Quasi-Hopf Superalgebras and Elliptic Quantum Supergroups

Yao-Zhong Zhang and Mark D. Gould

Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia
Email: yzz@maths.uq.edu.au

Abstract

We introduce the quasi-Hopf superalgebras which are $\mathbb{Z}_2$ graded versions of Drinfeld’s quasi-Hopf algebras. We describe the realization of elliptic quantum supergroups as quasi-triangular quasi-Hopf superalgebras obtained from twisting the normal quantum supergroups by twistors which satisfy the graded shifted cocycle condition, thus generalizing the quasi-Hopf twisting procedure to the supersymmetric case. Two types of elliptic quantum supergroups are defined, that is the face type $\mathcal{B}_{q,\lambda}(\mathcal{G})$ and the vertex type $\mathcal{A}_{q,p}[\widehat{\mathfrak{sl}(n|n)}]$ (and $\mathcal{A}_{q,p}[\widehat{\mathfrak{gl}(n|n)}]$), where $\mathcal{G}$ is any Kac-Moody superalgebra with symmetrizable generalized Cartan matrix. It appears that the vertex type twistor can be constructed only for $U_q[\widehat{\mathfrak{sl}(n|n)}]$ in a non-standard system of simple roots, all of which are fermionic.

Mathematics Subject Classifications (1991): 81R10, 17B37, 16W30
1 Introduction

One of the aims of this paper is to introduce $\mathbb{Z}_2$ graded versions of Drinfeld’s quasi-Hopf algebras [1], which are referred to as quasi-Hopf superalgebras. We then introduce elliptic quantum supergroups, which are defined as quasi-triangular quasi-Hopf superalgebras arising from twisting the normal quantum supergroups by twistors which satisfy the graded shifted cocycle condition, thus generalizing Drinfeld’s quasi-Hopf twisting procedure [2, 3, 4, 5, 6] to the supersymmetric case. We adopt the approach in [4] and construct two types of twistors, i.e. the face type twistor associated to any Kac-Moody superalgebra $\mathcal{G}$ with a symmetrizable generalized Cartan matrix and the vertex type twistor associated to $sl(\hat{n}|n)$ in a non-standard simple root system in which all simple roots are odd (or fermionic). It should be pointed out that the face type twistors for certain classes of non-affine simple superalgebras were also constructed in [5].

The elliptic quantum groups [7, 8] are believed to provide the underlying algebraic structures for integrable models based on elliptic solutions of the (dynamical) Yang-Baxter equation, such as Baxter’s 8-vertex model [9], the ABF model [10] and their group theoretical generalizations [11][12]. The elliptic quantum supergroups described in this paper are expected to play a similar role in supersymmetric integrable models based on elliptic solutions [13, 14] of the graded (dynamical) Yang-Baxter equation.

2 Quasi-Hopf Superalgebras

Definition 1 : A $\mathbb{Z}_2$ graded quasi-bialgebra is a $\mathbb{Z}_2$ graded unital associative algebra $A$ over a field $K$ which is equipped with algebra homomorphisms $\epsilon : A \to K$ (co-unit), $\Delta : A \to A \otimes A$ (co-product) and an invertible homogeneous element $\Phi \in A \otimes A \otimes A$ (co-associator) satisfying

\[
(1 \otimes \Delta)\Delta(a) = \Phi^{-1}(\Delta \otimes 1)\Delta(a)\Phi, \quad \forall a \in A, \tag{2.1}
\]

\[
(\Delta \otimes 1 \otimes 1)\Phi \cdot (1 \otimes 1 \otimes \Delta)\Phi = (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)\Phi \cdot (1 \otimes \Phi), \tag{2.2}
\]

\[
(\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta, \tag{2.3}
\]

\[
(1 \otimes \epsilon \otimes 1)\Phi = 1. \tag{2.4}
\]

(2.2), (2.3) and (2.4) imply that $\Phi$ also obeys

\[
(\epsilon \otimes 1 \otimes 1)\Phi = 1 = (1 \otimes 1 \otimes \epsilon)\Phi. \tag{2.5}
\]
The multiplication rule for the tensor products is $\mathbb{Z}_2$ graded and is defined for homogeneous elements $a, b, a', b' \in A$ by

$$(a \otimes b)(a' \otimes b') = (-1)^{[b][a']} (aa' \otimes bb'),$$  \hspace{1cm} (2.6)$$

where $[a] \in \mathbb{Z}_2$ denotes the grading of the element $a$.

**Definition 2**: A quasi-Hopf superalgebra is a $\mathbb{Z}_2$ graded quasi-bialgebra $(A, \Delta, \epsilon, \Phi)$ equipped with a $\mathbb{Z}_2$ graded algebra anti-homomorphism $S : A \to A$ (anti-pode) and canonical elements $\alpha, \beta \in A$ such that

$$m \cdot (1 \otimes \alpha)(S \otimes 1)\Delta(a) = \epsilon(a)\alpha, \quad \forall a \in A, \hspace{1cm} (2.7)$$

$$m \cdot (1 \otimes \beta)(1 \otimes S)\Delta(a) = \epsilon(a)\beta, \quad \forall a \in A, \hspace{1cm} (2.8)$$

$$m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1)\Phi^{-1} = 1, \hspace{1cm} (2.9)$$

$$m \cdot (m \otimes 1) \cdot (S \otimes 1 \otimes 1)(1 \otimes \alpha \otimes \beta)(1 \otimes 1 \otimes S)\Phi = 1. \hspace{1cm} (2.10)$$

Here $m$ denotes the usual product map on $A$: $m \cdot (a \otimes b) = ab$, $\forall a, b \in A$. Note that since $A$ is associative we have $m \cdot (m \otimes 1) = m \cdot (1 \otimes m)$. For the homogeneous elements $a, b \in A$, the antipode satisfies

$$S(ab) = (-1)^{[a][b]}S(b)S(a),$$  \hspace{1cm} (2.11)$$

which extends to inhomogeneous elements through linearity.

Applying $\epsilon$ to definition (2.3), (2.10) we obtain, in view of (2.4), $\epsilon(\alpha)\epsilon(\beta) = 1$. It follows that the canonical elements $\alpha, \beta$ are both even. By applying $\epsilon$ to (2.7), we have $\epsilon(S(a)) = \epsilon(a), \forall a \in A$.

In the following we show that the category of quasi-Hopf superalgebras is invariant under a kind of gauge transformation. Let $(A, \Delta, \epsilon, \Phi)$ be a quasi-Hopf superalgebra, with $\alpha, \beta, S$ satisfying (2.7)-(2.10), and let $F \in A \otimes A$ be an invertible homogeneous element satisfying the co-unit properties

$$(\epsilon \otimes 1)F = 1 = (1 \otimes \epsilon)F. \hspace{1cm} (2.12)$$

It follows that $F$ is even. Throughout we set

$$\Delta_F(a) = F\Delta(a)F^{-1}, \quad \forall a \in A, \hspace{1cm} (2.13)$$

$$\Phi_F = (F \otimes 1)(\Delta \otimes 1)F \cdot \Phi \cdot (1 \otimes \Delta)F^{-1}(1 \otimes F^{-1}).$$  \hspace{1cm} (2.14)$$
Theorem 1: \((A, \Delta_F, \epsilon, \Phi_F)\) defined by (2.13, 2.14) together with \(\alpha_F, \beta_F, S_F\) given by

\[
S_F = S, \quad \alpha_F = m \cdot (1 \otimes \alpha)(S \otimes 1)F^{-1}, \quad \beta_F = m \cdot (1 \otimes \beta)(1 \otimes S)F, \tag{2.15}
\]
is also a quasi-Hopf superalgebra. The element \(F\) is referred to as a twistor, throughout.

The proof of this theorem is elementary. For demonstration we show in some details the proof of the anti-pode properties. Care has to be taken of the gradings in tensor product multiplications and also in extending the antipode to the whole algebra. First of all let us state

Lemma 1: For any elements \(\eta \in A \otimes A\) and \(\xi \in A \otimes A \otimes A\),

\[
m \cdot (1 \otimes \alpha_F)(S \otimes 1)\eta = m \cdot (1 \otimes \alpha)(S \otimes 1)(F^{-1}\eta), \tag{2.16}
\]

\[
m \cdot (1 \otimes \beta_F)(1 \otimes S)\eta = m \cdot (1 \otimes \beta)(1 \otimes S)(\eta F), \tag{2.17}
\]

\[
m \cdot (m \otimes 1) \cdot (1 \otimes \beta_F \otimes \alpha_F)(1 \otimes S \otimes 1)\xi
\]

\[
= m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1)[(1 \otimes F^{-1}) \cdot \xi \cdot (F \otimes 1)], \tag{2.18}
\]

\[
m \cdot (m \otimes 1) \cdot (S \otimes 1 \otimes 1)(1 \otimes \alpha_F \otimes \beta_F)(1 \otimes 1 \otimes S)\xi
\]

\[
= m \cdot (m \otimes 1) \cdot (S \otimes 1 \otimes 1)(1 \otimes \alpha \otimes \beta)(1 \otimes 1 \otimes S)
\]

\[
\cdot [(F^{-1} \otimes 1) \cdot \xi \cdot (1 \otimes F)]. \tag{2.19}
\]

Proof: Write \(F = f_i \otimes f^i\) and \(F^{-1} = \bar{f}_i \otimes \bar{f}^i\). Here and throughout, summation convention on repeated indices is assumed. Then (2.15) can be written as

\[
\alpha_F = S(\bar{f}_i)\alpha \bar{f}^i, \quad \beta_F = f_i \beta S(f^i). \tag{2.20}
\]

Further write \(\eta = \eta_k \otimes \eta^k\) and \(\xi = \sum_i x_i \otimes y_i \otimes z_i\). Then

\[
\text{l.h.s. of (2.16)} = m \cdot (1 \otimes S(\bar{f}_i)\alpha \bar{f}^i)(S(\eta_k) \otimes \eta^k) = m \cdot (S(\eta_k) \otimes S(\bar{f}_i)\alpha \bar{f}^i \eta^k) = S(\eta_k)S(\bar{f}_i)\alpha \bar{f}^i \eta^k = S(\bar{f}_i \eta_k)\alpha \bar{f}^i \eta^k \times (-1)^{[\eta_k][\bar{f}_i]},
\]

\[
\text{r.h.s. of (2.16)} = m \cdot (1 \otimes \alpha)(S \otimes 1)(\bar{f}_i \eta_k \otimes \bar{f}^i \eta^k) \times (-1)^{[\bar{f}^i][\eta_k]} = S(\bar{f}_i \eta_k)\alpha \bar{f}^i \eta^k \times (-1)^{[\bar{f}^i][\eta_k]},
\]

thus proving (2.16). (2.17) can be proved similarly. As for (2.18) we have:

\[
\text{l.h.s. of (2.18)} = \sum_i x_i \beta_F S(y_i) \alpha_F z_i = \sum_i x_i f_j \beta S(f^j)S(y_i)S(\bar{f}_k)\alpha \bar{f}^k z_i = \sum_i x_i f_j \beta S(\bar{f}_k y_i f^j) \alpha \bar{f}^k z_i \times (-1)^{[y_i][\{f_j\}]+[\bar{f}_k][f_j]},
\]

\[
\text{r.h.s. of (2.18)} = m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1)
\]

\[
\cdot \sum_i [x_i f_j \otimes \bar{f}_k y_i f^j \otimes \bar{f}^k z_i] \times (-1)^{[y_i][\{f_j\}]+[\bar{f}_k][f_j]} = \sum_i x_i f_j \beta S(\bar{f}_k y_i f^j) \alpha \bar{f}^k z_i \times (-1)^{[y_i][\{f_j\}]+[\bar{f}_k][f_j]},
\]

\[
= \sum_i x_i f_j S(\bar{f}_k y_i f^j) \alpha \bar{f}^k z_i \times (-1)^{[y_i][\{f_j\}]+[\bar{f}_k][f_j]},
\]
where we have used the fact that the element $F$ is even. (2.19) is proved similarly.

Now let us prove the property (2.7) for $\alpha_F$ and $\Delta_F$. We write, following Sweedler,

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}. \quad (2.21)$$

Then, in view of lemma 1,

$$m \cdot (1 \otimes \alpha_F)(S \otimes 1)\Delta_F(a) = m \cdot (1 \otimes \alpha)(S \otimes 1)(F^{-1}\Delta_F(a))$$

$$= m \cdot (1 \otimes \alpha)(S \otimes 1)((\Delta(a)F^{-1})$$

$$= m \cdot (1 \otimes \alpha) \sum_{(a)} (S(a_{(1)} \bar{f}_i) \otimes a_{(2)} \bar{f}_i) \times (-1)^{[\bar{f}_i][a_{(a)}]}$$

$$= S(\bar{f}_i) \sum_{(a)} S(a_{(1)})\alpha a_{(2)} \bar{f}_i \times (-1)^{[\bar{f}_i][a_{(a)}]}$$

$$= S(\bar{f}_i) \sum_{(a)} S(a_{(1)})\alpha a_{(2)} \bar{f}_i \times (-1)^{[\bar{f}_i][a]}$$

$$= (-1)^{[\bar{f}_i][a]} S(\bar{f}_i) \sum_{(a)} S(a_{(1)})\alpha a_{(2)} \bar{f}_i$$

$$\overset{(2.7)}{=} S(\bar{f}_i)\epsilon(a)\alpha \bar{f}_i \times (-1)^{[\bar{f}_i][a]}$$

$$= S(\bar{f}_i)\epsilon(a)\alpha \bar{f}_i \overset{(2.21)}{=} \epsilon(a)\alpha_F, \quad (2.22)$$

where we have used the fact that

$$\epsilon(a) = 0, \quad \text{if} \ [a] = 1. \quad (2.23)$$

The property (2.8) for $\beta_F$ and $\Delta_F$ is proved similarly. We then prove property (2.9), which reads in terms of the twisted objects

$$m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha_F)(1 \otimes S \otimes 1)\Phi^{-1} = 1. \quad (2.24)$$

Let us write

$$\Phi^{-1} = \sum_{\nu} X_{\nu} \otimes Y_{\nu} \otimes Z_{\nu}. \quad (2.25)$$

Then, in view of (2.18),

l.h.s. of (2.24) $$= m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1)[(1 \otimes F^{-1})\Phi^{-1} (F \otimes 1)]$$

$$= m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1)[(1 \otimes \Delta)F \cdot \Phi^{-1} \cdot (\Delta \otimes 1)F^{-1}]$$

$$= m \cdot (m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1)$$

$$\cdot \sum_{\nu, (f), (f)} \sum_{f_1} f_1 X_{\nu} \bar{f}_j(1) \otimes f_1(1) Y_{\nu} \bar{f}_j(2) \otimes f_1(2) Z_{\nu} \bar{f}_j$$

$$\times (-1)^{[X_{\nu}][f_1(j_1)]+[f_1(j_2)]+[Z_{\nu}][f_1(j_1)]+[Y_{\nu}][f_1(j_2)]+[Y_{\nu}][f_1(j_2)]+[Y_{\nu}][f_1(j_2)]}$$

$$= \sum_{\nu} \sum_{(f)} f_1 X_{\nu} \bar{f}_j(1) \beta S(f_1(2)) S(Y_{\nu}) \sum_{(f)} S(f_1(2)) f_1(2) Z_{\nu} \bar{f}_j$$
The property (2.10) for the twisted objects, which reads,

\[
(\Delta^T(a)) \mathcal{R} = \mathcal{R} \Delta(a), \quad \forall a \in A,
\]

is proved in a similar way.

**Definition 3**: A quasi-Hopf superalgebra \((A, \Delta, \epsilon, \Phi)\) is called quasi-triangular if there exists an invertible homogeneous element \(\mathcal{R} \in A \otimes A\) such that

\[
\Delta^T(a) \mathcal{R} = \mathcal{R} \Delta(a), \quad \forall a \in A,
\]

\[
(\Delta \otimes 1) \mathcal{R} = \Phi^{-1}_{231} \mathcal{R}_{13} \Phi_{132} \mathcal{R}_{23} \Phi^{-1}_{123},
\]

\[
(1 \otimes \Delta) \mathcal{R} = \Phi_{312} \mathcal{R}_{13} \Phi^{-1}_{213} \mathcal{R}_{12} \Phi_{123}.
\]

Throughout, \(\Delta^T = T \cdot \Delta\) with \(T\) being the graded twist map which is defined, for homogeneous elements \(a, b \in A\), by

\[
T(a \otimes b) = (-1)^{|a||b|} b \otimes a;
\]

and \(\Phi_{132} \ etc\) are derived from \(\Phi \equiv \Phi_{123}\) with the help of \(T\)

\[
\Phi_{132} = (1 \otimes T) \Phi_{123},
\]

\[
\Phi_{312} = (T \otimes 1) \Phi_{132} = (T \otimes 1)(1 \otimes T) \Phi_{123},
\]

\[
\Phi^{-1}_{231} = (1 \otimes T) \Phi^{-1}_{213} = (1 \otimes T)(T \otimes 1) \Phi^{-1}_{123},
\]

and so on. We remark that our convention differs from the usual one which employs the inverse permutation on the positions (c.f. [3]).
It is easily shown that the properties (2.27)-(2.28) imply the graded Yang-Baxter type equation,

\[ R_{12} \Phi_{231}^{-1} R_{13} \Phi_{132} R_{23} \Phi_{123}^{-1} = \Phi_{321}^{-1} R_{23} \Phi_{312} R_{13} \Phi_{213}^{-1} R_{12}, \]  

(2.31)

which is referred to as the graded quasi-Yang-Baxter equation, and the co-unit properties of \( R \):

\[ (\epsilon \otimes 1)R = 1 = (1 \otimes \epsilon)R. \]  

(2.32)

**Theorem 2**: Denoting by the set \( (A, \Delta, \epsilon, \Phi, R) \) a quasi-triangular quasi-Hopf superalgebra, then \( (A, \Delta_F, \epsilon, \Phi_F, R_F) \) is also a quasi-triangular quasi-Hopf superalgebra, with the choice of \( R_F \) given by

\[ R_F = F^T R F^{-1}, \]  

(2.33)

where \( F^T = T \cdot F \equiv F_{21} \). Here \( \Delta_F \) and \( \Phi_F \) are given by (2.13) and (2.14), respectively.

The proof of this theorem is elementary computation. As an example, let us illustrate the proof of the property (2.28) for \( \Delta_F, R_F \) and \( \Phi_F \). Applying the homomorphism \( T \otimes 1 \) to \( (\Phi_F^{-1})_{123} \), one obtains

\[ (\Phi_F^{-1})_{213} = F_{13} (T \otimes 1)(1 \otimes \Delta) F \cdot \Phi_{213}^{-1} \cdot (\Delta^T \otimes F^{-1}) \cdot (F^T)_{12} \]

\[ = F_{13} \sum (\mathcal{f}^{(1)}_{(i)} \mathcal{f}^{(2)}_{(i)} \mathcal{f}^{(3)}_{(i)}) f^{(1)}_{(i)} \otimes f^{(2)}_{(i)} \Phi^{-1}_{231} (1 \otimes 1) \Delta^T \otimes F^{-1} \cdot (F^T)_{12}, \]  

(2.34)

which gives rise to, by applying the homomorphism \( 1 \otimes T \) to both sides,

\[ (\Phi_F^{-1})_{231} = F_{12} \sum (\mathcal{f}^{(1)}_{(i)} \mathcal{f}^{(2)}_{(i)} \mathcal{f}^{(3)}_{(i)}) f^{(1)}_{(i)} \otimes f^{(2)}_{(i)} \Phi^{-1}_{231} (1 \otimes 1) \Delta^T \otimes F^{-1} \cdot (F^T)_{12}, \]  

(2.35)

Then,

\[ (\Delta_F \otimes 1)R_F = (F \otimes 1)(\Delta \otimes 1)R_F \cdot (F^{-1} \otimes 1) \]

\[ = F_{12} (\Delta \otimes 1)(F^T R F^{-1}) \cdot F_{12} \]

\[ = F_{12} (\Delta \otimes 1)F^T (\Delta \otimes 1)R (\Delta \otimes 1)F^{-1} \cdot F_{12} \]

\[ = F_{12} (\Delta \otimes 1)F^T (\Phi^{-1}_{231} R_{13} \Phi_{132} R_{23} \Phi^{-1}_{123} (\Delta \otimes 1)F^{-1} \cdot F_{12} \]

\[ = F_{12} (\Delta \otimes 1)F^T (\Phi^{-1}_{231} (F^T)_{13} (1 \otimes T)(\Delta^T \otimes 1)F \cdot R_{13} \Phi_{132} R_{23} \Phi^{-1}_{123} (\Delta \otimes 1)F^{-1} \cdot F_{12} \]

\[ = (\Phi^{-1}_{F})_{231} F^T (\Phi^{-1}_{F})_{231} (1 \otimes T)(\Delta^T \otimes 1)F \cdot R_{13} \Phi_{132} R_{23} \Phi^{-1}_{123} (\Delta \otimes 1)F^{-1} \cdot F_{12} \]

\[ = (\Phi^{-1}_{F})_{231} (F^T)_{13} (1 \otimes T)(\Delta^T \otimes 1)F \cdot R_{13} \Phi_{132} R_{23} \Phi^{-1}_{123} (\Delta \otimes 1)F^{-1} \cdot F_{12} \]

\[ = (\Phi^{-1}_{F})_{231} (F^T)_{13} (1 \otimes T)(\Delta^T \otimes 1)F \cdot R_{13} \Phi_{132} R_{23} \Phi^{-1}_{123} (\Delta \otimes 1)F^{-1} \cdot F_{12} \]

\[ = (\Phi^{-1}_{F})_{231} (F^T)_{13} (1 \otimes T)(\Delta^T \otimes 1)F \cdot R_{13} \Phi_{132} R_{23} \Phi^{-1}_{123} (\Delta \otimes 1)F^{-1} \cdot F_{12} \]
The twisting procedure is particularly interesting when the twistor
on an element $\lambda \in Z$ triangular quasi-Hopf superalgebra with
Definition 4: A twistor $\lambda$ is assumed to depend on one (or possible several) parameters.

Let us now consider the special case that $A$ arises from a normal quasi-triangular
Hopf superalgebra via twisting with $F$. A quasi-triangular Hopf superalgebra is a quasi-
triangular quasi-Hopf superalgebra with $\alpha = \beta = 1$, $\Phi = 1 \otimes 1 \otimes 1$. Hence $A$ has the
following $Z_2$ graded quasi-Hopf algebra structure,

$$
\Delta_F(a) = F\Delta(a)F^{-1}, \quad \forall a \in A,
\Phi_F = F_{12} \cdot (\Delta \otimes 1)F \cdot (1 \otimes \Delta)F^{-1} \cdot F_{23}^{-1},
\alpha_F = m \cdot (S \otimes 1)F^{-1}, \quad \beta_F = m \cdot (1 \otimes S)F,
\mathcal{R}_F = F^T\mathcal{R}F^{-1}.
$$

The twisting procedure is particularly interesting when the twistor $F \in A \otimes A$ depends
on an element $\lambda \in A$, i.e. $F = F(\lambda)$, and is a shifted cocycle in the following sense. Here
$\lambda$ is assumed to depend on one (or possible several) parameters.

**Definition 4**: A twistor $F(\lambda)$ depending on $\lambda \in A$ is a shifted cocycle if it satisfies the
graded shifted cocycle condition:

$$
F_{12}(\lambda) \cdot (\Delta \otimes 1)F(\lambda) = F_{23}(\lambda + h^{(1)}) \cdot (1 \otimes \Delta)F(\lambda),
$$

where $h^{(1)} = h \otimes 1 \otimes 1$ and $h \in A$ is fixed.

Let $(A, \Delta_\lambda, \epsilon, \Phi(\lambda), \mathcal{R}(\lambda))$ be the quasi-triangular quasi-Hopf superalgebra obtained
from twisting the quasi-triangular Hopf superalgebra by the twistor $F(\lambda)$. Then
Proposition 1: We have

\[ \Phi(\lambda) \equiv \Phi_F = F_{23}(\lambda + h^{(1)})F_{23}(\lambda)^{-1}, \quad (2.39) \]
\[ \Delta_\lambda (a)^T R(\lambda) = R(\lambda) \Delta_\lambda (a), \quad \forall a \in A, \quad (2.40) \]
\[ (\Delta_\lambda \otimes 1) R(\lambda) = \Phi_{231}(\lambda)^{-1} R_{13}(\lambda) R_{23}(\lambda + h^{(1)}), \quad (2.41) \]
\[ (1 \otimes \Delta_\lambda) R(\lambda) = R_{13}(\lambda + h^{(2)}) R_{12}(\lambda) \Phi_{123}(\lambda). \quad (2.42) \]

As a corollary, \( R(\lambda) \) satisfies the graded dynamical Yang-Baxter equation

\[ R_{12}(\lambda + h^{(3)}) R_{13}(\lambda) R_{23}(\lambda + h^{(1)}) = R_{23}(\lambda) R_{13}(\lambda + h^{(2)}) R_{12}(\lambda). \quad (2.43) \]

3 Quantum Supergroups

Let \( G \) be a Kac-Moody superalgebra \[ [G, G] \] with a symmetrizable generalized Cartan matrix \( A = (a_{ij})_{i,j \in I} \). As is well-known, a given Kac-Moody superalgebra allows many inequivalent systems of simple roots. A system of simple roots is called distinguished if it has minimal odd roots. Let \( \{ \alpha_i, i \in I \} \) denote a chosen set of simple roots. Let \((,\,\,\,)\) be a fixed invariant bilinear form on the root space of \( G \). Let \( \mathcal{H} \) be the Cartan subalgebra and throughout we identify the dual \( \mathcal{H}^* \) with \( \mathcal{H} \) via \((,\,\,\,)\). The generalized Cartan matrix \( A = (a_{ij})_{i,j \in I} \) is defined from the simple roots by

\[ a_{ij} = \begin{cases} \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, & \text{if } (\alpha_i, \alpha_i) \neq 0 \\ (\alpha_i, \alpha_j), & \text{if } (\alpha_i, \alpha_i) = 0 \end{cases} \quad (3.1) \]

As we mentioned in the previous section, quantum Kac-Moody superalgebras are quasi-triangular quasi-Hopf superalgebras with \( \alpha = \beta = 1, \, \Phi = 1 \otimes 1 \otimes 1 \). We shall not give the standard relations obeyed by the simple generators (or Chevalley generators) \( \{ h_i, e_i, f_i, \, i \in I \} \) of \( U_q(G) \), but mention that for certain types of Dynkin diagrams extra \( q \)-Serre relations are needed in the defining relations. We adopt the following graded Hopf algebra structure

\[ \Delta(h) = h \otimes 1 + 1 \otimes h, \]
\[ \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \]
\[ \epsilon(e_i) = \epsilon(f_i) = \epsilon(h) = 0, \]
\[ S(e_i) = -t_i^{-1} e_i, \quad S(f_i) = -f_i t_i, \quad S(h) = -h, \quad (3.2) \]

where \( i \in I, \, t_i = q^{h_i} \) and \( h \in \mathcal{H} \).
The canonical element $\mathcal{R}$ is called the universal R-matrix of $U_q(\mathfrak{g})$, which satisfies the basic properties (e.g. $(2.27)-(2.29)$ with $\Phi = 1 \otimes 1 \otimes 1$ and $(2.32)$)

$$
\Delta^T(a)\mathcal{R} = \mathcal{R}\Delta(a), \quad \forall a \in U_q(\mathfrak{g}),
$$

$$(\Delta \otimes 1)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23},$$

$$(1 \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12},$$

$$(\epsilon \otimes 1)\mathcal{R} = (1 \otimes \epsilon)\mathcal{R} = 1,$$

and the graded Yang-Baxter equation (c.f. $(2.31)$ with $\Phi = 1 \otimes 1 \otimes 1$)

$$(\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}).$$

(3.3)

(3.4)

The Hopf superalgebra $U_q(\mathfrak{g})$ contains two important Hopf subalgebras $U^+_q$ and $U^-_q$ which are generated by $e_i$ and $f_i$, respectively. By Drinfeld’s quantum double construction, the universal R-matrix $\mathcal{R}$ can be written in the form

$$
\mathcal{R} = \left(1 \otimes 1 + \sum_t a^t \otimes a_t \right) \cdot q^{-T}
$$

(3.5)

where $\{a^t\} \in U^+_q$, $\{a_t\} \in U^-_q$. The element $T$ is defined as follows. If the symmetrical Cartan matrix is non-degenerate, then $T$ is the usual canonical element of $\mathcal{H} \otimes \mathcal{H}$. Let $\{h_l\}$ be a basis of $\mathcal{H}$ and $\{h^l\}$ be its dual basis. Then $T$ can be written as

$$
T = \sum_l h_l \otimes h^l.
$$

(3.6)

In the case of a degenerate symmetrical Cartan matrix, we extend the Cartan subalgebra $\mathcal{H}$ by adding some elements to it in such a way that the extended symmetrical Cartan matrix is non-degenerate [17]. Then $T$ stands for the canonical element of the extended Cartan subalgebra. It still takes the form $(3.6)$ but now $\{h_l\}$ ($\{h^l\}$) is understood to be the (dual) basis of the extended Cartan subalgebra. After such enlargement, one has $h = \sum_l (h^l, h) h_l = \sum_l (h_l, h) h^l$ for any given $h$ in the enlarged Cartan subalgebra.

For later use, we work out the explicit form of the universal R-matrix for the simplest quantum affine superalgebra $U_q[\hat{sl}(1|1)]$. This algebra is generated by Chevalley generators $\{e_i, f_i, h_i, d, i = 0, 1\}$ with $e_i$, $f_i$ odd and $h_i$, $d$ even. Here and throughout $d$ stands for the derivation operator. Let us write $h_i = \alpha_i$. Then we have $h_0 = \delta - \varepsilon_1 + \delta_1$, $h_1 = \varepsilon_1 - \delta_1$, where $\{\varepsilon_1, \delta_1, \delta\}$ satisfy $(\varepsilon_1, \varepsilon_1) = 1 = -(\delta_1, \delta_1)$, $(\varepsilon_1, \delta_1) = (\delta, \delta) = (\delta, \varepsilon_1) = (\delta, \delta_1) = 0$. We extend the Cartan subalgebra by adding to it the element $h_{ex} = \varepsilon_1 + \delta_1$. A basis for the enlarged Cartan subalgebra is thus $\{h_{ex}, h_0, h_1, d\}$. It is easily shown that the dual basis is

$\{h^{ex}, h^0, h^1, c\}$, where $h^{ex} = \frac{1}{2}(\varepsilon_1 - \delta_1) = \frac{1}{2} h_1$, $h^0 = d$, $h^1 = \varepsilon_1 + d - \frac{1}{2}(\varepsilon_1 - \delta_1) = d + \frac{1}{2} h_{ex}$,
As is well-known, $U_q[sl(1|1)]$ can also be realized in terms of the Drinfeld generators $\{X_n^\pm, H_n, H_n^ex, n \in \mathbb{Z}, c, d\}$, where $X_n^\pm$ are odd and all other generators are even. The relations satisfied by the Drinfeld generators read \[\{19\}\]

\[
[c, a] = [H_0, a] = [d, d] = [H_n, H_m] = [H_n^ex, H_m^ex] = 0, \quad \forall a \in U_q[sl(1|1)],
\]

\[
q^{H_0^ex} X_n^\pm q^{-H_0^ex} = q^{\pm 2} X_n^\pm,
\]

\[
[d, X_n^\pm] = n X_n^\pm, \quad [d, H_n] = n H_n, \quad [d, H_n^ex] = n H_n^ex,
\]

\[
[H_n, H_m^ex] = \delta_{n+m,0} [2n]_q [nc]_q,
\]

\[
[H_n^ex, X_m^\pm] = \pm \frac{[2n]_q}{n} X_{n+m}^\pm q^{\pm |n|c/2},
\]

\[
[H_n, X_m^\pm] = 0 = [X_n^\pm, X_m^\pm],
\]

\[
[X_n^+, X_m^-] = \frac{1}{q - q^{-1}} \left( q^{\frac{c}{2}(n-m)} \psi_{n+m}^+ - q^{-\frac{c}{2}(n-m)} \psi_{n+m}^- \right),
\]

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$, $[a, b] \equiv ab - (-1)^{|a||b|} ba$ denotes the supercommutator and $\psi_{n}^\pm$ are related to $H_{\pm n}$ by relations

\[
\sum_{n \geq 0} \psi_{n}^\pm z^{\mp n} = q^{\pm H_0} \exp \left( \pm (q - q^{-1}) \sum_{n > 0} H_{\pm n} z^{\mp n} \right). \tag{3.8}
\]

The relationship between the Drinfeld generators and the Chevalley generators is

\[
e_1 = X_0^+, \quad f_1 = X_0^-, \quad h_1 = H_0, \quad h_{ex} = H_0^ex, \]

\[
e_0 = X_1^- q^{-H_0}, \quad f_0 = -q^{H_0} X_1^+, \quad h_0 = c - H_0. \tag{3.9}
\]

With the help of the Drinfeld generators, we find the following universal R-matrix

\[
\mathcal{R} = \mathcal{R}' \cdot q^{-\mathcal{T}}, \tag{3.10}
\]

where

\[
\mathcal{T} = h_{ex} \otimes h_{ex} + h_0 \otimes h_0 + h_1 \otimes h_1 + d \otimes c
\]

\[
= \frac{1}{2} (H_0 \otimes H_0^ex + H_0^ex \otimes H_0) + c \otimes d + d \otimes c,
\]

\[
\mathcal{R}' = \mathcal{R}^\prec \mathcal{R}^0 \mathcal{R}^>,
\]

\[
\mathcal{R}^\prec = \prod_{n \geq 0} \exp \left[ (q - q^{-1}) (q^{-nc/2} X_n^+ \otimes q^{nc/2} X_n^-) \right],
\]

\[
\mathcal{R}^0 = \exp \left[ -(q - q^{-1}) \sum_{n=1}^{\infty} \frac{n}{[2n]_q} (H_n \otimes H_n^ex + H_n^ex \otimes H_n^-) \right],
\]

\[
\mathcal{R}^> = \prod_{n \geq 0} \exp \left[ -(q - q^{-1}) (X_n^- q^{nc/2 - H_0} \otimes q^{-nc/2 + H_0} X_n^+) \right]. \tag{3.11}
\]

Here and throughout,

\[
\prod_{k \geq 0} A_k = A_0 A_1 A_2 \cdots, \quad \prod_{k \geq 0} A_k = \cdots A_2 A_1 A_0. \tag{3.12}
\]
It seems to us that even for this simplest quantum affine superalgebra $U_q[sl(1|1)]$ the universal R-matrix has not been written down in its explicit form before.

Let us compute the image of $\mathcal{R}$ in the 2-dimensional evaluation representaion $(\pi, V)$ of $U_q[sl(1|1)]$, where $V = C^{1|1} = Cv_1 \oplus C v_2$ with $v_1$ even and $v_2$ odd. Let $e_{ij}$ be the $2 \times 2$ matrix whose $(i, j)$-element is unity and zero otherwise. In the homogeneous gradation, the simple generators are represented by

\[
e_1 = \sqrt{[\theta]_q} e_{12}, \quad f_1 = \sqrt{[\theta]_q} e_{21}, \quad h_1 = \theta(e_{11} + e_{22}), \quad h_{\text{ex}} = 2e_{11} + c_0(e_{11} + e_{22}),
\]

\[
e_0 = z\sqrt{[\theta]_q} e_{21}, \quad f_0 = -z^{-1}\sqrt{[\theta]_q} e_{12}, \quad h_0 = -\theta(e_{11} + e_{22}),
\]

(3.13)

where $\theta$ and $c_0$ are arbitrary constants. Then it can be shown that the Drinfeld generators are represented by

\[
H_n = z^n \frac{[n\theta]}{n} (e_{11} + e_{22}), \quad H_{\text{ex}}^n = z^n \frac{[2n]}{n} q^{n\theta} e_{11} + z^n c_n(e_{11} + e_{22}),
\]

\[
X_+ = z^n q^{n\theta} \sqrt{[\theta]_q} e_{12}, \quad X_- = z^n q^{n\theta} \sqrt{[\theta]_q} e_{21},
\]

(3.14)

where again $c_n$ are arbitrary constants. In the following we set $c_n$ to be zero. Then the image $R_{V V}(z; \theta, \theta') = (\pi_\theta \otimes \pi_{\theta'}) \mathcal{R}$ depends on two extra non-additive parameters $\theta$, $\theta'$, and is given by

\[
R_{V V}(z; \theta, \theta') = \frac{q^{-\theta-\theta'} - z}{1 - z q^{-\theta-\theta'}} e_{11} \otimes e_{11} + e_{22} \otimes e_{22} + \frac{q^{-\theta'} - z q^{-\theta}}{1 - z q^{-\theta-\theta'}} e_{11} \otimes e_{22} + \frac{q^{\theta} - z q^{-\theta} q^{-\theta'}}{1 - z q^{-\theta-\theta'}} e_{22} \otimes e_{11} + \sqrt{[\theta]_q [\theta']_q} q^{-\theta} e_{12} \otimes e_{21} \quad (3.15)
\]

(3.15) is nothing but the R-matrix obtained in [22] by solving the Jimbo equation.

4 Elliptic Quantum Supergroups

Following Jimbo et al [4], we define elliptic quantum supergroups to be quasi-triangular quasi-Hopf superalgebras obtained from twisting the normal quantum supergroups (which are quasi-triangular quasi-Hopf superalgebras with $\alpha = \beta = 1$, $\Phi = 1 \otimes 1 \otimes 1$) by twistors which satisfy the graded shifted cocycle condition.

4.1 Elliptic Quantum Supergroups of Face Type

Let $\rho$ be an element in the (extended) Cartan subalgebra such that $(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2$ for all $i \in I$, and

\[
\phi = \text{Ad}(q^{\frac{1}{2}} \sum h_i h_i^l - \rho),
\]

(4.1)
be an automorphism of $U_q(\mathcal{G})$. Here $\{h_i\}$, $\{h^l\}$ are as in (3.6) and are the dual basis of the (extended) Cartan subalgebra. Namely,

$$\phi(e_i) = e_i t_i, \quad \phi(f_i) = t_i^{-1} f_i, \quad \phi(q^h) = q^h. \quad (4.2)$$

In the following we consider the special case in which the element $\lambda$ introduced before belongs to the (extended) Cartan subalgebra. Let

$$\phi_\lambda = \phi^2 \cdot \text{Ad}(q^{2\lambda}) = \text{Ad}(q^{\sum_l h_l^l - 2\rho + 2\lambda}) \quad (4.3)$$

be an automorphism depending on the element $\lambda$ and $\mathcal{R}$ be the universal $R$-matrix of $U_q(\mathcal{G})$. Following Jimbo et al [4], we define a twistor $F(\lambda)$ by the infinite product

$$F(\lambda) = \prod_{k \geq 1} (\phi^k \otimes 1) \left( q^T \mathcal{R} \right)^{-1}. \quad (4.4)$$

It is easily seen that $F(\lambda)$ is a formal series in parameter(s) in $\lambda$ with leading term 1. Therefore the infinite product makes sense. The twistor $F(\lambda)$ is referred to as face type twistor. It can be shown that $F(\lambda)$ satisfies the graded shifted cocycle condition

$$F_{12}(\lambda)(\Delta \otimes 1)F(\lambda) = F_{23}(\lambda + h^{(1)}) (1 \otimes \Delta) F(\lambda), \quad (4.5)$$

where, if $\lambda = \sum_l \lambda_l h^l$, then $\lambda + h^{(1)} = \sum_l (\lambda_l + h_l^{(1)}) h^l$. The proof of (4.5) is identical to the non-super case given by Jimbo et al [4], apart from the use of the graded tensor products. Moreover, it is easily seen that $F(\lambda)$ obeys the co-unit property

$$(\epsilon \otimes 1) F(\lambda) = (1 \otimes \epsilon) F(\lambda) = 1. \quad (4.6)$$

We have

**Definition 5 (Face type elliptic quantum supergroup):** We define elliptic quantum supergroup $\mathcal{B}_{q,\lambda}(\mathcal{G})$ of face type to be the quasi-triangular quasi-Hopf superalgebra $(U_q(\mathcal{G}), \Delta_{\lambda}, \epsilon, \Phi(\lambda), \mathcal{R}(\lambda))$ together with the graded algebra anti-homomorphism $S$ defined by (3.2) and $\alpha_\lambda = m \cdot (S \otimes 1) F(\lambda)^{-1}$, $\beta_\lambda = m \cdot (1 \otimes S) F(\lambda)$. Here $\epsilon$ is defined by (3.2). Let

$$\Delta_{\lambda}(a) = F(\lambda) \Delta(a) F(\lambda)^{-1}, \quad \forall a \in U_q(\mathcal{G}),$$

$$\mathcal{R}(\lambda) = F(\lambda)^T \mathcal{R} F(\lambda)^{-1},$$

$$\Phi(\lambda) = F_{23}(\lambda + h^{(1)}) F_{23}(\lambda)^{-1}. \quad (4.7)$$

12
We now consider the particularly interesting case where $\mathcal{G}$ is of affine type. Then $\rho$ contains two parts

$$\rho = \bar{\rho} + gd, \quad (4.8)$$

where $g = (\psi, \psi + 2\bar{\rho})/2$, $\bar{\rho}$ is the graded half-sum of positive roots of the non-affine part $\mathcal{G}$ and $\psi$ is highest root of $\mathcal{G}$; $d$ is the derivation operator which gives the homogeneous gradation

$$[d, e_i] = \delta_{i0} e_i, \quad [d, f_i] = -\delta_{i0} f_i, \quad i \in I. \quad (4.9)$$

We also set

$$\lambda = \bar{\lambda} + (r + g)d + s'c, \quad r, s' \in \mathbb{C}, \quad (4.10)$$

where $\bar{\lambda}$ stands for the projection of $\lambda$ onto the (extended) Cartan subalgebra of $\bar{\mathcal{G}}$. Denoting by $\{\bar{h}_j\}$, $\{\tilde{h}^j\}$ the dual basis of the (extended) Cartan subalgebra of $\bar{\mathcal{G}}$ and setting $p = q^{2r}$, we can decompose $\phi_\lambda$ into two parts

$$\phi_\lambda = \text{Ad}(p^d q^{2cd}) \cdot \bar{\phi}_\lambda, \quad \bar{\phi}_\lambda = \text{Ad}(q^{\sum_j \bar{h}_j \tilde{h}^j + 2(\bar{\lambda} - \bar{\rho})}). \quad (4.11)$$

Introduce a formal parameter $z$ (which will be identified with spectral parameter) into $\mathcal{R}$ and $F(\lambda)$ by setting

$$R(z) = \text{Ad}(z^d \otimes 1) \mathcal{R},$$
$$F(z, \lambda) = \text{Ad}(z^d \otimes 1) F(\lambda),$$
$$\mathcal{R}(z, \lambda) = \text{Ad}(z^d \otimes 1) \mathcal{R}(\lambda) = F(z^{-1}, \lambda)^T \mathcal{R}(z) F(z, \lambda)^{-1}. \quad (4.12)$$

Then it can be shown from the definition of $F(\lambda)$ that $F(z, \lambda)$ satisfies the difference equation

$$F(pq^{2^{(1)}}, \lambda) = (\bar{\phi}_\lambda \otimes 1)^{-1}(F(z, \lambda)) \cdot q^T \mathcal{R}(pq^{2^{(1)}}, z),$$
$$F(0, \lambda) = F_{\bar{\mathcal{G}}} (\bar{\lambda}). \quad (4.13)$$

The initial condition follows from the fact that $\mathcal{R}(z) q^{d \otimes e + c \otimes d}|_{z=0}$ reduces to the universal $R$-matrix of $U_q(\bar{\mathcal{G}})$.

Let us give some examples.

**The case $B_{q, \lambda}[sl(1|1)]$:**

In this case the universal $R$-matrix is given simply by

$$\mathcal{R} = \exp[(q - q^{-1}) e \otimes f] q^{-T} = [1 + (q - q^{-1}) e \otimes f] q^{-T},$$
$$T = \frac{1}{2} (h \otimes h_{\text{ex}} + h_{\text{ex}} \otimes h). \quad (4.14)$$
Let us write
\[ \lambda = (s' + 1) \frac{1}{2} h + s \frac{1}{2} h_{\text{ex}}, \quad s', s \in \mathbb{C}. \] (4.15)

Since \( h \) commutes with everything, \( \phi_\lambda \) is independent of \( s' \). Set \( w = q^{2(s+h)} \), we have
\[ \phi_\lambda = \text{Ad}(w^{\frac{1}{2} h_{\text{ex}}}). \] (4.16)

The formula for the twistor becomes
\[
F(w) = \prod_{k \geq 1} \left( 1 - (q - q^{-1}) w^k q^{-h} e \otimes f q^h \right)
= 1 - (q - q^{-1}) \sum_{k=1}^{\infty} w^k q^{-h} e \otimes f q^h
= 1 - (q - q^{-1}) \frac{w}{1 - w} q^{-h} e \otimes f q^h.
\] (4.17)

The case \( B_{q,\lambda}[\hat{sl}(1|1)] \):

Taking a basis \( \{ c, d, h, h_{\text{ex}} \} \) of the enlarged Cartan subalgebra of \( \hat{sl}(1|1) \), we write
\[ \lambda = rd + s' c + (s'' + 1) \frac{1}{2} h + s \frac{1}{2} h_{\text{ex}}, \quad r, s', s'' \in \mathbb{C}. \] (4.18)

Then \( \phi_\lambda \) is independent of \( s' \) and \( s'' \). Set
\[ p = q^{2r}, \quad w = q^{2(s+h)}, \] (4.19)

Set \( F(z; p, w) \equiv F(z, \lambda) \). Then (4.13) take the form
\[
F(pq^2 z; p, w) = (\bar{\phi}_w^{-1} \otimes 1)(F(z; p, w)) \cdot q^T R(pq^2 z),
\] (4.20)
\[
F(0; p, w) = F_{sl(1|1)}(w),
\] (4.21)

where \( \bar{\phi}_w = \text{Ad}(w^{\frac{1}{2} h_{\text{ex}}}) \).

The image of (4.20) in the two-dimensional representation \( (\pi, V) \) given by (3.13) (by setting \( \theta = 1 \)) yields a difference equation for \( F_{VV}(z; p, w) = (\pi \otimes \pi) F(z; p, w) \). Noting that \( \pi \cdot \bar{\phi}_w = \text{Ad}(D_w^{-1}) \cdot \pi \), where \( D_w = e_{11} + we_{22} \), we find
\[
F_{VV}(pz; p, w) = \text{Ad}(D_w \otimes 1)(F_{VV}(z; p, w)) \cdot KR_{VV}(pz),
\] (4.22)

where \( K = (\pi \otimes \pi) q^T = q^2 e_{11} \otimes e_{11} + q e_{11} \otimes e_{22} + q e_{22} \otimes e_{11} + e_{22} \otimes e_{22} \) and \( R_{VV}(pz) \) is given by (3.15) (with \( \theta = \theta' = 1 \)). (4.22) is a system of difference equations of \( q \)-KZ equation type \( [23] \), and can be solved with the help of the \( q \)-hypergeometric series. The solution with the initial condition (4.21) is given by
\[
F_{VV}(z; p, w) = 1 \phi_0(z; p, w)e_{11} \otimes e_{11} + e_{22} \otimes e_{22}
+ f_{11}(z; p, w)e_{11} \otimes e_{22} + f_{22}(z; p, w)e_{22} \otimes e_{11}
+ f_{12}(z; p, w)e_{12} \otimes e_{21} + f_{21}(z; p, w)e_{21} \otimes e_{12},
\] (4.23)
where

\[1_{\phi_0}(z; p, w) = \frac{(pq^{-2}z; p)_\infty}{(pq^2z; p)_\infty},\]

\[f_{11}(z; p, w) = 2\phi_1 \left( \begin{array}{c} wq^{-2} \quad q^{-2} \\ \quad w \end{array}; p, pq^2z \right),\]

\[f_{12}(z; p, w) = -\frac{w(q-q^{-1})}{1-w} 2\phi_1 \left( \begin{array}{c} wq^{-2} \quad pq^{-2} \\ \quad pw \end{array}; p, pq^2z \right),\]

\[f_{21}(z; p, w) = \frac{zp^{-1}(q-q^{-1})}{1-pw^{-1}} 2\phi_1 \left( \begin{array}{c} pw^{-1}q^{-2} \quad pq^{-2} \\ \quad pw^{-1}p^2w^{-1} \end{array}; p, pq^2z \right),\]

\[f_{22}(z; p, w) = 2\phi_1 \left( \begin{array}{c} pw^{-1}q^{-2} \quad q^{-2} \\ \quad pw^{-1} \end{array}; p, pq^2z \right).\]  

(4.24)

Here

\[2\phi_1 \left( \begin{array}{c} q^a \quad q^b \\ q^c \end{array}; p, x \right) = \sum_{n=0}^{\infty} \frac{(q^n;p)_n(q^b;p)_n}{(p;p)_n(q^c;p)_n} x^n,\]

\[(a; p)_n = \prod_{k=0}^{n-1} (1 - ap^k), \quad (a; p)_0 = 1.\]  

(4.25)

### 4.2 Elliptic Quantum Supergroups of Vertex Type

As we mentioned before, a given Kac-Moody superalgebras \(G\) allows many inequivalent simple root systems. By means of the “extended” Weyl transformation method introduced in [20], one can transform from one simple root system to another inequivalent one [21]. For \(G = sl(\hat{n}|n)\), there exists a simple root system in which all simple roots are odd (or fermionic). This system can be constructed from the distinguished simple root system by using the “extended” Weyl operation repeatedly. We find the following simple roots, all of which are odd (or fermionic)

\[\alpha_0 = \delta - \varepsilon_1 + \delta_n,\]

\[\alpha_{2j} = \delta_j - \varepsilon_{j+1}, \quad j = 1, 2, \cdots, n - 1,\]

\[\alpha_{2i-1} = \varepsilon_i - \delta_i, \quad i = 1, 2, \cdots, n\]  

(4.26)

with \(\delta, \\{\varepsilon_i\}_{i=1}^n\) and \(\{\delta_i\}_{i=1}^n\) satisfying

\[(\delta, \delta) = (\delta, \varepsilon_i) = (\delta, \delta_i) = 0, \quad (\varepsilon_i, \varepsilon_j) = \delta_{ij},\]

\[(\delta_i, \delta_j) = -\delta_{ij}, \quad (\varepsilon_i, \delta_j) = 0.\]  

(4.27)
Such a simple root system is usually called non-standard. It seems to us that \( \hat{sl}(n|n) \) is the only non-twisted affine superalgebra which has a non-standard system of simple roots, all of which are fermionic.

As will be shown below, for \( G = \hat{sl}(n|n) \) with the above fermionic simple roots, one can construct a different type of twistor. Following Jimbo et al [4], we say this twistor is of \textit{vertex type}.

Let us write \( h_i = \alpha_i \) \( (i = 0, 1, \ldots, 2n-1) \) with \( \alpha_i \) given by (4.26). We extend the Cartan subalgebra of \( \hat{sl}(n|n) \) by adding to it the element \( h_\text{ex} = \sum_{i=1}^n (\varepsilon_i + \delta_i) \). A basis of the extended Cartan subalgebra is \( \{ h_\text{ex}, h_0, h_1, \ldots, h_{2n-1}, d \} \). Denote by \( \{ h_\text{ex}, h_0, h_1, \ldots, h_{2n-1}, c \} \) the dual basis. We have

\[
\begin{align*}
h^\text{ex} &= \frac{1}{2n} \sum_{i=1}^n (\varepsilon_i - \delta_i), \\
h^{2k} &= d + \sum_{i=1}^k (\varepsilon_i - \delta_i) - \frac{k}{n} \sum_{i=1}^n (\varepsilon_i - \delta_i), \\
h^{2k+1} &= d + \sum_{i=1}^{k+1} (\varepsilon_i - \sum_{i=1}^k \delta_i) - \frac{2k+1}{2n} \sum_{i=1}^n (\varepsilon_i - \delta_i),
\end{align*}
\]

where \( k = 0, 1, \ldots, n-1 \). The canonical element \( T \) in the extended Cartan subalgebra reads

\[
T = h_\text{ex} \otimes h^\text{ex} + \sum_{i=0}^{2n-1} (h_i \otimes h^i) + d \otimes c.
\]

Let \( \tau \) be the diagram automorphism of \( U_q[\hat{sl}(n|n)] \) such that

\[
\tau(e_i) = e_{i+1} \mod 2n, \quad \tau(f_i) = f_{i+1} \mod 2n, \quad \tau(h_i) = h_{i+1} \mod 2n.
\]

Obviously, the automorphism \( \tau \) is non-graded since it preserves the grading of the generators and moreover, \( \tau^{2n} = 1 \). Then we can show

\[
\begin{align*}
\tau(h_\text{ex}) &= -h_\text{ex} + \xi c, \\
\tau(c) &= c, \\
\tau(h^\text{ex}) &= -h^\text{ex} + \frac{1}{2n} c,
\end{align*}
\]

\[
\begin{align*}
\tau(h^{2k}) &= h^{2k+1} \mod 2n + \frac{\xi}{2n} \sum_{i=1}^n (\varepsilon_i - \delta_i) - \frac{\xi + n - 2k - 1}{2n} c, \\
\tau(h^{2k+1}) &= h^{2k+2} \mod 2n + \frac{\xi}{2n} \sum_{i=1}^n (\varepsilon_i - \delta_i) - \frac{n - 2k - 1}{2n} c,
\end{align*}
\]

where \( k = 0, 1, \ldots, n-1 \) and \( \xi \) is an arbitrary constant. Introduce element

\[
\bar{\rho} = \sum_{i=0}^{2n-1} h^i + \xi nh^\text{ex},
\]

which gives the principal gradation

\[
[\bar{\rho}, e_i] = e_i, \quad [\bar{\rho}, f_i] = f_i, \quad i = 0, 1, \ldots, 2n-1.
\]
It is easily shown that
\[ \tau(\tilde{\rho}) = \tilde{\rho}, \quad (\tau \otimes \tau)T = T. \]  
\hfill (4.34)

Notice also that
\[ (\tau \otimes \tau) \cdot \Delta = \Delta \cdot \tau, \]
\[ (\tau \otimes \tau)R = R. \]  
\hfill (4.35)

Here the second relation is deduced from the uniqueness of the universal R-matrix of $U_q[sl(n|n)]$. It can be shown that
\[ \sum_{k=1}^{2n}(\tau^k \otimes 1)T = \tilde{\rho} \otimes c + c \otimes \tilde{\rho} - \frac{2(n^2 - 1) - 3\xi}{6}c \otimes c. \]  
\hfill (4.36)

Therefore, if we set
\[ \tilde{T} = \frac{1}{2n} \left( \tilde{\rho} \otimes c + c \otimes \tilde{\rho} - \frac{2(n^2 - 1) - 3\xi}{6}c \otimes c \right), \]  
\hfill (4.37)

then we have
\[ \sum_{k=1}^{2n}(\tau^k \otimes 1)(T - \tilde{T}) = 0. \]  
\hfill (4.38)

Introduce an automorphism
\[ \tilde{\phi}_r = \tau \cdot \text{Ad} \left( q^{\frac{r+c}{2n}} \tilde{\rho} \right), \]  
\hfill (4.39)

which depends on a parameter $r \in \mathbb{C}$. Then the 2n-fold product
\[ \prod_{2n \geq k \geq 1}^{\tilde{\phi}} (\tilde{\phi}_r^k \otimes 1) \left( q^{\tilde{T}} R \right)^{-1} \]  
\hfill (4.40)

is a formal power series in $p^{\frac{1}{2n}}$ where $p = q^{2r}$. Moreover, it has leading term 1 thanks to the relation (4.38). Following Jimbo et al [4], we define the vertex type twistor
\[ E(r) = \lim_{N \to \infty} \prod_{2nN \geq k \geq 1}^{\tilde{\phi}} \left( \tilde{\phi}_r^k \otimes 1 \right) \left( q^{\tilde{T}} R \right)^{-1}. \]  
\hfill (4.41)

Then one can show that $E(r)$ satisfies the graded shifted cocycle condition
\[ E_{12}(r)(\Delta \otimes 1)E(r) = E_{23}(r + c^{(1)})(1 \otimes \Delta)E(r). \]  
\hfill (4.42)

Moreover, $E(r)$ obeys the co-unit property
\[ (\epsilon \otimes 1)E(r) = (1 \otimes \epsilon)E(r) = 1. \]  
\hfill (4.43)

We have
Definition 6 (Vertex type elliptic quantum supergroup): We define elliptic quantum supergroup \( \mathcal{A}_{q,p}[sl(n|m)] \) of vertex type to be the quasi-triangular quasi-Hopf superalgebra \((U_q[sl(n|m)], \Delta_r, \epsilon, \Phi(r), \mathcal{R}(r))\) together with the graded algebra anti-homomorphism \( S \) defined by (3.2) and \( \alpha_r = m \cdot (S \otimes 1)E(r)^{-1}, \beta_r = m \cdot (1 \otimes S)E(r) \). Here \( \epsilon \) is defined by (3.2), and

\[
\Delta_r(a) = E(r)\Delta(a)E(r)^{-1}, \quad \forall a \in U_q(\mathcal{G}),
\]

\[
\mathcal{R}(r) = E(r)^T \mathcal{R} E(r)^{-1},
\]

\[
\Phi(r) = E_{23}(r + c^{(1)})E_{23}(r)^{-1}.
\]

(4.44)

Similar to the face type case, introduce a formal parameter \( \zeta \) (or spectral parameter) into \( \mathcal{R} \) and \( E(r) \) by the formulae

\[
\tilde{\mathcal{R}}(\zeta) = \text{Ad}(\zeta \otimes 1)\mathcal{R},
\]

\[
E(\zeta, r) = \text{Ad}(\zeta \otimes 1)E(r),
\]

\[
\tilde{\mathcal{R}}(\zeta, r) = \text{Ad}(\zeta \otimes 1)\mathcal{R}(r) = E(\zeta^{-1}, r)^T \tilde{\mathcal{R}}(\zeta)E(\zeta, r)^{-1}.
\]

(4.45)

Then it can be shown from the definition of \( E(r) \) that \( E(\zeta, r) \) satisfies the difference equation

\[
E(p^{\frac{1}{n}} q^{\frac{1}{n}c^{(1)}} \zeta, r) = (\tau \otimes 1)^{-1}(E(\zeta, r)) \cdot q^{\hat{T}} \tilde{\mathcal{R}}(p^{\frac{1}{n}} q^{\frac{1}{n}c^{(1)}} \zeta),
\]

\[
E(0, r) = 1.
\]

(4.46)

The initial condition follows from (4.38) and the fact that we are working in the principal gradation. (4.46) implies that

\[
E \left( \left( p^{\frac{1}{n}} q^{\frac{1}{n}c^{(1)}} \right)^{2n} \zeta, r \right) = E(\zeta, r) \cdot \prod_{2n-1 \geq k \geq 0} q^{\hat{T}} (\tau \otimes 1)^{2n-k} \tilde{\mathcal{R}} \left( \left( p^{\frac{1}{n}} q^{\frac{1}{n}c^{(1)}} \right)^{2n-k} \zeta \right).
\]

(4.48)

Some remarks are in order. In non-super case \( \mathbb{N} \), \( \pi \) and \( \tau \) are commutable in the sense that \( \pi \cdot \tau = \text{Ad}(h) \cdot \pi \) with \( h \) obeying \( hv_i = v_{i+1} \mod m \), where \( \{v_i\} \) are basis of the vector module \( V = \mathbb{C}^m = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_m \) of \( \mathcal{A}_{q,p}[\hat{sl}_m] \) and \( \tau \) is the cyclic diagram automorphism of \( \hat{sl}_m \). In the super (or \( \mathbb{Z}_2 \) graded) case, however, \( \pi \) and \( \tau \) are not “commutable” in the above sense. This is because \( \tau \) is grading-preserving while the 2\( n \)-dimensional defining representation space \( V = \mathbb{C}^{n|m} = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_{2n} \) is graded. So to compute the image, one has to work out the action of \( \tau \) at the universal level and then apply the representation \( \pi \). Therefore, the knowledge of the universal R-matrix in its explicit form is required. This makes the image computation of the twistor more involved in the supersymmetric case.

As an example, consider the simplest case of elliptic quantum affine superalgebra \( \mathcal{A}_{q,p}[sl(1|1)] \). Let us calculate the image in the two-dimensional representation \( (\pi, V) \),
\( V = \mathbb{C}^{11} \). As remarked above, we have to work at the universal level first and then apply the representation. We have

**Lemma 2**: In the principal gradation, the action of \( \tau \) on the Drinfeld generators is represented on \( V \) by

\[
\tau(X^+_n) = (-1)^n z^{2n+1} q^{-n} e_{12}, \quad \tau(X^-_n) = (-1)^{n+1} z^{2n-1} q^{-n} e_{21},
\]
\[
\tau(H_n) = (-1)^{n+1} z^n \frac{[n]_q}{n} (e_{11} + e_{22}),
\]
\[
\tau(H^\otimes_n) = (-1)^{n+1} z^{2n} \frac{[2n]_q}{n} \left( q^{-n} e_{11} + \frac{q - q^{-1}}{2} [n]_q (e_{11} + e_{22}) \right). \tag{4.49}
\]

Applying \( \pi \otimes \pi \) to the both side of (4.48) and writing \( E_{VV} (\zeta; p) \equiv (\pi \otimes \pi) E(\zeta, r) \), where \( p = q^{2r} \), we get

\[
E_{VV}(p\zeta; p) = E_{VV}(\zeta; p) \cdot (\pi \otimes \pi) (\tau \otimes 1) \widetilde{R}(p^{\frac{1}{2}}\zeta) \cdot \widetilde{R}_{VV}(p\zeta), \tag{4.50}
\]

where \( \widetilde{R}_{VV}(\zeta) = (\pi \otimes \pi) \widetilde{R}(\zeta) \). In view of (4.49) and the explicit formula (3.11) of the universal R-matrix, (4.50) is a system of eight difference equations.

We can also proceed directly. We have, with the help of lemma 2

\[
(\pi \otimes \pi) (\tau^{2k} \otimes 1) \left( \text{Ad}(p^k \zeta) \hat{p} \otimes 1 \right) \mathcal{R}^{-1} q^{-\hat{r}} = K \cdot \bar{E}_{2k},
\]
\[
(\pi \otimes \pi) (\tau^{2k-1} \otimes 1) \left( \text{Ad}(p^{k-\frac{1}{2}} \zeta) \hat{p} \otimes 1 \right) \mathcal{R}^{-1} q^{-\hat{r}} = \rho_{2k-1} \cdot K^{-1} \cdot \bar{E}_{2k-1}, \tag{4.51}
\]

where \( K = (\pi \otimes \pi) q^{T} \) and

\[
\rho_{2k-1} = \frac{(1 + q^2 p^{2k-1} \zeta^2) (1 + q^{-2} p^{2k-1} \zeta^2)}{(1 + p^{2k-1} \zeta^2)^2},
\]
\[
\bar{E}_{2k} = \frac{1}{1 - q^2 p^{2k} \zeta^2} \left( (1 - q^{-2} p^{2k} \zeta^2) e_{11} \otimes e_{11} + (1 - q^2 p^{2k} \zeta^2) e_{22} \otimes e_{22} \right.
\]
\[\left. + (1 - p^{2k} \zeta^2) e_{11} \otimes e_{22} + (1 - p^{2k} \zeta^2) e_{22} \otimes e_{11} \right.
\]
\[\left. - (q - q^{-1}) p^{k} \zeta e_{12} \otimes e_{21} + (q - q^{-1}) p^{k} \zeta e_{21} \otimes e_{12} \right), \tag{4.52}
\]
\[
\bar{E}_{2k-1} = \frac{1}{1 + q^{-2} p^{2k-1} \zeta^2} \left( (1 + q^2 p^{2k-1} \zeta^2) e_{11} \otimes e_{11} + (1 + q^{-2} p^{2k-1} \zeta^2) e_{22} \otimes e_{22} \right.
\]
\[\left. + (1 + p^{2k-1} \zeta^2) e_{11} \otimes e_{22} + (1 + p^{2k-1} \zeta^2) e_{22} \otimes e_{11} \right.
\]
\[\left. + (q - q^{-1}) p^{k-\frac{1}{2}} \zeta e_{12} \otimes e_{21} - (q - q^{-1}) p^{k-\frac{1}{2}} \zeta e_{21} \otimes e_{12} \right). \tag{4.53}
\]

Then

\[
E_{VV}(\zeta; p) = \prod_{k \geq 1} \rho_{2k-1} K \bar{E}_{2k} K^{-1} \bar{E}_{2k-1} = \rho(\zeta; p) \left( E_{VV}^1 (\zeta; p) + E_{VV}^2 (\zeta; p) \right). \tag{4.54}
\]
where
\[
\rho(\zeta; p) = \frac{(-pq^2\zeta^2; p^2)_{\infty}}{(pq\zeta; p)_{\infty}(-pq\zeta; p)_{\infty}},
\]
\[
E_{VV}^1(\zeta; p) = \prod_{k \geq 1} \left( \frac{1}{1 + p^{2k-1}\zeta^2} \right) \left( (1 - q^{-2}p^{2k}\zeta^2)(1 + q^{-2}p^{2k-1}\zeta^2)_{e11} \otimes e_{11}
\right.
\left. + (1 - q^{-2}p^{2k}\zeta^2)(1 + q^{-2}p^{2k-1}\zeta^2)_{e22} \otimes e_{22}
\right.
\left. + (q - q^{-1})p^{k-\frac{1}{2}}(1 - q^{-2}p^{2k}\zeta^2)_{e12} \otimes e_{12}
\right.
\left. - (q - q^{-1})p^{k-\frac{1}{2}}(1 - q^{-2}p^{2k}\zeta^2)_{e21} \otimes e_{21}\right),
\]
\[
E_{VV}^2(\zeta; p) = \prod_{k \geq 1} \left( \frac{1}{1 + p^{2k-1}\zeta^2} \right) \left( (1 - p^{2k}\zeta^2)_{e11} \otimes e_{22} + (1 - p^{2k}\zeta^2)_{e22} \otimes e_{11}
\right.
\left. - (q - q^{-1})p^{k}\zeta_{e12} \otimes e_{21} + (q - q^{-1})p^{k}\zeta_{e21} \otimes e_{12}\right).
\]

The infinite product in \( E_{VV}^2(\zeta; p) \) can be calculated directly and we find
\[
E_{VV}^2(\zeta; p) = b_E(\zeta)(e_{11} \otimes e_{22} + e_{22} \otimes e_{11}) + c_E(\zeta)(e_{12} \otimes e_{21} - e_{21} \otimes e_{12}),
\]
where,
\[
b_E(\zeta) = c_E(\zeta) = \frac{(pq^{\pm 1}\zeta; p)_{\infty}(-pq^{\mp 1}\zeta; p)_{\infty}}{(-p\zeta^2; p^2)_{\infty}}.
\]

As for \( E_{VV}^1(\zeta; p) \), it can be written as
\[
E_{VV}^1(\zeta; p) = X_{11}(\zeta; p)e_{11} \otimes e_{11} + X_{22}(\zeta; p)e_{22} \otimes e_{22}
\]
\[
+ X_{12}(\zeta; p)e_{12} \otimes e_{12} + X_{21}(\zeta; p)e_{21} \otimes e_{21},
\]
where \( X_{ij}(\zeta; p) \) are the solution to the following system of four difference equations
\[
X_{11}(p\zeta; p) = \frac{1}{1 - q^{-2}p^{2}\zeta^2} \left( (1 + q^{-2}p^{2}\zeta^2)_{X_{11}(\zeta; p)} - p^{\frac{1}{2}}\zeta(q - q^{-1})_{X_{12}(\zeta; p)} \right),
\]
\[
X_{12}(p\zeta; p) = \frac{1}{1 - q^{2}p^{2}\zeta^2} \left( -p^{\frac{1}{2}}\zeta(q - q^{-1})_{X_{11}(\zeta; p)} + (1 + q^{2}p^{2}\zeta^2)_{X_{12}(\zeta; p)} \right),
\]
\[
X_{21}(p\zeta; p) = \frac{1}{1 - q^{-2}p^{2}\zeta^2} \left( p^{\frac{1}{2}}\zeta(q - q^{-1})_{X_{22}(\zeta; p)} + (1 + q^{-2}p^{2}\zeta^2)_{X_{21}(\zeta; p)} \right),
\]
\[
X_{22}(p\zeta; p) = \frac{1}{1 - q^{2}p^{2}\zeta^2} \left( (1 + q^{2}p^{2}\zeta^2)_{X_{22}(\zeta; p)} + p^{\frac{1}{2}}\zeta(q - q^{-1})_{X_{21}(\zeta; p)} \right).
\]

**Acknowledgements.** Our interest in quasi-Hopf algebras was ignited by Bo-Yu Hou’s one beautiful lecture on Jimbo et al paper [4] when Y.-Z.Z. was visiting Northwest University, Xi’an, in December 1997. Y.-Z.Z. thanks Bo-Yu Hou for interesting him the subject and for helpful discussions. The financial support from Australian Research Council through a Queen Elizabeth II Fellowship Grant for Y.-Z.Z is also gratefully acknowledged.
References

[1] V.G. Drinfeld, Quasi-Hopf algebras, Leningrad Math. J. 1 (1990) 1419.

[2] O. Babelon, D. Bernard, E. Billey, Phys. Lett. B375 (1996) 89.

[3] C. Fronsdal, Lett. Math. Phys. 40 (1997) 134.

[4] M. Jimbo, H. Konno, S. Odake, J. Shiraishi, e-print q-alg/9712029.

[5] D. Arnaudon, E. Buffenoir, E. Ragoucy, Ph. Roche, e-print q-alg/9712037.

[6] B. Enriquez, G. Felder, e-print q-alg/9703018.

[7] O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa, H. Yan, Lett. Math. Phys. 32 (1994) 259.

[8] G. Felder, Elliptic quantum groups, Proc. ICMP Paris 1994, pp. 211, 1995.

[9] R.J. Baxter, Ann. Phys. 70 (1972) 193.

[10] G.E. Andrews, R.J. Baxter, P.J. Forrester, J. Stat. Phys. 35 (1984) 193.

[11] A. Belavin, Nucl. Phys. B180 (1981) 189.

[12] M. Jimbo, T. Miwa, M. Odake, Commun. Math. Phys. 116 (1988) 507.

[13] V.V. Bazhanov, Yu.G. Stroganov, Theor. Math. Phys. 62 (1985) 253.

[14] T. Deguchi, A. Fujii, Mod. Phys. Lett. A6 (1991) 3413.

[15] V.G. Kac, Adv. Math. 26 (1977) 8.

[16] V.G. Kac, Lect. Notes in Math. 676 (1978) 597.

[17] S.M. Khoroshkin, V.N. Tolstoy, Commun. Math. Phys. 141 (1991) 599.

[18] V.G. Drinfeld, Sov. Math. Dokl. 36 (1988) 212.

[19] Y.-Z. Zhang, J. Phys. A30 (1997) 8325, q-alg/9703020.

[20] D.A. Leites, M.V. Saveliev, V.V. Serganova, Serpukhov preprint 85-81, 1985.

[21] L. Frappat, A. Sciarrino, P. Sorba, Commun. Math. Phys. 121 (1989) 457.

[22] A.J. Bracken, G.W. Delius, M.D. Gould, Y.-Z. Zhang, J. Phys. A27 (1994) 6551.

[23] I.B. Frenkel, N.Yu. Reshetikhin, Commun. Math. Phys. 146 (1992) 1.