ON EXTREMISERS TO A BILINEAR STRICHARTZ INEQUALITY

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ABSTRACT. In this paper, we show that a pair of Gaussian functions are extremisers to a bilinear Strichartz inequality on \( \mathbb{R} \times \mathbb{R}^2 \), and unique up to the symmetry group of the inequality.

1. INTRODUCTION

We consider the free Schrödinger equation

\[
\begin{align*}
    i \partial_t u + \Delta u &= 0, \\
    u(0, x) &= f(x)
\end{align*}
\]

with initial data \( u(0, x) = f(x) \) where \( u: \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) is a complex-valued function and \( d \geq 1 \). We denote the solution \( u \) by using the Schrödinger evolution operator \( e^{it\Delta} \):

\[
    u(t, x) := e^{it\Delta} f(x) := \int_{\mathbb{R}^d} e^{ix \cdot \xi + it|\xi|^2} \hat{f}(\xi) d\xi,
\]

where \( \hat{f} \) is the spatial Fourier transform of \( f \) defined via

\[
    \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(x) dx,
\]

where \( x \cdot \xi \) denotes the Euclidean inner product of \( x \) and \( \xi \) in the spatial space \( \mathbb{R}^d \). When \( f \in L^2(\mathbb{R}^d) \), the solution \( e^{it\Delta} f \) enjoys a space-time estimate

\[
    \|e^{it\Delta} f\|_{L^{2+4/d}_{t,x}(\mathbb{R} \times \mathbb{R}^d)} \leq C_d \|f\|_{L^2(\mathbb{R}^d)},
\]

for some \( C_d > 0 \), see e.g. [8] or [13]. Define

\[
    C_d := \sup\{ \|e^{it\Delta} f\|_{L^{2+4/d}_{t,x}(\mathbb{R} \times \mathbb{R}^d)} / \|f\|_{L^2(\mathbb{R}^d)} : f \in L^2(\mathbb{R}^d), f \neq 0 \}.
\]

Several authors have investigated the extremal problem for (4), which asks whether there is an extremal function \( f \in L^2(\mathbb{R}^d) \) such that

\[
    \|e^{it\Delta} f\|_{L^{2+4/d}_{t,x}(\mathbb{R} \times \mathbb{R}^d)} = C_d \|f\|_{L^2(\mathbb{R}^d)},
\]

and what properties the extremal functions have. More precisely, Kunze [10] treated the \( d = 1 \) case and showed that extremisers exist by a longer concentration-compactness argument; when \( d = 1, 2 \), Foschi [7] explicitly determined the best constants and showed that the extremisers are Gaussians, and they are unique up...
Strichartz inequality has been shown, see e.g. [11]. Similar extremal questions wave equation in [2, 7]. Euclidean spaces such as the sphere in [5, 6], and the Strichartz inequality for the Fourier restriction inequality for the hyper-surfaces in the Euclidean spaces.

By the Cauchy-Schwarz inequality and (4), we have

\( \|e^{it\Delta}f\|_{L^2_t(L^2_x)} \leq B \|f\|_{L^2_t(L^2_x)} \|g\|_{L^2_t(L^2_x)}. \)

We define an extremiser or an extremal function to (5) is a pair of nonzero functions \((f, g) \in L^2 \times L^2\) such that

\( \|e^{it\Delta}f\|_{L^2_t(L^2_x)} = B \|f\|_{L^2_t(L^2_x)} \|g\|_{L^2_t(L^2_x)}. \)

By the Cauchy-Schwarz inequality and (4), we have

\( \|e^{it\Delta}f\|_{L^2_t(L^2_x)} \leq \|e^{it\Delta}f\|_{L^2_t(L^2_x)} \|e^{it\Delta}g\|_{L^2_t(L^2_x)} \leq C_2^2 \|f\|_{L^2_t(L^2_x)} \|g\|_{L^2_t(L^2_x)}. \)

So by the definition in (6), we see that

\( B \leq C_2^2. \)

Equating the inequality signs in (7) yields

\( \|e^{it\Delta}f\|_{L^2_t(L^2_x)} = C_2 \|f\|_{L^2_t(L^2_x)}, \)

\( \|e^{it\Delta}g\|_{L^2_t(L^2_x)} = C_2 \|g\|_{L^2_t(L^2_x)}. \)

Thus in (7), \((f, g)\) is a pair of Gaussian functions. As a consequence of Theorem 1, it turns out that \(C_2^2 = B\).

It is well known that the linear Strichartz inequality [1] is invariant under the following symmetry group \(G\) generated by

- Translation. \(e^{it\Delta}f(x) \rightarrow e^{i(t-t_0)\Delta}f(x-x_0)\) for any \((t_0, x_0) \in \mathbb{R} \times \mathbb{R}^2\).
- Scaling. \(e^{it\Delta}f(x) \rightarrow \lambda^2e^{i\lambda^2\Delta}f(\lambda x)\) for any \(\lambda > 0\).
- Galilean transform. \(e^{it\Delta}f(x) \rightarrow e^{ix\cdot\xi_0 + it|\xi_0|^2}f(x+2t\xi_0)\) for any \(\xi_0 \in \mathbb{R}^2\).
- Phase transition. \(e^{it\Delta}f(x) \rightarrow \alpha e^{it\Delta}f(x)\) for \(\alpha \in \mathbb{C} \setminus \{0\}\).
- Space rotation. \(e^{it\Delta}f(x) \rightarrow e^{it\Delta}f(Rx)\) for any \(R \in SO(2)\).

The bilinear Strichartz inequality is invariant when the same symmetry group acts simultaneously on \((f, g)\) in (5). It is additionally allowed the following phase transition

\( e^{it\Delta}f(x) \rightarrow \alpha e^{it\Delta}f(x), e^{it\Delta}g(x) \rightarrow \beta e^{it\Delta}g(x) \)

for \(\alpha \neq 0\) and \(\beta \neq 0\). For the extremal problem for the linear Strichartz inequality, it is true that the symmetry group \(G\) changes an extremal function to another; so
an extremal function $f$ will generate a family of extremal functions under the action of $G$. This family of functions is called the orbit of $f$. Because the inequality (5) is invariant under the symmetry in $G$, it is also the case for the extremal function. Now we state the following result.

**Theorem 1.1.** The pair of Gaussian functions 

$$(f, g) = \left( \exp \left( -|x|^2 \right), \exp \left( -|x|^2 \right) \right)$$

is an extremiser to the bilinear Strichartz inequality (5), and $B = \frac{1}{2}$. Moreover, the set of extremisers for which (5) holds coincides with the orbit of 

$$(f, g) = \left( \exp \left( A|x|^2 + b \cdot x + C_1 \right), \exp \left( A|x|^2 + b \cdot x + C_2 \right) \right),$$

where $A \in \mathbb{C}$ with the real part $\Re(A) < 0$ and $C_1, C_2 \in \mathbb{C}$ and $b \in \mathbb{C}^2$.

To prove it, we follow closely Foschi’s simple argument in [7], where it is shown that Guassians are the only extremisers to the linear Strichartz inequality (9)

$$\| e^{it\Delta} f \|_{L^4_t(L^4_x)} \leq C \| f \|_{L^2_\xi(L^2_x)}$$

up to the symmetry in $G$.

**Remark 1.2.** An analogous theorem to Theorem 1.1 can be established for a trilinear Strichartz inequality in the one-dimensional case, 

$$\| e^{it\Delta} f e^{it\Delta} g e^{it\Delta} h \|_{L^4_t(L^2_x)} \leq C \| f \|_{L^2_\xi(L^2_x)} \| g \|_{L^2_\xi(L^2_x)} \| h \|_{L^2_\xi(L^2_x)}.$$

**Remark 1.3.** Recently it comes to our attention that M. Charalambides [4] systematically investigated the question of characterizing functions $f, g$ and $h$ such that the Cauchy-Pexider functional equation $f(x)g(y) = h(x + y)$ with $x, y$ on some hyper-surface in $\mathbb{R}^{d+1}$. The solutions to such functional equation are uniquely determined to be exponential affine functions. This is closely connected to Theorem 1.1 because the functional equation characterizing the sharpness of the bilinear Strichartz inequality (5) is in the same form, see Section 3.

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## 2. Notation and preliminary

We begin with some notation. Define the Fourier transform,

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^3.$$

The inverse of the Fourier transform,

$$\mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^3.$$

We also use these notations to indicate the Fourier transform or the inverse Fourier transform in space-time $(t, x)$.

The Plancherel theorem states that

$$\| f \|_{L^2_\xi(L^2_x)} = \frac{1}{(2\pi)^{3/2}} \| \hat{f} \|_{L^2_\xi(L^2_x)}.$$

Moreover the Parseval identity states that

$$\int_{\mathbb{R}^3} f(x)\overline{g}(x) dx = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi.$$
Let $\sigma$ be the endowed measure on the paraboloid $P := \{(\tau, \xi) : \tau = |\xi|^2\}$ in $\mathbb{R}^3$, defined to be the pullback of the Lebesgue measure under the projection map: $((|\xi|^2, \xi) \mapsto \xi$. Then it follows that

$$e^{it\Delta}f(x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi + it|\xi|^2} \hat{f}(\xi)d\xi.$$  

We lift $\hat{f}$ onto the paraboloid $P := \{((\tau, \xi) : \tau = |\xi|^2\}$ via

$$\tilde{F}(\xi, |\xi|^2) = \hat{f}(\xi).$$  

Then by using the notation of the adjoint Fourier restriction operator for the paraboloid \[12\],

$$e^{it\Delta}f(x) = \tilde{F}\sigma(t, x).$$  

We define the convolution of $f$ and $g$,

$$f * g(x) = \int_{\mathbb{R}^3} f(x - y)g(y)dy,$$

and record a useful identity about convolution under the action of Fourier transform

$$\hat{f} * \hat{g} = \hat{f} \hat{g} \text{ and } \hat{f + g} = (2\pi)^3 f * g.$$  

We also record several lemmas from Foschi \[7\].

The following lemma shows that the convolution of the surface measure $\sigma$ is constant on its corresponding support $\Omega := \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 : 2\tau \geq |\xi|^2\}$, see e.g. \[7\, Lemma 3.2\].

**Lemma 2.1.**

$$\text{If } x \in \Omega, \quad \sigma * \sigma(x) = \pi/2.$$  

The proof uses that the convolution of measure supported on paraboloid is invariant under the mapping $(x', x_3) \rightarrow (0, x_3 - \frac{|x'|^2}{2})$, and the dilation symmetry of the paraboloid.

Finally we cite a lemma \[7\, Proposition 7.15\] on characterizing the following functional inequality,

$$f(x)f(y) = H(|x|^2 + |y|^2, x + y), \text{ for almost everywhere } (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2.$$  

**Lemma 2.2.** If $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ and $H : \Omega \rightarrow \mathbb{C}$ are nontrivial locally integrable functions which satisfy the functional equation \[15\], then there exists constants $A \in \mathbb{C}$, $b \in \mathbb{C}^2$ and $C \in \mathbb{C}$ such that

$$f(x) = \exp(A|x|^2 + b \cdot x + C), \quad H(t, x) = \exp(At + b \cdot x + 2C)$$  

for almost all $(t, x) \in \Omega$.

### 3. The proof

Before we prove Theorem \[13\] we recall Foschi’s argument in \[7\]. Foschi establishes the inequality \[9\] with an explicit constant by the Cauchy-Schwarz inequality. The only place where an inequality sign occurs is due to the Cauchy-Schwarz inequality. Then the question reduces to what functions make the Cauchy-Schwarz inequality sharp in the sense that the inequality becomes equal. This yields a functional equation, whose solutions uniquely determine extremisers. We will apply this idea.
Proof of Theorem 1.1. By the Plancherel theorem,
\[ \| e^{it\Delta} f e^{it\Delta} g \|_{L^2} = \| \hat{F} \sigma \hat{G} \|_{L^2} = \frac{1}{(2\pi)^{3/2}} \| \hat{F} \sigma \hat{G} \|_{L^2}. \]
where \( F(\xi, |\xi|^2) = \hat{f}(\xi), \ G(\xi, |\xi|^2) = \hat{g}(\xi), \) respectively; \( \hat{F} \) denotes the space-time Fourier transform of \( F \) in \((t, x)\). Then by (13), the above
\[ = (2\pi)^{3/2} \| F \sigma \ast G \sigma \|_{L^2}. \]

Since the measure \( \sigma \) on paraboloid \( P \) is defined to be the pull-back of the Lebesgue measure \( dx' \) on \( \mathbb{R}^3 \), then for Schwartz functions \( F, G \) on \( P \),
\[ F \sigma = F(|x'|^2, x')dx', \ G \sigma = G(|y'|^2, y')dy'. \]
In this case, the convolution of measures \( F \sigma \ast G \sigma \) is a function on \( \mathbb{R}^3 \),
\[ F \sigma \ast G \sigma(x) = \int \int \delta(x)d(F \sigma \ast G \sigma) \]
\[ = \int \int F(|x' - y'|^2, x' - y')G(|y'|^2, y') \delta \left( \frac{|x' - y'|^2 + |y'|^2}{(x' - y')^2 + y'^2} \right) dx'dy' \]
By using the Cauchy-Schwarz inequality and denoting by \( 1(x) \) the identity function on \( \mathbb{R}^3 \),
\[ (16) \]
\[ F \sigma \ast G \sigma(x) \]
\[ = \int \int (F(|x' - y'|^2, x' - y')G(|y'|^2, y'))(1(|x' - y'|^2, x' - y')1(|y'|^2, y')) \delta \left( \frac{|x' - y'|^2 + |y'|^2}{(x' - y')^2 + y'^2} \right) dx'dy' \]
\[ \leq \left( \int \int |F(|x' - y'|^2, x' - y')G(|y'|^2, y')|^2 \delta \left( \frac{|x' - y'|^2 + |y'|^2}{(x' - y')^2 + y'^2} \right) dx'dy' \right)^{1/2} \]
\[ \times \left( \int \int [1(|x' - y'|^2, x' - y')1(|y'|^2, y')]^2 \delta \left( \frac{|x' - y'|^2 + |y'|^2}{(x' - y')^2 + y'^2} \right) dx'dy' \right)^{1/2} \]
\[ \leq (|F|^2 \sigma \ast |G|^2 \sigma)(x)^{1/2} (\sigma \ast \sigma(x))^{1/2}. \]

On the other hand, there holds the convolution inequality
\[ \| F \sigma \ast G \sigma \|_{L^2(\mathbb{R}^3)} \leq C \| F \|_{L^2(\sigma)} \| G \|_{L^2(\sigma)} \]
for some \( C > 0 \). Thus by limiting arguments, for general \( F, G \in L^2(\mathbb{R}^3) \),
\[ (17) \]
\[ \| F \sigma \ast G \sigma \|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |F \sigma \ast G \sigma(x)|^2 dx \]
\[ \leq \int_{\mathbb{R}^3} \| F \|^2 \sigma \ast \| G \|^2 \sigma dx \]
\[ = \frac{\pi}{2} \int_{\mathbb{R}^3} |F|^2 \sigma \ast |G|^2 \sigma dx \]
\[ = \frac{\pi}{2} \| F \|^2 \| G \|^2 \|_{L^2}, \]
where we have used Lemma 2.1.

\[ (18) \]
\[ \sigma \ast \sigma(x) = \pi/2, \text{ for all } x \in \Omega. \]
To conclude so far, we have established the bilinear Strichartz inequality (5) with an explicit constant $2\pi^2$.

If the Cauchy-Schwarz inequality used in (17) is sharp in the sense that an equal sign occurs, then all the inequalities in (17) become equal; that is to say

$$\|F \sigma * G \sigma\|_{L^2} = \sqrt{\frac{\pi}{2}} \|F\|_{L^2} \|G\|_{L^2}.$$  

An examination of sharpness of the Cauchy-Schwarz inequality in (16) shows that, there exists $\alpha \in \mathbb{C}$ such that

$$F(|x' - y'|^2, x' - y')G(|y'|^2, y') = \alpha F(|z' - w'|^2, z' - w')G(|w'|^2, w').$$

if

$$|x' - y'|^2 + |y'|^2 = |w' - z'|^2 + |z'|^2,$$

$$x' - y' + y' = (w' - z') + z'.$$

Thus by reducing $(F, G)$ to $(\hat{f}, \hat{g})$,  

$$(22) \quad \hat{f}(x')\hat{g}(y') = H(|x|^2 + |y|^2, x + y) \text{ for a.e. } (x', y') \in \mathbb{R}^2 \times \mathbb{R}^2.$$ 

for some measurable function $H$.

Since $\hat{f}$ is not identically 0, without loss of generality, we may assume that $\hat{f}(0) \neq 0$. Let $x = 0$ and $y = 0$ in (22), respectively,

$$\hat{f}(0)\hat{g}(x) = \hat{f}(x)\hat{g}(0) \text{ for a.e. } x \in \mathbb{R}^2.$$ 

Then

$$\hat{g}(x) = \frac{\hat{g}(0)}{\hat{f}(0)} \hat{f}(x).$$

In view of this, we may assume that $\hat{f} = \hat{g}$ up to constants. This assumption reduces (22) to

$$(23) \quad \hat{f}(x')\hat{f}(y') = H(|x'|^2 + |y'|^2, x' + y') \text{ for a.e. } (x', y') \in \mathbb{R}^2 \times \mathbb{R}^2.$$ 

Then by Lemma 2.2 there exists constants $A \in \mathbb{C}$, $b \in \mathbb{C}^2$, and $C \in \mathbb{C}$ such that

$$(24) \quad \hat{f}(x) = \exp(A|x|^2 + b \cdot x + C), \quad H(t, x) = \exp(A^2 + b \cdot C + 2C)$$

for almost everywhere $(t, x) \in P$. Since the Fourier transform of a Gaussian function is a Gaussian function, we have shown that, the extremisers to (5) are Gaussian functions, which are unique up to the symmetry specified in the first section. This completes the proof of Theorem 1.1. 

\[ \square \]

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