Decay time integrals in neutral meson mixing and their efficient evaluation

Till Moritz Karbach\textsuperscript{1}, Gerhard Raven\textsuperscript{2}, and Manuel Schiller\textsuperscript{2}

\textsuperscript{1}CERN, Switzerland
\textsuperscript{2}NIKHEF, The Netherlands

4th July 2014

Abstract

In neutral meson mixing, a certain class of convolution integrals is required whose solution involves the error function $\text{erf}(z)$ of a complex argument $z$. We show the general shape of the analytic solution of these integrals, and give expressions which allow the normalisation of these expressions for use in probability density functions. Furthermore, we derive expressions which allow a (decay time) acceptance to be included in these integrals, or allow the calculation of moments.

We also describe the implementation of numerical routines which allow the numerical evaluation of $w(z) = e^{-z^2}(1 - \text{erf}(-iz))$, sometimes also called Faddeeva function, in \texttt{C++}.

These new routines improve over the old CERNLIB routine(s) \texttt{WWERF}/\texttt{CWERF} in terms of both speed and accuracy. These new routines are part of the \texttt{RooFit} package, and have been distributed with it since \texttt{ROOT} version 5.34/08.
1 Introduction

When dealing with a time-dependent analysis of neutral mesons, one encounters the effect of meson mixing, leading to decay rate equations of the form

\[ \frac{d\Gamma_{\text{theo}}(t)}{dt} \sim e^{-\Gamma t} (A \cosh(\Delta \Gamma t/2) + B \sinh(\Delta \Gamma t/2) + C \cos(\Delta m t) + D \sin(\Delta m t)), \] (1)

for \( t > 0 \) with real coefficients \( A, B, C, \) and \( D \), where \( \Gamma \) is the average width of the two meson mass eigenstates and \( \Delta \Gamma \) and \( \Delta m \) are the width and mass difference between the mass eigenstates, respectively. Usually, the decay time resolution of the detector is finite, so this has to be convoluted with a resolution model, e.g. a Gaussian, to give the experimentally observable decay rate

\[ \frac{d\Gamma_{\text{exp}}(t)}{dt} = \int_{-\infty}^{+\infty} dt' \theta(t') \frac{d\Gamma_{\text{theo}}(t')}{dt'} G(t - t', \mu, \sigma), \] (2)

where

\[ G(t - t', \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t - t' - \mu)^2}{2\sigma^2}}, \] (3)

and \( \theta(t') \) is the Heavyside (step) function. The parameter \( \mu \) represents a potential bias in the reconstructed decay time, and \( \sigma \) is the decay time resolution. Due to the linearity of Eq. 2 it is trivial to extend to a more realistic, multi-Gaussian resolution function. In addition to a finite time resolution, most detectors show detection, reconstruction and/or trigger efficiency variations as a function of decay time, which need to be modeled by an acceptance function \( a(t) \). The final acceptance-corrected decay rate equation becomes:

\[ \frac{d\Gamma_{\text{acc}}(t)}{dt} = \frac{d\Gamma_{\text{exp}}(t)}{dt} a(t). \] (4)

To use either Eq. 2 or Eq. 4 as building blocks for a probability density function, the equations need to be normalised by dividing by their integral over the observable (the decay time \( t \) in this case).

The purpose of this note is to demonstrate how to solve the relevant integrals analytically, and to collect the resulting expressions for future reference. In addition, a numerical implementation of the the Faddeeva function is provided, which is, as will be shown, an essential part of the expressions.

This document is organised as follows. Section 2 defines the error function and some related functions and reviews their properties. The convolution integral in Eq. 2 and its normalisation integral are solved analytically in Sect. 3. Section 4 deals with the normalisation of Eq. 4 and the calculation of moments of Eq. 2. Section 5 discusses a computer program to compute the Faddeeva function numerically.

2 Definitions

2.1 Error Function

The error function is defined for real argument \( x \) as

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}. \] (5)
It is an odd function, i.e.
\[ \text{erf}(\text{-}x) = -\text{erf}(x). \] (6)

This definition can be continued into the complex plane with a complex argument \( z \) taking the place of \( x \). The resulting function is analytic over the entire complex plane, and in general takes complex values. There is an additional symmetry in the complex plane:
\[ \text{erf}(\bar{z}) = \overline{\text{erf}(z)}. \] (7)

The integral and derivative of the error function are given by
\[ \int dz \text{ erf}(z) = z \text{ erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}. \] (8)

### 2.2 Complementary Error Function

The complementary error function is defined as
\[ \text{erfc}(x) = 1 - \text{erf}(x). \] (9)

Its symmetry property is
\[ \text{erfc}(\text{-}x) = 2 - \text{erfc}(x). \] (10)

The continuation into the complex plane yields an analytic function with the same symmetry with respect to complex conjugation as the error function itself:
\[ \text{erfc}(\bar{z}) = \overline{\text{erfc}(z)}. \] (11)

The integral of the complementary error function is given by
\[ \int dz \text{ erfc}(z) = z \text{ erfc}(z) - \frac{e^{-z^2}}{\sqrt{\pi}}. \] (12)

### 2.3 Faddeeva Function

The Faddeeva function \( w(z) \) is closely related to the error function, it is defined as
\[ w(z) = e^{-z^2} \text{erfc}(-iz). \] (13)

This function has the symmetries
\[ w(-x + iy) = \overline{w(x + iy)}, \quad w(x - iy) = 2e^{-z^2} - w(x + iy). \] (14)

Its derivative is given by
\[ \frac{d}{dz} w(z) = \frac{2i}{\sqrt{\pi}} - 2z w(z). \] (15)
3 Neutral Meson Mixing in the Presence of Decay Time Resolution

In Eq. 2, there are three intrinsic time scales: the decay time \(1/\Gamma > 0\), the oscillation period \(1/\Delta m > 0\) and the time resolution \(\sigma > 0\). The numerical stability of the expressions we are about to derive depends on the relative orders of magnitude of \(1/\Gamma\), \(1/\Delta m\) and \(\sigma\). We consider the solution of the convolution integral in Eq. 2 in three cases:

1. the general case,
2. \(\min(1/\Gamma, 1/\Delta m) \gg \sigma\) (i.e. the detector resolution is much better than either lifetime or oscillation frequency demand), and
3. \(\min(\sigma, 1/\Delta m) \gg 1/\Gamma\) (i.e. the decay is so fast that all decaying particles can be said to decay at the same time).

The general case is discussed within the main text, the two special cases 2 and 3 have been moved to Appendix A, as the matter is dry enough as is.

3.1 General Case

Since \(\sin(\Delta m t) = \mathcal{I}(e^{i\Delta m t})\) and \(\cos(\Delta m t) = \mathcal{R}(e^{i\Delta m t})\), and both the \(\cosh\) and \(\sinh\) terms in Eq. 2 can be written as the sum and difference of exponentials, it is sufficient to consider the following convolution:

\[
f(t; \Gamma, \Delta m, \sigma, \mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{+\infty} dt' e^{-(\Gamma-i\Delta m)t'} e^{-(t-t'-\mu)^2/2\sigma^2} \tag{16}
\]

where we have substituted \(z = (\Gamma - i\Delta m)\sigma/\sqrt{2}\), \(x = \frac{t-\mu}{\sqrt{2}\sigma}\) and \(y = \frac{t'-\mu}{\sqrt{2}\sigma}\). Completing the square in the exponent and absorbing the shift in the boundaries of the integral, we find:

\[
f(x; z) = e^{-x^2+(z-x)^2} \frac{1}{\sqrt{\pi}} \int_0^{+\infty} dy e^{-(y+(z-x))^2} \tag{18}
\]

\[= e^{-x^2+(z-x)^2} \frac{1}{\sqrt{\pi}} \int_{z-x}^{+\infty} dy e^{-y^2} \tag{19}
\]

\[= e^{-x^2+(z-x)^2} \frac{1}{2} \text{erfc} (z-x) . \tag{20}
\]

Using the Faddeeva function \(w(z)\), this can be written as

\[
f(x; z) = \frac{1}{2} e^{-x^2} w(i(z-x)) . \tag{21}
\]

The corresponding normalisation integral is given by:

\[
I_0(t_1, t_2; \Gamma, \Delta m, \sigma, \mu) = \int_{t_1}^{t_2} dt \ f(t; \Gamma, \Delta m, \sigma, \mu) \sigma \sqrt{2} \int_{x_1}^{x_2} dx \ f(x; z) = \frac{\sigma}{\sqrt{2}} I_0(x_1, x_2; z) , \tag{22}
\]

One may argue that there is a fourth scale, \(\Delta \Gamma\), entering the problem. However, this can trivially be reduced to two different lifetimes \(\Gamma_1\) and \(\Gamma_2\) for the two mass eigenstates of the problem. Each mass eigenstate is then treated separately, leaving only three time scales.
where the latter $\hat{I}_0(x_1,x_2;z)$ is defined as
\begin{align}
\hat{I}_0(x_1,x_2;z) &\equiv \int_{x_1}^{x_2} dx \, e^{-x^2} w(i(z-x)) \\
&= \frac{1}{2z} \left[ \text{erf}(x) - e^{-x^2} w(i(z-x)) \right]_{x_1}^{x_2},
\end{align}
where we have used Eq. (12).

4 Calculating Moments and Including the Effect of an Acceptance Function

To describe a non-trivial decay time acceptance, one generally approximates $a(t)$ in some way, e.g. by piecewise constant or linear functions, or by piecewise polynomials such as splines. In these cases, it is sufficient to restrict the problem to functions $a(t)$ which are of the form $a(t) = \sum_k a_k t^k$. To normalise $\sum_k a_k t^k \cdot f(t; \Gamma, \Delta m, \sigma, \mu)$, one needs to compute the integrals
\begin{align}
I_k(t_1,t_2; \Gamma, \Delta m, \sigma, \mu) &= \int_{t_1}^{t_2} t^k f(t; \Gamma, \Delta m, \sigma, \mu) dt. 
\end{align}
These integrals also define the moments $m_k$,
\begin{align}
m_k &= \frac{\int dt t^k f(t)}{\int dt f(t)} = \frac{I_k}{I_0}.
\end{align}
When computing these integrals we again consider the three cases from the last section, where the two special cases 2 and 3 from the last section can be found in Appendix [B].

4.1 General Case

In the general case, it is again useful to go to the reduced coordinates $x$ and $z$ defined in the last section. The integral in Eq. (25) then becomes:
\begin{align}
I_k(x_1,x_2;z) &= \int_{x_1}^{x_2} (\sqrt{2} \sigma x + \mu)^k f(x;z) \sqrt{2} \sigma \, dx \\
&= \sqrt{2} \sigma \sum_{n=0}^{k} \binom{k}{n} (\sqrt{2} \sigma)^n \mu^{k-n} \int_{x_1}^{x_2} dx \, x^n f(x;z) \\
&= \frac{\sigma}{\sqrt{2}} \sum_{n=0}^{k} \binom{k}{n} (\sqrt{2} \sigma)^n \mu^{k-n} \hat{I}_n(x_1,x_2;z).
\end{align}
The required integrals are thus
\begin{align}
\hat{I}_n(x_1,x_2;z) &= \int_{x_1}^{x_2} dx \, x^n e^{-x^2} w(i(z-x)) .
\end{align}
They can be computed using the following method:
\begin{align}
\hat{I}_n(x_1,x_2;z) &\equiv \int_{x_1}^{x_2} dx \, x^n e^{-x^2} w(i(z-x)) \\
&= \frac{1}{2^n} \left. \frac{d^n}{d\lambda^n} \right|_{\lambda=0} \int_{x_1}^{x_2} dx \, e^{2\lambda x} e^{-x^2} w(i(z-x)) \\
&= \frac{1}{2^n} \left. \frac{d^n}{d\lambda^n} \right|_{\lambda=0} \hat{f}(x_1,x_2;z,\lambda).
\end{align}
Thus, we rewrite the term \( x^n \) as the slightly more complicated expression

\[
x^n = \frac{1}{2^n} \left. \frac{d^n}{d\lambda^n} \right|_{\lambda=0} e^{2\lambda x},
\]

to obtain an expression where the integration and the derivative with respect to \( \lambda \) commute. This facilitates the treatment of the integral enormously. Once again we complete the square, and shift the integrand to obtain:

\[
\hat{I}(x_1, x_2; z, \lambda) = \int_{x_1}^{x_2} dx \, e^{2\lambda x} e^{-x^2} w(i(z-x))
\]

\[
= e^{\lambda^2} \int_{x_1-\lambda}^{x_2-\lambda} dx \, e^{-x^2} w(i(z-\lambda-x))
\]

\[
= \frac{e^{\lambda^2}}{2(z-\lambda)} \left[ \text{erf}(x)-e^{-x^2} w(i(z-\lambda-x)) \right]_{x_1-\lambda}^{x_2-\lambda}
\]

\[
\equiv K(\lambda, z) \left[ J(x_2; \lambda, z) - J(x_1; \lambda, z) \right],
\]

where

\[
K(\lambda, z) = \frac{e^{\lambda^2}}{z-\lambda}, \quad J(x; \lambda, z) = \text{erf}(x-\lambda) - e^{-\lambda^2} w(i(z-x)).
\]

In order to simplify the computation of the \( \hat{I}_n \), we compute the \( n \)th order derivatives at \( \lambda = 0 \), \( K_n(z) \) and \( M_n(x; z) \), as follows:

| \( n \) | \( K_n(z) \) | \( M_n(x; z) \) |
|---|---|---|
| 0 | \( \frac{1}{\sqrt{\pi}} \) | \( \text{erf}(x)-e^{-x^2} w(i(z-x)) \) |
| 1 | \( \frac{1}{2z^2} \) | \( 2e^{-x^2} \left[ -\sqrt{\frac{1}{\pi}} - xw(i(z-x)) \right] \) |
| 2 | \( \frac{1}{2} \left( 1 + \frac{1}{z^2} \right) \) | \( 2e^{-x^2} \left[ -2x \sqrt{\frac{1}{\pi}} - (2x^2-1)w(i(z-x)) \right] \) |
| 3 | \( \frac{3}{2z} \left( 1 + \frac{1}{z^2} \right) \) | \( 4e^{-x^2} \left[ -(2x^2-1) \sqrt{\frac{1}{\pi}} - x(2x^2-3)w(i(z-x)) \right] \) |

The normalisation integrals in terms of \( K_n(z) \) and \( M_n(x; z) \) are thus

\[
\hat{I}_n(x_1, x_2, z) = \left[ \frac{1}{2^n} \left. \frac{d^n}{d\lambda^n} \right|_{\lambda=0} K(\lambda, z) J(x; \lambda, z) \right]_{x_1}^{x_2}
\]

\[
= \left[ \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} K_k(z) M_{n-k}(x, z) \right]_{x_1}^{x_2}
\]

\[
\equiv \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} K_k(z) M_{n-k}(x_1, x_2, z),
\]

where we have defined the abbreviation \( M_n(x_1, x_2; z) \equiv M_n(x_2; z) - M_n(x_1; z) \). Given that the typical use requires the computation of the sum over several \( \hat{I}_n \), e.g.

\[
N(x_1, x_2, z) = \sum_{k=0}^{n} a_k \hat{I}_k(x_1, x_2, z),
\]

it is advantageous to reorder the implied double sum:

\[
N(x_1, x_2, z) = \sum_{i=0}^{n} \sum_{j=0}^{i} A_{ij} M_i(x_1, x_2; z) K_j(z),
\]
where the matrix $A$ is defined by

$$
A_{ij} \equiv \begin{cases} 
\frac{a_i + j}{2} & \text{for } i + j \leq n \\
0 & \text{otherwise}
\end{cases}
$$

(32)

Now the dependence on the coefficients $a_k$ of Eq. (30) can be fully absorbed in the definition of the matrix $A$. For example, in case of $n = 3$, when written in a vector notation, this results in:

$$
N(x_1, x_2, z) = \left( \begin{array}{c}
M_0(x_1, x_2; z) \\
M_1(x_1, x_2; z) \\
M_2(x_1, x_2; z) \\
M_3(x_1, x_2; z)
\end{array} \right) \left( \begin{array}{cccc}
a_0 & a_1 & a_2 & a_3 \\
a_1 & \frac{a_1}{2} & \frac{a_2}{2} & \frac{a_3}{8} \\
a_2 & \frac{a_2}{4} & \frac{3a_2}{8} & 0 \\
a_3 & \frac{a_3}{8} & 0 & 0
\end{array} \right) \left( \begin{array}{c}
K_0(z) \\
K_1(z) \\
K_2(z) \\
K_3(z)
\end{array} \right).
$$

(33)

5 Evaluation of the Faddeeva Function in software

5.1 Implementation

To implement the Faddeeva function, we largely follow the ideas in [1], which we will sketch briefly below. Our code is also included in Appendix C.

The aim is to implement a full precision version which yields results that are accurate to within a few times the machine precision of a C++ double (64 bits, about $2 \cdot 10^{-16}$), and a faster version which is accurate to a few times the machine precision of a C++ float (32 bits, about $1 \cdot 10^{-7}$). We start from an alternative formulation of the Faddeeva function by representing it with a Fourier-style integral:

$$
w(z) = \frac{1}{\sqrt{\pi}} \int_0^\infty d\tau e^{-\frac{\tau^2}{4}} e^{i\tau z}.
$$

(34)

The idea is now to approximate the term $e^{-\tau^2/4}$ as a Fourier series

$$
e^{-\frac{\tau^2}{4}} \approx \sum_{n=0}^N \frac{a_n}{2} \left( e^{\frac{in\pi \tau}{\tau_m}} + e^{-\frac{in\pi \tau}{\tau_m}} \right) - \frac{a_0}{2}, \quad a_n \approx 2\sqrt{\pi} \frac{\tau^2}{\tau_m^2} e^{-\frac{\tau^2}{4}},
$$

(35)

in the interval $-\tau_m \leq \tau \leq \tau_m$, where the $a_n$ are the with Fourier coefficients. The resulting equation is

$$
w(z) \approx \frac{1}{\sqrt{\pi}} \int_0^\infty d\tau \left( \sum_{n=0}^N \frac{a_n}{2} \left( e^{\frac{in\pi \tau}{\tau_m}} + e^{-\frac{in\pi \tau}{\tau_m}} \right) - \frac{a_0}{2} \right) e^{i\tau z}
$$

$$
= \frac{i}{2\sqrt{\pi}} \sum_{n=0}^N a_n \tau_m \left( \frac{1 - e^{i(n\pi + \tau_m z)}}{n\pi + \tau_m z} - \frac{1 - e^{-i(n\pi + \tau_m z)}}{n\pi - \tau_m z} \right) - \frac{a_0}{2} \frac{1 - e^{i\tau_m z}}{z}.
$$

(36)

(37)

In the following we discuss how

1. to choose the integration cutoff $\tau_m$,
2. to choose $N$, and
3. the singularities at $z_n = \pm \frac{n\pi}{\tau_m}$ in Eq. (37) can be treated.
The choice of $\tau_m$ is easiest: Since $\tau_m$ cuts off the integral in Eq. [34] one needs to ensure that the portion of the integral that is neglected is sufficiently small. To obtain double (float) precision, $e^{-\tau_m^2/4}$ should be on the order of the machine precision of these data types, i.e. around $2 \cdot 10^{-16}$ ($1 \cdot 10^{-7}$). This leads to the choices of $\tau_m = 12$ for a full precision version of the routine, and $\tau_m = 8$ for a faster version with reduced precision.

Next, we chose $N$. It has to be large enough that the Fourier series in Eq. [35] is a good approximation. This is the case when the highest Fourier coefficient is smaller than the machine precision of the data type in question. For the full precision version, this means $N = 23$, for the fast version with reduced precision it means $N = 10$.

Finally, the singularities in Eq. [37] at $z_n = \pm \frac{n\pi}{\tau_m}$ are handled by using Taylor expansions of $w(z)$ in a tiny disc $|z - z_n| < 3 \cdot 10^{-3}$ around the singularities. To achieve the required precision, one has to take into account terms up to the fifth (second) order in $(z - z_n)$ for the slow (fast) version of the routine. Outside the discs around the singularities, the code thus uses Eq. [37] for $\Re(z)$, $\Im(z) \geq 0$, i.e. in the first quadrant of the complex plane. For arguments $z$ outside the first quadrant of the complex plane, the symmetries of the Faddeeva function (Eq. [14]) can be used. Thus, only $N + 1$ Taylor expansions of $w(z)$ need to be saved (and not $2N + 1$), and the numerical instability of $w(z)$ for $\Im(z) \ll 0$ due to its divergent nature in this regime can largely be avoided.

The code execution can also be optimised:

- The term $e^{in\pi}$ in Eq. [37] is a constant, so there is no need to compute it.
- The term $e^{i\tau_m z}$ in Eq. [37] depends only on $z$, so it can be precomputed at the beginning of the routine, avoiding a computationally expensive complex exponential inside the loop implementing the sum.
- The subexpressions $n\pi$ and coefficients $a_n$ in Eq. [37] can be precomputed before the code is compiled, and provided by small lookup tables.
- On the x86_64 architecture, the GNU C++ compiler produces suboptimal code for the complex exponentiation: Exponential and sine and cosine of a real argument are implemented in hardware and executed on the x87 unit of the CPU. Normal floating point operations like multiplication typically happen in another functional unit of the CPU, however. Both units have their separate floating point register sets, and moving values between the two involves a store to, and subsequent load from, the main memory (RAM). There are thus five of these load-store instruction pairs (real and imaginary part of the input argument to the complex error function, exponential of the real part, and sine and cosine of the imaginary part) which copy around input/output values. For this reason, the code includes a hand-coded inline assembly version of the complex exponential function, which saves at least one store-load instruction pair by computing the result entirely in the x87 unit of the CPU. It also does away with the subroutine calls into the math library of the system. On all other systems, the code automatically uses the less optimal version in the math library.
- The code contains two versions of the loop to compute the sum in Eq. [37]: A naïve implementation, and one that is at least partially vectorisable with modern compilers. The latter will use SIMD instructions when available. The code chooses the version to use based on architecture-specific macros being defined during the compilation phase.
Due to the divergent nature of \( w(z) \) for \( \Im(z) \ll 0 \), the fast version of the routine also needs to use \texttt{double} calculations internally to avoid loss of precision beyond the level we aim for based on our choices of \( \tau_m \) and \( N \).

The C++ code of our implementation is included in the Appendix. It has been part of the \texttt{RooFit} package \cite{2} since ROOT \cite{3} version 5.34/08.

5.2 Performance

In this subsection, we compare several packages to compute the Faddeeva, \( \text{erf} \) and \( \text{erfc} \) functions for complex arguments. \texttt{RooFit} is a fitting package in the ROOT framework which makes heavy use of the Faddeeva function, so it makes sense to check the accuracy and speed of various implementations. Specifically, we investigate:

- the original \texttt{CERNLIB} \texttt{WWERF} implementation \cite{4} written in 1970 in \texttt{FORTRAN77} (using an older algorithm),
- the old code in \texttt{RooFit} (before ROOT version 5.34/08); there is a slow version of the routine based on the \texttt{CERNLIB} implementation, ported to C++, and a fast version, which is based on a 12.5 Megabyte lookup table and interpolation in the rectangle defined by \( |\Re(z)| < 4 \) and \(-4 \leq \Im(z) \leq 6\) (which falls back on the slow version outside that area),
- our code (in \texttt{RooFit} since ROOT version 5.34/08), as described in the last subsection,
- code based on the \texttt{libcerf} library \cite{5} written in C.

The \texttt{libcerf} library provides special implementations for the \( \text{erf} \) and \( \text{erfc} \) functions, the other packages use the following relations to define these functions in terms of \( w(z) \):

\[
\text{erf}(z) = \begin{cases} 
1 - e^{-z^2} w(iz) & \text{for } \Re(z) \geq 0 \\
e^{-z^2} w(-iz) - 1 & \text{otherwise}
\end{cases},
\]

\[
\text{erfc}(z) = \begin{cases} 
e^{-z^2} w(iz) & \text{for } \Re(z) \geq 0 \\
2 - e^{-z^2} w(-iz) & \text{otherwise}
\end{cases}.
\]

5.2.1 Performance evaluation method

To judge the numerical accuracy of these routines, we compare the results of the implementations to those obtained with the computer algebra system Maxima \cite{6}. In Maxima, one can calculate these functions using a special “bigfloat” floating point data type for which one can chose the length of the mantissa at runtime. With 48 significant decimal digits in the mantissa, the results of Maxima can be trusted to full \texttt{double} precision. More specifically, we calculate the absolute value of the relative difference between the implementations under study and the result obtained with Maxima, \( \epsilon = |1 - w(z)/w_{\text{Maxima}}(z)| \).

To have an indication about the relative speed of the implementations under study, we measure the number of CPU cycles needed for the execution of the different routines using a hardware register incremented with each CPU clock (on \texttt{x86/x86_64}, the \texttt{TSC} register).

We consider two areas from which to choose \( z \):

- the “big square” \(-8 \leq \Re(z), \Im(z) \leq 8\), in which we test \( 2^{16} \) random points distributed uniformly over that square, and
Table 1: Performance of the various implementations of $w(z)$, erf$(z)$ and erfc$(z)$ for $2^{16}$ values of $z$ from the “big square” region (see text). Time is measured in CPU cycles per evaluation (with variations of 5-10% between different runs of the program on the same machine). $\epsilon$ is the average over the relative errors of all points tested, $\epsilon_{\text{max}}$ is the maximum relative error seen.

|       | CERNLIB | libcerf (precise) | libcerf (fast) | RooFit old (precise) | RooFit old (fast) | our code (precise) | our code (fast) |
|-------|---------|------------------|---------------|----------------------|------------------|--------------------|-----------------|
| $w(z)$ |         |                  |               |                      |                  |                    |                 |
| time  | $2.4 \cdot 10^3$ | $9.8 \cdot 10^2$ | $2.8 \cdot 10^3$ | $2.1 \cdot 10^3$   | $6.8 \cdot 10^2$ | $5.3 \cdot 10^2$   |
| $\epsilon$ | $6.7 \cdot 10^{-15}$ | $1.6 \cdot 10^{-15}$ | $6.7 \cdot 10^{-15}$ | $6.2 \cdot 10^{-8}$ | $6.1 \cdot 10^{-16}$ | $4.1 \cdot 10^{-9}$ |
| $\epsilon_{\text{max}}$ | $2.5 \cdot 10^{-12}$ | $8.4 \cdot 10^{-14}$ | $2.5 \cdot 10^{-12}$ | $5.1 \cdot 10^{-5}$ | $8.4 \cdot 10^{-14}$ | $1.8 \cdot 10^{-7}$ |
| erf$(z)$ |         |                  |               |                      |                  |                    |                 |
| time  | $2.5 \cdot 10^3$ | $1.2 \cdot 10^3$ | $2.9 \cdot 10^3$ | $2.0 \cdot 10^3$   | $7.9 \cdot 10^2$ | $6.4 \cdot 10^2$   |
| $\epsilon$ | $1.3 \cdot 10^{-14}$ | $1.4 \cdot 10^{-15}$ | $1.3 \cdot 10^{-14}$ | $6.4 \cdot 10^{-9}$ | $1.1 \cdot 10^{-15}$ | $3.5 \cdot 10^{-9}$ |
| $\epsilon_{\text{max}}$ | $5.6 \cdot 10^{-11}$ | $8.4 \cdot 10^{-14}$ | $5.6 \cdot 10^{-11}$ | $6.0 \cdot 10^{-6}$ | $8.4 \cdot 10^{-14}$ | $1.9 \cdot 10^{-7}$ |
| erfc$(z)$ |         |                  |               |                      |                  |                    |                 |
| time  | $2.5 \cdot 10^3$ | $1.1 \cdot 10^3$ | $2.8 \cdot 10^3$ | $2.0 \cdot 10^3$   | $7.7 \cdot 10^2$ | $6.2 \cdot 10^2$   |
| $\epsilon$ | $7.1 \cdot 10^{-15}$ | $2.0 \cdot 10^{-15}$ | $7.1 \cdot 10^{-15}$ | $5.7 \cdot 10^{-9}$ | $1.7 \cdot 10^{-15}$ | $4.0 \cdot 10^{-9}$ |
| $\epsilon_{\text{max}}$ | $2.6 \cdot 10^{-12}$ | $9.6 \cdot 10^{-14}$ | $2.6 \cdot 10^{-12}$ | $3.7 \cdot 10^{-7}$ | $7.0 \cdot 10^{-14}$ | $1.9 \cdot 10^{-7}$ |

The “singularity” areas where our algorithm has to switch to the Taylor expansions around $z_n = \frac{\pi}{2n}$, specifically, the area considered is $\max(\Re(|z - z_n|), \Im(|z - z_n|)) < 4 \cdot 10^{-3}$. For each of these $N$ squares, we test 1024 points distributed uniformly in that area.

5.2.2 Results

Tables 1 and 2 show the performance figures obtained on a typical laptop running a Linux system with an Intel Core i7-2620M CPU running at 2.7 GHz. The compiler suite used was the GNU compiler collection version 4.7.2 with optimisation options “-O3 -ffast-math -fno-math-errno -mtune=native -mmmx -msse -msse2 -msse3 -msse4.1 -msse4.2 -mavx”. We have also run the benchmarks on a different platform (Linux PowerPC G3, a 32 bit machine with big endian byte order) to make sure that there are no hidden portability pitfalls in our code. The accuracy is practically unchanged for all implementations but libcerf, which seems to produce slightly different results for infinite arguments, arguments with very large $|z|$, or arguments containing NaNs on the PowerPC machine. Timings appear to be slightly different, but the general trends are similar to those shown in Tables 1 and 2. Our code has been compiled and tested as part of the ROOT releases on many different platforms giving confidence that the code is quite portable, and delivers the same accuracy independent of the particular IEEE754 floating point implementation used.

5.2.3 Interpretation

In terms of accuracy, it seems that the libcerf implementation and our code give the best results with relative errors below $10^{-14}$. The old implementation in RooFit and the one in CERNLIB (on which the old RooFit one is based) behave very similarly, and their relative
We have presented the calculations needed to obtain analytic expressions for integrals of the form
\[
\frac{1}{\sqrt{2\pi\sigma^2}} \int_{t_1}^{t_2} dt \int_{0}^{+\infty} dt' e^{-(t-t')^2/2\sigma^2} \cdot e^{-(t-t') \Delta m}. \tag{38}
\]
These integrals have an important application in the description of the time evolution of neutral mesons, which exhibit particle-antiparticle mixing. There, sine and cosine terms are multiplied by a decaying exponential, convolved with a Gaussian experimental resolution function, and multiplied by a polynomial time acceptance function. Our analytic expressions permit the fast calculation the relevant terms. We also provide new fast and accurate routines to calculate the Faddeeva function of a complex argument numerically. This function is needed to evaluate many of the above integrals. We include the source code in Appendix C.

| w(z) | CERNLIB | libcerf | RooFit old | our code |
|------|---------|---------|------------|---------|
|      | (precise) | (fast) | (precise) | (fast) |
| time [cycles] | 3.0 \cdot 10^3 | 1.1 \cdot 10^4 | 3.4 \cdot 10^3 | 1.5 \cdot 10^3 | 5.9 \cdot 10^2 | 5.0 \cdot 10^2 |
| \epsilon | 2.9 \cdot 10^{-12} | 3.4 \cdot 10^{-16} | 2.9 \cdot 10^{-12} | 1.2 \cdot 10^{-7} | 4.1 \cdot 10^{-16} | 3.7 \cdot 10^{-9} |
| \epsilon_{\text{max}} | 3.0 \cdot 10^{-12} | 1.6 \cdot 10^{-15} | 3.0 \cdot 10^{-12} | 3.8 \cdot 10^{-7} | 2.5 \cdot 10^{-15} | 2.0 \cdot 10^{-8} |

Table 2: Performance of the various implementations of w(z), erf(z) and erfc(z) for (N+1) \cdot 10^{24} values of z from the “singularity” region (see text). Time is measured in CPU cycles per evaluation (with variations of 5-10\% between different runs of the program on the same machine). \epsilon is the average over the relative errors of all points tested, \epsilon_{\text{max}} is the maximum relative error seen.

error is two orders of magnitude larger. The two “fast” implementations offer a relative error of about 10^{-5} for the old implementation in RooFit and about 10^{-7} for our code.

Concerning the speed of the different algorithms: The CERNLIB based implementations are slowest. The libcerf implementation is about a factor 2.4 faster than the CERNLIB implementation, whereas our full-precision code is about a factor of 3.5 faster.

The fast version of the old RooFit code (our implementation) is a factor of 2 (6) faster than the original CERNLIB implementation.

6 Conclusion

We have presented the calculations needed to obtain analytic expressions for integrals of the form

\[
\frac{1}{\sqrt{2\pi\sigma^2}} \int_{t_1}^{t_2} dt \int_{0}^{+\infty} dt' e^{-(t-t')^2/2\sigma^2} \cdot e^{-(t-t') \Delta m}. \tag{38}
\]
Acknowledgements

The authors would like to thank Wouter Hulsbergen and Vladimir Gligorov for useful discussions and comments on earlier versions of the text.

A Neutral Meson Mixing in the Presence of Decay Time Resolution, Special Cases

This section contains the expressions for the special cases mentioned in Section 3.

A.1 Solution for \( \min(1/\Gamma, 1/\Delta m) \gg \sigma \)

If \( \min(1/\Gamma, 1/\Delta m) \gg \sigma \), the Gaussian \( G(t - t', \mu, \sigma) \) in Eq. 16 becomes too narrow to be observed, and can be replaced by a delta distribution \( \delta(t - t' - \mu) \). Thus Eq. 16 becomes

\[
\begin{align*}
  f(t; \Gamma, \Delta m, \sigma, \mu) &= \int_0^{+\infty} dt' e^{-\Gamma t'} \delta(t - t' - \mu) \\
  &= \begin{cases} 
    e^{-(\Gamma - i\Delta m)(t - \mu)} & \text{for } t \geq \mu \\
    0 & \text{otherwise}
  \end{cases}.
\end{align*}
\]  

The normalisation integral is

\[
I_0(t_1, t_2; \Gamma, \Delta m, \sigma, \mu) = \int_{t_1}^{t_2} dt f(t; \Gamma, \Delta m, \sigma, \mu) 
\]

\[
= \int_{t_1}^{t_2} dt e^{-(\Gamma - i\Delta m)(t - \mu)} 
\]

\[
= \left[ -\frac{e^{-(\Gamma - i\Delta m)(t - \mu)}}{\Gamma - i\Delta m} \right]_{\max(t_1, \mu)}^{\max(t_2, \mu)}. \tag{42}
\]

A.2 Solution for \( \min(\sigma, 1/\Delta m) \gg 1/\Gamma \)

In this case, the lifetime is short compared to any other processes, and we replace \( e^{-\Gamma t'} \) by \( \delta(t' - 1/\Gamma)/\Gamma \) (the delta distribution is shifted to 1/\( \Gamma \), the time expectation value of \( e^{-\Gamma t'} \), and scaled to account for the different normalisations). Eq. 16 thus becomes:

\[
\begin{align*}
  f(t; \Gamma, \Delta m, \sigma, \mu) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{+\infty} dt' \delta(t' - 1/\Gamma) e^{-i\Delta m t'} e^{-\frac{(t - t' - \mu)^2}{2\sigma^2}} \\
  &= \frac{1}{\Gamma} e^{-i\Delta m/\Gamma} G(t, 1/\Gamma + \mu, \sigma), \tag{43}
\end{align*}
\]

and the normalisation integral is

\[
I_0(t_1, t_2; \Gamma, \Delta m, \sigma, \mu) = \int_{t_1}^{t_2} dt f(t; \Gamma, \Delta m, \sigma, \mu) 
\]

\[
= \int_{t_1}^{t_2} dt \frac{e^{-i\Delta m/\Gamma}}{\Gamma} G(t, 1/\Gamma + \mu, \sigma) 
\]

\[
= \frac{e^{-i\Delta m/\Gamma}}{2\Gamma} \left[ \text{erf} \left( \frac{t - \frac{1}{\Gamma} - \mu}{\sqrt{2}\sigma} \right) \right]_{t_1}^{t_2}. \tag{46}
\]
B Calculating Moments and Including the Effect of an Acceptance Function, Special Cases

This section contains the expressions for the special cases mentioned in Section 4.

B.1 Solution for $\min (1/\Gamma, 1/\Delta m) \gg \sigma$

For $\min (1/\Gamma, 1/\Delta m) \gg \sigma$, $f(t; \Gamma, \Delta m, \sigma, \mu)$ simplifies to

$$f(t; \Gamma, \Delta m, \sigma, \mu) = \begin{cases} e^{-(\Gamma-i\Delta m)(t-\mu)} & \text{for } t \geq \mu \\ 0 & \text{otherwise} \end{cases}.$$  \hspace{1cm} (47)

One is thus interested in the integrals

$$I_k(t_1, t_2; \Gamma, \Delta m, \sigma, \mu) = \int_{t_1}^{t_2} dt \ t^k e^{-(\Gamma-i\Delta m)(t-\mu)}.$$  \hspace{1cm} (48)

Abbreviating $u = \Gamma - i\Delta m$, this can be written as

$$I_k(t_1, t_2; \Gamma, \Delta m, \sigma, \mu) = e^{u\mu} \int_{t_1}^{t_2} dt \ t^k e^{-ut} = e^{u\mu} \int_{t_1}^{t_2} dt \ \frac{d^k}{d\lambda^k} \bigg|_{\lambda=0} e^{(\lambda-u)t} \bigg|_{t_1}.$$  \hspace{1cm} (49)

The newly introduced functions $G_n(u) = -n!/u^n$ and $H_n(t; u) = t^n e^{-ut}$ are easily computed. This leaves us with:

$$I_k(t_1, t_2; u, \mu) = e^{u\mu} \sum_{j=0}^{k} \binom{k}{j} G_j(u) \ [H_{k-j}(t; u)]_{\max(t_1)}^{t_2}.$$  \hspace{1cm} (50)

B.2 Solution for $\min (\sigma, 1/\Delta m) \gg 1/\Gamma$

For $\min (\sigma, 1/\Delta m) \gg 1/\Gamma$, Eq. 16 simplifies to

$$f(t; \Gamma, \Delta m, \sigma, \mu) = \frac{e^{-i\Delta m/\Gamma}}{\Gamma} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-1/\Gamma-\mu)^2}{2\sigma^2}},$$  \hspace{1cm} (51)

as shown in Eq. 43. We are again interested in the integrals

$$I_k(t_1, t_2; \Gamma, \Delta m, \sigma, \mu) = \frac{e^{-i\Delta m/\Gamma}}{\Gamma} \cdot \int_{\max(0,t_1)}^{\max(0,t_2)} dt \ t^k \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-1/\Gamma-\mu)^2}{2\sigma^2}}.$$  \hspace{1cm} (52)
Substituting $s = t - \frac{1}{t} - \mu$ and adjusting the limits to $s_1 = -1/\Gamma - \mu + \max(0, t_1)$ and $s_2 = -1/\Gamma - \mu + \max(0, t_2)$, we obtain

$$I_k(s_1, s_2; \Gamma, \Delta m, \sigma, \mu) = \frac{e^{-i\Delta m/\Gamma}}{\Gamma} \cdot \int_{s_1}^{s_2} ds \left(s + \frac{1}{\Gamma} + \mu\right)^k \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{s^2}{2\sigma^2}}$$ \hspace{1cm} (50)

Substituting $s = \frac{\sqrt{2} \sigma}{\sqrt{\pi} \sigma}$ yields

$$I_k(s_1, s_2; \Gamma, \Delta m, \sigma, \mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \int_{s_1}^{s_2} ds e^{-\frac{s^2}{2\sigma^2}}$$ \hspace{1cm} (54)

The term $s^n$ can be rewritten in an analogous way as before, completing the square as well

$$I_n(s_1, s_2; \sigma) = \frac{\sigma^{2n}}{\sqrt{2\pi\sigma^2}} \frac{d^n}{d\lambda^n} \bigg|_{\lambda=0} \int_{s_1}^{s_2} ds e^{-\frac{s^2 - 2\lambda s + \lambda^2 - \lambda^2}{2\sigma^2}}$$ \hspace{1cm} (55)

Substituting $r = \frac{s - \lambda}{\sigma \sqrt{2}}$, yields

$$I_n(s_1, s_2; \sigma) = \frac{\sigma^{2n}}{\sqrt{\pi}} \frac{d^n}{d\lambda^n} \bigg|_{\lambda=0} e^{-\frac{x^2}{2\sigma^2}} \int_{r_1}^{r_2} dr e^{-r^2}$$ \hspace{1cm} (56)

Using the definitions

$$P_n(\sigma) = \frac{d^l}{d\lambda^l} \bigg|_{\lambda=0} e^{-\frac{x^2}{2\sigma^2}}, \hspace{1cm} Q_n(\sigma) = \frac{d^{n-l}}{d\lambda^{n-l}} \bigg|_{\lambda=0} \text{erf} \left( \frac{s - \lambda}{\sigma \sqrt{2}} \right)$$ \hspace{1cm} (60)

we tabulate $P_n$ and $Q_n$ for $0 \leq n \leq 3$:

| $n$ | $P_n(\sigma)$ | $Q_n(s, \sigma)$ |
|-----|---------------|-------------------|
| 0   | 1             | erf $\left( \frac{s}{\sigma \sqrt{2}} \right)$ |
| 1   | 0             | $-\frac{\sqrt{2}}{\sigma \sqrt{\pi} \sigma} e^{-\frac{x^2}{2\sigma^2}}$ |
| 2   | $-\frac{1}{\sigma^2}$ | $-\frac{\sqrt{2}}{\sigma \sqrt{\pi} \sigma} e^{-\frac{x^2}{2\sigma^2}}$ |
| 3   | 0             | $\frac{\sqrt{2}}{\sigma \sqrt{\pi} \sigma} e^{-\frac{x^2}{2\sigma^2}} \cdot (1 - \frac{s^2}{\sigma^2})$ |

13
Eq. 49 can thus be written as:

\[ I_k(t_1, t_2; \Gamma, \Delta m, \sigma, \mu) = e^{-i\Delta m/\Gamma} \cdot \sum_{j=0}^{k} \binom{k}{j} \left( \frac{1}{\Gamma} + \mu \right)^{k-j} \cdot \sigma^{2j} \cdot \sum_{l=0}^{j} \binom{j}{l} P_l(\sigma) \cdot \left[ Q_{j-l}(t - \frac{1}{\Gamma} - \mu; \sigma) \right]_{\text{max}(0,t_2)}^{\text{max}(0,t_1)}. \] (61)

C C++ source code

This section contains the source code of our Faddeeva function implementation. It has been slightly modified with respect to what is included in ROOT version 5.34/08 to allow standalone builds.

C.1 File cerf.h

```cpp
#ifndef CERF_H
#define CERF_H

#include <cmath>
#include <complex>

namespace Cerf {

/** @brief evaluate Faddeeva function for complex argument */

  * @author Manuel Schiller
  * @date 2013-02-21
  * @copyright
  *
  * Calculate the value of the Faddeeva function \( \exp(-z^2) \cdot \text{erfc}(-iz) \).
  *
  * The method described in
  *
  * S.M. Abrarov, B.M. Quine: "Efficient algorithmic implementation of
  * Voigt/complex error function based on exponential series approximation"
  * published in Applied Mathematics and Computation 218 (2011) 1894–1902
  * doi:10.1016/j.amc.2011.06.072
  *
  * is used. At the heart of the method (equation (14) of the paper) is the
  * following Fourier series based approximation:
  *
  * \[
  * w(z) \approx \frac{i}{2\sqrt{\pi}} \left( \sum_{n=0}^{N} a_n \cdot \left( \frac{1 - e^{i(n\pi + \tau_m z)}}{n\pi + \tau_m z} - \frac{1 - e^{-i(n\pi - \tau_m z)}}{n\pi - \tau_m z} \right) - a_0 \cdot \frac{1 - e^{i\tau_m z}}{z} \right)
  * \]
  *
  * The coefficients \( a_n \) are given by:
  *
  * \[
  * a_n = \frac{2\sqrt{\pi}}{\tau_m} \exp\left(-\frac{n^2\pi^2}{\tau_m^2}\right)
  * \]
  *
  * To achieve machine accuracy in double precision floating point arithmetic
  * for most of the upper half of the complex plane, choose \( \tau_m = 12 \) and
  * \( N = 23 \) as is done in the paper.
  *
  * There are two complications: For \( \text{Im}(z) \) negative, the exponent in the
  * equation above becomes so large that the roundoff in the rest of the
  * calculation is amplified enough that the result cannot be trusted.

```
Therefore, for \( \Im(z) < 0 \), the symmetry of the \( \text{erfc} \) function under the transformation \( z \to -z \) is used to avoid accuracy issues for \( \Im(z) < 0 \) by formulating the problem such that the calculation can be done for \( \Im(z) > 0 \). where the accuracy of the method is fine, and some postprocessing then yields the desired final result.

Second, the denominators in the equation above become singular at \( \Re(z) = \frac{n \pi}{12} \) (for \( 0 \leq n < 24 \)). In a tiny disc around these points, Taylor expansions are used to overcome that difficulty.

This routine precomputes everything it can, and tries to write out complex operations to minimise subroutine calls, e.g. for the multiplication of complex numbers.

In the square \(-8 \leq \Re(z) \leq 8, -8 \leq \Im(z) \leq 8\), the routine is accurate to better than \( 4 \times 10^{-13} \) relative, the average relative error is better than \( 7 \times 10^{-16} \). On a modern x86_64 machine, the routine is roughly three times as fast than the old CERNLIB implementation and offers better accuracy.

```cpp
std::complex<double> faddeeva(std::complex<double> z);
```

/* @brief evaluate Faddeeva function for complex argument (fast version)
*
* @author Manuel Schiller <manuel.schiller@nikhef.nl>
* @date 2013-02-21
*
* Calculate the value of the Faddeeva function \( w(z) = \exp(-z^2) \text{erfc}(-iz) \).
* \n* This is the "fast" version of the faddeeva routine above. Fast means that it takes roughly half the amount of CPU of the slow version of the routine, but is a little less accurate.
*
* To be fast, chose \( t_m = 8 \) and \( N = 11 \) which should give accuracies around \( 1 \times 10^{-7} \).
*
* In the square \(-8 \leq \Re(z) \leq 8, -8 \leq \Im(z) \leq 8\), the routine is accurate to better than \( 4 \times 10^{-7} \) relative, the average relative error is better than \( 5 \times 10^{-9} \). On a modern x86_64 machine, the routine is roughly five times as fast than the old CERNLIB implementation, or about 30% faster than the interpolation/lookup table based fast method used previously in RooFit, and offers better accuracy than the latter (the relative error is roughly a factor 280 smaller than the old interpolation/table lookup routine).
*/

```cpp
std::complex<double> faddeeva_fast(std::complex<double> z);
```

/* @brief complex erf function
*
* @author Manuel Schiller <manuel.schiller@nikhef.nl>
* @date 2013-02-21
*
* Calculate erf(z) for complex z.
*/

```cpp
std::complex<double> erf(const std::complex<double> z);
```

/* @brief complex erf function (fast version)
*
* @author Manuel Schiller <manuel.schiller@nikhef.nl>
* @date 2013-02-21
*
* Calculate erf(z) for complex z. Use the code in faddeeva_fast to save some time.
*/

```cpp
std::complex<double> erf_fast(const std::complex<double> z);
```
* Calculate erfc(z) for complex z.
* std::complex<double> erfc(const std::complex<double> z);
/* @brief complex erfc function (fast version)
* @author Manuel Schiller <manuel.schiller@nikhef.nl>
* @date 2013-02-21
* Calculate erfc(z) for complex z. Use the code in faddeeva_fast to save some time.
*/
std::complex<double> erfc_fast(const std::complex<double> z);

C.2 File cerf.cc

#include <complex>
#include <cmath>
#include <algorithm>
#include "cerf.h"

namespace faddeeva_impl {
    static inline void cexp(double& re, double& im)
    {
        // with gcc on unix machines and on x86_64, we can gain by hand-coding
        // exp(z) for the x87 coprocessor; other platforms have the default
        // routines as fallback implementation, and compilers other than gcc on
        // x86_64 generate better code with the default routines; also avoid
        // the inline assembly code when the compiler is not optimizing code, or
        // is optimizing for code size
        // (we insist on __unix__ here, since the assemblers on other OSs
        // typically do not speak AT&T syntax as gas does...)
    #if !defined(__GNUC__) || !defined(__unix__) || !defined(__x86_64__) ||
        !defined(__OPTIMIZE__) || defined(__OPTIMIZE_SIZE__) ||
        defined(__INTEL_COMPILER) || defined(__clang__) ||
        defined(__OPEN64__) || defined(__PATHSCALE__)
        const double e = std::exp(re);
        re = e * std::cos(im);
        im = e * std::sin(im);
    #else
        __asm__ ("fxam\n	"
                // examine st(0): NaN? Inf?
                "fstsw/uni2423%%ax\n	"
                "movb/uni2423$0x45,%%dh\n	"
                "andb/uni2423%%ah,%%dh\n	"
                "cmpb/uni2423$0x05,%%dh\n	"
                "jz/uni24231f\n	"
                // have NaN or infinity, handle below
                "fldl2e\n	"
                // load log2(e)
                "fmulp\n	"
                // re * log2(e)
                "fld,%%st(0)\n	"
                // duplicate re * log2(e)
                "frndint\n	"
                // int(re * log2(e))
                "fsubr,%%st,%%st(1)\n	"
                // st(1) = x = frac(re * log2(e))
                "fxch\n	"
                // swap st(0), st(1)
                "f2xm1\n	"
                // 2^x - 1
                "fild\n	"
                // st(0) = 1
                "faddp\n	"
                // st(0) = 2^x
                "fscale\n	"
                // 2^ (int(re * log2(e)) + x)
                "fstp,%%st(1)\n	"
                // pop st(1)
                "jmp\n	"
                // handle NaN, Inf...
                "testl,%%eax\n	"
                // -infinity?
                "jz\n	"
                "fstp,%%st\n	"
            )
        #endif
    }
}
"fldz\n\t" // st(0) = 0
"2:\n\t" // here, we have st(0) == exp(re)
"fsub\n\t" // st(0) = st(0) - exp(re)
"fsincos\n\t" // st(0) = cos(im), st(1) = sin(im)
"fnstsw/uni2423%%ax
\t"
"testl/uni2423$0x400,%%eax
\t"
"jz/uni24234f
\t" // | im | too large for fsincos?
"fldpi\n\t" // st(0) = pi
"fadd/uni2423%%st(0)
\t" // st(0) *= 2;
"fxch/uni2423%%st(1)
\t" // st(0) = im, st(1) = 2 * pi
"3:\n\t"
"fprem1\n\t" // st(0) = fmod(im, 2 * pi)
"fnstsw/uni2423%%ax
\t"
"testl/uni2423$0x400,%%eax
\t"
"jnz/uni24233b
\t" // fmod done?
"fstp/uni2423%%st(1)
\t" // yes, pop st(1) == 2 * pi
"fsincos\n\t" // st(0) = cos(im), st(1) = sin(im)
"4:\n\t" // all fine, fsincos succeeded
"fmul/uni2423%%st(2)
\t" // st(0) *= st(2)
"fxch/uni2423%%st(1)
\t" // st(2) = exp(re)*cos(im), st(0) = exp(im)
"4:\n\t" // st(1) = exp(re)*sin(im), pop st(0)
"fsub\n\t" // st(0) = st(0) - st(1)
"fprem1\n\t" // st(1) = fmod(st(1), 2 * pi)
"fstp/uni2423%%st(1)
\t" // yes, pop st(1) == 2 * pi
"fsincos\n\t" // st(0) = cos(st(1)), st(1) = sin(st(1))
"5:\n\t"
"fsub\n\t" // st(0) = st(0) - st(1)
"fprem1\n\t" // st(1) = fmod(st(1), 2 * pi)
"fstp/uni2423%%st(1)
\t" // yes, pop st(1) == 2 * pi
"fsincos\n\t" // st(0) = cos(st(1)), st(1) = sin(st(1))
"6:\n\t"
"fsub\n\t" // st(0) = st(0) - st(1)
"fprem1\n\t" // st(1) = fmod(st(1), 2 * pi)
"fstp/uni2423%%st(1)
\t" // yes, pop st(1) == 2 * pi
"fsincos\n\t" // st(0) = cos(st(1)), st(1) = sin(st(1))
"7:\n\t"
"fsub\n\t" // st(0) = st(0) - st(1)
"fprem1\n\t" // st(1) = fmod(st(1), 2 * pi)
"fstp/uni2423%%st(1)
\t" // yes, pop st(1) == 2 * pi
"fsincos\n\t" // st(0) = cos(st(1)), st(1) = sin(st(1))
"8:\n\t"
"fsub\n\t" // st(0) = st(0) - st(1)
"fprem1\n\t" // st(1) = fmod(st(1), 2 * pi)
"fstp/uni2423%%st(1)
\t" // yes, pop st(1) == 2 * pi
"fsincos\n\t" // st(0) = cos(st(1)), st(1) = sin(st(1))
"9:\n\t"
"fsub\n\t" // st(0) = st(0) - st(1)
"fprem1\n\t" // st(1) = fmod(st(1), 2 * pi)
"fstp/uni2423%%st(1)
\t" // yes, pop st(1) == 2 * pi
"fsincos\n\t" // st(0) = cos(st(1)), st(1) = sin(st(1))

 template <class T, unsigned N, unsigned NTAYLOR>
 static inline std::complex<T> faddeeva_smabmq_impl(
     T zre, T zim, const T tm,
     const T (&a)[N], const T (&npi)[N],
     const T (&taylorarr)[N * NTAYLOR * 2]) {
     // catch singularities in the Fourier representation At
     // z = n pi / tm, and provide a Taylor series expansion in those
     // points, and only use it when we're close enough to the real axis
     // that there is a chance we need it
     const T zim2 = zim * zim;
     const T maxnorm = T(9) / T(1000000);
     if (zim2 < maxnorm) {
         // we're close enough to the real axis that we need to worry about
         // singularities
         const T dnsing = tm * zre / npi[1];
         const T dnsingmax2 = (T(N) - T(1) / T(2)) * (T(N) - T(1) / T(2));
         if (dnsing + dnsing < dnsingmax2) {
             // we're in the interesting range of the real axis as well...
             // deal with Re(z) < 0 so we only need N different Taylor
             // expansions; use w(-x+iy) = conj(w(x+iy))
             const bool negrez = zre < 0.;
             // figure out closest singularity
             const int nsing = int(std::abs(dnsing) + T(1) / T(2));
             const T zmnpire = std::abs(zre) - npi[nsing];
             const T zmnpinorm = zmnpire * zmnpire + zim2;
             // close enough to one of the singularities?
             if (zmnpinorm < maxnorm) {
                 const T* coeffs = &taylorarr[nsing * NTAYLOR * 2];
                 // calculate value of taylor expansion...
                 // (note: there's no chance to vectorize this one, since
                 // the value of the next iteration depend on the ones from
                 // the previous iteration)
                 T sumre = coeffs[0], sumim = coeffs[1];
                 for (unsigned i = 1; i < NTAYLOR; ++i) {
                     const T re = sumre * zmnpire - sumim * zim;
                     const T im = sumim * zmnpire + sumre * zim;
                     sumre = re + coeffs[2 * i + 0];
                     sumim = im + coeffs[2 * i + 1];
                 }
             }
         }
     }
 }
// undo the flip in real part of z if needed
if (negrez) return std::complex<T>(sumre, -sumim);
else return std::complex<T>(sumre, sumim);

// negative Im(z) is treated by calculating for -z, and using the
// symmetry properties of erfc(z)
const bool negimz = zim < 0.;
if (negimz) {
    zre = -zre;
    zim = -zim;
}

const T twosqrtpi = 3.54490770181103205e+00;
const T tmzre = tm * zre, tmzim = tm * zim;
// calculate exp(i tm z)
T eitmzre = -tmzim, eitmzim = tmzre;
faddeeva_impl::cexp(eitmzre, eitmzim);
// form 1 +/- exp (i tm z)
const T numerarr[4] = {
    T(1) - eitmzre, -eitmzim, T(1) + eitmzre, +eitmzim
};
// form tm z * (1 +/- exp (i tm z))
const T numertmz[4] = {
    tmzre * numerarr[0] - tmzim * numerarr[1],
    tmzre * numerarr[1] + tmzim * numerarr[0],
    tmzre * numerarr[2] - tmzim * numerarr[3],
    tmzre * numerarr[3] + tmzim * numerarr[2]
};

// common subexpressions for use inside the loop
const T reimtmzm2 = T(-2) * tmzre * tmzim;
const T reimtmzm22 = reimtmzm2 * reimtmzm2;

#ifndef __x86_64__
    const T znorm = zre * zre + zim * zim;
    T sumre = (-a[0] / znorm) * (numerarr[0] * zre + numerarr[1] * zim);
    T sumim = (-a[0] / znorm) * (numerarr[1] * zre - numerarr[0] * zim);
    for (unsigned i = 0; i < N; ++i) {
        const T wk = imtmz2 + (npi[i] + tmzre) * (npi[i] - tmzre);
        const T norm = wk * wk + reimtmzm22;
        const T f = T(2) * tm / norm;
        sumre -= f * (numertmz[1] + wk * numertmz[0]) + reimtmzm2;
        sumim -= f * (numertmz[0] - wk * numertmz[1]) + reimtmzm2;
    }
#else
    // BEGIN fully vectorisable code - enjoy reading... ;)
    T tmp[2 * N];
    for (unsigned i = 0; i < N; ++i) {
        const T wk = imtmz2 + (npi[i] + tmzre) * (npi[i] - tmzre);
        tmp[2 * i + 0] = wk;
        tmp[2 * i + 1] = T(2) * tm / a[i];
    }
#endif
for (unsigned i = 0; i < N / 2; ++i) {
    T wk = tmp[4 * i + 0], f = tmp[4 * i + 1];
    tmp[4 * i + 0] = -f * (numertmz[0] * wk + numertmz[1] * reimtmzm2);
    tmp[4 * i + 1] = -f * (numertmz[1] * wk - numertmz[0] * reimtmzm2);
    wk = tmp[4 * i + 2], f = tmp[4 * i + 3];
    tmp[4 * i + 2] = -f * (numertmz[2] * wk + numertmz[3] * reimtmzm2);
    tmp[4 * i + 3] = -f * (numertmz[3] * wk - numertmz[2] * reimtmzm2);
}
if (N & 1) {
    // we may have missed one element in the last loop; if so, process
    // it now...
    const T wk = tmp[2 * N - 1], f = tmp[2 * N - N - 1];
    tmp[2 * (N - 1) + 0] = -f * (numertmz[0] * wk + numertmz[1] * reimtmzm2);
    tmp[2 * (N - 1) + 1] = -f * (numertmz[1] * wk - numertmz[0] * reimtmzm2);
}

const T znorm = zre * zre + zim2;
T sumre = (-a[0] / znorm) * (numerarr[0] * zre + numerarr[1] * zim);
T sumim = (-a[0] / znorm) * (numerarr[1] * zre - numerarr[0] * zim);
for (unsigned i = 0; i < N; ++i) {
    sumre += tmp[2 * i + 0];
    sumim += tmp[2 * i + 1];
}
// END fully vectorisable code
#endif

// prepare the result
if (negimz) {
    // use erfc(-z) = 2 - erfc(z) to get good accuracy for
    // Im(z) < 0; 2 / exp(z^2) - w(z)
    const T z2im = T(2) * zre * zim;
    const T z2re = (zre + zim) * (zre - zim);
    T ez2re = z2re, ez2im = z2im;
    faddeeva_impl::cexp(ez2re, ez2im);
    const T twoez2norm = T(2) / (ez2re * ez2re + ez2im * ez2im);
    return std::complex<T>(twoez2norm * ez2re + sumim / twosqrtpi,
                           -twoez2norm * ez2im - sumre / twosqrtpi);
} else {
    return std::complex<T>(-sumim / twosqrtpi, sumre / twosqrtpi);
}
}

static const double npi24[24] = {
    0.00000000000000000e+00,
    3.14159265358979324e+00,
    6.28318530717958648e+00,
    9.42477796076937972e+00,
    1.25663706143591730e+01,
    1.57079632679489662e+01,
    1.88495559215387594e+01,
    2.19911485751285527e+01,
    2.51327412287183459e+01,
    2.82743338823081391e+01,
    3.14159265358979324e+01,
    3.45575191894877256e+01,
    3.76991184307751890e+01,
    4.08407044966667121e+01,
    4.39822971502571053e+01,
    4.71238898038468966e+01,
    5.02654824574366918e+01,
    5.34070751110264851e+01,
    5.65486677646162783e+01,
    5.96902604182060715e+01,
    6.28318530717958648e+01,
    6.59734457253856580e+01,
    6.91150383789754512e+01,
    7.22566310325652445e+01,
};

static const double a24[24] = {
    2.95408975150919338e-01,
    2.75840233292177084e-01,
    2.24573955224615866e-01,
    1.59414938273911723e-01,
    9.86576641545418919e-02,
    5.32441078763941202e-02,
    2.75355660822107093e-02,
    8.37835871901245058e-03,
    2.75395660822107093e-03,
    7.39143342960301485e-04,
    2.75734393280083635e-05,
    2.75734393280083635e-06,
    7.39143342960301485e-07,
    5.930030404858103e-08,
    7.08449030774820423e-09,
    3.73952063581678038e-10,
    6.702171600200763e-11,
    5.30726516347079017e-12,
    3.66432411346763916e-13,
    2.0589494494103134e-14,
    1.57826862628558781e-15,
    5.2987114296730482e-17,
};

static const double taylorarr24[24 * 12] = {
    0.00000000000000000e+00,
    3.09011112254700200e+01,
    5.00000000000000000e+00,
};
| nsing  | x1       | x2       | x3       | x4       | x5       |
|--------|----------|----------|----------|----------|----------|
| 1      | -2.2243508493755319e-01 | 1.87966717746229718e-01 | 3.4180519240637628e-01 | 4.66574321730757753e-01 | -5.9649213591058097e-01 |
| 2      | -4.6052258651312894e-01 | -6.5649702008863349e-03 | 4.39243227763478846e-01 | 5.0591493715480700e-01 | -5.3898366115426093e-01 |
| 3      | -2.18249853727020186e-01 | -1.90471659765411376e-01 | -2.5904466869706639e-01 | 4.99077838344125714e-01 | 2.6114512111568373e-01 |
| 4      | 9.49825297066481941e-02  | -1.71428212158876197e-01 | -2.977667111471585e-01  | 1.8811421032460682e-01  | 3.9854061329390984e-02  |
| 5      | 1.8136707848232332e-02   | -6.0953306357908685e-02  | -7.7462998833038991e-01  | 3.67153465598415549e-02  | -4.6332190352216715e-01 |
| 6      | 8.8769809605071290e-02   | 2.8433935490894902e-02   | 3.1894303837066399e-01   | -1.9394688797704586e-02  | 1.70734367410600348e-01 |
| 7      | 2.99767046276705077e-02  | 5.3465969501718247e-02   | 4.53131032601822568e-02  | -3.7915401977138648e-02  | 3.82170507060760740e-02 |
| 8      | 7.3592494372035395e-03   | 5.7982359988083777e-03   | 5.0591493715480700e-01   | -1.0030873782517255e-02  | -2.5769148077963515e-02 |
| 9      | 2.95724850588435368e-02  | 1.8174625238337507e-01   | 6.72948505688435368e-02  | -5.21266530264988508e-02  | 3.24443217440050976e-02 |

0.0000000000000000e-00, -7.52252778063675049e-01, 1.2837916709551257e+00, 0.0000000000000000e-00, 0.0000000000000000e-00.
| Column 1 | Column 2 | Column 3 | Column 4 |
|---------|---------|---------|---------|
| 3.8810361995563741e-03 | 2.72090310854550347e-01 |
| 1.2124506816286880e-02 | 1.5980224240074898e-03 |
| 1.9116222508366035e-02 | 1.0587959419053302e-02 |
| -1.9728428219695318e-02 | -3.1692607712639397e-02 |
| 3.4110372628315158e-02 | 6.1804564429108837e-02 |
| -5.2574921865441838e-03 | -1.1425966380456945e-01 |
| 1.05534036292203489e-03 | 2.3732653489981828e-01 |
| -5.9685002183177493e-03 | -2.4259431567031205e-03 |
| 7.4475381747689418e-03 | 9.3345080757639438e-03 |
| -5.2649522783026481e-03 | -2.0816580620793201e-02 |
| 3.8998065678848650e-03 | 4.1278431345154913e-02 |
| -1.4411072110612792e-03 | -6.7648478299775472e-02 |
| 2.5021018490812137e-04 | 2.1131066219336647e-01 |
| -2.2450521223503419e-03 | -2.3814524227619446e-03 |
| 2.3673519870708934e-03 | 5.9732404060380626e-03 |
| 8.1333199366455318e-03 | -1.2812625072044051e-02 |
| 9.6925158618720835e-03 | 2.8305567987458973e-02 |
| -5.1723186203123061e-05 | 1.9068111717959752e-01 |
| -6.7688760754977909e-04 | -4.1858965249767064e-04 |
| 6.2254836947204695e-04 | 3.4387115674644686e-04 |
| -2.6557147166379929e-04 | -7.9885414500856400e-04 |
| 2.0664460199355524e-04 | 2.0310715258835321e-02 |
| -6.3456392941085698e-05 | -5.7142514491011583e-02 |
| 9.3225217914050245e-06 | 1.7416766378502582e-01 |
| -6.7596437777156162e-04 | -8.0538419386903178e-04 |
| 1.3762727777023791e-04 | 1.9765296202724093e-03 |
| 8.5439226487945971e-05 | -5.2308890641597716e-03 |
| 3.7896557756649351e-05 | 5.2191559129376332e-03 |
| -1.0739019481856466e-05 | -4.7934786215336629e-02 |
| 1.4650397062886179e-05 | 0.6047101168374778e-01 |
| -3.4571576630307877e-05 | -3.1089554210205493e-04 |
| 5.7535013801055497e-05 | 1.1449262097417514e-03 |
| -1.4632222751737226e-05 | -3.6130376679909378e-03 |
| 5.9905765658739226e-06 | 1.1799830501730890e-02 |
| -5.5600578509262722e-06 | -4.0916502374366970e-02 |
| 2.0073968320415217e-07 | 1.4887934855856267e-01 |
| -5.9973518885753734e-06 | -2.4294218855805052e-04 |
| 4.0924909093626972e-06 | 7.6740015272712817e-04 |
| -1.4290611287648034e-06 | -2.6071051957554623e-03 |
| 8.175916949568640978e-07 | 9.3858164013739053e-03 |
| -0.0091901402737743e-07 | -3.5040558123653803e-02 |
| 3.9819738182594508e-08 | 1.3981644940561371e-01 |
| -5.8070850515596665e-08 | -1.4647947515521504e-02 |
| 5.5569320739187190e-04 | 5.1916558784461541e-05 |
| -2.7131941259882675e-05 | -1.4433942758009957e-03 |
| 9.6641799664928425e-08 | 7.6153695720735798e-03 |
| -2.2357616066943296e-07 | -3.0976293485679078e-02 |
| 4.9806692045821258e-09 | 1.3024740712293206e-01 |
| -1.0007111030476390e-07 | -9.3588615088669178e-05 |
| 6.4624409699782439e-08 | 3.6526719341847904e-03 |
| -2.9517585699329524e-02 | -1.4873095954396108e-03 |
| 9.8494251974555378e-09 | 6.2782467914807017e-03 |
| -2.2672277074781576e-09 | -7.3547766571797965e-02 |
| 2.2687772443353217e-10 | 1.2262715881089526e-01 |
| -1.1730243996765755e-08 | -6.2489095672205333e-05 |
6.45231881609786173e-09, 2.64799907072561543e-04, -2.76943921343316549e-09, -1.16094178475938354e-03, 8.710768956648074e-10, 5.24514377390761210e-03, -1.78730768958639407e-10, -2.43490323901951382e-02, 1.79658233413659888e-11, 1.15870972518909888e-01, // nsing = 20
-1.07084502471985403e-09, -4.31515421260633319e-05, 5.54152563270547927e-10, 1.96606443937168357e-04, -2.44234744315423384e-10, -9.21550077887211094e-04, 6.7736377576211580e-11, 4.43201203646827019e-03, -2.986907717633162e-12, -2.18236356404862774e-02, 1.24042097336785166e-12, 0.98362769681518480e-01, // nsing = 21
-8.381652556690606e-11, -3.06091807093959821e-05, 4.10033961556230842e-11, 1.46830689216469994e-04, -9.531613905394895e-13, -1.78416955716514289e-02, 0.90548154647399962e-02, // nsing = 22
-5.6484922712870507e-12, -2.2021942382507691e-05, 2.61729537775838587e-12, 1.146830689216469994e-04, -5.05215735659161914e-03, -2.18236356404862774e-02, 3.9190413568087086e-15, 2.5132741228713459e-02, // nsing = 23
-3.2748357793897640e-13, -1.64138890390689871e-05, 4.4278798346454523e-13, 8.96362542918263938e-05, -5.0524303437266481e-13, -4.98699756861136124e-04, 1.28274026095767213e-14, 2.82359118537843994e-03, -1.60095399939171098e-16, -6.2538325704327487e-02, 1.79368667683653770e-16, 9.59473084594884184e-02
};
const double npi11[11] = { // precomputed values n * pi
 0.00000000000000000e+00, 3.14159265358979324e+00, 6.28318530717958648e+00, 9.42477766769397972e+00, 1.25663706144899662e+01, 1.57079632674989662e+01, 1.88495592153875594e+01, 2.20119922875066056e+01, 2.5132741228713459e+01, 2.82359118537843994e-01
};
const double a11[11] = { // precomputed Fourier coefficient prefactors
 4.4311346276379007e-01, 3.79788034073635143e-01, 2.39122407410867584e-01, 1.0599187402166972e-02, 8.73582250808904725e-02, 9.37693014302504859e-03, 1.71974046189940767e-03, 3.11635590005234616e-04, 2.29124010420125452e-05, 1.6658959213940077e-06, 5.89504561311882155e-08
};
const double taylorarr11[11 * 6] = { // real part imaginary part, low order coefficients last
 0.00000000000000000e+00, 3.14159265358979324e+00, 6.28318530717958648e+00, 9.42477766769397972e+00, 1.25663706144899662e+01, 1.57079632674989662e+01, 1.88495592153875594e+01, 2.20119922875066056e+01, 2.5132741228713459e+01, 2.82359118537843994e-01
};
}
3.36907922296469441e-01, 3.97048587678703930e-02, -2.66422678503135697e-01, -3.18469797424381480e-01, 8.4804972471137773e-02, 4.60546329221462864e-01, 4.8049724711377773e-02, 4.60546329221462864e-01, // nsing = 5
1.42043544696751869e-01, 1.24094227867032671e-01, -8.31224229982140323e-02, -2.40766729258442100e-01, 2.1169512031059302e-02, 3.4865019354945097e-01, 2.92113167048952835e-02, 9.0306084789976219e-02, -1.82889636251263500e-02, -1.5381621544915245e-01, 3.88103861995663741e-03, 2.72093010854550347e-01, // nsing = 7
7.37741897722738503e-03, 5.04625223970221539e-02, -2.87394336989990770e-03, -9.96122819257496929e-02, 2.2274878296428248e-04, 2.23361039070721010e-01, // nsing = 8
9.0625158617208358e-04, 2.8305667874589732e-02, -3.24986363596307374e-04, -6.97056268370209313e-02, 1.57231862038123061e-05, 1.90881117197597520e-01, // nsing = 9
9.01625563468897100e-05, 1.74661124275657019e-02, -2.65474612767337342e-05, -5.2207035634932341e-02, 3.75952450449394116e-06, 1.67018782142871146e-01, // nsing = 10
5.99057675687392260e-06, 1.17998305017130890e-02, -1.57660578509526722e-06, -4.09165023743669707e-02, 2.00739683204152177e-07, 1.48879348585662670e-01);

```cpp
std::complex<double> Cerf::faddeeva(const std::complex<double> z) {
    static std::complex<double> a24 = {12., 3.36907922296469441e-01, 4.8049724711377773e-02};
    static std::complex<double> a11 = {8., 7.37741897722738503e-03, 2.2274878296428248e-04};
    static std::complex<double> a11 = {9., 1.42043544696751869e-01, 1.24094227867032671e-01};
    static std::complex<double> a11 = {10., 1.57231862038123061e-05, 5.99057675687392260e-06};
    double re = -z.real() * z.real() + z.imag() * z.imag();
    double im = -2. * z.real() * z.imag();
    return (z.real() >= 0.) ?
        (std::complex<double>(re, im) *
            faddeeva(std::complex<double>(re, im)) *
            faddeeva(std::complex<double>(re, im)))
        : (2. - std::complex<double>(re, im) *
            faddeeva(std::complex<double>(re, im)) *
            faddeeva(std::complex<double>(re, im)));
}
```
std::complex<double> Cerf::erf(const std::complex<double> z)
{
    double re = -z.real() * z.real() + z.imag() * z.imag();
    double im = -2. * z.real() * z.imag();
    faddeeva_impl::cexp(re, im);
    return (z.real() >= 0.) ?
        (1. - std::complex<double>(re, im) *
         faddeeva(std::complex<double>(-z.imag(), z.real()))) :
        (std::complex<double>(re, im) *
         faddeeva(std::complex<double>(z.imag(), -z.real())) - 1.);
}

std::complex<double> Cerf::erf_fast(const std::complex<double> z)
{
    double re = -z.real() * z.real() + z.imag() * z.imag();
    double im = -2. * z.real() * z.imag();
    faddeeva_impl::cexp(re, im);
    return (z.real() >= 0.) ?
        (1. - std::complex<double>(re, im) *
         faddeeva_fast(std::complex<double>(-z.imag(), z.real()))) :
        (std::complex<double>(re, im) *
         faddeeva_fast(std::complex<double>(z.imag(), -z.real())) - 1.);
}

References

[1] S.M. Abrarov and B.M. Quine. “Efficient algorithmic implementation of the Voigt/complex error function based on exponential series approximation”. In: Applied Mathematics and Computation 218.5 (2011), pp. 1894 –1902. ISSN: 0096-3003.

[2] Wouter Verkerke and David P. Kirkby. “The RooFit toolkit for data modeling”. In: eConf C0303241 (2003), MOLT007. arXiv:physics/0306116 [physics].

[3] R. Brun and F. Rademakers. “ROOT — An Object Oriented Data Analysis Framework”. In: Nucl. Instr. and Meth. A389 (1997). See also http://root.cern.ch/ pp. 81–86. URL: http://root.cern.ch

[4] K. S. Köllbig et al. CERNLIB C335: Complex Error Function. http://wwwasdoc.web.cern.ch/wwwasdoc/shortwrupsdir/c335/top.html Geneva, 1993.

[5] S.G. Johnson, A. Cervellino and J. Wuttke. libcerf, a numeric library providing complex error functions, version 1.1 2013.

[6] Maxima. Maxima, a Computer Algebra System. Version 5.27.0 2012.