On optimal approximability results for computing the strong metric dimension

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Abstract

In this short note, we observe that the problem of computing the strong metric dimension of a graph can be reduced to the problem of computing a minimum node cover of a transformed graph within an additive logarithmic factor. This implies both a 2-approximation algorithm and a \((2 - \varepsilon)\)-inapproximability for the problem of computing the strong metric dimension of a graph.

1 Introduction

The strong metric dimension of a graph was introduced in [7] as an alternative to the previously introduced (weak) metric dimension of graphs [2, 8]. Subsequently, the strong metric dimension has been investigated in several research papers such as [5, 6, 10]. Let \( G = (V,E) \) be a given undirected graph of \( n \) nodes. To define the strong metric dimension, we will use the following notations and terminologies:

- \( \text{Nbr}(u) = \{ v \mid \{u,v\} \in E \} \) is the set of neighbors of \((i.e., \text{nodes adjacent to})\) a node \( u \).
- \( u \leftrightarrow v \) denotes a shortest path from between nodes \( u \) and \( v \) of length (number of edges) \( d_{u,v} = \ell(u \leftrightarrow v) \).
- \( \text{diam}(G) = \max_{u,v \in V} \{d_{u,v}\} \) denotes the diameter of a graph \( G \).
- A shortest path \( u \leftrightarrow v \) is maximal if and only if it is not properly included inside another shortest path, \(i.e., \text{if and only if}\)
  \[(\forall x \in \text{Nbr}(u) \ d(x,v) \leq d(u,v)) \land (\forall y \in \text{Nbr}(v) \ d(y,u) \leq d(u,v))\]
- A node \( x \) strongly resolves a pair of nodes \( u \) and \( v \), denoted by \( x \triangleright \{u,v\} \), if and only if either \( v \) is on a shortest path between \( x \) and \( u \) or either \( u \) is on a shortest path between \( x \) and \( v \).
- A set of nodes \( V' \subseteq V \) is a strongly resolving set for \( G \), denoted by \( V' \triangleright G \), if and only if every distinct pair of nodes of \( G \) is strongly resolved by some node in \( V' \), \(i.e., \text{if and only if}\)
  \[\forall (u,v \in V, u \neq v) \exists x \in V': x \triangleright \{u,v\}\]

Then, the problem of computing the string metric dimension of a graph is defined as shown below:

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**Problem name:** Strong Metric Dimension (STR-MET-DIM)

**Instance:** an undirected graph \( G = (V, E) \).

**Valid Solution:** a set of nodes \( V' \subseteq V \) such that \( V' \triangleright G \).

**Goal:** minimize \(|V'||.

**Related notation:** \( \text{sdim}(G) = \min_{V' \subseteq V \triangleright G} \{ |V'| \} \).

In this short note, we observe that the problem of computing the strong metric dimension of a graph can be reduced to the problem of computing a minimum node cover of a transformed graph within an additive logarithmic factor. This implies both a 2-approximation algorithm and a \((2 - \epsilon)\)-inapproximability for the problem of computing the strong metric dimension of a graph. More precisely, our result is summarized by the following Lemma.

**Lemma 1.1.**

(a) **STR-MET-DIM** admits a polynomial-time 2-approximation.

(b) Assuming the unique games conjecture\(^1\) (UGC) is true, **STR-MET-DIM** does not admit any polynomial-time \((2 - \epsilon)\)-approximation for any constant \( \epsilon > 0 \) even if the given graph is restricted in the sense that

(i) \( \text{diam}(G) \leq 2 \), or

(ii) \( G \) is bipartite and \( \text{diam}(G) \leq 4 \).

**Remark 1.2.** If instead of assuming the correctness of UGC the standard assumption of \( P \neq \text{NP} \) is made, then the part (b) of the above theorem still holds provided one replaces \((2 - \epsilon)\)-inapproximability by \(1.36\)-inapproximability. This is easily obtained by a similar proof in which we use the \((10 \sqrt{5} - 21 \approx 1.3606)\)-inapproximability result.

## 2 Proof of Theorem 1.1

The standard minimum node cover (MNC) problem for a graph is defined as follows:

| **Instance:** | an undirected graph \( G = (V, E) \). |
|--------------|----------------------------------|
| **Valid Solution:** | a set of nodes \( V' \subseteq V \) such that \( V' \cap \{u, v\} \neq \emptyset \) for every edge \( \{u, v\} \in E \). |
| **Goal:** | minimize \(|V'|| . |
| **Related notation:** | \( \text{MNC}(G) = \min_{\forall \{u, v\} \in E : V' \cap \{u, v\} \neq \emptyset} \{ |V'| \} \). |

Let \( G = (V, E) \) denote the input graph of \( n \) nodes. We recall the following result from [5].

**Theorem 2.1.** [5]

(a) Let \( \tilde{G} = (V, \tilde{E}) \) be the graph obtained from \( G \) in the following manner:

\[ \{u, v\} \in \tilde{E} \iff u \neq v \text{ and } u \xleftrightarrow{5} v \text{ is a maximal shortest path in } G \]

Then \( \text{sdim}(G) = \text{MNC}(\tilde{G}) \) and \( V' \subseteq V \) is a valid solution of **STR-MET-DIM** on \( G \) if and only if \( V' \) is a valid solution of MNC on \( \tilde{G} \).

\(^1\)See [3] for a definition of the unique games conjecture.
(b) Let \( \tilde{G} = (\tilde{V}, \tilde{E}) \) be the graph from \( G \) built in the following manner:

- Let \( u_1, u_2, \ldots, u_\kappa \) be the nodes in \( G \) such that, for every \( u_i \) (\( 1 \leq i \leq \kappa \)), there is a node \( v_i \neq u_i \) in \( G \) with the property that \( Nbr(u_i) = Nbr(v_i) \).

- Let \( \overline{G} = (V, \overline{E}) \) be the (edge) of \( G \), i.e., \( \{u, v\} \in \overline{E} \iff \{u, v\} \notin E \). Then \( \overline{G} \) is constructed as follows:
  \[
  \overline{V} = V \cup \{x_1, x_2, \ldots, x_k, y\} \text{ where } x_1, x_2, \ldots, x_k, y \notin V.
  \]
  \[
  \overline{E} = \overline{E} \cup \left( \cup_{j=1}^{k} \{x_j, u_j\} \right) \cup \left( \cup_{y' \in \overline{V} \setminus \{y\}} \{y', y\} \right).
  \]

Then, \( \text{diam}(\tilde{G}) = 2 \) and \( sdim(\tilde{G}) = \kappa + \text{MNC}(G) \).

\textbf{Proof of Lemma 1.1(a)}

Follows from Fact 2.1(a) and a well-known 2-approximation algorithm for \( \text{MNC} \) [9, Theorem 1.3].

\textbf{Proof of Lemma 1.1(b)}

Consider the standard Boolean satisfiability problem (SAT) and let \( \Phi \) be an input instance of SAT. Our starting point is the following inapproximability result proved Khot and Regev [4]:

\[ [4, \text{setting } k = 2] \text{ Assuming UGC is true, there exists a polynomial time algorithm that transforms a given instance } \Phi \text{ of SAT to an input instance graph } G = (V, E) \text{ of MNC with } n \text{ nodes such that, for any arbitrarily small constant } \varepsilon > 0, \text{ the following holds:} \]

\[ \begin{array}{ll}
  (\text{YES case}) & \text{if } \Phi \text{ is satisfiable then } \text{MNC}(G) \leq \left( \frac{1}{2} + \varepsilon \right) n, \text{ and} \\
  (\ast) & \text{if } \Phi \text{ is not satisfiable then } \text{MNC}(G) \geq (1 - \varepsilon)n. \\
\end{array} \]

Consider such an instance \( G \) of \( \text{MNC} \) as generated by the above transformation. We first construct the following graph \( G^+ = (V^+, E^+) \) from \( G \). Let \( k = 1 + \lceil \log_2 n \rceil \) and let \( b(j) = b_{k-1}(j)b_{k-2}(j) \ldots b_1(j)b_0(j) \)

be the binary representation of an integer \( j \in \{1, 2, \ldots, n\} \) using exactly \( k \) bits (e.g., if \( n = 5 \) then \( b(3) = 011 \)). Let \( u_1, u_2, \ldots, u_n \) be an arbitrary ordering of the nodes in \( V \). Then,

- \( V^+ = V \cup V_1^+ \) where \( V_1^+ = \{v_1, v_2, \ldots, v_{k-1}, y\} \) is a set of \( k \) new nodes, and

- \( E^+ = E \cup \left( \cup_{j=1}^{k} \{u_j, v_j\} \mid b_1(j) = 1 \right) \cup \left( \cup_{j=1}^{k-1} \{y, v_j\} \right). \)

Thus \( |V^+| = n + k \) and \( |E^+| < |E| + \frac{n^2}{2} + k \). Now, note that if \( V' \subseteq V \) is a solution of \( \text{MNC} \) on \( G \), then \( V' \cup V_1^+ \) is a solution of \( \text{MNC} \) on \( G^+ \), implying \( \text{MNC}(G^+) \leq \text{MNC}(G) + k \), and, conversely, if \( V' \subseteq V^+ \) is a solution of \( \text{MNC} \) on \( G^+ \), then \( V' \setminus V_1^+ \) is a solution of \( \text{MNC} \) on \( G \), implying \( \text{MNC}(G) \leq \text{MNC}(G^+) \). Combining the above inequalities with that in (\ast), we have

\[ \begin{array}{ll}
  (\text{YES case}) & \text{if } \Phi \text{ is satisfiable then } \text{MNC}(G^+) < \left( \frac{1}{2} + \varepsilon \right) n + \log_2 n + 1, \text{ and} \\
  (\ast\ast) & \text{if } \Phi \text{ is not satisfiable then } \text{MNC}(G^+) \geq (1 - \varepsilon)n. \\
\end{array} \]

We now build the graph \( \tilde{G}^+ = (\tilde{V}^+, \tilde{E}^+) \) from \( G \) using the construction in Fact 2.1 and observe the following:

- For any \( i \neq j \), since \( b(i) \neq b(j) \), there exists an index \( t \) such that \( b_t(i) \neq b_t(j) \), say \( b_t(i) = 0 \) and \( b_t(j) = 1 \). Thus, \( Nbr(u_i) \neq Nbr(u_j) \) since \( v_t \in Nbr(u_j) \) but \( v_t \notin Nbr(u_i) \).

- Since \( b(i) \neq 0 \) for any \( i \) and \( b(1), b(2), \ldots, b(n) \) are distinct binary numbers each of exactly \( k \) bits, for any \( t \neq t' \) there is an index \( i \) such that \( b_t(i) \neq b_{t'}(i) \), say \( b_t(i) = 0 \) and \( b_{t'}(i) = 1 \). Thus, \( Nbr(v_t) \neq Nbr(v_{t'}) \) since \( u_t \in Nbr(v_{t'}) \) but \( u_t \notin Nbr(v_t) \).
• For any \( i \) and \( j \), \( \text{Nbr}(u_i) \neq \text{Nbr}(v_j) \) since \( y \in \text{Nbr}(v_j) \) but \( y \notin \text{Nbr}(u_i) \).

• For any \( i \), \( b(i) \neq 0 \) and thus there exists an index \( j \) such that \( b_j(i) = 1 \). This implies \( u_j \in \text{Nbr}(v_i) \) but \( u_j \notin \text{Nbr}(y) \) and therefore \( \text{Nbr}(v_i) \neq \text{Nbr}(y) \).

• Since \( G \) is a connected graph, for every node \( u_i \) there exists a node \( u_j \) such that \( \{u_i, u_j\} \in E^+ \). Thus, \( u_j \in \text{Nbr}(u_i) \) but \( u_j \notin \text{Nbr}(y) \), implying \( \text{Nbr}(u_i) \neq \text{Nbr}(y) \).

Thus, no two nodes in \( G^+ \) have the same neighborhood, implying \( \kappa = 0 \) and \( s\text{dim}(\hat{G}^+) = \text{MNC}(G^+) \). Thus, setting \( \varepsilon' = \varepsilon + \frac{\log_2 n + 1}{n} > \varepsilon \) to be any arbitrarily small constant, it follows from (**) that

\[
(\text{YES case}) \quad \text{if } \Phi \text{ is satisfiable then } \text{MNC}(G^+) < \left(1 + \varepsilon'\right)n, \quad \text{and}
\]

\[
(\text{NO case}) \quad \text{if } \Phi \text{ is not satisfiable then } \text{MNC}(G^+) \geq (1 - \varepsilon')n.
\]

This proves the claim in (b)(i). To prove (b)(ii), we modify the graph \( \hat{G}^+ \) to a new graph \( G' = (V', E') \) by splitting every edge into a sequence of two edges, i.e., for every edge \( \{u, v\} \) in \( \hat{G}^+ \) we add a new node \( x_{uv} \) in \( G' \) and replace the edge \( \{u, v\} \) by the two edges \( \{u, x_{uv}\} \) and \( \{v, x_{uv}\} \). Clearly \( G' \) is bipartite since all its cycles are of even length and \( \text{diam}(G') \leq 2 + \text{diam}(\hat{G}^+) = 4 \). To show that \( \text{sdim}(\hat{G}^+) = \text{sdim}(G') \), by Fact 2.1(a) it suffices to show that no maximal shortest path ends at a node \( x_{uv} \). Indeed, if a maximal shortest path \( \mathcal{P} \) from some node \( z \) ends at \( x_{uv} \), it must use one of the two edges \( \{u, x_{uv}\} \) and \( \{v, x_{uv}\} \), say \( \{u, x_{uv}\} \). Then adding the edge \( \{v, x_{uv}\} \) to the path provide a shortest path between \( v \) and \( z \) and thus \( \mathcal{P} \) was not maximal. As a result, the inapproximability result for \( \hat{G}^+ \) directly translates to that for \( G' \).

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