REVERSIBLE FILTERS

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Abstract. A space is reversible if every continuous bijection of the space onto itself is a homeomorphism. In this paper we study the question of which countable spaces with a unique non-isolated point are reversible. By Stone duality, these spaces correspond to closed subsets in the Čech-Stone compactification of the natural numbers βω. From this, the following natural problem arises: given a space X that is embeddable in βω, is it possible to embed X in such a way that the associated filter of neighborhoods defines a reversible (or non-reversible) space? We give the solution to this problem in some cases. It is especially interesting whether the image of the required embedding is a weak P-set.

1. Introduction

A topological space X is reversible if every time that \( f : X \rightarrow X \) is a continuous bijection, then f is a homeomorphism. This class of spaces was defined in [10], where some examples of reversible spaces were given. These include compact spaces, Euclidean spaces \( \mathbb{R}^n \) (by the Brouwer invariance of domain theorem) and the space \( \omega \cup \{p\} \), where p is an ultrafilter, as a subset of \( \beta\omega \). This last example is of interest to us.

Given a filter \( \mathcal{F} \subset \mathcal{P}(\omega) \), consider the space \( \xi(\mathcal{F}) = \omega \cup \{\mathcal{F}\} \), where every point of \( \omega \) is isolated and every neighborhood of \( \mathcal{F} \) is of the form \( \{\mathcal{F}\} \cup A \) with \( A \in \mathcal{F} \).

Spaces of the form \( \xi(\mathcal{F}) \) have been studied before, for example by García-Ferreira and Uzcátegi ([6] and [7]). When \( \mathcal{F} \) is the Fréchet filter, \( \xi(\mathcal{F}) \) is homeomorphic to a convergent sequence, which is reversible; when \( \mathcal{F} \) is an ultrafilter it is easy to prove that \( \xi(\mathcal{F}) \) is also reversible, as mentioned above. Also, in [2, section 3], the authors of that paper study when \( \xi(\mathcal{F}) \) is reversible for filters \( \mathcal{F} \) that extend to precisely a finite family of ultrafilters (although these results are expressed in a different language).

Let us say that a filter \( \mathcal{F} \subset \mathcal{P}(\omega) \) is reversible if the topological space \( \xi(\mathcal{F}) \) is reversible. It is the objective of this paper to study reversible filters. First, we give some examples of filters that are reversible and others that are non-reversible, besides the trivial ones considered above. Due to Stone duality, every filter \( \mathcal{F} \) on \( \omega \) gives rise to a closed subset \( K_\mathcal{F} \subset \omega^* = \beta\omega \setminus \omega \) (defined below). Then our main concern is to try to find all possible topological types of \( K_\mathcal{F} \) when \( \mathcal{F} \) is either reversible or non-reversible. Our results are as follows.

- Given any compact space X embeddable in \( \beta\omega \), there is a reversible filter \( \mathcal{F} \) such that X is homeomorphic to \( K_\mathcal{F} \). (Theorem 3.2)
• Given any compact, extremally disconnected space $X$ embeddable in $\beta\omega$, there is a non-reversible filter $F$ such that $X$ is homeomorphic to $K_F$. (Theorem 3.5)

• If $X$ is a compact, extremally disconnected space that can be embedded in $\omega^*$ as a weak $P$-set and $X$ has a proper clopen subspace homeomorphic to itself, then there is a non-reversible filter $F$ such that $X$ is homeomorphic to $K_F$ and $K_F$ is a weak $P$-set of $\omega^*$. (Theorem 4.1)

• There is a compact, extremally disconnected space $X$ that can be embedded in $\omega^*$ as a weak $P$-set and every time $F$ is a filter with $X$ homeomorphic to $K_F$ and $K_F$ is a weak $P$-set, then $F$ is reversible. (Theorem 4.2)

• Given any compact, extremally disconnected space $X$ that is a continuous image of $\omega^*$, there is a reversible filter $F$ such that $X$ is homeomorphic to $K_F$ and $K_F$ is a weak $P$-set of $\omega^*$. (Theorem 4.4)

Also, in section 5, using Martin’s axiom, we improve some of the results above by constructing filters $F$ such that $K_F$ is a $P$-set.

2. Preliminaries and a characterization

Recall that $\beta\omega$ is the Stone space of all ultrafilters on $\omega$ and $\omega^* = \beta\omega \setminus \omega$ is the space of free ultrafilters. We will assume the reader’s familiarity of most of the facts about $\beta\omega$ from [9]. Recall that a space is an $F$-space if every cozero set is $C^*$-embedded. Since $\omega^*$ is an $F$-space, we obtain some interesting properties. For example, every closed subset of $\omega^*$ of type $G_\delta$ is regular closed and every countable subset of $\omega^*$ is $C^*$-embedded. We will also need the more general separation property.

2.1. Theorem [3, 3.3] Let $B$ and $C$ be collections of clopen sets of $\omega^*$ such that $B \cup C$ is pairwise disjoint, $|B| < b$ and $C$ is countable. Then there exists a non-empty clopen set $C$ such that $\bigcup B \subset C$ and $\bigcap (\bigcup C) \cap C = \emptyset$.

We will be considering spaces embeddable in $\beta\omega$. There is no ZFC characterization of spaces embeddable in $\beta\omega$ but we have the following embedding results. A space is extremally disconnected (ED, for short) if the closure of every open subset is open.

2.2. Theorem • [9, 1.4.4] Under CH, any closed suspace of $\omega^*$ can be embedded as a nowhere dense $P$-set.

• [9, 1.4.7] Every compact, 0-dimensional ED space of weight $\leq \mathfrak{c}$ embeds in $\omega^*$.

• [9, 3.5], [4] If $X$ is an ED space and a continuous image of $\omega^*$, then $X$ can be embedded in $\omega^*$ as a weak $P$-set.

Given $A \subset \omega$, we denote $\text{cl}_{\beta\omega}(A) \cap \omega^* = A^*$. Also, if $f : \omega \to \omega$ is any bijection, there is a continuous extension $\beta f : \beta\omega \to \beta\omega$ which is a homeomorphism; denote $f^* = \beta f|_{\omega^*}$.

The Fréchet filter is the filter $\mathcal{F}_\mathfrak{c} = \{A : \omega \setminus A \in [\omega]^{<\omega}\}$ of all cofinite subsets of $\omega$ and we will always assume that our filters extend the Fréchet filter. Each filter $\mathcal{F} \subset \mathcal{P}(\omega)$ defines a closed set $K_\mathcal{F} = \{p \in \beta\omega : \mathcal{F} \subseteq p\}$ that has the property that $A \in \mathcal{F}$ iff $K_\mathcal{F} \subset A^*$ and moreover, $K_\mathcal{F} = \bigcap\{A^* : A \in \mathcal{F}\}$. Notice that $\xi(\mathcal{F})$ is the
quotient space of $\omega \cup K_\mathcal{F} \subset \beta \omega$ when $K_\mathcal{F}$ is shrunk to a point. The first thing we will do is to find a characterization of reversible filters in terms of continuous maps of $\beta \omega$.

2.3. **Lemma** Let $\mathcal{F}$ be a filter on $\omega$. Then $\mathcal{F}$ is not reversible if and only if there is a bijection $f : \omega \to \omega$ such that $f^* [K_\mathcal{F}]$ is a proper subset of $K_\mathcal{F}$.

*Proof.* First, assume that $g : \xi(\mathcal{F}) \to \xi(\mathcal{F})$ is a continuous bijection that is not open. Then, $g[\omega] = \omega$ so let $f = g|_\omega : \omega \to \omega$, which is a bijection.

Let $A \subset \omega$ such that $K_\mathcal{F} \subset A^\ast$. Then $A \cup \{\mathcal{F}\}$ is open, so by continuity of $g$ we obtain that $g^\ast [A \cup \{\mathcal{F}\}] = f^\ast [A] \cup \{\mathcal{F}\}$ is also open. Thus, $f^\ast [A] \in \mathcal{F}$ which implies that $K_\mathcal{F} \subset f^\ast [A]^\ast$. This implies that $f^\ast [K_\mathcal{F}] \subset A^\ast$. Thus, we obtain that

$$f^\ast [K_\mathcal{F}] \subset \bigcap \{A^\ast : K_\mathcal{F} \subset A^\ast\} = K_\mathcal{F}.$$ 

Now, since $g$ is not open, there is $B \in \mathcal{F}$ such that $f[B] \notin \mathcal{F}$. Thus, $K_\mathcal{F} \notin f[B]^\ast$. Since $f^\ast [K_\mathcal{F}] \subset f[B]^\ast$, it follows that $K_f \not\subset f^\ast [K_\mathcal{F}]$ so $K_f \neq f^\ast [K_\mathcal{F}]$. We have proved that $f^\ast [K_\mathcal{F}] \subseteq K_\mathcal{F}$.

Now, assume that $f : \omega \rightarrow \omega$ is a bijection such that $f^\ast [K_\mathcal{F}] \not\subseteq K_\mathcal{F}$. Let $g = f \cup \{(\mathcal{F}, \mathcal{F})\}$, let us prove that this function is continuous but not open.

We first prove that $g$ is continuous. Clearly, continuity follows directly for points of $\omega$ so let us consider neighborhoods of $\mathcal{F}$ only. Any neighborhood of $\mathcal{F}$ is of the form $A \cup \{\mathcal{F}\}$ with $A \in \mathcal{F}$. Then $K_\mathcal{F} \subset A^\ast$ and $f^\ast [K_\mathcal{F}] \subset A^\ast$ too, so $K_\mathcal{F} \subset f^\ast [A]$. This implies that $f^\ast [A] \in \mathcal{F}$. We obtain that $g^\ast [A \cup \{\mathcal{F}\}] = f^\ast [A] \cup \{\mathcal{F}\}$ is a neighborhood of $\mathcal{F}$.

Now, let us prove that $g$ is not open. Since $f^\ast [K_\mathcal{F}] \not\subseteq K_\mathcal{F}$, there exists $B \subset \omega$ such that $f^\ast [K_\mathcal{F}] \subset B^\ast$ and $K_\mathcal{F} \not\subset B^\ast$. But $K_\mathcal{F} \subset f^\ast [B]$. Then $f^\ast [B] \cup \{\mathcal{F}\}$ is a neighborhood of $\mathcal{F}$ with image $g[f^\ast [B] \cup \{\mathcal{F}\}] = B \cup \{\mathcal{F}\}$ that is not open. □

So from now on we will always use Lemma 2.3 when we want to check whether a filter is reversible.

According to [10, Section 6], a space is hereditarily reversible if each one of its subspaces is reversible. Given a filter $\mathcal{F}$ on $\omega$, every subspace of $\xi(\mathcal{F})$ is either discrete or of the form $\xi(\mathcal{F} \upharpoonright A)$ for some $A \in [\omega]^\omega$. Here $\mathcal{F} \upharpoonright A = \{A \cap B : B \in \mathcal{F}\}$. So call a filter $\mathcal{F}$ hereditarily reversible if $\mathcal{F} \upharpoonright A$ is reversible for all $A \in [\omega]^\omega$.

We present some characterizations of properties of $\mathcal{F}$ and their equivalences for $K_\mathcal{F}$. The proof of these properties is easy and left to the reader.

2.4. **Lemma** Let $\mathcal{F}$ be a filter on $\omega$.

(a) $\xi(\mathcal{F})$ is a convergent sequence if and only if $\mathcal{F} = \mathcal{F}_\mathcal{F}$ if and only if $K_\mathcal{F} = \omega^\ast$.

(b) $\xi(\mathcal{F})$ contains a convergent sequence if and only if $\text{int}_{\omega^\ast} (K_\mathcal{F}) \neq \emptyset$.

(c) $\xi(\mathcal{F})$ is Fréchet-Urysohn if and only if $K_\mathcal{F}$ is a regular closed subset of $\omega^\ast$.

(d) $\mathcal{F}$ is an ultrafilter if and only if $|K_\mathcal{F}| = 1$.

(e) Let $A \subset \omega$. Then $K_{\mathcal{F} \upharpoonright A} = K_\mathcal{F} \cap A^\ast$.

3. **First results**

From Lemma 2.4, we can easily find all reversible filters that have convergent sequences. Notice that Proposition 3.1 follows from [2, Theorem 2.1]. However, we include a proof to illustrate a first use of Lemma 2.3.
3.1. Proposition Let $\mathcal{F}$ be a filter on $\omega$ such that $\xi(\mathcal{F})$ has a convergent sequence. Then the following are equivalent

(a) $\mathcal{F}$ is the Fréchet filter,
(b) $\mathcal{F}$ is hereditarily reversible, and
(c) $\mathcal{F}$ is reversible.

Proof. From Lemma 2.4 we immediately get that (a) implies (b). That (b) implies (c) is clear so let us prove that (c) implies (a). So assume that $\mathcal{F} \neq \mathcal{F}_r$ has a convergent sequence. By Lemma 2.4 there is $A \subseteq \omega$ such that $A^* \subseteq \mathcal{F}_d$. And since $\mathcal{F} \neq \mathcal{F}_r$, there is $B \in [\omega]^\omega$ with $\omega \setminus B \in \mathcal{F}$. Thus, $K_f \cap B^* = \emptyset$.

Let $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$ be partitions into two infinite subsets. Now, let $f : \omega \rightarrow \omega$ be a bijection such that $f$ is the identity restricted to $\omega \setminus (A \cup B)$, $f[B_1] = B$, $f[B_0] = A_0$ and $f[A] = A_1$. Then it easily follows that $f^*[K_F] = K_F \setminus A_0^* \subseteq K_F$, which shows that $\mathcal{F}$ is not reversible by Lemma 2.3. \hfill $\Box$

Clearly, every ultrafilter is hereditarily reversible by Lemmas 2.3 and 2.4 (this is known from [10, Example 9]). By considering ultrafilters with different Rudin-Keisler types, we may find many other examples with isolated points. So naturally the question is whether there exists a reversible filter $\mathcal{F}$ that is different from these examples. More precisely, we consider the following formulation of the problem.

Let $X$ be a space that can be embedded in $\omega^*$ and consider a filter $\mathcal{F}$ such that $K_f$ is homeomorphic to $X$. Is it possible to choose $\mathcal{F}$ in such a way that $\mathcal{F}$ is reversible? or not reversible?

For $X = \omega^*$, both questions have a positive answer. If $\mathcal{F} = \mathcal{F}_r$, then $K_\mathcal{F}$ is homeomorphic to $\omega^*$ and $\mathcal{F}$ is reversible. Now, say $\omega = A \cup B$ is a partition into infinite subsets and $\mathcal{F} = \{ C \subseteq A : |A \setminus C| = \omega \}$; then $K_\mathcal{F}$ is homeomorphic to $\omega^*$ and $\mathcal{F}$ is not reversible (Proposition 3.1). In the next result, we shall show that there are many reversible filters that are non-trivial and in fact, any closed subset of $\omega^*$ can be realized by one of them.

3.2. Theorem There exists a filter $\mathcal{F}_0$ on $\omega$ with the following properties

(a) any filter that extends $\mathcal{F}_0$ is reversible,
(b) $K_{\mathcal{F}_0}$ is crowded and nowhere dense, and
(c) if $X$ is any closed subset of $\omega^*$, there exists a filter $\mathcal{F} \supseteq \mathcal{F}_0$ such that $K_\mathcal{F}$ is homeomorphic to $X$.

Proof. Let $\{ p_n : n < \omega \} \subseteq \omega^*$ be a sequence of weak $P$-points with different RK types; that such a collection exists follows from [11]. Let $\omega = \bigcup \{ A_n : n < \omega \}$ be a partition into infinite subsets, we may assume that $p_n \in A_n$ for all $n < \omega$. Define $\mathcal{F}_0$ to be the filter of all subsets $B \subseteq \omega$ such that there is $n < \omega$ with $B \cap A_n = \emptyset$, if $m \leq n$; and $B \cap A_m \in p_m$, if $m > n$. It is easy to see that $K_f = \text{cl}_{\omega^*}(\{ p_n : n < \omega \}) \setminus \{ p_n : n < \omega \}$, notice that this implies that $K_f$ is nowhere dense. Also, since every countable subset of $\omega^*$ is $C^*$-embedded, it follows that $K_f$ is homeomorphic to $\omega^*$. From this, parts (b) and (c) follow.

So we are left to prove part (a). Let $\mathcal{F} \supseteq \mathcal{F}_0$ be any filter and let $f : \omega \rightarrow \omega$ be a bijection such that $f^*[K_\mathcal{F}] \subseteq K_\mathcal{F}$, according to Lemma 2.3 we have to prove that
f^*[K_F] = K_F. Consider the set

\[ B = \{ n < \omega : f^*(p_n) \in \{ p_k : k < \omega \} \}. \]

Notice that \( \{ p_n : n < \omega \} \) and \( \{ f^*(p_n) : n \in \omega \setminus B \} \) are disjoint sets of weak P-points of \( \omega^* \). Thus, \( \{ p_n : n < \omega \} \cup \{ f^*(p_n) : n \in \omega \setminus B \} \) is a discrete set. But countable sets in an \( F \)-space such as \( \omega^* \) are \( C^* \)-embedded so \( \text{cl}_{\omega^*}(\{ p_n : n < \omega \}) \cap \text{cl}_{\omega^*}(\{ f^*(p_n) : n \in \omega \setminus B \}) = \emptyset \). Since \( f^*[K_F] \subset K_F \), we obtain that \( f^*[K_F] \cap \text{cl}_{\omega^*}(\{ f^*(p_n) : n \in \omega \setminus B \}) = \emptyset \). Thus, \( K_F \subset \text{cl}_{\omega^*}(\{ p_n : n \in B \}) \).

From the fact that the ultrafilters chosen have different RK types, we obtain that \( f^*(p_n) = p_n \) for all \( n \in B \). From this it follows that in fact, \( f \) restricted to \( \text{cl}_{\omega^*}(\{ p_n : n \in B \}) \) is the identity function. Thus, \( f^*[K_F] = K_F \). \( \square \)

Next we will produce a non-reversible filter \( F \) with \( K_F \) homeomorphic to any closed subset of \( X \) that is ED. First, we will need two lemmas. Notice that an infinite, compact, 0-dimensional and ED space \( X \) has weight \( \geq \mathfrak{c} \). To see this, consider any pairwise disjoint family \( \{ U_n : n < \omega \} \) of pairwise disjoint clopen sets and for every \( A \subset \omega \), let \( V_A = \text{cl}_X(\{ U_n : n \in A \}) \), which is clopen. Then \( \{ V_A : A \subset \omega \} \) is a family of \( \mathfrak{c} \) different clopen subsets of \( X \).

3.3. Lemma Let \( \{ X_n : n < \omega \} \) be infinite, compact, 0-dimensional, ED spaces of weight \( \mathfrak{c} \). Then there exists a 0-dimensional, ED space \( Y \) such that \( Y = \bigcup \{ Y_n : n < \omega \} \), where

(a) \( Y_n \subset Y_{n+1} \) whenever \( n < \omega \), and
(b) \( Y_n \) is homeomorphic to \( X_n \) for each \( n < \omega \).

Moreover, \( Y \) is normal and has exactly \( \mathfrak{c} \) clopen sets.

Proof. Recall that in every 0-dimensional, ED space, all countable subsets are \( C^* \)-embedded. Thus, every infinite, compact, 0-dimensional, ED space has a copy of \( \beta \omega \). Also, every compact, 0-dimensional, ED space of weight at most \( \mathfrak{c} \) can be embedded in \( \omega^* \). This implies that for every \( n < \omega \), there exists a topological copy of \( X_n \) embedded in \( X_{n+1} \).

So for each \( n < \omega \), let \( e_n : X_n \to X_{n+1} \) an embedding. If \( n \leq m < \omega \), denote by \( e^n_m : X_n \to X_m \) the composition of all such appropriate embeddings. In the union \( \bigcup_{n < \omega} (X_n \times \{ n \}) \), define an equivalence relation \( \langle x, n \rangle \sim \langle y, m \rangle \) and \( n \leq m \) if and only if \( y = e^n_m(x) \). Let \( Y \) be the quotient space under this relation and for each \( n < \omega \), let \( Y_n \) be the image of \( X_n \times \{ n \} \) under the corresponding quotient map. It is easy to see that each \( Y_n \) is homeomorphic to \( X_n \) for each \( n < \omega \). Notice that a set \( U \) is open in \( Y \) if and only if \( U \cap Y_n \) is open in \( Y_n \) for all \( n < \omega \).

First, let us see that \( Y \) is normal and 0-dimensional. In fact, we will argue that if \( F \) and \( G \) are disjoint closed subsets of \( Y \), they can be separated by a clopen subset. For each \( n < \omega \), let \( F_n = F \cap Y_n \) and \( G_n = G \cap Y_n \). Since \( F_0 \cap G_0 = \emptyset \) and \( Y_0 \) is compact and 0-dimensional, there is a clopen set \( U_0 \subset Y_0 \) with \( F_0 \subset U_0 \) and \( G_0 \cap U_0 = \emptyset \). Assume that \( k < \omega \) and for each \( n \leq k \) we have found \( U_n \) clopen in \( Y_n \) such that if \( n \leq k \), then \( F_n \subset U_n \), \( G_n \cap U_n = \emptyset \) and if \( n < m < k \) then \( U_m \cap Y_n = U_n \). Now, the two sets \( F_{k+1} \cup U_k \) and \( G_{k+1} \cup (Y_k \setminus U_k) \) are disjoint and closed in \( Y_{k+1} \). Then choose a clopen subset \( U_{k+1} \subset F_{k+1} \cup U_k \subset U_{k+1} \) and \( [G_{k+1} \cup (Y_k \setminus U_k)] \setminus U_{k+1} = \emptyset \). This concludes the recursive construction of \( \{ U_n : n < \omega \} \). Finally, let \( U = \bigcup \{ U_n : n < \omega \} \), notice that \( F \subset U \) and \( G \cap U = \emptyset \). Also, \( U \) is clopen because \( U \cap Y_n = U_n \) is clopen in \( Y_n \) for each \( n < \omega \).
To see that $Y$ is ED, let $U \subset Y$ be open, we have to prove that $\text{cl}_Y(U)$ is clopen. We will define a sequence of open sets $U_\alpha \subset \text{cl}_Y(U)$ for all ordinals $\alpha$. Let $U_0 = U$ and if $\alpha$ is a limit ordinal, define $U_\alpha = \bigcup_{\beta < \alpha} U_\beta$. Now assume that $U_\alpha$ is defined and let $U_{\alpha+1} = \bigcup \{ \text{cl}_Y(U_\alpha \cap Y_n) : n < \omega \}$. Since $Y_\alpha$ is closed in $Y$ for every $n < \omega$, $U_{\alpha+1} \subset \text{cl}_Y(U)$. Moreover, $Y_\alpha$ is ED so $\text{cl}_Y(U_\alpha \cap Y_n)$ is open in $Y_n$ for each $n < \omega$. Also, clearly $\text{cl}_Y(U_\alpha \cap Y_n) \subset \text{cl}_Y(U_\alpha \cap Y_m)$ whenever $n < m < \omega$. From this it follows that $U_{\alpha+1}$ is open and we have finished our recursive construction. Notice that $U_\alpha \subset U_\beta$ whenever $\alpha < \beta$. So there exists some $\gamma < |Y|^+$ such that $U_\gamma = U_{\gamma+1}$.

Notice that for all $n < \omega$, $\text{cl}_Y(U_\gamma \cap Y_n) \subset U_{\gamma+1} \cap Y_n = U_\gamma \cap Y_n$ is clopen in $Y_n$. From this it follows that $U_\gamma$ is closed. Since $U \subset U_\gamma \subset \text{cl}_Y(U)$, we obtain that $U_\gamma = \text{cl}_Y(U)$. Then $Y$ is ED.

Since every clopen set $U$ of $Y$ is a union of the clopen subsets $U \cap Y_n$, for $n < \omega$, it follows that there are at most $\epsilon$ clopen subsets of $Y$. Also, since $Y$ is normal, $Y_0$ is $C^*$-embedded in $Y$ so $Y$ has at least $\epsilon$ clopen sets. This completes the proof. \[\square\]

3.4. Lemma Let $\{ A_n : n < \omega \}$ be pairwise disjoint infinite subsets of $\omega$ and for each $n < \omega$, let $K_n$ be a closed subset of $A^*_n$. Then $\bigcup \{ K_n : n < \omega \}$ is $C^*$-embedded in $\beta \omega$.

Proof. Let $f : \bigcup \{ K_n : n < \omega \} \to [0, 1]$ be a continuous function. Given $n < \omega$, since $K_n$ is closed in $\beta A_n$, there is a function $g_n : A_n \to [0, 1]$ such that $\beta g_n|_{K_n} = f|_{K_n}$. So if $g : \omega \to [0, 1]$ is any function extending $\bigcup \{ g_n : n < \omega \}$, then $\beta g : \beta \omega \to [0, 1]$ is an extension of $f$. \[\square\]

3.5. Theorem Let $X$ be any compact, 0-dimensional, ED space of weight $\leq \epsilon$. Then there is a non-reversible filter $\mathcal{F}$ on $\omega$ such that $K_{\mathcal{F}}$ is homeomorphic to $X$.

Proof. Let $\{ X_n : n < \omega \}$ be a family of pairwise disjoint clopen subsets of $X$. Let $B \subset \omega$ with $|B| = |\omega \setminus B| = \omega$, let $\{ A_n : n \in \mathbb{Z} \}$ be a partition of $\omega \setminus B$ into infinite subsets and let $f : \omega \to \omega$ be a bijection such that $f|_B$ is the identity function in $B$ and for all $n \in \mathbb{Z}$, $f[A_n] = A_{n+1}$. By Lemma 3.3, there is an 0-dimensional, ED space $Y$ with exactly $\epsilon$ clopen sets that is equal to the increasing union of spaces $\{ Y_n : n < \omega \}$ such that $Y_n$ is homeomorphic to $X_n$ and $Y_n \subset Y_{n+1}$ for all $n < \omega$. Recall that $\beta Y$ is also ED ([9, 1.2.2(a)]). Also, in a compact and 0-dimensional space the weight is equal to the number of clopen sets so $\beta Y$ has weight $\epsilon$. Thus, there is an embedding $e : Y \to A^*_0$ ([9, 1.4.7]).

Let $e_0 = e$ and if $n < \omega$, let $e_{n+1} = f^* \circ e_n : Y \to A^*_n$. For each $n < \omega$, let $Z_n = e_n[X_n]$. Define $Z = \bigcup \{ Z_n : n < \omega \}$ and let $W$ be a subset of $B^*$ homeomorphic to the set $X \setminus \text{cl}_X(\bigcup \{ X_n : n < \omega \})$. Notice that $\bigcup \{ X_n : n < \omega \}$ is a $C^*$-embedded subset of $X$ because $X$ is ED and $\bigcup \{ Z_n : n < \omega \}$ is $C^*$-embedded in $Z$ by Lemma 3.4. Thus, there is an embedding $h : \text{cl}_X(\bigcup \{ X_n : n < \omega \}) \to \omega^*$ such that $h[X_n] = Z_n$ for all $n < \omega$. Since $X$ is extremally disconnected, we may extend $h$ to an embedding $H : X \to \omega^*$ in such a way that $H[X \setminus \text{cl}_X(\bigcup \{ X_n : n < \omega \})] = W$.

So let $\mathcal{F}$ be the filter of all $A \subset \omega$ with $Z \cup W \subset A^*$. We will prove that $\mathcal{F}$ is not reversible by showing that $f^*[Z \cup W] \subseteq Z \cup W$. First, notice that $f^*[W] = W$ and $f^*[Z_n] = e_{n+1}[X_n] \subset Z_{n+1}$ for all $n < \omega$. Finally, $f^*[Z \cup W] \cap A^*_0 = \emptyset$ so $f^*[Z \cup W] \cap Z_0 = \emptyset$. This completes the proof. \[\square\]
4. Embedding as weak $P$-sets

Recall that every ED space that is a continuous image of $\omega^*$ can be embedded in $\omega^*$ as a weak $P$-set (\cite{9, 3.5, 4}). So now we study a problem similar to the one in the previous section, adding the requirement that the embedded space is a weak $P$-set. More carefully stated, we want the following.

Let $X$ be a space that can be embedded in $\omega^*$ as a weak $P$-set and consider a filter $F$ such that $K_f$ is a weak $P$-set homeomorphic to $X$. Is it possible to choose $F$ in such a way that $F$ is reversible? or not reversible?

First, we start finding filters that are not reversible. The construction is similar to that in Theorem 3.5. However, it needs an extra hypothesis.

4.1. Theorem Let $X$ be a compact ED space that can be embedded in $\omega^*$ as a weak $P$-set. Moreover, assume that there exists a proper clopen subspace of $X$ homeomorphic to $X$. Then there is a non-reversible filter $F$ on $\omega$ such that $K_F$ is a weak $P$-set homeomorphic to $X$.

Proof. From the hypothesis on $X$, it is easy to find a collection of non-empty, pairwise disjoint clopen sets $\{X_n : n < \omega\}$ of $X$ that are pairwise homeomorphic. Let $B \subset \omega$ with $|B| = |\omega \setminus B| = \omega$, let $\{A_n : n \in \mathbb{Z}\}$ be a partition of $\omega \setminus B$ into infinite subsets and let $f : \omega \to \omega$ be a bijection such that $f|B$ is the identity function in $B$ and for all $n \in \mathbb{Z}$, $f[A_n] = A_{n+1}$.

It is not hard to argue that there is an embedding $e : \bigcup\{X_n : n < \omega\} \to \bigcup\{A^*_n : n < \omega\}$ in such a way that for each $n < \omega$, $e[X_n]$ is a weak $P$-set of $A^*_n$ and $f^*[e[X_n]] = e[X_{n+1}]$. Since $\bigcup\{X_n : n < \omega\}$ is $C^*$-embedded in $X$ and $X$ is ED, we may assume that $X \subset \omega^*$, $e$ is the identity function and $X \setminus \text{cl}_X(\bigcup\{X_n : n < \omega\})$ is a weak $P$-set of $B^*$.

Now let us see that with these conditions, $X$ is in fact a weak $P$-set. Let $\{x_n : n < \omega\}$ be disjoint from $X$. Then for each $n < \omega$, $X_n$ is a weak $P$-set so $\text{cl}_{\omega^*}(\{x_n : n < \omega\}) \cap X_n = \emptyset$. Thus, the family $\{X_n : n < \omega\} \cup \{x_n : n < \omega\}$ is discrete and countable so it can be separated by pairwise disjoint clopen sets. By Lemma 3.4, it easily follows that $\bigcup\{X_n : n < \omega\}$ can be separated from $\{x_n : n < \omega\}$ by a continuous function. Also, $\text{cl}_{\omega^*}(\{x_n : n < \omega\}) \cap (X \setminus \text{cl}_X(\bigcup\{X_n : n < \omega\})) = \emptyset$. So in fact $\text{cl}_{\omega^*}(\{x_n : n < \omega\}) \cap X = \emptyset$, which is what we wanted to prove.

Finally, let $F$ be the neighborhood filter of $X$ so that $K_F = X$. It remains to notice that $f^*[X] \subset X \setminus A^*_0 \subseteq X$. Thus, the statement of the theorem follows.

Next, we would like to show that the extra hypothesis of Theorem 4.1 is really necessary.

4.2. Theorem There exists a compact ED space $X$ that can be embedded in $\omega^*$ as a weak $P$-set and such that every time $F$ is a filter with $K_F$ a weak $P$-set homeomorphic to $X$ then $F$ is reversible.

Proof. In \cite{5} it was shown that there exists a separable, ED, compact space $X$ that is rigid in the sense that the identity function is its only autohomeomorphism. Using very similar arguments, it can be easily proved that no clopen subset of $X$
is homeomorphic to $X$. Since $X$ is separable and crowded, it is easy to see that $X$ is a continuous image of $\omega^*$. This in turn implies that $X$ can be embedded in $\omega^*$ as a weak $P$-set.

Assume now that $F$ is any filter on $\omega$ such that $K_F$ is a weak $P$-set homeomorphic to $X$. Let $f : \omega \to \omega$ be a bijection such that $f^*[K_F] \subset K_F$ and assume that $U = K_F \setminus f^*[K_F] \neq \emptyset$. Then, since $X$ is separable, there is a countable set $D \subset U$ with $\text{cl}_{\omega^*}(D) = \text{cl}_{K_F}(U)$. Since $D \cap f^*[K_F] = \emptyset$ and $f^*[K_F]$ is a weak $P$-set, it follows that $\text{cl}_{\omega^*}(D) \cap f^*[K_F] = \emptyset$. Thus, $\text{cl}_{K_F}(U) \cap f^*[K_F] = \emptyset$ which shows that $U = \text{cl}_{K_F}(U)$ and $f^*[K_F]$ is clopen in $K_F$. So $f^*[K_F]$ is a clopen set of $K_F$ homeomorphic to itself, which implies $f^*[K_F] = K_F$. This is a contradiction so in fact $U = \emptyset$ and $f^*[K_F] = K_F$. This shows that $F$ is reversible. □

We finally consider filters that are reversible. In order to make the corresponding spaces weak $P$-sets of $\omega^*$, we will need to use Kunen’s technique of a construction of a weak $P$-point ([8]). We shall use Dow’s approach from [4].

First, let us recall the concept of a $c$-OK set. So let $\kappa$ be an infinite cardinal, $X$ a space and $K$ closed in $X$. Given an increasing sequence $\{C_n : n < \omega\}$ of closed subsets of $X$ disjoint from $K$, we will say that $K$ is $\kappa$-OK with respect to $\{C_n : n < \omega\}$ if there is a set $\mathcal{U}$ of neighborhoods of $K$ such that $|\mathcal{U}| = \kappa$ and every time $0 < n < \omega$ and $\mathcal{U}_n \in |\mathcal{U}|^n$, $\bigcap \mathcal{U}_n \cap C_n = \emptyset$. Then $K$ is $\kappa$-OK if it is $\kappa$-OK with respect to every countable increasing sequence of closed subsets of $X$. It easily follows that if a closed set is $\kappa$-OK for $\kappa$ uncountable, then it is a weak $P$-set (see [8, Lemma 1.3]).

In [4, Lemma 3.2], Dow proves that if $\omega^*$ maps onto $X$, then there is a continuous surjection $\varphi : \omega^* \to X \times (\varepsilon + 1)^I$, where $\varepsilon + 1 = \varepsilon \cup \{\varepsilon\}$ is taken as the one-point compactification of the discrete space $\varepsilon$. This map $\varphi$ will replace Kunen’s independent matrices from [8]. Lemma 3.4 in [4] gives a method to construct $c$-OK points in $\omega^*$ using this map $\varphi$. We will use the following modification mentioned by Dow by the end of the proof of [4, Theorem 3.5]. For any set $I \subset \varepsilon$, we denote by $\pi_I : X \times (\varepsilon + 1)^I \to X \times (\varepsilon + 1)^I$ the projection. To be consistent with notation, $\pi_\emptyset$ will denote the projection of $X \times (\varepsilon + 1)^\emptyset \to X$.

4.3. Lemma Let $\psi : \omega^* \to X \times (\varepsilon + 1)^I$ be continuous and onto, where $I \subset \varepsilon$ is an infinite set. Assume that $K \subset \omega^*$ is a closed set with $\psi[K] = X \times (\varepsilon + 1)^I$ and $\{C_n : n < \omega\}$ is a sequence of closed subsets of $\omega^*$ disjoint from $K$. Then there is a countable set $J \subset I$ and a closed subset $K' \subset K$ such that

- $(\pi_I^{-1} \circ \psi)[K'] = X \times (\varepsilon + 1)^J$ and
- $K'$ is $c$-OK with respect to $\{C_n : n < \omega\}$.

Recall in $(\varepsilon + 1)^J$, where $J$ any set, there is a base of clopen subsets of the form

$\prod\{U_\xi : \xi \in J\}$ where each factor $U_\xi$ is clopen and the support $\{\xi \in J : U_\xi \neq \varepsilon + 1\}$ is finite.

4.4. Theorem Let $X$ be a compact ED space that is a continuous image of $\omega^*$. Then there is a reversible filter $F$ such that $K_F$ is a weak $P$-set homeomorphic to $X$.

Proof. Let $\varphi : \omega^* \to X \times (\varepsilon + 1)^I$ be the surjection from [4, Lemma 3.2]. Our objective is to recursively construct a closed set $K \subset \omega^*$ such that $(\pi_\emptyset \circ \varphi)[K] = X$.
and \((\pi_0 \circ \varphi)|_K: K \to X\) is irreducible. By a classic result by Gleason (see, for example, the argument in [9, 1.4.7]) it follows that \((\pi_0 \circ \varphi)|_K\) is a homeomorphism. So it only remains to take \(\mathcal{F}\) to be the filter of neighborhoods of \(K\).

We will define \(K\) as the intersection of a family \(\{K_\alpha : \alpha < \varsigma\}\) of closed subsets of \(\omega^*\), ordered inversely by inclusion. We will also define a decreasing sequence \(\{I_\alpha : \alpha < \varsigma\} \subset \varsigma\) such that \(I_0 = \varsigma\) and \(|\varsigma \setminus I_\alpha| < |\alpha| \cdot \omega\) for all \(\alpha < \varsigma\). We will have the following conditions:

(a) If \(\beta < \varsigma\) is a limit, \(K_\beta = \bigcap\{K_\alpha : \alpha < \varsigma\}\) and \(I_\beta = \bigcap\{I_\alpha : \alpha < \varsigma\}\).

(b) For each \(\alpha < \varsigma\), \((\pi_\alpha \circ \varphi)[K_\alpha] = X \times (\varsigma + 1)^I_\alpha\).

We need to do accomplish three things in our construction: make \(K\) a weak \(P\)-set, that the map \((\pi_0 \circ \varphi)|_K: K \to X\) is irreducible and make sure that the filter of neighborhoods \(\mathcal{F}\) is reversible. So we will partition ordinals into three sets. For \(i \in \{0, 1, 2\}\), let \(\Lambda_i\) be the set of ordinals \(\alpha < \varsigma\) such that \(\alpha = \beta + n\), \(\beta\) is a limit ordinal and \(n < \omega\) is congruent to \(i\) modulo 3. Let \(\{C_\alpha : n < \omega\} : \alpha \in \Lambda_0\) be an enumeration of all countable increasing of clopen sets where each sequence is repeated cofinally often.

Finally, we need to take care of reversibility using the \(c\) chances we get from \(\Lambda_2\). Let \(\{f_\alpha : \alpha \in \Lambda_2\}\) be an enumeration of all bijections from \(\omega\) onto itself, each one repeated cofinally often. We will require the following condition.

(c) Let \(\alpha \in \Lambda_0\). If \(K_\alpha\) is disjoint from all the members of the sequence \(\{C_\alpha : n < \omega\}\), then \(|I_\alpha \setminus I_{\alpha + 1}| \leq \omega\) and \(K_{\alpha + 1}\) is \(c\)-OK with respect to \(\{C_\alpha : n < \omega\}\).

(d) Let \(\alpha \in \Lambda_1\). If \((\pi_\alpha \circ \varphi)[K_\alpha \cap B_\alpha] = X \times \varsigma^I_\alpha\), then \(I_{\alpha + 1} = I_\alpha \) and \(K_{\alpha + 1} = K_\alpha \cap B_\alpha\). Otherwise, there are clopen sets \(C \subset X\) and \(D \subset (\varsigma + 1)^\varsigma\) such that the support of \(D\) is equal to \(I_\alpha \setminus I_{\alpha + 1}\), \(\varphi[K_\alpha \cap B_\alpha] \cap (C \times D) = \emptyset\) and \(K_{\alpha + 1} = K_\alpha \cap \varphi^+(X \times D)\).

Clearly, (c) follows from Lemma 4.3 and implies that \(K\) is a weak \(P\)-set of \(\omega^*\). Also, it is not hard to see that condition (d) implies that \((\pi_0 \circ \varphi)|_K: K \to X\) is irreducible. Conditions (c) and (d) are taken from the proof of [4, Theorem 3.5].

Finally, we need to take care of reversibility using the \(c\) chances we get from \(\Lambda_2\). Let \(\{f_\alpha : \alpha \in \Lambda_2\}\) be an enumeration of all bijections from \(\omega\) onto itself, each one repeated cofinally often. We will require the following condition.

(e) Let \(\alpha \in \Lambda_2\). Assume that there exists a clopen sets \(U \subset \omega^*\) and \(V \subset X\) such that \((\pi_\alpha \circ \varphi)[K_\alpha \cap U] = V \times (\varsigma + 1)^I_\alpha\) and \((\pi_0 \circ \varphi)[f_\alpha[K_\alpha \cap U]] \subset X \setminus V\). Then \(|I_\alpha \setminus I_{\alpha + 1}| < \omega\) and there is \(x \in X\) such that \(f_\alpha[K_\alpha + 1 \cap (\pi_0 \circ \varphi)^+ (x)] \cap K_{\alpha + 1} = \emptyset\).

Before we show how to prove that (e) can be obtained, let us show why it implies that the filter of neighborhoods of \(K\) is reversible. Assume that after our construction, \(K\) is not reversible. Then by Lemma 2.3, there is a bijection \(f : \omega \to \omega\) such that \(f^*[K] \subseteq K\). By property (d) above we know that \((\pi_0 \circ \varphi)|_K: K \to X\) is irreducible so \((\pi_0 \circ \varphi \circ f^*)[K]\) is a proper closed subset of \(X\). Let \(V\) be a clopen set disjoint from \((\pi_0 \circ \varphi \circ f^*)[K]\). Now consider the clopen subset \(W = (\pi_0 \circ \varphi)^+ [V]\) of \(\omega^*\). From the definition of \(K\) and the facts that \(f^*\) is a homeomorphism and \(f^*[K] \cap W = \emptyset\), there is \(\beta < \varsigma\) such that \(f^*[K_\beta] \cap W = \emptyset\) every time \(\beta \leq \gamma < \varsigma\). So fix \(\alpha \in \Lambda_2\) such that \(\beta \leq \alpha\) and \(f_\alpha = f\). Now define

\[U = W \cap ((f^*)^+[\omega^* \setminus W])\]

which is a clopen set of \(\omega^*\) with the property that \((\pi_0 \circ \varphi \circ f^*)[U] \subset X \setminus V\). From the choice of \(\alpha\) we obtain that \(K_\alpha \cap W \subset U\). Also, \((\pi_\alpha \circ \varphi)[K_\alpha \cap W] = V \times (\varsigma + 1)^I_\alpha\) by property (b). Finally, notice that \(K_\alpha \cap W = K_\alpha \cap U\). Thus, our choice of \(\alpha, U\)
and $V$ take the hypothesis of condition (e). So let $x$ as in the conclusion of (e) and take $p \in K$ such that $(\pi_0 \circ \varphi)(p) = x$. Then $p \in K_{\alpha+1} \cap (\pi_0 \circ \varphi)^\sim(x)$ implies that $f^\star_{\alpha}(p) \notin K_{\alpha+1}$. Thus, $p \in K \setminus f^\star[K]$, a contradiction. This contradiction comes from the fact that we assumed that $K$ was not reversible.

So we are left to prove that condition (e) can be achieved. So assume we have $\alpha$, $U$ and $V$ like in the hypothesis of (e). Choose any $i \in I_\alpha$ and let $J = I_\alpha \setminus \{i\}$. For each $\xi \in \mathcal{C}$, let $U_\xi = (\pi_{\{i\}} \circ \varphi)^\sim(X \times \{\xi\})$, which is a clopen set in $X$. Then \{\{U_\xi : \xi \in \mathcal{C}\}$ is a pairwise disjoint collection of clopen subsets of $X$ such that $(\pi_j \circ \varphi)(K_\alpha \cap U_\xi) = X \times (\epsilon + 1)^J$ for all $\xi \in \mathcal{C}$. For each $\xi \in \mathcal{C}$, consider the set $V_\xi = (f^\star_{\alpha} \circ \varphi)(U_\xi) \cap K_\alpha \cap U$, which is a clopen set of $K_\alpha \cap U$. Here we will have two cases.

\textbf{Case 1:} There exists $\xi_0 \in \mathcal{C}$ such that $(\pi_j \circ \varphi)(V_{\xi_0}) = V \times (\epsilon + 1)^J$. Choose any $\xi_1 \in \mathcal{C} \setminus \{\xi_0\}$, let $I_{\alpha+1} = J$ and define

$$K_{\alpha+1} = V_{\xi_0} \cup (K_\alpha \cap U_{\xi_1} \cap (\pi_0 \circ \varphi)^\sim[X \setminus V]).$$

Notice that $K_{\alpha+1} \subset K_\alpha$ and $(\pi_{I_{\alpha+1}} \circ \varphi)(K_{\alpha+1}) = X \times (\epsilon + 1)^{I_{\alpha+1}}$. Now take any $x \in V$. Then $K_{\alpha+1} \cap (\pi_0 \circ \varphi)^\sim(x) \subset V_{\xi_0}$ so $f^\star[K_{\alpha+1} \cap (\pi_0 \circ \varphi)^\sim(x)] \subset V_{\xi_0}$. Since $U_{\xi_0} \cap U_{\xi_1} = \emptyset$, then $f^\star[K_{\alpha+1} \cap (\pi_0 \circ \varphi)^\sim(x)] \cap K_{\alpha+1} = \emptyset$ and we are done.

\textbf{Case 2:} Not Case 1. Take $\xi_0 \in \mathcal{C}$, then there exists clopen sets $C \subset V$ and $D \subset (\epsilon + 1)^J$ such that $C \times D$ is disjoint from $(\pi_j \circ \varphi)(V_{\xi_0})$. Let $J' \subset J$ be the support of $D$ and define $I_{\alpha+1} = J \setminus J'$. In this case, define

$$K_{\alpha+1} = (K_\alpha \cap U \cap (\pi_j \circ \varphi)^\sim[C \times D]) \cup (K_\alpha \cap U_{\xi_0} \cap (\pi_j \circ \varphi)^\sim[(X \setminus C) \times D]).$$

Clearly, $K_{\alpha+1} \subset K_\alpha$. It is not hard to see that and $(\pi_j \circ \varphi)(K_{\alpha+1}) = X \times D$, which implies that $(\pi_{I_{\alpha+1}} \circ \varphi)(K_{\alpha+1}) = X \times (\epsilon + 1)^{I_{\alpha+1}}$. Now let $x \in C$. Assume there is $p \in K_{\alpha+1}$ such that $(\pi_0 \circ \varphi)(p) = x$ and $q = f^\star_{\alpha}(p) \in K_{\alpha+1}$, we will reach a contradiction. Notice that $p \in U$, which implies that $q \in (\pi_0 \circ \varphi)^\sim[X \setminus V]$. So from the definition of $K_{\alpha+1}$ we obtain that $q \notin U_{\xi_0}$. This in turn implies that $p \notin V_{\xi_0}$. By the choice of $C \times D$ we obtain that $(\pi_j \circ \varphi)(p) \notin C \times D$. But since $x \in C$, $p \in K_{\alpha+1} \cap U \cap (\pi_j \circ \varphi)^\sim[C \times D]$ so $(\pi_j \circ \varphi)(p) \in C \times D$. So we obtain a contradiction and we obtain the negation of our assumption. Thus, $f^\star[K_{\alpha+1} \cap (\pi_0 \circ \varphi)^\sim(x)] \cap K_{\alpha+1} = \emptyset$, which is what we wanted to prove.

These two cases complete the proof of condition (e) and finish the proof of the Theorem.

\section{5. Filters generated by towers}

It is well known that Martin’s axiom (henceforth, \textbf{MA}) implies that there are filters that are $P$-filters (see, for example, [1, Theorem 4.4.5]). Equivalently, there is a filter $F$ such that $K_F$ is a $P$-set. It is not hard to see that by changing all instances of “weak $P$-set” to just “$P$-set” in Theorem 4.1, we obtain a valid statement. Also, every $P$-set in a weak $P$-set so the $P$-set version of Theorem 4.2 is in fact implied by Theorem 4.2.

As for the $P$-set version of Theorem 4.4, we will do something stronger, but only for separable, first countable spaces. Recall that a tower is a set $\{A_\alpha : \alpha < \kappa\} \subset \mathcal{P}(\omega)$, for some $\kappa$, such that

- $A_\beta \setminus A_\alpha$ is finite every time $\alpha < \beta < \kappa$, and
In this case, \( \{ A_\alpha : \alpha < \kappa \} \) is a decreasing chain of clopen subsets of \( \omega^* \) with nowhere dense intersection. Clearly, every tower generates a filter and every filter generated by a tower of height \( \kappa = \kappa \) is a \( P \)-filter. In fact, every filter generated by a tower of height \( \kappa \) is a \( P \)-filter.

In what follows we will assume the reader’s familiarity with \( \text{MA} \) and small uncountable cardinals from [3]. One fact that we will use several times is that \( \text{MA} \) implies that every intersection of less than \( \kappa \) many clopen sets is a regular closed set (this follows from Theorem 2.1).

5.1. Lemma Let \( X_0, X_1 \) be compact separable spaces of weight \( < \kappa \) and let \( \psi_0 : \omega^* \to X_0 \) be a continuous onto function. Assume that there is a continuous function \( \pi : X_1 \to X_0 \) and a partition \( X_1 = V_0 \cup V_1 \) into two clopen sets such that \( \pi|_{V_i} : V_i \to X_0 \) is an embedding for \( i < 2 \). Then \( \text{MA} \) implies there exists a clopen set \( W \subset \omega^* \) and a continuous onto function \( \psi_1 : W \to X_1 \) such that \( \pi \circ \psi_1 = \psi_0|_W \).

Proof. Let \( F_i = \pi[V_i] \) for \( i < 2 \), this is a closed subset of \( X_0 \). Choose a countable dense set \( \{ d_n : n < \omega \} \) of \( X_0 \) that is contained in the dense open set \( (X_0 \setminus F_0) \cup (X_0 \setminus F_1) \cup (\text{int}_{X_0}(F_0 \cap F_1)) \). Let \( N_i = \{ n < \omega : d_n \in X_0 \setminus F_i \} \) for \( i < 2 \) and \( N_2 = \omega \setminus (N_0 \cup N_1) \).

Since \( F_0 \) is an intersection of \( < \kappa \) many clopen sets of \( X_0 \), there is a collection \( G_0 \) of clopen sets of \( \omega^* \) such that \( \bigcap G_0 = \psi_0^\ast[F_0] \) and \( |G_0| < \kappa \). For each \( n \in N_0 \cup N_2 \), let \( U_n \) be a clopen subset of \( \omega^* \) such that \( \psi_0[U_n] = \{ d_n \} \). Clearly, \( U_n \subset \bigcap G_0 \) for all \( n \in N_0 \cup N_2 \). Thus, considering the collection \( G_0 \cup \{ U_n : n \in N_0 \cup N_2 \} \), by \( \text{MA} \) and Lemma 2.1, there exists an clopen set \( W_0 \subset \omega^* \) such that \( W_0 \subset \bigcap G_0 \) and \( W_n \subset W_0 \) for all \( n \in N_0 \cup N_2 \). It follows that \( W_0 \) is a clopen set of \( \omega^* \) such that \( d_n \in \psi[W_0] \) for all \( n \in N_0 \cup N_2 \). Since \( \{ d_n : n \in N_0 \cup N_2 \} \) is dense in \( W_0 \) we obtain that \( \psi_0[W_0] = F_0 \).

Now we will find a clopen set \( W_1 \) with \( \psi_0[W_1] = F_1 \). However, we will have to be more careful because of possible intersections with \( W_0 \). Let \( G_1 \) be a collection of clopen sets of \( \omega^* \) such that \( \bigcap G_1 = \psi_0^\ast[F_1] \) and \( |G_1| < \kappa \). For each \( n \in N_1 \), let \( U_n \) be a clopen subset of \( \omega^* \) such that \( \psi_0[U_n] = \{ d_n \} \). If \( n \in N_2 \), we choose two disjoint non-empty clopen subsets \( U_0^n \) and \( U_1^n \) of \( \omega^* \) such that \( U_0^n \subset W_0 \) and \( \psi_0[U_0^n] = \{ d_n \} \) for \( i < 2 \). Clearly, \( U_n \subset \bigcap G_1 \) for \( n \in N_1 \) and \( U_0^n \cup U_1^n \subset \bigcap G_1 \) for \( n \in N_2 \). So using \( \text{MA} \) and Lemma 2.1 again, we can find a clopen set \( W_1 \subset \omega^* \) such that \( W_1 \subset \bigcap G_1 \), \( U_n \subset W_1 \) for all \( n \in N_1 \), \( U_0^n \subset W_1 \) for all \( n \in N_2 \) and \( U_0^n \cap W_1 = \emptyset \) for all \( n \in N_2 \). Again, it easily follows that \( \psi_0[W_1] = F_1 \).

So now consider \( W_0 \setminus W_1 \). If \( n \in N_1 \), \( U_n \subset W_0 \setminus W_1 \) so \( d_n \in \psi_0[W_0 \setminus W_1] \). If \( n \in N_2 \), \( U_0^n \subset W_0 \setminus W_1 \) so \( d_n \in \psi_0[W_0 \setminus W_1] \). From this it follows that \( \psi_0[W_0 \setminus W_1] = F_0 \).

Let \( W = W_0 \cup W_1 \), we now define \( \psi_1 : W \to X_1 \) such that

\[
\psi_1(x) = \begin{cases} 
(\pi|_{V_0})^{-1}(\psi_0(x)), & \text{if } x \in W_0 \setminus W_1, \\
(\pi|_{V_1})^{-1}(\psi_0(x)), & \text{if } x \in W_1.
\end{cases}
\]

It is easy to see that \( \psi_1 \) is as required.

5.2. Theorem Let \( X \) be a separable, compact, ED space. Then \( \text{MA} \) implies that there is a reversible filter \( \mathcal{F} \) that is generated by a tower of height \( \kappa \) such that \( K_\mathcal{F} \) is a \( P \)-set homeomorphic to \( X \).
Proof: If our hypothesis, we may assume that $X \subset \omega$. For all pairs $\alpha \leq \beta \leq \epsilon$, let $\pi_\alpha^\beta : \beta^\alpha \to \alpha^\beta$ be the projection. Let $\{d_n : n < \omega\}$ be an enumeration of a countable dense set in $X$. By permuting the elements of $\epsilon$ and then renaming them if necessary, we may assume that if $n, m < \omega$ and $\pi_\omega^n(d_n) = \pi_\omega^m(d_m)$, then $n = m$. Let $X_\alpha = \pi_\alpha^n\{X\}$ for every $\alpha < \epsilon$.

We will recursively construct a decreasing sequence $\{K_\alpha : \omega \leq \alpha < \epsilon\}$ of clopen sets of $\omega^*$ and a sequence of continuous functions $\varphi_\alpha : K_\alpha \to X_\alpha$, for $\omega \leq \alpha \leq \epsilon$, in such a way that $\pi_\alpha^\beta \circ \varphi_\beta = \varphi_\alpha$ whenever $\omega \leq \alpha \leq \beta < \epsilon$. Once we have done this, let $K = \bigcap\{K_\alpha : \alpha < \epsilon\}$ and define $\varphi : K \to X$ by $\varphi(x) = \bigcup\{\varphi_\alpha(x) : \omega \leq \alpha < \epsilon\}$ for all $x \in K$. Notice that $\varphi$ is continuous and $\pi_\alpha^\beta \circ \varphi = \varphi_\alpha$ for every $\omega \leq \alpha < \epsilon$.

Let $\{\Phi_n : \omega \leq \alpha < \epsilon\}$ be an enumeration of all clopen subsets of $\omega^*$. Let $\{\phi_n : \omega \leq \alpha < \epsilon\}$ be the collection of all bijections from $\omega$ onto itself such that each one is repeated cofinally often. Let $\Lambda_0$ be the set of infinite even ordinals $< \epsilon$ and let $\Lambda_1$ be the set of infinite odd ordinals $< \epsilon$. We will carry out our construction respecting the following conditions.

(a) $K_\omega = \omega^*$.
(b) For all $\omega \leq \alpha < \epsilon$ and $n < \omega$, $\varphi_\alpha[K_n] = X_\alpha$.
(c) For all $\alpha \in \Lambda_0$, if $\varphi_\alpha[K_\alpha \cap B_\alpha] = X_\alpha$, then $K_{\alpha+1} \subset B_\alpha$.
(d) Let $\alpha \in \Lambda_1$. Assume that there exists a clopen sets $U \subset K_\alpha$ and $V \subset X_\alpha$ such that $\varphi_\alpha[U] = V$ and $\varphi_\alpha[f_\alpha(U)] \subset X \setminus V$. Then there is $x \in X_\alpha$ such that $f_\alpha[K_{\alpha+1} \cap \varphi_\alpha^{-1}(x)] \cap K_{\alpha+1} = \emptyset$.

It is not hard to prove that (c) implies that $\varphi : K \to X$ is irreducible, thus, a homeomorphism. And the proof that (d) implies that the filter $\mathcal{F}$ of neighborhoods of $K$ is reversible is analogous to the corresponding one in the proof of Theorem 4.4. Since any separable subspace of $\omega^*$ is nowhere dense, we obtain that $\{K_\alpha : \alpha < \epsilon\}$ is a tower that generates $\mathcal{F}$. So we will only show how to carry out this construction.

Let $\beta < \epsilon$ be a limit ordinal, let us show how to find $K_\beta$ and $\varphi_\beta$. Let $T = \bigcap\{K_\alpha : \alpha < \beta\}$ and define $\psi : T \to X_\beta$ by $\psi(x) = \bigcup\{\varphi_\alpha(x) : \omega \leq \alpha < \beta\}$ for all $x \in T$. Notice that $\psi$ is continuous and $\mathbf{MA}$ implies that $T$ is a regular closed set of $\omega^*$. Because $X_\beta$ has weight $\leq |\beta| < \epsilon$, $\psi^+|\pi_\beta^\alpha(d_n))$ is an intersection of $< \epsilon$ many clopen sets for each $n < \omega$. By $\mathbf{MA}$, we can choose for each $n < \omega$ a clopen set $U_n \subset T$ such that $\psi[U_n] = \{\pi_\beta^\alpha(d_n)\}$. Then by considering the sets $\{U_n : n < \omega\} \cup \{K_\alpha : \alpha < \beta\}$, by $\mathbf{MA}$ and Lemma 2.1, we obtain that there is a clopen set $V \subset T$ such that $U_n \subset V$ for every $n < \omega$. Let $K_\beta = V = \varphi_\beta = \psi|V$.

Now assume that $\alpha < \epsilon$ and we want to define $K_{\alpha+1}$ and $\varphi_{\alpha+1}$. First, assume that $\alpha \in \Lambda_0$. Let $T = K_\alpha \cap B_\alpha$ if $\varphi_\alpha[K_\alpha \cap B_\alpha] = X_\alpha$; otherwise, let $T = K_\alpha$. Then $\varphi_\alpha[T] = X_\alpha$. Notice that $V_0 = \{x \in X_{\alpha+1} : x(\alpha) = i\}$ for $i < 2$ is a pair of clopen sets of $X_{\alpha+1}$ where $\pi_\alpha^{i+1}$ is one-to-one and $X_{\alpha+1} = V_0 \cup V_1$. Thus, we can apply Lemma 5.1 to find a clopen set $W \subset T$ and a continuous function $\psi : W \to X_{\alpha+1}$ such that $\pi_\alpha^{i+1} \circ \psi = \varphi_\alpha$. So let $K_\alpha = W$ and $\varphi_{\alpha+1} = \psi$.

Finally, assume that $\alpha \in \Lambda_1$. If the hypothesis of (d) does not hold, just use Lemma 5.1 like in the previous paragraph to define $K_{\alpha+1}$ and $\varphi_{\alpha+1}$. So assume otherwise. By $\mathbf{MA}$ and the fact that all points of $X_\alpha$ have character $\leq |\alpha| < \epsilon$, we may assume that for each $n < \omega$, there exists a clopen set $U_n \subset K_\alpha$ such that $\varphi_\alpha[U_n] = \{\pi_\alpha^n(d_n)\}$. Let $N_0$ be the set of $n < \omega$ such that $d_n \in V$. We may assume that $U_n \subset V$ for all $n \in N_0$. For $n \in \omega \setminus N_0$, we may assume that either $U_n \subset f_\alpha^n(U)$ or $U_n \cap f_\alpha^n(U) = \emptyset$. Let $N_1$ the set of all $n \in \omega \setminus N_0$ such that $U_n \subset f_\alpha^n(U)$ and $N_2 = \omega \setminus (N_0 \cup N_1)$. 
For each $n \in N_1$, choose $p_n \in U_n$. Then $\{p_n : n \in N_1\}$ is a discrete (possibly empty) set contained in $f^*_\alpha[U]$. For each $n \in N_1$, let $q_n = (f^*_\alpha)^-(p_n)$. Then $\{q_n : n \in N_1\}$ is a discrete set contained in $U$. Since no clopen subset of $\omega^*$ is separable, it is possible to choose, for each $n \in N_0$, $p_n \in U_n \setminus cl_{\omega^*}(\{q_m : m \in N_1\})$. Then the set $\{p_n : n \in N_0\} \cup \{q_n : n \in N_1\}$ is a discrete subset of $U$. Then, since countable subsets are $C^*$-embedded in $\omega^*$, there exists a clopen set $W \subset U$ such that $\{p_n : n \in N_0\} \subset W$ and $\{q_n : n \in N_1\} \cap W = \emptyset$. With this, we can define

$$T = W \cup (K_\alpha \cap f^*_\alpha[U \setminus W]) \cup (\varphi^-_\alpha[X \setminus V] \cap (K_\alpha \setminus f^*_\alpha[U]))$$

Clearly, $T$ is a clopen subset of $K_\alpha$. If $n \in N_0$, then $p_n \in W$ so $d_n \in \varphi_\alpha[T]$. If $n \in N_1$, then $p_n \in K_\alpha \cap f^*_\alpha[U \setminus W]$ so $d_n \in \varphi_\alpha[T]$. Finally, if $n \in N_2$, $U_n \subset \varphi^-_\alpha[X \setminus V] \cap (K_\alpha \setminus f^*_\alpha[U])$ so $d_n \in \varphi_\alpha[T]$. Thus, $\{d_n : n < \omega\} \subset \varphi_\alpha[T]$, which implies that $\varphi_\alpha[T] = X$.

By an application of Lemma 5.1, there is a clopen set $T' \subset T$ and a continuous function $\psi : T' \to X_{\alpha+1}$ such that $\pi^\alpha_{\alpha+1} \circ \psi = \varphi_\alpha$. Let $K_{\alpha+1} = T'$ and $\varphi_{\alpha+1} = \psi$. Finally, choose $x \in V$ arbitrarily. Since $K_{\alpha+1} \cap \varphi^-_\alpha(x) \subset T \cap \varphi^-_\alpha(x) \subset W$, $f^*_\alpha[K_{\alpha+1} \cap \varphi^-_\alpha(x)] \subset f^*_\alpha[W] \subset \omega^* \setminus T \subset \omega^* \setminus K_{\alpha+1}$. Thus, these choices satisfy the conclusion of (d), so we have finished the proof. \hfill \square

5.3. Question Is the conclusion of Theorem 5.2 still valid if $X$ is not necessarily separable?

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