CAPABILITY OF NILPOTENT PRODUCTS OF CYCLIC GROUPS II

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Abstract. In Part I it was shown that if $G$ is a $p$-group of class $k$, generated by elements of orders $1 < p^{\alpha_1} \leq \cdots \leq p^{\alpha_r}$, then a necessary condition for the capability of $G$ is that $r > 1$ and $\alpha_r \leq \alpha_{r-1} + \left\lfloor \frac{k-1}{p-1} \right\rfloor$. It was also shown that when $G$ is the $k$-nilpotent product of the cyclic groups generated by those elements and $k = p = 2$ or $k < p$, then the given conditions are also sufficient. We make a correction related to the small class case, and extend the sufficiency result to $k = p$ for arbitrary prime $p$.

Recall that a group $G$ is said to be capable if and only if there exists a group $H$ such that $G \cong H/Z(H)$, where $Z(H)$ is the center of $H$. In [2] we proved that if $G$ is a capable $p$-group of class $k$, generated by $x_1, \ldots, x_r$, with $x_i$ of order $p^{\alpha_i}$, $1 \leq \alpha_1 \leq \cdots \leq \alpha_r$, then $r > 1$ and $\alpha_r \leq \alpha_{r-1} + \left\lfloor \frac{k-1}{p-1} \right\rfloor$. We also proved that if $G$ is the $k$-nilpotent product of the cyclic $p$-groups generated by the $x_i$, then the conditions are also sufficient for the cases $k < p$ and $k = p = 2$.

The purpose of this note is twofold: first, we will note an error in a lemma that was used in the proof of the small class case and make the necessary corrections to justify that result. Second, we will extend the result to the case $k = p$ with $p$ an arbitrary prime. Since we follow closely on [2], we refer the reader there for the relevant definitions and conventions.

I am extremely grateful to Prof. T. C. Hurley who brought to my attention the results from [1, 7]; these results allowed the correction of the error noted above, as well as simplifying my argument for the $k = p$ case.

1. Shoving commutators

In [2], the last clause of Lemma 4.2(ii) is incorrect. Because of this error, the last assertion in Lemma 4.3 is also incorrect; the proof of Theorem 4.4, which describes the center of a $k$-nilpotent product of cyclic $p$-groups when $k \leq p$, relied on that incorrect assertion and so has a gap. In this section we will provide the necessary correction to justify the conclusion of that theorem. Once it is established, the rest of the proof of the small class case will follow.

The error in question is the following: we start with the free group $F$ on $x_1, \ldots, x_r$, and a basic commutator $[u, v]$ of weight equal to $k \geq 2$. Then we considered $[u, v, x_r]$; when $v \leq x_r$, this is a basic commutator. If $v > x_r$, then we rewrite $[u, v, x_r]$ modulo $F_{k+2}$ as $[u, x_r, v][v, x_r, u]^{-1}$. The incorrect clause asserted that this expresses $[u, v, x_r]$ modulo $F_{k+2}$ as a product of basic commutators and their inverses, but this is not necessarily the case; there is no warrant for asserting that $[u, x_r]$ or $[v, x_r]$ will necessarily be basic commutators (though they are for small values of $k$), nor that $[v, x_r] > u$, another requirement. The main idea is

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Lemma 1.4. Among basic commutators of the same weight; the proofs are straightforward.

Lemma 1.5 (cf. [7, Lemma 18.3], [1, Lemma 1.2]). Let \( c \) be a basic commutator. Then \( \text{wt}(c) \) is the weight of \( c \) in the \( x_i \).

Definition 1.1. Let \( c \) be a basic commutator on \( x_1, \ldots, x_r \). Then \( \text{wt}(c) \) is the weight of \( c \) in the \( x_i \).

Definition 1.2 (cf. [7, §18]). Let \( u \) and \( v \) be basic commutators on \( x_1, \ldots, x_r \). The commutator \( [u \leftarrow v] \) is defined recursively as:

1. If \( v > u \), then \( [u \leftarrow v] = [v \leftarrow u] \).
2. If \( v < u \) and \( u = [c_1, c_2] \) and \( c_2 > v \), then \( [u \leftarrow v] = [c_1 \leftarrow v, c_2] \).
3. Otherwise, \( [u \leftarrow v] = [u, v] \).

Here is an explicit description, easily established:

Lemma 1.3 (cf. [4, Lemma 2.4]). Let \( u \) and \( v \) be basic commutators on \( x_1, \ldots, x_r \) with \( u > v \). If \( \text{wt}(u) = 1 \), then \( [u \leftarrow v] = [u, v] \). If \( \text{wt}(u) > 1 \), then letting \( u = [c_1, \ldots, c_n] \) where \( c_i \) is a basic commutator and \( \text{wt}(c_1) = 1 \), we have:

1. If \( c_2 > v \), then \( [u \leftarrow v] = [c_1, v, c_2, \ldots, c_n] \).
2. Otherwise, \( [u \leftarrow v] = [c_1, \ldots, c_j, v, c_{j+1}, \ldots, c_n] \), where \( j \) is the largest index such that \( c_j \leq v \).

Given a commutator \( c = [r, s] \), we will say informally that \( r \) is the “left entry” of \( c \), and that \( s \) is the “right entry” of \( c \).

The following lemma is due to Ward (modulo a different definition of the order among basic commutators of the same weight); the proofs are straightforward.

Lemma 1.4 (cf. [7, Lemma 18.1]). Let \( u \) and \( v \) be basic commutators on \( x_1, \ldots, x_r \), and assume that \( u > v \).

1. \([u \leftarrow v]\) exists and is a basic commutator on \( x_1, \ldots, x_r \).
2. \( \text{wt}([u \leftarrow v]) = \text{wt}(u) + \text{wt}(v) \).
3. \( u < [u \leftarrow v] \).
4. If \( \text{wt}(u) > 1 \), then the right entry of \([u \leftarrow v]\) is equal to the larger of \( v \) and the right entry of \( u \).
5. If \( v < u_1 < u_2 \), then \([u_1 \leftarrow v] < [u_2 \leftarrow v] \).
6. If \( v < u_1 < u_2 < u \), then \([u \leftarrow v_1] < [u_2 \leftarrow v_2] \).
7. \( [u_1 \leftarrow v] = [u_2 \leftarrow v] \), then \( u_1 = u_2 \).

We can think of \([u \leftarrow v]\) as the basic commutator which results from “shoving” \( v \) into its correct position inside of \( u \) (hence the title of this section).

Some of the results above, in particular (v) and (vi), were also obtained independently by Waldinger [5].

The following result is also essentially contained in [5,7]. However, both authors use a preferred ordering among basic commutators that is different from ours, so their conclusions also read differently. Because of this, we provide a proof.

Lemma 1.5 (cf. [7, Lemma 18.3], [1, Lemma 1.2]). Let \( F \) be the free group on \( x_1, \ldots, x_r \). Let \( u \) and \( v \) be basic commutators and let \( k = \text{wt}(u) + \text{wt}(v) \). Then

\[
[u, v] \equiv [u \leftarrow v]^k c_1^{a_1} c_2^{a_2} \cdots c_m^{a_m} \pmod{F_{k+1}},
\]
where $F_{k+1}$ is the $(k + 1)$st term of the lower central series of $F$, $\epsilon = \pm 1$, $\alpha_i$ are integers, each $c_i$ is a basic commutator of weight $k$, and $[u \leftarrow v] < c_1 < \cdots < c_m$. Moreover, if $u \geq v$, then we may choose $\epsilon = 1$.

**Proof.** It is enough to establish the result when $u > v$: if $u = v$, then $[u, v]$ and $[u \leftarrow v]$ are both trivial, so setting $\epsilon = 1$ and $m = 0$ proves the result. And if $v > u$, then $[u, v] = [v, u]^{-1}$; assuming the result holds when the left entry is greater than the right entry, and since $F_k/F_{k+1}$ is abelian, we obtain

\[
[u, v] = [v, u]^{-1} \equiv ([u \leftarrow u]c_1^{\alpha_1} \cdots c_n^{\alpha_n})^{-1} \pmod{F_{k+1}} \\
\equiv [u \leftarrow u]^{-1} c_1^{-\alpha_1} \cdots c_n^{-\alpha_n} \pmod{F_{k+1}} \\
\equiv [u \leftarrow v]^{-1} c_1^{-\alpha_1} \cdots c_n^{-\alpha_n} \pmod{F_{k+1}}.
\]

So we assume without loss of generality that $u > v$.

We proceed by induction on $k$. If $k = 2$, then $u = x_j$ and $v = x_i$ with $i < j$; hence $[u, v] = [u \leftarrow v]$. If $k = 3$, then $u = [x_i, x_j]$ and $v = x_k$, with $1 \leq i < j < k$; if $i \leq k$ then $[u, v] = [u \leftarrow v]$ and we are done. If $i > k$, then from [2 Prop. 2.2(iv)] we have that $[u, v] \equiv [x_j, x_k, x_i][x_i, x_j, x_k]^{-1} \pmod{F_4}$, and $[u \leftarrow v] = [x_j, x_k, x_i]$, which is strictly smaller than $[x_i, x_j, x_k]$, so the result also holds.

Assume that $k > 3$ and the result is true for all commutators $[c_1, c_2]$ where $c_1$ and $c_2$ are basic commutators with $\text{wt}(c_1) + \text{wt}(c_2) < k$ and $c_1 > c_2$. We will now argue by “descending induction” on $v$. Picking the largest possible weight for $v$ for which $\text{wt}(u) + \text{wt}(v) = k$ and $u > v$ yields $\text{wt}(u) = \text{wt}(v)$ or $\text{wt}(u) = \text{wt}(v) + 1$. Write $u = [a, b]$ (which we can do since $\text{wt}(u) \geq 2$). Then $\text{wt}(b) \leq \frac{1}{2}\text{wt}(u)$. If $\text{wt}(b) \geq \text{wt}(v)$, then we would have $\text{wt}(v) \leq \frac{1}{2}\text{wt}(u) \leq \frac{1}{2}(\text{wt}(u) + 1)$; for this to hold we must have $\text{wt}(v) = 1$ and $\text{wt}(u) \leq 2$, contradicting the assumption that $k > 3$. Hence $\text{wt}(b) < \text{wt}(v)$, so $b < v$ and $[u, v] = [u \leftarrow v]$; thus the result holds in this case.

Suppose then the result holds for $[c_1, c_2]$ whenever $c_1$ and $c_2$ are basic commutators, $c_1 > c_2$, and either $\text{wt}(c_1) + \text{wt}(c_2) < k$ or $\text{wt}(c_1) + \text{wt}(c_2) = k$ and $c_1 > c_2 > v$. Write $u = [a, b]$; if $b \leq v$, then $[u, v] = [u \leftarrow v]$ and we are done. Otherwise, again from [2 Prop. 2.2(iv)] we have that $[u, v] \equiv [a, v, b][b, v, a]^{-1} \pmod{F_{k+1}}$. Let $\kappa = \text{wt}(a) + \text{wt}(v)$ and $\lambda = \text{wt}(b) + \text{wt}(v)$. By induction, we know that:

\[
[a, v] \equiv [a \leftarrow v] \prod_{i=1}^{r} c_i^a \pmod{F_{k+1}}, \quad \text{and} \quad [b, v] \equiv [b \leftarrow v] \prod_{j=1}^{s} d_j^b \pmod{F_{\lambda+1}},
\]

where $\alpha_i, \beta_j$ are integers, the $c_i$ are basic commutators of weight $\kappa$, the $d_j$ are basic commutators of weight $\lambda$, and the inequalities $[a \leftarrow v] < c_1 < \cdots < c_r$ and $[b \leftarrow v] < d_1 < \cdots < d_s$ hold. Since $\kappa + \text{wt}(b) = \lambda + \text{wt}(a) = k$, from well-known commutator identities (e.g., those in [2 Prop. 2.2]) we obtain:

\[
[u, v] \equiv [a \leftarrow v, b] \left( \prod_{i=1}^{r} [c_i, b]^{a_i} \right) [b \leftarrow v, a]^{-1} \left( \prod_{j=1}^{s} [d_j, a]^{-\beta_j} \right) \pmod{F_{k+1}}.
\]

Note that $[a \leftarrow v, b] = [a \leftarrow v]$, that $\text{wt}(c_i) + \text{wt}(b) = \text{wt}(d_j) + \text{wt}(a) = k$, and likewise $\text{wt}(b \leftarrow v) + \text{wt}(a) = k$. Our result will therefore follow if we can prove that each of $[c_i, b]$, $[b \leftarrow v, a]$ and $[d_j, a]$ is congruent modulo $F_{k+1}$ to a product of powers of basic commutators of weight $k$, each strictly larger than $[u \leftarrow v]$; then we can invoke the fact that $F_k/F_{k+1}$ is abelian to obtain an expression for $[u, v]$ modulo $F_{k+1}$ of
the desired form. We remove any commutators that are trivial, and consider each of the remaining ones in turn.

Since \( b > v \), the induction hypothesis allows us to rewrite each \([c, b]\) as a product of powers of basic commutators, each greater than or equal to \([c, b]\). We know that \( c_i > [a \leftarrow v] \), so we have that \([c, b] > ([a \leftarrow v] \leftarrow a)\); since \( b \) is no smaller than the right entry of \([a \leftarrow v]\) we know that \([a \leftarrow v] \leftarrow b = [a \leftarrow v, b] = [u \leftarrow v]\). Therefore, \([c, b] > [u \leftarrow v] \) and so all basic commutators that appear in the expression for \([c, b]\) are also strictly larger than \([u \leftarrow v]\). So we can certainly replace each of the \([c, b]\) as needed.

If \([b \leftarrow v] < a\), then we replace \([b \leftarrow v, a]^{-1}\) with \([a, b \leftarrow v]\). Since \( b < [b \leftarrow v]\) and the right entry of \( a\) is less than or equal to \( b\), we have that \([a, b \leftarrow v]\) is a basic commutator; also since \( b < [b \leftarrow v]\) we deduce that \([u \leftarrow v] = [a \leftarrow v, b] < [a, b \leftarrow v]\). On the other hand, if \([b \leftarrow v] > a\), then we know that the right entry of \([b \leftarrow v]\) is strictly less than \( b\) (equal to either \( v\) or to the right entry of \( b\)), hence strictly smaller than \( a\); thus, \([b \leftarrow v, a]\) is already a basic commutator. The right entry of this latter commutator is \( a\), so \([b \leftarrow v, a] > [a \leftarrow v, b] = [u \leftarrow v]\). This shows the commutator \([b \leftarrow v, a]\) is either a basic commutator greater than \([u \leftarrow v]\), or the inverse of a basic commutator greater than \([u \leftarrow v]\).

Finally, we come to the commutators \([d_j, a]\). If \( a > d_j\), then we replace \([d_j, a]\) with \([a, d_j]^{-1}\). Since \( d_j > [b \leftarrow v] > b\), it follows that the right entry of \( a\) is strictly smaller than \( d_j\), so \([a, d_j]\) is a basic commutator; and \( d_j > b\) also implies that \([a, d_j] > [a \leftarrow v, b] = [u \leftarrow v]\). On the other hand, if \( d_j < a\), since \( a > v\) we can again apply induction to replace \([d_j, a]\) with a product of \([d_j \leftarrow a]\) times powers of basic commutators strictly larger than \([d_j \leftarrow a]\). The right entry of \([d_j \leftarrow a]\) is no smaller than \( a\), and hence is strictly larger than \( b\), the right entry of \([u \leftarrow v]\). Thus, we can also replace each \([d_j, a]\) with a product of powers of basic commutators, each larger than \([u \leftarrow v]\). This proves the lemma.

\[ \]

\textbf{Lemma 1.6} (cf. \[1\] Lemma 1.3). Let \( F \) be the absolutely free group on \( x_1, \ldots, x_m\), and suppose that \( c \equiv b_1^\alpha_1 \cdots b_\ell^\alpha_\ell \) (mod \( F_{k+1}\)), where \( \alpha_i \) are integers, \( b_i \) are basic commutators of weight exactly \( k\), and \( b_1 < b_2 < \cdots < b_\ell\). If \( a\) is a basic commutator of weight \( \ell\), then \([c, a]\) is of \([b_1 \leftarrow a]^\pm \cdot u_1^\beta_1 \cdots u_\ell^\beta_\ell \) (mod \( F_{k+\ell+1}\)) where the \( \beta_i\) are integers, \( u_i\) are basic commutators with \( wt(u_i) = k + \ell\), and \([b_1 \leftarrow a] < u_1 < \cdots < u_\ell\). Moreover, if \( c > a\) then the exponent of \([b_1 \leftarrow a]\) may be taken to be \( \alpha_1\).

\textbf{Proof.} We have that \([c, a] \equiv [b_1, a]^\alpha_1 \cdots [b_\ell, a]^\alpha_\ell \) (mod \( F_{k+\ell+1}\)); since the \( b_i\) are in increasing order, the corresponding \([b_i \leftarrow a]\) are also in increasing order; the result now follows from the fact that \( F_{k+\ell}/F_{k+\ell+1}\) is abelian and from Lemma 1.6. \[ \]

With this result, we can replace the argument based on the erroneous Lemma 4.3 and prove Theorem 4.4 from \[2\]:

\textbf{Theorem 1.7} (\[2\] Theorem 4.4). For a positive integer \( k\) and a prime \( p\) with \( p \geq k\), let \( C_1, \ldots, C_r\) be cyclic \( p\)-groups generated by \( x_1, \ldots, x_r\), respectively, with \( p^{\alpha_i}\) being the order of \( x_i\), and assume that \( 1 \leq \alpha_1 \leq \cdots \leq \alpha_r\). If \( G\) is the \( k\)-nilpotent product of the \( C_i\), \( G = C_1 \amalg \cdots \amalg C_r\), then \( Z(G) = \langle x_r^{p^{\alpha_r-1}}, G_k \rangle\).

\textbf{Proof.} The center contains both \( x_r^{p^{\alpha_r-1}}\) and \( G_k\) by \[2\] Lemma 3.11 and the properties of the nilpotent product. The reverse inclusion is established by induction, the case \( k = 1\) being trivial and the case \( k = 2\) having been proven in \[2\] Lemma 4.1. If we consider \( G/G_k\) we obtain the \((k-1)\)-nilpotent product of the \( C_i\), from which
we have that \((x_i^{p^{n_i-1}}, G_k) \subseteq Z(G) \subseteq (x_i^{p^{n_i-1}}, G_{k-1})\) by induction. To prove equality, it is enough to show that if \(g \in G_{k-1} \cap Z(G)\), then \(g \equiv e \pmod{G_k}\). Write \(g \equiv b_1^{\beta_1} \cdots b_t^{\beta_t} \pmod{G_k}\), with each \(b_i\) a basic commutator of weight \(k - 1\), and \(b_1 < \cdots < b_t\); from \(3\) Theorem 3] we know this expression is unique if we require that each \(\beta_i\) satisfy \(0 \leq \beta_i < p^{n_i}\), where \(s_i\) is the smallest index of a generator that appears in the full expression for \(b_i\). If \(t = 0\) then trivially \(g \in G_k\). Assuming the conclusion holds for expressions with fewer terms, by Lemma 1.6 we have that \(e = [g, x, r] = [b_1 \cdots b_t]^{x_r} \prod u_j^{y_j}\) (equality since \(G_{k+1} = 1\) is trivial), where \(u_j\) are basic commutators of weight \(k\), with \([b_1 \cdots b_t] < u_1 < u_2 < \cdots\). Again, by the normal form proven in \(3\) Theorem 3, and since the order of \([b_1 \cdots b_t]\) must be equal to the order of \(b_1\), we deduce \(\beta_1 = 0\) so we may express \(g\) modulo \(G_k\) using fewer than \(t\) powers of basic commutators, and by induction we deduce \(g \in G_k\), as claimed.  

\[ \text{2. The case } k = p \]

In this section we will extend the main result from \(2\) to the case \(k = p\) with \(p\) an arbitrary prime. We will do so by computing the center of a \((p + 1)\)-nilpotent product of cyclic \(p\)-groups much in the same way as above, using a normal form for the elements of such a product that was obtained by R.R. Struik in her detailed study \(4\). Lemma 2.2 will also play a key part.

Definition 2.1. Let \(G\) be a group, and let \(x, y \in G\). We define \([x, y] = [x, y]\) and \([x, n+y] = [x, n, y, y]\), where \(n > 1\) is an integer.

The main difficulty in a straightforward extension of the result lies in the fact that the basic commutators are no longer a good choice for a “basis” for the normal form in the case of the \((p + 1)\)-nilpotent product of cyclic \(p\)-groups, because there are nontrivial relations between them; for example, a sufficiently high power of \([b, a]\) will be nontrivial and equal to a power of \([b, p, a]\). In order to bypass this difficulty, one chooses a slightly different set of distinguished commutators, by replacing the basic commutators \([b, p, a]\) and \([b, a, p, b]\) with the (nonbasic) commutators \([b, a^p]\) and \([b^p, a]\), respectively. The normal form result appears in \(4\) Theorem 6, and is as follows: every element \(g\) of the \((p + 1)\)-nilpotent product of cyclic groups generated by elements \(x_1, \ldots, x_r\) with \(x_i\) of order \(p^{\alpha_i}\), \(1 \leq \alpha_1 \leq \cdots \leq \alpha_r\), can be written uniquely as \(g = \prod c_i^{\beta_i}\), where \(c_1 < c_2 < \cdots\) is the sequence of basic commutators of weight at most \(p+1\) in \(x_1, \ldots, x_r\), except that the basic commutator \([x_j, p, x_i]\) is replaced by the commutator \([x_j, x_j^{p_i}]\), and the basic commutator \([x_j, x_i, p-1 x_j]\) is replaced by the commutator \([x_j, x_j^{p_i}: x_i]\); the \(\beta_i\) are nonnegative integers satisfying \(0 \leq \beta_i < N_i\), where:

\begin{equation}
\begin{aligned}
N_i &= \begin{cases}
  p^{\alpha_i} & \text{if } wt(c_i) = 1 \text{ and } c_i = x_i; \\
  p^{\alpha_k+1} & \text{if } c_i = [x_j, x_k], 1 \leq k < j \leq r; \\
  p^{\alpha_i-1} & \text{if } c_i = v_j^{p_k} = [x_j, a_k^{p_i}], 1 \leq k < j \leq \alpha_r; \\
  p^{\alpha_k-1} & \text{if } c_i = v_j^{p_k} = [x_j, x_k], 1 \leq k < j \leq r \text{ and } \alpha_k = \alpha_j; \\
  p^{\alpha_k} & \text{if } c_i = v_j^{p_k} = [x_j, x_k], 1 \leq k < j \leq r \text{ and } \alpha_k < \alpha_j; \\
  p^{\alpha_i} & \text{if } c_i \text{ is any other basic commutator} \\
  & \text{and } s_i \text{ is the smallest index occurring} \\
  & \text{in the full expression for } c_i.
\end{cases}
\end{aligned}
\end{equation}

Remark 2.3. There is a slight inconsistency between the above and the statement of \(4\) Theorem 6; in the latter, the range for the exponents of \([x_j, x_k]\) is not
explicitly specified, and would be 0 to $p^{\alpha_k}$ following the general case. However, the discussion leading up to the theorem, and in particular [4 Equation 60] states that the exponent will be taken modulo $p^{\alpha_k+1}$; and this is explicitly the case in [3 Theorem 4] which deals with $p = 2$. So it seems clear that this is an inadvertent omission in the statement of Theorem 6. Nonetheless, our argument will avoid consideration of the specific exponent of these commutators except in the case $p = 2$.

We want to describe the center of a $(p+1)$-nilpotent product of cyclic $p$-groups. The idea is the same one as was used above: if we let $G$ be the $(p+1)$-nilpotent product of cyclic $p$-groups, then it is easy to show that $Z(G)$ has upper and lower bounds determined by a power of $x_r$ and $G_{p+1}$ below, and a power of $x_r$ and $G_p$ above. At this point we have two extra difficulties we did not encounter above: the first is that the power of $x_r$ is not the same in the two bounds, whereas it was the same in the proof of Theorem 1. This can be dealt with in a straightforward way and we do so first; we will return to the second difficulty after this lemma:

**Lemma 2.4.** Let $p$ be a prime, and let $\alpha, \beta$ be positive integers with $\alpha < \beta$. Let $G = \langle x \rangle \Pi \mathbb{Z}_{p+1}^m \langle y \rangle$, where $x$ generates a cyclic group of order $p^\alpha$ and $y$ generates a cyclic group of order $p^\beta$. Then $[y^{p^\alpha}, x] = [y^p, x]^{p^{\alpha-1}}$. In particular, $y^{p^\alpha}$ is not central in $G$.

**Proof.** All basic commutators in $x$ and $y$ are of exponent $p^\alpha$ in $G$, except for $y$ and perhaps $[y, x]$. This can be easily established using for example [3 Lemma H2]. Thus, from [4 Lemma 4] we obtain that:

$$[y^{p^\alpha}, x] = [y^p, x]^{p^\alpha} [y, x, p-1 y]^{(p^\alpha)}.$$

Since $[y, x, p-1 y]$ is of exponent $p^\alpha$ and $(p^\alpha) \equiv p^{\alpha-1} \equiv (p^\alpha) \pmod{p^\alpha}$, we obtain

$$[y^{p^\alpha}, x] = [y^p, x]^{p^\alpha} [y, x, p-1 y]^{p^{\alpha-1}}.$$

On the other hand, from [4 Equation 58] we have:

$$(2.5) \quad [y^p, x] = [y, x]^{p \prod u_{i}^{p g_i}} [y, x, p-1 y],$$

where the $g_i$ are integers, and $u_i$ are basic commutators of weight at least three and at most $p+1$ in $x$ and $y$, omitting both $[y, x, p-1 y]$ and $[y, p, x]$. From this, since all $u_i$ are of exponent $p^\alpha$, we obtain by [3 Theorem H3] that:

$$(2.6) \quad [y^{p^\alpha}, x]^{p^{\alpha-1}} = [y^p, x]^{p^\alpha} [y, x, p-1 y]^{p^{\alpha-1}}.$$

Therefore, $[y^{p^\alpha}, x] = [y^p, x]^{p^{\alpha-1}}$, as claimed. Since we are assuming $\alpha \neq \beta$, the normal form described above ensures that $[y^p, x]^{p^{\alpha-1}} \neq e$, and so $[y^{p^\alpha}, x] \neq e$, as claimed. \qed

The second difficulty alluded to above is a bit more subtle. Once again the result will come down to proving that if $g \in G_p \cap Z(G)$, then $g \equiv 0 \pmod{G_{p+1}}$. If we write $g$ modulo $G_{p+1}$ as a product of basic commutators of weight exactly $p$ (which can be done since $G/G_{p+1}$ is the $p$-nilpotent product and the usual normal form works), and apply Lemma 1.8 to compute $[g, x_r]$, we will obtain $[g, x_r]$ as a product of powers of basic commutators of weight exactly $p+1$. However, this may not be in the normal form for elements of $G$; e.g., if any of the basic commutators $[x_j, x_i, p-1 x_j]$ or $[x_j, p, x_i]$ occur in that expansion then we must replace them by
expressions using identity (2.5) and a similar identity for \([g, x^p] \quad \text{Equation (57)}\]. After replacing the occurrences, we must again apply the collection process to the resulting expression before it will be in normal form.

During all of these modifications it might be, at least in principle, that we modify the exponent of the leading factor in the expression for \([g, x_r]\) (or even completely replace this leading factor if it is one of the troublesome basic commutators); thus the argument becomes more involved. In addition, it may be that the range for the exponents for the leading factors of \(g, x\) becomes more involved. In addition, it may be that the range for the exponents for the leading factors of \(g, x\) are different. However, by being careful about just what modifications may be needed and what they would entail, and sometimes considering \([g, x_{r-1}]\) instead of \([g, x_r]\), we can nonetheless push the argument through to a happy conclusion.

**Theorem 2.7.** Let \(p\) be a prime and let \(C_1, \ldots, C_r\) be cyclic \(p\)-groups generated by \(x_1, \ldots, x_r\), of orders \(1 < p^{\alpha_1} \leq \cdots \leq p^{\alpha_r}\) respectively. If \(G\) is the \((p+1)\)-nilpotent product of the \(C_i\), \(G = C_1 \Pi^{\beta_1} \cdots \Pi^{\beta_r} C_r\), then \(Z(G) = \langle x_1^{p^{\alpha_1}} + 1, G_{p+1}\rangle\).

**Proof.** That \(x_1^{p^{\alpha_1}} + 1\) lies in the center follows from Lemma 3.11; the fact that \(G\) is of class \(p+1\) guarantees that \(G_{p+1} \subseteq Z(G)\).

To prove the reverse inclusion, consider \(G/G_{p+1}\). By Theorem 1.7 we know the center is generated by the images of \(x_1^{p^{\alpha_1}} + 1\) and \(G_p\). Pulling back to \(G\) we obtain the inclusions \(\langle x_1^{p^{\alpha_1}} + 1, G_{p+1}\rangle \subseteq Z(G) \subseteq \langle x_1^{p^{\alpha_1}} + 1, G_p\rangle\). By Lemma 2.2 if \(\alpha_{r-1} < \alpha_r\), then \(x_1^{p^{\alpha_r}} + 1\) is not central; if \(\alpha_{r-1} = \alpha_r\), then both \(x_1^{p^{\alpha_r}} + 1\) are trivial. So in either case we have \(\langle x_1^{p^{\alpha_r}} + 1, G_{p+1}\rangle \subseteq Z(G) \subseteq \langle x_1^{p^{\alpha_r}} + 1, G_p\rangle\).

The theorem will be established if we can show that for any \(g \in G_p\), if \(g \in Z(G)\) then \(g \equiv e \mod G_{p+1}\). Indeed, if \(g \in G_p \cap Z(G)\), then we can write

\[
g \equiv c_1^{\beta_1} \cdots c_m^{\beta_m} \mod G_{p+1},
\]

where \(c_1 < \cdots < c_m\) are basic commutators of weight exactly \(p\), and \(\beta_i\) are nonnegative integer that satisfy \(0 \leq \beta_i < p^{\alpha_i}\), where \(s_i\) is the smallest index of a generator that occurs in the full expression of \(c_i\). We wish to show that \(g \equiv e \mod G_{p+1}\), and we will do so by induction on \(m\). The result is trivial if \(m = 0\); assume then the result holds for all \(g\) expressed as a product of \(k\) powers of basic commutators of weight exactly \(p\), with \(0 \leq k < m\). We consider several cases depending on the nature of the basic commutator \(c_1\).

**Case 1:** The right entry of \(c_1\) is of weight at least two. Consider \([g, x_r]\). By Lemma 1.6 we have:

\[
[g, x_r] = [c_1 \leftarrow x_r]^{\beta_1} d_1^{\gamma_1} \cdots d_n^{\gamma_n},
\]

where \(\gamma_i\) are integers, the \(d_i\) are basic commutators, and \([c_1 \leftarrow x_r] < d_1 < \cdots < d_n\). We may assume that \(0 < \gamma_i < p^{s_i}\), where \(s_i\) is the smallest index of a generator that occurs in the full expression for \(d_i\); this equals the corresponding \(N_i\) from (2.2) except in the case where \(d_i\) is one of the troublesome commutators. Since the right entry of \(c_1\) is of weight at least two, so is the right entry of \([c_1 \leftarrow x_r]\), and the same holds for each \(d_i\). Thus, this expression is already in normal form and no replacements need to be made. The range of exponents for \([c_1 \leftarrow x_r]\) goes from 0 to \(p^{s_1}\), where \(s\) is the smallest index that occurs in the full expression of \([c_1 \leftarrow x_r]\), which is the same as the smallest index that occurs in the full expression for \(c_1\), namely \(s_1\). Since \(g\) is central, we must have \(\beta_1 \equiv 0 \mod p^{s_1}\); and from our assumption that
$0 \leq \beta_1 < p^{\alpha_1}$ we deduce that $\beta_1 = 0$. Thus we have $g \equiv e^{\beta_2} \cdots e^{\beta_m} \pmod{G_{p+1}}$, and by induction we deduce that $g \equiv e \pmod{G_{p+1}}$, as desired.

Case 2: The right entry of $c_1$ is of weight 1, and $c_1$ involves at least two generators other than $x_r$. We again consider $[g, x_r]$. Note that since the right entry of $c_1$ is of weight 1, then $[c_1 \leftarrow x_r] = [c_1, x_r]$. Since $[c_1, x_r] < d_i$ for each $d_i$ in (2.4), the only basic commutators that may need to be replaced occur among the $d_i$ and are of the form $[x_r, x_{i, p-1} x_r]$, which are replaced using (2.5); each of the commutators that are introduced involve only two generators, and so will not equal $c_1$. After doing the replacement we must apply the collection process to rewrite the entire expression in normal form. During the collection, since in the expression all factors are commutators of weight at least two, we will only introduce commutators $[b, a]$ in which $a$ is of weight at least two; again, they will not equal $c_1$. Thus, after rewriting (2.4) in normal form, the exponent of $[c_1 \leftarrow x_r]$ will remain $\beta_1$. Since $[g, x_r] = e$, we must have $\beta_1 \equiv 0 \pmod{p^{\alpha_1}}$, which as above yields the conclusion that $g \in G_{p+1}$, as desired.

Case 3: The right entry of $c_1$ is of weight 1, and $c_1$ involves only the generators $x_r$ and $x_i$ for some $i < r - 1$. Note that we will have $0 \leq \beta_1 < p^{\alpha_1}$. This time we consider $[g, x_{r-1}]$. We have

$$
(2.10) \quad [g, x_{r-1}] = [c_1 \leftarrow x_{r-1}]^{\beta_1} d_1^{\gamma_1} \cdots d_n^{\gamma_n}
$$

for some basic commutators $d_1 < \cdots < d_n$, with $[c_1 \leftarrow x_{r-1}] < d_1$. We may assume that $\gamma_i$ is positive in each case, and less than the corresponding $N_i$ defined as in (2.4). Since $[c_1 \leftarrow x_{r-1}]$ involves at least three generators, if any replacement need to be made they will be among the $d_i$, and none of the replacements nor the commutators introduced after collecting will be equal to $[c_1 \leftarrow x_{r-1}]$, which has right term of weight one and involves three generators; thus the exponent of $[c_1 \leftarrow x_{r-1}]$ in the normal form expression for $[g, x_{r-1}]$ is $\beta_1$. As above, this implies that $\beta_1 \equiv 0 \pmod{p^{\alpha_1}}$, and so we conclude $\beta_1 = 0$ and $g \in G_{p+1}$ by induction.

Case 4: The commutator $c_1$ involves only the generators $x_{r-1}$ and $x_r$, and $c_1 < [x_r, x_{r-1}, p-2 x_r]$. We have $0 \leq \beta_1 < p^{\alpha_r - 1}$. We consider $[g, x_r]$; the only basic commutator that may need to be replaced in the expression (2.4) is $[x_r, x_{r-1}, p-1 x_r]$, which may appear as one of the $d_i$, but not as $[c_1 \leftarrow x_r]$. If such a replacement is necessary, the exponent of $[x_r^p, x_{r-1}]$ in the normal form expression will be equal to $\gamma_i$, the exponent of $d_i$ before the rewriting; this follows from (2.5). See also [4] Equation (59).

If $\alpha_{r-1} < \alpha_r$, then $[g, x_r] = e$ implies that $\gamma_i \equiv 0 \pmod{p^{\alpha_r - 1}}$, which contradicts our assumption on the $\gamma_i$ (which we assumed to be positive and strictly smaller than $p^{\alpha_r - 1}$). Thus, if $\alpha_{r-1} < \alpha_r$, then (2.4) is already in normal form; since $g$ is central we must have $\beta_1 \equiv 0 \pmod{p^{\alpha_r - 1}}$, and so we deduce $\beta_1 = 0$ and $g \in G_{p+1}$.

If, on the other hand, $\alpha_{r-1} = \alpha_r$ then we can only conclude that $\gamma_i \equiv 0 \pmod{p^{\alpha_r - 1}}$. Writing $\gamma_i = kp^{\alpha_r - 1}$, then using the same argument as in (2.6) we have that we will replace $d_1^{\gamma_1}$ with $[x_r, x_{r-1}]^{p \gamma_1}$ (using the fact that $[x_r^p, x_{r-1}]$ is of order $p^{\alpha_r - 1}$). To write this in normal form we just need to move $[x_r, x_{r-1}]^{p \gamma_1}$ to the left, which introduces no new commutators since all other terms are already central. Thus, the exponent of $[c_1 \leftarrow x_r]$ remains $\beta_1$ in the new expression. Again we conclude that $\beta_1 \equiv 0 \pmod{p^{\alpha_r - 1}}$ and so $\beta_1 = 0$; induction now gives us that $g \in G_{p+1}$.

Case 5: The only remaining case, $c_1 = [x_r, x_{r-1}, p-2 x_r]$. 
Our assumption on $\beta_1$ is $0 \leq \beta_1 < p^{r-1}$. If $p = 2$, then $g \equiv c_{1}^{\beta} = [x_r, x_{r-1}]^{\beta}$ (mod $G_3$) with $0 \leq \beta < p^{r-1}$. Thus we have $e = [g, x_r] = [x_r, x_{r-1}]^{-2\beta}[x_r^{\beta}, x_{r-1}]^{\beta}$, so by [3, Theorem 4] we conclude that $-2\beta \equiv 0 \pmod{2^{r-1}+1}$. From this once again we obtain that $\beta = 0$ and $g \in G_3$.

If $p > 2$, then consider $[g, x_{r-1}]$. As above, the expression in (2.10) will be in normal form unless one of the commutators $d_i$ is equal to $[x_r, x_{r-1}, p-1 x_r]$; we now proceed as above to conclude that if $\alpha_{r-1} < \alpha_r$ then no $d_i$ needs to be replaced; and if $\alpha_{r-1} = \alpha_r$, then we deduce that $\gamma_i \equiv 0 \pmod{p^{\alpha_r-1}}$, and so we simply replace $d_i^{\gamma_i}$ with $[x_r, x_{r-1}]^{\gamma_i}$ and then shift this commutator to the left, without changing the exponent of $[x_1 \leftarrow x_{r-1}]$. Both cases imply $\beta_1 = 0$ and so $g \in G_{p+1}$ by induction.

Thus we conclude that if $g \in G_{p} \cap Z(G)$, then $g \in G_{p+1}$. This proves that $Z(G) = \langle x^{p^{r-1}+1}_r, G_{p+1} \rangle$, as claimed. $\square$

This yields the desired result:

**Theorem 2.11** (cf. [2, Theorem 5.2]). Let $p$ be a prime, and let $C_1, \ldots, C_r$ be cyclic $p$-groups generated by $x_1, \ldots, x_r$, respectively; assume that the order of $x_i$ is $p^{\alpha_i}$ and $1 \leq \alpha_1 \leq \cdots \leq \alpha_r$. If $G$ is the $p$-nilpotent product of the $C_i$,

$$G = C_1 \rtimes \cdots \rtimes C_r,$$

then $G$ is capable if and only if $r > 1$ and $\alpha_r \leq \alpha_{r-1} + 1$.

**Proof.** Necessity follows from [2, Theorem 3.12]. For sufficiency, let $K$ be the $(p+1)$-nilpotent product of the $C_i$, $K = C_1 \rtimes \cdots \rtimes C_r$. By Theorem 2.11 $Z(K)$ is generated by $x^{p^{r-1}+1}_r$ and $K_{p+1}$. Since $\alpha_r \leq \alpha_{r-1} + 1$, the former is trivial, so $Z(K) = K_{p+1}$. Thus $K/Z(K) = K/K_{p+1} \cong G$, so $G$ is capable. $\square$

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