$$M_{31} = \frac{1}{2} (\cos^2 \frac{\Theta_a}{2} \sin^2 \frac{\Theta_b}{2} + \sin^2 \frac{\Theta_a}{2} \cos^2 \frac{\Theta_b}{2}) - \frac{1}{4} \sin \Theta_a \sin \Theta_b \cos (\varphi_a + \varphi_b);$$
and takes maximal value 2. One can see that matrix corresponding to angles $\Theta_a, \Theta_c, \Theta_d$ for the second matrix components state discussed in previous sections. If one has the probability distribution vector with three nonnegative representation of quantum states. The idea of the construction is to find the qubit portrait of the qutrit tomograms of all the quantum states. Violation of Bell inequalities signals that the state is entangled. The product $W$ for two qubits.

One can see that matrix $M$ violates the condition which is Bell inequality for some angles and takes maximal value $2\sqrt{2}$ which is Cirelson bound. It is due to entanglement of the state. The product $M$ of two matrices corresponding to angles $\Theta_a, \Theta_b, \Theta_c, \Theta_d, \varphi_a, \varphi_b, \varphi_c, \varphi_d$ for the first matrix $M_1$ and $\Theta_{a'}, \Theta_{b'}, \Theta_{c'}, \Theta_{d'}, \varphi_{a'}, \varphi_{b'}, \varphi_{c'}, \varphi_{d'}$ for the second matrix $M_2$, i.e. $M = M_1 M_2$ must satisfy Bell inequality for separable state. These matrices form semigroup which is sub-semigroup of all the stochastic matrices constructed of means of tomograms of all the quantum states.

\begin{align}
M_{11} &= \frac{1}{2} (\cos^2 \Theta_a \cos^2 \Theta_b + \sin^2 \Theta_a \sin^2 \Theta_b) + \frac{1}{4} \sin \Theta_a \sin \Theta_b \cos (\varphi_a + \varphi_b); \\
M_{12} &= \frac{1}{2} (\cos^2 \Theta_a \cos^2 \Theta_c + \sin^2 \Theta_a \sin^2 \Theta_c) + \frac{1}{4} \sin \Theta_a \sin \Theta_c \cos (\varphi_a + \varphi_c); \\
M_{22} &= \frac{1}{2} (\cos^2 \Theta_a \sin^2 \Theta_c + \sin^2 \Theta_a \cos^2 \Theta_c) - \frac{1}{4} \sin \Theta_a \sin \Theta_c \cos (\varphi_a + \varphi_c); \\
M_{23} &= \frac{1}{2} (\cos^2 \Theta_a \sin^2 \Theta_d + \sin^2 \Theta_a \cos^2 \Theta_d) + \frac{1}{4} \sin \Theta_a \sin \Theta_d \cos (\varphi_a + \varphi_b); \\
M_{33} &= \frac{1}{2} (\cos^2 \Theta_d \cos^2 \Theta_b + \sin^2 \Theta_d \cos^2 \Theta_b) - \frac{1}{4} \sin \Theta_d \sin \Theta_b \cos (\varphi_a + \varphi_b); \\
M_{34} &= \frac{1}{2} (\cos^2 \Theta_d \sin^2 \Theta_c + \sin^2 \Theta_d \cos^2 \Theta_c) + \frac{1}{4} \sin \Theta_d \sin \Theta_c \cos (\varphi_a + \varphi_c); \\
M_{44} &= \frac{1}{2} (\cos^2 \Theta_d \sin^2 \Theta_d + \sin^2 \Theta_d \cos^2 \Theta_d) + \frac{1}{4} \sin \Theta_d \sin \Theta_d \cos (\varphi_a + \varphi_c); \\
\end{align}

One can see that matrix $M$ violates the condition which is Bell inequality for some angles and takes maximal value $2\sqrt{2}$ which is Cirelson bound. It is due to entanglement of the state. The product $M$ of two matrices corresponding to angles $\Theta_a, \Theta_b, \Theta_c, \Theta_d, \varphi_a, \varphi_b, \varphi_c, \varphi_d$ for the first matrix $M_1$ and $\Theta_{a'}, \Theta_{b'}, \Theta_{c'}, \Theta_{d'}, \varphi_{a'}, \varphi_{b'}, \varphi_{c'}, \varphi_{d'}$ for the second matrix $M_2$, i.e. $M = M_1 M_2$ must satisfy Bell inequality for separable state. These matrices form semigroup which is sub-semigroup of all the stochastic matrices constructed of means of tomograms of all the quantum states.

12 Reduction of the qubit-qutrit separability property to Bell inequalities for two qubits.

Here we demonstrate the new necessary condition of separability of qubit-qutrit state using the probability representation of quantum states. The idea of the construction is to find the qubit portrait of the qutrit state discussed in previous sections. If one has the probability distribution vector with three nonnegative components

\[ \vec{W} = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} \]  

\[ (97) \]
where \( W_1 + W_2 + W_3 = 1 \) the new probability distribution vector \( \vec{\rho} \) can be constructed

\[
\vec{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 + W_3 \end{pmatrix}
\]

(98)

It means that each three-dimensional distribution induces two-dimensional ones. One can use all vectors.

\[
\vec{\rho}' = \begin{pmatrix} \rho_1' \\ \rho_2' \end{pmatrix} = \begin{pmatrix} W_1 + W_2 \\ W_3 \end{pmatrix}
\]

(99)

and

\[
\vec{\rho}'' = \begin{pmatrix} \rho_1'' \\ \rho_2'' \end{pmatrix} = \begin{pmatrix} W_1 + W_3 \\ W_2 \end{pmatrix}
\]

(100)

Let us consider simply separable state of qubit-qutrit system with density operator \( \hat{\rho}(1, 2) = \hat{\rho}(1) \otimes \hat{\rho}(2) \).

Then the tomogram of this state is the probability distribution of the form

\[
w(m_1, \vec{n}_1, m_2, \vec{n}_2) = w_1(m_1, \vec{n}_1)W(m_2, \vec{n}_2).
\]

(101)

Here the spin projections \( m_1 \) take values \(-1/2, +1/2\) and spin projections \( m_2 \) take values \(-1, +1, 0\). In the form of 6-dimensional vector the tomogram (101) can be rewritten as

\[
\bar{w}(\vec{n}_1, \vec{n}_2) = \bar{w}_1(\vec{n}_1) \otimes \bar{W}_1(\vec{n}_2),
\]

(102)

where

\[
\bar{w}_1(\vec{n}_1) = \begin{pmatrix} w_1(\vec{n}_1) \\ w_2(\vec{n}_1) \end{pmatrix},
\]

(103)

and

\[
\bar{W}_1(\vec{n}_2) = \begin{pmatrix} W_1(\vec{n}_2) \\ W_2(\vec{n}_2) \\ W_3(\vec{n}_2) \end{pmatrix}.
\]

(104)

Thus one has

\[
\bar{w}(\vec{n}_1, \vec{n}_2) = \begin{pmatrix} w_1(\vec{n}_1)W_1(\vec{n}_2) \\ w_1(\vec{n}_1)W_2(\vec{n}_2) \\ w_1(\vec{n}_1)W_3(\vec{n}_2) \\ w_2(\vec{n}_1)W_1(\vec{n}_2) \\ w_2(\vec{n}_1)W_2(\vec{n}_2) \\ w_2(\vec{n}_1)W_3(\vec{n}_2) \end{pmatrix}.
\]

(105)

Now we apply the described ansatz of reduction of three dimensional distributions to two dimensional ones. We get from \( m \) (104) the vector

\[
\bar{\rho}_1(\vec{n}_2) = \begin{pmatrix} W_1(\vec{n}_2) \\ W_2(\vec{n}_2) + W_3(\vec{n}_2) \end{pmatrix}.
\]

(106)
This reduction induces the reduction of the 6-vector \((105)\) to the 4-vector

\[
\vec{\rho}(\vec{n}_1, \vec{n}_2) = \begin{pmatrix} w_1(\vec{n}_1)W_1(\vec{n}_2) \\ w_1(\vec{n}_1)(W_2(\vec{n}_2) + W_3(\vec{n}_2)) \\ w_2(\vec{n}_1)W_1(\vec{n}_2) \\ w_2(\vec{n}_1)(W_2(\vec{n}_2) + W_3(\vec{n}_2)) \end{pmatrix}.
\]

One has the simple observation. If the tomogram is simply separable the reduced distribution vector \(\vec{\rho}(\vec{n}_1, \vec{n}_2)\) is also simply separable distribution. From this property if follows the same property for a convex sum of simply separable distributions. One has for separable quantum state of qubit-qutrit system the following property of its spin tomogram. Let this spin tomogram be given by a probability distribution \(w(m_1, \vec{n}_1, m_2, \vec{n}_2)\) which corresponds either to separable or entangled state. Let us denote this tomogram by the vector

\[
\vec{w}(\vec{n}_1, \vec{n}_2) = \begin{pmatrix} w(\frac{1}{2}, \vec{n}_1, +1, \vec{n}_2) \\ w(\frac{1}{2}, \vec{n}_1, 0, \vec{n}_2) \\ w(\frac{1}{2}, \vec{n}_1, -1, \vec{n}_2) \\ w(-\frac{1}{2}, \vec{n}_1, +1, \vec{n}_2) \\ w(-\frac{1}{2}, \vec{n}_1, 0, \vec{n}_2) \\ w(-\frac{1}{2}, \vec{n}_1, -1, \vec{n}_2) \end{pmatrix}.
\]

Then we introduce the 4-vector

\[
\vec{\rho}(\vec{n}_1, \vec{n}_2) = \begin{pmatrix} w(\frac{1}{2}, \vec{n}_1, +1, \vec{n}_2) \\ w(\frac{1}{2}, \vec{n}_1, 0, \vec{n}_2) + w(\frac{1}{2}, \vec{n}_1, -1, \vec{n}_2) \\ w(-\frac{1}{2}, \vec{n}_1, +1, \vec{n}_2) \\ w(-\frac{1}{2}, \vec{n}_1, 0, \vec{n}_2) + w(-\frac{1}{2}, \vec{n}_1, -1, \vec{n}_2) \end{pmatrix}.
\]

Now we apply the criterion of separability used for two qubit states discussed in previous sections. It means that we construct stochastic 4x4-matrix where in the column one has the components of the vectors \((109)\) with corresponding vectors \(\vec{n}_1, \vec{n}_2\)

\[
P(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = ||\vec{\rho}(\vec{a}, \vec{b})\rho(\vec{c}, \vec{d})||.
\]

We get the result. If the matrix elements of the matrix \((110)\) violate the Bell inequality the qubit-qutrit state is entangled. The fulfilling of the Bell inequality \((92)\) is necessary condition of the separability of the qubit-qutrit state.

### 13 Qubit-qutrit and two qutrits

We present here two examples of entangled states. Let density matrix of qubit-qutrit state in standard basis \(|1/2, m_1 > |1, m_2 >\) have the form

\[
\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\]

\[\text{(111)}\]
Two unitary matrices transforming qubits

\[
\begin{align*}
U_{11} &= e^{\frac{i\varphi_1}{2}} \cos \frac{\theta_1}{2}; \\
U_{12} &= i e^{\frac{i\varphi_1}{2}} \sin \frac{\theta_1}{2}; \\
U_{21} &= i e^{-\frac{i\varphi_1}{2}} \sin \frac{\theta_1}{2}; \\
U_{22} &= e^{-\frac{i\varphi_1}{2}} \cos \frac{\theta_1}{2};
\end{align*}
\] (112)

and qutrits

\[
\begin{align*}
V_{11} &= e^{i\varphi_2} \cos^2 \frac{\theta_2}{2}; \\
V_{12} &= i e^{i\varphi_2} \sin \Theta_2 \sqrt{2}; \\
V_{13} &= -e^{i\varphi_2} \sin^2 \frac{\theta_2}{2}; \\
V_{21} &= \sin \Theta_2 \sqrt{2}; \\
V_{22} &= \cos \Theta_2; \\
V_{23} &= i \sin \Theta_2 \sqrt{2}; \\
V_{31} &= -e^{-i\varphi_2} \sin^2 \frac{\theta_2}{2}; \\
V_{32} &= i e^{-i\varphi_2} \sin \Theta_2 \sqrt{2}; \\
V_{33} &= e^{-i\varphi_2} \cos^2 \frac{\theta_2}{2};
\end{align*}
\] (113)

can be used to construct the 6x6-matrices \(U \otimes V\) and \(U^\dagger \otimes V^\dagger\). The diagonal matrix elements of the matrix

\[
[(U^\dagger \otimes V^\dagger) \rho(U \otimes V)]_{m_1 m_2, m_1 m_2} = w(m_1, \vec{m}_1, m_2, \vec{m}_2)
\] (114)

provide the spin tomogram of the state \(\text{(111)}\). Here the two vectors are determined by angles \(\theta_1, \varphi_1, \theta_2, \varphi_2\) as \(\vec{m}_1 = (\sin \Theta_1 \cos \varphi_1, \sin \Theta_1 \sin \varphi_1, \cos \Theta_1)\), \(\vec{m}_2 = (\sin \Theta_2 \cos \varphi_2, \sin \Theta_2 \sin \varphi_2, \cos \Theta_2)\).

One has

\[
\begin{align*}
w(\frac{1}{2}, \vec{m}_1, +1, \vec{m}_2) &= \frac{1}{2} |U_{11} V_{11} + U_{21} V_{31}|^2; \\
w(\frac{1}{2}, \vec{m}_1, 0, \vec{m}_2) &= \frac{1}{2} |U_{11} V_{12} + U_{21} V_{32}|^2; \\
w(\frac{1}{2}, \vec{m}_1, -1, \vec{m}_2) &= \frac{1}{2} |U_{11} V_{13} + U_{21} V_{33}|^2; \\
w(-\frac{1}{2}, \vec{m}_1, +1, \vec{m}_2) &= \frac{1}{2} |U_{12} V_{11} + U_{22} V_{31}|^2;
\end{align*}
\]
Applying the reduction ansatz we get the 4x4-matrix \((110)\). Calculating the modulus of trace of product of this matrix and the matrix \(I\) given by \((79)\) we get the expression which we denote as

\[
B = |\sin \Theta_a (\sin^2 \Theta_b \sin \Phi_{ab} + \sin^2 \Theta_c \sin \Phi_{ac}) + \sin \Theta_d (\sin^2 \Theta_b \sin \Phi_{db} - \sin^2 \Theta_c \sin \Phi_{dc})|.
\]  

(116)

Here \(\Phi_{ab} = \varphi_a + 2\varphi_b, \ \Phi_{ac} = \varphi_a + 2\varphi_c, \ \Phi_{db} = \varphi_d + 2\varphi_b, \ \Phi_{dc} = \varphi_d + 2\varphi_c\). One can check that for parameters

\[
\Theta_a = \frac{\pi}{2}, \ \Theta_b = \frac{\pi}{2}, \ \Theta_c = \frac{\pi}{2}, \ \Theta_d = \frac{\pi}{2}, \\
\Phi_{ab} = \frac{\pi}{2}, \ \Phi_{dc} = -\frac{\pi}{4}, \ \Phi_{ac} = \frac{\pi}{4}, \ \Phi_{db} = 0
\]  

(117)

the value \(B\) (116) is larger than 2, namely

\[
B = 1 + \sqrt{2}.
\]  

(118)

If means that the qubit-qutrit state is entangled. We know this fact because the density matrix \((111)\) corresponds to pure entangled state \(|\Psi\rangle = \frac{1}{\sqrt{2}}(|+\frac{1}{2}| + 1) + |\frac{3}{2}| - 1\rangle\). For two qutrit entangled state with 9x9-density matrix with 72 matrix elements equal to zero except 9 matrix elements

\[
\rho_{11} = \rho_{15} = \rho_{19} = \rho_{51} = \rho_{55} = \rho_{59} = \rho_{91} = \rho_{95} = \rho_{99} = \frac{1}{3}
\]  

(119)

the spin tomogram can be calculated by the same method using two 3x3-matrices \(U\) and \(V\) given by the same relations \((113)\). But the matrix elements of the matrix \(U\) are taken to depend on angles \(\varphi_1\) and \(\Theta_1\). We get the vector \(\vec{w}(\vec{n}_1, \vec{n}_2)\) with nine components:

\[
\begin{align*}
w(+1, \vec{n}_1, +1, \vec{n}_2) &= \frac{1}{3} \left| \sum_{j=1}^{3} U_{j1} V_{j1} \right|^2; \\
w(+1, \vec{n}_1, 0, \vec{n}_2) &= \frac{1}{3} \left| \sum_{j=1}^{3} U_{j1} V_{j2} \right|^2; \\
w(+1, \vec{n}_1, -1, \vec{n}_2) &= \frac{1}{3} \left| \sum_{j=1}^{3} U_{j1} V_{j3} \right|^2; \\
w(0, \vec{n}_1, +1, \vec{n}_2) &= \frac{1}{3} \left| \sum_{j=1}^{3} U_{j2} V_{j1} \right|^2; \\
w(0, \vec{n}_1, 0, \vec{n}_2) &= \frac{1}{3} \left| \sum_{j=1}^{3} U_{j2} V_{j2} \right|^2; \\
w(0, \vec{n}_1, -1, \vec{n}_2) &= \frac{1}{3} \left| \sum_{j=1}^{3} U_{j2} V_{j3} \right|^2; \\
w(0, \vec{n}_1, +1, \vec{n}_2) &= \frac{1}{3} \left| \sum_{j=1}^{3} U_{j3} V_{j1} \right|^2; \\
w(0, \vec{n}_1, 0, \vec{n}_2) &= \frac{1}{3} \left| \sum_{j=1}^{3} U_{j3} V_{j2} \right|^2; \\
w(0, \vec{n}_1, -1, \vec{n}_2) &= \frac{1}{3} \left| \sum_{j=1}^{3} U_{j3} V_{j3} \right|^2;
\end{align*}
\]
\[ w(-1, \vec{n}_1, +1, \vec{n}_2) = \frac{1}{3} \left| \sum_{j=1}^{3} U_{3j} V_{j1} \right|^2; \]
\[ w(-1, \vec{n}_1, 0, \vec{n}_2) = \frac{1}{3} \left| \sum_{j=1}^{3} U_{3j} V_{j2} \right|^2; \]
\[ w(-1, \vec{n}_1, -1, \vec{n}_2) = \frac{1}{3} \left| \sum_{j=1}^{3} U_{3j} V_{j3} \right|^2. \] (120)

We construct the qubit portrait of this state. One of 4-vectors \( \vec{P}(\vec{n}_1, \vec{n}_2) \) of this portrait has the components

\[ P_1(\vec{n}_1, \vec{n}_2) = w(+1, \vec{n}_1, +1, \vec{n}_2) \]
\[ P_2(\vec{n}_1, \vec{n}_2) = w(+1, \vec{n}_1, 0, \vec{n}_2) + w(+1, \vec{n}_1, -1, \vec{n}_2) \]
\[ P_3(\vec{n}_1, \vec{n}_2) = w(0, \vec{n}_1, +1, \vec{n}_2) + w(-1, \vec{n}_1, +1, \vec{n}_2) \]
\[ P_4(\vec{n}_1, \vec{n}_2) = w(0, \vec{n}_1, 0, \vec{n}_2) + w(0, \vec{n}_1, -1, \vec{n}_2) + w(-1, \vec{n}_1, 0, \vec{n}_2) + w(-1, \vec{n}_1, -1, \vec{n}_2) \] (121)

Using (120) and (121) and taking pairs \( (\vec{n}_1) = \vec{a}, (\vec{n}_2) = \vec{b}, \quad (\vec{n}_1) = \vec{a}, (\vec{n}_2) = \vec{c}, \quad (\vec{n}_1) = \vec{d}, (\vec{n}_2) = \vec{b}, \quad (\vec{n}_1) = \vec{d}, (\vec{n}_2) = \vec{c} \) one can construct the 4x4-matrix \( [110] \). Calculating the modulus of trace of product of matrix \( [110] \) with the obtained matrix we get the value of \( B \) of the form

\[ B = \frac{1}{2} |(\cos \Theta_b + 1)^2 - 2)(\cos \Theta_a + \cos \Theta_d) + \]
\[ + (\cos \Theta_c + 1)^2 - 2)(\cos \Theta_a - \cos \Theta_d) - \]
\[ - \sin^2 \Theta_b (\sin \Phi_{ab} \sin \Theta_a + \sin \Phi_{ab} \sin \Theta_d) - \]
\[ - \sin^2 \Theta_c (\sin \Phi_{ac} \sin \Theta_a + \sin \Phi_{ac} \sin \Theta_d) | \] (122)

One can check that for angles

\[ \varphi_a = 2\pi, \quad \varphi_b = -\frac{\pi}{8}, \quad \varphi_c = \frac{\pi}{8}, \quad \varphi_d = 0, \]
\[ \Theta_a = 0, \quad \Theta_b = \frac{\pi}{2}, \quad \Theta_c = \frac{\pi}{2}, \quad \Theta_d = \frac{\pi}{2}. \] (123)

the value of \( B \) is \( (1 + \sqrt{2}) > 2 \). It corresponds to entangled two qudit state.

### 14 General reduction criterion of separability

Now we use the experience with discussed qubit-qutrit system to formulate a general criterion of separability for a state of bipartite quantum system. The criterion is based on the property of a separable state tomogram of a bipartite system. Let us take for simplicity a two qudit separable state with the tomogram of the form \( [31] \). Let us associate with this tomogram the joint probability distribution given as four nonnegative numbers

\[ \tilde{w}(M_1 = j_1, M_2 = j_2, \vec{n}_1, \vec{n}_2) = w(j_1, j_2, \vec{n}_1, \vec{n}_2); \]
\[ \tilde{w}(M_1 = j_1, M_2 = j_2 - 1, \vec{n}_1, \vec{n}_2) = \sum_{m_2 = -j_2}^{j_2-1} w(j_1, m_2, \vec{n}_1, \vec{n}_2); \]
\[ \hat{w}(M_1 = j_1 - 1, M_2 = j_2, \vec{n}_1, \vec{n}_2) = \sum_{m_1 = -j_1}^{j_1 - 1} w(m_1, j_2, \vec{n}_1, \vec{n}_2); \]
\[ \hat{w}(M_1 = j_1 - 1, M_2 = j_2 - 1, \vec{n}_1, \vec{n}_2) = \sum_{m_1 = -j_1}^{j_1 - 1} \sum_{m_2 = -j_2}^{j_2 - 1} w(m_1, m_2, \vec{n}_1, \vec{n}_2). \]

(124)

Here \( M_1 \) takes two values \( j_1 \) and \( j_1 - 1 \) and \( M_2 \) takes the values \( j_2 \) and \( j_2 - 1 \). We will reinterpret the obtained joint probability distribution as a two-qubit "tomogram". Due to this the Bell inequality is fulfilled for the probability distribution if the initial two-qudit state is separable. We used ansatz of obtaining the reduced joint probability distribution by summing the probabilities in initial probability distribution with larger number of possible events (or measurements). But the separability of the initial quantum state is preserved in process of such summing in the sense that if initial tomographic probability distribution looks as a convex sum of products of two distributions the reduced distribution is also the convex sum of the product of two probability distributions. The obtained result can be formulated as the following reduction criterion of separability. The necessary condition of separability of bipartite system state is the separability property of the reduced state tomogram. The fulfilling of Bell inequalities for reduced state tomogram is necessary condition of separability of the quantum state under study. One can give a recipe for studying the separability of a given state of bipartite system. First step is to obtain the tomogram of the state. Than one has to reduce this tomogram by summing over all such events to get the "tomogram" of two qubit. Then one checks the fulfilling the Bell inequality for the obtained reduced tomogram. If it is violated the initial state is entangled.

15 Conclusion

To conclude we summarize the main results of our work. We shown that the qudit states can be mapped onto probability distributions which are the points on the simplex. The probability distributions can be considered as vectors. The stochastic and bistochastic matrices can be constructed using these vectors as columns of the matrices. Both stochastic and bistochastic matrices form semigroups. The invertible map of probability distributions onto bistochastic matrix was used to construct star-product of the probability distributions. For qudit tomograms we introduced the notion of qubit portrait. We shown that the necessary condition of separability of bipartite qudit state is separability of its qubit portrait. The Bell inequality violation for qubit portrait of bipartite system state (both for qudit states and for continuous variables) means that the system state is entangled. Examples of entangled qubit-qutrit state and two-qutrit state were considered using the method of constructing the qubit portrait of the states. The method can be generalized for multiqudit systems.

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