Planar graph characterization of NDSS graphs

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Abstract. Planar graph characterization is always of interest due to its complexity in characterization. In this paper, we obtain a necessary and sufficient condition for a graph to be NDSS and hence characterize the planarity and outer – planarity of its complement \( \overline{G} \).

1. Introduction
Dominating sets has been used in graph theory for characterizing graphs based on various properties. In [1], Magda Dettlaff, Joanna Raczek and Jerzy Topp have proved that the decision problem of the domination subdivision number is NP – complete even for bipartite graphs. In [2], B. Sharada et.al have provided the problem of domination subdivision number of grid graphs \( P_{m,n} \) and determine the domination subdivision numbers of grid graphs \( P_{m,n} \) for \( m = 2, 3 \) and \( n \geq 2 \).

Characterizing planar graphs based on graph properties is a general problem discussed by different authors. In [3], M. Yamuna et al have provided a characterization of planar graphs when \( G \) and its complement are \( \gamma \) – stable. In [4], Val Pinciu showed that for outer planar graphs where all bounded regions are 3 – cycles, the problem of identifying the connected domination number is equal to an art gallery problem, which is identified to be NP – hard. In [5], By Joseph Battle, Frank Harary and Yukihiro Kodama have proved that every planar graph with nine vertices has a non – planar complement. In [6], Jin Akiyama and Frank Harary have characterized all graphs for which \( G \) and its complement are outer planar. In [7], Enciso and Dutton have classified planar graph based on \( \overline{G} \) and also they have proved the following result.

R1. If \( G \) is a planar graph, then \( \gamma (\overline{G}) \leq 4 \).
R2. If \( u \) is an up vertex for a graph in \( G \), then \( u \) must be included in every possible \( \gamma \) – set [ 8 ].

2. Terminology
We consider only simple connected undirected graphs \( G = (V, E) \) with \( n \) vertices and \( m \) edges. The open neighbourhood of \( v \in V(G) \) is defined by \( N(v) = \{ u \in V(G) | uv \in E(G) \} \), while its closed neighborhood is \( N[v] = N(v) \cup \{ v \} \). \( H \) is a sub graph of \( G \), if \( V(H) \subseteq V(G) \) and \( uv \in E(H) \) implies \( uv \in E(G) \). If \( H \) satisfies the added property that for every \( uv \in E(H) \) if and only if \( uv \in E(G) \), then \( H \) is said to be an induced sub graph of \( G \) and is denoted by \( \langle H \rangle \). Two graphs are homeomorphic if one can be obtained from the other by the creation of edges in series or by the merging the edges in series. In graph theory, \( K_5 \) and \( K_{3,3} \) are called Kuratowski’s graph. A path is a trail in which all vertices ( except perhaps the first and last ones ) are distinct, \( P_n \) denotes the path with \( n \) vertices. A cycle is a circuit in which no vertex except the first ( which is also the last ) appears more than once. \( C_n \) is a cycle with \( n \) vertices. \( K_n \) is a complete graph with \( n \) vertices. A star \( S_n \) is the
A graph \( G \) is NDSS if and only if
\[ \gamma ( G ) + 1 \leq \gamma ( G - u ) \leq \gamma ( G ) \]
for every \( u \in V ( G ) \). A graph \( G \) is defined as DSS, if
\[ \gamma ( G - u ) = \gamma ( G ) \]
for \( u \in V ( G ) \). A vertex \( v \) is said to be a down vertex if
\[ \gamma ( G - v ) > \gamma ( G ) \]
and it is denoted by \( \gamma ( G - v ) \leq \gamma ( G ) \). A vertex \( v \) is said to be a level vertex if
\[ \gamma ( G - v ) = \gamma ( G ) \]
A vertex \( v \) is known to be a support vertex. If there is a
\[ \gamma - \text{set of } G \]
then \( v \) is said to be a good vertex. If \( v \) does not belong to any of the
\[ \gamma - \text{set of } G, \text{then } v \text{ is said to be a bad vertex}. \]
A vertex \( v \) is known to be a down vertex if
\[ \gamma ( G - u ) < \gamma ( G ) \]
A vertex \( v \) is known to be a level vertex if
\[ \gamma ( G - v ) = \gamma ( G ) \]
It is denoted by \( \gamma ( G - v ) \leq \gamma ( G ) \). A vertex \( v \) is said to be a up vertex if
\[ \gamma ( G - u ) > \gamma ( G ) \]
For properties related to domination we refer Haynes et al [8].

A subordination of a graph \( G \) is a graph obtained from the subordination of edges in \( G \). The subordination of some edge \( e \) with end vertices \( \{ u, v \} \) generate a graph with one new vertex \( w \), and with an edge set replacing \( e \) by two new edges, \( \{ u, w \} \) and \( \{ w, v \} \) and it is denoted by \( G_{sd} \). Let \( w \) be the vertex introduced by subdividing \( uv \). We shall denote this by \( G_{sd} = w \). If \( G \) is any graph and \( D \) is a \( \gamma - \text{set for } G \), then \( D \cup \{ w \} \) is a \( \gamma - \text{set for } G_{sd} \) implies \( \gamma ( G_{sd} ) \geq \gamma ( G ) \), \( \forall u, v \in V ( G ) \), \( u \perp v \). A graph \( G \) is defined as DSS, if \( \gamma ( G_{sd} ) = \gamma ( G ) \), \( \forall u, v \in V ( G ) \), \( u \perp v \). In [10], the following result is proved.

R3. A graph \( G \) is domination subdivision stable if and only if \( \forall u, v \in V ( G ) \), either \( \exists \ a \ \gamma - \text{set containing } u \text{ and } v \) or \( \exists \gamma - \text{set } D \) such that
\[ \begin{align*}
1 & \quad \text{pn ( } u, D ) = \{ v \} \text{ or} \\
2 & \quad v \text{ is } \gamma - \text{dominated.}
\end{align*} \]

In this paper we consider graphs for which \( \gamma ( G_{sd} ) = \gamma ( G ) + 1 \).

3. Results and Discussions
In this section, we provide a necessary and sufficient condition for a graph to be NDSS and characterize the planarity and outer – planarity of NDSS graph.

3.1 Non- domination subdivision stable graph

Theorem 1
A graph \( G \) is NDSS if and only if

- every \( \gamma - \text{set of } G \) is independent.
- \( G \) has no two dominated vertices.

Proof
Assume that \( G \) is NDSS
\[ \begin{align*}
& \text{If } G \text{ has a } \gamma - \text{set } D \ni u, v \in D, u \perp v, \text{then } D \text{ itself is a } \gamma - \text{set for } G_{sd}uv, \text{a contradiction as } G \\
& \text{is NDSS, implies every } \gamma - \text{set of } G \text{ is independent.}
\end{align*} \]
\[ \begin{align*}
& \text{If } u \in D \ni v \text{ is } 2 - \text{dominated, } u \text{ adjacent to } v, \text{then } D \text{ itself is a } \gamma - \text{set for } G_{sd}uv, \text{a} \\
& \text{contradiction as } G \text{ is NDSS, implies } G \text{ has no } 2 - \text{ dominated vertices.}
\end{align*} \]
Conversely, assume that the conditions of the theorem are satisfied. If \( G \) is not NDSS, then by R3 \( \exists \ a \ \gamma - \text{set } D \ni \)
\[ \begin{align*}
& \text{either } u, v \in D, \\
& \text{pn ( } u, D ) = \{ v \} \\
& v \text{ is } 2 - \text{dominated.}
\end{align*} \]
a contradiction to our assumption, implies \( G \) is NDSS.
Observations

O1. If G is a NDSS graph, then any v ∈ V(G) is not selfish.

Proof
If possible, assume that ∃ one v ∈ V(G) such that v is selfish. Let D be any γ−set for G. D′ = D − {v} ∪ {w} is a γ−set for G 删除v, implies G is not NDSS, a contradiction.

O2. If G is a NDSS graph, then G has no down vertices.

Proof
If possible, assume that ∃ one v ∈ V(G), v a down vertex. We know that if v is a down vertex, then ∃ a γ−set D for G including v such that v is selfish, a contradiction, (by O1) implies G has no down vertices.

O3. If G is a NDSS graph, then a pendant vertex is always a level vertex.

Proof
Since pn(u, D) ≥ 2 for any NDSS graph, deg(v) = 1, there exist no γ−set containing v. Also an up vertex is included in every γ−set, [R2] implies v is not an up vertex. By (O2) v is always a level vertex.

O4. If G is a NDSS graph, then ⟨pn(u, D)⟩ is not complete for every u ∈ D, G ≠ K_n.

Proof
If possible assume that there exist one u ∈ D, such that ⟨pn(u, D)⟩ is a clique. Let pn(u, D) = {u_1, u_2, ..., u_k}. Since ⟨pn(u, D)⟩ is a clique, D − {u} ∪ {u_i}, i = 1, 2, ..., k is a γ−set for G for any v ∈ N(u) is 2−dominated, a contradiction.

O5. If G is a NDSS graph, then no v_i ∈ N(u, D) adjacent to every v_j ∈ N(u, D), i ≠ j, deg(v_j) ≥ 2.

Proof
Let u ∈ D. Let N(u) = {u_1, u_2, ..., u_k}. If ∃ one v_i, v_j, adjacent to every v_j, i ≠ j, j = 1, 2, ... k then D − {u} ∪ {v_j} is a γ−set for G, every v_i ∈ N(u_i) is 2−dominated, a contradiction.

O6. If G is a NDSS graph, then pn(u, D) ≥ 2.

Proof
If pn(u, D) = v for some u ∈ D, then D′ = D − {u} ∪ {w} is a γ−set for G, |D′| = |D|, a contradiction as G is NDSS, implies pn(u, D) ≥ 2.

3. 2. Planarity
We recollect the following theorems on planar graphs.

R4. A graph is planar if and only if it does not contain either K_5 or K_3,3 or any graph homeomorphic to either of them.

R5. A graph G is planar if and only if it does not have a subgraph contractible to Kuratowski’s graph[9].

R6. A necessary and sufficient condition for a graph G to be outer planar if it has no subgraph homeomorphic to K_4 or K_3,3 except K_4 − x [9].

We shall prove that a NDSS graph is planar, non−planar, or non−outer planar using R4, R5 and R6. If γ(G) = 1, then complement of G is disconnected and hence complement of G is not a NDSS graph. Also by R1, if G is a planar graph, then − ≤ 4. So in the remaining part of this section we limit our discussion to cases where 1 ≤ γ(G) ≤ 4, 1 ≤ γ(Ḡ) ≤ 4. In all graphs, in the remaining part of the discussion,

i. 代表 the newly added edges in the current discussion.

ii. When we apply edge contraction, a vertex receives a label of the contracted vertices. For example y: b_b_1, x_1, x_2, x_3, x_4 means that the contracted edges are b_b_1, b_1, x_1, x_2, x_3 and is assigned the new label as y.

Theorem 2
If G is a NDSS graph, then ⟨V − D⟩ is not complete, where D is a γ−set for G.

Proof
Let $D = \{ u_1, u_2, \ldots, u_k \}$ be a $\gamma$ - set for $G$. Let $N(u_i) = \{ a_{i1}, a_{i2}, \ldots, a_{im_i} \}$, $N(u_j) = \{ b_{j1}, b_{j2}, \ldots, b_{jm_j} \}$.

Since $G$ is NDSS, $|k_i| \geq 2$, for all $i = 1, 2, \ldots, k$. If $\langle V - D \rangle$ is complete, then $D' = \{ a_1, b_1, c_1, d_1 \}$ is a $\gamma$ - set for $G$, $(a_1$ dominates $N(u_1)$, $N(u_2)$, $\ldots$, $N(u_k)$), $b_1$ dominates $u_2$, $c_1$ dominates $u_3$, $d_1$ dominates $u_4$ such that $\langle D' \rangle$ is complete, a contradiction as $G$ is NDSS.

**Theorem 3**

Let $G$ be a NDSS graph. Let $\gamma(G) = 3$. Let $D = \{ u_1, u_2, u_3 \}$ be a $\gamma$ - set for $G$. Let $X_1 = \text{pn}(u_1, D) = \{ a_1, a_2, \ldots, a_{k_1} \}$, $X_2 = \text{pn}(u_2, D) = \{ b_1, b_2, \ldots, b_{k_2} \}$, $X_3 = \text{pn}(u_3, D) = \{ c_1, c_2, \ldots, c_{k_3} \}$.

Then the following statements are true together

1. $X_1$ is collectively not adjacent to at least $k_1$ vertices in $X_2$, $X_3$.
2. $X_2$ is collectively not adjacent to at least $k_2$ vertices in $X_1$, $X_3$.
3. $X_3$ is collectively not adjacent to at least $k_3$ vertices in $X_1$, $X_2$.

**Proof**

Since $G$ is NDSS, $|k_i| \geq 2$, $i = 1, 2, 3$. If $k_1$ vertices in $X_1$ are collectively not adjacent to at least $k_1$ vertices in $X_2$, then we can find $k_1$ non adjacent pairs $(a_{i1}, b_{j1})$, $i = 1$ to $k_1$, $j = 1$ to $k_2$. If this is not true then there exist at least one $a_{i1}$ adjacent to every $b_{j1}$.

Similarly if $k_1$ vertices in $X_1$ are not collectively not adjacent to at least $k_1$ vertices in $X_2$, then there exist at least one $a_{i1}$ that is adjacent to every $c_{k_1}$, that is there exist one $a_{i1}$ adjacent to every $b_{j1}$ and some $a_{i1}$ adjacent to every $c_{k_1}$ ($a_{i1}$ may be equal to $a_{i2}$).

Similarly there exist one $b_{i2}$ adjacent to every $a_{i1}$ and some $b_{i2}$ adjacent to every $c_{k_1}$ ($b_{i1}$ may be equal to $b_{i2}$).

Similarly there exist one $c_{k_1}$ adjacent to every $a_{i1}$ and some $c_{k_2}$ adjacent to every $b_{j1}$ ($c_{k_1}$ may be equal to $c_{k_2}$).

The following statements are true together

1. Generalizing Theorem 3 if $D = \{ u_1, u_2, \ldots, u_m \}$, $X_1 = \text{pn}(u_1, D) = \{ a_1, a_2, \ldots, a_{k_1} \}$, $X_2 = \text{pn}(u_2, D) = \{ b_1, b_2, \ldots, b_{k_2} \}$, $X_3 = \text{pn}(u_3, D) = \{ c_1, c_2, \ldots, c_{k_3} \}$, $X_4 = \text{pn}(u_4, D) = \{ d_1, d_2, \ldots, d_{k_4} \}$, then the following statements are true together

2. For every $i_1 = 1$ to $k_1$, $i_2 = 1$ to $k_2$, $i_3 = 1$ to $k_3$, there exist at least one pair $(a_{i_1}, b_{i_2})$, $(a_{i_1}, c_{i_3})$, $\ldots$, $(a_{i_1}, d_{i_4})$ of vertices that are not adjacent. This means that every $a_{i_1}$ not adjacent to at least one $b_{i_2}$, $c_{i_3}$, $d_{i_4}$.

**Theorem 4**

If $G$ is a NDSS graph such that $\gamma(G) = 4$, then $G$ is non planar.

**Proof**

Let $D = \{ u_1, u_2, u_3, u_4 \}$ be a $\gamma$ - set for $G$. Let $\text{pn}(u_1, D) = \{ a_1, a_2, \ldots, a_{k_1} \}$, $\text{pn}(u_2, D) = \{ b_1, b_2, \ldots, b_{k_2} \}$, $\text{pn}(u_3, D) = \{ c_1, c_2, \ldots, c_{k_3} \}$, $\text{pn}(u_4, D) = \{ d_1, d_2, \ldots, d_{k_4} \}$.

Since $G$ is NDSS, $|k_i| \geq 2$ for all $i = 1, 2, 3, 4$. Since $\langle D \rangle$ is independent in $G$, $\langle D \rangle$ is complete in $G$. 


By Theorem 3, we know that there exist at least 2 vertices in $V - D$ which are not adjacent. Arbitrarily let us assume that some $a_i$, $i = 1, 2, \ldots, k_1$ not adjacent to some $b_j$, $j = 1, 2, \ldots, k_2$. Also $a_i$ is adjacent to $\{ u_2, u_3, u_4 \}$. Since in $G$ $a_i$ not adjacent to $b_j$, and $b_j$ not adjacent to $u_1$, in $\overline{G}$ there exist an edge from $a_i$ to $b_j$ and $b_j$ to $u_1$. Contracting edge $a_i b_j$, $a_i b_j$ is adjacent to $u_1$. $\langle u_1, u_2, u_3, u_4, u_5: a_i b_j \rangle$ is $K_5$, implies $\overline{G}$ is non-planar.

**Theorem 5**

Let $G$ be a NDSS graph such that $\gamma(G) = 3$, then $\overline{G}$ is non-planar.

**Proof**

Let $D = \{ u_1, u_2, u_3 \}$ be a $\gamma$-set for $G$. Let $pn(u_1, D) = \{ a_1, a_2, \ldots, a_{k_1} \}$, $pn(u_2, D) = \{ b_1, b_2, \ldots, b_{k_2} \}$, $pn(u_3, D) = \{ c_1, c_2, \ldots, c_{k_3} \}$. Since $G$ is NDSS, $|k_i| \geq 2$ for all $i = 1, 2, 3$. Since $\langle D \rangle$ is independent in $G$, $\langle D \rangle$ is complete in $\overline{G}$. 
Since $G$ is NDSS by Theorem 2, $(V - D)$ is not complete, implies $\exists$ at least two vertices in $V - D$ which are not adjacent. Arbitrarily let us assume that some $a_i$ not adjacent to some $b_j$.

Since in $G$, $a_i$ not adjacent to $b_j$ they are $\perp$ in $\overline{G}$. Also $b_j \perp u_1$. We know that $a_i$ is adjacent to $u_2$ and $b_j$ is adjacent to $u_3$. Contracting edge $a_ib_j$, $a_i$ adjacent to $u_1$. $\langle u_1, u_2, u_3, u_4 : a_ib_j \rangle$ is $K_4$.

By $O6$ and remark 2 of Theorem 3, there exist one $b_k$ not adjacent to $b_j$, $k \neq j$ and some $c_l$ not adjacent to $b_k$ in $G$. In $\overline{G}$, $c_l$ adjacent to $b_k$, $c_l$ adjacent to $u_4$. We know that $b_k$ adjacent to $u_2$, $u_1$, $u_3$ and $c_l$ adjacent to $u_1$. Contracting edges $b_kc_l$, $c_lu_2$, $\langle u_1, u_2, u_3, u_4, u_5 : b_kc_l \rangle$ is $K_5$, implies $\overline{G}$ is non planar.
If Fig. 8 (a) $\gamma(G) = 2$, $\overline{G}$ non planar. For the graph in Fig. 8 (b) $\gamma(G) = 2$, $\overline{G}$ planar. So when $\gamma(G) = 2$, $\overline{G}$ may or may not be planar. Contracting edges 63, 52 we see that $\langle 1, 4, 63, 52 \rangle$ is $K_4$, implies $G$ is non outer–planar. We generalize this result in Theorem 6.

**Theorem 6**

Let $G$ be a NDSS graph such that $\gamma(G) = 2$, then $\overline{G}$ is non–outer planar.

**Proof**

Let $D = \{ u_1, u_2 \}$ be a $\gamma$–set for $G$. Let $pn(u_1, D) = \{ a_1, a_2, \ldots, a_{k_1} \}$, $pn(u_2, D) = \{ b_1, b_2, \ldots, b_{k_2} \}$.

Since $G$ is NDSS, $|k_i| \geq 2$ for all $i = 1, 2$. Since $\langle D \rangle$ is independent in $G$, $\langle D \rangle$ is complete in $\overline{G}$.
Since G is NDSS by Theorem 2, \( \langle V - D \rangle \) is not complete, implies \( \exists \) at least two vertices in \( V - D \) which are not adjacent. Arbitrarily, let us assume that some \( a_i \) not adjacent to some \( b_j \).

Since in \( G \), \( a_i \) not adjacent to \( b_j \), they are \( \notin \text{in} \overline{G} \). Also \( b_j \) adjacent to \( u_1 \). Contracting edge \( a_i b_j \), \( a_i \) adjacent to \( u_1 \) implies \( \langle u_1, u_2, u_3; a_i b_j \rangle \) is \( K_3 \).

By O6 and remark 2 of Theorem 3, there exist one \( b_k \) not adjacent to \( b_j \), \( k \neq j \) and some \( a_l \), \( l \neq i \) not adjacent to \( b_k \) in \( G \). In \( \overline{G} \), \( b_k \) adjacent to \( b_j \), \( a_i \) not adjacent to \( u_2 \), \( b_k \) not adjacent to \( u_1 \). Contracting \( a_l b_k \), \( \langle u_1, u_2, u_3, u_4 : a_l b_k \rangle \) is \( K_4 \), implies \( \overline{G} \) is non–outer planar.

4. Conclusion
This paper contributes to planarity characterization of NDSS graphs. We conclude that if \( G \) is an NDSS graph then,

- \( \overline{G} \) is non–planar if \( 2 < \gamma(G) \leq 4 \).
- \( \overline{G} \) is non–outer planar if \( \gamma(G) = 2 \).
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