Mutual Chern-Simons Theory of Spontaneous Vortex Phase

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We apply the mutual Chern-Simons effective theory (Phys. Rev. B 71, 235102) of the doped Mott insulator to the study of the so-called spontaneous vortex phase in the low-temperature pseudogap region, which is characterized by strong unconventional superconducting fluctuations. An effective description for the spontaneous vortex phase is derived from the general mutual Chern-Simons Lagrangian, based on which the physical properties including the diamagnetism, spin paramagnetism, magneto-resistance, and the Nernst coefficient, have been quantitatively calculated. The phase boundaries of the spontaneous vortex phase which sits between the onset temperature \( T_c \) and the superconducting transition temperature \( T_{c1} \), are also determined within the same framework. The results are consistent with the experimental measurements of the cuprates.

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I. INTRODUCTION

Since the 1986 discovery\(^{1}\), the high-\( T_c \) cuprate superconductors have attracted strong interest both theoretically and experimentally. However, a well-accepted understanding is still elusive after two decades’ efforts. The main difficulty comes from the strongly correlated nature of the electronic dynamics and the complex phenomena observed in experiments. Although there have been a lot of theoretical proposals available in describing some aspects of the experimental observations, the most important challenge to a microscopic theory of the cuprates is how to provide a consistent understanding of the global and universal features of the whole phase diagram.

The mutual Chern-Simons gauge theory proposed in Ref.\(^{2}\) aims at facing this challenge. This theory is based on the phase string formalism of the \( t - J \) model\(^{3,4}\), in which the charge and spin degrees of freedom are both described by bosonic fractionalized fields—called holon and spinon, respectively. The fermionic statistics of the electron is taken into account by the mutual topological interaction between each spinon and holon, mediated by the mutual Chern-Simons gauge fields. The advantages of this approach are i) naturally including two ordered phases, \( i.e., \) antiferromagnetic (AF) and superconducting (SC), in the phase diagram; ii) explicitly incorporating the strong mutual influence between the charge and spin degrees of freedom via the mutual Chern-Simons gauge structure. In this theory, the bosonic spinons form the singlet pairing at a characteristic temperature \( T_0 \), whose maximum \( \sim J \approx 1,480 \) K at half-filling while monotonically decreases with doping. Physically, \( T_0 \) stands for the temperature scale below which the short range AF correlations start to grow from the length scale of the lattice constant\(^{5}\). The regime at \( T < T_0 \) is called the upper pseudo-gap phase (UPP), which is the “matrix” of all the lower temperature phases for the underdoped system, including the AF and SC ordered states. With each spinon playing the role of a \( \pi \)-vortex of the holon field and vice versa, the superconductivity is described by the holon condensation with the spinon-pair confinement, and the AF ordered state is depicted by the spinon condensation with the holon self-localization\(^{6}\). Here the global phase diagram is essentially decided by the mutual duality of the mutual Chern-Simons theory.

In this paper, we will apply this theory to a low-temperature phase embedded in the UPP, which is described by the holon condensate with \textit{unconfined} spinons. Physically, such a regime corresponds to an unconventional SC fluctuation region, in which strong SC fluctuations are driven by the low-lying spin excitations. In other words, such a phase is both a spin liquid and a vortex liquid, named by spontaneous vortex phase (SVP). A semi-classical mathematical description of such a region in the language of the generalized Ginzburg-Landau equation has been given in Refs.\(^{7,8}\). The main goal of the present work is to provide a mathematically self-contained effective theory, based on the mutual Chern-Simons gauge theory, which can work beyond the Ginzburg-Landau formalism. Starting from this effective theory, the physical quantities including the diamagnetism, spin paramagnetism, magneto-resistance, and the Nernst effect are studied, which are compared with the experimental results. The phase boundary of the SVP in the \( T - H \) (temperature-magnetic field) plane is decided by the superconducting transition temperature and characteristic magnetic field, \( T_c \) and \( H_m \), and the holon condensation onset temperature and critical magnetic field \( T_v, H_c \). All these four quantities and their doping dependence are calculated based on the leading order approximation in this effective theory, which show qualitative consistency with the experiments.

The remainder of this paper is organized as follows. In Sec. II, the effective description for the SVP is derived from the mutual-Chern-Simons gauge field theory of the phase string model. Sec. III is contributed to the calculation of physical quantities at the mean-field approximation level of the effective theory, which include the diamagnetism, paramagnetism, magneto-resistance and Nernst coefficient. Then the calculation of the phase boundary beyond the mean-field approximation is given in Sec. IV. Finally, the conclusion and discussions are
II. MUTUAL CHERN-SIMONS THEORY OF SPONTANEOUS VORTEX PHASE

A. Mutual Chern-Simons Effective Theory for a doped Mott Insulator

The mutual Chern-Simons theory\(^2\) is a field theory description of a doped Mott insulator. In such a theory, a mutual topological interaction between spin and charge degrees of freedom, due to the phase string effect\(^2\) in the \(t - J\) model, is simply captured by a mutual-Chern-Simons term. The total lattice Euclidean Lagrangian is given as follows\(^2\):

\[
L = L_h + L_s + L_{\text{CS}},
\]

\[
L_h = \sum_i \hbar^2 \left[ \partial_\alpha - i A_\alpha^h(I) \right] h_I - t_h \sum_{\langle ij \rangle} \left( \hbar^2 e^{i A_{ij}^h} h_J + h.c. \right) + \mu \left( \sum_i h_i^h h_J - N \delta \right) + \frac{g}{2} \sum_i \left( h_i^h h_J \right)^2,
\]

\[
L_s = \sum_{i\sigma} \hbar^2 \left[ \partial_\alpha - i \sigma A_\alpha^s(i) \right] b_{i\sigma} - \frac{j}{2} \sum_{\langle ij \rangle \sigma} \Delta_{ij}^s \left( b_{i\sigma}^\dagger e^{i \sigma A_{ij}^s} b_{j-\sigma}^\dagger + h.c. \right) + \lambda \left( \sum_{i\sigma} b_{i\sigma}^\dagger b_{i\sigma} - N \left( 1 - \delta \right) \right),
\]

\[
L_{\text{CS}} = \frac{i}{\pi} \sum_i e^{\mu_\alpha \lambda_\mu} A_\mu^s(I) \partial_\nu A_\nu^s(i),
\]

In which \(L_h\) and \(L_s\) describe the dynamics of the matter fields — bosonic spinless holon, \(h_I\), and bosonic neutral spinon, \(b_{i\sigma}\), respectively. The chemical potential \(\lambda\) and \(\mu\) terms are included to enforce the total number constraint of spinon and holon, respectively. In the \(\mu\) term, \(\delta\) denotes the doping concentration and \(N\) the total number of lattice sites, and the last term in \(L_h\) is introduced to account for the on-site repulsion between the holons, which may be regarded as a softened hard-core condition.

The two matter fields \(h_I\) and \(b_{i\sigma}\) minimally couple to two \(U(1)\) gauge fields \(A^s\) and \(A^h\), respectively, in \(L_h\) and \(L_s\). Physically, the mutual-Chern-Simons coupling \(L_{\text{CS}}\) [Eq. (4)] entangles two otherwise independent gauge fields, \(A^s\) and \(A^h\), to realize the topological constraint due to the phase string effect. This may be directly understood by considering the equations of motion for the temporal components \(A^s_\mu\) and \(A^h_\mu\):

\[
\frac{\partial L}{\partial A^s_\mu(i)} = 0 \Rightarrow \epsilon^{\alpha\beta} \Delta_\alpha A^s_\beta(i) = \pi n^h_i
\]

\[
\frac{\partial L}{\partial A^h_\mu(i)} = 0 \Rightarrow \epsilon^{\alpha\beta} \Delta_\alpha A^h_\beta(i) = \pi \sum_{\sigma} n^b_{i\sigma}
\]

In other words, the holon (spinon) number field \(n^h_i\) (\(n^b_{i\sigma}\)) determines the gauge-field strength of \(A^s\) (\(A^h\)), as each matter particle (holon or spinon) is attached to a fictitious \(\pi\) flux tube seen by the different species. Note that in the lattice version of \(L_{\text{CS}}\) [Eq. (4)], \(\partial_\alpha = \Delta_\alpha \) for the spatial components with \(\Delta_\alpha A^h_\beta(i) \equiv A^h_\beta(i + \alpha) - A^h_\beta(i)\) and \(\Delta_\alpha A^s_\beta(I) \equiv A^s_\beta(I) - A^h_\beta(I - \alpha)\). Here the indices \(\alpha\) and \(\beta\) will be always used to denote the spatial components \((\alpha, \beta = x, y)\) in the present formalism, and the lattice gauge fields \(A^s_{ij} \equiv A^s_\alpha(I)\) \((J = I - \alpha)\) and \(A^h_{ij} \equiv A^h_\beta(j)\) \((i = j + \alpha)\), with the indices \(i, I\) standing for a square lattice site and its dual lattice site, respectively.

In this theory, the order parameter \(\Delta_{ij}^s = \sum_{\sigma} \left\langle b_{i\sigma} e^{-i \sigma A_{ij}^s} b_{j-\sigma}^\dagger \right\rangle \neq 0\) in \(L_s\) [Eq. (3)] characterizes the short-range bosonic RVB pairing onset at an upper pseudogap temperature \(T_0\). At \(T < T_0\), the AF correlations start to develop concomitant with the bosonic RVB pairing condensation, as described by \(L_s\). This theory predicts two ordered phases in the lower temperature regions at \(T \ll T_0\), namely, the AF long range order (AFLRO) phase with spinon condensation and holon self-localization at low doping, and the SC phase with holon condensation and spinon confinement at higher doping.

In the backdrop of the RVB pairing and holon condensation, the SC phase coherence is realized when spinons are confined in pairs. The opposite case is also allowed, in which the spinons are not confined, namely, single spinons are present as free neutral objects. This defines the regime known as the spontaneous vortex phase (SVP) or lower pseudogap phase (LPP), previously discussed based on the phase string model in the Hamiltonian formalism at a generalized mean-field (Ginzburg-Landau) level.\(^7\)

In the present work, we will reformulate the description of the SVP based on the above mutual Chern-Simons gauge theory. Since the gauge fluctuations beyond the
mean-field level can be more faithfully incorporated and treated in this Lagrangian approach, the present formalism will be more suitable for describing the transitions from the SVP to, e.g., the SC phase, as well as the non-linear effects like magnetic field dependence of magnetization, etc.

B. Description of Spontaneous Vortex Phase

The SVP is defined as the holon condensed phase, in which the holong field can be decomposed as \( h_I = \)

\[
L_h = \sum_I (n^h_i - \bar{n}^h) + \frac{u}{2} \sum_I \left( n^h_i - \bar{n}^h \right)^2 - \cos \theta_I \sum_{I(\alpha = \bar{x} \bar{y})} (\hat{\alpha} - A_0^\alpha(I) - eA^\alpha_0(I))
\]

\[
\simeq \sum_I (n^h_i - \bar{n}^h)^2 + 2t_b \bar{n}^h \sum_{I(\alpha = \bar{x} \bar{y})} \left( \Delta_\alpha \theta_I - A_0^\alpha(I) - eA^\alpha_0(I) - 2\pi N_\alpha(I) \right)^2
\]

in which \( A^\alpha_\mu \) is the external electromagnetic gauge vector and \( \bar{n}^h = \delta \). Note that in obtaining the last line of the above expression, the following Villain approximation has been used in the partition function

\[
e^{-\frac{\pi}{2} \cos \theta_I - A^\alpha_\alpha(I) - eA^\alpha_0(I)} \simeq \text{const.} \times \sum_{\{N_\alpha \in \mathbb{Z}\}} e^{-\frac{\pi}{2} \theta_I - A^\alpha_\alpha(I) - eA^\alpha_0(I) - 2\pi N_\alpha(I)}
\]

at large \( \gamma \) or low temperature. Consequently the resulting effective holon Lagrangian becomes quadratic in \( A^\alpha_\mu \).

Since the mutual Chern-Simons term is linear in \( A^\alpha_\mu \) while the spinon part is independent of \( A^\alpha_\mu \), one can thus integrate out \( A^\alpha_\mu \) in the total partition function to obtain the effective dual Lagrangian \( L^d_{\text{h}} \) dual as:

\[
L^d_{\text{h}} = \frac{u}{2 \pi^2} \sum_i \left[ B^b(i) - \pi \bar{n}^h \right]^2 + \frac{1}{4\pi^2 t_b \bar{n}^h} \sum_{i, \alpha} E^b_{\alpha}(i)^2
\]

\[
-2t_b \sum_{i, \mu, \nu} A^\mu_{\nu} J^\nu_{\alpha} \bar{J}^\alpha_{\mu} - \frac{e^2}{\pi} \sum_i \epsilon^{\mu \nu \tau} A^e_{\mu} \Delta^e_{\nu} A^b_{\tau}
\]

where \( J^\nu_{\alpha} \equiv \epsilon^{\alpha \beta} \Delta^e_{\beta} N_{\beta} \in \mathbb{Z} \) is the temporal component of the current for the vortices of the holon condensate and \( J^\nu_{\alpha} = -\epsilon^{\alpha \beta} \partial_\beta \Delta^e_{\beta} \) denotes the spatial component, which satisfy the conservation equation \( \partial_\nu J^\nu_{\alpha} + \Delta_{\alpha} J^\nu_{\alpha} = 0 \).

The above procedure leading to Eq. (8) is similar to the standard boson-vortex duality transformation. Here the field strengths of the Maxwell gauge field \( A^\mu_\alpha \) are represented by \( B^b(i) = \epsilon^{\alpha \beta} \Delta^e_{\beta} A^b_\beta(i) \) and \( E^b_{\alpha}(i) = \partial_\beta A^b_\beta(i) - \Delta_{\alpha} A^b_\alpha(i) \) and the Gaussian fluctuations of \( B^b(i) - \pi \bar{n}^h \) and \( E^b_{\alpha}(i) \) are determined by the Maxwell terms in Eq. (8). Note that the integration over \( A^\alpha_0(I) \) results in the same constraint as in Eq. (8): \( B^b(i) = \pi n^b_i \) which will be further constrained to \( \pi \bar{n}^h \) in the limit of \( u \to \infty \) when no density fluctuations are allowed. Due to such a background “magnetic” field \( B^h \simeq \pi \bar{n}^h \), the spin state as governed by \( L_s \) will be driven into a spin liquid phase at finite doping concentration \( \bar{n}^h > 0 \) as opposed to an AFLRO ground state at half-filling.

The effective Lagrangian \( L^d_{\text{h}} \) dual also describes the vortices of the holon condensate as “charge” \( \pm 2 \) particles interacting with the Maxwell gauge field \( A^b_\mu \) via the minimal coupling \( A^b_{\mu} J^\nu_{\alpha} \). By contrast, the spinons in \( L_s \) also act as the source of the gauge field \( A^b_\mu \) with a charge \( \sigma = \pm 1 \). This corresponds to the fact that in the original holon language, each vortex with \( J^\nu_{\alpha} = \pm 1 \) has a phase winding \( \pm 2\pi \), while each spinon carries a half-vortex with a phase winding \( \pm \pi \), known as the spinon-vortex. Energetically one expects that the charge-2 vortices cost more than the charge-1 spinon-vortices and thus, to leading order approximation, the former excitations can be neglected unless when the temperature is very close to the upper boundary of the SVP where the phase winding starts to lose rigidity.

But before dropping the term of \( A^b_{\mu} J^\nu_{\alpha} \) from \( L^d_{\text{h}} \) dual, one has to be careful about the case when a \( \pm 2 \) vortex is bound to a \( \pm \pi \) spinon-vortex, resulting a \( \pm \pi \) vortex which is energetically the same as the original spinon-vortex. Generally, including both \( L_s \) and \( L^d_{\text{h}} \) dual, the total charge coupled to \( A^b_\mu(i) \) is given by \( q^\text{tot}_0 = n^b_i - n^b_i + 2J^\nu_{\alpha}(i) \). Consequently, the total charge can be \( q^\text{tot}_0 = -1 \) if there is an up spinon co-existent with an anti-vortex of the holon condensate at site \( i \), i.e., \( n^b_i = 1 \) \( n^b_i = 0 \) and \( J^\nu_{\alpha}(i) = -1 \). Similarly a down spinon can carry a \( q^\text{tot}_0 = +1 \) total charge by binding with an vortex with \( J^\nu_{\alpha}(i) = 1 \). By defining \( \Phi_1 \) and \( \Phi_i \) as the rising and lowering operators of the holon vortices, with the commutation

\[ \sqrt{n^b_i e^{i\theta I}} \]. Noting that the amplitude fluctuation \( n^b_i \) of \( h_I \) is gapped, with the phase fluctuation \( \theta_I \) as the most relevant mode, the Lagrangian \( L_h \) in Eq. (2) can be approximately reexpressed, up to a constant, as

\[ L_h = \sum_I (n^h_i - \bar{n}^h)^2 - 2t_b \bar{n}^h \sum_{I(\alpha = \bar{x} \bar{y})} \left( \Delta_\alpha \theta_I - A_0^\alpha(I) - eA^\alpha_0(I) - 2\pi N_\alpha(I) \right)^2 \]
relations \( [\Phi_i, J_0^{\alpha \sigma}(j)] = \Phi_i \delta_{ij} \) and \( \Phi_i^\dagger \Phi_i = 1 \), \( \Phi_i, \Phi_i^\dagger = 0 \), the creation operators of the above-discussed spinon-vortex bound states can be written as \( b_i^\dagger \Phi_i \) and \( b_i \Phi_i^\dagger \). Then, at each site, there will be 4 states that carry the minimal total topological charge \( q^{\text{tot}} \equiv \pm 1 \):

\[
\begin{align*}
b_i^\dagger |0\rangle &= |i \uparrow \rangle, \quad b_i |0\rangle &= |i \downarrow \rangle, \\
b_i^\dagger \Phi_i |0\rangle &= |i \uparrow - \rangle, \quad b_i \Phi_i^\dagger |0\rangle &= |i \downarrow + \rangle
\end{align*}
\]

where the sign \( \pm \) denotes that of \( q^{\text{tot}} \). The energy of each state above is composed of the superfluid vortex energy and the spinon excitation. Since the superfluid energy only depends on the total dual charge, \( |q^{\text{tot}}| = 1 \), the four types of spinon-vortices in Eq. (9) are degenerate in energy. In other words, the spinon-vortex states defined in Eq. (9), rather than the two-component spinons \( b_i \sigma \) and the holon vortex field \( J^{\alpha \sigma}_v \), describe the true low-energy spin/vortex excitations in the SVP. Similar conclusion has been also reached previously in the different mean-field (Ginzburg-Landau) approach for the SVP.

Based on the above discussion, by introducing \( (\bar{b}_i, b_i) \) to stand for the two new spinon-vortex states in Eq. (9) and neglect independent \( \pm 2\pi \) vortices, the low-lying effective theory of the SVP in the mutual Chern-Simons gauge theory description can be finally written down as

\[
L_{\text{eff}} = \tilde{L}_s + \tilde{L}^h_{\text{dual}}
\]

where

\[
\tilde{L}_s = \sum_{i \sigma} b_i^{\dagger \upsilon} \left[ \partial_0 - iA_0^\upsilon(i) \right] b_i^{\upsilon} + \sum_{i \sigma} \tilde{b}_i^{\dagger \upsilon} \left[ \partial_0 + iA_0^\upsilon(i) \right] \tilde{b}_i^{\upsilon} - \frac{J}{2} \sum_{(ij) \sigma} \Delta_{ij} \left( \left( b_i^{\upsilon} b_j^{\sigma - \upsilon} e^{i\sigma A_i^\upsilon + \tilde{b}_i^{\sigma} \tilde{b}_j^{\sigma - \upsilon} e^{-i\sigma A_i^\upsilon} + h.c. \right) + \frac{1}{2} \mu_B B^\upsilon \sum_{i \sigma} \left( \tilde{b}_i^{\sigma} b_i^{\upsigma} + \tilde{b}_i^{\upsigma} b_i^{\sigma} \right) \right)
\]

\[
\tilde{L}^h_{\text{dual}} = \frac{u}{2\pi^2} \sum_i \left( B^h(i) - \pi \tilde{n}^h \right)^2 + \frac{1}{4\pi^2 t_h n_h} \sum_{i,\sigma} E^h_{\alpha}(i)^2 - \frac{i}{\pi} \sum_{i,\sigma} e^{i\sigma \tau} A_\mu \Delta_{\alpha} A^\mu_\tau.
\]

in which the Zeeman coupling between the spinon and external magnetic field \( B^\upsilon \) is also included, with \( g \) as the Lande g-factor and \( \mu_B \) the Bohr magneton. Such an effective theory describes 4-flavor spinons minimally coupled to the Maxwell gauge field. There will be two possible phases as the consequences: a confined phase in which all spinon-vortices form short-range neutral pairs and a deconfined phase in which they are free. Physically, the former corresponds to the superconducting phase as discussed in Ref.2 and the latter is a spin liquid and at the same time a vortex liquid, which corresponds to the SVP to be further explored in this work.

III. MEAN-FIELD APPROXIMATION AND PHYSICAL PROPERTIES

A. Mean-field Approximation

The physical properties of the SVP is described by the effective Lagrangian (10), in which \( \tilde{L}_s \) in Eq. (11) determines the spinon degrees of freedom minimally coupled to the gauge field \( A_\mu^h \), while the latter is governed by \( \tilde{L}^h_{\text{dual}} \) in Eq. (12). Note that the external electromagnetic field \( A_\mu^h \) does not directly couple to the spinons except for the Zeeman coupling, indicating that the spinons indeed do not carry electromagnetic charge. On the other hand, \( A_\mu^c \) and \( A_\mu^h \) are coupled by a mutual Chern-Simons term in Eq. (12), implying that the spinons do carry superfluid vortices, which will become clear in the following mean-field solution.

\( \tilde{L}^h_{\text{dual}} \) indicates the following mean-field solution for \( A^h_\alpha \):

\[
\tilde{B}^h(i) = \pi \tilde{n}^h, \quad \tilde{E}^h_{\alpha}(i) = 0
\]

while \( \tilde{A}^h_\alpha \) couples to the external magnetic field by \( -iN \xi^h \tilde{A}^h_\alpha B^\upsilon \). To the leading order approximation, the fluctuations of \( A^h_\alpha \) and \( A^h_\beta \) around \( \tilde{A}^h_\alpha \) and \( \tilde{A}^h_\beta \) may be neglected here if the temperature is not very close to \( T_c \) or \( T_v \), namely, the phase boundaries. When \( T \) approaches \( T_c \), for example, the long-range interaction between spinons mediated via the gauge field \( A^h_\mu \) will become important and the fluctuation of \( E^h_{\alpha} \) in \( \tilde{L}^h_{\text{dual}} \) can no longer be omitted. We shall leave the discussion of the phase boundaries to the next section. In the following, we first focus on the physical consequences of the SVP at the mean-field level, where the spinon-vortices proliferate and the long-range interaction between them is well screened.

The effective Lagrangian (13) is then reduced to a mean-field Lagrangian

\[
L_{\text{eff}}^{MF} = \tilde{L}_s (\Delta^s, \tilde{A}^h_\mu, \lambda) - iN \frac{\xi^h}{\pi} \tilde{A}^h_\alpha B^\upsilon
\]
which can be diagonalized to \( L_{\text{eff}}^{\text{MF}} = \sum_{\sigma} \left[ \gamma_{\sigma}^+ (\partial_x + (E_0 + E_{l\text{MF}}) \gamma_{\sigma} + \gamma_{\sigma}^+ \{ \partial_y + (E_0 + E_{l\text{MF}}) \gamma_{\sigma} \} \right] \) constant by the standard Bogoliubov transformation. 

\[
\begin{align*}
    b_\sigma &= \sum_m w_{m\sigma}(i) \left( u_m \gamma_{m\sigma} - v_m \gamma_{m\sigma} \right) \\
    \bar{b}_\sigma &= \sum_m \bar{w}_{m\sigma}(i) \left( u_m \bar{\gamma}_{m\sigma} - v_m \bar{\gamma}_{m\sigma} \right)
\end{align*}
\]  

(15)

with

\[
E_{m\sigma} = E_m + \sigma \left( \frac{g \mu_B B^e}{2} - i \bar{A}_0^h \right)
\]

\[
\bar{E}_{m\sigma} = E_m + \sigma \left( \frac{g \mu_B B^e}{2} + i \bar{A}_0^h \right)
\]

(16)

\[
E_m = \sqrt{\lambda^2 - \xi_m^2}
\]

(17)

The wave function \( \bar{w}_{m\sigma}(i) = w_{m\sigma}^*(i) \) and \( w_{m\sigma}(i) \) satisfies the Schrödinger equation 

\[
-\frac{\hbar^2}{2} \Delta^s \sum_{j=m}\epsilon_{ij} e^{i \bar{\alpha}_j} \bar{w}_{m\sigma}(j) = \xi_m w_{m\sigma}(i).
\]

By minimizing \( L_{\text{eff}}^{\text{MF}} \) with regard to the parameters \( \lambda, \Delta^s \), and \( \bar{A}_0^h \), the following self-consistent equations are obtained

\[
1 - \delta = \frac{1}{N} \sum_{i,\sigma} \left\langle b_{i\sigma}^\dagger b_{i\sigma} + \bar{b}_{i\sigma}^\dagger \bar{b}_{i\sigma} \right\rangle
\]

(18)

\[
\Delta^s = \sum_{\sigma} \left\langle b_{j-\sigma} e^{i \bar{\alpha}_j} \bar{b}_{i\sigma} + \bar{b}_{j-\sigma} e^{-i \bar{\alpha}_j} b_{i\sigma} \right\rangle
\]

(19)

\[
-\frac{B^e a^2}{\phi_0} = \frac{1}{N} \sum_{i,\sigma} \left\langle b_{i\sigma}^\dagger b_{i\sigma} - \bar{b}_{i\sigma}^\dagger \bar{b}_{i\sigma} \right\rangle
\]

(20)

the first two of which self-consistently determine the average spinon number and the RVB order parameter, and the last equation describes the polarization of the total vorticity of the spinon-vortices by the external magnetic field. Here and below, \( \phi_0 = \frac{\hbar c}{2 e} \) stands for the flux quanta. Once \( \lambda, \Delta^s \), and \( \bar{A}_0^h \) are decided self-consistently, physical properties of the SVP can be calculated straightforwardly.

### B. Magnetization

The total magnetization can be decided by

\[
M_{\text{tot}} = \frac{1}{V} \frac{\partial F}{\partial B^e}
\]

(21)

with \( F \) the mean-field free energy

\[
F = -k_B T \ln \left[ \int d[b] d[b^*] \exp(-\int_0^\beta L_{\text{eff}}^{MF} d\tau) \right]
\]

and \( V = Na^2 d \) the average volume per CuO₂ layer (\( d \) is the interlayer spacing). From Eqs. (14) and (11), the total magnetization can be then expressed explicitly as

\[
M_{\text{tot}} = \frac{e}{\pi a^2} i \bar{A}_0^h - \frac{g \mu_B}{a^2} \langle S^z \rangle \equiv M_{\text{dia}} + M_{\text{para}}
\]

(22)

in which \( M_{\text{dia}} \propto i \bar{A}_0^h \), \( M_{\text{para}} \propto -\langle S^z \rangle \) stand for the orbital diamagnetism from the vortices and the paramagnetism from the Zeeman coupling, respectively, where \( i \bar{A}_0^h \) is decided by the self-consistent equations (13). Compared to the conventional vortex liquid theory, the origin of both the diamagnetism and paramagnetism here is intrinsically related to the same spin degrees of freedom, which is a unique feature of the present mutual Chern-Simons theory.

The magnetic field and temperature dependence of the total magnetization at different doping concentrations as well as the diamagnetism part \( M_{\text{dia}} \) at \( \delta = 0.125 \) are shown in Fig. 1 based on the above mean-field theory. The contour plot of \( M_{\text{dia}} \) in the temperature and doping space at \( B^e = 2 \) Tesla is presented in Fig. 2. Note that the diamagnetism disappears at \( \delta \to 0 \) where the density of the condensate vanishes, as well as at \( \delta \to x_{\text{RVB}} \simeq 0.25 \) (see Ref 7) where the RVB pairing vanishes with the proliferation of the unpaired spins at \( T = 0 \). The magnitude of diamagnetism we obtained is comparable to the experimental observations \( 8 \) in the weak-field region, but over-estimated under strong field \( B^e \sim H_{c2} \), where the upper phase boundary of the SVP is reached and the mean-field approximation is not applicable.

The paramagnetic magnetization determines the spin paramagnetic susceptibility \( \chi_s = M_s/B^e \) whose temperature dependence at \( \delta = 0.125 \) and \( B^e = 2 \) Tesla is shown in Fig. 3. \( \chi_s \) exhibits a prominent “spin gap” behavior with reducing temperature below 100 K similar to the pseudogap behavior in the zero field limit. Furthermore a Curie-type upturn emerges at lower temperature within the “spin gap”, which counts for the contribution from the free moments in the vortex cores. According to Eq. (15), the total vorticity density satisfies \( n_+ - n_- = B^e a^2/\phi_0 \), and at low temperature when the thermal excitations of spinons are suppressed, one finds \( n_- \simeq 0, n_+ \simeq B^e a^2/\phi_0 \), which means there are on average \( n_s = B^e a^2/\phi_0 \) unpaired spinons per site. At weak magnetic field \( B^e \ll \phi_0/ a^2, n_s \ll 1 \) and the unpaired spinons are very dilute and thus are nearly-independent to each other. Consequently, a Curie-type spin susceptibility \( \chi_s = n_s/k_B T \) is expected as shown in Fig. 4. Experimentally, the NMR spin-lattice relaxation rate for the in-plane oxygen nuclear spins, which is related to \( \chi_s \) due to the hyperfine coupling constant, does show a Curie-type temperature behavior in the vortex cores in the superconducting phase\( 11,12 \), before it eventually decreases and vanishes at very low temperature which is presumably due to the Kondo screening effect of the free moments by the nodal quasiparticles.
FIG. 1: The $B^c$ dependence of the total magnetization $M = M_{\text{dia}} + M_{\text{para}}$ at various doping concentrations: (a) $\delta = 0.078$, (b) $\delta = 0.125$, (c) $\delta = 0.188$; (d) the diamagnetism $M_{\text{dia}}$ at $\delta = 0.125$. The parameters in the mean-field theory are chosen as $a = 5.5\AA$, $d = 7.7\AA$, $J = 120\text{meV}$.

FIG. 2: The contour plot of the diamagnetism $M_{\text{dia}}$ (in units of $\text{A/m}$) under a weak field $B^c = 2T$.

C. Magneto-Resistivity

The charge resistivity is non-vanishing in the SVP, although the holon condensation is still present. The origin of the dissipation is due to the flow of the spinon-vortices, whose response to an external electric field can be seen by taking the classical equation of motion for $A^h_\alpha$ in the dual Lagrangians, [11] and [12]. $\frac{\partial L_\alpha}{\partial A^h_\alpha} = 0$, which leads to the spinon-vortex current:

$$J^x_\alpha = i \frac{\partial L_\alpha}{\partial A^h_\alpha} = \frac{e}{\pi} \epsilon_{\alpha\beta} E^h_\beta + i \left( \frac{1}{2\pi^2 \hbar} \sigma_\alpha E^h_\alpha - \frac{e}{2\pi} \epsilon_{\alpha\beta} \Delta^h B^h \right)$$

(23)
In a uniform stationary state, the last two terms vanish and thus one obtains
\[ J_{sv}^\alpha = \frac{\epsilon_{\alpha\beta} c E_\beta^s}{\phi_0} \] (24)

after recovering the full units.

In a similar way, the equation of motion for \( A^s_{\alpha} \) based on the original Lagrangian leads to the following relation
\[ J_{h}^\alpha = -\frac{\epsilon_{\alpha\beta} c E_\beta^h}{\phi_0} \] (25)

where \( J_{h}^\alpha \equiv ie\frac{\partial A^h}{\partial \phi_0} \) is the holon charge current which decides the strength of ”electric” field \( E_\beta^h \) for the gauge field \( A^h_{\beta} \).

Since the spinon-vortices directly see \( A^h \) in the Lagrangian, there is generally a linear response relation between \( E_\beta^h \) and \( J_{sv}^\alpha \):
\[ J_{sv}^\alpha = \sigma_{\alpha\beta}^{sv} E_\beta^h \] (26)

where \( \sigma_{\alpha\beta}^{sv} \) denotes the spinon-vortex conductivity. By combining (24)-(26) and noting that the holon current is equal to the electric current in the mutual Chern-Simons theory, one finally obtains the electric resistivity as follows
\[ \rho_{\alpha\beta} = -\left( \phi_0/\epsilon \right)^2 \epsilon_{\alpha\gamma} \sigma_{\gamma\delta}^{sv} \epsilon_{\delta\beta} \]

If one uses a simple semi-classical approximation by expressing the spinon-vortex conductivity as \( \sigma_{\alpha\beta}^{sv} \approx \frac{2}{\eta_0} \delta_{\alpha\beta} \), with \( n_v \) and \( \eta_0 \) the total vortex number and viscosity, respectively, then the charge resistivity in the SVP is given by
\[ \rho = \rho_{xx} = \frac{n_v}{\eta_0} \left( \frac{\phi_0}{\epsilon} \right)^2 \] (27)

The resistivity in is similar to the flux-flow resistivity in a Type II superconductor except that \( n_v \) in general is not simply proportional to the external magnetic field \( B^c \). Namely, in the SVP the spinon-vortices can be spontaneously (thermally) generated with \( n_v \neq 0 \), such that \( \rho \neq 0 \) even at \( B^c = 0 \). The resistivity \( \rho(B^c) \) can be expanded as
\[ \rho(B^c) = \rho(0) \left[ 1 + \gamma B^c + o(B^c) \right] \] (28)

where the odd power terms of \( B^c \) vanish due to the symmetry \( \rho(B^c) = \rho(-B^c) \). Suppose that the dependence of the viscosity \( \eta_0 \) on \( B^c \) is negligible, then the quadratic coefficient \( \gamma \) can be expressed as
\[ \gamma = \frac{\rho(B^c) - \rho(0)}{\rho(0) B^c} \approx \frac{n_v(B^c) - n_v(0)}{n_v(0) B^c}. \] (29)

The spinon-vortex density \( n_v \) is determined in the mean-field approximation by
\[ n_v = \frac{1}{N} \sum_{m,\sigma} \left\langle \gamma^\dagger_{m\sigma} \gamma_{m\sigma} + z^\dagger_{m\sigma} z_{m\sigma} \right\rangle \]
\[ = \frac{1}{N} \sum_{m,\sigma} \left( \frac{1}{e^{\beta E_{m\sigma}} - 1} + \frac{1}{e^{\beta E_{m\sigma}} - 1} \right) \] (30)

With Eqs. (29) and (30), the coefficient \( \gamma_\parallel \) and \( \gamma_\perp \), with the external magnetic field \( B^c \) perpendicular and parallel to the 2D plane, respectively, can be calculated numerically as shown in Fig. 4. An important prediction of the present theory, as shown by Fig. 4 is that \( \gamma_\parallel \) is comparable to \( \gamma_\perp \) in the SVP. This is a rather unusual case for a vortex-flow-induced resistivity, since normally the in-plane vortices are always created by the perpendicular magnetic field in a Type II superconductor, where the vortex-flow-induced resistivity only exhibits field-dependent magneto-resistivity for the component of \( B^c \) which is perpendicular to the plane. Experimentally, the c-axis resistivity shows an insulating behavior in the pseudo-gap phase until \( T \sim T_c \) at low doping, which implies that the interlayer quantum coherence is not important. Thus a magnetic field parallel to the ab plane is not expected to contribute significantly to the in-plane resistivity based on a conventional flux-flow picture, which would predict \( 0 \approx \gamma_\parallel \ll \gamma_\perp \).

But in the present theory, vortices are tied to the free spinons in the SVP. Since the latter can be created by the Zeeman term with the external magnetic field pointing at any direction, the former can thus be created by the in-plane field as well, although the total vorticity of the 2D orbital supercurrents still satisfies the constraint in which \( B^c \) should be replaced by \( B^c_{\perp} \). Although the present mean-field result of \( \rho \) may not be expected to be quantitatively accurate in view of possible corrections from the fluctuations, the existence of an anomalous transverse magneto-resistivity with \( \gamma_\parallel \) comparable to \( \gamma_\perp \) remains a peculiar prediction based on the mutual Chern-Simons theory, which is qualitatively consistent with the experimental results for the underdoped YBCO.
D. Nernst Effect and Vortex Entropy

The Nernst effect refers to a transverse electric field $E_y$ induced by applying a temperature gradient $-\nabla_x T$. The Nernst coefficient is defined as

$$e_y = \frac{E_y}{-\nabla_x T}. \quad (31)$$

One important mechanism that can lead to a significant Nernst signal is the flux flow in the vortex liquid phase, the contribution of which can be calculated by using Eq. (24). Here the vorticity current driven by a temperature gradient $-\nabla_x T$ is

$$J_{\alpha}^v = \nu_{\alpha\beta}^v (-\nabla_\beta T) \quad (32)$$

which, combined with Eq. (24), decides the Nernst coefficient

$$e_y = \frac{\phi_0}{c} \frac{J_{\sigma}^v}{-\nabla_x T} = \frac{\phi_0}{c} \nu_{xx}^v \quad (33)$$

The vortex thermo-conductivity formula (22) provides a systematic way to calculate the Nernst coefficient. As in the last section, we can introduce a simple drift approximation usually used in a vortex liquid phase to obtain a leading order estimation of the Nernst effect in the SVP, and leave the more microscopic calculation to future works.

If the vortices drift with a velocity $v_x$ under the temperature gradient, the vortex current $J_{\sigma}^v$ can be approximated by $J_{\sigma}^v \approx (n_+ - n_-)v_x = \frac{\phi_0}{c\eta_s} v_x$, in which the last equality comes from vorticity constraint (20), and the magnetic field is understood as applied in the perpendicular direction in the following. The velocity $v_x$ can be decided by the equation $-s_\phi \nabla_x T = \eta_s v_x$, in which $s_\phi$ is the transport entropy per vortex, and $\eta_s$ is the same vortex viscosity as in the resistivity formula (27). Thus we have $\nu_{xx}^v = B^c s_\phi / \phi_0 \eta_s$, which leads to the Nernst signal

$$e_y = \frac{B^c s_\phi}{c \eta_s} \quad (33)$$

According to Eqs. (21) and (27), the viscosity $\eta_s$ can be eliminated by the ratio

$$S_\phi = \frac{\phi_0 e_y}{c \rho} \equiv \frac{B^c s_\phi}{\phi_0 \eta_v} \quad (34)$$

which has the dimension of entropy and relates the transport entropy and the density of vortices to the observables $e_y$ and $\rho$. For the conventional Abrikosov vortex liquid, one has $B^c = \phi_0 n_v$ and thus $S_\phi = s_\phi$, as usually used in the analysis of experiments. However, in the present SVP, both $n_+$ and $n_-$ are non-vanishing such that generally $S_\phi < s_\phi$. In the mean-field theory, $s_\phi$ can be estimated by $s_\phi = S/n_v$, in which $S$ is the entropy density of the spinons, and the numerical results are presented in Fig. 5 at different magnetic fields. Note that such an estimation does not distinguish the transport entropy and those which do not contribute to the transport (such as the vortex configuration entropy), and may generally lead to an overestimate of $s_\phi$ and thus $S_\phi$. Furthermore, $S_\phi$ should be further reduced due to the vortex-pinning effect at low temperature in the superconducting phase, which is not included in the present mean-field theory. Experimentally, the Nernst effect in the SVP has been studied systematically in Ref. 13, the Nernst signal and resistivity are measured for the underdoped La$_{1.90}$Sr$_{0.08}$CuO$_4$, which shows $e_y \approx 7 \mu V/K$ and $\rho \approx 2 \times 10^{-6} \Omega \cdot m$ at $H = 10T$, $T = 25K$. Taking the c-axis lattice constant $d \approx 13A$ for LSCO, we obtain $S_\phi \approx 0.7k_B$, which is about one order of magnitude smaller than our estimation. Nevertheless, Fig. 5 provides an upper bound of the spinon transport entropy, which drives the spinons downward the temperature gradient.

IV. PHASE BOUNDARIES OF THE SPONTANEOUS VORTEX PHASE

In the last section, the mean-field approximation of the mutual Chern-Simons gauge theory is applied to describe the bulk properties of the spontaneous vortex phase. The fluctuations beyond the mean-field solution will become important when the phase boundary of the SVP is considered.

The lower phase boundary of the SVP at low temperature corresponds to the superconducting phase transition. With reducing temperature, the number of thermally excited vortices in the SVP decreases and the screening of the vortex interaction becomes weakened. Eventually at $T \to T_c^+$ a vortex-antivortex confining transition will take place and the system becomes SC phase coherent, which will be discussed firstly below.
based on the mutual Chern-Simons theory.

A. Superconducting transition

In order to study the superconducting phase transition, let us start with the general effective Lagrangian of the SVP. For simplicity, we shall consider the limit \( u \to \infty \) which means that the holon density fluctuations are not important and can be neglected. In this limit the phase transition will be solely driven by vortices as expected. Under such an approximation, the spatial component \( A^h_0 \) cannot fluctuate and is constrained by

\[
B^h = \pi \tilde{n}^h. \tag{35}
\]

Thus the only important dynamical variable will be the temporal component \( A^h_0 \).

By integrating out \( b_{i\sigma}, \tilde{b}_{i\sigma} \) in Lagrangian (10),

\[
e^{-S_{\text{eff}}[A^0_0]} \equiv \int D[b_{i\sigma}] D[\tilde{b}_{i\sigma}] \exp \left[ -\int_0^\beta d\tau L_{\text{eff}} \right] \tag{36}
\]

one obtains a classical action for \( A^h_0 \) in uniform external field \( \tilde{E}/(\nabla \times A^0) = B^e \) and \( E^e = 0 \)

\[
S_{\text{eff}}[A^0_0, B^e] = \beta \left( F_s[A^0_0, B^e] - i \frac{\epsilon}{\pi} \sum_i A^0_0(i) B^e \right). \tag{37}
\]

In which \( e^{-\beta F_s} \equiv \int D[b_{i\sigma}] D[\tilde{b}_{i\sigma}] \exp \left( -\int_0^\beta d\tau L_{\text{eff}} \right) \)

Under the standard RPA approximation, \( F_s \) can be expanded to the quadratic order of \( A^0_0 \) based on Eq. (11):

\[
F_s[A^0_0, B^e] \simeq -i A^0_0(q = 0) N_v + \frac{1}{2} \sum_q \chi_q(B^e) A^0_0(q) A^0_0(-q) \tag{38}
\]

in which the total vortex number \( N_v = \sum_i \langle n_v(i) \rangle \) with \( n_v(i) \equiv \sum_\sigma \langle b^\dagger_{i\sigma} b_{i\sigma} - \tilde{b}^\dagger_{i\sigma} \tilde{b}_{i\sigma} \rangle \) and the susceptibility \( \chi_q \) is defined by

\[
\chi_q = \frac{1}{N} \int_0^\beta \sum_{i,j} e^{i q(r_i - r_j)} \left( \langle n_v(i, \tau) n_v(j, 0) \rangle - \langle n_v(i) \rangle \langle n_v(j) \rangle \right) \tag{39}
\]

which is equal to the static spin susceptibility according to the spinon-vortex binding. Note that here the dynamic part of \( A^0_0(\iota \omega_n) \) for \( \omega_n \neq 0 \) is omitted. By further using the expansion

\[
\chi_q \simeq \chi_0 \left( 1 + \lambda_0 q^2 \right) \tag{40}
\]

in the long-wavelength limit, one finds

\[
F_s[A^0_0, B^e] \simeq -i A^0_0(q = 0) N_v + \frac{\chi_0}{2} \left( \sum_i (A^0_0(i))^2 + \lambda_0 \sum_{i\alpha} (\Delta^0_{\alpha A} A^0_0)^2 \right). \tag{41}
\]

However, by noting that the total vortex number \( N_v \) is an integer and is conserved in the system described by Lagrangian (11), the exact free energy \( F_s \) should be a periodical function of \( A^0_0(i) \) in the following sense

\[
\exp \left\{ -\beta F_s[A^0_0 + \frac{2\pi}{\beta}, B^e] \right\} = \exp \left\{ -\beta F_s[A^0_0, B^e] \right\} \tag{42}
\]

Consequently, the correct form of \( F_s \) satisfying (42) as well as (41) in the weak limit of \( A^h_0(i) \) should have the compact version

\[
F_s[A^0_0, B^e] \simeq i \frac{\epsilon}{\pi} \sum_i A^0_0 B^e + \frac{1}{2} \chi_0 \lambda_0 \sum_{i\alpha} (\Delta^0_{\alpha A} A^0_0)^2 - \frac{1}{\beta^2 \chi_0} \sum_i \cos \left( \beta A^0_0(i) \right) \tag{43}
\]

in which the self-consistent equation (13) for \( N_v \) is used, and \( \chi_0, \lambda_0 \) can be calculated based on Eq. (13).

Finally, combining Eqs. (37) and (43), and redefining \( \beta A^0_0(i) = \phi(r_i) \), we find that \( S_{\text{eff}} \) reduces to a sine-Gordon action

\[
S_{\text{eff}} \simeq \int d^2 r \left\{ \frac{K}{2} (\nabla \phi)^2 - 2y \cos \phi \right\} \tag{44}
\]

with

\[
K \equiv \beta^{-1} \left( \frac{1}{2 \pi^2 t_h \tilde{n}^h} + \chi_0 \lambda_0 \right), \tag{45}
\]

\[
y \equiv \frac{\chi_0(B^e)}{2 \beta}. \tag{46}
\]

Such a sine-Gordon effective action describes the fluctuations beyond the mean-field theory at low temperature where the holon density fluctuations are negligible and holon 2\( \pi \) vortices are not important.

In the following, we discuss the superconducting phase transition based on this action at low temperature, where the phase boundary of the SVP is characterized by the superconducting temperature \( T_c \) and melting magnetic field \( H_m \).

First, consider the case at \( B^e = 0 \). If \( \chi_0 \) is negligible (the suppression of spin fluctuations) and \( y \simeq 0 \), the
Kosterlitz-Thouless (KT) phase transition temperature is decided by the universal value

\[ K = \frac{1}{8\pi} \Rightarrow T_{\text{KT}} = \frac{\pi}{4} t_h h^h \]  

which reproduces the conventional KT physics of the holon condensate when the long-wavelength spin fluctuations are absent. Note that \( T_{\text{KT}} \) in this limit is only 1/4 of the usual XY model with the same stiffness \( t_h h^h \) because the spinon vortices are \( \pi \)-vortices instead of conventional \( 2\pi \) vortices and the latter are neglected at low temperature.

However, the spin susceptibility \( \chi_0 \), although suppressed at low temperature in the present pseudogap phase, can be greatly enhanced at the temperature comparable to the spin gap energy (cf. Fig. 3), which can then suppress the phase transition temperature from the universal value \( T_{\text{KT}} \) if the latter is higher than the characteristic spin gap scale. To the linear order of \( y \) and \( K \), the equation of critical line on the \( K - y \) plane can be written as

\[ y = 2(1 - 8\pi K) \]  

which results in

\[ T_c = \frac{1}{\chi_0(T_c) \left( \frac{1}{4} + 8\pi \lambda_0 \right) + \frac{4}{\pi t_h h^h}} \]  

This is a self-consistent equation for \( T_c \) as \( \chi_0(T) \) is strongly temperature dependent as shown in Fig. 3. The numerical solution of the superconducting transition temperature is presented in Fig. 4 whose value is comparable with the experimental results. By contrast, the corresponding unrenormalized \( T_{\text{KT}} \) is shown in the same figure for comparison. It is should be noted that in the cuprate superconductors, the superconducting transition can deviate from a pure 2D KT transition due to quantum fluctuations (e.g., the dynamic fluctuations of \( A_0^2 \)) as well as the interlayer coupling which may lead to a 3D critical behavior near \( T_c \). Nevertheless, the present mutual Chern-Simons theory clearly illustrates that the low-lying spin correlations play a crucial role in determining the temperature scale of the phase coherence.

Finally, the number of spinon-vortices can be significantly induced by the magnetic field via the Zeeman coupling as discussed in Sec. III, which in turn leads to an enhancement of the low energy spin fluctuations as indicated by the low-temperature Curie behavior shown in Fig. 3. Then it is expected that \( T_c \) get quickly reduced by such an enhancement of the low-lying spin fluctuations via the magnetic-field dependence of \( \chi_0 \). Such a suppression of \( T_c \) by magnetic field is naturally included in the self-consistent equation. The critical ("melting") magnetic field \( H_m \) defined by \( T_c(H_m) = 0 \) marks a quantum KT transition, where the vortices melt and proliferate at \( T = 0 \) due to quantum fluctuations. \( H_m \) is plotted as a function of \( \delta \) in Fig. 7 with \( T_c(H_m) = 0.01J \) in the numerical calculation.
B. Upper Phase Boundary

The upper phase boundary of the SVP is defined by the temperature and magnetic field scales $T_v$ and $H_{c2}$, respectively, at which the holon condensation disappears and the effective theory (10) breaks down. As has been discussed in the Ginzburg-Landau description in Ref. [11], such a phase boundary can be estimated by the core touching of spinon vortices, which completely destroy the short range phase coherence of the holon condensate. For completeness, in the following we provide the estimation of $T_v$ and $H_{c2}$ in the effective theory (10), which is qualitatively consistent with the Ginzburg-Landau result but quantitatively different.

The core touching condition can be simply written as:

$$n_v = \delta$$  (50)

which means the number of vortices is comparable to that of holons. Since we are now focusing on the upper phase boundary where the spinon-vortices proliferate, the phase fluctuations discussed in Sec. IV A is no longer important. Thus we can use the mean-field equation (30) of Sec. III C to determine the number of spinon-vortices. The temperature scale $T_v$ and magnetic field scale $H_{c2}$ of the upper phase boundary are numerically obtained as shown in Fig. 6 and 7 respectively. The value of $T_v$ is reasonable as compared to the experiment, whereas $H_{c2}$ is about one order of magnitude larger, possibly due to the omission of the holon amplitude fluctuations in the dual theory (10), which are crucial near the upper phase boundary.

V. CONCLUSION AND DISCUSSIONS

In conclusion, we have constructed an effective field theory description of the spontaneous vortex phase, based on the mutual Chern-Simons theory of the doped Mott insulator. By introducing a dual transformation in the holon Lagrangian, the spontaneous vortex phase, which corresponds to a holon condensate without the superconducting phase coherence, is described as a spinon-vortex liquid, in which strong superconducting fluctuations are determined by the quantum spinon-vortex dynamics. Consequently, both the residual diamagnetism and spin paramagnetism can be calculated from the same spin free energy under mean-field approximation, and the entropy of each vortex can be estimated in this theory, which are compared with the results of Nernst experiments. The key difference between the SVP described in the mutual Chern-Simons theory and the other proposals of vortex liquid phase is the quantum and spinful nature of vortices, which leads to a closed relation between charge and spin properties. As shown in Sec. III C, the magneto-resistance under transverse magnetic field ($H \parallel ab$ plane) is of comparable size as that under the perpendicular field $H_{c1}$, which shows that even the in-plane magnetic-field can change the vortex number due to the spin Zeeman effect. Another prediction from the spinon-vortex picture is the existence of spin Hall effect in the SVP, which suggests that the transverse vortex flow under external electric field $E_x$ and magnetic field $B_z$ carries a spin current, as has been discussed in Ref. [24].

When considering the physics in the neighborhood of superconducting phase transition, fluctuations beyond the mean-field approximation should be included, which leads to a KT type critical theory with coupling constants decided by the spinon-vortex correlation functions. The transition temperature $T_v$ calculated from this critical theory is shown to be comparable to the experiments. The magnetic field $H_{c2}$ needed to kill superconductivity can be calculated in a similar way. The upper phase boundary $T_v$ and $H_{c2}$ are crossover temperature and magnetic field scales where the vortex number becomes comparable to the holon number, and the short range phase coherence of holons is finally destroyed. In the global phase diagram of mutual Chern-Simons theory, the SVP is a wide fluctuation region on top of the superconductivity dome, both of which are embedded in the upper pseudogap phase with short-range antiferromagnetic correlations.

There are also several issues that are not included in the present theory: i) the amplitude fluctuation of holon is not included, so that the suppression of holon superfluidity under strong magnetic field is underestimated, which leads to an overestimation of high-field diamagnetism and $H_{c2}$. ii) the fermionic quasiparticle is described as a bound state of holon and spinon, which is well-defined at low energy, long wavelength regime when spinon excitation is gapped. But in the SVP the spinon is unconfined and the quasiparticle will decay into spinon and holon, just like the usual RPA collective mode merging into the particle-hole continuum. Consequently, the contribution of quasiparticle is not important in the SVP, although it is important for some low energy features in the superconducting phase.

In the present work, the onset doping of superconductivity is $\delta = 0$, which is a consequence of ignoring quantum fluctuations of the holon density. Future work is needed to derive a more accurate critical theory for the superconducting phase transition— both the quantum one in the doping axis and the classical one in the temperature axis.

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