Cosmological constant and equation of state of the quantum vacuum

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Recent studies of quantum field theory in FLRW spacetime suggest that the cause of the speeding up of the universe is the quantum vacuum, no need of ad hoc quintessence fields (see [11] [12]). Appropriate renormalization of the energy-momentum tensor shows that the vacuum energy density is a smooth function of the Hubble rate and its derivatives: \( \rho_{\text{vac}} = \rho_{\text{vac}}(H, \dot{H}, \ddot{H}, ...) \). This is because in QFT the quantum scaling of \( \rho_{\text{vac}} \) with the renormalization point turns into cosmic evolution with \( H \). As a result, any two nearby points of the cosmic expansion during the standard FLRW epoch are smoothly related through \( \delta \rho_{\text{vac}} \sim O(H^2) \). In this scenario, no fine tuning is needed at all. What we call the ‘cosmological constant’ \( \Lambda \) is just the nearly sustained value of \( 8\pi G \rho_{\text{vac}}(H) \) around (any) given epoch, where \( G(H) \) is the running gravitational coupling. In the very early universe, higher (even) powers \( \rho_{\text{vac}} \sim O(H^N) \) \((N = 4, 6, ...)\) triggered fast inflation during a short period in which \( H = \text{const} \), no need of ad hoc inflatons. In that period, the equation of state (EoS) of the vacuum is very close to \( w_{\text{vac}} = -1 \), but this ceases to be true during the FLRW era. Amazingly, the quantum vacuum acts as a formidable cosmic chameleon: it subsequently adopts the EoS of matter during the relativistic \((w_{\text{vac}} = 1/3)\) and non-relativistic \((w_{\text{vac}} = 0)\) epochs, and in the late universe it mimics quintessence, \( w_{\text{vac}} \gtrsim -1 \), only to tend again to \(-1\) in the remote future. In the transit, the quantum vacuum helps to solve the \( H_0 \) and \( \sigma_8 \) tensions.

I. INTRODUCTION: \( \Lambda \) AND THE COSMOLOGICAL CONSTANT PROBLEM

After 105 years of history [1], one of the most perplexing aspects of the cosmological constant (CC), \( \Lambda \), in Einstein’s gravitational field equations is that we still don’t know what it is and why it has the value that we have measured. It is usually associated to the energy density \( \rho_{\text{vac}}(t) \), which is usually associated to the energy density \( \rho \) of the quantum vacuum, or maybe the quantum mechanical vacuum, or else? In addition, we naively assume that it remains strictly constant throughout the cosmic evolution. There is actually no need for that, since a (direct and/or indirect) dependence on the cosmic time, i.e. \( \rho_{\text{vac}}(t, \zeta) \), is perfectly compatible with the Cosmological Principle, where \( \zeta = \zeta(t) \) is some dynamical variable. Still, we prefer to believe that \( \Lambda \) is a fundamental constant of Nature, maybe because we feel that in this way Occam’s razor is safely on our side. But soon we come across a really nasty surprise: measurements show that its current value is of order \( \rho_{\text{vac}} \sim 10^{-47} \text{ GeV}^4 \sim (10^{-3} \text{eV})^4 \) [2] [3] in natural units. Such a value turns out to be far too smaller than any typical energy density in particle physics or quantum field theory (QFT), and hence we have not the slightest chance to provide a fundamental explanation for it. We realize that we are up against an unsurmountable brick wall: the ‘cosmological constant problem’ (CCP), which smashes Occam’s razor to pieces in our hands, and with it all our hopes for a possible understanding of the universe on fundamental grounds. The CCP is indeed the baffling realization that the successful QFT methods applied to the world of the elementary particles seem to predict an effective value for \( \rho_{\text{vac}} \) which is excruciatingly much larger than the current critical density of the universe \( \rho_c \) (which \( \rho_{\text{vac}} \) should be comparable to) [4] [7]. The Higgs boson, whose discovery (with a mass \( M_H \sim 125 \text{ GeV} \)) was made just 10 years ago certified the existence of the electroweak vacuum from spontaneous symmetry breaking (SSB) [8]. However, it presumably contributes a huge (positive) amount \( M_Z^2 \sim 10^8 \text{ GeV}^4 \) to the zero-point energy (ZPE) of the quantum vacuum, and also as much as (V) \( \propto -M_h^2 v^2 \sim -10^9 \text{ GeV}^4 \) (negative) from SSB, with \( v \sim 250 \text{ GeV} \) the Higgs vacuum expectation value (VEV). No less significant is the ZPE part from the top quark, which is \( \propto -m_t^4 \sim -10^9 \text{ GeV}^4 \) (negative because it is a fermion). With no a priori correlation between ZPE and SSB,
we expect that our QFT estimates are wrong by a factor of \((10^9/10^{-47}) \sim 10^{56}\). Yet, this blatant fiasco pales when compared to the VED yield from quantum gravity: \(M_{\text{Pl}}^4/\rho_{\text{vac}} \sim 10^{120}\), where \(M_{\text{Pl}} = (8\pi G_N)^{-1/2} = 2.43 \times 10^{18} \text{ GeV}\) is the (reduced) Planck mass. In the face of it, we are left flabbergasted and impotent!

In the next section, we fly over some of the troublesome issues that the notion of vacuum energy and cosmological constant faces in the context of flat spacetime. A proper treatment is only possible in curved spacetime, and this is what the rest of the paper is about.

II. VACUUM ENERGY IN FLAT SPACETIME

Because of the CCP, the quantum vacuum option for explaining dark energy (DE) with a \(\Lambda\)-term became outcast and was blamed of all evils, particularly of the acute fine tuning problem. This is unfair, of course, as all existing forms of DE are actually plagued with the same tuning illness and to a degree which is no lesser than that of the quantum vacuum itself \([7]\). Moreover, the vacuum is a most fundamental notion in QFT: we should expect that a description of the CCP and of the DE from first principles should actually come from the quantum vacuum and the machinery of QFT. A simple calculation on renormalizing the VED of a single free scalar field of the CCP and of the DE from first principles should actually come from the quantum vacuum and the machinery

\[\rho_{\text{vac}} = \rho_\Lambda(\mu) + \frac{m^4}{64\pi^2} \left(\ln\frac{m^2}{\mu^2} + C_{\text{vac}}\right).\]

Here \(\rho_\Lambda(\mu)\) is the renormalized cosmological term in the Einstein-Hilbert (EH) action and \(\mu\) is the usual 't Hooft’s mass unit of DR \([9]\). The second term on the r.h.s is the MS-renormalized ZPE at one-loop. In a symbolic way, we can write VED = \(\rho_\Lambda + \text{ZPE}\). This expression was made finite by the usual counterterm procedure: \(\rho_\Lambda^{(b)} = \rho_\Lambda(\mu) + \delta\rho_\Lambda\), wherein \(\rho_\Lambda^{(b)}\) is the starting (bare) coupling in the EH action and \(\delta\rho_\Lambda\) is the MS-counterterm in any of its variants, which leaves an arbitrary constant \(C_{\text{vac}}\) in the result after cancelling a pole in \(n = 4\) spacetime dimensions. This is prima facie all very simple in Minkowski space, but simplicity is not at all an advantage here, for Eq. (1) carries already the whole drama of the CCP. If that expression is interpreted as the VED, the ZPE part is proportional to \(m^4\), and hence (for any typical mass in particle physics) we have to fine tune \(\rho_\Lambda(\mu)\) to an incommensurable level (from 55 to 120 decimal places, see above) to produce \(\rho_\Lambda + \text{ZPE} \sim 10^{-47} \text{ GeV}^4\). Sheer nonsense. Not to mention the mandatory (hyperfine) retuning to be made at higher and higher orders of perturbation theory \([6, 7]\).

It is important not to confuse VED with CC. The former may exist in Minkowski spacetime, as given e.g. in Eq. (1), whereas the latter can only exist in the context of Einstein’s equations of curved spacetime and hence in the presence of gravity. Only in the last case the CC is physically meaningful and its value becomes inextricably intertwined with the VED through Einstein’s equations, as follows: \(\rho_{\text{vac}} = \Lambda/(8\pi G_N)\). We should not confuse the physical \(\Lambda\) defined in this way with the corresponding bare parameter in the EH-action, which is related to \(\rho_\Lambda\) given above in a similar way (but in this case the relation involves only the bare values of all the parameters involved). The problem with the above calculation is that it is of no use at all in curved spacetime, say in the cosmological Friedman-Lemaitre-Robertson-Walker (FLRW) background. There is no sense in associating the scale \(\mu\) to any physical quantity in that context since, if Einstein’s equations are invoked, the \(\Lambda\) term as such cannot exist in Minkowski space unless the VED is exactly \(\rho_\Lambda + \text{ZPE} = 0\). So there is no cosmology to do with the simple-minded expression \([1]\), despite some stubborn attempts in the literature. This point has been driven home recently in \([7]\), and in \([10]\). A realistic approach to the VED within QFT in curved spacetime is quite another and it has recently been put forward in extenso in the work \([11]\), which further expands the preceding one \([12]\). See also the review \([7]\) for a summarized account. We shall adopt this same framework here in order to investigate the equation of state (EoS) of the quantum vacuum, which as we shall see does not reduce to just the traditional (and naive) result \(w_{\text{vac}} = -1\). It turns out that the EoS of the quantum vacuum is dynamical and evolves as a nontrivial function of the cosmic expansion, \(w_{\text{vac}} = w_{\text{vac}}(H, \dot{H}, \ddot{H}, ...),\) where dots indicate differentiation with respect to cosmic time \(t\), i.e. \((\cdot) \equiv d(\cdot)/dt\).

III. COMPUTING THE VACUUM ENERGY DENSITY IN FLRW SPACETIME

Before we can face the computation of the EoS of the quantum vacuum in curved spacetime, we need to compute the vacuum energy density (VED) and vacuum pressure. This is sooner said than done, and we should not presume that they are related in the simple way \(P_{\text{vac}} = -\rho_{\text{vac}}\), which is valid only in the classical theory without quantum matter fields. In this section and the next, we summarize the approach and the main results presented at length in
insofar as concerns the calculation and renormalization of the VED in FLRW spacetime\(^1\). The reader mainly interested on the phenomenological results may now wish to jump directly to Sec.\(^V\) and skip some QFT technicalities.

To simplify the (usually arduous) computations in curved spacetime, and also to minimize the number of parameters involved, we use just a single quantum matter scalar field \(\phi\) with mass \(m\), nonminimally coupled to curvature and without effective potential, hence with the action:

\[
S[\phi] = - \int d^n x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} (m^2 + \xi R) \phi^2 \right).
\]  

(2)

Even with this relatively simple system, in which \(\phi\) has no interactions with other fields nor with itself, QFT calculations become already quite cumbersome [11]. The ZPE associated to \(\phi\) is, of course, an UV-divergent quantity. Parameter \(\xi\) in the above action is the non-minimal coupling of \(\phi\) to gravity. It is well-known that in the special case \(\xi = 1/6\), the massless \((m = 0)\) action is (locally) conformal invariant in \(n = 4\) spacetime dimensions. Although \(\xi\) is not necessary for the QFT renormalization of the above action at one-loop, it is convenient to keep \(\xi\) arbitrary. In general, the presence of a nonminimal coupling is expected in a variety of contexts, e.g., in extended gravity theories [13–16].

There is also a fermionic contribution to the VED, of course, but it is not necessary for the present considerations [17].

In this Letter, therefore, we shall focus on the scalar contribution only.

First of all, we must compute the ZPE of \(\phi\) in FLRW spacetime. However, in contrast to the previous section, rather than using MS-renormalization to deal with the UV divergences also in the curved spacetime case (which proves inappropriate to deal with the CCP [6–7]), we adhere to adiabatic renormalization [18–20], although with a crucial nuance: we renormalize the energy-momentum-tensor (EMT) off-shell, meaning that we define its renormalized VEV (associated to the fluctuations \(\delta \phi\) of the fields) as follows [11][12]:

\[
\langle T^{\delta \phi}_{\mu\nu} \rangle_{\text{Ren}}(M) = \langle T^{\delta \phi}_{\mu\nu}(m) \rangle - \langle T^{\delta \phi}_{\mu\nu}(0-4)(M) \rangle.
\]  

(3)

The latter, as can be seen, is obtained by performing an appropriate substraction from its on-shell value (i.e., the value defined on the mass \(m\) of the quantized field), specifically we subtract the vacuum EMT value (i.e., its VEV) computed at an arbitrary scale \(M\). The result is finite because we subtract adiabatic orders up to order 4 (the only ones that can be divergent in \(n = 4\)). This is entirely different from MS since we subtract both UV-divergent and convergent parts at \(M\). The renormalization point \(M\) will be used later on as a renormalization group (RG) tool to explore the cosmic evolution at the expansion history time \(H(t)\) by setting \(M = H\). But here is left arbitrary. Let us note that the above renormalized EMT must be related with the (renormalized) effective action of the quantum vacuum, namely the action \(W\) describing the vacuum fluctuations of the quantized matter fields of QFT in FLRW spacetime [18–20]:

\[
\langle T^{\delta \phi}_{\mu\nu} \rangle = -\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}}.
\]  

(4)

This relation offers us a precious opportunity for a nontrivial cross-check. In fact, one can choose any pathway: we may either compute (3) directly by expanding the solution of the Klein-Gordon equation \((\Box - m^2 - \xi R)\phi = 0\) (satisfied by the quantum field operator \(\phi\) in FLRW spacetime) in Fourier field modes and letting the creation and annihilation operators to act on the vacuum with the usual commutation relations; or, alternatively, we may compute the (renormalized) effective action \(W\) through the DeWitt-Schwinger expansion [18–20] (upon carefully correcting their coefficients to account for the off-shell effects at the scale \(M\)), and then use Eq. (4) to retrieve the renormalized EMT. Let us note, in particular, that the Fourier field modes of the first method must be computed using the WKB expansion assuming the notion of adiabatic vacuum (which all of the annihilation operators must destroy) [21]. The gory details of this lengthy calculation can be found in the comprehensive studies [11][12], and we shall spare the reader from these thick technicalities here, of course (see [7] for a summarized exposition). The important point is that the two pathways must converge, and do indeed converge, exactly to the same result. Once the renormalized EMT is accounted for by any of these procedures, we must extract the renormalized VED out of it. We perform the calculation in the conformally flat metric, \(ds^2 = a^2(\tau) \eta_{\mu\nu} dx^\mu dx^\nu\), where \(\eta_{\mu\nu} = \text{diag}(-1,+1,+1,+1)\) is the Minkowski metric (\(\tau\) being the conformal time and \(a\) the scale factor of the FLRW line element). Since the renormalized VEV of the EMT at the scale \(M\) takes the form \(\langle T^{\text{vac}}_{\mu\nu} \rangle_{\text{Ren}}(M) = -\rho_A(M) g_{\mu\nu} + \langle T^{\delta \phi}_{\mu\nu} \rangle_{\text{Ren}}(M)\), the renormalized VED at that scale reads

\[
\rho_{\text{vac}}(M) = \frac{\langle T^{\text{vac}}_{00} \rangle_{\text{Ren}}(M)}{a^2} = \rho_A(M) + \frac{\langle T^{\delta \phi}_{00} \rangle_{\text{Ren}}(M)}{a^2}.
\]  

(5)

\(^1\) We adopt the same conventions as in [11]. See, in particular, Appendix A of that reference.
We can see that the above expression also adopts the structure \( \rho_\Lambda = \rho_{\Lambda} + \text{ZPE} \), where the 00th component (the ZPE) emerges from the explicit calculation of (3) in the FLRW metric [11] [12]. Up to 4th adiabatic order, a lengthy calculation yields the following compact result:

\[
\langle T_{00}^{(0-4)} \rangle_{\text{Ren}} = \frac{a^2}{128\pi^2} \left( -M^4 + 4m^2M^2 - 3m^4 + 2m^4 \ln \frac{m^2}{M^2} \right) \\
- \left( \xi - \frac{1}{6} \right) \frac{3H^2}{16\pi^2} \left( m^2 - M^2 - m^2 \ln \frac{m^2}{M^2} \right) + \left( \xi - \frac{1}{6} \right) \frac{9}{16\pi^2} \frac{2}{a^2} (2H^nH - \dot{H}^2 - 3\dot{\mathcal{H}}^2) \ln \frac{m^2}{M^2}.
\]

(6)

This expression is finite and explicitly dependent on the scale \( M \), the mass \( m \) of the particle and the conformal Hubble rate \( \mathcal{H} \) and its time derivatives in conformal time (related to the ordinary Hubble rate in cosmic time \( t \) simply as \( \mathcal{H}(\eta) = aH(t) \)). Primes denote differentiation with respect to conformal time: \( (\cdot)' = \partial(\cdot)/\partial\tau \). With \( \langle T_{\mu\nu} \rangle_{\text{Ren}}(M) \) given as above, the renormalized vacuum part of the generalized Einstein’s equations within QFT in curved spacetime is

\[
\rho_{\text{vac}}(M, H) - \rho_{\text{vac}}(M_0, H_0) = \frac{3}{16\pi^2} \left[ H^2 \left( M^2 - m^2 + m^2 \ln \frac{m^2}{M^2} \right) \\
- H_0^2 \left( M_0^2 - m^2 + m^2 \ln \frac{m^2}{M_0^2} \right) \right] + \mathcal{O}(H^4).
\]

(7)

The above result is a perfectly smooth function with no quartic mass terms \( \sim m^4 \) (see next section). In addition, the quadratic ones \( \sim m^2 \) become completely smoothed by the \( H^2 \) factor. Whence, the terms \( \sim m^2H^2 \) are fully innocuous for the CCP; and, finally, those of order \( \mathcal{O}(H^4) \) are irrelevant for the current universe.

IV. \( \beta \)-FUNCTION FOR THE VACUUM ENERGY DENSITY

A chief result which can be derived from equations (5) and (6) is the expression for the \( \beta \)-function driving the RG-running of the vacuum energy density, \( \rho_{\text{vac}} \). This important result was not know until very recently [11]:

\[
\beta_{\rho_{\text{vac}}} = M \frac{\partial \rho_{\text{vac}}(M)}{\partial M} = \left( \xi - \frac{1}{6} \right) \frac{3H^2}{8\pi^2} (M^2 - m^2) + \left( \xi - \frac{1}{6} \right) \frac{9}{8\pi^2} \frac{2}{a^2} (2H^nH - \dot{H}^2 - 6H^2\dot{H}) \nonumber.
\]

(8)

where we have used \( \mathcal{H}' = a^2(\dot{H}^2 + \dot{H}) \) and \( \mathcal{H}'' = a^3 \left( 2\dot{H}^3 + 4H\dot{H}^2 + \ddot{H} \right) \), and also the fact that the \( \beta \)-function for the renormalized parameter \( \rho_{\Lambda} \) in the EH-action is

\[
\beta_{\rho_{\Lambda}}(M) = M \frac{\partial \rho_{\Lambda}(M)}{\partial M} = \frac{1}{2(4\pi)^2} (M^2 - m^2)^2.
\]

(9)

The latter ensues from the fact that in Minkowski space \( (H = 0) \) the expression (5) must be RG invariant, as it is indeed the case with (11) in the MS scheme [7]. In both renormalization schemes, the flat spacetime expressions correspond originally to renormalizing a bare coupling and hence they are globally independent of \( M \) (the renormalization point)\(^3\). Again the terms \( \mathcal{O}(H^4) \) are irrelevant for the present universe. The obtained \( \beta \)-function of the VED is thus very softly

\(^2\) The scale \( M \) should not be confused with ’t Hooft’s mass unit \( \mu \) in DR [9]. Both scales may appear simultaneously in the calculations, with \( \mu \) playing here (optionally) a mere auxiliary role in intermediate steps (e.g. if one opts for using DR to deal with the divergent integrals), see [11] [12]. Since, however, we are not using at all the MS scheme as a renormalization procedure, the renormalized results cannot depend on \( \mu \), only on \( M \), and hence the setting \( M = H \) at the end of the renormalized calculation can only be performed with \( M \). Needless to say, the full effective action does not depend on \( M \) either, but the renormalized VED does [11].

\(^3\) In Minkowski spacetime there is nothing else in the vacuum action apart from the term \( \rho_{\Lambda} \). In curved spacetime, in contrast, we have also the curvature scalar plus the geometric HD terms. The renormalization of the VED is then not just the renormalization of a bare term, as in fact the VED becomes explicitly dependent on \( H \) as well as on \( M \), as we have just seen. In this case, only the full effective action (involving the classical part plus the nontrivial quantum vacuum effects) is scale- (i.e. RG-) independent, as previously noted.
FIG. 1: The plot on the left shows the evolution of the different energy densities with the expansion in the canonical RVM context, Eq. (11). The inner window serves to magnify the low redshift region. The right plot provides a complementary view using a (vertical) logarithmic scale. The VED exhibits a very mild dynamics up to the radiation dominated epoch.

dependent on the mass scale, just as $\beta_{\rho_{\text{vac}}} \propto H^2 (M^2 - m^2) + \mathcal{O}(H^4)$ rather than the traditional (and troublesome!) form $\beta_{\rho_{\text{vac}}} \propto m^4$. This explains the cancellation of quartic terms in performing the subtraction (7) in the previous section [11]. It follows that when one considers the evolution of the VED in this approach, there is no influence whatsoever from the dangerous quartic mass terms in (6). It is worth noticing that the result (8) is exact and does not depend on the fact that (6) was computed up to 4\textsuperscript{th} adiabatic order, as the higher order terms (order 6\textsuperscript{th} and above) are finite and hence do not depend on $M$.

It is remarkable that the $\beta$-function (8) for the VED running is crucially involved also in the local conservation law of the VED:

$$\dot{\rho}_{\text{vac}} + 3H (\rho_{\text{vac}} + P_{\text{vac}}) = \frac{\dot{M}}{M} \beta_{\rho_{\text{vac}}}.$$ (10)

This, truly elegant, equation has a clear interpretation: it shows that the non-conservation of the VED is due to both the running of $\rho_{\text{vac}}$ with $M$ (i.e. the fact that $\beta_{\rho_{\text{vac}}} \neq 0$) and also to the cosmic time dependence of $M$ (viz. $\dot{M} \neq 0$) [11]. This novel feature of the VED in curved spacetime is in contradistinction to ordinary gauge theories of strong and electroweak interactions in flat spacetime. Interestingly, it makes possible to probe the effect of the (cosmic) time-dependence of the running couplings and masses in the particle and nuclear physics world, and hence it may ultimately provide a possible theoretical explanation [22, 23] for the purported evolution of the fundamental ‘constants’ of Nature, as claimed in some experiments [24]. Modern attempts at challenging the stability of the fundamental ‘constants’ can be seen e.g. in [25] and references therein.

V. THE EVOLVING VED IN THE PRESENT EPOCH: THE CANONICAL RVM

What about VED physics? The measurable difference between the VED values at different epochs of the cosmic evolution, say $H_0 = H(t_0)$ and $H = H(t)$ within our observational range, now follows from the usual RG prescription, based on choosing the renormalization points near the corresponding values of the energy scales, in this case $M_0 = H_0$ and $M = H$ (hence bringing them near the physical state of the FLRW spacetime at each epoch). For simplicity, we denote these VED values as $\rho_{\text{vac}}(H_0)$ and $\rho_{\text{vac}}(H)$, respectively. Using (7), the leading result can be cast as follows [11, 12]:

$$\rho_{\text{vac}}(H) = \rho_{\text{vac}}^0 + \frac{3\nu_{\text{eff}}(H)}{8\pi} \rho_{\text{vac}}^0 m_{\text{Pl}}^2 \left( H^2 - H_0^2 \right) + \mathcal{O}(H^4),$$ (11)

with $m_{\text{Pl}} = \sqrt{G_N^{-1/2}}$ the usual Planck mass. As usual, we shall neglect the $\mathcal{O}(H^4)$ terms for all the considerations referring to the current universe (and for that matter for the entire FLRW regime, which is well away from the
early inflationary period). The effective running parameter $\nu_{\text{eff}}(H)$ is a (mildly evolving) function of $H$ during the FLRW regime and is given in the Appendix, but for the late time universe it suffices to take it constant, namely $\nu_{\text{eff}} \equiv \nu_{\text{eff}}(H_0)$:

$$
\nu_{\text{eff}} \simeq \epsilon \ln \frac{m^2}{H_0^2}, \quad \epsilon \equiv \frac{1}{2\pi} \left( \xi - \frac{1}{6} \right) \frac{m^2}{m_{\text{Pl}}^2}.
$$

(12)

Both $\epsilon$ and $\nu_{\text{eff}}$ are small parameter since $m^2 \ll m_{\text{Pl}}^2$ for any particle mass. Clearly, the dominant contribution to the VED running stems from the largest masses $m \sim M_X$, presumably from fields of a typical GUT scale $M_X \sim 10^{16}$ GeV (possibly including a large multiplicity factor). In the expression (11) we have identified $\rho_{\text{vac}}(H_0)$ with today’s VED value, $\rho_{\text{vac}}^0$, while $\rho_{\text{vac}}(H)$ stands for the VED at a nearby point $H$. Equation (11) constitutes the canonical form of the ‘running vacuum model’ RVM, yet freshly emerging here within the QFT framework expounded in [11,12].

From the foregoing discussion, we learn that QFT in curved spacetime predicts that the VED is a slowly evolving function of the cosmological expansion, and hence the $\Lambda$-term too. We can better appraise the evolution of the VED in a graphical way in Fig. 1. Parameters are taken from the best-fit values of [3]. On the left plot of Fig. 1 we show the evolution of the matter densities (relativistic and nonrelativistic) together with the slow evolution of the vacuum density. On the right plot we depict a logarithmic representation of the various densities such that the differences can be better appreciated, especially in the case of the VED. The curves are displayed for different typical values of $\nu_{\text{eff}}$. Despite of the fact that the VED evolution is very mild, of course, its EoS is nevertheless potentially observable, see Sec. VIII.

VI. RUNNING GRAVITATIONAL COUPLING

The evolution of the VED preserves the Bianchi identity provided there is an exchange with another dynamical variable [6,7]. If we assume local matter conservation (i.e. no exchange between the vacuum and all matter components such as dust or radiation density, collectively represented by $\rho_m$), then the gravitational coupling $G$ must vary with the cosmic expansion to compensate for the VED running. Furthermore, if we consider the late universe, in which we can neglect the $\mathcal{O}(H^4)$ renormalization effects on the VED. Using the formalism of [11] we find that the cosmic time evolution of the VED is connected to that of $G$ as follows:

$$
\dot{\rho}_{\text{vac}} + 3H (\rho_{\text{vac}} + P_{\text{vac}}) = -\frac{G}{H} (\rho_m + \rho_{\text{vac}}) = -\frac{\dot{G}}{G} \frac{3H^2}{8\pi G},
$$

(13)

where Friedmann’s equation has been called for under the assumption that the higher order gravitational terms do not contribute in the current universe. The above equation brings forward again the subject of the possible variation of the fundamental ‘constants’, as previously mentioned in Sec. IV. In fact, all ‘constants’ vary if one does! In the present QFT framework, all of them would be functions of the Hubble rate, as first noted in [22]. Now since the r.h.s.’s of equations (10) and (13) must obviously be equal, we can use our prescription $M = H$ to explore the evolution of the VED at the cosmic history time $H(t)$. Taking the leading terms of $\beta_{\rho_{\text{vac}}}$ from the r.h.s. of (8) for the present universe, we readily obtain the following differential equation

$$
\frac{1}{G^2} \frac{dG}{dt} = \frac{\pi}{2\epsilon} \frac{m^2 dH}{H^2 dt},
$$

(14)

in which the explicit time dependence drops from both sides. Integrating it by simple quadrature from the present time (characterized by the Hubble rate $H_0$ and gravitational coupling $G(H_0) = G_N$) up to an arbitrary point within the FLRW regime, $(H(t), G(H(t)))$, we obtain the desired running law for the gravitational coupling as a function of the Hubble rate:

$$
G(H) = \frac{G_N}{1 - \epsilon \ln \frac{H^2}{H_0^2}}.
$$

(15)

Notice that $G_N = G(H_0)$ is the current local gravity value (Newton’s ‘constant’), usually associated to the inverse Planck mass squared: $G(H_0) = G_N = 1/m_{\text{Pl}}^2$ (in natural units). The parameter $\epsilon$ in (15) is the same, of course, as the one previously defined in (12). It is apparent that for $\epsilon = 0$ (hence $\nu_{\text{eff}} = 0$), both $\rho_{\text{vac}}$ and $G$ cease to be running quantities since they do not feel the quantum vacuum effects. But for $\epsilon \neq 0$ ($\nu_{\text{eff}} \neq 0$) there is indeed a dynamical exchange between the two quantities which insures the perfect fulfilment of the Bianchi identity and shows the consistency of the obtained result. One can also determine the explicit form of the running couplings for the gravitational HD terms [11,12,27], but the most relevant running laws for our purposes are those for $\rho_{\text{vac}}$ and $G$. They are both necessary to compute the EoS of the quantum vacuum (see Appendix A for details).
FIG. 2: Inflationary period. On the left it is shown the evolution of the energy densities of vacuum and relativistic matter before and after the transition point \( a_\ast \sim 10^{-29} \) (where \( \hat{a} \equiv a/a_\ast \) takes the value 1) from inflation to the early radiation epoch (see the text). The (constant) VED during inflation decays into radiation and the standard FLRW regime starts. On the right we can see the evolution of the vacuum EoS from \( w_{\text{vac}} \simeq -1 \) up to \( \hat{a} = 1 \). Once this point is left well behind (\( \hat{a} \gg 1 \)), the vacuum evolves into an incipient radiation phase and adopts its EoS: \( w_{\text{vac}} \rightarrow 1/3 \).

VII. EOS OF THE QUANTUM VACUUM IN THE INFLATIONARY EPOCH

As it was recently shown in [11], inflation is another consequence of the QFT-driven universe. There is no need to introduce explicit, ad hoc, inflaton fields in the classical action. In truth, inflation in the very early universe can be produced by pure quantum effects in QFT in curved spacetime. To bring about inflation we need (even) powers of \( H \) beyond \( \sim H^2 \), i.e. \( H^N (N = 4, 6, \ldots) \). Inflation then proceeds through a short period where \( H = \text{const.} \) We call this mechanism RVM-inflation\([11, 12]\), see also\([28]\). The needed powers of \( H \) emerge from calculating the ZPE up to 6th adiabatic order (not shown in Eq. (6)). The \( \sim H^4 \) ones disappear in the adiabatic subtraction procedure, see however\([29, 30]\) and references therein for a related (stringy) approach\(^4\). In the present context, therefore, the \( H^6 \) terms take over during inflation. Their computation is rather cumbersome\([11]\), but these terms are finite and do not require renormalization. The final result can be condensed as follows:

\[
\rho_{\text{vac}}^\text{inf} = \frac{(T_{00}^{(6)})_{\text{Ren}}(m)}{a^2} = \frac{\tilde{\xi}}{80\pi^2 m^2} H^6 + f(\dot{H}, \ddot{H}, \ldots),
\]

where we have defined the parameter \( \tilde{\xi} = (\xi - \frac{1}{6}) - \frac{2}{3\xi} - 360 \left( \xi - \frac{1}{6} \right)^3 \). The remaining terms are collected in the complicated function \( f(\dot{H}, \ddot{H}, \ldots) \). They carry along many different combinations of powers of \( H \) accompanied in all cases with time derivatives of \( H \), and hence they all vanish for \( H = \text{const.} \). This means that a short period where \( H = \text{const.} \) can trigger inflation only from the \( \sim H^6/m^2 \) term indicated above, where \( m \sim M_X \sim 10^{16} \text{ GeV} \). During this initial phase, we find that the quantum vacuum behaves as ‘true’ vacuum with equation of state (EoS) \( w_{\text{vac}} = -1 \) (as we shall comment in a moment). Moreover, the Hubble rate evolves very little around an initial (big) value \( H_I \sim M^{1/2}_P m^{1/2}\xi^{-1/4} \), namely \( H(\hat{a}) = H_I \left( 1 + \hat{a}^8 \right)^{-1/4} \simeq H_I \) for \( 0 < \hat{a} < a_\ast \), where \( a_\ast \) is the transition point from the regime of vacuum dominance into that of radiation dominance (see Eq. (17) below), and we have defined \( \hat{a} \equiv a/a_\ast \). The point \( a_\ast \) is estimated to be around \( a_\ast \sim 10^{-29} \) in\([31]\). Since \( \dot{H} = -2H^2 \hat{a}^8/(1 + \hat{a}^8) \), we have \( |\dot{H}/H^2| \propto \hat{a}^8 \ll 1 \) for \( \hat{a} \ll 1 \) and we can safely neglect \( \dot{H} \approx 0 \), and successive derivatives, during inflation. The vacuum state rapidly

\(^4\) These powers may reappear in our QFT framework after we perform the setting \( M = H \) in the inflationary epoch. However, here we wish to focus on the genuine \( H^6 \)-powers left over from our subtraction procedure. We will investigate this matter further elsewhere.
decays into radiation and one may compute the explicit form of the fast evolving energy densities:

\[ \rho_r(\dot{a}) = \rho_I \dot{a}^8 (1 + \dot{a}^8)^{-\frac{7}{2}}, \quad \rho_{\text{vac}}(\dot{a}) = \rho_I (1 + \dot{a}^8)^{-\frac{3}{2}}. \]  

(17)

In Fig. 2 (left) we depict the evolution of the vacuum and radiation densities. Remarkably, we can see that at the beginning \( (a = 0) \) there is no radiation at all \( (\rho_r(0) = 0) \), whilst the VED at this point is maximal, namely \( \rho_{\text{vac}}(0) = \rho_I \propto M_{\text{Pl}}^4 H_0^2 \), but finite. Whence, there is no initial singularity in this approach. For \( \dot{a} \gg 1 \) (i.e. \( a \gg a_* \)), it is reassuring to see that we retrieve the standard decaying behavior of radiation, \( \rho_r(a) \sim a^{-4} \). In the meantime, the primordial VED decreases very fast. Thus, we predict RVM-inflation followed by a standard FLRW radiation epoch from QFT. This type of scenario was assumed phenomenologically in [34–38], where the precise formula is given. A sufficiently accurate approximation to the quantum vacuum EoS during the entire FLRW cosmic stretch reads as follows:

\[ P_{\text{vac}}(M) = -\rho_{\text{vac}}(M) + f_2(M, \dot{H}) + f_4(M, H, \dot{H}, ..., \ddot{H}) + f_6(\dot{H}, ..., \dddot{H}) + \cdots, \]

(18)

in which the functions \( f_2, f_4 \) and \( f_6 \) involve adiabatic contributions of second, fourth and sixth order, respectively, and all of them carry at least one time derivative of \( H \). Therefore, all these functions vanish for \( H = \text{const.} \) (\( \dot{a} \ll 1 \)) and we get \( P_{\text{vac}} = -\rho_{\text{vac}} \) to a very good approximation. The RVM inflationary period is thus characterized by the traditional EoS of vacuum, \( w_{\text{vac}} = -1 \). This can be appreciated in Fig. 2 (right).

VIII. EOS OF THE QUANTUM VACUUM IN THE FLRW REGIME

We have just seen that the vacuum EoS, \( w_{\text{vac}} \), during the inflationary epoch is very close to \(-1\), but the more we approach the radiation epoch the more it departs from \(-1\) and transmutes into \(+1/3\), as it can also be clearly seen in Fig. 2 (right). Quantum effects (starting at the end of the inflationary epoch for \( \dot{a} \gtrsim 1 \)) trigger a fully dynamical behavior of \( w_{\text{vac}} \) during the subsequent cosmic evolution within the conventional FLRW regime. As a result, the EoS of the quantum vacuum does not remain stuck to the classical value \( w_{\text{vac}} = -1 \) and indeed changes throughout different epochs. Such an evolution can be explicitly derived from the QFT framework of [11, 12]. Some details of the calculation are provided in the Appendix A where the precise formula is given. A sufficiently accurate approximation to the quantum vacuum EoS during the entire FLRW cosmic stretch reads as follows:

\[ w_{\text{vac}}(z) = -1 + \frac{\nu_{\text{eff}} (\Omega_m^0 (1 + z)^3 + 4^3 \Omega_r^0 (1 + z)^4)}{\Omega_{\text{vac}}^0 + \nu_{\text{eff}} [-1 + \Omega_m^0 (1 + z)^3 + \Omega_r^0 (1 + z)^3 + \Omega_\text{vac}^0]}, \]

(19)

where \( \Omega_{\text{vac}}^0 = \rho_{\text{vac}}^0 / \rho_c^0 \simeq 0.7 \) is the current vacuum cosmological parameter, whereas \( \Omega_m^0 = \rho_m^0 / \rho_c^0 \simeq 0.3 \) and \( \Omega_r^0 = \rho_r^0 / \rho_c^0 \sim 10^{-4} \) are the corresponding matter and radiation parts. Since \( |\nu_{\text{eff}}| \ll 1 \) and \( \Omega_r^0 \ll \Omega_m^0 \), it is readily seen that for small \( z \) the previous formula boils down to

\[ w_{\text{vac}}(z) \simeq -1 + \nu_{\text{eff}} \frac{\Omega_m^0}{\Omega_{\text{vac}}^0} (1 + z)^3 \quad (z < O(1)), \]

(20)

thus recovering the approximate result first advanced in [11]. Here, however, we have generalized this result into the more complete formula (19) for the full FLRW regime (cf. Appendix A).

The above EoS formulas depend on the crucial coefficient \( \nu_{\text{eff}} \), which we have computed in QFT but it must ultimately be fitted to the cosmological data [34, 35]. These analyses show that \( \nu_{\text{eff}} \sim 10^{-2} - 10^{-3} \) and that \( \nu_{\text{eff}} > 0 \) is the preferred sign. From the foregoing considerations, we find that the quantum vacuum never has the exact EoS \( w_{\text{vac}} = -1 \) during the FLRW stage, not even at \( z = 0 \), where

\[ w_{\text{vac}}(0) \simeq -1 + \nu_{\text{eff}} \frac{\Omega_m^0}{\Omega_{\text{vac}}^0} \gtrsim -1 \quad (\nu_{\text{eff}} > 0). \]

(21)

Thus, amazingly, the quantum vacuum currently behaves as quintessence. Such an effective behavior is triggered by

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5 Equation (20) resembles previous effective EoS forms for the dynamical VED derived phenomenologically in [39, 40], although it is fundamentally different from them since it predicts a quintessence behavior of the quantum vacuum already at \( z = 0 \), in contrast to the aforementioned forms which predict the conventional behavior \( w_{\text{vac}} = -1 \) at \( z = 0 \). Not to mention, of course, that the EoS we have found here is derived from explicit QFT calculations.
FIG. 3: Vacuum EoS for different (positive) values of $\nu_{\text{eff}}$ [34]. Some regimes to be noted: i) $w_{\text{vac}} \simeq -1$ for very low redshift, ii) $-1 < w_{\text{vac}} < -1/3$, vacuum mimics quintessence for low and intermediate redshift (the horizontal dotted line marks off the DE threshold $w_{\text{vac}} = -1/3$), iii) $w_{\text{vac}} = 0$ plateau, vacuum imitates dust matter, and iv) $w_{\text{vac}} = 1/3$ plateau, vacuum mimics radiation. The quantum vacuum behaves as a cosmic chameleon.

the quantum effects and there is no need to introduce ad hoc quintessence fields (nor ad hoc inflatons, as shown in the previous section). In Fig. 3 we provide a detailed plot of the EoS (19) for a large window of the FLRW regime well beyond the inflationary epoch. The plot is performed for different values of $\nu_{\text{eff}}$ in a wide redshift range spanning from the present time up to the radiation epoch. The approximate EoS (20) is only valid for the most recent universe and deviates significantly from the more accurate one (19) for intermediate or large values of $z$. This can be clearly seen in Fig. 4 where the two formulas are plotted to ease the comparison and to evince the large deviation at higher and higher redshifts. Notice that the detailed plot of the vacuum EoS in Fig. 3 interpolates in a numerical way the results that can be directly inferred analytically from Eq. (19) for the different redshift intervals all the way from the radiation epoch, down to the matter-dominated epoch until reaching the current epoch. Denoting by $z_{\text{eq}} = \Omega_0^m / \Omega_0^r - 1 \simeq 3300$ the equality point between matter and radiation, we find

$$w_{\text{vac}}(z) = \begin{cases} 
\frac{1}{3} & \text{for } z \gg z_{\text{eq}} \text{ with } \Omega_r^0 (1 + z) \gg \Omega_m^0, \text{ radiation behavior (} \nu_{\text{eff}} \neq 0 \text{)}, \\
0 & \text{for } \mathcal{O}(1) < z \ll z_{\text{eq}} \text{ with } \Omega_m^0 \gg \Omega_r^0 (1 + z), \text{ dust behavior (} \nu_{\text{eff}} \neq 0 \text{)}, \\
-1 + \nu_{\text{eff}} \Omega_r^0 (1 + z)^3 & \text{for } -1 < z < \mathcal{O}(1), \text{ quintessence behavior (} \nu_{\text{eff}} > 0 \text{)}. 
\end{cases} \tag{22}$$

As we can see, the quantum vacuum EoS follows the EoS of relativistic matter in the radiation-dominated epoch, then the EoS of non-relativistic (dust) matter in the matter-dominated epoch, the EoS of quintessence at present (for $\nu_{\text{eff}} > 0$) and asymptotes to de Sitter era in the future ($z \rightarrow -1$).

In the presence of the quantum vacuum effects, the deceleration parameter $q = -1 - \dot{H}/H^2$ can be easily derived. Using the expression for the quantum corrected $H$ derived in Appendix A up to order $\mathcal{O}(\nu_{\text{eff}})$ and requiring that $q = 0$ we find that the transition redshift from deceleration to acceleration becomes slightly shifted with respect to that of the concordance model (aka $\Lambda$CDM), as follows:

$$z_t = \left(\frac{2 \Omega_r^0 (1 - \nu_{\text{eff}}) \Omega_m^0}{\Omega_m^0 (1 + \nu_{\text{eff}})}\right)^{1/3} - 1. \tag{23}$$

As expected, the $\Lambda$CDM result is recovered for $\nu_{\text{eff}} = 0$. Since, however, $\nu_{\text{eff}}$ is small and $z_t$ cannot be measured with precision yet, it is not the ideal signature. What it really acts as a useful signature of the quantum vacuum is its
effective behavior as quintessence in the low redshift range, as we have seen above. Indeed, amazingly enough, the quantum vacuum is kind of ‘chameleonic’. It behaves as ‘true’ vacuum \( w_{\text{vac}} = -1 \) only in the very early times when it triggers inflation. It then remains silent for eons (hidden as if being relativistic or pressureless matter). Today, it appears as (dynamical) dark energy (DE), specifically as quintessence \( -1 < w_{\text{vac}} \lesssim -1/3 \), cf. Fig. 3. As a result of this multifaceted behavior, it may crucially help in solving the \( \sigma_8 \) and \( H_0 \) tensions afflicting the \( \Lambda \text{CDM} \) model, see e.g. \[41–45\] and the long list of references therein on different models and alternative points of view. In fact, in \[34\] it was shown that if there is a ‘DE threshold’ \( z^* \) near our time where the DE dynamics of the vacuum gets suddenly activated, this can be extremely helpful for solving the \( \sigma_8 \) tension within the RVM. At the same time, it was shown that if the gravitational coupling runs slowly (logarithmically) with the expansion, this can help to fix the \( H_0 \) tension. In Fig. 3 we can see that a continuous (i.e. not abrupt) DE ‘threshold’ window with low \( z^* = O(1) \) does indeed exist for the quantum vacuum, in the sense that for \( z < z^* \) the vacuum gets progressively activated as DE \( (w_{\text{vac}} < -1/3) \), whereas for \( z > z^* \) the EoS of the quantum vacuum transmutes successively into that of dust matter and radiation. Additionally, in \[11\] we found that the dynamics of the vacuum is intertwined with that of the gravitational coupling through a log of the Hubble rate: \( G = G(\ln H) \). Being these two crucial factors simultaneously present in our QFT approach, they combine constructively to relieve both tensions at a time. Incidentally, we note that the vacuum EoS for the current universe \( (20) \) in our QFT approach is similar to the EoS of the effective dark energy (DE) in a Brans-Dicke (BD) theory in the presence of a cosmological constant, see Ref. \[46\] for details.

IX. COSMOLOGICAL ‘CONSTANT’ AND VACUUM ENERGY DENSITY

The nontrivial modification of the EoS of the quantum vacuum with respect to the classical result \( w_{\text{vac}} = -1 \), as we have found in the current study, is a clear sign that a proper renormalization of the quantum matter effects was mandatory in the study of the QFT vacuum in a curved background. Not only so, it serves as an effective phenomenological signature to test this approach. Unfortunately, for some time the widespread confusion in the literature about cosmological constant, \( \Lambda \), and vacuum energy density (VED), \( \rho_{\text{vac}} \), has prevented to achieve a proper treatment of the renormalization of these quantities in cosmological spacetime. Perhaps the most pernicious practice has been the reiterated attempts to relate these concepts in the context of flat spacetime calculations, which is meaningless \[7,10\]. As indicated in Sec. II if we speak of \( \Lambda \) as the physically measured value, then its relation with the current \( \rho_{\text{vac}} \) is totally straightforward: \( \rho_{\text{vac}}^0 = \Lambda/(8\pi G_N) \). However, at a more formal level where these quantities are to be derived from a gravitational action in curved spacetime, a lot more of care needs to be exercised. Leaving for the moment quantum gravity considerations for a better future (when the quantum treatment of the gravitational field becomes, hopefully, accessible), the more pedestrian renormalization of \( \rho_{\text{vac}} \) in QFT in curved spacetime proves to be already quite helpful at present \[11,12\]. It shows, for example, that the VED is a mild dynamical quantity evolving with the cosmic expansion, in which \( \rho_{\text{vac}}^0 \) is just its value at present. In general, \( \rho_{\text{vac}} = \rho_{\text{vac}}(H) \) is a function of the Hubble rate and its time derivatives \[11,12\]. Because of inappropriate renormalization schemes and computational procedures, for a long time we have deluded ourselves into believing that we are forced to perform unnatural fine
tuning of the parameters in the computation of the vacuum energy density (VED) owing to the presence of \(\sim m^4\) terms in these schemes \[7\]. Let us note that even working in a curved background, not all computational procedures are granted a successful and meaningful result. For instance, Feynman diagram techniques, which are necessarily based on weak-field expansion around flat space (\(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\)) are inappropriate here since they do not have the ability to capture the more subtle details of the vacuum renormalization \[13\]–\[20\]. This has been one of the main stumbling blocks for real progress in vacuum renormalization and for understanding the cosmological ‘constant’ term and the cosmological constant problem (CCP) within QFT. Only an approach based on the effective action and/or an appropriate renormalization of the EMT has the capability to catch all the essential physical features. The equivalence between both procedures in the context of adiabatic renormalization of a non-minimally coupled scalar field has been proven recently in \[11\]–\[12\]. Moreover, the correct definition of the VED is also of paramount importance. It hinges on an appropriate calculation and renormalization of the ZPE in curved spacetime together with the \(\rho_\Lambda\) parameter in the action. Indeed, the VED is given schematically by \(\rho_\Lambda + \text{ZPE}\), and upon renormalizing the energy-momentum tensor directly, or indirectly through its relation with the effective action of vacuum (cf. Sec. \[III\]), the two procedures converge to the very same result \[11\]. Namely, we find that although both \(\rho_\Lambda\) and ZPE bring forth quartic powers \(\sim m^4\), they do not affect the running of the VED. Formally, this is because the \(\beta\)-function of the VED does not depend on \(m^4\) terms, as we have seen in Sec. \[IV\]. Thus, there is no need of fine tuning. The two independent procedures are based on adiabatic regularization and subsequent subtraction at an arbitrary scale \(M\), which allows to probe the VED evolution at any given cosmic epoch \(M = H\).

From this QFT point of view, the CCP is no longer the problem of understanding how on earth the huge contributions \(\sim m^4\) from the quantum fields ‘magically’ cancel each other, as they are not really there! What is left is only to measure the precise value of \(\rho_{\text{vac}}(H_0)\), which we don’t know a priori, but once measured it is not to be disrupted anymore by huge \(\sim m^4\) effects from any sort of quantum field we come across in the universe. Furthermore, the currently measured value \(\rho_{\text{vac}}(H_0) = \rho_{\text{vac}}^0\) is not to be thought of as a ‘fundamental constant’. QFT actually tells us that there is no strict cosmological constant conceived as an everlasting fundamental entity of Nature. Instead, at any expansion history time \(H(t)\), there is a smoothly evolving VED, \(\rho_{\text{vac}}(H)\), changing so little with the expansion that mimics the behavior of a ‘cosmological constant’ \(\Lambda = 8\pi G N \rho_{\text{vac}}(H)\) for a large stretch of cosmic time. At the vicinity of any given point \(H\) the change is only of order \(\sim \nu_{\text{eff}} H^2\), where the coefficient \(\nu_{\text{eff}}\) is computable from QFT and is responsible for the running of the VED. In the remote past, however, the higher powers of \(H\) (predicted in this approach) became extremely active and triggered fast inflation during a short period in which \(H \approx \text{const}\). At present, on the other hand, a new and much placid de Sitter epoch takes over gradually.

X. CONCLUSIONS

In this Letter, we have studied the equation of state (EoS) of the quantum vacuum, \(w_{\text{vac}}\), within the quantum field theoretical (QFT) approach recently expounded in \[11\]–\[12\]. We have explicitly computed \(w_{\text{vac}}\) for the whole FLRW regime and extracted potentially significant phenomenological implications. A pivotal conclusion is that \(\rho_{\text{vac}} = \rho_{\text{vac}}(H_0)\) is not to be interpreted as a rigid CC term, but only as the value of the vacuum energy density (VED) at present (\(H = H_0\)). In fact, there is no immutable cosmological constant in the ‘QFT-driven’ universe, but a VED mildly evolving \(H_0\) (the current value, although it could be any other one) to another value \(H\), and without any fine tuning since there are no \(\sim m^4\) terms driving the vacuum evolution. The latter is triggered by pure quantum renormalization effects. During the FLRW regime, it is given by Eq. \(\[11\]\). Likewise, being the evolution so slow (owing to it being proportional to \(H^2\) and \(\nu_{\text{eff}} \ll 1\)), any given \(\rho_{\text{vac}}(H_0)\) appears as an approximate cosmological constant – defined through \(\Lambda = 8\pi G N \rho_{\text{vac}}(H_0)\) – for a long cosmic span. Notwithstanding such mild effects, the EoS of the quantum vacuum carries two important signatures worth being mentioned owing to their possible phenomenological significance. First, the quantum vacuum EoS is not characterized by the rigid value \(w_{\text{vac}} = -1\) that has been typical of the classical vacuum; rather, it is a dynamical quantity, which can be expressed in a close form as an explicit function of the redshift: \(w_{\text{vac}} = w_{\text{vac}}(z)\). Second, the EoS dynamics carries a measurable imprint at present since it behaves as quintessence: \(w_{\text{vac}}(z) > -1\). There are no quintessence fields at all here, of course; the effective quintessence behavior is just the consequence of the underlying quantum vacuum effects. Thus, no ad hoc fields should be needed to explain the cosmic acceleration, it all can be the fine work of the quantum vacuum made out of the fluctuations of the quantized matter fields \[11\]–\[12\]. The QFT-vacuum acts, however, as a formidable cosmic chameleon: early on, it triggers inflation as ‘true vacuum’ \((w_{\text{vac}} = -1)\), no need of inflatons: right next, it hides behind matter for aeons (even adopting its EoS: \(w_{\text{vac}} = 1/3\) first, and \(w_{\text{vac}} = 0\) later); and, finally, it reappears disguised as quintessence in our days. Only in the remote future it will become ‘true vacuum’ again. The quantum vacuum, therefore, reveals itself as a time-evolving entity whose EoS is also dynamical and changes significantly over the cosmic evolution. Remarkably, in the late universe, it acts as (dynamical) dark energy. After all the admonitions and even severe criticisms that it has had to endure over time, the quantum vacuum might well be the ultimate raison d’être for dark energy within
the fundamental framework of QFT in curved spacetime, and even the clue to solving the cosmological tensions.

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Appendix A: Derivation of the quantum vacuum EoS for the FLRW regime

Our goal in this appendix is to provide some details about the derivation of the important EoS formula (19) given in the main text, which is valid for the post-inflationary epoch, i.e. for the whole FLRW regime. For this we will be using the approach and formulae from [11]. In the latter reference the quantum vacuum EoS was disclosed as a function of the redshift only within the approximation $z \ll 1$, but here we wish to provide a close expression for the EoS as a function of $z$ valid for the entire FLRW epoch. As previously warned, for all the considerations made during the FLRW regime we will neglect the quantum corrections of order $O(H^4)$ or above, which can only be relevant for the inflationary epoch. Thus, for the EoS determination during the post-inflationary epoch, it suffices to keep the terms of adiabatic orders 2 in Eq. (18) only. We find

$$w_{\text{vac}}(H) = \frac{P_{\text{vac}}(H)}{\rho_{\text{vac}}(H)} = -1 + \frac{1}{\rho_{\text{vac}}(H)} \left( \frac{\xi - 1}{8\pi^2} \right) \dot{H} m^2 \left( 1 - \ln \frac{m^2}{H^2} \right) + O(H^4)$$

(A1)

where $\rho_{\text{vac}}(H)$ in the denominator of the above formula is given by Eq. (11). The $O(H^4)$ terms are to be neglected hereafter. We can see from Eq. (A1) that at leading order the vacuum EoS is coincident with that of the $\Lambda$CDM ($w_{\text{vac}} = -1$), as it could not be otherwise. Up to second adiabatic order, it reads

$$w_{\text{vac}}(H) = -1 + \frac{\epsilon m^2_{\text{Pl}}}{4\pi \rho_{\text{vac}}(H)} \dot{H} \left( 1 - \ln \frac{m^2}{H^2} \right) \simeq -1 - \nu_{\text{eff}} m^2_{\text{Pl}} \frac{\dot{H}}{4\pi \rho_{\text{vac}}}$$

(A2)

where the small parameter $\epsilon$ is defined by Eq. (12). We have set $H = H_0$ in the log since the change is extremely slow within long cosmological periods, for example around our time, and used $\ln \frac{m^2}{H^2} \gg 1$ in the last step. This expression is the result at $O(\nu_{\text{eff}})$ for very low redshift and coincides with the result already reported in [11]. Upon using (12) and the $\Lambda$CDM form for $\dot{H}$ (which is consistent at this order) it can be immediately be written in terms of the redshift as indicated in Eq. (20) of the current work.

However, we would like to generalize that formula for arbitrarily large redshift within the FLRW epoch and for this we cannot approximate the denominator of (A1) through the constant $\rho_{\text{vac}}^0 = \rho_{\text{vac}}(H_0)$ as we did before. We need to use now the dynamical form of the VED during the FLRW epoch, i.e. Eq (11), in which the parameter $\nu_{\text{eff}}$ itself is running [11]:

$$\nu_{\text{eff}}(H) \equiv \epsilon \left( -1 + \ln \frac{m^2}{H^2} - \frac{H_0^2}{H^2 - H_0^2} \ln \frac{H^2}{H_0^2} \right).$$

(A3)

Its approximately constant form for $H$ in the late time universe is given by (12) in the main text. To find out the vacuum EoS such that it be valid for any redshift from now up to the initial stages of the radiation-dominated epoch, we have to insert Eq. (A3) into the canonical RVM form for the VED, i.e. Eq. (11), and use the latter in the denominator of the EoS equation (A1). To further proceed we need an explicit form for $H$. For $\nu_{\text{eff}}$ strictly constant, the RVM can be solved analytically [35, 36]. However, the QFT form of the RVM is more complicated since the effective parameter (A3) is a function of $H$ and then an exact analytical solution is not feasible. Even so, taking into account that $\nu_{\text{eff}}(H)$ is a slowly varying function of $H$ and that $|\epsilon| \ll 1$, the function $\nu_{\text{eff}}(H)$ remains always small, and hence we can obtain a very good approximate solution for the full FLRW regime by expanding the solution in the small parameter $\epsilon$. In this way we will be able to split the corrected $H^2$ (involving the QFT effects) into the leading $\Lambda$CDM part plus $O(\epsilon)$ corrections or higher. The standard or concordance $\Lambda$CDM model part of $H^2$ is simply

$$H^2_{\Lambda\text{CDM}}(z) = H_0^2 \left[ \Omega_m^0 (1 + z)^3 + \Omega_r^0 (1 + z)^4 + \Omega_{\text{vac}}^0 \right].$$

(A4)

Now upon inserting Eq. (11) into Friedmann’s equation and separating the $\Lambda$CDM contribution, we find the following result:

$$H^2 = \frac{8\pi G(H)}{3} (\rho_m(z) + \rho_{\text{vac}}(H) + \cdots) \simeq H^2_{\Lambda\text{CDM}} + \epsilon \left( H^2_{\Lambda\text{CDM}} - H_0^2 \right) \left( -1 + \ln \frac{m^2}{H_0^2} \right) + O(\epsilon^2),$$

(A5)
where the dots in the first equality stand for the neglected $\mathcal{O}(H^4)$ corrections to Friedmann’s equation in the present universe (the interested reader can find their explicit form in [11]). In the above expression, the term departing from the $\Lambda$CDM result has been calculated up to order $\mathcal{O}(\epsilon)$, but we should remark that $G(H)$ in (A5) is given by Eq. (15) and hence it had also to be expanded to $\mathcal{O}(\epsilon)$ so as to obtain the complete $\mathcal{O}(\epsilon)$ correction indicated in Eq. (A5). In a similar way we find

$$\dot{H} = \dot{H}_{\Lambda\text{CDM}} + \epsilon \dot{H}_{\Lambda\text{CDM}} \left(-1 + \ln \frac{m^2}{H_0^2}\right) + \mathcal{O}(\epsilon^2).$$

(A6)

Finally, introducing the above equations in Eq. (A1), we arrive after some calculations at the formula

$$w_{\text{vac}}(z) \simeq -1 + \frac{\nu_{\text{eff}}}{\Omega_0^\nu + \nu_{\text{eff}}} \left[-1 + E_{\text{vac}}^2(z) \left(1 - \frac{\ln E_{\Lambda\text{CDM}}^2(z)}{\ln H_0^2}\right)\right],$$

(A7)

in which $E_{\Lambda\text{CDM}}^2(z) = \frac{H_{\Lambda\text{CDM}}^2(z)}{H_0^2}$, with $\nu_{\text{eff}}$ given by (12). Once more we have used $\ln \frac{m^2}{H_0^2} \gg 1$ to simplify the final result. In practice, it is sufficient to use the even more simplified form

$$w_{\text{vac}}(z) = -1 + \frac{\nu_{\text{eff}} (\Omega_m^0 (1 + z)^3 + \frac{4}{3} \Omega_\nu^0 (1 + z)^4)}{\Omega_0^\nu + \nu_{\text{eff}} \left[-1 + E_{\Lambda\text{CDM}}^2(z)\right]},$$

(A8)

since $\frac{\ln E_{\Lambda\text{CDM}}^2(z)}{\ln H_0^2} \ll 1$ in the entire FLRW regime, as it can be easily checked. We immediately recognize that the obtained Eq. (A8) is just our EoS formula (19) in the main text (q.e.d.). It is fully model-independent as the mass of the scalar particle has been absorbed by the generalized coefficient $\nu_{\text{eff}}$ (within the very good approximation used to derive it). Moreover, as indicated in Sec. VIII for small redshift values Eq. (A8) trivially reduces to the much simpler form (20). Recall that the three distinct qualitative behaviors implied by the quantum vacuum EoS during the various epochs of the FLRW regime are summarized in Eq. (22).

The above EoS formula for the quantum vacuum can still be further refined to include the next-to-leading $\mathcal{O}(\nu_{\text{eff}}^2)$ terms. This implies more work since we need to consistently collect all of $\epsilon^2$ terms and in particular also those from expanding up to that order the running gravitational coupling ($\nu_{\text{eff}}$). We shall omit the details of this lengthier calculation. The result stays, however, rather compact and we find that up to the next-to-leading order in $\epsilon$ we have

$$H^2(z) = H_{\Lambda\text{CDM}}^2 + \epsilon \left(H_{\Lambda\text{CDM}}^2(z) - H_0^2\right) \left(-1 + \ln \frac{m^2}{H_0^2}\right) + \epsilon^2 \left(H_{\Lambda\text{CDM}}^2(z) - H_0^2\right) \left(-1 + \ln \frac{m^2}{H_0^2}\right)^2$$

(A9)

or

$$E_{\Lambda\text{CDM}}^2(z) \equiv \frac{H_{\Lambda\text{CDM}}^2(z)}{H_0^2} \simeq E_{\Lambda\text{CDM}}^2(z) + \nu_{\text{eff}} \left(E_{\Lambda\text{CDM}}^2(z) - 1\right) + \nu_{\text{eff}}^2 \left(E_{\Lambda\text{CDM}}^2(z) - 1\right)$$

(A10)

and

$$\dot{H} = \dot{H}_{\Lambda\text{CDM}} + \epsilon \dot{H}_{\Lambda\text{CDM}} \left(-1 + \ln \frac{m^2}{H_0^2}\right) + \epsilon^2 \dot{H}_{\Lambda\text{CDM}} \left(-1 + \ln \frac{m^2}{H_0^2}\right)^2 \simeq \dot{H}_{\Lambda\text{CDM}} + \nu_{\text{eff}} \dot{H}_{\Lambda\text{CDM}} + \nu_{\text{eff}}^2 \dot{H}_{\Lambda\text{CDM}}.$$

(A11)

These expressions obviously extend the previous ones up to $\mathcal{O}(\epsilon^2)$. We can use them to compute the EoS at this order. Once more we see that the expansion in $\epsilon$ is such that at leading order it can be expressed as an expansion in $\nu_{\text{eff}}$. The final result for the EoS to $\mathcal{O}(\nu_{\text{eff}}^2)$ takes on the form in Eq. (A7) with only the replacement $\nu_{\text{eff}} \rightarrow \nu_{\text{eff}} (1 + \nu_{\text{eff}})$ in the parameter $\nu_{\text{eff}}$ of its numerator. Thus, since $0 < \nu_{\text{eff}} \ll 1$, the next-to-leading $\mathcal{O}(\nu_{\text{eff}}^2)$ terms obviously imply a tiny correction to the $\mathcal{O}(\nu_{\text{eff}})$ formula, which in practice can be neglected.

We remark that the model at this point is solved. Indeed, from Eq. (A10) the quantum correction to the ordinary $\Lambda$CDM parameter $\Omega_{\text{vac}}$ can be expressed directly in terms of the redshift as follows:

$$\Omega_{\text{vac}}(z) \simeq \Omega_{\text{vac}}^0 + \nu_{\text{eff}} \left(E_{\Lambda\text{CDM}}^2(z) - 1\right) + \nu_{\text{eff}}^2 \left(E_{\Lambda\text{CDM}}^2(z) - 1\right).$$

(A12)

Obviously $\Omega_{\text{vac}}(z = 0) = \Omega_{\text{vac}}^0$ is satisfied, as it should be. Interestingly enough, to within $\mathcal{O}(\nu_{\text{eff}})$ this expression is similar to the one found in previous calculations based on the phenomenological RVM, see e.g. [35, 36], except
that here we have derived the fundamental RVM formulas, including the quantum vacuum EoS, from QFT in curved spacetime within the framework recently put forward in \cite{11,12}. The above equation can be written to $O(\nu_{\text{eff}})$ in terms of the vacuum energy density itself as follows:

$$\rho_{\text{vac}}(z) \simeq \rho_{\text{vac}}^0 + \nu_{\text{eff}} \rho_c^0 \left( E_{\Lambda\text{CDM}}^2(z) - 1 \right),$$

(A13)

where $\rho_c^0 = 3H_0^2/(8\pi G_N)$ is the current critical density. This expression has been used for the VED plots in Fig.1.

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