THE MANGOLDT FUNCTION AND THE NON-TRIVIAL ZEROS OF THE Riemann Zeta Function

JESÚS GUILLERA

Abstract. We prove a formula for the Mangoldt function which relates it to a sum over all the non-trivial zeros of the Riemann zeta function, in addition we analyze a truncated version of it.

1. Notation

We use the notation \( \rho = \beta + i \gamma \) for the non-trivial zeros of the zeta function. Following Riemann, we define \( \alpha = -i(\rho - \frac{1}{2}) \). Observe that \( \rho = \frac{1}{2} + i \alpha \) with \( \text{Re}(\alpha) = \gamma \) and \( \text{Im}(\alpha) = \frac{1}{2} - \beta \), and that the Riemann Hypothesis is the statement \( \alpha = \text{Re}(\alpha) \). It is known that \( 0 < \beta < 1 \) (critical band), and therefore \( -1/2 < \text{Im}(\alpha) < 1/2 \). This notation simplifies the appearance of our formulas. As usual in Number Theory, \( \log \) denotes the neperian logarithm. We denote with \( C \) the Euler’s constant to avoid confusing with the zeros of zeta.

2. Introduction

In 1911 E. Landau proved that for any fixed \( t > 1 \)

\[
\sum_{0 < \gamma \leq T} t^{\rho} = -\frac{T}{2\pi} \Lambda(t) + O(\log T),
\]

(1)

where \( \rho \) runs over the non-trivial zeros of the Riemann zeta function \( \zeta(s) \) and \( \Lambda(t) \) is the Mangoldt function which is equal to \( \log p \) if \( x \) is a power of a prime number \( p \) and 0 otherwise. Since the use of (1) is limited by its lack of uniformity in \( t \), Gonek was interested in a version of it uniform in both variables and in [5, 6], he gives the remarkable formula

\[
\sum_{0 < \gamma \leq T} t^{\rho} = -\frac{T}{2\pi} \Lambda(t) + E(t, T),
\]

where the error term \( E(t, T) \) has the estimation

\[
E(t, T) = O(t \log 2t \log \log 3t) + O(\log t \min(T; \frac{t}{(t)})) + O(\log 2T \min(T; \frac{1}{\log t})),
\]

with \( (t) \) denoting the distance between \( t \) and the nearest prime power other than \( t \). Gonek’s formula is also commented in [8]. The aim of this paper is to approximate \( \Lambda(t) \) in a good way. Of course we can do it with the Landau-Gonek’s formula:

\[
\Lambda(t) = -\frac{2\pi}{T} \sqrt{t} \sum_{0 < \gamma \leq T} \cos(\alpha \log t) + \frac{E(t, T)}{T},
\]

(2)

where we have used the Riemann’s notation \( \alpha = -i(\rho - 1/2) \). Observe that the formula (1) or either (2) imply

\[
\Lambda(t) = -2\pi \sqrt{t} \lim_{T \to +\infty} \frac{1}{T} \sum_{0 < \gamma \leq T} \cos(\alpha \log t).
\]

(3)
which has the surprising property that neglecting a finite number of zeros of zeta we still recover the Mangoldt’s function. Also surprising are the self-replicating property of the zeros of zeta observed recently in the statistics of [10], and later proved in [4]; and the property of the zeros discovered by Y. Matiyasevich [2]. In this paper we will prove the new formula:

\[
\Lambda(t) = -4\pi \sqrt{t} \cot \frac{x}{2} \sum_{\gamma > 0} \frac{\sinh x\alpha}{\sinh \pi\alpha} \cos(\alpha \log t) + 2\pi \cot \frac{x}{2} \left( t - \frac{1}{t^2 - 1} \right) + \varepsilon(t, x),
\]

and find bounds for the error term \(\varepsilon(t, x)\). In addition, letting \(\cot(x/2) = (\log T)/T\) we will prove that for integers \(t > 2\), the following truncated version of it holds

\[
\Lambda(t) = -4\pi \sqrt{t} \sum_{0 < \gamma < T} \frac{\sinh x\alpha}{\sinh \pi\alpha} \cos(\alpha \log t) \frac{\log T}{T} + 2\pi \left( t - \frac{1}{t^2 - 1} \right) \frac{\log T}{T} + O\left( t^2 \frac{\log^2 T}{T^2} \right),
\]

and we also will get the estimation of the error for non-integers \(t\). Finally, observing that \(\Lambda(t) = -4\pi \sqrt{t} \lim_{x \to \pi} \left( \cot \frac{x}{2} \sum_{\gamma > 0} \frac{\sinh x\alpha}{\sinh \pi\alpha} \cos(\alpha \log t) \right)\), we see that it shares with (3) the property of invariance when we neglect a finite number of zeros.

### 3. Series involving the Mangoldt function

The formulas that we will prove in this section involve the Mangoldt’s function and a sum over the non-trivial zeros of the Riemann-zeta function.

**Theorem 3.1.** Let \(\Omega = \mathbb{C} - (-\infty, 0]\) (the plane with a cut along the real negative axis). We shall denote by \(\log z\) the main branch of the log function defined on \(\Omega\) taking \(|\arg(z)| < \pi\). We also denote by \(z^s = \exp(s \log(z))\), the usual branch of \(z^s\) defined also on \(\Omega\). For all \(z \in \Omega\) we have

\[
\sum_{n=1}^{\infty} \frac{\Lambda(n)z}{\pi \sqrt{n(1 + nz)}} - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\pi \sqrt{n(z + n)}} = \sqrt{z} - \frac{\zeta'(1/2)}{\pi \zeta(1/2)} - 2 \sum_{\gamma > 0} \frac{\sin(\alpha \log z)}{\sinh \pi\alpha} + h(z),
\]

where

\[
h(z) = \frac{1}{\sqrt{z}(z^2 - 1)} - \frac{1}{2z - 2} + \frac{\log(8\pi)}{\pi} \frac{1}{z + 1} - \frac{2}{\pi} \frac{\sqrt{z}}{z + 1} \arctan \frac{1}{\sqrt{z}}.
\]

**Proof.** We consider the function

\[
f(s) = \frac{\zeta'(s + 1/2)}{\zeta(s + 1/2)} \frac{\pi}{\sin \pi s} z^s,
\]

and let \(I_0, I_r, I_\ell\), where \(I_r = I_1 + I_2 + I_3\) and \(I_\ell = I_4 + I_5 + I_6\), be the analytic continuation of the integral

\[
I = \frac{1}{2\pi i} \int f(s) ds,
\]

along the indicated sides of the contour of the figure. It is a known result that all the zeros of \(\zeta(s + 1/2)\) are in the band among the lines red and green.
we will arrive at (4).

Hence, by analytic continuation, we have that for all $z$

\[
-\infty + iT \quad I_4 \quad -\frac{1}{2} + iT \quad I_1 \quad +\infty + iT
\]

Integrals extended to all $z$ by analytic continuation

\[
-\infty - iT \quad I_6 \quad -\frac{1}{2} - iT \quad I_3 \quad +\infty - iT
\]

We will follow this scheme of proof: The integral along the line $\sigma = -1/2$ is calculated for $|z| > 1$ integrating to the left and for $|z| < 1$ integrating to the right. Both expressions are different but valid for $z \in \Omega$ by analytic continuation. Finally, equating both expressions we will arrive at (1).

Indeed, if $|z| < 1$, integrating to the right hand side, we get by applying the residues theorem that

\[
I_0 + I_r = -\text{res}_{s=\frac{1}{2}} \left( \frac{\zeta'(s + \frac{1}{2})}{\zeta(s + \frac{1}{2})} \frac{\pi}{\sin \pi s} z^s \right) - \sum_{\nu=0}^{\infty} \text{res}_{s=\nu} \left( \frac{\zeta'(s + \frac{1}{2})}{\zeta(s + \frac{1}{2})} \frac{\pi}{\sin \pi s} z^s \right) - \sum_{|\gamma| < T} \text{res}_{s=\rho+\frac{1}{2}} \left( \frac{\zeta'(s + \frac{1}{2})}{\zeta(s + \frac{1}{2})} \frac{\pi}{\sin \pi s} z^s \right) = \pi \sqrt{z} - \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\zeta'(n + \frac{1}{2})}{\zeta(n + \frac{1}{2})} z^n - \pi \sum_{|\gamma| < T} \frac{z^{\rho - \frac{1}{2}}}{\sin \pi (\rho - \frac{1}{2})}.
\]

Hence, by analytic continuation, we have that for all $z \in \Omega$

\[
I_0 + I_r = \pi \sqrt{z} - \frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} - \sum_{n=1}^{\infty} \frac{\Lambda(n)z}{\sqrt{n(z+n)}} - \pi \sum_{|\gamma| < T} \frac{z^{\rho - \frac{1}{2}}}{\sin \pi (\rho - \frac{1}{2})}.
\]

If $|z| > 1$, then following the way to the left hand side, we deduce that

\[
I_0 + I_l = \sum_{n=1}^{\infty} \text{res}_{s=-2n-\frac{1}{2}} \left( \frac{\zeta'(s + \frac{1}{2})}{\zeta(s + \frac{1}{2})} \frac{\pi}{\sin \pi s} z^s \right) + \sum_{\nu=0}^{\infty} \text{res}_{s=-\nu} \left( \frac{\zeta'(s + \frac{1}{2})}{\zeta(s + \frac{1}{2})} \frac{\pi}{\sin \pi s} z^s \right)
\]

\[= \sum_{n=1}^{\infty} \frac{\zeta'(-2n)}{\zeta(-2n)} \sin(2\pi n) z^{-2n-\frac{1}{2}} + \sum_{n=1}^{\infty} (-1)^n \frac{\zeta'(-n)}{\zeta(-n)} z^{-n},
\]

where we understand the expression inside the first sum of (7) as a limit based on the identity

\[
\lim_{s \to -2n} \frac{\zeta'(s)}{\zeta(s)} = \lim_{s \to -2n} \frac{\zeta'(s) \pi s + \zeta(s) \sin \pi s}{\pi s} = \lim_{s \to -2n} \frac{\sin \pi s \zeta'(s)}{\pi s} \zeta(s).
\]

We use the functional equation (which comes easily from the functional equation of $\zeta(s)$):

\[
\frac{\zeta'(1-s)}{\zeta(1-s)} = \log 2\pi - \psi(s) + \frac{\pi}{2} \tan \frac{\pi s}{2} - \frac{\zeta'(s)}{\zeta(s)},
\]

to simplify the sums in (7). For the first sum in (7), we obtain

\[
\sum_{n=1}^{\infty} \frac{\zeta'(-2n)}{\zeta(-2n)} \sin(2\pi n) z^{-2n-\frac{1}{2}} = \pi \sum_{n=1}^{\infty} z^{-2n-\frac{1}{2}},
\]
and for the last sum in (7), we have
\[
\log 2\pi \sum_{n=1}^{\infty} (-1)^n z^{-n} - \sum_{n=1}^{\infty} (-1)^n \psi \left( \frac{1}{2} + n \right) z^{-n} + \frac{\pi}{2} \sum_{n=1}^{\infty} z^{-n} - \sum_{n=1}^{\infty} (-1)^n \frac{\zeta'(n + \frac{1}{2})}{\zeta(n + \frac{1}{2})} z^{-n},
\]
where \(\psi\) is the digamma function, which satisfies the property
\[
\psi \left( \frac{1}{2} + n \right) = 2h_n - C - 2 \log 2, \quad h_n = \sum_{j=1}^{n} \frac{1}{2j - 1}.
\]
Using the identity, due to Hongwei Chen \([1, p.299, exercise 34]\)
\[
2 \sum_{n=1}^{\infty} (-1)^n h_n z^{-n} = i \sqrt{z} \log \frac{\sqrt{z} + i}{\sqrt{z} - i} = -2 \sqrt{z} \arctan \frac{1}{\sqrt{z}},
\]
we get that for \(|z| > 1\)
\[
I_0 + I_\ell = -\frac{\pi}{\sqrt{z}(z^2 - 1)} - \log 2\pi \frac{z + 1}{z + 1} + \frac{C + \log 4}{z + 1} + \frac{\pi}{2z - 2} + 2 \sqrt{z} \frac{1}{z + 1} \arctan \frac{1}{\sqrt{z}} - \sum_{n=1}^{\infty} (-1)^n \frac{\zeta'(n + \frac{1}{2})}{\zeta(n + \frac{1}{2})} z^{-n}.
\]
Then, by analytic continuation, we obtain that for all \(z \in \Omega\):
\[
I_0 + I_\ell = -\frac{\pi}{\sqrt{z}(z^2 - 1)} - \frac{C + \log 8\pi}{z + 1} + \frac{\pi}{2z - 2} + 2 \sqrt{z} \frac{1}{z + 1} \arctan \frac{1}{\sqrt{z}} - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n(1 + zn)}}. \quad (9)
\]
It is easy to deduce that \(I_2 = I_5 = 0\), and we will prove that \(I_\ell\) and \(I_\ell\) tend to 0 as \(T \to \infty\) in the Section 8 of this paper. Hence, by identifying (6) and (9), and observing that the pole at \(z = 1\) is removable, we complete the proof.

**Theorem 3.2.** The following identity
\[
\sum_{\gamma > 0} \frac{\sinh \gamma z}{\sinh \pi \alpha} - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{2\pi \sqrt{n}} \left( \frac{ie^{iz}}{e^{iz} + n} - \frac{ie^{-iz}}{e^{-iz} + n} \right) = f(z), \quad (10)
\]
where
\[
f(z) = \sin \frac{z}{2} - \frac{1}{8} \tan \frac{z}{4} - \frac{C + \log 8\pi}{4\pi} \tan \frac{z}{2} - \frac{1}{4\pi \cos \frac{z}{2}} \log \frac{1 - \tan \frac{z}{4}}{1 + \tan \frac{z}{4}},
\]
holds for \(|\text{Re}(z)| < \pi\).

**Proof.** Let
\[
H(z) = \sqrt{z} - \frac{\zeta'(\frac{1}{2})}{\pi \zeta(\frac{1}{2})} + h(z).
\]
That is
\[
H(z) = \sqrt{z} - \frac{\zeta'(\frac{1}{2})}{\pi \zeta(\frac{1}{2})} + \frac{1}{\sqrt{z}(z^2 - 1)} - \frac{1}{2z - 2} + \frac{\log(8\pi) + C}{\pi} \frac{1}{z + 1} + i \frac{\sqrt{z}}{\pi z + 1} \log \frac{\sqrt{z} + i}{\sqrt{z} - i}.
\]
From (4), we see that the function \(H(z)\) has the property \(H(z) = -H(z^{-1})\). Hence
\[
H(z) = \frac{H(z) - H(z^{-1})}{2} = \frac{1}{2} \left( \sqrt{z} - \frac{1}{\sqrt{z}} \right) + \frac{1}{2 \sqrt{z} - 1} \left( \frac{1}{\sqrt{z} + z^2 \sqrt{z}} \right) - \frac{z + 1}{4 z - 1} \log \frac{\sqrt{z} + i}{\sqrt{z} - i}.
\]
From (11), we see that the function \(H(z)\) has the property \(H(z) = -H(z^{-1})\). Hence
\[
H(z) = \frac{H(z) - H(z^{-1})}{2} = \frac{1}{2} \left( \sqrt{z} - \frac{1}{\sqrt{z}} \right) + \frac{1}{2 \sqrt{z} - 1} \left( \frac{1}{\sqrt{z} + z^2 \sqrt{z}} \right) - \frac{z + 1}{4 z - 1} \log \frac{\sqrt{z} + i}{\sqrt{z} - i}.
\]
(11)
When $|\text{Re}(z)| < \pi$ we have $e^{iz} \in \Omega$ so that, we may put $e^{iz}$ instead of $z$ in Theorem 3.1. If in addition we multiply by $-i/2$, we get

$$\sum_{\gamma > 0} \frac{\sinh z\alpha}{\sinh \pi\alpha} - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{2\pi \sqrt{n}} \left( \frac{ie^{iz}}{e^{iz} + n} - \frac{ie^{-iz}}{e^{-iz} + n} \right) = -\frac{i}{2} H(e^{iz}).$$

From (11), we have

$$-\frac{i}{2} H(e^{iz}) = -\frac{i}{4} \left( e^{iz/2} - e^{-iz/2} \right) - \frac{i}{4} \left( e^{-iz/2} - e^{iz/2} \right) + \frac{i}{e^{iz} - e^{-iz}} \left( \log 8\pi + C e^{iz/2} - e^{-iz/2} \right)$$

which we can write as

$$-\frac{i}{2} H(e^{iz}) = -\frac{i}{4} \left( e^{iz/2} - e^{-iz/2} \right) - \frac{i}{4} \left( e^{3iz/2} + e^{-3iz/2} \right)$$

which simplifies to

$$-\frac{i}{2} H(e^{iz}) = \frac{1}{2} \sin z \cdot \frac{1}{4} \cos \left( \frac{z + \frac{\pi}{2}}{2} \right) + \frac{1}{8} \cot \frac{z}{2} - \frac{\log 8\pi + C}{4\pi} \tan \frac{z}{2} + \frac{1}{4\pi \cos \frac{z}{2}} \log \left( \frac{i \tan \frac{z + \pi}{4}}{4} \right) - \frac{i}{8 \cos \frac{z}{2}}.$$

As

$$\frac{1}{4\pi} \frac{1}{\cos \frac{z}{2}} \log \left( \frac{i \tan \frac{z + \pi}{4}}{4} \right) - \frac{i}{8 \cos \frac{z}{2}} = \frac{1}{4\pi \cos \frac{z}{2}} \log \frac{z + \pi}{4},$$

and using elementary trigonometric formulas we arrive at (10).

4. New formulas for the Mangoldt function

In this section we relate the Mangoldt’s function to a sum over all the non-trivial zeros of the Riemann-zeta function and find bounds of the error term.

**Theorem 4.1.** If $x \in [0, \pi)$ and $t > 1$, then

$$-4\pi \sqrt{t} \cot \frac{x}{2} \sum_{\gamma > 0} \frac{\sinh x\alpha}{\sinh \pi\alpha} \cos(\alpha \log t) + 4\pi \sqrt{t} g(x, t) \cot \frac{x}{2}$$

$$= 4\pi \sqrt{t} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{(t - n)^2 + 4nt \cos^2 \frac{\pi}{2}} - 4\pi \sqrt{t} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{(nt - 1)^2 + 4nt \cos^2 \frac{\pi}{2}},$$

where $g(x, t)$ is the function

$$g(x, t) = \frac{(1 + t) \sin \frac{x}{2}}{2\sqrt{t}} - \frac{\sqrt{t} \sin \frac{x}{2}}{8\sqrt{t} \cos \frac{x}{2} + 4(1 + t)} \frac{t \sin x(C + \log 8\pi)}{2\pi(1 + t^2 + 2t \cos x)}$$

$$- \frac{(1 + t) \sqrt{t} \cos \frac{x}{2}}{4\pi(1 + t^2 + 2t \cos x)} \log \frac{1 + t - 2\sqrt{t} \sin \frac{x}{2}}{1 + t + 2\sqrt{t} \sin \frac{x}{2}} - \frac{(t - 1) \sqrt{t} \sin \frac{x}{2}}{2\pi(1 + t^2 + 2t \cos x)} \arctan \frac{t - 1}{2\sqrt{t} \cos \frac{x}{2}}. \quad (12)$$
Proof. Replace $z$ with $x - i \log t$ and take real parts. The function $g(x, t)$ is the real part of $f(x - i \log t)$.

It is interesting to expand $g(x, t)$ in powers of $\pi - x$, and we get

$$g(x, t) = \frac{1}{2} \left( \frac{t+1}{t} - \frac{t}{t^2-1} \right) \sqrt{t}$$

$$+ \left( \frac{-t}{4(t+1)^2} + \frac{t(C + \log 8\pi)}{2\pi(t-1)^2} - \frac{t}{2\pi(t-1)^2} + \frac{(1+t)\sqrt{t}}{4\pi(t-1)^2} \log \frac{\sqrt{t}-1}{\sqrt{t}+1} \right)(\pi-x)$$

$$+ O(\pi-x) = \frac{1}{2} \left( \frac{t+1}{t} - \frac{t}{t^2-1} \right) \sqrt{t} + O\left( \frac{\pi-x}{t} \right),$$

which shown that $g(x, t)$ tends to a simple function as $x \to \pi^-$.

Theorem 4.2. If $x \in [0, \pi)$, then

$$0 < \left( -4\pi \sqrt{t} \cot \frac{x}{2} \sum_{\gamma > 0} \frac{\sinh x\alpha}{\sinh \pi\alpha} \cos(\alpha \log t) + 4\pi \sqrt{t} g(x, t) \cot \frac{x}{2} \right) - \Lambda(t)$$

$$< F(t) + 4 \cos^2 \frac{x}{2} \left( 3t^2 \log t + \frac{\pi^2}{2} t + \frac{1}{4} \log t + 0.6 \right).$$

where $g(x, t)$ is the function (13), and

$$F(t) = E(t) \cdot 4t \sqrt{t} \left( \frac{\Lambda([t]) \sqrt{n} \cos^2 \frac{x}{2}}{(\{t\})^2 + 4t [t] \cos^2 \frac{x}{2}} + 4t \sqrt{t} \frac{\Lambda([t] + 1) \sqrt{[t] + 1} \cos^2 \frac{x}{2}}{(1 - \{t\})^2 + 4t ([t] + 1) \cos^2 \frac{x}{2}} \right),$$

where $E(t) = 0$ if $t$ is and integer and 1 otherwise.

Proof. Let

$$S = 4t \sqrt{t} \sum_{n=1}^{\infty} \frac{\Lambda(n) \sqrt{n} \cos^2 \frac{x}{2}}{(t-n)^2 + 4nt \cos^2 \frac{x}{2}} - 4t \sqrt{t} \sum_{n=1}^{\infty} \frac{\Lambda(n) \sqrt{n} \cos^2 \frac{x}{2}}{(nt-1)^2 + 4nt \cos^2 \frac{x}{2}}.$$

First, we see that

$$S < 4t \sqrt{t} \sum_{n=1}^{\infty} \frac{\Lambda(n) \sqrt{n} \cos^2 \frac{x}{2}}{(t-n)^2 + 4nt \cos^2 \frac{x}{2}}.$$

The contribution of the values $n = [t]$ and $n = [t] + 1$ to the above summation is equal to $\Lambda(t) + F(t)$, and the contribution of $n = [t] - 1$ is bounded by $4t^2 \log t \cos^2 \frac{x}{2}$. Hence

$$S < \Lambda(t) + F(t) + 4 \left( t^2 \log t + t \sqrt{t} \sum_{|t| = 2}^{\infty} \frac{\Lambda(n) \sqrt{n}}{(t-n)^2} + t \sqrt{t} \sum_{n=[t]+2}^{\infty} \frac{\Lambda(n) \sqrt{n}}{(t-n)^2} \right) \cos^2 \frac{x}{2}.$$

Then, as the Mangoldt function is bounded by the logarithm, we obtain

$$S < \Lambda(t) + F(t) + 4 \left( t^2 \log t + t \sqrt{t} \sum_{n=1}^{\infty} \frac{\log(n) \sqrt{n}}{(t-n)^2} + t \sqrt{t} \sum_{n=[t]+2}^{\infty} \frac{\log(n) \sqrt{n}}{(t-n)^2} \right) \cos^2 \frac{x}{2}.$$

Then we can deduce that

$$S < \Lambda(t) + F(t) + 4 \left( t^2 \log t + t \sqrt{t} \int_{1}^{t-1} \frac{\log(u) \sqrt{u}}{(t-u)^2} du + t \sqrt{t} \int_{t+1}^{\infty} \frac{\log(u) \sqrt{u}}{(t-u)^2} du \right) \cos^2 \frac{x}{2},$$
by observing that the integrands are increasing and decreasing functions of $u$ respectively. With the help of Maple, we get

$$
\sqrt{t} \int_1^{t-1} \frac{\log(u) \sqrt{u}}{(t-u)^2} \, du = \sqrt{t} \sqrt{t-1} \log(t-1) + \frac{1}{2} \log(t-1) \log \frac{\sqrt{t}-\sqrt{t-1}}{\sqrt{t}+\sqrt{t-1}}
+ \log \left(1 + \frac{1}{\sqrt{t}}\right) - \log \left(1 - \frac{1}{\sqrt{t}}\right) + \log \left(1 - \sqrt{1 - \frac{1}{t}}\right) - \log \left(1 + \sqrt{1 - \frac{1}{t}}\right)
+ \text{dilog} \left(1 + \frac{1}{\sqrt{t}}\right) - \text{dilog} \left(1 - \frac{1}{\sqrt{t}}\right) + \text{dilog} \left(1 - \sqrt{1 - \frac{1}{t}}\right) - \text{dilog} \left(1 + \sqrt{1 - \frac{1}{t}}\right),
$$

and

$$
\sqrt{t} \int_{t+1}^{\infty} \frac{\log(u) \sqrt{u}}{(t-u)^2} \, du = \sqrt{t} \sqrt{t+1} \log(t+1) + \frac{1}{4} \log^2 t - \frac{1}{4} \log t \log(t+1) + \frac{\pi^2}{3}
+ \frac{1}{2} \log(t+1) \log(\sqrt{t} + \sqrt{t+1}) - \frac{1}{2} \log(t) \log(\sqrt{t+1} - \sqrt{t}) + \log \frac{\sqrt{t+1} + \sqrt{t}}{\sqrt{t+1} - \sqrt{t}}
+ \text{dilog} \left(1 + \frac{1}{\sqrt{t}}\right) + \text{dilog} \left(\sqrt{1 + \frac{1}{t}}\right),
$$

where dilog denotes the dilogarithm. Finally, by expanding asymptotically and bounding each of the terms of (15) and (16), we can derive that

$$
t \sqrt{t} \int_1^{t-1} \frac{\log(u) \sqrt{u}}{(t-u)^2} \, du + t \sqrt{t} \int_{t+1}^{\infty} \frac{\log(u) \sqrt{u}}{(t-u)^2} \, du = 2t^2 \log t + \frac{\pi^2}{2} t + \frac{1}{4} \log t + h(t),
$$

where $h(t)$ is a positive decreasing function. Therefore $h(t) < h(2) < 0.6$ for $t > 2$. \hfill \Box

**Corollary 4.3.** If $x \in [0, \pi)$ and $t \geq 2$ is an integer, then

$$
0 < \left(-4 \pi \sqrt{t} \cot \frac{x}{2} \sum_{\gamma > 0} \frac{\sinh x \alpha}{\sinh \pi \alpha} \cos(\alpha \log t) + 4 \pi \sqrt{t} g(x, t) \cot \frac{x}{2}\right) - \Lambda(t)
< 4 \cos^2 \frac{x}{2} \left(4t^2 \log t + \frac{\pi^2}{2} t + \frac{1}{4} \log t + 0.6\right),
$$

where $g(x, t)$ is the function (12).

**Lemma 4.4.** If $x$ and $T$ are related by

$$
\cot \frac{x}{2} = \frac{\log T}{T},
$$

then for $T \geq 2$, we have

$$
\left|\sum_{\gamma \geq T} \frac{\sinh x \alpha}{\sinh \pi \alpha} \cos(\alpha \log t)\right| < 3 \sqrt{t^2 + \log T} \frac{T}{T}.
$$

**Proof.** Let $\eta \equiv \text{Im}(\alpha)$. It is well known that $-1/2 < \eta < 1/2$ (critical band). As $\alpha = \gamma + i \eta$, we see that $|\cos(\alpha \log t)| < \cosh(|\eta| \log t) + \sinh(|\eta| \log t)$, and we get

$$
\left|\sum_{\gamma \geq T} \frac{\sinh x \alpha}{\sinh \pi \alpha} \cos(\alpha \log t)\right| \leq \left|\sum_{\gamma \geq T} \frac{\sinh x \gamma}{\sinh \pi \gamma} \cos(\alpha \log t)\right| \leq \sum_{\gamma \geq T} \frac{\sinh x \gamma \log |\eta|}{\sinh \pi \gamma}
\leq \sqrt{t} \sum_{\gamma \geq T} \frac{\sinh x \gamma}{\sinh \pi \gamma} \leq \sqrt{t} \sum_{\gamma \geq T} e^{-(\pi - x) \gamma}. 
$$
As $x$ and $T$ are related by

$$x = 2 \arccot \frac{\log T}{T},$$

we see that

$$\pi - x = \frac{2 \log T}{T} - \frac{2 \log^3 T}{3 T^3} + O \left( \frac{1}{T^4} \right).$$

Hence

$$\left| \sum_{\gamma \geq T} \frac{\sinh x \alpha}{\sinh \pi \alpha} \cos(\alpha \log t) \right| < \sqrt{t} \sum_{\gamma \geq T} \exp \left( -2 \gamma \log \frac{T}{T} \right).$$

We subdivide the interval into intervals of length 1. Hence, the left hand side is also less or equal that

$$\sqrt{t} \left( \sum_{\gamma \in [T,T+1]} \exp \left( -2 \gamma \log \frac{T}{T} \right) + \sum_{\gamma \in [T+1,T+2]} \exp \left( -2 \gamma \log \frac{T+1}{T+1} \right) + \cdots \right).$$

From [12, Corollary 1] we get that for $T \geq 2$ the number of zeros in an interval $[T, T+1]$ is less than $3 \log T$. Hence

$$\left| \sum_{\gamma \geq T} \frac{\sinh x \alpha}{\sinh \pi \alpha} \cos(\alpha \log t) \right| < \sqrt{t} \sum_{n=1}^{\infty} 3(\log n) \exp(-2 \log n)$$

$$< 3 \sqrt{t} \int_{T-1}^{+\infty} \frac{\log u}{u^2} du < 3 \sqrt{T^2 + \log \frac{T}{T}},$$

which is the stated bound. $\square$

**Corollary 4.5.** If $T \geq 2$ and $t \geq 2$ is a positive integer number, then

$$\left| -4 \pi \sqrt{t} \left( \sum_{\gamma < T} \frac{\sinh x \alpha}{\sinh \pi \alpha} \cos(\alpha \log t) \right) \frac{\log T}{T} + 2 \pi \left( t - \frac{1}{t^2 - 1} \right) \frac{t}{T} - \Lambda(t) \right|$$

$$< 4 \left( 4t^2 \log t + \frac{\pi^2}{2} t + 3 \pi t + \frac{1}{4} \log t + 0.6 \right) \frac{\log^2 T}{T^2} + 24 \pi t \frac{\log T}{T^2}. \quad (19)$$

**Proof.** It is a consequence of the Corollary 4.3 and Lemma 4.4. $\square$

## 5. Graphics

As all the known zeros of the zeta-function satisfy the Riemann Hypothesis we will assume it in our calculations, hence

$$\Lambda(t) \approx 4 \pi \sqrt{t} \left( \sum_{0 < \gamma < T} \frac{\sinh x \gamma}{\sinh \pi \gamma} \cos(\gamma \log t) \right) \frac{\log T}{T} + 2 \pi \left( t - \frac{1}{t^2 - 1} \right) \frac{\log T}{T}. \quad (20)$$

We use Sagemath [11] to draw the graphics. In Figure 1 we see the graphic obtained with the formula (20) summing over the 10000 first non-trivial zeros of zeta, that is taking $T = 9877.782654004$. The following estimations

$$\varepsilon(t, T) = O \left( t \log t \frac{\log^2 T}{T^2} \right), \quad \varepsilon(t, T) = O \left( t \log 2t \log \log 3t \right),$$

are respectively the errors that we get in the Mangoldt’s function for integers $t > 1$ if we use either our formula or either the Landau’s formula.
In this figure we have represented the function \( \log(t) \) with the color red and the Mangoldt’s function \( \Lambda(t) \) with color blue.

6. ANOTHER BOUND

In this section we get another bound for

\[
\left| \sum \frac{\sinh x\alpha}{\sinh \pi\alpha} \cos(\alpha \log t) \right|,
\]

From (6) and (9), we get

\[
\sum_{\rho} \frac{z^{\rho - \frac{1}{2}}}{\sin \pi(\rho - \frac{1}{2})} - \sum_{|\gamma| < T} \frac{z^{\rho - \frac{1}{2}}}{\sin \pi(\rho - \frac{1}{2})} = \sum_{|\gamma| \geq T} \frac{z^{\rho - \frac{1}{2}}}{\sin \pi(\rho - \frac{1}{2})} = \frac{1}{\pi} (I_r - I_\ell),
\]

where \( I_r \) and \( I_\ell \) are the analytic continuation of the integral

\[
I = \frac{1}{2i} \int \frac{\zeta'(s + \frac{1}{2})}{\zeta(s + \frac{1}{2})} \frac{z^s}{\sin \pi s} ds,
\]

along the corresponding routes.

**Lemma 6.1.** Let \( T > 1 \), then we have

\[
\left| \frac{z^s}{\sin \pi s} \right| \leq 4 e^{\sigma \log |z|} e^{-T(\pi + \arg(z))} \quad \text{if} \quad s = \sigma + iT,
\]

\[
\left| \frac{z^s}{\sin \pi s} \right| \leq 4 e^{\sigma \log |z|} e^{-T(\pi - \arg(z))} \quad \text{if} \quad s = \sigma - iT,
\]

in case that \( \sigma > 0 \) and \( |z| < 1 \) or in case \( \sigma < 0 \) and \( |z| > 1 \).

**Proof.**

\[
\left| \frac{z^s}{\sin \pi s} \right| = 2 \left| \frac{e^{(\sigma + iT)(\log |z| + i \arg(z))}}{e^{\pi(\sigma + iT)} - e^{-\pi(\sigma + iT)}} \right| \leq 2 \left| \frac{e^{\sigma \log |z| - T \arg(z)}}{e^{-\pi \sigma} e^{\pi T} |e^{\pi \sigma} e^{-\pi T}|} \right| < 4 \left| \frac{e^{\sigma \log |z|} e^{-T\arg(z)}}{e^{\pi T} - e^{-\pi T}} \right| < 4 e^{\sigma \log |z|} e^{-T(\pi + \arg(z))}.
\]

The proof for \( s = \sigma - iT \) is similar. \( \Box \)
In the following lemma we get bounds of the function $\zeta'(s + 1/2)/\zeta(s + 1/2)$:

**Lemma 6.2.** For $\sigma \geq 2$, we have

$$\left| \frac{\zeta'(\sigma + iT)}{\zeta(\sigma + iT)} \right| = \left| \sum_{n=1}^{\infty} \Lambda(n) \right| \leq \sum_{n=1}^{\infty} \left| \frac{\Lambda(n)}{n^{\sigma + iT}} \right| = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2} < 0.57.$$  

For $\sigma < -1$ and $T > 1$, using the above bound for $\sigma \geq 2$, the inequalities

$$|\Psi(\sigma + iT)| < 3.2 + \frac{1}{2} \log(\sigma^2 + T^2), \quad \left| \tan \frac{\pi(\sigma + iT)}{2} \right| < 1.72,$$

and the functional equation (3), we get

$$\left| \frac{\zeta'(\sigma + iT)}{\zeta(\sigma + iT)} \right| < 7.33 + \log \sqrt{\sigma^2 + T^2} < 7.33 + \log |\sigma| + \log T.$$  

If $-1 < \sigma \leq 2$, then for every real number $T \geq 2$, there exist $T' \in [T, T + 1]$ such that uniformly one has

$$\left| \frac{\zeta'(\sigma + iT')}{\zeta(\sigma + iT')} \right| < 9 \log^2 T + 2 \log T < 11 \log^2 T.$$  

To prove it we first deduce from [12, Corollary 1] that the number of zeros $\rho$ such that $\gamma \in [T, T + 1]$ is less than $|3 \log T|$. If we subdivide the interval into $1 + |3 \log T|$ equal parts, then the length of each part is $(1 + |3 \log T|)^{-1}$. As the number of parts exceeds the number of zeros, we deduce applying the Dirichlet pigeon-hole that there is part that contains no zeros. Hence, for $T'$ lying in this part, we see that

$$|T' - \gamma| > \frac{1}{1 + |3 \log T|}.$$

Hence, we infer that each summand in [3, Proposition 3.89] is less than $1 + |3 \log T|$, and since the number of summands of this kind is less than $|3 \log T|$, we finally get

$$\left| \frac{\zeta'(\sigma + iT')}{\zeta(\sigma + iT')} \right| < 3(\log T) (1 + 3 \log T).$$

Remark: As

$$\left| \sum_{\gamma \in [T, T+1]} \frac{\sinh x \alpha}{\sinh \pi \alpha} \cos(\alpha \log t) \right| \leq \sqrt{t} \sum_{\gamma \in [T, T+1]} \exp \frac{-2\gamma \log T}{T} < 3\sqrt{t} \frac{\log T}{T^2},$$

the error that we are making in the above left sum when we take $T$ instead of $T'$ is less than $3\sqrt{t} T^{-2} \log T$.

**Corollary 6.3.** If $x$ and $T$ are related by

$$\cot \frac{x}{2} = \frac{\log T}{T},$$

then for $t \geq 2$ and $T \geq 2$, we have

$$|I_r - I_l| < 44 \frac{t^{3/2} + \log^2 T}{\log t} \frac{T}{T^2}.$$  

**Proof.** As $x$ and $T$ are related by

$$x = 2 \arccot \frac{\log T}{T},$$
we see that
\[ \pi - x = \mathcal{O}\left( \frac{2 \log T}{T} \right), \quad e^{-T(\pi - x)} = \mathcal{O}\left( \frac{1}{T^2} \right). \]

Let \( x \in [0, \pi) \), replacing \( z \) with \( e^{x - \log t} \), we see that \(|z| = t\) and \( \arg(z) = x \). Hence
\[ |I_3| < 44(\log^2 T) e^{-T(\pi - x)} \int_{-\frac{1}{2}}^{\frac{3}{2}} e^\sigma \log t \, d\sigma + 2.28 e^{-T(\pi - x)} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^\sigma \log t \, d\sigma. \]

As we can generalize the integral for \(|z| = t > 1\) by analytic continuation, for \( t \geq 2 \) and \( T \geq 2 \), we get
\[ |I_3| < e^{-T(\pi - x)} \left[ 44 \log^2 T \left( \frac{t^{3/2}}{\log t} - \frac{t^{-1/2}}{\log t} \right) - 2.28 \frac{t^{3/2}}{\log t} \right]. \]

Hence
\[ |I_3| < 44 \frac{t^{3/2}}{\log t} \frac{\log^2 T}{T^2}. \]

For \(|I_6|\), we have
\[ |I_6| < 4e^{-T(\pi - x)} \int_{-\infty}^{\frac{3}{2}} (7.33 + \log |\sigma|) e^\sigma \log t \, d\sigma + 4e^{-T(\pi - x)} \log T \int_{-\infty}^{\frac{1}{2}} e^\sigma \log t \, d\sigma \]
\[ + 44e^{-T(\pi - x)} \log^2 T \int_{-\frac{1}{2}}^{\frac{1}{2}} e^\sigma \log t \, d\sigma, \]

and as \( \log |\sigma| < |\sigma| \), and extending the integrals by analytic continuation, for \( t \geq 2 \) and \( T \geq 2 \), we get
\[ |I_6| < e^{-T(\pi - x)} \left[ \frac{29.4 t^{-3/2}}{\log t} + \frac{6 t^{-3/2}}{\log^2 t} - \frac{t^{-3/2}}{\log^2 t} + \frac{4(\log T)t^{-3/2}}{\log t} + 44(\log^2 T) \left( \frac{t^{-1/2}}{\log t} - \frac{t^{-3/2}}{\log t} \right) \right]. \]

Hence, for \( t \geq 2 \) and \( T \geq 2 \), we have
\[ |I_6| < \frac{\log^2 T}{T^2} \frac{44}{\sqrt{t} \log t}. \]

In a similar way we can evaluate the order of \(|I_1|\) and \(|I_4|\), and we get that they are of order much smaller. \( \square \)

**Corollary 6.4.** For \( T \geq 2 \) and integers \( t \geq 2 \), we have
\[ \left| \sum_{\gamma \geq T} \frac{\sinh \alpha \gamma}{\sinh \pi \alpha \gamma} \cos(\alpha \log t) \right| < 45 \frac{t^{3/2} \log^2 T}{\log t} \frac{1}{T^2}. \]

Compare this bound with that of (18).

**Final Remark**

In this paper we have continued our research initiated in [7] concerning the Mangoldt’s function. However this paper is self-contained. In [11] we also got some new formulas for the Moebius’ \( \mu \) and Euler’s \( \varphi \) functions but we only gave the error in the variable \( T \) and not its dependence on the variable \( t \). In our opinion finding \( \varepsilon(T, t) \) could be interesting. This has been done in this paper but only for the Mangoldt’s function.
Acknowledgements

Thanks a lot to Olivier Bordellès for informing me that an upper bound for the number of zeros of zeta such that $\gamma \in [T, T+1]$ can be obtained from [12, Corollary 1]. Also, many thanks to Juan Arias de Reyna for very interesting comments.

References

[1] D. Bailey, J. Borwein, N. Calkin, R. Girgensohn, D. Russell Luke and V. Moll, Experimental Mathematics in Action, A.K. Peters, Ltd, Wellesley, Massachusetts, (2007).
[2] G. Beliakov and Y. Matiyasevich, Approximation of Riemann’s zeta function by finite Dirichlet series: multiprecision numerical approach, Experimental Mathematics 24, 150-161, (2015).
[3] O. Bordellès, Arithmetic Tales, Universitext, Springer Verlag, London (2012).
[4] K. Ford and A. Zaharescu, Marco’s repulsion phenomenon between zeros of $L$-functions, preprint at arXiv:1305.2520 (2013).
[5] S.M. Gonek, A formula of Landau and mean values of $\zeta(s)$, Topics in Analytic Number Theory, ed. by S.W. Graham and J.D. Vaaler, 92–97, Univ. Texas Press 1985.
[6] S.M. Gonek, An explicit formula of Landau and its applications to the theory of the zeta-function, Contemporary Math. 143 (1993), 395-413.
[7] J. Guillera, Some sums over the non-trivial zeros of the Riemann zeta function, Unpublished paper available at arXiv:1307.5723 (2013-2014).
[8] J. Kaczorowski, A. Languasco and A. Perelli, A note on Landau’s formula, Funct. Approx. Comment. Math. 28, 173-186, (2000).
[9] E. Landau, Über die Nullstellen der Zetafunction, Math. Annalen 71, 548-564, (1911).
[10] R. Pérez Marco, Statistics on Riemann zeros; arXiv:1112.0346 (2011).
[11] W. Stein, Sage: a free open-source mathematics software system licensed under the GPL.
[12] T. Trudgian, Improved upper bound for the argument of the Riemann zeta function on the critical line II. J. Number Theory 134, 280-292, (2014).

Department of Mathematics, University of Zaragoza, 50009 Zaragoza, SPAIN
E-mail address: jguillera@gmail.com