Generic uniqueness of area minimizing disks for extreme curves

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GENERIC UNIQUENESS OF AREA MINIMIZING DISKS
FOR EXTREME CURVES

By BARIS COSKUNUZER

Abstract. We show that for a generic nullhomotopic simple closed curve \( \Gamma \) in the boundary of a compact, orientable, mean convex 3-manifold \( M \) with \( H_2(M, \mathbb{Z}) = 0 \), there is a unique area minimizing disk \( D \) embedded in \( M \) with \( \partial D = \Gamma \). We also show that the same is true for nullhomologous curves in the absolutely area minimizing surface case.

1. Introduction. The Plateau problem asks about the existence of an area minimizing disk for a given curve in the ambient manifold \( M \). This problem was solved for \( \mathbb{R}^3 \) by Douglas [Do], and Rado [Ra] in early 1930s. Later, it was generalized by Morrey [Mo] for Riemannian manifolds. Then, regularity (nonexistence of branch points) of these solutions was shown by Osserman [Os], Gulliver [Gu] and Alt [Al]. In the early 1960s, the same question was studied for absolutely area minimizing surfaces, i.e., for surfaces that minimize area among all oriented surfaces with the given boundary (without restriction on genus). The geometric measure theory techniques proved to be quite powerful, and De Georgi, Federer-Fleming solved the problem for area minimizing surfaces [Fe].

Later, the question of embeddedness of the solution was studied by many experts. First, Gulliver-Spruck showed embeddedness for the extreme curves with total curvature less than \( 4\pi \) in [GS]. Tomi-Tromba [TT] and Almgren-Simon [AS] showed the existence of embedded minimal (not necessarily area minimizing) disks for extreme curves. Then, Meeks-Yau [MY1] showed that, for extreme boundary curves, area minimizing disks must be embedded. Recently, Ekholm, White, and Wienholtz generalized Gulliver-Spruck embeddedness result by removing extremeness condition from the curves [EWW].

On the other hand, the number of the solutions was also an active area of research. First, Rado showed that if a curve can be projected bijectively to a convex plane curve, then it bounds a unique minimal disk. Then, Nitsche proved uniqueness of minimal disks for the boundary curves with total curvature less than \( 4\pi \) in [Ni]. Then, Tromba [Tr] showed that a generic curve in \( \mathbb{R}^3 \) bounds a unique area minimizing disk. Then, Morgan [M] proved a similar result for area minimizing surfaces. Later, White proved a very strong generic uniqueness result...
for fixed topological type in any dimension [Wh1]. In particular, he showed that a
generic $k$-dimensional, $C^{\infty}$ submanifold of a Riemannian manifold cannot bound
two smooth, minimal $(k + 1)$-manifolds having the same area.

In this paper, we will give a new generic uniqueness result for both versions
of the Plateau problem. Our techniques are simple and topological. The first main
result is the following:

**Theorem 3.2.** Let $M$ be a compact, orientable, mean convex $3$-manifold with
$H_2(M, \mathbb{Z}) = 0$. Then for a generic nullhomotopic (in $M$) simple closed curve $\Gamma$ in
$\partial M$, there exists a unique area minimizing disk $D$ in $M$ with $\partial D = \Gamma$.

This theorem is also true for compact, irreducible, orientable, mean convex
$3$-manifolds (See Remark 3.2). The second main result is a similar theorem for
absolutely area minimizing surfaces.

**Theorem 4.3.** Let $M$ be a compact, orientable, mean convex $3$-manifold with
$H_2(M, \mathbb{Z}) = 0$. Then for a generic nullhomologous (in $M$) simple closed curve $\Gamma$ in
$\partial M$, there exists a unique absolutely area minimizing surface $\Sigma$ in $M$ with
$\partial \Sigma = \Gamma$.

These results naturally generalize to noncompact homogeneously regular $3$-
manifolds (see the last section).

The short outline of the technique for generic uniqueness is the following:
For simplicity, we will focus on the case of the area minimizing disks in a mean
convex manifold $M$. Let $\Gamma_0$ be a nullhomotopic (in $M$) simple closed curve in
$\partial M$. First, we will show that either there exists a unique area minimizing disk $D_0$
in $M$ with $\partial D_0 = \Gamma_0$, or there exist two disjoint area minimizing disks $D_0^+, D_0^-$ in
$M$ with $\partial D_0^\pm = \Gamma_0$.

Now, take a small neighborhood $N(\Gamma_0) \subset \partial M$ which is an annulus. Then
foliate $N(\Gamma_0)$ by simple closed curves $\{\Gamma_t\}$ where $t \in (-\epsilon, \epsilon)$, i.e. $N(\Gamma_0) \simeq \Gamma \times
(-\epsilon, \epsilon)$.
By the above fact, for any $\Gamma_t$ either there exists a unique area minimizing disk $D_t$
or there are two area minimizing disks $D_t^\pm$ disjoint from each other.
Also, since these are area minimizing disks, if they have disjoint boundary, then
they are disjoint by [MY2]. This means, if $t_1 < t_2$, then $D_{t_1}$ is disjoint and below
$D_{t_2}$ in $M$. Consider this collection of area minimizing disks. Note that for curves
$\Gamma_t$ bounding more than one area minimizing disk, we have a canonical region $N_t$
in $M$ between the disjoint area minimizing disks $D_t^\pm$.

Now, take a finite curve $\beta \subset M$ which is transverse to the collection of these
area minimizing disks $\{D_t\}$ whose boundaries are $\{\Gamma_t\}$. Let the length of this
line segment be $C$.

Now, the idea is to consider the thickness of the neighborhoods $N_t$ assigned
to the boundary curves $\{\Gamma_t\}$. When $\Gamma_t$ bounds a unique area minimizing disk $D_t$,
let $N_t = D_t$ be a degenerate canonical region for $\Gamma_t$. Let $s_t$ be the length of the
segment $I_t$ of $\beta$ between $D_t^+$ and $D_t^-$, which is the width of $N_t$ assigned to $\Gamma_t$. Then, the curves $\Gamma_t$ bounding more than one area minimizing disk have positive width, and contribute to total thickness of the collection, and the curves bounding a unique area minimizing disk have 0 width and do not contribute to the total thickness. Since $\sum_{t \in (-\epsilon, \epsilon)} s_t < C$, the total thickness is finite. This implies for only countably many $t \in (-\epsilon, \epsilon)$, $s_t > 0$, i.e. $\Gamma_t$ bounds more than one area minimizing disk. For the remaining uncountably many $t \in (-\epsilon, \epsilon)$, $s_t = 0$, and there exists a unique area minimizing disk for those $t$. This proves the space of simple closed curves of uniqueness is dense in the space of Jordan curves in $\partial M$. Then, we will show this space is not only dense, but also generic.

The organization of the paper is as follows: In the next section we will cover some basic results which will be used in the following sections. In Section 3, we will prove the first main result of the paper. Then in Section 4, we will show the area minimizing surfaces case. Finally in Section 5, we will have some final remarks.

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2. Preliminaries. In this section, we will overview the basic results which we use in the following sections. First, we should note that Hass-Scott’s very nicely written paper [HS] would be a great reference for a good introduction to the notions in this paper. We will start with some basic definitions.

Definition 2.1. An area minimizing disk is a disk which has the smallest area among the disks with the same boundary. An absolutely area minimizing surface is a surface which has the smallest area among all orientable surfaces (with no topological restriction) with the same boundary.

Definition 2.2. Let $M$ be a compact Riemannian 3-manifold with boundary. Then $M$ is a mean convex (or sufficiently convex) if the following conditions hold:
- $\partial M$ is piecewise smooth.
- Each smooth subsurface of $\partial M$ has nonnegative curvature with respect to inward normal.
- There exists a Riemannian manifold $N$ such that $M$ is isometric to a submanifold of $N$ and each smooth subsurface $S$ of $\partial M$ extends to a smooth embedded surface $S'$ in $N$ such that $S' \cap M = S$.

Definition 2.3. A simple closed curve is an extreme curve if it is on the boundary of its convex hull. A simple closed curve is called an H-extreme curve if it is a curve in the boundary of a mean convex manifold $M$. 

Remark 2.1. Note that our results in this paper are for $H$-extreme curves which are in the boundary of a fixed 3-manifold $M$. Since any extreme curve is also $H$-extreme, our results apply to this case as well. Note also that for any smooth embedded curve $\Gamma$, one can find a mean convex (sufficiently thin) solid torus $T_\Gamma$ such that $\Gamma \subset \partial T_\Gamma$, hence $\Gamma$ is $H$-extreme. So, an $H$-extreme curve should be understood with the mean convex manifold which comes with the definition. However, being extreme for a curve is the property of the curve alone (depending only on the ambient manifold).

Now, we state the main facts which we use in the following sections.

**Lemma 2.1.** [MY2], [MY3] Let $M$ be a compact, mean convex 3-manifold, and $\Gamma \subset \partial M$ be a nullhomotopic simple closed curve. Then, there exists an area minimizing disk $D \subset M$ with $\partial D = \Gamma$. Moreover, all such disks are properly embedded in $M$ and they are pairwise disjoint. Also, if $\Gamma_1, \Gamma_2 \subset \partial M$ are disjoint simple closed curves, then the area minimizing disks $D_1, D_2$ spanning $\Gamma_1, \Gamma_2$ are also disjoint.

There is an analogous fact for area minimizing surfaces, too.

**Lemma 2.2.** [Fe], [HSi], [Wh2] Let $M$ be a compact, mean convex 3-manifold, and $\Gamma \subset \partial M$ be a nullhomologous simple closed curve. Then, there exists a smoothly embedded absolutely area minimizing surface $\Sigma \subset M$ with $\partial \Sigma = \Gamma$.

Now, we state a lemma about the limit of area minimizing disks in a mean convex manifold. Note that we mean that the boundary of the disk is in the boundary of the manifold by being properly embedded.

**Lemma 2.3.** [HS] Let $M$ be a compact, mean convex 3-manifold and let $\{D_i\}$ be a sequence of properly embedded area minimizing disks in $M$. Then there is a subsequence $\{D_{i_j}\}$ of $\{D_i\}$ such that $D_{i_j} \to \widehat{D}$, a countable collection of properly embedded area minimizing disks in $\Omega$.

**Convention.** Throughout the paper, all the manifolds will be assumed to be compact, orientable, mean convex and having trivial second homology, i.e., $H_2(M, \mathbb{Z}) = 0$. We will also assume that all the surfaces are orientable as well.

### 3. Generic uniqueness for area minimizing disks.

In this section, we will prove the generic uniqueness of area minimizing disks for $H$-extreme curves. For this, we first show that for any nullhomotopic simple closed curve in the boundary of a mean convex 3-manifold, either there exists a unique area minimizing disk spanning the curve, or there are two canonical extremal area minimizing disks which bound a region containing all other area minimizing disks with the same boundary. Similar results also appear in [MY3], [Li], [Wh3] and [Co].

**Lemma 3.1.** Let $M$ be a compact, orientable, mean convex 3-manifold with $H_2(M, \mathbb{Z}) = 0$. Let $\Gamma$ be a nullhomotopic (in $M$) simple closed curve in $\partial M$. Then
either there is a unique area minimizing disk \( D \) in \( M \) with \( \partial D = \Gamma \), or there are two canonical area minimizing disks \( D^+ \) and \( D^- \) in \( M \) with \( \partial D^\pm = \Gamma \), and any other area minimizing disk in \( M \) with boundary \( \Gamma \) must belong to the canonical region \( N \) bounded by \( D^+ \) and \( D^- \) in \( M \).

**Proof.** Let \( M \) be a mean convex 3-manifold and let \( \Gamma \subset \partial M \) be a nullhomotopic simple closed curve. Take a small neighborhood \( A \) of \( \Gamma \) in \( \partial M \), which will be a thin annulus where \( \Gamma \) is the core. \( \Gamma \) separates the annulus \( A \) into two parts, say \( A^+ \) and \( A^- \) by giving a local orientation. Define a sequence of pairwise disjoint simple closed curves \( \{ \Gamma_i^+ \} \subset A^+ \subset \partial M \) such that \( \lim \Gamma_i^+ = \Gamma \). Now, by Lemma 2.1, for any curve \( \Gamma_i^+ \), there exists an embedded area minimizing disk \( D_i^+ \) with \( \partial D_i^+ = \Gamma_i^+ \). This defines a sequence of area minimizing disks \( \{ D_i^+ \} \) in \( M \). By Lemma 2.3, there exists a subsequence \( \{ D_i^+ \} \) converging to a countable collection of area minimizing disks \( \hat{D}^+ \) with \( \partial \hat{D}^+ = \Gamma \).

We claim that this collection \( \hat{D}^+ \) consists of only one area minimizing disk. Assume that there are two disks in the collection say \( D^+_a \) and \( D^+_b \), and say \( D^+_a \) is above \( D^+_b \) (in the positive side of \( D^+_b \) in the local orientation). By Lemma 2.1, \( D^+_a \) and \( D^+_b \) are embedded and disjoint. They have the same boundary \( \Gamma \subset \partial M \). \( D^+_b \) is also limit of the sequence \( \{ D_i^+ \} \). But, since for any area minimizing disk \( D_i^+ \subset M \), \( \partial D_i^+ = \Gamma_i^+ \) is disjoint from \( \partial D^+_a = \Gamma \), \( D^+_a \) disjoint from \( D^+_b \), again by Lemma 2.1. This means \( D^+_b \) is a barrier between the sequence \( \{ D_i^+ \} \) and \( D^+_a \), and so, \( D^+_b \) cannot be limit of this sequence. This is a contradiction. So \( \hat{D}^+ \) is just one area minimizing disk, say \( D^+ \). Similarly, \( \hat{D}^- = D^- \).

Now, we claim these area minimizing disks \( D^+ \) and \( D^- \) are canonical, depending only on \( \Gamma \) and \( M \), and independent of the choice of the sequence \( \{ \Gamma_i \} \) and \( \{ D_i \} \). Let \( \{ \gamma_i^+ \} \) be another sequence of simple closed curves in \( A^+ \). Assume that there exists another area minimizing disk \( E^+ \) with \( \partial E^+ = \Gamma \) and \( E^+ \) is a limit of the sequence of area minimizing disks \( E_i^+ \) with \( \partial E_i^+ = \gamma_i^+ \subset A^+ \). By Lemma 2.1, \( D^+ \) and \( E^+ \) are disjoint. Then one of them is above the other one. If \( D^+ \) is above \( E^+ \), then \( D^+ \) between the sequence \( E_i^+ \) and \( E^+ \). This is because, all \( E_i^+ \) are disjoint and above \( E^+ \) as \( \partial E_i^+ = \gamma_i \) are disjoint and above \( \Gamma \). Similarly, \( D^+ \) is below \( E_i^+ \) for any \( i \) (in the negative side of \( E_i^+ \) in the local orientation), as \( \partial D^+ = \Gamma \) is below the curves \( \gamma_i^+ \subset A^+ \). Now, since \( D^+ \) is between the sequence \( \{ E_i^+ \} \) and its limit \( E^+ \), and \( E^+ \) and \( D^+ \) are disjoint, \( D^+ \) will be a barrier for the sequence \( \{ E_i^+ \} \), and so they cannot limit on \( E^+ \). This is a contradiction. Similarly, \( D^+ \) cannot be below \( E^+ \), so they must be same. Hence, \( D^+ \) and \( D^- \) are canonical area minimizing disks for \( \Gamma \).

Now, we will show that any area minimizing disk in \( M \) with boundary \( \Gamma \) must belong to the canonical region \( N \) bounded by \( D^+ \) and \( D^- \) in \( M \), i.e., \( \partial N \supseteq D^+ \cup D^- \) (1.2). Let \( E \) be any area minimizing disk with boundary \( \Gamma \). By Lemma 2.1, \( E \) is disjoint from \( D^+ \) and \( D^- \). Hence, if \( E \) is not in \( N \), then it must be completely outside of \( N \). So, \( E \) is either above \( D^+ \) or below \( D^- \). However, \( D^+ = \lim D_i^+ \) and \( \Gamma_i^+ \to \Gamma \) from above. Moreover, again by Lemma 2.1, \( E \) must
be disjoint from $D_i^+$. Hence, $E$ would be a barrier between the sequence $\{D_i^+\}$ and $D'$ like in previous paragraph. This is a contradiction. Similarly, the same is true for $D^-$. Hence, any area minimizing disk in $M$ with boundary $\Gamma$ must belong to the canonical region $N$ bounded by $D^+$ and $D^-$ in $M$. This also shows that if $D^+ = D^-$, then there exists a unique area minimizing disk in $M$ with boundary $\Gamma$.

Remark 3.1. The results in [MY3], [Li], [Wh3] are similar to this one in some sense. In those papers, the authors show the ”strong uniqueness” property, which says that either an $H$-extreme curve bounds more than one minimal disk in the mean convex manifold $M$ or there is a unique minimal surface bounding the curve which is indeed an area minimizing disk in $M$. Our result is relatively different than the others. In the above lemma, we proved that either there exists a unique area minimizing disk in $M$ bounding the $H$-extreme curve, or there are two canonical extremal area minimizing disks in $M$ which bound a region containing all other area minimizing disks with the same boundary.

Now, we prove the main result of the paper.

**Theorem 3.2.** Let $M$ be a compact, orientable, mean convex 3-manifold with $H_2(M, \mathbb{Z}) = 0$. Then for a generic nullhomotopic (in $M$) simple closed curve $\Gamma$ in $\partial M$, there exists a unique area minimizing disk $D$ in $M$ with $\partial D = \Gamma$. In other words, let $\mathcal{A}$ be the space of nullhomotopic (in $M$) simple closed curves in $\partial M$ and let $\mathcal{A}' \subset \mathcal{A}$ be the subspace containing the curves bounding a unique area minimizing disk in $M$. Then, $\mathcal{A}'$ is generic in $\mathcal{A}$, i.e., $\mathcal{A}'$ is countable intersection of open dense subsets.

**Proof.** We will prove this theorem in 2 steps.

**Claim 1.** $\mathcal{A}'$ is dense in $\mathcal{A}$ as a subspace of $C^0(\mathbb{S}^1, \partial M)$ with the supremum metric.

**Proof.** Let $\mathcal{A}$ be the space of nullhomotopic simple closed curves in $\partial M$. We parametrize this space with $C^0$ parametrizations, and use supremum metric, i.e.,

$$\mathcal{A} = \{ \alpha \in C^0(\mathbb{S}^1, \partial M) \mid \alpha(\mathbb{S}^1) \text{ is an embedding, and nullhomotopic in } M \}.$$  

Now, let $\Gamma_0 \in \mathcal{A}$ be a nullhomotopic simple closed curve in $\partial M$. Since $\Gamma_0$ is simple, there exists a small closed neighborhood $N(\Gamma_0)$ of $\Gamma_0$ which is an annulus in $\partial M$. Let $\Gamma: [-\epsilon, \epsilon] \to \mathcal{A}$ be a small path in $\mathcal{A}$ through $\Gamma_0$ such that $\Gamma(0) = \Gamma_i$ and $\{ \Gamma_i \}$ foliates $N(\Gamma)$ with simple closed curves $\Gamma_i$. In other words, $\{ \Gamma_i \}$ are pairwise disjoint simple closed curves, and $N(\Gamma_0) = \bigcup_{i \in [-\epsilon, \epsilon]} \Gamma_i$.

By Lemma 3.1, for any $\Gamma_i$ either there exists a unique area minimizing disk $D_i$ in $M$, or there is a canonical region $N_i$ in $M$ between the canonical area minimizing disks $D_i^+$ and $D_i^-$. With abuse of notation, if $\Gamma_i$ bounds a unique area minimizing disk $D_i$ in $M$, define $N_i = D_i$ as a degenerate canonical neighborhood for $\Gamma_i$. Clearly, a degenerate neighborhood $N_i$ means $\Gamma_i$ bounds a unique area
minimizing disk, and a nondegenerate neighborhood \( N_t \) means that \( \Gamma_t \) bounds more than one area minimizing disk. Note that by Lemma 3.1 and Lemma 2.1, all canonical neighborhoods in the collection are pairwise disjoint.

Now, let \( \tilde{N} \) be the union of these canonical neighborhoods \( \{ N_t \} \), i.e., \( \tilde{N} = \bigcup_{t \in [-\epsilon, \epsilon]} N_t \). Then, \( \partial \tilde{N} \supseteq D_t^+ \cup N(0) \cup D^-_\epsilon \). Let \( p^+ \) be a point in \( D_t^+ \) and \( p^- \) be a point in \( D^-_\epsilon \). Let \( \beta \) be a finite curve from \( p^+ \) to \( p^- \) intersecting transversely all the canonical neighborhoods in the collection \( \tilde{N} \).

Now, for each \( t \in [-\epsilon, \epsilon] \), we will assign a real number \( s_t \geq 0 \). Let \( I_t = \beta \cap N_t \) and \( s_t \) be the length of \( I_t \). Then, if \( N_t \) is degenerate (there exists a unique area minimizing disk \( D_t \) in \( M \) for \( \Gamma_t \)), then \( s_t \) would be 0. If \( N_t \) is nondegenerate (\( \Gamma_t \) bounds more than one area minimizing disk), then \( s_t > 0 \). Also, it is clear that for any \( t \), \( I_t \subset \beta \) and \( I_t \cap I_s = \emptyset \) for any \( t \neq s \). Then, \( \sum_{t \in [-\epsilon, \epsilon]} s_t < C \) where \( C \) is the length of \( \beta \). This means for only countably many \( t \in [-\epsilon, \epsilon] \), \( s_t > 0 \). So, there are only countably many nondegenerate \( N_t \) for \( t \in [-\epsilon, \epsilon] \). Hence, for all other \( t \), \( N_t \) is degenerate. This means there exist uncountably many \( t \in [-\epsilon, \epsilon] \), where \( \Gamma_t \) bounds a unique area minimizing disk. Since \( \Gamma_0 \) is arbitrary, this proves \( \mathcal{A}' \) is dense in \( \mathcal{A} \).

**Claim 2.** \( \mathcal{A}' \) is generic in \( \mathcal{A} \).

**Proof.** We will prove that \( \mathcal{A}' \) is a countable intersection of open dense subsets of \( \mathcal{A} \). Then the result will follow by the Baire category theorem.

Since the space of continuous maps from circle to boundary of \( M \), \( C^0(S^1, \partial M) \), is complete with supremum metric, then the closure of \( \mathcal{A} \) in \( C^0(S^1, \partial M) \), \( \bar{\mathcal{A}} \subset C^0(S^1, \partial M) \), is also complete.

Now, we will define a sequence of open dense subsets \( U_i \subset \mathcal{A} \) such that their intersection will give us \( \mathcal{A}' \). Let \( \Gamma \in \mathcal{A} \) be a simple closed curve in \( \partial M \). As in the Claim 1, let \( N(\Gamma) \subset \partial M \) be a neighborhood of \( \Gamma \) in \( \partial M \), which is an open annulus. Then, define an open neighborhood \( U_\Gamma \) of \( \Gamma \) in \( \mathcal{A} \), such that \( U_\Gamma = \{ \alpha \in \mathcal{A} \mid \alpha(S^1) \subset N(\Gamma), \ \alpha \text{ is homotopic to } \Gamma \} \). Clearly, \( \mathcal{A} = \bigcup_{\Gamma \in \mathcal{A}} U_\Gamma \).

Now, define a finite curve \( \beta_\Gamma \) as in Claim 1, which intersects transversely all the area minimizing disks bounding the curves in \( U_\Gamma \).

Now, for any \( \alpha \in U_\Gamma \), by Lemma 3.1, there exists a canonical region \( N_\alpha \) in \( M \) (which can be degenerate if \( \alpha \) bounds a unique area minimizing disk). Let \( I_{\alpha, \Gamma} = N_\alpha \cap \beta_\Gamma \). Then let \( s_{\alpha, \Gamma} \) be the length of \( I_{\alpha, \Gamma} \) \((s_{\alpha, \Gamma} = 0 \text{ if } N_\alpha \text{ degenerate})\). Hence, for every element \( \alpha \) in \( U_\Gamma \), we assign a real number \( s_{\alpha, \Gamma} \geq 0 \).

Now, we define the sequence of open dense subsets in \( U_\Gamma \). Let \( U^+_\Gamma = \{ \alpha \in U_\Gamma \mid s_{\alpha, \Gamma} < 1/i \} \). We claim that \( U^+_\Gamma \) is an open subset of \( U_\Gamma \) and \( \mathcal{A} \). Let \( \alpha \in U^+_\Gamma \), and let \( s_{\alpha, \Gamma} = \lambda < 1/i \). So, the interval \( I_{\alpha, \Gamma} \subset \beta_\Gamma \) has length \( \lambda \). Let \( I' \subset \beta_\Gamma \) be an interval containing \( I_{\alpha, \Gamma} \) in its interior, and has length less than \( 1/i \). By the proof of Claim 1, we can find two simple closed curves \( \alpha^+, \alpha^- \in U_\Gamma \) with the following properties:

- \( \alpha^+ \) are disjoint from \( \alpha \),
- \( \alpha^+ \) are lying in opposite sides of \( \alpha \) in \( \partial M \),
\begin{itemize}
  \item $\alpha^\pm$ bounds a unique area minimizing disk $D_{\alpha^\pm}$,
  \item $D_{\alpha^+} \cap \beta_T \subset I'$.
\end{itemize}

The existence of such curves is clear from the proof of Claim 1, as if one takes any foliation $\{\alpha_i\}$ of a small neighborhood of $\alpha$ in $\partial M$, there are countably many curves in the family bounding a unique area minimizing disk, and one can choose a sufficiently close pair of curves to $\alpha$, to ensure the conditions above.

After finding $\alpha^\pm$, consider the open annulus $F_{\alpha}$ in $\partial M$ bounded by $\alpha^+$ and $\alpha^-$. Let $V_{\alpha} = \{ \gamma \in U_\Gamma \mid \gamma(S^1) \subset F_{\alpha}, \gamma \text{ is homotopic to } \alpha \}$. Clearly, $V_{\alpha}$ is an open subset of $U_\Gamma$. If we can show $V_{\alpha} \subset U_\Gamma^i$, then this proves $U_\Gamma^i$ is open for any $i$ and any $\Gamma \in A$.

Let $\gamma \in V_{\alpha}$ be any curve, and $N_\gamma$ be its canonical neighborhood given by Lemma 3.1. Since $\gamma(S^1) \subset F_{\alpha}$, $\alpha^+$ and $\alpha^-$ lie in opposite sides of $\gamma$ in $\partial M$. This means $D_{\alpha^+}$ and $D_{\alpha^-}$ lie in opposite sides of $N_\gamma$. By choice of $\alpha^\pm$, this implies $N_\gamma \cap \beta_T = i_{\gamma, \Gamma} \subset I$. So, the length $s_{\gamma, \Gamma}$ is less than $1/i$. This implies $\gamma \in U_\Gamma^i$, and so $V_{\alpha} \subset U_\Gamma^i$. Hence, $U_\Gamma^i$ is open in $U_\Gamma$ and $A$.

Now, we can define the sequence of open dense subsets. Let $U^i = \bigcup_{\Gamma \in A} U_\Gamma^i$ be an open subset of $A$. Since, the elements in $A'$ represent the curves bounding a unique area minimizing disk, for any $\alpha \in A'$, and for any $\Gamma \in A$, $s_{\alpha, \Gamma} = 0$. This means $A' \subset U^i$ for any $i$. By Claim 1, $U^i$ is open dense in $A$ for any $i > 0$.

As we mention at the beginning of the proof, since the space of continuous maps from circle to boundary of $M$, $C^0(S^1, \partial M)$ is complete with supremum metric, then the closure $\bar{A}$ of $A$ in $C^0(S^1, \partial M)$ is also a complete metric space. Since $A'$ is dense in $\bar{A}$, it is also dense in $\bar{A}$. As $A$ is open in $C^0(S^1, \partial M)$, this implies $U^i$ is a sequence of open dense subsets of $\bar{A}$. On the other hand, since $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n \supseteq \cdots$ and $\bigcap_{i=1}^{\infty} U_i = A'$, $A'$ is generic in $A$. \hfill \square

Remark 3.2. Notice that we use the homology condition just to make sure that $D^+ \cup D^-$ is a separating sphere in $M$, and hence to define the canonical region between them in Lemma 3.1. So, if we replace the $H_2(M, \mathbb{Z}) = 0$ condition with the irreducibility of a 3-manifold (any embedded 2-sphere bounds a 3-ball in $M$), the same proof for Lemma 3.1 and Theorem 3.2 would go through. In other words, Theorem 3.2 is also true for compact, irreducible, orientable, mean convex 3-manifolds.

4. Generic uniqueness for area minimizing surfaces. In this section, we will prove the generic uniqueness result for $H$-extreme curves in the absolutely area minimizing case. The technique is basically the same as with the area minimizing disk case. First, we will prove an analogous version of Lemma 2.1 [MY2, Theorem 6] for absolutely area minimizing surfaces. However, the analogous version of Lemma 2.1 is not true in general for the global version. Hence, we will prove it for a local version which suffices for our purposes. See Remark 4.1.

Lemma 4.1. Let $M$ be a compact, orientable, mean convex 3-manifold with $H_2(M, \mathbb{Z}) = 0$. Let $A$ be an annulus in $\partial M$ whose core $\Gamma$ is nullhomologous in $M$. If
\( \Gamma_1 \) and \( \Gamma_2 \) are two disjoint simple closed curves in \( A \) which are homotopic to \( \Gamma \) in \( A \), then any absolutely area minimizing surfaces \( \Sigma_1 \) and \( \Sigma_2 \) in \( M \) with \( \partial \Sigma_1 = \Gamma_1 \) are disjoint, too. Moreover, if \( \Sigma \) and \( \Sigma' \) are two absolutely area minimizing surfaces in \( M \) where \( \partial \Sigma = \partial \Sigma' = \Gamma \), then they must be disjoint, too.

**Proof.** Let \( M \) be a mean convex 3-manifold, and \( A \) is an annulus in \( \partial M \) whose core \( \Gamma \) is nullhomologous in \( M \). Let \( \Gamma_1 \) and \( \Gamma_2 \) are two disjoint simple closed curves in \( A \) which are homotopic to \( \Gamma \) in \( A \). Let \( \Sigma_1 \) and \( \Sigma_2 \) be absolutely area minimizing surfaces in \( M \) with \( \partial \Sigma_1 = \Gamma_1 \). We want to show that \( \Sigma_1 \) and \( \Sigma_2 \) are disjoint.

Assume on the contrary that \( \Sigma_1 \cap \Sigma_2 \neq \emptyset \). Now, let \( \tilde{N} \) be the convex hull of \( A \) in \( M \). Then, by the maximum principle, \( \Sigma_1 \) and \( \Sigma_2 \) are in \( \tilde{N} \). Moreover, as \( \Gamma_1 \) separates the annulus \( A \), then \( \Sigma_1 \) is separating in \( \tilde{N} \). Similarly, \( \Sigma_2 \) is separating, too. Now, if \( \Sigma_1 \cap \Sigma_2 = \gamma \) where \( \gamma \) is a collection of closed curves, then \( \Sigma_1 \) separates \( \Sigma_2 \) into two subsurfaces \( S_1^1 \) and \( S_1^2 \) where \( \partial S_1^1 = \gamma \) and \( \partial S_1^2 = \gamma \cup \Gamma_1 \). Similarly, \( \Sigma_2 \) separates \( \Sigma_1 \) into two subsurfaces \( S_2^1 \) and \( S_2^2 \) where \( \partial S_2^1 = \gamma \) and \( \partial S_2^2 = \gamma \cup \Gamma_2 \).

Now, we will use the Meeks-Yau exchange roundoff trick to get a contradiction [MY2].

As \( \Sigma_1 \) and \( \Sigma_2 \) are absolutely area minimizing surfaces in \( M \), \( |S_1^1| = |S_2^2| \) where \( |S| \) is the area of \( S \). Now define a new surface by swapping the subsurfaces \( S_1^1 \) and \( S_2^2 \). In other words, let \( T_1 = (\Sigma_1 - S_1^1) \cup S_2^2 \). As \( T_1 \) and \( \Sigma_1 \) have same area, then \( T_1 \) is also absolutely area minimizing surface. However, \( \gamma \) is a folding curve in \( T_1 \) as in [MY2]. This is a contradiction (One can also argue with the regularity of the absolutely area minimizing surfaces [Fe]). Hence, this shows that \( \Sigma_1 \) and \( \Sigma_2 \) in \( M \) with \( \partial \Sigma_i = \Gamma_i \) are disjoint absolutely area minimizing surfaces in \( M \).

Now, we will consider the same boundary case. Let \( A \) and \( \Gamma \) be as in the statement of the theorem. Let \( \Sigma \) and \( \Sigma' \) be two absolutely area minimizing surfaces where \( \partial \Sigma = \partial \Sigma' = \Gamma \). Let \( \tilde{N} \) be as above. Then, \( \Sigma_1 \) and \( \Sigma_2 \) are separating in \( \tilde{N} \). As in the previous paragraph, \( \Sigma_1 \) and \( \Sigma_2 \) separate each other, and by the swapping argument again, we get a contradiction. The proof follows. \( \square \)

**Remark 4.1.** The techniques for Lemma 2.1 (or [MY2, Theorem 6]) not working for an analogous theorem in the absolutely area minimizing surfaces case in general. In other words, if we just assume \( \Gamma_1 \cap \Gamma_2 = \emptyset \), and do not require them to be in the annulus \( A \), then the techniques of the above lemma do not apply. This is because if, for example, \( \Gamma_1 \) or \( \Gamma_2 \) are not separating in \( \partial M \), then the intersection of absolutely area minimizing surfaces \( \Sigma_1 \) and \( \Sigma_2 \) might contain a nonseparating curve \( \gamma \) in one of the surfaces, say \( \Sigma_1 \). Hence, we cannot make any surgery there because \( \gamma \) may not bound a subsurface in \( \Sigma_1 \). So, we went to a local version of this theorem (which is enough for our purposes) by restricting \( \partial M \) to a small subannulus \( A \) in \( \partial M \) to make sure that each essential curve is separating in \( A \), and we can make the surgery in the intersection of surfaces.

Now, we will give a generalization of Lemma 3.1 in the absolutely area minimizing surface case.
Lemma 4.2. Let \( M \) be a compact, orientable, mean convex 3-manifold with \( H_2(M, \mathbb{Z}) = 0 \). Let \( \Gamma \) be a nullhomologous (in \( M \)) simple closed curve in \( \partial M \). Then either there is a unique absolutely area minimizing surface \( \Sigma \) in \( M \) with \( \partial \Sigma = \Gamma \), or there are uniquely defined two canonical extremal absolutely area minimizing surfaces \( \Sigma^+ \) and \( \Sigma^- \) in \( M \) with \( \partial \Sigma^\pm = \Gamma \), and any other absolutely area minimizing surface in \( M \) with boundary \( \Gamma \) must belong to the canonical region \( N \) bounded by \( \Sigma^+ \) and \( \Sigma^- \) in \( M \).

Proof. Let \( M \) be a mean convex 3-manifold and let \( \Gamma \subset \partial M \) be a nullhomologous simple closed curve. Take a small neighborhood \( A \) of \( \Gamma \) in \( \partial M \), which will be a thin annulus where \( \Gamma \) is the core. \( \Gamma \) separates the annulus \( A \) into two parts, say \( A^+ \) and \( A^- \) by giving a local orientation. Define a sequence of pairwise disjoint simple closed curves \( \{ \Gamma_i^+ \} \subset A^+ \subset \partial M \) such that \( \lim \Gamma_i^+ = \Gamma \). Now, by Lemma 2.2, for any curve \( \Gamma_i^+ \), there exists an embedded absolutely area minimizing surface \( \Sigma_i^+ \) with \( \partial \Sigma_i^+ = \Gamma_i^+ \). This defines a sequence of absolutely area minimizing surfaces \( \{ \Sigma_i^+ \} \) in \( M \). By [Fe], there exists a subsequence \( \{ \Sigma_i^+ \} \) converging to an absolutely area minimizing surface \( \Sigma^+ \) with \( \partial \Sigma^+ = \Gamma \). Similarly, by defining a similar sequence \( \{ \Gamma_i^- \} \) in \( A^- \) and similar construction, an absolutely area minimizing surface \( \Sigma^- \) with \( \partial \Sigma^- = \Gamma \) can be defined.

Now, we claim these absolutely area minimizing surfaces \( \Sigma^+ \) and \( \Sigma^- \) are canonical, depending only on \( \Gamma \) and \( M \), and independent of the choice of the sequence \( \{ \Gamma_i \} \) and \( \{ \Sigma_i \} \). Let \( \{ \gamma_i^+ \} \) be another sequence of simple closed curves in \( A^+ \). Assume that there exists another absolutely area minimizing surface \( S^+ \) with \( \partial S^+ = \Gamma \) and \( S^+ \) is a limit of the sequence of absolutely area minimizing surfaces \( S_i^+ \) with \( \partial S_i^+ = \gamma_i^+ \subset A^+ \). As \( \Sigma^+ \) and \( S^+ \) are absolutely area minimizing surfaces with the same boundary \( \Gamma \), they are disjoint by Lemma 4.1. Then one of them is above the other one (in the positive side of the other one in the local orientation). If \( \Sigma^+ \) is above \( S^+ \), then \( \Sigma^+ \) between the sequence \( S_i^+ \) and \( S^+ \). This is because all \( S_i^+ \) are disjoint and above \( S^+ \) as \( \partial S_i^+ = \gamma_i \) are disjoint and above \( \Gamma \). Similarly, \( \Sigma^+ \) is below \( S_i \) for any \( i \), as \( \partial \Sigma^+ = \Gamma \) is below the curves \( \gamma_i^+ \subset A^+ \).

Now, since \( \Sigma^+ \) is between the sequence \( \{ S_i^+ \} \) and its limit \( S^+ \), and \( S^+ \) and \( \Sigma^+ \) are disjoint, \( \Sigma^+ \) will be a barrier for the sequence \( \{ S_i^+ \} \), and so they cannot limit on \( S^+ \). This is a contradiction. Similarly, \( \Sigma^+ \) cannot be below \( S^+ \), so they must be the same. Hence, \( \Sigma^+ \) and \( \Sigma^- \) are canonical absolutely area minimizing surfaces for \( \Gamma \).

Now, we will show that any absolutely area minimizing surface in \( M \) with boundary \( \Gamma \) must belong to the canonical region \( N \) bounded by \( \Sigma^+ \) and \( \Sigma^- \) in \( M \), i.e., \( \partial N \supset \Sigma^+ \cup \Sigma^- \) (\( H_2(M, \mathbb{Z}) = 0 \)). Let \( T \) be any absolutely area minimizing surface with boundary \( \Gamma \). By Lemma 4.1, \( T \) is disjoint from \( \Sigma^+ \) and \( \Sigma^- \). Hence, if \( T \) is not in \( N \), then it must be completely outside of \( N \). So, \( T \) is either above \( \Sigma^+ \) or below \( \Sigma^- \). Assume \( T \) is above \( \Sigma^+ \). However, \( \Sigma^+ = \lim \Sigma_i^+ \) and \( \Gamma_i^+ \rightarrow \Gamma \) from above. Moreover, again by Lemma 4.1, \( T \) must be disjoint from \( \Sigma_i^+ \). Hence, \( T \) is a barrier between the subsequence \( \Sigma_i^+ \) and its limit \( \Sigma \). As in the previous
paragraph, this is a contradiction. Similarly, the same is true for $\Sigma^-$. Hence, any absolutely area minimizing surface in $M$ with boundary $\Gamma$ must belong to the canonical region $N$ bounded by $\Sigma^+$ and $\Sigma^-$ in $M$. This also shows that if $\Sigma^+ = \Sigma^-$, then there exists a unique absolutely area minimizing surface in $M$ with boundary $\Gamma$.

Now, we can prove the generic uniqueness result for absolutely area minimizing surfaces.

**Theorem 4.3.** Let $M$ be a compact, orientable, mean convex 3-manifold with $H_2(M, \mathbb{Z}) = 0$. Then for a generic nullhomologous (in $M$) simple closed curve $\Gamma$ in $\partial M$, there exists a unique absolutely area minimizing surface $\Sigma$ in $M$ with $\partial \Sigma = \Gamma$. In other words, let $A$ be the space of nullhomologous (in $M$) simple closed curves in $\partial M$ and let $A' \subset A$ be the subspace containing the curves bounding a unique absolutely area minimizing surface in $M$. Then, $A'$ is generic in $A$, i.e. $A'$ is countable intersection of open dense subsets.

**Proof.** The idea is basically the same as with Theorem 3.2. We will imitate the same proof in this context. Again, we will prove this theorem in 2 steps.

**Claim 1.** $A'$ is dense in $A$ as a subspace of $C^0(S^1, \partial M)$ with the supremum metric.

**Proof.** Let $A$ be the space of nullhomologous simple closed curves in $\partial M$. We parametrize this space with $C^0$ parametrizations, and use supremum metric, i.e., $A = \{ \alpha \in C^0(S^1, \partial M) \mid \alpha(S^1) \text{ is an embedding, and nullhomologous in } M \}$.

Now, let $\Gamma_0 \in A$ be a nullhomologous simple closed curve in $\partial M$. As in the proof of Theorem 3.2, let $N(\Gamma_0)$ be an annulus in $\partial M$ and let $\Gamma: [-\epsilon, \epsilon] \rightarrow A$ foliates $N(\Gamma)$ with simple closed curves $\Gamma_t$.

By Lemma 4.2, for any $\Gamma_t$ either there exists a unique absolutely area minimizing surface $\Sigma_t$ in $M$, or there is a canonical region $N_t$ in $M$ between the canonical area minimizing surfaces $\Sigma_t^+$ and $\Sigma_t^-$. As in the proof of Theorem 3.2, if $\Gamma_t$ bounds a unique absolutely area minimizing surface $\Sigma_t$ in $M$, define $N_t = \Sigma_t$ as a degenerate canonical neighborhood for $\Gamma_t$. Clearly, degenerate neighborhood $N_t$ means $\Gamma_t$ bounds a unique absolutely area minimizing surface, and nondegenerate neighborhood $N_t$ means that $\Gamma_t$ bounds more than one absolutely area minimizing surface. Note that by Lemma 4.1, all canonical neighborhoods in the collection are pairwise disjoint.

As before, let $\hat{N} = \bigcup_{t \in [-\epsilon, \epsilon]} N_t$. Let $p^+$ be a point in $\Sigma_{-\epsilon}^+$ and $p^-$ be a point in $\Sigma_{-\epsilon}^-$. Let $\beta$ be a finite curve from $p^+$ to $p^-$ intersecting transversely all the canonical neighborhoods in the collection $\hat{N}$. For each $t \in [-\epsilon, \epsilon]$, assign a real number $s_t$ to be the length of $I_t = \beta \cap N_t$. Clearly if $N_t$ is nondegenerate ($\Gamma_t$ bounds more than one absolutely area minimizing surface), then $s_t > 0$. Then, $\sum_{t \in [-\epsilon, \epsilon]} s_t < C$ where $C$ is the length of $\beta$. This means for only countably
many \( t \in [-\epsilon, \epsilon] \), \( s_t > 0 \). So, there are only countably many nondegenerate \( N_t \) for \( t \in [-\epsilon, \epsilon] \). Hence, for all other \( t \), \( N_t \) is degenerate. This means there exist uncountably many \( t \in [-\epsilon, \epsilon] \), where \( \Gamma_t \) bounds a unique absolutely area minimizing surface. Since \( \Gamma_0 \) is arbitrary, this proves \( \mathcal{A}' \) is dense in \( \mathcal{A} \).

\[
\text{Claim 2.} \quad \mathcal{A}' \text{ is generic in } \mathcal{A}.
\]

**Proof.** Let \( \mathcal{A} \) be as in the proof of Theorem 3.2. Again, we will define a sequence of open dense subsets \( U^i \subset \mathcal{A} \) such that their intersection will give us \( \mathcal{A}' \). Let \( \Gamma \in \mathcal{A} \) be a simple closed curve in \( \partial M \). As in the Claim 1, let \( N(\Gamma) \subset \partial M \) be a neighborhood of \( \Gamma \) in \( \partial M \), which is an open annulus. Then, define an open neighborhood \( U_\Gamma \) of \( \Gamma \) in \( \mathcal{A} \), such that \( U_\Gamma = \{ \alpha \in \mathcal{A} \mid \alpha(S^3) \subset N(\Gamma) \} \). Clearly, \( \mathcal{A} = \bigcup_{\Gamma \in \mathcal{A}} U_\Gamma \). Now, define a finite curve \( \beta_\Gamma \) as in Claim 1, which intersects all the absolutely area minimizing surfaces bounding the curves in \( U_\Gamma \).

Now, for any \( \alpha \in U_\Gamma \), by Lemma 4.2, there exists a canonical region \( N_\alpha \) in \( M \). Let \( I_{\alpha, \Gamma} = N_\alpha \cap \beta_\Gamma \). Then let \( s_{\alpha, \Gamma} \) be the length of \( I_{\alpha, \Gamma} \) (\( s_{\alpha, \Gamma} \) is 0 if \( N_\alpha \) degenerate). Now, we define the sequence of open dense subsets in \( U_\Gamma \).

Let \( U^i_\Gamma = \{ \alpha \in U_\Gamma \mid s_{\alpha, \Gamma} < 1/i \} \). We claim that \( U^i_\Gamma \) is an open subset of \( U_\Gamma \) and \( \mathcal{A} \). Let \( \alpha \in U^{i}_{\Gamma} \), and let \( s_{\alpha, \Gamma} = \lambda < 1/i \). So, the interval \( I_{\alpha, \Gamma} \subset \beta_\Gamma \) has length \( \lambda \). Let \( I' \subset I_\Gamma \) be an interval containing \( I_{\alpha, \Gamma} \) in its interior, and has length less than \( 1/i \). Now, let \( \alpha^+ \) and \( \alpha^- \) be as in the proof of Theorem 3.2. Consider the open annulus \( F_\alpha \) in \( \partial M \) bounded by \( \alpha^+ \) and \( \alpha^- \). Let \( V_\alpha = \{ \gamma \in U_\Gamma \mid \gamma(S^3) \subset F_\alpha \} \). Clearly, \( V_\alpha \) is an open subset of \( U_\Gamma \). If we can show \( V_\alpha \subset U^i_\Gamma \), then this proves \( U^i_\Gamma \) is open for any \( i \) and any \( \Gamma \in \mathcal{A} \).

Let \( \gamma \in V_\alpha \) be any curve, and \( N_\gamma \) be its canonical neighborhood given by Lemma 4.2. Since \( \gamma(S^3) \subset F_\alpha \), \( \alpha^+ \) and \( \alpha^- \) lie in opposite sides of \( \gamma \) in \( \partial M \). This means \( \Sigma^+ \) and \( \Sigma^- \) lie in opposite sides of \( N_\gamma \). By choice of \( \alpha^\pm \), this implies \( N_\gamma \cap \beta_\Gamma = I_\gamma \subset I' \). So, the length \( s_{\gamma, \Gamma} \) is less than \( 1/i \). This implies \( \gamma \in U^i_\Gamma \), and so \( V_\alpha \subset U^i_\Gamma \). Hence, \( U^i_\Gamma \) is open in \( U_\Gamma \) and \( \mathcal{A} \). The remaining part of the proof is just like Theorem 3.2.

\[
\text{5. Concluding remarks.} \quad \text{In this paper, we showed that for a generic null-homotopic, simple closed curve in the boundary of a mean convex 3-manifold } M, \text{ there exists a unique area minimizing disk in } M. \text{ We also prove a similar theorem for absolutely area minimizing surfaces. In many senses, the techniques used in this paper are purely topological and simple. They are quite original and can be applied to many similar settings.}
\]

Note that all the results of this paper are for compact 3-manifolds with mean convex boundary. For the noncompact case, like in [MY3] and [HS], with the additional condition of being homogeneously regular on the manifold \( M \), all the results of this paper will go through easily by using the analogous theorems from the same references.
There have been many embeddedness and uniqueness results for the Plateau problem. In the extreme and $H$-extreme curve cases, there have been many embeddedness results like [TT], [AS], [MY1]. There are also “strong uniqueness” results for $H$-extreme curves like [MY3], [Li], [Wh3]. However, those results do not say anything about the number of area minimizing disks bounded by an $H$-extreme curve. In those papers, the authors gave a dichotomy that either an $H$-extreme curve bounds more than one minimal disk, or the only minimal surface bounded by that curve is an area minimizing disk. One should not combine this result with ours in a wrong way. Our result says that a generic $H$-extreme curve bounds a unique area minimizing disk. However, bounding a unique area minimizing disk does not prohibit bounding other minimal surfaces. So, it is not true that for a generic $H$-extreme curve, the only minimal surface bounded by that curve is an area minimizing disk.

On the other hand, generic uniqueness for area minimizing disks and generic uniqueness for absolutely area minimizing surfaces might seem contradictory at first glance. This is because if we have an absolutely area minimizing surface (which is not a disk) in $M$, we can construct two different area minimizing disks in different sides of the surface. There are two points to consider here. The first obvious thing is that an absolutely area minimizing surface might be a disk. The other less obvious fact is that having two different disks in different sides of the surface does not mean that the curve has more than one area minimizing disk. This is because they are area minimizing disks in that part of $M$, not the whole $M$. The area minimizing disk in $M$ could be completely different than the others, and it still can be a unique area minimizing disk in $M$ bounded by that curve.

Another important point here is that these techniques may not work for surfaces which are area minimizing in a fixed topological class. If they are not absolutely area minimizing in the homology class, or area minimizing disk, then Lemma 2.1 and its local generalization Lemma 4.1 are not true in general. One should keep in mind that two just minimal surfaces with the same extreme boundary curve can intersect in a certain way, but two area minimizing disks, or two absolutely area minimizing surfaces must stay disjoint because of area constraints (intersection implies area reduction). In those lemmas, we are essentially using the Meeks-Yau exchange roundoff trick, and a surgery argument. However, two surfaces which are area minimizing in a fixed topological class may not give a surface in the same topological class after surgery. Hence, the key point in our technique (disjointness for the summation argument) fails in this case. However, as we pointed out in the introduction, White [Wh1] already gave a strong generic uniqueness result for this case in any dimension and codimension with some smoothness condition.

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