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Algebraic proofs for shallow water bi–Hamiltonian systems for three cocycle of the semi-direct product of Kac–Moody and Virasoro Lie algebras

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Abstract: We prove new theorems related to the construction of the shallow water bi-Hamiltonian systems associated to the semi-direct product of Virasoro and affine Kac–Moody Lie algebras. We discuss associated Verma modules, coadjoint orbits, Casimir functions, and bi-Hamiltonian systems.

Keywords: Affine Kac–Moody Lie algebras, Bi-Hamiltonian systems, Verma modules, Coadjoint orbits

MSC: 17B69, 17B08, 70G60, 82C23

1 Introduction: The semi-direct product of Virasoro algebra with the Kac–Moody algebra

This paper is a continuation of the paper [1] where we studied bi-Hamiltonian systems associated to the three-cocycle extension of the algebra of diffeomorphisms on a circle. In this note we show that certain natural problems (classification of Verma modules, classification of coadjoint orbits, determination of Casimir functions) [2–5] for the central extensions of the Lie algebra \( \text{Vect}(S^1) \times LG \) reduce to the equivalent problems for Virasoro and affine Kac–Moody algebras (which are central extensions of \( \text{Vect}(S^1) \) and \( LG \) respectively). Let \( G \) be a Lie group and \( \mathcal{G} \) its Lie algebra. The group \( \text{Diff}(S^1) \) of diffeomorphisms of the circle is included in the group of automorphisms of the Loop group \( LG \) of smooth maps from \( S^1 \) to \( G \). For any pairs \((\phi, \psi) \in \text{Diff}(S^1)^2 \) and \((g, h) \in LG^2 \) the composition law of the group \( \text{Diff}(S^1) \times LG \) is

\[
(\phi, a) \cdot (\psi, b) = (\phi \circ \psi, a \cdot b \circ \phi^{-1}).
\]

The Lie algebra of \( \text{Diff}(S^1) \times LG \) is the semi-direct product \( \text{Vect}(S^1) \ltimes LG \) of the Lie algebras \( \text{Vect}(S^1) \) and \( LG \).

Let \( \mathcal{G} \) be a Lie algebra and \( \langle , \rangle \) a non-degenerated invariant bilinear form. \( \text{Vect}(S^1) \) is the Lie algebra of vector fields on the circle and \( LG \) the loop algebra (i.e., the Lie algebra of smooth maps from \( S^1 \) to \( \mathcal{G} \)).

\( \text{Vect}(S^1)_C \) is the Lie algebra over \( C \) generated by the elements \( L_n, n \in \mathbb{Z} \) with the relations

\[
[L_m, L_n] = (n - m)L_{n+m}.
\]

We denote by \( LG_C \) the Lie algebra over \( C \) generated by the elements \( g_n, n \in \mathbb{Z}, g \in \mathcal{G} \) where \((\lambda g + \mu h)_n\) is identified with \( \lambda g_n + \mu h_n \) with the relations

\[
[g_n, h_m] = [g, h]_{n+m}.
\]

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The semi-direct product of \( \text{Vect}(S^1) \) with \( \mathcal{L}G \) is as a vector space isomorphic to \( C^\infty(S^1, \mathbb{R}) \oplus C^\infty(S^1, \mathcal{G}) \) [6]. The Lie bracket of \( \overline{\mathcal{U}}(\mathcal{G}) \) has the form

\[
[(u, a), (v, b)] = ([\cdot, \cdot], u \otimes v, va' - ub' + [a, b]),
\]

for any \((u, v) \in C^\infty(S^1, \mathbb{R})^2\) and any \((a, b) \in C^\infty(S^1, \mathcal{G})^2\), where prime denote derivative with respect to a coordinate on \( S^1 \). The Lie algebra \( \text{Vect}(S^1) \rtimes \mathcal{L}G \) can be extended with a universal central extension \( \overline{\mathcal{U}}(\mathcal{G}) \) by a two-dimensional vector space. Let us denote by \( \mathcal{J}(u) = \int u \). Two independent cocycles are given by

\[
\omega_{\mathcal{U}}((u, a), (v, b)) = \mathcal{J}(u''v), \quad \omega_{\mathcal{K}-\mathcal{M}}((u, a), (v, b)) = \mathcal{J}((a', b)).
\]

We denote by \((u, a, \chi, \alpha)\) the elements of \( \mathcal{U}(\mathcal{G}) \) with \( u \in C^\infty(S^1, \mathbb{R}), a \in C^\infty(S^1, \mathcal{G}) \) and \((\chi, \alpha) \in \mathbb{R}^2\). The algebra \( \mathcal{U}(\mathcal{G}) \) can be also represented as the semi-direct product of Virasoro algebra on the affine Kac–Moody algebra. We denote by \( c_{\mathcal{V}i} \) and \( c_{\mathcal{K}-\mathcal{M}} \) the elements \((0, 0, 1, 0)\) and \((0, 0, 0, 1)\) respectively. If \( \mathcal{G} = \mathbb{R} \), then the Lie algebra \( \text{Vect}(S^1) \rtimes \mathcal{L}R \) has a universal central extension \( \overline{\mathcal{U}}(\mathbb{R}) \) by a three-dimensional vector space. The third independent cocycle is given by

\[
\omega_{\mathcal{U}}((u, a), (v, b)) = \mathcal{J}(ub'' - va'').
\]

We denote by \((u, a, \chi, \alpha, \gamma, \delta)\) elements of \( \overline{\mathcal{U}}(\mathbb{R}) \) with \( u \in C^\infty(S^1, \mathbb{R}), a \in C^\infty(S^1, \mathcal{G}) \), and \((\chi, \alpha, \gamma) \in \mathbb{R}^3\). The Lie bracket of \( \overline{\mathcal{U}}(\mathbb{R}) \) is given by

\[
[(u, a, \phi, \chi, \alpha, \gamma, (v, b, \xi, \beta, \delta)) = (vu' - u'v, [a, b] - ub' + va', \mathcal{J}(u''v), \mathcal{J}((a', b)), \mathcal{J}(ub'' - va'')).
\]

In this paper we discuss a few questions. Let us mention the main results. First, in Section 2 we consider Kirillov-Kostant Poisson brackets [7] of the regular dual of the semi-direct product of Virasoro Lie algebra with the Affine Kac–Moody Lie algebra. Let us denote by \( \overline{\mathcal{U}}(\mathcal{G})' \) the subset of \( \overline{\mathcal{U}}(\mathcal{G}) \) of elements \((u, a, \xi, \beta)\) with non-vanishing \( \beta \). We denote by \( \text{Vect}(S^1) \oplus \mathcal{L}G \)' the subset of \( \text{Vect}(S^1) \oplus \mathcal{L}G \) composed of elements \((u, a, \xi, \beta)\) with \( \beta \neq 0 \). Then introduce two new maps \( \mathcal{I}(u, a, \xi, \beta) \) from \( \overline{\mathcal{U}}(\mathcal{G})' \) to \( \text{Vect}(S^1) \oplus \mathcal{L}G \)', and \( \overline{\mathcal{I}}(u, a, \xi, \beta, \gamma) \) from \( \overline{\mathcal{U}}(\mathcal{G}) \) to \( \text{Vect}(S^1) \oplus \mathcal{L}R \). We prove that \( \mathcal{I}(u, a, \xi, \beta) \) and \( \overline{\mathcal{I}}(u, a, \xi, \beta, \gamma) \) are Poisson maps. In Section 3 we discuss coadjoint orbits and Casimir functions for \( \overline{\mathcal{U}}(\mathcal{G}) \). Let \( \mathcal{H} \) be a central extension of a Lie algebra \( \mathcal{H} \) and \( \mathcal{H} \) be a Lie group with Lie algebra is \( \mathcal{H} \). We find explicit form for the the coadjoint actions of the groups \( \text{Diff}(S^1) \times \mathcal{L}G \) and \( \text{Diff}(S^1) \times \mathcal{L}R \). As a result we obtain the following new theorem. We prove that a coadjoint orbit of \( \overline{\mathcal{U}}(\mathcal{G}) \) is mapped by \( \mathcal{I} \) to a coadjoint orbit of \( \text{Vect}(S^1) \oplus \mathcal{L}G \) to a coadjoint orbits of \( \text{Vect}(S^1) \). We prove that the map \( \overline{\mathcal{I}} \) sends the coadjoint orbits of \( \overline{\mathcal{U}}(\mathcal{G}) \) to coadjoint orbits of \( \text{Vect}(S^1) \oplus \mathcal{L}G \). Previously, we determined Casimir functions on \( \overline{\mathcal{U}}(\mathcal{G})' \) and \( \overline{\mathcal{U}}(\mathbb{R}) \). We then prove new propositions concerning the explicit form of Casimir functions on \( \overline{\mathcal{U}}(\mathcal{G}) \), and in particular on \( \overline{\mathcal{U}}(\mathbb{R}) \). This paper was partially inspired by the construction of bi-Hamiltonian systems as natural generalization of the classical Korteweg-de Vries equation. [1, 8–11]. It has been showed in [1], that the dispersive water waves system equation \([9, 10, 12]\) is a bi–Hamiltonian system related to the semi-direct product of a Kac–Moody and Virasoro Lie algebras, and the hierarchy for this system was found. In Section 4 some results of [1] are obtained from another point of view. We prove new proposition for pairwise commuting functions under certain brackets. In section 5 we discuss properties of the universal enveloping algebra of \( \overline{\mathcal{U}}(\mathcal{G}) \). In subsection 5.1 we consider a decomposition of the enveloping algebra of a semi-direct product. We introduce the notion of realizability of the action of \( \mathcal{K} \) on \( \mathcal{H} \) in \( \mathcal{U}_{\omega_{\mathcal{K}-\mathcal{M}}}(\mathcal{H}) \). Then we show (Theorem 5.1) that the realizability of the action of \( \mathcal{K} \) in \( \mathcal{U}_{\omega_{\mathcal{K}-\mathcal{M}}}(\mathcal{H}) \) leads to the isomorphism

\[
\mathcal{U}_{\omega_{\mathcal{K}-\mathcal{M}}}(\mathcal{K} \ltimes \mathcal{H}) \cong \mathcal{U}_{\omega_{\mathcal{K}-\mathcal{M}}}(\mathcal{K}) \otimes \mathcal{U}_{\omega_{\mathcal{K}-\mathcal{M}}}(\mathcal{H})
\]

In subsection 5.2 the case of \( \mathcal{U}_{\mathcal{L}}(\mathcal{G}) \) is considered. In subsection 5.3 we discuss representations of \( \overline{\mathcal{U}}(\mathcal{G}) \). We prove that positive energy representation \( V \) of \( \mathcal{U}_{\mathcal{L}}(\mathcal{G}) \) with non-vanishing \( \beta \partial \delta \)-action of the cocyle \( c_{\mathcal{K}-\mathcal{M}} \) delivers a pair of commuting representations of Virasoro and affine Kac–Moody Lie algebras. This proposition determines whether a \( \mathcal{U}_{\mathcal{L}}(\mathcal{G}) \) Verma module is a sub-module of another Verma module of \( \mathcal{U}_{\mathcal{L}}(\mathcal{G}) \). We also prove a proposition regarding a linear form over \( \mathfrak{h} \) with non-vanishing \( \lambda(c_{\mathcal{K}-\mathcal{M}}) \). In this paper we present proofs for corresponding theorems and lemmas.
2 The Kirillov-Kostant structure of $SU(G)$

Now we consider Kirillov-Kostant Poisson brackets of the regular dual of the semi-direct product of Virasoro Lie algebra with the Affine Kac–Moody Lie algebra. Let $K$ be a Lie algebra with a non-degenerated bilinear form $(\cdot, \cdot)$. A function $f : K \to \mathbb{R}$ is called regular at $x \in K$ if there exists an element $\nabla f(x)$ such that

$$f(x + \epsilon a) = f(x) + \epsilon (\nabla f(x), a) + o(\epsilon),$$

for any $a \in K$. For two regular functions $f, g : K \to \mathbb{R}$, we define the Kirillov-Kostant structure as a Poisson structure on $K$ with

$$\{f, g\}(x) = \langle x, [\nabla f(x), \nabla g(x)] \rangle.$$

Then for any $e \in G$, the second Poisson structure $\{f, g\}_e(x)$ compatible with the Kirillov-Kostant Poisson structure is defined by

$$\{f, g\}_e(x) = \langle e, [\nabla f(x), \nabla g(x)] \rangle.$$

A non-degenerated bilinear form on $SU(G)$ and $\overline{\text{Vect}}(\tilde{S}^1) \oplus \overline{\mathbb{C}G}$ is defined by

$$\langle (u_1, a_1, \beta_1, \xi_1), (u_2, a_2, \beta_2, \xi_2) \rangle = \int_S u_1 u_2 + \int_S \langle a_1, a_2 \rangle + \xi_1 \xi_2 + \beta_1 \beta_2.$$

We denote by $SU(G)'$ the subset of $SU(G)$ of elements $(u, a, \xi, \beta)$ with non-vanishing $\beta$. Let $u' = u - \frac{1}{\beta} a$. We denote by $(\overline{\text{Vect}}(\tilde{S}^1) \oplus \overline{\mathbb{C}G})'$ the subset of $\overline{\text{Vect}}(\tilde{S}^1) \oplus \overline{\mathbb{C}G}$ composed of elements $(u, a, \xi, \beta)$ with $\beta \neq 0$. Let us introduce a new map $\overline{\mathcal{I}}(u, a, \xi, \beta) = (u', a, \xi, \beta)$ from $SU(G)'$ to $(\overline{\text{Vect}}(\tilde{S}^1) \oplus \overline{\mathbb{C}G})'$. Then for non-vanishing $\beta$, let us introduce another new map $\overline{\mathcal{I}}(u, a, \xi, \beta, \gamma) = (u' + \frac{\gamma}{\beta} a', a, \xi - \frac{\gamma^2}{\beta^2}, \beta)$ from $SU(G)$ to $\overline{\text{Vect}}(\tilde{S}^1) \oplus \overline{\mathbb{C}G}$. Here we give a proof for the following new theorem:

**Theorem 2.1.** $\mathcal{I}$ and $\overline{\mathcal{I}}$ are Poisson maps.

**Proof.** For any regular function $f(u, a, \xi, \beta)$ from $\overline{\text{Vect}}(\tilde{S}^1) \oplus \overline{\mathbb{C}G}$ to $\mathbb{R}$ let us define a regular function $\overline{f}$ from $SU(G)'$ to $\mathbb{R}$ by $\overline{f}(u, a, \xi, \beta) = f(u', a, \xi, \beta)$. For $f(u, a, \xi, \beta)$ a function on $SU(G)$ or $(\overline{\text{Vect}}(\tilde{S}^1) \oplus \overline{\mathbb{C}G})$, let us denote by $f_a$ the function of the variables $a$ and $\xi$ that we get when we fix $u$ and $\xi$. Let us denote $f_a$ the function of the variables $u$ and $\xi$ that we get when we fix $a$ and $\beta$. With the previous notations, one has for $\beta \neq 0$ for the bracket $\langle \cdot, \cdot \rangle_{SU(G)}$

$$\{f, g\}_{SU(G)}(u, a, \xi, \beta) = \left[ \{f_a, g_a\} + \{f_u, g_u\} + \{f_\xi, g_\xi\} \right](u, a, \xi, \beta),$$

and for the bracket $\langle \cdot, \cdot \rangle_{\overline{\text{Vect}}(\tilde{S}^1) \oplus \overline{\mathbb{C}G}}$ we have

$$\{f, g\}_{\overline{\text{Vect}}(\tilde{S}^1) \oplus \overline{\mathbb{C}G}}(u, a, \xi, \beta) = \{f_u, g_u\} + \{f_\xi, g_\xi\}. $$

Then the map $\pi_1$ from $SU(G)$ onto $\overline{\text{Vect}}(\tilde{S}^1)$ which sends $(u, a, \xi, \beta)$ onto $(u', \xi)$ is a Poisson morphism. The map $\pi_2$ from $SU(G)$ onto $\overline{\mathbb{C}G}$ which sends $(u, a, \xi, \beta)$ to $(a, \beta)$ is a Poisson morphism. For any regular function $f$ on $\overline{\text{Vect}}(\tilde{S}^1)$ and any regular function $g$ on $\overline{\mathbb{C}G}$ we have

$$\{\pi_1^* f, \pi_2^* g\}_U = 0.$$

Indeed, for $i = 1, 2$, \( \delta_a - \frac{g}{\beta} \delta_u \) $f_i(\overline{u}, \xi, 0) = 0$. We have:

$$\{f_1(\overline{u}, \xi), f_2(\overline{u}, \xi)\}_{\xi, \beta}(u, a, \xi, \beta) = \mathcal{J}(\xi(\delta f_1, u)(\overline{u}, \xi), \xi)\delta f_2, u(\overline{u}, \xi) + 2(\delta f_1, u)(\overline{u}, \xi) x \delta f_2, u(\overline{u}, \xi) u$$

$$+ \delta f_1, u(\overline{u}, \xi) u_x \delta f_2, u(\overline{u}, \xi) - \beta^{-1}(\delta f_1, u)(\overline{u}, \xi) x \parallel a \|^2 \delta f_2, u(\overline{u}, \xi)$$

$$- \langle (\delta f_1, u)(\overline{u}, \xi) a, \delta f_2, u(\overline{u}, \xi), \delta f_2, u(\overline{u}, \xi) a \rangle).$$

This gives

$$\{f_1(\overline{u}, \xi), f_2(\overline{u}, \xi)\}_{\xi, \beta}(u, a) = \mathcal{J}(\xi(\delta f_1, u)(\overline{u}, \xi)).$$
Proposition 3.1. Let \( g_i(\alpha, \beta), i = 1, 2 \) be two regular functions on the affine Kac–Moody algebra. One notes that \( \delta g_{1, u} = \delta g_{2, u} = 0 \). Therefore,

\[
\{g_1, g_2\}^U_{\xi, \beta}(u, a, \xi, \beta) = \{f, g\} \mathcal{C}_{\mathcal{G}}(a, \beta).
\]

We have:

\[
\{f(u, \xi), g(a, \beta)\}^U = \mathcal{J}(\{(\delta f_u(u, \xi), a, \xi), \delta g(a, \beta)\} - x a, g(a, \beta) + [a, g(a, \beta)])).
\]

The sum of the first two terms is equal to 0. The last term is \( \mathcal{J}(\delta f_u([a, a], \delta g_a)) \), and is equal to zero. One can proceed similarly for \( \mathcal{I} \).

\[\Box\]

3 Coadjoint orbits Casimir functions and for \( SU(\mathcal{G}) \)

Let \( \mathcal{H} \) be a central extension of a Lie algebra \( \mathcal{H} \), and \( H \) be a Lie group with Lie algebra is \( \mathcal{H} \). Then \( H \) acts on \( \mathcal{H}^* \) by the coadjoint action along coadjoint orbits.

Proposition 3.1. The coadjoint actions of the groups \( Diff(S^1) \times \mathcal{L}G \) and \( \mathcal{L}G \) are given by

\[
Ad^*(\phi, g)^{-1}(u, a, \xi, \beta) = \left( (u \circ \phi)^{\phi'^2} + \xi S(\phi) + (g^{-1}g', a) \phi'^2 + \frac{1}{2} \left\| g^{-1}g' \right\|^2, \phi' \mathcal{A}(g^{-1}) a \circ \phi + g^{-1}g', \xi, \beta \right),
\]

\[
(\{u \circ \phi\}^{\phi'^2} + \xi S(\phi) + (g'g^{-1}, a) \phi'^2 + \frac{1}{2} \beta (g'g^{-1})^2 + \gamma g'g^{-1}, \phi' \mathcal{A}(g^{-1}) a \circ \phi + \beta g^{-1}g' - \gamma g'g^{-1}, \xi, \beta, \gamma).
\]

The classification of coadjoint orbits of \( \mathcal{V}_{\mathcal{L}G} \) can be known from the classification of coadjoint orbits of the Virasoro and affine Kac-moody algebra. Here we obtain the following new

Theorem 3.2. A coadjoint orbit of \( SU(\mathcal{G}) \) is mapped by \( \mathcal{I} \) to a coadjoint orbit of \( \mathcal{V}_{\mathcal{L}G} \) to a coadjoint orbit of \( \mathcal{V}_{\mathcal{L}G} \).

In other words, this means that if \( \beta_1 \neq 0 \), the elements \( (u_1, a_1, \xi_1, \beta_1) \) and \( (u_1, a_1, \xi_2, \beta_2) \) are in the same coadjoint orbit if and only if: \( \xi_1 = \xi_2, \beta_1 = \beta_2, (a_1, \beta_1) \) and \( (a_2, \beta_2) \) are on the same coadjoint orbit of \( \mathcal{L} \), \( (u_1 - \frac{\phi_1}{\beta_1}, \xi_1) \) and \( (u_2 - \frac{\phi_2}{\beta_2}, \xi_2) \) are elements of the same coadjoint orbit of \( \mathcal{V}_{\mathcal{L}G} \).

Proof. For any \( \phi \in Diff(S^1) \), there exists \( h \in \mathcal{L}G \) such that

\[
hah^{-1} + \beta \frac{\partial h(x)}{\partial x} \cdot h^{-1} = a \circ \phi' \circ h.
\]

By direct computation we check that

\[
\mathcal{I}(Ad^*(\phi, g)(u, a, \xi, \beta)) = (Ad^*(\phi, g, h)\mathcal{I}(u, a, \xi, \beta).
\]

This implies Theorem 3.2. \(\Box\)
Proposition 3.3. The map \( \tilde{f} \) sends the coadjoint orbits of \( \tilde{SU}(\mathcal{G}) \) to coadjoint orbits of \( \text{Vect}(S^1) \otimes \tilde{\mathcal{G}} \).

In other words, this means that if \( \beta_1 \neq 0 \) the elements \( (u_1, a_1, \xi_1, \beta_1, \gamma_1) \) and \( (u_1, a_1, \xi_2, \beta_2, \gamma_2) \) are in the same coadjoint orbit if and only if \( \gamma_1 = \gamma_2, \xi_1 = \xi_2, \beta_1 = \beta_2, (a_1, \beta_1) \) and \( (a_2, \beta_2) \) are on the same coadjoint orbit of \( \tilde{\mathcal{G}} \), \( (u_1 - \frac{d_1}{2 \pi}, \xi_1 - \frac{\gamma_1}{2}) \) and \( (u_2 - \frac{d_1}{2 \pi}, \xi_2 - \frac{\gamma_2}{2}) \) are elements of the same coadjoint orbit of \( \text{Vect}(S^1) \). In a particular case, if \( \beta_1 = \beta_2 = 0 \), then:

Proposition 3.4. If the elements \( (u_1, a_1, \xi_1, \beta_1, \gamma_1) \) and \( (u_1, a_1, \xi_2, \beta_2, \gamma_2) \) are in the same coadjoint orbit then \( \gamma_1 = \gamma_2, (a_1^2 + \gamma_1 a_1^3, \gamma_1) \) and \( (a_2^2 + \gamma_2 a_2^3, \gamma_2) \) are in the same coadjoint orbit of the Virasoro Lie algebra.

Proof. We have: \( \text{Ad}(\phi, g)(a_1^2 + \gamma_1 a_1^3) = (a_1^2 + \gamma_1 a_1^3) \circ \phi + \gamma_1 \mathcal{S}(\phi) \). \( \square \)

Previously, we determined Casimir functions on \( \tilde{SU}(\mathcal{G})' \) and \( \tilde{SU}(\mathbb{R})' \). We gave the following proposition:

Proposition 3.5. Let \( C_{\text{Vir}}, C_{\text{K-M}}, C_{\text{A}} \) be Casimir functions for Virasoro, affine Kac–Moody, and the Heisenberg Lie algebras \( A \) correspondingly. Let \( \tilde{SU}(\mathcal{G})', \tilde{SU}(\mathbb{R})' \) be Poisson submanifolds of \( \tilde{SU}(\mathcal{G}) \) and \( \tilde{SU}(\mathbb{R}) \) defined by \( \xi = 0 \). Then the functions \( C_{\text{Vir}}(u', \xi), C(u, a, \beta, \xi) = C_{\text{K-M}}(a, \beta), \) and \( \int_{S^1} |u'|^{1/2} \), are Casimir functions on \( \tilde{SU}(\mathcal{G})' \). In particular, the functions \( C_{\text{A}}(u, a, \beta, \xi) = C_{\text{A}}(a, \beta), C_{\text{Vir}}(u' - \frac{\xi}{\beta} a', \xi), \) and \( \int_{S^1} |u' - \frac{\xi}{\beta} a'|^{1/2} \), are Casimir functions on \( \tilde{SU}(\mathbb{R})' \).

## 4 Bi–Hamiltonian dispersive water waves systems associated to \( SU(\mathcal{G}) \)

It has been showed in [1], that the dispersive water waves system equation [9, 10, 12] is a bi–Hamiltonian system related to the semi–direct product of a Kac–Moody and Virasoro Lie algebras, and the hierarchy for this system was found. In this section some results of [1] are obtained from another point of view. We obtain new

Proposition 4.1. The functions \( \{ \phi_i(A(u + B \frac{d a}{d x} + C)) | \lambda \in \mathbb{R} \} \) commute pairwise for the Sugawara {\( \{ \cdot \} \)} Sug and e–braket \( \{ \cdot, \cdot \} \) with \( e = (1, 0, 0, 2, 0) \), and \( A = (\xi - \frac{\gamma}{\pi - 2 \lambda}), \ B = -\frac{\gamma}{\pi - 2 \lambda}, \) \( C = -\frac{|\vec{a}|^2}{2|\bar{a}|^2-\lambda} \). \( \square \)

The function \( \lambda \mapsto \phi_1(A(u + B \frac{d a}{d x} + C)) \) has an asymptotic development. The coefficients of this development form a hierarchy. The first term of this development is \( \int_{S^1} u \), and the second one is \( \int_{S^1} (u^2 + \gamma u + \parallel a \parallel^2) \).

A linear combination of these two terms gives the Hamiltonian of equations \( H(u, a) = \int_{S^1} (u^2 + \parallel a \parallel^2) \).

Let \( \{ \phi_i, i \in I \} \) be a set of Casimir functions and \( e \in \mathcal{G} \). Define \( x_\chi = x - \chi e \), for some \( \chi \in \mathbb{R} \).

**Lemma 4.2.** For any \( (i, j) \in I^2 \) and any \( (\lambda, \mu) \in \mathbb{R}^2 \) we have \( \{ \phi_i(x_\lambda), \phi_j(x_\mu) \} = \{ \phi_i(x), \phi_j(x) \} \cdot e = 0 \).

**Lemma 4.3.** Suppose \( \phi_i(x_\lambda) \) can be expanded in terms of inverse powers of \( \lambda \) with some extra function \( f(\lambda) \), and modes \( F_{i,k}(x) \), i.e.,

\[
\phi_i(x_\lambda) = f(\lambda) \sum_{k \in \mathbb{R}} \lambda^{-k} F_{i,k}(x),
\]

then \( \{ F_{i,k+1}, f \} = \{ F_{i,k}, f \} \cdot e \). We can choose \( e \) so that the Hamiltonian \( H(x) = \frac{1}{2} (\chi, x, x) \) commute with these functions.

**Lemma 4.4.** If an element \( e \in \mathcal{G} \) satisfies two conditions: (i) \( ad^*(\cdot) e = 0 \); (ii) for any \( u \in \mathcal{G}, \) \( ad^*(u) e \) belongs to the tangent space to the coadjoint orbit of \( u \) (i.e., for any \( u \in \mathcal{G} \) there exists \( v \in \mathcal{G} \) such that \( ad^*(u) e = ad^*(v) u \)). then the functions \( \phi(a - \lambda e) \) commute with the Hamiltonian of the geodesics \( H(a) = \frac{1}{2} (a, a) \) with respect to the brackets \( \{ \cdot, \cdot \} \) and \( \{ \cdot, \cdot \} \).
5 The universal enveloping algebra of $\mathcal{SU}(\mathcal{G})$

When $\mathcal{H} = \sum_{k \in \mathbb{Z}} \mathcal{H}_k$ has a structure of graded algebra, its universal enveloping algebra $\mathcal{U}(\mathcal{H})$ is also naturally endowed with a structure of a graded Lie algebra. Indeed, the weight of a product $h_1, \ldots, h_n \in \mathcal{U}(\mathcal{H})$ of homogeneous elements is defined to be the sum of the weights of the elements $h_i$, $i = 1, \ldots, n$. The universal enveloping algebra $\mathcal{U}(\mathcal{H})$ admits a filtration $\mathcal{U}(\mathcal{H}) = \bigcup_{n=0}^{\infty} F_n$ where $F_k$ is the vector space generated by the products of at most $k$ elements of $\mathcal{H}$. The generalized enveloping algebra is the algebra of the elements of the form $\sum_{k \leq n} u_k$ where $u_k$ is an element of weight $k$ of $\mathcal{U}(\mathcal{H})$. The product of two such elements is defined by:

$$
\sum_{k \leq n} u_k \sum_{k \leq m} v_k = \sum_{k \leq n} w_k,
$$

where $w_k = \sum_{i \leq k} u_i v_k$, which is a finite sum. Let $\omega_1, \ldots, \omega_n$ be two-cocycles on the Lie algebra $\mathcal{H}$, let $\widetilde{\mathcal{H}}$ be the central extension associated with and let $e_1, \ldots, e_n$ be the central elements associated with these cocycles.

The modified generalized enveloping algebra $\mathcal{U}_{\omega_1, \ldots, \omega_n}(\mathcal{H})$ is defined to be the quotient of the generalized enveloping algebra of $\widetilde{\mathcal{H}}$ by the ideal generated by the elements $\{e_1 - 1, \ldots, e_n - 1\}$. We denote again by $1$ the neutral element of $\mathcal{U}_{\omega_1, \ldots, \omega_n}(\mathcal{H})$. The algebra $\mathcal{U}_{\omega_1, \ldots, \omega_n}(\mathcal{H})$ is by construction a graded algebra and a filtered algebra.

We denote by $F_n$, $n \in \mathbb{N}$ its filtration. Let us recall shortly the main properties of the modified generalized enveloping algebra. Let $V$ be a module over $\mathcal{H}$ such that for any $v \in V$, there exists $n_0 \in \mathbb{Z}$ such that for any $n > n_0$ and any $h \in \widetilde{\mathcal{H}}_n$ we have $h.v = 0$. Such modules are called representations of positive energy, and $e_i$ acts on $V$ by $\lambda_i 1d$. Then $V$ is a module over $\mathcal{U}_{\omega_1, \ldots, \omega_n}(\mathcal{H})$. Such modules are named modules of positive energy. The anticommutator provides a structure of Lie algebra on $\mathcal{U}_{\omega_1, \ldots, \omega_n}(\mathcal{H})$. For this bracket $F_1$ is a Lie sub-algebra isomorphic to the central extension of $\mathcal{H}$ by the cocycle $\omega = \sum_{i=1}^n \omega_i$. We denote by $i$ be the natural inclusion of $\mathcal{H}$ into $\mathcal{U}_{\omega_1, \ldots, \omega_n}(\mathcal{H})$ given by this identification.

5.1 Decomposition of the enveloping algebra of a semi-direct product

In some very particular cases, the modified generalized enveloping algebra of a semi-direct product $\mathcal{K} \rtimes \mathcal{H}$ of two Lie algebras is isomorphic to the tensor product of some modified generalized enveloping algebras of $\mathcal{K}$ and of $\mathcal{H}$. Let $\widetilde{\mathcal{H}}$ be the central extension of $\mathcal{H}$ with the two-cocycle $\omega_\mathcal{H}$. Denote by $\cdot$ the action of the Lie algebra $\mathcal{K}$ on the Lie algebra $\widetilde{\mathcal{H}}$. Let us introduce the semi-direct product $\mathcal{K} \rtimes \mathcal{H}$ which is a central extension of $\mathcal{K} \rtimes \mathcal{H}$ by a two-cocycle $\omega_{\mathcal{K} \rtimes \mathcal{H}}$ with

$$
\omega_{\mathcal{K} \rtimes \mathcal{H}}((0, h_1), (0, h_2)) = \omega_\mathcal{H}(h_1, h_2).
$$

A two-cocycle $\omega_\mathcal{K}$ on $\mathcal{K}$ defines also a two-cocycle $\omega'_{\mathcal{K}}$ by

$$
\omega'_{\mathcal{K}}((g_1, h_1), (g_2, h_2)) = \omega_\mathcal{K}(g_1, g_2),
$$

of $\mathcal{K} \rtimes \mathcal{H}$. Let $I$ be the natural inclusion of $\mathcal{H}$ into $\mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})$ and $J$ be the natural inclusion of $\mathcal{H}$ into $\mathcal{U}_{\omega_\mathcal{K}, \omega'_{\mathcal{K}}}(\mathcal{K} \rtimes \mathcal{H})$.

We call the action of $\mathcal{K}$ on $\mathcal{H}$ realizable in $\mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})$ when there exists a map $F : \mathcal{K} \rightarrow \mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})$ and a two-cocycle $\alpha$ on $\mathcal{K}$ such that for any pair $(g_1, g_2)$ in $\mathcal{K}^2$

$$
F([g_1, g_2]) = [F(g_1), F(g_2)] + \alpha(g_1, g_2) 1,
$$

and the map $F$ satisfies the compatibility condition, i.e., for any $g \in \mathcal{K}$ and $h \in \mathcal{H}$ with the anti-commutator $[F(g), I(h)] = I(g \cdot h)$, of the algebra $\mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})$.

**Theorem 5.1.** If the action of $\mathcal{K}$ is realizable in $\mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})$ then

$$
\mathcal{U}_{\omega_\mathcal{K}, \omega'_{\mathcal{K}}}(\mathcal{K} \rtimes \mathcal{H}) = \mathcal{U}_{\omega_\mathcal{K}, -\alpha}(\mathcal{K}) \otimes \mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})..
$$
Proof. Let \( \mathcal{U}_g = \{ g \in K \} \) with be the unitary subalgebra of \( \mathcal{U}_{\omega_K, \omega_H} (K \ltimes H) \) generated by the elements \( g = g - F(g) \), and \( \mathcal{U}_l = \{ j(h), h \in H \} \) be the unitary subalgebra of \( \mathcal{U}_{\omega_K, \omega_H} (K \ltimes H) \). For any \( (g, h) \) this implies that the generators of \( \mathcal{U}_g \) and \( \mathcal{U}_l \) commute, i.e., \([g, j(h)] = 0\). The subalgebras \( \mathcal{U}_g \) and \( \mathcal{U}_l \) therefore commute. The subalgebra \( \mathcal{U}_g \) is isomorphic to \( \mathcal{U}_{\omega_K - \alpha} (K) \). Let us check that the generators \( \{ g \} \) of this algebra satisfy the relations of the generators of \( \mathcal{U}_{\omega_K - \alpha} (K) \):

\[
[g_1, g_2] = [g_1, g_2] + \omega_K (g_1, g_2) 1 + [F(g_1), F(g_2)] - [F(g_1), g_2] - [g_1, F(g_2)].
\]

Since \( F(g_1) \) is an element of \( \mathcal{U}_g \) and since the algebras \( \mathcal{U}_g \) and \( \mathcal{U}_l \) commute \([F(g_1), g_2] = [F(g_1), F(g_2)] \) and \([g_1, F(g_2)] = [F(g_1), F(g_2)] \). Therefore:

\[
[g_1, g_2] = [g_1, g_2] + \omega_K (g_1, g_2) 1 - [F(g_1), F(g_2)],
\]

and finally

\[
[g_1, g_2] = [g_1, g_2] - F([g_1, g_2]) + (\omega_K (g_1, g_2) - \alpha (g_1, g_2)) 1.
\]

The subalgebra \( \mathcal{U}_l \) is obviously isomorphic to \( \mathcal{U}_{\omega_K} (H) \). The generalized modified enveloping algebra \( \mathcal{U}_{\omega_K + \omega_H} (K \ltimes H) \) is therefore isomorphic to the tensor product over \( \mathbb{C} \) of \( \mathcal{U}_{\omega_K - \alpha} (K) \) with \( \mathcal{U}_{\omega_K} (H) \).

\[\square\]

5.2 The case of \( \mathcal{SU}_C (G) \)

Let \( G \) be a complex Lie algebra and \( C_p \), its dual Coxeter number. Introduce the \( \{ K_1, \ldots, K_n \} \) a basis of \( G \), and the dual basis \( \{ K_1^*, \ldots, K_n^* \} \) with respect to the Killing form \( (\cdot, \cdot) \). We apply Theorem 5.1 for \( K = \text{Vect}(S^1) \), \( H = LG \), \( \omega_K = \xi \omega_{Vir} \), and \( \omega_H = \beta \omega_{K-M} \). In this case, \( \omega_H'^* = \beta \omega_{K-M} \). For \( \eta = \beta + C_p \neq 0 \), the Sugawara construction, delivers a map \( F: \text{Vect}(S^1) \rightarrow \mathcal{U}_g (LG \otimes C) \) defined by

\[
(\beta + \eta) F(L_n) = K \cdot K^*,
\]

where

\[
K \cdot K^* = \sum_{i \in \mathbb{Z}, j = 1, \ldots, n} : (K_j) (K_i^*)_{n-i} :,
\]

(here dots denote the normal ordering), i.e., the action of \( \text{Vect}(S^1) \) is realizable in \( \mathcal{U}_{\beta \omega_{K-M}} (LG) \), with \( \alpha = \beta \omega_{Vir} / 12 \eta \). Thus we obtain

Proposition 5.2. If \( \eta \neq 0 \), then \( \mathcal{U}_{\xi \omega_{Vir}, \beta \omega_{K-M}} (\mathcal{SU}_C (G)) = \mathcal{U}_{\beta \omega_{K-M}} (\text{Vect}(S^1) \otimes C_{(\xi - \alpha)}) \otimes \mathcal{U} (LG) \).

The Lie algebra \( \text{Vect}_C (S^1) \) acts on the Heisenberg algebra by

\[
L_n \cdot a_m = ma_{n+m} + \delta_{n-m} m^2 c_{K-M}.
\]

In this case, on has \( \omega_H'^* = \beta \omega_H + \gamma \omega_{Sp} \). The map \( F: \text{Vect}(S^1) \rightarrow \mathcal{SU}_C (C) \) defined by

\[
\beta F(L_n) = \frac{1}{2} \sum_{i \in \mathbb{Z}} : a_i a_{n-i} : + \gamma a_n,
\]

for a cocycle \( \alpha = (\alpha + \gamma^2 \beta^{-1}) \omega_{Vir} \). For \( \mathcal{SU}_C (C) \) we obtain

Proposition 5.3. For \( \beta \neq 0 \), we have

\[
\mathcal{U}_{\xi \omega_{Vir}, \beta \omega_{K-M}, \gamma \omega_{Sp}} (\mathcal{SU}_C (C)) = \mathcal{U}_{\theta \omega_{Sp}} (\text{Vect}(S^1) \otimes \mathcal{U}_{\omega_{K-M}} (LG)),
\]

with \( \theta = \xi - \gamma^2 / \beta - 1 / 12 \).
5.3 Representations of $\mathfrak{sl}(\mathfrak{g})$

**Proposition 5.4.** A positive energy representation $V$ of $\mathfrak{sl}(\mathfrak{g})$ with non-vanishing $\beta$Id-action of the cocyle $c_{K-M}$ brings about a pair of commuting representations of Virasoro and affine Kac–Moody Lie algebras.

This proposition determines whether a $\mathfrak{sl}(\mathfrak{g})$ Verma module is a sub-module of another Verma module of $\mathfrak{sl}(\mathfrak{g})$. Let $\mathfrak{h}$ be a Cartan algebra of $\mathfrak{g}$ with a basis $\{h_1, \ldots, h_l\}$. The Lie subalgebra $\mathfrak{t}$ of $\mathfrak{sl}(\mathfrak{g})$ is generated by the elements $\{c_{\text{Vir}}, c_{K-M}, u_0, (h_1)_0, \ldots, (h_l)_0\}$. A Verma module $V_{\lambda}(\mathfrak{sl}(\mathfrak{g}))$ of $\mathfrak{sl}(\mathfrak{g})$ is associated to any linear form $\lambda \in \mathfrak{h}^\ast$. Verma modules $V^{\text{Vir}}_\mu, V^{\text{K-M}}_\mu$, are associated to linear forms $\mu, \nu$ over the spaces generated by $c_{\text{Vir}}$ and $u_0$, $c_{K-M}$ and $\{(h_1)_0, \ldots, (h_l)_0\}$ correspondingly. For any $\lambda \in \mathfrak{t}^\ast$, the Verma module $V_{\lambda}(\mathfrak{sl}(\mathfrak{g}))$ is a positive energy representation. Thus, $V_{\lambda}(\mathfrak{sl}(\mathfrak{g}))$ is Virasoro and affine Kac–Moody algebra module. The generator $e$ of $V_{\lambda}(\mathfrak{sl}(\mathfrak{g}))$ brings about a Verma module $V^{\text{Vir}}_\nu$ for Virasoro algebra. It generates also a Verma module $V^{\text{Vir}}_{\mu}$ for the affine Kac–Moody algebra. The linear form $\nu$ satisfies $\nu(u_0) = \lambda(u_0 - F(u_0))e$, i.e.,

$$\lambda(u_0) = (\beta + \eta)^{-1} K \cdot K^\ast e = \nu(u_0) e.$$

Suppose the action of a Casimir element of $\mathfrak{g}$ is given by acts by $D(\lambda)\text{Id}$ for $D(\lambda) \in \mathbb{C}$. We then have

$$\lambda(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta} \cdot e.$$ 
This implies $\nu(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta}$. The other values of $\mu$ and $\nu$ can be computed by the same method.

**Proposition 5.5.** Let $\lambda$ be a linear form over $\mathfrak{h}$ with non-vanishing $\lambda(c_{K-M})$. Then

$$V_{\lambda}(\mathfrak{sl}(\mathfrak{g})) = V^{\text{Vir}}_{\nu} \otimes V^{\text{K-M}}_{\mu},$$

where $\mu(e_i) = \lambda(e_i), i = 1, \ldots, n$, defines $\mu, \mu(c_{K-M}) = \lambda(c_{K-M})$, and $\nu(c_{\text{Vir}}) = \lambda(c_{\text{Vir}}) - \frac{\beta}{2\eta}$ defines $\nu$,

$$\nu(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta}.$$ 

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