On some groupoids of small orders with Bol-Moufang type of identities

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Abstract

We count number of groupoids of order 3 with some Bol-Moufang type identities.

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1 Introduction

A binary groupoid \((G, \cdot)\) is a non-empty set \(G\) together with a binary operation \(\cdot\). This definition is very general, therefore usually groupoids with some identities are studied. For example, groupoids with identity associativity (semi-groups) are researched.

We continue the study of groupoids with some Bol-Moufang type identities \([9, 2, 14]\). Here we present results published in \([5, 4]\).

Definition. Identities that involve three variables, two of which appear once on both sides of the equation and one of which appears twice on both sides are called Bol-Moufang type identities.

Various properties of Bol-Moufang type identities in quasigroups and loops are studied in \([7, 11, 6, 1]\).

Groupoid \((Q, \ast)\) is called a quasigroup, if the following conditions are true \([2]\): \((\forall u, v \in Q)(\exists! x, y \in Q)(u \ast x = v \& y \ast u = v)\).

For groupoids the following natural problems are researched: how many groupoids with some identities of small order there exist? A list of numbers of semigroups of orders up to 8 is given in \([12]\); a list of numbers of quasigroups up to 11 is given in \([10, 15]\).

2 Some results

Original algorithm is elaborated and corresponding program is written for generating of groupoids of small (2 and 3) orders with some Bol-Moufang identities, which are well known in quasigroup theory.

To verify the correctness of the written program the number of semigroups of order 3 was counted. Obtained result coincided with well known, namely, there exist 113 semigroups of order 3.
The following identities have the property that any of them define a commutative Moufang loop in the class of loops: left (right) semimedial identity, Cote identity and its dual identity, Manin identity and its dual identity or in the class of quasigroups (identity (2.4) and its dual identity).

2.1 Groupoids with left semi-medial identity

Left semi-medial identity in a groupoid \((Q, \cdot)\) has the following form: \(xx \cdot yz = xy \cdot xz\). Bruck uses namely this identity to define commutative Moufang loops in the class of loops.

There exist 10 left semi-medial groupoids of order 2. There exist 7 non-isomorphic left semi-medial groupoids of order 2. The first five of them are semigroups.

\[\begin{array}{ccc}
\star & 1 & 2 \\
1 & 1 & 1 \\
2 & 1 & 1 \\
\odot & 1 & 2 \\
1 & 2 & 1 \\
2 & 1 & 2 \\
\cdot & 1 & 2 \\
1 & 1 & 2 \\
2 & 2 & 2 \\
\end{array}\]

There exist 399 left semi-medial groupoids of order 3.

The similar results are true for groupoids with right semi-medial identity \(xy \cdot zz = xz \cdot yz\). It is clear that the identities of left and right semi-mediality are dual. In other language they are (12)-parastrophes of each other.

It is clear that groupoids with dual identities have similar properties, including the number of groupoids of a fixed order.

2.2 Groupoids with Cote identity

Identity \(x(xy \cdot z) = (z \cdot xx)y\) is discovered in \([6]\). Here we name this identity Cote identity.

There exist 6 groupoids of order 2 with Cote identity. There exist 3 non-isomorphic in pairs groupoids of order 2 with Cote identity.

There exist 99 groupoids of order 3 with Cote identity.

The similar results are true for groupoids with the following identity \((z \cdot yx)x = y(xx \cdot z)\). The last identity is (12)-parastrophe of Cote identity.

2.3 Groupoids with Manin identity

The identity \(x(y \cdot xz) = (xx \cdot y)x\) we call Manin identity \([8]\). The following identity is dual identity to Manin identity: \((zx \cdot y)x = y(xx \cdot z)\).

There exist 10 groupoids of order 2 with Manin identity. There exist 7 non-isomorphic in pairs groupoids of order 2 with Manin identity.

There exist 167 groupoids of order 3 with Manin identity.
2.4 Groupoids with identity \((xy \ast x)z = (y \ast xz)x\) (identity (2.4))

Some properties of identity (2.4) are given in [13, 14]. The following identity is dual identity to identity (2.4):
\[ z(x \ast yx) = x(zx \ast y). \]

There exist 6 groupoids of order 2 with identity (2.4). There exist 3 non-isomorphic in pairs groupoids of order 2 with (2.4) identity. Any of these groupoids is a semigroup.

There exist 117 groupoids of order 3 with identity (2.4).

2.5 Number of groupoids of order 3 with some identities

We count number of groupoids of order 3 with some identities. We use list of Bol-Moufang type identities given in [6]. In Table 1 we present number of groupoids of order 3 with the respective identity.

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Table 1: Number of groupoids of order 3 with some identities.

| Name                      | Abbreviation | Identity                      | Number |
|---------------------------|--------------|-------------------------------|--------|
| Semigroups                | SCR          | $x(yz) = (xy)z$               | 115    |
| Extra                     | EL           | $x(y(xz)) = ((xy)z)x$         | 239    |
| Moufang                   | ML           | $(xy)(xz) = (x(yz))z$         | 196    |
| Left Bol                  | LB           | $x(y(xz)) = (xy)(yz)$         | 215    |
| Right Bol                 | RB           | $y((xz)x) = ((yz)x)z$         | 215    |
| C - loops                 | CL           | $y(x(xz)) = ((yx)x)z$         | 133    |
| LC - loops                | LC           | $(xz)(yz) = (z(xy))z$         | 220    |
| RC - loops                | RC           | $y((xz)x) = (yz)(xx)$         | 220    |
| Middle Nuclear Square     | MN           | $y((xz)x) = (y(xz))z$         | 350    |
| Right Nuclear Square      | RN           | $y(z(xz)) = (yz)(xx)$         | 352    |
| Left Nuclear Square       | LN           | $((xz)y)z = (zx)(yz)$         | 932    |
| Comm. Moufang             | CM           | $(xy)(xz) = (x(yz))(zy)$      | 297    |
| Abelian Group             | AG           | $x(yz) = (yx)z$               | 91     |
| Comm. C - loop            | CC           | $y(xy)z = x(ygz)$             | 169    |
| Comm. Alternative         | CA           | $(xx)y = z(xy)z$              | 110    |
| Comm. Nuclear Square      | CN           | $(xx)y = (xx)(zy)$            | 472    |
| Comm. loops               | CP           | $(yz)z = z(yz)(zy)$           | 744    |
| Cheban 1                  | C1           | $x((xy)z) = (yz)(xz)$         | 219    |
| Cheban 2                  | C2           | $x((xy)z) = (yz)(xz)$         | 153    |
| Lonely I                  | L1           | $(x(xy))z = g((zx)z)$         | 117    |
| Cheban I Dual             | CD           | $(yr)(xz) = (yx)(zx)$         | 219    |
| Lonely II                 | L2           | $(x(xy))z = g((zx)z)$         | 157    |
| Lonely III                | L3           | $(y(xy))z = g((zx)z)$         | 157    |
| Mate I                    | M1           | $(x(xy))z = ((yz)x)z$         | 111    |
| Mate II                   | M2           | $(y(xz))z = ((yz)x)z$         | 196    |
| Mate III                  | M3           | $x(y(gz)) = g((zx)z)$         | 111    |
| Mate IV                   | M4           | $x(y(gz)) = g((zx)z)$         | 196    |
| Triad I                   | T1           | $(x(yz))z = g((zx)z)$         | 162    |
| Triad II                  | T2           | $(x(yz))z = g((zx)z)$         | 180    |
| Triad III                 | T3           | $(x(yz))z = g((zx)z)$         | 162    |
| Triad IV                  | T4           | $(x(yz))z = g((zx)z)$         | 132    |
| Triad V                   | T5           | $x(y(gz)) = g((zx)z)$         | 132    |
| Triad VI                  | T6           | $(x(yz))z = g((zx)z)$         | 1419   |
| Triad VII                 | T7           | $(x(yz))z = g((zx)z)$         | 428    |
| Triad VIII                | T8           | $(x(yz))z = g((zx)z)$         | 120    |
| Triad IX                  | T9           | $(x(yz))z = g((zx)z)$         | 102    |
| Frute                     | FR           | $(x(yz))z = (yz)(xz)$         | 129    |
| Crazy Loop                | CR           | $(x(yz))z = (yz)(xz)$         | 136    |
| Krypton                    | KL           | $(x(yz))z = (xyz)z$           | 268    |
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