TESTING SPHERICAL TRANSITIVITY IN ITERATED WREATH PRODUCTS OF CYCLIC GROUPS

BENJAMIN STEINBERG

Abstract. We give a partial solution to a question of Grigorchuk, Nekrashevych, Sushchanskii and Šunič by giving an algorithm to test whether a finite state element of an infinite iterated (permutational) wreath product \( \hat{G} = \mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}/k\mathbb{Z} \wr \mathbb{Z}/k\mathbb{Z} \wr \cdots \) of cyclic groups of order \( n \) acts spherically transitively. We can also decide whether two finite state spherically transitive elements of \( \hat{G} \) are conjugate. For general infinite iterated wreath products, an algorithm is presented to determine whether two finite state automorphisms have the same image in the abelianization.

1. Introduction and main results

The purpose of this note is to offer a partial solution to a question of Grigorchuk, Nekrashevych, Sushchanskii and Šunič [4, 5]. Let \( T_k \) be the rooted regular \( k \)-ary tree. We view it as the Cayley graph of the free monoid \( A_k^* \), where \( A_k = \{0, \ldots, k-1\} \) is the standard alphabet of size \( k \). In particular, we identify vertices with words. It is well known that \( \text{Aut}(T_k) \) is a profinite group. In fact, there is a permutational wreath product decomposition \( (\text{Aut}(T_k), T_k) = (S_k, A_k) \wr (\text{Aut}(T_k), T_k) \wr \cdots \) [1, 2, 4]. Thus \( \text{Aut}(T_k) = (S_k, A_k) \wr (S_k, A_k) \wr \cdots \). For more on this group see [1, 2, 4, 5, 7].

An element \( f \in \text{Aut}(T_k) \) is said to be spherically transitive if, for each \( n \), \( \langle f \rangle \) acts transitively on the set of vertices at distance \( n \) from the root, i.e. transitively on the set of words of length \( n \) [1, 2, 4, 5, 7]. This is equivalent to topological transitivity and ergodicity of the action on the boundary \( \partial T_k \) [4].

If \( f \in \text{Aut}(T_k) \) has wreath product decomposition \( \lambda_f(f_0, \ldots, f_{k-1}) \), then \( f_i \) is called the section of \( f \) at \( i \in A_k \). We shall use the notation \( \lambda_f \) throughout for the element of \( S_k \) associated to \( f \). One can the define inductively, for any word \( w \in A_k^* \), the section \( f_w \) by the formula \( f_w = (f_u)_a \) where \( a \in A_k \) and \( u \in A_k^* \). Of course, \( f_\varepsilon = f \), where \( \varepsilon \) is the empty word. One then has the formula \( f(uv) = f(u)f_v \) for any words \( u, w \in A_k^* \). An element \( f \in \text{Aut}(T_k) \) is said to be finite state if it has only finitely many distinct sections. This is the same as saying that \( f \) can be computed by a finite state automaton.

Date: Version of March 18, 2022.

Key words and phrases. Automata, spherical transitivity, iterated wreath products, rooted trees, rational power series.

The author acknowledges the support of NSERC.
A finite state automaton over an alphabet $A$ is a 4-tuple $A = (Q, A, \delta, \lambda)$ where $Q$ is a finite set of states, $\delta : Q \times A \to A$ is the transition function and $\lambda : Q \times A \to A$ is the output function. We set $q_a = \delta(q, a)$ and $q(a) = \lambda(q, a)$ for $q \in Q, a \in A$. We extend this to words by the formulas:

$$
q_{au} = (q_a)_u \quad (1)
$$

$$
q(au) = q(a)q_a(u) \quad (2)
$$

So each state $q \in A$ gives rise to a function, via $(2)$, from $A^* \to A^*$ (in fact an endomorphism of the rooted Cayley tree of $A^*$), also denoted by $q$. An automaton with a distinguished state is called an initial automaton.

Automata are usually represented by Moore diagrams. The Moore diagram for $A$ is a directed graph with vertex set $Q$. The edges are of the form $q_a| \longrightarrow q_a$. Figure 1 gives the Moore diagram for a certain two-state automaton studied by Grigorchuk and Žuk [6].

![Moore diagram for the lamplighter automaton](image)

Figure 1. Moore diagram for the lamplighter automaton

It is sometimes convenient to define, for $q \in Q$, the state function $\lambda_q : A \to A$ given by

$$
\lambda_q(a) = q(a) = \lambda(q, a)
$$

If, for each $q \in Q$, the state function $\lambda_q$ is a permutation, that is belongs to $S_A$, then one can easily verify that each state $q$ computes a permutation of $A^*$ [4, 7]. We call such an automaton invertible. In particular, if the alphabet of the invertible automaton is $A_k$ and $q$ is a state, then the function $q$ belongs to $\text{Aut}(T_k) = S_k \wr \text{Aut}(T_k)$. The wreath product coordinates of $q$ are:

$$
q = \lambda_q(q_0, \ldots, q_{k-1}) \quad (3)
$$

and so our two uses of the notations $\lambda_q$ and $q_i$ are consistent. For instance, the automaton from Figure 1 is described in wreath product coordinates by $a = (a, b), b = (01)(a, b)$. More generally, if $w \in A_k^*$, then the section of $q$ at $w$ is exactly the state $q_w$ and in particular the transformation $q$ is finite state. One can show [4, 7] that the inverse of $q$ is given by the finite state automaton obtained by switching the two sides of the labels of the Moore diagram and choosing as the initial state the state corresponding to $q$. If $A$ is an invertible automaton, then $G(A)$ denotes the group generated by the

\[G(A) = \langle \tau_{q_0}, \ldots, \tau_{q_{k-1}} \rangle\]
states of $A$. Such groups are called automaton groups and constitute the main examples of finitely generated self-similar groups \[7\]. For instance the group generated by the states of the automaton in Figure 1 is the lamplighter group $\bigoplus \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}$\[4, 6, 10\].

If $f \in \text{Aut}(T_k)$ is finite state, then it can be computed by the initial automaton whose state set is $Q = \{ f_w \mid w \in A^* \}$ (note: this set is finite by assumption). The transition and output functions are given by $\delta(f_w, a) = f_{wa}$ and $\lambda(f_w, a) = f_w(a)$. The initial state is $f_\varepsilon = f$. We remark that the composition of finite state transformations is also finite state \[3, 4, 7\] and so the collection of invertible finite state maps is a subgroup of $\text{Aut}(T_k)$.

If $H$ is a profinite group, we denote by $[H, H]$ the closure of the commutator subgroup of $H$. The abelianization $H/[H, H]$ of $H$ shall be denoted $H_{ab}$ and is again a profinite group. Let $(G, A_k)$ be a transitive permutation group. Then the infinite permutational wreath product

$$\hat{G} = \wr \infty(G, A_k) = (G, A_k) \wr (G, A_k) \wr \cdots \quad (4)$$

is a closed subgroup of $\text{Aut}(T_k)$. Moreover, it acts spherically transitively on $T_k$ \[3\]. The abelianization $\hat{G}_{ab}$ is well known to be isomorphic to the infinite direct product $G_{ab} \times G_{ab} \times \cdots$ \[2\] Chapter 4, Proposition 4.3]. To describe the map, we think about $\hat{G}_{ab}$ in a different way. Since $G_{ab}$ is a finite abelian group, it is a finite direct product of cyclic groups of prime power order in a unique way. Hence we can view it as the additive group of a finite commutative ring via this decomposition. In particular, if $G_{ab}$ is cyclic of prime order $p$, we view it as the additive group of the field of $p$ elements. We can then identify $\hat{G}_{ab}$ with the additive group of the ring of formal power series $G_{ab}[t]$ over $G_{ab}$ in a single variable $t$. If $s \in G_{ab}[t]$, we use the notation $\langle s, t^n \rangle$ to denote the coefficient of $t^n$ in $s$. The abelianization map, with this notation, is given by \[2\]:

$$\langle g[\hat{G}, \hat{G}], t^n \rangle = \sum_{|w|=n} \lambda_{gw}[G, G] \quad (5)$$

The importance of the abelianization map is reflected in the following theorem \[2\] Chapter 4, Propositions (4.6) and (4.7]).

**Theorem 1** \[2\]. Let $\hat{G} = \infty(G/k\mathbb{Z}, A_k)$. Then:

1. an element $g \in \hat{G}$ is spherically transitive if and only if its abelianization $g[\hat{G}, \hat{G}] \in \mathbb{Z}/k\mathbb{Z}[t]$ satisfies $\langle g[\hat{G}, \hat{G}], t^n \rangle \in \mathbb{Z}/k\mathbb{Z}^\times$, for all $n \geq 0$;
2. two spherically transitive elements $f, g \in \hat{G}$ are conjugate if and only if they have the same image in $\hat{G}_{ab} = \mathbb{Z}/k\mathbb{Z}[t]$.

We sketch a proof of the first part of the theorem. The proof goes by induction on the levels of the tree and we merely illustrate how the inductive step works. The key point is that $\langle g \rangle$ acts transitively on $A_k^n$ if and only if it acts transitively on $A_k^{n-1}$ and, for each word $u \in A_k^{n-1}$, the stabilizer of $u$
acts transitively on $uA_k$. Now if we assume that $g$ acts as a $k^{n-1}$-cycle $\sigma$ on $A_k^{n-1}$, then $g^{k^{n-1}}$ generates the stabilizer of every word in $A_k^{n-1}$. Using the iterated wreath product decomposition, we can write $g = \sigma(g_{w_1}, \ldots, g_{w_{kn}})$ where $A_k^n = \{w_1, \ldots, w_{kn}\}$. A straightforward calculation then shows that $g^{k^{n-1}} = (h_1, \ldots, h_{kn})$ where $h_i = g_{w_{i-1}}g_{w_{i-2}}\cdots g_{w_1}g_{w_{kn}}g_{w_{kn-1}}\cdots g_{w_i}$. In particular, $\lambda_{h_i} = \sum_{|w|=n} \lambda_g = \langle g(\hat{G}, \hat{G}), t^n \rangle$, for all $i$. It follows that $g^{k^{n-1}}$ acts transitively on $uA_k$ for all $u \in A_k^{n-1}$ if and only if $\langle g(\hat{G}, \hat{G}), t^n \rangle \in \mathbb{Z}/k\mathbb{Z}^\times$.

Let us return to the setting where $(G, A_k)$ is a transitive permutation group and let $\hat{G}$ be as in (4). It is easy to see from (3) that if $A = (Q, A_k, \delta, \lambda)$ is a finite state automaton, then $G(A) \leq \hat{G}$ if and only if $\lambda_g \in G$ for all $g \in Q$. We are now in a position to present our results. Our first result provides a partial solution to a problem of Grigorchuk, Nekrashevych, Sushchanskii and Šuník from [4] and [5].

**Theorem 2.** Let $g \in \ell^\infty(\mathbb{Z}/k\mathbb{Z}, A_k)$ be a finite state transformation, given by a finite state initial automaton. Then it is decidable whether $f$ is spherically transitive.

Our second theorem concerns conjugacy of finite state elements.

**Theorem 3.** Let $f, g \in \hat{G} = \ell^\infty(\mathbb{Z}/k\mathbb{Z}, A_k)$ be finite state transformations, given by finite state initial automata. Then it is decidable whether $f$ and $g$ are conjugate in $\hat{G}$.

Theorem 3 can be deduced from Theorem 1 and our next theorem.

**Theorem 4.** Let $(G, A_k)$ be a transitive permutation group and let $\hat{G} = \ell^\infty(G, A_k)$. Let $f, g \in \hat{G}$ be finite state transformations, given by finite state initial automata. Then it is decidable whether $f$ and $g$ are equal in $\hat{G}^{ab}$.

The key idea for proving these results was inspired by Schützenberger’s theory of automata and rational power series [8, 9]. In fact, a biproduct of the proofs is:

**Theorem 5.** Let $(G, A_k)$ be a transitive permutation group and let $\hat{G} = \ell^\infty(G, A_k)$. Let $f \in \hat{G}$ be a finite state transformation. Then $f[\hat{G}, \hat{G}] \in G^{ab}[\langle t \rangle]$ is a rational power series.

2. **Proofs of the theorems**

All the theorems rely on the following simple lemma.

**Lemma 6.** Let $(G, A_k)$ be a transitive permutation group and let $\hat{G}$ be as in (4). Let $g \in \hat{G}$ be computed by an automaton $A$ with state set $\{1, \ldots, n\}$ and initial state 1. Let $A$ be the incidence matrix of $A$ and let $v_A$ be the vector whose entries are given by $(v_A)_i = \lambda_i [G, G]$, $i = 1, \ldots, n$. Then

$$g[\hat{G}, \hat{G}] = \sum_{j=0}^\infty \langle A^j v_A \rangle t^j$$
Proof. As $(A^j)_rs$ counts the number of paths in $A$ of length $j$ from $r$ to $s$:
\[
(A^jv_A)_1 = \sum_{|w|=j} \lambda_w [G,G] = \sum_{|w|=j} \lambda_w [G,G] = \langle g[G,G]_i, t^j \rangle
\]
where the last equality follows from \textbf{(5)}.

\textbf{Proof of Theorem 2} By Theorem 1 it follows that $g$ is spherically transitive if and only if each coefficient of $g[H,G]$ belongs to $\mathbb{Z}/k\mathbb{Z}$ for $j \geq 0$. Since $\mathbb{Z}/k\mathbb{Z}^n$ has $k^n$ elements, $A^jv_1 = A^sv_1$ for some $0 \leq r < s \leq k^n$ and so the above condition is a finite check.

\textbf{Proof of Theorem 3} Let $(G, A_k)$ be a transitive permutation group and let $\hat{G}$ be as in \textbf{1}. Let $A$ and $B$ be initial automata computing $f$ and $g$, respectively. Say that $A$ has $m$ states and $B$ has $n$ states. Let $A$ and $B$ be the respective incidence matrices of $A$ and $B$. Let $v_A$ and $v_B$ be the associated vectors, as per Lemma 6. Consider the matrix

\[
M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}
\]

Let $\{e_1, \ldots, e_{m+n}\}$ be the standard basis of row vectors for $(G^{ab})^{m+n}$ and set $v = \begin{pmatrix} v_A \\ v_B \end{pmatrix}$. Then, applying Lemma 6 we have for $j \geq 0$:

\[
(e_1 - e_{m+1})(M^jv) = (A^jv_A)_1 - (B^jv_B)_1 = \langle f[\hat{G}, \hat{G}], t^j \rangle - \langle g[\hat{G}, \hat{G}], t^j \rangle
\]

Hence $f[\hat{G}, \hat{G}] = g[\hat{G}, \hat{G}]$ if and only if $(e_1 - e_{m+1})(M^jv) = 0$ for all $j \geq 0$. But again, $M^jv = M^sv$ some $0 \leq r < s \leq k^{m+n}$, so we can check this.

If $G^{ab}$ is a finite field, then we can do better. Indeed, since the vectors $v, Mv, \ldots, M^{m+n}v$ in $\mathbb{Z}/k\mathbb{Z}^{m+n}$ must be linearly dependent, it follows that for some $0 \leq i \leq m+n$, $M^jv = c_0v + c_1Mv + \cdots + c_{i-1}M^{i-1}v$. Such a recursion implies that $M^jv$ is a linear combination of $v, Mv, \ldots, M^{m+n-1}v$ for all $j \geq n+m$. Hence $(e_1 - e_{m+1})(M^jv) = 0$ for all $j \geq 0$ if and only if $(e_1 - e_{m+1})(M^jv) = 0$ for $0 \leq j \leq m+n-1$.

\textbf{Remark 7.} The proof of Theorem 4 allows for an alternative algorithm for testing spherical transitivity for $\text{Aut}(T_2)$. By Theorem 1 $g \in \text{Aut}(T_2)$ is spherically transitive if and only if $g[\text{Aut}(T_2), \text{Aut}(T_2)] = \sum_{n=0}^{\infty} t^n$, and all spherically transitive elements are conjugate. The so-called \textit{odometer} $a = (01)(1,a)$ is one such spherically transitive element and it has two distinct sections, that is, it can be computed by a two-state automaton. It follows from the proof of Theorem 4 that if $g \in \text{Aut}(T_2)$ is computed by an $n$-state initial automaton with incidence matrix $A$, then one needs only to verify $(A^jv_A)_1 \neq 0$ for $0 \leq j \leq n+1$.

\textbf{Proof of Theorem 5} From Lemma 4 that we have $g[\hat{G}, \hat{G}] = ((I-At)^{-1}v_A)_1$. Since

\[
(I-At)^{-1} = \frac{1}{\det(I-At)} \text{Adj}(I-At)
\]
and the entries of the adjoint $\text{Adj}(I - At)$ are polynomials in $t$, while $\det(I - At)$ is a polynomial in $t$, it follows that the entries of $(I - At)^{-1}$ are rational power series in $t$. Since $((I - At)^{-1}v_A)_1$ is a linear combination of entries of $(I - At)^{-1}$, it follows that $g[\hat{G}, \hat{G}]$ is a rational power series. □

ACKNOWLEDGMENTS

We would like to thank Zoran Šunić for some helpful comments and his careful reading of an earlier draft of this paper.

REFERENCES

1. L. Bartholdi, R. I. Grigorchuk and Z. Šunić, Branch groups in: “Handbook of Algebra”, Vol. 3, 989–1112, North-Holland, Amsterdam, 2003.
2. H. Bass, M. V. Otero-Espinar, D. Rockmore and C. Tresser, “Cyclic Renormalization and Automorphism Groups of Rooted Trees”, Lecture Notes in Mathematics, 1621. Springer-Verlag, Berlin, 1996.
3. S. Eilenberg, “Automata, Languages and Machines”, Academic Press, New York, Vol. A, 1974; Vol. B, 1976.
4. R. I. Grigorchuk, V. V. Nekrashevich and V. I. Sushchanskiii, Automata, dynamical systems, and groups, Tr. Mat. Inst. Steklova 231 (2000), 134–214. English translation in: R. I. Grigorchuk, (ed.), “Dynamical systems, automata, and infinite groups.” Proc. Steklov Inst. Math. 231 (2000), 128–203.
5. R. I. Grigorchik and Z. Šunić, On self-similarity and branching in group theory , to appear in London Mathematical Society Lecture Note Series.
6. R. I. Grigorchuk and A. Žuk, The lamplighter group as a group generated by a 2-state automaton, and its spectrum, Geom. Dedicata 87 (2001), 209–244.
7. V. Nekrashevych, “Self-similar groups,” Mathematical Surveys and Monographs, 117. American Mathematical Society, Providence, RI, 2005.
8. M. P. Schützenberger, On the definition of a family of automata, Information and Control 4, 245–270.
9. M. P. Schützenberger, On a theorem of R. Jungen, Proc. Amer. Math Soc. 13, 885–889.
10. P. V. Silva and B. Steinberg, On a class of automata groups generalizing lamplighter groups, Internat. J. Algebra Comput. 15 (2005), 1213–1234.

School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, Ontario K1S 5B6, Canada

E-mail address: bsteinbg@math.carleton.ca