Flatness of the mod $p$ period morphism for the moduli space of principally polarized abelian varieties

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Abstract. In this note it is shown that associating to a principally polarized abelian variety its de Rham cohomology defines a faithfully flat morphism of the moduli space of principally polarized abelian varieties in positive characteristics to the moduli space of symplectic $F$-Zips as defined in \cite{MW}.

Introduction

Let $A_g$ be the moduli space of $g$-dimensional principally polarized abelian varieties in characteristic $p > 0$. If $S$ is any scheme of characteristic $p$, an $S$-valued point consists of an abelian scheme $f: A \to S$ of dimension $g$ and a principal polarization $\lambda$ of $A$. We can associate to this pair the following “linear algebra” datum over $S$:

We set $M = H^1_{\text{DR}}(A/S)$ which is a locally free $\mathcal{O}_S$-module of rank $2g$. The polarization $\lambda$ induces a perfect alternating bilinear form on $M$.

There are two canonical spectral sequences converging against the de Rham cohomology, namely the Hodge spectral sequence and the conjugate spectral sequence. Both of them induce a filtration on $M$. Moreover, they both degenerate and the filtrations are of the form

\[ 0 \longrightarrow C := f_\ast \Omega^1_{A/S} \longrightarrow M \longrightarrow R^1f_\ast \Omega^0_{A/S} \longrightarrow 0, \]
\[ 0 \longrightarrow D := R^1f_\ast (\mathcal{H}^0(\Omega^\bullet_{A/S})) \longrightarrow M \longrightarrow f_\ast (\mathcal{H}^1(\Omega^\bullet_{A/S})) \longrightarrow 0. \]

Finally, the Cartier isomorphism induces isomorphisms

$\varphi_0: (M/C)^{(p)} \sim D$, $\varphi_1: C^{(p)} \sim M/D$.

This tuple is a “symplectic $F$-zip of type $(g,g)$” in the terminology of \cite{MW}. In other words, we have constructed a morphism $\bar{\zeta}$ from $A_g$ into the moduli space $\bar{Z}_g$ of symplectic $F$-zips of type $(g,g)$. We show:
Theorem. The morphism $\bar{\zeta}$ is faithfully flat.

In [MW] it was proved that $\bar{Z}_g$ is the quotient (in the sense of algebraic stacks) of a smooth variety $Z_J$ by the action of the symplectic group $G = Sp_{2g}$ and that the $G$-orbits on $Z_J$ are in natural bijection with elements in $W_J \backslash W$ where $W$ is the Weyl group of $G$ and $W_J$ is a certain Levi subgroup of $W$. Moreover, a formula for the codimension of each stratum was given.

In this particular case, $W_J \backslash W$ can be identified with $\{0, 1\}^g$ and the codimension of the $G$-orbit corresponding to $u = (\varepsilon_i) \in \{0, 1\}^g$ is equal to

$$g(g+1)/2 - \sum_{i=1}^{g} \varepsilon_i(g+1-i).$$

Every $G$-orbit $Z^u_J$ in $Z_J$ defines a locally closed substack $\bar{Z}^u_g$ of $\bar{Z}_g$. The inverse images of these substacks in $A_g$ are the Ekedahl-Oort strata $A^u_g$ of $A_g$ as defined in [Oo]. As $\dim(A_g) = g(g+1)/2$, the flatness of $\bar{\zeta}$ implies:

Corollary. For $u = (\varepsilon_i) \in \{0, 1\}^g$ let $A^u_g$ be the corresponding Ekedahl-Oort stratum. Then $A^u_0$ is equi-dimensional and we have

$$\dim(A^u_g) = \sum_{i=1}^{g} \varepsilon_i(g+1-i).$$

This has also been shown by Oort in [Oo] by different methods (the elementary sequence $\varphi$ in the terminology of loc. cit. is given by $\varphi(i) = \varepsilon_1 + \cdots + \varepsilon_i$).

We will now give a short overview over the structure of the article. In the first section we fix some notations from the theory of reductive groups. In section 2 the moduli space of principally polarized abelian varieties is defined. Section 3 contains the development of the main technical tool for the proof of the flatness. Here we define split Dieudonné displays. On one hand they are related to Dieudonné displays defined by Zink [Zi] and we will use Zink’s theory for our proof. On the other hand we can express the datum of a split Dieudonné display in a purely group theoretical way (see [KS]). In the last section we define the morphism of (a cover of) our moduli spaces to (a cover of) the moduli space of $F$-zips with additional structures and then show that this morphism is flat and surjective.

Notation. Let $G$ be a group, $X \subset G$ a subset and $g \in G$. Then we set $gX = gXg^{-1}$.

1 Preliminaries on reductive groups

(1.1) Let $k$ be a field and let $G$ be a connected reductive group over $k$. We fix an algebraic closure $\bar{k}$ of $k$. We denote by $\text{Bor}_G$ the scheme of Borel groups of $G$. 

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In $\text{Bor}_G \times \text{Bor}_G$ we define the subscheme $\text{SP}$ whose $S$-valued points consists of those pairs $(B, B')$ of Borel subgroups of $G_S$ such that fppf-locally there exists a maximal torus $T$ of $G$ which is contained in $B$ and $B'$. In this case we say that $B$ and $B'$ are in standard position.

The group $G$ acts on $\text{SP}$ by simultaneous conjugation and the fppf-quotient of this action is representable by a finite étale $k$-scheme $W_G$. We set $W := W_G := W(\bar{k})$ and call it the Weyl group of $G$. For any $w \in W$ we denote the corresponding $G(\bar{k})$-orbit of $\text{SP}(\bar{k})$ by $\text{SP}_w$.

The set $W$ can be endowed with a group structure: For $w, w' \in W$ choose $(B_1, B_2) \in \text{SP}_w$ and $(B_1', B_2') \in \text{SP}_{w'}$ such that $B_2 = B_1'$ and such that the three Borel subgroups $B_1, B_2 = B_1'$ and $B_2'$ contain a common maximal torus (this always can be done). Then we define the product $ww'$ as the $G$-orbit of $(B_1, B_2')$.

For any $w \in W$ we define the length of $w$ as
$$\ell(w) = \dim(\text{SP}_w) - \dim(\text{Bor}).$$

We call the set $I := I_G := \{w \in W \mid \ell(w) = 1\}$ the set of simple elements in $W$.

It is well known that $(W, I)$ is a Coxeter group and that $\ell$ is the usual length function with respect to the Coxeter base $I$ (see [Lu] 7.3).

The canonical map $\text{SP}(\bar{k}) \to W$ is denoted by relpos and $\text{relpos}(B, B')$ is called the relative position of $B$ and $B'$.

(1.2) Let $\varphi: G \to G'$ be a homomorphism of reductive groups such that the induced homomorphism $\varphi^{\text{ad}}: G^{\text{ad}} \to G'^{\text{ad}}$ of adjoint groups is an isomorphism. Then $\varphi$ induces an isomorphism $(W_G, I_G) \sim (W_{G'}, I_{G'})$ of Coxeter groups.

Indeed, it suffices to show that the canonical homomorphism $G \to G^{\text{ad}}$ induces an isomorphism of Weyl groups. But for every $k$-scheme $S$ every Borel subgroup of $G_S$ contains the center of $G_S$ and therefore $\varphi$ induces an isomorphism $\text{Bor}_G \sim \text{Bor}_{G'}$. As the center of $G$ acts trivially on $\text{Bor}_G$, the claim follows.

(1.3) We denote by $\text{Par} = \text{Par}_G$ the scheme of parabolic subgroups of $G$. The algebraic group $G$ acts on $\text{Par}$ via conjugation. The $G_k$-orbits of $\text{Par}_k$ are the connected components of $\text{Par}_k$.

For $i \in I$ we say that $P \in \text{Par}(\bar{k})$ is of type $\{i\}$ if for all pairs $(B, B')$ of Borel subgroups $B \neq B'$ which are contained in $P$ we have $\text{relpos}(B, B') = i$.

If $P$ is any parabolic subgroup of $G_k$, we define the type $J$ as the subset of $I$ which consists of those $i \in I$ such that $P$ contains a parabolic subgroup of type $\{i\}$. This sets up a bijection between $G(k)$-conjugacy classes in $\text{Par}(\bar{k})$ and the set of subsets of $I$.

For every such subset $J$ we denote by $\text{Par}_J$ the variety of parabolics of type $J$. This variety is defined over the field extension of $k$ in $\bar{k}$ over which
\( J \subset W = W(\bar{k}) \) is defined. We denote by \( P_J \) the universal parabolic of type \( J \) over \( \text{Par}_J \) and by \( U_J \) its unipotent radical.

\((1.4)\) For any subset \( J \) we denote by \( W_J \) the subgroup of \( W \) generated by \( J \). Alternatively, \( W_J \) can be defined as the set of \( \text{relpos}(B, B') \) where \( (B, B') \) runs through all pairs of Borel subgroups which are contained in a common parabolic subgroup of type \( J \).

For \( w \in W \) and \( J, K \subset I \) the double coset \( W_J w W_K \) contains a unique element of minimal length and we set

\[
J^W K = \{ w \in W \mid w \text{ is of minimal length in } W_J w W_K \}.
\]

We define \( J^W = J^W \emptyset \) and \( W^K = \emptyset W^K \).

For any two parabolic subgroups \( P \in \text{Par}_J \) and \( Q \in \text{Par}_K \) we consider the set

\[
\{ \text{relpos}(B, C) \mid B \subset P, \ C \subset Q \text{ Borel subgroups} \} \subset W.
\]

Clearly \( W_J \) acts from the left and \( W_K \) acts from the right on this set, and it consists only of a single orbit under this \((W_J, W_K)\)-action. Therefore, it contains a unique element of minimal length which we call the relative position of \( P \) and \( Q \) and which we denote by \( \text{relpos}(P, Q) \in J^W K \).

\((1.5)\) As an example we consider a symplectic space \( (V, \langle \ , \ \rangle) \) of dimension \( 2g \) over a field \( k \) and denote by \( G = GSp(V, \langle \ , \ \rangle) \) the group of symplectic similitudes of \( (V, \langle \ , \ \rangle) \). Moreover, we denote by \( G' = Sp(V, \langle \ , \ \rangle) \subset G \) the group of symplectic isomorphisms.

As \( G/G' \cong \mathbb{G}_m \) is abelian, we have an isomorphism \( \text{Par}_G \xrightarrow{\sim} \text{Par}_{G'} \) given on points by \( P \mapsto P \cap G' \). This identifies in particular \( \text{Bor}_G \) with \( \text{Bor}_{G'} \). As \( G \) is the product of \( G' \) and \( \text{Cent}(G) \), this isomorphism also induces an isomorphism \( \mathcal{W}_{G'} \xrightarrow{\sim} \mathcal{W}_G \). In the sequel we will only consider \( G' \) and set \( \text{Bor} = \text{Bor}_{G'} \), \( W = \mathcal{W}_{G'} \) and so on.

Let \( S \) be any \( k \)-scheme. Then \( V_S = V \otimes_k \mathcal{O}_S \) is a free \( \mathcal{O}_S \)-module of rank \( 2g \), and the base change of \( (\ , \ ) \) is a perfect alternating form on \( V_S \).

Let \( M \) be any locally free \( \mathcal{O}_S \)-module locally of finite rank. An \( \mathcal{O}_S \)-submodule \( F \subset M \) is called locally direct summand if one of the following equivalent conditions holds:

1. For every open affine subset \( U \) of \( S \) there exists a complement \( F' \) of \( F|_U \) in \( M|_U \) (i.e. an \( \mathcal{O}_U \)-submodule \( F' \) of \( M|_U \) such that \( F|_U \oplus F' = M|_U \)).
2. Locally for the fpqc-topology \( F \) admits a complement in \( M \).
3. The quotient \( M/F \) is a locally free \( \mathcal{O}_S \)-module.
(4) There exists an open affine covering \((U_i = \text{Spec}(A_i))_{i \in I}\) of \(S\) such that for all \(i \in I\) and every maximal ideal \(m\) of \(A_i\) the base change

\[ \iota \otimes \kappa(m) : F_{\kappa(m)} \to M_{\kappa(m)} \]

is injective where \(\iota : F \to M\) is the inclusion homomorphism.

If \(F\) is locally a direct summand of \(V_S\), its orthogonal dual \(F^\vee \subset V_S^*\) is locally a direct complement and therefore its orthogonal \(F^\perp \subset V_S\) is locally a direct summand. Here we denote by \((\_)^*\) the \(\mathcal{O}_S\)-linear dual.

A flag of \(V_S\) is a sequence

\[ 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq F_r = V_S \]

of \(\mathcal{O}_S\)-submodules of \(V_S\) which are all locally direct summands of \(V_S\). This implies that \(F_i\) is also a locally direct summand of \(F_{i+1}\). Such a flag is called complete if \(F_{i+1}/F_i\) is locally free of rank 1 and it is called symplectic if for all \(i\) there exists a \(j\) such that \(F_i^\perp = F_j\). Any symplectic flag is already uniquely determined by those members which are totally isotropic.

We denote by \(\text{Par}'\) the \(k\)-scheme whose \(S\)-valued points are the symplectic flags and by \(\text{Bor}'\) the subscheme whose \(S\)-valued points are those which consist of complete symplectic flags. These are projective and smooth \(k\)-schemes. By associating to every symplectic flag over \(S\) its stabilizer in \(G'_S\) we obtain an isomorphism \(\text{Par}' \xrightarrow{\sim} \text{Par}\) which induces an isomorphism \(\text{Bor}' \xrightarrow{\sim} \text{Bor}\).

We use this isomorphism to identify symplectic flags of \(V_S\) and parabolic subgroups of \(G'_S\). Two complete symplectic flags \((F_i)\) and \((F'_i)\) are in standard position if and only if \(F_i + F'_j\) is locally a direct summand of \(V_S\) for all \(i, j = 1, \ldots, 2g\). By passage to the orthogonal it suffices to check this condition for those \(F_i\) and \(F'_j\) which are totally isotropic, i.e. for \(i, j = 1, \ldots, g\).

Two pairs of flags \(((F_i), (G_i))\) and \(((F'_i), (G'_j))\) of \(V_S\) are in the same \(G'(S)\)-orbit if the rank of \(F_i + G_j\) equals the rank of \(F'_i + G'_j\) for all \(i, j = 1, \ldots, g\). From this it is easily seen that every \(G(\overline{k})\)-orbit of a pair of complete symplectic flags in \(V_{\overline{k}}\) contains a pair of complete symplectic flags in \(V\). Therefore, the finite étale scheme \(W\) is the constant scheme associated to the set \(W\).

We describe \(W\): Let \(((F_i), (G_j))\) be a pair of complete symplectic flags in \(V\). For all \(i\) there is a unique \(j = \pi(i)\) such that \(gr_{F_i}^G gr_{F_j}^F \neq (0)\). The rule \(i \mapsto \pi(i)\) defines a permutation of the set \(\{1, \ldots, 2g\}\), i.e. an element \(\pi \in S_{2g}\). Clearly, two pairs of flags are in the same \(G\)-orbit if and only if the associated permutations are equal. Moreover, the condition for the flags to be symplectic implies that the permutation satisfies the condition

\[(1.5.1) \quad \pi(i) + \pi(2g + 1 - i) = 2g + 1 \quad \text{for all } i = 1, \ldots, g.\]
Hence we can identify $W$ with the set of $\pi \in S_{2g}$ satisfying (1.5.1). One can check that this defines a group isomorphism.

The simple elements in $W$ correspond to those $G$-orbits of pairs of symplectic complete flags $((F_i), (G_j))$ such that there exists an $i_0 \in \{1, \ldots, g\}$ with $F_{i_0} \neq G_i$ and $F_i = G_i$ for all $i \in \{1, \ldots, g\}$ with $i \neq i_0$. As we have $F_{j}^\perp = F_{2g-j}$ for all $j$, this implies $F_{2g-i_0} \neq G_{2g-i_0}$ and $F_i = G_i$ for all $i \in \{1, \ldots, 2g\} \setminus \{i_0, 2g - i_0\}$. Therefore, the set of simple elements $I$ consists of $\{s_1, \ldots, s_g\}$ with $s_i = \begin{cases} \tau_i \tau_{2g-i}, & \text{for } i = 1, \ldots, g-1; \\ \tau_g, & \text{for } i = g. \end{cases}$

where $\tau_j \in S_{2g}$ denotes the transposition of $j$ and $j+1$.

Let $J = \{s_{i_1}, \ldots, s_{i_r}\}$ be a subset of $I$. If $P$ is a parabolic of $G$ corresponding to a symplectic flag $(F_j)$, then $P$ is of type $J$ if and only if for all $\rho = 1, \ldots, r$ and for all $j$ the rank of $F_j$ is not equal to $i_\rho$.

(1.6) Consider the special case $J = \{s_1, \ldots, s_{g-1}\}$. Then $W_J$ consists of those permutation $\pi \in W$ such that $\pi(\{1, \ldots, g\}) = \{1, \ldots, g\}$. The map $W_J \to S_g, \quad \pi \mapsto \pi|_{\{1, \ldots, g\}}$ is a group isomorphism. A symplectic flag is of type $J$ if and only if it is of the form $(0) \subset F \subset V_S$ where $F$ is locally a direct summand with $F^\perp = F$.

The set $JW$ consists in this case of those elements $\pi \in W$ such that $\pi^{-1}(1) < \pi^{-1}(2) < \cdots < \pi^{-1}(g)$. Of course, this implies $\pi^{-1}(g+1) < \cdots < \pi^{-1}(2g)$. If $\Sigma = \{j_1 < \cdots < j_g\} \subset \{1, \ldots, 2g\}$ is a subset of $g$ elements such that either $i \in \Sigma$ or $2g+1-i \in \Sigma$ for all $i = 1, \ldots, g$, we get a corresponding element $\pi_\Sigma \in JW$ by setting $\pi^{-1}(i) = j_i$.

The sets of these $\Sigma$’s is in bijection with $\{0,1\}^g$ by associating to $\Sigma$ the tuple $(\epsilon_1, \ldots, \epsilon_g)$ with $\epsilon_i = \begin{cases} 0, & \text{if } i \in \Sigma; \\ 1, & \text{otherwise.} \end{cases}$

The length of such an element $(\epsilon_1, \ldots, \epsilon_g)$ is equal to $\sum_{i=1}^{g} \epsilon_i (g+1-i)$. 

6
2 The moduli space

(2.1) We fix a symplectic vector space \((V, \langle \cdot, \cdot \rangle)\) over \(\mathbb{Q}\) and denote by \(2g\) its dimension. We assume that \(g > 0\). Moreover, we fix a prime \(p > 0\). Let \(\Lambda\) be a \(\mathbb{Z}_p\)-lattice in \(V_{\mathbb{Q}_p}\) such that the restriction of \(\langle \cdot, \cdot \rangle_{\mathbb{Q}_p}\) to \(\Lambda\) is perfect.

Let \(G = \text{GSp}(V, \langle \cdot, \cdot \rangle)\) be the group of symplectic similitudes of \((V, \langle \cdot, \cdot \rangle)\) and denote by \(G' \subset G\) the subgroup of symplectic isomorphisms. We consider \(G\) and \(G'\) as reductive groups over \(\mathbb{Q}\).

We set \(G = \text{GSp}(\Lambda, \langle \cdot, \cdot \rangle)\) and \(G' = \text{Sp}(\Lambda, \langle \cdot, \cdot \rangle)\). These are reductive group schemes over \(\mathbb{Z}_p\) whose generic fibres is equal to \(G_{\mathbb{Q}_p}\) and \(G'_{\mathbb{Q}_p}\), respectively. Their special fibres are denoted by \(\bar{G}\) and \(\bar{G}'\), respectively.

(2.2) We denote by \(A = A_g\) the \(\mathbb{Z}\)-groupoid whose fibres over a scheme \(S\) consists of the category of tuples \((A, \lambda)\) where

- \(A\) is an abelian scheme over \(S\) of dimension \(g\);
- \(\lambda\) is a principal polarization of \(A\).

Then \(A\) is a smooth algebraic Deligne-Mumford stack over \(\mathbb{Z}\) of relative dimension \(g(g + 1)/2\). We denote by \(A_0 = A_{g, 0}\) the reduction \(A \otimes_{\mathbb{Z}} \mathbb{F}_p\) at \(p\).

(2.3) Let \(S\) be a \(\mathbb{F}_p\)-scheme and let \((A, \lambda)\) be an \(S\)-valued point of \(A_0\). We set \(M = H^1_{\text{DR}}(A/S)\). Then \(M\) is a locally free \(\mathcal{O}_S\)-module of rank \(2g\) endowed with a perfect alternating form \(\langle \cdot, \cdot \rangle\) induced by \(\lambda\). Let \(\beta: M \rightarrow M' = \mathcal{H}\text{om}_{\mathcal{O}_S}(M, \mathcal{O}_S)\) be the isomorphism associated to \(\langle \cdot, \cdot \rangle\).

Moreover, \(M\) has two locally direct summands, namely \(C := f_*(\Omega^1_{A/S})\) given by the Hodge spectral sequence and \(D := R^1 f_*(\mathcal{H}\text{om}_{\mathcal{O}_S}(M, \mathcal{O}_S))\) given by the conjugate spectral sequence. Both \(C\) and \(D\) are locally direct summands of rank \(g\). We have \(D = D^\perp\) and \(C = C^\perp\), i.e. \(\beta\) induces isomorphisms \(M/C \cong C^*\) and \(C \cong (M/C)^*\) (similarly for \(D\)).

Finally, the Cartier isomorphism induces isomorphisms

\[
\varphi_0: (M/C)^{(p)} \xrightarrow{\sim} D, \quad \varphi_1: C^{(p)} \xrightarrow{\sim} M/D
\]

such that the diagram

\[
\begin{array}{ccc}
(M/C)^{(p)} & \xrightarrow{\varphi_0} & D \\
\downarrow{\beta^{(p)}} & & \\
(C^*)^{(p)} & \xrightarrow{\varphi_1} & (M/D)^*
\end{array}
\]

commutes.

(2.4) We denote by \((W, I)\) the Weyl group of \(G\) together with its set of simple reflections \([1, 1]\). Let \(J \subset I\) be the subset of simple reflections corresponding
to the conjugacy class of those parabolic subgroups which are the stabilizers of symplectic flags of the form $(0) \subseteq F \subseteq V_S$. As $G'$ and $G$ have the same adjoint group, the Weyl group of $G'$ is canonically identified with the Weyl group of $G$.

Via the reductive group scheme $G'$ we have a canonical isomorphism of $(W, I)$ with the Weyl group of $G'$. (2.5) Lemma. Let $T$ be any scheme and let $M_1$ and $M_2$ be two locally free $\mathcal{O}_T$-modules of the same rank with a symplectic form $(\ , \ )_1$. Then Zariski locally on $T$, the symplectic modules $(M_1, (\ , \ )_1)$ and $(M_2, (\ , \ )_2)$ are isomorphic. Moreover, the scheme of symplectic isomorphisms

$$\text{Isom}_{\mathcal{O}_T}((M_1, (\ , \ )_1), (M_2, (\ , \ )_2))$$

is smooth.

Proof. This is clear. □

(2.6) We define two smooth coverings $A_0^\#$ and $\tilde{A}_0$ of $A_0$ as follows:

For every $\mathbb{F}_p$-scheme $S$ the $S$-valued points of $A_0^\#$ are given by tuples $(A, \lambda, \alpha)$ where $(A, \lambda) \in A_0(S)$ and where $\alpha$ is an $\mathcal{O}_S$-linear symplectic isomorphism $H^1_{\text{DR}}(A/S) \sim \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$.

Therefore, $A_0^\#$ is a torsor for the Zariski topology over $A_0$ under the smooth group scheme $\check{G}'$.

The $S$-valued points of $\tilde{A}_0$ are given by tuples $(A, \lambda, \alpha, C', D')$ with $(A, \lambda, \alpha) \in A_0^\#$ and where $C'$ and $D'$ are totally isotropic complements of $\alpha(C)$ and of $\alpha(D)$ in $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$, respectively.

3 Dieudonné displays with additional structures

(3.1) In this section we will always denote by $R$ a complete local Noetherian ring $R$ with perfect residue field of characteristic $p$. If $p = 2$, we also assume that $pR = 0$.

We will endow Dieudonné displays in the sense of Zink [Zi] with additional structures. We use freely the terminology of loc. cit.. In particular, we have the ring $\check{W}(R)$ which is endowed with Frobenius $\sigma$ and Verschiebung $\tau$ (which are denoted by $F$ and $V$, respectively, in loc. cit.). The kernel of the canonical homomorphism $\check{W}(R) \rightarrow R$ is denoted by $\check{I}_R$. Note that we have $\check{W}(k) = W(k)$ if $k$ is a perfect field. There is a unique structure of a $\mathbb{Z}_p$-algebra on $\check{W}(R)$.

For every $\check{W}(R)$-module $M$ we set $M^\sigma = \check{W}(R) \otimes_{\check{W}(R)} M$. If $M$ is of the form $M = \Lambda \otimes_{\mathbb{Z}_p} \check{W}(R)$ for some $\mathbb{Z}_p$-module $\Lambda$, we have a canonical isomorphism $M^\sigma \cong M$ which we use to identify these two $\check{W}(R)$-modules.
Let $X$ be any $\hat{W}(R)$-scheme. Then the ring endomorphism $\sigma$ of $\hat{W}(R)$ induces a map $\sigma : X(\hat{W}(R)) \to X(\hat{W}(R))$. We will use this notation in particular for the group scheme $X = G \otimes_{\mathbb{Z}_p} \hat{W}(R)$ and for the scheme of parabolics of $G \otimes_{\mathbb{Z}_p} \hat{W}(R)$.

**Definition.** We set $M := \Lambda \otimes_{\mathbb{Z}_p} \hat{W}(R)$. Then $M$ carries a symplectic form $\langle \ , \ \rangle$. A split symplectic Dieudonné display over $R$ consists of a tuple $(S,T,F,V^{-1})$ where $S$ and $T$ are totally isotropic $\hat{W}(R)$-submodules of $M$ such that $S \oplus T = M$. Further $F : M^p \to M$ and $V^{-1} : Q^p \to M$ are $\hat{W}(R)$-linear maps where $Q := S \oplus \hat{I}_R T = S + \hat{I}(R)M \subset M$. The following properties are satisfied:

(a) $V^{-1}$ is surjective.

(b) For all $x \in M$ and $w \in \hat{W}(R)$ we have

$$V^{-1}(1 \otimes \tau(w)x) = wF(1 \otimes x).$$

(c) For all $y, y' \in Q$ we have

$$\tau(\langle V^{-1}(1 \otimes y), V^{-1}(1 \otimes y') \rangle) = \langle y, y' \rangle.$$

By conditions (a) and (b), the map

$$V^{-1} \oplus F : S^p \oplus T^p \longrightarrow M$$

is a surjective $\hat{W}(R)$-linear map of free $\hat{W}(R)$-modules of the same rank, hence it is an isomorphism.

As $S$ and $T$ are both totally isotropic and as $M = S \oplus T$, we have $\text{rk}_{\hat{W}(R)}(S) = \text{rk}_{\hat{W}(R)}(T) = g$.

Using the identity $F(1 \otimes x) = V^{-1}(1 \otimes \tau(1)x)$ it is easy to check that (3.3) implies

$$\langle V^{-1}(1 \otimes y), F(1 \otimes x) \rangle = \sigma(\langle y, x \rangle) \quad \text{for all } y \in Q \text{ and } x \in P,$$

$$\langle F(1 \otimes x), F(1 \otimes x') \rangle = p\sigma(\langle x, x' \rangle) \quad \text{for all } x, x' \in P.$$

Let $(A, \lambda, \alpha)$ be an $R$-valued point of $A_{0}^{\#}$ (2.6). Let $(M, Q, F, V^{-1})$ be the Dieudonné display associated to the $p$-divisible group of $A$ by the theory of Zink [Zi].

Moreover, $\lambda$ induces a perfect alternating form $\langle \ , \ \rangle$ on the free $\hat{W}(R)$-module $M$ such that $\tau(\langle V^{-1}(1 \otimes y), V^{-1}(1 \otimes y') \rangle) = \langle y, y' \rangle$ for all $y, y' \in Q$.
By (2.5) we can find a $\hat{W}(R)$-linear symplectic isomorphism

$$\tilde{\alpha}: M \xrightarrow{\sim} \Lambda \otimes \hat{W}(R)$$

whose reduction modulo $\hat{I}(R)$ is equal to $\alpha$. We use $\tilde{\alpha}$ to identify $M$ and $\Lambda \otimes \hat{W}(R)$.

By definition of a Dieudonné display we have a split exact sequence of free $R$-modules

$$0 \longrightarrow C \longrightarrow M/\hat{I}R M \longrightarrow M/Q \longrightarrow 0.$$  

We choose a totally isotropic direct summand $S$ of $M$ such that its reduction modulo $\hat{I}(R)$ is equal to $C$ and we choose a totally isotropic $\hat{W}(R)$-complement $T$ of $S$ in $M$. Then $(S, T, F, V^{-1})$ is a split symplectic Dieudonné display such that $Q = S + \hat{I}(R)M$.

(3.7) We are now going to give a group theoretic reformulation of the split Dieudonné displays with additional structures defined in (3.3): For any ring $R$ as in (3.1) we define $\tilde{Y}_J(\hat{W}(R))$ to be the set of triples $(\tilde{P}, \tilde{M}, \tilde{g})$ where $\tilde{P} \subset G'_{\hat{W}(R)}$ is a parabolic of type $J$, where $\tilde{M} \subset \tilde{P}$ is a Levi subgroup and where $\tilde{g} \in G'(\hat{W}(R))$.

(3.8) Let $(S, T, F, V^{-1})$ be a split symplectic Dieudonné display over $R$. We associate an element $(\tilde{P}, \tilde{L}, \tilde{g})$ in $\tilde{Y}_J(\hat{W}(R))$ as follows: We define $\tilde{P}$ as the stabilizer of the flag $0 \subset S \subset M$ in $G'_{\hat{W}(R)}$. Then $\tilde{P}$ is a parabolic of type $J$ by (3.3) (d).

Furthermore, $\tilde{L}$ is by definition the stabilizer of the decomposition $M = S \oplus T$ in $G'_{\hat{W}(R)}$. Clearly, $\tilde{L}$ is a Levi subgroup of $\tilde{P}$.

Finally let $\tilde{g}$ be the composition

$$M \xrightarrow{\sim} M^\sigma = S^\sigma \oplus T^\sigma \xrightarrow{V^{-1}|S^\sigma \oplus F|T^\sigma} M.$$

Lemma. The map constructed above defines a bijection between the set of all split symplectic Dieudonné displays over $R$ and the set $\tilde{Y}_J(\hat{W}(R))$.

Proof. Clearly $\tilde{g}$ is a $\hat{W}(R)$-linear map. By (3.3) it is an isomorphism. Now we use (3.5) to check that $\tilde{g}$ respects the alternating form $\langle \ , \ \rangle$ (and therefore $\tilde{g} \in G'(\hat{W}(R))$):

Let $t, t' \in T$ and write $t = \sum_i w_i \otimes \lambda_i$ and $t' = \sum_j w'_j \otimes \lambda'_j$ with $w_i, w'_j \in \hat{W}(R)$ and $\lambda_i, \lambda'_j \in \Lambda$. As $T$ is totally isotropic, we have

$$\langle t, t' \rangle = 0.$$
On the other hand
\[ \langle \tilde{g}(t), \tilde{g}(t') \rangle = \sum_{i,j} w_i w'_j \langle F(1 \otimes (1 \otimes \lambda_i)), F(1 \otimes (1 \otimes \lambda'_j)) \rangle \]
\[ = \sum_{i,j} w_i w'_j p\sigma((1 \otimes \lambda_i, 1 \otimes \lambda'_j)) \]
\[ = p\langle \sum_i w_i \otimes \lambda_i, \sum_j w'_j \otimes \lambda'_j \rangle \]
\[ = 0. \]

A similar argument shows that \( \langle \tilde{g}(s), \tilde{g}(s') \rangle = \langle s, s' \rangle \) for all \( s, s' \in S \). For \( s = \sum_i w_i \otimes \lambda_i \in S \) and \( t = \sum_j w'_j \otimes \lambda'_j \in T \) we have
\[ \langle \tilde{g}(s), \tilde{g}(t) \rangle = \sum_{i,j} w_i w'_j \langle V^{-1}(1 \otimes (1 \otimes \lambda_i)), F(1 \otimes (1 \otimes \lambda'_j)) \rangle \]
\[ = \sum_{i,j} w_i w'_j \sigma((1 \otimes \lambda_i, 1 \otimes \lambda'_j)) \]
\[ = \langle \sum_i w_i \otimes \lambda_i, \sum_j w'_j \otimes \lambda'_j \rangle \]
\[ = \langle s, t \rangle. \]

This shows that \((\tilde{P}, \tilde{L}, \tilde{g}) \in \tilde{Y}_J(\tilde{W}(R))\).

We construct an inverse map: Let \((\tilde{P}, \tilde{L}, \tilde{g})\) be in \( \tilde{Y}_J(\tilde{W}(R)) \). We let \( S \) be the unique direct summand of \( M \) such that its stabilizer is equal to \( \tilde{P} \) and let \( T \subset M \) be the unique direct complement of \( S \) such that the stabilizer of the decomposition \( S \oplus T \) is equal to \( \tilde{L} \). Further we set for \( t \in T, s \in S, \)
\[ w \in \hat{W}(R) \]
\[
F(1 \otimes t) = \tilde{g}(t), \quad F(1 \otimes s) = p\tilde{g}(s), \]
\[ V^{-1}(1 \otimes \tau(w)t) = w\tilde{g}(t), \quad V^{-1}(1 \otimes s) = \tilde{g}(s), \]

Clearly \( V^{-1} \) is surjective as \( \tilde{g} \) is surjective. Moreover, we have for \( w \in \hat{W}(R), t \in T \) and \( s \in S \)
\[ V^{-1}(1 \otimes \tau(w)t) = w\tilde{g}(t) = wF(t), \]
\[ V^{-1}(1 \otimes \tau(w)s) = V^{-1}(pw \otimes s) = pw\tilde{g}(s) = wF(1 \otimes s) \]

which shows that condition (b) of (3.3) holds. A similar although much more lengthy calculation shows that condition (c) is also satisfied.

This shows that \((S, T, F, V^{-1})\) is a split symplectic Dieudonné display. Clearly this construction defines an inverse. \( \square \)
4 Flatness of the mod $p$ period morphism

(4.1) Let $w_0$ be the element of maximal length in $W$ and let $x$ be the element of minimal length in $W_J w_0 W_J = w_0 W_J$.

We denote by $Z_J$ the functor on $F_p$-schemes which is the Zariski-sheafification of the functor $Z_J'$ which associates to an $F_p$-scheme $S$ the set of triples

$$(P, Q, U_Q g U_F(P))$$

where $P \subset \bar{G}_S$ and $Q \subset \bar{G}_S$ are parabolics of type $J$, and where $g \in G(S)$ is an element such that relpos($Q, g F(P)$) = $x$. By [MW] 3.12 this functor is representable by a scheme. For any affine scheme $S$ we have $Z_J(S) = Z'_J(S)$.

By definition of $x$ we have relpos($Q, g F(P)$) = $x$ if and only if $Q \cap g F(P)$ is a common Levi subgroup of $Q$ and $g F(P)$, i.e. $Q$ and $g F(P)$ are in opposition.

(4.2) The forgetful morphism $Z_J \rightarrow \text{Par}_J \times \text{Par}_J$ which is defined on points by $(P, Q, U_Q g U_F(P)) \mapsto (P, Q)$ makes $Z_J$ into a torsor over $\text{Par}_J \times \text{Par}_J$ under a reductive group scheme of dimension $\dim(P/U_P)$ for any parabolic $P$ of $G$ of type $J$ ([MW] 3.11). In particular, $Z_J$ is a smooth $F_p$-scheme whose dimension equals $\dim(G)$.

(4.3) We denote by $\tilde{Z}_J$ the scheme which represents the functor on $F_p$-schemes which associates to $S$ the set of triples $(P, Q, g)$ where $P \subset \bar{G}_S$ and $Q \subset \bar{G}_S$ are parabolics of type $J$ and where $g \in G(S)$ is an element such that relpos($Q, g F(P)$) = $x$.

Via the forgetful morphismus $(P, Q, g) \mapsto (P, Q)$ we will consider $\tilde{Z}_J$ as a scheme over $\text{Par}_J \times \text{Par}_J$.

We consider $U_J \times F(U_J)$ as a group scheme over $\text{Par}_J \times \text{Par}_J$. Then $(u, v) \cdot (P, Q, g) = (P, Q, u g v^{-1})$ defines an action of $U_J \times F(U_J)$ on $\tilde{Z}_J$ over $\text{Par}_J \times \text{Par}_J$. The fppf-quotient of this action is $Z_J$.

(4.4) Lemma. The action of $U_J \times F(U_J)$ on $\tilde{Z}_J$ is free and hence $\tilde{Z}_J$ is a torsor under the smooth group scheme $U_J \times F(U_J)$ over $Z_J$.

Proof. Let $(P, Q, g) \in \tilde{Z}_J$ and let $u \in U_Q$ and $v \in U_F(P)$ such that $u g v^{-1} = g$. This implies that $g v^{-1} g^{-1} = u \in U_Q \cap g U_F(P)$. But by definition of $\tilde{Z}_J$, $g F(P)$ and $Q$ are in opposite position. Therefore, $U_Q \cap g U_F(P) = (1)$ which implies $u = v = 1$.

(4.5) The group $\bar{G}$ acts on $\tilde{Z}_J$ by the rule

$$h \cdot (P, Q, g) = (h P, h Q, h g F(h)^{-1}).$$

This induces an action of $\bar{G}$ on $Z_J$. 

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We denote by $\tilde{Y}_J$ the $\mathbb{F}_p$-scheme which represents the functor which associates to every $\mathbb{F}_p$-scheme $S$ the set of triples $(P, L, g)$ where $P \in \text{Par}_J(S)$, $g \in \tilde{G}(S)$ and $L$ is a Levi subgroup of $P$.

We construct a morphism

$$\tilde{Y}_J \longrightarrow \tilde{Z}_J.$$ 

For every $S$-valued point $(P, L, g)$ of $\tilde{Y}_J$ we define $Q$ as the unique parabolic of type $J$ such that $g^{-1}Q \cap F(P) = F(L)$ ([SGA3] Exp. XXVI, 4.3.). Then $(P, Q, g) \in \tilde{Z}_J(S)$.

Now we relate the moduli spaces defined in (2.6) and the varieties $\tilde{Y}_J$ and $\tilde{Z}_J$.

We define a morphism

$$\tilde{\zeta}: \tilde{A}_0 \rightarrow \tilde{Y}_J$$

as follows: To every $S$-valued point $(A, \lambda, \alpha, C', D')$ we associate the triple $(P, L, g)$ where $P$ is the stabilizer of $\alpha(C)$ in $\tilde{G}_S$, where $L$ is the stabilizer of the decomposition $\alpha(C) \oplus \alpha(C') = \Lambda_S$, and where $g$ is the composition

$$\Lambda_S \xrightarrow{\sim} \Lambda_S^{(p)} = \alpha(C)^{(p)} \oplus \alpha(C')^{(p)} \xrightarrow{\phi_1 \oplus \phi_0} \alpha(D) \oplus \alpha(D') = \Lambda_S.$$

By definition $L$ is a Levi subgroup of $P$, hence $(P, L, g) \in \tilde{Y}_J(S)$.

It follows from the definitions that $\tilde{\zeta}$ induces a morphism

$$\zeta: \tilde{A}_0^# \rightarrow Z_J.$$

**Theorem.** The morphism $\zeta$ is flat.

We will show that $\zeta$ is universally open. As $Z_J$ and $\tilde{A}_0^#$ are both regular, it then follows from [EGA] IV, (15.4.2), that $\zeta$ is flat. To show that $\zeta$ is universally open, we use the following criterion.

**Proposition.** Let $Y$ be a noetherian geometrically unibranch scheme, let $X$ be a scheme, and let $f: X \rightarrow Y$ be a morphism of finite type. Assume that for every commutative diagram

$$\begin{array}{ccc}
\text{Spec}(k) & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\text{Spec}(R) & \longrightarrow & Y
\end{array}$$

(4.9.1)

We have $\zeta: \tilde{A}_0^# \rightarrow Z_J$. 

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where $R$ is a complete discrete valuation ring with algebraically closed residue field $k$, there exists a surjective morphism $\text{Spec}(\tilde{R}) \to \text{Spec}(R)$ of discrete valuation rings and a morphism $\tilde{g}: \text{Spec}(\tilde{R}) \to X$ which commutes with

$$
\begin{array}{ccc}
\text{Spec}(\kappa(\tilde{R})) & \to & \text{Spec}(k) \\
| & | & \\
\text{Spec}(\tilde{R}) & \to & \text{Spec}(R) \\
\downarrow f & & \downarrow g \\
& & \\
& & \text{Spec}(R)'
\end{array}
$$

Then $f$ is universally open.

**Proof.** By [EGA] IV, 14.4.1 it suffices to show that $f$ is open. Let $U \subset X$ be an open subset. By Chevalley’s theorem $f(U)$ is constructible. By [AK] V, 4.4 it suffices therefore to show that $f(U)$ is stable under generization. Let $x_0 \in U$ and $y_0 = f(x_0) \in f(U)$ and let $y_1 \in Y$ be a generization with $y_1 \neq y_0$.

By (4.10) below, there exists a diagram like in (4.9.1). We apply the hypothesis and find a morphism $\tilde{g}: \text{Spec}(\tilde{R}) \to X$ such that $f \circ \tilde{g}$ is the composition

$$
\begin{array}{ccc}
\text{Spec}(\tilde{R}) & \to & \text{Spec}(R) \\
\downarrow \tilde{g} & & \downarrow g \\
& & \text{Spec}(R)'
\end{array}
$$

The image $x_1$ of the generic point of $\text{Spec}(\tilde{R})$ under $\tilde{g}$ is a generization of $x_0$ and hence lies in $U$ as $U$ is open, and therefore $y_1 = f(x_1) \in f(U)$. □

**(4.10) Lemma.** Let $Y$ be a locally noetherian scheme and let $f: X \to Y$ be a morphism of schemes. Let $x_0 \in X$, $y_0 := f(x_0)$ and let $y_1 \neq y_0$ be a generization of $y_0$. Then there exists a commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(\kappa) & \to & X \\
\downarrow i & & \downarrow f \\
\text{Spec}(R) & \to & Y
\end{array}
$$

where $R$ is a discrete valuation ring with algebraically closed residue field $i: \text{Spec}(\kappa) \to \text{Spec}(R)$ such that the image of the generic (resp. special) point of $\text{Spec}(R)$ under $g$ is $y_1$ (resp. $y_0$) and such that the image of $h$ is $x_0$.

**Proof.** There exists a morphism $g': \text{Spec}(R') \to Y$ where $R'$ is a discrete valuation ring such that $g'(s') = y_0$ and $g'(\eta') = y_1$ where $s'$ (resp. $\eta'$) is the closed (resp. generic) point of $\text{Spec}(R')$. 

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Let $\mathfrak{m}'$ be the maximal ideal of $R'$ and let $\kappa$ be an algebraically closed field extension of $\kappa(y_0)$ such that there exist $\kappa(y_0)$-embeddings $\kappa(x_0) \hookrightarrow \kappa$ and $\kappa(s') \hookrightarrow \kappa$ and let $R' \to R$ be a flat local homomorphism of $R'$ into a complete discrete valuation ring $R$ with residue field $\kappa$ such that $\mathfrak{m}' R$ is the maximal ideal of $R$ (this exists by [EGA] 0.8.3). We set $g$ as the composition
\[
\Spec(R) \longrightarrow \Spec(R') \overset{g'}{\longrightarrow} Y
\]
and $h$ as the composition
\[
\Spec(\kappa) \longrightarrow \Spec(\kappa(x_0)) \longrightarrow X.
\]
\[\square\]

(4.11) Proof of (4.8). By (4.9) it suffices to show the following lemma.

**Lemma.** Let $k$ be an algebraically closed field of characteristic $p$, let $R = k[[\varepsilon]]$ be the ring of formal power series in one variable $\varepsilon$ and set $R_1 = k[[\varepsilon^{1/p}]]$. We denote by $h: \Spec(R_1) \longrightarrow \Spec(R)$ the natural morphism.

Let $x = (A, \lambda, \alpha)$ be a $k$-valued point of $A^\#_0$. Let $(P, Q, [g]) \in Z_J(k)$ be the image of $x$ under $\zeta$. Denote by $(P_\varepsilon, Q_\varepsilon, [g_\varepsilon]) \in Z_J(R)$ any deformation of $(P, Q, g)$ to $R$. Then there exists a deformation $x_1 = (A_1, \lambda_1, t_1, \alpha_1) \in A^\#_0(R_1)$ of $x$ such that $h(\zeta(x_1)) = (P_\varepsilon, Q_\varepsilon, [g_\varepsilon]).$

**Proof.** Let $\mathcal{P} = (M, Q, F, V^{-1})$ be the Dieudonné display of the $p$-divisible group of the abelian variety $A$. The free $W(k)$-module $M$ is equipped with a perfect alternating form via $\lambda$. Moreover, we can fix an identification $\tilde{\alpha}$ of $M$ as a symplectic $W(k)$-module with $\Lambda \otimes W(k)$ which lifts the isomorphism $\alpha \overset{\lambda}{\longrightarrow} \Lambda \otimes \hat{W}(R)$ and $M_{\varepsilon_1} = \Lambda \otimes \hat{W}(R_1)$.

We choose submodules $S \subset M$ and $T \subset M$ such that $(S, T, F, V^{-1})$ is a split symplectic Dieudonné display over $k$. Let $(\bar{P}, \bar{L}, \bar{g})$ be the associated element in $\bar{Y}_J(W(k))$.

Let $g \in [g]$ be the reduction of $\bar{g}$ modulo $p$ and choose $g_\varepsilon \in [g_\varepsilon]$ such that the reduction of $g_\varepsilon$ modulo $\varepsilon$ is equal to $g$. Set $P_{\varepsilon,1} := P_\varepsilon \otimes_R R_1$, $Q_{\varepsilon,1} := Q_\varepsilon \otimes_R R_1$ and let $g_{\varepsilon,1}$ be the element $g_\varepsilon$ considered as an $R_1$-valued point of $\mathcal{P}'$.

Let $L_{\varepsilon,1}$ be a Levi subgroup of $P_{\varepsilon,1}$ such that
\[
F(L_{\varepsilon,1}) = (g_{\varepsilon,1}^{-1})Q_{\varepsilon,1} \cap F(P_{\varepsilon,1}).
\]

For any smooth $W(k)$-scheme $X$ the canonical map
\[
X(\hat{W}(R_1)) \to X(R_1) \times_{X(k)} X(W(k))
\]

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is surjective. Applying this to the scheme of parabolic subgroups of $G_W(k)$ of type $J$, to the group scheme $G_W(k)$ itself and to the smooth schemes of Levi subgroups of fixed parabolic subgroup of $G_W(k)$, we see that there exists an element
\[
(\tilde{P}_\varepsilon, 1, \tilde{L}_\varepsilon, 1, \tilde{g}_\varepsilon, 1) \in \tilde{Y}_J(\hat{W}(R_1))
\]
whose reduction to $R_1$ equals $(P_\varepsilon 1, L_\varepsilon 1, g_\varepsilon 1)$ and whose reduction to $W(k)$ equals $(\tilde{P}, \tilde{L}, \tilde{g})$.

Let $(S_\varepsilon 1, T_\varepsilon 1, F_\varepsilon 1, V_{\varepsilon 1}^{-1})$ be the split symplectic Dieudonné display associated to $(\tilde{P}_\varepsilon, 1, \tilde{L}_\varepsilon, 1, \tilde{g}_\varepsilon, 1)$ (3.8). We set $Q_\varepsilon 1 = S_\varepsilon 1 \oplus \hat{I}_RT_\varepsilon 1$. Then $(M_\varepsilon 1, Q_\varepsilon 1, F_\varepsilon 1, V_{\varepsilon 1}^{-1})$ is a Dieudonné display which lifts $P$. Via the correspondence of $p$-divisible groups over $R_1$ and Dieudonné displays over $R_1$, the theorem follows from Serre-Tate theory.

(4.12) The morphism $\zeta: \mathcal{A}_0^\# \longrightarrow Z_J$ induces a flat morphism
\[
\tilde{\zeta}: \mathcal{A}_0 \rightarrow [G\backslash Z_J]
\]
where $[G\backslash Z_J]$ denotes the stack quotient of $Z_J$ by the action of $G$. For every $G$-orbit $O$ of $Z_J$, the quotient stack $[G\backslash O]$ is a locally closed substack of $[G\backslash Z_J]$ whose underlying topological space consists of only one point. From [MW] 3.25 we know that the $G$-orbits of $Z_J$ are in natural bijection to $JW$. Moreover, we know that the $G$-orbit $O^u$ corresponding to $u \in JW$ has codimension $\dim(\text{Par}_J) - \ell(u)$ in $Z_J$. Therefore, the same holds for the substack $[G\backslash O^u]$ of $[G\backslash Z_J]$.

The inverse image of $[G\backslash O^u]$ in $\mathcal{A}_0$ is denoted by $\mathcal{A}_0^u$. These are just the Ekedahl-Oort strata.

(4.13) Corollary. The $\mathcal{A}_0^u$ for $u \in JW$ form a stratification of $\mathcal{A}_0$ (i.e. they are locally closed and the closure of one stratum is the union of strata). All strata $\mathcal{A}_0^u$ are equi-dimensional and we have
\[
\dim(\mathcal{A}_0^u) = \ell(u).
\]

Proof. Clearly the $O^u$ (for $u \in JW$) form a stratification of $Z_J$ as they are just the $G$-orbits. As $\tilde{\zeta}$ is open, this is true for the $\mathcal{A}_0^u$ as well.

Moreover, the unique closed point of the underlying topological space of $[G\backslash Z_J]$ is contained in the image of $\tilde{\zeta}$ (any superspecial principally polarized abelian variety is mapped to this point). Therefore the openness of $\tilde{\zeta}$ implies that $\tilde{\zeta}$ is surjective. In other words, all strata $\mathcal{A}_0^u$ are nonempty.

As $\tilde{\zeta}$ is flat, it also respects codimension, and hence we have
\[
\dim(\mathcal{A}_0^u) = \dim(\mathcal{A}_0) - \text{codim}(\mathcal{A}_0^u, \mathcal{A}_0)
\]
\[
= \dim(\text{Par}_J) - (\dim(\text{Par}_J) - \ell(u))
\]
\[
= \ell(u).
\]
(4.14) Corollary. The morphism
\[ \tilde{\zeta} : \mathcal{A}_0 \to [G \setminus Z_J] \]
is faithfully flat.

(4.15) The dimension formula (4.13) has also been shown by Oort in [Oo]. Here we give a new proof which can be carried over to arbitrary good reductions of Shimura varieties of PEL-type. We will come back to this in [Wd3].

(4.16) We can identify \( \mathcal{J}^W \) as a set with \( \{0,1\}^g \) (16). Further we know by [Wd2] that all Ekedahl-Oort strata are nonempty (although this was certainly known before). Now the corollary (4.13) tells us that the Ekedahl-Oort stratum corresponding to \( u = \{\epsilon_1, \ldots, \epsilon_g\} \in \mathcal{J}^W \) is equidimensional of dimension \( g\epsilon_1 + (g-1)\epsilon_2 + \cdots + \epsilon_g \).

(4.17) Now consider the forgetful morphism \( \eta : Z_J \to \text{Par}_J \) which is defined on points by \( (P,Q,g) \mapsto P \). We know from (4.2) that \( \eta \) is smooth. Moreover, by Grothendieck-Messing theory we know that the composition \( \theta := \eta \circ \tilde{\zeta} : \mathcal{A}_0^\# \longrightarrow \text{Par}_J \) is smooth. We therefore have a diagram of morphisms

\[ \begin{array}{ccc}
\mathcal{A}_0^\# & \xrightarrow{\tilde{\zeta}} & Z_J \\
\text{Par}_J, & \searrow & \\
& \nearrow & \\
& \text{Par}_J, & \end{array} \]

where \( \eta \) and \( \theta \) are smooth and where \( \tilde{\zeta} \) is flat.

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