Reliability Application Using Discrete Gamma Distribution

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Abstract

This research deals with discrete counter-part of continuous gamma distribution. In fact, the statistical and reliability properties of this distribution are discuss and some interesting interrelationships. Furthermore, an estimation of the underlying parameter and reliability for this distribution are utilized using different samples sizes, that’s done through different simulation experiments by use (R3.5.1) program, the simulation outputs proved that the Maximum likelihood method gives small bias estimators. An application done at two Soap production machines belongs to the Vegetable Oil Plant. The results show that the second machine which follows DGD (3) is more reliable from the first one.

Keywords : Discrete Gamma distribution, Maximum Likelihood, Reliability function.

I. Introduction

In the past years, gorgeous works have been achieved in the field of reliability and survival. Almost published studies are usually considering continuous data and little works have been done for discrete models. The discrete data sets are the most generally observable in real situations, since components and systems have discrete measures for times. Many discrete distributions have been proposed in the lifetime field, this includes Poisson, Geometric, Binomial,…etc. But the discrete distributions defined based on their continuous counter-parts started by (Nakagawa & Osaki,1975) [XI] that is the continuous Weibull distribution. (Salvia & Bollinger,1982) [XIII] are presented a basic result about discrete reliability by a simple discrete distribution with one parameter. The discrete Weibull type II was
proposed by (Stein & Dattero, 1984) and it is derived from the failure rates for discrete and continuous Weibull. The third type was proposed by (Padgett & Spurrier, 1985), (Ali Khan et al., 1989) were discuss the estimation of the parameters for these distributions by using different methods. The discrete Weibull type II was proposed by (Stein & Dattero, 1984) [XIV] they derived from the failure rates for discrete and continuous Weibull. The third type was proposed by (Padgett & Spurrier 1985) [XII]. (Ali Khan et al. 1989) [III] were discuss the estimation of the parameters for these distributions by using different methods. The discrete Gamma distribution (DGD) was define to be the discrete counter-part of the continuous gamma. In fact, many authors have accepted the negative binomial distribution as an approximation or has similarity with the expected discrete gamma, for more details, one can see (Johnson et al., 1992) [VIII]. Todays, with growing in the collected and storage of data due to technological advances, count data have become immensely available in many fields such as transportation and clinical research.

In this research we will discuss the soap machines plant in the general company for vegetable oils suffers from the large number of malfunctions of the production and the inaccuracy in limiting the operating time of these machines, which affects the quantity of production in general. Therefore, two machines were selected and the number of malfunctions and monthly stops, which follows the discrete Gamma distribution as a beneficial model in the lifetime and reliability models, which are certainly continuous variables, but if we want to deal with the number of machines failures that can occur during the operation, we will resort to the discrete distributions in estimating the reliability function. As well as the process of recording the number of failures is much easier than the process of recording failure times. Therefore, to satisfy this aim in this research we will include the counter variables of the reliability function, the Maximum likelihood method (MLM) is used to estimate the parameter of the distributions.

Application in real-life data will be given and their goodness of fit to the distributions using the Kolmogorov-Smirnov test are examined. Moreover, graphical analysis will be used in conjunction with the formal test to confirm the results. In this piece of work, we restrict our effort to point estimation and investigate some properties of estimators by simulation. In fact, estimation of the underlying parameters of any probability model is necessary for further real-life applications. Also, samples generation is discussed briefly for each of the distributions.

II. Theoretical and Methods:
II.i. Discrete Gamma Distribution:

The pdf of the regular continuous Gamma distribution is given by:

\[ f(x) = \frac{\beta^\theta}{\Gamma(\theta)} x^{\theta-1} e^{-\beta x}; \quad x \geq 0, \theta, \beta > 0 \]

Many authors have been used it in the modeling and applied probability. The discrete type of Gamma distribution is defined by discernment x and \( \theta \) and substitute \( q = e^{-\beta} \). Note that, \( \theta \) in the DG(\( \theta, q \)) can be any positive integer. A discrete random variable \( X \) that follow a discrete Gamma distribution with parameters (\( \theta, q \)), have the following pmf (2):

\[ P(X = k) = \frac{\beta^k}{\Gamma(\theta) q^k} \frac{1}{k!}; \quad k = 0, 1, 2, \ldots \]

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\( p(\theta, q) = Bp^\theta x^{\theta-1}q^x \quad ; \ x = 0,1,2, \ldots \ , \theta > 0 \ , 0 < q < 1 \) \quad (1)

Where:

\[
B = \begin{cases} 
1 & ; \ \theta = 1 \\
\frac{1}{q \sum_{i=0}^{\theta-2} A(\theta - 1, i)q^i} & ; \ \theta \geq 2 
\end{cases}
\]

Here, \( A(n, m) \) is known by the Eulerian number and it is given by:

\[
A(n, m) = \sum_{j=0}^{m} (-1)^j \binom{n-1}{j} (m + 1 - j)^n
\]

For properties of the Eulerian number see Hirzebruch (2008)(7)

It is clearly from (1) if \( \theta = 1 \), pmf of the DG(1, q) is the Geometric distribution \( G(q) \):

\[ P(1, q) = (1 - q)q^x ; x = 0,1,2, \ldots \ , \theta > 0 \ , 0 < q < 1 \] \quad (2)

In this research we will studying special cases when \( \theta = 2,3 \). The pmf of the DG(2, q) is given by:

\[ P(2, q) = \frac{1}{q} x (1 - q)^2q^x ; \ x = 0,1,2, \ldots \ , \theta > 0 \ , 0 < q < 1 \] \quad (3)

The Cumulative distribution function (cdf) of the DG(2, q), denoted by:

\[ F(x) = \Pr(X \leq x) = \sum_{i=0}^{x} P_i F(x) = \sum_{i=0}^{x} \frac{1}{q} x (1 - q)^2q^x F(x) = \frac{(1 - q)^2}{q} \sum_{i=0}^{x} xq^x \]

Recall that:\( \sum_{i=0}^{n} a_i (\frac{1-r^n}{1-r}) \)

\[ F(x) = \frac{(1 - q)^2}{q} \left( \frac{x - xq^x}{1 - q} \right) \]

\( (1 - q)(x - xq^x) = 1 + xq^x - xq + xq^{x+1} \)

\[ F(x) = (1 - q)(x - xq^x) \]

\[ F(x) = 1 + xq^x - xq \]

\[ + xq^{x+1} \] \quad (4)

Then the reliability function (R) is:

\[ R(x) = 1 - F(x) \]

\[ = q^x[(1 + x) - xq] \] \quad (5)
Note that the random variable \( x \) will represent the number of jams (faults) instead of time as in the continuous distributions.

The hazard function (Failure rate) given by:

\[
 h(x) = \frac{P(x)}{R(x)} = \frac{x(1 - q)^2}{x - (x - 1)q}
\]  

the pmf of the \( DG(3, q) \) is given by:

\[
 (3, q) = \frac{1}{q(q + 1)} x^2(1 - q)^3 q^x
\]

; \( x = 0,1,2, ... \), \( \theta > 0 \), \( 0 < q < 1 \)  

The Cumulative distribution function, Reliability function and Hazard function of \( DG(3, q) \), is defined respectively as:

\[
 F(x) = \frac{1 + q - (1 + x)^2 q^x}{(1 + q)} - \frac{(1 - 2x - 2x^2)q^{x+1} - x^2q^{x+2}}{(1 + q)}
\]

\[
 R(x) = q^x \left[ \frac{(1 + x)^2}{(1 + q)} + \frac{(1 - 2x - 2x^2)q + x^2q^2}{(1 + q)} \right]
\]

\[
 h(x) = \frac{x^2(1 - q)^3}{x^2 - (1 - 2x - 2x^2)q + (x - 1)^2 q^2}
\]
II.ii. Statistical properties of the DGD(2), (6):

Now we will provide a review of some statistical properties for the DG(θ, q):

The $L^i$ moment of the DG(θ, q) is:

$$E[X^i] = \sum_{i=0}^{\theta-1} A(\theta + k - 1, i)q^i$$

(11)

The mean of the DG(θ, q) is:

$$\mu = E(X) = \frac{\sum_{i=0}^{\theta-2} A(\theta, i)q^i}{p \sum_{i=1}^{\theta-1} A(\theta - 1, i)q^i}$$

(12)

The second moment of the DG(θ, q) is:

$$E(X^2) = \frac{\sum_{i=0}^{\theta-2} A(\theta + 1, i)q^i}{p^2 \sum_{i=1}^{\theta-1} A(\theta - 1, i)q^i}$$

(13)

The variance is evaluated based on:

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

(14)

The DG(θ, q) is a unimodal distribution, since the mode is given by:

$$\text{Mode} = \frac{1 - \theta}{\ln(q)}$$

(15)

Thus, the mean, the second moment and the variance of the DG(2, q) can be obtained respectively as follow:

$$E(X) = \frac{q + 1}{1 - q}$$

(16)

$$E(X^2) = \frac{1 + 4q + q^2}{(1 - q)^2}$$

(17)

$$\text{Var}(X) = \frac{2q}{(1 - q)^2}$$

(18)

Whereas, for the DGD(3, q) the mean, the second moment and the variance respectively given by:

$$E(X^2) = \frac{1 + 11q + 11q^2 + q^3}{(1 + q)(1 - q)^2}$$

(20)

$$\text{Var}(X) = \frac{4q(1 + q + q^2)}{(1 - q^2)^2}$$

(21)

II.iii. Maximum Likelihood (ML) Method(1) (4):

The maximum likelihood method of the DG(θ, q) parameters when both θ and q are unknown is complicated, therefore we will estimate q when θ is known.

The Likelihood function of the DG(2, q) for a sample of size n is given by:

$$L(x_1, 2, q) = p(x_1, 2, q) = (1 - q)^2n \left(\prod_{i=1}^{n} x_i\right) q^\sum x_{i1}^n$$

(22)

Thus, the log likelihood function is

$$L(x_i, q) = 2n \ln(1 - q) \sum_{i=1}^{n} \ln(x_i) + \left(\sum_{i=1}^{n} x_i - n\right) \ln(q)$$

(23)
Differentiating (23) with respect to $q$ and then equating to zero, we get:

$$\frac{\partial L(x_i, q)}{\partial q} = -\frac{2n}{1 - \hat{q}} - \frac{\sum_{i=1}^{n} x_i}{\hat{q}} - \frac{n}{\hat{q}} = 0$$

The MLE of $q$ is:

$$\hat{q} = \frac{x - 1}{x + 1} \quad (24)$$

Similarly, the Likelihood function of the DG(3, q) for a sample of size $n$ is given by:

$$L(x_i, 3, q) = (1 + q)^{-n} \left( \prod_{i=1}^{n} x_i^2 \right) (1 - q)^{3n} q^{\sum_{i=1}^{n} x_i - n} \quad (25)$$

Then, the log likelihood function is:

$$\ln L(x_i, q) = -n \ln (1 + q) + 3n \ln (1 - q) + 2 \sum_{i=1}^{n} \ln(x_i) + \left( \sum_{i=1}^{n} x_i - n \right) \ln(q) \quad (26)$$

Differentiating (26) with respect to $q$ and then equating to zero gives the following equation:

$$\frac{\partial L(x_i, q)}{\partial q} = -\frac{n}{1 + \hat{q}} - \frac{3n}{1 - \hat{q}} + \frac{\sum_{i=1}^{n} x_i - n}{\hat{q}} = 0$$

This leads to the MLE of $q$:

$$\hat{q} = -2 \pm \sqrt{3 + \bar{x}^2} \overline{\hat{x} + 1} \quad (27)$$

**II.iv. Minimum Variance Bound Estimator (MVBE) (4) :**

We can drive the Minimum Variance Bound Estimator (MVBE) can be found based on the Cramer-Rao Lower Bound quantity, so if $g(q) = \frac{2q}{1-q}$ represent a consistent function of $q$, then (MVBE) can be obtained as follows:

$$\frac{\partial L(x_i, q)}{\partial q} = -\frac{2n}{1 - q} - \frac{\left( \sum_{i=1}^{n} x_i \right) - n}{q}$$

$$\frac{\partial L(x_i, q)}{\partial q} = \frac{n}{q} \left( \bar{x} - 1 \right) \left[ \bar{x} - 1 - \frac{2q}{1 - q} \right]$$

$$\frac{\partial L(x_i, q)}{\partial q} = A(q) [T(X) + g(q)]$$

Where $A(q)$ is a function that independents on the observations, so based on the result of Cramer-Rao theorem, $T(X) = \bar{x} - 1$ is a (MVBE) of $g(q) = \frac{2q}{1-q}$. Furthermore, the variance of $\bar{x} - 1$ is:
\[
\text{Var}(T(X)) = \frac{g'(q)}{A(q)}
\]

\[
\text{Var} (\bar{x} - 1) = \text{Var}(\bar{x}) = \frac{2q}{n(2 - q)^2}
\]

Since \( \text{Var}(\bar{x}) \to 0 \) as \( n \to \infty \), then \( \bar{x} \) is a consistent estimator of \( \frac{2q}{1-q} \).

For a random sample from the \((3, q)\), \((\bar{x} - 1)\) is also a (MVBE) of the function \( g(q) = \frac{2q(2 + q)}{(1 + q)(1 - q)} \).

Based on the result of Cramer-Rao theorem, where:

\[
\frac{\partial L(x_i, q)}{\partial q} = -\frac{n}{1 + q} \left( \frac{3}{1 - q} + \frac{1}{1 + q} \right) \]

\[
\frac{\partial L(x_i, q)}{\partial q} = \frac{n}{q} \left[ (\bar{x} - 1) - \frac{2q(2 + q)}{(1 + q)(1 - q)} \right]
\]

III. Simulation Study:

By using (R3.5.1) program we apply and obtain a simulation study included the following steps(5),(10):

**Step1: Choose the default values:**

The study included the following stages:

We choose the default values for the parameters \( \theta, q \), as it has been

\( \theta = 2, 3 \) and \( q = 0.2, 0.5, 0.8 \).

2-Three different samples sizes were selected \( n = 10, 25, 50, 100 \).

**Step2: Generation DG(\( \theta, q \)) data:**

The random variable \( z_i \) is generate from the \( \text{DG}(\theta, q) \) based on following algorithms:

Generate the random variable \( u_i \) from the uniform distribution.

Solve the following equation:

\[
F(z_i) - u_i = 0 \quad ; \quad i = 1, 2, ..., n
\]

consider \( y_i = |z_i| \), even \( z_i \) be negative and integer

Solving equation 28 to get samples of the \( \text{DG}(2, q) \), \( \text{DG}(3, q) \) respectively as follows:

\[
1 - q^{z_i} - z_iq^{z_i} + z_iq^{z_i+1} - u_i = 0
\]
\[ 1 + q - \frac{(1 + z_i)^2 q^{z_i}}{(1 + q)} - \]
\[ \frac{(1 - 2z_i - 2xz_i^2)q^{z_i+1} - z_i^2q^{z_i+2}}{(1 + q)} - u_i = 0 \]

Repeat the points (1-4) \(D\) times, where \(D=1000\).

**Step3: Estimations:**

At this stage we will obtained all properties of the \(DG(2, q), DG(3, q)\) that were discussed previously at \(n = 10\). Furthermore the parameter \(q\) will be estimated using MLE from the generating data based on equations (24), (27) for all experiments. Note that every evaluated feature will represent the average of 1000 repetition, i.e., if \(\hat{q}_d\) represent the \((MLE_d)\) for the one sample generated, then the MLE for the parameter \(q\) will be:

\[ \hat{q}_{ML} = \frac{\sum_{d=1}^{D} \hat{q}_{MLE_d}}{D} \] (29)

That will be circularization for all properties.

**IV. Results and Discussion:**

The simulation gives an idea about the shape of the \(DG(\theta, q)\), we plot pmf's for different supposed values of \(q\), figures (1), (2) and (3) show the pmf's of the \(DG(2, q)\), figures (4), (5) and (6) show the pmf's of the \(DG(3, q)\).

![Figure (1). pmf of DG (2, 0.2)](image)

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Figure (5). pmf of DG (3, 0.5)

Figure (5). pmf of DG (3, 0.8)

Table (1)

Parameter estimated values and absolute bias

| $\theta = 2$ | n  | $\hat{q}$   | $|Bias|$          |
|-------------|----|-------------|-------------------|
| q=0.2       | 10 | 0.4082840   | 0.208284          |
|             | 25 | 0.3220338   | 0.1220339         |
|             | 50 | 0.3055555   | 0.1055556         |
|             | 100| 0.2307692   | 0.0307692         |
| q=0.5       | 10 | 0.6153846   | 0.1153846         |
|             | 25 | 0.586776    | 0.0867769         |
|             | 50 | 0.5454545   | 0.0454545         |
|             | 100| 0.5370370   | 0.037037          |
| q=0.8       | 10 | 0.411764706 | 0.3882353         |
|             | 25 | 0.82078853  | 0.0207885         |
The text contains a table titled "Estimated of the reliability function and absolute bias". The table entries are as follows:

| no | θ = 2 | n | Real | \( \hat{R}(x) \) | [Bias] | no | θ = 3 | n | real | \( \hat{R}(x) \) | [Bias] |
|----|------|---|------|-----------------|-------|----|------|---|------|-----------------|-------|
|    |      | 50| 0.6948722 | 0.2898722 | 0.0186763 | 100| 0.810964083 | 0.0109641 | |
| 1  | q=0.2| 10| 0.4137573 | 0.2137574 |  |
|    |      | 25| 0.3854696 | 0.1854697 |  |
|    |      | 50| 0.3758848 | 0.1758848 |  |
|    |      | 100| 0.3401093 | 0.1401093 |  |
|    | q=0.5| 10| 0.6702826 | 0.1702826 |  |
|    |      | 25| 0.6381378 | 0.1381378 |  |
|    |      | 50| 0.5917391 | 0.0917392 |  |
|    |      | 100| 0.5851121 | 0.0851122 |  |
|    | q=0.8| 10| 0.3401093 | 0.1401093 |  |
|    |      | 25| 0.8409601 | 0.0409601 |  |
|    |      | 50| 0.7890142 | 0.0109858 |  |
|    |      | 100| 0.8080990 | 0.0080991 |  |

Table (2)

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| q=0.2 | q=0.5 | q=0.8 |
|-------|-------|-------|
| 10    | 10    | 10    |
| 50    | 0.9671217 | 0.5019432 | 0.6553 |
| 100   | 0.9642654 | 0.4524836 | 0.8192 |
| 100   | 0.0071217 | 0.1894432 | 0.8449 |
| 100   | 0.0042654 | 0.1399836 | 0.8471803 |
| 100   | 0.0016 | 0.0710804 | 0.0166039 |
| 100   | 0.0071217 | 0.0575059 | 0.0134498 |
| 100   | 0.0042654 | 0.0134498 | 0.0103117 |
| 100   | 0.0016 | 0.0134498 | 0.0103117 |
| 100   | 0.0071217 | 0.0134498 | 0.0103117 |
| 100   | 0.0042654 | 0.0134498 | 0.0103117 |
| 100   | 0.0016 | 0.0134498 | 0.0103117 |

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From table (2) which represents the estimated values of the reliability function for the DGD. We found that the estimates are closer to the real (assumed) values as the sample size increases and these estimation values decrease over the number of jams (faults), also the same behaviors appear on the values of the absolute bias, that is the absolute bias be smaller when the sample size being larger.

V. Application:

The following data represents the number of the monthly malfunctions for two soap production machines which belong to the Vegetable Oil Plant, this data recorded for 18 months starting from 1/1/2017. Note that the failures resulting from the power outage were exclude.

| Month | 1st machine | 2nd machine |
|-------|-------------|-------------|
| 1     | 4           | 8           |
| 2     | 5           | 8           |
| 3     | 1           | 7           |
| 4     | 1           | 5           |
| 5     | 4           | 4           |
| 6     | 5           | 14          |
| 7     | 4           | 10          |
| 8     | 2           | 5           |
| 9     | 4           | 7           |
| 10    | 7           | 6           |
| 11    | 6           | 6           |
| 12    | 8           | 15          |
| 13    | 5           | 6           |
| 14    | 1           | 13          |
| 15    | 3           | 8           |
| 16    | 6           | 9           |
| 17    | 3           | 3           |
| 18    | 4           | 4           |

To determine whether the above data follows the DGD, one can use the Kolmogorov-Smirnov test according to the following hypothesis under (0.05) significance level:

H0: Data have not follow DGD
H1: Data have follow DGD
Table (4)

Kolmogorov- Smirnov test

| Machine     | K-S Statistic | P-Value |
|-------------|---------------|---------|
| Machine 1   | 0.0611        | 0.24    |
| Machine 2   | 0.05          | 0.221   |

Clearly from table (4) that the P-value is greater than the significance level, that mean the Machine’s data are follows DGD.

Now we will find the MLE,s for the $(\hat{q})$, $R(x)$ by using (R) program, table (5) shown these results.

Table (5)

Parameter and Reliability estimation of the real data

| Machine         | $\hat{q}$  | no | $R(x)$       |
|-----------------|------------|----|--------------|
| 1st machine     | 0.544304   | 1  | 0.79234097   |
|                 |            | 5  | 0.38171446   |
|                 |            | 10 | 0.15663164   |
| 2nd machine     | 0.6761405  | 1  | 0.97973439   |
|                 |            | 5  | 0.84154209   |
|                 |            | 10 | 0.63542563   |

We notice that the parameter value for the 1st machine was equal to (0.544304) and the reliability function at the 1st jam was (0.79234)

then it will gradually descend exceed to (0.156631) at the 10th jam. For the 2nd machine the parameter value was equal to (0.6761405)

and the reliability function at the 1st jam was (0.979734) then it will descend to (0.63542) at the 10th jam.
VI. Conclusion:
Through the results achieved in both experimental and applied sides, we can conclude:

1- All the estimate values that used in the estimation of the parameter and the reliability function of the DGD were identical to the statistical theory. Thus when the sample sizes being larger, the estimated values approached to the default values, also the estimated absolute values were reducing, furthermore the estimated values of the reliability function decreased over the number of jams (faults).

2- The calculated values of the reliability function for soap production facilities indicate that these machines, despite their relative novelty, have decreased over the faults number.

3- The value of the reliability function for the first machine at the tenth fault was (0.15663164), that means the first machine will stop at this fault, while the value of the second machine at the same time is (0.63542563), that means that the second machine will work after the tenth fault with moderate efficiency.

4- The company does not have modern electronic engineering systems to assist in the identification of faults combined, as we noticed that there are many stops for most machines and at all stage.

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