Abstract—In this article, the joint estimation of state and noise covariance for linear systems with unknown covariance of multiplicative noise is considered. The measurement likelihood is modeled as a mixture of two Gaussian distributions and a Student’s t distribution, respectively. The unknown covariance of multiplicative noise is modeled as an inverse Gamma/Wishart distribution and the initial condition is formulated as the nominal covariance. By using robust design and choosing hierarchical priors, two variational Bayesian-based robust Kalman filters are proposed. The stability and convergence of the proposed filters and the covariance parameters are analyzed. The lower and upper bounds are also provided to guarantee the performance of the proposed filters. A target tracking simulation is provided to validate the effectiveness of the proposed filters.

Index Terms—Kalman filter, multiplicative noise, unknown covariance, variational Bayesian (VB).

I. INTRODUCTION

The Kalman filter (KF) is widely used in state estimation problems and is shown to be optimal in the sense of minimum variance for linear systems with Gaussian white additive measurement and process noises [1]. On the one hand, the usefulness of the KF is limited by the prior information of noise statistics. The inaccurate usage of prior information may result in bad performance and even divergence. On the other hand, in many practical applications, the prior information may be unknown and then the robust adaptive KF is introduced to solve such a problem, which can be classified into four branches including maximum likelihood, Bayesian, covariance matching, and correlation approaches. The typical implements of adaptive KF include innovation-based adaptive filter [2], interacting multiple model filter, and the Sage–Husa adaptive filter [3]. In particular, innovation-based adaptive filter takes advantage of the innovation sequence and a maximum likelihood criterion to estimate the noise covariance. Interacting multiple model filter is a Bayesian approach, while other methods can be seen as its approximations. Sage–Husa adaptive filter leverages the covariance matching approach and the maximum-a-posterior criterion to estimate the noise statistics recursively. This class of adaptive KF may suffer from issues of convergence, practical applications, and computation burden [4]. Therefore, many improved robust adaptive filters were subsequently introduced. In particular, generalized robust noise-identification filters are proposed by using Huber’s theory and the expectation maximization algorithm [5]. A maximum correntropy KF is proposed to address the heavy-tailed noise problem [6]. However, the aforementioned filters need the known nominal covariance as prior information. Then, Student’s t filters [7], robust Student’s t filters [8], and Kullback–Leibler divergence-based Student’s t filters [9] are proposed to address this problem.

More recently, a variational Bayesian (VB) based adaptive filter has received much attention since it can be used to perform approximate posterior inference and to estimate uncertain hidden parameters or state variables [10] in broad applications including machine learning, visual tracking, signal processing, etc. [11]. In fact, using appropriate conjugate prior distributions, the existing VB approaches are provided to approximate the additive measurement/process noise covariance matrices [4], [12]. Due to the strong adaptability of VB, a large number of research results are subsequently obtained. These existing VB-based filters usually model heavy-tailed non-Gaussian/Gaussian additive process and measurement noises with the Student’s t-distribution (Std) and model additive noise covariance with the Wishart distribution, the inverse Wishart distribution (IW), or the inverse Gamma distribution [13], [14].

To the best of the authors’ knowledge, all aforementioned robust adaptive KFs are provided to address state and noise covariance joint estimation problems for additive noises but are not applicable to the case of multiplicative noise. In fact, multiplicative noise is ubiquitous in practical applications including uncertain measurements and fading or reflection of the transmitted signal over channels [15], [16], [17]. Since the product of two Gaussian distributions (multiplicative noise and state) is shown to be a compressed or amplified Gaussian distribution (for the same variable) and non-Gaussian distribution (for different variables) [18], [19], this difficult issue has attracted widespread attention in different fields [17]. Although there are many robust adaptive KFs that can estimate the unknown covariance of additive noise, ones still have no suitable solution to estimate the unknown covariance of multiplicative noise.

The main contributions of this article are summarized as follows.

1) In this article, the Std and a mixture of two Gaussian (MtG) distributions, in which one is used to depict the additive noise and the other is to depict the multiplicative noise, are proposed to model measurement likelihood and to perform state estimation with unknown covariance of multiplicative noise. The distribution of multiplicative covariance is chosen as an inverse Gamma/Wishart distribution. The unknown parameters (states and covariance parameters) are jointly estimated in the VB framework.

2) Compared with the existing VB filters for additive noises [7], we propose an improved Std based KF [8] that is applicable to multiplicative noise. To solve the technical issue raised by the multiplicative noise, the VB inference is employed. Moreover, the proposed filter is capable of estimating the covariances of multiplicative and additive measurement noises as a whole, while the filters proposed in [17] often operate in a separate way. Toward this end, two robust KFs are presented to jointly estimate the covariance of multiplicative noise and noise states based on the VB inference.

3) The stability and convergence of the proposed filters, the covariance parameters, and the VB inference are analyzed. The lower and upper bounds are also provided to guarantee the performance of the proposed filters.

The rest of this article is organized as follows. Sections II and III state the considered problem and the proposed filters, respectively. Then, Section IV evaluates the performance and Section V gives a numerical simulation example. Finally, Section VI concludes this article.

Notation: \( \mathbf{N}(\mathbf{z}_k; x_k, \mathbf{R}_k) \) denotes that stochastic vector \( \mathbf{z}_k \) obeys a Gaussian distribution with a mean vector \( x_k \) and a covariance
matrix $\mathbf{R}_k$. $G(\lambda_k; \frac{1}{2}v, \frac{1}{2}v)$ denotes that variable $\lambda_k$ obeys a Gamma distribution with a dof-parameter variable $v$. $\text{IG}(\sigma_k; \alpha_k, \beta_k)$ denotes that variable $\sigma_k$ obeys an inverse Gamma distribution with a shape parameter $\alpha_k$ and a scale parameter $\beta_k$. $\text{IW}(\mathbf{R}_k; \mathbf{m}_k, \mathbf{U}_k)$ denotes that stochastic matrix $\mathbf{R}_k$ obeys an IW with a dof-parameter variable $\mathbf{m}_k$ and a scale matrix $\mathbf{U}_k$. $\text{St}(\mathbf{z}_k; \mathbf{x}_k, \mathbf{R}_k, v)$ denotes that stochastic variable $\mathbf{z}_k$ obeys an $\text{St}$ with a location parameter $\mathbf{x}_k$, a scale parameter $\mathbf{R}_k$, and a dof-parameter variable $v$. $\sigma_k, \mathbf{F}_k, \mathbf{P}_k, \mathbf{z}_k$ denote the estimated values of state $\mathbf{x}_k$ in the time-update step, the values in the measurement-update step, and their corresponding estimation error covariances, respectively. $E[\cdot], \log[\cdot]$, and $\text{tr}(\cdot)$ denote the expectation operation, the logarithmic function, and the trace operation, respectively. $p(x)$ and $g(x)$, respectively, denote the distribution of $x$ and the approximated distribution of $p(x)$. Finally, $\|\cdot\|$ and $|\cdot|$ denote, respectively, the 2-Euclidian-norm and the determinant of a matrix or a vector.

II. Problem Formulation

Consider a discrete stochastic system contaminated by multiplicative noise described by the following state-space model:

$$
\mathbf{x}_k = \mathbf{Fx}_{k-1} + \mathbf{w}_{k-1}
$$

$$
\mathbf{z}_k = \mathbf{m}_k \mathbf{Hx}_k + \mathbf{v}_k
$$

where $\mathbf{x}_k \in \mathbb{R}^n$ denotes the state vector, $\mathbf{z}_k \in \mathbb{R}^m$ denotes the measurement vector, $\mathbf{F} \in \mathbb{R}^{n \times n}$ and $\mathbf{H} \in \mathbb{R}^{m \times n}$ denote the given state transition matrix and the observation matrix, respectively, and $\mathbf{w}_k \in \mathbb{R}^n$ and $\mathbf{v}_k \in \mathbb{R}^m$ are mutually uncorrelated Gaussian white additive noises with zero means and covariance matrices $\mathbf{Q}_k$ and $\mathbf{R}_k$, respectively. In addition, $\mathbf{m}_k \in \mathbb{R}$ is an unknown Gaussian multiplicative noise with mean $\mathbf{m}_k$ and covariance $\sigma_k$.

A. Measurement Likelihood Model

Here, $\mathbf{m}_k$ is introduced and leads to the non-Gaussian change of measurement likelihood. To characterize this non-Gaussian property, in this article, the measurement likelihood is modeled as an MIG distribution model or an $\text{Std}$ model, respectively, given by

$$
p(\mathbf{z}_k | \mathbf{x}_k) = \text{N}(\mathbf{z}_k; 0, \mathbf{R}_k) + \text{N}(\mathbf{z}_k; \mathbf{m}_k \mathbf{Hx}_k, \mathbf{R}_k)
$$

(3)

$$
p(\mathbf{z}_k | \mathbf{x}_k) = \text{St}(\mathbf{z}_k; \mathbf{m}_k \mathbf{Hx}_k, \mathbf{R}_k, v)
$$

(4)

where $v$ is the dof-parameter variable, $\lambda_k$ denotes scale-parameter variable and $\lambda_k \ll 1$. Here, $\mathbf{R}_k$, $\mathbf{R}_k$, and $\mathbf{R}_k$ are, respectively, the covariances of the mixing total measurement noise, multiplicative measurement noise (in MIG), and the additive measurement noise as

$$
\mathbf{R}_k = \mathbf{R}_k + \mathbf{R}_k + \mathbf{R}_k = \sigma_k \mathbf{Hsx}_k \mathbf{H}^T
$$

(5)

$$
\mathbf{R} = \mathbf{m}_k^2 \mathbf{H}^T \mathbf{H} + \mathbf{R}_k + \mathbf{S}_k = \mathbf{E} \left( \mathbf{x}_k | \mathbf{x}_k \right).
$$

(6)

In order to better regulate the likelihood, the prior on $\lambda_k$ is chosen as $p(\lambda_k) = G(\lambda_k; \frac{1}{2}v, \frac{1}{2}v)$. In addition, Gamma prior on $v$ is introduced as a Gamma distribution $p(v) = G(v; \alpha_k, \beta_k)$.

Remark 1: The reason that we choose the likelihood $p(\mathbf{z}_k | \mathbf{x}_k)$ as an $\text{Std}$ or an MIG distribution are as follows. In the first place, it is well known that the product of Gaussian probability density functions (PDFs) for different variables is generally the Meijer G function [19], [20]. This means that the PDF of $\mathbf{m}_k \times \mathbf{x}_k$ in (2) is Meijer G and, therefore, the likelihood $p(\mathbf{z}_k | \mathbf{x}_k)$ is non-Gaussian. Meijer G functions are usually not analytically tractable. Therefore, we choose an approximation method. On the other hand, we also know that an infinite mixture of Gaussian PDFs can approach arbitrary distribution [21]. Student’s $t$ is an infinite mixture of Gaussian PDFs and, therefore, can be used to represent a Meijer G-function. In the second place, according to the definition of covariance of measurement equation, MIG is an MIG distributions, where $N(\mathbf{z}_k; 0, \mathbf{R}_k)$ is used to model the disturbance of multiplicative noise and the other $N(\mathbf{z}_k; 0, \mathbf{R}_k)$ is to model the influence of additive measurement noise.

B. Covariance Model

For the two models (3) and (4), we characterize the multiplicative noise covariance as a random variable, which, respectively, obeys an inverse Gamma distribution and an $\text{IW}$ distribution. For model (3), we directly model the multiplicative noise covariance $\sigma_k$. While, for model (4), we model the unknown covariance of measurement noises ($\mathbf{m}_k$ and $\mathbf{v}_k$) as a whole $\mathbf{R}_k$. First, we define $p(\sigma_k | \mathbf{z}_k; \mathbf{x}_k) = \text{IG}(\sigma_k; \alpha_k, \beta_k)$ (if $\mathbf{m}_k \in \mathbb{R}^{m \times m}$, then choose inverse Wishart distribution (IW), since IW is the general matrix case of IG [4]). Let the posterior of $\sigma_k$ be $p(\sigma_k | \mathbf{z}_k; \mathbf{x}_k) = \text{IG}(\sigma_k; \alpha_k + 1, \beta_k + \mathbf{z}_k^T \mathbf{z}_k - 1)$. This article uses the similar heuristics as given in [4], i.e.,

$$
\hat{\sigma}_k = \frac{\beta_k + 1}{\beta_k + m - 1}
$$

(7)

$$
\hat{\beta}_k = \hat{\sigma}_k \hat{\sigma}_k
$$

(8)

where $\rho \in (0, 1]$. The initial value is chosen as $\hat{\beta}_{0,0} = \sigma_0$, where $\sigma_0$ is an empirical constant.

Then, similarly, define $p(\mathbf{R}_k | \mathbf{z}_k; \mathbf{x}_k) = \text{IW}(\mathbf{R}_k; \hat{\mathbf{U}}_k, \hat{\mathbf{U}}_k)$ and choose the similar heuristics in [13] as

$$
\hat{\mathbf{U}}_k = \rho \hat{\mathbf{U}}_k
$$

(9)

$$
\hat{\mathbf{U}}_k = \rho \mathbf{U}_k
$$

(10)

with the initial value being $\hat{\mathbf{U}}_{0,0} = \mathbf{R}_0$, where $\mathbf{R}_0$ is also a chosen initial value.

Before presenting the main results, we formally provide the following assumption.

Assumptions 1: Both the additive process noise $\mathbf{w}_k$ and the additive measurement noise $\mathbf{v}_k$ are zero-mean Gaussian white noises. The multiplicative measurement noise $\mathbf{m}_k$ is a nonzero mean Gaussian noise. The initial state vector $\mathbf{x}_0$ is assumed to be a Gaussian distribution with mean $\mathbf{x}_0$ and covariance matrix $\mathbf{P}_0$. The initial state $\mathbf{x}_0$ and the noise sequences ($\{\mathbf{w}_k\}, \{\mathbf{v}_k\}$, and $\{\mathbf{m}_k\}$) are mutually independent.

C. One-Step Prediction

The derivation of the desired filter includes one-step prediction (time update) step and measurement update step. Since the existence of multiplicative noise does not affect the state (1), according to the standardKF, the one-step prediction of state is modeled as a Gaussian distribution

$$
p(\mathbf{x}_k | \mathbf{z}_k; \mathbf{x}_k = \mathbf{Hx}_k | \mathbf{P}_k; \mathbf{k} | \mathbf{k})
$$

(11)

where one-step prediction $\mathbf{x}_k$ and its corresponding error covariance matrix $\mathbf{P}_k$ are given by

$$
\mathbf{Q}_k = \mathbf{F} \hat{\mathbf{R}} \mathbf{F}^T + \mathbf{Q}_k
$$

(12)

$$
\mathbf{P}_{k | k} = \mathbf{F} \mathbf{P}_{k | k} \mathbf{F}^T + \mathbf{Q}_k
$$

(13)

The system model, the proposed unknown covariance model, and the one-step prediction are formulated as in (1)–(13). We next present the measurement update and then form the whole filter in the following section.

Remark 2: The considered model (2) can be used to model many practical applications in the fields of communication, signal processing, petroleum seismic exploration, target tracking, and so on [22], [23]. Different from additive measurement noise, the second- and high-order statistics of multiplicative measurement noise are usually unknown and difficult to estimate because they trigger additional difficulties and lead to great fluctuations for state signals.
III. PRESENTED FILTERS

In this section, we first introduce the VB inference, which will be used together with the fixed-point iteration method in the derivation of the desired filter. Then, we present two filters to address the problem of unknown multiplicative noise covariance, one based on the assumption that the likelihood function is the MtG model and the other on the Std model.

A. VB Inference

Based on Bayes’ rule, we have

\[ p(\Phi_k|z_{1:k}, \Psi_k) = \frac{p(z_{1:k}|\Phi_k)p(\Phi_k|z_{1:k})}{p(z_{1:k})} \]

where \( \Phi_k = \{x_k, \sigma_k, \hat{R}_k\} \) and \( \Psi_k = \{\hat{\alpha}_{k-1}, \hat{\beta}_{k-1}, \hat{U}_{k-1}\} \). Since we cannot obtain the analytical solution of \( p(\Phi_k|z_{1:k}, \Psi_k) \), \( p(\Phi_k|z_{1:k}, \Psi_k) \) is approximated using a distribution \( q(\Phi_k) \). Based on variational inference [21], it yields

\[ \log q(\theta) = F_{\Phi_k}(\cdot) \log p(z_{1:k}|\Phi_k) p(\Phi_k|z_{1:k}) (14) \]

where \( \theta \) is a member of \( \Phi_k \) and \( \theta \) represents its complementary set. Then, by taking \( \Phi_k = \{x_k, \sigma_k\} \), we have

\[ p(\Phi_k|z_{1:k}) = p(z_{1:k} | \Phi_k) = \frac{1}{Z_k} \exp \{ -\frac{1}{2} \|z_{1:k} - \mu(z_{1:k})\|^2 \} \]

(15)

Since (15) cannot be solved directly, we apply the fixed-point iteration approach to solve it, i.e., \( q(\theta) \) is updated as \( q^{(i+1)}(\theta) \) at the \( (i+1) \)th iteration by \( q^{(i)}(\theta) \) under total \( L \) iterations [21].

We next specify the likelihood function (3) to find the first proposed filter (Algorithm 1), and then specify the likelihood function (4) to obtain the second proposed filter (Algorithm 2).

B. Proposed Filters

1) Measurement Update Based on MtG Assumption: Rewrite (15) as

\[ p(z_{1:k}|\Phi_k) = \mathbb{N}(\hat{x}_k; \hat{\alpha}_{k-1}, \hat{\beta}_{k-1}) \]

(16)

First, we let \( \theta = x_k \) and substitute (16) into (14). It then follows that

\[ \log q^{(i+1)}(x_k) = N(z_k; m_k \hat{H} \hat{X}_k, \hat{R}_k) \]

\[ = N(x_k; \hat{x}_k, \hat{P}_{k|k-1}) \]

Therefore, \( q^{(i+1)}(x_k) \) can be approximated using \( q^{(i+1)}(x_k) = N(x_k; \hat{x}_{k|k-1}, \hat{P}_{k|k-1}) \), where \( \hat{x}_{k|k-1} \) and \( \hat{P}_{k|k-1} \) are given by

\[ \begin{align*}
\hat{R}_{k|k-1} &= \sigma_k^2 H_k^T \hat{R}_k H_k + m_k^2 \hat{H} P_{k|k-1} - m_k H \hat{x}_{k|k-1} \\
K_{k|k-1} &= m_k \hat{H} \hat{X}_k - m_k \hat{H} \hat{x}_{k|k-1} \\
S_{k|k-1} &= \mathbb{E}[x_{k|k-1}^T (x_{k|k-1})] \\
\end{align*} \]

(21)

where \( \hat{x}_{k|k-1} \) and \( \hat{P}_{k|k-1} \) are given in (12) and (13).

Letting \( \theta = \sigma_k \) and substituting (16) into (14) yields

\[ \log q^{(i+1)}(\sigma_k) = N(z_k; m_k \hat{H} \hat{X}_k, \sigma_k^2) \]

(17)

where \( \hat{x}_{k|k-1} \) and \( \hat{P}_{k|k-1} \) are given in (12) and (13).

Finally, the proposed MtG filter is summarized as Algorithm 1, where \( L \) denotes the iteration number and \( \eta \) denotes the threshold.

2) Measurement Update Based on Std Assumption: We see from the abovementioned section that the MtG filter estimates multiplicative noise directly. However, the difference between multiplicative \( m_k \) and additive \( v_k \) measurement noise actually cannot be identified in reality. Therefore, we take them as a whole to estimate. In this case, as mentioned above, the measurement likelihood is modeled as Std (4). Note that the proposed Std filter is different from the existing Std filters [8], where both process and measurement equations are modeled and heavy-tailed additive noise is considered. On the other hand, the proposed Student’s \( t \) filter in this section is applied to model the non-Gaussian property of likelihood caused by multiplicative noise.

Similar to the MtG filter, we rewrite (15) as

\[ p(z_{1:k}|\Phi_k) = \frac{1}{Z_k} \exp \{ -\frac{1}{2} \|z_{1:k} - \bar{\mu}(z_{1:k})\|^2 \} \]

(22)

Then, log \( \log p(z_{1:k}) = \log p(z_{1:k}|\Phi_k) = \log \bar{\mu}(z_{1:k}) \]

(23)

(24)

The proposed MtG filter is summarized as Algorithm 1, where \( L \) denotes the iteration number and \( \eta \) denotes the threshold.

Algorithm 1: Proposed MtG Filter.

Input: \( F, H, Q, R, m_k, \hat{x}_{k|k}, P_{k|k-1}, z_{k-1}, z_k, \rho, \hat{\alpha}, \hat{\beta}, \eta, L \); 
1: Initialize \( \hat{\alpha}_{k|k-1} \) and \( \hat{\beta}_{k|k-1} \) as in (7) and (8); 
2: Update \( \hat{x}_{k|k-1} \) and \( P_{k|k-1} \) as in (12) and (13); 
3: for \( i = 0 \) to \( L - 1 \) do 
4: Update \( \hat{x}_{k|k-1}^{(i+1)}, \hat{R}_{k|k-1}^{(i+1)} \), and \( S_k^{(i+1)} \) as in (17)–(21); 
5: if \( \|S_k^{(i+1)}\| \leq \eta \), stop iteration; 
6: Update \( \hat{\alpha}_{k|k-1}^{(i+1)}, \hat{\beta}_{k|k-1}^{(i+1)} \) as in (22)–(24); 
7: end for 
8: return \( \hat{x}_{k|k} = \hat{x}_{k|k}^{(L)}, \hat{R}_{k|k} = \hat{R}_{k|k}^{(L)}, \hat{\alpha}_{k|k} = \hat{\alpha}_{k|k}^{(L)}, \) \( \hat{\beta}_{k|k} = \hat{\beta}_{k|k}^{(L)} \).

Output: \( \hat{x}_{k|k}, \hat{R}_{k|k}, \hat{\alpha}_{k|k}, \text{and} \hat{\beta}_{k|k} \).
First, letting \( \theta = \lambda_k \) and substituting (26) into (14), we obtain

\[
\log q^{(+)}(\lambda_k) = \left( \frac{m + v}{2} - 1 \right) \log \lambda_k - \frac{1}{2} \left\{ v + \text{tr} \left( E_k u_k^{-1} \right) \right\} \lambda_k
\]

where

\[
E_k^{(0)} = E^{(0)} \left[ (z_k - m_k H x_k)(z_k - m_k H x_k)^\top \right] = (z_k - m_k H x_k^{(0)})(z_k - m_k H x_k^{(0)})^\top + m_k^2 H P_{k|k-1} H^\top.
\]

Then, \( q^{(+)}(\lambda_k) \) can be updated as a Gamma distribution with shape parameter \( \gamma_k^{(+)} \) and rate parameter \( \delta_k^{(+)} \) being

\[
\gamma_k^{(+)} = \frac{1}{2} (m + v)
\]

\[
\delta_k^{(+)} = \frac{1}{2} \left\{ v + \text{tr} \left( E_k u_k^{-1} \right) \right\}.
\]

Second, letting \( \theta = \hat{R}_k \) and using (26) in (14), we obtain

\[
\log q^{(+)}(\hat{R}_k) = \frac{1}{2} \left( m + \hat{u}_{k|k-1} + 2 \right) \log |\hat{R}_k| - \frac{1}{2} \text{tr} \left( B_k^{(0)} E^{(+)}[|\hat{R}_k|] \right) + C
\]

where \( B_k^{(0)} \) is

\[
B_k^{(0)} = E^{(0)} \left[ (z_k - m_k H x_k)(z_k - m_k H x_k)^\top \right] = (z_k - m_k H x_k^{(0)})(z_k - m_k H x_k^{(0)})^\top + m_k^2 H P_{k|k-1} H^\top.
\]

Employing (31), \( \hat{R}_k \) is updated as \( q^{(+)}(\hat{R}_k) = \text{IW}(\hat{R}_k;\hat{u}_{k|k-1}^{(L)},\hat{U}_{k|k-1}) \) with shape parameters being

\[
\hat{u}_{k|k-1}^{(L)} = \hat{u}_{k|k-1} + 1
\]

\[
\hat{U}_{k|k-1}^{(L)} = B_k^{(0)} E^{(+)}[\hat{R}_k] + \hat{U}_{k|k-1}
\]

Third, letting \( \theta = x_k \) and using (26) into (14), we obtain

\[
\log q^{(+)}(x_k) = -0.5 E^{(+)}[\lambda_k] (z_k - m_k H x_k)^\top \times E^{(+)}[\hat{R}_k] (z_k - m_k H x_k) - 0.5 (x_k - \hat{x}_{k|k-1})^\top \times P_{k|k-1}^{-1} (x_k - \hat{x}_{k|k-1}) + C
\]

where \( E^{(+)}[\lambda_k] = \frac{n}{\lambda_k} \) and \( E^{(+)}[\hat{R}_k] \) are

\[
E^{(+)}[\lambda_k] = \frac{1}{\gamma_k^{(+)}} \text{ and } E^{(+)}[\hat{R}_k] = \frac{1}{\delta_k^{(+)}} \text{ with } \gamma_k^{(+)} = \frac{1}{2} \left\{ v + \text{tr} \left( E_k u_k^{-1} \right) \right\}.
\]

Define the modified measurement likelihood

\[
p^{(+)}(z_k|x_k) = N(z_k; m_k H x_k, R_k^{(+)})(37)
\]

where \( R_k^{(+)} \) is formulated as

\[
R_k^{(+)} = \left\{ E^{(+)}[\hat{R}_k] \right\}^{-1} E^{(+)}[\lambda_k]
\]

Then, we obtain \( q^{(+)}(x_k) = N(x_k; \hat{x}_{k|k}^{(+)}, P^{(+)}_{k|k}) \), where \( \hat{x}_{k|k}^{(+)} \), \( P^{(+)}_{k|k} \) and gain matrix \( K^{(+)}_{k|k} \) are given by

\[
K^{(+)}_{k|k} = m_k P_{k|k} H^\top \left\{ R_k^{(+)} \right\}^{-1}
\]

\[
\hat{x}_{k|k}^{(+)} = \hat{x}_{k|k-1} + K^{(+)}_{k|k} (z_k - m_k H x_k^{(0)})
\]

\[
P^{(+)}_{k|k} = P_{k|k-1} - m_k K^{(+)}_{k|k} H P_{k|k-1}
\]

Finally, after L-step iteration, the posterior \( q(\lambda_k), q(\hat{R}_k), \) and \( q(x_k) \) can be approximated as

\[
q(\lambda_k) \approx q^{(L)}(\lambda_k) = G(\lambda_k; \gamma_k^{(L)}, \delta_k^{(L)})
\]

\[
q(\hat{R}_k) = \text{IW}(\hat{R}_k; \hat{u}_{k|k}^{(L)}, \hat{U}_{k|k}^{(L)}) \approx q^{(L)}(\hat{R}_k) = \text{IW}(\hat{R}_k; \hat{u}_{k|k}^{(L)}, \hat{U}_{k|k}^{(L)})
\]

\[
q(x_k) = N(x_k; \hat{x}_{k|k}^{(L)}, P^{(L)}_{k|k}) \approx q^{(L)}(x_k) = N(x_k; \hat{x}_{k|k}^{(L)}, P^{(L)}_{k|k})
\]

The proposed Student’s t filter is summarized in Algorithm 2.

Remark 3: The current state-of-the-art filters are proposed either for known multiplicative noise or only for additive time-varying noise covariance [4], [13]. Since multiplicative noise leads the likelihood function to be a non-Gaussian distribution, the existing filters are not applicable to the case of multiplicative noise. The proposed filters can well depict the non-Gaussian property of likelihood, so as to estimate the states and noise covariance more accurately.

IV. PERFORMANCE ANALYSIS

In this section, we first analyze the influence of the chosen parameters on the convergence of the proposed filters. Then, we calculate the upper and lower bounds of the estimation error covariance. Finally, we analyze the convergence of VB inference using a fixed-point iteration.

A. Parameter Influence

In this section, we analyze the influence of the chosen covariance parameters on the convergence of the proposed filters.

1) Convergence Analysis for the Cases of Different Parameters: The proposed two filters characterize the distribution of noise covariance as \( p(\sigma_k) = IG(\sigma_k; \theta_k, \bar{\theta}_k) \) and \( p(R_k) = \text{IW}(R_k; \theta_k, \bar{\theta}_k) \), respectively. We next analyze the influence of these parameters on covariance estimation.

a) Convergence analysis of \( \theta_k \) and \( \bar{\theta}_k \): We first show the convergence of the sequence \( \{\hat{\theta}_k\} \) and \( \rho \hat{\theta}_k \gg 1/2 \) when \( 0.6 < \rho < 1 \).

According to (7) and (22), sequence \( \{\hat{\sigma}_k\} \) can be obtained from the following recursive form:

\[
\hat{\theta}^{k+1} = 1/2 + \rho \hat{\theta}_k
\]
\[ \hat{\alpha}_{k+2|k+2} = 1/2 + \rho \hat{\alpha}_{k+1|k+1}. \]  

Subtracting (a2) from (a1) yields
\[ \hat{\alpha}_{k+1|k+1} - \hat{\alpha}_{k|k} = \rho \hat{\alpha}_{k+1|k+1} - \hat{\alpha}_{k|k}. \]  

Let \( b_{k|k} \) be \( \hat{\alpha}_{k+1|k+1} - \hat{\alpha}_{k|k} \). Then, (a3) can be rewritten as
\[ b_{k|k} = \hat{\alpha}_{k|k} \]  

where \( b_{k|k} = \hat{\alpha}_{k+1|k+1} - \hat{\alpha}_{k|k} \). Therefore, the sequence \( \{b_{k|k}\} \) can be organized as
\[ b_{k|k} = \rho^{k} b_{1|1}. \]  

From (a3), it follows that
\[ \hat{\alpha}_{k+1|k+1} - \hat{\alpha}_{k|k} = \rho^{k} b_{1|1}. \]  

This yields
\[ \hat{\alpha}_{k+1|k+1} - \hat{\alpha}_{k|k} = \rho^{k} b_{1|1} \]  

\[ \ldots \]  

\[ \hat{\alpha}_{2|2} - \hat{\alpha}_{1|1} = b_{1|1}. \]  

By summing from (a6) to (a9), we obtain
\[ \hat{\alpha}_{k+1|k+1} - \hat{\alpha}_{1|1} = b_{1|1} \sum_{j=0}^{k-1} \rho^{j} = b_{1|1} \frac{1 - \rho^{k}}{1 - \rho}. \]  

If \( 0.6 < \rho < 1 \), it follows that
\[ \lim_{k \to \infty} \hat{\alpha}_{k+1|k+1} = b_{1|1} \frac{1 - \rho}{1 - \rho} + \hat{\alpha}_{1|1} = \frac{1}{2} \hat{\alpha}_{1|1} - \hat{\alpha}_{1|1} + \hat{\alpha}_{1|1} = \frac{1}{2} \hat{\alpha}_{1|1} \]  

Thus, \( \{\hat{\alpha}_{k|k}\} \) converges to \( \frac{1}{2} \hat{\alpha}_{1|1} \). In addition, (a11) indicates that \( \rho \hat{\alpha}_{k|k} \to 1/2 \).

Similarly, \( \lim_{k \to \infty} \hat{\alpha}_{k+1|k+1} = m + 2 - \rho(m + 1) \)  

which proves that \( \{\hat{\alpha}_{k|k}\} \) converges to \( m + 2 - \rho(m + 1) \).

**b) Convergence analysis of \( \hat{\beta}_{k|k}, \hat{U}_{k|k}, \hat{\sigma}_{k}, \) and \( \hat{R}_{k} \):**

From (8) and (23), we obtain
\[ \hat{\beta}_{k|k} = \rho \hat{\beta}_{k+1|k+1} + \frac{1}{2} \text{tr}(A_k) \]  

where
\[ A_k = \left[ (z_k - \tilde{m}_k \hat{H}_{k|k}^{(1)}) (\hat{H}_{k|k}^{(1)} H_k^{\top})^{-1} (z_k - \tilde{m}_k \hat{H}_{k|k}^{(1)})^{\top} + (\tilde{m}_k)^2 \hat{H} \right] \]  

From (10) and (33), we can obtain
\[ \hat{U}_{k+1} = \hat{B}_k^{(1)} E_k^{(1)} |_{\hat{R}_{k|k}} + \rho \hat{U}_{k-1} \]  

From (a13) and (a14), we know that \( \hat{\beta}_{k|k} \) and \( \hat{U}_{k+1|k+1} \) are estimated by using the state estimation and its error covariance at each iteration, which is crucial for the estimation performance of noise covariances thereby. \( \hat{\beta}_{k|k} \) and \( \hat{U}_{k|k} \) directly affect the covariance estimations \( \hat{\sigma}_{k} \) and \( \hat{R}_{k} \), respectively. Meanwhile, they are conversely affected by the state estimation. Therefore, this establishes the relation between state estimation and noise covariance estimation.
Furthermore, since \( \delta_k^{i+1} = \frac{1}{2} \left( v + \text{tr}(E^{(i)}(R_k^{-1})) \right) \), by using the similar approach on \( \{ A_k^{i+1} \} \), we can obtain that \( \delta_k^{i+1} \) converges to a bounded value.

Finally, since \( E^{(i+1)}[A_k] = \bar{A}_k^{(i+1)}/\delta_k^{(i+1)} \), we can easily prove that \( \lambda_k \) is convergent and bounded.

2) Relation Between State Estimation and Noise Covariance Estimation: The following theorem explains the relation between state estimation error and noise covariance estimation.

**Theorem 2:** Let \( \hat{x}_{k|k} = \hat{x}_{k|k} - \hat{X}_{k} \) be the estimation error between the proposed state estimation \( \hat{x}_{k|k} \) and the optimal state estimation \( \hat{X}_{k} \), and \( \tilde{R} = \hat{R} - \tilde{R} \) be the noise derivation between the real noise covariance (R) and the estimation (\( \hat{R} \)). If the innovation \( (z_k - \tilde{m}_k \hat{H} \hat{x}_{k|k-1}) \) does not change dramatically, the state estimation error \( \hat{x}_{k|k} \) and the noise estimation derivation \( \tilde{R} \) are positively interrelated. In particular

\[
\hat{x}_{k|k} = \tilde{m}_k P_{k|k-1} H^T \left[ \left( U^{-1}(U^{-1} + \tilde{R}^{-1})^{-1}U \right) \right] \times (z_k - \tilde{m}_k \hat{H} \hat{x}_{k|k-1}) \quad \text{(b1)}
\]

where \( U = \tilde{m}_k^2 \hat{H} \hat{H}^T \hat{x}_{k|k-1} + \tilde{R} \). This means that the accuracy of state estimation increases with the improvement in estimation performance of noise covariance, and vice versa.

**Proof:** Please refer to [24] for details. ■

3) Performance With Different \( \rho \): Then, we discuss the influence of parameter \( \rho \) (forgetting factor). From (36) and (38), we obtain

\[
\tilde{R}^{(i+1)} = \left[ \frac{U^{(i+1)}}{(u^{(i+1)}_k - m - 1)}E^{(i+1)}[A_k] \right].
\]

Based on (32) and (33), the abovementioned equation can be rewritten as

\[
\tilde{R}^{(i+1)} = \frac{\rho(u^{(i+1)}_k - m - 1) + B^{(i)}}{\rho(u^{(i+1)}_k - m - 1) + 1}.
\]

According to \( p(\tilde{R}_k|\hat{x}_{k|k-1}) = \text{IW}(\tilde{R}_k; \hat{u}_{k|k-1}, \tilde{U}_{k|k-1}) \), \( \tilde{R}^{(i+1)} \) is formulated as \( \tilde{R}^{(i+1)} = \frac{\rho(u^{(i+1)}_k - m - 1) + B^{(i)}}{1 + \rho(u^{(i+1)}_k - m - 1)} \).

Combining the abovementioned equations yields

\[
\tilde{R}^{(i+1)} = \frac{B^{(i)} + \rho(u^{(i+1)}_k - m - 1) + B^{(i)}}{\rho(u^{(i+1)}_k - m - 1) + 1} \quad \text{(c1)}
\]

Then, \( \tilde{R}^{(i+1)} \) can be rewritten as

\[
\tilde{R}^{(i+1)} = \frac{B^{(i)} + \rho(u^{(i+1)}_k - m - 1) + B^{(i)}}{\rho(u^{(i+1)}_k - m - 1) + 1} \quad \text{(c2)}
\]

This yields

\[
\lim_{k \to +\infty} \tilde{R}^{(i+1)} = \rho/(1 - \rho).
\]

4) Performance With \( R_k \): We next discuss the effect of \( R_0 \) (\( \sigma_0 \) can be similarly available). In the fixed-point iteration, the initial value is

\[
\tilde{R}_0 = R_{k=1}.
\]

Set \( \tau_k = \frac{\zeta^{(k)}(\tilde{R}_0)}{\tilde{R}_0} \), \( C_k = \frac{\tilde{R}_0^{(k-1)}}{\tilde{R}_0^{(k-1)}} \), where \( 0 < \tau_k < 1 \) and \( \tilde{C}_k \geq 0 \). (c1) is rewritten as

\[
\tilde{R}_k = \frac{\tau_k \tilde{R}_{k=1} + C_k}{\tilde{R}_0}.
\]

It then follows that

\[
\tilde{R}_{k=1} = \sum_{i=1}^{k-1} \tilde{C}_i + \left( \prod_{i=1}^{k-1} \tau_i \right) \tilde{R}_0.
\]

From (d3), we know that the initial value has effects on \( \tilde{R}^{(0)} \). \( R_0 \) proportionally affects on \( \tilde{R}_k \) as \( k \) increases. To ensure that \( \tilde{R}^{(k)} \) converges to the true value, we need to choose appropriate initial value \( \tilde{R}^{(0)} \) to ensure local convergence of VB inference. Therefore, these parameters need to be selected near their true values and we rely on our engineering experience. Thanks to the diagonal form, we can select \( \tilde{R}_0 \) as \( \tilde{R}_0 = \text{diag}[r_1, \ldots, r_i, \ldots, r_m] \), where \( r_j > 0 \).

5) Performance With \( R_k \): Finally, we prove the stability of the presented filter with an initial value \( \tilde{R}_0 \). Based on (d3), \( R_0 > 0 \), \( 0 < \tau_k < 1 \) and \( \tilde{C}_j \geq 0 \), we obtain \( R_{k=1} > 0 \). Note that (31) yields \( B_1^{(1)} \geq 0 \). Then, substituting the abovementioned equations into (d3) yields \( \tilde{R}^{(i+1)} > 0 \). This means that \( \tilde{R}_k \) is positive definite and indicates that the proposed filter is stable.

B. Bounds of the Error Covariance

This section calculates the upper and lower bounds of the estimation error covariance.

In order to obtain the error bounds, it is convenient to define the equivalent system of (1) and (2) as

\[
x_k = Fx_{k-1} + w_{k-1} \quad \text{and} \quad z_k = Hx_k + v_k
\]

where the state equation is the same as (1), \( H_m \) is defined as \( H_m = \tilde{m}_k H_k (I_k) < \infty \), and \( v_k \) is a zero-mean Gaussian white noise and satisfies \( E[v_k^{(m)}(v_k^{(n)})^T] = R_m \delta_{kn} \), where \( R_m = \sigma_m^2 H_k S_k (H_k^T) + R_k \). \( S_k \) is defined as \( \tilde{S}_k = E[x_k x_k^T] = F S_k F^T + Q_{k-1} \) with a boundary condition \( S_0 = P_{0|0} \). Since \( \tilde{R}_k \) is positive definite, so is \( \tilde{R}_k \). Then, we define

\[
M(T, T-l) = \sum_{k=T-l}^{T} F_k^T (H_m^{(m)})^{-1}(R_k)^{-1} (H_m^{(m)}) F_k
\]

\[
W(T, T-l) = \sum_{k=T-l}^{T} F_k^T Q_k F_k
\]

where \( F = F_k, F_{k-1} = F_{k-1}, \ldots, F_1, F_0, F_{T-1}, F_{T-2} \cdots F_{1,1,k} (k \leq T) \). Equations (g2) and (g3) are the stochastic observability matrix and stochastic controllability matrix, respectively. The following theorem on the bounds of the error covariance matrix is established.

**Theorem 3:** If system (1) and (g1) is uniformly controllable and uniformly observable, i.e.,

\[
\alpha_1 I \leq \tilde{W}(k, k - N + 1) \leq \beta_1 I
\]

\[
\alpha_2 I \leq \tilde{M}(k, k - N + 1) \leq \beta_2 I
\]

where \( \alpha_1, \alpha_2, \beta_1, \beta_2 > 0 \) and \( N \) is a positive integer, and assume \( P_{0|0} > 0 \), the estimation error covariance has a uniform upper bound and a uniform lower bound, i.e.,

\[
\frac{\alpha_1}{1 + n^2 \beta_1 \beta_2} I \leq P_{k|k} \leq \frac{1 + n^2 \beta_1 \beta_2}{\alpha_2} I.
\]

**Proof:** Please refer to [24] for details. ■
C. Convergence Analysis of VB Inference

This section investigates the convergence of the fixed-point iteration of the iterative VB procedure (14), which is used in the derivation process. Since \( \Phi_k = \{x_k, \sigma_k, R_k\} \) is considered, the proposed filters take the form

\[
q^{(i)} = T(q^{(i-1)})
\]

for \( i = 1, \ldots, L \), where \( T \) denotes a certain continuous mapping and \( \{q^{(i)}\} \) denotes the sequence of \( q \). Since (e1) is difficult to analyze, we take another form

\[
\phi^{(i)} = (1 - \varepsilon)\phi^{(i-1)} + \varepsilon T(\phi^{(i-1)}) \triangleq \Omega_i(\phi^{(i-1)})
\]

where \( \varepsilon > 0 \) when \( \varepsilon = 1 \), (e2) becomes (e1) and \( \Omega_i(s) \) is a continuous mapping with respect to \( s \) and the time index \( n \).

We use the parameter set \( \Theta = \{\sigma_k, R_k\} \) and the state \( x_k \) in the fixed-point iteration. Then, the iterative stage in the algorithm corresponding to (16) is

\[
\Theta^{(i)} = (1 - \varepsilon)\Theta^{(i-1)} + \varepsilon T(\Theta^{(i-1)}) \triangleq \Omega_i(\Theta^{(i-1)})
\]

In the fixed-point iteration of the VB procedure, we apply an approximation probability density \( q(x_k, \Theta) \) for \( p(x_k, \Theta | y_k) \), which can be factorized as \( q(x_k, \Theta) = q(x_k)q(\Theta) \). The states \( q(x_k) \) and the values of the hyperparameters of \( q(\Theta) \) are obtained by an iterative procedure. For the \((i)\)th iteration, \((i-1)\)th state estimation \( q^{(i-1)}(x_k) \) is given. Then, we perform the following two steps of VB procedure.

Step 1: Optimize the hyperparameters in \( q^{(i)}(\Theta) \) for fixed \( q^{(i)}(x_k) \).

Step 2: Optimize \( q^{(i)}(x_k) \) for fixed \( q^{(i)}(\Theta) \), which is calculated by Step 1.

Now, we investigate the convergence of the aforementioned two steps. Despite the convergence of a fixed-point algorithm is given in [25], the convergence of VB inference using the fixed-point iteration is still an open issue.

Suppose that the true value of the parameter \( \Theta = \Theta^* \), we next investigate whether the iterative algorithm (e3) with the aforementioned two steps is convergent. The following theorem can be established.

Theorem 4: With probability 1 as \( n \) approaches infinity, the iterative procedure (e3) converges locally to the true value \( \Theta^* \) whenever \( 0 < \varepsilon < 2 \), i.e., (e3) converges to the true value \( \Theta^* \) whenever the starting value is sufficiently near to \( \Theta^* \).

Proof: Please refer to [24] for details.

V. SIMULATIONS

In this section, we use a simulation example to verify the presented filters. A target moves with a constant velocity in 2-D space following a motion model given by \( x_k = \begin{bmatrix} I_2 & \Delta tI_2 \\ 0 & I_2 \end{bmatrix} x_{k-1} + w_{k-1} \) and \( x_k = m_k[I_2 \ 0]x_k + v_k \), where \( x_k = [x_k, y_k, \dot{x}_k, \dot{y}_k]^T \) denotes the cartesian coordinates and corresponding velocities, \( I_2 \) is a 2-D identity matrix and \( \Delta t = 1 \) is the sampling interval. Multiplicative noise \( m_k \) is set to be a Gaussian distribution with mean \( m_k = 5.5 \) and covariance \( \sigma_m = 2 + 0.05 \cos(\frac{k}{20}) \), where \( T = 500 \) is the total time.

The additive process and measurement noises are set to be Gaussian with zero means and covariance matrices \( Q_k = q_{11}^2 \Delta t I_2 \) and \( R_k = r_{11}^2 + 0.5 \eta \), where \( q \) and \( r \) are, respectively, given as \( q = 1 \text{ m}^2/\text{s}^3 \) and \( r = 0.02 \text{ m}^2/\text{s}^2 \).

Because no prior information on multiplicative noise covariance is available, \( \sigma_m \sim \text{IG}(1, 1) \) was used. In addition, the initial nominal covariance is selected as \( \hat{R}_0 = \varepsilon I_2 \). The nominal covariance KF (KF), the robust Student’s t KFs (RSTKF 1 and RSTKF 2 [7], [8]), the VB adaptive Kalman filter (VBAKF) [13], the VB particle filter (VBPF) [200 particles] [17], and the optimal Kalman filter for multiplicative noise (given true covariance) are compared. In the proposed filters, we set \( \rho = 0.8, \alpha_0 = 0.1 = c_0 = d_0 = 0.1, \eta = 10^2 \), and \( L = 20 \). The parameter of initial covariance \( \varepsilon \) is set as \( \varepsilon = 3 \). Besides, performance metrics are chosen as the root mean square error (RMSE), the average RMSE (ARMSE), the square root of the normalized Frobenius norm (SRNFN) of the measurement noises, and the average SRNFN (ASRNFN). In particular, \( \text{RMSE} \triangleq \sqrt{\frac{1}{M} \sum_{i=1}^M (x_k^i - \hat{x}_k^i)^2 + (y_k^i - \hat{y}_k^i)^2} \), \( \text{ARMSE} \triangleq \sqrt{\frac{1}{M} \sum_{k=1}^M \sum_{i=1}^T \| \hat{R}_{k,i} - R_{k,i} \|^2} \), and \( \text{ASRNFN} \triangleq \frac{1}{m^2 \sum_{i=1}^M \sum_{k=1}^T \| \hat{R}_{k,i} - R_{k,i} \|^2} \), where \( (x_k^i, y_k^i) \) and \( (\hat{x}_k^i, \hat{y}_k^i) \) are, respectively, the true and the estimated variables (position or velocity) at the \( t \)th Monte Carlo run. \( \hat{R}_{k,i} \) and \( R_{k,i} \) denote, respectively, the estimated and the true total measurement noise covariances, and \( M = 100 \) denotes the total number of Monte Carlo runs. The initial state is given as \( x_0 \sim N(x_0, P_0) \) [initialization (sampling) for VBPF], where \( x_0 = [100, 100, 10, 10, 10] \) and \( P_0 = 100 \times I_4 \).

The RMSEs of positions and velocities from the state-of-the-art filters and the presented filters are shown in Fig. 1. It can be seen that the presented filters have smaller RMSEs than those of state-of-the-art filters, including VBAKF, VBPF, and RSTKF. It can also be seen that the existing filters diverge eventually, while the proposed filters have robust convergence. Fig. 2 illustrates the ARMSEs of positions and velocities when \( L = 1 \) [20]. It can be seen that the presented filters have smaller ARMSEs and converge to the minimum when \( L = 2 \). Fig. 3 shows the quantitative SRNFN and ASRNFN of the existing filters. It is shown that the proposed filters estimate the noise covariance much better than the state-of-the-art filters. Therefore, the proposed filters have a faster convergence rate than those of state-of-the-art filters, including VBAKF, VBPF, and RSTKF. The Std filter (Sf) deals with the whole observation noise as a whole, therefore, it can get more accurate results. However, the MTG filter only deals with the case of multiplicative noise as an individual one and ignores the possible relationship with the whole. The results are, therefore, relatively poor but still better than the state-of-the-art filters.

Remark 4: According to [7], [9], [13], [26], we can see that the model used in the simulation is a typical example of the problem of noise statistics estimation. In fact, the model (1) and (2) considered in this article is a more generalized form of the model in [7], [9], [13], and [26]. Besides, the existing state-of-the-art filters work either for time-invariant multiplicative noise or only for additive time-varying noise covariances. The proposed filters, on the other hand, are applicable
to the case of time-varying multiplicative noise covariance. Since the mean of multiplicative noise in simulation is 5.5, which indicates that the true state signal is amplified by 5.5 times, the existing filters diverge eventually. However, the proposed filters can effectively eliminate the influence of multiplicative noise.

VI. CONCLUSION

In this article, we studied the joint estimation problem of state and noise covariance for linear systems with unknown covariance of multiplicative noise. Based on assumptions that an Std and an MtG distributions as the non-Gaussian likelihood functions, two novel VB-based robust filters were developed, where the states together with noise covariances were deduced by choosing the inverse Gamma/Wishart priors. The stability and convergence of the noise covariance parameters and the proposed filters were analyzed. Simulation results illustrated that the presented filters had a better performance and were robust enough to resist multiplicative noise.

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