Higher Order Correlations in Quantum Chaotic Spectra

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Abstract

The statistical properties of the quantum chaotic spectra have been studied, so far, only up to the second order correlation effects. The numerical as well as the analytical evidence that random matrix theory can successfully model the spectral fluctuations of these systems is available only up to this order. For a complete understanding of spectral properties it is highly desirable to study the higher order spectral correlations. This will also inform us about the limitations of random matrix theory in modelling the properties of quantum chaotic systems. Our main purpose in this paper is to carry out this study by a semiclassical calculation for the quantum maps; however results are also valid for time-independent systems.

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I. INTRODUCTION

In generic hamiltonian systems with many degrees of freedom, the classical dynamics shows an enormous richness in structure, increasing with the interaction between degrees of freedom. The classical motion is mainly of two types, integrable and chaotic; see [1] for details. This paper deals with the quantum properties of hamiltonians whose classical limit is chaotic.

The strongly chaotic nature of underlying classical dynamics intuitively suggest us to expect some kind of random behaviour in quantum dynamics too. This is because the classical dynamics is indeed a limit ($\hbar = 0$) of quantum dynamics and therefore the nature of former should be somehow reflected in the latter. In fact, various analytical and numerical studies (see [2] and references there in) have confirmed that the manifestation of chaotic behaviour in quantum dynamics occurs through randomization (partial or full) of matrices of associated quantum operators. The spectral and strength fluctuations of these operators can be well-modeled (up to second order correlations) by one of the various universality classes of random matrices. Most common among these are the Gaussian orthogonal and unitary ensembles, GOE and GUE, the analogous circular ensemble models COE and CUE [3]. The former pertain to autonomous systems whereas the latter have application in the study of non-autonomous systems such as quantum maps.

The presence of RMT type spectra in quantum chaotic systems can be explained by the Gutzwiller-emiclassical quantization scheme [4] for time-independent systems which uses the elegant technique of path integral sum given by Feynman and relates the chaotic manifolds of classical dynamics to the eigenfunctions of quantum dynamics. A similar formulation is given for time-evolution operators of quantum maps too [5]. The spectral fluctuation measures can then be determined approximately by using the principle of uniformity [6] which is based on the uniform distribution of periodic orbits at large time scales and gives a technique to evaluate the sum of periodic orbit contributions. Using this technique for autonomous hamiltonians, Berry [7] provided an explicit expression for the semiclassical form factor.
$K_2(\tau)$ -the fourier transform of the 2-level spectral correlation function-for values of $\tau$ in the range $\tau << 1$ (the time measured in units of $2\pi\hbar\overline{\sigma}$, where $\overline{\sigma}$ is the mean spectral density).

This result has an exact analogy with the corresponding RMT behaviour; following essentially the same technique as used by Berry for autonomous hamiltonians, this analogy can be proved for quantum maps too [2]. In the region $\tau >> 1$ too, the limiting behaviour was analysed by Berry using a semiclassical sum-rule which makes use of the properties of the function related to the quantum-mechanical density of states.

Notwithstanding the good agreement between RMT and statistical quantum chaos up to 2$^{nd}$ order correlations (long and very long), still there is no reason to believe that the Random Matrix Theory (RMT) can model all $n^{th}$ order spectral as well as strength correlations. The numerical studies for many systems (e.g Baker map [8], Quantum Kicked Rotor [2], etc.) have already indicated that even second order correlation effects, when considered on short time scales (i.e very long range correlations), do not follow random matrix prediction and are non-universal. This is the range where the classical dynamics is still diffusive and periodic orbits are not yet uniformly distributed. The deviation from RMT in this range agrees well with our intuition as one should expect RMT to be applicable only on those time-scales where the variables associated with classical dynamics are random enough to fully randomize the matrices associated with corresponding quantum operators. Moreover, the sum rules for the matrix elements of quantum chaotic operators [9] have already been found, differing from those of RMT. But a study of higher order correlations between zeros of Reimann-zeta function shows a good agreement with RMT [10,11].

Thus it is relevant to know as to what properties and up to what order the behaviour of quantum operators can be modeled by RMT and when it ultimately breaks down. Our attempt, in this paper, is to make a comparative study of one such property, namely the $n^{th}$ order spectral correlation as all $n$-level spectral fluctuation measures can be expressed in its terms. We fulfill this goal by carrying out a semiclassical study of the fourier transform of $n$-level correlation function $R_n$; the reason to consider the fourier transform lies in the convenience of its analytical as well as numerical calculability. We proceed as follows.
The Gutzwiller formulation gives us the density of states as a sum over periodic orbits and this gives rise to periodic orbit-interaction terms in $n$-level density correlation function. Berry, in order to obtain result for 2-level form factor, neglected the contribution from these interacting terms as a first order approximation (the so called diagonal approximation). But, for a complete evaluation of form factor, one has to calculate the contribution due to interacting terms. The lack of the knowledge of action correlations handicaps us from doing so. One attempt in this direction was made in ref.[12], in which by assuming the complete validity of random matrix theory, the periodic orbit correlations were calculated from the RMT form factor. These when compared with numerically obtained correlations (for Baker map, Hyperbola billiard and perturbed Cat map) showed a good agreement. The numerical study of these actions also indicated the presence of an uncorrelated component (exponentially larger than correlated part); in this paper we use this fact. We assume that, on long time scales as a first order approximation, actions are uncorrelated and calculate the $n^{th}$ order form factor. Under this approximation, the result turns out to be same as that of RMT which is also confirmed from the numerical analysis for at least two higher order fluctuation measures, given in this paper. For a complete calculation of $n^{th}$ order form factor, the action correlations should also be taken into account which will determine the higher order terms.

In this paper, we present our semiclassical study for quantum maps but the method can easily be generalized for time-independent systems and one obtains the same results. Similarly the RMT is given only for Circular ensembles but, once again, the final results are valid for Gaussian ensembles too which follows due to GE-CE equivalence for large dimensions).

This paper is organized as follows:

In section II A, we briefly review the definition of various random matrix ensembles. For later use, we also discuss the relation between the $n$-level form factor and correlation functions. The section II B deals with a brief review of the fundamentals of quantum maps and the earlier obtained results for 2-level form factor. Both the section II A and section II B
are included in this paper so as to clarify the ideas used in section III which deals with the higher order correlations and form factors for the quantum spectra with exact symmetry. In section IV, we numerically study the higher order fluctuation measures, namely, skewness and excess for a prototype quantum chaotic system, that is, Kicked Rotor and compare them with those of RMT. We summarize our results in section V.

II. PRELIMINARIES

A. Random Matrix Results

We briefly outline here the results for \( n^{th} \) order form factor (i.e fourier transform of \( n \)-point density correlation function) for the eigenvalues of the equilibrium circular ensembles of the random matrix theory.

1. The Circular ensembles

The circular type equilibrium ensembles are ensembles of unitary matrices \( U_\beta \) in which the distinct non-zero matrix elements of \( U \) are distributed independently as zero-centered random variables; \( \beta \) defines the number of independent components of the matrix elements of \( U \). There are three such ensembles, characterized by \( \beta \), namely COE, CUE and CSE for \( \beta = 1, 2 \) and 4 respectively. These Universality classes are determined by the invariance of the system under time-reversal transformation (TR) (or more generally antiunitary transformation) and are described by the invariance of the ensemble measure: invariance under orthogonal or symplectic transformations for TR-invariant systems and under unitary transformations for TR-non-invariant ones. The invariance restricts the allowed space of matrices, for example, to that of symmetric unitary matrices for orthogonal invariance.
2. n-point Correlators

The afore-mentioned unitarity of $U$ implies that its eigenvalues $\exp(iE_j)$ lie on the unit circle in the complex plane, where exponents $E_j$’s are termed as eigen-angles. The density of states is then defined by

$$\rho(E) = \sum_{j=1}^{N} \sum_{k=-\infty}^{\infty} \delta(E - 2\pi k - E_j)$$

and has the mean value $<\rho> = N/2\pi$.

For analytical studies of the spectrum, it is a usual practice to calculate the level density correlations. For cases where the level density $\rho(E)$ can be written as the sum of a smooth part $<\rho(E)>$ and a fluctuating component $\delta\rho(E)$, it is preferable to study the correlations $R_k$ between the fluctuating parts of the density. The $R_k$’s can be defined as follows,

$$R_k(E_1, \ldots, E_k) = \frac{<\delta\rho(E_1)\delta\rho(E_2)\ldots\delta\rho(E_k)>}{<\rho(E_1)> \ldots <\rho(E_k)>}$$

Here $\delta\rho(E) = \rho(E) - <\rho(E)>$, $E_j = E + l_jD$ for $j = 1, 2, \ldots, k - 1$ and $E_k = E$ with $D$ as the mean spacing and $<>$ implying the averaging over variable $E$ for ranges containing sufficient number of mean spacings. By substracting $<\rho(E)>$ from eq.(2.1) and using notation $Tr(U^n) = t_n$, $\delta\rho(E)$ can further be written as follows,

$$\delta\rho(E) = \frac{1}{2\pi} \sum_{n=-\infty,\neq 0}^{\infty} t_n \exp(inE)$$

The substitution of eq.(2.4) in eq.(2.3) gives

$$R_k(E_1, \ldots, E_k) = \frac{1}{Nk} \sum_{j} <t_jt_{j_2} \ldots t_{j_k}> \left[ \exp[i\left( \sum_{m=1}^{k} j_mE \right)] \right]_{E} \exp[D\left( \sum_{m=1}^{k-1} j_ml_m \right)]$$

Here $\sum_j$ implies the summation over all indices $j_1, j_2, \ldots, j_k$ with each index varying from $-\infty$ to $\infty$ except zero (that is, none of the indices take value zero). The averaging over $E$ reduces the eq.(2.5) in following form,
\[ R_k(E_1, ..., E_k) = \frac{1}{N^k} \sum_J < t_{j_1} t_{j_2} \cdots t_{j_k} > \delta \left( \sum_{m=1}^{k-1} j_m \right) \exp \left[ iD \left( \sum_{m=1}^{k-1} j_m l_m \right) \right] \] (2.6)

\[ = \frac{1}{N^k} \sum_J' < t_{j_1} t_{j_2} \cdots t_{-(\sum_{m=1}^{k-1} j_m)} > \exp \left[ iD \left( \sum_{m=1}^{k-1} j_m l_m \right) \right] \] (2.7)

Here \( \sum' \) implies the \( \sum_J \) subjected to condition that \( \sum_{m=1}^{k-1} j_m \neq 0 \). In semiclassical analysis, instead of dealing directly with \( R_k \), it is easier to calculate the \( k^{th} \) order form factor, defined as follows,

\[ K_k(\tau_1, ..., \tau_{k-1}) = \int \exp \left[ 2\pi i \sum_{j=1}^{k-1} (E_j - E_k) \tau_j \right] R_k(E_1, ..., E_k) dE_1 ... dE_{k-1} \] (2.8)

\[ = \int \exp \left[ 2\pi i \sum_{j=1}^{k-1} l_j \tau_j \right] R_k(l_1, ..., l_{k-1}) dl_1 ... dl_{k-1} \] (2.9)

Substitution of eq.(2.5) in eq.(2.7) gives \( K_k \) in terms of the traces,

\[ K_k = \frac{1}{N^k} \sum_J' < t_{j_1} t_{j_2} \cdots t_{-(\sum_{m=1}^{k-1} j_m)} > \prod_{m=1}^{k-1} \delta(\tau_m + \frac{j_m}{N}) \] (2.10)

As in this study we confine ourselves to calculation of \( K_k \) only for \( |\tau_m| < 1, m = 1, 2, ..,(k - 1) \), for these values of \( |\tau_m| \)'s only those terms in the \( \sum_J \) contribute which have indices \( \{j_1, j_2, ..., j_k\} \) much less than \( N \). Therefore, the \( \sum'_J \) in eq.(2.5) can be replaced by \( \sum'_G \) in which the indices vary from some value \(-n\) to \( n\) where \( n < N \) with all other conditions same.

The above result for \( K_k \) can further be simplified by using the recently obtained result for the statistics of the traces [13] which indicates that the first few traces \( t_1, t_2, ..., t_n \) of large unitary matrices (\( N \) large) taken from any of the circular ensembles display no noticeable correlation. The ensemble average of each of the traces vanishes (that is, \( < t_n > = 0 \)) for all the circular ensembles, due to uniformity of the distribution of \( E_i \)'s, the eigenangles; see ref.[13] for details. But note that \( t_n \) and \( t_{-n} \) are not independent of each other and one can show that[3,13]

\[ < |t_n|^2 > = \beta n/N \quad n << N \] (2.11)

where \( \beta \) is 1 or 2 depending on whether the ensemble is COE or CUE respectively. Therefore,
in general, \(< \prod_{j=1}^{n} t_j > = 0 \) if at least one \( t_j \) is such that its opposite \( t_{-|j|} \) is not present in the product. This product exists only if the following condition is satisfied,

\[
< \prod_j t_j > = < \prod_l |t_l|^2 > = \prod_l < |t_l|^2 > \tag{2.12}
\]

Now, as can be seen from eq.(2.10), for \( k \)-odd, every product appearing in the sum, contains odd number of \( t_j \)’s and therefore above condition can never be satisfied. This gives, for \( k \)-odd,

\[
K_{k-\text{odd}}(\tau_1, \tau_2, \ldots, \tau_{k-1}) \simeq 0, \quad |\tau_i|_{i=1,\ldots,k} < 1 \tag{2.13}
\]

on the other hand, the application of the condition (2.10) gives the following result for \( k \) even,

\[
K_{k-\text{even}}(\tau_1, \tau_2, \ldots, \tau_{k-1}) \simeq \alpha \sum_P \left[ \delta(\tau_{p_1} + \tau_{p_2})\delta(\tau_{p_3} + \tau_{p_4})\cdots \delta(\tau_{p_{k-3}} + \tau_{p_{k-2}}) \right] |\tau_{p_1}| |\tau_{p_3}| |\tau_{p_5}| \cdots |\tau_{p_{k-1}}| \tag{2.14}
\]

where \( \alpha = 1 \) for CUE and 2 for COE. The \( \sum_P \) implies the sum over all possible permutations of indices \( p_1, p_2, \ldots, p_{k-1} \) over the set \( 1, 2, \ldots, k-1 \).

The result given by eq.(2.13) and (2.14) are valid only when each \( |\tau| < 1 \). For cases with \( |\tau| \simeq 1 \) or \( >1 \), (i.e \( n \simeq N \)) one has to take into account the correlation between traces. Furthermore though the method adopted here for the derivation of \( K_k \) result is applicable only for ensembles of unitary matrices but the final results are valid also for Gaussian ensembles. This follows due to the equivalence of fluctuation measures of the circular and Gaussian ensemble in large dimensionality limit.

### B. Quantum Map Vs Classical Map

A classical map can be described by a canonical mapping \( M \) of coordinate variable \( q \) and momenta variable \( p \) at discrete time-step \( t_n \) to those at \( t_{n+1} \).

\[
\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = M \begin{pmatrix} q_n \\ p_n \end{pmatrix} \tag{2.15}
\]
with \( W(q_{n+1}, q_n) \) as the generator of the map such that

\[
p_n = -\frac{\partial W(q_{n+1}, q_n)}{\partial q_n}, \quad p_{n+1} = \frac{\partial W(q_{n+1}, q_n)}{\partial q_{n+1}}
\]  

(2.16)

The nature of the time-step considered can give rise to different kinds of maps [1]. For example, for time-periodic Hamiltonians, it is easier to study the dynamics in terms of fixed time steps (i.e., the period of the Hamiltonian), the related mapping known as stroboscopic mapping. For time-independent systems sometimes it is sufficient to consider only those steps of dynamics which occur on a definite plane that is, intersections of trajectory with a plane (instead of equal time-steps), known as Poincaré mapping.

The quantization of a two-dimensional classical map when the phase space upon which it acts is compact leads to the construction of unitary matrices \( U \) of a finite dimension \( N \), and their semiclassical limit is obtained for \( N \to \infty \). For example, for a canonical mapping on a two-dimensional torus (here taken to be a two-dimensional phase space with periodicities \( Q \) and \( P \) in \( q \)- and \( p \)-directions respectively), the corresponding quantum propagator acts in an \( N \)-dimensional Hilbert space and is represented by an \( N \times N \) unitary matrix \( U \). This follows as the number of states \( N \) allowed to be associated, by quantization, with the finite classical space is restricted (Uncertainty Principle); \( N \) is determined by the following relation,

\[
2\pi \hbar N = Q \times P
\]  

(2.17)

here \( N \) plays the role of the inverse of Planck’s constant, with \( N \to \infty \) as semiclassical limit.

C. Semiclassical Form Factor for Quantum Maps : Symmetry Preserving Cases

For quantum maps acting in a finite Hilbert space, eqs.(2.3) and (2.9) can be used to write the quantum-mechanical two-level form factor \( K_2(\tau) \),

\[
K_2(\tau) = \frac{2\pi}{N^2} \int_0^1 dr \, \left\langle \delta \rho \left( E + \frac{r\pi}{N} \right) \delta \rho \left( E - \frac{r\pi}{N} \right) \exp(2\pi ir\tau) \right\rangle_E
\]  

(2.18)

Using eq.(2.4) in eq.(2.18), \( K_2(k) \) can further be reduced in the following form,
\[ K_2(\tau) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \exp[ik(E_n - E_m)] - N\delta_{n,0} \]

\[ = \frac{1}{N} |\text{Tr}(U^n)|^2 - N\delta_{n,0} \]  

(2.19)  

(2.20)

Where \( n = N\tau \). Now the semiclassical expression of \( K_2(\tau) \) can be obtained from above equation by using semiclassical form of \( \text{Tr}(U^n) \) which can be expressed as a sum over periodic orbits in classical phase-space \( (r = (q, p)) \) [5],

\[ \text{Tr}(U^n) = g \sum_j \sum_{m_j} A_j^{\{m_j\}} \exp\left[ \frac{m_j W_j}{\hbar} - i\pi\nu_j/2 \right] \]  

(2.21)

here the amplitude \( A_j = n_j |\partial(r_n - r_0)/\partial r_0|^{-1/2} = n_j (\sinh \alpha_j)^{-1} \) of the contribution from each (multiply traversed) periodic orbit \( j \), with period \( n \), depends on the stability \( \alpha_j \) of the orbit; for long periodic orbits \( A_j \) can be approximated as \( A_j \simeq n_j \exp(-\alpha_j) = n_j \exp(-\gamma n_j) \) with \( \gamma \) as entropy of the classical motion. \( W_j \) is the action for one traversal of the orbit, \( m_j \) is the number of traversals, \( n_j = n/m_j \) is the period of the orbit with single forward traversal and \( \nu_j \) is the Maslov index. The index \( g \) refers to the number of symmetric analogs, existing in the phase-space, for the periodic orbit. While considering the long-range correlations which are mainly affected by the long periodic orbits, it is sufficient to consider \( |m_j| = 1 \) (and therefore \( n_j = n \)); this follows due to the principle of uniformity which states that on large time scales periodic orbits tend to distribute uniformly in phase space, their density increasing exponentially while intensity decreasing. Thus, on large time-scales, long periodic orbits which are almost all primitive dominate the phase space.

The evaluation of \( |\text{Tr}(U^n)|^2 \), in semiclassical limit \( N \rightarrow \infty \), can be done as follows. As obvious from eq.(2.21), \( |\text{Tr}(U^n)|^2 \) contains terms of the type \( \exp[i(W_j - W_k)/\hbar] \) and therefore, for a complete evaluation of \( K_2(\tau) \), it becomes important to study the distribution of amplitudes \( A_j \) and actions \( W_j \). But, in semiclassical limit \( \hbar \rightarrow 0 \), the significant contributions comes only from those orbit-interactions for which \( W_j - W_i \leq o(\hbar) \). The contributions from other orbit-interactions becomes negligible, in this limit, due to the presence of the rapid oscillations leading to destructive interferences. Thus, for the leading order semiclassical asymptotics of \( |\text{Tr}(U^n)|^2 \), one needs to consider only "diagonal" terms [7] with \( W_i \simeq W_j \)
which, in large-\(n\) limit (such that \(n/N = \tau << 1\)) can be evaluated by invoking the Hannay’s sum-rule for the amplitudes [6],

\[
|Tr(U^n)|^2 \simeq g^2 \sum_j A_j^2
\]  

(2.22)

Now by using Hannay’s sum rule for the intensities, which comes from the principle of uniformity [5] and is given by

\[
\sum_i A_i^2 \delta \left( |\tau| - \frac{n_i}{N} \right) = \frac{N^2 |\tau|}{g} \quad |\tau| << 1
\]  

(2.23)

one can obtain the two level form factor \(K_2(\tau)\) [7,2] which turns out to be same as that for random matrix ensembles, under small- \(\tau\) approximation (eq.(2.14)),

\[
K_2(\tau) \simeq g|\tau|
\]  

(2.24)

Note that above result is valid only for \(|\tau| << 1\) i.e \(|n| << N\). This limit of validity comes into existence due to considerations of only diagonal terms in the evaluation of \(|Tr(U^n)|^2\).

For cases \(|\tau| \simeq 1\), one needs to consider the contributions due to the constructive interference of very long periodic orbits (with \(W_i - W_j \simeq o(\hbar)\)) too which once again requires an understanding of the distribution of periodic orbit actions. Moreover, in the derivation of spectral density in terms of periodic orbits, the quasi-energy is assumed to be complex with a very small imaginary part \(\epsilon\) (required to avoid the divergence of the formula, occurring for real energies). Due to the finiteness of \(\epsilon\), the periodic orbits with period \(n > n^*\) (that is the oscillations with energies \(\delta E < \epsilon\) where \(\delta E = \hbar/n\) and \(\epsilon = \hbar/n^*\)) can not be taken into account in this formulation.

The result obtained in eq.(2.12) for semiclassical form factor is same as that for exact symmetry classes of RMT, under the same limit. For example, \(g = 2\) and \(1\) give corresponding COE and CUE results respectively [2].

### III. HIGHER ORDER SPECTRAL CORRELATIONS

In this paper, we restrain ourselves to the study of the \(k\)-level correlation functions in short time limits (where period \(n\) of the longest periodic orbit, in phase space, is much
greater than unity but much less than $N$ i.e $1 \ll n \ll N$ or $|\tau| = n/N \ll 1$ which will allow us to use principle of uniformity in the evaluation of cross multiplication of $k$ periodic orbit contributions, thus simplifying the calculations. For simplicity and to explain our method, we first calculate the $3^{rd}$ and $4^{th}$ order correlations and then generalize it to $k^{th}$ order correlations.

A. $3^{rd}$-Order Correlation

For simplicity, let us first calculate the $3^{rd}$ order correlation function. The substitution of eq.(2.4) and eq.(2.21) in eq.(2.3), with $k = 3$, gives us

$$ R_3(E_1, E_2, E_3) = \frac{g^3}{N^3} \sum_{ijk} \sum_{m_i, m_j, m_k = \pm 1} A_i A_j A_k \left\langle \left[ \exp \left( i m_i (n_i E + n_i \ell_1 \frac{2\pi}{N} + \frac{W_i}{\hbar}) \right) \exp \left( i m_j (n_j E + n_j \ell_2 \frac{2\pi}{N} + \frac{W_j}{\hbar}) \right) \right. \left. \exp \left( i m_k (n_k E + \frac{W_k}{\hbar}) \right) \right\rangle \right.$$ (3.1)

Here $<>$ implies a local averaging with respect to $E$, that is, the energy averaging over ranges which are classically small but quantum mechanically large so that a large number of levels are included. For example, a good choice is to take the size of the averaging range to be $E$ itself i.e to define

$$ < f(E) >_E = \frac{1}{E} \int_0^E f(E') dE' $$ (3.2)

Therefore the variation of amplitude, in the above equation, with respect to energy is very small (amplitude being a classical quantity) and can be ignored. This gives

$$ R_3(E_1, E_2, E_3) = \frac{g^3}{N^3} \sum_{ijk} \sum_{m_i, m_j, m_k = \pm 1} A_i A_j A_k \exp \left[ (m_i n_i \ell_1 + m_j n_j \ell_2) \frac{2\pi i}{N} \right] \langle \exp \left[ (m_i W_i + m_j W_j + m_k W_k) \frac{i}{\hbar} \right] \rangle $$ (3.3)

Due to averaging over $E$, the contribution of various terms, in eq.(3.3), will be determined by the fact whether their exponents contain $E$ or not; the terms containing a factor of type
exp[iEn] will not make any contribution. Thus we can divide all the terms into following two classes.

Case (1): Terms with all \( n_i, n_j, n_k \) of the same sign (i.e either all positive or all negative)

On averaging over \( E \), the contribution of these terms to \( R_3 \) turns out to be zero due to presence of a factor of type \( \exp[\pm i (|n_i| + |n_j| + |n_k|) E] \)

Case (2) Terms with any two among \( (n_i, n_j, n_k) \) with same sign (+ or −) and third one with opposite sign.

The terms under this case contain a factor \( \exp[\pm (n_i + n_j - n_k) iE] \) (and its permutations). As mentioned above, these terms will make a non zero contribution iff \( n_k = n_i + n_j \) (or \( n_i = n_j + n_k \), \( n_j = n_i + n_k \)). Thus eq.(3.3) can be reduced to the following form

\[
R_3 (E_1, E_2, E_3) = \frac{g^3}{N^3} \sum_{ijk} A_i A_j A_k \\
\sum_{m=\pm 1} \exp\left(\frac{2\pi m i}{N} (n_i \ell_1 + n_j \ell_2)\right) < \exp\left(\frac{im}{\hbar} (W_i + W_j - W_k)\right) > \\
+ \exp\left(\frac{2\pi m i}{N} (n_i \ell_1 - n_j \ell_2)\right) < \exp\left(\frac{im}{\hbar} (W_i - W_j + W_k)\right) > \\
+ \exp\left(\frac{2\pi m i}{N} (n_j \ell_2 - n_i \ell_1)\right) < \exp\left(\frac{im}{\hbar} (W_j - W_i + W_k)\right) > \tag{3.4}
\]

with 2nd and 3rd terms corresponding to \( n_i = n_j + n_k \) and \( n_j = n_i + n_k \) respectively.

To evaluate terms of type \( < e^{i(W_i+W_j-W_k)/\hbar} > \) we proceed as follows. Here \( W_i \), the action of a periodic orbit with period \( n_i \), can also be written as a sum of \( n_i \) single step actions \( W_i = \sum_{t=0}^{n_i-1} W_i(q_{t+1}, q_t)|_{q_{n_i}=q_0}. \) For strongly chaotic dynamics and on large time scales, these single step actions can be regarded as independent variables with a pair-correlation coefficient decaying exponentially to zero. An extension of central limit theorem therefore implies that \( W_i \)’s are Gaussian random variable on large time-scales. Hence \( \theta = W_i + W_j - W_k \) will also be a Gaussian random variable with mean zero, the variance of which is given as follows,

\[
Var\theta = < (W_i + W_j - W_k)^2 > \tag{3.5}
\]
Here $T$ is the average time-period of periodic orbits given by $T = \hbar/\delta E$ where $\delta E$ is the energy range over which average is taken.

As on large time-scales, the phase-space is densely and uniformly covered by periodic orbits and a typical trajectory can be approximated by a very long periodic orbit. This permits us to approximate the average of $\exp(iW_j)$ over all periodic orbits by a phase-space average. This gives $\langle \exp(i\theta) \rangle = \exp(-Var\theta) = \exp(-3T)$. Here, in eq.(3.6), the correlations between various actions i.e terms of type $< W_iW_j >$ has been approximated to zero ($W_i$ assumed to be random variable). But, as mentioned in section, the correlation between actions are not entirely zero, that is, $W_i$’s are not exactly random variables. The distribution function $P(\theta)$ of these actions can be written as $P(\theta) = P_{\text{random}} + P_{\text{correlated}}$, where random part of the distribution $P_{\text{random}}$ dominates the non-random part $P_{\text{correlated}}$. Therefore, on large time scales $P(\theta)$ can be approximated by a Gaussian which gives us first order term of $R_3$. To calculate higher order terms which are not negligible on very long time scales the correlations between actions must also be taken into account.

To further simplify the calculation of $R_3$, $A_k(\simeq n_k e^{-\gamma n_k})$, in eq.(3.4), can be replaced by $A_iA_j(n_i^{-1} + n_j^{-1})$ for the terms which survive due to $n_k = n_i + n_j$. Similarly for terms with $n_i = n_j + n_k$ or $(n_j = n_i + n_k)$, $A_k$ can be replaced by $A_iA_j^{-1}n_j(1 - n_jn_i^{-1})$ and $A_iA_j^{-1}n_i(1 - n_in_j^{-1})$ respectively. This leads us to following form of $R_3$,

$$R_3 = \frac{g^3}{N^3} \sum_{ij} A_i^2A_j^2 \left( \frac{1}{n_i} + \frac{1}{n_j} \right) \sum_{m=\pm 1} \exp \left( \frac{2\pi mi}{N} (n_i\ell_1 + n_j\ell_2) \right) + \sum_{ij} A_i^2n_j(1 - n_jn_i^{-1}) \sum_{m=\pm 1} \exp \left( \frac{2\pi mi}{N} (n_i\ell_1 - n_j\ell_2) \right) + \sum_{ij} A_j^2n_i(1 - n_in_j^{-1}) \sum_{m=\pm 1} \exp \left( \frac{2\pi mi}{N} (-n_i\ell_1 + n_j\ell_2) \right) e^{-3T} \quad (3.7)$$

Here the 2nd term corresponds to $n_i = n_j + n_k$ or $n_k = n_i - n_j$, the 3rd term corresponds to $n_j = n_i + n_k$ or $n_k = n_j - n_i$.

The Fourier transform of $R_3$ gives us the 3rd order form factor $K_3$,
\[ K_3(\tau_1, \tau_2) = \int e^{2\pi i [(r_1-r_3)\tau_1 + (r_2-r_3)\tau_2]} R_3(r_1, r_2, r_3) \, dr_1 dr_2 dr_3 \quad (3.8) \]
\[ = \int e^{2\pi i [\ell_1 \tau_1 + \ell_2 \tau_2]} R_3(\ell_1, \ell_2) \, d\ell_1 d\ell_2 \quad (3.9) \]

(where \( r_1 - r_3 = \ell_1 \) and \( r_2 - r_3 = \ell_2 \)). The eq.(3.9) follows from eq.(3.8) as \( R_3 \) depends only on differences \( r_1 - r_3, r_2 - r_3 \).

The further calculation of \( K_3 \) can be done by substituting eq.(3.7) in eq.(3.9) and by making use of following equalities (Appendix A) which follow from principle of uniformity,

\[ \sum_j n_j^a \delta \left( |\tau_2| - \frac{n_j}{N} \right) = \frac{< n^a >}{g} = \frac{f(0, a)}{g} \quad (3.10) \]

and

\[ \sum_i A_{2i}^2 \frac{\delta \left( |\tau_1| - \frac{n_i}{N} \right)}{n_i} = \frac{N}{g} \quad (3.11) \]

The result obtained depends on whether \( \tau \)'s are greater or less than zero. This gives rise to following three possibilities. **Case (1): both \( \tau_1, \tau_2 > 0 \) or \( \tau_1, \tau_2 < 0 \)**

\[ K_3 = \frac{g^3}{N^3} \left( \sum_{i,j} \left( A_{2i}^2 n_j + A_{2j}^2 n_i - A_{2i}^2 n_j^2 n_i^{-1} - A_{2j}^2 n_i^2 n_j^{-1} \right) \delta \left( |\tau_1| - \frac{n_i}{N} \right) \delta \left( |\tau_2| - \frac{n_j}{N} \right) \right) e^{-3T} \quad (3.12) \]
\[ \simeq g \left( |\tau_1| + |\tau_2| \right) e^{-3T} \quad (3.13) \]

It is obvious from above equation that \( K_3 \) falls very rapidly to zero for large \( T \)-values, that is, for time-scales on which sufficiently long periodic orbits exist in the phase space. On long time-scales, therefore this is similar to the RMT result (eq.(2.13)). **Case (2): \( \tau_1 > 0, \tau_2 < 0 \) or \( \tau_1 < 0, \tau_2 > 0 \)**

\[ K_3 = \frac{g^2}{N^3} \sum_{i,j} \left( A_{2i}^2 n_i + A_{2j}^2 n_j - A_{2i}^2 n_j^2 n_i^{-1} - A_{2j}^2 n_i^2 n_j^{-1} \right) \delta \left( |\tau_1| - \frac{n_i}{N} \right) \delta \left( |\tau_2| - \frac{n_j}{N} \right) e^{-3T} \quad (3.14) \]
\[ \simeq g \left[ |\tau_1| f_1(|\tau_2|) + |\tau_2| f_1(|\tau_1|) - f_2(|\tau_1|) - f_2(|\tau_2|) \right] e^{-3T} \quad (3.15) \]

On substituting values of \( f_1 = f(0,1) \) and \( f_2 = f(0,2) \) (Appendix A) in above equation, we get \( K_3 \simeq g(|\tau_1| - |\tau_2|)(e^{N\tau_2} - e^{N\tau_1})e^{-3T} \) where \( N|\tau_1| \) and \( N|\tau_2| \) are of the same order as that
of $T$. This results into a nearly zero $K_3$ on large-time scales which is again similar to RMT result.

Note the above-mentioned similarity between $K_3$ results for quantum maps and RMT has been shown here only for those time-scales at which principle of uniformity is well-applicable to the distribution of periodic orbits. No conclusion can be drawn about the short-time scales from the above analysis although the deviation of two-point fluctuation measures for quantum maps from those of RMT [2,7] suggests us to expect the same for higher orders too.

### B. 4th-Order Correlation

To calculate the 4th order correlation function, We substitute eq.(2.4) and eq.(2.21) in eq.(2.3), with $k = 4$. This gives us

$$R_4(E_1, E_2, E_3, E_4) = \frac{g^4}{N^4} \sum_{ijk} \sum_{m_i,m_j,m_k=\pm 1} A_i A_j A_k A_r \langle \exp \left[ (m_i n_i + m_j n_j + m_k n_k + m_r n_r) i E \right] \rangle$$

$$\times \exp \left[ \left( m_i n_i \ell_1 + m_j n_j \ell_2 + m_k n_k \ell_3 \right) \frac{2\pi i}{N} \right] \langle \exp \left[ (m_i W_i + m_j W_j + m_k W_k + m_r W_r) \frac{i}{\hbar} \right] \rangle$$

(3.16)

Again the significant contributions to $R_4$ come from following four types of terms, Case (1): Terms where pairwise cancellation occurs i.e terms with $n_i = n_k$, $n_j = n_r$ and $W_i = W_k, W_j = W_r$ (and their permutations)

The contribution $R_{4i}$ from such terms can be written as follows,

$$R_{4i} = \frac{g^4}{N^4} \sum_{\text{perm}} \sum_{ij} A_i^2 A_j^2 \sum_{m=\pm 1} \exp \left( \frac{2\pi mi}{N} (n_i (\ell_1 - \ell_3) + n_j \ell_2) \right)$$

(3.17)

here $\sum_{\text{perm}}$ refers to the sum over all possible permutations of pairs.

Case (2): Terms with $n_i - n_j - n_k - n_r = 0$ (and other such permutations)

The contributions to $R_4$ from terms with $n_i - n_j - n_k - n_r = 0$ can be written as follows,
\[
\frac{g^4}{N^4} \sum_{ijk} A_i^2 A_j^2 A_k^2 \left( n_j n_k - \frac{n_j^2 n_k}{n_i} - \frac{n_j n_k^2}{n_i} \right) \sum_{m=\pm 1} \exp \left( \frac{2\pi m i}{N} (n_i \ell_1 - n_j \ell_2 - n_k \ell_3) \right) \\
\left\langle \exp \left( \frac{im}{\hbar} (W_i - W_j - W_k - W_r) \right) \right\rangle
\]  
(3.18)

Similarly one can write the contributions from terms with \(n_j - n_i - n_k - n_r = 0\) and \(n_k - n_i - n_j - n_r = 0\). The symbol \(R_{4ii}\) will refer to the sum of contributions of all such terms.

The contribution \(R_{4iii}\) from term with \(n_r - n_i - n_j - n_k = 0\) is

\[
R_{4iii} = \frac{g^4}{N^4} \sum_{ij} A_i^2 A_j^2 A_k^2 \left( \frac{1}{n_i n_j} + \frac{1}{n_i n_k} \right) \sum_{m=\pm 1} \exp \left( \frac{2\pi m i}{N} (n_i \ell_1 + n_j \ell_2 + n_k \ell_3) \right) \\
\left\langle \exp \left( \frac{-im}{\hbar} (W_r - W_i - W_j - W_k) \right) \right\rangle
\]  
(3.19)

Case (3): Terms with \(n_i + n_j - n_k - n_r = 0\) (and other such permutations)

The contribution to \(R_4\) from terms with \(n_i + n_j - n_k - n_r = 0\) can be written as follows,

\[
\frac{g^4}{N^4} \sum_{ijk} A_i^2 A_j^2 A_k^2 \left( \frac{1}{n_i n_j} + \frac{1}{n_i n_k} \right) \sum_{m=\pm 1} \exp \left( \frac{2\pi m i}{N} (n_i \ell_1 + n_j \ell_2 - n_k \ell_3) \right) \\
\left\langle \exp \left( \frac{-im}{\hbar} (W_i + W_j - W_k - W_r) \right) \right\rangle
\]  
(3.20)

Similarly one can write the contributions from terms with from terms \(n_i - n_i - n_k + n_r = 0\) and \(n_i - n_j + n_k - n_r = 0\). The symbol \(R_{4iv}\) refers to the sum of contributions of all such terms.

Thus \(R_4\) can be written as follows

\[
R_4 = R_{4i} + R_{4ii} + R_{4iii} + R_{4iv}
\]  
(3.21)

where in each of the contributions \(R_{4i}...R_{4iv}\) the terms of type \(e^{W_i+W_j+W_k-W_r}\) can be replaced by \(e^{-4\tau}\) (as done earlier for \(K_3\)). The Fourier transform of \(R_4\) gives us the 4th order form factor \(K_4\),

\[
K_4 (\tau_1, \tau_2, \tau_3) = (2\pi)^3 \int e^{2\pi i (\ell_1 \tau_1 + \ell_2 \tau_2 + \ell_3 \tau_3)} R_4 (\ell_1, \ell_2, \ell_3) d\ell_1 d\ell_2 d\ell_3
\]  
(3.22)

\[
= K_i + K_{ii} + K_{iii} + K_{iv}
\]  
(3.23)
where \( K_{i,ii,iii,iv} = \int e^{2\pi i[\ell_1 \tau_1 + \ell_2 \tau_2 + \ell_3 \tau_3]} R_{4(ii,iii,iv)}(\ell_1, \ell_2, \ell_3) \, d\ell_1 d\ell_2 d\ell_3. \)

Now \( K_4 \) can be calculated by substituting eqs.(3.17-3.20) in eq.(3.23) and using equalities eqs.(3.10,3.11). Again as for \( K_3 \), the result depends on whether \( \tau \)'s are greater or less than zero. This gives rise to following three possibilities.

**Case (1) \( \tau_1, \tau_2, \tau_3 > 0 \) or \( \tau_1, \tau_2, \tau_3 < 0 \)**

In this case, except for \( K_{4iii} \), the contributions from all others, namely, \( K_{4i}, K_{4ii} \) and \( K_{4iv} \) are zero. Thus

\[
K_4 = K_{4iii} = \frac{g^4}{N^4} \sum_{i,j,k} \sum_{m=\pm 1} A_i^2 A_j^2 A_k^2 \left( \frac{1}{n_j n_k} + \frac{1}{n_j n_k} + \frac{1}{n_j n_k} \right) \delta \left( \tau_1 - \frac{mn_j}{N} \right) \\
\delta \left( \tau_2 - \frac{mn_j}{N} \right) \delta \left( \tau_3 - \frac{mn_k}{N} \right) e^{-4T} \\
\simeq g (|\tau_1| + |\tau_2| + |\tau_3|) e^{-4T} \tag{3.24}
\]

Due to presence of the exponentially decaying factor, \( K_4 \) turns out to be approximately zero for large \( T \)-values which is again similar to RMT results.

**Case (2) Any two of \( \tau_1, \tau_2, \tau_3 \) positive (negative) and the third one negative (positive)**

Let \( \tau_i, \tau_j > 0 \) and \( \tau_k < 0 \) (where \( i, j, k \) can take any of the values 1, 2, 3). In this case,

\[
K_{4i} \simeq g [\delta (\tau_i + \tau_k) |\tau_i||\tau_j| + \delta (\tau_j + \tau_k) |\tau_i||\tau_j|] \tag{3.26}
\]

\[
K_{4ii} = g [|\tau_k|f_1(|\tau_i|)f_1(|\tau_j|) - f_1(|\tau_j|)f_2(|\tau_i|) - f_1(|\tau_i|)f_2(|\tau_j|)] e^{-4T} \tag{3.27}
\]

\[
K_{4iii} = 0 \tag{3.28}
\]

\[
K_{4iv} = g [|\tau_i|f_1(|\tau_k|) + |\tau_j|f_1(|\tau_k|) - f_2(|\tau_k|)] e^{-4T} \tag{3.29}
\]

As obvious from above equations, only \( K_{4i} \) does not contain an exponentially decaying factor and therefore makes a non-zero contribution to \( K_4 \) in large time-limits. The above results are valid also if \( \tau_i, \tau_j < 0 \) and \( \tau_k > 0 \). A comparison with RMT results shows that this lowest order contribution to \( K_4 \) is same as that in RMT for \( |\tau_1|, |\tau_2|, |\tau_3| \ll 1. \)
C. \textit{k}^{th}-Order Correlation

The method used in calculation of 3\textsuperscript{rd} and 4\textsuperscript{th} order form factors can further be generalized to the \textit{k}^{th} order correlation function. The substitution of eqs.(2.4) and (2.21) in eq.(2.3) gives us

\[ R_k = \frac{g_k}{N_k} \sum_{i_1,..,i_k} A_{i_1}A_{i_2}...A_{i_k} \sum_{m_1,..,m_k=\pm 1} \left\langle \exp \left[ im_1 \left( n_{i_1} E + \frac{2\pi}{N} n_{i_1} \ell_1 + \frac{W_{i_1}}{h} \right) \right] \right. \]...
\[ \left. \exp \left[ im_k \left( n_{i_k} E + \frac{W_{i_k}}{h} \right) \right] \right\rangle \tag{3.30} \]

It can further be rearranged as follows,

\[ R_k = \frac{g_k}{N_k} \sum_{i_1,..,i_k} \prod_{l=1}^k A_{i_l} \sum_{m_1,..,m_k=\pm 1} \left\langle \exp \left[ iE \sum_{l=1}^k m_l n_{i_l} \right] \right. \]...
\[ \left. \exp \left[ \frac{2i\pi}{N} \sum_{l=1}^{k-1} m_l n_{i_l} \ell_l \right] \right\rangle \left\langle \exp \left[ i \sum_{l=1}^k \frac{m_l W_{i_l}}{h} \right] \right\rangle \tag{3.31} \]

As due to \pm 1 values taken by each \( m_l \), \( l = 1, 2, .., k \) there can be \( 2^k \) different combinations of \( n_{i_l} \)'s in the 1\textsuperscript{st} exponent of eq.(3.31). Let \( M(r) \) be the set of a particular choice of values for each \( m_l \) in the set \{\( m_1, m_2, .., m_k \)\}. Therefore there can exist \( 2^k \) such sets, denoted by \( M(r) \) with \( r = 1 \rightarrow 2^k \), of which only \( 2^{k-1} \) sets are distinct. Here two sets \( M(r) \) and \( M(r') \) are considered indistinct if the values of each \( m_l \) in \( M(r) \) is opposite (in sign) to that in \( M(r') \). Now eq.(3.31) can be rewritten as

\[ R_k = \frac{g_k}{N_k} \sum_{r=1}^{2^k} F_{M_r} \tag{3.32} \]

where \( F_{M_r} \) is given as follows,

\[ F_{M_r} = \sum_{i_1,..,i_k} \prod_{l=1}^k A_{i_l} \left\langle \exp \left[ iE \sum_{l=1}^k m_l n_{i_l} \right] \right. \]...
\[ \left. \exp \left[ \frac{2i\pi}{N} \sum_{l=1}^{k-1} m_l n_{i_l} \ell_l \right] \right\rangle \left\langle \exp \left[ i \sum_{l=1}^k \frac{m_l W_{i_l}}{h} \right] \right\rangle \tag{3.33} \]

here, in above equation, the values taken by \( m_l \)'s are the same as in \( M(r) \).

Now for the same reason, as given for 3\textsuperscript{rd} order correlation function, only those terms of eq.(3.33) will contribute significantly to \( R_k \) for which the multiplying factor of \( E \) in the
exponent is zero. Due to $k$-summations over periodic orbits, each summation containing a large number of them, there are many possibilities, resulting in a zero coefficient of $E$. These various possibilities may arise, due to the "group-wise cancellation of periods" in the $1^\text{st}$ exponent of eq.(3.32), containing various groups of periods in the exponent, where in each group the positive traversals of few orbits are cancelled by the negative traversals of few other orbits. Let $G$ stand for any division of indices $1, 2, \ldots, k$ into $q$ subgroups $(G_1, G_2, \ldots, G_q)$ then a term appearing in eq.(3.33) will make a non-zero contribution to $F_M$ if it satisfies following condition,

$$\sum_{G_j} m_l n_{i_l} = 0 \quad (j = 1, 2, \ldots q)$$  \hspace{1cm} (3.34)

where the summation is over indices $l$ present in subgroup $G_j$ and so on. As obvious, one of these subgroups will contain index $i_k$. Later we will need to distinguish it form other subgroups; let us call it $G'_j$. Thus $F_M$ can be rewritten as

$$F_M = \sum_G \left\langle \exp \left[ i \frac{\hbar}{\hbar} \sum_{l=1}^p m_l W_{j_l} \right] \right\rangle \prod_{j=1}^q C_{G_j}$$  \hspace{1cm} (3.35)

here $\sum_G$ implies the summation over all possible divisions of indices $i_1, i_2, \ldots, i_k$ into various subgroups, and $\prod_j$ implies the product of the contributions $C_{G_j}$ from all $G_j$’s for one such division, where

$$C_{G_j} = \sum_{j_1, \ldots, j_k} \left( \prod_{l=1}^p A_{j_l} \right) \delta(\sum_{l=1}^p m_l n_{j_l}) \exp \left[ \frac{2i\pi}{N} \sum_{l=1}^p m_l n_{j_l} b_l \right]$$  \hspace{1cm} (3.36)

here the $p$ indices contained in subgroup $G_j$ are denoted by $j_1, j_2, \ldots, j_p$ with $m_l$ as the coefficient of $n_{j_l}$ and $b_l$’s refer to $\ell$’s associated with these indices (e.g if $j_l = i_1$ then $b_l = \ell_1$ and so on); if $j_l = i_k$ then $b_l = b_k = 0$. Note that, in the group $G'_j$, the index $j_1$ is always chosen to be $i_k$ (so as to simplify the presentation) with $b_1 = b_k = 0$. Now by using $n_{j_l} = \sum_{l=2}^p \alpha_l n_{j_l}$, with $\alpha_l = -m_l/m_1$, and $A_{j_l} \simeq n_{j_l} e^{-\gamma n_{j_l}}$, one can show that

$$\prod_{l=1}^p A_{j_l} \simeq \sum_{l=2}^p \alpha_l n_{j_l}^{\alpha_l} \prod_{s=2 \neq l}^p A_{j_s}^{\alpha_s+1} n_{j_s}^{\alpha_s+1}$$  \hspace{1cm} (3.37)

By using eq.(3.37), $C_{G_j}$ can further be reduced as follows
\[ C_{Gj} = \sum_{j_1,\ldots,j_l} \left( \sum_{l=2}^{p} \alpha_l \frac{A_{j_l}^{\alpha_l+1}}{n_{j_l}^{\alpha_l-1}} \prod_{s=2}^{p} \frac{A_{j_s}^{\alpha_s+1}}{n_{j_s}^{\alpha_s}} \right) \exp \left[ \frac{2i\pi}{N} \left( \sum_{l=2}^{p} (b_l - b_1) m_l n_{j_l} \right) \right] \] (3.38)

Note here that for \( j = j', b_1 = 0 \). It can be seen from above equations that the most significant contribution to \( R_k \) comes from those terms where a pairwise cancellation of time-periods \( n_i, n_j, n_k, n_s, \ldots \) (appearing as a factor of \( E \) in the exponent) as well as actions \( W_i, W_j, W_k, W_s, \ldots \) (so that there is no exponential decay) occurs which is possible only when \( k \) is even. The contribution \( R_{ki} \) from such terms can be written as follows,

\[ R_{ki} = \frac{g^k}{N^k} \sum_{\text{perm} j_1,\ldots,j_k} \sum_{A_1^2, A_2^2, \ldots, A_{k-1}^2} \sum_{m=\pm 1} \exp \left[ \frac{2\pi mi}{N} \left( \sum_{l=1}^{(k-2)/2} n_{j_{2l-1}} (b_{2l-1} - b_2) + n_{j_{2l-1}} b_{k-1} \right) \right] \] (3.39)

The \( \sum_{\text{perm}} \) implies the summation over all possible permutations of indices \( j_1, j_2, \ldots, j_k \) taken from set \( i_1, i_2, \ldots, i_k \). The contributions from all other terms contain an exponential term, with sum over actions (in units of \( \hbar \)), as its exponent (i.e term \( < \exp(\sum m_i W_i) > \)). In large time limits, the application of CLT again permits us to replace this term by \( e^{-kT} \). The contribution of such terms to \( R_k \) can be given as follows

\[ R_{kii} = \frac{g^k}{N^k} \sum_r \sum_G \sum_{j_1,\ldots,j_p} \sum_{l=2}^{p} \alpha_l \frac{A_{j_l}^{\alpha_l+1}}{n_{j_l}^{\alpha_l-1}} \prod_{s=2}^{p} \frac{A_{j_s}^{\alpha_s+1}}{n_{j_s}^{\alpha_s}} \exp \left[ \frac{2i\pi}{N} \left( \sum_{l=2}^{p} (b_l - b_1) m_l n_{j_l} \right) \right] e^{-kT} \] (3.40)

The substitution of eqs.(3.39) and (3.40) in eq.(2.9) gives us following result for \( K_k \)

\[ K_k = K_{ki} + K_{kii} \] (3.41)

where

\[ K_{ki} = \frac{g^k}{N^k} \sum_{\text{perm} j_1,\ldots,j_k} A_1^2 A_2^2 \ldots A_{k-1}^2 \]

\[ \sum_{m=\pm 1} \delta(t_{2k-1} - \frac{mn_{j_{2k-1}}}{N}) \prod_{l=1}^{k-1} \delta(t_{2l-1} - \frac{mn_{j_{2l-1}}}{N}) \delta(t_{2l-1} + \tau_{2l}) \] (3.42)

and
\[ K_{kii} = \frac{g^k}{N^k} \sum_r \sum_G \prod_j \sum_{j_1, ..., j_p} \left( \sum_{l=2}^p \alpha_l \frac{A_{\alpha_l}^{n_{l}}}{n_{l}^{n_{l}}} \prod_{s=2}^p \frac{A_{\alpha_s}^{n_{s}}}{n_{s}^{n_{s}}} \right) \left[ \prod_{l=2}^p \delta(t_l + \frac{m_{l} n_{l}}{N}) \right]^G \delta(\sum_{l} t_l) e^{-k\tau} \] (3.43)

where \( t_l \) is the \( \tau \)-variable associated with \( b_l \) with \( t_l = 0 \) if \( b_l = \ell_k \). By using eq.(A3), one can further reduce above equations in following form,

\[ K_{kii} = \sum_{p} \left[ \delta(\tau_{p_1} + \tau_{p_2}) \delta(\tau_{p_3} + \tau_{p_4}) \right] (k/2 - 1) \text{terms} |\tau_{p_1}| |\tau_{p_3}| |\tau_{p(k-1)}| \delta_{k,\text{even}} \] (3.44)

where \( \sum_p \) refers to sum over all possible permutations of indices \( \{p_1, p_2, ..., p_4\} \) taken from set \( \{1, 2, ..., k-1\} \), and

\[ K_{kii} = g \sum_r \sum_G \prod_j D_{G_j} e^{-kT} \] (3.45)

with \( D_{G_j} \) given as follows,

\[ D_{G_j} = \delta(\sum_{l=1}^p t_l) \left[ \prod_{l=2}^p \delta(h_l + m_l) \right] \sum_{l=2}^p \alpha_l f(\alpha_l + 1, \alpha_l - 1) \prod_{s=2,\neq l}^p f(\alpha_s + 1, \alpha_s) \quad G_j \neq G_j' \] (3.46)

\[ = \left[ \prod_{l=2}^p \delta(h_l - m_l) \right] \sum_{l=2}^p \alpha_l f(\alpha_l + 1, \alpha_l - 1) \prod_{s=2,\neq l}^p f(\alpha_s + 1, \alpha_s) \quad G_j \equiv G_j' \] (3.47)

with \( h_l = \text{sign}(t_l) \). Here, as obvious, the contribution \( K_{kii} \) exist only for for all even order form factors. For odd order form factors, only \( K_{kii} \) contributes. But, on large time scales, the presence of an exponential decaying factor makes this contribution very small. Thus, on large time-scales (i.e \( n \to \infty, N \to \infty, n/N < 1 \)) one can write

\[ K_{k-\text{odd}} \simeq 0 \] (3.48)

A comparison of eqs.(3.44) and (3.47) with eqs.(2.14) and (2.13), informs us that the results obtained for both odd as well as even order form factors agree well with that of RMT.

IV. NUMERICAL VERIFICATION

This section contains a numerical study of 3\textsuperscript{rd} and 4\textsuperscript{th} order fluctuation measures, namely, skewness \( \gamma_1 \) and excess \( \gamma_2 \).
We choose the kicked rotor system for this purpose as it has been an active model of research, containing a variety of features such as localization, resonance, dependence of the spectra on number theoretical properties etc and has been used as a model for a very wide range of physical systems. For a better understanding, we briefly review the quantum and classical mechanics of kicked rotor in this section.

A. The Kicked Rotor: Classical and Quantum Dynamics

The kicked rotor can be described as a pendulum subjected to periodic kicks (with period T) with the following Hamiltonian

$$H = \frac{(p + \gamma)^2}{2} + K \cos(\theta + \theta_0) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$  \hspace{1cm} (4.1)

where $K$ is the stochasticity parameter. The parameter $\gamma$ and $\theta_0$ are introduced in the Hamiltonian in order to mimic the effects of the time-reversal ($T$) and the parity ($P$) symmetry breaking in the quantum system.

The related quantum dynamics can be described, by using Floquet’s theorem, by a discrete time evolution operator $U = BG$ where $B = \exp(-iK\cos(\theta + \theta_0)/\hbar)$ and $G = \exp(-i(p + \gamma)^2/4\hbar)$. The nature of the quantum dynamics and therefore the statistical properties of the associated quantum operators depend on $\hbar$ and $K$. For a rational value of $\hbar T/2\pi$, the dynamics can be confined to a torus while for irrational value it takes place on a cylinder. We employ torus boundary conditions ($q' = q + 2\pi, p' = p + 2\pi M/T$) by taking $\hbar T/2\pi = M/N$; both $p$ and $\theta$ then have discrete eigenvalues and $U$ can be reduced to a finite $N$-dimensional matrix of the form [32-34]

$$U_{mn} = \frac{1}{N} \exp \left[ -i \frac{K}{\hbar} \cos \left( \frac{2\pi m}{N} + \theta_0 \right) \right] \sum_{l=-N_1}^{N_1} \exp \left[ -i \left( \frac{\pi^2 l^2 - \pi \gamma l}{\hbar} \right) \right] \exp \left[ -i \left( \frac{l(m - n)}{N} \right) \right]$$  \hspace{1cm} (4.2)

where $N_1 = (N - 1)/2$ (with $N$ odd) and $m, n = -N_1, -N_1 + 1, ..., N_1$.

The quantum dynamics has a time reversal symmetry $T$ for $\gamma = 0$ and a parity symmetry $P$ for $\theta_0 = 0$. Though the $T$-symmetry may be violated for $\gamma \neq 0$, still a more generalized
antiunitary symmetry $S = TP = PT$ can be preserved in the system if $\theta_0 = 0$ [14,2]. For $K^2 >> N\hbar$, the quantum dynamics is delocalized in the phase space and two point spectral fluctuation measures have been shown to be well modeled by various symmetry classes of RMT [14,2]. In the opposite limit of weak chaos, namely, $K^2 << N\hbar$, the eigenstates localize in the momentum space and one obtains a Poisson distribution for the spectrum [2]. For numerical comparison of higher order measures with RMT, therefore, we choose various parameters such that condition $K^2 >> N\hbar$ is always satisfied.

**B. Numerical Study of Skewness and Excess**

In this section, we numerically study skewness and excess, that is, the 3$^{rd}$ and 4$^{th}$ order spectral fluctuation measure, for QKR-spectra and compare them with corresponding RMT results. Although the $k^{th}$ order form factor can easily be calculated for quantum maps (see eq.(2.10)), we choose to study $\gamma_1$ and $\gamma_2$ as corresponding numerical results (for form factors) in RMT are not available. But as both $\gamma_1$ and $\gamma_2$ can analytically be expressed in terms of 3$^{rd}$ and 4$^{th}$-order correlations $R_3$ and $R_4$ (see ref.[15] for these expressions) and, therefore, are related to form factors $K_3$ and $K_4$ (eq.2.8), any conclusion about the validity of RM model for the former measure will be applicable for the latter too.

Both skewness and excess are the functions of 3$^{rd}$ and 4$^{th}$ order central moment of a distribution respectively which contain the information about the probability of higher order events as compared to the lower order. More precisely, the skewness denotes the absence of symmetry in the distribution and can be defined as follows,

$$\gamma_1(r) = \frac{\mu_3(r)}{\sigma^3(r)}$$

(4.3)

The excess $\gamma_2$ describes the difference between the kurtosis values (i.e the fourth central moment calculated in units of the square of second central moment) of the distribution and that of a normal distribution (kurtosis=3),

$$\gamma_2(r) = \frac{\mu_4(r)}{\sigma^4(r)} - 3$$

(4.4)
where $\sigma^2$ is the variance in the number of levels in a length of r mean spacings and $\mu_3$ and $\mu_4$ are corresponding 3rd and 4th central moments. If excess is less than zero, the curve is platykurtic and if it is positive, leptokurtic.

For both $\gamma_1$ and $\gamma_2$ studies, the spectral data consists of the eigenvalues of 50 matrices of dimension $N = 199$ obtained by diagonalizing $U$-matrix (eq.(4.2)) for various values of $K$ in the neighbourhood of $K \simeq 20000$. The choice of such a high value of $K$ is made to ensure the delocalization of quantum dynamics which makes $U$ a full random matrix (see ref.[2]). Due to strong sensitivity of the eigenvalues to small changes in $K$, these sequences of quasienergies can be regarded as mutually independent.

The results obtained for $\gamma_1$ and $\gamma_2$ of the QKR-spectra are displayed, as function of $r$ in figures 1 and 2. For comparison, the corresponding RMT-results (taken from ref.[15]) are also given in each of the figures. The good agreement between QKR and RMT results, indicated by each of these figures reconfirms our semiclassical results obtained in section III. Furthermore, as indicated by these figures, the higher order correlations e.g 3rd and 4th order, seem to be very weak, nearly zero, at very long energy ranges and levels seem to be uncorrelated while strong correlation seem to be existing for short ranges. A continuously decreasing value of $\gamma_1$ implies the tendency of the distribution to appear more and more symmetric as range of the distribution increases and finally acquire the Gaussian form for long energy ranges. This conclusion is also supported by our $\gamma_2$ study which shows a larger probability of higher order events for QKR at small energy ranges (as compared to Gaussian case) while, for large energy ranges, it acquires the same form for both.

V. CONCLUSION

We conclude this article with a summary of our principle results and a brief discussion of the open problems.

We have shown that, in long time limits, the higher order spectral correlations in quantum chaotic maps can be well modeled by corresponding ones in RMT. By using the example
of kicked rotor, we have verified this numerically too. These results are also valid for con-
ser-vative Hamiltonians; this follows due to the similar expressions for the level density in terms
of classical periodic orbits for both the cases (see Ref.[4] for level density for autonomous
case). The various summation formulas used to evaluate fluctuation measures such as sum
over amplitudes (Appendix A), whose derivation depends on the uniform distribution of
periodic orbits and therefore only on the strongly chaotic nature of dynamics, still remain
valid. Thus one obtains the similar results for form factors in autonomous systems. This
should not be surprising, as both Gaussian ensembles (the ensembles of hermitian matrices
and therefore of conservative hamiltonians) and circular ensembles are known to have the
same statistical behaviour in semiclassical limit [16].

Although the study presented here deals with higher order correlations at fixed value of
parameter, we expect the validity of random matrix model also for higher order parametric
correlations in quantum chaotic spectra, but again only on long time-scales. The intuition,
based on the analogy of 2nd order parametric density correlations for mesoscopic systems
with a disordered potential, one dimensional many body hamiltonian (Calogero-Sutherland
model), scattering systems, quantum chaotic systems and random matrix models, as well
as the existence of a common mathematical base (nonlinear \(\sigma\)-model and supersymmetry
approach) further encourages us to hope the extension of this analogy for higher order
correlations too.

The ignorance about action correlations on very long time scales handicaps us from doing
the same analysis on these scales. It will also be of interest to study these correlations on
short time scales. It is on these scales where 2nd order spectral correlations deviate from
RMT and show non-universal behaviour. One expects to see similar deviations for higher
orders too. For a complete understanding of higher order correlations in quantum chaotic
systems, therefore, one should study the action and periodic orbit correlations; this will give
us some insight in the behaviour of fluctuations in other systems (above-mentioned) too.
We intend to do so in future.
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APPENDIX A: CALCULATION OF $F(A, B)$

To evaluate the following sum

$$f(a, b) = \sum_j \frac{A_j^a}{n_j^b} \delta \left( |\tau| - \frac{n_j}{N} \right)$$

(A1)

where $A_j$ is the amplitude and $n_j$ is the period of the orbit in the classical phase space. We remind ourselves that, in the phase space, the orbits proliferate exponentially with time and the density of distribution of periods over long orbits is given by $\exp(2\gamma |n|)/|n|$ with $\gamma$ as the entropy of the classical motion. In large time-limits. This, along with the approximation $A_j \simeq n_j \exp(-\gamma |n_j|)$, enables us to make the following replacement

$$\sum_j \frac{A_j^a}{n_j^b} \delta \left( |\tau| - \frac{n_j}{N} \right) \simeq \sum_{n=0}^{\infty} \exp(\gamma n(a - 2)) n^{a-b-1} \delta \left( |\tau| - \frac{n}{N} \right)$$

(A2)

which gives

$$f(a, b) = \exp(-\gamma N |\tau| (a - 1)) |\tau|^{a-b-1} N^{a-b}$$

(A3)
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FIGURES

FIG. 1. The behaviour of $\gamma_1(r)$ with respect to $r$, with $N = 199$, $\hbar = 1$, $T = 1$, $K = 20002 \rightarrow 20050$ and for (a) $\gamma = 0.0, \theta_0 = \pi/2N$, TR-preserved (b) $\gamma = 0.7071, \theta_0 = \pi/2N$, TR-broken. The solid curve depicts the corresponding RMT behaviour, namely COE (in 1(a)) and CUE limit (in 1(b)).

FIG. 2. The behaviour of $\gamma_2(r)$ with respect to $r$, with $N = 199$, $\hbar = 1$, $T = 1$, $K = 20002 \rightarrow 20050$ and for (a) $\gamma = 0.0, \theta_0 = \pi/2N$, TR-preserved (b) $\gamma = 0.7071, \theta_0 = \pi/2N$, TR-broken. The solid curve depict the corresponding RMT behaviour, namely COE (in 2(a)) and CUE limit (in 2(b)).
