DIVISION ALGORITHMS FOR THE FIXED WEIGHT SUBSET SUM PROBLEM

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Abstract. Given positive integers $a_1, \ldots, a_n, t$, the fixed weight subset sum problem is to find a subset of the $a_i$ that sum to $t$, where the subset has a prescribed number of elements. It is this problem that underlies the security of modern knapsack cryptosystems, and solving the problem results directly in a message attack. We present new exponential algorithms that do not rely on lattices, and hence will be applicable when lattice basis reduction algorithms fail. These algorithms rely on a generalization of the notion of splitting system given by Stinson [18]. In particular, if the problem has length $n$ and weight $\ell$ then for constant $k$ a power of two less than $n$ we apply a $k$-set birthday algorithm to the splitting system of the problem. This randomized algorithm has time and space complexity that satisfies $T \cdot S^{\log k} = \tilde{O}(n^\ell)$ (where the constant depends uniformly on $k$). In addition to using space efficiently, the algorithm is highly parallelizable.

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While the present paper was being refereed, [5] came out with an improvement to the main result. The most interesting aspect that remains is the idea of a $k$-set splitting system.

1. Problem Statement

Let $a_1, \ldots, a_n$ and a target $t$ be positive integers. The $\ell$-weight subset sum problem is to find a subset of the $a_i$ that sum to $t$, where the subset has $\ell$ elements. Equivalently, the problem is to find a bit vector $x$ of length $n$ and Hamming weight $\ell$ such that

$$
\sum_{i=1}^{n} a_i x_i = t.
$$

The corresponding decision problem is to determine whether or not a solution exists. We will refer to the integer subset sum problem as seeking a solution for (1) over the integers, while solving the modular subset sum problem involves solving (1) over some ring $\mathbb{Z}/m\mathbb{Z}$. A modular subset sum problem is random if we assume that the $a_i$ are chosen uniformly at random from $\mathbb{Z}/m\mathbb{Z}$.

The most important quantity associated with a subset sum problem is its density, defined to be $\frac{n}{\log A}$ in the integer case where $A = \max_{1 \leq i \leq n} a_i$. In the modular case we define density to be $\frac{n}{\log m}$ and will refer to it as modular density. Inspired by [7],

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we define the **information density** to be \( \frac{\log(n)}{\log A} \) (for the integer case) and **modular information density** to be \( \frac{\log(n)}{\log m} \).

The fixed weight subset sum problem is interesting both because it is NP-complete and because it has applications to knapsack cryptosystems (see Section 7). A brute force attack on the fixed weight subset sum problem takes \( \tilde{O}(\binom{n}{\ell}) \) bit operations. Here \( \tilde{O} \) is “Soft-Oh” notation. For functions \( f \) and \( g \), we say \( f \) is \( \tilde{O}(g) \) if there exist \( c, N \in \mathbb{N} \) such that \( f(x) \leq g(n)(\log(3 + g(n)))^c \) for all \( n \geq N \).

Throughout this paper all logarithms will have base 2. Suppose \( L \) is a set of integers and \( a \) is an integer. Then \( L - a \) is the set given by \( \{b - a : b \in L\} \) and \( L - a \mod m \) is the set given by \( \{b - a \mod m : b \in L\} \).

### 2. Prior Work and New Results

It is a nontrivial matter to apply the standard algorithmic technique of divide-and-conquer to problems with fixed weight bit vectors. One solution is to employ a \( k \)-set splitting system. Throughout most of this paper we assume that \( n \) and \( \ell \) are divisible by \( k \). See Section 9 for a discussion of the general case.

**Definition 2.1.** An \((n, \ell, k)\)-splitting system is a set \( X \) of \( n \) indices along with a set \( D \) of divisions, where each division is itself a set \( \{I_1, \ldots, I_k\} \) of subsets of indices, with \( I_1 \cup \cdots \cup I_k = X \) and \( |I_1| = \cdots = |I_k| = n/k \). These objects have the property that for every \( Y \subseteq X \) such that \( |Y| = \ell \), there exists a division \( \{I_1, \ldots, I_k\} \in D \) such that \( |Y \cap I_j| = \ell/k \) for \( 1 \leq j \leq k \). We call this division a good division with respect to \( Y \).

All splitting systems will appear in the context of a fixed weight subset problem with unknown solution \( Y \). With \( n \) and \( \ell \) understood from context, we will refer to an \((n, \ell, k)\)-splitting system as a \( k \)-set splitting system. With \( Y \) understood from context, we will call a division such that \( |Y \cap I_j| = \ell/k \) for all \( I_j \) a good division.

This is a generalization of 2-set splitting systems presented by Stinson in [18], which he called \((N; n, \ell)\)-splitting systems. In that paper design theory was utilized to minimize \( N \), the number of divisions.

Two set splitting systems allow for the application of the baby-step-giant-step algorithm to attain a square root time-space tradeoff. This had been done before Stinson, but without formalizing the notion of splitting systems. A version of this algorithm that searches for a good division randomly is presented in [1, Section 7.3] and applied to the fixed weight subset sum problem as a message attack against knapsack cryptosystems. Coppersmith developed the same algorithm for use on the fixed weight discrete logarithm problem, as well as a version that found a good 2-division deterministically rather than randomly. Both are presented in [18] along with an average case analysis.

Another line of attack on the fixed weight subset sum problem was revealed by the work of Nguyen and Stern [11]. They modified the lattice basis reduction technique of [2] to also work for problems of small pseudo-density \( \frac{\ell \log n}{\log A} \). Thus problems can be reduced to the closest vector problem on lattices. In practice this means that any problem with information density less than one and \( n \) less than 300 or so can be solved by current lattice reduction algorithms.

We present new algorithms for the fixed weight subset sum problem, which in the case of Theorem 2.2 is also a new algorithm for the fixed weight discrete logarithm...
problem. We use $T$ and $S$ to refer to the exponential term of an algorithm’s time and space usage.

**Theorem 2.2.** There is an algorithm for the fixed weight subset sum problem whose time and space constraints lie on the curve $T \cdot S^2 = \binom{n}{2}$. The deterministic version takes $\tilde{O}(n^{3}(\binom{n/4}{\ell/4})^2)$ bit operations and the randomized version is expected to take $\tilde{O}(\ell^{3/2}(\binom{n/4}{\ell/4})^2)$ bit operations. Both have space complexity $\tilde{O}(\binom{n}{4})$.

**Theorem 2.3.** Choose parameters $m$ and $k$ so that $k$ is a power of 2, $m < \binom{n/k}{\ell/k}$, and $\log m \geq 2(\log k)^2$. Assume that when reduced modulo $m$, the $a_i$ are uniformly random elements of $\mathbb{Z}/m\mathbb{Z}$. Then there is a randomized algorithm for the fixed weight subset sum problem whose expected running time is $\tilde{O}(m^{1/(\log k+1)} \cdot (n)/m)$ and which uses $\tilde{O}(m^{1/(\log k+1)})$ space. This gives a point on the time/space tradeoff curve $T \cdot S^{\log k} = \binom{n}{2}$.

Note that the assumption $m < \binom{n/k}{\ell/k}$ implies that the modular information density is greater than $k$. Also note that random fixed weight subset sum problems require information density greater than one to ensure a solution exists with high probability. This makes Theorem 2.3 a counterpoint to the lattice reduction technique employed in [11]. Finally note that the hidden polynomial terms include $\Theta(\ell^{1+k})$, the expected cost of finding a good $k$-division (See Proposition 3.1). This limits the applicability of Theorem 2.3 to practical settings.

The key ingredient of the first theorem is the general decomposition algorithm of Schroeppel-Shamir [15], while the second theorem relies on the $k$-set birthday algorithm of Wagner [20]. The application of these algorithms to the fixed weight setting relies on splitting systems to perform the necessary decomposition. Another candidate for the $k$-division algorithm is the generalization of Schroeppel-Shamir outlined in [19] (but see [3] for a rebuttal).

The general idea behind the algorithm of Theorem 2.3 is the following. We pick a parameter $m$ so that the corresponding modular problem has high enough modular density for the $k$-set birthday algorithm to be successful. Noting that the sought for integer solution is included in the set of solutions to the modular problem, we construct a modular oracle which outputs one of the modular solutions (nearly) uniformly at random. By repeating the modular oracle enough times, we expect to eventually find a solution to the original problem over the integers. The choice of $m$ determines the point on the time-space tradeoff curve, with larger choices being better in the sense that $T$ is smaller.

The importance of this new work is in improving the space complexity of the fixed weight subset sum problem. Theorem 2.2 is a direct improvement of the work given in [18], while Theorem 2.3 is the first to give a time/space tradeoff curve better than $T \cdot S^2$ for this problem. Although the time bound for the algorithm of Theorem 2.3 will nearly always be worse than $\tilde{O}(\binom{n/2}{\ell/2})$ due to the limitations on the choice of $m$, the algorithm is highly parallelizable by simply running the modular oracle on several processors at once. Thus with enough processors each will have less than $\tilde{O}(\binom{n/2}{\ell/2})$ work to do. An interesting open problem is to generalize the work in [10] to the subset sum problem, and then to explore the improvements to Theorem 2.3 that result from loosening the upper bound on $m$. 

In Section 3 we present the notion of an \((n, \ell, k)\)-splitting system and prove that they exist assuming \(k\) divides \(n\) and \(\ell\). We prove Theorem 2.2 in Section 4, develop the modular oracle in Section 5, and prove Theorem 2.3 in Section 6. We discuss application to message attacks on knapsack cryptosystems in Section 7, experimentally seek the optimal choice of \(m\) in Section 8, and finish by proving \((n, \ell, k)\)-splitting systems exist in general in Section 9.

3. Splitting Systems

Recall the definition of \(k\)-set splitting system given in the previous section, and that for now we assume both \(n\) and \(\ell\) are divisible by \(k\). In [18] it is proved that the probability of a random 2-division being good is \(\Omega(\ell^{-1/2})\) and that there is a trivial construction that yields a 2-set splitting system with \(n\) divisions. In this section we generalize these results for \(k\)-set splitting systems. Note that design theory may yield a construction of a \(k\)-set splitting system with fewer divisions, as Stinson showed that a 2-set splitting system exists with at most \(\ell^3/2\) divisions in [18].

The first result is a polynomial bound on the probability of choosing a good \(k\)-division randomly. One important note is that the constant depends exponentially on \(k\), so it is important that \(k\) be a fixed parameter.

**Proposition 3.1.** The probability of choosing a good \(k\)-division is bounded below by a constant times \(\ell^{-k/2}\).

**Proof.** First consider the number of ways of choosing \(k\) sets of \(n/k\) items from a total of \(n\) items. It is

\[
\frac{1}{k!} \left( \binom{n}{n/k} \binom{n - (k - 2)n/k}{n/k} \right) \cdots \left( \binom{n - (k - 2)n/k}{n/k} \right) = \frac{n!}{k! (\frac{n}{k})^k}
\]

where the extra \(1/k!\) term offsets the double counting that results from the \(k\) sets being indistinguishable.

This is also the number of \(k\) divisions. The number of good \(k\) divisions is counted by choosing \(k\) equal sized sets from \(Y\) and choosing \(k\) equal sized sets from \(X \setminus Y\). Thus the probability of choosing a good \(k\)-division is

\[
\frac{1}{k!} \cdot \frac{\ell!}{(\frac{n}{k})^k (\frac{n - \ell}{k})^k} \cdot \frac{n!}{(\frac{n}{k})^k}.
\]

We next find upper and lower bounds on \((n!)/(\frac{n}{k})^k\). Stirling’s formula gives us

\[
2n^ne^{-n}\sqrt{2\pi n} \geq n! \geq n^ne^{-n}\sqrt{2\pi n}.
\]

For the lower bound this implies

\[
\frac{n!}{(\frac{n}{k})^k} \leq \frac{2n^ne^{-n}\sqrt{2\pi n}}{(\frac{n}{k})^nk e^{-n/k}\sqrt{\frac{2\pi n}{k}}} = k^n \cdot 2^{k/2} (2\pi n)^{1-k/2}
\]

while similarly for the upper bound we have

\[
\frac{n!}{(\frac{n}{k})^k} \geq \frac{n^ne^{-n}\sqrt{2\pi n}}{(\frac{n}{k})^nk e^{-n/k}\sqrt{\frac{2\pi n}{k}}} = k^n \cdot 2^{-k} k^{k/2} (2\pi n)^{1-k/2}.
\]
This requires first proving that one of the sets $B_i$ satisfies $|B_i \cap \{0, \ldots, n-1\}| = \ell/k$ for some constant $n$ in $\mathcal{O}(\ell)$ time. We will assume for ease of exposition that $n$ is even.

By Proposition 3.3, there exists a $B_i$ such that $|B_i \cap \{0, \ldots, n-1\}| = \ell/k$. Call it $I_1$, and reorder the $a_i$ so that the indices in $I_1$ are the last $n/k$ indices.

Redefine the $B_i$ so that they still have size $n/k$, but now wrap modulo $n - n/k$ rather than $n$. Proposition 3.2 is still valid, and so there exists a $B_i \subseteq \{0, \ldots, n-1\}$ such that $|B_i \cap \{0, \ldots, n-1\}| = \ell/k$. Call it $I_2$, and reorder the $a_i$ so that the indices in $I_2$ are the last $n - n/k$ indices.

By continuing in this fashion, we find a good division. Only $I_1, \ldots, I_{k-1}$ need to be searched for, since $I_k$ consists of the leftover indices.

The number of divisions is the product of the number of $B_i$ searched for each of $I_1, \ldots, I_{k-1}$, which is

$$n \left( n - \frac{n}{k} \right) \left( n - \frac{2n}{k} \right) \cdots \left( n - \frac{(k-2)n}{k} \right) < n^{k-1}.$$ 

□

4. Applying Schroeppel-Shamir

Chor and Rivest in [11, Section 7.3] proposed that the general algorithm of Schroeppel and Shamir [14] may be applicable to the fixed weight subset sum problem. In this section we accomplish this, giving a square root time and fourth root space algorithm. The only missing ingredient was the idea of a 4-set splitting system. We will assume for ease of exposition that $n$ and $\ell$ are divisible by 4. See Section 3 for the general case.
We review the theory of problem decomposition presented in \cite{15}, though we specialize to the case of using a good 4-division to solve the \(\ell\)-weight subset sum problem.

The fixed weight subset sum problem has length \(n\) and weight \(\ell\). By Section 3 the problem can be decomposed into subproblems of length \(\frac{n}{4}\) and weight \(\frac{\ell}{4}\). As with all subset sum problems, this decomposition is sound, complete, and polynomial (see \cite{15} for definitions). However, it is not additive, and thus does not satisfy Schroeppel-Shamir’s definition of a composition operator. Fortunately, this lack does not affect the analysis of their algorithm, only the expression of the complexity.

In order to apply the Schroeppel-Shamir algorithm, our decomposition must have two essential properties.

**Definition 4.1.** A set of problems is *polynomially enumerable* if there is a polynomial time algorithm which finds for each bit string \(x\) the subset of problems which are solved by \(x\).

**Definition 4.2.** A composition operator \(\oplus\) is *monotonic* if the problems of each size can be totally ordered in such a way that \(\oplus\) behaves monotonically: if \(|P'| = |P''|\) and \(P' < P''\) then \(P \oplus P' < P \oplus P''\) and \(P' \oplus P' < P'' \oplus P'\).

Define a problem on set \(j, 1 \leq j \leq 4\) by \(\{b, \{a_i \mid i \in I_j\}\}\) where \(x\) of weight \(\ell/4\) is a solution if

\[
\sum_{i \in I_j} a_i x_i = b. 
\]

Define a composition operator by

\[
P_j \oplus P_j' = (b + b', \{a_i \mid i \in I_j \cup I_j'\}).
\]

This is polynomial and polynomially enumerable since addition is polynomial time. It is sound since if \(\sum_{i \in I_j} a_i x_i = b\) and \(\sum_{i \in I_j'} a_i x_i = b'\) then \(\sum_{i \in I_j \cup I_j'} a_i x_i = b + b'\). It is complete by the definition of a good division.

Finally, \(\oplus\) is monotonic if we order problems by their solution \(b\), and if this is equal then lexicographically by their sets \(\{a_i \mid i \in I_j\}\). For suppose that \((b', \{a_1', \ldots, a_n'/4\}) < (b'', \{a_1'', \ldots, a_n''/4\})\). Then

\[
(b' + b, \{a_1', \ldots, a_n'/4, a_1, \ldots, a_n\}) < (b'' + b, \{a_1'', \ldots, a_n''/4, a_1, \ldots, a_n\}) \quad \text{and} \quad (b + b', \{a_1, \ldots, a_n/4, a_1', \ldots, a_n'/4\}) < (b + b'', \{a_1, \ldots, a_n/4, a_1'', \ldots, a_n''/4\}).
\]

We now state the main theorem in the context of the \(\ell\)-weight subset sum problem.

**Theorem 4.3** (Schroeppel and Shamir \cite{15}). *If a set of problems is polynomially enumerable and has a monotonic composition operator, then instances can be solved in time \(\tilde{O}(n^{\ell/4})^2\) and space \(\tilde{O}(n^{\ell/4})\).*

The algorithm is summarized as follows. Let \(P\) be a problem of length \(n\) and weight \(\ell\) for which we seek a solution, and assume we are given a good division. For \(I_1, I_2, I_3, I_4\) enumerate all subproblems and store in tables \(T_j, 1 \leq j \leq 4\).

Sort \(T_2\) in increasing order and sort \(T_4\) in decreasing order. Make two queues (with arbitrary polynomial time insertions and deletions), with the first containing pairs \((P_1, \text{smallest } P_2)\) for all \(P_1 \in T_1\) and the other containing pairs \((P_3, \text{largest } P_4)\) for all \(P_4 \in T_4\). Now repeat the following until either a solution is found or both queues are empty (in which case there is no solution): compute \(S = (P_1 \oplus P_2) \oplus\)
\((P_3 \oplus P_4)\) and output \(S\) if \(S = P\). If \(S < P\) delete \((P_1, P_2)\) from the first queue and add \((P_1, P'_2)\) where \(P'_2\) is the successor of \(P_2\). If \(S > P\) delete \((P_3, P_4)\) from the second queue and add \((P_3, P'_4)\) where \(P'_4\) is the successor of \(P_4\). 

We conclude that if we have a good 4-division, the algorithm of Schroeppel and Shamir will solve the problem. By Propositions 3.1 and 3.3 we know that a good 4-division can be found in \(O(n^3)\) trials deterministically or expected \(O(\ell^{3/2})\) trials randomly. This inspires the following algorithm for the fixed weight subset sum problem.

**Algorithm 1** Schroeppel-Shamir for fixed weight subset sum

1. **Input:** positive integers \(a_1, \ldots, a_n, t, \ell\)
2. **Output:** \(x \in \{0, 1\}^n\) of weight \(\ell\) such that \(\sum_{i=1}^{n} a_i x_i = t\)
3. **while** no solution **do**
4. choose division \(D = \{I_1, I_2, I_3, I_4\}\)
5. for \(1 \leq j \leq 4\) form table \(T_j\) of problems, one for each weight \(\ell/4\) subset of \(I_j\)
6. apply Schroeppel-Shamir to \(T_1, T_2, T_3, T_4\)
7. **end while**

**Proof of Theorem 2.2.** The correctness follows from the monotonicity of \(\oplus\), see [15] for details. From [15], the maximum number of elements in either queue at any one time is \(\left(\frac{n}{\ell/4}\right)^2\) and the maximum number of steps needed is the number of pairs \((P_i, P_j) = \left(\frac{n}{\ell/4}\right)^2\). Thus the space complexity of Algorithm 1 is \(\tilde{O}\left(\left(\frac{n}{\ell/4}\right)^2\right)\) and the time complexity is \(\tilde{O}(n^{3/4}\left(\frac{n}{\ell/4}\right)^2)\) using deterministic splitting and \(\tilde{O}(\ell^{3/2}\left(\frac{n}{\ell/4}\right)^2)\) using randomized splitting.

As this work was inspired by Stinson’s paper [18] on the fixed weight discrete logarithm problem, it is worth noting that Algorithm 1 applies directly to that problem as well.

Also note that given a brute force running time of \(O(n^\ell)\), Algorithm 1 is a square root time and fourth root space algorithm, and hence lies on the tradeoff curve \(T \cdot S^2 = \binom{n}{\ell}\). This is justified by Stirling’s formula, which gives

\[
\left(\frac{n}{\ell/4}\right) = \Theta\left(\left(\frac{n}{\ell}\right)^{1/4} \left(\frac{n}{\ell(n-\ell)}\right)^{3/8}\right).
\]

5. **Modular Oracle**

Having proved Theorem 2.2, our task in the next two sections is to prove Theorem 2.3. Along with the notion of a \(k\)-division, the new ingredient needed is an oracle that for a given \(m\), returns a random solution of the modular subset sum problem over \(\mathbb{Z}/m\mathbb{Z}\). This oracle will be the multi-set birthday algorithm of Wagner [20], modified for the subset sum problem by Lyubashevsky [8] and proven correct in [17] (with complete proofs in [16]). In this section we present the multi-set birthday algorithm, modified to output a modular solution uniformly at random. In Section 6 we demonstrate how this applies to the integer fixed weight subset sum problem to finish the proof of Theorem 2.3.

Suppose we have lists \(L_1, \ldots, L_k\) of \(N\) elements drawn uniformly and independently from \(\mathbb{Z}/m\mathbb{Z}\) and a target \(t\). The \(k\)-set birthday problem is to find \(s_i \in L_i\)
with \( \sum s_i = t \mod m \). We can assume without loss of generality that our target is 0, since if it is not we can replace \( L_k \) with \( L_k - t \mod m \) and the elements will still be uniformly generated from \( \mathbb{Z}/m\mathbb{Z} \). Use the representation that places elements in the interval \( \left[ -\frac{m}{2}, \frac{m}{2} \right) \).

We will now briefly describe the original \( k \)-set algorithm from [20]. Assume that \( k \) is a power of 2, and define parameter \( p = m^{-1/(\log k + 1)} \). Let \( I_0 \) denote the interval \( \left[ -\frac{m}{2}, \frac{m}{2} \right) \) and in general let \( I_\lambda \) denote the interval \( \left[ -\frac{mp^\lambda}{2}, \frac{mp^\lambda}{2} \right) \). Denote by \( \triangleright_1 \) the list merging operator, so that \( L_1 \triangleright_1 L_2 \) is the set of elements \( a + b \in I \) where \( a \in L_1, b \in L_2 \) and addition is in \( \mathbb{Z} \). Let \( \triangleleft \) be the matching operator, so that \( L_1 \triangleleft L_2 \) outputs pairs \( (a, b) \) with \( a \in L_1, b \in L_2 \) such that \( a + b = 0 \) (over \( \mathbb{Z} \)).

These operators are instantiated as follows. For \( \triangleright_1 \), start by sorting \( L_1 \) and \( L_2 \). For each \( a \in L_1 \), search for \( b \) from \( L_2 \) that fall in the interval \( I - a \) and place all such \( a + b \) in the output list. Note that if \( L_1 \) and \( L_2 \) have size \( N \), then the complexity of this operator is \( O(N \log N) \) time and space. For \( \triangleleft \), sort \( L_1 \) and apply a random permutation to \( L_2 \). Then for each \( b \in L_2 \), search for \( -b \) in \( L_1 \). The complexity is again \( O(N \log N) \) time and space.

The \( k \)-set birthday algorithm proceeds as follows. For level \( \lambda, 1 \leq \lambda \leq \log k - 1 \), we denote lists by \( L^{(\lambda)} \) and apply the operator \( \triangleright_1 \) to pairs of lists. At level \( \log k \) we apply \( \triangleleft \) to the remaining pair of lists, and every element of \( L_1^{(\log k)} \triangleright_1 L_2^{(\log k)} \) is a solution to the problem. Here we deviate from Wagner slightly and have the algorithm output a random element from the result of \( \triangleleft \) to ensure that the output is a random modulo \( m \) solution. Pseudocode for this algorithm appears as Algorithm 2.

**Algorithm 2 Modular \( k \)-set Oracle**

1: **Input:** Lists \( L_1, \ldots, L_k \) of size \( N \), modulus \( m \), target \( t \)
2: **Output:** \( s_1, \ldots, s_k \) with \( s_i \in L_i \) such that \( s_1 + \cdots + s_k \equiv t \mod m \)
3: Set \( p = m^{-1/(\log k + 1)} \), ensure that \( N > 1/p \)
4: For all list elements use representation in \( \left[ -\frac{m}{2}, \frac{m}{2} \right) \)
5: for level \( \lambda = 1 \) to \( \log k - 1 \) do
6: apply \( \triangleright_1 \) to pairs of lists
7: end for
8: apply \( \triangleleft \) to the final pair of lists \( (L_1^{(\log k)}, L_2^{(\log k)}) \)
9: output an element of \( L_1^{(\log k)} \triangleleft L_2^{(\log k)} \) at random

We assume that with \( N = 1/p \), the size of \( L_1^{(\lambda)} \triangleleft L_2^{(\lambda)} \) is again a list of size \( 1/p \) for \( 1 \leq \lambda \leq \log k - 1 \). In [17] it is proven that list elements at all levels are close to uniform. Furthermore, if we assume the initial lists have size \( \alpha/p \) and modify the listmerge operator so that for each \( a \in L_1 \), exactly one \( b \) from \( L_2 \) is chosen so that \( a + b \in I \), then \( L_1^{(\lambda)} \triangleright_1 L_2^{(\lambda)} \) again has \( \alpha/p \) elements (with exponentially small failure probability). Here \( \alpha \) is a parameter chosen that depends on the requested chance of failure; for our purposes it suffices to know it is bounded by a polynomial in \( n \).

Now, our stated implementation of the listmerge operator keeps all sums \( a + b \in I \) because we want all solutions to have a chance at being found. Since having more elements at each level only increases the probability of the \( k \)-set algorithm succeeding, we have a rigorously analyzed algorithm if we accept an additional complexity factor of \( \alpha^{O(1)} = n^{O(1)} \).
Lemma 5.2 starting with the uniform elements in the interval \([-\frac{m^2}{2^{(\log k + 1)}}, \frac{m^2}{2^{(\log k + 1)}}]\), we conclude by the work in \([12]\) that \(L_1^{(\log k)} \gg L_2^{(\log k)}\) contains at least one element with positive probability, and thus that Algorithm \(2\) outputs a solution with positive probability. The complexity of the algorithm is the complexity of running \(\gg j\) a total of \(2k\) times, for a total of \(O(m^{1/(\log k + 1)})\) time and space.

5.1. Randomizing the Modular Oracle. Note that not every solution to the modular subset sum problem could be output by Algorithm \(2\). Inspired by a suggestion from \([20]\), our focus for the rest of this section will be on using Algorithm \(2\) to generate a random solution to the \(k\)-set birthday problem, one which has a nearly uniform distribution.

Define the 2-sums of the problem to be \(L_1 + L_2, L_3 + L_4, \ldots, L_{k-1} + L_k\), the 4-sums to be \(L_{4i+1} + L_{4i+2} + L_{4i+3} + L_{4i+4}\) for \(0 \leq i \leq \frac{k-4}{4}\), and so on up to the two \(k/2\)-sums \(L_1 + \cdots + L_{k/2}\) and \(L_{k/2+1} + \cdots + L_k\). This term will also be used for the corresponding sums of a particular solution \((s_1, \ldots, s_k)\). We refer to both integer sums and modular sums depending on whether the addition is over \(\mathbb{Z}\) or over \(\mathbb{Z}/m\mathbb{Z}\).

Let \(R\) be a set of \(\frac{2k}{3} - 1\) elements of \(\mathbb{Z}/m\mathbb{Z}\) generated uniformly at random. For each of the 4-sums, replace the lists \(L_1, L_2, L_3, L_4\) with \(L_1 + r_1, L_2 + r_2, L_3 - r_1, L_4 - r_2\) where \(r_1\) and \(r_2\) are two elements of \(R\). For each of the 8-sums, replace \(L_{8i+4}\) with \(L_{8i+4} + r\) and \(L_{8i+8}\) with \(L_{8i+8} - r\). In general, for each of the \(2^j\)-sums \((3 \leq j \leq \log k)\), replace \(L_{2^j i + 2^j - 1}\) with \(L_{2^j i + 2^j - 1} + r\) and \(L_{2^j i + 2^j}\) with \(L_{2^j i + 2^j} - r\).

All these operations are in \(\mathbb{Z}/m\mathbb{Z}\).

In the example of the 8-set algorithm \(R = \{r_1, r_2, r_3, r_4, r_5\}\) and lists \(L_1, \ldots, L_8\) are replaced by

\[
L_1 + r_1, L_2 + r_2, L_3 - r_1, L_4 - r_2, L_5 + r_3, L_6 + r_4, L_7 - r_3, L_8 - r_4 - r_5\ .
\]

We seek to prove that applying Algorithm \(2\) to lists modified in this way results in a solution drawn almost uniformly at random from the space of all solutions to the \(k\)-set birthday problem on fixed lists \(L_1, \ldots, L_k\). To classify which solutions are possible output we make the following definition.

Definition 5.1. Let a solution \(s_1 + \cdots + s_k\) modified in the above manner by a randomizing set \(R\) be denoted \(s'_1 + \cdots + s'_k\). Call a solution to the modulo \(m\) subset sum problem viable with respect to a randomizing set \(R\) if for \(1 \leq i \leq \log k - 1\), all integer \(2^i\)-sums \(s'\) satisfy \(s' \in I_i\).

We will also refer to an individual integral or modular \(2^i\)-sum \(s'\) as viable if \(s' \in I_i\).

Algorithm \(2\) performs additions in \(\mathbb{Z}\) despite the fact that a modular solution is sought. Our goal is to prove that the number of randomizing sets making a solution \(s\) viable is roughly equal. We first prove this for modular \(2^i\) sums with \(i \geq 2\) in Lemma \(5.2\) starting with the \(\frac{k}{2}\)-sums and working down. The integer 2-sums are analyzed in Lemma \(5.3\) from which the main theorem quickly follows. The key observation is that a modular solution with viable modular \(2^i\)-sums for all \(i > 2\) and viable integral 2-sums must also have viable integral \(2^i\)-sums for all \(i > 2\).

Lemma 5.2. Let \(\frac{k}{2^i} > 2\) and consider \(s + t\), the sum of two \(\frac{k}{2^i}\)-sums. Assuming that \(s + t \mod m \in I_{\log k - i + 1}\), the number of \(r\) such that \(s + r \mod m\) and \(t - r\)
\[ \text{mod } m \text{ simultaneously fall in } I_{\log k-i} \text{ is at least } mp^{\log k-i}(1 - p) - 1 \text{ and at most } mp^{\log k-i}. \]

**Proof.** Call an \( r \) value good if \( s + r \mod m \in I_{\log k-i} \) and \( t - r \mod m \in I_{\log k-i} \).

The maximum number of good \( r \) values occurs when \( s + t = 0 \mod m \). The size of \( I_{\log k-i} \) is \( |mp^{\log k-i}| \), and so this is the number of \( r \) such that \( s + r \mod m \in I_{\log k-i} \). Since \( s = -t \mod m \), the same set of \( r \) place \( t - r \mod m \in I_{\log k-i} \), and the same set of \( r \) place \( t - r \mod m \in I_{\log k-i} \) since the interval is symmetric.

The minimum occurs when \( s + t \leq \pm mp^{\log k-i+1} \). The number of \( r \in \mathbb{Z}/m\mathbb{Z} \) that place \( s + r \in I_{\log k-i} \) is \( |mp^{\log k-i}| \). The same set of \( r \) values place \( r - t + mp^{\log k-i+1} \in I_{\log k-i} \), but a total of \( mp^{\log k-i+1} \) of the \( r \) values are lost when we instead ask for \( t - r \mod m \in I_{\log k-i} \). So the number of valid \( r \) values is at least \( |mp^{\log k-i} - mp^{\log k-i+1}| \geq mp^{\log k-i}(1 - p) - 1. \)

Suppose that randomizers have been found that place the modular 4-sums of a solution in \( I_2 \). We now seek to place the integer 2-sums in \( I_1 \). Since we will be mixing integer addition and modular addition, we use \( \oplus \) to signify the latter. Recall that we are using \([-\frac{m}{2}, \frac{m}{2}]\) as the set of representatives for elements of \( \mathbb{Z}/m\mathbb{Z} \).

**Lemma 5.3.** Suppose that \( s_1 + s_2 + s_3 + s_4 \mod m \) is in \( I_2 \). Then the number of pairs \((r_1, r_2)\) such that

\[ (s_1 \oplus r_1) + (s_2 \oplus r_2) \in I_1 \text{ and } (s_3 \oplus r_1) + (s_4 \oplus r_2) \in I_1 \]

is at most \( m^2p \) and at least \( (m - 2mp)(mp - mp^2 - 1) \).

**Proof.** First, consider a fixed \( r_1 \), and let \( s_1' = s_1 \oplus r_1 \) and \( s_3' = s_3 \oplus r_1 \). Then we need \( s_2 \oplus r_2 \in I_1 - s_1' \), where the interval subtraction is over \( \mathbb{Z} \). The size of \( I_1 - s_1' \) might be as small as \( \frac{m}{2} \) if \( s_1' = \pm \frac{m}{2} \). Since we can choose \( r_2 \) such that \( s_2 \oplus r_2 \) is any element in \([-\frac{m}{2}, \frac{m}{2}]\), the number of such \( r_2 \) is the size of \( I_1 - s_1' \). Simultaneously \( r_2 \) must satisfy \( s_4 \oplus r_2 \in I_1 - s_3' \). There are two extremes, depending on whether \( s_1 + s_3 + s_4 = 0 \mod m \) or \( s_1 + s_2 + s_3 + s_4 = \pm mp^2 \mod m \).

In the first case, \( s_1' \oplus s_2 = -(s_3' \oplus s_4) \) and \( I_1 \) symmetric implies that there are at most \( mp \) values of \( r_2 \) such that \( s_1' \oplus s_2 \oplus r_2 \), \( s_3' \oplus s_4 \oplus r_2 \) are in \( I_1 \). Since switching to \( s_3' + s_2 \oplus r_2 \) and \( s_3' + s_4 \oplus r_2 \) can only reduce the number of valid \( r_2 \), \( mp \) is an upper bound.

However, if \( s_1' \oplus s_2 = mp^2 \oplus (s_3' \oplus s_4) \), then by the same argument from Lemma 5.2 the number of valid \( r_2 \) for the modular sums is \( |mp - mp^2| \). The number of valid \( r_2 \) for the integer sums could be smaller depending on the sizes of \( I_1 - s_1' \) and \( I_1 - s_3' \).

Now consider the size of \( I_1 - s_1' \) and \( I_1 - s_3' \) depending on \( r_1 \). When \( r_1 \) shifts by one, the intervals shift by one as well. The intervals will have less than full size when \( s_1 \oplus r_1 \) or \( s_3 \oplus r_1 \) is less than \( -\frac{m}{2} + mp \) or greater than \( \frac{m}{2} - mp \). Hence the number of \( r_1 \) that make for one of the intervals to have less than full size is at most \( 2mp \).

Thus the number of valid pairs \((r_1, r_2)\) is at most \( m^2p \) (assuming intervals full size for all \( r_1 \) and in case one above) and is at least \( (m - 2mp)(mp - mp^2 - 1) \) (assuming interval size taken from case two).

**Theorem 5.4.** Assume that \( \frac{m}{2} < \frac{1}{k} \). Let \( A \) be the event that a solution \( s = s_1 + \cdots + s_k \) is output by the modular oracle, given that some solution is output. Then the distribution of \( A \) is uniform within a factor of \((1 - 2p)^{3k/4} \).
Proof. We have $\Pr[s \text{ solution}] = \Pr[s \text{ viable}] \Pr[s \text{ solution} \mid s \text{ viable}]$, where we leave unwritten the assumption that some solution is output. We first bound $\Pr[s \text{ viable}]$.

We have $s_1 + \cdots + s_k = 0 \mod m$. Using the same argument as in the first case of Lemma 5.2, there are $mp\log k - 1$ values of $r$ such that $s_1 + \cdots + s_{k/2} + r \mod m$ and $s_{k/2+1} + \cdots + s_k - r \mod m$ both fall in $I_{\log k - 1}$.

Using this as the base case and Lemma 5.2 as the inductive step, we have upper and lower bounds on the number of randomizers at each level. Given randomizers that place modular sums in the proper interval, and in particular that place modular 4-sums in $I_2$, Lemma 5.3 gives us the number of randomizers that place integer 2-sums in $I_1$. Thus our modified solution $s'_1 + \cdots + s'_k$ is a modular solution with integer 2-sums, which since $k \cdot \frac{mp}{2} < m$ implies that all integer 2-sums lie in $I_i$, and hence that the solution is viable with respect to those randomizing sets.

There are a total of $m^{3k/4 - 1}$ randomizing sets. Combining the bounds from Lemmas 5.2 and 5.3 gives the following bounds on the number for which $s$ is viable.

Setting $N = \frac{k}{4} + \frac{2k}{8} + \frac{3k}{16} + \cdots + (\log k - 1)\frac{k}{p}$ an upper bound is given by

$$(m^2p)^{k/4} \cdot (mp^{2})^{k/8} \cdot (mp^{3})^{k/16} \cdots (mp)^{\log k - 1)} = m^{3k/4 - 1} \cdot p^N.$$

Noting that $mp^{\log k - i} - mp^{\log k - i+1} - 1 \geq mp^{\log k - i}(1 - 2p)$ a lower bound is given by

$$(m^2p(1 - 2p)^2)^{k/4} \cdot (mp^2(1 - 2p))^{k/8} \cdot (mp^3(1 - 2p))^{k/16} \cdots (mp^{\log k - 1})(1 - 2p)$$

$$= m^{3k/4 - 1} \cdot (1 - 2p)^{3k/4 - 1} \cdot p^N.$$

Thus $\Pr[s \text{ viable}]$ is uniform on the upper bound and uniform within a factor of $(1 - 2p)^{3k/4}$ on the lower bound.

We now consider the second term. Algorithm 2 is written so that for a given set of randomizers, a solution is output uniformly at random from the set of viable solutions. Since the number of viable solutions is bounded by $\Pr[s \text{ viable}]$ times the number of solutions, the fact that $\Pr[s \text{ viable}]$ is close to uniform makes $\Pr[s \text{ solution} \mid s \text{ viable}]$ close to uniform, but with the factors on the upper and lower bounds switched.

Thus upper and lower bounds for the probability of the event $A$ are separated from uniform by a factor of $(1 - 2p)^{3k/4}$.

6. The k-Set Algorithm

In this section we utilize the $k$-set modular oracle in designing an algorithm for the fixed weight subset sum problem. Lyubashevsky [8] was the first to leverage an algorithm for the modular subset sum problem out of an algorithm for the $k$-set birthday problem. Our modifications include dealing with the fixed weight nature of the problem by employing a $k$-division, and dealing with the integral nature of the problem by looping on the modular oracle until an integer solution is found. The pseudocode appears as Algorithm 3.

If $\ell$ is small compared to $k$, one could instead solve the $(n - \ell)$-weight subset sum problem with target $(\sum_{i=1}^{n} a_i) - t$.

Algorithm 2 takes as input uniformly distributed elements of $\mathbb{Z}/m\mathbb{Z}$. By the work in [6], if $a_1 \mod m, \ldots, a_n \mod m$ are uniformly distributed over $\mathbb{Z}/m\mathbb{Z}$, then random $n/k$-length, $\ell/k$-weight subsets of these elements will be exponentially close to uniform as long as $m < \binom{n}{\ell/k}$. If in addition we seed the lists with $\text{poly}(n)$.
Algorithm 3 Multi-set Algorithm for Fixed Weight Subset Sum

1: **Input:** positive integers $a_1, \ldots, a_n$, target $t$, weight $\ell$, parameters $k$, $m$
2: **Output:** $x \in \{0, 1\}^n$ of weight $\ell$ with $\sum_{i=1}^n a_i x_i = t$
3: while no integer solution do
4: choose random $k$-division $(I_1, \ldots, I_k)$
5: choose set $R$ of $\frac{n}{k} - 1$ random elements of $\mathbb{Z}/m\mathbb{Z}$.
6: form lists $L_1, \ldots, L_k$ of size $\frac{m}{1/(\log k+1)}$ whose elements are random subsets of weight $\ell/k$ from appropriate $I_j$, reduced modulo $m$
7: apply randomizers from $R$ to lists as described in Section 5.1
8: apply Algorithm 2 to $L_1, \ldots, L_k$
9: if success then
10: check if integer solution
11: end if
12: end while

$m^{1/(\log k+1)}$ elements, then combined with the work of Section 5.1 we get a rigorous analysis of Algorithm 3.

We now prove Theorem 2.3 (restated here for convenience) by analyzing Algorithm 3. To solve the integer fixed weight subset sum problem, we make an appropriate choice of $m$, which determines the resulting point on the time-space tradeoff curve. The necessary assumption that $\frac{k}{2} < \frac{1}{k}$ in Theorem 5.4 is satisfied by choosing $m$ and $k$ so that $\log m \geq 2(\log k)^2$.

**Theorem 6.1.** Choose parameters $m$ and $k$ so that $k$ is a power of 2, $m < \binom{n}{\ell/k}$, and $\log m \geq 2(\log k)^2$. Assume that when reduced modulo $m$, the $a_i$ are uniformly random elements of $\mathbb{Z}/m\mathbb{Z}$. Then the expected running time of Algorithm 3 is $O(m^{1/(\log k+1)} \cdot \binom{n}{\ell}/m)$ and the algorithm uses $O(m^{1/(\log k+1)})$ space. This gives a point on the time/space tradeoff curve $T \cdot S^{\log k} = \binom{n}{\ell}$.

**Proof.** The probability that Algorithm 3 finds a solution on a particular iteration of the while loop is the product of three probabilities: the probability that the $k$-division is good with respect to some unknown solution, the probability that Algorithm 2 succeeds, and the probability that the modular solution found by Algorithm 2 is also the integer solution.

By Proposition 3.1 the first term is greater than $\ell^{1-k}$. The second probability is greater than some fixed $\epsilon$ by the previous work outlined in Section 5. For the third term, we first call upon a theorem of Impagliazzo and Naor [6] (proven using the leftover hash lemma) which tells us that with the $a_i$ drawn uniformly at random from $\mathbb{Z}/m\mathbb{Z}$ and $m < \binom{n}{\ell}$, the distribution of random $\ell$-weight subsets is exponentially close to uniform. Thus we expect the number of modular solutions to be a constant times $\binom{n}{\ell}/m$. By Theorem 5.4 we conclude that the third probability factor is greater than $(1 - 2p)^{3k/4} \cdot m/\binom{n}{\ell}$. Note that $(1 - 2p)^{3k/4} \geq 1 - \frac{3k}{2}p \geq \frac{1}{2}$ since $\log m \geq 2(\log k)^2$ implies $p = m^{-1/(\log k+1)} \leq \frac{1}{3k}$.

Thus the expected number of iterations of the while loop is

$$O\left(\epsilon \ell^{k-1} \cdot 2\binom{n}{\ell}/m\right).$$
The cost of each iteration is dominated by Algorithm 2, which takes $\tilde{O}(m^{1/(\log k + 1)})$ time and space.

Thus Algorithm 3 takes expected time $\tilde{O}\left(\frac{m^{1}}{\log k + 1}\cdot \binom{n}{\ell}/m\right)$ and space $\tilde{O}(m^{1/(\log k + 1)})$, which is a point on the time and space tradeoff curve $T \cdot S^{\log k} = \binom{n}{\ell}$.

As an example of parameter choices in action, suppose we wish to solve an integer fixed weight subset sum problem with an 8-set birthday algorithm. Our conjectural maximal choice of $m$ is $\frac{(\ell/8)^4}{\ell} \approx (\ell^{1/2})$. Thus we expect the problem to be solved in time $\tilde{O}\left(\binom{n}{\ell}^{1/8}\binom{n}{\ell}^{1/2}\right)$ and space $\tilde{O}\left(\binom{n}{\ell}^{1/8}\right)$.

Note that Algorithm 3 is highly parallelizable, since running it simultaneously on $N$ processors increases the probability of success by a factor of $N$.

7. Application to Knapsack Cryptosystems

Knapsack cryptosystem is the term used for a class of public key cryptosystems whose underlying hard problem is the integer subset sum problem. Though few have remained unbroken, the search for knapsack cryptosystems remains popular due to their fast encryption and easy implementation.

A knapsack cryptosystem is defined abstractly as follows. We have a public key $(a_1, \ldots, a_n)$ defining a hard subset sum problem, and a private key which transforms the hard problem into an easy subset sum problem. To send a message $x \in \{0, 1\}^n$, a user computes $t = \sum_{i=1}^{n} a_i x_i$ and sends it. The receiver, who has the private key, transforms the problem and then solves the easy subset sum problem to recover $x$.

There are two main attacks on knapsack cryptosystems. First, there are key attacks which attempt to recover the easy subset sum problem from the public key. Second, there are message attacks which attempt to recover the message by solving the hard subset sum problem $a_1 x_1 + \cdots + a_n x_n = t$. Key attacks are not our concern in this paper, we simply note that many systems have succumbed to such attacks, the seminal cryptosystem of Merkle-Hellman [9] among them. We focus instead on message attacks, which are equivalent to solving the subset sum problem or its variants.

The most successful message attack in theory and in practice is the low-density attack that reduces the subset sum problem to the shortest vector problem or the closest vector problem, discussed in Section 2. Since unique decryption requires $2^n \leq \sum_{i=1}^{n} a_i$, and hence that the density be no more than a little above 1, these results pose a conundrum for the knapsack designer. As a result, modern designs have relied on fixing the hamming weight of allowed messages, so that the underlying hard problem becomes the fixed weight subset sum problem. This began with Chor-Rivest [1] and continues into the present with the notable OTU scheme [14] and its non-quantum variant [4]. In this way $n$ can be made great enough so that the density is above one, while the information density stays below one to preserve unique decryption. As an added bonus, the fixed weight subset sum problem has received much less attention in the literature, and so message attacks remain in a primitive state. Until recently the only known algorithm was the square root time-space tradeoff algorithm in [1, Section 7.3].

Here we have only scratched the surface of the vast literature on knapsack cryptosystems. For further information consult the survey [13].

The new result in this paper is Theorem 2.2 from which we immediately get a message attack that takes square root time and fourth root space. Theorem 2.3
is less interesting from this perspective because the large constant and polynomial terms, along with the sharp upper bound on the size of $m$, mean that seldom would the $k$-division algorithm reach even square root time in practice.

8. Data and Conclusions

In this section we explore experimentally two questions related to Algorithm 3. The first is to measure the number of times Algorithm 2 succeeds before an integer solution is found, and to compare that to the expected number $\binom{n}{\ell} / m$. The second is to measure the success probability of Algorithm 2 when the modular information density is pushed lower than Theorem 2.3 requires. In particular, $m$ cannot be larger than $\binom{n/k}{\ell/k} \log k + 1$ since otherwise there will not be enough weight $\ell/k$ subsets to fill the lists $L_j$, so we choose $m$ between $\binom{n/k}{\ell/k}$ and $\binom{n/k}{\ell/k} \log k + 1$.

We implemented 2-set, 4-set, and 8-set algorithms for the modular subset sum problem and applied them to the integer subset sum problem. We chose not to explore the additional impact of searching for a $k$-division, since the probability calculation is straightforward. We ran these algorithms on a desktop workstation on problems with $n$ equal to 24 and an integer density of 0.9.

In the tables that follow $d_m$ denotes the modular density. Each entry represents the mean over ten trials, except those marked with a * which represent the result after one trial. Let $N_o$ be the number of modular oracle successes before an integer solution is found.

| $d_m$ | $N_o$ | $E[N_o]$ | $N_o$ | $E[N_o]$ | $N_o$ | $E[N_o]$ |
|-------|-------|----------|-------|----------|-------|----------|
| 1.5   | 209   | 256      | 1955  | 4096     | 33000 | 262000   |
| 2     | 168   | 256      | 5436  | 4096     | 26000 | 262000   |
| 4     | 265*  | 256      | 1831  | 4096     | 33000 | 262000   |

The next table explores the effect that parameters $m$ and $k$ have on Algorithm 3.

| $d_m$ | success % | time (s) | success % | time (s) | success % | time (s) |
|-------|------------|----------|------------|----------|------------|----------|
| 1.5   | 58.9 %     | 15       | 61.4 %     | 28       | 58.1 %     | 466      |
| 2     | 19.8 %     | 121      | 40.5 %     | 336      | 46.7 %     | 1594     |
| 4     | 0.7* %     | 11058*   | 11.9 %     | 945      | 57.2 %     | 6069     |

Taken together, this data supports our heuristic analysis of Algorithm 3. We see that the modular oracle succeeds with some constant probability, and that the number of successful oracle calls needed is roughly the expected number (though the variance is quite large).

We also see that despite a lower success percentage, choosing $d_m$ as small as possible results in a faster running time. There is a boundary beyond which the algorithm succeeds too rarely to be of any use, as exemplified by the 8-set algorithm with $d_m = 1.5$. A reasonable conjecture places this boundary at $d_m = \frac{k}{\log k + 1}$, since below this point, there are not enough subsets to fill the lists $L_1, \ldots, L_k$ with $m^{1/(\log k + 1)}$ elements.
As $k$ increases the overhead associated with the more complicated algorithms outstrips their asymptotic improvement, at least for $n = 24$. It is unclear how large $n$ will have to be before the 8-set algorithm is faster than the 2-set algorithm for $d_m = 4$.

9. Splitting Systems in the Indivisibility Case

In Section 8 we presented $(n, \ell, k)$-splitting systems and proved their existence under the assumption that $n$ and $\ell$ were divisible by $k$. In this section we relax this restriction, showing that splitting systems exist when $n$, $\ell$ are any positive integers greater than $k$. Let positive integers $r_1$, $r_2$ be defined by $n = k \cdot \lceil n/k \rceil + r_1$ and $\ell = k \cdot \lceil \ell/k \rceil + r_2$.

**Definition 9.1.** A $(n, \ell, k)$-splitting system is a set $X$ of $n$ indices along with a set $D$ of divisions, where each division is itself a set $\{I_1, \ldots, I_k\}$ of subsets of indices. Here the $I_j$ partition $X$ and their sizes satisfy $|I_1| = \cdots = |I_{k-1}| = \lceil n/k \rceil$, $|I_k| = \lceil n/k \rceil + r_1$. A splitting system has the property that for every $Y \subseteq X$ such that $|Y| = \ell$, there exists a division $\{I_1, \ldots, I_k\} \in D$ such that $|Y \cap I_k| = |\ell/k|$ for $1 \leq j \leq k - 1$ and $|Y \cap I_k| = |\ell/k| + r_2$.

Again, with $n, \ell, Y$ understood as parameters of a fixed weight subset sum problem we are interested in solving, we refer to an $(n, \ell, k)$-splitting system as a $k$-set splitting system.

Most likely a better strategy in practice would be to spread the extra weight among the $I_j$ rather than assigning it all to $I_k$. This definition was chosen to quickly demonstrate that nondivision poses no barrier in theory. The key result is to prove the existence of this more general structure. In order to do this, we will first find $I_1, \ldots, I_{k-1}$, and leave the remainder of $X$ to $I_k$. Our candidates will be

$$B_i^{(n)} = \{i + j \mod n \mid 0 \leq j \leq \lceil n/k \rceil\}.$$

Given a fixed $Y \subset X$ of size $\ell$, we define a function $\nu$ be $\nu(i) = |B_i \cap Y| - \lfloor \ell/k \rfloor$.

**Proposition 9.2.** There exists a $k$-set splitting system with fewer than $n^{k-1}$ divisions.

**Proof.** Our initial goal is to prove that there must exist an $i$ with $\nu(i) = 0$. Consider $B_0$, $B_{\lfloor n/k \rfloor}$, $B_{\lfloor n/k \rfloor + 1}$, $\ldots$, $B_{(k-1)\lfloor n/k \rfloor}$. Define $B$ to be the remainder of the indices of $X$. If $\nu(i) = 0$ for one of $i = 0, \lfloor n/k \rfloor, \ldots, (k-2)\lfloor n/k \rfloor$ then we are done. If not, we wish to find $i, i'$ such that $\nu(i), \nu(i')$ have opposite signs.

If $\nu(i) > 0$ for each of $i = 0, \lfloor n/k \rfloor, \ldots, (k-2)\lfloor n/k \rfloor$, then the combined weight of the corresponding $B_i$ is at least $(k-1)\lfloor \ell/k \rfloor + k - 1$ and so $B$ must have weight less than $\lfloor \ell/k \rfloor + r_2 - (k-1) \leq \lfloor \ell/k \rfloor$. Thus in particular $B_{(k-1)\lfloor n/k \rfloor}$, the first $\lfloor n/k \rfloor$ indices of $B$, must have weight less than $\lfloor \ell/k \rfloor$.

If $\nu(i) < 0$ for $i = 0, \lfloor n/k \rfloor, \ldots, (k-2)\lfloor n/k \rfloor$, then the combined weight of the corresponding $B_i$ is at most $(k-1)\lfloor \ell/k \rfloor - (k-1)$ and so $B$ must have weight greater than $\lfloor \ell/k \rfloor + r_2 + k - 1$. Then $B_{(k-1)\lfloor n/k \rfloor}$, the first $\lfloor n/k \rfloor$ indices of $B$, must have weight greater than $\lfloor \ell/k \rfloor$. For if not, the weight of $B$ is at most $\ell/k + r_1 \leq \lfloor \ell/k \rfloor + r_2 + k - 1$, a contradiction.

In either case there is an $i$ with $\nu(i) > 0$ and an $i'$ with $\nu(i') < 0$. Since $|\nu(i) - \nu(i+1)| \leq 1$, there must be an $i$ with $\nu(i) = 0$. Label the corresponding set $I_1$. 

We now remove the indices in $I_1$ from consideration, relabel the indices $0,\ldots,n-[n/k]$, and seek an $i$ such that $B^i_{n-[n/k]}$ has weight $\lfloor \ell/k \rfloor$. Using the same reasoning as above, one must exist.

In this fashion $I_1,\ldots,I_{k-1}$ can be found. The remaining indices make up $I_k$. The number of divisions needed to satisfy this process is

$$n(n-[n/k])(n-2[n/k])\cdots(n-(k-2)[n/k]) < n^{k-1}.$$  

□

Next we discuss the effect on running times. For the Shroeppe–Shamir algorithm, the main terms of the complexity bounds become

$$\left(\frac{n}{4}\right)^3 \left(\frac{1}{4}\right)^3$$

space. Since

$$\left(\frac{n}{4}+3\right)^3 = \left(\frac{n}{4}\right)^3 \left(\frac{1}{4}\right)^3$$

the complexity is worse by at most a polynomial factor. A similar result holds for the modular oracle. The polynomial factor becomes $(n/\ell)^k$, which is polynomial for constant $k$.

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