Calculation and Implementation of Crofton Formula

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Abstract. This paper introduces the Crofton formula of differential geometry, and clarifies the meaning of each variable. Furthermore, in order to verify the identity of the Crofton formula, the lengths of several special curves have been accurately calculated by Matlab programming, and have been compared with the actual lengths. At the same time, the Crofton formula is transformed from the calculation of the double integral to the calculation of the micro-element, so that it is easier to find the approximate solution to the general curve by Matlab programming. It is concluded that this method can be used to find an approximate solution for the length of any general curve.

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Introduction

The Crofton formula has attracted much attention in calculus geometry and has been applied in the study of hyperbolic space surfaces [1,2] and Minkovski tensors [3]. In the Euclidean plane, the length of the curve can be calculated using the Crofton formula. It is defined as: a regular curve $C$ with an arc length $L$ on the real Euclidean plane, then the measure of all the lines intersecting $C$ is $2L$, that is

$$\iint_{D} n(\theta, p) d\theta dp = 2L.$$  

Where $d\theta$ represents the change in the angle of rotation of the line, $0 \leq \theta < 2\pi$, $dp$ represents the density of motion of the line, $p \geq 0$, and $n(\theta, p)$ represents the number of intersections of $p$ and $C$ [4]. In this paper, the exact calculation of several special curve lengths is given by using the Crofton formula. At the same time, the Crofton formula of the Euclidean plane reveals the relationship between the line measure of all points on a plane curve and the length of the curve, thus a kind of approximate method was given for the arc length of a plane curve.

Calculation on the Length of Special Curves

Calculation on the Length of Straight Line (Line Segment)

For a given line $C$, note that the length of a segment is $L$. $D = \{ (\theta, p) | 0 < p \leq L | \cos \theta |, 0 \leq \theta < 2\pi \}$. 

Figure 1. Schematic diagram of the number of intersections between a set of straight lines and the line segment.
By analysing, the intersection of each line $L$ and $C$ is only one, that is $n(\theta, p) = 1$, so that
\[
\int_D n(\theta, p) d\theta dp = \int_D 1 d\theta dp = 2\int_0^{\frac{\pi}{2}} 2\int_0^L \cos \theta \, 1 dp = 2L.
\]

The pseudo code implemented by Matlab is as follows:

```matlab
% curve equation
y = x
p = cos(theta)+sin(theta) % solve the direction p
xp = p*cos(theta)
yp = p*sin(theta)
```

Figure 2. The number of intersections a set of straight lines and the line segment realized by Matlab.

**Calculation on the Length of the Arc**

For a given line $C$, note that the radius is $R$. $D = \{(\theta, p) | 0 < p \leq R, 0 \leq \theta < 2\pi\}$.

![Diagram of the number of intersections between a set of straight lines and the arc.](image)

By analyzing the number of intersections between the straight lines and the curve, there are two cases:

\[
n(\theta, p) = \begin{cases} 
2, & \theta \in [0, 2\pi), p \in [0, R) \\
1, & \theta \in [0, 2\pi), p = R
\end{cases}
\]

Because the length of a full circle curve is $L = 2\pi R$,
\[
\int_D n(\theta, p) d\theta dp = \int_D 2 d\theta dp = 2\int_0^{\frac{\pi}{2}} 2\int_0^R \cos \theta \, 1 dp = 4\pi R = 2L.
\]

The pseudo code implemented by Matlab is as follows:

```matlab
% curve parametric equation
x = cos(t)
y = 1+sin(t)
% solve the direction p
p = cos(theta)+sin(theta)
xp = p*cos(theta)
yp = p*sin(theta)
% solve the center line
xp1 = xp/2
yp1 = yp/2
```
% find the intersection of the center line and the curve
y = y1+k(x-x1)
x^2+(y-1)^2 = 1
% a b c are the quadratic term, the first term and the constant term coefficient of the equation
det = b^2-4a*c
xp0 = (-b+sqrt(det))/(2a)
or xp0 = (-b-sqrt(det))/(2a)
yp0 = yp1+k*(xp0-xp1)

Figure 4. The number of intersections between a set of straight lines and the arc realized by Matlab.

**Calculation on the Length of the Parabola**

For a given line $C$, note that the parabola equation is $y = x^2$, $x \in (0,1)$.

Based on the integral, the formula of the length of the parabola is $ds = \sqrt{1+y'^2}dx$, so that

\[ L = \int_0^1 \sqrt{1+(2x)^2} dx = \frac{\sqrt{5}}{2} + \frac{1}{8} \ln(9+4\sqrt{5}) = 1.479. \]

By analyzing the number of intersections between the straight lines and the curve, there are such cases:

- $\theta \in \left(0, \frac{3\pi}{4}\right)$, $0 < p \leq \cos \theta + \sin \theta$, $n = 1$
- $\theta \in \left(\frac{3\pi}{4}, \frac{\pi}{2}\right)$, $0 < p \leq \cos \theta + \sin \theta$, $n = 0$
- $\theta \in \left(\frac{\pi}{2}, \frac{7\pi}{4}\right)$, $0 < p \leq -\frac{\cos^2 \theta}{4\sin \theta}$, $n = 2$
- $\theta \in \left(\frac{7\pi}{4}, 2\pi - \frac{\pi}{2} + \arctan 2\right)$, \begin{align*}
\begin{cases}
  n = 2, & \cos \theta + \sin \theta < p \leq -\frac{\cos^2 \theta}{4\sin \theta} \\
  n = 1, & 0 < p \leq \cos \theta + \sin \theta
\end{cases}
\end{align*}
- $\theta \in \left(2\pi - \frac{\pi}{2} + \arctan 2, 2\pi\right)$, $0 < p \leq \cos \theta + \sin \theta$, $n = 1$

So that:

\[
\begin{align*}
\int_{D_1} n(\theta, p)d\theta dp &= \int_0^{\frac{3\pi}{4}} \int_0^{\cos \theta + \sin \theta} dp \, d\theta = 2.4141 \\
\int_{D_1} n(\theta, p)d\theta dp &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta + \sin \theta} dp \, d\theta = 0 \\
\int_{D_1} n(\theta, p)d\theta dp &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \int_0^{\frac{\cos^2 \theta}{4\sin^2 \theta}} 2dp \, d\theta = 0.0871
\end{align*}
\]
\[
\int_{\theta_1}^{\theta_2} n(\theta, p) d\theta dp = \int_{\theta_1}^{\theta_2} \frac{2\pi - \frac{\pi}{2} \arctan \frac{2}{7}}{\theta_2 - \theta_1} d\theta dp = 0.0726
\]

\[
\int_{\theta_1}^{\theta_2} n(\theta, p) d\theta dp = \int_{\theta_1}^{\theta_2} \frac{2\pi - \frac{\pi}{2} \arctan \frac{2}{7}}{\theta_2 - \theta_1} \cos^2 \theta d\theta dp = 0.0424
\]

\[
\int_{\theta_1}^{\theta_2} n(\theta, p) d\theta dp = \int_{\theta_1}^{\theta_2} \frac{2\pi - \frac{\pi}{2} \arctan \frac{2}{7}}{\theta_2 - \theta_1} \cos \theta d\theta dp = 0.3416
\]

\[
\sum_{i=1}^{n} \int_{\theta_1}^{\theta_2} n(\theta, p) d\theta dp = 2.4141 + 0.0871 + 0.0726 + 0.0424 + 0.3416 = 2.9578 = 2L
\]

The pseudo code implemented by Matlab is as follows:

```matlab
% parabola equation
y = x.^2
%solve the direction p
p = cos(theta)+sin(theta)
or p = -cos(theta)*cos(theta)/4/sin(theta)
xp = p*cos(theta)
yp = p*sin(theta) %solve the center line
%solve the center line
xp1 = xp/2
yp1 = yp/2
%find the intersection of the center line and the curve
y = y1+k*(x-x1)
y = x^2
% a b c are the quadratic term, the first term and the constant term coefficient of the equation
det = b^2-4a*c
xp0 = (-b+sqrt(det))/(2a)
or xp0 = (-b-sqrt(det))/(2a)
yp0 = yp1+k*(xp0-xp1)
```

Figure 5. The number of intersections between a set of straight lines and the parabola realized by Matlab.

Approximate calculation on the length of a general curve

**Approximate Calculation On the Length of General Curves**

For a general curve, it is difficult to accurately calculate the length of a curve by finding the intersection of a set of lines and the curve. Therefore, we use the idea of differentiation to convert the Crofton formula to:
\[ L = \frac{1}{2} \int_0^n n(p, \theta)dpd\theta \]
\[ \approx \frac{1}{2} \sum_i n(p_i, \theta_i)\Delta p_i \Delta \theta_i \]

note \( \Delta p_i = \Delta p, \Delta \theta_i = \Delta \theta \)

\[ = \frac{1}{2} \Delta p \Delta \theta \sum_i n(p_i, \theta_i) = \frac{1}{2} n \Delta p \Delta \theta \]

where \( \Delta p \) is the line spacing, \( \Delta \theta \) is the angular interval, and \( n \) is the number of intersections.

![Figure 6. Schematic diagram of the number of intersections between a set of straight lines and a general curve.](image)

**Approximate Calculation on the Length of the Parabola**

For the approximate calculation on the length of the parabola \( y = x^2 \), note that \( \Delta p = \frac{2}{30} \).

\( \Delta \theta = \frac{2\pi}{180} \), Matlab calculates \( n = 1256 \).

So that: \( L \approx \frac{1}{2} n \Delta p \Delta \theta = 1.461 \), relative error is 1.21%.

The pseudo code implemented by Matlab is as follows:

```matlab
% scan circle radius
nR=30
%delta_p and its coordinates
sR=linspace(0,2*pi,nR)
xR=Ra*cos(sR)
yR=Ra*sin(sR)
%delta_theta and its coordinates
ns=181
s=linspace(0,2*pi,ns)
rsit=0.1
xsit=rsit*cos(s)
yosit=rsit*sin(s)
% Find the intersection of a set of lines and a parabola
s0 = s[i]
ap=1+(cot(s0))^2
bp=-2*ps*cot(s0)*csc(s0)
cp=ps.^2*(csc(s0))^2-Ra^2
detp=max(bp.^2-4*ap.*cp,0)
 xp1=((-bp-sqrt(detp))./(2*ap))
```

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xp2=(-bp+sqrt(detp))/(2*ap)
yp1=ps*csc(s0)-xp1*cot(s0)
yp2=ps*csc(s0)-xp2*cot(s0)

Figure 7. The number of intersections between a set of straight lines and the parabola realized by Matlab (approximate calculation).

Approximate Calculation on the Length of the General Curve

For the approximate calculation on the length of the general curve, note that \( \Delta p = \frac{2}{30} \), \( \Delta \theta = \frac{2\pi}{90} \).

Matlab calculates \( n = 4257 \).

So that: \( L \approx \frac{1}{2} n\Delta p\Delta \theta = 9.976 \), relative error is 0.12%.

The results achieved by Matlab are as follows:

Figure 8. The number of intersections between a set of straight lines and the general curve realized by Matlab (fix \( \Delta p \), change \( \theta \), approximate calculation).

The above solution is to fix \( \Delta p \), and change \( \theta \) in the process of rotation.

On the other hand, we fix \( \Delta \theta \), change the value of \( p \) every time, and the results achieved by Matlab are as follows:

Figure 9. The number of intersections between a set of straight lines and the general curve realized by Matlab (fix \( \Delta \theta \), change \( p \), approximate calculation).
Summary

Through the exploration of the Crofton formula, for a special curve, as long as a corresponding set of straight lines is given, the direction and rotation angle of the curve are changed, and the number of intersections of the curve and the straight line in each case can be obtained. Therefore, the exact solution of the curve length can be calculated using the Crofton formula. For the general curve, it is difficult to directly give the exact solution of the curve length by the Crofton formula. By converting Crofton into a fixed form of \( \theta \) or \( p \), the formula for approximate the length of the curve can be obtained.

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