COUNTING LATTICE TRIANGULATIONS:
FREDHOLM EQUATIONS IN COMBINATORICS

S. Yu. Orevkov

Abstract. Let \( f(m, n) \) be the number of primitive lattice triangulations of \( m \times n \) rectangle. We compute the limits \( \lim_{n} f(m, n)^{1/n} \) for \( m = 2 \) and \( 3 \). For \( m = 2 \) we obtain the exact value of the limit which is equal to \((611 + \sqrt{73})/36\). For \( m = 3 \), we express the limit in terms of certain Fredholm’s integral equation on generating functions. This provides a polynomial time algorithm for computation of the limit with any given precision (polynomial with respect to the number of computed digits).

1. Introduction

A lattice triangulation of a (lattice) polygon in \( \mathbb{R}^2 \) is a triangulation with all vertices in \( \mathbb{Z}^2 \). As it was discovered in [3], lattice triangulations are important in algebraic geometry (see also [9]). A lattice triangulation is called primitive (or unimodular) if each triangle is primitive, i.e., has the minimal possible area 1/2. We denote the number of primitive lattice triangulations of the rectangle \( m \times n \) by \( f(m, n) \). Let

\[
\begin{align*}
  c(m, n) &= \frac{\log_2 f(m, n)}{mn}, \\
  c_m &= \sup_n c(m, n) = \lim_{n \to \infty} c(m, n), \\
  c &= \sup_m c_m = \lim_{m \to \infty} c_m = \sup_n c(n, n) = \lim_{n \to \infty} c(n, n).
\end{align*}
\]

The existence of the limits is proven in [4; Proposition 3.6]. The number \( c(m, n) \) is called in [4] the capacity of the rectangle \( m \times n \). In [8] I gave an upper bound \( c < 6 \) (which can be easily improved by the same arguments up to \( c < \log_2 27 = 4.755 \): it is enough just not to distinguish the cases \( v_j = 1 \) and \( v_j = 2 \) in the notation of [8]). Later on, a much better estimate \( c < 3 \) was obtained by Anclin [1] as well as \( c_m < 3 - 1/m \). A yet better upper bound \( c < 4 \log_2 \frac{1 + \sqrt{5}}{2} = \log_2 6.854 = 2.777 \) is obtained in [7] and announced in [12] (I have not seen the manuscript [7] but Professor Welzl kindly sent me the slides of his talk [13] where the proof of this bound is clearly exposed).

Easy to see that

\[
f(1, n) = \binom{2n}{n} \quad \text{whence} \quad c_1 = 2
\]

which yields a lower bound \( c > 2 \). It is also computed in [4] that \( c \geq c_4 \geq c(4, 32) = 2.055702 \). It is written in [4; §2.1]: “For \( f(2, n) \) we have no explicit formula, and we cannot evaluate the asymptotics precisely”. We still have no explicit formula for \( f(2, n) \) but we give here the principal term of the asymptotics:
Theorem 1. \( \lim_{n \to \infty} f(2,n)^{1/n} = \alpha \) where
\[
\alpha = \frac{611 + \sqrt{73}}{36}, \quad \text{hence} \quad c_2 = \frac{1}{2} \log_2 \alpha = 2.05256897
\]

An exact value of \( c_3 \), in a sense, is given in Proposition 4.5 where we express \( c_3 \) in terms of Fredholm’s integral equations on certain generating functions. In particular, Proposition 4.5 provides an algorithm to compute \( c_3 \) up to \( n \) digits in a polynomial time in \( n \). A Mathematica code implementing the main step of this algorithm is presented in Figure 7 below.

Theorem 2. \( \lim_{n \to \infty} f(3,n)^{\frac{1}{3n}}, \) up to 360 digits, is equal to
\[
4.239369481548025671877625742045235772100695711251795499830801
\]
\[
6878333582382767289878370548317633412767088855553395893005289
\]
\[
580195934799338289257489707990192054275721787374165246347114
\]
\[
46609624174115181432691478002150133793833581314241896953051
\]
\[
597942032082556780952912032761797534112146994900056374798271
\]
\[
988378451540168358202181556482461979420039542105330977266751
\]
and hence \( c_3 = 2.0838497... \)

We computed \( c_3 \) with this high precision hoping to find an algebraic equation for it, or to relate it with some known constants, but we did not succeed so far.

In §2.2 we present the results of computations of exact values of the numbers \( f(m,n) \) for some small \( m \) and \( n \). These computations show in particular that \( c \geq c(5,115) = 2.10449551... \)

In §6 we give an asymptotic upper bound for the number of all (not necessarily primitive) lattice triangulations. However it seems to be far from optimal.

2. Recurrent relations for strips of fixed width

2.1. Recurrent relations. Given a polygon \( P \subset \mathbb{R}^2 \), the upper part of its boundary is the set \( \{(x,y) \in P \mid y' > y \Rightarrow (x,y') \notin P\} \). A vertical side of \( P \) is a side of \( P \) contained in a line \( \{x = x_0\} \). Let \( T \) be a triangulation of a polygon \( P \) in \( \mathbb{R}^2 \). We say that \( Q \) is a tile of \( T \) in the following three cases:

(1) \( Q \) is a triangle of \( T \) without vertical sides;
(2) \( Q \) is a triangle of \( T \) whose vertical side lies on the boundary of \( P \);
(3) \( Q \) is a union of two triangles of \( T \) which share a common vertical side.

A polygon is called \( y \)-convex if its intersection with any line \( x = \text{const} \) is either the empty set, or a point, or a segment.

Lemma 2.1. Let \( T \) be a triangulation of a \( y \)-convex polygon \( P \) in \( \mathbb{R}^2 \). Then there exists a tile \( Q \) of \( T \) such that the upper part of the boundary of \( Q \) is contained in the upper part of the boundary of \( P \).

Proof. Let \( \Gamma_P \) be the upper part of the boundary of \( P \). Let \( Q_1,\ldots,Q_n \) be all the tiles of \( T \) which have at least one side lying on \( \Gamma_P \). Let \( \Gamma_i \) be the union of the sides of \( Q_i \) lying on \( \Gamma_P \). It is clear that each \( \Gamma_i \) is either a side of \( Q_i \) or a union of two sides with a common vertex. It is also clear that the projections of the \( \Gamma_i \) onto
the $x$-axis have pairwise disjoint interiors, hence we may assume that $\Gamma_1, \ldots, \Gamma_n$ are numbered from the left to the right. We say that a tile $Q_i$ is shadowed on the left (resp. shadowed on the right) if the upper part of the boundary of $Q_i$ contains a segment $I$ such that $I \not\subset \Gamma_P$ and $I$ is on the left (resp. on the right) of $\Gamma_i$; see Figure 1. It is clear that none of the tiles $Q_1, \ldots, Q_n$ can be shadowed on the left and on the right simultaneously. Hence, without lost of generality we may assume that at least one of these tiles is not shadowed on the right. Let $i_0$ be the minimal number such that $Q_{i_0}$ is not shadowed on the right. Then $Q_{i_0}$ is the desired tile with the upper part contained in $\Gamma_P$. Indeed, it is not shadowed on the right by its definition. It cannot be shadowed on the left neither because otherwise $Q_{i_0-1}$ would not be shadowed on the right which contradicts the minimality of $i_0$. $\square$

![Figure 1](image)

Now we fix an integer $m > 0$ and we consider primitive lattice triangulations of polygons contained in the vertical strip $\{0 \leq x \leq m\}$ bounded by two graphs of continuous piecewise linear functions.

By analogy with the terminology introduced in [4, §2.2], we say that $\varphi : [0, m] \to \mathbb{R}$ is an admissible function if it is a continuous piecewise linear function whose graph is a union of segments with endpoints at $\mathbb{Z}^2$. Let us fix an admissible function $\varphi_0$ and say that a function $\varphi : [0, m] \to \mathbb{R}$ is $\varphi_0$-admissible if it is admissible and $\varphi(x) \geq \varphi_0(x)$ for any $x \in [0, m]$. A $\varphi_0$-admissible shape is a polygon $S$ of the form $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq m, \varphi_0(x) \leq y \leq \varphi(x)\}$ for some $\varphi_0$-admissible function $\varphi$.

As in the above definition of a tile of a triangulation, we say that $Q$ is a primitive lattice tile in the following three cases:

1. $Q$ is a primitive lattice triangle without vertical sides;
2. $Q$ is a primitive lattice triangle whose vertical side is contained in the boundary of the strip $0 \leq x \leq m$;
3. $Q = \Delta_1 \cup \Delta_2$ where $\Delta_1$ and $\Delta_2$ are primitive lattice triangles such that $\Delta_1 \cap \Delta_2$ is a common vertical side of $\Delta_1$ and $\Delta_2$.

A primitive lattice tile $Q$ is $P$-maximal for a polygon $P$ if $Q \subset P$ and the upper part of the boundary of $Q$ is contained in the upper part of the boundary of $P$. We say that $S'$ is a $\varphi_0$-admissible subshape of a $\varphi_0$-admissible shape $S$, if $S'$ is the closure of $S \setminus (Q_1 \cup \cdots \cup Q_n)$, where $Q_1, \ldots, Q_n$ are $S$-maximal primitive lattice tiles with pairwise disjoint interiors. Following [4], in this case we set $\#(S', S) = n$.

Let us denote the number of primitive lattice triangulations of a polygon $P$ by $f^*(P)$. When $P$ sits in the strip $\{0 \leq x \leq m\}$, we also define $f(P)$ as the number of primitive lattice triangulations of $P$ which do not have any interior edge whose projection onto the $x$-axis is the whole segment $[0, m]$ (we choose a simpler notation for a more complicated notion because the numbers $f(P)$ will be used more often than $f^*(P)$).
The following lemma is the inclusion-exclusion formula in our setting. The proof is the same as for [4, Lemma 2.2].

**Lemma 2.2.** For any \( \varphi_0 \)-admissible shape \( S \), we have
\[
f^*(S) = \sum_{S'} (-1)^{\#(S',S)-1} f^*(S'), \quad \text{and} \quad f(S) = \sum_{S'} (-1)^{\#(S',S)-1} f(S'),
\]
where the left sum is taken over all proper \( \varphi_0 \)-admissible subshapes of \( S \), and the right sum is taken over those proper \( \varphi_0 \)-admissible subshapes of \( S \) whose upper part of the boundary contains a point from \( \mathbb{Z}^2 \cap \{0 < x < m\} \).

**Example 2.3.** Let \( m = 2 \) and \( \varphi_0 = 0 \). For non-negative integers \( a, b, c \), let \( S_{a,b,c} \) be the \( \varphi_0 \)-admissible shape bounded from above by the segment \( [(0, a), (1, b)] \) and \( [(1, b), (2, c)] \). Let \( f_{a,b,c} = f(S_{a,b,c}) \). We set also \( f_{a,b,c} = 0 \) when \( \min(a, b, c) < 0 \). Then (see Figure 2) the recurrent formula of Lemma 2.2 reads
\[
f_{a,b,c} = \begin{cases} f_{a-1,b,c} + f_{a,b-1,c} + f_{a,b,c-1} - f_{a-1,b,c-1} & \text{if } (a, b, c) \neq (0, 0, 0), \\ 1 & \text{if } (a, b, c) = (0, 0, 0). \end{cases}
\]
Let \( F(x, y, z) = \sum_{a,b,c} f_{a,b,c} x^a y^b z^c \) be the generating function. Then, by summing the recurrent relation over all triples \((a, b, c) \neq (0, 0, 0)\), we obtain
\[
F(x, y, z) - 1 = \sum f_{a-1,b,c} x^a y^b z^c + \sum f_{a,b-1,c} x^a y^b z^c + \ldots
\]
\[
= \sum f_{a,b,c} x^{a+1} y^b z^c + \sum f_{a,b,c} x^a y^{b+1} z^c + \ldots
\]
\[
= F(x, y, z)(x + y + z - xz)
\]
whence \( F(x, y, z) = 1/(1 - x - y - z + xz) \).

![Figure 2](image-url)

**Example 2.4.** Let \( \varphi_0 \) and \( S_{a,b,c} \) be as in Example 2.3. For non-negative \( a, c \) such that \( a \equiv c + 1 \mod 2 \), we define \( S'(a, c) \) as the \( \varphi_0 \)-admissible shape bounded from above by the segment \( [(0, a), (2, c)] \). Let \( f^*_{a,b,c} = f^*(S_{a,b,c}) \) and \( g^*(a, c) = f^*(S'_{a,c}) \). We set also \( f^*_{a,b,c} = 0 \) when \( \min(a, b, c) < 0 \) and \( g^*(a, c) = 0 \) when \( \min(a, c) < 0 \) or \( a \equiv c \mod 2 \). Then, for \((a, b, c) \neq (0, 0, 0)\), the recurrent formula of Lemma 2.2 applied to \( S_{a,b,c} \) reads
\[
f^*_{a,b,c} = f^*_{a-1,b,c} + f^*_{a,b-1,c} + f^*_{a,b,c-1} - f^*_{a-1,b,c-1} + \chi_{a,b,c} g^*_{a,c}
\]
where \( \chi_{a,b,c} = 1 \) if \( 2b + 1 = a + c \), and \( \chi_{a,b,c} = 0 \) otherwise. Let \( F^*(x, y, z) \) and \( G^*(x, z) \) be the respective generating functions. Then (cf. Example 2.3) we have
\[
F^*(x, y, z) - 1 = F^*(x, y, z)(x + y + z - xz) + \sum \chi_{a,b,c} g^*_{a,c} x^a y^b z^c
\]
and the last sum is equal to
\[
\sum_{a,c} g_{a,c}^* x^a y^{(a+c-1)/2} z^c = y^{-1/2} \sum_{a,c} g_{a,c}^* (xy^{1/2})^a (y^{1/2} z)^c = y^{-1/2} G^*(xy^{1/2}, y^{1/2} z)
\]
which gives us the relation
\[
F^*(x, y, z)(1 - x - y - z + xz) = 1 + y^{-1/2} G^*(xy^{1/2}, y^{1/2} z).
\]

Now let us apply the recurrent relation to \(S'_a,c\). The only admissible subshape of \(S'(a, c)\) is \(S(a, (a+c-1)/2, c)\), hence the relation for \(S'_a,c\) reads \(g_{a,c}^* = f_{a,(a+c-1)/2,c}^*\).

In terms of the generating functions this means that
\[
G^*(x, z) = \sum_{a,c} f_{a,(a+c-1)/2,c}^* x^a z^c = \text{coef}_{u^0} \left[ \sum_{a,b,c} f_{a,b,c}^* u^a (u^2)^b (z/u)^c \right] = \text{coef}_{u^0} (uF^*(x/u, u^2, z/u)).
\]

### 2.2. Some exact values of \(f(m, n)\).

The recurrent relations in Lemma 2.2 provide an algorithm of computation of exact values of \(f(m, n)\) for small \(m\) and \(n\). The algorithm is similar to the one described in [4, §2.2]. We performed computations using this algorithm and one can see in Table 1 that we advanced much further with respect to the computations in [4]. There are three reasons for this which have more or less equal impact.

| Table 1 |
| Capabilities computed in [4] | Capabilities computed in this paper |
|-----------------|-----------------|
| \(c_1 = 2.0000\) | \(c_1 = 2.0000\) |
| \(c_{4,32} = 2.0557\) | \(c_{4,200} = 2.0946\) |
| \(c_{2,375} = 2.0441\) | \(c_{5,12} = 2.0175\) |
| \(c_{5,115} = 2.1045\) | \(c_{8,13} = 2.0669\) |
| \(c_{3,60} = 2.0275\) | \(c_{6,7} = 1.9841\) |
| \(c_{6,50} = 2.1024\) | \(c_{9,9} = 2.0490\) |
| \(c_{3,600} = 2.07966\) | \(c_{9,12} = 2.0490\) |

The first reason (an evident one) is that the computers became more powerful. The second reason is that we used another definition of admissible shapes which allowed us to divide the amount of used memory by \(3^{m-1}\) which is rather important when \(m = 9\) (as it is pointed out in [4], for this kind of algorithms, “the bottleneck in the computations is always memory”). The third reason is that instead of long arithmetics, we used computations mod different primes and then recovered the results with the Chinese Remainder Theorem. This trick allowed us to “convert” memory to time whose lack was not so crucial.

We have computed \(f(3, n)\) till \(n = 600\) and \(f(4, n)\) till \(n = 200\). The exact value of \(f(3, 600)\) has 1127 digits and it yields \(c_{3,600} = 2.07966\ldots\) Comparing this with the limit value \(c_3 = 2.08385\) we see that the convergence is very slow. For \(m = 4\),
the last computed exact value is

\[
f(4,200) = 26219933430396507314052141167072596609151907003573304927487
419128543906730659218480439253346584137204205604500628092962
69799742609554540340483027164194339979807927812812142668569
09756020384393594728621308903256950859658838687531965864231
570521446370439565640979852878302993978768696718322811686043
3077495106765406132102076783816460247478162699981105797912
38534626596601164596410043968216134349971638142523003353406
530183843913302635663917084864069175263416748948835535483336
471730901812541550646500; \quad c(4,200) = 2.09455... 
\]

In Tables 2–6 we present some other results of computations in the same format as in [4]. All the computed exact values are available on the webpage https://www.math.univ-toulouse.fr/~orevkov/tr.html

| \(n\) | \# primitive triangulations of rectangle \(5 \times n\) | \(c(5,n)\) |
|---|---|---|
| 1 | 252 | 1.5954 |
| 2 | 182132 | 1.7474 |
| 3 | 182881520 | 1.8297 |
| 4 | 208902766788 | 1.8802 |
| 5 | 26042054814496 | 1.9155 |
| 6 | 34181648962552032 | 1.9415 |
| 7 | 464476385680935656240 | 1.9615 |
| 8 | 645855159466371391947660 | 1.9773 |
| 9 | 913036902513499041820702784 | 1.9902 |
| 10 | 1306520849733616781789190513820 | 2.0008 |
| 11 | 18875911658916512539043093432371172 | 2.0098 |
| 12 | 2747848427721241461905176361078147168 | 2.0174 |
| 13 | 4024758386310801427793602374466243714608 | 2.0240 |
| 14 | 5924744736041718687622958191829471010847132 | 2.0298 |
| 15 | 8757956199571261116690226598764501142088496860 | 2.0348 |
| 16 | 12991215957916577635251095613859465176216530106080 | 2.0394 |
| 17 | 19327902156972014645215931908930612218954616366464668 | 2.0434 |
| 18 | 28828843648796117963238681180919362090157971920576213992 | 2.0470 |
| \vdots | \vdots | \vdots |
| 115 | 18700706608364882730712710491937598381242505216572196 | \vdots |
| 116 | 74626658766824095096227084981348969054292582022965697 | \vdots |
| 117 | 9753620934745513435765184618673616197344892595460029612 | \vdots |
| 118 | 596693103995831984101084647892901181818014289189323068 | \vdots |
| 119 | 3143599559630624502282111212186222320544399050742600358 | \vdots |
| 120 | 3142647588605075767408153732325783413307209633451618 | \vdots |
| 121 | 73035677107305109076541667755690839416820326596 | 2.1044 |
### Table 3

| $n$ | # primitive triangulations of rectangle $6 \times n$ | $c(6,n)$ |
|-----|---------------------------------|---------|
| 1   | 924                             | 1.6419  |
| 2   | 2801708                         | 1.7848  |
| 3   | 1224484472                      | 1.8617  |
| 4   | 61756221742966                  | 1.9088  |
| 5   | 34181648625522032               | 1.9415  |
| 6   | 1999206934751133055518          | 1.9655  |
| 7   | 1216940954414198870186052       | 1.9840  |
| 8   | 760833633294751365554918994     | 1.9987  |
| 9   | 48477251216726668498399632918196| 2.0107  |
| 10  | 3131521959869770128138491287826065904| 2.0206 |
| 11  | 20443767611927599823217291769468449848548| 2.0289 |
| 12  | 13455855036840096364589064704536849131736024| 2.0360 |
| 13  | 891513898740246853038326950483812868791208442016| 2.0421 |
| 14  | 5938780824869668513059568892370775952933721743377354| 2.0474 |
| 15  | 39738456660509411434285642370153959115525603844258515860| 2.0521 |
|     |                                 |         |
| 50  | 733088849377871573475229677373109896289395791929 |          |
|     | 2888922927798932074203013116473882328714681504398 |          |
|     | 803902969400882970235141773360945092837017232937 |          |
|     | 1864995986534063127990363531908201551410584718 | 2.1023 |

### Table 4

| $n$ | # primitive triangulations of rectangle $7 \times n$ | $c(7,n)$ |
|-----|---------------------------------------------------|---------|
| 1   | 3432                                              | 1.6778  |
| 2   | 43936824                                          | 1.8134  |
| 3   | 839660660268                                       | 1.8862  |
| 4   | 18792896208387012                                  | 1.9307  |
| 5   | 464476385680935656240                              | 1.9615  |
| 6   | 1216940954414198870186052                          | 1.9840  |
| 7   | 332633840844113103751597995920                     | 2.0014  |
| 8   | 9369363517501208819530429967280708                 | 2.0152  |
| 9   | 269621109753732518252493257828413137272           | 2.0264  |
| 10  | 7880009979020501614060394747170100093057300        | 2.0357  |
| 11  | 233031642883906149386619647304562977586311372556   | 2.0435  |
| 12  | 6953609830304518024125545674642770582274167760568260| 2.0501  |
| 13  | 2089809948331032668557716536086203015983854051275967612| 2.0559 |
|     |                                                   |         |
| 20  | 52066212145180734892042606757684021681422119       | 2.0813  |

2.3. Convexity conjecture for the numbers $f(m,n)$. The following conjecture is confirmed by all the computed exact values of the numbers $f(m,n)$ (we set by convention $f(m,0) = 1$).

**Conjecture 2.5.** One has $f(m, n - 1)f(m, n + 1) \geq f(m, n)^2$ for any $m, n \geq 1$. 
Table 5

| n  | # primitive triangulations of rectangle $8 \times n$ | $c(8, b)$ |
|----|--------------------------------------------------|----------|
| 1  | 12870                                            | 1.7064   |
| 2  | 698607816                                       | 1.8362   |
| 3  | 58591381296256                                   | 1.9056   |
| 4  | 5831528022482629710                              | 1.9480   |
| 5  | 9369363517501208819530429967280708               | 2.0152   |
| 6  | 11910648128826855397857137454009344308           | 2.0282   |
| 7  | 155023302820254133629368881178138076738462112    | 2.0388   |
| 8  | 2052733723876903231579633200716710298474517344046 | 2.0476   |
| 9  | 2753810232976351788081274786378733309236298426977203848 | 2.0550   |
| 10 | 373119178357778061717948090998001460229206030805799398500854 | 2.0613   |
| 11 | 5095132675357773696400958035190445392087069512323700346738258636 | 2.0668   |

Table 6

| n  | # primitive triangulations of rectangle $9 \times n$ | $c(9, n)$ |
|----|--------------------------------------------------|----------|
| 1  | 48620                                            | 1.7299   |
| 2  | 11224598424                                     | 1.8547   |
| 3  | 4140106747178292                                 | 1.9214   |
| 4  | 183593338491453312                               | 1.9621   |
| 5  | 913036902513499041820702784                     | 1.9902   |
| 6  | 48477251216726688498399632918196                 | 2.0107   |
| 7  | 26962110975373518252493257828413137272           | 2.0264   |
| 8  | 155023302820254133629368881178138076738462112    | 2.0388   |
| 9  | 913765124094622356941511119897052344522006298310908 | 2.0489   |

Proposition 2.6. If Conjecture 2.5 holds true, then $c_m \geq (n + 1)c(m, n + 1) - nc(m, n)$ for any $m, n \geq 1$. In particular, Conjecture 2.5 would imply that $c \geq c_{115} \geq 5c(115, 5) - 4c(115, 4) = 2.1684837\ldots$

Proof. Let us set $d(m, n) = \log_2 f(m, n + 1) - \log_2 f(m, n)$. Then Conjecture 2.5 implies $d(m, n) \leq d(m, n + 1) \leq d(m, n + 2) \leq \ldots$ whence $\log_2 f(m, n + k) - \log_2 f(m, n) \geq kd(m, n)$. Dividing by $km$ and passing to the limit when $k \to \infty$, we obtain $c_m \geq d(m, n)/m = (n + 1)c(m, n + 1) - nc(m, n)$. □

3. The exact value of $c_2$ (proof of Theorem 1)

For $a, c \geq 0$, $a \equiv c \mod 2$, let $g_{a,c}^*$ be the number of primitive lattice triangulations of the trapezoid $T(a, c)$ spanned by $(0, 0), (a, 0), (1, 2), (1 + c, 2)$ (if $a = 0$ or $c = 0$, then $T(a, c)$ degenerates to a triangle). When $a \not\equiv c \mod 2$, we set $g_{a,c}^* = 0$. We also set $g_{0,0}^* = 1$. Let $G^*(x, z)$ be the generating function for $g_{a,c}^*$:

$$G^*(x, z) = \sum_{a, c \geq 0} g_{a,c}^* x^a z^c$$

$$= 1 + (x^2 + xz + z^2) + (6x^4 + 10x^3z + 12x^2z^2 + 10xz^3 + 6z^4) + \ldots$$
Let $g_n^*$ be the coefficient of $x^{2n}$ in the series $G^*(x,y) = \sum_{n \geq 0} g_n^* x^{2n}$, i.e.

$$g_n^* = g_{0,2n} + g_{1,2n-1} + g_{2,2n-2} + \cdots + g_{2n,0}.$$  

Then Theorem 1 follows immediately from Lemmas 3.1 and 3.2 below.

**Lemma 3.1.** $\lim_{n \to \infty} f(2, n)^{1/n} = \lim_{n \to \infty} (g_n^*)^{1/n}$.

**Proof.** The rectangle $2 \times (n - 1)$ can be placed into $T(n,n)$, hence $f(2, n-1) < g_n^*$. On the other hand, the union of $T(2,2n-1)$ with its image under the central symmetry with center $(1/2, 1)$ is $T(2, a+c, a+c)$, and it can be placed into the rectangle $2 \times (a + c + 1)$, hence $(g_n^*)^2 < f(2, a + c + 1)$. Therefore

$$g_n^* \geq \frac{1}{n+1} \max_{a+c=2n} g_{a,c}^2 \leq f(2, 2n+1)^{1/2} \leq (g_{2n+2}^*)^{1/2},$$

whence $\frac{1}{n} \log g_n^* - \log(2n) \leq \frac{1}{2n} f(2, 2n+1) \leq \frac{1}{2n} g_{2n+2}^*$ and the result follows because $\frac{1}{n} \log(2n) \to 0$. \qed

**Lemma 3.2.** $\lim_{n \to \infty} (g_n^*)^{1/n} = \alpha$ where $\alpha$ is as in Theorem 1.

**Proof.** For $a, c \geq 0$, $a \equiv c \mod 2$, let $g_{a,c}$ be the number of those primitive lattice triangulations of the trapezoid $T(a,c)$ which do not contain interior edges of the form $[(k,0),(l,2)]$, in other words, primitive lattice triangulations which agree with the subdivision of $T(a,c)$ into two triangles and two trapezoids depicted in Figure 3(left). If $a + c$ is odd, we set $g_{a,c} = 0$. By convention, we set $g_{0,0} = 0$. Let $G(x,z) = \sum_{a,c \geq 0} g_{a,c} x^a z^c$ be the generating function.

\[ g_{a,c}^* = \sum_{a_1 + \cdots + a_k = a \atop c_1 + \cdots + c_k = c} \prod_{j=1}^k g_{a_j,c_j} \]  

thus $G^*(x,z) = \frac{1}{1 - G(x,z)}$. \hspace{1cm} (2)

Easy to see (cf. (1)) that the number of primitive lattice triangulations of the narrow (i.e. of width 1) trapezoids in Figure 3 are binomial coefficients, hence

$G(x,z) = (x^2 + xz + z^2) + (5x^4 + 8x^3z + 9x^2z^2 + 8xz^3 + 5z^4) + \ldots.$

One can also check that $g_{a,c} = f_{a,(a+c)/2-1,c}$ where $f_{a,b,c} = f(S_{a,b,c})$ are the numbers discussed in Example 2.3. Hence (cf. Example 2.4)

$$G(x,z) = \sum_{a,c} f_{a,(a+c)/2-1,c} x^a z^c = \cof_{u^0} \left[ \sum_{a,b,c} f_{a,b,c} x^a u^{2b-a-c+2} z^c \right]$$

$$= \cof_{u^0} \left[ u^2 \sum_{a,b,c} f_{a,b,c} (x/u)^a u^{2b} (z/u)^c \right] = \cof_{u^{-1}} \left[ u F(x/u, u^2, z/u) \right].$$
Since the function \(1/(1 - x - y - z + xz) = 1/((1 - x)(1 - z) - y)\) is analytic in the domain \([x, |y|, |z|] < 1/2\), its power series \(\sum f_{a,b,c} x^a y^b z^c\) (see Example 2.3) converges to it in this domain. Therefore, for \(0 < \varepsilon \ll r < 1/2\), the Laurent series of \(F(x/u, u^2, z/u)\) converges in the domain \([x, |z|] < \varepsilon, r - \varepsilon < |u| < r + \varepsilon\).

Hence, for \(x\) small enough, we have
\[
G(x, x) = \text{coef}_{u^{-1}} [F(x/u, u^2, x/u)] = \frac{1}{2\pi i} \oint_{|u|=r} \frac{u \, du}{(1 - x/u)^2 - u^2}
\]
and
\[
\frac{u}{(1 - x/u)^2 - u^2} = -\frac{u}{2(u^2 + u - x)} - \frac{u}{2(u^2 - u + x)} = \sum_{j=1}^{2} \frac{1}{2(u_j^+ - u_j^-)} \left( \frac{u_j^+}{u - u_j^+} + \frac{u_j^-}{u - u_j^-} \right),
\]
where, for \(|x|\) small enough,
\[
u_1^+ = -\frac{1}{x}(1 + \sqrt{1 + 4x}), \quad u_2^+ = \frac{1}{x}(1 + \sqrt{1 - 4x}); \quad |u_j^+| > r, |u_j^-| < r.
\]

Thus
\[
G(x, x) = \sum_{j=1}^{2} \text{Res}_{u=u_j} \left( \ldots \right) = \sum_{j=1}^{2} \frac{u_j^-}{2(u_j^+ - u_j^-)} = \frac{1}{4\sqrt{1 - 4x}} + \frac{1}{4\sqrt{1 + 4x}} - \frac{1}{2}.
\]

The graph of the function \(y = G(x, x)\) sits in the algebraic curve
\[
(2y + 1)^2(16x^2 - 1)(4x^2 + (y^2 + y)(16x^2 - 1)) + x^2 = 0.
\]

By (2), the poles of \(G^*(x, x)\) are the \(x\)-coordinates of the intersections of this curve with the line \(y = 1\), i.e., the roots of \(5184x^4 - 611x^2 + 18\) (the smallest ones being \(\pm \sqrt{1/\alpha}\)), and the branching points are \(\pm 1/4\). Hence the radius of convergence of the series \(G^*(x, x) = \sum g_n^* x^{2n}\) is \(\sqrt{1/\alpha}\) whence \(\lim_{n \to \infty} (g_n^*)^{1/n} = \alpha\). \(\square\)

4. Computation of \(c_3\) (proof of Theorem 2)

4.1. Preparation. For \(a, d \geq 0\) such that \(a \not\equiv d + 1 \mod 3\), let \(h_{a,d}^*\) be the number of primitive lattice triangulations of the trapezoid \(T_3(a, d)\) spanned by \((0, 0), (1, 3), (1 + d, 3), (a, 3)\). We set \(h_{0,0}^* = 1\) and \(h_{a,d}^* = 0\) when \(a \equiv d + 1 \mod 3\), and we consider the generating function
\[
H^*(x) = \sum_n h_n^* x^n = \sum_{a,d \geq 0} h_{a,d}^* x^{a+d} = 1 + x + 3x^2 + 19x^3 + 125x^4 + \ldots
\]

Similarly to the beginning of proof of Lemma 3.2, we define \(h_{a,d}\) as the number of the triangulations of \(T_3(a, d)\) which do not have edges of the form \([(k, 0), (l, 3)]\) and we consider the generating function
\[
H(x) = \sum_n h_n x^n = \sum_{a,b \geq 0} h_{a,d} x^{a+d} = x + 2x^2 + 14x^3 + 86x^4 + 712x^5 + \ldots
\]
These functions satisfy the relation similar to (2) specialized for $x = z$:

$$H^*(x) = 1/(1 - H(x))$$

Indeed, the edges of the form $[(k, 0), (l, 3)]$ cut $T_3(a, d)$ into smaller trapezoids. Each of them can be mapped to a standard one by a unique lattice automorphism of the form $(x, y) \mapsto (x + py + q, y)$ with $p, q \in \mathbb{Z}$ (in contrary to §2, here the upper and lower horizontal sides of the trapezoids are mixed, so we do not have (2) for two-variable generating functions). In Figure 4 we illustrate the relation

$$h_3^* = h_{03}^* + h_{12}^* + h_{30}^* = h_{01}^3 + 2h_{01}(h_{11} + h_{20}) + (h_{03} + h_{12} + h_{30}) = h_1^3 + 2h_1h_2 + h_3.$$

![Figure 4](image)

Similarly to Lemma 3.1, we have $\lim_n f(3, n)^{1/n} = \lim_n (h_{2n}^*)^{1/n} = 1/\beta^2$ where $\beta$ is the real positive root of the equation $G(x) = 1$, hence $c_3 = -\frac{2}{3} \log \beta$.

4.2. Recurrent relations. Using the notation introduced in §2, let us set $m = 3$, $\varphi_0(x) = \frac{3}{2}x - 1$, and

$$F(x, y, z, w) = \sum_{a,b,c,d} f_{a,b,c,d} x^a y^b z^c w^d,$$

$$G_1(x, z, w) = \sum_{a,c,d} g_{a,c,d}^{(1)} x^a z^c w^d, \quad G_2(x, y, w) = \sum_{a,b,d} g_{a,b,d}^{(2)} x^a y^b w^d,$$

$$H_k(x, w) = \sum_{a,d} g_{a,d}^{(k)} x^a w^d, \quad (k = 1, 2)$$

where all the coefficients are of the form $f(S)$ (see §2) for the $\varphi_0$-admissible shapes in Figure 5 where $(0, a), (1, b), (2, c)$, and $(3, d)$ (if present) are the coordinates of integral points on the upper part of the boundary of $S$. The lower corners of $S$ are at the points $(0, -1)$ and $(3, 0)$. If the congruences given in Figure 5 are not satisfied, then the corresponding numbers are zero. If $\min(a + 1, b, c, d) < 0$, they are also zero (this case does not correspond to any $\varphi_0$-admissible shape). By convention, we also set $h_{-1,0}^{(2)} = 0$ (the case when $S$ degenerates to a segment).

![Figure 5](image)
In terms of the generating functions, the recurrent relations in Lemma 2.2 read (cf. Examples 2.3, 2.4):

\[
F(x, y, z, w)(1 - x - y - z - w + xz + yw + xw) = y^{1/2}(1 - w)G_1(xy^{1/2}, y^{1/2}z, w) + z^{1/2}(1 - x)G_2(x, yz^{1/2}, z^{1/2}w),
\]

\[
G_1(x, z, w)(1 - w) = \text{coef}_{u^{-1}}[F(x/u, u^2, z/u, w)(1 - w)] + x^{-1},
\]

\[
G_2(x, y, w)(1 - x) = \text{coef}_{u^{-1}}[F(x, y/u^2, w/u)(1 - x)]
\]

(the asymmetry between \(G_1\) and \(G_2\) is caused by the asymmetry of \(\varphi_0\)),

\[
H_1(x, w) = \text{coef}_{u^{-1}}[G_1(x/u, u^3, w/u^2)],
\]

\[
H_2(x, w) = \text{coef}_{u^{-1}}[G_2(x/u^2, u^3, w/u)].
\]

Notice that in this subsection, by generating functions we mean formal series. Let us consider the symmetrized generating functions

\[
\tilde{F}(x, y, z, w) = F(x, y, z, w) + F(w, z, y, x),
\]

\[
\tilde{G}(x, z, w) = G_1(x, z, w) + G_2(w, z, x),
\]

\[
\tilde{H}(x, w) = H_1(x, w) + H_2(w, x).
\]

The above relations for \(F, G_1, G_2, H_1, H_2\) imply immediately:

\[
\tilde{F}(x, y, z, w)(1 - x - y - z - w + xz + yw + xw) = y^{1/2}(1 - w)\tilde{G}(xy^{1/2}, y^{1/2}z, w) + z^{1/2}(1 - x)\tilde{G}(x, yz^{1/2}, z^{1/2}w),
\]

\[
\tilde{G}(x, z, w)(1 - w) = \text{coef}_{u^{-1}}[\tilde{F}(x/u, u^2, z/u, w)(1 - w)] + x^{-1}
\]

\[
\tilde{H}(x, w) = \text{coef}_{u^{-1}}[\tilde{G}(x/u, u^3, w/u^2)].
\]

4.3. The equation. We are going to obtain an equation for \(\tilde{G}(xt^{-1/2}, t^{3/2}, t^{-1}x)\) by expressing \(\tilde{F}\) via \(\tilde{G}\) from (3) and plugging it into (4). To this end we need to divide power series by polynomials. However, when some variables appear with powers varying from \(-\infty\) to \(+\infty\), the meaning of such division should be precised. To illustrate a possible ambiguity, let us consider the expression \(\text{coef}_{u^{-1}}[1/(x-uy)]\).

It can be understood either as

\[
\text{coef}_{u^{-1}}\left[\frac{x^{-1}}{1-uyx^{-1}}\right] = \frac{1}{x}\text{coef}_{u^{-1}}\left[1 + \frac{uy}{x} + \frac{u^2y^2}{x^2} + \ldots\right] = 0
\]

or as

\[
\text{coef}_{u^{-1}}\left[-\frac{(uy)^{-1}}{1-x(uy^{-1})}\right] = -\text{coef}_{u^{-1}}\left[\frac{1}{uy}\left(1 + \frac{x}{uy} + \frac{x^2}{u^2y^2} + \ldots\right)\right] = -\frac{1}{y}.
\]

To avoid this kind of ambiguity, we introduce a new formal variable \(q\) and consider the formal series

\[
F_q(x, y, z, w) = F(qx, yq^2, zq^2, wq),
\]

\[
G_{1,q}(x, z, w) = G_1(qx^2, zq^3, wq),
\]

\[
G_{2,q}(x, y, w) = G_2(qx, yq^3, wq^2),
\]

\[
H_{k,q}(x, w) = H_k(qx^3, wq^3), \quad k = 1, 2,
\]
and all the generating functions will be treated as elements of the ring
\[ \mathbb{Z}[x^{\pm 1}, y^{\pm 1/2}, z^{\pm 1/2}, w^{\pm 1}, u^{\pm 1/2}, t^{\pm 1/2}][(q)] \]
of formal power series in \( q \) (starting, maybe, with a negative power) whose coefficients are Laurent polynomials in \( x, y^{1/2}, \ldots \).

The geometric meaning of an exponent of \( q \) is twice the doubled signed area of the \( \varphi_0 \)-admissible shape corresponding a monomial, i.e., \( 2 \int_0^x \varphi(x) \, dx \) where the graph of \( \varphi \) is the upper boundary of the shape. One can easily check by hand that

\[
\begin{align*}
F_q &= (xq)^{-1} + (1 + x^{-1}w) + (x + w + x^{-1}y + x^{-1}z + x^{-1}w^2)q + \ldots \\
G_{1,q} &= x^{-1}q^{-2} + w(xq)^{-1} + w^2x^{-1} + w^3x^{-1}q + (x + w^4x^{-1})q^2 + \ldots \\
G_{2,q} &= x^{-1}wq + (w + yx^{-1})q^2 + (wx + 2y)q^3 + (wx^2 + 4xy)q^4 + \ldots \\
H_{1,q} &= wx^{-1} + xq^3 + 4w^2q^6 + (30wx^2 + 24w^4x^{-1})q^9 + \ldots \\
H_{2,q} &= wq^3 + 5(x^2 + w^3x^{-1})q^6 + 32w^2xq^9 + \ldots
\end{align*}
\]

Further, we define \( \tilde{F}_q, \tilde{G}_q, \tilde{H}_q \) by the same formulas as in §4.2 but with the subscript \( q \) everywhere. For example,

\[
\tilde{G}_q(x, z, w) = \frac{1}{xq^2} + \frac{w}{xq} + \frac{w^2}{x} + \left( \frac{w^3}{x} + \frac{x}{w} \right)q + \left( 2x + \frac{w^4}{x} + \frac{z}{w} \right)q^2 + \ldots
\]

Then the relations (3)–(5) take the form

\[
\tilde{F}_q = \frac{qy^{1/2}(1 - wq)}{1 - xq - yq^2 - zq^2 - wq + xzq^3 + ywq^3 + xwq^2} \tilde{G}_q(x, y^{1/2}, z^{1/2}, w) + qz^{1/2}(1 - xq)\tilde{G}_q(wz^{1/2}, z^{1/2}y, x),
\]

\[
\tilde{G}_q(x, z, w) = \text{coef}_{u^{-1}}[q\tilde{F}_q(x/u, u^2, z/u, w)] + \frac{1}{x(1 - wq)q^2},
\]

\[
\tilde{H}_q(x, w) = \text{coef}_{u^{-1}}[q\tilde{G}_q(x/u, u^3, w/u^2)].
\]

Let us set

\[
\begin{align*}
g_q(x, t) &= t^{1/2}x^2q^2\tilde{G}_q(x^2t^{-1/2}, x^3t^{3/2}, xt^{-1}) \\
&= t + xq + t^{-1}x^2q^2 + (t^{-2} + t)x^3q^3 + (t^{-3} + 2 + t^3)x^4q^4 + \ldots
\end{align*}
\]

The parity condition on the indices of nonzero coefficients of \( G_1 \) and \( G_2 \) (see Figure 5) ensures that the series \( g_q(x, t) \) does not have fractional powers. Moreover, \( x \) and \( q \) appear in each monomial of \( g_q \) with the same power, thus we have \( g_q(x, t) = g(qx, t) \) with \( g(x, t) \in \mathbb{Z}[t^{\pm 1}][(x)] \).

By plugging (3) into (4), denoting the denominator in (6) by \( Q_q(x, y, z, w) \), and observing that

\[
\text{coef}_{u^{-1}}[\mathcal{F}(x, t, u)] = \text{coef}_{u^{-1}}[xt^{-1/2}\mathcal{F}(x, t, uxt^{-1/2})]
\]

(9)
for any formal Laurent series in $u$, we obtain

\[ g_q(x, t) = \text{coef}_{u^{-1}} \left[ t^{1/2} x^2 q^3 \tilde{F}_q \left( \frac{x^2}{u t^{1/2}}, u, x^3 t^{3/2}, x, t \right) \right] + \frac{t^2}{t - x q} \]

\[ = x^2 q^2 \text{coef}_{u^{-1}} \left[ \left( \frac{1}{u} \left( 1 - \frac{x}{t} \right) g_q(x, t) + \frac{1}{u} \left( 1 - \frac{x}{t} \right) g_q(x, u) \right) Q(x/u, x^2 u^2/t, x^2 t^2/u, x/t) \right] + \frac{t^2}{t - x q} \]

where

\[ P(x, t, u) = u^2 t^2 - (u + t) u t x + (1 - t^3 - u^3) u t x^2 + (t^4 + u^4) x^3. \]  

We see that the variables $x$ and $q$ are “synchronized” in the right hand side of the obtained equation: they occur with the same power in each monomial of each power series in this expression. Hence we obtain the following identity in the ring $\mathbb{Z}[t^{\pm 1}, u^{\pm 1}]((x))$:

\[ g(x, t) \Psi(x, t) = \frac{t^2}{t - x} + \text{coef}_{u^{-1}} \left[ \frac{t^3 x^2 (u - x) g(x, u)}{P(x, t, u)} \right] \]

where

\[ \Psi(x, t) = 1 - x^2 (t - x) \Phi(x, t), \quad \Phi(x, t) = \text{coef}_{u^{-1}} \left[ u^3 / P(x, t, u) \right]. \]

Here are several initial terms of these series:

\[ \Phi(x, t) = t^{-2} x^2 + (t^{-3} + 1) x^3 + (t^{-4} + 2 t^{-1} + t^2) x^4 + (t^{-5} + 3 t^{-2} + 3 t) x^6 + \ldots \]

\[ \Psi(x, t) = 1 - t^{-1} x^4 - t x^5 - (1 + t^3) x^6 - (t^{-1} + 2 t^2) x^7 - (6 t^{-2} + 3 t) x^8 - \ldots \]  

Having found $g$ form (11), we can compute $\tilde{H}(x, x)$. Indeed, by (5) we have

\[ x \tilde{H}_q(x^3, x^3) = \text{coef}_{t_0} \left[ t x \tilde{G}_q(x^3/t, t^3, x^3/t^2) \right] = \text{coef}_{t_0} \left[ t^{1/2} x \tilde{G}_q(x^3/t^{1/2}, t^{3/2}, x^3/t) \right]. \]

Replacing $t$ by $x^2 t$ (cf. (9)) and setting $q = 1$, we obtain

\[ x \tilde{H}(x^3, x^3) = \text{coef}_{t_0} \left[ g(x, t) \right]. \]  

---

1 All coefficients of $\Phi$ and $1 - \Psi$ that I have computed are positive. If they are really all positive, it would be interesting to find their combinatorial meaning.
4.4. Computation. In this subsection we study the analytic functions defined by the series discussed in the previous subsection.

By §4.1, we need to find the smallest positive pole of $H^*(x)$, that is the smallest positive zero $\beta$ of $1 - H(x)$. One can check that

$$H(x) = x\tilde{H}(x, x).$$

Being the sum of a power series with positive coefficients, the function $x\tilde{H}(x, x)$ is increasing when $x > 0$, thus it is enough know how to compute with any given precision the value of $\tilde{H}(x, x)$ for any fixed $x$ in an interval containing $\beta$. By (13), this can be done by numerical integration of the function $g(x^{1/3}, t)$ along a suitable contour $\Gamma_x$ (cf. the proof of Lemma 3.2). Thus we need to be able to compute $g(x, t)$ for any $x \in [0, x_0^+]$ and $t \in \Gamma_x$ for some $x_0^+ > x_0 = \beta^{1/3}$. This can be done because for a fixed $x$, after replacing $\operatorname{coef}_{u - 1}[\ldots]$ by $\frac{1}{2\pi i} \int_{\Gamma_x} (\ldots) du$, the equation (11) becomes a Fredholm equation for the function $g$ restricted to $\Gamma_x$. Now we pass to more detailed explanations.

Let

$$\Gamma = \{(x, t, u) \in \mathbb{R} \times \mathbb{C}^2 \mid 0 < x < 1/2, \ |t| = |u| = 1\},$$

$$\Gamma' = \{(x, t) \in \mathbb{R} \times \mathbb{C} \mid 0 < x < 1/2, \ |t| = 1\}.$$

Lemma 4.1. The polynomial $P(x, t, u)$ defined in (10) does not vanish on $\Gamma$. For any fixed $(x, t) \in \Gamma'$, the polynomial $P(x, t, u)$ has two simple roots $u_k(x, t)$, $k = 1, 2$, in the unit disk $|u| < 1$ and two simple roots outside it.

Proof. The first statements can be checked using any software for symbolic computations. This can be done, for example, as follows. Let $S^1$ be the unit circle in $\mathbb{C}$. Then $\Gamma = (0, 1/2) \times S^1 \times S^1$. We can identify $S^1$ with $\mathbb{RP}^1$ by some rational parametrization. Then $\operatorname{Re} P$ and $\operatorname{Im} P$ become real rational functions on the variety $\Gamma$ and, by computing resultants, discriminant, etc., one can check that the real algebraic curve given by the equations $\operatorname{Re} P = \operatorname{Im} P = 0$ does not enter in the layer $0 < r < 1/2$. More precisely, let $p(x, T, U)$ and $q(x, T, U)$ be real polynomials such that

$$P(x, \zeta(T), \zeta(U)) = \frac{p(x, T, U) + iq(x, T, U)}{(i + T)^4(i + U)^4}, \quad \zeta(X) = \frac{i - X}{i + X}.$$

Note that $\zeta(\mathbb{R}) = S^1 \setminus \{-1\}$, hence $(x, T, U)$ are coordinates on the affine chart $\Gamma \setminus \{(t + 1)(u + 1) = 0\}$ of $\Gamma$. The projection of the real algebraic curve $\Gamma \cap \{P = 0\}$ onto the plane $(x, T)$ is given by the equation $R(x, T) = 0$ where $R(x, T)$ is the resultant of $p$ and $q$ with respect to $U$. To prove that the curve $R(x, T) = 0$ does not have real points with $0 < x < 1/2$, we compute the real roots of $D(x) = 0$ on this interval where $D(x)$ is the discriminant of $R$ with respect to $T$, and we check that the equations $R(x_k, T) = 0$ for each $k = 1, \ldots, 2n + 1$ do not have real roots where $0 < x_1 < \cdots < x_{2n+1} < 1/2$ and $x_k$ with even $k$ are all the real roots of $D(x)$ on the interval $0 < x < 1/2$. This computation shows that $P(x, u, t) \neq 0$ when $(x, u, t) \in \Gamma$ and $(t + 1)(u + 1) \neq 0$. Then we check that $P(x, \zeta(T), -1) \neq 0$, $P(x, -1, \zeta(U)) \neq 0$, and $P(x, -1, -1) \neq 0$ for $0 < x < 1/2$, $T \in \mathbb{R}$.

Similarly one can check that for any fixed $(x, t) \in \Gamma'$, the discriminant of $P$ with respect to the variable $u$ does not vanish, hence for any fixed $(x, t) \in \Gamma'$, all the four roots of $P$ (viewed as a polynomial in $u$) are pairwise distinct.
Therefore, the number of roots of $P$ in the unit disk $|u| < 1$ is constant. Thus, to prove the second statement, it is enough to check it for some value of $x$ and $t$, for example, for $t = 1$ and a very small $x$. □

**Lemma 4.2.** (a). The formal power series $1/P(x, t, u) \in \mathbb{Z}[t^\pm 1, u^\pm 1]((x))$ converges to the function $1/P(x, t, u)$ in a neighborhood of $\Gamma \cap \{ x < \frac{1}{4} \}$.

(b). The formal power series $\Phi(x, t) \in \mathbb{Z}[t^\pm 1]((x))$ converges to an analytic function (which we also denote by $\Phi(x, t)$) in a neighborhood of $\Gamma' \cap \{ x < \frac{1}{4} \}$. The function $\Phi(x, t)$ admits an analytic continuation to a neighborhood of $\Gamma'$ defined by the Cauchy integral

$$
\Phi(x, t) = \frac{1}{2\pi i} \oint_{|u|=1} \frac{u^3 \, du}{P(x, t, u)} = \sum_{k=1}^{2} \frac{u_k(x, t)^3}{P'_u(x, t, u_k(x, t))}
$$

where $u_1(x, t)$ and $u_2(x, t)$ are the roots of $P$ in the unit disk $|u| < 1$; see Lemma 4.1.

**Proof.** The power series $1/P(x, t, u)$ involved in the definition $\Phi(x, t)$ is a power series expansion with respect to $x$, hence $1/P = a_0^{-1}(1 + X + X^2 + \ldots)$ where $X = (a_1 + a_2 + a_3)/a_0$ and $a_k = x^k \text{coeff}_X^k[P]$. If $(x, t, u) \in \Gamma$, then $|a_0| = 1$, $|a_1| \leq 2x$, $|a_2| \leq 3x^2$, $|a_3| \leq 2x^3$, and thus $|X| \leq 2x + 3x^2 + 2x^3$. Therefore $|X| < 1$ for $x < 1/4$, whence the convergence of $1/P$ in the required domain. This fact combined with Lemma 4.1 implies all the other assertions of the lemma. □

Mathematica function $\text{Psi}$ in Figure 7 computes $\Psi(x, t)$ for $(x, t) \in \Gamma'$ with any given precision.

Notice that one of the functions $u_1(x, t)$ or $u_2(x, t)$ has a ramification point at $(x, t) = (1/2, 1)$, and hence the functions $\Phi$ and $\Psi$ are ramified in this point as well. The Laurent-Puiseux expansion of $\Psi(x, 1)$ in powers of $s = \sqrt{1/2 - x}$ is

$$
\Psi(x, 1) = -\frac{1}{4\sqrt{6}} s^{-1} + \frac{12 - \sqrt{2}}{8} - \frac{3}{8\sqrt{6}} s - \frac{3}{8\sqrt{2}} s^2 + \frac{103}{96\sqrt{6}} s^3 - \frac{87}{32\sqrt{2}} s^4 + \frac{2635}{192\sqrt{6}} s^5 + \ldots
$$

Let $x_0^- = \frac{16}{33}$ and $x_0^+ = \frac{17}{35}$. We shall see later that $x_0 \in [x_0^-, x_0^+]$; in fact, $x_0^\pm$ are given by initial segments of the continued fraction of $x_0$.

Using the expansion of $\Psi$ at $(1/2, 1)$ and computing the values of $\Psi(x, t)$ (with the program in Figure 7) on a sufficiently dense grid on $\Gamma'$, one can check that $\Psi$ does not vanish on $\Gamma' \cap \{ x < x_0^+ \}$ and

$$
\min_{0 \leq x \leq x_0^+, |t|=1} |\Psi(x, t)| = \min_{0 \leq x \leq x_0^+, |t|=1} \text{Re} \Psi(x, t) = \Psi(x_0^+, 1) = 0.44768...
$$

See the level lines of $\text{Re} \Psi$ in Figure 6; we omit the details of the error estimate.

Using Lemma 5.2 applied to the function $|P(x/4, e^{i\tau}, e^{i\theta})|^2$ with an appropriately chosen $h$, we find

$$
\min_{x<x_0^+, |t|=|u|=1} |P| = P(x_0^+, 1, 1) = 0.02183...
$$

(here we rescaled $x$ to equilibrate the partial derivatives). The computation can be fastened by choosing different grid in different zones of $\Gamma$. In our computation, the grid step varied from $h = 1/300$ near the point of minimum to $h = 1/20$ far from
Figure 6. Level lines of $\text{Re}\Psi(x,t)$ for $|t| = 1$. The shown vertical line is $x = x_0$ or $x = x_0^+$ (no difference with this resolution).

it. To estimate the error, we used evident coarse bounds for the fourth derivatives and, using them, computed finer upper bounds for the second derivatives in each zone again using Lemma 5.2.

**Lemma 4.3.** The formal series $g(x,t)$ (introduced in §4.2) converges in some neighborhood of $\Gamma' \cap \{|x| < 2^{-3/2}\}$.

**Proof.** By Anclin’s theorem [1], the number of primitive lattice triangulations of a lattice polygon $\Pi$ is bounded above by $2^N$ for $N = \#(\Pi \cap (\mathbb{Z}^2 \setminus \frac{1}{2}\mathbb{Z}^2))$, and it is easy to derive from Pick’s formula that $N < 3\text{Area}(\Pi) - 3/2$. The area of the shape corresponding to $g_{a,c,d}^{(k)}$ is $(2a + 3c + d + 3)/2$. Hence $\tilde{g}_{a,c,d} < c_0 2^{3(2a+3c+d)/2}$ for some constant $c_0$ and, for $|t| = 1$, we obtain

$$|g(x,t)| \leq x^2 \sum_{a,c,d} |g_{a,c,d} x^{2a} x^{3c} x^d| \leq c_0 x^2 \sum_{a,c,d} |2^{3(2a+3c+d)/2} x^{2a+3c+d}| = c_0 x^2 \sum_n 2^{3/2n} A_n x^n,$$

where $A_n = \#\{(a,c,d) \in \mathbb{Z}_+^3 \mid 2a + 3c + d = n\}$. Since $A_n$ is bounded by a polynomial function of $n$, the series converges for $x < 2^{-3/2}$. $\square$

Lemmas 4.2 and 4.3 combined with (11) and (16) imply that the function $g(x,t)$ is analytic in a neighborhood of $\Gamma' \cap \{x < 2^{-3/2}\}$, and it satisfies the condition

$$g(x,t) = \frac{t^2}{(t-x)\Psi(x,t)} + \frac{1}{2\pi i} \oint_{|u|=1} \frac{x^2 t^3(u-x)g(x,u)du}{P(x,t,u)\Psi(x,t)}.$$  \hspace{1cm} (18)

For any fixed $x$, this is a Fredholm equation of the second kind for $g(x,t)$ considered as a function of $t$.

**Lemma 4.4.** The function $g(x,t)$ analytically extends to a neighborhood of $\Gamma' \cap \{x < x_0^+\}$ and it satisfies the equation (18) in this domain.
Proof. Let us rewrite (18) in a more conventional form

$$\varphi_g(x, \tau) = f(x, \tau) + \int_0^1 K(x, \tau, \theta) \varphi_g(x, \theta) \, d\theta$$

where we set $t = e^{2\pi i \tau}$, $u = e^{2\pi i \theta}$, and

$$\varphi_g(x, \tau) = g(x, t), \quad f(x, \tau) = \frac{t^2}{(t-x) \Psi(x, t)}, \quad K(x, \tau, \theta) = \frac{x^2 t^3 u(u-x)}{P(x, t, u) \Psi(x, t)}.$$

As we already pointed out, $g$ satisfies (18) and thus $\varphi_g$ satisfies (19) for small $x$. Thus, thanks to the Identity Theorem for analytic functions, it is enough to show that for any $x \in [0, x_0^+]$ there exists a unique solution of (19) and that it is analytic with respect to $(x, \tau)$. Hence, by Lemma 5.6, it suffices to show that 1 is not an eigenvalue of $K_x$ for any $x \in [0, x_0^+]$ where $K_x : C[0, 1] \to C[0, 1]$ is the the Fredholm integral operator which takes $\varphi(\tau)$ to $\psi(\tau) = \int_0^1 K(x, \tau, \theta) \varphi(\theta) \, d\theta$. The latter fact, in its turn, follows from the bound

$$\max_{0 \leq x \leq x_0^+} N_2(x) = N_2(x_0^+) = 0.88525$$

where $N_2(x) = \int_{[0,1]^2} |K(x, \tau, \theta)|^2 \, d\tau \, d\theta$. This bound is computed by numerical integration. To estimate the approximation error, one needs upper bounds of partial derivatives of $K$. They can be easily obtained using the lower bounds (16) and (17) of $|\Psi|$ and $|P|$, and upper bounds of the derivatives of $\Psi$ obtained from its integral form in (15). For upper bounds of the derivatives of polynomials involved in the definition of $K$ one can use just the sums of upper bounds of monomials. □

Replacing the integrals by integral sums, equation (18) can be solved with any given precision. Then, due to (13) and (14) we can numerically compute $H(x)$ using the Cauchy integral

$$H(x^3) = \frac{x^2}{2\pi i} \oint_{|t|=1} \frac{g(x, t) \, dt}{t} = x^2 \int_0^1 \varphi_g(x, \tau) \, d\tau$$

(recall that $\varphi_g(x, \tau) := g(x, e^{2\pi i \tau})$; see (19)). We can summarize the content of this section as follows (recall that $f(m, n)$ is the number of primitive lattice triangulations of the rectangle $m \times n$).

**Proposition 4.5.** $\lim_{n \to \infty} f(3, n)^{1/n} = 1/x_0^2$ where:

- $x_0$ is a unique solution of the equation $H(x^3) = 1$ on the interval $[0, x_0^+]$ with $x_0^+ = 17/3$;
- $H(x)$ is defined via $g(x, t)$ by (20) and it is monotone on $[0, x_0^+]$;
- $g(x, t)$ is the solution of the Fredholm equation (18) whose ingredients $P$ and $\Psi$ are defined by (10) and by $\Psi(x, t) = 1 - x^2(t-x)\Phi(x, t)$ with $\Phi$ defined by (15); for any $x \in [0, x_0^+]$ the equation (18) has a unique solution.

In Figure 7 we present a Mathematica function $H$ which computes $H(x)$ with any given precision. The approximating error can be estimated using Lemma 5.4. One can check that the functions $P(x, t, u)$ and $\Psi(x, t)$ do not vanish when $x < x_0^+$. 
\( P = u^2 t^2 - (u+t)u t x + (1-t^3-u^3)u t x^2 + (t^4+u^4)x^3; \)
\[ \Psi = \text{Function}\{x0,t0,\text{prec}\}, \text{Module}\{\text{P0},u0,i\}, \]
\[ \text{P0} = P/.\{x->x0,t->t0\}; u0 = \text{NRoots}\{0==P0,u,\text{prec}\}; \]
\[ u0 = \text{Sort}\{\text{Table}\{\{\text{Abs}[u0[i,2]],u0[i,2]\}\},\{i,4\}\}\}; \]
\[ 1-x0^2(t0-x0)\sum\left(\frac{u^3}{\partial P0/u}\right)/.u->u0[i,2]; \]
\[ \]
\[ H = \text{Function}\{x3,n,\text{prec}\}, \text{Module}\{x0,z,P0,Id,K,F,G,j,k,Tj,Uk,PsiTj,Pjk\}, \]
\[ x0 = \text{N}[x3^{1/3},\text{prec}]; z = \text{N}[\text{Exp}[2\Pi*I/n],\text{prec}]; \]
\[ K = \text{Id} = \text{IdentityMatrix}[n]; F = K[[1]]; P0 = P/.x->x0; \]
\[ \text{Do}[ Tj = z^j; \PsiTj = \Psi[x0,Tj,\text{prec}]; \]
\[ F[[j]] = Tj^2/(Tj-x0)/PsiTj; \]
\[ \text{Do}[ Uk = r*z^k; Pjk = P0/.\{t->Tj,u->Uk\}; \]
\[ \quad K[[j,k]] = x0^2*Tj^3(Uk-x0)Uk/Pjk/PsiTj/n, \]
\[ \quad \{k,n\}\}; \]
\[ \{j,n\}; \]
\[ G = \text{Inverse}[\text{Id}-K].F; x0^2*(\text{Plus}@@G)/n; \]

**Figure 7.** Mathematica code for computation of \( H(x) \)

\(|u| = 1, \text{and } 10/13 < |t| < 13/10\). In Figures 8 and 9 we show the image of the annulus \( 10/13 < |t| < 13/10 \) under the mapping \( t \mapsto \Psi(x_0,t) \). Thus we can apply the error estimate (27) with \( r = 10/13 \) and hence \( a = -\frac{\log r}{2\pi} = 0.04176 \). When estimating the error of \( H(x) \) with \( x \approx x_0 \), we can set in (27)

\[ C \leq 1; \quad \frac{1}{n}\|B\|_1 \leq 3.05; \quad M \leq 3910; \quad M' \leq 94.6; \quad M_f \leq 258. \]

Then we obtain the error estimate presented in the last column of Table 7. We see that it is reasonably close to the actual error which is given in the 4th column.

| \( n \) | \( \text{prec.} \) | \( \text{time} \) (sec.) | \( n\)-th approx. of \( H(x_0) - 1 \) | error estimate |
|---|---|---|---|---|
| 100 | 24 | 0.299391 | \( 1.44 \times 10^{-10} \) | \( 6.95 \times 10^{-4} \) |
| 200 | 36 | 6.759046 | \( 5.01 \times 10^{-22} \) | \( 5.60 \times 10^{-15} \) |
| 300 | 48 | 21.77949 | \( 1.73 \times 10^{-33} \) | \( 3.39 \times 10^{-26} \) |
| 400 | 60 | 51.22560 | \( 6.02 \times 10^{-45} \) | \( 1.82 \times 10^{-37} \) |
| 500 | 72 | 115.5499 | \( 2.09 \times 10^{-56} \) | \( 9.19 \times 10^{-49} \) |
| 600 | 84 | 231.5893 | \( 7.26 \times 10^{-68} \) | \( 4.45 \times 10^{-60} \) |
| 700 | 96 | 380.6020 | \( 2.52 \times 10^{-79} \) | \( 2.09 \times 10^{-71} \) |
| 800 | 108 | 608.9937 | \( 8.78 \times 10^{-91} \) | \( 9.65 \times 10^{-83} \) |
| 900 | 120 | 869.7188 | \( 3.06 \times 10^{-102} \) | \( 4.38 \times 10^{-94} \) |
| 1000 | 132 | 1072.923 | \( 1.06 \times 10^{-113} \) | \( 1.96 \times 10^{-105} \) |
| 1100 | 144 | 1456.021 | \( 3.72 \times 10^{-125} \) | \( 8.70 \times 10^{-117} \) |
| 1200 | 156 | 1852.763 | \( 1.29 \times 10^{-136} \) | \( 3.83 \times 10^{-128} \) |
Figure 8. A realistic drawing of the image of the circles $|t| = \frac{10}{13}$, $|t| = 1$, and $|t| = \frac{13}{10}$ by the mapping $t \mapsto \Psi(x_0, t)$. The left zoom is stretched in the vertical direction.

Figure 9. A schematic drawing of the image of the upper half-annulus \( \{ \frac{10}{13} \leq |t| \leq \frac{13}{10}, \text{Im} t \geq 0 \} \) by the mapping $t \mapsto \Psi(x_0, t)$. The lower half-annulus is mapped symmetrically. The gray part is $|t| < 1$.

5. Approximate solutions of Fredholm integral equations with analytic kernels

5.1. Error estimates. Generalities. The notation in this subsection is independent of the notation in the rest of the paper.

Lemma 5.1. Let $f$ be a holomorphic function in a neighborhood of the annulus $R_1 < |z| < R_2$, and let $f(z) = \sum_{n \in \mathbb{Z}} c_n z^n$ be its Laurent series. Then, for $R_1 < r < R_2$ and for any $n > 0$,

$$
\left| c_0 - \frac{1}{n} \sum_{k=1}^{n} f(r \omega^k) \right| = \left| \int_0^1 f(re^{2\pi it}) \frac{dt}{n} \sum_{k=1}^{n} f(r \omega^k) \right| \leq \frac{M_1 q_1^n}{1 - q_1^n} + \frac{M_2 q_2^n}{1 - q_2^n} \tag{21}
$$

where $\omega = e^{2\pi i/n}$, $q_1 = R_1/r$, $q_2 = r/R_2$, and $M_j = \max_{|z| = R_j} |f(z)|$ for $j = 1, 2$.

Proof. We have

$$
\sum_{k=1}^{n} f(r \omega^k) = \sum_{k=1}^{n} \sum_{m \in \mathbb{Z}} c_m (r \omega^k)^m \quad \text{and} \quad \sum_{k=1}^{n} \omega^{km} = \begin{cases} 
  n, & n \text{ divides } m, \\
  0, & \text{otherwise}
\end{cases}
$$

hence the left hand side of (21) is equal to $\sum_{p \in \mathbb{Z} \setminus \{0\}} c_{pn} r^{pn}$ and the coefficients can be estimated using the Cauchy integrals. \(\square\)
Lemma 5.2. Let $h > 0$ and $D \subset \mathbb{R}^d$ be a product of segments $[0, n_1h] \times \cdots \times [0, n_dh]$ with positive integers $n_1, \ldots, n_d$. Let $f : D \to \mathbb{R}$ be a function of class $C^2$ and $M = \max_{i,j} \max_D |\partial_i \partial_j f|$. Then

$$
\min_D f \geq \min_{h\mathbb{Z}^d} f - \frac{1}{8} Md^2h^2
$$

where $h\mathbb{Z}^d = \{h\bar{n} \mid \bar{n} \in \mathbb{Z}^n\}$. A similar estimate holds for $\max_D f$.

Proof. Induction on $d$. Let the minimum be attained at $x_0 \in D$. If $x_0$ is in the interior of $D$, then we estimate $|f(x) - f(x_0)|$ for the nearest to $x_0$ grid point $x$ using the Taylor–Lagrange formula for the second order expansion of $f(x_0 + t(x - x_0))$ at $t = 0$. If $x_0$ is on the boundary of $D$, then we apply the induction hypothesis to the restriction of $f$ to the facet of $D$ containing $x_0$. □

5.2. Error estimates for approximate solutions of Fredholm equations.

Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a continuous solution of the Fredholm integral equation

$$
\varphi(x) = \int_0^1 K(x, y)\varphi(y) \, dy + f(x) \quad (22)
$$

with analytic complex-valued functions $K$ and $f$ which are (bi)-periodic with period 1, i.e., $K(x, y) = K(x + 1, y) = K(x, y + 1)$ and $f(x) = f(x + 1)$. Assume that $K$ and $f$ extend to complex analytic functions in a neighborhood of $(D \times \mathbb{R}) \cup (\mathbb{R} \times D_1)$ in $\mathbb{C}^2$ and in a neighborhood of $D$ in $\mathbb{C}$ respectively where

$$
D = \{ z \in \mathbb{C} \mid -a \leq \text{Im} \, z \leq a \}, \quad D_1 = \{ z \in \mathbb{C} \mid -a_1 \leq \text{Im} \, z \leq a_1 \}, \quad 0 < a_1 < a.
$$

Let us set

$$
C = \int_0^1 |\varphi(x)| \, dx, \quad M = \max_{D \times \mathbb{R}} |K|, \quad M' = \max_{\mathbb{R} \times D_1} |K|, \quad M_f = \max_D |f|.
$$

Lemma 5.3. The function $\varphi$ analytically extends to a neighborhood of $D$ and

$$
M_{\varphi} := \max_{D_1} |\varphi| \leq \frac{a(CM + M_f)}{a - a_1}. \quad (23)
$$

Proof. For any $(x_0, y_0) \in \mathbb{R}^2$ and any $n$, we have

$$
|\partial^n_x K(x_0, y_0)| \leq \left| \frac{n!}{2\pi i} \int_{|z| = a} \frac{K(z, y_0) \, dz}{z^{n+1}} \right| \leq \frac{Mn!}{a^n}
$$

and similarly $|f^{(n)}(x_0)| \leq M_f n!/a^n$. Then, derivating $(22)$ $n$ times with respect to $x$, we obtain

$$
|\varphi^{(n)}(x_0)| = \left| \int_0^1 \partial^n_x K(x_0, y)\varphi(y) \, dy + f^{(n)}(x_0) \right| \leq \frac{(CM + M_f)n!}{a^n}. \quad (24)
$$
Hence the Taylor series of $\varphi$ at $x_0$ converges in the disk $|z - x_0| < a$ and, for $|z - x_0| \leq a_1$, we have

$$|\varphi(z)| = \left| \sum_{n \geq 0} \frac{\varphi^{(n)}(x_0)}{n!} (z - x_0)^n \right| \leq \sum_{n \geq 0} \frac{(CM + M_f)a^n}{n!} = \frac{a(CM + M_f)}{a - a_1},$$

whence the required bound for $M_\varphi$. □

For a positive integer $n$, let us see what happens if we replace the integral in (22) by the $n$-th integral sum. Namely, consider the vectors $\varphi^{[n]} = (\varphi_1^{[n]}, \ldots, \varphi_n^{[n]})$, $f^{[n]} = (f_1^{[n]}, \ldots, f_n^{[n]})$, and the $n \times n$ matrix $K^{[n]} = (K_{jk}^{[n]})_{jk}$ defined by

$$\varphi_j^{[n]} = \varphi(j/n), \quad f_j^{[n]} = f(j/n), \quad K_{jk}^{[n]} = \frac{1}{n}K(j/n, k/n).$$

Let $\hat{\varphi}^{[n]} = (\hat{\varphi}_1^{[n]}, \ldots, \hat{\varphi}_n^{[n]})$ be a solution of the equation

$$\hat{\varphi}^{[n]} = K^{[n]}\hat{\varphi}^{[n]} + f^{[n]}, \quad (25)$$

This equation is a discretization of (22) and it is natural to expect that $\hat{\varphi}^{[n]}$ well approximates $\varphi$. Now, following the approach from [5], we estimate the rate of the convergence. Our final purpose is to find a good upper bound for the approximating error

$$E_n := \left| \int_0^1 \varphi(x) \, dx - \frac{1}{n} \sum_{j=1}^n \hat{\varphi}_j^{[n]} \right|.$$

We define the norms $\| \cdot \|_p, 1 \leq p \leq \infty$, on $\mathbb{C}^n$ in the usual way. For a square matrix $A = (a_{jk})_{jk}$ with complex entries we set

$$\|A\|_1 = \sum_{j,k} |a_{jk}|, \quad \|A\|_2 = \left( \sum_{j,k} |a_{jk}|^2 \right)^{1/2}.$$

**Lemma 5.4.** (a) Suppose that the matrix $A^{[n]} = I - K^{[n]}$ is invertible and denote its inverse by $B^{[n]}$. Then

$$E_n \leq \frac{2a(CM + M_f)r_1^n}{(a - a_1)(1 - r_1^n)} \left( 1 + \frac{1}{n} \|B^{[n]}\|_1 M_f^n \right), \quad r_1 = e^{-2\pi a_1}. \quad (26)$$

If $K$ analytically extends to a neighborhood of $\mathbb{R} \times D$ and $M' = \max_{\mathbb{R} \times D} K$,

$$E_n \leq 4\pi e(CM + M_f) \left( 1 + \frac{1}{n} \|B^{[n]}\|_1 M' \right) \frac{nar^n}{1 - er^n}, \quad r = e^{-2\pi a}. \quad (27)$$

For $n > \alpha_n$, we have

$$C \leq \frac{\|\hat{\varphi}^{[n]}\|_1 + \alpha_n M_f}{n - \alpha_n} \quad \text{where} \quad \alpha_n = \frac{2M_1 a\|B^{[n]}\|_1 r_1^n}{(a - a_1)(1 - r_1^n)} + \frac{1}{4a}. \quad (28)$$

(b) Suppose that $\|K^{[n]}\|_2 = M_2 < 1$. Then $A^{[n]}$ is invertible and $\frac{1}{n} \|B^{[n]}\|_1 \leq 1/(1 - M_2)$ which implies in particular that $\alpha_n < \alpha_0$ for some constant $\alpha_0 = \alpha_0(a, a_1, M, M'_1, M_2, M_f)$ and hence $C$ can be estimated using (28) for $n > \alpha_0$. 
Proof. (a). Let $J = \int_0^1 \varphi(x) \, dx$, $S = \frac{1}{n} \sum_j \varphi(j/n)$, $\hat{S} = \frac{1}{n} \sum_j \hat{\varphi}(j/n)$, $\rho = \varphi[n] - \hat{\varphi}[n]$, and $\sigma = A[n] \rho$. In this notation, $E_n = |J - \hat{S}|$. We have

$$
\|\sigma\|_\infty = \|A[n] \varphi[n] - A[n] \hat{\varphi}[n]\|_\infty \overset{(25)}{=} \|A[n] \varphi[n] - f[n]\|_\infty = \|K[n] \varphi[n] - (\varphi[n] - f[n])\|_\infty
$$

By (22), we have $\varphi^n[j] - f^n[j] = \int_0^1 K(j/n, y) \varphi(y) \, dy$, and the $j$-th component of the vector $K[n] \varphi[n]$ is the $n$-th integral sum for this integral. Hence, applying Lemma 5.1 to the functions $K(j/n, z(\zeta)) \varphi(z(\zeta))$ after the change of variable $\zeta = e^{2\pi i z}$, we obtain $\|\sigma\|_\infty \leq M_1' C_1$ with $C_1 = 2M_2r_1^n/(1 - r_1^n)$ and then

$$
\|\rho\|_1 = \|B[n] \sigma\|_1 \leq \|B[n]\|_1 \times \|\sigma\|_\infty \leq M_1' C_1 \|B[n]\|_1.
$$

Lemma 5.1 applied to $\varphi(z(\zeta))$ yields $|J - S| \leq C_1$. We also have $|S - \hat{S}| \leq \frac{1}{n} \|\rho\|_1$, hence

$$
E_n = |J - \hat{S}| \leq |J - S| + |S - \hat{S}| \leq C_1 + \frac{1}{n} \|\rho\|_1 \leq C_1 + \frac{1}{n} M_1' C_1 \|B[n]\|_1
$$

which yields (26) after applying (23). Setting $a_1 = a - \frac{1}{2\pi n}$ (hence $r_1 = e^{1/n} r$) and $M_1' < M'$ in (26), we obtain (27).

Let us prove (28). It is easy to check that

$$
nC \leq \|\varphi[n]\|_1 + \frac{1}{4} \max_{\mathbb{R}} |\varphi'| \leq \|\hat{\varphi}[n]\|_1 + \|\rho\|_1 + \frac{1}{4} \max_{\mathbb{R}} |\varphi'|.
$$

Using the estimates (30) and (24) for $\|\rho\|_1$ and $|\varphi'|$ respectively, we obtain

$$
nC \leq \|\hat{\varphi}[n]\|_1 + \frac{2M_1 M_2 \|B[n]\|_1 r_1^n}{1 - r_1^n} + \frac{CM + M_f}{4a} \overset{(23)}{\leq} \|\hat{\varphi}[n]\|_1 + (CM + M_f)\alpha_n.
$$

(b). Suppose now that $\|K[n]\|_2 = M_2 < 1$. Then $\|B[n]\|_2 = \|(I - K[n])^{-1}\|_2 = \|I + K[n] + (K[n])^2 + \ldots\|_2 \leq 1/(1 - M_2)$. By Cauchy Inequality we also have $\|B[n]\|_1 \leq n \|B[n]\|_2$. □

5.3. A numerical criterion of existence and uniqueness of solutions. Here we keep the above assumptions about $K(x, y)$ and $f(x)$ except that we no longer assume a priori that equation (22) has a continuous solution $\varphi$. Let $\mathcal{K}: \mathcal{C}([0, 1]) \to \mathcal{C}([0, 1])$ be the Fredholm integral operator with kernel $K(x, y)$, i.e., the operator $\varphi \mapsto \psi$ where $\psi(x) = \int_0^1 K(x, y) \varphi(y) \, dy$.

Lemma 5.5. (cf. [5, Ch.II, §1, Eq. (26)]). Suppose that there exists $n$ such that the matrix $I - K[n]$ is invertible and $\alpha_n < n$ where $\alpha_n$ is defined in (28) (note that neither $f$ nor $\varphi$ is used in the definition of $\alpha_n$). Then $1$ is not an eigenvalue of $\mathcal{K}$ and hence, for any given continuous function $f$, equation (22) has a unique continuous solution $\varphi$.

Proof. Let $n$ be such that $\alpha_n < n$. Let us apply Lemma 5.4(a) when $f = 0$ and hence $\hat{\varphi}[n] = 0$. Then (28) reads $C \leq 0$ which means that there are no non-zero solutions of the equation $\mathcal{K} \varphi = \varphi$, i.e. $1$ is not an eigenvalue of $\mathcal{K}$. By Fredholm Theorem [2], in this case (22) has a unique continues solution for any $f$. □
5.4. Analyticity of solutions with respect to a parameter. Let $\Lambda$ be a domain in $\mathbb{C}$ and $U = \{ z \in \mathbb{C} \mid -a \leq \text{Im} \, z \leq a \}$, $a > 0$. Let $K(\lambda, x, y)$ be an analytic function in a neighborhood of $\Lambda \times U^2$ in $\mathbb{C}^3$ and $f(\lambda, y)$ be an analytic function in a neighborhood of $\Lambda \times U$ in $\mathbb{C}^2$. We assume that $K(\lambda_0, x, y)$ is $(1, 1)$-biperiodic and $f(\lambda_0, x)$ is 1-periodic for any fixed $\lambda_0 \in \Lambda$.

For $\lambda \in \Lambda$, let $K_\lambda : C([0, 1]) \to C([0, 1])$ be the Fredholm integral operator $\varphi \mapsto \psi$, $\psi(x) = \int_0^1 K(\lambda, x, y) \varphi(y) \, dy$. The next lemma immediately follows from Fredholm’s results in his seminal paper [2] (a more general fact is proven in [11]).

Lemma 5.6. Suppose that 1 is not an eigenvalue of $K_\lambda$ for any $\lambda \in \Lambda$. Then, for any $\lambda \in \Lambda$, there exists a unique solution $\varphi(\lambda, x)$ of the equation

$$\varphi(\lambda, x) = \int_0^1 K(\lambda, x, y) \varphi(\lambda, y) \, dy + f(\lambda, x)$$

(31)

and the function $\varphi(\lambda, x)$ is analytic in a neighborhood of $\Lambda \times U$.

Proof. By Fredholm’s results [2] (see also [6]), for any $\lambda \in \Lambda$, the solution $\varphi(\lambda, t)$ is unique under our assumptions and it can be written as

$$\varphi(\lambda, t) = f(\lambda, t) + \int_0^1 \frac{D(\lambda, x, y)}{D(\lambda)} f(\lambda, y) \, dy$$

where

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n A_n(\lambda)}{n!}, \quad D(\lambda, x, y) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n(\lambda, x, y)}{n!},$$

$$A_n(\lambda) = \int_{[0, 1]^n} K(\lambda, x, x) \, dx, \quad B_n(\lambda, x, y) = \int_{[0, 1]^n} K(\lambda, x, y, x) \, dx,$$

$$K(\lambda, x_1, \ldots, x_n, y_1, \ldots, y_n) = \det \left( K(\lambda, x_i, y_j) \right)_{i,j=1}^n.$$}

It is shown in [2] that $D(\lambda)$ does not vanish on $U$ (because 1 is not an eigenvalue of $K_\lambda$ for any $\lambda \in \Lambda$). It is clear that the functions $A_n$ and $B_n$ are analytic in $\Lambda$ and in $\Lambda \times U^2$ respectively and the Hadamard Inequality $|\det N| \leq n^{n/2} \max_{i,j} |N_{ij}|$ implies the upper bounds (cf. [2, p. 368, line 4]):

$$|A_n(\lambda)| \leq n^{n/2} M(\lambda)^n, \quad |B_{n-1}(\lambda, x, y)| \leq n^{n/2} M(\lambda)^n$$

where $M(\lambda) = \sup_{(x, y) \in U^2} |K(\lambda, x, y)|$. Hence the series (32) converge to analytic functions whence the result. \(\square\)

6. Non-primitive lattice triangulations

Denote the number of all (not necessarily primitive) lattice triangulations of the $m \times n$ rectangle by $f_{\text{np}}(m, n)$, and set

$$c^{\text{np}} = \lim_{n \to \infty} \frac{\log_2 f_{\text{np}}(n, n)}{n^2}. $$
Proposition 6.1. \( c_{\text{np}} \leq 4.735820221... \)

Proof. Let \( N = n^2 \). Any lattice triangulation can be subdivided up to a primitive lattice triangulation. Hence a lattice triangulation is completely determined by a choice of a primitive lattice triangulation and a set of its edges to be removed. Let \( f_{k\text{np}}^n(n, n) \) be the number of lattice triangulations of the \( n \times n \) square with \( k \) interior vertices and hence with \( \approx 3k \) edges. Then

\[
f_{k\text{np}}^n(n, n) \leq \left( \frac{3N}{3k} \right)^{2cN} \tag{33}
\]

(recall that \( 2^{cN} \) is a bound for the number of primitive lattice triangulations). On the other hand the number of triangulations with vertices in an arbitrary fixed set of \( k \) points on a plane is \( O(30^k) \) (see [10]), hence

\[
f_{k\text{np}}^n(n, n) \leq \left( \frac{N}{k} \right)^{30^k}. \tag{34}
\]

Combining (33) and (34) with Stirling formula, we obtain

\[
c_{\text{np}} \leq \max_{0 \leq x \leq 1} \min (3h(x) + c, h(x) + x \log_2 30), \tag{35}
\]

\[
h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x).
\]

Using the bound \( c \leq 4 \log_2 \frac{1+\sqrt{5}}{2} \) (see [7], [12], [13]), we obtain the result (the maximum in (35) is attained at \( x = 0.83206855 \)). \( \square \)

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Steklov Mathematical Institute, Gubkina 8, Moscow, Russia

IMT, l’université Paul Sabatier, 118 route de Narbonne, Toulouse, France

AGHA Laboratory, Moscow Institute of Physics and Technology, Russia

E-mail address: orevkov@math.ups-tlse.fr