Minimum-error discrimination between symmetric mixed quantum states

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Abstract

We provide a solution of finding optimal measurement strategy for distinguishing between symmetric mixed quantum states. It is assumed that the matrix elements of at least one of the symmetric quantum states are all real and nonnegative in the basis of the eigenstates of the symmetry operator.
I. INTRODUCTION

The theory of quantum information and communication is a well-developed field of research [1–3]. It concerns the transmission of information using quantum states and channels. The transmission party encodes a message onto a set of quantum states \( \{ \rho_k \} \) with prior probability \( p_k \) for each of the states \( \rho_k \). The set of signal states and the prior probabilities are also known to the receiving party. The task of the receiving party is to decode the received message, i.e., finding the best measurement strategy based upon the knowledge of the signal states and their prior probabilities. One possibility is to choose the strategy that minimizes the probability of detection error. In this paper we will consider the minimization of the probability of error for a certain class of quantum ensembles.

In general, the measurement strategy is described in terms of a set of nonnegative-definite operators called the probability operator measure (POM) [1,2]. The measurement outcome labeled by "k" is associated with the element \( (\pi_k) \) of POM that has all the eigenvalues be either positive or zero. The POM elements must sum into the identity operator \( \sum_k \pi_k = \hat{1} \).

The probability that the receiver will observe the outcome \( k \) given that the transmitted signal is \( \rho_j \) is \( P(k|j) = \text{tr}(\pi_k\rho_j) \). Here \( \text{tr} \) denotes the trace operation. It follows that the error probability is given by

\[
P_{\text{error}} = 1 - \sum_k p_k \text{tr}(\pi_k\rho_k).
\]  

(1)

The necessary and sufficient conditions that lead to the minimum error probability are known to be [1,2,4,5]

\[
\pi_k(p_k\rho_k - p_j\rho_j)\pi_j = 0,
\]

(2)

\[
\sum_k p_k \pi_k \rho_k - p_j \rho_j \geq 0.
\]

(3)

The first condition holds for all \( j \) and \( k \). The second condition means that all the eigenvalues of the operator at the left-hand side are nonnegative and it holds for all \( j \). These conditions are highly nontrivial such that the required POM elements for the best measurement strategy
are not easily derived from the conditions. In fact, only some classes of quantum ensembles are known for their best measurement strategies. These include the cases of only two signal states [1], symmetric states [5,6], mirror-symmetric states [7], linearly independent states [8], and equiprobable states that are complete in the sense that a weighted sum of projectors onto the states equals the identity operator [4].

In this paper, we will provide the optimal measurement strategy for a set of \( N \) mixed symmetric quantum states \( \{ \rho_k \} \). These states are of equal prior probabilities \( p_k = 1/N \) and assumed to respect the \( Z_N \) symmetry

\[
\rho_k = R^k \rho_0 R^{ik}, \quad k = 0, 1, \ldots, (N - 1),
\]

\[
R^N = \pm \hat{1},
\]

where the operator \( R \) denotes the relevant part of the symmetry operator that lives in the same Hilbert subspace of the signal states \( \{ \rho_k \} \). \( \hat{1} \) denotes the identity operator of the Hilbert subspace of the signal states. We also assume \( R \) to be unitary (\( RR^\dagger = R^\dagger R = \hat{1} \)) and nondegenerate, i.e., all its eigenvalues \( \{ b_k \} \) are different for different eigenstates \( \{ |\lambda\rangle \} \). Therefore the dimensionality of \( R \) cannot be larger than the number of the signal states \( N \) otherwise at least two of the eigenvalues of \( R \) will be the same. Besides, we also assume that at least one of the the signal states (assigned to be \( \rho_0 \)) can be made to have all its matrix elements be real and nonnegative, i.e., \( \langle \lambda | \rho_0 | \lambda' \rangle \geq 0 \) for some chosen set of the eigenstates \( \{ |\lambda\rangle \} \) of the operator \( R \).

II. THE OPTIMAL MEASUREMENT STRATEGY

In many of the cases [5,9] where the optimal strategies are known to be the square-root measurements with POM elements

\[
\pi_k = \Phi^{-\frac{1}{2}} (p_k \rho_k) \Phi^{-\frac{1}{2}},
\]

\[
\Phi \equiv \sum_k p_k \rho_k
\]
where $\rho_k$ denotes the $k$-th quantum signal states to be discriminated, and $\Phi$ is invariant under the transformation $R$. In this paper, we assume the invariant operator $\Phi$ as

$$\Phi \equiv \sum_{k=0}^{N-1} R^k \Gamma_0 R^\dagger k,$$

(8)

where $\Gamma_0 \equiv |\varphi_0\rangle\langle\varphi_0|$ is the rank one operator that is formed by some normalized pure quantum state $|\varphi_0\rangle$. From equation (8), $\Phi$ is Hermitian and nonnegative-definite, and commutes with $R$. This implies that both $R$ and $\Phi$ can be expanded in terms of the same orthonormal basis $\{|\lambda\rangle\}$ as

$$\Phi = \sum_\lambda a_\lambda |\lambda\rangle\langle\lambda|,$$

(9)

$$R = \sum_\lambda b_\lambda |\lambda\rangle\langle\lambda|,$$

(10)

where $a_\lambda = N|\langle\lambda|\varphi_0\rangle|^2$ for all $\lambda$. In general, it is difficult to obtain the POM elements that satisfy the conditions (2) and (3). However, we can obtain a solution to these conditions for the symmetric mixed quantum states described in the equations (4) and (5).

**Proposition.** Given the mixed symmetric quantum states as described in the equations (4) and (5), the optimal measurement strategy that minimizes the error probability $P_{\text{error}}$ is described by the POM $\{\pi_k\}$ that is defined by

$$\pi_k \equiv R^k \Phi_2 \Gamma_0 \Phi_2 R^\dagger k, \quad k = 0, 1, \ldots, N - 1.$$

(11)

$$\Gamma_0 \equiv |\varphi_0\rangle\langle\varphi_0|,$$

(12)

where $|\varphi_0\rangle$ is chosen such that $\langle\lambda|\varphi_0\rangle$ is real for all $|\lambda\rangle$ and satisfies $\langle\lambda|\varphi_0\rangle \neq 0$. The operator $\Phi_2$ is defined by $\Phi_2 \equiv \sum_\lambda c_\lambda |\lambda\rangle\langle\lambda|$ with $c_\lambda \equiv N^{-\frac{1}{2}} \langle\lambda|\varphi_0\rangle^{-1}$.

It is noted that $\Phi_2$ is Hermitian and commutes with the operator $R$. The square of $\Phi_2$ equals the inverse of $\Phi$, i.e., $\Phi_2^2 = \Phi^{-1}$. The operator $\Phi_2$ becomes the inverse square-root of $\Phi$ only when all $\langle\lambda|\varphi_0\rangle$ are real and positive.

**Proof of the Proposition.** We need to prove that the POM elements defined in equations (11) and (12) are indeed POM elements and satisfy the necessary and sufficient conditions.
in equations (2) and (3). From equations (11) and (12), we can prove that all \( \pi_k \geq 0 \) as follows

\[
\langle \phi | \pi_k | \phi \rangle = |\langle \phi | R^k \Phi_2 | \varphi_0 \rangle|^2 \geq 0, \quad \text{for arbitrary } k, |\phi\rangle.
\] (13)

We can also see that \( \pi_0 \geq 0 \) by expanding \( \pi_0 \) in the basis \( \{|\lambda\rangle\} \), \( \pi_0 = \frac{1}{N} \sum_{\lambda, \lambda'} |\lambda\rangle \langle \lambda'| \). Under the basis all the matrix elements of \( \pi_0 \) equals \( 1/N \), thus \( \pi_0 \) has only one non-vanishing eigenvalue 1. The requirement that all eigenvalues of \( R \) are different guarantees that all the POM elements sum into identity operator

\[
\sum_{k=0}^{N-1} \pi_k = \frac{1}{N} \sum_{k} \sum_{\lambda, \lambda'} (\frac{b_{\lambda}}{b_{\lambda'}})^k |\lambda\rangle \langle \lambda'| = \sum_{\lambda} |\lambda\rangle \langle \lambda| = 1.
\] (14)

We proceed to prove that the POM given by equation (11) does satisfy the necessary and sufficient conditions listed in equations (2) and (3). By taking equation (11) into (2), we find

\[
\pi_k (p_k \rho_k - p_j \rho_j) \pi_j = \frac{1}{N} R^k \Phi_2 |\varphi_0\rangle \langle \varphi_0 | \Phi_2 (\rho_0 R^{j-k} - R^{j-k} \rho_0) \Phi_2 |\varphi_0\rangle \langle \varphi_0 | \Phi_2 R^{-j}.
\] (15)

By using equations (9) and (10) and that all \( \langle \lambda|\rho_0|\lambda'\rangle \) and \( \langle \lambda'|\varphi_0 \rangle \) are real, we derive the following identity thus prove that equation (15) actually equals zero

\[
\langle \varphi_0 | \Phi_2 (\rho_0 R^{j-k} - R^{j-k} \rho_0) \Phi_2 |\varphi_0 \rangle = \sum_{\lambda, \lambda'} c_{\lambda} c_{\lambda'} \langle \varphi_0 |\lambda\rangle \langle \lambda|\rho_0|\lambda'\rangle \langle \lambda'|\varphi_0 \rangle (b_{\lambda'}^{j-k} - b_{\lambda}^{j-k}) = 0.
\] (16)

The condition in equation (3) is proved as follows. First we observe that \( \sum_k \pi_k \rho_k \) is Hermitian by
Therefore the operators \( \sum_k \pi_k \rho_k - \rho_j \) are also Hermitian for all \( j \). By sandwiching \( \sum_k \pi_k \rho_k - \rho_0 \) using an arbitrary state \( |\phi\rangle \), we have

\[
\langle \phi | \sum_k \pi_k \rho_k - \rho_0 |\phi\rangle = \sum_{\lambda, \lambda'} (|\lambda\rangle \langle \lambda| - \langle \lambda' \rho_0 | \lambda\rangle \langle \lambda'| \rho_0 | \lambda\rangle)
\]

\[
= \frac{1}{2} \sum_{\lambda, \lambda'} (|\langle \phi | \lambda\rangle|^2 + |\langle \phi | \lambda'\rangle|^2 - \langle \phi | \lambda\rangle \langle \lambda | \rho_0 | \lambda\rangle - \langle \phi | \lambda'\rangle \langle \lambda' | \rho_0 | \lambda\rangle)
\]

\[
= \frac{1}{2} \sum_{\lambda, \lambda'} \varepsilon_{\lambda\lambda'}(\phi) \langle \lambda' | \rho_0 | \lambda\rangle \geq 0,
\]

\[
\varepsilon_{\lambda\lambda'}(\phi) \equiv (\langle \lambda | \phi\rangle - \langle \lambda' | \phi\rangle)(\langle \lambda | \phi\rangle - \langle \lambda' | \phi\rangle)^\dagger \geq 0.
\]

From equation (18) we conclude that \( \sum_k \pi_k \rho_k - \rho_0 \) is a Hermitian operator and nonnegative-definite. This then leads to the fact that \( \sum_k \pi_k \rho_k - \rho_j \) are also nonnegative-definite and Hermitian for all possible \( j \) since \( \sum_k \pi_k \rho_k - \rho_j \) = \( R_j (\sum_k \pi_k \rho_k - \rho_0) R_j^\dagger \).

### III. EXAMPLES

It is instructive to consider some examples of symmetric quantum signals and solve for the optimal discrimination strategies by using the proposition provided in the previous section.

**Ex. 1: Signals as pure quantum states**

Although our proposition aims at providing optimal discrimination strategy for mixed quantum states, it can also be applied to the case that has only symmetric pure quantum states. Given that \( \rho_k = R_k(\theta) |\Psi_0\rangle \langle \Psi_0| R_k^\dagger(\theta) \) with

\[
|\Psi_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

(20)
$$R(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}, \quad \theta = \frac{2\pi}{N}. \quad (21)$$

we find that $R(\theta)$ has two eigenstates $|\lambda_1\rangle = \frac{1}{\sqrt{2}} (1, -i)$, $|\lambda_2\rangle = \frac{1}{\sqrt{2}} (1, i)$ with eigenvalues $\lambda_1 = e^{i\theta/2}$ and $\lambda_2 = e^{-i\theta/2}$, respectively. The matrix elements $\langle \lambda | \rho_0 | \lambda' \rangle$ are found to be real and nonnegative for all $\lambda$ and $\lambda'$ in the basis $\{|\lambda_1\rangle, |\lambda_2\rangle\}$. On the other hand, we may choose the operator $\Gamma_0 \equiv |\varphi_0\rangle \langle \varphi_0|$ by assigning $|\varphi_0\rangle = |\Psi_0\rangle$ so that all $\langle \lambda | \varphi_0 \rangle$ are positive real numbers. With such choice of $|\varphi_0\rangle$, the operator $\Phi_2$ becomes the inverse square-root of $\Phi$ ($\Phi_2 = \Phi^{-\frac{1}{2}}$). Therefore, we have

$$\Phi = \sum_{k=0}^{N-1} R^k |\Psi_0\rangle \langle \Psi_0 | R^\dagger k = \frac{N}{2} \hat{1}, \quad (22)$$

$$\pi_k = \Phi^{-\frac{1}{2}} |\Psi_k\rangle \langle \Psi_k | \Phi^{-\frac{1}{2}} = \frac{2}{N} |\Psi_k\rangle \langle \Psi_k|, \quad (23)$$

$$(P_{\text{error}})_{\text{min}} = 1 - |\langle \Psi_0 | \Phi^{-\frac{1}{2}} |\Psi_0 \rangle|^2 = 1 - \frac{2}{N}. \quad (24)$$

Equations (22-24) are exactly the same results obtained in the literatures [1,5]. In this example, our method is identical to the square-root measurement [5].

**Ex. 2: Signals as mixed quantum states (I)**

Consider three symmetric mixed quantum states that satisfy equations (4, 5) with

$$\rho_0 = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}. \quad (25)$$

The rotation operator $R(\theta)$ is also given in equation (21) with $\theta = 2\pi/3$. We find that if we choose $\{|\lambda_1\rangle = \frac{1}{\sqrt{2}} (-i, 1), |\lambda_2\rangle = \frac{1}{\sqrt{2}} (i, 1)\}$ as the basis that spans the Hilbert space of the signal states, all the matrix elements of $\rho_0$ will be real and nonnegative in the basis. According to the proposition in the previous section, the pure quantum state $|\varphi_0\rangle$ must be chosen such that all $\langle \lambda | \varphi_0 \rangle$ are nonzero and real. It is easy to see that any pure quantum state $(a|\lambda_1\rangle + b|\lambda_2\rangle)$ with nonzero real coefficients $a, b$ that satisfy $a^2 + b^2 = 1$ could be a candidate for $|\varphi_0\rangle$. By choosing $|\varphi_0\rangle = \frac{1}{\sqrt{2}} (|\lambda_1\rangle + |\lambda_2\rangle)$, we have $\Phi = \frac{2}{3} \hat{1}$ and

$$\pi_0 = \frac{2}{3} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (26)$$

$$\pi_0 = \frac{2}{3} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (26)$$
\[(P_{\text{error}})_{\text{min}} = 1 - \text{tr}(\pi_0 \rho_0) = \frac{5}{9}. \] (27)

Although we have just obtained the results by using the proposition, we can also solve for the optimal measurement strategy in a direct way. Let us expand the mixed quantum state \(\rho_0\) in terms of Pauli matrices \(\{\sigma_1, \sigma_2, \sigma_3\}\) and the identity operator \(\hat{1}_2\) in the spin-1/2 Hilbert space, \(\rho_0 = (1/2 \hat{1}_2 - 1/3 \sigma_3)\). Consider that \(R\) is the rotation about the \(\hat{2}\)-direction by an angle \(\theta = 2\pi/3\), therefore the optimal POM elements \(\pi_k\) should be of the following general forms:

\[
\pi_k = R^k (b_0 \hat{1}_2 + 2b_1 \sigma_1 + 2b_3 \sigma_3) R^k, \quad k = 0, 1, 2. \tag{28}
\]

where \(b_0, b_1\) and \(b_2\) are the coefficients to be determined. It is found that \(b_0 = 1/3\) by requiring \(\sum_k \pi_k = \hat{1}_2\). By considering that all the POM elements are Hermitian and nonnegative we get the constraint on \(b_1\) and \(b_3\), \(\sqrt{b_1^2 + b_3^2} \leq 1/3\). We then use the constraint to find the minimum of the error probability \(P_{\text{error}}\)

\[
P_{\text{error}} = 1 - \frac{1}{3} \sum_{k=0}^{2} \text{tr}(\pi_k \rho_k)
= 1 - \text{tr}(\pi_0 \rho_0)
= \frac{2}{3} + \frac{b_3}{3} \geq \frac{5}{9}. \tag{29}
\]

From equation (29), the probability of error \(P_{\text{error}} = 5/9\) is optimal at \(b_1 = 0\) and \(b_3 = -1/3\), which is exactly the same measurement strategy as described in equation (26).

**Ex. 3: Signals as mixed quantum states (II)**

In the previous examples we discussed the optimal discrimination among single-qubit quantum states. Here we would like to discuss how to discriminate the mixed quantum states with higher dimensions of Hilbert space.

Let \(|0\rangle, |1\rangle\) and \(|2\rangle\) be the trine states \((1, 0), (1/2, \sqrt{3}/2)\) and \((1/2, -\sqrt{3}/2)\) that respect the \(Z_3\) symmetry \(R(\theta = 2\pi/3)\), respectively.

\[
R(\theta)^3 = -\hat{1}_2, \quad \theta = \frac{2\pi}{3}
\]

\[
|k\rangle = R(\theta)^k |0\rangle, \quad k = 0, 1, 2. \tag{30}
\]
The quantum states to be discriminated are not single-qubit states but the two-qubit quantum states that also respect the $Z_3$ symmetry

$$\rho_0 = \frac{1}{2} \{(|1\rangle \otimes |2\rangle)(\langle 1| \otimes \langle 2|) + (|2\rangle \otimes |1\rangle)(\langle 2| \otimes \langle 1|)\},$$

$$\rho_1 = \frac{1}{2} \{(|2\rangle \otimes |0\rangle)(\langle 2| \otimes \langle 0|) + (|0\rangle \otimes |2\rangle)(\langle 0| \otimes \langle 2|)\},$$

$$\rho_2 = \frac{1}{2} \{(|0\rangle \otimes |1\rangle)(\langle 0| \otimes \langle 1|) + (|1\rangle \otimes |0\rangle)(\langle 1| \otimes \langle 0|)\}. \tag{31}$$

It is obvious that these quantum states are reducible mixed states. They can be decomposed into direct sums of spin-1 and spin-0 parts by $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$. The spin-1 Hilbert subspace is spanned by the spin-1 states $|1, 1\rangle$, $|1, 0\rangle$ and $|1, -1\rangle$. The spin-0 subspace is of dimension one, and is spanned by the spin-0 state $|0, 0\rangle$. By assigning $|1, 1\rangle = (1, 0, 0)$, $|1, 0\rangle = (0, 1, 0)$ and $|1, -1\rangle = (0, 0, 1)$ we can rewrite the mixed signals in the following matrix forms

$$\rho_k = R_3^k \tilde{\rho}_0 R_3^{k*} \oplus \frac{3}{8} |0, 0\rangle \langle 0, 0|, \quad k = 0, 1, 2.$$ 

$$\tilde{\rho}_0 \equiv \begin{pmatrix}
1/16 & 0 & -3/16 \\
0 & 0 & 0 \\
-3/16 & 0 & 9/16
\end{pmatrix},$$

$$R_3 \equiv \begin{pmatrix}
\cos^2\left[\frac{\theta}{2}\right] & \frac{1}{\sqrt{2}} \sin[\theta] & \sin^2\left[\frac{\theta}{2}\right] \\
\frac{1}{\sqrt{2}} \sin[\theta] & \cos[\theta] & \frac{1}{\sqrt{2}} \sin[\theta] \\
\sin^2\left[\frac{\theta}{2}\right] & \frac{1}{\sqrt{2}} \sin[\theta] & \cos^2\left[\frac{\theta}{2}\right]
\end{pmatrix}, \quad \theta = \frac{2\pi}{3}. \tag{32}$$

It is easy to verify that the rotation operator $R_3$ does respect the $Z_3$ symmetry by $R_3^3 = \hat{1}$, and has three different eigenvalues. The best measurement strategy $\{\pi_k\}$ can also be decomposed into direct sums as $\pi_k = \tilde{\pi}_k \oplus \frac{1}{3} |0, 0\rangle \langle 0, 0|$. Again, the operators $\tilde{\pi}_k$ denote the POM elements in spin-1 subspace.

The probability of error can also be viewed as being contributed from different Hilbert subspaces as

$$P_{error} = 1 - \sum_{\text{subspaces}} \left( \sum_k \text{tr}(p_k \tilde{\pi}_k \tilde{\rho}_k) \right)$$

$$= 1 - \frac{1}{3} \sum_k (\tilde{\pi}_k \tilde{\rho}_k)_{\text{spin1}} - \frac{1}{8}, \tag{33}$$

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where 1/8 in equation (33) comes from tracing over spin-0 subspace. As seen from equation (33), only the measurement in spin-1 subspace needs to be optimized. We will solve the optimization problem by using the proposition given in the previous section.

First we note that the rotation operator $R_3$ has three different eigenvalues $1$, $e^{-i2\pi/3}$ and $e^{i2\pi/3}$ with normalized eigenstates $|\lambda_1\rangle = \frac{1}{\sqrt{2}}(1, 0, 1)$, $|\lambda_2\rangle = \frac{1}{2}(-1, i\sqrt{2}, 1)$ and $|\lambda_3\rangle = \frac{1}{2}(-1, -i\sqrt{2}, 1)$, respectively. In the basis formed by $\{|\lambda_1\rangle, |\lambda_2\rangle, |\lambda_3\rangle\}$, all matrix elements of $\tilde{\rho}_0$ are real and nonnegative. According to the proposition, the pure quantum state $|\varphi_0\rangle$ must be chosen such that $\langle \lambda | \varphi_0 \rangle$ are real and nonzero for all possible $|\lambda\rangle$. A convenient choice for $|\varphi_0\rangle$ is $|\varphi_0\rangle = (0, 0, 1)$. We then obtain the operator $\Phi_2$ in spin-1 Hilbert subspace

$$\Phi_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 + \sqrt{2} & 0 & 1 - \sqrt{2} \\ 0 & 2\sqrt{2} & 0 \\ 1 - \sqrt{2} & 0 & 1 + \sqrt{2} \end{pmatrix}.$$  \hspace{1cm} (34)$$

Therefore we get

$$\tilde{\pi}_0 = \Phi_2 \Gamma_0 \Phi_2$$
$$= \frac{1}{6} \begin{pmatrix} 3 - 2\sqrt{2} & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 3 + 2\sqrt{2} \end{pmatrix}, \hspace{1cm} (35)$$

and the optimal error probability is $(3 - \sqrt{2})/6$. This result coincides with our previous calculation that uses von Neumann measurement for signal discrimination [10]. This coincidence is reasonable. Since the dimension of Hilbert space is larger than the number of the signal states in this example, both the optimal POM and the optimal von Neumann measurement may have the same optimal probability of error.

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