Cosmological perturbations from multi-field inflation in generalized Einstein theories

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We study cosmological perturbations generated from quantum fluctuations in multi-field inflationary scenarios in generalized Einstein theories, taking both adiabatic and isocurvature modes into account. In the slow-roll approximation, explicit closed-form long-wave solutions for field and metric perturbations are obtained by the analysis in the Einstein frame. Since the evolution of fluctuations depends on specific gravity theories, we made detailed investigations based on analytic and numerical approaches in four generalized Einstein theories: the Jordan-Brans-Dicke (JBD) theory, the Einstein gravity with a non-minimally coupled scalar field, the higher-dimensional Kaluza-Klein theory, and the $R + R^2$ theory with a non-minimally coupled scalar field. We find that solutions obtained in the slow-roll approximation show good agreement with full numerical results except around the end of inflation. Due to the presence of isocurvature perturbations, the gravitational potential $\Phi$ and the curvature perturbations $\zeta$ and $\zeta$ do not remain constant on super-horizon scales. In particular, we find that negative non-minimal coupling can lead to strong enhancement of $\zeta$ in both the Einstein and higher-derivative gravity, in which case it is difficult to unambiguously decompose scalar perturbations into adiabatic and isocurvature modes during the whole stage of inflation.

I. INTRODUCTION

The beauty of the inflationary paradigm is that it both 1) explains why the present-day Universe is approximately homogeneous, isotropic and spatially flat, so that it may be described by the Friedmann-Robertson-Walker (FRW) model in the zero approximation, \cite{1} and 2) makes detailed quantitative predictions about small deviations from homogeneity and isotropy including density perturbations which produce gravitationally bound objects (such as galaxies, quasars, etc.) and the large-scale structure of the Universe. It is the latter predictions that make it possible to test and falsify this paradigm (for each its concrete realization) like any other scientific hypothesis. Fortunately, all existing and constantly accumulating data, instead of falsifying, continue to confirm these predictions (within observational errors). Historically, among models of inflation making use of a scalar field (called an inflaton), the original model or the first-order phase transition model \cite{3} failed due to the graceful exit problem, which was taken over by the new \cite{4} and the chaotic \cite{5} inflation scenarios where the inflaton is slowly rolling during the whole de Sitter (inflationary) stage. The latter property was shared by the alternative scenario with higher-derivative quantum gravity corrections \cite{6} (where the role of an inflaton is played by the Ricci scalar $R$) just from the beginning. Note that a simplified version of this scenario - the $R + R^2$ model - was even shown to be mathematically equivalent to some specific version of the chaotic scenario \cite{7} (see also a review in \cite{8}).

Turning to inhomogeneous perturbations on the FRW background, the inflationary paradigm generically predicts two kinds of them: scalar perturbations and tensor ones (gravitational waves) which are generated from quantum-gravitational fluctuations of the inflaton field and the gravitational field respectively during an inflationary quasi-de Sitter stage in the early Universe. The spectrum of tensor perturbations generated during inflation was first derived in \cite{9}, while the correct expression for the spectrum of scalar perturbations after the end of inflation was obtained in \cite{10}. For completeness, one should also cite two papers \cite{11} and \cite{12} where two important intermediate steps on the way to the right final answer for scalar perturbations were made, in particular, in the latter paper the spectrum of scalar perturbations during inflation was calculated for the Starobinsky inflationary model \cite{13}. In order to obtain a small enough amplitude of density perturbations in all the above mentioned slow-roll inflationary models, the inflaton should be extremely weakly coupled to other fields. It is therefore not easy to find sound motivations to have such a scalar field in particle physics (see, however, \cite{14,15}).
Reflecting such a situation, the extended inflation scenario was proposed to revive a GUT Higgs field as the inflaton by adopting non-Einstein gravity theories. Although the first version of the inflation model, which considers a first-order phase transition in the Jordan-Brans-Dicke (JBD) theory, resulted in failure again due to the graceful-exit problem, it triggered further study of more generic class of inflation models in non-Einstein theories, in particular extended chaotic inflation where both the inflaton and the Brans-Dicke scalar fields are in the slow rolling regime during inflation. Note that the natural source of Brans-Dicke-like theories of gravity is the low-energy limit of the superstring theory with the Brans-Dicke scalar being the dilaton.

Several analyses have been done on the density perturbations produced in extended new or chaotic inflation models, all of which made use of the constancy of the gauge-invariant quantity or its equivalent on super-horizon scales, and matched them directly to quantum field fluctuations at the moment of horizon crossing which would be the correct procedure in a single component inflationary model. However, in the presence of two sources of quantum fluctuations (i.e., the inflaton and the Brans-Dicke scalar field), ζ does not remain constant during inflation due to the appearance of isocurvature perturbations. In such a case, mixing between adiabatic and isocurvature perturbations may occur due to ambiguity in the definition of the latter ones (see the discussion in the Sec. III A below).

Note that it is nothing unusual nor unexpected about non-conservation of the quantities ζ and R even in the long-wave, or super-horizon, case, if we follow the meaning of this term used in classical mechanics. Here, a(t) is the FRW scale factor, H ≡ a/a, where a dot denotes the time derivative, and k is the conserved covariant momentum of a perturbation Fourier mode. Namely, the “conservation” of ζ in the latter case is restricted to a part of initial conditions for perturbations for which the decaying mode is not strongly dominating. In the opposite case, since then ζ = O(k²Φ), the k² term in equations for perturbations, e.g. in Eq. (3.16) below, may not be neglected even in the k ≪ aH case. Of course, any classical dynamical system with N degrees of freedom has exactly 2N conserved combinations of its coordinates and conjugate momenta irrespective of the fact if it is integrable or not (or even chaotic), but the functional form of these combinations is generically not universal and strongly depends on initial conditions. That is why no special attention is paid to such constant quantities in mechanics. The quantity ζ is just the examples of such a conserved combination for the growing mode. For the purely decaying mode, another conserved combination may be introduced (see Sec. III below).

It is the specifics of the inflationary scenario where the case of strongly dominated decaying mode is excluded that leads to the impression of the universal conservation of ζ in the one-field, k ≪ aH case. Of course when more modes for each k appear in a multi-field case, generic non-conservation of ζ becomes more transparent. This general remark explains numerous findings of non-conservation of this quantity in particular cases. However, this circumstance does not affect the predictive power of the inflationary paradigm at all since metric perturbations, in particular the gravitational potential Φ, may be calculated during and after inflation without any reference to ζ conservation or non-conservation. The only problem is that evolution of isocurvature modes of scalar perturbations (in contrast to adiabatic ones) is not universal, and its knowledge requires some additional assumptions about behavior of matter after the end of inflation (in particular, isocurvature modes disappear completely if the total thermodynamic equilibrium is reached at some moment of time).

A very important point in the derivation of spectra of primordial perturbations generated in the inflationary scenario is played by exact solutions of perturbations equations in the long-wave, or super-horizon limit k → 0. It is these solutions that give us a possibility to match quantum inflaton perturbations inside the de Sitter horizon during inflation to scalar perturbations during matter or radiation dominated eras, bypassing the study of physical processes in the Universe in between. That is why such solutions were sought and found for more and more complicated cases in numerous papers. In particular, in [29], two of the present authors made the first correct analysis of the issue in the case of slow-roll inflation in the original Brans-Dicke gravity, extending the method used in [30,31] to find spectra of all modes of adiabatic and isocurvature fluctuations. This was further used to constrain the parameters of general scalar-tensor theories (e.g., the Damour-Nordvedt model) from the spectrum of the CMB anisotropy in [32,33]. General analytic formula for evaluating the spectral index in multi-field inflation was developed in [34], which neglected isocurvature modes. k → 0 solutions for all modes in the case of a factorizable potential U = V₁(φ₁)V₂(φ₂) were found in [35]. Finally, a general k → 0 solution for perturbations in the two-field case with an arbitrary potential U(φ₁,φ₂) was found in [36] in the form of a functional over an inflationary background solution, from which explicit solutions for perturbations can be obtained in any case when the background solution may be explicitly integrated in the slow-roll approximation.

However, there exists an old and completely different way to obtain super-horizon (k → 0) solutions for perturbations, even without writing corresponding equations for them: the Lagrange method of variation of the background FRW solution a(t), φᵣ(t), n = 1, ..., N with respect to all constants entering in it (here n numerates different scalar fields). In the case of matter in the form of classical fields, this method always directly produces all 2N physical modes of the k → 0 solution for some quantities (in particular, for some gauge-invariant quantities, too), though to obtain the full k → 0 solution for all gauge-invariant quantities one has additionally either to use the 0 – i Einstein
equations, or to integrate the \( i - j \ (i \neq j), i, j = 1, 2, 3 \) Einstein equations. The latter necessity is related to the fact that all gauge-invariant quantities constructed from a space-time metric and its first derivatives (including a Newtonian potential \( \Phi \)) are necessarily non-local, in accordance with the Einstein equivalence principle. As a result, their expressions in terms of quantities which may be defined and measured locally contain \( k^2 \) in the denominator. So, the next order of the series in powers of \( k^2 \) (beyond that which follows from the Lagrange method) has to be computed for their determination.

Since the background FRW space-time is in the synchronous form, the Lagrangian method yields \( k \to 0 \) perturbations in the synchronous gauge, too. Of course, this method produces solutions in the explicit from only if the background solution is known explicitly, too. Thus, its power in the two-field case is the same as that of the Mukhanov-Steinhardt functional. On the other hand, it is applicable for any number of scalar fields and during any stage of the Universe evolution, not necessarily at slow-roll inflation.

In the case of adiabatic modes of perturbations (both growing and decaying ones), this method was not only known for many years, but has been already used by Lifshitz and Khalatnikov in a more advanced form – to produce non-homogeneous solutions near singularity without assuming inhomogeneous perturbations to be small. This is achieved by taking integration constants in the background FRW solution as arbitrary functions of spatial coordinates. That was low how their quasi-isotropic solution [38], or the so called 7-functional solution [39] were constructed. In the case of isocurvature modes of perturbations, this method was recently considered in details in [10] [12].

We will use this method in the case of adiabatic modes of perturbations, since it is especially simple in this case. Also, it provides a simple reason for the universal constancy of the properly defined growing adiabatic mode at super-horizon scales (different from that recently proposed in [43]) which does not depend on any local physical process, in particular, on presence or absence of preheating. On the other hand, since the Lagrange method and the Mukhanov-Steinhardt functional are equally powerful in the two-field case in the slow-roll approximation (so that we may use any of them), we will use the latter formula to derive explicit expressions for non-decreasing isocurvature modes of perturbations.

At present there are many generalized Einstein theories which can provide inflationary solutions, e.g. generalized scalar-tensor theory, Einstein gravity with a non-minimally coupled scalar field, higher-dimensional Kaluza-Klein theories, \( f(R) \) gravity theories. Making use of the conformal equivalence between these theories, cosmological perturbations can be analyzed in a unified manner (see, e.g., [14]). In the present paper we analyze density perturbations generated in slow-roll inflationary models for a general class of generalized Einstein theories in the presence of two scalar fields. We make use of the conformal transformation [13] which transforms the original, or the Jordan frame to the Einstein frame in which equations are somewhat simpler. In particular, the \( i - j \ (i \neq j) \) Einstein equations directly lead to the universal relation [33] for all models considered in our paper. If this class of gravity theories is considered as a low-energy limit of the superstring theory, then the Jordan frame is also called the string frame.

Though it is always possible to write \( k \to 0 \) solutions for perturbations in a functional form, actual evolution of perturbations depends on specific gravity theories. In the JBD theory, where the Brans-Dicke parameter is constrained to be \( \omega > 3500 \) from observations [4], the gravitational potential is dominated by adiabatic perturbations [29] [24], in which case the variation of \( R \) is restricted to be small as we will see later. On the other hand, it was recently found that negative non-minimal coupling with a second scalar field other than inflaton can lead to significant growth of \( \xi \) by the analysis in Jordan frame [17]. In this case it is not obvious whether the slow-roll analysis provides correct amplitudes of field and metric perturbations, since these exhibit strong enhancement during inflation by negative instability. In this paper we will investigate the validity of slow-roll approximations by numerical simulations in the Einstein frame. We will also work on the evolution of cosmological perturbations in a higher-dimensional theory and the \( R + R^2 \) theory with a non-minimally coupled scalar field. The latter corresponds to the case where explicit and closed forms of solutions in the super-horizon limit are obtained by slow-roll analysis in spite of a coupled form of the effective potential.

The rest of the present article is organized as follows. In Sec. II we present the Lagrangian in the Einstein frame and introduce several generalized Einstein theories which can be recasted to the Lagrangian by conformal transformations. Then in Sec. III basic equations and closed form solutions for super-horizon perturbations are given. In Sec. IV we apply the results of Sec. III to specific gravity theories, namely, the JBD theory, the Einstein gravity with a non-minimally coupled scalar field, the higher-dimensional Kaluza-Klein theory and the \( R^2 + (1/2)\xi R \chi^2 \) theory. In order to confirm analytic estimates, we also show numerical results by solving full equations of motion. We present conclusions and discussions in the final section.

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1For matter in the form of \( N \) hydrodynamic fluids, only \( N \) non-decreasing modes, including the most important growing adiabatic mode, may be obtained in this way.
II. INFLATION IN GENERALIZED EINSTEIN THEORIES

Consider the following two-field model with scalar fields $\varphi_1$ and $\varphi_2$:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} (\nabla \varphi_1)^2 - \frac{1}{2} e^{-2F(\varphi_1)} (\nabla \varphi_2)^2 - U(\varphi_1, \varphi_2) \right],$$

(2.1)

where $\kappa^2/8\pi = G$ is the Newton’s gravitational constant, $F(\varphi_1)$ is a function of $\varphi_1$, and $U(\varphi_1, \varphi_2)$ is a potential of scalar fields. Many of the generalized Einstein theories are reduced to the Lagrangian (2.1) via conformal transformations [45]. We have the following theories, which may provide inflationary solutions.

1. Theories with a scalar field $\psi$ coupled to gravity whose action is written by

$$S = \int d^4x \sqrt{-\hat{g}} \left[ f(\psi)R(\hat{g}) - h(\psi)(\nabla \psi)^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right],$$

(2.2)

where $V(\phi)$ is a potential of inflaton, $\phi$. In this work we consider the following theories.

(a) Jordan-Brans-Dicke (JBD) theory with a Brans-Dicke field, $\psi$ [15]. In this case $f$ and $h$ are

$$f = \frac{\psi}{16\pi}, \quad h = \frac{\omega}{16\pi \psi},$$

(2.3)

where $\omega$ is the Brans-Dicke parameter which is restricted as $\omega > 3500$ from observations [46]. Making a conformal transformation,

$$g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu},$$

(2.4)

where

$$\Omega^2 = \frac{\kappa^2}{8\pi \psi} \exp \left( \frac{\kappa \chi}{\sqrt{\omega + 3/2}} \right),$$

(2.5)

we obtain the action in the Einstein frame (2.1) with replacement,

$$\varphi_1 \to \chi, \quad \varphi_2 \to \phi,$$

(2.6)

and

$$F = (\beta/4)\kappa \chi, \quad U(\chi, \phi) = e^{-\beta \kappa \chi} V(\phi),$$

(2.7)

with $\beta = \sqrt{8/(2\omega + 3)}$.

(b) Non-minimally coupled massless scalar field, $\psi$, with an interaction, $(1/2)\xi R\psi^2$ [47][48]. In this case $f$ and $h$ read

$$f = \frac{1 - \xi \kappa^2 \psi^2}{2\kappa^2}, \quad h = \frac{1}{2},$$

(2.8)

Applying the conformal transformation (2.4) with $\Omega^2 = 1 - \xi \kappa^2 \psi^2$ and defining a new field $\chi$ in order for the kinetic term to be canonical as

$$\chi = \int \sqrt{\frac{1 - (1 - 6\xi)\xi \kappa^2 \psi^2}{(1 - \xi \kappa^2 \psi^2)^2}} d\psi,$$

(2.9)

we obtain the action (2.1) with replacement (2.6) and

$$F = \frac{1}{2} \ln |1 - \xi \kappa^2 \psi^2|, \quad U(\chi, \phi) = e^{-4F(\chi)} V(\phi) = \frac{V(\phi)}{(1 - \xi \kappa^2 \psi^2)^2},$$

(2.10)
The induced gravity theory \[49\] is also described by the action, (2.2). In this theory the scalar field \(\psi\) has its own potential of the form, \(V(\psi) = (\lambda/8)(\psi^2 - \eta^2)^2\) with \(f = (\epsilon/2)\psi^2\) and \(h = 1/2\).

2. The higher-dimensional theories where the inflaton, \(\bar{\phi}\), is introduced in \(N = D + 4\) dimensions

\[
S = \int d^N x \sqrt{-\bar{g}} \left[ \frac{\bar{R}}{2\bar{\kappa}^2} - \frac{1}{2} (\nabla\bar{\phi})^2 - \bar{V}(\bar{\phi}) \right],
\]

where \(\bar{\kappa}^2\) and \(\bar{R}\) are the \(N\)-dimensional gravitational constant and a scalar curvature, respectively. We compactify the \(N\)-dimensional spacetime into the four-dimensional spacetime and the \(D\)-dimensional internal space with length scale, \(b\). Then the metric can be expressed as

\[
ds^2_N = \hat{g}_{\mu\nu} dx^\mu dx^\nu + b^2 ds^2_D,
\]

where \(\hat{g}_{\mu\nu}\) is a four-dimensional metric. Assuming that extra dimensions are compactified on a torus which has zero curvature, one gets the following action after dimensional reduction \[21\]:

\[
S = \int d^4 x \sqrt{-\hat{g}} \left[ \hat{R} + d(d-1) \frac{\partial b \partial b}{b^2} \hat{g}^{\mu\nu} - \frac{\bar{\kappa}^2}{2} \left\{ \frac{1}{2} (\nabla\hat{\phi})^2 - \bar{V}(\hat{\phi}) \right\} \right],
\]

where \(b_0\) is the present value of \(b\), and \(\hat{R}\) is the scalar curvature with respect to \(\hat{g}_{\mu\nu}\). In order to obtain the Einstein-Hilbert action, we make the conformal transformation (2.4) with a conformal factor,

\[
\Omega^2 = \exp \left( D \frac{\chi}{\chi_0} \right),
\]

where a new scalar field, \(\chi\), is defined by

\[
\chi = \chi_0 \ln \left( \frac{b}{b_0} \right), \quad \text{with} \quad \chi_0 = \left[ \frac{D(D+2)}{2\bar{\kappa}^2} \right]^{1/2}.
\]

Then the four-dimensional action in the Einstein frame can be described as (2.1) with replacement (2.6) and

\[
F = 0, \quad U(\chi, \phi) = \exp \left( -\beta \kappa \chi \right) V(\phi), \quad \text{with} \quad \beta = \sqrt{\frac{2D}{D+2}}.
\]

Note that when inflaton is introduced in the four-dimensional action after compactification, we find \(F = e^{-\beta/2} \kappa \chi\) and \(U(\chi, \phi) = e^{-\beta \kappa \chi} V(\phi)\) with \(\beta = \sqrt{8D/(D+2)}\). In this case, however, we do not have inflationary solutions since the effective potential does not satisfy the condition: \(\beta < \sqrt{2}\), which is required for power-law inflation to occur (see the next section).

3. The \(f(R)\) theories where the Lagrangian includes the higher-order curvature terms, i.e., \(\partial f/\partial R\) depends on the scalar curvature \(R\) \[44\]:

\[
S = \int d^4 x \sqrt{-g} \left[ f(R) - \frac{1}{2} (\nabla\chi)^2 \right].
\]

In this case the conformal factor

\[
\Omega^2 = 2\bar{\kappa}^2 \left| \frac{\partial f}{\partial R} \right|,
\]

\[
\text{Note that there exist other methods of compactifications. One of them is the compactification on the sphere \[50\], in which case stability of extra dimensions and the evolution of cosmological perturbations during inflation are studied in \[51, 52\].}
\]

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describes a dynamical freedom in the Einstein-Hilbert action. Introducing a new scalar field
\[ \phi = \sqrt{\frac{3}{2\kappa^2}} \ln \left[ 2\kappa^2 \left| \frac{\partial f}{\partial R} \right| \right], \] (2.19)
the action in the Einstein frame is described as (2.1) with replacement,
\[ \varphi_1 \to \phi, \quad \varphi_2 \to \chi, \] (2.20)
and
\[ F = \frac{\kappa \phi}{\sqrt{6}}, \quad U(\phi, \chi) = (\text{sign}) \exp \left( -\frac{2\sqrt{6}}{3} \kappa \phi \right) \left[ \frac{(\text{sign})}{2\kappa^2} R(\phi, \chi) \exp \left( \frac{\sqrt{6}}{3} \kappa \phi \right) - f(\phi, \chi) \right], \] (2.21)
where sign = \( (\partial f/\partial R)/|\partial f/\partial R| \). For example, in the \( R^2 \) theory with a non-minimally coupled massless \( \chi \) field, i.e.,
\[ f(R) = \frac{1}{2\kappa^2} R + qR^2 - \frac{1}{2} \xi R\chi^2, \] (2.22)
the effective two-field potential is described as
\[ U(\phi, \chi) = \frac{m_{\text{pl}}^4}{(32\pi)^2 q} e^{-2(\sqrt{6}/3)\kappa \phi} \left( e^{(\sqrt{6}/3)\kappa \phi} - 1 + \xi \kappa^2 \chi^2 \right)^2, \] (2.23)
where we have chosen a positive sign. In this case the \( \phi \) field behaves as an inflaton and leads to an inflationary expansion of the Universe.

III. COSMOLOGICAL PERTURBATIONS IN TWO-FIELD INFLATION

A. Basic equations and solution for the adiabatic mode

Let us first derive the background equations. Variation of the action (2.1) yields the following background equations for the cosmic expansion rate \( H = \dot{a}/a \) and homogeneous parts of scalar fields:
\[ H^2 = \frac{\kappa^2}{3} \left( \frac{1}{2} \dot{\varphi}_1^2 + \frac{1}{2} e^{-2F} \dot{\varphi}_2^2 + U \right), \] (3.1)
\[ \dot{H} = -\frac{\kappa^2}{2} \left( \dot{\varphi}_1^2 + e^{-2F} \dot{\varphi}_2^2 \right), \] (3.2)
\[ \dot{\varphi}_1 + 3H \varphi_1 + U_{,\varphi_1} + F_{,\varphi_1} e^{-2F} \dot{\varphi}_2 = 0, \] (3.3)
\[ \dot{\varphi}_2 + 3H \varphi_2 + e^{2F} U_{,\varphi_2} - 2F_{,\varphi_1} \varphi_1 \varphi_2 = 0, \] (3.4)
where a prime denotes a derivative with respect to \( \varphi_1 \). A generic solution of this system contains 5 arbitrary integration constants (two constants appear from the solution of Eq. (3.3), two – from Eq. (3.4), and one from Eq. (3.1), while Eq. (3.2) is a consequence of the other equations). However, one of these constants corresponds to a trivial shift of the cosmic time \( t \). The Lagrange variation with respect to these constant yields a gauge mode. So, variation with respect to only 4 constants may be used to produce physical solutions for perturbations in the long-wave limit.

Moving to perturbations now, we restrict ourselves to the spatially flat FRW background, first, for simplicity and, second, because recent data on angular fluctuations of the cosmic microwave background (CMB) convincingly confirm the absence of any significant spatial curvature of the Universe within a few percent accuracy (see, e.g., [34]). Then a perturbed space-time metric has the following form for scalar perturbation in an arbitrary gauge:
\[ ds^2 = -(1 + 2A)dt^2 + 2a(t)B_i dx^i dt + a^2(t)(1 + 2D)\delta_{ij} + 2E_{i,j}dx^i dx^j, \quad i, j = 1, 2, 3, \]  

where a comma means usual flat space coordinate derivative and \( \Delta \) is the flat 3D Laplacian. In the synchronous gauge, \( A = B = 0, D = (\lambda + \mu)/6, \Delta E = -\lambda/2 \) in the Lifshitz notations. In the longitudinal gauge, \( B = E = 0, A = \Phi, D = -\Psi, \) and \( \Phi, \Psi \) are gauge-invariant potentials [43–46]. Further, we assume the \( \exp(ikx) \) dependence for each Fourier mode \( k \) and omit the subscript \( k = |k| \) in expressions for time-dependent parts of perturbations. Note the useful relations between \( \lambda, \mu, \Phi, \Psi, \) and \( \delta \Phi, \delta \Psi \) in variables like \( H, t \) for perturbations. This is possible since one of integration constants of a background FRW solution, namely, \( \frac{q_n}{\Delta} \), follows from the \( \lambda, \mu, \Phi, \Psi \) dependence:

\[ \Phi = -\frac{1}{2k^2} \frac{d}{dt}(a^2 \dot{\lambda}), \quad \Psi = -\frac{1}{6}(\lambda + \mu) + \frac{a\dot{a}}{2k^2} \dot{\lambda}, \quad \delta \Phi_n = \delta \Phi_{S,n} - \frac{\dot{\phi}_n}{2H}. \]  

Another useful gauge-invariant scalar field perturbation is given by the Mukhanov-Sasaki variable [57]:

\[ q_n = \delta \phi_n + \frac{\dot{\phi}_n}{H} \Psi = \delta \phi_{S,n} - \frac{\dot{\phi}_n}{6H}(\lambda + \mu). \]  

For all models considered in our paper, we have the following relation in the Einstein frame

\[ \Phi = \Psi, \]  

which follows from the \( i - j (i \neq j) \) Einstein equations taking into account the fact that anisotropic stresses vanish at the linear order there. Transforming back to the Jordan frame, this relation does not hold generically [28,44].

Now we can derive one solution for super-horizon perturbations using the Lagrange method without writing equations for perturbations. This is possible since one of integration constants of a background FRW solution, namely, \( \frac{q_n}{\Delta} \), is simple a multiplier \( a_0 \) of \( a(t) \). Of course, due to invariance of measurable quantities with respect to equal rescaling of all 3 spatial coordinates \( x^i \), this constant does not appear in variables like \( H(t) \) and \( \phi_n(t) \). So, in the \( k \to 0 \) limit,

\[ \mu = \frac{6}{a_0} \frac{\delta a_0}{a_0} \equiv 3h = \text{const}; \quad \lambda, \delta \phi_{S,n} = \mathcal{O}(k^2h); \quad q_n = -\frac{h}{2H} \frac{\dot{\phi}_n}{2H}. \]  

This formula is valid both in the Jordan and the Einstein frames. By definition, this partial solution will be called the growing adiabatic mode (we will discuss the ambiguity of this definition later). It is clear that this solution exists for any form of the gravity Lagrangian and the matter energy-momentum tensor. In particular, presence of fast oscillations in a background solution does not affect it, too. The only things which are needed for its existence are the spatial flatness and isotropy of the background metric. That is why the derivation of the spectrum of adiabatic oscillations in a background solution does not affect it, too. The only things which are needed for its existence are the spatial flatness and isotropy of the background metric. That is why the derivation of the spectrum of gravitational waves produced during inflation in [8].

Let us now calculate the gauge-invariant quantities \( \Phi \) and \( \delta \phi_n \) for the solution (3.9) in the Einstein frame where the relation (3.8) holds. In the synchronous gauge, the latter relation reads

\[ \dot{\lambda} + 3H \dot{\lambda} - \frac{k^2}{3a^2}(\lambda + \mu) = 0. \]  

This thus, in the \( k \to 0 \) limit, we get

\[ \lambda = \frac{h}{a^3} \int_{t_1}^t a dt, \quad \Phi = \Psi = -\frac{h}{2} \left( 1 - \frac{H}{a} \int_{t_1}^t a dt \right), \quad \delta \phi_n = -\frac{h}{2a} \int_{t_1}^t a dt. \]  

\((t_1 \text{ depends on } k \text{ generically). A shift in the } t_1 \text{ produces one more superhorizon solution -- the decaying adiabatic mode:}

\[ \lambda = h_1 a^{-3}, \quad \mu = \mathcal{O}(h_1 k^2), \quad \Phi = \Psi = h_1 H/2a, \quad \delta \phi_n = -h_1 \dot{\phi}_n/2a. \]  

Thus, the growing adiabatic mode (3.9) is defined up to an addition of some amount of the decaying adiabatic mode (3.12). In the inflationary scenario, \( t_1 \) is the moment of the first Hubble radius crossing during inflation for each
k, so there is no ambiguity at all. Thus, as a whole the adiabatic solution (mode) is defined unambiguously by Eqs. (3.9, 3.11). On the other hand, one may always add some amount of the adiabatic mode to other, isocurvature solutions (modes). So, the definition of isocurvature modes is not unique. In particular, this ambiguity may be used to make \( \Phi = \Psi = 0 \) for an isocurvature mode in the Einstein frame at some chosen moment of time, e.g., at the end of inflation or at the moment when the full thermal equilibrium is reached (if the latter occurs at all). Of course, this choice does not affect any observable quantities.

A useful gauge-invariant quantity is the comoving curvature perturbation \( \mathcal{R} \) [10,27]:

\[
\mathcal{R} = -\frac{1}{6}(\lambda + \mu) + \frac{\dot{\lambda} + \dot{\mu}}{6H} = \Psi - \frac{H}{H}(\dot{\Psi} + H \Phi) .
\]  

A similar quantity is the curvature perturbation on uniform density hypersurfaces introduced in [26]:

\[
\zeta = D - H \frac{\delta \rho}{\rho} = \left( \frac{1}{6} - \frac{k^2}{18a^2H} \right) (\lambda + \mu) - \frac{H}{6H} \dot{\mu} = -\mathcal{R} + \frac{k^2}{3a^2H} \Psi
\]  

where \( \rho \) is the total energy density of matter and \( \delta \rho \) – its perturbation. For the solution (3.9), we get

\[
\mathcal{R} = -\zeta = -\frac{h}{2} = \text{const} \neq 0.
\]  

Thus, we have the theorem:

In the superhorizon limit \( k \to 0 \) and for \( \dot{H} \neq 0 \) identically, there always exists one solution for perturbations (the growing adiabatic mode) for which \( \mathcal{R} = -\zeta \) is conserved in the leading order in \( k^2 \) (apart from the vicinity of points where \( H = 0 \)).

Let us emphasize that this statement is valid both in the Jordan and the Einstein frames.

Pathological behavior of \( \mathcal{R} \) and \( \zeta \) at the moments of time when \( H = 0 \) where they diverge (if terms of the next order in \( k^2 \) are taken into account) is solely an artifact of their definition. The potentials \( \Phi \) and \( \Psi \) remain regular and small near these points, so no consideration of non-linear effects is required there. Moreover, it immediately follows from the formulas (3.9, 3.11) that the same constant value (3.15) of \( \mathcal{R} \) and \( \zeta \) is restored after passing through any point where \( H = 0 \). This behavior of \( \zeta \) is clearly seen in our numerical calculations in Figs. 1 and 3 below.

What about conservation of \( \mathcal{R} \) and \( \zeta \) for the decaying adiabatic mode (3.12)? Here, there is a subtle point. In the leading order, \( \mathcal{R} = \zeta = 0 \) for this mode. However, one may not say that these quantities are conserved (even approximately). Real (approximate) conservation would require \( H|\mathcal{R}/\mathcal{R}| \ll 1 \) that is not valid if the decaying adiabatic mode is strongly dominating [58]. Also, the relation \( \mathcal{R} \approx -\zeta \) is not correct for this mode. On the other hand, in the Einstein frame one can introduce the gauge invariant quantity \( T \equiv a\Phi/H \) which does conserve in the super-horizon limit in the leading order in \( k^2 \).

Finally, for other solutions (isocurvature modes) \( \mathcal{R} \) and \( \zeta \) are not conserved in the super-horizon limit, too. This can be easily seen by considering the time derivative of \( \mathcal{R} \) [33,36]:

\[
\mathcal{R} = \frac{H k^2}{H a^2} \frac{\delta \varphi_1}{\varphi_1} + H \left( \frac{\delta \varphi_1}{\varphi_1} - \frac{\delta \varphi_2}{\varphi_2} \right) Z,
\]  

where

\[
Z \equiv \frac{2e^{-2F} \varphi_1 \dot{\varphi}_2 (\dot{\varphi}_1 \varphi_2 - \dot{\varphi}_1 \varphi_2 + \dot{F} \varphi_1 \varphi_2) + F_{,\varphi_1} \varphi_1 e^{-4F} \dot{\varphi}_2^2}{(\dot{\varphi}_1^2 + e^{-2F} \dot{\varphi}_2^2)^2}.
\]  

The quantity

\[
S_{\varphi_1 \varphi_2} \equiv H \left( \frac{\delta \varphi_1}{\varphi_1} - \frac{\delta \varphi_2}{\varphi_2} \right),
\]  

represents a generalized entropy perturbation between \( \varphi_1 \) and \( \varphi_2 \) fields [59,60]. In the multi-field case, \( S_{\varphi_1 \varphi_2} \neq 0 \) and \( Z \neq 0 \) generically. So, the presence of isocurvature perturbations leads to the variation of \( \mathcal{R} \approx -\zeta \).

Note also that we use the terms "adiabatic perturbations" and "isocurvature perturbations", as is commonly done, to denote different solutions (modes) of the same physical variables. This should be contrasted with the approach of the recent paper [59] where "adiabatic" and "entropy" fields are introduced which are certain linear combinations of initial fields \( \varphi_n(t) \), and then adiabatic and entropy perturbations mean perturbations of these fields. However, for our adiabatic solution (3.9) the perturbed entropy field \( \delta s \propto (\dot{\varphi}_1 \delta \varphi_2 - \dot{\varphi}_2 \delta \varphi_1) = 0 \) in the super-horizon limit,
even if the background field trajectory in the scalar field space is curved. Thus, the adiabatic mode is not sourced by isocurvature (entropy) modes, in contrast to results of [59].

That is all what may be obtained without solving equations for perturbations. So, to proceed further, equations for time-dependent parts of metric and field fluctuations have to be written. They read:

\[
\ddot{\Phi} + 4H\dot{\Phi} + \kappa^2 U\Phi = \frac{\kappa^2}{2} \left[ \dot{\phi}_1 \delta \phi_1 - (U, \dot{\phi}_1 + F, \dot{\phi}_1 e^{-2F} \dot{\phi}_2^2) \delta \phi_1 + e^{-2F} \dot{\phi}_2 \delta \phi_2 - U, \delta \phi_2 \right],
\]

(3.19)

\[
\left( \frac{k^2}{a^2} - H \right) \Phi = -\frac{\kappa^2}{2} \left[ \dot{\phi}_1 \delta \phi_1 + (3H \dot{\phi}_1 + U, \dot{\phi}_1 - F, \dot{\phi}_1 e^{-2F} \dot{\phi}_2^2) \delta \phi_1 + e^{-2F} \dot{\phi}_2 \delta \phi_2 + (U, \delta \phi_2 + 3H \dot{\phi}_2 e^{-2F}) \delta \phi_2 \right],
\]

(3.20)

\[
\frac{\delta \phi_1}{\dot{\phi}_1} + 3H \delta \phi_1 + \left[ \frac{k^2}{a^2} + U, \dot{\phi}_1 \dot{\phi}_1 - (e^{-2F})_{\dot{\phi}_1 \dot{\phi}_1} \frac{\dot{\phi}_1^2}{2} \right] \frac{\delta \phi_1}{\dot{\phi}_1} + 2F, \dot{\phi}_1 e^{-2F} \dot{\phi}_2 \delta \phi_2 + U, \delta \phi_2 \delta \phi_2 = 4\dot{\phi}_1 \dot{\phi}_1 - 2U, \dot{\phi}_1 \dot{\phi}_1,
\]

(3.21)

\[
\frac{\delta \phi_2}{\dot{\phi}_2} + (3H - 2\dot{\Phi}) \delta \phi_2 + \left[ \frac{k^2}{a^2} + e^{2F} U, \dot{\phi}_2 \dot{\phi}_2 \right] \frac{\delta \phi_2}{\dot{\phi}_2} - 2\dot{\Phi} \delta \phi_1 + e^{2F} (2F, \dot{\phi}_1 U, \dot{\phi}_2 + U, \dot{\phi}_1 \dot{\phi}_2) \delta \phi_1 = 4\dot{\phi}_2 \dot{\phi}_2 - e^{2F} U, \dot{\phi}_2 \dot{\phi}_2.
\]

(3.22)

The relation (3.20) clearly indicates that metric perturbations are determined when the evolution of scalar fields is known.

### B. Closed form solutions in slow-roll approximations

The use of the slow-roll approximation allows us to obtain closed form solutions for isocurvature perturbations in the long-wave limit [29][32][33][37]. Under this approximation, the background equations are simplified as

\[
H^2 = \frac{\kappa^2}{3} U,
\]

(3.24)

\[
3H \dot{\phi}_1 + U, \dot{\phi}_1 = 0,
\]

(3.25)

\[
3H \dot{\phi}_2 + e^{2F} U, \dot{\phi}_2 = 0.
\]

(3.26)

Combining Eqs. (3.24), (3.26) with Eq. (3.2), we find

\[
\dot{\phi}_1 = -\frac{H U, \dot{\phi}_1}{\kappa^2 U}, \quad \dot{\phi}_2 = -\frac{H e^{2F} U, \dot{\phi}_2}{\kappa^2 U},
\]

(3.27)

\[
-\frac{\dot{H}}{H^2} = \frac{1}{2\kappa^2} \left[ \left( \frac{U, \dot{\phi}_1}{U} \right)^2 + e^{2F} \left( \frac{U, \dot{\phi}_2}{U} \right)^2 \right].
\]

(3.28)

In JBD and higher-dimensional theories where the potentials take the form, \( U(\phi_1, \phi_2) = e^{-\beta \kappa \phi_1} V(\phi_2) \), it is straightforward to show that the scale factor evolves as power-law. In this case integrating \( \dot{\phi}_1 = \beta H / \kappa \) over \( t \), we find

\[
\varphi_1(t) = \frac{\beta}{\kappa} \ln \frac{a(t)}{a_f} + \varphi_{1f} \equiv -\frac{\beta}{\kappa} z + \varphi_{1f},
\]

(3.29)
where a subscript $f$ denotes the value of each quantity at the end of inflation and $z$ is the number of $e$-folds of inflationary expansion after the time $t$. Assuming $V(\varphi_2)$ takes a constant value $V_0$ during inflation, one finds

$$a = a_0 \left[ \frac{\kappa^2}{3} e^{-\beta \kappa \varphi_1(0)} V_0 \right]^{1/2} \frac{\beta^2}{2} (t - t_0)^{2/\beta^2},$$  

(3.30)

where $a_0$ and $t_0$ are constants. Then we have a power-law inflationary solution when $\beta < \sqrt{2}$. For example, in the JBD case with potential (2.7), inflation takes place with a large power exponent because $\beta$ is constrained to be $\beta = \sqrt{8/2\omega + 3} \lesssim 0.034$ from observations (46). In higher-dimensional theories with potential (2.16), inflation is realized for arbitrary extra dimensions $D$, because the condition, $\beta < \sqrt{2}$, always holds.

Let us consider large scale perturbations with $k \ll aH$. Neglecting $\Phi$ and those terms which include second order time derivatives in Eqs. (3.21)-(3.23), one finds

$$\Phi = \frac{\kappa^2}{2H} (\varphi_1 \delta \varphi_1 + e^{-2F} \varphi_2 \delta \varphi_2),$$  

(3.31)

$$3H \delta \varphi_1 + U_{,\varphi_1} \delta \varphi_1 + U_{,\varphi_1} \delta \varphi_2 + 2U_{,\varphi_2} \Phi = 0,$$  

(3.32)

$$3H \delta \varphi_2 + (e^{2F} U_{,\varphi_1})_{,\varphi_1} \delta \varphi_1 + (e^{2F} U_{,\varphi_2})_{,\varphi_2} \delta \varphi_2 + 2e^{2F} U_{,\varphi_2} \Phi = 0.$$  

(3.33)

Note that this approximation may not be always valid especially when perturbations exhibit nonadiabatic growth during inflation. We will check its validity in the next section.

In the slow-roll approximation, the expression (3.11) for the growing adiabatic mode is simplified:

$$\Phi = \Psi = \frac{H}{2H}, \quad \delta \varphi_n = -\frac{\dot{\varphi}_n}{2H}.$$  

(3.34)

To find a generic solution, we introduce new variables, $x$ and $y$, with $\delta \varphi_1 = U_{,\varphi_1} x$ and $\delta \varphi_2 = e^{2F} U_{,\varphi_2} y$. Then Eqs. (3.32) and (3.33) yield

$$3H \dot{x} + \frac{U_{,\varphi_1} \varphi_2 e^{2F}}{U_{,\varphi_1}} (y - x) + 2\Phi = 0,$$  

(3.35)

$$3H \dot{y} + \frac{(e^{2F} U_{,\varphi_1})_{,\varphi_1} U_{,\varphi_1} e^{-2F}}{U_{,\varphi_2}} (x - y) + 2\Phi = 0.$$  

(3.36)

Subtracting Eq. (3.35) from Eq. (3.36), we obtain the following integrated solution

$$y - x = Q_3 \exp \left[ \int \frac{A}{3H} dt \right],$$  

(3.37)

where $Q_3$ is a constant, and

$$A \equiv \frac{U_{,\varphi_1} \varphi_2 U_{,\varphi_2} e^{2F}}{U_{,\varphi_1}} + \frac{(e^{2F} U_{,\varphi_2})_{,\varphi_1} U_{,\varphi_1} e^{-2F}}{U_{,\varphi_2}}.$$  

(3.38)

Taking notice of the relation

$$\Phi = \frac{\kappa^2}{2H} \left[ \dot{U} x + U_{,\varphi_2} \varphi_2 (y - x) \right] = \frac{\kappa^2}{2H} \left[ \dot{U} y + U_{,\varphi_1} \varphi_1 (x - y) \right],$$  

(3.39)

and making use of Eq. (3.37) and background equations (3.24)-(3.26), we find

$$x = -\frac{Q_3}{U} \int \left[ \frac{H U_{,\varphi_1} \varphi_2 U_{,\varphi_2} e^{2F}}{U_{,\varphi_1}} + U_{,\varphi_2} \varphi_2 \right] \frac{J}{U} dt,$$  

(3.40)

$$y = -\frac{Q_3}{U} \int \left[ \frac{H (e^{2F} U_{,\varphi_1})_{,\varphi_1} U_{,\varphi_1} e^{-2F}}{U_{,\varphi_2}} + U_{,\varphi_1} \varphi_1 \right] \frac{J}{U} dt.$$  

(3.41)
In this case, we have an integration constant by making use of Eq. (3.27):

\[ J \equiv U \exp \left[ \int \frac{A}{3H} dt \right]. \tag{3.42} \]

Then the final closed-form solutions for long-wave perturbations are expressed as

\[ \delta \varphi_1 = (\ln U)_{\varphi_1} \left[ Q_1 + Q_3 \int_{t_*}^t (\ln(\ln U)_{\varphi_1})_{\varphi_2} J d\varphi_2 \right], \tag{3.43} \]

\[ \delta \varphi_2 = e^{2F}(\ln U)_{\varphi_2} \left[ Q_2 - Q_3 \int_{t_*}^t (\ln(e^{2F}(\ln U))_{\varphi_2})_{\varphi_1} J d\varphi_1 \right], \tag{3.44} \]

\[ \Phi = -\frac{1}{2} [ (\ln U)_{\varphi_1} \delta \varphi_1 + (\ln U)_{\varphi_2} \delta \varphi_2 ] , \tag{3.45} \]

with

\[ J = \exp \left\{ - \int_{t_*}^t \left[ (\ln (e^{2F}(\ln U))_{\varphi_2})_{\varphi_1} d\varphi_1 + (\ln(\ln U)_{\varphi_1})_{\varphi_2} d\varphi_2 \right] \right\}. \tag{3.46} \]

Here integration constants, \( Q_1 \), \( Q_2 \), and \( Q_3 \) satisfy the relation \( Q_2 = Q_1 + Q_3 \), which comes from Eq. (3.37). The \( Q_3 \) terms appear due to the presence of isocurvature perturbations. These constants are evaluated by the amplitudes of quantum fluctuations of scalar fields at horizon crossing, \( t_* \). The fluctuations are generated by small scale perturbations \((k > aH)\), so that they can be considered as free massless scalar fields which are described by independent random variables \[29,31\]. Then the field perturbations when they crossed the Hubble radius \((k \simeq aH)\) are written in the form:

\[ \delta \varphi_1(k)|_{t=t_*} = \frac{H(t_*)}{\sqrt{2k^3}} e_{\varphi_1}(k), \quad \delta \varphi_2(k)|_{t=t_*} = \frac{H(t_*)}{\sqrt{2k^3}} e^{F(t_*)} e_{\varphi_2}(k). \tag{3.47} \]

Here \( e_{\varphi_1} \) and \( e_{\varphi_2} \) are classical stochastic Gaussian quantities, described by

\[ \langle e_{\varphi_i}(k) \rangle = \langle e_{\varphi_2}(k) \rangle = 0, \quad \langle e_{\varphi_i}(k)e_{\varphi_j}^*(k') \rangle = \delta_{ij}\delta^{(3)}(k-k'), \tag{3.48} \]

where \( i, j = \varphi_1, \varphi_2 \). From Eqs. (3.27), (3.43), and (3.44), we find

\[ \frac{\delta \varphi_1}{\varphi_1} = -\frac{\kappa^2}{H} \left[ Q_1 + Q_3 \int_{t_*}^t (\ln(\ln U)_{\varphi_1})_{\varphi_2} J d\varphi_2 \right], \tag{3.49} \]

\[ \frac{\delta \varphi_2}{\varphi_2} = -\frac{\kappa^2}{H} \left[ Q_2 - Q_3 \int_{t_*}^t (\ln(e^{2F}(\ln U))_{\varphi_2})_{\varphi_1} J d\varphi_1 \right]. \tag{3.50} \]

Making use of Eqs. (3.47), (3.49), and (3.50), the integration constants are expressed as

\[ Q_1 = -\frac{H^2(t_*)}{\kappa^2 \sqrt{2k^3}} \left( \frac{e_{\varphi_1}(k)}{\varphi_1} \right)_{t_*}, \quad Q_3 = -\frac{H^2(t_*)}{\kappa^2 \sqrt{2k^3}} \left( \frac{e_{\varphi_1}(k)}{\varphi_1} - e^{F} \frac{e_{\varphi_2}(k)}{\varphi_2} \right)_{t_*}. \tag{3.51} \]

In the next section, we will investigate the evolution of large-scale perturbations in specific gravity theories.

### IV. APPLICATIONS TO SPECIFIC GRAVITY THEORIES

Among the generalized Einstein theories which we presented in Sec. II, most of them take the following separated form of potentials except for the \( f(R) \) theories:

\[ U(\varphi_1, \varphi_2) = V_1(\varphi_1)V_2(\varphi_2). \tag{4.1} \]

In this case, we have an integration constant by making use of Eq. (3.27):
The gravitational potential can also be decomposed in a different way. For example, let us introduce the integration
\[ C \equiv \frac{V_1^\prime}{V_1} \epsilon_1^2 + \frac{V_2^\prime}{V_2} \delta_2, \]
(4.2)
This characterizes the trajectory in field space in two-field inflation.

Since \( J = e^{-2(F-F_c)} \) in the separated potential, Eqs. (3.43) and (3.44) are easily integrated to give
\[ \delta_1 = \frac{V_1^\prime}{V_1} Q_1, \quad \delta_2 = \frac{V_2^\prime}{V_2} [Q_1 e^{2F} + Q_3 e^{2F_c}] \quad (4.3) \]

together with the gravitational potential,
\[ \Phi = -\frac{1}{2} \left( \frac{V_1^\prime}{V_1} \right)^2 Q_1 - \frac{1}{2} \left( \frac{V_2^\prime}{V_2} \right)^2 (Q_1 e^{2F} + Q_3 e^{2F_c}). \]
(4.4) Introducing new integration constants, \( C_1 \equiv -\kappa^2 (Q_1 + Q_3 e^{2F_c}) \) and \( C_3 \equiv -\kappa^2 Q_3 e^{2F_c} \), and making use of Eq. (3.28), one finds
\[ \delta_1 = -(C_1 - C_3) \frac{V_1^\prime}{\kappa^2 V_1}, \quad \delta_2 = -\left[ C_1 e^{2F} - C_3 (e^{2F} - 1) \right] \frac{V_2^\prime}{\kappa^2 V_2}, \quad (4.5) \]

\[ \Phi = C_1 \left( \epsilon_1 + e^{2F} \epsilon_2 \right) - C_3 \left[ \epsilon_1 + (e^{2F} - 1) \epsilon_2 \right] = -C_1 \frac{\dot{H}}{H^2} - C_3 \left[ \epsilon_1 + (e^{2F} - 1) \epsilon_2 \right], \quad (4.6) \]

where \( \epsilon_1 \) and \( \epsilon_2 \) are given by
\[ \epsilon_1 \equiv \frac{1}{2\kappa^2} \left( \frac{V_1^\prime}{V_1} \right)^2, \quad \epsilon_2 \equiv \frac{1}{2\kappa^2} \left( \frac{V_2^\prime}{V_2} \right)^2. \]
(4.7)

From Eq. (3.51) \( C_1 \) and \( C_3 \) are expressed as
\[ C_1 = \frac{H^2(t_\ast)}{\sqrt{2k^3}} \left[ 1 - e^{2F} \frac{\epsilon_1}{\phi_1} + e^{3F} \frac{\epsilon_2}{\phi_2} \right]_{t_\ast}, \quad C_3 = \frac{H^2(t_\ast)}{\sqrt{2k^3}} \left[ e^{3F} \frac{\epsilon_2}{\phi_2} - e^{2F} \frac{\epsilon_1}{\phi_1} \right]_{t_\ast}. \]
(4.8)

The gravitational potential can also be decomposed in a different way. For example, let us introduce the integration constants \( \tilde{C}_1 \) and \( \tilde{C}_3 \), defined by
\[ \tilde{C}_1 \equiv -\kappa^2 Q_1 - \frac{\kappa^2 e^{2(F - F_c)}}{1 + \alpha_f} Q_3 = \frac{H^2(t_\ast)}{\sqrt{2k^3}} \left[ \left( 1 - \frac{e^{2(F - F_c)}}{1 + \alpha_f} \right) \frac{\epsilon_1}{\phi_1} + \frac{e^{2F - 2F_c}}{1 + \alpha_f} \frac{\epsilon_2}{\phi_2} \right]_{t_\ast}, \]
\[ \tilde{C}_3 \equiv -\kappa^2 e^{2(F - F_c)} Q_3 = \frac{H^2(t_\ast)}{\sqrt{2k^3}} \left[ e^{3F - 2F_c} \frac{\epsilon_2}{\phi_2} - e^{2(F - F_c)} \frac{\epsilon_1}{\phi_1} \right]_{t_\ast}, \quad (4.9) \]

where the subscript \( f \) denotes the value at the end of inflation, and
\[ \alpha \equiv \frac{\epsilon_1}{e^{2F} \epsilon_2}. \]
(4.10)

Then Eq. (4.4) reads
\[ \Phi = -\frac{\tilde{C}_1}{\tilde{C}_3} \frac{\dot{H}}{H^2} - \tilde{C}_3 \left[ \frac{\epsilon_1}{1 + \alpha_f} + \left( \frac{e^{2F}}{1 + \alpha_f} - e^{2F_c} \right) \frac{\epsilon_2}{\phi_2} \right]. \]
(4.11)

This decomposition corresponds to the case where the second term in the rhs of Eq. (4.11) vanishes at the end of inflation.

During slow-roll we have that \( \dot{H} / H^2 \ll 1 \) in Eq. (3.28), which yields from Eqs. (3.13), (4.6), and (4.11) as
\[ \mathcal{R} \simeq -\frac{H^2}{H} \Phi = C_1 - C_3 \frac{\epsilon_1 + (e^{2F} - 1) \epsilon_2}{\epsilon_1 + e^{2F} \epsilon_2} \]
\[ = \tilde{C}_1 - \tilde{C}_3 \frac{\epsilon_1 + (e^{2F} - (1 + \alpha_f) e^{2F_c}) \epsilon_2}{(1 + \alpha_f)(\epsilon_1 + e^{2F} \epsilon_2)}. \]
(4.12)
Note that both decompositions coincide each other in the limit \( \alpha_f \to 0 \) and \( F_f \to 0 \). When \( \alpha_f \) and \( F_f \) are nonvanishing, the second term in Eq. (4.10) gives contributions to the gravitational potential, \( \Phi \). When this term is negligible relative to \(-C_1 \dot{H}/H^2\) during the whole stage of inflation, the first and second terms in Eq. (4.10) can be identified as adiabatic and isocurvature modes, respectively. However, in some generalized Einstein theories which we discuss in the following subsections, the final \( \Phi \) is dominated by the second term in Eq. (4.10). In those cases we can no longer regard the second term as the isocurvature mode at the end of inflation.

The relation (4.12) or (4.13) indicates that \( R \) is not generally conserved during inflation when both fields are evolving due to the presence of isocurvature perturbations. The generalized entropy perturbations are written as

\[
S_{\varphi_1 \varphi_2} = -C_3 e^{-2F} = -\tilde{C}_3 e^{2(F_f - F)}. \tag{4.14}
\]

Since \( S_{\varphi_1 \varphi_2} \) and \( Z \) do not vanish generally, these work as a source term for the change of \( R \). The evolution of \( R \) depends on specific gravity theories as we will show below.

### A. JBD theory

Let us first apply the results in the previous section to the JBD theory with Eq. (2.7). In this case Eqs. (4.6) and (4.12) are

\[
\Phi = -C_1 \frac{\dot{H}}{H^2} - C_3 \left[ \frac{\beta^2}{2} + (e^{(\beta/2)\kappa \chi} - 1) \epsilon_f \right], \tag{4.15}
\]

\[
R = C_1 - C_3 \left[ 1 - \frac{1}{e^{(\beta/2)\kappa \chi} + \beta^2/(2\epsilon_f)} \right]. \tag{4.16}
\]

In the JBD theory, \( \beta \) is required to be \( \beta \lesssim 0.034 \) from observational constraints, which yields \( \epsilon_f = \beta^2/2 \ll 1 \). In addition to this, the value of \( \chi \) should be practically vanishing, \( \chi_f \simeq 0 \), at the end of inflation in order to reproduce the present value of the gravitational constant, because \( \chi \) generally grows only as logarithm of \( t \) after inflation and is even constant during radiation-dominated stage in this theory. These lead to \( F_f = (\beta/4)\kappa \chi_f \simeq 0 \) and \( \alpha_f = e^{-2F_f} \epsilon_f / \epsilon_f \simeq 0 \), which implies that the decomposition of Eq. (4.11) looks almost the same form as that of (4.10). Since the second term in the rhs of Eq. (4.15) is negligible relative to the first term after inflation, the term proportional to \( C_1 \) represents the growing adiabatic mode [55], while the one proportional to \( C_3 \) corresponds to the isocurvature mode [28][34]. During inflation, \( R \) evolves due to the change of the term,

\[
W \equiv e^{(\beta/2)\kappa \chi} + \beta^2/(2\epsilon_f). \tag{4.17}
\]

As a specific model of inflation, let us first consider chaotic inflation driven by a potential, \( V(\phi) = \lambda_2 n \phi^{2n}/(2n) \). In this model, inflation occurs at large \( \phi \) and it is terminated when \( |\dot{\phi}|/|\phi| \) becomes as large as \( H \) at \( \phi = \phi_f = \sqrt{2n}/\kappa \). Making use of Eqs. (3.20) and (4.2) we find

\[
\kappa^2 \int_{\phi_f}^{\phi} \frac{V(\phi)}{V'(\phi)} d\phi = \kappa^2 \frac{\phi^2 - 2n}{4n} \left( \frac{\phi^2}{\kappa^2} \right) = \frac{2}{\beta^2} \frac{1 - e^{\beta \kappa \chi/2}}{1 - e^{-\beta^2 z/2}} \simeq z. \tag{4.18}
\]

The error in the last expression is less than 1.7% for \( z \leq 60 \) and \( \beta \leq 0.034 \). Then Eq. (4.17) is rewritten as

\[
W = \left( 1 - \frac{2}{n} \right) e^{(\beta/2)\kappa \chi} + \frac{n}{2} + \frac{\beta^2}{2n}. \tag{4.19}
\]

Since \( W \) is constant for \( n = 2 \), \( R \) is conserved in the quartic potential, \( V(\phi) = \lambda_4 \phi^4/4 \). In other cases such as the quadratic potential, \( R \) evolves during inflation due to the presence of isocurvature perturbations, although its change is typically small. At the end of inflation, \( R \) takes almost constant value, \( R \simeq C_1 \).
One may worry that the above results are obtained by imposing slow-roll conditions, which are only approximations to the full equations of motion. In order to answer such suspicions, we numerically solved full equations (3.19)-(3.23) along with the background equations (3.1)-(3.4). We adopt the quadratic potential $V(\phi) = m^2 \phi^2 / 2$, and start integrating from about 60 e-folds before the end of inflation. We found that the evolution of field and metric perturbations are well described by analytic estimations except around the end of inflation. The evolution of $\Phi$ for the adiabatic and isocurvature mode is plotted in Fig. 1, which shows that the contribution of the isocurvature mode is small as estimated by Eq. (4.15). The adiabatic growth of the gravitational potential terminates for $m t > \sim 20$, after which the system enters a reheating stage. During reheating no additional growth of super-Hubble metric perturbations occurs in this scenario, unless some interactions between inflaton and other field, $\sigma$, such as $g^2 \phi^2 \sigma^2 / 2$ are not introduced.

The evolution of $\mathcal{R}$ for $\beta = 0.09$ (corresponding to $\omega = 500$) is plotted in Fig. 1, which shows that the change of $\mathcal{R}$ is small during inflation. We have also confirmed that the variation of $\mathcal{R}$ is negligible in the case of $\beta \lesssim 0.034$. Since its growth is sourced by the $e^{(\beta/2)\kappa^2}$ term in Eq. (4.19), we find in Fig. 1 that $\mathcal{R}$ approaches almost constant value around the end of inflation. During reheating $\mathcal{R}$ is conserved except for the short period when $\dot{H}$ passes through zero.

Thus one can use analytic expressions for adiabatic curvature perturbations based on slow-roll approximations in order to constrain the model parameters of the potential. Note that the number of e-folds, $z_k$, after the comoving wave-number $k$ leaves the Hubble radius during inflation satisfies

$$\frac{k}{k_f} = e^{-(1-\beta^2/2)z_k} (2z_k)^{n/2}, \quad \text{for } 1 \ll z_k \lesssim 60. \quad (4.20)$$

We can then express the amplitude of curvature perturbation on scale $l = 2\pi/k$ as [29]

$$\Phi(l) = \left[ 1 + \frac{2}{3(1 + w)} \right]^{-1} \sqrt{2k^3 (|C_1|^2)} \frac{\lambda_{2n}}{2\pi} \left[ 2 \frac{e^{(4n\beta^2z_k)^{1/2}}}{6\pi} \frac{4n\beta^2z_k^{n/2}}{\kappa^2} \right]^{n/2} \frac{z_k}{n} e^{-\beta^2z_k/4} \left[ 1 + \frac{1}{\beta} (1 - e^{-\beta^2z_k/2}) \right], \quad (4.21)$$

which is valid until the second horizon crossing after inflation and also for $1 \ll z_k \lesssim 60$. Here $w$ is the ratio of the pressure to the energy density.
Since the large-angular-scale anisotropy of background radiation due to the Sachs-Wolfe effect is given by \( \frac{\delta T}{T} = \frac{1}{3} \Phi(z_k \simeq 60) \simeq \begin{cases} 9\lambda_2/m_{\text{pl}} \equiv 9m/m_{\text{pl}} & \text{for } n = 1 \\ 28\sqrt{\lambda_4} & \text{for } n = 2 \end{cases} \) (4.22), we can normalize the value of \( \lambda_2 \) by the COBE normalization \([13]\). For \( \beta = 0.034 \), since this gives the relation,

\[ \frac{\delta T}{T} = 1.1 \times 10^{-5}, \]

one finds

\[ m = 1 \times 10^{13} \text{ GeV}, \quad n = 1, \]
\[ \lambda_4 = 2 \times 10^{-13}, \quad n = 2, \]

(4.24)

(4.25)

which is not much different from the values obtained assuming the Einstein gravity \([11]\). Since the behavior of the system approaches to that in the Einstein gravity as we increase \( \omega \), we can conclude that in Brans-Dicke theory, model parameters of chaotic inflation should take the same value as in the Einstein gravity.

Next we consider new inflation with a potential

\[ V(\phi) = V_0 - \frac{\lambda}{4}\phi^4, \]  

(4.26)

for which we find,

\[ \kappa^2 \int_{\phi_f}^{\phi} \frac{V(\varphi)}{V'(\varphi)} d\varphi \approx \frac{\kappa^2 V_0}{2\lambda} \left( 1 - \frac{1}{2\phi_f^2} \right) \approx \frac{\kappa^2}{2\lambda \phi^2} \simeq z. \]  

(4.27)

We can again express the amplitude of curvature fluctuation as a function of \( z_k \), which is now related with \( k \) as,

\[ \frac{k}{k_f} = e^{-(1-\beta^2/2)z_k}, \]  

(4.28)

\[ \Phi(l) = \left[ 1 + \frac{2}{3(1+w)} \right]^{-1} \left[ e^{\beta^2 z_k/4} \sqrt{\frac{\lambda}{3}} (2z_k)^{3/2} + \kappa H_f e^{\beta^2 z_k/2} \beta (1 - e^{-\beta^2 z_k/2}) \right], \]  

(4.29)

with \( H_f \equiv \sqrt{\kappa^2 V_0/3} \). Taking \( \beta = 0.034 \) again, it predicts the amplitude of \( \delta T/T \) to be compared with COBE data as

\[ \frac{\delta T}{T}(z_k \simeq 60) \simeq 30\sqrt{\lambda} + 0.21 \frac{H_f}{m_{\text{pl}}}. \]  

(4.30)

Since \( H_f \) should also satisfy

\[ \frac{H_f}{m_{\text{pl}}} \lesssim 10^{-5}, \]  

(4.31)

to suppress long-wave gravitational radiation of quantum origin \([62]\), we obtain

\[ \lambda \lesssim 1 \times 10^{-13}, \]  

(4.32)

from Eq. (4.30). Again its amplitude is practically no different from the case of the Einstein gravity.

**B. Non-minimally coupled scalar field case**

Let us first briefly review the single-field inflationary scenario with a non-minimally coupled scalar field \((\xi R\phi^2/2)\). In chaotic inflationary models, Futamase and Maeda \([63]\) found that the non-minimal coupling is constrained as \( |\xi| \lesssim 10^{-3} \) in the quadratic potential, by the requirement of sufficient amount of inflation \([1]\). In the quartic potential,

\[^3\text{This constraint is loosened by considering topological inflation, see \([64]\).} \]
such a constraint is absent for negative $\xi$, and as a bonus, the fine tuning problem of the self-coupling $\lambda$ in the minimally coupled case can be relaxed by considering large negative values of $\xi$. Several authors evaluated scalar and tensor perturbations generated during inflation [65–67]. Several authors evaluated scalar and tensor perturbations generated during inflation [68–72] and preheating [72] in this model. Since the system is reduced to the single-field case with a modified inflaton potential by a conformal transformation, we cannot expect nonadiabatic growth of $\mathcal{R}$ on large scales.

We shall proceed to the case of the non-minimally coupled $\psi$ field with Eq. (2.10) in the presence of inflaton, $\phi$. In this theory the evolution of field and metric perturbations was studied in [47] in the Jordan frame. It was found that $\mathcal{R}$ can grow nonadiabatically during inflation on super-horizon scales for negative $\xi$. Here we will show that similar results are obtained by the analysis in the Einstein frame.

From Eqs. (4.13), (4.16), and (4.12), we obtain the following explicit solutions:

\[
\delta \chi = -4(C_1 - C_3) \frac{\xi \psi}{\sqrt{1 - (1 - 6\xi)\xi^2 \psi^2}}, \quad \delta \phi = -\frac{V'(\phi)}{\kappa^2 V(\phi)} \left[ C_1 (1 - \xi)\kappa^2 \psi^2 + C_3 (\xi^2 \psi^2) \right], \tag{4.33}
\]

\[
\Phi = -C_1 \frac{\dot{H}}{H^2} - C_3 \left[ \varepsilon \psi - (\xi \kappa^2 \psi^2) \varepsilon_\phi \right], \tag{4.34}
\]

\[
\mathcal{R} = C_1 - C_3 \frac{\varepsilon \psi - (\xi \kappa^2 \psi^2) \varepsilon_\phi}{\varepsilon \psi + (1 - \xi \kappa^2 \psi^2) \varepsilon_\phi}, \tag{4.35}
\]

with

\[
\varepsilon_\psi = \frac{8(\xi \kappa \psi)^2}{1 - (1 - 6\xi)\xi \kappa^2 \psi^2}, \quad \varepsilon_\phi = \frac{1}{2\kappa^2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2. \tag{4.36}
\]

When $\xi$ is negative, the coefficient of $\psi$ in the rhs of Eq. (3.23) is always positive, which leads to the rapid growth of $\psi$ (and $\chi$). Eq. (4.33) indicates that long wave $\delta \chi$ fluctuations are amplified with $|\psi|$ being increased. This is due to the fact that the effective mass of $\delta \chi$ becomes negative after horizon crossing [77], whose property is different from the JBD case. In the JBD case, $\delta \chi$ is almost constant during inflation [see Eq. (4.5) with $V_1 = e^{-2\beta \kappa \chi}$], which restricts the nonadiabatic growth of large scale metric perturbations. In contrast, in the present model, $\Phi$ and $\mathcal{R}$ exhibit strong amplification due to the excitation of low momentum field perturbations unless $|\psi|$ is initially very small.

The second terms in Eqs. (4.34) and (4.35) appear in the presence of non-minimal coupling, whose contributions are negligible when $|\xi| \kappa^2 \psi^2 < 1$. With the increase of $|\psi|$, however, isocurvature perturbations are generated during inflation, which can lead to nonadiabatic growth of $\Phi$ and $\mathcal{R}$. When the second term in Eq. (4.34) grows to of order the first term, the adiabatic mode includes the isocurvature mode partially. In this case one cannot completely decompose adiabatic and isocurvature modes in the final results by the expression, Eq. (4.34).

Let us consider the massive chaotic inflationary scenario with initial conditions, $\phi_0 = 3m_{pl}$ and $\chi_0 = 10^{-2}m_{pl}$. We plot in Fig. 2 the evolution of field perturbations for $\xi = -0.02$ in two cases, i.e., (i) solving directly the perturbed equations, (5.19)–(5.22), (ii) using the slow-roll solutions, Eq. (4.33). In spite of the rapid growth of field perturbations, slow-roll analysis agrees reasonably well with full numerical results. The enhancement of $\delta \chi_k$ fluctuations stimulates the amplification of $\delta \phi_k$ fluctuations for $mt \gtrsim 10$. Inflationary period ends around $mt \simeq 17$, after which the slow-roll results begin to fail. In Fig. 3, the evolution of $\Phi$ and $\mathcal{R}$ is depicted. We also plot the first and the second term in the rhs of Eq. (4.34), where we denote $\Phi_1^{(s)}$ and $\Phi_2^{(s)}$, respectively. Although $\Phi$ is dominated by the $\Phi_1^{(s)}$ term in the initial stage, $\Phi_2^{(s)}$ catches up $\Phi_1^{(s)}$ around $mt \simeq 7$, after which the $\Phi_2^{(s)}$ term completely determines the evolution of $\Phi$. We find in Fig. 3 that slow-roll approximations are valid right up until the end of inflation. The growth of metric perturbations stops when the system enters a reheating stage.

If we use the decomposition of Eq. (4.14), the second term gives negligible contribution to the gravitational potential around the end of inflation. In the early stage of inflation, however, its contribution is comparable to the first term, in which case the isocurvature mode cannot be completely separated from the adiabatic mode. When fluctuations are sufficiently amplified, it is inevitable that both adiabatic and isocurvature modes mix each other with the growth of fluctuations, which means that complete decomposision is difficult during the whole inflationary stage.
FIG. 2: The evolution of background fields, $\tilde{\phi} = \phi/m_{\text{pl}}$ and $\tilde{\chi} = \chi/m_{\text{pl}}$, and large-scale field perturbations, $\delta \tilde{\phi} \equiv k^{3/2} \delta \phi/m_{\text{pl}}$ and $\delta \tilde{\chi} \equiv k^{3/2} \delta \chi/m_{\text{pl}}$, in the massive chaotic inflationary scenario with a non-minimally coupled $\chi$ field for $\xi = -0.02$. The initial values of scalar fields are chosen as $\phi_* = 3m_{\text{pl}}$ and $\chi_* = 10^{-2}m_{\text{pl}}$. The slow-roll results of Eq. (4.33) are also plotted, where we denote $\delta \tilde{\phi}^{(s)}$ and $\delta \tilde{\chi}^{(s)}$ in the figure. We find that slow-roll approximations are valid except in the final stage of inflation ($mt > 17$). The evolution of $\delta \tilde{\phi}^{(s)}$ and $\delta \tilde{\chi}^{(s)}$ after inflation is not shown.

FIG. 3: The evolution of the metric perturbation $\tilde{\Phi} \equiv k^{3/2} \Phi$ and the curvature perturbation $\tilde{\mathcal{R}} \equiv k^{3/2} \mathcal{R}$ with same parameters as in Fig. 2. The amplification of field perturbations leads to the nonadiabatic growth of $\Phi$ and $\mathcal{R}$. We also plot $\tilde{\Phi}_1^{(s)} \equiv k^{3/2} \Phi_1^{(s)}$ and $\tilde{\Phi}_2^{(s)} \equiv k^{3/2} \Phi_2^{(s)}$, where $\Phi_1^{(s)}$ and $\Phi_2^{(s)}$ denote the first and the second terms in Eq. (4.34), respectively. $\Phi$ is mainly sourced by the $\Phi_2^{(s)}$ term, after $\Phi_2^{(s)}$ catches up $\Phi_1^{(s)}$. The evolution of $\Phi_1^{(s)}$ and $\Phi_2^{(s)}$ after inflation is not shown.
In Fig. 3 the curvature perturbation, $R$, is nonadiabatically amplified sourced by the second term in Eq. (4.35). Whether this occurs or not depends upon the strength of the coupling, $\xi$, and the initial $\chi$. When both are small and the second term in Eq. (4.35) is negligible relative to the $C_1$ term during inflation, we can regard the first and second terms in Eq. (4.35) as adiabatic and isocurvature modes, respectively. In the simulations of Figs. 2 and 3, we take $\chi_* = 10^{-2}m_{pl}$, in which case numerical calculations imply that the conservation of $R$ is violated for $\xi \lesssim -0.01$. When $\xi \lesssim -1$, strong amplification of $R$ is inevitable even for very small values of $\chi$ far less than $m_{pl}$. For positive $\xi$, conservation of $R$ is typically preserved due to an exponential suppression of $\chi$ during inflation [47,48,49]. Regarding detailed investigation about the observational constraints of the strength of $\xi$, see [47] whose results are similar to those in the analysis of the Einstein frame.

C. Higher-dimensional theories

In the higher-dimensional theory with Eq. (2.16), the kinetic term takes a canonical form. In this theory the condition, $D > 1$, gives the constraint, $\sqrt{2/3} < \beta < \sqrt{2}$, which is different form the JBD theory with $\beta \ll 1$. Larger values of $\beta$ correspond to the steep exponential potential of the $\chi$ field, which leads to the rapid evolution toward the $\chi$ direction. In this case inflaton decreases slowly relative to the $\chi$ field. Then the expansion of the universe is described by the power-law solution, Eq. (3.30).

For example, in the polynomial inflaton potential, $V(\phi) = \lambda_{2n} \phi^{2n}/(2n)$, classical trajectories of scalar fields are given by Eq. (4.2) as

$$C = \frac{\kappa}{\beta} \chi + \frac{\kappa^2}{4n} \phi^2.$$  
(4.37)

Differentiating Eq. (4.37) with respect to $t$ yields

$$\left|\frac{\dot{\chi}}{\dot{\phi}}\right| = \frac{\beta \kappa}{2n} \left|\frac{\sqrt{2\pi \beta}}{n} \frac{\phi}{m_{pl}}\right|.$$  
(4.38)

This relation indicates that for the values of $\phi$ greater than $m_{pl}$ with $\beta$ and $n$ being of order unity, $|\ddot{\chi}|$ is typically larger than $|\dot{\phi}|$, in which case $\chi$ rapidly evolves along the exponential potential. The power-law inflation continues until $|\dot{\phi}|$ grows comparable to $|\ddot{\chi}|$, corresponding to $\phi/m_{pl} \approx n/(\sqrt{2\pi \beta})$. After $\phi$ falls down this value, $\phi$ begins to evolve faster than $\chi$ toward the local minimum at $\phi = 0$ in the $\phi$ direction. In this stage the system deviates from the power-law expansion, (3.30).

From Eqs. (4.6), (4.11), (4.12), and (4.13), $\Phi$ and $R$ evolve during inflation as

$$\Phi = -C_1 \frac{\dot{H}}{H^2} - C_3 \frac{\beta^2}{2} = -\tilde{C}_1 \frac{\dot{H}}{H^2} - \tilde{C}_3 \frac{\beta^2}{2(1 + \alpha_f)},$$  
(4.39)

$$R = C_1 - C_3 \frac{\beta^2}{\beta^2 + 2\epsilon_\phi} = \tilde{C}_1 - \tilde{C}_3 \frac{\beta^2 - 2\alpha_f \epsilon_\phi}{(1 + \alpha_f)(\beta^2 + 2\epsilon_\phi)},$$  
(4.40)

where $\alpha_f = \beta^2/(2\epsilon_\phi)$ is of order unity for $\sqrt{2/3} < \beta < \sqrt{2}$. Since the condition, $\beta^2 \gg \epsilon_\phi$, holds during power-law inflation, the curvature perturbation in this stage takes almost a constant value, $R \approx C_1 - C_3 = \tilde{C}_1 - \tilde{C}_3/(1 + \alpha_f)$. As $|\dot{\phi}|$ grows relative to $|\ddot{\chi}|$, $R$ begins to evolve due to the change of $\epsilon_\phi$ in Eq. (4.40). This corresponds to the stage where deviations from power-law inflation become relevant. When $\epsilon_\phi$ in Eq. (4.40) becomes comparable to $\beta^2/2$ (i.e., $\alpha_f \approx 1$) around the end of inflation, we have $R \approx C_1 - C_3 \alpha_f/(1 + \alpha_f) = \tilde{C}_1$. After inflation $R$ takes this conserved value.

D. The $R^2$ theory with a non-minimally coupled $\chi$ field

In the $f(R)$ theories, effective potentials do not generally take separated forms, Eq. (1.1), as found in Eq. (2.21). Nevertheless we have closed form solutions, Eqs. (3.43)-(3.46), by which the evolution of cosmological perturbations can be studied analytically.

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Let us analyze the $R^2$ inflationary scenario with a non-minimally coupled $\chi$ field as one example of the $f(R)$ theory [see Eqs. (2.22) and (2.23)]. Note that when $\chi = 0$ the system has an effective potential,

$$U = \frac{m_\phi^4}{(32\pi)^2} \left( 1 - e^{-\sqrt{\frac{2}{3}}\kappa\phi} \right)^2. \tag{4.41}$$

The $\phi$ field defined by Eq. (2.19) plays the role of an inflaton and leads to inflationary expansion of the Universe in the region, $\phi \gtrsim m_{\text{pl}}$ [23]. In the absence of the non-minimally coupled $\chi$ field, the resulting spectrum of density perturbations after the end of inflation was found in [7] (using equations for perturbations of the FRW model for the Einstein gravity with one-loop quantum corrections derived in [27] and then rederived in [78]). Here we study how the effect of non-minimal coupling alters the adiabatic evolution of cosmological perturbations in the single field case.

In the presence of non-minimal coupling, Eq. (3.46) is reduced to

$$J = \frac{[1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}](1 - \xi\kappa^2\chi^2)}{[1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}]} \tag{4.42}$$

Then Eqs. (3.43) and (3.44) are integrated to give

$$\delta\phi = -\frac{2\sqrt{6}(1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}}{3\kappa[1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}]C_1 - \frac{\xi\kappa^2(\chi^2 - \chi^2)}{1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}}C_3}, \tag{4.43}$$

$$\delta\chi = -\frac{4\xi\kappa}{1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}} \left[ C_1 + C_3 \frac{1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}}{1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}} \right], \tag{4.44}$$

where $C_1 = -\kappa^2 Q_1$ and $C_3 = -\kappa^2 Q_3$. Therefore $\Phi$ and $R$ are expressed as

$$\Phi = -C_1 \frac{\dot{H}}{H^2} - \frac{C_3}{1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}} \left[ \xi\kappa^2(\chi^2 - \chi^2) + \frac{\xi\kappa\phi}{1 - \xi\kappa^2\chi^2} \right], \tag{4.45}$$

$$R = C_1 - \frac{C_3}{1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}} \left[ \xi\kappa\phi e^{-\sqrt{\frac{2}{3}}\kappa\phi} - \xi\kappa\phi \frac{\xi\kappa\phi}{1 - \xi\kappa^2\chi^2} \right], \tag{4.46}$$

where $\epsilon_{\phi}$ and $\epsilon_{\chi}$ are defined by

$$\epsilon_{\phi} = \frac{1}{2\kappa^2} \left( \frac{U_{\phi}}{U} \right)^2 = \frac{4}{3} \left( \frac{1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}}{1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}} \right)^2, \tag{4.47}$$

$$\epsilon_{\chi} = \frac{1}{2\kappa^2} \left( \frac{U_{\chi}}{U} \right)^2 = \frac{8}{3} \left( \frac{\xi\kappa\phi e^{-\sqrt{\frac{2}{3}}\kappa\phi}}{1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}} \right)^2. \tag{4.48}$$

In the absence of non-minimal coupling, one finds adiabatic results, $\Phi = -C_1 \dot{H}/H^2$ and $R = C_1$. In two-field inflation with a non-minimally coupled $\chi$ field, the presence of isocurvature perturbations can lead to nonadiabatic growth of $\Phi$ and $R$ as found in the second terms in Eqs. (4.45) and (4.46). Their contributions are negligible when the conditions, $|\xi|\kappa^2\chi^2 \ll 1$ and $\epsilon_{\chi} \ll 1$, holds during inflation. The latter condition is similar to the former one when $|\xi| \lesssim 1$.

From Eq. (3.27), since $\dot{\chi}$ is approximately written as

$$\dot{\chi} = -\frac{4\xi\kappa}{1 - (1 - \xi\kappa^2\chi^2)e^{-\sqrt{\frac{2}{3}}\kappa\phi}} \chi, \tag{4.49}$$

the $\chi$ field exhibits exponential decrease for positive values of $\xi$. For negative $\xi$, however, $\chi$ is exponentially amplified during inflation, which means that the condition, $|\xi|\kappa^2\chi^2 \ll 1$, can be violated. The long-wave $\delta\chi$ fluctuation grows with the increase of $\chi$ as found in Eq. (4.44). On the other hand, the growth of $\delta\phi$ begins only when the second term in Eq. (4.43) becomes comparable to the first term.
FIG. 4: The evolution of background fields, $\tilde{\phi} = \phi/m_{\text{pl}}$ and $\tilde{\chi} = \chi/m_{\text{pl}}$, and long-wave field perturbations, $\delta\tilde{\phi} \equiv k^{3/2}\delta\phi/m_{\text{pl}}$ and $\delta\tilde{\chi} \equiv k^{3/2}\delta\chi/m_{\text{pl}}$ as a function of time, $\tilde{t} \equiv m_{\text{pl}}t/\sqrt{96\pi q}$ in the $R^2$ inflationary scenario with a non-minimally coupled $\chi$ field for $\xi = -0.025$. The initial values of scalar fields are chosen as $\phi_* = 1.1m_{\text{pl}}$ and $\chi_* = 10^{-3}m_{\text{pl}}$. We also plot the slow-roll results, where we denote $\delta\tilde{\phi}(s)$ and $\delta\tilde{\chi}(s)$ in the figure. The evolution of $\delta\tilde{\phi}(s)$ and $\delta\tilde{\chi}(s)$ after inflation is not shown.

FIG. 5: The evolution of the metric perturbation $\tilde{\Phi} \equiv k^{3/2}\Phi$ and the curvature perturbation $\tilde{R} \equiv k^{3/2}R$ in the $R^2$ inflationary scenario with same parameters as in Fig. 4. We also plot $\Phi_1(s) \equiv k^{3/2}\Phi_1(s)$ and $\Phi_2(s) \equiv k^{3/2}\Phi_2(s)$, where $\Phi_1(s)$ and $\Phi_2(s)$ denote the first and the second terms in Eq. (4.45), respectively. The evolution of $\Phi_1(s)$ and $\Phi_2(s)$ after inflation is not shown.
We plot the evolution of $\delta \chi$, $\delta \phi$, and the slow-roll results (4.43) and (4.44) for $\xi = -0.025$ with initial conditions, $\phi_* = 1.1m_{pl}$ and $\chi_* = 10^{-3}m_{pl}$. In this case inflation ends around $\tilde{t} \equiv m_{pl}\sqrt{32\pi q} \approx 130$ with $e$-foldings, $N \approx 63$. Again the slow-roll analysis is quite reliable except around the end of inflation.

In Fig. 5 we also depict the evolution of $\Phi$ and $R$, and the first ($= \Phi^{(s)}$) and second ($= \Phi^{(s)}$) terms in Eq. (4.45). In the initial stage of inflation where $\chi$ and $\delta \chi$ are not sufficiently amplified, the gravitational potential is dominated by the $\Phi^{(s)}$ term, in which case $\Phi^{(s)}$ may be regarded as the isocurvature mode. However, after $\tilde{t} \approx 70$ where $\Phi^{(s)}$ catches up $\Phi^{(s)}$, we find in Fig. 5 that $\Phi^{(s)}$ mainly contributes to the gravitational potential. As is similar to the case of the chaotic inflationary scenario with a non-minimally coupled $\chi$ field, adiabatic and isocurvature modes mix each other with the growth of the $\chi$ fluctuation. If one defines the isocurvature mode as the one which gives negligible contribution to the gravitational potential, it can not be completely separated from the adiabatic mode during the whole stage of inflation.

The conservation of $R$ is typically violated when the term proportional to $C_{2}$ in Eq. (4.46) surpasses the one proportional to $C_{1}$. For the initial values, $\phi_* = 1.1m_{pl}$ and $\chi_* = 10^{-3}m_{pl}$, $R$ exhibits nonadiabatic growth for $\xi \lesssim 0.02$. Negative large non-minimal coupling such as $\xi \lesssim -1$ leads to strong amplification of $R$ unless $\chi$ is initially very small. Although we do not make detailed analysis here, the basic property is quite similar to the case of the subsection B. These results are also expected to hold for other inflationary models with a non-minimally coupled $\chi$ field, since the scalar curvature is proportional to the potential energy of inflaton which slowly decreases during inflation.

V. CONCLUSIONS

We have studied generation and evolution of adiabatic and isocurvature perturbations during multi-field inflation in generalized Einstein theories. Most of the generalized Einstein theories recast to the Lagrangian (2.1) in the Einstein frame by conformal transformations. While behavior of adiabatic perturbations is universal and is given by Eqs. (2.2) and (3.11) in the long-wave limit ($k \to 0$), isocurvature perturbations behave differently depending on specific gravity theories. Making use of slow-roll approximations, we have obtained closed form solutions for all non-decaying field and metric perturbations in the long-wave limit. The existence of isocurvature perturbations may lead to significant variations of the curvature perturbations $R$ and $\zeta$.

In this work we considered the following four gravity theories.

(1) The Jordan-Brans-Dicke theory with a Brans-Dicke field $\chi$ and inflaton $\phi$. Using the Brans-Dicke parameter constrained by observations, the isocurvature mode in the gravitational potential $\Phi$ is negligible relative to the adiabatic mode. Therefore the variation of $R$ is typically small in this theory. In particular, for the quartic potential, $R$ is conserved in the slow-roll analysis.

(2) A non-minimally coupled scalar field $\chi$ in the presence of inflaton $\phi$. When the coupling $\xi$ is negative, $\chi$ and its long-wave fluctuations exhibit exponential increase during inflation, leading to the nonadiabatic amplification of $\Phi$ and $R$ due to the existence of isocurvature perturbations. Even in this case we find that slow-roll analysis agrees well with full numerical results. When field and metric perturbations are sufficiently amplified, adiabatic and isocurvature modes of the gravitational potential mix each other.

(3) Higher-dimensional Kaluza-Klein theory with dilaton $\chi$ and inflaton $\phi$. In this theory the inflationary period can be divided into two stages: the first is the power-law inflationary stage where $\chi$ evolves along the exponential potential and the second is the deviation from power-law inflation due to the rapid evolution of $\phi$ around the end of inflation. In the former stage $R$ is nearly constant but its change occurs at the transition between two stages.

(4) $R^{2}$ theory with a non-minimally coupled scalar field $\chi$. This system has an additional scalar field $\phi$ playing the role of inflaton after conformal transformations. Although this theory has a coupled effective potential which is different from the above theories (1)-(3), we have integrated forms of long-wave field and metric perturbations by slow-roll analysis, which is found to be quite reliable right up until the end of inflation. Negative non-minimal coupling again leads to the nonadiabatic growth of $\Phi$ and $R$, in which case complete decomposition between adiabatic and isocurvature modes is difficult.

While we analyzed generalized Einstein theories involving two scalar fields, there exist other multi-field inflationary scenarios such as hybrid inflation [74] and models of two interacting scalar fields [80]. During preheating after inflation, there has been growing interest about the evolution of cosmological perturbations for the simple two-field model with potential $U(\phi, \chi) = \lambda \phi^{4}/4 + g^{2} \phi^{2} \chi^{2}/2$ [73, 81]. It is certainly of interest to constrain realistic multi-field inflationary models based on particle physics using density perturbations generated during inflation, together with constraints by gravitinos [58] and possible primordial black hole over-production during preheating [3].
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[1] E.W. Kolb and M.S. Turner, The Early Universe (Addison-Wesley, Redwood City, California, 1990); A.D. Linde, Particle Physics and Inflationary Cosmology (Harwood, Chur, Switzerland, 1990); A.R. Liddle and D.H. Lyth, Cosmological inflation and Large-Scale Structure (Cambridge University Press, 2000).

[2] A.H. Guth, Phys. Rev. D 23, 347 (1981); K. Sato, Mon. Not. R. Astr. Soc. 195, 467 (1981).

[3] A.D. Linde, Phys. Lett. B108, 389 (1982); A. Albrecht and P.J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).

[4] A.D. Linde, Phys. Lett. B129, 177 (1983).

[5] A.A. Starobinsky, Phys. Lett. B91, 99 (1980).

[6] B. Whitt, Phys. Lett. B145, 176 (1984).

[7] S. Gottlöber, V. Müller, H.-J. Schmidt, and A.A. Starobinsky, Int. J. Mod. Phys. D1, 257 (1992).

[8] A.A. Starobinsky, JETP Lett. 30, 682 (1979).

[9] S.W. Hawking, Phys. Lett. B115, 295 (1982); A.A. Starobinsky, Phys. Lett. B117, 175 (1982); A.H. Guth and S.Y. Pi, Phys. Rev. Lett. 49, 1110 (1982).

[10] V.N. Lukash, Sov. Phys. JETP 52, 807 (1980).

[11] V.F. Mukhanov and G.V. Chibisov, JETP Lett. 33, 532 (1981).

[12] H. Murayama, H. Suzuki, T. Yanagida, and J. Yokoyama, Phys. Rev. Lett. 70, 1912 (1993); Phys. Rev. D 50, R2356 (1994).

[13] D.H. Lyth and A. Riotto, Phys. Rept. 314, 1 (1999).

[14] D. La and P.J. Steinhardt, Phys. Rev. Lett. 62, 376 (1989); P.J. Steinhardt and F.S. Accetta, Phys. Rev. Lett. 64, 2470 (1990).

[15] C. Brans and R.H. Dicke, Phys. Rev. 24, 925 (1961).

[16] D. La, P.J. Steinhardt, and E.W. Bertschinger, Phys. Lett. B231, 231 (1989); E. Weinberg, Phys. Rev. D 40, 3950 (1989).

[17] A.L. Berkin, K. Maeda, and J. Yokoyama, Phys. Rev. Lett. 65, 141 (1990).

[18] A.D. Linde, Phys. Lett. B238, 160 (1990).

[19] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B158, 316 (1985); Nucl. Phys. B261, 1 (1985).

[20] C.G. Callan, D. Friedan, E.J. Martinec, and M.J. Perry, Nucl. Phys. B262, 593 (1985); C.G. Callan, I.R. Klebanov, and M.J. Perry, Nucl. Phys. B278, 78 (1986).

[21] A.L. Berkin and K. Maeda, Phys. Rev. D 44, 1691 (1991).

[22] J. McDonald, Phys. Rev. D 44, 2314 (1991).

[23] S. Mollerach and S. Matarrese, Phys. Rev. D 45, 1961 (1992).

[24] N. Deruelle, C. Gundlach, and D. Langlois, Phys. Rev. D 46, 5537 (1992).

[25] J. García-Bellido, A.D. Linde and D.A. Linde, Phys. Rev. D 50, 730 (1994); J. García-Bellido, Nucl. Phys. B423, 221 (1994).

[26] J.M. Bardeen, P.J. Steinhardt, and M.S. Turner, Phys. Rev. D 28, 679 (1983).

[27] D.H. Lyth, Phys. Rev. D 31, 1792 (1985).

[28] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. 78, 1 (1984).

[29] A.A. Starobinsky and J. Yokoyama, in: Proc. of The Fourth Workshop on General Relativity and Gravitation, ed. by K. Nakao et al. (Kyoto University, 1994), p. 381 [gr-qc/9502003].

[30] A.A. Starobinsky, JETP Lett. 42, 152 (1985).

[31] D. Polarski and A.A. Starobinsky, Nucl. Phys. B385, 623 (1992); Phys. Rev. D 50, 6123 (1994).

[32] T. Damour and K. Nordtvedt, Phys. Rev. Lett. 70, 2217 (1993); Phys. Rev. D 48, 3436 (1993).

[33] J. García-Bellido and D. Wands, Phys. Rev. D 52, 6739 (1995).

[34] T. Chiba, N. Sugiyama, and J. Yokoyama, Nucl. Phys. B 530, 304 (1998).

[35] M. Sasaki and E. Stewart, Prog. Theor. Phys. 95, 71 (1996).

[36] J. García-Bellido and D. Wands, Phys. Rev. D 53, 437 (1996).

[37] V.F. Mukhanov and P.J. Steinhardt, Phys. Lett. 422, 52 (1998).
[58] V.F. Mukhanov, JETP Lett. 34, 1882 (1981).
[59] V.A. Rubakov, M.V. Sazhin, and A.V. Veryaskin, Phys. Lett. B115, 189 (1982).
[60] T. Futamase and K. Maeda, Phys. Rev. D 39, 399 (1989).
[61] N. Sakai and J. Yokoyama, Phys. Lett. B456, 113 (1999).
[62] B.L. Spokoiny, Phys. Lett. B147, 39 (1984).
[63] D.S. Salopek, J.R. Bond, and J.M. Bardeen, Phys. Rev. D 40, 1753 (1989).
[64] R. Fakir and W.G. Unruh, Phys. Rev. D 41, 1783 (1990).
[65] N. Makino and M. Sasaki, Prog. Theor. Phys. 86, 103 (1991).
[66] D.I. Kaiser, Phys. Rev. D 52, 4295 (1995).
[67] J. Hwang and H. Noh, Phys. Rev. D 60, 123001 (1999).
[68] E. Komatsu and T. Futamase, Phys. Rev. D 58, 023004 (1998); Phys. Rev. D 59, 064029 (1999).
[69] S. Tsujikawa and B.A. Bassett, Phys. Rev. D 62, 043510 (2000); S. Tsujikawa, K. Maeda, and T. Torii, Phys. Rev. D 61, 103501 (2000).
[70] B.A. Bassett and F. Viniegra, Phys. Rev. D 62, 043507 (2000).
[71] B.A. Bassett, G. Pollifrone, S. Tsujikawa, and F. Viniegra, Phys. Rev. D 63, 103515 (2001).
[72] S. Tsujikawa, K. Maeda, and T. Torii, Phys. Rev. D 60, 123505 (1999).
[73] A.A. Starobinsky, Sov. Astron. Lett. 9, 302 (1983).
[74] A.A. Starobinsky, JETP Lett. 34, 438 (1981).
[75] L.A. Kofman and V.F. Mukhanov, JETP Lett. 44, 619 (1986); L.A. Kofman, V.F. Mukhanov, and D.Yu. Pogosyan, Sov. Phys. JETP 66, 433 (1987); L.A. Kofman, V.F. Mukhanov, and D.Yu. Pogosyan, Phys. Lett. B193, 427 (1987).
[76] A.D. Linde, Phys. Lett. B259, 38 (1991); Phys. Rev. D 49, 748 (1994); E.J. Copeland, A.R. Liddle, D.H. Lyth, E.D. Stewart, and D. Wands, Phys. Rev. D 49, 6410 (1994).
[77] L.A. Kofman and A.D. Linde, Nucl. Phys. B282, 555 (1987); L.A. Kofman and D.Yu. Pogosyan, Phys. Lett. B214, 508 (1988).
[78] F. Finelli and R.H. Brandenberger, Phys. Rev. D 62, 083502 (2000); S. Tsujikawa, B.A. Bassett, and F. Viniegra, JHEP 0008, 019 (2000); J.P. Zibin, R.H. Brandenberger, and D. Scott, Phys. Rev. D 63, 043511 (2001); F. Finelli and S. Khlebnikov, Phys. Lett. B504, 309 (2001); hep-ph/0107143.
[79] A.L. Maroto and A. Mazumdar, Phys. Rev. Lett. 84, 1655 (2000); J.P. Zibin, R.H. Brandenberger, and D. Scott, Phys. Rev. D 63, 043511 (2001); F. Finelli and S. Khlebnikov, Phys. Lett. B504, 309 (2001); hep-ph/0107143.
[80] A.L. Maroto and A. Mazumdar, Phys. Rev. Lett. 84, 1655 (2000); R. Kallosh, L. Kofman, A. Linde, and A. Van Proeyen, Phys. Rev. D 61, 103503 (2000); G.F. Giudice, A. Riotto and I.I. Tkachev, JHEP 9911, 036 (1999); D.H. Lyth and H.B. Kim, hep-ph/0011202; H.P.Nilles, M. Peloso, and L. Sorbo, hep-ph/0102264; JHEP 0104, 004 (2001).