$O(9, 9)$ symmetry of NS-NS couplings

at order $\alpha'^3$

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Abstract

Recently, imposing the $O(1, 1)$ symmetry on the circle reduction of the classical effective action of string theory, we have found all NS-NS couplings of type II superstring theories at order $\alpha'^3$. In this paper we use the cosmological reduction on the couplings and show that, up to one-dimensional field redefinitions and total derivative terms, they are invariant under the $O(9, 9)$ transformations.
1 Introduction

A theory of gravity in a spacetime manifold with/without boundary which is consistent with
the rules of quantum mechanics is the string theory. This theory includes the finite number
of massless modes and the tower of infinite number of massive modes of the string excitations.
At low energies, however, the massive modes are integrated out to produce an effective theory
which includes only the massless fields. The effective action has a double expansions. The
genus-expansion which includes the classical tree-level \( i.e., \) \( S_{\text{eff}} + \partial S_{\text{eff}} \) and a tower of quantum
loop-level corrections, and the stringy-expansion which is an expansion in terms of higher
derivative couplings at each loop level. The classical effective action has the following higher-
derivative or \( \alpha' \)-expansion:

\[
S_{\text{eff}} = \sum_{m=0}^{\infty} \alpha'^m S_m = S_0 + \alpha' S_1 + \alpha'^2 S_2 + \alpha'^3 S_3 + \cdots \tag{1}
\]

\[
\partial S_{\text{eff}} = \sum_{m=0}^{\infty} \alpha'^m \partial S_m = \partial S_0 + \alpha' \partial S_1 + \alpha'^2 \partial S_2 + \alpha'^3 \partial S_3 + \cdots
\]

The leading order bulk action \( S_0 \) includes the Hilbert-Einstein term and the boundary action
\( \partial S_0 \) includes the Hawking-Gibbons term \([1, 2]\). These actions and their appropriate higher
derivative extensions should be found by specific techniques in the string theory. Since the
effective action includes the couplings at all orders of derivative, one complication in finding
the couplings in the string theory is the freedom of the field redefinitions that include the higher
derivatives of fields \([3]\). As a result, the effective action in the string theory can appear in many
different equivalent schemes.

One of the most exciting discoveries in perturbative string theory is T-duality \([4, 5]\) which
appears when one compactifies theory on a torus, \( e.g., \) the compactification of the full bosonic
string theory on tours \( T^d \) is invariant under \( O(d, d, Z) \) transformations. After integrating out
the massive modes, however, the T-duality should appears as symmetry in the effective actions.
It has been shown in \([6, 7]\) that the dimensional reduction of the classical effective actions of
the bosonic and heterotic string theories on a torus \( T^d \) are in fact invariant under \( O(d, d, R) \)
transformations.

When one reduces the effective action on a circle, the invariance of the reduced action under
the \( \mathbb{Z}_2 \)-subgroup of the \( O(1, 1, R) \)-group constrains greatly the couplings in the effective action.
In fact there is only one T-dual multiplet in the effective action of the bosonic string theory,
and there are two T-dual multiplets in the effective action of type II superstring theories at
the leading order of \( \alpha' \), one for NS-NS couplings and one for R-R couplings \([5]\). By the T-dual
multiplet we means the set of couplings in the effective action which are related into each others
under the \( \mathbb{Z}_2 \)-transformations after reducing them on the circle. The \( \mathbb{Z}_2 \)-transformations or T-
duality transformations are the Buscher rules \([9, 10]\) and some higher derivative corrections at
each order of \( \alpha' \) which depend on the scheme that one uses for the gauge invariant couplings in
the effective action at that order of \( \alpha' \) \([11, 12]\). The corrected transformations, however, should
satisfy the \( \mathbb{Z}_2 \)-symmetry. There is no scheme for the higher-derivative couplings in the original
action in which the T-duality transformations are the standard Buscher rules \([12]\).
Since the T-duality transformations have higher derivative corrections, the other T-dual multiplets include couplings at all orders of $\alpha'$. In fact it has been observed in [13] that the couplings in the effective action of the bosonic string theory at order $\alpha'$ are related by the T-duality transformations to the couplings at order $\alpha'^2$. They belong to one T-dual multiplet. If one extends the calculations in [13] to the order $\alpha'^3$, one would find the couplings at order $\alpha'$ and $\alpha'^2$ are related to some of the couplings at order $\alpha'^3$. They belong to the same T-dual multiplet. However, there are couplings at this order that are not connected to the couplings at order $\alpha'$, $\alpha'^2$ by the T-duality transformations. They belong to another T-dual multiplet. In other words, if one finds the couplings in the bosonic string theory at order $\alpha'^3$ by S-matrix method, one would find they have two factors $a_1, a_2$, i.e., $S_3 = a_1 S_3^1 + a_2 S_3^2$. One factor should be the same as the one appears in the couplings at order $\alpha', \alpha'^2$, and another one which is proportional to $\zeta(3)$. These two factors should appear in the T-duality transformations as well. At the order $\alpha'^4$, the couplings in the bosonic theory should have three factors $a_1, a_2, a_3$, i.e., $S_4 = a_1 S_4^1 + a_2 S_4^2 + a_3 S_4^3$. One is the same as the factor in $\alpha', \alpha'^2$, one which is proportional to $\zeta(3)$ and another independent factor. Similar pattern should appear for the higher orders of $\alpha'$. Schematically, the bulk action (1) has the following expansion in terms of the T-dual multiplets:

$$S_{\text{eff}} = \sum_{n=0}^{\infty} a_n T_n = a_0 T_0 + a_1 T_1 + a_2 T_2 + a_3 T_3 + \cdots \tag{2}$$

where $a_0, a_1, \cdots$ are some coefficients that can not be fixed by the T-duality. They should be fixed by the S-matrix calculations, e.g., $a_0 = 1$, $a_1 = 1$, $a_3 = \zeta(3)$. In the type II superstring theory, $a_1 = 0$ and there are T-dual multiplets which include the R-R couplings. The T-dual multiplets in the bosonic string theory are

$$
T_0 = S_0 \\
T_1 = \alpha' S_1 + \alpha'^2 S_2 + \alpha'^3 S_3^1 + \alpha'^4 S_4^1 + \alpha'^5 S_5^1 + \cdots \\
T_2 = \alpha'^3 S_3^2 + \alpha'^4 S_4^2 + \alpha'^5 S_5^2 + \alpha'^6 S_6^2 + \alpha'^7 S_7^2 + \cdots \\
T_3 = \alpha'^4 S_4^3 + \alpha'^5 S_5^3 + \alpha'^6 S_6^3 + \alpha'^7 S_7^3 + \alpha'^8 S_8^3 + \cdots \tag{3}
$$

There are no parameters in these multiplets. If the spacetime manifold has no boundary in which the total derivative terms can be ignored, then each multiplet should be invariant under the T-duality transformations after reducing it on the circle. In principle, this constraint may fix all couplings in each T-dual multiplet. The couplings in the multiplet $T_1$ at orders $\alpha'$ and $\alpha'^2$ in a particular scheme have been found in [12, 13]. The couplings in the multiplet $T_2$ at order $\alpha'^3$ have been also found in [14, 15].

When the spacetime has boundary, however, one should keep the total derivative terms before and after reduction and use the Stokes’s theorem to transfer them to the boundary. They dictate that the invariance under the T-duality transformations requires some couplings on the boundary as well [16]. Hence, the bulk T-dual multiplets should be accompanied with
appropriate boundary couplings to be fully invariant under the T-duality. Schematically, the boundary action (1) should have the following expansion in terms of the boundary T-dual multiplets:

$$\partial S_{\text{eff}} = \sum_{n=0}^{\infty} a_n \partial T_n = a_0 \partial T_0 + a_1 \partial T_1 + a_2 \partial T_2 + a_3 \partial T_3 + \cdots$$  \hspace{1cm} (4)$$

where $a_0, a_1, \cdots$ are the same coefficients that appear in the bulk T-dual multiplets (2). The boundary multiplets are

$$\partial T_0 = \partial S_0$$

$$\partial T_1 = \alpha' \partial S_1 + \alpha^2 \partial S_2 + \alpha^3 \partial S_3 + \alpha^4 \partial S_4 + \alpha^5 \partial S_5 + \cdots$$

$$\partial T_2 = \alpha^2 \partial S_2 + \alpha^4 \partial S_3 + \alpha^5 \partial S_4 + \cdots$$

$$\partial T_3 = \alpha^4 \partial S_3 + \alpha^5 \partial S_4 + \cdots$$

$$\vdots$$  \hspace{1cm} (5)$$

The combination of bulk and boundary multiplets, i.e., $T_i + \partial T_i$, are then invariant under the T-duality transformations. In other words, neither the bulk multiplets nor the boundary multiplets are invariant separately under the T-duality transformations. Their anomalies cancel each other. There are, however, boundary couplings that are invariant under the T-duality transformations without anomaly. Some of them are related to the anomalous boundary multiplets by imposing the principle of the least action in the presence of the boundary with appropriate boundary values for the massless fields. Using these constraints, the boundary coupling in the multiplet $\partial T_0$ has been found in [16, 17]. The boundary couplings in the multiplet $\partial T_1$ at order $\alpha'$ in a particular scheme have been found in [17].

When one uses the cosmological reduction on the classical effective action, the resulting one-dimensional effective action should have $O(d, d, R)$ symmetry [6, 7]. This symmetry has been first observed for the leading order bulk couplings in [18, 19, 20] and for the couplings at order $\alpha'$ in a specific scheme in [21]. The T-duality transformations or $O(d, d, R)$ transformations in this case also recive higher derivative corrections. The corrected transformations satisfy the $O(d, d, R)$ symmetry [21]. In this case also there is no scheme for the original couplings in which the T-duality transformation are the standared $O(d, d, R)$-transformations of the leading order. Unlike the circle reduction, some of the couplings in the original action disappear upon the reduction [22]. Hence, this symmetry is not appropriate for fixing the couplings in the original action. However, this symmetry is useful for classifying the couplings in the one-dimensional effective action at all orders of $\alpha'$ [22, 23].

Using the most general corrections for the T-duality transformations, including the transformations for the lapse function, and using integration by part, it has been shown in [22, 23] that the cosmological reduction of the bulk action (1) at order $\alpha'$ and higher, can be written in a scheme in which only the first time-derivative of the generalized metric $\mathcal{S}$ appears. Trace of odd number of the first-derivative of $\mathcal{S}$ is zero. It has been shown in [23] that the couplings which include $\text{tr}(\mathcal{S}^2)$ can be removed by the lapse function transformation. Then the one-dimensional
bulk action can be written in a specific scheme as the following expansion [22, 23]:

\[
S_{\text{eff}}^c = S_0^c + \int dt e^{-\phi} \left( \alpha' c_{2,0} \text{tr}(\dot{S}^4) + \alpha'^2 c_{3,0} \text{tr}(\dot{S}^6) \\
+ \alpha'^3 [c_{4,0} \text{tr}(\dot{S}^8) + c_{4,1} (\text{tr}(\dot{S}^4))^2] \\
+ \alpha'^4 [c_{5,0} \text{tr}(\dot{S}^{10}) + c_{5,1} \text{tr}(\dot{S}^6) \text{tr}(\dot{S}^4)] + \cdots \right) \quad (6)
\]

where the coefficients \(c_{m,n}\) depends on the specific theory, e.g., \(c_{2,0}\) is non-zero for the bosonic string theory whereas this number is zero for the superstring theory.

To find the cosmological reduction of the corresponding boundary action in (1), one has to take into account the one-dimensional total derivative terms and the T-duality transformations or the field redefinitions that have been used in [22, 23]. However, it has been observed in [17] that if one adds the total derivative term resulting from the cosmological reduction of the leading order action to the boundary by using the Stokes’s theorem, it cancels the cosmological reduction of the Hawking-Gibbons term, i.e., \(S_{\text{eff}}^c = 0\). Since the cosmological reduction of the boundary term at the leading order of \(\alpha'\) is zero, we expect it should be zero at all higher orders of \(\alpha'\) as well, i.e.,

\[
\partial S_{\text{eff}}^c = 0 \quad (7)
\]

This may be used for conforming the boundary couplings in the effective action (1) at each order of \(\alpha'\) that can be found by the \(Z_2\)-symmetry. Since the cosmological reduction of the leading order action is zero, then for studying the cosmological reduction of the boundary action at order \(\alpha'\), one does not need the field redefinitions used in the bulk action. It has been shown in [17] that if one adds the total derivative terms at order \(\alpha'\) to the reduction of the boundary couplings at order \(\alpha'\), they become zero which is consistent with the above conjecture.

The cosmological bulk action (6) may also be used for confirming the bulk couplings in the effective action (1) at each order of \(\alpha'\) that can be found by the \(Z_2\)-symmetry. The NS-NS couplings in the effective action of type II superstring theory at order \(\alpha'^3\) have been found in [14, 15] by imposing the \(Z_2\)-symmetry on the effective action. In fact it has been shown in [24] that there are 872 independent couplings at this order. The \(Z_2\)-symmetry is imposed on these couplings in a particular scheme. Interestingly, all parameters are fixed up to an overall factor in [14]. In that scheme there are 445 non-zero couplings which include derivatives of dilaton. A field redefinition has been used in [15] to write them in terms of 251 couplings in which the dilaton appears only as the overall factor \(e^{-2\phi}\). In this paper we are going to show that the cosmological reduction of the couplings in [14] or in [15] can be written in the form of the cosmological action (6) at order \(\alpha'^3\). The gravitational sector of these couplings which have been found a long time ago by the S-matrix and sigma model calculations [25, 26, 27, 28], has been shown in [29] that satisfies the \(O(9,9)\) symmetry.

The outline of the paper is as follows: In section 2, we review the observation that the cosmological reduction of the leading order bulk and boundary actions are invariant under the \(O(d,d)\) transformations. In section 3, we first reduce the bulk NS-NS couplings at order
α^3 that have been found in [15] to find its corresponding one-dimensional bulk action. We add to it all one-dimensional total derivative terms and all possible field redefinitions with arbitrary coefficients to write the action in a scheme which has all the arbitrary parameters of the field redefinitions and the total derivative terms. We then impose the constraint on the parameters that the cosmological action has no derivative of the one-dimensional dilaton, no second and higher derivatives on metric and B-field and we impose the constraint that the couplings involving the first derivative of metric which are not consistent with the O(9, 9) symmetry to be zero. Moreover, we impose the condition that the terms which have contribution to the O(9, 9)-invariant structure tr(\dot{S}^2) to be zero. We have found that in fact there is a solution for the parameters for such conditions. After imposing the resulting relations between the parameters, we find the action in the scheme which can be written explicitly as the α^3-order terms of (6).

2 Cosmological reduction at the leading order

In this section, we review the cosmological reduction of the leading order bulk and boundary actions. These actions are given as

\[
S_0 + \partial S_0 = -\frac{2}{\kappa^2} \left[ \int d^D x \sqrt{-G} e^{-2\phi} \left( R + 4 \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{12} H^2 \right) + 2 \int d^{D-1} \sigma \sqrt{|g|} e^{-2\phi} K \right] \tag{8}
\]

where \( G \) is determinant of the bulk metric \( G_{\mu\nu} \) and boundary is specified by the functions \( x^\mu = x^\mu(\sigma^\bar{\nu}) \). In the boundary action, \( g \) is determinant of the induced metric on the boundary

\[
g_{\bar{\mu}\bar{\nu}} = \frac{\partial x^\mu}{\partial \sigma^{\bar{\mu}}} \frac{\partial x^\nu}{\partial \sigma^{\bar{\nu}}} G_{\mu\nu} \tag{9}
\]

The extrinsic curvature of boundary, i.e., \( K_{\mu\nu} \), is defined as \( K_{\mu\nu} = \nabla_\mu n_\nu - n_\mu a_\nu \) where \( n^\mu \) is the unite vector orthogonal to the boundary which is outward-going (inward-going) if the boundary is spacelike (timelike), and \( a_\nu = n^\rho \nabla_\rho n_\nu \) is acceleration. It satisfies the relation \( n^\mu a_\mu = 0 \). The extrinsic curvature is symmetric and satisfies \( n_\mu K_{\mu\nu} = 0 \).

When fields depend only on time, using the gauge symmetries it is possible to write the metric, B-field and dilaton as

\[
\begin{pmatrix}
-n^2(t) & 0 \\
0 & G_{ij}(t)
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & B_{ij}(t)
\end{pmatrix},
2\phi = \Phi + \frac{1}{2} \log \det(G_{ij}) \tag{10}
\]

where the lapse function \( n(t) \) can also be fixed to \( n = 1 \). The cosmological reduction of the bulk action then becomes

\[
S_0^c = -\frac{2}{\kappa} \int dt e^{-\Phi} \left[ \frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} - \frac{3}{4} \dot{G}_{ij} \dot{G}^{ij} - G^{ij} \dot{G}_{ij} \dot{\Phi} - \dot{\Phi}^2 + G^{ij} \ddot{G}_{ij} \right] \tag{11}
\]

where \( \dot{G}^{ij} \equiv G^{ik} G^{jl} \ddot{G}_{kl} \). Using the following total derivative term:

\[
\int dt \frac{d}{dt} \left[ e^{-\Phi} G^{ij} \dot{G}_{ij} \right] = \int dt e^{-\Phi} \left[ -G^{ij} \dot{G}_{ij} \dot{\Phi} - \dot{G}^{ij} \dot{G}_{ij} + G^{ij} \ddot{G}_{ij} \right] \tag{12}
\]
one can write $S_0^c$ as

$$ S_0^c = -\frac{2}{\kappa^2} \int dt e^{-\Phi} \left[ \frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} + \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} - \dot{\Phi}^2 \right] - \frac{2}{\kappa} \int dt \frac{d}{dt} \left[ e^{-\Phi} G^{ij} \dot{G}_{ij} \right] $$

The total derivative term can be transferred to the boundary by using the Stokes’s theorem.

On the other hand, the cosmological boundary is specified by $x^i = \sigma^i$, and $x^0 = t$ is independent of $\sigma^i$. Hence, $\sqrt{|g|} e^{-2\phi} = e^{-\Phi}$. The unit vector to the boundary is fixed, i.e., $\dot{n} = 0$, and the reduction of the trace of the extrinsic curvature becomes

$$ K^c = \frac{1}{2} G^{ij} \dot{G}_{ij} $$

Therefore, the reduction of the boundary term is cancelled with the total derivative term in the bulk action, i.e.,

$$ S_0^c = -\frac{2}{\kappa^2} \int dt e^{-\Phi} \left[ \frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} + \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} - \dot{\Phi}^2 \right] - \frac{2}{\kappa} \int dt \frac{d}{dt} \left[ e^{-\Phi} G^{ij} \dot{G}_{ij} \right] $$

Similar cancellation between the reduction of boundary action and the total derivative terms in the bulk action has been observed for the couplings at order $\alpha'$ in [17].

Using the generalized metric $S$

$$ S \equiv \eta \left( \begin{array}{cc} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{array} \right) $$

where $\eta$ is the metric of the $O(d,d)$ group which in the non-diagonal form is

$$ \eta = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), $$

one can write the bulk action as

$$ S_0^c = -\frac{2}{\kappa^2} \int dt e^{-\Phi} \left[ -\frac{1}{8} \text{tr}(\dot{S}^2) - \dot{\Phi}^2 \right] $$

which is invariant under the global $O(d,d,R)$ transformations because the one-dimensional dilaton is invariant and the generalized metric transforms as

$$ S \rightarrow \Omega^T S \Omega $$

where $\Omega$ belong to the $O(d,d,R)$ group, i.e., $\Omega^T \eta \Omega = \eta$. Note that $\text{tr}(\dot{S}) = 0$. Hence the reduction of the extrinsic curvature [14] can not be written in $O(9,9)$ invariant form. So it was necessary that this term was cancelled with the total derivative term in the bulk action. In other words, there is no way to write the boundary action in $O(d,d)$ invariant form unless it is zero.
3 Cosmological reduction at order $\alpha'^3$

The NS-NS couplings in the bulk effective action of type II superstring theory at order $\alpha'^3$ have been found in [14, 15] by imposing the $Z_2$-symmetry on the minimal gauge invariant couplings. In the particular scheme used in [15], the dilaton appears only as the overall factor $e^{-2\phi}$, and metric and $B$-field appear in the Riemann curvature, $H$ and the first covariant derivative of $H$, i.e.,

$$S_3 = -\frac{2c}{\kappa^2} \int d^{10}x \sqrt{-g} \left[ e^{-2\phi} L_3(G, B) + \cdots \right]$$

(20)

where dots represent the R-R and fermion fields in which we are not interested, and $c$ is an overall factor which can not be fixed by the T-duality constraint. The gravitational sector is

$$L_3(G) = 2R_\alpha^\epsilon \gamma R^\alpha_{\beta\gamma\delta} R^\beta_{\epsilon\zeta\xi\mu} + R_\alpha^\epsilon \gamma R^\alpha_{\beta\gamma\delta} R^\beta_{\xi\epsilon\zeta\mu}$$

(21)

The couplings in this sector are exactly the couplings that have been found by the S-matrix and sigma-model calculations [25, 26, 27, 28] provided that one chooses the overall parameter to be $c = -\zeta(3)/2^6$. There are 249 couplings which involve $H$. They appear in 8 structures. There are two couplings with structure $H^8$, i.e.,

$$L^H_3 = \frac{1}{48} H_\alpha^\delta H^\alpha_{\beta\gamma} H_\beta^\epsilon H_\gamma^\zeta H_\delta^\eta H_\xi^\theta H_\epsilon^\iota H_\mu^\kappa$$

(22)

There is one coupling with structure $RH^6$, i.e.,

$$L^RH^6_3 = \frac{9}{128} H_\alpha^\delta H^\alpha_{\beta\gamma} H_\beta^\epsilon H_\gamma^\zeta H_\delta^\eta H_\xi^\theta H_\epsilon^\iota R_{\mu\eta\theta}$$

(23)

There are 7 couplings with structure $R^2H^4$, i.e.,

$$L^{R^2H^4}_3 = \frac{7}{2} H_\alpha^\delta H^\alpha_{\beta\gamma} H_\beta^\epsilon H_\gamma^\zeta H_\delta^\eta H_\epsilon^\theta R_{\mu\nu\theta}$$

(24)

where dots refer to the other 6 couplings in this structure. There are 22 couplings with structure $R^3H^2$, i.e.,

$$L^{R^3H^2}_3 = -\frac{15}{2} H^\alpha_{\beta\gamma} H_\beta^\epsilon H_\epsilon^\theta R_{\alpha\delta\mu\kappa} R_{\beta\mu\epsilon\nu} R_{\gamma\eta\xi\zeta}$$

(25)

There are 77 couplings with the structure $(\nabla H)^2H^4$, i.e.,

$$L^{(\nabla H)^2H^4}_3 = \frac{5}{8} H_\alpha^\delta H^\alpha_{\beta\gamma} H_\beta^\epsilon H_\epsilon^\theta \nabla_\gamma H_\eta^\psi \nabla_\zeta H_{\mu\nu\theta}$$

(26)

There are 106 couplings with the structure $R(\nabla H)^2H^2$, i.e.,

$$L^{R(\nabla H)^2H^2}_3 = \frac{457}{48} H_\alpha^\delta H^\alpha_{\beta\gamma} R_{\xi\eta\mu\nu} \nabla_\delta H_\beta^\epsilon \nabla_\epsilon H_\gamma^\zeta$$

(27)
There are 22 couplings with the structure \( R^2(\nabla H)^2 \), i.e.,

\[
L_3^{R^2(\partial H)^2} = -\frac{5}{24} R_{\mu\nu\zeta} \varepsilon^\mu\nu\zeta \nabla_\delta H_{\alpha\beta\gamma} \nabla^\delta H^{\alpha\beta\gamma} + \ldots
\]  

(28)

And finally, there are 12 couplings with the structure \((\nabla H)^4\):

\[
L_3^{(\partial H)^4} = \frac{1}{8} \nabla^\delta H^{\alpha\beta\gamma} \nabla_\delta H_{\gamma} \nabla_\mu \nabla_\nu H_{\delta\mu\zeta} \nabla^\varepsilon H^{\alpha\beta} + \ldots
\]  

(29)

We refer the interested readers to [15] for the explicit form of all couplings.

To find the cosmological reduction of these couplings we first find the cosmological reduction of the Riemann curvature, \( H \) and \( \nabla H \). They are

\[
R_{ijkl} = -\frac{1}{4} \hat{G}_{il} \hat{G}_{jk} + \frac{1}{4} \hat{G}_{ik} \hat{G}_{jl} ; \quad R_{i0jk} = 0 ; \quad R_{d0j0} = \frac{1}{4} \hat{G}_{ik} \hat{G}_{j}^k - \frac{1}{2} \hat{G}_{ij}
\]

\[
H_{ijk} = 0 ; \quad H_{ij0} = \hat{B}_{ij} ; \quad \nabla_0 H_{ijk} = 0 ; \quad \nabla_k H_{ij0} = 0
\]  

(30)

\[
\nabla_i H_{ijk} = -\frac{1}{2} \hat{B}_{jk} \hat{G}_{il} + \frac{1}{2} \hat{B}_{ik} \hat{G}_{jl} - \frac{1}{2} \hat{B}_{ij} \hat{G}_{kl} ; \quad \nabla_0 H_{ij0} = -\frac{1}{2} \hat{B}_j^k \hat{B}_{ik} - \frac{1}{2} \hat{B}_i^k \hat{G}_{jk} + \hat{B}_{ij}
\]

Using the above reductions, one finds the following reduction for the Lagrangians \( L_3^{H^8} \):

\[
L_3^{H^8} = 13(\text{Tr}(M^4))^2/64 + 61\text{Tr}(M^8)/128
\]  

(31)

where the \( 9 \times 9 \) matrices \( M = G^{-1} \hat{B} \) and \( L = G^{-1} \hat{G} \). The reduction of the Lagrangian \( L_3(G) \) is the following

\[
L_3(G) = (\text{Tr}(L^4))^2/64 + 5\text{Tr}(L^8)/128 + \ldots
\]  

(32)

where dots represent terms which have \( \hat{G} \), \( \text{Tr}(L^2) \), \( \text{Tr}(L) \) or \( \text{Tr}(L^3) \). The reduction of all other Lagrangians are

\[
L_3^{RH^6} = 9\text{Tr}(L^2 M^6)/16 + 9\text{Tr}(L M^2 L M^4)/16 - 9\text{Tr}(L^3 L M^3)/32
\]

\[+9\text{Tr}(L^2 M^2)\text{Tr}(M^4)/16 + \ldots
\]

\[
L_3^{R^2 H^4} = 19\text{Tr}(L^2 M^2)^2/64 + 45\text{Tr}(L^2 M^2 L^2 M^2)/64 - 15\text{Tr}(L^2 M^2 L M LM)/16
\]

\[+17\text{Tr}(L^2 M^3 L^2 M)/32 + 7\text{Tr}(L^2 M^2 L M^2 LM)/16 - 5\text{Tr}(L^3 M^2 L M^2)/32
\]

\[-7\text{Tr}(L^3 M^3 LM)/16 + 17\text{Tr}(L^4 M^4)/32 - 11\text{Tr}(L^2 M^2)\text{Tr}(L M LM)/32
\]

\[+\text{Tr}(L^4)\text{Tr}(M^4)/128 + \ldots
\]

\[
L_3^{R^2 H^2} = -81\text{Tr}(L^3 M L^3 M)/64 - 89\text{Tr}(L^2 M^2)\text{Tr}(L^4)/128 + 147\text{Tr}(L^4 M^2 L^2 M)/64
\]

\[-99\text{Tr}(L^5 L M LM)/64 + 23\text{Tr}(L^6 M^2)/32 + 75\text{Tr}(L^4)\text{Tr}(L M LM)/256 + \ldots
\]

\[
L_3^{(\partial H)^2 H^4} = 2257\text{Tr}(L^2 M^6)/192 - 509\text{Tr}(L^2 M^2 L^4)/64 + 91\text{Tr}(L M LM)\text{Tr}(M^4)/96
\]

\[+169\text{Tr}(L M^3 LM)/192 + 33\text{Tr}(L^2 M^2)\text{Tr}(M^4)/64 - 389\text{Tr}(L^3 M^3 LM)/192 + \ldots
\]

\[
L_3^{R(\partial H)^2 H^2} = 3(\text{Tr}(L^2 M^2))^2/8 - 23\text{Tr}(L M LM LM LM)/48 + 1969\text{Tr}(L^2 M^2 L M LM)/192
\]
We add the following total derivative terms:

\[
L_3^{R(\partial H)^2} = 17\text{Tr}(L^3M^3M^3)/16 - 17\text{Tr}(L^3M^3L^2M)/16 + 29\text{Tr}(L^4ML^2M)/16
\]

\[
L_3^{(\partial H)^4} = 3\text{Tr}(L^2M^2)^2/64 + 3\text{Tr}(L^2M^2L^2M^2)/64 - 113\text{Tr}(LMLMLML)/192
\]

\[
+163\text{Tr}(L^2M^3L^2M)/48 - 865\text{Tr}(L^2ML^2L^2M)/96 - 1247\text{Tr}(L^3M^2L^2M^2)/192
\]

\[
-41\text{Tr}(L^3M^3LM)/64 + 2383\text{Tr}(L^4M^4)/192 + 1439\text{Tr}(L^2M^2)\text{Tr}(LMLM)/384
\]

\[
+113\text{Tr}(LMML)^2/96 - 2485\text{Tr}(L^2M^2LMLM)/192
\]

\[
-277\text{Tr}(L^4)\text{Tr}(M^4)/128 + \cdots
\]

(33)

where dots represent terms which have \(\dot{G}, \dot{B}, \text{Tr}(L^2), \text{Tr}(M^2), \text{Tr}(L)\) or \(\text{Tr}(L^3)\). As it has been argued in [22], using the field redefinitions and total derivative terms, the couplings involving these structures can be converted to the other couplings which have no such structures. Before showing how this works, let us add all above reductions to find the cosmological reduction of the Lagrangian \(L_3(G, B)\), i.e.,

\[
L_3(G, B) = 23\text{Tr}(L^2M^2)^2/32 + 455\text{Tr}(L^2M^2L^2M^2)/96 - 29\text{Tr}(L^2M^2LMLM)/8
\]

\[
+3553\text{Tr}(L^2M^3L^2M)/192 + 2365\text{Tr}(L^2M^6)/192 - 27\text{Tr}(L^2ML^2L^2M)/2
\]

\[
-1517\text{Tr}(L^3ML^2M)/96 - 613\text{Tr}(L^3M^2LM)/64 - 13\text{Tr}(L^3ML^3M)/64
\]

\[
-97\text{Tr}(L^2M^2)\text{Tr}(L^4)/128 + (\text{Tr}(L^4))^2/64 + 323\text{Tr}(L^4M^4)/16
\]

\[
+205\text{Tr}(LMLMLML)/192 + 51\text{Tr}(L^4)\text{Tr}(LMLM)/128 - 29\text{Tr}(LMMLM)^2/24
\]

\[
-63\text{Tr}(L^2M^2)\text{Tr}(LMML)/64 + 51\text{Tr}(L^4)\text{Tr}(LMLM)/128 - 29\text{Tr}(LMMLM)^2/24
\]

\[
-205\text{Tr}(LMLMLML)/192 + 69\text{Tr}(L^2M^2)\text{Tr}(M^4)/64 - 587\text{Tr}(L^4)\text{Tr}(M^4)/128
\]

\[
+91\text{Tr}(LMML)\text{Tr}(M^4)/96 + 13\text{Tr}(M^4)^2/64 + 61\text{Tr}(M^8)/128 + \cdots
\]

(34)

which has 28 structures which have no \(\dot{G}, \dot{B}, \text{Tr}(L^2), \text{Tr}(M^2), \text{Tr}(L)\) or \(\text{Tr}(L^3)\). The cosmological reduction of the effective action (20), then is

\[
S_3^\varepsilon = -\frac{2c}{\kappa^2} \int dt \left[ e^{-\Phi} L_3(G, B) + \cdots \right]
\]

(35)

where \(L_3(G, B)\) is given in (34).

To remove the total derivatives terms and field redefinition freedom from (35), we add all total derivative terms at order \(\alpha^3\) and all field redefinitions with arbitrary coefficients to (35). We add the following total derivative terms:

\[
-\frac{2c}{\kappa^2} \int dt e^{-\Phi} \mathcal{F}_3 \equiv -\frac{2c}{\kappa^2} \int dt \frac{d}{dt} (e^{-\Phi} \mathcal{I}_3)
\]

(36)
where $\mathcal{I}_3$ is all possible terms at seven-derivative level with even parity which are constructed from $\Phi, B, G, \dot{\Phi}, \dot{B}, \dot{G}, \cdots$. Using the package "xAct" [30], one finds there are 2288 such terms, i.e.,

$$
\mathcal{I}_3 = j_1 \dot{B}_i^j \dot{B}_j^k \dot{B}_k^l \dot{G}_l^m \dot{B}_m^n \dot{B}_n^p \dot{B}_p^i + \cdots
$$

(37)

where the coefficients $J_1, \cdots, J_{2288}$ are 2288 arbitrary parameters.

One can change the field variables in (10) as

$$
G_{ij} \rightarrow G_{ij} + \alpha^3 \delta G_{ij}^{(3)}
$$

$$
B_{ij} \rightarrow B_{ij} + \alpha^3 \delta B_{ij}^{(3)}
$$

$$
\Phi \rightarrow \Phi + \alpha^3 \delta \Phi^{(3)}
$$

$$
n \rightarrow n + \alpha^3 \delta n^{(3)}
$$

(38)

where the matrices $\delta G_{ij}^{(3)}$, $\delta B_{ij}^{(3)}$ and $\delta \Phi^{(3)}$, $\delta n^{(3)}$ are all possible terms at 6-derivative level constructed from $\Phi, B, G, \dot{\Phi}, \dot{B}, \dot{G}, \cdots$. The perturbations $\delta G_{ij}^{(3)}$, $\delta \Phi^{(3)}$, $\delta n^{(3)}$ contain even-parity terms and $\delta B_{ij}^{(3)}$ contains odd-parity terms, i.e.,

$$
\delta n^{(3)} = n_1 \dot{B}_i^j \dot{B}_j^k \dot{B}_k^l \dot{G}_l^m \dot{B}_m^n \dot{B}_n^i + \cdots
$$

$$
\delta \Phi^{(3)} = e_1 \dot{B}_i^j \dot{B}_j^k \dot{B}_k^l \dot{B}_l^m \dot{B}_m^n \dot{B}_n^i + \cdots
$$

$$
\delta G_{ij}^{(3)} = d_1 \dot{B}_i^j \dot{B}_j^k \dot{B}_k^l \dot{B}_l^m \dot{B}_m^n \dot{B}_n^i + \cdots
$$

$$
\delta B_{ij}^{(3)} = f_1 \dot{G}_i^j \dot{B}_j^k \dot{B}_k^l \dot{B}_l^m \dot{B}_m^n \dot{B}_n^i \dot{B}_p^l + \cdots
$$

(39)

The coefficients $n_1, \cdots, n_{748}$, $e_1, \cdots, e_{748}$, $d_1, \cdots, d_{1105}$ and $f_1, \cdots, f_{665}$ are arbitrary parameters. When the field variables in $\mathbf{S}_3$ are changed according to the above field redefinitions, they produce some couplings at orders $\alpha^6$ and higher in which we are not interested in this paper. However, when the NS-NS field variables in $\mathbf{S}_6^0$ are changed, the following couplings at order $\alpha^3$ are produced:

$$
\delta \mathbf{S}_6^0 = -\frac{2\alpha^3}{\kappa^2} \int dt e^{-\Phi} \left[ \delta n^{(3)} \left( -\frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} - \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} + \dot{\Phi}^2 \right) + \delta \Phi^{(3)} \left( -\frac{1}{4} \dot{B}_{ij} \dot{B}^{ij} - \frac{1}{4} \dot{G}_{ij} \dot{G}^{ij} + \dot{\Phi}^2 \right) - 2\dot{\Phi} \frac{d}{dt} \delta \Phi^{(3)} + \delta G_{ij}^{(3)} \left( -\frac{1}{2} \dot{B}_i^j \dot{B}^{ki} - \frac{1}{2} \dot{G}_k^j \dot{G}^{ki} \right) + \frac{1}{2} \dot{G}_{ij} \frac{d}{dt} \delta G_{ij}^{(3)} + \frac{1}{2} \dot{B}_i^j \frac{d}{dt} \delta B_{ij}^{(3)} \right]
$$

$$
\equiv -\frac{2\alpha^3}{\kappa^2} \int dt e^{-\Phi} \mathcal{K}_3
$$

(40)

where we have used the fact that the lapse function appears in the action (15) by replacing $dt \rightarrow dt/n$ [22].
Adding the total derivative terms and the field redefinition terms to the action (35), one finds new action $S^c_3$, i.e.,

$$S^c_3 = - \frac{2c}{\kappa^2} \int dt \left[ e^{-\Phi} \mathcal{L}_3(G, B, \Phi) + \cdots \right]$$

(41)

where the Lagrangian $\mathcal{L}_3(G, B, \Phi)$ is related to the Lagrangian $\mathcal{L}_3(G, B)$ as

$$\mathcal{L}_3 = L_3 + J_3 + K_3$$

(42)

The action $S^c_3$ and $S^c_5$ are physically equivalent. They appear in different schemes. Choosing different values for the arbitrary parameters in $J_3$, $K_3$, one would find different forms of couplings for the Lagrangian $\mathcal{L}_3$. We choose these parameters such that all terms that have any derivative of $\Phi$, second and higher derivatives of $G, B$, and all terms that have $\text{Tr}(L^2)$, $\text{Tr}(M^2)$, $\text{Tr}(L)$ or $\text{Tr}(L^3)$ to be zero. Inserting the resulting relations between the parameters into (42), we find the following scheme for the Lagrangian $\mathcal{L}_3$:

$$\mathcal{L}_3 = (\text{Tr}(L^2 M^2))^2/4 - 3\text{Tr}(L^2 M^2 L^2 M^2)/32 + 3\text{Tr}(L^2 M^2 L M L M)/8 - 3\text{Tr}(L^2 M^2 L^2 M)/16 + 3\text{Tr}(L^2 M^2)/16 - 3\text{Tr}(L^2 M L M L M)/16 - 3\text{Tr}(L^2 M^2 L M^2)/16 + 3\text{Tr}(L^3 M^2 L M)/8 - 3\text{Tr}(L^3 M L^3 M)/32 - \text{Tr}(L^2 M^2)\text{Tr}(L^4)/8 + (\text{Tr}(L^4))^2/64 - 3\text{Tr}(L^4 M^4)/16 + 3\text{Tr}(L^4 M^2 L M)/16 + 3\text{Tr}(L^4 M^2)/16 - 3\text{Tr}(L^8)/128 + 3\text{Tr}(L^2 M^2 L M^2)/16 - 3\text{Tr}(L^2 M^2 L M^3)/32 - 3\text{Tr}(L M^2 M L M)/16 - \text{Tr}(L^2 M^2)\text{Tr}(L M L M)/4 + \text{Tr}(L^4)\text{Tr}(L M L M)/16 + (\text{Tr}(L M L M))^2/16 - 3\text{Tr}(L M M L M L M)/64 - \text{Tr}(L^2 M^2)\text{Tr}(M^4)/8 + \text{Tr}(L^4)\text{Tr}(M^4)/32 + \text{Tr}(L M L M)\text{Tr}(M^4)/16 + (\text{Tr}(M^4))^2/64 - 3\text{Tr}(M^8)/128$$

(43)

which has the same 28 structures as in (34) but with different coefficients. Note that in this scheme, there is no term in $\mathcal{L}_3$ other than the above 28 couplings.

Now using the definition of the generalized metric in (16), one finds

$$\text{tr}(\mathcal{S}^4) = 2\text{Tr}(L^4) + 2\text{Tr}(M^4) - 8\text{Tr}(L^2 M^2) + 4\text{Tr}(L M L M)$$

$$\text{tr}(\mathcal{S}^8) = 8\text{Tr}(L^2 M^2 L^2 M^2) - 32\text{Tr}(L^2 M^2 L M L M) + 16\text{Tr}(L^2 M^3 L^2 M) - 16\text{Tr}(L^2 M^6) + 16\text{Tr}(L^2 M L M^2 L M) + 16\text{Tr}(L^3 M^2 L M^2) - 32\text{Tr}(L^3 M^3 L M) + 8\text{Tr}(L^3 M L M^3) + 16\text{Tr}(L^4 M^4) - 16\text{Tr}(L^4 M L M^2 L M) + 16\text{Tr}(L^5 M L M) - 16\text{Tr}(L^6 M^2) + 2\text{Tr}(L^8) - 16\text{Tr}(L M^2 M L M^4) + 8\text{Tr}(L M^3 L M^3) + 16\text{Tr}(L M^5 L M) + 4\text{Tr}(L M L M L M L M) + 2\text{Tr}(M^8)$$

(44)

Using the above $O(9, 9)$-invariant expressions, one can write (43) as

$$\mathcal{L}_3 = \frac{1}{256}(\text{tr}(\mathcal{S}^4))^2 - \frac{3}{256}\text{tr}(\mathcal{S}^8)$$

(45)

which is consistent with the cosmological action (5). We have done the same calculations with couplings in (14), and found exactly the same result. The form of the one-dimensional field redefinitions and total derivative terms, however, are different in the two cases. The above calculations confirm the NS-NS couplings at order $\alpha'^3$ which has been found in (14) (15).
References

[1] J. W. York, Jr., Phys. Rev. Lett. 28, 1082 (1972). doi:10.1103/PhysRevLett.28.1082

[2] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2752 (1977). doi:10.1103/PhysRevD.15.2752

[3] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B 293, 385 (1987). doi:10.1016/0550-3213(87)90077-0

[4] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rept. 244, 77 (1994) doi:10.1016/0370-1573(94)90070-1 [hep-th/9401139].

[5] E. Alvarez, L. Alvarez-Gaume and Y. Lozano, Nucl. Phys. Proc. Suppl. 41, 1 (1995) doi:10.1016/0920-5632(95)00429-D [hep-th/9410237].

[6] A. Sen, Phys. Lett. B 271, 295-300 (1991) doi:10.1016/0370-2693(91)90090-D

[7] O. Hohm, A. Sen and B. Zwiebach, JHEP 02, 079 (2015) doi:10.1007/JHEP02(2015)079 [arXiv:1411.5696 [hep-th]].

[8] M. R. Garousi, Phys. Rev. D 102, no.8, 086019 (2020) doi:10.1103/PhysRevD.102.086019 [arXiv:1908.06627 [hep-th]].

[9] T. H. Buscher, Phys. Lett. B 194, 59 (1987). doi:10.1016/0370-2693(87)90769-6

[10] T. H. Buscher, Phys. Lett. B 201, 466 (1988). doi:10.1016/0370-2693(88)90602-8

[11] N. Kaloper and K. A. Meissner, Phys. Rev. D 56, 7940 (1997) doi:10.1103/PhysRevD.56.7940 [hep-th/9705193].

[12] M. R. Garousi, Phys. Rev. D 99, no. 12, 126005 (2019) doi:10.1103/PhysRevD.99.126005 [arXiv:1904.11282 [hep-th]].

[13] M. R. Garousi, Eur. Phys. J. C 79, no.10, 827 (2019) doi:10.1140/epjc/s10052-019-7357-4 [arXiv:1907.06500 [hep-th]].

[14] M. R. Garousi, JHEP 02, 157 (2021) doi:10.1007/JHEP02(2021)157 [arXiv:2011.02753 [hep-th]].

[15] M. R. Garousi, “On NS-NS couplings at order $\alpha^3$,” [arXiv:2012.15091 [hep-th]].

[16] M. R. Garousi, Phys. Lett. B 809, 135733 (2020) doi:10.1016/j.physletb.2020.135733 [arXiv:1907.09168 [hep-th]].

[17] M. R. Garousi, “Effective action of string theory at order $\alpha'$ in the presence of boundary,” [arXiv:2103.13682 [hep-th]].
[18] G. Veneziano, Phys. Lett. B 265, 287-294 (1991) doi:10.1016/0370-2693(91)90055-U

[19] K. A. Meissner and G. Veneziano, Phys. Lett. B 267, 33-36 (1991) doi:10.1016/0370-2693(91)90520-Z

[20] J. Maharana and J. H. Schwarz, Nucl. Phys. B 390, 3-32 (1993) doi:10.1016/0550-3213(93)90387-5 [arXiv:hep-th/9207016 [hep-th]].

[21] K. A. Meissner, Phys. Lett. B 392, 298-304 (1997) doi:10.1016/S0370-2693(96)01556-0 [arXiv:hep-th/9610131 [hep-th]].

[22] O. Hohm and B. Zwiebach, JHEP 04, 101 (2016) doi:10.1007/JHEP04(2016)101 [arXiv:1510.00005 [hep-th]].

[23] O. Hohm and B. Zwiebach, Phys. Rev. D 100, no.12, 126011 (2019) doi:10.1103/PhysRevD.100.126011 [arXiv:1905.06963 [hep-th]].

[24] M. R. Garousi, Eur. Phys. J. C 80, no.11, 1086 doi:10.1140/epjc/s10052-020-08662-9 [arXiv:2006.09193 [hep-th]].

[25] D. J. Gross and E. Witten, Nucl. Phys. B 277, 1 (1986).

[26] D. J. Gross and J. H. Sloan, Nucl. Phys. B 291, 41 (1987).

[27] M. T. Grisaru and D. Zanon, Phys. Lett. B 177, 347 (1986).

[28] M. D. Freeman, C. N. Pope, M. F. Sohnius and K. S. Stelle, Phys. Lett. B 178, 199 (1986).

[29] T. Codina, O. Hohm and D. Marques, Phys. Rev. Lett. 126, no.17, 171602 (2021) doi:10.1103/PhysRevLett.126.171602 [arXiv:2012.15677 [hep-th]].

[30] T. Nutma, Comput. Phys. Commun. 185, 1719 (2014) doi:10.1016/j.cpc.2014.02.006 [arXiv:1308.3493 [cs.SC]].