STATIONARY SOLUTIONS FOR STOCHASTIC DAMPED NAVIER-STOKES EQUATIONS IN $\mathbb{R}^d$

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Abstract. We consider the stochastic damped Navier-Stokes equations in $\mathbb{R}^d$ ($d = 2, 3$), assuming as in our previous work [4] that the covariance of the noise is not too regular, so Itô calculus cannot be applied in the space of finite energy vector fields. We prove the existence of an invariant measure when $d = 2$ and of a stationary solution when $d = 3$.

1. Introduction

We consider the stochastic damped Navier-Stokes equations, that is the equations of motion of a viscous incompressible fluid with two forcing terms, one is random and the other one is deterministic. These equations are

\[
\begin{cases}
\partial_t v + [-\nu \Delta v + \gamma v + (v \cdot \nabla)v + \nabla p] \, dt = G(v) \, \partial_t w + f \, dt \\
\text{div} \, v = 0
\end{cases}
\]  

where the unknowns are the vector velocity $v = v(t, \xi)$ and the scalar pressure $p = p(t, \xi)$ for $t \geq 0$ and $\xi \in \mathbb{R}^d$. By $\nu > 0$ we denote the kinematic viscosity and by $\gamma > 0$ the sticky viscosity, see [16], [10], [1]. When $\gamma = 0$ they reduce to the stochastic Navier-Stokes equations. On the right hand side $\partial_t w$ is a space-time white noise and $f$ is a deterministic forcing term; we consider a multiplicative term $G(v)$ keeping track of the fact that the noise may depend on the velocity. The low regularity of this term $G(v)$ is the peculiarity of our problem.

The current paper is a natural continuation of our recent work [4] in which we dealt with SNSEs (1.1) when $\gamma = 0$. A common feature of both these papers is that the diffusion coefficient $G(v)$ is not regular enough to allow to use the Itô formula.
in the space of finite energy velocity vector fields. In [4] we proved the existence of martingale solutions, for $d = 2$ or $d = 3$ and, for $d = 2$, the pathwise uniqueness of solutions.

Our aim here is to investigate the existence of invariant measures in $\mathbb{R}^2$ and stationary solutions in $\mathbb{R}^3$ for these stochastic damped Navier-Stokes equations (1.1). On one side we follow the work of [15], where the existence of stationary martingale solutions has been proved in two dimensional bounded domains when the noise term is of low regularity. On the other side we follow the method used by the first named authour with Motyl and Ondreját [7] for two dimensional unbounded domains but with more regular noise. The proof of the existence of an invariant measure given in the latter paper is based on the technique working with weak topologies introduced by Maslowski and Seidler [21]. This technique has been successful also in the study of invariant measures for stochastic nonlinear beam and wave equations considered in [9].

We would like to point out that similarly to our previous paper [4] by using $\gamma$-radonifying (instead of Hilbert-Schimdt) operators and Itô integral in suitable Banach spaces we were able to weaken the assumptions from [15], even in the bounded domain cases, on the covariance of the noise, for the existence of an invariant measure and a stationary solution.

Let us recall that the existence of an invariant measure for 2d stochastic Navier-Stokes equations driven by an additive noise in unbounded domains satisfying the Poincaré condition has been proved (as a byproduct of the existence of an invariant compact random set) by the first named authour and Li in [5], see also [3]. The same proof would work, when the noise is additive, for the damped stochastic Navier-Stokes equations in $\mathbb{R}^2$ considered in the present article and for stochastic reaction diffusion equations. Finally, we point out that recently Wang in [28] considered the 2d stochastic Navier-Stokes equations in unbounded Poincaré domains and with a linear multiplicative noise in Stratonovich form (for which the Doss-Sussman transformation is applicable). So with respect to these papers, our work concerns more general noise terms.

As far as the contents of the paper are concerned, in Section 2 we introduce the spaces and operators appearing in the abstract formulation of system (1.1), the assumptions on the noise term, some compactness results and definitions. In Section 3 we work in $\mathbb{R}^2$ and obtain the existence of an invariant measure. In Section 4 we work in $\mathbb{R}^3$ and obtain the existence of a stationary solution, as limit of a sequence of stationary martingale solutions solving a smoothed equation.

### 2. Mathematical framework

We recall the basic notations and results. For more details we refer to the monograph [27] by Temam and our paper [4].
2.1. Spaces and operators. For $1 \leq p < \infty$ let $L^p = [L^p(\mathbb{R}^d)]^d$ with norm

$$
\|v\|_{L^p} = \left(\sum_{k=1}^{d} \|v^k\|_{L^p(\mathbb{R}^d)}^p\right)^{1/p}
$$

where $v = (v^1, \ldots, v^d)$. For $p = \infty$, we set $\|v\|_{L^\infty} = \sum_{k=1}^{d} \|v^k\|_{L^\infty(\mathbb{R}^d)}$.

Set $J^s = (I - \Delta)^{\frac{s}{2}}$. We define the generalized Sobolev spaces of divergence free vector distributions as

\begin{align}
H^{s,p} & = \{u \in \mathcal{S}'(\mathbb{R}^d)^d : \|J^s u\|_{L^p} < \infty\}, \\
H^{s,p}_{\text{sol}} & = \{u \in H^{s,p} : \text{div } u = 0\}
\end{align}

for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. $J^s$ is an isomorphism between $H^{s,p}$ and $H^{s-s,p}$ for $s \in \mathbb{R}$ and $1 < p < \infty$. Moreover $H^{s_2,p} \subset H^{s_1,p}$ when $s_1 < s_2$. In particular, for the Hilbert case $p = 2$ we set $H = H^{0,2}_{\text{sol}}$ and, for $s \neq 0$, $H^s = H^{s,2}_{\text{sol}}$; that is

$$H = \{v \in [L^2(\mathbb{R}^d)]^d : \text{div } v = 0\}$$

with scalar product inherited from $[L^2(\mathbb{R}^d)]^d$. Moreover, $H_{\text{loc}}$ is the space $H$ with the topology generated by the family of semi-norms $\|v\|_{H^s} = \left(\int_{|\xi| < N} |v(\xi)|^2 d\xi\right)^{1/2}$, $N \in \mathbb{N}$, and $L^2(0, T; H_{\text{loc}})$ is the space $L^2(0, T; H)$ with the topology generated by the family of semi-norms $\|v\|_{L^2(0, T; H^s)}$, $N \in \mathbb{N}$. By $H_w$ we denote the space $H$ with the weak topology and by $C([0, T]; H_w)$ the space of $H$-valued weakly continuous functions with the topology of uniform weak convergence on $[0, T]$; in particular $v_n \to v$ in $C([0, T]; H_w)$ means

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} |(v_n(t) - v(t), h)_H| = 0$$

for all $h \in H$. Notice that $v(t) \in H$ for any $t$ if $v \in C([0, T]; H_w)$.

From [17] one knows that there exists a separable Hilbert space $U$ such that $U$ is a dense subset of $H^1$ and is compactly embedded in $H^1$. We also have that

$$U \subset H^1 \subset H \subset H' \subset H^{-1} \subset U'$$

with dense and continuous embeddings, but in addition $H^{-1}$ is compactly embedded in $U'$.

Now we define the operators appearing in the abstract formulation. Let $A = -\Delta$; then $A$ is a linear unbounded operator in $H^{s,p}_{\text{sol}}$ as well as in $H^{s,p}_{\text{sol}}$ ($s \in \mathbb{R}$, $1 \leq p < \infty$), which generates a contractive and analytic $C_0$-semigroup $\{e^{-tA}\}_{t \geq 0}$. Moreover, for $t > 0$ the operator $e^{-tA}$ is bounded from $H^{s,p}_{\text{sol}}$ into $H^{s',p}_{\text{sol}}$ with $s' > s$ and there exists a constant $M$ (depending on $s' - s$ and $p$) such that

$$\|e^{-tA}\|_{\mathcal{L}(H^{s,p}_{\text{sol}}, H^{s',p}_{\text{sol}})} \leq M(1 + t^{-(s'-s)/2}).$$
We have $A : H^1 \to H^{-1}$ as a linear bounded operator and

$$\langle Av, v \rangle = \| \nabla v \|^2_{L^2}, \quad v \in H^1,$$

where

$$\| \nabla v \|^2_{L^2} = \sum_{k=1}^{d} \| \nabla v^k \|^2_{L^2}, \quad v \in H^1$$

and, in general, $\langle \cdot, \cdot \rangle$ denotes the $(H^{s,p})' - H^{s,p}$ duality bracket.

Moreover

(2.4) $\| v \|^2_{H^1} = \| v \|^2_{L^2} + \| \nabla v \|^2_{L^2}.$

We define the bilinear operator $B : H^1 \times H^1 \to H^{-1}$ via the trilinear form

$$\langle B(u, v), z \rangle = \int_{\mathbb{R}^d} (u(\xi) \cdot \nabla) v(\xi) \cdot z(\xi) \ d\xi.$$ 

This operator $B$ is bounded and

(2.5) $\langle B(u, v), z \rangle = -\langle B(u, z), v \rangle, \quad \langle B(u, v), v \rangle = 0$

for any $u, v, z \in H^1$. Moreover, $B$ can be extended to a bounded bilinear operator from $H^{0,4}_{sol} \times H^{0,4}_{sol}$ to $H^{-1}$ with

(2.6) $\| B(u, v) \|_{H^{-1}} \leq \| u \|_{L^4} \| v \|_{L^4}$

and, for any $a > \frac{d}{2} + 1$, $B$ can be extended to a bounded bilinear operator from $H \times H$ to $H^{-a}$ with

(2.7) $\| B(u, v) \|_{H^{-a}} \leq C \| u \|_{L^2} \| v \|_{L^2}$.

Once and for all we denote by $C$ a generic constant, which may vary from line to line; we number it if we need to identify it.

Finally, we define the noise forcing term. Given a real separable Hilbert space $Y$ we consider a $Y$-cylindrical Wiener process $\{w(t)\}_{t \geq 0}$ defined on a stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t \geq 0}, \mathbb{P})$ where the filtration is right continuous.

For the covariance of the noise we make the following assumptions: there exists $g \in (0, 1)$ such that

(G1) the mapping $G : H \to \gamma(Y; H^{-g})$ is well defined and

$$\sup_{v \in H} \| G(v) \|_{\gamma(Y; H^{-g})} =: K_{g,2} < \infty$$

(G2) the mapping $G : H \to \gamma(Y; H^{g,4}_{sol})$ is well defined and

$$\sup_{v \in H} \| G(v) \|_{\gamma(Y; H^{g,4}_{sol})} =: K_{g,4} < \infty$$

(G3) if assumption (G1) holds, then for any $\phi \in H^{-g}$ the mapping $H \ni v \mapsto G(v)^{\ast} \phi \in Y$ is continuous when in $H$ we consider the Fréchet topology inherited from the space $H_{loc}$ or the weak topology of $H$.
(G4) if assumption (G1) holds, then $G$ extends to a Lipschitz continuous map $G : H^{-g} \to \gamma(Y; H^{-g})$, i.e.

$$\exists L_g > 0 : \|G(v_1) - G(v_2)\|_{\gamma(Y; H^{-g})} \leq L_g \|v_1 - v_2\|_{H^{-g}}$$

for any $v_1, v_2 \in H^{-g}$.

We remind that the case $g = 0$ in (G1) would give a Hilbert-Schmidt operator in $H$, which is the case considered in [6, 7].

2.2. Abstract formulation. Projecting equation (1.1) onto the space of divergence free vector fields, we get the abstract form of the stochastic damped Navier-Stokes equations (1.1)

$$dv(t) + [Av(t) + \gamma v(t) + B(v(t), v(t))]dt = G(v(t))dw(t) + f(t)dt$$

with initial velocity $v(0) : \Omega \to H$ having the law $\mu$. Here $\gamma > 0$ is fixed and for simplicity we have put $\nu = 1$. The case $\gamma = 0$ was considered in our previous paper [4]. Now we add the damping term in order to investigate the existence of a stationary solution; this is not necessary in Poincaré (in particular bounded) domains.

We define a martingale solution on a finite time interval, assuming (G1) and $f \in L^1(0, T; H^{-1})$, $\mu(H) = 1$.

Definition 2.1 (martingale solution). We say that there exists a martingale solution of equation (2.8) on the time interval $[0, T]$ with initial velocity of law $\mu$ if there exist

- a stochastic basis $(\hat{\Omega}, \hat{\mathbb{F}}, \{\hat{\mathbb{F}}_t\}_{t \in [0, T]}, \hat{\mathbb{P}})$
- a $Y$-cylindrical Wiener process $\hat{w}$
- a progressively measurable process $\hat{v} : [0, T] \times \hat{\Omega} \to H$ with $\hat{\mathbb{P}}$-a.e. path $\hat{v} \in C([0, T]; H_\omega) \cap L^2(0, T; L^4)$

and $\hat{v}(0)$ has law $\mu$; moreover for any $t \in [0, T], \psi \in H^2$, $\hat{\mathbb{P}}$-a.s.,

$$\langle \hat{v}(t), \psi \rangle_H + \int_0^t \langle A\hat{v}(s), \psi \rangle ds + \gamma \int_0^t \langle \hat{v}(s), \psi \rangle_H ds$$

$$+ \int_0^t \langle B(\hat{v}(s), \hat{v}(s)), \psi \rangle ds$$

$$= \langle \hat{v}(0), \psi \rangle_H + \int_0^t \langle f(s), \psi \rangle ds + \int_0^t G(\hat{v}(s)) d\hat{w}(s), \psi \rangle.$$  

(2.9)

All the terms in (2.9) make sense; in particular the trilinear term is well defined thanks to (2.6) which provides the following estimate

$$\left| \int_0^t \langle B(\hat{v}(s), \hat{v}(s)), \psi \rangle ds \right| \leq \|\psi\|_{H^1} \int_0^t \|\hat{v}(s)\|_{L^4}^2 ds.$$

However, in the following sections we shall prove the existence of a martingale solution with a time integrability higher than $\hat{v} \in L^2(0, T; L^4)$. As far as the stochastic integral
is concerned, by assuming (G1) we obtain that the random variable \( \int_0^t G(\hat{v}(s)) \, d\hat{w}(s) \) belongs to \( H^{-g} \) (\( \hat{P} \)-a.s.), see Proposition 5.2 in [4].

Initial deterministic velocity \( x \in H \) corresponds to \( \mu = \delta_x \).

As far as stationary martingale solutions are concerned, we give the definition involving the time interval \( \mathbb{R}_+ = [0, \infty) \). Again, we assume (G1) and \( f \in L^1_{\text{loc}}(\mathbb{R}_+; H^{-1}) \), \( \mu(H) = 1 \).

**Definition 2.2 (stationary martingale solution).** We say that there exists a stationary martingale solution of equation (2.8) on the time interval \( \mathbb{R}_+ \) with initial velocity of law \( \mu \) if there exist

- a stochastic basis \((\hat{\Omega}, \hat{\mathbb{F}}, \{\hat{\mathbb{F}}_t\}_{t \geq 0}, \hat{\mathbb{P}})\)
- a \( Y \)-cylindrical Wiener process \( \hat{w} \)
- a stationary and progressively measurable process \( \hat{v} : \mathbb{R}_+ \times \hat{\Omega} \to H \) with \( \hat{\mathbb{P}} \)-a.e. path

\[
\hat{v} \in C(\mathbb{R}_+; H_w) \cap L^2_{\text{loc}}(\mathbb{R}_+; L^4)
\]

and \( \hat{v}(0) \) has law \( \mu \); moreover, for any \( t > 0 \) and \( \psi \in H^2 \) equation (2.9) holds \( \hat{\mathbb{P}} \)-a.s.

The first result is about the existence of a martingale solution. Moreover for \( d = 2 \) there is pathwise uniqueness. Now, our aim is to prove the existence of a martingale stationary solution for \( d = 2, 3 \); actually, for \( d = 2 \) we prove something more, that is the existence of an invariant measure.

### 2.3. Compactness results

In this section we fix \( p \in (1, \infty) \) and \( \beta, \delta \in (0, \infty) \). For any \( T > 0 \) let us define the space

\[
Z_T = C([0, T]; H_w) \cap L^2(0, T; H_{\text{loc}}) \cap L^p_w(0, T; L^4) \cap C([0, T]; U')
\]

which is a locally convex topological space with the topology \( \mathcal{T}_T \) given by the supremum of the corresponding topologies. Let us also define a function \( |\cdot| : Z_T \to [0, \infty] \) by

\[
|v|_T = \|v\|_{L^\infty(0,T;H)} + \|v\|_{L^2(0,T;H^\beta)} + \|v\|_{L^p(0,T;L^4)} + \|v\|_{C^\beta([0,T];H^{-1})},
\]

if the RHS above is finite, and, otherwise, \( |v|_T = \infty \). Notice that the function \( |\cdot|_T \) is not the natural norm of \( Z_T \). But it is used in the following compactness result.

**Lemma 2.3.** For any \( a > 0 \) the set

\[
K_T(a) = \{v \in Z_T : |v|_T \leq a\}
\]

is a metrizable compact subset of \( Z_T \).

**Proof.** Since \( K_T(a) \) is a bounded subset of \( Z_T \), it is metrizable for each of the four topologies involved in the definition of \( \mathcal{T}_T \). The compactness result comes from Lemma 5.4 in [4]. \( \square \)
Let us point out that we considered $L^2(0,T; H_{\text{loc}})$ to be the space $L^2(0,T; H)$ with the topology generated by the semi-norms $\| v \|_{L^2(0,T; H_N)}$, $N \in \mathbb{N}$, since the space $K_T(a)$ involves a boundedness in the $L^2(0,T; H^3)$-norm; we know that any bounded sequence in $\| v \|_{L^2(0,T; H^3)}$ has a subsequence weakly converging and its limit belongs to $L^2(0,T; H^3)$ which is contained in $L^2(0,T; H)$.

Working in the unbounded time interval $\mathbb{R}_+$, we have a similar compactness result. We consider the following locally convex topological spaces:

1. $C(\mathbb{R}_+; H_w)$ with the topology generated by the family of semi-norms
   \[ \| v \|_{N,h} = \sup_{0 \leq t \leq N} |\langle v(t), h \rangle|, \quad N \in \mathbb{N}, h \in H; \]
   - $L^2_{\text{loc}}(\mathbb{R}_+; H_{\text{loc}})$ is the space $L^2_{\text{loc}}(\mathbb{R}_+; H)$ with metric
     \[ d(u, v) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{1}{1 + \| u - v \|_{L^2(0,N; H_N)}}; \]
   - $L^p_{\text{loc},w}(\mathbb{R}_+; L^4)$ is the space $L^p_{\text{loc}}(\mathbb{R}_+; L^4)$ with the topology generated by the family of semi-norms, with $\frac{1}{p} + \frac{1}{p'} = 1$,
     \[ \| v \|_{N,h} = \left| \int_0^N \int_{\mathbb{R}^3} v(t, \xi) h(t, \xi) dt d\xi \right|, \quad N \in \mathbb{N}, h \in L^{p'}(0,N; L^4); \]
   - $C(\mathbb{R}_+; U')$ with metric $d(u, v) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{1}{1 + \| u - v \|_{C([0,N]; U')}}.$

We define the space
\[ Z = C(\mathbb{R}_+; H_w) \cap L^2_{\text{loc}}(\mathbb{R}_+; H_{\text{loc}}) \cap L^p_{\text{loc},w}(\mathbb{R}_+; L^4) \cap C(\mathbb{R}_+; U'). \]
It is a locally convex topological space with the topology $\mathcal{T}$ given by the supremum of the corresponding topologies.

We have this compactness result.

**Lemma 2.4.** For any sequence $\alpha = \{ \alpha_N \}_{N \in \mathbb{N}}$ of positive numbers, the set
\[ K(\alpha) = \{ v \in Z : \| v \|_{N} \leq \alpha_N \text{ for any } N \in \mathbb{N} \} \]
is a metrizable compact subset of $Z$.

**Proof.** It is enough to know that each $K_N(\alpha_N)$ is a metrizable compact subset of $Z_N$, see, e.g., a similar case in Corollary B.2 of [24]. But this is Lemma 2.3. \hfill $\Box$

2.4. **Invariant measures.** Here we suppose that given the initial velocity $x \in H$ there exists a unique solution to equation (2.8). We denote by $v(t; x)$ this solution at time $t > 0$ and assume that the deterministic forcing term $f$ is independent of time. Therefore we define the family $\{ P_t \}_{t \geq 0}$,
\[ (P_t \phi)(x) = E[\phi(v(t; x))] \]
for any $\phi \in B_b(H)$, i.e. $\phi : H \to \mathbb{R}$ is a bounded and Borel measurable function.
It is obvious that $P_t\phi$ is bounded for every $\phi \in B_b(H)$. Moreover, it is also measurable when the weak existence and uniqueness in law hold for equation (2.8), see Corollary 23 in [23] (which generalizes to the infinite dimensional setting the finite dimensional result of Stroock and Varadhan [26]). Let us point out that if this unique solution to equation (2.8) has a.e. path in $C([0,T];H)$, then it is also a Markov process, see Theorem 27 in [23].

First we introduce four classes of functions $\phi : H \to \mathbb{R}$ which are continuous with respect to different topologies:

(i) A function $\phi \in C(H)$ iff $\phi$ is continuous w.r.t. to strong topology on $H$;

(ii) a function $\phi \in C(H_w)$ iff $\phi$ is continuous w.r.t. to weak topology on $H$;

(iii) a function $\phi \in SC(H_w)$ iff $\phi$ is sequentially continuous w.r.t. to weak topology on $H$;

(iv) a function $\phi \in C(H_{bw})$ iff $\phi$ is continuous w.r.t. to bounded weak topology on $H$, which is the finest topology on $H$, whose family of closed sets agrees with the family of closed sets from the weak topology on closed balls in $H$.

It is known, see e.g. [22], that

(2.13) \[ SC(H_w) = C(H_{bw}). \]

Note also that the following inclusions hold

\[ C(H_w) \subset C(H_{bw}) \subset C(H). \]

When we add the subscript $b$ we mean that the function is also bounded, for instance $C_b(H_{bw}) = \{ \phi \in C(H_{bw}) : \phi \text{ is bounded} \}$.

Thus, we get the following inclusions

(2.14) \[ C_b(H_w) \subset C_b(H_{bw}) \subset C_b(H). \]

Let us also notice that because $H$ is a separable space, the weak Borel and the (strong) Borel $\sigma$ fields on $H$ are equal, i.e. $\mathcal{B}(H) = \mathcal{B}(H_w)$, see Theorem 7.19 in [29] and [14] for more general claims. On the other hand, since the strong topology is finer than the bw-topology which in turn is finer than the weak topology, we infer that $\mathcal{B}(H disc) \supseteq \mathcal{B}(H_{bw}) \supseteq \mathcal{B}(H_w)$. Thus we proved that

(2.15) \[ \mathcal{B}(H) = \mathcal{B}(H_{bw}) = \mathcal{B}(H_w). \]

We infer that the sets of $\mathcal{B}(H_w)$, $\mathcal{B}(H_{bw})$ and $\mathcal{B}(H)$-measurable functions are equal. Hence

(2.16) \[ B_b(H) = B(H_{bw}) = B_b(H_w). \]

Let us now recall the following fundamental definition, see [21]. We say that a family $\{ P_t \}_{t \geq 0}$ is sequentially weakly Feller iff

(2.17) \[ P_t : SC_b(H_w) \to SC_b(H_w), \quad t \geq 0. \]

Thanks to the results of Ondreját [23] quoted before, we deduce that the family $\{ P_t \}_{t \geq 0}$ is also a Markov semigroup.
Given a sequentially weakly Feller Markov semigroup on a separable Hilbert space $H$, we can define an invariant measure $\mu$ for equation (2.8) as a Borel probability measure on $H$ such that for any time $t \geq 0$

\begin{equation}
\int_H P_t \phi \, d\mu = \int_H \phi \, d\mu, \quad \forall \phi \in C_b(H_w).
\end{equation}

This definition is meaningful as the LHS of (2.18) makes sense. Indeed, if $\phi \in C_b(H_w) \subset SC_b(H_w)$, then by (2.17) and (2.13) $P_t \phi \in SC_b(H_w) = C_b(H_{bw}) \subset B_b(H_{bw}) = B_b(H)$; therefore the integral in the LHS is well defined.

Moreover, it is well known that the set of continuous functions is a determining set for a Borel measure. In particular, the set $C_b(H_w)$ is a determining set for the measure $\mu$, i.e. if $\mu_i$, $i = 1, 2$ are two Borel probability measures on $B_b(H_w)$ such that $\int_H \phi \, d\mu_1 = \int_H \phi \, d\mu_2$, for every $\phi \in C_b(H_w)$, then $\mu_1 = \mu_2$. Therefore, relationship (2.18) corresponds to the invariance (in time) of the law of the random variable $v(t; x)$.

**Remark 2.5.** Often the definition of invariant measure is given when the Markov semigroup is Feller, i.e. $P_t : C_b(H) \to C_b(H)$ for any $t \geq 0$, see, e.g. [13]. Indeed, the set $C_b(H_w)$ is still a determining set for the measure $\mu$ and the LHS of (2.18) makes sense since $P_t \phi \in C_b(H_w) \subset B_b(H_w)$.

3. **Invariant measures for $d = 2$**

We consider equation (2.8) in $\mathbb{R}^2$. In [4] we proved the existence and uniqueness of solutions for this equation when $\gamma = 0$. Dealing now with the case $\gamma > 0$, we can prove in the same way the same result. Indeed, as in [4] we split the analysis of equation (2.8) for $v$ in two subproblems involving the processes $z$ and $u$ (with $v = z + u$), where

\begin{align}
(3.1) & \quad dz(t) + Az(t) \, dt + \gamma z(t) \, dt = G(v(t)) \, dw(t), \quad t \in (0, T]; \quad z(0) = 0 \\
(3.2) & \quad \frac{du}{dt}(t) + Au(t) + \gamma u(t) + B(v(t), v(t)) = f(t), \quad t \in (0, T]; \quad u(0) = x
\end{align}

The solution of equation (3.1) is

\[ z(t) = \int_0^t e^{-\gamma(t-s)} e^{-(t-s)A} G(v(s)) \, dw(s) \]

and the basic energy equality for equation (3.2) is

\begin{align}
(3.3) & \quad \frac{1}{2} \frac{d}{dt} ||u(t)||_H^2 + ||\nabla u(t)||_H^2 + \gamma ||u(t)||_H^2 \\
& \quad \quad = - \langle B(z(t) + u(t), z(t) + u(t)), u(t) \rangle + \langle f(t), u(t) \rangle.
\end{align}

This shows that we can work on equations (3.1) and (3.2) as we did in [4] when $\gamma = 0$. In this way, we can prove first the existence of a martingale solution and then
the pathwise uniqueness, see Theorem 4.2 in [4]. Hence by invoking the Yamada-Watanabe Theorem, see [18], we deduce the existence of a strong solution too (in the probabilistic sense).

**Theorem 3.1.** Let \( d = 2 \). If \( x \in H \), \( f \in L^p_{\text{loc}}([0, \infty); H^{-1}) \) for some \( p > 2 \) and assumptions (G1)-(G2)-(G3) are satisfied, then there exists a martingale solution \( (\tilde{\Omega}, \tilde{\mathbb{F}}, \{\tilde{\mathbb{F}}_t\}_{t \in [0, \infty)}, \tilde{\mathbb{P}}), \tilde{w}, \tilde{v} \) of equation (2.8) on the time interval \([0, \infty)\) with deterministic initial velocity \( x \). Moreover, the solution \( \tilde{v} \) satisfies the following

\[
\tilde{v} \in C([0, \infty); H) \cap L^4_{\text{loc}}([0, \infty); H^{1-g}) \cap L^4_{\text{loc}}([0, \infty); L^4) \quad \tilde{\mathbb{P}} \text{-a.s.}
\]

Finally, assuming also (G4) there is pathwise uniqueness for equation (2.8).

**Remark 3.2.** The above result has been proved on a fixed time interval. But it is not difficult to modify the proof in the spirit of our proofs of Theorems 4.4 and 4.6 so that the result holds true also on the whole time interval. We have formulated our result in such a form.

The same comment applies to Theorem 4.2.

For deterministic initial velocity \( x \in H \), we denote by \( v(t; x) \) the solution at time \( t > 0 \).

Let us recall a result of Maslowski-Seidler [21] about the existence of an invariant measure. This is a modification of the Krylov-Bogoliubov technique, see [2] and [12], the latter being successful in bounded domains.

**Theorem 3.3.** Assume that
i) the semigroup \( \{P_t\}_{t \geq 0} \) is sequentially weakly Feller in \( H \);
ii) for any \( \varepsilon > 0 \) there exists \( R > 0 \) such that

\[
\sup_{T \geq 1} \frac{1}{T} \int_0^T \mathbb{P}(\|v(t; 0)\|_H > R) dt < \varepsilon.
\]

Then there exists at least one invariant measure for equation (2.8).

Below we will verify the two assumptions i) and ii) in order to prove

**Theorem 3.4.** Let \( d = 2 \). If \( f \in H^{-1} \) and assumptions (G1)-(G2)-(G3)-(G4) are satisfied, then there exists at least one invariant measure \( \mu \) for equation (2.8) and \( \mu(H) = 1 \).

3.1. Sequentially weakly Feller. We need to verify, under the assumptions of Theorem 3.4, that the Markov semigroup \( \{P_t\}_{t \geq 0} \) is sequentially weakly Feller in \( H \). This means that for any \( t > 0 \) and any bounded and sequentially weakly continuous function \( \phi : H \to \mathbb{R} \),

\[
\text{if } x_k \to x \text{ in } H, \text{ then } P_t \phi(x_k) \to P_t \phi(x).
\]
Let us fix $0 < t < T < \infty$. We are given a sequence $\{x_k\}_{k \in \mathbb{N}}$ weakly convergent in $H$ to $x$. For each $k \in \mathbb{N}$, let $v_k$ be the strong solution of (2.8) on the time interval $[0, T]$ with initial velocity $x_k$, given by Theorem 3.1. Set $v_k = z_k + u_k$, with

$$
\begin{align*}
\frac{dz_k(t)}{dt} + Az_k(t) dt + \gamma z_k(t) dt &= G(v_k(t)) dw(t), t \in [0, T], \\
z_k(0) &= 0
\end{align*}
$$

and

$$
\begin{align*}
\frac{du_k(t)}{dt} + Au_k(t) dt + \gamma u_k(t) dt &= B(v_k(t), v_k(t)) f, t \in [0, T]; \\
u_k(0) &= x_k.
\end{align*}
$$

First, we look for bounds in probability for $u_k$ and $z_k$, uniform in $k$ in order to get tightness and then convergence of the sequence $\{v_k\}_k$. This will lead to prove that $P_t \phi(x_k) \equiv \mathbb{E}[\phi(v_k(t; x_k))] \to \mathbb{E}[\phi(v(t; x))] \equiv P_t \phi(x)$ for any bounded and sequentially weakly continuous function $\phi : H \to \mathbb{R}$.

We proceed as in [4]; actually, the role of $\gamma > 0$ is negligible in this section.

As far as the processes $z_k$ are concerned, we appeal to Lemma 3.2 and Lemma 3.3 in [4]; indeed

$$z_k(t) = \int_0^t e^{-\gamma(t-s)} e^{-(t-s)A} G(v_k(s)) dw(s)$$

so each process $z_k$ depends on $k$ through the process $v_k$ only (now the operator $G$ is fixed) and we can proceed as in [4]; the operator $A + \gamma$ is no worse than the operator $A$. This means that, given $\beta, \delta \geq 0$ such that

$$\beta + \frac{\delta}{2} < \frac{1 - g}{2},$$

we have

$$\sup_k \mathbb{E}\|z_k\|_{C^\beta([0,T]; H^s)} < \infty \quad \text{and} \quad \sup_k \mathbb{E}\|z_k\|_{L^4(0,T; L^4)} < \infty.$$ 

This implies that for any $\varepsilon > 0$ there exist positive constants $R_i = R_i(\varepsilon), i = 1, 2, 3, 4$, such that

$$\sup_k \mathbb{P}(\|z_k\|_{C^\beta([0,T]; H^{-1})} > R_1) \leq \varepsilon,$$

$$\sup_k \mathbb{P}(\|z_k\|_{L^2(0,T; H^s)} > R_2) \leq \varepsilon,$$

$$\sup_k \mathbb{P}(\|z_k\|_{L^\infty(0,T; H)} > R_3) \leq \varepsilon,$$

$$\sup_k \mathbb{P}(\|z_k\|_{L^4(0,T; L^4)} > R_4) \leq \varepsilon.$$
Hence, for any \( \varepsilon > 0 \), with \( u_k \), we get
\[
\frac{1}{2} \frac{d}{dt} \| u_k(t) \|_H^2 + \| \nabla u_k(t) \|_H^2 + \gamma \| u_k(t) \|_H^2 = - \langle B(z_k(t) + u_k(t), z_k(t) + u_k(t)), u_k(t) \rangle + \langle f, u_k(t) \rangle.
\]

Estimating the RHS as in the proof of Proposition 3.4 in [4] and neglecting the positive term \( \gamma \| u_k(t) \|_H^2 \) in the LHS, we obtain
\[
\sup_{0 \leq t \leq T} \| u_k(t) \|_H^2 \leq \| x_k \|_H^2 e^{\int_0^T \phi_k(r)dr} + \int_0^T e^{\int_s^T \phi_k(r)dr} \psi_k(s)ds \leq e^{\int_0^T \phi_k(r)dr} \left[ \| x_k \|_H^2 + \int_0^T \psi_k(s)ds \right]
\]
where \( \phi_k(t) = 1 + C_1 \| z_k(t) \|_L^4 \) and \( \psi_k(t) = C_2(\| z_k(t) \|_L^4 + \| f \|_H^{-1}) \) for suitable constants \( C_1 \) and \( C_2 \) independent of \( k \), and
\[
\int_0^T \| \nabla u_k(t) \|_{L^2} dt \leq \| x_k \|_H^2 + \int_0^T (\phi_k(t) \| u_k(t) \|_H^2 + \psi_k(t)) dt \leq \| x_k \|_H^2 + E \int_0^T \phi_k(t) dt + \int_0^T \psi_k(t) dt.
\]

Since the sequence \( \{ x_k \}_k \) is weakly convergent in \( H \), we have \( \sup_k \| x_k \|_H < \infty \); then, from (3.10), (3.12), (3.13) we get uniform bound for \( \| u_k \|_{L^\infty(0,T;H)} \) and \( \| u_k \|_{L^2(0,T;H^1)} \). Proceeding as in the proof of Proposition 3.4 in [4], we also get uniform bounds for \( \| u_k \|_{C^{1,2}([0,T];H^{-1})} \) and \( \| u_k \|_{L^4(0,T;L^4)} \).

Now we sum up our results for \( z_k \) and \( u_k \). Let us choose \( \beta \in (0, \frac{1}{2}] \) and \( \delta \in (0,1] \) fulfilling (3.6) so we also have \( C^2([0,T];H^{-1}) \subseteq C^\beta([0,T];H^{-1}) \) and \( L^2(0,T;H^1) \subseteq L^2(0,T;H^\delta) \). Hence for \( v_k = z_k + u_k \) we have the following result:

for any \( \varepsilon > 0 \) there exist positive constants \( R_i = R_i(\varepsilon), i = 5, \ldots, 8 \), such that
\[
\sup_k \mathbb{P}(\| v_k \|_{L^\infty(0,T;H)} > R_5) \leq \varepsilon,
\]
\[
\sup_k \mathbb{P}(\| v_k \|_{L^2(0,T;H^\delta)} > R_6) \leq \varepsilon,
\]
\[
\sup_k \mathbb{P}(\| v_k \|_{L^4(0,T;L^4)} > R_7) \leq \varepsilon,
\]
\[
\sup_k \mathbb{P}(\| v_k \|_{C^\beta([0,T];H^{-1})} > R_8) \leq \varepsilon.
\]

Hence, for any \( \varepsilon > 0 \) there exist \( R = R(\varepsilon) > 0 \) such that
\[
\sup_k \mathbb{P} \left( \| v_k \|_{L^\infty(0,T;H)} + \| v_k \|_{L^2(0,T;H^\delta)} + \| v_k \|_{L^4(0,T;L^4)} + \| v_k \|_{C^\beta([0,T];H^{-1})} > R \right) \leq \varepsilon.
\]
Bearing in mind Lemma 2.3, we obtain that the sequence of laws of the processes \(v_k\) is tight in \(Z_T\), see also Lemma 5.5 in [4]. Now we appeal to the Jakubowski’s [19] generalization of the Skorohod Theorem to nonmetric spaces. We can do so since there exists a countable family \(\{f_i : Z_T \to \mathbb{R}\}\) of \(T_T\)-continuous functions, which separate points of \(Z_T\), see the proof of Corollary 3.12 in [6]. Therefore, there exist a subsequence \(\{v_{k_j}\}_{j=1}^{\infty}\), a stochastic basis \((\tilde{\Omega}, \tilde{\mathbb{P}}, \{\tilde{\mathbb{P}}\}_{0 \leq t \leq T}, \mathbb{P})\), \(Z_T\)-valued Borel measurable variables \(\tilde{v}\) and \(\{\tilde{v}_j\}_{j=1}^{\infty}\) such that for any \(j\) the laws of \(v_{k_j}\) and \(\tilde{v}_j\) are the same and \(\tilde{v}_j\) converges to \(\tilde{v}\) (\(\mathbb{P}\)-a.s.) with the topology \(T_T\). Moreover, one proves as in [4] that \(\tilde{v}\) coincides with the solution of (2.8) with initial velocity \(v\).

In particular, for fixed \(t\), \(\tilde{v}_j(t; x_k)\) weakly converges in \(H\) to \(\tilde{v}(t; x)\), \(\mathbb{P}\)-a.s., according to the fact that \(\tilde{v}_j \to \tilde{v}\) in \(C([0, T]; H_w)\). Hence, given any sequentially weakly continuous function \(\phi : H \to \mathbb{R}\), we have that \(\phi(\tilde{v}_j(t; x_k)) \to \phi(\tilde{v}(t; x))\) \(\mathbb{P}\)-a.s. and therefore, when \(\phi\) is also bounded, by invoking the Lebesque Dominated Convergence (LDC) theorem we deduce that \(\tilde{\mathbb{E}}[\phi(\tilde{v}_j(t; x_k))] \to \tilde{\mathbb{E}}[\phi(\tilde{v}(t; x))]\). Using that \(\tilde{v}\) has the same law as \(v\) and \(\tilde{v}_j\) has the same law as \(v_{k_j}\), we infer that \(\mathbb{E}[\phi(v_{k_j}(t; x_k))] \to \mathbb{E}[\phi(v(t; x))]\). Moreover, by the uniqueness the whole sequence converges. This proves the sequentially weakly Feller property in \(H\).

3.2. Boundedness in probability. We need to verify, under the assumptions of Theorem 3.1, that for any \(\varepsilon > 0\) there exists \(R = R(\varepsilon) > 0\) satisfying

\[
\sup_{T \geq 1} \frac{1}{T} \int_0^T \mathbb{P}(\|v(t; 0)\|_H > R) \, dt < \varepsilon.
\]

We define the modified Ornstein-Uhlenbeck equation

\[
dz(t) + Az(t) \, dt + (\gamma + \alpha)z(t) \, dt = G(v(t)) \, dw(t)
\]

with an additional damping \(\alpha > 0\), to be chosen later on. Here \(v\) is the strong solution to equation (2.8) with zero initial velocity, given in Theorem 3.1. We consider the stochastic convolution integral

\[
\zeta^\alpha(t) = \int_0^t e^{-(\gamma + \alpha)(t-s)} e^{-(t-s)A} G(v(s)) \, dw(s)
\]

solving (3.15) with vanishing initial data and find good estimates on it. This result is independent of the space dimension \(d\).

**Lemma 3.5.** Assume conditions (G1)-(G2) and let \(v\) be a continuous \(H\)-valued process. Then for any \(\alpha \geq 0\) and \(q \geq 2\) there exist positive constants \(C_{\alpha,2}, C_{\alpha,q,4}\) (depending also on \(g\) and \(\gamma\)) such that for the process \(\zeta^\alpha\) given by (3.16) we have

\[
\mathbb{E}\|\zeta^\alpha(t)\|^2_H \leq C_{\alpha,2}, \quad \mathbb{E}\|\zeta^\alpha(t)\|^q_{H^0_\text{sol}} \leq C_{\alpha,q,4}
\]

for any \(t \geq 0\). Moreover

\[
\lim_{\alpha \to +\infty} C_{\alpha,2} = 0, \quad \lim_{\alpha \to +\infty} C_{\alpha,q,4} = 0.
\]
Proof. First, we find the estimate for $\mathbb{E}\|\zeta^\alpha(t)\|_H^2$. Using inequality (2.3) and assumption (G1) we get

$$
\mathbb{E}\|\zeta^\alpha(t)\|_H^2 \leq \mathbb{E}\left[\int_0^t e^{-\gamma(t-s)} e^{-(t-s)A} G(v(s)) \|_{\gamma(Y; H)}^2 ds\right]
$$

$$
\leq K^2_{g,2} \int_0^t e^{-\gamma(t-s)} e^{-(t-s)A} \| J^g e^{-(t-s)A} \|_{L(H; H)} \| J^g G(v(s)) \|_{\gamma(Y; H)}^2 ds
$$

$$
= 2K^2_{g,2} M^2 \int_0^t e^{-\gamma(t-s)} \left[1 + \frac{1}{r^g}\right] dr
$$

$$
\leq 2K^2_{g,2} M^2 \int_0^\infty e^{-\gamma(t-s)} \left[1 + \frac{1}{r^g}\right] dr = 2K^2_{g,2} M^2 \int_0^\infty e^{-\gamma(t-s)} \left[1 + \frac{1}{r^g}\right] dr
$$

Calling $C_{\alpha,2}$ the expression on the RHS, because $g < 1$ we deduce the first limit behaviour (3.18) by the LDC theorem as $\alpha \to \infty$.

The second estimate in (3.18) is obtained in the same way along the lines of the proof of Lemma 3.2 in [4]. Indeed, using again inequality (2.3) and assumption (G1) we get

$$
\mathbb{E}\|\zeta^\alpha(t)\|_{H^{sol}_{0,4}}^2 \leq C(q) \mathbb{E}\left[\int_0^t e^{-\gamma(t-s)} \| e^{-(t-s)A} G(v(s)) \|_{\gamma(Y; H^{sol}_{0,4})}^2 ds\right]^{q/2}
$$

$$
\leq C(q) \mathbb{E}\left[\int_0^t e^{-\gamma(t-s)} \| J^g e^{-(t-s)A} \|_{L(H^{sol}_{0,4}; H^{sol}_{0,4})} \| J^g G(v(s)) \|_{\gamma(Y; H^{sol}_{0,4})}^2 ds\right]^{q/2}
$$

$$
\leq C(q) (K^2_{g,4})^q \left[\int_0^t e^{-\gamma(t-s)} \left(2 \left(M^2 + \frac{M^2}{(t-s)^g}\right)\right) ds\right]^{q/2}
$$

We estimate the time integral as in the previous case to conclude the proof. □

By the Chebyshev inequality, from (3.17) we get

$$
\sup_{t \geq 0} \mathbb{P}\left(\|\zeta^\alpha(t)\|_H > R\right) \leq \frac{C_{\alpha,2}}{R^2}
$$

for any $R > 0$. This gives the bound (3.14) for the process $\zeta^\alpha$.

Now we look for a similar result for the process $u^\alpha = v - \zeta^\alpha$ solving

$$
\frac{du}{dt}(t) + Au(t) + \gamma u(t) + B(v(t), v(t)) = \alpha \zeta^\alpha(t) + f, \quad u(0) = 0.
$$

For this aim we need the following result
Proposition 3.6. Let $f \in H^{-1}$ and let $u^\alpha$ be the solution of

$$
\frac{du}{dt}(t) + Au(t) + \gamma u(t) + B(u(t) + \zeta(t), u(t) + \zeta(t)) = \alpha \zeta(t) + f
$$

with $u^\alpha(0) = 0$ and $\zeta$ given by (3.16) under the assumptions of Lemma 3.5. Then for any $\varepsilon > 0$ there exist $\alpha, R > 0$ such that

$$
\frac{1}{T} \int_0^T \mathbb{P}(\|u^\alpha(t; 0)\|_H > R) dt < \varepsilon
$$

for any $T > 0$.

Proof. We proceed as in the proof of Proposition 3.4 in [4]. We take the $H$-scalar product of equation (3.20) with $u^\alpha$ and get

$$
\frac{1}{2} \frac{d}{dt} \|u^\alpha(t)\|_H^2 + \|\nabla u^\alpha(t)\|_{L^2}^2 + \gamma \|u^\alpha(t)\|_H^2 = -\langle B(u^\alpha(t) + \zeta(t), u^\alpha(t) + \zeta(t)), u^\alpha(t) \rangle + \alpha \langle \zeta(t), u^\alpha(t) \rangle + \langle f, u^\alpha(t) \rangle
$$

$$
= \langle B(u^\alpha(t), u^\alpha(t)), \zeta(t) \rangle + \langle B(\zeta(t), u^\alpha(t)), \zeta(t) \rangle + \alpha \langle \zeta(t), u^\alpha(t) \rangle + \langle f, u^\alpha(t) \rangle
$$

$$
\leq C \|u^\alpha(t)\|_H^2 \|\nabla u^\alpha(t)\|_{L^2}^2 \|\zeta(t)\|_{L^4}^4 + \|\nabla u^\alpha(t)\|_{L^2} \|\zeta(t)\|_{L^4}^2 + \alpha \|u^\alpha(t)\|_H \|\zeta(t)\|_H + \|u^\alpha(t)\|_{H^1} \|f\|_{H^{-1}}
$$

$$
\leq \frac{1}{2} \|\nabla u^\alpha(t)\|_{L^2}^2 + \frac{\gamma}{2} \|u^\alpha(t)\|_H^2 + C \|\zeta(t)\|_{L^4}^4 \|u^\alpha(t)\|_H^2 + C \|\zeta(t)\|_{L^4}^4 \|u^\alpha(t)\|_H^2 + C \|\zeta(t)\|_{L^4}^4
$$

$$
+ C \|\zeta(t)\|_{H^1}^2 + C \|f\|_{H^{-1}}^2.
$$

Hence

$$
\frac{d}{dt} \|u^\alpha(t)\|_H^2 \leq -\gamma \|u^\alpha(t)\|_H^2 + C_3 \left( \|\zeta(t)\|_{L^4}^4 \|u^\alpha(t)\|_H^2 + \|\zeta(t)\|_{L^4}^4 + \alpha^2 \|\zeta(t)\|_H^2 + \|f\|_{H^{-1}}^2 \right)
$$

for a constant $C_3$ independent of $\alpha$. Therefore, following the Da Prato-Gątarek technique from [4], we infer that for every $R > 0$ the following happens

$$
\frac{d}{dt} \ln(\|u^\alpha(t)\|_H^2 \vee R)
$$

$$
= 1_{\{\|u^\alpha(t)\|_H^2 \geq R \}} \frac{1}{\|u^\alpha(t)\|_H^2} \frac{d}{dt} \|u^\alpha(t)\|_H^2
$$

$$
\leq 1_{\{\|u^\alpha(t)\|_H^2 \geq R \}} \left( -\gamma + C_3 \|\zeta(t)\|_{L^4}^2 \right)
$$

$$
+ 1_{\{\|u^\alpha(t)\|_H^2 \geq R \}} C_3 \|\zeta(t)\|_{L^4}^4 + \alpha^2 \|\zeta(t)\|_H^2 + \|f\|_{H^{-1}}^2
$$

$$
\leq 1_{\{\|u^\alpha(t)\|_H^2 \geq R \}} \left( -\gamma + C_3 \|\zeta(t)\|_{L^4}^2 \right)
$$

$$
+ \frac{C_3}{R} \left( \|\zeta(t)\|_{L^4}^4 + \alpha^2 \|\zeta(t)\|_H^2 + \|f\|_{H^{-1}}^2 \right).
$$
We integrate in time and take expectation; since \( u^\alpha(0) = 0 \) we get that the time integral in the LHS is non negative and therefore
\[
\gamma \int_0^T \mathbb{P}(\|u^\alpha(t)\|_H^2 > R) \, dt \leq C_3 T \sup_{t \geq 0} \mathbb{E}[\|\zeta^\alpha(t)\|_{L^4}^4] \\
+ \frac{C_3}{R} T \left( \sup_{t \geq 0} \mathbb{E}[\|\zeta^\alpha(t)\|_{L^4}^4] + \alpha^2 \sup_{t \geq 0} \mathbb{E}[\|\zeta^\alpha(t)\|_H^2] + \|f\|_{H^{-1}}^2 \right) \\
\leq TC_3 \left( C_{\alpha,4,4} + \frac{C_{\alpha,4,4} + \alpha^2 C_{\alpha,2} + \|f\|_{H^{-1}}^2}{R} \right).
\]

Now, bearing in mind (3.18) we find that for \( \alpha \) and \( R \) suitably chosen the quantity
\[
\frac{1}{T} \int_0^T \mathbb{P}(\|u^\alpha(t)\|_H^2 > R) \, dt
\]
can be as small as we want, uniformly in time. \( \square \)

Therefore, merging (3.19) and Proposition 3.6 we get the bound (3.14) for the process \( v = \zeta^\alpha + u^\alpha \).

4. Stationary solutions for \( d = 3 \)

Working in the whole space, we can prove only the existence but not uniqueness of martingale solutions for equation (2.8). Hence we cannot define the Markov semigroup and a fortiori even the invariant measures. But if we regularise the equation, we can prove similar results as in \( \mathbb{R}^2 \); this way of approaching the three dimensional Navier-Stokes equation by regularizing the nonlinearity in order to get an equation with the same level of difficulty as the two dimensional one, goes back to the work of Leray [20]. So we first approximate equation (2.8) by
\[
(4.1) \quad dv(t) + [Av(t) + \gamma v(t) + B_m(v(t), v(t))] \, dt = G(v(t)) \, dw(t) + f(t) \, dt
\]
with initial velocity \( v(0) = x \). The smoother operator \( B_m \) will be defined in the next section. We shall prove the existence of a unique solution for the approximating equation (4.1). Moreover, as in the \( \mathbb{R}^2 \) case, there exists at least one invariant measure \( \mu_m \). Considering the sequence \( \{v_m\}_{m \in \mathbb{N}} \) of stationary solutions of (4.1) whose marginals at a fixed time are \( \mu_m \), we shall pass to the limit as \( m \to \infty \) in order to get a stationary solution for the original equation (2.8).

4.1. Smoothing. In this section we investigate the smoothed equation (4.1). We define the regularization as follows. For any \( m > 0 \) let \( \rho_m(\xi) = \left(\frac{m}{2\pi}\right)^\frac{3}{2} e^{-\frac{m}{4}|\xi|^2} \), \( \xi \in \mathbb{R}^3 \), and
\[
B_m(u, v) = B(\rho_m \ast u, v)
\]
where \( \ast \) denotes the convolution. For any \( 1 \leq p < \infty \) we have \( \|\rho_m\|_{L^p} = \left(\frac{m}{2\pi}\right)^\frac{3}{2} \left(\frac{2\pi}{mp}\right)^\frac{1}{p} \); in particular \( \|\rho_m\|_{L^1} = 1 \).

By property (2.3) of the bilinear map \( B \) we have
\[
(4.2) \quad \langle B_m(u, v), z \rangle = -\langle B_m(u, z), v \rangle, \quad \langle B_m(u, v), v \rangle = 0
\]
for any \( u, v, z \in H^1 \). In addition the following estimates hold.

**Lemma 4.1.** For any \( m > 0 \) we have

\[
\| B_m(u, v) \|_{H^{-\alpha}} \leq \| u \|_{L^4} \| v \|_{L^4}
\]

(4.3)

\[
\| B_m(u, v) \|_{H^{-\alpha}} \leq \| \rho_m \|_{L^2} \| u \|_{H} \| v \|_{H}
\]

(4.4)

\[
\| B_m(u, v) \|_{H^{-\alpha}} \leq C \| \rho_m \|_{L^{\frac{6}{4+\alpha}}} \| u \|_{H} \| v \|_{H^{-\alpha}}^{\frac{6}{2+\alpha}}
\]

(4.5)

\[
\| B_m(u, v) \|_{H^{-\alpha}} \leq C \| \rho_m \|_{L^{\frac{6}{4+\alpha}}} \| u \|_{H} \| v \|_{H^{-\alpha}}^{\frac{6}{2+\alpha}}
\]

(4.6)

*Proof.* We use repeatedly Young and Sobolev inequalities and that

\[
\| B(u, v) \|_{H^{-\alpha}} = \sup_{\| \phi \|_{H^\alpha} \leq 1} \| B(u, v, \phi) \| = \sup_{\| \phi \|_{H^\alpha} \leq 1} \| B(u, \phi, v) \|
\]

for smooth enough vectors. Then, by density, the same holds for the regularity involved at each step.

We prove (4.3), which is the only uniform estimate.

\[
\| B_m(u, \phi, v) \| = \| B(\rho_m * u, \phi, v) \|
\]

\[
\leq \| \rho_m * u \|_{L^4} \| \nabla | \|_{L^2} \| v \|_{L^4}
\]

\[
\leq \| \rho_m \|_{L^2} \| u \|_{L^4} \| \phi \|_{H^1} \| v \|_{L^4}
\]

We prove (4.4).

\[
\| B_m(u, \phi, v) \| = \| B(\rho_m * u, \phi, v) \|
\]

\[
\leq \| \rho_m * u \|_{L^\infty} \| \nabla \phi \|_{L^2} \| v \|_{L^2}
\]

\[
\leq \| \rho_m \|_{L^2} \| u \|_{L^2} \| \phi \|_{H^1} \| v \|_{L^2}
\]

We prove (4.5)

\[
\| B_m(u, \phi, v) \| = \| B(\rho_m * u, \phi, v) \|
\]

\[
\leq \| \rho_m * u \|_{L^{\frac{6}{4+\alpha}}} \| \nabla \phi \|_{L^{\frac{6}{2+\alpha}}} \| v \|_{L^{\frac{6}{2+\alpha}}}
\]

\[
\leq C \| \rho_m \|_{L^{\frac{6}{4+\alpha}}} \| u \|_{L^2} \| \nabla \phi \|_{H^\alpha} \| v \|_{H^{-\alpha}}^{\frac{6}{2+\alpha}}
\]

and similarly (4.6)

\[
\| B_m(u, \phi, v) \| = \| B(\rho_m * u, \phi, v) \|
\]

\[
\leq \| \rho_m * u \|_{L^{\frac{6}{4+\alpha}}} \| \nabla \phi \|_{L^{\frac{6}{2+\alpha}}} \| v \|_{L^2}
\]

\[
\leq C \| \rho_m \|_{L^{\frac{6}{4+\alpha}}} \| u \|_{L^2} \| \nabla \phi \|_{H^\alpha} \| v \|_{L^2}
\]

\[
\leq C \| \rho_m \|_{L^{\frac{6}{4+\alpha}}} \| u \|_{H^{-\alpha}} \| \phi \|_{H^1} \| v \|_{H}
\]

□

In the limit as \( m \to \infty \) we recover the operator \( B \). Bearing in mind (2.7) with given \( a > \frac{5}{2} \), we get for all \( u, v \in H \)

\[
\| B_m(u, v) - B(u, v) \|_{H^{-\alpha}} = \| B(\rho_m * u - u, v) \|_{H^{-\alpha}} \leq C \| \rho_m * u - u \|_{L^2} \| v \|_{L^2}.
\]

(4.7)
Note that the RHS in (4.7) converges to 0 as \( m \to \infty \). Indeed, by the Plancherel equality

\[
\|\hat{\rho}_m * u - u\|_{L^2} = \|\hat{\rho}_m \hat{u} - \hat{u}\|_{L^2}
\]

with the Fourier transform \( \hat{\rho}_m(\xi) = (2\pi)^{-d/2} e^{-\|\xi\|^2/2m} \to 1 \) pointwise as \( m \to \infty \) and \( \|\hat{\rho}_m \hat{u}\|_{L^2} \leq (2\pi)^{-d/2} \|\hat{u}\|_{L^2} \). Hence, \( \hat{\rho}_m \hat{u} - \hat{u} \to 0 \) pointwise and, by dominated convergence, \( \|\hat{\rho}_m \hat{u} - \hat{u}\|_{L^2} \to 0 \) for any given \( u \in L^2 \).

Here is our first result on the smoothed equation for any \( m > 0 \). It involves the norm

\[
|v|_T = \|v\|_{L^\infty(0,T;H)} + \|v\|_{L^2(0,T;H^4)} + \|v\|_{L^3(0,T;L^4)} + \|v\|_{C^0([0,T];H^{-1})}
\]

with \( \beta \in (0, \frac{1}{4}] \) and \( \delta \in (0, 1) \) such that \( \beta + \frac{\delta}{2} < \frac{1-\delta}{2} \).

**Theorem 4.2.** Let \( d = 3 \). If \( x \in H, f \in L^p_{\text{loc}}([0,\infty);H^{-1}) \) for some \( p > 2 \) and assumptions (G1)-(G2)-(G3)-(G4) are satisfied, then there exists a unique solution \( v_m \) of equation (4.11) on the time interval \([0,\infty)\) with initial velocity \( x \); in addition, there exist \( \beta \in (0, \frac{1}{4}] \) and \( \delta \in (0, 1) \) with \( \beta + \frac{\delta}{2} < \frac{1-\delta}{2} \) such that \( \mathbb{P}\)-a.s.

\[
v_m \in C([0,\infty);H) \cap L^\frac{8}{5}_{\text{loc}}([0,\infty);L^4) \cap L^2_{\text{loc}}([0,\infty);H^\delta) \cap C^\beta_{\text{loc}}([0,\infty);H^{-1}).
\]

and, for each \( T > 0 \),

\[
(4.8) \quad |v_m|_T^2 \leq C_4 \left[ (1 + T^2) \left( 1 + \Psi(z_m, T)^2(1 + \Phi(z_m, T)^2) e^{2\Phi(z_m, T)} \right) + (1 + T) \|z_m\|_{L^4(0,T;L^4)} + T^{\frac{4}{7}} \|f\|_{L^2(0,T;H^{-1})} \right] \quad \mathbb{P}\text{-a.s.},
\]

where

\[
\Psi(z_m, T) = \|x\|^2_H + C_5 \|z_m\|_{L^4(0,T;L^4)}^4 + C_5 \|f\|_{L^2(0,T;H^{-1})}^2,
\]

\[
\Phi(z_m, T) = C_0 \|z_m\|^8_{L^8(0,T;L^4)}.
\]

the positive constants \( C_4, C_5, C_6 \) are independent of \( m \) and \( T \), and the process \( z_m \) is the solution of equation

\[
(4.11) \quad \left\{ \begin{array}{l}
dz(t) + A\z(t) \dt + \gamma\z(t) \dt = G(v_m(t)) \, dw(t), \ t > 0; \\
z(0) = 0.
\end{array} \right.
\]

**Proof.** The proof is based on our previous paper [4]. We present a proof on a fixed time interval \([0, T]\), see however Remark 3.2. First we prove the existence of a martingale solution; then the pathwise uniqueness. Hence by a result of [18] there exists a unique strong solution in the probabilistic sense. More precisely, the existence of a martingale solution is obtained with the procedure used in [4] for the 3d stochastic Navier-Stokes equation. More regularity and pathwise uniqueness come from the techniques used in [4] for the 2d stochastic Navier-Stokes equation.

We provide some details for the reader’s convenience.
As usual, we set \( v_m = z_m + u_m \) with

\[
z_m(t) = \int_0^t e^{-\gamma(t-s)} e^{-(t-s)A} G(v_m(s)) \, dw(s)
\]

solving equation (4.11) and \( u_m \) solving

\[
\begin{cases}
\frac{du_m(t)}{dt} + Au_m(t) + \gamma u_m(t) + B_m(v_m(t), v_m(t)) = f(t), \; t \in (0, T]; \\
u_m(0) = x.
\end{cases}
\]

(4.12)

Keeping in mind (4.3) and (4.2), we have that for any finite \( m \) the operator \( B_m \) enjoys the same properties as \( B \) necessary to prove the existence of martingale solutions for the three dimensional stochastic Navier-Stokes equation (2.8); the additional damping term \( \gamma \| u_m(t) \|^2_H \) appearing in the energy estimate has no effect on the apriori estimates as we have noticed in the previous section, see (3.11). Hence, assuming (G1)-(G2)-(G3) we obtain the same result as Theorem 3.6 in [3]: there exists a martingale solution to equation (4.1), and a.e. path of this process \( v_m \) is in \( L^\infty(0, T; H) \cap C([0, T]; H_w) \cap L^8(0, T; L^4) \).

Let us show that a.e. path of this solution process is in \( C([0, T]; H) \); here we need the smoothing in order to proceed as in the two dimensional case. We work pathwise. First, we have \( z_m \in C([0, T]; H) \) as in Lemma 3.3 of [4]. Since \( v_m \in L^\infty(0, T; H) \), (4.4) implies that \( B_m(v_m(t), v_m(t)) \in L^\infty(0, T; H^{-1}) \). Moreover, \( u_m \in L^2(0, T; H^1) \).

Hence

\[
\frac{du_m}{dt}(t) = -Au_m(t) - \gamma u_m(t) - B_m(v_m(t), v_m(t)) + f(t) \in L^2(0, T; H^{-1}).
\]

We conclude by means of a classical result, see Ch III Lemma 1.2 of [27]: if \( u_m \in L^2(0, T; H^1) \) and \( \frac{du_m}{dt} \in L^2(0, T; H^{-1}) \) then \( u_m \in C([0, T]; H) \). Therefore \( v_m = z_m + u_m \in C([0, T]; H) \) a.s.

Let us prove the estimate (4.8). Taking the H-scalar product of equation (4.12) with \( u_m \) and using the bilinearity of \( B_m \) and (4.2), we get

\[
(4.13) \quad \frac{1}{2} \frac{d}{dt} \| u_m(t) \|^2_H + \| \nabla u_m(t) \|^2_{L^2} + \gamma \| u_m(t) \|^2_H = -\langle B_m(u_m(t) + z_m(t), u_m(t) + z_m(t)), u_m(t) \rangle + \langle f(t), u_m(t) \rangle.
\]

Using (4.3), we estimate the trilinear term as in the proof of Proposition 3.4 in [4]:

\[
(4.14) \quad -\langle B_m(u_m + z_m, u_m + z_m), u_m \rangle \leq \frac{1}{4} \| \nabla u_m \|^2_{L^2} + C \| u_m \|^2_H \| z_m \|^8_{L^4} + C \| z_m \|^4_{L^4}
\]

for some positive constant \( C \) independent of \( m \). Moreover

\[
|\langle f, u_m \rangle| \leq \| f \|_{H^{-1}} \| u_m \|_{L^2} + \| \nabla u_m \|_{L^2} \leq \gamma \| u_m \|^2_H + \frac{1}{4} \| \nabla u_m \|^2_{L^2} + \left( \frac{1}{4\gamma} + 1 \right) \| f \|_{H^{-1}}^2.
\]
Inserting these estimates in (4.13) we get

\[
\frac{d}{dt} \|u_m(t)\|_{L^2}^2 + \|\nabla u_m(t)\|_{L^2}^2 \leq C_5 \|z_m(t)\|_{L^4}^4 + C_5 \|f(t)\|_{L^2}^2 + C_6 \|u_m(t)\|_{L^2}^2 \|z_m(t)\|_{L^4}^8
\]

for some constants $C_5, C_6$ independent of $T$ and $m$. Gronwall Lemma applied to

\[
\frac{d}{dt} \|u_m(t)\|_{L^2}^2 \leq C_6 \|z_m(t)\|_{L^4}^8 \|u_m(t)\|_{L^2}^2 + C_5 \|z_m(t)\|_{L^4}^4 + C_5 \|f(t)\|_{L^2}^2
\]

gives

\[
\sup_{0 \leq t \leq T} \|u_m(t)\|_{L^2}^2 \leq \|x\|_{H^1}^2 e^{\int_0^T C_6 \|z_m(r)\|_{L^4}^8 \, dr}
\]

\[
+ C_5 \int_0^T e^{\int_s^T C_6 \|z_m(r)\|_{L^4}^8 \, dr} \left( \|z_m(s)\|_{L^4}^4 + \|f(s)\|_{L^2}^2 \right) \, ds
\]

\[
\leq \Psi(z_m, T)e^{\Phi(z_m,T)}
\]

where $\Psi$ and $\Phi$ are defined in (4.9) and (4.10), respectively. Integrating in time (4.15) we get

\[
\int_0^T \|\nabla u_m(t)\|_{L^2}^2 \, dt \leq \Psi(z_m, T) + \Phi(z_m, T) \|u_m\|_{L^\infty(0,T;L^2)}
\]

\[
\leq \Psi(z_m, T) + \Phi(z_m, T) \Psi(z_m, T)e^{\Phi(z_m,T)}.
\]

Now, we recall the continuous embedding $H^{1,\frac{4}{3}}(0, T; H^{-1}) \subset C^{\frac{1}{3}}([0, T]; H^{-1})$; using (4.3) we proceed as in Proposition 3.4 of [4] to get

\[
\|u_m\|_{C^{\frac{1}{3}}([0,T];H^{-1})}
\]

\[
\leq C \|u_m\|_{H^{1,\frac{4}{3}}(0,T;H^{-1})}
\]

\[
= C \left( \|u_m\|_{L^\frac{4}{3}(0,T;H^{-1})} + \| - Au_m - \gamma u_m - B_m(v_m, v_m) + f \|_{L^\frac{4}{3}(0,T;H^{-1})} \right)
\]

\[
\leq C \left( \|u_m\|_{L^\frac{4}{3}(0,T;H)} + \|u_m\|_{L^\frac{4}{3}(0,T;H^1)} + \|u_m\|_{L^\frac{4}{3}(0,T;H^1)} \right)
\]

\[
+ \|B_m(u_m + z_m, u_m + z_m)\|_{L^\frac{4}{3}(0,T;H^{-1})} + \|f\|_{L^\frac{4}{3}(0,T;H^{-1})}
\]

\[
\leq C \left( 1 + \gamma \right) T^{\frac{3}{2}} \|u_m\|_{L^\infty(0,T;H)} + T^{\frac{1}{2}} \|u_m\|_{L^2(0,T;H^1)} + \|u_m\|_{L^2(0,T;L^4)}^2
\]

\[
+ \|z_m\|_{L^\frac{8}{3}(0,T;L^4)}^{\frac{2}{3}} + T^{\frac{1}{2}} \|f\|_{L^2(0,T;H^{-1})} \right).
\]

The Gagliardo-Nirenberg inequality

\[
\|u_m(t)\|_{L^4} \leq C \|u_m(t)\|_{L^2}^{\frac{1}{2}} \|\nabla u_m(t)\|_{L^2}^{\frac{1}{2}}
\]
this difference satisfies

\( (4.19) \)

\[
\| u_m \|_{L^2(0,T;L^4)}^2 \leq C \| u_m \|_{L^\infty(0,T;L^2)} \| \nabla u_m \|_{L^2(0,T;L^2)}^2 \\
\leq C (\| u_m \|_{L^\infty(0,T;H)}^2 + \| \nabla u_m \|_{L^2(0,T;H)}^2).
\]

Summing up, we can bound \( |u_m|_T \). First,

\[
|u_m|_T \leq C \left( (\| u_m \|_{L^\infty(0,T;H)} + \| u_m \|_{L^2(0,T;H)} + \| u_m \|_{L^\infty(0,T;L^4)}^4) + (1 + T^{\frac{4}{3} - \beta}) \| u_m \|_{C^{\frac{1}{4}}([0,T];H^{-1})} \right);
\]

then by (4.16), (4.17), (4.18) and (4.19) we get

\[
|u_m|_T^2 \leq C \left( (1 + T^2) \| u_m \|_{L^\infty(0,T;H)}^2 + (1 + T) \| \nabla u_m \|_{L^2(0,T;L^2)}^2 + \| u_m \|_{L^\infty(0,T;H)}^4 \right. \\
+ \| \nabla u_m \|_{L^2(0,T;L^2)}^4 + \| z_m \|_{L^\infty(0,T;L^4)}^4 + T^{\frac{4}{3}} \| f \|_{L^2(0,T;H^{-1})}^2 \\
\leq C \left( (1 + T^2)[1 + \Psi(z_m, T) e^{2\Phi(z_m, T)}(1 + \Phi(z_m, T)) \right] \\
+ \| z_m \|_{L^\infty(0,T;L^4)}^4 + T^{\frac{4}{3}} \| f \|_{L^2(0,T;H^{-1})}^2 \right)
\]

for some positive constant \( C \) independent of \( T \) and \( m \).

For the process \( z_m \) we have

\[
|z_m|_T \leq C \left( (1 + T^{\frac{4}{3}}) \| z_m \|_{C^{\frac{1}{4}}([0,T];H^\delta)} + \| z_m \|_{L^\infty(0,T;L^4)} \right)
\]

for some positive constant \( C \) independent of \( T \) and \( m \).

We finally get (4.8) for \( v_m = u_m + z_m \).

As far as the pathwise uniqueness is concerned, we proceed in a similar way we did in Theorem 4.1 of [4] for the two dimensional stochastic Navier-Stokes equation. Let \( v_m \) and \( \tilde{v}_m \) be two processes solving (4.1) on the same stochastic basis with the same Wiener process, initial velocity and deterministic force \( f \). Set \( V = v_m - \tilde{v}_m \); this difference satisfies

\[
dV(t) + [AV(t) + \gamma V(t) + B_m(v_m(t), v_m(t)) - B_m(\tilde{v}_m(t), \tilde{v}_m(t))] \, dt \\
= [G(v_m(t)) - G(\tilde{v}_m(t))] \, dw(t)
\]

with \( V(0) = 0 \); the equation is equivalent to

\[
dV(t) + [AV(t) + \gamma V(t) + B_m(V(t), v_m(t)) + B_m(\tilde{v}_m(t), V(t))] \, dt \\
= [G(v_m(t)) - G(\tilde{v}_m(t))] \, dw(t).
\]

We will apply the Itô formula for the process \( e^{-\int_0^t \frac{\sigma(s)}{2} \, ds} \left\| V(t) \right\|_{H^{-\gamma}}, \quad t \in [0,T], \) by choosing \( \sigma \) as it has been done in paper [25] by B. Schmalfuss:

\[
\sigma(s) = L_g^2 + 2M_m, \gamma \left( \left\| v_m(s) \right\|_H + \left\| \tilde{v}_m(s) \right\|_H \right)^{\frac{4}{3 - \gamma}}
\]
with $L_9$ the Lipschitz constant given in (G4) and $M_{m, \gamma}$ the constant appearing later on in (4.20). We have $\sigma \in L^1(0, T) \ \mathbb{P}$-a.s. since $v_m, \tilde{v}_m \in C([0, T]; H) \ \mathbb{P}$-a.s.

First

$$d \left( e^{-\int_0^t \sigma(s)ds} \|V(t)\|^2_{H^{-g}} \right) = -\sigma(t) e^{-\int_0^t \sigma(s)ds} \|V(t)\|^2_{H^{-g}} dt + e^{-\int_0^t \sigma(s)ds} d\|V(t)\|^2_{H^{-g}}$$

and the latter differential is well defined and given by

$$\frac{1}{2} d\|V(t)\|^2_{H^{-g}} = -\|\nabla V(t)\|^2_{H^{-g}} dt - \gamma\|V(t)\|^2_{H^{-g}} dt$$

$$- \langle J^{-g}[B_m(V(t), v_m(t)) + B_m(\tilde{v}_m(t), V(t))], J^{-g}V(t) \rangle dt$$

$$+ \langle J^{-g}[G(v_m(t)) - G(\tilde{v}_m(t))] dw(t), J^{-g}V(t) \rangle$$

$$+ \frac{1}{2} \|G(v_m(t)) - G(\tilde{v}_m(t))\|^2_{\gamma(Y; H^{-g})} dt.$$ 

We deal with the trilinear terms:

$$\langle J^{-g}[B_m(V, v_m) + B_m(\tilde{v}_m, V)], J^{-g}V \rangle$$

$$\leq \|B_m(V, v_m)\|_{H^{-g}} + \|B_m(\tilde{v}_m, V)\|_{H^{-g}} \|V\|_{H^{-g}}$$

$$\leq C \|\rho_m\|_{L^{\frac{4}{1+\gamma}}} (\|v_m\|_H + \|\tilde{v}_m\|_H) \|V\|_{H^{-g}} \|V\|_{H^{-g}} \text{ by (1.5) - (1.6)}$$

$$\leq C \|\rho_m\|_{L^{\frac{4}{1+\gamma}}} (\|v_m\|_H + \|\tilde{v}_m\|_H) \|V\|_{H^{-g}}^{\frac{4}{2+\gamma}} \|V\|_{H^{-g}} \frac{4}{2+\gamma} \text{ by interpolation}$$

$$\leq \frac{\min(1, \gamma)}{2} \|V\|_{H^{-g}}^2 + M_{m, \gamma}(\|v_m\|_H + \|\tilde{v}_m\|_H) \|V\|_{H^{-g}}$$

by Young inequality

$$= \frac{\min(1, \gamma)}{2} \|V\|_{H^{-g}}^2 + \frac{\min(1, \gamma)}{2} \|V\|_{H^{-g}}^2 + M_{m, \gamma}(\|v_m\|_H + \|\tilde{v}_m\|_H) \|V\|_{H^{-g}}.$$

Therefore, using (G4) we get

$$\frac{1}{2} d\left( e^{-\int_0^t \sigma(s)ds} \|V(t)\|^2_{H^{-g}} \right)$$

$$\leq e^{-\int_0^t \sigma(s)ds} \langle J^{-g}[G(v_m(t)) - G(\tilde{v}_m(t))], J^{-g}V(t) \rangle dt.$$ 

The RHS is a local martingale; indeed if we define the stopping time

$$\tau_N = T \wedge \inf\{t \in [0, T] : \|V(t)\|_{H^{-g}} > N\}$$

and

$$M_N(t) = \int_0^{\tau_N} e^{-\int_0^r \sigma(s)ds} \langle J^{-g}V(r), J^{-g}[G(v_m(r)) - G(\tilde{v}_m(r))] \rangle dw(r)$$

then

$$\mathbb{E}[M_N(t)^2] \leq \mathbb{E} \int_0^{\tau_N} e^{-2\int_0^r \sigma(s)ds} \|V(r)\|^2_{H^{-g}}^2 |G(v_m(r)) - G(\tilde{v}_m(r))|^2_{\gamma(Y; H^{-g})} dr$$

$$\leq L_9^2 \mathbb{E} \int_0^{\tau_N} \|V(r)\|^4_{H^{-g}} dr \leq L_9^2 N^4 t.$$
Hence, \( M_N \) is a square integrable martingale; in particular \( \mathbb{E}[M_N(t)] = 0 \) for any \( t \).

Therefore, by integrating (4.21) over \([0, t \wedge \tau_N]\) and taking the expectation we get
\[
\mathbb{E} \left[ e^{-\int_0^{t \wedge \tau_N} \sigma(s)ds} \| V(t \wedge \tau_N) \|^2_{H^{-g}} \right] \leq 0.
\]

So
\[
e^{-\int_0^{t \wedge \tau_N} \sigma(s)ds} \| V(t \wedge \tau_N) \|^2_{H^{-g}} = 0 \quad \mathbb{P} \text{-a.s.}
\]

Since \( \lim_{N \to \infty} \tau_N = T \mathbb{P}\text{-a.s.} \), we get in the limit that for any \( t \in [0, T] \)
\[
e^{-\int_0^t \sigma(s)ds} \| V(t) \|^2_{H^{-g}} = 0 \quad \mathbb{P} \text{-a.s.}
\]

Thus, if we take a sequence \( \{t_k\}_{k=1}^\infty \) which is dense in \([0, T]\) we have
\[
\mathbb{P} (\| V(t_k) \|_{H^{-g}} = 0 \text{ for all } k \in \mathbb{N}) = 1.
\]

Since a.e. path of the process \( V \) belongs to \( C([0, T]; H) \subset C([0, T]; H^{-g}) \), we get
\[
\mathbb{P} (\| V(t) \|_{H^{-g}} = 0 \text{ for all } t) = 1.
\]

This gives pathwise uniqueness. \( \square \)

Now, for each \( m \in \mathbb{N} \) we define the Markov semigroup \( \{P_t^{(m)}\}_{t \geq 0} \) associated to (4.1) as
\[
P_t^{(m)} \phi(x) = \mathbb{E}[\phi(v_m(t; x))]
\]
for any bounded Borel function \( \phi : H \to \mathbb{R} \).

Let us point out that the result of Theorem 4.2 holds also for general initial distribution, see Corollary 22 in [23] (based on the finite dimensional case considered in [18]). Therefore, we have the following

**Corollary 4.3.** Let \( d = 3 \). If \( \mu \) is a probability measure on the Borelian subsets of \( H \), \( f \in L^p(0,T; H^{-1}) \) for some \( p > 2 \) and assumptions (G1)-(G2)-(G3)-(G4) are satisfied, then there exists a unique solution \( v_m \) of equation (4.1) on the time interval \([0, T]\) with initial velocity of law \( \mu \) and the same properties given in Theorem 4.2 hold.

As in the previous section, by means of Maslowski and Seidler result of Theorem 3.3, we can prove the existence of at least one invariant measure for the smoothed equation (4.1).

**Theorem 4.4.** Let \( d = 3 \). If \( f \in H^{-1} \) and assumptions (G1)-(G2)-(G3)-(G4) are satisfied, then there exists at least one invariant measure \( \mu_m \) for equation (4.1) and \( \mu_m(H) = 1 \).

**Proof.** The proof is the same as that of Theorem 3.4. Indeed, both the sequentially weakly Feller property and the boundedness in probability result are based on the results of [4], which hold for the stochastic damped Navier-Stokes equation (2.8) as
well as for the smoothed version \((4.1)\). In particular the estimates for the boundedness in probability of the sequence of \(v_m = \zeta_m^\alpha + u_m^\alpha\) come from those for \(\zeta_m^\alpha\) and \(u_m^\alpha\), where

\[
\zeta_m^\alpha(t) = \int_0^t e^{-(\gamma+\alpha)(t-s)} e^{-(t-s)A} G(v_m(s)) \, dw(s), \quad t \in [0, T],
\]

and

\[
(4.22)
\begin{align*}
\frac{du_m^\alpha}{dt}(t) + Au_m^\alpha(t) + \gamma u_m^\alpha(t) + B_m(v_m(t), v_m(t)) &= \alpha \zeta_m^\alpha(t) + f, \\
u_m^\alpha(0) &= 0.
\end{align*}
\]

In particular, Lemma 3.5 gives

\[
\sup_{t \geq 0} \mathbb{E}\|\zeta_m^\alpha(t)\|_H^2 \leq C_{\alpha,2}, \quad \sup_{t \geq 0} \mathbb{E}\|\zeta_m^\alpha(t)\|^q_{H^q_{\text{vol}}} \leq C_{\alpha,q,4}
\]

(here we need \(q = 2\) and \(q = 4\) and by the way we notice that the estimates are uniform in \(m\)). As far as the estimate for \(u_m^\alpha\) is concerned, we proceed as follows. By taking the \(H\)-scalar product of equation \((4.22)\) with \(u_m^\alpha(t)\) we get

\[
\frac{1}{2} \frac{d}{dt} \|u_m^\alpha(t)\|_H^2 + \|\nabla u_m^\alpha(t)\|_2^2 + \gamma \|u_m^\alpha(t)\|_H^2 \\
= -\langle B_m(u_m^\alpha(t) + \zeta_m^\alpha(t)m, u_m^\alpha(t) + \zeta_m^\alpha(t)), u_m^\alpha(t) \rangle \\
+ \alpha \langle \zeta_m^\alpha(t), u_m^\alpha(t) \rangle + \langle f, u_m^\alpha(t) \rangle.
\]

We estimate the trilinear term as in \((4.14)\) (but with different constants) and get

\[
\frac{d}{dt} \|u_m^\alpha(t)\|_H^2 \leq -\gamma \|u_m^\alpha(t)\|_H^2 + C_7 \|\zeta_m^\alpha(t)\|_{L^4}^4 \|u_m^\alpha(t)\|_H^2 \\
+ \alpha^2 C_7 \|\zeta_m^\alpha(t)\|_H^2 + C_7 \|\zeta_m^\alpha(t)\|_{L^4}^4 + C_7 \|f\|_{H^{-1}}^2
\]

providing, as in the proof of Theorem 3.6 that

\[
\gamma \frac{1}{T} \int_0^T \mathbb{P}\left(\|u_m^\alpha(t)\|_H^2 > R\right) \, dt \leq C_7 \sup_{t \geq 0} \mathbb{E}[\|\zeta_m^\alpha(t)\|_{L^4}^4] \\
+ \frac{1}{R} C_7 \left( \alpha^2 \sup_{t \geq 0} \mathbb{E}[\|\zeta_m^\alpha(t)\|_H^2] + \sup_{t \geq 0} \mathbb{E}[\|\zeta_m^\alpha(t)\|_{L^4}^4] + \|f\|_{H^{-1}}^2 \right) \\
\leq C_7 \left( C_{\alpha,8,4} + \frac{\alpha^2 C_{\alpha,2} + C_{\alpha,4,4} + \|f\|_{H^{-1}}^2}{R} \right).
\]

From here we conclude as in the two dimensional case considered in the previous section, since the RHS above can be as small as we want for a suitable choice of \(\alpha\) and \(R\). \(\square\)
4.2. Existence of stationary martingale solutions. Now we go back to the original equation (2.8). For each \( m \in \mathbb{N} \), we fix an invariant measure \( \mu_m \) for the smoothed equation (4.1) as given by Theorem 4.4. Therefore we denote by \( v_m \) the stationary solution of (4.1) whose marginal at any fixed time is \( \mu_m \); this is the solution given in Corollary 4.3 with initial velocity of law \( \mu_m \), given a probability space rich enough to support a random variable with law \( \mu_m \). In the limit as \( m \to \infty \), we shall get a stationary solution of equation (2.8).

First, we prove a tightness result.

**Proposition 4.5.** Let \( d = 3 \), \( f \in H^{-1} \) and assume (G1)-(G2)-(G3)-(G4). For any \( m \in \mathbb{N} \), let \( \mu_m \) be an invariant measure for equation (4.1) as given in Theorem 4.4. Then the sequence of the laws of the stationary processes \( v_m \), solving equation (4.1) with initial velocity of law \( \mu_m \), is tight in the space

\[
Z = C(\mathbb{R}_+; H_w) \cap L^2_{\text{loc}}(\mathbb{R}_+; H_{\text{loc}}) \cap \left( \frac{1}{2} L^2_{\text{loc}}(\mathbb{R}_+; L^4) \right)_w \cap C(\mathbb{R}_+; U')
\]

with the topology \( T \) given by the supremum of the corresponding topologies.

**Proof.** We have to prove that for every \( \varepsilon > 0 \) we can find a compact subset \( K_\varepsilon \) of \( Z \) such that

\[
\sup_m \Pr(v_m \notin K_\varepsilon) < \varepsilon.
\]

Let us fix \( \varepsilon > 0 \). Choose \( \beta \in (0, \frac{1}{4}] \) and \( \delta \in (0, 1] \) such that \( \beta + \frac{\delta}{2} < \frac{1-g}{2} \). We shall take a set of the form \( K(\alpha^{(\varepsilon)}) = \{ v \in Z : |v|_N \leq \alpha^{(\varepsilon)}_N \text{ for any } N \in \mathbb{N} \} \) (which is compact according to Lemma 2.4) with

\[
|v|_N = \|v\|_{L^\infty(0,N; H)} + \|v\|_{L^2(0,N; H^4)} + \|v\|_{L^\infty(0,N; L^4)} + \|v\|_{C^g([0,N]; H^{-1})}.
\]

As in [8], it is sufficient to find a sequence \( \alpha^{(\varepsilon)} = (\alpha_1^{(\varepsilon)}, \alpha_2^{(\varepsilon)}, \ldots) \) such that

\[
(4.23) \quad \sup_m \Pr(|v_m|_N > \alpha_N^{(\varepsilon)}) < \frac{1}{2N\varepsilon}.
\]

Indeed, this implies tightness, since

\[
\sup_m \Pr(v_m \notin K(\alpha^{(\varepsilon)})) = \sup_m \Pr \left( \bigcup_{N=1}^{\infty} \{ |v_m|_N > \alpha_N^{(\varepsilon)} \} \right) \leq \sum_{N=1}^{\infty} \sup_m \Pr \left( |v_m|_N > \alpha_N^{(\varepsilon)} \right) < \varepsilon.
\]

Bearing in mind (4.8), we will estimate \( \Pr(|v_m|_N > \alpha_N^{(\varepsilon)}) \). From Lemma 3.5 we infer that there exists a constant \( C_8 \) depending on \( p, g, \gamma, K_{g,4} \) such that for all \( m \) and \( T \)

\[
(4.24) \quad \mathbb{E}\|z_m\|_{L^p(0,T; L^4)}^p \leq C_8(1 + T).
\]
From the proof of Lemma 3.3 of [4] we infer that there exists a constant $C_0$ depending on $p, \beta, \delta, K_{g,2}$ such that for all $m$ and $T$

$$
(4.25) \quad \mathbb{E}\|z_m\|_{C^\beta([0,T];H^\delta)} \leq C_0(1 + T^{1-\beta/2-\delta/2}).
$$

Therefore, by the Chebyshev inequality for any $b > 0$ and $m \in \mathbb{N}$ we have

$$
(4.26) \quad \mathbb{P}\left(\|z_m\|_{L^p(0,T;L^4)} > b\right) \leq \frac{C_0(1 + T)}{b^p}
$$

and

$$
(4.27) \quad \mathbb{P}\left(\|z_m\|_{C^\beta([0,T];H^\delta)} > b\right) \leq \frac{C_0(1 + T^{1-\beta/2-\delta/2})}{b}.
$$

Bearing in mind the definition of $\Phi(z_m,N)$ given in (4.10), the estimate (4.24) provides that $\mathbb{P}(\Phi(z_m,N) < \infty) \equiv \mathbb{P}\{C_0\|z_m\|_{L^8(0,N;L^4)}^8 < \infty\} = 1$ and by (4.26) we have that for any $\varepsilon, N > 0$ there exists $M = M(\varepsilon, N) > 0$ such that

$$
\mathbb{P}(\Phi(z_m,N) > M) < \frac{\varepsilon}{22N}.
$$

Now we define the subset $S_{z,T,b} = \{\|z\|_{L^8(0,N;L^4)}^8 \leq b, \|x\|_H^2 \leq b\} \subset \Omega$. So for any $\varepsilon, N > 0$ there exists $b = b^{(e)}_N$ such that for any $m$

$$
(4.28) \quad \mathbb{P}\{S_{z_m,N,b}\} > 1 - \frac{\varepsilon}{22N}.
$$

Notice that, on the set $S_{z_m,N,b}$, the process $\Phi(z_m,N)$ given in (4.10) is bounded by the constant $C_0b$ and the process $\Psi(z_m,N)$ given in (4.3) is bounded by the constant $b + C_5(N + b + N\|f\|_{L^{H-1}}^2)$, by using $\|z\|_{L^4(0,N;L^4)}^4 \leq \frac{N}{4} + \|z\|_{L^8(0,N;L^4)}^8$. Hence, on the set $S_{z_m,N,b}$ the process $1 + \Psi(z_m,N)^2(1 + \Phi(z_m,N)^2)e^{2\Phi(z_m,N)}$ appearing in the estimate (4.8) for $\|v_m\|_N$ is bounded, $\mathbb{P}$-a.s., by the constant

$$
\begin{align*}
C(b,N) := 1 + [b + C_5(N + b + N\|f\|_{L^{H-1}}^2)]^2(1 + (C_0b)^2)e^{2Cb}.
\end{align*}
$$

This allows to find each element $\alpha^{(e)}_N$ of the sequence $\alpha^{(e)}$ satisfying (4.23) as follows. For short, we denote by $\eta(z_m,T)$ the RHS of (4.8).

By (4.28), we have

$$
\mathbb{P}\left(\|v_m\|_N > \alpha^{(e)}_N\right) \leq \mathbb{P}\left(\eta(z_m,N) > \alpha^{(e)}_N\right)^2 \leq \mathbb{P}\left(\Omega \setminus S_{z_m,N,b^{(e)}_N}\right)
$$

$$
+ \mathbb{P}\left(S_{z_m,N,b^{(e)}_N} \cap \lbrace \eta(z_m,N) > \alpha^{(e)}_N\rbrace\right)
$$

$$
\leq \frac{\varepsilon}{22N} + \mathbb{P}\left(S_{z_m,N,b^{(e)}_N} \cap \lbrace \eta(z_m,N) > \alpha^{(e)}_N\rbrace\right).
$$

So, we are left to find $\alpha^{(e)}_N$ such that the latter probability is bounded by $\frac{\varepsilon}{22N}$. 
By Chebyshev inequality we get
\[
\mathbb{P}(1_{S_{z_m,N,b}^{(\epsilon)}} \eta(z_m, N) > (\alpha_N^{(\epsilon)})^2) \\
\leq \frac{1}{(\alpha_N^{(\epsilon)})^2} \mathbb{E}[1_{S_{z_m,N,b}^{(\epsilon)}} \eta(z_m, N)] \\
\leq \frac{1}{(\alpha_N^{(\epsilon)})^2} \left( (1 + N^2)C \|u_N^{(\epsilon)}\|, N + 1 + \mathbb{E}\|z_m\|_L^4(0,N;L^4) \\
+ (1 + N)\mathbb{E}\|z_m\|^2_{C^\beta([0,N];H^\delta)} + N^2 \|f\|_{H^{-1}}^2 \right) \\
\leq \frac{1}{(\alpha_N^{(\epsilon)})^2} \left( (1 + N^2)C \|u_N^{(\epsilon)}\|, N + 1 + C^3_N(1 + N)^{3/2} \right) \\
+ (1 + N)C^3_N(1 + N^{1/2-\beta-\delta/2})^2 + N^2 \|f\|_{H^{-1}}^2 \right) \\
\leq \frac{1}{(\alpha_N^{(\epsilon)})^2} \left( (1 + N^2)C \|u_N^{(\epsilon)}\|, N + 1 + C^3_N(1 + N^{1/2-\beta-\delta/2})^2 + N^2 \|f\|_{H^{-1}}^2 \right)
\]
where we used (4.24)-(4.25) to estimate the mean values involving the processes \(z_m\).
And finally we ask the latter quantity to be smaller than \(\epsilon^2_{12}\) to determine \(\alpha_N^{(\epsilon)}\) so that (4.23) holds true. This proves the tightness.

Now, we prove

**Theorem 4.6.** Let \(d = 3\). If \(f \in H^{-1}\) and (G1)-(G2)-(G3)-(G4) are satisfied, then there exists at least one stationary martingale solution for (2.8).

**Proof.** For each \(m \in \mathbb{N}\), we fix an invariant measure \(\mu_m\) for equation (4.1) as given by Theorem 4.4 and denote by \(v_m\) the stationary solution of (4.1) whose marginal at any fixed time is \(\mu_m\). According to Proposition 4.5, the sequence of the laws of the stationary processes \(v_m\) is tight in the space
\[
Z = C(\mathbb{R}_+; H_w) \cap L^2_{\text{loc}}(\mathbb{R}_+; H_{\text{loc}}) \cap L^{3/2}_{\text{loc,w}}(\mathbb{R}_+; L^4) \cap C(\mathbb{R}_+; U').
\]

As in Section 3.1, we use the Jakubowski’s generalization of the Skorokhod Theorem to nonmetric spaces, see [19]. Hence, there exist a subsequence \(\{v_{m_j}\}_{j=1}^\infty\), a stochastic basis \((\tilde{\varOmega}, \tilde{\mathbb{F}}, \{\tilde{\mathbb{F}}_t\}_{t \geq 0}, \tilde{\mathbb{P}})\), \(Z\)-valued Borel measurable variables \(\tilde{v}\) and \(\{\tilde{v}_j\}_{j=1}^\infty\) such that for any \(j \in \mathbb{N}\) the laws of \(v_{m_j}\) and \(\tilde{v}_j\) are the same and \(\tilde{v}_j\) converges to \(\tilde{v}\) \(\tilde{\mathbb{P}}\)-a.s. with the topology \(\mathcal{T}\). Moreover, this limit process \(\tilde{v}\) is stationary in \(H\). Indeed, by the a.s. convergence in \(C(\mathbb{R}_+; U')\) the process \(\tilde{v}\) is stationary in \(U'\). But, \(\tilde{v} \in C(\mathbb{R}_+; H_w)\) so each \(\tilde{v}(t)\) is an \(H\)-valued random variable and therefore \(\tilde{v}\) is stationary in \(H\).

Since each \(\tilde{v}_j\) has the same law as \(v_{m_j}\), it is a martingale solution to the smoothed equation (4.1); therefore for any \(\phi \in H^a\) with \(a > \frac{5}{2}\) (so that \(\|\nabla \phi\|_{L^\infty} \leq C\|\phi\|_{H^a}\)),


we consider the process

\[
M_j^\phi(t) = (\tilde{v}_j(t), \phi)_H - (\tilde{v}_j(0), \phi)_H + \int_0^t (\tilde{v}_j(s), A\phi)_H ds + \int_0^t (B_{m_j}(\tilde{v}_j(s), \tilde{v}_j(s)), \phi) ds - \langle f, \phi \rangle t
\]

Since \( M_j^\phi(t) \) has the same law as \( \langle \int_0^t G(v_{m_j}(s)) dw(s), \phi \rangle \), it is a martingale; in particular

\[
(4.29) \quad \mathbb{E}[(M_j^\phi(t) - M_j^\phi(s))h(\tilde{v}_j|_{[0,s]})] = 0
\]

\[
(4.30) \quad \mathbb{E}[(M_j^\phi(t)M_j^\psi(t) - M_j^\phi(s)M_j^\psi(s))h(\tilde{v}_j|_{[0,s]})]
\]

\[
= \mathbb{E}[(\int_s^t (G(\tilde{v}_j(r))^* J^{2g}\phi, G(\tilde{v}_j(r))^* J^{2g}\psi)_Y dr)h(\tilde{v}_j|_{[0,s]})]
\]

for any \( 0 < s < t \), any \( \phi, \psi \in H^a \) and any bounded and continuous function \( h : C([0,s]; U'') \to \mathbb{R} \).

When \( j \to \infty \), for any \( 0 < t \) we have \( M_j^\phi(t) \to M^\phi(t) \) \( \mathbb{P} \)-a.s., where

\[
M^\phi(t) = (\tilde{v}(t), \phi)_H - (\tilde{v}(0), \phi)_H + \int_0^t (\tilde{v}(s), A\phi)_H ds
\]

\[
+ \gamma \int_0^t (\tilde{v}(s), \phi)_H ds + \int_0^t \langle B(\tilde{v}(s), \tilde{v}(s)), \phi \rangle ds - \langle f, \phi \rangle t.
\]

Indeed, the convergence of each term is done as in [4] except for the trilinear term. For this, we take \( \phi \) regular enough and with compact support (hence \( \nabla \phi \) is bounded), let us say that the support is contained in a centered ball of radius \( R \); we have

\[
|\langle B_{m_j}(\tilde{v}_j(s), \tilde{v}_j(s)), \phi \rangle - \langle B(\tilde{v}(s), \tilde{v}(s)), \phi \rangle|
\]

\[
\leq |\langle B_{m_j}(\tilde{v}_j(s) - \tilde{v}(s), \phi), \tilde{v}_j(s) \rangle|
\]

\[
+ |\langle B_{m_j}(\tilde{v}(s), \phi), \tilde{v}_j(s) - \tilde{v}(s) \rangle| + |\langle B(\rho_{m_j} * \tilde{v}(s) - \tilde{v}(s), \phi), \tilde{v}(s) \rangle|
\]

\[
\leq C\|\tilde{v}_j(s) - \tilde{v}(s)\|_{H_R} \|\tilde{v}_j(s)\|_{H_R} \|\nabla \phi\|_{L^\infty}
\]

\[
+ C\|\tilde{v}(s)\|_{H_R} \|\tilde{v}_j(s) - \tilde{v}(s)\|_{H_R} \|\nabla \phi\|_{L^\infty}
\]

\[
+ C\|\rho_{m_j} * \tilde{v}(s) - \tilde{v}(s)\|_{H_R} \|\tilde{v}(s)\|_{H_R} \|\nabla \phi\|_{L^\infty}.
\]

Using the convergence in the space \( L^2(0, T; H_{loc}) \), we pass into the limit as \( m_j \to \infty \). Since each \( \phi \in H^a \) can be approximated by a smooth and compactly supported vector field, we obtain the same limit for any \( \phi \in H^a \), see Lemma B.1 in [3].
Therefore, to get the convergence of the LHS of (4.29) and (4.30) we appeal to Vitali theorem, using the pathwise convergence and the uniform estimate
\[
\mathbb{E} \left[ |M_j^\phi(t)|^4 \right] = \mathbb{E} \left[ |\int_0^t G(v_{m_j}(r))dw(r), \phi|^4 \right]
\]
\[
\leq \mathbb{E} \left\| \int_0^t G(v_{m_j}(r))dw(r) \right\|^4_{H^{-g}} \|\phi\|^4_{H^g}
\]
\[
\leq C \mathbb{E} \left( \int_0^t \|G(v_{m_j}(r))\|^2_{\gamma(Y;H^{-g})}dr \right)^2 \|\phi\|^4_{H^g}
\]
\[
\leq C(K_{g,2}^2)^2 \|\phi\|^4_{H^g}
\]
thanks to (G1).

As far as the convergence of the RHS of (4.30) is concerned, we recall from [4] the pathwise convergence of the cross variance process
\[
\int_0^t (G(\tilde{v}_j(r))^* f_1, G(\tilde{v}_j(r))^* f_2)_Y dr \to \int_0^t (G(\tilde{v}(r))^* f_1, G(\tilde{v}(r))^* f_2)_Y dr
\]
for any $f_1, f_2 \in H^{-g}$, thanks to assumption (G3). Again by means of Vitali theorem we get the convergence for the mean value in the RHS of (4.30) since
\[
\|G(\tilde{v}_j(r))^*\|_{\gamma(H^{-g}, Y)} = \|G(\tilde{v}_j(r))\|_{\gamma(Y;H^{-g})} \leq K_{g,2}.
\]

Therefore relationships (4.29) and (4.30) hold also for the limit process $M^\phi$; they show that this is a martingale. Therefore, with usual martingale representation theorem, see e.g. [13], we conclude that there exists a $Y$-cylindrical Wiener process $\tilde{w}$ such that
\[
\langle M^\phi(t), \phi \rangle = \langle \int_0^t G(\tilde{v}(s)) d\tilde{w}(s), \phi \rangle.
\]
This concludes the proof that the limit process $\tilde{v}$ is a stationary martingale solution of equation (2.8). □

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