Dissipativity and optimal control

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1 Introduction

The close link between dissipativity and optimal control is already apparent in Jan C. Willems’ first papers on the subject. Particularly, the paper [46], which appeared one year before his famous dissipativity papers [47, 48], already contains a lot of insight on this connection for linear quadratic optimal control problems. In recent years, research on this link has been revived with a particular focus on nonlinear problems and applications in model predictive control (MPC).

This development was initiated about ten years ago in the paper [9] by Diehl, Amrit, and Rawlings, in which strict dissipativity with linear storage function was used in order to construct a Lyapunov function for the closed-loop solution resulting from an economic MPC scheme. Soon after it was realized in [1] that linearity of the storage function is not really needed for this result, meaning that the same Lyapunov function construction can be carried out for general nonlinear strict dissipativity. In both cases, the link between dissipativity and optimal control lies in the fact that the running cost (or the stage cost in discrete time) serves as the supply rate in the dissipativity formulation.

While these results use strict dissipativity of the optimal control problem within the MPC scheme in a formal way in order to obtain stability results via an appropriate Lyapunov function, the effect of strict dissipativity on optimal trajectories can also be analyzed in a more geometrical fashion. More precisely, generalizing techniques that were already around in the 1990s (see, e.g., [6, Theorem 4.2]), it was observed in [17, Theorems 5.3 and 5.6] that strict dissipativity plus a suitable controllability property is sufficient for the occurrence of the so-called turnpike property. This property, first observed in mathematical economy in the works of Ramsey and von Neumann in the 1920s and 1930s [39, 45], and first called this way by Dorfman, Samuelson and Solow in the 1950s [10], describes the fact that optimal trajectories most of the time stay close to an optimal equilibrium.

Based on this observation, in [29, 11, 22, 18] (see also [23, Chapter 7] and [13]) the stability results from [1] could be extended to larger classes of MPC schemes and, in addition, non-averaged or transient approximate optimality of the MPC closed-loop could be established. Motivated by these new applications, the relation of strict dissipativity to classical notions
of detectability for linear and nonlinear systems was also clarified [19, 20, 31]. This paper surveys these recent developments and some of Willems’ early results.

2 Optimal control problems and (strict) dissipativity

2.1 Optimal control problems

We consider optimal control problems either in continuous time

\[
\text{minimize } J_T(x_0, u) = \int_0^T \ell(x(t), u(t)) dt
\]

(2.1)

with respect to \( u \in \mathcal{U} \), \( u(t) \in \mathcal{U} \) and \( x(t) \in \mathcal{X} \) for all \( t \in [0, T] \), \( T \in \mathbb{R}>0 \), where \( \mathcal{U} \) is an appropriate space of functions, \( \mathcal{X} \subset \mathbb{R}^n \) and \( \mathcal{U} \subset \mathbb{R}^m \) are the sets of admissible states and admissible control inputs, respectively, and

\[
\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0,
\]

(2.2)

or in discrete time

\[
\text{minimize } J_T(x_0, u) = \sum_{k=0}^{T-1} \ell(x(t), u(t))
\]

(2.3)

with respect to \( u \in \mathcal{U} \), \( u(t) \in \mathcal{U} \) and \( x(t) \in \mathcal{X} \) for all \( t = 0, \ldots, T \), \( T \in \mathbb{N} \), where \( \mathcal{U} \) is an appropriate space of sequences, \( \mathcal{X} \) and \( \mathcal{U} \) are as above, and

\[
x(t+1) = f(x(t), u(t)), \quad x(0) = x_0,
\]

(2.4)

which we briefly write as \( x^+ = f(x, u) \). Here \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is either the vector field in continuous time or the iteration map in discrete time, while \( \ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is called the running cost in continuous time and the stage cost in discrete time. In order to unify the notation, we use the symbol \([a, b]\) both in continuous and discrete time. In continuous time it denotes the usual closed interval \( \{ t \in \mathbb{R} \mid a \leq t \leq b \} \), while in discrete time it denotes \( \{ t \in \mathbb{Z} \mid a \leq t \leq b \} \), where \( \mathbb{Z} \) is the set of integers. For simplicity of exposition we limit ourselves to finite dimensional state space but we mention that some of the results discussed in this paper are also available in infinite dimensional settings. Given an initial value \( x_0 \in \mathcal{X} \), we denote the set of control functions \( u \in \mathcal{U} \) for which \( x(t) \in \mathcal{X} \) holds for all \( t \in [0, T] \) by \( \mathcal{U}(x_0, T) \). The optimal value function is then defined as

\[
V_T(x_0) := \inf_{u \in \mathcal{U}(x_0, T)} J_T(x_0, u)
\]

and a control \( u^* \in \mathcal{U}(x_0, T) \) with corresponding trajectory \( x^*(\cdot) \) is called optimal control for initial condition \( x_0 \) and time horizon \( T \) if

\[
J_T(x_0, u^*) = V_T(x_0).
\]

The corresponding trajectory \( x^*(\cdot) \) is then called an optimal trajectory.
2.2 Dissipativity and optimal control

Dissipativity in the sense of Willems as we use it in this paper involves an abstract notion of energy that is stored in the system. For each admissible state \( x \in \mathbb{R}^n \), we denote the energy in the system by \( \lambda(x) \). The function \( \lambda : \mathbb{R}^n \to \mathbb{R} \) is called storage function and it is usually assumed that \( \lambda \) is bounded from below, in order to avoid that an infinite amount of energy can be extracted from the system. In continuous time, dissipativity then demands that there exists another function, the so called supply rate \( s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \), such that the inequality

\[
\lambda(x(\tau)) \leq \lambda(x_0) + \int_0^\tau s(x(t), u(t)) dt
\]

holds for all \( \tau \geq 0 \), all control functions \( u \in \mathcal{U}(x_0, \tau) \) and all initial conditions \( x_0 \in \mathcal{X} \). As in the optimal control problem (2.1), \( x(t) \) is supposed to satisfy (2.2). In discrete time, the demanded inequality completely analogously reads

\[
\lambda(x(\tau)) \leq \lambda(x_0) + \sum_{t=0}^{\tau-1} s(x(t), u(t)),
\]

with \( x(t) \) satisfying (2.4). It appears that this discrete-time variant of (2.5) was first used by Byrnes and Lin in [4].

The interpretation of these inequalities is as follows: They demand that the energy \( \lambda(x(\tau)) \) in the system after a certain time \( \tau \) is not larger than the initial energy \( \lambda(x_0) \) plus the integral or sum over the supplied energy, expressed at each time instant by \( s(x(t)) \). Note that \( s(x(t)) \) can be negative, which means that negative energy is supplied, i.e. that energy is extracted from the system.

In continuous time, if \( \lambda \) is continuously differentiable, inequality (2.5) can equivalently be rewritten in infinitesimal form

\[
D\lambda(x)f(x,u) \leq s(x,u),
\]

while in discrete time equivalently to (2.6) one can use the one-step form

\[
\lambda(x^+) \leq \lambda(x) + s(x,u),
\]

where we use the common brief notation \( x^+ = f(x,u) \).

This dissipativity concept can be related to the optimal control problem from the previous section by setting \( s(x,u) := \ell(x,u) \) for the running or stage cost \( \ell \) from (2.1) and (2.3), respectively. It appears that this connection was first made in Willems’ paper [46], which remarkably appeared in the year before his seminal papers [47, 48], which introduced and discussed dissipativity in a comprehensive way. More precisely, in [46] continuous-time, linear, controllable dynamics \( f(x,u) = Ax + Bu \) and quadratic running cost \( \ell(x,u) = x^TQx + 2u^TCx + u^TRu \) without any definiteness assumption on \( Q \) and \( R \) are studied. The paper provides necessary and sufficient conditions for the existence of a storage function \( \lambda \) in terms of the finiteness of optimal value functions of certain related optimal control problems. These characterizations led to the concepts of available storage and required supply, that are described in Section 6 in the appendix. What is important here that these
characterizations involve constraints on the asymptotic behavior of the optimal control problems under consideration. This already indicates the fundamental connection between dissipativity and the long-term behavior of optimal trajectories, which was the key reason for the recent renaissance of this theory. Before we turn to the description of this recent development, we introduce a stricter variant of the dissipativity property.

2.3 Strict dissipativity

Strict dissipativity is nothing but dissipativity with storage function
\[ s(x,u) - \alpha(\|x\|) \]
in place of \( s(x,u) \), where \( \alpha \in K_\infty \) with
\[ K_\infty := \{ \alpha : [0, \infty) \to [0, \infty) \mid \alpha \text{ continuous, strictly decreasing, unbounded, and } \alpha(0) = 0 \} \]
and \( x^e \) is an equilibrium of the control system, i.e. there is a control value \( u^e \) with \( f(x^e, u^e) = 0 \) in continuous time and \( f(x^e, u^e) = x^e \) in discrete time. Written explicitly, the corresponding dissipation inequalities read
\[ \lambda(x(t)) \leq \lambda(x_0) + \int_0^t s(x(\tau), u(\tau)) - \alpha(\|x(\tau) - x^e\|) d\tau \]  
and
\[ \lambda(x(k)) \leq \lambda(x_0) + \sum_{j=0}^{k-1} s(x(j), u(j)) - \alpha(\|x(j) - x^e\|), \]
The short versions \[ (2.7) \] and \[ (2.8) \] of these inequalities then become
\[ D\lambda(x)f(x,u) \leq s(x,u) - \alpha(\|x - x^e\|), \]
and
\[ \lambda(x^+) \leq \lambda(x) + s(x,u) - \alpha(\|x - x^e\|), \]
respectively.

For some applications, it is necessary to use \( \alpha(\|(x, u) - (x^e, u^e)\|) \) in place of \( \alpha(\|x - x^e\|) \). In this case, one speaks about strict \((x, u)\)-dissipativity.

One easily sees that \[ (2.9) \] and \[ (2.10) \] are more demanding than \[ (2.5) \] and \[ (2.6) \], respectively, since they do not only demand that the energy difference \( \lambda(x(t)) - \lambda(x_0) \) is bounded by the integral or sum over the supplied energy \( s \), but that actually some of the energy is dissipated if the system is not in the equilibrium \( x^e \), and the amount of dissipated energy increases the further away the state \( x(\tau) \) is from \( x^e \).

At a first glance it seems that the difference between strict and non-strict dissipativity is only quantitative. Indeed, it follows readily from the definition that if the system is dissipative with supply \( s(x,u) \) then it is strictly dissipative with supply \( s(x,u) + \alpha(\|x\|) \). However, if the supply function \( s \) is not merely a parameter we can play with but a function that results from some modelling procedure, then there is not only a quantitative but also a qualitative difference between strict and non-strict dissipativity. This in particular applies
for the setting in which \( s \) is derived from the running or stage cost \( \ell \) of an optimal control problem, i.e., when \( s = \ell \).

When using strict dissipativity for the analysis of the control system behavior, it is often necessary to demand \( s(x^e, u^e) = 0 \), as we will see in the next section. This is in general a restrictive condition, but it is actually not restrictive in case that \( s \) is derived from the optimal control problem. This is because the optimal trajectories for the cost \( \ell(x, u) \) are the same as for the cost \( \ell(x, u) - \ell(x^e, u^e) \). It is thus convenient and not restrictive to use the supply rate

\[
 s(x, u) = \ell(x, u) - \ell(x^e, u^e). \tag{2.13}
\]

### 3 The turnpike property

The turnpike property demands that optimal trajectories (and in some variants also near-optimal trajectories) stay in the vicinity of an equilibrium \( x^e \) most of the time. Of course, this statement needs to be made mathematically precise in order to be able to analyze it mathematically. The meaning of “most of the time” is that the amount of time the trajectory spends outside a given neighborhood of the equilibrium \( x^e \) is bounded, and the bound is independent of the optimization horizon and of the initial condition, at least for initial conditions which itself are contained in a bounded set.

Formally, we say that an optimal control problem has the turnpike property, if there is an equilibrium \( x^e \), such that for any \( \varepsilon > 0 \) and \( K > 0 \) there is a \( C > 0 \) such that for all time horizons \( T > 0 \) and all optimal trajectories \( x^*(\cdot) \) with \( x^*(0) \in B_K(x^e) \cap X \) the set of times

\[
 Q := \{ t \in [0, T] \mid x^*(t) \notin B_\varepsilon(x^e) \}
\]

satisfies \( |Q| \leq C \). Here \( |Q| \) is the Lebesgue measure of \( Q \) in continuous time or the number of elements contained in \( Q \) in discrete time, and \( B_K(x^e) \) and \( B_\varepsilon(x^e) \) denote the balls around \( x^e \) with radii \( K \) and \( \varepsilon \), respectively.

Figure 3.1 shows a sketch of a trajectory exhibiting the turnpike phenomenon. Here the set \( Q \) contains 7 time instants, which are marked with short red vertical lines on the \( t \)-axis. The equilibrium \( x^e \) is represented by the dashed blue line.

Figure 3.2 shows optimal trajectories for varying time horizons \( T = 5, 7, \ldots, 19 \) for two simple one-dimensional discrete-time optimal control problems. The first problem is given by

\[
 x^+ = u, \quad \ell(x, u) = -\log(5x^{0.34} - u), \quad X = [0, 10], \quad U = [0.01, 10] \tag{3.1}
\]

and the second by

\[
 x^+ = 2x + u, \quad \ell(x, u) = u^2, \quad X = [-2, 2], \quad U = [-3, 3]. \tag{3.2}
\]

The model (3.1) is an optimal investment problem from [3], in which \( x \) denotes the investment in a company and \( 5x^{0.34} \) is the return from this investment (including the investment itself) after one time period. As \( x^+ = u \) is the investment in the next time period, \( 5x^{0.34} - u \) is the amount of money that can be used for consumption in the current time period and the optimal control problem thus models the maximization of the sum of the logarithmic
utility function $\log(5x^{0.34} - u)$ over the time periods. Clearly, it is optimal to spend all the available money until the end of the time horizon $T$, which is why all optimal trajectories end at $x = 0$. However, in between the trajectories spend most of the time near the equilibrium $x^e \approx 2.2344$, i.e. it exhibits the turnpike property. We will explain later why this is the case.

The task modelled in the second example is to keep the state of the system in $X$ with as little quadratic control effort $\ell(x, u) = u^2$ as possible. In the long run it is beneficial to stay near $x^e = 0$, because the control effort for staying in $X$ is very small near this equilibrium. Thus, it makes sense for the optimal trajectories to stay near $x^e = 0$ for most of the time. At the end of the horizon, however, it is beneficial to turn off the control completely, because this entirely reduces the control effort to 0 and does not violate the state constraint provided the control is turned off sufficiently late.

In both examples in Figure 3.2 it is clearly visible that the number of states outside a neighborhood of the respective equilibrium (indicated by the dashed blue line) remains constant with increasing horizon length, which is exactly what the turnpike property demands.

These examples show that the turnpike property occurs already in very simple problems. It is, however, known that it also occurs in many much more complicated problems, including optimal control problems governed by partial differential equations; \cite{37, 30, 51, 36, 42, 27}.
Interestingly, the two examples exhibit a particular form of the turnpike property, in which the exceptional points, i.e., the points corresponding to the times in $Q$, lie only at the beginning and at the end of the time interval. One speaks of the *approaching arc* at the beginning and the *leaving arc* at the end of the time horizon. While the general definition of the turnpike property allows for excursions from $x^e$ also in the middle of the time interval (as indicated in the sketch in Figure 3.1), these do not appear in the two examples. The reason for this will be explained in the next section.

The optimal trajectories in Figure 3.2 nicely illustrate the source of the name “turnpike property”, which was coined in [10]: the behavior of the optimal trajectory is similar to a car driving from the initial to the end point, where the equilibrium—i.e., the dashed blue line—plays the role of a highway, or turnpike, as highways are called in parts of the USA. If the time is sufficiently large, it pays off to first go to the highway (even if this is associated with some additional cost), stay there for most of the time and then leave the highway at the end. There may be different reasons for the occurrence of the leaving arc. We may impose a constraint $x(N) = \hat{x}$ for the terminal state that forces the trajectory to leave the turnpike, or it may simply be beneficial to leave the turnpike because this reduces the cost of the overall trajectory. As we did not impose any terminal constraints, the latter must be the case in our examples, and it is actually easy to convince yourself that this is the case.

Besides the definition given above, there are a number of alternative definitions of the turnpike property. A widely used variant is the *exponential turnpike property*, which demands an inequality of the type

$$\|x^*(t) - x^e\| \leq C(e^{-\sigma t} + e^{-\sigma(T-t)})$$

for all $t \in [0, T]$ with constants $C, \sigma > 0$. On easily sees that this property implies the turnpike property above, since for each $\varepsilon > 0$ there is $\tau > 0$ with $C(e^{-\sigma_1 t} + e^{-\sigma_2(T-t)}) < \varepsilon$ for all $t \in [\tau, T - \tau]$ holds, regardless of how large $T$ is. Another variant demands the turnpike behavior not only for the optimal trajectories $x^*(\cdot)$, but for all trajectories corresponding to control functions satisfying $J(x_0, u) \leq V(x_0) + \delta$, i.e. for all *near optimal trajectories*. 

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**Figure 3.2:** Optimal trajectories for the examples (3.1) and (3.2) for time horizon length $T = 5, 7, \ldots, 19$. 

[2] [28] is just a small selection of recent references where this is investigated.
For even more variants and a historic discussion of the turnpike property we refer to the recent survey [12].

3.1 Strict dissipativity and the turnpike property

Under a boundedness assumption on the optimal value functions it is fairly easy to prove that strict dissipativity implies the turnpike property. The boundedness assumption demands that there are a $K_{\infty}$-function $\gamma$ and a constant $C > 0$, such that the optimal value and the storage functions satisfy

$$|V_T(x) - V_T(x^e)| \leq \gamma(\|x - x^e\|) + C \quad \text{and} \quad |\lambda(x)| \leq \gamma(\|x - x^e\|) + C$$

(3.3)

for all $x \in X$ and all $T \geq 0$. The first condition can be ensured by a reachability condition: if the equilibrium $x^e$ can be reached from every $x \in X$ with costs that are bounded in bounded subsets of $X$, then the inequality for $|V_T(x) - V_T(x^e)|$ can be established for all $T > 0$; for details see [17, Section 6].

We illustrate the derivation of the turnpike property under these boundedness assumptions in the discrete time setting. Using (2.10) and (2.13) and choosing $D > 0$ such that $\lambda(x) \geq -D$ for all $x \in X$ we obtain

$$J(x_0, u) = \sum_{k=0}^{T-1} \ell(x(t), u(t)) \geq \sum_{k=0}^{T-1} \alpha(\|x(t) - x^e\|) - \lambda(x_0) + \lambda(x(T)) \geq \sum_{k=0}^{T-1} \alpha(\|x(t) - x^e\|) + T\ell(x^e, u^e) - \lambda(x_0) - D$$

Moreover, using the constant control $u \equiv u^e$ we obtain

$$V_T(x^e) \leq J_T(x^e, u) = \sum_{t=0}^{T-1} \ell(x^e, u^e) = T\ell(x^e, u^e).$$

Together this implies for $x = x^*$ with $x^*(0) = x_0$

$$\gamma(\|x_0 - x^e\|) + C \geq |V_T(x_0) - V_T(x^e)| \geq \sum_{k=0}^{T-1} \alpha(\|x^*(t) - x^e\|) - \lambda(x_0) - D.$$

Given an arbitrary $\varepsilon > 0$, if $x^*$ spends too much time outside $B_\varepsilon(x^e)$, then this inequality is violated. From this fact the existence of the constant $C$ in the turnpike property can be concluded. Here the inequalities in (3.3) are needed in order to make sure that the constant $C$ in the turnpike property only depends on the size $K$ of the ball $B_K(x^e)$ containing $x_0$ and not on the individual initial value $x_0$.

One easily checks that both examples (3.1) and (3.2) are strictly dissipative, the former with the storage function $\lambda(x) = \alpha(x - x^e)/x^e$ and the latter with the storage function
$\lambda(x) = -x^2/2$. This explains why we see the turnpike behavior in these examples. Note that in both examples boundedness of $X$ is important to ensure that $\lambda$ is bounded from below on $X$.

With a more involved proof, strict dissipativity with suitable bounds on the problem data can also be used in order to obtain an exponential turnpike property, see [8]. Alternatively, the exponential turnpike property can be established using necessary optimality conditions (we refer to [34, 33, 42, 28] for a selection of papers in this direction), but the strict dissipativity based approach has the advantage that it works for arbitrary nonlinearities, while the optimality condition-based approach is typically limited to linear systems or to small nonlinearities via linearization arguments.

The fact that strict dissipativity implies the turnpike property immediately raises the question how much stronger strict dissipativity is than the turnpike property. The answer lies in the observation that the inequality chain, above, can also be used for establishing the turnpike property for trajectories and controls that are not strictly optimal but only satisfy the inequality $J_T(x_0, u) \leq V_T(x_0) + \delta$, i.e. they are near-optimal. In other words, we obtain a near-optimal turnpike property. For systems that are locally controllable in a neighborhood of $x^e$ and controllable to this neighborhood from all $x \in X$, it was shown in [21] that strict dissipativity is equivalent to the near-optimal turnpike property.

Interestingly, the proof of this equivalence relies on a similar equivalence result for plain (i.e., nonstrict) dissipativity: In [34] it was shown that, again under a local controllability assumption, dissipativity is equivalent to the fact that the average cost for all admissible $u$ satisfies the inequality

$$
\limsup_{T \to \infty} \frac{1}{T} J_T(x_0, u) \geq \ell(x^e, u^e), \tag{3.4}
$$

a property that is known under the name of optimal operation at steady state.

Strict dissipativity also explains why optimal trajectories exhibiting the turnpike property typically look as in Figure 3.2 and hardly ever as in Figure 3.1. The reason is that a similar inequality as above shows that an excursion from the equilibrium $x^e$ followed by a return to $x^e$ causes costs that are significantly higher than staying in $x^e$. Under the assumption that the optimal values are continuous near $x^e$ (in the sense that the optimal value function near $x^e$ has about the same values as in $x^e$ — a property that can again be rigorously established for locally controllable systems), this implies that such excursions will not happen in (near-)optimal trajectories.

In contrast to that, an excursion from the equilibrium $x^e$ can be cheaper then staying in $x^e$ if the solution does not return to $x^e$ after the excursion. This is precisely the effect that creates the leaving arc of the optimal trajectories. As the gain that can be obtained from the leaving arc is bounded by the value of the storage function $\lambda(x(T))$ at the terminal state of the trajectory, it is important that $\lambda$ is bounded from below. Indeed, the example [3.2] would cease to be strictly dissipative if we changed $X = [-2, 2]$ to $X = \mathbb{R}$. In this case, it is easily seen that the optimal trajectories would tend to $\pm \infty$ (with optimal control $u^* \equiv 0$) instead of staying near $x^e = 0$ most of the time, i.e. the turnpike property also ceases to exist.
3.2 Related properties

Strict dissipativity generalizes a lot of well-known properties for optimal control problems ensuring a certain asymptotic behavior for the optimal trajectories. To begin with, it is easily seen that strict dissipativity holds with storage function \( \lambda \equiv 0 \) for any cost function \( \ell \) satisfying \( \ell(x^e, u^e) = 0 \) and \( \ell(x, u) \geq \alpha(\|x - x^e\|) \) for all \( x \in X \) and \( u \in U \). In case of a quadratic cost

\[
\ell(x, u) = (x - x^e)^T Q(x - x^e) + (u - u^e)^T R(u - u^e)
\]

this amounts to requiring that \( Q \) is positive definite. For linear quadratic problems with dynamics \( f(x, u) = Ax + Bu \), generalized quadratic cost

\[
\ell(x, u) = x^T Q x + u^T R u + q^T x + r^T u
\]

with \( Q = C^T C \), and no state constraints, i.e., \( X = \mathbb{R}^n \), strict dissipativity is equivalent to detectability of the pair \( (A, C) \), i.e. to the fact that all unobservable eigenvalues \( \lambda \) of \( A \) satisfy \( \text{Re} \lambda < 0 \) in continuous time or \( |\lambda| < 1 \) in discrete time. If \( X \) is bounded with \( x^e \) in its interior, then strict dissipativity is equivalent to the fact that all unobservable eigenvalues \( \lambda \) of \( A \) satisfy \( \text{Re} \lambda \neq 0 \) in continuous time or \( |\lambda| \neq 1 \) in discrete time (see [19] and [20] for proofs in discrete and continuous time, respectively). We note that the last criterion applies to example (3.2). Moreover, strict dissipativity holds if \( f \) is affine and \( \ell \) is strictly convex (see [8] for a proof in discrete time that can also be carried over to continuous time); this criterion applies to example (3.1). Finally, as shown in [31], strict dissipativity also follows from nonlinear detectability notions, such as the one introduced in [15] or input-output-to-state stability (IOSS) [5].

4 An application: Model Predictive Control

Model Predictive Control (MPC), as described in Section 7 in the appendix, is a highly popular control method, in which the computationally challenging solution of an infinite horizon optimal control problem is replaced by the successive solutions of optimal control problems on finite time horizons. As described in the sidebar, this induces an error that can be analyzed employing strict dissipativity of the optimal control problem.

4.1 Stability analysis

Strict dissipativity turns out to be useful for two important aspects of this analysis. The first aspect is the stability analysis. In general, it is not clear, at all, that the MPC closed-loop solutions will exhibit stability-like behaviour. However, under the assumption that strict dissipativity holds, we can define an auxiliary optimal control problem by using the modified or rotated running or stage cost

\[
\tilde{\ell}(x, u) := \ell(x, u) - \ell(x^e, u^e) - D\lambda(x)f(x, u)
\]

in continuous time or

\[
\tilde{\ell}(x, u) := \ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \lambda(f(x, u))
\]
in discrete time. It is immediate from the strict dissipativity inequalities \( 2.11 \) or \( 2.12 \) that this modified stage cost is positive definite at \( x^e \), i.e., it satisfies \( \ell(x^e, u^e) = 0 \) and \( \ell(x, u) \geq \alpha\|x - x^e\| \). Thus, if we consider the optimal control problems \( 2.1 \) or \( 2.3 \) with \( \tilde{\ell} \) in place of \( \ell \), it forces the optimal solutions to approach \( x^e \). Denote the corresponding functional and optimal value function by \( \tilde{J}_T \) and \( \tilde{V}_T \), respectively, and the optimal solution by \( \tilde{x}^*(\cdot) \). Then by either adding a terminal cost \( \tilde{F} \) with suitable properties to \( \tilde{J}_T \) (cf., e.g., \[7, 33\], \[40, Chapter 2\], or \[23, Chapter 5\]) or by imposing conditions that ensure that \( \tilde{V}_T \) is close to \( \tilde{V}_\infty \) for sufficiently large \( T \) (cf., e.g., \[15\] or \[23, Chapter 6\]), one can prove that \( \tilde{V}_T \) is a Lyapunov function for the MPC closed loop. The trouble is, though, that this is only true if \( \tilde{\ell} \) is used as running cost or stage cost in the MPC scheme. This, however, is often not practical because the storage function may not be known and difficult to compute. In this case, one would like to keep the original cost \( \ell \) in the optimal control problem.

In case the running or stage cost \( \ell \) together with the terminal cost \( F \) is used, the trick is to define \( F = \tilde{F} - \lambda \) in order to arrive at an equivalent optimal control problem that uses \( \ell \) and \( F \) and produces the same optimal solutions as the one using \( \tilde{\ell} \) and \( \tilde{F} \). The conditions on the terminal cost needed to make \( \tilde{V}_T \) a Lyapunov function can be formulated directly for \( F \) without having to use (or even to know) \( \lambda \). The resulting required inequality for \( F \) is of the form

\[
DF(x)f(x, u) \leq -\ell(x, u) + \ell(x^e, u^e)
\]

in continuous time and

\[
F(f(x, u)) \leq F(x) - \ell(x, u) + \ell(x^e, u^e)
\]

in discrete time, which must hold for all \( x \) in the terminal constraint set \( \mathcal{X}_0 \) and a control value \( u \) (depending on \( x \)), which is such that the solution does not leave \( \mathcal{X}_0 \). Since the optimal control problems with \( \tilde{\ell} \) and \( \tilde{F} \) on the one side and with \( \ell \) and \( F \) on the other side produce the same optimal solutions, the MPC closed-loops resulting from the two problems also coincide and \( \tilde{V}_T \) can again be used as a Lyapunov function to conclude asymptotic stability. This trick was first proposed in \[9\] and then refined in \[1\].

Figure \( 4.1 \) show the MPC closed-loop trajectories and the corresponding predictions for example \( 3.2 \), using the terminal constraint set \( \mathcal{X}_0 = \{0\} \) and the terminal cost \( F \equiv 0 \).

In many cases it may be difficult to find a terminal cost \( F \) that meets the required condition \( 4.1 \) or \( 4.2 \). While the trivial choice \( \mathcal{X}_0 = \{x^e\} \) and \( F \equiv 0 \) always satisfies \( 4.1 \) or \( 4.2 \), this choice of \( \mathcal{X}_0 \) may cause problems in the numerical optimization and result in a small set of feasible states for \( 2.1 \) or \( 2.3 \) when the terminal condition \( x(T) \in \mathcal{X}_0 \) is added. It may thus be attractive to drop the terminal constraint \( \mathcal{X}_0 \) and cost \( F \). In this case, however, the trick with passing from \( \tilde{F} \) to \( F \) is not applicable. In fact, without terminal costs the optimal trajectories and controls of the problem with \( \ell \) and with \( \tilde{\ell} \) do not coincide anymore. However, they may still coincide approximately in a suitable sense.

In order to establish this property, we need to assume that the storage function \( \lambda \) and the optimal value functions \( V_T \) and \( \tilde{V}_T \) are continuous in \( x^e \), uniformly in \( T \) (the former is not very restrictive since often \( \lambda \) is a polynomial and the latter can again be ensured by local controllability, see \[17, Section 6\]). Next we use that both the problem with cost \( \ell \) and the problem with cost \( \tilde{\ell} \) are strictly dissipative and thus exhibit the turnpike property. From this we can conclude that for any two optimal trajectories \( x^*(\cdot) \) and \( \tilde{x}^*(\cdot) \) starting
in the same initial value \(x_0\), there will be a time \(\tau\) such that they satisfy \(\tilde{x}^*(\tau) \approx x^e\) and \(x^*(\tau) \approx x^e\), where the error hidden in “≈” tends to 0 as the horizon \(T\) increases. Together with the continuity assumption on the optimal value function this implies that the cost of the two trajectories on the time interval \([0,\tau]\) is almost identical. This, finally, can be used to conclude that \(\tilde{V}_T\) is an approximate Lyapunov function, from which practical asymptotic stability, i.e., asymptotic stability of a neighborhood of \(x^e\), which shrinks down to \(\{x^e\}\) when \(T\) increases, can finally be concluded. The details of this reasoning were originally derived in the papers \([17, 29, 11]\). A concise presentation can be found in \([23, \text{Section 8.6}]\) or in \([13, \text{Section 4}]\).

Figure 4.2 show the MPC closed-loop trajectories and the corresponding predictions for example (3.2) without any terminal conditions. In comparison to Figure 4.1 the merely practical asymptotic stability of the equilibrium \(x^e = 0\) is clearly visible, since the red closed-loop solution does not converge to the equilibrium \(x^e = 0\) but only to a neighborhood of this equilibrium, which becomes smaller as \(T\) increases.

We note that under stronger assumptions than strict dissipativity stronger stability statements can be made for MPC without terminal conditions. For instance, under a nonlinear detectability condition and suitable bounds on the optimal value function, it was shown in \([15]\) that true (as opposed to merely practical) asymptotic stability of \(x^e\) for the MPC closed loop can be expected for sufficiently large \(T\). For positive definite cost, different ways of estimating the length of the horizon \(T\) needed for obtaining asymptotic stability were proposed in \([44, 16, 24, 41]\) (see also \([23, \text{Chapter 6}]\)).

4.2 Performance analysis

Since MPC relies on the solution of optimal control problems, it seems reasonable to expect that the MPC closed loop also enjoys certain optimality properties. To this end, we denote the MPC closed-loop trajectory by \(x_{MPC}(t)\) and the corresponding control by \(u_{MPC}(t)\). Then we can define different measures for the closed loop cost (we only give the
Figure 4.2: MPC closed-loop trajectories (red solid) and predictions (black dashed) for example (3.2) without terminal conditions with horizon $T=5$ (left) and $T=10$ (right). Without any terminal constraints all predictions end in $x=2$, cf. also Figure 3.2(right).

discrete-time formulations here as so far the corresponding results have only be derived for discrete-time systems): The infinite horizon performance

$$J_{\infty}^{MPC}(x_0) := \sum_{t=0}^{\infty} \ell(x_{MPC}(t), u_{MPC}(t))$$

would be the “natural” measure if we consider MPC as an approximation to an infinite horizon problem. However, as the infinite sum may not converge, we also look at other measures. We also consider the finite horizon closed-loop performance

$$J_{S}^{MPC}(x_0) := \sum_{t=0}^{S-1} \ell(x_{MPC}(t), u_{MPC}(t)) \quad (4.3)$$

and the averaged infinite horizon performance

$$\overline{J}_{\infty}^{MPC}(x_0) := \limsup_{S \to \infty} \frac{1}{S} J_{S}^{MPC}(x_0).$$

These last two performance measures complement each other, as the first measures the performance on finite intervals $[0,S]$ while the second measures the performance in the limit for $S \to \infty$.

The derivation of estimates for these quantities heavily relies on strict dissipativity and the turnpike property, which are exploited for MPC both with and without terminal conditions, and on the stability properties described in the previous section. The key idea is to use the similarity of the initial pieces of optimal trajectories until they reach the optimal equilibrium in order to derive approximate versions of the dynamic programming principle. From these, estimates on the above quantities can be derived by induction over $t$. The following estimates were originally developed in the papers [17, 29, 22]. Concise presentations can be found in [18] or [23, Chapter 8].

For MPC with terminal conditions satisfying (4.2), the identity

$$\overline{J}_{\infty}^{MPC}(x_0) = \ell(x^e, u^e) \quad (4.4)$$
holds for all \( N \in \mathbb{N} \), and because of (3.4) this is the best possible value the average performance functional can attain. In case \( V_\infty \) assumes finite values, there exists a function \( \delta_1 : \mathbb{N} \to [0, \infty) \) with \( \delta_1(T) \to 0 \) as \( T \to \infty \) such that the inequality
\[
J^{\infty}_{\text{MPC}}(x_0) \leq V_\infty(x_0) + \delta_1(T)
\] (4.5)
holds. In case \( V_\infty \) does not assume finite values, for each \( S \in \mathbb{N} \) we can obtain the estimate
\[
J^{S}_{\text{MPC}}(x_0) \leq \inf_{u \in \tilde{U}^S} J_S(x_0, u) + \delta_1(T) + \delta_2(S) + \delta_1(T)
\] (4.6)
where \( \delta_2 \) is the same type of function as \( \delta_1 \). Here \( \tilde{U}^S \) denotes the set of admissible controls for which \( \| x_u(S, x_0) - x^e \| \leq \| x^{\text{MPC}}(S, x_0) - x^e \| \) holds. As \( x^{\text{MPC}}(S, x_0) \to x^e \) holds for \( S \to \infty \), for large \( S \) the quantity \( \inf_{u \in \tilde{U}^S} J_S(x_0, u) \) measures the optimal transient cost for trajectory going from \( x_0 \) to a small neighborhood of \( x^e \). Thus, the estimates show that MPC produces trajectories that produce optimal averaged cost and approximately optimal transient cost.

Without terminal conditions the results become somewhat weaker. Particularly, we can in general no longer ensure that \( J^{\infty}_{\text{MPC}}(x_0) \) is finite, even if \( V_\infty(x_0) \) is finite. However, we can still establish counterparts of (4.4) and (4.6), namely
\[
J^{\infty}_{\text{MPC}}(x_0) \leq \ell(x^e, u^e) + \delta_1(T)
\] (4.7)
and
\[
J^{S}_{\text{MPC}}(x_0) \leq \inf_{u \in \tilde{U}^S} J_S(x_0, u) + S\delta_1(T) + \delta_2(S),
\] (4.8)
with \( \delta_1 \) and \( \delta_2 \) of the same type as above. We note that the fact that the error term \( \delta_1(T) \) in (4.6) increases to \( S\delta_1(T) \) in (4.8) is not an effect of an insufficiently precise analysis but actually a natural consequence of the mere practical asymptotic stability of \( x^e \) without terminal conditions: If the closed-loop solution does not converge to \( x^e \), the stage cost will typically not converge to 0 but to a nonzero residual value, which keeps accumulating over the time \( S \). In order to illustrate this effect, Figure 4.3 shows the values of \( J^{S}_{\text{MPC}}(x_0) \) for example (3.2) with terminal conditions (\( \circ \)) and without terminal conditions (\( \times \)) for different \( S \). In the left figure with horizon \( T = 5 \), the effect of the additional factor \( S \) in front of \( \delta_1(T) \) in (4.8) is clearly visible. In the right figure with horizon \( T = 10 \), the error term \( \delta_1(T) \) in (4.8) is already so small that the effect of the factor \( S \) is not visible anymore. For this horizon, MPC with and without terminal conditions yield almost the same performance.

Just as for stability, under stronger assumptions stronger statements can be made. For instance, for positive definite cost \( \ell \) finiteness of \( J^{\infty}_{\text{MPC}}(x_0) \) and an estimate of the form (4.5) can be obtained also for MPC without terminal conditions, see [24, 41].

### 5 Summary, extensions and outlook

Dissipativity and strict dissipativity are important systems theoretic properties with multiple applications. In this paper we have shown that they naturally link to optimal control,
Figure 4.3: MPC closed-loop cost $J_{\text{MPC}}^S(x_0)$ for example (3.2) with $x_0 = 2$ for varying $S$. The solid line with circles shows the values with terminal conditions $X_0 = \{0\}$ and $F \equiv 0$, the dashed line with crosses the values without terminal conditions, with horizon $T = 5$ (left) and $T = 10$ (right).

a fact that is already prominently present in Jan C. Willems’ earliest publications on the subject. Here we provided a survey on recent results in this direction, which establish a close link between strict dissipativity and the turnpike property and showed how these concepts can be used for analyzing stability and performance of MPC schemes.

The present results have already been extended into various directions. In particular, dissipativity has been extended to optimal control problems that do not exhibit an optimal equilibrium but an optimal periodic orbit [49, 35, 50, 32] or general time-varying optimal trajectories [26, 25]. In this context, the concept of overtaking optimality can be used in order to define a meaning for optimality also in the case that the infinite horizon optimal value function $V_\infty$ is not finite.

A major open question is the relation between strict dissipativity and detectability-like notions for infinite dimensional systems. Even for linear-quadratic optimal control problems this relation is not yet fully understood.

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6 Appendix: Available Storage and Required Supply

Optimal control can be used in order to compute storage functions for dissipative systems, provided the supply rate \( s \) and—in case of strict dissipativity—the \( K_\infty \)-function \( \alpha \) are known. More precisely, the system is strictly dissipative if and only if the optimal value function
\[
V(x_0) := \sup_{T \geq 0, u \in U} \int_0^T -s(x(\tau), u(\tau)) + \alpha(\|x(\tau) - x^e\|) d\tau
\]
is finite for all initial values \( x_0 \). In this case, \( \lambda = V \) is a storage function called the *available storage*. An analogous construction works without \( \alpha \) in case of non-strict dissipativity and with a sum instead of the integral in case of discrete-time systems.

That \( V \) is indeed a storage function follows for any \( t > 0 \) and \( \hat{u} \in U \) from the inequalities
\[
V(x_0) = \sup_{T \geq 0, u \in U} \int_0^T -s(x(\tau), u(\tau)) + \alpha(\|x(\tau) - x^e\|) d\tau
\geq \sup_{T \geq t, u \in U} \int_t^T -s(x(\tau), u(\tau)) + \alpha(\|x(\tau) - x^e\|) d\tau
\geq \int_t^0 -s(x(\tau), \hat{u}(\tau)) + \alpha(\|x(\tau) - x^e\|) d\tau
+ \sup_{T \geq 0, u \in U} \int_{T-t}^T -s(x(\tau), u(\tau)) + \alpha(\|x(\tau) - x^e\|) d\tau
= \int_0^t -s(x(\tau), \hat{u}(\tau)) + \alpha(\|x(\tau) - x^e\|) d\tau + V(x(t)),
\]
which implies (2.9) for \( \lambda = V \).

If each \( x \in \mathbb{R}^n \) can be reached from the equilibrium \( x^e \), then another optimal control characterization of a storage function is given by the *required supply*. In this case one defines
\[
V(x) := \inf_{T \geq 0, x \in \mathbb{R}^n} \int_0^T s(x(\tau), u(\tau)) - \alpha(\|x(\tau) - x^e\|) d\tau.
\]

Then strict dissipativity holds if and only if \( V \) is bounded from below and then, again, \( \lambda = V \) is a storage function. Here, the storage function property follows from the fact that steering from \( x^e \) to \( x(t) \) via \( x_0 \) cannot be cheaper than steering from \( x^e \) to \( x(t) \) in the optimal way. For details on both constructions we refer to [47].
Appendix: Model Predictive Control

Model Predictive Control (MPC) is one of the most successful optimization based control techniques with ample applications in industry [38, 14]. We describe it here in discrete time, noting that the adaptation to continuous time is relatively straightforward. For further reading we recommend, e.g., the monographs [40, 23].

In many regulation problems, one would like to solve the infinite horizon optimal control problem

$$\min_{u \in U} J_{\infty}(x_0, u) = \sum_{k=0}^{\infty} \ell(x(t), u(t))$$

(7.1)

with respect to $u \in U$, $u(t) \in U$ and $x(t) \in X$ for all $t = 0, 1, 2, \ldots$, where $U$ is an appropriate space of sequences, $X$ and $U$ are as above, and $x(t)$ satisfies $x(0) = 0$ and (2.4). Unless the problem has a very particular structure (as, e.g., linear dynamics, quadratic costs and no constraints), a closed-form solution of (7.1) is usually not available and due to the infinite time horizon a numerical solution is very costly to obtain, particularly if one wants to obtain the optimal control in feedback form, i.e., in the form $u^*(t) = F(x(t))$ for a suitable map $F$.

The key idea of MPC now consists in truncating the optimization horizon and instead of (7.1) solve (2.3) or a variant thereof. This variant may include additional terminal constraints of the form $x(T) \in X_0$ and/or a terminal cost of the form $F(x(T))$ as an additional summand in $J_T$. The MPC loop then proceeds as follows:

1) Pick an initial value $x_{MPC}(0)$ and a time horizon $T$. Set $k := 0$.

2) Set $x_0 := x_{MPC}(k)$ and solve (2.3) with this initial value. Denote the resulting optimal control sequence by $u^*(\cdot)$ and set $F_{MPC}(x_{MPC}(k)) := u^*(0)$.

3) Apply $F_{MPC}(x_{MPC}(k))$, measure $x_{MPC}(k+1)$, set $k := k + 1$ and go to 2).

The resulting solution $x_{MPC}(t)$ is called the MPC closed-loop solution while the individual open-loop finite horizon optimal solutions computed by solving (2.3) in Step 2 are called the predictions.

Figure 7.1 shows a schematic sketch of the resulting solutions loop. Here the red solid line depicts the MPC closed-loop solution while the black dashed lines indicate the predictions. The figure shows the ideal case in which the closed-loop dynamics exactly coincides with the dynamics used to compute the predictions.

We note that by means of the dynamic programming principle (see, e.g., [23, Theorem 4.6]) the solution strategy “solve the optimal control problem and use the first element of the resulting optimal control sequence as feedback control value” would yield an optimal feedback law if we solved the infinite horizon problem (7.1) in Step 2. However, by resorting to the (numerically much more easy to obtain) solution of the finite horizon problem (2.3) in Step 2, we make an error that needs to be analyzed. MPC for stage costs considered in this paper, which do not merely penalize the distance of the state from a desired steady state, is often denoted as economic MPC, although the term general MPC would probably be more fitting. In any case, strict dissipativity of the optimal control problem plays an important role in the analysis of such MPC schemes.
Figure 7.1: Sketch of the solutions generated in the MPC loop. The red solid line depicts the MPC closed-loop solution while the black dashed lines indicate the predictions. The figure sketches the ideal case in which the closed-loop dynamics exactly coincides with the dynamics used to compute the predictions.