Extreme Cosmic String

Pedro F. González-Díaz.
Centro de Física "Miguel Catalán",
Instituto de Matemáticas y Física Fundamental
Consejo Superior de Investigaciones Científicas
Serrano 121, 28006 Madrid (SPAIN)

Abstract

This paper deals with the geometry of supermassive cosmic strings. We have used an approach that enforces the spacetime of cosmic strings to also satisfy the conservation laws of a cylindric gravitational topological defect, that is a spacetime kink. In the simplest case of kink number unity the entire energy range of supermassive strings becomes then quantized so that only cylindrical defects with linear energy density \( G_\mu = \frac{1}{4} \) (critical string) and \( G_\mu = \frac{1}{2} \) (extreme string) are allowed to occur in this range. It has been seen that the internal spherical coordinate \( \theta \) of the string metric embedded in an Euclidean three-space also evolves on imaginary values, leading to the creation of a covering shell of broken phase that protects the core with trapped energy, even for \( G_\mu = \frac{1}{2} \). Then, the conical singularity becomes a removable horizon singularity. We re-express the extreme string metric in the Finkelstein-McCollum standard form and remove the geodesic incompleteness by using the Kruskal technique. The \( z=\text{const} \) sections of the resulting metric are the same as the hemispherical section of the metric of a De Sitter kink. Some physical consequences from these results, including the possibility that the extreme string drives inflation and thermal effects in its core, are also discussed.
1 Introduction

Topological defects consisting of confined regions of false vacuum can occur in gauge theories with spontaneous symmetry breaking [1]. Among such defects, cosmic strings trapped during the breaking of a local $U(1)$ gauge symmetry are of particular interest [1,2]. If local strings appeared in phase transitions in the early universe, they could have served as seeds for the formation of galaxies and other larger scale structures we now are able to observe [3]. Recently, the possibility that inflation can be driven in the core of topological defects has also been advanced [4,5]. In the case of local cosmic strings this extremely interesting possibility will critically depend [5] on the strength of their gravity coupling $G\mu$, where $\mu$ is the string mass per unit length.

Static defect solutions occur in simple models of the form

$$V(\varphi) = \frac{1}{4} \lambda (\varphi_a \varphi_a - \eta^2)^2, \ a = 1, ..., N,$$

where $\eta$ is the symmetry breaking scale, and $N = 2$ for cosmic strings. Equalizing [5] the core radius $\delta_0 \sim \eta V_0^{-\frac{1}{2}}$, where $V_0 = \frac{1}{4} \lambda \eta^4$, and the horizon size corresponding to the vacuum energy $V_0$, $H_0^{-1} = (\frac{3}{8\pi G V_0})^{\frac{1}{2}}$, one obtains using $\mu \sim \eta^2$ a value for the string mass per unit length $\mu_I \sim \frac{1}{G}$, such that for $\mu > \mu_I$ inflation could be generated in the core.

Nevertheless, the concepts of radius and mass per unit length for a source like the string core are not unambiguously defined [6], and one would expect that even for the extreme case where $\mu = \mu_e = \frac{1}{2G}$ inflation could not be ensured to be driven in the core of the cosmic string. On the other hand, an extreme supermassive string with $2G\mu_e = 1$ does not seem to exist because it would correspond to a situation where all the exterior broken phase is collapsed into the core, leaving a pure false-vacuum phase in which the picture of a cosmic string with a core region of trapped energy is lost [6-9]. This happens in all considered string metrics, i.e. for the Hiscock-Gott metric [10,11] and for the Laguna-Garfinkle [8] and Ortiz [9] metrics, in all the cases the spacetime possesses an unwanted singularity which cannot be smoothed out.

In this paper we enforce the interior string metric to describe a cylindrically-symmetric gravitational kink, i.e. an allowed gravitational topological defect which can move about spacetime but cannot be removed without cutting [12], and is characterized by a conserved kink number measuring the number of times the light cone tips over on a boundary. We show that in this case the picture of a cosmic string with a core region of trapped energy is still retained even at the extreme value $2G\mu = 1$, and that the conical singularity becomes then the apparent singularity (event horizon) of a De Sitter kink. This hori-
zon singularity can be removed by a suitable Kruskal extension. The resulting extreme supermassive string would then be able to drive a gravitational inflationary process, without any fine tuning of the initial conditions.

The paper is outlined as follows. In section 1 we construct an explicit metric for the supermassive cosmic string kink, and discuss the constraints that the existence of the kink imposes on the internal geometry of the string. As a result of conservation of kink number, the energy of the supermassive cosmic string becomes quantized so that it can only take on values $4G\mu = 1$ and $2G\mu = 1$. We then re-express the metric of the cosmic string kink in standard form and obtain an analytical expression for the relevant time parameter entering that metric in section 3. The geodesic incompleteness of the standard metric is removed in section 4 by maximally-extending this metric using the Kruskal technique. In section 4 we also show that the $z = \text{const.}$ sections of the geodesically complete metric of a $2G\mu = 1$ cosmic-string kink describes a hemispherical section of a kinky De Sitter spacetime which may allow an eternal inflationary process [13], and study the quantum creation of particles in the one-kink extreme cosmic string. Throughout the paper we use units so that $\hbar = c = 1$.

2 The Spacetime of a Cosmic String Kink

2.1 Static Metric

The static, cylindrically symmetric internal metric of a straight cosmic string, which is an exact solution of Einstein equation, is given by [10,11]

$$ds^2 = -d\tau^2 + d\rho^2 + dz^2 + r_*^2 \sin^2 \frac{\rho}{r_*} d\phi^2,$$

with $-\infty < \tau < \infty$, $-\infty < z < \infty$, $0 \leq \phi < 2\pi$, $0 \leq \rho \leq r_\ast \arccos(1 - 4G\mu)$, and

$$r_\ast = \left(8\pi G\epsilon\right)^{-\frac{1}{2}},$$

where $\epsilon$ is the uniform string density, out to some cylindrical radius $\rho_0$.

Both the interior metric (2.1) and the exterior metric [10,11],

$$ds^2 = -d\tau^2 + d\rho'^2 + dz^2 + (1 - 4G\mu)^2 \rho'^2 d\phi^2,$$

define two-surfaces at $z = \text{const}$, $\tau = \text{const}$, which can be simultaneously visualized by embedding the metrics in an Euclidean three-space [11]. Then, the geometries of such surfaces are, respectively, that of a spherical cap (interior region) and that of a cone with deficit angle $\Delta = 8\pi G\mu$ in the exterior vacuum region. The interior spherical cap
becomes a hemisphere for $G\mu = \frac{1}{4}$ and a sphere for $G\mu = \frac{1}{2}$. In what follows, these embeddings will be denoted as embedded hemispherical or spherical geometries.

We shall write now the internal metric (2.1) in a form which is best suited for performing a kink extension. Thus, the change of coordinate

$$ r = r_* \sin \frac{\rho}{r_*} $$

shows more transparently the invariance of the line element (2.1) under rotation about $\phi$ and translation along $z$ of the underlying cylindrical symmetry (i.e. under the two Killing vectors $\xi = \partial_\phi$ and $\zeta = \partial_z$ [14]), and gives rise to a singularity at $r = r_*$; i.e.:

$$ ds^2 = -d\tau^2 + \frac{dr^2}{1 - \frac{r_*^2}{r^2}} + dz^2 + r^2 d\phi^2. $$

The divergence that appears in (2.5) as $r$ tends to $r_*$ would correspond to an apparent singularity and should therefore be removable by an appropriate coordinate transformation. It follows from (2.2) and (2.4) that values of coordinate $r$ larger than $r_*$ should be associated with purely imaginary values of $r_*$ and, therefore with negative string densities, $\epsilon < 0$. It will be seen in section 3 that, if the internal spacetime of a cosmic string satisfies the conservation law of a kink, then such imaginary values of $r_*$ would imply Euclidean continuation of time $\tau$ so that it becomes generally complex, and hence instantonic "tunneling" into the exterior broken-symmetry phase. One would regard the kink extension of metric (2.5) as a way to determine the precise extent of such a continuation.

A coordinate change which still transparently shows invariance under the Killing vectors and leads to no apparent interior singularity is

$$ u = \frac{\tau}{r_*} + \arcsin \frac{r}{r_*}, \quad v = \frac{\tau}{r_*} - \arcsin \frac{r}{r_*}. $$

In terms of coordinates $u$ and $v$ we obtain for the interior metric

$$ ds^2 = -r_*^2 du dv + dz^2 + r^2 d\phi^2, $$

where

$$ r = r_* \sin \left[ \frac{1}{2} (u - v) \right]. $$

In any case, the exterior metric still keeps a conical singularity.

2.2 Kink Nonstatic Extension

The regular metric (2.7) would trivially describe the maximally extended interior region of a cosmic string. The structure of the string
core is then determined by the matter content of the gauge theory (e.g. the Abelian Higgs model of Nielsen and Olesen [15]) and the coupling $G\mu$, and should only lodge real energy from the unbroken gauge phase. Nevertheless, metric (2.5) can be analytically extended into some limited region with the broken phase if we make this metric allow the lightcones to tip over on the boundaries so that one conserved kinky topological defect describable by relativistic metric twists is present [12].

The relationship between a possible cosmic string kink and the apparent singularity in metric (2.5) can be appreciated by considering [16] a cylindrically-symmetric spacetime with manifest invariance under the action of the Killing vectors $\xi$ and $\zeta$ of the form

$$ds^2_0 = \cos 2\alpha (-dt^2 + dr^2) - 2\sin 2\alpha dt dr + dz^2 + r^2 d\phi^2,$$

where $\alpha$ denotes the angle of tilt of the spacetime lightcones. We shall then ensure the presence of one kink (i.e. a gravitational topological defect with kink number unity) by requiring $\alpha$ to monotonously vary from 0 to $\pi$, starting from $\alpha(0) = 0$.

Metric (2.9) is nonstatic (as it is not invariant under time reversal $t \rightarrow -t$) and can be transformed into the static metric (2.5) if we set

$$\sin \alpha = \frac{r}{2\pi r_*},$$

(10)

and change the time coordinate so that $\tau = t + G(\omega)$, where

$$d\omega = d\tau + F(r) dr,$$

(11)

with $F(r)$ a given function of $r$. Denoting $\frac{dG(\omega)}{d\omega}$ by $G'$ it follows

$$dt = d\tau (1 - G') - G' F(r) dr.$$

(12)

Metric (2.5) can then be obtained from (2.9) if

$$G' = 1 - \frac{k_1}{\cos^{1/2} 2\alpha}, \quad k_1 = \pm 1$$

(13)

$$G' F(r) = \tan 2\alpha.$$  

(14)

Note that both $F(r)$ and $G'$ are singular at $r = r_*$. This still reflects the singular character of metric (2.5). Moreover, since this metric can be transformed into metric (2.9) by (2.10)-(2.14), the latter metric becomes also a solution of the same Einstein equation as for (2.5), expressed in terms of the new time coordinate $t$. 

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2.3 The Broken-Phase Shell

Let us consider the limitations that the functional form (2.10) imposes on the internal geometry of a supermassive cosmic string. From (2.4) and (2.10) it is obtained

$$
\cos^2 \theta = \cos 2\alpha,
$$

(15)

where we have denoted \[\theta = \frac{\rho}{r^*},\] so that the \(\tau,z = \text{const.}\) sections of metric (2.1) becomes \(r^2_\tau (d\theta^2 + \sin^2 \theta d\phi^2)\) for some \(\theta\)-interval, and \(\alpha\) monotonously varies from 0 to \(\pi\).

The question now is, how does \(\theta\) vary under the complete one-kink variation of \(\alpha\) from 0 to \(\pi\)? From (2.15) it follows that monotonous variation of \(\alpha\) from 0 to \(\pi/4\), and from \(3\pi/4\) to \(\pi\) induces monotonous variation of \(\theta\), respectively, from 0 to \(\pi/2\) and from \(\pi/2\) to \(\pi\) (or, likewise, from \(\pi\) to \(3\pi/2\) and from \(3\pi/2\) to 0), i.e. the entire two-sphere. Note that the corresponding induced variations of \(\theta\) in the interval \((\pi,2\pi)\) are also allowed but, since \(\theta\) appears in the metric in the form \(\sin^2 \theta\), these variations would lead to the same geometrical situations as from variations of \(\theta\) in the interval \((0,\pi)\). Variation of \(\alpha\) from \(\pi/4\) to \(3\pi/4\) induces variation of \(\theta\) only along its imaginary axis, first from \(\pi/2\) to \(\pi/2 + i\kappa \ln(\sqrt{2} + 1)\) (at \(\alpha = \pi/2\)), and then to \(\pi/2\) again. The choice of sign in the argument of the \(\ln\) at the extremum value of \(\theta\) corresponding to \(\alpha = \pi/2\) should be made as follows. On the interval where \(\theta\) is complex, i.e. \(\theta = \frac{\pi}{2} + i\theta_i\), with \(\ln(\sqrt{2} - 1) \leq \theta_i \leq \ln(\sqrt{2} + 1)\), we have (see Refs. [10,11])

\[
G_{\mu} = \frac{1}{4} + \frac{i}{4} \sinh \theta_i
\]

(16)

\[
r = i r_* \coth \theta_i
\]

(17)

Taking into account (2.17) it follows that the imaginary part of (2.16) corresponds to some real mass per unit imaginary length, or equivalently, to some tachyonic mass per real unit length. Thus, once the critical string mass \(G_{\mu} = \frac{1}{4}\) is reached, the interior of the kinky string starts developing a new real region up to a maximum width \((2^{1/2} - 1)r_*\), covering the entire embedded hemisphere of the \(\tau,z = \text{const.}\) sections, where the gauge symmetry is broken at the given scale \(\varphi = \pm \eta\). Then, the symmetry \(\eta \rightarrow -\eta\) of the broken phase will imply an identification \(\frac{\pi}{2} + i \ln(2^{3/2} + 1) \rightarrow \frac{\pi}{2} + i \ln(2^{3/2} - 1)\) on the extremum value of \(\theta_i\) that corresponds to \(\alpha = \frac{\pi}{2}\). We can therefore choose complex \(\theta\) to vary first from \(\frac{\pi}{2}\) to \(\frac{\pi}{2} + i\kappa \ln(2^{1/2} + 1)\) as \(\alpha\) goes from \(\frac{\pi}{4}\) to \(\frac{\pi}{2}\), and then from \(\frac{\pi}{2} + i\kappa \ln(2^{3/2} - 1)\) to \(\frac{\pi}{2}\) again as \(\alpha\) goes from \(\frac{\pi}{2}\) to \(\frac{3\pi}{4}\), either for \(\kappa = \pm 1\).

This strip corresponds to the maximum analytical extension beyond \(r = r_*\) of the real region described by the internal string metric (2.5) which is compatible with the presence of one kink.
Now, the maximum value of the internal spherical coordinate \( \theta \), \( \theta_M \), is related to the string mass per unit length \( \mu \) by [10,11]
\[
\theta_M = \arccos(1 - 4G\mu).
\]

Then, in order to ensure the occurrence of one kink in the cosmic string interior, we must have \( \theta_M = \pi \) and hence \( G\mu = \frac{1}{2} \); or in other words, the presence of one kink implies a quantization of the supermassive cosmic string so that only the critical string at \( G\mu = \frac{1}{2} \) with internal embedded hemispherical geometry (no kink present), and the extreme string at \( G\mu = \frac{1}{2} \) with internal embedded spherical geometry (one kink present) are allowed to exist along the entire interval, \( \frac{\pi}{2} \leq \theta \leq \pi \), of possible classical supermassive cosmic strings.

If we ensure the presence of one kink, then no string exterior (except the portion which is incorporated into the interior metric by the complex evolution of \( \theta \)) is possible or needed [10,11], and the conical singularity becomes a removable horizon singularity on the core surface at \( r = r_* \), separating the two gauge phases that make up the extended string interior. Thus, all of the possible geometry of the extreme string can be regarded as describable by a spacetime which is \( \mathbb{R}^2 \times S^2 \), showing still the picture of a cosmic string with a core region of trapped energy surrounded by a shell of true vacuum protecting the string from dissolving in the background phase with unbroken symmetry.

### 3 The Lightcone Configuration

The results of section 2 allow us to deal with the interior metric of an extreme supermassive cosmic string kink, \( 2G\mu = 1 \), in a similar fashion to as it is done in the cases of the black hole kink [16] or the De Sitter kink [17].

The interior spacetime metric of one such extreme string has still a geodesic incompleteness at \( r = r_* \), and can only be described by using a number of different coordinate patches. Since \( \sin \alpha \) is given by Eq. (2.10) and cannot exceed unity, it follows that \( 0 \leq r \leq A \equiv \sqrt{2}r_* \), so that \( \alpha \) varies only from 0 to \( \frac{\pi}{2} \). In order to have an entire one-kink gravitational defect, we need then a second coordinate patch which would describe the other half of the \( \alpha \) interval, \( \frac{\pi}{2} \leq \alpha \leq \pi \). The identification and distinction of such two patches can be achieved by transforming (2.9) into the standard metrical form proposed by Finkelstein and McCollum [16], adapted here to cylindric symmetry. To accomplish such a transformation, it is convenient to introduce a new time coordinate, such that
\[
\bar{t} = t + G(\sigma) \quad (1)
\]
\[
\cos 2\alpha (1 - G^2 F^2_\sigma (r)) + 2 \sin 2\alpha G F_\sigma (r) = 0, \tag{2}
\]
where the new variable \(\sigma\) and the new functionals \(F_\sigma (r)\) and \(G \equiv G(\sigma)\) are defined as follows

\[
 d\sigma = d\bar{t} + F_\sigma (r) dr
\]

\[
 G^i \equiv G(\sigma)^i = \frac{dG}{d\sigma}.
\]

We obtain then

\[
 G^i F_\sigma (r) = \frac{k_2}{\cos 2\alpha}, k_2 = \pm 1, \tag{3}
\]
so that metric (2.9) becomes

\[
 ds^2 = -dt^2 - \frac{2k_1 k_2}{\cos^2 2\alpha} d\bar{t} dr + dz^2 + r^2 d\phi^2, \tag{4}
\]
where \(k_1\) is defined in (2.13).

Metric (3.4) still is not the kink metric in standard form. This is obtained by making the re-definition

\[
 d\hat{t} = k_1 \frac{d\bar{t}}{\cos 2\alpha} = dt + (\tan 2\alpha - \frac{k_2}{\cos 2\alpha}) dr, \tag{5}
\]
so that we finally obtain

\[
 ds^2 = -\cos 2\alpha d\hat{t}^2 - 2k_2 d\hat{t} dr + dz^2 + r^2 d\phi^2. \tag{6}
\]

Metric (3.6) has the same form as the general line element in standard form given by Finkelstein and McCollum [16], after replacing spherical symmetry for cylindric symmetry, and becomes the solution of the same Einstein equation as for (2.1) when this equation is expressed in terms of the new time coordinate \(\hat{t}\). On the other hand, using (2.10) it can be readily seen that any \(z=\text{const}\) section of metric (3.6) is not but the hemispherical section of the standard De Sitter kink metric [17], for a positive cosmological constant \(\Lambda = \frac{3}{r^2}\).

The choice of sign in (3.3) is adopted for the following reason. The zeros of the denominator of \(F_\sigma G^i = (\sin 2\alpha \mp 1)/\cos 2\alpha\) correspond to the two horizons where \(r = r^*_\pm\), one per patch. For the first patch, the horizon occurs at \(\alpha = \frac{\pi}{4}\) and therefore the upper sign is selected so that \(F_\sigma G^i\) remains well defined and hence the kink is preserved in the transformation from (2.9) to (3.6). For the second patch the horizon occurs at \(\alpha = \frac{3\pi}{4}\) and therefore the lower sign is selected. \(k_2 = +1\) will then correspond to the first coordinate patch, and \(k_2 = -1\) to the second one. Thus, it is the parameter \(k_2\), but not \(k_1\), which defines the two coordinate patches needed for a complete description of the spacetime of a cosmic string kink. One would then expect an analytical
expression for the time $\hat{t}$ entering metric (3.6) which contains the sign ambiguity $k_2$, but not $k_1$. Taking

$$\sin \alpha = \frac{r}{A}, \cos \alpha = k_2(1 - \frac{r^2}{2r_*^2})^{\frac{1}{2}}, \cos 2\alpha = k_1(1 - \frac{r^2}{r_*^2})^{\frac{1}{2}},$$ \hspace{1cm} (7)

by directly integrating (3.5) we in fact obtain [20]

$$\hat{t} = k_1 \int_{0/A}^{r} \frac{d\ell}{\cos^{\frac{1}{2}} 2\alpha} = t - r_* k_2 \left\{ \frac{A}{r_*} \left(1 - \frac{r^2}{2r_*^2}\right)^{\frac{1}{2}} \right. \right.$$

$$\left. - \frac{1}{2} \ln \left[ \frac{(A(1 - \frac{r^2}{2r_*^2})^{\frac{1}{2}} + r_*)((r_* - r)/(r_* + r))}{(A(1 - \frac{r^2}{2r_*^2})^{\frac{1}{2}} - r_*)((r_* + r)/(r_* - r))} \right] \right\}, \hspace{1cm} (8)
$$

where the lower limit $0/A$ refers to the choices $r = 0$ and $r = A \equiv 2^{\frac{1}{2}} r_*$, depending on whether the case $k_2 = +1$ or the case $k_2 = -1$ is being considered [17]. Note that all sign ambiguity arising from the square root in the argument of the ln has been omitted in Eqn. (3.8). Such an ambiguity, so as some constant term coming from the lower integration limit, does not affect the discussion to follow as it only manifests as an additive constant term which leaves metric (3.6) unchanged. This ambiguity will be of decisive importance however for the consideration of the thermal properties of the extreme string which we consider in section 4.

The embedded geometry of a $t=$-const, $z=$-const section of an extreme string kink is that of two hemispheres, each corresponding to a coordinate patch, whose mutual matching at their equators lies on the extremum imaginary values of the spherical coordinate $\theta$, $\ln(2^{\frac{1}{2}} + 1) \rightarrow \ln(2^{\frac{1}{2}} - 1)$. The imaginary values of $\theta$ are mapped onto real values of $\alpha$ in the kink, giving rise to an equator identification at $r = A$ which is ensured by continuity of the tilt angle $\alpha$ at $\alpha = \frac{\pi}{2}$. Thus, the lightcone configuration that corresponds to the one-kink cosmic string is that given in Fig. 1.

We finally note that, as mentioned in section 2, time $\tau$ must become generally complex for $r > r_*$,

$$d\tau = (1 - \frac{r^2}{r_*^2})^{\frac{1}{2}} dt + 2k_2 \frac{r}{A} \left( \frac{1 - \frac{r^2}{2r_*^2}}{1 - \frac{r^2}{r_*^2}} \right)^{\frac{1}{2}} dr, \hspace{1cm} (9)$$

so that by enforcing the system to lie parallel to the $r$ axis, that is taking time $t = t_0$ constant, and using (3.7), we have

$$\tau = t_0 - \frac{k_2 r_*^2}{A} \left\{ \left[ (1 - \frac{r^2}{r_*^2})(1 - \frac{r^2}{r_*^2}) \right]^{\frac{1}{2}} \right. \right.$$

$$\left. + k_1 \frac{r_*}{A} \ln \left[ \frac{2k_1 A}{r_*} (1 - \frac{r^2}{r_*^2})^{\frac{1}{2}} + 4(1 - \frac{r^2}{2r_*^2})^{\frac{1}{2}} \right] \right\} 0/A. \hspace{1cm} (10)$$
We can check that in fact for real $t_0$, both $d\tau$ and $\tau$ are real for $r \leq r_*$ and complex for $A \geq r > r_*$. 

4 Kruskal Extension of Extreme String Metric

4.1 Removal of Geodesic Incompleteness

The geodesic incompleteness at $r = r_*$, which occurs in each of the two coordinate patches described by metric (3.6), can be removed by the usual Kruskal technique [19]. Thus, we define the metric

$$ds^2 = -2F(U, V)dUdV + dz^2 + r^2d\phi^2,$$

in this way straightening the null geodesics into lines parallel to the new $U$ and $V$ axis, and identify it with the standard metric (3.6), with $g_{\hat{t}\hat{t}} = -\cos 2\alpha$, $g_{\hat{t}r} = -k_2$ and $g_{UV} = -F$, in such a way that $F$ be finite, nonzero and depend on $r$ and $k_2$ alone. All of these requirements can be met by the choice

$$U = \mp e^{\beta i} \exp(2\beta k_2 \int_{0/A}^r \frac{dr}{\cos 2\alpha})$$

(2)

$$V = \mp \frac{1}{\beta r_*} e^{-\beta i}$$

(3)

$$F = -\frac{r_* \cos 2\alpha}{2\beta} \exp(-2\beta k_2 \int_{0/A}^r \frac{dr}{\cos 2\alpha}),$$

(4)

with $\beta$ a constant which should be chosen so that $F$ has a finite limit as $r \to r_*$. Using [17]

$$\int_{0/A}^r \frac{dr}{\cos 2\alpha} = \frac{1}{2}r_* \ln\left(\frac{r_* + r}{r_* - r}\right),$$

where we have omitted a constant term coming from the lower integration limit because it is canceled by the similar constant term which appears in the Kruskal coordinate $U$, and

$$\cos 2\alpha = 1 - \frac{r^2}{r_*^2},$$

we obtain from (4.4)

$$F = -\frac{(r_*^2 - r^2)(r_* + r)^2}{2\beta r_*} \left[\frac{r_*}{r_*^2 - r^2}\right]^{-\beta k_2 r_*}.$$ 

To avoid $F$ being either 0 or $\infty$ at $r = r_*$, we then choose $\beta = -\frac{1}{k_2 r_*}$, and arrive therefore at

$$F = \frac{1}{2} k_2 (r_* + r)^2$$

(5)
\[ U = \mp e^{-\frac{k_2^2}{r_2^*} \left( \frac{r_2^* - r}{r_2^* + r} \right)} \]  

\[ V = \pm k_2 e^{-\frac{k_2^2}{r_2^*}} \]  

so that

\[ UV = -k_2 \frac{(r_2^* - r)}{(r_2^* + r)}. \]  

Since \( \hat{t} \) depends on \( k_2 \), but not \( k_1 \), it follows that \( F, U \) and \( V \) contain \( k_2 \), but not \( k_1 \), as well.

In terms of the coordinate product (4.8) we have finally

\[ F = \frac{2r_2^2k_2}{(k_2 - UV)^2} \]  

\[ r = r_2^* \left( \frac{k_2 + UV}{k_2 - UV} \right) \]  

\[ \hat{t} = t - k_2 r_2^* \left\{ (1 - \frac{4k_2 UV}{(k_2 - UV)^2})^{\frac{1}{2}} \right\} \]  

\[ + \frac{1}{2} \ln \left\{ \frac{1 + (1 - \frac{4k_2 UV}{(k_2 - UV)^2})^{\frac{1}{2}}}{1 - (1 - \frac{4k_2 UV}{(k_2 - UV)^2})^{\frac{1}{2}}} \right\}. \]  

The metric (4.1) becomes then

\[ ds^2 = -\frac{4k_2 r_2^*^2}{(k_2 - UV)^2} dU dV + dz^2 + \frac{r_2^2 (k_2 + UV)^2}{(k_2 - UV)^2} d\phi^2, \]  

which, as expected, does not contain any sign parameter other than \( k_2 \).

We notice that the \( z=\text{const} \) sections of this metric actually coincide with that is obtained for the hemispherical section of a De Sitter kink with positive cosmological constant \( \Lambda = \frac{3}{r_2^*} \) [17]. In Fig. 2 we give a representation in terms of coordinate \( U, V \) of the two different coordinate patches that occur in the one-kink geodesically-complete extreme string spacetime. These Kruskal diagrams do not show any spatial infinities and can only be extended beyond \( r = r_2^* \) up to the surfaces \( r = A \), both on the original and new regions created by the Kruskal extension. Because of the continuity of the angle of tilt \( \alpha \) on \( \alpha = \frac{\pi}{2} \), such surfaces should be identified in passing from one patch to another, either on the original regions \( I \) and \( II \) or on the new regions \( III \) and \( IV \).

The maximally-extended extreme string metric (4.12) does not possess any singularity and covers all the space of an extreme string kink, including both an unbroken-phase interior of radius \( r_2^* \) and a broken-phase covering shell of width \( (2^{\frac{1}{2}} - 1)r_2^* \). The first of these facts is in
sharp contrast with the unavoidability of an unwelcome singularity found by other authors for supermassive cosmic strings [8,9]. The fact that (4.12) also describes an exterior shell that protects the core from dissolving is, in turn, in contrast with the trivial maximal extension (2.7) of the interior metric of cosmic strings.

In order to analyse possible physical processes taking place in the spacetime of a cosmic string kink, it is illustrative to consider the paths followed by null geodesics on the Kruskal diagrams. For the solid geodesic on Fig. 2, the segment in the original region $I_+^+$ starts at the pole $r = 0$, crosses the $V$ axis on the event horizon $r = r_*$ and passes into the original region $II_+^+$; it then continues along that region to pass over into the original region $I_-^-$ of the second patch at the surface $r = A$ on which the two coordinate patches are identified. The null geodesic crosses finally the $U$ axis at $t = \infty$ to get in the new region $III_-$, propagating along it to die at the pole $r = 0$. Since identification of surfaces at $r = 0$ of the two patches is disallowed, null geodesics can never complete a closed itinerary.

4.2 Physical Consequences

The existence of an extreme string kink which on each $z=$const section exactly possesses the symmetry of a hemispherical section of the De Sitter kink, may have consequences of interest. First of all, in order for the event horizon of the De Sitter kink at $r = r_*$ to be a cosmological horizon [20] with size $H^{-1}_0 = (\frac{3}{8\pi GV_0})^{\frac{1}{2}}$, according to (2.2), one should take $\epsilon = \frac{V_0}{3}$ (with $\epsilon$ the uniform string density and $V_0$ the vacuum energy of the model), so that the cosmological constant becomes $\Lambda = \frac{3}{r_*^2} = 8\pi GV_0$. Then the radius of a spherical extreme string, $\sqrt{2}r_*$, would exceed the size of its corresponding cosmological horizon, $H_0^{-1} = r_*$, by precisely the width $r_*(\sqrt{2} - 1)$ of the true-vacuum shell of the string, and therefore the extreme cosmic string kink can quite naturally drive a De Sitter inflationary process [21], without any fine tuning of the initial conditions [4,5] or the kind of uncertainties pointed out in the Introduction. On the other hand, since we have now $r_* \sim \eta/\sqrt{V_0}$, and hence a symmetry-breaking scale $\eta \sim \sqrt{\mu}$ of the order the Planck scale, the inflationary expansion driven in the extreme string kink should be primordial (i.e. taking place in the Planck era) and essentially unique. It is worth noting, moreover, that if we would interpret our one-kink cosmic strings as incipient baby universes, then the above implication would lead to quite comfortably accommodate the concept of an eternal process of continually self-reproducing inflating universes [4,13] in the present picture.

On the other hand, identification between surfaces in $III_{k_2}$ and $II_{k_2}$ (or between surfaces in $IV_{k_2}$ and $I_{k_2}$) will simply come from explicitly
displaying the sign ambiguity of the square root of the argument for the ln in Eqn. (3.10). Thus, if we want to express that argument as an absolute value, then one has as the most general expression for metrical time

\[ \hat{t}(k_3) = t - r_\ast k_2 \left\{ \frac{A}{r_\ast} \left( 1 - \frac{r^2}{2r_\ast^2} \right)^{\frac{1}{2}} - \ln \left\{ \left\lfloor \left( \frac{A(1 - \frac{r^2}{2r_\ast^2})^{\frac{1}{2}} + r_\ast(r_\ast - r)}{A(1 - \frac{r^2}{2r_\ast^2})^{\frac{1}{2}} - r_\ast(r_\ast + r)} \right\rfloor^{\frac{1}{2}} \right\} + i \frac{k_2 k_3 (1 - k_3) \pi r_\ast}{2} \right\} \right\} + \frac{i}{2} k_2 k_3 (1 - k_3) \pi r_\ast, \]

(13)

where \( \hat{t} \equiv \hat{t}(k_2) \) is the same as in (3.10), and \( k_3 = \pm 1 \) is a new sign parameter which unfolds the two coordinate patches given in Fig. 2 into still two sets of two patches. Time (4.13) is the most general expression for the time \( \hat{t} \) entering the standard metric (3.6). One would again recover metric (3.6) from metric (4.1) with the same requirements as before using \( \hat{t}(k_3) \) instead of \( \hat{t} \) if we re-define the Kruskal coordinates \( U, V \) such that

\[ U = \pm k_3 e^{-k_3 e^{\frac{k_2 t_{\ast}}{r_\ast} (r_\ast - r)}} \]

(14)

\[ V = \mp k_3 e^{k_2 e^{\frac{k_2 t_{\ast}}{r_\ast} (r_\ast + r)}} \]

(15)

where

\[ t_{\ast} = \hat{t} + i k_2 k_3 \pi r_\ast. \]

(16)

This choice leaves expressions (4.8), (4.9), (4.10) and, of course, the Kruskal metric (4.12) real and unchanged.

For \( k_3 = -1 \), (4.14) and (4.15) become the sign-reversed to (4.6) and (4.7); i.e. the points \( (\hat{t} - i k_2 \pi r_\ast, r, z, \phi) \) on the coordinate patches of Fig. 2 are the points on the new region \( III_{k_2} \), on the same figure, obtained by reflecting in the origins of the respective \( U, V \) planes, while keeping metric (4.12) and the physical time \( t \) real and unchanged. This periodicity in the metric should result [22] in the appearance of a thermal bath of particles which, in the present case, entails no unjustified extension of the physical time \( t \) into the imaginary axis. Our approach offers therefore a physically reasonable resolution of the paradox posed by the Euclidean gravity method [23].

Let us consider then the evolution of a field along any null geodesics on Fig. 2 described by a quantum propagator. If the field has mass \( m \), such a propagator will be the same as the propagator \( G(x', x) \) used by Gibbons and Hawking [20], and satisfy therefore the Klein-Gordon equation

\[ (\square_x^2 - m^2)G(x', x) = -\delta(x, x'). \]

(17)
For metric (4.12), the propagator $G(x', x)$ becomes analytic [20] on precisely the strip of width $\pi r_*$ predicted by (4.16), here without any need to extend $t$ to the Euclidean regime.

From (4.16), we can write for the amplitude for detection of a detector sensitive to particles of energy $E$ in regions $I I_{k_2}$ [20,22]

$$\Pi_E = e^{k_2 k_3 \pi r_* E} \int_{-\infty}^{+\infty} dt e^{-iE \hat{t}} G(0, \vec{R}'; \hat{t} + ik_2 k_3 \pi r_*, \vec{R}),$$

(18)

where $\vec{R}'$ and $\vec{R}$ denote respectively $(r', z', \phi')$ and $(r, z, \phi)$. Note that the time parameter entering the amplitude for detection is $\hat{t}$, rather than $t$, as both the matter field and the detector must evolve in the spacetime described by metrics (3.6) and (4.12). Since time $\hat{t}$ (but not $t$) already contains the imaginary term which is exactly required for the thermal effect to appear, we have not need to make the physical time $t$ complex. Following Gibbons and Hawking [20], we now investigate the different particle creation processes that can take place on the extreme hyperbolae at $r = 0$ and $r = 2^{1 \over 2} r_*$. Let us first consider the case $k_3 = -1$ for the original regions $I I_{k_2}$ on the patches of Fig. 2. For $k_2 = +1$, if $x'$ is a fixed point on the hyperbola at $r = 0$ of region $I_+$, and $x$ is a point on the hyperbola at $r = 2^{1 \over 2} r_*$ of region $I I_+$, we obtain from (4.18)

$$P^{I I_+}_{a}(E) = e^{-2 \pi r_* E} P^{I I_+}_{e}(E),$$

(19)

where $P^{I I_{k_2}}_{a}(E)$ generically denotes the probability for detector to absorb a particle with positive energy $E$ from region $I I_{k_2}$, and $P^{I I_{k_2}}_{e}(E)$ accounts for the similar probability for detector to emit the same energy also to region $I I_{k_2}$ [20,22].

An observer on the extreme hyperbola of the exterior original region of patch $k_2 = +1$ will then measure an isotropic background of thermal positive-energy radiation at a temperature

$$T_s = \frac{1}{2 \pi r_*}.$$  

(20)

If, in turn, $x'$ and $x$ are fixed points on the extreme hyperbolae in regions $I_-$ and $I I_-$, respectively, we obtain then for an observer on the hyperbola $r = 0$ in the interior original region

$$P^{I I_-}_{a}(-E) = e^{2 \pi r_* E} P^{I I_-}_{e}(-E).$$

(21)

According to (4.21), there will appear an isotropic background of thermal radiation which is formed by exactly the antiparticles to the particles contained in the thermal bath detected in region $I I_+$, at the same temperature $T_s$ given by (4.20). Thus, we can regard the joint
process as the propagation of thermal particles with positive energy from the second to the first patch through just original regions.

For \( k_3 = +1 \) we obtain similar hypersurface identifications as for \( k_3 = -1 \). In this case, the identification comes about by simply exchanging the mutual positions of the original regions \( I_{k_2} \) and \( II_{k_2} \) for, respectively, the new regions \( III_{k_2} \) and \( IV_{k_2} \), on the coordinate patches given in Fig. 2, while keeping the sign of coordinates \( U, V \) unchanged with respect to those in \( (4.6) \) and \( (4.7) \); i.e. the points \((\hat{t} + ik_2 \pi r^*, r, z, \phi)\) on the so-modified regions are the points on the original regions \( II_{k_2} \), on the same patches, again obtained by reflecting in the origins of the respective \( U,V \) planes, while keeping metric \( (4.12) \) and the physical time \( t \) real and unchanged.

Expressions for the relation between probabilities of absorption and emission for \( k_3 = +1 \) are obtained by simply replacing regions \( II_{k_2} \) for regions \( III_{k_2} \) (or regions \( I_{k_2} \) for regions \( IV_{k_2} \)), and energy \( E \) for \( -E \), in \( (4.19) \) and \( (4.21) \), keeping the same radiation temperature \( (4.20) \) in all the cases. Thus, we obtain that observers on extreme hyperbolae will detect an isotropic thermal bath of particles with energy: (i) \( E < 0 \) in the exterior new region \( III_+ \), and (ii) \( E > 0 \) in the interior new region \( III_- \), in both cases at an equilibrium temperature \( (4.20) \). The joint process would now represent propagation of a thermal bath of particles with positive energy from the first to the second patch just through the new regions created by Kruskal extension.

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Legends for Figures.

- Fig. 1. One-kink lightcone configuration for the extreme string kink. Also represented are geodesics passing through the different regions $I_{k_2}$ and $II_{k_2}$.

- Fig. 2. The two coordinate patches for the one-kink extended extreme string. In the figure, $A = 2\pi r_*$. Each point on the diagrams represents an infinite cylinder. The null geodesics starting at $North_+$ (solid line) and at $South_-$ (broken line) are not closed as any pole identification is disallowed. The curved lines at $t = 0$ and all similar $t = \text{const.}$ curves can neither close up and, since their slope $dU/dV$ changes sign at the horizons, they are not spacelike everywhere. Thus, the extended spacetime of the extreme cosmic string kink do not admit a complete foliation.
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