In this communication, we present an efficient method for computation of energy and wave function of weakly bound nuclei by the application of supersymmetric quantum mechanics (SSQM) and bound states in continuum (BIC) technique. As a case study the scheme is implemented to the two-body \((^{30}\text{Ne} + n)\) cluster model calculation of neutron-rich nucleus \(^{31}\text{Ne}\). Woods-Saxon central potential with spin-orbit component is used as the core-nucleon interaction. The two-body Schrödinger equation in relative coordinate is solved numerically to get the energy and wave function of the low-lying bound states. A one-parameter family of isospectral potential (IP) is constructed from the bound state solutions following algebra of SSQM to find energies and wave functions of the resonance states. In addition to the \(2p_{3/2}^{-}\) (-0.33 MeV) ground state, two bound excited states: \(s_{1/2}\) (-0.30 MeV), \(p_{1/2}\) (-0.15 MeV) are also obtained. Few low-lying resonance states: \(f_{7/2}\) (2.57 MeV), \(f_{7/2}\) (4.59 MeV), \(f_{5/2}\) (5.58 MeV), \(p_{1/2}\) (1.432 MeV), \(p_{1/2}\) (4.165 MeV), \(p_{3/2}\) (1.431 MeV), \(p_{3/2}\) (4.205 MeV) are predicted. Among the predicted resonance states, the \(f_{7/2}^{-}\) state having resonance energy \(E_R \approx 4.59\) MeV is in excellent agreement with the one found in the literature.

**Keywords** Nuclear dripline · Halo nuclei · Resonance · Isospectral potential (IP) · Bound states in continuum (BIC) · Supersymmetric quantum mechanics (SSQM)

1 Introduction

One of the most exciting areas of research in nuclear physics involves exotic nuclei that appear away from the nuclear stability line but closer to the neutron (n) or proton (p) driplines in the nuclear chart. Limiting lines on either side of the nuclear stability line, along which the single nucleon separation energy is zero, are called nuclear driplines. Panin et al. 2021 [1] have recently commented on the significance of the dripline nuclei in areas nuclear physics, nuclear medicine, nuclear astrophysics, defence research, material science, etc. The discovery of halo nuclei near the drip lines after the advent of radioactive ion beam (RIB) facilities is considered one of the most significant breakthrough events in nuclear physics. Typical halo nuclei consist of a dense core having a low-density envelope of the loosely bound nucleon(s). These weakly bound halo nuclei seldom have any excited bound state except the ground state having energy typically less than 1 MeV, near the continuum. Halo nuclei have matter radii more than the liquid drop model (LDM) prediction of \(R_A \propto A^{1/3}\), small valence nucleon(s) separation energies, and high probability of occupation to low-\(l\) shells. Another exciting and scientifically valued characteristic of halo nuclei is their resonance state(s) just above the binding threshold [2,3,4].

Tanihata et al. 1985 [5] is credited for the first confirmation of 2n-halo structure in neutron-rich \(^{11}\text{Li}\). One-neutron halo has been affirmed in \(^{11}\text{Be}\) [6,7], \(^{19}\text{C}\) [8-10], \(^{31}\text{Ne}\) [11,12,13], while two-neutron halo are observed in \(^{8}\text{He}\) [14,15], \(^{11}\text{Li}\) [16,17], \(^{14}\text{Be}\) [17] and in \(^{22}\text{C}\) [18,19]. Nuclei \(^{8}\text{B}\) [20], \(^{26}\text{P}\) [21], \(^{17}\text{F}\) [17], etc have one proton halo structure while \(^{27}\text{S}\)
According to shell-model scheme, $^{31}$Ne ground state should have $^{30}$Ne$(0^+) \otimes 1f_{7/2}$ configuration. But, Poves and Retamosa 1994 [31], Descouvemont 1999 [32] predicted a possible inversion between the shell model levels $1f_{7/2}$ and $2p_{3/2}$ for the ground state of $^{31}$Ne. They predicted valance neutron occupying the $2p_{3/2}$ intruder orbit with a dominant $^{30}$Ne$(0^+)$ single-particle configuration. Ren et al. 2001 [33] predicted $3/2^-$ as the ground state of $^{31}$Ne instead of shell model label 7/2$^-$ using DDRMF concept. Hamamoto 2010 [34] using scattering phase shift method found $1f_{7/2}$ resonance of width 0.224 MeV at 2.40 MeV. Using the particle-rotor model, Urata et al. 2011 [25] found the ground state configuration of $^{31}$Ne as $J^\pi=3/2^-$. Liu et al. 2012 [35] reported the energy and widths of low-lying neutron resonances in $^{31}$Ne using a complex scaling technique and identified three resonances namely $1f_{7/2}$, $1f_{5/2}$, and 1$g_{9/2}$ among which the acceptable one is the $7/2^-$ resonance close to 4.592 MeV. Using analytic continuation approach for resonances Zhang et al. 2014 [37] predicted one-neutron p-orbit halo structure in $^{31}$Ne. By complex momentum representation technique, Tian et al. 2017 [38] found $2p_{3/2}$ resonance together with $1f_{7/2}$ resonance, as well as a $p-f$ inversion in single-particle levels. Very recently, He 2019 [39] using halo effective field theory (HEFT) with effective range method obtained $S_n$ for the $3/2^-$ ground state and predicted $1/2^+$ state in-line with the findings of Nakamura et al. 2014 [11].

Literature survey indicated a $^{30}$Ne$(0^+) \otimes p_{3/2}$ ground configuration of $^{31}$Ne instead of the shell model configuration $^{30}$Ne$(0^+) \otimes f_{7/2}$. However, in this work all possible configurations for the bound and resonance states of $^{31}$Ne will be explored in the framework of ($^{30}$Ne+n) two-body cluster model. Resonance energy computation in weakly bound systems can be accomplished by the application of supersymmetric quantum mechanics (SSQM). The scheme is based on the fact that for any arbitrarily chosen potential (say, $V$), one can generate a one-parameter family of isospectral potential ($\hat{V}$) with a free parameter ($\eta$). The present method has an edge over several other methods providing resonance properties, namely- the Complex Scaling Method reported by Moiseyev 1998 [40], the R-matrix method of Descouvemont and Vincke 1990 [41], the ACCC method of Kukulin et al. 1989 [42], the Numerov method reported by Baluja et al. 1982 [43] or the one using phase shifts to extract the resonance parameters, because of the advantages of the robustness of the algebra of supersymmetric quantum mechanics. When the original potential has a shallow well following a wide skinny barrier (poorly supporting any resonance state), $\eta$ can be selected judiciously to increase the depth of the well and height of the barrier (in $\hat{V}$) simultaneously. It is accomplished by decreasing the value of $\eta$ in suitable steps and checking its effect to the potential. The optimized well-barrier combination in $\hat{V}$ facilitates trapping of the particle inside it, resulting in an accurate computation of the resonant state exactly at the same energy, as that in the original shallow potential $V$. This is because, $V$ and $\hat{V}$ are strictly isospectral. Earlier, the scheme has been implemented successfully in two-body model calculation for $^{13}$C by Khan et al., 2021 [10], in three-body cluster model calculation of resonances in $^{22}$C by Hasan et al., 2019 [19] and in $^{42,44}$Mg by Khan et al. 2021 [44].

2 Theoretical Scheme

For core-nucleon two-body model of $^{A-1}X$ ($=A-X+N$) nucleus, the relative motion can be described by the Schrödinger equation

$$
\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{L(L+1)\hbar^2}{2\mu r^2} + V(r) - E\right] \psi(r) = 0.
$$

(1)
Resonance energy and wave functions of $^{31}\text{Ne}$.

where $\mu = \frac{(A, m)(m)}{A + m + m} = \frac{A}{A + m}$ is the reduced mass of the core-N two-body system, $m$ is the nucleon mass and $V(r)$ is the core-nucleon potential. In terms of the effective potential,

$$U(r) = V(r) + \frac{\hbar^2}{2\mu} \frac{L(L + 1)}{r^2},$$

Eq. (1) can be rewritten as

$$\left\{ \frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + U(r) - E \right\} \psi(r) = 0.$$  

Numerical solution of Eq. (3) by the re-normalized Numerov method (RNM) prescribed by Johnson 1978 [45] subject to appropriate boundary conditions yield the ground state energy $E = E_0 < 0$ and wave function $\psi_0(r)$.

2.1 Construction of one-parameter family of IP using SSQM

After obtaining the ground state energy ($E_0$) and wave function ($\psi_0(r)$), a one-parameter family of IP

$$\hat{U}(\eta; r) = U(r) - E_0 - \frac{\hbar^4}{2\mu} \frac{d^2}{dr^2} \log[I_0(r) + \eta]$$

can be constructed following Nieto 1984 [46], Khare and Sukhatme 1989 [47], and Cooper et al. 1995 [48] applying formalism of SSQM. In Eq. (4), $\eta$ is a free parameter and $I_0(r) = \int_{r'}^{r} [\psi_0(r')]^2 dr'$. As $I_0(r)$ lies between 0 and 1, the interval $-1 \leq \eta \leq 0$ is forbidden, to bypass singularities in $\hat{U}(\eta; r)$. For $\eta \to \pm\infty$, $\hat{U} \to U$ and for $\eta \to 0+$, $\hat{U}$ develops a narrow and deep attractive well near the origin following a high barrier. This improved well-barrier combination facilitates trapping of the particle giving rise to sharp resonance at the same energy as that in the original effective potential $U(r)$. However, a delta-function like behaviour of the improved well-barrier combination may add large computation error in the observables, hence a judicious choice of $\eta$ is important. The above procedure holds for resonance state of same spin-parity ($J^P$) as that of the bound state but fails for states of different $J^P$. In the latter case one needs to combine the bound states in continuum (BIC) technique of Pappademos et al. 1993 [49] with the formalism of SSQM to accomplish the desired goal.

When the original effective potential of Eq. (2) has a shallow well followed by a low and wide barrier, an accurate computation of energy and width is not feasible numerically. Though, in principle, for such a barrier of finite height, a system may be temporarily trapped inside the well, when its energy matches the resonance energy. But, in these cases penetrability through the barrier will be large enough to give a broad resonance width. In our scheme, such numerical obstacles can be bypassed and a reliable energy and width can be obtained by generating a one parameter family of IP having optimum depth to suppress the tunnelling probability of the system, yet providing the correct position of the obstacles can be bypassed and a reliable energy and width can be obtained by generating a one parameter family of IP having optimum depth to suppress the tunnelling probability of the system, yet providing the correct position of the resonance state. To accomplish this, Eq. (3) is solved for some positive energies $E (> 0)$ subject to the initial condition $\psi(0) = 0$. This positive energy $\psi(r)$ is non-normalizable and oscillates with constant amplitude in the asymptotic region where the potential $U(r)$ vanishes. Using this $\psi(r)$ one can readily construct the potential

$$\hat{U}(\eta; r) = U(r) - \frac{\hbar^2}{2\mu} \left[ \frac{4\psi(r)\psi'(r)}{I(r) + \eta} - \frac{2(\psi(r))^4}{(I(r) + \eta)^2} \right]$$

where $I(r) = \int_{r'}^{r} [\psi(r')]^2 dr'$. Eq. (3) has the solution $\psi(\eta; r) = \frac{\psi(r)}{I(r) + \eta}$ at the same energy $E$ when $\hat{U}(\eta; r)$ replaces $U(r)$. As $\psi(r)$ is non-normalizable and oscillates in the asymptotic region, $I(r) grows approximately linearly with r making $\psi(\eta; r)$ normalizable at larger r as described by Pappademos et al [49]. In this way, $\psi(\eta; r)$ represents a bound state in continuum (BIC) of $\hat{U}(\eta; r)$, since $\hat{U}(\eta; r)$ is strictly isospectral with $U(r)$. BIC is a less-familiar result of SSQM [50,46,47,49]. For $\eta \to \pm\infty$, $\hat{U} \to U$ and for $\eta \to 0+$, $\hat{U}$ develops a narrow and deep attractive well near the origin following a high surface barrier as can be seen from the data presented in Table 2 (see later) and also from the schematics of Fig. 3 (see later). As the improved well-barrier combination effectively traps the system resulting in a quasi-bound state, we may define a trapping probability as

$$\xi(E) = \int_{r' = 0}^{r_B} [\psi(\eta; r')]^2 dr'$$

where $r_B$ indicates the location of the top of the potential barrier. A plot of $\xi(E)$ versus $E$ ($E > 0$) shows prominent peak at the resonance energy, $E = E_R$. WKB approximation is used to compute resonance width ($\Gamma$) in terms of
the barrier transmission coefficient and time of flight between the classical turning points corresponding to \( E_R \) in the potential-well represented by \( \hat{U}(\eta; r) \) (see Eq. (5)):

\[
\Gamma = 2 \sqrt{\frac{\hbar^2}{2\mu}} \exp \left[ -\frac{2}{\hbar} \int_a^b \sqrt{\frac{2\mu}{\hat{U}(\eta; r) - E_R}} \, dr \right] \quad \text{(7)}
\]

In Eq. (7), \( a, b, c \) represent the classical turning points corresponding to \( E_R \) in the potential of Eq. (5) for the chosen \( \eta \) satisfying the condition: \( \hat{U}(\eta; a) = \hat{U}(\eta; b) = \hat{U}(\eta; c) = E_R \).

### 3 Application to \(^{31}\text{Ne}\)

The above scheme is tested against bound and resonance states of \(^{31}\text{Ne}\) which has weakly bound excited state of spin-parity \( J^p = 1/2^+ \) in addition to the 3/2\(^-\) ground state (see Nakamura et al 2014 [11] and He 2019 [39]). The system also exhibits low-lying resonances of spin-parity \( J^p = 1/2^- \) as reported by Hamamoto 2010 [34] and Liu et al. 2012 [36]. The nucleus \(^{31}\text{Ne}\) being strong one-neutron halo candidate can be treated as a two-body system consisting of a structure-less core (point like particle) plus one valence neutron moving around the \(^{30}\text{Ne}\) core. Hence, 3/2\(^-\) ground state may be regarded as a p-wave state, while the predicted 1/2\(^+\) state may be regarded an s-wave state. These s- and p-wave states are the outcome of the interaction between valence neutron and core [39] nucleus. Here, Pauli restrictions arising due valence neutron and the core nucleons is suppressed by considering the core as point like particle.

Chosen Woods-Saxon potential having a spin-orbit term, adopted from works of Pahlavani [51] for the core-n pair is given by \( V_{\text{core-n}}(r) = -V_c f(r) + V_{LS} \frac{1}{\hbar} \frac{df(r)}{dr} (L,S) \), where \( f(r) = [1 + \exp(-r/R_0)]^{-1/2} \). Energy and wave function of these states are obtained by numerical solution of Eq. (3) using RNM algorithm prescribed by Johnson 1978 [45]. Calculated bound state energies and root mean squared (RMS) matter radii have been listed in Table 1.

For a fruitful demonstration of the resonances, we adopted the technique of the bound states in the continuum (BIC) introduced in the preceding section. BIC represents the solution of Eq. (3) for the IP \( \hat{U}(\eta; r) \) represented by Eq. (5), in which \( \eta \) manages the strength of \( \hat{U}(\eta; r) \). It is seen that resonance energy is independent of \( \eta \) and an appropriate choice of \( \eta \), apart from protecting the stability of the resonant state, also preserves the spectrum of the original potential \( U(r) \), while adding a discrete BIC at specified energy. Here, we have constructed the IP following Eq. (5) for some of the low-lying unbound states: 1/2\(^-\), 3/2\(^-\), 5/2\(^-\), 7/2\(^-\) of \(^{31}\text{Ne}\) and obtained the corresponding resonant wave functions. The tuning parameter \( \eta \) produced a dramatic effect in the IP as indicated by the representative schematics.

### 4 Results and Discussions

Adjusted values of \( V_c \) used to reproduce observed neutron separation energy \( S_n \) for the low-lying states- 3/2\(^-\), 1/2\(^+\), 1/2\(^-\) of \(^{31}\text{Ne}\) are listed row-wise in Table 1 along with corresponding minimum of \( U(r) \). Calculated \( S_n \) together with some of them found in the literature are presented in Table 1. It is to be noted that \( S_n \) for the state 1/2\(^-\) is a mere prediction only. The RMS matter radius calculated using wave functions of 3/2\(^-\), 1/2\(^+\), 1/2\(^-\) states are also listed in Table 1. The original shallow potential (\( \eta \rightarrow \infty \)) represents a sufficiently broad and spatially extended resonance profile (not shown in the plot) as compared to the uppermost curve in Fig. 5. Data in Table 2 reflects that smaller \( \eta \) causes
Among the predicted low-lying resonances in \(^{31}\)Ne together with some of them found in the literature. Calculated energies of the bound states: \(0.33\, \text{MeV}\), \(0.30\, \text{MeV}\). The SSQM scheme adopted in this work has an edge over others because, this is the only theoretical approach by which the scheme applies also on two-body bound or unbound nucleus with excellent outcomes. 

Table 1: Potential parameters, energies and RMS matter radius obtained for the low-lying states of \(^{31}\)Ne together with some of those found in the literature.

| State, \(J^\pi\) | Gr. st.: \(3/2^-\) | Ex. st.: \(1/2^+\) | Ex. st.: \(1/2^-\) |
|---------------|----------------|----------------|----------------|
| \(E (\text{MeV})\) | \(V_c\) | -15.1170 | -5.4100 | -18.9910 |
| \(U_{\text{min}}\) | -7.9449 | -5.3973 | -8.6139 |
| \(S_n (\text{Calc.})\) | 0.3317 | 0.3005 | 0.1503 |
| \(S_n (\text{Expt.})\) | \(0.15^{+0.16}_{-0.10}\text{[11]}\) | \(0.30^{+0.26}_{-0.17}\text{[11]}\) | - |
| \(\text{Radius (fm)}\) | \(R(^{31}\text{Ne})_{\text{Calc.}}\) | 3.6109 | 3.6090 | 3.6094 |
| | \(R(^{31}\text{Ne})_{\text{Expt.}}\) | 3.47(AMD)\text{[35]} | 3.62(AMD-RGM)\text{[35]} |

enhancement in the depth of the well and height of the barrier simultaneously. Position of the minimum of the well and maximum of the barrier are found to shift towards the origin as \(\eta\) decreases. For sufficiently large \(\eta\) (\(\eta \rightarrow \infty\)), the potential well has a depth of -6.79 MeV near 3.32 fm while the barrier has a height of 6.15 MeV near 3.73 fm. Which, for \(\eta = 5 \times 10^{-5}\), changes to -268.81 MeV near 1.15 fm and 130.44 MeV near 1.79 fm respectively. These changes in the IP may be seen as a dramatic effect in it (See Table 2). One of the major advantages of the present scheme is that the resonance energy \((E_R)\) and width \((\Gamma)\) are independent of \(\eta\), except the computational error that creeps in \(\Gamma\) as \(\eta \rightarrow 0^+\), due to delta-function like behaviour of the IP. Calculated energies and widths of the resonances are presented in columns 4 and 5 of Table 3, together with some of them found in the literature. Calculated energies of the bound states: \(3/2^-\) (0.33 MeV), \(1/2^+\) (0.30 MeV) presented in Table 1 are in excellent agreement with the experimental values within error bars. Calculated RMS matter radius of the bound state: \(3/2^-\) (3.6 fm) also agrees fairly with the experimental values. Among the predicted low-lying resonances in \(^{31}\)Ne viz: \(f_{7/2}\) \((E_R=2.570 \text{ MeV}, \Gamma = 0.863 \pm 0.087), f_{7/2}\) \((E_R=4.59 \text{ MeV}, \Gamma = 0.803 \pm 0.083), p_{1/2}\) \((E_R=1.432 \text{ MeV}, \Gamma = 0.866 \pm 0.282), p_{1/2}\) \((E_R=4.165 \text{ MeV}, \Gamma = 2.080 \pm 0.565), p_{3/2}\) \((E_R=1.431 \text{ MeV}, \Gamma = 0.828 \pm 0.265), p_{3/2}\) \((E_R=4.205 \text{ MeV}, \Gamma = 1.586 \pm 0.523), f_{5/2}\) \((E_R=5.580 \text{ MeV}, \Gamma = 0.821 \pm 0.138),\) the state \(f_{7/2}\) having resonance energy \(E_R=4.59 \text{ MeV}\) are in excellent agreement with the one found in the literature \[36\].

5 Conclusion

In this communication, we have reproduced the energy and wave functions of the low-lying bound and unbound states of \(^{31}\)Ne neutron halo nucleus using a spherically-symmetric two-body potential having a spin-orbit component. The SSQM formalism together with the BIC technique is successfully used to generate resonant state wave functions, resonance energy, and width of the resonance. The technique confirmed the existence of the unbound \(7/2^-\) state in \(^{31}\)Ne and its resonance energy \(E_R=4.592 \text{ MeV}\)[35]. In this technique, the parameter \(\eta\) facilitates a fruitful demonstration of the resonance effect without affecting the exact location of the resonance energy or the position of the peak in the trapping probability versus energy plot as shown in Fig. 5. Physics of exotic nuclei formed near the drip-lines having quasi-bound or unbound states will continue to dominate the field of nuclear physics in the coming years. A sound theoretical framework involving less numerical uncertainties is inevitable for a meaningful description of their structure. Because, they are characterized by numerous types of long-lived (quasi-stationary) states, some of which include one particle shape resonances, one-particle virtual states, Effimov states, three-particle near-threshold long-lived states appearing due to the existence of bound, virtual or resonance states in their two-body subsystems, and compound states or quasi-bound states embedded in a continuum-an example of which is the Feshbach resonances in atomic physics. The SSQM scheme adopted in this work has an edge over others because, this is the only theoretical approach by which resonant state wave functions can be extracted and exploited to reproduce the experimental observable \(\Gamma\). In our earlier works on exotic three-body systems- \(^{22}\)C[19], \(^{42,44}\)Mg[44], we have applied this procedure successfully. The scheme has also been used for the unbound nucleus \(^{15}\)Be by Dutta et al. 2018[52]. Thus, we may conclude that the present scheme applies also on two-body bound or unbound nucleus with excellent outcomes.
Resonance energy and wave functions of $^{31}\text{Ne}$.

Figure 1: Plot of original potential $U(r)$ as a function of radial distance $r$ for $1/2^+$, $1/2^-$ and $3/2^-$ bound states of $^{31}\text{Ne}$.

Figure 2: Plot of wave function $\psi(r)$ as a function of radial distance $r$ for $1/2^+$, $1/2^-$ and $3/2^-$ bound states of $^{31}\text{Ne}$. 
Resonance energy and wave functions of $^{31}\text{Ne}$. 

Figure 3: Plot of isospectral potential (IP) $\hat{U}(\eta; r)$ as a function of radial distance $r$ for $7/2^-$ resonant state of $^{31}\text{Ne}$. 

Figure 4: Plot of resonant wave function ($\hat{\psi}(\eta; r)$) as a function of radial distance $r$ for $7/2^-$ resonant state of $^{31}\text{Ne}$ corresponding to the resonance energy $E_R = 4.592$ MeV.
Resonance energy and wave functions of $^{31}\text{Ne}$.

![Graph showing trapping probability $\xi(E)$ against energy $E$ for $7/2^-$ resonant state of $^{31}\text{Ne}$.](image)

**Figure 5**: Plot of the trapping probability $\xi(E)$ against energy $E$ for $7/2^-$ resonant state of $^{31}\text{Ne}$.

| Potential Well | Potential Barrier |
|----------------|-------------------|
| Tuning factor, $\eta$ | Depth | Pos. of minimum | Height | Pos. of maximum |
| $\infty$ | -6.79 | 3.32 | 6.15 | 5.73 |
| 50.00 | -9.01 | 2.30 | 6.81 | 5.39 |
| $1 \times 10^{-1}$ | -18.14 | 2.70 | 8.86 | 4.76 |
| $5 \times 10^{-2}$ | -24.03 | 2.54 | 10.96 | 4.48 |
| $1 \times 10^{-3}$ | -110.52 | 1.63 | 48.94 | 2.60 |
| $1 \times 10^{-4}$ | -221.97 | 1.24 | 105.99 | 1.94 |
| $5 \times 10^{-5}$ | -268.81 | 1.15 | 130.44 | 1.79 |
| $5 \times 10^{-6}$ | -489.03 | 0.88 | 246.34 | 1.36 |

**Table 2**: Parameters of the isospectral potential (IP) for some representative values of $\eta$ derived for obtaining energy $E_R$ of $7/2^-$ resonance state in $^{31}\text{Ne}$. 
Declaration of competing interest

The authors state that they have no known competing financial or personal interests that could have influenced the work presented in this paper.

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