Geometry of Spaces of Orthogonally Additive Polynomials on $C(K)$

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Abstract
We study the space of orthogonally additive $n$-homogeneous polynomials on $C(K)$. There are two natural norms on this space. First, there is the usual supremum norm of uniform convergence on the closed unit ball. As every orthogonally additive $n$-homogeneous polynomial is regular with respect to the Banach lattice structure, there is also the regular norm. These norms are equivalent, but have significantly different geometric properties. We characterise the extreme points of the unit ball for both norms, with different results for even and odd degrees. As an application, we prove a Banach-Stone theorem. We conclude with a classification of the exposed points.

1 Introduction
A real function $f$ on a Banach lattice is said to be orthogonally additive if $f(x + y) = f(x) + f(y)$ whenever $x$ and $y$ are disjoint. Non-linear orthogonally additive functions on function spaces often have useful integral representations — see, for example the papers of Chacon, Friedman and Katz [8, 12, 13], Mizel [21] and Rao [24]. In 1990, Sundaresan [26] initiated the study of orthogonally additive $n$-homogeneous polynomials with particular reference to the spaces $L_p[0,1]$ and $\ell_p$ for $1 \leq p < \infty$. Building on the work of Mizel, he showed that, for every orthogonally additive $n$-homogeneous polynomial $P$ on $L_p[0,1]$ with $n \leq p$, there exists a unique function $\xi$ in $L_\tilde{p}$, where $\tilde{p} = p/(p - n)$, such that

$$P(x) = \int_0^1 \xi x^n \, d\mu$$

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for every $x \in L_p[0,1]$. When $n > p$, there are no non-zero orthogonally additive $n$-homogeneous polynomials on $L_p[0,1]$. He went on to show that the Banach space of orthogonally additive $n$-homogeneous polynomials on $L_p[0,1]$ is isometrically isomorphic to $L_{\tilde{p}}$ where the latter space is equipped not with the usual norm, but with the equivalent norm $\|x\| = \max\{\|x^+\|_{\tilde{p}}, \|x^-\|_{\tilde{p}}\}$.

The next significant development was the discovery of an integral representation for orthogonally additive $n$-homogeneous polynomials on $C(K)$ spaces by Pérez and Villanueva [23] and by Benyamini, Lassalle and Llavona [3], who proved a representation of the form

$$P(x) = \int_K x^n d\mu$$

where $\mu$ is a regular Borel signed measure on $K$. The integral representations (1) and (2) have been extended and generalized in various directions in recent years. See, for example, [22, 17, 1, 30].

Orthogonally additive $n$-homogeneous polynomials are also of interest in the study of multilinear operators on Banach lattices and, more generally, on vector lattices. If $E, F$ are vector lattices, an $n$-linear mapping $A: E^n \to F$ is orthosymmetric if $A(x_1, \ldots, x_n) = 0$ whenever $x_i$ and $x_j$ are disjoint for some pair of distinct indices $i, j$. Orthosymmetric multilinear mappings are automatically symmetric [4]. In [5], Bu and Buskes prove that an $n$-linear function is orthosymmetric if and only if the associate $n$-homogeneous polynomial is orthogonally additive.

Let $E$ be a Banach lattice. For every positive element $a$ of $E$, we may form the principal ideal

$$E_a = \{x \in E : |x| \leq na \text{ for some } n \in \mathbb{N}\}$$

with lattice structure inherited from $E$ and the norm defined by $\|x\|_a = \inf\{C > 0 : |x| \leq Ca\}$. With this norm, $E_a$ is a Banach lattice. By virtue of the Kakutani representation theorem [16], the Banach lattice $E_a$ is canonically Banach lattice isometrically isomorphic to $C(K)$ for some compact Hausdorff topological space $K$, with $a$ being identified with the unit function on $K$. The Banach lattice structure of $E$ is uniquely determined by its principal ideals. It follows that an analysis of the orthogonally additive $n$-homogeneous polynomials on $C(K)$ is central to an understanding of the behaviour of orthogonally additive $n$-homogeneous polynomials on general Banach lattices.

In this paper, we focus on the geometric properties of the spaces $\mathcal{P}_n(\mathcal{P}(K))$ of orthogonally additive $n$-homogeneous polynomials on $C(K)$. There are two phenomena that are of particular interest. The first is that there are two natural ways to norm the space
\( P_\circ(E) \). The first is the norm of uniform convergence on the unit ball of \( E \), given by
\[
\| P \|_\infty = \sup \{ |P(x)| : x \in E, \| x \| \leq 1 \}.
\]
In this norm, \( P_\circ(E) \) is a Banach space. Now \( P_\circ(E) \) also has a lattice structure and so another choice of norm is the \textit{regular norm}, defined by \( \| P \|_{r} = \| |P| \|_\infty \), where \( |P| \) is the absolute value of \( P \). In this norm, \( P_\circ(E) \) is a Banach lattice. The existence of these two norms was first observed by Bu and Buskes [5] and is hinted at in the paper of Sundaresan [26]. These norms are equivalent, but we shall see that they have significantly different geometric properties.

The second phenomenon is the influence of the parity of the degree \( n \) on the structure of the space \( P_\circ(nE) \) for the two norms. Bu and Buskes [5] showed that, when \( n \) is odd, the supremum and regular norms on \( P_\circ(nE) \) are the same and that they are equivalent when \( n \) is even. We sharpen their results, using the strategy of working first on \( P_\circ(nC(K)) \) and then extending to general Banach lattices. The integral representation (2) gives a canonical isomorphism between \( P_\circ(nC(K)) \) and \( M(K) \), the space of regular Borel signed measures on \( K \). The regular norm on \( P_\circ(nC(K)) \) corresponds to the usual variation norm on \( M(K) \), but the supremum norm is identified with a different norm on \( M(K) \), given by
\[
\| \mu \|_0 = \max \{ \| \mu^+ \|_1, \| \mu^- \|_1 \}.
\]
We show that \( (M(K), \| \cdot \|_0) \) is isometrically isomorphic to the dual space of \( C(K) \), where \( C(K) \) is endowed with the norm \( \| x \|_d = \| x^+ \|_\infty + \| x^- \|_\infty \) and we show that this norm is closely related to the \textit{diameter seminorm} (see, for example, [6]). We use these identifications to give a complete description of the extreme points of the unit ball of \( P_\circ(nC(K)) \) for both norms, extending the results in [7]. Our starting point is a characterisation of the extreme points in \( C(K) \) for the norm \( \| \cdot \|_d \) and \( M(K) \) for the norm \( \| \cdot \|_0 \). This allows us to prove a Banach-Stone theorem for \( (C(K), \| \cdot \|_d) \).

We finish with a study of the exposed points of the unit ball of the space \( P_\circ(nC(K)) \).

The identification of this space with the space of measures \( M(K) \), which is a dual space for both norms, allows us to use the theory of Šmul’yan [28, 29]. Using this machinery, we characterise the weak* exposed and the weak* strongly exposed points of the unit ball.

**Preliminaries**

Let \( E \) be a real Banach space and let \( n \) be a natural number. A function \( P: E \to \mathbb{R} \) is an \textit{n-homogeneous polynomial} if there exists a necessarily unique, bounded \( n \)-linear function \( A: E^n \to \mathbb{R} \) such that \( P(x) = A(x, \ldots, x) \) for all \( x \in E \). We write \( P = \hat{A} \) if \( P \) and \( A \) are related in this way. The space \( P(nE) \) of \( n \)-homogeneous polynomials is a Banach space with the supremum norm,
\[
\| P \|_\infty = \sup \{ |P(x)| : x \in E, \| x \| \leq 1 \}.
\]
The Banach space \((\mathcal{P}(^nE), \| \cdot \|_\infty)\) is a dual space. We refer to the book by Dineen [10] for this and other facts about \(n\)-homogeneous polynomials.

Now assume that \(E\) is a Banach lattice. A partial order is defined on \(\mathcal{P}(^nE)\) by \(P = \widehat{A} \leq Q = \widehat{B}\) if \(A(x_1, \ldots, x_n) \leq B(x_1, \ldots, x_n)\) for all \(x_1, \ldots, x_n \geq 0\). In particular, an \(n\)-homogeneous polynomial \(P\) is said to be \textit{positive} if \(P \geq 0\) in the sense of this order and \(P\) is \textit{regular} if it is the difference of two positive \(n\)-homogeneous polynomials. The regular polynomials are precisely those that have an \textit{absolute value}, which is given by the formula

\[
|P|(x) = \sup \left\{ \sum_{i_1, \ldots, i_n} |A(u_{i_1}^1, \ldots, u_{i_n}^n)| : u_1^1, \ldots, u^n \in \Pi(x) \right\},
\]

where \(\Pi(x)\) denotes the set of partitions of \(x\), namely, all finite sets of positive elements of \(E\) whose sum is \(x\) [5].

The space \(\mathcal{P}_r(^nE)\) of regular \(n\)-homogeneous polynomials on \(E\) is a Banach lattice with the \textit{regular norm},

\[
\|P\|_r = \| |P| \|_\infty.
\]

We have \(\|P\|_\infty \leq \|P\|_r\), and in general these norms are not equivalent on \(\mathcal{P}_r(^nE)\). Every regular \(n\)-homogeneous polynomial \(P\) can be decomposed canonically as the difference of two positive \(n\)-homogeneous polynomials, so that \(P = P^+ - P^-\) and \(|P| = P^+ + P^-\). We refer to the paper of Bu and Buskes [5] for further details. For example, they show that \((\mathcal{P}_r(^nE), \| \cdot \|_r)\) is a dual Banach lattice.

Let \(K\) be a compact, Hausdorff space. The space \(C(K)\) of continuous real functions on \(K\) is a Banach lattice with the supremum norm, \(\|x\|_\infty = \sup \{|x(t)| : t \in K\}\). We denote by \(\mathcal{M}(K)\) the space of regular Borel signed measures on \(K\). Then the Banach lattice dual of \(C(K)\) can be identified with \(\mathcal{M}(K)\) under the variation norm, which we denote by \(\| \cdot \|_1\). Thus,

\[
\|\mu\|_1 = |\mu|(K) = \mu^+(K) + \mu^-(K) = \|\mu^+\|_1 + \|\mu^-\|_1,
\]

where \(\mu^+, \mu^-\) are the positive and negative parts of \(\mu\).

\section{Orthogonally additive \(n\)-homogeneous polynomials}

Let \(E\) be a Banach lattice and \(n\) a positive integer. A function \(P : E \to \mathbb{R}\) is called an \textit{orthogonally additive \(n\)-homogeneous polynomial} if \(P\) is a bounded \(n\)-homogeneous polynomial with the property that \(P(x + y) = P(x) + P(y)\) whenever \(x, y \in E\) are
disjoint. The space of orthogonally additive \( n \)-homogeneous polynomials on \( E \) is denoted by \( \mathcal{P}_o(nE) \). It is easy to see that \( \mathcal{P}_o(nE) \) is a closed subspace of the space \( (\mathcal{P}(nE), \| \cdot \|_\infty) \) of bounded \( n \)-homogeneous polynomials with the supremum norm. Thus \( \mathcal{P}_o(nE) \), with this norm, is a Banach space. When \( n = 1 \), this space is simply the dual space \( E' \), since every bounded linear functional is orthogonally additive.

We have the following integral representation for orthogonally additive \( n \)-homogeneous polynomials on \( C(K) \) spaces, due to P´erez-Garc´ıa and Villanueva [23] and Benyami, Lassalle and Llavona [3] (see also [7]).

**Theorem 1.** Let \( K \) be a compact, Hausdorff topological space. For every orthogonally additive \( n \)-homogeneous polynomial \( P \) on \( C(K) \) there is a regular Borel signed measure \( \mu \) on \( K \) such that

\[
P(x) = \int_K x^n \, d\mu
\]

for all \( x \in C(K) \).

In general, there is no guarantee that a Banach lattice supports any non-trivial orthogonally additive polynomials of degree greater than one. Sundaresan [26] showed that there are no non-zero orthogonally additive \( n \)-homogeneous polynomials on \( L_1[0, 1] \) for \( n > 1 \). In the case of \( \ell_1 \), it is easy to see that an \( n \)-homogeneous polynomial \( P \) is orthogonally additive if and only if there exists a bounded sequence of real numbers, \( (a_j) \), such that

\[
P(x) = \sum_{j=1}^{\infty} a_j x_j^n
\]

for every \( x \in \ell_1 \), and that \( \|P\|_\infty = \sup_j |a_j| \). Thus \( \mathcal{P}_o(n\ell_1) \) is isometrically isomorphic to \( \ell_\infty \) for every \( n \).

To put the results of the previous paragraph in a general context, we recall that a Banach lattice \( E \) is an AL-space if the norm is additive on the positive cone: \( \|x + y\| = \|x\| + \|y\| \) for all \( x, y \geq 0 \). The Kakutani representation theorem [15, 18] states that every AL-space \( E \) can be decomposed into a disjoint sum of copies of \( \ell_1 \) and \( L_1 \) spaces. Accordingly, \( E \) is Banach lattice isometrically isomorphic to a space of the form

\[
[\ell_1(\Gamma) \oplus \bigoplus_{\alpha \in A} L_1[0, 1]^{m_\alpha}]_1
\]

In this representation, the unit basis vectors \( e_\gamma \) in \( \ell_1(\Gamma) \) are in one-to-one correspondence with the atoms in \( E \) of unit norm. We recall that a positive element \( x \) of \( E \) is said to be an *atom* if \( 0 \leq y \leq x \) implies that \( y \) is a scalar multiple of \( x \). We can write the
second component in this representation as $L_1(\mu)$, where $\mu$ is the product of the Lebesgue measures on the sets $[0,1]^m$. Thus, we see that $E$ can be represented as the disjoint sum $\ell_1(\Gamma) \oplus_1 L_1(\mu)$, where the measure $\mu$ is nonatomic.

**Proposition 1.** Let $E$ be an AL-space and let $n > 1$. There is a non-zero orthogonally additive $n$-homogeneous polynomial on $E$ if and only if $E$ contains at least one atom.

**Proof.** Let $\ell_1(\Gamma) \oplus_1 L_1(\mu)$ be the Kakutani representation of $E$ as described above.

Suppose that $E$ contains an atom. Then the set $\Gamma$ in the Kakutani representation is non-empty. Choose $\gamma_0 \in \Gamma$ and define $P(x) = x_{\gamma_0}^n$ for $x = (x_\gamma) \in \ell_1(\Gamma)$ and $P(x) = 0$ for $x \in L_1(\mu)$. Then $P$ is a non-zero orthogonally additive $n$-homogeneous polynomial.

Conversely, suppose that $P$ has no atoms. Then the Kakutani representation of $P$ is $L_1(\mu)$ where the measure $\mu$ is nonatomic. The proof in this case can be gleaned from [26], but we can give a direct proof as follows. We treat the case $n = 2$ for simplicity. Suppose that $P$ is an orthogonally additive 2-homogeneous polynomial on $L_1(\mu)$, where $\mu$ is nonatomic. Let $A$ be the bounded, symmetric bilinear form that generates $P$. Then $A$ is orthosymmetric: if $x, y$ are disjoint, then $A(x, y) = 0$ [5, Lemma 4.1]. It follows from the fact that $L_1(\mu) \hat{\otimes}_\pi L_1(\mu)$ is isometrically isomorphic to $L_1(\mu^2)$ that there exists $g \in L_\infty(\mu^2)$ such that

$$A(x, y) = \int x(s)y(t)g(s, t)\,d\mu^2(s, t)$$

If we take $x, y$ to be the characteristic functions of arbitrary disjoint measurable sets, this integral is zero and so we have

$$A(x, y) = \int_D x(t)y(t)g(t, t)\,d\mu^2$$

for all $x, y \in L_1(\mu)$, where $D$ is the diagonal. However, if $\mu$ has no atoms, then the product measure of the diagonal is zero. Hence $P(x) = 0$ for every $x$. \hfill $\square$

The Banach lattices $L_1(\mu)$, where $\mu$ is nonatomic, do not support any real valued lattice homomorphisms. Our next result indicates that the existence of non-trivial $n$-homogeneous orthogonally additive polynomials on a Banach lattice is closely related to the existence of lattice homomorphisms.

**Proposition 2.** Let $E$ be a Banach lattice, let $\varphi \in E'$ and let $n \geq 2$. The $n$-homogeneous polynomial defined by $P(x) = \varphi(x)^n$ is orthogonally additive if and only if either $\varphi$ or $-\varphi$ is a lattice homomorphism.
Proof. Suppose that \( \varphi \) or \(-\varphi\) is a lattice homomorphism. Then if \( x \) and \( y \) are disjoint, we have either \( \varphi(x) = 0 \) or \( \varphi(y) = 0 \) and so \( P(x + y) = P(x) + P(y) \).

Conversely, suppose that \( P = \varphi^n \) is orthogonally additive. For every \( x \in E \), the vectors \( x^+ \) and \( tx^- \) are disjoint for all \( t \in \mathbb{R} \). Therefore

\[
\varphi(x^+)^n + t^n \varphi(x^-)^n = P(x^+ + tx^-) = \sum_{j=0}^{n} \binom{n}{j} \varphi(x^+)^{n-j} \varphi(x^-)^j t^j.
\]

for every \( t \in \mathbb{R} \). Hence either \( \varphi(x^+) = 0 \) or \( \varphi(x^-) = 0 \). If we can show that \( \varphi \) (or \(-\varphi\)) is positive, then it follows that \( \varphi \) (or \(-\varphi\)) is a lattice homomorphism.

Let \( a \) be a positive element of \( E \). The principal ideal \( E_a \) generated by \( a \) is isometrically Banach lattice isomorphic to \( C(K) \) for some compact Hausdorff topological space \( K \). The functional \( \varphi \) is represented by a regular Borel signed measure \( \mu \) on \( K \) and the fact that \( \varphi(x^+) = 0 \) or \( \varphi(x^-) = 0 \) for all \( x \in E_a \) implies that the support of \( \mu \) consists of a single point. It follows that either \( \varphi \) or \(-\varphi\) is positive on \( E_a \). Now \( E \) is the union of the principal ideals \( E_a \), which are upwards directed by inclusion. Thus, if \( \varphi \) (or \(-\varphi\)) is positive on one \( E_a \), then \( \varphi \) (or \(-\varphi\)) is positive on all of \( E \).

A Banach lattice \( E \) is an AM-space if the norm has the property that \( x \wedge y = 0 \) implies \( \|x \vee y\| = \max\{\|x\|, \|y\|\} \). In contrast with AL-spaces, there is a good supply of orthogonally additive \( n \)-homogeneous polynomials on every AM-space. The Kakutani representation theorem for AM-spaces [16] shows that the real valued lattice homomorphisms on an AM-space \( E \) separate the points of \( E \). It follows that there is a rich supply of orthogonally additive \( n \)-homogeneous polynomials of every degree on \( E \).

We now look at some properties of orthogonally additive polynomials on general Banach lattices. Our starting point is the fact that every orthogonally additive \( n \)-homogeneous polynomial on a Banach lattice \( E \) is regular. This has been shown by Toumi [27, Theorem 1]. One may also argue as follows. Let \( P \) be an orthogonally additive \( n \)-homogeneous polynomial on a Banach lattice \( E \). As \( E \) is the upwards directed union of its principal ideals, it suffices to show that \( P \) is regular on each of them. Since each principal ideal is Banach lattice isometrically isomorphic to a \( C(K) \), we can use the integral representation in Theorem 1. Then the Jordan decomposition of the representing measure gives a decomposition of the polynomial into the difference of two positive orthogonally additive \( n \)-homogeneous polynomials. Therefore \( P \) is regular.

Let \( P = \hat{A} \) be a regular \( n \)-homogeneous polynomial \( P \) on \( E \). The absolute value of \( P \)
is given by \([5, 19]\)

\[
|P|(x) = \sup \left\{ \sum_{i_1, \ldots, i_n} |A(u_{i_1}^1, \ldots, u_{i_n}^n)| : u^1, \ldots, u^n \in \Pi(x) \right\}
\tag{4}
\]

for \(x \geq 0\), where \(\Pi(x)\) denotes the set of partitions of \(x\), namely, all finite sets of positive vectors whose sum is \(x\). In general, we have

\[
|P(x)| \leq |P|(x)
\tag{5}
\]

for every \(x \in E\) and \(|P|\) is the smallest positive \(n\)-homogeneous polynomial, in the sense of the lattice structure of \(\mathcal{P}_r(nE)\), with this property. The space \(\mathcal{P}_r(nE)\) is a Banach lattice in the regular norm,

\[
\|P\|_r = \|P\|_\infty.
\]

It follows from (5) that \(\|P\|_\infty \leq \|P\|_r\) for every \(P \in \mathcal{P}_r(nE)\). In general, these norms are not equivalent.

Now \(\mathcal{P}_\infty(nE)\) is complete in the regular norm; indeed, it is even a dual Banach lattice [5, Theorem 5.4]. It follows that the supremum and regular norms are equivalent on this space. Thus, there is a sequence \((C_n)\) of positive real numbers such that \(\|P\|_r \leq C_n \|P\|_\infty\) for every \(n\) and every \(P \in \mathcal{P}_\infty(nE)\). Bu and Buskes [5] show that the two norms are the same for odd values of \(n\). For even values of \(n\), they show that \(C_n \leq n^n/2!\), the polarization constant. We shall show that, in fact, \(C_n = 2\) for even values of \(n\) and that this is sharp. This will follow from estimates we give for the value of \(|P|\) at positive points in \(E\).

If \(\varphi\) is a bounded linear functional on \(E\), then [20]

\[
|\varphi|(x) = \sup \{|\varphi(y)| : |y| \leq x\}
\]

for every \(x \geq 0\). It would be surprising if there were such a simple formula for \(|P|(x)\) when \(P\) is a regular \(n\)-homogeneous polynomial. As a linear functional, \(P\) acts on an \(n\)-fold symmetric tensor power of \(E\) and the set of vectors \(y\) satisfying \(|y| \leq x\) is now a set of tensors, rather than elements of \(E\). However, if \(P\) is orthogonally additive, it is possible to establish a relatively simple estimate for the values of \(|P|\).

**Theorem 2.** Let \(P\) be an orthogonally additive \(n\)-homogeneous polynomial on the Banach lattice \(E\).

(a) If \(n\) is odd, then

\[
|P|(x) = \sup \{|P(y)| : |y| \leq x\}.
\]

for every \(x \geq 0\) in \(E\).
If \( n \) is even, then
\[
|P|(x) \leq 2 \sup \{|P(y)| : |y| \leq x\}.
\]
for every \( x \geq 0 \) in \( E \).

**Proof.** Let \( x \geq 0 \). It follows from (3) that the value \( |P|(x) \) is unchanged if we consider \( P \) as an \( n \)-homogeneous polynomial on the principal ideal \( E_x \) generated by \( x \). Now \( E_x \) is Banach lattice isomorphic to \( C(K) \) for some compact topological space \( K \). Since \( P \) is orthogonally additive there exists a regular signed Borel measure \( \mu \) on \( K \) such that
\[
P(y) = \int_K y^n d\mu.
\]
for every \( y \in E_x \cong C(K) \). The symmetric \( n \)-linear form on \( C(K)^n \) that generates \( P \) is given by
\[
A(x_1, \ldots, x_n) = \int_K x_1 \ldots x_n d\mu.
\]
Thus, for \( x_1, \ldots, x_n \geq 0 \),
\[
|A(x_1, \ldots, x_n)| \leq \int_K x_1 \ldots x_n d|\mu|
\]
and it follows that
\[
|P|(x) \leq \int_K x^n d|\mu|
\]
for \( x \geq 0 \).

Now in general, for a nonnegative function \( w \in C(K) \) we have
\[
\int_K w d|\mu| = \sup \left\{ \left| \int_K g d\mu \right| : g \in C(K), |g| \leq w \right\},
\]
Therefore
\[
|P|(x) \leq \sup \left\{ \int_K y d\mu : y \in C(K), |y| \leq x^n \right\}.
\]
where we are identifying elements of \( E \) with continuous functions on \( K \). We now consider separately the cases where \( n \) is odd and even.

(a) We first consider the case when \( n \) odd.
If \( |y| \leq x^n \), let \( v = y^{1/n} \). Then \( |v| \leq x \) and \( \int_K y d\mu = \int_K v^n d\mu \). Therefore
\[
|P|(x) \leq \sup \left\{ \left| \int_K v^n d\mu \right| : v \in E, |v| \leq x \right\}.
\]
Thus we have
\[
|P|(x) \leq \sup \{|P(y)| : |y| \leq x\}.
\]
and it is easy to see that the reverse inequality also holds.

(b) We now consider the case when \( n \) even. We have

\[ |P|(x) \leq \sup\left\{ \left| \int_K y \, d\mu \right| : |y| \leq x^n \right\}. \]

Given \( v \in E_x \cong C(K) \) satisfying \( |v| \leq x^n \), we define \( v_1, v_2 \in C(K) \) by

\[ v_1(t) = \begin{cases} v(t)^{1/n} & \text{if } v(t) \geq 0 \\ 0 & \text{if } v(t) < 0 \end{cases} \quad v_2(t) = \begin{cases} 0 & \text{if } v(t) \geq 0 \\ |v(t)|^{1/n} & \text{if } v(t) < 0 \end{cases} \]

Then \( v = v_1^n - v_2^n \), and so

\[ \left| \int_K v \, d\mu \right| \leq \left| \int_K v_1^n \, d\mu \right| + \left| \int_K v_2^n \, d\mu \right| = |P(v_1)| + |P(v_2)|. \]

It follows from \( |v| \leq x^n \) that \( 0 \leq v_1, v_2 \leq x \). Therefore

\[ |P|(x) \leq 2 \sup\{|P(y)| : 0 \leq y \leq x\} = 2 \sup\{|P(y)| : |y| \leq x\}, \tag{6} \]

since \( n \) is even.

\[ \square \]

To see that the bound in (6) for even values of \( n \) is sharp, consider the example \( P(x) = x_1^n - x_2^n \) on \( \mathbb{R}^2 \) with any Banach lattice norm. The bound is attained for the vector \( x = (1, 1) \).

**Corollary 1.** Let \( P \) is an orthogonally additive \( n \)-homogeneous polynomial on a Banach lattice \( E \). Then \( \|P\|_r = \|P\|_\infty \) if \( n \) is odd and \( \|P\|_\infty \leq \|P\|_r \leq 2 \|P\|_\infty \) if \( n \) is even. These inequalities are sharp.

## 3 Orthogonally additive polynomials on \( C(K) \)

In this section, we study the supremum and regular norms on the spaces of orthogonally additive \( n \)-homogeneous polynomials on \( C(K) \).

The integral representation for orthogonally additive \( n \)-homogeneous polynomials on \( C(K) \) allows us to identify the vector space \( P_r(^nC(K)) \) with \( M(K) \), the space of regular Borel signed measures on \( K \). The natural norm on \( M(K) \cong C(K)' \) is the dual norm. This is the variation norm for measures: \( \|\mu\|_1 = |\mu|(K) \). We shall see that this norm corresponds to the regular norm on the spaces of orthogonally additive \( n \)-homogeneous
polynomials. However, the supremum norm on $P_0(nK)$ corresponds to a different, but equivalent norm on the space of regular Borel signed measures.

The space $P_r(nC(K))$ is a Banach lattice with the regular norm, as is the dual Banach lattice $M(K)$ with the variation norm. We shall see that the lattice structures of these two Banach lattices are the same. We note that the lattice structure of $M(K)$ as the dual of $C(K)$ is the same as the lattices structure of $M(K)$ considered as a sublattice of the lattice of Borel signed measures on $K$. In other words, a measure $\mu \in M(K)$ is positive, in the sense that $\int_K f \, d\mu \geq 0$ for every nonnegative $x \in C(K)$, if and only if $\mu(E) \geq 0$ for every Borel subset $E$ of $K$ [25, Theorem 2.18].

**Proposition 3.** Let $K$ be a compact Hausdorff topological space and let $P$ be an orthogonally additive $n$-homogeneous polynomial on $C(K)$, given by

$$P(x) = \int_K x^n \, d\mu.$$ 

Then the absolute value of $P$ is given by

$$|P|(x) = \int_K x^n \, d|\mu|.$$ 

**Proof.** We have seen in the proof of Theorem 2 that

$$|P|(x) \leq \int_K x^n \, d|\mu|$$

for every $x \geq 0$.

To prove the reverse inequality, we start with the definition of the absolute value:

$$|P|(x) = \sup \left\{ \sum_{i_1, \ldots, i_n} |A(u_{i_1}^1, \ldots, u_{i_n}^n)| : u^1, \ldots, u^n \in \Pi(x) \right\}$$

for $x \geq 0$ Taking each of $u^2, \ldots, u^n$ to be the trivial partition $\{x\}$ gives

$$|P|(x) \geq \sup \left\{ \sum_i |A(u_i^1, x, \ldots, x)| : u^1 \in \Pi(x) \right\}$$

$$= \sup \left\{ \sum_i \left| \int_K u_i^1 x^{n-1} \, d\mu \right| : u^1 \in \Pi(x) \right\} = \int_K x^n \, d|\mu|,$$

applying the partition form of the Riesz-Kantorovich formula for the absolute value of a linear functional [2, Theorem 1.16] to the measure $d\lambda = x^{n-1} \, d\mu$ in $M(K)$ and using the fact that $d|\lambda| = x^{n-1} \, d|\mu|$.
Theorem 3. Let $K$ be a compact, Hausdorff space. Let $J_n : \mathcal{M}(K) \to \mathcal{P}_o(^nC(K))$ be given by

$$(J_n \mu)(x) = \int_K x^n \, d\mu.$$ 

(a) For every $n$, $J_n$ is a Banach lattice isometric isomorphism from $(\mathcal{M}(K), \| \cdot \|_1)$ onto $(\mathcal{P}_o(^nC(K)), \| \cdot \|_r)$.

(b) If $n$ is odd, then the regular and supremum norms coincide on $\mathcal{P}_o(^nC(K))$ and so $J_n$ is an isometric isomorphism for the supremum norm on $\mathcal{P}_o(^nC(K))$.

(c) If $n$ is even, then $J_n$ is an isometric isomorphism for the norm on $\mathcal{M}(K)$ defined by

$$\| \mu \|_0 := \max\{ \| \mu^+ \|_1, \| \mu^- \|_1 \}$$

and the supremum norm on $\mathcal{P}_o(^nC(K))$.

Proof. Clearly, $J_n$ is linear and surjective. To see that it is injective, suppose that the $n$-homogeneous polynomial $P(x) = \int_K x^n \, d\mu$ is zero. The associated symmetric $n$-linear form is

$$A(x_1, \ldots, x_n) = \int_K x_1 \ldots x_n \, d\mu$$

and so $A(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in C(K)$. Taking $x_2 = \cdots = x_n = 1$, we have $\int_K x \, d\mu = 0$ for every $x \in C(K)$ and so $\mu = 0$.

(a) Clearly, $J_n$ is positive. If we show that $J_n^{-1}$ is also positive, then it will follow that $J_n$ is a lattice homomorphism [2, Theorem 7.3]. Let $P = \hat{A}$ be a positive element of $\mathcal{P}_r(^nC(K))$, with $\mu \in \mathcal{M}(K)$ satisfying $J_n \mu = P$. Then, for every nonnegative $x \in C(K)$, we have $\int_K x \, d\mu = A(x, 1, \ldots, 1) \geq 0$ and so $\mu$ is positive. Therefore $J_n$ is a lattice isomorphism for every $n$.

By Proposition 3, the regular norm of $P = J_n \mu$ is $\|P\|_r = \|P\|_{\infty} = \|P\|(1) = |\mu|(K) = \|\mu\|_1$, since $|P|$ is increasing on the positive cone of $C(K)$. Therefore $J_n$ is both a lattice isomorphism and an isometry.

(b) This has already been proved in Corollary 1.

(c) Let $\mu \in \mathcal{M}(K)$ and let $P = J_n \mu$. It follows from (a) that $P^+ = J_n \mu^+$ and $P^- = J_n \mu^-$. We have

$$P(x) = \int_K x^n \, d\mu^+ - \int_K x^n \, d\mu^-$$

for every $x \in C(K)$. As $|a - b| \leq \max\{|a|, |b|\}$ for $a, b \in \mathbb{R}^+$ and $n$ is even, it follows that $\|P\|_\infty \leq \max\{\|\mu^+\|_1, \|\mu^-\|_1\}$. 

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Now let \( \{A, B\} \) be a Hahn decomposition of \( \mu \), with \( \mu \) positive on \( A \) and negative on \( B \). If \( F \subset A \) is compact, then by a standard argument using Urysohn’s lemma (see, for example, [14, Theorem 12.41]) there is a decreasing sequence \( (x_k) \) of continuous functions on \( K \) with values in \([0, 1]\) that converges almost everywhere with respect to \(|\mu|\) to \( 1_F \), the characteristic function of \( F \). Then, by the bounded convergence theorem,

\[
\|P\|_\infty \geq \lim_{k \to \infty} \left| \int_K x_k^n \, d\mu \right| = \left| \int_K 1_F \, d\mu \right| = \mu^+(F).
\]

It follows from the regularity of \( \mu^+ \) that \( \|P\|_\infty \geq \mu^+(A) = \|\mu^+\|_1 \). Similarly, \( \|P\|_\infty \geq \|\mu^-\|_1 \). Therefore \( \|P\|_\infty = \|\mu\|_0 \).

We summarize the identifications of the various norms, bearing in mind that the supremum and regular norms coincide for positive polynomials.

**Corollary 2.** Let \( P \) be an orthogonally additive \( n \)-homogeneous polynomial on \( C(K) \), with corresponding measure \( \mu \in \mathcal{M}(K) \). Then

(a) \( \|P\|_r = \|P^+\|_r + \|P^-\|_r = \|\mu^+\|_1 + \|\mu^-\|_1 = \|\mu\|_1 \).

(b) If \( n \) is odd, then \( \|P\|_\infty = \|P\|_r \).

(c) If \( n \) is even, then \( \|P\|_\infty = \max\{\|P^+\|_r, \|P^-\|_r\} = \max\{\|\mu^+\|_1, \|\mu^-\|_1\} = \|\mu\|_0 \).

We note that the norm \( \| \cdot \|_0 \) is easily seen to be equivalent to the dual (variation) norm on \( \mathcal{M}(K) \). In fact, we have

\[
\|\mu\|_0 \leq \|\mu\|_1 \leq 2 \|\mu\|_0
\]

for every \( \mu \in \mathcal{M}(K) \).

It will be useful to have an alternative expression for the norm \( \| \cdot \|_0 \) on \( \mathcal{M}(K) \). Using the identity \( \max\{a, b\} = \frac{1}{2}(a + b + |a - b|) \) for non-negative real numbers, we have

\[
\|\mu\|_0 = \frac{1}{2}(\|\mu^+\|_1 + \|\mu^-\|_1 + \|\mu^+\|_1 - \|\mu^-\|_1)
\]

\[
= \frac{1}{2}(\|\mu\|_1 + \|\mu^+(K) - \mu^-(K)\|) = \frac{1}{2}(\|\mu\|_1 + |\mu(K)|)
\]

Thus, we have

\[
\|\mu\|_0 = \max\{\|\mu^+\|_1, \|\mu^-\|_1\} = \frac{1}{2}(\|\mu\|_1 + |\mu(K)|)
\]  \( (7) \)
These results clarify the geometric properties of the spaces $P_o(nC(K))$; for the regular norm, these spaces are all essentially the same as the dual space $M(K)$ with the variation norm. The case of $P_o(nC(K))$ with the supremum norm and $n$ even is substantially different. To understand this, we must study the extreme point structure of the unit ball of $M(K)$ for the norm $\| \cdot \|_0$.

4 Extreme points in $P_o(nC(K))$

In this section, we study the extreme points of the unit ball of the space $P_o(nC(K))$. We begin with the regular norm. We have seen in Proposition 3 that there is an isometric isomorphism

$$(P_o(nC(K)), \| \cdot \|_r) \cong (M(K), \| \cdot \|_1)$$

where $\| \cdot \|_1$ denotes the variation norm on $M(K)$, the space of regular Borel signed measures on $K$. Furthermore, when the degree $n$ is odd, the supremum and regular norms on $P_o(nC(K))$ coincide.

It is a classical result that the extreme points of the unit ball of $M(K)$ for the variation norm are the measures of the form $\pm \delta_t$, where $t \in K$ (see, for example, [11, V.8.6]). The isomorphism between $P_o(nC(K))$ and $M(K)$ associates the polynomial $P(x) = x(t)^n$ with the measure $\delta_t$. Thus, we have

**Proposition 4.** Let $K$ be a compact Hausdorff topological space. The extreme points of the closed unit ball of the space $(P_o(nC(K)), \| \cdot \|_r)$ are the $n$-homogeneous polynomials $\pm \delta_t^n$, where $t \in K$ and $\delta_t^n(x) = x(t)^n$.

This result is given in [7] for the supremum norm, but the proof given there is not valid for polynomials of even degree. However, this does not affect the results that follow in [7]. In particular, their elegant proof of the integral representation still stands. Essentially, all that is required for their arguments to work is that $P_o(nC(K))$ is a dual space and that the extreme points of the unit ball are as described above.

We now turn to the geometry of $P_o(nC(K))$ for the supremum norm, where the degree $n$ is even. We have the isometric isomorphism

$$(P_o(nC(K)), \| \cdot \|_\infty) \cong (M(K), \| \cdot \|_0)$$

where $\| \mu \|_0 = \max\{\| \mu^+ \|_1, \| \mu^- \|_1\}$. We will show that $\| \cdot \|_0$ is the dual of a norm on $C(K)$ that is equivalent to the supremum norm.
The norm we seek is related to the diameter seminorm on $C(K)$, which is defined by

$$\rho(x) = \text{diam}(x) = \sup\{|x(s) - x(t)| : s, t \in K\}.$$  

It is easy to see that we also have

$$\rho(x) = 2 \inf\{\|x - \alpha 1_K\|_{\infty} : \alpha \in \mathbb{R}\}.$$  

The kernel of $\rho$ is the one dimensional subspace of constant functions. As in [6], we use $C_{\rho}(K)$ to denote the quotient space $C(K)/\ker \rho$. It is a Banach space under the norm

$$\|\pi(x)\|_{\rho} = \rho(x)$$  

where $\pi : C(K) \to C(K)/\ker \rho$ is the quotient map. Following Cabello-Sanchez [6], we note that this means that $(C_{\rho}(K), \| \cdot \|_{\rho})$ is isometrically isomorphic, up to a constant factor 2, to the quotient space of $(C(K), \| \cdot \|_{\infty})$ by the subspace of constant functions. Therefore the dual space $(C_{\rho}(K), \| \cdot \|_{\rho})'$ is isometrically isomorphic, up to a constant factor 1/2, to a subspace of $(C(K), \| \cdot \|_{\infty})'$, the space of regular Borel signed measures with the variation norm. This subspace is the space of measures $\mu$ satisfying $\mu(K) = 0$ and on it we have [6]

$$\|\mu\|_1 = 2\|\mu\|_{(C_{\rho}(K), \| \cdot \|_{\rho})}'.$$

**Theorem 4** (Cabello-Sanchez [6]). Let $K$ be a compact Hausdorff topological space. A regular Borel signed measure $\mu$ is an extreme point of the unit ball of the dual space $(C_{\rho}(K), \| \cdot \|_{\rho})'$ if and only if $\mu = \delta_s - \delta_t$, where $s$ and $t$ are distinct points of $K$.

In order to apply this result, we first need to identify the predual of the norm $\| \cdot \|_1$ on $\mathcal{M}(K) \cong C(K)'$.

**Theorem 5.** Let $K$ be a compact Hausdorff topological space. Let $\| \cdot \|_d$ be the norm on $C(K)$ defined by

$$\|x\|_d := \|x^+\|_\infty + \|x^-\|_\infty = \max\{|\|x\|_\infty, \rho(x)\}$$  

where $\rho$ is the diameter seminorm. Then the dual space of $(C(K), \| \cdot \|_d)$ is isometrically isomorphic to the space of regular Borel signed measures on $K$ with the norm $\|\mu\|_0 = \max\{\|\mu^+\|_1, \|\mu^-\|_1\}$.

**Proof.** A routine calculation shows that the formula $\|x\|_d = \|x^+\|_\infty + \|x^-\|_\infty$ defines a norm on $C(K)$. To establish the second equality in (8), we consider two cases.
(a) Suppose the function $x$ has constant sign. Then $\rho(x) \leq \|x\|_\infty$ and one of $\|x^+\|_\infty$, $\|x^-\|_\infty$ is zero. Therefore $\|x\|_d = \|x\|_\infty$.

(b) If $x$ changes sign, then $\|x\|_\infty \leq \|x^+\|_\infty + \|x^-\|_\infty = \rho(x)$. Therefore $\|x\|_d = \max\{\|x\|_\infty, \rho(x)\}$ for every $x \in C(K)$.

Let us denote the dual norm of $\| \cdot \|_1$ by $\| \cdot \|_1'$. If $x \in C(K)$ and $\mu \in \mathcal{M}(K)$, then

$$\int_K x \, d\mu = \left( \int_K x^+ \, d\mu^+ + \int_K x^- \, d\mu^- \right) - \left( \int_K x^+ \, d\mu^- + \int_K x^- \, d\mu^+ \right).$$

Now

$$0 \leq \int_K x^+ \, d\mu^+ + \int_K x^- \, d\mu^- \leq \|x^+\|_\infty \|\mu^+\|_1 + \|x^-\|_\infty \|\mu^-\|_1 \leq \|x\|_d \|\mu\|_0$$

and similarly

$$0 \leq \int_K x^+ \, d\mu^- + \int_K x^- \, d\mu^+ \leq \|x\|_d \|\mu\|_0.$$

Therefore

$$\left| \int_K x \, d\mu \right| \leq \|x\|_d \|\mu\|_0$$

and so $\|\mu\|'_1 \leq \|\mu\|_0$.

Fix $\mu \in \mathcal{M}(K)$ and let $\varepsilon > 0$. Let $\{A, B\}$ be a Hahn decomposition for $\mu$, where $A$ is a positive set and $B$ a negative set. Since $\mu$ is regular, there exist compact sets $C \subset A$ and $D \subset B$ such that $|\mu|(A \setminus C), |\mu|(B \setminus D) < \varepsilon$. By Urysohn’s lemma, there is a continuous function $y: K \to [0, 1]$ that takes the values 1 and 0 on the sets $C$ and $D$ respectively. Then $\|y\|_1 = 1$ and

$$\int_K y \, d\mu = \int_C y \, d\mu + \int_{A \setminus C} y \, d\mu + \int_{B \setminus D} y \, d\mu.$$

It follows that

$$\left| \int_K y \, d\mu \right| \geq \mu^+(C) - 2\varepsilon \geq \mu^+(A) - 3\varepsilon = \|\mu^+\|_1 - 3\varepsilon.$$

Similarly,

$$\left| \int_K y \, d\mu \right| \geq \|\mu^-\|_1 - 3\varepsilon$$

Thus, $\|\mu\|'_1 \geq \|\mu\|_0 - 3\varepsilon$ for every $\varepsilon > 0$.

Therefore $\|\mu\|'_1 = \|\mu\|_0$ for every $\mu \in \mathcal{M}(K)$. \qed
4.1 The extreme points of the unit ball of \((C(K), \| \cdot \|_d)\)

The extreme points of the closed unit ball of \(C(K)\) with the supremum norm are the constant functions \(\pm 1\). Our next result shows that changing to the equivalent norm given in the preceding proposition leads to a different set of extreme points.

**Theorem 6.** A function \(x\) is an extreme point of the closed unit ball of \((C(K), \| \cdot \|_d)\) if and only if either

(i) \(x(t) = 1\) or \(0\) for every \(t \in K\), or

(ii) \(x(t) = -1\) or \(0\) for every \(t \in K\)

(and \(\{t : x(t) \neq 0\} \neq \emptyset\) in each case.)

**Proof.** To show that every such function is extreme, let \(\|x\|_d = 1\), with \(x(t) = 1\) for \(t \in A\) and \(x(t) = 0\) for \(t \in A^c\), where \(A\) is a nonempty subset of \(K\). Suppose that

\[
x = ay + bz,
\]

where \(a, b \in (0, 1)\) with \(a + b = 1\) and \(\|y\|_d = \|z\|_d = 1\). Then, for \(t \in A\), \(ay(t) + bz(t) = 1\). But \(|y(t)|, |z(t)| \leq 1\) and it follows that \(y(t) = z(t) = 1\) for every \(t \in A\).

Now, if \(t \in A^c\), then \(ay(t) + bz(t) = 0\). But \(\text{diam}(y), \text{diam}(z) \leq 1\) and \(\|y\|_\infty, \|z\|_\infty = 1\) imply that \(0 \leq y(t), z(t) \leq 1\) for every \(t \in K\) and hence \(y(t) = z(t) = 0\) for every \(t \in A^c\). Therefore \(y(t) = z(t) = x(t)\) for every \(t \in K\) and so \(x\) is an extreme point. The case in which \(x\) takes values \(-1\) and \(0\) is done in exactly the same way.

We now show that every extreme point is of this type. Let \(x\) be an extreme point. Since \(\|x\|_d = \max\{\|x\|_\infty, \text{diam}(x)\} = 1\), there are two cases to consider.

Case 1: \(\|x\|_\infty = 1\) and \(\text{diam}(x) \leq 1\). Then \(x\) takes its values either in \([-1, 0]\) or \([0, 1]\). Suppose it is the latter. Then there is at least one point at which \(x(t) = 1\). Suppose there is a point \(s \in K\) for which \(0 < x(s) < 1\). Then, by a standard argument, there is a function \(y \in C(K)\) with values in \([0, 1]\) and supported by a neighbourhood of \(s\), such that \(\|x \pm y\|_\infty \leq 1\). Clearly, we also have \(\text{diam}(x \pm y) \leq 1\). This implies that \(x\) is not extreme and so we have a contradiction. Therefore \(x\) can only have values \(0\) or \(1\).

Case 2: \(\text{diam}(x) = 1\) and \(\|x\|_\infty < 1\). There exist points \(s, t\) in \(K\) such that \(|x(t) - x(s)| = \text{diam}(x) = 1\). Without loss of generality we may assume that \(x(t) > x(s)\). Then \(x\) takes its values in the interval \([x(s), x(t)]\). If there exists \(u \in K\) such that \(x(s) < x(u) < x(t)\), then, using the same perturbation argument as in the proof of Case 1, it follows that \(x\) is not extreme. Therefore \(x\) has precisely two distinct values, \(x(s)\) and \(x(t)\).
Suppose that $-1 < x(s) < x(t) < 1$. Then, for sufficiently small $\varepsilon > 0$, we have $\|x \pm \varepsilon 1_K\|_d = 1$, which implies that $x$ is not extreme. Therefore either $x(t) = 1$ and $x(s) = 0$, or $x(s) = -1$ and $x(t) = 0$.

Note that if $K$ is connected, then the closed unit ball of $C(K)$ for both the supremum norm and the norm $\| \cdot \|_d$ has the same extreme points — the constant functions 1 and $-1$. However, if $K$ has more than one connected component, then there are functions that are extreme for $\| \cdot \|_\infty$ but not $\| \cdot \|_d$, and vice versa.

### 4.2 The extreme points of the unit ball of $(\mathcal{M}(K), \| \cdot \|_0)$

The isometric isomorphism

$$(C(K), \| \cdot \|_d)' \cong (\mathcal{M}(K), \| \cdot \|_0).$$

now enables us to identify the extreme points of the unit ball of $(\mathcal{M}(K), \| \cdot \|_0)$.

**Theorem 7.** Let $K$ be a compact Hausdorff topological space. A regular Borel signed measure $\mu$ on $K$ is an extreme point of the unit ball of $(\mathcal{M}(K), \| \cdot \|_0)$ if and only if it is one of the following:

(a) $\mu = \pm \delta_t$, where $t \in K$;

(b) $\mu = \delta_s - \delta_t$, where $s, t$ are distinct points in $K$.

**Proof.**

Step 1. We show that every extreme point must be one of the types described in the statement. Let $\bar{K}$ be the space $K \cup K^2$, with the sum topology, where $K^2$ carries the product topology. For $x \in C(K)$, let $\bar{x}$ be the continuous function on $\bar{K}$ defined by $\bar{x}(u) = x(u)$ for $u \in K$ and $\bar{x}(s, t) = x(s) - x(t)$ for $(s, t) \in K^2$. The fact that

$$\|\bar{x}\|_\infty = \max\{\sup\{|x(u)| : u \in K\}, \sup\{|x(s, t)| : s, t \in K\}\} = \|x\|_d$$

shows that the mapping $x \mapsto \bar{x}$ is an isometric embedding of $(C(K), \| \cdot \|_d)$ into a closed subspace of $(C(\bar{K}), \| \cdot \|_\infty)$. It follows from [11, V.8.6] that every extreme point of the unit ball of $(C(K), \| \cdot \|_d)' \cong (\mathcal{M}(K), \| \cdot \|_0)$ is either $\pm \delta_u$ for some $u \in K$, or $\delta_s - \delta_t$ for some $s, t \in K$. 

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Step 2. ± δ_τ are extreme points: Suppose that

[ \delta_τ = aµ_1 + bµ_2 , ]

where µ_1, µ_2 ∈ M(K), \|µ_1\|_0 = \|µ_2\|_0 = 1, a, b ∈ (0, 1) and a + b = 1. Applying δ_τ to the function 1_K, we have

aµ_1(1_K) + bµ_2(1_K) = 1.

On the other hand, \|1_K\|_d = 1 implies that |µ_i(K)| ≤ 1 for i = 1, 2. Therefore µ_1(K) = µ_2(K) = 1 and it follows from \|µ_i\|_0 = \frac{1}{2}(\|µ_i\|_1 + |µ_i(K)|) = 1 that \|µ_1\|_1 = \|µ_2\|_1 = 1. Since δ_τ is an extreme point of the unit ball of M(K) for the variation norm, it follows that µ_1 = µ_2 = δ_τ. Therefore δ_τ is an extreme point of the unit ball of (M(K), \| · \|_0).

Step 3. δ_s − δ_τ is extreme for every pair of distinct points s, t ∈ K: Suppose that

[ δ_s − δ_τ = aµ_1 + bµ_2 , ]

where µ_1, µ_2 ∈ M(K), \|µ_1\|_0 = \|µ_2\|_0 = 1, a, b ∈ (0, 1) and a + b = 1. Without loss of generality, we may assume that a = b = \frac{1}{2}. As \|δ_s − δ_τ\|_1 = 2, we have 4 ≤ \|µ_1\|_1 + \|µ_2\|_1.

On the other hand,

\|µ_i\|_0 = \frac{1}{2}(\|µ_i\|_1 + |µ_i(K)|) = 1 \quad \text{for } i = 1, 2

and it follows that \|µ_i\|_1 = 2 and µ_i(K) = 0 for i = 1, 2. Therefore δ_s − δ_τ, µ_1 and µ_2 all lie in (C_ρ(K), \| · \|_ρ)', the space of regular Borel signed measures on K that are zero on K. Furthermore, these measures are all unit vectors in this space, since the variation norm is exactly twice the dual norm in (C_ρ(K), \| · \|_ρ)'. It follows from the result of Cabello-Sanchez (Theorem 4 above) that µ_1 = µ_2 = δ_s − δ_τ. Therefore δ_s − δ_τ is an extreme point of the unit ball of (M(K), \| · \|_0).

We can now describe the extreme points of the unit ball of P_o(^nC(K)) for the supremum norm. Recall that, when n is odd, the supremum and regular norms coincide. Thus, by Propositions 3, 4 and Theorem 7 we have the following result.

**Corollary 3.** Let K be a compact, Hausdorff space.

(a) If n is odd, then P ∈ P_o(^nC(K)) is an extreme point of the closed unit ball of the space (P_o(^nC(K)), \| · \|_∞) if and only if P = ±δ^n_t, for some t ∈ K, where

\[ δ^n_t(x) = x(t)^n. \]
(b) If \(n\) is even, then \(P \in \mathcal{P}_o(\mathbb{C}(K))\) is an extreme point of the closed unit ball of the space \((\mathcal{P}_o(\mathbb{C}(K)), \| \cdot \|_\infty)\) if and only if \(P\) is one of the following:

(i) \(P = \pm \delta^n_t\) for some \(t \in K\), where \(\delta^n_t(x) = x(t)^n\);

(ii) \(P = \delta^n_s - \delta^n_t\), where \(s, t\) are distinct points in \(K\), and

\((\delta^n_s - \delta^n_t)(x) = x(s)^n - x(t)^n\).

**Example 1.** Suppose that the compact, Hausdorff space \(K\) has just two points, \(\alpha, \beta\). Then the vector lattice \(\mathbb{C}(K)\) can be identified with \(\mathbb{R}^2\), where \((x_1, x_2) \in \mathbb{R}^2\) corresponds to the function \(\alpha \mapsto x_1, \beta \mapsto x_2\). The supremum norm on \(\mathbb{C}(K)\) is identified with the supremum norm on \(\mathbb{R}^2\). The orthogonally additive \(n\)-homogeneous polynomials on \(\mathbb{R}^2\) have the form \(P(x) = a_1x_1^n + a_2x_2^n\). The regular and supremum norms are

\[
\|P\|_r = |a_1| + |a_2|,
\]

\[
\|P\|_\infty = \begin{cases} |a_1| + |a_2|, & \text{if } n \text{ is odd,} \\ \max\{|a_1|, |a_2|, |a_1 + a_2|\}, & \text{if } n \text{ is even.} \end{cases}
\]

The diagrams below show the unit balls for both norms.

**4.3 The isometries of \((\mathbb{C}(K), \| \cdot \|_d)\)**

We would like next to determine the isometries of the spaces \(\mathcal{P}_o(\mathbb{C}(K))\), both for the regular and the supremum norms. Our results show that this reduces to the problem of finding the isometries between the spaces \(M(K)\) for the variation norm and the equivalent norm \(\| \cdot \|_0\).

The Banach-Stone theorem [11, V.8.8] uses the classification of the extreme points of the space of regular Borel signed measures to determine the isometries of \(\mathbb{C}(K)\) spaces
with the supremum norm. We recall the statement of this theorem: if $T$ is an isometric isomorphism between $C(K)$ and $C(L)$, then there exists a homeomorphism $\varphi: L \to K$ and a function $\alpha \in C(L)$ with values $\pm 1$, such that

$$ (Tx)(s) = \alpha(s)x(\varphi(s)) $$

for all $x \in C(K)$, $s \in L$. We shall say that an linear bijection, $T$, from $C(K)$ to $C(L)$ is *canonical* if it has this form. In other words,

$$ Tx = \alpha x \circ \varphi, $$

where $\alpha, \varphi$ are as described above.

Consider the space $(M(K), \| \cdot \|_0) \cong (C(K), \| \cdot \|_d)'$. By Theorem 7, the set of extreme points of the unit ball of $(M(K), \| \cdot \|_0)$ is $\{ \pm \delta_u, \delta_t - \delta_s : u, t, s \in K, t \neq s \}$. The crucial step in showing that an isometry $T$ of from $(C(K), \| \cdot \|_d)$ to $(C(L), \| \cdot \|_d)$ is canonical is to establish that $T^t$, the transpose of $T$, maps each $\delta_t$ to $\pm \delta_s$ for some $s$ in $K$. This leads to the following proposition.

**Proposition 5.** Let $K$ and $L$ be compact Hausdorff topological spaces and let $T: (C(K), \| \cdot \|_d) \to (C(L), \| \cdot \|_d)$ be an isometric isomorphism. Let $S_L = \{ t \in L : T^t(\delta_t) = \pm \delta_s, \text{ for some } s \in K \}$. If $S_L$ contains more than one point, then $T$ is canonical. Moreover, in addition, $\alpha$ will either take the constant value 1 or $-1$ on $L$.

**Proof.** Assume that $|S_L| \geq 2$ and $S_L^c$ is non-empty. Choose $r \in S_L^c$. Then we have that $T^t(\delta_r) = \delta_u - \delta_v$, for some $u$ and $v$ in $K$. Since $|S_L| \geq 2$, there are $t$ and $s$ in $L$ with $t \neq s$ so that $T^t(\delta_t) = \pm \delta_w$ and $T^t(\delta_s) = \pm \delta_p$ for some $w$ and $p$ in $K$ with $w \neq p$. We now claim that $\{ w, p \} \neq \{ u, v \}$ and so there $t$ in $L$ so that $T^t(\delta_t) = \pm \delta_w$ with $w \neq u, v$. Without loss of generality suppose that $w = u$ and $p = v$. Then we have $T^t(\delta_t) = \pm \delta_u$ and $T^t(\delta_s) = \pm \delta_v$. Hence

$$ T^t(\delta_t - \delta_s) = \pm \delta_u \mp \delta_v $$

and therefore $(T^t)^{-1}(\delta_u - \delta_v) = \pm (\delta_t - \delta_s)$. Since $(T^t)^{-1}$ is a bijection we have $\delta_r = \pm (\delta_t - \delta_s)$ which is impossible. Let $t$ in $L$ be such that $T^t(\delta_t) = \pm \delta_w$ with $w \neq u, v$. Then $\delta_r - \delta_t$ is an extreme point of the unit ball of $(M(L), \| \cdot \|_0)$. However,

$$ T^t(\delta_r - \delta_t) = \delta_u - \delta_v \pm \delta_w. $$

Since $w \neq u, v$, $\delta_u - \delta_v \pm \delta_w$, is not an extreme point of the unit ball of $(M(K), \| \cdot \|_0)$. This is a contradiction. Hence, if $|S_L| \geq 2$, then $S_L = L$. 

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Note that $T: (C(K), \| \cdot \|_\infty) \to (C(L), \| \cdot \|_\infty)$ is an isomorphism since the norms $\| \cdot \|_\infty$ and $\| \cdot \|_d$ are equivalent. Further, since $S_L = L$, we have that for every $t$ in $L$ there is $s$ in $K$ such that $T^t(\delta_t) = \pm \delta_s$. Hence $T^t$ maps extreme points of the unit ball of $(\mathcal{M}(L), \| \cdot \|_1)$ to the extreme points of the unit ball of $(\mathcal{M}(K), \| \cdot \|_1)$ in one to one manner. Hence $T^t(B_{\mathcal{M}(L)}) \subseteq B_{\mathcal{M}(K)}$ and $(T^t)^{-1}(B_{\mathcal{M}(K)}) \subseteq B_{\mathcal{M}(L)}$. This gives us that $T: (C(K), \| \cdot \|_\infty) \to (C(L), \| \cdot \|_\infty)$ is an isometric isomorphism. Hence we can now apply Banach-Stone theorem to find a homeomorphism $\varphi$ from $L$ to $K$ and a function $\alpha \in C(K)$ with $\alpha(t) = \pm 1$ for all $t \in K$ such that

$$T(x) = \alpha x \circ \varphi.$$  

Now let us see that $\alpha$ is constant on $L$. To see this suppose that $S_L^+ = \{ t \in L : T^t(\delta_t) = \delta_s \text{ for some } s \in K \}$ and $S_L^- = \{ t \in L : T^t(\delta_t) = -\delta_s \text{ for some } s \in K \}$ are both non empty. Choose $t$ in $S_L^+$ and $r$ in $S_L^-$. Suppose that $T^t(\delta_t) = \delta_u$ and that $T^r(\delta_r) = -\delta_v$. Then $\delta_t - \delta_r$ is an extreme point of the unit ball of $(\mathcal{M}(L), \| \cdot \|_0)$ yet $T^t(\delta_t - \delta_r) = \delta_u + \delta_v$ is not extreme point of the unit ball of $(\mathcal{M}(K), \| \cdot \|_0)$. The result now follows and we get that

$$T(x) = \pm x \circ \varphi.$$  

Let us now consider the case when $|S_L| = 1$ and show that we can construct a non canonical isometry in this case. To help understand this result, we first consider the following example.

**Example 2.** Let $K = \{a, b\}$ and $L = \{\alpha, \beta\}$. We observe that we can identify both $(C(K), \| \cdot \|_d)$ and $(C(L), \| \cdot \|_d)$ with $\mathbb{R}^2$. Let $x$ in $C(K)$ and set $x_1 = x(a)$ and $x_2 = x(b)$. Then $(x_1, x_1) \in \mathbb{R}^2$ and the norm of $(x_1, x_1)$ is given by

$$\| (x_1, x_1) \|_d = \max\{ |x_1|, |x_2|, |x_1 - x_2| \}.$$  

Now define $T: (\mathbb{R}^2, \| \cdot \|_d) \to (\mathbb{R}^2, \| \cdot \|_d)$ by

$$T(x_1, x_2) = (x_1, x_1 - x_2)$$  

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Clearly, $T$ is a continuous linear bijection. We can also show that
\[ T^t(\delta_\alpha) = \delta_a, \]
\[ T^t(\delta_\beta) = \delta_a - \delta_b. \]

We have that
\[ \|T(x_1, x_2)\|_d = \max\{|x_1|, |x_1 - x_2|, |x_2|\} = \|(x_1, x_2)\|_d \]
and hence $T$ is an isometry. However, $T$ is not canonical since
\[ (Tx)(\alpha) = x(a), \]
\[ (Tx)(\beta) = x(a) - x(b). \]

Guided by Proposition 5 and the above example, we now have the following result.

**Theorem 8.** Let $K$ and $L$ be compact Hausdorff topological spaces.

(a) Suppose that $K$ and $L$ do not contain isolated points. Then every isometric isomorphism $T$ from $(C(K), \| \cdot \|_d)$ onto $(C(L), \| \cdot \|_d)$ has the form
\[ T(x) = \pm x \circ \varphi. \]
for some homeomorphism $\varphi: L \to K$.

(b) Suppose that either $K$ or $L$ contains an isolated point. Let $T : (C(K), \| \cdot \|_d) \to (C(L), \| \cdot \|_d)$ be an isometric isomorphism. Then $T$ is one of the following types.

(i) \[ T(x) = \pm x \circ \varphi. \]
for some homeomorphism $\varphi: L \to K$.

(ii) There exist $p$ in $K$ and $t$ in $L$ and a homeomorphism $\varphi: L \setminus \{t\} \to K \setminus \{p\}$ such that $T = \pm T_1$, where
\[ (T_1 x)(t) = x(p) \]
\[ (T_1 x)(s) = x(p) - x(\varphi(s)) \quad \text{for } s \neq t. \]
Proof. (a) Note that $L = S_L \cup S^c_L$. We claim that, if $|S_L| = 1$, then $L$ contains an isolated point. Suppose that $S_L = \{t\}$ and, without loss of generality, $T^t \delta_t = \delta_s$. Then

$$(T^1_1)(t) = \delta_t(T^1_1) = (T^t_1)(1) = \delta_s(1) = 1.$$ 

For any $r \in S^c_L$, a similar calculation shows that $(T^1 r)(0) = 0$. As $S_L = (T^1_1)^{-1}(1)$ and $S^c_L = (T^1_1)^{-1}(0)$ and $T^1_1$ is continuous, it follows that $S_L$ and $S^c_L$ are disjoint closed sets. Therefore $S_L = \{t\}$ is an isolated point of $L$. Therefore, if $L$ does not contain isolated points then $|S_L| \geq 2$ and Proposition 5 gives us that $T$ is canonical.

(b) We only need to consider the case $|S_L| = 1$ as otherwise Proposition 5 gives us that $T$ is canonical. Suppose $L$ contains an isolated point $t$ and $K$ an isolated point $p$. For each $x$ in $C(K)$ the function $Tx$ as defined in (b) is continuous and the mapping $x \rightarrow Tx$ is easily seen to be an isometry.

Let us see that if $T$ is not canonical then this is the form that an isometry can take. By definition and the fact that $T$ is invertible we have that $|S_K| = 1$, where $S_K$ is the set of points $q$ in $K$ for which $T^q(\delta_t) = \pm \delta_q$ for some $t \in L$. Let $S_L = \{t\}$ and $S_K = \{p\}$. Then $T^t(\delta_t) = \pm \delta_p$. We claim that for each $s \in S^c_L$ we have $T^t(\delta_s) = \delta_p - \delta_q$ for some $q \in K \setminus \{p\}$. Otherwise we have that $\delta_t - \delta_s$ is extreme but $T^t(\delta_t - \delta_s) = \delta_p - \delta_u + \delta_v$ is not. The mapping $T^t(\delta_s) = \delta_p - \delta_q$ now induces a bijection $\varphi: L \setminus \{t\} \rightarrow K \setminus \{p\}$ so that $T^t(\delta_s) = \delta_p - \delta_{\varphi(s)}$. Since the mapping $L \setminus \{p\} \rightarrow (\mathcal{M}(K), \sigma(\mathcal{M}(K), C(K)))$, $s \mapsto \delta_p - \delta_{\varphi(s)}$, is continuous, $\varphi$ will be continuous. As $\varphi$ is a continuous bijection from the compact space $L \setminus \{t\}$ to the Hausdorff space $K \setminus \{p\}$ it is a homeomorphism. Rewriting $s \mapsto \delta_p - \delta_{\varphi(s)}$ in terms of $x$, we see that $(Tx)(s) = x(p) - x(\varphi(s))$. When $T^t(\delta_t) = -\delta_s$, we obtain $(Tx)(s) = x(\varphi(s)) - x(p)$. \hfill \Box

Our characterisation of the isometries of $(C(L), \| \cdot \|_d)$ onto $(C(K), \| \cdot \|_d)$ allows us to construct isometries of $(\mathcal{P}_o^n C(K)), \| \cdot \|_\infty)$ onto $(\mathcal{P}_o^n C(L)), \| \cdot \|_\infty)$. Given a homeomorphism $\varphi: K \rightarrow L$ we use $C_\varphi$ to denote the composition operator $C_\varphi: C(L) \rightarrow C(K)$ defined by $C_\varphi(f) = f \circ \varphi$ for each $f$ in $C(L)$. The transpose of the canonical isometry of $(C(K), \| \cdot \|_d)$ onto $(C(L), \| \cdot \|_d)$ determined by $\varphi$ now gives rise to the isometry $T: (\mathcal{P}_o^n C(K)), \| \cdot \|_\infty) \rightarrow (\mathcal{P}_o^n C(L)), \| \cdot \|_\infty))$ given by $T(P) = P \circ C_\varphi$.

To understand the isometries from $(\mathcal{P}_o^n C(K)), \| \cdot \|_\infty)$ to $(\mathcal{P}_o^n C(L)), \| \cdot \|_\infty)$ induced by non canonical isometries of $(C(L), \| \cdot \|_d)$ onto $(C(K), \| \cdot \|_d)$ we note that if $K$ and $L$ have isolated points $t$ and $p$ respectively then we have that $(C(K), \| \cdot \|_\infty)$ is isometrically isomorphic to $(C(\{t\}), \| \cdot \|_\infty) \oplus_\infty (C(K \setminus \{t\}), \| \cdot \|_\infty)$ while $(C(L), \| \cdot \|_\infty)$ is isometrically isomorphic to $(C(\{p\}), \| \cdot \|_\infty) \oplus_\infty (C(L \setminus \{p\}), \| \cdot \|_\infty)$. Hence, if $P$ is an $n$-homogeneous
orthogonally additive polynomial on \((C(K), \| \cdot \|_\infty)\) then we can write \(P\) as \(P = \lambda \delta_t^n + P_2\) where \(P_2 = P|_{C(K \setminus \{t\})}\). It follows that the transpose of each non canonical isometry from \((C(K), \| \cdot \|_d)\) onto \((C(L), \| \cdot \|_d)\) gives an isometry from \((P_o^nC(K)), \| \cdot \|_\infty)\) onto \((P_o^nC(L)), \| \cdot \|_\infty)\) of the form

\[
T(P) = P(1) \delta_p^n - P_2 \circ C_\varphi
\]

where \(\varphi\) is a homeomorphism of \(K \setminus \{t\}\) to \(L \setminus \{p\}\).

In a similar manner, we can construct canonical isometries from \((P_o^nC(K)), \| \cdot \|_r)\) onto \((P_o^nC(L)), \| \cdot \|_r)\).

5 Exposed points in \(P_o^nC(K)\)

In this section we shall characterise the weak* exposed and weak* strongly exposed point of the unit ball of \((C(K), \| \cdot \|_d)'\). We have an upper bound for this set. We know that it is contained in the set of extreme points of the unit ball of \((C(K), \| \cdot \|_d)' \cong (\mathcal{M}(K), \| \cdot \|_0)\) and that the set of extreme points of this set is equal to \(\{ \pm \delta_p, \delta_t - \delta_s : p, t, s \in K, t \neq s \}\).

Let us begin with some definitions.

**Definition 1.** Let \(E\) be a Banach space. A point \(x\) in the closed unit ball of \(E\) is said to be an *exposed point* if there exists \(\varphi \in E'\) with \(\| \varphi \| = 1\) such that

\[
\varphi(x) = 1 \text{ and } \varphi(y) < 1 \text{ for } y \in \overline{B}_E \setminus \{x\}.
\]

If this is the case then we say that \(\varphi\) exposes \(x\).

**Definition 2.** We say that \(x\) is a *strongly exposed point* of the closed unit ball of \(E\) if there exists \(\varphi \in E'\) such that

\[
\varphi(x) = 1
\]

and whenever \((x_n)_n\) is a sequence in \(\overline{B}_E\) with \(\lim_{n \to \infty} \varphi(x_n) = 1\) then \((x_n)_n\) converges to \(x\) in norm. We will say that \(\varphi\) strongly exposes \(x\).

If \(E = F'\) is a dual Banach space and the point \(x \in E\) is exposed (respectively, strongly exposed) by \(\varphi\) in \(F\) we say that \(x\) is a *weak* exposed (respectively, *weak* strongly exposed) point of \(E\) and that \(\varphi\) weak* exposes (respectively, weak*-strongly exposes) the unit ball of \(E\) at \(x\).
We also observe that if each \( \delta_t, t \in K \) and each \( \delta_t - \delta_s, t, s \in K \) with \( t \neq s \) are of norm 1 in \( (C(K), \| \cdot \|_d)' \). Hence, if \( \delta_t \) is exposed by \( x \) then we must have \( \operatorname{diam}(x) < 1 \). Conversely, if \( \delta_t - \delta_s \) is exposed by \( x \) then we must have \( \|x\|_\infty < 1 \).

We note that if \( K \) is a compact Hausdorff topological space then a net \( (t_\alpha)_\alpha \) converges to \( t \) in \( K \) if and only if \( y(t_\alpha) \) converges to \( y(t) \) for every \( y \) in \( C(K) \).

### 5.1 Gâteaux differentiability of the norm

We start with a characterisation of Gâteaux differentiability of the norm on \( (C(K), \| \cdot \|_d) \).

**Theorem 9.** Let \( K \) be a compact Hausdorff topological space. Let \( t \in K, x \in C(K) \) with \( \|x\|_d = 1 \). Then the following are equivalent

(a) The norm of \( (C(K), \| \cdot \|_d) \) is Gâteaux differentiable at \( x \) with derivative \( \delta_t \).

(b) (i) \( \|x\|_d = x(t) = 1 \) and \( \operatorname{diam}(x) < 1 \).

(ii) If \( (t_n)_n \) is a sequence of points in \( K \) such that \( \lim_{n \to \infty} x(t_n) = 1 \) then \( (t_n)_n \) has a subnet, \( (t_\alpha)_\alpha \) such that \( (t_\alpha)_\alpha \) converges to \( t \).

(c) \( t \) is the unique point in \( K \) with \( x(t) = 1 \) and \( \operatorname{diam}(x) < 1 \).

**Proof.** First observe that Šmul’yan [28, 29] (see also [9]) showed that a point \( x \) in \( B_{C(K)} \) weak* exposes the unit ball of \( (C(K), \| \cdot \|_d)' \) at \( \delta_t \) if and only if the norm of \( C(K) \) is Gâteaux differentiable at \( x \) with derivative \( \delta_t \). Hence we have that (a) implies (c).

Let us see that (c) implies (b). Clearly we have that (c) implies (b) (i).

Suppose that (c) is true and that (b) part (ii) fails. Then there is a sequence \( (t_n)_n \) in \( K \) with \( \lim_{n \to \infty} x(t_n) = x(t) = 1 \) but that for all subnets \( (t_\alpha)_\alpha \) of \( (t_n)_n \) there is \( y \) in \( C(K) \) such that \( y(t_\alpha) \not\rightarrow y(t) \). As \( K \) is compact, we can choose a subnet \( (t_\alpha)_\alpha \) of \( (t_n)_n \) and \( s \) in \( K \) so that \( \lim_{\alpha \to \infty} t_\alpha = s \). We claim that \( t \neq s \). Suppose \( t = s \). Then for every \( y \) in \( C(K) \) we have that \( \lim_{\beta} y(t_\beta) = y(t) \) contrary to what we have assumed. As \( s \neq t \) and \( x \) is continuous we have \( x(s) = \lim_{\beta} x(t_\beta) = 1 \) which contradicts (c). Hence, we see that (c) implies (b).

Next suppose that (b) is true and that (a) is false. Then we can find \( y \) in \( C(K), \varepsilon > 0 \) and a sequence of positive numbers \( (\lambda_n)_n \) converging to 0 so that

\[
\left| \|x + \lambda_n y\|_d - \|x\|_d - \lambda_n y(t) \right| \geq \varepsilon \lambda_n
\]

for every positive integer \( n \).
Note that as $\|x + \lambda_n y\|_d \geq (x + \lambda_n y)(t)$ we actually have that $\|x + \lambda_n y\|_d - \|x\|_d - \lambda_n y(t)$ is non-negative and therefore we have

$$\|x + \lambda_n y\|_d - \|x\|_d - \lambda_n y(t) \geq \lambda_n \varepsilon$$

for every positive integer $n$.

Each of the functions $x + \lambda_n y$ attains its norm either at a point of the form $\delta_t$ or at a point $\delta_u - \delta_v$. As $\text{diam}(x) < 1$ choosing $n$ sufficiently large we can assume that $x + \lambda_n y$ attains its norm at a point of the first type. Hence, for each $n$ in $\mathbb{N}$, we can find $t_n$ in $K$ and $\beta_n = \pm 1$ so that

$$\beta_n (x + \lambda_n y)(t_n) = \|x + \lambda_n y\|_d.$$

Then we have

$$1 = \|x\|_d \geq \beta_n x(t_n) = \beta_n (x + \lambda_n y)(t_n) - \beta_n \lambda_n y(t_n)$$

$$\geq \|x + \lambda_n y\|_d - |\lambda_n| \|y\|_d.$$  

As $(\lambda_n)_n$ is a null sequence we have that $\|x + \lambda_n y\|_d - |\lambda_n| \|y\|_d$ converges to $\|x\|_d$ as $n$ tends to $\infty$. Hence we have that $\beta_n x(t_n) \to 1$. However, as $\text{diam}(x) < 1$, we have $x(t_n) > 0$ for all $n$. Hence, without loss of generality, we may assume that $\beta_n = 1$ for all $n$ and therefore we have $\lim_{n \to \infty} x(t_n) = 1$.

Then we have that

$$\varepsilon \lambda_n \leq \|x + \lambda_n y\|_d - \|x\|_d - \lambda_n y(t)$$

$$= (x + \lambda_n y)(t_n) - x(t) - \lambda_n y(t)$$

$$= x(t_n) - x(t) + \lambda_n (y(t_n) - y(t))$$

$$\leq \lambda_n (y(t_n) - y(t))$$

However this means that there is no subnet $(t_\alpha)$ of $(t_n)_n$ so that $y(t_\alpha)$ converges to $y(t)$ and so (b) (ii) is false. $\square$

We recall that a function $x \in C(K)$ is said to peak at a point $t \in K$ if $t$ is the unique point at which $x$ attains its maximum.

**Lemma 1.** Let $K$ be a compact Hausdorff topological space and $t \in K$. Then there is $x$ in $C(K)$ which peaks at $t$ if and only if $\{t\}$ is a $G_\delta$ subset of $K$. 

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Theorem 10. We first suppose that \( \{t\} \) is a \( G_\delta \) subset of \( K \). Then we can find a sequence of open sets \( (U_n)_n \) so that \( \{t\} = \bigcap_{n=1}^{\infty} U_n \). As \( K \) is compact and Hausdorff it is completely regular. Hence, for each \( n \in \mathbb{N} \) we can find a continuous function \( x_n \): \( K \rightarrow [0,1] \) such that \( x_n(t) = 1 \) and \( x_n(U_n') = 0 \). Now let \( x: K \rightarrow [0,1] \) be defined by

\[
x(t) = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} x_n(t).
\]

Then we have \( x(t) = 1 \) and \( x(s) < 1 \) for \( s \in K \), \( s \neq t \). So \( x \) peaks at \( t \).

Conversely, if there is \( x \) in \( C(K) \) which peaks at \( t \), for each \( n \in \mathbb{N} \) let \( U_n = \{ s \in K : x(s) > 1 - \frac{1}{n} \} \). Then \( \{t\} = \bigcap_{n=1}^{\infty} U_n \). As each \( U_n \) is open, \( \{t\} \) is a \( G_\delta \) set.

The weak* exposed points of the ball of the form \( \delta_t \) are characterised by the following proposition.

**Proposition 6.** Let \( K \) be a compact Hausdorff topological space. Then \( \{\pm \delta_t : t \in K\} \) is contained in the set of weak* exposed points of the unit ball of \( (\mathcal{M}(K), \| \cdot \|_0) \) if and only if \( K \) is first countable.

Just as we have characterised the weak* exposed points of the ball of the form \( \delta_t \) we now characterise weak* exposed points of the form \( \delta_t - \delta_s \). Replacing \( \delta_t \) with \( \delta_t - \delta_s \) in Theorem 9 we obtain the following result.

**Theorem 10.** Let \( K \) be a compact Hausdorff topological space. Let \( t, s \in K, \ x \in C(K) \) with \( \|x\|_d = 1 \). Then the following are equivalent

(a) The norm of \( (C(K), \| \cdot \|_d) \) is Gâteaux differentiable at \( x \) with derivative \( \delta_t - \delta_s \).

(b) \( \|x\|_d = x(t) - x(s) = 1 \) and \( \|x\|_\infty < 1 \).

\( (ii) \) If \( (t_n)_n \) and \( (s_n)_n \) are sequences of points in \( K \) such that \( \lim_{n \rightarrow \infty} x(t_n) - x(s_n) = 1 \) then \( (t_n)_n \) and \( (s_n)_n \) have subnets \( (t_\alpha)_\alpha \) and \( (s_\alpha)_\alpha \) which converge to \( t \) and \( s \) respectively.

(c) \( t, s \) is the unique pair of points in \( K \) with \( x(t) - x(s) = 1 \) and \( \|x\|_\infty < 1 \).

As the proof of the following lemma is similar to that of Lemma 1 we omit it.

**Lemma 2.** Let \( K \) be a compact Hausdorff topological space and \( t, s \in K \) with \( t \neq s \). Then there is \( x \) in \( C(K) \) such that \( x(t) = \frac{1}{2}, \ x(s) = -\frac{1}{2} \) and \( -\frac{1}{2} < x(u) < \frac{1}{2} \) for \( u \in K \setminus \{t, s\} \) if and only if \( \{t\} \) and \( \{s\} \) are \( G_\delta \) sets.
The weak* exposed points of the ball of the form $\delta_t - \delta_s$ are now characterised by the following proposition.

**Proposition 7.** Let $K$ be a compact Hausdorff topological space and $n$ be an even integer. Then $\{\delta_t - \delta_s : t, s \in K, t \neq s\}$ is contained in the set of weak* exposed points of the unit ball of $(\mathcal{M}(K), \| \cdot \|_0)$ if and only if $K$ is first countable.

Propositions 6 and 7 can be rephrased in terms of spaces of orthogonally additive polynomials. Since we have canonically identified the space $\mathcal{P}_o(^nC(K))$ with the space $\mathcal{M}(K)$, we may transfer the weak* topology on $\mathcal{M}(K) = C(K)'$ to the space $\mathcal{P}_o(^nC(K))$. References to the weak* topology on $\mathcal{P}_o(^nC(K))$ should be understood in this sense. It is easy to see that this is the topology of pointwise convergence on $\mathcal{P}_o(^nC(K))$.

**Proposition 8.** Let $K$ be a compact Hausdorff topological space and $n$ be an even integer. Then $\{\pm \delta^n_p, \delta^n_t - \delta^n_s : p, t, s \in K, t \neq s\}$ is equal to the set of weak* exposed points of the unit ball of $(\mathcal{P}_o(^nC(K)), \| \cdot \|_{\infty})$ if and only if $K$ is first countable.

### 5.2 Fréchet differentiability of the norm

We now characterise Fréchet differentiability of the norm on $(C(K), \| \cdot \|_d)$.

**Theorem 11.** Let $K$ be a compact Hausdorff topological space. Let $t \in K$, $x \in C(K)$ with $\|x\|_d = 1$. Then the following are equivalent.

(a) The norm of $(C(K), \| \cdot \|_d)$ is Fréchet differentiable at $x$ with derivative $\delta_t$.

(b) (i) $\|x\|_d = x(t) = 1$ and $\text{diam}(x) < 1$.

(ii) If $(t_n)_n$ is a sequence of points in $K$ such that $\lim_{n \to \infty} x(t_n) = 1$ then $(t_n)_n$ is eventually equal to $t$.

(c) $x$ weak* strongly exposes the unit ball of $(C(K), \| \cdot \|_d)'$ at $\delta_t$.

**Proof.** First observe that Šmul’yan [28, 29] (see also [9]) showed that a point $x$ in $B_{C(K)}$ weak* strongly exposes the unit ball of $(C(K), \| \cdot \|_d)'$ at $\delta_t$ if and only if the norm of $(C(K)$ is Fréchet differentiable at $x$ with derivative $\delta_t$. Thus (a) and (c) are equivalent.

If the norm of $(C(K), \| \cdot \|_d)$ is Fréchet differentiable at $x$ with derivative $\delta_t$ then it is Gâteaux differentiable at $x$ with derivative $\delta_t$. Theorem 9 now implies that (b) (i) holds.
Suppose that (c) is true. Then \( x \in B_{C(K)} \) weak* strongly exposes the unit ball of \((C(K), \| \cdot \|_d)'\) at \( \delta_t \). If \( \lim_{n \to \infty} x(t_n) = 1 \) then \( \lim_{n \to \infty} \delta_{t_n}(x) = \delta_t(x) = 1 \). As \( f \) weak*-strongly exposes the unit ball of \((C(K), \| \cdot \|_d)'\) at \( \delta_t \) we have that \( \lim_{n \to \infty} \delta_{t_n} = \delta_t \) in norm. However, as \( \| \delta_u - \delta_v \|_0 = 1 \) whenever \( u \neq v \) we see that only way we can have \( (\delta_{t_n})_n \) converge to \( \delta_t \) is that the sequence \((t_n)_n\) is eventually equal to \( t \).

The implication (b) implies (a) is similar to the corresponding part of the proof of Theorem 9 where instead of using the fact that \((t_n)_n\) has a subsequence that converges to \( t \) we use the fact that \((t_n)_n\) has a subsequence so that it is eventually equal to \( t \).

**Corollary 4.** Let \( K \) be a compact Hausdorff topological space and \( t \in K \). Then \( \delta_t \) is a weak* strongly exposed point of the unit ball of \((C(K), \| \cdot \|_d)'\) if and only if \( t \) is an isolated point of \( K \).

**Proof.** If \( t \) is an isolated point of \( K \) then the function given by

\[
x(s) = \begin{cases} 
1, & s = t \\
1/2 & \text{otherwise}
\end{cases}
\]

is continuous on \( K \). Moreover, if \( x(t_n) \to 1 \) then \((t_n)_n\) is eventually equal to \( t \).

Conversely, if \( t \) is not an isolated point of \( K \). Choose a sequence of points \((t_n)_n\) with \( t_n \neq t \), all \( n \), so that \( t_n \) converges to \( t \). Let \( x \) be any function in \( C(K) \) with \( \| x \|_d = 1 \) and \( x(t) = 1 \). Then we have that \( x(t_n) \to x(t) = 1 \). However, as \((t_n)_n\) is not eventually equal to \( t \) we see that condition (b) (ii) of Theorem 11 is not satisfied and therefore no \( x \) in \( C(K) \) with \( \| x \|_d = 1 \) can expose the unit ball \((C(K), \| \cdot \|_d)'\) at \( \delta_t \).

**Theorem 12.** Let \( K \) be a compact Hausdorff topological space. Let \( t, s \in K, x \in C(K) \) with \( \| x \|_d = 1 \). Then the following are equivalent

(a) The norm of \((C(K), \| \cdot \|_d)\) is Fréchet differentiable to \( x \) with differential \( \delta_t - \delta_s \).

(b) \( (i) \| x \|_d = x(t) - x(s) = 1 \) and \( \| x \|_\infty < 1 \).

\( (ii) \) If \((t_n)_n\) and \((s_n)_n\) are sequences of points in \( K \) such that \( \lim_{n \to \infty} x(t_n) - x(s_n) = 1 \) then \((t_n)_n\) and \((s_n)_n\) are eventually the constant sequences \( t \) and \( s \) respectively.

(c) \( x \) weak* strongly exposes the unit ball of \((C(K), \| \cdot \|_d)'\) at \( \delta_t - \delta_s \).

**Proof.** The proof is similar to Theorem 11 and therefore omitted.
Corollary 5. Let $K$ be a compact Hausdorff topological space and $t, s \in K$ with $t \neq s$. Then $\delta_t - \delta_s$ is a weak$^*$ strongly exposed point of the unit ball of $(C(K), \| \cdot \|_d)'$ if and only if $t$ and $s$ are isolated points of $K$.

We can rephrase these results in terms of spaces of orthogonally additive polynomials as follows.

Proposition 9. Let $K$ be a compact Hausdorff topological space, let $n$ be an even integer and let $s, t$ be distinct points in $K$.

(a) $\delta^n_t$ is a weak$^*$ strongly exposed point of the unit ball of $(P_o(nC(K)), \| \cdot \|_\infty)$ if and only if $t$ is an isolated point of $K$.

(b) $\delta^n_t - \delta^n_s$ ($s \neq t$) is a weak$^*$ strongly exposed point of the unit ball of $(P_o(nC(K)), \| \cdot \|_\infty)$ if and only if $t$ and $s$ are isolated points of $K$.

In particular, we see that if $K$ has no isolated points, then the unit ball of $P_o(nC(K))$ does not contain any weak$^*$ strongly exposed points.

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