Comments on the nonlinear Schrödinger equation

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Abstract

A proof is given that if the nonlinear Schrödinger wave function is constrained to have support over only a finite volume in configuration space, then the total energy is bounded from below for either sign of the logarithmic term in the Hamiltonian. It is concluded that the usual assumption about the sign of the logarithmic term made by Bialynicki-Birula and Mycielski is not the only possibility, and that a sensible theory can be made with the opposite sign as well.

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1 Introduction

A generalization of Schrödinger’s equation to include a nonlinear logarithmic potential was first proposed by Bialynicki-Birula and Mycielski [1]. In this work, only one sign of the logarithmic term was considered, as the other sign led to a Hamiltonian which was not bounded from below. Soliton solutions were exhibited in [1], and it was proved that the Hamiltonian was bounded from below for one sign choice.

A logarithmic term was also proposed long ago within the context of stochastic quantum mechanics [2,3]. In [2] the logarithmic term could have either sign. In [3] a speculative stochastic-electromagnetic model of Schrödinger’s equation was proposed which led to the existence of the logarithmic term with a positive sign (opposite in sign to [1]), which could be interpreted as a diffusion force. Weinberg [4] also considered the implications of a general class of nonlinear Schrödinger equations including the logarithmic one.

The nonlinear Schrodinger equation considered here is:

\[
\left[ -\frac{\hbar^2}{2m}\Delta + V + kT\ln(\psi^*\psi) \right] \psi = i\hbar\frac{\partial \psi}{\partial t} \tag{1}
\]

Here \(k\) is Boltzman’s constant, \(T\) has units of temperature, and \(v_0\) is an arbitrary constant volume to make the dimensions of the logarithm argument dimensionless. Only the single particle equation is considered, as the generalization to other Hamiltonians is straightforward. Note that if we add a multiplicative

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constant term inside the logarithm, it has no physical effect as it just adds a constant to the Hamiltonian. Thus changing $v_0$ just adds a different constant to the Hamiltonian. To conform with [1] $T$ would be negative, and to conform with [2,3] it could be positive.

There has been much theoretical and experimental analysis looking for evidence of a nonlinear term. Lamb shift calculations [1] led to a limit of $|kT| < 4 \times 10^{-10}$eV for it. Shimony [5] proposed an experiment using coherent thermal neutron interferometry. These experiments were performed by Shull et al.[6,7] with the result that $|kT| < 3.4 \times 10^{-13}$eV, improving the bound by three orders of magnitude. Gähler et al. [8] measured the Fresnel diffraction of coherent thermal neutrons at a sharp edge and were able to lower the bound to $|kT| < 3.3 \times 10^{-15}$eV. The theoretical analysis used in these coherent neutron papers ignored the combined temporal and spatial incoherence of the thermal neutron beam, and so their validity might be questioned considering the nonlinear interaction term. This shortcoming was pointed out in [8]. However, an analysis of the fully incoherent case which is not presented here for brevity has concluded that the coherent approximations and the results claimed in [5-8] appear to be justified. Therefore, it seems that the logarithmic term, if it exists at all, is rather small. It is important nevertheless to continue the search for a nonlinear term, because any such term would help resolve the deep conceptual problems which the various interpretations of quantum theory pose. If Schrödinger’s equation is exactly linear then this is hard to explain in a stochastic model of quantum mechanics where extra terms tend to appear.

Gisin [9,10] has further argued that if there is a nonlinear term of any kind in Schrödinger’s equation, then signals could in principle be sent faster than the speed of light, leading to a causality dilemma. Gisin’s argument seems sound, but the subject is somewhat challenging conceptually, and it is possible that some modification in Gisin’s assumptions might allow a logarithmic nonlinearity to peacefully coexist with causality.

The logarithmic nonlinearity has a number of unique properties that make it the best candidate for a nonlinear correction to Schrödinger’s equation aside from the causality issue [2].

Property 1 - The total integrated force caused by the logarithmic term vanishes.

Property 2 - The total integrated torque about any center caused by the logarithmic term also vanishes.

Property 3 - If a wave function satisfies (1) then any constant times that wave function also yields a solution. The normalization adjustment simply shifts the logarithmic term by a constant energy factor which can be ignored.

Property 4 - The logarithmic perturbation preserves factorization properties since the log of a product is a sum of logs. So factorizing solutions can still be found for Coulomb potentials in spherical coordinates for example. Also, in multiparticle systems, it allows the particles to be independent of one another.

Property 5 - A plane wave is still an exact solution to the wave equation, although linear superposition no longer holds.

Properties 1 and 2 are necessary in order to agree with the classical corre-
spondence limit. If the nonlinear term led to a net force or torque this would be apparent with large objects. Properties 3 and 4 are desirable for a probability interpretation.

2 The Hamiltonian Bounds

The quantum average of the Hamiltonian is

\[ H = \int \psi^* \left[ -\frac{\hbar^2}{2m} \Delta + V + kT \ln(\psi^* \psi v_0) \right] \psi d^3x \]  

(2)

Bialynicki-Birula and Mycielski [1] reject the positive sign for \( T \) because the integrated Hamiltonian for (1) is not bounded from below in this case. To see why this is, consider a wave function which is very constant and slowly varying in the absence of any external force. Then the kinetic term is insignificant and we have that

\[ H = kT \int \rho \ln(\rho v_0) d^3x, \quad \rho = \psi^* \psi \]  

(3)

This expression is clearly not bounded from below for positive \( T \) as we see below. The integrand in (3) is bounded from below however since we have

\[ f(\rho) \equiv \rho \ln(\rho v_0), \quad \frac{df}{d\rho} = 1 + \ln(\rho v_0) = 0 \text{ at the extremum} \]  

(4)

and therefore

\[ \rho \ln(\rho v_0) \geq -\frac{1}{e v_0} \]  

(5)

This must be true at the extremum, no matter what units we had chosen. Now suppose that the wave function is constrained to have support over only a finite volume. This volume could be large, say the size of the observable universe. No matter how large it is, the Hamiltonian is then bounded from below by:

\[ H \geq -kT \frac{\text{volume}}{e v_0} \]  

(6)

A stronger and better bound can be derived since \( \rho \) satisfies a normalization condition

\[ \int_{\text{volume}} \rho d^3x = 1 \]  

(7)

We can use the calculus of variations to minimize \( f \) subject to the constraint (7). To achieve this we use a Lagrange multiplier \( \lambda \). Let \( S \) denote the functional to be rendered extremal.
\[ S = \int_{volume} \left[ \rho \ln \left( \frac{\rho v_0}{\text{volume}} \right) + \lambda \left( \rho - \frac{1}{\text{volume}} \right) \right] d^3x \] (8)

The solution to the extremum problem is simply that \( \rho \) is independent of \( x \) and therefore \( \rho = 1/\text{volume} \) and so

\[ H \geq -kT \ln \left( \frac{\text{volume}}{v_0} \right) \] (9)

So we see from (9) that if the volume tends to infinity, the Hamiltonian is not bounded from below, but it is bounded as long as the volume is constrained to be a finite value.

The requirement that the wave function be constrained to have support over at most only a fixed finite volume is a very weak one. Certainly if wave functions are allowed to have support only over the observable universe then this would not have any adverse consequences for any quantum mechanical calculations. No experimental predictions of quantum mechanics would be in any way be affected by such a constraint. And therefore for all practical purposes the Hamiltonian is bounded from below for both signs of \( T \), and the positive sign for the logarithmic term should not be dismissed out of hand as has been done up till now. The negative sign was shown in [1] to have soliton solutions. The positive sign for the logarithmic term does not have soliton solutions. It causes a slightly faster spreading of wave packets than quantum mechanics would predict. It has the interpretation of a diffusion force.

### 3 Other Nonlinear Models

Other nonlinear forms have been proposed within the context of stochastic quantum mechanics such as the stochastic electrodynamic calculation in [11] where a phase space model based on the theory of electron diffusion in gases is presented and applied to the problem of an electron subject to classical zero point radiation. There it was argued that the Schrödinger equation is a good approximation to the diffusion equations. The leading correction term in the Hamiltonian has the form

\[ H_1(\rho) = -b \left[ \frac{1}{\rho} \nabla^2 \rho + \nabla^2 \left[ \frac{1}{\rho} \nabla^2 \rho - \frac{1}{\rho^2} (\nabla \rho)^2 \right] \right] \] (10)

where \( b \) is a constant. Remarkably properties 1, 2, 3, and 5 are still true for this nonlinear form, but property 4 (separability) fails. The contribution to the Lamb shift was also estimated for this nonlinear form [11]. A difficulty for this type of model was pointed out in [12] where the spectral absorption of a Hydrogen atom was found not to have sharp line structure.
4 Conclusion

The possibility of a logarithmic term in Schrödinger’s equation with a positive sign, opposite to that chosen by Bialynicki-Birula and Mycielski should not be ruled out as unphysical when considering further experiments to look for evidence of a nonlinear term.

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