The historical significance of the problem of relativistic rigid rotation is reviewed in light of recently published correspondence between Einstein and the mathematician Vladimir Varićak from the years 1909 to 1913.

1 Introduction

The rigidly rotating disk has long been recognized as a crucial ‘missing link’ in our historical reconstruction of Einstein’s recognition of the non-Euclidean nature of spacetime in his path toward general relativity. Relativistic rigid rotation combines several different but related problems: the issue of a Lorentz-covariant definition of rigid motion, the number of degrees of freedom of a rigid body, the reality of length contraction, as well as Ehrenfest’s paradox and the introduction of non-Euclidean geometric concepts into the theory of relativity.

2 Relativistic rigid motion

A relativistic definition of rigid motion was first given by Max Born. The definition was given in the context of a theory of the dynamics of a model of an extended, rigid electron, and defined a rigid body as one whose infinitesimal volume elements appear undeformed for any observer that is comoving instantaneously with the (center of the) respective volume element. The definition and its implications were discussed at the 81st meeting of the Gesellschaft Deutscher Naturforscher und Ärzte in Salzburg in late September 1909.

Gustav Herglotz and Fritz Noether, in papers received by the Annalen der Physik on 7 and 27 December, respectively, further elaborated on the mathematical consequences of Born’s definition. Herglotz, in particular, reformulated...
the definition in more geometric terms: A continuum performs rigid motion if the world lines of all its points are equidistant curves. The analysis showed that Born’s infinitesimal condition of rigidity can only be extended to the motion of a finite continuum in special cases. It implied that a rigid body has only three degrees of freedom. The motion of one of its points fully determines its motion. Translation and uniform rotation are special cases. In particular, the definition does not allow for \textit{acceleration} of a rigid disk from rest to a state of uniform rotation with finite angular velocity.

In view of these consequences, various other definitions of a rigid body were suggested, e.g. by Born and Noether,\cite{Born1, Born2} until it became clear that special relativity does not allow for the usual concept of a rigid body. In other words, a relativistic rigid body necessarily has an infinite number of degrees of freedom.\cite{RelativisticRigidBody}

On 22 November 1909, a short note appeared by Paul Ehrenfest pointing to a paradox that follows from Born’s relativistic definition of rigid motion of a continuum.\cite{Ehrenfest} He considered a rigid cylinder rotating around its axis and contended that its radius would have to meet two contradictory requirements. The periphery must be Lorentz-contracted, while its diameter would show no Lorentz contraction. The difficulty became known as the “Ehrenfest paradox.” In a polemic exchange with von Ignatowsky,\cite{EhrenfestI} Ehrenfest devised the following thought experiment to illustrate the difficulty. He imagined the rotating disk to be equipped with markers along the diameter and the periphery. If their positions were marked onto tracing paper in the rest frame at a fixed instant, with the disk both at rest and in uniform rotation, the two images should show the same radius but different circumferences.

\section{The Einstein-Varičak correspondence}

Immediately after the 1909 Salzburg meeting, Einstein wrote to Arnold Sommerfeld that “the treatment of the uniformly rotating rigid body seems to me of great importance because of an extension of the relativity principle to uniformly rotating systems.”\cite{Einstein-Sommerfeld} This was a necessary step for Einstein following the heuristics of his equivalence hypothesis, but only in spring 1912, a few weeks before he made the crucial transition from a scalar to a tensorial theory of gravitation based on a general spacetime metric,\cite{Einstein-Gravitation} do we find another hint at the problem in his writings.\cite{Einstein-Schwarzschild, Einstein-Relativitat}

The \textit{Collected Papers of Albert Einstein} recently published\cite{CollectedPapers} nine letters by Einstein to Vladimir Varičak (1865–1942), professor of mathematics at Agram (now Zagreb, Croatia). Varičak had published on non-Euclidean geometry\cite{Varičak} and is known for representing special relativistic relations in terms of real hyperbolic geometry.\cite{VaričakHyperbolic, VaričakGeometric} The correspondence seems to have been initiated by Varičak asking for offprints of Einstein’s papers. In his response, Einstein added a personal tone to it with his wife Mileva Marić, a native Hungarian Serb, writing the address in Cyrillic script in order to raise Varičak’s curiosity. After exchanging publications, Varičak soon commented on Einstein’s (now) famous 1905 special relativity paper, pointing to misprints but also raising doubts about his treat-
ment of reflection of light rays off moving mirrors. These were rebutted by
Einstein in a response of 28 February 1910 in which he also, with reference to
Ehrenfest’s paradox, referred to the rigidly rotating disk as the “most interesting
problem” that the theory of relativity would presently have to offer. In his next
two letters, dated 5 and 11 April 1910 respectively, Einstein argued against the
existence of rigid bodies invoking the impossibility of superluminal signalling,
and also discussed the rigidly rotating disk. A resolution of Ehrenfest’s paradox,
suggested by Varticak, in terms of a distortion of the radial lines so as to preserve
the ratio of $\pi$ with the Lorentz contracted circumference, was called interesting
but not viable. The radial and tangential lines would not be orthogonal in spite
of the fact that an inertial observer comoving with a circumferential point would
only see a pure rotation of the disk’s neighborhood.

About a year later, Einstein and Varticak corresponded once more. Varticak
had contributed to the polemic between Ehrenfest and von Ignatowsky by sug-
gestng a distinction between ‘real’ and ‘apparent’ length contraction. The real-
ity of relativistic length contraction was discussed in terms of Ehrenfest’s tracing
paper experiment, but for linear relative motion. According to Varticak, the ex-
periment would show that the contraction is only a psychological effect whereas
Einstein argued that the effect will be observable in the distance of the recorded
marker positions. When Varticak published his note, Einstein responded with a
brief rebuttal.\textsuperscript{17}

Despite their differences in opinion, the relationship remained friendly. In
1913, Einstein and his wife thanked Varticak for sending them a gift, commented
favorably on his son who stayed in Zurich at the time, and Einstein announced
sending a copy of his recent work on a relativistic theory of gravitation. The
Einstein-Varticak correspondence thus gives us additional insights into a signifi-
cant debate. It shows Einstein’s awareness of the intricacies of relativistic rigid
rotation and bears testimony to the broader context of the conceptual clarifica-
tions in the establishment of the special and the genesis of the general theory
of relativity.

References

[1] J. Stachel, Einstein and the Rigidly Rotating Disk, in General Relativity
and Gravitation: One Hundred Years after the Birth of Albert Einstein.
Vol. 1, ed. A. Held (Plenum, 1980), 1–15; see also “The First Two Acts,”
in J. Stachel. Einstein from ‘B’ to ‘Z’ (Birkhauser, 2002), 261–292.

[2] G. Maltese and L. Orlando. Stud. Hist. Phil. Mod. Phys. 26, 263 (1995).

[3] M. Klein et al. (ed.) The Collected Papers of Albert Einstein. Vol. 3. The
Swiss Years: Writings, 1909–1911. (Princeton University Press, 1993),
478–480.

[4] M. Klein. Paul Ehrenfest: The Making of a Theoretical Physicist. (North-
Holland, 1970), 152–154.
[5] M. Janssen, J. Norton, J. Renn, T. Sauer, J. Stachel. The Genesis of General Relativity: Einstein’s Zürich Notebook. Vol. 1. Introduction and Source. Vol. 2. Commentary and Essays. (Springer, 2007).

[6] M. Born. Ann. Phys. 30, 1 (1909); Phys. Zs. 10, 814 (1909).

[7] G. Herglotz, Ann. Phys. 31, 393 (1910); F. Noether, Ann Phys. 31, 919 (1910).

[8] M. Born, Nachr. Königl. Ges. d. Wiss. (Göttingen) 161 (1910).

[9] A. Einstein, Jahrb. Radioaktive. Elektr. 4, 411 (1907); M. Laue, Phys. Zs. 12, 85 (1911).

[10] P. Ehrenfest, Phys. Zs. 10, 918 (1909).

[11] P. Ehrenfest, Phys. Zs. 11, 1127 (1910); 12, 412 (1911); V. Ignatowsky, Ann. Phys. 33, 607 (1910); Phys. Zs. 12, 164, 606 (1911).

[12] M. Klein et al. (ed.) The Collected Papers of Albert Einstein. Vol. 5. The Swiss Years: Correspondence, 1902–1914. (Princeton University Press, 1993).

[13] D. Buchwald et al. (ed.) The Collected Papers of Albert Einstein. Vol. 10. The Berlin Years: Correspondence, May–December 1920 and Supplementary Correspondence, 1909–1920. (Princeton University Press, 2006).

[14] V. Varičak. Jahresber. dt. Math. Ver. 17, 70 (1908); Atti del Cong. internat. del Mat. 2, 213 (1909).

[15] V. Varičak. Phys. Zs. 11, 93, 287, 586 (1910); Jahresber. dt. Math. Ver. 21, 103 (1912).

[16] S. Walter. The Non-Euclidean Style of Minkowskian Relativity, in The Symbolic Universe. ed. J. Gray (Oxford University Press, 1999), 91–127.

[17] V. Varičak, Phys. Zs. 12, 169 (1911); A. Einstein. Phys. Zs. 12, 509 (1911).