Free Surface Deformation and Cusp Formation During the Drainage of a Very Viscous Fluid

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Abstract

We report an experimental study of the deformation of the free surface of a very viscous fluid when drained out of a container through a circular orifice. At some critical height of the fluid, the free surface deforms and forms a dimple. At later times, this dimple becomes a cusp. We found that the height of the dimple measured from the bottom of the tank as well as the curvature of the dimple before the cusp formation, follows power laws behaviors. We found that this behavior was due to an interplay between viscous pressure and surface tension.

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When a normal fluid, such as water, is drained out of a container such as a bath tub, the free surface deforms and, if inertia dominates, a whirlpool forms. This whirlpool is in the form of a vortex or a dimple which apex will be at the level of the bottom surface of the tank just above the draining hole \( h_f \). We have noticed that if the liquid we drain is very viscous, or in other words, when viscous effects dominate the flow, the dynamics and the form of the deformed free surface are no longer the same. The dimple appears far away from the drainage hole in the vertical direction, and it evolves into a sharp dimple at early stages of the draining. In this paper, we report such effect where the free surface of a very viscous fluid deforms into a dimple when drained from a cylindrical container. At later times this dimple evolves into a cusp. Such free surface deformations are also found in selective withdrawal\([2, 3, 4, 5, 6, 7, 8]\), where the flow is steady whereas our flow is gravity driven and non-steady. We, however, fitted the profile of the dimple to a power law of the form \( z \propto x^\alpha \) and found that the power \( \alpha \) is a linear function of time. This might be an effect of the non-steadiness of the interface motion.

A scheme of the set up and the axis notation is displayed in figure 1. We used viscous liquids (from the PDMS family: called also silicon oils) which viscosity ranges from 10 to 1000 Pa.s. (viscosity of water is 0.001 Pa. s.). The fluids are drained out of the container under the effect of gravity through an axisymmetrically placed circular orifice of diameter ranging from 1 to 4 mm at the bottom of a 107 mm diameter cylindrical container.

As the liquid drains, the liquid-air interface dimples above the drain hole. In figure 2, we show a sequence of how the smooth dimple evolves in time to become cusp-like. Notice the transition from a smooth dimple in figure 2a to a sharp interface in figure 2b. In figure 2c: the free surface enters the hole and the bottom of the cusp is far below the bottom of the container. As the level of the liquid falls during the liquid drainage, the dimple becomes sharper. The dimple in the liquid-air interface breaks up at some critical height \( h_f \) measured from the bottom surface of the container and begins to entrain a thread of air. The moment where the thread of air bubbles appear will be called \( t_f \) and will be associated to \( h_f \). In other words, at \( h_f \) the interface undergoes a topological transition in a manner analogous to the pinch-off of a viscous thread. At the transition, the curvature \( \kappa \) of the interface at the dimple becomes infinite, i.e. a cusp forms.

We measured the height of the dimple \( h_f \) and its corresponding time \( t_f \) with an error of 1 mm and 1 sec respectively. This is checked by looking at the video frame just before
some air bubbles start nucleating at the dimple’s apex. We use standard video with a frame frequency of 30 frames per second. We define the non dimensional number $\tilde{h} = \frac{h - h_f}{D}$ where $D$ is the hole diameter as shown in figure 1. The flow rate is varied by varying the hole diameter. We measured the dependence of $\tilde{h}$ versus the dimensionless time scale defined in our case as $\tau = \frac{\nu(t-t_f)}{D^2}$. In this expression $\nu$ is the kinematic viscosity $\mu/\rho$. The result is displayed in figure 3. The Log-Log plot shows a power law dependence and the line is best fit by a $2/3$ power law: $\tilde{h} \sim \tau^{2/3}$.

We monitored, $h_{\infty}$, which is the height of the liquid far away from the cusp region as shown in Fig. 1, versus time and found the flow rate which gives the velocity at the draining hole. The Reynolds number defined as $\frac{\rho U R}{\mu}$ is equal to $3.3 \times 10^{-3}$ for the liquid of viscosity $10$ Pa. s. and is equal to $2.7 \times 10^{-6}$ for a liquid of viscosity $100$ Pa.s. Where $\rho$ is the density of the liquid, $U$ is the flow velocity of the liquid at the hole and $R$ is the exit-hole radius.

This cusp formation is analogous to the selective withdrawal and to drop pinch off. Here the critical dimple height $h_f$ is analogous to the point $z_0$ for thread pinch-off [7]. The distance between the current dimple height and $h_f$, $dh = h - h_f$, is analogous to the minimal thread radius for thread pinch-off [5, 6]. As the interface’s dimple sharpens to a cusp, the capillary stress $\gamma \kappa$ diverges. To check this ideas we measured the radius of curvature $R$ of the dimple, by fitting it to a polynomial, while it is still smooth before the appearance of the bubbles, defined in the magnified area of figure 1. The plot in figure 4 is the value of the curvature $\tilde{R} = (R - R_f)/D$ as defined in fig 1 versus the quantity $\tau$. Similarly $R_f$ is the radius of curvature just before the nucleation of the bubbles. From figure 4 we notice that the cusp radius of curvature scales like $\tau^2$. Notice that the smallest radius we could measure is around $200 \mu m$. Although it is expected that the cusp curvature increases above this value, we do not have enough video resolution to go beyond this value at which numerical noise becomes important. The inset of figure 4 shows how fast the radius of curvature drops in time and is indeed a signature of a finite time singularity process.

If we suppose that, before the cusp is developed, the only forces present in the system are the pressure drop at the hole and the pressure difference across the interface, then the scaling can be self consistently derived as the following: The divergence in capillary stress is balanced by a divergence in the viscous stress at the exit hole and which scales as $\beta \mu Q/(dh)^3$, where $Q$ is the flux of the fluid out of the drain hole, $\beta$ is a number of the order of unity and $\mu$ is the viscosity. Here we are assuming that the flow associated with the dimple
formation is a sink flow which is verified experimentally by monitoring the flow far and close to the hole by monitoring the flow using PIV (Particle Image Velocimetry) and we used passive tracers such as small bubbles to monitor the displacement of a volume element of the liquid. The air bubble we followed is around 200 µm in diameter with a rise velocity smaller than 10 µm/sec measured by tracking the bubble path. Since the flow at the hole has a velocity of 1-2 mm/sec, we suppose that the ascending velocity of the tracer bubble is negligible over the time scale of the observation and the bubble follows essentially the flow.

In figure 5, we depict the position $r$ of the tracer versus the time before it reaches the hole. Here $r$ is the position of a point in the fluid where the origin $r = 0$ is at the center of the hole at the level of the bottom surface and $t^*$ is the time at which the tracer passes through the hole. We plot $r$ versus $t^* - t$. The fit to the plot in figure 5 shows that the particle’s position scales like $(t^* - t)^{33}$ for long times and far away from the hole, and it scales like $(t^* - t)^{5}$ for short times or closer to the hole. The scaling of $r$ versus $t^* - t$ closer to the hole can be explained by writing the equation for a Stokes flow, knowing that the pressure drop at the hole is given by the Poiseuille formula which gives the expression of the pressure gradient along the hole and the flow rate, the geometry and the fluid properties [11].

From this equation [11] we know that the only quantity on which $v$ can depend, other than $z$ and $t$ is $\nu$. From these three quantities only one dimensionless combination can be found which is $\eta = z/(\nu t)^2$. From this scaling we note that the vertical distance $z$ goes like $t^{1/2}$. This explain the behavior $r \propto (t^* - t)^{1/2}$.

We interpret the second result with a simple (heuristic) argument as the following: When the tracer is far from the singularity, the velocity field is the due to a dipole, since the singularity has an image across the horizontal plate, due to the no-slip condition. We know from low Reynolds number hydrodynamics [9] that the dipolar contribution to the velocity field, due to a sink flow, is such that $v \sim 1/r^2$, and since the tracer is passive as we pointed it out earlier in the text, the velocity is simply $dr/dt$, solving in $t$, we find that $r \sim (t^* - t)^{1/3}$, which is the behavior found experimentally. This proves experimentally that the flow in the tank is indeed a sink flow. This property will allow us to use the results of sink flow mainly the pressure drop at the sink which is given by $\beta \mu Q/(dh)^3$. We will suppose our process as quasi-static and we can balance capillary forces and viscous forces due to the sink flow at every step of the drainage. From the balance $\gamma k \approx \beta \mu Q/(dh)^3$ we see that close to
the formation of the cusp, $\kappa \approx 1/(dh)^3$. Measurements of $\kappa$ and $dh$ show that they have the following power law scaling with the time remaining before the formation of the cusp: $\kappa \propto (t_f - t)^{-2}$ and $dh \propto (t_f - t)^{2/3}$. Although the scaling is explained self-consistently, it is unclear at present what determines the values of these exponents. These exponents and the curvature drop in the inset of Fig. 4 are a signature of finite time singularity.

In summary, we have examined the motion of an air-liquid interface due to drainage at very low reynolds numbers where viscosity is dominant. We found that the prior to cusp formation, the interface height and the dimple curvature follow scaling laws dictated by the interplay between surface tension and viscous pressure at the drainage hole.

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[11] The pressure gradient along the $z$ axis of a cylinder if given by: $\frac{\partial P}{\partial z} = \frac{8\mu Q}{\pi D^4}$ where the $z$ axis is the vertical axis along gravity, $Q$ is the flow rate, $\mu$ is the viscosity and $D$ is the radius of the cylinder. The Navier-Stokes equation becomes $\frac{\partial v}{\partial t} = \frac{8\mu Q}{\pi D^4} + \nu \frac{\partial^2 v}{\partial z^2}$.
Fig. 1: S. Chaieb and G.H. McKinley "How to encapsulate"

FIG. 1: The sketch of the set up and the definition of $R$ and $h_f$ which is the radius of curvature of the cusped surface and the height of the hollow dimple that evolves to a cusp respectively.
FIG. 2: Sketch of the time series of the drainage process. In **a**, we display early time of the drainage process. Notice that the surface is smooth and the dimple is not singular. In **b**, Intermediate time in the draining process. Notice the singular aspect of the dimple. This picture was taken just before the break-up into a thread of droplets. In picture **c**, the tip of the dimple is now entering the hole and is following the thread which is a hollow tube where air is being encapsulated. Another liquid can be encapsulated too (later in the text.). The viscosity of the liquid used in these pictures is $10^3$ Pa. s.
FIG. 3: The normalized height of the dimpled region just before “cusping” versus the normalized time scale. The closed circles correspond to viscosity of $10^5$ cSt and the open circles correspond to a viscosity of $10^4$ cSt. The data are best fit by a $2/3$ power law.
FIG. 4: The normalized radius of curvature of the dimple, as defined in figure 1, versus the normalized time scale. The line is the best fit of the data to the power 2. Inset: The drop of the radius of curvature versus time.
FIG. 5: Position of a passive tracer versus the time elapsed between the onset of the draining and the time $t^*$ when the tracer disappears in the hole of the container and the coordinates origin ($r = 0$) corresponds to the center of the hole at the level of the bottom of the plate. The lines are 1/3 power law (open circles) and 1/2 power law (opened squares). The crossover between these two tendencies lies at the hole diameter. These data points were obtained for the same run.