OPTIMAL ARITHMETIC STRUCTURE IN EXPONENTIAL RIESZ SEQUENCES

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Abstract. We consider exponential systems $E(\Lambda) = \{e^{i\lambda t}\}_{\lambda \in \Lambda}$ for $\Lambda \subset \mathbb{Z}$. It has been shown by Londner and Olevskii in [9] that there exists a subset of the circle, of positive Lebesgue measure, so that every set $\Lambda$ which contains, for arbitrarily large $N$, an arithmetic progressions of length $N$ and step $\ell = O(N^\alpha)$, $\alpha < 1$, cannot be a Riesz sequence in the $L^2$ space over that set. On the other hand, every set admits a Riesz sequence containing arbitrarily long arithmetic progressions of length $N$ and step $\ell = O(N)$.

In this paper we show that every set $S \subset \mathbb{T}$ of positive measure belongs to a unique class, defined through the optimal growth rate of the step of arithmetic progressions with respect to the length that can be found in Riesz sequences in the space $L^2(S)$. We also give a partial geometric description of each class.

1. Introduction

1.1. Preliminaries. Let $S \subset \mathbb{R}$ be a bounded set of positive Lebesgue measure ($|S| > 0$), and $\Lambda \subset \mathbb{R}$ a uniformly discrete set. The exponential system $E(\Lambda) = \{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is called a Riesz sequence in $L^2(S)$ (we denote $\Lambda \in RS(S)$) if there exist positive constants $A, B$ such that

$$\sum_{\Lambda} a_\lambda^2 \leq \left\| \sum_{\Lambda} a_\lambda e^{i\lambda t} \right\|^2_{L^2(S)} \leq B \sum_{\Lambda} a_\lambda^2$$

for every finite sequence of coefficients $\{a_\lambda\}$.

Any constant satisfying the left inequality (1.1) will be called a lower Riesz bound for $E(\Lambda)$ in $L^2(S)$.

The problem of determining the exact relationship between sets of frequencies on the one hand, and their corresponding spectra on the other, in general, is very difficult. Heuristically, it can be stated that for a fixed spectrum $S$, the more dense a set of points is, the harder it becomes to verify that it is a Riesz sequence. Analogously, for a fixed set of frequencies $\Lambda$, the more complicated the structure of the spectrum, and the smaller its measure, the less likely it is for $E(\Lambda)$ to be a Riesz sequence for this spectrum. In past years, a large body of work has been dedicated to the analysis of the interplay between $S$ and $\Lambda$ (See, for instance, [5, 13, 15] and the references therein). Below we survey some of the most relevant results.

Throughout this paper, unless stated otherwise, we will always assume $S$ is a subset of the circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $\Lambda \subset \mathbb{Z}$. Also, in various places, $c$ and
C will denote absolute constants which might be different from one another, even within the same line.

1.2. Density. The case where $S = I$ is an interval is classical. In order to verify that $E(\Lambda)$ is indeed a Riesz sequence in $L^2(I)$, one essentially needs to know the upper Beurling density of $\Lambda$.

**Theorem (Kahane, [6]).** Let $\Lambda \subset \mathbb{R}$. If

$$D^+(\Lambda) := \lim_{r \to \infty} \max_{a \in \mathbb{R}} \frac{\#(\Lambda \cap (a, a + r))}{r} < \frac{|I|}{2\pi}$$

then $\Lambda \in RS(I)$, while if $D^+(\Lambda) > \frac{|I|}{2\pi}$ then it is not.

For arbitrary sets $S$ of positive (and finite) measure, the situation is much more complicated and only a necessary condition exists.

**Theorem (Landau, [7]).** Let $\Lambda \subset \mathbb{R}$. If $\Lambda \in RS(S)$ then $D^+(\Lambda) \leq \frac{|S|}{2\pi}$.

1.3. Simultaneous Riesz sequences. With regards to the Riesz sequence property it is only natural to ask whether there exists a set $\Lambda$ such that $\Lambda \in RS(S)$ simultaneously for all subsets $S \subset \mathbb{T}$ of positive measure. We call such sequences simultaneous Riesz sequences. By Landau’s necessary condition, it is clear that such sets must have zero upper density.

Zygmund essentially proved the following

**Theorem ([16]).** Let $\Lambda = \{\lambda_j\} \subset \mathbb{Z}$ be a Hadamard lacunary sequence, i.e. for some $q > 1$ we have

$$\lambda_{j+1}/\lambda_j \geq q > 1 \quad \forall j,$$

then $\Lambda$ is a simultaneous Riesz sequence.

Later, this result has been further generalized (see [12]).

1.4. Riesz sequences with positive density. Since Riesz sequence cannot be too dense, one may ask whether a given set $S$ admits a Riesz sequence $E(\Lambda)$ such that $\Lambda$ has positive density. This question may be considered under various notions of density. Bourgain and Zafriri, as a consequence of their “restricted invertibility” theorem, answered this question positively proving

**Theorem ([1]).** Every set $S \subset \mathbb{T}$ of positive measure admits a Riesz sequence $E(\Lambda)$, $\Lambda \subset \mathbb{Z}$, with positive asymptotic density

$$\text{dens}(\Lambda) := \lim_{r \to \infty} \frac{\#(\Lambda \cap (-r, r))}{2r} > C|S|,$$

where $C$ is an absolute positive constant, independent of $S$.

1.5. Syndetic Riesz sequences. An even stronger notion is the following; we say that a set $\Lambda$ is syndetic if $\Lambda + \{0, 1, \ldots, d\} = \mathbb{Z}$, for some $d \in \mathbb{N}$.

Given a set $S$, Lawton proved ([8]) that the existence of a syndetic Riesz sequence is equivalent to the Feichtinger conjecture for exponentials. The Feichtinger conjecture in its general form has been proved by Casazza et al ([1]) to be equivalent to the Kadison-Singer problem. The latter has been solved recently by Marcus, Spielman and Srivastava ([11]), and so the existence of syndetic Riesz sequences holds unconditionally.
It should be mentioned that the aforementioned solution was used in [2] to prove the existence of a syndetic Riesz sequence with a sharp asymptotic bound on the quantity $d$.

1.6. Riesz sequences and arithmetic progressions. In this paper we restrict our attention to the situation in which the set of frequencies $\Lambda \subset \mathbb{Z}$ contains arbitrarily long arithmetic progressions. More accurately, suppose that for some increasing sequence $N_j$, one can find $\ell_j \in \mathbb{N}$ and $M_j \in \mathbb{Z}$ such that

$$M_j + \{\ell_j, 2\ell_j, \ldots, N_j\ell_j\} \subset \Lambda \quad \forall j$$

What can be said about the different sets $S$ for which $E(\Lambda)$ is, or is not a Riesz sequence in $L^2(S)$?

Remark. From hereon, given an arithmetic progression, we will denote by $N$ its length, and its step by $\ell$.

Since every set $S$ admits a Riesz sequence of positive density, in particular, it follows from Szemerédi’s theorem ([14]) that this set contains arbitrarily long arithmetic progressions. We emphasize that the existence of Riesz sequences containing arbitrarily long arithmetic progressions may also be proved independently, we leave it as an exercise for the curious reader.

On the other hand Miheev proved

Theorem ([12]). Given a set $\Lambda$ which contains arbitrarily long arithmetic progressions, there exists a set $S$ of positive measure such that $\Lambda \notin RS(S)$, i.e. $\Lambda$ cannot be a simultaneous Riesz sequence.

In case $\ell$ grows slowly with respect to $N$, one can choose the set $S$ independently of $\Lambda$. This question was considered initially by Bownik and Speegle who gave a quantified version of this result.

Theorem ([3]). There exists a set $S$ such that $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$ whenever $\Lambda$ contains arithmetic progressions of length $N_j$ and step

$$\ell_j = o\left(N_j^{1/2}\log^{-3} N_j\right) \quad (N_1 < N_2 < \ldots).$$

Later, using a different approach, this was improved by Londner and Olevskii.

Theorem 1 ([9]). There exists a set $S \subset \mathbb{T}$ such that if a set $\Lambda \subset \mathbb{Z}$ contains arithmetic progressions of length $N (= N_1 < N_2 < \ldots)$ and step $\ell = O(N^\alpha)$, $\alpha < 1$, then $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$.

It should be mentioned that the set $S$ in Theorem 1 was constructed independently of the choice of $\alpha$, and with arbitrarily small measure of the complement.

As indicated earlier, every set of positive measure admits a Riesz sequence containing arbitrarily long arithmetic progressions. It had been asked in [9] what can be said about the growth rate of $\ell$ with respect to $N$ in those systems? By Theorem 1 we know that, in general, $\ell$ cannot grow sublinearly. In [9] it was proved that Theorem 1 is essentially sharp.

Theorem 2 ([9]). Given a set $S \subset \mathbb{T}$ of positive measure, there exists a set $\Lambda \subset \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz sequence in $L^2(S)$, and for infinitely many $N$’s $\Lambda$ contains an arithmetic progression of length $N$ and step $\ell = O(N)$.
Theorem 2 was proved by explicitly constructing the set \( \Lambda \), which takes the form
\[
\Lambda = \bigcup_{j \in \mathbb{N}} (M_j + \{N_j, 2N_j, \ldots, N_j^2\})
\]
for some specially chosen increasing sequence \( \{N_j\}_{j \in \mathbb{N}} \) (depending on \( S \)) and translations \( \{M_j\}_{j \in \mathbb{N}} \), so in reality we have here \( \ell_j = N_j \).

We note that the latter results were considered in the multidimensional setting as well (see [10]).

2. Results

2.1. The classes \( \mathcal{A}(\alpha) \). Theorems 1 and 2 suggest that one can distinguish between subsets of the circle through Riesz sequences containing arbitrarily long arithmetic progressions. The goal of this paper is to show that this phenomenon occurs at every scale, i.e. every set admits an optimal asymptotic growth rate which controls the step in any arithmetic progression of a given length in Riesz sequences over that set. To this end we define the following classes of sets.

Definition. Let \( \alpha \in [0, 1] \). We say that a measurable set \( S \subset \mathbb{T} \) belongs to the class \( \mathcal{A}(\alpha) \), if \( S \) admits a Riesz sequence \( E(\Lambda) \) such that \( \Lambda \) contains, for infinitely many \( N \)'s, an arithmetic progression of length \( N \) and step \( \ell = O(N^\alpha) \).

It follows directly from the definition that \( \mathcal{A}(\beta) \subseteq \mathcal{A}(\alpha) \) for all \( \beta < \alpha \). We observe that by Theorem 2, the class \( \mathcal{A}(1) \) contains all measurable subsets of the circle. When combined with Theorem 1, it implies that \( \mathcal{A}(1) \setminus \bigcup_{\beta < 1} \mathcal{A}(\beta) \) is nonempty.

Also, by a trivial argument, the class \( \mathcal{A}(0) \) contains all sets with nonempty interior (but not only those).

When \( \beta < \alpha \), which inclusions are proper? can we, in fact, separate these classes from one another?

Our main result answers this question positively.

Theorem 3. For any \( \alpha \in (0, 1] \) we have \( \mathcal{A}(\alpha) \setminus \bigcup_{\beta < \alpha} \mathcal{A}(\beta) \) is nonempty.

Such set \( S \in \mathcal{A}(\alpha) \setminus \bigcup_{\beta < \alpha} \mathcal{A}(\beta) \) would admit a Riesz sequence containing arbitrarily long arithmetic progressions of length \( N \) and step \( \ell = O(N^\alpha) \), but any system \( E(\Lambda) \) for which the step in arithmetic progressions of length \( N \) in \( \Lambda \) grow at a rate bounded by \( C \cdot N^\beta \) cannot be Riesz sequences in \( L^2(\mathbb{S}) \). We prove Theorem 3 by generalizing the construction from section 3 in [9], showing that the constructed set admits a Riesz sequence having the desired properties. It is important to emphasize that the set constructed in the proof of Theorem 3 can be chosen to have arbitrarily small measure of the complement, independently from \( \alpha \).

2.2. On the structure of sets in \( \mathcal{A}(\alpha) \). Theorem 3 suggests that the arithmetic structure that can be found inside a Riesz sequence can help to sort subsets of the circle. It is therefore in our interest to obtain a description of the structure of sets in a given class. This is done using the following
Definition. Given a set $S \subset \mathbb{T}$ with $|S| > 0$ and a positive integer $\ell$, we define the $\ell$-th multiplicity function $\nu_{\ell, S} : [0, \frac{2\pi}{\ell}) \to \{0, 1, \ldots, \ell\}$ associated with $S$

$$\nu_{\ell, S}(t) := \left(1_S * \delta_{\frac{2\pi}{\ell}}\right)(t).$$

Our next result conveys the idea that some type of order inside a set $S$ is reflected in another type of order in Riesz sequences this set admits.

Theorem 4. Given a set $S \subset \mathbb{T}$ with $|S| > 0$ and $\alpha \in [0, 1]$. If there exist constants $c, \delta \in (0, 1)$, which might depend on $\alpha$, such that for infinitely many values of $\ell \in \mathbb{N}$, the $\ell$-th multiplicity function satisfies

(2.1) $$|\{t \in \mathbb{T} | \nu_{\ell, S}(t/\ell) < \delta\ell\}| < \frac{c}{\ell^{\ell/\alpha}}$$

then $S \in A(\alpha)$. Moreover, if $\alpha = 0$ then this condition is also necessary.

Here $\alpha = 0$ means $|\{t \in \mathbb{T} | \nu_{\ell, S}(t/\ell) < \delta\ell\}| = 0$.

Simply put, a set $S$ satisfies condition (2.1) with a fixed $\delta$ if for infinitely many values of $\ell$ and for every $t \in \mathbb{T}$, up to some exceptional set of small measure, $S$ contains a $\delta$-proportion of the arithmetic progression of length $\ell$ and step $\frac{2\pi}{\ell}$, passing through $t$.

3. The classes $A(\alpha)$

The proof of Theorem 3 is composed of 3 parts. In the first part we generalize our construction from section 3 in [9], constructing the set $S_\alpha$. In the second, we show that this set does not belong to any of the “lower” classes, i.e. $S_\alpha \notin A(\beta)$ when $\beta < \alpha$. In the third part we prove that it does, in fact, belong to $A(\alpha)$.

Since the case $\alpha = 1$ has already been covered in [9], we restrict to $\alpha \in (0, 1)$.

3.1. Construction of the set $S_\alpha$. We start by generalizing the construction from [9]. Let $\alpha \in (0, 1)$ and $\varepsilon > 0$. Fix a positive constant

(3.1) $$c_0 = c_0(\alpha, \varepsilon) < 1$$

to be specified later and define

$$\delta(\ell) = \frac{c_0}{\ell^{\ell/\alpha}}, \quad \ell = 1, 2, \ldots$$

so that $\sum_{\ell=1}^{\infty} 2\delta(\ell) < \varepsilon$. The constant $c_0$ is chosen to satisfy the latter inequality as well as other properties that will be specified in section 3.3.

For every $\ell \in \mathbb{N}$ set

$$I_\ell = (-\delta(\ell), \delta(\ell)),$$
$$I_{\ell} = \left(-\frac{\delta(\ell)}{\ell}, \frac{\delta(\ell)}{\ell}\right)$$

and let $\tilde{I}_\ell$ its $2\pi$-periodic extension, i.e.

$$\tilde{I}_\ell = \bigcup_{k \in \mathbb{Z}} (I_\ell + 2\pi k).$$

We define

(3.2) $$I_{[\ell]} = \frac{1}{\ell} \cdot \tilde{I}_\ell \cap [-\pi, \pi) \text{ and } S_\alpha = T \setminus \bigcup_{\ell \in \mathbb{N}} I_{[\ell]} = \left(\bigcup_{\ell \in \mathbb{N}} I_{[\ell]}\right)^c.$$
and see that
\[ |S_\alpha|/2\pi \geq 1 - \sum_{\ell=1}^{\infty} |I_\ell| = 1 - \sum_{\ell=1}^{\infty} 2\delta(\ell) > 1 - \varepsilon. \]

Observe that by definition we may identify \( I_\ell \) with the following union
\[(3.3) \quad I_\ell = \bigcup_{j=0}^{\ell-1} \left( \frac{2\pi j}{\ell} + I_\ell \right)\]
considered as a subset of \( \mathbb{T} \). Since the sequence \( \{\delta(\ell)\} \) is decreasing, it is easily seen that in the definition of \( S_\alpha \), considering the union \( \bigcup_{\ell \in \mathbb{N}} I_\ell \), any of the sets \( I_\ell \) may be replaced by a set \( J_\ell \), where \( I_{[1]} = J_{[1]} \) and
\[(3.4) \quad J_\ell = \bigcup_{j=1}^{\ell-1} \left( \frac{2\pi j}{\ell} + I_\ell \right) \text{ for } \ell > 1. \]

In some cases the sets \( J_\ell \) would be easier to handle. We prove

**Lemma 5.** \( S_\alpha \notin A(\beta) \) for any \( \beta \in [0, \alpha) \).

**Proof.** Fix \( \beta \in [0, \alpha) \) and let \( \Lambda \subset \mathbb{Z} \) be such that one can find arbitrarily large \( N \in \mathbb{N} \) for which
\[ \{M + \ell, \ldots, M + N\ell \} \subset \Lambda \]
with some \( M = M(N) \in \mathbb{Z}, \ell = \ell(N) \in \mathbb{N} \) and
\[(3.5) \quad \ell < C(\beta) N^\beta. \]

Recall that by (3.2) we have \( t \in I_\ell \) if and only if \( t\ell \in I_\ell \cap [-\pi\ell, \pi\ell] \). Since \( S_\alpha \) lies inside the complement of \( I_\ell \), we get
\[
\int_{S_\alpha} \left| \sum_{k=1}^{N} a(k) e^{i(M + k\ell)t} \right|^2 dt = \int_{I_\ell} \left| \sum_{k=1}^{N} a(k) e^{i(M + k\ell)t} \right|^2 dt = \int_{[-\pi\ell, \pi\ell] \setminus I_\ell} \left| \sum_{k=1}^{N} a(k) e^{ik\tau} \right|^2 \frac{d\tau}{\ell} \leq \int_{I_\ell} \left| \sum_{k=1}^{N} a(k) e^{ik\tau} \right|^2 d\tau
\]
Next we set \( P_N(t) = \frac{1}{N} \sum_{k=1}^{N} e^{ikt} \), so \( \|P_N\|_{L^2(\mathbb{T})} = 1 \). Moreover, for every \( t \in \mathbb{T} \) we have \( |P_N(t)| \leq \frac{1}{\sqrt{N \sin \frac{\beta}{2}}} \), hence, by (3.5)
\[
\int_{I_\ell} |P_N(t)|^2 dt \leq \frac{1}{N} \int_{-\pi}^{\pi} \frac{dt}{\sin \frac{\beta}{2}} < \frac{C}{\delta(\ell)^N} \int_{-\pi}^{\pi} \frac{dt}{\delta(\ell)^N} \leq \frac{C(\beta)}{\delta(\ell)^N} \leq \frac{C(\beta)}{\delta(\ell)^{\ell^\beta}}
\]
where the last inequality holds for every \( N \) for which (3.5) holds. It now follows from the choice of \( \delta(\ell) \) that the last term can be made arbitrarily small, and so \( E(\Lambda) \) is not a Riesz sequence in \( L^2(S_\alpha) \). \( \square \)
3.2. **Uniting blocks.** In this section we lay out a basic, yet rather useful, principle that enables the construction of Riesz sequences containing arbitrarily long arithmetic progressions. Whenever this principle will be applied, it will always lead to the conclusion that some countable union of finite arithmetic progressions, which we sometimes refer to as “blocks”, form a Riesz sequence over some set.

We require the following lemma.

**Lemma 6** ([7], Lemma 8). Let \( \gamma > 0 \), \( S \subset \mathbb{T} \) with \( |S| > 0 \) and \( A_1, A_2 \) finite sets of integers such that \( \gamma \) is a lower Riesz bound in \( L^2(S) \) for \( E(A_k) \), \( k = 1, 2 \). Then for any \( 0 < \gamma' < \gamma \) there exists \( M \in \mathbb{Z} \) such that the system \( E(A_1 \cup (M + A_2)) \) is a Riesz sequence in \( L^2(S) \) with lower Riesz bound \( \gamma' \).

Given a countable collection of finite sets \( \{A_k\}_{k \in \mathbb{N}} \) with a common lower Riesz bound, positioning them distant enough from one another, we can take their union while keeping the Riesz sequence property.

**Corollary 7.** Let \( S \subset \mathbb{T} \) with \( |S| > 0 \) and \( \{A_k\}_{k \in \mathbb{N}} \) be a collection of finite sets of integers. Suppose that \( E(A_k) \) is a Riesz sequence in \( L^2(S) \), with lower Riesz bound \( \gamma \) for all \( k \). Then there exists a sequence of integers \( \{M_k\}_{k \in \mathbb{N}} \) such that the system \( E(A) \), where \( A = \bigcup_{k \in \mathbb{N}} (M_k + A_k) \), is a Riesz sequence in \( L^2(S) \) with lower Riesz bound \( \frac{\gamma}{2} \).

**Proof.** By induction. Take \( M_1 = 0 \).

Suppose we have chosen \( M_1, \ldots, M_K \) such that for

\[
A(K) = \bigcup_{k=1}^{K} (M_k + A_k)
\]

the corresponding exponential system \( E(A(K)) \) has lower Riesz bound \( \frac{\gamma}{2} \left(1 + \frac{1}{K}\right) \).

Note that \( A(K) \) and \( A_{K+1} \) are both finite sets satisfying the assumptions of Lemma 6 with \( \tilde{\gamma} = \frac{\gamma}{2} \left(1 + \frac{1}{K}\right) \). It follows that we can find \( M_{K+1} \) such that \( E(A(K+1)) \) is a Riesz sequence in \( L^2(S) \) with lower Riesz bound \( \frac{\gamma}{2} \left(1 + \frac{1}{K+1}\right) < \tilde{\gamma} \). \( \square \)

3.3. **Proving \( S_\alpha \in A(\alpha) \).** By Corollary 7 in order to prove Theorem 3 it’s enough to prove the following

**Lemma 8.** Let \( \alpha \in (0,1) \), \( \varepsilon > 0 \). For \( S_\alpha \), the set constructed in section 2.4, there exists \( \gamma > 0 \) and \( N = N(\alpha) \in \mathbb{N} \) such that for every prime \( p > N \), the exponential system \( E(B_p^\alpha) \) is a Riesz sequence in \( L^2(S_\alpha) \) with lower Riesz bound \( \gamma \). Where we define

\[
B_p^\alpha := \{p, 2p, \ldots, N_p p\}, \quad N_p = N(\alpha, p) = \left\lfloor p^{1/\alpha} \right\rfloor.
\]

Given \( \alpha \) and \( \varepsilon \), let \( c_0 \) be the constant chosen in (3.1). Let

\[
Q_{N_p}(t) = \sum_{k=1}^{N_p} a(k) e^{ikt}, \quad \sum_{k=1}^{N_p} |a(k)|^2 = 1
\]

so that \( Q_{N_p}(t) \) has spectrum in \( B_p^\alpha \). We prove Lemma 8 by bounding from above each of the integrals

\[
\int_{\mathbb{T}} |Q_{N_p}(pt)|^2 dt, \quad \ell \in \mathbb{N}
\]
separately. Given that our estimates sum up to something smaller than 1 (in fact, it will depend on \( c_0 \) and \( N \)), we will write
\[
\int_{S_n} |Q_{N_p}(pt)|^2 \, dt \geq 1 - \sum_{t} \int_{J_{[t]}} |Q_{N_p}(pt)|^2 \, dt.
\]

For convenience we shall denote \( f_\ell(t) = 1_{J_{[t]}}(t) \). Notice that since \( f_\ell \) is a \( \frac{2\pi}{\ell} \)-periodic function, we have
\[
(3.6) \quad \hat{f}_\ell(n) \neq 0 \iff n \in \ell \mathbb{Z}.
\]

Let \( \mathcal{N} = \mathcal{N}(\alpha) \in \mathbb{N} \) to be chosen later and \( p > \mathcal{N} \) a prime number.

For \( 1 \leq \ell \leq \lfloor c_0 p \rfloor \) we write
\[
\int_{J_{[t]}} \frac{|Q_{N_p}(pt)|^2}{\ell} \, dt \leq \int_{J_{[t]}} \frac{|Q_{N_p}(pt)|^2}{\ell} \, dt = \int_{pI_{[\ell]}} \frac{|Q_{N_p}(\tau)|^2}{\ell} \, d\tau
\]
We use the following key observation: considering the change of variables \( pt = \tau \) as above, we notice that if, as in (3.3), we identify \( I_{[t]} \) as a subset of \([0, 2\pi)\) which is a union of \( \ell \) disjoint intervals (counting \([0, \delta(\ell)/\ell) \cup (2\pi - \delta(\ell)/\ell, 2\pi)\) as one interval), each of length \( \frac{2\delta(\ell)}{\ell} \), and centered at points \( 0, \frac{2\pi}{\ell}, \ldots, \frac{2\pi(\ell - 1)}{\ell} \). We get that \( pI_{[\ell]} \) is a subset of \([0, 2\pi p] \) composed of \( \ell \) disjoint intervals, each of length \( p \frac{2\delta(\ell)}{\ell} \), and centered at points \( 0, \frac{2\pi p}{\ell}, \ldots, \frac{2\pi p(\ell - 1)}{\ell} \).

Since the mapping from \( \mathbb{Z}/\ell \mathbb{Z} \) to itself defined by \( j \mapsto pj \) is an isomorphism (this is so since \( p \) is invertible \( \mod \ell \)), residues modulo \( \ell \) are mapped to themselves under this mapping and so we conclude that there exists a permutation \( \sigma \) of \( \{0, 1, \ldots, \ell - 1\} \) such that \( \sigma(0) = 0 \) and
\[
\left\{ \frac{pj}{\ell} \right\} = \frac{\sigma(j)}{\ell}, \quad j \in \{0, 1, \ldots, \ell - 1\}.
\]

Since \( Q_{N_p} \) is a trigonometric polynomial of period \( 2\pi \), we have
\[
\int_{\frac{2\pi p}{\ell} + pI_{[\ell]}} |Q_{N_p}(t)|^2 \, dt = \int_{\frac{2\pi p}{\ell} + pI_{[\ell]}} |Q_{N_p}(t)|^2 \, dt, \quad j \in \{0, 1, \ldots, \ell - 1\}.
\]

By symmetry, we notice that the collection of intervals
\[
\bigcup_{j=0}^{\ell-1} \left( \frac{2\pi j}{\ell} + pI_{[\ell]} \right) = \bigcup_{j=0}^{\ell-1} \left( \frac{2\pi j}{\ell} + pI_{[\ell]} \right)
\]
covers every point of the circle at most \( |2p\delta(\ell)| + 2 \) times. This is so since if we fix an interval from this union and consider its center point, then any of the centers of the other intervals is covered by this interval if and only if their associated interval covers the center of our fixed interval. Since the centers are uniformly distributed on the circle, indeed every point of the circle is covered at most
\[
\ell \cdot p \frac{2\delta(\ell)}{\ell} + 2 = |2p\delta(\ell)| + 2
\]
times. From this we get
\[
\int_{pI_{[\ell]}} \frac{|Q_{N_p}(\tau)|^2}{\ell} \, d\tau \leq \frac{1}{p} \left(|2p\delta(\ell)| + 2\right) \leq 2\delta(\ell) + \frac{2}{p}.
\]
Making summation over all \( \ell \) in the current range, we get
\[
\sum_{\ell=1}^{[c_0p]} \int_{J_{[\ell]}} |Q_Np (pt)|^2 \, dt \leq \sum_{\ell=1}^{[c_0p]} \int_{pI_{[\ell]}} |Q_Np (\tau)|^2 \, \frac{d\tau}{p} \leq
\]
\[
\sum_{\ell=1}^{[c_0p]} \left( 2\delta (\ell) + \frac{2}{p} \right) < \sum_{\ell=1}^{[c_0p]} |I_{[\ell]}| + 2c_0.
\]
(3.7)

For \( \ell \) in the range \( A = \{ [c_0p] \leq \ell \leq N_p \mid p \mid \ell \} \) we will require the following

**Definition.** Given \( \rho \in (0,1) \), for every \( x \in [0,1] \) and \( N \in \mathbb{N} \) we denote by \( M_\rho (x, N) \) the function that counts the number of pairs of integers \((m, n)\) satisfying
\[
|x - \frac{m}{n}| < \frac{1}{n^\rho}
\]
where \( 1 \leq n \leq N, 1 \leq m < n \) and \( \gcd (m, n) = 1 \).

The connection of the counting function \( M_\beta \) to our problem can expressed as follows. Write
\[
\sum_{\ell=1}^{[c_0p]} \int_{J_{[\ell]}} |Q_Np (pt)|^2 \, dt = \sum_{\ell=1}^{[c_0p]} \int_{pI_{[\ell]}} |Q_Np (\tau)|^2 \, \frac{d\tau}{p}
\]
by the same reasoning as we had for the previous case, since for all \( \ell \in A \) we have \( \gcd (\ell, p) = 1 \), we can conclude that
\[
\sum_{\ell=1}^{[c_0p]} \int_{pI_{[\ell]}} |Q_Np (\tau)|^2 \, \frac{d\tau}{p} = \frac{1}{p} \sum_{\ell=1}^{[c_0p]} \int_{J_{[\ell]}} |Q_Np (\tau)|^2 \, d\tau
\]
(3.9)
where
\[
J_{p,[\ell]} = \bigcup_{j=1}^{\ell-1} \left( \frac{2\pi j}{\ell} + pI_\ell \right).
\]

Notice that the sets \( J_{p,[\ell]} \) may overlap, and so one way of estimating the integrals in the above sum is by bounding, for every \( \tau \in \mathbb{T} \), the number of sets \( \{ J_{p,[\ell]} \}_{\ell \in A} \) to which \( \tau \) belongs. We will show that when \( 0 < \alpha < 1/2 \) this number is bounded independently of \( p \), i.e. every \( \tau \) belongs to at most \( d \) of the sets \( \{ J_{p,[\ell]} \}_{\ell \in A} \), and \( d \) depends only on \( \alpha \). While for \( 1/2 \leq \alpha < 1 \) we shall prove that even though this number depends on \( p \), it grows slow enough to allow control over the latter sum. Moreover, when \( \alpha \in (0, 1/2) \) note that
\[
\sum_{\ell \in A} |J_{p,[\ell]}| \leq p \sum_{\ell=[c_0p]}^{N_p} |I_{[\ell]}| = c_0p \sum_{\ell=[c_0p]}^{N_p} \frac{1}{p^{1/2}} \leq c_0p \frac{1}{(c_0p)^{1/2} - 1} = o(1) \text{ as } N \to \infty
\]

This observation should serve as an intuitive justification for Lemma 9 below.

In addition, for every \( \ell \in A \) the intervals forming the set \( J_{p,[\ell]} \) are pairwise disjoint, that is
\[
\left( \frac{2\pi j_1}{\ell} + pI_\ell \right) \cap \left( \frac{2\pi j_2}{\ell} + pI_\ell \right) = \emptyset \text{ for all } 1 \leq j_1 < j_2 \leq \ell - 1.
\]
(3.10)
Indeed \([3.11]\) holds if and only if
\[
\frac{p^\delta (\ell)}{\ell} \leq \frac{\pi}{\ell} \iff (cp)^\alpha \leq \ell,
\]
which holds for all \(\ell \in A\). Moreover, the following holds true.

**Lemma 9.** Let \(\alpha \in (0, 1/2)\), \(\eta \in \left(\frac{2}{1-\alpha}, 1\right)\) and set \(\eta' = \frac{2}{1-\alpha} (\eta - \alpha/2)\). Then for all \(\ell \geq 1\)
\[
P^n \leq \ell_1 < \ell_2 \leq P^{\eta'} \text{ with } P \nmid \ell_1, \ell_2
\]
we have \(J_{p,[\ell_1]} \cap J_{p,[\ell_2]} = \emptyset\).

**Proof.** Fix \(\alpha \in (0, 1/2)\), \(\eta \in \left(\frac{2}{1-\alpha}, 1\right)\) and let \(\ell_1, \ell_2\) satisfy \((3.11)\) First notice that
\[
\eta' = \frac{2}{1+\alpha} (\eta - \alpha/2) > \eta \iff \eta > \frac{\alpha}{1-\alpha}.
\]
Next, the sets \(J_{p,[\ell_1]}\) and \(J_{p,[\ell_2]}\) are disjoint if and only if for all \(j_k \in \{1, \ldots, \ell_k - 1\}\) with \(\gcd (j_k, \ell_k) = 1\), for \(k = 1, 2\), we have
\[
\frac{p^\delta (\ell_1)}{\ell_1} + \frac{p^\delta (\ell_2)}{\ell_2} \leq 2\pi \left| \frac{j_1}{\ell_1} - \frac{j_2}{\ell_2} \right| = 2\pi \left| \frac{j_1 \ell_2 - j_2 \ell_1}{\ell_1 \ell_2} \right|
\]
since \(j_1 \ell_2 - j_2 \ell_1\) is a positive integer, it's enough to verify
\[
\frac{p^\delta (\ell_1)}{\ell_1} + \frac{p^\delta (\ell_2)}{\ell_2} \leq 2\pi \left| \frac{j_1}{\ell_1} - \frac{j_2}{\ell_2} \right| = 2\pi \left| \frac{j_1 \ell_2 - j_2 \ell_1}{\ell_1 \ell_2} \right|
\]
By plugging the assumption \((3.11)\), we get that
\[
c_0 P^{(1+1/\alpha)\ell_2^{1+1/\alpha} + \ell_1^{1+1/\alpha}} \leq c_0 P^{(1+1/\alpha)\ell_2^{1+1/\alpha} \leq c_0},
\]
and the last inequality holds true whenever
\[
\eta' (1+1/\alpha) + 1 \leq 2\eta/\alpha \iff \eta' \leq \frac{2}{1+\alpha} (\eta - \alpha/2).
\]
\[\square\]

As a corollary, we can break the range \(A\) into finitely many sub-ranges, so that within every sub-range the sets \(J_{p,[\ell]}\) are pairwise disjoint.

**Corollary 10.** Let \(\alpha \in (0, 1/2)\). Then there exists a positive integer \(d = d(\alpha)\) and a sequence \(\eta_1 < \eta_2 < \ldots < \eta_d\) satisfying
1. \(P^{\eta_i} < c_0 p\) for all primes \(p > N\)
2. \(\eta_d > 1/\alpha\)
and for every \(i \in \{1, \ldots, d-1\}\) and every \(P^{\eta_i} \leq \ell_1 < \ell_2 \leq P^{\eta_{i+1}}\) so that \(p \nmid \ell_1, \ell_2\), the sets \(J_{p,[\ell_1]}\) and \(J_{p,[\ell_2]}\) are disjoint.

**Proof.** Fix \(\alpha \in (0, 1/2)\). We prove Corollary \((10)\) by iterating Lemma \((9)\) setting
\[
\eta_{i+1} = \frac{2}{1+\alpha} (\eta_i - \alpha/2) \text{ for } i \geq 1
\]
once comdpute
\[
\eta_{i+1} = \left(\frac{2}{1+\alpha}\right)^i \left(\eta_i - \frac{\alpha}{1-\alpha}\right) + \frac{\alpha}{1-\alpha}, \quad i \geq 1.
\]
Taking $\eta_1 \in \left( \frac{\alpha}{1-\alpha}, 1 \right)$ we get as a consequence that the sequence $\{\eta_i\}$ is monotonically increasing and clearly property (1) is satisfied for all primes $p > N$. Since $\{\eta_i\}$ grows, roughly, as a geometric series we can deduce there exists $d$ such that

$$\eta_d > \frac{1}{\alpha}.$$

\[\square\]

In order to get an estimate on the contribution of the sets $J_{p, |\ell|}$ when $\ell$ varies through $A$, we write

$$A = \bigcup_{i=1}^{d-1} A_{p,i}$$

where

$$A_{p,i} = \{ [p^n] \leq \ell < [p^{n+1}] \mid p \nmid \ell \}.$$

By Corollary 10

$$\int \bigcup_{\ell \in A_{p,i}} |Q_{N_p}(\tau)|^2 \, d\tau \leq \int \bigcup_{\ell \in A_{p,i}} |Q_{N_p}(\tau)|^2 \, d\tau = 1$$

Taking into consideration the change of variables (3.9) we deduce

$$\sum_{\ell \in A_{J_p, |\ell|}} |Q_{N_p}(pt)|^2 \, dt \leq \frac{1}{p} \sum_{i=1}^{d-1} \int \bigcup_{\ell \in A_{p,i}} |Q_{N_p}(\tau)|^2 \, d\tau \leq \frac{d-1}{p} = o(1).$$

When $\alpha \in [1/2, 1)$ we require the following estimate on the counting function $M_\beta$.

**Lemma 11.** Let $\rho \in (0, 1)$. Then there exists $C > 0$ such that

$$M_\rho(x, N) \leq CN^{1-\rho}, \quad x \in [0, 1].$$

**Proof.** Fix $x \in [0, 1]$. Note that for any two fractions $\frac{m_1}{n_1}, \frac{m_2}{n_2}$ where $2^{-k}N \leq n_j \leq 2^{-k+1}N$, $1 \leq m_j < n_j$, $\gcd(m_j, n_j) = 1$ for $j = 1, 2$ and $k \geq 1$ we have

$$\frac{|m_1 - m_2|}{n_1 n_2} \geq \frac{2k}{N^2}.$$

Hence the maximal number of pairs $(m, n)$ with $2^{-k}N \leq n \leq 2^{-k+1}N$ and which satisfy (3.8) is at most

$$\frac{2 \cdot 2^{k(1+\rho)}}{N^{1+\rho}} = 2 \cdot 2^{k(1+\rho)-2} N^{1-\rho}.$$

When summing over all dyadic intervals the lemma follows. \[\square\]

Going back to the sum from (3.9)

$$\frac{1}{p} \sum_{\ell \in A_{J_p, |\ell|}} |Q_{N_p}(\tau)|^2 \, d\tau,$$

we wish to apply the estimate from Lemma 11. Since $\tau \in J_{p, |\ell|}$ if and only if there exists $1 \leq j < \ell$ with $\gcd(j, \ell) = 1$ such that

$$\left| \frac{\tau}{2\pi} - j \right| < \frac{p\delta(\ell)}{\ell} \leq \frac{c_0 p}{\ell^{1+1/\alpha}} \leq \frac{\ell}{\ell^{1+1/\alpha}} = \frac{1}{\ell^{1+1/\alpha-1}}$$

we have

$$\sum_{\ell \in A_{J_p, |\ell|}} |Q_{N_p}(\tau)|^2 \, d\tau \leq 1$$

as claimed.
and that last inequality holds for all \( \ell \in A \). We conclude

\[
\frac{1}{p} \sum_{\ell \in A} \int_{Q_N_p} |Q_{N_p}(\tau)|^2 d\tau \leq \frac{1}{p} \int_{T} |Q_{N_p}(\tau)|^2 M_{1/\alpha-1} \left( \frac{\tau}{2\pi} N_p \right) d\tau \leq \frac{1}{p} \sum_{\ell \in A} \int_{Q_N_p} |Q_{N_p}(\tau)|^2 d\tau \leq \frac{1}{p} CN_p^{1-(1/\alpha-1)} = \frac{1}{p} CN_p^{2-1/\alpha}
\]

(3.13)

Observe that

\[
\frac{1}{p} CN_p^{2-1/\alpha} = C p^{1/\alpha(2-1/\alpha)-1} = o(1) \iff 0 < \alpha^2 - 2\alpha + 1 = (\alpha - 1)^2
\]

which hold for \( \alpha \in [1/2, 1) \).

Remark. It is evident that the bound attained in Lemma 11 can be used when \( \alpha \in (0, 1/2) \), but what we have actually proved for this case is that the counting function is bounded in uniformly bounded both in \( N \) and in \( x \).

For \( \ell \geq N_p \) we write

\[
\int_{J_{[\ell]}} |Q_{N_p}(pt)|^2 dt \leq \int_{J_{[\ell]}} |Q_{N_p}(pt)|^2 dt = \sum_{k,k' = 1}^{N_p} a(k) a(k') \int_{T} f_{\ell}(t) e^{-ip(k'-k)t} dt = |I_{[\ell]}| \sum_{k = 1}^{N_p} |a(k)|^2 + \sum_{k,k' = 1}^{N_p} a(k) a(k') \hat{f}_{\ell}(p(k' - k)) = |I_{[\ell]}| + \sum_{k,k' = 1}^{N_p} a(k) a(k') \hat{f}_{\ell}(p(k' - k)).
\]

(3.14)

Notice that if \( \ell \geq pN_p \), in the second term of (3.14) \( k, k' \in \{1, \ldots, N_p\} \) and \( k \neq k' \), hence

\[
p(k' - k) \in \{ p, 2p, \ldots, (N_p - 1)p \},
\]

which, by (3.15), implies that \( \hat{f}_{\ell}(p(k' - k)) = 0 \) whenever \( \ell \geq pN_p \) and so

\[
\sum_{\ell = pN_p}^{\infty} \int_{J_{[\ell]}} |Q_{N_p}(pt)|^2 \frac{dt}{2\pi} = \sum_{\ell = pN_p}^{\infty} |I_{[\ell]}|.
\]

(3.15)

This also applies when \( N_p < \ell < pN_p \) and \( p \nmid \ell \). Indeed, assuming without loss of generality that \( k' > k \), we have \( \hat{f}_{\ell}(p(k' - k)) \neq 0 \) only if

\[
m\ell = p(k' - k)
\]

with some \( m \in \mathbb{N} \). Since \( p \) does not divide \( \ell \) it must divide \( m \), in which case \( m = pm' \) with some \( m' \in \mathbb{N} \), and

\[
m'\ell = k' - k.
\]

But this is not possible since \( m'\ell \geq \ell > N_p \) while \( k' - k < N_p \). It follows that

\[
\sum_{\ell = N_p + 1}^{pN_p - 1} \int_{J_{[\ell]}} |Q_{N_p}(pt)|^2 \frac{dt}{2\pi} = \sum_{\ell = N_p + 1}^{pN_p - 1} |I_{[\ell]}|.
\]

(3.16)
For \( p \leq \ell \leq pN_p \) and \( p|\ell \), applying the estimate

\[
(3.17) \quad \int \left| \sum_{k=1}^{N} a(k) e^{i\lambda k t} \right|^2 dt \leq |E| \left( \sum_{k=1}^{N} |a(k)| \right)^2 \leq |E| |N| \sum_{k=1}^{N} |a(k)|^2,
\]

which holds for any measurable set \( E \), any \( N \in \mathbb{N} \), sequence of scalars \( \{a(k)\}_{k=1}^{N} \) and sequence of real numbers \( \{\lambda_k\}_{k=1}^{N} \). We get

\[
\sum_{p|\ell}^{\ell=p} |Q_{N_p}(pt)|^2 dt = \sum_{j=1}^{N_p} \int |Q_{N_p}(pt)|^2 dt \leq N_p \sum_{j=1}^{N_p} |I_{|jp|}| = N_p \sum_{j=1}^{N_p} \delta(jp) = c_0 N_p \sum_{j=1}^{N_p} \frac{1}{(jp)^{\gamma/a}} < c_0 \frac{N_p}{p^{\gamma/a}} \leq c_0.
\]

(3.18)

We gather all our estimates (3.7), (3.12), (3.13), (3.15), (3.16) and (3.18), which hold for every prime \( p \geq \mathcal{N} \), we finally have

\[
\int_{S_{\alpha}} \left| Q_{N_p}(pt) \right|^2 dt \geq 1 - \sum_{\ell} \int \left| Q_{N_p}(pt) \right|^2 dt \geq 1 - \left( \sum_{\ell} |I_{|\ell|}| + 3c_0 + o(1) \right) \geq 1 - (\varepsilon + 3c_0 + o(1))
\]

hence given that we have chosen \( \varepsilon \) and \( c_0 \) to be small enough and \( \mathcal{N} \) large enough, one can find some positive constant \( \gamma \) such that \( E(\{p, 2p, \ldots, N_p, p\}) \) is a Riesz sequence in \( L^2(S_{\alpha}) \) with lower Riesz bound \( \gamma \), for all primes \( p > \mathcal{N} \). This completes the proof of Theorem 3.

4. The multiplicity function

In this section we prove a structural sufficient condition to be a member of the class \( \mathcal{A}(\alpha) \).

Proof of Theorem 5, the case \( \alpha > 0 \). Fix \( \alpha \in (0, 1], c, \delta \in (0, 1) \) and let \( \ell \in \mathbb{N} \) a value for which (2.1) holds. Notice that

\[
\int_{S} \left| Q_{N}(\ell t) \right|^2 dt = \int_{[0, 2\pi]} \left| Q_{N}(\ell t) \right|^2 \nu_{\ell, S}(t) dt = \int_{T} \left| Q_{N}(\tau) \right|^2 \nu_{\ell, S}(\tau/e) \frac{d\tau}{\ell} \geq \delta \int_{\{\nu_{\ell, S}(\tau/e) \geq \delta \ell\}} \left| Q_{N}(\tau) \right|^2 d\tau,
\]

clearly we can write

\[
\int_{\{\nu_{\ell, S}(\tau/e) \geq \delta \ell\}} \left| Q_{N}(\tau) \right|^2 d\tau = 1 - \int_{\{\nu_{\ell, S}(\tau/e) < \delta \ell\}} \left| Q_{N}(\tau) \right|^2 d\tau,
\]

applying the estimate (3.17) we can bound the last term

\[
\int_{\{\nu_{\ell, S}(\tau/e) < \delta \ell\}} \left| Q_{N}(\tau) \right|^2 d\tau \leq N |\{\nu_{\ell, S}(\tau/e) < \delta \ell\}|
\]
By assumption we get

\[
\int_S |Q_N(\ell t)|^2 \, dt \geq \delta (1 - N |\{ \nu_{\ell,S}(\gamma/\ell) < \delta \ell \}|) \geq \delta (1 - c)
\]

\[\square\]

For the class \(A(0)\) we have a complete characterization in terms of the multiplicity function.

**Lemma 12.** Given \(\ell \in \mathbb{N}\), the exponential system \(E(\ell \mathbb{Z})\) is a Riesz sequence in \(L^2(S)\) if and only if

\[
|\{ t \in \mathbb{T} \mid \nu_{\ell,S}(t/\ell) = 0 \}| = 0.
\]

**Proof.** Suppose \(|\{ \nu_{\ell,S}(t/\ell) = 0 \}| = 0\) for some \(\ell \in \mathbb{N}\). This implies that for almost every point \(t \in [0, 2\pi]\) one can find \(j \in \{0, \ldots, \ell - 1\}\) such that

\[
t + \frac{2\pi j}{\ell} \in S.
\]

Now, take any positive integer \(N\) and consider \(Q_N(\ell t)\) which is a \(2\pi\) periodic function and so

\[
1 = \int_{\mathbb{T}} |Q_N(\ell t)| \frac{dt}{2\pi} = \ell \int_{[0,2\pi/\ell]} |Q_N(\ell t)|^2 \, dt
\]

By (4.2) we have

\[
\int_S |Q_N(\ell t)|^2 \, dt \geq \int_{[0,2\pi/\ell]} |Q_N(\ell t)|^2 \, dt = \frac{1}{\ell}
\]

and so \(E(\ell \mathbb{Z})\) is a Riesz sequence in \(L^2(S)\).

For the other direction, suppose \(|\{ \nu_{\ell,S}(t/\ell) = 0 \}| > 0\) for some \(\ell \in \mathbb{N}\). This means that there exists a set \(E_\ell \subset [0, 2\pi]\) of positive measure such that for all \(j \in \{0, \ldots, \ell - 1\}\) we have

\[
E_\ell + \frac{2\pi j}{\ell} \in \mathbb{T}\backslash S.
\]

Let \(\varepsilon > 0\) and find \(N\) such that the polynomial

\[
P(\ell t) = \frac{1}{\sqrt{\ell |E_\ell|}} \sum_{|k| \leq N} \hat{1}_S(k) e^{ikt}\]

satisfies

\[
\int_S |P(\ell t)|^2 \frac{dt}{2\pi} < \varepsilon, \quad \int_{\mathbb{T}} |P(\ell t)|^2 \frac{dt}{2\pi} > \frac{1}{2}
\]

This is possible due to the fact that the \(L^2(\mathbb{T})\) function \(\sum_{k \in \mathbb{Z}} \hat{1}_S(k) e^{ikt}\) vanishes at almost every point of \(S\). Since \(\varepsilon > 0\) is arbitrary we deduce that \(E(\ell \mathbb{Z})\) is not a Riesz sequence in \(L^2(S)\).

\[\square\]
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