Non-Hermitian heat engine with all-quantum-adiabatic-process cycle

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Abstract

As a quantum device, a quantum heat engine (QHE) is described by a Hermitian Hamiltonian. However, since it is an open system, reservoirs must be imposed phenomenologically without any description in the context of quantum mechanics. A non-Hermitian system is expected to describe an open system that exchanges energy and particles with external reservoirs. Correspondingly, such an exchange can be adiabatic in the context of quantum mechanics. We first propose a non-Hermitian QHE by a concrete simple two-level system, which is an \( S = 1/2 \) spin in a complex external magnetic field. The non-Hermitian \( \mathcal{PT} \)-symmetric Hamiltonian, as a self-contained one, describes both the working medium and reservoirs. A heat engine cycle is composed of completely quantum adiabatic processes. Surprisingly, the heat efficiency is obtained to be the same as that of the Hermitian quantum Otto cycle. A classical analog of this scheme is also presented. Our finding paves the way for revealing the role of a non-Hermitian Hamiltonian in physics.

Keywords: quantum heat engine, non-Hermitian, thermodynamics

(Some figures may appear in colour only in the online journal)

1. Introduction

Non-Hermitian systems have received intensive studies due to two reasons: it can possess full real eigenvalues [1], which is fundamental in the quantum world, and a non-Hermitian potential breaks the probability conservation, which is supposed to characterize the source or drain of particles and energy. The connection between the imaginary potentials and the environment has been investigated in discrete systems [2]. A non-Hermitian system could be employed to describe an open system that exchanges energy and particles with external reservoirs. An alternative way to deeply understand the exact meaning of a non-Hermitian
Hamiltonian in physics is to find out which functions are similar to some phenomena in practice. An exemplified example is that an imaginary potential can be the function of a laser or anti-laser [3, 4] when spectral singularity is reached [5].

The topic of the quantum heat engine (QHE) [6–13] has attracted a lot of interest since it was first proposed by Scovil and Schultz–Dubois [14]. A QHE converts heat into useful mechanical work from quantum working mediums, such as multilevel systems [15], harmonic oscillator systems [16–19], spins or coupled spins [20], optomechanical systems [21], relativistic particles [22], and so on. Many investigations have been conducted to explore various possible improvements so that QHE efficiency can surpass that of a classical Carnot heat engine, and also extract work from a single heat bath [23]. Due to the quantum properties of the working medium, or the effects of the quantum heat bath, some unusual and exotic phenomena manifested when considering and using a squeezed reservoir [24, 25], quantum coherence [23], or coupled spins [26–29]. Moreover, via constructing a QHE that is a two-level quantum system and undergoes quantum adiabatic processes and energy exchanges with heat baths at different stages in a work cycle, some important aspects of the second law of thermodynamics have been clarified by Kieu [30]. Very recently, many other investigations have been conducted about the Carnot statement of the second law of thermodynamics and the quantum Jarzynski equality in quantum systems described by pseudo-Hermitian Hamiltonians [31].

In contrast to those quantum devices, external reservoirs are inevitable. QHEs considered in the literature are mainly described by a Hermitian Hamiltonian and imposed reservoirs. With the proposal and development of theoretical explorations [32–34], non-Hermitian quantum theories arise as effective descriptions of certain open quantum systems, and therefore induce an effective non-Hermitian QHE in the presence of absorption and gain. It is natural to establish a non-Hermitian description for QHEs.

In this paper, we employ a non-Hermitian \( PT \)-symmetric Hamiltonian to study a QHE. It is a two-level system that describes an \( S = 1/2 \) spin in a complex external magnetic field. The non-Hermitian QHE behaves quite differently from a Hermitian one in all approaches, providing an alternative description. Although we do not provide a Hamiltonian to describe the working medium, a full description of working medium and external reservoirs is included. The process of exchanging heat can be adiabatic in the context of quantum mechanics. We will show that such a non-Hermitian system can fully describe a QHE without imposed external reservoirs, which can operate at Otto efficiency for an optimal cycle. To get a clear physical picture of the non-Hermitian engine we construct a classical analog of this scheme, which is also a variable-mass cycle.

2. Model

We consider a non-Hermitian spin system in an external magnetic field, which can be described by the following Hamiltonian

\[
H = \vec{B} \cdot \vec{\sigma},
\]

where \( \vec{B} = (B_x, B_y, i\gamma J_z) \) is a time-dependent complex magnetic field and \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) is Pauli matrices. Taking

\[
B_x = J \cos \phi, \quad B_y = -J \sin \phi,
\]

\[
\cos \theta = i\gamma / \sqrt{1 - \gamma^2},
\]

(2)
we rewrite the Hamiltonian in the form

\[
H = J(t) \sqrt{1 - \gamma^2} \begin{pmatrix} \cos \theta & \sin \theta e^{i\phi(t)} \\ \sin \theta e^{-i\phi(t)} & -\cos \theta \end{pmatrix}
\]

(3)

where \( J(t) \) and \( \phi(t) \) are real functions of time, and \( \theta(\gamma) \) is a complex (real) constant. Obviously, the Hamiltonian \( H \) is \( \mathcal{PT} \)-symmetric, i.e., \([\mathcal{P}, H] = 0 \) and \([\mathcal{T}, H] = 0 \), but \([\mathcal{PT}, H] = 0 \) in the sense of \( \mathcal{T} \mathcal{I} \mathcal{T} = -i \) and \( \mathcal{P} = \sigma_z \). The diagonal elements of the matrix can be regarded as imaginary on-site potentials of a two-site tight-binding model, the existence of which violates the law of conservation of mass and energy. This system is considered an open system with source and drain. The aim of this paper is to establish a full quantum mechanical description of the QHE. We will consider a non-Hermitian two-level quantum engine in which the source and drain are thought of as the channel to the heat bath. To this end, we will seek the solution of the Schrödinger equation of the system.

Considering the adiabatic time evolution of an initial eigenstate of \( H(0) \) under the quantum adiabatic condition, we have

\[
|\Psi_\lambda(t)\rangle = e^{i\Lambda_\lambda} |\psi_\lambda(t)\rangle,
\]

(4)

where \( |\psi_\lambda(t)\rangle \) (\( \lambda = \pm \)) is the instantaneous eigenstate of \( H(t) \),

\[
|\psi_\lambda\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{-i\phi} \end{pmatrix},
|\psi_\lambda\rangle = \begin{pmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} e^{-i\phi} \end{pmatrix},
\]

(5)

with eigenvalue \( \varepsilon_\lambda = \lambda J(t) \sqrt{1 - \gamma^2} \) and \( J(t) > 0 \) (see appendix). Phase \( \Lambda_\lambda \) is the key quantity of this work, which can be expressed as

\[
\Lambda_\lambda = -\lambda \sqrt{1 - \gamma^2} \int_0^t J(t) dt + \frac{1}{2} \left[ \phi(t) - \phi(0) \right] (1 - i\lambda \gamma / \sqrt{1 - \gamma^2}).
\]

(6)

In the rest of the paper, our investigation does not involve a coherent superposition of \( |\psi_\lambda(t)\rangle \). We neglect the real part of \( \Lambda_\lambda \) and take

\[
\Lambda_\lambda = -\frac{i\lambda \gamma}{2} (\phi(t) - \phi(0)) / \sqrt{1 - \gamma^2}.
\]

(7)

We note that \( \Lambda_\lambda \) is path independent and proportional to the difference of \( \phi \). Furthermore, the nonzero imaginary adiabatic phase is quite essential to the thermal process of QHE, since it can vary the amplitude of instantaneous eigenstates [35], leading to changes in population distributions. The non-Hermitian Hamiltonian is rather different from a Hermitian one, as it treats the system as a whole when describing the working medium and reservoirs.

Before dealing with a heat engine cycle, we would like to briefly remark on the time evolution of a non-Hermitian system, which involves both non-unitary evolution and the complex geometrical state phase. The Schrödinger equation is the common basis of both conventional and non-Hermitian systems. A quantum state evolves along the solution of the Schrödinger equation. According to the solution, the Dirac probability is no longer conservative. In this sense, the non-unitary evolution and the complex geometric phase are nothing but the direct and natural results of the Schrödinger equation.
3. All-quantum-adiabatic cycle

Now we investigate a thermodynamic process for an adiabatic passage of the non-Hermitian system. For a traditional (Hermitian) QHE, the temperature of a quantum working medium is introduced based on the thermalisation assumption \[9, 10, 30, 36\], that is, the working medium has the same distribution of occupation probability as the heat bath in thermodynamic equilibrium. Then, for such a small quantum system, the probability distribution is related to a parameter called spectral temperature. In this paper, we still assume such a concept for the non-Hermitian two-level system. Consider an initially prepared mixed state, which has Dirac probability \( P_0 (t = 0) \) on the eigenstate \( |\psi_0 (t = 0)\rangle \) of \( H (0) \) with eigenenergy \( \lambda J_0 \sqrt{1 - \gamma^2} \). The system parameters are taken as \( (J_0, \phi_0) \) initially, where \( J_0 = J (0) \) and \( \phi_0 = \phi (0) \). And the initial populations for the upper and lower levels are \( P_u (0) = P_0 \) and \( P_l (0) = 1 - P_0 \), respectively. We express this mixed state by a density matrix

\[
\rho (0) = \sum_{\lambda = \pm} P_\lambda (0) |\psi_\lambda (0)\rangle \langle \psi_\lambda (0)|,
\]

where eigenstate \( |\psi_\lambda (0)\rangle \) is normalized in the context of the Dirac inner product (see appendix). Here we assume that the Boltzmann distribution still refers to the Dirac probability even for a non-Hermitian system. When a two-level system couples to a heat bath, both systems obey the thermal equilibrium Boltzmann distributions, and then the two-level mixed state has an even temperature. A density matrix would represent a thermal equilibrium state with temperature

\[
T_0 = \frac{2J_0 \sqrt{1 - \gamma^2}}{k_B \ln (p_0^{-1} - 1)},
\]

where \( k_B \) is the Boltzmann constant. In quantum statistics, the definition of temperature does not require the preservation of particle probability. It is only determined by the distribution of particle probability on each level. For a two-level system, one can always find a temperature that depends only on the ratio of the probabilities and the energy difference of the two levels, i.e.,

\[
\ln \frac{P_+}{P_-} = - \frac{\varepsilon_+ - \varepsilon_-}{k_B T},
\]

where \( P_\pm \) and \( \varepsilon_\pm \) are the particle probability and the energy of the two levels. At time \( t \), state \( |\psi_\lambda (0)\rangle \) evolves into state \( |\Psi_\lambda (t)\rangle \) under the time-dependent Hamiltonian \( H (t) \). We note that the initial state does not involve a coherent superposition of \( |\psi_+ (0)\rangle \) and \( |\psi_- (0)\rangle \). Then, the evolved mixed state does not involve a coherent superposition of \( |\Psi_+ (t)\rangle \) and \( |\Psi_- (t)\rangle \). We are only concerned about the time evolution of the pure state \( |\Psi_\lambda (t)\rangle \).

It is an acceptable consensus that the Schrödinger equation is the common basis for both Hermitian and non-Hermitian quantum mechanics. Whether the operator \( H (t) \) is Hermitian or not, an evolved state, vector \( |\Psi_\lambda (t)\rangle \), should be the solution of the Schrödinger equation

\[
i \frac{\partial}{\partial t} |\Psi_\lambda (t)\rangle = H (t) |\Psi_\lambda (t)\rangle,
\]

with the initial condition \( |\Psi_\lambda (0)\rangle = |\psi_\lambda (0)\rangle \). Accordingly, the density matrix of the evolved mixed state becomes

\[
\rho (t) = \sum_{\lambda = \pm} P_\lambda (0) |\Psi_\lambda (t)\rangle \langle \Psi_\lambda (t)|,
\]
where $|\psi_y(t)\rangle$ is probably not Dirac normalized so as to contribute to the population $P_y(t)$. However, it is difficult to obtain an analytical solution for an arbitrary time-dependent operator $H(t)$. Fortunately, when we consider only the time evolution under the quantum adiabatic condition, i.e., there is no transition between the two levels, an approximate solution can be obtained. Then, the density matrix of the evolved mixed state can be expressed as

$$\rho(t) = \sum_{\lambda=\pm} P_{\lambda}(t)|\psi_{\lambda}(t)\rangle \langle \psi_{\lambda}(t)|,$$

where

$$P_{\lambda}(t) = P_{\lambda}(0) \exp[-2\text{Im}(\Lambda_{\lambda})].$$

Thus, the corresponding temperature of the thermal state $\rho(t)$ is

$$T(t) = \frac{2J(t)\sqrt{1-\gamma^2}}{k_B \ln\{|P_0^{-1} - 1|\xi^{-2}\}},$$

where $\xi = \exp\{\gamma [\phi(t) - \phi_0]/\sqrt{1-\gamma^2}\}$ characterizes the amplification or attenuation of the whole probability in the working medium.

Under the quantum adiabatic condition, the solution of equation (4) tells us that any evolution along an arbitrary path in the $J-\phi$ plane is always adiabatic in the context of quantum mechanics, i.e., there is no transition between the two levels. Nevertheless, it does not mean that the occupation probabilities for the levels and their distributions are always invariant due to the contribution of the $\Lambda_{\lambda}$ in equation (7). These are unusual and exotic properties in the non-Hermitian system, contrasting with the Hermitian system. Moreover, we demonstrate that the adiabatic process in our paper for the non-Hermitian Hamiltonian may be different from that in the Hermitian system because the occupation probability for each level is not invariant due to the image part of $\Lambda_{\lambda}$.

However, what both adiabatic processes have in common is that there is no tunneling between each level. Therefore, we could call it ‘generalized adiabatic’ in our non-Hermitian system. Since all processes considered in our work are ‘generalized adiabatic’, we only write it as ‘adiabatic’ for simplification. Now, we consider a loop in the $J-\phi$ plane as a thermal cycle.

According to the first law of thermodynamics, the total heat transferred in the cycle is

$$\Delta Q = \oint c \, dQ = \oint c \, \sum_{\lambda} \delta_{\lambda} \, dP_{\lambda}$$

$$= \frac{2\gamma}{Z} \int \int_{\Lambda} \cosh \left( \ln \frac{1}{\sqrt{P_0}} - 1 - \ln \xi \right) dJ d\phi,$$

where $Z = 2 \cosh \left( \ln \frac{1}{\sqrt{P_0}} - 1 \right)$ is the partition function. Here, $A$ denotes the area enclosed by the loop $c$ in the $J-\phi$ plane. We can see that $\Delta Q$ is nonzero for a non-trivial cycle, since a hyperbolic cosine function is positive definite. Similarly, the net work done in the cycle is

$$\Delta W = \oint c \, dW = \oint c \, \sum_{\lambda} P_{\lambda} \, d\delta_{\lambda} = -\Delta Q,$$

which leads to

$$\Delta U = \Delta Q + \Delta W = 0.$$
Furthermore, a straightforward derivation shows that the entropy variation in the loop is

\[ \Delta S = \oint \frac{1}{T} dQ = \oint \frac{1}{T} \sum_\lambda \delta \lambda d\lambda = 0. \]  

(19)

According to traditional thermodynamics, these indicate that the internal energy \( U \) and entropy \( S \) are state functions, which are only determined by the system parameters \((J, \phi)\). We would like to point out that \( \Delta Q = 0 \) and \( \Delta W = 0 \) in the loop are exclusive for a non-Hermitian system. Taking \( \gamma = 0 \), the system is reduced to a Hermitian one. We can see that \( \Delta Q = \Delta W = 0 \), which indicates that the Hermitian system is trivial as a QHE. In more detail, as aforementioned, the complex parameter represents the effect of reservoirs phenomenologically. Once \( \gamma = 0 \), the working medium is decoupled from reservoirs. There is no exchange between the working medium and the reservoirs. Then, the heat engine becomes trivial since we do not assume extra reservoirs imposed on the system. This is different from the conventional Hermitian QHE, which is always accompanied by reservoirs. This reflects the key point of our work, replacing the imposed reservoirs with the imaginary parameter \( \gamma \).

In the context of quantum mechanics, processes for an arbitrary loop is spontaneously adiabatic without any extra condition. This may enhance the feasibility of experimental realizations of the present scheme.

4. Cycle with Otto efficiency

So far we have established an alternative description for QHEs via a non-Hermitian Hamiltonian. Remarkably, all thermal processes are adiabatic in a quantum mechanics manner. By this we mean that irrespective of whether there is energy transfer between the heat engine and reservoirs, all processes are adiabatic evolutions driven by a time-dependent Hamiltonian. To understand such quantum processes, it is useful to contrast them to classical thermal processes. In the following, we try to connect such a description to a classical engine, showing that the non-Hermitian QHE is not a toy model.

To this end, we consider a specific cycle consisting of four processes connecting four points, \( A: (J_1, \phi_1) \), \( B: (J_1, \phi_2) \), \( C: (J_2, \phi_2) \), and \( D: (J_2, \phi_1) \). There are two types of processes
Table 1. State parameters for a quantum variable-mass Otto cycle illustrated in figure 1. Here, $T$, $\varepsilon$, and $U$ are in the unit of $\sqrt{1 - \gamma^2}$.

|   | $J$   | $\phi$ | $T$        | $\varepsilon$ | $P_0$ | $P$          | $U$          | $S$          |
|---|-------|--------|------------|----------------|-------|--------------|--------------|--------------|
| A | $J_1$ | $\phi_1$ | $\frac{2\hbar}{k_B \ln(p_0^{-1} - 1)}$ | $J_1$ | $p_0$ | $1 - p_0$ | $J_1(2p_0 - 1)$ | $- k_B p_0 \ln p_0$ | $- k_B (1 - p_0) \ln (1 - p_0)$ |
| B | $J_1$ | $\phi_2$ | $\frac{2\hbar}{k_B \ln[(p_0^{-1} - 1)\xi^{-1}]}$ | $J_1$ | $p_0\xi$ | $(1 - p_0)\xi^{-1}$ | $p_0(\xi + \xi^{-1}) J_1$ | $- k_B p_0 \xi \ln (p_0 \xi)$ | $- k_B (1 - p_0) \xi^{-1} \ln [(1 - p_0) \xi^{-1}]$ |
| C | $J_2$ | $\phi_2$ | $\frac{2\hbar}{k_B \ln[(p_0^{-1} - 1)\xi^{-1}]}$ | $J_2$ | $p_0\xi$ | $(1 - p_0)\xi^{-1}$ | $p_0(\xi + \xi^{-1}) J_2$ | $- k_B p_0 \xi \ln (p_0 \xi)$ | $- k_B (1 - p_0) \xi^{-1} \ln [(1 - p_0) \xi^{-1}]$ |
| D | $J_2$ | $\phi_1$ | $\frac{2\hbar}{k_B \ln(p_0^{-1} - 1)}$ | $J_2$ | $p_0$ | $1 - p_0$ | $J_2(2p_0 - 1)$ | $- k_B p_0 \ln p_0$ | $- k_B (1 - p_0) \ln (1 - p_0)$ |
Table 2. Variances of state parameters related to four processes in a quantum variable-mass Otto cycle illustrated in figure 1. Here $\Delta T$ and $\Delta U$ are in the unit of $\sqrt{1 - \gamma^2}$.

| Processes | $\Delta T$ | $\Delta U$ | $\Delta Q$ | $\Delta W$ | $\Delta S$ |
|-----------|------------|------------|------------|------------|------------|
| A $\rightarrow$ B | $\frac{2J_1 \ln \xi}{k_B \ln (\gamma \rho_0^{\frac{1}{3}} - 1)}$ | $p_0 (\xi + \xi^{-1} - 2) J_1$ | $\Delta U$ | 0 | $- k_B p_0 \ln (\rho_0^{\frac{1}{3}} \xi) - k_B (1 - p_0) \times \ln [(1 - p_0)^{\xi^{-1} - 1}(\xi^{-1})^{1+}]$ |
| B $\rightarrow$ C | $\frac{2(J_2 - J_1)}{k_B \ln (\rho_0^{\frac{1}{3}} - 1)}$ | $p_0 (\xi + \xi^{-1})(J_2 - J_1)$ | $- \xi^{-1}(J_2 - J_1)$ | 0 | $\Delta U$ | 0 |
| C $\rightarrow$ D | $\frac{2J_2 \ln \xi}{k_B \ln (\rho_0^{\frac{1}{3}} - 1)}$ | $- p_0 (\xi + \xi^{-1} - 2) J_2$ | $- (1 - \xi^{-1}) J_2$ | $\Delta U$ | 0 | $k_B p_0 \ln (\rho_0^{\frac{1}{3}} \xi) + k_B (1 - p_0) \times \ln [(1 - p_0)^{\xi^{-1} - 1}(\xi^{-1})^{1+}]$ |
| D $\rightarrow$ A | $\frac{2(J_1 - J_2)}{k_B \ln (\rho_0^{\frac{1}{3}} - 1)}$ | $(J_1 - J_2)(2p_0 - 1)$ | 0 | $\Delta U$ | 0 |
involved: (I) Isospectrum processes, $A \rightarrow B$ and $C \rightarrow D$; and (II) Adiabatic processes, $B \rightarrow C$ and $D \rightarrow A$. A schematic illustration is listed in figures 1(a) and (b). In type I processes, varying $\phi$ can change the populations on the two levels due to the imaginary adiabatic phase $\Lambda_l$, but leave the level structure unchanged. This level structure can be completely characterized by quantity $\varepsilon_+$. In contrast, in type II processes, varying $J$ can change the level structure $\varepsilon_+$, but leave the populations on the two levels unchanged. Based on the adiabatic solution of equation (4), quantities, including the state parameters at each point and their variances in each process, are obtained and listed in tables 1 and 2, respectively. According to the definition in [9, 10, 30], our calculations indicate that such an engine can operate at Otto efficiency

$$\eta = \frac{Q_1 + Q_2}{Q_2} = 1 - \frac{J_2}{J_1}. \quad (20)$$

Such a non-Hermitian QHE does not violate the Carnot efficiency limit, which is of great significance in thermodynamics. This indicates that the non-Hermitian Hamiltonian still obeys some underlying rules in nature.

We note that the probability is not conservative in the cycle, which differs from that in a Hermitian system. We refer to this cycle as a variable-mass Otto cycle. However, we would like to point out that the particle-number conservation still holds in such a cycle\(^1\). Here, we use the term variable-mass to correspond with its classical analog. This has been investigated for a particle-exchange heat engine in [38]. Moreover, to understand such a non-Hermitian cycle and its related issues, we propose a variable-mass cycle for classical ideal gases in the following.

5. Classical analog

To obtain a clear physical picture of the non-Hermitian QHE we construct a classical analog of this scheme. It is a variable-mass cycle and consists of the following four steps, as schematically illustrated in figure 2.

\(^1\) For a non-Hermitian Hamiltonian, conservations of probability and particle are two different concepts and are investigated in [37].
1. Reversible isothermal addition of gas at the high temperature $T_1$. During this step a volume $\Delta V_1$ of the gas with the same temperature $T_1$ and the same density as the working medium is added. It does no work on the surroundings. The temperature of the gas does not change during the process, and thus the addition is isothermal. The gas addition is associated by absorption of heat energy $Q_1$ and increase of entropy $\Delta S = Q_1/T_1$ from the high temperature reservoir.

2. Reversible adiabatic expansion of gas. For this step, the gas is allowed to expand and does work on the surroundings. This expansion causes the gas to cool to the low temperature $T_2$, with entropy remaining unchanged.

3. Reversible isothermal deduction of gas at the low temperature $T_2$. During this step a volume $\Delta V_2$ of the gas with the same temperature $T_2$ and the same density as the working medium is removed. Such an amount of the gas contains heat energy $Q_2$ and entropy $\Delta S = Q_2/T_2$, which flows to the low temperature reservoir.

4. Reversible adiabatic compression of gas. During this step, the surroundings do work on the gas, increasing its internal energy and compressing it, causing the temperature to rise to $T_1$. The gas then returns to its initial state.

For a standard cycle, mass is conserved, i.e., the working medium does not lose or gain mass. The present cycle transfers mass from hot to cool reservoirs, associated with heat transfer. The essence of this cycle is that there is a volume of gas transferred from hot to cool reservoirs rather than only heat transfer. Thus, the heat efficiency is always $1 - T_2/T_1$, which is independent of the mass transferred. Therefore, the proposed variable-mass cycle helps us better understand the non-Hermitian cycle.

6. Summary and discussion

We have studied a time-dependent non-Hermitian $\mathcal{PT}$-symmetric two-level system, which could be regarded as a QHE. A non-Hermitian Hamiltonian turns out to be quite different from a Hermitian one, as it does not solely describe the working medium, but offers a full description including the working medium and external reservoirs. We have shown that any cycle can be implemented by an adiabatic time evolution along a quantum adiabatic passage. As an example, a specific cycle operating at Otto efficiency is presented. Moreover, we have also discussed a classical analog of this scheme, which corresponds to a variable-mass cycle, transferring not only heat but also mass from hot to cool reservoirs. Our main conclusion is that the proposed non-Hermitian Hamiltonian can provide an alternative description for the QHE, leading to a deeper and better understanding of the role of non-Hermitian Hamiltonian systems in physics.

Finally, we would like to point out that there is a long way to go for connecting the current non-Hermitian two-level system to a Hermitian description of QHE using a rigorous mathematical formulation. This is due to the subtle relationship between Hermitian and non-Hermitian Hamiltonians: on one hand, a non-Hermitian Hamiltonian is usually thought to be a better candidate for describing some natural processes in an open system that do not follow the law of conservation of mass and energy. However, on the other hand, it is still a challenge to achieve a perfect connection between a non-Hermitian Hamiltonian and a Hermitian one with a clear physical picture, although many efforts have been dedicated to this topic [2].
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Appendix

The instantaneous eigenstates of $H(t)$ and $H^+(t)$ with eigenvalue $\lambda = \lambda J(t)\sqrt{1 - \gamma^2}$ are

$$|\psi_\lambda\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{-i\phi} \end{pmatrix}, \quad |\psi_-\rangle = \begin{pmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} e^{-i\phi} \end{pmatrix}, \quad (21)$$

and

$$|\eta_\lambda\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}, \quad |\eta_-\rangle = \begin{pmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} e^{i\phi} \end{pmatrix}, \quad (22)$$

respectively, which can construct biorthonormal complete set $\{|\psi_\lambda\rangle, |\eta_\lambda\rangle\}$ ($\lambda = \pm$), i.e.,

$$\langle \eta_\lambda | \psi_\lambda \rangle = \delta_{\lambda \lambda}, \quad \sum_\lambda |\psi_\lambda\rangle \langle \eta_\lambda| = 1. \quad (23)$$

Here, we demonstrate that when computing the density matrix or detecting the probability in experiments, both eigenstates of $H(t)$ must be normalized in the Dirac inner product, i.e., we must take $|\psi_\lambda\rangle \rightarrow |\psi_\lambda\rangle/\sqrt{\Theta}$ with $\Theta = |\cos \frac{\theta}{2}|^2 + |\sin \frac{\theta}{2}|^2$. Now, we extend the adiabatic theorem to the non-Hermitian Hamiltonian and show that such an extension is correct. This extension can be simply applied by replacing the Dirac inner product with the biorthonormal inner product. Then under the quantum adiabatic condition

$$\left| \frac{\langle \eta_\lambda | \frac{dH}{dt} | \psi_\lambda \rangle}{(\varepsilon_+ - \varepsilon_-)^2} \right| = \left| \frac{\dot{\phi}}{4J(t)(1 - \gamma^2)} \right| \ll 1, \quad (24)$$

phase $\Lambda_\lambda$ can be expressed as

$$\Lambda_\lambda = -\int_0^t \varepsilon_\lambda dt + i \int_0^t \langle \eta_\lambda | \frac{\partial}{\partial t} | \psi_\lambda \rangle dt$$

$$= -\lambda \sqrt{1 - \gamma^2} \int_0^t J(t)dt$$

$$+ \frac{1}{2} [\phi(t) - \phi_0] (1 - i\lambda \gamma / \sqrt{1 - \gamma^2}). \quad (25)$$

Similar works have been done in many systems [39–43], and this conclusion can also be obtained by an exact solution of the Schrödinger equation

$$i \frac{\partial}{\partial t} |\Psi\rangle = H |\Psi\rangle, \quad (26)$$
in this concrete example. Here, our $\mathcal{PT}$-symmetric Hamiltonian $H$ could be used to describe an open system, which is different from that used in [44] to describe a closed system. Actually, the propagator of the Hamiltonian $H$

$$U = T\exp\left(-i \int_0^t H(t) \, dt\right) \quad (27)$$

can be obtained as the form

$$
\begin{align*}
U_{11} &= \left[ \cos (\Omega \tau) - \frac{i \Delta}{2\Omega} \sin (\Omega \tau) \right] e^{i \omega \tau / 2}, \\
U_{22} &= \left[ \cos (\Omega \tau) + \frac{i \Delta}{2\Omega} \sin (\Omega \tau) \right] e^{-i \omega \tau / 2}, \\
U_{12} &= -\frac{i}{\Omega} \sin (\Omega \tau) e^{i \omega \tau / 2}, \\
U_{21} &= -\frac{i}{\Omega} \sin (\Omega \tau) e^{-i \omega \tau / 2},
\end{align*}
$$

(28)

when we take $\phi(\tau) = \omega \tau$. Here, $T$ is the time-ordering operator and

$$\tau = \int_0^t J(t) \, dt, \quad \Omega = \frac{1}{2} \sqrt{4 + \Delta^2},$$

(29)

$$\Delta = 2\gamma + \omega.$$  

(30)

The adiabatic condition in equation (24) reduces to

$$\left| \frac{\omega}{1 - \gamma^2} \right| \ll 1,$$  

(31)

under which the Taylor expansion gives

$$- \Omega \tau = - \sqrt{1 - \gamma^2} \left[ 1 + \frac{i \gamma \omega}{1 - \gamma^2} \right] \tau$$

$$\approx - \int_0^t \sqrt{1 - \gamma^2} J(t) \, dt - \frac{i \gamma \omega \tau}{2 \sqrt{1 - \gamma^2}}.$$  

(32)

This solution indicates that the adiabatic process exists under a certain condition, i.e., there is no tunneling between instantaneous eigenstates during the time evolution. Then, the corresponding geometrical phase can be obtained naturally. In [45], dynamics near the exceptional point are considered and it is natural that the adiabatic theorem cannot hold in that situation. The process cannot hold when the system closes to the exceptional point (or quantum phase transition point for breaking $\mathcal{PT}$ symmetry). Furthermore, similar behavior can occur in a Hermitian system. For example, an adiabatic process cannot be achieved near a quantum phase transition point. A simple demonstration is seen with the Landau–Zener formula.

Moreover, in the biorthonormal formalism, compared with right ket $|\Psi\rangle$, which obeys equation (26), the left bra $\langle \Phi|$ follows a different evolution equation
and correspondingly the propagator

\[ U = T \exp \left( -i \int_0^t H^\dagger(t') dt' \right) \]

can be obtained in the form

\[
\begin{align*}
\hat{U}_{11} &= \left[ \cos \left( \Omega^* \tau \right) - i \frac{\Delta^*}{2\Omega^*} \sin \left( \Omega^* \tau \right) \right] e^{i\omega_\tau/2}, \\
\hat{U}_{22} &= \left[ \cos \left( \Omega^* \tau \right) + i \frac{\Delta^*}{2\Omega^*} \sin \left( \Omega^* \tau \right) \right] e^{-i\omega_\tau/2}, \\
\hat{U}_{12} &= -\frac{i}{\Omega^*} \sin \left( \Omega^* \tau \right) e^{i\omega_\tau/2}, \\
\hat{U}_{21} &= -\frac{i}{\Omega^*} \sin \left( \Omega^* \tau \right) e^{-i\omega_\tau/2},
\end{align*}
\]

and then we can prove

\[ \hat{U}^\dagger U = 1. \]

As a result, it is unitary according to the biorthonormal inner product. However, we would like to point out that the complex phase for a non-Hermitian system is a natural result (see references \[39–43\]). The Berry phase and the unitarity of time evolution are compatible with each other.

Next, explicit derivations of equation (16), (17), and (19) are given. Together with \( P_c = p_0 \xi, P_\gamma = (1 - p_0) \xi^{-1} \) (here, \( \xi \) is only the function of \( \phi \)) and \( \delta_\lambda \), we obtain the heat exchange, the work done, and the variation of entropy of any arbitrary path in the \( J - \phi \) plane as follows:

\[
\begin{align*}
\Delta Q &= \oint_c dQ = \oint_c \sum_\lambda \delta_\lambda dP_\lambda \\
&= \sqrt{1 - \gamma^2} \oint_c \left[ \frac{\chi_e}{J} \left( \frac{1}{\sqrt{p_0 - 1 - \ln \xi}} - \frac{1}{\sqrt{p_0 - 1 - \ln \xi}} \right) \right] \\
&= -\frac{2\sqrt{1 - \gamma^2}}{Z} \oint_c \left[ \frac{\chi_e}{J} \left( \sinh \left( \frac{1}{\sqrt{p_0 - 1 - \ln \xi}} \right) - \sinh \left( \frac{1}{\sqrt{p_0 - 1 - \ln \xi}} \right) \right) \right] \\
&= \frac{2\gamma}{Z} \oint_c \left[ \cosh \left( \frac{1}{\sqrt{p_0 - 1 - \ln \xi}} \right) \right] d\phi \\
&= \frac{2\gamma}{Z} \oint \left[ \cosh \left( \frac{1}{\sqrt{p_0 - 1 - \ln \xi}} \right) \right] dJ d\phi.
\end{align*}
\]

(37)
\[ \Delta W = \oint_c dW = \oint_c \sum_{\lambda} P_\lambda d\xi = 0 \]
\[ = \sqrt{1 - \gamma^2} \oint_c \sum_{\lambda} \left[ \lambda e^{-\lambda \left( \ln \sqrt{\frac{1}{P_0} - 1 - \ln \xi} \right)} \right] dJ \]
\[ = \sqrt{1 - \gamma^2} \oint_c \sum_{\lambda} \left[ e^{-\left( \ln \sqrt{\frac{1}{P_0} - 1 - \ln \xi} \right)} - e^{\left( \ln \sqrt{\frac{1}{P_0} - 1 - \ln \xi} \right)} \right] dJ \]
\[ = -2\sqrt{1 - \gamma^2} \oint_c \sum_{\lambda} \sinh \left( \ln \sqrt{\frac{1}{P_0} - 1 - \ln \xi} \right)dJd\phi. \]

Obviously we have
\[ \Delta Q + \Delta W = 0. \] (39)

Similarly, the variation of entropy is
\[ \Delta S = \oint_c \frac{1}{T} dQ = \oint_c \frac{1}{T} \sum_{\lambda} \sum_{\lambda} dP_\lambda \]
\[ = -2k_B \oint_c \sum_{\lambda} \left[ \ln \sqrt{\frac{1}{P_0} - 1 - \ln \xi} \right] dJ \left[ \sinh \left( \ln \sqrt{\frac{1}{P_0} - 1 - \ln \xi} \right) \right] \]
\[ = -2k_B\gamma \oint_c \sum_{\lambda} \left[ \ln \sqrt{\frac{1}{P_0} - 1 - \ln \xi} \right] \cosh \left( \ln \sqrt{\frac{1}{P_0} - 1 - \ln \xi} \right) dJd\phi \]
\[ = 0. \] (40)

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