Homotopy theory in a quasi-abelian category

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October 15, 2015

Abstract

In order to set up a convenient setting to undertake the quantization of field theories, one requires
the homotopy theory of smooth differential graded algebras in the infinite dimensional setting. One
approach, which is explored in this article, is to study complexes of complete bornological vector spaces
or equivalently, complete locally convex topological vector spaces which are moreover $c^\infty$-complete. We
use Quillen’s theory of model categories to make the idea precise. In particular we prove that the
categories of chain complexes of (convex) bornological vector spaces and commutative monoid objects
therein are endowed with closed symmetric monoidal model category structures. These results hold on
the subcategory of complete objects. Both cases follow from a more general theorem for model structures
on modules and algebras in chain complexes in an arbitrary Grothendieck quasi-abelian category.

One application is that the Chevalley-Eilenberg resolution of a commutative monoid object in a quasi-
elian category is a cofibrant replacement in the model category of dg-modules over a dg-Lie algebra
in the quasi-abelian category. Similarly, the Koszul resolution of the commutative monoid is a cofibrant
replacement in the model category of commutative dg-algebras over the symmetric algebra of the dual of
a certain dg-module in the quasi-abelian category. Examples of these results, included here, is to derived
quotients (by an infinite dimensional Lie algebra) and to the derived critical locus of a function in the
$\infty$-dimensional context.

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Introduction

It has been appreciated for some time now that the correct setting to undertake the quantization of field theories is within the realm of homotopical algebra. In particular, the construction of gauge invariant observables and on-shell gauge invariant observables should be understood cohomologically. Doing so clarifies the roles played by seemingly exotic structures in the theory such as ghost fields and antifields. Working homotopically also enables one to properly understand the dependence on certain structures, such as the contractibility of the space of gauge fixing conditions, and prove universal properties.

In order to undertake this program rigorously, one should work in an appropriate ∞-category. For full control over this ∞-category, it is advantageous that it arises as the localization of a closed model category. In this case many constructions such as limits and colimits can be induced from the underlying model category and in many cases proofs simplify considerably. Since collections of objects in field theories form infinite dimensional spaces we must develop a homotopy theory of infinite dimensional spaces.

The model category we are interested in for the study of field theories is the category of chain complexes of complete bornological vector spaces, or equivalently, chain complexes of convenient vector spaces, with a model structure whose weak equivalences are reduced quasi-isomorphisms. Complete bornological and convenient vector spaces are two possible approaches to infinite dimensional smooth spaces whose theory has now reached a level of maturity [M2][KM].

There are other categories of vector spaces one may wish to consider. They sit naturally in a diagram of adjunctions

\[
\begin{array}{cccccc}
\text{Ind}(\text{SNorm}_k) & \xrightarrow{\text{colim}} & \text{Born}_k & \xrightarrow{\gamma} & \text{TVS}_k & \xrightarrow{i} & \text{BTVS}_k \\
\text{Ind}(\text{Norm}_k) & \xrightarrow{\text{colim}} & \text{SBorn}_k & \xrightarrow{\gamma} & \text{STVS}_k & \xrightarrow{i} & \text{BSTVS}_k \\
\text{Ind}(\text{Ban}_k) & \xrightarrow{\text{colim}} & \text{CBorn}_k & \xrightarrow{\gamma} & \text{CTVS}_k & \xrightarrow{i} & \text{Conv}_k \\
\end{array}
\]

originating from the categories of convex bornological vector spaces Born\_k and locally convex topological vector spaces TVS\_k. The vertical downward pointing arrows are separation and completion functors respectively and their right adjoints are inclusions. The categories in the left hand column are ind-categories of semi-normed, normed and Banach spaces respectively and are related to the other categories through the dissection functor diss. The categories in the right hand column are the essential images of the left adjoint γ to the functor vN which associates to a locally convex (resp. separated locally convex, complete locally convex) topological vector space its von Neumann bornology.

One convenient arena in which to do homotopical algebra is abelian categories. However, the examples we are interested in, in particular those in the diagram above, are not abelian. Nevertheless, there does exist a weaker notion in which our examples do reside and whose theory we will exploit. This is the theory of quasi-abelian categories.

One may wish to start with the category CTVS\_k of complete locally convex topological vector spaces to set up a theory of differential graded infinite dimensional vector spaces. It turns out that this category has a number of shortcomings which hinder its applicability. Firstly, it is not a closed symmetric monoidal category (with respect to the complete projective tensor product) and it is not quasi-abelian. However, the category CBorn\_k of complete bornological vector spaces is. It then follows that the category Conv\_k of convenient vector spaces is quasi-abelian. In fact all of the categories listed in the diagram above are quasi-abelian except for CTVS\_k.
Instead of proving the existence of a model category structure on the category of chain complexes in some of these quasi-abelian categories separately, we will prove a general existence theorem for the category of chain complexes in any quasi-abelian category which is moreover locally presentable and whose strict monomorphisms are closed under small filtered colimits. This mimics the standard definition of a Grothendieck abelian category to the quasi-abelian setting. All our examples of interest satisfy these conditions but this result may also be useful for other quasi-abelian categories not included in the diagram above. Our model categories will then be shown to have the advantages of being closed, symmetric monoidal and stable.

An application of this work to the setting of quantization can be found in recent work by the author [Wa].

Remark 0.1. Model structures on dg-modules, dg-algebras, and dg-Lie algebras in the abelian setting already exist in the literature from a number of sources [Qu][Hi][Ho]. The proofs of the model structures we construct will in fact follow those of Lurie in [L2] and [L3] (using formal results contained in the appendix of [L1]) with small modifications to include the more general setting of quasi-abelian categories.

Notation

In this paper we will often be confronted with set theoretic issues. In order not to burden the notation, we will take the pragmatic approach of fixing here a Grothendieck universe $U$ and calling elements therein $U$-small. We then fix universes $V$ and $W$ such that $U \in V \in W$ and refer to elements in $V$ as $V$-small or large and those in $W$ as very large. We will leave it to the reader to supplement the terms small limits, small colimits and locally presentable category to $U$-small limits, $U$-small colimits and $U$-locally presentable category and likewise for $V$ and $W$. We also assume the Vopenka principle which ensures that a full reflective subcategory of a locally presentable category is itself locally presentable.

Acknowledgements

I would like to thank Bertrand Toën for comments. This work was supported by the World Premier International Research Center Initiative (WPI), MEXT, Japan.

1 Quasi-abelian categories

Let $k$ be the field of real or complex numbers. Recall that a convex bornological vector space is a $U$-small vector space $V$ endowed with a bornology $\mathcal{B}_V$ (which we will often omit from the notation) whose elements are called bounded subsets [HN]. A convex bornological vector space is said to be separated if all bounded disks are norming and complete if each bounded subset is contained in a complete bounded disk. We will denote the large category of convex bornological vector spaces and bounded linear maps by $\text{Born}_k$ and the full subcategories of separated and complete objects by $\text{SBorn}_k$ and $\text{CBorn}_k$ respectively. The latter is equivalent to the large category $\text{CTVS}_k$ of complete locally convex topological vector spaces over $k$ with bounded (as opposed to continuous) linear maps.

All bornological vector spaces will be henceforth convex so objects in $\text{Born}_k$ will be simply called bornological vector spaces. For any object $V$ in $\text{Born}_k$, the canonical morphism

$$\text{colim}_{B \in \mathcal{B}_V} V_B \rightarrow V$$

is an isomorphism where $V_B$ is the linear hull of $B$ endowed with the gauge semi-norm and $\mathcal{B}_V$ is the collection of disks in $V$. Let $(V, \mathcal{B}_V)$ and $(W, \mathcal{B}_W)$ be bornological vector spaces. A subset $S$ of the vector space $\text{Hom}(V, W)$ is called equibounded if $\{f(x) | f \in S, x \in B\} \in \mathcal{B}_W$ for all $B \in \mathcal{B}_V$. The bornology of equibounded maps turns the category of bornological (resp. separated bornological,
complete bornological (vector spaces) into a cartesian closed category. We denote by $\text{Hom}(V, W)$ the internal hom object between $V$ and $W$.

Let $\text{TVS}_k$ denote the large category of locally convex $U$-small topological vector spaces, hereafter called topological vector spaces, over $k$ and continuous linear maps. Let

$$vN : \text{TVS}_k \to \text{Born}_k$$

denote the functor which associates to $M$ its von Neumann bornology where a subset is bounded if and only if it is absorbed by every neighborhood of the origin in $M$. The bornological vector space $vN(M)$ is often called the bornologification of $M$. There exists a fully faithful left adjoint which associates to $V$ a topological vector space $\gamma(V)$ with basis those subsets of the neighborhood of the origin that absorb bounded sets. This is the finest topology whose von Neumann bornology coincides with the original one, or equivalently, the unit map of the adjunction is bounded. Therefore, we define a bornological topological vector space $M$ such that $\gamma \circ vN(M) \simeq M$ is an isomorphism. Denote the essential image of $\gamma$ by $\text{BTVS}_k$. The essential image of $\gamma$ on the subcategory $\text{SBorn}_k$ of separated bornological vector spaces will be denoted $\text{BSTVS}_k$ and on the subcategory $\text{CBorn}_k$ by $\text{Conv}_k$. The objects in $\text{BSTVS}_k$ will be called bornological separated topological vector spaces and objects in $\text{Conv}_k$, convenient vector spaces.

Recall that there exists a dissection functor

$$\text{diss} : \text{Born}_k \to \text{Ind}(\text{SNorm}_k)$$

which sends a bornological vector space to an inductive system of $U$-small semi-normed spaces as follows. Let $V$ be a bornological vector space and $(\mathcal{D}_V, \leq)$ the directed set of bounded disks $B$ in $V$ partially ordered by absorption. Then $\text{diss}(V) := (V_B)_{B \in \mathcal{D}_V}$ is an inductive system of semi-normed spaces. The dissection functor is fully faithful.

There exists a left adjoint to the dissection functor which sends an inductive system $(V_B)_{B \in \mathcal{D}_V(V)}$ to $\text{colim}_{B \in \mathcal{D}_V} V_B$. The dissection functor on the subcategory of separated objects defines a full embedding into the category $\text{Ind}(\text{Norm}_k)$ of $V$-small ind-objects in the large category $\text{Norm}_k$ of $U$-small normed spaces and bounded linear maps. Moreover, on the subcategory of complete objects, it defines a full embedding into the category $\text{Ind}(\text{Ban}_k)$ of ind-objects in the large category $\text{Ban}_k$ of Banach spaces when we replace the directed set $(\mathcal{D}_V, \leq)$ by the directed set of complete bounded disks in $V$. The essential image of these dissection functors consists of inductive systems $(V_{\alpha}, \alpha)$ such that each $\alpha$ is a monomorphism.

A majority of the examples above share an underlying structure. Recall that a quasi-abelian category is an additive category with kernels and cokernels wherein pullbacks preserve strict epimorphisms and pushouts preserve strict monomorphisms [Sc]. Strict morphisms are those whose coimage is isomorphic to its image, ie. a map $f : x \to y$ such that

$$\text{coker}(\text{Ker}f \to x) = : \text{Coim}f \to \text{Im}f := \text{ker}(y \to \text{Coker}f)$$

is an isomorphism. Quasi-abelian categories are exact categories where we deem all conflations to be short exact sequences. For example, in the category of bornological vector spaces, inflations are strict monomorphisms and correspond to bornological isomorphisms onto their image with the subspace bornology. On the other hand, deflations are strict epimorphisms which are maps such that any bounded subset in the codomain is the image of a bounded subset. Every morphism in a quasi-abelian category has a canonical decomposition into a strict epimorphism (resp. epimorphism) followed by a monomorphism (resp. strict monomorphism).

In order to prove that there exists a model structure on the category of chain complexes in a quasi-abelian category we will need to introduce some further assumptions. An abelian category is said to be Grothendieck if it is locally presentable and its collection of monomorphisms is stable under filtered colimits. The analogue in our case is closure under filtered colimits of strict monomorphisms. Therefore we make the following definition.

**Definition 1.1.** Let $C$ be a quasi-abelian category. Then $C$ is said to be Grothendieck if it is locally presentable and its collection of strict monomorphisms is closed under filtered colimits.
Proposition 1.2. Let $C$ be a cocomplete quasi-abelian category with exact filtered colimits. Then the category $\text{Ind}(C)$ of ind-objects in $C$ is a Grothendieck quasi-abelian category.

Proof. Let $C$ be a quasi-abelian category. By duality in the definition of a quasi-abelian category, the opposite of a quasi-abelian category is quasi-abelian. Therefore, by Proposition 7.1.5 and Proposition 7.1.7 of [P1], the category $\text{Ind}(C)$ is quasi-abelian and cocomplete. Any cocomplete category of the form $\text{Ind}(C)$ is locally presentable by definition. Closure under small filtered colimits of strict monomorphisms follows from the assumption on $C$. Indeed, if filtered colimits are exact in $C$ they are exact in $\text{Ind}(C)$. Let $\text{Ind}(C)_{\Delta^1}$ denote the category of morphisms in $\text{Ind}(C)$ and consider the colimit functor

$$\text{colim}_I : (\text{Ind}(C)_{\Delta^1})^I \to \text{Ind}(C)_{\Delta^1}$$

for a filtered category $I$. Since filtered colimits are exact in $\text{Ind}(C)$, we have a diagram

$$\text{Coi}(\text{colim}_IF) \to \text{Im}(\text{colim}_IF)$$

$$\downarrow$$

$$\text{colim}_I\text{Coi}(F) \to \text{colim}_I\text{Im}(F)$$

of objects in $\text{Ind}(C)$ for $F \in (\text{Ind}(C)_{\Delta^1})^I$ where the vertical arrows are equivalences. Since the bottom arrow is an equivalence by assumption, the top horizontal arrow is an equivalence and thus the collection of strict monomorphisms in $\text{Ind}(C)$ is closed under filtered colimits as required.

Lemma 1.3. Let $C$ be a (symmetric) monoidal Grothendieck quasi-abelian category and $R$ a (commutative) monoid object in $C$. Then the category $\text{Mod}(R)$ of $R$-modules in $C$ is a Grothendieck quasi-abelian category.

Proof. This follows from the fact that the forgetful functor from $\text{Mod}(R)$ to $C$ preserves all limits and colimits.

Let $\text{Fre}_k$ denote the large category of Fréchet spaces over $k$ and continuous $k$-linear maps. Recall that a subcategory is said to be reflective if the natural inclusion admits a left adjoint.

Proposition 1.4. The category $\text{Fre}_k$ is Grothendieck quasi-abelian.

Proof. The category $\text{Fre}_k$ is quasi-abelian by Proposition 4.4.5 of [P2] and is closed under colimits. The bornologification functor $\text{vN} : \text{CTVS}_k \to \text{CBorn}_k$ is fully faithful on the subcategory of Fréchet spaces. Therefore the composition of functors $\text{dis} \circ \text{vN}$ exhibits $\text{Fre}_k$ as a full reflective subcategory of the locally presentable category $\text{Ind}(\text{Ban}_k)$ and the result follows from Proposition 1.39 of [AR].

All of the ind-categories of interest to us satisfy the conditions of Definition 1.1.

Proposition 1.5. The category $\text{Ind}(\text{SNorm}_k)$ is Grothendieck quasi-abelian.

Proof. Recall that $\text{SNorm}_k$ is the category whose objects are semi-normed spaces $(V, \rho_V)$ and a morphism between $(V, \rho_V)$ and $(W, \rho_W)$ is a morphism $f : V \to W$ of vector spaces such that $|\rho_W \circ f| \leq c \rho_V$ for some $c > 0$. Equivalently, a morphism is a continuous morphism of locally convex topological vector spaces for the canonical topology induced by the semi-norm. We will emit the semi-norm from the notation and refer simply to $V$.

We first show that filtered colimits are exact in $\text{SNorm}_k$. We will show that the functor $\text{colim}_I : (\text{SNorm}_k)^I \to \text{SNorm}_k$ preserves monomorphisms for a filtered category $I$ and leave the remaining steps to the reader. Let $\alpha : F \to G$ be a monomorphism in $(\text{SNorm}_k)^I$ and $v \in \text{colim}_IF$ such that $\text{colim}_I(\alpha)(v) = 0$ in $\text{colim}_IG$. We need to show that $v = 0$. We know that $v$ is the image of some $v_i \in F(i)$ and therefore $\alpha_i(v_i) \in G(i)$ vanishes in $\text{colim}_IG$. Therefore there exists a map $u : i \to j$ such that $G(u)(\alpha_i(v_i)) = 0$ and $\text{colim}_I(\alpha)(u)(\alpha_i(v_i)) = 0$ in $\text{colim}_IG$.
\[ \alpha_j(F(u)(v_i)) = 0 \text{ in } G(j). \] Since \( \alpha_j \) is a monomorphism, then \( F(u)(v_i) = 0 \) in \( F(j) \) and \( v = 0 \) in \( \operatorname{colim}_I F \) as required.

The category \( \text{SNorm}_k \) is quasi-abelian by Proposition 3.2.4 of \([\text{Sc}]\). Since the category \( \text{SNorm}_k \) is cocomplete, the result follows from Proposition \([\text{L2}]\).

**Proposition 1.6.** The categories \( \text{Ind}(\text{Norm}_k) \) and \( \text{Ind}(\text{Ban}_k) \) are Grothendieck quasi-abelian.

**Proof.** The category \( \text{Norm}_k \) is quasi-abelian by Proposition 3.2.17 of \([\text{Sc}]\). The category \( \text{Ban}_k \) is quasi-abelian since it is a full subcategory of \( \text{Fre}_k \) which is closed under subobjects, quotients and conflations. Since \( \text{Norm}_k \) and \( \text{Ban}_k \) are cocomplete, it follows from Proposition \([\text{L2}]\) that \( \text{Ind}(\text{Norm}_k) \) and \( \text{Ind}(\text{Ban}_k) \) are locally presentable and quasi-abelian. They are moreover Grothendieck since the separation and completion functors are left adjoints and thus preserve colimits.

The remaining categories in the diagram in the introduction satisfy our conditions except for the category \( \text{CTVS}_k \) of complete locally convex vector spaces which was shown in Proposition 4.1.14 of \([\text{P2}]\) not to be quasi-abelian.

**Proposition 1.7.** The categories \( \text{Born}_k, \text{SBorn}_k, \text{CBorn}_k, \text{BTVS}_k, \text{BSTVS}_k \) and \( \text{Conv}_k \) are Grothendieck quasi-abelian categories.

**Proof.** The category of bornological (separated bornological, complete bornological) vector spaces over \( k \) is quasi-abelian by Proposition 1.8 (resp. Proposition 4.10, Proposition 5.6) of \([\text{PS}]\). From Proposition \([\text{L5}]\) and Proposition \([\text{L6}]\) the categories \( \text{Ind}(\text{SNorm}_k), \text{Ind}(\text{Norm}_k) \) and \( \text{Ind}(\text{Ban}_k) \) are locally presentable and the dissection functor is fully faithful. By Proposition 1.9 (resp. Proposition 4.12, Proposition 5.6) of \([\text{PS}]\) the category \( \text{Born}_k \) (resp. \( \text{SBorn}_k, \text{CBorn}_k \)) is cocomplete. Every full reflective subcategory of a locally presentable category which is closed under colimits is locally presentable. Therefore \( \text{Born}_k, \text{SBorn}_k \) and \( \text{CBorn}_k \) are locally presentable quasi-abelian. It then follows by definition that \( \text{BTVS}_k, \text{BSTVS}_k \) and \( \text{Conv}_k \) are locally presentable quasi-abelian categories. These categories are moreover Grothendieck since the dissection functor is right adjoint and thus its left adjoint preserves colimits.

### 2 Differential graded modules

In this section we will set up the homotopy theory of infinite dimensional differential graded vector spaces. We will use the category of complete bornological vector spaces as an example for illustration. However, all the results here hold for any Grothendieck quasi-abelian category, in particular, the categories listed in Proposition \([\text{L6}]\), Proposition \([\text{L6}]\) and Proposition \([\text{L7}]\).

Let \( C \) be a quasi-abelian category and \( \operatorname{Ch}(C) \) the category of cochain complexes in \( C \). We define a cohomology functor
\[ H^n : \operatorname{Ch}(C) \to C \]
by sending \( M \) to \( \operatorname{coker}(M_n \to \operatorname{Ker}d_{n+1}) \). We call this functor the reduced cohomology.

**Example 2.1.** Every abelian category \( C \) is quasi-abelian. In this case the reduced cohomology of \( C \) coincides with the usual cohomology of \( C \).

**Example 2.2.** A subspace \( N \) of a complete bornological vector space \( M \) is said to be closed if limits of sequences in \( N \) which converge in \( M \) belong to \( N \). The closure \( \overline{N} \) of \( N \) is the intersection of all the closed subspaces of \( M \) containing \( N \). Now let \( M \) be a chain complex of complete bornological vector spaces. Then the kernel \( \operatorname{Ker}(d_n) \) is complete. Therefore the quotient vector space \( H_n(M) = \operatorname{Ker}(d_n)/\operatorname{Im}(d_{n+1}) \) with the quotient bornology is complete \([\text{H-N}]\).

**Definition 2.3.** Let \( C \) be a quasi-abelian category. A map in \( \operatorname{Ch}(C) \) will be called a reduced quasi-isomorphism if it is an isomorphism in \( C \) on the reduced cohomology.

The following is a weaker version of the analogous statement for cochain complexes in a Grothendieck abelian category (Proposition 1.3.5.3 of \([\text{L2}]\)).
Proposition 2.4. Let $C$ be a Grothendieck quasi-abelian category. There exists a combinatorial model structure on the category $\text{Ch}(C)$ of cochain complexes in $C$ with the following classes of morphisms:

$(\mathcal{C})$ The cofibrations are the degreewise strict monomorphisms.

$(\mathcal{W})$ The weak equivalences are the reduced quasi-isomorphisms.

$(\mathcal{F})$ The fibrations are the those maps with the right lifting property with respect to trivial cofibrations.

**Proof.** We first need to construct a small set of generating cofibrations $\mathcal{C}_0$ such that each cofibration belongs to the weakly saturated class generated by $\mathcal{C}_0$. Since $C$ is locally presentable we can define an object $x := \bigoplus x_i$ of $C$ given by the coproduct of the objects $x_i$ which generate $C$ under small colimits. Then for every strict monomorphism $u : y \to x$, let $M(u, n) := \ldots 0 \to y \xrightarrow{u} x \to 0 \to \ldots$ be the complex with $y$ in degree $n$ and $x$ in degree $(n + 1)$. We define $\mathcal{C}_0$ to be the set of all strict monomorphisms $M(u, n) \to M(\text{id}_x, n)$ for $n \in \mathbb{Z}$. Arguing as in Proposition 1.3.5.3 of [L2], using the fact that strict monomorphisms are stable under filtered colimits, we find that every cofibration belongs to the smallest weakly saturated class of morphisms containing $\mathcal{C}_0$. Conversely, it is clear that $\mathcal{C}$ contains $\mathcal{C}_0$ and is weakly saturated (in particular, strict monomorphisms are closed under the formation of pushouts by definition).

One must now check the conditions of Proposition A.2.6.13 of [L1]. Since $C$ is Grothendieck, the reduced cohomology functor commutes with filtered colimits. The class of isomorphisms in $C$ is perfect since $C$ is locally presentable, therefore it follows from Corollary A.2.6.12 of [L1] that the weak equivalences in $\text{Ch}(C)$ are a perfect class.

Now consider the pushout diagram

$$
\begin{array}{ccc}
M & \xrightarrow{g} & N' \\
\downarrow{f} & & \downarrow{g'} \\
N & \xrightarrow{g} & P 
\end{array}
$$

in $\text{Ch}(C)$ where $f$ is a cofibration and $g$ is a weak equivalence. To show that $g' : N \to P$ is a weak equivalence one can use the same argument as the Grothendieck abelian case in Proposition 1.3.5.3 of [L2] using the reduced cohomology. Finally, we need to show that if $f : M \to N$ is a morphism in $C$ with the right lifting property with respect to every morphism in $\mathcal{C}$, then $f \in \mathcal{W}$. One can again employ the same arguments as in *loc. cit.* using reduced cohomology and strict monomorphisms as the cofibrations.

We will call the model structure of Proposition 2.4 the *injective model structure*.

**Example 2.5.** Let $A$ be a complete bornological algebra over $k$. Denote the category of complete bornological $A$-modules by $\text{Mod}_A^{\text{cborn}}$. The category of chain complexes in $\text{Mod}_A^{\text{cborn}}$ will be denoted $\text{dg}_A^{\text{cborn}}$. The objects in this category will be called complete bornological dg-modules over $A$.

The category $\text{CBorn}_k$ of complete bornological vector spaces over $k$ is an additive category with kernels and cokernels. By Proposition 1.7 it is Grothendieck quasi-abelian. Therefore, by Proposition 1.3 the category $\text{Mod}_A^{\text{cborn}}$ is Grothendieck quasi-abelian. It now follows from Proposition 2.4 that the category $\text{dg}_A^{\text{cborn}}$ of complete bornological dg-modules admits an injective model structure.

**Definition 2.6.** Let $A$ be a Banach algebra over $k$. Denote the category of Banach $A$-modules by $\text{Mod}_A^{\text{Ban}}$. A chain complex in $\text{Mod}_A^{\text{Ban}}$ will be called a *Banach dg-module* over $A$ and the category of Banach dg-modules over $A$ will be denoted $\text{dg}_A^{\text{Ban}}$. The category of inductive systems in $\text{dg}_A^{\text{Ban}}$ will be denoted $\text{Ind}(\text{dg}_A^{\text{Ban}})$.

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Proposition 2.7. The category of inductive systems of Banach dg-modules admits an injective model structure.

Proof. The category $\text{Ban}_k$ is an additive category which is Grothendieck quasi-abelian by Proposition 1.6. By Proposition 1.3, the category $\text{Mod}^{\text{Ban}}_A$ is Grothendieck quasi-abelian. It then follows from Proposition 1.2 that $\text{Ind}(\text{Mod}^{\text{Ban}}_A)$ is also Grothendieck quasi-abelian. Since there exists a canonical equivalence

$$\text{Ch}(\text{Ind}(\text{Mod}^{\text{Ban}}_A)) \to \text{Ind}(\text{dg}^{\text{Ban}}_A)$$

of categories, it follows from Proposition 2.4 that $\text{Ind}(\text{dg}^{\text{Ban}}_A)$ admits an injective model structure.

We now give some examples of dg-modules in a quasi-abelian category which are simple extensions of familiar examples.

Example 2.8. Let $\text{dg}_k$ denote the category of dg-vector spaces over $k$. Then there exists a fully faithful functor

$$\text{Fine} : \text{dg}_k \to \text{dg}^{\text{chorn}}_k$$

sending a dg-vector space $M$ to the dg-vector space $M$ endowed with the fine bornology. A subset $N$ of $M$ is bounded in $\text{Fine}(M)$ if and only if there exists a finite dimensional subspace $(M_N)_n \subseteq M_n$ for each $n \in \mathbb{Z}$ such that $N_n$ is a bounded subset of $(M_N)_n \simeq k^n$. It is the finest possible bornology on $M$. The functor $\text{Fine}$ is left adjoint to the forgetful functor and the adjunction is a Quillen adjunction of model categories where $\text{dg}_k$ is endowed with its injective model structure.

Example 2.9. Let $\text{dg}^{\text{ctvs}}_k$ denote the category of chain complexes in the category of complete topological vector spaces over $k$. Let $\nu N : \text{dg}^{\text{ctvs}}_k \to \text{dg}^{\text{chorn}}_k$ denote the functor which sends a complete topological dg-vector space $M$ to its bornologification, ie. $\nu N(M)$ is endowed with the von Neumann bornology in which for all $n \in \mathbb{Z}$, a subset $N_n \subset M_n$ is bounded if and only if it is absorbed by every neighbourhood of the origin in $M_n$. This functor admits a fully faithful left adjoint $\gamma$ which associates to a bornological dg-vector space $V$, the complete topological dg-vector space $\gamma(V)$ such that for each $n \in \mathbb{Z}$, a basis of $\gamma(V)_n$ consists of those subsets of the neighborhood of the origin that absorb bounded sets. This is the finest locally convex topology whose von Neumann bornology coincides with the original one. The restriction of the bornologification functor to Fréchet dg-vector spaces

$$\nu N : \text{dg}^{\text{Fre}}_k \to \text{dg}^{\text{chorn}}_k$$

is fully faithful.

Example 2.10. There is another interesting bornology one can attach to an object in $\text{dg}^{\text{ctvs}}_k$. Let

$$\text{PCpt} : \text{dg}^{\text{ctvs}}_k \to \text{dg}^{\text{chorn}}_k$$

denote the functor which sends a complete topological dg-vector space $M$ to the precompact bornologification, ie. $\nu N(M)$ is endowed with the precompact bornology in which for all $n \in \mathbb{Z}$, a subset $N_n \subset M_n$ is bounded if and only if its closure is compact. Again, the restriction functor to Fréchet dg-vector spaces is fully faithful.

Example 2.11. Let $E$ be a complete topological dg-vector space. Then the $c^\infty$-completion of $E$ is a convenient dg-vector space $c^\infty(E)$ together with a bounded linear map $f : E \to c^\infty(E)$ satisfying the following universal property: for any convenient dg-vector space $F$ there exists an isomorphism

$$\text{Hom}(c^\infty(E), F) \to \text{Hom}(E, F)$$

(Theorem 4.29 of [KM]). The category of convenient dg-vector spaces is equivalent to the category of $c^\infty$-complete topological dg-vector spaces. This construction defines a functor

$$c^\infty : \text{dg}^{\text{ctvs}}_k \to \text{dg}^{\text{conv}}_k$$

which is left adjoint to the fully faithful inclusion.
According to Example 2.5, the category of complete bornological dg-vector spaces is endowed with an injective model structure. However, to define algebras in these categories, it is convenient to introduce another model structure with the same weak equivalences.

We let \( \text{dg}_R(C) \) denote the category of chain complexes in the category of modules over a monoid object \( R \) in a locally presentable symmetric monoidal quasi-abelian category \( C \). We call \( \text{dg}_R(C) \) the category of dg-modules over \( R \).

**Proposition 2.12.** Let \( C \) be a monoidal quasi-abelian category and \( R \) a monoid object in \( C \). There exists a combinatorial model structure on the category \( \text{dg}_R(C) \) of dg-modules over \( R \) with the following classes of morphisms:

- (\( \mathcal{F} \)) The fibrations are the degreewise strict epimorphisms.
- (\( \mathcal{W} \)) The weak equivalences are the reduced quasi-isomorphisms.
- (\( \mathcal{C} \)) The cofibrations are those maps with the left lifting property with respect to trivial fibrations.

**Proof.** One can follow exactly the same argument as Proposition 7.1.2.8 of \( \textbf{[L2]} \) replacing monomorphisms and epimorphisms using their strict notions and cohomology by reduced cohomology. The cofibrations in this model structure coincide with the smallest weakly saturated class of morphisms containing generating cofibrations \( \mathcal{C}_0 \) given by the collection of strict monomorphisms \( \{R[n-1] \to M(\text{id}_R, n)\}_{n \in \mathbb{Z}} \) (using the notation in the proof of Proposition 2.4).

The model structure of Proposition 2.12 will be called the projective model structure.

**Proposition 2.13.** Let \( C \) be a symmetric monoidal quasi-abelian category and \( R \) a commutative monoid object in \( C \). Then the category \( \text{dg}_R(C) \) of dg-modules over \( R \) is a symmetric monoidal model category with respect to the projective model structure.

**Proof.** We need to show that the tensor product functor is a left Quillen bifunctor and that the unit object is cofibrant. The second part is clear and the first follows the same argument in the abelian case of Proposition 7.1.2.11 of \( \textbf{[L2]} \) using the fact that strict monomorphisms are stable under pushouts.

Recall that a model category is stable if it is a pointed category (ie. its initial and terminal objects coincide) and the suspension functor is invertible on the homotopy category. The model category \( \text{dg}_R(C) \) with respect to the projective model structure is stable. Here, as usual, the suspension and loop space functors are shift functors. It follows that the homotopy category \( \text{h}(\text{dg}_R(C)) \) is triangulated.

The tensor products we are interested in are the following. There exists a tensor product called the bornological tensor product satisfying the property that for bornological dg-vector spaces \( E \) and \( F \) there exists an isomorphism

\[
\text{Hom}(E \otimes F, G) \to \text{Hom}(E \times F, G)
\]

for any bornological dg-vector space \( G \). Equivalently, the functor

\[
E \otimes - : \text{dg}_k^{\text{born}} \to \text{dg}_k^{\text{born}}
\]

is left adjoint to \( \text{Hom}(E, -) \) where \( \text{dg}_k^{\text{born}} \) is the category of bornological dg-vector spaces. The fully faithful inclusion \( \text{dg}_k^{\text{cborn}} \hookrightarrow \text{dg}_k^{\text{born}} \) admits a left adjoint completion functor

\[
-^c : \text{dg}_k^{\text{born}} \to \text{dg}_k^{\text{cborn}}.
\]

The complete tensor product is the completion of the bornological tensor product. The category of complete bornological dg-vector spaces is then closed under the complete tensor product. It follows from Proposition 2.12 that the category \( \text{dg}_A^{\text{born}} \) of bornological dg-modules over a bornological algebra and the category \( \text{dg}_A^{\text{cborn}} \) of complete bornological dg-modules over a complete bornological algebra are endowed with projective model structures. Moreover, they are both closed symmetric monoidal model categories.
Lemma 2.14. Every object in $\text{dg}_{A}^{\text{born}}$ and $\text{dg}_{A}^{\text{born}}$ is cofibrant with respect to the projective model structure.

Proof. The category of bornological dg-modules is cocomplete. Also, the class of generating cofibrations in the injective model structure are pushouts of coproducts of generating cofibrations in the projective model structure. Any model category with the same generating cofibrations and weak equivalences define the same model structure. Therefore the injective and projective model structure coincide for the category of bornological dg-modules. It follows that every object is cofibrant. The same argument applies to complete bornological dg-modules.

We now consider some examples of symmetric monoidal functors between model categories in the quasi-abelian setting.

Example 2.15. Let $\text{dg}_{k}^{\text{born}}$ be endowed with its bornological tensor product and projective model structure. When the category of dg-vector spaces is endowed with its canonical symmetric monoidal model structure, then the functor $\text{Fine}$ of Example 2.8 is symmetric monoidal functor between symmetric monoidal model categories.

Example 2.16. Let $\text{dg}_{k}^{\text{top}}$ be endowed with its complete projective tensor product. Then the bornologification functor $\text{vN}$ and $\text{PCpt}$ of Example 2.9 are not symmetric monoidal. However, the restriction of $\text{vN}$ to the subcategory of Banach dg-vector spaces and the restriction of $\text{PCpt}$ to Fréchet dg-vector spaces are symmetric monoidal functors between model categories.

Example 2.17. By construction, the $c^\infty$-completion functor of Example 2.11 is a symmetric monoidal functor between symmetric monoidal model categories.

Example 2.18. There exists a differential graded dissection functor

$$\text{diss} : \text{dg}_{k}^{\text{born}} \to \text{Ind}(\text{dg}_{k}^{\text{Ban}})$$

which sends a complete bornological dg-vector space to an inductive system of dg-Banach spaces as in Section 1. The dissection functor is fully faithful and its essential image consists of reduced inductive systems, i.e. those diagrams for which each map in the inductive system is a monomorphism [M2]. The left adjoint to the dissection functor sends an inductive system $\{M_B\}_{B \in \mathcal{E}(M)}$ to $\text{colim}_{B \in \mathcal{E}(M)} M_B$.

Let $\text{Ind}(\text{dg}_{k}^{\text{Ban}})$ be endowed with the canonical extension of the complete projective tensor product on $\text{dg}_{k}^{\text{Ban}}$ to $\text{Ind}(\text{dg}_{k}^{\text{Ban}})$. This makes $\text{Ind}(\text{dg}_{k}^{\text{Ban}})$ a symmetric monoidal category and we endow it with a symmetric monoidal model structure with respect to the projective model structure. Then the differential graded dissection functor $\text{diss}$ is not symmetric monoidal. However, the composition

$$\text{diss} \circ \text{PCpt} : \text{dg}_{k}^{\text{Pre}} \to \text{Ind}(\text{dg}_{k}^{\text{Ban}})$$

of the functor PCpt restricted to Fréchet dg-vector spaces with the differential graded dissection functor is a symmetric monoidal functor between symmetric monoidal model categories.

Proposition 2.19. There exists a Quillen equivalence

$$\text{colim} : \text{Ind}(\text{dg}_{k}^{\text{norm}}) \rightleftarrows \text{dg}_{k}^{\text{born}} : \text{diss}$$

of model categories. This Quillen equivalence holds between ind-objects of normed spaces and separated bornological vector spaces and between ind-objects in Banach spaces and complete bornological vector spaces.

Proof. We endow $\text{Ind}(\text{dg}_{k}^{\text{Ban}})$ and $\text{dg}_{k}^{\text{born}}$ with the injective model structure. The left adjoint $\text{colim}$ clearly preserves strict monomorphisms and reduced quasi-isomorphisms. Therefore the adjunction is a Quillen adjunction. It remains to show that the left derived functor

$$L\text{colim} : h(\text{Ind}(\text{dg}_{k}^{\text{norm}})) \to h(\text{dg}_{k}^{\text{born}})$$

is an equivalence of categories. This follows from Proposition 3.10 of [PS]. The same is true utilizing Proposition 4.16 and Proposition 5.16 of loc.cit. for the other two cases.

9
3 Differential graded algebras

In this section we will set up the homotopy theory of differential graded algebras in a quasi-abelian category. As in Section 2, we will use the quasi-abelian category of complete bornological vector spaces as our primary example.

Definition 3.1. Let $\text{dga}_A^{\text{born}}$ denote the category of monoid objects in $\text{dg}_A^{\text{born}}$ endowed with its complete bornological tensor product. We call objects in this category complete bornological dg-algebras over $A$. Let $\text{cdga}_A^{\text{born}}$ denote the category of commutative monoid objects in $\text{dg}_A^{\text{born}}$ endowed with its complete bornological tensor product. We call objects in this category commutative complete bornological dg-algebras over $A$.

It follows from Proposition 2.13, Lemma 2.14 and Theorem 4.1 of [SS] that the category of (commutative) complete bornological dg-algebras is endowed with a combinatorial model structure induced from Proposition 2.12, i.e. a fibration is a map if it is a fibration of complete bornological dg-modules and a weak equivalence if it is a weak equivalence of complete bornological dg-modules. In the spirit of Section 2, we prove a general theorem which encompasses this model structure, together with analogous structures on the examples in Proposition 1.6 and Proposition 1.7, by proving that a model structure exists on the category of algebras in an arbitrary locally presentable quasi-abelian category.

Let $C$ be a combinatorial symmetric monoidal model category and $D$ the collection of all morphisms in $C$ of the form $\id_x \otimes g : x \otimes y \to x \otimes y'$ where $g$ is a trivial cofibration. Let $\mathcal{D}$ denote the weakly saturated class of morphisms generated by $D$. Recall that $C$ is said to satisfy the monoid axiom if every morphism in $\mathcal{D}$ is a weak equivalence.

Proposition 3.2. Let $C$ be a quasi-abelian category and $R$ a commutative monoid object in $C$. Let $\text{dg}_R(C)$ be endowed with the structure of a symmetric monoidal model category of Proposition 2.12. Then $\text{dg}_R(C)$ satisfies the monoid axiom.

Proof. It suffices to prove that every morphism in $\mathcal{D}$, the weakly saturated class of morphisms generated by $D$ in $\text{dg}_R(C)$, is a trivial cofibration in the projective model structure. This in turn can be deduced if such morphisms in $D$ are trivial cofibrations.

Consider the strict exact sequence

$$0 \to y \xrightarrow{g} y' \xrightarrow{g'} y'' \to 0$$

which by definition means that it is exact in the usual sense and $g$ is strict. Since the hom-functor $\text{Hom}(-, z)$ preserves strict exact sequences, the sequence

$$0 \to x \otimes y \xrightarrow{\id_x \otimes g} x \otimes y' \xrightarrow{\id_x \otimes g'} x \otimes y'' \to 0$$

is strict exact in $\text{dg}_R$. Therefore $\id_x \otimes g$ is a strict monomorphism.

Therefore we need to show that $x \otimes y''$ is an acyclic dg-module over $R$. This follows from the fact that $y''$ admits a contracting homotopy. \qed

Let $R$ be a commutative monoid object in $C$ and let $\text{dga}_R(C)$ denote the category of monoid objects in $\text{dg}_R(C)$. The objects in $\text{dga}_R(C)$ will be called differential graded algebras over $R$.

Proposition 3.3. Let $C$ be a monoidal locally presentable quasi-abelian category and $R$ a commutative monoid object in $C$. Then there exists a combinatorial model structure on the category $\text{dga}_R(C)$ of differential graded algebras over $R$ with the following classes of morphisms:

$$(\mathcal{F})\text{ A morphism is a fibration if and only if it is a fibration in $\text{dg}_R(C)$}.$$
A morphism is a weak equivalences if and only if it is a weak equivalence in $dg_R(C)$.

The cofibrations are the those maps with the left lifting property with respect to trivial fibrations.

Proof. This follows from Theorem 4.1 of [SS], Proposition 3.2 and Proposition 2.12.

Let $M$ be a left proper combinatorial symmetric monoidal model category. Recall that $M$ is said to be freely powered if it satisfies the monoid axiom, the collection of cofibrations is generated by cofibrations between cofibrant objects and for every cofibration $f : x \to y$ and every $n \geq 0$, the induced map

$$\wedge^n(f) : (x \otimes y) \coprod_{x \otimes x} (x \otimes y) \cdots (x \otimes y) \coprod_{x \otimes x} y^\otimes n,$$

(with $n$ factors of brackets in the domain), is a cofibration in the projective model category of objects of $M$ endowed with an action of the symmetric group.

Proposition 3.4. Let $C$ be locally presentable quasi-abelian category and $R$ a commutative monoid object in $C$ containing the field of rational numbers. Let $dg_R(C)$ be endowed with the structure of a symmetric monoidal model category of Proposition 2.12. Then $dg_R(C)$ is freely powered.

Proof. It follows from Proposition 3.2 that $dg_R(C)$ satisfies the monoid axiom and we know that in the projective model structure, cofibrations between cofibrant objects generate the class of cofibrations. Let $dg_R(C)^{\Sigma_n}$ denote the category of dg-modules over $R$ equipped with an action of the symmetric group (on $n$ letters) which is endowed with the projective model structure. It remains to check that for every cofibration $f : x \to y$ in $dg_R(C)$, the map $\wedge^n(f)$ is a cofibration in $dg_R(C)^{\Sigma_n}$. One can use the same argument as Proposition 7.1.4.7 of [L2].

Let $R$ be a commutative monoid object in $C$ and let cdga$_R(C)$ denote the category of commutative monoid objects in $dg_R$. The objects in cdga$_R(C)$ will be called commutative differential graded algebras over $R$.

Proposition 3.5. Let $C$ be a symmetric monoidal quasi-abelian category and $R$ a commutative monoid object in $C$ containing the field of rational numbers. Then there exists a combinatorial model structure on the category cdga$_R(C)$ of commutative differential graded algebras over $R$ with the following classes of morphisms:

(\text{F}) A morphism is a fibration if and only if it is a fibration in $dg_R(C)$.

(\text{W}) A morphism is a weak equivalences if and only if it is a weak equivalence in $dg_R(C)$.

(\text{C}) The cofibrations are the those maps with the left lifting property with respect to trivial fibrations.

Proof. This follows from Proposition 4.5.4.6 of [L2], Proposition 3.4 and Proposition 2.12.

4 Chevalley-Eilenberg resolutions

We now explain Chevalley-Eilenberg resolutions in the quasi-abelian setting. Our main result shows that given a dg-Lie algebra in a quasi-abelian category over some commutative ring object, then the Chevalley-Eilenberg complex is a cofibrant resolution of the ring object with its trivial dg-Lie algebra structure. For example, given a bornological dg-Lie algebra $g$ over a bornological algebra $A$, then the Chevalley-Eilenberg complex associated to $g$ is a cofibrant replacement of $A$ in the model category of modules over the universal enveloping algebra of $g$.

Definition 4.1. Let $C$ be a monoidal quasi-abelian category and $R$ a commutative monoid object in $C$. A dg-Lie algebra over $R$ in $C$ is a dg-module $(g, d)$ over $R$ equipped with a bracket

$$[\cdot, \cdot] : g_p \otimes_R g_q \to g_{p+q}$$

satisfying the following conditions:
1. For $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$, the relation $[x, y] + (-1)^{pq}[y, x] = 0$ holds.

2. For $x \in \mathfrak{g}_p$, $y \in \mathfrak{g}_q$ and $z \in \mathfrak{g}_r$, the relation

\[-1]^{pq}[x, [y, z]] + (-1)^{pr}[y, [z, x]] + (-1)^{qr}[z, [x, y]] = 0

holds.

3. For $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$, the relation $d[x, y] = [dx, y] + (-1)^{|x| + n}[x, dy]$ holds, i.e. the differential $d$ on $\mathfrak{g}$ is a derivation with respect to the bracket.

The category of dg-Lie algebras over $R$ in $C$ and bracket preserving maps of complexes will be denoted $\text{dgLie}_R(C)$. We have an obvious forgetful functor

\[ f : \text{dgLie}_R(C) \to \text{dg}_R(C) \]

to the category of dg-modules over $R$.

Every associative dg-algebra $A$ has a primordial dg-Lie algebra structure with Lie bracket

\[ [\cdot, \cdot] : A_p \otimes_k A_q \to A_{p+q} \]

given by $[x, y] = xy - (-1)^{pq}yx$. The left adjoint

\[ U : \text{dgLie}_R(C) \to \text{dga}_R(C) \]

to the forgetful functor associates to $\mathfrak{g}$ its universal enveloping algebra $U(\mathfrak{g})$. This algebra has the following concrete form. Let $M$ be a dg-module over $R$ and denote by

\[ T : \text{dg}_R(C) \to \text{dga}_R(C) \]

the left adjoint to the forgetful functor which associates to $M$ the tensor algebra $T(M)$ of $M$. Explicitly, $T(M) = \bigoplus_{n \geq 0} M \otimes^n$ where $M^0$ is $R$ in degree 0 by convention. Then $U(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ by the two-sided ideal generated by the relations $[x, y] = x \otimes y - (-1)^{pq}(y \otimes x)$ where $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$.

One can endow $U(\mathfrak{g})$ with a filtration

\[ U(\mathfrak{g})^{\leq 0} \xrightarrow{i_0} U(\mathfrak{g})^{\leq 1} \xrightarrow{i_1} \ldots \]

where $U(\mathfrak{g})^{\leq n}$ is the image of $\bigoplus_{0 \leq i \leq n} \mathfrak{g}^{\otimes i}$ in $U(\mathfrak{g})$. The associated graded dg-algebra $\text{gr}(U(\mathfrak{g}))$ of this filtered object is given by $\text{gr}(U(\mathfrak{g})) := \bigoplus_{n \in \mathbb{N}} U(\mathfrak{g})_n$ for which the underlying complex of $U(\mathfrak{g})_n$ is $\text{coker}(i_n)$. This graded dg-algebra is moreover a graded commutative dg-algebra. Therefore the following result holds where

\[ \text{Sym}_R : \text{dgLie}_R(C) \to \text{cdga}_R(C) \]

is left adjoint to the forgetful functor.

**Proposition 4.2. (Quasi-abelian Poincaré-Birkhoff-Witt).** Let $C$ be a monoidal locally presentable quasi-abelian category and $R$ a commutative monoid object in $C$ containing the rational numbers. Let $\mathfrak{g}$ be a dg-Lie algebra over $R$. Then there exists an isomorphism

\[ \text{Sym}_R(\mathfrak{g}) \to \text{gr}(U(\mathfrak{g})) \]

of commutative dg-algebras over $R$.

**Proof.** The map $f : \text{Sym}_R(\mathfrak{g}) \to \text{gr}(U(\mathfrak{g}))$ is induced from the canonical morphism $\mathfrak{g} \to U(\mathfrak{g})^{\leq 1}$ and one can follow the general proof given in Section 1.3.7 of [DM].

In particular, there exists an isomorphism $U(\mathfrak{g}) \simeq \text{Sym}_R(\mathfrak{g})$ of dg-modules over $R$. 

Proposition 4.3. Let $C$ be a monoidal locally presentable quasi-abelian category and $R$ a commutative monoid object in $C$. Then there exists a combinatorial model structure on the category $\text{dgLie}_R(C)$ of differential graded Lie algebras over $R$ with the following classes of morphisms:

(F) A morphism is a fibration if and only if it is a fibration in the projective model structure on $\text{dg}_R(C)$.

(W) A morphism is a weak equivalences if and only if it is a weak equivalence in the projective model structure on $\text{dg}_R(C)$.

(C) The cofibrations are the those maps with the left lifting property with respect to trivial fibrations.

Proof. Let $g : \text{dg}_R(C) \to \text{dgLie}_R(C)$ denote the free functor which is left adjoint to the forgetful functor $f$. We then define the collection of generating cofibrations $\mathcal{C}_0$ to be $\{g(R[n - 1]) \to g(M(id_R, n))\}_{n \in \mathbb{Z}}$ using the notation in the proof of Proposition 2.12. It suffices to show that our classes of generating cofibrations and weak equivalences satisfy the conditions in Proposition A.2.6.13 of [L1]. Moreover, one must prove that a morphism in $\text{dgLie}_k(C)$ is a fibration if and only if it is a degreewise strict epimorphism. One can follow the proof in Proposition 2.1.10 of [L3] using the maps in the projective model structure of Proposition 2.12. □

We will call the model structure of Proposition 4.3 the projective model structure.

Example 4.4. It is clear that the category $\text{dgLie}_A^{\text{cborn}} := \text{dgLie}_A(C\text{Born}_k)$ of complete bornological dg-Lie algebras admits a projective model structures due to Proposition 2.12.

For the remainder of this paper we will use the notation $\bigwedge g := \text{Sym}(g[1])$.

Proposition 4.5. Let $C$ be a monoidal locally presentable quasi-abelian category and $R$ a commutative monoid object in $C$. Let $g$ be a dg-Lie algebra over $R$. Then $U(g) \otimes \bigwedge g$ is a cofibrant replacement for $R$ in the model category $\text{Mod}(U(g)) := \text{Mod}_{U(g)}(\text{dg}_R(C))$ of $U(g)$-modules.

Proof. Define the cone $g \oplus g[1]$ of $g$ to be the dg-Lie algebra with differential

$$d_n(x + \epsilon y) := dx + y - \epsilon dy$$

and Lie bracket

$$[x + \epsilon y, x' + \epsilon y'] = [x, x'] + \epsilon([y, x'] + (-1)^n[x, y'])$$

for any $x \in g_n$ and $y \in g_{n+1}$. The underlying dg-module of $g \oplus g[1]$ is the mapping cone of the identity on $g$. Therefore $g \oplus g[1]$ is contractible and thus there exists a reduced quasi-isomorphism $0 \to g \oplus g[1]$ in $\text{dg}_R(C)$. Since $U$ preserves reduced quasi-isomorphisms, the map $U(0) = R \to U(g \oplus g[1])$ is a reduced quasi-isomorphism. By the Poincaré-Birkhoff-Witt theorem, there exists an equivalence $U(g \oplus g[1]) \simeq \bigwedge g \otimes_k U(g)$ in $\text{Mod}(U(g))$ and the result follows. □

Example 4.6. Let $C$ be a closed symmetric monoidal locally presentable quasi-abelian category and $A$ an object in $\text{cdga}_R(C)$. Assume $A$ is endowed with an action of a dg-Lie algebra $g$ over $R$. Then a model for the quotient space $A/g$ is the dg-algebra $\underline{\text{Hom}}_g(k, A)$. Then we have a chain

$$\underline{\text{Hom}}_g(k, A) \simeq \underline{\text{Hom}}_{U(g)}(k, A) \simeq \underline{\text{Hom}}_{U(g)}(\bigwedge g \otimes_k U(g), A) \simeq \underline{\text{Hom}}_g(\bigwedge g, A) \hookrightarrow \bigwedge g' \otimes_k A$$

of equivalences. Let $A \in \text{cdga}_k^{\text{cborn}}$. Then the last inclusion is dense when $\bigwedge g$ satisfies the bornological approximation property [KM]. This is satisfied, for example, when $\bigwedge g$ is a nuclear Fréchet space.
5 Koszul-Tate resolutions

Let $C$ be a symmetric monoidal quasi-abelian category and $R$ a commutative monoid object in $C$. For any dg-$R$-module $P$, we will define an object $K(R, P)$ which we prove is a cofibrant replacement of $R$ in a certain model category of commutative dg-algebras. This dg-algebra is closely related to the Koszul complex (associated to an element in $P$) in the setting of dg-algebras in the quasi-abelian setting.

We endow dg$_R$ with the symmetric monoidal model structure of Proposition 2.12 and define the dual of a $R$-dg-module $P$ to be

$$P' := \text{Hom}(P, 1)$$

where 1 is the unit object of the monoidal structure. Note that this notion of dual differs from the algebraic dual in the model categories of bornological and convenient dg-modules. Denote by

$$Q := \text{Sym}(P')$$

the symmetric algebra on $P'$. Let $\wedge P'$ be the exterior algebra of $P'$ considered as a graded $R$-dg-algebra. The $Q$-dg-module $Q \otimes_R \wedge P'$ is then a (non-positively) graded commutative $Q$-dg-algebra where the grading is given by

$$(Q \otimes_R \wedge P')_m := Q \otimes_R \wedge^{-m} P'$$

for $m \leq 0$. We endow this graded commutative $Q$-dg-algebra with a differential as follows. Firstly, consider the map

$$h_{n+1} : Q \otimes_R \wedge^{n+1} P' \to Q \otimes_R \wedge^{n} P' \otimes_R \wedge^{n+1} P'$$

induced from the canonical map $i : R \to \text{Hom}(P, P) \simeq P' \otimes_R P$ sending $1_R$ to $\text{id}_P$. Secondly, consider the map

$$c_{n+1} : Q \otimes_R P' \otimes_R \wedge^{n+1} P' \to Q \otimes_R \wedge^{n} P'$$

induced by the action of $P' \subset Q$ on $Q$ by right multiplication and the action of $P \subset \wedge^{n+1} P'$ on $\wedge^{n} P'$ defined by $b \cdot (f_1 \wedge \ldots \wedge f_{n+1}) := \sum_j (-1)^j f_j (b f_1 \wedge \ldots \wedge \hat{f}_j \wedge \ldots \wedge f_{n+1})$. We then define the differential

$$d_{n+1} : Q \otimes_R \wedge^{n+1} P' \to Q \otimes_R \wedge^{n} P'$$

by $d_{n+1} := c_{n+1} \circ h_{n+1}$.

**Definition 5.1.** The dg-algebra over $Q$ given by

$$K(R, P) := (Q \otimes_R \wedge P', d)$$

is called the fancy Koszul algebra of $(R, P)$.

The zero section $R \to Q$ defines a natural map given by the composition

$$K(R, P) \to (Q \otimes_R \wedge P')_0 = Q \to R$$

which we call the augmentation map.

**Proposition 5.2.** The augmentation map $K(R, P) \to R$ is a cofibrant replacement of $R$ in the model category cdga$_Q$ of commutative dg-algebras over $Q$ with respect to the projective model structure.

**Proof.** Since every object is cofibrant, it suffices to check that $K(R, P) \to R$ is a trivial fibration. It is clearly a fibration so we will check that it is a weak equivalence. By definition, we need to show that

$$H^n(K(R, P)) \to H^n(R)$$
is an isomorphism of objects in $C$. The underlying dg-module of $P' \oplus P'[1]$ is the mapping cone of the identity on $P'$. Therefore $0 \to P' \oplus P'[1]$ is a reduced quasi-isomorphism. A map between dg-modules $M \to N$ over $R$ is a reduced quasi-isomorphism if and only if $\text{Sym}_R(M) \to \text{Sym}_R(N)$ is a quasi-isomorphism. Therefore

$$\text{Sym}_R(0) = R \to \text{Sym}_R(P' \oplus P'[1]) \simeq K(R, P)$$

is a reduced quasi-isomorphism and the result follows.

The usual Koszul algebra is

$$K(R, P; m) := (\bigwedge P', d_m)$$

which is a dg-algebra over $R$ for any $m \in P$. The differential $d_m$ is induced by contraction along $m$. This choice of element $m$ also induces a map $Q \to R$ given by evaluation at $m$. We denote by $R_m$ the dg-algebra (concentrated in degree 0) with this $Q$-algebra structure.

**Lemma 5.3.** There exists an equivalence

$$R_m \otimes_Q K(R, P) \to K(R, P; m)$$

of dg-algebras over $Q$.

**Proof.** We view $\bigwedge P'$ as an $S$-module via the composite morphism $Q \to R \to \bigwedge P'$ of dg-algebras over $k$ where the first map is given by evaluation at $m$. The underlying graded $Q$-module of $R_m \otimes_Q K(R, P)$ is then $R_m \otimes_Q Q \otimes_Q P' \simeq \bigwedge P'$. Show that the induced differential on $R_m \otimes_Q K(R, P)$ is $d_m$. □

Lemma 5.3 extends to an equivalence of dg-algebras over $R$ via the canonical map $R \to Q$.

**Example 5.4.** We denote by $\text{Aff}^{\text{born}}_k := (\text{cdga}^{\text{born}}_k)^{\circ}$ the opposite of the category of complete bornological dg-algebras over $k$. One can define an étale topology on the homotopy category of complete bornological dg-algebras over $k$ and thus a model category $\text{Sh}_{\text{Set}}(\text{Aff}^{\text{born}}_k, \text{et})$ of stacks on the model site $(\text{cdga}^{\text{born}}_k, \text{et})$ (see [Wa] for more details).

Let $A$ be a complete bornological dg-algebra over $k$ and $X = \text{Spec} A$ the image of $A$ in $\text{Sh}_{\text{Set}}(\text{Aff}^{\text{born}}_k, \text{et})$. Set $T^* X := \text{Spec}(\text{Sym}(T_X))$ where $T_X := T_A$ is the tangent complex (dual to the cotangent complex) of the bornological dg-algebra $A$ defined in loc.cit. Let $d f : X \to T^* X$ be the differential for a function $f$ on $X$. Then the complete bornological stack given by the homotopy pullback

$$
\begin{array}{ccc}
X & \xrightarrow{d f} & T^* X \\
\downarrow & & \downarrow \\
\text{Crit}(f) & \rightarrow & X
\end{array}
$$

in the model category of complete bornological stacks, called the critical locus of $f$, can be calculated explicitly as follows. There exists a chain of equivalences

$$\text{Crit}(f) \simeq X \times_{(d f, T^* X, 0)} QX \simeq X \times_{(d f, T^* X, 0)} \text{Spec}(K(A, L_X)) \simeq \text{Spec}(K(A, L_X; d f))$$

of complete bornological stacks by Proposition 5.2 and Lemma 5.3. Therefore, the Koszul bornological stack

$$T^* X[-1] := \text{Spec}(\text{Sym}(T_X[1]), d_{d f})$$

is a model for the critical locus of $f$ and $(\text{Sym}(T_X[1]), d_{d f})$ for the complete bornological dg-algebra of functions on $\text{Crit}(f)$.  

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