Weak forms of topological and measure-theoretical equicontinuity: relationships with discrete spectrum and sequence entropy

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Abstract. We define weaker forms of topological and measure-theoretical equicontinuity for topological dynamical systems, and we study their relationships with sequence entropy and systems with discrete spectrum. We show that for topological systems equipped with ergodic measures having discrete spectrum is equivalent to \(\mu\)-mean equicontinuity. In the purely topological category we show that minimal subshifts with zero topological sequence entropy are strictly contained in diam-mean equicontinuous systems; and that transitive almost automorphic subshifts are diam-mean equicontinuous if and only if they are regular (i.e. the maximal equicontinuous factor map is one–one on a set of full Haar measure). For both categories we find characterizations using stronger versions of the classical notion of sensitivity. As a consequence, we obtain a dichotomy between discrete spectrum and a strong form of measure-theoretical sensitivity.

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1. Introduction
A topological dynamical system (TDS), \((X, T)\), is a continuous action \(T\) on a compact metric space \(X\). The dynamical behaviour of these systems can range from very rigid
to very chaotic. Equicontinuity represents predictability. A TDS is equicontinuous if the family \( \{ T^i \} \) is equicontinuous or, equivalently, if whenever two points \( x, y \in X \) are close, then \( T^i(x), T^i(y) \) stay close for all \( i \). The prototype for an equicontinuous TDS is a rotation on a compact abelian group, and it is well known that any transitive equicontinuous TDS is topologically conjugate to such a rotation. Sensitive dependence on initial conditions (sensitivity) is considered a weak form of chaos. Auslander and Yorke [4] showed that a minimal TDS is either sensitive or equicontinuous. In [11], Fomin introduced a weaker form of equicontinuity called mean-L-stable (or mean equicontinuity), which requires that if \( x, y \in X \) are close, then \( T^i(x), T^i(y) \) stay close for many \( i \).

A classical result of Halmos and von Neumann [21] states that an ergodic measure-preserving transformation (MPT) \( T \) has discrete spectrum if and only if it is measure-theoretically isomorphic to a rotation \( S \) on a compact abelian group; here, the measure on the group is the Haar probability measure, and the spectrum refers to the spectrum of the operator induced by \( T \) on \( L^2 \).

Consider the hybrid situation of a TDS that is also an MPT, i.e. a continuous map \( T \) on a compact metric space \( X \) endowed with a Borel probability measure \( \mu \) such that \( T \) preserves \( \mu \). Physical models of systems at very low temperatures, like quasicrystals, can be modelled by TDSs with discrete spectrum [20]. If an ergodic TDS \( T \) has discrete spectrum, it is natural to ask how much of the equicontinuity of a rotation, as a TDS, must be preserved by the isomorphism between \( T \) and the rotation.

Gilman [14, 15] introduced a notion of \( \mu \)-equicontinuity for cellular automata and later Huang et al [24] introduced a different definition of \( \mu \)-equicontinuity (which under some conditions are equivalent; see [12]) and showed that \( \mu \)-equicontinuous systems have discrete spectrum. We introduce a weakening of both \( \mu \)-equicontinuity and mean equicontinuity that we call \( \mu \)-mean equicontinuity and we show that if \( (X, \mu, T) \) is an ergodic system, then it has discrete spectrum if and only if it is \( \mu \)-mean equicontinuous (Corollary 39).

For this result, we make use of Kushnirenko’s characterization of MPTs with discrete spectrum as those with zero measure-theoretical sequence entropy [30].

We also define \( \mu \)-mean sensitivity and we show that ergodic topological systems are either \( \mu \)-mean equicontinuous or \( \mu \)-mean sensitive. This implies that every ergodic TDS \( (X, \mu, T) \) is either \( \mu \)-mean sensitive or has discrete spectrum (Corollary 39). These results can be interpreted in two different ways. First, that systems with discrete spectrum are predictable in the sense that they are \( \mu \)-mean equicontinuous; and that systems that do not have pure discrete spectrum are chaotic in the sense that they are \( \mu \)-mean sensitive.

As a corollary, we can develop a notion of sensitivity for purely measure-preserving transformations. We show that an ergodic MPT is either measurably sensitive or has discrete spectrum (Theorem 41).

We may ask if some of these results hold at the topological level. The topological version of the Halmos–von Neumann theorem states that for transitive TDSs, equicontinuous maps can be characterized as those with topological discrete spectrum, i.e. the induced operator on \( C(X) \) has discrete spectrum (see e.g. [39]). It is easy to see that any equicontinuous TDS has zero topological entropy. Similar to the measure-theoretical
sequence entropy, one can define topological sequence entropy. A null system is a TDS that has zero topological sequence entropy. It is well known that equicontinuity implies nullness, but the converse is false [18]. Nevertheless, one can ask if there is a sense in which every null TDS is ‘nearly’ equicontinuous. Indeed, in the minimal case there is. Any TDS has a unique maximal equicontinuous factor [3], and Huang et al [23] showed that for any minimal null TDS \((X, T)\), the factor map from \(X\) to its maximal equicontinuous factor is one–one on a residual set (i.e. \((X, T)\) is almost automorphic). We strengthen this result for subshifts in Corollary 67, by showing that the factor map is one–one on a set of full Haar measure (i.e. \((X, T)\) is regular).

In order to establish Corollary 67, we introduce another weak topological form of equicontinuity that we call diam-mean equicontinuity (stronger than mean equicontinuity). We show that for a minimal TDS, nullness implies a form of diam-mean equicontinuity (Corollary 66) and that an almost automorphic subshift is diam-mean equicontinuous if and only if it is regular (Theorem 54).

In conclusion, for minimal subshifts we have the following implications:

\[
\text{Top. discrete spectrum } = \text{ equicontinuity} \\
\quad \implies \text{nullness} \\
\quad \implies \text{diam-mean equicontinuity} \\
\quad \implies \text{mean equicontinuity} \\
\quad \implies \mu\text{-mean equicontinuity (for every ergodic measure }\mu) \\
\quad = \text{ every ergodic measure has discrete spectrum.}
\]

In §5, we explain how these results can be generalized to amenable semigroup actions.

Mean equicontinuous \(\mathbb{Z}\)-systems were recently studied by Li et al [31]. They independently obtained Theorem 8 and they proved that if \((X, T)\) is mean equicontinuous and transitive and \(\mu\) is ergodic, then the system has discrete spectrum. This was an open question from [37]. This result can also be obtained with Corollary 39.

2. Topological dynamical systems

A \(\mathbb{G}\)-topological dynamical system (\(\mathbb{G}\)-TDS) is a pair \((X, T)\), where \(X\) is a compact metric space, \(\mathbb{G}\) is a semigroup, and \(T := \{T^i : i \in \mathbb{G}\}\) is a \(\mathbb{G}\)-continuous action on \(X\). If \(\mathbb{G} = \mathbb{Z}_+^d\), we simply say that \((X, T)\) is a TDS. In §§2–4, we use \(\mathbb{G} = \mathbb{Z}_+^d\). All the results hold for countable discrete abelian actions; some are more general: see §5 for details.

The metric and \(\varepsilon\)-closed balls on a compact metric space \(X\) will be denoted by \(d\) and \(B_\varepsilon(x)\), respectively.

Mathematical definitions of chaos have been widely studied. Most of them require the system to be sensitive. A TDS \((X, T)\) has sensitive dependence on initial conditions (or is sensitive) if there exists \(\varepsilon > 0\) such that for every open set \(A \subset X\) there exist \(x, y \in A\) and \(i \in \mathbb{G}\) such that \(d(T^i x, T^i y) > \varepsilon\). On the other hand, equicontinuity represents predictable behaviour. A TDS is equicontinuous if \(T\) is an equicontinuous family. Auslander and Yorke showed that a minimal TDS is either sensitive or equicontinuous [4]. A problem with this classification is that equicontinuity is a strong property and not adequate for subshifts; a subshift is equicontinuous if and only if it is finite (see for example [13]).
2.1. Mean equicontinuity and mean sensitivity. In this section, we define mean equicontinuity and mean sensitivity and we adapt Auslander and Yorke’s dichotomy to this setup.

Definition 1. Let \( S \subset G \). We denote with \( F_n \) the \( n \)-cube \([0, n]^d\). We define the lower density of \( S \) as
\[
\underline{D}(S) := \liminf_{n \to \infty} \frac{|S \cap F_n|}{|F_n|}
\]
and the upper density of \( S \) as
\[
\overline{D}(S) := \limsup_{n \to \infty} \frac{|S \cap F_n|}{|F_n|}.
\]

The following properties are easy to prove and will be used throughout the paper.

Lemma 2. Let \( S, S' \subset G, i \in G, \) and \( F \subset G \) be a finite set. We have that:

- \( \underline{D}(S) = \underline{D}(i + S) \) and \( \overline{D}(S) = \overline{D}(i + S) \);
- \( \underline{D}(S) + \overline{D}(S') = 1 \);
- if \( \underline{D}(S) + \overline{D}(S') > 1 \), then \( S \cap S' \neq \emptyset \);
- \( \underline{D}(S) := \liminf_{n \to \infty} \frac{|S \cap F_n \setminus F|}{|F_n |} \);
- \( \overline{D}(S) := \limsup_{n \to \infty} \frac{|S \cap F_n \setminus F|}{|F_n |} \).

Definition 3. Let \((X, T)\) be a TDS. We say that \( x \in X \) is a mean equicontinuous point if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( y \in B_\delta(x) \), then
\[
\overline{D}(i \in G : d(T^i x, T^i y) > \varepsilon) < \varepsilon
\]
(equivalently, \( \underline{D}(i \in G : d(T^i x, T^i y) \leq \varepsilon) \geq 1 - \varepsilon \)). We say that \((X, T)\) is mean equicontinuous (or mean-L-stable) if every \( x \in X \) is a mean equicontinuous point. We say that \((X, T)\) is almost mean equicontinuous if the set of mean equicontinuity points is residual.

Mean equicontinuous systems were introduced by Fomin [11]. They have been studied in [2, 4, 31, 37].

Using the fact that a continuous function on a compact set is uniformly continuous, we will see that \((X, T)\) is mean equicontinuous if and only if it is uniformly mean equicontinuous, i.e. for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( d(x, y) \leq \delta \), then \( \overline{D}(i \in G : d(T^i x, T^i y) > \varepsilon) < \varepsilon \) (see Remark 10).

Definition 4. We denote the set of mean equicontinuity points by \( E^m \) and we define
\[
E^m_\varepsilon := \{ x \in X : \exists \delta > 0 \forall y, z \in B_\delta(x), \underline{D}(i \in G : d(T^i y, T^i z) \leq \varepsilon) \geq 1 - \varepsilon \}.
\]

Note that \( E^m = \bigcap_{\varepsilon > 0} E^m_\varepsilon \).

Lemma 5. Let \((X, T)\) be a TDS. The sets \( E^m, E^m_\varepsilon \) are inversely invariant (i.e. \( T^{-j}(E^m) \subseteq E^m, T^{-j}(E^m_\varepsilon) \subseteq E^m_\varepsilon \) for all \( j \in G \)) and \( E^m_\varepsilon \) is open.
Proof. Let \( j \in \mathbb{G}, \varepsilon > 0 \), and \( x \in T^{-j}E^m. \) There exists \( \eta > 0 \) such that if \( d(T^jx, z) \leq \eta, \) then \( D[i : d(T^{i+j}x, T^iz) \leq \varepsilon] \geq 1 - \varepsilon. \) There exists \( \delta > 0 \) such that if \( d(x, y) < \delta, \) then \( d(T^jx, T^jy) < \varepsilon \) (and hence \( D[i : d(T^{i+j}x, T^iy) \leq \varepsilon] \geq 1 - \varepsilon). \) We conclude that \( x \in E^m. \) This implies that \( E^m \) is also inversely invariant.

Let \( x \in E^m \) and \( \delta > 0 \) be a constant that satisfies the property of the definition of \( E^m. \) If \( d(x, w) < \delta/2, \) then \( w \in E^m. \) Indeed, if \( y, z \in B_{\delta/2}(w), \) then \( y, z \in B_{\delta}(x). \)

Definition 6. A TDS \((X, T)\) is mean sensitive if there exists \( \varepsilon > 0 \) such that for every open set \( U \) there exist \( x, y \in U \) such that

\[
\overline{D}(i \in \mathbb{G} : d(T^ix, T^iy) > \varepsilon) > \varepsilon.
\]

Definition 7. Let \((X, T)\) be a TDS. We say that \((X, T)\) is transitive if for every open sets \( U \) and \( V \) there exists \( i \in \mathbb{G} \) such that \( T^iU \cap V \neq \emptyset. \)

We say that \( x \in X \) is a transitive point if \( \{ T^ix : i \in \mathbb{G} \} \) is dense. If every \( x \in X \) is transitive, then we say that the system is minimal.

If \((X, T)\) is transitive, then \( X \) contains a residual set of transitive points. If \( X \) has no isolated points and \((X, T)\) has a transitive point, then \((X, T)\) is transitive [4]. If \((X, T)\) is sensitive, then \( X \) has no isolated points.

It is not hard to see that mean sensitive systems have no mean equicontinuity points; as a matter of fact, we have the following dichotomies.

Theorem 8. A transitive system is either almost mean equicontinuous or mean sensitive. A minimal system is either mean equicontinuous or mean sensitive.

Proof. Let \((X, T)\) be a transitive TDS.

If \((X, T)\) is not sensitive, then, by [1], it is almost equicontinuous and hence almost mean equicontinuous.

Let \((X, T)\) be a sensitive TDS (and hence \( X \) has no isolated points). We will show that \( E^m \) is either empty or dense.

Assume that \( E^m \) is non-empty and not dense. Then \( U = X \setminus \overline{E^m} \) is a non-empty open set. Since the system is transitive and \( E^m \) is open (Lemma 5), there exists \( t \in \mathbb{G} \) such that \( U \cap T^{-t}(E^m) \) is non-empty. By Lemma 5, we have that \( U \cap T^{-t}(E^m) \subset U \cap E^m = \emptyset, \) which is a contradiction.

If \( E^m \) is non-empty for every \( \varepsilon > 0, \) then we have that \( E^m = \bigcap_{n \geq 1} E^m_{1/n} \) is a residual set and hence the system is almost mean equicontinuous.

If there exists \( \varepsilon > 0 \) such that \( E^m \) is empty, then, for any open ball \( U = B_\delta(x), \) there exist \( y, z \in B_\delta(x) \) such that \( D[i \in \mathbb{G} : d(T^iy, T^iz) \leq \varepsilon] \leq 1 - \varepsilon; \) this means that \( \overline{D}[i \in \mathbb{G} : d(T^iy, T^iz) > \varepsilon] > \varepsilon. \) It follows that \((X, T)\) is mean sensitive.

Now suppose that \((X, T)\) is minimal and almost mean equicontinuous. For every \( x \in X \) and every \( \varepsilon > 0, \) there exists \( t \in \mathbb{G} \) such that \( T^tx \in E^m. \) Since \( E^m \) is inversely invariant, \( x \in E^m \) and hence \( x \in E^m. \)

An analogous result appeared in [31] for \( \mathbb{G} = \mathbb{Z}_+. \)

It will be useful to describe mean equicontinuity in terms of the Besicovitch pseudometric.
Definition 9. We define $\Delta_\delta(x, y) := \{ i \in \mathbb{G} : d(T^i x, T^i y) > \delta \}$ and the Besicovitch pseudometric as $d_b(x, y) := \inf\{ \delta > 0 : \overline{D}(\Delta_\delta(x, y)) < \delta \}$. By identifying points that are at pseudo-distance zero, we obtain a metric space $(X/d_b, d_b)$ that will be called the Besicovitch space. The projection $f_b : (X, d) \to (X/d_b, d_b)$ will be called the Besicovitch projection. The $\varepsilon$-closed balls of the Besicovitch pseudometric will be denoted by $B^b_\varepsilon(x)$.

One can check that in fact this is a pseudometric using that $\overline{D}(S) + \overline{D}(S') \geq \overline{D}(S \cup S')$.

It is not difficult to see that if $x \in X$ is a mean equicontinuous point, then $f_b$ is continuous at $x$. This implies that the Besicovitch projection is continuous if and only if $(X, T)$ is mean equicontinuous.

Remark 10. If $(X, T)$ is mean equicontinuous, then $f_b$ is continuous and hence $f_b$ is uniformly continuous; this means that $(X, T)$ is uniformly mean equicontinuous, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) \leq \delta$ then $D(i \in \mathbb{G} : d(T^i x, T^i y) \leq \varepsilon) \geq 1 - \varepsilon$.

Remark 11. The Besicovitch pseudometric is sometimes expressed with an equivalent metric using averages. For example, if $\mathbb{G} = \mathbb{Z}_+$, then

$$d_b(x, y) := \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d(T^i x, T^i y).$$

In [6], equicontinuity with respect to the Besicovitch pseudometric (of the shift) was studied for cellular automata; this is a different property than mean equicontinuity.

It is well known that transitive equicontinuous systems are minimal. We give a similar result by weakening one hypothesis and strengthening the other.

Definition 12. A TDS $(X, T)$ is strongly transitive if for every open set $U$ there exists a transitive point $x \in U$ that returns to $U$ with positive lower density.

Theorem 13. Every strongly transitive mean equicontinuous system is minimal.

Proof. Let $x, y \in X$ and $\varepsilon > 0$. Since the system is strongly transitive, there exists a transitive point $z \in B_{\varepsilon/2}(y)$ such that $a := D\{ i : T^i z \in B_{\varepsilon/2}(y) \} > 0$. Since the system is mean equicontinuous, there exists $\delta > 0$ such that if $w \in B_\delta(x)$, then $d_b(x, w) \leq \min\{ \varepsilon/2, a \}$. There exists $t_1 \in \mathbb{G}$ such that $T^{t_1} z \in B_\delta(x)$. By Lemma 2 (third bullet), there exists $t_2 \in \mathbb{G}$ such that $T^{t_2} z \in B_{\varepsilon/2}(y)$ and $d(T^{t_2} x, T^{t_2} z) \leq \varepsilon/2$; thus, $T^{t_2} x \in B_{\varepsilon}(y)$. This means that the system is minimal.

A similar result is known for a null system (Definition 63), i.e. every Banach transitive null system is minimal [23]. It is an open question whether every transitive null $\mathbb{Z}_+$-system is minimal (see [17, 23]).

3. Measure-theoretical results

Measure-theoretical equicontinuity for TDSs with respect to Borel probability measures has been studied in [7, 12, 15, 24]. A natural question is to ask how this concept relates to other known forms of rigidity for ergodic systems, for example discrete spectrum (see Definition 37). In [24], it was shown that $\mu$-equicontinuous systems have discrete
spectrum. However, the converse is not true; for example Sturmian and regular Toeplitz subshifts (equipped with their unique ergodic measure) are not \(\mu\)-equicontinuous but have discrete spectrum.

In this section, we introduce \(\mu\)-mean equicontinuity (a measure-theoretical form of mean equicontinuity) and \(\mu\)-mean sensitivity. The main result of this section states that an ergodic TDS has discrete spectrum if and only if it is \(\mu\)-mean equicontinuous if and only if it is not \(\mu\)-mean sensitive (Corollary 39).

A \(G\)-measure-preserving transformation (\(G\)-MPT) is a triplet \((M, \mu, T)\), where \((M, \mu)\) is a standard probability space and \(T := \{T^i : i \in G\}\) is a \(G\)-measure-preserving action on \(M\). When we say that a system is ergodic, we mean that it is measure preserving and ergodic.

3.1. \(\mu\)-mean equicontinuity. We denote Borel probability measures of \(X\) by \(\mu\) and we define \(B_+^X := \{A \text{ is Borel : } \mu(A) > 0\}\).

Definition 14. Let \((X, T)\) be a TDS and \(\mu\) a (not necessarily invariant) Borel probability measure on \(X\). We say that \((X, T)\) is \(\mu\)-mean equicontinuous if for every \(\kappa > 0\) there exists a compact set \(M\) such that \(\mu(M) > 1 - \kappa\) and \(T\res M\) is mean equicontinuous, i.e. for every \(x \in M\) and every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(y \in B_\delta(x) \cap M\), then \(D(i \in G : d(T^i x, T^i y) > \varepsilon) < \varepsilon\) (this implies that \(f_{b\res M}\) is continuous).

Remark 15. By approximation arguments we could equivalently define \(\mu\)-mean equicontinuity by asking that \(M\) is simply Borel (and not necessarily compact).

This definition may remind the reader of Lusin’s theorem. In fact, we can use this to obtain information about \(\mu\)-mean equicontinuous systems.

Definition 16. Let \(X\) be a compact metric space, \(\mu\) a Borel probability measure on \(X\), and \(Y\) a metric space.

A set \(A \subset X\) is \(\mu\)-measurable if \(A\) is in the sigma-algebra generated by the completion of \(\mu\).

A function \(f : X \to Y\) is \(\mu\)-measurable if for every Borel set \(B\) we have that \(f^{-1}(B)\) is \(\mu\)-measurable.

A function \(f : X \to Y\) is \(\mu\)-Lusin (or Lusin measurable) if for every \(\kappa > 0\) there exists a compact set \(M \subset X\) such that \(\mu(M) > 1 - \kappa\) and \(f\res M\) is continuous.

It is not difficult to see that \((X, T)\) is \(\mu\)-mean equicontinuous if and only if \(f_{b}\) (see Definition 9) is \(\mu\)-Lusin.

Every \(\mu\)-Lusin function is \(\mu\)-measurable. The converse is true if \(Y\) is separable (Lusin’s theorem); this fact is generalized in the following result.

Theorem 17. (Lusin’s theorem [38, p. 63/145]) Let \(X\) be a compact metric space, \(\mu\) a Borel probability measure on \(X\), \(Y\) a metric space, and \(f : X \to Y\) a function such that there exists \(X' \subset X\) such that \(\mu(X') = 1\) and \(f(X')\) is separable. We have that \(f\) is \(\mu\)-Lusin if and only if for every open ball \(B\), \(f^{-1}(B)\) is \(\mu\)-measurable.
Remark 18. Since
\[ d_b(x, y) = \inf \left\{ \delta > 0 : \limsup_{n \to \infty} \frac{|\{ i \in \mathbb{G} | d(T^i x, T^i y) > \delta \} \cap F_n |}{|F_n|} < \delta \right\} \]
and \( \mu \) is Borel, \( d_b(x, y) \) is a Borel function. This implies that for every \( \varepsilon > 0 \) and every \( x \in X \) we have that \( B_\varepsilon(x) \) is \( \mu \)-measurable.

Proposition 19. Let \( (X, T) \) be a TDS and \( \mu \) a Borel probability measure. We have that \( (X, T) \) is \( \mu \)-mean equicontinuous if and only if there exists \( X' \subset X \) such that \( \mu(X') = 1 \) and \( (X' / d_b, d_b) \) is separable.

Proof. Define \( f := f_b \).

If there exists \( X' \subset X \) such that \( \mu(X') = 1 \) and \( (X' / d_b, d_b) \) is separable, apply Theorem 17 to obtain that \( f_b \) is \( \mu \)-Lusin and hence \( (X, T) \) is \( \mu \)-mean equicontinuous.

If \( f_b \) is \( \mu \)-Lusin, it means that for every \( \kappa > 0 \) there exists a compact set \( M_\kappa \subset X \) such that \( \mu(M_\kappa) > 1 - \kappa \) and \( f_b |_{M_\kappa} \) is continuous. This implies that \( X' = \bigcup_{n \in \mathbb{N}} M_{1/n} \) satisfies the desired conditions. \( \square \)

Under some circumstances we can describe \( \mu \)-mean equicontinuous systems using \( \mu \)-mean equicontinuity points.

Definition 20. We say that \( x \in X \) is a \( \mu \)-mean equicontinuous point if for every \( \varepsilon > 0 \),
\[ \lim_{\delta \to 0} \frac{\mu(B_\delta(x) \cap B_\varepsilon^b(x))}{\mu(B_\delta(x))} = 1. \]

We can apply [12, Theorem 16] to obtain the following result.

Theorem 21. Let \( (X, T) \) be a TDS and \( \mu \) a Borel probability measure. Consider the following properties.

1. \( (X, T) \) is \( \mu \)-mean equicontinuous.
2. Almost every \( x \in X \) is a \( \mu \)-mean equicontinuous point.
   - If \( (X, \mu) \) satisfies the Lebesgue density theorem, then (1) \( \implies \) (2).
   - If \( (X, \mu) \) is Vitali (i.e. satisfies Vitali’s covering theorem), then (2) \( \implies \) (1).

If \( X \) is a Cantor space or \( X \subset \mathbb{R}^d \), then, for every Borel measure \( \mu \), \( (X, \mu) \) is Vitali and satisfies the Lebesgue density theorem. For more information, see [12].

Measure-theoretical equicontinuity has been studied under non-invariant measures; for example the existence and ergodicity of limit measures of \( \mu \)-equicontinuous cellular automata was studied in [13]. From now on we will only study measure-preserving systems.

3.2. \( \mu \)-mean sensitivity. Measure-theoretical forms of sensitivity have been studied in [7, 15, 24]. In particular, in [24], it was shown that ergodic TDSs are either \( \mu \)-equicontinuous or \( \mu \)-sensitive.

We also show that \( \mu \)-mean equicontinuity is a measurable invariant property for a TDS (this is not satisfied by a \( \mu \)-equicontinuous TDS).
Definition 22. A TDS \((X, T)\) is \(\mu\)-mean sensitive if there exists \(\varepsilon > 0\) such that for every \(A \in \mathcal{B}_X^+\) there exist \(x, y \in A\) such that \(d_b(x, y) > \varepsilon\) (and hence \(\overline{D}[i \in G : d(T^i x, T^i y) > \varepsilon] > \varepsilon\)). In this case we say that \(\varepsilon\) is a \(\mu\)-mean sensitivity constant.

Definition 23. A TDS \((X, T)\) is \(\mu\)-mean expansive if there exists \(\varepsilon > 0\) such that \(\mu \times \mu\{(x, y) : d_b(x, y) > \varepsilon\} = 1\).

The following fact is well known. We give a proof for completeness.

Lemma 24. Let \((Y, d_Y)\) be a metric space. Suppose that there is no uncountable set \(A \subset Y\) and \(\varepsilon > 0\) such that \(d_Y(x, y) > \varepsilon\) for every \(x, y \in A\) with \(x \neq y\); then \((Y, d_Y)\) is separable.

Proof. For every \(\varepsilon > 0\) rational, we define

\[
\mathcal{F}_\varepsilon := \{A \subset Y : d_Y(x, y) > \varepsilon \forall x \neq y \in A\}.
\]

Using Zorn’s lemma, we obtain that \(\mathcal{F}_\varepsilon\) admits a maximal element \(M_\varepsilon\), which by hypothesis must be countable. Then \(M := \bigcup_{\varepsilon \in \mathbb{Q}_+} M_\varepsilon\) is also countable. We have that for every \(x \in X\) and \(\varepsilon > 0\) there exists \(y \in M\) such that \(d_Y(x, y) \leq \varepsilon\) \(\Box\)

Lemma 25. Let \((X, \mu, T)\) be an ergodic TDS. Then \(f(x) := \mu(B_b^\varepsilon(x))\) is constant for almost every \(x \in X\) and equal to \(\mu \times \mu\{(x, y) : d_b(x, y) \leq \varepsilon\}\).

Proof. By Remark 18, \(d_b(x, y)\) is \(\mu \times \mu\)-measurable. This means that \(\{(x, y) : d_b(x, y) \leq \varepsilon\}\) is \(\mu \times \mu\)-measurable for every \(\varepsilon > 0\). Using Fubini’s theorem, we obtain that

\[
\mu \times \mu\{(x, y) : d_b(x, y) < \varepsilon\} = \int_X \int_X 1_{\{(x, y) : d_b(x, y) \leq \varepsilon\}} d\mu(y) \, d\mu(x)
= \int_X \mu\{y : d_b(x, y) \leq \varepsilon\} \, d\mu(x)
= \int_X \mu(B_b^\varepsilon(x)) \, d\mu(x).
\]

Since \(f\) is \(T\)-invariant, we conclude that \(f(x)\) is constant for almost every \(x \in X\) and equal to \(\mu \times \mu\{(x, y) : d_b(x, y) < \varepsilon\}\) \(\Box\)

Theorem 26. Let \((X, \mu, T)\) be an ergodic TDS. The following are equivalent:

1. \((X, T)\) is \(\mu\)-mean sensitive;
2. \((X, T)\) is \(\mu\)-mean expansive;
3. there exists \(\varepsilon > 0\) such that for almost every \(x\), \(\mu(B_b^\varepsilon(x)) = 0\);
4. \((X, T)\) is not \(\mu\)-mean equicontinuous.

Proof. (2) \(\Rightarrow\) (1) Let \(A \in \mathcal{B}_X^+\). This means that \(A \times A \in \mathcal{B}_X^+\). By hypothesis, we can find \((x, y) \in A \times A\) such that \(\overline{D}[i : d(T^i x, T^i y) > \varepsilon] > \varepsilon\).

(1) \(\Rightarrow\) (3) Suppose that \((X, T)\) is \(\mu\)-mean sensitive (with \(\mu\)-mean sensitivity constant \(\varepsilon\)) and that (3) is not satisfied. This means that there exists \(x \in X\) such that \(B_b^\varepsilon(x) \in \mathcal{B}_X^+\). For any \(y, z \in B_b^\varepsilon(x)\), we have that \(d_b(y, z) < \varepsilon\). This contradicts the assumption that \(T\) is \(\mu\)-mean sensitive.

(3) \(\Rightarrow\) (2) Using Lemma 25, we obtain that \(\mu \times \mu\{(x, y) : d_b(x, y) \leq \varepsilon\} = 0\).
(2) ⇒ (4) If \((X, T)\) is \(\mu\)-mean expansive, then there exists \(\varepsilon > 0\) such that \(\mu \times \mu\{x, y : d_b(x, y) > \varepsilon\} = 1\). Suppose that \((X, T)\) is \(\mu\)-mean equicontinuous. This implies that there exists a compact set \(M\) such that \(\mu(M) > 0\) and \(f_b|_M\) is continuous (and hence uniformly continuous). This implies that there exists \(\delta > 0\) such that if \(x, y \in M\) and \(d(x, y) \leq \delta\), then \(d_b(x, y) \leq \varepsilon/2\). We can cover \(M\) with finitely many \(\delta/2\)-balls; this implies that one of them must have positive measure. Using this and \(\mu\)-mean expansiveness, we would obtain that there exist \(p, q \in M\) such that \(d(p, q) \leq \delta\) and \(d_b(p, q) > \varepsilon\), which is a contradiction to the continuity of \(f_b|_M\).

(4) ⇒ (3) Suppose that (3) is not satisfied. By Lemma 25, we have that for every \(n \in \mathbb{N}\) there exist a set of full measure \(Y_n\) and \(a_n > 0\) such that \(\mu(B_{1/n}(x)) = a_n\) for all \(x \in Y_n\). Let \(Y := \bigcap_{n \in \mathbb{N}} Y_n\). If \((Y / d_b, d_b)\) is not separable, then, by Lemma 24, there exist \(\varepsilon > 0\) and an uncountable set \(A\) such that for every \(x, y \in A\) such that \(x \neq y\) we have that \(B^b(x) \cap B^b(y) = \emptyset\). This is a contradiction and hence \((Y / d_b, d_b)\) is separable. Using Proposition 19, we conclude that \((X, T)\) is \(\mu\)-mean equicontinuous.

**Definition 27.** Two measure-preserving transformations, \((M, \mu, T)\) and \((M', \mu', T')\), are isomorphic (measurably) if there exists an almost everywhere bijective and measure-preserving function \(f : (M, \mu) \rightarrow (M', \mu')\) such that \(T'^i \circ f = f \circ T^i\) for every \(i \in \mathbb{G}\) and \(f^{-1}\) is also measure preserving.

We say that \((M', \mu', T')\) is a factor of \((M, \mu, T)\) if there exists a surjective and measure-preserving function \(f : (M, \mu) \rightarrow (M', \mu')\) such that \(T'^i \circ f = f \circ T^i\) for every \(i \in \mathbb{G}\).

**Proposition 28.** Let \((X, \mu, T)\) and \((X', \mu', T')\) be two ergodic topological systems. If \((X, T)\) is \(\mu\)-mean equicontinuous and \((X', \mu', T')\) is a factor of \((X, \mu, T)\), then \((X', T')\) is \(\mu'\)-mean equicontinuous.

**Proof.** We will denote by \(d\) and \(d'\) the metrics of \(X\) and \(X'\), respectively.

Suppose that \((X', T')\) is not \(\mu'\)-mean equicontinuous. By the previous theorem, we have that \((X', T')\) is \(\mu'\)-mean expansive, i.e. there exist \(\varepsilon > 0\) and a set \(Y' \subset X' \times X'\) such that for every \((x', y') \in Y'\) we have that \(d_b(x', y') > \varepsilon\) and \(\mu' \times \mu'(Y') = 1\).

Let \(f : X \rightarrow X'\) be the factor map. By Lusin’s theorem, we know that there exists a compact set \(K \subset X\) such that \(\mu(K) \geq 1 - \varepsilon/4\) and \(f |_K\) is continuous. This implies that there exists \(\varepsilon_1 > 0\) such that if \(f(x), f(y) \in f(K)\) and \(d'(f(x), f(y)) > \varepsilon\), then \(d(x, y) > \varepsilon_1\). This implies that \(\overline{D}[i \in \mathbb{G} : d'(T'^i f(x), T'^i f(y)) > \varepsilon] \geq \varepsilon\). We define

\[
E_1(x, y) := \{i \in \mathbb{G} : T^i x, T^i y \in K\}
\]

and

\[
E_2(x, y) := \{i \in \mathbb{G} : d(T^i x, T^i y) > \varepsilon_1\}.
\]

Using that \(\mu(K) \geq 1 - \varepsilon/4\) and the ergodic theorem, we have that for almost every \(x, y \in X\) we have that \((f(x), f(y)) \in Y'\) and \(\overline{D}(E_1(x, y)) \geq 1 - \varepsilon/2\). Since \(\{i \in \mathbb{G} : d'(T'^i f(x), T'^i f(y)) > \varepsilon\} \subset E_2(x, y)\), we have that \(\overline{D}(E_2(x, y)) \geq \varepsilon\). This implies that for almost every \(x, y \in X\) we have that \(d(T^i x, T^i y) > \varepsilon_1\) for every \(i \in E_1(x, y) \cap E_2(x, y)\), and that \(\overline{D}(E_1(x, y) \cap E_2(x, y)) \geq \varepsilon/2\). Hence, \((X, \mu, T)\) is \(\mu\)-mean expansive (and hence \(\mu\)-mean sensitive).

\[\square\]
3.3. \(\mu\)-mean sensitive pairs. The notion of entropy pairs was introduced in [5]. Different kinds of pairs have been studied, in particular sequence entropy pairs [25] and \(\mu\)-sensitive pairs [24]. In [24], \(\mu\)-sensitive pairs were used to characterize \(\mu\)-sensitivity; we introduce \(\mu\)-mean sensitive pairs to characterize \(\mu\)-mean sensitivity.

**Definition 29.** [25] Let \((X, \mu, T)\) be a measure-preserving TDS. We say that \((x, y)\) is a \(\mu\)-sequence entropy pair if for any finite partition \(\mathcal{P}\), such that there is no \(P \in \mathcal{P}\) such that \(x, y \in \overline{P}\), there exists \(S \subset \mathcal{G}\) such that \(h^S_{\mu}(\mathcal{P}, T) > 0\).

The following result was proven for \(\mathbb{Z}_+\)-systems in [25, Theorem 4.3], and was generalized in [27, Propositions 4.7, 4.9, and Theorem (2)].

**Theorem 30.** An ergodic TDS \((X, \mu, T)\) is \(\mu\)-null if and only if there are no \(\mu\)-sequence entropy pairs.

**Definition 31.** We say that \((x, y) \in X^2\) is a \(\mu\)-mean sensitive pair if \(x \neq y\) and for all open neighbourhoods \(U_x\) of \(x\) and \(U_y\) of \(y\), there exists \(\varepsilon > 0\) such that for every \(A \in \mathcal{B}_X^+\) there exist \(p, q \in A\) with \(\overline{D}(i \in \mathcal{G} : T^i p \in U_x\) and \(T^i q \in U_y) > \varepsilon\). We denote the set of \(\mu\)-mean sensitive pairs as \(S^m_{\mu}(X, T)\).

**Theorem 32.** Let \((X, \mu, T)\) be an ergodic TDS. Then \(S^m_{\mu}(X, T) \neq \emptyset\) if and only if \((X, T)\) is \(\mu\)-mean sensitive.

**Proof.** If \((x, y) \in S^m_{\mu}(X, T)\), then there exist open neighbourhoods \(U_x\) of \(x\) and \(U_y\) of \(y\) (with \(d(U_x, U_y) > 0\)) and \(\varepsilon > 0\) such that for every \(A \in \mathcal{B}_X^+\) there exist \(p, q \in A\) and \(S \subset \mathcal{G}\) with \(\overline{D}(S) > \varepsilon\) such that \(T^i p \in U_x\) and \(T^i q \in U_y\) for every \(i \in S\). This implies that \(\overline{D}(i \in \mathcal{G} : d(T^i x, T^i y) \geq d(U_x, U_y)) > \varepsilon\). Thus, \((X, T)\) is \(\mu\)-mean sensitive.

Let \((X, T)\) be \(\mu\)-mean sensitive with sensitive constant \(\varepsilon_0\) and \(0 < \varepsilon < \varepsilon_0\).

For \(\varepsilon > 0\), we define the compact set

\[X^\varepsilon := \{(x, y) \in X^2 \mid d(x, y) \geq \varepsilon\}.
\]

Suppose that \(S^m_{\mu}(X, T) = \emptyset\). In particular, this implies that for every \((x, y) \in X^\varepsilon\) there exist open neighbourhoods of \(x\) and \(y\), \(U_x\) and \(U_y\), such that for every \(\delta > 0\) there exists a set \(A_\delta(x, y) \in \mathcal{B}_X^+\) such that

\[\overline{D}(i \in \mathcal{G} : (T^i p, T^i q) \in U_x \times U_y) \leq \delta
\]

for all \(p, q \in A_\delta(x, y)\).

There exists a finite set of points \(F \subset X^\varepsilon\) such that

\[X^\varepsilon \subseteq \bigcup_{(x, y) \in F} U_x \times U_y.
\]

Let \(\delta = \varepsilon/|F|\). Since \(\mu\) is ergodic, for every \((x, y) \in F\) there exists \(n(x, y) \in \mathcal{G}\) such that \(A := \bigcap_{(x, y) \in F} T^{n(x, y)} A_\delta(x, y) \in \mathcal{B}_X^+\). Thus, for every \((x, y) \in F\),

\[\overline{D}(i \in \mathcal{G} : (T^i p, T^i q) \in U_x \times U_y) \leq \delta
\]

for every \(p, q \in A\).
Since $\varepsilon$ is smaller than a sensitive constant, there exist $p, q \in A$ such that
\[
\overline{D}(i \in \mathbb{G} : (T^i p, T^i q) \in X^c) > \varepsilon
\]
and hence
\[
\overline{D}\left\{i \in \mathbb{G} : (T^i p, T^i q) \in \bigcup_{(x, y) \in F} U_x \times U_y \right\} > \varepsilon.
\]
We have a contradiction, since this means that there exists $(x', y') \in F$ such that
\[
\overline{D}(i : (T^i p, T^i q) \in U_{x'} \times U_{y'}) > \varepsilon/|F| = \delta.
\]
□

The relationship between entropy (and sequence entropy) and independence was studied in [27]. The following result shows that there is a relationship between $\mu$-mean sensitive pairs and a different kind of measure-theoretical independence pairs.

**Lemma 33.** Let $(X, \mu, T)$ be an ergodic TDS. Suppose that $x \neq y$ and that for all open neighbourhoods $U_x$ of $x$ and $U_y$ of $y$, there exists $\delta > 0$ such that for every $N$ there exists $S_N \subset \mathbb{G}$, with $|S_N| \geq N$, such that for all $s_i, s_j \in S_N$ we have that $\mu(T^{-s_i}U_x \cap T^{-s_j}U_y) > \delta$. Then $(x, y) \in S^m_\mu(T)$.

**Proof.** Let $A \in \mathcal{B}_X^+$. There exist $N > 0$ and $s_1 \neq s_2 \in S_N$ such that
\[
\mu(T^{-s_1}A \cap T^{-s_2}A) > 0.
\]

Let $W := T^{-s_1}U_1 \cap T^{-s_2}U_2$. By the pointwise ergodic theorem, there exists a point $z \in T^{-s_1}A \cap T^{-s_2}A$ such that $\overline{D}(S) = \mu(W) > \delta$, where $S = \{i > s_1, s_2 | T^i z \in W\}$. Let $p := T^i z$ and $q := T^j z$. We have that $p, q \in A$ and $T^i p \in U_x$ and $T^j q \in U_y$ for every $i \in S$. This means that $(x, y) \in S^m_\mu(T)$. □

### 3.4. Discrete spectrum and sequence entropy.

#### 3.4.1. Sequence entropy.

**Definition 34.** [30] Let $(M, \mu, T)$ be a measure-preserving transformation. Given a finite measurable partition $\mathcal{P}$ of $X$ and $S = \{s_n\} \subset \mathbb{G}$, we define $h^S_\mu(\mathcal{P}, T) := \limsup_{n \to \infty} \frac{1}{n} H(\bigvee_{i=1}^n T^{-s_i} \mathcal{P})$, and the **sequence entropy** of $(X, \mu, T)$ with respect to $S$ as $h^S_\mu(T) := \sup_{\mathcal{P}} h^S_\mu(\mathcal{P}, T)$. The system is said to be $\mu$-null (or zero sequence entropy) if $h^S_\mu(T) = 0$ for every $S \subset \mathbb{G}$.

The following remarkable lemma by Kushnirenko provides a connection between entropy and functional analysis (which is in part responsible for the connection between sequence entropy and discrete spectrum; see §3.4.2).

**Lemma 35.** [30] Let $(M, \mu)$ be a probability space and $\{\xi_n\}$ be a sequence of two-set partitions of $M$, with $\xi_n = (P_n, P_n^\complement)$. The closure of $\{1_{P_1}, 1_{P_2}, \ldots\} \subset L^2(M)$ is compact if and only if for all subsequences,
\[
\lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=1}^n \xi_m) = 0.
\]
Given a measure-preserving system and a non-trivial measurable partition \( \{B, B^c\} \), we can associate a shift invariant measure \( \mu_B \) on \( \{0, 1\}^\mathbb{Z} \) as follows. We first define the function \( \phi_B : X \to \{0, 1\}^\mathbb{G} \) with \( (\phi(x))_i = 0 \) if and only if \( T_i x \in B \) and we define \( \mu_B = \phi_B \mu \).

**Theorem 36.** Let \( (X, \mu, T) \) be an ergodic TDS. If \( (X, T) \) is \( \mu \)-mean equicontinuous, then it is \( \mu \)-null.

**Proof.** We will show that there exists a factor map \( \phi_B : (X, \mu, T) \to (\{0, 1\}^\mathbb{G}, \mu_B, \sigma) \) such that \( (\{0, 1\}^\mathbb{G}, \sigma) \) is not \( \phi_B \mu \)-mean equicontinuous. By Proposition 28, we conclude that \( (X, T) \) cannot be \( \mu \)-mean equicontinuous.

Since the system is not \( \mu \)-null, there exist a two-set partition \( \mathcal{P} = (B, B^c) \) and a sequence \( S = \{s_n\} \subset \mathbb{G} \) such that \( h^\mu_\mathcal{P}(\mathcal{P}, T) > 0 \).

Now we consider the factor \( (\{0, 1\}^\mathbb{Z}, \mu_B, \sigma) \).

Define the partition \( \xi \mathbb{Z} = (\{x_i = 0\}, \{x_i = 1\}) \). By Lemma 35, we have that the closure of \( \{1_{\{x_i = 0\}}, 1_{\{x_i = 1\}}, \ldots\} \) is not compact and hence it is not totally bounded. So, there exists \( \varepsilon > 0 \) such that for every \( N \in \mathbb{N} \) there exists \( S_N \subset S \) with \( |S_N| = N \) with

\[
\mu(\{x_i = 0\} \Delta \{x_j = 0\}) = \int |1_{\{x_i = 0\}} - 1_{\{x_j = 0\}}|^2 \, d\mu \geq \varepsilon
\]

for every \( i \neq j \in S' \).

Now we want to show that \( (\{0, 1\}^\mathbb{Z}, \sigma) \) is \( \mu_B \)-mean sensitive. Let \( A \in \mathcal{B}_\{0, 1\}^\mathbb{Z} \) and \( N \) be sufficiently large. There exist \( s \neq t \in S_N \) such that

\[
\mu(\sigma^{-s} A \cap \sigma^{-t} A) > 0.
\]

Let \( W := \{x : x_s \neq x_t\} \) and \( k \in \mathbb{N} \) be such that \( s, t \in F_k \). By the pointwise ergodic theorem, there exists \( z \in \sigma^{-s} A \cap \sigma^{-t} A \) such that \( \overline{D}(S_W) = \mu(W) \), where

\[
S_W := \{i \notin F_k \mid \sigma^iz \in W\}.
\]

By (1), we have that \( \mu_B(W) > \varepsilon \); hence, \( \overline{D}(S_W) > \varepsilon \). Let \( p := \sigma^s z \) and \( q := \sigma^t z \). Since \( \mathbb{G} \) is commutative, we have that \( p_i \neq q_i \) for every \( i \in S_W \).

We conclude that for every \( A \in \mathcal{B}_\{0, 1\}^\mathbb{Z} \) there exist \( p, q \in A \) such that \( \overline{D}\{i \in \mathbb{G} : p_i \neq q_i\} > \varepsilon \). This means that \( (\{0, 1\}^\mathbb{Z}, \sigma) \) is \( \mu_B \)-mean sensitive and hence not \( \mu_B \)-mean equicontinuous. \( \square \)

### 3.4.2. Discrete spectrum.

A measure-preserving transformation is a measurable isometry if it is isomorphic to an isometry (a TDS that satisfies \( d(x, y) = d(T^i x, T^i y) \) for all \( i \)).

A measure-preserving transformation on a probability space \( (M, \mu, T) \) generates a unitary linear operator on the Hilbert space \( L^2(M, \mu) \), by \( U_T : f \mapsto f \circ T \).

**Definition 37.** We say that \( (M, \mu, T) \) has discrete spectrum if there exists an orthonormal basis for \( L^2(M, \mu) \) which consists of eigenfunctions of \( U_T \).

The Halmos–von Neumann theorem states that an ergodic system is a measurable isometry if and only if it has discrete spectrum [21].
Kushnirenko’s theorem states that an ergodic system is $\mu$-null if and only if it has discrete spectrum [30].

A $\mathbb{G}$-measure-preserving transformation on a probability space $(M, \mu, T)$ generates a family of unitary linear operators on the Hilbert space $L^2(M, \mu)$, by $U_{T^i}: f \mapsto f \circ T^i$. We say that $(M, \mu, T)$ has discrete spectrum if $L^2(M, \mu)$ is the direct sum of finite-dimensional $U_T$-invariant subspaces. Mackey showed that Halmos and von Neumann’s theorem holds for locally compact group actions [33].

Kushnirenko’s result was generalized for discrete actions by Kerr and Li [27].

**Theorem 38.** Let $(X, \mu, T)$ be an ergodic TDS. If $(X, \mu, T)$ has discrete spectrum, then it is $\mu$-mean equicontinuous.

**Proof.** This follows from the Halmos–von Neumann theorem and Proposition 28. \hfill \Box

We have seen several characterizations of $\mu$-mean equicontinuity throughout the paper; we now choose the ones we consider the most interesting.

**Corollary 39.** Let $(X, \mu, T)$ be an ergodic TDS. Then $(X, \mu, T)$ satisfies either a property on the left or on the right (which are all equivalent per column):

| $\mu$-mean equicontinuity | $\mu$-mean sensitivity |
|---------------------------|------------------------|
| discrete spectrum         | $\mu$-mean expansivity |
| $\mu$-null                | there exists a $\mu$-mean sensitive pair |

**Proof.** Apply Theorems 38, 26, 36, Kushnirenko’s theorem, and Theorem 32. \hfill \Box

In [37], it was asked if every mean equicontinuous system equipped with an ergodic measure has discrete spectrum. Corollary 39 implies that the answer of this question is positive. This question has been independently solved directly in [31] using different tools.

### 3.5. Sensitivity for measure-preserving transformations.

Using our results we can develop a notion of sensitivity for purely measure-preserving transformations.

A topological model of a measure-preserving transformation is a TDS that is isomorphic to the transformation.

**Definition 40.** Let $(M, \mu, T)$ be a $\mathbb{G}$-measure-preserving transformation. We say that $(M, \mu, T)$ is measurably sensitive if there exists a topological model that is $\nu$-mean sensitive (where $\nu$ is the image of $\mu$ under the isomorphism).

**Theorem 41.** Let $(M, \mu, T)$ be an ergodic $\mathbb{G}$-measure-preserving transformation. The following conditions are equivalent:

1. $(M, \mu, T)$ is measurably sensitive;
2. every topological model is $\nu$-mean sensitive ($\nu$ is the image of $\mu$ under the isomorphism);
3. every minimal topological model is mean sensitive;
4. $(M, \mu, T)$ does not have purely discrete spectrum.
Proof. (1) $\iff$ (2) Apply Proposition 28 and Corollary 39.
(2) $\iff$ (4) Apply Corollary 39.
(3) $\Rightarrow$ (4) If $(M, \mu, T)$ has discrete spectrum, then, by the Halmos–von Neumann theorem, there exists a minimal equicontinuous topological model.
(4) $\Rightarrow$ (3) If a minimal topological model is not mean sensitive, then it is mean equicontinuous (by Theorem 8). This would imply that the system is $\mu$-mean equicontinuous and hence has discrete spectrum.

4. Topological results
In this section, we will see that the characterizations of Corollary 39 do not hold in the topological setting.

It is known that equicontinuous systems have zero topological sequence entropy with respect to every subsequence (also known as null systems; see Definition 63) and that there exist sensitive null systems [18].

In this section, we define another weak form of equicontinuity (diam-mean equicontinuity) and another strong form of sensitivity (diam-mean sensitivity).

We will see that for (not necessarily minimal) subshifts we have the following picture:

\[
\text{mean sensitivity} \implies \text{diam-mean sensitivity} \implies \text{not null} \implies \text{sensitivity}
\]

and that each implication is strict. The implications for minimal subshifts are stated in the introduction of the paper.

We will also show that almost automorphic subshifts are regular if and only if they are diam-mean equicontinuous (Theorem 54). Since minimal null systems are almost automorphic [23], we obtain that minimal null subshifts are regular almost automorphic (Corollary 67).

4.1. Diam-mean equicontinuity and diam-mean sensitivity.

**Definition 42.** Let $A \subset X$. We denote the diameter of $A$ as $\text{diam}(A)$.

We introduce the following notion.

**Definition 43.** Let $(X, T)$ be a TDS. We say that $x \in X$ is a **diam-mean equicontinuous point** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

\[
\overline{D}\{i \in \mathbb{G} : \text{diam}(T^i B_\delta(x)) > \varepsilon\} < \varepsilon.
\]

We say that $(X, T)$ is **diam-mean equicontinuous** if every $x \in X$ is a diam-mean equicontinuous point. We say that $(X, T)$ is **almost diam-mean equicontinuous** if the set of diam-mean equicontinuity points is residual.

**Remark 44.** Equivalently, $x \in X$ is a diam-mean equicontinuous point if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\overline{D}\{i \in \mathbb{G} : \text{diam}(T^i B_\delta(x)) \leq \varepsilon\} \geq 1 - \varepsilon$. 
By adapting the proof that a continuous function on a compact space is uniformly continuous, one can show that a TDS is diam-mean equicontinuous if and only if it is uniformly diam-mean equicontinuous, i.e. for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\overline{D}\{ i \in \mathbb{G} : \text{diam}(T^i B_\delta(x)) > \varepsilon \} < \varepsilon.
\]

**Definition 45.** We denote the set of diam-mean equicontinuity points by \( E^w \) and we define
\[
E^w_\varepsilon := \{ x \in X : \exists \delta > 0, \text{s.t. } D\{ i \in \mathbb{G} : \text{diam}(T^i B_\delta(x)) \leq \varepsilon \} \geq 1 - \varepsilon \}.
\]

Note that \( E^w = \bigcap_{\varepsilon > 0} E^w_\varepsilon \).

**Lemma 46.** Let \((X, T)\) be a TDS. The sets \( E^w \) and \( E^w_\varepsilon \) are inversely invariant (i.e. \( T^{-j}(E^w) \subseteq E^w \) and \( T^{-j}(E^w_\varepsilon) \subseteq E^w_\varepsilon \) for all \( j \in \mathbb{G} \)) and \( E^w_\varepsilon \) is open.

**Proof.** Let \( \varepsilon > 0 \) and \( T^j x \in E^w_\varepsilon \). There exist \( \eta > 0 \) and \( S \subseteq \mathbb{G} \) such that \( D(S) \geq 1 - \varepsilon \) and \( d(T^{i+j} x, T^{i+j} z) \leq \varepsilon \) for every \( i \in S \) and \( z \in B_\delta(T^j x) \). There exists \( \delta > 0 \) such that if \( d(x, y) < \delta \), then \( d(T^j x, T^j y) < \eta \). We conclude that \( x \in E^w_\varepsilon \). Thus, \( T^{-j}(E^w_\varepsilon) \subseteq E^w_\varepsilon \).

Let \( x \in E^w_\varepsilon \) and take \( \delta > 0 \) a constant that satisfies the property of the definition of \( E^w_\varepsilon \).

If \( d(x, w) < \delta/2 \), then \( w \in E^w_\varepsilon \). Indeed, if \( y, z \in B_\delta(w) \), then \( y, z \in B_\delta(x) \).

**Definition 47.** A TDS \((X, T)\) is diam-mean sensitive if there exists \( \varepsilon > 0 \) such that for every open set \( U \) we have
\[
\overline{D}\{ i \in \mathbb{G} : \text{diam}(T^i U) > \varepsilon \} > \varepsilon.
\]

Other strong forms of ‘diameter’ sensitivity have been studied in [34]: the times of separation are taken to be cofinite (complement is finite) or syndetic (bounded gaps).

The proof of the following result is analogous to that of Theorem 8 (using \( E^w_\varepsilon \) instead of \( E^m_\varepsilon \)).

**Theorem 48.** A transitive system is either almost diam-mean equicontinuous or diam-mean sensitive. A minimal system is either diam-mean equicontinuous or diam-mean sensitive.

4.2. Almost automorphic systems. Let \((X_1, T_1)\) and \((X_2, T_2)\) be two TDSs and \( f : X_1 \rightarrow X_2 \) a continuous function such that \( T_2^i \circ f = f \circ T_1^i \) for every \( i \in \mathbb{G} \).

If \( f \) is surjective, we say that \( f \) is a factor map and \((X_2, T_2)\) is a factor of \((X_1, T_1)\). If \( f \) is bijective, we say that \( f \) is a conjugacy and \((X_1, T_1)\) and \((X_2, T_2)\) are conjugate (topologically).

Any TDS \((X, T)\) has a unique (up to conjugacy) maximal equicontinuous factor, i.e. an equicontinuous factor \( f_{eq} : (X, T) \rightarrow (X_{eq}, T_{eq}) \) such that if \( f_2 : (X, T) \rightarrow (X_2, T_2) \) is a factor map such that \((X_2, T_2)\) is equicontinuous, then there exists a factor map \( g : (X_{eq}, T_{eq}) \rightarrow (X_2, T_2) \) such that \( g \circ f_{eq} = f_2 \). The equivalence relation whose equivalence classes are the fibers of \( f_{eq} \) is called the equicontinuous structure relation. This relation can be characterized using the regionally proximal relation. We say that \( x, y \in X \) are regionally proximal if there exist sequences \( \{x_n\}_{n=1}^\infty \subseteq \mathbb{G} \) and \( \{y_n\}_{n=1}^\infty \subseteq \mathbb{G} \) such that
\[
\lim_{n \to \infty} x^n = x, \quad \lim_{n \to \infty} y^n = y \quad \text{and} \quad \lim_{n \to \infty} d(T^{in} x^n, T^{in} y^n) = 0.
\]
The equicontinuous structure relation is the smallest closed equivalence relation containing the regional proximal relation [3, Ch. 9].

For mean equicontinuous systems we can characterize the maximal equicontinuous factor map using the Besicovitch pseudometric (Definition 9).

**Proposition 49.** Let \((X, T)\) be mean equicontinuous. Then \(f_{eq} = f_{b}\).

**Proof.** We have that \(f_{b}\) is continuous and \((X/d_{b}, d_{b}, T)\) is equicontinuous, so, if \(f_{eq}(x) = f_{eq}(y)\), then \(f_{b}(x) = f_{b}(y)\). If \(f_{b}(x) = f_{b}(y)\), then \(d_{b}(x, y) = 0\), so there exists a sequence \(\{t_{n}\}\) such that \(\lim_{n \to \infty} d(T^{t_{n}}x, T^{t_{n}}y) = 0\); hence, \(x\) and \(y\) are regionally proximal; hence, \(f_{eq}(x) = f_{eq}(y)\).

A transitive equicontinuous system is conjugate to a system where \(G\) acts as translations on a compact metric abelian group. If \((X, T)\) is a transitive TDS, we denote the maximal equicontinuous factor by \(G_{eq}\) (since it is a group). The TDS \((G_{eq}, T_{eq})\) has a unique ergodic invariant probability measure, the normalized Haar measure on \(G_{eq}\); this measure has full support and will be denoted by \(v_{eq}\).

**Definition 50.** We say that a TDS is almost automorphic if it is an almost one–one extension of its maximal equicontinuous factor, i.e. if \(f_{eq}^{-1} f_{eq}(x) = \{x\}\) on a residual set.

A transitive almost automorphic TDS is regular if

\[
\nu_{eq}\{g \in G_{eq} : f_{eq}^{-1}(g) \text{ is a singleton} \} = 1.
\]

It is not difficult to see that if there exists a transitive point \(x \in X\) such that \(f_{eq}^{-1} f_{eq}(x) = \{x\}\), then \((X, T)\) is almost automorphic. Transitive almost automorphic systems are minimal.

Two well-known families of almost automorphic systems are the Sturmian subshifts (maximal equicontinuous factor is an irrational circle rotation) and Toeplitz subshifts (maximal equicontinuous factor is an odometer; see §4.4 for definition and examples).

An important class of TDSs are the shift systems. Let \(A\) be a compact metric space (with metric \(d_{A}\)). For \(x \in A^{G}\) and \(i \in G\), we use \(x_{i}\) to denote the \(i\)th coordinate of \(x\) and

\[
\sigma := \{\sigma^{i} : A^{G} \to A^{G} | (\sigma^{i}x)(j) = x_{i+j} \forall x \in A^{G} \text{ and } j \in G\}
\]

to denote the shift maps. Using the (Cantor) product topology generated by the topology of \(A\), we have that \(A^{G}\) is a compact metrizable space. A subset \(X \subset A^{G}\) is a general shift system if it is closed and \(\sigma\)–invariant; in this case \((X, \sigma)\) is a TDS. Every TDS is conjugate to a general shift system (by mapping every point to its orbit).

A general shift system is a subshift if \(A\) is finite.

**Remark 51.** Let \(X \subset A^{G}\) be a general shift system. We have that \((X, \sigma)\) is diam-mean equicontinuous if and only if for all \(\epsilon > 0\) there exists \(\delta > 0\) such that for all \(x \in X\) there exists \(S \subset G\) with \(D(S) \geq 1 - \epsilon\), such that if \(d(x, y) \leq \delta\), then \(d_{A}(x_{i}, y_{i}) \leq \epsilon\) for all \(i \in S\).

**Definition 52.** Let \((X, \sigma)\) be a transitive general shift system. We define

\[
D := \{g \in G_{eq} : \exists x, y \in X \text{ such that } f_{eq}(x) = f_{eq}(y) = g \text{ and } x_{0} \neq y_{0}\},
\]

where \(x_{0}\) and \(y_{0}\) represent the 0th coordinates (or the value at the identity of the semigroup) of \(x\) and \(y\), respectively.
In general, \( D \) is first category. If \( \mathcal{A} \) is finite and \( f_{eq}(x) \notin D \), there exists a neighbourhood \( U_{f_{eq}(x)} \) such that if \( f_{eq}(y) \in U_{f_{eq}(x)} \), then \( x_0 = y_0 \). So, if \((X, \sigma)\) is an almost automorphic subshift, then \( D \) is closed and nowhere dense. Also, note that \((X, \sigma)\) is regular if and only if \( \nu_{eq}(D) = 0 \).

**Lemma 53.** Let \((X, \sigma)\) be a minimal almost automorphic subshift and \( F \subset \mathbb{G} \) a finite set. Then, for all \( g \in G_{eq} \), there exists \( k \in \mathbb{G} \) such that \( T^{k+i}_{eq}(g) \notin D \) for all \( i \in F \).

**Proof.** Let \( D' := \bigcup_{i \in F} T^{-i}_{eq}(D) \). This means that \( D' \) is also closed and nowhere dense. Thus, there exists \( k \) such that \( T^k_{eq}(g) \in G_{eq} - D' \) and hence \( T^{k+i}_{eq}(g) \notin D \) for all \( i \in F \).

**Theorem 54.** Let \((X, \sigma)\) be a minimal almost automorphic subshift. Then \((X, \sigma)\) is regular if and only if it is diam-mean equicontinuous.

**Proof.** Suppose that \( \nu_{eq}(D) = 0 \). Hence, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( U := \{ x \mid d(g, D) \leq \delta \} \), then \( \nu_{eq}(U) < \varepsilon \).

Let \( x \in X \). We define \( S_x := \{ i \in \mathbb{G} \mid T^i_{eq}f_{eq}x \notin U \} \); hence, \( D(S) = 1 - \nu(U) \geq 1 - \varepsilon \) (using the pointwise ergodic theorem). For every \( i \in S_x \), we have that \( B_\delta(T^i_{eq}f_{eq}(x)) \cap D = \emptyset \). This implies that if \( d(T^i_{eq}f_{eq}(x), T^i_{eq}f_{eq}(y)) \leq \delta \), then \( x_i = y_i \). There exists \( \delta' > 0 \) such that if \( d(x, y) \leq \delta' \), then \( d(T^i_{eq}f_{eq}(x), T^i_{eq}f_{eq}(y)) \leq \delta \). We can now prove that \( x \) is a diam-mean equicontinuous point. If \( d(x, y) \leq \delta' \), then \( d(T^i_{eq}f_{eq}(x), T^i_{eq}f_{eq}(y)) \leq \delta \) for every \( i \) and hence \( x_i = y_i \) for every \( i \in S_x \). Using Remark 51, we conclude that \((X, T)\) is diam-mean equicontinuous.

Now suppose that \((X, \sigma)\) is diam-mean equicontinuous. We have that \( f_b = f_{eq} \) (Proposition 49).

Let \( \varepsilon > 0 \) and \( \delta > 0 \) be given by Remark 51.

Let \( x \in X \). Note that \( \sigma^i(x) \in f^{-1}_{eq}(D) \) if and only if there exists \( y \in f^{-i}_b \circ f_b(x) \) such that \( y_i \neq x_i \).

Using Lemma 53, there exists \( k \in \mathbb{G} \) such that \( f_{eq}(x) = f_{eq}(y) \), then \( d(\sigma^k y, \sigma^k x) \leq \delta \).

Since \((X, \sigma)\) is diam-mean equicontinuous, there exists \( S \subset \mathbb{G} \), with \( D(S) \geq 1 - \varepsilon \), such that if \( d(\sigma^k x, \sigma^k y) \leq \delta \), then \( d(x_i, y_i) \leq \varepsilon \) for all \( i \in S \).

This means that for every \( \varepsilon > 0 \),

\[
\overline{D}[i : \exists y \in f^{-1}_b \circ f_b(x) \text{ s.t. } d_{\mathcal{A}}(x_i, y_i) > \varepsilon] \leq \varepsilon.
\]

Using that

\[
D = \bigcup_{n \in \mathbb{N}} \{ g \in G_{eq} : \exists x, y \text{ s.t. } f_{eq}(x) = f_{eq}(y) = g \text{ and } d_{\mathcal{A}}(x_0, y_0) > 1/n \}
\]

and the pointwise ergodic theorem, we conclude that \( \nu_{eq}(D) = 0 \).

We do not know if every minimal diam-mean equicontinuous TDS is almost automorphic.

**Definition 55.** A TDS is **diam-mean equicontinuous** if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every \( x \in X \), we have \( \overline{D}[i \in \mathbb{G} : \text{diam}(T^i B_\delta(x)) > \varepsilon] < \varepsilon \).
It is clear that every diam-

**Definition 56.** A TDS \((X, T)\) is diam-mean sensitive if there exists \(\varepsilon > 0\) such that for every open set \(U\) we have \(D\{i \in G : \text{diam}(T^i U) > \varepsilon\} > \varepsilon\).

The proof of the following theorem is analogous to the proofs of Theorems 8 and 48.

**Proposition 57.** A minimal TDS is either diam-mean equicontinuous or diam-mean sensitive.

**Definition 58.** Let \((X, T)\) be a TDS and \(x \in X\). We denote the orbit of \(x\) with \(o_T(x)\).

**Proposition 59.** Let \((X, \sigma)\) be a transitive almost automorphic subshift. The function \(h : \mathcal{D}^c \to A\) defined as \(h(g) = (f_{eq}^{-1}(g))_0\) is continuous and there exists \(y \in X\) such that:

- is transitive;
- \(f_{eq}^{-1}f_{eq}(y)\) is a singleton, in other words, \(o_{eq}(y) \cap D = \emptyset\).

**Proof.** This follows from [9, Theorem 6.4].

**Lemma 60.** Let \((X, \sigma)\) be a transitive almost automorphic subshift, \(w = a_0 \cdots a_{n-1} \in \mathcal{A}^n\), and \(U = \{x : x_0 \cdots x_{n-1} = w\} \subset X\) a non-empty set. There exists \(p \in U\) such that \(p' = f_{eq}(p)\) is generic for \(D\) (with respect to \(v_{eq}\)) and \(\sigma_{eq}^i p' \in \mathcal{D}^c\) for \(i = 0, \ldots, n - 1\).

**Proof.** Let \(U_a = \{g \in X_{eq} : g \notin D, (f_{eq}^{-1}(g))_0 = a\}\) and \(h : \mathcal{D}^c \to A\) be the continuous function from Proposition 59. This implies that for every \(a\), \(U_a\) is an open set. Hence, \(\cap_{i=0}^{n-1} \sigma_{eq}^i U_{a_i}\) is an open set.

Let \(y \in X\) be the point given by Proposition 59. Since \(U\) is non-empty and \(y\) transitive, there exists \(z \in o_{eq}(y) \cap U\). Considering that \(o_{eq}(y) \cap D = \emptyset\), we obtain \(f_{eq}(z) \in \cap_{i=0}^{n-1} \sigma_{eq}^i U_{a_i}\). Thus, \(\cap_{i=0}^{n-1} \sigma_{eq}^i U_{a_i}\) is a non-empty open set. Since \(v_{eq}\) is fully supported, it contains a generic point for \(D\).

**Proposition 61.** Let \((X, \sigma)\) be a transitive almost automorphic subshift. If \((X, \sigma)\) is not regular, then it is diam-mean sensitive.

**Proof.** Assume that \((X, \sigma)\) is not regular. This means that \(v_{eq}(D) > 0\).

Let \(w \in \mathcal{A}^n\) and \(U = \{x : x_0 \cdots x_{n-1} = w\} \subset X\) be non-empty (these sets form a base of the topology). Let \(p \in U\) be the point given by the previous lemma. Let \(S := \{i \in G : \sigma_{eq}^i p' \in D\}\). Since \(p'\) is generic for \(D\), we have that \(D(S) = v_{eq}(D)\). Furthermore, for every \(i \in S\), there exists \(q \in X\) such that \(f_{eq}(p) = f_{eq}(q)\) and \(p_i \neq q_i\). Since \(\sigma_{eq}^i p' \in \mathcal{D}^c\) for \(j = 0, \ldots, n - 1\), we have that \(q \in U\). Hence, \((X, \sigma)\) is diam-mean sensitive.

**Corollary 62.** Let \((X, \sigma)\) be a minimal almost automorphic subshift. The following conditions are equivalent:

1. \((X, \sigma)\) is diam-mean equicontinuous;
2. \((X, \sigma)\) is not diam-mean sensitive;
(3) \((X, \sigma)\) is regular;
(4) \((X, \sigma)\) is \textit{diam-mean} equicontinuous;
(5) \((X, \sigma)\) is not \textit{diam-mean} sensitive.

\textbf{Proof.} Apply Theorem 48 to get \((1) \iff (2)\).

Apply Theorem 54 to get \((2) \iff (3)\).

By definition, \((1) \Rightarrow (4)\).

Proposition 61 implies \((5) \Rightarrow (3)\).

Proposition 57 implies \((4) \iff (5)\). \(\Box\)

We do not know if this result holds in general for TDSs.

4.3. \textit{Topological sequence entropy.} Let \(\mathcal{U}\) and \(\mathcal{V}\) be two open covers of \(X\). We define \(\mathcal{U} \vee \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}\) and \(N(\mathcal{U})\) as the minimum cardinality of a subcover of \(\mathcal{U}\).

\textbf{Definition 63.} \cite{18} Let \((X, T)\) be a TDS, \(S = \{s_m\}_{m=1}^{\infty} \subset \mathcal{G}\), and \(\mathcal{U}\) an open cover. We define

\[
h_{\text{top}}^{S}(T, \mathcal{U}) := \lim_{n \to \infty} \sup \frac{1}{n} \log N\left(\bigcup_{m=1}^{n} T^{-s_m}(\mathcal{U})\right).
\]

The \textit{topological entropy along the sequence} \(S\) \textit{is defined by}

\[
h_{\text{top}}^{S}(T) := \sup_{\text{open covers } \mathcal{U}} h_{\text{top}}^{S}(T, \mathcal{U}).
\]

A TDS is \textit{null} if the topological entropy along every sequence is zero.

\textbf{Lemma 64.} Let \(K\) be a finite set, \(\epsilon > 0\), and \(h : K \to 2^\mathcal{G}\) be such that \(D(h(k)) > \epsilon\) for every \(k \in F\). There exist \(K' \subset K\) and \(i \in \mathcal{G}\) with \(|K'| \geq \epsilon |K|/2\) such that \(i \in h(k)\) for every \(k \in K'\).

\textbf{Proof.} There exists \(n_0 \in \mathbb{N}\) such that

\[
\frac{|h(k) \cap F_{n_0}|}{|F_{n_0}|} \geq \frac{\epsilon}{2} \quad \text{for every } k \in K.
\]

This means that

\[
\sum_{j \in F_{n_0}} |k \in K : j \subset h(k)| = \sum_{k \in K} |j \in F_{n_0} : j \in h(k)| \geq \frac{\epsilon}{2} |K||F_{n_0}|.
\]

Hence, there exists \(j \in F_{n_0}\) such that \(|k \in K : j \subset h(k)| \geq (\epsilon/2)|K|\). \(\Box\)

\textbf{Theorem 65.} Let \((X, T)\) be a TDS. If \((X, T)\) is \textit{diam-mean} sensitive, then there exists \(S \subset \mathcal{G}\) such that \(h_{\text{top}}^{S}(T) > 0\).

\textbf{Proof.} Let \((X, T)\) be \textit{diam-mean} sensitive with sensitive constant \(\epsilon\). Let \(\mathcal{U} := \{U_1, \ldots, U_N\}\) be a finite open cover with balls with diameter smaller than \(\epsilon/2\).

We will define the sequence \(S^n = \{s_1, \ldots, s_n\}\) inductively with \(s_1 = 1\). For every \(n \in \mathbb{N}\), we define \(L_n := \{\bigcap_{i=1}^{n} T^{-s_i} U_v \neq \emptyset : v \in \{1, \ldots, N\}\} \) \((N\) is the size of the
cover). We denote by $L'_n := \{A_k'\}_{k \leq N(L_n)}$ a subcover of $L_n$ of minimal cardinality. We define the function $f : L'_n \to 2^\mathbb{N}$ as follows: $m \in f(A_k)$ if and only if there exist $x, y \in A_k \setminus \bigcup_{j<k} \overline{A}_j$ such that $d(T^m x, T^m y) > \epsilon$.

Assume that $S^n$ is defined. Since $(X, T)$ is diam-mean sensitive, we have that $D(f(U)) > \epsilon$ for every $U \in L'_n$. By Lemma 64, there exists $g \in 2^\mathbb{N}$ such that $\{(U \in L'_n : g \neq f(U))/N(L_n) > \epsilon/2\}$; we define $s_{n+1} := g$. The definition of $f$ implies that $(N(L_{n+1}))/N(L_n) > 1 + \epsilon/2$. Let $S^\infty := \bigcup_{n \in \mathbb{N}} S^n$. Since $N(L_n) = N(\bigvee_{i \in S^n} T^{-i}(U))$, we conclude that $h_{\text{top}}(T, U) > 0$.

**Corollary 66.** Let $(X, T)$ be a minimal TDS. If $(X, T)$ is null, then it is diam-mean equicontinuous.

**Proof.** Apply Theorem 65 and Proposition 57.

Every minimal null TDS is almost automorphic (see [22, 23]). Using the previous corollary and Corollary 62, we obtain a stronger result for subshifts.

**Corollary 67.** Let $(X, \sigma)$ be a minimal subshift. If $(X, \sigma)$ is null, then it is a regular almost automorphic subshift and hence diam-mean equicontinuous.

The converse of this result is not true (Proposition 74).

In Corollary 39, we saw that an ergodic TDS is $\mu$-null if and only if it is $\mu$-mean equicontinuous. If $(X, T)$ is mean equicontinuous and $\mu$ is an ergodic measure, then $(X, T)$ is $\mu$-mean equicontinuous and hence it has zero entropy. This implies that mean equicontinuous and diam-mean equicontinuous systems have zero topological entropy.

Surprisingly, it was shown in [31] that transitive almost mean equicontinuous subshifts can have positive entropy.

### 4.3.1. Tightness and mean distality.

**Definition 68.** Let $(X, T)$ be a TDS and $\mu$ an invariant measure. We say that $(X, T)$ is mean distal if $d_{\text{b}}(x, y) > 0$ for every $x \neq y \in X$, and we say that $(X, T)$ is $\mu$-tight if there exists $X'$ such that $\mu(X') = 1$ and $d_{\text{b}}(x, y) > 0$ for every $x \neq y \in X'$.

Mean distal systems were studied by Ornstein and Weiss [35]. They showed that any tight measure-preserving $\mathbb{Z}$-TDS has zero entropy (assuming the system has finite entropy). A $\mathbb{Z}_+$-TDS has zero topological entropy if and only if it is the factor of a mean distal TDS (see [10, 35]).

Let $(X, T)$ be a mean equicontinuous TDS. Since $f_{\text{eq}} = f_{\text{b}}$, we have that if $(X, T)$ is mean distal then $f_{\text{eq}}$ is one–one and hence $(X, T)$ is equicontinuous. So, mean equicontinuity and mean distality are both considered rigid properties, and a TDS satisfies both properties if and only if it is equicontinuous.

**Proposition 69.** A mean equicontinuous TDS is mean distal if and only if it is equicontinuous.
A measure-theoretical version of mean distality was also defined. Let \((X, T)\) be a transitive almost automorphic diam-mean equicontinuous TDS and \(\mu\) an ergodic measure. Using Theorem 54, it is not hard to see that \((X, T)\) is \(\mu\)-tight.

It is curious that \(\mu\)-tightness represents rigid motion and that \(\mu\)-mean expansiveness \((\mu \times \mu \{((x, y) : dB(x, y) > \varepsilon)\} = 1)\) represents very sensitive behaviour.

4.4. Counterexamples. The following example shows that there are mean equicontinuous not diam-mean equicontinuous TDSs. We do not have a transitive counterexample.

Example 70. Let \(X \subset \{0, 1\}^{\mathbb{Z}_+}\) be the subshift consisting of sequences that contain at most one 1. For every \(x, y \in X\), \(dB(x, y) = 0\), so \((X, \sigma)\) is mean equicontinuous. Nonetheless, for every \(\varepsilon > 0\), \(\mathcal{D}\{i \in \mathbb{Z}_+ : \exists x \in B_\varepsilon(0^\infty) \text{ s.t. } x_i = 1\} = 1\), so \(0^\infty\) is not a diam-mean equicontinuous point (even not a diam-equicontinuous point).

In Kerr and Li [26], the relationship between independence and entropy was studied. We will make use of their characterization of null systems.

Definition 71. Let \((X, T)\) be a TDS and \(A_1, A_2 \subset X\). We say that \(S \subset \mathbb{G}\) is an independence set for \((A_1, A_2)\) if for every non-empty finite subset \(F \subset S\) we have
\[
\bigcap_{i \in F} T^{-i} A_{v(i)} \neq \emptyset
\]
for any \(v \in \{1, 2\}^F\).

Theorem 72. [26] Let \((X, T)\) be a TDS. The system \((X, T)\) is not null if and only if there exist \(x, y \in X\) (with \(x \neq y\)) such that for all neighbourhoods \(U_x\) of \(x\) and \(U_y\) of \(y\) there exists an arbitrarily large finite independence set for \((U_x, U_y)\).

The following example shows that there are transitive non-null mean equicontinuous systems.

Example 73. Let \(S = \{2^n\}_{n=1}^\infty, Y := \{x \in \{0, 1\}^{\mathbb{Z}_+} : x_i = 0 \text{ if } i \notin S\},\) and \(X\) the shift closure of \(Y\). For every \(x, y \in X\), we have \(\text{dist}_b(x, y) = 0\) and hence \((X, \sigma)\) is mean equicontinuous. Nonetheless, since \(S\) is an infinite independence set for \((\{x_0 = 0\}, \{x_0 = 1\})\), we conclude that \((X, \sigma)\) is not null.

A \(\mathbb{Z}_+\)-subshift is Toeplitz if and only if it is the orbit closure of a regularly recurrent point, i.e. \(x \in X\) such that for every \(j > 0\) there exists \(m > 0\) such that \(x_j = x_{j+i}\) for all \(i \in \mathbb{Z}_+\). Toeplitz subshifts are precisely the minimal subshifts that are almost one–one extensions of odometers (for \(\mathbb{Z}_+\)-actions, see [9]; for finitely generated discrete group actions, see [8]).

Given a Toeplitz subshift and a regularly recurrent point \(x \in X\), there exists a set of pairwise disjoint arithmetic progressions \(\{S_n\}_{n \in \mathbb{N}}\) (called the periodic structure) such that \(\bigcup_{n \in \mathbb{N}} S_n = \mathbb{Z}_+, x_i\) is constant for every \(i \in S_n\), and every \(S_n\) is maximal in the sense that
there is no larger arithmetic progression where \( x_i \) is constant. Let \( x \) be a regularly recurrent point and \( X \) the orbit closure. We have that \((X, \sigma)\) is regular if and only if \( \sum_{n \in \mathbb{N}} D(S_n) = 1 \) (see [9]).

The following proposition shows that the converse of Corollary 67 does not hold.

**Proposition 74.** There exists a regular Toeplitz subshift (and hence diam-mean equicontinuous) with positive sequence entropy.

**Proof.** For every \( n \in \mathbb{N} \), let \( w^n \) be a finite word that contains all binary words of size \( n \). We denote the concatenation of \( w^n \) by \( w^{n, \infty} \in \{0, 1\}^{\mathbb{Z}_+} \).

We define the sequence \( \{j_n\} \subset \mathbb{N} \) inductively with \( j_1 = 0 \) and \( j_{n+1} := \min(\bigcup_{m \leq n} \{k2^m + j_m\}) \).

Let \( x \in \{0, 1\}^{\mathbb{Z}_+} \) be the point such that for every \( n \in \mathbb{N} \) we have that \( x_{j_n+1+2^n} = w^{n, \infty}_i \) for all \( i \in \mathbb{Z}_+ \).

We define \( X \) as the orbit closure of \( x \). Since \( x \) is regularly recurrent, we obtain that \( X \) is a Toeplitz subshift (and hence almost automorphic). By using the condition for regularity using the periodic structure (see comments regarding regular Toeplitz systems below Example 73), we obtain that \((X, \sigma)\) is regular. Hence, by Theorem 54, we get that \((X, \sigma)\) is diam-mean equicontinuous.

On the other hand, \( w^k \) contains all the binary words of size \( k \). This implies that there exist arbitrarily long independence sets for \((\{x_0 = 0\}, \{x_0 = 1\})\). Using Theorem 72, we conclude that \((X, \sigma)\) is not null. \( \square \)

Another class of rigid TDSs are the tame systems introduced in [16, 28]. These systems were characterized in [26] similarly to Theorem 72 but with infinitely large independence sets. This means that Example 73 is also not tame (note that it is not minimal). The example in [26, §11] is a tame non-regular Toeplitz subshift; this means that there are tame minimal systems that are not diam-mean equicontinuous. Nonetheless, we do not know if every minimal mean equicontinuous system is tame.

5. Amenable semigroup actions

All of our results can be stated for more general group actions. In this section, we state the generality of the results. All the results need amenability and all the results hold for countable abelian discrete actions.

Amenable groups are usually defined with invariant means (see [36]). We give an equivalent definition that is more useful for our paper.

**Definition 75.** Let \((G, +)\) be a locally compact semigroup that has an invariant measure \( v \). We say that \( G \) is amenable if there exists a Følner sequence, i.e. a sequence of measurable sets with finite measure \( \{F_n\} \subset G \), such that for any \( i \in G \) we have that

\[
\lim_{n \to \infty} \frac{v(i + F_n) \triangle F_n}{v(F_n)} = 0.
\]

If the group is countable, then any invariant measure is a counting measure. In this case if \( F \) is compact, then \( F \) is finite and \( v(F) = |F| \).

A semigroup \((G, +)\) is left cancellative if whenever \( a + b = a + c \) we have that \( b = c \). Every group is a left cancellative semigroup.
Throughout this section, we assume that \((G, +)\) is an amenable left cancellative locally compact semigroup with identity \((0)\) that has an invariant measure \(\nu\) (a group is always left cancellative and there exists (left) invariant measures known as Haar measures).

From now on, \(G\) implicitly represents a pair consisting of a semigroup and a Følner sequence.

Abelian semigroups are amenable. In particular, \(\mathbb{Z}^d_+\) and \(\mathbb{R}^d_+\) are amenable (and left cancellative); in these cases we associate the cubes \([0, n]^d\) as the Følner sequence. For \(\mathbb{Z}^d\) and \(\mathbb{R}^d\), we associate \([-n, n]^d\).

**Definition 76.** Let \(S \subset G\). We define lower density as

\[
D(S) := \liminf_{n \to \infty} \frac{\nu(S \cap F_n)}{\nu(F_n)}
\]

and upper density as

\[
\overline{D}(S) := \limsup_{n \to \infty} \frac{\nu(S \cap F_n)}{\nu(F_n)}.
\]

**Lemma 77.** Let \(S, S' \subset G\), \(i \in G\), and \(F \subset G\) a finite set. We have that:

- \(D(S) = D(i + S)\) and \(\overline{D}(S) = \overline{D}(i + S)\);
- \(\overline{D}(S) + \overline{D}(S') = 1\);
- if \(\overline{D}(S) + \overline{D}(S') > 1\), then \(S \cap S' \neq \emptyset\);
- if \(G\) is countable, then

\[
\begin{align*}
D(S) &:= \liminf_{n \to \infty} \frac{|S \cap F_n \setminus F|}{|F_n \setminus F|} \\
\overline{D}(S) &:= \limsup_{n \to \infty} \frac{|S \cap F_n \setminus F|}{|F_n \setminus F|}.
\end{align*}
\]

**Proof.** To prove the first property, we have that

\[
D(S) = \liminf_{n \to \infty} \frac{\nu(S \cap F_n)}{\nu(F_n)}
\]

\[
= \liminf_{n \to \infty} \frac{\nu(-i + i + S \cap F_n)}{\nu(F_n)}
\]

\[
= \liminf_{n \to \infty} \frac{\nu(i + S \cap i + F_n)}{\nu(F_n)}
\]

\[
= \liminf_{n \to \infty} \frac{\nu(i + S \cap i + F_n)}{\nu(i + F_n)} \frac{\nu(i + F_n)}{\nu(F_n)}
\]

\[
= D(i + S).
\]

The other properties are also easy to show. \(\square\)

A \(G\)-measure-preserving transformation (\(G\)-MPT) is a triplet \((M, \mu, T)\), where \((M, \mu)\) is a probability space and \(T := \{T^i : i \in G\}\) is a \(G\)-measure-preserving action on \(M\). When we say that a system is ergodic, it means that it is measure preserving and ergodic.

For some of the results in this section, we will assume that the pointwise ergodic theorem holds.
Let $(M, \mu, T)$ be an ergodic $\mathbb{G}$-MPT. Let $A$ be a $\mu$-measurable set. We say that $x \in M$ is a \textit{generic point for $A$} if

$$\lim_{n \to \infty} \frac{\nu\{i \in F_n : T^i x \in A\}}{\nu(F_n)} = \mu(A).$$

We say that a $\mathbb{G}$-MPT satisfies the \textit{pointwise ergodic theorem} if for every measurable set $A$ almost every point is a generic point for $A$. The pointwise ergodic theorem was originally proved for $\mathbb{Z}_+$-systems by Birkhoff. It also holds if $\mathbb{G} = \mathbb{Z}_d^+$. Every second countable amenable group has a Følner sequence that satisfies the pointwise ergodic theorem \cite{29} (note that this is not satisfied for every Følner sequence). For other conditions when this holds, see \cite[Ch. 6.4]{29}.

5.1. \textit{Results that hold for any $\mathbb{G}$.} The following results hold for any $\mathbb{G}$ (that is, amenable left cancellative locally compact with identity (0)): Theorems 21, 26, Proposition 28, Theorems 32, 38, 48, and Lemma 64.

The proofs of Theorems 21, 26, Proposition 28, and Theorems 32, 48 are identical. Lemma 64 also holds in general, but the proof is slightly different.

\textbf{Lemma 78.} Let $K$ be a finite set, $\varepsilon > 0$, and $h : K \to 2^\mathbb{G}$ be such that $D(h(k)) > \varepsilon$ for every $k \in F$. There exist $K' \subset K$ and $i \in \mathbb{G}$ with $|K'| \geq \varepsilon|K|/2$ such that $i \in h(k)$ for every $k \in K'$.

\textbf{Proof.} There exists $n_0 \in \mathbb{N}$ such that

$$\frac{\nu(h(k) \cap F_{n_0})}{\nu(F_{n_0})} \geq \varepsilon/2$$

for every $k \in K$.

Let $\mathcal{B}$ be a finite family of disjoint subsets of $F_{n_0}$ such that $\nu(B)$ is constant for every $B \in \mathcal{B}$ and for every $k \in K$ there exists $\mathcal{B}_k \subset \mathcal{B}$ such that

$$\nu(h(k) \cap F_{n_0}) = \nu\left( \bigcup_{B \in \mathcal{B}_k} B \right).$$

This means that

$$\sum_{B \in \mathcal{B}} |k \in K : B \subset h(k)| = \sum_{k \in K} |B \in \mathcal{B} : B \subset h(k)| \geq \frac{\varepsilon}{2}|K||\mathcal{B}|.$$

Using this, we obtain Theorem 65 and Corollary 67 for any $\mathbb{G}$ such that $(\mathbb{G}_{eq}, \nu_{eq}, T_{eq})$ satisfies the pointwise ergodic theorem.

A $\mathbb{G}$-measure-preserving transformation on a probability space $(M, \mu, T)$ generates a family of unitary linear operators on the Hilbert space $L^2(M, \mu)$, by $U_{T^i} : f \mapsto f \circ T^i$. We say that $(M, \mu, T)$ has \textit{discrete spectrum} if $L^2(M, \mu)$ is the direct sum of finite-dimensional $U_T$-invariant subspaces. Mackey proved that the Halmos–von Neumann theorem holds for locally compact group actions \cite{33}. This implies that Theorem 38 holds for any $\mathbb{G}$. 


5.2. Results that hold for countable discrete abelian \(\mathbb{G}\). The following results hold for countable discrete abelian \(\mathbb{G}\): Corollary 39, Theorem 54, and Corollary 67.

For \(\mathbb{Z}_+\)-actions, Kushnirenko proved that an ergodic system is \(\mu\)-null if and only if it has discrete spectrum \([30]\). This result was generalized for discrete actions by Kerr and Li \([27]\). With this, we obtain that Corollary 39 holds for discrete countable abelian actions that satisfy the pointwise ergodic theorem.

Generalized shift systems are defined for countable groups. Theorem 54 holds whenever \(\mathbb{G}\) is countable.

If \(\mathbb{G}\) is an abelian group, then every minimal null \(\mathbb{G}\)-TDS is almost automorphic (see \([22, 23]\)). This implies that Corollary 67 holds for countable abelian groups.

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