Constructive error analysis of a full-discrete finite element method for the heat equation

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Abstract

In this paper, we present a new full-discrete finite element method for the heat equation, and show the numerical stability of the method by verified computations. Since, in the error analysis, we use the results in the paper [5], this work is considered as an extension of the paper. We emphasize that concerned scheme seems to be a quite normal Galerkin method and easy to implement for evolutionary equations comparing with scheme in [5]. In the constructive error estimates, we effectively use the numerical computations with guaranteed accuracy, namely, a computer assisted technique is adopted.

1 Introduction

The purpose of this paper is to establish the constructive a priori error estimates for a full-discrete approximations $Q_h^k u$ which is defined in this section, of the solution $u$ to the following linear heat equation:

\begin{align}
\frac{\partial}{\partial t} u - \nu \Delta u &= f \quad \text{in } \Omega \times J, \\
 u(x, t) &= 0 \quad \text{on } \partial \Omega \times J, \\
 u(0) &= 0 \quad \text{in } \Omega.
\end{align}

(1.1)

Here, $\Omega \subset \mathbb{R}^d$, $(d \in \{1, 2, 3\})$ is a bounded polygonal or polyhedral domain; $J := (0, T) \subset \mathbb{R}$, (for a fixed $T < \infty$) is a bounded open interval; the diffusion coefficient $\nu$ is a positive constant; and $f \in L^2(J; L^2(\Omega))$, where, in general for any normed space $Y$, we define the time-dependent Lebesgue space $L^2(J; Y)$ as a space of square integrable $Y$-valued functions on $J$. Namely,

\[ f \in L^2(J; Y) \iff \int_J \|f(t)\|_Y^2 \, dt < \infty. \]

In the discussion below, we refer to the a priori estimates as ‘constructive’ if all the constants can be numerically determined.

1.1 Notations

We denote by $L^2(\Omega)$ and $H^1(\Omega)$ the usual Lebesgue and the first order $L^2$-Sobolev spaces on $\Omega$, respectively, and by $\langle u, v \rangle_{L^2(\Omega)} := \int_\Omega u(x)v(x) \, dx$ the natural inner product of $u, v$ in $L^2(\Omega)$. By
considering the boundary and initial conditions, we define the following subspaces of $H^1(\Omega)$ and $H^1(J)$ as
\[ H^1_0(\Omega) := \{ u \in H^1(\Omega) \, ; \, u = 0 \text{ on } \partial \Omega \} \quad \text{and} \quad V^1(J) := \{ u \in H^1(J) \, ; \, u(0) = 0 \}, \]
respectively. These are Hilbert spaces with inner products
\[ \langle u, v \rangle_{H^1_0(\Omega)} := \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} \quad \text{and} \quad \langle u, v \rangle_{V^1(J)} := \left\langle \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right\rangle_{L^2(J)}. \]
Let $X(\Omega)$ be a subspace of $L^2(\Omega)$ defined by $X(\Omega) := \{ u \in L^2(\Omega) \, ; \, \Delta u \in L^2(\Omega) \}$. We define the time-dependent Sobolev spaces as usual, and define
\[ V^1(J; L^2(\Omega)) := \left\{ u \in L^2(J; L^2(\Omega)) \, ; \, \frac{\partial u}{\partial t} \in L^2(J; L^2(\Omega)) \, \right. \quad \text{and} \quad \left. u(x, 0) = 0 \text{ in } \Omega \right\}, \]
with inner product $\langle u, v \rangle_{V^1(J; L^2(\Omega))} := \left\langle \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right\rangle_{L^2(J; L^2(\Omega))}$. In the following discussion, abbreviations like $L^2H^1_0$ for $L^2(J; H^1_0(\Omega))$ will often be used. We set $V(\Omega, J) := V^1(J; L^2(\Omega)) \cap L^2(J; H^1_0(\Omega))$. Moreover, we denote the partial differential operator $\Delta_t : V(\Omega, J) \cap L^2(J; X(\Omega)) \to L^2(J; L^2(\Omega))$ by $\Delta_t := \frac{\partial^2}{\partial t^2} - \nu \Delta$.

Now let $S_h(\Omega)$ be a finite-dimensional subspace of $H^1_0(\Omega)$ dependent on the parameter $h$. For example, $S_h(\Omega)$ is considered to be a finite element space with mesh size $h$. Let $n$ be the degrees of freedom for $S_h(\Omega)$, and let $\{ \phi_i \}_{i=1}^n \subset H^1_0(\Omega)$ be the basis of $S_h(\Omega)$. Similarly, let $V^1_k(J)$ be an approximation subspace of $V^1(J)$ dependent on the parameter $k$. Let $m$ be the degrees of freedom for $V^1_k(J)$, and let $\{ v_i \}_{i=1}^m \subset V^1_k(J)$ be the basis of $V^1_k(J)$. Let $V^1(J; S_h(\Omega))$ be a subspace of $V$ corresponding to the semidiscretized approximation in the spatial direction, and the space $V^1_k(J; S_h(\Omega))$ is defined as the tensor product $V^1_k(J) \otimes S_h(\Omega)$, which corresponds to a full discretization. We define the $H^1_0$-projection $P^1_h u \in S_h(\Omega)$ of any element $u \in H^1_0(\Omega)$ by the following variational equation:
\[ \langle \nabla (u - P^1_h u), \nabla v_h \rangle_{L^2(\Omega)} = 0, \quad \forall v_h \in S_h(\Omega). \tag{1.2} \]
The $V^1$-projection $P^1_k : V^1(J) \to V^1_k(J)$ is similarly defined.

Now let $\Pi_k : V^1(J) \to V^1_k(J)$ be an interpolation operator. Namely, if the nodal points of $J$ are given by $0 = t_0 < t_1 < \cdots < t_m = T$, then for an arbitrary $u \in V^1(J)$, the interpolation $\Pi_k u$ is defined as the function in $V^1_k(J)$ satisfying
\[ u(t_i) = (\Pi_k u)(t_i), \quad \forall i \in \{1, \ldots, m\}. \tag{1.3} \]
We know that there exist constants $C_{\Omega}(h) > 0$, $C_J(k) > 0$ and $C_{\text{inv}}(h) > 0$ satisfying
\[ \| u - P^1_h u \|_{H^1_0(\Omega)} \leq C_{\Omega}(h) \| \Delta u \|_{L^2(\Omega)}, \quad \forall u \in H^1_0(\Omega) \cap X(\Omega), \]
\[ \| u - \Pi_k u \|_{L^2(J)} \leq C_J(k) \| u \|_{V^1(J)}, \quad \forall u \in V^1(J), \]
\[ \| u_h \|_{H^1_0(\Omega)} \leq C_{\text{inv}}(h) \| u_h \|_{L^2(\Omega)}, \quad \forall u_h \in S_h(\Omega). \]
Moreover, there exists a Poincaré constant $C_p > 0$ satisfying
\[ \| u \|_{L^2(\Omega)} \leq C_p \| u \|_{H^1_0(\Omega)}, \quad \forall u \in H^1_0(\Omega). \]
For example, if $\Omega$ is a bounded open interval in $\mathbb{R}$, and $S_h(\Omega)$ is the piecewise linear (P1) finite element space, then they can be taken by $C_{\Omega}(h) = \frac{\bar{h}}{\pi}$ (see, e.g., [4]) and $C_{\text{inv}}(h) = \frac{\sqrt{2}}{h_{\text{min}}}$, where $h_{\text{min}}$
is the minimum mesh size for $\Omega$ (see, e.g., [7] Theorem 1.5). Moreover, if $V^1_k(J)$ is the P1-finite element space, then it can be taken by $C_J(k) = \frac{1}{h}$ (see, e.g., [7] Theorem 2.4).

From [1, Lemma 2.2], if $V^1_k(J)$ is P1-finite element space (i.e., the basis functions $\psi_i$ are piecewise linear functions), then $P^1_k$ coincides with $\Pi_k$. For any element $u \in V(\Omega, J)$, we define the semidiscrete projection $P_h u \in V^1(J; S_h(\Omega))$ by the following weak form:

$$\left\langle \frac{\partial}{\partial t}(u(t) - P_h u(t)), v_h \right\rangle_{L^2(\Omega)} + \nu \left\langle \nabla (u(t) - P_h u(t)), \nabla v_h \right\rangle_{L^2(\Omega)^d} = 0,$$

$$\forall v_h \in S_h(\Omega), \text{ a.e. } t \in J,$$

where a.e. means an abbreviation for 'almost everywhere'. Moreover, we define the full discretization operator $P^k_h : V(\Omega, J) \to V^1_k(J; S_h(\Omega))$ by $P^k_h := \Pi_k P_h$.

Finally, we denote an approximation subspace of $V(\Omega, J)$ by $S^k_h(\Omega, J) \equiv V^1_k(J) \cap S_h(\Omega)$, and let $\{\varphi_i\}_{i=1}^{m_n}$ be the basis of $S^k_h(\Omega, J)$. In addition, we denote the matrix norm induced from the Euclidean 2-norm by $\| \cdot \|_F$ and denote the transposed matrix of the matrix $X$ by $X^T$.

We show known results for the equation (1.11) below.

**Theorem 1** (see Theorem 5.5, 5.6, and proof of Theorem 4.6 in [5]) For an arbitrary $u \in V(\Omega, J) \cap L^2(J; X(\Omega))$, we have the following estimations.

$$\| u - P^k_h u \|_{L^2(J, H^1_0(\Omega))} \leq C_1(h, k) \| \Delta_t u \|_{L^2(J, L^2(\Omega))},$$

$$\| u - P^k_h u \|_{L^2(J, L^2(\Omega))} \leq C_0(h, k) \| \Delta_t u \|_{L^2(J, L^2(\Omega))},$$

$$\| u(T) - P_h u(T) \|_{L^2(\Omega)} \leq c_0(h) \| \Delta_t u \|_{L^2(J, L^2(\Omega))},$$

where $C_1(h, k) := \frac{2}{\nu} C_{\Omega}(h) + C_{\nu}(h) C_{J}(k)$, $C_0(h, k) = \frac{\nu}{\nu} C_{\Omega}(h)^2 + C_{J}(k)$ and $c_0(h) = \sqrt{\frac{2}{\nu}} C_{\Omega}(h)$.

### 1.2 The full-discrete finite element method

We define the bi-linear form $a_0(\cdot, \cdot)$ by

$$a_0(\phi, \psi) := \left\langle \frac{\partial}{\partial t}(\phi), \frac{\partial}{\partial t}(\psi) \right\rangle_{L^2(J, L^2(\Omega))} + \nu \left\langle \nabla \phi, \frac{\partial}{\partial t}(\nabla \psi) \right\rangle_{L^2(J, L^2(\Omega)^d)},$$

for $\phi, \psi \in V(\Omega, J)$. Then, for any element $u \in V(\Omega, J)$, we define the full-discrete projection $Q^k_h u \in S^k_h(\Omega, J)$ by the following weak form:

$$a_0(u - Q^k_h u, v^k_h) = 0, \quad \forall v^k_h \in S^k_h(\Omega, J).$$

First, we have the following estimation from above definition. Note that the scheme in [5] is based on the finite element Galerkin method with an interpolation in time that uses the fundamental solution for semidiscretization in space. Since, in the derivation procedure, it uses the fundamental matrix of solutions for the ODEs associated with the semidiscrete approximation, it is necessary to implement the complicated verified computations on matrix functions. But the present scheme by (1.18) need not any such kind of procedures at all.

**Lemma 2** For an arbitrary $u \in V(\Omega, J) \cap L^2(J; X(\Omega))$, the full-discrete projection $Q^k_h u \in S^k_h(\Omega, J)$ satisfies the following $V^1$-stability:

$$\| Q^k_h u \|_{V^1(J, L^2(\Omega))} \leq \| \Delta_t u \|_{L^2(J, L^2(\Omega))}.$$
Proof: From (1.8) and $Q^k_h u(x, 0) = 0$ in $\Omega$, if we take $v_h^k = Q^k_h u$ then

$$
\|Q^k_h u\|_{V^1(J; L^2(\Omega))}^2 + \nu \|Q^k_h u(T)\|_{H^1_0(\Omega)}^2 = a_0(Q^k_h u, Q^k_h u)
$$

$$
= \left\langle \Delta_t u, \frac{\partial}{\partial t} Q^k_h u \right\rangle_{L^2(J; L^2(\Omega))}
$$

$$
\leq \|\Delta_t u\|_{L^2(J; L^2(\Omega))} \|Q^k_h u\|_{V^1(J; L^2(\Omega))}.
$$

Therefore, the proof is completed.

Next, we consider the estimation $\|Q^k_h u\|_{L^2(J; H^1_0(\Omega))}$ for $u \in V(\Omega, J) \cap L^2(J; X(\Omega))$. From (1.8), it follows that

$$
a_0(Q^k_h u, v^k_h) = \left\langle \Delta_t u, \frac{\partial}{\partial t} v^k_h \right\rangle_{L^2(J; L^2(\Omega))}, \quad \forall v_h^k \in S^k_h(\Omega, J).
$$

(1.9)

Now we define $\alpha_h^k \in S^k_h(\Omega, J)$ satisfying

$$
\left\langle \frac{\partial}{\partial t} \alpha_h^k, \frac{\partial}{\partial t} v^k_h \right\rangle_{L^2(J; L^2(\Omega))} = \left\langle \Delta_t u, \frac{\partial}{\partial t} v^k_h \right\rangle_{L^2(J; L^2(\Omega))}
$$

for all $v_h^k \in S^k_h(\Omega, J)$. Note that by taking $v_h^k = \alpha_h^k$, it follows that $\|\alpha_h^k\|_{V^1(J; L^2(\Omega))} \leq \|\Delta_t u\|_{L^2(J; L^2(\Omega))}$.

We now define the matrices $A$ and $M$ in $\mathbb{R}^{mn \times mn}$ by

$$
A_{i,j} := \left\langle \frac{\partial}{\partial t} \varphi_j, \frac{\partial}{\partial t} \varphi_i \right\rangle_{L^2(J; L^2(\Omega))}, \quad M_{i,j} := \left\langle \nabla \varphi_j, \nabla \varphi_i \right\rangle_{L^2(J; L^2(\Omega))}, \quad \forall i, j \in \{1, \ldots, mn\},
$$

respectively. Since matrices $A$ and $M$ are the symmetric and positive definite, we denote the Cholesky decomposition such that $A = A^T A^T$ and $M = M^T M^T$, respectively. Moreover, we define the matrix $B$ in $\mathbb{R}^{mn \times mn}$ by

$$
B_{i,j} := \left\langle \nabla \varphi_j, \frac{\partial}{\partial t} \nabla \varphi_i \right\rangle_{L^2(J; L^2(\Omega))}, \quad \forall i, j \in \{1, \ldots, mn\}.
$$

From the fact that $Q^k_h u$ and $\alpha_h^k$ are in $S^k_h(\Omega, J)$, there exist coefficient vectors $u := (u_1, \ldots, u_{mn})^T$ and $a := (a_1, \ldots, a_{mn})^T$ in $\mathbb{R}^{mn}$ such that $Q^k_h u = \sum_{i=1}^{mn} u_i \varphi_i = \varphi^T u$ and $\alpha_h^k = \sum_{i=1}^{mn} a_i \varphi_i = \varphi^T a$ where $\varphi := (\varphi_1, \ldots, \varphi_{mn})^T$. Then, the variational equation (1.9) is equivalent to the following.

$$
(A + \nu B)u = A a.
$$

(1.10)

Thus we have the following result.

**Lemma 3** For an arbitrary $u \in V(\Omega, J) \cap L^2(J; X(\Omega))$, the full-discrete projection $Q^k_h u \in S^k_h(\Omega, J)$ satisfies

$$
\|Q^k_h u\|_{L^2(J; H^1_0(\Omega))} \leq \eta \|\Delta_t u\|_{L^2(J; L^2(\Omega))},
$$

where $\eta := \|M^{\frac{1}{2}} (A + \nu B)^{-1} A^T\|_E$.
Proof: From (1.10), we can obtain
\[ \|Q_h^k u\|_{L^2(J,H^1_0(\Omega))} = \|M^T u\|_E = \|M^T (A + \nu B)^{-1} A a\|_E \]
\[ \leq \|M^T (A + \nu B)^{-1} A \|_E \|\Delta t a\|_E \]
\[ = \eta \|\Delta t a\|_{L^2(J,L^2(\Omega))} \]
\[ \leq \eta \|\Delta t u\|_{L^2(J,L^2(\Omega))}. \]

Therefore, the proof is completed. \[\square\]

Note that our proposed scheme (1.8) has the feature of that the test function is differentiated of the time direction. For comparison of our scheme and the simple finite element scheme for the heat equation, we also define the bi-linear form \( \hat{a}_0(\cdot,\cdot) \) by
\[ \hat{a}_0(\phi,\psi) := \left\langle \frac{\partial}{\partial t} \phi, \psi \right\rangle_{L^2(J,L^2(\Omega))} + \nu \left\langle \nabla \phi, \nabla \psi \right\rangle_{L^2(J,L^2(\Omega))^d}, \]
for \( \phi,\psi \in V(\Omega,J) \). Then, for any element \( u \in V(\Omega,J) \), we define the full-discrete projection \( \hat{Q}_h^k u \in S_h^k(\Omega,J) \) by the following weak form:
\[ \hat{a}_0(u - \hat{Q}_h^k u, v_h^k) = 0, \quad \forall v_h^k \in S_h^k(\Omega,J). \quad (1.11) \]

For an arbitrary \( u \in V(\Omega,J) \cap L^2(J;X(\Omega)) \), if we take \( v_h^k = \hat{Q}_h^k u \) then
\[ \frac{1}{2} \left\| \hat{Q}_h^k u(T) \right\|^2_{L^2(\Omega)} + \nu \left\| \hat{Q}_h^k u \right\|^2_{L^2(J,H^1_0(\Omega))} = \hat{a}_0(\hat{Q}_h^k u, \hat{Q}_h^k u) \]
\[ = \left\langle \Delta t u, \hat{Q}_h^k u \right\rangle_{L^2(J,L^2(\Omega))} \]
\[ \leq \|\Delta t u\|_{L^2(J,L^2(\Omega))} \left\| \hat{Q}_h^k u \right\|_{L^2(J,L^2(\Omega))} \]
\[ \leq C_p \|\Delta t u\|_{L^2(J,L^2(\Omega))} \left\| \hat{Q}_h^k u \right\|_{L^2(J,H^1_0(\Omega))}. \]

Thus we have the following \( L^2 H_0^1 \)-stability:
\[ \left\| \hat{Q}_h^k u \right\|_{L^2(J,H^1_0(\Omega))} \leq \frac{C_p}{\nu} \|\Delta t u\|_{L^2(J,L^2(\Omega))}. \]

Moreover, we consider the estimation \( \left\| \hat{Q}_h^k u \right\|_{V^1(J,L^2(\Omega))} \) for \( u \in V(\Omega,J) \cap L^2(J;X(\Omega)) \). From (1.11), it follows that
\[ \hat{a}_0(\hat{Q}_h^k u, v_h^k) = \left\langle \Delta t u, v_h^k \right\rangle_{L^2(J,L^2(\Omega))}, \]

Now we define \( \hat{a}_h^k \in S_h^k(\Omega,J) \) satisfying \( \left\langle \hat{a}_h^k, v_h^k \right\rangle_{L^2(J,L^2(\Omega))} = \left\langle \Delta t u, v_h^k \right\rangle_{L^2(J,L^2(\Omega))} \) for all \( v_h^k \in S_h^k(\Omega,J) \). Note that by taking \( v_h^k = \hat{a}_h^k \), it follows that \( \left\| \hat{a}_h^k \right\|_{L^2(J,L^2(\Omega))} \leq \|\Delta t u\|_{L^2(J,L^2(\Omega))} \). Then we obtain
\[ \hat{a}_0(\hat{Q}_h^k u, v_h^k) = \left\langle \hat{a}_h^k, v_h^k \right\rangle_{L^2(J,L^2(\Omega))}. \quad (1.12) \]
We now define the matrices $G$ and $U$ in $\mathbb{R}^{mn \times mn}$ by

$$G_{i,j} := \left\langle \frac{\partial}{\partial t} \varphi_j, \varphi_i \right\rangle_{L^2(J;L^2(\Omega))}, \quad U_{i,j} := \left\langle \varphi_j, \varphi_i \right\rangle_{L^2(J;L^2(\Omega))^d}, \quad \forall i, j \in \{1, \ldots, mn\},$$

respectively. Since the matrix $U$ is the symmetric and positive definite, we denote the Cholesky decomposition such that $U = U^\frac{1}{2}U^\frac{1}{2}$. From the fact that $\hat{Q}_k u$ and $\beta_h$ in $S_k^1(\Omega, J)$, there exist coefficient vectors $\hat{u} := (\hat{u}_1, \ldots, \hat{u}_m)^T$ and $\hat{\alpha} := (\hat{\alpha}_1, \ldots, \hat{\alpha}_m)^T$ in $\mathbb{R}^{mn}$ such that $\hat{Q}_k u = \sum_{i=1}^m \hat{u}_i \varphi_i = \varphi^T \hat{u}$ and $\hat{\alpha}_h = \sum_{i=1}^m \hat{\alpha}_i \varphi_i = \varphi^T \hat{\alpha}$, respectively. Then, the variational equation (1.12) is equivalent to the following.

$$(G + \nu M)\hat{u} = U\hat{\alpha}.$$ 

Letting $\hat{\eta} := \|A^\frac{1}{2}(G + \nu M)^{-1}U^\frac{1}{2}\|_E$, it follows that

$$\left\| \hat{Q}_k u \right\|_{V^1(J;L^2(\Omega))} = \|A^\frac{1}{2}\hat{u}\|_E = \|A^\frac{1}{2}(G + \nu M)^{-1}U^\frac{1}{2}\hat{\alpha}\|_E \\
\leq \|A^\frac{1}{2}(G + \nu M)^{-1}U^\frac{1}{2}\|_E \|U^\frac{1}{2}\hat{\alpha}\|_E \\
= \hat{\eta} \|\hat{\alpha}_h\|_{L^2(J;L^2(\Omega))} \\
\leq \hat{\eta} \|\Delta u\|_{L^2(J;L^2(\Omega))}.$$ 

**Remark 4** All computations in Tables are carried out on the Dell Precision 5820 Intel Xeon CPU 4.0GHz by using INTLAB, a tool box in MATLAB developed by Rump [6] for self-validating algorithms. Therefore, all numerical values in these tables are verified data in the sense of strictly rounding error control. Moreover, we take the basis of finite element subspaces $S_h(\Omega)$ and $V_k^1(J)$ are taken as P1-function with uniform mesh on $\Omega$ and $J$, respectively. The numerical value shown on the table cuts off the numerical value by which the precision was guaranteed by the suitable effective number.

For $\nu = 1$, $\nu = 0.1$ and $\nu = 0.01$ in $\Omega = (0, 1)$ and $J = (0, 1)$, Table 1 and 2 show verified results of $\eta$ and $\hat{\eta}$, respectively. By the verified computing results, we can say that the projection $\hat{Q}_k$ is not $V^1$-stable, and our proposed projection $Q_k^1$ satisfies $V^1$-stability and as well as it has $L^2H^1_0$-stability.

## 2 Constructive error estimates

In this section, we consider a constructive error estimates of the projection $Q_k^1$ for the finite element approximation. For an arbitrary $u \in V(\Omega, J) \cap L^2(J;X(\Omega))$, we define the projection $\hat{P}_h u \in S_h^k(Q)$ satisfying the following weak form:

$$\left\langle \frac{\partial}{\partial t} \hat{P}_h u, \frac{\partial}{\partial t} v^k_h \right\rangle_{L^2(J;L^2(\Omega))} + \nu \left\langle \nabla \hat{P}_h u, \frac{\partial}{\partial t} \nabla v^k_h \right\rangle_{L^2(J;L^2(\Omega))^d} = \left\langle \Delta t u, \frac{\partial}{\partial t} v^k_h \right\rangle_{L^2(J;L^2(\Omega))}, \quad (2.1)$$

for all $v^k_h \in S_h^k(\Omega, J)$. Note that from (1.4) and take $v_h$ as $v_h = \frac{\partial}{\partial t} v^k_h$ for fixed $v^k_h$, it follows that

$$\left\langle \frac{\partial}{\partial t} P_h u, \frac{\partial}{\partial t} v^k_h \right\rangle_{L^2(J;L^2(\Omega))} + \nu \left\langle \nabla P_h u, \frac{\partial}{\partial t} v^k_h \right\rangle_{L^2(J;L^2(\Omega))^d} = \left\langle \Delta t u, \frac{\partial}{\partial t} v^k_h \right\rangle_{L^2(J;L^2(\Omega))}. \quad (2.2)$$
Table 1: The numerical results $\eta$ in $\Omega = (0, 1)$, $J = (0, 1)$.

| $k$ | $\nu = 1$ | $\nu = 0.1$ | $\nu = 0.01$ |
|-----|-----------|-------------|-------------|
|     | $h$       |             |             |
| 1/40| 0.3014    | 0.3047      | 0.3055      |
| 1/80| 0.3014    | 0.3046      | 0.3054      |
| 1/120| 0.3014   | 0.3046      | 0.3054      |
| 1/160| 0.3014   | 0.3046      | 0.3054      |
| 1/200| 0.3014   | 0.3046      | 0.3054      |
| 1/240| 0.3014   | 0.3046      | 0.3054      |
| 1/280| 0.3014   | 0.3046      | 0.3054      |
| 1/320| 0.3014   | 0.3046      | 0.3054      |
| 1/360| 0.3014   | 0.3046      | 0.3054      |
| 1/400| 0.3014  | 0.3046      | 0.3054      |

Table 2: The numerical results $\hat{\eta}$ in $\Omega = (0, 1)$, $J = (0, 1)$.

| $k$ | $\nu = 1$ | $\nu = 0.1$ | $\nu = 0.01$ |
|-----|-----------|-------------|-------------|
|     | $h$       |             |             |
| 1/40| 10.92     | 11.12       | 11.16       |
| 1/80| 21.86     | 22.25       | 22.34       |
| 1/120| 32.80   | 33.37       | 33.52       |
| 1/160| 43.74   | 44.50       | 44.69       |
| 1/200| 54.67   | 55.63       | 55.87       |
| 1/240| 65.61   | 66.75       | 67.04       |
| 1/280| 76.55   | 77.88       | 78.21       |
| 1/320| 87.48   | 89.00       | 89.39       |
| 1/360| 98.42   | 100.13      | 100.56      |
| 1/400| 109.35  | 111.26      | 111.73      |

From (2.3) and (2.4), we obtain

\[ \left\langle \frac{\partial}{\partial t} P_k^k u, \frac{\partial}{\partial t} v_k^k \right\rangle_{L^2(J, L^2(\Omega))} = \left\langle \frac{\partial}{\partial t} \bar{P}_k^k u, \frac{\partial}{\partial t} v_k^k \right\rangle_{L^2(J, L^2(\Omega))}. \]  

Moreover, from the definition of $V^1$-projection, we have

\[ \left\langle \frac{\partial}{\partial t} P_k^k u, \frac{\partial}{\partial t} v_k^k \right\rangle_{L^2(J, L^2(\Omega))} = \left\langle \frac{\partial}{\partial t} P_1^k P_k^k u, \frac{\partial}{\partial t} v_k^k \right\rangle_{L^2(J, L^2(\Omega))}. \]

From (2.3) and (2.4), it follows that $\bar{P}_k^k = P_1^k P_k^k$ since $P_k^k P_k^k u \in S_k^k(\Omega, J)$.

Remark 5 [1, Lemma 2.2] If $V^1_k(J)$ is the $P1$-finite element space, then $P_k^k$ coincides with $\Pi_k$, it follows that $\bar{P}_k^k = P_k^k (= \Pi_k P_k)$.
For the projection $Q_h^k$, we have the following estimations.

\[
\|u - Q_h^k u\|_{L^2(J;H_0^1(\Omega))} \leq \|u - P_h^k u\|_{L^2(J;H_0^1(\Omega))} + \|P_h^k u - Q_h^k u\|_{L^2(J;H_0^1(\Omega))},
\]

(2.5)
\[
\|u - Q_h^k u\|_{L^2(J;L^2(\Omega))} \leq \|u - P_h^k u\|_{L^2(J;L^2(\Omega))} + \|P_h^k u - Q_h^k u\|_{L^2(J;L^2(\Omega))},
\]

(2.6)
\[
\|u(T) - Q_h^k u(T)\|_{L^2(\Omega)} \leq \|u(T) - P_h u(T)\|_{L^2(\Omega)} + \|P_h u(T) - Q_h^k u(T)\|_{L^2(\Omega)},
\]

(2.7)

where we have used the fact that $P_h^k u(T) = \Pi_k P_h u(T) = P_h u(T)$. Thus we now describe the estimation for $P_h^k u - Q_h^k u \in S_h^k(\Omega, J)$ below.

From (1.8) and (2.1) and letting $\delta_h^k := P_h^k u - Q_h^k u \in S_h^k(\Omega, J)$, we can obtain

\[
a_0(\delta_h^k, v_h^k) = \nu \left\langle \nabla \xi, \frac{\partial}{\partial t} \nabla v_h^k \right\rangle_{L^2(J;L^2(\Omega))},
\]

(2.8)

where $\xi := \hat{P}_h^k u - P_h u \in V$. Here we define $\beta_h^k \in S_h^k(\Omega, J)$ satisfying

\[
\left\langle \frac{\partial}{\partial t} \nabla \beta_h^k, \frac{\partial}{\partial t} \nabla v_h^k \right\rangle_{L^2(J;L^2(\Omega))} = \left\langle \nabla \xi, \frac{\partial}{\partial t} \nabla v_h^k \right\rangle_{L^2(J;L^2(\Omega))}, \quad \forall v_h^k \in S_h^k(\Omega, J).
\]

Note that by taking $v_h^k = \beta_h^k$, it follows that $\|\frac{\partial}{\partial t} \beta_h^k\|_{L^2(J;H_0^1(\Omega))} \leq \|\xi\|_{L^2(J;H_0^1(\Omega))}$. Then we obtain

\[
a_0(\delta_h^k, v_h^k) = \nu \left\langle \frac{\partial}{\partial t} \nabla \beta_h^k, \frac{\partial}{\partial t} \nabla v_h^k \right\rangle_{L^2(J;L^2(\Omega))},
\]

(2.9)

We now define matrices $W$ and $Y$ in $\mathbb{R}^{mn \times mn}$ by

\[
W_{i,j} := \left\langle \frac{\partial}{\partial t} \nabla \varphi_j, \frac{\partial}{\partial t} \nabla \varphi_i \right\rangle_{L^2(J;L^2(\Omega))}, \quad Y_{i,j} := \left\langle \varphi_j(\cdot, T), \varphi_i(\cdot, T) \right\rangle_{L^2(\Omega)}, \quad \forall i, j \in \{1, \ldots, mn\},
\]

respectively. Since the matrix $W$ is the symmetric and positive definite, we denote the Cholesky decomposition such that $W = W^{1/2}W^{1/2}$. Moreover, if $m \neq 1$ the matrix $Y$ is the symmetric and positive semi-definite, but we can decompose such that $Y = Y^{1/2}Y^{1/2}$ by the definition of $Y$. From the fact that $\delta_h^k$ and $\beta_h^k$ in $S_h^k(\Omega, J)$, there exist coefficient vectors $d := (d_1, \ldots, d_{mn})^T$ and $b := (b_1, \ldots, b_{mn})^T$ in $\mathbb{R}^{mn}$ such that $\delta_h^k = \sum_{i=1}^{mn} d_i \varphi_i = \varphi^T d$ and $\beta_h^k = \sum_{i=1}^{mn} b_i \varphi_i = \varphi^T b$. Then, the variational equation (2.9) is equivalent to the following.

\[
(A + \nu B)d = \nu Wb.
\]

(2.10)

Let

\[
\gamma_1 := \nu \|M^{1/2}(A + \nu B)^{-1}W^{1/2}\|_E,
\]

\[
\gamma_0 := \nu \|U^{1/2}(A + \nu B)^{-1}W^{1/2}\|_E,
\]

\[
\gamma_T := \nu \|Y^{1/2}(A + \nu B)^{-1}W^{1/2}\|_E.
\]

Then we have the following main result in this paper.
Theorem 6 Assume that $V^1_k(J)$ is the P1 finite element space. For an arbitrary $u \in V(\Omega, J) \cap L^2(J; X(\Omega))$, we have the following estimations.

$$
\| u - Q_h^k u \|_{L^2(J; H_0^1(\Omega))} \leq \tilde{C}_1(h, k) \| \triangle_t u \|_{L^2(J; L^2(\Omega))},
$$
$$
\| u - Q_h^k u \|_{L^2(J; L^2(\Omega))} \leq \tilde{C}_0(h, k) \| \triangle_t u \|_{L^2(J; L^2(\Omega))},
$$
$$
\| u(T) - Q_h^k u(T) \|_{L^2(\Omega)} \leq \tilde{c}_0(h) \| \triangle_t u \|_{L^2(J; L^2(\Omega))},
$$

where

$$
\tilde{C}_1(h, k) \equiv C_1(h, k) + C_J(h, k) C_{inv}(h) \gamma_1,
$$
$$
\tilde{C}_0(h, k) \equiv C_0(h, k) + C_J(h, k) C_{inv}(h) \gamma_0,
$$
$$
\tilde{c}_0(h, k) \equiv c_0(h) + C_J(h, k) C_{inv}(h) \gamma_T.
$$

Proof: From (2.10), we can obtain

$$
\| \delta_h^k \|_{L^2(J; H_0^1(\Omega))} = \| M^{T} u \|_E = \nu \| M^{T} (A + \nu B)^{-1} W b \|_E \leq \gamma_1 \| W_{\gamma_1} \|_E
$$
$$
\| \delta_h^k \|_{L^2(J; L^2(\Omega))} = \| U^{T} u \|_E = \nu \| U^{T} (A + \nu B)^{-1} W b \|_E \leq \gamma_0 \| W_{\gamma_0} \|_E.
$$

Moreover, we have

$$
\| W_{\gamma} \|_E = \left\| \frac{\partial}{\partial t} \delta_h^k \right\|_{L^2(J; H_0^1(\Omega))} \leq \| \xi \|_{L^2(J; H_0^1(\Omega))}.
$$

Note that $\tilde{P}_h^k = P_h^k (= \Pi_k P_h)$ from Remark 4. Then it follows that

$$
\| \xi \|_{L^2(J; H_0^1(\Omega))} = \| \tilde{P}_h^k u - P_h^k u \|_{L^2(J; H_0^1(\Omega))} = \| \Pi_k P_h u - P_h^k u \|_{L^2(J; H_0^1(\Omega))} \leq C_{inv}(h) \| \Pi_k P_h u - P_h^k u \|_{L^2(J; L^2(\Omega))} \leq C_J(h, k) C_{inv}(h) \| P_h^k u \|_{L^1(J; L^2(\Omega))} \leq C_J(h, k) C_{inv}(h) \| \triangle_t u \|_{L^2(J; L^2(\Omega))},
$$

where we have used the fact that $\| P_h^k u \|_{L^1(J; L^2(\Omega))} \leq \| \triangle_t u \|_{L^2(J; L^2(\Omega))}$ in [3]. Therefore, the proof is completed from (2.5), (2.6), (2.7), Theorem 1 and the fact $\delta_h^k = P_h^k u - Q_h^k u$.

The same assumptions in Remark 4, Table 3, 4 and 5 show verified computations of $\gamma_1$, $\gamma_0$ and $\gamma_T$ for $\nu = 1$, $\nu = 0.1$ and $\nu = 0.01$ in $\Omega = (0, 1)$ and $J = (0, 1)$. From the verified results in Table 3, 4 and 5, we may conclude that $\gamma_0$ and $\gamma_T$ are dependent on the parameter $\nu$, but asymptotically converge to some constants when $h$ and $k$ tend to zero.

Conclusion

We present a new full-discrete finite element projection $Q_h^k$ for the heat equation, and derived the constructive stability by the numerical computations with guaranteed accuracy. Our scheme is closely related to that in [5] as well as the error estimates is established by using the results in the
same paper. Therefore, it is considered as an extended version of [5], but the present scheme should be more familiar method to people working on numerical analysis. Namely, it is not necessary any complicated manipulation at all for verified computation of matrix function.

Moreover, we have the constructive a priori error estimates for the projection $Q^k_h$ which also implies the numerical stability. In particular, we can obtain the estimate $\| u(T) - Q^k_h u(T) \|_{L^2(\Omega)}$ of the boundary at $T$, we expect that we could apply our method to the verified computation for nonlinear problems by using similar discussion in [2]. Thus, our method will play an important contribution in the numerical verification method of exact solutions for the nonlinear parabolic equations.

References

[1] T. Kinoshita, T. Kimura, and M. T. Nakao, A posteriori estimates of inverse operators for initial value problems in linear ordinary differential equations, J. Comput. Appl. Math., 236 (2011), pp. 1622–1636.

[2] T. Kinoshita, T. Kimura, M. T. Nakao, On the a posteriori estimates for inverse operators of linear parabolic equations with applications to the numerical enclosure of solutions for nonlinear problems, Numerische Mathematik, 126 (2014), pp. 679–701.

[3] M.T.Nakao, Solving nonlinear parabolic problems with result verification. Part I: One-spacedimensional case, J. Comput. Appl. Math., 3 (1991), 323–334.

[4] M. T. Nakao, N. Yamamoto, and S. Kimura, On the Best Constant in the Error Bound for the $H^1_0$-Projection into Piecewise Polynomial Spaces, J. Approx. Theory, 93 (1998), pp. 491–500.

[5] M.T.Nakao, T.Kimura, T.Kinoshita, Constructive a priori error estimates for a full discrete approximation of the heat equation, SIAM J. Numer. Anal., 51, 3 (2013), 1525–1541.

[6] S. M. Rump, INTLAB–INTerval LABoratory, in Developments in Reliable Computing, Tibor Csendes(ed.), Kluwer Academic Publishers, Dordrecht, 1999, pp. 77–104. http://www.ti3.tu-harburg.de/rump/intlab/

[7] M. H. Schultz, Spline Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.
| $\nu = 0.1$ | $k$ | $\gamma_1$ | $\gamma_0$ | $\gamma_T$ | $\gamma_1$ | $\gamma_0$ | $\gamma_T$ | $\gamma_1$ | $\gamma_0$ | $\gamma_T$ | $\gamma_1$ | $\gamma_0$ | $\gamma_T$ |
|-------------|-----|-----------|----------|------------|-----------|----------|------------|-----------|----------|------------|-----------|----------|------------|
| $1/40$      | 0.9915 | 0.1402 | 0.2236 | 0.9998 | 0.1396 | 0.2236 | 3.3302 | 0.1395 | 0.2236 |
| $1/80$      | 0.9914 | 0.1402 | 0.2236 | 0.9996 | 0.1396 | 0.2236 | 1.6986 | 0.1395 | 0.2236 |
| $1/120$     | 0.9914 | 0.1402 | 0.2236 | 0.9996 | 0.1396 | 0.2236 | 1.1335 | 0.1395 | 0.2236 |
| $1/160$     | 0.9914 | 0.1402 | 0.2236 | 0.9996 | 0.1396 | 0.2236 | 0.9999 | 0.1395 | 0.2236 |
| $1/200$     | 0.9913 | 0.1402 | 0.2236 | 0.9996 | 0.1396 | 0.2236 | 0.9999 | 0.1395 | 0.2236 |
| $1/240$     | 0.9913 | 0.1402 | 0.2236 | 0.9996 | 0.1396 | 0.2236 | 1.0000 | 0.1395 | 0.2236 |
| $1/280$     | 0.9913 | 0.1402 | 0.2236 | 0.9996 | 0.1396 | 0.2236 | 1.0000 | 0.1395 | 0.2236 |
| $1/320$     | 0.9913 | 0.1402 | 0.2236 | 0.9996 | 0.1396 | 0.2236 | 1.0000 | 0.1395 | 0.2236 |
| $1/400$     | 0.9914 | 0.1402 | 0.2236 | 0.9996 | 0.1396 | 0.2236 | 1.0000 | 0.1395 | 0.2236 |

$$(k, \gamma_1) = (1/500, 2.7210), (k, \gamma_1) = (1/700, 1.9451), (k, \gamma_1) = (1/900, 1.5170)$$ for $h = 1/20$. 

Table 5: The numerical results in $\Omega = (0, 1), J = (0, 1)$. 

| $\nu = 0.01$ | $k$ | $\gamma_1$ | $\gamma_0$ | $\gamma_T$ | $\gamma_1$ | $\gamma_0$ | $\gamma_T$ | $\gamma_1$ | $\gamma_0$ | $\gamma_T$ | $\gamma_1$ | $\gamma_0$ | $\gamma_T$ |
|-------------|-----|-----------|----------|------------|-----------|----------|------------|-----------|----------|------------|-----------|----------|------------|
| $1/40$      | 0.6972 | 0.0461 | 0.0697 | 0.9682 | 0.0466 | 0.0707 | 0.9981 | 0.0465 | 0.0707 |
| $1/80$      | 0.6972 | 0.0461 | 0.0697 | 0.9681 | 0.0466 | 0.0707 | 0.9979 | 0.0465 | 0.0707 |
| $1/120$     | 0.6972 | 0.0461 | 0.0697 | 0.9681 | 0.0466 | 0.0707 | 0.9979 | 0.0465 | 0.0707 |
| $1/160$     | 0.6972 | 0.0461 | 0.0697 | 0.9681 | 0.0466 | 0.0707 | 0.9979 | 0.0465 | 0.0707 |
| $1/200$     | 0.6972 | 0.0461 | 0.0697 | 0.9681 | 0.0466 | 0.0707 | 0.9979 | 0.0465 | 0.0707 |
| $1/240$     | 0.6972 | 0.0461 | 0.0697 | 0.9681 | 0.0466 | 0.0707 | 0.9979 | 0.0465 | 0.0707 |
| $1/280$     | 0.6972 | 0.0461 | 0.0697 | 0.9681 | 0.0466 | 0.0707 | 0.9979 | 0.0465 | 0.0707 |
| $1/320$     | 0.6972 | 0.0461 | 0.0697 | 0.9681 | 0.0466 | 0.0707 | 0.9979 | 0.0465 | 0.0707 |
| $1/400$     | 0.6972 | 0.0461 | 0.0697 | 0.9681 | 0.0466 | 0.0707 | 0.9979 | 0.0465 | 0.0707 |