A classification of nilpotent 3-BCI groups

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Abstract

Given a finite group \( G \) and a subset \( S \subseteq G \), the bi-Cayley graph \( \text{BCay}(G, S) \) is the graph whose vertex set is \( G \times \{0, 1\} \) and edge set is \( \{(x, 0), (sx, 1)\} : x \in G, s \in S \}. \) A bi-Cayley graph \( \text{BCay}(G, S) \) is called a BCI-graph if for any bi-Cayley graph \( \text{BCay}(G, T) \), \( \text{BCay}(G, S) \cong \text{BCay}(G, T) \) implies that \( T = gS^\alpha \) for some \( g \in G \) and \( \alpha \in \text{Aut}(G) \). A group \( G \) is called an \( m \)-BCI-group if all bi-Cayley graphs of \( G \) of valency at most \( m \) are BCI-graphs. In this paper we prove that, a finite nilpotent group is a 3-BCI-group if and only if it is in the form \( U \times V \), where \( U \) is a homocyclic group of odd order, and \( V \) is trivial or one of the groups \( \mathbb{Z}_2^r \), \( \mathbb{Z}_2^r \), and \( \mathbb{Q}_8 \).

Keywords: bi-Cayley graph, BCI-group, graph isomorphism.

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1 Introduction

In this paper every group and every (di)graph will be finite. Given a group \( G \) and a subset \( S \subseteq G \), the bi-Cayley graph \( \text{BCay}(G, S) \) of \( G \) with respect to \( S \) is the graph whose vertex set is \( G \times \{0, 1\} \) and edge set is \( \{(x, 0), (sx, 1)\} : x \in G, s \in S \}. \) We call two bi-Cayley graphs \( \text{BCay}(G, S) \) and \( \text{BCay}(G, T) \) bi-Cayley isomorphic if \( T = gS^\alpha \) for some \( g \in G \) and \( \alpha \in \text{Aut}(\Gamma) \) (here and in what follows for \( x \in G \) and \( R \subseteq G \), \( xR = \{xr: r \in R\} \)).
It is an easy exercise to show that bi-Cayley isomorphic bi-Cayley graphs are isomorphic as usual graphs. The converse implication is not true in general, and this makes the following definition interesting (see [21]): a bi-Cayley graph $BCay(G, S)$ is a BCI-graph if for any bi-Cayley graph $BCay(G, T)$, $BCay(G, S) \cong BCay(G, T)$ implies that $T = gS^\alpha$ for some $g \in G$ and $\alpha \in Aut(G)$. A group $G$ is called an $m$-BCI-group if all bi-Cayley graphs of $G$ of valency at most $m$ are BCI-graphs, and an $|G|$-BCI-group is simply called a BCI-group.

**Remark 1.1.** It should be remarked that each of the above concepts has a natural analog in the theory of Cayley digraphs. Recall that, for a group $G$ and a subset $S \subseteq G$, the Cayley digraph $Cay(G, S)$ is the digraph whose vertex set is $G$ and arc set is $\{(x, sx) : x \in G, s \in S\}$. A Cayley digraph $Cay(G, S)$ is called a CI-graph if for any Cayley digraph $Cay(G, T)$, $Cay(G, S) \cong Cay(G, T)$ implies that $T = S^\alpha$ for some $\alpha \in Aut(G)$. A group $G$ is called an $m$-DCI-group if all Cayley digraphs of $G$ of valency at most $m$ are CI-graphs, and an $(|G| - 1)$-CI-graph is simply called a CI-group. Finite CI-groups and $m$-DCI-groups have attracted considerable attention over the last 40 years. For more information on these groups, the reader is referred to the survey [14].

The study of $m$-BCI-groups was initiated in [21], where it was shown that every group is a 1-BCI-group, and a group is a 2-BCI-group if and only if it has the property that any two elements of the same order are either fused or inverse fused (these groups are described in [15]). The problem of classifying all 3-BCI-groups is still open. Up to our knowledge, it is only known that every cyclic group is a 3-BCI-group (this is a consequence of [20, Theorem 1.1], see also [8]), and that $A_5$ is the only non-Abelian simple 3-BCI-group (see [9]). The purpose of this paper is to make a further step by classifying the nilpotent 3-BCI-groups.

In fact, we have a relatively short list of candidates for nilpotent 3-BCI groups, which arises from the earlier works of W. Jin and W. Liu [9, 10] on the Sylow $p$-subgroups of 3-BCI-groups. In particular, a Sylow 2-subgroup of a 3-BCI-group is $Z_2^r$, $Z_4^2$ or the quaternion group $Q_8$ (see [9]), while a Sylow $p$-subgroup for $p > 2$ is homocyclic (see [10]). A group is said to be homocyclic if it is a direct product of cyclic groups of the same order. Consequently, if $G$ is a nilpotent 3-BCI-group, then $G$ decomposes as $G = U \times V$, where $U$ is a homocyclic group of odd order, and $V$ is trivial or one of the groups $Z_2^r$, $Z_2^2$ and $Q_8$. In this paper, we prove that the converse implication also holds, and by this complete the classification of nilpotent 3-BCI-groups. Our main result is the following theorem:

**Theorem 1.2.** Every finite group $U \times V$ is a 3-BCI-group, where $U$ is a homocyclic group of odd order, and $V$ is trivial or one of the groups $Z_2^r$, $Z_2^2$ and $Q_8$.

**Remark 1.3.** It is interesting to compare the known 3-BCI-groups with the class of 2-DCI-groups. It follows from the classification of finite 2-DCI-groups [12, Theorem 1.3] that, $A_5$ is the only non-Abelian simple 2-DCI-group, and the nilpotent 2-DCI-groups are exactly those given in Theorem [12]. However, a 3-BCI-group is not always a 2-DCI-group. A rather exhausted analysis shows that the Frobenius group $Z_3^3 \rtimes Z_4$ is a 3-BCI-group, while it is not a 2-DCI group, which can be seen from [12, Theorem 1.3].
We prove Theorem 2.1 in two parts. In Section 2 we treat the case when BCay\((G, S)\) is not arc-transitive, \(G = U \times V\), where \(U\) is a homocyclic group of odd order, and \(V\) is trivial or one of \(\mathbb{Z}_2\), \(\mathbb{Z}_2^r\) and \(Q_8\), and \(|S| = 3\). The rest will be done in Section 3.

## 2 Non-arc-transitive BCI-graphs

We start by fixing the relevant notation and terminology. Let \(G\) be a finite group acting on a finite set \(V\). For a subset \(U \subseteq V\), denote by \(G_U\) the elementwise stabilizer of \(U\) in \(G\), while by \(G_{\{U\}}\) the setwise stabilizer of \(U\) in \(G\). If \(U = \{u_1, \ldots, u_k\}\), then \(G_{u_1, \ldots, u_k}\) will be written for \(G_{\{u_1, \ldots, u_k\}}\), in particular, we write \(G_u\) for \(G_{\{u\}}\). We say that the subset \(U\) is \(G\)-invariant if \(G\) leaves \(U\) setwise fixed, or equivalently, when \(G_{\{U\}} = G\).

Suppose, in addition, that \(G\) acts transitively on \(V\). A subset \(\Delta \subseteq V\) is called a block for \(G\) if for every \(g \in G\), \(\Delta^g = \Delta\) or \(\Delta \cap \Delta^g = \emptyset\). The sets \(\Delta^g, g \in G\), form a partition of \(V\), which is called the system of blocks for \(G\) induced by \(\Delta\). Denoted this partition by \(\delta\), \(G\) acts on \(\delta\) naturally. The corresponding kernel will be denoted by \(G_\delta\), i.e., \(G_\delta = \{g \in G : \Delta^g = \Delta'\ \text{for all } \Delta' \in \delta\}\).

For a graph \(\Gamma\), we let \(V(\Gamma), E(\Gamma), A(\Gamma),\) and \(\text{Aut}(\Gamma)\) denote the vertex set, the edge set, the arc set, and the full group of automorphisms of \(\Gamma\), respectively. For a subset \(U \subseteq V(\Gamma)\), we let \(\Gamma[U]\) denote the subgraph of \(\Gamma\) induced by \(U\). A graph \(\Gamma\) is called arc-transitive when \(\text{Aut}(\Gamma)\) is transitive on \(A(\Gamma)\). We let \(K_n\) and \(K_{n,n}\) denote the complete graph on \(n\) vertices and the complete bipartite graph on \(2n\) vertices respectively. By a cubic graph we simply mean a regular graph of valency 3.

Throughout the paper \(\mathcal{C}\) denotes the set of all groups \(U \times V\), where \(U\) is a homocyclic group of odd order, and \(V\) is either trivial or one of \(\mathbb{Z}_2\), \(\mathbb{Z}_2^r\) and \(Q_8\); and \(\mathcal{C}_{\text{sub}}\) denotes the set of all groups that have an overgroup in \(\mathcal{C}\).

The main result in this section is the following theorem, which we are going to prove in the end of the section:

**Theorem 2.1.** Let \(\Gamma = \text{BCay}(G, S), G \in \mathcal{C}, |S| = 3\), and suppose that \(\Gamma\) is not arc-transitive. Then \(\text{BCay}(G, S)\) is a BCI-graph.

Given a group \(G\) with identity element \(1_G\), we shall use the symbols \(0\) and \(1\) to denote the elements \((1_G, 0)\) and \((1_G, 1)\), respectively, from \(G \times \{0, 1\}\); and for a subset \(S \subseteq G\), we write \((S, 0) = \{(s, 0) : s \in S\}\) and \((S, 1) = \{(s, 1) : s \in S\}\). For \(g \in G\), let \(\hat{g}\) be the permutation of \(G \times \{0, 1\}\) defined by \((x, i)^\hat{g} = (xg, i)\) for every \(x \in G\) and \(i \in \{0, 1\}\), and let \(\hat{G} = \{\hat{g} : g \in G\}\). Obviously, \(\hat{G} \leq \text{Aut}(\text{BCay}(G, S))\) always holds.

**Lemma 2.2.** Let \(\Gamma\) be a cubic bipartite graph with bipartition classes \(\Delta_i, i = 1, 2,\) and \(X \leq \text{Aut}(\Gamma)\) be a semiregular subgroup whose orbits are \(\Delta_i, i = 1, 2,\) and \(X \in \mathcal{C}_{\text{sub}}\). Then \(\text{Aut}(\Gamma)\) has an element \(\tau_X\) which satisfies:

(i) every subgroup of \(X\) is normal in \(\langle X, \tau_X \rangle\);

(ii) \(\langle X, \tau_X \rangle \leq \text{Aut}(\Gamma)\) is regular on \(V(\Gamma)\).
Proof. It is straightforward to show that $\Gamma \cong \text{BCay}(X, S)$ for some subset $S \subseteq X$ with $|S| = 3$. Moreover, there is an isomorphism from $\Gamma$ to $\text{BCay}(G, S)$ which induces a permutation isomorphism from $X$ to $\hat{X} \leq \text{Aut}(\text{BCay}(G, S))$. Therefore, it is sufficient to find $\tau \in \text{Aut}(\text{BCay}(X, S))$ for which every subgroup of $\hat{X}$ is normal in $\langle X, \tau \rangle$; and $\langle X, \tau \rangle \leq \text{Aut}(\text{BCay}(X, S))$ is regular on $V(\text{BCay}(X, S))$.

Since $X \in \mathcal{C}_{\text{sub}}$, $X = U \times V$, where $U$ is an Abelian group of odd order, and $V$ is trivial or one of $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $Q_8$. We prove below the existence of an automorphism $\iota \in \text{Aut}(X)$, which maps the set $S$ to its inverse $S^{-1} = \{x^{-1} : x \in S\}$. Let $\pi_U$ and $\pi_V$ denote the projections $U \times V \to U$ and $U \times V \to V$ respectively. It is sufficient to find an automorphism $\iota_1 \in \text{Aut}(U)$ which maps $\pi_U(S)$ to $\pi_U(S)^{-1}$, and an automorphism $\iota_2 \in \text{Aut}(V)$ which maps $\pi_V(S)$ to $\pi_V(S)^{-1}$. Since $U$ is Abelian, we are done by choosing $\iota_1$ to be the automorphism $x \mapsto x^{-1}$. If $V$ is Abelian, then let $\iota_2 : x \mapsto x^{-1}$. Otherwise, $V \cong Q_8$, and since $|\pi_V(S)| \setminus \{1_V\} = 2$, it follows that $\pi_V(S)$ is conjugate to $\pi_V(S)^{-1}$ in $V$. This ensures that $\iota_2$ can be chosen to be some inner automorphism. Now, define $\iota$ by setting its restriction $\iota|_U$ to $U$ as $\iota|_U = \iota_1$, and its restriction $\iota|_V$ to $V$ as $\iota|_V = \iota_2$.

Define the permutation $\tau$ of $X \times \{0, 1\}$ by

$$(x, i)\tau = \begin{cases} (x^i, 1) & \text{if } i = 0, \\
(x^i, 0) & \text{if } i = 1. \end{cases}$$

The vertex $(x, 0)$ of $\text{BCay}(X, S)$ has neighborhood $(Sx, 1)$. This is mapped by $\tau$ to the set $(S^{-1}x^i, 0)$, which is equal to the neighborhood of $(x^i, 1)$. We have proved that $\tau \in \text{Aut}(\text{BCay}(X, S))$.

It follows from its construction that $\tau$ is an involution. Fix an arbitrary subgroup $Y \leq X$, and pick $y \in Y$. We may write $y = y_Uy_V$ for some $y_U \in U$ and $y_V \in V$. Then $\langle y_U, y_V \rangle \leq Y$, since $y_U$ and $y_V$ commute and $\gcd(|U|, |V|) = 1$. Also, $(y_U)^{i_1} = y_U^{-1}$ and $(y_V)^{i_2} \in \langle y_V \rangle$, implying that $y^i = (y_U)^{i_1}(y_V)^{i_2} \in \langle y_U, y_V \rangle \leq Y$. We conclude that $\iota$ maps $Y$ to itself. Thus $\tau^{-1}y\tau = \tau y\tau = y^i$ is in $\hat{Y}$, and $\tau$ normalizes $\hat{Y}$. Since $X \in \mathcal{C}_{\text{sub}}$, $\hat{Y}$ also normal in $\hat{X}$, and part (i) follows.

For part (ii), observe that $|\langle \hat{X}, \tau \rangle| = 2|X| = |V(\text{BCay}(X, S))|$. Clearly, $\langle \hat{X}, \tau \rangle$ is transitive on $V(\text{BCay}(X, S))$, so it is regular. \hfill $\Box$

The following result about Cayley digraphs is a special case of [11] Lemma 3.1:

**Lemma 2.3.** The following are equivalent for every Cayley digraph $\text{Cay}(G, S)$.

(i) $\text{Cay}(G, S)$ is a CI-graph.

(ii) Every two regular subgroups of $\text{Aut}(\text{Cay}(G, S))$, isomorphic to $G$, are conjugate in $\text{Aut}(\text{Cay}(G, S))$.

We prove next an analog of the previous lemma for cubic bi-Cayley graphs on groups $G \in \mathcal{C}_{\text{sub}}$. For a permutation group $H \leq \text{Sym}(G \times \{1, 0\})$, we denote by $\mathcal{G}(H)$ the set of all semiregular subgroups of $H$ whose orbits are $(G, 0)$ and $(G, 1)$.

**Lemma 2.4.** The following are equivalent for every bi-Cayley graph $\Gamma = \text{BCay}(G, S)$, where $G \in \mathcal{C}_{\text{sub}}$ and $|S| = 3$.
(i) BCay($G, S$) is a BCI-graph.

(ii) Every two subgroups in $G(\text{Aut}(\Gamma))$, isomorphic to $X$, are conjugate in $\text{Aut}(\Gamma)$.

Proof. We start with the part $(i) \Rightarrow (ii)$. Let $X \in G(\text{Aut}(\Gamma))$ such that $X \cong G$. We have to show that $X$ and $G$ are conjugate in $\text{Aut}(\Gamma)$. Let $i \in \{0, 1\}$, and set $X^{(G,i)}$ and $\hat{G}^{(G,i)}$ for the permutation groups of the set $(G,i)$ induced by $X$ and $G$ respectively. The groups $X^{(G,i)}$ and $\hat{G}^{(G,i)}$ are conjugate in $\text{Sym}((G,i))$, because these are isomorphic and regular on $(G,i)$. Thus $X$ and $G$ are conjugate by a permutation $\phi \in \text{Sym}(G \times \{0, 1\})$ such that $(G,0)$ is $\phi$-invariant, and we write $X = \phi \hat{G} \phi^{-1}$. Consider the graph $\Gamma$, the image of $G$ under $\phi$. Then $\hat{G} = \phi^{-1} X \phi \leq \text{Aut}(\Gamma)$. Using this and that $(G,0)$ is $\phi$-invariant, we obtain that $\Gamma^\phi = \text{BCay}(G,T)$ for some subset $T \subseteq G$. Then $\Gamma \cong \text{BCay}(G,T)$, and by $(i)$, $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \text{Aut}(G)$. Define the permutation $\sigma$ of $G \times \{0, 1\}$ by

$$(x,i)^\sigma = \begin{cases} (x^\alpha,0) & \text{if } i = 0, \\ (gx^\alpha,1) & \text{if } i = 1. \end{cases}$$

Notice that, $\sigma$ normalizes $\hat{G}$. The vertex $(x,0)$ of $\text{BCay}(X, S)$ has neighborhood $(Sx,1)$. These are mapped by $\sigma$ to the vertex $(x^\alpha,0)$ and the set $(gS^\alpha x^\alpha,1) = (Tx^\alpha,1)$. This proves that $\sigma$ induces an isomorphism from $\Gamma$ to $\Gamma^\phi$, and it follows in turn that, $\Gamma^\sigma = \Gamma^\phi = G^\alpha$, $\phi \sigma^{-1} \in \text{Aut}(\Gamma)$, and thus $\phi = \rho \sigma$ for some $\rho \in \text{Aut}(\Gamma)$. Finally, $X = \phi \hat{G} \phi^{-1} = \rho \sigma \hat{G} \sigma^{-1} \rho^{-1} = \rho \hat{G} \rho^{-1}$, i.e., $X$ and $G$ are conjugate in $\text{Aut}(\Gamma)$.

We turn to the part $(i) \Leftarrow (ii)$. Let $\Gamma' = \text{BCay}(G,T)$ such that $\Gamma' \cong \Gamma$. We have to show that $T = gS^\alpha$ for some $g \in G$ and $\alpha \in \text{Aut}(G)$. We claim the existence of an isomorphism $\phi : \Gamma \to \Gamma'$ for which $\phi : 0 \to 0$ and $(G,0)$ is $\phi$-invariant (here $\phi$ is viewed as a permutation of $G \times \{0,1\}$). We construct $\phi$ in a few steps. To start with, choose an arbitrary isomorphism $\phi_1 : \Gamma \to \Gamma'$. Let $\tau_\hat{G}$ be the automorphism of $\Gamma'$ defined in Lemma 2.2. Since $(\hat{G}, \tau_\hat{G})$ is regular on $V(\Gamma')$, there exists $\rho \in (\hat{G}, \tau_\hat{G})$ which maps $0^{\phi_1}$ to $0$. Let $\phi_2 = \phi_1 \rho$. Then $\phi_2$ is an isomorphism from $\Gamma$ to $\Gamma'$, and also $\phi_2 : 0 \to 0$. The connected component of $\Gamma'$ containing the vertex $(x,0)$ is equal to the induced subgraph $\Gamma'[(xH,0) \cup (sxH,1)]$, where $s \in S$ and $H \leq G$ is generated by the set $s^{-1}S$. It can be easily checked that

$$\Gamma'[(xH,0) \cup (sxH,1)] \cong \text{BCay}(H,s^{-1}S).$$

Similarly, the connected component of $\Gamma'$ containing the vertex $(x,0)$ is equal to the induced subgraph $\Gamma'[(xK,0) \cup (txK,1)]$, where $t \in T$ and $K \leq G$ is generated by the set $t^{-1}T$, and

$$\Gamma'[(xK,0) \cup (txK,1)] \cong \text{BCay}(K,t^{-1}T).$$

Since $\phi_2$ fixes $0$, it induces an isomorphism from $\Gamma[(H,0) \cup (sH,1)]$ to $\Gamma[(K,0) \cup (tK,1)]$; denote this isomorphism by $\phi_3$. It follows from the connectedness of these induced subgraphs that $\phi_3$ preserves their bipartition classes, moreover, $\phi_3$ maps $(H,0)$ to $(K,0)$, since it fixes $0$. Finally, take $\phi : \Gamma \to \Gamma'$ to be the isomorphism whose restriction to each component of $\Gamma$ equals $\phi_3$. It is clear that $\phi : 0 \to 0$ and $(G,0)$ is $\phi$-invariant.

Since $\hat{G} \leq \Gamma'$, $\phi \hat{G} \phi^{-1} \leq \text{Aut}(\Gamma)$. The orbit of $0$ under $\phi \hat{G} \phi^{-1}$ is equal to $(G,0)^{\phi^{-1}} = (G,0)$, and hence $\phi \hat{G} \phi^{-1} \in G(\text{Aut}(\Gamma))$. By $(ii)$, $\phi \hat{G} \phi^{-1} = \sigma^{-1} \hat{G} \sigma$ for some $\sigma \in \text{Aut}(\Gamma)$. 5
By Lemma 2.2, the normalizer of $\hat{G}$ in $\text{Aut}(\Gamma)$ is transitive, implying that $\sigma$ can be chosen so that $\sigma : 0 \mapsto 0$. To sum up, we have an isomorphism $(\sigma \phi) : \Gamma \mapsto \Gamma'$ which fixes $0$ and also normalizes $\hat{G}$. Thus $(\sigma \phi)$ maps $(G, 1)$ to itself. Let $G_{\text{right}} \leq \text{Sym}(G, 1)$ be the permutation group induced by the action of $\hat{G}$ on $(G, 1)$. Then the permutation of $(G, 1)$ induced by $(\sigma \phi)$ belongs to the holomorph of $G_{\text{right}}$ (cf. [6, Exercise 2.5.6]), and therefore, there exist $g \in G$ and $\alpha \in \text{Aut}(G)$ such that $(\sigma \phi) : (x, 1) \mapsto (gx^\alpha, 1)$ for all $x \in G$. On the other hand, being an isomorphism from $\Gamma$ to $\Gamma'$, $(\sigma \phi)$ maps $(S, 1)$ to $(T, 1)$. These give that $(T, 1) = (S, 1)^{\sigma \phi} = (gS^\alpha, 1)$, i.e., $T = gS^\alpha$.

In the following lemma we connect the BCI-property with the CI-property.

Lemma 2.5. Let $\Gamma = \text{BCay}(G, S), G \in C_{\text{sub}}, |S| = 3$, and suppose, in addition, that $\text{Aut}(\Gamma)_0 = \text{Aut}(\Gamma)_1$. Then $\text{Cay}(G, S)$ is a CI-graph $\iff$ $\text{BCay}(G, S)$ is a BCI-graph.

Proof. Set $A = \text{Aut}(\Gamma)$ and $A^+ = A_{\{G, 0\}}$. Obviously, $X \leq A^+$ for every $X \in \mathcal{G}(A)$. Let $\tau_G \in A$ be the automorphism defined in Lemma 2.2 Then $A = A^+ \rtimes \langle \tau_G \rangle$. By Lemma 2.2(i), $\tau_G$ normalizes $\hat{G}$, hence the conjugacy class of subgroups of $A$ containing $\hat{G}$ is equal to the conjugacy class of subgroups of $A^+$ containing $\hat{G}$. Using this and Lemma 2.4(ii) follows if every $X \in \mathcal{G}(A)$, isomorphic to $G$, is conjugate to $\hat{G}$ in $A^+$.

Let $\Delta = \{0, 1\}$ and consider the setwise stabilizer $A_{\{\Delta\}}$. Since $A_0 = A_1, A_0 \leq A_{\{\Delta\}}$. By [6, Theorem 1.5A], the orbit of $0$ under $A_{\{\Delta\}}$ is a block for $A$. Since $\tau_G$ switches $0$ and $1$, this orbit is equal to $\Delta$, and the system of blocks induced by $\Delta$ is

$$\delta = \{\Delta^x : x \in G\} = \{\{(x, 0), (x, 1]\} : x \in G\}.$$ 

This allows us to define the action of $A$ on $G$ by letting $x^\sigma = x'$ if $\sigma$ maps the block $\{(x, 0), (x, 1]\}$ to the block $\{(x', 0), (x', 1]\}$. We write $\tilde{\sigma}$ for the image of $\sigma$ under the corresponding permutation representation, and let $\tilde{X} = \{\tilde{\sigma} : \sigma \in X\}$ for a subgroup $X \leq A$. Notice that, the subgroup $A^+ < A$ is faithful in this action. In particular, two subgroups $X$ and $Y$ of $A^+$ are conjugate in $A^+$ exactly when $\tilde{X}$ and $\tilde{Y}$ are conjugate in $\tilde{A}^+$. Also, for every $X \leq A^+$, $X \in \mathcal{G}(A)$ and $X \cong G$ if and only if $\tilde{X}$ is regular on $G$ and $X \cong G$.

We prove next that $\tilde{A}^+ = \text{Aut}(\text{Cay}(G, S))$. Pick an automorphism $\sigma \in A^+$ and an arc $(x, sx)$ of $\text{Cay}(G, S)$. The edge $\{(x, 0), (sx, 1]\}$ of $\Gamma$ is mapped by $\sigma$ to an edge $\{(x', 0), (s'x', 1]\}$ for some $x' \in G$ and $s' \in S$. Hence $\tilde{\sigma} : x \mapsto x'$ and $sx \mapsto s'x'$, i.e., it maps the arc $(x, sx)$ to the arc $(x', s'x')$. We have just proved that $\tilde{\sigma} \in \text{Aut}(\text{Cay}(G, S))$, and hence $\tilde{A}^+ \leq \text{Aut}(\text{Cay}(G, S))$. In order to establish the relation $\geq$, for an arbitrary automorphism $\rho \in \text{Aut}(\text{Cay}(G, S))$, define the permutation $\pi$ of $G \times \{0, 1\}$ by $(x, i)\pi = (x^\rho, i)$ for all $x \in G$ and $i \in \{0, 1\}$. It is easily checked that $\pi \in A^+$ and $\tilde{\pi} = \rho$. Thus $\tilde{A}^+ \geq \text{Aut}(\text{Cay}(G, S))$, and so $\tilde{A}^+ = \text{Aut}(\text{Cay}(G, S))$.

Now, the desired equivalence follows along the following lines:

(i) $\iff$ every two regular subgroup of $\tilde{A}^+$, isomorphic to $G$, are conjugate in $\tilde{A}^+$

(ii) $\iff$ every two subgroups in $\mathcal{G}(A)$, isomorphic to $G$, are conjugate in $G$

(iii) $\iff$ (ii),
where (a) is Lemma 2.3, (b) is proved above, and (c) is Lemma 2.4.

Now it is easy to prove Theorem 2.1.

Proof of Theorem 2.1. Since $\Gamma$ is vertex-transitive (see Lemma 2.2), but not arc-transitive, we have $A_0 = A_{(s,1)}$ for some $s \in S$. We show below that $\text{BCay}(G, s^{-1}S)$ is a BCI-graph, this obviously yields that the same holds for $\text{BCay}(G, S)$. Define the permutation $\phi$ of $G \times \{0, 1\}$ by

$$(x, i) \phi = \begin{cases} (x, 0) & \text{if } i = 0, \\ (s^{-1}x, 1) & \text{if } i = 1. \end{cases}$$

We showed before that $\phi$ induces an isomorphism from $\Gamma$ to $\Gamma' = \text{BCay}(G, s^{-1}S)$. Then we have $\text{Aut}(\Gamma) \phi = \phi^{-1}A_0 \phi = \phi^{-1}A_{(1,0)} \phi = \text{Aut}(\Gamma)'$. Thus Lemma 2.5 applies to $\Gamma'$, as a result, it is sufficient to show that $\text{Cay}(G, s^{-1}S)$ is a CI-graph. This follows because $|s^{-1}S \setminus \{1G\}| = 2$ and that $G$ is a 2-DCI-group (see [12, Theorem 1.3]).

3 Proof of Theorem 1.2

Let $\Gamma$ be an arbitrary finite graph and $G \leq \text{Aut}(\Gamma)$ which is transitive on $V(\Gamma)$. For a normal subgroup $N \triangleleft G$ which is not transitive on $V(\Gamma)$, the quotient graph $\Gamma_N$ is the graph whose vertices are the $N$-orbits on $V(\Gamma)$, and two $N$-orbits $\Delta_i, i = 1, 2$, are adjacent if and only if there exist $v_i \in \Delta_i, i = 1, 2$, which are adjacent in $\Gamma$. For a positive integer $s$, an $s$-arc of $\Gamma$ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of $\Gamma$ such that, for every $i \in \{1, \ldots, s\}$, $v_{i-1}$ is adjacent to $v_i$, and for every $i \in \{1, \ldots, s-1\}$, $v_{i-1} \neq v_{i+1}$. The graph $\Gamma$ is called $(G,s)$-arc-transitive ($(G,s)$-arc-regular) if $G$ is transitive (regular) on the set of $s$-arcs of $\Gamma$. If $G = \text{Aut}(\Gamma)$, then a $(G,s)$-arc-transitive ($(G,s)$-arc-regular) graph is simply called $s$-transitive ($s$-regular). The proof of the following lemma is straightforward, hence it is omitted (it can be also deduced from [17, Theorem 9]).

Lemma 3.1. Let $\Gamma = \text{BCay}(G, S)$ be a connected arc-transitive graph, $G$ be any finite group, $|S| = 3$, and $N < \hat{G}$ be a subgroup which is normal in $\text{Aut}(\Gamma)$. Then the following hold:

(i) $\Gamma_N$ is a cubic connected arc-transitive graph.

(ii) $N$ is equal to the kernel of $\text{Aut}(\Gamma)$ acting on the set of $N$-orbits.

(iii) $\Gamma_N$ is isomorphic to a bi-Cayley graph of the group $\hat{G}/N$.

Remark 3.2. Let $\Gamma$ and $N$ be as described in Lemma 3.1. The group $\text{Aut}(\Gamma)$ acts on the set of $N$-orbits, i.e., on the vertex set $V(\Gamma_N)$. Lemma 3.1(ii) implies that, the induced permutation group on $V(\Gamma_N)$ is isomorphic to $\text{Aut}(\Gamma)/N$, and therefore, by some abuse of notation, this permutation group will also be denoted by $\text{Aut}(\Gamma)/N$. In what follows we shall write $\text{Aut}(\Gamma)/N \leq \text{Aut}(\Gamma_N)$. Also note that, if $\Gamma$ is $s$-transitive, then $\Gamma_N$ is $(\text{Aut}(\Gamma)/N, s)$-arc-transitive.
The proof of Theorem 1.2 will be based on three lemmas about cubic connected arc-transitive bi-Cayley graphs, to be proved below. In order to simplify the formulations, we keep the following notation in all lemmas:

\((*)\) \(\Gamma = \text{BCay}(G, S)\) is a connected arc-transitive graph, where \(G \in \mathcal{C}_{\text{sub}}\) and \(|S| = 3\).

**Lemma 3.3.** With notation (\(*\)), let \(\delta\) be a system of blocks for \(\text{Aut}(\Gamma)\) induced by a block properly contained in \((G, 0)\), and \(X\) be in \(\mathcal{G}(\text{Aut}(\Gamma))\) such that \(X \in \mathcal{C}_{\text{sub}}\). Then \(A_\delta < X\).

**Proof.** Set \(A = \text{Aut}(\Gamma)\). Let \(Y = X \cap A_\delta\), where \(\Delta \in \delta\) with \(\Delta \subset (G, 0)\). Then \(\Delta\) is equal to an orbit of \(Y\), and \(|Y| = |\Delta|\) because \(\Delta \subset (G, 0)\) and \(X\) is regular on \((G, 0)\).

Formally, \(\Delta = \text{Orb}_Y(v)\) for some vertex \(v \in \Delta\).

Let \(\tau_X \in A\) be the automorphism defined in Lemma 2.2 and set \(L = \langle X, \tau_X \rangle\). The group \(L\) is regular on \(V(\Gamma)\), and \(Y \leq L\). These yield

\[\delta = \{\Delta^l : l \in L\} = \{\text{Orb}_Y(v)^l : l \in L\} = \{\text{Orb}_Y(v^l) : l \in L\}.\]

From this \(Y \leq A_\delta\). Since \(\delta\) has more than 2 blocks, and \(\Gamma\) is a connected and cubic graph, it is known that \(A_\delta\) is semiregular. These imply that \(A_\delta = Y < X\). \(\square\)

**Corollary 3.4.** With notation (\(*\)), let \(N < \hat{G}\) be normal in \(\text{Aut}(\Gamma)\), and \(X\) be in \(\mathcal{G}(\text{Aut}(\Gamma))\) such that \(X \in \mathcal{C}_{\text{sub}}\). Then \(N < X\).

**Proof.** Let \(\delta\) be the system of blocks for \(\text{Aut}(\Gamma)\) consisting of the \(N\)-orbits. Then \(A_\delta = N\) by Lemma 3.1(ii), and the corollary follows directly from Lemma 3.3. \(\square\)

We denote by \(Q_3\) the graph of the cube and by \(\mathcal{H}\) the Heawood graph. Recall that, the core of a subgroup \(H \leq K\) in the group \(K\) is the largest normal subgroup of \(K\) contained in \(H\).

**Lemma 3.5.** With notation (\(*\)), suppose that \(\hat{G}\) is not normal in \(\text{Aut}(\Gamma)\), and let \(N\) be the core of \(\hat{G}\) in \(\text{Aut}(\Gamma)\). Then \((\hat{G}/N, \Gamma_N)\) is isomorphic to one of the pairs \((\mathbb{Z}_3, K_{3,3}), (\mathbb{Z}_4, Q_3)\), and \((\mathbb{Z}_7, \mathcal{H})\).

**Proof.** Set \(A = \text{Aut}(\Gamma)\). Consider the quotient graph \(\Gamma_N\), and let \(M \leq \hat{G}\) such that \(N \leq M\) and \(M/N \leq \text{Aut}(\Gamma_N)\) (here \(M/N \leq A/N \leq \text{Aut}(\Gamma_N)\), see Remark 3.2). This implies in turn that, \(M/N \leq A/N\), \(M \leq A\), and \(M = N\). We conclude that, \(\Gamma_N\) is a bi-Cayley graph of \(\hat{G}/N\), \(\hat{G}/N\) is in \(\mathcal{C}_{\text{sub}}\), and \(\hat{G}/N\) has trivial core in \(\text{Aut}(\Gamma_N)\). This shows that it is sufficient to prove Lemma 3.5 in the particular case when \(N\) is trivial. For the rest of the proof we assume that the core \(N\) is trivial, and we write \(N = 1\).

By Tutte’s theorem [19], \(\Gamma\) is \(k\)-regular for some \(k \leq 5\). Set \(A^+ = \text{Aut}(\Gamma)_{((G,0))}\). Then \(A = \langle A^+, \tau_G \rangle\), where \(\tau_G \in A\) is the automorphism defined in Lemma 2.2. Let \(M\) be the core of \(\hat{G}\) in \(A^+\). Then \(M \leq A\), since \(M\) is normalized by \(\tau_G\) by Lemma 2.2(i), and \(A = \langle A^+, \tau_G \rangle\). Thus \(M \leq N = 1\), hence \(M\) is also trivial. Consider \(A^+\) acting on the set \([A^+ : \hat{G}]\) of right \(\hat{G}\)-cosets in \(A^+\). This action is faithful because \(M\) is trivial. Equivalently, \(G\) is embedded into \(S_{3,2k-1-1}\), and we will write below that \(G \leq S_{3,2k-1-1}\).
Also, $A_0$ is determined uniquely by $k$, and we have, respectively, $A_0 \cong \mathbb{Z}_3$, or $S_3$, or $D_{12}$, or $S_4$, or $S_4 \times Z_{12}$. We go through each case.

CASE 1. $k = 1$.

This case can be excluded at once by observing that we have $G \leq S_2$ by the above discussion, which contradicts the obvious bound $|G| \geq 3$.

CASE 2. $k = 2$.

In this case $G \leq S_5$. Using also that $G \in C_{\text{sub}}$, we see that $G$ is Abelian, hence $|G| \leq 6$, $|V(\Gamma)| \leq 12$. We obtain by [4, Table] that $\Gamma \cong Q_3$, and $G \cong Z_3$.

CASE 3. $k = 3$.

Then $A^+ = \hat{G}A_0 = \hat{G}D_{12}$, a product of a nilpotent and a dihedral subgroup. Thus $A^+$ is solvable by Huppert-Itô’s theorem (cf. [18, 13.10.1]). Since the core $N = 1$, $A^+$ is primitive on $(G, 0)$, see Lemma 5.3. Therefore, $G$ is a $p$-group. We see that $G$ is either Abelian or it is $Q_8$. In the latter case $|V(\Gamma)| = 16$, and $\Gamma$ is isomorphic to the Moebius-Kantor graph, which is, however, 2-regular (see [4, Table]). Therefore, $G$ is an Abelian $p$-group. Let $S = \{s_1, s_2, s_3\}$. Since $G$ is Abelian, for $\Gamma$ we have:

$$0 \sim (s_1, 1) \sim (s_2^{-1}s_1, 0) \sim (s_3s_2^{-1}s_1, 1) \sim (s_1s_2^{-1}s_3, 1) \sim (s_2^{-1}s_3, 0) \sim (s_3, 1) \sim 0.$$ 

Thus $\Gamma$ is of girth at most 6. It was proved in [5, Theorem 2.3] that the Pappus graph on 18 points and the Deargues graph on 20 points are the only 3-regular cubic graphs of girth 6. For the latter graph $|G| = 10$, contradicting that $G$ is a $p$-group. We exclude the former graph by the help of the computer package Magma [3]. We compute that the Pappus graph has no Abelian semiregular automorphism group of order 9 which has trivial core in the full automorphism group. Thus $\Gamma$ is of girth 4 (3 and 5 are impossible as the graph is bipartite). It is well-known that there are only two cubic arc-transitive graphs of girth 4 (see also [11, page 163]): $K_{3,3}$ and $Q_8$. We get at once that $\Gamma \cong K_{3,3}$ and $G \cong Z_3$.

CASE 4. $k = 4$.

It is sufficient to show that $G$ is Abelian. Then by the above reasoning $\Gamma$ is of girth 6, and as the Heawood graph is the only cubic 4-regular graph of girth 6 (see [5, Theorem 2.3]), we get at once that $\Gamma \cong H$ and $G \cong Z_7$.

Assume, towards a contradiction, that $G$ is non-Abelian. Thus $G = U \times V$, where $U$ is an Abelian group of odd order, and $V \cong Q_8$. Since the core $N = 1$, $A^+$ is primitive on $(G, 0)$, see Lemma 5.3. In other words, $\Gamma$ is a 4-transitive bi-primitive cubic graph. Two possibilities can be deduced from the list of 4-transitive bi-primitive graphs given in [13, Theorem 1.4]:

- $\Gamma$ is the standard double cover of a connected vertex-primitive cubic 4-regular graph, in which case $A = A^+ \times \langle \eta \rangle$ for an involution $\eta$; or
- $\Gamma$ isomorphic to the sextet graph $S(p)$ (see [22]), where $p \equiv \pm 7(\text{mod } 16)$, in which case $A \cong PGL(2, p)$, and $A^+ \cong PSL(2, p)$.
The second possibility cannot occur, because then \( A^+ \cong PSL(2, p) \), whose Sylow 2-subgroup is a dihedral group (cf. [2, Satz 8.10]), which contradicts that \( V \leq G \leq A^+ \), and \( V \cong Q_8 \). It remains to exclude the first possibility. We may assume, by replacing \( S \) with \( xS \) for a suitable \( x \in G \) if necessary, that \( \eta \) switches 0 and 1. Since \( \eta \) commutes with \( \hat{G} \), we find \( (x, 0)^{n} = 0^{\eta^{x}} = 0^{\eta^{x}} = 1^{x} = (x, 1) \) for every \( x \in G \). Let \( s \in S \). Then \( 0 \sim (s, 1) \), hence \( 1 = 0^{n} \sim (s, 0)^{n} = (s, 1) \), which shows that \( s \in S^{-1} \), and thus \( S = S^{-1} \). Thus there exists \( s \in S \) with \( o(s) \leq 2 \). Put \( T = s^{-1}S = sS \). Then \( 1_{G} \in T \), and since \( \Gamma \) is connected, \( G = \langle T \rangle \). Notice that \( s \in Z(G) \). This implies that \( T^{-1} = T \), and thus \( \pi_{V}(T) \) satisfies \( 1_{V} \in \pi_{V}(T) \) and \( \pi_{V}(T) = \pi_{V}(T)^{-1} \). Since \( V \cong Q_8 \), this implies that \( \langle \pi_{V}(T) \rangle \neq V \), a contradiction to \( G = \langle T \rangle \). This completes the proof of this case.

CASE 5. \( k = 5 \).

In this case \( \Gamma \) is a 5-transitive bi-primitive cubic graph. It was proved in [13, Corollary 1.5] that \( \Gamma \) is isomorphic to either the PTL(2, 9)-graph on 30 points (also known as the Tutte’s 8-Cage), or the standards double cover of the \( PSL(3, 3) \mathbb{Z}_2 \)-graph on 468 points. These graphs are of girth 8 and 12 respectively (see [4, Table]). Also, in both cases \( 8 \not| |G| \), hence \( G \) is Abelian. By this, however, \( \Gamma \) cannot be of girth larger than 6. This proves that this case does not occur. \( \Box \)

For a group \( A \) and a prime \( p \) dividing \(|A|\), we let \( A_{p} \) denote a Sylow \( p \)-subgroup of \( A \).

**Lemma 3.6.** With notation (*), let \( X \in \mathcal{G}(\text{Aut}(\Gamma)) \) such that \( X \in \mathcal{G}_{\text{sub}} \) and \( X_{2} \cong G_{2} \). Then \( X \) and \( \hat{G} \) are conjugate in \( \text{Aut}(\Gamma) \).

**Remark 3.7.** We remark that, the assumption \( X_{2} \cong G_{2} \) cannot be deleted. The Moebius-Kantor graph is a bi-Cayley graph of the group \( Q_{8} \), which has a semiregular cyclic group of automorphism of order 8 which preserves the bipartition classes.

**Proof.** Set \( A = \text{Aut}(\Gamma) \). The proof is split into two parts according to whether \( \hat{G} \) is normal in \( A \).

CASE 1. \( \hat{G} \) is not normal in \( A \).

Let \( N \) be the core of \( \hat{G} \) in \( A \). By Corollary 3.3 \( N < X \cap \hat{G} \). Therefore, it is sufficient to show that

\[
\frac{X}{N} \text{ and } \frac{\hat{G}}{N} \text{ are conjugate in } \frac{A}{N}.
\]

Recall that, the group \( A/N \leq \text{Aut}(\Gamma_N) \) for the quotient graph \( \Gamma_N \) induced by \( N \) (see Remark 3.2 and the preceding paragraph). Both groups \( X/N \) and \( \hat{G}/N \) are semiregular whose orbits are the bipartition classes of \( \Gamma_N \). Also notice that, \( \hat{G}/N \) cannot be normal in \( A/N \), otherwise \( \hat{G} \) were normal in \( A \).

According to Lemma 5.2 \( \frac{\hat{G}/N, \Gamma_N}{\cong} (\mathbb{Z}_3, K_{3,3}) \), or \((\mathbb{Z}_4, Q_8)\), or \((\mathbb{Z}_7, H)\). Thus (1) follows immediately from Sylow’s theorem when \( \hat{G}/N, \Gamma_N \cong (\mathbb{Z}_7, H) \).

Let \( \frac{\hat{G}/N, \Gamma_N}{\cong} (\mathbb{Z}_3, K_{3,3}) \). Since \( \hat{G}/N \) is not normal in \( A/N \), and \( \Gamma_N \) is \((A/N, 1)\)-arc-transitive, we compute by Magma that \( A/N = \text{Aut}(\Gamma_N) \), or it is a unique subgroup of \( \text{Aut}(\Gamma_N) \) of index 2. In both cases \( A/N \) has one conjugacy classes of semiregular subgroups whose orbits are the bipartition classes of \( \Gamma_N \). Thus (1) holds.
Let \((\hat{G}/N, \Gamma_N) \cong (\mathbb{Z}_4, Q_8)\). Since \(X_3 \cong G_2\), \(N/X \cong \hat{G}/N \cong \mathbb{Z}_4\). Using this and that \(\Gamma_N\) is \((A/N, 1, 1)\)-arc-transitive, we compute by \texttt{Magma} that \(A/N = \text{Aut}(\Gamma_N)\), and that \(\text{Aut}(\Gamma_N)\) has one conjugacy classes of semiregular cyclic subgroups whose orbits are the bipartition classes of \(\Gamma_N\). Thus (1) holds also in this cases.

CASE 2. \(\hat{G}\) is normal in \(A\).

We have to show that \(X = \hat{G}\). Notice that, \(X\) contains every proper subgroup \(K < \hat{G}\) which is characteristic in \(\hat{G}\). Indeed, since \(\hat{G} \triangleleft A\), \(K \triangleleft A\), and hence \(K < X\) follows by Corollary 3.4. This property will be used often below.

In particular, \(\hat{G}_p < X\) if \(G\) is not a p-group, hence \(X = \hat{G}\). Let \(G\) be a p-group. If \(p > 3\), then both \(\hat{G}\) and \(X\) are Sylow p-subgroups of \(A\), and the statement follows by Sylow’s theorem. Notice that, since \(\Gamma\) is connected, \(G\) is generated by the set \(s^{-1}S\) for \(s \in S\), hence it is generated by two elements.

Let \(p = 2\). Assume for the moment that \(G\) is cyclic. Then \(\hat{G}\) has a characteristic subgroup \(K\) such that \(\hat{G}/K \cong \mathbb{Z}_4\). Then \(K \triangleleft A\), \(\Gamma_K \cong Q_3\), \(\Gamma_K\) is a bi-Cayley graph of \(\hat{G}/K\), which is is normal in \(A/K \leq \text{Aut}(\Gamma_K)\). A simple computation shows that this situation does not occur. Let \(G\) be a non-cyclic group in \(\mathcal{C}_{\text{sub}}\). Therefore, \(G \cong \mathbb{Z}_p^2\) and \(\Gamma \cong Q_3\), or \(G \cong Q_8\) and \(\Gamma\) is the Moebius-Kantor graph. Now, \(X = X_2 \cong G_2 = G\). Using this, \(X = \hat{G}\) can be verified by the help of \texttt{Magma} in either case.

Let \(p = 3\). Observe first that \(|G| > 3\). For otherwise, \(\Gamma \cong K_{3,3}\), but no semiregular automorphism group of order 3 is normal in \(\text{Aut}(K_{3,3})\). Assume for the moment that \(G \not\cong \mathbb{Z}_3^e \times \mathbb{Z}_3^e\) for any \(e \geq 1\). In this case \(\hat{G}\) has a characteristic subgroup \(K\) such that \(\hat{G}/K \cong \mathbb{Z}_3\), or \(\mathbb{Z}_9 \times \mathbb{Z}_3\). It follows that \(K \triangleleft A\), and \(\Gamma_K\) is the Pappus graph, or the unique cubic arc-transitive graph on 54 points (see \[4\], Table). Moreover, it is a bi-Cayley graph of \(\mathbb{Z}_9\) in the first, and of \(\mathbb{Z}_9 \times \mathbb{Z}_3\) in the second case. However, we have checked by \texttt{Magma} that none of these is possible; and therefore, \(G \cong \mathbb{Z}_3^e \times \mathbb{Z}_3^e\) for some \(e \geq 1\). If \(e = 1\), then \(G \cong \mathbb{Z}_3^2\), and \(\Gamma\) is the Pappus graph. However, this graph has no automorphism group which is isomorphic to \(\mathbb{Z}_3^2\) and also normal in the full automorphism group. Therefore, \(e > 1\), and thus the subgroup \(K = \langle x^3 : x \in G \rangle\) is characteristic in \(\hat{G}\) of index 81. It follows that \(K < X\), and \(\Gamma_K\) is the unique cubic arc-transitive graph on 162 points (see \[4\], Table). A direct computation gives that \(X/K = \hat{G}/K\), which together with \(K < X \cap \hat{G}\) yield that \(X = \hat{G}\).

Recall that, a group \(H\) is homogeneous if every isomorphism between two subgroups of \(H\) can be extended to an automorphism of \(H\). The following result is \[12\] Proposition 3.2:

**Proposition 3.8.** Every 2-DCI-group is homogeneous.

Since every group in \(\mathcal{C}\) is a 2-DCI-group (see \[12\] Theorem 1.3)], we have the corollary that every group in \(\mathcal{C}\) is homogeneous. Everything is prepared to complete the proof of Theorem \[12\].

**Proof of Theorem \[12\]** Let \(G \in \mathcal{C}\) and \(\Gamma = \text{BCay}(G, S)\) such that \(|S| \leq 3\). We have to show that \(\Gamma\) is a BCI-graph. This holds trivially when \(|S| = 1\), and follows from the
homogeneity of $G$ when $|S| = 2$. Let $|S| = 3$. The claim is proved in Theorem 2.4 when $\Gamma$ is not arc-transitive. For the rest of the proof we assume that $\Gamma$ is an arc-transitive graph.

Let $\text{BCay}(G, S) \cong \text{BCay}(G, T)$ for some subset $T \subseteq G$. We may assume without loss of generality that $1_G \in S \cap T$. Let $H = \langle S \rangle$ and $K = \langle T \rangle$. Then $H, K \in \mathcal{C}_{\text{sub}}$, both bi-Cayley graphs $\text{BCay}(H, S)$ and $\text{BCay}(K, T)$ are connected, and $\text{BCay}(H, S) \cong \text{BCay}(K, T)$. We claim that $\text{BCay}(H, S)$ is a BCI-graph. In view of Lemma 2.4 this holds if for every $X \in G(\text{Aut}(\text{BCay}(H, S)))$, isomorphic to $H$, $X$ and $\hat{H}$ are conjugate in $\text{Aut}(\text{BCay}(H, S))$. Now this follows directly from Lemma 3.6.

Let $\phi$ be an isomorphism from $\text{BCay}(K, T)$ to $\text{BCay}(H, S)$, and consider the group $X = \phi^{-1}K\phi \leq \text{Sym}(H)$. Since $\phi$ maps the bipartition classes of $\text{BCay}(K, T)$ to the bipartition classes of $\text{BCay}(H, S)$, we have $X \in G(\text{Aut}(\text{BCay}(H, S)))$. Also, $X_2 \cong H_2$, because $X \cong K$, $|H| = |K|$ and $H$ and $K$ are both contained in the group $G$ from $\mathcal{C}$. Thus Lemma 3.6 is applicable, as a result, $X$ and $\hat{H}$ are conjugate in $G(\text{Aut}(\text{BCay}(H, S)))$. In particular, $H \cong K$, and since $G$ is homogeneous, there exists $\alpha_1 \in \text{Aut}(G)$ such that $K^{\alpha_1} = H$. This $\alpha_1$ induces an isomorphism from $\text{BCay}(K, T)$ to $\text{BCay}(H, T^{\alpha_1})$. Therefore, $\text{BCay}(H, S) \cong \text{BCay}(H, T^{\alpha_1})$, and since $\text{BCay}(H, S)$ is a BCI-graph, $T^{\alpha_1} = gS^{\alpha_2}$ for some $g \in H$ and $\alpha_2 \in \text{Aut}(H)$. By homogeneity, $\alpha_2$ extends to an automorphism of $G$, implying eventually that $\text{BCay}(G, S)$ is a BCI-graph. This completes the proof of the theorem.

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