C metric: the equatorial plane and Fermi coordinates

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Abstract

We discuss geodesic motion in the vacuum C metric using Bondi-like spherical coordinates, with special attention to the role played by the ‘equatorial plane’. We show that the spatial trajectory of photons on such a hypersurface is formally the same as that of photons on the equatorial plane of the Schwarzschild spacetime, apart from an energy shift involving the spacetime acceleration parameter. Furthermore, we show that photons starting their motion from this hypersurface with vanishing component of the momentum along \( \theta \), remain confined on it, differently from the case of massive particles. This effect is shown to have a counterpart also in the massless limit of the C metric, i.e. in Minkowski spacetime. Finally, we give the explicit map between Bondi-like spherical coordinates and Fermi coordinates (up to the second order) for the world line of an observer at rest at a fixed spatial point of the equatorial plane of the C metric, a result which may be useful to estimate both the mass and the acceleration parameter of accelerated sources.

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1. Introduction

The vacuum C metric was first discovered by Levi-Civita [1] in 1918 and belongs to the class of Petrov type D (degenerate) static vacuum metrics [2]. However, over the years it has been rediscovered many times: by Newman and Tamburino [4] in 1961, by Robinson and Trautman [5] in 1961 and again by Ehlers and Kundt [6]—who called it the C metric—in 1962. It is worth mentioning that a charged and a spinning extension of the C metric exist: the former has been studied in detail by Kinnersley and Walker [7, 8], the latter instead has been deeply investigated by Bicak, Pravda, Pravdová, Farhoosh and Zimmerman [9–11]. In general, the spacetime represented by the C metric contains one or, via an extension, two uniformly accelerated particles as explained in [8, 12] and a description of the geometric properties as
well as the various extensions of the C metric is contained in [2], which should be consulted for a more complete list of references. Furthermore, we note that there exist many representations of the C metric in different coordinate systems, which better exhibit some geometrical or physical properties. Of interest, for example, is the form recently proposed by Hong and Teo [3] which allows an explicit factorization of the metric coefficients; however, for the purpose of the present work this form does not lead to great simplifications.

The main property of the C metric is the existence of two hypersurface-orthogonal Killing vectors, one of which is timelike (showing the static property of the metric) in the spacetime region of interest in this work, as indicated below.

The C metric written in Bondi-like spherical coordinates $x^0 = u$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ has the form [13]

$$ds^2 = -H du^2 - 2 du dr + 2Ar^2 \sin \theta du d\theta + \frac{r^2 \sin^2 \theta}{G} d\theta^2 + r^2 G d\phi^2,$$  \hspace{1cm} (1.1)

where $G$ and $H$ are given by

$$G(\theta) = \sin^2 \theta - 2mA \cos^3 \theta,$$

$$H(r, \theta) = 1 - \frac{2m}{r} - A^2 r^2 (\sin^2 \theta - 2mA \cos^3 \theta)$$  \hspace{1cm} (1.2)

$$- 2Ar \cos \theta (1 + 3mA \cos \theta) + 6mA \cos \theta.$$

The constants $m \geq 0$ and $A \geq 0$ denote the mass and acceleration of the source, respectively. The metric (1.1) can be thought of as the nonlinear superposition of the flat spacetime metric written in uniformly accelerating coordinates (which is recovered in the limit $m = 0$) and the Schwarzschild black hole metric (corresponding to vanishing acceleration parameter $A = 0$), as discussed in [7, 8, 11]; the exterior vacuum C metric thus describes the gravitational background associated with a uniformly accelerated static black hole. Moreover, the C metric in the form (1.1) is assumed to have signature $+2$ with $\tilde{F} > 0$ and $\tilde{G} > 0$, limiting the ranges of the nonignorable coordinates. To avoid a naked singularity, one must restrict the two parameters by the condition $mA < 1/(3\sqrt{3})$ [11, 14, 15]. Units here are chosen so that the speed of light is taken to be 1.

The C metric has event horizons (which are also Killing horizons) given by hypersurfaces of the form $r = r(\theta)$ that are solutions of the equation $H = 0$ and can be determined exactly [16]. To study the location of the horizons, it is useful to introduce an acceleration length scale based on $A > 0$ given by $L_A = 1/(3\sqrt{3}A)$ as well as the quantities:

$$U = -\frac{1}{\sqrt{3}} \cos \left(\frac{1}{3} \arccos \frac{m}{L_A}\right), \quad V = \frac{1}{\sqrt{3}} \sin \left(\frac{1}{3} \arccos \frac{m}{L_A}\right).$$  \hspace{1cm} (1.3)

It turns out that there is an inner horizon which is a deformation of the Schwarzschild one, located at

$$r_S = \frac{1}{A} \frac{\sqrt{3}V - U}{[1 + (\sqrt{3}V - U) \cos \theta]}$$  \hspace{1cm} (1.4)

and an outer horizon which is a deformation of the Rindler one, located at

$$r_R = \frac{1}{A} \frac{2U}{(1 + 2U \cos \theta)}.$$  \hspace{1cm} (1.5)

while the third real radial root of $H = 0$ corresponds to a negative value of $r$. In the polar coordinate plane of the nonignorable coordinates $(r, \theta)$, depicted in terms of the auxiliary
C metric: the equatorial plane and Fermi coordinates

Figure 1. The accessible spacetime region (unshaded) in the polar coordinate representation of the $r$-$\theta$ plane ($\theta$ is measured from the upward vertical $Z$-axis) is shown for $mA = 0.1$ (implicitly assumed is the rotational symmetry about the $Z$-axis). The upper boundary curve $H_R$ represents the modified Rindler horizon, while the ‘circle’ $H_S$ is the modified Schwarzschild horizon. The forbidden conical region corresponds to negative values of the metric function $G$, i.e. to signature changes which are not considered here.

Cartesian-like coordinates $X = r \sin \theta$ (horizontal), $Z = r \cos \theta$ (vertical), the accessible region of spacetime is shown as the unshaded region in figure 1 for $mA = 0.1$, satisfying the condition $mA < 1/(3\sqrt{3})$ (implicitly assumed is the rotational symmetry about the $Z$-axis). As has been deeply discussed in [17], the spacetime region between the modified Schwarzschild and Rindler horizons contains a conical singularity at the only meaningful root of the equation $G = 0$ (see also [18]).

The C metric also admits a conformal Killing tensor [7, 8],

$$P = r^2 \left[ H du^2 + 2 du dr - 2 A \sin \theta r^2 du d\theta + \frac{r^2 \sin^2 \theta}{G} d\theta^2 + r^2 G d\phi^2 \right],$$  \hspace{1cm} (1.6)

symmetric and traceless, satisfying the equation $P_{(\alpha\beta;\gamma)} - \frac{1}{2} g_{(\alpha\beta} P_{\gamma);\delta} = 0$ with which a conserved quantity for null geodesics (with 4-momentum $P^\mu$)

$$Q = \frac{1}{2} P_{\alpha\beta} P^\alpha P^\beta,$$  \hspace{1cm} (1.7)

is associated.

A natural family of observers in the C metric is represented by the (static) timelike congruence of the coordinate time world lines. Let us denote the observer 4-velocity by $e_a = \frac{1}{M} \partial_a$ and use the lapse-shift notation for the metric components:

$$g_{uu} = -M^2, \quad g_{ua} = M^2 M_a, \quad g_{ab} = \gamma_{ab} + M^2 M_a M_b,$$  \hspace{1cm} (1.8)

where $M$ is the lapse factor and $M_a dx^a$ is the shift 1-form field, $a = 1, 2, 3$; the line element (1.1) then takes the form

$$ds^2 = -M^2 (du - M_a dx^a)^2 + \gamma_{ab} dx^a dx^b,$$  \hspace{1cm} (1.9)
where the functions $M, M_a$ and $\gamma_{ab}$ are identified by using the same equation (1.1):

\[
M = \sqrt{H} > 0, \quad M_r = -\frac{1}{H} < 0, \quad M_\theta = \frac{Ar^2 \sin \theta}{H} > 0, \quad M_\phi = 0,
\]

\[
\gamma_{rr} = H, \quad \gamma_{\theta\theta} = r^2 \sin^2 \theta \left[ \frac{1}{G} + \frac{A^2 r^2}{H} \right], \quad \gamma_{r\theta} = -\frac{Ar^2 \sin \theta}{H}, \quad \gamma_{\phi\phi} = r^2 G,
\]

and all the other metric components vanish. The dual of $e^{\hat{u}}$ is denoted by $\omega^{\hat{u}} = M(e^{\hat{u}} - M_r \partial_r - M_\theta \partial_\theta)$ and an orthonormal triad adapted to $e^{\hat{u}}$ is given by

\[
e^{\hat{r}} = -\frac{1}{M} \left[ \frac{1}{M_r} \partial_r + \partial_u \right], \quad e^{\hat{\theta}} = \frac{1}{\sqrt{g_{\theta\theta}}} \left[ \partial_\theta - \frac{M_\theta}{M_r} \partial_r \right], \quad e^{\hat{\phi}} = \frac{1}{\sqrt{g_{\phi\phi}}} \partial_\phi,
\]

(1.10)

with dual

\[
\omega^{\hat{r}} = -M(e^{\hat{r}} - M_r \partial_r + M_\theta \partial_\theta), \quad \omega^{\hat{\theta}} = \sqrt{g_{\theta\theta}} \partial_\theta, \quad \omega^{\hat{\phi}} = \sqrt{g_{\phi\phi}} \partial_\phi.
\]

(1.11)

Hence, when expressed in terms of this frame, the spacetime metric (1.1) can be written as $g^{\hat{\alpha} \hat{\beta}} = \eta^{\hat{\alpha} \hat{\beta}} \omega^{\hat{\alpha}} \otimes \omega^{\hat{\beta}}$.

In this paper, we present a novel study of the geodesic motion of photons on the equatorial plane of the exterior vacuum C metric. It is well known [7, 8] that these orbits can be integrated (in general and not only on the equatorial plane), due to the existence of the conformal Killing tensor (1.6). We will show that the radial equation for equatorial motion can be put in exact correspondence with the case of equatorial geodesic motion of photons in the Schwarzschild background, apart from an energy shift of the form:

\[
E_0 \text{ (Schwarzschild)} \rightarrow \sqrt{E_0^2 + A^2 L_z^2} \text{ (C metric)},
\]

(1.13)

which is evidence of a coupling between the background acceleration $A$ and the (conserved) angular momentum ($L_z$) of photons. Furthermore, photons starting their motion from the equatorial plane (with zero component of momentum along $\theta$: $P_\theta = 0$) remain confined on it; the same property does not hold for massive particles which, instead, are forced to escape from it due to effects of the background acceleration itself. Inspecting the massless limit of the C metric, we will show that this effect actually has its origin in Minkowski spacetime endowed with the ‘Bondi coordinate system’, but it is lost if the Schwarzschild solution is considered, i.e. in the zero acceleration limit of the C metric.

In connection with the Pioneer anomaly, it has been suggested in [16] that the centre of mass of the Sun could be accelerating, so that—ignoring the rotation of the Sun as well as the deviation of its shape from spherical symmetry—the exterior of the accelerating Sun could be described by the C metric. This acceleration could in principle be detected by studying the motion of planets as well as light in this gravitational field. We explore the latter possibility by using a Fermi coordinate system that is established about an observer $O$ at rest at a fixed spatial point of the equatorial plane, giving also the transformation law between Bondi-like spherical coordinates and Fermi coordinates explicitly (up to the second order). We then suggest an experiment of light deflection as seen from the lab of the observer $O$ which can be used to estimate the background acceleration parameter as well as the mass of the source. We argue that these results can be applied to suitable astrophysical systems to set upper limits to a spacetime acceleration.

2. Geodesics in Bondi-like spherical coordinates

The geodesic equations in $\{u, r, \theta, \phi\}$ coordinates are given by
\[ \ddot{u} - \frac{H}{2} \dot{u}^2 + 2Ar \sin \theta \dot{u} \dot{\theta} + \frac{r \sin^2 \theta}{G} \dot{\theta}^2 + rG \dot{\phi}^2 = 0, \]  
(2.1) 
\[ \ddot{\phi} + \phi \left[ \frac{2}{r} - \frac{G_\theta}{G} \right] = 0, \]  
(2.2) 
\[ \ddot{r} + \frac{1}{2} \left( (H + \Lambda^2r^2G)H_r + \frac{AG}{\sin \theta} H_\theta \right) \dot{u}^2 - \left[ r^3 \Lambda^2 \sin^2 \theta + \frac{A^2 \sin \theta G_\theta}{2G} + \frac{r \sin^2 \theta H}{G} \right] \dot{\theta}^2 \] 
\[ - \left[ r^3 \Lambda^2 G^2 + \frac{A^2 r^2 G H}{2 \sin \theta} + rG \dot{H} \right] \dot{\phi}^2 + \left[ H_r + 2A^2 r G \right] \dot{u} \dot{r} \] 
\[ + [H_\theta - 2 \sin \theta Ar (H + A^2 r^2 G)] \dot{u} \dot{\theta} = 0, \]  
(2.3) 
\[ \ddot{\theta} + \frac{G}{2r^2 \sin^2 \theta} \left[ Ar \sin \theta H_r + H_\theta \right] \dot{u}^2 + \left[ \cos \theta - Ar \sin^2 \theta \sin \theta \right] \frac{G_\theta}{2G} \dot{\theta}^2 \] 
\[ - \left[ G_\theta \cos \theta + 2GAr \sin \theta \right] \dot{\phi}^2 + \frac{2GA}{r \sin \theta} \dot{u} \dot{r} - 2A^2 r G \ddot{u} \dot{\theta} + \frac{2r^2 G \ddot{u}}{r} = 0, \]  
(2.4) 
where a dot means differentiation with respect to an affine parameter \( \lambda \) along the curve.

By using the Killing symmetries of the metric one gets the conserved quantities: the angular momentum \( L_z \), associated with the Killing vector \( \partial_\phi \), and the energy \( E_0 \), associated with the Killing vector \( \partial_u \)

\[ \phi = \frac{L_z}{r^2 G}, \quad H \dot{u} + \dot{r} - r^2 A \sin \theta \dot{\theta} = E_0. \]  
(2.5) 
Furthermore, one has to specify if the particles are massive or null, i.e. one has the additional constraint

\[ H \ddot{u}^2 - 2E_0 \dot{u} + \frac{r^2 \sin^2 \theta}{G} \dot{\theta}^2 + \frac{L_z^2}{G} = -\mu^2. \]  
(2.6) 
For timelike (spacelike) geodesics, \( \lambda \) can be identified with the proper time (spacelike curvilinear abscissa) and \( \mu \) can be taken as 1 (i). Timelike geodesics was first discussed by Pravda and Prámová [17], by using various coordinate systems (Weyl coordinates as well as symmetry-adapted coordinate systems, different from the present one too). They constructed an effective potential, whose properties allowed them to distinguish three different types of timelike geodesics, corresponding to particles (a) falling under the black hole horizon, (b) crossing the acceleration horizon and reaching future infinity, (c) spinning around the symmetry axis, co-accelerating with the black holes and then reaching future infinity.

In the \( \{u, r, \theta, \phi\} \) coordinate system, the effective potential can be introduced as well, rewriting the mass constraint (2.6) as follows:

\[ (H \dot{u} - E_0)^2 + \frac{r^2 \sin^2 \theta}{G} \dot{\theta}^2 = E_0^2 - V_{(\text{eff})}^2, \]  
(2.7) 
with

\[ V_{(\text{eff})}^2 = H \left( \frac{L_z^2}{r^2 G} + \mu^2 \right). \]  
(2.8)

Null geodesics, instead, have been formally integrated by Kinnersley and Walker in 1970 [8]. However, their main properties including the limiting behaviour in the case of vanishing mass as well as acceleration parameter have been poorly investigated in the literature.

Following the results of [8], the vector \( P \) tangent to a null geodesic can be written as follows:

\[ P = \frac{1}{H} \left[ E_0 + R(r, \theta) \right] \partial_u - \left[ R(r, \theta) + AV(\theta) \right] \partial_r - \frac{1}{r^2 \sin \theta} V(\theta) \partial_\theta + \frac{L_z}{r^2 G} \partial_\phi, \]  
(2.9)
where
\[ R(r, \theta) = \pm \sqrt{E_0^2 - \frac{QH}{r^2}}, \quad V(\theta) = \pm \sqrt{QG - L_z^2}. \] (2.10)

We will focus below on the equatorial geodesic motion of null particles, exploring its characteristic features also in connection with the limiting cases of the Schwarzschild and Rindler spacetimes.

### 2.1. Equatorial orbits: \( \theta = \pi/2 \)

Let us specialize our discussion to the equatorial plane \( \theta = \pi/2 \), considering the static spacetime portion lying in between the deformed Schwarzschild and Rindler horizons, located at
\[ r_S = -U + \sqrt{3V} A, \quad r_R = 2U A, \]
respectively (see equations (1.4) and (1.5)). We denote the momentum of photons as \( P^\mu = dx^\mu / d\lambda \). The geodesic equations in this case \( \theta(\lambda) = \pi/2 \) reduce to
\[ \ddot{u} - \frac{H_r}{2} \dot{u}^2 + r \dot{\phi}^2 = 0, \quad \dot{\phi} + \frac{2}{r} \dot{\phi} \dot{r} = 0, \]
\[ \dot{r} + \frac{1}{2} [H + A^2 r^2] \dot{u}^2 - r [H + r^2 A^2] \dot{\phi}^2 + [H + 2A^2 r] \ddot{r} = 0, \]
\[ \frac{1}{2r^2} [A^2 r H_r + H_\theta] \dot{u}^2 - A r \dot{\phi}^2 + \frac{2A}{r} \dot{u} \dot{r} = 0, \]
where we have used the \( \theta \to \pi/2 \) limit of \( G, H \) and their derivatives
\[ G = 1, \quad G_\theta = 0, \quad H = 1 - \frac{2m}{r} - A^2 r^2, \quad H_r = \frac{2m}{r^2} - 2A^2 r, \quad H_\theta = 2A (r - 3m). \] (2.12)

The conserved quantities (2.5) and the mass constraint (2.6) become
\[ \phi = \frac{L_z}{r^2}, \quad H \dot{u} + \dot{r} = E_0, \quad H \dot{u}^2 - 2E_0 \dot{u} + \frac{L_z^2}{r^2} = -\mu^2. \] (2.13)

Equation (2.11) is equivalent to equation (2.13)1. Equation (2.11)3 can be rewritten as
\[ \dot{r} + \frac{1}{2} [(H + A^2 r^2) \dot{u}^2 - r[H + r^2 A^2] \dot{\phi}^2] + \frac{AH_\theta}{2} \dot{u}^2 + [H_r + 2A^2 r] \ddot{r} = 0; \] (2.14)

using equation (2.11), one has
\[ \dot{r} + [(H + A^2 r^2) \dot{u} + (H_r + 2A^2 r) \dot{\phi}] + \frac{AH_\theta}{2} \dot{u}^2 = 0, \]
\[ \frac{d}{d\lambda} \left[ \dot{r} + (H + A^2 r^2) \dot{u} \right] + \frac{AH_\theta}{2} \dot{u}^2 = 0. \] (2.16)

Using the Killing constraint (2.13)2, the latter equation becomes
\[ A[2r \dot{r} \dot{u} + r^2 \ddot{u}] + \frac{H_\theta}{2} \dot{u}^2 = 0. \] (2.17)

Replacing \( \dot{r} \) using again the Killing constraint (2.13)2 and \( \dot{u} \) using equation (2.11), we find
\[ 2r AE_0 \ddot{u} - r A H \dot{u}^2 - \frac{AL_z^2}{r} = 0, \] (2.18)
which can be compatible only with photons: \( \mu = 0 \), as follows by direct comparison with equation (2.13). Using the relation \((Ar^2 H_r + H_\theta) = 2Ar H\), equation (2.11) can be rewritten as

\[
Hu^2 - r^2 \dot{\phi}^2 + 2a \dot{r} = 0, \tag{2.19}
\]

which again coincides with equation (2.13), with \( \mu = 0 \). Hence, the first important result is that on the equatorial plane of the C metric only photons can be geodesic. Massive particles on this plane should necessarily be accelerated.

Summarizing, the motion of geodesic photons on the equatorial plane of the C metric \((\theta = \pi/2)\) is described by

\[
\dot{u} = \frac{1}{H}[E_0 - \dot{r}], \quad \dot{\phi} = \frac{L_z}{r^2},
\]

\[
\dot{r} = \pm \left( E_0^2 - \frac{HL_z^2}{r^2} \right)^{1/2} = \pm \left[ E_0^2 + A^2 L_z^2 - \frac{L_z^2}{r^2} \left( 1 - \frac{2m}{r} \right) \right]^{1/2}, \tag{2.20}
\]

in agreement with the solution of general null geodesic motion as described by Kinnersley and Walker (and rederived in equation (2.9)).

Finally, it is convenient to rewrite the radial equation in terms of the impact parameter \( b = L_z/E_0 \) and an effective potential, so that

\[
\left( \frac{dr}{d\lambda} \right)^2 = \pm L_z^2 + \frac{1}{b^2} - V_{(ed)}^2 \left( \frac{1}{b} \right)^{1/2}, \quad V_{(ed)}^2 = \frac{1}{r^2} \left( 1 - \frac{2m}{r} \right) - A^2 \equiv \frac{H}{r^2}. \tag{2.21}
\]

The effective potential coincides with the Schwarzschild one modulo a constant shift, \(-A^2\). Therefore, as in the Schwarzschild black hole case, the potential is peaked at \( r = 3m \) with a value

\[
V_{(ed)}(3m)^2 = \frac{1}{27m^2} - A^2.
\]

It is worth noting that the radial equation (2.20) can be put in exact correspondence with the analogous radial equation for null geodesics of the equatorial plane of the Schwarzschild solution by defining

\[
\frac{1}{b_5^2} = \frac{1}{b^2} + A^2 \quad \rightarrow \quad b_5 = \frac{b}{\sqrt{1 + A^2 b^2}}. \tag{2.22}
\]

Hence, as all possible kinds of equatorial null geodesics in the Schwarzschild spacetime have been completely determined in terms of elliptic functions [19], this problem here is also completely solved. In fact, let us consider the orbit as parametrized by \( \phi \), i.e.

\[
\left( \frac{dr}{d\phi} \right)^2 = r^4 \left[ \frac{1}{b_5^2} - \frac{1}{r^2} \left( 1 - \frac{2m}{r} \right) \right], \tag{2.23}
\]

and introduce the new variable: \( v(\phi) = 2m/r(\phi) \). Equation (2.23) thus becomes

\[
\left( \frac{dv}{d\phi} \right)^2 = [\Omega^2 - v^2(1 - v)], \quad \Omega = \frac{2m}{b_5} = \frac{2mb}{\sqrt{1 + A^2 b^2}}. \tag{2.24}
\]

whose solutions have been fully studied in [19], where all the details can be found.

### 2.2. The equatorial plane in the massless (flat) limit of the C metric

In the case \( m = 0 \), the C metric reduces to the Minkowski spacetime in Bondi coordinates

\[
ds^2 = -(1 - Ar \cos \theta)^2 - A^2(r^2) \, du^2 - 2 \, du \, dr + 2Ar^2 \sin \theta \, du \, d\theta + r^2 \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2.
\]
The map between Bondi and Cartesian coordinates is easily obtained passing first to Rindler coordinates. In detail, start from the form of the metric in standard Cartesian coordinates \( \{ t, x, y, z \} \)

\[
d s^2 = -dt^2 + dx^2 + dy^2 + dz^2 \equiv \eta_{\mu \nu} \, dx^\mu \, dx^\nu.
\]

Then, pass to Rindler coordinates \( \{ T, X, Y, Z \} \) using the map

\[
t = \left( \frac{1}{A} - Z \right) \sinh(AT), \quad x = X, \quad y = Y, \quad z = \left( \frac{1}{A} - Z \right) \cosh(AT),
\]

so that the metric has the Rindler form

\[
d s^2 = -(1 - AZ)^2 dT^2 + dX^2 + dY^2 + dZ^2, \tag{2.27}
\]

with the limitations \( X \in (-\infty, +\infty), Y \in (-\infty, +\infty), Z \in (-\infty, +1/A) \). Finally, pass to Bondi-like coordinates \( \{ u, r, \theta, \phi \} \) with

\[
T = -u + \frac{1}{2A} \ln \left[ 1 - Ar(1 + \cos \theta) \right], \quad X = r \sin \theta \cos \phi, \quad Y = r \sin \theta \sin \phi, \quad Z = \frac{1}{A} - \sqrt{\left( \frac{1}{A} - r \cos \theta \right)^2 - r^2}, \tag{2.28}
\]

which brings the metric (2.27) in the Bondi form (2.25). The interpretation of \( r \) as a radial coordinate gives the limitation

\[
r < \frac{1}{A(1 + \cos \theta)}; \tag{2.29}
\]

for the remaining coordinates, instead, the limitations are the usual: \( u \in (-\infty, \infty), \theta \in (0, \pi), \phi \in (0, 2\pi) \).

We are now ready to determine the shape of the hypersurface \( \theta = \pi/2 \), once represented in Cartesian coordinates, as well as to study geodesics confined on this surface, having in mind what happens in the case of the C metric. We have the result

\[
cot \theta = \frac{A}{2 \sqrt{x^2 + y^2}} \left( \frac{1}{A^2} + t^2 - x^2 - y^2 - z^2 \right); \tag{2.30}
\]

hence \( \theta = \pi/2 \) corresponds to the hyperboloid

\[
\frac{1}{A^2} + t^2 - x^2 - y^2 - z^2 = 0. \tag{2.31}
\]

Moreover, the geodesics for Minkowski spacetime in Cartesian coordinates are straight lines:

\[
t = t_0 + \lambda E_0, \quad x = x_0 + \lambda P_x, \quad y = y_0 + \lambda P_y, \quad z = z_0 + \lambda P_z. \tag{2.32}
\]

In order to remain confined on the hyperboloid, corresponding to the equatorial plane \( \theta = \pi/2 \), equations (2.31) and (2.32) imply that the equation

\[
\frac{1}{A^2} + t_0^2 - x_0^2 - y_0^2 - z_0^2 - (x_0 + \lambda P_x)^2 - (y_0 + \lambda P_y)^2 - (z_0 + \lambda P_z)^2 = 0 \tag{2.33}
\]

must be satisfied for any \( \lambda \), leading to the following conditions:

\[
0 = \frac{1}{A^2} + t_0^2 - x_0^2 - y_0^2 - z_0^2, \quad E_0^2 = P_x^2 + P_y^2 + P_z^2, \quad 0 = t_0 E_0 - x_0 P_x - y_0 P_y - z_0 P_z. \tag{2.34}
\]
The interpretation follows immediately: (1) the starting point \( P_0 = (t_0, x_0, y_0, z_0) \) must belong to the hyperboloid itself, so that the parametrization
\[
t_0 = \frac{1}{A} \sinh \psi_1, \quad x_0 = \frac{1}{A} \cosh \psi_1 \sin \psi_2 \cos \psi_3, \quad y_0 = \frac{1}{A} \cosh \psi_1 \sin \psi_2 \sin \psi_3, \quad z_0 = \frac{1}{A} \cosh \psi_1 \cos \psi_2,
\]
can be useful (see equation (2.34)1); (2) the geodesics must be null (see equation (2.34)2); (3) the momentum must be orthogonal to the normal to the hyperboloid in the point \( P_0 \), i.e., tangent to the hyperboloid (see equation (2.34)3). As all the null vectors emanating from \( P_0 \) generate a null cone, geodesics on the equatorial plane are defined by those null vectors which are obtained intersecting the null cone and the hyperboloid itself. Explicitly, one rewrites the spatial components of the momentum as follows
\[
P_x = E_0 \sin \alpha \cos \beta, \quad P_y = E_0 \sin \alpha \sin \beta, \quad P_z = E_0 \cos \alpha,
\]
and looks for the solutions \( \alpha \) and \( \beta \) of the equation:
\[
t_0 - \sin \alpha (x_0 \cos \beta - y_0 \sin \beta) - z_0 \cos \alpha = 0, \quad (2.35)
\]
which are in general \( \infty^1 \). It is worth noting that when the Schwarzschild limit of the C metric \((A = 0)\) is instead considered, both photons and massive particles can be confined on the equatorial plane \( \theta = \pi/2 \). Therefore, the equatorial plane of the C metric seems to inherit the corresponding property of Minkowski spacetime in Bondi-like coordinates.

3. ‘Lab’ on the equatorial plane: Fermi coordinates

In the previous section, we have shown that a peculiar role is played by the equatorial plane of the C metric for what concerns photons. We have also mentioned that this property still survives for the equatorial plane of the Minkowski solution written in Bondi-like coordinates (the massless limit of the C metric), but it is lost for the zero acceleration limit of the C metric (i.e., the Schwarzschild solution): here, in fact, condition (2.18), being proportional to \( A \), is automatically satisfied for \( A = 0 \), and it does not give restrictions to the mass of the particles.

Therefore, it is quite natural to study the possibility of putting an observer with his own ‘Lab’ on the equatorial plane of the C metric and considering the ‘observer proper frame’, naturally defined in terms of Fermi coordinates adapted to his world line. For instance, such an observer could measure in principle the acceleration of the centre of mass of the Sun, since, as has been suggested in [16], the exterior of the accelerating Sun could be described by the C metric.

The coordinate transformation from the Bondi-like spherical coordinates \( \{u, r, \theta, \phi\} \) to the Fermi coordinates \( \{T, X, Y, Z\} \) adapted to \( e_\alpha \) can be obtained following the procedure described in [20]:
\[
u = u_0 + \frac{1}{\sqrt{H_0}}(T - X) + \frac{AX}{\sqrt{H_0}} + \frac{1}{2r_0} \left( \frac{1}{r_0} - \frac{3m}{r_0H_0} \right) X^2 - Y^2 - Z^2,
\]
\[
r = r_0 + \sqrt{H_0}X + Ar_0Y + \frac{m}{2r_0^2}X^2 + \frac{r_0 - 2m}{2r_0^2} (Y^2 + Z^2),
\]
\[
\theta = \frac{\pi}{2} + \frac{Y}{r_0} \left( 1 - \frac{\sqrt{H_0}}{r_0} X \right) + \frac{A}{2r_0} (X^2 - Y^2 + Z^2),
\]
\[
\phi = \phi_0 + \frac{Z}{r_0} \left( 1 - AY - \frac{\sqrt{H_0}}{r_0} X \right).
\]

(3.1)
where \( H_0 = 1 - 2m/r_0 - A^2 r_0^2 \). This coordinate transformation maps the Bondi-like form (1.1) of the C metric into the (flat) ‘Rindler’ form, up to the second order in the coordinates \( T, X, Y, Z \):

\[
ds^2 = -(1 - 2G_1 X - 2G_2 Y) \,dT^2 + dX^2 + dY^2 + dZ^2 + O(2),
\]

where

\[
G_1 = -\frac{\sqrt{H_0}}{r_0} + \frac{r_0 - 3m}{r_0^3 \sqrt{H_0}^3}, \quad G_2 = -A
\]

denote the negative of the coordinate components of the observer’s acceleration \( \nabla_{\hat{e}_i} \varepsilon_{\hat{e}_j} \). In order to give a geometrical interpretation of equation (3.1), we find it convenient to introduce the following linear combinations:

\[
\Delta u = M(u - u_0) - MM(r - r_0) - MM_\theta(\theta - \theta_0),
\]
\[
\Delta r = -MM(r - r_0) - MM_\theta(\theta - \theta_0),
\]
\[
\Delta \theta = \frac{1}{\sqrt{G_0}}(\theta - \theta_0), \quad \Delta \phi = \frac{1}{\sqrt{G_0}}(\phi - \phi_0),
\]

so that

\[
\Delta u = T + \kappa(r, \varepsilon_\theta) \,T^2 + 2AXY - \hat{e}_\theta(\theta, \varepsilon_\theta)Y^2,
\]
\[
\Delta r = X + AXY + \frac{1}{2} \kappa(r, \varepsilon_\theta) \,T^2 - \frac{1}{2} \kappa(\phi, \varepsilon_\phi) \,(Y^2 + Z^2),
\]
\[
\Delta \theta = Y + \frac{A}{2} \,(X^2 - Y^2 + Z^2) + \kappa(\phi, \varepsilon_\phi) \,XY,
\]
\[
\Delta \phi = Z - AYZ + \kappa(\phi, \varepsilon_\phi) \,XZ,
\]

where we have introduced the curvatures \( \kappa(x^\alpha, \varepsilon_\theta) \equiv -e_\varepsilon(\ln \sqrt{G_0}) \) associated with the coordinate lines:

\[
\kappa(r, \varepsilon_\theta) = \frac{m - A^2 r_0^3}{r_0 \sqrt{H_0}}, \quad \kappa(r, \varepsilon_\phi) = 0,
\]
\[
\kappa(\theta, \varepsilon_\theta) = \frac{(r_0 - 2m)^2 - m A^2 r_0^2}{r_0^3 (r_0 - 2m) \sqrt{H_0}}, \quad \kappa(\theta, \varepsilon_\phi) = 0,
\]
\[
\kappa(\phi, \varepsilon_\theta) = -\frac{\sqrt{H_0}}{r_0}, \quad \kappa(\phi, \varepsilon_\phi) = 0
\]

together with another useful quantity

\[
\hat{\kappa}(\theta, \varepsilon_\theta) \equiv -\frac{1}{\sqrt{G_0}} e_\theta(\kappa M_\theta) = -\frac{A^2 r_0}{\sqrt{H_0}},
\]

which plays the same role as a curvature. The two limits \( m = 0 \) and \( A = 0 \) of these relations immediately follow.

In the new form (3.2) of the C metric, the solution for the geodesic equations is

\[
T(\lambda) = C \lambda^2 (G_1 X_1 + G_2 Y_1) + C \lambda(1 + 2G_1 X_0 + 2G_2 Y_0) + T_0,
\]
\[
X(\lambda) = \frac{C^2 G_1}{2} \lambda^2 + X_1 \lambda + X_0,
\]
\[
Y(\lambda) = C^2 G_2 \lambda^2 + Y_1 \lambda + Y_0,
\]
\[
Z(\lambda) = Z_1 \lambda + Z_0,
\]

where \( C = \sqrt{(\mu^2 + X_1^2 + Y_1^2 + Z_1^2)(1 - 2G_1 X_0 - 2G_2 Y_0)}, \) and \( \mu = 0 \) for photons (\( \mu = 1 \) for massive particles if proper time parametrization of the orbits is chosen).
Let us confine our attention to the motion in the $X$–$Y$ plane ($Z_0 = 0 = Z_1$) of a particle emitted at the origin ($X_0 = 0 = Y_0$). By eliminating the affine parameter from equations (3.8)$_2$ and (3.8)$_4$, i.e.
\[
\lambda = \frac{G_1 X - G_2 Y}{G_1 X_1 - G_2 Y_1},
\]
and then substituting for instance into equation (3.8)$_3$, we obtain the following implicit form for the parametric equation of the trajectory:
\[
0 = \frac{G_1 X - G_2 Y}{G_1 X_1 - G_2 Y_1} \left[ \frac{G_1 X - G_2 Y}{G_1 X_1 - G_2 Y_1} \frac{C^2 G_1}{2} + Y_1 \right] - Y.
\]
The vertical deflection of a particle horizontally emitted ($Y_1 = 0$) at the origin is given by the coordinate
\[
Y = \frac{C^2}{2X_1^2} G_1(\delta X)^2, \quad C^2 = \mu^2 + X_1^2,
\]
where $\delta X$ denotes a small displacement along the horizontal direction. From the previous equation, the light deflection turns out to be
\[
Y = \frac{1}{2} G_1(\delta X)^2,
\]
as expected. Analogously, the horizontal deflection of a particle vertically emitted ($X_1 = 0$) at the origin is given by the coordinate
\[
X = \frac{C^2}{2Y_1^2} G_2(\delta Y)^2, \quad C^2 = \mu^2 + Y_1^2,
\]
where $\delta Y$ denotes a small displacement along the vertical direction; the corresponding light deflection is
\[
X = \frac{1}{2} G_2(\delta Y)^2.
\]
Being $G_2 = -A$, it is clear that experiments of light deflection in labs, placed on the equatorial plane of a C metric solution, may be useful to determine the background acceleration parameter $A$. Moreover, the combined measurements of vertical and horizontal deflections would allow us in principle to estimate both the mass and the acceleration parameters of the source. However, the effective feasibility of such experiments is beyond the scope of the present paper.

4. Concluding remarks

We have presented a novel discussion of geodesics in the vacuum C metric, using Bondi-like spherical coordinates, which provide evidence of a special role played by the equatorial plane. We have first shown that the exact integrability of photon orbits lying (with their spatial trajectory) on the equatorial plane can be put in exact correspondence with the analogous null orbits on the equatorial plane of the Schwarzschild spacetime, apart from an energy shift involving the spacetime acceleration parameter. Furthermore, we have shown that photons starting their motion from this hypersurface and with initial momentum contained in the tangent hyperplane, remain confined on it, whereas the same is not true for massive particles.

Finally, we have given the explicit map between Bondi-like spherical coordinates and Fermi coordinates (up to the second order) for the world line of an observer at rest at a fixed spatial point of the equatorial plane of the C metric, possibly a building-block for the construction of any lab and measurement process. We have then suggested a possible procedure to estimate both the mass and acceleration parameters of the background source by measuring light deflection. We expect that these results might be applied to accelerated sources of astrophysical interest.
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