On the strong metric dimension of corona product graphs and join graphs

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Abstract

Let \(G\) be a connected graph. A vertex \(w\) strongly resolves a pair \(u, v\) of vertices of \(G\) if there exists some shortest \(u - w\) path containing \(v\) or some shortest \(v - w\) path containing \(u\). A set \(W\) of vertices is a strong resolving set for \(G\) if every pair of vertices of \(G\) is strongly resolved by some vertex of \(W\). The smallest cardinality of a strong resolving set for \(G\) is called the strong metric dimension of \(G\). It is known that the problem of computing this invariant is NP-hard. It is therefore desirable to reduce the problem of computing the strong metric dimension of product graphs, to the problem of computing some parameter of the factor graphs. We show that the problem of finding the strong metric dimension of the corona product \(G \odot H\), of two graphs \(G\) and \(H\), can be transformed to the problem of finding certain clique number of \(H\). As a consequence of the study we show that if \(H\) has diameter two, then the strong metric dimension of \(G \odot H\) is obtained from the strong metric dimension of \(H\) and, if \(H\) is not connected or its diameter is greater than two, then the strong metric dimension of \(G \odot H\) is obtained from the strong metric dimension of \(H\).
of $K_1 \odot H$, where $K_1$ denotes the trivial graph. The strong metric
dimension of join graphs is also studied.

**Keywords:** Strong metric dimension, strong resolving set, strong metric
basis, clique number, corona product graph, join graph.

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1 Introduction

Generators of metric spaces are sets of points with the property that every
point of the space is uniquely determined by the distances from their ele-
ments. Such generators put a light on some kinds of problems in graph theory
that apparently are not directly related to metric spaces. Given a simple and
connected graph $G = (V, E)$, we consider the metric $d_G : V \times V \to \mathbb{R}^+$,
where $d_G(x, y)$ is the length of a shortest path between $u$ and $v$. $(V, d_G)$ is clearly
a metric space. A vertex $v \in V$ is said to distinguish two vertices $x$ and $y$
if $d_G(v, x) \neq d_G(v, y)$. A set $S \subset V$ is said to be a metric generator for $G$
if any pair of vertices of $G$ is distinguished by some element of $S$. A minimum
generator is called a metric basis, and its cardinality the metric dimension of
$G$, denoted by $\text{dim}(G)$. Motivated by the problem of uniquely determining
the location of an intruder in a network, the concept of metric dimension
of a graph was introduced by Slater in [19, 20], where the metric generators
were called locating sets. The concept of metric dimension of a graph was
also introduced by Harary and Melter in [9], where metric generators were
called resolving sets. Applications of this invariant to the navigation of robots
in networks are discussed in [13] and applications to chemistry in [11, 12].
Once the first article in this topic was published several papers have been
appearing in the literature, e.g. [1, 3, 4, 6, 8, 10, 14, 17, 21]. Remarkable
variations of the concept of metric generators are resolving dominating sets
[2], independent resolving sets [5], local metric sets [16], strong resolving sets
[18], etc.

In this article we are interested in the study of strong resolving sets
[15, 18]. For two vertices $u$ and $v$ in a connected graph $G$, the interval
$I_G[u, v]$ between $u$ and $v$ is defined as the collection of all vertices that belong
to some shortest $u - v$ path. A vertex $w$ strongly resolves two vertices $u$
and $v$ if $v \in I_G[u, w]$ or $u \in I_G[v, w]$. A set $S$ of vertices in a connected graph $G$
is a strong resolving set for $G$ if every two vertices of $G$ are strongly resolved
by some vertex of $S$. The smallest cardinality of a strong resolving set of $G$ is called strong metric dimension and is denoted by $\dim_s(G)$. So, for example, $\dim_s(G) = n - 1$ if and only if $G$ is the complete graph of order $n$. For the cycle $C_n$ of order $n$ the strong dimension is $\dim_s(C_n) = \lceil n/2 \rceil$ and if $T$ is a tree, its strong metric dimension equals the number of leaves of $T$ minus 1 (see [18]). We say that a strong resolving set for $G$ of cardinality $\dim_s(G)$ is a strong metric basis of $G$.

It is known that the problem of computing the strong metric dimension of a graph is NP-hard [15]. It is therefore desirable to reduce the problem of computing the strong metric dimension of product graphs, to the problem of computing some parameters of the factor graphs. We show that the problem of finding the strong metric dimension of the corona product $G \odot H$, of two graphs $G$ and $H$, can be transformed to the problem of finding certain clique number of $H$. As a consequence of the study we show that if $H$ has diameter two, then the strong metric dimension of $G \odot H$ is obtained from the strong metric dimension of $H$ and, if $H$ is not connected or its diameter is greater than two, then the strong metric dimension of $G \odot H$ is obtained from the strong metric dimension of $K_1 \odot H$, where $K_1$ denotes the trivial graph. The strong metric dimension of join graphs is also studied.

We begin by giving some basic concepts and notations. For two adjacent vertices $u$ and $v$ of $G = (V, E)$ we use the notation $u \sim v$. For a vertex $v$ of $G$, $N_G(v)$ denotes the set of neighbors that $v$ has in $G$, i.e., $N_G(v) = \{ u \in V : u \sim v \}$. The set $N_G(v)$ is called the open neighborhood of $v$ in $G$ and $N_G[v] = N_G(v) \cup \{ v \}$ is called the closed neighborhood of $v$ in $G$. The degree of a vertex $v$ of $G$ will be denoted by $\delta_G(v)$, i.e., $\delta_G(v) = |N_G(v)|$. Recall that the clique number of a graph $G$, denoted by $\omega(G)$, is the number of vertices in a maximum clique in $G$. Two distinct vertices $x$, $y$ are called true twins if $N_G[x] = N_G[y]$. We say that $X \subseteq V$ is a twin-free clique in $G$ if the subgraph induced by $X$ is a clique and for every $u$, $v \in X$ it follows $N_G[u] \neq N_G[v]$, i.e., the subgraph induced by $X$ is a clique and it contains no true twins. We say that the twin-free clique number of $G$, denoted by $\varpi(G)$, is the maximum cardinality among all twin-free cliques in $G$. So, $\omega(G) \geq \varpi(G)$. We refer to a $\varpi(G)$-set in a graph $G$ as a twin-free clique of cardinality $\varpi(G)$.

We say that a vertex $u$ of $G$ is maximally distant from $v$ if for every $w \in N_G(u)$, $d_G(v, w) \leq d_G(u, v)$. If $u$ is maximally distant from $v$ and $v$ is maximally distant from $u$, then we say that $u$ and $v$ are mutually maximally distant. Since no vertex of $G$ strongly resolves two mutually maximally distant vertices of $G$, we have the following remark which will be useful later.
Remark 1. For every pair of mutually maximally distant vertices $x, y$ of a connected graph $G$ and for every strong metric basis $S$ of $G$, it follows that $x \in S$ or $y \in S$.

Let $G$ and $H$ be two graphs of order $n_1$ and $n_2$, respectively. Recall that the corona product $G \odot H$ is defined as the graph obtained from $G$ and $H$ by taking one copy of $G$ and $n_1$ copies of $H$ and joining by an edge each vertex from the $i^{th}$-copy of $H$ with the $i^{th}$-vertex of $G$. We will denote by $V = \{v_1, v_2, ..., v_n\}$ the set of vertices of $G$ and by $H_i = (V_i, E_i)$ the copy of $H$ such that $v_i \sim v$ for every $v \in V_i$. The join $G + H$ is defined as the graph obtained from disjoined graphs $G$ and $H$ by taking one copy of $G$ and one copy of $H$ and joining by an edge each vertex of $G$ with each vertex of $H$. Notice that the corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$.

2 Main results

We shall start studying the relationship between the strong metric dimension of a connected graph and its twin-free clique number.

Theorem 2. Let $H$ be a connected graph of order $n \geq 2$. Then

$$\text{dim}_s(H) \leq n - \varpi(H).$$

Moreover, if $H$ has diameter two, then

$$\text{dim}_s(H) = n - \varpi(H).$$

Proof. Let $W$ be a maximum twin-free clique in $H = (V, E)$. We will show that $V - W$ is a strong resolving set for $H$. Since $W$ is a twin-free clique, for any two distinct vertices $u, v \in W$ there exists $s \in V - W$ such that either $(s \in N_H(u) \text{ and } s \notin N_H(v))$ or $(s \in N_H(v) \text{ and } s \notin N_H(u))$. Without loss of generality, we consider $s \in N_H(u) \text{ and } s \notin N_H(v)$. Thus, $u \in I_H[v, s]$ and, as a consequence, $s$ strongly resolves $u$ and $v$. Therefore, $\text{dim}_s(H) \leq n - \varpi(H)$.

Now, suppose that $H$ has diameter two. Let $X$ be a strong metric basis of $H$ and let $u, v$ be two distinct vertices of $H$. If $d_H(u, v) = 2$ or $N_H[u] = N_H[v]$, then $u$ and $v$ are mutually maximally distant vertices of $H$, so $u \in X$ or $v \in X$. Hence, for any two distinct vertices $x, y \in V - X$ we have $x \sim y$ and $N_H(x) \neq N_H(y)$. As a consequence, $|V - X| \leq \varpi(H)$. Therefore, $\text{dim}_s(H) \geq n - \varpi(H)$ and the result follows.
Corollary 3. Let $H$ be a graph of diameter two and order $n$. Let $\omega(H)$ be the clique number of $H$ and let $c(H)$ be the number of vertices of $H$ having degree $n-1$. If the only true twins of $H$ are vertices of degree $n-1$, then
\[
dim_s(H) = n + c(H) - \omega(H) - 1.
\]
Moreover, if $H$ has no true twins, then
\[
dim_s(H) = n - \omega(H).
\]

Lemma 4. Let $G$ and $H$ be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, and maximum degree $\Delta_1$ and $\Delta_2$, respectively.

(i) If $\Delta_1 \neq n_1 - 1$ or $\Delta_2 \neq n_2 - 1$, then
\[
\varpi(G + H) = \varpi(G) + \varpi(H).
\]

(ii) If $\Delta_1 = n_1 - 1$ and $\Delta_2 = n_2 - 1$, then
\[
\varpi(G + H) = \varpi(G) + \varpi(H) - 1.
\]

Proof. Given a $\varpi(G+H)$-set $Z$ we have that for every $u_1, u_2 \in U = Z \cap V(G)$ it follows $N_{G+H}[u_1] \neq N_{G+H}[u_2]$. So, $N_G[u_1] \neq N_G[u_2]$ and, as a consequence, $U$ is a twin-free clique in $G$. Analogously we show that $W = Z \cap V(H)$ is a twin-free clique in $H$. Hence, $\varpi(G + H) = |Z| = |U| + |W| \leq \varpi(G) + \varpi(H)$.

Now, if $\Delta_1 = n_1 - 1$ and $\Delta_2 = n_2 - 1$, then every $\varpi(G)$-set ($\varpi(H)$-set) contains exactly one vertex of degree $\Delta_1 = n_1 - 1$ ($\Delta_2 = n_2 - 1$) and every $\varpi(G + H)$-set contains exactly one vertex of degree $n_1 + n_2 - 1$. Hence, in this case $|U| < \varpi(G)$ or $|W| < \varpi(H)$ and, as a consequence, $\varpi(G + H) = |Z| = |U| + |W| \leq \varpi(G) + \varpi(H) - 1$.

On the other hand, let $U'$ be a $\varpi(G)$-set and let $W'$ be a $\varpi(H)$-set.

In order to complete the proof of (i), we assume, without loss of generality, that $\Delta_1 \neq n_1 - 1$. Let $u \in U'$ and $w \in W'$. Since $\delta_G(u) \neq n_1 - 1$, there exists a vertex $x \in V(G) - U'$ such that $u \not\sim x$. From the definition of $G + H$ we have $w \sim x$ and the subgraph induced by $U' \cup W'$ is a clique in $G + H$. So, $u$ and $w$ are not true twins in $G + H$ and, as a consequence, $U' \cup W'$ is a twin-free clique in $G + H$. Hence, $\varpi(G + H) \geq |U' \cup W'| = \varpi(G) + \varpi(H)$. The proof of (i) is complete.

Now, if $\Delta_1 = n_1 - 1$, then we take $x \in U'$ such that $\delta_G(x) = n_1 - 1$ and as above we see that two vertices $v, w \in U' \cup W' - \{x\}$ are not true
twins in $G + H$. Hence, $U' \cup W' - \{x\}$ is a twin-free clique in $G + H$. So, $\overline{\omega}(G + H) \geq |U'| + |W'| - 1 = \overline{\omega}(G) + \overline{\omega}(H) - 1$. Therefore, the proof of (ii) is complete. \hfill \Box

If $G$ and $H$ are two complete graphs of order $n_1$ and $n_2$, respectively, then $G + H = K_{n_1 + n_2}$ and $\text{dim}_s(G + H) = \text{dim}_s(K_{n_1 + n_2}) = n_1 + n_2 - 1$. From Theorem 2 and Lemma 4 we obtain the following results.

**Theorem 5.** Let $G$ and $H$ be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, and maximum degree $\Delta_1$ and $\Delta_2$, respectively.

(i) If $\Delta_1 \neq n_1 - 1$ or $\Delta_2 \neq n_2 - 1$, then

$$\text{dim}_s(G + H) = n_1 + n_2 - \overline{\omega}(G) - \overline{\omega}(H) \geq \text{dim}_s(G) + \text{dim}_s(H).$$

(ii) If $G$ and $H$ are graphs of diameter two where $\Delta_1 \neq n_1 - 1$ or $\Delta_2 \neq n_2 - 1$, then

$$\text{dim}_s(G + H) = \text{dim}_s(G) + \text{dim}_s(H).$$

(iii) If $\Delta_1 = n_1 - 1$ and $\Delta_2 = n_2 - 1$, then

$$\text{dim}_s(G + H) = \text{dim}_s(G) + \text{dim}_s(H) + 1.$$

The following lemma shows that the problem of finding the strong metric dimension of a corona product graph can be transformed to the problem of finding the strong metric dimension of a graph of diameter two.

**Lemma 6.** Let $G$ be a connected graph of order $n$ and let $H$ be a graph. Let $H_i$ be the subgraph of $G \odot H$ corresponding to the $i^{th}$-copy of $H$. Then

$$\text{dim}_s(G \odot H) = \text{dim}_s(K_1 + \bigcup_{i=1}^{n} H_i).$$

*Proof.* As the result is obvious for $n = 1$, we take $n \geq 2$. Let $v$ be the vertex of $K_1$ and let $S'$ be a strong resolving set for $G \odot H$. We will show that $S = \bigcup_{i=1}^{n} (S' \cap V_i)$ is a strong resolving set for $K_1 + \bigcup_{i=1}^{n} H_i$. We consider $x, y$ are two different vertices of $K_1 + \bigcup_{i=1}^{n} H_i$ not belonging to $S$. We differentiate the following cases.

Case 1: $x = v$ and $y \in V_i$, for some $i$. For any $u \in V_j$, $j \neq i$, we have $x \in I_{K_1 + \bigcup_{i=1}^{n} H_i}[u, y]$ and since $y$ and $u$ are mutually maximally distant in $G \odot H$, we have $y \in S$ or $u \in S$. 


Case 2: \(x, y \in V_i\). Let \(u\) be a vertex of \(S'\) which strongly resolves \(x\) and \(y\) in \(G \circ H\). As no vertex of \(G \circ H\) not belonging to \(V_i\) strongly resolves \(x\) and \(y\), we have that \(u \in V_i\) and \(u \in S\). Hence, \(u\) strongly resolves \(x\) and \(y\) in \(K_1 + \cup_{i=1}^n H_i\).

Note that in the case \(x \in V_i\) and \(y \in V_j, i \neq j\), we have that \(x\) and \(y\) are mutually maximally distant in \(G \circ H\). Thus, we have \(x \in S\) or \(y \in S\). Hence, \(S\) is a strong resolving set for \(K_1 + \cup_{i=1}^n H_i\) and, as a consequence, \(\dim_s(G \circ H) \geq \dim_s(K_1 + \cup_{i=1}^n H_i)\).

Now, given a strong resolving set for \(K_1 + \cup_{i=1}^n H_i\) denoted by \(W'\), let us show that \(W = W' - \{v\}\) is a strong resolving set for \(G \circ H\). Let \(x, y\) be two different vertices of \(G \circ H\) not belonging to \(W\). We denote by \(V = \{v_1, v_1, ..., v_n\}\) the vertex set of \(G\), where \(v_i\) is the vertex of \(G\) adjacent to every vertex of \(V_i\) in \(G \circ H, i \in \{1, ..., n\}\). We differentiate the following cases.

Case 1: \(x = v_i \in V\) and \(y \in V_i\). Let \(u \in V_j, j \neq i\). In this case we have \(x \in I_{G \circ H}[u, y]\) and, since \(y\) and \(u\) are mutually maximally distant in \(K_1 + \cup_{i=1}^n H_i\), we have \(y \in W\) or \(u \in W\).

Case 2. \(x = v_i \in V\) and \(y \in V_j, j \neq i\). For every \(u \in V_i\) we have \(x \in I_{G \circ H}[u, y]\) and, since \(y\) and \(u\) are mutually maximally distant in \(K_1 + \cup_{i=1}^n H_i\), we have \(y \in W\) or \(u \in W\).

Case 3: \(x, y \in V\). Let \(x = v_i, y = v_j, u_i \in V_i\) and \(u_j \in V_j\). We have \(x \in I_{G \circ H}[u_i, y]\) and \(y \in I_{G \circ H}[u_j, x]\). As \(u_i\) and \(u_j\) are mutually maximally distant in \(K_1 + \cup_{i=1}^n H_i\), we have \(u_i \in W\) or \(u_j \in W\).

Finally, note that the case \(x \in V_i\) and \(y \in V_j\), where \(i, j \in \{1, 2, ..., n\}\), leads to \(x \in W\) or \(y \in W\). Therefore, \(W\) is a strong resolving set for \(G \circ H\) and, as a consequence, \(\dim_s(G \circ H) \leq \dim_s(K_1 + \cup_{i=1}^n H_i)\).

**Corollary 7.** For any connected graph \(G\) of order \(n\), \(\dim_s(G \circ K_1) = n - 1\).

**Proof.** For \(H \cong K_1\) Lemma 6 leads to \(\dim_s(G \circ K_1) = \dim_s(K_1 + \cup_{i=1}^n K_1) = \dim_s(K_{1,n}) = n - 1\). \(\square\)

Our next result is obtained from Lemma 6 and Theorem 2.

**Theorem 8.** Let \(G\) be a connected graph of order \(n_1\). Let \(H\) be a graph of order \(n_2\) and maximum degree \(\Delta\).

(i) If \(\Delta = n_2 - 1\), then \(\dim_s(K_1 + H) = n_2 + 1 - \varpi(H)\).

(ii) If \(\Delta \leq n_2 - 2\) or \(n_1 \geq 2\), then \(\dim_s(G \circ H) = n_1 n_2 - \varpi(H)\).
Proof. Since (i) is trivial, we will prove (ii). For $\Delta = n_2 - 1$ we have $\bar{\omega}(K_1 + \cup_{i=1}^{n_1} H_i) n_1 \geq 1 = \bar{\omega}(K_1 + H) + 1$, while for $\Delta \leq n_2 - 2$ we have $\bar{\omega}(K_1 + \cup_{i=1}^{n_1} H_i) = \bar{\omega}(K_1 + H) = \bar{\omega}(H) + 1$. So, by Lemma 6 and Theorem 2 we conclude the proof. 

Let us derive some consequences of the above result.

**Corollary 9.** Let $G$ be a connected graph of order $n_1$ and let $H$ be a graph of order $n_2$, clique number $\omega(H)$ and maximum degree $\Delta$. Let $c(H)$ be the number of vertices of $H$ having degree $n_2 - 1$.

(i) If $H$ has no true twins and $\Delta = n_2 - 1$, then

$$
\dim_s(K_1 + H) = n_2 + 1 - \omega(H).
$$

(ii) If $H$ has no true twins and $\Delta \leq n_2 - 2$,

$$
\dim_s(K_1 + H) = n_2 - \omega(H).
$$

(iii) If $H$ has no true twins and $n_1 \geq 2$, then

$$
\dim_s(G \odot H) = n_1 n_2 - \omega(H).
$$

(iv) If the only true twins of $H$ are vertices of degree $n_2 - 1$, then

$$
\dim_s(K_1 + H) = n_2 + c(H) - \omega(H).
$$

(v) If the only true twins of $H$ are vertices of degree $n_2 - 1$ and $n_1 \geq 2$, then

$$
\dim_s(G \odot H) = n_1 n_2 + c(H) - 1 - \omega(H).
$$

As our next result shows, when $H$ is a triangle free graph we obtain the exact value for the strong metric dimension of $G \odot H$.

**Corollary 10.** Let $G$ be a connected graph of order $n_1$ and let $H$ be a triangle free graph of order $n_2 \geq 3$ and maximum degree $\Delta$. If $n_1 \geq 2$ or $\Delta \leq n_2 - 2$, then

$$
\dim_s(G \odot H) = n_1 n_2 - 2.
$$

Our next result is an interesting consequence of Theorem 2 and Theorem 8.
Theorem 11. Let $G$ be a connected graph of order $n_1$. Let $H$ be a graph of order $n_2$ and maximum degree $\Delta$.

(i) If $\Delta = n_2 - 1$, then
\[ \dim_s(K_1 + H) = \dim_s(H) + 1. \]

(ii) If $H$ has diameter two and either $\Delta \leq n_2 - 2$ or $n_1 \geq 2$, then
\[ \dim_s(G \odot H) = (n_1 - 1)n_2 + \dim_s(H). \]

(iii) If $H$ is not connected or its diameter is greater than two, then
\[ \dim_s(G \odot H) = (n_1 - 1)n_2 + \dim_s(K_1 + H). \]

Note that the above theorem allow us to derive results on the strong metric dimension of some join graphs.

Corollary 12. Let $H$ be a graph of order $n$ and maximum degree $\Delta$.

(i) If $\Delta = n - 1$, then
\[ \dim_s(K_r + H) = \dim_s(H) + r. \]

(ii) If $\Delta \leq n - 2$ and $H$ has diameter two, then
\[ \dim_s(K_r + H) = \dim_s(H) + r - 1. \]

(iii) If $H$ is not connected or its diameter is greater than two, then
\[ \dim_s(K_r + H) = \dim_s(K_1 + H) + r - 1. \]

2.1 Bounds

It is well known that the second smallest Laplacian eigenvalue of a graph is probably the most important information contained in the spectrum. This eigenvalue, frequently called algebraic connectivity, is related to several important graph invariants and imposes reasonably good bounds on the values of several parameters of graphs which are very hard to compute.

The following theorem shows the relationship between the algebraic connectivity of a graph and the clique number.
Theorem 13. Let $G$ be a connected non-complete graph of order $n$, maximum degree $\Delta$ and algebraic connectivity $\mu$. The clique number of $\omega(G)$ is bounded by

$$\omega(G) \leq \frac{n(\Delta - \mu + 1)}{n - \mu}.$$ 

Proof. The algebraic connectivity of $G$, satisfies the following equality shown by Fiedler [7],

$$\mu = 2n \min \left\{ \frac{\sum_{v_i \sim v_j} (w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} \right\},$$

where not all the components of the vector $(w_1, w_2, ..., w_n) \in \mathbb{R}^n$ are equal. Let $S$ be a clique of $G = (V, E)$ of cardinality $\omega(G)$. The vector $w \in \mathbb{R}^n$ associated to $S$ is defined as,

$$w_i = \begin{cases} 1 & \text{if } v_i \in S; \\ 0 & \text{otherwise}, \end{cases} \quad (2)$$

Considering the 2-partition $\{S, V - S\}$ of the vertex set $V$ we have $(w_i - w_j)^2 = 1$ if $v_i$ and $v_j$ are in different sets of the partition, and 0 if they are in the same set. Then,

$$\sum_{v_i \sim v_j} (w_i - w_j)^2 = 2|S|(n - |S|). \quad (3)$$

By (1) and (3) we have

$$\mu \leq \frac{n \sum_{v_i \sim v_j} (w_i - w_j)^2}{|S|(n - |S|)}. \quad (4)$$

Moreover, since $\sum_{v_i \sim v_j} (w_i - w_j)^2$ is the number of edges of $G$ having one end-point in $S$ and the other one in $V - S$, we have $\sum_{v_i \sim v_j} (w_i - w_j)^2 = \sum_{v \in S} |N_{V - S}(v)|$,

where $N_{V - S}(v)$ denotes the set of neighbors that $v$ has in $V - S$. Thus, since $S$ is a clique of $G$, we have that for every $v \in S$, $|N_{V - S}(v)| = \delta_G(v) - (|S| - 1)$.

Hence,

$$\mu \leq n \sum_{v \in S} (\delta_G(v) - |S| + 1) \leq \frac{n(\Delta - |S| + 1)}{|S|(n - |S|)} \leq \frac{n(\Delta - |S| + 1)}{n - |S|}. \quad (5)$$

The result follows directly by inequality (5). \qed
The above bound is tight, it is achieved, for instance, for the Cartesian product graph $G = K_r \square K_2$, where $\mu = 2$, $n = 2r$, $\Delta = r$ and $\omega(G) = r$.

Notice that the above result and the inequality $\omega(H) \geq \omega(H)$ combined with Theorem 2, Theorem 5 or Theorem 8, lead to lower bounds on the strong metric dimension. For instance, by Theorem 2 we derive the following tight bound on the strong metric dimension of graphs with diameter two.

**Theorem 14.** Let $H$ be a connected graph of diameter two, order $n \geq 2$, maximum degree $\Delta$ and algebraic connectivity $\mu$. Then

$$\dim_s(H) \geq \left\lceil \frac{n(n - \Delta - 1)}{n - \mu} \right\rceil.$$ 

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