Relativistic cosmological perturbation scheme on a general background: scalar perturbations for irrotational dust

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Abstract
In standard perturbation approaches and N-body simulations, inhomogeneities are described to evolve on a predefined background cosmology, commonly taken as the homogeneous–isotropic solutions of Einstein’s field equations (Friedmann–Lemaître–Robertson–Walker (FLRW) cosmologies). In order to make physical sense, this background cosmology must provide a reasonable description of the effective, i.e. spatially averaged, evolution of structure inhomogeneities also in the nonlinear regime. Guided by the insights that (i) the average over an inhomogeneous distribution of matter and geometry is in general not given by a homogeneous solution of general relativity, and that (ii) the class of FLRW cosmologies is not only locally but also globally gravitationally unstable in relevant cases, we here develop a perturbation approach that describes the evolution of inhomogeneities on a general background being defined by the spatially averaged evolution equations. This physical background interacts with the formation of structures. We derive and discuss the resulting perturbation scheme for the matter model ‘irrotational dust’ in the Lagrangian picture, restricting our attention to scalar perturbations.

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1. A new approach to perturbation theory
Perturbation theory is a key tool in cosmology to describe the formation of structures in the weakly nonlinear regime and to initialize the N-body simulations of cosmic structures.

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The standard motivation to describe perturbations on a homogeneous–isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) background comes from the known gravitational instability of the latter. However, this background is also supposed, implicitly, to describe the average evolution of an inhomogeneous universe model in the nonlinear regime, and this is implemented as a construction principle in most known relativistic, quasi-Newtonian and Newtonian perturbation schemes and Newtonian $N$-body simulations [8]. We remark that such a construction is correct in Newtonian cosmology: if structures evolve on a Euclidean geometry, and if they are subjected to periodic boundary conditions on some large scale, then the average of inhomogeneities is given by the assumed background (see [7] for the proofs).

Obviously, a Newtonian cosmology has to be considered as highly restrictive when one moves to the framework of general relativity, and the previous construction cannot be expected to work there because of: (i) the relevance of the spatial intrinsic curvature (the second derivatives of the metric may be significant even if the metric perturbations are small [17, 37–39, 48, 49]), together with (ii) the fact that inhomogeneities are coupled to the spatial curvature evolution [10], and finally (iii) the absence of a conservation law for the averaged intrinsic curvature [16]. Also, it has been recently shown [51], in the special class of scaling laws for the spatially averaged inhomogeneities (the so-called backreaction terms), that FLRW backgrounds are not only locally but also globally unstable as a result of structure formation and accelerated expansion, when subjected to perturbations whose global contribution does not vanish. As soon as a homogeneous and isotropic self-gravitating system is perturbed, the related inhomogeneities invoke a departure of its average from a FLRW background. This property is of great significance for a theory of perturbations: in the standard approach, the background evolution is known to impact on the evolution of perturbations, but the converse effect, namely that the structure inhomogeneities also affect the evolution of the background, has been neglected thus far by construction\(^4\). Note that such a property is expected from first principles: it expresses the fact that, in Einstein’s theory, the formation of structures and the evolution of geometry are mutually (and generically) coupled. An evolution of structures on a predefined background is, in light of these remarks, an \textit{a priori} restricted approach.

Looking at perturbations in the universe, we can only apprehend their strength and their evolution correctly if we know with respect to which background they have to be considered. Previous work that has addressed the issue in Newtonian cosmology [59, 60] faces the drawback that, on some large scale, all averages are strictly free of backreaction from inhomogeneities due to the restriction to a non-dynamical geometry and the necessity of a torus architecture, as explained above. The same drawback remains in quasi-Newtonian relativistic perturbation schemes for the evolution of gauge-invariant variables on a predefined background, even if they take into account backreaction effects [40–43, 52, 53, 67]. A related discussion by Räsänen can be found in [47]. For example, the assumption of periodic boundary conditions on initially flat-space sections or the restriction of the scalar curvature to a constant curvature suppresses any interaction effect between structures and the background. Beyond the usual FLRW perturbation scheme, Clarkson \textit{et al.} have furnished in [23] a complete system of master equations that represent the general linear perturbations to Lemaître–Tolman–Bondi (LTB) cosmologies. In [32, 44], a fully gauge-invariant relativistic perturbation theory has been given that holds for any background metric.

The framework presented in this paper provides the needed tools to implement a background as the average over fluctuating fields, which is not based on the introduction of a predefined background and deviations thereof; it furnishes the evolution of the scalar parts

\(^4\) Or it has been claimed to be a negligible effect. However, arguments for a small backreaction rely on a weakly perturbed FLRW cosmology and only address the issue within a limited framework (see, e.g., the review [25] for a clear presentation of this issue and references therein).
of the deviation fields off this general background in a non-perturbative way. We expect from this improvement that we shall be able to explicitly imprint structure inhomogeneities into the background that eventually describes the expansion history of the universe without the need to invoke, e.g., a dark energy fundamental component. A further aim is to understand initial conditions for relativistic numerical simulations of inhomogeneities that are not restricted to vanish on average on a FLRW background.

We proceed as follows. In section 2, we briefly recall the local and averaged evolution and constraint equations for the description of inhomogeneous dust cosmologies, and we provide a system of equations for the deviation fields off a general background. This is followed, in section 3, by a thorough discussion of the properties of this new deviation scheme. We conclude the paper in section 4 with some prospects.

2. Perturbation scheme on a general background

2.1. Local and averaged equations for inhomogeneous cosmologies

Let us consider a globally hyperbolic four-dimensional manifold, endowed with some metric tensor $g$. An irrotational fluid congruence, defined by a unit time-like vector field $u$, will be used to foliate the spacetime into a family of flow-orthogonal space-like hypersurfaces. We shall restrict ourselves, in what follows, to the case of a pressureless irrotational fluid (irrotational dust), $T_{\mu\nu} = \varrho u\mu u\nu$, with $\varrho$ being its energy density, as described in the Lagrangian picture. In the canonical bases $(\partial_t, \partial_i)$ and $(dt, dX^i)$, the 4-velocity of the fluid assumes the form $u^{\mu} = (1, 0, 0, 0), \quad u_{\mu} = (-1, 0, 0, 0)$, and the line element is written as $ds^2 = g_{\mu\nu} dX^\mu dX^\nu = -dt^2 + h_{ij} dX^i dX^j$, with $X^i$ being the Lagrangian spatial coordinates (coordinates comoving with the fluid), and $h$ the inhomogeneous 3-metric of the $t$-constant hypersurfaces. The foliation of Einstein’s equations with respect to $u$ implies the well-known Raychaudhuri equation and Hamilton constraint:

$$\dot{\Theta} + \frac{1}{3} \Theta^2 = -4\pi G \varrho - 2\sigma^2 + \Lambda,$$ (2a)

$$\frac{1}{3} \Theta^2 = 8\pi G \varrho - \frac{\mathcal{R}}{2} + \sigma^2 + \Lambda.$$ (2b)

Throughout our study, the overdot will stand for the covariant derivative (here identical to the partial time derivative $\partial_t$). $\Theta := h^{ij} \dot{h}_{ij}$ and $\sigma := (\frac{1}{2} \sigma^{ij} \sigma_{ij})^{1/2}$ are the local expansion and shear rates, respectively, while $\mathcal{R}$ is the local three-Ricci scalar curvature of the hypersurfaces, and $\Lambda$ is the cosmological constant that we carry along for the sake of generality. Those relations may be supplemented by the fluid continuity equation and a balance relation between $\mathcal{R}$ and $\sigma$:

$$\dot{\varrho} + \Theta \varrho = 0,$$ (2c)

$$\mathcal{R} + \frac{2}{3} \Theta \mathcal{R} = 2(\sigma^2) - 4\Theta \sigma^2,$$ (2d)

where the latter is obtained upon requiring equation (2b) to be an integral of (2a) (we shall simply call, hereafter, 'integrability condition' the result of this procedure). The local system

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5 Greek indices refer to spacetime components, they run in $\{0, 1, 2, 3\}$, and Latin indices denote space components, running in $\{1, 2, 3\}$. For a presentation of the (3+1)-splitting of Einstein’s equations, see e.g. [1, 33, 55].
will be used, later on, to derive the evolution of the deviation fields inside any spatial domain. For notational ease, we have omitted the time and space dependences, but the reader should bear in mind that all variables at stake are inhomogeneous.

Note that for a locally isotropic cosmology, \( \sigma = 0 \), the system (2) becomes homogeneous (Schur–Trümper’s theorem [56, 61]). We recover a FLRW cosmology. In particular, writing the expansion rate as \( \Theta_H := 3a_H/a_H \), with \( a_H \) being the scale factor, the integrability condition (2d) implies that the spatial Ricci scalar curvature follows the evolution law: \( \mathcal{R}_H = 6k/a_H^2 \), where \( k \) is a constant of integration.

Let us now consider a scalar field \( \psi \). Its spatial average performed on some compact domain \( D \), contained within the hypersurfaces and transported along the fluid flow lines (Lagrangian averaging), is defined as (see [9] for details)\(^6\)

\[
\langle \psi \rangle_D := \frac{1}{V_D} \int_D \psi \sqrt{\det h_{ij}} \, d^3X, \tag{3a}
\]

with \( V_D := \int_D \sqrt{\det h_{ij}} \, d^3X \) being the volume of the domain under consideration satisfying \( V_D/V_D = \langle \theta \rangle_D \). We shall also make frequent use of the commutation rule between the spatial averaging and differentiation with respect to time:

\[
\langle \psi \rangle_D - \langle \dot{\psi} \rangle_D = \langle \theta \psi \rangle_D - \langle \theta_D \psi \rangle_D, \tag{3b}
\]

where the right-hand side reduces to zero for a homogeneous domain. Equipped with these relations, we can provide the Lagrangian averaging on \( D \) of Raychaudhuri’s equation and Hamilton’s constraint\(^7\):

\[
\langle \theta \rangle_D + \frac{1}{3} \langle \theta \rangle_D^2 = -4\pi G \langle \varrho \rangle_D + \mathcal{Q}_D + \Lambda, \tag{4a}
\]

\[
\frac{1}{3} \langle \theta \rangle_D^2 = 8\pi G \langle \varrho \rangle_D - \frac{\langle \mathcal{R} \rangle_D}{2} - \frac{\dot{\mathcal{Q}}_D}{2} + \Lambda, \tag{4b}
\]

and we obtain the conservation law for the total rest mass within \( D \) and the integrability condition for the averaged variables as

\[
\langle \varrho \rangle_D + \langle \dot{\varrho} \rangle_D = 0; \tag{4c}
\]

\[
\langle \mathcal{R} \rangle_D + \frac{2}{3} \langle \theta \rangle_D \langle \mathcal{R} \rangle_D + \dot{\mathcal{Q}}_D + 2\langle \theta \rangle_D \mathcal{Q}_D = 0, \tag{4d}
\]

with \( \mathcal{Q}_D \) being the kinematical backreaction,

\[
\mathcal{Q}_D := \frac{2}{3} \langle (\theta - \langle \sigma \rangle_D) \mathcal{R} \rangle_D - 2\langle \sigma \rangle_D \mathcal{Q}_D. \tag{5}
\]

Equations (4), the averaged counterpart of (2), will also be used in the following to obtain the evolution of the deviation fields in the interior of \( D \). \( \mathcal{Q}_D \) determines how the fluid inhomogeneities inside the domain globally contribute to the evolution of its background (equations (4a) and (4b)), and this variable is dynamically coupled to the averaged scalar curvature (equation (4d)). Equation (4d) also shows that the averaged curvature does not individually obey a conservation law like the fluid density; rather a combined expression of intrinsic and extrinsic curvature invariants is conserved\(^8\). It is important to note that the

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6 Note that such a domain is ‘frozen’ into the metric. It is simply connected for regular solutions, but it changes its morphology due to the time dependence of the metric. Since the domain encloses during its evolution the same collection of fluid elements, there are no fluxes across its boundary, unlike in the corresponding Newtonian model where fluid elements move with respect to an external embedding space [7].

7 For comprehensive reviews on averaged inhomogeneous cosmologies in general relativity, we recommend the reading of, e.g., [15, 19, 20, 24, 25, 28, 46, 47, 63], and references therein.

8 \( \mathcal{Q}_D \) can be written in terms of the extrinsic curvature of the hypersurfaces, \( K_{ij} = -\theta_{ij} \), as \( \mathcal{Q}_D = \langle K^2 - K^2 \rangle_D + \frac{2}{3} \langle K \rangle_D^2 \).
background is scale dependent: for another domain we have in general a different background, due to the unconstrained distribution of inhomogeneities. There so exists a deep correlation between the background of any domain and the inhomogeneities inside. This feature is habitually absent in the usual cosmological perturbation schemes on a global scale where perturbations are assumed to average out on a predefined background.

Note that if we continuously shrink the compact domain to a point (null-homotopy), \((\Theta - \langle \Theta \rangle_\mathcal{D})^2\) \(\to 0\) and \(\mathcal{Q}_\mathcal{D} \to -2\sigma^2\). The system (4) then reduces to (2).

Finally, it is also convenient for later discussion to introduce two of the scalar invariants of the expansion tensor \(\Theta^{ij} := \sigma^i_j + \frac{1}{2}\Theta h^j_i\), its trace and the dispersion of its non-diagonal components:

\[
I := \text{tr} (\Theta^i_j) = \Theta, \quad II := \frac{1}{2}(\text{tr}^2 (\Theta^i_j) - \text{tr} (\Theta^i_j \Theta^j_i)) = \frac{1}{2}\Theta^2 - \sigma^2. \tag{6}
\]

The systems (2) and (4) assume the same form with these variables [9]:

\[
I + I^2 = 2\Pi - 4\pi G\rho + \Lambda, \quad II = 8\pi G\langle \rho \rangle_\mathcal{D} - \frac{\mathcal{R}}{2} + \Lambda, \tag{7a}
\]

\[
\mathcal{R} + \frac{2}{3}I\mathcal{R} + \left(2\Pi - \frac{2}{3}I^2\right) + 2I\left(2\Pi - \frac{2}{3}I^2\right) = 0, \tag{7b}
\]

for the local one, and

\[
(I)_{\mathcal{D}} = 2\langle I \rangle_{\mathcal{D}} - 4\pi G\langle \rho \rangle_{\mathcal{D}} + \Lambda, \quad (II)_{\mathcal{D}} = 8\pi G\langle \rho \rangle_{\mathcal{D}} - \frac{(\mathcal{R})_{\mathcal{D}}}{2} + \Lambda, \tag{7c}
\]

\[
(\mathcal{R})_{\mathcal{D}} + \frac{2}{3}(I)_{\mathcal{D}}(\mathcal{R})_{\mathcal{D}} + \left(2\Pi - \frac{2}{3}(I)_{\mathcal{D}}^2\right) + 2(I)_{\mathcal{D}}\left(2\Pi - \frac{2}{3}(I)_{\mathcal{D}}^2\right) = 0, \tag{7d}
\]

for the averaged one. We have reformulated the kinematical backreaction as \(\mathcal{Q}_{\mathcal{D}} = 2\langle I \rangle_{\mathcal{D}} - \frac{2}{3}(I)_{\mathcal{D}}^2\) to obtain the second set of expressions. This statement obviously holds true for the local and averaged continuity equations.

### 2.2. Equations for the deviation fields

We now provide the whole set of equations for the deviation fields off the background of a comoving dust domain. As is customary, we shall designate the deviation (or peculiar) field of any scalar field \(\psi\) from its background value by \(\delta \psi := \psi - \langle \psi \rangle_{\mathcal{D}}\). One would prefer to write \(\delta_{\mathcal{D}} \psi\) in order to make explicit the scale dependence of the deviations; however, we drop this index for notational ease. Note finally that, \(\langle \psi \rangle_{\mathcal{D}}\) being a scalar (refer to [31] for a proof), the deviation \(\delta \psi\) is also a scalar.

In this paragraph, we only add a few remarks about each proposition. A thorough discussion follows in section 3.

#### 2.2.1. Deviations in density

Using the local and averaged conservation laws (2c) and (4c), we find the following continuity equation for the fluid density deviations, which we formulate in the form of a first proposition.

**Proposition 1a.** The evolution equation for the density deviations on a compact domain \(\mathcal{D}\) is given by

\[
(\delta \rho)_\mathcal{D} + \langle \Theta \rangle_{\mathcal{D}} \delta \rho = -\delta \Theta (\langle \rho \rangle_{\mathcal{D}} + \delta \rho). \tag{8a}
\]

or, equivalently, in terms of scalar invariants by

\[
(\delta \rho)_\mathcal{D} + (I)_{\mathcal{D}} \delta \rho = -\delta I (\langle \rho \rangle_{\mathcal{D}} + \delta \rho). \tag{8b}
\]
Remark. The density deviation field does not obey a conservation law like the local and averaged densities. We are faced with a source term involving the deviation of the expansion rate from its background value. By making use of the commutation rule, the average on \( D \) of these equations results in identities.

For later discussion, we shall prefer to use an alternative form of this proposition. Consider to this end the scale-dependent contrast density
\[
\Delta_{\delta} = \frac{\delta \mathcal{\rho}}{\langle \mathcal{\rho} \rangle_D} \quad \text{when} \quad \Delta_{\delta} < 1,
\]
which is more adapted to the Lagrangian picture and the nonlinear situation \([3, 4, 14]\), than the conventional definition used in Eulerian perturbation theory,
\[
\Delta_D = \frac{\delta \mathcal{\rho}}{\langle \mathcal{\rho} \rangle_D} \quad \text{restricted to the same domain}.
\]

By means of the local and averaged Raychaudhuri equations (2a) and (4a), we end up with the following evolution equations for \( \Delta_D \).

Proposition 1b. The evolution equations for the contrast density on a compact domain \( D \) are written as
\[
\begin{align*}
\dot{\Delta}_D &= \delta \Theta (\Delta_D - 1), \\
\ddot{\Delta}_D + \frac{2}{3} \langle \Theta \rangle_D \dot{\Delta}_D - 4\pi G \langle \mathcal{Q} \rangle_D \Delta_D &= \delta \mathcal{Q}(\Delta_D - 1),
\end{align*}
\]

or, equivalently, in terms of scalar invariants as
\[
\begin{align*}
\dot{\Delta}_D &= \delta I (\Delta_D - 1), \\
\ddot{\Delta}_D + \frac{2}{3} \langle I \rangle_D \dot{\Delta}_D - 4\pi G \langle \mathcal{Q} \rangle_D \Delta_D &= 2(\delta II - \frac{2}{3} \langle I \rangle_D \delta I)(\Delta_D - 1).
\end{align*}
\]

Remark. We have introduced here the local contribution of fluid inhomogeneities within the domain, \( \mathcal{Q} := \frac{1}{2}(\Theta - \langle \Theta \rangle_D)^2 - 2\sigma^2 = 2I - \frac{2}{3} \langle I \rangle_D (2I - \langle I \rangle_D) \). By construction we have \( \langle \mathcal{Q} \rangle_D = \mathcal{Q}_D \), and \( \delta \mathcal{Q} \) stands for the deviation of the kinematical backreaction. Taking the averages of these relations and using the definition of \( \Delta_D \) and the commutation rule, we obtain identities, as it should be for a proper definition of deviation fields.

2.2.2. Deviations in kinematical variables. Using the local and averaged systems, ((2a) and (2b)) and ((4a) and (4b)) respectively, we find the following equations for the kinematical deviations.

Proposition 2. The evolution and constraint equations for the kinematical deviations on a compact domain \( D \) read
\[
\begin{align*}
(\delta \Theta)’ + (\delta \Theta)^2 + \frac{2}{3} \langle \Theta \rangle_D \delta \Theta &= -4\pi G \delta \mathcal{Q} + \delta \mathcal{Q}, \\
\frac{2}{3} \langle \Theta \rangle_D \delta \Theta &= 8\pi G \delta \mathcal{Q} - \frac{1}{2} \delta \mathcal{R} - \frac{1}{2} \delta \mathcal{Q},
\end{align*}
\]
or, equivalently, in terms of scalar invariants
\[
\begin{align*}
(\delta I)’ + (\delta I)^2 + 2\langle I \rangle_D \delta I &= 2\delta II - 4\pi G \delta \mathcal{Q}, \\
\delta II &= 8\pi G \delta \mathcal{Q} - \frac{1}{2} \delta \mathcal{R}.
\end{align*}
\]

Remark. Contrary to the relation (9d), the shape of (9c) is not identical to that of its local and averaged counterparts, (7a) and (7e). The nonlinear character of the latter makes the extra term \( 2\langle I \rangle_D \delta I \) appear. Taking the averages of (9c) and (9d), and using the commutation rule for the first one, we end up with identities.
2.2.3. Integrability condition. Finally, demanding equation (9b) to be an integral of (9a), we obtain the integrability condition between the kinematical deviations and the intrinsic curvature, which we formulate in the form of a last proposition.

**Proposition 3.** The integrability condition on a compact domain $\mathcal{D}$ reads

\[
(\delta R)\prime + \frac{2}{3} (\Theta)\mathcal{D} \delta R + (\delta Q)\prime + 2(\Theta)\mathcal{D} \delta Q = -\delta \Theta (\hat{\Theta} (R)\mathcal{D} + 2Q\mathcal{D} + R + Q), \tag{10a}
\]

or, equivalently, in terms of scalar invariants

\[
(\delta R)\prime + (I)\mathcal{D} \delta R + 2(\delta II)\prime + 2(I)\mathcal{D} \delta II = -16\pi G \delta I (\langle \rho \rangle\mathcal{D} + \delta \rho). \tag{10b}
\]

**Remark.** The curvature and backreaction deviations are coupled through this integrability condition. Here again a source term involving the deviation of the expansion rate is present; in terms of scalar invariants, this term is the same as that of proposition 1a. The average of these relations also results in identities.

3. Properties of the deviation scheme

3.1. Discussion

We have generalized the usual dust perturbation scheme off a predefined background to deviations off a general background that was obtained through the spatial average of inhomogeneous fields on a generic domain. The background of a pressureless self-gravitating system is then not restricted to follow a predefined evolution, but rather an evolution depending on the inhomogeneous distribution of matter and geometry. As we shall see further, deviations off FLRW backgrounds are recovered for globally isotropic domains having a vanishing kinematical backreaction; they constitute a subclass of solutions within the present framework.

Here, we prefer to speak in terms of deviations rather than perturbations since, by construction, this scheme is non-perturbative (no approximation or linearization has been performed). A consequence of this property, and accordingly a second interesting feature of this approach, is that we do not get any constraint on the strength of deviations, apart from the requirement of regularity of the solutions\(^9\).

It is also worth noting that this scheme only functionally depends on a metric (via the domain of averaging); all the equations outlined above keep the same form for any spatial metric. The role of the 3-metric was indeed entirely implicit for the derivation of our scheme: we did not need to compute an averaged metric from the local one in order to eventually obtain the dynamics of the deviation fields off a general background. This is a nice feature, and one can choose any spatial metric and end up with a solution. We recognize that the disadvantage is to be able to deal only with the scalar modes of the deviations, and not the vectorial and tensorial ones. However, we think that the results expounded here clearly constitute a first useful step for an understanding of cosmological deviations off a general background.

The generalization we have proposed shows that the kinematical backreaction, which encodes the global contribution of fluid inhomogeneities within a spatial domain, impacts

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\(^9\) Regularity is violated in the presence of shell-crossing singularities. Shell crossing happens generically due to the chosen matter model ‘irrotational dust’, since dispersion and vorticity are not taken into account to regularize the solutions (for generalizations and discussions in the Newtonian theory, see [11]). The inclusion of backreaction effects improves on this situation in so far as large perturbations on a FLRW background would be eventually mirrored by small perturbations on the general background, so that the scale of the perturbations undergoing shell crossing could be smaller. Note also that for volume averages on sufficiently large scales the inclusion of dispersion and vorticity is not expected to be quantitatively relevant, since those effects are sizable within small fractions of the volume only.
in several ways on the dynamics of the deviations: not only $Q_D$ affects the evolution of deviations through that of the background, as it is implied by the coefficients like $\langle \Theta \rangle_D$ on the left-hand side of equations (8d), (9a) and (9b), but it also acts as a source through the peculiar backreaction $\delta Q$ (right-hand side of (8d), (9a) and (9b)). Comparing the evolution of averaged inhomogeneous cosmologies to that of (scale-dependent) FLRW models, it appears natural that the extra term involved in the evolution of density deviations off a general background is precisely the kinematical backreaction (see subsection 3.4). We also stress the difference between a general domain and a FLRW domain: the spatial curvature of the former is in general not given by a constant-curvature model. The different evolution histories of the background curvature influence the dynamics of the background, which modifies in turn the evolution of the deviation fields—again in comparison to that of a FLRW background. In addition, inhomogeneities in geometry also act as a source in the evolution equations (the term $\delta R$ in equation (9b)).

3.2. Steps toward an exact resolution

Let us first consider the systems for the local variables, the averaged variables and the deviation variables individually.

- The local system (2) needs one additional relation to be solved, since we deal with three independent expressions for four variables ($\rho$, $\Theta$, $\sigma$ and $R$). The hierarchy would continue with the evolution equation for the shear, but it will never be closed on the level of ordinary differential equations (see [9] and also [35]).
- The averaged system (4) also requires a last relation (one on the averaged variables), since in this situation we also have three independent expressions for four variables ($\langle \rho \rangle_D$, $\langle \Theta \rangle_D$, $Q_D$ and $\langle R \rangle_D$) (see ibid.).
- Finally, the deviation field system ((8), (9) and (10)) calls for three additional relations, since we have three independent equations for now six variables ($\delta \rho$, $\langle \rho \rangle_D$, $\delta \Theta$, $\langle \Theta \rangle_D$, $\delta Q$ and $\delta R$).

However, the last two systems are obviously related to the first one. Given any closure relation for the local system, we can solve all sets of equations and eventually obtain an exact solution for the deviation scheme. One can think of choosing a specific form for the spatial metric, or of giving a constraint on kinematical or geometrical variables (e.g. considering a ‘silent universe model’ [2]). For instance, considering a domain endowed with a spatial LTB metric, we are able to construct its background and compute its averaged evolution. According to our scheme, deviations from this background have to be understood as local perturbations giving back the local LTB metric. This can be applied to any synchronous metric. Although maybe not suited to furnish relevant physical models, this first procedure may provide interesting toy models.

If we instead specify a constraint on the averaged variables on a chosen scale, we cannot end up with a solution for the deviation scheme. Nevertheless, by doing so we reduce the needed number of extra relations to one in order to exactly resolve it, since there will remain four variables ($\delta \rho$, $\delta \Theta$, $\delta Q$ and $\delta R$) for the three-equation system ((8), (9), and (10)). The second approach to obtain a solution may then be realized by reducing the space of possible backgrounds, e.g. with scaling laws [13, 51], or particular effective state equations for the backreaction terms [18, 50], or multi-scale partitionings combined with closure assumptions, e.g. [62, 64, 65], and then by considering the resulting equations for the deviation fields. Although being straightforward to get working models of structure formation, this latter approach shall always call for physical verification of the closure relations used.
3.3. Definition of a global physical background and deviations thereof

We dedicate this subsection to the reformulation of the deviation field system ((8), (9), and (10)) with the help of a spatial metric comoving with the global physical background, that we shall define in the ensuing paragraph\(^{10}\).

3.3.1. The global physical background in cosmology. We have discussed thus far the evolution and constraint equations for the deviation fields off a general physical background, obtained from the spatial averaging procedure. Let us now consider a compact spatial domain of the universe—we shall call it \(\Sigma\)—that we may assume to cover the homogeneity scale, namely that spatial scale beyond which all averages do no longer depend on scale. Such an assumption, common to cosmology, is not necessary, but it enables us to define a scale over which we think the universe is representative (any larger scale would not provide new insights) and to have a more transparent frame of comparison with the standard cosmological model and its usual perturbation schemes. We emphasize, again, that the following reformulation does not rely on the existence of a homogeneity scale, and it may equivalently be employed to describe globally inhomogeneous models, in which case \(\Sigma\) would cover the whole spatial manifold, e.g. a spherical space without boundary (see [12] for the average properties of such universe models).

We shall identify the scale of homogeneity as the one where the global physical background is defined (in practice such a scale would correspond to statistical homogeneity—see [58] for an examination of the subject). The idea realized here corresponds to the ‘average background solution’ discussed by Kolb and collaborators [38, 39]. All averages indexed by \(\Sigma\) shall then refer to ‘global’ averages that define our background.

3.3.2. A globally volume-preserving metric. Let us introduce, on \(\Sigma\), the conformally rescaled Riemannian 3-metric \(\tilde{h}\) as

\[
\tilde{h}_{ij} = a_{\Sigma}^2 h_{ij}, \quad \tilde{h}^{ij} = a_{\Sigma}^{-2} \tilde{h}^{ij}, \quad \tilde{h}_i^j = \delta_i^j, \quad (11a)
\]

where \(a_{\Sigma}\) is the dimensionless effective scale factor of the global domain \(\Sigma\), defined as, and satisfying

\[
a_{\Sigma} := \left(\frac{V_{\Sigma}}{V_{\Sigma}}\right)^{1/3}, \quad \frac{3}{2} \frac{\dot{a}_{\Sigma}}{a_{\Sigma}} = \frac{\dot{V}_{\Sigma}}{V_{\Sigma}} = \langle \Theta \rangle_{\Sigma}, \quad (11b)
\]

with \(\Sigma_i\) being the domain at initial time. Definition (11a) guarantees that the metric \(\tilde{h}\) conserves the volume of the domain \(\Sigma\) : \(\tilde{V}_{\Sigma} := \int_\Sigma (\det \tilde{h}_{ij})^{1/2} d^3X = a_{\Sigma}^3 V_{\Sigma}\). We may say that \(\tilde{h}\) stands for the spatial metric ‘comoving’ with the global background. Remark that, the conformal 3-metric being inhomogeneous, we still have local variations of volume elements in \(\tilde{h}\).

We now propose to construct, in this metric, all the scalar fields we shall need for the reformulation of our scheme. Upon defining the corresponding expansion tensor as

\[
\tilde{\Theta}_j := \frac{1}{2} \tilde{h}_{ij} \tilde{h}^{ij}, \quad (12a)
\]

we are able to write, with the help of expressions (11a) and (11b):

\[
\tilde{\theta}_j = \theta_j - \frac{1}{2} \langle \Theta \rangle_{\Sigma} h^i_j = \sigma_j + \frac{1}{2} (\theta - \langle \Theta \rangle_{\Sigma}) h^i_j, \quad (12b)
\]

\(^{10}\)The formalism propounded hereafter may as well be viewed as a reformulation of our scheme for spatial coordinates rescaled by the scale factor of the global background. Although reminiscent of the standard procedure of introducing ‘comoving coordinates’, we shall not pursue this possibility since our approach is coordinate invariant in hypersurfaces, and such coordinate changes would add no physical insight here. Comoving coordinates make sense if a global coordinate system, e.g., on a constant-curvature domain, can be introduced. In general, this is not possible, and a conformal transformation of local coordinates seems unnecessary (it may be of technical help in calculations).
The trace of this equality and the average of the resulting expression, respectively, yield
\[ \tilde{\Theta} = \Theta - \langle \Theta \rangle_\Sigma = \delta \Theta, \quad \langle \tilde{\Theta} \rangle_\Sigma = 0. \] (12c)

The first relation reveals that \( \tilde{\Theta} \) pinpoints the deviation of the local expansion rate from the \( h \)-background expansion rate of the global domain, and it thus defines the peculiar expansion rate of the latter. These expressions are consistent with the stationarity of the \( h \)-background of \( \Sigma \), \( \langle \tilde{\Theta} \rangle_\Sigma = 0 \) and with the existence, in general, of local variations of volume elements evaluated with \( \tilde{h} \), \( \tilde{\Theta} \neq 0 \) (\( \tilde{\Theta} \) vanishes only if the global domain is homogeneous).

By means of equation (12b), the traceless part of the ‘tilde’ expansion tensor is the ‘tilde’ shear tensor of the fluid defined as \( \tilde{\sigma}^j := \sigma^j \), which implies \( \tilde{\sigma}^2 = \sigma^2 \). Concerning the spatial curvature of the global domain, a straightforward calculation, calling for the use of equation (11a), results in \( \tilde{R} = \tilde{a}_S^2 \tilde{R} \).

Finally, the tilde energy density \( \tilde{\rho} \) is obtained by considering the fluid conservation law
\[ \tilde{\rho} + \tilde{\Theta} \tilde{\rho} = 0, \] (13a)
which is simply the counterpart of (2c) in the spatial geometry generated from the conformally rescaled metric. Writing
\[ \tilde{\Theta} = \Theta - \langle \Theta \rangle_\Sigma = -\frac{\dot{\rho}}{\rho} + \frac{\langle \dot{\rho} \rangle_\Sigma}{\langle \rho \rangle_\Sigma} = -\left( \frac{\rho}{\langle \rho \rangle_\Sigma} \right) \cdot \frac{\langle \rho \rangle_\Sigma}{\langle \rho \rangle_\Sigma}, \] (13b)
where we have used the local and averaged conservation laws for the second equality, we obtain
\[ \left( \frac{\rho}{\langle \rho \rangle_\Sigma} \right) + \tilde{\Theta} \frac{\langle \rho \rangle_\Sigma}{\langle \rho \rangle_\Sigma} = 0. \] (13c)
Multiplying both sides by the initial averaged density, we conclude
\[ \tilde{\rho} := \frac{\rho}{\langle \rho \rangle_\Sigma} \langle \phi \rangle_\Sigma, \quad \langle \tilde{\rho} \rangle_\Sigma = \langle \rho \rangle_\Sigma = \text{const.} \] (13d)
The last relation is naturally expected: the total rest mass within the global domain being conserved, the matter tilde density has to remain \textit{globally} constant in the frozen volume \( \tilde{V}_\Sigma \).

(Consider the invariant total rest mass \( M = \tilde{M} = \tilde{V}_\Sigma \langle \tilde{\rho} \rangle_\Sigma \); its conservation indeed implies \( \langle \tilde{\rho} \rangle_\Sigma = 0 \).) We also remark that \( \sigma \) is the only scalar field we shall use that is not affected by the conformal rescaling. These characteristics are due to the fact that \( h \) is built such to provide a volume-preserving metric and thus only transforms quantities related to the trace part of tensors.

In terms of scalar invariants, we are able to write the following equalities:
\[ I = (l)_\Sigma + \tilde{l}, \quad \Pi = \tilde{\Pi} + \frac{1}{2} (l)_\Sigma, \] (14a)
where we have introduced the scalar invariants of \( \tilde{\Theta} \), as \( \tilde{l} := \text{tr} (\tilde{\Theta}^j) = \tilde{\Theta} \) and \( \tilde{\Pi} := \frac{1}{2} (\text{tr}^2 (\tilde{\Theta}^j) - \text{tr} (\tilde{\Theta}^j \tilde{\Theta}^k)) = \frac{1}{2} \tilde{\Theta}^2 - \sigma^2 \). Averaging over \( \Sigma \) the last relation and inserting the result back into it give\[ \Pi = (\Pi)_\Sigma + \tilde{\Pi} - (\tilde{\Pi})_\Sigma + \frac{1}{2} (l)_\Sigma \tilde{l}. \] (14b)
\( \tilde{I} = \delta I \) represents here the deviation field of the first local scalar invariant from its average on the global domain, and \( \tilde{\Pi} = (\Pi)_\Sigma + \frac{1}{2} (l)_\Sigma \tilde{l} - (\Pi) \tilde{l} = (\Pi) + \delta \Pi \) is that of the second local scalar invariant.

Using these relations, we also recast the local contribution of fluid inhomogeneities into \( \tilde{Q} = \frac{1}{2} (\tilde{\Theta}^2 - 2\sigma^2) = 2\Pi = Q \), and the global one into \( \tilde{Q}_\Sigma = \frac{1}{2} (\tilde{\Theta}^2)_\Sigma - 2(\sigma^2)_\Sigma = 2(\Pi)_\Sigma = Q_\Sigma \) (therefore \( \delta Q = \delta \tilde{Q} = \delta \tilde{\Pi} \)).

\[ 11 \text{To be rigorous we should write } \int \tilde{\rho}_\Sigma, \text{ the average over } \Sigma \text{ of any tilde scalar field, with } \int \cdot \Sigma := \langle 1/\tilde{V}_\Sigma \rangle_\Sigma \sqrt{\text{det} \tilde{h}_ij} \text{ d}^4X. \text{ However, one can easily check that, for any spacetime scalar field, } \int \cdot \Sigma = \langle \cdot \rangle_\Sigma. \]
3.3.3. Deviations off a global physical background. From the propositions given in section 2 a set of corollaries follows. It determines the evolution and constraint equations for the deviation fields off the global background, as expressed in a globally volume-preserving metric. From this point of view, deviations do not ‘see’ and are not affected by the volume deformation of the background. We write the system in terms of the tilde scalar invariants, which can be straightforwardly found by using the above definitions.

**Corollary 1.** The evolution equations for the contrast density on a global background $\Sigma$ read

\[
\dot{\Delta}_\Sigma = \tilde{I}(\Delta_\Sigma - 1),
\]

\[
\ddot{\Delta}_\Sigma + 2\frac{\dot{a}}{a} \Delta_\Sigma - 4\pi G \frac{\delta \bar{\rho}}{a^4} \Delta_\Sigma = 2\delta \tilde{\Pi}(\Delta_\Sigma - 1).
\]

**Remark.** We have used the equality $\Delta_\Sigma = \delta \rho / \rho = \delta \tilde{\rho} / \tilde{\rho} = \Delta_\Sigma$ and definition (11b).

**Corollary 2.** The evolution and constraint equations for the tilde scalar invariants on a global background $\Sigma$ are written as

\[
\dot{\tilde{I}} + \tilde{T}^2 + 2\frac{\dot{a}}{a} \tilde{I} = 2\delta \tilde{\Pi} - 4\pi G \frac{\delta \bar{\rho}}{a^4},
\]

\[
\delta \tilde{\Pi} + 2\frac{\dot{a}}{a} \tilde{I} = 8\pi G \frac{\delta \bar{\rho}}{a^4} - \frac{1}{2} \frac{\delta \tilde{R}}{a^2}.
\]

**Corollary 3.** The integrability condition on a global background $\Sigma$ is given by

\[
\frac{(\delta \tilde{R})}{a^2} + 2(\delta \tilde{\Pi})^2 + 12 \frac{\dot{a}}{a} \delta \tilde{\Pi} = -\tilde{I} \left( \frac{2}{3} \frac{(\bar{R})_\Sigma}{a^2} + 4(\bar{I})_\Sigma + \frac{\delta \bar{R}}{a^2} + 2\delta \tilde{\Pi} \right).
\]

3.4. The limit of a FLRW background

FLRW cosmologies are recovered only if, for any compact region $D$ lying in the interior of $\Sigma$, we ask for the vanishing of $Q_D$ (the integrability condition (4d) then imposes $\mathcal{R}$ to be the Friedmannian curvature). In this situation, any scalar field equals its background value; the local and averaged systems (2) and (4) are identical, and there do not exist deviations over $\Sigma$. The existence of deviations, and hence the possibility to form structures, then demands the global domain to not remain locally isotropic. In other words, we need to abandon the strong cosmological principle in favor of, for instance, a weaker version that defines a scale of homogeneity (see subsection 3.3). We may for example consider $\Sigma$ to follow globally, and not locally, a FLRW evolution, requiring the kinematical backreaction to vanish on the homogeneity scale $\Sigma$, and therefore on any larger scale. Such an assumption ensures that the background is globally Friedmannian, but at the same time does not prevent local inhomogeneities to live inside the global domain. In this picture, the cancellation of $Q_\Sigma$ would be the result of an exact compensation between the expansion variance and the average of the shear squared. (This is what happens, for instance, for spatially averaged zero-curvature LTB models, see [19] for the proofs and e.g. [29, 57] for further details.)

Facing such a global domain constrains the averaged system with the condition $Q_\Sigma = 2(\bar{I})_\Sigma = 0$, and allows us to write the second-order differential equation of the contrast density (15b) as follows:
\[
\ddot{\delta}_\Sigma + \frac{2}{a_\Sigma} \dot{\delta}_\Sigma - 4\pi G \frac{\langle \rho \rangle}{\Sigma_1} \delta_\Sigma = 2\ddot{\Pi} (\dddot{\delta}_\Sigma - 1), \tag{18a}
\]

where \(a_\Sigma\) is the scale factor of the background satisfying
\[
3 \frac{\dot{a}_\Sigma}{a_\Sigma} = -4\pi G \frac{\langle \rho \rangle}{\Sigma_1} + \Lambda, \quad 3 \left( \frac{\dot{a}_\Sigma}{a_\Sigma} \right)^2 = 8\pi G \frac{\langle \rho \rangle}{\Sigma_1} - \frac{1}{2} \frac{1}{\dot{a}_\Sigma^2} + \Lambda. \tag{18b}
\]

We recover with (18) the standard framework of density deviations off a FLRW background cosmology [6], which therefore constitutes a subclass of solutions of our deviation scheme. We leave it to the reader to simplify the other deviation field equations (corollaries 2 and 3) using the above condition (\(\ddot{\Pi} = \Pi\)). Let us further suppose that the fluid inhomogeneities weakly contribute to the local kinematics of the deviation fields: \(Q = \tilde{Q} \approx 0\). We can then neglect the quadratic invariant \(\Pi\) and end up with
\[
\ddot{\delta}_\Sigma + \frac{2}{a_\Sigma} \dot{\delta}_\Sigma - 4\pi G \frac{\langle \rho \rangle}{\Sigma_1} \delta_\Sigma = 0, \tag{19a}
\]

which gives the evolution of the first-order Lagrangian (relativistic) density perturbations off a FLRW background. The same expression is obtained for the evolution of the linear density perturbations in the standard perturbation theory. This equation corresponds to the linearization in \(\tilde{\delta}_\Sigma\) of (15b), and it is solved by the exact relativistic form of Zel’dovich’s approximation [34, 66], systematically derived in [21].

We finally take advantage of this small-deviation picture to make a digression about the usefulness of the contrast density \(\tilde{\delta}_\Sigma\) over the density contrast \(\delta_\Sigma = \tilde{\delta}_\Sigma / (1 - \tilde{\delta}_\Sigma)\). Expressing relation (19a) in terms of \(\delta_\Sigma\) yields
\[
\ddot{\delta}_\Sigma + \frac{2}{a_\Sigma} \dot{\delta}_\Sigma - 4\pi G \frac{\langle \rho \rangle}{\Sigma_1} \delta_\Sigma + \delta_\Sigma \left( \ddot{\delta}_\Sigma + \frac{2}{a_\Sigma} \dot{\delta}_\Sigma - 8\pi G \frac{\langle \rho \rangle}{\Sigma_1} \delta_\Sigma - 4\pi G \frac{\langle \rho \rangle}{\Sigma_1} \delta_\Sigma^2 \right) = 0. \tag{19b}
\]

This illustrates that the solution to the linear equation (19a) for \(\tilde{\delta}_\Sigma\) substantially goes beyond that for \(\delta_\Sigma\), \(\ddot{\delta}_\Sigma + \frac{2}{a_\Sigma} \dot{\delta}_\Sigma - 4\pi G \frac{\langle \rho \rangle}{\Sigma_1} \delta_\Sigma = 0\). Hence, first-order Lagrangian (relativistic) deviations off a FLRW background already involve nonlinearities in the dependent variable \(\delta_\Sigma\), which demonstrates the inherently nonlinear character of a Lagrangian perturbation approach. Remark at last that the density contrast coincides by construction with the density deviations \(\delta\langle \rho \rangle\) evaluated in the globally comoving metric; it is simply the relativistic extension of the density deviation field used in the standard Eulerian cosmological perturbation theory [45] (see [5, 6, 27] for other remarks on the Lagrangian picture versus the Eulerian one in Newtonian cosmology).

4. Concluding remarks and outlook

We have generalized, in the present paper, the dust scalar perturbation scheme off a predefined background to deviations off a general background. The kinematical backreaction, which determines the global contribution of fluid inhomogeneities within a generic domain, is at the very core of this non-perturbative scheme: it not only influences the dynamics of a general background, as is well known, but it also explicitly impacts on that of the deviation fields. Our long-term expectation from this improvement is to be able to describe large-scale structure formation uniquely from the existence of inhomogeneities and without the need to invoke a dark energy fundamental component.

Our scheme may be exactly solved either by considering a specific 3-metric, or by imposing local dynamical constraints, or by restricting attention to subclasses of backgrounds
and then constraining the deviation fields. Another, and we think the most promising, strategy to solve the deviation equations would be to develop an iterative procedure. The reason why an iterative procedure takes better care, compared to a perturbative approach, of the nonlinear character of the proposed scheme is obvious: a perturbation point of view runs into contradiction due to the fact that the background (the zeroth-order solution) is generally modified by the kinematical backreaction (a second-order term). Hence, for situations where the backreaction term does not vanish, the notion of, e.g., first-order deviations off a general background would be ill-defined. An iterative point of view also entails methods that are known to numerical simulations, and we expect that the simplest application of the presented scheme is numerical in nature. This relativistic Lagrangian procedure would consist, for the zeroth level, of taking the Zel’dovich approximation (19a) and computing the first-level backreaction term. This first implementation has been depicted in [19] (see section 7.3). Reinjecting this first-level solution, at each step of time, into the background, perturbing this latter and solving the equations of the above corollaries, would then drive the second-level kinematical backreaction, and so on for the ensuing levels of iteration. This process should clearly be carried beyond the first levels of iteration in order that the inherently nonlinear character of structure formation, and its effects, shall be taken into account. Another promising approach would be to use this procedure in the framework of the gradient expansion treatment of inhomogeneities. Developed in [26, 54] and used in [36] to study backreaction, this technique—contrary to perturbation theory—seems to take better care of the impact of the small-scale nonlinear effects on the larger ones. A recent study [30] has shown that, already for gradient expansion quantities of the fourth level, the effects of backreaction can grow up to 10% of the background. A realization of this iterative strategy is the subject of forthcoming work.

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