ON FINITENESS OF KLEINIAN GROUPS IN GENERAL DIMENSION

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ABSTRACT. In this paper we provide a criteria for geometric finiteness of Kleinian groups in general dimension. We formulate the concept of conformal finiteness for Kleinian groups in space of dimension higher than two, which generalizes the notion of analytic finiteness in dimension two. Then we extend the argument in the paper of Bishop and Jones to show that conformal finiteness implies geometric finiteness unless the set of limit points is of Hausdorff dimension $n$. Furthermore we show that, for a given Kleinian group $\Gamma$, conformal finiteness is equivalent to the existence of a metric of finite geometry on the Kleinian manifold $\Omega(\Gamma)/\Gamma$.

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\section{Introduction}

A discrete subgroup $\Gamma$ of the group of conformal transformations of the unit sphere $S^n$ is called a Kleinian group if there is a non-empty domain $\Omega(\Gamma)$ of discontinuity in $S^n$. The group $\Gamma$ also acts as a subgroup of the group of hyperbolic isometries of the unit ball $B^{n+1}$. Assuming the group has no torsion elements, the quotient is a hyperbolic manifold $B^{n+1}/\Gamma$, which is bounded at infinity by a Kleinian manifold $\Omega(\Gamma)/\Gamma$ which is a manifold with a locally conformally flat structure.

There is a useful notion of geometric finiteness for Kleinian groups that assures nice properties of the geometric quotient. There are several equivalent forms of this condition. According to one formulation, a Kleinian group $\Gamma$ is said to be geometrically finite if the limit set $\Lambda(\Gamma)$ consists only of conical limit points and cusped limit points. It is a natural problem to find useful criteria to assure geometric finiteness. In dimension two, a Kleinian group is said to be analytically finite if the Riemann surface $\Omega(\Gamma)/\Gamma$ is of finite type (that is to say a union of a finite number of closed Riemann surfaces each with finite number of punctures). Recently Bishop and Jones ([5]) showed that, for an analytically finite group $\Gamma$, $\Lambda(\Gamma)$ has Hausdorff

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dimension less than two if and only if \( \Gamma \) is geometrically finite. An important part of their work is the construction of an invariant Lipschitz graph which serves to relate the geometry of the hyperbolic manifold \( B^3/\Gamma \) to the geometry of the Riemann surface \( \Omega(\Gamma)/\Gamma \).

Recently we studied locally conformally flat 4-manifolds and obtained some finiteness for certain class of such manifolds [7] [8]. In those works [8] the holonomy representation of the fundamental group of such manifolds as Kleinian group played a key role in our understanding of the structure of such manifolds. In dimension higher than two, Kleinian groups have been studied mostly in conjunction with hyperbolic structure. Our motivation is to investigate the close relation between the geometry of the hyperbolic manifolds \( B^{n+1}/\Gamma \) and the geometry of the Kleinian manifold \( \Omega(\Gamma)/\Gamma \) for a given Kleinian group \( \Gamma \).

For our purpose, we introduce a notion of conformal finiteness (see Definition 3.2 in Section 3), which is the natural analogue of the notion of analytic finiteness to higher dimensions. Then we extend the theorem of Bishop and Jones to Kleinian groups in higher dimension.

**Theorem 0.1.** Suppose that \( \Gamma \) is a nonelementary, conformally finite Kleinian group on \( S^n \), then \( \Gamma \) is geometrically finite if and only if the limit set of \( \Gamma \) has Hausdorff dimension strictly smaller than \( n \).

Recall the celebrated finiteness theorem of Ahlfors [2] and Bers [3] [4], which states that a finitely generated Kleinian group in dimension two is analytically finite. This finiteness theorem fails to hold in higher dimensions as pointed out by the examples of Kapovich [11], and Kapovich and Potyagailo [12]. On the other hand, a result of Jarvi and Vuorinen [10] shows that in general dimensions, the limit set of a finitely generated Kleinian group is uniformly perfect. The latter condition is equivalent, in dimension two, to the condition of analytic finiteness of the group. Moreover in [1] Ahlfors showed some weak finiteness for Kleinian groups in higher dimension: if \( \Gamma \) is finitely generated, then the dimension of the space of certain class of mixed tensor densities, automorphic under \( \Gamma \), is finite (see also [14] of Hiromi Ohtake). Therefore, it is interesting to search for the appropriate version of the finiteness result in higher dimension, particularly for those Kleinian groups with small limit sets. We take a first step in that direction by characterizing the conformally finite ends by geometric conditions. As another application of the Lipschitz graph construction, we have

**Theorem 0.2.** Given a Kleinian group \( \Gamma \), the Kleinian manifold \( \Omega(\Gamma)/\Gamma \) is of finite geometry for some metric in the conformal class if and only if \( \Gamma \) is conformally finite.

By finite geometry here we mean that its curvature and covariant derivatives of curvature is bounded, and its volume is finite.

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§1. **Construction of the Lipschitz Graph**
The following construction, by completely elementary means, of the invariant Lipschitz graph over a domain of discontinuity $\Omega(\Gamma)$ of a Kleinian group $\Gamma$ is based on the idea of Bishop and Jones’ in [5]. Take a small positive number $\epsilon_0$ and consider a collection of balls $\{B_\alpha\}$ such that

$$B_\alpha = B(x_\alpha, d_\alpha) \quad \text{and} \quad d_\alpha = \epsilon_0 \cdot \text{dist}(x_\alpha, L(\Gamma))$$

for each point $x_\alpha \in \Omega(\Gamma)$, where diameters and distances are all measured on $S^n$ with the standard metric $g_0$. To construct an invariant graph we would enlarge the collection to take in all images of $B_\alpha$ under the group $\Gamma$ and denote the collection of balls by $B(\Gamma)$. Set

$$G(\Gamma) = \partial(\bigcup_\beta H_\beta) \bigcap B^{n+1}$$

where $H_\beta$ is the hyperbolic half space over each ball $B_\beta$ in $B(\Gamma)$, i.e. the dome whose boundary intersects perpendicularly at $\partial B_\beta$ with $S^n$. Clearly $G(\Gamma)$ is a graph over $\Omega(\Gamma)$ in the following sense

$$G(\Gamma) = \{f(x) : x \in \Omega(\Gamma)\}$$

where $f(x) : \Omega(\Gamma) \to (0,1)$. In fact $G(\Gamma)$ is a Lipschitz graph in the sense that

$$|f(x) - f(y)| \leq M \text{dist}(x,y)$$

for some $M > 0$ and all $x,y \in \Omega(\Gamma)$. Therefore

**Proposition 1.1.** Given a nonelementary Kleinian group $\Gamma$ and a small positive number $\epsilon_0$, the above constructed graph $G(\Gamma)$ is a $\Gamma$-invariant Lipschitz graph. Moreover

$$0 < C_1 \leq \frac{1 - f(x)}{\text{dist}(x, L(\Gamma))} \leq C_2$$

for all $x \in \Omega(\Gamma)$, where $C_1, C_2$ only depend on $\epsilon_0$.

We need to recall some facts about Möbius transformations. For any Möbius transformation $\gamma$ on $\mathbb{R}^n \cup \{\infty\}, n \geq 3$, we have

$$|\gamma(x) - \gamma(y)| = |\gamma'(x)|^{\frac{1}{2}} |\gamma'(y)|^{\frac{1}{2}} |x - y|$$

(see, for example, (1.3.2) in [13]). In addition if $\gamma$ is not a composition of just scalings, rotations, or translations, then

$$|\gamma'(x)|_e = \frac{1}{\lambda |x - b|^2}$$

for $b \in \mathbb{R}^n$ and $\gamma(b) = \infty$, where $|\cdot|_e$ denotes norm under Euclidean metric while we will use $|\cdot|_s$ for the norm under the standard metric of the sphere, i.e.

$$|\gamma'(x)|_s = \frac{1 + |x|^2}{1 + |\gamma x|^2} |\gamma'(x)|_e.$$

As a consequence (1.5) and (1.6) we have:
Lemma 1.2. Suppose that $\Gamma$ is a nonelementary Kleinian group, $\Omega(\Gamma)$ is its domain of discontinuity and $L(\Gamma) = \partial \Omega(\Gamma)$ is its set of limit points. Then there exists a positive number $C$ such that

\[ \frac{1}{C \dist(x, L(\Gamma))} \leq \dist(\gamma(x), L(\Gamma)) \leq C \frac{\dist(x, L(\Gamma))}{\dist(x, L(\Gamma))} \]

for all $x \in \Omega(\Gamma)$ and all $\gamma \in \Gamma$.

Proof. For any given $x \in \Omega(\Gamma)$ and $\gamma \in \Gamma$. Let

$\dist(x, L(\Gamma)) = \dist(x, a)$, and $\dist(\gamma x, L(\Gamma)) = \dist(\gamma x, b)$,

for some $a, b \in L(\Gamma)$. Since $L(\Gamma)$ contains more than two points, there exists a positive number $d_0 > 0$ (which is independent of $\{a, b\}$) such that we can always find a third point $p \in L(\Gamma)$ such that

$\dist(p, \{a, b\}) \geq d_0$

Then

$\dist(x, p) \geq \dist(p, a) - \dist(a, x) \geq \dist(p, a) - \dist(x, p)$.

Hence,

$\dist(x, p) \geq \frac{1}{2} d_0$.

Similarly,

$\dist(\gamma x, p) \geq \frac{1}{2} d_0$.

Let us fix $p$ as $\{\infty\}$ in $\mathbb{R}^n \cup \{\infty\}$. For convenience we continue to use the same notations for points in $\mathbb{R}^n \cup \{\infty\}$. By (1.5) and (1.6) we have

$|\gamma x - \gamma z| = |x - z| |\gamma'(x)|^{\frac{1}{2}} |\gamma'(z)|^{\frac{1}{2}}$

$= |x - z| \frac{1}{\lambda} \frac{1}{|x - c|} \frac{1}{|z - c|}$
for some \( c \in L(\Gamma) \) and all \( z \in \Omega(\Gamma) \) satisfying \( |x - z| = \frac{1}{2}|x - a| \). Therefore,

\[
|\gamma x - \gamma z| = \frac{|x - z| |x - c|}{\lambda |x - c|^2 |z - c|} \geq |x - z| |\gamma'(x)| e \frac{|x - c|}{|x - c| + |x - z|},
\]

Thus

\[
|\gamma x - b| > |\gamma x - \gamma z| \geq \frac{1}{2}|x - a||\gamma'(x)| e \frac{2}{3}.
\]

Due to the choice of \( p \) which is some fixed spherical distance away from all four points \( x, \gamma x, a, b \), we have, for some constant \( C \) (may depend on \( d_0 \)),

\[
(1.8) \quad \frac{\text{dist}(\gamma(x), L(\Gamma))}{\text{dist}(x, L(\Gamma))} \geq \frac{1}{C} |\gamma'(x)| s.
\]

Applying the same argument to \( \gamma^{-1} \) we find

\[
(1.9) \quad |\gamma'(x)| \geq \frac{1}{C} \frac{\text{dist}(\gamma(x), L(\Gamma))}{\text{dist}(x, L(\Gamma))}.
\]

For those element in the group \( \Gamma \) which is a composition of scalings, rotations, or translations, (1.7) is easily verified. Therefore the proof is complete.

**Remark 1.3.** For \( L(\Gamma) = \{\infty\} \) and \( \Gamma \) is simply generated by a translation \( \gamma x = x + h \), we have

\[
|\gamma'(x)| s = \frac{1 + |x|^2}{1 + |\gamma x|^2} = \frac{q(\gamma x, \infty)^2}{q(x, \infty)^2},
\]

where

\[
q(x, y) = \frac{2|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad \text{and} \quad q(x, \infty) = \frac{2}{\sqrt{1 + |x|^2}}
\]

which is called the chordal metric and is equivalent to the spherical distance \( d(x, y) \).

For \( L(\Gamma) = \{0, \infty\} \) and \( \Gamma \) is simply generated by an inversion \( \gamma x = \frac{x}{|x|^2} \), we also have

\[
|\gamma'(x)| s = \frac{q(\gamma x, \infty)^2}{q(x, 0)^2}.
\]

**Proof of Proposition 1.1.** It is clear from the construction that the graph \( G(\Gamma) \) is \( \Gamma \)-invariant. We need to verify that it is a Lipschitz graph. This follows easily as long as for all balls in \( B(\Gamma) \) we have

\[
(1.10) \quad C_0 \leq \frac{\text{diam}(D_\beta)}{\text{dist}(D_\beta, L(\Gamma))} \leq \frac{1}{2},
\]

for some positive constant \( C_0 \). But (1.10) is a consequence of (1.7) in Lemma 1.2 for some appropriate choice of \( \epsilon_0 \). So the proof is complete.

Given a Kleinian group \( \Gamma \), let \( C(L(\Gamma)) \subset B^{n+1} \) denote the convex hull of the set of limit points \( L(\Gamma) \subset S^n \) with respect to the hyperbolic metric on \( B^{n+1} \). Then \( C(B^{n+1}/\Gamma) = C(L(\Gamma))/\Gamma \) is called the convex core of the hyperbolic manifold \( B^{n+1}/\Gamma \). From the construction of the graph \( G(\Gamma) \) the following is clear.
Lemma 1.4. For any nonelementary Kleinian group, the above constructed $\Gamma$-invariant graph $G(\Gamma)$ separates the convex hull $C(L(\Gamma))$ from $\Omega(\Gamma)$.

§2 Compact Case

The Lipschitz graph constructed in the previous section has an induced metric from the hyperbolic metric. We would like to compare it with a suitable conformal metric on the Kleinian quotient $\Omega(\Gamma)/\Gamma$. In the case when the Kleinian quotient is compact, this is a relatively simple matter.

Theorem 2.1. Suppose that $\Gamma$ is a nonelementary Kleinian group, and that the Kleinian manifold $\Omega(\Gamma)/\Gamma$ is compact. Then, for any metric in the conformal class on $\Omega(\Gamma)/\Gamma$, we have a complete $\Gamma$-invariant metric $e^{2u}g_0$ on $\Omega(\Gamma)$ and

$$(2.1) \quad \frac{1}{K \operatorname{dist}(x, L(\Gamma))} \leq e^u(x) \leq \frac{1}{\operatorname{dist}(x, L(\Gamma))},$$

for all $x \in \Omega(\Gamma)$ and some positive number $K$.

Proof. This basically is a consequence of Lemma 1.2. Due to the invariance of the metric $e^{2u}g_0$, we have

$$e^u(x) = e^{u(y)}|\gamma'(x)|_s.\quad (2.2)$$

Now, fix a fundamental region $D$, whose closure $\bar{D}$ is compact in $\Omega(\Gamma)$, and for any $x \in \Omega(\Gamma)$, there exists $\gamma \in \Gamma$ such that $\gamma^{-1}x = y \in \bar{D}$. Then

$$e^u(y) = e^u(x)|\gamma'(y)|_s$$

where, by Lemma 1.2,

$$\frac{1}{C \operatorname{dist}(x, L(\Gamma))} \leq |\gamma'(y)|_s \leq C \frac{\operatorname{dist}(x, L(\Gamma))}{\operatorname{dist}(y, L(\Gamma))}.$$

Thus

$$(2.3) \quad \frac{1}{C \operatorname{dist}(x, L(\Gamma))} \leq e^u(x) \leq e^{u(y)} \frac{\operatorname{dist}(x, L(\Gamma))}{\operatorname{dist}(y, L(\Gamma))} \leq C \frac{e^{u(y)} \operatorname{dist}(y, L(\Gamma))}{\operatorname{dist}(x, L(\Gamma))},$$

that is

$$\frac{1}{K \operatorname{dist}(x, L(\Gamma))} \leq e^u(x) \leq K \frac{1}{\operatorname{dist}(x, L(\Gamma))},$$

for some positive constant $K$.

Remark 2.2. In particular, we may consider the Yamabe metric on $\Omega(\Gamma)/\Gamma$ in Theorem 2.1. Therefore, under the assumptions of Theorem 2.1, there is a complete $\Gamma$-invariant metric on $\Omega(\Gamma)$ with constant scalar curvature and satisfying (2.1).

We point out that the natural bounds (2.1) on the invariant metric on $\Omega(\Gamma)$ is the key to relate the hyperbolic geometry inside $B/\Gamma$ to the conformal geometry at infinity $\Omega(\Gamma)/\Gamma$ through the constructed $\Gamma$-invariant Lipschitz graph $G(\Gamma)$. Namely,
Proposition 2.3. Suppose that $\Gamma$ is a nonelementary Kleinian group and that the Kleinian manifold $\Omega(\Gamma)/\Gamma$ is compact. Then the map
\[
F(x) = f(x)x : \Omega(\Gamma) \to G(\Gamma)
\]
is a $\Gamma$-invariant bi-Lipschitz map with respect to the induced hyperbolic metric on the graph $G(\Gamma)$ and any metric on $\Omega(\Gamma)$ which is induced by a metric on $\Omega(\Gamma)/\Gamma$ in the conformal class.

§3 Geometric finiteness

In this section we discuss notions of finiteness for Kleinian groups. We begin with recalling notions of geometric finiteness for Kleinian groups in the study of hyperbolic manifolds. Then we give a definition of conformal finiteness as a generalization of analytic finiteness for Kleinian groups as discrete subgroups of conformal transformations on $S^2$. Then we will give some useful metrics on the Kleinian manifold $\Omega(\Gamma)/\Gamma$ for conformally finite group $\Gamma$.

A good reference for the discussion of geometric finiteness for Kleinian groups is the paper of Bowditch [6]. We also refer readers to the book of Ratcliffe [15] and Ahlfors’ lecture notes [1] for all basics about Kleinian groups.

Definition 3.1. A Kleinian group $\Gamma$, as a discrete subgroup of the group of hyperbolic isometries, is geometrically finite if its limit set $L(\Gamma)$ consists entirely of conical limit points and cusped limit points.

An equivalent formulation says $\Gamma$ is geometrically finite if and only if the thick part of the convex core $C(B^{n+1}/\Gamma)$ is compact. Thus, if $\Gamma$ is geometrically infinite, there must be a sequence of points $\{p_i\} \in C(B^{n+1}/\Gamma)$ for which the injective radius of $B^{n+1}/\Gamma$ at $p_i$ is bounded from below and $p_i$ tends to infinity in the convex core $C(B^{n+1}/\Gamma)$. This fact will be used later. Another equivalent definition of geometric finiteness says $\Gamma$ is geometrically finite if and only if $(B^{n+1} \cup \Omega(\Gamma))/\Gamma$ may be considered as a union of a compact set and a finite number of disjoint standard cusp ends. It follows that $\Omega(\Gamma)/\Gamma$ is the union of a compact set and a finite number of disjoint ends which we will call standard conformal cusp ends. Of course, a standard conformal cusp end $C_m$ is the ideal boundary of the standard hyperbolic cusp end. For a discrete subgroup $\Gamma_\infty$ of the group of Euclidean isometries of $R^n$, let $R^{n-m}$ be the maximal invariant subspace so that $R^{n-m}/\Gamma_\infty$ is compact. Suppose that $N(R^{n-m}, \epsilon)$ is an $\epsilon$-neighborhood of $R^{n-m}$ in $R^n$. Then $N(R^{n-m}, \epsilon)$ is also invariant under $\Gamma_\infty$ and a standard conformal cusp end is of the form
\[
(R^n \setminus N(R^{n-m}, \epsilon))/\Gamma_\infty.
\]

Therefore, a standard conformal cusp end is conformal to $(R^m \setminus B_R(0)) \times K$ where $K$ is a compact locally flat manifold of dimension $n - m$. Now we are ready to give the following definition.
Definition 3.2. Suppose that $\Gamma$ is a Kleinian group. Then we say $\Gamma$ is conformally finite if $\Omega(\Gamma)/\Gamma$ is a disjoint union of a compact set and a finite number of standard conformal cusp ends.

By definition, geometric finiteness implies conformal finiteness. In this terminology, our goal is to investigate when conformal finiteness implies geometric finiteness. It is clear that the notion of conformal finiteness is a higher dimension analogue of the analytic finiteness.

In the case that $\Gamma$ is an analytic finite group acting on $S^2$, the uniformization theorem yields a hyperbolic metric on $\Omega(\Gamma)/\Gamma$ and the Lemma of Schwarz and the Koebe distortion theorem show that the hyperbolic metric satisfies the bounds (2.1). In higher dimension there are several possible canonical metrics available, but the natural bounds (2.1) becomes an issue. When $\Gamma$ is conformally finite, we describe a metric on the Kleinian manifold $\Omega(\Gamma)/\Gamma$ by explicitly writing metrics on each conformal cusp end. Take $(x, y) \in R^m \times R^{n-m}$ and let

$$g_h = \frac{1}{|x|^2}(|dx|^2 + |dy|^2).$$

The set $\{(x, y)|x \neq 0\}$ is conformally the standard $H^{n-m+1} \times S^{m-1}$, it is the holonomy cover of our typical conformally finite end.

Lemma 3.3. $g_h$ induces a complete metric in the standard conformal class of $(R^m \setminus \{0\}) \times K$ with constant scalar curvature $\frac{n-2}{4}(2m-n-2)$. More importantly, if we write $g_h = e^{2u}g_0$ where $g_0$ is the standard metric on $S^n$, then

$$\frac{1}{C} \frac{1}{\text{dist}(p, \infty)} \leq e^{u(p)} \leq C \frac{1}{\text{dist}(p, \infty)}$$

for some constant $C > 0$ and all $p$ in a fundamental domain $R^m \times K'$ where $K'/\Gamma_{\infty} = K$. A standard conformal cusp end $((R^m \setminus B_R(0)) \times K, g_h)$ has a finite volume.

Proof. Let us verify (3.2), we again use the chordal distance instead of spherical distance. Recall

$$q((x, y), \infty) = \frac{2}{\sqrt{1 + |x|^2 + |y|^2}}.$$

Therefore, for $p = (x, y)$,

$$q(p, \infty)e^{u(p)} = \frac{\sqrt{1 + |x|^2 + |y|^2}}{|x|}.$$

Thus, when restrict $p$ in a fundamental domain, for instance, $R^m \times K'$, we have $|y|$ bounded and

$$\frac{1}{C} \leq q(p, \infty)e^{u(p)} \leq C'. $$
for some constant $C'$ depending on the size of the fundamental domain of $\Gamma_\infty$, which implies (3.2). To compute the volume, we have

\begin{align}
\text{vol}(R^m \setminus B_R(0) \times K) &= \int_{R^m \setminus B_R(0)} \int_K |x|^{-n} dy dx \\
&= \text{vol}(K) \int_{R^m \setminus B_R(0)} |x|^{-n} dx \\
&= \text{vol}(K) \int_{R} \frac{1}{t^{n-m+1}} dt \\
&= \text{vol}(K) \frac{1}{n-m} R^{n-m}.
\end{align}

(3.5)

So the proof is complete.

Now let us fix a conformal metric on $\Omega(\Gamma)/\Gamma$ which agrees with the $g_h$ on each conformal cusp end and arbitrary on the compact part. Let us denote it by $g_\Gamma$ (this is not intended to signify $g_\Gamma$ is an any way canonical).

Lemma 3.4. Suppose that $\Gamma$ is nonelementary, conformally finite Kleinian group, and that $g_\Gamma$ is a metric constructed as the above. Then the metric $e^{2u}g_0$ on $\Omega(\Gamma)$ lifted from $g_\Gamma$ satisfies

\begin{equation}
\frac{1}{C} \frac{1}{\text{dist}(x, L(\Gamma))} \leq e^{u(x)} \leq C \frac{1}{\text{dist}(x, L(\Gamma))}
\end{equation}

(3.6)

for some constant $C > 0$ and all $x \in \Omega(\Gamma)$.

Proof. In light of (2.3) in Section 2, we only need to verify (3.6) for all $x$ in a fundamental region. In a fundamental region for a conformally finite $\Gamma$, one only needs to verify (3.6) for all $x$ in the part corresponding to each conformal cusp end, which is supported by (3.2) in Lemma 3.3. Thus the proof is complete.

To relate the geometry of the hyperbolic manifold $B^{n+1}/\Gamma$ and the Kleinian manifold $\Omega(\Gamma)/\Gamma$, we find

Proposition 3.5. Suppose that $\Gamma$ is nonelementary, conformally finite, and that $g_\Gamma$ is a metric constructed as the above. Then the map

\[ F(x) = f(x)x : \Omega(\Gamma) \to G(\Gamma) \]

is a $\Gamma$-invariant bi-Lipschitz map with respect to the induced hyperbolic metric on the graph $G(\Gamma)$ and the above metric $g_\Gamma$ on $\Omega(\Gamma)$. Moreover the hypersurface $G(\Gamma)/\Gamma$ in $B^{n+1}/\Gamma$ has finite volume.

§4 Hyperbolically Harmonic Functions

This section is concerned with harmonic functions on the hyperbolic manifolds $B^{n+1}/\Gamma$ for a given Kleinian group $\Gamma$. Good references are Chapter V in Ahlfors’
lecture notes [1] and Chapter V in Nicholls’ book [13]. We begin with Green’s function on $B^{n+1}/\Gamma$. In this article we are only concerned with the Kleinian group of second kind, which simply means $L(\Gamma) \neq S^n$. According to Lemma 2 and Theorem 1 in Chapter VI in Ahlfors’ Lecture notes [1], $B^{n+1}/\Gamma$ always possesses a unique minimal positive Green’s function, which is of the form

$$G(x, y) = \sum_{\gamma \in \Gamma} g(x, \gamma y)$$

where $g(x, y)$ is the Green’s function on the hyperbolic $B^{n+1}$ and

$$g(x, y) = g(0, |Txy|) = \int_1^{\infty} \frac{(1-t^2)^{n-1}}{t^n} dt$$

(see, Chapter V in Ahlfors’ lecture notes [1]). On the other hand, the Green’s function on $B^{n+1}/\Gamma$ is also obtained by integrating the heat kernel:

$$G(x, y) = \int_0^\infty H(x, y, t) dt.$$

Therefore, based on bounds for the heat kernel given in Davies [9], we have the following upper bound for the Green’s function.

**Lemma 4.1.** Suppose that the first eigenvalue $\lambda_0$ of $B^{n+1}/\Gamma$ is positive. Then, for $0 < \delta < \frac{1}{2} \lambda_0$, we have

$$0 < G(x, y) < C \text{vol}(B_1(x))^{-\frac{1}{2}} \text{vol}(B_1(y))^{-\frac{1}{2}} e^{-\frac{\rho(x, y)^2}{4+(n+3)t}}$$

where $\rho(x, y)$ is the hyperbolic distance, for some constant $C > 0$ and $\rho(x, y) > 32\sqrt{\lambda_0}$.

**Proof.** Recall that

$$0 \leq H(x, y, t) \leq C \text{vol}(B_1(x))^{-\frac{1}{2}} \text{vol}(B_1(y))^{-\frac{1}{2}} e^{-\frac{\rho(x, y)^2}{4+(n+3)t}}$$

for all $0 < t < 1$, and

$$0 \leq H(x, y, t) \leq C \text{vol}(B_1(x))^{-\frac{1}{2}} \text{vol}(B_1(y))^{-\frac{1}{2}} e^{-(\lambda_0-\delta)t} e^{-\frac{\rho(x, y)^2}{4+(n+3)t}}$$

for all $1 \leq t < \infty$. First, by the nonincreasing property of the function

$$t^{-(n+1)} e^{-nt} \text{vol}(B_t(x))$$

proved in Proposition 4.3 in Chapter 1 of [16] we have

$$H(x, y, t) \leq C \text{vol}(B_1(x))^{-\frac{1}{2}} \text{vol}(B_1(y))^{-\frac{1}{2}} t^{-(n+1)} e^{-\frac{\rho(x, y)^2}{4+(n+3)t}}$$
for all $0 < t < 1$. Therefore

$$G(x, y) = \int_0^1 H(x, y, t)dt + \int_1^\infty H(x, y, t)dt$$

$$\leq C\text{vol}(B_1(x))^{-\frac{1}{2}}\text{vol}(B_1(y))^{-\frac{1}{2}}\left\{ \int_0^1 t^{-(n+1)}e^{-\frac{\lambda_0}{2}t}dt + \int_1^\infty e^{-\delta t}dt \right\}$$

$$\cdot e^{-\sqrt{\frac{4(\lambda_0-2\delta)}{4+\delta}}\rho(x,y)}$$

where we use the fact that

$$(\lambda_0 - 2\delta)t + \frac{\rho(x, y)^2}{(4 + \delta)t} \geq \sqrt{\frac{4(\lambda_0 - 2\delta)}{4 + \delta}}\rho(x, y).$$

So the proof is complete.

Now, let us consider the harmonic function on the hyperbolic $B^{n+1}$ with Dirichlet boundary condition on $S^n$. Given a $f \in L^1(S^n)$, according to Ahlfors (Chapter V of Ahlfors’ lecture notes [1]), we have a harmonic function $u$ on $B^{n+1}$ as

$$u(x) = \frac{1}{\text{vol}(S^n)} \int_{S^n} k(x, y)^n f(y) d\omega(y),$$

where $d\omega$ is the standard volume element for the sphere $S^n$. So, if we consider $\chi_{\Omega(\Gamma)}$: the characteristic function of the domain of discontinuity of $\Gamma$, then, its harmonic extension

$$u_\Gamma(x) = \frac{1}{\text{vol}(S^n)} \int_{S^n} k(x, y)^n \chi_{\Omega(\Gamma)}(y) d\omega(y)$$

is $\Gamma$-invariant, therefore descends to a harmonic function on the hyperbolic manifold $B^{n+1}/\Gamma$. We denote by $\omega_\Gamma(x)$ the descended harmonic function on $B^{n+1}/\Gamma$. Recall, for a nonelementary Kleinian group $\Gamma$, we have constructed a $\Gamma$-invariant Lipschitz graph $G(\Gamma)$ over $\Omega(\Gamma)$. Thus we have a hypersurface $S_\Gamma = G(\Gamma)/\Gamma$ in the hyperbolic manifold $B^{n+1}/\Gamma$ separating the convex core $C(B^{n+1}/\Gamma)$ from the ideal boundary $\Omega(\Gamma)/\Gamma$. We have the following representation formula:

**Lemma 4.2.** Suppose that $\Gamma$ is nonelementary Kleinian group. Then

$$\omega_\Gamma(x) = \frac{1}{2^{n-1}\text{vol}(S^n)} \int_{S_\Gamma} (-\frac{\partial G}{\partial n}(x, y)) d\sigma(y)$$

where $\frac{\partial}{\partial n}$ is the hyperbolic normal derivative of the hypersurface $S_\Gamma$ in $B^{n+1}/\Gamma$, and $d\sigma$ is the induced one from $B^{n+1}/\Gamma$.

**Proof.** The proof given by Bishop and Jones in [5] works even in higher dimension with little modifications. But for the convenience of the reader, we present the proof here. We start with $x = 0$, namely,

$$\omega_\Gamma(0) = u_\Gamma(0) = \frac{1}{\text{vol}(S^n)} \int_{\Omega(\Gamma)} d\omega = \frac{\text{vol}(\Omega(\Gamma))}{\text{vol}(S^n)}.$$
Recall that
\[
\left. -\frac{\partial g}{\partial n} d\sigma \right|_{\partial B_r(0)} = \frac{2^{n-1}}{r^n} d\omega |_{B_r(0)}.
\]

Let \(\Omega_r\) be the part of \(\partial B_r(0)\) which is between \(G(\Gamma)\) and \(\Omega(\Gamma)\) and \(G_r = G(\Gamma) \cap B_r\).

Clearly
\[
u_\Gamma(0) = \frac{1}{\text{vol}(S^n)} \lim_{r \to 1} \frac{\text{vol}(\Omega_r)}{r^n}
= \frac{1}{2^{n-1}\text{vol}(S^n)} \lim_{r \to 1} \int_{\Omega_r} (-\frac{\partial g}{\partial n}) d\sigma
= \frac{1}{2^{n-1}\text{vol}(S^n)} \lim_{r \to 1} \int_{G_r} (-\frac{\partial g}{\partial n}) d\sigma
\]
by the fact that \(g\) is harmonic in the region bounded by \(G_r\) and \(\Omega_r\). Thus
\[
u_\Gamma(0) = \frac{1}{2^{n-1}\text{vol}(S^n)} \int_{G(\Gamma)} (-\frac{\partial g}{\partial n}) d\sigma.
\]
Notice here that we have used the fact that the constructed \(\Gamma\)-invariant graph \(G(\Gamma)\) is Lipschitz, i.e.
\[
\int_{G(\Gamma)} |\frac{\partial g}{\partial n}| d\sigma < \infty.
\]
Then, by dominated convergence theorem and (4.1), we have, if let \(S\) be any fundamental region for \(G(\Gamma)\),
\[
\int_{G(\Gamma)} (-\frac{\partial g}{\partial n}) d\sigma = \sum_{\gamma \in \Gamma} \int_{\gamma S} (-\frac{\partial g}{\partial n}) d\sigma
= \sum_{\gamma \in \Gamma} \int_{S} (-\frac{\partial (\gamma 0, \gamma y)}{\partial n} g(\gamma 0, \gamma y)) d\sigma
= \int_{S_T} (-\frac{\partial G(0, y)}{\partial n}) d\sigma.
\]
This proves the lemma for \(x = 0\). For a general point \(x \in S_\Gamma\), take a conformal transformation \(T\) of \(S^n\) such that \(T0 = x\). Then
\[
\omega_\Gamma(x) = u_\Gamma(T0) = \frac{1}{\text{vol}(S^n)} \int_{S^n} \chi_{T\Omega(\Gamma)} d\omega = \frac{\text{vol}(T\Omega(\Gamma))}{\text{vol}(S^n)}.
\]
Therefore, similarly, we can verify (4.10) for all \(x \in S_\Gamma\).

**Remark 4.3.** The constant \(\frac{1}{2^{n-1}\text{vol}(S^n)}\) in the formula (4.10) depends on the choice of the Green function \(g(x, y)\).
§5 Proof of Theorem 0.1

We begin with the thick-thin decomposition for a hyperbolic space \( B^{n+1}/\Gamma \). Good references are Section 12.5 in Ratcliff’s book [15] and Section 3.3 in Bowditch’s paper [6]. We recall that, by Margulis lemma, there is a dimensional constant \( c_n > 0 \) such that, for any \( \epsilon < c_n \),

\[
V(\Gamma, \epsilon) = \{ x \in B^{n+1} : d_H(x, \gamma x) < \epsilon, \text{ for some } \gamma \in \Gamma \}
\]

is a disjoint union of connected components

\[
V(\Gamma_a, \epsilon) = \{ x \in B^{n+1} : d_H(x, \gamma x) < \epsilon, \text{ for some } \gamma \in \Gamma_a \}
\]

where \( \Gamma_a \) is either a maximum parabolic elementary subgroup or a maximum hyperbolic elementary subgroup of \( \Gamma \). Each connected component \( V(\Gamma_a, \epsilon) \) for \( \epsilon < c_n \) is called a Margulis region. Notice that

\[
\gamma V(\Gamma_a, \epsilon) \cap V(\Gamma_a, \epsilon) = \emptyset
\]

for all \( \gamma \in \Gamma \setminus \Gamma_a \). Because, otherwise, for some \( x \in V(\Gamma_a, \epsilon) \) and some \( \gamma_a \in \Gamma_a \),

\[
d_H(\gamma x, \gamma_a \gamma x) < \epsilon.
\]

Then

\[
d_H(x, \gamma^{-1} \gamma_a \gamma x) < \epsilon,
\]

which implies, by Margulis lemma, \( \gamma^{-1} \gamma_a \gamma \in \Gamma_a \), i.e. \( \gamma_a \gamma = \gamma \). So \( \gamma \in \Gamma_a \). This proves (5.3). Therefore the thin part \( V(\Gamma, \epsilon)/\Gamma \) is disjoint union of connected components where each component has the form \( V(\Gamma_a, \epsilon)/\Gamma_a \). A component \( V(\Gamma_a, \epsilon)/\Gamma_a \) is called a Margulis cusp if \( \Gamma_a \) is parabolic otherwise called a Margulis tube. Since Margulis cusps and Margulis tubes represent the thin part of the hyperbolic manifold \( B^{n+1}/\Gamma \) it is difficult to relate them to the Kleinian manifold \( \Omega(\Gamma)/\Gamma \) directly, in contrast to the standard cusped region for a cusped limit point. The Lipschitz hypersurface \( S_\Gamma \) is designed to make this comparison possible.

**Lemma 5.1.** Suppose that \( \Gamma \) is nonelementary, conformally finite Kleinian group, then

\[
\int_{S_\Gamma} \text{vol}(B_1(x))^{-\frac{1}{2}} d\sigma(x) < \infty.
\]

**Proof.** Let \( \epsilon \) be chosen to be small than the Margulis constant \( c_n \), then we decompose the hypersurface into the Margulis region \( S_\Gamma \cap V(\Gamma, \epsilon)/\Gamma \) and its complement \( S' \) which is compact. Then clearly

\[
\int_{S_\Gamma} \text{vol}(B_1(x))^{-\frac{1}{2}} d\sigma(x) \leq C + \int_{S_\Gamma \cap V(\Gamma, \epsilon)/\Gamma} \text{vol}(B_1(x))^{-\frac{1}{2}} d\sigma(x).
\]
In fact, for any Margulis tube $M_a = V(\Gamma_a, \epsilon)/\Gamma_a$, it is easily seen that $M_a \cap S_\Gamma$ is compact. Otherwise, the fixed point $a$ of a hyperbolic subgroup $\Gamma_a$ would be on the boundary of a fundamental region for $\Gamma$ on $S^n$, which is impossible. Now suppose that $\Omega(\Gamma)/\Gamma = M_c \cup \bigcup_k C_k$ where $M_c$ is compact and $\{C_k\}$ are finite number of conformal cusp ends. Suppose $a_k$ is the parabolic fixed point associated with the conformal cusp end $C_k$. Then the only Margulis cusps that has noncompact intersection with $S_\Gamma$ are those which is associated with the parabolic fixed point $a_k$ and its stabilizer $\Gamma_{a_k}$. Thus

$$\int_{S_\Gamma} \operatorname{vol}(B_1(x))^{-\frac{2}{n}} d\sigma(x) \leq C + \sum_k \int_{S_\Gamma \cap (V(\Gamma_{a_k}, \epsilon)/\Gamma_{a_k})} \operatorname{vol}(B_1(x))^{-\frac{2}{n}} d\sigma(x)$$

$$\leq C + \sum_k \int_{R^{m_k} \setminus B_k} \int_{K_k} \operatorname{vol}(B_1(F^{-1}(x, y)))^{-\frac{1}{n}} |x|^{-n} dy dx$$

where $F : G(\Gamma) \to \Omega(\Gamma)$ and $C_k$ is conformal to $K_k \times R^{m_k}$ with the metric as $g_h$ as given in Section 3. So

$$\leq C + C \sum_k \operatorname{vol}(K_k) \int_1^\infty |x|^{-\frac{n-m_k}{2} - 1} dx$$

$$\leq C + C \sum_k \int_1^\infty |x|^{-\frac{n-m_k}{2} - 1} dx$$

$$< \infty$$

since $m_k \leq n - 1$. Therefore the proof is complete.

The following lemma is adopted from Bishop and Jones’ paper (see, Lemma 3.6 in [5]). Their proof applies to higher dimension with little modifications.

**Lemma 5.2.** Suppose that $\Gamma$ is nonelementary, conformally finite Kleinian group and suppose that $\Gamma$ is geometrically infinite. Then, there exist $\epsilon > 0$ and a sequence of points $x_n \in C(B^{n+1}/\Gamma)$ such that $d_H(x_n, S_\Gamma) \to \infty$ and the injectivity radius $\text{inj}(x_n) > \epsilon$ for all $n$.

**Proof.** When $\Gamma$ is conformally finite, the noncompact part of the surface $S_\Gamma$ has to be inside the disjoint union of finite number of Margulis cusps, where the injectivity radius decays exponentially as the point moves towards infinity. Therefore the sequence of points tending to infinity in the thick part of the hyperbolic manifold must move away from the hypersurface.

With all preparations in place, we are now ready to state and prove the main theorem.

**Theorem 5.4.** Suppose that $\Gamma$ is nonelementary, conformally finite Kleinian group, then it is geometrically finite unless the Hausdorff dimension of its limit point set on $S^n$ is $n$. 
Proof. We will follow the argument in Bishop and Jones’ paper [5]. We consider the harmonic function $\omega_\Gamma$ discussed in Lemma 4.2. Then

\begin{equation}
\omega_\Gamma(x) = C \int_{S_\Gamma} \frac{\partial}{\partial n} G(x, y) d\sigma(y).
\end{equation}

Since $G(x, y)$ is a harmonic function of $y$ away from $x$. Therefore, by the gradient estimate (see, for example, Corollary 3.2 in [16]), we have

\begin{equation}
|\frac{\partial}{\partial n} G(x, y)| \leq CG(x, y).
\end{equation}

Before we apply the estimate for the Green function in Lemma 4.1, we notice that, first, Bishop and Jones proves in [5] that

\begin{equation}
\delta(\Gamma) = \dim(L_c(\Gamma))
\end{equation}

for all nonelementary Kleinian group $\Gamma$, where $\delta(\Gamma)$ is the Poincaré exponent and $L_c(\Gamma)$ is the set of all conical limit points (they only state (5.11) in 2-dimension, but as they pointed out their argument proves (5.11) in higher dimension too.); second, Sullivan [17] generalized Elstrodt-Patterson theorem in higher dimension as

\begin{equation}
\lambda_0(B^{n+1}/\Gamma) = \begin{cases} 
(n/2)^2 & \text{if } \delta(\Gamma) \leq n/2, \\
\delta(\Gamma)(n - \delta(\Gamma)) & \text{if } \delta(\Gamma) \geq n/2.
\end{cases}
\end{equation}

Therefore $\lambda_0 > 0$. By (4.4) and (5.10) we arrive at

\begin{equation}
\omega_\Gamma(x) \leq Ce^{-\sqrt{4(\lambda_0 - \delta^2)} d_H(x, y)} \int_{S_\Gamma} \text{vol}(B_1(x))^{-\frac{1}{2}} \text{vol}(B_1(y))^{-\frac{1}{2}} d\sigma(y)
\end{equation}

Now, if $\Gamma$ is geometrically infinite, we evaluate $\omega_\Gamma$ at the sequence of points $\{x_n\}$ given by Lemma 5.2,

\begin{equation}
\omega_\Gamma(x_n) \leq Ce^{-\sqrt{4(\lambda_0 - \delta^2)} d_H(x_n, \Sigma_\Gamma)} \text{vol}(B_1(x_n))^{-\frac{1}{2}} \int_{\Sigma_\Gamma} \text{vol}(B_1(y))^{-\frac{1}{2}} d\sigma(y).
\end{equation}

Therefore, by Lemma 5.1 and Lemma 5.2, $\omega_\Gamma(x_n) \to 0$ as $n \to \infty$. In light of the formula (4.8), this implies that $n$-dimensional Lebesgue measure of $L(\Gamma)$ has to be positive, which is a contradiction. Thus the proof of this theorem is finished.

\section{Conformal finiteness}

This section is concerned with the question: when is a Kleinian group conformally finite? We would like to give some geometric criteria for a Kleinian group to be conformally finite. The idea still is that the hypersurface $G(\Gamma)/\Gamma$ with the metric induced from the hyperbolic metric is the right geometric representative for the Kleinian manifold $\Omega(\Gamma)/\Gamma$. We first observe:
Theorem 6.1. Suppose that \( \Gamma \) is a nonelementary Kleinian group and \( G(\Gamma) \) is the Lipschitz graph constructed in Section 1. Then \( \Gamma \) is conformally finite if and only if the volume of the hypersurface \( G(\Gamma)/\Gamma \) in the hyperbolic manifold \( B^{n+1}/\Gamma \) is finite.

Remark 6.1. This condition has an analogue in another formulation of geometric finiteness: the thick part of the convex core be compact. The latter is equivalent to say that some neighborhood of the convex core for the hyperbolic manifold \( B^{n+1}/\Gamma \) has finite volume (cf. [6] [15]).

Proof of Theorem 6.1. Let us begin with the thick-thin decomposition of hyperbolic manifolds \( B^{n+1}/\Gamma \) with respect to a small number \( \epsilon \) which is smaller than the Margulis constant in the same dimension. The hypersurface \( G(\Gamma)/\Gamma \) is also decomposed into thick part \( W_\epsilon \) and thin part \( S_\epsilon \). Clearly at each point in the thick part \( W_\epsilon \) there is the hyperbolic geodesic ball \( B_{\frac{3}{2}} \) where \( G(\Gamma) \cap B_{\frac{3}{2}} \epsilon \) belongs to some fundamental domain for \( \Gamma \) on the graph \( G(\Gamma) \). Because \( G(\Gamma) \) is Lipschitz graph over the unit sphere (or any sphere with the same center), the volume of \( G(\Gamma) \cap B_{\frac{3}{2}} \) under the metric induced from the hyperbolic metric on \( B^{n+1} \) is bounded from below by some constant only depending on \( \epsilon \). Therefore, if \( G(\Gamma)/\Gamma \) has a finite volume with the metric, then the thick part \( W_\epsilon \) has to be compact. Now we may conclude that the number of the noncompact connected components has to be finite. Because the finite boundary of each noncompact component, which is the connecting region of the end to the thick part, has a size again bounded from below (depending on \( \epsilon \)). Notice that, each of those noncompact thin ends corresponds to a maximum parabolic subgroup \( P_i \) whose fixed point is \( p_i \). Then, we find that, for some fundamental domain for \( \Gamma \) in its domain of discontinuity \( \Omega(\Gamma) \), there are only finite number of limit points \( p_i \) on its boundary. Moreover, those parabolic fixed points therefore have to be bounded, i.e. have to be so-called cusped limit points. This means precisely that the Kleinian manifold \( \Omega(\Gamma)/\Gamma \) is a disjoint union of a compact part and a finite number of standard conformal cusp ends. So \( \Gamma \) is conformally finite.

On the other hand if \( \Gamma \) is conformally finite, it follows from Theorem 3.5 that the hypersurface \( G(\Gamma)/\Gamma \) has finite volume. So the proof is completed.

As a consequence we have the following criterion to tell when a Kleinian group is conformally finite.

Theorem 6.2. Suppose that \( \Gamma \) is a nonelementary Kleinian group. Then \( \Gamma \) is conformally finite if and only if the Kleinian manifold \( \Omega(\Gamma)/\Gamma \) possesses a conformal metric such that

1. its volume is finite;
2. \( |R| + |\nabla R| \leq C \) and \( \text{Ric} \geq -C \).

Proof. First of all, if \( \Gamma \) is conformally finite, it is clear the Kleinian manifold possesses a metric satisfying (1) and (2), in light of the discussion in Section 3. The converse part of this theorem is a consequence of above Theorem 6.1 and Theorem 2.12, Chapter VI in [16]. We remark that, a stronger assumption that \( M \) has bounded curvature is listed for the above result, but it is clear from the proof (e.g. applying method of gradient estimate), that assumptions as (2) given here are
sufficient for the conclusion. More precisely, on the domain of discontinuity $\Omega(\Gamma)$ the metric is complete and satisfies (2). The Harnack estimate in [16] shows that, denoting the metric by $e^{2u}g_0$ where $g_0$ is the standard metric on the sphere,

$$u(x) \geq C\frac{1}{d(x)} \text{ for all } x \in \Omega(\Gamma).$$

where $d(x) = \text{dist}(x, \partial\Omega(\Gamma))$. Then for the metric induced from the hyperbolic metric on the hypersurface $G(\Gamma)/\Gamma$, its volume is controlled by the volume of the Kleinian manifold with the given metric, therefore is finite. In light of the above Theorem 6.1, the proof is finished.

**Remark 6.2.** Conditions (1) and (2) in Theorem 6.2 can be replaced by a simpler one: finite volume and all curvature and their derivatives bounded.

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