An elliptic partial differential equation and its application

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Abstract
This paper deals with the following elliptic equation

\[-2 |\sigma|^2 \Delta z + |\nabla z|^2 + 4 \alpha z = 4 |x|^2 \text{ for } x \in \mathbb{R}^N, (N \geq 1),\]

where \(\alpha > 0\) is a real parameter and \(\sigma\) is a vector from \(\mathbb{R}^N\). The solution method is based on the sub- and super-solution method. The case \(N > 1\) seemed not considered before. This equation models a stochastic production planning problem.

1 Introduction

In this article, we look for positive solutions of the following partial differential equation

\[-2 |\sigma|^2 \Delta z + |\nabla z|^2 + 4 \alpha z = 4 |x|^2, \ x \in \mathbb{R}^N . \]

Here \(N \geq 1\) is the space dimension, \(|\sigma|\) is the Euclidean norm of \(\mathbb{R}^N\), \(\alpha > 0\), and \(\sigma > 0\) are some real constants.

This equation has received much attention in the last decades since it is related with several models that arises in production planning problems; for more on this see the papers of Akella and Kumar [1], Alvarez [2], Bensoussan, Sethi, Vickson and Derzko [3] and Lasry and Lions [7].

Our motivation in studying this equation comes from the recent work of [4], where the author obtained non-positive radial solutions for the equation [1] and where we postulate an open problem regarding the existence of positive solution for this equation. Another goal of this paper is to improve the production planning model given in [3], [4], and to give a verification result, i.e., show that the solution of the equation yields the optimal production.
To the best of our knowledge, the first mathematical result about the existence of positive solution for the semilinear equation (1) seems to be due to [3] for the case $N = 1$ and no results on existence of positive solutions are known for the case $N > 1$. This should not surprise us since there are some difficulties in analyzing this class of problems in $\mathbb{R}^N$, with $N \geq 1$, which will be revealed in the following sections organized as follows. In Section 2 we give our main theorem regarding the existence of positive solution for the problem (1) and its proof. The Section 3 contains the application to a production planning problem. Section 4 presents a verification result. In Section 4.1 we obtain a closed form solution for our equation in a special case.

2 Main Result

The following Theorem is the main result of our paper.

**Theorem 1.** There exists a positive function $z(x) \in C^2(\mathbb{R}^N)$ satisfying (1). Moreover, the following estimate holds

$$0 \leq z(x) \leq |x|^2 + 1 + \frac{N}{\alpha} \sigma^2, \text{ for } x \in \mathbb{R}^N.$$

(2)

To achieve our goal, which is establishing the above theorem, we prove the following lemma:

**Lemma 2.** The partial differential equation with gradient term (1) is equivalent to the semilinear elliptic equation

$$-\Delta u(x) + \frac{1}{\sigma^4} |x|^2 u(x) = -\frac{2\alpha}{\sigma^2} u(x) \ln u(x), \text{ for } x \in \mathbb{R}^N.$$ 

(3)

**Proof.** With the change of variable $z(x) = -v(x)$,

we obtain

$$2\sigma^2 \Delta v(x) + |\nabla v(x)|^2 = 4 |x|^2 + 4\alpha v(x), \text{ for } x \in \mathbb{R}^N,$$

or, equivalently

$$\Delta v(x) = \frac{4 |x|^2 + 4\alpha v(x) - |\nabla v(x)|^2}{2\sigma^2}, \text{ for } x \in \mathbb{R}^N.$$ 

(4)

Again, after changing the variable $u(x) = e^{\frac{\sigma^2}{\sigma^4} v(x)}$, the equation (4) becomes (1). Notice that the above process is invertible, so (1) and (3) are equivalent.

In the next theorem we prove the existence of a solution $u(x) \in C^2(\mathbb{R}^N)$ for the problem (3), such that $0 < u(x) \leq 1$ for all $x \in \mathbb{R}^N$.

**Theorem 3.** There exist functions $u, \overline{u} : \mathbb{R}^N \to (0, 1]$ of class $C^2(\mathbb{R}^N)$ such that

$$\begin{cases}
-\Delta u(x) + \frac{1}{\sigma^4} |x|^2 u(x) \leq -\frac{2\alpha}{\sigma^2} u(x) \ln u(x), & x \in \mathbb{R}^N, \\
-\Delta \overline{u}(x) + \frac{1}{\sigma^4} |x|^2 \overline{u}(x) \geq -\frac{2\alpha}{\sigma^2} \overline{u}(x) \ln \overline{u}(x), & x \in \mathbb{R}^N, \\
u(x) \leq \overline{u}(x) & \text{ for } x \in \mathbb{R}^N.
\end{cases}$$ 

(5)
Moreover, for such functions \( u, \overline{u} \) there exists a function \( u(x) \in C^2(\mathbb{R}^N) \) with \( \underline{u}(x) \leq u(x) \leq \overline{u}(x) \) in \( \mathbb{R}^N \) and satisfying (3).

Before giving the proof of the above theorem, we point that the function \( u \) (resp. \( \overline{u} \)) is called a sub-solution (resp. super-solution) for the problem (3).

Proof. In the following we construct the functions \( u, \overline{u} \). We adopt the idea of Bensoussan, Sethi, Vickson and Derzko [3], for the one dimensional case. More exactly, we find \( a, b \) and \( c \) such that

\[
\underline{u}(x) = e^{a|x|^2 + b|x| + c},
\]

is a sub-solution for the problem (3). A simple calculation shows that we can take \( a = -\frac{1}{2\sigma^2}, b = 0 \) and \( c = -\frac{N}{2\alpha} \), to provide the sub-solution mentioned.

To construct a super-solution it is useful to remember that \( \ln 1 = 0 \) and then a simple calculation shows that

\[
\overline{u}(x) = 1,
\]

is a super-solution of the problem (3).

Until now, we constructed the corresponding sub- and super-solutions employed in the one dimensional case by [3]. Clearly, (5) holds and then in Theorem 3 it remains to prove that there exists \( u(x) \in C^2(\mathbb{R}^N) \) with \( \underline{u}(x) \leq u(x) \leq \overline{u}(x) \) in \( \mathbb{R}^N \) satisfying (3).

To do this, let \( B_k = \{ x \in \mathbb{R}^N \mid |x| < k \} \) be the ball centered at the origin and of radius \( k = 1, 2, \ldots \). We consider the problem

\[
\begin{aligned}
-\Delta u(x) + \frac{1}{4} |x|^2 u(x) &= -\frac{2\alpha}{2\sigma^2} u(x) \ln u(x) , & x \in B_k, \\
u(x) &= \underline{u}_k(x) , & x \in \partial B_k,
\end{aligned}
\]

where \( \underline{u}_k = u|_{B_k} \). In a similar way, we define \( \overline{u}_k = \overline{u}|_{B_k} \) and then \( \underline{u}_k, \overline{u}_k \in C^2(\overline{B}_k) \).

Observing that

\[
\inf_{x \in \mathbb{R}^N} u(x) \leq \min_{x \in \overline{B}_k} u_k(x) \text{ and } \sup_{x \in \mathbb{R}^N} \overline{u}(x) \geq \max_{x \in \overline{B}_k} \overline{u}_k(x),
\]

a result of Kazdan and Kramer [5], proves the existence of a solution \( u_k \in C^2(B_k) \cap C(\overline{B}_k) \) satisfying the problem (3). The function \( u_k \) also satisfies

\[
\underline{u}_k(x) \leq u_k(x) \leq \overline{u}_k(x) , \quad x \in \overline{B}_k.
\]

By a standard regularity argument based on Schauder estimates, see Nous-sair and Swanson [8] Lemma 3.2, p. 124 for details, we can see that for all integers \( k \geq n + 1 \) there are \( \alpha \in (0,1) \) and a positive constant \( C_1 \) depending on \( \alpha, N \), \( \min_{x \in \overline{B}_k} u(x) \) and \( \max_{x \in \overline{B}_k} \overline{u}(x) \), but independent of \( k \), such that

\[
u_k \in C^{2,\alpha}(\overline{B}_k) \text{ and } |u_k|_{C^{2,\alpha}(\overline{B}_k)} < C_1,
\]
where $|\cdot|_{C^2,\alpha}$ is the usual norm of the space $C^{2,\alpha}(\overline{B}_n)$. Moreover, there exists a constant $C_2$ independent of $u_k$ such that

$$\max_{x \in \overline{B}_n} |\nabla u_k(x)| \leq C_2 \max_{x \in \overline{B}_n} |u_k(x)|. \quad (8)$$

The information from (7) and (8) implies that $\{\nabla u_k\}$ as well as $\{u_k\}$ are uniformly bounded on $\overline{B}_n$. Using the compactness of the embedding $C^{2,\alpha}(\overline{B}_n) \hookrightarrow C^2(\overline{B}_n)$, enables us to define the subsequence

$$u_k := u_k|_{\partial B_n}, \text{ for all } k \geq n + 1.$$

Then for $n = 1, 2, 3, \ldots$ there exist a subsequence $\{u_n^{k_{nj}}\}_{k \geq n + 1, j \geq 1}$ of $\{u_n\}_{k \geq n + 1}$ and a function $u_n$ such that

$$u_n^{k_{nj}} \to u_n, \quad (9)$$

uniformly in the $C^2(\overline{B}_n)$ norm. More exactly, we get through a well-known diagonal process that

$$u_1^{k_{11}}, u_1^{k_{12}}, u_1^{k_{13}}, \ldots \to u_1 \text{ in } C^2(\overline{B}_1),$$
$$u_2^{k_{21}}, u_2^{k_{22}}, u_2^{k_{23}}, \ldots \to u_2 \text{ in } C^2(\overline{B}_2),$$
$$u_3^{k_{31}}, u_3^{k_{32}}, u_3^{k_{33}}, \ldots \to u_3 \text{ in } C^2(\overline{B}_3),$$
$$\ldots$$

Since $\mathbb{R}^N = \bigcup_{n=1}^\infty B_n$, we can define the function $u : \mathbb{R}^N \to [0, \infty)$ such that

$$u(x) = \lim_{n \to \infty} u_n(x).$$

From the regularity theory the solution $u$ belongs to $C^2(\mathbb{R}^N)$ and satisfies (3).

Let us give the construction of the solution $u$ for the problem (3). This is obtained by considering the sequence $(u_d^{k_{nd}})_{d \geq 1}$ and the sequence $(u_n^{k_{nd}})_{k \geq n + 1}$, restricted to the ball $B_n$, which are such that

$$u_n^{k_{nd}} \xrightarrow{d \to \infty} u_n := u(x) \text{ for all } x \in B_n,$$

and then, for $d \to \infty$ we obtain

$$u_d^{k_{dd}} \xrightarrow{d \to \infty} u(x) \text{ in } C^2(\mathbb{R}^N),$$

according with the diagonal process. Moreover, since

$$u(x) \leq u_d^{k_{dd}} \leq \overline{u}(x), \text{ for } x \in \mathbb{R}^N,$$

and for each $d = 1, 2, 3, \ldots$ the function $u$ also satisfies

$$u(x) \leq u(x) \leq \overline{u}(x), \text{ for } x \in \mathbb{R}^N.$$
Proof of Theorem 1: The existence of solutions is proved by Lemma 2 and Theorem 3. Then it remains to prove (2).

A recapitulation of the changes of variables says that

\[ z(x) = -2\sigma^2 \ln u(x), \]

is a solution for (1). Observing that

\[ u(x) = e^{-|x|^2/2\sigma^2} \leq u(x) \leq u(x) = 1, \quad x \in \mathbb{R}^N, \]

it follows that

\[ -|x|^2/2\sigma^2 - \frac{1}{2\sigma^2} - \frac{N}{2\alpha} \leq \ln u(x) \leq \ln 1, \]

and then

\[ 0 \leq -2\sigma^2 \ln u(x) \leq |x|^2 + 1 + \frac{N}{\alpha} \sigma^2, \quad \text{for} \quad x \in \mathbb{R}^N, \]

or equivalently

\[ 0 \leq z(x) \leq |x|^2 + 1 + \frac{N}{\alpha} \sigma^2, \quad \text{for} \quad x \in \mathbb{R}^N, \]

and the proof is completed.

3 Production Planning problem

As we already mentioned the equation studied is appearing in a stochastic production planning problem. Indeed, let \( \mathbb{R}^N (N \geq 1) \) be the \( N \)-dimensional Euclidean space and consider a factory producing \( N \) homogeneous goods and having an inventory warehouse. Define the following quantities:

1. \( p(t) = (p_1(t), \ldots, p_N(t)) \) represents the production at time \( t \) (control variable);
2. \( p^0 = (p_1^0, \ldots, p_N^0) \) represents factory optimal production level;
3. \( y(t) = (y_1(t), \ldots, y_N(t)) \) denotes the inventory level for production rate at time \( t \) (state variable);
4. \( l = (l_1, \ldots, l_N) \) denote the factory-optimal inventory level;
5. \( c \) represents production cost coefficient;
6. \( h \) is the inventory holding cost coefficient;
7. \( \xi = (\xi_1, \ldots, \xi_N) \) represents the constant demand rate at time \( t \);
8. \( \sigma \) is a positive diffusion coefficient;
9. \( \alpha > 0 \) is the constant discount rate;
10. \( y_0^i \) is the initial inventory level;
11. \( w = (w_1, \ldots, w_N) \) is a \( N \)-dimensional Brownian motion on a complete probability space \( (\Omega, \mathcal{F}, P) \) endowed with the natural completed filtration \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \), where \( T = \infty \) is the length of planning period (we deal with the infinite horizon case), and the filtration is generated by the standard Brownian motion process \( w \).

We now state the conditions of the model. The first condition is the state dynamic equation for the inventory level stated as an Itô stochastic differential equation

\[ dy_i(t) = (p_i - \xi_i) dt + \sigma dw_i, \quad y_i(0) = y_0^i, \quad i = 1, \ldots, N. \]

(10)
The diffusion part of this equation is interpreted as “sales returns,” “inventory spoilage,” etc. which are random in nature.

The inventory production control problem is to choose the factory production as to minimize the following cost functional

$$J(p_1,\ldots,p_N) := E \int_0^\infty \left( c_f(1(p(t) - p^0)) + h_f(y(t) - l) \right) e^{-\alpha t} dt,$$

where $f_1(x) = f_2(x) = |x|^2$ is the quadratic loss function.

To simplify the presentation we assume that $p^0(t) = l = (0,\ldots,0)$ and $h = c = 1.$

This assumption makes perfect sense if we consider the deviations from the factory-optimal production level and deviations from the factory-optimal inventory level. In light of this the deviations are allowed to be negative. The aim is to minimize the stochastic production planning problem

$$\inf \{ J(p_1,\ldots,p_N) \mid p_i \forall i = 1,2,\ldots,N \},$$

with the inventory level subject to the Itô equation (10).

Let $z(x) = z(x_1,\ldots,x_N)$ denote the expected current-valued value of the control problem (10) with initial value $(x_1,\ldots,x_N).$ In order to solve this stochastic production planning problem we apply the martingale principle: that is, we search for a function $U(x)$ such that the stochastic process $M^c(t)$ defined below

$$M^c(t) = e^{-\alpha t} U(y(t)) - \int_0^t [c_f(p(s) - p^0) + h_f(y(s) - l)] e^{-\alpha s} ds,$$

is supermartingale for all $p(t) = (p_1(t),\ldots,p_N(t))$ and martingale for the optimal control $p^*(t) = (p^*_1(t),\ldots,p^*_N(t)).$ Then, it can be shown that $-U(x) = z(x)$ is $C^2[0,\infty)$ and satisfies the Hamilton-Jacobi-Bellman equation (HJB) formally associated with the problem (10-12)

$$\alpha z - \frac{\sigma^2}{2} \Delta z + \xi \nabla z - |x|^2 = \inf \{ p \nabla z + |p|^2 \mid p_i \forall i = 1,2,\ldots,N \},$$

where $z := z(x_1,\ldots,x_N)$ is the corresponding value function. The first order conditions yield the optimal candidate $p^*_i(t) = (p^*_1(t),\ldots,p^*_N(t)),$ by

$$p^*_i(t) = -\frac{1}{2} \frac{\partial z}{\partial x_i}(y^*_1(t),\ldots,y^*_N(t)) \quad \text{for} \quad i = 1,\ldots,n.$$ (14)

and

$$dy^*_i(t) = (p^*_i - \xi_i) dt + \sigma dw_i, \quad y^*_i(0) = y^*_i, \quad i = 1,\ldots,N.$$ (15)

We point that the solution of (13) equation is used to test controller for optimality and equation (14) is used to construct a feedback controller.

We consider the case $\xi = (0,\ldots,0)$ which makes sense if deviation from the constant demand rate is taken into account. Then, this equation (13) can be simplified by noting that

$$\inf \{ p \nabla z + |p|^2 \mid p_i \geq 0 \forall i = 1,2,\ldots,N \} = -\frac{1}{4} |\nabla z|^2,$$
so that equation (13) can be written as
\[ \alpha z - \frac{\sigma^2}{2} \Delta z - |x|^2 = - \frac{1}{4} |\nabla z|^2 \quad \text{for} \quad x \in \mathbb{R}^N, \]
or, equivalently
\[ -2\sigma^2 \Delta z + |\nabla z|^2 + 4\alpha z = 4 |x|^2 \quad \text{for} \quad x \in \mathbb{R}^N. \quad (16) \]
which is the same as equation (1).

4 A Verification Result

In this section we establish the optimality of control \((p_1^*, ..., p_N^*)\) given by (14) and (15). In order to achieve this we need the following assumption.

**Standing Assumption.** Assume that the solution of (1) satisfies the following gradient estimate
\[ |\nabla z(x)| \leq C(1 + |x|). \quad (17) \]

Then, the verification theorem proceeds with the following steps:

**First Step:** The system of SDEs (15) with \((p_1^*, ..., p_N^*)\) given by (14) has a weak solution via Girsanov Theorem in light of the standing assumption (see Section 3.5 in [6] for more on this). Indeed this is true since in light of the assumption the Novikov condition holds true on small intervals, and Girsanov Theorem can be extended by an induction argument to arbitrarily large intervals.

**Second Step:** Let \(y_t^*\) be the inventory level corresponding to \((p_1^*, ..., p_N^*)\) given by (14). In light of the standing assumption one can get using the arguments appearing in the proof of Theorem 5.2.1 from [9], we get for the optimal control candidate
\[ E |y_t^*|^2 \leq C_1 e^{C_2 t}, \quad (18) \]
for some positive constants \(C_1, C_2\). Indeed, by integrating (15), and by applying the expectation operator to \(|y_t|^2\), using Cauchy Schwarz inequality, employing the standing assumption, and Gronwall inequality yields (18).

**Third Step:** The set of acceptable production rates that we consider is encompassing production rates and inventory levels for which
\[ J(p_1, ..., p_N) := E \int_0^\infty (f_1(p(t)) + f_2(y(t))) e^{-\alpha t} dt < \infty, \]
and the following transversality condition
\[ \lim_{t \to \infty} E e^{-\alpha t} |y_t|^2 = 0, \]
is met. The set of acceptable production rates is non empty. Because of the standing assumption and estimate (18), the candidate optimal control \((p_1^*, ..., p_N^*)\) verifies that \(J(p_1^*, ..., p_N^*) < \infty\), for \(\alpha\) large enough. Moreover, for \(\alpha\) large enough the transversality condition is met because of (2) and (18). Also the control \(p_1 = 0, ..., p_N = 0\), is an acceptable control. In
light of the quadratic estimate on the value function (see (2) in the main
theorem), the transversality condition implies that
\[
\lim_{t \to \infty} E e^{-\alpha t} U(y^*_t) = 0.
\]

**Fourth Step:** Recall that \( U(x) = -z(x) \), where \( z \) is the solution of \( (1) \), and
\[
M^c(t) = e^{-\alpha t} U(y(t)) - \int_0^t (f_1(p(u)) + f_2(y(u))) e^{-\alpha u} du.
\]
Therefore, the Itô’s Lemma yields for the optimal control candidate
\[
dM^c(t) = e^{-\alpha s} \sigma \nabla z(y^*(s)) dw(s).
\]
Consequently \( M^c(t) \) is a local martingale. Moreover,
\[
E \int_0^t e^{-2\alpha s} \sigma^2 |\nabla z(y^*(s))|^2 ds \leq CE \int_0^t e^{-2\alpha s} \sigma^2 |y^*(s)|^2 ds + C_3 \\
\leq CC_1 \int_0^t e^{-2\alpha s} e^{C_2 s} + C_3 < C_4,
\]
for some positive constants \( C, C_1, C_2, C_3, C_4 \), and large enough \( \alpha \). This in turn makes \( M^c(t) \) a (true) martingale.

**Fifth Step:** This step establishes the optimality of \((p_1^*, ..., p_N^*)\). The martingale/supermartingale principle yields
\[
E e^{-\alpha t} U(y^*(t)) - E \int_0^t (f_1(p^*(u)) + f_2(y^*(u))) e^{-\alpha u} du = U(x)
\]
and
\[
E e^{-\alpha t} U(y(t)) - E \int_0^t (f_1(p(u)) + f_2(y(u))) e^{-\alpha u} du \leq U(x)
\]
By passing \( t \to \infty \) and using transversality condition we get the optimality of \((p_1^*, ..., p_N^*)\).

### 4.1 Special Case

In the following we manage to obtain a closed form solution for our equation given a special discount \( \alpha \). That is, assume \( \alpha = N \sigma^2 \). Then, two solutions for the problem \( (19) \) are
\[
u(|x|) = e^{m(|x|^2 + 1)}, \quad m = \frac{1}{4\sigma^2} \left( \alpha + \sqrt{\alpha^2 + 4} \right), \quad (19)
\]
\[
u(|x|) = e^{m(|x|^2 + 1)}, \quad m = \frac{1}{4\sigma^2} \left( \alpha - \sqrt{\alpha^2 + 4} \right). \quad (20)
\]
Let us point out that \( (19) \) implies
\[
z(x) = -2\sigma^2 m (|x|^2 + 1) < 0 \text{ for all } x \in \mathbb{R}^N.
\]
and then $z(x)$ is the negative solution obtained in the paper [4] and (20) implies that

$$z(x) = -2\sigma^2 m (|x|^2 + 1) > 0 \text{ for all } x \in \mathbb{R}^N,$$

(21)
i.e. $z(x)$ is the positive solution obtained with the above procedure. For the production planning problem we choose the positive solution, i.e., the one given in (21). Let us notice that $z(x)$ given in (21) satisfies the standing assumption (17), thus the verification holds true.

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