Leading Temperature Corrections to Fermi Liquid Theory in Two Dimensions

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We calculate the basic parameters of the Fermi Liquid: the scattering vertex, the Landau interaction function, the effective mass, and physical susceptibilities for a model of two-dimensional (2D) fermions with a short ranged interaction at non-zero temperature. The leading temperature dependences of the spin components of the scattering vertex, the Landau function, and the spin susceptibility are found to be linear. $T$-linear terms in the effective mass and in the “charge-sector”-quantities are found to cancel to second order in the interaction, but the cancellation is argued not to be generic. The connection with previous studies of the 2D Fermi-Liquid parameters is discussed.

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The question of the low-energy behavior of two-dimensional (2D) Fermi Liquid (FL) is of long-standing and of fundamental importance. One important motivation has been the non-Fermi-Liquid behavior observed in high-$T_c$ superconductors above $T_c$. Interest has grown in recent years also because these leading corrections provide the “bare” temperature dependence of the parameters in theories describing quantum critical phenomena in metals Indeed a number of surprising experimental results have been argued to imply an unusual underlying temperature, momentum or frequency dependence of electronic susceptibilities.

Rather surprisingly, the issue of the leading temperature corrections to Fermi Liquid Theory (FLT) remains controversial at certain points. For example, the textbook Sommerfeld expansion suggests that physical quantities may in general be expanded in powers of $T^2$. However, this is known not to be true. In particular, it was found that the leading temperature correction to the specific heat coefficient $\gamma = C/T$ was $T^2 \ln T$ in d=3 spatial dimensions and $T$ in d=2. Whether the spin and charge susceptibilities display similarly anomalous (i.e., non-$T^2$) temperature dependence is a subject of a contradictory literature: see, e.g., Ref. [13], discussion and references there in. For a most recent reassessment of such results see Ref. [14]. The prevailing conclusion was that of Carneiro and Pethick who found no leading $T^2 \ln T$ correction to the spin susceptibility of the 3D FL. Their arguments imply that terms $\sim T$ are absent in 2D.

The heuristic argument for the absence of anomalous terms in $T$ or in $q$ in response functions is that although these terms are known to occur in individual diagrams, they cancel in physical quantities due to Ward identities. We note that in the existing literature on this point it is assumed that the crucial coupling is between quasi-particles and long-wavelength collective modes. However the possibility of “2$k_F$ singularities”, i.e., anomalous temperature terms coming from processes involving large ($\sim 2k_F$) momentum transfers, have been discussed in the context of semiconductor physics. Stern was the first to note that in a 2D electron gas the electron scattering rate was proportional to $T$ due to 2$k_F$ effects. The consequences of the 2$k_F$ effects for the leading $T$-dependence of 2D FL quantities seem not to have been considered in the literature. In this paper we present an analysis taking into account both 2$k_F$ effects and Ward identities.

The issue of the leading correction to FLT have recently been revived by two different papers. Belitz, Kirkpatrick and Vojta presented perturbative calculations, mode-coupling arguments and power counting estimates which showed that the leading $q$ dependence of the spin susceptibility (but not the charge susceptibility) was $|q|$ in 2D and $q^2 \ln q$ in 3D. They did not find the analogous $T$-correction explicitly, but concluded however that one should generally expect a linear $T$-term in the 2D FL susceptibility ($T^2 \ln T$ in 3D). This dependence has important implications for the theory of the quantum critical metallic ferromagnet. They also focused on long-wavelength contributions.

In the other study, Sénéchal and one of us predicted the occurrence of the linear $T$-corrections to the FL vertices from one-loop Renormalization Group (RG) calculations based on a 2D effective action. However, the behavior of other FL quantities was not determined.

To elucidate the question in the most transparent way, we apply perturbation theory for 2D contact-interacting spin-$\frac{1}{2}$ fermions, starting from a microscopic action. Although Landau FLT is not a perturbative theory, for sufficiently weak repulsive interactions one should be able to find the parameters of a stable FL in terms of coupling series. Since both papers predict the linear $T$-terms appearing in the second order of the effective interaction, this effect should be seen perturbatively at second order in the microscopic interaction coupling. We present what is apparently the first exact calculation of the leading $T$-dependence of the effective mass, Landau parameters and response functions of a 2D electron gas, to second order in the interaction strength, including all channels and all momentum processes.

- **The model:** We treat interacting fermions at finite temperature in the standard path integral Grassmannian formalism. The partition function is given by the path integral $Z = \int \mathcal{D}\psi/\mathcal{D}\overline{\psi} \exp(S_0 + S_{\text{int}})$, where
We have adopted the condensed notations: \( (i) \equiv (k_i, \omega_i) \) and \( \int (i) \equiv \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \sum_{\omega_i, \beta} \). \( \beta \) is the inverse temperature, \( \mu \) the chemical potential, \( \omega_i \) the fermion Matsubara frequencies and \( \psi_{\alpha}(i) \) a two-component Grassmann field with a spin index \( \alpha \). Summation over repeated indices is implicit throughout this paper. We set \( k_B = 1 \) and \( \hbar = 1 \). We consider mainly electrons with the bare spectrum of a free gas \( \epsilon(k) = k^2/2m \) and the circular Fermi surface, but discuss the consequences of generic spectra and a non-circular Fermi surface below. We take

\[
S_0 = \int \left( 1 \right) \bar{\psi}_\alpha(1) \left[ i\omega_1 + \mu - \epsilon(k_1) \right] \psi_\alpha(1)
\]  

(1)

We should mention that taken separately, each contribution of the ZS' or BCS bubble gives a leading \( T \)-term to the Fourier components of the vertices. However, a cancellation of such terms coming from two graphs occurs in the "charge sector" (i.e., in \( A, F \) components), while the linear \( T \)-terms survive in the "spin sector" (\( B, G \) components). The temperature dependence of the ZS' contribution to the Fourier components of the vertices comes from integration around the "effective transfer" through the loop \( |k_1 - k_2| \sim 2k_F \), i.e., when incoming momenta \( k_1 \sim -k_2 \). In the same vein, the temperature dependence of the BCS contribution into the Fourier components of the vertices comes from regions of small \( k_1 + k_2 \), i.e., again when \( k_1 \sim -k_2 \). In other words, the temperature dependence comes from what we called previously "2\( k_F \)-effects".

We expect that the cancellation of the \( T \)-linear terms in the charge sector of the vertices \( \Re \) is an artefact of our simple model calculation in which all three one-loop terms have the same factor \( u^2 \) in front of the bubble contributions. Had we had a bare coupling function of, say, two incoming momenta and transfer, then the coupling factors would have been different in each of the three graphs, and, the anomalous \( T \)-corrections would not have cancelled.

The linearity of the leading \( T \)-corrections to the vertices seems to be generic. The same \( T \)-dependence of the BCS contribution into the Fourier components of the vertices seems to be generic. The same \( T \)-dependence of the BCS contribution into the Fourier components of the vertices seems to be generic.

\( m^* \) is the effective mass, and the uniform response functions.

- **Four-point 1PI vertex and the FL vertices:**

The 1PI vertex \( \Gamma(1, 2; \Omega) \) is defined in the standard way (see, e.g., Ref.\[3\]), where the transfer vector \( \Omega = (\mathbf{q}, \Omega) \), and \( \Omega \) is a bosonic Matsubara frequency. To shorten notations we denote operators in spin space with a circumflex. For this model we calculate the vertices, effective mass, and the uniform response functions.

- **Scattering vertex and Landau function:**

The one-loop approximation for \( \Gamma(1, 2; \Omega) \) in diagrammatic form is given in Fig. 1. In this approximation we calculate the FL vertices (scattering vertex, Landau function) using definition \( \Re \). At the one-loop level we can put \( \nu_R = \nu_0 \) (\( m^* = m \)) and \( Z = 1 \). By doing the direct analytic evaluation of the each diagram’s contribution to the vertex we find the Fourier components of the angular-dependent FL vertices in terms of the temperature series. This series comes from the contributions of the ZS' and BCS loops. The details will be given in a companion paper.

We find the leading temperature corrections to first two Fourier components of the vertices:

\[
\delta A_0 = \delta F_0 = \delta A_1 = \delta F_1 = -u^2 \pi^2 \frac{T^2}{4 \pi} \frac{1}{k_F^4}
\]

(4)

\[
\delta B_0 = \delta G_0 = -\delta B_1 = -\delta G_1 = -u^2 \frac{T}{E_F} \ln 2
\]

(5)

To second order we have

\[
\frac{m^*}{m} = 1 - \left[ \frac{\partial \Sigma(1)}{\partial \omega} \frac{m}{k_F} + \frac{\partial \Sigma(1)}{\partial k} \frac{k_1}{k_F} \right] \bigg|_{k_1 \in S_F}
\]

(7)

Using then two Ward identities following from the charge conservation and Galilean invariance, the above equation can be written as:

\[
\frac{m^*}{m} = 1 - \frac{1}{2} \int \frac{k_1 k_2}{k_F^2} \Gamma_\alpha^{\beta}(1, 2; \Omega \to 0) \Delta(2) \bigg|_{k_1 \in S_F}
\]

(8)
\[ \Delta(n) = \frac{\beta \cdot \delta(\omega_n - \xi_{k_n})}{4 \cosh^2\left(\frac{1}{2} \beta \xi_{k_n}\right)} \]  

(9)

Within our accuracy we can use the one-loop approximation for the vertex in Eq. (6). One can easily verify that in the the zero-temperature limit Eq. (6) recovers the standard result of the FLT if, i.e., \( m^*(T = 0)/m = 1 + F_1(T = 0) \). A straightforward extension of this relationship to finite temperatures like \( m^*(T)/m = 1 + F_1(T) \) is not valid since according to Eq. (6) \( m^*(T) \) contains an extra contribution from the “off-shell” integration over \( k_2 \) normal to the Fermi surface, albeit the factor \( \beta / \cosh(\beta \xi_{k_n}/2) \) makes this contribution well-localized near the Fermi surface. In other words the vertex entering the r.h.s. of Eq. (6) is not exactly the FL vertex \( F(T) \) (up to the normalization factor) as it is defined in the FLT, since one of its momenta (namely, \( k_2 \)) is not confined to the Fermi surface. After calculations we find that the linear-temperature terms, coming essentially from two one-loop contributions (ZS’, BCS) to the vertex, cancel, resulting in

\[ \frac{m^*}{m} = 1 + \frac{1}{2} u^2 + \mathcal{O}(u^2T^2) \]  

(10)

In a close analogy with the cancellation of the linear-temperature terms in the Fourier components of the FL vertices \( A(F) \), here the cancellation occurs between additive linear-\( T \) corrections coming from both “on-shell” (i.e., linear \( T \)-term coming from the \( 2k_F \)-contribution to the vertex) and “off-shell” (i.e., the small-momentum component) integrations in two diagrams. Moreover, the “on-shell” (“off-shell”) \( T \)-term of the ZS’ graph cancels the “on-shell” (“off-shell”) \( T \)-term of the BCS graph, correspondingly. We expect the cancellation does not occur at higher orders in the interaction. We have also evaluated [6] for a generic 2D Fermi surface without explicitly using Ward identities, finding a \( T \)-linear term [3,4].

The result may be expressed as the sum of two terms, one arising from \( 2k_F \) processes and one from long-wavelength processes. The two contributions cancel for a circular Fermi surface, but not generically.

Calculations of the order \( u^2 \) term in the free energy show that analogous cancellations occur and there is no \( T \)-linear term in the specific heat coefficient \( \gamma \), contrary to the results of Ref. [3].

• Response functions: Using the same Ward identities as in the effective mass calculation, we found for the dynamic zero-transfer limit (\( \Omega = 0 \), \( q \to 0 \)) of the density response function:

\[ \chi = \frac{m}{\pi} \left\{ 1 + u^2 + f_1 - f_0 \right\} \]  

(11)

\[ f_1 = -\frac{\pi}{m} \int_{(1,2)} \Delta(1) \Gamma^{\alpha\beta}_{\alpha\beta}(1, 2; \Omega \to 0) \Delta(2) \]  

\[ f_0 = -\frac{\pi}{m} \int_{(1,2)} \Delta(1) \Gamma^{\alpha\beta}_{\alpha\beta}(1, 2; \Omega \to 0) \Delta(2) \]  

At \( T = 0 \) we can read off from Eq. (11) \( \chi(T = 0) = \frac{m}{\pi} \frac{1}{1 + F_1^2 + F_1 - F_0} \) which is nothing but the FLT result [2].

\[ \chi_{FLT} = m \frac{1 + F_1}{n^2} \]  

expanded up to the third order over the interaction. Adding into consideration the Ward identity following from the total spin conservation, we derived for the uniform spin susceptibility (for details see Ref. [17]):

\[ \chi = \frac{m g^2}{4\pi} \left\{ 1 + u^2 + f_1 - g_0 \right\} \]  

(12)

\[ g_0 = -\frac{\pi}{3m} \int_{(1,2)} \Delta(1) \sigma^{\alpha\gamma}_{\gamma\alpha} \sigma^{\beta\delta}_{\beta\delta} \Gamma^{\alpha\beta}_{\gamma\delta}(1, 2; \Omega \to 0) \Delta(2) \]  

where \( g \) stands for the gyromagnetic ratio. Once again, one can see that in the zero-temperature limit the above equation gives \( \chi(T = 0) = \frac{m g^2}{4\pi} (1 + G_0^2 + F_1 - G_0) \) reproducing thus the second-order expansion of the FLT result [2].

We were able to analytically calculate the integrals on the r.h.s. of Eqs. (11) [3] in the leading order of their temperature dependence. We found that the leading linear \( T \)-corrections, which can be traced back to the ZS and BCS-loop contributions to the vertex, cancel in each of the integral terms \( f_0 \) and \( f_1 \) in Eq. (11) separately. The result for the density response is:

\[ \chi = \frac{m}{\pi} \left( 1 - \frac{1}{2} u^2 + \frac{u^2 \ln 2\Lambda_{k_F}}{k_F} \right) + \mathcal{O}(u^2T^2) \]  

(13)

where \( \Lambda > k_F \) is the ultraviolet cutoff we introduced to regularize the BCS loop. We finally remind [4] that the compressibility \( K = \chi/n^2 \), where \( n \) is electron density.

We may calculate the spin susceptibility in the same way. In this case the second integral term \( g_0 \) in Eq. (12) does not contain the contribution of the ZS’ loop, so the linear \( T \)-term coming from the BCS loop survives. Thus the spin susceptibility has a linear leading temperature correction:

\[ \frac{\chi(T)}{\chi(0)} \approx 1 + \frac{u^2 T}{E_F} \]  

(14)

where \( \chi(0) = \frac{m g^2}{4\pi} [1 + u^2 \ln (\frac{2\Lambda_{k_F}}{k_F} - \frac{3}{2})] \), and the first omitted term is \( \mathcal{O}(u^4T^2) \).

It is useful to keep in mind that albeit the response functions in Eqs. (11) [3] are explicitly expressed in terms of the vertex only, those contributions indeed entangle both “purely vertex” corrections and self-energy corrections. The latter are just expressed in terms of the vertex via the Ward identities.

Let us come back to the argument for the cancellation of temperature terms in the response functions, appealing to the Ward identities. These identities should work at each level in terms of the coupling expansion. We have calculated the vertices at the one-loop level \( \mathcal{O}(u^2) \). Through the Ward identities the self-energy correction were taken into account with the same accuracy. There are no more terms of the order \( \mathcal{O}(u^2) \) to cancel the temperature dependence [14]. Thus, the linear temperature dependence of susceptibility (or weaker temperature dependence of the compressibility) does not contradict the
exact Ward identities known to us, moreover in our results for the response functions both vertex and self-energy corrections are included on the same footing by using the Ward identities.

• Conclusions: In this paper we have systematically examined the leading temperature corrections to FLT in two spatial dimensions. Our results reveal the crucial importance of $2k_F$ processes mainly neglected by other workers. We find for a 2D Galilean-invariant FL that to order $u^2$ the leading $T$-dependence of the parameters in the spin sector is $T$, for the others it is $T^2$.

We find that the standard relationship between the effective mass and the Landau parameter in a Galilean-invariant FL is violated by finite-temperature terms. The structure of the formulas shows that those terms arise in invariant FL is violated by finite-temperature terms. The effective mass and the Landau parameter in a Galilean-invariant FL is violated by finite-temperature terms. The effective mass and the Landau parameter in a Galilean-invariant FL is violated by finite-temperature terms. The effective mass and the Landau parameter in a Galilean-invariant FL is violated by finite-temperature terms. The effective mass and the Landau parameter in a Galilean-invariant FL is violated by finite-temperature terms. The effective mass and the Landau parameter in a Galilean-invariant FL is violated by finite-temperature terms. The effective mass and the Landau parameter in a Galilean-invariant FL is violated by finite-temperature terms.

The particularly interesting new result we found is the leading linear temperature dependence of the spin susceptibility \[\chi(q, T = 0)\]. According to the perturbative calculations of Belitz et al, the 2D FL susceptibility has a leading linear correction in $|q|$ at $T = 0$ with a positive coefficient which is of the second order in interaction, i.e., their result has a structure of Eq. (13). Comparison between our Eq. (13) and the prediction of Ref. [10] for $\chi(q, T = 0)$ meets heuristic expectations of a reciprocity between small $T$- and $q$-dependencies.

For more realistic models of electrons in (quasi)-2D crystals, i.e., for various tight-binding spectra and fillings, the free-gas-like square-root $2k_F$ singularities (with $k_F$ depending on a chosen direction in $q$-space) are known to exist in the Lindhard functions. We think this is enough to result in linear $T$-terms in physical quantities analogous to what we found in this study. We argue that the rather accidental cancellation of the $T$-terms in some FL parameters is special to second order perturbation theory and a circular Fermi surface, while the leading linear temperature corrections are a generic feature of the 2D FL.

We hope our results may be experimentally tested in real 2D FL systems. For example, a very naive fit of the temperature dependence of the spin susceptibility in Sr$_2$RuO$_4$ system when it is in the 2D metallic regime (above 3D crossover temperature) shows that the data are compatible with the form (14). We hope our results stimulate a more detailed examination of the leading temperature dependences of response functions in 2D systems.

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