AN EXISTENCE THEOREM FOR THE MAGNETO-VISCOELASTIC PROBLEM

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Abstract. The dynamics of magneto-viscoelastic materials is described by a nonlinear system which couples the equation of the magnetization, given in Gilbert form, and the viscoelastic integro-differential equation for the displacements. We study the general three-dimensional case and establish a theorem for the existence of weak solutions. The existence is proved by compactness of the approximated penalty problem.

1. Introduction. We consider the initial value problem

\[
\begin{cases}
\gamma^{-1} \dot{m} - m \times (a \Delta m - \dot{m} - Lm \otimes \nabla u) = 0 \\
\rho \ddot{u} - \text{div} \left( G(0) \nabla u + \int_0^t (\dot{G}(t-\tau) \nabla u(\tau) d\tau + \frac{1}{2} Lm \otimes m) \right) = f \\
m(0) = m^0, \quad |m^0| = 1, \quad u(0) = u^0, \quad \dot{u}(0) = u^1.
\end{cases}
\]

The system (1) describes the evolution of a magneto-visco-elastic material. We denote by \(m\) and \(u\) the magnetization and the displacement respectively; by \(G\) and \(L\) the tensors of visco-elasticity and magneto-elasticity respectively; by \(\rho, \gamma\) and \(a\) three positive constants. The function \(f\) is given and it takes also into account the history.

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Let $\Omega$ an open bounded set of $\mathbb{R}^3$ and $\nu$ the outer normal at the boundary $\partial \Omega$, we assume the boundary conditions

$$m_\nu|_{\partial \Omega} = 0, \quad u|_{\partial \Omega} = 0.$$  \hfill (3)

The first equation (1) is the the well known Gilbert-Landau-Lifschitz equation (see [9], [12]) introduced for describing the dynamics of micro-magnetic processes. The magnetization is constant in modulus but not in direction and here we assume $|m| = 1$.

We study the existence of solutions to the nonlinear integro differential problem (1), (2), (3). Our reference here is the paper [14] where a theorem of existence for the general three dimensional magnetoelastic problem is proved.

In particular, we first introduce a penalty auxiliary problem depending on a small parameter $\varepsilon$, to remove the constraint $|m| = 1$, then we approximate the new problem by the Faedo-Galerkin method. The crucial point in the proof of the existence theorem is the characterization of some a-priori estimates which allow us to obtain convergence results. For this we need some constitutive assumptions on the quadratic form associated to the tensor $G(s)$, that is positive, monotonically non-increasing and convex. In the following section 2. we detail our assumptions and some notations; the section 3. is devoted to the proof of the existence result. Note that the existence and uniqueness of the solution to the one-dimensional magneto-viscoelastic problem is proved in [2] via the fixed point theory.

We want also to quote some references on the subjects of magnetoelasticity ([1],[5],[6],[10],[11]) and viscoelasticity ([3],[4],[7],[8],[13],[15],[16]) that inspired this paper.

### 2. Assumptions and main result.

Let $x_i, i = 1, 2, 3$ be the position of a point $x$ of $\Omega$, we denote by

$$u_i = u_i(x, t), \quad i = 1, 2, 3$$

the components of the displacement vector $u$ and by

$$\epsilon_{kl}(u) = \frac{1}{2}(u_{k,l} + u_{l,k}), \quad k, l = 1, 2, 3$$

the components of the deformation tensor $\epsilon(u)$ where, as a common praxis, $u_{k,l}$ stands for $\partial u_k / \partial x_l$. Moreover let

$$m_j = m_j(x, t), \quad j = 1, 2, 3$$

be the components of the magnetization vector $m$. In the sequel, where not specified, the Latin indices vary in the set $\{1, 2, 3\}$ and the summation of the repeated indices is assumed. We define the exchange energy due to the magnetization

$$E_{ex}(m) = \frac{1}{2} \int_\Omega a_{ij} m_{k,i} m_{k,j} d\Omega$$  \hfill (4)

where $(a_{ij})$ is a symmetric positive definite matrix which is supposed diagonal for most materials with all diagonal elements equal to a positive number $a$.

The magneto-elastic energetic term is assumed as

$$E_{em}(m, u) = \frac{1}{2} \int_\Omega \lambda_{ijkl} m_i m_j \epsilon_{kl}(u) d\Omega$$  \hfill (5)

where $\lambda_{ijkl} = \lambda_1 \delta_{ijkl} + \lambda_2 \delta_{ij} \delta_{kl} + \lambda_3 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$ with $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. $\lambda_1$, $\lambda_2$, $\lambda_3$ are three given constants.
Finally we introduce the tensor field \( \{ G_{klmn}(t) \} \) satisfying for any \( t \in [0, T] \) and for any symmetric tensor \( e_{kl} \) the properties

\[
G_{klmn} = G_{mnkl} = G_{lkmn},
\]

\[
G_{klmn} e_{kl} e_{mn} \geq \beta \sum |e_{kl}|^2, \quad \beta > 0,
\]

\[
\dot{G}_{klmn} e_{kl} e_{mn} \leq 0,
\]

\[
\ddot{G}_{klmn} e_{kl} e_{mn} \geq 0.
\]

We define the following viscoelastic contribution

\[
E_{ve}(u) = \frac{1}{2} \int_{\Omega} G_{klmn}(0) e_{kl} e_{mn} d\Omega + \frac{1}{2} \int_0^t d\tau \left( \int_{\Omega} \dot{G}_{klmn}(t-\tau) e_{kl} e_{mn}(\tau) d\Omega \right).
\]

Using the tensorial notations

- \( \mathbb{G}(s) = \{ G_{klmn}(s) \} \), \( s \in [0, T] \)
- \( \mathbb{L} = \{ \lambda_{klmn} \} \)

and taking into account the symmetry hypotheses on the tensor fields \( \mathbb{G}(s) \) and \( \mathbb{L} \) and putting

- \( \{ G_{klmn}(s) e_{kl}(u) \} = \mathbb{G}(s) \nabla u, \quad s \in [0, T] \)
- \( G_{klmn}(s) e_{kl}(u) e_{mn}(v) = \mathbb{G}(s) \nabla u \cdot \nabla v, \quad s \in [0, T] \)
- \( \{ \lambda_{klmn} m_{klm} \} = \mathbb{L} m \otimes m \)
- \( \{ \lambda_{klmn} m_{klm} e_{mn}(u) \} = \mathbb{L} m \otimes m \cdot \nabla u \)
- \( \lambda_{klmn} m_{klm} e_{mn}(u) = \mathbb{L} m \otimes m \cdot \nabla u, \)

from (4), (5) and (10) we can get to the equations of the magneto-viscoelasticity in the form (1).

The main result of the paper is

**Theorem 2.1** Given \( u^0 \in H^1_0(\Omega; \mathbb{R}^3) \), \( u^1 \in L^2(\Omega; \mathbb{R}^3) \), \( m^0 \in H^1(\Omega; \mathbb{R}^3) \) with \( |m^0| = 1 \) a.e. in \( \Omega \) and let \( Q = \Omega \times [0, T] \). Assume \( f \in L^2(Q; \mathbb{R}^3) \) and \( \mathbb{G}(s) \in C^2[0, T] \) verifying the assumptions (6)-(9), then there exists a weak solution \( (m, u) \) to the problem (1), (2), (3) in the sense that:

- \( m \in H^1(Q; \mathbb{R}^3) \) with \( |m| = 1 \) a.e. in \( Q \)
- \( u \in L^2(0, T; H^1_0(\Omega; \mathbb{R}^3)) \) and \( \dot{u} \in L^2(Q; \mathbb{R}^3) \)

for each couple \( (p, g) \) such that \( g \in H^1_0(Q; \mathbb{R}^3) \) and \( p \in H^{1, \infty}(Q; \mathbb{R}^3) \) vanishing at \( t = 0 \) and \( t = T \), one has

\[
\int_Q \left[ \gamma^{-1} m \cdot p + a(m \times \nabla m) \cdot \nabla p + m \times (m + \mathbb{L} m \otimes p \cdot \nabla u) \right] d\Omega dt = 0 \quad (11)
\]

\[
\int_Q \left[ -\rho \dot{u} \cdot g + (\mathbb{G}(0) \nabla u + \frac{1}{2} \mathbb{L} m \otimes m) \cdot \nabla g \right] d\Omega dt + \int_Q \left( \int_0^t \dot{G}(t-\tau) \nabla u(\tau) \cdot \nabla g(\tau) d\tau \right) d\Omega dt - \int_Q f \cdot g d\Omega dt = 0 \quad . \quad (12)
\]
3. Proof of Theorem 2.1.

3.1. The approximated penalty problem. In this section we focus our attention to the penalty version of the previous system (1), (2), (3). First we introduce a small positive parameter $\varepsilon$ and consider in $\mathcal{Q}$ the problem

\[
\begin{align*}
\begin{cases}
\gamma^{-1}\mathbf{m}^\varepsilon \times \mathbf{m}^\varepsilon + \mathbf{m}^\varepsilon - a\Delta \mathbf{m}^\varepsilon + \|\mathbf{m}^\varepsilon\| \otimes \nabla \mathbf{u}^\varepsilon + \varepsilon^{-1}(\|\mathbf{m}^\varepsilon\|^2 - 1)\mathbf{m}^\varepsilon = 0 \\
\rho\mathbf{u}^\varepsilon - \text{div} \left( \mathcal{G}(0)\nabla \mathbf{u}^\varepsilon + \int_0^t (\hat{\mathcal{G}}(t - \tau)\nabla \mathbf{u}^\varepsilon(\tau)d\tau + \frac{1}{2}\mathbf{L}\mathbf{m}^\varepsilon \otimes \mathbf{m}^\varepsilon \right) = \mathbf{f}
\end{cases}
\end{align*}
\]  

with initial and boundary conditions

\[
\mathbf{u}^\varepsilon(\cdot, 0) = \mathbf{u}^0, \quad \mathbf{u}^\varepsilon(\cdot, 0) = \mathbf{u}^1, \quad \mathbf{m}^\varepsilon(\cdot, 0) = \mathbf{m}^0, \quad |\mathbf{m}^0| = 1 \quad \text{in } \Omega, \quad \mathbf{u}^\varepsilon = 0, \quad \mathbf{m}^\varepsilon = 0 \quad \text{on } \Sigma = \partial\Omega \times [0, T].
\]

Then we consider the Faedo-Galerkin approximation for the problem (13), (14), (15) obtained taking

\[
\mathbf{u}^{\varepsilon,N} = \sum_{j=1}^N \mathbf{b}_j(t)\phi_j(x), \quad \mathbf{m}^{\varepsilon,N} = \sum_{j=1}^N \mathbf{a}_j(t)\psi_j(x)
\]

where the functions $\phi_j, \psi_j$ both in $H^2(\Omega)$ are the eigenfunctions of the eigenvalue problems

\[
\begin{align*}
\begin{cases}
-\Delta \phi_j = \sigma_j \phi_j, & \text{in } \Omega \quad j = 1, 2, \ldots, N \\
\phi_j = 0, & \text{on } \partial\Omega, \quad j = 1, 2, \ldots, N
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
-\Delta \psi_j = \lambda_j \psi_j, & \text{in } \Omega \quad j = 1, 2, \ldots, N \\
\frac{\partial \psi_j}{\partial \nu} = 0, & \text{on } \partial\Omega, \quad j = 1, 2, \ldots, N
\end{cases}
\end{align*}
\]

respectively and $\|\phi_j\|_{L^2(\Omega)} = \|\psi_j\|_{L^2(\Omega)} = 1$, for $j = 1, 2, \ldots, N$.

We get to the following approximated penalty problem

\[
\begin{align*}
\begin{cases}
\gamma^{-1}\mathbf{m}^{\varepsilon,N} \times \mathbf{m}^{\varepsilon,N} + \mathbf{m}^{\varepsilon,N} - a\Delta \mathbf{m}^{\varepsilon,N} + \|\mathbf{m}^{\varepsilon,N}\| \otimes \nabla \mathbf{u}^{\varepsilon,N} + \\
+ \varepsilon^{-1}(\|\mathbf{m}^{\varepsilon,N}\|^2 - 1)\mathbf{m}^{\varepsilon,N} = 0 \\
\rho\mathbf{u}^{\varepsilon,N} - \text{div} \left( \mathcal{G}(0)\nabla \mathbf{u}^{\varepsilon,N} + \int_0^t (\hat{\mathcal{G}}(t - \tau)\nabla \mathbf{u}^{\varepsilon,N}(\tau)d\tau + \frac{1}{2}\mathbf{L}\mathbf{m}^{\varepsilon,N} \otimes \mathbf{m}^{\varepsilon,N} \right) = \mathbf{f}^N
\end{cases}
\end{align*}
\]

with the components of the vector $\mathbf{f}^N(x, t)$ defined by

\[
f_i^N(x, t) = \sum_{j=1}^N (f_i(\cdot, t), \phi_j(\cdot))_{L^2(\Omega)} \phi_j(x), \quad i = 1, 2, 3
\]

and the associated initial and boundary conditions defined by

\[
\begin{align*}
\mathbf{u}^{\varepsilon,N}(x, 0) = \mathbf{u}^N(x, 0), \quad \mathbf{u}^{\varepsilon,N}(x, 0) = \mathbf{u}^N(x, 0), \quad \text{in } \Omega \\
\mathbf{m}^{\varepsilon,N}(x, 0) = \mathbf{m}^N(x, 0), \quad \text{in } \Omega \\
\mathbf{u}^N = 0, \quad \mathbf{m}^N = 0 \quad \text{on } \Sigma
\end{align*}
\]

3.2. The approximated constraint problem.
where \( u^N(x, 0), \dot{u}^N(x, 0), m^N(x, 0) \) are three vectors of components

\[
\begin{align*}
  u^N_i(x, 0) &= \sum_{j=1}^{\infty} \left( u^1_j(\cdot), \phi_j(\cdot) \right)_{L^2(\Omega)} \phi_j(x), \\
  \dot{u}^N_i(x, 0) &= \sum_{j=1}^{\infty} \left( u^1_j(\cdot), \dot{\phi}_j(\cdot) \right)_{L^2(\Omega)} \phi_j(x), \\
  m^N_i(x, 0) &= \sum_{j=1}^{\infty} \left( m^0_j(\cdot), \psi_j(\cdot) \right)_{L^2(\Omega)} \psi_j(x), \quad i = 1, 2, 3.
\end{align*}
\]

We take \( u^0 \in H^1(\Omega, \mathbb{R}^3), u^1 \in L^2(\Omega, \mathbb{R}^3) \) and \( m^0 \in H^1(\Omega; \mathbb{R}^3) \) with \( |m^0| = 1 \) a.e. \( \Omega \). Since the sequence \( \{ \phi_j \}_{j \in \mathbb{N}} \) forms a basis for \( L^2(\Omega) \) and the space of finite linear combination of \( \phi_j, j = 1, \ldots, N \) is dense in \( H^1(\Omega) \cap H^2(\Omega) \), we have as \( N \to \infty \)

\[
\begin{align*}
  u^N(\cdot, 0) &\to u^0 \quad \text{in} \quad H^1(\Omega; \mathbb{R}^3), \\
  \dot{u}^N(\cdot, 0) &\to u^1 \quad \text{in} \quad L^2(\Omega; \mathbb{R}^3)
\end{align*}
\]

and analogously for the basis \( \{ \psi_j \}_{j \in \mathbb{N}} \) we have that space of finite linear combination of \( \psi_j, j = 1, \ldots, N \) is dense in \( H^2(\Omega) \) hence

\[
\begin{align*}
  m^N(\cdot, 0) &\to m^0 \quad \text{in} \quad H^1(\Omega; \mathbb{R}^3)
\end{align*}
\]

and

\[
|m^N(\cdot, 0)| \to |m^0| = 1 \quad \text{in} \quad L^2(\Omega; \mathbb{R}).
\]

If we multiply each scalar equation of the first vector equation (19) by \( \psi_j(x) \) and the second equation (19) by \( \dot{\phi}_j(x) \) and integrate in \( \Omega \) we get to a system of integro-differential equations in the unknown \( (a_j(t), b_j(t)), j = 1, 2, \ldots, N \).

First of all we observe that

\[
\gamma^{-1} m^{N \varepsilon} \times m^{N \varepsilon} + \dot{m}^{N \varepsilon} = M(m^{N \varepsilon}) \dot{m}^{N \varepsilon},
\]

where \( M(m^{N \varepsilon}) \) is a \( 3 \times 3 \) matrix with

\[
\det M(m^{N \varepsilon}) = 1 + \gamma^{-3} |m^{N \varepsilon}|^2 \geq 1,
\]

that implies the matrix \( M \) is invertible and we can put our integro-differential system in the following integral system

\[
\begin{align*}
  m^{N \varepsilon} &= \int_0^t M^{-1}(m^{N \varepsilon}) (a \Delta m^{N \varepsilon} - L m^{N \varepsilon} \otimes \nabla u^{N \varepsilon} - \varepsilon^{-1} |m^{N \varepsilon}|^2 - 1)m^{N \varepsilon} d\tau + m^N(0), \\
  \rho \dot{u}^{N \varepsilon} &= \int_0^t \left[ \operatorname{div} \left( \left( \dot{G}(t - \tau) \nabla u^{N \varepsilon}(\tau) + \frac{1}{2} L m^{N \varepsilon} \otimes m^{N \varepsilon} \right) + f^N \right) d\tau + \dot{u}^N(0), \\
  u^{N \varepsilon} &= \int_0^t \dot{u}^{N \varepsilon}(\tau) d\tau + u^N(0)
\end{align*}
\]

that is in a single integral equation

\[
\mathbf{v}(t) = \int_0^t 2M(t, \mathbf{v}(\tau)) d\tau + \mathbf{v}(0) + f(t),
\]
Lemma 3.1
established in the following lemma 3.2.

\[ \text{From the fixed point theorem we deduce the local existence of a unique solution to} \]

\[ \text{the problem (19)-(21),} \]

\[ \text{and hence the local existence of a unique solution to} \]

\[ \text{the above system (29). Indeed we put} \]

\[ \text{First of all we prove the following result} \]

\[ \text{In order to extend the solution to any finite time} \]

\[ \text{we have that, for} \]

\[ \text{where} \]

\[ \text{Moreover let} \]

\[ \text{we have} \]

\[ \text{Since} \]

\[ \text{From the fixed point theorem we deduce the local existence of a unique solution to} \]

\[ \text{and hence the local existence of a unique solution to the problem (19)-(21),} \]

\[ \text{In order to extend the solution to any finite time} \]

\[ \text{First of all we prove the following result} \]

**Lemma 3.1** Let \( u^0 \in H^1(\Omega; \mathbb{R}^3) \), \( u^1 \in L^2(\Omega; \mathbb{R}^3) \), \( m^0 \in H^1(\Omega; \mathbb{R}^3) \) with \( |m^0| = 1 \) a.e. in \( \Omega \) and \( f \in L^2(Q, \mathbb{R}^3) \) then we have

\[
\frac{a}{2} \frac{d}{dt} \int_{\Omega} |\nabla u^{\varepsilon,N}|^2 d\Omega + \frac{\varepsilon - 1}{4} \frac{d}{dt} \int_{\Omega} (|u^{\varepsilon,N}|^2 - 1)^2 d\Omega + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \mathbb{G}(t) \nabla u^{\varepsilon,N} \cdot \nabla u^{\varepsilon,N} d\Omega -
\]

\[
- \frac{1}{2} \frac{d}{dt} \int_{\Omega} \mathbb{G}(s) (\nabla u^{\varepsilon,N}(t) - \nabla u^{\varepsilon,N}(t-s)) \cdot (\nabla u^{\varepsilon,N}(t) - \nabla u^{\varepsilon,N}(t-s)) d\Omega +
\]

\[
+ \frac{\rho}{2} \frac{d}{dt} \int_{\Omega} |u^{\varepsilon,N}|^2 d\Omega + \int_{\Omega} |m^{\varepsilon,N}|^2 d\Omega + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \mathbb{L} m^{\varepsilon,N} \otimes m^{\varepsilon,N} \cdot \nabla u^{\varepsilon,N} d\Omega =
\]

\[
= \int_{\Omega} f^{N} \hat{u}^{\varepsilon,N} d\Omega + \frac{1}{2} \int_{\Omega} \mathbb{G}(t) \nabla u^{\varepsilon,N} \cdot \nabla u^{\varepsilon,N} d\Omega -
\]

\[
- \frac{1}{2} \int_{0}^{t} ds \int_{\Omega} \mathbb{G}(s) (\nabla u^{\varepsilon,N}(t) - \nabla u^{\varepsilon,N}(t-s)) \cdot (\nabla u^{\varepsilon,N}(t) - \nabla u^{\varepsilon,N}(t-s)) d\Omega
\]

\[ (30) \]
Proof. We put the above system (19) in the following equivalent form

\[
\begin{cases}
\gamma^{-1}\dot{m}^{\varepsilon,N} \times m^{\varepsilon,N} + \dot{m}^{\varepsilon,N} - a\Delta m^{\varepsilon,N} + L m^{\varepsilon,N} \otimes \nabla u^{\varepsilon,N} +
\quad + \varepsilon^{-1}(|m^{\varepsilon,N}|^2 - 1)m^{\varepsilon,N} = 0
\end{cases}
\]

\[
\begin{cases}
\rho\ddot{u}^{\varepsilon,N} - \text{div}\left(\mathbb{G}(t)\nabla u^{\varepsilon,N} - \int_0^t \mathbb{G}(s) \left[\nabla u^{\varepsilon,N}(t) - \nabla u^{\varepsilon,N}(t-s)\right] ds\right) +
\quad + \frac{1}{2}\text{div}\left(\mathbb{L}m^{\varepsilon,N} \otimes m^{\varepsilon,N}\right) = f^N
\end{cases}
\] (31)

then we multiply the first equation (31) by \(\dot{m}^{\varepsilon,N}\) and the second one by \(\dot{u}^{\varepsilon,N}\), integrating in \(\Omega\) and omitting, for sake of simplicity, the indices \(\varepsilon\) and \(N\) we get

\[
\frac{a}{2} \frac{d}{dt} \int_{\Omega} |\nabla m|^2 \, d\Omega + \frac{\varepsilon^{-1}}{4} \frac{d}{dt} \int_{\Omega} (|m|^2 - 1)^2 \, d\Omega + \frac{1}{2} \int_{\Omega} \mathbb{G}(t) \nabla u \cdot \nabla \dot{u} \, d\Omega +
\]

\[
+ \int_{\Omega} \dot{u}(t) \, d\Omega \int_0^t \mathbb{G}(s) \left[\nabla u(t) - \nabla u(t-s)\right] ds + \rho \frac{d}{dt} \int_{\Omega} |\dot{u}|^2 \, d\Omega +
\]

\[
+ \int_{\Omega} \dot{m}^2 \, d\Omega + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \mathbb{L} m \otimes m \cdot \nabla u \, d\Omega = \int_{\Omega} f \cdot \dot{u} \, d\Omega
\] (32)

that is

\[
\frac{a}{2} \frac{d}{dt} \int_{\Omega} |\nabla m|^2 \, d\Omega + \frac{\varepsilon^{-1}}{4} \frac{d}{dt} \int_{\Omega} (|m|^2 - 1)^2 \, d\Omega + \frac{d}{dt} \int_{\Omega} \mathbb{G}(t) \nabla u \cdot \nabla u \, d\Omega -
\]

\[
- \frac{1}{2} \int_{\Omega} \mathbb{G}(t) \nabla u \cdot \nabla u \, d\Omega - \int_{\Omega} \int_0^t \mathbb{G}(s) \left[\nabla u(t) - \nabla u(t-s)\right] ds \, d\Omega ds +
\]

\[
+ \rho \frac{d}{dt} \int_{\Omega} |\dot{u}|^2 \, d\Omega + \int_{\Omega} |\dot{m}|^2 \, d\Omega + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \mathbb{L} m \otimes m \cdot \nabla u \, d\Omega = \int_{\Omega} f \cdot \dot{u} \, d\Omega.
\] (33)
Now we observe that

\[-\int_0^t \int_\Omega \partial_s \mathcal{G}(s) \nabla \hat{u}(t) \cdot [\nabla u(t) - \nabla u(t-s)] \, d\Omega \, ds = \]

\[= -\frac{1}{2} \frac{d}{dt} \int_0^t ds \int_\Omega \mathcal{G}(s)[\nabla u(t) - \nabla u(t-s)] \cdot [\nabla u(t) - \nabla u(t-s)] \, d\Omega + \]

\[+ \frac{1}{2} \int_\Omega \mathcal{G}(s)[\nabla u(t) - \nabla u(0)] \cdot [\nabla u(t) - \nabla u(0)] \, d\Omega - \]

\[- \int_\Omega \int_0^t \mathcal{G}(s) \nabla \hat{u}(t-s) \cdot [\nabla u(t) - \nabla u(t-s)] \, d\Omega \, ds = \]

\[= -\frac{1}{2} \frac{d}{dt} \int_0^t ds \int_\Omega \mathcal{G}(s)[\nabla u(t) - \nabla u(t-s)] \cdot [\nabla u(t) - \nabla u(t-s)] \, d\Omega + \]

\[+ \frac{1}{2} \int_\Omega \mathcal{G}(s)[\nabla u(t) - \nabla u(0)] \cdot [\nabla u(t) - \nabla u(0)] \, d\Omega - \]

\[- \int_\Omega \int_0^t \mathcal{G}(s) \nabla \hat{u}(t-s) \cdot \frac{d}{ds} [\nabla u(t) - \nabla u(t-s)] \, d\Omega \, ds = \]

\[= -\frac{1}{2} \frac{d}{dt} \int_0^t ds \int_\Omega \mathcal{G}(s)[\nabla u(t) - \nabla u(t-s)] \cdot [\nabla u(t) - \nabla u(t-s)] \, d\Omega + \]

\[+ \frac{1}{2} \int_\Omega \mathcal{G}(s) \nabla \hat{u}(t-s) \cdot \frac{d}{ds} [\nabla u(t) - \nabla u(t-s)] \, d\Omega \, ds . \]

Substituting in (33) we get to the proof of the lemma. \(\square\)

Now we introduce the functional

\[\mathcal{F}^{\varepsilon,N}(T) = \int_0^T \int_\Omega |\mathbf{m}^{\varepsilon,N}|^2 \, d\Omega \, dt + \frac{a}{2} \int_\Omega |\nabla \mathbf{m}^{\varepsilon,N}|^2 \, d\Omega + \]

\[+ \frac{b}{4} \int_\Omega |\nabla (\mathbf{u}^{\varepsilon,N})|^2 \, d\Omega + \frac{\rho}{2} \int_\Omega |\mathbf{u}^{\varepsilon,N}|^2 \, d\Omega + \frac{\varepsilon^{-1}}{8} \int_\Omega (|\mathbf{m}^{\varepsilon,N}|^2 - 1)^2 \, d\Omega \]

(34)

where the last four integrals are computed for \(t = T\). We introduce the positive parameter \(\lambda\) such that

\[2 \lambda > \sup_{ijkl} |\lambda_{ijkl}| \]

we have

**Lemma 3.2** Let \(\mathbf{u}^0 \in H^1_0(\Omega; \mathbb{R}^3)\), \(\mathbf{u}^1 \in L^2(\Omega; \mathbb{R}^3)\), \(\mathbf{m}^0 \in H^1(\Omega; \mathbb{R}^3)\) with \(|\mathbf{m}^0| = 1\) a.e. in \(\Omega\), \(\mathbf{f} \in L^2(\Omega; \mathbb{R}^3)\) and \(\mathcal{G}\) verifying the assumptions (6), (7), (8), then there exists a positive constant \(C\) depending on the data \(\mathbf{m}^0, \mathbf{u}^0, \mathbf{u}^1, \mathbf{f}\) and \(T\) but independent of \(N\) and \(\varepsilon\), and a positive constant \(C_{\varepsilon,N}\) such that for positive \(\varepsilon\) small
enough (i.e. $16\varepsilon \leq \beta \lambda^{-2}$) the following estimate holds

$$F^{\varepsilon,N} \leq C(m^0, u^0, u^1, f, T) + C_{\varepsilon,N} \quad (35)$$

where $C_{\varepsilon,N} \to 0$, as $N \to \infty$.

**Proof.** We multiply the first equation (19) by $\dot{m}^{\varepsilon,N}$ and the second one by $\dot{u}^{\varepsilon,N}$, integrating in $\Omega \times [0,T]$ from Lemma 3.1 and taking into account the hypothesis on the operator $G(t)$, we obtain the inequality

$$\frac{a}{2} \int_{\Omega} |\nabla m^{\varepsilon,N}|^2 d\Omega + \frac{1}{2} \int_{\Omega} G(t) \nabla u^{\varepsilon,N} \cdot \nabla u^{\varepsilon,N} d\Omega + \int_{0}^{T} \int_{\Omega} |m^{\varepsilon,N}|^2 d\Omega dt +$$

$$+ \frac{1}{2} \int_{\Omega} \mathbb{L} m^{\varepsilon,N} \otimes m^{\varepsilon,N} \cdot \nabla u^{\varepsilon,N} d\Omega + \frac{\varepsilon^{-1}}{4} \int_{\Omega} (|m^{\varepsilon,N}|^2 - 1)^2 d\Omega +$$

$$+ \frac{b}{2} \int_{\Omega} |\dot{u}^{\varepsilon,N}|^2 d\Omega \leq \frac{b}{2} \int_{\Omega} |\dot{u}^{\varepsilon,N}|^2 d\Omega + \frac{b}{2} \int_{\Omega} |u^{\varepsilon,N}(0)|^2 d\Omega +$$

$$+ \frac{1}{2} \int_{\Omega} \mathbb{G}(0) \nabla u^{N}(0) \cdot \nabla u^{N}(0) d\Omega + \frac{1}{2} \int_{\Omega} \mathbb{L} m^{\varepsilon,N}(0) \otimes m^{\varepsilon,N}(0) \cdot \nabla u^{\varepsilon,N}(0) d\Omega +$$

$$+ \varepsilon^{-1} \int_{\Omega} (|m^{\varepsilon,N}(0)|^2 - 1)^2 d\Omega + \frac{b}{2} \int_{\Omega} |\dot{u}^{\varepsilon,N}(0)|^2 d\Omega. \quad (36)$$

Since for all $b \in \mathbb{R}^+$ we have

$$\frac{1}{2} \int_{\Omega} \mathbb{L} m^{\varepsilon,N} \otimes m^{\varepsilon,N} \cdot \nabla u^{\varepsilon,N} d\Omega \leq \frac{\lambda}{2b} \int_{\Omega} |m^{\varepsilon,N}|^4 d\Omega + \frac{b}{2} \int_{\Omega} |\nabla u^{\varepsilon,N}|^2 d\Omega \leq$$

$$\frac{\lambda b}{2} \int_{\Omega} (|m^{\varepsilon,N}|^2 - 1)^2 d\Omega + \frac{\lambda}{b} \operatorname{vol}(\Omega) + \frac{\lambda b}{2} \int_{\Omega} |\nabla u^{\varepsilon,N}|^2 d\Omega,$$

taking $2b = \beta / \lambda$ and recalling the inequality (7) we obtain

$$\frac{1}{2} \int_{\Omega} \mathbb{L} m^{\varepsilon,N} \otimes m^{\varepsilon,N} \cdot \nabla u^{\varepsilon,N} d\Omega \leq$$

$$\frac{2\lambda^2}{\beta} \int_{\Omega} (|m^{\varepsilon,N}|^2 - 1)^2 d\Omega + \frac{2\lambda^2}{\beta} \operatorname{vol}(\Omega) + \frac{\beta}{4} \int_{\Omega} |\nabla u^{\varepsilon,N}|^2 d\Omega$$

and hence for $\varepsilon$ small enough (i.e. $\varepsilon < \beta/(16\lambda^2)$) one has

$$\frac{1}{2} \int_{\Omega} \mathbb{L} m^{\varepsilon,N} \otimes m^{\varepsilon,N} \cdot \nabla u^{\varepsilon,N} d\Omega \leq$$

$$\frac{1}{8\varepsilon} \int_{\Omega} (|m^{\varepsilon,N}|^2 - 1)^2 d\Omega + \frac{2\lambda^2}{\beta} \operatorname{vol}(\Omega) + \frac{1}{4} \int_{\Omega} \mathbb{G}(t) |\nabla u^{\varepsilon,N}|^2 d\Omega$$

using this last inequality in (36) we get

$$F^{\varepsilon,N}(T) \leq 3F^{\varepsilon,N}(0) + \frac{4\lambda^2}{\beta} \operatorname{vol}(\Omega) + \frac{\beta^{-1}}{2} \|f^N\|_{L^2(\Omega)}^2 + \frac{b}{2} \int_{0}^{T} |\dot{u}^{\varepsilon,N}|^2 d\Omega dt$$
and hence
\[
\mathcal{F}^{\varepsilon,N}(T) \leq 3\mathcal{F}^{\varepsilon,N}(0) + \frac{4\lambda^2}{\beta} \|\mathbf{v}\|_{\Omega} + \frac{\rho^{-1}}{2} \|f^N\|_{L^2(Q)} + \int_0^T \mathcal{F}^{\varepsilon,N}(t) \, dt
\]
where
\[
\mathcal{F}^{\varepsilon,N}(0) = \frac{a}{2} \int_{\Omega} |\nabla \mathbf{m}^N(\cdot, 0)|^2 \, d\Omega + \frac{1}{4} \int_{\Omega} \mathbb{G}(0) \nabla \mathbf{u}^N(\cdot, 0) \cdot \nabla \mathbf{u}^N(\cdot, 0) \, d\Omega + \frac{\rho}{2} \int_{\Omega} |\mathbf{u}^N(\cdot, 0)|^2 \, d\Omega + \frac{\varepsilon^{-1}}{8} \int_{\Omega} (|\mathbf{m}^N(\cdot, 0)|^2 - 1)^2 \, d\Omega.
\]
From the Gronwall Lemma we obtain
\[
\mathcal{F}^{\varepsilon,N}(T) \leq \exp(T) \left( 3\mathcal{F}^{\varepsilon,N}(0) + \frac{4\lambda^2}{\beta} \|\mathbf{v}\|_{\Omega} + \frac{\rho^{-1}}{2} \|f^N\|_{L^2(Q)}^2 \right).
\]
Now writing
\[
\mathcal{F}^{\varepsilon,N}(0) = \mathcal{F}(\mathbf{m}^N(\cdot, 0), \mathbf{u}^N(\cdot, 0), \mathbf{u}^N(\cdot, 0))
\]
we have
\[
\mathcal{F}(\mathbf{m}^N(\cdot, 0), \mathbf{u}^N(\cdot, 0), \mathbf{u}^N(\cdot, 0)) \leq \bar{\mathcal{C}}(\mathbf{m}^0, \mathbf{u}^0, \mathbf{u}^1) + \bar{\mathcal{C}}_{\varepsilon,N}
\]
where \(\bar{\mathcal{C}}(\mathbf{m}^0, \mathbf{u}^0, \mathbf{u}^1)\) is a positive constant independent of \(N\) and \(\varepsilon\) (since \(|\mathbf{m}^0| = 1\)) i.e.
\[
\bar{\mathcal{C}}(\mathbf{m}^0, \mathbf{u}^0, \mathbf{u}^1) = a \int_{\Omega} |\nabla \mathbf{m}^0|^2 \, d\Omega + \frac{\beta}{2} \int_{\Omega} |\nabla \mathbf{u}^0|^2 \, d\Omega + \rho \int_{\Omega} |\mathbf{u}^1|^2 \, d\Omega
\]
and
\[
\bar{\mathcal{C}}_{\varepsilon,N} = a \int_{\Omega} |\nabla (\mathbf{m}^0 - \mathbf{m}^N(\cdot, 0))|^2 \, d\Omega + \frac{\beta}{2} \int_{\Omega} |\nabla (\mathbf{u}^0 - \mathbf{u}^N(\cdot, 0))|^2 \, d\Omega + \rho \int_{\Omega} |(\mathbf{u}^1 - \mathbf{u}^N(0))|^2 \, d\Omega + \frac{\varepsilon^{-1}}{4} \int_{\Omega} (|\mathbf{m}^N(0)|^2 - 1)^2 \, d\Omega.
\]
In the definition of the constant \(\bar{\mathcal{C}}\) and \(\bar{\mathcal{C}}_{\varepsilon,N}\) we have assumed
\[
\beta \nabla \mathbf{u} \leq \bar{\mathcal{C}}(\mathbf{m}^0, \mathbf{u}^0, \mathbf{u}^1) \leq \beta \|\nabla \mathbf{u}\|^2.
\]
So we have
\[
\mathcal{F}^{\varepsilon,N}(T) \leq e^T \left( 3\bar{\mathcal{C}}(\mathbf{m}^0, \mathbf{u}^0, \mathbf{u}^1) + \bar{\mathcal{C}}_{\varepsilon,N} + \frac{4\lambda^2}{\beta} \|\mathbf{v}\|_{\Omega} + \rho^{-1} \|f^N\|_{L^2(Q)}^2 + \rho^{-1} \|f^N - f\|_{L^2(Q)}^2 \right)
\]
from the convergence of the initial data (24), (25), (26), (27) and the convergence of the function \(f\) we have
\[
\mathcal{C}_{\varepsilon,N} = e^T \left( \mathcal{C}_{\varepsilon,N} + \rho^{-1} \|f^N - f\|_{L^2(Q)}^2 \right) \to 0 \quad N \to \infty
\]
and setting
\[
C = e^T \left( 3\bar{\mathcal{C}}(\mathbf{m}^0, \mathbf{u}^0, \mathbf{u}^1) + \frac{4\lambda^2}{\beta} \|\mathbf{v}\|_{\Omega} + \rho^{-1} \|f^N\|_{L^2(Q)}^2 \right)
\]
the proof of the Lemma easily follows. \(\square\)
3.2. Convergence of the approximate solutions. For proving the existence theorem 2.1, we look again at the variational formulation of the Faedo-Galerkin approximation to the penalty problem (19), (21), (22), (23) and put
\[ \varepsilon = \varepsilon(N) = \|(m^N(\cdot, 0))^2 - 1\|_{L^2(\Omega)} \] (40)
from (26), we have \( \varepsilon(N) \to 0 \) as \( N \to \infty \).

With this choice of the parameter \( \varepsilon \) the approximate functions depend only on \( N \), so we can replace \( m^{(\varepsilon)(N), N} = m^N, u^{(\varepsilon)(N), N} = u^N \) and \( \hat{u}^{(\varepsilon)(N), N} = \hat{u}^N \). We assume \( g \in H^1_0(Q; \mathbb{R}^3) \) and \( h = m^N \times p \) with \( p \in C^\infty(\Omega \times [0, T]; \mathbb{R}^3) \) vanishing at \( t = 0 \) and at \( t = T \). We obtain
\[ \gamma^{-1} \int_Q [m^N \times (m^N \times p)] \cdot \dot{m}^N \, d\Omega \, dt + \int_Q \dot{m}^N \cdot (m^N \times p) \, d\Omega \, dt + \] + \( a \int_Q \nabla m^N \cdot \nabla (m^N \times p) \, d\Omega \, dt + \int_Q \|m^N \otimes \nabla u^N \cdot (m^N \times p) \| \, d\Omega \, dt = 0 \) , \] (41)
\[ - \rho \int_Q \dot{u}^N \cdot \dot{g} \, d\Omega \, dt + \int_Q \|\dot{u}^N \cdot \nabla g \| \, d\Omega \, dt + \int_Q \nabla g \, d\Omega \, dt \cdot \int_0^T \tilde{g}(t - \tau) \nabla \dot{u}^N(\tau) \, d\tau \] + \( \frac{1}{2} \int_Q \|\nabla g \| \, d\Omega \, dt = \int_Q f^N \cdot g \, d\Omega \, dt \) . \] (42)

As a consequence of the estimate established in Lemma 3.2 we have
\[ \int_\Omega |\nabla m^N|^2 \, d\Omega \leq C_2 , \quad \int_\Omega |\nabla u^N|^2 \, d\Omega \leq C_2 , \] \[ \rho \int_\Omega |\dot{u}^N|^2 \, d\Omega \leq C_2 , \quad \int_0^T |\dot{m}^N|^2 \, d\Omega \, dt \leq C_2 , \] \[ \int_\Omega (|m^N|^2 - 1)^2 \, d\Omega \leq \varepsilon(N) C_2 , \] where \( C_2 \) is a positive constant depending on the data of the problem but bounded for \( N \to \infty \) (see (37), (40)); then from compactness lemmas and passing to subsequences we have, as \( N \to \infty \)
\[ m^N \to m \quad \text{weakly in } H^1(Q; \mathbb{R}^3) , \] (43)
\[ m^N \to m \quad \text{strongly in } L^2(Q; \mathbb{R}^3) , \] (44)
\[ |m^N| \to 1 \quad \text{strongly in } L^2(Q; \mathbb{R}) , \text{ i.e. } |m| = 1, \text{ a.e. in } Q , \] (45)
\[ u^N \to u \quad \text{weakly in } L^2(0, T; H^1_0(\Omega))^3 , \] (46)
\[ \dot{u}^N \to \dot{u} \quad \text{weakly in } L^2(0, T; L^2(\Omega))^2 , \] (47)
\[ u^N \to u \quad \text{strongly in } L^2(Q; \mathbb{R}^3) . \] (48)

Moreover, from the Sobolev embedding theorem, the further compactness result holds
\[ m^N_i m^N_j \to m_i m_j \quad \text{strongly in } L^2(Q; \mathbb{R}) , \quad i, j = 1, 2, 3 . \] (49)

From (46)-(48) the convergence of the second variational equation (42) easily follows. For the convergence of first equation (41) we need further remarks. First of all we rewrite (41) in the form
\[
\gamma^{-1} \int_{\Omega} [(\dot{m}^N \times m^N) \times m^N] \cdot p \, d\Omega dt - \int_{\Omega} p \cdot (m^N \times \dot{m}) \, d\Omega dt - \]
\[
- a \int_{\Omega} (m^N \times \partial_t m^N) \cdot \partial_t p \, d\Omega dt - \int_{\Omega} (m^N \times Lm^N \otimes \nabla u^N) \cdot p \, d\Omega dt = 0 .
\]

The convergence of the second and third term in the left side of (50) directly follows from (49) thanks to the strong convergence of \(m^N\) and the weak convergence of \(\nabla m^N\). So we have
\[
\int_{\Omega} p \cdot (m^N \times m^N) \, d\Omega dt \to \int_{\Omega} p \cdot (m \times \dot{m}) \, d\Omega dt ,
\]
\[
a \int_{\Omega} (m^N \times \partial_t m^N) \cdot \partial_t p \, d\Omega dt \to a \int_{\Omega} (m \times \partial_t m) \cdot \partial_t p \, d\Omega dt .
\]

For the first term of (50) we have
\[
\gamma^{-1} \int_{\Omega} [(\dot{m}^N \times m^N) \times m^N] \cdot p \, d\Omega dt = \gamma^{-1} \int_{\Omega} [(\dot{m}^N \cdot m^N)m^N - (|m^N|^2 \cdot \dot{m})] \cdot p \, d\Omega dt
\]
and hence
\[
\gamma^{-1} \int_{\Omega} [(\dot{m}^N \times m^N) \times m^N] \cdot p \, d\Omega dt = - \gamma^{-1} \int_{\Omega} \dot{m}^N \cdot p \, d\Omega dt + P^N ,
\]
where
\[
P^N = \gamma^{-1} \left\{ \frac{1}{2} \int_{\Omega} (|m^N|^2 - 1) \dot{m}^N \cdot p \, d\Omega dt - \int_{\Omega} (|m^N|^2 - 1)|\dot{m}^N| \cdot p \, d\Omega dt \right\} .
\]

Since \(p\) vanishes at \(t=0\) and at \(t=T\), integrating by part the first integral we have
\[
P^N = - \gamma^{-1} \left\{ \frac{1}{2} \int_{\Omega} (|m^N|^2 - 1)m^N \cdot \dot{p} \, d\Omega dt + \frac{3}{2} \int_{\Omega} (|m^N|^2 - 1)|m^N| \cdot p \, d\Omega dt \right\}
\]
from the regularity of the function \(p\) and from (43), (44), (45) we get
\[
P^N \to 0, \quad as \quad N \to \infty
\]
and hence
\[
\gamma^{-1} \int_{\Omega} [(\dot{m}^N \times m^N) \times m^N] \cdot p \, d\Omega dt \to - \gamma^{-1} \int_{\Omega} \dot{m} \cdot p \, d\Omega dt .
\]

The convergence of the last term in (50) follows from (49)
\[
\int_{\Omega} (m^N \times Lm^N \otimes \nabla u^N) \cdot p \, d\Omega dt \to \int_{\Omega} (m \times Lm \otimes \nabla u) \cdot p \, d\Omega dt
\]
and hence the proof of theorem 2.1 easily follows. Indeed from (50) and from (54), (51), (52), (55) we have that the couple \((m, u)\) satisfies the weak form of the system (1). Moreover from the above convergence results we obtain
\[
F = \frac{a}{2} \int_{\Omega} |\nabla m|^2 \, d\Omega + \frac{\beta}{4} \int_{\Omega} |\nabla (u)|^2 \, d\Omega + \frac{\nu}{2} \int_{\Omega} |\dot{u}|^2 \, d\Omega \leq C
\]
where \(C\) (see (39)) is the constant independent of \(\varepsilon\) defined in lemma 3.1.
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