Research Article
New Bounds for the Generalized Distance Spectral Radius/Energy of Graphs

Yuzheng Ma,1 Yubin Gao,2 and Yanling Shao2

1School of Data Science and Technology, North University of China, Taiyuan, Shanxi 030051, China
2School of Mathematical Sciences, North University of China, Taiyuan, Shanxi 030051, China

Correspondence should be addressed to Yuzheng Ma; 1002631851@qq.com, Yubin Gao; ybgao@nuc.edu.cn, and Yanling Shao; ylshao@nuc.edu.cn

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Let G be a simple connected graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( d_v \) be the degree of the vertex \( v \). Let \( D(G) \) be the distance matrix and \( T(G) \) be the diagonal matrix of the vertex transmissions of \( G \). The generalized distance matrix of \( G \) is defined as \( D_\alpha(G) = \alpha T(G) + (1 - \alpha)D(G) \), where \( 0 \leq \alpha \leq 1 \). If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of \( D_\alpha(G) \), then the generalized distance spectral radius of \( G \) is defined as \( \rho(D_\alpha(G)) = \max\{\lambda_i\} \). The generalized distance energy of \( G \) is \( ED_\alpha(G) = \sum_{i=1}^{n} |\lambda_i - 2\pi W(G)/n| \), where \( W(G) \) is the Wiener index of \( G \). In this paper, we give some bounds of the generalized distance spectral radius and the generalized distance energy.

1. Introduction

Throughout this paper, we consider simple, connected, and finite graphs. Let \( G \) be such graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( d_v \) denote the degree of vertex \( v \) and \( N(v_i) \) denote the neighbor set of \( v_i \). The distance between vertices \( v_i \) and \( v_j \) in \( G \) is the length of the shortest path connecting \( v_i \) to \( v_j \), which is denoted as \( d(v_i, v_j) \). The distance matrix \( [1, 2] \) of \( G \) is an \( n \times n \) matrix \( D(G) = (d_{ij}) \), where \( d_{ij} = d(v_i, v_j) \) for \( i, j = 1, 2, \ldots, n \).

Definition 1. Let \( G \) be a graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \). The transmission of vertex \( v_i \), denoted by \( T_{rc}(v_i) \) or \( T_{ri} \), is defined to be the sum of the distances from \( v_i \) to all vertices in \( G \), that is, \( T_{rc}(v_i) = T_{ri} = \sum_{v_j \in V(G)} d(u, v_j) \). The sequence \( \{T_{r1}, T_{r2}, \ldots, T_{rn}\} \) is the transmission degree sequence of \( G \), and \( Tr(G) = \text{diag}(Tr_1, Tr_2, \ldots, Tr_n) \) is the diagonal matrix of vertex transmissions of \( G \).

Note that

1. Transmission of a vertex \( v \) is also called the distance degree or the first distance degree of \( v \).

2. If \( Tr_G(v_i) = k \) for \( i = 1, 2, \ldots, n \), then \( G \) is called a \( k \)-transmission regular graph.

Definition 2. Let \( G \) be a graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \), distance matrix \( D(G) \), and transmission degree sequence \( \{T_{r1}, T_{r2}, \ldots, T_{rn}\} \) such that \( T_{r1} \geq T_{r2} \geq \ldots \geq T_{rn} \). Then, the second transmission of vertex \( v_i \), denoted by \( T_{r1} \), is defined to be \( T_{r1} = \sum_{j=1}^{n} d_{ij} \), and \( \{T_{r1}, T_{r2}, \ldots, T_{rn}\} \) is called the second transmission degree sequence of \( G \).

Definition 3 (see [3]). Let \( G \) be a graph of order \( n \). The Wiener index of \( G \) is defined as

\[
W(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} = \frac{1}{2} \sum_{v_i, v_j \in V(G)} d_{ij} = \frac{1}{2} \sum_{v_i \in V(G)} Tr_G(v_i).
\]

In 1970, Gutman first proposed the concept of graph energy in [4]. The adjacency matrix \( A(G) \) of a graph \( G \) is a matrix of order \( n \) whose \( (i, j) \)-entry is equal to unity if the vertices \( v_i \) and \( v_j \) are adjacent and is equal to zero otherwise. Since \( A(G) \) is real and symmetric, all eigenvalues of \( A(G) \)
are real, denoted by $\mu_1, \mu_2, \ldots, \mu_n$, also known as the eigenvalues of $G$. The energy of graph $G$ is $e(G) = \sum_{i=1}^{n} |\mu_i|$. Let $\text{Deg}(G) = \text{diag}(d_1, d_2, \ldots, d_n)$, where $d_i = d_{v_i}$ for $i = 1, 2, \ldots, n$. The matrices $L(G) = \text{Deg}(G) - A(G)$ and $Q(G) = \text{Deg}(G) + A(G)$ are called the Laplacian matrix and the signless Laplacian matrix of graph $G$, respectively. For more research on Laplacian matrix and signless Laplacian matrix, refer to [5–8]. Aouchiche and Hansen [9, 10] introduced the distance Laplacian matrix $D_1(G) = \text{Tr}(G) - D(G)$ and the distance signless Laplacian matrix $D_Q(G) = \text{Tr}(G) + D(G)$ of graph $G$.

In 2019, Cui et al. [11] proposed the generalized distance matrix $D_\alpha(G) = a\text{Tr}(G) + (1 - a)D(G)$ by using the convex linear combination of $\text{Tr}(G)$ and $D(G)$, where $0 \leq a \leq 1$. As you can see, $D_0(G) = D(G), 2D_{1/2}(G) = D_Q(G), D_1(G) = \text{Tr}(G), \text{ and } D_\alpha(G) - D_\beta(G) = (a - \beta)D_1(G)$. Since the matrix $D_\alpha(G)$ is real and symmetric, all its eigenvalues are real, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, which are called the generalized distance eigenvalues of $G$, and the generalized distance spectral radius of $G$ is defined as $\rho(D_\alpha(G)) = \max_{1 \leq i \leq n} |\lambda_i|$.

Definition 4 (see [12]). Let $G$ be a graph of order $n$. The generalized distance energy of $G$ can be thought of as the mean deviation of the values of the generalized distance eigenvalues of $G$, namely, $E_{D_\alpha}(G) = \sum_{i=1}^{n} |\lambda_i - 2aW(G)/2n|$. The study of generalized distance spectrum was proposed by Cui et al. in [11]. They established some basic spectral properties of the generalized distance matrix of graphs, obtained the bounds of the generalized distance spectral radius, and determined the graphs with the minimum generalized distance spectral radius in all connected bipartite graphs with fixed vertices. In [13], the authors obtained some upper and lower bounds for the second largest eigenvalue of the generalized distance matrix of graphs, in terms of various graph parameters, and the graphs attaining the corresponding bounds are characterized. In [14], the authors obtained an upper bound for the smallest generalized distance eigenvalue $\lambda_n$ in terms of different graph parameters. In particular, they showed that this upper bound is better than the upper bound obtained by Cui et al. In [14], the authors established the relations between the smallest eigenvalues of $D_\alpha(G), D(G), \text{ and } D_Q(G)$ and obtained sharp bounds for the smallest eigenvalue $\lambda_n$ of $D_\alpha(G)$ in terms of various graph parameters. In [12, 15–17], Alhevaz et al. established some new sharp bounds for the generalized distance spectral radius of $G$ by using different graph parameters and characterized the extremal graphs. Then, they obtained new bounds for the $k$-th generalized distance eigenvalue. Moreover, through the eigenvalues of adjacency matrix and some auxiliary matrices, they studied the generalized distance spectrum of graphs obtained by generalization of the join graph operation. In 2020, they defined the generalized distance energy in [12] and gave the upper and lower bounds of the generalized distance energy. In [18, 19], Pirzada et al. obtained the bounds of the generalized distance spectral radius of bipartite graphs by using some parameters of graphs and characterized their extremal graphs. It was proved that for $\alpha \in (1/2, 1)$, the complete bipartite graph has the minimum generalized distance energy among all connected bipartite graphs, and for $\alpha \in (0, 2n/3n - 2)$, the star has the minimum generalized distance energy among all trees. In addition, the generalized distance spectrum, generalized distance energy and the spectral spread of generalized distance matrix are also studied. For some recent results on spread of generalized distance matrix, we refer the readers to [20, 21] and the references therein. Inspired by the above literature, in this paper, we further study the generalized distance spectral radius and generalized distance energy.

The rest of the paper is organized as follows. In Section 2, several lemmas are given. In Section 3, the new lower and upper bounds of the generalized distance spectral radius are obtained according to the distance between the vertices and some parameters of the graph. In Section 4, we obtain new bounds on the generalized distance energy in terms of spectral radius and parameters that depend on the distance between the vertices and the order of the graph.

2. Lemmas

In this section, we give some definitions and lemmas to prepare for subsequent proofs.

Lemma 1 (see [22]). If $A$ is a nonnegative real matrix of order $n$, then its spectral radius $\rho(A)$ is an eigenvalue of $A$ and it has an associated nonnegative eigenvector. Furthermore, if $A$ is irreducible, then $\rho(A)$ is a simple eigenvalue of $A$ with an associated positive eigenvector.

Lemma 2 (Rayleigh–Ritz theorem [23]). If $B$ is a real symmetric matrix of order $n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, then for a nonzero vector $X$,

$$\lambda_1 \geq \frac{X^T B X}{X^T X},$$

with equality holding if and only if $X$ is an eigenvector of $B$ corresponding to $\lambda_1$.

Lemma 3 (Cauchy–Schwartz inequality). Let $a_k$ and $b_k$ be real numbers for all $1 \leq k \leq n$. Then,

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 \leq \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right).$$

Equality holds if and only if $a_j b_k = a_k b_j$ for all $1 \leq k, j \leq n$.

Lemma 4 (see [11]). Let $G$ be a graph with distance degree sequence \{Tr_1, Tr_2, \ldots, Tr_n\}. Then,

$$\rho(D_\alpha(G)) \geq \frac{2W(G)}{n}.$$  

The equality holds if and only if $G$ is distance regular.

Lemma 5 (see [11]). Let $G$ be a simple connected graph, $Tr_i$ be the transmission of vertex $v_i$, and $T_i$ be the second transmission of $v_i$. Then,
\[
\rho(D_a(G)) \geq \sqrt{\sum_{i=1}^{n} \left( \alpha Tr^2_i + (1 - \alpha)T_i \right)^2 / T_1 + T_2 + \ldots + T_n}
\] (5)

The equality holds if and only if \( \alpha Tr_i + (1 - \alpha)T_i / Tr_i \) is a constant for all \( i = 1, 2, \ldots, n \).

Lemma 6 (see [24]). Let the transmission degree sequence of graph \( G \) be \( \{Tr_1, Tr_2, \ldots, Tr_n\} \) and the second transmission degree sequence of \( G \) be \( \{T_1, T_2, \ldots, T_n\} \). Then,

\[
\rho(D_a(G)) \leq \max_{1 \leq i \leq n} \left\{ \frac{\alpha(Tr_i + Tr_j) + \sqrt{\alpha^2(Tr_i - Tr_j)^2 + 4(1 - \alpha)^2(Tr_i Tr_j / Tr_j)} }{2} \right\}.
\]

\[
\rho(D_a(G)) \geq \min_{1 \leq i \leq j \leq n} \left\{ \frac{\alpha(Tr_i + Tr_j) + \sqrt{\alpha^2(Tr_i - Tr_j)^2 + 4(1 - \alpha)^2(Tr_i Tr_j / Tr_j)} }{2} \right\}.
\]

Each equality holds if and only if \( G \) is a transmission regular graph.

3. Lower and Upper Bounds of Generalized Distance Spectral Radius

In this section, the matrix sequence is introduced according to the relationship between the transmission and the second transmission, and the bounds of the generalized distance spectral radius in Lemmas 5 and 6 are generalized by using the matrix sequence in Theorems 1–3.

Definition 5 (see [25]). For \( i = 1, 2, \ldots, n \), the matrix sequence \( M_i^{(1)}, M_i^{(2)}, \ldots, M_i^{(t)} \) is defined as follows: fix \( \beta \in \mathbb{R} \), let \( M_i^{(1)} = (Tr_i)^{\beta} \), and for each \( t \geq 2 \), let \( M_i^{(t)} = \sum_{j=1}^{n} d_j M_i^{(t-1)} \).

Note that for \( \beta = 1 \), \( M_i^{(1)} = Tr_i \), and \( M_i^{(2)} = T_i \) for \( i = 1, 2, \ldots, n \).

Theorem 1. Let \( G \) be a connected graph of order \( n \), \( \beta \) be a real number, and \( t \) be an integer. Then,

\[
\rho(D_a(G)) \geq \sqrt{\frac{\sum_{i=1}^{n} (\alpha Tr_i M_i^{(t)} + (1 - \alpha)M_i^{(t+1)})^2}{\sum_{i=1}^{n} (M_i^{(t)})^2}}.
\] (7)

The equality holds (for particular values of \( \beta \) and \( t \)) if and only if \( \alpha Tr_i + (1 - \alpha)M_i^{(t+1)}/M_i^{(t)} \) is a constant for all \( i = 1, 2, \ldots, n \).

Proof. Let \( X = (x_1, x_2, \ldots, x_n)^T \) be the unit positive Perron eigenvector of \( D_a(G) \) corresponding to \( \rho(D_a(G)) \). Let \( Y \) be the unit positive vector defined by

\[
Y = \frac{1}{\sqrt{\sum_{i=1}^{n} (M_i^{(t)})^2}} \left( M_i^{(1)}, M_i^{(2)}, \ldots, M_i^{(t)} \right)^T.
\] (8)

Note that

\[
\rho(D_a(G)) = \sqrt{\rho(D_a(G))^2} = \sqrt{X^T (D_a(G))^2 X} \geq \sqrt{Y^T (D_a(G))^2 Y},
\] (9)

\[
D_a(G)Y = \frac{1}{\sqrt{\sum_{i=1}^{n} (M_i^{(t)})^2}} \left( \begin{array}{cccc}
\alpha Tr_1 & (1 - \alpha)d_{12} & \cdots & (1 - \alpha)d_{1n} \\
(1 - \alpha)d_{21} & \alpha Tr_2 & \cdots & (1 - \alpha)d_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
(1 - \alpha)d_{n1} & (1 - \alpha)d_{n2} & \cdots & \alpha Tr_n
\end{array} \right) \left( \begin{array}{c} M_1^{(1)} \\
M_2^{(1)} \\
\vdots \\
M_n^{(1)}
\end{array} \right)
\]

\[
= \frac{1}{\sqrt{\sum_{i=1}^{n} (M_i^{(t)})^2}} \left( \begin{array}{c}
\alpha Tr_1 M_1^{(1)} + (1 - \alpha)(M_2^{(1)} d_{12} + \cdots + M_n^{(1)} d_{1n}) \\
\alpha Tr_2 M_2^{(1)} + (1 - \alpha)(M_2^{(1)} d_{21} + \cdots + M_n^{(1)} d_{2n}) \\
\vdots \\
\alpha Tr_n M_n^{(1)} + (1 - \alpha)(M_1^{(1)} d_{n1} + \cdots + M_{n-1}^{(1)} d_{nn})
\end{array} \right)
\]

\[
= \frac{1}{\sqrt{\sum_{i=1}^{n} (M_i^{(t)})^2}} \left( \begin{array}{c}
\alpha Tr_1 M_1^{(1)} + (1 - \alpha)M_1^{(t+1)} \\
\alpha Tr_2 M_2^{(1)} + (1 - \alpha)M_2^{(t+1)} \\
\vdots \\
\alpha Tr_n M_n^{(1)} + (1 - \alpha)M_n^{(t+1)}
\end{array} \right).
\] (10)
We obtain
\[
Y^T (D_a(G))^2 Y = \frac{\sum_{i=1}^{n} (\alpha Tr_i M_i^{(1)} + (1-\alpha) M_i^{(t+1)})^2}{\sum_{i=1}^{n} (M_i^{(t)})^2}. \tag{11}
\]

Therefore,
\[
\rho(D_a(G)) \geq \frac{\sum_{i=1}^{n} (\alpha Tr_i M_i^{(t)} + (1-\alpha) M_i^{(t+1)})^2}{\sum_{i=1}^{n} (M_i^{(t)})^2}. \tag{12}
\]

Now we assume that the equality holds in (7). By (9), \(Y\) is a positive eigenvector corresponding to \(\rho(D_a(G))\). From \(D_a(G)Y = \rho(D_a(G))Y\), we obtain that
\[
\alpha Tr_i + (1-\alpha) M_i^{(t+1)} = \rho(D_a(G))\text{ for } i = 1, \ldots, n.
\]
Conversely, if \(\alpha Tr_i + (1-\alpha) M_i^{(t+1)} = k\), then \(\alpha Tr_i M_i^{(t)} + (1-\alpha) M_i^{(t+1)} = k M_i^{(t)}\) for all \(i = 1, \ldots, n\). Hence,
\[
D_a(G)Y = \frac{1}{\sqrt{\sum_{i=1}^{n} (M_i^{(t)})^2}} (k M_1^{(t)} \cdots k M_n^{(t)}) = k Y. \tag{13}
\]

We observe that \(G_1\) is a 7-transmission regular graph and \(G_6\) is a 12-transmission regular graph. In Table 1, we show the lower bounds for \(\rho(D_a(G))\), using four decimal places.

**Theorem 2.** Let \(G\) be a connected graph of order \(n\) and \(t\) be an integer. Then,
\[
\rho(D_a(G)) \leq \max_{1 \leq i, j \leq n} \left\{ \frac{\alpha (Tr_i + Tr_j) + \sqrt{\alpha^2 (Tr_i - Tr_j)^2 + 4(1-\alpha)^2 (M_i^{(t+1)} M_j^{(t)})^2}}{2} \right\}. \tag{15}
\]

The equality holds if and only if \(t = 1\), \(\beta = 1\), and \(M_i^{(1)}\) is a constant for all \(i = 1, \ldots, n\).

**Proof.** Let \(X = (x_1, x_2, \ldots, x_n)^T\) be the unit positive Perron eigenvector of \(M^{-1} D_a(G)M\) corresponding to \(\rho(D_a(G))\), where
\[
M = \begin{pmatrix}
M_1^{(t)} & 0 & \cdots & 0 \\
0 & M_2^{(t)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M_n^{(t)}
\end{pmatrix}. \tag{16}
\]

Let \(x_p = \max \{|x_i|; i = 1, 2, \ldots, n\}\), \(x_q = \max \{|x_i|; x_i \neq x_p, i = 1, 2, \ldots, n\}\), and the \((i, j)\)-entry of matrix \(M^{-1} D_a(G)M\) be
\[
\begin{pmatrix}
\alpha Tr_i, & i = j, \\
(1-\alpha) M_j^{(t)} M_i^{(t)} & i \neq j.
\end{pmatrix}
\]

Therefore,
\[
(M^{-1} D_a(G)M)X = \rho(D_a(G))X. \tag{18}
\]
The $p$ and $q$ lines in (18) can be expressed by

\[
(\rho(D_a(G)) - \alpha \Tr_p)x_p = (1 - \alpha) \sum_{i=1}^{n} \frac{M^{(t)}_{p,i}}{M^{(t)}_{p}} d_{p,i}x_i \leq (1 - \alpha) \frac{X_p}{M^{(t)}_{p}} \sum_{i=1}^{n} d_{p,i}M^{(t)}_{i},
\]

\[
= (1 - \alpha) \frac{X_p}{M^{(t)}_{p}} M^{(t+1)}_{p}.
\]

\[\text{(19)}\]

\[
(\rho(D_a(G)) - \alpha \Tr_q)x_q = (1 - \alpha) \sum_{i=1}^{n} \frac{M^{(t)}_{q,i}}{M^{(t)}_{q}} d_{q,i}x_i \leq (1 - \alpha) \frac{X_q}{M^{(t)}_{q}} \sum_{i=1}^{n} d_{q,i}M^{(t)}_{i},
\]

\[
= (1 - \alpha) \frac{X_q}{M^{(t)}_{q}} M^{(t+1)}_{q}.
\]

\[\text{(20)}\]

Then,

\[
\left(\rho(D_a(G)) - \alpha \Tr_p\right) \left(\rho(D_a(G)) - \alpha \Tr_q\right)x_p x_q \leq (1 - \alpha)^2 \frac{M^{(t+1)}_{p}}{M^{(t)}_{p}} \frac{M^{(t+1)}_{q}}{M^{(t)}_{q}} X_p X_q,
\]

\[\text{(21)}\]

that is,

\[
\rho^2(D_a(G)) - \alpha \left(\Tr_p + \Tr_q\right) \rho(D_a(G)) + \alpha^2 \Tr_p \Tr_q - (1 - \alpha)^2 \frac{M^{(t+1)}_{p}}{M^{(t)}_{p}} \frac{M^{(t+1)}_{q}}{M^{(t)}_{q}} \leq 0.
\]

\[\text{(22)}\]
\[
\rho(D_a(G)) \leq \frac{\alpha (Tr_i + Tr_j) + \sqrt{\alpha^2 (Tr_i - Tr_j)^2 + 4(1 - \alpha)^2 (M_i^{(t+1)}M_j^{(t+1)}/M_i^{(t)}M_j^{(t)})}}{2}.
\] (23)

Therefore,
\[
\rho(D_a(G)) \leq \max_{1 \leq i, j \leq n} \frac{\alpha (Tr_i + Tr_j) + \sqrt{\alpha^2 (Tr_i - Tr_j)^2 + 4(1 - \alpha)^2 (M_i^{(t+1)}M_j^{(t+1)}/M_i^{(t)}M_j^{(t)})}}{2}.
\] (24)

For \( \beta = 1 \), for \( i = 1, 2, \ldots, n \): \( M_i^{(1)} = Tr_i, M_i^{(2)} = T_i \). Let \( M_i^{(1)} = k \). Then, \( M_i^{(2)} = k^2 \). We know that \( Tr_i = k \) and \( T_i = k^2 \). According to Lemma 1, \( k \) is the largest eigenvalue of \( D_a(G) \), and \( \rho(D_a(G)) = k \), so the equality holds in (15). On the contrary, if inequality (15) is equal, then \( x_1 = x_2 = \ldots = x_n \) can be obtained from (19) and (20), that is, \( \rho(D_a(G)) = \alpha Tr_i + (1 - \alpha)M_i^{(t+1)}/M_i^{(t)} = \alpha Tr_1 + (1 - \alpha)M_1^{(t+1)}/M_1^{(t)} = \alpha Tr_n + (1 - \alpha)M_n^{(t+1)}/M_n^{(t)} = k \). It means when \( t = 1 \) and \( \beta = 1 \), \( M_i^{(1)} \) is a constant for \( i = 1, 2, \ldots, n \). This completes the proof. \( \square \)

**Theorem 3.** Let \( G \) be a connected graph and \( t \) be an integer. Then,
\[
\rho(D_a(G)) \geq \frac{X^T D_a(G)X}{X^T X} = \frac{1}{n} \sum_{i=1}^{n} Tr_i = \frac{1}{n} \sum_{i=1}^{n} (2n - d_i - 2) = \frac{2n^2 - 2m - 2n}{n}.
\] (27)

That is, (26) holds.

If \( G \) is a transmission regular graph, then \( Tr_1 = Tr_2 = \ldots = Tr_n \), and so the equality in (26) holds. Conversely, if the equality in (26) holds, it is clear that \( G \) is a transmission regular graph. \( \square \)

### 4. Upper Bound of Generalized Distance Energy

In this section, we obtain new bounds on the generalized distance energy in terms of spectral radius and parameters that depend on the distance between the vertices and the order of the
Theorem 5. Let $G$ be a connected graph of order $n$. Then,

$$E_{D_2}(G) = \begin{cases} 
|\theta| + \sqrt{(n-1)(T-|\theta|)}, & \text{if } |\theta| \leq \sqrt{T/n}, \\
\sqrt{T/n} + \sqrt{(n-1)(T-\sqrt{T/n})}, & \text{if } |\theta| \geq \sqrt{T/n}, \\
\sqrt{T/n} + \sqrt{(n-1)(T-\sqrt{T/n})}, & \text{if } |\theta| \geq \sqrt{T/n}, \\
\sqrt{T/n} + \sqrt{(n-1)(T-\sqrt{T/n})}, & \text{if } |\theta| \geq \sqrt{T/n}.
\end{cases} \tag{29}$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the generalized distance eigenvalues of $G$, and $\theta_i = \lambda_i - 2\alpha W(G)/n$ for $i = 1, 2, \ldots, n$. By Lemma 3, for $i = 1, 2, \ldots, n$, we get

$$\left( E_{D_2}(G) - |\theta| \right)^2 \leq (n-1)(T-\theta^2), \tag{30}$$

so

$$E_{D_2}(G) \leq |\theta| + \sqrt{(n-1)(T-\theta^2)}. \tag{31}$$

We now consider the function $f(x) = x + \sqrt{(n-1)(T-x^2)}$ with $0 < x \leq \sqrt{T}$. For $\sqrt{T/n} \leq x \leq \sqrt{T}$, $f'(x) = 1 + (-2nx + 2x)/2\sqrt{(n-1)(T-x^2)} < 0$ and $f(x)$ is monotonically decreasing. For $0 \leq x \leq \sqrt{T/n}$, $f'(x) = 1 + (-2nx + 2x)/2\sqrt{(n-1)(T-x^2)} > 0$ and $f(x)$ is monotonically increasing. Thus, we have the following results.

(i) If $|\theta| \leq \sqrt{T/n}$,

$$E_{D_2}(G) \leq f(|\theta|) = |\theta| + \sqrt{(n-1)(T-\theta^2)}. \tag{32}$$

(ii) If $|\theta| \geq \sqrt{T/n}$, then

$$E_{D_2}(G) \geq f(|\theta|) = |\theta| + \sqrt{(n-1)(T-\theta^2)}. \tag{33}$$

(iii) If $|\theta| \leq \sqrt{T/n} \leq |\theta|$, then

$$E_{D_2}(G) \leq f(\sqrt{T/n}) = \sqrt{T/n} + \sqrt{(n-1)(T-T/n)}. \tag{34}$$

This completes the proof. \qed

Lemma 7. If $G$ is a graph of diameter 2, then

$$\sum_{i=1}^{n} \theta_i^2 = (1-\alpha)^2 \sum_{i=1}^{n} d_{i}^2 + \alpha^2 \sum_{i=1}^{n} T r_i^2 - \alpha \sum_{i=1}^{n} T r_i^2 \tag{35}$$

Proof. Since there are $2m$ elements equal to 1 and $n(n-1)-2m$ elements equal to 2 in $D(G)$, we have

$$\sum_{i=1}^{n} \theta_i^2 = (1-\alpha)^2 \sum_{i=1}^{n} d_{i}^2 + \alpha^2 \sum_{i=1}^{n} (2n-2-d_i)^2 - \frac{1}{n} \left( \alpha \sum_{i=1}^{n} (2n-2-d_i)^2 \right)^2 \tag{36}$$

$$= (1-\alpha)^2 \left( 2m + \left( n^2 - n - 2m \right) \cdot 2^2 \right) + \alpha^2 \left( 4(n-1)^2 n^2 - 8m(n-1) + \sum_{i=1}^{n} d_i^2 \right) \tag{37}$$

$$= \frac{\alpha^2}{n} (4(n-1)^2 n^2 - 8m(n-1) + 4m^2) \tag{38}$$

$$= (1-\alpha)^2 \left( 4(n(n-1)-6m) + \alpha^2 \left( \sum_{i=1}^{n} d_i^2 - 4mn^2 \right) \right). \tag{39}$$

The lemma holds. \qed

Theorem 6. If $G$ is a $r$-regular graph of diameter 2, then

$$E_{D_2}(G) \leq (1-\alpha)(2n-2-r) + \sqrt{(n-1)\left[ (1-\alpha)^2 \left( r + 2 \right)^2 - n(12-2r) \right]} \tag{40}$$

Proof. Since $G$ is a $r$-regular graph of diameter 2, by Theorem 4, $\rho(D_1(G)) = 2n^2 - 2rn - 2m/n$, and $\theta = \rho(D_1(G)) - 2\alpha W(G)/n = (2n+2) - \alpha(2n-2-r) = (1-\alpha)(2n-2-r)$. Applying Lemma 3 to $(1, 1, \ldots, 1)^T$ and $(\theta_2, \theta_3, \ldots, \theta_n)^T$, we get

$$\left( E_{D_2}(G) - \theta \right) \leq (n-1) \sum_{i=2}^{n} \theta_i^2 \tag{41}$$

By Lemma 7,
That is, the authors declare that they have no conflicts of interest.

Conflicts of Interest

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Table 2: Bounds for $\rho(D_\alpha(G))$, using four decimal places, where $G_i$ is given in Figure 1.

| $G_i$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ | $G_6$ |
|-------|-------|-------|-------|-------|-------|
| 1/2 ($\beta = 1/\alpha = 1$) | 7/7 | 3.9/3.1333 | 10.6545/9.0889 | 10.9091/7.0000 | 20/13.2 | 12/12 |
| 1/2 ($\beta = 1/\alpha = 2$) | 7/7 | 3.9/3.1333 | 10.6545/9.0889 | 10.9091/7.0000 | 20/13.2 | 12/12 |
| 1/2 ($\beta = 1/\alpha = 3$) | 7/7 | 3.9/3.1333 | 10.6545/9.0889 | 10.9091/7.0000 | 20/13.2 | 12/12 |
| 1/2 ($\beta = 1/\alpha = 4$) | 7/7 | 3.9/3.1333 | 10.6545/9.0889 | 10.9091/7.0000 | 20/13.2 | 12/12 |
| 1/2 ($\beta = 1/\alpha = 5$) | 7/7 | 3.9/3.1333 | 10.6545/9.0889 | 10.9091/7.0000 | 20/13.2 | 12/12 |
| 1/2 ($\beta = 1/\alpha = 6$) | 7/7 | 3.9/3.1333 | 10.6545/9.0889 | 10.9091/7.0000 | 20/13.2 | 12/12 |
| 1/2 ($\beta = 1/\alpha = 7$) | 7/7 | 3.9/3.1333 | 10.6545/9.0889 | 10.9091/7.0000 | 20/13.2 | 12/12 |

$\left( E_{D_\alpha}(G) - \theta_1 \right)^2 \leq (n-1) \left[ (1-\alpha)^2 (4n(n-1)-6m) + \alpha^2 \left( \sum_{i=1}^{n} d_i^2 - \frac{4m^2}{n} \right) \right] - \theta_1^2$, \hspace{1cm} (36)

that is,

$E_{D_\alpha}(G) \leq \theta_1 + \sqrt{(n-1) \left[ (1-\alpha)^2 (4n(n-1)-6m) + \alpha^2 \left( \sum_{i=1}^{n} d_i^2 - \frac{4m^2}{n} \right) \right] - \theta_1^2}$. \hspace{1cm} (37)

Since $\theta_1 = (1-\alpha)(2n-2-r)$ and $2m = nr$, (37) gives

$E_{D_\alpha}(G) \leq (1-\alpha)(2n-2-r) + \sqrt{(n-1) \left[ (1-\alpha)^2 (r+2)^2 - n(12-7r) \right]}$. \hspace{1cm} (38)

The theorem follows.

5. Conclusions

In this paper, some new lower and upper bounds of the generalized distance spectral radius are obtained in terms of the distance between the vertices and some parameters of the graph. Meanwhile, we obtain new bounds on the generalized distance energy in terms of spectral radius and parameters that depend on the distance between the vertices and the order of the graph.

Data Availability

All data, models, and codes generated or used during the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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