A Unification of the Coding Theory and OAQEC Perspectives on Hybrid Codes

Shayan Majidy¹,²

Received: 4 June 2023 / Accepted: 1 August 2023 / Published online: 7 August 2023
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2023

Abstract
There is an advantage in simultaneously transmitting both classical and quantum information over a quantum channel compared to sending independent transmissions. The successful implementation of simultaneous transmissions of quantum and classical information will require the development of hybrid quantum-classical error-correcting codes, known as hybrid codes. The characterization of hybrid codes has been performed from a coding theory perspective and an operator algebra quantum error correction (OAQEC) perspective. First, we demonstrate that these two perspectives are equivalent and that the coding theory characterization is a specific case of the OAQEC model. Second, we include a generalization of the quantum Hamming bound for hybrid error-correcting codes. We discover a necessary condition for developing non-trivial hybrid codes—they must be degenerate. Finally, we construct an example of a non-trivial degenerate 4-qubit hybrid code.

Keywords Quantum error correction · Quantum information

1 Introduction

The simultaneous transmission of quantum and classical information over a quantum channel was initially explored in Ref. [1]. There it was shown that there exists an advantage in transmitting both quantum and classical information simultaneously over a quantum channel when compared to independent transmissions. This work has since been followed by many authors [2–8]. Different error-correcting codes for the simultaneous transmission of quantum and classical information (“hybrid codes”) have been developed. There exist two characterizations of hybrid codes in the literature. One was developed from a coding theory perspective [9] and the other from an operator algebra quantum error correction (OAQEC) perspective [10].
A simple example can illustrate the basic structure of a hybrid code. Consider a two-qubit system. In the computational basis, these two qubits will have four orthogonal states available to them; \( |00\rangle, |01\rangle, |10\rangle \) and \( |11\rangle \). From these states, we can construct two pairs of logical states, and each pair can be labelled with the classical bits 0 or 1:

\[
|\bar{0}\rangle_0 := |00\rangle, \quad |\bar{1}\rangle_0 := |01\rangle, \quad |\bar{0}\rangle_1 := |10\rangle, \quad |\bar{1}\rangle_1 := |11\rangle.
\]  

(1)

These four codewords can move through a two-qubit channel and be immune to \( ZI \) errors. In this simple example, two unprotected qubits are used to transmit one classical bit and one qubit of information with some degree of error protection. The different hybrid codes which have been proposed (see [9] for examples) go well beyond this simple construction. The main text explains when hybrid code constructions provide an advantage over independent quantum and classical information transmissions. The objective of a quantum error correcting code is to (i) maximize the number of logical qubits which can be transmitted, (ii) maximize the robustness of their protection, and (iii) minimize the number of physical qubits that are required to transmit them. Hybrid codes add an additional dimension—maximizing the amount of classical information which is being transmitted.

We first demonstrate that the two perspectives on hybrid codes are equivalent and that the coding theory characterization is a specific case of the OAQEC model. This allows for each perspective to benefit from the results developed in the other. Next, we study the quantum Hamming bound. This bound is a seminal result in quantum error correction and loadstone for developing quantum error-correcting codes [11]. Extending such bounds to hybrid codes is thus an important step in developing hybrid codes. We generalize the quantum Hamming bound for hybrid error-correcting codes. Third, we discover a necessary condition for developing non-trivial hybrid codes—non-trivial hybrid codes must be degenerate. This is a clear restriction that all future non-trivial hybrid codes must adhere to. Finally, proof-of-principle experimental tests of hybrid codes have not been performed. One barrier to doing so is that the non-trivial codes to date require more qubits than are available in most academic labs [9]. We present an example of a non-trivial 4-qubit hybrid code.

The rest of this paper is organized as follows. Section 2 reviews the two perspectives on hybrid codes, and Section 3 presents their unification. We present a generalization of the Hamming bound on quantum error correcting codes for hybrid codes in Section 4. We demonstrate here that non-trivial hybrid codes must be degenerate and exemplify a 4-qubit example of such a code.

### 2 Two Perspectives

#### 2.1 Coding Theory Perspective

The characterization and construction of hybrid codes were formulated from a coding theory perspective in Ref. [9]. A quantum error-correcting code that encodes \( k \) qubits into \( n \) qubits with a distance \( d \) can be denoted by \( C = [n, k, d] \) or equivalently as \( C = (n, K, d) \) where \( K \) is the dimensions of the subspace of the \( k \) qubits, \( K = 2^k \). A classical code can be denoted identically, except with \( m \) and \( M \) representing the number of classical bits and dimension of the bits subspace respectively, \( C = [n, m, d] \) or \( C = (n, M, d) \). In this notation, a hybrid code can be denoted as \( C = [n, k : m, d] \) or \( C = (n, K : M, d) \).
Using this notation, we can describe three hybrid code constructions which do not provide an advantage over the independent solutions. First, given a quantum code, \( C = ((n, K M, d)) \), one can factor the code space into two subsystems of dimension \( K \) and \( M \), and use these subspaces to transmit \( K \) dimensional quantum and \( M \) dimensional classical information separately. Since this effectively sacrifices quantum bits for classical bits, there is no advantage to using such codes. A second construction comes from assuming one already has a hybrid code \( C = [[n, k : m, d]] \) and constructing a new code \( C' = [[n, k - 1 : m + 1, d]] \). A qubit can always be used to transmit classical information, making this construction also trivial. Finally, given a quantum code \( C_q = [[n_1, k, d]] \) and classical code \( C_c = [[n_2, m, d]] \) one can form a hybrid code \( C = [[n_1 + n_2, k : m, d]] \), which again would not provide any advantage. Our goal in developing hybrid codes is to find codes with better parameters than those provided.

A hybrid quantum code \( C = ((n, K : M, d)) \) can be described by a collection of \( M \) quantum codes \( \{ C^{(v)} : v = 1, ..., M \} \), where \( v \) is the classical information that determines which of the \( C^{(v)} \) is used. Each code has an orthonormal basis \( \{ | c_i^{(v)} \rangle : i = 1, 2, ..., K \} \).

Denote by \( \{ E_A \} \) the set of errors each code can correct. To correct such errors, each code must obey a modified version of the Knill–Laflamme condition [12]:

\[
| c_i^{(v)} \rangle E_k^\dagger E_l | c_j^{(v)} \rangle = \alpha_{kl}^{(v)} \delta_{ij},
\]

where \( \alpha_{kl}^{(v)} \) is a complex constant. Equation (2) differs from the original Knill–Laflamme condition in that the constant \( \alpha_{kl}^{(v)} \) depends on the classical information being transmitted as well. A second condition hybrid codes must satisfy is that each quantum code must be simultaneously distinguishable from all others. This is necessary for retrieving the classical information. This provides a second error correction condition on hybrid codes

\[
| c_i^{(v)} \rangle E_k^\dagger E_l | c_j^{(\mu)} \rangle = 0, \quad \text{for} \quad \mu \neq v
\]

Equations (2) and (3) can be written succinctly as

\[
| c_i^{(v)} \rangle E_k^\dagger E_l | c_j^{(\mu)} \rangle = \alpha_{kl}^{(v)} \delta_{ij} \delta_{\nu\mu}
\]

The proof for this condition is outlined in Ref. [9].

### 2.2 OAQEC Perspective

To introduce OAQEC, we must first introduce its predecessor—operator quantum error correction (OQEC) [13]. OQEC unifies the standard model and the noiseless subsystems model of error correction. The standard model [12, 14–16] consists of the 3-tuple \( (\mathcal{R}, \mathcal{E}, \mathcal{C}) \). Consider a quantum system whose states are elements of the Hilbert space \( \mathcal{H} \) and \( \mathcal{B}(\mathcal{H}) \) are the set of operators acting on \( \mathcal{H} \). Denote by \( \mathcal{C} \) the quantum code, a subspace of \( \mathcal{H} \). \( \mathcal{E} \) and \( \mathcal{R} \) are the error and recovery operations acting on \( \mathcal{B}(\mathcal{H}) \), such that \( \mathcal{R} \) corrects \( \mathcal{E} \),

\[
(\mathcal{R} \circ \mathcal{E})(\rho) = \rho, \quad \forall \rho \in \mathcal{C}.
\]

Using the Kraus operator notation, we can write \( \mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger \). Denote by \( P_{\mathcal{C}} \) the projection operator onto \( \mathcal{C} \). Such an \( \mathcal{R} \) exists for a given \( \mathcal{E} \) and \( \mathcal{C} \) if

\[
P_{\mathcal{C}} E_a^\dagger E_b P_{\mathcal{C}} = \lambda_{ab} P_{\mathcal{C}}, \quad \forall a, b,
\]

where \( \lambda_{ab} \) is a constant [12, 16].
In the noiseless subsystem model [17–20], one considers the error set they wish to correct \( \{ E_a \} \). Denote by \( \mathcal{A} \) the algebra generated by \( \{ E_a, E_a^\dagger \} \). Since \( \mathcal{A} \) is a \( C^* \)-algebra, it is unitarily equivalent to a direct sum of full matrix algebras, \( \mathcal{A} = \bigoplus_j M_{m_j} \otimes I_n \). This decomposes the Hilbert space into a noisy and noiseless subsystem, which we denote as subsystems \( A \) and \( B \), respectively. Elements in the noiseless subsystem (the “noise commutant”) will commute with \( \mathcal{A} \) and be immune to errors. The information one wishes to protect is encoded in the noise and be immune to errors. The information one wishes to protect is encoded in the noiseless subsystem.

A necessary condition for the existence of such a \( U \) exists such a semigroup \( P_k \) only if

\[
\Gamma(\rho) = \sum_{k,l} P_{k,l} \rho P_{k,l}^\dagger \in \mathcal{A}', \quad (7)
\]

\[
\Gamma(\rho \otimes \rho_B) \propto I_A \otimes \rho_B. \quad (8)
\]

This can be further generalized since it is not necessary to protect the entire space \( \rho_A \otimes \rho_B \), only \( \rho_B \). Thus instead of being confined to \( \mathcal{A}' = I_A \otimes \rho_B \) we consider the space \( \mathcal{U} = \rho_A \otimes \rho_B \).

We define \( P_k := P_{kk} \), \( P_{\mathcal{U}} := \sum_k P_k \), and \( P_{\mathcal{U}}^\perp = I - P_{\mathcal{U}} \). With these, we can give three equivalent definitions for a noiseless subsystem:

\[
\forall \rho_A, \rho_B, \exists \sigma_A : \mathcal{E}(\rho_A \otimes \rho_B) = \sigma_A \otimes \rho_B, \quad (9)
\]

\[
\forall \rho_B \exists \sigma_A : \mathcal{E}(I_A \otimes \rho_B) = \sigma_A \otimes \rho_B, \quad \text{and} \quad (10)
\]

\[
\forall \rho \in \mathcal{U} : (\text{tr}_A \circ \mathcal{P}_\mathcal{U} \circ \mathcal{E})(\rho) = \text{tr}_A(\rho). \quad (11)
\]

The subspace \( \mathcal{H}_B \) is noiseless if it satisfies any, and thus all, of the above conditions. There exists such a semigroup \( \mathcal{U} \) for a channel \( \mathcal{E} \) if and only if,

\[
P_k E_a P_l = \lambda_{a k l} P_{k l}, \quad \forall a, k, l, \quad (12)
\]

\[
P_{\mathcal{U}}^\perp E_a P_{\mathcal{U}} = 0, \quad \forall a. \quad (13)
\]

OQEC also consists of the 3-tuple \((\mathcal{R}, \mathcal{E}, \mathcal{C})\). The noiseless subspace model is a specific case where \( \mathcal{R} = I \), and the standard model is a specific case where \( \mathcal{C} = \mathcal{U} \). In OQEC, such a 3-tuple is correctable if

\[
(\text{tr}_A \circ \mathcal{P}_\mathcal{U} \circ \mathcal{R} \circ \mathcal{E})(\rho) = \text{tr}_A(\rho) \quad (14)
\]

A necessary condition for the existence of such a \( \mathcal{U} \) is

\[
P_k E_i^\perp E_j P_l = \alpha_{ijkl} P_{kl} \quad \forall i, j, k, l. \quad (15)
\]

We now move from OQEC to OAQEC. In the Schrödinger picture, our states change with time. Thus the expectation value of an operator, \( A \), acting on a system, \( \rho \), which is experiencing an error, \( E_i \), is given by \( \text{tr}(A E_i \rho E_i^\dagger) = \text{tr}(E_i^\perp A E_i \rho) \). The right-hand side of the equation implies that the error is acting on the observable (a Heisenberg picture of the error). For every trace-preserving channel in the Schrödinger picture that acts on state \( \rho \), there exists a corresponding dual map which is unital acting on the observable \( A \).

In the Schrödinger picture a subspace was noiseless for a channel \( \mathcal{E} \) if \( \mathcal{E}(\rho_A \otimes \rho_B) = \sigma_A \otimes \rho_B \), equivalently a space is noiseless for an error channel \( \mathcal{E}^\dagger \) in the Heisenberg picture if and only if \( P \mathcal{E}^\dagger (X \otimes I) P = X \otimes I \) for all \( X \) which are observables. Where \( P \) is the projector of the Hilbert space onto the subspace \( A \otimes B \). This gives two equivalent definitions for a noiseless subsystem; when one is satisfied, the other is also.

We say a set of operators \( S \) on \( \mathcal{H} \) are conserved by \( \mathcal{E} \) for states on some subspace \( \mathcal{H}_S \) if every \( X \in S \) satisfies \( P \mathcal{E}^\dagger(X) P = P X P \). These observables can generate an algebra that
we wish to protect from errors. This can equivalently be done via a theorem from [10], which states: Let \( \mathcal{A} \) be a subalgebra of \( \mathcal{B}(\mathcal{H}_S) \), \( \mathcal{A} \) is conserved by \( \mathcal{E} \) if and only if \( PE_a^E E_b P \) commutes with every element of the algebra. This can be expanded to the subalgebra \( \mathcal{A} \) being correctable for \( \mathcal{E} \) if and only if \( PE_a^E E_b P \) commutes with every element of the algebra for every combination of errors. This generalization, when now considered from the perspective of the Schrödinger picture, shows that the algebra \( \mathcal{A} \) is correctable for \( \mathcal{E} \) for subspaces of the Hilbert space \( \mathcal{H}_S \) on one condition. The condition is that there exists a recovery operation \( \mathcal{R} \) such that, for any density operator which can be separated into a sum of tensor products of operators in the separate spaces \( \rho = \sum_k \alpha_k (\rho_k \otimes \tau_k) \) for \( \sum_k \alpha_k = 1 \), the following equation holds

\[
(\mathcal{R} \circ \mathcal{E})(\rho) = \sum_k \alpha_k \mathcal{R}(\mathcal{E}(\rho_k \otimes \tau_k)) = \sum_k \alpha_k (\rho_k \otimes \tau_k').
\] (16)

Not that for \( \alpha_1 = 1 \), this reduces to the OQEC condition. \( (\mathcal{R} \circ \mathcal{E})(\rho_k \otimes \tau_k) = \rho_k \otimes \tau_k' \). Denote by \( X_{abk} \) operators acting on subspace \( B \). There exists such a correction operation if and only if for all \( a, b \) there exists \( X_{abk} \) such that:

\[
PE_a^E E_b P = \sum_k I_{Ak} \otimes X_{abk}.
\] (17)

### 3 Unified Perspective

We now show that the coding theory perspective is a special case of the OAQEC formulation. The coding theory construction included three restrictions that do not exist in the OAQEC model. First, the error set in the coding theory construction is restricted to containing only unitary errors, particularly the Pauli channel. Second, in the coding theory model each quantum channel is viewed as a subspace where the OAQEC model deals with subsystems. Finally, the coding theory model restricts each quantum channel to be of equal dimension. These three restrictions can be summarized as:

1. \( \mathcal{E} \subseteq P_n \)
2. \( C^{(v)} \subseteq \mathcal{H}, \forall \ 1 \leq v \leq M \) subspaces
3. \( \dim C^{(v)} = K \ \forall \ v, C^{(v)} = \text{span}\{c^{(v)}_i : 1 \leq i \leq K\} \)

The coding theory error correction condition acts on codewords in Hilbert space while the OAQEC models acts on operators. In unifying these two models it is necessary to rewrite the coding theory condition in terms of operators. This can be done by considering the two equivalent forms of the regular Knill–Laflamme condition.

\[
\langle c_i | E_a^E E_b | c_j \rangle = \alpha_{ab} \delta_{ij} \iff PE_a^E E_b P = \alpha_{ab} P.
\]

Either form can be used. With this, we can define the projector onto the hybrid codeword space as:

\[
P = \sum_{i,v} |c_i^{(v)}\rangle \langle c_i^{(v)}|\]

(18)

This projector can then be used to rewrite (4) as

\[
P E_a^E E_b P = \sum_{i,v} \alpha_{kl}^{(v)} |c_i^{(v)}\rangle \langle c_i^{(v)}|\]

(19)
Unlike the general Knill–Laflamme condition, $\alpha$ depends on the codewords, so it must be included within the projector on the right-hand side of (18). Consider substituting (18) into (19):

$$PE_a^\dagger E_b P = \sum_{i,j,v,\mu} \sum_{\nu,\mu} \alpha(\nu) \alpha(\mu) \delta_{ij} \delta_{\nu\mu} \langle c_i^{(v)} | c_j^{(\mu)} \rangle,$$

(20)

$$PE_a^\dagger E_b P = \sum_{i,j,v,\mu} \alpha(\nu) \alpha(\mu) \langle c_i^{(v)} | c_j^{(\mu)} \rangle,$$

(21)

$$PE_a^\dagger E_b P = \sum_{i,j,v,\mu} \alpha(\nu) \alpha(\mu) \langle c_i^{(v)} | c_j^{(\mu)} \rangle.$$

(22)

By starting from the operator form and applying the two requirements outlined in having a hybrid quantum code we arrive at a condition similar to the regular Knill–Laflamme condition but with $\alpha$ contained in the summations. Finally, if we consider the OAQEC model for the case where the entire space is correctable, namely that the noisy subspace is 1. Then Eq. 15 simplifies to

$$PE_a^\dagger E_b P = \sum_k \alpha(\nu) \alpha(\mu) P_{Ak},$$

(23)

which is equivalent to (22)

4 Hybrid Hamming Bound

An important question in the discussion of hybrid codes is when hybrid codes will provide an advantage over codes which transmit quantum and classical information separately. Constructing the Hamming bound [21] for hybrid codes provides us with one means with which to compare the parameters of a hybrid and quantum code. The quantum Hamming bound applies to non-degenerate codes with the error set consisting of the Pauli matrices. For qubits, the bound can be stated as

$$t \sum_{j=1}^{t} \binom{n}{j} 3^jt^k \leq 2^n,$$

(24)

where $t = \frac{d-1}{2}$. The bound can be reconstructed for hybrid codes. The quantum Hamming bound is essentially a packing argument. The argument is that the total space available to the system of qubits must be greater than the total space the errors can map codewords to plus the amount of space taken by the codewords themselves.

A code with $n$ physical qubits will have $2^n$ orthogonal subspaces available. Some of this space will be used by the logical codewords themselves, and the rest can be used by the space that errors map these codewords to. A code with $k$ qubits will have $2^k$ codewords. If the code can correct up to $j$ errors then there are $\binom{n}{j}$ sets of locations where an error can occur. At each location any of the three possible Pauli errors can occur, giving $3^j$ possible errors for each set of locations. These errors can occur on any of the $2^k$ codewords. This gives a total of $\sum_{j=1}^{t} \binom{n}{j} 3^j 2^k$ possible errors. Thus we arrive at the quantum Hamming bound,

$$\sum_{j=1}^{t} \binom{n}{j} 3^j 2^k + 2^k \leq 2^n,$$

(25)
\[ \sum_{j=0}^{t} \binom{n}{j} 3^j 2^k \leq 2^n. \] (26)

A hybrid code with \( n \) physical qubits will also have a total of \( 2^n \) orthogonal subspaces available. For a hybrid code with \( M \) codes, there will be \( M 2^k \) logical codewords. Since each quantum code making up the hybrid code must correct the same error set, then the number of locations an error can occur and the number of possible errors does not change for the non-degenerate case. The number of codewords this error can occur on has changed though to \( M 2^k \), thus the total number of errors which can occur is \( \sum_{j=1}^{t} \binom{n}{j} 3^j M 2^k \). Therefore the quantum Hamming bound for hybrid codes is given by:

\[ \sum_{j=1}^{t} \binom{n}{j} 3^j M 2^k + M 2^k \leq 2^n \] (27)

\[ M \sum_{j=0}^{t} \binom{n}{j} 3^j 2^k \leq 2^n \] (28)

This bound highlights another important result of hybrid codes. A non-degenerate hybrid code can not provide an advantage over an equivalent quantum code. Therefore, degeneracy will be necessary for constructing non-trivial hybrid codes. We construct an example of a degenerate hybrid code. The code we construct has parameters \([4, 1 : 1, 2]\). This code can detect the error set \( E = \{X_i, Y_i, Z_i, Z_1 Z_2, Z_3 Z_4\} \quad \forall i = 1, 2, 3, 4 \), or equivalent in can correct the given error set, given the location of errors are known. This hybrid code has codewords

\[ |\bar{0}\rangle_0 = |0000\rangle + |1111\rangle, \quad |\bar{1}\rangle_0 = |0011\rangle - |1100\rangle, \] (29)

\[ |\bar{0}\rangle_1 = |0101\rangle + |1010\rangle, \quad |\bar{1}\rangle_1 = |1001\rangle - |0110\rangle. \] (30)

### 5 Outlook

We demonstrated that the two perspectives on hybrid codes are equivalent and that the coding theory characterization is a specific case of the OAQEC model. Furthermore, we generalized the quantum Hamming bound for hybrid error-correcting codes. In doing so, we found that a non-degenerate hybrid code can not provide an advantage over an equivalent quantum code. We then designed a four-qubit non-degenerate hybrid code. This code transmits one qubit and one classical bit and detects any single Pauli error.

Our results present natural opportunities for future work. First, there have been no physical implementations of hybrid codes in the literature. The code we constructed contained few enough qubits to be readily achieved on various quantum hardware available in academic labs. For instance, similar quantum states as those presented in (29) and (30) have been prepared using nuclear magnetic resonance spectrometers in, for example, tests of quantum error correction [22–24] and quantum foundations [25–27]. Second, developing the hybrid forms of other quantum bounds, particularly the quantum singleton bound and the quantum Gilbert–Varshamov bound [28, 29], is important for understanding the limitations of hybrid code. Finally, OAQEC has recently been used for various applications in the study of black holes [30–33]. Using the connection established in this work, it may be possible to cast a coding theory perspective on these black-hole physics results and benefit from the tools therein.
Author Contributions I carried out this work independently.

Declarations

Competing interests The authors declare no competing interests.

References

1. Devetak, I., Shor, P.W.: The capacity of a quantum channel for simultaneous transmission of classical and quantum information. Commun. Math. Phys. 256, 287 (2005)
2. Hsieh, M.-H., Wilde, M.M.: Entanglement-assisted communication of classical and quantum information. IEEE Trans. Inf Theory, 56, 4682 (2010)
3. Hsieh, M.-H., Wilde, M.M.: Trading classical communication, quantum communication, and entanglement in quantum shannon theory. IEEE Trans. Inf. Theory 56, 4705 (2010)
4. Kremsky, I., Hsieh, M.-H., Brun, T.A.: Classical enhancement of quantum-error-correcting codes. Phys. Rev. A 78, 012341 (2008)
5. Nemec, A.S.: Hybrid and Nonadditive Quantum Codes, Ph.D. thesis (2022)
6. Nemec, A., Klappenecker, A.: Hybrid codes, In: IEEE International Symposium on Information Theory (ISIT), pp. 796–800 (2018)
7. Nemec, A., Klappenecker, A.: Nonbinary error-detecting hybrid codes. Am. J. Sci. & Eng. 1, 1 (2020)
8. Nemec, A., Klappenecker, A.: Infinite families of quantum-classical hybrid codes. IEEE Trans. Inf. Theory 67, 2847 (2021)
9. Grassl, M., Lu, S., Zeng, B.: Codes for simultaneous transmission of quantum and classical information, in Information Theory (ISIT), IEEE Int. Symp. pp. 1718–1722 (2017)
10. Bény, C., Kempf, A., Kribs, D.W.: Quantum error correction of observables, Physical Review A 76, 042303 (2007)
11. Gottesman, D.: Class of quantum error-correcting codes saturating the quantum hamming bound. Phys. Rev. A, 54, 1862 (1996)
12. Knill, E., Laflamme, R.: Theory of quantum error-correcting codes. Phys. Rev. A, 55, 900 (1997)
13. Kribs, D., Laflamme, R., Poulin, D.: Unified and generalized approach to quantum error correction. Phys. Rev. lett, 94, 180501 (2005)
14. Shor, P.W.: Scheme for reducing decoherence in quantum computer memory. Phys. Rev. A 52, R2493 (1995)
15. Steane, A.M.: Simple quantum error-correcting codes. Phys. Rev. A 54, 4741 (1996)
16. Bennett, C.H., DiVincenzo, D.P., Smolin, J.A., Wootters, W.K.: Mixed-state entanglement and quantum error correction. Phys. Rev. A, 54, 3824 (1996)
17. Palma, G.M., Suominen, K.-A., Ekert, A.: Quantum computers and dissipation, Proceedings of the Royal Society of London. Series A: Math. Phys. Eng. Sci. 452, 567 (1996)
18. Duan, L.-M., Guo, G.-C.: Preserving coherence in quantum computation by pairing quantum bits. Phys. Rev. Lett. 79, 1953 (1997)
19. Zanardi, P., Rasetti, M.: Noiseless quantum codes. Phys. Rev. Lett. 79, 3306 (1997)
20. Lidar, D.A., Chuang, I.L., Whaley, K.B.: Decoherence-free subspaces for quantum computation. Phys. Rev. Lett. 81, 2594 (1998)
21. Hamming, R.W.: Error detecting and error correcting codes. The Bell Syst. Tech. J. 29, 147 (1950)
22. Cory, D.G., Price, M., Maas, W., Knill, E., Laflamme, R., Zurek, W.H., Havel, T.F., Somaroo, S.S.: Experimental quantum error correction. Phys. Rev. Lett. 81, 2152 (1998)
23. Leung, D., Vanderven, L., Zhou, X., Sherwood, M., Yannoni, C., Kubinec, M., Chuang, I.: Experimental realization of a two-bit phase damping quantum code. Phys. Rev. A 60, 1924 (1999)
24. Knill, E., Laflamme, R., Martinez, R., Negrevergne, C.: Benchmarking quantum computers: the five-qubit error correcting code. Phys. Rev. Lett. 86, 5811 (2001)
25. Majidy, S.-S., Katiyar, H., Amikeeva, G., Halliwell, J., Laflamme, R.: Exploration of an augmented set of leggett-garg inequalities using a noninvasive continuous-in-time velocity measurement. Phys. Rev. A 100, 042325 (2019)
26. Majidy, S.-S.: Violation of an augmented set of Leggett-Garg inequalities and the implementation of a continuous in time velocity measurement, Master’s thesis, University of Waterloo (2019)
27. Majidy, S., Halliwell, J.J., Laflamme, R.: Detecting violations of macrorealism when the original leggett-garg inequalities are satisfied. Phys. Rev. A 103, 062212 (2021)
28. Gilbert, E.N.: A comparison of signalling alphabets. The Bell Syst. Tech. J. 31, 504 (1952)
29. Varshamov, R.R.: Estimate of the number of signals in error correcting codes, Doklady Akad. Nauk, SSSR 117, 739 (1957)
30. Penington, G.: Entanglement wedge reconstruction and the information paradox. J. High Energy Phys. 2020, 1 (2020)
31. Kibe, T., Mandayam, P., Mukhopadhyay, A.: Holographic spacetime, black holes and quantum error correcting codes: a review. Eur. Phys. J. C 82, 463 (2022)
32. Hayden, P., Penington, G.: Learning the alpha-bits of black holes. J. High Energy Phys. 2019, 1 (2019)
33. Kim, I., Tang, E., Preskill, J.: The ghost in the radiation: Robust encodings of the black hole interior. J. High Energy Phys. 2020, 1 (2020)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.