Double Lynden-Bell Structure of Low-Energy Quasi-Stationary Distributions in the Hamiltonian Mean Field Model

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In the Hamiltonian mean field model, we study the core-halo structure of low-energy quasi-stationary states under the water-bag type initial condition. The core-halo structure results in the superposition of two independent Lynden-Bell distributions. We examine the completeness of the Lynden-Bell relaxation and the relaxation between these two Lynden-Bell distributions.

Introduction. — One of the most interesting feature of long-range systems is the existence of a non-equilibrium and long-lived quasi-stationary state (QSS).[1] The life time of this QSS diverges with the number of particles, and the QSS temporally separates the Vlasov collisionless regime from the collisional regime of the system. There have been many investigations revealing the various type structures of QSSs appearing in self-gravitating systems, plasma systems and other simplified models in diverse dimensions.[1, 2] Recently, it has been a great challenge to understand the non-equilibrium core-halo structure in QSSs beyond the paradigmatic Lynden-Bell model of QSSs in such long-range systems.[1, 2]

The Hamiltonian mean field (HMF) model is an accepted toy model of long-range systems.[1, 4] In the one-dimensional HMF model, Pakter and Levin explained the origin of their proposed uniform core-halo structure of QSSs at low total energies by the parametric resonance of the system with the initial (first one or two periods of) strong oscillation of magnetization.[5] The resonant particles form the high-energy halo and, at the same time, rest particles condense into the low-energy dense core. The latter Fermi degeneration-like phenomenon is due to the Vlasov incompressibility. The resultant distribution is obviously different from the Lynden-Bell equilibrium one.[2, 3]. Benetti et al.[1] advanced Pakter and Levin’s argument to global ergodicity breaking and introduced the generalized virial condition (GVC) on the HMF model, deviation from which reflects the deviation from the Lynden-Bell equilibrium.

Pakter and Levin modeled the core-halo structure in QSSs in the HMF model as the attachment of two energy water-bag distributions.[5] In contrast, we investigate the possibility of describing the core-halo structure in QSSs as a superposition of two independent Lynden-Bell energy distributions, that is, a double Lynden-Bell structure. The halo matches the deviation from the Lynden-Bell equilibrium in general cases and, remarkably, this structure conserves local ergodicity independently for the core and the halo. In this letter, we intend to corroborate this scenario in the HMF model at low energies and examine the completeness of the relaxation[8, 9] in two ways: by using the Lynden-Bell entropy and by using the double Lynden-Bell entropy.

The model. — The N-body HMF model treats N-identical particles with unit mass interacting on a circle. Their dynamics is governed by the Hamiltonian

$$H = \sum_{j=1}^{N} \frac{p_j^2}{2} + \frac{1}{2N} \sum_{j,k=1}^{N} \left[ 1 - \cos(\theta_j - \theta_k) \right],$$

(1)

where angle $\theta_j$ is the orientation of the j-th particle and $p_j$ is its conjugate momentum.[1, 4] Throughout this letter, both of the particle number N and the simulation time t are assumed to be 10^4, and the initial distributions are the water-bag type. We choose the phase constant of the one-body mean-field potential to be $\Phi(\theta, M) = 1 + M \cos \theta$, where $M$ denotes the magnetization of the system. The one-particle energy function is $\varepsilon(\theta, p, M) = p^2/2 + \Phi(\theta, M)$, and the self-consistency condition on the energy distribution $f(\varepsilon)$ is

$$M = -\frac{1}{N} \int d\theta dp \cos \theta f(\varepsilon(\theta, p, M)).$$

(2)

Now, we describe the formation process of the core-halo structure of an HMF system at low total energy. The formation process consists of four steps. Firstly, as the result of a trigger, which creates a chemical potential gap, by the parametric resonance of the system with the initial strong oscillation of the magnetization, the core-halo structure starts forming.[5] Secondly, after dynamical process facilitated by particle and energy exchanges between the core and the halo, the distribution relaxes to a steady superposition of two components, that is, the core and the halo. (Note that due to the long-range nature, the potential $\Phi(\theta, M)$ is common between the core and halo distributions.) Here, we denote the fine-grained core and halo distributions by $f_c$ and $f_h$, respectively. The dynamical relaxation between the core and the halo

$$df_a/dt \to 0, \quad a = c, h$$

(3)

plays the role of the Vlasov fluid property of incompressibility for each component $a = c, h$. This relaxation converges the total mass $N_a$ and the diluted phase-space...
density $\eta_a$ of each $f_a$ under the condition $\eta = \eta_c + \eta_h$, where $\eta$ denotes the fine-grained phase-space density of the system. Thirdly, the magnetization stabilizes, and the system enters the QSS regime. Finally, the system completes phase-mixing. Namely, $f$ and $f_a$ become functions of $\varepsilon$ only. Then, due to Eq. (3), $\partial f_a(\varepsilon)/\partial t \to 0$ holds. Consequently, the total energy $E_a$ of each $f_a$ converges.

At this time, the core-halo formation is complete.

Based on this process, we derive the core-halo QSS distribution and its corresponding entropy, by following the discussion of the collisionless ergodic relaxation by Lynden-Bell.\[3\] The phase space is divided into macro-cells, i.e., assemblies of micro-cells that are occupied by incompressible Vlasov elements. From now on, we denote the coarse-grained (macro-cell level) core and halo distributions by

$$f_c(\theta, p) = f_{c,i} = \frac{\eta m_i \omega_{\nu_{\omega}}}{\nu_{\omega}} = \eta_c, \quad \eta_c = \frac{\eta}{\nu} \nu_c, \quad (4)$$

$$f_h(\theta, p) = f_{h,i} = \frac{\eta m_i \omega_{\nu_{\omega}}}{\nu_{\omega}} = \eta_h, \quad \eta_h = \frac{\eta}{\nu} \nu_h, \quad (5)$$

where $i$ labels macro-cells ($i = 1, \ldots, p$), $m_i$ and $n_i$ are the numbers of Vlasov elements occupying the $i$-th macro-cell, $\nu$ is the number of micro-cells in each macro-cell, and $\omega$ is the area of each micro-cell. In the process described above, the following partitions are fixed:

$$N = N_c + N_h, \quad E = E_c + E_h, \quad \nu = \nu_c + \nu_h. \quad (6)$$

The third partition is kept for the ratios in the continuum limit $\nu \to 0$. The total partition number of the configurations of Vlasov elements in the phase space is

$$W = W_{mix} W^{(c)} W^{(h)} , \quad (7)$$

$$W_{mix} = \frac{N!}{N_c! N_h!} \prod_{i=1}^{p} \nu_{\omega}^{n_i}, \quad (8)$$

$$W^{(c)} = \frac{N_c!}{\prod_{i=1}^{p} m_i! \prod_{i=1}^{p} (\nu_c - m_i)!}, \quad (9)$$

$$W^{(h)} = \frac{N_h!}{\prod_{i=1}^{p} n_i! \prod_{i=1}^{p} (\nu_h - n_i)!}, \quad (10)$$

where $W_{mix}$ is the partition number of mixing the core and the halo and $W^{(a)}$ are the Lynden-Bell partition numbers for the core and the halo. Using Eqs. (3) and (4), the total partition number $W$ can be expressed as a functional of coarse-grained distributions $f_{c,i}$ and $f_{h,i}$. The procedure for the maximization of the entropy in terms of these distributions is\[10\]

$$\delta f_{c,i} \ln W = 0, \quad \delta f_{h,i} \ln W = 0 \quad (11)$$

under the constraints in Eq. (3). We introduce two kinds of Lagrange multiplier, $\alpha_a$ and $\beta_a$, where $a = c, h$, for fixed particle number $N_a$ and energy $E_a$ respectively. Under the continuum limit ($\nu \to 0$), the total entropy reduces to

$$S = S^{(c)} + S^{(h)}, \quad (12)$$

where each $S^{(a)}$ is the Lynden-Bell entropy:\[3\]

$$S^{(a)} = - \int d\theta dp \left( \frac{f_a}{\eta_a} \ln \frac{f_a}{\eta_a} + \left( \frac{f_a}{\eta_a} - 1 \right) \ln \left( 1 - \frac{f_a}{\eta_a} \right) \right) \quad (13)$$

for $a = c, h$. The maximization solution of Eq. (12) is the double Lynden-Bell distribution

$$f(\theta, p) = \sum_{a=c,h} \exp\left( \frac{\eta_a}{\mu_a} (\varepsilon(\theta, p, M) - \mu_a) \right) + 1, \quad (14)$$

where $\mu_a = -\alpha_a/\beta_a$ is the chemical potential of the core or the halo. Here, readers may think that since the partition number $W$ is the product in Eq. (7), the distribution function Eq. (14) would also be a product. However, the fine grains of distribution functions $f_c$ and $f_h$ do not share any micro-cell. Thus, $f$ is a superposition, i.e., $f = f_c + f_h$.

Simulation results and discussion.— In the $N$-body simulation described in this letter, the initial phase-space distribution function $f_0(\theta, p)$ was the uniform water-bag type distribution over the rectangle $[-\theta_0, \theta_0] \times [-p_0, p_0]$, namely

$$f_0(\theta, p) = \hat{\eta} \Theta(\theta - |\theta|) \Theta(p - |p|), \quad (15)$$

where $\Theta$ is the Heaviside unit one-step function. The parameters $\theta_0$ and $p_0$ of Eq. (15) satisfy the relations $|\theta_0| = M_0$, $p_0 = \sqrt{6E - 3(1 - M_0^2)}$, and $\hat{\eta} = 1/(4\theta_0 p_0)$ for initial magnetization $M_0$ and energy per particle $E$. Using these relations, we can fix $\hat{\eta}$ so that we can deduce $E$ from $M_0$. In order to take advantage of the Vlasov incompressibility, that is, the dynamical conservation of $\hat{\eta}$, we classify simulation data by the common number of each $\mu_a = -\alpha_a/\beta_a$ is the chemical potential of the core or the halo. Here, readers may think that since the partition number $W$ is the product in Eq. (7), the distribution function Eq. (14) would also be a product. However, the fine grains of distribution functions $f_c$ and $f_h$ do not share any micro-cell. Thus, $f$ is a superposition, i.e., $f = f_c + f_h$.

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In this letter, we consider $\hat{\eta} = 0.15$.

Figure 1 shows that $M_0 = M_{\text{min}}$ gives the global minimum of the function $E(M_0)$ and in the neighborhood of this point, the function is convex. This also holds for other values of $\hat{\eta}$. Thus, it is natural to cast some properties of the system in its total energy $E$. Accordingly, we introduce the residual total energy $E_{\text{res}}$ which is equal to the total energy $E$ of the system minus the total energy $E_{\text{res}}$ (the red line in Fig.1) of the Vlasov stationary state $f_0(\varepsilon) = \eta \Theta(\varepsilon - \varepsilon)$ for $\eta = 1500$ (i.e., $E_{\text{res}} = E - E_{\text{res}}$).\[12\] To clarify the meaning of $E_{\text{res}}$, we consider the dynamics of the system on the phase space. When the dynamics start from $f_0$, its center $E_{\text{res}}$ is Vlasov stationary, and there is a total energy gap $E_{\text{res}} > 0$ between them. So, by using $E_{\text{res}}$ the system creates the halo of high-energy particles in the outer site, then, the inner part approaches the Vlasov stationary state due to energy conservation. Thus, $E_{\text{res}}$ measures the degree of
the creation of the high-energy tail of the halo, which causes the system to deviate from the Lynden-Bell equilibrium. That is, we argue that $E_{\text{res}}$ is a measure of the deviation of the system from the Lynden-Bell equilibrium. For $M_0 \leq M_{\text{min}}$, the Vlasov stationary distributions that inscribe and circumscribe the initial distribution Eq.(15) are close to each other. So, in these cases, the validity of this argument weakens. In Fig.2, we illustrate this argument by the almost monotone correspondence between the residual total energy and the residue of the Lynden-Bell entropy of the Lynden-Bell equilibrium against that of the system. As easily confirmed, this deviation part is fitted by the halo part of the distribution, $f_h$. As an illustration of this argument, in Fig.4 we show the theoretical semi-predictions by the double Lynden-Bell distributions for the three initial magnetizations $M_0 = 0.53, M_{\text{min}}$ and 0.78. (Note that a double Lynden-Bell distribution has seven degrees of freedom. In Fig.4, by adjusting three parameters by hand, we solve the four conditions, that is, three conservation laws for mass, energy and phase-space density, and the self-consistency condition Eq.(4), and derive the double Lynden-Bell distributions. So, Fig.4 is not just a fitting but is also a theoretical result.) The three parameters adjusted by hand to produce Fig.4 include the Lynden-Bell entropy. By setting the Lynden-Bell entropy to be lower than that of the Lynden-Bell equilibrium, we accurately reproduce the $N$-body simulation results. These accurate reproductions corroborate the double Lynden-Bell scenario.

As a basic question, it is worthwhile to examine whether or not the simulation resultant QSSs complete the relaxation between the core and halo Lynden-Bell distributions, which is a weaker criterion than Lynden-Bell relaxation. The relaxation criterion to be considered is cast in the on-shell maximization of the double Lynden-

\[ \dot{\eta} = 1500, \text{respectively.} \]

\[ \text{FIG. 1: The } \dot{\eta} = 0.15 \text{ configuration of four data } M_0 = 0.53, M_{\text{min}}, 0.72, 0.78 \text{ on the } (M_0, E) \text{ plane to be used in Figs.3 and 4. Here, } M_{\text{min}} \sim 0.6556 \text{ is the magnetization of the initial water-bag distribution Eq.(15) at the minimum } E. \text{ The blue and red curves represent the initial water-bag states and the energy per particle of the Vlasov stationary state } f_{\text{LP}} \text{ for } \eta = 1500, \text{ respectively.} \]

\[ \text{FIG. 2: This figure shows the residue of the Lynden-Bell entropy of the Lynden-Bell equilibrium } S_{\text{eq}} \text{ for given } E \text{ and } \eta = 1500 \text{ against that of the simulation result } S_{\text{sim}} \text{ as a function of the residual energy per particle } \dot{E}_{\text{res}}. \text{ Blue and purple plotted dots represent the cases of } M_0 = 0.66 \sim 0.78 \text{ (with 0.01 increments) and } M_0 = 0.41 \sim 0.65 \text{ (with 0.02 increments) using the distributions averaged over 10 runs, respectively.} \]

\[ \text{FIG. 3: These figures show the deviation of the simulation resultant } f(\varepsilon) \text{ (red dots) averaged over 20 runs from the single Lynden-Bell equilibrium (gray curve) for } M_0 = M_{\text{min}}, 0.72, 0.78 \text{ (from top to bottom).} \]
The former is constrained to satisfy the three conservation laws and the self-consistency condition Eq. (2) and by adjusting values of the Lynden-Bell entropy, stationary magnetization \( M_s \) and \( \eta_s \) by hand. Gray dashed curves represent the Lynden-Bell equilibrium for given \( E \) and \( \eta = 1500 \).

The case \( M_0 = 0.72 \) fulfills this criterion for \( M_s = 0.6345 \), which is within the oscillation range of the stationary magnetization (see Fig. 5). However, in other values of \( M_0 \), the simulation results do not fulfill the on-shell entropy maximization criterion and are regarded as cases of incomplete relaxation of Eq. (12).

Summary and outlook.— In the HMF model, we have systematically studied the core-halo structure of the QSSs for the initial rectangle water-bag distributions with \( \eta = 0.15 \) and corroborated the double Lynden-Bell scenario at low total energies. We also examined the completeness of relaxation by considering two entropies. By using the Lynden-Bell entropy, we found that the systems being considered do not reach equilibrium and for higher total energy the degree of incompleteness increases. By using the double Lynden-Bell entropy, in the case of \( M_0 = 0.72 \), the system completes the relaxation; however, for other values of \( M_0 \) this does not happen.

We suggest two open issues. First a measure of the deviation from complete relaxation between the core and halo Lynden-Bell distributions needs to be found. Second, it is important to determine the total energy limit for the application of the double Lynden-Bell scenario and extend our ideas to systems at high total energies.

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[10] We omit the details of the calculations in the following few sentences, since these are parallel to those in the single Lynden-Bell case \( ^{[3]} \).
[11] In this letter, we use a hat to denote normalization by the factor \( 1/N \).
[12] In this context, the Vlasov stationary state \( f_{\text{vs}} \) depends on three parameters, that is, the Fermi energy \( \varepsilon_F \), the magnetization and the total energy \( E_{\text{vs}} \). These are determined by the two conservation laws and the self-consistency condition Eq. (4).