Conformal Skorokhod embeddings of the uniform distribution and related extremal problems

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Abstract

Let \( \mu \) be a probability distribution with zero mean and finite nonzero variance. The conformal Skorokhod embedding problem (CSEP) asks for a simply connected domain \( D \subset \mathbb{C} \) containing 0 such that the real part of standard complex Brownian motion at its first exit time from \( D \) has distribution \( \mu \) with the exit time having finite mean. The CSEP was posed and solved by Gross in [Gro19] where the author gives an explicit construction of a solution domain for any \( \mu \). In this paper we give an example of a solution domain \( U \) for the uniform distribution \( U[-1,1] \) which differs from Gross’ and possesses the following extremal property: If \( D \) is any solution to the CSEP for \( U[-1,1] \), then the principal Dirichlet eigenvalue of \( D \) is at least that of \( U \). We also give general upper and lower bounds on the principal Dirichlet eigenvalue of a solution domain to the CSEP for any \( \mu \). The proofs rely on a recent spectral upper bound of the torsion function as well as a precise relationship between the widths of the orthogonal projections of a simply connected planar domain and the support of its harmonic measure which is developed in the paper.

1 Introduction

Let \( W = (W_t: t \geq 0) \) be a complex Brownian motion starting at 0, and for any open set \( D \subset \mathbb{C} \) containing 0, let \( \tau_D \) denote the first exit time of \( W \) from \( D \). Moreover, let \( \mu \) be a probability distribution with zero mean and finite nonzero variance. The conformal Skorokhod embedding problem (CSEP) was introduced in [Gro19] where the author shows that there exists a simply connected domain \( D \subset \mathbb{C} \) containing 0 such that

\[
\begin{cases}
\text{Re} \, W_{\tau_D} \sim \mu \\
E_0[\tau_D] < \infty
\end{cases}
\]

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This result was recently generalized in [BM19]. There the authors showed that if \( \mu \) is a probability distribution with zero mean and finite nonzero \( p \)-th moment for some \( 1 < p < \infty \), then there exists a simply connected domain \( D \subset \mathbb{C} \) containing 0 such that \( \text{Re} W_{\tau_D} \sim \mu \) and \( \mathbb{E}_0 [\tau_D^{p/2}] < \infty \). They also give conditions on the domain which ensure that a solution domain is unique.

The original Skorokhod embedding problem (SEP) asks the following: Given a standard Brownian motion \( (B_t : t \geq 0) \) and a probability distribution \( \mu \) with zero mean and finite variance, find a stopping time \( T \) such that \( B_T \sim \mu \) and \( \mathbb{E}_0 [T] < \infty \). It was first posed and solved by Skorokhod in 1961 [Sko61, Sko65] and since then a veritable zoo of varied and interesting solutions have appeared, see [Ob104] for a thorough survey. The similarity between these problems is clear, and as Gross points out, his solution resembles Root’s barrier hitting solution [Roo69] with additional randomness, that of the other Brownian motion in the second coordinate.

Gross’s paper is more than just an existence result and his method gives a relatively explicit construction of a domain \( D \) which solves (1). In fact, his example of a bounded domain which solves the CSEP for the uniform distribution on \([-1,1]\) is what inspired the present paper as it complements an unbounded domain \( U \) known to the authors which achieves the same objective. Besides bringing to light this additional example, we also aim to compare some aspects of these two solution domains.

This brings us to the second goal of the paper which is to pose related extremal problems. For a domain \( D \subset \mathbb{C} \) which solves the CSEP for \( \mu \), define the rate of the solution by

\[
\lambda(D) = -\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_0(\tau_D > t).
\]

The limit exists by Fekete’s subadditivity lemma and is clearly finite and non-negative. Additionally, \( 2 \lambda(D) \) is equal to the bottom of the spectrum of the semigroup generated by the Laplacian on \( D \) with Dirichlet boundary conditions and is usually referred to as the principal Dirichlet eigenvalue of \( D \), see Section 3.1 of [Szn98]. Two questions regarding the rate naturally present themselves:

A. Find upper and lower bounds on the rate (or principal Dirichlet eigenvalue) in terms of \( \mu \).

B. For a specific \( \mu \), find extremal domains that attain the highest and lowest possible rate (or principal Dirichlet eigenvalue).

In this paper, we answer Question A by giving an upper bound in terms of the variance of \( \mu \) and a lower bound in terms of the width of the support of \( \mu \). We give a partial answer to Question B in the case of the uniform distribution on \([-1,1]\), henceforth denoted by \( U[-1,1] \), by presenting a minimal rate solution domain. Finding a maximal rate solution domain to the CSEP for \( U[-1,1] \) remains an open question and one wonders if the domain given by Gross’ construction in [Gro19, Section 3.2] might be a maximal rate solution.
Question B. is in the same spirit as the original SEP where different solutions often have their own extremal property. For instance, given a probability distribution $\mu$ with zero mean and finite variance, Root’s embedding minimizes the variance of $T$ over all stopping times $T$ such that $B_T \sim \mu$ and $\mathbb{E}_0 [T] < \infty$, while Rost’s reversed barrier embedding maximizes the variance of $T$ over the same class of stopping times. Again, we refer to [Ol94] for more on this topic.

Extremal problems for Brownian motion apart from the SEP have also been a popular topic of study. A common theme is optimizing the principal Dirichlet eigenvalue or the maximum expected exit time of a domain taken over all starting points when various constraints are given, see [BnC94, BnC00, MH02, BnC11, HS19] and reference therein for some examples.

The CSEP also has a connection to Walden and Ward’s harmonic measure distributions [WW96] that is worth mentioning. Indeed, if the domain $D$ solves the CSEP for $\mu$, then the orthogonal projection of the harmonic measure of $D$ with pole at 0 on to the real axis has distribution $\mu$. If instead we are given a distribution function $F : (0, \infty) \to [0, 1]$, then one might ask whether there exists a domain $D \subset \mathbb{C}$ containing 0 such that the circular projection of the harmonic measure of $D$ with pole at 0 on to the positive real axis has distribution function $F$. This is a topic of current interest and the reader is directed to the recent survey [SW16] for more on this question. Perhaps an adaptation of Gross’ construction will prove fruitful in this direction.

Part of our answer to Question A relies on a recent spectral upper bound of the torsion function which is described at the beginning of Section 2. To answer the other part of Question A we need a precise relationship between the width of the orthogonal projection of a simply connected planar domain and the width of the support of the orthogonal projection of its harmonic measure. This result is developed in Section 3. As far as the authors know, it hasn’t appeared in the literature before. The domain $U$ which partially answers Question B was discovered through a naive ansatz which uses the conformal invariance of Brownian motion and the Cauchy-Riemann equations to prescribe a conformal deformation of the upper half-plane $\mathbb{H}$, whose harmonic measure is Cauchy distributed, such that the orthogonal projection of the harmonic measure of the deformed domain has distribution $U[-1, 1]$. This ansatz is then verified in Section 4.3 using an explicit conformal map from $U$ onto $\mathbb{H}$. Unfortunately, it seems that this technique only works for embedding the uniform distribution.

2 Main Results

Our first main result is an upper bound on $\lambda(D)$ for $d$-dimensional Brownian motion in terms of the second moments of the components at time $\tau_D$. When applied to a domain $D \subset \mathbb{C}$ which solves the CSEP, this general result provides a partial answer to Question A. The proof involves a spectral upper bound of the torsion function of Brownian motion. Recall that the torsion function of a domain $D$ is nothing but the expected exit time of Brownian motion from $D$ as
a function of the starting point and is given by
\[ u_D(x) = \mathbb{E}_x [\tau_D], \quad x \in D \]
\[ = \begin{cases} -\frac{1}{2} \Delta u_D = 1 \\ u_D \in H^1_0(D). \end{cases} \]

The best constant in the spectral upper bound of the torsion function depends on the dimension \( d \) and is defined by
\[ C_d = \sup \{ \lambda(D) \| u_D \|_\infty : D \subset \mathbb{R}^d \text{ is a domain with } \lambda(D) > 0 \} . \tag{2} \]

It was shown in [vdBC09] that \( \| u_D \|_\infty < \infty \) if and only if \( \lambda(D) > 0 \), so it follows from (2) that the spectral upper bound of the torsion function
\[ \| u_D \|_\infty \leq C_d \lambda(D)^{-1} \]
holds for all domains \( D \).

Bounds on \( C_d \) for Brownian motion and related processes have been studied extensively via several techniques under various assumptions on \( D \), see for instance [BnC94, vdBC09, GS10, vdB17, Vog19, Pan20]. The current best bound in the Brownian case was derived by H. Vogt in [Vog19] and states that
\[ C_d \leq \frac{d}{8} + \frac{1}{4} \sqrt{5 \left( 1 + \frac{1}{4} \log 2 \right) d + 1} . \tag{3} \]

**Theorem 1.** Suppose \( D \) is an open subset of \( \mathbb{R}^d \) which contains \( 0 \) and let \( W = (W_t : t \geq 0) \) be a \( d \)-dimensional Brownian motion starting at \( 0 \) with \( W_t = (W_t^{(1)}, \ldots, W_t^{(d)}) \). Then for each \( 1 \leq i \leq d \) we have
\[ \lambda(D) \leq C_d \mathbb{E}_0 \left[ \left( W_{\tau_D}^{(i)} \right)^2 \right]^{-1} \]

In particular, if \( D \) solves the CSEP [1] for a probability distribution \( \mu \) with zero mean and finite variance, then
\[ \lambda(D) \leq \frac{C_2}{\Var \mu} \leq \frac{2.1063}{\Var \mu} . \tag{4} \]

**Remark 1.** Under the added condition that \( D \) is convex and if we assume that the equilateral triangle conjecture of [HLPT18] is true, then the inequality (4) can be improved to
\[ \lambda(D) \leq \frac{4\pi^2}{27 \Var \mu} \approx 1.4622 \frac{\Var \mu}{\Var \mu} . \]

**Corollary 1.** If \( D \) solves the CSEP for \( U[-1,1] \), then
\[ \lambda(D) \leq 3 C_2 \leq 6.319. \]
Our next result is a lower bound for $\lambda(D)$ in terms of the support of $\mu$. When paired with Theorem 1, this provides a full answer to Question A.

**Theorem 2.** Suppose $D \subset \mathbb{C}$ solves the CSEP (1) for a probability distribution $\mu$. If $[\alpha, \beta]$ is the smallest interval containing the support of $\mu$, then

$$\lambda(D) \geq \frac{\pi^2}{2(\beta - \alpha)^2}. \quad (5)$$

**Corollary 2.** If $D$ solves the CSEP for $U[-1,1]$, then

$$\lambda(D) \geq \frac{\pi^2}{8} \approx 1.2337.$$  

Our final result partially answers Question B, by exhibiting a domain $U$ which is a minimal rate solution to the CSEP for $U[-1,1]$. See Figure 1 for a comparison of $U$ and Gross’ solution domain to the CSEP for $U[-1,1].$

Figure 1: $U$ is the region above the U-shaped graph. Gross’ solution to the CSEP for $U[-1,1]$ is the region bounded by the closed curve.
Theorem 3. Let

\[ U = \{ z \in \mathbb{C} : |\text{Re } z| < 1, \text{Im } z > h(\text{Re } z) \} \quad (6) \]

where

\[ h(x) = -\frac{2}{\pi} \log \left( 2 \cos \frac{\pi}{2} \right), \quad |x| < 1. \]

Then under \( P_0 \) we have

\[ \text{Re } (W_{\tau_U}) \sim U[-1, 1]. \]

Moreover, \( U \) is a minimal rate solution to the CSEP for \( U[-1, 1] \). That is, if \( D \subset \mathbb{C} \) is another solution to the CSEP \( (\Pi) \) for \( U[-1, 1] \), then \( \lambda(U) \leq \lambda(D) \).

Remark 2. The simply connected domain \( U \) is a rotation and scaling of a domain studied in \cite{Mar11, Example 4} where the author points out that its boundary curve has been referred to as the “catenary of equal resistance”. In that paper the expected exit time is computed but no mention is made of the distribution of the real or imaginary parts of the stopped Brownian motion.

Remark 3. The family of scaled domains \( aU \) with \( a > 0 \) gives minimal rate solutions to the CSEP for \( U[-a,a] \).

3 Orthogonal Projection of Harmonic Measure

The goal of this section is to prove a relationship between the width of the projection of \( D \) on to the real axis and the width of the support of \( \text{Re } W_{\tau_D} \). More specifically, suppose \( D \subset \mathbb{C} \) is a simply connected domain containing 0 and let \( \text{Re } D \) denote its projection on to the real axis, that is, \( \text{Re } D = \{ \text{Re } z : z \in D \} \). Since \( D \) is both open and connected and the projection map is both open and continuous, we know that \( D = (a, b) \) for some \( a \in [-\infty, 0) \) and \( b \in (0, \infty] \).

The boundary \( \partial D \) is nonpolar under the assumptions on \( D \) so \( \tau_D \) is almost surely finite. Define

\[ \alpha = \sup \{ x \in \mathbb{R} : P_0(\text{Re } W_{\tau_D} \in [x, \infty)) = 1 \} \]

and

\[ \beta = \inf \{ x \in \mathbb{R} : P_0(\text{Re } W_{\tau_D} \in (-\infty, x]) = 1 \}. \]

It is clear that \( \alpha \leq \beta \) by definition. Moreover, since \( W_{\tau_D} \in \partial D \) almost surely, it follows that \( [\alpha, \beta] \subset [a, b] \). Hence if \( (a, b) \) is bounded, then \( [\alpha, \beta] \) is the smallest compact interval (possibly degenerate) such that \( P_0(\text{Re } W_{\tau_D} \in [\alpha, \beta]) = 1 \).

Note that \( D \) may still be unbounded in this case. The following theorem gives conditions under which the reverse inclusion also holds.

Theorem 4. Suppose \( D \subset \mathbb{C} \) is a simply connected domain containing 0 and let \( a, \alpha, b, \beta \) be defined as above. Then \( a > -\infty \) implies \( \alpha = a \) and \( b < \infty \) implies \( \beta = b \). Moreover, if \( E_0[\tau_D] < \infty \), then \( \alpha = a \) and \( \beta = b \) regardless of whether \( a \) and \( b \) are finite.
Remark 4. Considering domains such as the interior of \( U^c \) (translated so that it contains 0) show that an additional condition is needed for \( \alpha = a \) and \( \beta = b \) to hold when \( a \) and \( b \) are infinite. The sufficient condition \( \mathbb{E}_0[\tau_D] < \infty \) is one that happens to fit well with the CSEP.

The proof of Theorem 4, which we postpone until the end of the section, relies on the following lemma which states that there is positive probability of \( W \) exiting \( D \) through any neighborhood of any boundary point. Here we denote by \( B_r(x) \) and \( \overline{B}_r(x) \) the open and closed ball, respectively, of radius \( r > 0 \) centered at \( x \).

**Lemma 1.** Suppose \( D \subset \mathbb{C} \) is a simply connected domain and let \( x \in D \) and \( \xi \in \partial D \). Then for all \( \epsilon > 0 \) we have
\[
\mathbb{P}_x \left( W_{\tau_D} \in \overline{B}_\epsilon(\xi) \cap \partial D \right) > 0.
\]

**Proof.** Fix \( \epsilon > 0 \) and pick \( y \in B_\epsilon(\xi) \cap D \). Since \( D \) is a domain, \( x \) and \( y \) can be connected by a polygonal path in \( D \). This path is a compact subset of \( D \) so it has a positive distance from \( \partial D \). Hence the Harnack inequality implies
\[
\mathbb{P}_x \left( W_{\tau_D} \in \overline{B}_\epsilon(\xi) \cap \partial D \right) > 0 \text{ iff } \mathbb{P}_y \left( W_{\tau_D} \in \overline{B}_\epsilon(\xi) \cap \partial D \right) > 0 \quad (7)
\]
so we can focus on proving that the latter inequality holds. Since \( W \) has continuous paths, it will hit \( \partial D \) before it hits the interior of \( D^c \), hence the latter probability appearing in (7) is equal to
\[
\mathbb{P}_y \left( W_{\tau_D} \in \overline{B}_\epsilon(\xi) \cap D^c \right).
\]

Next, we can bound (8) from below by
\[
\mathbb{P}_y \left( W_{\tau_{B_\epsilon(\xi) \cap D}} \in \overline{B}_\epsilon(\xi) \cap D^c \right) \quad (9)
\]
since (9) excludes paths that leave \( B_\epsilon(\xi) \) and return to hit \( \overline{B}_\epsilon(\xi) \cap D^c \). Letting \( K = B_\epsilon(\xi) \cap D^c \), we can write (9) as
\[
\mathbb{P}_y \left( W_{\tau_{B_\epsilon(\xi) \cap K}} \in K \right).
\]

Define the affine transformation \( f(w) = -\frac{w-\xi}{y-\xi}(w-\xi) \) which maps \( \xi \) to 0 and \( y \) to \(-|y-\xi| \in (0, \epsilon)\). For a set \( E \subset \mathbb{C} \), let \( E^* \) denote the circular projection of \( E \). That is, \( E^* = \{|w| : w \in E\} \). Now we can use conformal invariance of Brownian motion and the version of Beurling’s projection theorem from [Øks83, Theorem 1] to get
\[
\mathbb{P}_y \left( W_{\tau_{B_\epsilon(\xi) \cap K}} \in K \right) = \mathbb{P}_f(y) \left( W_{\tau_{B_\epsilon(0) \cap (f(K))}} \in f(K) \right) \\
\geq \mathbb{P}_f(y) \left( W_{\tau_{B_\epsilon(0) \cap (f(K))^*}} \in f(K)^* \right).
\]

In order to produce a meaningful estimate, \( f(K)^* \) must contain a proper interval. We claim \( f(K)^* = [0, \epsilon] \). To see that this is true, consider the connected component of \( K \) that contains \( \xi \), call it \( E \). Clearly \( E \) contains some point \( z \)
such that $|z - \xi| = \epsilon$, for otherwise $E$ would be a bounded connected component of $D^c$ which contradicts $D$ being simply connected. This implies both 0 and $\epsilon$ are elements of $f(E)^*$. Since $E$ is connected and both $f$ and circular projection are continuous, we know that $f(E)^*$ is connected. Hence

$$[0, \epsilon] \supset f(K)^* \supset f(E)^* \supset [0, \epsilon]$$

and the claim follows.

Finally, we can use $w \mapsto \sqrt{w/\epsilon}$ to conformally map $B_\epsilon(0) \setminus [0, \epsilon]$ onto the upper half-disk $\mathbb{D} \cap \mathbb{H}$, thereby sending $f(y)$ to $\sqrt{f(y)/\epsilon}$ and the boundary set $[0, \epsilon]$ to $[-1, 1]$. Using the above inequalities along with the explicit formula for the harmonic measure of $\mathbb{D} \cap \mathbb{H}$ [Ran95, Table 4.1] while noting that $\sqrt{f(y)/\epsilon}$ is purely imaginary, we can write

$$\mathbb{P}_y(W_{\tau_D} \in \overline{B}_\epsilon(\xi) \cap \partial D) \geq 1 - \frac{2}{\pi} \arg \left( \frac{1 + \sqrt{f(y)/\epsilon}}{1 - \sqrt{f(y)/\epsilon}} \right)$$

$$= 1 - \frac{2}{\pi} \arctan \left( \frac{\sqrt{|f(y)/\epsilon|}}{1 - |f(y)/\epsilon|} \right)$$

$$> 0$$

where the last inequality follows from $|f(y)| = |y - \xi| < \epsilon$. In conjunction with (7), this proves the lemma.

With Lemma 1 in hand, we can now give a proof of Theorem 4.

**Proof of Theorem 4** Suppose $b < \infty$ and let $\epsilon > 0$. Then there exists $x \in D$ such that $\text{Re } x - b < \epsilon$. Hence $B_\epsilon(x) \cap D^c$ is nonempty, for otherwise we could increase $b$. It follows that $B_\epsilon(x)$ must contain some $\xi \in \partial D$ and that $|\text{Re } \xi - b| < 2\epsilon$.

By Lemma 1, we know that $\mathbb{P}_0(W_{\tau_D} \in \overline{B}_\epsilon(\xi) \cap \partial D) > 0$, from which it follows that $\mathbb{P}_0(\text{Re } W_{\tau_D} > b - 3\epsilon) > 0$. Hence $\mathbb{P}_0(\text{Re } W_{\tau_D} \in (-\infty, b - 3\epsilon)) < 1$. This implies that $b - 3\epsilon < \beta \leq b$ so we can conclude that $\beta = b$. The proof that $\alpha = a$ follows similarly.

Now suppose that $\mathbb{E}_0[\tau_D] < \infty$. If $b < \infty$, we have already shown that $\beta = b$, so assume $b = \infty$. We will show that $b < \infty$ leads to a contradiction.

A result of Burkholder [Bur77, Equation 3.13] shows that

$$\mathbb{E}_0[\tau_D] < \infty \text{ implies } \mathbb{E}_x[\tau_D] < \infty \text{ for all } x \in D. \quad (10)$$

By hypothesis, there exists $x \in D$ such that $\text{Re } x > \beta$. Note that $\text{Re } \xi \leq \beta$ for all $\xi \in \partial D$, for otherwise we could use Lemma 1 to increase $\beta$. Since $\tau_D \in \partial D$ almost surely, this implies

$$\mathbb{E}_x[\tau_D] \geq \mathbb{E}_x[\inf \{t \geq 0 : \text{Re } W_t \leq \beta \}] = \infty.$$  

In light of (10), this leads to the desired contradiction. The proof that $\alpha = a$ follows similarly.
4 Proofs of Main Results

4.1 Proof of Theorem 1

Proof of Theorem 1. If \( \lambda(D) = 0 \) there is nothing to prove, so assume that \( \lambda(D) > 0 \). In this case we can use (2) to write
\[
\lambda(D) \leq C_d \| u_D \|_1 \leq C_d E_0 \left( \tau_D \right)^{-1}.
\] (11)

Since the components of \( W \) are all independent, each \( W^{(i)} \) is a Brownian motion in the natural filtration of \( W \), with respect to which \( \tau_D \) is a stopping time. Hence it follows from Davis’ inequality for Brownian motion [Dav76, Equation 1.1] with best constant \( A_2 = 1 \) that
\[
E_0 \left( \frac{(W^{(i)})^2}{\tau_D} \right)^2 \leq E_0 \left( \tau_D \right)
\] (12)

for each \( 1 \leq i \leq d \). Combining (11) and (12) proves the theorem. The numerical estimates follow from (3). \( \square \)

4.2 Proof of Theorem 2

Proof of Theorem 2. If \( \beta - \alpha = \infty \), then (5) gives the trivial lower bound of 0 so there is nothing to prove in this case. Hence we can assume that \( \alpha > -\infty \) and \( \beta < \infty \). Since \( D \) is a solution domain to the CSEP (1), we know that \( E_0 [\tau_D] < \infty \). By Theorem 4 this implies that \( \{ \text{Re} z : z \in D \} = (\alpha, \beta) \). Hence \( D \) is contained in the infinite strip \( S_{\alpha,\beta} = \{ z \in \mathbb{C} : \alpha < \text{Re} z < \beta \} \). Since \( \lambda(S_{\alpha,\beta}) = \frac{\pi^2}{4(\beta - \alpha)^2} \), the result follows by domain monotonicity of Dirichlet Laplacian eigenvalues. \( \square \)

4.3 Proof of Theorem 3

The first step in proving Theorem 3 is to calculate the rate of \( U \). This will be essential in establishing that it is indeed a minimal rate solution to the CSEP for \( U[-1,1] \). We do this in the following lemma.

Lemma 2.
\[
\lambda(U) = \frac{\pi^2}{8}
\]

Proof. Since \( U \) is contained in the infinite strip \( \{ z \in \mathbb{C} : |\text{Re} z| < 1 \} \), it follows from domain monotonicity of Dirichlet Laplacian eigenvalues that \( \lambda(U) \geq \frac{\pi^2}{8} \). From (6), we see that \( U \) contains a sequence of rectangles with arbitrarily large height and whose width is approaching 2. Hence domain monotonicity implies \( \lambda(U) \leq \frac{\pi^2}{8} \) as well. \( \square \)
Now that we know $\lambda(U)$, it remains to show that $U$ is actually a solution to the CSEP for $U[-1, 1]$. The finite width of $U$ implies $E_0[\tau_U] < \infty$ and we can show that $\text{Re} W_{\tau_U} \sim U[-1, 1]$ by way of an explicit conformal map from $U$ onto the upper half-plane $\mathbb{H}$ while exploiting conformal invariance of Brownian motion.

**Proof of Theorem 3.** Consider the holomorphic function

$$f(z) = 2i e^{-\frac{\pi i z}{2}} - i.$$ 

Since $z \mapsto e^z$ is injective on the infinite strip $\{z \in \mathbb{C} : \text{Im} z < \frac{\pi}{2}\}$ and since $U$ is contained in the infinite strip $\{z \in \mathbb{C} : \text{Re} z < 1\}$, it follows that $f$ is injective on $U$.

From [4], we know that $\partial U = \{x + i h(x) : x \in (-1, 1)\}$. It will be convenient to foliate $U$ by vertical translates of $\partial U$. More specifically, for each $y \geq 0$, define $h_y = \{x + i(h(x) + y) : x \in (-1, 1)\}$.

Since $f$ is injective on $U$, we can foliate $f(U)$ by the images of $h_y$ under $f$.

Towards this end, we compute

$$f(h_y) = \left\{2i \exp \left( -\frac{\pi i x}{2} - \log \left( 2 \cos \frac{\pi x}{2} + \frac{\pi}{2} y \right) - i : x \in (-1, 1) \right) \right\}$$

$$= \left\{2i e^{\frac{\pi y}{2}} \frac{\cos \frac{\pi x}{2} - i \sin \frac{\pi x}{2}}{2 \cos \frac{\pi x}{2}} - i : x \in (-1, 1) \right\}$$

$$= \left\{e^{\frac{\pi y}{2}} \tan \frac{\pi x}{2} + i \left( e^{\frac{\pi y}{2}} - 1 \right) - i : x \in (-1, 1) \right\}$$

$$= \left\{z \in \mathbb{C} : \text{Im} z = e^{\frac{\pi y}{2}} - 1 \right\}.$$ \hfill (13)

This shows that $f(U) = \bigcup_{y>0} f(h_y) = \mathbb{H}$, hence $f$ maps $U$ conformally onto $\mathbb{H}$.

Now for $0 \leq x < 1$, the conformal invariance of Brownian motion and the fact that $\partial U$ is given by the graph of the function $h$ imply that

$$\mathbb{P}_0 \left( 0 \leq \text{Re} W_{\tau_U} \leq x \right) = \mathbb{P}_f(0) \left( \text{Re} f \left( i h(0) \right) \leq \text{Re} W_{\tau_U} \leq \text{Re} f \left( x + i h(x) \right) \right)$$

$$= \mathbb{P}_i \left( 0 \leq \text{Re} W_{\tau_U} \leq \tan \frac{\pi}{2} x \right)$$

$$= \frac{1}{2} x.$$ 

We used [13] with $y = 0$ in the middle equation and the well-known fact that $\text{Re} W_{\tau_U}$ has the standard Cauchy distribution under $\mathbb{P}_i$ in the last equation. Together with symmetry considerations, this shows that $\text{Re} W_{\tau_U} \sim U[-1, 1]$ under $\mathbb{P}_0$. In light of Corollary [2] and Lemma [2], it follows that $U$ is a minimal rate solution to the CSEP for $U[-1, 1]$.

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