Some bidouble planes with \( p_g = q = 0 \) and 
\[ 4 \leq K^2 \leq 7 \]

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Abstract

We construct smooth minimal bidouble planes \( S \) of general type with \( p_g = 0 \) and \( K^2 = 4, \ldots, 7 \) having involutions \( i_1, i_2, i_3 \) such that \( S/i_1 \) is birational to an Enriques surface, \( S/i_3 \) is rational and the bicanonical map of \( S \) is not composed with \( i_1, i_2 \) and is composed with \( i_3 \).

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1 Introduction

Smooth minimal surfaces of general type with \( p_g = q = 0 \) have been studied by several authors in the last years, but a classification is still missing. We refer the surveys [MP2] and [BCP] for information on these surfaces.

There is, to my knowledge, only one example of a smooth minimal surface \( S \) of general type with \( p_g = 0 \) and \( K^2 = 7 \) ([In]). This surface has an alternative description as a bidouble cover of a rational surface ([MP1]). Its bicanonical map is of degree two onto a rational surface and is not composed with the other two involutions \( i_1, i_2 \) associated with the covering. Moreover, \( S/i_1 \) and \( S/i_2 \) are also rational (cf. [LS]).

In this paper we consider the case where \( S \) has an involution \( i \) such that the bicanonical map is not composed with \( i \) and \( S/i \) is not a rational surface. We construct examples with \( K^2 = 4, \ldots, 7 \) such that \( S/i \) is birational to an Enriques surface. This answers a question of Lee and Shin ([LS]) about the existence of the cases with \( K^2 = 5, 6, 7 \) and \( S/i \) birational to an Enriques surface. In all cases \( S \) has another involution \( j \) such that \( S/j \) is rational and the bicanonical map of \( S \) is composed with \( j \).

The paper is organized as follows. First we recall some facts on involutions. Secondly we note that minor modifications to [Ri1, Theorems 7, 8 and 9] give a list of possibilities for the branch curve in the quotient surface \( S/i \). Then we construct some examples as double covers of an Enriques surface obtained as a quotient of a Kummer surface. In Section 4.2 we describe these surfaces as bidouble covers of the plane. Finally we give some other bidouble plane examples.

Notation

We work over the complex numbers; all varieties are assumed to be projective algebraic. An involution of a surface \( S \) is an automorphism of \( S \) of order 2. We say that a map is composed with an involution \( i \) of \( S \) if it factors through the double cover \( S \to S/i \). A \((-2)\)-curve or nodal curve \( N \) on a surface is a curve
isomorphic to \( \mathbb{P}^1 \) such that \( N^2 = -2 \). An \((m_1, m_2, \ldots)\)-point of a curve, or point of type \((m_1, m_2, \ldots)\), is a singular point of multiplicity \( m_1 \), which resolves to a point of multiplicity \( m_2 \) after one blow-up, etc. The rest of the notation is standard in Algebraic Geometry.

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2 General facts on involutions

The following is according to [CM].

Let \( S \) be a smooth minimal surface of general type with an involution \( i \). Since \( S \) is minimal of general type, this involution is biregular. The fixed locus of \( i \) is the union of a smooth curve \( R'' \) (possibly empty) and of \( t \geq 0 \) isolated points \( P_1, \ldots, P_t \). Let \( S/i \) be the quotient of \( S \) by \( i \) and \( p : S \rightarrow S/i \) be the projection onto the quotient. The surface \( S/i \) has nodes at the points \( Q_i := p(P_i), i = 1, \ldots, t \), and is smooth elsewhere. If \( R'' \neq \emptyset \), the image via \( p \) of \( R'' \) is a smooth curve \( B'' \) not containing the singular points \( Q_i, i = 1, \ldots, t \).

Let now \( h : V \rightarrow S \) be the blow-up of \( S \) at \( P_1, \ldots, P_t \) and set \( R' = h^*(R'') \). The involution \( i \) induces a biregular involution \( \tilde{i} \) on \( V \) whose fixed locus is \( R := R' + \sum_1^t h^{-1}(P_i) \). The quotient \( W := V/\tilde{i} \) is smooth and one has a commutative diagram:

\[
\begin{array}{c}
V \\
\downarrow \pi
\end{array} \quad \begin{array}{c}
\rightarrow \\
\downarrow p
\end{array} \quad \begin{array}{c}
S \\
\downarrow \pi
\end{array}
\]

\[
\begin{array}{c}
W \\
\downarrow g
\end{array} \quad \begin{array}{c}
\rightarrow \\
\downarrow \pi
\end{array} \quad \begin{array}{c}
S/i
\end{array}
\]

where \( \pi : V \rightarrow W \) is the projection onto the quotient and \( g : W \rightarrow S/i \) is the minimal desingularization map. Notice that

\[
A_i := g^{-1}(Q_i), \quad i = 1, \ldots, t,
\]

are \((-2)\)-curves and \( \pi^*(A_i) = 2 \cdot h^{-1}(P_i) \).

Set \( B' := g^*(B'') \). Since \( \pi \) is a double cover with branch locus \( B' + \sum_1^t A_i \), it is determined by a line bundle \( L \) on \( W \) such that

\[
2L \equiv B := B' + \sum_1^t A_i.
\]

Proposition 1 ([CM], [CCM]) The bicanonical map of \( S \) (given by \( |2K_S| \)) is composed with \( i \) if and only if \( h^0(W, \mathcal{O}_W(2K_W + L)) = 0 \).
3 List of possibilities

Let $P$ be a minimal model of the resolution $W$ of $S/i$, let $\rho : W \to P$ be the corresponding projection and denote by $B$ the projection $\rho(B)$.

**Theorem 2** (cf. [Ri1]) Let $S$ be a smooth minimal surface of general type with $p_g = 0$ having an involution $i$ such that the bicanonical map of $S$ is not composed with $i$ and $S/i$ is not rational.

With the previous notation, one of the following holds:

a) $P$ is an Enriques surface and:
   - $B^2 = 0$, $t - 2 = K_S^2 \in \{2, \ldots, 8\}$, $B$ has a $(3, 3)$-point or a 4-uple point and at most one double point.

b) $\text{Kod}(P) = 1$ and:
   - $K_P B = 2$, $B^2 = -12$, $t - 2 = K_S^2 \in \{2, \ldots, 8\}$, $B$ has at most two double points, or
   - $K_P B = 4$, $B^2 = -16$, $t = K_S^2 \in \{4, \ldots, 8\}$, $B$ is smooth.

c) $\text{Kod}(P) = 2$ and:
   - $K_S^2 = 2 K_P^2$, $K_P^2 = 1, \ldots, 4$, $B$ is a disjoint union of four $(-2)$-curves, or
   - $K_P B = 2$, $K_P^2 = 1$, $B^2 = -12$, $t = K_S^2 \in \{4, \ldots, 8\}$, $B$ has at most one double point, or
   - $K_P B = 2$, $K_P^2 = 2$, $B^2 = -12$, $t + 2 = K_S^2 \in \{6, 7, 8\}$, $B$ is smooth.

Moreover there are examples for a), b) and c).

**Proof:** This follows from the proof of [Ri1 Theorems 7, 8 and 9] taking in account that:

- $p_g(P) = q(P) = 0$ (because $p_g(P) \leq p_g(S)$, $q(P) \leq q(S)$);
- $h^0(W, \mathcal{O}_W(2K_W + L)) \leq \frac{1}{2} K_W^2 + 2$ (see [Ri1 Proposition 4, b]));
- $K_S^2 \neq 9$ (see [DMP] Theorem 4.3);
- We can have $B^2 > 0$ (unlike the case $p_g = q = 1$).

Examples for a) and b) are given below. Rebecca Barlow ([Ba]) has constructed a surface of general type with $p_g = 0$ and $K^2 = 1$ containing an even set of four disjoint $(-2)$-curves. This gives an example for c).

4 Examples

4.1 $S/i$ birational to an Enriques surface

Consider the involution of $\mathbb{P}^1 \times \mathbb{P}^1$

$$ j : [x : y, a : b] \mapsto [y : x, b : a] $$
and denote by $f, g$ the projections onto the first and second factors, respectively. Let $F_1, \ldots, F_4, G_1, \ldots, G_4$ be fibres of $f, g$ such that

$$C := F_1 + \cdots + F_4 + G_1 + \cdots + G_4$$

is preserved by $j$ and does not contain the fixed points $[1 : \pm 1, 1 : \pm 1]$ of $j$.

Let

$$\pi : Q \to \mathbb{P}^1 \times \mathbb{P}^1$$

be the double cover with branch locus $C$ and let $k$ be the corresponding involution. It is well known that $Q$ is a Kummer surface and

$$E := Q / k \circ j$$

is an Enriques surface with 8 nodes.

### 4.1.1 $\mathbb{B}$ with a 4-uple point

Let $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a generic curve of bi-degree $(1, 2)$ tangent to $C$ at smooth points $p_1, p_2$ of $C$ such that $p_2 = j(p_1)$. The pullback $\pi^*(D + j(D)) \subset Q$ is a reduced curve with two 4-uple points, corresponding to the $(2, 2)$-points of $D + j(D)$ (which are tangent to the branch curve $C$). These points are identified by the involution $k \circ j$, thus the projection of $\pi^*(D + j(D))$ into $E$ is a reduced curve $\mathbb{B}$ with one 4-uple point.

Now let $\tilde{E}$ be the minimal smooth resolution of the Enriques surface $E$, $A_1, \ldots, A_8 \subset \tilde{E}$ be the nodal curves corresponding to the nodes of $E$ and $\mathbb{B} \subset \tilde{E}$ be the strict transform of $\mathbb{B}$. If $D$ does not contain one of the 16 double points of $C$, the divisor

$$\mathbb{B} := \mathbb{B} + \sum_{i=1}^{8} A_i$$

is reduced, divisible by 2 in the Picard group and satisfies $\mathbb{B}^2 = 0$. Let $S$ be the smooth minimal model of the double cover of $\tilde{E}$ ramified over $\mathbb{B}$. One can show that $S$ is a surface of general type with $p_g = 0$ and $K_S^2 = 6$. Moreover, $D$ can be chosen through one or two double points of $C$. This provides examples with $K_S^2 = 5$ or 4, corresponding to branch curves

$$\mathbb{B} + \sum_{i=1}^{7} A_i \quad \text{or} \quad \mathbb{B} + \sum_{i=1}^{6} A_i.$$

### 4.1.2 $\mathbb{B}$ with a (3,3)-point

Let $D_1$ be a curve of bi-degree $(0, 1)$ through $p$, $D_2$ be a general curve of bi-degree $(1, 1)$ through $p$ and $j(p)$ and set $D := D_1 + D_2$. Then $D + j(D)$ is a reduced curve with triple points at $p$ and $j(p)$. Now we proceed as in Section 4.1.1. In this case the branch curve $\mathbb{B} \subset \tilde{E}$ has a (3,3)-point instead of a 4-uple point. This gives an example of a surface of general type $S$ with $p_g = 0$ and $K_S^2 = 7$ (notice that the resolution of the (3,3)-point gives rise to an additional nodal curve in the branch locus). As above, $D$ can be chosen containing one or two double points of $C$, providing examples with $K_S^2 = 6$ or 5.
4.2 Bidouble plane description

Here we obtain the examples of Section 4.1 as bidouble covers of the plane.

4.2.1 Construction

Let \( T_1, \ldots, T_4 \subset \mathbb{P}^2 \) be distinct lines through a point \( p \) and \( C_1, C_2 \) be distinct smooth conics tangent to \( T_1, T_2 \) at points \( p_1, p_2 \neq p \), respectively. The smooth minimal model \( \tilde{E} \) of the double cover of \( \mathbb{P}^2 \) with branch locus \( T_1 + \ldots + T_4 + C_1 + C_2 \) is an Enriques surface with 8 disjoint nodal curves \( A_1, \ldots, A_8 \), which correspond to the 8 double points of

\[ G := T_3 + T_4 + C_1 + C_2. \]

Now let \( p_3 \) be a generic point in \( T_3 \) and consider the pencil \( l \) generated by \( 2H_i + T_i, i = 1, 2, 3 \), where \( H_i \) is a conic through \( p_i \) tangent to \( T_j, T_k \) at \( p_j, p_k \), for each permutation \((i, j, k)\) of \((1, 2, 3)\). Let \( L \) be a generic element of \( l \). Notice that the quintic curve \( L \) contains \( p \), it has a \((2, 2)\)-point at \( p_i \) and the intersection number of \( L \) and \( T_i \) at \( p_i \) is 4, \( i = 1, 2, 3 \).

The strict transform of \( L \) in \( \tilde{E} \) is a reduced curve \( B' \) with a 4-uple point (at the pullback of \( p_3 \)) such that the divisor

\[ B := B' + \sum A_i \]

is reduced, satisfies \( B^2 = 0 \) and is divisible by 2 in the Picard group (because \( L + T_1 \) is divisible by 2). Let \( S \) be the smooth minimal model of the double cover of \( \tilde{E} \) ramified over \( B \). One can verify that \( K_S^2 = 6 \). As in Section 4.1.1 choosing \( L \) through 1 or 2 double points of \( T_3 + T_4 + C_1 + C_2 \) one obtains examples with \( K_S^2 = 5 \) or 4, respectively.

To obtain a branch curve \( B \subset \tilde{E} \) with a \((3, 3)\)-point as in Section 4.2 it suffices to change the \((2, 2)\)-point of the quintic \( L \) at \( p_3 \) to an ordinary triple point. In this case \( L \) is the union of a conic through \( p_3 \) with a cubic having a double point at \( p_3 \). Choosing \( C_1 \) and \( C_2 \) so that \( L \) passes through 0, 1 or 2 double points of \( T_3 + T_4 + C_1 + C_2 \) one obtains examples with \( K_S^2 = 7, 6 \) or 5.

4.2.2 Involutions on \( S \)

We refer [Ca] or [Pa] for information on bidouble covers.

Each surface \( S \) constructed in Section 4.2.1 is the smooth minimal model of the bidouble cover of \( \mathbb{P}^2 \) determined by the divisors

\[ D_1 := L, \]
\[ D_2 := T_1 + C_1 + C_2, \]
\[ D_3 := T_2 + T_3 + T_4. \]

Let \( i_g \) be the involution of \( S \) corresponding to \( D_g + D_k \), for each permutation \((g, j, k)\) of \((1, 2, 3)\). We have that \( S/i_1 \) is birational to an Enriques surface, \( S/i_3 \) is a rational surface and the bicanonical map of \( S \) is not composed with \( i_1, i_2 \) and is composed with \( i_3 \). Moreover, \( S \) has an hyperelliptic fibration of genus 3.

We omit the proof for these facts: it is similar to the one given in [NC3] for an example with \( K_S^2 = 3 \).
4.3 More bidouble planes

In the examples above, $S/i_1$ is birational to an Enriques surface with 8 disjoint \((-2)\)-curves, corresponding to the 8 nodes of the sextic $G = T_3 + T_4 + C_1 + C_2$, which contains 2 lines. Now we give examples with $G$ containing only one line and with $G$ without lines.

4.3.1 $G$ with one line, $K_S^2 = 4, 5, 6$

Let $T_1, T_2, T_3$ and $L$ be as in Section [4.2] and $p_4$ be a smooth point of $L$. There exists a plane curve $J$ of degree 5 through $p$ with $(2, 2)$-points tangent to $T_1, T_2, L$ at $p_1, p_2, p_4$, respectively (notice that we are imposing 19 conditions to a linear system of dimension 20; such a curve can be easily computed using the Magma function LinSys given in [Ri2]).

Let $S$ be the smooth minimal model of the bidouble cover of $\mathbb{P}^2$ determined by the divisors

$$D_1 := L,$$
$$D_2 := T_3,$$
$$D_3 := T_1 + T_2 + J.$$

Notice that the double plane with branch locus $D_2 + D_3$ is an Enriques surface $E$ with 6 disjoint nodal curves $A_1, \ldots, A_6$ (two of them are contained in the pullback of $p_4$) and that the strict transform $L$ of $L$ in $E$ has a 4-uple point at the pullback of $p_3$. Moreover, the divisor $B := L + \sum A_i$ satisfies $B^2 = 0$ and is even (because $L + T_3$ is even). This gives an example for Theorem [42 a) with $K_S^2 = 4$.

To obtain an example with $K_S^2 = 5$ it suffices to choose the quintic $J$ with a triple point at $p_4$ instead of a $(2, 2)$-point. In this case $J$ is the union of a conic with a singular cubic. Here the Enriques surface contains 7 disjoint nodal curves, three of them contained in the pullback of $p_4$.

Finally, choosing $L$ with a triple point at $p_3$ one obtains $L \subset E$ with a $(3, 3)$-point. This gives examples for Theorem [42 a) with $K_S^2 = 5, 6$.

4.3.2 $G$ without lines, $K_S^2 = 4$

Consider, in affine plane, the points $p_0, \ldots, p_5$ with coordinates $(0, 0), (1, 1), \ldots, (2, 3)$, respectively, and let $T_1$ be the line through $p_1, p_2$. Let $C_1$ be the conic tangent to $T_{01}, T_{02}$ at $p_1, p_2$ which contains $p_5$ and let $C_2$ be the conic tangent to $T_{01}, T_{02}$ at $p_1, p_2$ which contains $p_3, p_4$. Let $l$ be the linear system generated by $T_{01} + T_{02} + 2T_{34}$ and $T_{03} + T_{04} + C_2$.

The element $Q$ of $l$ through $p_5$ is an irreducible quartic curve with double points at $p_0, p_3, p_4$ and tangent to $T_{01}, T_{02}$ at $p_1, p_2$. Moreover, because of the symmetry with respect to $T_{05}$, the line $H$ tangent to $Q$ at $p_5$ is horizontal.

There is a cubic $F$ through $p_0, p_3, p_4$ tangent to $T_{01}, T_{02}, H$ at $p_1, p_2, p_5$, respectively (notice that we are imposing 9 conditions to a linear system of dimension 9). One can verify that $F$ contains no line, thus it is irreducible.

The surface $S$ is the smooth minimal model of the bidouble cover of $\mathbb{P}^2$ determined by the divisors

$$D_1 := C_1 + F,$$
$$D_2 := T_{01},$$
$$D_4 := T_{02} + C_2 + Q.$$
Notice that the double plane with branch locus $D_2 + D_3$ is an Enriques surface $E$ with 6 disjoint nodal curves (contained in the pullback of the triple points $p_1, p_4$ of $D_3$) and that the strict transform of $D_1$ in $E$ has a 4-uple point (at the pullback of $p_5$). The double plane with branch locus $D_1 + D_3$ is a surface with Kodaira dimension 1. This gives an example for Theorem 2, a), b) with $K_S^2 = 4$.

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