UV/IR Mixing and Anomalies in Noncommutative Gauge Theories

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Abstract

Using path integral method (Fujikawa’s method) we calculate anomalies in noncommutative gauge theories with fermions in the bi-fundamental and adjoint representations. We find that axial and chiral gauge anomalies coming from non-planar contributions are derived in the low noncommutative momentum limit \( \tilde{p}^\mu (\equiv \theta^{\mu\nu} p_\nu) \to 0 \). The adjoint chiral fermion carries no anomaly in the non-planar sector in \( D = 4k (k = 1, 2, \ldots) \) dimensions. It is naturally shown from the path integral method that anomalies in non-planar sector originate in UV/IR mixing.

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1 Introduction

Gauge theories on noncommutative space (or simply noncommutative gauge theories) are the subject of much recent interest. (See for recent reviews [1, 2] and references therein.) One of the interesting features of noncommutative gauge theories at the quantum level is UV/IR mixing [3]. Although the planar diagrams are essentially the same as those in the corresponding ordinary field theories, the non-planar diagrams can also be seen to exhibit an interesting stringy phenomenon. The planar diagrams control the UV properties, while the non-planar diagrams generally lead to new IR phenomena through the mixing.

Anomalies in noncommutative gauge theories have been discussed by several authors [5]-[20]. There are two kinds of anomalies one of which comes from the planar contributions and the other of which from the non-planar contributions. The anomalies in planar sector can be evaluated by several different methods. Axial anomalies have been calculated by the path integral formulation (Fujikawa’s method) besides the perturbative analysis [3, 9]. Chiral gauge anomalies can also be described using cohomological methods [7, 11]. It is known that these anomalies take the form of the straightforward Moyal deformation in the corresponding anomalies in ordinary gauge theories. The theta (noncommutative) parameter does not explicitly appear in the final formula except in the appearance of the Moyal star product.

The anomalies in non-planar sector have been studied from different points of view [9, 12, 19, 20]. When the chiral fermions of the noncommutative gauge theories are in the bi-fundamental representation and the adjoint representation, there are non-planar contributions from the non-planar diagrams. For the non-vanishing noncommutative momentum \( \hat{p}^\mu \equiv \theta^{\mu\nu} p^\nu \), however, the non-planar triangle diagrams can be expressed in terms of the (modified) Bessel functions and they are UV-finite without any regularizations. Hence, there are no anomalous contributions from the non-planar diagrams [4, 13]. On the other hand, it was shown that axial anomaly in non-planar sector does not vanish. For noncommutative QED with fermions in the fundamental representation, there are two kinds of axial currents in which the order of the product of the fermions differs. One of these currents leads to the anomaly of the non-planar contributions when the noncommutative momentum is very small [12]. Anomalies in non-planar sector in the case of zero noncommutative momentum have been discussed in detail in [20].

These arguments on the anomalies in non-planar sector are based on the perturbative analysis, while the anomalies in planar sector can be evaluated by the path integral method and cohomological approach besides the perturbative analysis. Therefore, it will be natural to consider approaches other than the perturbative analyze in the evaluation of the anomalies in non-planar sector. In this paper we would like to derive axial and chiral gauge anomalies in non-planar sector by path integral method. The path integral method will be found to be suited for the calculation of the anomalies in non-planar sector. The paper is organized as follows. In Sec. 2, we consider a noncommutative gauge theory with fermion in the bi-fundamental
and the adjoint representation and derive the axial anomaly in non-planar sector by the path integral method. In Sec. 3, the path integral method is also applied in deriving the chiral gauge anomaly in a noncommutative chiral gauge theory with chiral fermion in the bi-fundamental and the adjoint representation. Sec. 4 is devoted to conclusions.

2 The axial anomaly for bi-fundamental and adjoint fermion

We first discuss the gauge theories with fermions in the bi-fundamental representation in noncommutative Euclidean space. Let us consider a bi-fundamental Dirac fermion $\psi^i_j(x)$ interacting with a $U(N_A)$ gauge field $A^{i_1}_{\mu} x(x)$ and a $U(N_B)$ gauge field $B^j_{\mu} x(x)$. Here the index $i$ runs from 1 to $N_A$ and $j$ from 1 to $N_B$, respectively. The classical action of this theory on 2n-dimensional noncommutative (Euclidean) space is given by

$$S[\bar{\psi}, \psi, A, B] = \int d^{2n}x \bar{\psi}^i_j(x) \ast (i\mathcal{D}[A, B]) \psi^j_i(x).$$  \hspace{1cm} (2.1)

Here the operator $\mathcal{D}[A, B]$ denotes the Dirac operator whose concrete form is given by,

$$\mathcal{D}[A, B] \psi^{i_1}_{j_1}(x) = \partial \psi^{i_1}_{j_1}(x) + A^{i_1}_{\mu} \gamma^\mu \psi^{i_1}_{j_1}(x) - \gamma^\mu A^j_{\mu} \psi^{i_1}_{j_1}(x) \ast B^{j_2}_{\mu} \psi^{i_1}_{j_1}(x)$$

$$= (\partial \delta^{i_1}_{i_2} \delta^{j_1}_{j_2} + A^{i_1}_{\mu} \gamma^\mu ) \psi^{i_1}_{j_1}(x) \ast \gamma^\mu A^j_{\mu} \psi^{i_1}_{j_1}(x) + \gamma^\mu B^j_{\mu} \psi^{i_1}_{j_1}(x) \ast \gamma^\mu A^j_{\mu} \psi^{i_1}_{j_1}(x),$$ \hspace{1cm} (2.2)

with the notations $(A_{\mu})^* \psi \equiv A_{\mu}^* \psi$ and $(B_{\mu})^* \psi \equiv \psi^* B_{\mu}[8]$. Since we have chosen the gamma matrices $\gamma^\mu$, $\mu = 1, 2, \ldots, 2n$ as Hermitian matrices, the matrix $\gamma^{2n+1} \equiv (-i)^n \prod_{k=1}^{2n} \gamma_{\mu_k}$ remains Hermitian. The symbol $\ast$ stands for the Moyal star product defined as follows,

$$f(x) \ast g(x) = e^{i\theta^{\mu\nu} \frac{\partial}{\partial x} \frac{\partial}{\partial \xi}} f(x + \xi)g(x + \xi) \bigg|_{\xi = \zeta = 0}$$

$$= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{i(p_{\mu} + q_{\nu}) x_{\mu}} \tilde{f}(p) \tilde{g}(q),$$ \hspace{1cm} (2.3)

where $\theta^{\mu\nu} = -\theta^{\nu\mu}$ denotes an antisymmetric real matrix.

We begin with the evaluation of the axial anomalies. We perform a infinitesimal (local) chiral transformation:

$$\delta^{2n+1}_{2n+1} \psi^{i}_j(x) = i 2n+1 \lambda_2(x) \ast \gamma^{2n+1} \psi^{i}_j(x) + i \gamma^{2n+1} \psi^{i}_j(x) \ast \lambda_2(x),$$ \hspace{1cm} (2.4)

$$\delta^{2n+1}_{2n+1} \bar{\psi}^i_j(x) = i \bar{\psi}^i_j(x) \gamma^{2n+1} \ast \lambda_2(x) + i \bar{\psi}^i_j(x) \ast \lambda_2(x) \gamma^{2n+1},$$

where $\lambda_2(x)$ and $\lambda_2(x)$ denote some infinitesimal functions. For the infinitesimal transformation, the action (2.1) changes to

$$\delta^{2n+1}_{2n+1} S = - \int d^{2n}x \left\{ \lambda_2 \ast D^{(A)}_{\mu} j^{(A)}_{2n+1} + \lambda_2 \ast D^{(B)}_{\mu} j^{(B)}_{2n+1} \right\}.$$
where the currents \( j^{\mu(A)}_{2n+1}(= j^{\mu(A)}_{2n+1,i}) \) and \( j^{\mu(B)}_{2n+1}(= j^{\mu(B)}_{2n+1,j}) \) are defined, respectively, by the identities

\[
j^{\mu(A)}_{2n+1}(x) \equiv (\psi_{\beta}^i \gamma_j \bar{\psi}_{\alpha}^j)(\gamma^\mu \gamma_{2n+1})_{\alpha\beta},
\]

\[
j^{\mu(B)}_{2n+1}(x) \equiv (\bar{\psi}_{\alpha}^i \gamma_j \psi_{\beta}^j)(\gamma^\mu \gamma_{2n+1})_{\alpha\beta},
\]

(2.5)

and the covariant derivative of these current are given by

\[
D^{(A)}_{\mu} j^{\mu(A)}_{2n+1}(x) \equiv \partial_{\mu} j^{\mu(A)}_{2n+1}(x) + A_{\mu} i^i_{i_2}(x) * j^{\mu(A)}_{2n+1} i^i_{i_1}(x) - j^{\mu(A)}_{2n+1} i^i_{i_2}(x) * A_{\mu} i^i_{i_1}(x),
\]

\[
D^{(B)}_{\mu} j^{\mu(B)}_{2n+1}(x) \equiv \partial_{\mu} j^{\mu(B)}_{2n+1}(x) + B_{\mu} j^j_{j_2}(x) * j^{\mu(B)}_{2n+1} j^j_{j_1}(x) - j^{\mu(B)}_{2n+1} j^j_{j_2}(x) * B_{\mu} j^j_{j_1}(x).
\]

The partition function with classical action given in the expression (2.1) is defined as

\[
Z[A, B] = \int D\bar{\psi} D\psi \exp \left(-S[\bar{\psi}, \psi, A, B]\right).
\]

In order to compute the change of the path integral measure under the chiral transformation (2.4), we introduce an orthonormal and complete set of eigenfunctions \( \{\varphi_n\} \) of the Dirac operator \( D[A, B] \). The fermions can be expanded in the orthonormal basis of eigenfunctions \( \{\varphi_n\} \) as \( \psi(x) = \sum_n a_n \varphi_n(x) \) and \( \bar{\psi}(x) = \sum_n \bar{b}_n \varphi^\dagger_n(x) \), where the coefficients \( a_n \) and \( \bar{b}_n \) are Grassmann numbers. Under the infinitesimal transformations (2.4), the integration measure of the fermionic fields transform as \( D\bar{\psi} D\psi = J_{axial}[\lambda_A, \lambda_B] D\bar{\psi} D\psi \) with the Jacobian,

\[
J_{axial}[\lambda_A, \lambda_B] = \exp \left(-2i \mathcal{A}_{axial}[\lambda_A, \lambda_B]\right),
\]

(2.6)

where \( \mathcal{A}_{axial} \) includes the sum over the eigenstates \( n \). Since the sum is ill-defined, we must regularize the sum in a gauge invariant way. This is done by introducing a Gaussian damping factor,

\[
\mathcal{A}_{axial}^{reg}[A, B, \lambda_A, \lambda_B] = \lim_{\varepsilon \to 0} \sum_n \int d^2n x \left[ \lambda_A(x) * \gamma_{2n+1} e^{-\varepsilon D[A, B]^2} \varphi_n(x) * \varphi_n^\dagger(x) - \lambda_B(x) * \varphi_n^\dagger(x) * \gamma_{2n+1} e^{-\varepsilon D[A, B]^2} \varphi_n(x) \right].
\]

(2.7)

We now evaluate the regularized sum in Fourier space \( \varphi_n(x) = \int \frac{d^2n k}{(2\pi)^2} e^{ik \cdot x} \varphi_n(k) \). Then we have

\[
\mathcal{A}_{axial}^{reg}[A, B, \lambda_A, \lambda_B] = \lim_{\varepsilon \to 0} \int d^2n x \int \frac{d^2k}{(2\pi)^2} \text{Tr} \left[ \lambda_A(x) * \gamma_{2n+1} e^{-\varepsilon D[A, B]^2} \{i k + D[A, B]\} e^{ik \cdot x} e^{-ik \cdot x} - \lambda_B(x) * e^{-ik \cdot x} * \gamma_{2n+1} e^{-\varepsilon D[A, B]^2} \{i k + D[A, B]\} e^{ik \cdot x} \right],
\]

(2.8)
where $k \cdot x \equiv k_\mu x^\mu$ and the notation “\( \text{Tr} \)” denotes the trace over the $U(N_A)$, $U(N_B)$ and the Dirac matrices. Notice that multiplication by a plane wave $e^{ik \cdot x}$ translates a general function as $e^{-ik \cdot x} \ast f(x) \ast e^{ik \cdot x} = f(x - \tilde{k})$, where $\tilde{k}_\mu \equiv \theta^{\mu \nu} k_\nu$. This exhibits that large momenta will lead to large nonlocality of the theory. Taking this into account and inserting decomposition $\mathcal{D} = D_\mu D_\mu + \frac{1}{2} \gamma^{\mu \nu} \gamma^\nu \left[ D_\mu, D_\nu \right]$ into the expression (2.8), we obtain

$$
\begin{align*}
\mathcal{A}^\text{reg}_{\text{axial}}[A, B, \lambda_A, \lambda_B] &= \lim_{\varepsilon \to 0} \int d^2 k \int \frac{d^n k}{(2\pi)^n} e^{k \cdot k} \\
\text{Tr} \left[ \lambda_A(x) \gamma^{2n+1} \exp \left( -\varepsilon \left\{ 2ik_\mu \left( \partial^\mu + A^\mu(x) \ast - B^\mu(x + \tilde{k}) \right) \right. \right. \\
& \quad \left. \left. + \left( \partial^\mu + A^\mu(x) \ast - B^\mu(x + \tilde{k}) \right)^2 + \frac{1}{2} \gamma^{\mu \nu} \left( F^A_{\mu \nu}(x) - \ast F^B_{\mu \nu}(x + \tilde{k}) \right) \right) \right] \\
& \quad - \lambda_B(x) \gamma^{2n+1} \exp \left( -\varepsilon \left\{ 2ik_\mu \left( \partial^\mu + A^\mu(x - \tilde{k}) \ast - B^\mu(x) \right) \right. \right. \\
& \quad \left. \left. + \left( \partial^\mu + A^\mu(x - \tilde{k}) \ast - B^\mu(x) \right)^2 + \frac{1}{2} \gamma^{\mu \nu} \left( F^A_{\mu \nu}(x - \tilde{k}) - \ast F^B_{\mu \nu}(x) \right) \right) \right] ,
\end{align*}
$$

(2.9)

with $F^A_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and $F^B_{\mu \nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$. We expand the exponential and utilize the trace properties of the Dirac matrices:

$$
\text{Tr} \left( \gamma^{2n+1}(\gamma^\mu_1 \gamma^\mu_2 \cdots \gamma^\mu_{2n}) \right) = (-2i)^n \varepsilon^{\mu_1 \mu_2 \cdots \mu_{2n}},
$$

where $\varepsilon^{\mu_1 \mu_2 \cdots \mu_{2n}}$ is the Levi-Civita tensor. Then only the term of $n$-th order in $F_{\mu \nu}$ remains under the limit $\varepsilon \to 0$. Performing the rescaling $k_\mu \to (1/\sqrt{\varepsilon}) k_\mu$, we have

$$
\begin{align*}
\mathcal{A}^\text{reg}_{\text{axial}}[A, B, \lambda_A, \lambda_B] &= \lim_{\varepsilon \to 0} \int d^2 x \int \frac{d^n k}{(2\pi)^n} e^{k \cdot k} \\
& \quad \{ \lambda_A(x) \ast \text{Tr}_A \text{Tr}_B \left[ (F^A_{\mu_1 \mu_2}(x) \ast - F^B_{\mu_1 \mu_2}(x + \frac{\tilde{k}}{\sqrt{\varepsilon}})) \cdots (F^A_{\mu_{2n-1} \mu_{2n}}(x) \ast - F^B_{\mu_{2n-1} \mu_{2n}}(x + \frac{\tilde{k}}{\sqrt{\varepsilon}})) \right] \\
& \quad - \lambda_B(x) \ast \text{Tr}_A \text{Tr}_B \left[ (F^A_{\mu_1 \mu_2}(x - \frac{\tilde{k}}{\sqrt{\varepsilon}}) \ast - F^B_{\mu_1 \mu_2}(x)) \cdots (F^A_{\mu_{2n-1} \mu_{2n}}(x - \frac{\tilde{k}}{\sqrt{\varepsilon}}) \ast - F^B_{\mu_{2n-1} \mu_{2n}}(x)) \right] \}
\end{align*}
$$

(2.10)

where the notations $\text{Tr}_A$ and $\text{Tr}_B$ denote the traces over the $U(N_A)$ and $U(N_B)$ matrices, respectively.

Before advancing the calculation of axial anomaly in arbitrary dimensions, we will examine the case of two and four dimensions concretely.

**Two dimensions**

First we consider the case of two dimensions. The explicit form of the expression (2.10) is given by

$$
\begin{align*}
\mathcal{A}^\text{reg}_{\text{axial}}[A, B, \lambda_A, \lambda_B] &= \lim_{\varepsilon \to 0} \int d^2 x \int \frac{d^2 k}{(2\pi)^2} e^{k \cdot k} \epsilon^{\mu \nu} \left\{ N_B \lambda_A(x) \ast \text{Tr}_A F^A_{\mu \nu}(x) + N_A \lambda_B(x) \ast \text{Tr}_B F^B_{\mu \nu}(x) \\
& \quad - N_B \lambda_A(x) \ast \text{Tr}_A (F^A_{\mu \nu}(x - \frac{\tilde{k}}{\sqrt{\varepsilon}}) - N_A \lambda_B(x) \ast \text{Tr}_B F^B_{\mu \nu}(x + \frac{\tilde{k}}{\sqrt{\varepsilon}}) \}
\end{align*}
$$

(2.11)
where the coefficients \( N_A(\equiv \text{Tr}_A I_{N_A \times N_A}) \) and \( N_B(\equiv \text{Tr}_B I_{N_B \times N_B}) \) come from the trace of the unit matrices \( I_{N_A \times N_A} \) and \( I_{N_B \times N_B} \), respectively. Performing integration over the momentum \( k_\mu \), we find

\[
\mathcal{A}_{\text{axial}}^{\text{reg}}[A, B, \lambda_A, \lambda_B] = \lim_{\varepsilon \to 0} \int d^2x \frac{1}{4\pi} i \varepsilon^{\mu\nu} \left\{ N_B \lambda_A(x) * \text{Tr}_A F^{A}_{\mu\nu}(x) + N_A \lambda_B(x) * \text{Tr}_B F^{B}_{\mu\nu}(x) - N_B \lambda_A(x) * \text{Tr}_A \tilde{F}^{A}_{\mu\nu}(0) - N_A \lambda_B(x) * \text{Tr}_B \tilde{F}^{B}_{\mu\nu}(0) \right\},
\]

(2.12)

where \( \tilde{F}^{A}_{\mu\nu}(p) \) is the Fourier coefficients of the field strength \( F_{\mu\nu}(x) \) and the notation \( p \circ p = \eta_{\rho\sigma} p^\rho p^\sigma (= \tilde{p}_\mu \tilde{p}^\mu) \) denotes the square of the noncommutative momentum \( \tilde{p} \). It can be regarded that the third and fourth term in the right-hand side of the expression (2.12) come from the non-planar contributions, while the first and the second term come from the planar contributions. The quantity \( p \circ p \) satisfies the condition \( p \circ p < 0 \) for arbitrary non-zero momentum \( p_\mu \) in two dimensions, the expression (2.12) becomes as follows

\[
\mathcal{A}_{\text{axial}}^{\text{reg}}[A, B, \lambda_A, \lambda_B] = \int d^2x \frac{1}{4\pi} i \varepsilon^{\mu\nu} \left\{ N_B \lambda_A(x) * \text{Tr}_A F^{A}_{\mu\nu}(x) + N_A \lambda_B(x) * \text{Tr}_B F^{B}_{\mu\nu}(x) - N_B \lambda_A(x) * \text{Tr}_A \tilde{F}^{A}_{\mu\nu}(0) - N_A \lambda_B(x) * \text{Tr}_B \tilde{F}^{B}_{\mu\nu}(0) \right\},
\]

(2.13)

by taking the limit \( \varepsilon \to 0 \). If the Fourier coefficients \( \tilde{F}^{A}_{\mu\nu}(0) \) and \( \tilde{F}^{B}_{\mu\nu}(0) \) are non-zero, then there are the non-planar contributions to the axial anomaly \( [20] \). Note that the terms coming from the non-planar sector in the expression (2.13) do not diverge under the “local” chiral transformations.

**Four dimensions**

Next we consider the case of four dimensions. The explicit form of the expression (2.10) is given by

\[
\mathcal{A}_{\text{axial}}^{\text{reg}}[A, B, \lambda_A, \lambda_B] = \lim_{\varepsilon \to 0} \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{k \cdot k - \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma}}
\]

\[
\left\{ \lambda_A(x) * \text{Tr}_A \text{Tr}_B \left[ (F^{A}_{\mu\nu}(x) * - * F^{B}_{\mu\nu}(x + \frac{k}{\sqrt{\varepsilon}}))(F^{A}_{\rho\sigma}(x) * - * F^{B}_{\rho\sigma}(x + \frac{k}{\sqrt{\varepsilon}})) \right] - \lambda_B(x) * \text{Tr}_A \text{Tr}_B \left[ (F^{A}_{\mu\nu}(x) - \frac{k}{\sqrt{\varepsilon}}) * - * F^{B}_{\mu\nu}(x))(F^{A}_{\rho\sigma}(x) - \frac{k}{\sqrt{\varepsilon}}) * - * F^{B}_{\rho\sigma}(x)) \right] \right\}.
\]

(2.14)
Performing the integration over the momentum $k_{\mu}$, we find

\[
\mathcal{A}_{\text{axial}}^{\text{reg}}[A, B, \lambda_A, \lambda_B] = - \lim_{\varepsilon \to 0} \int d^4x \frac{1}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} \tag{2.15}
\]

\[
\left\{ N_B \lambda_A(x) * \text{Tr}_A F^{A}_{\mu\nu}(x) * F^{A}_{\rho\sigma}(x) - N_A \lambda_B(x) * \text{Tr}_B F^{B}_{\mu\nu}(x) * F^{B}_{\rho\sigma}(x) \right. \\
-2\lambda_A(x) * \text{Tr}_A F^{A}_{\mu\nu}(x) * \text{Tr}_B \tilde{F}^{B}_{\rho\sigma}(p) e^{ip\cdot x} \exp \left( \frac{p \circ p}{4\varepsilon} \right) \\
+2\lambda_B(x) * \text{Tr}_A \tilde{F}^{A}_{\mu\nu}(p) e^{ip\cdot x} \exp \left( \frac{p \circ p}{4\varepsilon} \right) * \text{Tr}_B F^{B}_{\rho\sigma}(x) \\
+N_A \lambda_A(x) * \text{Tr}_B \tilde{F}^{B}_{\mu\nu}(p) e^{ip\cdot x} * e^{iq\cdot x} \exp \left( \frac{(p + q) \circ (p + q)}{4\varepsilon} \right) \\
-N_B \lambda_B(x) * \text{Tr}_A \tilde{F}^{A}_{\mu\nu}(p) e^{ip\cdot x} * e^{iq\cdot x} \exp \left( \frac{(p + q) \circ (p + q)}{4\varepsilon} \right) \right\}.
\]

The factors $\exp \left( \frac{p \circ p}{4\varepsilon} \right)$ and $\exp \left( \frac{(p + q) \circ (p + q)}{4\varepsilon} \right)$ in the right-hand side of the expression (2.15) are generated from the non-planar contributions. The quantities $p \circ p$ and $(p + q) \circ (p + q)$ satisfy the condition $p \circ p \leq 0$ and $(p + q) \circ (p + q) \leq 0$ under the Euclidean metric, respectively. Although $p \circ p = 0$ is equivalent to $\tilde{p}^{\mu} = 0$, it is not equivalent to $p_{\mu} = 0$ in four or higher dimensions. By using the notations $\tilde{F}^{A}_{\mu\nu}(x) \equiv \lim_{\varepsilon \to 0} \int \frac{d^4p}{(2\pi)^4} \tilde{F}^{A}_{\mu\nu}(p) e^{ip\cdot x} \exp \left( \frac{p \circ p}{4\varepsilon} \right)$ and $\tilde{F}^{A}_{\mu\nu}(x) * \tilde{F}^{A}_{\rho\sigma}(x) \equiv \lim_{\varepsilon \to 0} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \tilde{F}^{A}_{\mu\nu}(p) \tilde{F}^{A}_{\rho\sigma}(q) e^{ip\cdot x} * e^{iq\cdot x} \exp \left( \frac{(p + q) \circ (p + q)}{4\varepsilon} \right)$, we can rewrite the expression (2.15) as

\[
\mathcal{A}_{\text{axial}}^{\text{reg}}[A, B, \lambda_A, \lambda_B] = - \int d^4x \frac{1}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} \tag{2.16}
\]

\[
\left\{ N_B \lambda_A(x) * \text{Tr}_A F^{A}_{\mu\nu}(x) * F^{A}_{\rho\sigma}(x) - N_A \lambda_B(x) * \text{Tr}_B F^{B}_{\mu\nu}(x) * F^{B}_{\rho\sigma}(x) \right. \\
-2\lambda_A(x) * \text{Tr}_A F^{A}_{\mu\nu}(x) * \text{Tr}_B \tilde{F}^{B}_{\rho\sigma}(x) + 2\lambda_B(x) * \text{Tr}_B F^{B}_{\mu\nu}(x) * \text{Tr}_A \tilde{F}^{A}_{\rho\sigma}(x) \\
+N_A \lambda_A(x) * \text{Tr}_B \tilde{F}^{B}_{\mu\nu}(x) * \tilde{F}^{B}_{\rho\sigma}(x) - N_B \lambda_B(x) * \text{Tr}_A \tilde{F}^{A}_{\mu\nu}(x) * \tilde{F}^{A}_{\rho\sigma}(x) \right\}.
\]

In deriving the expression (2.16), we have utilized the commutativity of the product: $\text{Tr} \tilde{F}^{\mu\nu} \cdot \text{Tr} F_{\rho\sigma} = \text{Tr} F_{\rho\sigma} \cdot \text{Tr} \tilde{F}^{\mu\nu}$. Note that the Moyal star product $\text{Tr} F_{\mu\nu} * \text{Tr} \tilde{F}^{\rho\sigma}$ (or $\text{Tr} \tilde{F}^{\mu\nu} * \text{Tr} F_{\rho\sigma}$) results in the normal (commutative) product under the vanishing noncommutative momentum.

The axial anomaly for the fermion in the adjoint representation can be obtained by setting $A_{\mu}(x) = B_{\mu}(x)$, $\lambda_A(x) = \lambda_B(x)$ and $N_A = N_B$ in the expressions (2.16). Since the expression of anomaly is antisymmetric under the exchange of the subscripts A and B, the axial anomaly for the fermion in the adjoint representation vanishes in four dimensions.

**Arbitrary even dimensions**

Let us now return to the case of arbitrary even dimensions. The axial anomaly in arbitrary even dimensions are derived by taking the limit $\varepsilon \to 0$, after performing the integration over
the momentum $k_\mu$ in the expression (2.10). The explicit expression of the anomaly is given as follows,

$$
\mathcal{A}_\text{axial}^{\text{reg}}[A, B, \lambda_A, \lambda_B] = \int d^{2n}x \frac{\varepsilon^n}{n!(4\pi)^n} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4 \cdots \mu_{2n-3} \mu_{2n-2} \mu_{2n-1} \mu_{2n}} (2.17)
$$

$$
\left\{ N_B \lambda_A(x) \cdot \text{Tr}_A F_{\mu_1 \mu_2}^A(x) \cdot F_{\mu_3 \mu_4}^A(x) \cdot \cdots \cdot F_{\mu_{2n-3} \mu_{2n-2}}^A(x) \cdot F_{\mu_{2n-1} \mu_{2n}}^A(x) \\
+ (-1)^n N_A \lambda_A(x) \cdot \text{Tr}_A F_{\mu_1 \mu_2}^A(x) \cdot F_{\mu_3 \mu_4}^A(x) \cdot \cdots \cdot F_{\mu_{2n-3} \mu_{2n-2}}^A(x) \cdot \text{Tr}_B \hat{F}_{\mu_{2n-1} \mu_{2n}}(x) \\
+ (-1)^n F_{\mu_1 \mu_2}^B(\hat{x}) \cdot F_{\mu_3 \mu_4}^B(\hat{x}) \cdot \cdots \cdot F_{\mu_{2n-3} \mu_{2n-2}}^B(\hat{x}) \cdot F_{\mu_{2n-1} \mu_{2n}}^B(\hat{x}) \\
- (-1)^n \times (\text{All of the terms with subscript } A \leftrightarrow B) \right\},
$$

where we have used the notations $n C_r \equiv \frac{n!}{r!(n-r)!}$ and

$$
\hat{F}_{\mu_1 \mu_2}(x) \cdot \hat{F}_{\mu_3 \mu_4}(x) \cdot \cdots \cdot \hat{F}_{\mu_{2r-1} \mu_{2r}}(x) \\
\equiv \lim_{\varepsilon \to 0} \int \frac{d^{2n}p}{(2\pi)^{2n}} \hat{F}_{\mu_1 \mu_2}(p) e^{ip \cdot x} \times \int \frac{d^{2n}q}{(2\pi)^{2n}} \hat{F}_{\mu_3 \mu_4}(q) e^{iq \cdot x} \times \cdots \times \int \frac{d^{2n}s}{(2\pi)^{2n}} \hat{F}_{\mu_{2r-1} \mu_{2r}}(s) e^{is \cdot x}
$$

$$
\times \exp \left\{ \frac{1}{4\varepsilon} (p + q + \cdots + s) \circ (p + q + \cdots + s) \right\}. \quad (2.18)
$$

The terms without $\hat{F}_{\mu \nu}$ in the right-hand side of the expression (2.17) come from the non-planar contributions, while all the other terms in the same expression come from the non-planar contributions. In deriving the expression (2.17), we have utilized again the commutativity of the product between $\text{Tr}(\hat{F}_{\mu \nu} \cdots \hat{F}_{\rho \sigma})$ and the other fields. Notice that the expression (2.17) is antisymmetric under the exchange of the subscript $A$ for $B$ in $D = 4k(k = 1, 2, \ldots,)$ dimensions, while it is symmetric in $D = 4k - 2(k = 1, 2, \ldots,)$ dimensions. Therefore, the chiral anomalies in $D = 4k(k = 1, 2, \ldots,)$ dimensions vanish in the noncommutative gauge theories with fermions in adjoint representation.

In general, the limit of the cutoff parameter $\varepsilon(\sim 1/\Lambda) \to 0$ and that of the noncommutative momentum $\tilde{p}_\mu \to 0$ do not commute in noncommutative quantum field theories. This phenomenon is known as UV/IR mixing. When we take the limit $\varepsilon \to 0$ after integrating over the momentum $k_\mu$ in the expression (2.10), and next take the limit $\tilde{p}_\mu \to 0$, then we obtain the axial anomaly (2.17). On the other hand, when we take the limit $\tilde{k}_\mu \to 0$ before integrating over the momentum $k_\mu$ in the expression (2.10), and next take the limit $\varepsilon \to 0$, then we obtain the axial anomaly comes from the planar contributions only. This phenomenon can be regarded as a UV/IR mixing for the axial anomalies in noncommutative gauge theories.
3 The chiral gauge anomaly for bi-fundamental and adjoint chiral fermion

We next discuss the non-abelian anomalies for bi-fundamental chiral fermions in noncommutative Euclidean space. Let us consider a bi-fundamental chiral fermion $P_R \psi^i_j(x) \equiv \frac{1+\gamma_{2n+1}}{2} \psi^i_j(x)$ interacting with a $U(N_A)$ gauge field $A_\mu^{i_1 i_2}(x)$ and a $U(N_B)$ gauge field $B_\mu^{j_1 j_2}(x)$. The classical action on $2n$-dimensional noncommutative (Euclidean) space is given by

$$S[\bar{\psi}, \psi, A, B] = \int d^{2n}x \bar{\psi}^j_i((i\slashed{\partial}_R[A, B]) \psi^i_j(x) ,$$

with the Dirac operator $\slashed{\partial}_R[A, B]

$$i\slashed{\partial}_R[A, B] \psi^i_j(x) = (\bar{\psi}^i_j \delta_{j_1}^j + A_\mu^{i_1 i_2} \gamma^\mu P_R - \delta_{i_1}^{i_2} B_\mu^{j_1 j_2} \gamma^\mu P_R) \psi^i_j(x) .$$

Here the index $i$ runs from 1 to $N_A$ and $j$ from 1 to $N_B$, respectively. Although the Dirac operator $\slashed{\partial}_R[A, B]$ is not Hermitian in Euclidean space, the operators $\slashed{\partial}_R \slashed{\partial}_R$ and $\slashed{\partial}_R \slashed{\partial}_R^\dagger$ are Hermitian and positive definite:

$$\slashed{\partial}_R \slashed{\partial}_R \phi_n(x) = \lambda_n^2 \phi_n(x) , \quad \slashed{\partial}_R \slashed{\partial}_R \phi_n(x) = \lambda_n^2 \phi_n(x) ,$$

then we can introduce the orthonormal and complete systems $\{ \phi_n(x) \}$ and $\{ \phi_n(x) \}$:

$$\int d^{2n}x \bar{\phi}^m(x) \phi_n(x) = \int d^{2n}x \bar{\phi}^m(x) \phi_n(x) = \delta_{mn} .$$

The infinitesimal gauge transformations for the fermions are given as follows,

$$\delta \psi^{i_1 j_1}(x) = \Lambda^{i_1 i_2}_A(x) P_R \psi^{i_2 j_1}(x) - P_R \psi^{i_1 j_2}(x) \Lambda^{j_1 j_2}_B(x) ,$$

$$\delta \bar{\psi}^{i_1 j_1}(x) = -\bar{\psi}^{i_1 j_2}(x) P_L \Lambda^{i_2 i}_A(x) + \Lambda^{j_1 j_2}_B(x) \bar{\psi}^{i_2 j_1}(x) P_L ,$$

with $P_L \equiv \frac{1-\gamma_{2n+1}}{2}$. Here $\Lambda^{i_1 i_2}_A(x)$ and $\Lambda^{i_1 i_2}_B(x)$ denote some infinitesimal functions. Let us consider the effective action for the gauge fields derived from the classical action (3.1):

$$W[A, B] = \ln \int D\bar{\psi} D\psi \exp \left(-\int d^{2n}x \bar{\psi}^j_i((i\slashed{\partial}_R[A, B]) \psi^i_j(x) \right) ,$$

The invariance of the effective action (3.5) under the infinitesimal transformations (3.4) leads to

$$\int d^{2n}x \left\{ \Lambda^{i_1 i_2}_A(x) (D_R^{(A)} \mu J_{2n+1}^{(A)} i_2 i_1(x) + \Lambda^{j_1 j_2}_B(x) (D_R^{(B)} \mu J_{2n+1}^{(B)} j_2 j_1(x) \right)$$

$$= A_{\text{chiral}}[A, B, \Lambda_A, \Lambda_B] ,$$

8
where $J_{2n+1}^{(A)} i_1 i_2 (x)$ and $J_{2n+1}^{(B)} j_1 j_2 (x)$ are the (right-handed) nonabelian currents:

$$J_{2n+1}^{(A)} i_1 i_2 (x) \equiv (\bar{\psi}_\beta^{i_1} j_1 (x) \ast \bar{\psi}_\alpha^{i_2} j_2 (x)) (\gamma^\mu P_R)_{\alpha\beta},$$

(3.7)

and the covariant derivative of these current are given by

$$D^\mu_{(A)} J_{2n+1}^{(A)} i_1 i_2 (x) \equiv \partial^\mu J_{2n+1}^{(A)} i_1 i_2 (x) + A^\mu i_1 i_2 (x) * J_{2n+1}^{(A)} i_3 i_2 (x) - J_{2n+1}^{(A)} i_1 i_2 (x) * A^\mu i_3 i_2 (x),$$

$$D^\mu_{(B)} J_{2n+1}^{(B)} j_1 j_2 (x) \equiv \partial^\mu J_{2n+1}^{(B)} j_1 j_2 (x) + B^\mu j_1 j_2 (x) * J_{2n+1}^{(B)} j_3 j_2 (x) - J_{2n+1}^{(B)} j_1 j_2 (x) * B^\mu j_3 j_2 (x).$$

The right-hand side of the expression (3.6) is contained in the Jacobian factor of the path integral measure in the effective action (3.5). We evaluate the Jacobian factor with respect to the gauge transformation of the fermions. Under the infinitesimal gauge transformation the path integral measure in the effective action (3.5) transforms as $D\bar{\psi} D\psi = J_{chiral}[\Lambda_A, \Lambda_B] D\bar{\psi} D\psi$ with the Jacobian

$$J_{chiral}[\Lambda_A, \Lambda_B] = \exp (-A_{chiral}[\Lambda_A, \Lambda_B]).$$

(3.8)

Since the Jacobian is ill defined, we have to regularize it by inserting a damping factor. Then we obtain two types of chiral anomaly reflecting the two different regularization procedures. We perform the regularization in a gauge covariant way \[4, 8\]. Inserting the Gaussian factor $\exp(-\bar{\epsilon} D_R^\mu D_R^\mu)$ and $\exp(-\bar{\epsilon} D_R^\mu D_R^\mu)$, we have

$$A_{chiral}^{reg}[A, B, \Lambda_A, \Lambda_B] = \lim_{\epsilon \to 0} \sum_n \int d^{2n}x \left[ \Lambda_A (x) * \left( P_R (e^{-\bar{\epsilon} D_R^\mu D_R^\mu} \varphi_n (x)) \ast \varphi_n^\dagger (x) - P_L (e^{-\bar{\epsilon} D_R^\mu D_R^\mu} \phi_n (x)) \ast \phi_n^\dagger (x) \right) ight.$$

$$- \Lambda_B (x) * \left( \varphi_n^\dagger (x) \ast P_R (e^{-\bar{\epsilon} D_R^\mu D_R^\mu} \varphi_n (x)) - \phi_n^\dagger (x) \ast P_L (e^{-\bar{\epsilon} D_R^\mu D_R^\mu} \phi_n (x)) \right) \right] .$$

(3.9)

where the differential operator $D_R$ is the Dirac operator given in the expression (2.2). In deriving the expression (3.9), we have used the property of the projection operators $P_R$ and $P_L$. We now evaluate the regularized sum in Fourier space $\varphi_n (x) = \int \frac{d^{2n}k}{(2\pi)^{2n}} e^{i k \cdot x} \varphi_n (k)$. Then we have

$$A_{chiral}^{reg}[A, B, \Lambda_A, \Lambda_B] = \lim_{\epsilon \to 0} \int d^{2n}x \int \frac{d^{2n}k}{(2\pi)^n} \left( \Lambda_A (x) * P_R \exp \left( -\bar{\epsilon} (i k + \bar{\psi})^2 \right) e^{i k \cdot x} - P_L \exp \left( -\bar{\epsilon} (i k + \bar{\psi})^2 \right) e^{i k \cdot x} \right)$$

$$- \Lambda_B (x) * e^{-i k \cdot x} \ast P_R \exp \left( -\bar{\epsilon} (i k + \bar{\psi})^2 \right) e^{i k \cdot x} = \Lambda_B (x) * P_L \exp \left( -\bar{\epsilon} (i k + \bar{\psi})^2 \right) e^{i k \cdot x} \left( 2 \right) ,$$

(3.10)
where the notation "\(\text{Tr}\)" denotes the trace over the \(U(N_A), U(N_B)\) and the Dirac matrices. Inserting the decomposition \(\mathcal{D}^2 = D_\mu D^\mu + \frac{1}{2} \gamma^\mu \gamma^\nu [D_\mu, D_\nu]\) into the expression (3.10) and performing the rescaling \(k_\mu \to (1/\sqrt{\varepsilon} k_\mu)\), we have

\[
A_{\text{chiral}}^{\text{reg}}[A, B, \Lambda_A, \Lambda_B] = \lim_{\varepsilon \to 0} \int d^2 x \int \frac{d^2 k}{(2\pi)^n} \frac{1}{\varepsilon^{\mu\nu}} e^{k \cdot k} \frac{1}{n!} \varepsilon^{\mu_1 \mu_2 \cdots \mu_n} \tag{3.11}
\]

\[
\left\{ \text{Tr}_A \text{Tr}_B \left[ \Lambda_A(x) \ast (F^A_{\mu_1 \mu_2}(x) \ast \cdots \ast F^A_{\mu_{n-1} \mu_n}(x)) \right] \right\} + N_B \text{Tr}_A \Lambda_A(x) \ast F^A_{\mu \nu}(x) + N_A \text{Tr}_B \Lambda_B(x) \ast F^B_{\mu \nu}(x)

- \text{Tr}_A \Lambda_A(x) \ast F^B_{\mu \nu}(x - \frac{k}{\sqrt{\varepsilon}}) - \text{Tr}_B \Lambda_B(x) \ast \text{Tr}_A F^A_{\mu \nu}(x - \frac{k}{\sqrt{\varepsilon}}) \right\},
\]

where the notations \(\text{Tr}_A\) and \(\text{Tr}_B\) denote the traces over the \(U(N_A)\) and \(U(N_B)\) matrices, respectively. We shall advance to the calculation in arbitrary dimensions after examining the case of two and four dimensions.

**Two dimensions**

The explicit form of the expression (3.11) in two dimensions is given by

\[
A_{\text{chiral}}^{\text{reg}}[A, B, \Lambda_A, \Lambda_B] = \lim_{\varepsilon \to 0} \int d^2 x \int \frac{d^2 k}{(2\pi)^n} \frac{1}{\varepsilon^{\mu\nu}} e^{k \cdot k} \frac{1}{2} \varepsilon^{\mu\nu} \tag{3.12}
\]

\[
\left\{ N_B \text{Tr}_A \Lambda_A(x) \ast F^A_{\mu \nu}(x) + N_A \text{Tr}_B \Lambda_B(x) \ast F^B_{\mu \nu}(x)

- \text{Tr}_A \Lambda_A(x) \ast \text{Tr}_B F^B_{\mu \nu}(x - \frac{k}{\sqrt{\varepsilon}}) - \text{Tr}_B \Lambda_B(x) \ast \text{Tr}_A F^A_{\mu \nu}(x - \frac{k}{\sqrt{\varepsilon}}) \right\},
\]

where the coefficients \(N_A\) and \(N_B\) arise from \(N_A \equiv \text{Tr}_A I_{N_A \times N_A}\) and \(N_B \equiv \text{Tr}_B I_{N_B \times N_B}\), respectively. Performing the integration over the momentum \(k\), we find that the factor \(\exp(\frac{1}{4\varepsilon} p \circ p)\) is generated from the non-planar contributions. Since \(p \circ p\) takes a negative value for arbitrary non-zero momentum \(p_\mu\) in two dimensions, we obtain the following result under the limit \(\varepsilon \to 0\),

\[
A_{\text{chiral}}^{\text{reg}}[A, B, \Lambda_A, \Lambda_B] = \int d^2 x \frac{1}{4\pi} \varepsilon^{\mu\nu} \left\{ N_B \text{Tr}_A \Lambda_A(x) \ast F^A_{\mu \nu}(x) \ast + N_A \text{Tr}_B \Lambda_B(x) \ast F^B_{\mu \nu}(x)

- \text{Tr}_A \Lambda_A(x) \ast \text{Tr}_B \tilde{F}^B_{\mu \nu}(0) - \text{Tr}_B \Lambda_B(x) \ast \text{Tr}_A \tilde{F}^A_{\mu \nu}(0) \right\}. \tag{3.13}
\]

If the Fourier coefficients \(\tilde{F}^A_{\mu \nu}(0)\) and \(\tilde{F}^B_{\mu \nu}(0)\) are non-zero, there are non-planar contributions to the chiral gauge anomaly.

**Four dimensions**
The chiral gauge anomaly in four dimensions is also derived by taking the limit \( \varepsilon \to 0 \) after performing the integration over the momentum \( k_\mu \) in the expression (3.11) with \( n = 2 \). The concrete form of the chiral gauge anomaly is given by

\[
\mathcal{A}^{\text{reg}}_{\text{chiral}}[A, B, \Lambda_A, \Lambda_B] = -\int d^4x \frac{1}{(2\pi)^4} \varepsilon^{\mu\nu\rho\sigma}
\]

(3.14)

\[
\left\{ N_B \text{Tr}_A \Lambda_A(x) \ast F^A_{\mu_1 \mu_2}(x) \ast F^A_{\mu_3 \mu_4}(x) \ast \cdots \ast F^A_{\mu_{2n-1} \mu_{2n}}(x) \ast F^A_{\mu_{2n-1} \mu_{2n}}(x) \\
-2 \text{Tr}_A \Lambda_A(x) \ast F^A_{\mu_1 \mu_2}(x) \ast \text{Tr}_B \hat{F}^B_{\rho \sigma}(x) + 2 \text{Tr}_A \Lambda_B(x) \ast F^B_{\mu_1 \mu_2}(x) \ast \text{Tr}_A \hat{F}^A_{\rho \sigma}(x) \\
+ \text{Tr}_A \Lambda_A(x) \ast \text{Tr}_B \hat{F}^B_{\mu_1 \mu_2}(x) \ast \hat{F}^B_{\rho \sigma}(x) - \text{Tr}_B \Lambda_B(x) \ast \text{Tr}_A \hat{F}^A_{\mu_1 \mu_2}(x) \ast \hat{F}^A_{\rho \sigma}(x) \right\},
\]

where we have used the notations introduced in the expression (2.18). In deriving the expression (3.14), we have utilized the commutativity of the product between \( \text{Tr}\hat{F}_{\mu \nu} \) and the other fields. We see that the mixed \( U(N)U(M)^2 \) or \( U(N)^2U(M) \) anomaly comes from the non-planar contributions. The chiral gauge anomaly for an adjoint fermion can be obtained by setting \( A_\mu(x) = B_\mu(x) \), \( \Lambda_A(x) = \Lambda_B(x) \) and \( N_A = N_B \) in the expression (3.14). Since the expression of anomaly is antisymmetric under the exchange of subscript \( A \) and \( B \), the chiral gauge anomaly vanish in four dimensions. Therefore noncommutative gauge theories with adjoint chiral fermions are anomaly free in four dimensions.

**Arbitrary even dimensions**

Let us now return to the arbitrary even dimensions. By the same calculations as the case of two and four dimensions, we obtain the concrete form of the chiral gauge anomaly in arbitrary even dimensions:

\[
\mathcal{A}^{\text{reg}}_{\text{axial}}[A, B, \Lambda_A, \Lambda_B] = \int d^{2n}x \frac{\varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4 \cdots \mu_{2n-3} \mu_{2n-2} \mu_{2n-1} \mu_{2n}}}{n!(4\pi)^n}
\]

(3.15)

\[
\left\{ N_B \text{Tr}_A \Lambda_A(x) \ast F^A_{\mu_1 \mu_2}(x) \ast F^A_{\mu_3 \mu_4}(x) \ast \cdots \ast F^A_{\mu_{2n-3} \mu_{2n-2} \mu_{2n-1} \mu_{2n}}(x) \ast F^A_{\mu_{2n-1} \mu_{2n}}(x) \\
+(-1)^n C_1 \text{Tr}_A \Lambda_A(x) \ast F^A_{\mu_1 \mu_2}(x) \ast F^A_{\mu_3 \mu_4}(x) \ast \cdots \ast F^A_{\mu_{2n-3} \mu_{2n-2} \mu_{2n-1} \mu_{2n}}(x) \ast \text{Tr}_B \hat{F}^B_{\mu_{2n-1} \mu_{2n}}(x) \\
+(-1)^n C_2 \text{Tr}_A \Lambda_A(x) \ast F^A_{\mu_1 \mu_2}(x) \ast F^A_{\mu_3 \mu_4}(x) \ast \cdots \ast \text{Tr}_B \hat{F}^B_{\mu_{2n-1} \mu_{2n}}(x) \\
\vdots \\
+(-1)^{n-1} C_{n-1} \text{Tr}_A \Lambda_A(x) \ast F^A_{\mu_1 \mu_2}(x) \ast \text{Tr}_B \hat{F}^B_{\mu_3 \mu_4} \ast \cdots \ast \hat{F}^B_{\mu_{2n-3} \mu_{2n-2} \mu_{2n-1} \mu_{2n}}(x) \\
+(-1)^n C_n \text{Tr}_A \Lambda_A(x) \ast \text{Tr}_B \hat{F}^B_{\mu_1 \mu_2} \ast \hat{F}^B_{\mu_3 \mu_4} \ast \cdots \ast \hat{F}^B_{\mu_{2n-3} \mu_{2n-2} \mu_{2n-1} \mu_{2n}}(x) \\
-(-1)^n \times (\text{All of the terms with subscript } A \leftrightarrow B) \right\},
\]

with the notation given in the expression (2.18). Here we have utilized again the commutativity of the product between \( \text{Tr}(\hat{F}_{\mu \nu} \ast \cdots \ast \hat{F}_{\rho \sigma}) \) and the other fields. The terms without \( \hat{F}_{\mu \nu} \) in the
right-hand side of the expression (3.15) come from the non-planar contributions, while all the other terms in the same expression come from the non-planar contributions.

We notice the expression (3.15) is antisymmetric under the exchange of the subscript \( A \) for \( B \) in \( D = 4k(k = 1, 2, \ldots) \) dimensions, while it is symmetric in \( D = 4k - 2(k = 1, 2, \ldots) \) dimensions. Therefore, the chiral anomalies in \( D = 4k(k = 1, 2, \ldots) \) dimensions vanish in the theories with adjoint chiral fermions. Namely, the noncommutative gauge theories with adjoint chiral fermions are anomaly free in \( D = 4k(k = 1, 2, \ldots) \) dimensions.

When we take the limit \( \varepsilon \to 0 \) after integrating over the momentum \( k_\mu \) in the expression (3.11), and next take the limit \( \tilde{p}_\mu \to 0 \), then we obtain the chiral gauge anomaly (3.15). On the other hand, when we take the limit \( \tilde{k}_\mu \to 0 \) before integrating over the momentum \( k_\mu \) in the expression (3.11), and next take the limit \( \varepsilon \to 0 \), then we obtain the chiral gauge anomaly comes from the planar contributions only. This phenomenon can be regarded as a UV/IR mixing for the chiral gauge anomalies in noncommutative gauge theories.

4 Conclusions

In this paper, we have calculated the axial and chiral gauge anomalies emerging from non-planar sector in noncommutative gauge theories. In noncommutative gauge theories with fermions in the bi-fundamental and the adjoint representation, there are non-planar contributions to the anomalies. These anomalies in non-planar sector can be evaluated not only in perturbative analysis but also in path integral formulation. When the regularization by introducing a Gaussian cut-off is performed in path integral formulation, non-planar sector includes the damping factor depending on the noncommutative momentum \( \tilde{p}^\mu \equiv p_\nu \tilde{\theta}^{\mu\nu} \). Therefore, the anomaly with the non-zero noncommutative momentum in non-planar sector vanishes. This fact has been shown also by the perturbative analysis about the chiral gauge anomalies [8]. The argument, however, breaks down for the zero noncommutative momentum, since in this case the non-planar sector is not regularized by the damping factor. Therefore the anomalies in non-planar sector remain for the zero noncommutative momentum. This result is consistent with the result obtained in perturbative analysis [20].

In the noncommutative gauge theories with adjoint chiral fermion, the chiral gauge anomalies in planar sector vanish in \( D = 4k(k = 1, 2, \ldots) \) dimensions [7, 8]. The adjoint chiral fermion can be regarded as the product of fundamental and anti-fundamental chiral fermions. Since the fundamental and anti-fundamental chiral fermions give opposite contributions to the chiral gauge anomalies in \( D = 4k(k = 1, 2, \ldots) \) dimensions, the anomalies cancel out. The cancellation mechanism is also valid in non-planar sector. The chiral gauge anomalies in non-planar sector cancel out in \( D = 4k(k = 1, 2, \ldots) \) dimensions.

Noncommutative quantum field theories exhibit an intriguing mixing of the ultraviolet and
infrared regions. The limit of the cut-off parameter $\varepsilon \to 0$ and that of the noncommutative momentum $\tilde{p}^\mu \to 0$ do not commute. In deriving the axial and chiral gauge anomaly by path integral formulation, we find that the limits of the cutoff parameter and that of the noncommutative momentum do not commute, either. This phenomenon can be interpreted as UV/IR mixing for the anomalies in noncommutative gauge theories. Namely, the IR singularity (at the vanishing noncommutative momentum) in non-planar sector leads to the anomalies on the UV behavior via the intriguing UV/IR mixing in noncommutative gauge theories.

It was well known that there are two types of chiral gauge anomaly: the covariant anomaly and the consistent anomaly. Although the consistent anomaly is not gauge covariant, it is a solution of the Wess–Zumino consistency condition. Hence the cohomological method is applicable to deriving the consistent anomaly $[7, 11]$. The consistent anomaly and the covariant anomaly are related to the Moyal star polynomial of the Bardeen–Zumino type $[10, 8]$. It will be an interesting subject to investigate the Moyal star polynomial of the Bardeen–Zumino type coming from non-planar contributions in noncommutative gauge theories. We hope to discuss this subject in the future.

**Acknowledgments**

I am grateful to S. Deguchi for helpful suggestions. I would also like to thank A. Sugamoto for careful reading of the manuscript and useful comments.
References

[1] Michael R. Douglas and Nikita A. Nekrasov, “Noncommutative Field Theory” *Rev. Mod. Phys.* **73** (2002) 977-1029, [hep-th/0106048].

[2] Richard J. Szabo, “Quantum Field Theory on Noncommutative Spaces” [hep-th/0109162].

[3] Shiraz Minwalla, Mark Van Raamsdonk and Nathan Seiberg, “Noncommutative Perturbative Dynamics” *JHEP* **0002** (2000) 020, [hep-th/9912072].

[4] K. Fujikawa, “Evaluation of the chiral anomaly in gauge theories with $\gamma_5$ couplings” *Phys. Rev.* **D29** (1984) 285-292.

[5] Farhad Ardalan and Neda Sadooghi, “Axial Anomaly in Noncommutative QED on $R^4$” *Int. J. Mod. Phys.* **A16** (2001) 3151-3178, [hep-th/0002143].

[6] J. M. Gracia-Bondia and C. P. Martin, “Chiral Gauge Anomalies on Noncommutative $R^4$” *Phys. Lett.* **B479** (2000) 321-328, [hep-th/0002171].

[7] L. Bonora, M. Schnabl and A. Tomasiello, “A note on consistent anomalies in noncommutative YM theories” *Phys. Lett.* **B485** (2000) 311-313, [hep-th/0002210].

[8] C. P. Martin, “The covariant form of the gauge anomaly on noncommutative $R^{2n}$” *Nucl. Phys.* **B623** (2002) 150-164, [hep-th/0110040].

[9] C. P. Martin “The UV and IR Origin of Nonabelian Chiral Gauge Anomalies on Noncommutative Minkowski Space-time” *J. Phys.* **A34** (2001) 9037-9056, [hep-th/0008126].

[10] C. P. Martin, “Chiral Gauge Anomalies on Noncommutative Minkowski Space-time” *Mod. Phys. Lett.* **A16** (2001) 311-320, [hep-th/0102066].

[11] L. Bonora and A. Sorin “Chiral anomalies in noncommutative gauge theories” *Phys. Lett.* **B521** (2001) 421-428, [hep-th/0109204].

[12] Farhad Ardalan and Neda Sadooghi, “Anomaly and Nonplanar Diagrams in Noncommutative Gauge Theories” *Int. J. Mod. Phys.* **A17** (2002) 123, [hep-th/0009233].

[13] J. Nishimura and M. A. Vazquez-Mozo, “Noncommutative Chiral Gauge Theories on the Lattice with Manifest Star-Gauge Invariance” *JHEP* **0108** (2001) 033, [hep-th/0107110].
[14] Rabin Banerjee and Subir Ghosh, “Seiberg Witten Map and the Axial Anomaly in Noncommutative Field Theory” Phys. Lett. B533 (2002) 162, [hep-th/0110177].

[15] D.H. Correa, G.S. Lozano, E.F. Moreno and F.A. Schaposnik, “Anomalies in Noncommutative Dipole Field Theories” JHEP 0202 (2002) 031, [hep-th/0202040].

[16] Marcus T. Grisaru and Silvia Penati, “Noncommutative Supersymmetric Gauge Anomaly” Phys. Lett. B504 (2001) 89-100, [hep-th/0010177].

[17] L.O. Chekhov and A.K. Khizhnyakov, “Gauge anomalies and the Witten-Seiberg correspondence for N=1 supersymmetric theories on noncommutative spaces” [hep-th/0103048].

[18] T. Nakajima, “Conformal Anomalies in Noncommutative Gauge Theories” to appear in Phys. Rev. D66 (2002) 0850xx, [hep-th/0108158].

[19] Ken Intriligator and Jason Kumar, “*-Wars Episode I: The Phantom Anomaly” Nucl. Phys. B620 (2002) 315-330, [hep-th/0107199].

[20] Adi Armoni, Esperanza Lopez and Stefan Theisen, “Nonplanar Anomalies in Noncommutative Theories and the Green-Schwarz Mechanism” [hep-th/0203165].