GROWTH OF PSEUDO-ANOSOV CONJUGACY CLASSES IN TEICHMÜLLER SPACE

JIAWEI HAN

Abstract. Athreya, Bufetov, Eskin and Mirzakhani [2] have shown the number of mapping class group lattice points intersecting a closed ball of radius $R$ in Teichmüller space is asymptotic to $e^{hR}$, where $h$ is the dimension of the Teichmüller space. We show for any pseudo-Anosov mapping class $f$, there exists a power $n$, such that the number of lattice points of the $f^n$ conjugacy class intersecting a closed ball of radius $R$ is coarsely asymptotic to $e^{h^2R}$.

1. Introduction

One can study a group by understanding its “growth” in various ways. Consider $G$ acting on a metric space $S$ by isometries, one can measure the number of orbit or lattice points of $G$ in a ball of radius $R$ as $R$ goes to infinity. For example, consider $\mathbb{Z}^3$ acting on $\mathbb{R}^3$ in the standard way, the number of lattice points of $\mathbb{Z}^3$ in a ball of radius $R$ is roughly the volume of this ball, see [7] for example. In this paper, we study mapping class groups by understanding the lattice points of pseudo-Anosov conjugacy classes in Teichmüller space.

Let $M$ be a compact, negatively curved Riemannian manifold and let $\tilde{M}$ denote its universal cover. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ by isometries. Let $B_R(x)$ denote the ball of radius $R$ in $\tilde{M}$ centered at $x$. G.A. Margulis studied the growth rate of any orbit $\pi_1(M) \cdot y$ by intersecting with any metric balls $B_r(x)$.

**Theorem 1.1** (Margulis [9]). There is a function $c: M \times M \to \mathbb{R}^+$ so that for every $x, y \in \tilde{M}$,

$$|\pi_1(M) \cdot y \cap B_R(x)| \sim c(p(x), p(y))e^{hR}$$

where $h$ equals to the dimension of the manifold, which is the topological entropy of the geodesic flow on the unit tangent bundle of $M$.

The notation $f(R) \sim g(R)$ means $\lim_{R \to \infty} \frac{f(R)}{g(R)} = 1$.

Inspired by this result, Athreya, Bufetov, Eskin and Mirzakhani studied lattice point asymptotics in Teichmüller space. Let $S_{g,n}$ denote a closed surface of genus $g$ with $n$ punctures such that $3g - 3 + n > 0$, and we let $\text{Mod}_{g,n}$ and $(\mathcal{T}_{g,n}, d)$ denote the corresponding mapping class group and Teichmüller space with Teichmüller metric. Then $\text{Mod}_{g,n}$ acts on $\mathcal{T}_{g,n}$ by isometries. We use $\text{Mod}_g, \mathcal{T}_g$ to denote $\text{Mod}_{g,0}, \mathcal{T}_{g,0}$ for simplicity. They showed the orbits of mapping class group have analogous asymptotics.

**Theorem 1.2** (Athreya, Bufetov, Eskin and Mirzakhani [2]). For any $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_g$, we have

$$|\text{Mod}_g \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \sim e^{hR}$$
Note in their original paper, there is a factor of \( \Lambda(\mathcal{X})\Lambda(\mathcal{Y}) \) in front of \( e^{\frac{\Lambda}{2}} \), \( \Lambda \) is called the Hubbard-Masur function. Mirzakhani later showed that \( \Lambda \) is a constant function, see \([3]\). Moreover, we recall the following result from Parkkonen and Paulin \([12]\) about the lattice point asymptotics for conjugacy classes of \( \pi_1(M) \).

**Theorem 1.3** (Parkkonen, Paulin \([12]\)). Let \( G \) be a nontrivial conjugacy class of \( \pi_1(M) \), for any \( x \in M \), we have

\[
\lim_{R \to \infty} \frac{1}{R} \ln |G \cdot X \cap B_R(X)| = \frac{h}{2}.
\]

Inspired by this result, we wish to explore the lattice point asymptotics for conjugacy classes of \( \text{Mod}_{g,n} \). The Nielsen-Thurston Classification \([13]\) says every element in \( \text{Mod}_g \) is one of the three types: periodic, reducible, or pseudo-Anosov. When \( f \in \text{Mod}_{g,n} \) is a Dehn twist, a special kind of reducible element, we prove in \([5]\) that the lattice point growth for the conjugacy class of \( f \) is “coarsely” asymptotic to \( e^{\frac{\Lambda}{2}} \). In this paper, we are interested in pseudo-Anosov elements. Let \( PA \subset \text{Mod}_g \) denote the subset of pseudo-Anosov elements. Maher showed pseudo-Anosov elements are generic in the following sense.

**Theorem 1.4** (Maher \([8]\)). For any \( \mathcal{X}, \mathcal{Y} \in \mathcal{T}_g \), we have

\[
\frac{|PA \cdot \mathcal{Y} \cap B_R(\mathcal{X})|}{|\text{Mod}_g \cdot \mathcal{Y} \cap B_R(\mathcal{X})|} \sim 1.
\]

The above Theorems \([1,3,14]\) motivate us to explore the lattice point asymptotics for pseudo-Anosov conjugacy class subgroups. For any mapping class \( \phi \in \text{Mod}_{g,n} \), we use \( [\phi] = \{ f \phi f^{-1} \mid f \in \text{Mod}_{g,n} \} \) to denote its conjugacy class. For simplicity of notation, we denote \( \Gamma_R(\mathcal{X}, \mathcal{Y}, \phi) = |[\phi] \cdot \mathcal{Y} \cap B_R(\mathcal{X})| \).

Let \( C > 0 \), we say \( f(R) \leq g(R) \) if for any \( \delta > 1 \), there exists a \( M(\delta) \) such that

\[
\frac{1}{C \delta^2} \cdot f(R) \leq g(R) \text{ for any } R \geq M(\delta).
\]

We say \( f(R) \overset{C}{\leq} g(R) \) if \( f(R) \leq C g(R) \) and \( g(R) \overset{C}{\leq} f(R) \), thus \( f(R) \overset{1}{\sim} g(R) \) is the same as \( f(R) \sim g(R) \). Accordingly, we simply write \( \overset{C}{\preceq} \) when \( C = 1 \). The main results of this paper are the following.

**Theorem 1.5.** Fix \( S_{g,n} \) and \( \epsilon > 0 \), there exists a constant \( A > 0 \) such that given any \( \epsilon \)-thick pseudo-Anosov element \( \phi \) with translation distance \( \lambda \geq A \) and given any \( \mathcal{X}, \mathcal{Y} \) in \( \mathcal{T}_{g,n} \), there exists a corresponding \( G(\mathcal{X}, \mathcal{Y}, \phi) \) such that

\[
\Gamma_R(\mathcal{X}, \mathcal{Y}, \phi) \overset{G(\mathcal{X}, \mathcal{Y}, \phi, k)}{\sim} e^{\frac{\lambda}{2}}.
\]

**Corollary 1.6.** Fix \( S_{g,n} \), given any pseudo-Anosov element \( \phi \) and given any \( \mathcal{X}, \mathcal{Y} \) in \( \mathcal{T}_{g,n} \). There exists a power \( N \) depending on \( \phi \) such that for any \( k \geq N \), there is a corresponding \( G(\mathcal{X}, \mathcal{Y}, \phi, k) \) so that the following holds:

\[
\Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) \overset{G(\mathcal{X}, \mathcal{Y}, \phi, k)}{\sim} e^{\frac{\lambda}{2}}.
\]

In parallel with the Theorem \([1,3]\) above, we note the above Theorem \([1,5]\) and Corollary \([1,6]\) imply the following.

**Corollary 1.7.** Fix \( S_{g,n} \), given any pseudo-Anosov element \( \phi \) and given any \( \mathcal{X}, \mathcal{Y} \) in \( \mathcal{T}_{g,n} \), for all sufficiently large \( k \) we have

\[
\lim_{R \to \infty} \frac{1}{R} \ln \Gamma_R(\mathcal{X}, \mathcal{Y}, \phi^k) = \frac{h}{2}.
\]
These results again indicate the similarity of Teichmüller spaces and hyperbolic spaces in terms of lattice point asymptotics.

1.1. Acknowledgments. I would like to thank my advisor, Spencer Dowdall, for suggesting this project, for his guidance and consistent support throughout.

2. Background

Let $\text{Homeo}_g,n^+$ denote the group of all the orientation-preserving homeomorphisms of $S_{g,n}$ preserving the set of punctures, and let $\text{Homeo}_g,n^0$ denote the connected component of the identity. The mapping class group of $S_{g,n}$ is defined to be the group of isotopy classes of orientation-preserving homeomorphisms:
\[
\text{Mod}_{g,n} = \text{Homeo}_g,n^+ / \text{Homeo}_g,n^0 = \text{Homeo}_g,n^+ / \text{isotopy}
\]

A hyperbolic structure $X$ on $S_{g,n}$ is a pair $(X, \phi)$ where $\phi: S_{g,n} \to X$ is a homeomorphism and $X$ is a hyperbolic surface. We say two hyperbolic structures $X = (X, \phi), Y = (Y, \psi)$ are isotopic if there is an isometry $I: X \to Y$ isotopic to $\psi \circ \phi^{-1}$. The Teichmüller space $T_{g,n}$ is the set of hyperbolic structures on $S_{g,n}$ modulo isotopy. We let $X = (X, \phi), Y = (Y, \psi)$ denote elements in $T_{g,n}$. Given any $X, Y \in T_{g,n}$, the Teichmüller distance between them is defined to be
\[
d_T (X, Y) = \frac{1}{2} \inf_{f \sim \phi \circ \psi^{-1}} \log(K_f)
\]
where the infimum is over all quasi-conformal homeomorphisms $f$ isotopic to $\phi \circ \psi^{-1}$ and $K_f$ is the quasi-conformal dilatation of $f$. Equipped with the Teichmüller metric, the Teichmüller space is a complete, unique geodesic metric space.

Given any $X = (X, \phi) \in T_{g,n}$ and given any isotopy class $\gamma$ of nontrivial simple closed curves on $S_{g,n}$, there exists a unique geodesic in the free homotopy class of $\phi(\gamma)$ on $X$. We define $\ell_X(\phi(\gamma))$ to the length of this unique geodesic and define $\ell_X(\gamma) = \ell_X(\phi(\gamma))$. A pants decomposition of the surface $S_{g,n}$ is a collection of pairwise disjoint non-trivial simple closed curves $\gamma_1, \ldots, \gamma_{3g-3+n}$ on $S_{g,n}$, together they decompose the surface $S_{g,n}$ into $2g + n - 2$ pairs of pants. Using pants decomposition and by introducing Fenchel-Nielsen coordinates, Fricke [4] showed that $T_{g,n}$ is homeomorphic to $\mathbb{R}^{6g+2n-6}$.

The mapping class group acts isometrically on $T_{g,n}$ by changing the marking $(f, (X, \phi)) \mapsto (X, \phi \circ f^{-1})$. This action is properly discontinuous but not cocompact. The quotient $M_{g,n} = T_{g,n} / \text{Mod}_{g,n}$ is called the moduli space, and it is a non-compact orbifold parameterizing hyperbolic surfaces homeomorphic to $S_{g,n}$.

Given any $\epsilon > 0$, the $\epsilon$-thick part of Teichmüller space is defined to be
\[
T_{g,n}^\epsilon = \{ X \in T_{g,n} \mid \ell_X(\alpha) \geq \epsilon \text{ for any simple closed curve } \alpha \text{ on } S_{g,n} \}
\]
and consequently the $\epsilon$-thick part of moduli space is $M_{g,n}^\epsilon = T_{g,n}^\epsilon / \text{Mod}_{g,n}$. The Mumford compactness criterion [11] says $M_{g,n}^\epsilon$ is compact for any $\epsilon > 0$.

Similar to hyperbolic isometrics acting on hyperbolic space, each pseudo-Anosov element $\phi \in \text{Mod}_{g,n}$ acts on $T_{g,n}$ by translating along its corresponding bi-infinite geodesic axis, denoted as axis($\phi$) with translation distance denoted as $\lambda(\phi)$. Moreover, we say a pseudo-Anosov element $\phi \in \text{Mod}_{g,n}$ is called $\epsilon$-thick if its axis axis($\phi$) $\subset T_{g,n}^\epsilon$. 


For any $r > 0$ and for every closed set $W \subset \mathcal{T}_{g,n}$, denote $N_r(W)$ the $r$-neighborhood of $W$. For every closed set $C \subset \mathcal{T}_{g,n}$, the closest point projection map is defined as follows

$$
\pi_{C}(x) = \{y \in C \mid d(x, y) = d(x, C) = \inf_{z \in C} d(x, z)\}.
$$

As one of the early works exploring negative curvature in Teichmüller space, the result below from Minsky [10] says that $\epsilon$-thick geodesics in Teichmüller space satisfy the strongly contracting property.

**Theorem 2.1** (Minsky [10]). There exists a constant $A > 0$ depending on $\epsilon, \chi(S)$ such that if $\mathcal{L}$ is an $\epsilon$-thick geodesic in $\mathcal{T}_{g,n}$ and $d(\mathcal{X}, \mathcal{L}) > A$, then we have

$$
\text{diam} \left( \pi_{\mathcal{L}} \left( N_{d(\mathcal{X}, \mathcal{L})-A}(\mathcal{X}) \right) \right) \leq A
$$

for any $\mathcal{X} \in \mathcal{T}_{g,n}$.

For $\mathcal{L}$ a geodesic in $\mathcal{T}_{g,n}$, we let $d^\mathcal{L}(C, W) = \text{diam} \left( \pi_{\mathcal{L}}(C) \cup \pi_{\mathcal{L}}(W) \right)$. We can pick the constant $A$ in Theorem 2.1 in a way so that the following holds.

**Corollary 2.2** (Arzhantseva, Cashen, and Tao, [1]). Let $\mathcal{L}$ be an $\epsilon$-thick geodesic in $\mathcal{T}_{g,n}$ and let $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_{g,n}$ be such that $d^\mathcal{L}(\mathcal{X}, \mathcal{Y}) > A$, then

$$
d(\mathcal{X}, \mathcal{Y}) \geq d(\mathcal{X}, \pi_{\mathcal{L}}(\mathcal{X})) + d^\mathcal{L}(\mathcal{X}, \mathcal{Y}) + d(\pi_{\mathcal{L}}(\mathcal{Y}), \mathcal{Y}) - A.
$$

Moreover, if $\mathcal{Y}$ happens to be on the geodesic $\mathcal{L}$, then $\pi_{\mathcal{L}}(\mathcal{Y}) = \{\mathcal{Y}\}$ and

$$
d(\mathcal{X}, \mathcal{Y}) \geq d(\mathcal{X}, \pi_{\mathcal{L}}(\mathcal{X})) + d(\pi_{\mathcal{L}}(\mathcal{X}), \mathcal{Y}) - A.
$$

For any pseudo-Anosov element $\phi \in \text{Mod}_{g,n}$, we denote $\pi_{\text{axis}(\phi)}$ as $\pi_{\phi}$. Since $\phi$ acts by translation along its axis, it commutes with the projection map $\pi_{\phi}$. That is, for any $\mathcal{X} \in \mathcal{T}_{g,n}$, we have $\pi_{\phi}(\phi(\mathcal{X})) = \phi(\pi_{\phi}(\mathcal{X}))$.

By using Theorem 2.1 and Corollary 2.2 one can show if an $\epsilon$-thick pseudo-Anosov element $\psi$ has sufficiently large translation length, then the distance it translates a point is roughly twice the distance from the point to the axis. See Figure 1 for an illustration.

**Figure 1.** Shaded area are $\epsilon$-thin parts. Given a $\epsilon$-thick pseudo-Anosov element $\psi$ with $\lambda(\psi) > A$, the diameter of projection of any balls like $B$ to axis($\psi$) is bounded by $A$, see Theorem 2.1. The geodesic from $\mathcal{X}$ to $\psi(\mathcal{X})$ fellow travels axis($\psi$), see Corollary 2.3.
Corollary 2.3. Let $\phi$ be a $c$-thick pseudo-Anosov element with translation distance $\lambda(\phi) > A$. Then for any $x \in T_{g,n}$ and for any $\psi \in [\phi]$, we have
\[ 2d(x, \pi_\psi(x)) + \lambda(\phi) - A \leq d(x, \pi_\psi(x)) \leq 2d(x, \pi_\psi(x)) + \lambda(\phi) + 2A. \]

Proof. Since translation distance is invariant under conjugation, $\lambda(\psi) = \lambda(\phi) > A$ for any $\psi \in [\phi]$. Thus we have
\[ d_\psi^p(x, \psi(x)) = \text{diam}(\pi_\psi(x) \cup \pi_\psi(\psi(x))) = \text{diam}(\pi_\psi(x) \cup \psi(\pi_\psi(x))) \]
where $\lambda(\phi) \leq \text{diam}(\pi_\psi(x) \cup \psi(\pi_\psi(x))) \leq \lambda(\phi) + 2A$. Take any $x \in T_{g,n}$, by the triangle inequality, we have
\[ d(x, \psi(x)) \leq d(x, \pi_\psi(x)) + d_\psi^p(x, \psi(x)) + d(\psi(x), \pi_\psi(\psi(x))) \]
\[ \leq 2d(x, \pi_\psi(x)) + \lambda(\phi) + 2A. \]

Meanwhile we can apply the previous Corollary 2.2 and get
\[ d(x, \psi(x)) \geq d(x, \pi_\psi(x)) + d_\psi^p(x, \psi(x)) + d(\psi(x), \pi_\psi(\psi(x))) - A \]
\[ \geq 2d(x, \pi_\psi(x)) + \lambda(\phi) - A. \]

The result follows.  \hfill \Box

3. Proof of the main theorem

By Theorem 1.2 for any $x \in T_{g,n}$, we have
\[ |\text{Mod}_{g,n} \cdot x \cap B_r(x)| \sim e^{hr}. \]

For any $r > 0$, define the set
\[ \Omega_r(x) = \{ f \in \text{Mod}_{g,n} \mid d(x, f x) \leq r \} \]
and denote $N$ the maximal order of point stabilizer subgroups in $\text{Mod}_{g,n}$. It follows that
\[ |\text{Mod}_{g,n} \cdot x \cap B_r(x)| \leq |\Omega_r(x)| \leq N \cdot |\text{Mod}_{g,n} \cdot x \cap B_r(x)|, \]
\[ e^{hr} \leq |\Omega_r(x)| \leq N \cdot e^{hr}. \]

Moreover, given any $\phi \in \text{Mod}_{g,n}$, we have
\[ \Gamma_r(x, \mathcal{X}, \phi) \leq |[\phi] \cap \Omega_r(x)| \leq N \cdot \Gamma_r(x, \mathcal{X}, \phi). \]

Combining things together, we have
\[ (1) \quad \frac{1}{N} \cdot |[\phi] \cap \Omega_r(x)| \leq \Gamma_r(x, \mathcal{X}, \phi) \leq |[\phi] \cap \Omega_r(x)|. \]

We first prove a simplified version of the main theorem.

Theorem 3.1. For any $S_{g,n}$ and $\epsilon > 0$, there exists a constant $A > 0$ such that given any $c$-thick pseudo-Anosov element $\phi$ with translation distance $\lambda \geq A$ and given any $x \in \text{axis}(\phi)$, there exists a corresponding constant $G(x, \phi) > 0$ such that
\[ \Gamma_R(x, \mathcal{X}, \phi)^{G(x, \phi)} \sim e^{\frac{1}{2}hr}. \]
Proof. Given \( \phi, \mathcal{X} \) satisfying the assumptions. For any \( R \), define

\[
\begin{align*}
P_R^+ &= \left\{ \psi \in [\phi] \mid d(\mathcal{X}, \pi_\psi(\mathcal{X})) \leq \frac{R + A - \lambda}{2} \right\}, \\
P_R^- &= \left\{ \psi \in [\phi] \mid d(\mathcal{X}, \pi_\psi(\mathcal{X})) \leq \frac{R - 2A - \lambda}{2} \right\}.
\end{align*}
\]

Denote \( \Omega_r(\mathcal{X}) \) as \( \Omega(r) \) for simplicity, by Corollary 2.3 we have

\[
P_R^- \subset [\phi] \cap \Omega(R) \subset P_R^+.
\]

(2) We now work towards obtaining an upper bound for \( |P_R^+| \). Take any \( \psi \in P_R^+ \), there exists a \( f \in \text{Mod}_{g,n} \) such that \( \psi = f\phi f^{-1} \). Since \( \mathcal{X} \in \text{axis}(\phi) \), \( f(\mathcal{X}) \) therefore lies on the axis(\( \psi \)). In particular, this means there exists a \( k \in \mathbb{Z} \) such that

\[
d \left( \psi^k \circ f(\mathcal{X}), \pi_\psi(\mathcal{X}) \right) \leq \frac{\lambda}{2},
\]

\[
d \left( \psi^k \circ f(\mathcal{X}), \mathcal{X} \right) \leq \frac{R + A}{2}.
\]

See Figure 2 for an example.

![Figure 2](image_url)

**Figure 2.** Each \( x_i \) denotes \( \psi^i \circ f(x) \) and distance between any two adjacent \( x_i \) is \( \lambda \). The injective map maps \( \mathcal{X} \) to \( x_3 \) since \( x_3 \) is the closest point to \( \pi_\psi(\mathcal{X}) \) in \( \{x_i\}_{i\in\mathbb{Z}} \).

We claim one can define an injective map from \( P_R^+ \rightarrow \Omega_r(\mathcal{X}) \) by sending \( \psi \) to \( \psi^k f \). Indeed, if there is any another \( \eta \in P_R^+, \eta \neq \psi \), \( \eta = h\phi h^{-1} \) for some \( h \in \text{Mod}_{g,n} \), then \( h(\mathcal{X}) \in \text{axis}(\eta) \) and there exists a \( m \in \mathbb{Z} \) such that

\[
d(\eta^m \circ h(\mathcal{X}), \pi_\eta(\mathcal{X})) \leq \frac{\lambda}{2},
\]

\[
d(\eta^m \circ h(\mathcal{X}), \mathcal{X}) \leq \frac{R + A}{2}.
\]

We claim in this case \( \psi^k f \neq \eta^m h \). Indeed, suppose they are equal, then

\[
\psi = \psi^k \psi \psi^{-k} = \psi^k f \phi f^{-1} \psi^{-k} = \eta^m h \phi h^{-1} \eta^{-m} = \eta^m \eta^{-m} = \eta.
\]
However, this contradicts $\psi \neq \eta$. This means for $R$ large, we can inject $P_R^+$ into $\Omega(\frac{R+\lambda}{2})$, so that

$$|P_R^+| \leq \Omega\left(\frac{R+\lambda}{2}\right) \leq e^{\frac{R+\lambda}{2}} \cdot e^{\frac{R^2}{2}}. \quad (3)$$

To obtain the lower bound for $|P_R^+|$, we define $\mathcal{A}_R = \{\text{axis}(\psi) \mid \psi \in P_R^-\}$. This gives us a surjective map $F : P_R^- \to \mathcal{A}_R, \psi \mapsto \text{axis}(\psi)$, and each $\Theta \in \mathcal{A}_R$ has the form $\Theta = \text{axis}(f \phi f^{-1})$ for some $f \in \Omega(R-2A)$. For any $L < \frac{R-2A+\lambda}{2}$, we define $\mathcal{A}_R^L = \{\Theta \in \mathcal{A}_R \mid \text{d}(X, \pi_\Theta(X)) > \frac{R-2A+\lambda}{2} - L\}$ so that $\mathcal{A}_R^L \subset \mathcal{A}_R$. For each $\Theta \in \mathcal{A}_R$, we denote $H(\Theta) = \{f \in \Omega(R-2A) \mid \text{axis}(f \phi f^{-1}) = \Theta\}$, which is a subset of $\Omega(R-2A)$.

By Corollary 2.2, for any $\Theta \in \mathcal{A}_R^L$, there are at most $\frac{2(L+A)}{\lambda} + 2$ many $f \in H(\Theta)$ satisfying $\text{axis}(f \phi f^{-1}) = \Theta$ since $d(X, \pi_\Theta(X)) \in \left(\frac{R-2A+\lambda}{2} - L, \frac{R-2A+\lambda}{2}\right]$. In the example of Figure 3, there are six such $f$ for this $\Theta$. This means

$$|\mathcal{A}_R^L| \geq \frac{\lambda}{2(L+A+\lambda)} \cdot \sum_{\Theta \in \mathcal{A}_R^L} |H(\Theta)|. \quad (4)$$

For any element $f \in \Omega(\frac{R-2A+\lambda}{2})$, let’s denote $\Theta_f = \text{axis}(f \phi f^{-1})$, then each $f$ is exactly one of the following types.

(a) $\Theta_f$ never enters $B_{\frac{R-2A+\lambda}{2} - L}(X)$.
(b) $\Theta_f$ enters $B_{\frac{R-2A+\lambda}{2} - L}(X)$ and $d(X, f(X)) \leq \frac{R-2A+\lambda}{2} - L$.
(c) $\Theta_f$ enters $B_{\frac{R-2A+\lambda}{2} - L}(X)$ and $d(X, f(X)) > \frac{R-2A+\lambda}{2} - L$.

![Figure 3](image-url)  

**Figure 3.** $\Theta$ is of type (a) and $\Upsilon$ is of type (c). The lengths of $\Theta$ and $\Upsilon$ intersecting $B_{\frac{R-2A+\lambda}{2} - L}$ can be approximated by Corollary 2.2 which showed as the dotted geodesic segments.

The union of type (a) elements is $\bigcup_{\Theta \in \mathcal{A}_R^L} H(\Theta)$, and the union of type (b) elements are $\Omega\left(\frac{R-2A+\lambda}{2} - L\right) \subset \Omega\left(\frac{R-2A+\lambda}{2} - L\right)$. By Corollary 2.2 we notice there are at most $\frac{2(L+A)}{\lambda}$ many type (c) elements can share the same axis, and the
numbers of axes going through $B_{{R-2A-\lambda}}(X)$ is bounded by $|\Omega(\frac{R-2A}{2} - L)|$. In the example of Figure 3, there are six $f$ satisfying type (c) conditions sharing the axis $\Upsilon$. Notice there are two $f$ realize $\Upsilon = \Theta_f$ but not satisfy the type (c) assumption. Since type (a), (b), (c) elements compose $\Omega(\frac{R-2A-\lambda}{2})$, we have

$$\sum_{\Theta \in \mathcal{A}_L} |H(\Theta)| \geq \left| \Omega\left(\frac{R-2A-\lambda}{2}\right) \right| - \left(1 + \frac{2(L + A)}{\lambda}\right) \cdot \left| \Omega\left(\frac{R-2A}{2} - L\right) \right| .$$

Moreover, we let $L$ be a constant satisfy $e^{hL} > 2 \cdot e^{\frac{h}{2}} \cdot N \left(1 + \frac{2(L + A)}{\lambda}\right)$, then

$$\sum_{\Theta \in \mathcal{A}_L} |H(\Theta)| \geq e^{\frac{h(L-2A-\lambda)}{2}} - \left(1 + \frac{2(L + A)}{\lambda}\right) \cdot N \cdot e^{\frac{h(L-2A-\lambda)-hL}{2}} \geq e^{\frac{hL}{2}} \cdot e^{-hA} \cdot \left(\frac{1}{e^{h\frac{L}{2}}} - \frac{N \cdot \left(1 + \frac{2(L + A)}{\lambda}\right)}{e^{hL}}\right) \geq e^{\frac{hL}{2}} \cdot \frac{1}{2e^{h(\frac{L}{2} + A)}} ,$$

and this lower bound is nontrivial.

Thus, to construct the lower bound for $|P_R^-|$, we let $L$ be a constant satisfy $e^{hL} > 2 \cdot e^{\frac{h}{2}} \cdot N \left(1 + \frac{2(L + A)}{\lambda}\right)$. Apply formulas (4), (5) from above, for $R$ large we have

$$|P_R^-| \geq |\mathcal{A}_R| \geq |\mathcal{A}_L^-| \geq \frac{\lambda}{2(L + A + \lambda)} \cdot \sum_{\Theta \in \mathcal{A}_L} |H(\Theta)| \geq e^{\frac{hL}{2}} \cdot \frac{\lambda}{2(L + A + \lambda)e^{hA}} \cdot \frac{1}{2e^{h(\frac{L}{2} + A)}} .$$

Finally, combining formulas (1), (2), (6) we have

$$||\phi| \cdot X \cap B_R(X)| \geq \frac{1}{N} \cdot ||\phi| \cap \Omega(R)| \geq \frac{1}{N} \cdot |P_R^-| \geq G_L(X, \phi) \cdot e^{\frac{hL}{2}}$$

where

$$G_L(X, \phi) = \frac{\lambda}{2N(L + A + \lambda)e^{hA}} \cdot \frac{1}{2e^{h(\frac{L}{2} + A)}} .$$

And combining formulas (1), (2), (3) we have

$$||\phi| \cdot X \cap B_R(X)| \leq ||\phi| \cap \Omega(R)| \leq P_R^- \leq G_U(X, \phi) \cdot e^{\frac{hL}{2}}$$

where

$$G_U(X, \phi) = N e^{\frac{hA}{2}} .$$

Recall $f(R) \leq g(R)$ is the same as $f(R) \leq Ag(R)$. Thus we have

$$e^{\frac{hL}{2}} \cdot G_L^{-1}(X, \phi) \leq ||\phi| \cdot X \cap B_R(X)| \leq e^{\frac{hL}{2}} \cdot G_U(X, \phi) \leq ||\phi| \cap \Omega(R)| \leq P_R^- \leq G_U(X, \phi) \cdot e^{\frac{hL}{2}}$$

This means by setting

$$G(X, \phi) = \max\{G_L^{-1}(X, \phi), G_U(X, \phi)\}$$

we obtain the desired result. $\square$

Now we are ready to prove the general case.
Proof of Theorem 1.5. Take any \(X, Y \in T_{g,n}\), and let \(D\) be the maximum between 
\(d(X, \pi_\phi(X))\) and \(d(\pi_\phi(X), Y)\). We then have

\[
\begin{align*}
|\phi \cdot Y \cap B_R(X)| & \geq |\phi \cdot \pi_\phi(X) \cap B_{R-D}(X)| \geq |\phi \cdot \pi_\phi(X) \cap B_{R-2D}(\pi_\phi(X))|, \\
|\phi \cdot Y \cap B_R(X)| & \leq |\phi \cdot \pi_\phi(X) \cap B_{R+D}(X)| \leq |\phi \cdot \pi_\phi(X) \cap B_{R+2D}(\pi_\phi(X))|.
\end{align*}
\]

By applying these inequalities and by applying Theorem 3.1 to \(\phi\) and \(\pi_\phi(X)\), without loss of generality, we get the desired result by setting \(G(X, Y, \phi) = G(\pi_\phi(X), \phi)\cdot e^{hD}\).

\(\square\)

Proof of Corollary 1.6. Given \(\phi\), we pick \(\epsilon\) so that axis(\(\phi\)) is in \(T_{g,k}\). Since \(\lambda(\phi^k) = k \cdot \lambda(\phi)\) for any pseudo-Anosov element \(\phi\), there exists a \(N(\phi)\) such that \(\lambda(\phi^k) \geq A\) for any \(k \geq N(\phi)\). We now can apply Theorem 1.5 and the corresponding error constant \(G\) depends on \(X, Y, \phi, k\).

\(\square\)

Proof of Corollary 1.7. Assuming the conditions, we can apply the Corollary 1.6. This means for any \(k \geq N\) and for any \(\delta > 1\), there exists a \(M(\delta)\) such that

\[\frac{1}{\delta G(X, Y, \phi, k)} \cdot e^{\frac{h}{2}R} \leq \Gamma_R(X, Y, \phi^k) \leq \delta G(X, Y, \phi, k) \cdot e^{\frac{h}{2}R}\]

for any \(R \geq M(\delta)\). Let \(\epsilon > 0\), one can pick \(\delta > 0\) and pick \(M(\epsilon) \geq M(\delta)\) so that

\[\delta G(X, Y, \phi, k) \leq e^{\epsilon\frac{h}{2}R},
\]

\[e^{-\epsilon\frac{h}{2}R} \leq \frac{1}{\delta G(X, Y, \phi, k)}\]

for any \(R \geq M(\epsilon)\). This implies for any \(\epsilon > 0\), we have

\[e^{(1-\epsilon)\frac{h}{2}R} \leq \Gamma_R(X, Y, \phi^k) \leq e^{(1+\epsilon)\frac{h}{2}R},
\]

\[(1-\epsilon)\frac{h}{2} \leq \ln \Gamma_R(X, Y, \phi^k) \leq (1+\epsilon)\frac{h}{2},
\]

whenever \(R \geq M(\epsilon)\). That is,

\[\lim_{R \to \infty} \frac{1}{R} \ln \Gamma_R(X, Y, \phi^k) = \frac{h}{2}\]

This finishes the proof. \(\square\)

REFERENCES

[1] Goulnara N. Arzhantseva, Christopher H. Cashen, and Jing Tao, *Growth tight actions*, Pacific J. Math. 278 (2015), no. 1, 1–49. MR 3404665
[2] Jayadev Athreya, Alexander Bufetov, Alex Eskin, and Maryam Mirzakhani, *Lattice point asymptotics and volume growth on Teichmüller space*, Duke Math. J. 161 (2012), no. 6, 1055–1111. MR 2913101
[3] David Dumas, *Skinning maps are finite-to-one*, Acta Math. 215 (2015), no. 1, 55–126. MR 3413977
[4] Robert Fricke and Felix Klein, *Vorlesungen über die Theorie der automorphen Funktionen. Band I: Die gruppentheoretischen Grundlagen. Band II: Die funktionentheoretischen Ausführungen und die Anwendungen*, Bibliotheca Mathematica Teubneriana, Bände 3, vol. 4, Johnson Reprint Corp., New York; B. G. Teubner Verlagsgesellschaft, Stuttg art, 1965. MR 0183872
[5] Jiawei Han, *Growth rate of Dehn twist lattice points in Teichmüller space*, forthcoming.
[6] Steven P. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. (2) **117** (1983), no. 2, 235–265. MR 690845

[7] Peter D. Lax and Ralph S. Phillips, *The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces*, J. Functional Analysis **46** (1982), no. 3, 280–350. MR 661875

[8] Joseph Maher, *Asymptotics for pseudo-Anosov elements in Teichmüller lattices*, Geom. Funct. Anal. **20** (2010), no. 2, 527–544. MR 2671285

[9] Grigoriy A. Margulis, *On some aspects of the theory of Anosov systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004, With a survey by Richard Sharp: Periodic orbits of hyperbolic flows, Translated from the Russian by Valentina Vladimirovna Szulikowska. MR 2035655

[10] Yair N. Minsky, *Quasi-projections in Teichmüller space*, J. Reine Angew. Math. **473** (1996), 121–136. MR 1390685

[11] David Mumford, *A remark on Mahler’s compactness theorem*, Proc. Amer. Math. Soc. **28** (1971), 289–294. MR 276410

[12] Jouni Parkkonen and Frédéric Paulin, *On the hyperbolic orbital counting problem in conjugacy classes*, Math. Z. **270** (2015), no. 3-4, 1175–1196. MR 3318265

[13] William P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 2, 417–431. MR 956596

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37235

Email address: jiawei.han@vanderbilt.edu