The $\mathcal{P}(\phi)_2$ Model on the de Sitter Space

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Abstract

In 1975 Figari, Høegh-Krohn and Nappi \[57\] constructed the $\mathcal{P}(\phi)_2$ model on the two-dimensional de Sitter space. Here we complement their work with a number of new results. In particular, we show that

i.) the unitary irreducible representations of $SO_0(1, 2)$ for both the principal and the complementary series can be formulated on the Hilbert space formed by wave functions supported on the Cauchy surface;

ii.) for $m > -\frac{1}{2r}$ physical infrared problems are absent on de Sitter space;

iii.) the interacting quantum fields satisfy the equations of motion in their covariant form;

iv.) the Haag-Kastler and the time-slice axiom hold true. In fact, one can choose an arbitrary space-like geodesic and require that the local von Neumann algebras for all double cones with bases on this specific geodesic are the same for the both free and the interacting theory;

v.) the generators of the boosts and the rotation for the interacting quantum field theory arise by contracting the stress-energy tensor with the relevant Killing vector fields and integrating over the relevant line segments. They generate a reducible, unitary representation of the Lorentz group on the Fock space for the free field.

In addition, we provide a detailed discussion of the causality structure of de Sitter space and a brief review of the representation theory of $O(1, 2)$. We describe the free classical dynamical system in both its covariant and canonical form, and present the associated quantum one-particle KMS structures. The $\mathcal{P}(\phi)_2$ interaction is added on the Euclidean sphere and the Osterwalder-Schrader reconstruction is carried out in some detail.

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As shown in \[57\], the ultraviolet problems can be resolved in the same manner as on flat Minkowski space. The number $r > 0$ is the radius of the time-zero circle in de Sitter space.
List of Symbols

Space-Time

| Symbol         | Definition                                                                 |
|----------------|---------------------------------------------------------------------------|
| \((R^{1+2}, g)\) | Minkowski space-time in 1+2 dimensions                                  |
| \((dS, g)\)     | de Sitter space-time \(dS \subset R^{1+2}\)                             |
| \(g\)           | metric on Minkowski space \(R^{1+2}\)                                  |
| \(g = g|_{dS}\) | metric restricted to \(dS\)                                            |
| \(\mathcal{C}\) | a Cauchy surface                                                        |
| \(S^1\)         | time-zero circle                                                        |
| \(I_+\)         | the half-circle \(W_1 \cap S^1\)                                       |
| \(V^+\)         | forward light-cone in \(R^{1+2}\)                                      |
| \(\Gamma^\pm(x)\) | future and past of a space-time point \(x \in dS\)                    |
| \(\mathcal{O}\)  | a open, bounded space-time region                                       |
| \(\mathcal{O}'\) | space-like complement of \(\mathcal{O} \subset dS\)                   |
| \(W\)           | a wedge in \(dS\)                                                      |
| \(W_1\)         | the wedge \(\{ x \in dS \mid x_2 > |x_0| \}\)                         |
| \(W\)           | the double wedge \(W \cup W'\)                                         |
| \(\mathcal{O}_I\) | the double-cone \(I''\), with basis \(I \subset S^1\)                 |
| \(H_m\)         | mass hyperboloid in \(R^{1+2}\)                                        |
| \(P_t\)         | the wedge \(R_0(\alpha)W_1\)                                           |
| \(g|_{S^1}\)     | metric restricted to \(S^1\)                                           |
| \(dl(\psi)\)    | induced surface element on \(S^1\)                                      |

De Sitter Group

| Symbol         | Definition                                                                 |
|----------------|---------------------------------------------------------------------------|
| \(O(1, 2)\)    | de Sitter group, i.e., Lorentz group in 1+2 dimensions                    |
| \(SO_0(1, 2)\) | proper, orthochronous de Sitter group                                     |
| \(R_0\)        | a rotation around the \(x_0\)-axis                                       |
| \(\Lambda\)    | an arbitrary element in \(SO_0(1, 2)\)                                  |
| \(\Lambda_1(t)\) | the boost which leaves \(W_1\) invariant                                |
| \(\Lambda(\alpha)(t)\) | the boost which leaves \(W^{(\alpha)}\) invariant |
| \(K_0, L_1, L_2\) | generators of Lorentz transformations                                  |
| \(L^{(\alpha)}\) | generator of the boost \(t \mapsto \Lambda(\alpha)(t)\)                 |
| \(T: \mathbb{R}^3 \to \mathbb{R}^3\) | time reflection                                                          |
| \(P_1: \mathbb{R}^3 \to \mathbb{R}^3\) | parity reflection                                                        |
| \(\Theta_W\)   | reflection at the edge of the wedge \(W\)                              |
**List of Symbols**

### Test-Functions on dS
- \( D_{\mathbb{R}}(dS) \) \( \text{real } C^\infty \)-functions with compact support on \( dS \)
- \( f, g \) \( \text{elements of } D_{\mathbb{R}}(dS) \)

### Unitary irreducible representations of SO\(_{0}(1, 2)\)
- \( K_0, L_1, L_2 \) \( \text{generators of } \text{SO}_{0}(1, 2) \text{ on } L^2(\partial V^+, \langle \frac{\omega}{2\pi} \rangle d\rho) \)
- \( M \) \( \text{the Casimir operator on the light-cone} \)
- \( p = (p_0, p_1, p_2) \in \partial V^+ \) \( \text{coordinates on the light-cone} \)
- \( S(S + 1) \Phi = \mu^2 \Phi \) \( \text{KG equation on the light-cone } \partial V^+ \)
- \( \tilde{u}(\Lambda) \) \( \text{UIR of } \text{SO}_{0}(1, 2) \text{ on } \tilde{h}(\partial V^+) \)
- \( \Gamma \) \( \text{a path on the forward light cone } \partial V^+ \)
- \( d\mu_\Gamma \) \( \text{restriction of } d\mu_{\Lambda V^+} \text{ to a path } \Gamma \subset \partial V^+ \)

### (Pseudo-)Differential Operators
- \( \square_{dS} + \mu^2 \) \( \text{Klein–Gordon operator} \)
- \( n \) \( \text{future pointing normal vector field } n(t, \psi) = \cos^{\frac{1}{2}} \psi \partial_t \)
- \( \varepsilon \) \( \text{generator of the boosts } t \mapsto \Lambda_1(t) \)
- \( \cos \psi \) \( \text{multiplication operator by } \cos \psi \)

### Covariant Dynamical System
- \( \sigma \) \( \text{symplectic form associated to } \xi \)
- \( \delta \) \( \text{the commutator function for the Klein–Gordon equation} \)
- \( u(\Lambda) \) \( \text{representation of } \text{O}(1, 2) \text{ on } (\mathfrak{t}(dS), \sigma) \)
- \( \mathfrak{t}(dS) \) \( \text{space of solutions of the Klein–Gordon equation} \)
- \( \phi \) \( \text{a solution of the Klein–Gordon equation (an element in } \mathfrak{t}(dS)) \)
- \( \mathcal{P} \) \( \text{projection from } D_{\mathbb{R}}(dS) \text{ to } \mathfrak{t}(dS) \)
- \( \tilde{\mathcal{P}} \) \( \text{solution of the KG equation for } \tilde{f} \in D_{\mathbb{R}}(dS) \)

### Canonical Dynamical System
- \( \tilde{\mathfrak{t}}(S^1) \) \( \text{the space of Cauchy data for the Klein–Gordon equation} \)
- \( (\tilde{\psi}, \tilde{n}) \) \( \text{Cauchy data (an element of } \tilde{\mathfrak{t}}(S^1)) \)
- \( \tilde{\sigma} \) \( \text{the canonical symplectic form on } \tilde{\mathfrak{t}}(S^1) \)
- \( \tilde{u}(\Lambda) \) \( \text{representation of } \text{O}(1, 2) \text{ on } (\tilde{\mathfrak{t}}(S^1), \tilde{\sigma}) \)
- \( \tilde{\mathcal{P}} \) \( \text{a map from } D_{\mathbb{R}}(dS) \text{ to } \tilde{\mathfrak{t}}(S^1) \)

### Complex Space-Time
- \( dS_C \) \( \text{complex de Sitter space} \)
- \( \mathcal{T}_\pm \) \( \text{forward (backward) tuboid} \)
- \( S^2 \) \( \text{Euclidean sphere} \)

### Fourier Transformation
- \( (x_\pm \cdot p)^s \) \( \text{the Harish-Chandra plane-wave} \)
- \( \mathcal{F}_{\pm}(p, s) \) \( \text{Fourier transform} \)
- \( \mathcal{F}_{\pm|\nu} \) \( \text{FH-transformation restricted to the mass shell} \)
- \( \tilde{\mathcal{F}}_{\nu}(p) \) \( \text{restriction of the Fourier transformation to the mass shell} \)
### Covariant One-Particle Structure

- $\mathcal{h}(dS)$: completion of $\mathcal{D}_R(dS)/\ker(E_{\mu}\mathcal{F}_+)$
- $\langle \cdot, \cdot \rangle_{\mathcal{h}(dS)}$: scalar product on $\mathcal{h}(dS)$
- $\mathcal{u}(\Lambda)$: unitary irreducible representation of $SO_0(1,2)$ on $\mathcal{h}(dS)$
- $\mathcal{K}$: maps $\mathfrak{t}(S^1)$ into $\mathcal{h}(dS)$
- $(\mathcal{K}, \mathcal{h}(dS), \mathcal{u})$: one-particle structure for $(\mathfrak{t}(dS), \sigma, \mathcal{u})$

### Canonical One-Particle Structure

- $\hat{\mathcal{h}}(S^1)$: time-zero Hilbert space
- $\langle \cdot, \cdot \rangle_{\hat{\mathcal{h}}(S^1)}$: scalar product on $\hat{\mathcal{h}}(S^1)$
- $\mathcal{K}$: maps $\hat{\mathfrak{t}}(S^1)$ into $\hat{\mathcal{h}}(S^1)$
- $(\mathcal{K}, \hat{\mathcal{h}}(S^1), \hat{\mathcal{u}})$: one-particle structure for $(\hat{\mathfrak{t}}(S^1), \tilde{\sigma}, \tilde{\mathcal{u}})$
- $\hat{\mathcal{u}}(\Lambda)$: unitary irreducible representation of $SO_0(1,2)$ on $\hat{\mathcal{h}}(S^1)$

### Operator Algebras and States

- $\mathfrak{W}(\mathfrak{t}, \sigma)$: Weyl algebra
- $\alpha_\Lambda$: automorphic representation of $SO_0(1,2)$ on $\mathfrak{W}(\mathfrak{t}(dS), \sigma)$
- $(\mathfrak{W}(dS), \alpha_\Lambda^\circ)$: covariant quantum dynamical system
- $(\hat{\mathfrak{W}}(dS), \hat{\alpha}_\Lambda^\circ)$: canonical quantum dynamical system
- $\omega^\circ$: free de Sitter vacuum state
- $\hat{\omega}^\circ$: free de Sitter vacuum state
- $\mathfrak{L}(\hat{\mathfrak{W}})$: v. N. algebra for the free fields in a double cone $\mathfrak{O} \subset dS$
- $\mathfrak{R}(I)$: v. N. algebra for the free fields in the interval $I \subset S^1$

### Euclidean space-time

- $S^2$: Euclidean space-time
- $S_{\pm}$: upper (resp. lower) hemisphere
- $S^1$: time-zero circle
- $I_{\pm}$: half-circle formed by $W_1 \cap S^1$ or $W'_1 \cap S^1$
- $I_{\alpha}$: the half-circle $I_{\alpha} = \mathbb{R}_0(\alpha)I_+$

### Probability space

- $\mathfrak{Q} = \mathfrak{D}_S(S^2)$: distributions
- $\Sigma$: Borel $\sigma$-algebra on $\mathfrak{Q}$
- $C(f,g)$: covariance
- $d\Phi_C$: Gaussian measure
- $L^p(\mathfrak{Q}, \Sigma, d\Phi_C)$: $L^p$ spaces

### Time-zero fields

- $C_{|_{\mathfrak{t}}}(h_1, h_2)$: time-zero covariance
- $\Phi(\theta, h)$: sharp-time field
LIST OF SYMBOLS

Sobolev spaces
$H^{s \pm 1}(S^2)$ Sobolev spaces 120
$h(S^1)$ a subspace of $h^{s \pm 1}(S^2)$ 120
$e(S^1), e(S_\pm)$ orthogonal projections 120

Fock space
$\Gamma(h^{-1}(S^2))$ Fock space over the Sobolev space $h^{-1}(S^2)$ 121
$\mathcal{F}(S_1)$ a subspace of $H_{\mathcal{F}}(S^2)$ 121
$e(S_1), e(S_\pm)$ orthogonal projections 120

Interaction
$\Phi(f)^{n_c}$ Wick ordering 127
$V(S_+)$ interaction on the upper hemisphere 129
d$\mu_V$ perturbed measure on the sphere 129
$V_0(\cos\varphi)$ the interaction on the half-circle $I_+$ 129

Fock representations
$\pi_\mathcal{F}$ Fock representation on $\mathcal{F}(h^{s \pm 1}(S^2)) = L^2(Q, \Sigma, d\Phi_C)$ 121

Canonical von Neumann algebras
$\mathcal{U}(S^1)$ abelian algebra of time-zero fields in $\mathcal{H}$ 136

Osterwalder-Schrader Hilbert spaces
$\Theta := \Gamma(T_s)$ time reflection on $L^2(Q, \Sigma, d\Phi_C)$ 135
$\mathcal{H}$ Osterwalder-Schrader Hilbert space 136
$\mathcal{V}$ canonical map from $L^2(Q, \Sigma_{S^1 \setminus T^*}, d\Phi_C)$ to $\mathcal{H}$ 136
$\Omega$ free vacuum vector in $\mathcal{H}$ 136
$A_{os}$ multiplication operator on $L^2(Q, \Sigma_{S^1 \setminus T^*}, d\Phi_C)$ 136
$K_{os}$ generator of the rotation on $\mathcal{H}$ 137
$\Omega_{int}$ interacting vacuum vector in $\mathcal{H}$ 144

Generalized path spaces
$(\Omega, \Sigma, \Sigma_0, \mathcal{U}(t), \Theta, \mu)$ generalised path space 137
$U^{(\alpha)}(\theta)$ measure preserving automorphisms 138
$\Sigma = \bigvee_{\theta \in S^1} U^{(\alpha)}(\theta) \Sigma^{(\alpha)}$ the Borel $\sigma$-algebra on $\Omega$ 138
$\Sigma^{(\alpha)}$ smallest $\sigma$-algebra for which $\Phi(0, h)$ is measureable 138

Local symmetric semigroups
$(p^{(\alpha)}(\theta), \partial^{(\alpha)})$ local symmetric semigroup for the free dynamics 139
$L^{(\alpha)}_0$ generator of the free boost $\Lambda^{(\alpha)}$ 139
$J^{(\alpha)}_0$ modular conjugation associate to $(\mathcal{R}(I_\alpha), \Omega)$ 143
$(p^{(\alpha)}_V(\gamma), \partial_{\theta, V})$ local sym. semigroup for interacting dynamics 150
$L^{(\alpha)}_V$ generator of the interacting boost $\Lambda^{(\alpha)}$ 150
$J^{(\alpha)}_V$ modular conjugation associate to $(\mathcal{R}(I_\alpha), \Omega_{int})$ 150
Virtual representations

\((G, H, \varrho)\)  
\(g, \ell, m\)  
\(g^\varphi\)  
\(\varphi\)

Auxiliary Hilbert spaces for the interacting measure

\(L^2(\Omega, \Sigma, d\mu_V)\)  
\(\mathcal{K}_V\)  
\(\mathcal{V}_V\)  
\(\Omega_V\)  
\(A_V\)  
\(U_V(S^1)\)  
\(\mathcal{V}\)

Unitary groups

\(\hat{u}\)  
\(\hat{U}\)  
\(\hat{U}_{\text{int}}\)

Symbols Appendices

\((K, \sigma, T_1)\)  
\((h_{\text{AW}}, K_{\text{AW}}, U_{\text{AW}}(t))\)

| Symbol | Description | Page |
|--------|-------------|------|
| \((G, H, \varrho)\) | symmetric space | 150 |
| \(g, \ell, m\) | Lie algebras | 150 |
| \(g^\varphi\) | dual symmetric Lie algebra | 150 |
| \(\varphi\) | virtual representation | 151 |
| \(L^2(\Omega, \Sigma, d\mu_V)\) | \(L^p\) spaces for the interacting measure | 149 |
| \(\mathcal{K}_V\) | completion of \(L^2(\Omega, \Sigma, d\mu_V)/N_V\) | 149 |
| \(\mathcal{V}_V\) | canonical map from \(L^2(\Omega, \Sigma, d\mu_V)\) to \(\mathcal{K}_V\) | 149 |
| \(\Omega_V\) | interacting vacuum vector in \(\mathcal{K}_V\) | 149 |
| \(A_V\) | multiplication operator on \(L^2(\Omega, \Sigma, d\mu_V)\) | 149 |
| \(U_V(S^1)\) | abelian algebra of interacting time-zero fields in \(\mathcal{K}_V\) | 149 |
| \(\mathcal{V}\) | a unitary operator from \(\mathcal{K}_V\) to \(\mathcal{K}\) | 158 |
| \(\hat{u}\) | unitary irreducible representation of \(SO_{0,2}\) on \(\hat{h}(S^1)\) | 100 |
| \(\hat{U}\) | a unitary representation of \(SO_{0,2}\) on \(\mathcal{K}\) | 151 |
| \(\hat{U}_{\text{int}}\) | interacting representations of \(SO_{0,2}\) on \(\mathcal{K}\) | 158 |
| \((K, \sigma, T_1)\) | classical dynamical system | 173 |
| \((h_{\text{AW}}, K_{\text{AW}}, U_{\text{AW}}(t))\) | Araki-Woods one-particle structure | 173 |
Preface

There is well-founded trust in quantum field theory on curved space-times and its ability to predict and explain many of the exciting astrophysical and cosmological phenomena currently discovered in one of the most thriving branches of experimental physics. But despite substantial effort, little is known about the physics of quantum fields on general curved space-times beyond the scope of (re-normalised) perturbation theory\(^1\) (see, e.g., \([11, 32, 31, 102, 103, 101, 99, 100]\) and references therein).

However, for static space-times, i.e., solutions to Einstein’s equations for which the metric has the form

\[
g = \lambda \, dt \otimes dt - \sum_{i,j=1}^{3} \lambda^{-1} h_{ij} \, dx_i \otimes dx_j,
\]

with both \(\lambda\) and \(h_{ij}\) time-independent\(^2\), some progress has been made in recent years. These space-times allow analytic continuations to Riemannian manifolds, and Ritter and Jaffe \([110, 111, 112]\) pioneered a non-perturbative, constructive approach to interacting fields defined on them. They have shown that one can reconstruct a unitary representation of the isometry group of the static space-time under consideration, starting from the corresponding Euclidean field theory \([111]\). Some progress has also been made in case the space-time is asymptotically flat, see, e.g., \([49, 69, 70]\).

For maximally symmetric space-times, like the (two-dimensional) de Sitter space, the situation is even more favourable. In fact, in 1975 Figari, Høegh-Krohn and Nappi \([57]\) constructed the first (and up till now the only) interacting quantum field theory on a curved space-time, the so-called \(\mathcal{P}(\phi)^2\) model on the de Sitter space. In this work we reconsider the contribution of Figari, Høegh-Krohn and Nappi \([57]\) in the light of more recent work by Birke and Fröhlich \([19]\), Dimock \([50]\) and Fröhlich, Osterwalder and Seiler \([62]\). We provide a detailed and very explicit, non-perturbative description of the \(\mathcal{P}(\phi)^2\) model on de Sitter space. Euclidean methods play an essential role in our approach, despite the fact that they are not

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\(^1\)Even in flat space, the \(\mathcal{P}(\phi)^2\)-models, do not allow Borel summation of the perturbation series, unless the order of the polynomial is less or equal four, as the number of Feynman diagrams grows to rapidly for polynomials of higher order. Although in each order of perturbation theory there are no divergences, the Green’s functions are not analytic in the coupling constant, neither are the proper self energy and the two-particle scattering amplitude \([109]\). For the \(\phi^4\)-model on Minkowski space, perturbation theory yields a Borel summable asymptotic series for the Schwinger functions.

\(^2\)Strictly speaking, de Sitter space \(dS\) is not a static space-time, unless it is restricted to the past \(\Gamma^{-}(W)\) of a wedge \(W\) (which itself is conformally equivalent to Minkowski space).
available on general curved space-times. We, however, would like to mention that in forthcoming complementary work [9], they play a less significant role.

Let us add some comments on the relevance of this particular model. In our opinion, the role of $\mathcal{P}(\phi)_2$ model in quantum field theory may well be compared with the role the Ising model plays in (quantum) statistical mechanics or the role $\text{SL}(2,\mathbb{R})$ plays in harmonic analysis. The various $\mathcal{P}(\phi)_2$ models were the first interacting quantum field theories (in Minkowski space), which gained a precise mathematical meaning and up till now they remain the most thoroughly studied models in the axiomatic framework. The original construction of these models (without cutoffs) is due to Glimm and Jaffe [73, 74, 75, 76, 77, 78]. An enormous amount of work has been invested to understand the scattering theory, the bound states, the low energy particle structure and the properties of the correlation functions of these models (see the books by Glimm and Jaffe [79, 80] and Simon [175], and the references therein). The $\mathcal{P}(\phi)_2$ models are also the only interacting quantum field theories, for which the non-relativistic limit (including bound states) has been analysed in detail, demonstrating that the low energy regime of these models can be equally well described by non-relativistic bosons interacting with $\delta$-potentials [47, 180]. In addition, Hepp demonstrated how one can recover the classical field equations for the $\mathcal{P}(\phi)_2$ models by taking the classical limit [87].

Finally, we would like to express our deep gratitude to the mathematics and physics community. This work would not be possible without the continuing efforts of colleagues working in differential geometry [60, 127], harmonic analysis [51, 59, 98, 115, 187], complex analysis in several variables [53, 105, 189], operator algebras [24, 115, 184], the representation theory of semi-simple Lie groups [12, 13, 128, 140, 183, 190], the theory special functions [131, 179], axiomatic quantum field theory [181, 114], local quantum field theory [2, 86], constructive quantum field theory [79, 80, 175] and quantum statistical mechanics [24, 165]. Of course, the references given here can only serve as an entry point to the literature as the scope of the material, on which this work is firmly based on, is exceptionally wide.

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Part 1

De Sitter Space
CHAPTER 1

De Sitter Space as a Lorentzian Manifold

We start with a few historical remarks. In 1915 Albert Einstein published his theory of gravitation. The Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \quad \mu, \nu = 0, 1, \ldots, 3,$$

describe the curvature of space-time resulting from the distribution of matter fields in space-time. They relate the metric tensor $g_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$. The Ricci tensor $R_{\mu\nu}$ and the scalar curvature $R$ both depend only on the metric tensor $g_{\mu\nu}$. Given a particular model, one can obtain the stress-energy tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g_{\mu\nu}}$$

from Hilbert’s classical prescription of varying the action $S_m$ representing the matter fields with respect to the metric tensor.

In 1916 Einstein (re-) introduced a positive (i.e., repulsive) cosmological constant $\Lambda > 0$ in the Einstein equations, requesting

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} ,$$

in an attempt to ensure the existence of static solutions. Only a few months later, in 1917, de Sitter showed that for $T_{\mu\nu} = 0$ (i.e., the empty space), the new constant $\Lambda > 0$ leads to a universe, which undergoes accelerated expansion [177, 178]. Einstein first discarded de Sitter’s solution as physically irrelevant because it is not globally static [170]. However, experimental evidence soon suggested that on a large scale our universe is isotropic, homogeneous and indeed undergoing accelerated expansion. The latter may be attributed to the existence of a positive cosmological constant [164].

1.1. The metric and the isometry group

De Sitter space ($dS, g$) is the Lorentzian manifold analog of the Euclidean sphere. It is maximally symmetric and has constant negative curvature. In more than two space-time dimensions, it is simply-connected. In two dimensions, $dS$ can be

1. The Einstein equations themselves may be obtained by demanding that $S_g + \kappa S_m$ is stationary with respect to variations of $g_{\mu\nu}$, where $S_g = \frac{1}{2} \int \sqrt{|g|} R$.

2. In space-time dimension two, the Einstein tensor $G_{\mu\nu}$ is always zero. Nevertheless, $R$ may be non-zero. Note that there is no Bianchi identity in two dimensions.
viewed as a one-sheeted hyperboloid, embedded in $1 + 2$-dimensional Minkowski space $\mathbb{R}^{1+2}$.

### 1.1.1. Embedding de Sitter space into $\mathbb{R}^{1+2}$

Following [169], we identify de Sitter space with the submanifold

\[ dS \equiv \{ x = (x_0, x_1, x_2) \in \mathbb{R}^{1+2} | x_0^2 - x_1^2 - x_2^2 = -1, \, \, r > 0 \}. \]

Unless the radius $r$ of the time-zero circle plays a significant role, we will suppress the dependence on $r$. We denote the points of the $1 + 2$-dimensional Minkowski space $\mathbb{R}^{1+2}$ as either triples $(x_0, x_1, x_2)$ or column vectors $\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$, which ever is more convenient. The point $o \equiv (0,0,0) \in dS$ is called the origin of $dS$.

We denote by $S^1 \subset dS$ the “time zero” circle

\[ S^1 \equiv \{ (0, r \sin \psi, r \cos \psi) \in \mathbb{R}^{1+2} | -\frac{\pi}{2} \leq \psi < \frac{3\pi}{2} \} \]

and by $I_+$ (respectively, by $I_-$) the open subset of $S^1$ with positive (respectively, negative) coordinate:

\[ I_+ \equiv \{ (0, r \sin \psi, r \cos \psi) \in \mathbb{R}^{1+2} | -\frac{\pi}{2} < \psi < \frac{\pi}{2} \} \]

and $I_- \equiv \{ (0, r \sin \psi, r \cos \psi) \in \mathbb{R}^{1+2} | \frac{\pi}{2} < \psi < \frac{3\pi}{2} \}$. We identify de Sitter space $dS$ with the submanifold $dS \equiv \{ (x_0, x_1, x_2) \in \mathbb{R}^{1+2} | x_0^2 - x_1^2 - x_2^2 = -1, \, \, r > 0 \}$.

\[ \text{1.1.2. The metric.} \]

The metric on $dS$ equals the induced metric $g = g_{\|,dS}$, with

\[ g = dx_0 \otimes dx_0 - dx_1 \otimes dx_1 - dx_2 \otimes dx_2 \]

the metric of the ambient space $[\mathbb{R}^{1+2}, g]$. We denote the Minkowski product of two vectors $x, y \in \mathbb{R}^{1+2}$ by $x \cdot y \in \mathbb{R}$.

\[ \text{1.1.3. The Isometry Group.} \]

The isometry group of $dS$ is $O(1,2)$. Its linear action on the ambient space $\mathbb{R}^{1+2}$ is given by $3 \times 3$-matrices acting on vectors $\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^{1+2}$. The group

\[ O(1,2) = O^+_1(1,2) \cup O^+_2(1,2) \cup O^1_1(1,2) \cup O^1_2(1,2) \]

has four connected components [181], namely those (distinguished by $\pm$), which preserve or change the orientation and those (distinguished by $\uparrow \downarrow$), which preserve or change the time orientation. Group elements, which preserve the orientation, are called proper. Lorentz transformations, which preserve the time orientation, are called orthochronous. The connected component containing the identity is the

proper, orthochronous Lorentz group, denoted as $SO_0(1,2) \equiv O^+_1(1,2)$. The group $SO_0(1,2)$ acts transitively on the de Sitter space $dS$. $SO_0(1,2)$ has three uniparametric subgroups leaving the coordinate axes in $\mathbb{R}^{1+2}$ invariant: the rotation subgroup $\{ R_0(\alpha) \mid \alpha \in [0, 2\pi) \}$, with

\[ R_0(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}, \]

and the two subgroups of Lorentz boots $\{ \Lambda_1(t) \mid t \in \mathbb{R} \}$ and $\{ \Lambda_2(s) \mid s \in \mathbb{R} \}$, with

\[ \Lambda_1(t) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \quad \text{and} \quad \Lambda_2(s) = \begin{pmatrix} \cosh s & \sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]
According to our convention, the boosts $\Lambda_1(s)$ (respectively, $\Lambda_2(t)$) keep the $x_1$-axis (respectively, the $x_2$-axis) invariant, and therefore correspond to boosts in the $x_2$-direction (respectively, in the $x_1$-direction).

1.1.4. Generators. The generators of the boosts $\mathbb{R} \ni t \mapsto \Lambda_1(t)$, $\mathbb{R} \ni s \mapsto \Lambda_2(s)$ and the rotations $[0, 2\pi) \ni \alpha \mapsto R_0(\alpha)$ are

$$L_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $K_0 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, respectively. We will also refer to the generators

$$(1.1.5) \quad L^{(\alpha)} = \cos \alpha L_1 + \sin \alpha L_2, \quad \alpha \in [0, 2\pi),$$

of the boosts

$$\Lambda^{(\alpha)}(t) = R_0(\alpha)\Lambda_1(t)R_0(-\alpha), \quad t \in \mathbb{R}.$$ 

Note that $L^{(0)} = L_1$ and $L^{(\pi/2)} = L_2$. The Casimir operator

$$C^2 = -K_0^2 + L_1^2 + L_2^2 = 2 \cdot 1_3,$$

with $1_3$ the unit $3 \times 3$-matrix, is an element in the centre of the universal enveloping algebra of the Lie algebra $\mathfrak{so}(1, 2)$.

We will continue our discussion of the Lorentz group $SO_0(1, 2)$ in Chapter 2.

1.2. The causal structure

1.2.1. Time-like and space-like curves. The intrinsic causal structure of dS coincides with the one inherited from the ambient Minkowski space. This allows us to freely use the standard terminology. In particular, we call a smooth curve $t \mapsto \gamma(t)$ on dS (with nowhere vanishing tangent vector $\dot{\gamma}$) causal, time-like, light-like and space-like, according to whether the tangent vector satisfies

$$0 \leq \dot{\gamma} \cdot \dot{\gamma}, \quad 0 < \dot{\gamma} \cdot \dot{\gamma}, \quad 0 = \dot{\gamma} \cdot \dot{\gamma}, \quad \dot{\gamma} \cdot \dot{\gamma} < 0,$$

everywhere along the curve. Similarly, a point $y \in$ dS is called causal, time-like, light-like and space-like separated to $x \in$ dS, if $(y - x) \cdot (y - x)$ is larger or equal than, larger than, equal to or smaller than zero, respectively. Since $x \cdot x = y \cdot y = -r^2$, these notions are equivalent to

$$x \cdot y \leq -r^2, \quad x \cdot y < -r^2, \quad x \cdot y = -r^2, \quad -r^2 < x \cdot y,$$

respectively.

$^3$An analog result holds in $SO(3)$, where the Casimir operator equals $2 \cdot 1_3$ as well.
1.2.2. The future and the past. The future $\Gamma^+(x)$ and the past $\Gamma^-(x)$ of a point $x \in dS$ are given by

$$\Gamma^\pm(x) = \{ y \in dS \mid \pm(y - x) \in V^\pm \},$$

where the bar in (1.2.1) denotes the closure of the future cone

$$V^+ = \{ y \in \mathbb{R}^{1+2} \mid y \cdot y > 0, y_0 > 0 \}$$

in the ambient space ($\mathbb{R}^{1+2}, g$). The boundaries of the future (and the past) are given by two light rays, which form the intersection of $dS$ with a Minkowski forward (respectively, backward) light cone

$$C^\pm(x) = \{ y \in dS \mid (y - x) \cdot (y - x) = 0, \pm(y_0 - x_0) > 0 \}$$

with apex at $x$. As it turns out, these two light rays are also given by the intersection of $dS$ with the tangent plane at $x \in dS$. They separate the future, the past and the space-like regions relative to the point $x$. The forward light cone $C^+(0, 0, 0)$ with apex at the origin coincides with the boundary set $\partial V^+$ of the forward cone.

1.2.3. Cauchy surfaces. De Sitter space is globally hyperbolic, i.e., it has no timelike closed curves and for every pair of points $x, y \in dS$ the set

$$\Gamma^-(x) \cap \Gamma^+(y)$$

is compact (eventually empty). These two properties imply that $dS$ is diffeomorphic to $\mathcal{C} \times \mathbb{R}$, with $\mathcal{C}$ a Cauchy surface for $dS$ (see, e.g., [16]). It is convenient to choose $\mathcal{C} = S^1$; see (1.1.2). One may arrive at this choice by first choosing an arbitrary point $x \in dS$ and a space-like geodesic $\gamma$ passing through $x$, and then introducing coordinates in (1.1.) such that $\gamma$ equals (1.1.2).

1.2.4. Space-like complements and causal completions. The complement of the union $\Gamma^+(x) \cup \Gamma^-(x)$ consists of space-like points. The space-like complement of a simply connected set $\mathcal{O} \subset dS$ is the set

$$\mathcal{O}' = \{ y \in dS \mid y \notin \Gamma^+(x) \cup \Gamma^-(x) \ \forall x \in \mathcal{O} \} .$$

The causal completion $\mathcal{O}''$ of $\mathcal{O}$ is defined as the space-like complement of $\mathcal{O}'$. A subset $\mathcal{O} \subset dS$ is called causally complete, if $\mathcal{O}'' = \mathcal{O}$. (Note that one always has $\mathcal{O} \subset \mathcal{O}''$.)

Remark 1.2.1. These notions apply as well to subsets of lower dimension, e.g., line-segments in $dS$. For example, one can easily compute the causal completion of an open interval $I \subset S^1$: set

$$x(\psi) = (0, r \sin \psi, r \cos \psi), \quad \text{with} \quad 0 \leq \psi_- < \psi_+ < \pi .$$

4In particular, $\Gamma^+(0, 0, r) = \{ y \in dS \mid \pm y_0 > 0, y_2 > r \}$.  
5For the origin $o$, the light rays $\{ o + \lambda(\pm 1, 0, 1) \mid \lambda \in \mathbb{R} \}$ are given by the intersection of $dS$ with the plane $\{ x \in \mathbb{R}^{1+2} \mid x_2 = r \}$.  
6In the presence of a metric, a geodesic can be defined as the curve joining $x$ and $y$ with maximum possible length in time — for a time-like curve — or the minimum possible length in space — for a space-like curve. The null-geodesics on the de Sitter space are light rays, i.e., straight lines.
The two intersecting (half-) light rays passing through \((0, r \sin \psi_\pm, r \cos \psi_\pm)\) are

\[
R_0(\psi_\pm) \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -r \sin \psi_\pm + \lambda \cos \psi_\pm \\ -r \cos \psi_\pm + \lambda \sin \psi_\pm \\ r \end{bmatrix},
\]

with \(\lambda > 0\). They intersect at

\[
\lambda = r \tan \left( \frac{\psi_+ - \psi_-}{2} \right).
\]

Now, any space-like geodesic can be identified with \(S^1\) by applying a coordinate transformation. Therefore the causal completion of an open interval \(I\) on an arbitrary space-like geodesic is a bounded space-time region in \(dS\), if the length \(|I|\) (measured by inserting the endpoints of the interval \(I\) in (1.3.1), see below) is less than \(\pi r\).

### 1.3. Geodesics and geodesic distances

De Sitter space is geodesically complete, i.e., the affine parameter of any geodesic passing through an arbitrary point \(x \in dS\) can be extended to reach arbitrary values. However, given two points \(x, y \in dS\), one may ask whether there exist geodesics passing through both \(x\) and \(y\):

i.) if \(y\) is time- or light-like to the antipode \(-x\) of \(x\), then there is no geodesic passing through both points \(x\) and \(y\);

ii.) the case \(y = -x\) is degenerated, as every space-like geodesics passing through \(x\) also passes through \(-x\);

iii.) in all the other cases, there exists a unique geodesic passing through \(x\) and \(y\). It is a connected component of the intersection of \(dS\) with the plane in \(\mathbb{R}^{1+2}\) passing through \(x, y\) and \(0\) [156].

**Remark 1.3.1.** If a time-like curve is contained in the intersection of \(dS\) with a plane not passing through the origin, then it describes the trajectory of a uniformly accelerated observer.

#### 1.3.1. Geodesic distance

If two points are connected by a geodesic, a **geodesic distance** can be defined:

i.) if \(x\) and \(y\) are space-like to each other and \(|x \cdot y| < r^2\), a spatial distance

\[
d(x, y) \doteq r \arccos \left( -\frac{x \cdot y}{r^2} \right)
\]

is defined as the length of the arc on the ellipse connecting \(x\) and \(y\). Note\(^7\) that \(d(x, x) = 0\), iff \(x \cdot x = -r^2\);

ii.) if \(x\) and \(y\) are time-like to each other, a proper time-difference

\[
d(x, y) \doteq r \arccosh \left( -\frac{x \cdot y}{r^2} \right) = r \ln \left( -\frac{x \cdot y}{r^2} + \sqrt{\frac{|x \cdot y|^2}{r^4} - 1} \right)
\]

is defined as the length of the arc on the hyperbola connecting \(x\) and \(y\).

---

\(^7\)For a proof, we refer the reader to [156] Bemerkung (4.3.14).

\(^8\)If \(y\) is time- or light-like to the antipode \(-x\) (i.e., \(x \cdot y > r^2\)), then \(d(x, y)\) is not defined.

\(^9\)Recall that the principle values of the function \([-1, 1] \ni z \rightarrow \arccos(z)\) are monotonically decreasing between \(\arccos(-1) = \pi\) and \(\arccos(1) = 0\).
1.4. Wedges and double cones

If \( x \in \text{dS} \), the point \(-x\), called the antipode, is in \( \text{dS} \) as well. The light rays going through \( x \) and \(-x\) lie in the tangent planes at \( x \) and \(-x\), respectively. These tangent planes are parallel to each other. It follows that a point \( x \in \text{dS} \) determines four closed regions, namely \( \Gamma^\pm(\pm x) \). Since \( \Gamma^+(\pm x) \cap \Gamma^-(\pm x) = \{\pm x\} \), their union consists of two disjoint, connected components. The complement of the union of these two sets consists of two open and disjoint sets, which we call wedges.

1.4.1. Wedges. The points \((0, \pm r, 0) \in \text{dS}\) are the edges of the wedges

\[
W_1 \doteq \{ x \in \text{dS} | x_2 > |x_0| \} \quad \text{and} \quad W'_1 \doteq \{ x \in \text{dS} | x_2 < |x_0| \}.
\]

Since the proper, orthochronous Lorentz group \( \text{SO}_0(1, 2) \) is transitive[10] on the de Sitter space \( \text{dS} \), an arbitrary wedge \( W \) is of the form \( W = \Lambda W_1 \), \( \Lambda \in \text{SO}_0(1, 2) \).

A one-to-one correspondence[82, p. 1203] between points \( x \in \text{dS} \) and wedges is established by requiring that

— \( x \) is an edge of the wedge \( W_x \);
— for any point \( y \) in the interior of \( W_x \) the triple \( \{(1, 0, 0), x, y\} \) has positive orientation.

For example, \( y = o \) lies in the wedge \( W_1 = W_{(0,r,0)} \).

Remarks 1.4.1.

i.) Two wedges \( W_x \) and \( W_y \) have empty intersection, iff \( y \in \Gamma^+(\pm x) \cup \Gamma^-(\pm x) \)[82 Lemma 5.1].

ii.) Given a wedge \( W \), there is exactly one time-like geodesic \( G \), which lies entirely within \( W \). Indeed, the wedge \( W \) itself is the causal completion of \( G \), i.e., \( G'' = W \).

iii.) The union of \( \Gamma^+(W) \) with \( \Gamma^-(W') \) covers the de Sitter space \( \text{dS} \); the intersection of \( \Gamma^+(W) \) and \( \Gamma^-(W') \) are two light-rays.

iv.) The space-like complement \( W' \) of a wedge \( W \) is a wedge, called the opposite wedge. The double wedge

\[(1.4.1) \quad \mathcal{W} = W \cup W' \]

is uniquely specified by fixing (one of) its edges (the other one is just the antipode).

1.4.2. Double cones. Open, bounded, connected, causally complete space-time regions in \( \text{dS} \) are called double cones. We provide the following characterisation.

Proposition 1.4.2. Let \( \mathcal{O} \) be a double cone. Then there exist

i.) two[9] wedges such that \( \mathcal{O} \) is equal to their intersection;

ii.) a time-like geodesic \( G \) and an open bounded interval \( J \subset G \) such that the causal completion \( G'' \) (which lies entirely within the wedge \( G'' \)) equals \( \mathcal{O} \);

[10] In fact, the orbit \( \{gx | g \in \text{SO}_0(1, 2)\} \) of any point \( x \in \text{dS} \) is all of \( \text{dS} \).

[11] Note that every bounded non-empty region \( \mathcal{O} \) given as the intersection of wedges, is an intersection of two (canonically determined) wedges[82 Lemma 5.2].
iii.) two points \( x, y \in dS \) such that \( \Omega \) is the interior of the intersection of the future of \( x \) and the past of \( y \in \Gamma^+(x) \) (both \( x \) and \( y \) can be identified as boundary points of the segment \( J \) appearing in ii.));

iv.) an interval \( I \) of length \(|I| < \pi r\) on a space-like geodesic such that the causal completion \( I'' \) equals \( \Omega \).

For double cones with base \( I \) on \( S^1 \), we introduce the following notation:

\[
\Lambda_0 = \Lambda \subset dS, \quad |I| < \pi r, \quad I \in S^1.
\]

Note that any double cone is of the form

\[
\Lambda \subset \Omega, \quad \Lambda \in SO_0(1,2), \quad I \subset S^1, \quad |I| < \pi r.
\]

As \(|I| \to \pi r\), the light rays in (1.2.2) become parallel, and \( I'' \) itself becomes a wedge \( W \).

Wedges and double cones are causally complete. Wedges are also geodesically closed, in the sense that if \( x, y \in W \), then there is an interval \( I \) on some geodesic connecting these two points, which lies entirely in \( W \). In fact, the causal completion \( I'' \) of \( I \) automatically lies in \( W \) as well. A similar statement holds for double cones.

### 1.5. Finite speed of propagation

The support of Cauchy data that can influence events at some point \( x \equiv (x_0, x_1, x_2) \in dS \) with \( x_0 > 0 \) is given by the intersection \( \Gamma^- (x) \cap S^1 \) of the past \( \Gamma^- (x) \) of \( x \) with the Cauchy surface \( S^1 \). It will be of particular importance to describe the evolution of this set as the point \( x \) is subject to a Lorentz boost.

**Lemma 1.5.1.** Consider a point \( x(\tau, \psi) = (0, r \sin \psi, r \cos \psi) \in I_+ \). For \( \tau > 0 \) the intersection of the past of the point

\[
\Lambda_1 (\tau) x = \begin{pmatrix}
\cosh \tau & 0 & \sinh \tau \\
0 & 1 & 0 \\
\sinh \tau & 0 & \cosh \tau
\end{pmatrix}
\begin{pmatrix} 0 \\ r \sin \psi \\ r \cos \psi \end{pmatrix}
\]

with \( S^1 \),

\[
\Gamma^- (\Lambda_1 (\tau) x) \cap S^1 = \{ x(\tau, \psi) \in S^1 \mid \psi_- \leq \psi \leq \psi_+ \},
\]

is an interval of length

\[
(1.5.1) \quad r(\psi_+ - \psi_-) = 2r \arctan(\sinh \tau \cos \psi)
\]

centred at \( x(\tau, \frac{\psi_+ + \psi_-}{2}) \) with \( \frac{\psi_+ + \psi_-}{2} = \arcsin \frac{\sin \psi}{\sqrt{1 + (\sinh \tau \cos \psi)^2}} \).

**Proof.** We compute:

\[
\Lambda_1 (\tau) x = \begin{pmatrix}
r \sinh \tau \cos \psi \\
r \sin \psi \\
r \cosh \tau \cos \psi
\end{pmatrix}
\]

\[\text{We will later on show that the } \mathcal{P}(\varphi)_2 \text{ models respect both the finite speed of propagation and the particularities of de Sitter space (e.g., the presence of a cosmological horizon).}\]
Eq. (1.5.1) now follows directly from (1.2.3), and the localisation of the interval follows from
\[
\sin \left( \frac{\psi_+ - \psi_-}{2} \right) = \frac{\sin \psi}{\sqrt{\sin^2 \psi + \cosh^2 \tau \cos^2 \psi}} ,
\]
using \( \sin^2 \psi = 1 - \cos^2 \psi \) and \( \cosh^2 \tau = 1 + \sinh^2 \tau \). \( \square \)

The set \( I(\alpha, \tau) \) introduced in the following proposition describes the localisation region for the Cauchy data, which can influence space-time points in the set \( \Lambda^{(\alpha)}(\tau)I, \tau > 0 \), where \( I \subset S^1 \) is some open interval.

**Proposition 1.5.2.** Let \( I \) be an arbitrary interval in \( S^1 \). Consider the boosts \( \tau \mapsto \Lambda^{(\alpha)}(\tau)I \). It follows that the set
\[
I(\alpha, \tau) = S^1 \cap \left( \bigcup_{y \in \Lambda^{(\alpha)}(\tau)I} \Gamma^-(y) \cup \Gamma^+(y) \right)
\]
equals
\[
I(\alpha, \tau) = \bigcup_{(0, r \sin \psi, r \cos \psi) \in I} \{ x(\tau, \psi - \alpha) \mid \psi_-(\pm \tau, \psi + \alpha) \leq \psi \leq \psi_+(\pm \tau, \psi + \alpha) \}.
\]
As before, \( x(\tau, \psi) = (0, r \sin \psi, r \cos \psi) \). Explicit formulas for the angles \( \psi_\pm(\tau, \psi) \) are provided in Lemma 1.5.1.

**Remarks 1.5.3.**

i.) The speed of propagation
\[
v_\mp = \tau \frac{d \psi_\mp(\tau, \psi + \alpha)}{d \tau}
\]
(to the left and to the right) along the circle \( S^1 \) goes to zero as \( x \) approaches the fixed points \( R_0(\alpha)x \), with \( x = (0, \pm r, 0) \), for the boost \( \tau \mapsto \Lambda^{(\alpha)}(\tau) \).

ii.) For \( \tau \) small the interval \( I(\alpha, \mp) \) grows at most\(^1\) with the speed of light (on both sides), while for \( \tau \) large and increasing the growth rate decreases to zero. In fact, for any interval \( I \subset I_\alpha \) we have
\[
I(\alpha, \mp) \subset I_\alpha \quad \forall \tau \geq 0.
\]
Recall that \( \bigcup_{t \in \mathbb{R}} \Lambda^{(\alpha)}(t)I_\alpha = W^{(\alpha)} \) and \( W^{(\alpha)} \cap S^1 = I_\alpha \).

iii.) Let \( I \subset I_+ \) be an open interval. It follows that \( \lim_{\tau \to \infty} I(\alpha, \mp) = I_+ \). In fact, for every point \( x \in W^{(\alpha)} \) one has
\[
\lim_{\tau \to \infty} \Gamma^-((\Lambda^{(\alpha)}(\tau)x) \cap S^1 = I_\alpha.
\]

\(^1\)Note that speed refers to proper time and spatial geodesics distances as defined in (1.3.1).
\(^2\)This is the case for small \( \tau \), if the interval is centred at \( R_0(\alpha)x \), with \( x = (0, \pm r, 0) \).
CHAPTER 2

Space-time symmetries

Just like Minkowski space, de Sitter space is maximally symmetric. Just like the sphere and the plane, it has constant curvature.

2.1. The group $O(1,2)$

The isometry group of the ambient space $\mathbb{R}^{1+2}$ is the Poincaré group $E(1,2)$. The stabiliser of the zero vector $0 \equiv (0,0,0) \in \mathbb{R}^{1+2}$ is the subgroup $O(1,2)$ of $E(1,2)$. It is the group of isometries of $dS_2$.

**Lemma 2.1.1.** The action of the group $O(1,2)$ splits $\mathbb{R}^{1+2}$ into orbit\(^1\) i.e., the group $O(1,2)$ leaves the origin $(0,0,0)$ invariant; ii.) $\{g \begin{pmatrix} m \\ 0 \end{pmatrix} \mid g \in O(1,2)\} = H_m^+ \cup H_m^-$, where

$$H_m^\pm = \{x \in \mathbb{R}^{1+2} \mid x_0^2 - x_1^2 - x_2^2 = m^2, \pm x_0 > 0\}.$$ More generally, the orbit of any point in the interior of the forward light-cone is a two-sheeted hyperboloid $H_m^+ \cup H_m^-$ for some mass $m > 0$; iii.) $\{g \mid g \in O(1,2)\} = dS$. More generally, the orbit of any point, which is space-like to the zero vector $0$, is a de Sitter space $dS$ of some radius $r > 0$; iv.) $\{g \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid g \in O(1,2)\} = (\partial V^+ \cup \partial V^-) \setminus \{0\}$. More generally, the orbit of any point, which is light-like to the zero vector $0$, is $(\partial V^+ \cup \partial V^-) \setminus \{0\}$.

The Minkowski space $\mathbb{R}^{1+2}$ is the disjoint union of all of these sets.

**Proof.** If $X$ is $\partial V^+ \cup \partial V^-$, $dS$, or $H_m^+ \cup H_m^-$, then

$$\Lambda(\Lambda'x) = (\Lambda \circ \Lambda')x \in X, \quad \Lambda, \Lambda' \in O(1,2), \quad x \in X.$$ In particular, $\Lambda(\Lambda^{-1}x) = (\Lambda \circ \Lambda^{-1})x = x$ for all $x \in X$. Moreover, the group $O(1,2)$ acts transitively on $\partial V^+ \cup \partial V^-$, $dS$ and $H_m^+ \cup H_m^-$: \vspace{1ex}

$$\partial V^+ \cup \partial V^- = (T^k R_0(\alpha) \Lambda_1(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid k = 0, 1, \ t \in \mathbb{R}, \alpha \in [0, 2\pi]); \quad dS = (R_0(\alpha) \Lambda_1(t) \alpha \mid t \in \mathbb{R}, \alpha \in [0, 2\pi]); \quad H_m^+ \cup H_m^- = (T^k R_0(\alpha) \Lambda_1(t) \begin{pmatrix} m \\ 0 \end{pmatrix} \mid k = 0, 1, \ t \in \mathbb{R}, \alpha \in [0, \pi]).$$

\[\square\]

\(^1\)In other words, the sets $\partial V^+ \cup \partial V^-$, $dS$ and $H_m^+ \cup H_m^-$ are $G$-sets for the group $G = O(1,2)$. 

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2.1.1. The action of $SO_0(1,2)$ on the light-cone. In the sequel, the forward light cone
\[
\partial V^+ = \left\{ R_0(\alpha)\Lambda_1(t) \left( \begin{array}{c} 1 \\ 0 \\ \alpha \end{array} \right) \mid t \in \mathbb{R}, \alpha \in [0,2\pi) \right\}
\]
\[
= \left\{ \left( \begin{array}{c} p_0 \\ p_0 \sin \alpha \\ -p_0 \cos \alpha \end{array} \right) \mid p_0 > 0, \alpha \in [0,2\pi) \right\},
\]
will play an important role. In the second line, we have set $p_0 = e^{-t}$.

We will therefore provide explicit formulas for the action of the boosts $\Lambda_1(t)$, $\Lambda_2(s)$ and the rotations $R_0(\beta)$ on $\partial V^+$:
\[
\Lambda_1^{-1}(t) = \left( \begin{array}{c} p_0 \\ p_0 \sin \alpha \\ -p_0 \cos \alpha \end{array} \right) = p_0 \left( \begin{array}{c} \cosh t + \sinh t \cos \alpha \\ \sin \alpha \\ -\sinh t - \cosh t \cos \alpha \end{array} \right),
\]
and
\[
\Lambda_2^{-1}(s) = \left( \begin{array}{c} p_0 \\ p_0 \sin \alpha \\ -p_0 \cos \alpha \end{array} \right) = p_0 \left( \begin{array}{c} \cosh s - \sinh s \sin \alpha \\ -\sinh s + \cosh s \sin \alpha \\ -\cos \alpha \end{array} \right),
\]
Finally,
\[
R_0^{-1}(\beta) = \left( \begin{array}{c} p_0 \\ p_0 \sin \alpha \\ -p_0 \cos \alpha \end{array} \right) = p_0 \left( \begin{array}{c} 1 \\ \cos \beta \sin \alpha - \sin \beta \cos \alpha \\ -\sin \beta \sin \alpha - \cos \beta \cos \alpha \end{array} \right).
\]

2.1.2. Reflections. The time reflection and the parity transformation
\[
T = \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \quad P_1 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right) \in O(1,2),
\]
leave the Cauchy surface $S^1$ invariant. $P_1T$ is the reflection at the edge of the wedge $W_1$. Together $P_1$ and $T$ generate the Klein four group. The reflection at the edge of an arbitrary wedge $W = \Lambda W_1$, is
\[
\Theta_W = \Lambda P_1 T \Lambda^{-1}, \quad \Lambda \in SO_0(1,2).
\]
$\Theta_W$ is an isometry of both $W$ and $dS$. It preserves the orientation but inverts the time orientation, in other words, $\Theta_W$ is an element of $SO^+(1,2)$.

2.1.3. Boosts associated to wedges. The boost $t \mapsto \Lambda_1(t)$ leaves the wedge $W_1$ invariant. In fact, $W_1$ is the causal completion of the worldline $t \mapsto \Lambda_1(t)\circ$. For an arbitrary wedge $W = \Lambda W_1$, $\Lambda \in SO_0(1,2)$,
\[
\Lambda_w(t) = \Lambda \Lambda_1(t) \Lambda^{-1}, \quad t \in \mathbb{R},
\]
defines a boost leaving $W$ invariant, i.e.,
\[
\Lambda_w(t)W = W, \quad t \in \mathbb{R}.
\]
In particular, $\Lambda_1(t) = \Lambda_w(t)$ for all $t \in \mathbb{R}$.

The Killing vector field\(^2\) induced by $\Lambda_w(t)$ leaves the opposite wedge $W'$ invariant too. It is however past directed in $W'$. One may fix the scaling factor by

\(^2\)Killing fields are the infinitesimal generators of isometries; that is, flows generated by Killing fields are continuous isometries of the manifold.
normalising the Killing vector field on the time-like geodesic $\mathcal{G}$ satisfying $\mathcal{G}'' = W$. Uniqueness then implies

$$\Lambda_W(t) = \Lambda_{W'}(-t), \quad t \in \mathbb{R}.$$  

The double-wedge $\mathcal{W}$ introduced in (1.4.1) is invariant under both $\Lambda_W(t)$ and $\Lambda_{W'}(t), t \in \mathbb{R}$.

Another interesting property of the boost $t \mapsto \Lambda_1(t)$ is that it leaves the points $(0, \pm r, 0)$ invariant. In other words, it is the stabilizer — within the group $SO_0(1, 2)$ — of the point $(0, r, 0) \in dS$ (and, at the same time, the antipode $-(0, r, 0)$). Similarly, the origin $o$ and its antipode $-o$ are invariant under the boosts $\Lambda_2(s), s \in \mathbb{R}$.

More generally, the group $t \mapsto \Lambda_W(x)(t), t \in \mathbb{R}$, is the unique — up to rescaling — one-parameter subgroup of $SO_0(1, 2)$, which leaves the edges of the wedge $W_x$ invariant and induces a future directed Killing vector field in the wedge $W_x$. Clearly, it also leaves the light rays passing through $\pm x$ invariant.

**Remark 2.1.2.** A free falling observer passing through the origin $o$ interprets the boost $\nu \mapsto \Lambda_2(\nu) = \exp(\nu L_2)$ as a Lorentz transformation, the boost (re-scaled to proper time)

$$\tau \mapsto \Lambda_1(\frac{\tau}{r}) = e^{\frac{\tau}{r} L_1}$$

as his geodesic time evolution and the rotation $a \mapsto R_0(\frac{a}{r}) = \exp\left(\frac{a}{r} K_0\right), a \in [0, 2\pi r)$, as a spatial translation. Unless $x = o$, the path

$$\tau \mapsto \Lambda_1(\frac{\tau}{r}) x, \quad x \in I_+,$$

describes a uniformly accelerated observer. Note that such a path lies on the intersection of $dS$ with a plane parallel to the $\{x_2 = 0\}$-plane, passing through $x$.

### 2.1.4. Coordinates for the wedge $W_1$.

The chart

$$(2.1.4) \quad x(t, \psi) = \Lambda_1(t) R_0(-\psi) \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}, \quad t \in \mathbb{R}, \quad -\frac{\pi}{2} < \psi < \frac{\pi}{2},$$

provides coordinates for the wedge $W_1$. Allowing $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, these coordinates extend to the space-time region

$$(2.1.5) \quad \mathcal{W}_1 \cup \{(0, r, 0), (0, -r, 0)\} = \bigcup_{t \in \mathbb{R}} \Lambda_1(t) S^1.$$

The r.h.s. is the union of the boosted time-zero circles $\Lambda_1(t) S^1, t \in \mathbb{R}$.

---

3Alternatively, one may also view a motion along a horosphere, given by the map $q \mapsto D(q/r)$ as a translation; see (2.2.1) below.

4In the sequel, we will always take care of the fact that these coordinates are degenerated at the fixed-points $(0, r, 0), (0, -r, 0) \in dS$ for the boost $t \mapsto \Lambda_1(t)$.
The set of orbits of all maximally \textit{unipotent} subgroups — the so-called \textit{horospheres} — play a central role in the construction of induced representations of the Lorentz group. Gelfand and Gindikin (see, e.g., \cite{141, 142, 143}) have shown that the generalized Fourier transform on homogeneous spaces and the horospherical transform are connected by the (commutative) Mellin transform.

**Lemma 2.2.1.** The stabilizer — within the group \( \text{SO}_0(1, 2) \) — of the point \((1, 0, -1) \in \partial V^+\) is the one-parameter group \(D(q)\).

\[
D(q) = \begin{pmatrix}
1 + \frac{q^2}{2} & q & \frac{q^2}{2} \\
q & 1 & q \\
-\frac{q^2}{2} & -q & 1 - \frac{q^2}{2}
\end{pmatrix}, \quad q \in \mathbb{R}.
\]

It has the following properties:

i.) it leaves the light ray \(\lambda(1, 0, -1), \lambda \in \mathbb{R}\), pointwise invariant;

ii.) it is nilpotent. In fact,

\[
D(q) = e^{q(L_2 - K_0)} = \mathbb{1} + q(L_2 - K_0) + \frac{q^2}{2}(L_2 - K_0)^2, \quad q \in \mathbb{R};
\]

iv.) it leaves the half-spaces \(\Gamma^+(W_1)\) and \(\Gamma^-(W'_1)\) invariant. In particular, it leaves the two light rays

\[
D(q) \begin{pmatrix} 0 \\ \pm r \\ 0 \end{pmatrix} = \begin{pmatrix} \pm rq \\ \pm r \\ \mp rq \end{pmatrix}, \quad q \in \mathbb{R},
\]

which form the intersection of \(\Gamma^+(W_1)\) with \(\Gamma^-(W'_1)\), invariant;

iv.) it satisfies

\[
\Lambda_1(t)D(q)\Lambda_1(-t) = D(e^t q), \quad t, q \in \mathbb{R}.
\]

**2.2.1. Coordinates for the half-space \(\Gamma^+(W_1)\).** The boosts \(\Lambda_1(t)\), \(t \in \mathbb{R}\), together with the translations \(D(q), q \in \mathbb{R}\), give rise to the chart.\(^{10}\)

\[
x(\tau, \xi) = D\left(\frac{\xi}{\tau}\right) \Lambda_1 \begin{pmatrix} 0 \\ \frac{\tau}{\xi} \end{pmatrix} = \begin{pmatrix} r \sinh \frac{\tau}{2} + \frac{\xi^2}{4r^2}e^{\frac{2\tau}{r}} \\ \xi e^{\frac{\tau}{r}} \\ r \cosh \frac{\tau}{2} - \frac{\xi^2}{2r^2}e^{\frac{2\tau}{r}} \end{pmatrix}
\]

\(^5\)A unipotent element \(a\) of a ring \(\mathbb{R}\) is one such that \(a - 1\) is a nilpotent element. Any unipotent algebraic group is isomorphic to a closed subgroup of the group of upper triangular matrices with diagonal entries 1, and conversely any such subgroup is unipotent.

\(^6\)Horospheres previously appeared in hyperbolic geometry. They are spheres of infinite radius with centres at infinity and different from hyperbolic hyperplanes.

\(^7\)One verifies that \(D(q)D(q') = D(q + q')\) for all \(q, q' \in \mathbb{R}\).

\(^8\)See, e.g., \cite{190} Chapter 9.1.1, Equ. (11)].

\(^9\)These coordinates are frequently called \textit{Lemaître-Robinson} coordinates in the physics literature, see \cite{132}. In the mathematics literature they are called \textit{orispherical} coordinates, see, e.g., \cite{190} Chapter 9.1.5, Equ. (16)].
for the interior of the half-space $\Gamma^+(W_1)$. In particular,

\[
D \left( \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} e^{\frac{\alpha^2}{2r^2}} \\ e^{\frac{\alpha}{2r}} \\ 0 \end{bmatrix} \quad \text{for } \xi, \tau \in \mathbb{R}.
\]

The metric takes the form $g_{\Gamma^+(W_1)} = d\tau \otimes d\tau - e^{\frac{2\alpha}{r}} d\xi \otimes d\xi$.

### 2.2.2. Parabolas in $\Gamma^+(W_1)$

For $\tau$ fixed, $(2.2.3)$ parametrizes the horosphere (which actually is a parabola in $\mathbb{R}^{1+2}$)

\[
P_\tau = \{ x(\tau, \xi) | \xi \in \mathbb{R} \} \subset dS.
\]

General horospheres result from taking the intersection of $dS$ with a plane whose normal vector $p$ is light-like, i.e., $p \cdot p = 0$. In particular, the horosphere $P_\tau$ is given by

\[
P_\tau = \left\{ x \in dS | x \cdot \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = re^{\phi} \right\}.
\]

Thus general horospheres are of the form $R_0(\alpha)P_\tau$ with $\alpha \in [0, 2\pi]$ and $\tau \in \mathbb{R}$.

### 2.2.3. Horospheric distances

The proper time-difference—given by $(1.3.1)$—of the points $\Lambda_1(\frac{\tau_1}{r})o$ and $\Lambda_1(\frac{\tau_2}{r})o$ on the geodesic passing through the origin $o = (0, 0, r)$ is\(^{11}\)

\[
d(\Lambda_1(\frac{\tau_1}{r})o, \Lambda_1(\frac{\tau_2}{r})o) = r \text{arcosh} \left( - \left( \begin{bmatrix} \sinh \frac{\tau_1}{r} \\ 0 \\ \cosh \frac{\tau_1}{r} \end{bmatrix} \cdot \begin{bmatrix} \sinh \frac{\tau_2}{r} \\ 0 \\ \cosh \frac{\tau_2}{r} \end{bmatrix} \right) \right) = |\tau_1 - \tau_2|.
\]

In fact, $|\tau_1 - \tau_2|$ is the minimal distance of any two time-like points on the horospheres $P_{\tau_1}$ and $P_{\tau_2}$, respectively: if

\[
x = \left( r \sinh \frac{\tau_1}{r} + \frac{1}{2r} e^{\frac{\tau_1}{r}}, \xi e^{\frac{1}{2r}} - \frac{r e^{\frac{\tau_1}{r}}}{2r} \right)
\]

is a point in $P_{\tau_2}$, then the minimal distance to time-like points in the horosphere $P_{\tau_1}$, called the horospheric distance, is given by\(^{12}\)

\[
d(x, P_{\tau_1}) = r \text{arcosh} \left( \min_{y \in P_{\tau_1}} - \frac{x \cdot p}{r} \right) = |\tau_1 - \tau_2| = r \ln |e^{\phi} \cdot p(\frac{\tau_1}{r})|,
\]

with $p(t) = \left( \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-1} \end{bmatrix} \right) = \Lambda_1(t) \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$. Note that $P_{\tau} = \{ x \in dS | e^{\phi} \cdot p(\frac{\tau}{r}) = 1 \}$.

\(^{10}\)Note that the Lorentzian scalar product $x \cdot p, x \in \mathbb{R}^{1+2}, p \in \partial V^+$, equals the Euclidean scalar product of $x$ with $Pp \in \partial V^+$, with $P$ the space-reflection (see the final paragraph in Section 2. The plane defined by $x \cdot p = 0, x \in \mathbb{R}^{1+2}, p \in \partial V^+$ fixed, contains the point $x = p$.

\(^{11}\)Simply recall that $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$.

\(^{12}\)To show the second identity, one may use the invariance the Minkowski scalar product, i.e., $D(q) x \cdot D(q') y = D(q - q') x \cdot y$ for all $q, q' \in \mathbb{R}$ and $x, y \in \mathbb{R}^{1+2}$. 


2.3. The complex Lorentz group

The complex de Sitter group is defined as the group

$$O_C(1, 2) \doteq \{ \Lambda \in M_3(\mathbb{C}) | \Lambda g \Lambda^T = g \}.$$  

The elements in $M_3(\mathbb{C})$ are $3 \times 3$ matrices with complex entries and $g$ is the metric on Minkowski space $\mathbb{R}^{1+2}$ given in (1.1.4). The group $O_C(1, 2)$ has two connected components (distinguished by the sign of $\det \Lambda$, which takes the values $\det \Lambda = \pm 1$). Following standard terminology, we set

$$SO_C(1, 2) \doteq \{ \Lambda \in M_3(\mathbb{C}) | \Lambda g \Lambda^T = g, \ \det \Lambda = 1 \}.$$  

Note that $SO_C(1, 2)$ is isomorphic to $SO_C(3)$; the isomorphism from $SO_C(1, 2)$ to $SO_C(3)$ is given by the map

$$\Lambda \mapsto \begin{pmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Lambda \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

2.3.1. Analytic continuation. In $O_C(1, 2)$ the reflections

$$P_1 T \doteq \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P_2 T \doteq \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$  

are topologically connected to the identity $1$. In fact, the matrix-valued function $t \mapsto \Lambda_1(t)$ extends to an entire analytic function

$$\Lambda_1(t + i \theta) = \Lambda_1(t) \left[ \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos \theta \end{pmatrix} + i \begin{pmatrix} 0 & 0 & \sin \theta \\ 0 & 0 & 0 \\ \sin \theta & 0 & 0 \end{pmatrix} \right].$$  

The first matrix in the square brackets continuously deforms the unit $1$ to $P_1 T$, as $\theta$ takes values starting at $\theta = 0$ and ending at $\theta = \pm \pi$. The second matrix in the square brackets projects the wedge $W_1$ continuously into the $x_1 = 0$ section of the forward light cone, cf. [86].

2.4. The Cartan decomposition of $SO_0(1, 2)$

Since every point $\vec{x}$ of $H^+_m \cong SO_0(1, 2)/SO(2)$ has the form

$$\vec{x} = \begin{pmatrix} m \cosh \theta \\ -m \sinh \theta \sin \alpha \\ m \sinh \theta \cos \alpha \end{pmatrix},$$  

any element $g \in SO_0(1, 2)$ can be represented in the form

$$g = R_0(\alpha) \Lambda_1(t) R_0(\alpha'), \quad \alpha, \alpha' \in [0, 2\pi), \quad t \in \mathbb{R}.$$  

The corresponding decomposition

$$SO_0(1, 2) = KAK$$  

with $K = SO(2)$ and $A = SO(1, 1)$ is called the Cartan decomposition. Note that the decomposition (2.4.1) is not unique; see, e.g., [190] Chapter 9.1.5.
2.5. The Iwasawa decomposition of $\text{SO}_0(1, 2)$

A brief inspection shows that every point $x \in H^+$ can also be written in the form

$$x(t, q) = \text{D}(q)\Lambda_1(t) \begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix} = m \begin{pmatrix} \cosh t + \frac{q^2}{2} e^t \\ q e^t \\ \sinh t - \frac{q^2}{2} e^t \end{pmatrix}$$

(2.5.1)

Now, if $g \in \text{SO}_0(1, 2)$, then $g \begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix}$ is of the form (2.5.1) for some unique $t$ and $q$. It follows that this point can be carried back to the point $(m, 0, 0)$ by the action of $\Lambda_1(-t)D(-q)$, i.e.,

$$\Lambda_1(-t)D(-q) \begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix}$$

But the stabiliser of the point $\begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix}$ is the group $K \cong \text{SO}(2)$. Thus there exists some $\alpha \in [0, 2\pi)$ such that $R_0(-\alpha)\Lambda_1(-t)D(-q)g = 1$. Thus we arrive at the so-called Iwasawa decomposition:

**Lemma 2.5.1.** In case $G$ is the two-fold covering group$^{13}$ of $\text{SO}_0(1, 2)$, any element $g \in G$ can be written as

$$g = R_0(2\alpha)P^k\Lambda_1(t)D(q)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\alpha & -\sin 2\alpha \\ 0 & \sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & (-1)^k \end{pmatrix} \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}$$

$$\times \begin{pmatrix} 1 + \frac{q^2}{2} & q & \frac{q^2}{2} \\ q & 1 & q \\ -\frac{q^2}{2} & -q & 1 - \frac{q^2}{2} \end{pmatrix}, \quad \alpha \in [0, \pi), \ t, q \in \mathbb{R}, \ k \in \{0, 1\}.$$  

The resulting decomposition, $G = KAN$, provides

i.) a maximal compact subgroup $K$ (the two-fold covering group of $\text{SO}(2)$), consisting of the rotations $K/M \cong \text{SO}(2)$ and the group $M \cong \mathbb{Z}_2$ generated by the reflection $P$:

$$M = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} \cong \mathbb{Z}_2;$$

ii.) a Cartan maximal abelian subgroup $A \cong (\mathbb{R}, +)$, namely the boosts $\{\Lambda_1(t) | t \in \mathbb{R}\}$.

---

$^{13}$Consider a non-compact semi-simple Lie group $G$. If one chooses a maximal compact subgroup $K$, and a suitable abelian subgroup $A$, then there exists a nilpotent subgroup $N$, normalised by $A$, such that any group element $g \in G$ can be written uniquely as $kan$ with $k \in K$, $\alpha \in A$ and $n \in N$.

$^{15}$The two-fold covering group of $\text{SO}_0(1, 2)$ is $\text{SU}(1, 1)$. 
iii.) a nilpotent group \( N \cong (\mathbb{R}, +) \), namely the horospheric translations \( \{ D(q) \mid q \in \mathbb{R} \} \).

**Remark 2.5.2.** The subgroup \( M \) is the centralizer of \( A \), i.e.,

\[
M \triangleq \{ k \in K \mid ka = ak \ \forall a \in A \}.
\]

The group \( AN \) is a solvable subgroup and \( B = MAN \) is the minimal parabolic subgroup of \( G \).

### 2.6. The Hannabuss decomposition of \( SO_0(1, 2) \)

A decomposition, which is closely related to the Iwasawa decomposition, was discovered by Takahashi [183]:

**Lemma 2.6.1 (Hannabuss [88]).** Almost every element \( g \in SO_0(1, 2) \) can be written uniquely in the form of a product

\[
g = \Lambda_2(s)P^k \Lambda_1(t)D(q)
\]

\[
= \begin{pmatrix}
\cosh s & \sinh s & 0 \\
\sinh s & \cosh s & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & (-1)^k & 0 \\
0 & 0 & (-1)^k
\end{pmatrix}
\begin{pmatrix}
\cosh t & 0 & \sinh t \\
0 & 1 & 0 \\
\sinh t & 0 & \cosh t
\end{pmatrix}
\begin{pmatrix}
1 + \frac{q^2}{2} & q & -\frac{q^2}{2} \\
q & 1 & -q \\
\frac{q^2}{2} & r & 1 - \frac{q^2}{2}
\end{pmatrix},
\]

with \( s, t, q \in \mathbb{R} \) and \( k = \{0, 1\} \), i.e., almost every element \( g \in SO_0(1, 2) \) can be decomposed into a product, which consists of a Lorentz transformation \( s \mapsto \Lambda_2(s) \), possibly a reflection \( P^k \), a time translation \( t \mapsto \Lambda_1(t) \), and a spatial translation \( q \mapsto D(q) \).

**Proof.** Let \( g \in G \) be given in its Iwasawa decomposition, i.e.,

\[
g = R_0(\alpha)\Lambda_1(t)D(q), \quad \alpha \in [0, 2\pi), \quad t, q \in \mathbb{R}.
\]

We will show that, unless \( \alpha = \frac{\pi}{2} \) or \( \frac{3\pi}{2} \), \( R_0(\alpha) \) can be written in the form

\[
R_0(\alpha) = \Lambda_2(s)P^k D(-q')(\Lambda_1(-t')), \quad s, t', q' \in \mathbb{R}, \quad k = 0, 1.
\]

---

16 A nilpotent group is one that has a central series of finite length. Any unipotent group is a nilpotent group, though the diagonal matrices are only nilpotent in \( GL(n) \).

17 We recall that a normal series of a group \( G \) is a finite sequence \( A_1, \ldots, A_N \) of subgroups such that \( 1 \leq A_1 \subseteq \ldots \subseteq A_N \leq G \). A normal factor of \( G \) is a quotient group \( A_{k+1}/A_k \) for some index \( k < N \). The group \( G \) is a solvable group if all normal factors are abelian. Any maximal solvable subgroup \( B \subseteq G \) is called a Borel subgroup. (A maximal subgroup \( B \) of a group \( G \) is a proper subgroup, such that no proper subgroup \( K \) contains \( B \) strictly.) Given the Iwasawa decomposition, a parabolic subgroup of \( G \) is a closed subgroup containing some conjugate of \( MAN \); the conjugates of \( MAN \) are called minimal parabolic subgroups (see [125 Chapter V5]). In the case of the group \( SL(2, \mathbb{R}) \), there is up to conjugacy only one proper parabolic subgroup, the Borel subgroup of the upper-triangular matrices of determinant 1. The Iwasawa decomposition shows that it is enough to conjugate to \( K \).

18 Its relevance in the present context was first emphasised by Hannabuss [88].
Taking (2.2.2) into account, this will imply that
\[ g = \Lambda_2(s)P^k\Lambda_1(t''-t')D(q''-e^{-(t''-t')}q') \, . \]

Thus it remains to establish (2.6.1). Multiplying (2.6.1) with \( \Lambda_1(t')D(q') \) from the right yields
\[ \Lambda_2(s)P^k = R_0(\alpha)\Lambda_1(t')D(q'), \quad s \in \mathbb{R}, \quad k = 0, 1. \]

This is the Iwasawa decomposition of \( \Lambda_2(s)P^k, k = 0, 1 \), which is given by choosing
\[ \cosh s = (-1)^k \cos^{-1} \alpha, \quad e^{t'} = \frac{1}{\cos \alpha}, \quad e^{q'} = (-1)^{k+1} \sin \alpha. \]

In fact, unless \( \cos \alpha = 0 \),
\[
\begin{pmatrix}
\frac{1}{\cos \alpha} & -\frac{\sin \alpha}{\cos \alpha} & 0 \\
-\frac{\sin \alpha}{\cos \alpha} & \frac{1}{\cos \alpha} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & (-1)^k & 0 \\
0 & 0 & (-1)^k
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\cos \alpha} & 0 & 0 \\
0 & \frac{1}{\cos \alpha} & 0 \\
0 & 0 & \frac{1}{\cos \alpha}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\cos \alpha} & 0 \\
0 & 0 & \frac{1}{\cos \alpha}
\end{pmatrix}
\begin{pmatrix}
1 + \frac{\sin^2 \alpha}{2} & (-1)^k \sin \alpha & \frac{1}{2} \sin^2 \alpha \\
(-1)^k \sin \alpha & 1 & (-1)^k \sin \alpha \\
\frac{1}{2} \sin^2 \alpha & (-1)^k \sin \alpha & 1 - \frac{\sin^2 \alpha}{2}
\end{pmatrix},
\]

with
\[ k = \begin{cases} 
0 & \text{if } \cos \alpha > 0, \\
1 & \text{if } \cos \alpha < 0. 
\end{cases} \]

The exceptional group elements, which can not be represented in this form, contain the rotations \( R_0(\pm \frac{\pi}{2}) \) in their Iwasawa decomposition. \( \square \)

The resulting decomposition of \( G \) is of the form
\[ G = K'AN, \quad K' = \langle \Lambda_2(s), \Lambda_2(s)P \ | \ s \in \mathbb{R} \rangle. \]

The spatial reflection \( P \) is needed to account for elements whose Iwasawa decomposition contains a rotation \( R_0(\alpha) \), with \( \frac{\pi}{2} < \alpha < \frac{3\pi}{2} \).

### 2.7. Homogeneous spaces, cosets and orbits

Consider a closed subgroup \( H \) of a topological group \( G \). Let \( \Pi : G \to G/H \) denote the canonical mapping defined by
\[ \Pi(g) = gH. \]

We equip \( G/H \) with the quotient topology, i.e., a set \( O \subset G/H \) is open if \( \Pi^{-1}(O) \subset G \) is open. By construction, \( G \) acts transitively on \( G/H \):
\[ g\Pi(g') = \Pi(gg'). \]
We note that (locally) there is a continuous section \( \Xi : G/H \to G \), which satisfies \( \Pi \circ \Xi = \text{id} \).

**Lemma 2.7.1.** Let \( G \equiv \text{SU}(1, 1) \) be the two-fold covering group of \( \text{SO}_0(1, 2) \). Furthermore, let \( H \subset G \) be the stabiliser of an arbitrary point \( x \in \mathbb{R}^{1+2} \).

Then there exists a bijective map \( \Gamma : G/H \to X \),
\[
gH \mapsto gx, \quad g \in G,
\]
from the homogeneous space \( G/H = \{gH \mid g \in G\} \) to the orbit \( X = \{gx \in \mathbb{R}^{1+2} \mid g \in G\} \) such that
\[
(2.7.1) \quad \Gamma(gg'H) = g\Gamma(g'H) \quad \forall g, g' \in G;
\]
i.e., \( g(g'x) = (g \circ g')x \) for all \( g, g' \in G \).

**Proof.** One easily verifies that \( \Gamma \) is well-defined: if \( g_1H = g_2H \), then \( g_1 = g_2h \) for some \( h \in H \), and since the \( H \) leaves the point \( x \) invariant, the map is well-defined. On the other hand, if
\[
g_1x = g_2x,
\]
then \( g_2^{-1}g_1 \) fixes \( x \) and thus must be in \( H \). This implies \( g_1H = g_2H \). Thus the map \( \Gamma : G/H \to X \) is bijective. By construction, it satisfies (2.7.1).

In the following, we concentrate on the cases where \( H \) is the stabiliser (within \( \text{SO}_0(1, 2) \)) of a point \( x \in \mathbb{R}^{1+2} \).

**2.7.1. The forward light-cone.** Let us first consider the case \( x = \left( \frac{1}{0} \right) \). According to Lemma 2.2.1, \( x = D(q)x \) for all \( q \in \mathbb{R} \). Since the group \( \{D(q) \mid q \in \mathbb{R}\} \) is nilpotent, it is usually denoted by the letter \( N \). Clearly, the point \( x \) is also invariant under the reflection \( P \), which generates the subgroup \( M \). Thus the stabiliser of \( x \) in the two-fold covering group of \( \text{SO}_0(1, 2) \) is \( H = MN \). Now recall that the map
\[
(\alpha, t) \mapsto R_0(\alpha)A_1(t) \left( \frac{1}{0} \right) = R_0(\alpha) \left( \begin{smallmatrix} 1 & 0 \\ 0 & e^{-t} \end{smallmatrix} \right) \in \partial V^+, \quad t \in \mathbb{R}, \quad \alpha \in [0, 2\pi),
\]
provides coordinates for \( \partial V^+ \setminus \{(0, 0, 0)\} \). Note that \( A_1(t) \) leaves the light ray connecting the origin \( (0, 0, 0) \) and the point \( (1, 0, -1) \) invariant. Using Lemma 2.5.1, the canonical mapping \( \Pi : G \to G/MN \) is given by
\[
g \mapsto R_0(\alpha)A_1(t)MN, \quad \text{with} \quad g = R_0(\alpha)p^kA_1(t)D(q).
\]

**Lemma 2.7.2.** The homogeneous space \( G/MN \equiv \{gMN \mid g \in G\} \) can be naturally identified with \( \partial V^+ \setminus \{(0, 0, 0)\} \) by setting
\[
\Gamma(gMN) \equiv g \left( \frac{1}{0} \right), \quad g \in G.
\]
Moreover, \( g(g'x) = (g \circ g')x \) for all \( g, g' \in G \) and \( x \in \partial V^+ \setminus \{(0, 0, 0)\} \).
2.7.2. The mass hyperboloid. Next consider the point \( x = \left( \begin{array}{c} m \\ 0 \\ 0 \end{array} \right) \). Clearly, \( R_0(\alpha)x = x \) for all \( \alpha \in [0, 2\pi) \). In other words, the stabiliser of \( x \) is \( K \).

**Lemma 2.7.3.** The coset space \( SO_0(1, 2)/K \) can be naturally identified with a two-fold covering of \( H_m^+ \) by setting
\[
\Xi(gK) = g \left( \begin{array}{c} m \\ 0 \\ 0 \end{array} \right), \quad g \in SO_0(1, 2).
\]

**Proof.** The rotations \( K \) are the stabiliser of the point \( \left( \begin{array}{c} m \\ 0 \\ 0 \end{array} \right) \) in \( SO_0(1, 2) \). Note that
\[
\Lambda_1(t)x = \left( \begin{array}{c} m \cosh t \\ 0 \\ m \sinh t \end{array} \right) \in \partial V^+, \quad t \in \mathbb{R}.
\]
Applying the rotations \( R_0(\alpha), \alpha \in [0, 2\pi) \), to \( \Lambda_1(t)x \) results in a two-fold covering of the mass hyperboloid \( H_m^+ \). Thus the result follows from Lemma 2.7.1. \( \square \)

Clearly, this result gives rise to the Cartan decomposition, see (2.4.1).

2.7.3. De Sitter space. Finally, consider the case \( x = 0 \). Clearly, the boosts \( A'(t), t \in \mathbb{R} \), form the stabiliser \( A' \) of the origin \( 0 \). Moreover, the map
\[
(2.7.2) \quad x(\alpha, t) = R_0(\alpha)\Lambda_1(t)0 = R_0(\alpha) \left( \begin{array}{c} r \sinh t \\ 0 \\ r \cosh t \end{array} \right),
\]
with \( \alpha \in [0, 2\pi) \) and \( t \in \mathbb{R} \), provides coordinates \(^{19}\) for \( dS \).

**Lemma 2.7.4.** The coset space \( G/A' \) can be naturally identified with \( dS \), by setting
\[
\Lambda(gA') = g \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right), \quad g \in G.
\]
Moreover, \( g(g'x) = (g \circ g')x \) for all \( g, g' \in G \) and \( x \in dS \).

2.7.4. Circles. We next consider the choice \( H = MAN \). Clearly, \( H \) leaves the light ray
\[
\{ \lambda \left( \begin{array}{c} 1 \\ \lambda \end{array} \right) \mid \lambda > 0 \}
\]
passing through the origin and the point \( \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \) invariant. It is therefore natural to identify the factor group \( SO(2) \cong G/MAN \) with the projective space
\[
\partial V^+ = \{ \hat{y} \mid y \in \partial V^+ \},
\]
formed by the light rays \( \hat{y} = \{ \lambda y \mid \lambda > 0 \}, y \in \partial V^+ \). Each light ray in \( \partial V^+ \) intersects the circle
\[
(2.7.3) \quad \Gamma_0 \cong \{ R_0(\alpha) \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \mid \alpha \in [0, 2\pi) \}
\]
just once. However, it should be emphasised that the boosts in \( A = \{ \Lambda_1(t) \mid t \in \mathbb{R} \} \) do not leave the point \( \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \) invariant.

---

\(^{19}\)This should be compared with the chart introduced in (2.1.4), which only covers \( \mathbb{H}_1 \).
2.7.5. Mass shells. Using the Hannabuss decomposition of \( SO_0(1,2) \), almost every element \( g \) in \( SO_0(1,2) \) can be written in the form

\[
g = \Lambda_2(s) p^j \Lambda_1(t) D(q), \quad j \in \{0,1\}.
\]

Thus almost all of the cosets \( g \text{MAN}, \ g \in G \), (or light rays in \( \partial \dot{V}^+ \)) are in one-to-one correspondence to points in the two hyperbolas

\[
\{ \Lambda_2(s) \left( \begin{array}{c} \frac{m}{0} \\ \frac{0}{m} \end{array} \right) \ | \ s \in \mathbb{R} \} \cup \{ \Lambda_2(s) P \left( \begin{array}{c} \frac{m}{0} \\ \frac{0}{m} \end{array} \right) \ | \ s \in \mathbb{R} \}.
\]

Note that \( P \Lambda_2(s) P = \Lambda_2(-s) \) for all \( s \in \mathbb{R} \).

Remark 2.7.5. It is convenient to choose a parametrization such that

\[
m \cosh s = \sqrt{p_1^2 + m^2}, \quad m \sinh s = p_1.
\]

Then, using \( s = \arcsinh \frac{p_1}{m} \), the measure is \( \text{d}s = \frac{dp_1}{\sqrt{p_1^2 + m^2}} \) and \( \Lambda_2(s) \) is of the form

\[
\Lambda_2(s) = \left( \begin{array}{ccc}
\frac{\sqrt{p_1^2 + m^2}}{m} & \frac{p_1}{m} & 0 \\
\frac{p_1}{m} & \frac{\sqrt{p_1^2 + m^2}}{m} & 0 \\
0 & 0 & 1
\end{array} \right).
\]

2.8. Invariant measures

2.8.1. Haar measure. Using the Cartan decomposition, the Haar measure can be decomposed as [190] Chapter 9

\[
dg = d\alpha \sinh t \ dt \ d\alpha', \quad g = R(\alpha) \Lambda_1(t) R(\alpha'),
\]

with \( \alpha, \alpha' \in [0,2\pi) \) and \( t \in \mathbb{R} \).

On the other hand, using the Iwasawa decomposition, the Haar measure on \( SO_0(1,2) \) can be written as

\[
dg = \frac{d\alpha}{2\pi} e^{t} dt \ dq, \quad g = R_0(\alpha) p^k \Lambda_1(t) D(q),
\]

with \( \alpha \in [0,2\pi) \), \( t, q \in \mathbb{R} \) and \( k \in \{0,1\} \).

2.8.2. The invariant measure on the mass hyperboloid. The restriction of the measure (2.8.1) to the mass hyperboloid \( H^+_{m_0} \) equals \( d\alpha \sinh t dt \). The latter equals twice the measure

\[
\int d^3p \ \theta(p_0) \delta(p_0^2 - p_1^2 - p_2^2 - m^2) = \int p dp \ d\alpha \ dp_0 \ \theta(p_0) \frac{\delta(p - \sqrt{p_0^2 - m^2})}{2\sqrt{p_0^2 - m^2}}
\]

used by Bros and Moschella in [28]. This can be seen by setting \( p_0 = m \cosh t \), which implies \( dp_0 = m \sinh t dt \). In the last line we have changed coordinates, setting \( p_1 = \rho \sin \alpha \) and \( p_2 = \rho \cos \alpha \).
2.8.3. The invariant measure on the one sheeted hyperboloid. The measure used by Bros and Moschella in [28],
\[
\int d^3 x \delta(x_0^2 - x_1^2 - x_2^2 + r^2) = \int \rho d\rho \, d\psi \, dx_0 \frac{\delta(\rho - \sqrt{x_0^2 + r^2})}{2\sqrt{x_0^2 + r^2}} = \frac{1}{2} \int dS \, r \, d\psi \, dx_0
\]
differs from the measure we will use, namely
\[
d\mu_{ds} \doteq dx_0 \, r \, d\psi
\]
by a factor two. Taking (2.7.2) into account we find
\[
d\mu_{ds} = r^2 \cosh t \, dt \, d\psi.
\]

2.8.4. The invariant measure on the forward light cone. Setting \( p_0 = e^{-t} \), we find \( dp_0 = e^{-t} \, dt \) and, consequently, the invariant measure on the forward light cone
\[
\partial V^+ \setminus \{(0,0,0)\} = \left\{ \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \middle| p_0 > 0, \alpha \in [0, 2\pi) \right\} \\
\cong \{g^{\mathbb{M}} \middle| g \in G\}
\]
is given — up to normalisation — by the formula [190, Chapter 9.1.9, Equ. (13)]
\[
|p_0|^{-1} dp_1 dp_2 = dp_0 d\alpha,
\]
in agreement with taking the limit \( m \to 0 \) in (2.8.3). There one finds \( \frac{1}{2} d\alpha \, dp_0 \).

2.8.5. Measures on contours on the light-cone. Let \( \Gamma \) be a contour on the light cone which intersects every light ray of the light cone at one point. Following [190, Chapter 9.1.9], we denote by \( d\mu_\Gamma \) a measure on this contour such that
\[
|p_0|^{-1} dp_1 dp_2 = d\lambda d\mu_\Gamma (\eta), \quad \eta \in \Gamma,
\]
where \( p = \lambda \eta, \lambda > 0, \eta \in \Gamma \). It follows that if a function \( f(p) \) on \( \partial V^+ \) is homogeneous of degree \(-1\), that is \( f(\lambda p) = \lambda^{-1} f(p), \lambda > 0 \), then the integral
\[
\int_\Gamma d\mu_\Gamma (\eta) f(\eta)
\]
does not depend on the choice of \( \Gamma \); in agreement with [29, Proposition 10].
Part 2

Harmonic Analysis
CHAPTER 3

Induced Representations for the Lorentz Group

Assume we are given a quantum theory, formulated in terms of operators (which play the role of the observables) and normalised, positive linear functionals, i.e., states (which play the role of physical expectation values). Then a symmetry operation is realised by a so-called Wigner automorphism \[197\]. Given an invariant state \(\omega\), any such automorphism can be implemented by either a unitary or an anti-unitary operator in the GNS Hilbert space \(\mathcal{H}_\omega\). These operators are unique up to a factor. Thus, given a group of symmetries, one is led to study its projective representations. The latter are in one-to-one correspondence with the unitary representations of the universal covering group \(\widetilde{G}\). Unitary irreducible representations of the Lorentz group \(SO_0(1, 2)\) (and its two-fold covering group) have first\(^2\) been constructed by Bargmann \[12\], using multipliers. The latter had appeared in Schur’s theory of projective representations. The approach we will follow here was pioneered by Wigner \[196\] and Mackey \[140\], based on earlier work by Frobenius (see, e.g., \[187\]).

3.1. The general case

We briefly recall some key elements of the general theory of induced representations, following \[59\] (see also \[30, 65, 128, 129, 137, 140, 187, 190, 191, 192\]).

3.1.1. Modular functions. Let \(G\) be a locally compact topological group \(G\). Then \(G\) has a left invariant Haar measure \(\mu_G\) on the \(\sigma\)-algebra of Borel sets, which is unique up to a normalisation factor; see, e.g., \[59\], Theorems (2.10) and (2.20)]. In general, the left Haar measure \(\mu_G\) on \(G\) is not equal to the right-invariant Haar measure. However, there always exists a multiplicative \(\mathbb{R}^+\)-valued function \(\Delta_G\) on \(G\), called the modular function of \(G\), such that

\[
\int_G d\mu_G(g) f(gg') = \frac{1}{\Delta_G(g')} \int_G d\mu_G(g) f(g) \quad \forall g' \in G
\]

and for every \(\mu_G\)-integrable function \(f\) on \(G\). The modular function relates the left and the right invariant Haar measure:

\[
\int_G d\mu_G(g) f(g^{-1}) = \int_G d\mu_G(g) f(g)\Delta_G(g^{-1}) .
\]

\(^2\)The case \(SO_0(1, 2)\) is somewhat exceptional, as the universal covering group does not coincide with the two-fold covering group. However, for simplicity, here we deal only with the latter. These representations describing anyons will be discussed elsewhere.

\(^2\)At the same time, Gelfand and Naimark investigated the group \(SL(2, \mathbb{C})\) \[64\].
In case $\Delta_G(g) = 1$ for all $g \in G$, the left and the right Haar measure coincide, and $G$ is called unimodular.

**Lemma 3.1.1.** The twofold covering group of $SO_0(1, 2)$ is unimodular.

**Proof.** The two-fold covering group of $SO_0(1, 2)$ is isomorphic to $SL(2, \mathbb{R})$. For $SL(2, \mathbb{R})$, consider the action $\Delta : sl(2, \mathbb{R}) \to \mathbb{R}$ of the character $\Delta$ (given by the modular function) on the Lie algebra of $SL(2, \mathbb{R})$. Since $\mathbb{R}$ is abelian, $\Delta([X, Y]) = 0$ for all $X, Y \in sl(2, \mathbb{R})$. But every element of $sl(2, \mathbb{R})$ is of this form, hence $\Delta_* = 0$. Since $SL(2, \mathbb{R})$ is connected this determines $\Delta_G(g) = 1$ for all $g \in G$. \hfill \Box

**Remark 3.1.2.** Consider the group $AN = \{\Lambda(t)D(q) \mid t, q \in \mathbb{R}\}$. We compute (using (2.2.2))

$$
\int dt dq \ f(\Lambda(t)D(q)\Lambda(t')D(q')) = \int dt dq \ f(\Lambda(t + t')D(e^{-t'}q + q'))
$$

$$
= \int dt'' dq'' e^{-t'} \ f(\Lambda(t'')D(q'')) ;
$$

i.e., $\Delta_AN(\Lambda(t')D(q')) = e^{-t'}$. Thus AN is not unimodular.

**3.1.2. Invariant measures on the quotient space.** Let $H$ be a closed subgroup of a locally compact group $G$. Moreover, let $\Delta_G$ and $\Delta_H$ denote the modular functions of $G$ and $H$, respectively.

**Theorem 3.1.3 (Theorem 2.49, [59]).** The quotient space $G/H$ admits a nonzero positive $G$-invariant measure $\mu_{G/H}$, if and only if

\begin{equation}
\Delta_G \mid H = \Delta_H .
\end{equation}

If (3.1.1) holds, then the positive invariant measure $\mu_{G/H}$ is unique (up to multiplication by a positive constant). Moreover, one can normalize the invariant measure $\mu_{G/H}$ on $G/H$ such that for every $f$ in $C_0(G)$,

\begin{equation}
\int_{G/H} d\mu_{G/H}(gH) \int_H d\mu_H(h) \ f(gh) = \int_G d\mu_G(g) \ f(g) ,
\end{equation}

where $\mu_G$ and $\mu_H$ denote the Haar measures of $G$ and $H$, respectively.

**Remark 3.1.4.** One can define a linear map $C_0(G) \to C_0(G/H)$ by setting

$$
f^{H}(gh) = \int_H d\mu_H(h) \ f(gh) , \quad f \in C_0(G) .
$$

The identity (3.1.2) then takes the form

\begin{equation}
\int_{G/H} d\mu_{G/H}(gH) f^{H}(gH) = \int_G d\mu_G(g) \ f(g) .
\end{equation}

**Lemma 3.1.5.** Let $H \subseteq G$ be compact, then $\Delta_G \mid H = 1$. In particular, if $G$ is compact, then $G$ is unimodular.

**Proof.** As $\Delta_G$ is continuous, it follows that $\Delta_G(H)$ is a compact subgroup of $\mathbb{R}^+$ and hence equal to $\{1\}$. \hfill \Box
3.1.3. **Quasi-invariant measures.** In the case we are interested in, the condition (3.1.1) is not satisfied. Consequently, there is no $G$-invariant measure on $G/H$.

**Definition 3.1.6.** A regular Borel measure $\mu$ on $G/H$ is called

i.) *quasi-invariant*, if the measure $\mu$ and the measure

$$\mu^g(.):=\mu(g. )$$

are mutually absolutely continuous for all $g \in G$;

ii.) *strongly quasi-invariant*, if there exists a continuous $\mathbb{R}^+$-valued function $\lambda_g(g'H)$ on $G \times G/H$ such that

$$d\mu^g(g'H) = \lambda_g(g'H) d\mu(g'H) \quad \forall g \in G , \quad \forall g'H \in G/H ,$$

*i.e.,* $\lambda_g$ is the Radon-Nikodym derivative

$$\lambda_g(g'H) = \frac{d\mu^g}{d\mu}(g'H), \quad g, g' \in G .$$

Quasi-invariant measures send null sets into null sets under the action of $G$. Strongly quasi-invariant measures on $G/H$ are closely related to rho-functions on $G$.

**Definition 3.1.7.** A real-valued function $\rho$ on $G$ is a *rho-function* for $(G, H)$, if it is positive, continuous, and satisfies

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)} \rho(g)$$

for all $g \in G$ and $h \in H$.

**Remark 3.1.8.** Let $G = SO(1, 2)$ and $AN = \{A_1(t)D(q) \mid t, q \in \mathbb{R}\}$. It follows that $\Delta_G(h) = 1$ for all $h \in AN$. Consequently,

$$\rho(h) = \Delta_{AN}(h) \rho(1) \quad \forall h \in AN .$$

Here $1$ denotes the identity in $G$. We have already seen (in Remark 3.1.2) that

$$\Delta_{AN}(A_1(t)D(q)) = e^{-t}, \quad t, q \in \mathbb{R} .$$

Thus (possibly up to a constant) $\rho(A_1(t)D(q)) = e^{-t}$.

**Theorem 3.1.9 (see Theorem (2.56), [59]).** Let $\rho$ be a rho-function for $(G, H)$. It follows that

i.) *there exists a strongly quasi-invariant measure* $\mu$ on $G/H$ such that

$$\int_{G/H} d\mu(gH) f^\mu(gH) = \int_G d\mu_G(g) \rho(g)f(g) \quad \forall f \in C_0(G) ;$$

ii.) *the measures* $\mu^g$ and $\mu$ *are absolutely continuous to each other*;

iii.) *the Radon-Nikodym derivative is given by*

$$\lambda_g(g'H) = \frac{\rho(gg')}{\rho(g')} , \quad g, g' \in G ;$$

iv.) *the Radon-Nikodym derivative satisfies the cocycle relation*

$$\lambda_{g_1 g_2}(gH) = \lambda_{g_1}(g_2 gH)\lambda_{g_2}(gH) \quad \forall g_1, g_2, g \in G .$$
REMARKS 3.1.10.

i.) Clearly, (3.1.5) should be compared with (3.1.3).

ii.) In case $g'H = R_0(\alpha')\text{MAN}$, Remark 3.1.8 implies that $\rho(R_0(\alpha')) = 1$. Thus (3.1.4) implies that

$$\lambda_g(g'H) = \rho(g).$$

In other words,

$$\lambda_{R_0(\alpha)A_1(t)D(q)}(g'H) = e^{-t}$$

for all $\alpha \in [0, 2\pi)$ and $t, q \in \mathbb{R}$.

3.1.4. **Induced representations.** Let $G$ be a locally compact group, $H$ a closed subgroup, $\Gamma: G \to G/H$ the canonical quotient map, and $\pi: H \to \mathcal{B}(\mathcal{H})$ a representation of $H$ on some Hilbert space $\mathcal{H}$. Denote the norm and the inner product on $\mathcal{H}$ by $\|u\|_\mathcal{H}$ and $\langle u, v \rangle_\mathcal{H}$, and denote by $C(G, \mathcal{H})$ the space of continuous functions from $G$ to $\mathcal{H}$. Now consider the following space of vector valued functions $F_0$:

$$F_0 = \{f \in C(G, \mathcal{H}) \mid \Gamma(\text{supp } f) \text{ is compact}, \ f(gh) = \pi(h^{-1})f(g) \text{ for } h \in H, g \in G \}. $$

Note that if $\pi$ is unitary and $f \in F_0$, then $\|f\|_\mathcal{H}$ depends only on the equivalence classes $gH$, $g \in G$.

**DEFINITION 3.1.11.** Let $\mu$ be a strongly quasi-invariant measure on $G/H$.

i.) In case $\pi$ is unitary, define an *inner product* on $F_0$ by setting

$$\langle f, f' \rangle_\mu = \int_{G/H} d\mu(gH) \langle f(g), f'(g) \rangle_\mathcal{H}, \quad f, f' \in F_0.$$ (3.1.8)

ii.) In case $\pi$ is a non-unitary character of $H$, the scalar product (3.1.8) gives rise to a bounded (non-unitary) representation of $G$ on the completion of $F_0$. However, it may still be a possibility to find a new *inner product* on $F_0$ with respect to which the induced representation becomes unitary. If that is possible, we refer to the resulting unitary representation on the completion of $F_0$ as the complementary series representation (see, e.g., [137], p. 32).

In both cases, denote the completing of $F_0$ w.r.t. the norm $\|f\|_\mu \doteq \sqrt{\langle f, f \rangle_\mu}$ by $\mathcal{F}_\mu$.

The *induced representation* $\Pi_\mu(g)$ on the Hilbert space $\mathcal{F}_\mu$ is specified by setting

$$\langle \Pi_\mu(g)f, g' \rangle_\mathcal{F}_\mu = \lambda_g(g'H)f(g^{-1}g'), \quad g, g' \in G,$$ (3.1.9)

with $\lambda_g$ the Radon-Nikodym derivative defined in (3.1.4).

**REMARK 3.1.12.** Note that while $\Pi_\mu$ depends on $\mu$, its unitary equivalence class depends only on $\pi$.

The induced representation is equivalent to a representation on $C_0(G/H, \mathcal{H})$, as follows. Let $\Xi: G/H \to G$ be a smooth global section. Then $\mathcal{F}_0$ is, as a linear space,
isomorphic to \( C_0(G/H, \tilde{\mathcal{S}}) \) by identifying \( f \in \mathcal{F}_0 \) with \( \tilde{f} \in C_0(G/H, \tilde{\mathcal{S}}) \) defined by

\[
(3.1.10) \quad \tilde{f}(p) = f(\Xi(p)), \quad p \in G/H.
\]

The scalar product \( \| \|_2 \) in \( \mathcal{F}_0 \) goes over, under this equivalence, into the \( L^2 \)-product in \( C_0(G/H; \tilde{\mathcal{S}}) \):

\[
(3.1.11) \quad \| f \|^2 = \| \tilde{f} \|^2_{L^2(G/H; \tilde{\mathcal{S}})} = \int_{G/H} d\mu(gH) \| f(gH) \|^2_{\mathcal{S}}.
\]

Further, for \( g \in G \) and \( p \in G/H \), the group elements \( g^{-1}\Xi(p) \) and \( \Xi(g^{-1} \cdot p) \) differ by an element in \( H \), the so-called Wigner rotation \( \Omega(g, p) \):

\[
(3.1.12) \quad g^{-1}\Xi(p) = \Xi(g^{-1} \cdot p) \Omega(g, p)^{-1}, \quad \Omega(g, p) \equiv \Xi(p) \Xi(g^{-1} \cdot p) \in H.
\]

Using this fact, the induced representation \( \pi \) on \( \mathcal{F}_0 \) is equivalent, via the isomorphism \( (3.1.10) \), to the representation \( \tilde{\Pi}_\mu \) defined on \( C_0(G/H, \tilde{\mathcal{S}}) \) by

\[
(3.1.13) \quad (\tilde{\Pi}_\mu(g)h)(p) = \lambda_g(\Xi(p)H)^{\frac{1}{2}} \pi(\Omega(g, p)) h(g^{-1} \cdot p).
\]

As the isomorphism \( (3.1.10) \) intertwines the respective scalar products, an (anti-) unitary operator in \( \mathcal{F}_0 \) goes over into an (anti-) unitary operator in \( L^2(G/H; \tilde{\mathcal{S}}) \). Thus, if the representation \( \pi \) of \( H \) in \( \tilde{\mathcal{S}} \) is unitary, then the representation \( \tilde{\Pi}_\mu \) is unitary in \( L^2(G/H; \tilde{\mathcal{S}}) \).

### 3.2. Induced representations for \( SO_0(1, 2) \)

A representation \( \pi_\nu^\pm : MAN \to C \) of the closed solvable subgroup \( MAN \) of the two-fold covering group of \( SO_0(1, 2) \) on \( C \) is defined by lifting a character \( \chi_\nu \) of \( A \) to \( AN \), and taking its product with a representation of \( M \): set \( \pi_\nu^\pm = (\sigma_\pm \otimes \chi_\nu \otimes 1) \), where

\[
(3.2.1) \quad \sigma_\pm \otimes \chi_\nu \otimes 1 : \text{man} \mapsto \chi_\nu(a) \sigma_\pm(m),
\]

with \( \sigma_\pm(1_3) = 1 \),

\[
\sigma_\pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \pm 1, \quad \text{and} \quad \chi_\nu \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} = e^{ivt}.
\]

Thus

\[
\pi_\nu^\pm(p^kA_1(t)D(q)) = (\pm 1)^k e^{ivt}.
\]

The induced representation is initially defined on the space\( ^5 \) (see Definition \( 3.1.11) \)

\[
\mathcal{H}_{\nu,0}^\pm = \{ f \in C(G, C) \mid \mathcal{F}(\text{supp} f) \text{ is compact}, f(gh) = \pi_\nu^\pm(h^{-1}) f(g) \text{ for } h \in MAN \}.
\]

This definition implies that

\[
^5 \text{The inverse map } h \in C_0(G/H, \tilde{\mathcal{S}}) \mapsto \tilde{h} \in \mathcal{F}_0 \text{ is given by}
\]

\[
\tilde{h}(g) \equiv \pi(g^{-1}\Xi(p)) h(p), \quad p \equiv \Pi(g).
\]

\[
^4 \text{We denote the action of } G \text{ in } G/H \text{ by a dot, } g \cdot (gH) \equiv (gg')H.
\]

\[
^5 \text{As we have seen in Section } 3.1.1 \text{ the cosets } \{ gMAN \in g \in G \} \text{ can be identified either with a circle on the forward light cone (using the Iwasawa decomposition of the group) or with a pair of mass-hyperbolas on the forward light cone (using the Hannabus decomposition of } SO_0(1, 2)).
\]
We will explore these facts further in the next subsection.

\[ (3.2.2) \quad f(gmn) = f(g) \quad \forall g \in G, \forall m \in M, \forall n \in N; \]

\[ (3.2.3) \quad \text{if } f \text{ is a function in } h^+_\nu \text{ or } h^-_{\nu,0}, \text{ then} \]
\[ f(gA_1(t)D(q)) = p_0^{\nu}f(g), \quad \text{with } p_0 = e^{-t} > 0; \]

\[ (3.2.4) \quad \text{in case } \nu \in \mathbb{R}, \text{ the representation } \pi^+_\nu \text{ is unitary;} \]
\[ (3.2.5) \quad \text{in case } \nu \text{ is purely imaginary, the representation } (3.2.4) \text{ is no longer a unitary representation of MAN in } \mathbb{C}. \]

We will explore these facts further in the next subsection.

Let us denote by \( \Pi^+_{\nu} \) the representation of \( G \) induced from the representation \( \pi^+_\nu \) of the closed subgroup MAN,

\[ (3.2.4) \quad (\Pi^+_{\nu}(g)f)(g') = \sqrt{\Lambda_\nu(g'MAN)} f(g^{-1}g'), \quad f \in h^+_{\nu,0}. \]

**Remark 3.2.1.** To compute explicit expressions for the representation \( (3.2.4) \) (for specific choices of \( g \in G \)) one can take advantage of \( (3.2.2) \). According to Lemma 2.7.2, the map
\[ \Gamma(gMN) = g \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g \in G, \]
defines a bijection, which identifies the homogeneous space
\[ \{gMN \mid g \in SO_0(1,2)\} = \{R_0(\alpha)A_1(t)MN \mid \alpha \in [0,2\pi), t \in \mathbb{R}\} \]
with the forward light cone
\[ (3.2.5) \quad \partial V^+ = \{ (\alpha, e^{-t}) \in S^1 \times \mathbb{R}^+ \mid \alpha \in [0,2\pi), t \in \mathbb{R} \} \].

Setting \( p_0 = e^{-t} \), the action of \( SO_0(1,2) \) on the forward light cone \( \partial V^+ \) is given by \( (2.1.2) \), i.e.,
\[ \Lambda_2(s)^{-1}(\alpha', p'_0) = (\alpha_2, p_0'(\cosh s - \sinh s \sin \alpha')) \]
\[ \Lambda_1(t)^{-1}(\alpha', p'_0) = (\alpha_1, p_0'(\cosh t - \sinh t \cos \alpha')) \]
\[ R_0^{-1}(\alpha)(\alpha', p'_0) = (\alpha' + \alpha, p'_0), \]
\[ P(\alpha', p'_0) = (\alpha' + \pi, p'_0) \]
with
\[ (\sin \alpha_2, \cos \alpha_2) = \left( \frac{-\sinh s + \cosh s \sin \alpha'}{\cosh s - \sinh s \sin \alpha'}, \frac{\cos \alpha'}{\cosh s - \sinh s \sin \alpha'} \right), \]
\[ (\sin \alpha_1, \cos \alpha_1) = \left( \frac{\sin \alpha'}{\cosh t - \sinh t \cos \alpha'}, \frac{-\sinh t + \cosh t \sin \alpha'}{\cosh t - \sinh t \cos \alpha'} \right). \]

**Lemma 3.2.2.** The restriction of the Lorentz invariant measure on \( \mathbb{R}^{1+2} \) to the forward light-cone \( (3.2.5) \), given by
\[ (3.2.7) \quad d\mu(\alpha, p_0) = \frac{d\alpha}{2\pi} dp_0, \quad p_0 = e^{-t}, \]
defines a strongly quasi-invariant measure on the homogeneous space \( G/MN \cong \{ gMN \mid g \in SO_0(1, 2) \} \). Its Radon–Nikodym derivative is

\[
(3.2.8) \quad \frac{d\mu^g}{d\mu}(g'MN) = \chi_t(\Lambda_1(t)) , \quad \text{with} \quad g^{-1}g' = R_0(2\alpha)p^k\Lambda_1(t)D(q) .
\]

**Proof.** Set \( g^{-1}g' = R_0(2\alpha)p^k\Lambda_1(t)D(q) \) and let \( f \) be a measurable function on \((3.2.5)\). Then

\[
\int d\mu^g(\alpha', p_0') f(\alpha', p_0') = \int d\mu(\alpha', p_0') f(\alpha, p_0) \\
= \int d\mu(\alpha, p_0) p_0^{-1} f(\alpha, p_0) .
\]

The second identity follows from Remark 3.1.10 ii. Note that \( \chi_t(\Lambda_1(t)) = e^{-t} \). Thus (3.2.8) follows. As expected, \( \lambda_g(g'H) \) satisfies the cocycle relation stated in Proposition 3.1.9. This result is in agreement with [128, p.169, 170]. \( \square \)

### 3.3. Unitary representations on a circle on the light cone

The Iwasawa decomposition together with the definition of \( h^+_{x, \theta} \) imply that a function \( f \in h^+_{x, \theta} \) is determined by the restriction \( f_{xK} \) of \( f \) to \( K \). We have seen that \( \{ gMN \mid g \in G \} \) can be identified with \( \partial V^+ \), while \( \{ gMAN \mid g \in G \} \) can be identified with the projective space formed by the light rays on the forward light cone, see Subsection 2.7.4.

The latter can be identified with the subgroup \( SO(2) \) of \( G \), considered as a topological space, and we have

\[ G/MAN \cong SO(2) . \]

This can be also directly seen by considering the unique Iwasawa decomposition \( G = KAN = SO(2) \) MAN. The projection \( G \to G/MAN \) is then given by

\[ R_0(2\alpha)p^k\Lambda_1(t)D(q) \mapsto R_0(2\alpha) , \quad \alpha \in [0, \pi] , \]

and the embedding of \( SO(2) \) into \( G \) can be considered as a global smooth section

\[ \Xi: G/MAN \to G , \quad R_0(\alpha) \mapsto R_0(\alpha) . \]

We wish to translate the induced representation (3.2.4) as in Eq. (3.1.13) to a representation acting on \( C_0(SO(2)) \). For given \( g \in G \) and \( R_0(\alpha') \in G/MAN \cong SO(2) \) there are unique \( \alpha, k, t \) and \( q \) such that

\[
(3.3.1) \quad g^{-1}R_0(\alpha') = R_0(\alpha)p^k\Lambda_1(t)D(q) .
\]

Taking the class w.r.t. MAN, this implies that \( g^{-1}R_0(\alpha') = R_0(\alpha) \) in the sense of the action of \( G \) on \( G/MAN \cong SO(2) \) and that \( \Xi(g^{-1}R_0(\alpha')) = R_0(\alpha) \in G \). Eq. (3.3.1) then implies that \( P^k\Lambda_1(t)D(q) = \Omega(g, R_0(\alpha'))^{-1} \), see (3.1.12).
Let us denote by $\Pi^\pm_\nu$ the representation living on $C(\text{SO}(2))$ equivalent to the induced representation $\Pi^\pm_{\nu,0}$ (3.2.4). According to (3.1.13), it acts as

$$
(\Pi^\pm_\nu(g) f)_{|K}(R_0(\alpha')) = \sqrt{\lambda_g(R_0(\alpha')|\text{MAN})} \pi^\pm_\nu(\Omega(g, R_0(\alpha'))) f_{|K}(g^{-1} \cdot R_0(\alpha'))
$$

$$
= e^{-\frac{1}{2}t} \pi^\pm_\nu((p^k \Lambda_1(t)D(q))^{-1}) f_{|K}(R_0(\alpha))
$$

$$
= e^{(-\frac{1}{2} - iv)t} (\pm 1)^k f_{|K}(R_0(\alpha))
$$

We have used $\lambda_g(R_0(\alpha')|\text{MAN}) = \rho(g) = e^{-t}$ and

$$
\pi^\pm_\nu((p^k \Lambda_1(t)D(q))^{-1}) = e^{-ivt}(\pm 1)^k,
$$

as well as $g^{-1} \cdot R_0(\alpha') = R_0(\alpha)$. Note that if $\nu \in \mathbb{R}$, then $\pi^\pm_{\nu,0}$ (3.2.1) is a unitary representation of MAN in $\mathbb{C}$.

Identifying $\text{SO}(2)$ with the circle $\Gamma_0$ introduced in (2.7.3) by setting

$$
h(\alpha) = f_{|K}(R_0(\alpha)), \quad \alpha \in [0, 2\pi),
$$

the representation $\tilde{\Pi}^\pm_\nu$ extends to a unitary representation on $L^2(\Gamma_0, d\mu_{\Gamma_0})$, with $d\mu_{\Gamma_0} = \frac{da}{2\pi}$, the strongly quasi-invariant measure on $G/\text{MAN} \cong \Gamma_0$; see the remark after Eq. (3.1.13).

PROPOSITION 3.3.1. Let $\tilde{h}^\pm_\nu$ denote the completion of $C_0(\Gamma_0)$ with respect to one of the following norms:

i.) in case $0 < \zeta < \frac{1}{2}$, define for $\nu = \pm i\sqrt{\frac{1}{2} - \zeta^2}$ a norm on $\tilde{h}^\pm_{\nu,0}$ by setting

$$
||h||^2_{\nu} = \int_{\Gamma_0} \frac{d\alpha}{2\pi} \frac{h(\alpha)}{h(\alpha)} \int_{\Gamma_0} \frac{d\alpha'}{2\pi} \varrho_\nu(\alpha - \alpha') h(\alpha'),
$$

with $\varrho_\nu(\alpha) = \frac{r(\frac{1}{2} - iv)}{r(\frac{1}{2} + iv)} (\sin \frac{\alpha}{2})^{-\frac{1}{2} - iv} \pi$;

ii.) in case $\frac{1}{2} \leq \zeta$, define for $\nu = \pm i\sqrt{\zeta^2 - \frac{1}{4}}$ a norm on $\tilde{h}^\pm_{\nu,0}$ by setting

$$
||h||^2_{\nu} = \frac{1}{2\pi} \int_{\Gamma_0} \frac{d\alpha}{2\pi} |h(\alpha)|^2.
$$

It follows that for all $\zeta > 0$ the operators $\Pi^\pm_{\nu,0}(g), \ g \in G$, extend from $C_0(\Gamma_0)$ to a unitary representation

(3.3.2) $$(\tilde{u}^\pm_\nu(g)h)(\alpha') = (\pm 1)^k e^{(\frac{1}{2} - iv)t} h(\alpha)
$$

of the two fold covering group of the Lorentz group $\text{SO}(1,2)$. The parameters $\alpha, k, t, q$ on the r.h.s. are given by (3.3.1).

PROOF. The case $\nu \in \mathbb{R}$ follows from the discussion preceding the proposition. In the case $-\frac{1}{2} < i\nu < \frac{1}{2}$, note that the norm reads

$$
||h||^2_{\nu} = \langle h, A_\nu h \rangle_{L^2(\Gamma_0)},
$$

where $A_\nu$ is the operator acting on $C_0(\Gamma_0)$ as

$$(A_\nu h)(\alpha) \equiv \int_{\Gamma_0} \frac{d\alpha'}{2\pi} \varrho_\nu(\alpha - \alpha') h(\alpha'), \quad \alpha \in [0, 2\pi).$$
We show below that this map intertwines $\tilde{\Pi}_\nu$ and $\tilde{\Pi}_{-\nu}$; see (3.3.14). Using this fact and the fact that $\pi_\nu^\pm(\text{man}) \pi_\nu^{-\nu}(\text{man}) = 1$ for all $\text{man} \in \text{MAN}$, one verifies that $\tilde{\Pi}_\nu^\pm(g)$, $g \in G$, is a unitary operator in $h^\pm_{\nu,0}$.

**Remarks 3.3.2.**

i.) Note that in case $\frac{1}{2} \leq \zeta$, the norm does not depend on $\nu$.

ii.) In Bargmann's classification [12] of the unitary irreducible representations of $SO_0(1,2)$, the principle series and the complementary series are both denoted by $C^\infty_\nu$. They are distinguished by the eigenvalue of $\zeta^2$ of the Casimir operator $C^2$, with $\zeta^2$ being larger or equal or smaller than $1/4$.

Choosing $p_0 = 1$ in (3.2.6) and using the notation introduced in (3.3.1), one finds (see Equ. (4.41) and Equ. (4.42) in [28])

\[
\begin{align*}
(\tilde{\Pi}_\nu^\pm(\Lambda_2(s))h)(\alpha') &= e^{-\frac{i}{2} - i\nu t_2}h(\alpha_2) \\
(\tilde{\Pi}_\nu^\pm(\Lambda_1(t))h)(\alpha') &= e^{-\frac{i}{2} - i\nu t_1}h(\alpha_1) \\
(\tilde{\Pi}_\nu^\pm(R_0(\alpha))h)(\alpha') &= h(\alpha + \alpha')
\end{align*}
\]

with

\[
\begin{align*}
t_2 &= \ln(\cosh s - \sinh s \sin \alpha') \\
t_1 &= \ln(\cosh t - \sinh t \cos \alpha')
\end{align*}
\]

and

\[
\begin{align*}
e^{i\alpha_1} &= \frac{\cos \alpha' - i \sinh s + i \cosh s \sin \alpha'}{\cosh s - \sinh s \sin \alpha'}, \\
e^{i\alpha_2} &= \frac{\sinh t - i \cosh t \sin \alpha' - i \sin \alpha'}{\cosh t + \sinh t \cos \alpha'}.
\end{align*}
\]

**Theorem 3.3.3 (Bargmann, [12]).** The representations $\tilde{\Pi}_\nu^\pm$ given by (3.3.2) are irreducible. The representations for $\nu$ and $-\nu$, $\nu \in \mathbb{R}$, are unitarily equivalent both for the principal and the complementary series.\(^6\)

**Proof.** Let us consider $C^\infty_0$ functions on the forward light cone. It follows that the generators $L_2$, $L_1$ and $K_0$ take the form (see [12] §6a)

\[
\begin{align*}
iL_2 &= \cos \alpha \frac{\partial}{\partial \alpha} + \sin \alpha p_0 \frac{\partial}{\partial p_0}, \\
iL_1 &= \sin \alpha \frac{\partial}{\partial \alpha} - \cos \alpha p_0 \frac{\partial}{\partial p_0}, \\
iK_0 &= -\frac{\partial}{\partial \alpha}.
\end{align*}
\]

(3.3.4)

Note that $K_0^2 = -\frac{\partial^2}{\partial \alpha^2}$ is a positive operator. The eigenfunctions of $K_0^2$ on the light cone for the eigenvalue $k^2$ are of the form $h(p_0) e_k$ with

\[
e_k = \frac{e^{ik\alpha}}{\sqrt{2\pi}}, \quad k \in \mathbb{Z}.
\]

\(^6\)See, e.g., [161] p. 104].
The generator of the horospheric translations is \( i(L_2 - K_0) \) and the Casimir operator is
\[
(3.3.5) \quad C^2 = -K_0^2 + L_1^2 + L_2^2.
\]
The latter equals \[12\] Eq. (6.5)]
\[
(3.3.6) \quad C^2 = -S(S + 1) = -\partial_{p_0}p_0^2\partial_{p_0}, \quad \text{with} \quad S = p_0\partial_{p_0}.
\]
It is positive, since
\[
(3.3.7) \quad \langle g^* C^2 g \rangle = \int_0^\infty dp_0 \int_0^{2\pi} d\alpha \frac{g(p_0, \alpha)}{2\pi} \partial_{p_0}p_0^2\partial_{p_0}g(p_0, \alpha)
\]
\[
= \int_0^\infty dp_0 \int_0^{2\pi} d\alpha p_0^2|\partial_{p_0}g(p_0, \alpha)|^2 \geq 0.
\]
The eigenvalue equation \( \zeta^2 = -s(s + 1) \) has the solutions
\[
(3.3.8) \quad s^\pm = \frac{-1}{2} \mp iv, \quad \text{with} \quad v = \begin{cases} \frac{1}{4} - \zeta^2 & \text{if } 0 < \zeta < 1/2, \\ \sqrt{\zeta^2 - \frac{1}{4}} & \text{if } \zeta \geq 1/2. \end{cases}
\]
Eq. \( (3.3.6) \) implies \[12\] Eq. (6.6b)] that the generalised eigenfunctions for the eigenvalue \( \zeta^2 \) of \( C^2 \) are homogenous functions of the form
\[
(\alpha, p_0) \mapsto p_0^{-\frac{1}{2}-iv}f(\alpha, 1),
\]
in agreement with \( (3.3.2) \). Thus, in the representation \( \tilde{u}_v \) the Casimir operator is a multiple of the identity with eigenvalue \( s^+ = -\frac{1}{2} - iv \).

Now let \( A \) be a bounded linear operator \( A \) on \( \tilde{h}_v^\pm \), which commutes with all \( \tilde{u}_v \) \( (g), g \in SO_0(1, 2) \). It follows \[12\] p. 608\] that
\[
(3.3.9) \quad K_0A f_k = AK_0 f_k, \quad f_k \equiv p_0^{-\frac{1}{2}-iv}e_k, \\
L_iA f_k = AL_i f_k, \quad i = 1, 2, \quad k \in \mathbb{Z}.
\]
The first equation implies that \( A f_k = \alpha v, k \cdot f_k \) for some \( \alpha v, k \in \mathbb{C} \). To explore the content of the second and third equation in \( (3.3.9) \), we introduce the ladder operators \( L_\pm = L_1 \pm L_2 \). They satisfy
\[
(3.3.10) \quad L_+ f_k = c_{k+1}\sqrt{\zeta^2 + k(k + 1)} f_{k+1}, \\
L_- f_k = c_k^{-1}\sqrt{\zeta^2 + k(k - 1)} f_{k-1},
\]
with \( |c_k| = 1 \) some constants of absolute value 1. Since \( L_1 = \frac{1}{2}(L_+ + L_-) \) and \( L_2 = \frac{1}{2}(L_+ - L_-) \), we obtain from \( (3.3.10) \) a set of equations, which may be written in the form \[12\] Equ. (5.34)]
\[
L_i f_k = \sum_{k'} h_{k,k'} f_{k'},
\]
where \( h_{k,k'} = h_{k',k} \) and where \( h_{k,k'} = 0 \) if \( |k - k'| > 1 \). We therefore obtain from the second and third equation in \( (3.3.9) \) equations of the form
\[
(\alpha v, k - \alpha v, k') h_{k,k'} = 0 \quad \forall k, k' \in \mathbb{Z}.
\]
A brief inspection shows that all \( \alpha v, k \) have to be equal to each other (for \( v \) fixed), \( i.e., \) that \( A = \alpha v, 0, 1 \).
3.3.1. Representations of \( \text{SO}(0,1,2) \) on the forward light cone. For \( 0 < \zeta < 1/2 \), the eigenfunctions of the Casimir operator (3.3.5) are not in \( L^2(\partial V^+, \frac{dv}{2\pi} dp_0) \), as their decay in the variable \( p_0 \) is not fast enough to ensure the existence of the integral. Thus the unitary irreducible representations in the complementary series (corresponding to \( 0 < \zeta < 1/2 \)) do not appear, if one decomposes the reducible representation on \( L^2(\partial V^+, \frac{dv}{2\pi} dp_0) \) given by the pull-back:

**Theorem 3.3.4** (Spectral theorem). As an operator on \( L^2(\partial V^+, \frac{dv}{2\pi} dp_0) \) with domain \( \mathcal{D}_R(\partial V^+) \), the Casimir operator \( C^2 \) given in (3.3.6) is essentially self-adjoint and positive. The positive square root of its self-adjoint extension, denoted by \( \tilde{h} \), has spectrum \( \text{Sp}(C) = [1/2, \infty) \). The corresponding spectral decomposition is

\[
L^2(\partial V^+, \frac{dv}{2\pi} dp_0) = \int^\infty_{\zeta} d\zeta \mathcal{H}_\zeta, \quad \mathcal{H}_\zeta \cong L^2(S^1, \frac{d\alpha}{2\pi}) \otimes \mathbb{C}^2.
\]

**Proof.** We exploit the properties of the Mellin transform [17, 157]: for \( g \in L^2(0, \infty) \) one finds

\[
g(p_0) = \frac{1}{2\pi} \int_{\mathbb{R}} dv \int^\infty_0 dp' \frac{1}{p_0^{-\frac{\nu}{2}} - iv} \int^\infty_0 dp'' p_0''^{-\frac{\nu}{2} + iv} g(p_0').
\]

The integral w.r.t. \( dv \) is over the whole real axis. This implies:

i.) The identity operator on \( L^2(\partial V^+, \frac{dv}{2\pi} dp_0) \) is

\[
1 = \int_{\mathbb{R}} dv \left( \sum_j \langle p_0^{-\frac{\nu}{2} - iv} h_j | p_0''^{-\frac{\nu}{2} - iv} h_j \rangle \right),
\]

with \( \{ h_j \in L^2(S^1, \frac{d\alpha}{2\pi}) \mid j \in \mathbb{N} \} \) an orthonormal basis in \( L^2(S^1, \frac{d\alpha}{2\pi}) \). Thus \( L^2(\partial V^+, \frac{dv}{2\pi} dp_0) \) is the direct integral over \( \nu \in \mathbb{R} \) of the Hilbert spaces \( \tilde{h}_\nu \) consisting of functions

\[
(p_0, \alpha) \mapsto p_0^{-\frac{\nu}{2} - iv} h(\alpha).
\]

The scalar product in \( \tilde{h}_\nu \) is just the scalar product in \( L^2(S^1, \frac{d\alpha}{2\pi}) \);

ii.) the spectrum of \( C \) in \( L^2(\partial V^+, \frac{dv}{2\pi} dp_0) \) is \( [1/2, \infty) \);

iii.) for \( \zeta^2 = 1/2 + \nu^2 \),

\[
\mathcal{H}_\zeta = \tilde{h}_\nu \oplus \tilde{h}_{-\nu};
\]

i.e., homogeneous functions of degree \( s^+ \) and \( s^- \) (see (3.3.8)) both appear.

\[ \square \]

3.3.2. Intertwiners.

**Proposition 3.3.5.** Consider the representations described in (3.3.2). It follows that the map

\[
(A_\nu h)(\alpha) = \int_0^{2\pi} \frac{d\alpha'}{2\pi} \varphi_\nu(\alpha - \alpha') h(\alpha'), \quad \alpha \in [0, 2\pi),
\]

with

\[
\varphi_\nu(\alpha) = \frac{\Gamma(\frac{\nu}{2} - iv)}{\Gamma(\frac{\nu}{2}) \Gamma(-iv)} \left( \sin^2 \frac{\alpha}{2} \right)^{-\frac{\nu}{2} - iv} \tau,
\]

\[ ^7 \text{The Euler function } B(x, y) = \Gamma(x)\Gamma(y) / \Gamma(x + y) \text{ appears here.} \]
defines an operator \( A_\nu \), which intertwines \( \tilde{u}^\pm_\nu \) and \( \tilde{u}^\mp_\nu \), i.e.,

\[
A_\nu \tilde{u}^\pm_\nu (g) = \tilde{u}^\mp_\nu (g) A_\nu \quad \forall g \in G.
\]

**Remarks 3.3.6.**

i.) The integral kernels appearing in (3.3.12) were first derived by Bargmann [12]. In the literature they are frequently written in the following alternative form:

\[
\theta_\nu(\alpha) = \frac{\Gamma(\frac{1}{2} - i\nu)}{\Gamma(\frac{1}{2})} \frac{(1 - \cos \alpha)}{2}^{-\frac{1}{2} - i\nu} \pi.
\]

ii.) In case \( \nu = \pm i \sqrt{\frac{1}{4} - \zeta^2} \) with \( 0 < \zeta < \frac{1}{2} \), the sesquilinear form

\[
h, h' \mapsto \int_{\Gamma_0} d\alpha \; \overline{h(\alpha)} (A_\nu h')(\alpha)
\]

is positively definite [154]. This implies

\[
\int_{\Gamma_0} d\alpha \; \overline{h(\alpha)} (A_\nu h')(\alpha) = \int_{\Gamma_0} d\alpha \; \overline{A_\nu h(\alpha)} h'(\alpha),
\]

and, consequently, (3.3.14) defines a positive operator.

iii.) In case \( \nu \) is real, we have \( A_\nu^* = A_{-\nu} \) [166, Lemma 2.1]. In fact, \( A_\nu \) is unitary as \( A_\nu^* A_\nu = 1 \); see Remark 3.3.7 below.

iv.) The bilinear form-valued function \( \nu \mapsto (., A_\nu \cdot)_{L^2(S^1, d\alpha)} \) is meromorphic in \( \mathbb{C} \). The poles of this function are the points \( i\nu = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \).

v.) The integral (3.3.12) is convergent if \( i\nu < 0 \) [154, p. 605]. In this case

\[
\frac{\Gamma(\frac{1}{2}) \Gamma(1 - i\nu)}{\Gamma(\frac{1}{2} - i\nu)} (e^{in\psi}, A_\nu e^{im\psi})_{L^2(S^1, d\phi)} =

2^{i\nu} \frac{\pi}{4} B(1/2 - i\nu - n, 1/2 - i\nu + n) \delta_{n,m}.
\]

The beta function is

\[
B(1/2 - i\nu - n, 1/2 - i\nu + n) = \frac{\Gamma(1/2 - i\nu - n) \Gamma(1/2 - i\nu + n)}{\Gamma(1 - 2i\nu)}.
\]

Using \( \Gamma(z+1) = z\Gamma(z) \) and the Stirling formula for the \( \Gamma \)-functions implies that the pre-factor on the r.h.s. diverges as

\[
|n|^{2i\nu} \left( 1 + O \left( \frac{1}{n} \right) \right), \quad n \to \infty.
\]

Hence the form \( (., A_\nu \cdot)_{L^2(S^1, d\phi)} \) is well-defined on the Sobolev space \( H^{i\nu}(S^1) \); see Definition [D.0.17]

---

The operator \( A_\nu \) intertwines the pullback action of \( \text{SO}_0(1, 2) \) on homogeneous functions of degree \( -\frac{1}{2} + \sqrt{\frac{1}{4} - \mu^2} \) and \( -\frac{1}{2} - \sqrt{\frac{1}{4} - \mu^2} \), respectively.
PROOF. Inspecting (3.3.3) we find that

\[
\bar{u}^\pm_\nu(R_0(\alpha)) = \bar{u}^\pm_\nu(R_0(\alpha)), \quad \alpha \in [0, 2\pi].
\]

Moreover,

\[
\int_{t_0} \frac{d\beta'}{2\pi} \varphi_\nu(\beta - \beta') (\bar{u}^\pm_\nu(R_0(\alpha)) h(\beta'))
= \int_{t_0} \frac{d\beta'}{2\pi} \varphi_\nu(\beta - \beta') h(\beta' + \alpha)
= \bar{u}^\pm_\nu(R_0(\alpha)) \left( \int_{t_0} \frac{d\beta'}{2\pi} \varphi_\nu(\beta - \beta') h(\beta') \right).
\]

It remains to be shown that

\[
A_\nu \bar{u}^\pm_\nu(\Lambda_2(s)) = \bar{u}^\pm_\nu(\Lambda_2(s)) A_\nu \quad \forall s \in \mathbb{R}.
\]

Compute

\[
\int_{t_0} \frac{d\beta'}{2\pi} \varphi_\nu(\beta - \beta') (\bar{u}^\pm_\nu(\Lambda_2(s)) h(\beta'))
= \int_{t_0} \frac{d\beta'}{2\pi} \varphi_\nu(\beta - \beta') e^{s^{-1}h(\alpha)},
\]

with \( \alpha = \alpha(s, \beta') \) given by

\[
R_0(\alpha) \tilde{p}^k \Lambda_1(t) \tilde{D}(q) \equiv \Lambda_2(s)^{-1}R_0(\beta').
\]

Note that \( 1 + s^{-} = -s^{+} \). Thus \( \frac{d\alpha(s, \beta')}{ds} = (\cosh s - \sinh s \sin \beta')^{-1} \),

\[
\sin \beta' = \sinh s + \cosh s \sin \alpha, \quad \cos \beta' = \frac{\cosh \alpha}{\cosh s + \sinh s \sin \alpha},
\]

and \( \cosh s - \sinh s \sin \beta' = (\cosh s + \sinh s \sin \alpha)^{-1} \). This allows us to rewrite (3.3.17) as

\[
\int_{t_0} \frac{d\beta'}{2\pi} \varphi_\nu(\beta - \beta') (\bar{u}^\pm_\nu(\Lambda_2(s)) h(\beta'))
= \int_{t_0} \frac{d\alpha}{2\pi} \varphi_\nu(\beta - \beta'(s, \alpha)) (\cosh s + \sinh s \sin \alpha)^{s^+} h(\alpha).
\]

The kernel \( \varphi_\nu(\beta - \beta'(s, \alpha)) \) can be rewritten using the formula

\[
(1 - \cos(\beta - \beta'(s, \alpha)))^{s^+}
= \left(1 - \cos \beta \cos(\beta'(s, \alpha)) - \sin \beta \sin(\beta'(s, \alpha))\right)^{s^+}
= \left(\cosh s + \sinh s \sin \alpha - \frac{\cos \beta \cos \alpha - \sin \beta \sin \alpha}{\cosh s + \sinh s \sin \alpha}\right)^{s^+}.
\]
For (3.3.17) this yields

\[
(3.3.20) \int_{\Gamma_0} \frac{d\beta'}{2\pi} \varrho_{\nu}(\beta - \beta') (\bar{u}_{\nu}^- (A_2(s)) h)(\beta')
\]

\[=rac{\Gamma(\frac{1}{2} + \nu)}{\Gamma(\frac{1}{2}) \Gamma(\nu)} \int_{\Gamma_0} \frac{d\alpha}{2\pi} \left( \cosh s + \sinh s \sin \alpha - \cos \beta \cos \alpha \ight.
\]

\[- \sin \beta \sinh s - \sin \beta \cosh s \sin \alpha \left. \right)^{\nu+} h^- (\alpha). \]

On the other hand, using (3.2.6)

\[\bar{u}_{\nu}^+(A_2(s)) \left( \int_{\Gamma_0} \frac{d\beta'}{2\pi} \varrho_{\nu}(\beta - \beta') h(\beta') \right)
\]

\[= (\cosh s - \sinh s \sin \beta)^{\nu+} \times \int_{\Gamma_0} \frac{d\alpha}{2\pi} \varrho_{\nu}(\arccos(\cosh s - \sinh s \sin \beta) - \alpha) h(\alpha)
\]

(3.3.21) \[= (\cosh s - \sinh s \sin \beta)^{\nu+} \int_{\Gamma_0} \frac{d\alpha}{2\pi} \varrho_{\nu}(\beta'(s, \alpha) - \alpha) h(\alpha).
\]

Next we compute \(\varrho_{\nu}(\beta'(s, \alpha) - \alpha)\):

\[(1 - \cos(\beta'(s, \alpha) - \alpha)^{\nu+} = (1 - \cos(\beta'(s, \alpha))) \cos \alpha - \sin(\beta'(s, \alpha)) \sin \alpha)^{\nu+}
\]

\[= (\cosh s - \sinh s \sin \beta \cos \alpha + \sin \alpha \sinh s - \sin \beta \cosh s \sin \alpha)^{\nu+}.
\]

Inserting this result into (3.3.21) shows that

\[\int_{\Gamma_0} \frac{d\beta'}{2\pi} \varrho_{\nu}(\beta - \beta') (\bar{u}_{\nu}^+(A_2(s)) h)(\beta')
\]

(3.3.22) \[= \bar{u}_{\nu}^+(A_2(s)) \left( \int_{\Gamma_0} \frac{d\beta'}{2\pi} \varrho_{\nu}(\beta - \beta') h(\beta') \right).
\]

Since \(R_0(\alpha)\) and \(A_2(s)\) generate \(SO_0(1, 2)\), this verifies (3.3.14).

\[\square\]

**Remark 3.3.7.** Clearly (3.3.16) implies

\[[A_\nu, \bar{u}_{\nu}^+ (R_0(\alpha))] = 0, \quad \alpha \in [0, 2\pi).\]

Therefore \(A_\nu\) has diagonal form in the spectral representation of the generator of the rotations \(\alpha \mapsto R_0(\alpha)\). In fact [12 p. 619],

\[
\frac{\Gamma(\frac{1}{2} - iv)}{\Gamma(\frac{1}{2}) \Gamma(-iv)} \left( \sin^2 \frac{\alpha}{2} \right)^{-\frac{1}{2}+iv} \pi = 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left| k \right| \prod_{j=1}^{|k|} \frac{\Gamma\left( j + \frac{1}{2} + iv \right)}{\Gamma\left( j + \frac{1}{2} - iv \right)} e^{ik\alpha},
\]

and, consequently, the Fourier coefficients\(^7\) of \(\varrho_{\nu}\) are

\[\varrho_{\nu}(k) = \sqrt{2\pi} \frac{\Gamma\left( \left| k \right| + \frac{1}{2} + iv \right)}{\Gamma\left( \left| k \right| + \frac{1}{2} - iv \right)} \frac{\Gamma\left( \frac{1}{2} - iv \right)}{\Gamma\left( \frac{1}{2} + iv \right)}.
\]

\(^7\)These coefficients refer to the eigenfunctions \(\tilde{u}_{\nu}^k(\alpha)\).
Hence, for $\nu \in \mathbb{R}$,
\[ \int_{r_0} \frac{d\alpha'}{2\pi} \bar{\rho}_\nu(\alpha - \alpha') \rho_\nu(\alpha' - \alpha'') \] \[ = \int_{r_0} \frac{d\alpha'}{2\pi} \sum_k \bar{\rho}_\nu(k) e^{-ik(\alpha - \alpha')} \sum_j \rho_\nu(j) e^{-ij(\alpha' - \alpha'')} \] \[ = \sum_k \frac{\bar{\rho}_\nu(k)}{2\pi} \sum_j \frac{\rho_\nu(j)}{2\pi} \] \[ \times \int_{r_0} \frac{d\alpha'}{2\pi} e^{-ik(\alpha - \alpha')} e^{ij(\alpha' - \alpha'')} \] \[ = \sum_k e^{ik(\alpha - \alpha'')} \] \[ = 2\pi \delta(\alpha - \alpha''). \]

We have used that $|\bar{\rho}_\nu(k)|^2 = 2\pi$ for all $k \in \mathbb{Z}$. Note that $2\pi \delta(\alpha - \alpha'')$ is the kernel of the identity operator with respect to the measure $\frac{d\alpha''}{2\pi}$.

### 3.3.3. The time reflection.
Our next aim is to extend the unitary irreducible representations of the two fold covering group of $\text{SO}(1,2)$ discussed so far to (anti-)unitary representations of $\text{O}(1,2)$.

We start with the induced representation $\Pi^\pm_\nu$ [3.2.3], and consider first the case $\nu \in \mathbb{R}$, i.e., $\zeta \geq 1/2$. Let $\Pi^\pm_{\nu,0}(T)$ be the anti-linear map from $\mathfrak{h}^\pm_{-\nu,0}$ to $\mathfrak{h}^\pm_{\nu,0}$ defined by

\[ (\Pi^\pm_{\nu,0}(T)f)(g) = T\tilde{f}(P g), \quad f \in \mathfrak{h}^\pm_{\nu,0} \]

where $P$ is the space-reflection, $P = R_0(\pi) \in \text{SO}(1,2)$. Since $\lambda_P(\text{gMAN}) = 1$, this is an isometric map. Now pick an intertwiner

$A_\nu : \mathfrak{h}^\pm_{\nu,0} \rightarrow \mathfrak{h}^\pm_{-\nu,0}$

between the representations $\Pi^\pm_\nu$ and $\Pi^\pm_{-\nu}$ and define $\Pi^\pm_{\nu,\nu}(T) = \Pi^\pm_{\nu,0}(T) \circ A_\nu$:

\[ (\Pi^\pm_{\nu,\nu}(T)f)(g) = (A_\nu f)(P g). \]

Then one has

\[ \Pi^\pm_{\nu}(T) \Pi^\pm_{\nu}(g) \Pi^\pm_{\nu}(T)^{-1} = \Pi^\pm_{\nu,\nu}(T) \Pi^\pm_{\nu,\nu}(g) \Pi^\pm_{\nu,\nu}(T)^{-1} \]

and

\[ (\Pi^\pm_{\nu}(T) \Pi^\pm_{\nu}(g) \Pi^\pm_{\nu}(T)^{-1} f)(g') = \lambda_g(P g' H)^{1/2} f(P g^{-1} P g'), \]

while on the other hand

\[ (\Pi^\pm_{\nu}(T g^{-1} T)^{-1} f)(g') = \lambda_{P g P}(g' H)^{1/2} f((P g P)^{-1} g'), \]

where it has been used that the adjoint action of $T$ on $\text{SO}(1,2)$ coincides with that of the space-reflection $P$, $T g T = P g P$. Since $\lambda_{P g P}(g' H) = \lambda_g(P g^{-1} H)$, this proves that

\[ \Pi^\pm_{\nu}(T) \Pi^\pm_{\nu}(g) \Pi^\pm_{\nu}(T)^{-1} = \Pi^\pm_{\nu}(T g^{-1} T)^{-1}. \]

Thus, $\Pi^\pm_{\nu}(T)$ is a representor of $T$ which, in addition, is easily seen to be anti-unitary.

We wish to find the equivalent representor in the representation space $C_0(\text{G}/\text{H})$. The intertwiner $A_\nu$ corresponds uniquely to an operator $\tilde{A}_\nu$ in $C_0(\text{G}/\text{H})$ by the equivalence [3.1.10],

\[ \tilde{A}_\nu f = \tilde{A}_\nu f, \]
which intertwines the representations \( \tilde{u}^\pm \) and \( \tilde{u}^\pm_{\nu} \). Now this equivalence translates \( \Pi^\pm_{\nu}(T) \) into the anti-unitary operator \( \tilde{u}^\pm(T) \) in \( C_0(\text{SO}(2)) \) given by

\[
\begin{align*}
(\tilde{u}^\pm(T) f)(R_0(\alpha)) &\doteq (\Pi^\pm_{\nu}(T) f)(R_0(\alpha)) = (\Pi^\pm_{\nu}(T) f)(R_0(\alpha)) = (A_{\nu} f)(\text{PR}_0(\alpha)) \\
&= (A_{\nu} f)(\text{PR}_0(\alpha)) = (A_{\nu} f)(\text{PR}_0(\alpha)).
\end{align*}
\]

In the second and fourth equation we have used the fact that \( \Xi(R) = R \in G \) for any rotation \( R \in \text{SO}(2) \cong G/\text{MAN} \) and that \( \text{PR}_0(\alpha) \equiv R_0(\alpha + \pi) \) is a rotation. In short, \( \tilde{u}^\pm(T) \) acts on \( C_0(\text{SO}(2)) \) as

\[
(\tilde{u}^\pm(T) h)(R_0(\alpha)) \doteq (A_{\nu} h)(\text{PR}_0(\alpha)) .
\]

Note that \( \tilde{u}^\pm(T)^2 = A^*_\nu A_\nu = 1 \).

In the case \( \nu \in i\mathbb{R}, i.e., 0 < \zeta < 1/2 \), the anti-linear map \( \Pi^\pm_{\nu,0}(T) \) defined above leaves \( h^\pm_{\nu,0} \) invariant, and we take this operator to be the representer of \( T \) in \( h^\pm_{\nu,0} \). The proof of the representation property goes as above. We then define \( \tilde{u}^\pm(T) \) as the equivalent representer in the representation space \( C_0(\Gamma_0) \), namely,

\[
(\tilde{u}^\pm(T) h)(R_0(\alpha)) \doteq h(\text{PR}_0(\alpha)) .
\]

Anti-unitarity can be seen as follows:

\[
\|\tilde{u}^\pm(T) h\|_\nu = \langle \tilde{u}^\pm(T) h, A_{\nu} \tilde{u}^\pm(T) h \rangle_{L^2(\Gamma_0, d\mu_{\Gamma_0})} \\
= \int_{\text{SO}(2)} \frac{d\alpha}{2\pi} h(\text{PR}_0(\alpha)) \int_{\Gamma_0} \frac{d\alpha'}{2\pi} \rho_\nu(\alpha - \alpha') h(\text{PR}_0(\alpha')) \\
= \int_{\Gamma_0} \frac{d\alpha'}{2\pi} h(\text{PR}_0(\alpha')) \int_{\Gamma_0} \frac{d\alpha}{2\pi} \rho_\nu(\alpha - \alpha') h(\text{R}_0(\alpha)) \\
= \langle h, A_{\nu} h \rangle_{L^2(\Gamma_0, d\mu_{\Gamma_0})} = \|h\|_\nu.
\]

In the fourth equation we have used the symmetry \( \rho_\nu(\alpha) = \rho_\nu(-\alpha) \).

Note that the preceding discussion also shows that in both cases, \( \zeta < 1/2 \) and \( \zeta > 1/2 \), the representer of the space-reflection \( P \) is given by

\[
(\tilde{u}^\pm(P) h)(R_0(\alpha)) = h(\text{PR}_0(\alpha)) \equiv h(\text{R}_0(\alpha - \pi)).
\]

Summarising, our discussion leads to the following definition.

**DEFINITION 3.3.8.** For \( h(\alpha) \in \tilde{h}_\nu \), define an antilinear operator by setting

\[
(\tilde{u}^\pm(T) h)(\alpha) = \begin{cases} 
[A_{\nu} h](\alpha - \pi) & \text{if } 1/2 \leq \zeta , \\
h(\alpha - \pi) & \text{if } 0 < \zeta < 1/2 .
\end{cases}
\]

We have shown:

**PROPOSITION 3.3.9.** The anti-unitary operator \( \tilde{u}^\pm(T) \) is an anti-unitary representer of the time-reflection \( T \) on \( \tilde{h}_\nu \). Together with \( \tilde{u}^\pm(P_2) \) it extends the representation \( \tilde{u}^\pm \) from \( \text{SO}_0(1, 2) \) to \( \text{O}(1, 2) \).
3.4. Unitary representations on the mass shell

The Hannabuss decomposition implies that a function \( f \in h_0^+ \) is determined by \( f(k'MAN) \) with \( k' \in K' = \{A_2(s),A_2(s)P \mid s \in \mathbb{R}\} \). We have seen in Subsection \[2.7.5\] that we can identify the cosets \([\mathbb{P}]\) of the form

\[
k'MAN, \quad k' \in K',
\]

with points in the two hyperbolas

\[
\Gamma_1 = (p_+(s) \in \partial V^+ \mid s \in \mathbb{R}) \cup \{p_-(s) \in \partial V^+ \mid s \in \mathbb{R}\},
\]

(3.4.1)

where

\[
(3.4.2) \quad p_\pm(s) = \Lambda_2(s)p_\pm(0) = \begin{pmatrix} m \cosh s \\ \pm m \sinh s \\ \mp m \end{pmatrix}, \quad s \in \mathbb{R}.
\]

Note that \( p_\pm(0) = \begin{pmatrix} m \\ 0 \\ \pm m \end{pmatrix} \). By construction, the boosts \( s \mapsto \Lambda_2(s) \), \( s \in \mathbb{R} \), leave the curves \( \Gamma_\pm \) invariant. The reflections \( P_1, P_2 \) and \( P \) act on \( \Gamma_1 \):

\[
P_1\Lambda_2(s)p_\pm(0) = \begin{pmatrix} m \cosh s \\ m \sinh s \\ \mp m \end{pmatrix}, \quad P_2\Lambda_2(s)p_\pm(0) = \begin{pmatrix} m \cosh(-s) \\ m \sinh(-s) \\ \pm m \end{pmatrix},
\]

and, consequently, \( P\Lambda_2(s)p_\pm(0) = \begin{pmatrix} m \cosh(-s) \\ \pm m \sinh(-s) \\ \mp m \end{pmatrix} \).

Remark 3.4.1. Given a homogeneous function on the forward light cone, one can define a function on \( \Gamma_1 \) by restriction. On the contrary (see [28, Equ. 4.44]), given a pair of functions \( p_1 \mapsto (h_+(p_1),h_-(p_1)) \) on \( \Gamma_+ \cup \Gamma_- \),

\[
f(p) = \begin{cases} \left(\frac{p_1}{m}\right)^s h_+\left(\frac{mp_1}{p_1}\right) & \text{if } p_2 > 0, \\ \left(\frac{p_1}{m}\right)^s h_-\left(\frac{mp_1}{p_1}\right) & \text{if } p_2 < 0, \end{cases}
\]

defines a homogeneous function of degree \( s \) on the light cone \( \partial V^+ \).

The restriction of the quasi-invariant measure \( \frac{dm}{m^2}dp_0 \) to \( \Gamma_\pm \) is \( \frac{dm}{m} \). It follows that \( L^2(\Gamma_1,d\mu_{\Gamma_1}) \) consists of two copies of \( L^2(\mathbb{R},\frac{dm}{m^2}) \):

\[
(3.4.3) \quad L^2(\Gamma_1,d\mu_{\Gamma_1}) \cong L^2(\mathbb{R},\frac{dm}{m^2}) \oplus L^2(\mathbb{R},-\frac{dm}{m^2}).
\]

Note that the two disjoint parts of \( \Gamma_1 \) form a closed curve enclosing the origin; thus the minus sign in the second component.

Lemma 3.4.2. The generator of the boost \( s \mapsto \tilde{\mu}_{\Lambda_2(s)} \) on \( L^2(\Gamma_1,d\mu_{\Gamma_1}) \) is

\[
(3.4.4) \quad -i\left(\frac{\partial}{\partial s} \oplus \frac{\partial}{\partial s}\right).
\]

Its spectrum is absolutely continuous on the whole real line.

\[\text{10The cosets } k'MN, k' \in K', \text{ can be identified with points in } \partial V^+. \]
THEOREM 3.4.3. The induced representation \((\tilde{u}_\nu^{+)}(g)h_+)(s') = \chi_{\gamma-n}^+(\Lambda_1(t))h_{(-)}(s)\)
\[ (3.4.5) \]
\[ e^{-\frac{1}{2}i\nu t}h_{(-)}(s) , \]
where
\[ (3.4.6) \]
\[ \Lambda_2(s)p^j\Lambda_1(t)D(q) \doteq g^{-1}\Lambda_2(s') , \]
with \(s, j, t\) and \(q \in \mathbb{R}\).

REMARK 3.4.4. Recalling (2.2.6), we find
\[ (u_{\tilde{\nu}}^{+}) = \begin{pmatrix} \frac{q^2}{q^2} \\ \frac{-2}{-2} \end{pmatrix} \in P_{\tau=0} . \]
Thus (3.4.5) yields
\[ (3.4.7) \]
\[ t = d(x, P_t) \quad \forall x = \begin{pmatrix} \frac{q^2}{q^2} \\ \frac{r}{-2} \end{pmatrix} \in P_{\tau=0} . \]

Thus (3.4.5) takes the form
\[ (3.4.8) \]
\[ (u_{\tilde{\nu}}^{+})h_{+}^{-1}i\nu h_{(-)}(s) , \]
with \(s, j, t\) defined by (3.4.6) and \(x \in P_{\tau=0}\).

PROOF. If \(p \in \Gamma_1\) and \(p_{\pm} = \Lambda_2(s)p_{\pm}(0)\), then the cosets \(gMN\) can be identified with \(\Gamma_1\), and we may thus consider
\[ (\Pi_{\gamma-n}^{+}(g)f)(\Lambda_2(s')) = \sqrt{\lambda_g(\Lambda_2(s')MN)} f(g^{-1}\Lambda_2(s')) \]
Thus the induced representation takes the form (3.4.5).

REMARK 3.4.5. In particular [28 Equ. (4.45–4.47)], recalling (2.1.3), we find
\[ (u_{\tilde{\nu}}^{+}(R_0(\alpha))f)(p) = \begin{cases} \begin{pmatrix} \frac{p_1}{m} \\ \frac{m}{m} \end{pmatrix} \doteq \frac{1}{2}i\nu \begin{pmatrix} h_+((p_1 + m \sin \alpha)/p_2) \\ h_-((p_1 - m \sin \alpha)/p_2) \end{pmatrix} & \text{if } p_2 > 0 , \\ \begin{pmatrix} \frac{p_1}{m} \\ \frac{m}{m} \end{pmatrix} \doteq \frac{1}{2}i\nu \begin{pmatrix} h_-((p_1 + m \sin \alpha)/p_2) \\ h_+((p_1 - m \sin \alpha)/p_2) \end{pmatrix} & \text{if } p_2 < 0 , \end{cases} \]
with
\[ p_2 = p_1 \sin \alpha + m \cos \alpha . \]

Recalling (2.1.3), we find
\[ (u_{\tilde{\nu}}^{+}(\Lambda_1(t))f)(p) = \begin{cases} \begin{pmatrix} p_2 \end{pmatrix} \doteq \frac{1}{2}i\nu \begin{pmatrix} h_+(p_1/p_2') \\ h_-((p_1/p_2') \end{pmatrix} & \text{if } p_2' > 0 , \\ \begin{pmatrix} p_2 \end{pmatrix} \doteq \frac{1}{2}i\nu \begin{pmatrix} h_-((p_1/p_2') \\ h_+(p_1/p_2') \end{pmatrix} & \text{if } p_2' < 0 , \end{cases} \]
with
\[ p_2' = \frac{m \cosh t - \sqrt{p_t^2 + m^2 \sinh t}}{m} . \]

\[ ^{11}\text{In particular, } \Lambda_2(s)^{-1}\Lambda_2(s') = \Lambda_2(s' - s). \]
Finally,
\[
\left(\tilde{u}_\nu (A_2(s)) f \right)(p) = \begin{cases} 
  h_+ (p_1 \cosh s - \sqrt{p_1^2 + m^2} \sinh t) \\
  h_- (p_1 \cosh s - \sqrt{p_1^2 + m^2} \sinh t)
\end{cases}
\]

3.4.1. The group contraction $SO_0(1,2)$ to $E_0(1,1)$. On the two-dimensional Minkowski space, the plane waves
\[
(t, q) \mapsto e^{i(t,q) \cdot (\sqrt{p_1^2 + m^2}, p_1)}
\]
can be interpreted as improper common eigenvectors of the space-time translations $T(t', q'), (t', q') \in \mathbb{R}^{1+1}$. They form an improper basis in the eigenspace of the Casimir operator
\[
M^2 = \sqrt{p_0^2 - p_1^2}
\]
for the eigenvalue $m^2 > 0$.

The energy operator $P_0$ and the momentum operator $P_1$ act like multiplication operators in Fourier space:
\[
P_0 e^{i(t,q) \cdot (\sqrt{p_1^2 + m^2}, p_1)} = \sqrt{p_1^2 + m^2} e^{i(t,q) \cdot (\sqrt{p_1^2 + m^2}, p_1)},
\]
\[
P_1 e^{i(t,q) \cdot (\sqrt{p_1^2 + m^2}, p_1)} = p_1 e^{i(t,q) \cdot (\sqrt{p_1^2 + m^2}, p_1)}.
\]

We note that the inner product $(t, q) \cdot (\sqrt{p_1^2 + m^2}, p_1)$ equals $m$ times the Euclidean distance of the point $(t, q)$ from the line passing through the origin whose normal vector is $(\sqrt{p_1^2 + m^2}, -p_1)$.

Now let us compare this with the situation on de Sitter space. Consider a point $x \in \Gamma^+(W_1) \subset dS$, parametrized by $t$ and $q$, i.e.,
\[
x(t, q) = \Lambda_1 \left( \frac{\tau}{t} \right) \mathcal{D} \left( \frac{\tau}{t} \right) \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
\[
= \begin{pmatrix} \cosh \frac{r}{\tau} & 0 & \sinh \frac{r}{\tau} \\ 0 & 1 & 0 \\ \sinh \frac{r}{\tau} & 0 & \cosh \frac{r}{\tau} \end{pmatrix} \begin{pmatrix} \frac{q^2}{2\tau} e^{-\frac{r}{\tau}} + r \sinh \frac{r}{\tau} \\ \frac{q^2}{2\tau} e^{-\frac{r}{\tau}} + r \cosh \frac{r}{\tau} \end{pmatrix} = \begin{pmatrix} \frac{q^2}{2\tau} e^{-\frac{r}{\tau}} + r \sinh \frac{r}{\tau} \\ \frac{q^2}{2\tau} e^{-\frac{r}{\tau}} + r \cosh \frac{r}{\tau} \end{pmatrix}
\]

We have \( \lim_{r \to \infty} \left[ x(t, q) - \begin{pmatrix} t \\ q \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). In particular, for \( r \) very large the $x_2$-component of $x(\tau, \xi)$ approaches $r$.

**Lemma 3.4.6** (Bros & Moschella [28]). As $r \to \infty$, the generalised plane wave

\[
(3.4.9) \quad \mathbb{R}^{1+1} \ni (t, q) \mapsto \left( \frac{x(t, q)}{r} , \frac{p}{m} \right)^{-\frac{1}{2} + \text{i} m r}
\]

approaches—see (2.2.6) and (3.4.8)—the plane wave

\[
(3.4.10) \quad (t, q) \mapsto e^{\pm \text{i} t \sqrt{p_1^2 + m^2}} q_1 p_1,
\]

of a Minkowski space particle with mass $m$.

---

\(^{12}\)Note that in [28] p. 358] the order of the group elements in the parametrisation is reversed.

\(^{13}\)Note that in a neighbourhood of the origin the curves in $dS$, for which $x \cdot p$ is constant, become straight lines in $dS$ perpendicular to $p$ as the radius $r \to \infty$ in (3.4.1).
PROOF. We set \( p = \left( \begin{pmatrix} \frac{q^2}{2r} e^t - r \sinh \frac{1}{r} \\ q \end{pmatrix}, \frac{p^2}{2r} + m^2 \right) \), \( m > 0 \). Now consider (3.4.5). Note that \( x \cdot p_\pm(t) = \Lambda_1(-t)x \cdot p_\pm \). It follows that
\[
\lim_{r \to \infty} \frac{1}{m r} \left[ \left( \frac{q^2}{2r} e^t - r \sinh \frac{1}{r} \right) q \right] = \lim_{r \to \infty} e^{\ln \left( \frac{p^2}{2r} + m^2 \right)} (\frac{1}{r} \mp \im r).
\]
The result now follows by expanding the logarithm \([28\) Equ. (4.5)].

Let \( \mathcal{D}_{\pm m} \) be the unitary irreducible representation of the Poincaré group \( E_0(1, 1) \) for mass \( m \) and spin zero induced by the representation
\[
(t, q) \mapsto e^{\pm it\sqrt{p_1^2 + m^2} + iqp_1} 
\]
of the translation subgroup and by the representation \( \sigma_+ \) of the little group \( \{ 1, p_1 \} \), i.e.,
\[
\left( \mathcal{D}_m(\Lambda_2(s)\Lambda_1(\frac{1}{r})D(\frac{t}{r}))g_+ \right)(p_1) = e^{i(t, q)(\sqrt{p_1^2 + m^2} p_1)} g_+(p_1 + p_1'), \quad (t, q) \in \mathbb{R}^{1+1},
\]
and \( m \sinh s = p_1' \). Here \( T(t, q) \) denotes the space-time translations in \( \mathbb{R}^{1+1} \).

The following result is closely related to a theorem on group contractions by Mickelsson and Niederle \([144\) Theorem 2].

**Theorem 3.4.7.** Consider the representation \( \tilde{u}_{m_r} \equiv \tilde{u}_{mr} \) of \( SO_0(1, 2) \) acting on the de Sitter space \( dS \). Let \( g \in L^2(\Gamma_1, d\mu_1) \). Then
\[
\lim_{r \to \infty} \left\| \tilde{u}_{mr} (\Lambda_2(s)\Lambda_1(\frac{1}{r})D(\frac{t}{r}))g \right\|_{L^2(\Gamma_1, d\mu_1)}.
\]
Here \( \mathcal{D}_m \) is the reducible representation of the Poincaré group \( E_0(1, 1) \) given by
\[
\mathcal{D}_m = \mathcal{D}_+ \oplus \mathcal{D}_-.
\]

**Proof.** We have to show that
\[(3.4.11) \quad \left( \tilde{u}_{mr} (\Lambda_2(s)\Lambda_1(\frac{1}{r})D(\frac{t}{r}))g_\pm \right)(p_1) \to e^{\pm it\sqrt{p_1^2 + m^2} - qp_1} g_\pm(p_1 + p_1') \]
in \( L^2(\mathbb{R}, \frac{dp_1}{2\sqrt{p_1^2 + m^2}}) \) as \( r \to \infty \), with \( p_1' = m \sinh s \). Note that, by definition,
\[
\tilde{u}_{mr} (\Lambda_2(s)g_\pm)(p_1) = g_\pm(p_1 + p_1').
\]
Thus it is sufficient to show that
\[
\tilde{u}_{mr} (\Lambda_1(\frac{1}{r})D(\frac{t}{r}))g_\pm \to e^{\pm it\sqrt{p_1^2 + m^2} - qp_1} g_\pm \quad \text{in} \quad L^2(\Gamma_1, d\mu_1) \quad \text{as} \quad r \to \infty.
\]
In order to be able to interchange the limit with the integration, we approximate \( g_\pm \) with continuous functions with compact support. Set
\[
F_r^\pm(p_1) \equiv \left( (\tilde{u}_{mr} (\Lambda_1(\frac{1}{r})D(\frac{t}{r}))g_\pm)(p_1) - e^{\pm it\sqrt{p_1^2 + m^2} - qp_1} g_\pm(p_1) \right)^2.
\]
It follows that
\[(3.4.12) \quad \lim_{r \to \infty} \left( \int_{\Gamma_1} dp_1 F_r^\pm(p_1) \right)^{1/2} = \left( \int_{C} dp_1 \lim_{r \to \infty} F_r^\pm(p_1) \right)^{1/2} = 0.
\]
We have used (3.4.8) and the fact that \( F^\pm_r(p_1) \) is zero outside of some compact region \( C \subset \Gamma_1 \) for \( t, q \) fixed and \( r \) sufficiently large. (This follows form \( g_\pm \in C_0(\mathbb{R}) \).) Note that the set \( C_0(\mathbb{R}) \subset L^2(\mathbb{R}, \frac{dp_1}{2\sqrt{p_1^2 + m^2}}) \) of continuous functions with compact support is dense in \( L^2 \). (In Minkowski space, these functions would be called finite energy wave-functions.) Thus (3.4.11) follows for \( g_\pm \in L^2(\Gamma_1, d\mu_1) \). \( \square \)
Harmonic Analysis on the Hyperboloid

Harmonic analysis on semi-simple Lie groups originated with the monumental work of Harish-Chandra \[89\] – \[96\]. The subject has been developed further in particular by Helgason. But strictly speaking, the framework considered by Helgason \[98\], namely symmetric spaces \(G/H\), does not cover the case of the one-sheeted hyperboloid, as the stabilizer \(H \cong (\mathbb{R},+)\) of the Lie group \(G \cong SO_0(1,2)\) fails to be a compact subgroup. The necessary alterations can be found in the work of Molchanov \[145, 146\] and Faraut \[53\].

In this work we follow a more recent approach to Fourier(-Helgason) transformation on de Sitter space, due to Bros and Moschella \[29\], which emphasises the role of tuboids. The advantage of this approach is that the analyticity properties of the Fourier transform are evident from the definition.

4.1. Tuboids

Consider the complex de Sitter space
\[
\mathbb{dS}_C = \{z \in \mathbb{C}^{1+2} | z_0^2 - z_1^2 - z_2^2 = -r^2\}.
\]
A tuboid is a subset of \(\mathbb{dS}_C\), which is bordered by real de Sitter space \(\mathbb{dS}\) and whose shape (called its profile) near each point \(x\) of \(\mathbb{dS}\) can be mapped to a cone \(T_x\) in the tangent space \(T_x\mathbb{dS}\). The precise definition of a profile can be found in \[28\]; for the benefit of the reader we recall it in Appendix C.

Complex de Sitter space \(\mathbb{dS}_C\) is equipped \[29\] with four distinguished tuboids, which are invariant under the action of the proper orthochronous Lorentz group \(SO_0(1,2)\):
\[
\mathcal{T}_\pm = \{\Lambda(\theta,0,0) | 0 < \theta < \pi, \Lambda \in SO_0(1,2)\},
\]
\[
\mathcal{T}_+ = \{\Lambda(0,t) | t > 0, \Lambda \in SO_0(1,2)\}.
\]
The chiral tuboids \(\mathcal{T}_\pm\) and \(\mathcal{T}_+\) are not simply-connected. Their profiles at the origin \(o = (0,0,r)\) of \(\mathbb{dS}\) are the cones
\[
T_0 \mathbb{dS} \cap \{y \in \mathbb{R}^{1+2} | \mp y_1 > |y_0|\}.
\]
The chiral tuboids play a key role for quantum fields on anti-de Sitter space \[37, 27\], but are of no relevance for this work.

The Lorentzian tuboids \(\mathcal{T}_\pm\) are similar in many respects to the tubes \(\mathcal{T}_+\)
\[
\mathcal{T}_\pm = \mathbb{R}^{1+2} \mp iV^+.
\]

\[\text{Of course, } V_+ \text{ here denotes the future light cone in } \mathbb{R}^{1+2}. \text{ Note that our sign convention follows } [181], \text{ in contrast to the less common sign convention chosen in } [29].\]
defined in complex Minkowski space. In fact [29 Proposition 2],
\[ T_{\pm} = \mathcal{I}_{\pm} \cap dS_{C} . \]

**Proposition 4.1.1** (Proposition 1, [29]). *The tuboids \( T_{\pm} \) consists of all points \( z \in dS_{C} \) for which the inequality \( \mp \mathcal{I}z \cdot p > 0 \) holds for all \( p \in \partial V^{+} \), i.e.,

\[ T_{\pm} = \{ z \in dS_{C} \mid \mp \mathcal{I}z \cdot p > 0 \ \forall p \in \partial V^{+} \setminus \{(0,0,0)\} \} . \]

**Proof.** Consider the vectors \( p = (1,0,-1) \) and \( q = (0,r,0) \). Since \( p \cdot p = 0 \) and \( p \cdot q = 0 \), we find

\[ (\lambda p + \mu q) \cdot p = 0 , \quad \lambda, \mu \in \mathbb{R} . \]

The plane spanned by \( p \) and \( q \) separates the regions in \( \mathbb{R}^{3} \) for which \( x \cdot p < 0 \) and \( x \cdot p > 0 \), respectively. The latter half-space includes the positive \( x_{0} \)-axis. Rotating the vector \( p \) and taking the intersection of the resulting regions for which the scalar product \( x \cdot p \) is positive, yields the forward light cone \( \partial V_{+} \setminus \{(0,0,0)\} \). □

The profile of the *forward tuboid* \( T_{+} \) near each point \( x \) of \( dS \) (in the space of \( \mathcal{I}z \) and for \( \mathcal{I}z \setminus x \)) is the cone

\[ \mathcal{P}_{+}^{x} = \mathcal{T}_{x} \cap (-V_{+}) \]

in the tangent space \( \mathcal{T}_{x} dS \) at the point \( x \in dS \). (Note that in (4.1.3) the tangent space \( \mathcal{T}_{x} dS \cong \mathbb{R}^{2} \) at \( x \in dS \) is viewed as a subspace of \( \mathcal{T}_{x} \mathbb{R}^{3} \cong \mathbb{R}^{3} \).) For the origin \( o \in dS \) this yields \( \mathcal{P}_{+}^{o} = \{ y \in \mathbb{R}^{3} \mid -y_{0} > |y_{1}|, y_{2} = 0 \} \).

**4.1.1. The Euclidean sphere.** Applying the rotations \( R_{0}(\alpha) \), \( \alpha \in [0,2\pi) \), to the half-circles

\[ \{ (ir \sin \theta, 0, r \cos \theta) \in dS_{C} \mid 0 < \mp \theta < \pi \} \]

we find the (open) lower and upper hemispheres

\[ S_{\mp} = \{ (ir \lambda_{\phi}, x_{1}, x_{2}) \in (i\mathbb{R}) \times \mathbb{R}^{2} \mid \lambda_{0}^{2} + x_{1}^{2} + x_{2}^{2} = r^{2}, \mp \lambda_{0} > 0 \} \]

of the *Euclidean sphere* \(^{3}\)

\[ S^{2} = \left\{ (ir \sin \theta \cos \psi, r \sin \psi, r \cos \theta \cos \psi) \in \mathbb{C}^{3} \mid \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \psi \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \right\} . \]

Thus \( T_{\pm} = \{ \Lambda S_{\mp} \mid \Lambda \in SO_{0}(1,2) \} \) and, consequently,

\[ S^{2} \subset \mathcal{T}_{+} \cup \mathcal{T}_{-} \subset dS_{C} . \]

\(^{2}\)We note that this plane contains the light rays \( (0, \pm r, 0) + \lambda(1,0,-1), \lambda \in \mathbb{R} \), which form the horosphere \( P_{-\infty} \), see (4.2.4).

\(^{3}\)Applying the boosts \( \Lambda_{x}(t), t \in \mathbb{R} \), to the half-circles (4.1.4), followed by the rotations \( R_{0}(\alpha) \), \( \alpha \in [0,2\pi) \), yields the interior of the one-sheeted hyperboloid in \((i\mathbb{R}) \times \mathbb{R}^{2} \) with the (closed) past light cone and the interior of the future mass shell for the value \( m = r \) removed.

\(^{4}\)Note that the definition of \( S^{2} \) in (4.1.6) refers to the Lorentz metric (1.3.4).
4.1.2. Rotations. The rotations, which leave the Euclidean sphere (4.1.6) invariant, form the subgroup $\text{SO}(3)$ of $\text{SO}_C(1,2)$; the imaginary part in the square bracket on the right hand side of (2.3.1) is in agreement with (4.1.6). We denote the generators of the rotations around the three coordinate axis by $K_0$, $K_1$ and $K_2$, and set
\[ K^{(\alpha)} = \cos \alpha K_1 + \sin \alpha K_2, \quad \alpha \in [0, 2\pi), \]
in agreement with the definition of $L^{(\alpha)}$ in (1.1.5).

**Remark 4.1.2.** Clearly, the decomposition of the Euclidean sphere into a lower and an upper hemisphere in (4.1.5) distinguishes a Cauchy surface $S = S_1 \subset dS$. However, as $T_+^+$ is invariant under the action of $\text{SO}_0(1,2)$, one might as well consider the Lorentz transformed Cauchy surface $\Lambda S_1 \subset dS$ together with a Lorentz transformed sphere $\Lambda S_2 \subset T_+^+ \cup T_-^+ \subset dS_C$, $\Lambda \in \text{SO}_0(1,2)$.

**Lemma 4.1.3.** Let $x = R_0(\psi) o \in S^1$, $\psi \in [0, 2\pi)$. Then
\begin{equation}
\Gamma^\pm(x) = \left\{ \Lambda^{(\alpha)}(t)x \in dS \mid t \in \mathbb{R}^\pm, \alpha \in (\psi - \pi/2, \psi + \pi/2) \right\}.
\end{equation}
Moreover, the map
\begin{equation}
\tau \mapsto \Lambda^{(\alpha)}(\tau)x, \quad \alpha \in (\psi - \pi/2, \psi + \pi/2),
\end{equation}
is entire, and for $\tau \in S = \mathbb{R} \mp i(0,\pi]$ the map (4.1.8) takes values in $T_\pm$.

**Remark 4.1.4.** Given an arbitrary point $x \in dS$, formulas analogous to (4.1.7) and (4.1.8) hold true for all possible choices of space-like geodesics passing through the point $x$. Note that a space-like geodesic is used to define $\Lambda^{(\alpha)}$.

For $\alpha \neq 0$ the map $\mathbb{R} \ni t \mapsto \Lambda^{(\alpha)}(t)o$ no longer describes the geodesic motion of a free falling observer. As $\alpha \to \pm\pi/2$, the observer following the path $(\Lambda^{(\alpha)}(t)o \mid t \in \mathbb{R})$ is exposed to a *uniformly accelerated motion*, namely a boost, and will observe a temperature $((2\pi \cos \alpha)^{-1}$. This result follows by parameterising the path (4.1.7) in the proper time, see (1.1.5) and also [148]. In other words, the result of Bisognano-Wichmann [21, 22] and Unruh [186] remains valid on $dS$ (see also [18]).

**Lemma 4.1.5.** Let $M = \{(\psi, \alpha) \in S^1 \times S^1 \mid |\alpha - \psi| < \pi/2\}$ be the double twisted Möbius strip. Here $|\alpha - \psi|$ denotes the minimal distance on $S^1$. The map
\begin{equation}
S \times M \to T_+^+ \cup T_-^+ \subset dS, \quad (\tau, \psi, \alpha) \mapsto \Lambda^{(\alpha)}(\tau)R_0(\psi)o
\end{equation}
is surjective and, if restricted to $-\pi/2 < 3\tau < 0$, it is one-to-one onto $T_+^+ \setminus \{z \in dS_C \mid \Re z = 0\}$.
PROOF. Let $\tau = t + i\theta$, with $-\pi/2 < \theta < 0$. Then

\[(4.1.10) \quad \Lambda^{(\alpha)}(\tau) R_0(\psi) o = u + iy\]

with

\[(4.1.11) \quad u = \begin{pmatrix} \cos(\psi - \alpha) \cos \theta \sinh t \\ -\cos \alpha \sin(\psi - \alpha) - \sin \alpha \cos(\psi - \alpha) \cos \theta \cosh t \\ -\sin \alpha \sin(\psi - \alpha) - \cos \alpha \cos(\psi - \alpha) \cos \theta \cosh t \end{pmatrix}\]

and

\[(4.1.12) \quad y = \sin \theta \cos(\psi - \alpha) \begin{pmatrix} \cosh t \\ -\sin \sinh t \\ \cos \alpha \sinh t \end{pmatrix}.\]

The vector $y$ is time-like, i.e., $0 \leq y \cdot y \leq 1$, and

\[x = \frac{1}{\sqrt{1 - y \cdot y}} u \in dS.\]

Moreover, $u \cdot y = 0$. The equality $u \cdot y = 0$ implies that $u + iy \in dS_C$, as

\[(4.1.13) \quad dS_C = \{(u, y) \in \mathbb{R}^6 \mid u \cdot u - y \cdot y = -1, u \cdot y = 0\}.\]

Now assume that $u + iy$ can be written (see (4.1.10)) as $\Lambda^{(\alpha')}(\tau') R_0(\psi') o$. A short calculation, using (4.1.11) and (4.1.12), shows that $\psi' = \psi$, $\alpha' = \alpha$ and $\tau' = \tau$, using the restriction $-\pi < \tau, \tau' < 0$ to ensure the latter equality. Thus there are no further ambiguities, and uniqueness of the restriction follows. \(\square\)

The coordinates provided by the map (4.1.9) can not be extended to the whole tuboid $\mathcal{T}_+$: the south pole of the Euclidean sphere $(i, 0, 0) \in S^2$ would correspond to $\theta = -\pi/2$ and $\psi \in S^1$. Similarly the coordinate system would be degenerated at every single point in the purely imaginary negative unit mass-shell (compare to Eq. (4.1.13))

\[(z \in \mathcal{T}_+ \mid \Re z = 0) = \{\Lambda(-i, 0, 0) \mid \Lambda \in \SO(1, 2)\}.\]

For $-\pi < \theta < 0, \theta \neq -\pi/2$, the identity

\[\Lambda^{(\psi)}(i\theta) R_0(\psi) o = \Lambda^{(\psi + \pi)}(i(-\pi - \theta)) R_0(\psi + \pi) o\]

exemplifies the two possibilities to reach a single point on the Euclidean sphere (within the tuboid $\mathcal{T}_+$) from the circle $S^1$; the two points $R_0(\psi) o$ and $R_0(\psi + \pi) o$ are opposite to each other on the circle, and

\[\Lambda^{(\alpha + \pi)}(\tau) = \Lambda^{(\alpha)}(-\tau)\]

for all $\tau \in \mathbb{C}$ with $\Re \tau = 0$.

**Lemma 4.1.6.** For every point $z \in \mathcal{T}_+$ one can find two wedges $W_1, W_2$, close to each other\(^5\) and two angles $\theta_1, \theta_2 \in (0, \pi)$ as well as two points $x_1 \in W_1$ and $x_2 \in W_2$ such that

i.) $\Lambda_{W_1}(i\theta_0) x_1 = z = \Lambda_{W_2}(i\theta_2) x_2$;

---

5Two wedges are close to each other if their edges are close to each other in $dS$ w.r.t. the Euclidean metric.
ii.) the map

\[(\tau_1, \tau_2) \mapsto \Lambda_{W_1}(\tau_1)\Lambda_{W_1}(\tau_1)z\]  

(4.1.14)

gives rise to a holomorphic chart in a neighbourhood of 0.

**Proof.** It is known that for every \( z \in \mathcal{T}_+ \) there exists [147] Lemma A.2] an angle \( \theta_1 \in (0, \pi) \) and some \( \Lambda \in SO_0(1,2) \) such that

\[\Lambda^{-1}z = z_0 = \begin{pmatrix} 0 \\ r \end{pmatrix} \cos \theta_1 + i \begin{pmatrix} r \\ 0 \end{pmatrix} \sin \theta_1 \in S_+ ,\]

i.e., \( z_0 = \Lambda_1(\theta_1)0 \), with \( \alpha = (0,0, r) \) the origin; see also (2.3.1). Hence \( z \) is of the form as claimed in (i), namely \( z = \Lambda_{W_1}(i\theta_1)x_1 \), with \( W = \Lambda W_1 \) and \( x_1 = \Lambda x_0 \).

It is noteworthy that \( \theta_1, x_0 \) and \( W_1 \) can be directly characterized in a coordinate-independent manner: let \( z = u + iy \). The real part \( u \) satisfies \( \frac{\partial u}{\partial r} \in (-1,0] \) and is orthogonal to \( y \). Then

\[\theta_1 = \arccos \sqrt{\frac{u \cdot u}{r^2}}, \quad x_1 = \frac{u}{r \cos \theta_1} ,\]

and \( W_1 \) is the causal completion of the unique time-like geodesic in \( \text{dS} \) starting at \( x_1 \) with initial velocity \( y \). (Note that \( y \) is orthogonal to \( x_1 \) and can therefore be identified with a tangential vector at \( x_1 \).

By construction, the boosts \( \Lambda_{W_1}(t) \) leave the \( u-y \) plane in the ambient space \( \mathbb{R}^{1+2} \) invariant. Hence the generator \( L_W \) leaves the complex \( u-y \) plane in ambient \( \mathbb{C}^3 \) invariant. Now pick a different wedge \( W_2 \) sufficiently close to \( W_1 \) and such that the vector \( L_{W_2}z \) is not in the \( u-y \) plane. (This implies that \( W_2 \neq R_0(\alpha)W_1 \) for all \( \alpha \in (0,2\pi) \).) It follows that the vectors \( L_{W_1}z \) and \( L_{W_2}z \) are linearly independent and (4.1.14) is a holomorphic chart in a neighbourhood of \( z \). Furthermore, if \( W_2 \) is close enough to \( W_1 \), then the line segment

\[\{ \Lambda_{W_1}(-i\theta_2)z | 0 < \theta_2 < \pi \}\]

intersects \( \text{dS} \) in some point, say \( x_2 \in \text{dS} \) (just like the line segment \( \{ \Lambda_{W_1}(-i\theta_1)z | 0 < \theta_1 < \pi \} \), which intersects \( \text{dS} \) in \( x_1 \in \text{dS} \)). Thus there is some \( \theta_2 \) such that \( z = \Lambda_{W_2}(i\theta_2)x_2 \), as claimed in i.) \( \Box \)

Next we provide a flat tube theorem (see, e.g., [25][26]; an early result of this type is due to Malgrange and Zerner) for the de Sitter space.

**Theorem 4.1.17.** Let \( f \) be a tempered distribution on \( \text{dS} \) with the following property: for any wedge \( W \subset \text{dS} \) and any \( x \in W \), the map

\[(t) \mapsto f(\Lambda_W(t)x)\]

(4.1.15)

can be uniquely extended to a function defined and analytic in the strip \( S = \mathbb{R} + i(0,\pi) \), whose boundary values are described by (4.1.15).

Then \( f \) is the boundary value, in the sense of distributions, of a unique function \( F \), which is analytic in the tubeoid \( \mathcal{T}_+ \).

---

In fact, \( L_Wu \parallel y \) and \( L_Wy \parallel u \).
4. HARMONIC ANALYSIS ON THE HYPERBOLOID

PROOF. In a first step, assume that \( f \) is a continuous function. Let 

\[
z \in \mathcal{I}_+, \quad x_1 \in W_1, \quad x_2 \in W_2 \quad \text{and} \quad -\pi < \theta_1, \theta_2 < 0
\]
as in Lemma 4.1.6. By hypotheses, \( f \) can be analytically continued into the point \( z \) in the variable \( t+i\theta \) (defined in Eq. (4.1.14)) to \( z \) along the path 

\[
\theta \mapsto \Lambda_{W_j}(i\theta)x_1, \quad \theta_1 \leq \theta \leq 0.
\]
f can as well be analytically continued to \( z \) in the variables \( t_2+i\theta_2 \), namely along 

the path \( \theta' \mapsto \Lambda_{W_2}(i\theta')x_2 \). Both continuations coincide at \( z \), yielding a function \( F \), which is holomorphic separately in the variables \( t+i\theta \) and \( t'+i\theta' \) in a neighbourhood \( U \) of \( z \). By the flat tube theorem [163, Vol. I, Theorem IX.14.2], \( F \) is holomorphic on an open convex subset of \( U \). Since \( z \) was an arbitrary point in \( \mathcal{I}_+ \), it follows 

that \( F \) is holomorphic in \( \mathcal{I}_+ \). This proves the statement in case \( f \) is a continuous function.

The necessary generalization to tempered distributions, together with an appropriate generalization of the Bros-Epstein-Glaser Lemma [163, Vol. II, Theorem IX.15], will be given elsewhere. \( \square \)

The following result clarifies the relation between the tubo id \( \mathcal{I}_+ \), as described 
in Lemma 4.1.5, and the tangent bundle \( TdS \).

**Lemma 4.1.8** (Bros and Moschella [28], p. 339). The map

\[
\bigcup_{x \in dS} (x, T_x dS) \rightarrow dS_C
\]

(4.1.16)

\[
(x, y) \mapsto \sqrt{1-y \cdot y} \cdot x + iy
\]
is a one-to-one \( C^\infty \)-mapping from \( \bigcup_{x \in dS} (x, T_x dS) \) onto \( dS_C \setminus \{ z \in dS_C | \Re z = 0 \} \), and if 

\( y \in V^+ \), the map (4.1.16) defines a diffeomorphism from 

\[
\bigcup_{x \in dS} (x, P^+_x \cap \{ y \in V^+ | y \cdot y < 1 \})
\]
ono \( \mathcal{I}_+ \setminus \{ z \in dS_C | \Re z = 0 \} \). \( P^+_x \) is defined in (4.1.3).

4.2. Plane waves

The eigenfunctions of the Casimir operator on the light-cone \( \partial V^+ \) are homogeneous functions of degree \( s = s^\pm \); see (3.3.8). Thus, in order to construct a plane wave on \( dS \ni x \), one considers homogeneous functions of the scalar product

\[
(x \cdot p = (x + \lambda p + \mu q) \cdot p, \quad \lambda, \mu \in \mathbb{R}.
\]
In (4.2.1) we have used (4.1.2), with \( q \in S^1 \) such that \( q \cdot p = 0 \). Given a point \( x \in \Gamma(W_1) \), the intersection of the plane

\[
\left\{ x + \left[ \lambda \left( \begin{array}{c} 0 \\ 1 \\ -1 \end{array} \right) + \mu q \right] | \lambda, \mu \in \mathbb{R} \right\},
\]
with the de Sitter space \( dS \) is just the horosphere

\[
P_x = \left\{ y \in dS | \Re \tau = y \cdot \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) \right\},
\]

\footnote{Of course, the planes (4.2.2) for different \( x \in \Gamma(W_1) \) are all parallel to each other.}
where $\tau \in \mathbb{R}$ is fixed by requesting $\text{re}^{\tau} = x \cdot \left( \frac{1}{\sqrt{2}} \right)$. The angle between the plane (4.2.2) in $\mathbb{R}^{1+2}$ containing the horosphere $P_{\tau}$ and the $x_0$-axis is always $\pi/4$.

For the two light rays forming the horosphere $P_{-\infty}$, i.e., the intersection of $dS$ with the plane (4.1.2), the scalar $x \cdot p$ vanishes and powers with negative real part have to be defined in distributional sense. One possibility, which we will pursue, is to define them as the boundary values of analytic functions, using the principal value of the complex powers. The characterisation of the tuboid given in (4.1.1) guarantees that the functions

$$z \mapsto (z \cdot p)^{s}$$

are holomorphic both in $\mathcal{T}_{+}$ and $\mathcal{T}_{-}$. Their boundary values as $z \in dS_{C}$ tend to $x \in dS$ from within the respective tuboids $\mathcal{T}_{+}$ and $\mathcal{T}_{-}$ of $dS$ are denoted as

$$x \mapsto (x_{\pm} \cdot p)^{s}, \quad x \in dS.$$

As expected, we encounter a discontinuity as $3x_{+} \cdot p \nearrow 0$ or $3x_{-} \cdot p \searrow 0$, respectively. Another way of denoting the function (4.2.3) is

$$\lim_{t \rightarrow \pm \infty} x(t) = \begin{cases} x_{+} \cdot p, & t < 0, \\ x_{-} \cdot p, & t > 0, \\ \text{Heaviside step function}, & t = 0. \end{cases}$$

An explicit computation\(^8\) shows that the plane waves given in (4.2.3) satisfy the Klein–Gordon equation

$$\left( \Box_{dS} + \mu^2 \right) (x_{\pm} \cdot p)^s = r^{-2} \left( K_0^2 - L_1^2 - L_2^2 + \frac{1}{4} + m^2 r^2 \right) (x_{\pm} \cdot p)^{-\frac{1}{2} + imr},$$

(4.2.5)

$$= 0, \quad -s(s+1) = \mu^2 r^2 = \frac{1}{4} + m^2 r^2,$$

on the de Sitter space $dS$. As we have seen in Equ. (3.3.4)–(3.3.8) they also satisfy the Klein–Gordon equation on the forward light cone $\partial V^{+}$ (see also (3.3.6)):

$$\left( \Box_{dS} + \dot{z}^2 \right) (x_{\pm} \cdot p)^s = 0, \quad -s(s+1) = \dot{z}^2.$$

Note that in contrast to the Minkowski space case, the operators $K_0, L_1$ and $L_2$ do not commute, so they cannot be represented as commuting multiplication operators in Fourier space.

### 4.3. The Fourier-Helgason transformation

**Definition 4.3.1.** Let $p \in \partial V^{+}$ and $s \in \{ z \in \mathbb{C} \mid -z(z + 1) > 0 \}$. The *Fourier-Helgason transforms* $\mathcal{F}_{\pm}$ are defined \(^{29}\) see also Definition 2\] by

$$\mathcal{D}(dS) \ni f \mapsto \mathcal{F}_{\pm}(p, s) = \int_{dS} d\mu_{dS}(x) f(x) (x_{\pm} \cdot p)^s.$$"
For \( p \) fixed, the functions \( \tilde{\eta}_\pm(p, .) \) are holomorphic with respect to \( s \) in the strip \(-1 < 9s < 0\) [Bros und Moschella [29], Prop. 8.a].

**Lemma 4.3.2.** The function

\[
\nu \mapsto \tilde{\eta}_\pm(p, -\frac{1}{2} - i\nu)
\]

is analytic in the open strip \( \{\nu \in \mathbb{C} | |\Im \nu| < \frac{1}{2}\} \).

For \( s \) fixed, the two functions \( \tilde{\eta}_\pm(., s) \) are continuous, homogeneous functions of degree \( s \) on \( \partial V^+ \). Together with (3.3.6) this implies that \( \tilde{\eta}_\pm(., s) \) is an eigenfunction of the Casimir operator \( M^2 \) on \( \partial V^+ \) iff \( s \) lies on

(i) the symmetry axis

\[
(4.3.2) \quad s = -1/2 \mp iv, \quad \nu = \sqrt{\mu^2 r^2 - \frac{1}{4}} = m r \in \mathbb{R}_0^+, \quad \text{i.e., to a \textit{bare mass} } m \geq 0;
\]

of the strip \(-1 < 9s < 0\). This choice corresponds to \( \mu^2 = \frac{1}{4m^2} + m^2 > \frac{1}{4m^2} \),

(ii) the symmetry axis

\[
(4.3.3) \quad s = -1/2 \mp iv, \quad \nu = i \sqrt{\frac{1}{4} - \mu^2 r^2} = im r, \quad \text{i.e., to a \textit{negative bare mass} } -\frac{1}{2m} < m \leq 0.
\]

Thus the critical mass \( \mu_c \), which separates the two cases, is \( \mu_c = \sqrt{-s(s + 1)} = (2r)^{-1} \). Note that the factor \((2r)^{-1}\) may be interpreted as a contribution to the mass coming from the curvature of space-time (see, e.g., [63]).

**Remark 4.3.3.** Taking advantage of (4.2.4), the Fourier-Helgason transforms \( \mathcal{F}_\pm \)

can be written in the following form (see [29 Eq. (50)])

\[
\tilde{\eta}_\pm(p, -\frac{1}{2} - i\nu) = \int_{\{x \in dS | x \cdot p > 0\}} d\mu_{dS}(x) f(x) |x \cdot p|^{-1/2 - i\nu} + e^{\mp i\pi s +} \int_{\{x \in dS | x \cdot (-p) > 0\}} d\mu_{dS}(x) f(x) |x \cdot p|^{-1/2 - i\nu}.
\]

This identity is valid in the open strip \( \{\nu \in \mathbb{C} | |\Im \nu| < 1/2\} \). The second term can be viewed as a continuous, homogeneous function of degree \( s^+ \) on \( \partial V^- \).

### 4.4. The Plancherel theorem on the hyperboloid

Denote by \( H^2(\mathcal{T}_+), H^2(\mathcal{T}_-), H^2(\mathcal{T}_\pm) \) and \( H^2(\mathcal{T}_\rightarrow) \) the Hardy spaces of functions \( F(z) \) characterised by the following properties [29 Sect. 3.2][153 Sect. 3.3]:

i.) \( F \) is holomorphic in the tuboid considered;

ii.) the function \( F(z) \) admits boundary values \( f(x) \) on \( dS \) from this tuboid, which belong to \( L^2(dS, d\mu_{dS}) \);

iii.) \( F \) is ‘sufficiently regular at infinity in its domain’ (in the sense made precise in [29 p. 10]).

\[^{10}\text{Note that a function analytic in the strip } -1 < 9s < 0 \text{ is uniquely determined by its values on one of the two symmetry axis given in (4.3.2) and (4.3.3).} \]
4.4. THE PLANCHEREL THEOREM ON THE HYPERBOLOID

THEOREM 4.4.1 (Bros & Moschella [29], Theorem 1). Any given function \( f \in L^2(dS, d\mu_{dS}) \) admits a decomposition of the form

\[
(4.4.1) \quad f = f_+ + f_- + f_\leftrightarrow + f_\to \equiv \sum_{\text{tub}} f_{(\text{tub})}, \quad (\text{tub}) = +, -, \leftrightarrow, \to,
\]

where \( f_{(\text{tub})}(x) \in L^2(dS, d\mu_{dS}) \) is the boundary value of the function

\[
(4.4.2) \quad F_{(\text{tub})}(z) = e_{(\text{tub})} \frac{1}{\pi^2} \int_{dS} d\mu_{dS}(x) \frac{f(x)}{(x - z) \cdot (x - z)} \in H^2(\mathcal{T}_{(\text{tub})}).
\]

The sign function \( e_{(\text{tub})} \) takes the value \(-1\) for \( \mathcal{T}_\pm \), and \(+1\) for \( \mathcal{T}_+ \) and \( \mathcal{T}_- \).

REMARK 4.4.2. In Minkowski space \( \mathbb{R}^{1+1} \), a similar decomposition can be gained by simply dividing the support of the Fourier transform \( \tilde{f} \) into the four cones \( \{(E, p) \in \mathbb{R}^{1+1} \mid \pm E > |p|\} \) and \( \{(E, p) \in \mathbb{R}^{1+1} \mid \pm p > |E|\} \). Note that for \( m > 0 \) the boundary sets \( \{(E, p) \in \mathbb{R}^{1+1} \mid \pm p = |E|\} \) are of measure zero. The inverse Fourier transform of each of these functions is then the boundary of a function analytic in a tube. For the first two, the tube is \( T^+ = \mathbb{R}^2 \pm i \{ (x_0, x_1) \in \mathbb{R}^{1+1} \mid x_0 > |x_1| \} \). The situation is similar for the two other cases.

The Cauchy kernel on \( dS_C \) introduced in (4.4.2) is [29 Proposition 11]

\[
(4.4.3) \quad \frac{1}{(z' - z) \cdot (z' - z)} = -\frac{\pi^2}{2} \int_0^\infty d\mu_{\pm}(\nu) \int d\mu_{\Gamma}(p) (z \cdot p)^{-\frac{1}{2} + i\nu} (p \cdot z')^{-\frac{1}{2} - i\nu}.
\]

The integral in (4.4.3) is absolutely convergent for \((z, z') \in \mathcal{T}_+ \times \mathcal{T}_- \) for \( d\mu_{\pm} \) and for \((z, z') \in \mathcal{T}_- \times \mathcal{T}_+ \) for \( d\mu_{\pm} \), respectively. The measure \( d\mu_{\pm}(\nu) \) on \( \mathbb{R}^+ \) is (see [29 Sect. 4.1])

\[
d\mu_{\pm}(\nu) = \frac{1}{2\pi^2} \frac{\nu \tanh \pi\nu}{e^{\pi\nu} \cosh \pi\nu} d\nu.
\]

Combine (4.4.2), (4.4.3) and (4.3.1) to find the inversion formula [29 Eq. (80)]

\[
(4.4.4) \quad F_{\pm}(z) = -\int_0^\infty d\mu_{\pm}(\nu) \int d\mu_{\Gamma}(p) (z \cdot p)^{-\frac{1}{2} + i\nu} \tilde{f}_{\pm}(p, -\frac{1}{2} - i\nu).
\]

The functions \( f_{\pm}(x) \) introduced in (4.4.1) now appear as boundary values of the holomorphic functions \( F_{\pm}(z) \), \( z \in \mathcal{T}_{\pm} \).

REMARK 4.4.3. For every function \( F_{\pm} \) in the Hardy space \( H^2(\mathcal{T}_{\pm}) \) the transform \( \tilde{f}_{\pm}(p, -\frac{1}{2} - i\nu) \) vanishes [29 Proposition 8]. This follows from the analyticity properties stated in Proposition 6.1.7. A similar result holds true in the Minkowski space-time: The functions \( f \) on \( \mathbb{R}^{1+d} \), which are boundary values of holomorphic functions in either tube \( \mathcal{T}_{\pm} = \mathbb{R}^{1+d} \mp iV^\pm \) are exactly the functions characterised by the fact that their Fourier transforms

\[
\tilde{f}(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^{1+d}} dy \ f(y) \ e^{ik \cdot y}, \quad f \in \mathcal{D}(\mathbb{R}^{1+d}),
\]

have their support contained in the closure of either one of the cones \( V^\pm \); see, e.g., [171 Ch. 8].
THEOREM 4.4.4 (Molchanov [145]). For any pair of functions \( f, g \) in \( L^2(dS, d\mu_{dS}) \) and their corresponding decomposition given in (4.4.1), one has the Plancherel theorem\(^{11}\):

\[
\int_{dS} d\mu_{dS}(x) \overline{f_\pm(x)} g_\pm(x) = \int_0^\infty d\mu_{\pm}(\nu) \int_{\Gamma} d\mu_{\Gamma}(p) \overline{f_\pm(p, \frac{1}{2} - i\nu)} \overline{g_\pm(p, \frac{1}{2} - i\nu)}.
\]

The measures \( d\mu_{dS} \) and \( d\mu_{\Gamma} \) denote the Lorentz invariant measures on \( dS \) and the restriction of the Lorentz invariant measures on \( \partial V^+ \) to \( \Gamma \).

\(^{11}\)These are the Eq. (118) and Eq. (119) in [29].
Part 3

Classical Fields
In this chapter we consider the classical Lagrangian density
\[ \mathcal{L}(\Phi) = \frac{1}{2} (\nabla \Phi \cdot \nabla \Phi - \mu^2 \Phi^2 - P(\Phi)) . \]

Here \( \Phi \) is a real valued scalar field, \( \nabla \) denotes the Levi-Civita connection on \( \text{dS} \). The polynomial \( P \) is bounded from below and \( \mu > 0 \).

### 5.1. The classical equations of motion

Let \( K \) be a compact submanifold of \( \text{dS} \). The action associated to \( \mathcal{L} \) and \( K \) is
\[
S(K, \Phi) = \frac{1}{2} \int_K d^2 x \sqrt{|g|} \left( g^{\mu \nu} \partial_\mu \Phi(x) \partial_\nu \Phi(x) - \mu^2 \Phi^2 - P(\Phi(x)) \right).
\]

The (inhomogeneous) Klein–Gordon equation is obtained by demanding that for every such \( K \), the action \( S(K, \Phi) \) is stationary with respect to smooth variations \( \Phi \mapsto \Phi + \delta \Phi \) of \( \Phi \) that vanish on the boundary \( \partial K \) of \( K \). In other words, we require
\[ 0 = \frac{\delta S(K, \Phi)}{\delta \Phi(y)} = \int_K d\mu_{\text{dS}}(x) \left( \frac{\partial \mathcal{L}(\Phi(x))}{\partial \Phi(y)} + \frac{\partial \mathcal{L}(\Phi(x))}{\partial \Phi(x)} \frac{\delta \partial_\mu \Phi(x)}{\delta \Phi(y)} \right) \]
for every such \( K \). The resulting Euler-Lagrange equation
\[
\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0
\]
is the equation of motion\(^1\)
\[
\partial_\mu \left( \sqrt{|g|} g^{\mu \nu} \partial_\nu \Phi \right) + \sqrt{|g|} \left( \mu^2 \Phi + P'(\Phi) \right) = 0
\]
on \( \text{dS} \). In a more compact notation, this equation is rewritten as
\[
(\Box_{\text{dS}} + \mu^2) \Phi = -P'(\Phi), \quad \Phi \in C^\infty(\text{dS}), \quad \mu > 0 ,
\]
where \( \mu \) is a mass\(^2\) parameter. In the sequel we keep \( \mu > 0 \) fixed, and although almost all quantities we encounter depend on \( \mu \), we will suppress this dependence on \( \mu \) in the notation.

\(^1\)In local coordinates, the Laplace-Beltrami operator \( \Box_{\text{dS}} \) equals \( |g|^{-1/2} \partial_\mu g^{\mu \nu} |g|^{1/2} \partial_\nu \), with \( |g| \equiv |\text{det} g| \).

\(^2\)How \( \mu \) is related (or identical) to the physically observable mass of a particle on de Sitter space will be discussed elsewhere. See [63] for a discussion of several interpretations of \( \mu \) found in the literature.
5. CLASSICAL FIELD THEORY

5.2. Conservation Laws

The advantage of the Lagrangian formulation is that any one-parameter subgroup, which leaves the Lagrangian density invariant, gives rise to a conservation law\footnote{In \cite{[57]} p. 269, the authors have chosen $K = \mathcal{W}_1$, and thus the action $S(K)$ yields $S(\mathcal{W}_1) = \frac{1}{2} \int_{\mathcal{W}_1} r^2 \cos \psi \, d\psi \, dt \left( r^{-1} \cos^{-1} \psi - r^{-2} \left( \frac{\partial \psi}{\partial t} \right)^2 - \mu^2 \right)$. The invariance with respect to translations of $t$ yields the conserved quantity $L_{11t} = \int_{1+} r \cos \psi \, d\psi \, T_{00}$. In the last equation we have used $n = r^{-1} \cos^{-1} \psi \delta_t$ and $n = \psi$. Here $\mathcal{C}$ denotes unit normal, future pointing vector field, restricted to the Cauchy surface $\mathcal{C}$.}. If the 2-form $\mathcal{L}$ depends on the scalar $\Phi$, then its variation

$$\delta \mathcal{L} = \delta \Phi \wedge \left[ \frac{\partial \mathcal{L}}{\partial \Phi} - d \left( \frac{\partial \mathcal{L}}{\partial (d\Phi)} \right) \right] + d \left( \delta \Phi \wedge \frac{\partial \mathcal{L}}{\partial (d\Phi)} \right),$$

and the equations of motion

$$\frac{\partial \mathcal{L}}{\partial \Phi} - d \frac{\partial \mathcal{L}}{\partial (d\Phi)} = 0,$$

imply that

$$\delta \mathcal{L} = 0 \Rightarrow d \left( \delta \Phi \wedge \frac{\partial \mathcal{L}}{\partial (d\Phi)} \right) = 0.$$  

If the variation results from a Lie derivative $L_v$, with $v$ some vector field, then

$$\delta \mathcal{L} = L_v \mathcal{L} = d \left( i_v \mathcal{L} \right)$$

as the exterior derivative of the 2-form vanishes in two-dimensional space. It follows that

$$\sum_j \delta \Phi \wedge \frac{\partial \mathcal{L}}{\partial (d\Phi)}$$

is a closed 1-form, or, using the Hodge $\ast$, a conserved current.

**Theorem 5.2.1 (Noether).** Let $v \in E_1$ with $\delta \Phi = L_v \Phi$ and $\delta \mathcal{L} = L_v \mathcal{L}$. It follows that

$$d \left( L_v \Phi \wedge \frac{\partial \mathcal{L}}{\partial (d\Phi)} \right) =: -d \ast T_v = 0 .$$

**Remark 5.2.2.** If $L_v$ is a translation (i.e., $v = dx_\mu$, $\mu = 0, 1, \ldots, d$), on Minkowski space $\mathbb{R}^{1+d}$, then $T^\mu$ (as defined in Theorem 5.2.1) is called the energy-momentum tensor. Its components with respect to a basis define the energy-momentum tensor $T^\mu{}_{\nu}$:

$$T^\mu = T^\mu{}_{\nu} \, dx^\nu .$$

If one integrates over a space-like surface $N_d$, than one finds the energy $P_0$ and the momentum $P^1$:

$$P^\mu = \int_{N_d} \ast T^\mu .$$

$T^\mu{}_{\nu}$ describes the flux of the $\mu$-th component of the conserved energy-momentum vector across a surface with constant $x_\nu$ coordinate (see, e.g., \cite{[185]} p. 35).
A convenient basis to derive explicit expressions for the stress-energy tensor on de Sitter space is the following:

\[
\begin{pmatrix}
  x_0 \\
  x_1 \\
  x_2
\end{pmatrix} = \begin{pmatrix}
  \sqrt{r^2 + x_0^2 \sin \psi} \\
  \sqrt{r^2 + x_0^2 \cos \psi}
\end{pmatrix}, \quad x_0 \in \mathbb{R}, \quad \psi \in [0, 2\pi).
\]

The metric takes the form

\[ g = \text{d}x_0 \otimes \text{d}x_0 - \sqrt{r^2 + x_0^2} \text{d}\psi \otimes \text{d}\psi \]

and the stress-energy tensor is given by

\[
T^{\mu\nu} = \partial^\mu \Phi \partial^\nu \Phi - g^{\mu\kappa} g^{\nu\lambda} L(\Phi) = \partial^\mu \Phi \partial^\nu \Phi - \frac{1}{2} \delta^{\mu\nu} \left( \mu^2 \Phi^2 + P(\Phi) \right), \quad \mu, \nu = x_0, \psi.
\]

In particular,

\[ T_{00} = \frac{1}{2} \left( \pi^2 + r - 2 \left( \partial_\psi \Phi \right)^2 + \mu^2 \Phi^2 + P(\Phi) \right), \]

with \( \pi = \frac{\partial}{\partial x_0} \Phi \), and

\[ T_{0\psi} = \frac{1}{r} \partial_{x_0} \Phi \partial_\psi \Phi. \]

The Killing vector fields on \( dS \) are given by \( \partial_t \) (within the double cone \( W_1 \), using the coordinates introduced in (2.1.4)) and \( \partial_\psi \). The corresponding conserved quantities

\[ L^1 = \int_{S^1} *T^t \quad \text{and} \quad K_0 = \int_{S^1} *T^\psi \]

generate the \( \Lambda_1 \)-boosts and the rotations around the \( x_0 \)-axis, respectively. Rewriting \( \partial_t \Phi \) as

\[ \partial_t \Phi = \tau \cos \psi \ n \Phi \equiv \tau \cos \psi \ \pi, \]

where \( n \) is the future directed normal vector field to the time-circle \( x_0 = 0 \), we have

\[ L^1 = \frac{1}{2} \int_{S^1} \tau \cos \psi \ \text{d}\psi \left( \pi^2 + r^{-2} \left( \partial_\psi \Phi \right)^2 + \mu^2 \Phi^2 + P(\Phi) \right). \]

Using (5.2.1) yields the formula for the angular momentum

\[ K_0 = \int_{S^1} r \cos \psi \ \text{d}\psi \ T_{0\psi} = \int_{S^1} r \ \text{d}\psi \ \pi \left( \partial_\psi \Phi \right). \]

**Remark 5.2.3.** Integrating \( T_{00} \) over the time-zero circle \( S^1 \) yields a positive quantity,

\[ \int_{S^1} r \ \text{d}\psi \ T_{00}(\psi) > 0, \]

which may be interpreted as the energy density for the classical \( P(\psi)_2 \) model on the *Einstein universe* (see, e.g., [55][56]), i.e., the space-time of the form \( S^1 \times \mathbb{R} \).

Although there are interesting results concerning the non-linear Klein-Gordon equation in two space-time dimensions (see, e.g., [40][41][42][43][44][71][130]), we will concentrate on free fields for the rest of this chapter.

---

4Using the coordinates introduced in (2.1.3) we have \( g = r^2 (\cos^2 \psi \ \text{d}t \otimes \text{d}t - \text{d}\psi \otimes \text{d}\psi) \) and \( |g|^{1/2} = r^2 |\cos \psi| \).
5.3. The covariant classical dynamical system

As mentioned in Section 1.3, the de Sitter space-time \( dS \) is globally hyperbolic. Thus the inhomogeneous Klein–Gordon equation

\[
(\square_{dS} + \mu^2)\Phi = f, \quad f \in \mathcal{D}(dS),
\]

has smooth solutions, which are uniquely specified by fixing their support properties (see \([39, 48, 133, 136]\)):

**Theorem 5.3.1.** There exist unique operators

\[ E^\pm : \mathcal{D}(dS) \to C^\infty(dS) \]

such that \( E^\pm f \) is a solution of the inhomogeneous equation (5.3.1) with

\[
\text{supp } (E^\pm f) \subset \Gamma^\pm(\text{supp } f) \quad \text{and} \quad \text{supp } (E^\pm f) \cap \Gamma^\mp(\text{supp } f) \text{ compact}. 
\]

\( E^\pm f \) are called the **retarded** and the **advanced solution**, respectively. The difference between the retarded and the advanced solution of the inhomogeneous equation (5.3.1), namely

\[
\Phi = Ef, \quad \text{with } E = E^+ - E^-, 
\]

is a solution of the homogenous Klein–Gordon equation (5.1.1).

**Remark 5.3.2.** For comparison, we briefly recall the situation on Minkowski space \( \mathbb{R}^{1+1} \). After Fourier transformation, the inhomogeneous equation

\[
(\square_{\mathbb{R}^{1+1}} + m^2)G = -\delta 
\]

takes the simple form \((-P_0^2 + P_1^2 + m^2)\tilde{G} = -1\); the latter has the retarded and advanced propagators as its solution:

\[
\mathcal{E}_{\text{adv}}(x, y) = \lim_{\epsilon \downarrow 0} \frac{1}{(2\pi)^2} \int d^2p \frac{e^{-ip \cdot (x-y)}}{(p_0 - i\epsilon)^2 - p_1^2 - m^2},
\]

and

\[
\mathcal{E}_{\text{ret}}(x, y) = \lim_{\epsilon \downarrow 0} \frac{1}{(2\pi)^2} \int d^2p \frac{e^{-ip \cdot (x-y)}}{(p_0 + i\epsilon)^2 - p_1^2 - m^2},
\]

The difference \( \mathcal{E}(x, y) = \mathcal{E}_{\text{ret}}(x, y) - \mathcal{E}_{\text{adv}}(x, y) \) is a bi-solution of the Klein-Gordon equation. Note that if

\[
p \cdot (x - y) = 0
\]

the integral is defined in distributional sense only. We will soon encounter a similar problem on de Sitter space.

As we will see next, any smooth solution of (5.1.1) is of the type (5.3.2).
5.3. THE COVARIANT CLASSICAL DYNAMICAL SYSTEM

Theorem 5.3.3 (Bär, Ginoux and Pfäffle [10], Theorem 3.4.7).

i.) Any smooth solution \( \Phi \) of the Klein–Gordon equation (5.1.1) may be written in the form

\[
\Phi = E f , \quad \text{for some } f \in \mathcal{D}(dS) ;
\]

and, given any neighbourhood \( \mathcal{N} \) of a Cauchy surface \( \mathcal{C} \), one may choose such an \( f \in \mathcal{D}(\mathcal{N}) \).

ii.) We have

\[
\ker E = (\Box dS + \mu^2) \mathcal{D}(dS) .
\]

In consequence, the space of smooth real-valued solutions equals

\[
\mathcal{E} \mathcal{D}(dS) \cong \mathcal{D}(dS)/(\Box dS + \mu^2) \mathcal{D}(dS) \cong \mathfrak{t}(dS) .
\]

Taking advantage of the properties i.) and ii.), we can define a projection

\[
P : \mathcal{D}(dS) \to \mathfrak{t}(dS) \quad f \mapsto [f] .
\]

This yields a one-to-one correspondence between

\[
\mathfrak{t}(dS) \ni [f] \leftrightarrow f \in \mathcal{E} \mathcal{D}(dS) ,
\]

with \([f] \equiv \{f + (\Box dS + \mu^2) h | h \in \mathcal{D}(dS)\}\) and \(f \equiv Ef\).

Definition 5.3.4. Subspaces of \( \mathfrak{t}(dS) \) associated to open space-time regions \( \mathcal{O} \subset dS \) are defined by restricting \( P \) to \( \mathcal{D}(\mathcal{O}) \), i.e.,

\[
\mathfrak{t}(\mathcal{O}) \cong \mathcal{D}(\mathcal{O})/\ker E .
\]

\( \mathfrak{t}(\mathcal{O}) \) will be used in Chapter 8 to define local von Neumann algebras.

Embedding \( \mathcal{C}^\infty(dS) \) into \( \mathcal{D}'(dS) \) (see [55 Sect. 2][56]), the map \( E \) gives rise to a bidistribution \( \mathcal{E} \) on \( dS \times dS \)

\[
\mathcal{E}(f, g) \equiv \int d\mu_{dS}(x) f(x)(E g)(x)
\]

\[(5.3.3)\]

\[
\cong \int d\mu_{dS}(x) d\mu_{dS}(y) f(x) \mathcal{E}_d(x, y) g(y) ,
\]

antisymmetric in \( f, g \in \mathcal{D}(dS) \), whose kernel \( \mathcal{E}_d(x, y) \), called the fundamental solution, is a weak bisolution for the Klein–Gordon equation,

\[
\mathcal{E} (\Box dS + \mu^2) f, g) = \mathcal{E} (f, (\Box dS + \mu^2) g) = 0 ,
\]

with initial data\(^5\)

\[
(5.3.5) \quad \mathcal{E}_d|_{\mathcal{C} \times \mathcal{C}} = 0 ,
\]

\[
(5.3.6) \quad (\partial_t \mathcal{E}_d)|_{\mathcal{C} \times \mathcal{C}} = -\delta_{\mathcal{C}} .
\]

\(^5\)In [9] we will demonstrate that for the \( P(\varphi)_2 \) model on the de Sitter space, the expectation values of all observables can be predicted from the expectation values of observables, which can be measured within an arbitrarily small time interval. Thus the \( P(\varphi)_2 \) model on the de Sitter space satisfies the Time-Slice Axiom [38].

\(^6\)Micro-local analysis shows that \( \mathcal{E}_d \) and its normal derivatives can be restricted to \( \mathcal{C} \times \mathcal{C} \), see [106].
Here \( n_t \) denotes the vector field \( n \) acting on the left variable \( x \) in \( \mathcal{E}(x, y) \) and \( \delta_\mathcal{E} \) is the integral kernel of the unit operator with respect to the induced measure on \( \mathcal{E} \). The map

\[
\mathcal{D}(d\mathcal{S}) \ni f \mapsto \mathcal{E}f
\]

can now be viewed as a convolution\(^7\) of a test function \( f \) with the kernel \( \mathcal{E} \), i.e.,

\[
f(x) = \int \mathcal{D}_d\mathcal{S}(y) \mathcal{E}(x, y)f(y), \quad f \in \mathcal{D}_d\mathcal{S}(d\mathcal{S}).
\]

Eq. (5.3.7) implies that \( f(x) = 0 \) for all \( x \in d\mathcal{S}, \) iff

\[
f \in \ker \mathcal{E} \equiv \{ f \in \mathcal{D}_d\mathcal{S}(d\mathcal{S}) \mid \mathcal{E}(g, f) = 0 \quad \forall g \in \mathcal{D}_d\mathcal{S}(d\mathcal{S}) \}.
\]

In other words, \( \ker \mathcal{E} = \ker \mathcal{E} \). Consequently, the bidistribution \( \mathcal{E} \) provides a non-degenerated symplectic form \( \sigma \) on the space of solutions \( \mathcal{D}(d\mathcal{S}) \):

\[
\sigma([f], [g]) = \mathcal{E}(f, g), \quad f, g \in \mathcal{D}_d\mathcal{S}(d\mathcal{S}).
\]

As a consequence of (5.3.4), the right hand side does not dependent on the choice of the representatives in the equivalence classes \([f]\) and \([g]\). Thus \( (\mathcal{D}(d\mathcal{S}), \sigma) \) is a the symplectic vector space.

**Lemma 5.3.5.** Let \( f \in \mathcal{D}_d\mathcal{S}(\emptyset), \emptyset \subset d\mathcal{S} \) an open region. Then \( f = \mathcal{E}f \) is a solution of the Klein–Gordon equation with

\[
\text{supp } (f) \subset \Gamma^+(\emptyset) \cup \Gamma^-(\emptyset).
\]

In particular, if \( \emptyset \subset W \), then \( \text{supp } (f) \subset d\mathcal{S} \setminus W \).

**Proof.** The support properties of \( \mathcal{E}^\pm \) force \( \mathcal{E}(f, g) \) to vanish, whenever the support of \( f \) is space-like separated from that of \( g \). Thus, for \( g \in d\mathcal{S} \) fixed, the distribution \( x \mapsto \mathcal{E}(x, y) \) has support in \( \Gamma^+(y) \cup \Gamma^-(y) \). The final statement follows from this fact as well.

Exploring the one-to-one correspondence between \( \mathcal{D}(d\mathcal{S}) \ni [f] \) and \( f \in \mathcal{E}\mathcal{D}_d\mathcal{S}(d\mathcal{S}) \), this result can be rephrased in the following way.

**Lemma 5.3.6.** Let \( f \in \mathcal{D}_d\mathcal{S}(\emptyset), \emptyset \subset d\mathcal{S} \) a bounded open region, and \( g \in \mathcal{D}_d\mathcal{S}(\emptyset^\prime) \), where \( \emptyset^\prime \) denotes the space-like complement of \( \emptyset \). Then

\[
\sigma([f], [g]) = 0.
\]

**Proposition 5.3.7.** The symplectic space \( (\mathcal{D}(d\mathcal{S}), \sigma) \) carries a representation

\[
\Lambda \mapsto u(\Lambda), \quad \Lambda \in O(1, 2),
\]

of the Lorentz group.

**Proof.** The group of isometries \( \Lambda \in O(1, 2) \) of \( d\mathcal{S} \) gives rise to a group of symplectic transformations \( \Lambda \mapsto T_\Lambda \) on \( (\mathcal{D}(d\mathcal{S}), \sigma) \) induced by the pull-back \( \Lambda_* \), which maps

\[
f + \ker \mathcal{E} \mapsto \Lambda_* f + \ker \mathcal{E}.
\]

The map (5.3.11) is well-defined, because \( g \in \ker \mathcal{E} \) implies \( \Lambda_* g \in \ker \mathcal{E} \). \( \square \)

\(^7\)On Minkowski space, Fourier transformation converts a convolution in position space to a multiplication in momentum space. For the situation on \( d\mathcal{S} \), see Section [4].
5.4. The restriction of the KG equation to a (double) wedge

Our next objective is to provide an explicit formula for $\mathcal{E}(x, y)$. In some sense, it is sufficient to solve this problem in the causal dependence region of a half-circle: given an arbitrary point $x \in \mathcal{dS}$ and the Cauchy surface $S^1$, there exists a wedge $W^{(\alpha)} = R_0(\alpha)W_1$, which contains both $x$ and $\Gamma^-(x) \cap S^1$ (or, if this intersection is empty, $\Gamma^+(x) \cap S^1$). On the other hand, all the formulas we will derive in this section naturally extend to the double-wedge $\mathcal{W}^{(\alpha)} = W^{(\alpha)} \cup W^{(\alpha + \pi)}$, so it is natural to state them in their extended form.

In order to keep the notation simple, we work out explicit expressions for the double wedge $\mathcal{W}_1$ in the chart (2.1.4) for $x \equiv x(t, \psi)$ and $y \equiv y(t', \psi')$. (The points $\psi = \pm \frac{\pi}{2}$ in this chart correspond to the points $(0, \pm \tau, 0) \in \mathcal{dS}$.) However, we would like to emphasize that all computations in this subsection can be carried out for arbitrary double wedges $\Lambda W_1$, $\Lambda \in \text{SO}(1, 2)$.

The restriction of the metric $g$ to $\mathcal{W}_1$ is
$$g|_{\mathcal{W}_1} = r^2 \cos^2 \psi dt \otimes dt - r^2 d\psi \otimes d\psi.$$  

The restriction of the Lorentz invariant measure $d\mu_\mathcal{dS}$ to $\mathcal{W}_1$ is
$$d\mu_{\mathcal{W}_1}(t, \psi) = r dt d\psi,$$  

with $d\psi = |\cos \psi| r d\psi$.

The line element on the circle $S^1$ is
$$|g|_{S^1}^{1/2} r d\psi = r d\psi.$$  

Restricted to the double wedge $\mathcal{W}_1$ the Klein–Gordon operator is
$$\square_{\mathcal{W}_1} + \mu^2 = \frac{1}{r^2 \cos^2 \psi} (\partial_t^2 + \varepsilon^2),$$  

with
$$\varepsilon^2 \triangleq - (\cos \psi \partial_\psi)^2 + (\cos \psi)^2 \mu^2 r^2.$$  

Remark 5.4.1. For $\psi \in (-\pi/2, \pi/2)$ define a new spatial coordinate $\chi = \chi(\psi)$ by
$$\frac{d\chi}{d\psi} = \frac{1}{\cos \psi}, \quad \chi(0) = 0.$$  

Find $\chi(\psi) = \ln \tan(\psi/2 + \pi/4)$ and
$$\cos \psi = (\cosh \chi)^{-1}, \quad \sin \psi = \tanh \chi.$$  

$\chi$ is a diffeomorphism from $(-\pi/2, \pi/2)$ onto $\mathbb{R}$.

$$g|_{\mathcal{W}_1} = \frac{r^2}{\cosh^2 \chi}(dt \otimes dt - d\chi \otimes d\chi).$$  

Thus $W_1$ is conformally equivalent to Minkowski space $\mathbb{R}^{1+1}$ [57]. In these coordinates
$$\square_{\mathcal{W}_1} + \mu^2 = \frac{\cosh^2 \chi r^2}{\mu^2} (\partial_\chi^2 + \varepsilon^2),$$  

with $\varepsilon^2 \triangleq - \partial_\chi^2 + (\cosh \chi)^{-2} \mu^2 r^2$. 

Definition 5.3.8. The triple $(\mathfrak{t}(dS), \sigma, u)$ is the covariant classical dynamical system associated to the Klein–Gordon equation (5.1.1). 

5.4. The restriction of the KG equation to a (double) wedge
Lemma 5.4.2. Identify $S^1 \cong [\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)]$, $I_+ \cong \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $I_- \cong \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. It follows that $\varepsilon^2$ is positive and symmetric on

$$\mathcal{D}(S^1 \setminus \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}) \subset L^2(S^1, |\cos \psi|^{-1} \, r \, d\psi).$$

Denote its Friedrich extension by the same symbol. Then $\text{Sp}(\varepsilon^2) = [0, \infty)$.

Proof. Clearly, $\varepsilon^2$ is positive and symmetric on

$$\mathcal{D}(S^1 \setminus \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}) = \mathcal{D}(I_+) \oplus \mathcal{D}(I_-).$$

We next show that $\mathcal{D}(S^1 \setminus \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\})$ is dense in $L^2(S^1, |\cos \psi|^{-1} \, r \, d\psi)$. First note that $\mathcal{D}(S^1 \setminus \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\})$ is dense in $L^2(S^1, r \, d\psi)$. Moreover, a function

$$\cos^{1/2}_\psi h \in L^2(S^1, |\cos \psi|^{-1} \, r \, d\psi) \iff h \in L^2(S^1, r \, d\psi).$$

It follows that

$$\cos^{1/2}_\psi \mathcal{D}(S^1 \setminus \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}) = \mathcal{D}(S^1 \setminus \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\})$$

is dense in $L^2(S^1, |\cos \psi|^{-1} \, r \, d\psi)$. Thus [163, Theorem X.23, p.177] applies and defines the Friedrich extension.

Remark 5.4.3. Since the spectrum of $\varepsilon^2$ has no gap around the discrete eigenvalue zero, the choice of coordinates (2.1.4) may lead to artificial infrared problems if one adds an interaction, similar to the ones encountered in [57]. We will avoid this problem later on by working with functions in the Hilbert space $\mathfrak{h}(S^1)$, whose scalar product is rotation-invariant; see Section 6.2.

$\varepsilon^2$ is a differential operator, thus $\varepsilon^2$ acts locally and maps the subspaces

$$\mathcal{D}^\pm \doteq \mathcal{D}(\varepsilon^2) \cap L^2(I_\pm, |\cos \psi|^{-1} \, r \, d\psi)$$

into $L^2(I_\pm, |\cos \psi|^{-1} \, r \, d\psi)$, respectively. It therefore is consistent to define

$$\varepsilon(h_+ + h_-) \doteq \sqrt{\varepsilon^2 I_+} h_+ - \sqrt{\varepsilon^2 I_-} h_-, \quad h_\pm \in \mathcal{D}^\pm.$$

$\varepsilon$ is densely defined by (5.4.5), as $\mathcal{D}^+ \oplus \mathcal{D}^- = \mathcal{D}(\varepsilon^2)$. The pseudo-differential operator $\varepsilon$ is non-local, but does not mix functions supported on the half-circles $I_+$ and $I_-$. Denote the restrictions by $\varepsilon|_{I_+}$ and $\varepsilon|_{I_-}$.

Lemma 5.4.4. There exits a self-adjoint operator $\varepsilon$ on $L^2(S^1, |\cos \psi|^{-1} \, r \, d\psi)$ such that (5.4.5) holds. $\text{Sp}(\varepsilon) = \mathbb{R}$, $\text{Sp}(\varepsilon|_{I_+}) = [0, \infty)$ and $\text{Sp}(\varepsilon|_{I_-}) = (-\infty, 0]$. Moreover, zero is not an eigenvalue of $\varepsilon$.

Proof. Use the spectral theorem to define the square roots in (5.4.5) as self-adjoint operators. $\mathcal{D}^+ \cap \mathcal{D}^- = \{0\}$, in fact $\mathcal{D}^+ \text{ and } \mathcal{D}^-$ are orthogonal to each other in $L^2(S^1, |\cos \psi|^{-1} \, r \, d\psi)$. Thus the sum of the square roots is self-adjoint on the direct sum of their domains (see [163, Theorem VIII.6]) and $\text{Sp}(\varepsilon) = \mathbb{R}$. □
**Notation.** If $P$ is a pseudo-differential operator on $L^2(S^1, |\cos \psi|^{-1} \, d\psi)$, define its kernel $P(\psi, \psi')$ for all $h \in L^2(S^1, |\cos \psi|^{-1} \, d\psi) \cap \mathcal{D}(P)$, for which the following expressions exist, by

$$(Ph)(\psi) = \int_{S^1} \frac{rd\psi'}{|\cos \psi'|} P(\psi, \psi') h(\psi') = \int_{S^1} \frac{dl(\psi')}{|\cos \psi'|^2} P(\psi, \psi') h(\psi') .$$

dl(\psi') was defined in (5.4.1).

If $P$ is hermitian with domain $\mathcal{D} \subset L^2(S^1, |\cos \psi|^{-1} \, d\psi)$, then

$$P(\psi, \psi') = \overline{P(\psi', \psi)}, \quad \psi, \psi' \in S^1 .$$

In the next lemma, $(\sin \varepsilon (t-t'))^{(5.4.6)}$ is considered as such a pseudo-differential operator on $L^2(S^1, |\cos \psi|^{-1} \, d\psi)$.

**Lemma 5.4.5.** Use the coordinates (2.14). Then

$$(5.4.6) \quad \mathcal{S}'(x, y) = -\left( \frac{\sin \varepsilon (t-t')}{|\varepsilon|} \right) (\psi, \psi') .$$

**Proof.** For $f, g \in \mathcal{D}_{\mathbb{R}}(\mathcal{H}_1)$, set $f_t(\psi) \doteq f(t, \psi)$ and $g_t(\psi') \doteq g(t', \psi')$. Clearly, $f_t, g_t \in L^2(S^1, |\cos \psi|^{-1} \, d\psi)$. Consider

$$(5.4.7) \quad \mathcal{E}_{\mathcal{H}_1}(f, g) \doteq -\int \mathcal{R} \, dt \, dt' \left[ \cos^2 \varepsilon f_t, \frac{\sin \varepsilon (t-t')}{|\varepsilon|} \cos^2 g_t \right]_{L^2(S^1, |\cos \psi|^{-1} \, d\psi)} ,$$

with $\cos \psi$ the multiplication operator by $\cos \psi$. Clearly, $\mathcal{E}_{\mathcal{H}_1}$ is anti-symmetric with respect to permutation of $f$ and $g$. Moreover, according to (5.4.3)

$$\begin{align*}
\mathcal{E}_{\mathcal{H}_1}(f, (\Box_{\mathcal{H}_1} + \mu^2)h) \\
= \mathcal{E}_{\mathcal{H}_1} \left( f, r^{-2} \cos^2 \varepsilon (\partial_t^2 + \varepsilon^2)h \right) \\
= -\int \mathcal{R} \, dt \, dt' \left[ \cos^2 \varepsilon f_t, \frac{\sin \varepsilon (t-t')}{|\varepsilon|} r^{-2} (\partial_t^2 + \varepsilon^2)h_t \right]_{L^2(S^1, |\cos \psi|^{-1} \, d\psi)},
\end{align*}$$

where $h_t(\psi) \doteq h(t, \psi) \in \mathcal{D}(\mathbb{S}^1 \setminus \left\{ \mp \frac{\pi}{2}, \mp \frac{\pi}{2} \right\})$. Now

$$\int dt' \sin \varepsilon t' \partial_{t'} h_t = \int dt' (\partial_{t'} \sin \varepsilon t') h_t = \int dt' \sin \varepsilon t' (-\varepsilon^2) h_t,$$

by partial integration and using $(\partial_{t'} h_t)_{t'} = \partial_{t'}(h_{t'})$. Thus

$$\mathcal{E}_{\mathcal{H}_1}(f, (\Box_{\mathcal{H}_1} + \mu^2)h) = 0 .$$

A similar argument can be used to show $\mathcal{E}_{\mathcal{H}_1} \left( (\Box_{\mathcal{H}_1} + \mu^2)h, f \right) = 0$. It follows that the kernel $\mathcal{S}_{\mathcal{H}_1}(x, y)$, defined by

$$\int d\mu_{AS}(x) d\mu_{AS}(y) f(x) \mathcal{S}_{\mathcal{H}_1}(x, y) g(y) \doteq \mathcal{E}_{\mathcal{H}_1}(f, g) ,$$
is anti-symmetric and satisfies the Klein–Gordon equation in both entries. Furthermore,

\[
\mathcal{E}_{\mathcal{W}_1}(f, g) = -\int r^3 \, dt \, dt' \left\langle \cos_0^2 f_t, \frac{\sin(\epsilon(t-t'))}{|\epsilon|} \cos_0^2 g_{t'} \right\rangle_{L^2(S^1, |\cos \psi|^{-1} \, r \, d\psi)}
\]

\[
= -\int r^2 \, dt \, dt' \int \frac{\, r \, d\psi'}{|\cos \psi'|} \cos^2(\psi) f_t(\psi) \times \int \frac{\, r \, d\psi'}{|\cos \psi'|} \left( \frac{\sin(\epsilon(t-t'))}{|\epsilon|} \right) (\psi, \psi') \cos^2(\psi') g_{t'}(\psi')
\]

\[
= -\int \, r \, dt \, dl(\psi) \int \, r \, dt' \, dl(\psi') f_t(\psi) \left( \frac{\sin(\epsilon(t-t'))}{|\epsilon|} \right) (\psi, \psi') g_{t'}(\psi')
\]

In the last equation we used (5.4.1), i.e., \( dl(\psi) = r \, |\cos \psi| \, d\psi \). Thus

\[
\partial_t \mathcal{E}_{\mathcal{W}_1}(x, y) = -\left( \frac{\epsilon}{|\epsilon|} \right) (\psi, \psi')
\]

using \( x \equiv x(t, \psi) \) and \( y \equiv y(t', \psi') \). Clearly, \( \mathcal{E}_{\mathcal{W}_1} \) satisfies (5.3.5) with \( \epsilon = S^1 \).

The unit normal future pointing vector field on \( S^1 \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\} \) is

(5.4.8)

\[
n(t, \psi) = r^{-1} \cos^{-1}(x) \, \partial_t
t.
\]

In \( I_- \) the vector field \( \partial_t \) is past directed and \( \cos \psi < 0 \), thus equation (5.4.8) holds for both half-circles \( I_+ \) and \( I_- \). From (5.4.7) read off

(5.4.9)

\[
r^{-1} \partial_t \mathcal{E}_{\mathcal{W}_1}(x(t, \psi); y(0, \psi')) |_{t=0} = -\left( \frac{\epsilon}{|\epsilon|} \right) (\psi, \psi')
\]

where

\[
\Pi(\psi, \psi') = \frac{1}{2} |\cos \psi| \, \delta(\psi - \psi')
\]

is the kernel of the unit in \( L^2(S^1 \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}, |\cos \psi|^{-1} \, r \, d\psi) \). Now \( \cos^{-1}(x) |\epsilon| = |\cos \psi|^{-1} \). Hence the r.h.s. in (5.4.9) is \(- \cos \psi \, \delta(\psi - \psi') \) and (5.4.8) implies

\[
n(t, \epsilon x(t, \psi); y(0, \psi')) = -\delta(\psi - \psi')
\]

\( \delta(\psi - \psi') \) is the kernel of the unit with respect to the induced line element \( r \, d\psi \)

on \( S^1 \), see Equation (5.4.2). Thus (5.3.6) holds, and hence, by the uniqueness result mentioned, \( \mathcal{E}_{\mathcal{W}_1} = \mathcal{E}_{\mathcal{W}_1} \) within the double wedge \( \mathcal{W}_1 \). □

Thus, for \( f \in \mathcal{D}(I_+) \), \( x \equiv x(t, \psi) \in \mathcal{W}_1 \) and \( f_t(\psi) \equiv f(x(t', \psi)) \),

(5.4.10) \[
\mathcal{F}(x) = -\int r \, dt' \left( \frac{\sin(\epsilon(t-t'))}{|\epsilon|} \right) \cos_0^2 f_{t'}(\psi)
\]

Note that (5.4.10) describes \( \mathcal{F} \) only on a proper subset of its support, namely the intersection of its support with \( \mathcal{W}_1 \).

**Remark 5.4.6.** For \( h \in \mathcal{D}(I_+) \) one can extend the domain of \( \mathcal{F} \) to distributions of the form

\[
f(x) \equiv (\delta \otimes h)(x) = \delta(t) \frac{h(0, \psi)}{r \cos \psi}
\]

(5.4.11)

\[
g(x) \equiv (\delta' \otimes h)(x) = \left( \frac{\partial_t}{r \cos \psi} \delta \right) (t) \frac{h(0, \psi)}{r \cos \psi}
\]
with \( x \equiv x(t, \psi) \), using the coordinates introduced in (2.1.4), and
\[
d\mu_{\psi}(t, \psi) = r^2 \, dt \, d\psi \, \cos \psi.
\]

The properties of the convolution ensure that \( f, g \) are \( C^\infty \)-solutions of the Klein–Gordon equation (5.1.1), whose support is contained in \( dS \setminus W^r \). Within the region \( W_1 \) these solutions are given by
\[
f(x) = -\frac{\sin(\epsilon t)}{|\epsilon|} \cos \psi \cdot h(0, \psi),
\]
\[
g(x) = \frac{\cos(\epsilon t)}{r} \h(0, \psi).
\]

### 5.5. The canonical classical dynamical system

Let \( (n \Phi)|_e \) denote the Lie derivative of \( \Phi \) along the unit normal, future pointing vector field \( n \), restricted to the Cauchy surface \( \mathcal{C} \).

**Theorem 5.5.1** (Dimock [48], Theorem 1). Let \( \mathcal{C} \subset dS \) be a Cauchy surface and let \( (\hat{\psi}, \pi) \in C^\infty(\mathcal{C}) \times C^\infty(\mathcal{C}) \). Then there exists a unique \( \Phi \in C^\infty(dS) \) satisfying the Klein–Gordon equation (5.1.1) with Cauchy data
\[
(5.5.1) \quad \Phi|_e = \hat{\psi}, \quad (n\Phi)|_e = \pi.
\]

Furthermore, \( \text{supp} \, \Phi \subset \bigcup_{\pm} \Gamma_{\pm}(\text{supp} \, \hat{\psi} \cup \text{supp} \, \pi) \).

**Remark 5.5.2.** For functions in the Sobolev space \( H^2_{\text{loc}}(dS) \), this is the classical existence and uniqueness theorem of Leray [133].

Now consider the space
\[
\hat{\mathcal{T}}(S^1) \equiv C^\infty_{\text{loc}}(S^1) \times C^\infty_{\text{loc}}(S^1)
\]

**Proposition 5.5.3.** The symplectic space \( (\hat{\mathcal{T}}(S^1), \hat{\sigma}) \) carries a representation \( \Lambda \mapsto \hat{u}(\Lambda), \Lambda \in O(1, 2) \), defined by
\[
(5.5.3) \quad \hat{u}(\Lambda)|_{\hat{\mathcal{T}}(S^1)} \equiv ([\Lambda, f]|_{S^1}, [n\Lambda, f]|_{S^1}).
\]

The triple \( (\hat{\mathcal{T}}(S^1), \hat{\sigma}, \hat{u}) \) is the canonical classical dynamical system associated to the covariant classical dynamical system specified in Definition 5.5.8.

**Proof.** This result follows directly from Theorem 5.5.1 and the invariance of the Klein–Gordon operator under the adjoint pull-back action of \( O(1, 2) \). \( \square \)

**Proposition 5.5.4.** The map
\[
T : (\hat{\mathcal{T}}(dS), \sigma, u(\Lambda)) \to (\hat{\mathcal{T}}(S^1), \hat{\sigma}, \hat{u}(\Lambda))
\]
\[
[f] \mapsto (f|_{S^1}, (n\Phi)|_{S^1}) \equiv (\hat{\psi}, \pi)
\]

is symplectic.
5. CLASSICAL FIELD THEORY

PROOF. Let $f, g \in \mathcal{D}_R(dS)$. Then Stokes’ theorem implies (see Lemma A.1) that

$$\sigma([f], [g]) = \mathcal{E}(f, g)$$

$$= \int_{dS} d\mu_{dS}(x) f(x)(\mathcal{E}g)(x)$$

$$= \int_{S^i} r \, d\psi \left( (E f)_{|S^i}(\psi)(n \mathcal{E}g)_{|S^i}(\psi) - (n E f)_{|S^i}(\psi)(\mathcal{E}g)_{|S^i}(\psi) \right)$$

$$= \langle f_{|S^i}, (ng)_{|S^i} \rangle_{L^2(S^i, r \, d\psi)} - \langle n f_{|S^i}, g_{|S^i} \rangle_{L^2(S^i, r \, d\psi)}$$

$$= \hat{s} \left( \langle f_{|S^i}, (n\hat{f})_{|S^i} \rangle, \langle g_{|S^i}, (n\hat{g})_{|S^i} \rangle \right).$$

Thus $\mathbb{T}$ is symplectic. \hfill \qed

The canonical projection

$$\hat{\mathcal{F}}: \mathcal{D}_R(dS) \rightarrow C^\infty_{\mathcal{R}}(S^i) \times C^\infty_{\mathcal{R}}(S^i)$$

$$f \mapsto \langle f_{|S^i}, (n f)_{|S^i} \rangle \equiv \hat{f}$$

maps a smooth, real valued function $f \in \mathcal{D}_R(dS)$ with compact support to the Cauchy data of a $C^\infty$-solution $\hat{f}$ of the Klein–Gordon equation \eqref{5.1.1}.

REMARK 5.5.5. For the special case $f \in \mathcal{D}_{\mathcal{R}}(\mathbb{H}/\mathbb{I})$, Eq. \eqref{5.5.4} yields

$$\hat{f}_{|S^i}(\psi) = \int r \, dt' \left( \frac{\sin(t' \epsilon)}{r \cos \epsilon} \cos^2 \phi, f_{t'} \right)(\psi),$$

\eqref{5.5.6}

$$\langle n f \rangle_{|S^i}(\psi) = -\frac{1}{\epsilon \cos \epsilon} \int r \, dt' \left( \cos(t' \epsilon) \cos^2 \phi, f_{t'} \right)(\psi),$$

where $f_{t'}(\psi) := f(x(t, \psi))$, using again $\cos^{-1} \epsilon = |\cos \epsilon|^{-1}$. An explicit formula, which generalizes both \eqref{5.5.5} and \eqref{5.5.6} to $f \in \mathcal{D}_R(dS)$ will follow from Eq. \eqref{6.1.12} in Section 6.1.

PROPOSITION 5.5.6. Let $\hat{f} \in \mathfrak{h}(\mathbb{I})$, $I \subset S^i$. Then

$$\hat{u}(\Lambda) \hat{f} \in \mathfrak{f} \left( (\Gamma^+(\Lambda I) \cup \Gamma^-(\Lambda I)) \cap S^i \right).$$

PROOF. Let $[f] = T^{-1} \hat{f}$ be the element in $\mathfrak{f}(dS)$ associated to the smooth solution $f$ of the Klein–Gordon equation with Cauchy data given by $\hat{f}$. It follows that

$$\mathbb{T}^{-1} \left( \hat{u}(\Lambda) \hat{f} \right) = u(\Lambda)[f].$$

The smooth solution of the Klein–Gordon equation associated to $u(\Lambda)[f]$ has support in $\Gamma^+(\Lambda I) \cup \Gamma^-(\Lambda I)$; thus the Cauchy data of the solution associated to $u(\Lambda)[f]$ have support in $(\Gamma^+(\Lambda I) \cup \Gamma^-(\Lambda I)) \cap S^i$. \hfill \qed

\footnote{Note that Dimock’s operator $\mathcal{E}$ differs from our conventions by a sign, as can be seen by comparing Corollary 1.2 in \cite{48} with \cite{53.24}.}
PROPOSITION 5.5.7. The rotations \( \hat{u}(R_0(\alpha)), \alpha \in [0, 2\pi) \), which map

\[
(\hat{\psi}(\psi), \nu(\psi)) \mapsto (\hat{\psi}(\psi - \alpha), \nu(\psi - \alpha)),
\]
and the boosts \( \hat{u}(\Lambda_1(t)), t \in \mathbb{R} \), which map

\[
(\hat{\psi}, \nu) \mapsto (\hat{\psi}_t, \nu_t),
\]
with

\[
\hat{\psi}_t = \cos(\epsilon t)\hat{\psi} - \sin(\epsilon t) \epsilon^{-1} \cos \phi \nu
\]
\[
\nu_t = (\tau \cos \phi)^{-1}(\epsilon \sin(\epsilon t)\hat{\psi} + \cos(\epsilon t) \cos \phi \nu),
\]
generate the representation \( \Lambda \mapsto \hat{u}(\Lambda) \) of \( SO_0(1, 2) \) introduced in (5.5.3). The points \( (\hat{\psi}(\pm \frac{\pi}{2}), \nu(\pm \frac{\pi}{2})) \) are fixed points of the map \( t \mapsto (\hat{\psi}_t(\psi), \nu_t(\psi)) \). The representers of the reflections \( P_1 \) and \( T \) are

\[
\hat{u}(P_1) : (\hat{\psi}, \nu) \mapsto (\hat{\psi}_1, \nu),
\]
\[
\hat{u}(T) : (\hat{\psi}, \nu) \mapsto (\hat{\psi}, -\nu).
\]

PROOF. Recall (2.1.5) and consider the boosts \( t \mapsto \Lambda_1(t) \), acting on the Cauchy data on \( S^1 \). Now combine (5.5.3) and the definition of \( \Lambda_\star \) to conclude that the boosts \( \hat{u}(\Lambda_1(t))(\hat{\psi}, \nu), t \in \mathbb{R} \), are determined by \( \hat{\psi}(\Lambda_1(t)) \), where \( \hat{\psi} \) is the solution of the Klein–Gordon equation (5.1.1) specified in Theorem 5.5.1. Evaluate (5.5.7) with care—write it out explicitly and take advantage of the fact that \( \epsilon^2 \) is a differential operator which satisfies \( (\epsilon^2 \hat{h})(\psi \pm \frac{\pi}{2}) = 0 \) in \( \mathbb{C}^\infty(S^1) \), just as \( \cos(\psi \pm \frac{\pi}{2}) \). To show that the map is well-defined for \( \psi \to \pm \frac{\pi}{2} \) and

\[
(\hat{\psi}_t(\pm \frac{\pi}{2}), \nu_t(\pm \frac{\pi}{2})) = (\hat{\psi}(\pm \frac{\pi}{2}), \nu(\pm \frac{\pi}{2})) \quad \forall t \in \mathbb{R}.
\]

(This ensures that \( \hat{\psi}_t \) and \( \nu_t \) are both well-defined despite the fact that the coordinate system is degenerated at \( \psi = \pm \frac{\pi}{2} \).) It remains to construct \( \hat{u}(\Lambda_1(t)) \) in the space-time region \( \mathcal{W}_1 \). On \( \mathcal{W}_1 \), the Klein–Gordon equation (5.1.1) reads

\[
(\partial_t^2 + \epsilon^2 \Phi) = 0,
\]

using (5.4.3). The real valued solution of (5.5.11) in the region \( \mathcal{W}_1 \) with Cauchy data (see (5.4.8))

\[
\hat{\psi} = \Phi|_{S^1}, \quad \nu = \frac{1}{\tau \cos \psi} (\partial_t \Phi)|_{S^1},
\]
is \( \Phi(t, \cdot) = \cos(\epsilon t)\hat{\psi} + \sin(\epsilon t) \epsilon^{-1} \cos \phi \nu \). Hence

\[
\hat{\psi}_t \equiv (\Lambda_1(t) \ast \Phi)|_{S^1}, \quad \nu_t \equiv \nu(\Lambda_1(t) \ast \Phi)|_{S^1}
\]
are determined by

\[
(\Lambda_1(t) \ast \Phi)(t', \cdot) = \cos(\epsilon(t' - t))\hat{\psi} + \sin(\epsilon(t' - t)) \epsilon^{-1} \cos \phi \nu,
\]
\[
(\partial_t \ast \Lambda_1(t) \ast \Phi)(t', \cdot) = -\epsilon \sin(\epsilon(t' - t))\hat{\psi} + \cos(\epsilon(t' - t)) \cos \phi \nu.
\]

\( \epsilon^2 \) maps \( C^\infty(R)(S^1) \) to the functions in \( C^\infty(S^1) \), which vanish at \( \psi = \pm \frac{\pi}{2} \). Thus \( \hat{u}(\Lambda_1(1)) \) maps \( C^\infty(S^1) \times C^\infty(S^1) \to C^\infty(S^1) \times C^\infty(S^1) \).

The boosts \( \Lambda_1(t), t \in \mathbb{R} \), together with the rotations \( R_0(\alpha), \alpha \in [0, 2\pi) \), generate \( SO_0(1, 2) \).
(P₁)* and Tₙ commute with the restriction to S¹, and (P₁)* commutes with n while Tₙ anti-commutes with n. On the time-zero circle S¹, the spatial reflection P₁ acts as

\[ P₁ : \psi \mapsto \pi - \psi , \quad \psi \in \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right) . \]

Thus (5.5.9) and (5.5.10) follow.

The reflection at the edge of the wedge \( W₁ \)

\[ (P₁ T)ₙ : \ g(x₀, x₁, x₂) \mapsto g(-x₀, x₁, -x₂) , \quad g \in \mathcal{D}(dS) , \]
gives rise to a double classical system in the sense of Kay [117]:

**Proposition 5.5.8.** The symplectic space \( \hat{\mathfrak{f}}(S¹ \backslash \{ \pm \frac{\pi}{2} \}) \) is the direct sum of \( \hat{\mathfrak{f}}(I⁺) \) and \( \hat{\mathfrak{f}}(I⁻) \). Moreover,

i.) \( \hat{\sigma}(\hat{f}, \hat{g}) = 0 \) for all \( \hat{f} \in \hat{\mathfrak{f}}(I⁺) \) and \( \hat{g} \in \hat{\mathfrak{f}}(I⁻) \);

ii.) the maps \( \{ \hat{u}(Λ₁(t)) \}_{t \in \mathbb{R}} \) leave the subspaces \( \hat{\mathfrak{f}}(I⁺) \) and \( \hat{\mathfrak{f}}(I⁻) \) invariant;

iii.) \( \hat{u}(P₁T) \) is an anti-symplectic involution, which satisfies

\[ \hat{u}(P₁T)\hat{f}(I⁺) = \hat{f}(I⁻) \quad \text{and} \quad [\hat{u}(Λ₁(t)), \hat{u}(P₁T)] = 0 \quad \forall t \in \mathbb{R} . \]

Thus \( (\hat{\mathfrak{f}}(S¹ \backslash \{ \pm \frac{\pi}{2} \}), \hat{u}(Λ₁), \hat{u}(P₁T)) \) is a double classical linear dynamical system in the sense of A.0.6.

In other words, the following diagram commutes:

\[
\begin{array}{ccc}
(\hat{\mathfrak{f}}(I⁺), \hat{\sigma}) & \xrightarrow{\hat{u}(P₁T)} & (\hat{\mathfrak{f}}(I⁻), \hat{\sigma}) \\
\downarrow \hat{u}(Λ₁(t)) & & \downarrow \hat{u}(Λ₁(t)) \\
(\hat{\mathfrak{f}}(I⁺), \hat{\sigma}) & \xrightarrow{\hat{u}(P₁T)} & (\hat{\mathfrak{f}}(I⁻), \hat{\sigma}) \\
\end{array}
\]

**Remark 5.5.9.** The domain of the map \( \hat{f} \) extends to testfunctions \( f, g \) of the form given in (5.4.11). Use (5.5.5) and (5.5.6) to compute the corresponding Cauchy data:

(5.5.13) \( (\hat{f}|_{S¹}, (n \hat{f})|_{S¹}) = (0, -\frac{\hbar}{2}) \equiv (\psi, \pi) , \)

and, by partial integration,

(5.5.14) \( (g|_{S¹}, (ng)|_{S¹}) = (\frac{\psi}{\hbar}, 0) \equiv (\phi, n) . \)

All elements in \( \hat{\mathfrak{f}}(I⁺) \) are linear combinations of the Cauchy data arising from sharp-time testfunctions \( f, g \) of the form described in (5.4.11).
Part 4

Free Quantum Fields
CHAPTER 6

One-Particle Hilbert Spaces

6.1. The covariant one-particle Hilbert space

Our basic strategy is to use, just as in Minkowski space (see, e.g., \[163\]), the restriction of the Fourier-Helgason transform to the upper mass shell, \(\mathcal{F}_{+}\) \(\mathcal{D}_{R}(dS) \rightarrow \tilde{h}_{\nu}(\partial V^+)\)

\[
\mathcal{F}_{+}\colon \mathcal{D}_{R}(dS) \rightarrow \tilde{h}_{\nu}(\partial V^+);
\]

where \(s^+\) given by (3.3.8), to define a (complex valued) semi-definite quadratic form

\[
\mathcal{D}_{R}(dS) \ni f, g \mapsto \langle \tilde{f}_{\nu}, \tilde{g}_{\nu}\rangle \tilde{h}(\partial V^+) = \tilde{f}_{\nu}, \quad (6.1.1)
\]

with \(s^+\) given by (3.3.8), to define a (complex valued) semi-definite quadratic form

\[
\mathcal{D}_{R}(dS) \ni f, g \mapsto \langle \tilde{f}_{\nu}, \tilde{g}_{\nu}\rangle \tilde{h}(\partial V^+);
\]

on the test-functions. (We will suppress the index \(\nu\) when possible, for example, we will frequently write \(\tilde{h}(\partial V^+)\) instead of \(\tilde{h}_{\nu}(\partial V^+)\).) The value of the positive normalisation constant (see Harish-Chandra \[89, 90\])

\[
c_{\nu} = -\frac{1}{2 \sin(\pi s^+)} = \frac{1}{2 \cos(i\pi)}
\]

is chosen such that twice the imaginary part of the scalar product (6.1.2) equals the value of the symplectic form \(\sigma\) of the classical dynamical system given in (5.3.9); for further details, see the discussion preceding (6.1.12) below. Using (E.0.4), one can show that

\[
c_{\nu} = \frac{\Gamma(1 + s^+) \Gamma(1 + s^-)}{2\pi} = c_{-\nu}.
\]

The following result of Faraut was pointed out to us by J. Bros.

PROPOSITION 6.1.1 (Faraut \[53\], Prop. II.4). Let \(f \in \mathcal{D}_{R}(dS)\) and \(0 < -i\nu < \frac{1}{2}\). Then

\[
\int_{dS} d\mu_{dS}(x) f(x) (x_{\pm} \cdot p)^{-\frac{1}{2} - i\nu} = \frac{\Gamma(1-i\nu)}{\Gamma\left(\frac{3}{4} + \frac{i\nu}{2}\right)} \sqrt{\pi} \Gamma\left(\frac{3}{4} + i\nu\right) \int_{p} d\mu_{p}(p') (p \cdot p')^{-\frac{1}{2} + i\nu} \times \int_{dS} d\mu_{dS}(x) f(x) (x_{\pm} \cdot p')^{-\frac{1}{2} + i\nu},
\]

where \(\Gamma\) is a closed curve on the forward light cone \(\partial V^+\), which encloses the origin.

REMARK 6.1.2. Choosing \(p = (1, \cos \alpha, \sin \alpha)\) and \(p' = (1, \cos \alpha', \sin \alpha')\) we find

\[
p \cdot p' = 1 - \cos(\nu - \nu').
\]

Thus we have recovered the factor (3.3.15) first introduced by Bargmann; see the definition of the intertwined \(A_{\nu}\).
Lemma 6.1.3. Let $\mu^2 = \frac{1}{4\pi} + m^2$, i.e., $v^2 = m^2 r^2$. It follows that
\[ \ker \mathcal{F}_{+|\nu} = \ker P = (\Box_{dS} + \mu^2)\mathcal{D}_r(dS). \]

Proof. If $f \in \ker P$, then (5.3.4) implies that there exists $g \in \mathcal{D}_r(dS)$ such that $f = (\Box_{dS} + \mu^2)g$. Evaluate $\mathcal{F}_{+|\nu}((\Box_{dS} + \mu^2)g)$ using the definition of the Fourier-Helgason transform (see (4.2.5)) and
\[ \Box_{dS} + \mu^2(x_+ \cdot \rho) = 0 \]
for $s^\pm$ given by (3.3.8) with $\zeta^2 = \mu^2 r^2$. This shows that $\ker \mathcal{F}_{+|\nu} \subset \ker P$. The inclusion $\ker \mathcal{F}_{+|\nu} \subset \ker P$ will follow from the fact that
\[ \mathcal{E}(f, g) = 2\mathcal{F}(\tilde{f}, g)_{\mathcal{H}(O_{+V})}. \]
This will be verified in (6.1.12) below. \hfill \Box

6.1.1. Real Hilbert Spaces. The kernel of the quadratic form (6.1.2) equals $\ker \mathcal{F}_{+|\nu}$. This allows us to turn the real symplectic spaces $t(X)$ into real pre-Hilbert spaces
\[ \mathfrak{h}^o(X) = (t(X), \mathcal{R}(\mathcal{T}_{\nu}, \mathcal{T}_{\nu})_{\mathcal{H}(O_{+V})}), \quad X = dS, O, W. \]
The completion of $\mathfrak{h}^o(X)$ defines the real Hilbert spaces $\mathfrak{h}(X), X = dS, O, W$. Their real valued scalar product is given by the real part
\[ \mathcal{R}(f, g)_{\mathfrak{h}(dS)} = \frac{1}{2}||f + g||_{\mathfrak{h}(dS)} - ||f - g||_{\mathfrak{h}(dS)} \]
of the complex valued scalar product
\[ \langle [f_\nu, g_\nu] \rangle_{\mathfrak{h}(dS)} = \mathcal{F}(\tilde{f}_\nu, \tilde{g}_\nu)_{\mathcal{H}(O_{+V})}, \quad [f_\nu, g_\nu] \in t(X), \]
with $\nu = v(\mu)$ given by (3.3.8) with $\zeta^2 = \mu^2 r^2$.

6.1.2. Complex Hilbert Spaces. The question now arises, whether these real Hilbert spaces can be interpreted as complex Hilbert spaces, i.e., whether or not they carry an intrinsic complex structure. The answer to this question depends on the choice of $X \subset dS$. In case $X = dS$, the real-valued scalar product $f, g \mapsto \mathcal{R}(f, g)_{\mathfrak{h}(dS)}$ can be used to define an operator $\mathcal{S}$,
\[ \mathcal{R}(\mathcal{S}f, g)_{\mathfrak{h}(dS)} = 2\mathcal{I}(f, g)_{\mathfrak{h}(dS)} \quad \forall g \in \mathfrak{h}(dS). \]
The Riesz lemma (applied on the real Hilbert space $\mathfrak{h}(dS)$) fixes the vector
\[ \mathcal{S} f \in \mathfrak{h}(dS) \]
uniquely, since the symplectic form $\mathcal{I}(\cdot, \cdot)_{\mathfrak{h}(dS)}$ is non-degenerated on $\mathfrak{h}^o(dS)$. The operator $\mathcal{S}$ satisfies
\[ \mathcal{I}(\mathcal{S}f, g)_{\mathfrak{h}(dS)} = -\mathcal{I}(f, \mathcal{S}g)_{\mathfrak{h}(dS)} \]
and $\mathcal{S}^2 = -1$, and therefore defines a complex structure: for $f \in \mathfrak{h}(dS)$ we have
\[ \langle \lambda_1 + i\lambda_2 f = \lambda_1 f + \lambda_2 (\mathcal{S}f), \quad \lambda_1, \lambda_2 \in \mathbb{R}. \]
This turns the real Hilbert space $(\mathfrak{h}(dS), \mathcal{R}(\cdot, \cdot)_{\mathfrak{h}(dS)})$ into a complex Hilbert space $(\mathfrak{h}(dS), \langle \cdot, \cdot \rangle_{\mathfrak{h}(dS)})$. The scalar product $f, g \mapsto \langle f, g \rangle_{\mathfrak{h}(dS)}$ is anti-linear in $f$ and linear in $g$ with respect to the complex structure defined in (6.1.5).
Remark 6.1.4. In case \( X = \emptyset \) (with \( \emptyset \) bounded) or \( X = W \), the spaces \( \mathfrak{h}(\emptyset) \) and \( \mathfrak{h}(W) \) are only real subspaces of \( \mathfrak{h}(\mathcal{D}) \). We will later show that their complex linear span is dense in \( \mathfrak{h}(\mathcal{D}) \).

6.1.3. A representation of \( O(1,2) \). In [9] we identify the quantum one-particle space with some abstract Hilbert space \( \mathfrak{h} \) carrying a unitary irreducible representation of the Lorentz group \( SO_0(1,2) \). The real subspaces \( \mathfrak{h}(\emptyset) \) associated to open bounded subsets of \( \emptyset \) will be identified using the concept of modular localization [33]. Here we show that \( \mathfrak{h}(\mathcal{D}) \) carries a representation of \( O(1,2) \).

Proposition 6.1.5. There is a unitary representation \( u \) of \( SO_0(1,2) \) on \( \mathfrak{h}(\mathcal{D}) \) such that for \( f \in \mathcal{D}_R(\mathcal{D}) \)

\[
u(\Lambda)[f] = [\Lambda f], \quad \Lambda \in SO_0(1,2);
\]

and consequently, \( u(\Lambda)[\mathfrak{h}(\emptyset)] = \mathfrak{h}(\Lambda \emptyset), \Lambda \in SO_0(1,2) \). In other words, \( u \) acts geometrically on \( \mathfrak{h}(\mathcal{D}) \).

Proof. In order to extend the pull-back from \( \mathfrak{h}(\emptyset) \) to a unitary representation of \( SO_0(1,2) \) on \( \mathfrak{h}(\mathcal{D}) \), we have to show that \( \|[\Lambda f]\|_{\mathfrak{h}(\mathcal{D})} = \|[f]\|_{\mathfrak{h}(\mathcal{D})} \).

By construction,

\[
\|[\Lambda f]\|_{\mathfrak{h}(\mathcal{D})} = \int_{\mathcal{D}} d\mu_{\mathfrak{d}}(x) f(\Lambda^{-1}x) (x_+ \cdot p)^s^+ \|_{\mathfrak{h}(\partial V^+)}
\]

\[
= \int_{\mathcal{D}} d\mu_{\mathfrak{d}}(x) f(x) (\Lambda x_+ \cdot p)^s^+ \|_{\mathfrak{h}(\partial V^+)}
\]

\[
= \int_{\mathcal{D}} d\mu_{\mathfrak{d}}(x) f(x) (x_+ - \Lambda^{-1}p)^s^+ \|_{\mathfrak{h}(\partial V^+)}
\]

\[
= \|\tilde{u}_v(\Lambda)f_v\|_{\mathfrak{h}(\partial V^+)} = \|f_v\|_{\mathfrak{h}(\partial V^+)} = \|[f]\|_{\mathfrak{h}(\mathcal{D})},
\]

where \( s^+ \) is given by (3.3.8) with \( \zeta^2 = \mu^2 \).

Proposition 6.1.6. Let \( u(T) \) and \( u(P) \) be defined by

\[
\tilde{u}_v^+(T)\mathcal{F}_+\mathcal{F}_+ f = \tilde{u}_v^+(T,\tilde{T},\tilde{f}), \quad u(P_2)[f] = [P_2 f], \quad f \in \mathcal{D}_R(\mathcal{D})\).
\]

The operators \( u(T) \) and \( u(P_2) \) extend to well-defined (anti-)unitary operators on \( \mathfrak{h}(\mathcal{D}) \). They extend the representation \( u \) form \( SO_0(1,2) \) to \( O(1,2) \).

Proof. Let us calculate the action of \( \tilde{u}(T) \) in \( \mathfrak{h}(\partial V^+) \):

\[
(\mathcal{F}_+\mathcal{F}_+ T,\tilde{f})(p) = \int_{\mathcal{D}} d\mu_{\mathfrak{d}}(x) \tilde{f}(x) \mathcal{F}_+(x_+ \cdot p)^s^+.
\]

Using the fact that for \( t \in \mathbb{R} \)

\[
(6.1.6) \quad (t \pm i\epsilon)^s = e^{\mp i\pi \epsilon}(t \pm i\epsilon)^s,
\]

we write

\[
(Tx \cdot p + i\epsilon)^s = e^{i\pi \epsilon}(x \cdot (-Tp) + i\epsilon)^s = e^{i\pi \epsilon} (x \cdot (-Tp) - i\epsilon)^s^+ = e^{i\pi \epsilon}(x \cdot (-Tp) - i\epsilon)^s^+ = e^{i\pi \epsilon} (x \cdot (-Tp) + i\epsilon)^s^+.
\]
Now for $0 < m < 1/2$ the number $s^+$ is real, hence (note that $P = -T$ leaves the light cone invariant)

$$\mathcal{F}_{+|+} f(p) = \mathcal{F}_{-|+} f(-Tp).$$

For $\mu \geq 1/2\pi$, the complex conjugate of $s^+$ is $s^-$, hence

$$\mathcal{F}_{+|+} f(p) = e^{i\pi r^+} (\mathcal{F}_{-|+} f)(-Tp) = e^{i\pi r^+} (A_{s} \mathcal{F}_{+|+} f)(-Tp),$$

where we have used that (see Proposition 6.1.1)

$$A_{s} : h(\partial V) \rightarrow h(\partial V^+)$$

Comparing this result with the corresponding result for $\bar{u}_r^+(P)$, and inspecting (3.3.26), proves the claim. □

6.1.4. The Wightman two-point function. Finally, we apply the nuclear theorem to the quadratic form (6.1.2). It follows that there exist tempered distribution (3.3.26), proves the claim.

The distributions $\mathcal{F}_{+|+} f$ are called the two-point functions.

**Theorem 6.1.7** (Bros and Moschella [28], Theorem 4.1 & 4.2). The Wightman two-point function $\mathcal{W}^{(2)}(x_1, x_2)$ is a tempered distribution, which is the boundary value of the function

$$\mathcal{W}^{(2)}(z_1, z_2) = c_\gamma \frac{e^{-\pi r}}{\pi} \int p dp |p|^s (p \cdot z_2)^s (z_1 \cdot p)^s$$

defined and holomorphic for $(z_1, z_2) \in \mathcal{F}_+ \times \mathcal{F}_-$. The boundary values of (6.1.7) are taken as $\mathcal{F}_+ \not\supset 0$ and $\mathcal{F}_- \not\subset 0$, $(z_1, z_2) \in \mathcal{F}_+ \times \mathcal{F}_-$. As before, the exponents $s, \bar{s}$ are given by (3.3.8) and for the measure $d\mu_{\mathcal{F}}(p)$ one has

$$d\mu_{\mathcal{F}}(p) = \frac{d\alpha}{2},$$

in agreement with the normalisation used in [29] Section 4.2.

**Remark 6.1.8.** In Minkowski space, after Fourier transformation, the two-point function

$$\mathcal{W}^{(2)}_{m}(x, y) = \int_{\mathbb{R}^{d+1}} dk \delta(k^0) \delta(k \cdot m^2) e^{-ik \cdot x} e^{i\bar{k} \cdot y}$$

is the boundary value of a holomorphic function as $x \in \mathcal{F}^+$ and $y \in \mathcal{F}^-$ approach the reals. For $x = (x_0, \vec{x})$, $y = (y_0, \vec{y})$ and $p = (\sqrt{\vec{k}^2 + m^2}, \vec{k})$ this yields

$$\mathcal{W}^{(2)}_{m}(x_0, \vec{x}, y_0, \vec{y}) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \frac{dk}{2\sqrt{k^2 + m^2}} e^{i\bar{k}(x_0 - y_0) - i(x_0 - y_0)\sqrt{k^2 + m^2} \cdot \bar{\vec{x}}}. $$

A direct consequence of this result is the one-particle Reeh-Schlieder theorem:

**Theorem 6.1.9** (Bros and Moschella [28], Proposition 5.4). Let $\mathcal{O}$ be an open region in $dS$. It follows that $h(\mathcal{O}) + i\mathfrak{h}(\mathcal{O})$ is dense in $h(dS)$.
4.3.1

Moreover, the permuted Wightman function \( W \) is analytic and invariant under Lorentz transformations, i.e., for any vector in \( \mathfrak{h}(\mathcal{O}) \), the function \( W \) holds for \( (z_1, z_2) \) and \( (\Lambda z_1, \Lambda z_2) \) if \( \Lambda \in \text{SO}(1,2) \).

It follows that its boundary values vanish on \( dS \). This means that \(|f|\) is orthogonal to any vector in \( \mathfrak{h}(\mathcal{O}) \); thus it is the zero-vector. □

**Proposition 6.1.10** (Bros and Moschella [28], Proposition 2.2). *The Wightman two-point function \( W^{(2)}(z_1, z_2) \) can be analytically continued into the cut-domain

\[
\Delta = dS_C \times dS_C \setminus \Sigma
\]

where the cut \( \Sigma \) is the set

\[
\Sigma = \{ (z_1, z_2) \in dS_C \times dS_C \mid (z_1 - z_2) \cdot (z_1 - z_2) \geq 0 \}.
\]

Within \( \Delta \) the two point function is invariant under the transformation

\[
W^{(2)}(z_1, z_2) = W^{(2)}(\Lambda z_1, \Lambda z_2), \quad \Lambda \in \text{SO}(1,2).
\]

Moreover, the permuted Wightman function \( W^{(2)}(x_2, x_1) \) is the boundary value of the analytic function \( W^{(2)}(z_2, z_1) \) from its domain \( \{ (z_1, z_2) \mid z_1 \in \mathcal{F}^+, z_2 \in \mathcal{F}^- \} \).

**Proof.** Proposition 6.1.5 guarantees that the distribution \( W^{(2)}(z_1, z_2) \) is invariant under Lorentz transformations, i.e., if \( \Lambda \in \text{SO}(1,2) \). Invariance under the complexified group then follows by analytic continuation in the group parameter. For further details see [Bros and Moschella [28], Proposition 2.2]. □

**Remark 6.1.11.** The cut \( \Sigma \) contains all pairs of points \((x, y)\) in \( dS \times dS \), which are causal to each other. In other words,

\[
\Sigma \cap (dS \times dS) = \{ (x, y) \in dS \times dS \mid y \in \Gamma^+(x) \cup \Gamma^-(x) \}.
\]

The two-point function (6.1.7) can be expressed in terms of Legendre functions.

**Proposition 6.1.12** (Proposition 12 [29]). *The following integral representation holds for \((z_1, z_2) \in \mathcal{F}^+ \times \mathcal{F}^-\):

\[
W^{(2)}(z_1, z_2) = c_\nu \, P_\nu^+ \left( \frac{z_1 - z_2}{r} \right), \quad m > 0.
\]

The boundary values of (6.1.10) can be taken as \( \mathfrak{h} \) and \( \mathfrak{h}^{\infty} \) by linearity of (6.1.1).

**Remark 6.1.13.** The image of the domain \( \mathcal{F}^+ \times \mathcal{F}^- \) by the mapping

\[
(z_1, z_2) \mapsto \frac{z_1}{r} \frac{z_2}{r} \]

Note that \(|g| \in \mathfrak{h}(\mathcal{O}) + i\mathfrak{h}(\mathcal{O}) \) for \( g \in \mathcal{D}_C(\mathcal{O}) \), by linearity of (6.1.1).
The constants introduced in (6.1.1) were chosen to ensure that

\[
\begin{align*}
W_{29} = \cos(u_1 + u_2 + iv_1), & \quad 0 < u_1, u_2 < \pi, \quad v_1 \in \mathbb{R}.
\end{align*}
\]

It follows that

\[
\begin{align*}
\frac{\partial \gamma}{\partial \tau} &= -\cos(u_1 + u_2 + iv_1), & 0 < u_1 + u_2 < 2\pi, \quad v_1 \in \mathbb{R}.
\end{align*}
\]

Thus \( \mathbb{C} \setminus (-\infty, -1] \) is contained in the image. The fact that \( \mathbb{C} \setminus (-\infty, -1] \) equals the image follows from an argument in the ambient space, see \[29\) Proposition 3. The region \( \mathbb{C} \setminus (-\infty, -1] \) is exactly the domain of analyticity of the Legendre function \( P_{s+} \).

**Proof.** Let \( p \in \Gamma = \{(1, r \cos \alpha, r \sin \alpha) \in \partial \mathbb{V}^+ \mid -\pi \leq \alpha \leq \pi\} \). Because of the invariance properties of \( W^{(2)}(z_1, z_2) \) it is sufficient to consider the choice \( z_1 = (-ir \cosh \beta, 0, ir \sinh \beta) \), \( z_2 = (ir, 0, 0) \) such that \( \frac{z_1 \cdot z_2}{r} = \cosh \beta \in \mathbb{R}^+ \). It follows that

\[
\begin{align*}
W^{(2)}(z_1, z_2) &= c_v \int_{\Gamma} \frac{e^{-\pi r \tau}}{\pi} \, d\mu(p) \left( z_1 \cdot p \right)^{s^-} \left( p \cdot z_2 \right)^{s^+} \\
&= c_v \int_{\pi}^{\pi} \frac{d\alpha}{2\pi} (\cosh \beta + \sinh \beta \sin \alpha)^{s^-}.
\end{align*}
\]

In the second equality we have used (6.1.8). Finally, recall that according to \[131\) Eq. 7.4.2

\[
P_{s+}(\cosh \beta) = \frac{1}{\pi} \int_{\pi}^{\pi} \frac{d\alpha}{\pi} (\cosh \beta - \sinh \beta \cos \alpha)^{s^+ + 1},
\]

and \(-s^+ - 1 = s^- \).

Note that \( W^{(2)}(x_1, x_2) = W^{(2)}(x_2, x_1) \), and

\[
W^{(2)}(x_1, x_2) = W^{(2)}(x_2, x_1) = 2i \mathcal{J} W^{(2)}(x_1, x_2).
\]

The commutator function \( 2\mathcal{J} W^{(2)}(x_1, x_2) \) is an anti-symmetric distribution on \( dS \times dS \), which satisfies the Klein–Gordon equation in both entries, with initial conditions described in (5.3.5) and (5.3.6). In fact, for \( x_1, x_2 \) space-like, this is obvious and for \( x_1 = x_2 \) this follows from \[131\) page 199:

\[
P_{s+}(-1 + i0) - P_{s+}(-1 - i0) = 2i \sin s^+ \pi.
\]

In other words,

\[
W^{(2)}(x_+, x_-) - W^{(2)}(x_-, x_+) = c_v \cdot 2i \sin s^+ \pi = -i.
\]

The constants introduced in (6.1.11) were chosen to ensure that

\[
\frac{\partial}{\partial x_0} 2\mathcal{J} W^{(2)}(x, y) = -\delta_{S^1}.
\]

As before, \( \delta_{S^1} \) is the integral kernel of the unit operator with respect to the induced measure on \( S^1 \). It follows that

\[
(6.1.12)
\]

\[
\delta(x_1, x_2) = 2\mathcal{J} W(x_1, x_2)
\]

\[\text{See [29 Proposition 3].}\]

\[\text{The second line in the following formula is exactly the one given in [28 Eq. (4.18)].}\]
is the kernel of the commutator function defined in (6.3.3). Equation (6.1.12) extends the formula for the propagator given in (5.4.6) from $\mathcal{V}$ to $dS$. To show that $\mathcal{E}(x_1, x_2)$ as given in (6.1.12) is invariant under the rotations $R_0(\alpha), \alpha \in [0, 2\pi)$, choose a circle $\Gamma_0$ on $dV^+$ with $p_0 = 1$ in (6.1.7). Rotation invariance of the propagator now follows from $z_1 \cdot R_0 p = R_0^{-1} z_1 \cdot p$ and rotation invariance of the measure $d\mu_\Gamma = \frac{d\Gamma}{\Gamma}$; see (2.1.2).

6.2. The canonical one-particle Hilbert space

We now define a Hilbert space for functions supported on the time-zero circle $S^1$. As we have seen, the two-point function $W^{(2)}(x, y)$ is analytic for $x$ space-like to $y$. For $x = (0, r \sin \psi, r \cos \psi)$ and $y = (0, r \sin \psi', r \cos \psi')$, the (minimal) spatial distance $d$ define in (1.3.1) is given by

$$d(x, x') = r \arccos(-\frac{x' \cdot x}{r^2}) = |\psi' - \psi| r.$$ 

Thus $W^{(2)}(x, y) = c_\nu P^{s+}(-\cos(\psi' - \psi))$. Note that the singularity at $\psi = \psi'$ is integrable. This suggest the following definition.

**Definition 6.2.1.** The completion of $C^\infty(S^1)$ with respect to the scalar product

$$\langle h, h' \rangle_{\hat{h}(S^1)} = c_\nu \int_{S^1} r \, d\psi \int_{S^1} r \, d\psi' h(\psi) \overline{h(\psi')} P^{s+}(-\cos(\psi' - \psi)) \langle \cdot \rangle_{L^2(S^1, r d\psi)}$$

is a Hilbert space, which we denote by $\hat{h}(S^1)$. As before, $s^+$ is given by (3.3.8).

**Proposition 6.2.2.** The scalar product (6.2.1) can be

i.) expressed as

$$\langle h, h' \rangle_{\hat{h}(S^1)} = \langle h, \frac{1}{2\omega} h' \rangle_{L^2(S^1, r d\psi)} ,$$

with $\omega$ a strictly positive self-adjoint operator on $L^2(S^1, r d\psi)$ with Fourier coefficients

$$\tilde{\omega}(k) = r^{-1} (k + s^+) \left( \frac{k + s^+}{2} \right) \left( \frac{k + s^+}{2} \right), \quad k \in \mathbb{Z} ;$$

ii.) written as

$$\langle h, h' \rangle_{\hat{h}(S^1)} = \int \int \frac{h(\psi) e^{ik(\psi' - \psi)}}{\sqrt{2\pi r}} \frac{h'(\psi') e^{-ik(\psi' - \psi)}}{\sqrt{2\pi r}} \, d\psi \, d\psi' .$$

**Proof.** i.) Set

$$P^{s+}(-\cos(\psi' - \psi)) = \sum_{k \in \mathbb{Z}} P_k \frac{e^{ik(\psi' - \psi)}}{\sqrt{2\pi r}} .$$

This yields

$$\langle h, h' \rangle_{\hat{h}(S^1)} = \sqrt{2\pi r} c_\nu \sum_{k \in \mathbb{Z}} P_k \left( \int_{S^1} r \, d\psi \, h(\psi) \frac{e^{-ik\psi}}{\sqrt{2\pi r}} \right) \left( \int_{S^1} r \, d\psi' \, h'(\psi') \frac{e^{-ik\psi'}}{\sqrt{2\pi r}} \right)$$

$$= \sqrt{2\pi r} c_\nu \sum_{k \in \mathbb{Z}} P_k h_k \overline{h'_k} ,$$

(6.2.5)
where \( h_k \) and \( h'_k \) are the Fourier coefficients of \( h \) and \( h' \), respectively. Comparing (6.2.1) with (6.2.5), we see that

- \( \omega \) is a diagonal operator w.r.t. the orthonormal basis \( \{ e_k \in L^2(S^1, rd\psi) \mid e_k(\psi) = e^{ik\psi}, k \in \mathbb{Z} \} \).
- the Fourier coefficients \( \tilde{\omega}(k) \) of \( \omega \) are given by

\[
(6.2.6) \quad \tilde{\omega}(k) = \frac{2 \sin(\pi s^+)}{\sqrt{2\pi r}} \frac{1}{r \nu_k}, \quad k \in \mathbb{Z}.
\]

Using Proposition E.0.25, we arrive at (6.2.2).

\( \tilde{s} = 0 \) implies, from (6.2.2),

\[
\tilde{\omega}(k) = \tilde{\omega}(-k) \quad \text{for all} \quad k \in \mathbb{Z}.
\]

The details are given in the proof of Lemma 10.5.1. \( \square \)

**Remark 6.2.3.** In (E.0.24) we will establish that \( \tilde{\omega}(k) = \tilde{\omega}(-k) \) for all \( k \in \mathbb{Z} \). For the case of the principal series, one has \( s^+ = -\frac{1}{2} + iv \), with \( v \in \mathbb{R}^+ \), and

\[
\Gamma \left( \frac{k+i+iv}{2} \right) = \frac{k-i-iv}{2} \Gamma \left( \frac{k-i-iv}{2} \right)
\]

implies, from (6.2.2),

\[
(6.2.8) \quad \tilde{\omega}(k) = \frac{r^{-1} \left( \frac{(k-1/2)^2 + v^2}{4} \right) \left| \Gamma \left( \frac{k-i+iv}{2} \right) \right|^2}{\left| \Gamma \left( \frac{k+i+iv}{2} \right) \right|^2},
\]

showing that \( \tilde{\omega}(k) > 0 \) for all \( k \in \mathbb{Z} \). This positivity property also holds in the case of the complementary series. In that case, one has \( v = i \sqrt{\frac{1}{4} - \zeta^2} \), with \( 0 < \zeta \leq 1/2 \). Hence, \(-1/2 < s^- \leq 0\). Since \( \tilde{\omega}(k) = \tilde{\omega}(-k) \) for all \( k \in \mathbb{Z} \), it is enough to consider \( k \geq 0 \). We know from (7.4.8) that \( \tilde{\omega}(k) \tilde{\omega}(k+1) = r^{-2} k(k+1) + \mu^2 > 0 \). Hence \( \tilde{\omega}(k+1) \) and \( \tilde{\omega}(k) \) have the same sign and, therefore, in order to prove that \( \tilde{\omega}(k) > 0 \) for all \( k \geq 0 \), it is enough to establish that \( \omega(0) > 0 \). But, from (6.2.2), one has

\[
\tilde{\omega}(0) = r^{-1} s^+ \Gamma \left( \frac{s^+}{2} \right) \frac{\Gamma \left( \frac{1-s^+}{2} \right)}{\Gamma \left( \frac{1+s^+}{2} \right)} > 0,
\]

since \( s^+ < 0 \), and since \( \Gamma(x) > 0 \) for all \( x > 0 \) and \( \Gamma(x) < 0 \) for all \( x < 0 \) (one has \( \frac{s^+}{2} \in (-1, 0) \), but \( \frac{1-s^+}{2}, \frac{-s^+}{2}, \frac{1+s^+}{2} \) are all positive).
REMARK 6.2.4. As we shall see in (7.4.10), for both the principal and complementary series one has \( \frac{1}{2}(\bar{\omega}(k)\bar{\omega}(k+1) + \bar{\omega}(k)\bar{\omega}(k-1)) = \frac{k^2}{r^2} + \mu^2 \) and, hence, we conclude that \( \bar{\omega}(k) \) behaves for large \(|k|\) as \( \sqrt{\frac{k^2}{r^2} + \mu^2} \), approaching thus the dispersion relation of the Minkowski space-time.

COROLLARY 6.2.5. The operator \( \omega \) on \( L^2(S^1, rd\psi) \) satisfies the operator identity

\[
\omega = |r \cos \psi|^{-1} |\varepsilon| \left( \coth \pi \varepsilon \rho - \frac{(P_1)_{\omega}}{\sinh \pi \varepsilon} \right).
\]

PROOF. Comparing i.) and ii.) in Proposition 6.2.2 yields

\[
\omega^{-1} = \frac{1}{|\varepsilon|} \left( \coth \pi \varepsilon \rho + \frac{(P_1)_{\omega}}{\sinh \pi \varepsilon} \right) |r \cos \psi|,
\]

which is equivalent to (6.2.9). \( \square \)

PROPOSITION 6.2.6. For \( h, h' \in \mathcal{D}(\omega) \) have

\[
\langle \omega r h, \omega r h' \rangle_{\mathcal{H}(S^1)} = c_\nu \int_{S^1} r \, d\psi \int_{S^1} r \, d\psi' \mathcal{H}(\psi') P_\nu \left( -\cos(\psi' - \psi) \right) h'(\psi).
\]

Note that \( C^\infty(S^1) \subset \mathcal{D}(\omega) \).

PROOF. See Proposition E.0.28 \( \square \)

The following definition respects the causal structure of the globally hyperbolic manifold \( dS \supset S^1 \). This will become evident in the sequel.

DEFINITION 6.2.7. For \( I \) a bounded open interval in \( S^1 \), we define a real subspace of \( \mathcal{H}(S^1) \) by

\[
\mathcal{H}(I) \doteq \left\{ h \in \mathcal{H}(S^1) \mid \text{supp} \left( \mathfrak{R}h, \omega^{-1}\mathfrak{I}h \right) \subset I \times I \right\}.
\]

Clearly, \( \mathcal{H}(I) \) is in the symplectic complement of \( \mathcal{H}(I) \) if \( J \subset S^1 \setminus I \). This follows directly from the definition:

\[
\mathcal{I}(h, g)_{\mathcal{H}(S^1)} = \langle \mathfrak{R}h, \omega^{-1}\mathfrak{I}g \rangle_{L^2(S^1, rd\psi)} - \langle \omega^{-1}\mathfrak{I}f, \mathfrak{R}g \rangle_{L^2(S^1, rd\psi)} = 0
\]

for \( h \in \mathcal{H}(I) \) and \( g \in \mathcal{H}(J) \).

6.3. Time-symmetric and time-antisymmetric test-functions

The restriction of the Fourier transform to the mass shell allows an extension from \( \mathcal{D}(dS) \) to distributions supported on the time-zero circle. We shall identify \( dS \) with \( \mathbb{R} \times S^1 \) via the coordinate system

\[
x(x_0, \psi) = \left( \begin{array}{c} x_0 \cos \psi \\ \sqrt{r^2 + x_0^2} \sin \psi \\ \sqrt{r^2 + x_0^2} \cos \psi \end{array} \right) \in dS
\]

and write \( (f \otimes h)(x) := f(x_0)h(\psi) \) for \( f \in \mathcal{D}(\mathbb{R}) \) and \( h \in \mathcal{D}(S^1) \) if \( x = x(x_0, \psi) \). The metric on \( dS \) is

\[
g = \frac{1}{r^2 + x_0^2} \, dx_0 \otimes dx_0 - (r^2 + x_0^2) \, d\psi \otimes d\psi
\]

and \( |g| = 1 \). Thus \( d\mu_{dS}(x) = dx_0 rd\psi \).
THEOREM 6.3.1. Let \( h \in C_0^\infty(S^1) \) and let \( \delta_k \) be a sequence of absolutely integrable smooth functions, supported in a neighbourhood of the origin in \( \mathbb{R} \), approximating the Dirac \( \delta \)-function. It follows that for all \( \mu > 0 \) the limits

\[
\lim_{k \to \infty} \| [\delta_k \otimes h] \|_{\mathfrak{h}(dS)} \text{ and } \lim_{k \to \infty} \| [\mathfrak{n} (\delta_k \otimes g)] \|_{\mathfrak{h}(dS)}
\]

exist and equal \( \| h \|_{\mathfrak{h}(S^1)} \) and \( \| \mathfrak{w} r g \|_{\mathfrak{h}(S^1)} \), respectively. Here \( (\delta_k \otimes h) \) denotes the Lie derivative\(^4\) of \((\delta_k \otimes h)\) along the unit normal future pointing vector field \( \mathfrak{n} \).

PROOF. According to Proposition 6.1.12

\[
\lim_{k, k' \to \infty} \langle [\delta_k' \otimes h], [\delta_{k'}' \otimes h'] \rangle_{\mathfrak{h}(dS)} = \lim_{k, k' \to \infty} \int_{dS \times dS} d\mu_{dS}(x) d\mu_{dS}(x') \overline{\langle \delta_k \otimes h \rangle}(x) W^{(2)}(x, x') \overline{\langle \delta_{k'}' \otimes h' \rangle}(x')
\]

\[
= c_v \int_{S^1 \times S^1} r^2 d\psi d\psi' \int_{\mathbb{R} \times \mathbb{R}} dx_0 dx_0' \delta(x_0) \delta(x_0')
\]

\[
\times \overline{\langle h \rangle}(\psi)' (\overline{\frac{\partial}{\partial x_0} \frac{\partial}{\partial x_0'}} P_s (\frac{x_0 - x_0'}{r}))
\]

\[
= c_v \int_{S^1 \times S^1} r^2 d\psi d\psi' \int_{\mathbb{R} \times \mathbb{R}} dx_0 dx_0' \delta(x_0) \delta(x_0')
\]

\[
\times \overline{\langle h \rangle}(\psi)' (\overline{\frac{\partial}{\partial x_0} \frac{\partial}{\partial x_0'}} P_s (\frac{x_0 - x_0'}{r}))
\]

\[
= c_v \int_{S^1 \times S^1} r d\psi \int_{S^1} r d\psi' \overline{\langle h \rangle}(\psi)' P_s \left(-\cos(\psi' - \psi)\right) (\psi')
\]

(6.3.3) \(= \langle \mathfrak{w} h, \mathfrak{w} \mathfrak{r} h' \rangle_{\mathfrak{h}(S^1)} \).

The second but last equality follows from

\[
\frac{\partial}{\partial x_0'} (x \cdot x') = \frac{\partial}{\partial x_0} \left(x_0 x_0' - \sqrt{1 + x_0^2} \sqrt{1 + x_0'} \cos(\psi - \psi') \right)
\]

\[
= x_0 - \frac{x_0}{\sqrt{1 + x_0^2}} \sqrt{1 + x_0'} \cos(\psi - \psi')
\]

and \( \frac{\partial}{\partial x_0} \frac{\partial}{\partial x_0'} (x \cdot x') = 1 - \frac{x_0}{\sqrt{1 + x_0^2}} \frac{x_0'}{\sqrt{1 + x_0^2}} \cos(\psi - \psi') \). Thus

\[
\frac{\partial}{\partial x_0} (x \cdot x') |_{x_0 = x_0'} = 0, \quad \frac{\partial}{\partial x_0} (x \cdot x') |_{x_0 = x_0'} = 0, \quad \frac{\partial}{\partial x_0} (x \cdot x') |_{x_0 = x_0'} = 1.
\]

The last equality in (6.3.3) follows from Proposition 6.2.6 \(\square\)

\(^4\)Recall that \(\int_{S^1 \times \mathbb{R}} \overline{\delta'(t) h}(\mathfrak{s}) e^{i(\omega t - \mathfrak{s} \cdot \mathfrak{x})} = -i \omega \int_{S^1} d\mathfrak{s} h(\mathfrak{s}) e^{-i \mathfrak{s} \cdot \mathfrak{x}}\).
Remark 6.3.2. We can now extend the class of distributions considered in Remark 5.5.9. The domain of $E$ contains distributions of the form
\[ f(x) \equiv (\delta \otimes h)(x) = \delta(x_0)h(\psi), \]
\[ g(x) \equiv (\delta' \otimes h)(x) = \delta'(x_0)h(\psi), \]
with $h \in D_S(\mathbb{S}^1)$ and $x \equiv x(x_0,\psi)$, using the coordinates introduced in (6.3.1). The Lorentz invariant measure is $d\mu dS(x_0,\psi) = dx_0 r d\psi$. Using (6.1.12), the properties of the convolution ensure that there exist $C^\infty$-solutions $f, g$ of the Klein–Gordon equation (5.1.1) with Cauchy data:
\[ (f|_{\mathbb{S}^1}, (n f)|_{\mathbb{S}^1}) = (0, -h) \equiv (\phi, \pi), \]
and, by partial integration,
\[ (g|_{\mathbb{S}^1}, (n g)|_{\mathbb{S}^1}) = (h, 0) \equiv (\phi, \pi). \]
All elements in $\hat{\mathfrak{f}}(\mathbb{S}^1)$ are linear combinations of the Cauchy data arising from sharp-time testfunctions $f, g$ of the form described above.

The functions $\delta \otimes h$ and $\delta' \otimes h$ provide examples of test-functions, which are symmetric and anti-symmetric, respectively, under time-reflection. In fact, the time-reflection $T$ induces a conjugation $\kappa$ on $\mathfrak{h}(dS)$, as the map $f \mapsto T_\ast f$ leaves the kernel of $\mathcal{F}_{\pm 1}$ invariant. The subspace consisting of functions invariant under time-reflection is
\[ \mathfrak{h}^\kappa(dS) = \{ f \in \mathfrak{h}(dS) | \kappa f = f \}. \]
One can decompose any testfunction into a symmetric and an anti-symmetric part with respect to time-reflections:
\[ f = \frac{1}{2}(f + \kappa f) + \frac{1}{2}(f - \kappa f), \quad f \in \mathfrak{h}(dS). \]
For $[f], [g] \in \mathfrak{h}^\circ(dS) \cap \mathfrak{h}^\kappa(dS)$, polarisation yields
\[ \langle [f], [g] \rangle_{\mathfrak{h}(dS)} = \langle [T_\ast f], [T_\ast g] \rangle_{\mathfrak{h}(dS)} = \langle [g], [f] \rangle_{\mathfrak{h}(dS)}. \]
Since $\mathfrak{h}^\circ(dS)$ is dense in $\mathfrak{h}(dS)$, this implies
\[ \mathcal{J}(f, g)_{\mathfrak{h}(dS)} = 0 \quad \text{for all } f, g \in \mathfrak{h}^\kappa(dS). \]
Thus $\mathfrak{h}^\kappa(dS)$ is a real subspace of $\mathfrak{h}(dS)$.

The following result shows that the functions introduced above are already the most general elements in $\mathfrak{h}^\kappa(dS)$ and its symplectic complement $\mathfrak{h}^\kappa(dS)^\perp$, respectively.

---

5An anti-linear isometry $C$ satisfying $C^2 = 1$ is called a conjugation.
COROLLARY 6.3.3. Let \( I \subset S^1 \) be an open interval (or \( I = S^1 \)) and let \( h, g \in D_{\mathbb{R}}(I) \). It follows that

i.) \( \delta \otimes h \in h^\times(dS) \cap h(I) \) and \( h \in \hat{h}(I) \) is real valued;
ii.) \( \delta' \otimes g \in h^\times(dS)^\perp \cap h(I) \) and \( \imath \omega g \in \hat{h}(I) \) has purely imaginary values;
iii.) for every time-symmetric function \( f \in D_{\mathbb{R}}(\emptyset_I) \) there exists a function \( h \in D_{\mathbb{R}}(I) \) such that
\[
[f] = [\delta \otimes h] \; ;
\]
iv.) for every anti-time-symmetric function \( e \in D_{\mathbb{R}}(\emptyset_I) \) there exists a function \( g \in D_{\mathbb{R}}(I) \) such that
\[
[e] = [\imath n(\delta \otimes g)] \; .
\]

REMARK 6.3.4. The statements iii.) and iv.) imply that there is a one-to-one relation between the image of time-symmetric (time-antisymmetric) testfunctions in \( h(dS) \) and real (purely imaginary) valued functions in \( \hat{h}(S^1) \). The Minkowski space case of this result is proven in [163], Vol. II p. 217. It also follows directly by differentiation from Eq. (6.1.9).

PROOF. i.) By assumption, \( h \) is real valued, and we have already seen that \( \delta \otimes h \in h(dS) \) is equivalent to \( h \in \hat{h}(I) \); thus we have only to show that \( \delta \otimes h \in h^\times(dS) \). This can be achieved by approximation the delta function with a sequence of functions which are all symmetric around the origin.

ii.) By assumption, \( g \) is real valued, and we have already seen that \( \delta' \otimes g \in h(dS) \) is equivalent to \( \imath \omega g \in \hat{h}(S^1) \). Clearly, the definition of \( \hat{h}(I) \) together with \( g \in D_{\mathbb{R}}(I) \) implies that the function \( \imath \omega g \) takes purely imaginary values and lies in \( \hat{h}(I) \). Thus it only remains to show that \( \delta' \otimes g \in h^\times(dS)^\perp \). This can be achieved by approximation the derivative of the delta function with a sequence of functions which are all anti-symmetric around the origin.

iii.) For every time-symmetric function \( f \in D_{\mathbb{R}}(\emptyset_I) \), the \( C^\infty \)-solution \( \check{f} \) of the Klein–Gordon equation is time-symmetric. This implies that \( (\check{f} \mid_{S^1}) \) vanishes. On the other hand, we can define \( \check{h} \doteq \check{f} \mid_{S^1} \). It then follows from Theorem 5.5.1 that \( [\check{f}] = [\delta \otimes \check{h}] \).

iv.) For every anti-time-symmetric function \( e \in D_{\mathbb{R}}(\emptyset_I) \), the corresponding \( C^\infty \)-solution \( \check{e} \) of the KG equation is anti-time-symmetric. This implies that \( \check{e} \mid_{S^1} \) vanishes. On the other hand, we can define \( \check{g} \doteq -(\imath \omega \check{e}) \mid_{S^1} \). It then follows from Theorem 5.5.1 that \( [\check{e}] = [\imath n(\delta \otimes \check{g})] \). \( \square \)

PROPOSITION 6.3.5. The linear extension of the map
\[
(6.3.7) \quad h_1 + \imath \omega h_2 \mapsto [\delta \otimes h_1] + [\delta' \otimes h_2]
\]
defines a unitary map \( U : \hat{h}(S^1) \rightarrow h(dS) \), which respects the local structure, i.e.,
\[
U : \hat{h}(I) \rightarrow h(\emptyset_I) \; ,
\]
with \( \emptyset_I = 1'' \) the causal completion of \( I \subset S^1 \).

PROOF. We have already seen that the image of \( [\delta \otimes h_1] + [\delta' \otimes h_2] \) is dense in \( h(dS) \). Moreover,
\[
\| h_1 + \imath \omega h_2 \|_{\hat{h}(S^1)} = \| [\delta \otimes h_1] + [\delta' \otimes h_2] \|_{h(dS)} \; .
\]
The result now follows by linear extension. The local part follows from Corollary 6.3.3.

**Corollary 6.3.6.** Let $I$ be an open interval in $S^1$. Then $\hat{h}(I) + i\hat{h}(I)$ is dense in the Hilbert space $\hat{h}(S^1)$.

**Proof.** This result follows directly from Proposition 6.1.9:
\[
\hat{h}(I) + i\hat{h}(I) = U^{-1}\hat{h}(O_1) + i\hat{h}(O_1) = U^{-1}\hat{h}(dS) = \hat{h}(S^1).
\]
(A direct proof might be based on arguments similar to those given in [188]. However, we have not fully investigated this question.)

**Corollary 6.3.7.** For any double wedge $\mathcal{W}$, we have $\hat{h}(\mathcal{W}) = \hat{h}(dS)$.

**Proof.** The completion of $\mathcal{D}_c(S^1 \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\})$ with respect to the scalar product (6.2.1) is $\hat{h}(S^1)$. Thus, by Corollary 6.3.3 one has $\hat{h}(\mathcal{W}) = \hat{h}(dS)$. The general result follows from $\mathcal{W} = \Lambda\mathcal{W}_1$ for some $\Lambda \in SO(1,2)$.
CHAPTER 7

Quantum One-Particle Structures

Given a classical dynamical system for the Klein–Gordon equation on the de Sitter space (in either the covariant or the canonical formulation) there is a unique one-particle quantum system associated to it, characterised by the geodesic KMS condition.

7.1. The covariant one-particle structure

As we have seen, the Hilbert space \( \mathcal{H}(\text{dS}) \) carries an (anti-)unitary irreducible representation \( u \) of \( O(1,2) \).

\[ \text{THEOREM 7.1.1.} \quad \text{Consider the identity map} \]
\[ K: \quad \mathfrak{t}(\text{dS}) \to \mathcal{H}(\text{dS}) \]
\[ [f] \mapsto [f], \quad f \in \mathcal{D}_{\mathcal{F}}(\text{dS}) . \]

\[ \text{It follows that the triple} \quad (K, \mathcal{H}(\text{dS}), u) \quad \text{is a de Sitter one-particle structure for the classical dynamical system} \quad (\mathfrak{t}(\text{dS}), \sigma, u). \]

\[ \text{In other words,} \]
\[ \text{i.)} \quad K \text{ defines a symplectic map from} \quad (\mathfrak{t}(\text{dS}), \sigma) \quad \text{to} \quad (\mathcal{H}(\text{dS}), \langle \cdot, \cdot \rangle_{\mathcal{H}(\text{dS})}) \quad \text{and the image of} \quad \mathfrak{t}(\text{dS}) \quad \text{is dense in} \quad \mathcal{H}(\text{dS}); \]
\[ \text{ii.)} \quad \text{there exists a (anti-) unitary representation} \quad u \quad \text{of} \quad O(1,2) \quad \text{on} \quad \mathcal{H}(\text{dS}) \quad \text{satisfying} \]
\[ u(\Lambda) \circ K = K \circ u(\Lambda), \quad \Lambda \in O(1,2) ; \]
\[ \text{iii.)} \quad \text{for any wedge} \quad W, \quad \text{the geodesic KMS condition holds:} \]
\[ \text{(7.1.1)} \quad K \mathfrak{t}(W) \subset \mathcal{D}(u(\Lambda_W(i\pi))) , \]
\[ \text{and} \]
\[ \text{(7.1.2)} \quad u(\Lambda_W(i\pi))[f] = u(\Theta_W)[f] , \quad [f] \in K \mathfrak{t}(W) , \]
\[ \Theta_W \text{ is the reflection on the edge of the wedge} \quad W. \]

\[ \text{PROOF.} \quad K \text{ is well-defined, as ker} \quad \mathcal{F}_{+|\mathcal{V}} = \ker \mathcal{F}. \]

\[ \text{i.)} \quad \text{It follows from} \]
\[ \delta'(x_1, x_2) = 2m^2 \mathcal{W}_{\infty}(x_1, x_2) \]
\[ \text{that} \quad K \text{ is symplectic; see (6.1.12). Recall that} \quad \mathcal{H}(\text{dS}) \text{ is dense in} \quad \mathcal{H}(\text{dS}); \]
\[ \text{ii.)} \quad \text{Both} \quad u(\Lambda) \quad \text{and} \quad u(\Lambda) \quad \text{are induced by the pullback action on the test functions: for} \quad f \in \mathcal{D}_{\mathcal{F}}(\text{dS}) \]
\[ K \circ u(\Lambda) [f] = K \circ \mathcal{P}(\Lambda_* f) = [\Lambda_* f] \]
\[ = u(\Lambda)[f] = u(\Lambda) \circ K [f] , \quad \Lambda \in O(1,2) . \]
The second but last identity follows from the definition of the Fourier-Helgason transform (see \(6.1.1\)), and Proposition \(6.1.5\). One can read of from (2.3.1) that
\[
\Lambda_1(t + i\pi) = \Lambda_1(t)TP_1.
\]

Now, let \(f, g \in \mathcal{D}_F(W_1)\). It follows that the map
\[
t \mapsto \langle [f, u(\Lambda_1(t))g] \rangle_{\mathfrak{h}(dS)} = \langle [f, (\Lambda_1(t)_+)g] \rangle_{\mathfrak{h}(dS)}
\]
allows an analytic continuation into the strip \(\{ t \in \mathbb{C} \mid 0 < \Im t < \pi \} \) with continuous boundary values. The boundary values are
\[
\langle [f, \Lambda_1(i\pi t)_+/g] \rangle_{\mathfrak{h}(dS)}
\]
\[
= \int_{dS \times dS} d\mu_{dS}(x_1)d\mu_{dS}(x_2)f(x_1)W^{(2)}(x_1, x_2)g(\Lambda_1^{-1}(t)x_2)
\]
\[
= \int_{dS \times dS} d\mu_{dS}(x_1)d\mu_{dS}(x_2)f(x_1)W^{(2)}(x_1, \Lambda_1(t)x_2)g(x_2)
\]

This identity holds for the total set of vectors \([f] \in \mathfrak{h}(dS) \mid f \in \mathcal{D}_F(W_1)\). It follows that the identity \(7.1.2\) holds.

\[\Box\]

The space \(\mathfrak{h}(W)\) is a real subspace in \(\mathfrak{h}(dS)\). Moreover, \(\mathfrak{h}(W) + i\mathfrak{h}(W)\) is dense in \(\mathfrak{h}(dS)\) and \(\mathfrak{h}(W) \cap i\mathfrak{h}(W) = \{0\}\). Thus one can define, following Eckmann and Osterwalder \([52]\) (see also \([134]\)), a closeable operator
\[
s_w: \mathfrak{h}(W) + i\mathfrak{h}(W) \rightarrow \mathfrak{h}(W) + i\mathfrak{h}(W)
\]
\[
f + ig \mapsto f - ig.
\]

The polar decomposition of its closure \(\overline{s_w} = j_w \delta_{w}^{1/2}\) provides
- an anti-unitary involution (i.e., a conjugation)
\[
j_w: \mathfrak{h} \oplus \overline{\mathfrak{h}} \rightarrow \mathfrak{h} \oplus \overline{\mathfrak{h}}
\]
\[
f \oplus g \mapsto \overline{g} \oplus \overline{f}.
\]

- a complex linear, positive operator \(\delta_{w}^{1/2}\).

**Theorem 7.1.2** (One-particle Bisognano-Wichmann theorem). The one-particle
Tomita operator \(s_{w_1}\) has the polar decomposition
\[
s_{w_1} = u(TP_1)u(\Lambda_1(i\pi)).
\]

**Proof.** According to Theorem \(7.1.1\)iii. we have
\[
u(TP_1)([f] + i[g]) = u(\Lambda_1(i\pi))[f] + i[g], \quad [f], [g] \in \mathfrak{h}^*(W_1),
\]
Since \( u(TP_1) \) is idempotent and anti-linear, this implies
\[
(\langle f \rangle - i\langle g \rangle) = u(TP_1)u(\Lambda_t(i\pi))(\langle f \rangle + i\langle g \rangle), \quad [\langle f \rangle, [\langle g \rangle] \in \mathfrak{h}^\circ(\mathcal{W}).
\]
The left hand side coincides with \( s_{w_1}(\langle f \rangle + i\langle g \rangle) \). The space \( \mathfrak{h}^\circ(\mathcal{W}) \) is invariant under \( u(\Lambda_t(t)) \) and therefore is a core for \( u(\Lambda_t(i\pi)) \). Therefore the above equation implies that \( s_{w_1} \) has the polar decomposition (7.1.5).

**Corollary 7.1.3.** The quadrupel \( (K, h(\bar{\mathcal{W}}), u(\Lambda_W(\frac{x}{2})), u(\Theta_W)) \), with \( W \) an arbitrary wedge, forms a double \( 2\pi\tau \)-KMS one-particle structure for the classical double dynamical system \( (\mathbb{R}^2, \sigma, u(\Lambda_W(\frac{x}{2})), u(\Theta_W)) \) in the sense of \( \text{A.0.7} \).

**Proof.** We verify the properties listed in \( \text{A.0.7} \):

a.) \( \mathfrak{h}(\bar{\mathcal{W}}) \) is a complex Hilbert space; in fact, it equals \( \mathfrak{h}(dS) \), see Proposition 6.3.7.

b.) The map \( K : \mathbb{R}^2 \to \mathfrak{h}(\bar{\mathcal{W}}) \) is real linear and symplectic (Theorem 7.1.1(i)). Moreover,
\[
Kt(W) + iKt(W) = \mathfrak{h}^0(W) + i\mathfrak{h}^0(W)
\]
is dense in \( \mathfrak{h}(dS) = \mathfrak{h}(\bar{\mathcal{W}}) \). This follows from Theorem 6.1.9.

c.) \( t \mapsto u(\Lambda_W(t)) \) is a strongly continuous one-parameter group of unitaries, and
\[
u(\Lambda_W(t)) \circ K = K \circ u(\Lambda_W(t)) \tag{7.1.6}
\]
This is a special case of (ii). By construction, the generator of the boost \( t \mapsto \Lambda_W(t) \) is unitarily equivalent to \( \text{3.4.4} \). It has no zero eigenvalue and according to 7.1.1
\[
(Kt(W) + iKt(W)) \subset \mathcal{D}(u(\Lambda_W(i\pi)))
\]
d.) \( u(\Theta_W) \) is a conjugation, and
\[
u(\Theta_W) \circ K = K \circ u(\Theta_W) \tag{7.1.7}
\]
The pre-Bisognano-Wichmann condition (see [119, p. 75]) holds:
\[
u(\Lambda_W(i\pi))K|f\rangle = u(\Theta_W)K|f\rangle, \quad |f\rangle \in \mathcal{T}(W_1).
\]
Both properties follow from Theorem 7.1.1(iii.) and the fact that \( \Theta_W \in O(1,2) \).

### 7.2. One-particle structures with positive and negative energy

Let \( \hat{\mathcal{D}}(S^1) \) be the completion of \( \mathcal{D}_c(S^1 \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}) \) with respect to the scalar product
\[
\langle h_1, h_2 \rangle_{\hat{\mathcal{D}}(S^1)} = \langle h_1, (2|k|)^{-1}h_2 \rangle_{L^2(S^1, |\cos \psi|^2)} \tag{7.2.1}
\]
for \( h_1, h_2 \in \mathcal{D}_c(S^1 \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}) \). Let \( \hat{\mathcal{D}}(I_\pm) \) be the completion of \( \mathcal{D}_c(I_\pm) \) with respect to the scalar product (7.2.1). Then
\[
\hat{\mathcal{D}}(S^1) = \hat{\mathcal{D}}(I_+) \oplus \hat{\mathcal{D}}(I_-)
\]
This follows from Eq. (5.4.4) and Lemma 5.4.4.
PROPOSITION 7.2.1. Let \( \hat{\mathcal{K}}_\infty : \hat{\mathfrak{f}}(\mathbb{S}^1) \rightarrow \hat{\mathfrak{d}}(\mathbb{S}^1) \) be the map given by

\[
(\hat{\mathcal{K}}_\infty (\hat{\psi}, \pi))(\psi) = \cos \psi \, \pi(\psi) - i \, (\epsilon \hat{\psi})(\psi) .
\]

Then \( (\hat{\mathcal{K}}_\infty, \hat{\mathfrak{d}}(\mathbb{S}^1), e^{it\epsilon}) \) forms a one-particle structure for the classical dynamical system \( (\hat{\mathfrak{f}}(\mathbb{S}^1), \hat{\sigma}, \hat{\mathcal{U}}(\Lambda_1(t))) \).

**Proof.** The map \( (7.2.2) \) is well-defined for \( (\hat{\psi}, \pi) \in C^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1) \). This follows from the fact that \( \epsilon^2 \) is a differential operator, which satisfies

\[
(\epsilon^2 \hat{h})(\psi \pm \frac{\pi}{2}) = O(\psi) \quad \text{for} \quad h \in C^\infty(\mathbb{S}^1),
\]

just as \( \cos(\psi \pm \frac{\pi}{2}) \).

Use that \( \epsilon |\epsilon|^{-1} = \cos \psi |\cos \psi|^{-1} \) equals 1 on \( L^2(I_+) \) and \(-1\) on \( L^2(I_-) \) to show

\[
2\mathfrak{C} (\hat{\mathcal{K}}_\infty(\hat{\psi}_1, \pi_1), \hat{\mathcal{K}}_\infty(\hat{\psi}_2, \pi_2))(\pi)(\psi) = 2\mathfrak{C} (\psi_1 - i \epsilon \hat{\psi}_1, \frac{1}{2\epsilon}(\cos \psi \, \pi - i \epsilon \hat{\psi})) L^2(\mathbb{S}^1, |\cos \psi|^{-1} \, d\psi) = (\psi_1, \pi_2)L^2(\mathbb{S}^1, rd\psi) - (\psi_1, \pi_2)L^2(\mathbb{S}^1, rd\psi).
\]

Thus \( \hat{\mathcal{K}}_\infty \) is symplectic.

Moreover, \( \hat{\mathcal{K}}_\infty \) intertwines \( \hat{\mathcal{U}}(\Lambda_1(t)) \) and \( e^{it\epsilon} \); according to \( (5.5.8) \)

\[
\hat{\mathcal{U}}(\Lambda_1(t))(\hat{\psi}, \pi) = (\hat{\psi}_t, \pi_t)
\]

with

\[
\hat{\psi}_t(\psi) = (\cos(\epsilon t) \hat{\psi} - \sin(\epsilon t)|\epsilon|^{-1} \cos \psi \, \pi)(\psi)
\]

\[
\pi_t(\psi) = \cos^{-1}(\psi)(\epsilon \sin(\epsilon t) \hat{\psi} + \cos(\epsilon t) \cos \psi \, \pi)(\psi).
\]

Consequently,

\[
\hat{\mathcal{K}}_\infty \circ \hat{\mathcal{U}}(\Lambda_1(t))(\hat{\psi}, \pi) = \cos \psi \, \pi_t - i \epsilon \hat{\psi}_t
\]

\[
= ( \epsilon \sin(\epsilon t) \hat{\psi} + \cos(\epsilon t) \cos \psi \, \pi)
\]

\[
- i \epsilon (\cos(\epsilon t) \hat{\psi} - \epsilon^{-1} \sin(\epsilon t) \cos \psi \, \pi)
\]

\[
= \cos(\epsilon t)(\cos \psi \, \pi - i \epsilon \hat{\psi}) + i \sin(\epsilon t)(\cos \psi \, \pi - i \epsilon \hat{\psi})
\]

\[
= e^{it\epsilon} (\hat{\mathcal{K}}_\infty(\hat{\psi}, \pi)).
\]

Since \( |\cos \psi|^{-1} \) is finite away from the boundary of \( I_+ \), the set \( \hat{\mathcal{K}}_\infty(\hat{\mathfrak{f}}(\mathbb{S}^1)) \) is dense in \( \hat{\mathfrak{d}}(\mathbb{S}^1) \). \( \square \)

PROPOSITION 7.2.2. Consider the one-particle structure \( (\hat{\mathcal{K}}_\infty, \hat{\mathfrak{d}}(\mathbb{S}^1), e^{it\epsilon}) \). It follows that

i.) the restricted structure \( (\hat{\mathcal{K}}_\infty, \hat{\mathfrak{d}}(I_+), e^{it\epsilon;1_+}) \) is a positive energy one-particle structure for \( (\hat{\mathfrak{f}}(I_+), \hat{\sigma}, \hat{\mathcal{U}}(\Lambda_1(t))) \), i.e.,

- the group \( t \mapsto e^{it\epsilon;1_+} \) has a positive generator \( \epsilon|_{1_+} \geq 0 \);
- \( \hat{\mathcal{K}}_\infty \hat{\mathfrak{f}}(I_+) \) is dense in \( \hat{\mathfrak{d}}(I_+) \).

ii.) the restricted structure \( (\hat{\mathcal{K}}_\infty, \hat{\mathfrak{d}}(I_-), e^{it\epsilon;1_-}) \) is a negative energy one-particle structure for \( (\hat{\mathfrak{f}}(I_-), \hat{\sigma}, \hat{\mathcal{U}}(\Lambda_1(t))) \), i.e.,

- the group \( t \mapsto e^{it\epsilon;1_-} \) has a negative generator \( \epsilon|_{1_-} \leq 0 \);
- \( \hat{\mathcal{K}}_\infty \hat{\mathfrak{f}}(I_-) \) is dense in \( \hat{\mathfrak{d}}(I_-) \).
iii.) the parity and time-reflections are represented (anti-) unitarily, namely

\( \hat{K}_\infty \circ \hat{u}(P_1) = -(P_1)_* \circ \hat{K}_\infty ; \)  
(7.2.3)

\( \hat{K}_\infty \circ \hat{u}(T) = -C \circ \hat{K}_\infty , \)
(7.2.4)

where

\( (Ch)(\psi) = \hat{h}(\psi) , \) \( h \in C^\infty(S^1) , \)

extends to \( \hat{d}(S^1) ; \)

iv.) zero is not an eigenvalue of \( \epsilon \); thus the one-particle structures given in i.) and ii.) are unique, up to unitary equivalence.

PROOF. i.) and ii.) follow from (5.4.5) as well as the final statement in the proof of Proposition 7.2.1. For iii.) use that \( (P_1)_* \) anti-commutes with \( \epsilon \) and with the multiplication operator \( \cos \psi \). Eq. (7.2.3) follows from

\[ \hat{K}_\infty ((P_1)_* \psi, \pi) = -(P_1)_* \cos \pi \pi + i(P_1)_* \epsilon \psi \]

and Eq. (7.2.4) follows from \( \hat{K}_\infty (\psi, -\pi) = -(\cos \pi \pi - i\epsilon \psi) \). Finally, iv.) follows from Lemma 5.4.4 and Proposition A.0.4. \( \square \)

PROPOSITION 7.2.3. The operator \( j = C(P_1)_* \) acting on \( \hat{d}(S^1) \) is an anti-unitary involution (i.e., a conjugation), which implements the \( P_1 T \) transformation and anti-commutes with the generator \( \epsilon \) of the boosts \( t \mapsto \Lambda_t(t) : \)

\( j \circ \hat{K}_\infty = \hat{K}_\infty \circ \hat{u}(P_1 T) , \)
(7.2.5)

\( j \epsilon = -\epsilon j . \)
(7.2.6)

Note that Eq. (7.2.6) and anti-linearity imply

\( e^{it\epsilon} j = j e^{it\epsilon} \) and \( j |\epsilon| = |\epsilon| j . \)
(7.2.7)

PROOF. Clearly \( (P_1)_* \) commutes with \( \epsilon^2 \) and hence with its positive square root \( |\epsilon| \). Now \( \epsilon \) may be written

\( \epsilon = |\epsilon|(\chi_{1+} - \chi_{1-}) , \)

where \( \chi_{1\pm} \) denotes multiplication by the characteristic function of \( I_{1\pm} \). Since

\( (P_1)_* \circ \chi_{1\pm} = \chi_{1\mp} \circ (P_1)_* , \)

and pointwise complex conjugation commutes with \( \epsilon \), this proves (7.2.6). Equation (7.2.5) follows from Proposition 7.2.2iii.). \( \square \)

7.3. **One-particle KMS structures**

Define the real linear map \( \tilde{K}_\beta : \hat{d}(S^1) \rightarrow \hat{d}(S^1) , \beta > 0 \), by

\( \tilde{K}_\beta (\psi, \pi) = ((1 + \rho_\beta) + \rho_\beta \frac{1}{j}) \hat{K}_\infty (\psi, \pi) \)

with

\( \rho_\beta = \frac{e^{-\beta |\epsilon|}}{1 + e^{-\beta |\epsilon|}} \) and \( 1 + \rho_\beta = \frac{1}{1 + e^{-\beta |\epsilon|}} . \)
The domain of $\rho_\beta$ and $(1 + \rho_\beta)$ contains $\mathcal{D}(\|e\|^{1/2})$, as can be seen from the elementary bound [117] §A2

$$0 < \frac{e^{-\lambda}}{1 + e^{-\lambda}}, \quad \frac{1}{1 + e^{-\lambda}} \leq \max(1, \lambda^{1/2}), \quad \lambda \in \mathbb{R}^+.$$ 

Note that $\tilde{K}_\beta$ is not the Araki-Woods map $K_{\text{aw}}$ discussed in [A.0.5] as $K_{\text{aw}}$ would map $\tilde{t}(I_+)$ to $\tilde{\delta}(I_+) + \delta(I_+)$. 

**Proposition 7.3.1.** The quadruple $(\tilde{K}_\beta, \tilde{d}(S^1), e^{i\pi \epsilon}, j)$ is a double $\beta r$-KMS one-particle structure for the classical double dynamical system

$$\left( \tilde{t}(S^1), \tilde{\sigma}, \tilde{u}(\Lambda_1(t)), \tilde{u}(P_1 T) \right)$$

in the sense of Kay, see [A.0.7].

**Proof.** Let $\tilde{\phi}_i = (\tilde{e}_i, \tilde{\pi}_i) \in \tilde{d}(S^1)$, $i = 1, 2$ and denote the scalar product in $\tilde{d}(S^1)$ just by $\langle . , . \rangle$. Then

$$\mathcal{I}(\tilde{K}_\beta \tilde{\phi}_1, \tilde{K}_\beta \tilde{\phi}_2) = \mathcal{I}(\tilde{K}_\beta \tilde{\phi}_1, \tilde{K}_\beta \tilde{\phi}_2) + \langle j \tilde{K}_\beta \tilde{\phi}_1, \rho_\beta j \tilde{K}_\beta \tilde{\phi}_2 \rangle$$

$$= \mathcal{I}(\tilde{K}_\beta \tilde{\phi}_1, \tilde{K}_\beta \tilde{\phi}_2) + \mathcal{I}(\tilde{K}_\beta \tilde{\phi}_1, \tilde{K}_\beta \tilde{\phi}_2) - \mathcal{I}(\tilde{K}_\beta \tilde{\phi}_1, \tilde{K}_\beta \tilde{\phi}_2)$$

$$= \mathcal{I}(\tilde{K}_\beta \tilde{\phi}_1, \tilde{K}_\beta \tilde{\phi}_2) = \frac{1}{2} \tilde{\sigma}(\tilde{\phi}_1, \tilde{\phi}_2). \quad (7.3.1)$$

Now verify the properties listed in Definition [A.0.7]

i. ) $\tilde{d}(S^1)$ is a complex Hilbert space;

ii. ) the map $\tilde{K}_\beta : \tilde{d}(S^1) \to \tilde{d}(S^1)$ is real linear and symplectic, as can be seen from Eq. (7.3.1). Moreover, 

$$\tilde{K}_\beta \tilde{t}(I_+) + i \tilde{K}_\beta \tilde{t}(I_+)$$

$$= ((1 + \rho_\beta)^{1/2} + \rho_\beta^{1/2}) \tilde{R}(I_+) + i ((1 + \rho_\beta)^{1/2} + \rho_\beta^{1/2}) \tilde{R}(I_+)$$

$$= (1 + \rho_\beta)^{1/2} \tilde{R}(I_+) + i \tilde{R}(I_+)$$

$$+ \rho_\beta^{1/2} (\tilde{R} \circ \tilde{u}(P_1 T) \tilde{t}(I_+) + i \tilde{R} \circ \tilde{u}(P_1 T) \tilde{t}(I_+))$$

$$= (1 + \rho_\beta)^{1/2} \tilde{R}(I_+) + i \tilde{R}(I_+) + \rho_\beta^{1/2} (\tilde{R}_\beta \tilde{t}(I_-) + i \tilde{R}_\beta \tilde{t}(I_-)) \quad .$$

It follows from Proposition [7.2.2] (i.) and (ii.) that this set is dense in $\tilde{d}(S^1)$. We also used [7.2.6] and the fact that $(1 + \rho_\beta)$ and $\rho_\beta$ are strictly positive, and therefore invertible on $\tilde{R} \subset \mathcal{D}_C (S^1 \setminus (\pi, \pi/2 , \pi/2))$.

iii. ) $t \mapsto e^{it\epsilon}$ is a strongly continuous group of unitaries, and [7.2.7] implies

$$e^{it\epsilon} \circ \tilde{K}_\beta = e^{it\epsilon} \circ ((1 + \rho_\beta)^{1/2} + \rho_\beta^{1/2}) \circ \tilde{R}$$

$$= ((1 + \rho_\beta)^{1/2} + \rho_\beta^{1/2}) \circ e^{it\epsilon} \circ \tilde{R} \circ \tilde{u}(\Lambda_1(t)) \quad . \quad (7.3.2)$$
Let \( \hat{\Phi} = (\hat{\Phi}, \pi) \), where \( \hat{\Phi} \) and \( \pi \) have compact supports in the open half-circle \( I_+ \). Then

\[
\varepsilon \hat{K}_\infty \hat{\Phi} = |\varepsilon| \hat{K}_\infty \hat{\Phi} \quad \text{and} \quad \varepsilon j \hat{K}_\infty \hat{\Phi} = -|\varepsilon| j \hat{K}_\infty \hat{\Phi}.
\]

This implies

\[
e^{-\beta \varepsilon/2} \hat{K}_\beta \hat{\Phi} = e^{-\beta \varepsilon/2} \left( (1 + \rho_\beta)^{\frac{1}{2}} + \rho_\beta^{\frac{1}{2}} j \right) \hat{K}_\infty \hat{\Phi}
= e^{-\beta |\varepsilon|/2} (1 + \rho_\beta)^{\frac{1}{2}} \hat{K}_\infty \hat{\Phi} + e^{\beta |\varepsilon|/2} \rho_\beta^{\frac{1}{2}} j \hat{K}_\infty \hat{\Phi}
= \left( \frac{e^{-\beta |\varepsilon|}}{1-e^{-|\beta| \varepsilon}} \right)^{\frac{1}{2}} \hat{K}_\infty \hat{\Phi} + \left( \frac{1}{1-e^{-|\beta| \varepsilon}} \right)^{\frac{1}{2}} j \hat{K}_\infty \hat{\Phi}
= \rho_\beta^{\frac{1}{2}} \hat{K}_\infty \hat{\Phi} + (1 + \rho_\beta)^{\frac{1}{2}} j \hat{K}_\infty \hat{\Phi}.
\]

Thus (by linearity)

\[
(7.3.3) \quad \hat{K}_\beta \hat{t}(I_+) + i \hat{K}_\beta \hat{t}(I_+) \subset \mathcal{D}(e^{-\beta \varepsilon/2});
\]

Moreover, according to Lemma 5.4.4, zero is not an eigenvalue of the generator \( \varepsilon \).

iv. \( j \) is a conjugation, and

\[
j \circ \hat{K}_\beta = \hat{K}_\beta \circ \hat{u}(P_1 T)
\]

by Lemma 7.2.3 and the fact that \( j \) commutes with \( \rho_\beta \). The KMS condition holds: we have already seen that

\[
e^{-\beta \varepsilon/2} \hat{K}_\beta \hat{\Phi} = \rho_\beta^{\frac{1}{2}} \hat{K}_\infty \hat{\Phi} + (1 + \rho_\beta)^{\frac{1}{2}} j \hat{K}_\infty \hat{\Phi} = j \hat{K}_\beta \hat{\Phi}.
\]

\[\square\]

**Lemma 7.3.2.** Let \( h \in \mathcal{D}_c \left( S^1 \setminus \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right) \). Then

\[
(7.3.4) \quad \|((1 + \rho_\beta)^{\frac{1}{2}} + \rho_\beta^{\frac{1}{2}} (P_1)_+) h\|_{L^2(S^1)}^2 = \left\langle \left| \cos \theta \right| h, \frac{1}{2|\varepsilon|} \left( \coth \frac{\beta \varepsilon}{2} + \frac{\sinh \frac{\beta \varepsilon}{2}}{\sinh 2|\varepsilon|} \right) |\cos \theta| h \right\rangle_{L^2(S^1, r d\Phi)}.
\]

**Proof.** Write \( h = h_+ + h_- \), where the support of \( h_\pm \) is contained in \( I_\pm \), respectively. Note that \((P_1)_+ \) commutes with \( |\varepsilon| \) and with the multiplication operator \( |\cos \theta| \). It follows that

\[
\|((1 + \rho_\beta)^{\frac{1}{2}} + \rho_\beta^{\frac{1}{2}} (P_1)_+) h_\pm\|_{L^2(S^1)}^2 = \left\langle \left| \cos \theta \right| h_\pm, (1 + 2 \rho_\beta) |\cos \theta| h_\pm \right\rangle_{L^2(S^1, r d\Phi)}
= \left( \cos \theta h_\pm, \frac{\coth \beta |\varepsilon|}{2|\varepsilon|} \cos \theta h_\pm \right)_{L^2(S^1, r d\Phi)}.
\]

For the mixed terms, we find

\[
\left\langle \left( (1 + \rho_\beta)^{\frac{1}{2}} + \rho_\beta^{\frac{1}{2}} (P_1)_+ \right) h_+, \left( (1 + \rho_\beta)^{\frac{1}{2}} + \rho_\beta^{\frac{1}{2}} (P_1)_+ \right) h_- \right\rangle_{L^2(S^1)}
= \left( \cos \theta h_+, \frac{e^{-\frac{1}{2} |\varepsilon|}}{e^{\frac{1}{2} |\varepsilon|} (1 - e^{-|\beta| \varepsilon})} \cos \theta h_- \right)_{L^2(S^1, r d\Phi)}
= \left( \cos \theta h_+, \frac{1}{2|\varepsilon| \sinh \frac{\beta |\varepsilon|}{2}} \cos \theta h_- \right)_{L^2(S^1, r d\Phi)}.
\]

(7.3.5)

We have used the identities \( 1 + 2 \rho_\beta = \coth \frac{\beta |\varepsilon|}{2} \) and

\[
2(\rho_\beta (1 + \rho_\beta))^{\frac{1}{2}} = (\sinh \frac{\beta |\varepsilon|}{2})^{-1}.
\]
The term with \( h_+ \) and \( h_- \) interchanged yields a similar expression. Putting together the four terms, and noting that \( \epsilon \) leaves the subspaces \( L^2(1_\pm, \frac{\rd u}{\cos \varphi}) \) invariant, completes the proof.

### 7.4. The canonical one-particle structure

It was recognised by Borchers and Buchholz \[23\] that the proper, orthochronous Lorentz group \( \text{SO}_0(1, 2) \) group can be unitarily implemented iff \( \beta \) is equal to the Hawking\[1\] temperature \( 2\pi T \) \[97, 173, 174\]. In fact, we will now show that if \( \beta = 2\pi T \), then the unitary map

\[
u: \hat{\delta}(S^1) \to \hat{h}(S^1)
\]

\[
h \mapsto \frac{1}{\sqrt{\pi}} |\cos \varphi|^{-1} \left( \rho_{2\pi}(P_1)_* - (1 + \rho_{2\pi}) \right) h,
\]

allows us to implement the rotations \( R_0(\alpha), \alpha \in [0, 2\pi) \), in the double \((2\pi T)\)-KMS one-particle structure introduced in Proposition \[7.3.1\].

**Proposition 7.4.1.** The operator \( \nu \) is unitary, i.e.,

\[
\|\nu h\|_{\hat{h}(S^1)} = \|h\|_{\hat{\delta}(S^1)}.
\]

Its inverse \( \nu^{-1}: \hat{h}(S^1) \to \hat{\delta}(S^1) \) is given by

\[
(7.4.1) \quad \nu^{-1} = -\sqrt{\pi} (1 + \rho_{2\pi}) \frac{1}{\cos \varphi} + \rho_{2\pi}(P_1)_* |\cos \varphi|.
\]

**Proof.** Let \( h \in \hat{h}(S^1) \). Using again that \( (P_1)_* \) commutes with \( |\epsilon| \) and with the multiplication operator \( |\cos \varphi| \), we find

\[
\|\nu^{-1} h\|^2_{\hat{\delta}(S^1)} = \left\langle (1 + \rho_{2\pi}) \frac{1}{\cos \varphi} + \rho_{2\pi}(P_1)_* \right| \cos \varphi \|h\|^2_{\hat{\delta}(S^1)}
\]

\[= \frac{1}{\pi} \left\langle \cos \varphi |h,(1 + 2\rho_{2\pi})|\cos \varphi \|h\rangle_{\hat{\delta}(S^1)} + 2r \left\langle |\cos \varphi \|h,(\rho_{2\pi}(1 + \rho_{2\pi}) \frac{1}{\cos \varphi} + \rho_{2\pi}(P_1)_* |\cos \varphi \|h\rangle_{\hat{\delta}(S^1)}
\]

\[= \frac{1}{\pi} \left\langle h, \frac{1}{\sin \varphi} \left( \coth \pi |\epsilon| + \frac{(\rd u)}{\sin \pi |\epsilon|} \right) |\cos \varphi \|h\rangle_{L^2(S^1, \rd\varphi)}
\]

\[= \|h\|^2_{\hat{h}(S^1)}.
\]

The last equality follows from Proposition \[6.2.2\]. We have again used the identities

\[1 + 2\rho_{2\pi} = \coth \pi |\epsilon| \]

and

\[2(\rho_{2\pi}(1 + \rho_{2\pi}) \frac{1}{\cos \varphi}) = (\sinh \pi |\epsilon|)^{-1}.
\]

**Proposition 7.4.2.** Consider the map

\[
\hat{K}: \hat{\delta}(S^1) \to \hat{h}(S^1)
\]

\[
(\hat{\delta}, \pi) \mapsto \frac{1}{\sqrt{\pi}} (-\pi + i \omega r \hat{\delta})
\]

It follows that the quadruple

\[
(\hat{K}, \hat{h}(S^1), e^{it \omega \cos \varphi}, C(P_1)_*)
\]

\[\text{In the present context, the temperature } T = 2\pi T \text{ was first derived by Figari, Høegh-Krohn and Nappi}[37]. \text{ The article by Hawking was submitted soon afterwards.}\]
forms a double $2\pi\tau$-KMS one-particle structure for the classical double dynamical system \((\hat{\mathcal{K}}(\mathbb{S}^1 \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}), \hat{\sigma}, \hat{\Lambda}(\frac{1}{2}))\) in the sense of \(\text{[A.0.7]}\) unitarily equivalent to \((\hat{\mathcal{K}}_{2\pi\tau}, \hat{\delta}(\mathbb{S}^1), e^{it\omega \cos \phi})\), in agreement with Theorem \(\text{[A.0.8]}\).

**Proof.** We first show that \(\hat{\mathcal{K}} = u \circ \hat{\mathcal{K}}_{2\pi}\). Using \(j = C(\mathcal{P}_1)\), where \((\text{Ch})(\psi) \doteq \hat{h}(\psi)\), one gets

\[
\hat{u} \circ \hat{\mathcal{K}}_{2\pi}(\psi, \pi) = \hat{u} \circ \left((1 + \rho_{2\pi})^\frac{1}{\pi} + \rho_{2\pi}^\frac{1}{\pi} j\right) \hat{\mathcal{K}}_\infty(\psi, \pi)
\]

\[
= -\frac{1}{\sqrt{\pi}} |\cos \psi|^{-1}\left((1 + \rho_{2\pi})^\frac{1}{\pi} - \rho_{2\pi}^\frac{1}{\pi} (\mathcal{P}_1)^*\right)\left((1 + \rho_{2\pi})^\frac{1}{\pi} + \rho_{2\pi}^\frac{1}{\pi} j\right) \hat{\mathcal{K}}_\infty(\psi, \pi)
\]

\[
= -\frac{1}{\sqrt{\pi}} |\cos \psi|^{-1}\left(1 + (\rho_{2\pi} - (\rho_{2\pi}(1 + \rho_{2\pi}))^\frac{1}{\pi})(\mathcal{P}_1)^*(1 - C)\right) \hat{\mathcal{K}}_\infty(\psi, \pi).
\]

Taking \(1 + 2\rho_{2\pi} = \coth \pi|\epsilon|\) and \(2(\rho_{2\pi}(1 + \rho_{2\pi}))^\frac{1}{\pi} = (\sinh \pi|\epsilon|)^{-1}\) into account, we find

\[
u \circ \hat{\mathcal{K}}_{2\pi}(\psi, \pi) = \begin{cases} 
-\frac{1}{\sqrt{\pi}} |\cos \psi|^{-1} \hat{\mathcal{K}}_\infty(\psi, \pi) & \text{if } \hat{\mathcal{K}}_\infty(\psi, \pi) \in \hat{\delta}(\mathbb{S}^1, \mathbb{R}), \\
-\sqrt{\pi} \omega \epsilon^{-1} \hat{\mathcal{K}}_\infty(\psi, \pi) & \text{if } \hat{\mathcal{K}}_\infty(\psi, \pi) \in \hat{\delta}(\mathbb{S}^1, \mathbb{R}).
\end{cases}
\]

In the last equation we have used \(\text{[6.2.9]}\) and \((\mathcal{P}_1)^* = -\epsilon(\mathcal{P}_1)\). By \(\hat{\delta}(\mathbb{S}^1, \mathbb{R})\) we have denoted the real subspace of real valued functions in \(\hat{\delta}(\mathbb{S}^1)\). Use \(\hat{\mathcal{K}}_\infty(\psi, \pi) = \cos \psi - i\epsilon \psi\) to prove that

\[
(7.4.3) \quad \hat{\mathcal{K}} = u \circ \hat{\mathcal{K}}_{2\pi}.
\]

It remains to show that the unitary map \(u\) satisfies

\[
u \circ \epsilon \circ \omega^{-1} = \omega \cos \psi \quad \text{and} \quad u \circ j \circ \omega^{-1} = C(\mathcal{P}_1)^* \quad \text{on } \hat{h}(\mathbb{S}^1).
\]

Using again \((\mathcal{P}_1)^* = -\epsilon(\mathcal{P}_1)\), we can verify the first of these two identities:

\[
u \circ \epsilon \circ \omega^{-1} = |\cos \psi|^{-1}\left((1 + \rho_{2\pi})^\frac{1}{\pi} - \rho_{2\pi}^\frac{1}{\pi} (\mathcal{P}_1)^*\right)\left((1 + \rho_{2\pi})^\frac{1}{\pi} + \rho_{2\pi}^\frac{1}{\pi} j\right) |\cos \psi|
\]

\[
= |\cos \psi|^{-1}\epsilon\left((1 + 2\rho_{2\pi}) + 2\rho_{2\pi}^\frac{1}{\pi}(1 + \rho_{2\pi})^\frac{1}{\pi}(\mathcal{P}_1)^*\right) |\cos \psi|
\]

\[
= |\cos \psi|^{-1}\epsilon\left(\coth \pi|\epsilon| + \frac{(\mathcal{P}_1)^*}{\sinh \pi|\epsilon|}\right)^{-1} |\cos \psi|
\]

\[
= \omega \epsilon \cos \psi.
\]

In the second but last equality we have used the identity \(\text{[6.2.9]}\).

The second identity follows from the fact that \(j\) commutes with the multiplication operator \(|\cos \psi|\):

\[
u \circ j \circ \omega^{-1} = C(\mathcal{P}_1)^* \quad \text{on } \hat{h}(\mathbb{S}^1).
\]

We have thus established unitarily equivalence of the two double \(2\pi\tau\)-KMS one-particle structure under consideration, in agreement with Theorem \(\text{[A.0.8]}\).

It is now straightforward to verify that \((\hat{\mathcal{K}}, \hat{h}(\mathbb{S}^1), e^{it\omega \cos \phi}, C(\mathcal{P}_1)^*)\) forms a double \(2\pi\tau\)-KMS one-particle structure for the classical double dynamical system
\( \hat{\mathbf{k}}(S^1 \setminus \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)), \hat{\sigma}, \hat{u}(\Lambda_1(\frac{\pi}{2})), \hat{u}(\Lambda_1(\frac{\pi}{2})) \) in the sense of [A.0.7]

\[
\hat{K} \circ \hat{u}(\Lambda_1(t)) = u \circ \hat{K}_{2\pi} \circ \hat{u}(\Lambda_1(t))
= u \circ e^{it\epsilon} \hat{K}_{2\pi}
= e^{it \omega r \cos \psi} \circ u \circ \hat{K}_{2\pi}
\]
\[= \hat{u}(\Lambda_1(t)) \circ \hat{K} , \quad (7.4.4)\]

see Eq. (7.3.2); and
\[
\hat{K} \circ \hat{u}(P_1T) = u \circ \hat{K}_{2\pi r} \circ \hat{u}(P_1T) = u \circ j \circ \hat{K}_{2\pi r} = C(P_1) \circ \hat{K} .
\]

This also shows that \( \hat{u}(P_1T) = C(P_1) \) is anti-unitary. \( \square \)

**THEOREM 7.4.3.** The rotations
\[
(\hat{u}(R_0(\alpha))h)(\psi) = h(\psi - \alpha) , \quad \alpha \in [0, 2\pi) , \quad h \in \hat{h}(S^1) ,
\]
and the boosts
\[
\hat{u}(\Lambda_1(t)) = e^{it \omega r \cos \psi} , \quad t \in \mathbb{R} ,
\]
generate a representation of \( SO_0(1, 2) \) on \( \hat{h}(S^1) \).

**PROOF.** The generator of the rotations \( K_0 = -i\partial_\psi \) has purely discrete spectrum. Its eigenfunctions are
\[
e_k = \frac{e^{ik\psi}}{\sqrt{2\pi}} , \quad k \in \mathbb{Z} .
\]
The generators of the boosts,
\[
L_1 = \omega r \cos \psi , \quad L_2 = \omega r \sin \psi ,
\]
satisfy the commutator relations
\[
[K_0 , L_1] = iL_2 , \quad [L_2 , K_0] = iL_1 .
\]
The latter follow from
\[
[-i\partial_\psi , \omega r \cos \psi] = i\omega r \sin \psi , \quad [\omega r \sin \psi , -i\partial_\psi] = i\omega r \cos \psi .
\]
To verify the commutation relation \( [L_1 , L_2] = -iK_0 \), we consider the ladder operators
\[
L_\pm = L_1 \pm iL_2 = \omega r e^{\pm i\psi} .
\]
We will show that
\[
\langle e_{k'} , [L_+, L_-] e_k \rangle_{\hat{h}(S^1)} = -2\langle e_{k'} , K_0 e_k \rangle_{\hat{h}(S^1)} .
\]
The latter is equivalent to
\[
(7.4.6) \quad \tilde{\omega}(k)(\tilde{\omega}(k-1) - \tilde{\omega}(k+1)) = -\frac{2k}{r^2} , \quad \forall k \in \mathbb{Z} .
\]
In order to verify this identity, let us first consider only the \( \Gamma \)-factors occurring in (6.2.2). Define, for \( k \in \mathbb{Z} ,
\[
w(k) \doteq \frac{\Gamma\left(\frac{k+s^+}{2}\right) \Gamma\left(\frac{k-s^+ +1}{2}\right) \Gamma\left(\frac{k-s^-+1}{2}\right) \Gamma\left(\frac{k+s^-}{2}\right)}{\Gamma\left(\frac{k-s^-+1}{2}\right) \Gamma\left(\frac{k+s^- +1}{2}\right)} .
\]
Hence, we get the two following useful relations:

\[
\frac{w(k+1)}{w(k)} = \frac{k-s^+}{k+s^+}.
\]

One has

\[
\frac{w(k+1)}{w(k)} = \frac{\Gamma \left( \frac{k+s^+ + 1}{2} \right) \Gamma \left( \frac{k-s^+ + 1}{2} \right)}{\Gamma \left( \frac{k-1}{2} \right) \Gamma \left( \frac{k+s^+ + 1}{2} \right)} = \frac{k-s^+}{k+s^+}.
\]

as one easily verifies. Hence,

\[
(7.4.7) \quad w(k)w(k+1) = \frac{k-s^+}{k+s^+}.
\]

Since \( \tilde{\omega}(k) = \frac{(k+s^+)}{r}w(k) \), we have

\[
\tilde{\omega}(k)\tilde{\omega}(k+1) = r^{-2}(k+s^+)(k+s^+ + 1) \frac{k-s^+}{k+s^+} = r^{-2}(k-s^+)(k+s^+ + 1).
\]

Hence, we get the two following useful relations:

\[
(7.4.8) \quad \tilde{\omega}(k)\tilde{\omega}(k+1) = r^{-2}(k-s^+)(k+s^+ + 1) = r^{-2}k(k+1) + \mu^2
\]

\[
(7.4.9) \quad \tilde{\omega}(k)\tilde{\omega}(k-1) = r^{-2}(k+s^+)(k-s^+ - 1) = r^{-2}k(k-1) + \mu^2.
\]

The last one is obtained from the previous one by taking \( k \to k - 1 \). We note that

\[
(7.4.10) \quad \frac{1}{2}(\tilde{\omega}(k)\tilde{\omega}(k+1) + \tilde{\omega}(k)\tilde{\omega}(k-1)) = \frac{k^2}{r^2} + \mu^2,
\]

which allows us to establish the usual dispersion relation in the limit \( r \to \infty \).

Relation (7.4.6) can now be verified using (7.4.8–7.4.9):

\[
\tilde{\omega}(k)\tilde{\omega}(k-1) - \tilde{\omega}(k)\tilde{\omega}(k+1) = r^{-2}(k+s^+)(k-s^+ - 1)
\]

\[
- r^{-2}(k-s^+)(k+s^+ + 1)
\]

\[
= \frac{2k}{r^2},
\]

as desired. We conclude that

\[
[L_+, L_-] = -2K_0, \quad [t_0, L_+] = \pm K_0,
\]

in agreement with \([L_+, L_-] = -2K_0\)

\[\square\]

The operator

\[
\omega r \cos \varphi = (\omega r \cos \varphi)_{|1_+} + (\omega r \cos \varphi)_{|1_-}
\]

is the sum of a positive operator \((\omega r \cos \varphi)_{|1_+}\) acting on \(\hat{h}(1_+)\), and a negative operator \((\omega r \cos \varphi)_{|1_-}\) acting on \(\hat{h}(1_-)\). Both operators have absolutely continuous spectrum. Similar results hold for \(I_{\alpha}, \alpha \in [0,2\pi]\).

**Theorem 7.4.4.** The triple \((\hat{K}, \hat{h}(S^1), \tilde{\omega})\) is a one-particle de Sitter structure for the canonical classical dynamical system \((\hat{\mathcal{E}}(S^1), \varrho, \tilde{\omega})\) introduced in Proposition 5.5.3. In other words,

i.) \(\hat{K}\) defines a symplectic map from \((\hat{\mathcal{E}}(S^1), \varrho)\) to \((\hat{h}(S^1), \mathcal{J}(\cdot, \cdot))_{\hat{h}(S^1)}\) and \(\hat{K}\hat{\mathcal{E}}(S^1)\) is dense in \(\hat{h}(S^1)\);
ii.) there exists a unique (anti-) unitary representation of $O(1, 2)$ satisfying

\[(7.4.11) \quad \hat{u}(\Lambda) \circ \hat{K} = \hat{K} \circ \hat{u}(\Lambda).\]

Moreover, $\hat{u}(R_0(\alpha)) = R_0(\alpha)$ for $\alpha \in [0, 2\pi]$;

iii.) for any half-circle $I_\alpha$, the pre-Bisognano-Wichmann property \[119\] p. 75 holds:

\[(7.4.12) \quad \hat{K} \hat{f}(I_\alpha) \subset D(\hat{u}(\Lambda W(\alpha)(i\pi))),\]

and

\[(7.4.13) \quad \hat{u}(\Lambda W(\alpha)(i\pi))h = \hat{u}(\Theta W(\alpha))h, \quad h \in \hat{K} \hat{f}(I_\alpha).\]

PROOF.

i.) Clearly, $C^\infty(S^1) + i\omega C^\infty(S^1)$ is dense in $\hat{h}(S^1)$. To verify that $\hat{K}$ is a symplectic map, compute

\[
2\mathcal{J}(\hat{K}(\varphi_1, \pi_1), \hat{K}(\varphi_2, \pi_2)|_{\hat{h}(S^1)}) = 2\mathcal{J}(-\pi_1 + i\omega \varphi_1, -\pi_2 + i\omega \varphi_2)|_{\hat{h}(S^1)}
= (\varphi_1, \pi_2)_{L^2(S^1, \text{rd}\vartheta)} - (\varphi_2, \pi_1)_{L^2(S^1, \text{rd}\vartheta)}
= \hat{\sigma}((\varphi_1, \pi_1), (\varphi_2, \pi_2)).
\]

ii.) For $\Lambda = R_0$ a rotation, we have

\[
(\hat{u}(R_0) \circ \hat{K})(\hat{\varphi}, \hat{\pi}) = (R_0)^* (\varphi + i\omega \varphi) 
= -(R_0)^* \varphi + i\omega (R_0)^* \varphi
= \hat{K}((R_0)^* \varphi, (R_0)^* \pi) = (\hat{K} \circ \hat{u}(R_0))(\hat{\varphi}, \hat{\pi}),
\]

since $\omega$ commutes with the pullback $(R_0)^*$ of a rotation. For the boosts, see (7.4.4); and for the reflections, see (7.4.5).

iii.) For $(\hat{\varphi}, \hat{\pi}) \in \hat{f}(I_\alpha)$,

\[
\hat{u}(\Lambda W(i\pi)) \hat{K}(\hat{\varphi}, \hat{\pi}) = \hat{K} \circ \hat{u}(\Lambda W(i\pi))(\hat{\varphi}, \hat{\pi})
= \hat{K} \circ \hat{u}(\Theta W)(\hat{\varphi}, \hat{\pi})
= \hat{u}(\Theta W) \hat{K}(\hat{\varphi}, \hat{\pi}),
\]

which demonstrates both (7.4.12) and (7.4.13). The first equality follows from combining (7.3.3) and (7.4.3). \[\square\]
Proposition 7.4.5. There exists a unitary map $\mathcal{U}$ from $\widehat{\mathfrak{h}}(S^1)$ to $\mathfrak{h}(dS)$, which intertwines the representations $\hat{u}(\Lambda)$ and $u(\Lambda)$, $\Lambda \in \mathcal{O}(1,2)$, and the one-particle structures. In other words, the following diagram commutes:

Moreover, the restricted map

$\mathcal{U}$: $\widehat{\mathfrak{h}}(I) \mapsto \mathfrak{h}(\mathcal{O}_I)$, $I \subset S^1$,

is unitary too.

Proof. The existence of $\mathcal{U}$ follows from the uniqueness of the de Sitter one-particle structure. The latter is a direct consequence of the uniqueness of the $(2\pi r)$-KMS structure for the double wedge, see Appendix A.0.8. The local part, Eq. (7.4.14), follows from Lemma 6.3.3: for $f \in D(S^1)$ and $f(x) \equiv (\delta \otimes h)(x) = \delta(x_0) h(\psi)$,

$$g(x) \equiv (\delta' \otimes h)(x) = \left( \frac{\partial}{\partial x_0} \delta \right)(x_0) h(\psi),$$

with $x \equiv x(t,\psi)$ the coordinates introduced in (6.3.1), the Cauchy data for the corresponding solutions $f, g$ of the Klein–Gordon equation are:

$$f|_{S^1}, (n f)|_{S^1} = (0, -\hat{h}) \equiv (\hat{g}, \pi),$$

$$g|_{S^1}, (n g)|_{S^1} = (\hat{h}, 0) \equiv (\hat{g}, \pi).$$

Together with $\hat{K} (\hat{g}, \pi) = -\pi + i \omega r \hat{g}$ this gives

$$\hat{K} (f|_{S^1}, (n f)|_{S^1}) = \hat{h},$$

$$\hat{K} (g|_{S^1}, (n g)|_{S^1}) = i \omega r h,$$

both elements of $\mathfrak{h}(S^1)$. Finally, the unitary map $\mathcal{U}$: $\widehat{\mathfrak{h}}(S^1) \to \mathfrak{h}(dS)$ is the linear extension of the map

$$h_1 + i \omega r h_2 \mapsto [\delta \otimes h_1] + [\delta' \otimes h_2].$$

The latter shows that $\mathcal{U}$: $\widehat{\mathfrak{h}}(I) \to \mathfrak{h}(\mathcal{O}_I)$, with $\mathcal{O}_I = I''$ the causal completion of $I \subset S^1$. 

\[\square\]
COROLLARY 7.4.6. Let $I$ be an open subset in $S^1$. The unitary group $t \mapsto e^{it\omega r \cos \psi}$ maps $\hat{h}(I)$ to
\[
\hat{h}\left(\big(\Gamma^+(\Lambda_1(t)I) \cup \Gamma^-(\Lambda_1(t)I)\big) \cap S^1\right).
\]
In particular, the unitary group $t \mapsto e^{it\omega r \cos \psi} |_{I}$ leaves $\hat{h}(I_{\pm})$ invariant.

PROOF. This is a direct consequence of the fact that $e^{it\omega r \cos \psi}$ implements the Lorentz boost $\Lambda_1(t)$. The latter act geometrically on $\hat{h}(\partial V^+)$, i.e., a distribution supported at $I \subset S^1 \subset dS$ is mapped to a distribution supported at $\Lambda_1(t)I \subset dS$. This result extends by continuity to $\hat{h}(I)$. \hfill \Box

COROLLARY 7.4.7. The unitary representation $\hat{u}(\Lambda)$, $\Lambda \in SO_0(1,2)$, defined by Eq. (7.4.11), is irreducible.

PROOF. By construction, $\hat{u}$ is unitarily equivalent to the unitary irreducible representation $\tilde{u}$ on $\tilde{h}(\partial V^+)$. In fact, the Casimir operator takes the form
\[
C^2 = -k_0^2 + \frac{1}{2}(L_-L_- + L_+L_+).
\]
Its off-diagonal matrix elements vanish and the diagonal matrix elements equal $\zeta^2$:
\[
\frac{\langle e_k, C^2 e_k \rangle_{\hat{h}(S^1)}}{\|e_k\|_{\hat{h}(S^1)}} = -k^2 + \frac{r^2}{2}(\tilde{\omega}(k)\tilde{\omega}(k-1) + \tilde{\omega}(k)\tilde{\omega}(k+1))
\]
\[
= -k^2 + \frac{1}{2}((k+s)(k-s-1) + (k-s)(k+s+1))
\]
\[
= -s(s+1) = \frac{1}{4} + \nu^2 = \zeta^2,
\]
as expected. In the second but last equality we have used (7.4.8–7.4.9). \hfill \Box
CHAPTER 8

Second Quantization

Let \((\mathfrak{t}, \sigma)\) be a symplectic space. The unique \(C^*\)-algebra \(\mathcal{W}(\mathfrak{t}, \sigma)\) generated by nonzero elements \(W(f), f \in \mathfrak{t}\), satisfying
\[
W(f_1)W(f_2) = e^{-i\sigma(f_1, f_2)/2}W(f_1 + f_2),
\]
(8.0.1)
\[W^*(f) = W(-f) , \quad W(0) = 1 ,\]
is called the Weyl algebra associated to \((\mathfrak{t}, \sigma)\); see, e.g., [24]. In case \(\mathfrak{t}\) is a Hilbert space, we suppress the dependence on the symplectic form given by twice the imaginary part of the scalar product.

Definition 8.0.8. Set \(\mathcal{W}(X) \equiv \mathcal{W}(\mathfrak{t}(X), \sigma), \quad X = \emptyset, W, dS\).

Let \(\Lambda \mapsto u(\Lambda)\) be the representation of \(SO_0(1, 2)\) on \(\mathfrak{t}(dS)\); see Proposition 5.3.7. Define a group of automorphisms \(\alpha^\circ: \Lambda \mapsto \alpha^\circ_\Lambda\) acting on \(\mathcal{W}(dS)\) by
\[\alpha^\circ_\Lambda(W([f])) = W(u(\Lambda)[f]), \quad [f] \in \mathfrak{t}(dS) .\]
The pair \((\mathcal{W}(dS), \alpha^\circ)\) is called the covariant quantum dynamical system associated to the Klein–Gordon equation on the de Sitter space.

There is no global time evolution (in terms of a 1-parameter group of automorphisms) on \(\mathcal{W}(dS)\). In fact, there is not even a globally time-like Killing vector field on the de Sitter space. Nevertheless, the automorphisms \(\alpha^\circ\) respect the local structure:
\[\alpha^\circ_\Lambda(\mathcal{W}(\emptyset)) = \mathcal{W}(\Lambda \emptyset), \quad \emptyset \subset dS .\]
The map \(\alpha^\circ: \Lambda \mapsto \alpha^\circ_\Lambda\) fails to be strongly continuous in the \(C^*\)-norm; thus strictly speaking \((\mathcal{W}(dS), \alpha^\circ)\) is not a \(C^*\)-dynamical system.

Definition 8.0.9. Set \(\widehat{\mathcal{W}}(I) \equiv \mathcal{W}(\hat{\mathfrak{t}}(I), \hat{\sigma})\), \(I \subseteq S^1\).

Let \(\Lambda \mapsto \hat{u}(\Lambda)\) be the representation of \(O(1, 2)\) on \(\hat{\mathfrak{t}}(S^1)\); see Proposition 5.5.3. Define a group of automorphisms \(\hat{\alpha}^\circ: \Lambda \mapsto \hat{\alpha}^\circ_\Lambda\) acting on \(\widehat{\mathcal{W}}(S^1)\) by
\[\hat{\alpha}^\circ_\Lambda(\hat{W}(\hat{f})) = \hat{W}(\hat{u}(\Lambda)\hat{f}), \quad \hat{f} \in \hat{\mathfrak{t}}(S^1), \quad \Lambda \in O(1, 2) .\]
The pair \((\widehat{\mathcal{W}}(S^1), \hat{\alpha}^\circ)\) is the canonical quantum dynamical system associated to the Klein–Gordon equation on the de Sitter space.

Proposition 8.0.10. Let \(I \subseteq S^1\). Then
\[\hat{\alpha}^\circ_\Lambda(\widehat{\mathcal{W}}(I)) \subset \widehat{\mathcal{W}}((\Gamma^+\Lambda I) \cup \Gamma^-\Lambda I) \cap S^1) .\]
8.1. De Sitter vacuum states

Let \( \alpha : \Lambda \mapsto \alpha_{\Lambda} \) be a representation (in terms of automorphisms) of \( \text{SO}_0(1, 2) \) on the \( \mathbb{C}^* \)-algebra \( \mathcal{M}(\mathbb{dS}) \).

**Definition 8.1.1.** A normalised positive linear functional \( \omega \) is called a *de Sitter vacuum state* for the quantum dynamical system \( (\mathcal{M}(\mathbb{dS}), \alpha) \), if

i.) \( \omega \) is invariant under the action of the proper, orthochronous Lorentz group \( \text{SO}_0(1, 2) \), i.e.,

\[
\omega = \omega \circ \alpha_{\Lambda} \quad \forall \Lambda \in \text{SO}_0(1, 2) ;
\]

ii.) \( \omega \) satisfies the geodesic KMS condition: for every wedge \( W = \Lambda W_1, \Lambda \in \text{SO}_0(1, 2) \), the restricted (partial) state \( \omega_{|\mathcal{M}(W)} \) satisfies the KMS-condition at inverse temperature \( 2\pi \tau \) with respect to the one-parameter group \( t \mapsto \Lambda_W(\frac{t}{\tau}) \),

of boosts, which leaves the wedge \( W \) invariant. In other words: for each pair \( [f],[g] \in \mathcal{F}(W) \) there exists a function \( \mathfrak{f}_{f,g}(\tau) \) holomorphic in the strip \( \mathbb{C} \ni \tau \mid 0 < \Im \tau < 2\pi \tau \)

and continuous on \( \overline{\mathbb{C}}_{2\pi \tau} \) such that

\[
\mathfrak{f}_{f,g}(t) = \omega_{|\mathcal{M}(W)}(W([f]) \alpha_{\Lambda_W(\frac{t}{\tau})}(W([g]))) \quad \forall t \in \mathbb{R} .
\]

**Remark 8.1.2.** It is sufficient to verify the geodesic KMS condition for *one* wedge, as the invariance property i.) then implies that it holds for any wedge.

The *de Sitter vacuum state* for the free field is presented next.

**Theorem 8.1.3.** The state \( \omega^0 \) on \( \mathcal{M}(\mathbb{dS}) \) given by

\[
\omega^0(W([f])) = e^{-\frac{1}{2}\|f\|_{\mathfrak{F}(\mathbb{dS})}^2}, \quad f \in \mathcal{D}_{\mathfrak{F}}(\mathbb{dS}) ,
\]

is the unique *de Sitter vacuum state* for the pair \( (\mathcal{M}(\mathbb{dS}), \alpha^0) \). Moreover, the GNS representation \( \pi^0 \) associated to the pair \( (\mathcal{M}(\mathbb{dS}), \omega^0) \) is (unitarily equivalent to) the Fock representation (see Appendix B over the one-particle space \( \mathfrak{h}(\mathbb{dS}) \), i.e.,

\[
\pi^0(W([f])) = W_{\mathfrak{F}}([f]) \quad \forall f \in \mathcal{F}(\mathbb{dS}) = \Gamma(\mathfrak{g}(\mathbb{dS})) .
\]

Note that \( \ker \mathbb{P} = \ker \mathcal{F}_{+|\mathfrak{F}}, \) thus \( \omega^0 \) is well-defined.

**Proof.** Recall the commutative diagram in Proposition 7.4.5. By construction, \( \omega^0 \) is invariant under Lorentz transformations: for \( f \in \mathcal{D}_{\mathfrak{F}}(\mathbb{dS}) \), we have

\[
\omega^0(\alpha_{\Lambda}(W(f))) = e^{-\frac{1}{2}\|\Lambda \cdot f\|_{\mathfrak{F}(\mathbb{dS})}^2} = e^{-\frac{1}{2}\|\alpha(\Lambda)(f)\|_{\mathfrak{F}(\mathbb{dS})}^2} = e^{-\frac{1}{2}\|f\|_{\mathfrak{F}(\mathbb{dS})}^2} \quad \forall \Lambda \in \text{SO}_0(1, 2) .
\]
The geodesic KMS condition follows from Proposition 7.1.1: for \( f, g \in \mathcal{D}_R(W) \) we find
\[
\omega^\circ (W([f]) \alpha_{W(t)}(W([g]))) = e^{-i \sigma([f], \alpha_{W(t)}([g]))/2} \omega(W([f] + \Lambda_{W(t)}([g]))) \\
= e^{-i \Omega([f], u(\Lambda_{W(t)}([g]))/2)} e^{-\frac{i}{2} \| [f] + u(\Lambda_{W(t)}([g])) \|_{H(dS)}} \\
= e^{-i \Omega([f], u([g]))/2} e^{-\frac{i}{2} \| [f] \|_{H(dS)}}.
\]
Moreover, Proposition 7.1.1 implies that for \( f, g \in \mathcal{D}_R(W) \)
\[
\langle u(\Lambda_{W(t)}([f])), u(\Lambda_{W(t)}([g])) \rangle_{H(dS)} = \langle u(\Omega_{W}([f])), u(\Omega_{W}([g])) \rangle_{H(dS)} \\
= \langle [g], [f] \rangle_{H(dS)}.
\]
Together these two identities imply
\[
\omega^\circ (W([f]) \alpha_{W(t)}(W([g]))) = \omega^\circ (\alpha_{W(t)}(W([g])) W([f])).
\]
Uniqueness of the geodesic KMS state follows by expressing the n-point functions in terms of two-point functions, using the KMS condition for the wedge (see Vol. II, p. 49]). Uniqueness of the two-point function in the wedge follows from Proposition 7.1.1, i.e., the fact that zero is not an eigenvalue of \( \varepsilon_{1L_+} \gt 0 \).

The GNS representation \( \mathcal{H}(dS, \pi^\circ, \Omega) \) is characterised, up to unitary equivalence, by the following two properties:

i.) \( (\Omega, \pi^\circ(W([f])) = \omega^\circ(W([f])) \) for all \( W([f]) \in \mathcal{W}(dS) \); ii.) the vector \( \Omega \in \mathcal{H}(dS) \) is cyclic for \( \pi^\circ(\mathcal{W}(dS))'' \).

A short inspection of the Fock representation (see Appendix B) verifies that the properties i.) and ii.) hold.

**Notation.** Denote the generators implementing the automorphisms corresponding to the subgroups \( t \mapsto \lambda_1(t) \), \( s \mapsto \lambda_2(s) \) and \( \alpha \mapsto R_0(\alpha) \) in the GNS representation \( \pi^\circ \) associated to a de Sitter vacuum state by \( L_1, L_2 \) and \( K_0 \).

**Definition 8.1.4.** The local von Neumann algebras for the free covariant field are defined by setting
\[
\mathcal{A}_t(\Omega) \doteq \pi^\circ(\mathcal{W}(\Omega))'' , \quad \Omega \subset dS.
\]
It follows from Theorem 8.1.3 that the algebra \( \mathcal{A}(\Omega) \) is equal to the von Neumann algebra generated by \( W_f(\Omega), f \in \mathfrak{h}(\Omega) \).

**Definition 8.1.5 (Extension to the weak closure).** The (anti-) unitary operators
\[
U(\Lambda) \doteq \Gamma(u(\Lambda)), \quad \Lambda \in O(1,2),
\]
implement the free dynamics in the Fock space \( \mathcal{H}(dS) \): for \( f \in \mathcal{D}_R(dS) \) we have
\[
\pi^\circ(\alpha^\circ_W(W([f]))) = U(\Lambda) U_f([f]) U(\Lambda)^{-1}, \quad \Lambda \in O(1,2).
\]
The right hand side extends to arbitrary elements \( h \in \mathfrak{h}(dS) \), and, in the sequel, to arbitrary elements in the weak closure \( \pi^\circ(\mathcal{W}(dS))'' \) of \( \mathcal{W}(dS) \). We denote this extension of the automorphism \( \alpha^\circ_W \) by the same letter, i.e., for \( h \in \mathfrak{h}(dS) \),
\[
\alpha^\circ_W(W_f(h)) \doteq U(\Lambda) U_f(h) U(\Lambda)^{-1}, \quad \Lambda \in O(1,2).
\]
Similarly, the GNS vector can be used to extend the free de Sitter vacuum state \( \omega^o \) to the weak closure \( \pi^o(\mathfrak{M}(dS))'' \):

\[
(8.1.2) \quad \omega^o(A) = (\Omega, A\Omega), \quad A \in \pi^o(\mathfrak{M}(dS))''.
\]

Here \( \Omega \) denotes the vacuum vector in the Fock space \( \mathcal{H}(dS) \).

**Proposition 8.1.6.** The state (8.1.2) satisfies the geodesic KMS condition with respect to the automorphisms \( \pi \).

**Proof.** The geodesic KMS property for \((\mathfrak{M}(W), \alpha_{A\omega})\) is part of Theorem 8.1.3. The fact that the KMS property extends to the weak closure is a standard result, see, e.g., [24, Corollary 5.3.4]. \(\square\)

### 8.2. The canonical free field

As C*-algebras, the Weyl algebras \(\mathfrak{W}(S^1)\) and \(\mathfrak{M}(dS)\) are isomorphic, and can be identified using the map (see Proposition 5.5.3)

\[ \hat{W}(\hat{f}) \mapsto W(f), \quad f \in \mathcal{D}_E(dS). \]

Moreover, for \( f \in \mathcal{D}_E(dS) \) we have (see Proposition 7.4.5)

\[ e^{-\frac{1}{2}\|\hat{f}\|_{\mathfrak{W}(S^1)}} = e^{-\frac{1}{2}\|f\|_{\mathfrak{M}(dS)}}. \]

Consequently, the state

\[
(8.2.1) \quad \hat{\omega}^o(\hat{W}(\hat{f})) = e^{-\frac{1}{2}\|\hat{f}\|_{\mathfrak{W}(S^1)}}, \quad f \in \mathcal{D}_E(dS),
\]

describes the same (we will clarify exactly in which sense) state as the one given in Theorem 8.1.3.

**Theorem 8.2.1.** The state (8.2.1) is the unique normalised positive linear functional on \(\mathfrak{W}(S^1)\), which satisfies the following properties:

i.) \( \hat{\omega}^o \) is invariant under the action of \(SO_0(1, 2)\), i.e.,

\[ \hat{\omega}^o = \hat{\omega}^o \circ \hat{\alpha}_\Lambda, \quad \forall \Lambda \in SO_0(1, 2); \]

ii.) \( \hat{\omega}^o \) satisfies the geodesic KMS condition: for every half-circle \( 1_\alpha \) the restricted (partial) state

\[ \hat{\omega}^o_{|\mathfrak{W}(1_\alpha)} \]

satisfies the KMS-condition at inverse temperature \( 2\pi \tau \) with respect to the one-parameter group \( t \mapsto \Lambda^{(\alpha)}(\frac{t}{\tau}) \) of boosts.

**Proof.** Property i.) follows from the definition; property ii.) will follow from Proposition 8.3.1 and the properties of the time-zero covariance. \(\square\)

It is convenient to take the weak closure in the GNS representation \((\mathfrak{W}^o, \mathfrak{H}(S^1), \hat{\Omega})\) for the pair \((\mathfrak{W}(S^1), \hat{\omega}^o)\). The latter is (unitarily equivalent to) the Fock representation (see Appendix [3]) over the one-particle space \(\hat{\mathfrak{h}}(S^1)\), i.e.,

\[ \hat{\pi}^o(\hat{W}(\hat{f})) = W_F(\hat{K}\hat{f}), \quad \mathfrak{H}(S^1) = \Gamma(\hat{\mathfrak{h}}(S^1)). \]

The GNS vacuum vector \(\hat{\Omega}\) can be used to extend \(\hat{\omega}\) to the weak closure:

\[ \hat{\omega}(W_F(h)) = (\hat{\Omega}, W_F(h)\hat{\Omega}) = e^{-\frac{1}{2}\|h\|_{\mathfrak{W}(S^1)}}, \quad h \in \hat{\mathfrak{h}}(S^1). \]
DEFINITION 8.2.2. The local von Neumann algebras for the free canonical field are defined by
\[ \mathcal{R}(I) \doteq \pi^e(\widehat{\mathcal{H}}(\mathcal{I}))'' , \quad I \subset S^1. \]
A similar argument to the one given in the proof of Theorem 8.1.1 shows that the algebra \( \mathcal{R}(I) \) is equal to the von Neumann algebra generated by \( \widehat{W}_f(h), h \in \widehat{h}(I) \).

PROPOSITION 8.2.3.

i.) The local von Neumann algebras for the canonical free field are regular from the inside and regular from the outside:
\[ \bigcap_{J \supset I} \mathcal{R}(J) = \mathcal{R}(I) = \bigvee_{\mathcal{T} \subset I} \mathcal{R}(J) ; \]
ii.) The net \( I \mapsto \mathcal{R}(I) \) of local von Neumann algebras for the canonical free field is additive:
\[ \mathcal{R}(I) = \bigvee_{J_i \subset I} \mathcal{R}(J_i) \quad \text{if} \quad I = \bigcup J_i. \]
Moreover,
\[ \mathcal{R}(S^1) = \mathcal{B}(\Gamma(\widehat{h}(S^1))) , \quad \mathcal{R}(S^1)' = C \cdot 1 ; \]
iii.) For each open interval \( I \subset S^1 \), the local observable algebra \( \mathcal{R}(I) \) is \(*\)-isomorphic to the unique hyper-finite factor of type \( \text{III}_1 \).

PROOF. Clearly,
\[ (8.2.2) \quad \bigcap_{J \supset I} \widehat{h}(J) = \bigvee_{\mathcal{T} \subset I} \widehat{h}(J) = \widehat{h}(I) , \]
which together with Proposition B.0.11 (see Eq. (B.0.8), Appendix E) implies i.) and ii.). The proof of iii.) will be given in [9]. □

REMARK 8.2.4. A special case of i.) is the following: let \( I \) be an open interval contained in a half-circle. Then
\[ \mathcal{R}(I) = \bigcap_{I \subset I_{\alpha}} \mathcal{R}(I_{\alpha}) , \]
where the \( I_{\alpha} \)'s are the half-circles containing \( I \).

THEOREM 8.2.5 (Finite speed of propagation). Let \( I \subset S^1 \) be an open interval. Then
\[ (8.2.3) \quad \tilde{\alpha}_{\lambda(I_{\alpha}(t))}^0 : \mathcal{R}(I) \leftrightarrow \mathcal{R}(I(\alpha, t)) . \]

PROOF. This result follows directly from Proposition 1.5.2 and Corollary 7.4.6. □
8.3. Analyticity properties of the correlation functions

Consider the Weyl operators \( W(f) \), \( f \in \mathfrak{h}(W_1) \), \( 1 \leq j \leq n \), and the correlation functions

\[
G(t_1, \ldots, t_n; W(f_1), \ldots, W(f_n)) = \omega^\circ \left( \prod_{j=1}^{n} \alpha_{\Lambda_i(t_j)}(W(f_j)) \right).
\]

More specifically, let us first consider an element \([f] \in \mathfrak{t}(W_1)\), together with its Cauchy data \( \hat{f} \in \mathfrak{t}(I_+) \). It follows that

\[
\|f\|_{\mathfrak{h}(dS)} = \|\hat{K}_f\|_{\mathfrak{h}(S^1)} = \|u \circ \hat{K}_{2\pi t}\|_{\mathfrak{h}(S^1)} = \sqrt{t} \|\hat{K}_{2\pi t}\|_{\mathfrak{h}(S^1)}.
\]

We have used Eq. (7.4.3) and Proposition 7.4.1. Recall that

\[
\hat{K}_{2\pi t} = ((1 + p_{2\pi t})\hat{\rho} + p_{2\pi t})\hat{K}_\infty \hat{f},
\]

with \( p_{2\pi t} = \frac{e^{-2\pi t}}{1 + e^{-2\pi t}} \) and \( (1 + p_{2\pi t}) = \frac{1}{1 + e^{-2\pi t}} \). Since \( \hat{K}_\infty \hat{f} \subset \mathfrak{d}(I_+) \), no cross terms arise, and consequently

\[
\|\hat{K}_{2\pi t}\|_{\mathfrak{h}(S^1)} = \|(1 + 2p_{2\pi t})\hat{K}_\infty \hat{f}\|_{\mathfrak{h}(S^1)}.
\]

Now compute

\[
G(t_1, \ldots, t_n; W(f_1), \ldots, W(f_n)) = \omega^\circ \left( \prod_{j=1}^{n} W(u(\Lambda_1(t_j))f_j) \right)
\]

\[
= \left( \prod_{1 \leq i < j \leq n} e^{-i2\tau(u(\Lambda_1(t_i))f_i,u(\Lambda_1(t_j))f_j)_{\mathfrak{h}(dS)}} \right) \omega^\circ \left( \sum_{j=1}^{n} W(u(\Lambda_1(t_j))f_j) \right)
\]

\[
= \left( \prod_{1 \leq i < j \leq n} e^{-i2\tau(\hat{K}_\infty \hat{f}_i, e^{\frac{i}{2}(\hat{K}_\infty \hat{f}_i, f_j)}\hat{K}_\infty \hat{f}_j)_{\mathfrak{h}(I_+)} - \frac{i}{2} \sum_{j=1}^{n} e^{\frac{i}{2}(\hat{K}_\infty \hat{f}_i, (1 + 2p_{2\pi t})\hat{K}_\infty \hat{f}_j)_{\mathfrak{h}(I_+)}} \right)
\]

where

\[
R_{\frac{i}{2}}(\hat{K}_\infty \hat{f}_1, \hat{K}_\infty \hat{f}_2) = \tau \left( \hat{K}_\infty \hat{f}_1, \frac{e^{it\epsilon}}{\epsilon(1 - e^{-2\pi t})} \hat{K}_\infty \hat{f}_2 \right)_{L^2(I_+ \mid \cos \phi \mid^{-1} \mathrm{d}\phi)}
\]

\[
+ \tau \left( \hat{K}_\infty \hat{f}_2, \frac{e^{2\pi t\epsilon}}{\epsilon(1 - e^{-2\pi t})} e^{it\epsilon} \hat{K}_\infty \hat{f}_1 \right)_{L^2(I_+ \mid \cos \phi \mid^{-1} \mathrm{d}\phi)}.
\]

For \( f_1, f_2 \in \mathfrak{h}(W_1) \) the function \( t \mapsto R_{t/\tau}(\hat{K}_\infty \hat{f}_1, \hat{K}_\infty \hat{f}_2) \) allows a holomorphic extension to the strip \( \{ \tau \in \mathbb{C} \mid 0 < \Im \tau < 2\pi \} \). Consequently, the function

\[
(t_1, \ldots, t_n) \mapsto G(t_1, \ldots, t_n; W(f_1), \ldots, W(f_n))
\]

is holomorphic in the set

\[
\mathfrak{B}_{2\pi t}^n = \{ (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n \mid \Im \tau_1 < \Im \tau_{i+1}, \ Im \tau_n - \Im \tau_1 < 2\pi \}.
\]
and continuous on $\overline{T_{2\pi r}^{+}}$. The holomorphic extension is

$$
(\tau_1, \ldots, \tau_n) \mapsto \prod_{i=1}^{n} e^{-\frac{i}{\hbar} (\hat{\mathcal{F}} \tau_i, (1 + 2pr_n) \hat{\mathcal{F}} \tau_i)} \prod_{1 \leq i < j \leq n} e^{-\frac{i}{\hbar} \mathcal{R}_{\tau_i, \tau_j} \left( \hat{\mathcal{F}} \tau_i, \hat{\mathcal{F}} \tau_j \right)}.
$$

Hence the Euclidean Green's functions

$$
\mathbb{G}^{c}(s_1, \ldots, s_n; W(f_1), \ldots, W(f_n)) \doteq \mathbb{G}(is_1, \ldots, is_n; W(f_1), \ldots, W(f_n))
$$

$$
= \prod_{i=1}^{n} e^{-\frac{i}{\hbar} C_{\tau_i} (\cos \frac{1}{\hbar} \hat{\mathcal{F}} \tau_i,\cos \frac{1}{\hbar} \hat{\mathcal{F}} \tau_i)} \prod_{1 \leq i < j \leq n} e^{-\frac{i}{\hbar} C_{\tau_i, \tau_j} (\cos \frac{1}{\hbar} \hat{\mathcal{F}} \tau_i,\cos \frac{1}{\hbar} \hat{\mathcal{F}} \tau_j)},
$$

where

$$
C_{\tau_i} (\cos \frac{1}{\hbar} \hat{\mathcal{F}} \tau_i,\cos \frac{1}{\hbar} \hat{\mathcal{F}} \tau_i) = \tau \left( \hat{\mathcal{F}} \tau_i, \frac{e^{-i|s_i|} - e^{-i(2n - |s_i|)}}{2x(1 - e^{-2\pi r})} \hat{\mathcal{F}} \tau_i \right)_{L^2(1 + \frac{4}{\hbar} \pi r)}^{n},
$$

with $\varepsilon^2 = -\cos \psi \partial_\psi^2 + \cos \psi \partial_\psi + 2$. For $\hat{\mathcal{F}} \tau_j$ real valued and $1 \leq j \leq n$,

$$
\mathbb{G}^{c}(s_1, \ldots, s_n; W(f_1), \ldots, W(f_n)) = \prod_{1 \leq i, j \leq n} e^{-\frac{i}{\hbar} C_{\tau_i, \tau_j} (\cos \frac{1}{\hbar} \hat{\mathcal{F}} \tau_i,\cos \frac{1}{\hbar} \hat{\mathcal{F}} \tau_j)}.
$$

We now specialise this result to time-zero test-functions.

**Proposition 8.3.1.** Let $f_i = \delta \otimes h_i$ with $h_i \in \mathcal{D}(I_+)$, $i = 1, \ldots, n$. It follows that the map

$$
(t_1, \ldots, t_n) \mapsto G(t_1, \ldots, t_n; W(f_1), \ldots, W(f_n))
$$

$$
= \left( \Omega, e^{it_1 \hat{L}_1} W_f([f_1]) e^{it_2 \hat{L}_2} W_f([f_2]) \cdots e^{it_n \hat{L}_n} W_f([f_n]) \Omega \right)
$$

is holomorphic in the set

$$
T_{2\pi r}^{n+} = \{(z_1, \ldots, z_n) \in C^n \mid \mathcal{J}z_1 < \mathcal{J}z_{n+1}, \mathcal{J}z_n - \mathcal{J}z_1 < 2\pi r \},
$$

and continuous on $T_{2\pi r}^{n+}$. Moreover,

$$
G(i\theta_1, \ldots, i\theta_n; W([f_1]), \ldots, W([f_n])) = \prod_{1 \leq i, j \leq n} e^{-\frac{i}{\hbar} C_{i\theta_i - j\theta_j} (h_i, h_j)},
$$

where $C_{i\theta_i - j\theta_j} (h_i, h_j)$ is defined in Eq. (8.3.1).

**Proof.** Recall that for $h \in \mathcal{D}(I_+)$ one can extend the domain of $\mathcal{F}_{+r}$ to distributions of the form

$$
f(x) \equiv (\delta \otimes h)(x) = \delta(t) \frac{h(0, \psi)}{r \cos \psi},
$$

with $x \equiv x(t, \psi)$ the coordinates introduced in (2.1.4) and $d\mu_{\psi_i}(t, \psi) = r^2 dt d\psi \cos \psi$. As we have seen, this leads to

$$
(f_{|S^1}, (n \bar{n})_{|S^1}) = (0, -h) \equiv (\bar{h}, \bar{w}),
$$

As a consequence of Lemma 8.3.3, $f_1$ (and in the sequel $\hat{\mathcal{F}} \tau_j$) is real valued if $[f] \in \mathcal{B}^\ast$. 


and finally, recalling (7.2.2), i.e., \( \hat{K}_\infty(\hat{f}, \pi)(\psi) \triangleq \cos \psi \varpi(\psi) - i (\epsilon \Phi)(\psi) \), we find
\[
\hat{K}_\infty(f|_{S^1}, (n f)|_{S^1}) = -\cos \varphi \, h .
\]
Together with (8.3.2) this proves the result. \qed

□
Part 5

Euclidean Quantum Fields
CHAPTER 9

The Euclidean Sphere

We will now introduce Markov fields on the sphere, which later on will allow us to reconstruct free quantum field on the de Sitter space. In order to define a probability measure on the sphere, we briefly review the geometrical setting. We consider the Euclidean sphere
\[ S^2 \equiv \{ \vec{x} \in \mathbb{R}^3 \mid x_0^2 + x_1^2 + x_2^2 = r^2 \} , \]
embedded in \( \mathbb{R}^3 \). Let \( \vec{0} = (0,0,0) \) denote the origin in \( \mathbb{R}^3 \). The upper (resp. lower) hemisphere is \( S^\pm = \{ \vec{x} \in S^2 \mid \pm x_0 > 0 \} \).

The equator is \( S^1 = \{ \vec{x} \in S^2 \mid x_0 = 0 \} \). The hemispheres \( S^\pm \) are open, their boundaries \( \partial S^\pm = S^1 \) coincide with the equator, and \( S^2 \) is the disjoint union
\[ S^2 = S_- \cup S^1 \cup S_+ . \]

The Euclidean time reflection (9.0.3)
\[ T : (x_0, x_1, x_2) \mapsto (-x_0, x_1, x_2) \]
maps \( S^\pm \) onto \( S^\mp \) and leaves \( S^1 \) invariant. \( S^1 \) itself is the disjoint union
\[ I_+ \cup \{(0,-r,0),(0,r,0)\} \cup I_- , \]
with \( I_{\pm} = \{ \vec{x} \in S^1 \mid \pm x_2 > 0 \} \) open half-circles.

9.1. The symmetry group of the sphere

The group of rotations \( SO(3) \) leaves the sphere \( S^2 \) invariant. We denote the generators of the rotations
\[ R_0(\alpha) \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} , \quad R_1(\beta) \triangleq \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} , \]
and
\[ R_2(\gamma) \triangleq \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \alpha, \beta, \gamma \in [0, 2\pi) , \]
around the three coordinate axis by \( K_0, K_1 \) and \( K_2 \), and set
\[ K^{(\alpha)} \triangleq \cos \alpha K_1 + \sin \alpha K_2 , \quad \alpha \in [0, 2\pi) . \]

Denote by \( R^{(\alpha)} \) the rotations generated by \( K^{(\alpha)} \), namely the rotations
\[ R^{(\alpha)}(\theta) = R_0(\alpha)R_1(\theta)R_0(-\alpha) , \quad \alpha, \theta \in [0, 2\pi) , \]

\[ ^1 \text{We have changed the notation; see (4.1.6) for comparison.} \]
which leave the boundary points \( x_\alpha = (0, r \sin \alpha, r \cos \alpha) \) and \(-x_\alpha \) of the time-zero half-circles
\[
I_\alpha = R_0(\alpha) I_+
\]
invariant.

9.2. Geographical and path-space coordinates

We will now define two charts, which will be convenient in the sequel. Together they provide an atlas for the sphere.

9.2.1. Geographical coordinates. The chart
\[
\begin{pmatrix}
  x_0 \\
  x_1 \\
  x_2 \\
\end{pmatrix} = \begin{pmatrix}
  r \sin \vartheta \\
  r \cos \vartheta \sin \psi \\
  r \cos \theta \cos \psi \\
\end{pmatrix}, \quad -\pi \leq \vartheta < \pi, \quad 0 \leq \psi < 2\pi,
\]
covers the sphere, except for the geographical poles \((\pm r, 0, 0) \in \mathbb{R}^3\). Refer to \((\vartheta, \psi)\) as geographical coordinates. The equator \( S^1 \sim \{ (\vartheta, \psi) \mid \vartheta = 0 \} \) and the point \((0, 0)\) is mapped to the origin \( \mathbf{o} = (0, 0, r) \). The restriction of the Euclidean metric to this chart is
\[
g = r^2 d\vartheta \otimes d\vartheta + r^2 \cos^2 \vartheta \left( d\psi \otimes d\psi \right)
\]
and
\[
\Delta = \frac{1}{r^2 \cos^2 \vartheta} \left( \cos \vartheta \frac{\partial}{\partial \vartheta} \left( \cos \vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{\partial^2}{\partial \psi^2} \right).
\]
Here \( \cos \vartheta \) denotes the multiplication operator \((\cos \vartheta f) \vartheta = \cos \vartheta f(\vartheta)\) acting on functions of \( \vartheta \). The surface element on \( S^2 \) is \( d\Omega(\vartheta, \psi) = r^2 \cos \vartheta \, d\vartheta \, d\psi \).

9.2.2. Path-space coordinates. The chart
\[
\begin{pmatrix}
  x_0 \\
  x_1 \\
  x_2 \\
\end{pmatrix} = \begin{pmatrix}
  r \sin \psi \cos \vartheta \\
  r \sin \psi \sin \vartheta \\
  r \cos \theta \cos \psi \\
\end{pmatrix}, \quad 0 \leq \theta < 2\pi, \quad -\pi \leq \vartheta < \pi,
\]
covers the sphere with exception of the two points \((0, \pm r, 0) \in \mathbb{R}^3\). Refer to this chart as path-space coordinates. The point \((0, 0)\) is mapped to the origin \( \mathbf{o} = (0, 0, r) \). The restriction of the Euclidean metric to this chart is
\[
g = \cos^2 \psi (d\theta \otimes d\theta) + d\psi \otimes d\psi
\]
and
\[
\Delta = \frac{1}{r^2 \cos^2 \psi} \left( \frac{\partial^2}{\partial \theta^2} + \left( \cos \psi \frac{\partial}{\partial \psi} \right)^2 \right).
\]
The surface element on \( S^2 \) is \( d\Omega(\theta, \psi) = r^2 \cos \psi \, d\theta \, d\psi \).

9.2.3. The Laplace operator. The expressions in \((9.2.1)\) and \((9.2.3)\) both extend to the self-adjoint Laplace operator \( \Delta_{S^2} \) on \( L^2(S^2, d\Omega) \). \(-\Delta_{S^2}\) has non-negative discrete spectrum and an isolated simple eigenvalue at zero with eigenspace the constants.
CHAPTER 10

Gaussian Measures

10.1. The definition of the measure

Let $\Sigma$ be the $\sigma$-algebra generated by the Borel cylinder sets of $\mathcal{Q} := \mathcal{D}_R'(S^2)$, the dual space of $C_\infty^\infty(S^2)$. For $f \in C_\infty^\infty(S^2)$, let $\Phi(f) : \mathcal{Q} \to \mathbb{C}$ be the evaluation map

$$q \mapsto \langle q, f \rangle.$$  

(10.1.1)

$\langle ., . \rangle$ is the duality bracket. For $F$ a Borel function on $\mathbb{R}$, define

$$F(\Phi(f)) : \mathcal{Q} \to \mathbb{C}, q \mapsto F(\langle q, f \rangle).$$

The Fourier transform of a Gaussian measure $d\Phi_C$ on $(\mathcal{Q}, \Sigma)$ with covariance

$$C(f, g) = \langle \mathcal{T}, (-\Delta_{S^2} + \mu^2)^{-1} g \rangle_{L^2(S^2, d\Omega)}, \quad f, g \in C_\infty(S^2),$$  

(10.1.2)

is

$$\int_{\mathcal{Q}} d\Phi_C e^{i\Phi(f)} = e^{-C(f, f)/2}, \quad f \in C_\infty^\infty(S^2).$$  

(10.1.3)

According to Minlos’ theorem [79, 175, 20], Equ. (10.1.3) defines a unique probability measure, namely $d\Phi_C$, on $\mathcal{Q}$.

The mean of the Gaussian measure $d\Phi_C$ is zero, and the covariance $C$ coincides with the second momentum. More generally,

$$\int_{\mathcal{Q}} d\Phi_C \Phi(f)^p = \begin{cases} 0, & p \text{ odd} \\ (p - 1)!! C(f, f)^{p/2}, & p \text{ even} \end{cases}. $$  

(10.1.4)

It follows from (10.1.4) that

$$e^{\Phi(f)} \in L^1(\mathcal{Q}, \Sigma, d\Phi_C) \quad \text{if} \quad f \in C_\infty^\infty(S^2).$$

For $\mu > 0$ the covariance (10.1.2) defines a norm $\| . \|_{-1} = C(., .)^{1/2}$ on $C_\infty^\infty(S^2)$. The completion of $C_\infty^\infty(S^2)$ with respect to $\| . \|_{-1}$ is the Sobolev space $H^{-1}(S^2)$. 

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Proposition 10.1.1. Let \( f, g \in L^2(S^2) \). It follows that
\[
\langle f, g \rangle_{H^{-1}(S^2)} = \langle f, (-\Delta_{S^2} + \mu^2)^{-1} g \rangle_{L^2(S^2, d\Omega)}
\]
\[
= \frac{1}{2} \int_{S^2} d\Omega(x) \int_{S^2} d\Omega(y) f(x) c_{\nu} P_{\nu}^+ \left( -\frac{x \cdot y}{r^2} \right) g(y).
\]

Remark 10.1.2. This result should be compared with Proposition 6.1.12, which says that for two functions \( f, g \in C^\infty_0(dS) \) one has
\[
\langle [f], [g] \rangle_{H(dS)} = \int_{dS} d\mu_{dS}(x) \int_{dS} d\mu_{dS}(y) f(x) c_{\nu} P_{\nu}^+ \left( \frac{x \cdot y}{r^2} \right) g(y).
\]

Proof. We recall that the spherical harmonics
\[
(10.1.5) \quad Y_{l,k}(\theta, \psi) = \sqrt{\frac{2l + 1 \{l - k\}!}{4\pi (l + k)!}} P_l^k(\cos \theta) e^{ik\psi}, \quad \theta \in [0, \pi],
\]
with \( \theta = 0 \) at the north pole, are orthonormal
\[
\int_{S^2} d\Omega Y_{l',k'}(\theta, \psi) Y_{l,k}(\theta, \psi) = r^2 \delta_{l,l'} \delta_{k,k'},
\]
and satisfy
\[
\Delta_{S^2} Y_{l,k}(\theta, \psi) = -\frac{1}{r^2} (l + 1) \delta_{l,k} Y_{l,k}(\theta, \psi).
\]

Now consider two vectors \( \bar{x} = \bar{x}(\theta, \psi) \) and \( \bar{y} = \bar{y}(\theta', \psi') \) of length \( |\bar{x}| = |\bar{y}| = r \). It follows that the kernel of the operator
\[
(-\Delta_{S^2} + \mu^2)^{-1}(\bar{x}, \bar{y}) = \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \frac{Y_{l,k}(\theta', \psi') Y_{l,k}(\theta, \psi)}{l(l+1)+\mu^2 r^2}
\]
\[
= \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \frac{21 + 1}{l(l+1)+\mu^2 r^2} P_l \left( -\frac{x \cdot y}{r^2} \right)
\]
\[
= \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{k=-l}^{l} \frac{21 + 1}{l(l+1)+\mu^2 r^2} (-1)^l P_l \left( -\frac{x \cdot y}{r^2} \right).
\]

In the second equality we have used the summation formula [195, p. 395]:
\[
P_l \left( -\frac{x \cdot y}{r^2} \right) = \frac{4\pi}{21 + 1} \sum_{k=-l}^{l} Y_{l,k}(\theta', \psi') Y_{l,k}(\theta, \psi).
\]

In the third equality we have used \((-1)^l P_l(z) = P_l(-z)\).

The identity [155, Eq. (23), page 205]
\[
\int_{-1}^{1} dz P_l(z) P_{-1 + i\nu}(z) = \frac{2 \cos(i\nu\pi)}{\pi} \frac{(-1)^l}{(l + \frac{i}{2})^2 + \nu^2}
\]
\[
= \frac{2 \cos(i\nu\pi)}{\pi} \frac{(-1)^l}{l(l+1)+\mu^2 r^2}
\]

\[\text{This identity extends by analyticity from } \nu \in \mathbb{R} \text{ to the case } \nu = i\sqrt{\frac{4}{\mu^2 r^2} - \mu^2 r^2}, 0 < \mu < 1/2r.\]
shows that
\[
P_{\frac{1}{2} - iv}(z) = \sum_{l=0}^{\infty} \left( \frac{2l+1}{2} \int_{-1}^{1} \text{d}z' \ P_l(z')P_{\frac{1}{2} - iv}(z') \right) P_l(z)
\]
\[
= \cos(i\nu\pi) \frac{\sum_{l=0}^{\infty} (-1)^{l} \left( \frac{2l+1}{2} \int_{-1}^{1} d\frac{z'}{r^2} \ P_l(z') \right)}{4 \cos(i\nu\pi)}.
\]

Thus
\[
\left(-\Delta_{S^2} + \mu^2\right)^{-1}(x, y) = \frac{P_{\frac{1}{2} - iv}(\frac{x+y}{r})}{4 \cos(i\nu\pi)}.
\]

□

Next use [10.1.2] and [10.1.4] to show the following result.

**Lemma 10.1.3.** The map
\[
C_{\mu}^{\infty}(S^2) \to \bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\Phi C)
\]
\[
f \mapsto \Phi(f)
\]
extends to a continuous map from $H^{-1}(S^2)$ to $\bigcap_{1 \leq p < \infty} L^p(Q, \Sigma, d\Phi C)$.

**Lemma 10.1.4.** The cylindrical functions $F(\Phi(f_1), \ldots, \Phi(f_n))$, $f_i \in C_{\mu}^{\infty}(S^2)$, $F$ a Borel function on $\mathbb{R}^n$ and $n \in \mathbb{N}$, are dense in $L^p(Q, \Sigma, d\Phi C)$ for $1 \leq p < \infty$.

### 10.2. Properties of the covariance

The short distance behaviour of the covariance $C$ has been studied in [1]. We briefly present their findings. The covariance $C$ can be expressed [1] in terms of the heat kernel $p(t, x, y)$ on the sphere $S^2$:
\[
C(x, y) = \int_{0}^{\infty} dt \ e^{-t\mu^2} p(t, x, y)
\]
This allows us to introduce the usual multi-scale decomposition (see, e.g., [15]).

The covariance $C(x, y)$ is given by
\[
C(x, y) = \sum_{l=0}^{\infty} C_l(x, y),
\]
where, for some fixed constant $\gamma$ larger than 1,
\[
C_l(x, y) = \int_{0}^{\infty} dt \ \left( e^{-t\gamma^2 l^2} - e^{-t\gamma^2 (l+1)^2} \right) p(t, x, y)
\]
is the kernel of the operator
\[
C_l(x, y) = \frac{1}{-\Delta_{S^2} + \mu^2 \gamma^2 l^2} - \frac{1}{-\Delta_{S^2} + \mu^2 \gamma^2 (l+1)^2}
\]
Following [1], we introduce the regularized covariance
\[
C^{(k)}(x, y) = \sum_{l=0}^{\log_{\gamma} k-1} C_l(x, y),
\]
where, of course, $k$ is such that $\log_{\gamma} k$ ranges on the positive integers. $C^{(k)}$ represents the covariance of the field with length cutoff $\gamma(\mu k)^{-1}$, the analog, in the
flat case, of a momentum cutoff of order \( \mu k \). In this sense, \( C^{(k)} \) compares with the \( \delta_k \) in the equations LR1, LR2, and LR3 of [79, p. 160–161].

**Theorem 10.2.1** (De Angelis, de Falco and Di Genova [11]). Let \( 1 \leq q < \infty \). With the notation introduced above, we have

\[
\sup_{x \in S^2} \| C(x, .) \|_{L^q(S^2, d\Omega)} < +\infty ,
\]

\[
\| C^{(k)}(., .) - C(., .) \|_{L^1(S^2 \times S^2, d\Omega \otimes d\Omega)} \lesssim O(k^{-2/q}) ,
\]

\[
\sup_{x \in S^2} C^{(k)}(x, x) \lesssim O(\log_2 k) .
\]

**Remark 10.2.2.** The logarithmic nature of the singularity of the covariance \( C(x, y) \) at coinciding points

\[
C(x, y) \sim \frac{1}{2\pi} \log \mu d(x, y)
\]

follows from the asymptotic behaviour of the heat kernel as \( t \downarrow 0 \) [14]:

\[
p(t, x, y) \sim \frac{1}{4\pi t} H(x, y)e^{-\frac{d^2(x, y)}{4t}}
\]

uniformly on all compact sets in \( S^2 \times S^2 \), which do not intersect the cut locus of \( S^2 \). Here \( d(x, y) \) is the geodesic distance and \( H(x, y) \) the Ruse invariant (see, e.g., [194]). The latter is equal to one in the case of constant curvature.

### 10.3. Sobolev spaces

In [50] Dimock defines real Sobolev spaces over a Riemannian manifold \((M, g)\), with \( M \) an oriented compact connected finite-dimensional manifold and \( g \) a positive definite metric. The Euclidean sphere \( S^2 \) is a special case: define for \( m > 0 \) fixed, the spaces \( H^1(S^2) \) as the completion of \( C^\infty_c(S^2) \) in the norm

\[
\| h \|_1^2 = \langle h, (-\Delta_{S^2} + \mu^2)h \rangle_{L^2(S^2, d\Omega)} .
\]

The function spaces \( H^{\pm 1}(S^2) \) are real Hilbert spaces, \( \| (f, g) \| \leq \| f \|_1 \| g \|_{-1} \), and

\[
C^\infty_c(S^2) \subset H^1(S^2) \subset L^2(S^2, d\Omega) \subset H^{-1}(S^2) .
\]

The inner product extends to a bilinear pairing of \( H^1(S^2) \) and \( H^{-1}(S^2) \). In fact, \( H^1(S^2) \) and \( H^{-1}(S^2) \) are dual to each other with respect to this pairing, and the map \( f \mapsto (-\Delta_{S^2} + \mu^2)f \) is unitary from \( H^1(S^2) \) to \( H^{-1}(S^2) \).

For a closed subset \( C \subset S^2 \), which contains an open subset of \( S^2 \), we define a closed subspace \( H^{-1}_C(S^2) \) of \( H^{-1}(S^2) \):

\[
H^{-1}_C(S^2) = \{ f \in H^{-1}(S^2) \mid \text{supp} f \subset C \} .
\]

For the open sets \( S_+ \subset S^2 \), let \( H^1_0(S_+) \) be the closure of \( C^\infty_0(S_+) \) in \( H^1(S^2) \).

**Lemma 10.3.1** (Dimock [50], Lemma 1, p. 245).

\[
H^{-1}(S^2) = H^{-1}_S(S^2) \oplus (-\Delta_{S^2} + \mu^2)H^1_0(S_{\pm}) ,
\]

\[
H^{-1}(S^2) = (-\Delta_{S^2} + \mu^2)H^1_0(S_-) \oplus H^{-1}_S(S^2) \oplus (-\Delta_{S^2} + \mu^2)H^1_0(S_+) .
\]
Let \( e(S^1) \) and \( e(\overline{S^1}_+^\pm) \) denote the orthogonal projections from \( \mathbb{H}^{-1}(S^2) \) onto \( \mathbb{H}_{\overline{S^1}}^{-1}(S^2) \) and \( \mathbb{H}_{\overline{S^1}_+^\pm}(S^2) \), respectively.

**Lemma 10.3.2** (Dimock’s pre-Markov property [50], Lemma 2, p. 246).
\[
e(\overline{S^1}_+^\pm) e(\overline{S^1}_+^\pm) = e(S^1) \quad \text{on } \mathbb{H}^{-1}(S^2).
\]
Thus \( \mathbb{H}_{\overline{S^1}}^{-1}(S^2) = \mathbb{H}_{\overline{S^1}_+^\pm}(S^2) \cap \mathbb{H}_{\overline{S^1}_+^\pm}(S^2) \).

We note that the origins of Dimock’s work can be traced back to [85] and even further to [149] [150] [151] [152].

### 10.4. Conditional expectations

The Markov property for the sphere, presented in Theorem 10.4.2 vi.) below, is satisfied, iff for any function of the Euclidean field in \( \overline{S^1}_+^\pm \), conditioning to the fields in \( \overline{S^1}_+^\pm \) is the same as conditioning to the fields in \( \partial S_+^1 \).

**Theorem 10.4.1** (Simon [175], Theorem III.7, p. 91). Let \( (\mathcal{Q}, \Sigma, \mu) \) be a probability space and let \( \Sigma' \) be a sub-\( \sigma \)-algebra of \( \Sigma \). Let \( F \) be an element of \( \mathcal{L}_F(Q, \Sigma, \mu) \). Then there exists a unique function \( E_{\Sigma',F} \) such that

i.) \( E_{\Sigma',F} \) is \( \Sigma' \)-measurable;

ii.) for all \( G \in \mathcal{L}_F(Q, \Sigma, d\mu) \) which are \( \Sigma' \)-measurable
\[
\int_Q d\mu \ G E_{\Sigma',F} = \int_Q d\mu \ GF.
\]

\( E_{\Sigma',F} \) is called the conditional expectation of \( F \) given \( \Sigma' \).

Next consider (see [24], Vol. III) the Fock space \( \Gamma(\mathbb{H}^{-1}_C(S^2)) \) over the complexification \( \mathbb{H}^{-1}_C(S^2) \) of \( \mathbb{H}^{-1}(S^2) \). Let \( \Phi_\Sigma \) be the Fock space field operator and \( \Omega_\Sigma \) the vacuum vector in \( \Gamma(\mathbb{H}^{-1}_C(S^2)) \). The map
\[
e^{i\Phi(f)} \mapsto e^{i\Phi_\Sigma(f)} \Omega_\Sigma, \quad f \in \mathbb{H}^{-1}_C(S^2),
\]
extends to a unitary operator from \( \mathcal{L}_F(Q, \Sigma, d\Phi_\Sigma) \) to the Euclidean Fock space \( \Gamma(\mathbb{H}^{-1}_C(S^2)) \). Set
\[
E_{\Sigma,\Omega} \overset{\triangle}{=} \Gamma(e(\overline{S^1}_+)) \quad \text{and} \quad E_{\Sigma,\Omega_\Sigma} \overset{\triangle}{=} \Gamma(e(\overline{S^1}_+^\pm)).
\]

Use (10.4.1) and denote the linear operators on \( \mathcal{L}_F(Q, \Sigma, d\Phi_\Sigma) \) corresponding to (10.4.2) by \( E_{\Sigma,\Omega} \) and \( E_{\Sigma,\Omega_\Sigma} \), respectively.

**Notation.** For any closed and simply connected region \( X \subset S^2 \) denote by \( \Sigma_X \) the smallest sub-\( \sigma \)-algebra of \( \Sigma \) for which the functions \( \{ \Phi(f) \mid f \in \mathbb{H}^{-1}_C(S^2) \} \) are measurable.

**Theorem 10.4.2** (Dimock [50], Theorem 1, p. 247).

i.) \( E_{\Sigma,\Omega} \) is the conditional expectation for \( \Sigma' = \Sigma_{\Omega} \).

ii.) \( E_{\Sigma,\Omega_\Sigma} \) is the conditional expectation for \( \Sigma' = \Sigma_{\Omega_\Sigma} \).

iii.) If \( F \in \mathcal{L}_F(Q, \Sigma, d\Phi_\Sigma) \), then \( E_{\Sigma,\Omega_\Sigma} F = E_{\Sigma,\Omega} F \).

iv.) \( E_{\Sigma,\Omega} = E_{\Sigma,\Omega_\Sigma} E_{\Sigma,\Omega} \), acting on \( \mathcal{L}_F(Q, \Sigma, d\Phi_\Sigma) \) (Markov property).

**Proof.** Parts i.) and ii.) follow directly from the definitions. Parts iii.) and iv.) follow from Lemma [10.3.2]
10.5. Sharp-time fields

Let $\mathcal{H}_C^{-1}(S^2)$ be the complexification of $\mathcal{H}^{-1}(S^2)$. The Sobolev space $\mathcal{H}_C^{-1}(S^2)$ contains the distribution

$$(10.5.1) \quad (\delta \otimes h)(\bar{x}) = r^{-1} \delta(\theta) h(0, \varphi), \quad h(0, .) \in C^\infty(S^1), \quad \bar{x} \equiv \bar{x}(\theta, \varphi),$$

using geographical coordinates (see Section 9.2.1), which is supported on $S^1$. If supp $h$ does not contain $(0, \pm r, 0)$, then $(10.5.1)$ equals, as an element in $\mathcal{H}_C^{-1}(S^2)$,

$$(10.5.2) \quad (\delta \otimes h)(\bar{x}) = \delta(\theta) \frac{h(0, \psi)}{r \cos \psi} - \delta(\theta - \pi) \frac{h(\pi, \psi)}{r \cos \psi}, \quad \bar{x} \equiv \bar{x}(0, \psi),$$

in path-space coordinates. The sign on the right hand side is due to the different orientations of the one-forms $d\theta$ and $d\psi$ on $\bar{I}$.

**Lemma 10.5.1.** Consider distributions of the form $(10.5.1)$. It follows that the time-zero covariance

$$C_0(\overline{\mathcal{H}^1}, h_2) \equiv C(\delta \otimes \overline{\mathcal{H}^1}, \delta \otimes h_2)$$

$$(10.5.3) \quad = \frac{1}{2} \int_{S^1} r \, d\theta \int_{S^1} r \, d\theta' \overline{h_2(\theta)} c_\nu P_{\frac{1}{2} + iv}(-\cos(\varphi - \varphi')) h_2(\varphi')$$

exists as a positive quadratic form on $C^\infty(S^1)$. Moreover, $C_0$ is invariant under rotations around the axis connecting the geographical poles. The constant $c_\nu$ appearing in $(10.5.3)$ is given by

$$c_\nu = \frac{1}{2 \sin(\pi(-\frac{1}{2} + iv))} = \frac{1}{2 \cos(\nu \pi)}.$$ 

and—as in $(4.3.2)$ and $(4.3.3)$—

$$\nu = \begin{cases} 
\frac{i}{2} - \mu \frac{i}{2} & \text{if } 0 < \mu < \frac{1}{2}, \\
\sqrt{\mu^2 - \frac{1}{4}} & \text{if } \mu \geq \frac{1}{2}.
\end{cases}$$

**Proof.** Recall Proposition [10.1.1]. This allows us to compute

$$\langle \delta \otimes \overline{\mathcal{H}^1}(\Delta S^2 + \mu^2)^{-1} \delta \otimes h_2 \rangle_{L^2(S^2, d\Omega)}$$

$$= \int_{S^2} r \, d\psi' \, d\theta' \int_{S^2} r \, d\psi \, d\theta \sin \theta' \delta(\theta' - \frac{\pi}{2}) h_1(\psi')$$

$$\times \sum_{l=0}^{\infty} \sum_{k=-l}^{l} Y_{l,k}(\theta', \psi') Y_{l,k}(\theta, \psi) \frac{\sin \theta \delta(\theta - \frac{\pi}{2}) h_2(\psi)}{l(l+1) + \mu^2 r^2}$$

$$= \frac{1}{4 \cos(\nu \pi)} \int_{S^1} r \, d\theta' \int_{S^1} r \, d\psi' \, h_1(\psi') P_{\frac{1}{2} + iv}(-\cos(\psi - \psi')) h_2(\psi).$$

We have used that for $\bar{x} = (0, \sin \psi, \cos \psi)$ and $\bar{y} = (0, \sin \psi', \cos \psi')$

$$\langle \bar{x}, \bar{y} \rangle_{\nu} = \cos(\psi' - \psi),$$

as $\cos \psi \cos \psi' + \sin \psi \sin \psi' = \cos(\psi' - \psi)$.

---

2The tensor product notation will be used for functions as well.
An explicit formula for $C_0(h_1, h_2)$ (restricted to a half-circle) in path-space coordinates is provided next. Let, for $h \in D(I_+)$ and $0 \leq \theta_0 < 2\pi$,

$$
(\delta_{\theta_0} \otimes h)(\chi) \doteq \delta(\theta - \theta_0) \frac{h(\psi)}{r \cos \psi} ; \quad \chi \equiv \chi(\theta, \psi).
$$

For $\theta = 0$, $\delta_{\theta = 0} \otimes h$ coincides with $\delta \otimes h$ defined in Eq. (10.5.1), cf. Eq. (10.5.2).

**Lemma 10.5.2.** For $h_1, h_2 \in D(I_+)$, $0 \leq \theta_1, \theta_2 < 2\pi$,

$$
C_{[\theta_1 - \theta_2]}(h_1, h_2) \doteq C(\delta_{\theta_1} \otimes h_1, \delta_{\theta_2} \otimes h_2)
$$

$$
= r \left\langle \int \frac{e^{-|\theta_2 - \theta_1|\varepsilon} + e^{-(2\pi - |\theta_2 - \theta_1|)\varepsilon}}{2\varepsilon(1 - e^{-2\pi\varepsilon})} \right. \cos \theta \right. \left. h_1, \, \left. \cos \theta \right. h_2 \right\rangle_{L^2(I_+, r \cos \psi d\psi)},
$$

with $\varepsilon^2 = -|\cos \psi \partial_\psi|^2 + \mu^2 r^2 \cos^2 \psi$.

**Proof.** An approximation of the Dirac $\delta$-function is given by $\delta_k$, $k \in \mathbb{N}$, with

$$
\delta_k(\theta) = (2\pi)^{-1} \sum_{|\ell| \leq k} e^{i\ell \theta} \chi_{[0,2\pi)}(\theta), \quad \theta \in [0,2\pi),
$$

and $\chi_{[0,2\pi)}$ the characteristic function of the interval $[0,2\pi) \subset \mathbb{R}$. Use

$$
(-\Delta + \mu^2)^{-1} = (-\partial_\theta^2 + \varepsilon^2)^{-1} \frac{1}{1 + \varepsilon^2}
$$

and

$$
\sum_{\ell \in \mathbb{Z}} \int_0^{2\pi} d\theta \frac{e^{i\ell(0 - \theta_1)} (-\partial_\theta^2 + \varepsilon^2)^{-1} e^{i\ell'(0 - \theta_2)}}{1 + \varepsilon^2} = \frac{e^{i\ell(\theta_1 - \theta_2)}}{\ell^2 + \varepsilon^2}
$$

to show that for $0 \leq \theta_1, \theta_2 < 2\pi$

$$
\lim_{k \to \infty} \lim_{k' \to \infty} C(\delta_k(-, - \theta_1) \otimes h_1, \delta_k(-, - \theta_2) \otimes h_2)
$$

$$
= \lim_{k \to \infty} \lim_{k' \to \infty} \left\langle \delta_{k'}(-, - \theta_1) \otimes h_1, \cos \theta h_1, \left. \left. \cos \theta \right. h_2 \right\rangle_{L^2(I_+, \cos \psi d\psi)}
$$

$$
= \lim_{k \to \infty} \lim_{k' \to \infty} \left\langle \delta_{k'}(-, - \theta_1) \otimes h_1, \cos \theta h_1, \left. \left. \cos \theta \right. h_2 \right\rangle_{L^2(I_+, \cos \psi d\psi)}
$$

$$
= \lim_{k \to \infty} \left\langle \delta_k(-, - \theta_1) \otimes h_1, \cos \theta h_1, \left. \left. \cos \theta \right. h_2 \right\rangle_{L^2(I_+, \cos \psi d\psi)}
$$

$$
- \frac{e^{i\ell(\theta_1 - \theta_2)}}{\ell^2 + \varepsilon^2}.
$$

$\varepsilon^2$ is a differential operator, thus $\varepsilon^2$ acts locally and maps the subspaces

$$
\mathcal{D}_\pm \doteq \mathcal{D}(\varepsilon^2) \cap L^2(I_+, |\cos \psi|^2 d\psi)
$$

into $L^2(I_+, |\cos \psi|^2 d\psi)$, respectively. It therefore is consistent (see (5.4.5)) to define

$$
\varepsilon(h_+ + h_-) \doteq \sqrt{\varepsilon^2} \varepsilon \doteq \sqrt{\varepsilon^2} \varepsilon \doteq \sqrt{\varepsilon^2} \varepsilon , \quad h_+ = \mathcal{D}_+, \quad h_- = \mathcal{D}_-.
$$

$\varepsilon$ is $\varepsilon_{I_+} + \varepsilon_{I_-}$ is densely defined by (5.4.5), as $\mathcal{D}_+ \oplus \mathcal{D}_- = \mathcal{D}(\varepsilon^2)$. On the half-circle $I_+$, the operator $\varepsilon$ has, just like $\varepsilon^2$, purely a.c. spectrum on all of $\mathbb{R}^+$. Especially, $\varepsilon$ does not have a discrete eigenvalue at zero. Thus, despite the fact that $\varepsilon$ has no mass gap, one can apply the Poisson sum formula (see, e.g., [124])

$$
\frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} \frac{e^{i\ell \theta}}{\ell^2 + \varepsilon^2} = \frac{e^{-|\theta||\varepsilon|} + e^{-(2\pi - |\theta||\varepsilon|)}}{2|\varepsilon|(1 - e^{-2\pi|\varepsilon|})} \quad \text{for} \quad 0 \leq |\theta| < 2\pi
$$
to conclude that \([10.5.4]\) holds.

Note that for \( \vartheta = \pi \), we have

\[
\delta_\pi \otimes h = \delta \otimes P_1 \ast h,
\]

where \( P_1 \ast \) is the pull-back of the reflection at the \( x_0-x_1 \) plane, i.e.,

\[
(P_1 \ast h)(\psi) = h(\pi - \psi).
\]

**Remark 10.5.3.** Consider distributions of the form \([10.5.1]\). The closure of this set of distributions within the Sobolev space \( H^{-1}(S^2, \mathbb{R}) \) can be naturally identified with the Hilbert space \( H^{-1}(S^2, \mathbb{R}) \) introduced in Definition \([6.2.1]\), which itself is the completion of \( C^\infty(S^1) \) in the norm

\[
\| h \|_{\tilde{H}(S^1)}^2 = 2C(\delta \otimes \Pi, \delta \otimes h).
\]

We denote the scalar product in \( \tilde{H}(S^1) \) by

\[
\langle h_1, h_2 \rangle_{\tilde{H}(S^1)} \doteq \langle h_1, \frac{1}{r \cos \varphi} h_2 \rangle_{L^2(S^1, r \, d \varphi)}.
\]

**Corollary 10.5.4 (Time-zero Covariances).**

i.) Let \( h \in \tilde{H}(S^1) \). Then

\[
\| h \|_{\tilde{H}(S^1)} = r \langle \cos \varphi, \cos \varphi \rangle_{L^2(S^1, r \, d \varphi)}.
\]

ii.) The pseudo-differential operator \( \omega \) on \( L^2(S^1, r \, d \varphi) \) satisfies the operator identity

\[
\omega = |r \cos \varphi|^{-1} |\varepsilon| \langle \text{coth} \pi \varepsilon - \frac{P_1 \ast}{\sinh \pi \varepsilon} \rangle^{-1}.
\]

**Proof.** Write \( h = h_+ + h_- \), where the support of \( h_\pm \) is contained in \( I_\pm \), respectively. By Lemma \([10.5.1]\) and Lemma \([10.5.2]\)

\[
\langle h_+, \frac{1}{r \cos \varphi} h_+ \rangle_{L^2(S^1, r \, d \varphi)} = r \langle \cos \varphi, \cos \varphi \rangle_{L^2(S^1, r \, d \varphi)}.
\]

Now note that \( \langle h_+, \frac{1}{r \cos \varphi} h_- \rangle_{L^2(I, r \, d \varphi)} = \langle P_1 \ast h_-, \frac{1}{r \cos \varphi} P_1 \ast h_- \rangle_{L^2(I, r \, d \varphi)} \) and again apply Lemma \([10.5.1]\) and Lemma \([10.5.2]\) to find a similar expression. For the mixed term, use Eq. \([10.5.7]\) to write

\[
\langle h_+, \frac{1}{r \cos \varphi} h_- \rangle_{L^2(S^1, r \, d \varphi)} = C(\delta \otimes h_+, \delta \otimes h_-)
\]

\[
= C(\delta \otimes h_+, \delta_\pi \otimes P_1 \ast h_-) = C_\pi(h_+, P_1 \ast h_-)
\]

\[
= r \langle \cos \varphi, h_+, \frac{1}{r \cos \varphi} \cos \varphi \rangle_{L^2(I, r \, d \varphi)}.
\]

The term \( \langle h_+, \frac{1}{r \cos \varphi} h_+ \rangle \) yields a similar expression, with \( h_+ \) and \( h_- \) interchanged. We can now put the four terms together. The proof of Eq. \([10.5.8]\) is completed by noting that \( \varepsilon \) leaves the two subspaces \( L^2(I, r \, d \varphi) \) invariant. The latter implies, by polarization, the operator identity

\[
\omega^{-1} = |\varepsilon|^{-1} \langle \text{coth} \pi \varepsilon + \frac{P_1 \ast}{\sinh \pi \varepsilon} \rangle \, |r \cos \varphi|,
\]

which is equivalent to \([6.2.9]\).
The existence of Euclidean sharp-time fields now follows from (10.1.4) and Lemma 10.5.1.

**Proposition 10.5.5.** The time-zero fields

\[
(10.5.10) \quad \Phi(0, h) = \lim_{n \to \infty} \Phi(\delta_n(\cdot, \cdot) \otimes h), \quad h \in C_0^\infty(S^1),
\]
exist as elements of \(L^p(\Omega, \Sigma, d\Phi_C), 1 \leq p < \infty\).

### 10.6. Foliation of the Euclidean Field

Consider a foliation of the upper hemisphere in terms of half-circles

\[ R_1(\theta)I_+, \quad 0 < \theta < \pi. \]

The **Euclidean sharp-time fields**

\[
(10.6.1) \quad \Phi(\theta, h) = \lim_{k \to \infty} \Phi(\delta_k(\cdot, \cdot) \otimes h), \quad h \in D_R(I_+),
\]
belong to \(\bigcap_{1 \leq p < \infty} L^p(\Omega, \Sigma, d\Phi_C)\). Use (8.3.1) to show that the map

\[
(10.6.2) \quad S^1 \times C_0^\infty(I_+) \to \bigcap_{1 \leq p < \infty} L^p(\Omega, \Sigma, d\Phi_C), \quad (\theta, h) \mapsto \Phi(\theta, h)
\]
is continuous.

**Lemma 10.6.1.** The following identity holds on \(\bigcap_{1 \leq p < \infty} L^p(\Omega, \Sigma, d\Phi_C)\):

\[
(10.6.3) \quad \int_{S^1} r \cos \theta \, d\theta \, \Phi(\delta_k(\cdot, \cdot) \otimes f_0) = \Phi(f), \quad f_0 \equiv f(\theta, \cdot) \in C_0^\infty(I_+), \quad f \in C_0^\infty(S^2).
\]

**Proof.** Let \(f \in C_0^\infty(S^2)\) and consider the approximation of the Dirac \(\delta\)-function given in (10.5.5). For \(k \in \mathbb{N}\) fixed, the map

\[
S^1 \to \mathbb{H}^{-1}(S^2), \quad \theta \mapsto \delta_k(\cdot, \cdot) \otimes f_0
\]
is continuous. Since \(f \in C_0^\infty(S^2)\), the expression \(\|\delta_k(\cdot, \cdot) \otimes f_0\|_{\mathbb{H}^{-1}(S^2)}\) is bounded. Hence by (10.6.2) the map

\[
S^1 \to \bigcap_{1 \leq p < \infty} L^p(\Omega, \Sigma, d\Phi_C), \quad \theta \mapsto \Phi(\delta_k(\cdot, \cdot) \otimes f_0)
\]
is continuous and \(\|\Phi(\delta_k(\cdot, \cdot) \otimes f_0)\|_{L^p(\Omega, \Sigma, d\Phi_C)}\) is bounded. Therefore

\[
\int_0^{2\pi} r \cos \psi \, d\theta \, \Phi(\delta_k(\cdot, \cdot) \otimes f_0)
\]
is well defined as an element of \(\bigcap_{1 \leq p < \infty} L^p(\Omega, \Sigma, d\Phi_C)\). Moreover,

\[
\int_0^{2\pi} r \cos \psi \, d\theta \, \Phi(\delta_k(\cdot, \cdot) \otimes f_0) = \Phi\left(\int_0^{2\pi} r \cos \psi \, d\theta \, (\delta_k(\cdot, \cdot) \otimes f_0)\right) = \Phi(\delta_k * f),
\]
where the convolution product \(*\) acts only in the variable \(\theta\). Use (10.5.2) and

\[
\lim_{k \to \infty} \delta_k * f = \lim_{k \to \infty} \int_0^{2\pi} r d\theta \, r^{-1} \delta_k(\cdot, \cdot) f(\theta, \cdot) = f
\]
in $H^{-1}(S^2)$ to obtain from (10.1.6)

$$\lim_{k \to \infty} \int_0^{2\pi} r \cos \psi \, d\theta \, \Phi(\delta_k(-\theta) \otimes f_\theta) = \Phi(f) \quad \text{in} \quad \bigcap_{1 \leq p < \infty} L^p(\Omega, \Sigma, d\Phi_C).$$

Furthermore, it follows from (10.1.6) that

$$\lim_{k \to \infty} \sup_{\theta \in S^1} \| \Phi(\delta_k(-\theta) \otimes f_\theta) - \Phi(\theta, f_\theta) \|_{L^p(\Omega, \Sigma, d\Phi_C)} = 0$$

for $f \in C^\infty(S^2)$. Hence (10.6.3) follows. \qed
Non-Gaussian Measures

11.1. Wick ordering of random variables

Recall the normal ordering of Gaussian random variables: Let $(\mathcal{Q}, \mu)$ be a probability space and $X$ a real vector space equipped with a positive quadratic form $f \mapsto c(f, f)$, i.e., a covariance. Let $f \mapsto \Phi(f)$ be a $\mathbb{R}$-linear map from $X$ into the space of real measurable functions on $\mathcal{Q}$. Normal ordering $\Phi(f)_c$ with respect to a covariance $c$ is defined by

\begin{equation}
\Phi(f)_c = \frac{n!}{m!(n-2m)!} \Phi(f)_{n-2m} \left( -\frac{1}{2} c(f, f) \right)^m,
\end{equation}

where $\lfloor . \rfloor$ denotes the integer part.

Let, for $m \in \mathbb{N}$,

$$
\delta^{(2)}_m(\cdot, -\theta, -\psi) = \frac{1}{\pi} \sum_{\ell=0}^{m} \sum_{k=-\ell}^{-\ell} Y_{\ell,k}(\theta, \psi) Y_{\ell,k}(\cdot, \cdot), \quad \vec{x} \equiv \vec{x}(\theta, \psi) \in S^2,
$$

if $\vec{x} \neq (0, \pm \tau, 0)$. It follows from (D.0.2) that $\{\delta^{(2)}_m\}_{m \in \mathbb{N}}$ approximates the two-dimensional Dirac $\delta$-function $\delta^{(2)}_m$ for $m \to \infty$ unless $\vec{x} = (0, \pm \tau, 0)$. Note that $\delta^{(2)}$ is supported at the point $(0, 0, \tau) \in S^2$ and $\int_{S^2} d\Omega \delta^{(2)} = 1$.

**Theorem 11.1.1** (Ultraviolet renormalization). For $n \in \mathbb{N}$ and $f \in L^2(S^2, d\Omega)$ the following limit exists in $\bigcap_{1 \leq p < \infty} L^p(\mathcal{Q}, \Sigma, d\Phi_C)$:

$$
\lim_{m \to \infty} \int_{0}^{2\pi} r d\theta \int_{-\pi/2}^{\pi/2} r \cos \psi d\psi \, f(\theta, \psi) :\Phi(\delta^{(2)}_m(\cdot, -\theta, -\psi))_n :_{\mathcal{C}} .
$$

It is denoted by $\int_{S^2} d\Omega \, f(\theta, \psi) :\Phi(\theta, \psi)_n :_{\mathcal{C}}$.

**Proof.** Use the identification of $L^2(\mathcal{Q}, \Sigma, \mu)$ with $\Gamma(\mathcal{H}^{-1}(S^2))$ given in (10.4.1). Then Wick ordering with respect to $\mathcal{C}$ coincides with Wick ordering with respect to the Fock vacuum on $\Gamma(\mathcal{H}^{-1}(S^2))$ and

$$
:\Phi(g)_n :_{\mathcal{C}} = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} a^*(g)^j a(g)^{n-j}, \quad g \in \mathcal{H}^{-1}(S^2).
$$
In particular,
\[ :\Phi(\delta_m^{(2)})^n :_C = \frac{1}{2\pi r^{2n}} \sum_{j=0}^{n} \binom{n}{j} \sum_{\ell_i=0}^{m} \sum_{k_i=-\ell_i}^{\ell_i} \cdots \sum_{\ell_n=0}^{m} \sum_{k_n=-\ell_n}^{\ell_n} Y_{\ell_j,\ell_j}(\theta,\psi) \cdots Y_{\ell_j+k_{j+1},\ell_j}(\theta,\psi) \cdots Y_{\ell_n,\ell_n}(\theta,\psi) \]
(11.1.2)

Using \[ Y_{\ell,k}(\theta,\psi) = (-1)^k Y_{\ell,-k}(\theta,\psi) \], we find that (see, e.g., [176] or [45 Sect. 6] for a recent survey)
\[ p_m^{(n)}(f) \equiv \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} r \cos \psi \, d\psi \, f(\theta,\psi) \, :\Phi(\delta_m^{(2)})(-\theta, -\psi) :_C \]
is a linear combination of Wick monomials of the form
\[ \sum_{j=0}^{n} \binom{n}{j} \sum_{\ell_i=0}^{m} \sum_{k_i=-\ell_i}^{\ell_i} \cdots \sum_{\ell_n=0}^{m} \sum_{k_n=-\ell_n}^{\ell_n} (-1)^{\sum_{i=1}^{l_k} k_i} \]
(11.1.3)
with \[ a_{\ell_i,k_i}^{(n)} \equiv a^{(n)}(Y_{\ell_i,k_i}) \] and
\[ w^{(n)}(\ell_1,k_1,\ldots,\ell_j,k_j,\ell_{j+1},k_{j+1},\ldots,\ell_n,k_n) \]
\[ = \frac{1}{2\pi r^{2n}} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} r \cos \psi \, d\psi \, f(\theta,\psi) \sum_{i=1}^{n} Y_{\ell_i,k_i}(\theta,\psi) . \]

Next we apply the Wick monomials (11.1.3) to the Fock vacuum \[ \Omega_E \] in the Fock space over the one-particle space \([H^{-1}(S^2)]^2\); see Definition[4.0.19] Only the term with \( j = n \) contributes in the sum over \( j \). However, when removing the cut-off (i.e., for \( m \to \infty \)) there are still \( n \) infinite double sums. The ultraviolet divergencies, stemming from high (angular) momenta, are now visible from the sum
\[ \sum_{k_i=-\ell_i}^{\ell_i} (\ell_i + \frac{1}{2})^{-2} \]
over the Fourier coefficients, which goes like \( l_i^{-1} \) for large \( l_i \in \mathbb{N} \). However, we can use the bound [45 Lemma 6.1]
(11.1.4)
\[ \prod_{i=1}^{n} l_i^{-1} \leq \sum_{p=1}^{n} \left( \prod_{i \neq p} l_i^{-\frac{n}{n-1}} \right) \]
which follows from \( \left( \prod_{p=1}^{n} \lambda_p \right)^{1/n} \leq \sum_{p=1}^{n} \lambda_p \), applied to \( \lambda_p = \prod_{i \neq p} l_i^{-\frac{n}{n-1}} \). The remaining sum over the \( l_p \)'s can be estimated using the Plancherel theorem for the expansion into spherical harmonics which ensure that
(11.1.5)
\[ \sqrt{\sum_{l_p=0}^{\infty} \sum_{m_p=-l_p}^{l_p} |\tilde{f}_{l_p,m_p}|^2 = \|f\|_{L^2(S^2,d\Omega)}^2} \]
11.2. Sharp-time interactions

We conclude that $P_m^{(n)}(f)\Omega$ converges to a vector $P_\infty^{(n)}(f)\Omega$ in $\Gamma^{(n)}(H^{-1}(S^2))$, or equivalently that $P_m^{(n)}(f)$ converges to $P_\infty^{(n)}(f)$ in $L^2(Q, \Sigma, \mu)$. Since $P_m^{(n)}(f)\Omega$ is a finite particle vector, it follows from a standard argument (see, e.g., [175, Theorem 1.22] or [45, Lemma 5.12]) that

$$P_m^{(n)}(f) \to P_\infty^{(n)}(f) \in L^p(Q, \Sigma, \mu)$$

for all $1 \leq p < \infty$. □

If $P = P(\lambda)$ is a real valued polynomial, then

$$\int_{S^2} d\Omega \ f(\theta, \psi) : P(\Phi(\vec{x})) : C_0, \quad \vec{x} = \vec{x}(\theta, \psi) \in S^2,$$

is well defined, by linearity, for $f \in L^2(S^2, d\Omega)$. On subsets $K \subset S^2$ with non-empty interiors the interaction is defined by

$$V(K) = \int_K d\Omega : P(\Phi(\vec{x})) : C_0, \quad K \subset S^2.$$

$V(K)$ is a densely defined operator, but it is unbounded from below.

If, in addition, $P$ is bounded from below, then the function $e^{-tV(K)}$ is in $L^1$ for any $t > 0$ [176, Lemma 3.15]; see also [72][172]. In particular, one has

$$e^{-V(S^2)} \in L^1(\Omega, \Sigma, d\Phi_C).$$

Hence one can define the perturbed measure $d\mu_V$ on the sphere:

$$d\mu_V = \frac{e^{-V(S^2)} d\Phi_C}{\int_{S^2} e^{-V(S^2)} d\Phi_C}.$$

11.2. Sharp-time interactions

The results of this section will be used to define Feynman-Kac-Nelson kernels in Section 13.4.

**Lemma 11.2.1.** The following limit exists in $\bigcap_{1 \leq p < \infty} L^p(\Omega, \Sigma, d\Phi_C)$:

$$\lim_{k \to \infty} \int_{S^1} r d\psi \ h(\psi) : P(\Phi(0, \delta_k(\cos \chi, -\psi))) C_0, \quad h \in L^2(S^1, d\psi).$$

It is denoted by $\int_{S^1} r d\psi \ h(\psi) : P(\Phi(0, \psi)) C_0$.

**Proof.** See the proof of Theorem [11.1.1]. □

Thus for $\mathcal{P}$ a real valued polynomial, the expression

$$V_0(h) = \int_{S^1} r d\psi \ h(\vec{\psi}) : P(\Phi(0, \vec{\psi})) C_0, \quad h \in L^2(S^1, r d\psi),$$

is well defined, by linearity. The interaction

$$V(\theta) = V_0(\cos \chi, \chi_1),$$

is defined by rotating $V^{(0)}$ around the $x_0$-axis by an angle $\alpha$, see Equ. (13.4.1), Section 13.4.
with \( \chi_{I_1} \), the characteristic function of the interval \( I_+ \subset S^1 \), has two equivalent meanings: first of all (see Lemma [11.2.1]), it can be viewed as a \( \Sigma^{(0)} \)-measurable function

\[
V^{(0)} \in \bigcap_{1 \leq p < \infty} L^p(\Omega, \Sigma^{(0)}, d\Phi_C).
\]

Here \( \Sigma^{(0)} \) is the smallest sub-\( \sigma \)-algebra of \( \Sigma \) for which the functions \( \{ \Phi(0, h) \mid h \in D_T([I_+]) \} \) are measurable.

Secondly, \( V^{(0)} \) can be considered, applying (12.2.2), as a self-adjoint operator affiliated to the abelian von Neumann algebra \( \mathcal{U}(S^1) \) acting on the Hilbert space \( \mathcal{H} \). This second possibility will be discussed in Section 13.

### 11.3. Foliations of the interaction

The \((2\pi)\)-periodic one-parameter group \( \{0, 2\pi\} \ni \theta \mapsto R^{(\alpha)}(\theta)_* \) induces a representation

\[
(11.3.1) \quad \theta \mapsto U^{(\alpha)}(\theta)
\]

of \( \mathcal{U}(1) \) in terms of automorphisms of \( L^\infty(\Omega, \Sigma, d\Phi_C) \), which extends to a strongly continuous representation in terms of isometries of \( L^p(\Omega, \Sigma, d\Phi_C) \).

**Lemma 11.3.1.** For \( h \in L^2(I_+, r d\psi) \) and \( g \in C_0^\infty(S^1) \)

\[
(11.3.2) \quad \int_{S^1} r d\theta \; g(\theta)|U^{(0)}(\theta)V_0|\cos(h(\theta)) = \int_{S^2} d\Omega \; g(\theta) h(\theta) :\mathcal{P}(\Phi(\theta, \psi)) :^C
\]

as functions on \( \Omega \).

**Proof.** Consider path-space coordinates and follow an argument given in [68]; let \( F \in L^p(\Omega, \Sigma, d\Phi_C) \) for some \( 1 \leq p < \infty \) and \( g \in C_0^\infty(S^1) \). Then

\[
\int_{S^1} r d\theta \; g(\theta)|U^{(0)}(\theta)F
\]

belongs to \( L^p(\Omega, \Sigma, d\Phi_C) \). Together with Lemma [11.2.1] this implies that the functions given in (11.3.2) are in \( L^p(\Omega, \Sigma, d\Phi_C) \). Next prove that they are identical: by linearity, one may assume that \( \mathcal{P}(\lambda) = \lambda^n \). Use Lemma [11.2.1] and the identity (11.1.1) to derive

\[
\int_{S^2} d\Omega \; g(\theta) h(\psi) :\mathcal{P}(\Phi(\theta, \psi)) :^C = \lim_{(k, k') \to \infty} F(k, k') \text{ in } L^p(\Omega, \Sigma, d\Phi_C),
\]

where

\[
F(k, k') = \sum_{m=0}^{[m/2]} \frac{(-1)^m C(\delta_{k', k}^{(2)}, \delta_{k, k}^{(2)} m)}{m!(n-2m)!} \times \int_{S^2} d\Omega \; g(\theta) h(\psi) \Phi(\delta_k(.-\theta) \otimes \delta_k'(.-\psi))^{n-2m}
\]

and \( \delta_{k, k'}^{(2)} = \delta_k \otimes \delta_{k'} \) provides an approximation of the Dirac \( \delta \)-function \( \delta^{(2)} \) on \( S^2 \). According to Proposition [10.5.1]

\[
\lim_{k \to \infty} C(\delta_{k, k}^{(2)}, \delta_{k', k'}^{(2)}) = C_0(\delta_{k'}, \delta_{k'}).
\]
The definition of sharp-time fields in [10.5.10] implies that

$$\lim_{k, k' \to \infty} F(k, k') = \int_{S^1} r \, d\theta \, g(\theta) V_{k'}(\theta, \cos \psi, h) \ \text{in} \ L^p(\Omega, \Sigma, d\Phi_C),$$

where

$$V_{k'}(\theta, \cos \psi, h) = \sum_{m=0}^{[n/2]} \frac{n!}{m!(n-2m)!} \left(-\frac{1}{2} C_0(\delta_{k'}, \delta_{k'})\right)^m \times \int_{-\pi/2}^{\pi/2} r \, \cos \psi \, d\psi \, h(\psi) \Phi(\theta, \delta_{k'}(\theta - \psi))^{n-2m},$$

Note that $V_{k'}(\theta, \cos \psi, h) = U(0) | V_{k'}(0, \cos \psi, h)$. By Lemma [11.2.1]

$$\lim_{k' \to \infty} V_{k'}(0, \cos \psi, h) = \int_{-\pi/2}^{\pi/2} r \, \cos \psi \, d\psi \, h(\psi) \Phi(0, \psi) : c_0$$

in $L^p(\Omega, \Sigma, d\Phi_C)$ and hence

$$\lim_{k' \to \infty} \int_{S^1} r d\theta g(\theta) V_{k'}(\theta, \cos \psi, h) = \int_{S^1} r d\theta g(\theta) U(0)(\theta) V_0(\cos \psi, h)$$

in $L^p(\Omega, \Sigma, d\Phi_C)$. Apply Lemma [11.3.2] below with $E = L^p(\Omega, \Sigma, d\Phi_C)$ to obtain the identity in (11.3.2).

For completeness we recall a simple technical lemma from [68].

**Lemma 11.3.2.** Let $F: \mathbb{R}^2 \to E$ be a map with values in a metric space $E$.

i.) Assume that

$$\lim_{k, k' \to \infty} F(k, k') = F_\infty \text{ exists},$$

$$\lim_{k' \to \infty} F(k, k') = G(k) \text{ exists } \forall k \in \mathbb{N},$$

$$\lim_{k \to \infty} G(k) = G_\infty \text{ exists}.$$

Then $F_\infty = G_\infty$.

ii.) Assume that

$$\lim_{k' \to \infty} F(k, k') = G(k) \text{ exists},$$

$$\lim_{k \to \infty} G(k) = G_\infty \text{ exists},$$

$$\lim_{k \to \infty} F(k, k') = H(k') \text{ exists and the convergence is uniform w.r.t. } k'.$$

Then $\lim_{k' \to \infty} H(k') = G_\infty$. 
Part 6

The Osterwalder-Schrader Reconstruction
CHAPTER 12

The Reconstruction of Free Quantum Fields

The Markov property implies that the time-zero quantum fields acting on the vacuum vector generate the physical Hilbert space. In case it holds, Nelson’s reconstruction theorem (see [149][150][151][152]) applies, and the more sophisticated reconstruction theorem of Osterwalder and Schrader [158][159] is not necessary.

12.1. Reflection positivity

The time reflection diffeomorphism $T$ (see (9.0.3)) induces a map $T^*$ on $C^\infty(S^2)$:

$$T^*h = h \circ T^{-1}, \quad h \in C^\infty(S^2).$$

$T^*$ extends to a unitary operator on $H_{-1}C^\infty(S^2)$. Use (10.4.1) to define a unitary operator $\Theta$ on $L^2(Q, \Sigma, d\Phi_C)$ corresponding to $\Gamma(T_\ast)$. For $f_1, \ldots, f_n \in H^{-1}(S^2)$,

$$\Theta(\Phi(f_1) \cdots \Phi(f_n)) = \Phi(T_\ast f_1) \cdots \Phi(T_\ast f_n).$$

$\Theta$ induces a measure preserving automorphism of $L^\infty(Q, \Sigma, d\Phi_C)$, which extends to an isometry of $L^p(Q, \Sigma, d\Phi_C)$ for $1 \leq p < \infty$.

**Theorem 12.1.1** (Dimock [50], Theorem 2, p. 248). Let $F \in L^2(Q, \Sigma, d\Phi_C)$. Then

1. $\Theta(F) \in L^2(Q, \Sigma_{\overrightarrow{\Phi}} d\Phi_C)$;
2. $\int d\Phi_C \Theta(F) = 0$ (reflection positivity);
3. $\Theta E_{S^1} = E_{S^1} \Theta$.

**Proof.** Properties i.) and iii.) follow directly from the definitions; ii.) is a direct consequence (see [50][123]) of the Markov property (Theorem [10.4.2 iv.])). □

We note that an alternative proof of reflection positivity for Riemannian manifolds with a suitable symmetry (the sphere being in the class considered) was given in [85].

---

1Consider the Hilbert space $L^2(Q, \Sigma, d\Phi_C)$ and let $M$ be the von Neumann algebra of all multiplications by bounded measurable functions, acting on $L^2(Q, \Sigma, d\Phi_C)$. The Lebesgue measure lifts to a countably additive probability measure $\mu$ on the projections of $M$. A $*$-automorphism $\alpha$ of a von Neumann algebra $M$ is said to preserve the measure $\mu$ if $\mu \circ \alpha = \mu$ on the projections of $M$. See, e.g., [8].
12.2. The reconstruction of the Hilbert space

Define \( N \subset L^2(Q, \Sigma_S, d\Phi_C) \) as the kernel of the positive quadratic form

\[
\langle F, G \rangle_{\text{os}} = \int_Q d\Phi_C \overline{\Theta(F)} G, \quad F, G \in L^2(Q, \Sigma_S, d\Phi_C).
\]

In other words,

\[
N = \left\{ F \in L^2(Q, \Sigma_S, d\Phi_C) \mid \langle G, F \rangle_{\text{os}} = 0 \quad \forall G \in L^2(Q, \Sigma_S, d\Phi_C) \right\}.
\]

Complete the quotient space with respect to the positive definite scalar product (12.2.1):

\[
H = \text{completion of } L^2(Q, \Sigma_S, d\Phi_C)/N.
\]

Let \( V \) be the canonical projection \( V: L^2(Q, \Sigma_S, d\Phi_C) \to L^2(Q, \Sigma_S, d\Phi_C)/N \).

There is a distinguished unit vector

\[
\Omega = V(1) \in H.
\]

1 \( \in L^2(Q, \Sigma_S, d\Phi_C) \) is the constant function equal to 1 on \( Q \).

If \( A \in L^\infty(Q, \Sigma_S, d\Phi_C) \), multiplication by \( A \) preserves \( N \), since \( A \) is by assumption \( \Sigma_S \)-measurable. Define a bounded operator \( A^{\text{os}} \in \mathcal{B}(H) \) by

\[
A^{\text{os}} V(F) = V(AF), \quad F \in L^2(Q, \Sigma_S, d\Phi_C),
\]

and denote by \( \mathcal{U}(S^1) \subset \mathcal{B}(H) \) the abelian von Neumann algebra

\[
\mathcal{U}(S^1) = \{ A^{\text{os}} \in \mathcal{B}(H) \mid A \in L^\infty(Q, \Sigma_S, d\Phi_C) \}.
\]

**Lemma 12.2.1** (Klein & Landau [123], Lemma 8.1). The map \( A \mapsto A^{\text{os}} \) induces a weakly continuous \( * \)-isomorphism between \( L^\infty(Q, \Sigma_S, d\Phi_C) \) and \( \mathcal{U}(S^1) \).

**Lemma 12.2.2** (Klein & Landau [123], Theorem 11.2; see also [67]).

\[
\mathcal{U}(S^1) \Omega = H.
\]

In other words, application of bounded functions of time-zero fields to the vector \( \Omega \) generates a dense set in \( H \).

**Proof.** Let \( \Phi_F \) be the Fock space field operator and \( \Omega_F \) the vacuum vector in \( \Gamma(\hat{\mathfrak{h}}(S^1)) \). The decomposition of \( H^{-1}_C(S^2) \) provided by Lemma [10.3.1] allows us to restrict the map \( e^{i\Phi(h)} \) to the Euclidean time-zero fields,

\[
e^{i\Phi_{\hat{\mathfrak{h}}}(h)} \Omega_F, \quad h \in \hat{\mathfrak{h}}(S^1),
\]

thereby demonstrating that \( H \cong \Gamma(\hat{\mathfrak{h}}(S^1)) \) and that \( \mathcal{U}(S^1) \) can be identified with the von Neumann algebra generated by the Fock space field operators \( \Phi_F(h) \), with \( h \in \hat{\mathfrak{h}}(S^1, \mathbb{R}) \), defined in terms of creation and annihilation operators (see, e.g., [24]). This establishes the result. \( \square \)
The conditional projection \textup{(12.2.4)} identifies the time-zero fields \(\Phi(0,h) \in L^2(\Omega, \Sigma_{\Theta_0}, d\Phi_C)\) with unbounded operators \(\Phi^0(0,h)\) affiliated to the abelian algebra \(\mathcal{U}(S^1)\): recall
\[
\Phi(0,h) \in \bigcap_{1 \leq p < \infty} L^p(\Omega, \Sigma_{\Theta_0}, d\Phi_C), \quad h \in \mathbb{C}_R(\mathbb{S}^1),
\]
and approximate \(\Phi(0,h)\) by a sequence of \(L^\infty(\Omega, \Sigma_{\Theta_0}, d\Phi_C)\)-functions
\[
\Phi_n(0,h) = 1_{[n-n,n]}(h)\Phi(0,h), \quad n \in \mathbb{N}.
\]
\(1_{[n-n,n]}(h)\) is the characteristic function of the set \(\{q \in \mathbb{Q} : |\Phi(0,h)(q)| \leq n\}\). According to \textup{(12.2.4)}
\[
\Phi^0_n(0,h) \in \mathcal{U}(S^1). \tag{12.2.4}
\]
It follows that \(\Phi^0(0,h)\) is affiliated to \(\mathcal{U}(S^1)\).

\textbf{Proposition 12.2.3.} Consider the map \textup{(12.2.4)}. It follows that
\[
e^{it\alpha\Theta_0}A^0 \Omega \equiv \mathcal{V}(U(\mathbb{R}_0(\alpha))A), \quad A \in L^\infty(\Omega, \Sigma_{\Theta_0}, d\Phi_C),
\]
extends to a strongly continuous unitary representation of the rotation group \(\mathcal{U}(1)\) on the Hilbert space \(\mathcal{H}\).

\textbf{Proof.} The abelian algebra \(\mathcal{U}(S^1)\) and the vector \(\Omega\) are invariant w.r.t. rotations around the \(x_0\)-axis. Use Lemma \textup{12.2.2} to extend \textup{(12.2.5)} to a unitary representation of \(\mathcal{U}(1)\) on \(\mathcal{H}\). \(\square\)

\section{12.3. Generalised path spaces}

Recall the following notions from \textup{126} Definition 1.3, p. 47:

\textbf{Definition 12.3.1.} A generalised pathspace \((\Omega, \Sigma, \Sigma_0, \mathcal{U}(t), \Theta, \mu)\) consists of

i.) a probability space \((\Omega, \Sigma, \mu)\);

ii.) a distinguished sub \(\sigma\)-algebra \(\Sigma_0 \subset \Sigma\);

iii.) a one-parameter group \(t \mapsto \mathcal{U}(t), t \in \mathbb{R}\), of measure preserving automorphisms of \(L^\infty(\Omega, \Sigma, \mu)\), strongly continuous in measure, such that
\[
\Sigma = \bigvee_{t \in \mathbb{R}} \mathcal{U}(t)\Sigma_0;
\]

iv.) a measure preserving automorphism \(\Theta\) of \(L^\infty(\Omega, \Sigma, \mu)\) such that \(\Theta^2 = 1\),
\[
\Theta \mathcal{U}(t) = \mathcal{U}(-t)\Theta, \quad t \in \mathbb{R},
\]
and \(\Theta E_0 = E_0\Theta\), where \(E_0\) is the conditional expectation with respect to \(\Sigma_0\).

The generalised path space \((\Omega, \Sigma, \Sigma_0, \mathcal{U}(t), \Theta, \mu)\) is said to be \emph{supported} by the probability space \((\Omega, \Sigma, \mu)\).

The properties iii.) and iv.) imply that \(\mathcal{U}(t)\) extends to a strongly continuous group of isometries of \(L^p(\Omega, \Sigma, \mu)\) and \(\Theta\) extends to an isometry of \(L^p(\Omega, \Sigma, \mu)\) for \(1 \leq p < \infty\) \textup{123}. Two generalised path spaces \((\Omega, \Sigma, \Sigma_0, \mathcal{U}_i(t), \Theta, \mu_i)\), \(i = 1, 2\), are equivalent, if
\[
\int_{\mathcal{Q}} d\mu_1 \mathcal{U}_1(t_1)F_1 \cdots \mathcal{U}_1(t_n)F_n = \int_{\mathcal{Q}} d\mu_2 \mathcal{U}_2(t_1)F_1 \cdots \mathcal{U}_2(t_n)F_n
\]
for all \( t_1, \ldots, t_n \) and \( F_1, \ldots, F_n \in C_\mathbb{R}(\Omega) \), where \( C_\mathbb{R}(\Omega) \) is the space of real valued continuous functions on \( \Omega \). By convention,

\[
U(t_1)F_1 U(t_2) F_2 = U(t_1) \left( F_1 \left( U(t_2) F_2 \right) \right),
\]

which needs to be distinguished from \( U(t_1) | F_1 \rangle | U(t_2) F_2 \rangle \).

**Definition 12.3.2.** For \( I \subset \mathbb{R} \), denote by \( E^I \) the conditional expectation with respect to the \( \sigma \)-algebra \( \bigvee_{t \in I} \Sigma_t \), where \( \Sigma_t = U(t) \Sigma_0 \). The generalised path space \( R \subset E^I \), where \( \Sigma = \Theta \Sigma_0 \), satisfies the two-sided Markov property for semi-circles, if

- \( i. \) is periodic, if \( U(2\pi) = \mathbb{1} \);
- \( ii. \) is OS-positive, if \( E^{[0,\pi]} \Theta E^{[0,\pi]} \geq 0 \) as an operator on \( L^2(\Omega, \Sigma, \mu) \);
- \( iii. \) satisfies the two-sided Markov property for semi-circles, if

\[
E^{[0,\pi]} \Theta E^{[0,\pi]} = E^{[0,\pi]}.
\]

### 12.4. Path-spaces on the sphere

Recall the definition of the unitary group \( \theta \mapsto U^{(\alpha)}(\theta) \) from Equ. (11.3.1), where \( \alpha = 0, 2 \pi \).

**Lemma 12.4.1.** Let \( \Sigma \) be the Borel \( \sigma \)-algebra on \( \Omega = D^e_\mathbb{R}(S^2) \). Then

\[
U^{(\alpha)}(\theta) \Theta = \Theta U^{(\alpha)}(-\theta)
\]

and, for \( h \in D^e_\mathbb{R}(I_+) \),

\[
U^{(0)}(\theta) \Phi(0, h) = \Phi(\theta, h).
\]

**Remark 12.4.2.** Results similar to (12.4.1) hold for time-zero fields supported on arbitrary half-circles \( I_\alpha \), \( \alpha \in [0, 2\pi) \), with respect to the appropriate rotations \( \theta \mapsto R^{(\alpha)}(\theta) \).

Let \( \Sigma^{(\alpha)} \) be the smallest sub-\( \sigma \)-algebra of \( \Sigma \) for which the functions \( \{ \Phi(0, h) | h \in D^e_\mathbb{R}(I_\alpha) \} \) are measurable.

**Proposition 12.4.3.** For each \( \alpha \in [0, 2\pi) \), \( (\Omega, \Sigma, \Sigma^{(\alpha)}, U^{(\alpha)} \cdot , \cdot \Theta, d\Phi_C) \) is a \((2\pi)\)-periodic, OS-positive generalised path space (in the sense of Definition 12.3.1), which satisfies the two-sided Markov property for semi-circles.

**Proof.** Use Lemma 10.6.1 to deduce that for fixed \( \Sigma = \bigvee_{\theta \in \mathbb{S}^1} U^{(\alpha)}(\theta) \Sigma^{(\alpha)} \).

The two-sided Markov property for semi-circles follows from the Markov property, Theorem 10.4.2(iv.).

### 12.5. Local symmetric semigroups

The rotations \( \theta \mapsto R^{(\alpha)}(\theta) \) do not preserve the (closed) upper hemisphere \( \mathbb{S}^+ \). In fact, the map

\[
\mathcal{V}(F) \mapsto \mathcal{V}(U^{(\alpha)}(\theta) F),
\]

is only defined if both \( F \) and \( U^{(\alpha)}(\theta) F \) are in \( L^2(\Omega, \Sigma^{(\alpha)}, d\Phi_C) \). The domain problems which arise, if one tries to associate self-adjoint operators

\[
P^{(\alpha)}(\theta) : \mathcal{D} \to \mathcal{H}, \quad \mathcal{D} \subset \mathcal{H},
\]

where \( \mathcal{D} \) is a domain, and the initial conditions are satisfied, are discussed in Section 12.5.
to (12.5.1), are addressed by the theory of local symmetric semigroups developed by Fröhlich [61] and, independently, by Klein & Landau [123] [124]:

**Definition 12.5.1.** A pair \((P(\theta), \mathcal{D}_0)\) forms a local symmetric semigroup on a Hilbert space \(\mathcal{H}\), if

i.) for each \(\theta, 0 \leq \theta \leq \pi\), fixed, the set \(\mathcal{D}_0\) is a linear subset of \(\mathcal{H}\). The union

\[
\mathcal{D} = \bigcup_{0 < \theta \leq \pi} \mathcal{D}_0
\]

is dense in \(\mathcal{H}\) and \(\mathcal{D}_0 \supset \mathcal{D}_{\theta'}\) if \(\theta \leq \theta'\);

ii.) for each \(\theta, 0 \leq \theta \leq \pi\), \(P(\theta)\) is a linear operator on \(\mathcal{H}\) with domain \(\mathcal{D}_0\) and

\[
P(\theta') \mathcal{D}_0 \subset \mathcal{D}_{\theta - \theta'} \quad \text{for} \quad 0 \leq \theta' < \theta \leq \pi;
\]

iii.) \(P(0) = 1\), and the semi-group property

\[
P(\theta)P(\theta') = P(\theta + \theta')
\]

holds on \(\mathcal{D}_{\theta + \theta'}\), for \(\theta, \theta', 0 \leq \theta' \leq [0, \pi]\);

iv.) \(P(\theta)\) is symmetric, i.e.,

\[
(u, P(\theta)v)_{\mathcal{H}} = (P(\theta)u, v)_{\mathcal{H}} \quad \forall u, v \in \mathcal{D}_0, \quad 0 \leq \theta \leq \pi;
\]

v.) the map \(\theta \mapsto P(\theta)\) is weakly continuous, i.e., if \(u \in \mathcal{D}_{\theta'}\), \(0 \leq \theta' \leq \pi\), then

\[
\theta \mapsto (u, P(\theta)v)_{\mathcal{H}}
\]

is a continuous function for \(0 < \theta < \theta'\).

It is remarkable that a local symmetric semi-group has a unique self-adjoint generator:

**Theorem 12.5.2 (Fröhlich [61], Klein & Landau [124]).** Let \((P(\theta), \mathcal{D}_0)\) be a local symmetric semigroup, acting on a Hilbert space \(\mathcal{H}\). Then there exists a unique self-adjoint operator \(L\), the generator of the local symmetric semigroup \((P(\theta), \mathcal{D}_0)\) on \(\mathcal{H}\), such that

\[
P(\theta')\psi = e^{-\theta'L}\psi, \quad \psi \in \mathcal{D}_0, \quad 0 \leq \theta' \leq \theta.
\]

Return to (12.5.1). For \(0 \leq \theta \leq \pi\) fixed, set

\[
\mathcal{M}_0^{(\alpha)} = L^2\left(\Omega, \bigvee_{\theta' \in [0, \pi - \theta]} U^{(\alpha)}(\theta')\Sigma^{(\alpha)}, d\Phi_C, \right), \quad 0 \leq \theta \leq \pi.
\]

\(\mathcal{M}_0^{(\alpha)}\) is the set of all \(F \in L^2(\Omega, \Sigma^{(\alpha)}_C, d\Phi_C)\) for which \(U^{(\alpha)}(\theta)F \in L^2(\Omega, \Sigma^{(\alpha)}_C, d\Phi_C)\).

**Definition 12.5.3.** Set \(\mathcal{D}_0^{(\alpha)} = \mathcal{V}(\mathcal{M}_0^{(\alpha)})\). Define, for \(0 \leq \theta \leq \theta'\),

\[
P^{(\alpha)}(\theta') : \mathcal{D}_0^{(\alpha)} \rightarrow \mathcal{H}
\]

\[
\forall F \mapsto \mathcal{V}(U^{(\alpha)}(\theta')F), \quad F \in \mathcal{M}_0^{(\alpha)}.
\]

**Proposition 12.5.4.** \((P^{(\alpha)}(\theta), \mathcal{D}_0^{(\alpha)})\) is a local symmetric semigroup. Its generator \(L^{(\alpha)}\) satisfies

\[
P^{(\alpha)}(\theta')\psi = e^{-\theta'L^{(\alpha)}}\psi, \quad \psi \in \mathcal{D}_0^{(\alpha)}, \quad 0 \leq \theta' \leq \theta.
\]

**Proof.** Verify the conditions i.)–v.) of Definition 12.5.1. □
For the free dynamics, we can provide an explicit formula:

**Proposition 12.5.5.** Identifying \( \mathcal{H} \) with \( \Gamma(\hat{h}(\mathbb{S}^1)) \) the generator of the boost can be identified with

\[
L^{(\alpha)} = d\Gamma(\omega_r \cos \phi + \alpha).
\]

Moreover, the spectrum \( \text{Sp}(L^{(\alpha)}|_{1_\alpha}) \geq 0 \) and \( \text{Sp}(L^{(\alpha)}|_{1_\alpha + \pi}) \leq 0 \).

**Remark 12.5.6.** Note that for \( f, g \in L^2(\mathbb{S}^1) \)

\[
\langle f, \omega \cos \phi + \alpha g \rangle_{\hat{h}(\mathbb{S}^1)} = \frac{1}{2} \langle f, \cos \phi + \alpha g \rangle_{L^2(\mathbb{S}^1, rd\phi)} = \langle \omega \cos \phi + \alpha f, g \rangle_{\hat{h}(\mathbb{S}^1)}.
\]

This shows that \( \omega_r \cos \phi + \alpha \) is symmetric. In fact, it is self-adjoint by construction.

**Proof.** We proceed in several steps.

i.) First, we show that for \( f \in \mathcal{D}_R(\mathbb{S}^+) \)

\[
\int_0^\pi d\Phi C \Theta(\Phi(f)) \Phi(f) = C(T_\ast f, f)
\]

is equal to

\[
r \left\| \int_0^\pi r d\theta e^{-\theta} \frac{(1 + \rho_{2\pi}) + \rho_{2\pi}^2 (P_1)}{\sqrt{2|\theta|}} \cos \phi | f_\theta \rangle \right\|_{L^2(1_+, \cos \phi^{-1} rd\phi)}^2.
\]

with

\[
(12.5.3) \quad \rho_{2\pi} = \frac{\epsilon^{-2\pi|\epsilon|}}{1 - \epsilon^{-2\pi|\epsilon|}}, \quad 1 + \rho_{2\pi} = \frac{1}{1 - \epsilon^{-2\pi|\epsilon|}}.
\]

To show this, recall that according to Lemma 10.6.1

\[
\int_0^\pi d\theta r \cos \phi \Phi(\theta, f_\theta) = \Phi(f), \quad f_\theta = f(\theta, \cdot) \in \mathcal{D}_R(1_+).
\]

Lemma 10.5.2 implies that \( C(T_\ast f, f) \) equals

\[
\int_0^\pi r d\theta_1 \int_0^\pi r d\theta_2 \times
\]

\[
\times r \left\langle \cos \phi, f_{\theta_1}, e^{-2\pi(2\pi + \theta_1 + \epsilon)} \cos \phi, f_{\theta_2} \right\rangle_{L^2(1_+, \cos \phi^{-1} rd\phi)}.
\]

Using (12.5.3) we find that \( C(T_\ast f, f) \) equals

\[
\int_0^\pi r d\theta_1 \int_0^\pi r d\theta_2 \left\langle e^{-\theta_1 \epsilon} \cos \phi, f_{\theta_1}, (1 + \rho_{2\pi}) e^{-\theta_2 \epsilon} \cos \phi, f_{\theta_2} \right\rangle_{L^2(1_+, \cos \phi^{-1} rd\phi)}
\]

\[+ \int_0^\pi r d\theta_1 \int_0^\pi r d\theta_2 \left\langle e^{-\theta_1 \epsilon} + \cos \phi, f_{\theta_1}, \rho_{2\pi} e^{-\theta_2 \epsilon} + \cos \phi, f_{\theta_2} \right\rangle_{L^2(1_+, \cos \phi^{-1} rd\phi)}.
\]

This formula is symmetric up to a reflection. Note that \( \pi - \theta \) is the angle measured starting from \( 1_- \). The second term in this sum equals

\[
\int_0^\pi r d\theta_1 \int_0^\pi r d\theta_2 \left\langle e^{-\theta_1 \epsilon} - (P_1), \cos \phi, f_{\theta_1} \right\rangle_{L^2(1_-, \cos \phi^{-1} rd\phi)}
\]

\[
\times \frac{\rho_{2\pi}}{2|\epsilon|} e^{-\theta_2 \epsilon} - (P_1) \cos \phi, f_{\theta_2} \right\rangle_{L^2(1_-, \cos \phi^{-1} rd\phi)}.
\]
Consequently,

\[ C(T_s f, f) = r \left\| \int_0^\pi r d\theta \ e^{-\theta \epsilon \left( \frac{1}{2} + \rho_{2\pi} \frac{1}{2} + \rho_{2\pi}^2 (P_1)_s \right) \cos \psi} f_\theta \right\|_{L^2(S^1, \frac{rd\Phi}{|\cos \psi|})}^2. \]

ii.) Next, we show that

\[ C(T_s f, f) = r \left\| \int_0^\pi r d\theta \ e^{-2\theta \cos \psi} f_\theta \right\|_{\tilde{h}(S^1)}^2. \]

We first note that Corollary 10.5.4.i.) implies that

\[ \|f_\theta\|_{\tilde{h}(S^1)}^2 = r \left\| \frac{(1 + \rho_{2\pi}) \frac{1}{2} + \rho_{2\pi}^2 (P_1)_s \cos \psi}{\sqrt{2|\epsilon|}} \right\|_{L^2(S^1, \frac{rd\Phi}{|\cos \psi|})}^2. \]

The map \( u^{-1} : \tilde{h}(S^1) \to L^2(S^1, \frac{rd\Phi}{|\cos \psi|}) \) given by

\[ u^{-1} = -\sqrt{r} \frac{(1 + \rho_{2\pi}) \frac{1}{2} + \rho_{2\pi}^2 (P_1)_s \cos \psi}{\sqrt{2|\epsilon|}}. \]

is unitary with inverse

\[ u = \frac{1}{\sqrt{r|\cos \psi|}} \sqrt{2|\epsilon|} (\rho_{2\pi}^2 (P_1)_s - (1 + \rho_{2\pi}) \frac{1}{2}). \]

The generator of the boost is \( u \circ \epsilon \circ u^{-1} \). Using \( \{P_1\}_s \epsilon = -\epsilon (P_1)_s \), we can now compute

\[ u \circ \epsilon \circ u^{-1} = |\cos \psi|^{-1} \left( (1 + \rho_{2\pi}) \frac{1}{2} - \rho_{2\pi}^2 (P_1)_s \epsilon (1 + \rho_{2\pi}) \frac{1}{2} + \rho_{2\pi}^2 (P_1)_s \right) |\cos \psi| \]

\[ = |\cos \psi|^{-1} \epsilon \left( (1 + 2\rho_{2\pi}) + 2\rho_{2\pi}^2 (1 + \rho_{2\pi}) \frac{1}{2} (P_1)_s \right) |\cos \psi| \]

\[ = |\cos \psi|^{-1} \epsilon \left( \coth \pi|\epsilon| + \frac{\mp \rho_{2\pi}}{\mp 2\pi|\epsilon|} \right) |\cos \psi| \]

\[ = \omega \epsilon \cos \psi. \]

Thus

\[ \int_Q d\Phi C \Theta(\Phi(f)) \Phi(f) = r \left\| \int_0^\pi r d\theta \ e^{-\theta \omega \cos \psi} f_\theta \right\|_{\tilde{h}(S^1)}^2. \]

This identity verifies reflection positivity, and it also verifies that the generator of the boost on \( \tilde{h}(S^1) \) is \( \omega \epsilon \cos \psi. \)

Note that the map \( u^{-1} \) was defined in Section 12.5.4 as a map from \( \tilde{h}(S^1) \) to \( \tilde{h}(S^1) \), instead of to \( L^2(S^1, \frac{rd\Phi}{|\cos \psi|}) \).
iii.) Finally, second quantization yields
\[ L^{(\alpha)} | I_a \rangle = d\Gamma(\chi_{I_a \omega \cos \phi + \alpha} \chi_{I_a}) , \]
where \( \chi_{I_a} \) is the characteristic function of the half-circle \( I_a \). Since \( \omega > 0 \), the spectral properties now follow from the sign the cosine function takes on the circle:
\[ \langle h, \chi_{I_a \omega \cos \phi + \alpha} \chi_{I_a} | h \rangle_{\hat{h}(S^1)} = \langle h, r \cos \phi + \alpha \chi_{I_a} | h \rangle_{L^2(I_a, rd\phi)} \geq 0. \]
A similar result holds for \( I_{\alpha + \pi} \).

We complement this result with an explicit formula for the generator of rotations:

**PROPOSITION 12.5.7.** Identifying \( \mathcal{K} \) with \( \Gamma(\hat{h}(S^1)) \) the generator of the rotations \( R_0 \) can be identified with
\[ K_0 = d\Gamma(-i\partial \psi). \]

**PROOF.** By definition (see (13.1.7)) we have
\[ e^{i\alpha K_0} e^{i\Phi^m(0, h)} \Omega = e^{i\Phi^m(0, R_0(\alpha), h)} \Omega, \quad h \in \hat{h}(S^1). \]
Thus \( K_0 = d\Gamma(-i\partial \psi). \)

Finally, recall that according to Proposition 7.4.3 the rotations
\[ (\hat{u}(R_0(\alpha))h)(\psi) = h(\psi - \alpha), \quad \alpha \in [0, 2\pi), \quad h \in \hat{h}(S^1), \]
and the boosts
\[ \hat{u}(\Lambda_1(t)) = e^{it\omega \cos \phi}, \quad t \in \mathbb{R}, \]
generate a unitary representation of \( SO_0(1, 2) \) on \( \hat{h}(S^1) \).

**12.6. Tomita-Takesaki modular theory**

The abelian algebra
\[ \mathcal{U} = \{ A^{os} \in \mathcal{B}(\mathcal{H}) \mid A \in L^\infty(\Omega, \Sigma_{I_a}, d\Phi_C) \} \]

together with the group of unitary operators \( \{ e^{-itL^{(\alpha)}} \mid t \in \mathbb{R} \} \) generate the nonabelian von Neumann algebra
\[ \mathcal{R}(I_a) = \bigvee_{t \in \mathbb{R}} \{ e^{-itL^{(\alpha)}} \mathcal{U}(I_a) e^{itL^{(\alpha)}} \}. \]

The vector \( \Omega \) is cyclic and separating for \( \mathcal{R}(I_a) \): assume that for \( A, B \in \mathcal{U}(I_a) \) the maps
\[ t \mapsto \langle \Psi, A e^{itL^{(\alpha)}} B \Omega \rangle_{os} = 0 \]
vanish identically. Then analyticity of the maps (12.6.2) implies that
\[ \langle \Psi, A e^{-itL^{(\alpha)}} B \Omega \rangle_{os} = 0, \]
showing that \( \Psi \) vanishes, as its scalar product with a dense set of vectors is zero.

(As before, we use that \( e^{-itL^{(\alpha)}} \) maps \( \mathcal{U}(I_a) \) to the opposite algebra \( \mathcal{U}(I_{\alpha + \pi}) \), and the fact that \( \mathcal{U}(I_a) \vee \mathcal{U}(I_{\alpha + \pi}) = \mathcal{U}(S^1). \)

Since \( L^{(\alpha)} \) does not mix \( \hat{h}(I_a) \) and \( \hat{h}(I_{\alpha + \pi}) \), the same argument can be applied to the commutant \( \mathcal{R}(I_a)' = \mathcal{R}(I_{\alpha + \pi}) \). This implies that \( \Omega \) is separating for \( \mathcal{R}(I_a) \).
Thus the map

\[ A\Omega \mapsto A^*\Omega, \quad A \in \mathcal{R}(I_\alpha), \]

is well-defined and closeable, and one can study its polar decomposition directly (see, e.g., [23]); but alternatively, we can reconstruct it from the Euclidean theory:

**Definition 12.6.1.** Let \( J^{(\alpha)} \) denote the unique extension of \( (12.6.3) \)

\[ J^{(\alpha)} \mathcal{V}(F) = \mathcal{V}(\Theta^{(\alpha)}_{\pi/2}), \quad F \in L^2(\Omega,\Sigma_{\mathcal{S}^+},d\Phi_C), \]

with \( (12.6.4) \)

\[ \Theta^{(\alpha)}_{\pi/2} = U^{(\alpha)}(\pi/2)\Theta^{(\alpha)}(-\pi/2) \]

on \( L^2(\Omega,\Sigma,d\Phi_C) \). Thus \( J^1 = J^{(0)} \) stems from the reflection of \( \mathcal{S}^+ \) at the \((x_0,-x_1)\)-plane (which clearly preserves \( \mathcal{S}^+ \)).

**Theorem 12.6.2** (Klein and Landau, Theorem 12.1 [123]). The operator \( e^{-\pi L^{(\alpha)}} \)

is the Tomita-Takesaki modular operator for the pair \((\mathcal{R}(I_\alpha),\Omega)\), and \( J^{(\alpha)} \) is the corresponding modular conjugation, i.e.,

\[ J^{(\alpha)} e^{-\pi L^{(\alpha)}} A\Omega = A^*\Omega \quad \forall A \in \mathcal{R}(I_\alpha) \]

and \( J^{(\alpha)} \mathcal{R}(I_\alpha) J^{(\alpha)} = \mathcal{R}(I_\alpha)' \).

### 12.7. Multi-analyticity of the correlation functions

The following result is a direct consequence of the reconstruction theorem.

**Proposition 12.7.1** (Klein & Landau, Lemma 8.4 [123]). Let \( \theta_1,\ldots,\theta_n \geq 0 \) and \( \sum_{j=1}^n \theta_j \leq \pi \). Then, for all \( A_{1n}^{os},\ldots,A_{nn}^{os} \in \mathcal{U}(I_\alpha), \alpha \in [0,2\pi] \), the vector

\[ A_n^{os} e^{-\theta_n L^{(\alpha)}} A_{n-1}^{os} \cdots e^{-\theta_1 L^{(\alpha)}} A_1^{os} \Omega \in \mathcal{D}(e^{-\theta_n L^{(\alpha)}}). \]

The linear span of such vectors is dense in \( \mathcal{H} \) and

\[ e^{-\theta_n L^{(\alpha)}} A_n^{os} e^{-\theta_{n-1} L^{(\alpha)}} A_{n-1}^{os} \cdots e^{-\theta_1 L^{(\alpha)}} A_1^{os} \Omega \]

\[ = \mathcal{V}(U^{(\alpha)}(\theta_n)A_n^{os}U^{(\alpha)}(\theta_{n-1})A_{n-1} \cdots U^{(\alpha)}(\theta_1)A_1) \Omega. \]

**Remark 12.7.2.** It is easy to extend this result to Euclidean time-zero fields: for \( h_1,\ldots,h_n \in \mathcal{D}_\mathcal{E}(I_+ \mathcal{H}) \) and \( \theta_1,\ldots,\theta_n \geq 0 \),

\[ e^{-\theta_n L^{(0)}} \Phi^{os}(0,h_n) \cdots e^{-\theta_1 L^{(0)}} \Phi^{os}(0,h_1) \Omega \]

\[ = \mathcal{V}(U^{(0)}(\theta_n)\Phi(0,h_n) \cdots U^{(0)}(\theta_1)\Phi(0,h_1)) \Omega. \]

Formula \((12.7.1)\) can be justified by verifying that products of Euclidean sharp-time fields are in \( L^2(\Omega,\Sigma,d\Phi_C) \). Similar results hold for arbitrary halfcircles \( I_\alpha \), \( \alpha \in [0,2\pi] \).

**Corollary 12.7.3.** Let \( W(h_1) = e^{i\Phi^{os}(0,h_1)} \), with \( h_i \in \mathcal{D}_\mathcal{E}(I_+) \), \( i = 1,\ldots,n \). It follows that the map

\[ G_{os}(t_1,\ldots,t_n;W(h_1),\ldots,W(h_n)) \]

\[ = (\Omega,e^{-i\frac{t_n}{h_n} L^{(0)}} W(h_n) e^{-i\frac{t_{n-1}}{h_{n-1}} L^{(0)}} W(h_{n-1}) \cdots e^{-i\frac{t_1}{h_1} L^{(0)}} W(h_1)) \Omega \]
is holomorphic in the set
\[(12.7.2) \quad J_{2\pi}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \Im z_1 < \Im z_{i+1}, \Im z_n - \Im z_1 < 2\pi\},\]
and continuous on \(\overline{J_{2\pi}^n}\). Moreover,
\[
G_{\text{os}}(i\theta_1, \ldots, i\theta_n, W(h_1), \ldots, W(h_n)) = \prod_{1 \leq i,j \leq n} e^{-\frac{i}{2}C_{i\theta_i - j\theta_j}(h_i, h_j)},
\]
where \(C_{i\theta_i - j\theta_j}(h_i, h_j)\) is defined in Eq. (8.3.1).

**Proof.** Multi-analyticity has been shown by Araki [4]. To derive the final formula compute
\[
G_{\text{os}}(i\theta_1, \ldots, i\theta_n, W(h_1), \ldots, W(h_n)) = \left(\Omega_E, e^{i\Phi_E(\theta_n, h_n)} \cdots e^{i\Phi_E(\theta_1, h_1)} \Omega_E\right) = \int_\varnothing d\Phi_C e^{i\Phi(\sum_{i=1}^n \delta(-\theta_i) \otimes h_i)} = \prod_{1 \leq i,j \leq n} e^{-\frac{i}{2}C_{i\theta_i - j\theta_j}(h_i, h_j)}.
\]
We note that we have assumed that the \(h_i, i = 1, \ldots, n\), are real valued.  \(\Box\)

### 12.8. The interacting vacuum vector

Eq. (11.1.6) implies \(e^{-V(S^+)} \in L^2(\Omega, \Sigma, d\Phi_C)\). Thus the vector
\[
(12.8.1) \quad \frac{\mathcal{V}(e^{-V(S^+)} \Omega_{\text{int}})}{\|\mathcal{V}(e^{-V(S^+)}\Omega_{\text{int}})\|} = \Omega_{\text{int}} \in \mathcal{H}
\]
is well-defined. The corresponding vector state will be identified as the interacting de Sitter vacuum in the sequel. We emphasise that there is no need to distinguish a wedge to define the vector \(12.8.1\).

**Lemma 12.8.1.** Let \(h \in C^\infty(S^1)\). Then
\[
\int_{S^+} d\Omega(x) \int_{S^+} d\Omega(y) \langle \delta \otimes h \rangle(x) P_{-\frac{1}{2} - i\nu} \left(-\frac{x \cdot y}{r^2}\right) = \sqrt{\pi \tau} \cdot \frac{(2\pi)^2}{|\Gamma(\frac{1}{4} + i \frac{\tau}{2})|^4 |\Gamma(\frac{1}{4} + i \frac{\nu}{2})|^2} c_\nu \int_{S^1} r d\psi \int_{S^1} r d\psi' h(\psi) P_{-\frac{1}{2} - i\nu}(\cos(\psi - \psi')).
\]

**Proof.** We use geographical coordinates. As
\[
\frac{x \cdot y}{r^2} = \sin \theta \sin \theta' + \cos \theta \cos \theta' \cos(\psi - \psi'),
\]
we find
\[
\int_{S^+} d\Omega(x) \int_{S^+} d\Omega(y) \langle \delta \otimes h \rangle(x) P_{-\frac{1}{2} - i\nu} \left(-\frac{x \cdot y}{r^2}\right) = r^2 \int_{0}^{\frac{\pi}{2}} \cos \theta' d\theta' \int_{S^1} d\psi \int_{S^1} d\psi' h(\psi) P_{-\frac{1}{2} - i\nu}(\cos \theta' \cos(\psi - \psi')).
\]
A special case of \( (E.0.11) \) is the following formula.

\[
\begin{align*}
(12.8.2) \quad P_s(- \cos(\psi - \psi') \cos \theta') &= \\
&= \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - \frac{i}{2}\right)\Gamma\left(\frac{1}{2} + 1\right)} P_s(\sin(\psi - \psi')) + 2 \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(s - k + 1)}{\Gamma(s + k + 1)} \cos(k \theta') \ p_s^k(0) p_s^k(\sin(\psi - \psi')).
\end{align*}
\]

We have used \( P_s(0) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - \frac{i}{2}\right)\Gamma\left(\frac{1}{2} + 1\right)} = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}\right)}. \) Next recall that

\[
\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2}).
\]

Thus

\[
c_{\nu} = \frac{\Gamma\left(\frac{1}{2} - i \nu\right) \Gamma\left(\frac{1}{2} + i \nu\right)}{2\pi} = \frac{\Gamma\left(\frac{1}{2} - i \frac{\lambda}{2}\right) \Gamma\left(\frac{1}{2} + i \frac{\lambda}{2}\right) \Gamma\left(\frac{1}{2} + i \frac{\nu}{2}\right)}{(2\pi)^2}.
\]

When integrating out the \( \theta' \) variable, only the first term on the r.h.s. contributes. Thus

\[
\int_{S^1} d\Omega(x) \int_{S^1} d\Omega(y) \langle \delta \otimes h(x) P_{-\frac{1}{2} - i \nu \psi - \psi'} \rangle = \frac{\sqrt{\pi} r}{|\Gamma(\frac{1}{2} + i \frac{\lambda}{2})|^2 |\Gamma(\frac{1}{2} + i \frac{\nu}{2})|^2} c_{\nu} \int_{S^1} r \, d\psi \int_{S^1} r \, d\psi' \ h(\psi) P_{-\frac{1}{2} - i \nu \psi - \psi'}.
\]

The last equality follows from shifting the integration in the \( \psi' \) variable. \( \square \)

**Remark 12.8.2.** The integral over \( \psi' \) can be computed using the formula

\[
\lambda P_0(x) = x P_0(x) - P'_{0-1}(x),
\]

which implies that

\[
\lambda \left[ \int_0^1 dx \, P_\lambda(x) \right] = \left[ \int_0^1 dx \, x P_\lambda(x) - P_{\lambda-1}(1) + P_{\lambda-1}(0) \right] = P_\lambda(1) - \left[ \int_0^1 dx \, P_\lambda(x) - P_{\lambda-1}(1) + P_{\lambda-1}(0) \right].
\]

Note that \( P_\lambda(1) = P_{\lambda-1}(1) = 1 \) and \( P_\lambda(0) = \frac{\sqrt{\pi} r}{\Gamma(\frac{1}{2} + \frac{1}{2}\right)\Gamma(\frac{1}{2} + 1). \) Thus

\[
\int_0^1 du \, P_\lambda(u) = \frac{\sqrt{\pi} r}{(1 + s) \Gamma(1 - \frac{1}{2}\right)\Gamma(\frac{1}{2} + 1).}
\]

The function \( e^{-V(S^1)} \) is \( \Sigma_{S^1} \)-measurable. We now construct the conditional expectation value.

**Theorem 12.8.3 (Conditional Expectation).** Let

\[
V(S^1) = \int_{S^1} d\psi : \mathcal{O}(\kappa_0 \Phi(0, \psi)) ; c_0, \quad \kappa_0 = \frac{\sqrt{\pi} r}{|\Gamma(\frac{1}{2} + i \frac{\lambda}{2})|^2 |\Gamma(\frac{1}{2} + i \frac{\nu}{2})|^2}.
\]

\[ ^3 \text{Note that } \lim_{|y| \to \infty} |\Gamma(x + i y)| = \sqrt{2\pi} |y|^{x-\frac{1}{2}} e^{-\pi|y|^2/2}. \]
Here 1 is the function, which is constant on \( S^1 \) and equal to 1 at every point. It follows that

\[
\mathcal{V}(e^{-\mathcal{V}(\Sigma^+))} = e^{-\mathcal{V}(\Sigma^+)} \quad \text{on } L^2(Q, \Sigma_0, d\Phi_C).
\]

Remark 12.8.4. We note that \( \mathcal{V}(S^1) \) has to be distinguished from \( \mathcal{V}_0(\chi_{S^1}) \); the two expressions refer to different polynomials in \( \Phi(0, \psi) \) integrated over the time-zero circle \( S^1 \).

Proof. By linearity, one may assume that \( \mathcal{P}(\lambda) = \lambda^n \). We can then expand the exponential function, and identifying \( L^2(\Omega, \Sigma, d\Phi_C) \) with \( \Gamma(H^{-1}_C(S^2)) \) we may apply the projection (see (10.4.2))

\[
\mathcal{E}_{x^1} \triangleq \mathcal{P}(e(S^1))
\]
to the vacuum vector \( \Omega_E \) in \( \Gamma(H^{-1}_C(S^2)) \); i.e.,

\[
\mathcal{V}(e^{-\mathcal{V}(\Sigma^+))} = \mathcal{E}_{x^1} e^{-\mathcal{V}(\Sigma^+)} \Omega_E = \Omega_E - \mathcal{P}(e(S^1)) \mathcal{V}(\Sigma^+) \Omega_E + \mathcal{P}(e(S^1)) (V(S^+))^2 \Omega_E - \ldots
\]

Let us first consider the term \( \mathcal{P}(e(S^1)) \mathcal{V}(\Sigma^+) \Omega_E \) in some more detail. Let \( \chi_{S^+} \) denote the characteristic function of the closed upper hemisphere. We have seen in the proof of Theorem 12.1.1 that

\[
\lim_{m \to \infty} \int_0^\pi r \, d\theta \int_{-\pi/2}^{\pi/2} r \cos \psi \, d\psi \, d\phi(\delta^{(2)}_m (\theta, \psi - \psi)) \mathcal{G}_C \quad \text{is a linear combination of Wick monomials of the form}
\]

\[
\sum_{j=0}^n \binom{n}{j} \sum_{t_1=0}^{\infty} \ldots \sum_{t_n=0}^{\infty} \sum_{k_1=-\ell_1}^{\ell_1} \ldots \sum_{k_n=-\ell_n}^{\ell_n} \frac{1}{(-1)^n} \delta_{k_1, \ldots, k_n} \delta_{\ell_1, \ldots, \ell_n} w^{(n)}(\ell_1, k_1, \ldots, k_j, \ell_j, \ell_{j+1}, k_{j+1}, \ldots, \ell_n, k_n)
\]

(12.8.3)

where \( a_{t_i, k_i} \equiv a^{(s)}(\chi_{x_i, k_i}) \) and

\[
w^{(n)}(\ell_1, k_1, \ldots, k_j, \ell_j, \ell_{j+1}, k_{j+1}, \ldots, \ell_n, k_n) = \frac{1}{2^{2n}} \int_0^\pi r \, d\theta \int_{-\pi/2}^{\pi/2} r \cos \psi d\psi \, d\phi(\delta^{(2)}_m (\theta, \psi)) \mathcal{E}_{x^1} \mathcal{E}_{x^1} \Omega_E
\]

When (12.8.3) is applied to the free vacuum vector \( \Omega_E \), only the term \( j = n \) is non-zero. Moreover, we can undo the expansion into spherical harmonics:

\[
\lim_{m \to \infty} \int_0^\pi r \, d\theta \int_{-\pi/2}^{\pi/2} r \cos \psi \, d\psi \, d\phi(\delta^{(2)}_m (\theta, \psi)) \mathcal{G}_C \Omega_E
\]

\[
= \int_{S^2} d\Omega(\tilde{x}) \chi_{S^+}(\tilde{x}) \frac{a^{(s)}(\delta^2) \cdots a^{(s)}(\delta^2)}{n \text{ times}} \Omega_E.
\]

The resulting \( n \)-particle wave-function

\[
f(\theta_1, \psi_1, \ldots, \theta_n, \psi_n) = \delta(\theta_1 - \theta_2) \delta(\psi_1 - \psi_2) \cdots \delta(\theta_1 - \theta_n) \delta(\psi_1 - \psi_n)
\]

is in \( \mathcal{H}^{-1}(S^2) \otimes \cdots \otimes \mathcal{H}^{-1}(S^2) \).
In second order, a term of the form (12.8.3) is applied to the n-particle wave-function $f(\theta_1, \psi_1, \ldots, \theta_n, \psi_n)$. Now all terms in the sum over $j$ will contribute. A generic term is of the form
\[
\int_{S^2} d\Omega(y) \chi_{S^2}(y) a^*(\delta^2_\theta) \cdot \cdot \cdot a^*(\delta^2_\theta) a(\delta^2_\theta) \cdot \cdot \cdot a(\delta^2_\theta) f(\theta_1, \psi_1, \ldots, \theta_n, \psi_n).
\]
The resulting wave-function is in $\Gamma^{(2)}(\mathcal{H}^{-1}(S^2))$.

Higher orders clearly have a very involved structure. Nevertheless, we are able to apply the projection $e(S^1) \otimes \cdot \cdot \cdot \otimes e(S^1)$. In first order, we can compute the conditional expectation by integrating against wave-functions of the form
\[
(\delta(\theta_1) \otimes h_1(\psi_1)) \otimes \cdot \cdot \cdot \otimes (\delta(\theta_n) \otimes h_n(\psi_n))
\]
This yields
\[
\int_{S^2} d\Omega(x_1) \int_{S^2} d\Omega(y_1) \cdots \int_{S^2} d\Omega(y_n) (\delta \otimes h_1)(x_1) \cdots (\delta \otimes h_n)(x_n)
\times P_{-\frac{i}{\hbar} + iv} \left( -\frac{x_1 \cdot y_1}{r^2} \right) \cdots P_{-\frac{i}{\hbar} + iv} \left( -\frac{x_n \cdot y_n}{r^2} \right) \chi_{S^2}(y_1) \delta y_1(y_2) \cdots \delta y_1(y_n)
\int_{S^1} d\Omega(y_1) \int_{S^1} d\Omega(x_1) \cdots \int_{S^1} d\Omega(x_n) (\delta \otimes h_1)(x_1) \cdots (\delta \otimes h_n)(x_n)
\times P_{-\frac{i}{\hbar} + iv} \left( -\frac{x_1 \cdot y_1}{r^2} \right) \cdots P_{-\frac{i}{\hbar} + iv} \left( -\frac{x_n \cdot y_n}{r^2} \right) \chi_{S^1}(y_1)
\int_{S^1} r d\psi_1 \int_{S^1} r d\psi_1' \cdots \int_{S^1} r d\psi_1 \int_{S^1} r d\psi_1' h_1(\psi_1) \cdots h_n(\psi_n)
\times \kappa_0 c_\nu P_{-\frac{i}{\hbar} + iv} (-\cos(\psi_1 - \psi_1')) \cdots \kappa_0 c_\nu P_{-\frac{i}{\hbar} + iv} (-\cos(\psi_n - \psi_n'))
\times \chi_{S^1}(\psi_1') \delta(\psi_1' - \psi_1') \cdots \delta(\psi_n' - \psi_n').
\]
Note that the characteristic function $\chi_{S^1}(\psi_1')$ can be removed without alliterating the result. Moreover, note that symmetrisation is not required, as the expression is already symmetric. We have again used (12.8.3) and the fact that when integrating out the $\theta'$ variable, all the terms containing $\cos k\theta'$ will not contribute.

Thus
\[
(\underbrace{e(S^1) \otimes \cdot \cdot \cdot \otimes e(S^1)}_{n-times}) f(\psi_1, \ldots, \psi_n)
\]
(12.8.4)
\[
= \kappa_0^n \chi_{S^1}(\psi_1) \delta(\psi_1 - \psi_2) \cdots \delta(\psi_1 - \psi_n) \in \mathcal{H}_{\mathcal{S}^1}^{-1}(S^2) \otimes \cdot \cdot \cdot \otimes \mathcal{H}_{\mathcal{S}^1}^{-1}(S^2) \otimes \cdot \cdot \cdot \otimes \mathcal{H}_{\mathcal{S}^1}^{-1}(S^2).
\]
This shows that (12.8.4) is equal to
\[
\int_{S^1} r d\psi : (\kappa_0 \Phi(0, \psi))^{(n)} \cdot c_\nu \Omega_{E} ,
\]
where $\kappa_0$ is the constant first appearing in Lemma (12.8.1). Note that normal ordering is with respect to the time-zero covariance $c_\nu$. Similar arguments now show that
\[
\Gamma (e(S^1)) (V (S^1))^n \Omega_{E} = V(S^1)^n \Omega_{E} ,
\]
proofing the statement.
CHAPTER 13

The Reconstruction of Interacting Quantum Fields

We start by introducing an auxiliary Hilbert space $\mathcal{H}_V$, associated to the interacting measure $d\mu_V$ on the sphere $S^2$.

13.1. The Hilbert space for the interacting measure

Replace the Gaussian measure $d\Phi_C$ by the interacting measure $d\mu_V$ first introduced in (11.1.7), and repeat the Osterwalder-Schrader reconstruction:

i.) Since $d\mu_V$ is absolutely continuous with respect to the Gaussian measure $d\Phi_C$, reflection positivity extends to $L^2(\Omega, \Sigma_{S^2}, d\mu_V)$:

$$\int_{\Omega} d\mu_V \overline{\Theta(F)} F \geq 0, \quad F \in L^2(\Omega, \Sigma_{S^2}, d\mu_V).$$

Define

(13.1.1) $\mathcal{H}_V = \text{completion of } L^2(\Omega, \Sigma_{S^2}, d\mu_V)/N_V,$

where $N_V \subset L^2(\Omega, \Sigma_{S^2}, d\mu_V)$ is the kernel of the positive quadratic form

(13.1.2) $\langle F, G \rangle_V = \int_{\Omega} d\mu_V \overline{\Theta(F)} G, \quad F, G \in L^2(\Omega, \Sigma_{S^2}, d\mu_V),$

and the completion in (13.1.1) is with respect to the scalar product induced by (13.1.2) on the quotient space $L^2(\Omega, \Sigma_{S^2}, d\mu_V)/N_V$. Let $V_V$ denote the canonical map

$$V_V: L^2(\Omega, \Sigma_{S^2}, d\mu_V) \to L^2(\Omega, \Sigma_{S^2}, d\mu_V)/N_V$$

and let

(13.1.3) $\Omega_V \doteq V_V(1)$

denote the distinguished unit vector in $\mathcal{H}_V$.

ii.) For $A \in L^\infty(\Omega, \Sigma_{S^1}, d\mu_V)$ define $A^V \in B(\mathcal{H}_V)$ by

(13.1.4) $A^V V_V(F) = V_V(AF), \quad F \in L^2(\Omega, \Sigma_{S^2}, d\mu_V).$

Denote by $\mathcal{U}_V(S^1)$ the abelian von Neumann algebra

$$\mathcal{U}_V(S^1) \doteq \{ A^V \in B(\mathcal{H}_V) \mid A \in L^\infty(\Omega, \Sigma_{S^1}, d\mu_V) \}. $$

The vector $\Omega_V$ is cyclic for $\mathcal{U}_V(S^1)$ (see again Theorem 11.2 of [123]).
PROPOSITION 13.1.1. \((\Omega, \Sigma, \Sigma^{(\alpha)}, U^{(\alpha)}(\cdot, \cdot), \Theta, d\mu_V)\) is a \((2\pi)\)-periodic, \(\Theta\)-positive generalised path space in the sense of Definition [13.3.1].

For \(0 \leq \theta \leq \pi\) fixed, set
\[
\mathcal{M}_{0,\theta}^{(\alpha)} \doteq L^2(\Omega, \bigvee_{\theta'' \in [0,\pi-\theta]} U^{(\alpha)}(\theta'')\Sigma^{(\alpha)}, d\mu_V), \quad 0 \leq \theta \leq \pi.
\]

DEFINITION 13.1.2. Let \(\mathcal{D}_{0,V}^{(\alpha)} \doteq \mathcal{V}_V(\mathcal{M}_{0,V}^{(\alpha)})\) and define, for \(0 \leq \theta' \leq \theta\),
\[
P_V^{(\alpha)}(\theta') : \mathcal{D}_{0,V}^{(\alpha)} \rightarrow \mathcal{H}_V
\]
\[
\mathcal{V}_V(F) \mapsto \mathcal{V}_V(U^{(\alpha)}(\theta')F), \quad F \in \mathcal{M}_{0,V}^{(\alpha)}.
\]

PROPOSITION 13.1.3. \((P_V^{(\alpha)}(\theta), \mathcal{D}_{0,V}^{(\alpha)})\) is a local symmetric semigroup on \(\mathcal{H}_V\). Its generator \(L_V^{(\alpha)}\) satisfies
\[
P_V^{(\alpha)}(\theta') = e^{-\theta' L_V^{(\alpha)}}, \quad \Psi \in \mathcal{D}_{0,V}^{(\alpha)}, \quad 0 \leq \theta' \leq \theta.
\]

Define \(J_{V}^{(\alpha)}\) as the unique extension of the anti-linear operator
\[
J_{V}^{(\alpha)}\mathcal{V}_V(F) \doteq \mathcal{V}_V(\Theta_{\pi/2}^{(\alpha)}F), \quad F \in L^2(\Omega, \Sigma_{\alpha} d\mu_V),
\]
where \(\Theta_{\pi/2}^{(\alpha)}\) was introduced in (12.6.4). Furthermore, define the von Neumann algebra
\[
\mathcal{R}_V(I_\alpha) \doteq \bigvee_{t \in \mathbb{R}} \{ e^{-itL_{V}^{(\alpha)}} U_V(I_\alpha) e^{itL_{V}^{(\alpha)}} | A \in L^\infty(\Omega, \Sigma_{I_\alpha}, d\mu_V) \}.
\]

THEOREM 13.1.4. The positive operator \(e^{-\pi L_{V}^{(\alpha)}}\) is the Tomita-Takesaki modular operator for the pair \((\mathcal{R}_V(I_\alpha), \Omega_V)\), and \(J_{V}^{(\alpha)}\) is the modular conjugation for the pair \((\mathcal{R}_V(I_\alpha), \Omega_V)\), i.e.,
\[
J_{V}^{(\alpha)} e^{-\pi L_{V}^{(\alpha)}} \Omega_V = \Omega^* \Omega_V, \quad A \in \mathcal{R}_V(I_\alpha).
\]

The rotations can be implemented easily; however, we will have to show later on that the boosts and the rotations together generate a representation of \(SO_0(1,2)\).

PROPOSITION 13.1.5. Consider the map (13.1.4). It follows that
\[
e^{i\alpha K_Y^V} A^* \Omega_V \doteq \mathcal{V}_V(U(R_0(\alpha))A), \quad A \in L^\infty(\Omega, \Sigma_{\alpha}, d\mu_V),
\]
extends to a strongly continuous unitary representation of the rotation group \(SO(2)\) on the Hilbert space \(\mathcal{H}_V\).

### 13.2. Virtual representations

The basic object in the theory of virtual representations\footnote{See [113] for recent work and an extensive list of references on this topic.} is a symmetric space: let \(G\) be a Lie group, \(K\) a closed subgroup of \(G\), with Lie algebras \(\mathfrak{g}\) and \(\mathfrak{k}\), respectively. A Lie algebra \(\mathfrak{g}\) is symmetric (giving rise to a symmetric space \((G, K, \Sigma)\)), if it allows a decomposition
\[
\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}
\]
(where $\oplus$ indicates a direct sum of vector spaces) such that

\[(13.2.2) \quad [k, k] \subset k, \quad [k, m] \subset m, \quad [m, m] = k.\]

On a symmetric Lie algebra there exists (use (13.2.1)–(13.2.2) to derive this fact) a natural involutive automorphism $\mathcal{T}$ of $g$, such that

\[
\mathcal{T}|_k = 1, \quad \mathcal{T}|_m = -1.
\]

The dual symmetric Lie algebra $g^*$ is (see, e.g., [127])

\[(13.2.3) \quad g^* = k \oplus i m.\]

(13.2.2) implies that $g^*$ is the real Lie algebra of a simple connected Lie group $G^*$. 

**Lemma 13.2.1.** $\text{so}(3)$ is a symmetric Lie algebra with sub-Lie algebra $\text{SO}(2)$; the dual symmetric Lie algebra is $\text{so}(1, 2)$.

**Proof.** Decompose $\text{so}(3)$ according to (13.2.1) and verify (13.2.2). □

**Definition 13.2.2.** A virtual representation $(\varphi, \mathfrak{h})$ of a symmetric space $(G, K, \mathcal{T})$ consists of a separable Hilbert space $\mathfrak{h}$ together with a local group homomorphism $\varphi$ from $G$ into linear operators densely defined on $\mathfrak{h}$, with the following properties:

i.) $\varphi|_H$ is a continuous unitary representation of $K$ on $\mathfrak{h}$;

ii.) there exists a neighbourhood $N$ of $1 \in G$, invariant under right translation by $K$, and a linear subspace $\mathcal{D}$, dense in $\mathfrak{h}$, such that

--- $\mathcal{D} \subset \mathcal{D}(\varphi(g))$ for all $g \in N$; and

--- if $g_1, g_2$ and $g_1 \circ g_2$ are all in $N$, then

\[(13.2.4) \quad \varphi(g_2)\Psi \in \mathcal{D}(\varphi(g_1)), \quad \Psi \in \mathcal{D}, \]

and

\[\varphi(g_1)\varphi(g_2)\Psi = \varphi(g_1 \circ g_2)\Psi, \quad \Psi \in \mathcal{D};\]

iii.) if $\ell \in m$, $0 \leq t \leq 1$, and

\[e^{-t\ell} \in N, \quad 0 \leq t \leq 1,
\]

then $\varphi(e^{-t\ell})$ is a hermitian operator defined on $\mathcal{D}$ and

\[(13.2.5) \quad s - \lim_{t \to 0} \varphi(e^{-t\ell})\Psi = \Psi, \quad \Psi \in \mathcal{D}.
\]

The main result in the theory of virtual representations is the following:

**Theorem 13.2.3** (Fröhlich, Osterwalder, and Seiler [62]). Let $\varphi$ be a virtual representation of a symmetric space $(G, K, \mathcal{T})$, with $K$ compact. Then $\varphi$ can be analytically continued to a unitary representation $\varphi^*$ of $G^*$.

### 13.3. A unitary representation of the Lorentz group

We now apply the main result of the previous subsection to the special case relevant in the present context.

**Theorem 13.3.1.** The self-adjoint operators $\mathcal{K}_Y^\ell, L_Y^{(0)} = L^{(0)}_V$ and $L_Y^{(\pi/2)} = L^{(\pi/2)}_V$ defined in Proposition 13.1.2 and Proposition 13.1.3 respectively, generate a unitary representation $\Lambda \mapsto U_V(\Lambda)$ of $\text{SO}_0(1, 2)$ on $\mathcal{H}_V$. 


PROOF. It is sufficient to construct a virtual representation \( \psi \) of \( SO(3) \) on \( \mathcal{H}_V \) and then apply Theorem 13.2.3.

i.) The self-adjoint operator \( k^V_0 \) generates a unitary representation of the subgroup \( K = SO(2) \) of \( G = SO(3) \);

ii.) For \( 0 < \theta < \pi/2 \) let \( N_\theta \) be the subset of elements in \( SO(3) \) consisting of the rotations which (when acting on it) do not move the north pole \((r, 0, 0)\) outside of the polar cap

\[
\left\{ \left( \frac{r \cos \alpha}{r \cos \alpha \sin \theta'}, \frac{r \cos \alpha \sin \theta'}{r \cos \alpha \sin \theta'} \right) \mid 0 \leq \theta' < \theta, \alpha \in [0, 2\pi) \right\}.
\]

Now recall (12.5.2) and set, for \( 0 < \theta < \pi/2 \),

\[
\mathcal{D}_{\theta, V} = \mathcal{V}_V(M_\theta) \quad \text{with} \quad M_\theta = \bigcap_{\alpha \in (0, 2\pi]} M_0^{(\alpha)}.
\]

In other words, given the polar cap \( \sim \theta \) with distance \( \theta \) to the equator, the set \( M_\theta \) is the set of all \( F \in L^2(Q, \Sigma, d\mu) \), with \( \Sigma \) the smallest \( \sigma \)-algebra for which the functions \( \{ \Phi(f) \mid f \in \mathcal{D}_{\theta, V}(\sim \theta) \} \) are measurable. In Lemma 13.3.3 we will show that \( \mathcal{D}_{\theta, V} \) is dense in \( \mathcal{H}_V \).

It follows from the definitions of \( \Sigma \) and \( \Theta \) that for \( R \in SO(3) \)

\[
U(R)(\Theta F) = \Theta U(\Sigma(R)) F, \quad F \in C^\infty(S^2).
\]

Clearly, for \( R \in N_\theta \) and \( F, G \in M_\theta \)

\[
\langle G, U(R)F \rangle_V = \int_Q d\mu \Theta(G)^* U(R)F = \int_Q d\mu U(R^{-1}) \Theta(G) F = \int_Q d\mu \Theta(U(\Sigma(R^{-1})) G) F.
\]

Now use the Schwarz inequality to show that the intersection of the kernel \( N_V \) of \( \mathcal{V}_V \) with \( \mathcal{D}_{\theta, V} \) is invariant under the map \( F \mapsto U(R)F \) for \( R \in N_\theta \), \( \theta > 0 \). Thus, for each \( R \in N_\theta \subset SO(3) \), the map

\[
\psi(R) : \mathcal{D}_{\theta, V} \to \mathcal{H}_V
\]

\[
\mathcal{V}_V(F) \mapsto \mathcal{V}_V(U(R)F), \quad F \in M_\theta,
\]

is well-defined, verifying (13.2.4).

Now, if \( F \in M_\theta \) and \( R_1, R_2 \in N_\theta \) as well as \( R_1 R_2 \in N_\theta \), then

\[
\mathcal{V}_V(U(R_1)U(R_2)F) = \mathcal{V}_V(U(R_1R_2)F).
\]

For example, for \( \gamma \) and \( \theta \) sufficiently small and \( F \in M_\theta \)

\[
e^{i\nu_0} e^{-i\nu^{(\alpha)}_0} \mathcal{V}_V F = \mathcal{V}_V(U(R_0(\gamma)R^{(\alpha)}(\theta)) F).
\]

iii.) The group \( R \to U(R) \), \( R \in SO(3) \), acts continuously on \( L^2(Q, \Sigma, d\Phi_C) \) and \( \mathcal{V}_V \) is continuous on \( M_\theta \). Thus the vector valued function \( N_\theta \ni R \to \psi(R) \psi \) is continuous for each \( \psi \in \mathcal{D}_{\theta, V} \). Thus \( \psi(R) \to 1 \) as \( R \to \mathbb{I} \), verifying (13.2.5).

It follows that \( \psi \) is a virtual representation of \( SO(3) \) on \( \mathcal{H}_V \). \( \square \)
REMARK 13.3.2. The theory of virtual representations provides a general argument, which also applies in the free case. However, for the free field, we have already verified—by directly computing the Lie brackets on the eigenfunctions of \( K_0 \) in the proof of Proposition 7.4.3—that the generators \( \mathcal{L}_\alpha = d \Gamma (\omega \cos \Psi, \omega \alpha) \), \( \alpha \in [0, 2\pi) \), of the boosts together with the generator \( K_0 \) of the rotations generate a representation of \( SO(1, 2) \) on the Fock space \( \Gamma (\mathfrak{h}(S^1)) \). In the general case, such a direct proof is not available.

LEMMA 13.3.3. Let \( 0 < \theta < \pi/2 \). Then the set \( \mathcal{M}_\theta \) is a quantization domain [104]; i.e., the set \( \mathcal{D}_{\theta,V} \) is dense in \( \mathcal{H}_V \).

REMARK 13.3.4. In fact, if \( O \) is any open set in \( S_+ \), then the image of
\[
\mathcal{M}(O) = L^2(\Omega, \Sigma_O, d\mu_V)
\]
under \( \mathcal{V}_V \) is dense in \( \mathcal{H}_V \). Here \( \Sigma_O \) is the smallest \( \sigma \)-algebra for which the functions \( \{ \Phi(f) \mid f \in \mathcal{D}_\theta(O) \} \) are measurable.

PROOF. In order to show that \( \mathcal{D}_{\theta,V} \) is dense in \( \mathcal{H}_V \), it is sufficient to show that if \( \Psi \perp \mathcal{D}_{\theta,V} \) is a vector in the orthogonal complement of \( \mathcal{D}_{\theta,V} \subset \mathcal{H}_V \), then it equals the zero-vector. We have already seen that \( \mathcal{U}_V(S^1) \mathcal{O}_V \) is dense in \( \mathcal{H}_V \). Thus it is sufficient to show that
\[
\langle \Psi, e^{i\Phi_V(h)} \mathcal{O}_V \rangle_V = 0 \quad \forall e^{i\Phi_V(h)} \in \mathcal{U}_V(S^1),
\]
as this would imply that \( \Psi \) is the zero-vector. Moreover,
\[
\mathcal{U}_V(S^1) = \bigvee_I \mathcal{U}_V(I)
\]
with \( \bigvee I \) a covering of \( S^1 \) in terms of open intervals. Thus it is sufficient to show that
\[
\langle \Phi, e^{i\Phi_V(h)} \mathcal{O}_V \rangle_V = 0 \quad \forall e^{i\Phi_V(h)} \in \mathcal{U}_V(I),
\]
with \( I \) an arbitrary fixed interval in the covering of \( S^1 \). For the covering we choose sufficiently many circle segment
\[
I_{\alpha, \theta + \epsilon} = \{ \xi \in I_{\alpha} \mid \text{dist}(\xi, \partial I_{\alpha}) > \theta + \epsilon \}, \quad 0 < \epsilon \ll \theta,
\]
of equal size, consisting of points in the interior of the half-circle \( I_{\alpha} \), which are more than \( \theta + \epsilon \) away from the end points of the half-circle.

Now consider, for \( h \in \mathcal{D}_\theta(I_{\alpha, \theta + \epsilon}) \), fixed, the analytic function
\[
(13.3.1) \quad z \mapsto \langle \Psi, e^{-z L^{(\alpha)}_{\theta}} e^{i\Phi_V(h)} \mathcal{O}_V \rangle_V, \quad \{ z \in \mathbb{C} \mid 0 < \Re z < \pi \}.
\]
By construction there exists an open interval \( J \) (whose size depends on \( \epsilon \)) such that
\[
\mathcal{R}^{(\alpha)}(\theta') I_{\alpha, \theta + \epsilon} \subset \mathcal{N}_\theta, \quad \theta' \in J,
\]
and consequently
\[
e^{-\overline{\mathcal{N}} z L^{(\alpha)}_{\theta}} e^{i\Phi_V(h)} \mathcal{O}_V \in \mathcal{D}_{\theta,V}, \quad \Re z \in J,
\]
and, since \( \Psi \perp \mathcal{D}_{\theta,V} \), the analytic function \((13.3.1)\) vanishes on an open line segment in the interior of its domain, and is therefore identical zero. \( \square \)
13.4. Perturbation theory of generalised path spaces

In the previous section we have constructed a representation of \( \text{SO}_0(1,2) \) on a new Hilbert space \( \mathcal{H}_V \). In this section, we discuss the action of the interacting field on the Fock space of the free field. To do so, construct (see [123]) a generalised path space \( (\mathcal{Q}, \Sigma, \Sigma(\alpha), U^{(\alpha)}_{\text{int}}(\theta), \Theta, d\Phi_C) \), which is equivalent (see [123.1]) to the one given in Proposition [13.1.1].

Set

\[
V^{(\alpha)} = V_0(\cos_\theta + \alpha \chi_{I_\alpha}),
\]

where \( V_0 \) was defined in (11.2.1) and \( \chi_{I_\alpha} \) denotes the characteristic function of the half-circle \( I_\alpha \subset S^1 \).

Hence the Feyman-Kac-Nelson kernels \( \left\{ V^{(\alpha)}_{[\theta, \theta']} \right\}_{0 \leq \theta' \leq \pi} \) belong to \( L^2(\mathcal{Q}, \Sigma, d\Phi_C) \).

Next consider the sets

\[
M^{(\alpha)}_{\text{int}} = \text{linear span of } \bigcup_{0 \leq \theta' \leq \pi - \theta} V^{(\alpha)}_{[0, \theta')} \bigg[ \mathcal{Q}, \Sigma^{(\alpha)}_{[0, \pi - \theta]}, d\Phi_C \bigg].
\]

Here \( \Sigma^{(\alpha)}_{[0, \pi - \theta]} = \bigvee_{\theta' \in [0, \pi - \theta]} U^{(\alpha)}(\theta') \Sigma^{(\alpha)} \).

**DEFINITION 13.4.1.** Set, for \( 0 \leq \theta \leq \pi \),

\[
U^{(\alpha)}_{\text{int}}(\theta') : M^{(\alpha)}_{\text{int}} \to L^2(\mathcal{Q}, \Sigma^{(\alpha)}_{[0, \pi - \theta]}, d\Phi_C)
\]

\[
G \mapsto V^{(\alpha)}_{[0, \theta']} U^{(\alpha)}(\theta') G.
\]

The map \( \theta' \mapsto U^{(\alpha)}_{\text{int}}(\theta') \) defines the interacting rotations on the sphere.

The one-parameter group \( \theta' \mapsto U^{(\alpha)}_{\text{int}}(\theta') \) induces a local symmetric semigroup on \( \mathcal{H} \):

**PROPOSITION 13.4.2.** Let \( \mathcal{D}^{(\alpha)}_{\text{int}} = \mathcal{V}(M^{(\alpha)}_{\text{int}}) \) and set, for \( 0 \leq \theta' \leq \theta \),

\[
p^{(\alpha)}_{\text{int}}(\theta') : \mathcal{D}^{(\alpha)}_{\text{int}} \to \mathcal{H}
\]

\[
\mathcal{V}(G) \mapsto \mathcal{V}(U^{(\alpha)}_{\text{int}}(\theta') G).
\]

\( (p^{(\alpha)}_{\text{int}}(\theta), \mathcal{D}^{(\alpha)}_{\text{int}}) \) is a local symmetric semigroup on \( \mathcal{H} \) with generator \( H^{(\alpha)}_{\text{os}} \).
Part 7

Interacting Quantum Fields
CHAPTER 14

The \( \mathcal{P}(\varphi)_2 \) Model on the de Sitter Space

14.1. Identification of Hilbert spaces

The Radon-Nikodym theorem implies that the interacting measure \( d\mu_V \) is absolutely continuous with respect to the Gaussian measure \( d\Phi_C \). Consequently,

\[
L^\infty(\mathcal{Q}, \Sigma_S, d\mu_V) \cong L^\infty(\mathcal{Q}, \Sigma_S, d\Phi_C).
\]

Moreover, \( \mathcal{U}(S^1)\Omega \) is dense in \( \mathcal{H}_V \) and \( \mathcal{U}(S^1)\Omega \) is dense in \( \mathcal{H} \). As we will see next, \( \mathcal{U}(S^1)\Omega_{\text{int}} \) is dense in \( \mathcal{H} \) as well. The vectors \( \Omega_V \) and \( \Omega_{\text{int}} \) were defined in (13.1.3) and (12.8.1), respectively. Note that

\[
e^{-\pi H^{(\alpha)}_1} \Omega = \mathcal{V}(F_{[0,\pi]}^{(\alpha)} U^{(\alpha)}(\pi)_1) = \mathcal{V}(F_{[0,\pi]}^{(\alpha)}) = \mathcal{V}(e^{-\pi V(S^1)}).
\]

The first equality follows from the reconstruction theorem, but it can also be verified directly using the Trotter product formula:

\[
\mathcal{V}
\left(e^{-\int_0^\pi \mathcal{U}(\theta) V_0(\cos \theta) d\theta}\right)
= \lim_{n \to \infty} \mathcal{V}
\left(e^{-\sum_{k=1}^{n} \mathcal{U}(k\pi/n) V_0(\cos k\pi/n)}\right)
= \lim_{n \to \infty} \mathcal{V}
\left(e^{-\mathcal{U}(\pi/n) V_0(\cos \pi/n)} \cdots e^{-\mathcal{U}(\pi/n) V_0(\cos \pi/n)}\right)
= \lim_{n \to \infty} \mathcal{V}
\left(e^{-\frac{n \pi}{n} d\Gamma(\omega \pi) \cos k\pi/n} \cdots e^{-\frac{n \pi}{n} d\Gamma(\omega \pi) \cos k\pi/n}\right)\Omega
= e^{-\pi H^{(0)}_1} \Omega.
\]

In fact, the identity (14.1.2) holds for all \( \alpha \in [0,2\pi] \), i.e.,

\[
\frac{e^{-\pi H^{(\alpha)}_1} \Omega}{\|e^{-\pi H^{(\alpha)}_1} \Omega\|} = \Omega_{\text{int}}, \quad \alpha \in [0,2\pi).
\]

The vector \( \Omega_{\text{int}} \) was defined in Eq. (12.8.1).

**Theorem 14.1.1.** The vector \( \Omega_{\text{int}} \) is cyclic and separating for \( \mathcal{R}(1_\alpha), \alpha \in [0,2\pi) \).

**Proof.** Since \( F_{[0,\pi]}^{(\alpha)} \) is in \( L^2(\mathcal{Q}, \Sigma_S, d\Phi_C) \), it follows that for \( 0 \leq \theta_1 \leq \ldots \leq \theta_n \leq \pi \) the vectors

\[
U_{\text{int}}^{(\alpha)}(\theta_n) A_n U_{\text{int}}^{(\alpha)}(\theta_{n-1} - \theta_n) A_{n-1} \cdots U_{\text{int}}^{(\alpha)}(\theta_1 - \theta_2) A_1 F_{[0,\pi]}^{(\alpha)} \,
\]

with \( A_1, \ldots, A_n \in L^\infty(\mathcal{Q}, \Sigma_1, d\Phi_C) \) are in \( L^2(\mathcal{Q}, \Sigma_{[0,\pi]}, d\Phi_C) \). Since

\[
F_{[0,\pi]}^{(\alpha)} > 0 \quad \mu \text{ a.e.,}
\]

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they form a total set in $L^2(Q, \Sigma_{0, \pi}, d\Phi_C)$. Therefore the vectors
\begin{equation}
\begin{aligned}
e^{-\theta_n H^{(\alpha)}} A_n^{\infty} e^{-(\theta_{n-1} - \theta_n) H^{(\alpha)}} A_{n-1}^{\infty} \cdots e^{-(\theta_1 - \theta_2) H^{(\alpha)}} A_1^{\infty} e^{-(\pi - \theta_1) H^{(\alpha)}} \Omega \\
= \mathcal{V}(U_{\text{int}}^{(\alpha)}(t_n) A_n U_{\text{int}}^{(\alpha)}(t_{n-1}) A_{n-1} \cdots U_{\text{int}}^{(\alpha)}(\theta_1 - \theta_2) A_1 F_{[0, \pi - \theta_1]}^{(\alpha)})
\end{aligned}
\end{equation}

form a total set in $\mathcal{H}$. By analyticity it follows that the vectors
\begin{equation}
A_n^{\infty}(t_n) \cdots A_1^{\infty}(t_1) e^{-\pi H^{(\alpha)}} \Omega
\end{equation}

with $A_i^{\infty}(t) = e^{i t H^{(\alpha)}} A_i^{\infty} e^{-i t H^{(\alpha)}}$, $t \in \mathbb{R}$, form a total set in $\mathcal{H}$. 

We next show that $\mathcal{U}(S^1) \Omega_{\text{int}}$ is dense in $\mathcal{H}$ [123.15.4 Remark].

**Theorem 14.1.2.** The vector $\Omega_{\text{int}}$ is cyclic and separating for $\mathcal{U}(S^1)$.

**Proof.** Recall from (12.8.1) that
\begin{equation}
\Omega_{\text{int}} = \frac{\mathcal{V}(e^{-V(S^2)})}{\|\mathcal{V}(e^{-V(S^2)})\|}.
\end{equation}

Since $\Omega$ is cyclic for $\mathcal{U}(S^1)$, and $\mathcal{U}(S^1)$ is abelian, the result follows from the fact that $\mathcal{V}(e^{-V(S^2)})$ is affiliated to $\mathcal{U}(S^1)$ and strictly positive. 

**Proposition 14.1.3. The map**
\begin{equation}
A^V \Omega_V \mapsto A^{\infty} \Omega_{\text{int}}, \quad A \in L^\infty(Q, \Sigma_{S^1}, d\mu_V),
\end{equation}

extends to a unitary representation $\mathcal{V}: \mathcal{H}_V \rightarrow \mathcal{H}$.

**Proof.** We have already seen that $\mathcal{U}_V(S^1) \Omega_V$ is dense in $\mathcal{H}_V$ and $\mathcal{U}(S^1) \Omega_{\text{int}}$ is dense in $\mathcal{H}$. It remains to show that the map (14.1.4) is norm preserving. This follows from
\begin{equation}
\|A^V \Omega_V\|^2 = \int d\mu_V |A|^2 = \int d\Phi_C e^{-V(S^2)} |A|^2
\end{equation}

The second equality follows from the definition of the interacting measure $d\mu_V$, see (11.1.7). 

**Proposition 14.1.4. The map** $\mathcal{V}: \mathcal{H}_V \rightarrow \mathcal{H}$ introduced in Proposition 14.1.3
i.) intertwines $\mathcal{U}_V(S^1)$ and $\mathcal{U}(S^1)$; 
ii.) respects the local structure, i.e.,
\begin{equation}
\mathcal{U}(I) = \mathcal{V} \mathcal{U}_V(I) \mathcal{V}^{-1}, \quad I \subset S^1;
\end{equation}

iii.) and, intertwines $K_0^V$ and $K_0$, i.e., $\mathcal{V} K_0^V \mathcal{V}^{-1} = K_0$.

**Theorem 14.1.5.** Set
\begin{equation}
L_{\text{int}}^{(\alpha)} = \mathcal{V} L_{\text{int}}^{(\alpha)} \mathcal{V}^{-1}.
\end{equation}

It follows that the generators $K_0^{(\alpha)} L_{\text{int}}^{(\alpha)} \equiv L_{\text{int}}^{(\alpha)}$ and $L_{\text{int}}^{(\alpha/2)}$ generate a unitary representation $\mathcal{U}_{\text{int}}$ of $\text{SO}_0(1,2)$ on the Hilbert space $\mathcal{H}$. 
We end this section by defining the interacting automorphisms.

**Definition 14.1.6.** Given the unitary representation \( \hat{U}_\text{int} \) of \( SU(1,2) \) on the Hilbert space \( \mathcal{H} \), we define the corresponding automorphisms: set
\[
\hat{\alpha}^{\text{int}}_\alpha(A) = \hat{U}_\text{int}(\Lambda) A \hat{U}_\text{int}(\Lambda)^{-1}, \quad A \in \mathcal{B}(\mathcal{H}), \quad \Lambda \in SU(1,2).
\]
We call this group of automorphisms the *interacting dynamics* on the Cauchy surface \( S^1 \).

### 14.2. Perturbation theory for modular automorphisms

Next recall Araki’s perturbation theory for modular automorphisms [1][2], which has been generalised to unbounded perturbations by Derezinski, Jaksic and Pillet [46].

**Theorem 14.2.1.** Set
\[
V^{(\alpha)} = \int_{I_{\alpha}} d\psi \cos(\psi + \alpha) : \mathcal{D}(\Phi^\alpha(0,\psi)) : \mathcal{C}_0.
\]
It follows that
i.) the operator sum \( L^{(\alpha)} + V^{(\alpha)} \) is essentially self-adjoint on \( \mathcal{D}(L^{(\alpha)}) \cap \mathcal{D}(V^{(\alpha)}) \)
and
\[
L^{(\alpha)} + V^{(\alpha)} = H^{(\alpha)}, \quad \alpha \in [0,2\pi);
\]
ii.) the vector \( \Omega \) defined in Equ. (14.2.3) belongs to \( \mathcal{D}(e^{-\pi H^{(\alpha)}}) \), \( \alpha \in [0,2\pi) \), and
\[
e^{-\pi H^{(\alpha)}} \Omega = \Omega_{\text{int}}, \quad \alpha \in [0,2\pi).\]
The vector \( \Omega_{\text{int}} \) was defined in Equ. (12.8.1);
iii.) the vector \( \Omega_{\text{int}} \) satisfies the Peierls-Bogoliubov and the Golden-Thompson inequalities:
\[
e^{-\pi (\Omega, V^{(\alpha)} \Omega)} \leq \|e^{-\pi H^{(\alpha)}} \Omega\| \leq \|e^{-\pi V^{(\alpha)} \Omega}\| ;
\]
iv.) the operator \( H^{(\alpha)} - J^{(\alpha)} V^{(\alpha)} J^{(\alpha)} \) is essentially self-adjoint on the domain
\[
\mathcal{D}(H^{(\alpha)}) \cap \mathcal{D}(J^{(\alpha)} V^{(\alpha)} J^{(\alpha)});
\]
and the closure equals \( L^{(\alpha)} \)
\[
H^{(\alpha)} - J^{(\alpha)} V^{(\alpha)} J^{(\alpha)} = L^{(\alpha)}, \quad \alpha \in [0,2\pi),
\]
where \( L^{(\alpha)} \) is defined in Equ. (14.1.15). Moreover, \( L^{(\alpha)} \Omega_{\text{int}} = 0 \);
v.) the conjugation \( J^{(\alpha)} \) defined in (12.6.3) satisfies
\[
J^{(\alpha)} e^{-\pi L^{(\alpha)} \Omega_{\text{int}}} A \Omega_{\text{int}} = A^* \Omega_{\text{int}} \quad \forall A \in \mathcal{R}(I_{\alpha}).
\]
Thus \( J^{(\alpha)} \) is the modular conjugation for the pair \( \langle I_{\alpha}, \Omega_{\text{int}} \rangle \). Note that Equ. (14.2.4) implies \( J^{(\alpha)} V^{(\alpha)} J^{(\alpha)} = J^{(\alpha)} \), with \( J^{(\alpha)} \) defined in (13.1.5).

**Proof.** Most of the arguments rely on results from the literature:
i.) Essential selfadjointness follows from the results on local symmetric semigroups by Fröhlich [61] and Klein and Landau [123][124]. The key step in the proof is to show that $L(\alpha) + V(\alpha) = H(\alpha)$ on

$$\mathcal{D} = \bigcup_{0 < \theta \leq \pi} \left( \bigcup_{0 < \theta' < \theta} \mathcal{P}_{\text{int}}(\alpha) (\theta') \mathcal{D}_{\text{int}}(\alpha) \right),$$

which itself is a core for $H(\alpha)$.

ii.) The expression on the l.h.s. of (14.2.3) is a formula, which is well known from the perturbation theory of KMS states (see [46][123]). The identification (14.2.3) follows from (14.1.2). Note that the final expression on the r.h.s. is independent of $\alpha$.

iii.) The Peierls-Bogoliubov and the Golden-Thompson inequalities (see [6]) were generalised to the present case in [46] Theorem 5.5.

iv.) Since $e^{-2\pi V(\alpha)} \in L^1(\Omega, \Sigma(\alpha), d\Phi_C)$ and $V \in L^p(\Omega, \Sigma(\alpha), d\Phi_C)$, $e^{-\pi V(\alpha)} \in L^q(\Omega, \Sigma(\alpha), d\Phi_C)$, with $p^{-1} + q^{-1} = \frac{1}{2}$ and $2 \leq p, q \leq \infty$, property (iv) follows from [67] Theorem 6.12. $L(\alpha) \Omega_{\text{int}} = 0$ follows from (14.1.4) and (14.1.5).

v.) This result is due to Klein and Landau [123]; see also [67] Lemma 7.13.

Note that $H(\alpha)$ does not implement the Lorentz boosts on $\mathcal{H}$. However,

$$\hat{\omega}^{\text{int}}_{\mathcal{A}(\alpha)}(A) = e^{itH(\alpha)} A e^{-itH(\alpha)} \quad \forall A \in \mathcal{R}(I_\alpha).$$

The difference between $H(\alpha)$ and $L(\alpha)_{\text{int}}'$ is an unbounded operator affiliated to the commutant $\mathcal{R}(I_\alpha)'$ of $\mathcal{R}(I_\alpha)$. This is in agreement with the perturbation theory of modular automorphisms [24][46].

**THEOREM 14.2.2 (Uniqueness of the interacting de Sitter vacuum state).** For each $\alpha \in [0, 2\pi)$, the restricted state

$$\omega^{\text{int}}_{\mathcal{R}(I_\alpha)}(A) = \langle \Omega_{\text{int}}, A \Omega_{\text{int}} \rangle, \quad A \in \mathcal{R}(I_\alpha),$$

is the unique $\hat{\omega}^{\text{int}}_{\mathcal{A}(\alpha)}$-KMS state on $\mathcal{R}(I_\alpha)$ and (therefore) $\omega^{\text{int}}$ is the unique de Sitter vacuum state for the $W^*$-dynamical system $(\mathcal{R}(S^1), \hat{\omega}^{\text{int}}_{\mathcal{A}(\alpha)}).$

**PROOF.** Note that $\mathcal{R}(I_\alpha)$ and

$$\mathcal{R}(I_\alpha) = \bigvee_{t \in \mathbb{R}} \left( e^{-itH(\alpha)} U(I_\alpha) e^{itH(\alpha)} \right),$$

coincide: as consequence of (14.2.2), (14.2.5) and the Trotter product formula

$$\hat{\omega}^{\text{int}}_{\mathcal{A}(\alpha)}(A) = \lim_{n \to \infty} \left( e^{i \frac{t}{n} V(\alpha)} \hat{\omega}^{\text{int}}_{\mathcal{A}(\alpha)(t/n)}(A) e^{i \frac{t}{n} V(\alpha)} \right)^n, \quad A \in U(I_\alpha),$$

we find $\mathcal{R}(I_\alpha) = \mathcal{R}(I_\alpha)$, as $e^{itV(\alpha)} \in \mathcal{R}(I_\alpha)$ for $t \in \mathbb{R}$. For the free field the $\hat{\omega}^{\text{int}}_{\mathcal{A}(\alpha)}$-KMS state on $\mathcal{R}(I_\alpha)$ is unique, thus $\mathcal{R}(I_\alpha)$ is a factor, and uniqueness of the interacting state now is a direct consequence of [24] Proposition 5.3.29, as was kindly pointed out to us by Jan Derezenski.
Theorem 14.2.3. The operator sum \( L^{(\alpha)} + V_0(\cos \phi + \alpha) \) is essentially self-adjoint and the closure

\[
L^{(\alpha)} + V_0(\cos \phi + \alpha) = L_{\text{int}}^{(\alpha)},
\]

where \( V_0(h) \) was defined in (11.2.1).

Note that the integration in (11.2.1) is over the whole circle \( S^1 \), while the integration in (14.2.1) is restricted to the halfcircle \( I_\alpha \).

Proof. This result follows from the fact that \( J^{(\alpha)} \) implements the space-reflection \( P^{(\alpha)} = R_0(\alpha)P_1R_0(\alpha)^{-1} \) on \( \mathcal{U}(S^1) \) and \( \cos(\frac{\pi}{2} + \psi) = -\cos(\frac{\pi}{2} - \psi) \). Thus

\[
V_0(\cos \phi + \alpha) = \mathcal{V}^{(\alpha)} - J^{(\alpha)}\mathcal{V}^{(\alpha)}J^{(\alpha)},
\]

The statement now follows from Theorem (14.2.1) and iv.).

14.3. Local von Neumann algebras on the circle \( S^1 \)

Let \( \mathcal{A}^{(\alpha)}_r(I_\alpha) \) denote the von Neumann algebra generated by

\[
\left\{ \hat{\alpha}_{\mathcal{A}^{(\alpha)}_r(I_\alpha)}(A) \mid A \in \mathcal{U}(I_\alpha), \ |t| < r \right\}.
\]

Then

\[
\bigcap_{r>0} \mathcal{A}^{(\alpha)}_r(I_\alpha) = \mathcal{R}(I_\alpha).
\]

(This is a special case of Theorem (14.3.1) below). This suggest to define the local non-commutative von Neumann algebra \( \mathcal{R}(I) \) as the intersection over the von Neumann algebras \( \mathcal{A}^{(\alpha)}_r(I) \) generated by

\[
\left\{ \hat{\alpha}_{\mathcal{A}^{(\alpha)}_r(I)}(A) \mid A \in \mathcal{U}(I), \ |t| < r \right\}.
\]

There is however the question, whether this definition depends on \( \alpha \). This is not the case, as will be shown next.

Theorem 14.3.1. Consider the real subspace \( \tilde{h}(I) \subset \tilde{h}(S^1) \),

(14.3.1) \( \tilde{h}(I) \doteq \left\{ h \in \tilde{h}(S^1) \mid \text{supp } (\mathfrak{R}h, \omega^{-1}\mathfrak{R}h) \subset I \times I \right\} \),

first introduced in Section 6.2. Then

(14.3.2) \( \mathcal{R}(I) \doteq \bigcap_{r>0} \mathcal{A}^{(\alpha)}_r(I) = \mathfrak{M}(\tilde{h}(I))'' \), \( I \subset I^{(\alpha)} \).

In particular, \( \mathcal{R}(I) \) as defined in (14.3.2) does not dependent on \( \alpha \).

Proof. The following argument is similar to the one given in the proof of [68, Theorem 6.5]. To simplify the notation, set

(14.3.3) \( \mathcal{H}(I) \doteq \mathfrak{M}(\tilde{h}(I))'' \).

We first prove that \( \bigcap_{r>0} \mathcal{A}^{(\alpha)}_r(I) \subset \mathcal{H}(I) \). Using \( \mathcal{U}(I) \subset \mathcal{R}(I) \) and finite-speed-of-light (Theorem 8.2.5), we see that

\[
\mathcal{A}^{(\alpha)}_r(I) \subset \mathcal{H}(I(\alpha, r)) \quad \forall r > 0.
\]

According to Proposition (8.2.3) the von Neumann algebras \( \mathcal{H}(I), I \subset S^1 \), are regular from the outside. This implies \( \bigcap_{r>0} \mathcal{A}^{(\alpha)}_r(I) \subset \mathcal{H}(I) \).
Let us now prove that \( \mathcal{W}(I) \subset \bigcap_{\tau > 0} A^\tau_\alpha(I) \). Using that the local time-zero algebras are regular from the inside (Proposition 8.2.3), it suffices to show that for each \( \mathcal{F} \subset I \) there exists some positive real number \( r \ll 1 \) such that

\[
(14.3.4) \quad \mathcal{W}(J) \subset A^r_\alpha(I) .
\]

To this end we fix \( I \) and \( J \) with \( \mathcal{F} \subset I \) and set \( \delta = \frac{1}{2} \text{dist}(J, I^\circ) \). We first note that

\[
(14.3.5) \quad e^{itL_\alpha} A e^{-itL_\alpha} \in A^r_\alpha(I) , \quad A \in \mathfrak{u}(J) , \quad |t| < r ,
\]

if \( 0 < r < \delta \). Clearly, the Weyl operators \( W_I(h) = \exp(i\Phi^\alpha(0, h)) \), \( h \in \hat{h}(S^1) \) real valued, belong to \( \mathfrak{u}(J) \) if \( \text{supp} \ h \in J \) and hence to \( A^r_\alpha(I) \). Now \( (14.3.5) \) implies

\[
(14.3.6) \quad \hat{\alpha}_{A^r_\alpha(I)}(t)(W_I(h)) = W_I(e^{it\cos\phi+\alpha}h) \in A^r_\alpha(I) , \quad |t| < r .
\]

Hence

\[
W_I(t^{-1}(e^{it\cos\phi+\alpha}h - h)) \in A^r_\alpha(I) , \quad |t| < \epsilon .
\]

Letting \( t \to 0 \) and using the fact that the map \( h \mapsto W_I(h) \) is continuous for the strong operator topology, we obtain that \( W_I(\omega \cos\phi+\alpha \ h) \in A^r_\alpha(I) \). But any vector \( h \in \hat{h}(J) \) can be approximated in norm by vectors of the form

\[
h_1 + i \omega \cos\phi+\alpha \ h_2 ,
\]

with \( \text{supp} \ h_1 \in J, \ i = 1,2, \ \text{real} \) and \( \cos\phi+\alpha \ h_2 \in \mathcal{D}(\omega) \). Thus for all \( h \in \hat{h}(J) \) the operators \( W_I(h) \) belong to \( A^r_\alpha(I) \) and hence \( \mathcal{W}(I) \subset A^r_\alpha(I) \).

14.4. Finite speed of light for the \( \mathcal{D}(\phi)_2 \) model

The set (see Proposition 1.5.2 for an explicit formula)

\[
I(\alpha, t) = S^1 \cap \left( \bigcup_{y \in \Lambda^{\alpha}(t) \cap I} \Gamma^-(y) \cup \Gamma^+(y) \right)
\]

describes the localisation region for the Cauchy data, which can influence space-time points in the set \( \Lambda^{\alpha}(t) \cap I \), \( t \in \mathbb{R} \) fixed.

**Theorem 14.4.1.** Let \( I \subset S^1 \) be an open interval. Then

\[
(14.4.1) \quad \hat{\alpha}_{A^\alpha(\cdot)}^\text{int} : \mathcal{R}(I) \to \mathcal{R}(I(\alpha, t)) .
\]

**Proof.** The following argument is similar to the one given in the proof of [80, Theorem 4.1.2]. We have seen in Theorem 8.2.5 that

\[
(14.4.2) \quad \hat{\alpha}_{A^\alpha(\cdot)}^\circ : \mathcal{R}(I) \to \mathcal{R}(I(\alpha, t)) .
\]

We can now explore the fact that on the half-circle \( I_\alpha \) the automorphism \( \hat{\alpha}_{A^\alpha(\cdot)}^\text{int} \) is unitarily implemented by \( e^{itH_\alpha} \), where \( H_\alpha = H^{\alpha} + V^{\alpha} \) with

\[
V^{\alpha} = \int_{I_\alpha} r \text{d}\psi \cos(\psi - \alpha) : \mathcal{D}(\Phi^\alpha(0, \psi)) : C_0 .
\]

Trotter’s product formula yields

\[
e^{itH_\alpha} = s - \lim_{n \to \infty} \left( e^{itL_\alpha}/n e^{itV^{\alpha}/n} \right)^n .
\]
Hence
\begin{equation}
\hat{\alpha}_{\alpha \{ t \}}^{| \{ t \} } (\alpha) = s - \lim_{n \to \infty} \left( \hat{\alpha}_{\alpha \{ t/n \}}^{| \{ t/n \} } \circ \hat{\gamma}_{t/n}^{(\alpha)} \right)^n (\alpha), \quad A \in \mathcal{R}(I_\alpha),
\end{equation}
with
\[ \hat{\gamma}_{t/n}^{(\alpha)}(A) = e^{itV(\alpha)}Ae^{-itV(\alpha)}. \]
Note that \( \hat{\gamma}_{t/n}^{(\alpha)} \) has zero propagation speed \(^{[80]}\), as for every open interval \( J \subset I_\alpha \) there exists \( V(\alpha) \) affiliated \(^{[1]}\) to \( \mathcal{U}(J) \) such that for all \( t \in \mathbb{R} \)
\[ e^{itV(\alpha)}Ae^{-itV(\alpha)} = e^{itV(\alpha)} \mathcal{R}_{\text{loc}}(\alpha) = e^{itV(\alpha)}(\alpha) = A \in \mathcal{R}(J). \]

Consequently,
\begin{equation}
\hat{\gamma}_{t/n}^{(\alpha)}(\mathcal{R}(J)) = \mathcal{R}(J) \quad \forall t \in \mathbb{R}.
\end{equation}

Now \((14.4.1)\) follows from \((14.4.3)\) and \((14.4.2)\).

**Theorem 14.4.2.** For \( I \subset S^1 \), let \( \mathcal{B}_r^{(\alpha)}(I) \) denote the von Neumann algebra generated by
\[ \left\{ \hat{\alpha}_{\alpha \{ t \}}^{| \{ t \} } (\alpha) \mid A \in \mathcal{U}(I), \ |t| < r \right\}. \]
Then
\begin{equation}
\bigcap_{r > 0} \mathcal{B}_r^{(\alpha)}(I) = \mathcal{R}(I), \quad I \subset S^1.
\end{equation}

Both sides in \((14.4.5)\) are independent of \( \alpha \).

**Proof.** Let us first prove that \( \bigcap_{r > 0} \mathcal{B}_r^{(\alpha)}(I) \subset \mathcal{R}(I) \). Using \((8.2.3)\) and \( \mathcal{U}(I) \subset \mathcal{R}(I) \), we see that
\[ \mathcal{B}_r^{(\alpha)}(I) \subset \mathcal{R}(I(\alpha, r)) \quad r > 0. \]
According to Proposition \((8.2.3)\), the local time-zero algebras are regular from the outside. This implies \( \bigcap_{r > 0} \mathcal{B}_r^{(\alpha)}(I) \subset \mathcal{R}(I) \).

Let us now prove that \( \mathcal{R}(I) \subset \bigcap_{r > 0} \mathcal{B}_r^{(\alpha)}(I) \). Using again Proposition \((8.2.3)\) (this time using that the local time-zero algebras are regular from the inside), it suffices to show that for each \( I \subset I \) there exists some positive real number \( r < \delta \) such that
\begin{equation}
\mathcal{R}(I(\alpha, r)) \subset \bigcap_{r > 0} \mathcal{B}_r^{(\alpha)}(I).
\end{equation}

To this end we fix \( J \) and \( I \) with \( J \subset I \) and set \( \delta = \frac{1}{2} \text{dist}(J, I^c) \). For \( |t| \leq \delta \) the unitary group \( e^{itH(\alpha)} \) with
\[ H(\alpha) = \frac{L(\alpha) + V(\alpha)}{2} \]
induces the correct dynamics \( \hat{\alpha}_{\alpha \{ t \}}^{| \{ t \} } \) on \( \mathcal{R}(J) \). Apply \([68] Proposition 2.5\) to obtain
\[ e^{itH(\alpha)} = s - \lim_{n \to \infty} e^{itH_n(\alpha)}, \quad t \in \mathbb{R}, \]
for \( H_n(\alpha) = L(\alpha) + V(\alpha) - V_n(\alpha) \), where \( V_n(\alpha) = V(\alpha) \mathbb{1}_{\{ |V(\alpha)| \leq n \}} \). Since \( V_n(\alpha) \) is bounded,
\[ H_n(\alpha) = L(\alpha) + V(\alpha) - V_n(\alpha) = H(\alpha) - V_n(\alpha) \]

\(^{[1]}\)Let \( \mathcal{R} \) be a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \). A closed and densely defined operator \( A \) is said to be affiliated with \( \mathcal{R} \) if \( A \) commutes with every unitary operator \( U \) in the commutant of \( \mathcal{R} \).
and Trotter’s formula yields
\[ e^{iH_n s} = s - \lim_{p \to \infty} \left( e^{iH_n s} / p \right)^p \to e^{-itV_n / p} \] .

Hence, for \( A \in R(I) \),
\[ e^{itL} A e^{-itL} = s - \lim_{n \to \infty} s - \lim_{p \to \infty} \left( e^{itH_n s} / p \right)^p A \left( e^{itV_n s} / p e^{-itH_n s} / p \right)^p \] .

For \( |t| < \tau \) and \( p \in \mathbb{N} \)
\[ \left( e^{itH_n s} / p \right)^p \to e^{itV_n s} / p \to e^{-itH_n s} / p \] .

Thus implies
\[ \left( \hat{\gamma}_n \right) (t/p) \right)^p (A) = (\mathcal{A} \left( t/p \right)) \to \mathcal{B}(\alpha) (I) , \quad |t| < \tau , \quad p \in \mathbb{N} . \]

Take the limit \( n \to \infty \) and recall from (14.4.3) that \( \hat{\gamma} = \lim_{n \to \infty} \hat{\gamma}_n \) has zero propagation speed. Since \( \mathcal{B}(\alpha) (I) \) is weakly closed, we obtain (14.4.6). The result now follows from Proposition 14.4.3.

**Remark 14.4.3.** Thus
\[ R_{\text{int}} (I) = R(I) \quad \forall I \subset S^1 . \]

We have seen earlier that \( R_{\text{int}} (I) \) is associated to the half-circles \( I \). If \( I \) is contained in some half-circle, then it follows that
\[ R_{\text{int}} (I) = \bigcap_{I \subset I} R_{\text{int}} (I) \]

can be identified with the intersection of all algebras \( R_{\text{int}} (I) \) associated to the half-circles \( I \), which contain \( I \).

### 14.5. The stress-energy tensor

One may introducing canonical time-zero fields \( \phi \) and canonical momenta \( \pi \): they can be defined in terms of the Fock fields \( \phi_F \) on \( \Gamma (\tilde{h} (S^1)) \) (see, e.g., [163]):
\[ \tilde{\phi}(h) = \phi_F (h) , \quad \tilde{\pi}(g) = \phi_F (i \omega g) , \quad h, g \in \tilde{h} (S^1 , \mathbb{R}) . \]

Thus \( \tilde{\phi}(h) = \phi_F (0 , h) \) and \( \tilde{\pi}(g) = -i [\Gamma (\omega , \phi_F (0 , g)] . \) They satisfy the canonical commutation relations
\[ [\tilde{\phi}(\psi) , \tilde{\pi}(\psi')] = \frac{i}{\Gamma} \delta (\psi - \psi') , \]
\[ [\tilde{\phi}(\psi) , \tilde{\phi}(\psi')] = [\tilde{\pi}(\psi) , \tilde{\pi}(\psi')] = 0 , \]
in the sense of quadratic forms on \( \Gamma (\tilde{h} (S^1)) \).
However, it is now more convenient to work on the Fock space over $L^2(S^1, \mathrm{d}\psi)$, using the map
\[
\hat{f}(S^1) \ni f \mapsto \frac{1}{\sqrt{2\omega}} f \in L^2(S^1, \mathrm{r}d\psi)
\]
to identify the two realisations of the Fock space. The canonical fields and the canonical momenta take the form
\[
\varphi(\psi) = \frac{1}{\sqrt{2}} \left( (\omega^{-\frac{1}{2}} a)(\psi)^* + (\omega^{-\frac{1}{2}} a)(\psi) \right),
\]
\[
\pi(\psi) = \frac{i}{\sqrt{2}} \left( (\omega^{\frac{1}{2}} a)(\psi)^* - (\omega^{\frac{1}{2}} a)(\psi) \right)
\]
with
\[
a(\psi) = \sum_{k \in \mathbb{Z}} \frac{e^{-ik\phi}}{\sqrt{2\pi}} a_k \quad \text{and} \quad a(\psi)^* = \sum_{k \in \mathbb{Z}} \frac{e^{ik\phi}}{\sqrt{2\pi}} a_k^*
\]
Thus
\[
(\omega^{\pm \frac{1}{2}} a)(\psi) = \sum_{k \in \mathbb{Z}} \tilde{\omega}(k)^{\pm \frac{1}{2}} e^{-ik\phi} a_k \quad \text{and} \quad (\omega^{\pm \frac{1}{2}} a)(\psi)^* = \sum_{k \in \mathbb{Z}} \tilde{\omega}(k)^{\pm \frac{1}{2}} e^{ik\phi} a_k^*
\]
and $[\pi(\psi'), \varphi(\psi)] = -\frac{i}{\pi} \delta(\psi - \psi')$ still holds. Using
\[
a(\psi) = \frac{1}{\sqrt{2}} \left( (\omega^{\frac{1}{2}} \varphi)(\psi) - i(\omega^{-\frac{1}{2}} \pi)(\psi) \right),
\]
\[
a(\psi)^* = \frac{1}{\sqrt{2}} \left( (\omega^{\frac{1}{2}} \varphi)(\psi) + i(\omega^{-\frac{1}{2}} \pi)(\psi) \right).
\]
one verifies that
\[
[a(\psi')^*, a(\psi)] = \frac{i}{2} \{[\pi(\psi'), \varphi(\psi)] + [\pi(\psi), \varphi(\psi')]\} = \frac{1}{\pi} \delta(\psi - \psi')
\]
and $[a(\psi')^*, a(\psi)^*] = [a(\psi'), a(\psi)] = 0$.

**Lemma 14.5.1.** Consider the Fock space over $L^2(S^1, \mathrm{d}\psi)$. It follows that\footnote{Note that Definition 8.1.5 refers to the original Fock space $\hat{\Gamma}(\hat{f}(S^1))$.}
\[
L^{(\alpha)} = df(\sqrt{\omega} \, r \cos(\psi + \alpha) \, \sqrt{\omega})
\]
\[
= \frac{1}{2} \int_{S^1} \pi^2(\psi) + \frac{1}{\pi} \left( \frac{\partial}{\partial \theta} \right)^2(\psi) + \mu^2 \varphi^2(\psi)
\]
is the generator of the free boost $t \mapsto U(\Lambda^{(\alpha)}(t))$ first introduced\footnote{We note that normal ordering is not needed at this point.} in Definition 8.1.5.

**Proof.** We write, using the fact that $\tilde{\omega}(k) = \tilde{\omega}(-k)$,
\[
\varphi(\psi) = \frac{1}{\sqrt{4\pi}} \sum_{k \in \mathbb{Z}} \tilde{\omega}(k)^{-\frac{1}{2}} \left( e^{ik\phi} a_k^* + e^{-ik\phi} a_k \right),
\]
\[
\frac{\partial}{\partial \psi}(\psi) = \frac{i}{\sqrt{4\pi}} \sum_{k \in \mathbb{Z}} \tilde{\omega}(k)^{-\frac{1}{2}} k \left( e^{ik\phi} a_k^* - e^{-ik\phi} a_k \right),
\]
\[
\pi(\psi) = \frac{i}{\sqrt{4\pi}} \sum_{k \in \mathbb{Z}} \tilde{\omega}(k)^{\frac{1}{2}} \left( e^{ik\phi} a_k^* - e^{-ik\phi} a_k \right).
\]
One has
\[
\mu^2 \psi^2 = \frac{\mu^2}{4\pi} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \bar{\omega}(k)^{-\frac{n}{2}} \bar{\omega}(l)^{-\frac{n}{2}} \times \left( e^{i(k+1)\psi} a_k^* a_l^* + e^{i(k-1)\psi} a_k^* a_l + e^{-i(k-1)\psi} a_k a_l^* + e^{-i(k+1)\psi} a_k a_l \right)
\]
\[
\frac{1}{4\pi} \left( \frac{\partial \psi}{\partial t} \right)^2(\psi) = -\frac{1}{4\pi} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \bar{\omega}(k)^{-\frac{n}{2}} \bar{\omega}(l)^{-\frac{n}{2}} kl \times \left( e^{i(k+1)\psi} a_k^* a_l^* - e^{i(k-1)\psi} a_k^* a_l + e^{-i(k-1)\psi} a_k a_l^* + e^{-i(k+1)\psi} a_k a_l \right)
\]
\[
\pi(\psi)^2 = -\frac{1}{4\pi} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \bar{\omega}(k)^{\frac{n}{2}} \bar{\omega}(l)^{\frac{n}{2}} kl \times \left( e^{i(k+1)\psi} a_k^* a_l^* - e^{i(k-1)\psi} a_k^* a_l + e^{-i(k-1)\psi} a_k a_l^* + e^{-i(k+1)\psi} a_k a_l \right)
\]

Next, define, for \( j \in \mathbb{Z} \),
\[
S_j \equiv \int_{S^1} d\psi \cos \psi e^{i\psi} \frac{1}{2} \int_{S^1} d\psi e^{i(j+1)\psi} + \frac{1}{2} \int_{S^1} d\psi e^{i(j-1)\psi} = \pi(\delta_{j,-1} + \delta_{j,1}).
\]

It is clear that \( S_j = S_{-j} \) for all \( j \in \mathbb{Z} \). Hence, we may write
\[
\frac{1}{2} \int_{S^1} r d\psi \cos \psi \pi(\psi)^2 = -\frac{r}{8\pi} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \bar{\omega}(k)^{\frac{n}{2}} \bar{\omega}(l)^{\frac{n}{2}} \times \left( S_{k+1} a_k^* a_l^* - S_{k-1} a_k^* a_l - S_{k-1} a_k a_l^* + S_{k+1} a_k a_l \right)
\]
\[
= \frac{r}{8} \sum_{k \in \mathbb{Z}} \bar{\omega}(k)^{\frac{n}{2}} \left[ \bar{\omega}(k+1)^{\frac{n}{2}} a_k^* a_{k+1} - \bar{\omega}(k-1)^{\frac{n}{2}} a_k^* a_{k-1} \\
+ \bar{\omega}(k+1)^{\frac{n}{2}} a_k a_{k+1} + \bar{\omega}(k-1)^{\frac{n}{2}} a_k a_{k-1} \\
+ \bar{\omega}(k+1)^{\frac{n}{2}} a_k^* a_{k+1} + \bar{\omega}(k-1)^{\frac{n}{2}} a_k^* a_{k-1} \\
- \bar{\omega}(k+1)^{\frac{n}{2}} a_k a_{k+1} - \bar{\omega}(k-1)^{\frac{n}{2}} a_k a_{k-1} \right]
\]
\[
\frac{1}{2} \int_{S^1} r d\psi \cos \psi \left( \frac{\partial \psi}{\partial t} \right)^2(\psi) = -\frac{1}{8\pi} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \bar{\omega}(k)^{-\frac{n}{2}} \bar{\omega}(l)^{-\frac{n}{2}} kl \times \left( S_{k+1} a_k^* a_l^* - S_{k-1} a_k^* a_l - S_{k-1} a_k a_l^* + S_{k+1} a_k a_l \right)
\]
\[
= \frac{1}{8\pi} \sum_{k \in \mathbb{Z}} \bar{\omega}(k)^{-\frac{n}{2}} k \times \left[ \bar{\omega}(k+1)^{-\frac{n}{2}} (k+1) a_k^* a_{k+1} - \bar{\omega}(k-1)^{-\frac{n}{2}} (k-1) a_k a_{k-1} \\
+ \bar{\omega}(k+1)^{-\frac{n}{2}} (k+1) a_k a_{k+1} + \bar{\omega}(k-1)^{-\frac{n}{2}} (k-1) a_k a_{k-1} \\
+ \bar{\omega}(k+1)^{-\frac{n}{2}} (k+1) a_k^* a_{k+1} + \bar{\omega}(k-1)^{-\frac{n}{2}} (k-1) a_k^* a_{k-1} \\
- \bar{\omega}(k+1)^{-\frac{n}{2}} (k+1) a_k a_{k+1} - \bar{\omega}(k-1)^{-\frac{n}{2}} (k-1) a_k a_{k-1} \right]
\]
\[\frac{\mu^2}{2} \int_{S^1} r \, d\psi \cos \psi \left( \varphi(\psi) \right)^2\]

\[= \frac{\mu^2 r}{8\pi} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \bar{w}(k)^{-\frac{1}{2}} \bar{w}(l)^{-\frac{1}{2}} \times \left( S_{k+1} a^*_k a^*_l + S_{k-1} a^*_k a^*_l + S_k a_k a_{k+1} + S_{k+1} a_k a_l \right)\]

\[= \frac{r}{8} \sum_{k \in \mathbb{Z}} \mu^2 \bar{w}(k)^{-\frac{1}{2}} \times \left[ \bar{w}(-k+1)^{-\frac{1}{2}} a^*_k a^*_k a_{k+1} + \bar{w}(-k-1)^{-\frac{1}{2}} a^*_k a^*_k a_{k-1} + \bar{w}(k+1)^{-\frac{1}{2}} a^*_k a^*_k a_{k-1} + \bar{w}(k-1)^{-\frac{1}{2}} a^*_k a^*_k a_{k-1} + \bar{w}(-k+1)^{-\frac{1}{2}} a^*_k a_{k+1} + \bar{w}(-k-1)^{-\frac{1}{2}} a^*_k a_{k-1} \right] \]

Rearranging the terms in order to join terms having factors involving the operators \(a\) in common and using the fact that \(\bar{w}(-k) = \bar{w}(k)\) for all \(k \in \mathbb{Z}\), we get

\[L_1 = \frac{r}{8} \sum_{k \in \mathbb{Z}} \left[ -\bar{w}(k)^{-\frac{1}{2}} \bar{w}(k+1)^{-\frac{1}{2}} \left( \bar{w}(k) \bar{w}(k-1) + r^2(-k+1)k - \mu^2 \right) a_k^* a_{k+1}^* \right] \quad \text{(A)}\]

\[+ \bar{w}(k)^{-\frac{1}{2}} \bar{w}(k+1)^{-\frac{1}{2}} \left( \bar{w}(k) \bar{w}(k+1) + r^2(-k+1)k - \mu^2 \right) a_k^* a_{k+1}^* \quad \text{(B)}\]

\[+ \bar{w}(k)^{-\frac{1}{2}} \bar{w}(k-1)^{-\frac{1}{2}} \left( \bar{w}(k) \bar{w}(k-1) + r^2(-k+1)k + \mu^2 \right) a_k^* a_{k-1}^* \quad \text{(-A+2\bar{w}(k)\bar{w}(k-1))}\]

\[+ \bar{w}(k)^{-\frac{1}{2}} \bar{w}(k-1)^{-\frac{1}{2}} \left( \bar{w}(k) \bar{w}(k-1) + r^2(k+1)k + \mu^2 \right) a_k^* a_{k+1}^* \quad \text{(B)}\]

\[+ \bar{w}(k)^{-\frac{1}{2}} \bar{w}(k-1)^{-\frac{1}{2}} \left( \bar{w}(k) \bar{w}(k-1) + r^2(k+1)k + \mu^2 \right) a_k^* a_{k-1}^* \quad \text{(-A+2\bar{w}(k)\bar{w}(k-1))}\]

\[(14.5.1)\]

Due to (7.4.18) and (7.4.9), both A, B vanish.
Returning with these informations to (14.5.1), we get
\[
L_1 = \frac{r}{4} \sum_{k \in \mathbb{Z}} \left[ \tilde{\omega}(k) \tilde{\omega}^\dagger(k + 1) a_k^{+} a_{k+1}^* + \tilde{\omega}(k) \tilde{\omega}^\dagger(k - 1) a_k^* a_{k-1}^* + \tilde{\omega}(k) \tilde{\omega}^\dagger(k + 1) a_k^{+} a_{k+1}^* + \tilde{\omega}(k) \tilde{\omega}^\dagger(k - 1) a_k^* a_{k-1}^* \right]
\]  
(14.5.2)

Now, we have
\[
\sum_{k \in \mathbb{Z}} \tilde{\omega}(k) \tilde{\omega}(k \pm 1) a_k a_{k\pm1}^* = \sum_{k \in \mathbb{Z}} \tilde{\omega}(k) \tilde{\omega}(k \pm 1) a_{k\pm1} a_k^* \sum_{k \in \mathbb{Z}} \tilde{\omega}(k) \tilde{\omega}(k \pm 1) a_{k\pm1}^* a_k^*
\]  
(14.5.3)

In the first equality in (14.5.3) we have used the fact that \( a_k \) and \( a_{k\pm1}^* \) commute. Inserting (14.5.3) into (14.5.2), we get, finally,
\[
L_1 = \frac{r}{2} \sum_{k \in \mathbb{Z}} \left[ \tilde{\omega}(k) \tilde{\omega}^\dagger(k + 1) a_k^{+} a_{k+1}^* + \tilde{\omega}(k) \tilde{\omega}^\dagger(k - 1) a_k^* a_{k-1}^* \right].
\]

On the other hand, one has
\[
\int_{S^1} r \, d\psi (\omega^{1/2} a)(\psi)^* \cos \psi (\omega^{1/2} a)(\psi)
\]
\[
= \frac{r}{2} \sum_{k \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \left( \frac{1}{2\pi} \int_{S^1} d\psi \, e^{i(k-l+1)\psi} + e^{i(k-l-1)\psi} \right) \tilde{\omega}(k)^{1/2} \tilde{\omega}(l)^{1/2} a_k^* a_t
\]
\[
= \frac{r}{2} \sum_{k \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \left( \delta_{l,k+1} + \delta_{l,k-1} \right) \tilde{\omega}(k)^{1/2} \tilde{\omega}(l)^{1/2} a_k^* a_t
\]
\[
= \frac{r}{2} \sum_{k \in \mathbb{Z}} \left( \tilde{\omega}(k)^{1/2} \tilde{\omega}(k + 1)^{1/2} a_k^{+} a_{k+1}^* + \tilde{\omega}(k)^{1/2} \tilde{\omega}(k - 1)^{1/2} a_k^* a_{k-1}^* \right).
\]

Therefore,
\[
L_1 = \frac{1}{2} \int_{S^1} r \, d\psi \cos \psi \left( \pi(\psi)^2 + \left( \frac{\partial \pi}{\partial \psi}(\psi) \right)^2 + \mu^2 (\varphi(\psi))^2 \right)
\]
\[
= \int_{S^1} r \, d\psi (\omega^{1/2} a)(\psi)^* \cos \psi (\omega^{1/2} a)(\psi)
\]
\[
= d\Gamma(\sqrt{\omega} \cos \psi \sqrt{\omega}).
\]

\[\square\]

The energy density \( T_{00}(\psi) \) is the restriction of the energy density in the time-zero plane (in the ambient Minkowski space) to the Cauchy surface \( S^1 \), i.e., for \( \psi \in S^1 \),
\[
(14.5.5) \quad T_{00}(\psi) = \frac{1}{2} \left( \pi^2(\psi) + \frac{1}{\pi^2}(\frac{\partial \pi}{\partial \psi}(\psi))^2 + \mu^2 \varphi(\psi) + : P(\varphi(\psi)) : C_\phi \right).
\]

The following formulas should be compared with the classical expressions derived in Section 5.2.
14.5. THE STRESS-ENERGY TENSOR

**Lemma 14.5.2.** The following identities hold in the sense of quadratic forms on the Hilbert space \( \Gamma(L^2(S^1, r \, d\psi)) \):

\[
L^{(\alpha)}_{\text{int}} = \int_{S^1} r \cos(\psi + \alpha) \, d\psi \, T_{00},
\]

\[
K_0 = \int_{S^1} r^2 |\cos \psi| \, d\psi \, T_{0\psi},
\]

with \( T_{0\psi} = (r \cos \psi)^{-1} \pi (\partial_\psi \Phi) \).

**Proof.** Recall that \( L^{(\alpha)} = d\Gamma(\sqrt{\omega} \, r \cos \psi + \alpha \sqrt{\omega}) \). Moreover, according to Theorem [14.2.3]

\[
L^{(\alpha)}_{\text{int}} = d\Gamma(\omega^{-1/2}) \, d\Gamma(\omega \, r \cos \psi + \alpha) \, d\Gamma(\omega^{-1/2})
\]

with \( V^{(\alpha)} = \int_{S^1} r \, d\psi + (r \cos \psi + \alpha) \cdot P(\phi(\psi)) \cdot c_0 \).

Next consider the angular momentum operator:

\[
K_0 = \int_{S^1} r \, d\psi : \pi (\partial_\psi \Phi):
\]

\[
= \frac{i}{4\pi} \sum_{k \in Z} \sum_{j \in Z} \int_{S^1} d\psi : \bar{\omega}(k)^{\frac{1}{2}} \left( e^{ik\psi} a_k^* - e^{-ik\psi} a_k \right)
\]

\[
\times \partial_\psi \bar{\omega}(j)^{-\frac{1}{2}} \left( e^{ij\psi} a_j^* + e^{-ij\psi} a_j \right) :
\]

\[
= \frac{i}{4\pi} \sum_{k \in Z} \sum_{j \in Z} \bar{\omega}(k)^{\frac{1}{2}} \left( a_k^* a_j \int_{S^1} d\psi \, e^{ik\psi} \partial_\psi e^{-ij\psi} + a_j^* a_k \int_{S^1} d\psi \, e^{-ik\psi} \partial_\psi e^{ij\psi} \right)
\]

\[
= \frac{1}{2\pi} \sum_{k \in Z} k a_k^* a_k = d\Gamma(-i\partial_\psi).
\]

\( \square \)

**Theorem 14.5.3.** The operator \( L^{(0)}_{\text{int}} \) (and similar \( L^{(0)} \))

\[
L^{(0)}_{\text{int}} = \int_{L_+} d\psi \, \tau \cos \psi \, T_{00}(\psi) + \int_{L_-} d\psi \, \tau \cos \psi \, T_{00}(\psi)
\]

\[
= L^{+}_{\text{int}} + L^{-}_{\text{int}} + \langle \Omega | \Omega \rangle
\]

splits into a positive and negative part,

\[
\text{Sp}(\pm L^{\pm}_{\text{int}}) = [0, \infty).
\]

Moreover, \( \text{Sp}(L^{\pm}_{\text{int}}) \) is absolutely continuous.

**Proof.** The operator \( \omega \cos \psi = (\omega \cos \psi)_{L_+} + (\omega \cos \psi)_{L_-} \) is the sum of a positive operator \( (\omega \cos \psi)_{L_+} \) acting on \( \hat{h}(L_+) \), and a negative operator \( (\omega \cos \psi)_{L_-} \) on \( \hat{h}(L_-) \); see Proposition [12.5.5]

To show that

\[
L^{+}_{\text{int}|L_+} = d\Gamma(\tau \cos \psi)_{L_+} + \int_{L_+} d\psi \, \tau \cos \psi \cdot P(\phi(\psi)) \cdot c_0
\]

is bounded from below on \( \Gamma(\hat{h}(S^1)) \), one can follow the arguments given in [57].

The main idea is to cut off the support of the interaction at the
boundaries by some distance $\epsilon \ll \pi r$ form the boundary point. The bulk can then be bounded from below following standard arguments (see, e.g., \cite{149} and also the proof of Proposition 14.5.5 below.) It remains to estimate the two contributions from the interaction in $\epsilon$-neighbourhoods of the boundary. This does not cause a problem, as the edges of the wedge are fixed points under the action of the boosts. A detailed analytic argument will be presented elsewhere. □

**Remark 14.5.4.** We expect that the decomposition of $L_\omega^{\{0\}}$ given above is relevant in context of the dethermalization discussed by Guido and Longo in \cite{82}. However, further work is need to clarify this question.

In connection with Remark 5.2.3 it is worth while noting the following result:

**Proposition 14.5.5 ($\phi$-bounds).** For $c \gg 1$,

\begin{equation}
\left\| \phi(g) \left( \int_{S^1} r \, d\psi \, T_{00}(\psi) + c \right)^{-\frac{1}{2}} \right\| \leq C \|g\|_{\hat{h}(S^1)},
\end{equation}

and

\begin{equation}
\pm \phi(g) \leq C \|g\|_{\hat{h}(S^1)} \left( \int_{S^1} r \, d\psi \, T_{00}(\psi) + c \right)^{\frac{1}{2}},
\end{equation}

for all $g \in \hat{h}(S^1)$. In particular, $\int_{S^1} r \, d\psi \, T_{00}(\psi)$ is bounded from below.

**Proof.** One easily obtains (see, e.g., \cite{175} Theorem V.20] or \cite{45} Theorem 6.4 (ii)) that

\begin{equation}
(d\Gamma(\omega) + 1) \leq C \left( \int_{S^1} r \, d\psi \, T_{00}(\psi) + c \right) \text{ for } c \gg 1.
\end{equation}

Since $d\Gamma(\omega)$ has compact resolvent on $\Gamma(\hat{h}(S^1))$, it follows that

\begin{equation}
\int_{S^1} r \, d\psi \, T_{00}(\psi)
\end{equation}

is bounded from below with a compact resolvent and hence has a ground state. The uniqueness of this ground state of follows from a Perron-Frobenius argument (see e.g. \cite{175} Theorem V.17)). Since

\[ \omega \geq m^\circ > 0 \]

for some $m^\circ > 0$, we see that it suffices to check \(14.5.6\) with \(14.5.9\) replaced by the number operator $N$, which is immediate. To prove \(14.5.7\) we use \(14.5.8\) and the well known bound (see, e.g., \cite{66} Appendix)

\[ \pm \phi(g) \leq \|g\|_{\hat{h}(S^1)} (d\Gamma(\omega) + 1). \]

□
14.6. The equations of motion

Equations of motion for interacting quantum fields on Minkowski space were derived by Glimm and Jaffe [74], Schrader [168] and, in $2+1$ space-time dimensions, by Feldman and Raczka [54]. Formulas similar to the ones presented in this section were given in [57].

Use the coordinate system

$$x(t, \psi) = \Lambda^{(\alpha)}(\frac{1}{t}) \begin{pmatrix} 0 \\ r \sin \psi \\ r \cos \psi \end{pmatrix}, \quad x \in \mathcal{W}(\alpha),$$

with $t \in \mathbb{C}$ and $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2}) - \alpha$ and define

$$\Phi_{\text{int}}(x) := e^{i\frac{x}{t}L^{(\alpha)}} \varphi(\psi)e^{-i\frac{x}{t}L^{(\alpha)}}, \quad x \equiv x(t, \psi).$$

We note that the interacting field $\Phi_{\text{int}}(x)$, defined as an operator-valued distribution at a space-time point $x \in \text{dS}$ does not depend on the choice of coordinates in (14.6.1).

**THEOREM 14.6.1.** The interacting quantum field $\Phi_{\text{int}}(x), x \in \text{dS}$, satisfies the covariant equation of motion:

$$\left(\Box_{\text{dS}} + \mu^2\right) \Phi_{\text{int}}(x) = -:\varphi(\psi)\mathcal{P}^{(\alpha)}\pi(\psi):\text{c}.$$

**PROOF.** Without restriction of arbitrariness, we may consider the case $L^{(\alpha)}_{\text{int}} = L^{(0)}_{\text{int}}$ and compute (following [163], p. 224)

$$[L^{(0)}_{\text{int}}, \varphi(\psi)] = [L^{(0)}_{\text{int}}, \varphi(\psi)] = -i\tau \cos(\psi) \pi(\psi)$$

and

$$[L^{(0)}_{\text{int}}, [L^{(0)}_{\text{int}}, \varphi(\psi)]] = [L^{(0)}_{\text{int}}, [L^{(0)}_{\text{int}}, \varphi(\psi)]] + [V^{(0)}_{\text{int}}, [L^{(0)}_{\text{int}}, \varphi(\psi)]] .$$

The first term on the right hand side yields

$$[L^{(0)}_{\text{int}}, [L^{(0)}_{\text{int}}, \varphi(\psi)]] = -i\tau \cos(\psi) [L^{(0)}_{\text{int}}, \pi(\psi)]$$

(14.6.2)

$$= -\left(\tau \cos(\psi) \partial_{\psi}\right)^2 \varphi(\psi) + \tau^2 \cos(\psi) \mu^2 \varphi(\psi) .$$

The second equality follows from partial integration (see (14.5.5)), i.e.,

$$\frac{1}{2} \int d\psi' \tau \cos(\psi') \left[\partial_{\psi'} \varphi(\psi')\right]^2 \pi(\psi)$$

$$= -\int d\psi' \partial_{\psi'} \tau \cos(\psi') \partial_{\psi'} \varphi(\psi') \left[\varphi(\psi'), \pi(\psi)\right] .$$

The second term yields

$$[V^{(\alpha)}_{\text{int}}, [L^{(0)}_{\text{int}}, \varphi(\psi)]] = -i\tau \cos(\psi) [V^{(\alpha)}_{\text{int}}, \pi(\psi)]$$

$$= -i\tau \cos(\psi) \int d\psi' \cos(\psi') \left[\mathcal{P}(\varphi(\psi'))\mathcal{P}^{(\alpha)}\pi(\psi)\right]$$

$$= \tau^2 \cos^2(\psi) :\mathcal{P}(\varphi(\psi))\mathcal{P}^{(\alpha)}\pi(\psi): c_0 .$$
Set \( x \equiv x(t_\alpha, \psi) \). Use definition (14.6.1) and compute
\[
\frac{\partial^2}{\partial t^2} \Phi_{\text{int}}(x) = -[L_{\text{int}}^{(0)}, [L_{\text{int}}^{(0)}, \Phi_{\text{int}}(x)]]
\]
\[
= \left( \cos(\psi) \partial_{\psi} \right)^2 - \cos(\psi) \mu^2 e^{L_{\text{int}}^{(0)} t} \varphi(\psi) e^{-L_{\text{int}}^{(0)} t} - \cos^2(\psi) :P'(e^{L_{\text{int}}^{(0)} t} \varphi(\psi) e^{-L_{\text{int}}^{(0)} t}) :C_0
\]
\[
= \left( \cos(\psi) \partial_{\psi} \right)^2 \Phi_{\text{int}}(x) - r^2 \cos^2(\psi) \left( \mu^2 \Phi_{\text{int}}(x) - P'(\Phi_{\text{int}}(x)) :C \right),
\]
i.e., \((\partial_t^2 + \epsilon^2) \Phi_{\text{int}}(x) = r^2 \cos^2(\psi) :P'(\Phi_{\text{int}}(x)) :C\) with
\[
\epsilon^2 = -(\cos \psi \partial_{\psi})^2 + (\cos \psi)^2 \mu^2 r^2.
\]
Recalling from (5.4.3) that in the coordinates \( x = x(x_0, \psi) \),
\[
\Box_{\text{dS}} + \mu^2 = \frac{1}{r^2 \cos^2 \psi} (\partial_t^2 + \epsilon^2),
\]
we arrive at the equations of motion in their covariant form
\[
\left( \Box_{\text{dS}} + \mu^2 \right) \Phi_{\text{int}}(x) = - :P'(\Phi_{\text{int}}(x)) :C.
\]
One particle structures

Let G be a group. A (classical) linear dynamical system \((t, \sigma, \{T_g\}_{g \in G})\) is a real symplectic vector space \((t, \sigma)\) together with a group of symplectic transformations \(\{T_g\}_{g \in G}\). If \(h\) is a complex Hilbert space with scalar product \(\langle ., . \rangle\), then \((h, 2\mathbb{I}\langle ., . \rangle)\) is a symplectic space. If, in addition, a unitary representation \(\{u(g)\}_{g \in G}\) of G is given, then \((h, 2\mathbb{I}\langle ., . \rangle, \{u(g)\}_{g \in G})\) is a linear dynamical system.

**Definition A.0.2.** Given a linear dynamical system \((t, \sigma, \{T_g\}_{g \in G})\), a symplectic transformation \(K : h \rightarrow h\) defines a one-particle quantum structure on a Hilbert space \(h\), if there exists a group of unitary operators such that the following diagram commutes

\[
\begin{array}{ccc}
(t, \sigma) & \xrightarrow{K} & (h, 2\mathbb{I}\langle ., . \rangle) \\
\downarrow T_g & & \downarrow u(g) \\
(t, \sigma) & \xrightarrow{K} & (h, 2\mathbb{I}\langle ., . \rangle)
\end{array}
\]

By definition, \(K\) is injective. Kay \[116, 117, 118\] has shown that one can associate several essentially unique one-particle quantum structures to a given classical dynamical system.

**Definition A.0.3.** Given a linear dynamical system \((t, \sigma, \{T_t\}_{t \in \mathbb{R}})\), the symplectic transformation \(K\) specifies

- a one-particle structure with positive energy, if
  - i.) \(t \mapsto u(t)\) is strongly continuous and its generator \(\epsilon \geq 0\) is positive;
  - ii.) \(Kt\) is dense in \(h\).
- a one-particle \(\beta\)-KMS structure, if
  - iii.) the map \(t \mapsto \langle Kf, u(t)Kg \rangle, f, g \in t\), is analytic in the strip \(\{t \in \mathbb{C} | 0 < \Im t < \beta\}\), continuous at the boundary, and satisfies the one-particle \(\beta\)-KMS condition
  \[
  \langle Kf, u(t + i\beta)Kg \rangle = \langle u(t)Kg, Kf \rangle, \quad t \in \mathbb{R}, f, g \in t;
  \]
  - iv.) \(Kt + iKt\) is dense in \(h\).
Note that the Hilbert space $\mathfrak{h}$ and the one-parameter group $t \mapsto u(t)$ acting on it, although denoted by the same letters in i.)–ii.) and iii.)–iv.), are necessarily different in the two distinct cases.

**Proposition A.0.4 (Kay [117], Theorems 1a & 1b).** There exists a unique (up to unitary equivalence) one-particle structure with positive energy for which zero is not an eigenvalue of the generator of $t \mapsto u(t)$. Moreover, for each $\beta > 0$ there exists a unique (up to unitary equivalence) one-particle $\beta$-KMS structure for which zero is not an eigenvalue of the generator of $t \mapsto u(t)$.

_Notation._ If $\mathfrak{h}$ is a complex vector space, then the conjugate vector space $\overline{\mathfrak{h}}$ is the real vector space $\mathfrak{h}$ equipped with the complex structure $-i$. We denote by

$$\mathfrak{h} \ni h \mapsto \bar{h} \in \overline{\mathfrak{h}}$$

the *linear* identity operator. If $\mathfrak{h}$ is a Hilbert space, then the conjugate Hilbert space $\overline{\mathfrak{h}}$ is equipped with the scalar product $\langle \bar{h}_1, \bar{h}_2 \rangle = \langle h_2, h_1 \rangle$. If $a \in \mathcal{L}(\mathfrak{h})$, then we denote by $\overline{a\mathfrak{h}} = \overline{\mathfrak{a}} \mathfrak{h}$.

Given a one-particle structure with positive energy there exists an associated one-particle $\beta$-KMS structure:

**Proposition A.0.5.** Let $(\mathfrak{K}, \mathfrak{h}, \{u(t)\}_{t \in \mathbb{R}})$ be a one-particle structure with positive energy for a classical dynamical system $(\mathfrak{t}, \mathfrak{O}, \{T_t\}_{t \in \mathbb{R}})$. If $\mathfrak{K}t \in \mathcal{D}(e^{-1/2})$, then

$$K_{aw}f = (1 + \varrho)^{1/2}Kf + \varrho^{1/2}Kf, \quad \varrho \equiv (e^{\mathfrak{h}t} - 1)^{-1},$$

$$\mathfrak{h}_{aw} \equiv \mathfrak{h} \oplus \overline{\mathfrak{h}},$$

$$u_{aw}(t) \equiv u(t) \oplus \overline{u(t)} ,$$

defines a one particle $\beta$-KMS structure for $(\mathfrak{t}, \mathfrak{O}, \{T_t\}_{t \in \mathbb{R}})$.

_Remarks:_

i.) The subscripts used in $K_{aw}, \mathfrak{h}_{aw}$ and $u_{aw}(t)$ pay tribute to the fundamental work of Araki and Woods [7].

ii.) $(\mathfrak{h}_{aw}, \{u_{aw}(t)\}_{t \in \mathbb{R}})$ is a one-particle $\beta$-KMS structure for the dynamical system $(\mathfrak{h}, \mathfrak{O}, \{u(t)\}_{t \in \mathbb{R}})$, specified by $K_{aw}: \mathfrak{h} \rightarrow \mathfrak{h}_{aw},$

$$\mathfrak{h} \mapsto (1 + \varrho)^{1/2} \mathfrak{h} \oplus \varrho^{1/2} \mathfrak{h},$$

has negative spectrum.

iii.) $\overline{u(t)} = e^{it\mathfrak{h}} = e^{-it\mathfrak{h}}$, hence the generator of the one-parameter group $t \mapsto \overline{u(t)}$

iv.) The space $\mathfrak{h}^t \equiv \{K_{aw}f \mid f \in \mathfrak{h}\}$ is a real subspace in $\mathfrak{h}_{aw}$. Moreover, $\mathfrak{h}^t + i\mathfrak{h}^t$ is dense in $\mathfrak{h}_{aw}$ and $\mathfrak{h}^t \cap i\mathfrak{h}^t = \{0\}$. Thus one can define, following Eckmann and Osterwalder [52] (see also [134]), a closeable operator

$$s: \begin{array}{c} \mathfrak{h}^t + i\mathfrak{h}^t \rightarrow \mathfrak{h}^t + i\mathfrak{h}^t \\ f + ig \mapsto f - ig \end{array}$$

(A.0.4)

The polar decomposition of its closure $s = j\delta^{1/2}$ provides
an anti-unitary involution (i.e., a conjugation)

\[ j : \mathfrak{h} \oplus \overline{\mathfrak{h}} \rightarrow \mathfrak{h} \oplus \overline{\mathfrak{h}} \]

\[ f \oplus g \mapsto \overline{g} \oplus \overline{f} \]

— a complex linear, positive operator \( \delta^{1/2} \), such that

\[ \delta^{it} = u_{AW}(-t\beta), \quad t \in \mathbb{R}. \]

\( A.0.5 \) implies \( j\mathfrak{h}^{\perp} = \mathfrak{h}^{\perp} \) and \( A.0.6 \) implies that \( \{\delta^{it}\}_{t \in \mathbb{R}} \) leaves the subspaces \( \mathfrak{h}^{\perp} \) and \( \mathfrak{h}^{\perp} \) invariant.

v.) Sometimes we denote \( K_{aw} \) by \( K_{aw}^{L} \). This is useful as one encounters as well the map \( \delta_{aw} : k \rightarrow \mathfrak{h}_{aw} \),

\[ \delta_{aw} \mathfrak{g} = \frac{1}{2} \mathfrak{g} \oplus (1 + \beta)_{2} \mathfrak{g}, \]

which maps \( \mathfrak{k} \) to the symplectic complement \( \mathfrak{h}^{\perp} \subset \mathfrak{h}_{aw} \) of \( \mathfrak{h}^{\perp} \).

vi.) The triple \( \{K_{aw}, \mathfrak{h}_{aw}, \{u_{aw}(t)\}_{t \in \mathbb{R}}\} \) provides a \((-\beta)\)-KMS structure for the linear dynamical system \( (\mathfrak{t}, \sigma, \{T_{t}\}_{t \in \mathbb{R}}) \).

The existence of vi.) motivated Kay \([117][118]\) to investigate the possibility of doubling the classical dynamical system as well:

**Definition A.0.6.** Let \( \mathfrak{k} = \mathfrak{k}_{L} \oplus \mathfrak{k}_{R} \) be the direct sum of two symplectic subspaces \( \mathfrak{k}_{L} \) and \( \mathfrak{k}_{R} \) such that

\[ \mathfrak{g}(f, g) = 0 \quad \text{if} \quad f \in \mathfrak{k}_{L} \quad \text{and} \quad g \in \mathfrak{k}_{R}. \]

Let \( \{T_{t}\}_{t \in \mathbb{R}} \) be a one-parameter group of symplectic maps, which leaves \( \mathfrak{k}_{L} \) and \( \mathfrak{k}_{R} \) invariant. Furthermore, let \( j \) be an anti-symplectic involution such that

\[ [j, T_{t}] = 0 \quad \text{and} \quad jT_{t}j = T_{t}. \]

The quadruple \( \{\mathfrak{k}, \mathfrak{g}, \{T_{t}\}_{t \in \mathbb{R}}, j\} \) is called a double (classical) linear dynamical system.

It follows that \( jT_{t}j = T_{t} \). In other words, the following diagram commutes:

\[ \begin{array}{ccc}
\mathfrak{k}_{L} & \xrightarrow{j} & \mathfrak{k}_{R} \\
\downarrow T_{t} & & \downarrow T_{t} \\
\mathfrak{k}_{L} & & \mathfrak{k}_{R}
\end{array} \]

**Definition A.0.7.** (Kay \([117]\), Def. 3). A double \( \beta \)-KMS one-particle structure, i.e., a quadruple \( \{K, \mathfrak{h}, \{\delta_{aw}^{-it}/\beta\}_{t \in \mathbb{R}}, j\} \), associated to a double linear classical dynamical system \( \{\mathfrak{t}, \mathfrak{g}, \{T_{t}\}_{t \in \mathbb{R}}, j\} \) consists of

i.) a complex Hilbert space \( \mathfrak{h} \);

ii.) a real linear symplectic map \( K : \mathfrak{k} \rightarrow \mathfrak{h} \) such that \( K\mathfrak{k}_{L} + iK\mathfrak{k}_{R} \) is dense in \( \mathfrak{h} \);

iii.) a strongly continuous unitary group \( t \mapsto \delta_{aw}^{-it}/\beta \) such that
\[ \delta^{-it/\beta} \circ K = K \circ T_t \text{ for all } t \in \mathbb{R}; \]

\[ Kk + iKk \subset D(\delta^{1/2}); \]

iv.) an anti-unitary operator \( j \) such that \( j \circ K = K \circ 1 \) on \( \mathfrak{g} \) and

\[ j\delta^{1/2}f = f \quad \forall f \in Kk. \]

The operator \( \delta \) is positive, \( Kk + iKk \) is dense in \( \mathfrak{h} \),

\[ Kk + iKk \subset D(\delta^{-1/2}) \]

and \( j\delta^{-1/2}g = g \) for all \( g \in Kk. \).

**Theorem A.0.8** (Kay [117], Theorem 2). There exists a unique, up to unitary equivalence, double \( \beta \)-KMS-structure for which the generator \( \varepsilon \) of the one parameter group

\[ \delta^{it} = e^{-it\beta \varepsilon}, \quad \beta > 0, \]

has no zero eigenvalue.

\[ ^1 \text{This is in agreement with [A.0.6].} \]
Fock space

Consider a Hilbert space $h$ with scalar product $\langle ., . \rangle$. Let $\Gamma^{(n)}(h)$, $n \in \mathbb{N}$, be the $n$-fold totally symmetric tensor product $\otimes_s$ of $h$ with itself. The elements of $\Gamma^{(n)}(h)$ are of the form

$$P_+(f_1 \otimes \ldots \otimes f_n) \doteq \frac{1}{n!} \sum_\pi f_{\pi_1} \otimes f_{\pi_2} \otimes \ldots \otimes f_{\pi_n}, \quad f_1, \ldots, f_n \in h.$$ 

The sum is over all permutations $\pi: (1, 2, \ldots, n) \mapsto (\pi_1, \pi_2, \ldots, \pi_n)$. In other words, the symmetrisation operator $P_+$ takes care of the necessary symmetrisation required.

**Definition B.0.9.** The symmetric Fock-space $\Gamma(h)$ over $h$ is the direct sum of the $n$-particle spaces:

$$\Gamma(h) \doteq \bigoplus_{n=0}^{\infty} \Gamma^{(n)}(h),$$

with $\Gamma^{(0)}(h) \doteq \mathbb{C}$.

The vectors with finitely many components unequal to zero form a dense subspace

$$\Gamma^\circ(h) \doteq \text{Span}\left\{ \bigotimes_{n=0}^{N} \Gamma^{(n)}(h) \mid N \in \mathbb{N}\right\}$$

in $\Gamma(h)$. The vector $\Omega \doteq (1, 0, 0, \ldots)$ is called the Fock vacuum vector.

For $f \in h$ define the creation operator $a^*(f): \Gamma^\circ(h) \to \Gamma^\circ(h)$ by

$$a^*(f)\Psi(n) \doteq \sqrt{n+1} f \otimes_s \Psi(n), \quad \Psi(n) \in \Gamma^{(n)}(h).$$

The operator $a(f)$ denotes the adjoint of $a^*(f)$, and is called the annihilation operator. Both $a(f)$ and $a^*(f)$ are defined on $\Gamma^\circ(h)$ and can be extended to densely defined closed, unbounded operators on $\Gamma(h)$.

The map $f \mapsto a^*(f)$ is linear, while the map $f \mapsto a(f)$ is anti-linear. They satisfy the canonical commutation relations:

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0$$

and

$$[a(f), a^*(g)] = \langle f, g \rangle \quad \forall f, g \in h.$$ 

By applying the creation operators $a^*(f)$ to $\Omega$ we get $\Gamma^\circ(h)$ and by closure all of $\Gamma(h)$:

$$a^*(f_1) \ldots a^*(f_n)\Omega = \sqrt{n!} \left( f_1 \otimes_s \ldots \otimes_s f_n \right) \in \Gamma^{(n)}(h), \quad f_1, \ldots, f_n \in h.$$ 

\footnote{We follow [24] Vol. II.}
The symmetric operator $a^*(f) + a(f)$ is essentially self-adjoint on $\Gamma^0(\mathfrak{h})$, its closure is denoted by

$$\Phi(f) \doteq \frac{1}{\sqrt{2}} (a^*(f) + a(f))^\perp.$$  

The field operators $\Phi(f)$ satisfy, in the sense of quadratic forms on $\mathscr{D}(\Phi(f)) \cap \mathscr{D}(\Phi(g))$, the commutation relations

$$[\Phi(f), \Phi(g)] = i \mathcal{J}(f, g), \quad f, g \in \mathfrak{h}.$$  

The Weyl operators $W_f(f) \doteq e^{i\Phi(f)}$ satisfy

$$W_f(f)W_g(g) = e^{i\mathcal{J}(f, g)}W_{f + g}(g), \quad f, g \in \mathfrak{h}.$$  

$W_f(f)$ is related to the exponentials $e^{ia^*(f)}$ and $e^{ia(f)}$ by

$$W_f(f) = e^{ia^*(f)} e^{ia(f)} e^{-\frac{1}{2} \|f\|^2}$$  

on $\Gamma^0(\mathfrak{h})$. Thus

$$(\Omega, W_f(f)\Omega) = e^{-\frac{1}{2} \|f\|^2} \quad \forall f \in \mathfrak{h}.$$  

Moreover, the Weyl operators $\{W_f(f) \mid f \in \mathfrak{h}(dS)\}$ generate all of $\mathcal{B}(\Gamma(\mathfrak{h}))$.

Given a selfadjoint operator $H$ acting on the one-particle space $\mathfrak{h}$, one can define operators $H_n$ acting on the $n$-particle space $\Gamma^{(n)}(\mathfrak{h})$ by setting $H_0 = 0$ and

$$H_n(P_+(f_1 \otimes \ldots \otimes f_n)) \doteq P_+\left( \sum_i f_1 \otimes f_2 \otimes \ldots \otimes Hf_i \otimes \ldots \otimes f_n \right)$$

for all $f_i \in \mathscr{D}(H) \subset \mathfrak{h}$. The operator $H_n$ extends to a selfadjoint operator on $\Gamma^{(n)}(\mathfrak{h})$. The direct sum of all $H_n$ is symmetric and therefore closable, and essentially self-adjoint, because there exists a dense set of analytic vectors, which is formed by the finite sums of symmetrised products of analytic vectors of $H$. The selfadjoint closure of the direct sum $\oplus_{n \in \mathbb{N}_0} H_n$ of $H_n$ is called the second quantisation of $H$. It is denoted by

$$d\Gamma(\mathfrak{h}) \doteq \oplus_{n \in \mathbb{N}_0} H_n.$$  

If $U$ is a unitary operator acting on $\mathfrak{h}$, then $U_n$ acting on $\Gamma^{(n)}(\mathfrak{h})$ is defined by $U_0 \doteq 1$ and

$$U_n(P_+(f_1 \otimes \ldots \otimes f_n)) \doteq P_+\left( Uf_1 \otimes Uf_2 \otimes \ldots \otimes Uf_n \right)$$

and continuous extension. The second quantisation of $U$ is

$$\Gamma(U) \doteq \oplus_{n \in \mathbb{N}_0} U_n.$$  

$\Gamma(U)$ is a unitary operator acting on $\Gamma(\mathfrak{h})$.

**Lemma B.0.10.** If $t \mapsto U_t = e^{itH}$ is a strongly continuous group of unitary operators on $\mathfrak{h}$, then $\Gamma(U_t) = e^{itd\Gamma(H)}$ holds on $\Gamma(\mathfrak{h})$.

For the convenience of the reader we also recall the following results.
Theorem B.0.11 (Araki [3], Theorem 1). Let $\mathfrak{h}$ be a Hilbert space and let $d$ be a real subspace of $\mathfrak{h}$. Let $\mathfrak{W}(d) \subset \mathfrak{W}(\mathfrak{h})$ denote the abstract Weyl algebra generated by \{\mathfrak{W}(h) | h \in d\} and let $\pi: \mathfrak{W}(\mathfrak{h}) \to \mathfrak{W}_F(\mathfrak{h}) \in \mathfrak{B}(\Gamma(\mathfrak{h}))$ be the Fock representation. Then

\[
\bigcap_{\alpha} \pi(\mathfrak{W}(d_\alpha))'' = \pi(\mathfrak{W}(\bigcap_\alpha d_\alpha))'',
\]

and

\[
\bigvee_{\alpha} \pi(\mathfrak{W}(d_\alpha))'' = \pi(\mathfrak{W}(\bigvee_\alpha d_\alpha))'',
\]

where $d_\alpha$ is a family of real subspaces of $\mathfrak{h}$ and $d^\perp$ is the vector space orthogonal to $d$ for the symplectic form $\sigma(h_1, h_2) = 2\mathfrak{I} \langle h_1, h_2 \rangle$.

Theorem B.0.12 (Leyland, Roberts and Testard [134], Theorem I.3.2). Let $\mathfrak{h}$ be a Hilbert space and let $d$ be a real subspace of $\mathfrak{h}$. Let $\overline{d}$ denote the closure of $d$. Furthermore, let $\mathfrak{W}(d) \subset \mathfrak{W}(\mathfrak{h})$ denote the abstract Weyl algebra generated by \{\mathfrak{W}(h) | h \in d\} and let $\pi: \mathfrak{W}(\mathfrak{h}) \to \mathfrak{W}_F(\mathfrak{h}) \in \mathfrak{B}(\Gamma(\mathfrak{h}))$ be the Fock representation. It follows that

i.) $\pi(\mathfrak{W}(d))'' = \pi(\mathfrak{W}(\overline{d}))$;

ii.) $\Omega$ is cyclic for $\pi(\mathfrak{W}(d))''$ if and only if $d + id$ is dense in $\mathfrak{h}$;

iii.) $\Omega$ is separating for $\pi(\mathfrak{W}(d))''$ if and only if $\overline{d} \cap i\overline{d} = \{0\}$;

iv.) $\pi(\mathfrak{W}(d))' = \pi(\mathfrak{W}(d^\perp))'$;

v.) $\pi(\mathfrak{W}(d_1))'' \vee \pi(\mathfrak{W}(d_2))'' = \pi(\mathfrak{W}(d_1 + d_2))''$;

vi.) $\pi(\mathfrak{W}(d_1))'' \cap \pi(\mathfrak{W}(d_2))'' = \pi(\mathfrak{W}(\overline{d_1} \cap \overline{d_2}))''$ and consequently $\pi(\mathfrak{W}(d))''$ is a factor if and only if $\overline{d} \cap d^\perp = \{0\}$.

As in the previous theorem, $d^\perp$ denotes the symplectic complement of $d$ in $\mathfrak{h}$.

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2See, e.g., Eq. (8.0.1) with the symplectic form $\sigma$ given by twice the imaginary part of the scalar product in the Hilbert space $\mathfrak{h}$. 

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APPENDIX C

Tuboids

A tuboid for the de Sitter space is a subset of $dS_C$ which is i.) bordered by real de Sitter space $dS$ and ii.) allows boundary values on $dS$ of functions holomorphic in the tuboid to be controlled by methods of complex analysis (in several variables). We proceed in several steps, following closely [28].

DEFINITION C.0.13. A profile $\mathcal{P}$ is an open subset of the tangent bundle $TdS$ of the form

$$\mathcal{P} = \bigcup_{x \in dS} (x, \mathcal{P}_x),$$

where each fibre $\mathcal{P}_x$ is a non-empty cone with apex at the origin in $T_x dS$.

DEFINITION C.0.14 (Bros and Moschella [28], Def. A.1). A diffeomorphism $\Xi$ is an admissible local diffeomorphism at a point $x^\circ \in dS$, if it maps a neighbourhood $N_{TdS}(x^\circ, 0)$ of $(x^\circ, 0)$ in $TdS$ onto a corresponding neighbourhood $N_{C}(x^\circ) = \Xi(N_{TdS}(x^\circ, 0))$ in $dS_C$, considered as a 4-dimensional $C^\infty$-manifold, in such a way that the following properties hold:

i.) $\Xi(x, 0) = x \in N_{C}(x^\circ)$ if $(x, 0) \in N_{TdS}(x^\circ, 0)$;

ii.) for all $(x, y) \in N_{TdS}(x^\circ, 0)$ with $y \neq 0$ the differentiable function

$$t \mapsto f(t) = \Xi(x, ty) \in dS_C$$

is such that

$$\frac{1}{t} \left( \frac{df}{dt} \right)_{t=0} = \alpha y, \quad \alpha > 0.$$

In order to define admissible local diffeomorphisms, which respect the causal structure of $dS$, it is convenient to consider the projective representation $\hat{T}dS \doteq \bigcup_{x \in dS} (x, \hat{T}_x dS)$ of $TdS$, with

$$\hat{T}_x dS \doteq (T_x dS \setminus \{0\})/R^+.$$

The image of each point $y \in T_x dS$ in $\hat{T}_x dS$ is $\hat{y} = (\lambda y \mid \lambda > 0)$. The complement $\hat{\mathcal{P}}'$ of $\hat{\mathcal{P}} \doteq (\hat{\mathcal{P}} \setminus \{0\})/R^+$ in $\hat{T}dS$ is the open set

$$\hat{\mathcal{P}}' \doteq \hat{T}dS \setminus \hat{\mathcal{P}}.$$

This set equals $\bigcup_{x \in dS} (x, \hat{\mathcal{P}}'_{\hat{x}})$, where $\hat{\mathcal{P}}'_{\hat{x}} = \hat{T}_x dS \setminus \hat{\mathcal{P}}_x$. Taking advantage of these notions, one can now define tuboids for the de Sitter space:

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DEFINITION C.0.15 (Bros and Moschella [28]). A domain \( T \) (i.e., a connected open subset) in \( dS_C \) is called a tuboid with profile \( P \) above \( dS \) if it satisfies the following property: for every point \( x_o \) in \( dS \), there exists an admissible local diffeomorphism \( \Xi \) at \( x_o \) such that

i.) every point \((x_o, \dot{y}_1)\) in \( \dot{P} \) admits a compact neighbourhood \( \mathcal{K}(x_o, \dot{y}_1) \) in \( \dot{P} \) and, in the sequel, a sufficiently small neighborhood \( N_{t\!d\!S}(x_o,0) \) of \((x_o,0)\) in \( TdS \) such that

\[
\Xi(\{(x,y) \in N_{t\!d\!S}(x_o,0) \mid (x,\dot{y}) \in \mathcal{K}(x_o, \dot{y}_1)\}) \subset T;
\]

ii.) every point \((x_o, \dot{y}_2)\) in \( \dot{P}^c \) admits a compact neighbourhood \( \mathcal{K}^c(x_o, \dot{y}_2) \) in \( \dot{P}^c \) and, in the sequel, a sufficiently small neighborhood \( N_{t\!d\!S}^c(x_o,0) \) of \((x_o,0)\) in \( TdS \) such that

\[
\Xi(\{(x,y) \in N_{t\!d\!S}^c(x_o,0) \mid (x,\dot{y}) \in \mathcal{K}^c(x_o, \dot{y}_2)\}) \cap T = \emptyset.
\]

In i.) and ii.) \( N_{t\!d\!S}(x_o,0) \) and \( N_{t\!d\!S}^c(x_o,0) \) may depend on \( \dot{y}_1 \) and \( \dot{y}_2 \), respectively.
APPENDIX D

Sobolev spaces on the circle and on the sphere

If $h \in L^2(S^1, d\psi)$, then $h$ has a Fourier series

\[
(D.0.1) \quad h(\psi) = \sum_{k \in \mathbb{Z}} a_k e^{ik\psi}, \quad a_k = \frac{1}{2\pi} \int_{S^1} d\psi \ h(\psi) e^{-ik\psi}.
\]

The infinite sum on the r.h.s. converges in $L^2(S^1, d\psi)$. In fact, the infinite sum $\sum_{k \in \mathbb{Z}} a_k e^{ik\psi}$ exists, iff $|a_k| = o(k^{-N})$ for all $N \in \mathbb{N}$.

By the Weierstraß’ approximation theorem the polynomials $\sum_{k=-N}^{N} a_k e^{ik\psi}$, $N \in \mathbb{N}$, are dense in the sup norm in $C(S^1)$. Parseval’s identity states that

\[
\sum_{k \in \mathbb{Z}} |a_k|^2 = \frac{1}{2\pi} \int_{S^1} d\psi \ |h(\psi)|^2.
\]

In case $h \in C^1(S^1)$, this implies that the Fourier series converges uniformly and absolutely.

**Definition D.0.16.** Let $0 \leq p \leq \infty$. The Sobolev space of order $p$ is given by

\[
H^p(S^1) = \left\{ h \in L^2(S^1) \mid \sum_{k \in \mathbb{Z}} (1 + k^2)^p |a_k|^2 < \infty \right\},
\]

where the $\{a_k\}$ are the Fourier coefficients of $h$, see (D.0.1).

$H^p(S^1)$ is a Hilbert space with the inner product

\[
\left\langle \sum_{j \in \mathbb{Z}} a_j e^{ij\psi}, \sum_{k \in \mathbb{Z}} b_k e^{ik\psi} \right\rangle_{H^p(S^1)} = \sum_{k \in \mathbb{Z}} (1 + k^2)^p \ a_k \overline{b_k}
\]

for $h, g \in H^p(S^1)$ with Fourier coefficients $\{a_j\}, \{b_k\}$, respectively. The norm is given by

\[
\|h\|_{H^p(S^1)} = \left( \sum_{k \in \mathbb{Z}} (1 + k^2)^p |a_k|^2 \right)^{1/2}.
\]

The trigonometric polynomials are dense in $H^p(S^1)$.

**Definition D.0.17.** For $0 < p < \infty$, we denote by $H^{-p}(S^1)$ the dual space of $H^p(S^1)$, i.e., the space of bounded linear functionals on $H^p(S^1)$.

For $\xi \in H^{-p}(S^1)$ we have

\[
\|\xi\|_{H^{-p}(S^1)} = \left( \sum_{k \in \mathbb{Z}} (1 + k^2)^{-p} |b_k|^2 \right)^{1/2},
\]

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where \( b_k = \xi(e^{ik\psi}) \). Furthermore, for each sequence \( \{b_k\} \) satisfying
\[
\sum_{k \in \mathbb{Z}} (1 + k^2)^{-p} |b_k|^2 < \infty ,
\]
there exists a bounded linear functional \( \xi \in H^{-p}(S^1) \) with \( b_k = \xi(e^{ik\psi}) \).

**Proposition D.0.18.** The elements in \( H^p \) share the following properties:

1. If \( p < 0 \), then the elements in \( H^p \) are generalised functions;
2. If \( p > 1/2 \), then the functions \( f \in H^p \) are continuous;
3. If \( p \geq 1 \), then the functions \( f \in H^p \) are differentiable almost everywhere.

Next we consider the sphere. The surface element is
\[
d\Omega = \cos \psi d\psi d\theta .
\]
We denote by \( L^2(S^2, d\Omega) \) the set of measurable functions \( f \) on the sphere \( S^2 \) for which
\[
\|f\|_{L^2(S^2, d\Omega)}^2 = \int_{S^2} d\Omega |f(\theta, \psi)|^2 < \infty .
\]
A function \( f \in L^2(S^2, \cos \psi d\psi d\theta) \) can be expanded, in the \( L^2 \)-sense, into its Fourier (Laplace) series (with respect to spherical harmonics) where
\[
\begin{align*}
\hat{f}_{\ell, k} & \doteq \int_{S^2} d\Omega \ f(\theta, \psi) \overline{Y_{\ell, k}(\theta, \psi)} ,
\end{align*}
\]

**Definition D.0.19.** The Sobolev \( H^p(S^2) \), \( p \geq 0 \), is the closure of the set of \( C^\infty(S^2) \) functions with respect to the norm
\[
\|f\|_{H^p(S^2)} = \left( \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} (\ell + \frac{1}{2})^{2p} |\hat{f}_{\ell, k}|^2 \right)^{1/2} .
\]

The space \( H^p(S^2) \) is a Hilbert space with inner product
\[
\langle f, g \rangle_{H^p(S^2)} = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} (\ell + \frac{1}{2})^{2p} \overline{\hat{f}_{\ell, k}} \hat{g}_{\ell, k} , \quad f, g \in H^p(S^2) .
\]
By construction, \( H^0(S^2) = L^2(S^2, d\Omega) \).

**Definition D.0.20.** For \( 0 < p < \infty \), we denote by \( H^{-p}(S^2) \) the dual space of \( H^p(S^2) \), i.e., the space of bounded linear functionals on \( H^p(S^2) \).

For \( \xi, \in H^{-p}(S^2) \) we have
\[
\|\xi\|_{H^{-p}(S^2)} = \left( \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} (\ell + \frac{1}{2})^{-2p} |b_{\ell, k}|^2 \right)^{1/2} ,
\]
where \( b_{\ell, k} = \xi(Y_{\ell, k}) \). Furthermore, for each sequence \( \{b_{\ell, k}\} \) satisfying
\[
\sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} (\ell + \frac{1}{2})^{-2p} |b_{\ell, k}|^2 < \infty ,
\]
there exists a bounded linear functional \( \xi \in H^{-p}(S^2) \) with \( b_{\ell, k} = \xi(Y_{\ell, k}) \).
APPENDIX E

Some identities involving Legendre functions

In the sequel, we will use the following well-known properties of the Gamma function:

\begin{align}
\Gamma(z + 1) &= z\Gamma(z), \\
\Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin(\pi z)}, \\
\Gamma(z)\Gamma(-z) &= -\frac{\pi}{z}\sin(\pi z), \\
\Gamma(2z) &= \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \\
\Gamma(z) &= \Gamma(z).
\end{align}

They are valid except when the arguments are non-positive integers.

The Legendre function \( P_s \) solves \([81], 8.820]\ the differential equation

\[
\frac{d}{dz}(1-z^2)\frac{d}{dz}P_s(z) + s(s+1)P_s(z) = 0.
\]

It is analytic in \( \mathbb{C} \setminus (-\infty, -1) \), that means, it has a cut in \( (-\infty, -1) \).

**Remark E.0.21.** Setting \( z = -\cos \psi \) we find

\[
\frac{1}{\sin \psi} \frac{\partial}{\partial \psi} \sin \psi \frac{\partial}{\partial \psi} P_s(-\cos \psi) + s(s+1)P_s(-\cos \psi) = 0.
\]

The associated Legendre functions

\[
P_k^s(\cos \psi) = (-1)^k (\sin \psi)^k \frac{d^k}{d(\cos \psi)^k} (P_s(\cos \psi))
\]

\[
p^{-k}_s = (-1)^k (p^{-k}_s)^k P^k_s, \quad k = 0, 1, 2, \ldots ,
\]

are analytic in \( \mathbb{C} \setminus (-\infty, +1) \).

**Lemma E.0.22.** The function

\[
S(z) = \sqrt{z^2 - 1}
\]

is analytic in \( \mathbb{C} \setminus (-\infty, 1) \) and one has

\[
\lim_{\epsilon \to 0_+} S(\epsilon(1 \pm i)) = e^{\pm \pi/2}.
\]

**Lemma E.0.23.** The Fourier series of the Legendre function is given by

\[
P_s(-\cos \psi) = p(0) + 2 \sum_{k=1}^{\infty} p(k) \cos(k\psi),
\]

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where, for \( k \in \mathbb{N}_0 \),

\[
(E.0.10) \quad p(k) = (-1)^k \frac{\Gamma(s - k + 1)}{\Gamma(s + k + 1)} \left( \lim_{\epsilon \to 0^+} p_s^k(\epsilon (1 + i)) \right) \left( \lim_{\epsilon \to 0^+} p_s^k(\epsilon (1 - i)) \right).
\]

**Proof.** For \(|\arg(z - 1)| < \pi\) and \(|\arg(w - 1)| < \pi\) and \(\Re z > 0\) and \(\Im w > 0\), one has [131] page 202

\[
(E.0.11) \quad P_s \left( zw - \sqrt{z^2 - 1} \sqrt{w^2 - 1} \cos \psi \right) = P_s(z)P_s(w) + 2 \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(s - k + 1)}{\Gamma(s + k + 1)} \frac{P_s^k(z)}{p_s^k} \frac{P_s^k(w)}{p_s^k} \cos(k\psi). 
\]

(This relation is also found in [179], page 78.) Hence, setting \( z = \epsilon (1 + i) \), \( w = \epsilon (1 - i) \), and taking the limit \( \epsilon \to 0 \), we have for the l.h.s. of \( (E.0.11) \),

\[
\lim_{\epsilon \to 0^+} P_s \left( 2e^{2\epsilon} - S(\epsilon (1 + i))S(\epsilon (1 + i)) \cos \psi \right) = P_s(-\cos \psi). 
\]

Setting \( z = i\epsilon \), \( w = -i\epsilon \) and taking the limit \( \epsilon \to 0 \) on the r.h.s. of \( (E.0.11) \), the lemma follows. \( \square \)

**Lemma E.0.24.**

\[
(E.0.12) \quad \lim_{\epsilon \to 0^+} P_s^k(\epsilon (1 \pm i)) = \frac{e^{\pm ik\pi/2} \sqrt{\pi}}{2^k \Gamma(k + 1) \Gamma(1/2)} \frac{\Gamma(s + k + 1)}{\Gamma(s + k + 1 - k + s + 1)} \frac{1}{\Gamma(k + s + 1)}.
\]

**Proof.** According to [131] Eq. 7.12.27, page 198, one has, for \( k \in \mathbb{N}_0 \), \(|z - 1| < 2\) and \(\arg(z - 1) < \pi\),

\[
p_s^k(z) = \frac{\Gamma(z - 1)^{k/2} \Gamma(s + k + 1)}{2^k \Gamma(k + 1) \Gamma(s - k + 1)} F \left( k - s, k + s + 1, k + 1; \frac{1 - z}{2} \right)
\]

\[
= \frac{S(z)^k \Gamma(s + k + 1)}{2^k \Gamma(k + 1) \Gamma(s - k + 1)} \Gamma(k + s + 1) F \left( k - s, k + s + 1, k + 1; \frac{1 - z}{2} \right),
\]

where \( F \) is the hypergeometric function

\[
F(\alpha, \beta, \gamma, z) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} z^n
\]

\[
(E.0.13) \quad = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n) \Gamma(\beta + n)}{\Gamma(\gamma + n)} \frac{\Gamma(\alpha + n) \Gamma(\beta + n)}{n!} z^n,
\]

valid for \(|z| < 1\). Here \((\gamma)_n\) is the Pochhammer symbol, which is defined by

\[
(\gamma)_n = \begin{cases} 1 & \text{if } n = 0; \\ q(q + 1) \cdots (q + n - 1) & \text{if } n > 0. \\ \end{cases}
\]

\( F \) is analytic in the whole open unit disk \(|z| < 1\). Therefore,

\[
\lim_{\epsilon \to 0^+} P_s^k(\epsilon (1 \pm i)) = \frac{e^{\pm ik\pi/2} \sqrt{\pi}}{2^k \Gamma(k + 1) \Gamma(s + k + 1)} \Gamma(k + s + 1) F \left( k - s, k + s + 1, k + 1; \frac{1}{2} \right).
\]

\( (E.0.14) \)
The value of $F(\alpha, \beta, \gamma; z)$ at the point $z = 1/2$ cannot be easily computed from the power series definition (E.0.13). However, the hypergeometric function satisfies the following relation (see [131], Eq. 9.6.11, page 253):

(E.0.15) \[ F(2\alpha, 2\beta, \alpha + \beta + \frac{1}{2}; \frac{1-x}{2}) = \frac{\Gamma(\alpha + \beta + \frac{1}{2})}{\Gamma(\alpha + \frac{1}{2}) \Gamma(\beta + \frac{1}{2})} F(\alpha, \beta, \frac{1}{2}; z^2) + z \frac{\Gamma(\alpha + \beta + \frac{1}{2}) \Gamma(-\frac{1}{2})}{\Gamma(\alpha) \Gamma(\beta)} F(\alpha + \frac{1}{2}, \beta + \frac{1}{2}, \frac{3}{2}; z^2), \]

valid for all $z \in \mathbb{C} \setminus ((-\infty, -1) \cup (1, \infty))$ and for all $\alpha + \beta + \frac{1}{2} \not\in \mathbb{N}_0$ (i.e., for all $\alpha + \beta + \frac{1}{2} \neq 0, -1, -2, \ldots$). Taking $z = 0$ in (E.0.15), one gets

(E.0.16) \[ F\left(2\alpha, 2\beta, \alpha + \beta + \frac{1}{2}; \frac{1}{2}\right) = \frac{\Gamma(\alpha + \beta + \frac{1}{2})}{\Gamma(\alpha + \frac{1}{2}) \Gamma(\beta + \frac{1}{2})}, \]

since $F(\alpha, \beta, 1/2; 0) = 1$ (see (E.0.13)). By choosing

\[ \alpha = \frac{k-s}{2} \quad \text{and} \quad \beta = \frac{k+s+1}{2} \]

one has $\alpha + \beta + \frac{1}{2} = k+1$ (which is non-zero for $k \in \mathbb{N}_0$) and it follows from (E.0.16) that

(E.0.17) \[ F\left(k-s, k+s+1, k+1; \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma(k+1)}{\Gamma(\frac{k+1}{2}) \Gamma(\frac{k+s+1}{2})}, \]

since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Inserting (E.0.17) into (E.0.10), one gets (E.0.12). \hfill \Box

**PROPOSITION E.0.25.** The Fourier series of the Legendre function is given by

(E.0.18) \[ P_s(-\cos \psi) = p(0) + 2 \sum_{k=1}^{\infty} p(k) \cos(k\psi), \]

where, for $k \in \mathbb{N}_0$,

(E.0.19) \[ p(k) = -\frac{\sin(\pi s) \; 1}{\pi (k+s)} \frac{\Gamma\left(\frac{k-s}{2}\right) \Gamma\left(\frac{k+s+1}{2}\right)}{\Gamma\left(\frac{k+s}{2}\right) \Gamma\left(\frac{k+s+1}{2}\right)}, \quad \forall k \in \mathbb{N}_0. \]

**PROOF.** Inserting (E.0.12) into (E.0.10), one gets

(E.0.20) \[ p(k) = (-1)^k \frac{\pi \; \Gamma(s+k+1)}{2^{k+1} (s-k+1) \; \Gamma\left(\frac{k-s+1}{2}\right)^2 \Gamma\left(\frac{k+s+1}{2}\right)^2}, \quad k \in \mathbb{N}_0. \]

Now, using the well-known properties (E.0.3)–(E.0.7) of the Gamma function, we start a series of manipulations, in order to write $p(k)$ in a more convenient fashion.

In (E.0.20) we consider the factor

\[ \frac{\Gamma(s+k+1)}{\Gamma\left(\frac{k+s}{2}+1\right)} = \frac{\Gamma(s+k+1)}{\Gamma\left(\frac{k+s+1}{2}\right)^2 \Gamma\left(\frac{k+s+1}{2}+1\right)}. \]

by taking $z = \frac{k+s+1}{2}$. From (E.0.6), one has $\frac{\Gamma(z)}{\Gamma\left(\frac{z+1}{2}\right)} = \frac{\sinh^2(z)}{\sqrt{\pi}} \Gamma(z)$. Hence,

\[ \frac{\Gamma(s+k+1)}{\Gamma\left(\frac{k+s}{2}+1\right)} = \frac{2^{s-1} \Gamma(z)}{\sqrt{\pi}} = \frac{2^{k+s}}{\sqrt{\pi}} \Gamma\left(\frac{k+s+1}{2}\right). \]
Inserting this into (E.0.20), we get

\[ p(k) = (-1)^k \sqrt{\pi} \frac{2^{s-k}}{s^2 - k^2} \frac{\Gamma \left( \frac{k+s+1}{2} \right)}{\Gamma(s-k)\Gamma(k-s)\Gamma \left( \frac{k-s+1}{2} \right)} \]  \hspace{1cm} (E.0.21)

Now, we write \( \Gamma \left( \frac{k-s+1}{2} \right) = \Gamma \left( \frac{z + \frac{1}{2}}{2} \right) \) with \( z = \frac{k-s}{2} \) and, using (E.0.6) we get

\[ \Gamma \left( \frac{k-s+1}{2} \right) = \Gamma \left( z + \frac{1}{2} \right) \]  \hspace{1cm} (E.0.22)

We now write

\[ \Gamma(s-k)\Gamma(k-s) \]  \hspace{1cm} (E.0.5)

and inserting this into (E.0.22) we get

\[ p(k) = (-1)^k \frac{2^{k-s-1}}{\sqrt{\pi}(s-k)^2} \frac{\Gamma \left( \frac{k+s+1}{2} \right)\Gamma \left( \frac{k-s}{2} \right)}{\Gamma(k-s)\Gamma \left( \frac{k-s+1}{2} \right)} \]  \hspace{1cm} (E.0.23)

Taking \( z = \frac{k-s}{2} \), we have

\[ \frac{\Gamma \left( \frac{k-s}{2} \right)}{\Gamma(k-s)} = \frac{\Gamma(z)}{\Gamma(2z)} \]  \hspace{1cm} (E.0.6)

and inserting this to (E.0.23), we find (E.0.19). \hspace{1cm} \Box

Remark E.0.26. Comparing (E.0.9) with (6.2.4), we see that \( p_k = \sqrt{2\pi}r(p(k)) \),
for all \( k \in \mathbb{Z} \). Thus, from the definition (6.2.6) we get

\[ \omega(k) = \omega(-k) \]  \hspace{1cm} (E.0.24)

for all \( k \in \mathbb{Z} \). Actually, we can directly establish that \( p(k) = p(-k) \) for all \( k \). This is the content of the next lemma.

Lemma E.0.27. For all \( k \in \mathbb{Z} \), we have \( p(k) = p(-k) \).

Proof. Until now we considered \( k \in \mathbb{N}_0 \) but, for \( s \not\in \mathbb{Z} \), (E.0.19) is well-defined for all \( k \in \mathbb{Z} \) and will now show that \( p(k) = p(-k) \) for all \( k \in \mathbb{Z} \). Let

\[ \mathcal{F}(k) = \frac{1}{(k+s)\Gamma(k+s)} \frac{\Gamma \left( \frac{k-s}{2} \right)\Gamma \left( \frac{k+s+1}{2} \right)}{\Gamma \left( \frac{k-s+1}{2} \right)\Gamma \left( \frac{k+s+1}{2} \right)} \].

Then,

\[ \mathcal{F}(-k) = \frac{1}{(-k+s)\Gamma(-k+s)} \frac{\Gamma \left( \frac{-k-s}{2} \right)\Gamma \left( \frac{-k+s+1}{2} \right)}{\Gamma \left( \frac{-k-s+1}{2} \right)\Gamma \left( \frac{-k+s+1}{2} \right)} \].
Now, 
\[
\frac{\Gamma \left( \frac{-k-s}{2} \right)}{\Gamma \left( \frac{-k+s+1}{2} \right)} = \frac{(-k+s) \sin \left( \frac{\pi-k+s}{2} \right)}{(-k-s) \sin \left( \frac{\pi-k-s}{2} \right)} \Gamma \left( \frac{k-s}{2} \right) \Gamma \left( \frac{k+s+1}{2} \right)
\]
and
\[
\frac{\Gamma \left( \frac{-k+s+1}{2} \right)}{\Gamma \left( \frac{-k-s+1}{2} \right)} = \frac{\sin \left( \frac{\pi-k+s+1}{2} \right)}{\sin \left( \frac{\pi-k-s+1}{2} \right)} \Gamma \left( \frac{k-s+1}{2} \right) \Gamma \left( \frac{k+s+1}{2} \right).
\]
Therefore,
\[
(E.0.25) \quad \mathcal{F}(-k) = \frac{1}{(-k+s)} \frac{(-k+s) \sin \left( \frac{\pi-k+s}{2} \right)}{(-k-s) \sin \left( \frac{\pi-k-s}{2} \right)} \frac{\sin \left( \frac{\pi-k-s+1}{2} \right)}{\sin \left( \frac{\pi-k+s+1}{2} \right)} \frac{\Gamma \left( \frac{k-s}{2} \right) \Gamma \left( \frac{k+s+1}{2} \right)}{\Gamma \left( \frac{k-s+1}{2} \right) \Gamma \left( \frac{k+s+1}{2} \right)}.
\]

Using \( \sin(a) \sin(b) = \cos(a-b) - \cos(a+b) \), we get
\[
\frac{\sin \left( \frac{\pi-k+s}{2} \right) \sin \left( \frac{\pi-k-s+1}{2} \right)}{\sin \left( \frac{\pi-k-s}{2} \right) \sin \left( \frac{\pi-k+s+1}{2} \right)} = \frac{\cos \left( -\pi k - \frac{\pi}{2} \right) - \cos \left( \pi s + \frac{\pi}{2} \right)}{\cos \left( -\pi k - \frac{\pi}{2} \right) - \cos \left( -\pi s - \frac{\pi}{2} \right)} = \frac{\cos \left( \pi s + \frac{\pi}{2} \right)}{\cos \left( \pi s - \frac{\pi}{2} \right)} = -1.
\]
Hence, returning to \( E.0.25 \),
\[
\mathcal{F}(-k) = \frac{1}{k+s} \frac{\Gamma \left( \frac{k-s}{2} \right) \Gamma \left( \frac{k+s+1}{2} \right)}{\Gamma \left( \frac{k-s+1}{2} \right) \Gamma \left( \frac{k+s+1}{2} \right)} = \mathcal{F}(k).
\]
This establishes that \( p(k) = p(-k) \) for all \( k \in \mathbb{Z} \).

**Proposition E.0.28.** Let \( \hat{h}(S^1) \) be defined in \( 6.2.1 \), and let \( f, g \in \mathcal{D}(\omega) \). It follows that
\[
(E.0.26) \quad \langle \omega f, \omega g \rangle_{\hat{h}(S^1)} = -\frac{r^2}{2 \sin(\pi s^+)} \int_{S^1 \times S^1} d\psi d\psi' \bar{f}(\psi') P_s'(-\cos(\psi' - \psi)) g(\psi).
\]

**Proof.** In what follows we will denote the coefficients \( p(k) \), \( p_k \) and \( \omega(k) \) by \( p_s(k), p_{k,s} \) and \( \omega_s(k) \), respectively.

For \( s \in \mathbb{C} \setminus \mathbb{Z} \), define
\[
\langle \langle f, g \rangle \rangle_{\hat{h}(S^1)} = c \int_{S^1} r d\psi \int_{S^1} r d\psi' \bar{f}(\psi') g(\psi) P_s'(-\cos(\psi' - \psi))
\]
\[
= -\frac{1}{2 \sin(\pi s^+)} \int_{S^1} r d\psi \int_{S^1} r d\psi' \bar{f}(\psi') g(\psi) P_s'(-\cos(\psi' - \psi)).
\]
We write
\[
P_s'( -\cos(\varphi) ) = \sum_{k \in \mathbb{Z}} p_{k,s} \frac{e^{ik\varphi}}{\sqrt{2\pi^r}}.
\]
and, as in (6.2.5), we get

(E.0.27) \( \langle f, g \rangle = -\frac{\sqrt{2\pi}}{2\sin(\pi s^+)} \sum_{k \in \mathbb{Z}} p_{k, s} f_k g_k \),

where \( f_k \) and \( g_k \) are the Fourier coefficients of \( f \) and \( g \), respectively, i.e.,

\[
f_k \doteq \int_{S^1} r \, d\psi' \, f(\psi') \frac{e^{-ik\psi'}}{\sqrt{2\pi}} \quad \text{and} \quad g_k \doteq \int_{S^1} r \, d\psi' \, g(\psi) \frac{e^{-ik\psi'}}{\sqrt{2\pi}}.
\]

Taking the mixed derivatives \( \partial_z \partial_w \) of both sides in (E.0.11), we get

(E.0.28) \[
\begin{align*}
P''_s(zw - \sqrt{z^2 - 1} - \sqrt{w^2 - 1} \cos \psi) & \left( w \doteq \frac{zw - \sqrt{z^2 - 1} - \sqrt{w^2 - 1} \cos \psi}{\sqrt{z^2 - 1} - \sqrt{w^2 - 1} \cos \psi} \right) \\
& + P'_s(zw - \sqrt{z^2 - 1} \sqrt{w^2 - 1} \cos \psi) \left( 1 - \frac{zw}{\sqrt{z^2 - 1} - \sqrt{w^2 - 1} \cos \psi} \right)
\end{align*}
\]

Writing \( z = \epsilon(1 + i) \), \( w = \epsilon(1 - i) \) (with \( \epsilon > 0 \)) and taking the limit \( \epsilon \to 0_+ \), we get from (E.0.28)

\[
P'_s(-\cos \psi) = (P'_s(0))^2 + 2 \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(s - k + 1)}{\Gamma(s + k + 1)} \left( \lim_{\epsilon \to 0_+} P^k_s(\epsilon(1 + i)) \right)
\]

\[
\times \left( \lim_{\epsilon \to 0_+} P^k_s(\epsilon(1 - i)) \right) \cos(k\psi)
\]

\[
= p'_1(0) + 2 \sum_{k=1}^{\infty} p^s_k(k) \cos(k\psi),
\]

with

\[
p^s_k(k) \doteq (-1)^k \frac{\Gamma(s - k + 1)}{\Gamma(s + k + 1)} \left( \lim_{\epsilon \to 0_+} P^k_s(\epsilon(1 + i)) \right) \left( \lim_{\epsilon \to 0_+} P^k_s(\epsilon(1 - i)) \right).\]

Now, according to [131] Eq. (7.12.16), page 195, one has

\[
(z^2 - 1) P^k_s(z) = s z P^k_s(z) - (s + k) P^k_{s-1}(z), \quad k \in \mathbb{N}_0.
\]

Hence,

\[
\lim_{\epsilon \to 0_+} P^k_s(\epsilon(1 \pm i)) = (s + k) \lim_{\epsilon \to 0_+} P^k_{s-1}(\epsilon(1 \mp i)),
\]
and from this we have
\[
p_s^1(k) = (-1)^k (s+k)^2 \frac{\Gamma(s-k+1)}{\Gamma(s+k+1)} \left( \lim_{\epsilon \to 0^+} p_{s-1}^k(\epsilon(1+i)) \right) \\
	imes \left( \lim_{\epsilon \to 0^+} p_{s-1}^k(\epsilon(1-i)) \right) \\
= (-1)^k (s+k)(s-k) \frac{\Gamma(s-k)}{\Gamma(s+k)} \left( \lim_{\epsilon \to 0^+} p_{s-1}^k(\epsilon(1+i)) \right) \\
	imes \left( \lim_{\epsilon \to 0^+} p_{s-1}^k(\epsilon(1-i)) \right).
\]
(E.0.10) \((s+k)(s-k)p_{s-1}(k)\).

Since \(p_{s-1}(k) = p_{s-1}(\epsilon=-k), k \in \mathbb{Z}\), it follows from the last equality that \(p_s^1(k) = p_s^1(-k), k \in \mathbb{Z}\), and we have
\[
P_s'(\cos \psi) = \sum_{k \in \mathbb{Z}} p_{k,s}^1 \frac{e^{ik\psi}}{\sqrt{2\pi r}}
\]
with
\[
p_{k,s}^1 = \sqrt{2\pi r} p_s^1(k) = \sqrt{2\pi r} (s+k)(s-k)p_{s-1}(k), \quad k \in \mathbb{Z}.
\]
Now, we have
(E.0.29) \((s+k)(s-k)p_{s-1}(k) = -\frac{\sin(\pi s)}{\pi} \frac{(s+k)(s-k)}{(k+s-1)} \times \frac{\Gamma\left(\frac{k-s+1}{2}\right) \Gamma\left(\frac{k+s}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{k-s+1}{2}\right)}.
\]
Since \(\sin(\pi s) = -\sin(\pi s), \Gamma\left(\frac{k-s+2}{2}\right) = \Gamma\left(\frac{k+s}{2}\right) = \frac{k-s}{2} \Gamma\left(\frac{k-s}{2}\right)\) and \((k+s-1)\Gamma\left(\frac{k-s+1}{2}\right) = 2\Gamma\left(\frac{k-s+1}{2}\right) + 1 = 2\Gamma\left(\frac{k-s+1}{2}\right))\), relation (E.0.29) becomes
\[
(s+k)(s-k)p_{s-1}(k) = -\frac{\sin(\pi s)}{\pi} \frac{\Gamma\left(\frac{k-s+1}{2}\right) \Gamma\left(\frac{k+s}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{k-s+1}{2}\right)}.
\]
Comparing with (6.2.2), we find
\[
(s+k)(s-k)p_{s-1}(k) = -\frac{\sin(\pi s)}{\pi} \bar{\omega}_s(k) r.
\]
Here \(\omega_s = \omega\), with the index indicating the dependence of \(\omega\) on \(s\). The latter had been suppressed in the main text. Hence,
\[
p_{k,s}^1 = -\sqrt{\frac{2\pi}{\pi}} \sin(\pi s) \bar{\omega}_s(k) r.
\]
Returning to (E.0.27) we have
\[
\langle\langle f, g \rangle\rangle = -\frac{\sqrt{2\pi r}}{2 \sin(\pi s)^2} \sum_{k \in \mathbb{Z}} p_{k,s}^1 \bar{f}_k g_k
\]
\[
= r^2 \langle\langle f, \omega g \rangle\rangle_{1^2(S^1, t, d\psi)}
\]
\[
= r^2 \langle\langle \omega f, \omega g \rangle\rangle_{b(S^1)}.
\]
\(\square\)
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