The PGM-index: a multicriteria, compressed and learned approach to data indexing

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ABSTRACT

The recent introduction of learned indexes has shaken the foundations of the decades-old field of indexing data structures. Combining, or even replacing, classic design elements such as B-tree nodes with machine learning models has proven to give outstanding improvements in the space footprint and time efficiency of data systems. However, these novel approaches are based on heuristics, thus they lack any guarantees both in their time and space requirements.

We propose the Piecewise Geometric Model index (shortly, PGM-index), which achieves guaranteed I/O-optimality in query operations, learns an optimal number of linear models (such as RMI and FITing-tree). We show experimentally that the PGM-index improves the space of the best known learned index, i.e. FITing-tree, by 63.3\% and of the B-tree by more than four orders of magnitude, while achieving their same or even better query time efficiency.

We complement this result by proposing three variants of the PGM-index which address some key issues occurring in the design of modern big data systems. First, we design a compressed PGM-index that further reduces its succinct space footprint by exploiting the repetitiveness at the level of the learned linear models it is composed of. Second, we design a PGM-index that adapts itself to the distribution of the query operations, thus resulting in the first known distribution-aware learned index to date. Finally, given its flexibility in the offered space-time trade-offs, we propose the multicriteria PGM-index whose speciality is to efficiently auto-tune itself in a few seconds over hundreds of millions of keys to the possibly evolving space-time constraints imposed by the application of use.

1. INTRODUCTION

The ever-growing amount of information coming from the Web, social networks and Internet of Things severely impairs the management of available data. Advances in CPUs, GPUs and memories hardly solve this problem without properly devised algorithmic solutions. Hence, much research has been devoted to dealing with this enormous amount of data, particularly focusing on memory hierarchy utilisation [2, 31, 34], query processing on streams [10], space efficiency [23, 24], parallel and distributed processing [15]. But despite these formidable results, we still miss proper algorithms and data structures that are flexible enough to work under computational constraints that vary across users, devices and applications, and possibly evolve over time.

In this paper, we restrict our attention to the case of indexing data structures for internal or external memory which solve the so-called fully indexable dictionary problem. This problem asks to store a multiset of real keys in order to efficiently support the query rank(x), which returns for any possible key x the number of keys in S which are smaller than x. In formula, rank(x) = |{y ∈ S | y < x}|. Now, suppose that the keys in S are stored in a sorted array A. It is not difficult to deploy the rank primitive to implement the following classic queries:

- member(x) = TRUE if x ∈ S, FALSE otherwise. Just check whether A[rank(x)] = x, since A stores items from position 0.
- predecessor(x) = max{y ∈ S | y < x}. Return A[i], where i = rank(x) − 1.
- range(x, y) = S ∩ [x, y]. Scan from A[rank(x)] up to keys smaller than or equal to y.

Moreover, we notice that it is easy to derive from member(x) the implementation of the query lookup(x), which returns the satellite data of x ∈ S (if any), NIL otherwise.

In the following, we will use the generic expression query operations to refer to any of the previous kinds of pointwise queries, namely: member(x), predecessor(x) and lookup(x). On the contrary, we will be explicit in referring to range(x, y) because of its variable-size output.

Background and related work. Existing indexing data structures can be grouped into: (i) hash-based, which range from traditional hash tables to recent techniques, like Cuckoo hashing [25]; (ii) tree-based, such as B-trees and its variants [2, 31, 34]; (iii) bitmap-based [33, 7], which allow efficient set operations; and (iv) trie-based, which are commonly used for string keys. Unfortunately, hash-based indexes do not support predecessor or range searches; bitmap-based indexes can be expensive to store, maintain and decompress [35]; trie-based indexes are mostly pointer-based and, apart from recent results [12], keys are stored uncompressed thus taking space proportional to the dictionary size.

As a result, B-trees and their variations remain the predominant data structures in commercial database systems for these kinds of queries [29].

1For other related work we refer the reader to [18, 13], here we mention only the results which are closer to our proposal.
Very recently, this old-fashioned research field has been shaken up by the introduction of learned indexes [18], whose combination, or even replacement, of classic design elements, such as B-tree nodes, with machine-learned models have been shown to achieve outstanding improvements in the space footprint and time efficiency of all the above query operations. The key idea underlying these new data structures is that indexes are models that we can train to map keys to their location in the array $A$, given by rank. This parallel between indexing data structures and rank functions does not seem a new one, in fact any of the previous four families of indexes offers a specific implementation of it. But its novelty becomes clear when we look at the keys $k \in S$ as points $(k, \text{rank}(k))$ in the Cartesian plane. As an example, let us consider the case of a dictionary of keys $a, a + 1, \ldots, a + n - 1$, where $a$ is an integer. Here, $\text{rank}(k)$ can be computed exactly as $k - a$ (i.e. via a line of slope 1 and intercept $-a$), and thus it takes constant time and space to be implemented, independently of the number $n$ of keys in $S$. This trivial example sheds light on the potential compression opportunities offered by patterns and trends in the data distribution. However, we cannot argue that all datasets follow exactly a “linear trend”.

In general, we have to design Machine Learning (ML) techniques that learn rank by extracting the patterns in the data through succinct models, ranging from linear to more sophisticated ones, which admit some “errors” in the output of the model approximating rank and that, in turn, can be efficiently corrected to return the exact value of rank. This way, we can reframe the implementation of rank as a ML problem in which we search for the model that is fast to be computed, is succinct in space, and best approximates rank according to some criteria that will be detailed below.

This is exactly the design goal pursued by [18] with their Recursive Model Index (RMI), which uses a hierarchy of ML models organised as a Directed Acyclic Graph (DAG) and trained to learn the input distribution $(k, \text{rank}(k))$ for all $k \in S$. At query time each model, starting from the top one, takes the query key as input and picks the following model in the DAG that is “responsible” for that key. The output of RMI is the position returned by the last queried ML model, which is, however, an approximate position. A final binary search is thus executed within a range of neighbouring positions whose size depends on the prediction error of RMI.

One could presume that ML models cannot provide the guarantees ensured by traditional indexes, both because they can fail to learn the distribution and because they can be expensive to evaluate [17]. Unexpectedly, it was reported that RMI dominates the B$^+$-tree, being up to 1.5–3× faster and two orders of magnitude smaller in space [18].

This notwithstanding, the RMI introduces another set of space-time trade-offs between model size and query time which are difficult to control because they depend on the distribution of the input data, on its DAG structure and on the complexity of the ML models adopted. This motivated [19] to introduce the FITing-tree which uses only linear models, a B$^+$-tree to index them, and it provides an integer parameter $\varepsilon \geq 1$ controlling the size of the region in which the final binary search step has to be performed. Figure 1 shows an example of a linear model $f_s$ approximating 14 keys and its use in determining the approximate position of a key $k = 37$, which is indeed $f_s(k) \approx 7$ instead of the correct position 5, thus making an error $\varepsilon = 2$. Experiments showed that the FITing-tree improves the time performance of the B$^+$-tree with a space saving of orders of magnitude [13], but this result was not compared against the performance of RMI. Moreover, the computation of the linear models residing in the leaves of the FITing-tree is sub-optimal in theory and inefficient in practice. This impacts negatively on its final space occupancy (as we will quantify in Section 6.1) and slows down its query efficiency because of an increase in the height of the B$^+$-tree indexing those linear models.

**Our contribution.** In this paper, we contribute to the design of optimal linear-model learned indexes, to their compression and to the automatic selection of the best learned index that fits the requirements (in space, latency or query distribution) of an underlying application in five main steps.

1. We design the first learned index that solves the fully indexable dictionary problem with time and space complexities which are provably better than classic data structures for hierarchical memories, such as B-trees, and modern learned indexes. Our index is I/O-optimal according to the lower bound for predecessor search in external memory proved by [30]. We call it the Piece-wise Geometric Model index (PGM-index) because it turns the indexing of a sequence of keys into the coverage of a sequence of points via segments. Unlike previous work [13, 18], the PGM-index is built upon an optimal number of linear models, and its peculiar recursive construction makes it a purely learned data structure, rather than hybrid of traditional and learned data structures. This aspect allows the PGM-index to make the most of the constant space-time indexing feature offered by the linear models on which it is built upon (see Section 2.2, Theorem 1, and Table 1).

2. We test the experimental efficiency of the PGM-index through a large set of experiments over three known datasets (see Section 6). We show that the PGM-index improves the space occupancy of the FITing-tree by 63.3%, of the Cache-Sensitive Search tree (CSS-tree)
by a factor 82.7×, and of the B-tree by more than four orders of magnitude, while achieving their same or even better query efficiency (see Figure 6). Unlike the RMI, the PGM-index offers theoretical bounds on the query time and space occupancy, and it guarantees a 4× increase in the precision of approximating the position of the searched key which, in turn, induces a uniform improvement over all possible space-time trade-offs achieved by RMI.

3. We then show that the (succinct) space footprint of a PGM-index can be further reduced by designing novel compression algorithms for the building blocks of the linear models (i.e. slopes and intercepts) on which our index hinges upon. In particular, we provide an efficient algorithm that reduces the number of distinct slopes to be encoded to their optimal minimum number, which is an interesting algorithmic contribution in itself. In practice, in just 80 ms this algorithm improves the space occupancy of a PGM-index over a dataset of hundreds of million keys by up to 52.2%. This makes the PGM-index the first compressed learned index to date (see Section 3).

4. We also propose the first example of a distribution-aware learned index, namely one that adapts itself not only to the distribution of the dictionary keys but also to their access frequencies. The resulting distribution-aware PGM-index achieves the query time of biased data structures [1] [3] [20] [52], but with a space occupancy that adapts to the “regularity trend” of the input dataset thus benefiting of the succinctness of learned indexes (see Section 4 and Theorem 2).

5. Giving the flexibility in space-time trade-offs offered by the PGM-index (as shown in Section 6), we finally study the concept of Multicriteria Data Structures, which combines in a formal yet effective way multicriteria optimisation with data structures. A multicriteria data structure, for a given problem $P$, is defined by a pair $(\mathcal{F}, A)_P$ where $\mathcal{F}$ is a family of data structures, each one solving $P$ with a proper trade-off in the use of some resources (e.g. time, space, energy), and $A$ is an optimisation algorithm that selects in $\mathcal{F}$ the data structure that “best fits” an instance of $P$. We demonstrate the fruitfulness of this concept by introducing the Multicriteria PGM-index, which hinges upon an optimisation algorithm designed to efficiently explore $\mathcal{F}$ via a proper space-time cost model for our PGM-index (Section 5). In our experiments, we show that the Multicriteria PGM-index is fast, taking less than 20 seconds, to reorganise itself to best index a dataset of 750M keys within newly given space or time bound.

This supports the vision of a new generation of big data processing systems designed upon data structures that can be adjusted on-the-fly to the application, device and user needs, which may possibly change over time, as foreseen in [13]. In a way, the multicriteria PGM-index solves their ambitious research challenge within a design space in which $\mathcal{F}$ consists of variants of PGM-index, space-time constraints change continuously, and the data structure has to be optimised as fast as possible.

2. THE PGM-INDEX

Given a multiset $S$ of $n$ keys drawn from a universe $\mathcal{U}$ the PGM-index is a data structure parametric in an integer $\epsilon \geq 1$ which solves the fully indexable dictionary problem introduced in Section 1. Let $A$ be a sorted array storing the (possibly repeated) keys of $S$.

The first ingredient of the PGM-index is a Piecewise Linear Approximation model (PLA-model), namely a mapping between keys from $\mathcal{U}$ and their approximate positions in the array $A$. Specifically, we aim to learn a mapping that returns a position for a key $k \in \mathcal{U}$ which is at most $\epsilon$ away from the correct one in $A$. We say piecewise because one single linear model (i.e. a segment) could be insufficient to $\epsilon$-approximate the positions of all the keys from $\mathcal{U}$. As a consequence, the PGM-index learns a sequence of segments, each one taking constant space (two floats and one key) and constant query time to return the $\epsilon$-approximate position of $k$ in $A$. We show below in Lemma 1 that there exists a linear time and space algorithm which computes the optimal PLA-model namely one that consists of the minimum number of $\epsilon$-approximate segments. We also observe that the $\epsilon$-approximate positions returned by the optimal PLA-model can be turned into exact positions via a binary search within a range of $\pm \epsilon$ keys in $A$, thus taking time logarithmic in the parameter $\epsilon$, not in the size of $A$.

The second ingredient of the PGM-index is a recursive algorithm which adapts the index structure to the distribution of the input keys, thus resulting as much independent as possible from their number (see Figure 2 for pictorial example). More precisely, in order to make the most of the ability of a single segment to index in constant space and time an arbitrarily long range of keys, we turn the optimal PLA-model built over the array $A$ into a set of keys, and we proceed recursively by building another optimal PLA-model over these keys. This process continues until one single segment is obtained, which will form the root of our data structure. Overall, each PLA-model forms a level of the PGM-index, and each segment of that PLA-model forms a node of the data structure at that level. The speciality of this recursive construction with respect to known learned index proposals (cf. FITing-tree or RMI) is that the PGM-index is a pure learned index which does not hinge on classic data structures either in its structure (as in the FITing-tree) or as a fallback when the ML models err too much (as in RMI).

The net result are three main advantages in its space-time complexity. First, the PGM-index is built upon the minimum number of segments, while other learned indexes, such as FITing-tree and RMI compute a sub-optimal number of segments with a subsequent penalisation in their time and space efficiency. Second, the PGM-index uses these segments as constant-space routing tables at all levels of...
the data structure, while other indexes (e.g. FITing-tree, B-tree and variants) use space-consuming nodes storing a large number of keys which depends on the disk-page size only, thus resulting blind to the possible regularity present in the data distribution. Third, these routing tables of the PGM-index take constant time to restrict the search of a key in a node to a smaller subset of the indexed keys (of size \(\varepsilon\)), whereas nodes in the B\(^7\)-tree and the FITing-tree incur a search cost that grows with the node size, thus slowing the tree traversal during the query operations.

The following two subsections will detail the two main ingredients of the PGM-index described above.

2.1  The optimal PLA-model

Let us be given a sorted array \(A = [k_0, k_1, \ldots, k_{n-1}]\) of \(n\) real and possibly repeated keys drawn from a universe \(U\). In this section, we describe how an \(\varepsilon\)-approximate implementation of the mapping \(rank\) from keys to positions in \(A\) can be efficiently computed and succinctly stored via an optimal number of segments, which is one of the core design elements of a PGM-index. In the next section, we will comment on the recursive construction of the whole PGM-index and the implementation of the query operations.

A segment \(s\) is a triple \((k, \text{slope}, \text{intercept})\) that indexes a range of \(U\) through the function \(f_s(k) = k \times \text{slope} + \text{intercept}\), as depicted in Figure 1. An important characteristic of the PGM-index is the "precision" \(\varepsilon\) of its segments.

**Definition 1.** Let \(A\) be a sorted array of \(n\) keys drawn from a universe \(U\) and let \(\varepsilon \geq 1\) be an integer. A segment \(s = (k, \text{slope}, \text{intercept})\) is said to provide an \(\varepsilon\)-approximate indexing of the range of all keys in \([k_i, k_{i+r}]\), for some \(k_i, k_{i+r} \in A\), if \(|f_s(x) - rank(x)| \leq \varepsilon\) for all \(x \in U\) such that \(k_i \leq x \leq k_{i+r}\).

An \(\varepsilon\)-approximate segment can be seen as an approximate predecessor search data structure for its covered range of keys offering constant query time and constant occupied space. One single segment, however, could be insufficient to \(\varepsilon\)-approximate the \(rank\) function over the whole \(U\); hence, we look at the computation of a sequence of segments, also termed Piecewise Linear Approximation model (PLA-model).

**Definition 2.** Given \(\varepsilon \geq 1\), the piecewise linear \(\varepsilon\)-approximation problem consists of computing the PLA-model which minimises the number of its segments \(\{s_0, \ldots, s_{m-1}\}\), provided that each segment \(s_j\) is \(\varepsilon\)-approximate for its covered range of keys, these ranges are disjoint and together cover the entire universe \(U\).

A way to find the optimal PLA-model for an array \(A\) is by dynamic programming, but the \(O(n^3)\) time it requires is prohibitive. The authors of the FITing-tree [13] attacked this problem via a heuristic approach, called shrinking cone, which is linear in time but does not guarantee to find the optimal PLA-model and indeed it performs poorly in practice (as we will show in Section 6.1).

Interestingly enough, we found that this problem has been extensively studied for lossy compression and similarity search of time series (see e.g. [27, 5, 8, 9, 37] and refs therein), and it admits streaming algorithms which take \(O(n)\) optimal time. The key idea of this family of approaches is to reduce the piecewise linear \(\varepsilon\)-approximation problem to the one of constructing convex hulls of a set of points, which in our case is the set \([k_i, rank(k_i))\) grown incrementally for \(i = 0, \ldots, n - 1\). As long as the convex hull can be enclosed in a (possibly rotated) rectangle of height no more than \(2\varepsilon\), the index \(i\) is incremented and the set is extended. As soon as the rectangle enclosing the convex hull is higher than \(2\varepsilon\), we compute one segment of the PLA-model by taking the line which splits that rectangle into two equal-sized halves. Then, the current set of processed elements is emptied and

![Figure 2: Each segment in a PGM-index is “responsible” for routing the queried key to one of the segments of the level below. In the picture, the dotted lines show that the root segment \(s_0\) routes the queried key to one of the segments among \(\{s_7, s_6, s_9\}\) (which together cover the same range of keys as \(s_0\)), whereas \(s_8\) routes the queried key to either \(s_3\) or \(s_4\) (which together cover the same range of keys as \(s_8\)). Segments at the last level (i.e. levels[2]) are \(\varepsilon\)-approximate segments for the sub-range of keys in \(A\) depicted by wavy lines of which they are responsible for.](image-url)
the algorithm restarts from the rest of the input points. This greedy approach can be proved to be optimal and to have linear time and space complexity. We can rephrase this result in our context as follows.

**Lemma 1 (Optimal PLA-model).** Given a sequence \((x_i, y_i)_{i=0, \ldots, n-1}\) of points that are nondecreasing in their \(x\)-coordinate. There exists a streaming algorithm that in linear time and space computes the minimum number of segments that \(\varepsilon\)-approximate the \(y\)-coordinate of each point in that set.

For our application to the dictionary problem, the \(x, s\) of Lemma 1 correspond to the input keys \(k_i\), and the \(y, s\) correspond to their positions \(0, \ldots, n-1\) in the sorted input array \(A\). Therefore, Lemma 1 provides an algorithm which computes in linear time and space the optimal PLA-model for the keys stored in \(A\).

The next step is to prove a simple but very useful bound on the number of keys covered by a segment of the optimal PLA-model, which we will deploy in the analysis of the PGM-index.

**Lemma 2.** Given an ordered sequence of keys \(k_i \in \mathcal{U}\) and the corresponding sequence \(\{(k_i, i)\}_{i=0, \ldots, n-1}\) of points in the Cartesian plane that are nondecreasing in both their coordinates. The algorithm of Lemma 1 determines a (minimum) number \(m_{\text{opt}}\) of segments which cover at least \(2\varepsilon\) points each, so that \(m_{\text{opt}} \leq n/(2\varepsilon)\).

**Proof.** For any chunk of \(2\varepsilon\) consecutive keys \(k_i, k_{i+1}, \ldots, k_{i+2\varepsilon-1}\), let us take the horizontal segment \(y = i + \varepsilon\). It is easy to see that those keys generate the points \((k_i, i), (k_{i+1}, i+1), \ldots, (k_{i+2\varepsilon-1}, i+2\varepsilon-1)\) and each of these keys have \(y\)-distance at most \(\varepsilon\) from that line, which is then an \(\varepsilon\)-approximate segment for that range of \(2\varepsilon\)-keys. Hence, any segment of the optimal PLA-model covers at least \(2\varepsilon\) keys.

### 2.2 Indexing the PLA-model

The algorithm of Lemma 1 returns an optimal PLA-model for the input array \(A\) as a sequence \(M = [s_0, \ldots, s_{m-1}]\) of \(m\) segments. Now, in order to solve the fully indexable dictionary problem, we need a way to find the \(\varepsilon\)-approximate segment \(s_j = (k_{i_j}, \text{slope}_j, \text{intercept}_j)\) responsible for estimating the approximate position \(\text{pos}\) of a query key \(k\), i.e., this is the rightmost segment \(s_j\) such that \(k_{i_j} \leq k\). When \(m\) is large, we could perform a binary search on the sequence \(M\), or we could index it via a proper data structure, such as a multiway search tree (as done in the FITting-tree). In this case, the membership query could then be answered in three steps. First, the multiway search tree is queried to find the rightmost segment \(s_j\) such that \(k_{i_j} \leq k\). Second, that segment \(s_j\) is used to estimate the position \(\text{pos} = f_{s_j}(k)\) for the query key \(k\). Third, the exact position of \(k\) is determined via a binary search within \([\text{pos} - \varepsilon, \text{pos} + \varepsilon]\). The net consequence is that a query over this data structure would take \(O(\log_b m + \log \varepsilon)\) time, where \(B\) is the fan-out of the multiway tree and \(\varepsilon\) is the error incurred by \(s_j\) when approximating \(\text{rank}(k)\).

However, the indexing strategy above does not take full advantage of the key distribution because it resorts to a classic data structure with fixed fan-out to index \(M\). Therefore, we introduce a novel strategy which consists of repeating the piecewise linear approximation process recursively on a set of keys derived from the sequence of segments. More precisely, we start with the sequence \(M\) constructed over the whole input array \(A\), then we extract the first key of \(A\) covered by each segment and finally construct another optimal PLA-model over this reduced set of keys. We proceed in this recursive way until the PLA-model consists of one segment.

If we map segments to nodes, then this approach constructs a sort of multiway search tree but with three main advantages with respect to B-trees (and thus FITting-trees): (i) its nodes have variable fan-out driven by the (typically large) number of keys covered by the segments associated with those nodes; (ii) the segment in a node plays the role of a constant-space and constant-time \(\varepsilon\)-approximate routing table for the various queries to be supported; (iii) the search in each node corrects the \(\varepsilon\)-approximate position returned by that routing table via a binary search (see next), and thus it has a time cost that depends logarithmically on \(\varepsilon\), and hence it is independent of the number of keys covered by the corresponding segment.

Now, a query operation over this Recursive PGM-index works as follows. At every level, it uses the segment referring to the visited node to estimate the position of the searched key \(k\) among the keys of the lower level. The real position is then found by a binary search in a range of size \(2\varepsilon\) centred around the estimated position. Given that every key on the next level is the first key covered by a node on that level, we have identified the next node to visit, and thus the next segment to query, and the process continues until the last level is reached. An example of a query operation is depicted in Figure 2.

**Theorem 1.** Let \(A\) be an ordered array of \(n\) keys from a universe \(\mathcal{U}\), and \(\varepsilon \geq 1\) be a fixed integer parameter. The Recursive PGM-index with parameter \(\varepsilon\) indexes the array \(A\) taking \(\Theta(m)\) space and answers rank, membership and predecessor queries in \(O(\log m)\) time and \(O(\log_m m \log(\varepsilon/B))\) I/Os, where \(m\) is the minimum number of \(\varepsilon\)-approximate segments covering \(A\), \(c \geq 2\varepsilon\) denotes the variable fan-out of the data structure, and \(B\) is the block size of the External Memory model. Range queries are answered in extra (optimal) \(O(K)\) time and \(O(K/B)\) I/Os, where \(K\) is the number of keys satisfying the range query.

**Proof.** Each step of the recursion reduces the number of segments by a variable factor \(c\) which is nonetheless at least \(2\varepsilon\) because of Lemma 2. The number of levels is, therefore, \(L = O(\log \log m)\), and the total space required by the index is \(\sum_{i=0}^{L} m/(2\varepsilon)^i = \Theta(m)\). For the rank, membership and predecessor queries, the bounds on the running time and the I/O complexity follow easily by observing that a query performs \(L\) binary searches over intervals having size at most \(2\varepsilon\). In the case of range queries, we report the \(K\) keys by scanning \(A\) from the position returned by the rank query.

The main novelty of the PGM-index is that its space overhead does not grow linearly with \(n\), as in the traditional

\[\text{To correctly approximate the position of a key } k \text{ falling between the last key covered by a segment } s_j \text{ and the first key covered by } s_{j+1}, \text{ we compute } \min\{f_{s_j}(k), f_{s_{j+1}}(k_{i_{j+1}})\}.\]
indexes mentioned in Section 1 but it depends on the “regularity trend” of the input array \( A \) which also decreases with the value of \( \varepsilon \). Because of Lemma 2 the number of segments at the last level of a PGM-index cannot be more than \( n/(2\varepsilon) \), so that \( m < n \) since \( \varepsilon \geq 1 \). Since this fact holds for the recursive levels too, the PGM-index cannot be asymptotically worse in space and time than a \( 2\varepsilon \)-way tree, such as a FITing-tree, \( B^+ \)-tree or CSS-tree (just take \( \varepsilon = \Theta(B) \) in Theorem 1). According to the lower bound proved by [30], we can state that the PGM-index solves I/O-optimally the fully indexable dictionary problem with predecessor search, meaning that it can potentially replace any existing index with virtually no performance degradation.

Table 1 summarises these bounds for the PGM-index and its competitors both in the RAM model and in the External Memory (EM) model for the rank query and its derivatives: i.e. predecessor, membership and lookup.

The thorough experimental results of Section 6 will support further these theoretical achievements by showing that the PGM-index is much faster and succinct than FITing-tree, \( B^+ \)-tree and CSS-tree because, in practice, it will be \( m_{\text{opt}} \ll n \) and \( c \gg 2\varepsilon \).

### 3. The Compressed PGM-index

Compressing the PGM-index boils down to providing proper lossless compressors for the keys and the segments (i.e. intercepts and slopes) which constitute the building blocks of our learned data structure. In this section, we propose techniques specifically tailored to the compression of the segments, since the compression of the input keys is an orthogonal problem for which there exist a plethora of solutions (see e.g. [21] [23] for integer keys, and [6] [19] for floating-point keys).

For what concerns the compression of the intercepts, they can be made increasing by using the coordinate system of the segments, i.e. the one that computes the position of an element \( k \) as \( f_s(k) = (k-k_j) \times \text{slope}_j + \text{intercept}_j \). Then, since the result of \( f_s(k) \) has to be truncated to return an integer position in \( A \), we store the intercepts as integers \( \text{intercept}_j \).

Finally, we exploit the fact that intercepts are increasing integers smaller than \( n \) and thus use the succinct data structure of [25] to obtain the following result.

**Proposition 1.** Let \( m \) be the number of segments of a PGM-index indexing \( n \) keys drawn from a universe \( U \). The intercepts of those segments can be stored using \( m \log(n/m) + 1.92m + o(m) \) bits and be randomly accessed in \( O(1) \) time.

The compression of slopes is more involved, and we need to design a specific novel compression technique. The starting observation is that the algorithm of Lemma 1 computes the segments, computed by Lemma 1, and with \( m_{\text{greedy}} \) the number of \( \varepsilon \)-approximate segments computed by the greedy algorithm at the core of the FITing-tree. Of course, \( m_{\text{opt}} \leq m_{\text{greedy}} \). The learned index RMI is not included in the table because it lacks of guaranteed bounds.

| Data structure               | Space       | RAM model worst case time | EM model worst case I/Os | EM model best case I/Os |
|------------------------------|-------------|----------------------------|--------------------------|-------------------------|
| Plain sorted array           | \( O(1) \)  | \( O(\log n) \)           | \( O(\log n) \)          | \( O(\log n) \)        |
| Multiway tree (e.g. B-tree)  | \( \Theta(n) \) | \( O(\log n) \)           | \( O(\log n) \)          | \( O(\log n) \)        |
| FITing-tree                  | \( \Theta(m_{\text{greedy}}) \) | \( O(\log m_{\text{greedy}} + \log \varepsilon) \) | \( O(\log m_{\text{greedy}}) \) | \( O(\log m_{\text{greedy}}) \) |
| Recursive PGM-index          | \( \Theta(m_{\text{opt}}) \) | \( O(\log m_{\text{opt}} + \log \varepsilon) \) | \( O(\log m_{\text{opt}}) \) | \( O(1) \) |

The compression of slopes is more involved, and we need to design a specific novel compression technique. The starting observation is that the algorithm of Lemma 1 computes the segments, computed by Lemma 1, and with \( m_{\text{greedy}} \) the number of \( \varepsilon \)-approximate segments computed by the greedy algorithm at the core of the FITing-tree. Of course, \( m_{\text{opt}} \leq m_{\text{greedy}} \). The learned index RMI is not included in the table because it lacks of guaranteed bounds.

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Lemma 3. Let $m$ be the number of $\epsilon$-approximate segments of a PGM-index indexing $n$ keys drawn from a universe $U$. There exists a lossless compressor for the segments which computes the minimum number of distinct slopes $t \leq m$ while preserving the $\epsilon$-guarantee. The algorithm takes $O(m \log m)$ time and compresses the slopes into $64t + m \lceil \log t \rceil$ bits of space.

Proof. The compressed space occupancy of the $t$ distinct slopes in $T$ is, assuming double-precision floats, 64t bits. The new slopes $\rho_i$ are still $m$ in their overall number, but each of them can be encoded as the position $0, \ldots, t - 1$ into $T$ of its corresponding double-precision float. □

An interesting future work is to experiment how much the use of universal coders for reals [19], as an alternative to floating-point numbers, can further reduce the additive term $64t$.

4. THE DISTRIBUTION-AWARE PGM-index

The PGM-index of Theorem 1 implicitly assumes that the queries are uniformly distributed, but this seldom happens in practice. For example, queries in search engines are very well known to follow skewed distributions such as the Zipf’s law [33]. In such cases, it is desirable to have an index that answers the most frequent queries faster than the rare ones, so to achieve a higher query throughput. Previous work exploited query distribution in the design of binary trees [3], Trenoops [22], and skip lists [1], to mention a few.

In this section, we introduce an orthogonal approach that builds upon the PGM-index by proposing a variant that adapts itself not only to the distribution of the input keys but also to the distribution of the queries. This turns out to be the first distribution-aware learned index to date, with the additional positive feature of being very succinct in space.

Formally speaking, given a sequence $S = \{(k_i, p_i)\}_{i=1, \ldots, n}$, where $p_i$ is the probability to query the key $k_i$ that is assumed to be known, we want to solve the distribution-aware dictionary problem, which asks for a data structure that searches for a key $k_i$ in time $O(\log(1/p_i))$ so that the average query time coincides with the entropy of the query distribution $H = \sum_{i=1, \ldots, n} p_i \log(1/p_i)$.

We note that the algorithm of Lemma 1 can be modified so that, given a $y$-range for each one of $n$ points in the plane, finds also the set of all (segment) directions that intersect those ranges in $O(n)$ time (see [24]). This corresponds to find the optimal PLA-model whose individual segments guarantee an approximation which is within the $y$-range given for each of those points. Therefore, our key idea is to define, for every key $k_i$, a $y$-range of size $y_i = \min \{1/p_i, \epsilon\}$, and then apply the algorithm of Lemma 1 on that set of keys and $y$-races. Clearly, for the keys whose $y$-range is $\epsilon$ we can use Theorem 1 and derive the same space bound of $O(m)$; whereas for the keys whose $y$-range is $1/p_i < \epsilon$ we observe that these keys are no more than $\epsilon$ (in fact, the $p_i$s sum up to 1), but they are possible spread among all position in $A$ and thus they induce in the worst case $2\epsilon$ extra segments. Therefore, the total space occupancy of the bottom level of the index is $\Theta(m + \epsilon)$, where $m$ is the one defined in Theorem 1.

Now, let us assume that the search for a key $k_i$ arrived at the last level of this Distribution-Aware PGM-index, and thus we know in which segment to search for $k_i$: the final binary search step within the $\epsilon$-approximate range returned by that segment takes $O(\log \min \{1/p_i, \epsilon\}) = O(\log(1/p_i))$ as we aimed for.

We are left with showing how to find that segment in a distribution-aware manner. We proceed similarly to the Recursive PGM-index but with a careful design of the recursive step because of the probabilities (and thus the variable $y$-ranges) assigned to the recursively defined set of keys.

Let us consider the segment covering the sequence $S_{[a, b]} = \{(k_a, p_a), \ldots, (k_b, p_b)\}$, denote by $q_{a, b}$ the maximum probability of a key in $S_{[a, b]}$, and by $P_{a, b} = \sum_{i=a}^{b} p_i$ the cumulative probability of all keys in $S_{[a, b]}$ (which is indeed the probability to end up in that segment when searching for one of its keys). We create the new set $S' = \{(k_a, q_{a, b}/P_{a, b}), \ldots\}$ formed by the first key $k_a$ covered by each segment (as in the recursive PGM-index) and setting its associated probability to $q_{a, b}/P_{a, b}$. Then, we construct the next upper level of the Distribution-Aware PGM-index by applying the algorithm of Lemma 1 on $S'$. If we iterate the above analysis for this new level of weighted segments, we conclude that: if we know from the search executed on the levels above that $k_i \in S_{[a, b]}$, the time cost to search for $k_i$ in this level is $O(\log \min \{P_{a, b}/q_{a, b}, 1\}) = O(\log(1/p_i))$.

Let us repeat this argument for another upper level in order to understand the influence on the search time complexity. We denote the segment that covers the range of keys which include $k_i$, with $S_{[a', b']} \supset S_{[a, b]}$, the cumulative probability with $P_{a', b'}$, and thus assign to its first key $a'$ the probability $r/P_{a', b'}$, where $r$ is the maximum probability of the form $P_{a, b}$ of the ranges included in $[a', b']$. In other words, if $[a', b']$ is partitioned into $\{z_1, \ldots, z_e\}$, then $r = \max_{x \in [1, e]} P_{z_{i, i+1}}$. Reasoning as done previously, if we know from the search executed on the levels above that $k_i \in S_{[a', b']}$, the time cost to search for $k_i$ in this level is $O(\log(\min \{P_{a', b'}/r, \epsilon\}) = O(\log(P_{a', b'}/P_{a, b}))$ because $[a, b]$ is, by definition, one of these ranges in which $[a', b']$ is partitioned.

Repeating this design until one single segment is obtained (whose cumulative probability is one), we get a total time cost for the search in all levels of the PGM-index equal to a sum of logarithms whose arguments “cancel out” and get $O(\log(1/p_i))$.

As far as the space bound is concerned, we recall that the number of levels in the PGM-index is $L = O(\log m)$ with $\epsilon \geq 2\epsilon$ and that we have to account for the $\epsilon$ extra segments per level returned by the algorithm of Lemma 1. Consequently, this distribution-aware variant of the PGM-index takes $O(m + L\epsilon)$ space. This space bound is indeed $O(m)$ because $\epsilon$ is a constant parameter (see Section 6).

Theorem 2. Let $A$ be an ordered array of $n$ keys $k_i$ drawn from a universe $U$, which are queried with (known) probability $p_i$, and let $\epsilon \geq 1$ be a fixed integer parameter. The Distribution-Aware Recursive PGM-index with parameter $\epsilon$ indexes the array $A$ in $O(m)$ space and answers queries in $O(H)$ average time, where $H$ is the entropy of the query distribution, and $m$ is the number of segments of the optimal PLA-model for the keys in $A$ with error $\epsilon$. 


5. THE MULTICRITERIA PGM-index

Tuning a data structure to match the application’s needs is often a difficult and error-prone task for a software engineer, not to mention that these needs may change over time due to mutations in data distribution, devices, resource requirements, and so on. The typical approach is a grid search on the various instances of the data structure to be tuned until the one that matches the application’s needs is found. However, the data structure may be not flexible enough to adapt to those changes, or the search space can be so huge that the reorganisation of the data structure takes too much time.

In the rest of this section, we exploit the space-time flexibility of the PGM-index by showing that this tuning process can be efficiently automated over this data structure via an optimisation strategy that: (i) given a space constraint outputs the PGM-index that minimises its query time; or symmetrically, (ii) given a maximum query time outputs the PGM-index that minimises its space footprint.

The time-minimisation problem. According to Theorem 1, the query time of a Recursive PGM-index can be described as \( t(\varepsilon) = c \log_2(m) \log(2\varepsilon/B) \), where \( B \) is the page size of the PGM-model, \( m \) is the number of segments in the last level, and \( c \) depends on the access latency of the memory. For the space, we introduce \( s_L(\varepsilon) \), which denotes the number of segments needed to have precision \( \varepsilon \) over keys available at level \( L \) of the Recursive PGM-index, and compute the overall number of segments as \( s(\varepsilon) = \sum_{l=0}^{L} s_l(\varepsilon) \). By Lemma 2, we know that \( s_L(\varepsilon) = m \leq n/(2\varepsilon) \) for any \( \varepsilon \geq 1 \) and that \( s_{L-1}(\varepsilon) \leq s_L(\varepsilon)/(2\varepsilon) \). So that \( s(\varepsilon) \leq \sum_{l=0}^{L} m/(2\varepsilon)^l = (2m - 1)/(2\varepsilon - 1) \).

Given a space bound \( s_{\max} \), the “time-minimisation problem” consists of minimising \( t(\varepsilon) \) subject to \( s(\varepsilon) \leq s_{\max} \). The greatest challenge here is that we do not have a closed formula for \( s(\varepsilon) \), but only an upper bound which does not depend on the underlying dataset as \( s(\varepsilon) \) does. Section 6 will show that in practice we can model \( m = s_L(\varepsilon) \) with a simple power-law having the form \( a\varepsilon^{-b} \), whose parameters \( a \) and \( b \) will be properly estimated on the dataset at hand. The power-law covers both the pessimistic case of Lemma 2 and the best case in which the dataset is strictly linear.

Clearly, the space occupancy decreases with increasing \( \varepsilon \), whereas the query time \( t(\varepsilon) \) increases with \( \varepsilon \), since the number of keys on which it is executed a binary search at each level equals \( 2\varepsilon \). Therefore, the time-minimisation problem reduces to the problem of finding the value of \( \varepsilon \) for which \( s(\varepsilon) = s_{\max} \) because it is the lowest \( \varepsilon \) that we can afford. Such value of \( \varepsilon \) could be found by a binary search in the bounded interval \( \mathcal{E} = [B/2, n/2] \), which is derived by requiring that each model error at least a page size (i.e. \( 2\varepsilon \geq B \)), since lower \( \varepsilon \) values do not save I/Os, and by observing that one model is the minimum possible space (i.e. \( 2\varepsilon \leq n \), by Lemma 2). However, provided that our power-law approximation holds, we can speed up the search of that “optimal” \( \varepsilon \) by guessing the next value of \( \varepsilon \) rather than taking the midpoint of the current search interval. In fact, we can find the root of \( s(\varepsilon) - s_{\max} \), i.e. the value \( \varepsilon_0 \) for which \( s(\varepsilon_0) = s_{\max} \). We emphasise that such \( \varepsilon_0 \) may not be the solution of our problem, as it may be the case that the approximation or the fitting of \( s(\varepsilon) \) by means of a power-law is not precise. Thus, more iterations of the search may be needed to find the optimum \( \varepsilon \) value; anyway, we guarantee to be always faster than a binary search by gradually switching to it. Precisely, we bias the guess \( \varepsilon_0 \) towards the midpoint \( \varepsilon_m \) of the current search range via a simple convex combination of the two [14].

The space-minimisation problem. Given a time bound \( t_{\max} \), the space-minimisation problem consists of minimising \( s(\varepsilon) \) subject to \( t(\varepsilon) \leq t_{\max} \). As for the problem above, we could perform a binary search inside the interval \( \mathcal{E} = [B/2, n/2] \) and look for the maximum \( \varepsilon \) which satisfies that time constraint. Instead, we speed up this process by guessing the next iterate for \( \varepsilon \) via the equation \( t(\varepsilon) = t_{\max} \), that is solving \( c \log_2(s_L(\varepsilon)) \log(2\varepsilon/B) = t_{\max} \), in which \( s_L(\varepsilon) \) is replaced by the power-law approximation \( a\varepsilon^{-b} \), for proper \( a \) and \( b \), and \( c \) is replaced by the measured memory latency of the given machine.

Although effective, this approach raises a subtle issue, namely, the time model could not be a correct estimate of the actual query time because of hardware-dependent factors such as the presence of several caches and the CPU pre-fetching. To further complicate this issue, we note that both \( s(\varepsilon) \) and \( t(\varepsilon) \) depend on the power-law approximation \( a\varepsilon^{-b} \).

For these reasons, instead of using the time model \( t(\varepsilon) \) to steer the search, we measure and use the actual average query time \( t(\varepsilon) \) of the PGM-index over a fixed batch of random queries. Moreover, instead of performing a binary search inside the whole \( \mathcal{E} \), we run an exponential search starting from the solution of the dominating term \( c \log(2\varepsilon/B) \) i.e. the cost of searching the data. Eventually, we stop the search of the best \( \varepsilon \) as soon as the searched range is smaller than a given threshold because \( t(\varepsilon) \) is subject to measurement errors (e.g. due to an unpredictable CPU scheduler).

6. EXPERIMENTS

We experimented with an implementation in C++ of the PGM-index on a machine with a 2.3 GHz Intel Xeon Gold and 192 GiB memory. We used the following three standard datasets, each having different data distributions, regularities and patterns:

- Web logs [13] [14] contains timestamps of about 715M requests to a web server;
- Longitude [26] contains longitudes of about 166M points-of-interest from OpenStreetMap;
- IoT [13] [14] contains timestamps of about 26M events recorded by IoT sensors installed throughout an academic building.

We also generated some synthetic datasets according to the uniform distribution in the interval \([0, u]\), to the Zipf distribution with exponent \( s \), and to the lognormal distribution with standard deviation \( \sigma \), they will allow to test thoroughly the various indexes.

The implementation will be released on GitHub with the acceptance of this paper.
6.1 Space occupancy of the PGM-index

In this set of experiments, we estimated the size of the optimal PLA-model (see Section 2.1) returned by our implementation of [31], which provides the segments stored in the bottom level of the PGM-index, and compared it against the non-optimal PLA-model computed with the greedy shrinking cone algorithm [13, 33] used in the FITing-tree [13]. This comparison is important because the size of a PLA-model is the main factor impacting the space footprint of a learned index based on linear models.

6.2 Query performance of the PGM-index

We evaluated the query performance of the PGM-index and other indexing data structures on Web logs dataset, the biggest and most complex dataset available to us. We have dropped the comparison against the FITing-tree, because of the evident structural superiority of the Recursive PGM-index and its indexing of the optimal (minimum) number of segments in the bottom level (see Figure 4). Nonetheless, we will investigate the performance of some variants of the PGM-index that will provide a clear picture of the improvements determined by its recursive indexing, compared to the classic approaches based on multiway search trees (à la FITing-tree), CSS-tree [31] or B*-tree.

In this experiment, the dataset was loaded in memory as a contiguous array of integers represented with 8 bytes and with 128 bytes payload. Slopes and intercepts were stored as double-precision floats. Each index was presented with 10M queries randomly generated on the fly. The next three paragraphs present, respectively, the query performance of the three indexing strategies for the PGM-index, a comparison between the PGM-index and traditional indexes, and a comparison between the PGM-index and the RAM [13] under this experimental scenario.

**PGM-index variants.** The three indexing strategies experimented for the PGM-index are binary search, multiway search trees (à la FITing-tree), CSS-tree [31] and our novel recursive construction (see Section 2.2). We refer to

| Dataset                        | $\varepsilon$ |
|--------------------------------|---------------|
|                                | 8             | 32            | 128           | 512           | 2048          |
| Uniform $u = 2^{32}$           | 33.8          | 59.4          | 66.5          | 68.6          | 68.8          |
| Uniform $u = 2^{32}$           | 65.8          | 68.3          | 68.9          | 69.4          | 68.7          |
| Zipf $s = 1$                   | 47.9          | 59.0          | 62.8          | 44.7          | 29.0          |
| Zipf $s = 2$                   | 45.3          | 40.2          | 24.2          | 20.8          | 21.6          |
| Lognormal $\sigma = 0.5$      | 66.1          | 68.5          | 68.8          | 62.1          | 35.6          |
| Lognormal $\sigma = 1.0$      | 66.1          | 68.4          | 69.0          | 61.9          | 34.5          |

Table 2: Space savings of PGM-index with respect to a FITing-tree for a varying $\varepsilon$ on six synthetic datasets of 1G elements generated according to the specified distributions.

**Figure 4:** A log-log plot with the ratio between the number of segments $m$, stored in the last level of a PGM-index, and the size $n$ of the real-world datasets as a function of $\varepsilon$. For comparison, the plot shows with a dashed line the function $1/(2\varepsilon)$ which is the fraction of the number of keys stored in the level above the input data of $B^*$-tree with $B = 2^\varepsilon$ (see text). Note that $m$ is 2–5 orders of magnitude less than $n$. This reduction gets impressively evident for larger values of $\varepsilon$, reaching five orders of magnitude.

**Figure 3:** The PGM-index saved from 37.7% to 63.3% space with respect to a FITing-tree over the three real-world datasets. The construction time complexity of the two approaches is the same in theory (i.e. linear in the number of processed keys) and in practice (a couple of seconds, up to hundreds of millions of keys).
them with PGM BIN, PGM CSS and PGM REC, respectively. We set $\varepsilon = 4$ for all but the last level of PGM REC, that is the one that includes the segments built over the input dataset. Likewise, the node size of the CSS-tree was set to $B = \varepsilon \cdot L$ for a fair comparison with PGM REC. Figure 5 shows that PGM REC dominates PGM CSS for $\varepsilon \leq 256$, and has better query performance than PGM BIN. The advantage of PGM REC over PGM CSS is also evident in terms of index height since the former has five levels whereas the latter has seven levels, thus PGM REC experiences a shorter traversal time which is induced by a higher branching factor (as conjectured in Section 2.2). For $\varepsilon > 256$ all the three strategies behaved similarly because the index was so small to fit into the L2 cache.

**PGM-index vs traditional indexes.** We compared the PGM-index against the cache-efficient CSS-tree and the classic $B^+$-tree. For the former, we used our implementation. For the latter, we chose a well-known library [4, 13, 18].

The PGM-index dominated these traditional indexes, as shown in Figure 6 for page sizes of 4–16 KiB. Performances for smaller page sizes were too far (i.e. worse) from the main plot range, and thus are not shown. For example, the fastest CSS-tree in our machine had page size of 128 bytes, occupied 341 MiB and was matched in query performance by a PGM REC with $\varepsilon = 128$ which occupied only 4 MiB (82.7× less space). As another example, the fastest $B^+$-tree occupied 874 MiB and was matched in performance by a PGM-index which occupied only 87 KiB (four orders of magnitude less space).

![Figure 5](image5.png)

**Figure 5:** The query performance of several configurations of the PGM-index. The Recursive PGM-index, depicted as a pentagon, had better space and time performance than all the other configurations.

![Figure 6](image6.png)

**Figure 6:** The Recursive PGM-index improved uniformly with different second-stage sizes and traditional indexes with different page sizes over all possible space-time trade-offs. Results of traditional indexes for smaller page sizes are not shown because too far from the plot range. For example, the fastest CSS-tree occupied 341 MiB and was matched in performance by a PGM-index of only 4 MiB (82.7× less space); the fastest $B^+$-tree occupied 874 MiB and was matched in performance by a PGM-index which occupied only 87 KiB (four orders of magnitude less space).

Discussion. Overall, the experiments have shown that the PGM-index is fast in construction (about 3 seconds to index a real-world table of 91 GiB with 715M key-value pairs) and has space footprint that is up to 63.3% lower than what was achieved by a state-of-the-art FITing-tree. Moreover, the PGM-index dominates in space and time both the traditional and other learned index structures (e.g. the RMI).
particular, it improved the space footprint of the CSS-tree by a factor $82.7 \times$ and the one of the B-tree by more than four orders of magnitude, while achieving the same or even better query efficiency.

### 6.3 The Compressed PGM-index

We investigated the effectiveness of the compression techniques proposed in Section 3. Figure 7 shows that the slope compression algorithm reduced the number of distinct slopes significantly, up to 99.94%, still preserving the same optimal number of segments. As far as the space occupancy is concerned, and considering just the last level of a PGM-index which is the largest one, the reduction induced by the compression algorithm was up to 81.2%, as shown in Figure 8. Note that in the Longitude datasets for $\varepsilon \geq 2^6$ the slope compression is not effective enough. As a result, the mapping from segments to the slopes table causes an overhead that exceeds the original space occupancy of the segments. Clearly, a real-world application would turn off slope compression in such situations.

In information theory, the compressibility of data is measured with its entropy (as defined by Shannon). We conjecture that a similar measure, characterising “difficult” datasets like Longitude, also exists in our “geometric setting” and is worth studying. Another interesting future work is to explore the relation between the algorithm of Lemma 1 and the slope compression algorithm. To explain, recall that during the construction of the optimal PGM-index the range of slopes reduces each time a new point is added to the current convex hull (segment). Therefore “shortening” a segment, on the one hand, improves the performance of the slope compression algorithm (because it enlarges the sizes of the possible slope intervals), but on the other hand, it increases the overall number of segments. Given that Lemma 3 offers a compressed space bound which depends on $m$ (the overall number of segments) and $t$ (the number of distinct segments), balancing the above two effects to achieve better compression is an intriguing extension of this paper.

Afterwards, we measured the query performance of the Compressed PGM-index in which compression was activated over the intercepts and the slopes of the segments of all the levels. Table 3 shows that, with respect to the corresponding Recursive PGM-index, the space footprint is reduced by up to 52.2% at the cost of moderately slower queries (no more than 24.5%).

### 6.4 The Multicriteria PGM-index

Our implementation of the Multicriteria PGM-index operates in two modes: the time-minimisation mode (shortly, min-time) and the space-minimisation mode (shortly, min-space), which implement the algorithms described in Section 5. In min-time mode, inputs to the program are $s_{\text{max}}$ and a tolerance $\text{tol}$ on the space occupancy of the solution, and the output is the value of the error $\varepsilon$ which guarantees a space bound $s_{\text{max}} \pm \text{tol}$. In min-space mode, inputs to the program are $t_{\text{max}}$ and a tolerance $\text{tol}$ on the query time of the solution, and the output is the value of the error $\varepsilon$ which guarantees a time bound $t_{\text{max}} \pm \text{tol}$ in the query operations. We note that the introduction of a tolerance parameter allows us to stop the search earlier as soon as any further step would not appreciably improve the solution (i.e., we seek only improvements of several bytes or nanoseconds). So $\varepsilon$ is not a parameter that has to be tuned but rather a stopping criterion like the ones used in iterative methods.

To model the space occupancy of a PGM-index, we studied empirically the behaviour of the number of segments $n_{\text{opt}} = s_L(\varepsilon)$ forming the optimal PGM-index by varying $\varepsilon$ and by fitting ninety different functions over about two hundred points $(\varepsilon, s_L(\varepsilon))$ generated beforehand by a long-running grid search over our real-world datasets. Looking at the fittings, we chose to model $s_L(\varepsilon)$ with a power-law having the form $a \varepsilon^{-b}$. As further design choices we point out that: (i) the fitting of the power-law was performed with the Levenberg-Marquardt algorithm, while root-finding was performed with Newton’s method; (ii) the search space for $\varepsilon$ was set to $\varepsilon = [8, n/2]$ (since a cache line holds eight 64

| $\varepsilon$  | 64   | 128  | 256  | 512  | 1024 | 2048 |
|---------------|------|------|------|------|------|------|
| Space saving (%) | 52.2 | 50.8 | 48.5 | 46.0 | 41.5 | 35.5 |
| Time loss (%)  | 13.7 | 22.6 | 24.5 | 15.1 | 11.7 | 9.9  |

Table 3: Query performance of the Compressed PGM-index with respect to the corresponding Recursive PGM-index.
bits integers); and finally (iii) the number of guesses was set to \(2\log\log\varepsilon\).

The following experiments were executed by addressing some use cases in order to show the efficacy and efficiency of the multicriteria PGM-index.

**Experiments with the min-time mode.** Suppose that a database administrator wants the most efficient PGM-index for the *Web logs* dataset that fits into an L2 cache of 1 MiB. Our solver derived an optimal PGM-index matching that space bound by setting \(\varepsilon = 393\) and taking 10 iterations and a total of 19 seconds. This result was obtained by approximating \(s_1(\varepsilon)\) with the power-law \(46032135 \cdot \varepsilon^{-1.16}\) which guaranteed a mean squared error of no more than 4.8% over the range \(\varepsilon \in [8, 1024]\).

As another example, suppose that a database administrator wants the most efficient PGM-index for the *Longitude* dataset that fits into an L1 cache of 32 KiB. Our solver derived an optimal PGM-index matching that space bound by setting \(\varepsilon = 1050\) and taking 14 iterations and a total of 9 seconds.

**Experiments with the min-space mode.** Suppose that a database administrator wants the PGM-index for the IoT dataset with the lowest space footprint that answers any query in less than 500 ns. Our solver derived an optimal PGM-index matching that time bound by setting \(\varepsilon = 432\), occupying 74.55 KiB of space, and taking 9 iterations and a total of 6 seconds.

As another example, suppose that a database administrator wants the most compressed PGM-index for the *Web logs* dataset that answers any query in less than 800 ns. Our solver derived an optimal PGM-index matching that time bound by setting \(\varepsilon = 1217\), occupying 280.05 KiB of space, and taking 8 iterations and a total of 17 seconds.

**Discussion.** In contrast to the FITing-tree and the RAM, the Multicriteria PGM-index can trade efficiently query time with space occupancy, making it a promising approach for applications with rapidly-changing data distributions and space/time constraints. Overall, in both modes our approach ran in less than 20 seconds.

7. CONCLUSIONS AND FUTURE WORK

We have introduced the PGM-index, a learned data structure for the fully indexable dictionary problem which improves the query performance and the space occupancy of both traditional and modern learned indexes up to several orders of magnitude. We have also designed three variants of the PGM-index: one that improves its already succinct space footprint using ad-hoc compression techniques, one that adapts itself to the query distribution, and one that provides estimations of the number of occurrences of individual or range of keys. Finally, we have demonstrated that the PGM-index is a multicriteria data structure as it can be fast optimised within some user-given constraint on the space occupancy or the query time.

A possible research direction is to experiment with the performance of insertion and deletion of keys in a PGM-index. To this end, we mention classic techniques such as the split-merge strategy in B-tree nodes [2, 44], and the use of buffers that once full are merged into the index (cf. [13]). The possibility of orchestrating segmentations, nonlinear models and rank/select indexing techniques from the compression domain [23, 36] is another intriguing research direction, especially within our Multicriteria framework.

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