"Extended" Particles, Non Commutative Geometry and Unification

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Abstract
A reconciliation of gravitation and electromagnetism has eluded physics for nearly a century. It is argued here that this is because both quantum physics and classical physics are set in differentiable space time manifolds with point particles. Once we consider extended particles as in Quantum Superstring theory, and the consequent underlying Non-Commutative geometry, then a reconciliation is possible.

1 Introduction
Despite nearly a century of work, it has not been possible to achieve a unification of gravitation and electromagnetism. It must be borne in mind that the tools used, be it Quantum Theory or General Relativity are deeply entrenched in differentiable space time manifolds (and point particles) - the former with Minkowski space time and the latter with curved space time. The challenge has been, as Wheeler noted, the introduction of Quantum Mechanical spin half into General Relativity on the one hand and the introduction of curvature into Quantum Mechanics on the other.
More recent models including Quantum Superstrings on the contrary deal with extended and not point particles and lead to a non commutative geometry (NCG). Indeed this type of non commutativity arises if there is a minimum space time length as shown a long time ago by Snyder. What we will argue below is that once the underlying non commutative nature of
the geometry is recognized then it is possible to reconcile electromagnetism and gravitation.

2 NCG of Extended Particles

It is well known that once we consider non zero minimum space time intervals or equivalently extended particles as in Quantum Superstrings, then we have the following non commutative geometry (Cf.refs.[2]-[6]):

\[ [x, y] = 0(l^2), [p_x, p_y] \approx \frac{\hbar^2 0(1)}{l^2} \] (1)

(and similar equations) where \( l, \tau \) are the extensions of the space time coordinates.

In conventional theory the space time coordinates as also the momenta commute amongst themselves unlike in equation (1). It must be observed that the non commutative relations are self evident, in the sense that \( xy \) or \( yx \) is each of the order of \( l^2 \), and so is their difference because of the non commutativity.

Let us now introduce this effect into the usual distance formula in flat space

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \] (2)

Rewriting the product of the two coordinate differential in (2) in terms of the symmetric and non symmetric combinations, we get

\[ g_{\mu\nu} = \eta_{\mu\nu} + kh_{\mu\nu} \] (3)

where the first term on the right side of (3) denotes the usual flat space time and the second term denotes the effect of the non commutativity, \( k \) being a suitable constant.

It must be noted that if \( l, \tau \to 0 \) then equations (1) and also (3) reduce to the usual formulation.

The effect of the non commutative geometry is therefore to introduce a departure from flat space time, as can be seen from (3).

Infact remembering that the second term of the right side of (3) is small, this can straightaway be seen to lead to a linearized theory of General
Relativity\[\text{[4]}\]. Exactly as in this reference we could now deduce the General Relativistic relation
\[
\partial_\lambda \partial^\lambda h^{\mu\nu} - (\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda}) - \eta^{\mu\nu} \partial_\lambda \partial^\lambda h + \eta^{\mu\sigma} \partial_\lambda \partial_\sigma h^{\nu\lambda} = -kT^{\mu\nu}
\]
(4)
Let us now consider the non commutative relation (1) for the momentum components. Then, it can be shown using (1) and (3) that\[\text{[8]},\]
\[
\frac{\partial}{\partial x_\lambda} \frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\lambda} \quad \text{goes over to} \quad \frac{\partial}{\partial x_\lambda} \Gamma^\nu_{\mu\lambda} - \frac{\partial}{\partial x_\mu} \Gamma^\nu_{\lambda\nu}
\]
(5)
Normally in conventional theory the right side of (5) would vanish. Let us designate this nonvanishing part on the right by
\[
\frac{e}{c\hbar} F^{\mu\lambda}
\]
(6)
(5) can be written as
\[
Bl^2 \sim \frac{\hbar c}{e}
\]
(7)
where $B$ is the magnetic field, if we are to identify $F^{\mu\nu}$ with the electromagnetic tensor\[\text{[8]}\]. It will be recognized that (5) gives the celebrated expression for the magnetic monopole, and indeed it has also been shown that a non commutative space time at the extreme scale throws up the monopole\[\text{[8], [10]}\].
We have shown here that the non commutativity in momentum components leads to an effect that can be identified with electromagnetism and in fact from expression (6) we have
\[
A^\mu = \hbar \Gamma^\mu_{\nu}
\]
(8)
where $A^\mu$ is the electromagnetic four potential.
Thus non commutativity as expressed in equations (1) generates both gravitation and electromagnetism.

### 3 Discussion

1. It must be noted that equation (8) for the electromagnetic vector potential is mathematically identical to the formulation of Weyl\[\text{[11]}\]. However in Weyl’s formulation, the electromagnetic potential was put in by hand. In the
above case it is a consequence of the non commutative geometry at small scales, which again is symptomatic of the spinorial behaviour of the electron, as has been discussed in detail elsewhere\cite{4, 8}.

On the other hand the characterization of the metric in equations (2) and (3) in terms of symmetric and non symmetric components is similar to the tortional formulation of General Relativity\cite{12}. However in this latter case, there is no contribution to the differential interval from the tortional (that is non-commutative) effects. The non-commutative contribution is given by (1) and herein comes the extended, rather than point like particle.

In any case the above attempts at unification of electromagnetism and gravitation had made part headway, but unless the underpinning of a non commutative geometry is recognised, the full significance does not manifest itself. It is also well known that, if equation (8) holds, then in the absence of matter, the general relativistic field equations (4) reduce to Maxwell’s equations\cite{13}.

2. We now make the following remarks:

It can be seen from the transition to (3) from (2), that the curvature arises from the non commutativity of the coordinates. Indeed this is the classical analogue of a Quantum Mechanical result deduced earlier that the origin of mass is in the minimum space intervals and the non local Quantum Mechanical amplitudes within them as has been discussed in detail in references cited\cite{14, 15}. In Quantum Superstring theory also, the mass arises out of the tension of the string in this minimum interval. We see here the convergence of the Quantum Mechanical and classical approaches once the extension of particles is recognized.

We also know the the minimum space time intervals are at the Compton scale where the momentum $p$ equals $mc$. For a Planck mass $\sim 10^{-5}gms$, this is also the Planck scale, as in Quantum Superstring theory.

In Snyder’s original work, the commutation relations like (1) hold good outside the minimum space time intervals, and are Lorentz invariant. This is quite pleasing because in any case, even in Quantum Field Theory, we use Minkowski space time.

4 Appendix

We start with the effect of an infinitessimal parallel displacement of a vector\cite{13}.

$$\delta a^\sigma = -\Gamma^\sigma_{\mu\nu}a^\mu dx^\nu \quad (A1)$$
As is well known, (A1) represents the extra effect in displacements, due to the curvature of space - in a flat space, the right side would vanish. Considering partial derivatives with respect to the $\mu^{th}$ coordinate, this would mean that, due to (A1)

$$\frac{\partial a^\sigma}{\partial x^\mu} \rightarrow \frac{\partial a^\sigma}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} a^\nu \quad (A2)$$

where the $\Gamma$s are the Christoffel symbols. The second term on the right side of (A2) can be written as:

$$-\Gamma^\lambda_{\mu\nu} g_{\lambda} a^\sigma = -\Gamma^\nu_{\mu\nu} a^\sigma$$

where we have utilized the linearity property that in the above formulation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\eta_{\mu\nu}$ being the Minkowski metric and $h_{\mu\nu}$ a small correction whose square is neglected.

That is, (A2) becomes,

$$\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x^\mu} - \Gamma^\nu_{\mu\nu} \quad (A3)$$

From (A3) we get

$$\frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\lambda} \rightarrow \frac{\partial}{\partial x^\lambda} \Gamma^\nu_{\mu\nu} - \frac{\partial}{\partial x^\mu} \Gamma^\nu_{\lambda\nu}$$

as required.

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