GENERALIZATION OF SCHENSTED INSERTION ALGORITHM
TO THE CASES OF HOOKS AND SEMI-SHUFFLES.

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Abstract. Given an rc-graph $R$ of permutation $w$ and an rc-graph $Y$ of
permutation $v$, we provide an insertion algorithm, which defines an rc-graph
$R \leftarrow Y$ in the case when $v$ is a shuffle with the descent at $r$ and $w$ has no
descents greater than $r$ or in the case when $v$ is a shuffle, whose shape is a
hook. This algorithm gives a combinatorial rule for computing the generalized
Littlewood-Richardson coefficients $c_{uv}^w$ in the two cases mentioned above.

1. Introduction.

Rc-graphs were originally introduced by Fomin and Kirillov in [4]. They are
explicit combinatorial objects, which encode monomials in Schubert polynomials.
Rc-graphs proved to be very useful for providing combinatorial rules of computing
certain generalized Littlewood-Richardson (or just LR) coefficients (see [1], [8], [9]).
In this paper we extend these results to more general cases.

Denote by $S_w$ the Schubert polynomial of permutation $w \in S_n$. Then the
generalized LR coefficients $c_{uv}^w$ for $u, v, w \in S_n$ are defined by

$$S_w \cdot S_v = \sum_u c_{uv}^w S_w.$$ 

(If $u, v, w$ are shuffles (also called grassmanian permutations) with descents at $r$,
the coefficients $c_{uv}^w$ are just the LR coefficients.) It can be shown that all $c_{uv}^w$
are nonnegative integers. (Consider the Schubert basis of the cohomology ring of
the flag variety, then $c_{uv}^w$ are the structure constants, and they count the number
of points in certain intersections of algebraic varieties.) There are many totally
positive rules for computing LR coefficients (see [3] for further references), but
there is no know totally positive rule for generalized LR coefficients. (By a totally
positive rule we understand a construction of an explicit combinatorial set for each
triple $(u, v, w)$, such that $c_{uv}^w$ is equal to the number of elements in this set.)

In certain cases (see [2], [9]) a totally positive rule can by given by equating
generalized LR coefficients to LR coefficients. In other cases, such as Pieri formula
(see [14]), a totally positive rule is given in terms of paths in the Bruhat order.

Yet another approach to produce totally positive rule, adopted in [3], [9], [8]
and in this paper, is to generalize Schensted insertion algorithm to rc-graphs. The
rule, which we believe will be eventually generalized to the most general case, is
the following. An algorithm is constructed, which inserts an rc-graph $Y$ of $v$
into rc-graph $R$ of $w$ to produce an rc-graph $R \leftarrow Y$. Then for a fixed rc-graph $U$
of $u$, $c_{uv}^w$, is the number of tuples $(R, Y)$ with $U = R \leftarrow Y$. We present such an
algorithm in the cases, when $v$ is an $r$-shuffle and $w$ is an $r$-semi-shuffle, or when $v$

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is a shuffle, whose shape is a hook. (An $r$-shuffle is a shuffle with the descent at $r$, an $r$-semi-shuffle is a permutation with no descents greater than $r$.)

The first such algorithm was constructed by Bergeron and Billey \cite{1} to prove Monk’s formula (the case when $v$ is a simple transposition). The author \cite{8} showed that their algorithm works in the case when $w$ is an $r$-semi-shuffle and $v$ is an $r$-shuffle. A modified algorithm was constructed by Kumar and the author \cite{9} when $v$ is an $r$-shuffle whose shape is a row. Kumar \cite{10} constructed an analogous algorithm for the column case. The analogy between two algorithms in \cite{9} and \cite{10} is similar to the analogy between row and column insertion algorithms for Young tableaux.

This paper presents an algorithm, which works in all mentioned above cases as well as in the case when $v$ is a shuffle, whose shape is a hook. (A rule in this case written in terms of $r$-Bruhat chains was originally constructed by Sottile \cite{16}.) Our algorithm directly generalizes the algorithm of \cite{9}. In the case when both $w$ and $v$ are $r$-shuffles, the algorithm produces the same results as Schensted insertion algorithm. In all the cases (see \cite{8}, \cite{9}, \cite{10}), except for the case when $v$ is a shuffle whose shape is a hook, the algorithm can be simplified.

Using the new insertion algorithm we also provide a rule for computing generalized LR coefficients in the cases mentioned above using $r$-Bruhat chains. This rule can be thought of as a generalized RSK correspondence. In the case when the shape of $v$ is a hook, it is just a restatement of Pieri formula (see \cite{16}). In the case when $v$ is an $r$-shuffle and $w$ is an $r$-semi-shuffle it is a new result.

The paper is organized as follows. Section 2.1 introduces most of notations and definitions and contains the statements of main results in Theorems 2.2, 2.3, and 2.4. Theorem 2.2 states that the algorithm defined in Section 3 works, it is proved in Section 4. Theorem 2.3 states that the inverse algorithm defined in Section 2 works, it is given without a proof, since the proof is very similar to the proof of Theorem 2.2. Theorem 2.3 gives a rule of computing certain generalized LR coefficients in terms of $r$-Bruhat chains. Finally, Section 5 contains examples of the algorithm. Section 4 should be read together with Section 3 to understand how the algorithm works.

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2. Notation, Definitions and Main Results.

2.1. Permutations. Let $S_n$ be the group of permutations $w = (w(1), \ldots, w(n))$ and let $S_\infty = \bigcup_n S_n$ be the group of permutations on $\mathbb{N}$ which fix all but finitely many integers. For $1 \leq i < j$, denote by $t_{ij}$ the transposition, which exchanges $i$ and $j$. The simple transpositions $s_i = t_{i,i+1}$ for $1 \leq i \leq n-1$ generate $S_n$.

A word $i_1 \ldots i_l$ in the alphabet $\{1, 2, \ldots\}$ is a reduced word of $w \in S_\infty$, if $w = s_{i_1} \ldots s_{i_l}$ and $l$ is minimal. The length $l(w)$ of $w$ is set to be $l$. The longest permutation $w_0^n$ of the group $S_n$ is given by $w_0^n(i) = n + 1 - i$ for $i \leq n$.

$w \in S_\infty$ is an $r$-shuffle if $w(i) < w(i+1)$ for $i \neq r$. It is an $r$-semi-shuffle if $w(i) < w(i+1)$ for $i > r$. To each shuffle we associate a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ given by $\lambda_j = w(r+1-j) - r - 1 + j$ for $j \leq r$, where $\lambda_{r+1} = 0$. Then $w$ is uniquely determined by $\lambda$ and $r$ and we write $w = v(\lambda, r)$. A partition $\lambda$ is a row if $\lambda = (\lambda_1)$, it is a column if $\lambda = (1, \ldots, 1)$, and it is a hook if $\lambda = (\lambda_1, 1, \ldots, 1)$.

2.2. Rc-graphs. Let $W_0^n$ be the reduced word $(n-1 \ldots 1 n-1 n-2 n-1)$ of $w_0^n$. A subword $R$ of $W_0^n$ is called a graph. Each graph $R = i_1 \ldots i_m$ defines
a permutation \( w(R) = s_{i_1} \ldots s_{i_m} \). If \( R \) is a reduced word of \( w(R) \), it is called an \textit{rc-graph} of \( w(R) \). Note that two different subwords of \( W^n_0 \), which produce the same words are two different graphs. For example, if \( n = 3, w = s_2 \), then \( W^3_0 = 212 \) has two different subwords, whose permutation is \( w \), namely the subword \( 2 \) placed at the first or third slot. The set of all rc-graphs of \( w \) is denoted by \( RC(w) \).

We think of graphs using the following pictorial presentation. Think of \( W^n_0 \) as a triangular set of crossings shown in the first picture of Figure 1 for \( n = 5 \). Each crossing is labelled by a letter from the alphabet \([1, \ldots, n - 1]\). To get back \( W^n_0 \) we read those labels from top to bottom row, from right to left in each row. Then each subword \( R \) of \( W^n_0 \) is presented as a subset of the crossings for \( W^n_0 \). Two illustrations are provided in Figure 1, where the second picture corresponds to subword 2323, while the third picture corresponds to subword 4132.

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & + & + & + & + \\
2 & + & + & - & - \\
3 & + & - & - & - \\
4 & - & + & + & + \\
5 & - & - & - & - \\
\end{array}
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & \cdot & + & + & + \\
2 & + & + & - & - \\
3 & + & - & - & - \\
4 & - & + & + & + \\
5 & - & - & - & - \\
\end{array}
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & + & \cdot & \cdot & + \\
2 & + & + & - & - \\
3 & + & - & - & - \\
4 & - & + & + & + \\
5 & - & - & - & - \\
\end{array}
\]

\textbf{Figure 1.} Examples of graphs.

Connect the crossings of \( R \) by strands, which intersect at the places, where there is a crossing and do not intersect otherwise. For illustration see Figure 2, where the graphs correspond to the graphs from Figure 1. Notice that when we draw pictures of graphs we omit those parts of graphs, which have no crossings. So graphs from Figures 1 and 2 can be extended to the right and down by nonintersecting strands.

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

\textbf{Figure 2.} Examples of graphs.

It is easy to see that for each graph \( R \) and \( i \in \mathbb{N}, w(R)(i) \) is given by the column where the strand, which starts at row \( i \), ends. (Instead of referring to a strand as "a strand, which starts at row \( i \), we will say "strand \( i \)." So, the above statement transforms to: strand \( i \) ends in column \( w(R)(i) \).) This immediately leads to

(2.1) If graph \( R \) is constructed out of graph \( R' \) by adding or removing a crossing of strands \( c \) and \( d \) then \( w(R) = w(R')_{cd} \).
Denote by \(|R|\) the length of the corresponding subword, or, in other words, the number of crossings in \(R\). Clearly, \(|R| \geq l(w(R))\). The following two statements are very easy to check and are given without proofs.

1. \(R\) is an rc-graph if and only if no two strands intersect twice in \(R\).

2. \(R\) is an rc-graph if and only if \(|R| = l(w(R))\).

For example, the first and third graphs of Figure 2 are rc-graphs, since both graphs have no double crossings and for both \(|R| = l(w(R))\). But the second graph is not an rc-graph, since strands 3 and 4 intersect twice, or \(|R| > l(w(R)) = 2\).

Given a graph \(R\), and a subset \(I \subseteq \mathbb{N}\), define \(R_I\) to be the graph, which coincides with \(R\) in the rows labelled by elements of \(I\) and has no crossings outside these rows. For example, if \(I = \{\ell, \ell + 1, \ldots\}\), then \(R_I = R_{\geq \ell}\) is the graph, which coincides with \(R\) below or at row \(\ell\) and has no crossings above row \(\ell\). Or, if \(I = \{\ell\}\), then \(R_I = R_\ell\) coincides with \(R\) at row \(\ell\) and has no crossings outside row \(\ell\).

Given two graphs \(R, S\), the union \(R \cup S\) is defined to be the graph, which contains crossings of both \(R\) and \(S\). If \(R\) lies above row \(\ell\), while \(S\) lies at or below row \(\ell\), then it is easy to see \(w(R \cup S) = w(R)w(S)\).

A "place \((i, j)\)" of a graph \(R\) will refer to either crossing or non-crossing of strands in row \(i\) and column \(j\). For example, in the second graph from Figure 2 strands intersect at place \((2, 1)\), but do not intersect at place \((3, 2)\). We refer to those strands which intersect or do not intersect at place \((i, j)\) as strands, which pass the place \((i, j)\). For instance, strands 2, 4 pass places \((2, 2)\) and \((1, 3)\) in the third graph of Figure 2. It will be convenient to write \(a \boxplus b = \ell\), if strand \(a\) intersect strand \(b\) in row \(\ell\) and strand \(a\) is the horizontal strand of the crossing. For example, \(3 \boxplus 4 = 3\) and \(4 \boxplus 3 = 1\) for the second graph from Figure 2.

2.3. Schubert polynomials. For detailed discussions of Schubert polynomials \(S_w\) we refer the reader to [13] or [14]. The only property of Schubert polynomials used in this paper is stated in Theorem 2.1, proved in [5] and [3]. So, for purposes of this paper, we treat Theorem 2.1 as a definition.

For an rc-graph \(R\) define \(x^R = x_{R_1}^{w_1}x_{R_2}^{w_2} \cdots\) (recall that in our notations \(|R_i|\) is the number of crossings of \(R\) in row \(i\)).

**Theorem 2.1.** For \(w \in S_\infty\),

\[
S_w = \sum_{R \in RC(w)} x^R.
\]

If \(w\) is a shuffle \(v(\lambda, r)\), then the Schubert polynomial \(S_w\) is known to be equal to the Schur polynomial \(S_{\lambda}(x_1, \ldots, x_r)\) (for a definition of Schur polynomials see [12]).

Schubert polynomials \(S_w\) for all \(w \in S_n\) form a basis for \(\mathbb{C}[x_1, \ldots, x_n]\). Hence, for \(u, v, w \in S_{\infty}\), we can uniquely define the LR coefficients \(c_{uv}^w\) by

\[
S_w \cdot S_v = \sum_u c_{uv}^w S_u.
\]

2.4. Tableaux. To a partition \(\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)\) associate a Young diagram, which is given by \(\lambda_i\) boxes in row \(i\). If \(m = \sum_{i=1}^r \lambda_i\), label the boxes of the Young diagram by integers 1 to \(m\) starting with the bottom row going up and going from left to right in each row as shown in the first picture of Figure 3, for \(\lambda = (3, 3, 1)\).

Given a Young diagram \(D\), a tableau of shape \(D\) is a filling of boxes of the diagram \(D\) by elements of certain alphabets (we will talk later about the types of
tableaux we consider). For any tableau $U$, produce the word of $U$ by reading the content of the boxes 1 through $m$, denote it by $\text{word}(U)$. For example, the word of the second tableau of Figure 3 is 4356145. Set $|U|$ to be the number of boxes of $U$.

Sometimes we consider only partially filled tableaux: we say that a tableau is filled up to $i$, when boxes 1 to $i$ are filled, others are empty.

A tableau filled with positive integers is row (column) strict if the numbers increase (do not decrease) from left to right and do not decrease (increase) from top to bottom. For example, second tableau of Figure 3 is row and column strict.

Given a shuffle $v(\lambda, r)$, by a tableaux of transpositions of $v(\lambda, r)$ we will understand a tableaux of shape $\lambda$ filled by tuples $(ab)$ with $a \leq r < b$. For example, see the third picture of Figure 3, where $\lambda = (3, 3, 1)$ and $r = 3$. For a tableau of transpositions $T$, let its word $\text{word}(T)$ be $(a_1 b_1) \ldots (a_m b_m)$. Define the permutations $w_i(T) = t_{a_i b_i} \ldots t_{a_j b_i}$ for $1 \leq i \leq m$, and $w(T) = w_m(T)$.

For $w \in S_\infty$, we say that $T$ is an $r$-Bruhat chain of $w$, if for $1 \leq i \leq m$

\begin{equation}
    l(w t_{a_i b_i} \ldots t_{a_j b_i}) = l(w) + i.
\end{equation}

If $T$ is filled up to $j$, we say that $T$ is an $r$-Bruhat chain of $w$ if (2.4) holds for $1 \leq i \leq j$. For a discussion of $r$-Bruhat chains see [2], where any sequence $(a_1 b_1) \ldots (a_m b_m)$, satisfying $l(w t_{a_i b_i} \ldots t_{a_j b_i}) = l(w) + i$ and $a_i \leq r < b_i$ for all $i$, is an $r$-Bruhat chain.

We say that a triple $(w, R, T)$, consisting of a permutation $w$, an rc-graph $R$ and a tableau of transpositions $T$ is an $r$-Bruhat package if $w T = w(R)$ and $T$ is an $r$-Bruhat chain of $w$.

Associate to each permutation $w$ and each tableau of transpositions $T$ of $v(\lambda, r)$ another tableau $E(w, T)$ of the same shape. Fill box $i$ of $E(w, T)$ with $w t_{a_i b_i} \ldots t_{a_j b_i}$. (If $T$ is filled up to $j$, then $E(w, T)$ will be filled up to $j$.) For example, if $w = (1, 3, 4, 2, 5, 6, \ldots)$ and $T$ is the third tableau from Figure 3, then the second tableau from Figure 3 is $E(w, T)$.

2.5. Main Results. We are now ready to state our main results. Given $R \in \mathcal{RC}(w)$ and $Y \in \mathcal{RC}(v(\lambda, r))$ satisfying certain conditions, in Section 3 we will define a graph $R \leftarrow Y$ together with a tableaux of transpositions $T(R, Y)$ of $v(\lambda, r)$.

Theorem 2.2. Let $w, v(\lambda, r) \in S_\infty$ satisfy one the following

\begin{enumerate}
    \item[(2.5)] $w$ is an $r$-semi-shuffle,
    \item[(2.6)] $\lambda$ is a hook.
\end{enumerate}
Let $R \in \mathcal{R}C(w)$ and $Y \in \mathcal{R}C(v(\lambda, r))$. Then $U = R \leftrightarrow Y$ is an rc-graph and

\begin{equation}
(2.7) \quad E(w, T(R, Y)) \text{ is a row and column strict tableau,}
\end{equation}

\begin{equation}
(2.8) \quad (w, U, T(R, Y)) \text{ is an r-Bruhat package,}
\end{equation}

\begin{equation}
(2.9) \quad x^U = x^R x^Y.
\end{equation}

In Section 2, the inverse insertion algorithm will define graphs $U \to T$ and $Y(U, T)$ for certain rc-graphs $U$ and tableaux of transpositions $T$.

**Theorem 2.3.** Let $u, v(\lambda, r), w \in S_\infty$, $U \in \mathcal{R}C(u)$ and $T$ be a tableau of transposition of $v(\lambda, r)$. Assume $w, v(\lambda, r)$ satisfy (2.4) or (2.3), $(w, U, T)$ is an r-Bruhat package and $E(w, T)$ is row and column strict. Then $R = U \to T \in \mathcal{R}C(w)$ and $Y = Y(U, T) \in \mathcal{R}C(v(\lambda, r))$. Moreover, $U = R \leftrightarrow T$ and $T = T(Y, R)$.

The next theorem is an immediate corollary of Theorems 2.1, 2.2, 2.3.

**Theorem 2.4.** Assume $w, u, v(\lambda, r) \in S_\infty$ satisfy (2.4) or (2.3). Then $c_{wv(\lambda, r)}^u$ is equal to the number of tableaux of transpositions $T$ of $v(\lambda, r)$, such that $T$ is an r-Bruhat chain of $w$, $E(w, T)$ is row and column strict and $ww(T) = u$.

Let us restate Theorem 2.4 in the case when the shape of $v$ is a hook in the form it appeared in [10]. Given $w$ and a tableau of transpositions $T$ of $v$, define $w^{(i)} = w w_i(T)$. If the shape of $v$ is a hook $(p, 1^{q-1})$ then $E(w, T)$ is row and column strict if and only if

\begin{equation}
(2.10) \quad w^{(1)}(b_1) > \cdots > w^{(p)}(b_p) \quad \text{and} \quad w^{(p)}(b_p) < \cdots < w^{(m)}(b_m),
\end{equation}

where $m = l(v) = p + q - 1$. Using the fact that $t_{ab}$ and $t_{a'b'}$ commute as long as $a, b, a', b'$ are distinct, it can be shown that there is a one to one correspondence between r-Bruhat chains, which satisfy (2.10) and r-Bruhat chains, which satisfy

\begin{equation}
(2.11) \quad w^{(1)}(a_1) > \cdots > w^{(p)}(a_p) \quad \text{and} \quad w^{(p)}(a_p) < \cdots < w^{(m)}(a_m).
\end{equation}

This correspondence can be constructed by starting with a chain, which satisfies (2.10) and commuting transpositions of this chain to make sure (2.11) holds.)

So Theorem 2.4 in case (2.6) can be restated as it originally appeared in [10].

**Theorem 2.5.** Assume $v = ((p, 1^{q-1}), r)$. Then $\mathcal{S}_w \mathcal{S}_v = \sum \mathcal{S}_{w^{(m)}}$, the sum over all paths in k-Bruhat order $w <_w \cdots <_w <_w (m)$, which satisfy (2.11).

3. Insertion Algorithm

3.1. Preliminaries. We need some preliminaries before defining the algorithm.

First, let $Y$ be an rc-graph with $w(Y) = v(\lambda, r)$. Sometimes we will think of the Young diagram of $\lambda$ as the shape of $Y$, denoted by $sh(Y)$. It is easy to see that strand $s \leq r$ intersects exactly $\lambda_{r+1-s}$ other strands. Let these intersections be in rows $i_1 \geq \cdots \geq i_{\lambda_{r+1-s}}$ (one number for each crossing, so that repetitions are allowed), then define word($Y, s$) = $i_1 \ldots i_{\lambda_{r+1-s}}$. Define word($Y$) to be the concatenation word($Y, 1$) $\cdots$ word($Y, r$). Notice that if two strands $a, b$ intersect in $Y$ and $a < b$ then $a \leq r < b$. Hence every crossing of $Y$ correspond to a single letter in word($Y$). Also notice that for any $\ell$, the permutation of $Y_{\geq \ell}$ is again a shuffle. Moreover, the shape of $Y_{\geq \ell}$ is a subdiagram of the shape of $Y$.

Secondly, we need the following lemma and the construction after the lemma.
Lemma 3.1. (1) If $R$ is an rc-graph and $l(w(R)_{cd}) = l(w(R)) - 1$, then strands $c$ and $d$ intersect in $R$, and removing this crossing produces another rc-graph.
(2) Let $R$ be an rc-graph with strands $c$ and $d$ passing place $(\ell, j)$ but never intersecting in $R$. Then insertion a crossing into place $(\ell, j)$ produces an rc-graph.

Proof. For $w \in S_{\infty}$ its length is $l(w) = \#\{(i, j) : i < j, w(i) > w(j)\}$. Hence

(3.1) \hspace{1cm} \text{for } c < d, \ l(w_{cd}) = l(w) + 1 \text{ if and only if } w(c) < w(d) \text{ and there is no } i \text{ with } c < i < d, \ w(c) < w(i) < w(d).

To prove the first part of the lemma, notice that (3.1) applied to $w = w(R)_{cd}$ immediately implies that $w(R)(c) > w(R)(d)$. In particular, strands $c$ and $d$ must intersect in $R$. Remove their crossing to produce graph $R'$. Using (2.3), (2.4) and $l(w(R')) = l(w(R)) = l(w(R)) - 1 = |R'|$, we conclude $R'$ is an rc-graph.

To prove the second part, add the crossing of strand $c$ and $d$ in place $(\ell, j)$ of $R$ to produce graph $R'$. By (2.3) it is enough to check $l(w(R')) = l(w(R)) + 1$. Since strands $c$ and $d$ do not intersect in $R$, $w(R)(c) < w(R)(d)$, hence by (3.1), it is enough to check that there is no $i$ with $c < i < d$, $w(R)(c) < w(R)(i) < w(R)(d)$.

It is very easy to see that if such $i$ existed, strand $i$ would have to intersect either strand $c$ or strand $d$ twice in $R$, which is impossible. \hfill \Box

As a consequence to Lemma 3.1, let us present the following construction. Given an $r$-Bruhat package $P = (w, R, T)$, let $m = |T|$. Set $S_m(P) = R$. Then, by Lemma 3.1, rc-graphs $S_j(P)$ for $0 \leq j \leq m$ are uniquely defined, once we require that $w(S_j(P)) = w w_j(T)$, and $S_j(P)$ is constructed out of $S_{j+1}(P)$ by removing exactly one crossing. An example is provided in Section 6.1.

3.2. Outline. We now start the description of the algorithm. We begin with a general outline to better explain the procedure.

Assume we are given an rc-graph $R$ with $w = w(R)$ and an rc-graph $Y$ with $v(\lambda, r) = w(Y)$, satisfying (2.3) or (2.4). Our goal is to define a graph $R \leftarrow Y$ and a tableau of transpositions $T(R, Y)$ of $v(\lambda, r)$.

The algorithm starts with row $r$ and goes up. After the algorithm finishes with row $\ell$ it produces $R(\ell) = R_{\geq \ell} \leftarrow Y_{\geq \ell}$ and $T(\ell) = T(R_{\geq \ell}, Y_{\geq \ell})$.

It is convenient to think of $T(\ell)$ as "the history" of the algorithm up to $\ell$. Namely, $T(\ell)$ says how to go from $w(R_{\geq \ell})$ to $w(R(\ell))$ along a chain in Bruhat order. Moreover, since the shape of $Y_{\geq \ell}$ is the same as the shape of $T(\ell)$, word($Y_{\geq \ell}$) and word($T(\ell)$) are of the same shape, so each step in the chain corresponds to a crossing of $Y_{\geq \ell}$.

Assume the insertion has been performed up to row $\ell + 1$. The next row where it has to operate is row $\ell$. Clearly $\text{word}(Y_{\geq \ell})$ is constructed from $\text{word}(Y_{\geq \ell+1})$ by adding letter $\ell$ at some places. If $\text{word}(Y_{\geq \ell}) = \text{word}(Y_{\geq \ell+1})$, so there are no crossings in row $\ell$ of $Y$, then it looks like we can say $T(\ell) = T(\ell + 1)$. But since $T(\ell + 1)$ defines a chain which starts at $w(R_{\geq \ell + 1})$, it may not define a chain starting from $w(R_{\geq \ell})$. So, the algorithm goes through the rectification of both rc-graph and the chain to make sure $T(\ell)$ indeed defines the chain starting at $w(R_{\geq \ell})$.

If there are some crossings in row $\ell$, then the algorithm inserts them whenever necessary. The order in which insertions and rectifications are performed are determined by the order of letter in word($Y_{\geq \ell}$).
3.3. Sequence of Steps of the algorithm. Throughout the rest of this section all statements, which require proofs, will be underlined and then proved in Section 4.

For each \((\ell, i)\) with \(r \geq \ell \geq 1\) and \(0 \leq i \leq m_\ell\) (where \(m_\ell = |Y_{\geq \ell}|\)) the algorithm performs a step, which we call step \((\ell, i)\). The steps go in the following order. Step \((\ell, i + 1)\) goes after step \((\ell, i)\) if \(m_\ell - 1 \geq i \geq 0\). Step \((\ell, 0)\) follows step \((\ell + 1, m_{\ell + 1})\).

Before giving a detailed description of each step, let us present the data produced by each step and the conditions this data satisfies. Step \((\ell, i)\) constructs

(3.2) rc-graphs \(R(\ell, i)\), with no crossings above row \(\ell\),

(3.3) tableau of transpositions \(T(\ell, i)\) for the shuffle \(w(Y_{\geq \ell})\) filled up to \(i\).

Here, \(R(\ell, i)\) and \(T(\ell, i)\) play the role of the intermediate result of the algorithm. After each step, \(R(\ell, i)\) and \(T(\ell, i)\) must satisfy the following conditions

(3.4) \(E(w(R_{\geq \ell}), T(\ell, i))\) is a row and column strict tableau,

(3.5) \(P(\ell, i) = (w(R_{\geq \ell}), R(\ell, i), T(\ell, i))\) is an \(r\)-Bruhat package.

Remark 3.2. Conditions (3.4), (3.5) are analogues of (2.7), (2.8) from Theorem 2.2. That is why they have to be satisfied by the intermediate results of the algorithm. Condition (3.4) is the condition, which we do not know how to generalize to cases other than (2.5) and (2.6).

So for each row \(\ell\) we start with a row-to-row step \((\ell, 0)\), which sets up the data needed for performing insertion in this row. Then we perform a step for each letter of \(\text{word}(Y_{\geq \ell})\). If this letter is equal to \(\ell\) it an insertion step and we will insert a crossing in row \(\ell\) to the current rc-graph. If the letter is not \(\ell\), then we perform a rectification and rectify, if necessary, both rc-graph \(R(\ell, i)\) and the chain given by \(T(\ell, i)\) to guarantee both (3.4) and (3.5) are satisfied.

The rest of Section 3.3 introduces additional notation and states two additional conditions, which clarify certain parts of the algorithm and simplify certain proofs.

After we are finished with all the steps for row \(\ell\), we are given \(R(\ell, m_\ell)\) and \(T(\ell, m_\ell)\), which, to shorten the notations, we denote by \(R(\ell)\) and \(T(\ell)\). Denote by \(P(\ell)\) the \(r\)-Bruhat package \((w(R_{\geq \ell}), R(\ell), T(\ell))\).

Fix \(\ell\), let \(\text{word}(Y_{\geq \ell}) = k_1 \ldots k_{m_\ell}\). Each letter \(k_i\) of \(\text{word}(Y_{\geq \ell})\) corresponds to a crossing of \(Y_{\geq \ell}\) in row \(k_i\). If \(k_i > \ell\), then the letter \(k_i\) is also a part of \(\text{word}(Y_{\geq \ell + 1}) = k_1' \ldots k_{m_{\ell + 1}}'\), let the index of \(k_i\) inside \(\text{word}(Y_{\geq \ell + 1})\) be \(i_+\). Set \(i_+ = 0\), if \(i = 0\). So, if we think of \(sh(Y_{\geq \ell + 1})\) as a subdiagram of \(sh(Y_{\geq \ell})\), then box \(i\) of \(Y_{\geq \ell}\) coincides with box \(i_+\) of \(sh(Y_{\geq \ell + 1})\). The two additional conditions are

(3.6) \[ x^{R(\ell)} = x^{R_{\geq \ell}} x^{Y_{\geq \ell}}, \]

(3.7) If \(k_i < \ell\), then \(R(\ell, i)_{\geq \ell + 1} = S_{i_+}(P(\ell + 1))\).

It will be obvious from the description of the algorithm that these conditions are always satisfied. Condition (3.6) implies that (2.3) holds for the final result, while (3.7) indicates, that step \((\ell, i)\) only operates in row \(\ell\), as the part of \(R(\ell, i)\), which lies below row \(\ell\), is uniquely determined by \(P(\ell + 1)\).

3.4. Start of the algorithm. Set \(R(r, 0) = R_{\geq r}\) and let \(T(r, 0)\) be the empty tableau of shape \(sh(Y_{\geq r})\), then \(R(r, 0)\) and \(T(r, 0)\) satisfy (3.4), (3.5).
3.5. **Row-to-row steps.** Each step \((\ell, 0)\) is called a *row-to-row step*. This step sets \(T(\ell, 0)\) to be the empty tableau of shape \(sh(Y_{\geq \ell})\) and

\[
R(\ell, 0) = S_0(\mathcal{P}(\ell + 1)) \cup R_{\ell}.
\]

Then \(R(\ell, 0)\) is an rc-graphs and \(R(\ell, 0)\) and \(T(\ell, 0)\) satisfy (3.4), (3.5).

As mentioned before, this step sets up the data for performing the algorithm in row \(\ell\). \(T(\ell + 1)\) defines a chain from \(w(R_{>\ell + 1})\) to \(w(R(\ell + 1))\). On the level of rc-graph this chain is given by the chain \(S_0(\mathcal{P}(\ell + 1)), \ldots, S_{m+1}(\mathcal{P}(\ell + 1))\). So we can thing of \(R(\ell, 0)\) as backtracking the algorithm from \(R(\ell + 1)\) to \(S_0(\mathcal{P}(\ell + 1))\) and then adding those crossings of \(R\), which lie in row \(\ell\).

3.6. **Insertions.** Assume \(word(Y_{\geq \ell}) = k_1 \ldots k_{m_{+}}\). If \(k_i\) is the letter \(\ell\), then step \((\ell, i)\) is called an *insertion*.

During insertion step \((\ell, i)\), we say that insertion into a place \((\ell, j)\) is allowed, if strands \(c, d\) pass this place in \(R(\ell, i - 1)\) as shown in Figure 4 and \(c \leq r < d\).

\[
\begin{array}{c}
\text{c} \\
\downarrow \\
\text{d}
\end{array}
\quad \text{with } c \leq r < d,
\]

**Figure 4.** Place where insertion is allowed.

Find the rightmost place, where insertion is allowed. Let it be place \((\ell, j_0)\) and let strands \(c\) and \(d\) pass through it. Define \(R(\ell, i)\) by adding a crossing to \(R(\ell, i - 1)\) into place \((\ell, j_0)\). Define \(T(\ell, i)\) by adding \((cd)\) to box \(i\) of \(T(\ell, i - 1)\). Then \(R(\ell, i)\) is an rc-graph and (3.4) and (3.5) are satisfied.

3.7. **Rectifications.** If \(k_i > \ell\), then step \((\ell, i)\) is called a *rectification*. The first part of rectification is to define a graph \(R'\) and a tableau of transpositions \(T'\). The rc-graph \(S_{i+1}(\mathcal{P}(\ell + 1))\) has one more crossing than \(S_{i-1}(\mathcal{P}(\ell + 1))\), add this crossing to \(R(\ell, i - 1)\) to produce \(R'\). Then, since (3.7) holds for \(R(\ell, i - 1)\), \(R'\) coincides with \(S_{i+1}(\mathcal{P}(\ell + 1))\) below row \(\ell\) and row \(\ell\) of \(R'\) is the same as row \(\ell\) of \(R(\ell, i - 1)\). To produce \(T'\), add to box \(i\) of \(T(\ell, i - 1)\) the entry \((ab)\) of box \(i_{+}\) of \(T(\ell + 1)\).

If \(E(w(R_{>\ell}), T')\) is row and column strict and \((w(R_{>\ell}), R', T')\) is an r-Bruhat package, set \(R(\ell, i) = R'\) and \(T(\ell, i) = T'\) and move on to the next step. Otherwise, there is a crossing in \(R'\) in row \(\ell\), which fits the description in Figure 5.

\[
\begin{array}{c}
\text{b} \\
\downarrow \\
\text{a}
\end{array} \quad \text{where } (ab) \text{ is the entry of box } i \text{ of } T'.
\]

\[
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a}
\end{array} \quad \text{where } (ab) \text{ and } (af) \text{ are the entries of boxes } i \text{ and } i - 1 \text{ of } T'.
\]

**Figure 5.** One of these crossing in row \(\ell\) needs to be removed.

If \(R'\) has a crossing, which looks like the first crossing of Figure 5, remove this crossing to produce \(R''\) and remove \((ab)\) from box \(i\) of \(T'\) to produce \(T''\). Otherwise remove the crossing of strands \(b\) and \(f\) shown in Figure 5 to produce \(R''\), remove \((ab)\) from box \(i\) in \(T'\) and replace the entry of box \(i - 1\) of \(T'\) by \((ab)\) to produce \(T''\).

We say that insertions into places in \(R''\) shown in Figure 5 are allowed.

Remember that \(R''\) is constructed out of \(R'\) by removing a crossing from place \((\ell, j_0)\). Find rightmost place to the left of \((\ell, j_0)\), where insertion is allowed. Insert a crossing there to produce \(R(\ell, i)\). If this is the place of the first type from Figure 5, then insert \((cd)\) into box \(i\) of \(T''\) to define \(T(\ell, i)\). In the second case, replace the
entry of box \(i - 1\) of \(T''\) by \((ed)\) and place \((eg)\) into box \(i\) of \(T''\) to define \(T(\ell, i - 1)\). Then \(R(\ell, i)\) is an rc-graph and (3.4) and (3.5) are satisfied.

3.8. **End of the algorithm.** Set \(R \leftarrow Y = R(1, m_1)\) and \(T(R, Y) = T(1, m_1)\).

3.9. **Concluding remarks.** As mentioned before, the algorithm can be simplified in all the cases, except for the case when shape of \(v\) is a hook. For case (2.5) (see [8]) our algorithm produces the same result as inserting letters of word \((Y)\) one by one into \(R\) using algorithm of Bergeron and Billey [1]. Moreover, in the case when \(w\) is also an \(r\)-shuffle, it is just the Schensted insertion algorithm. So our algorithm is a direct generalization of Schensted insertion algorithm for Young tableaux.

For the case when shape of \(v\) is a row a simplified algorithm is given in [9]. In the case when shape of \(v\) is a column, the only simplification, which can be done, is omitting the second picture from Figure 6. Analogous algorithm in this case can also be found in [10].

4. **Proof of Theorem 2.2**

To prove Theorem 2.2, it is enough to prove all the statements underlined in Section 3. Let us list these statements again.

1. \(R(\ell, 0)\) and \(T(\ell, 0)\) satisfy conditions (3.4), (3.5).
2. For \(r > \ell \geq 1\), \(R(\ell, 0)\) is a rc-graph and \(R(\ell, 0), T(\ell, 0)\) satisfy (3.4), (3.5).
3. After insertion, \(R(\ell, i)\) is an rc-graph and (3.4), (3.5) are satisfied.
4. During rectification, if \(R'\) is not an rc-graph or \(E(w(R_{\geq \ell}, T))\) is not row and column strict, there is a crossing in \(R'\) shown in Figure 5.
5. After rectification, \(R(\ell, i)\) is an rc-graph and (3.4), (3.5) are satisfied.

4.1. **Proof of (1).** Since \(T(\ell, 0)\) is an empty tableau, condition (3.4) is vacuous, while (3.3) follows directly from \(R(\ell, 0) = R_{\geq \ell}\).

4.2. **Proof of (2).** Since \(S_0(P(\ell + 1))\) has no crossings above row \(\ell + 1\) and \(R_\ell\) has only crossings in row \(\ell\), we know from (3.8)

\[
w(R(\ell, 0)) = w(S_0(P(\ell + 1)))w(R_\ell) = w(R_{\geq \ell+1})w(R_\ell) = w(R_{\geq \ell+1} \cup R_\ell) = w(R_{\geq \ell}).
\]

On the other hand

\[
|R(\ell, 0)| = |S_0(P(\ell + 1))| + |R_\ell| = |R_{\geq \ell+1}| + |R_\ell| = |R_{\geq \ell}|.
\]

Hence \(l(w(R(\ell, 0))) = |R(\ell, 0)|\) and using (2.3) we conclude \(R(\ell, i)\) is an rc-graph.

Since \(T(\ell, 0)\) is an empty tableau, condition (3.4) is vacuous, while (3.5) follows immediately from \(w(R(\ell, 0)) = w(R_{\geq \ell})\).
4.3. **Proof of (3).** To show that the algorithm works properly during insertion, we must show that there are places in \( R(\ell, i-1) \), where insertions are allowed. To do this, define the sequence \( c_k \) by \( c_0 = \ell \) and

\[
(4.1) \quad a_k = w(R(\ell, i-1))(c_k), \quad c_{k+1} = w(R(\ell, i-1)|_{\geq \ell+1})^{-1}(a_k + 1).
\]

Here is another way of defining \( c_k \)’s. Look at all strands of \( R(\ell, i-1) \), which have horizontal parts in row \( \ell \), in other words, which do not cross row \( \ell \) vertically. These strands do not cross each other in row \( \ell \). So, we let the sequence \( c_k \) be the labels of these strands, read from left to right. For example, if \( R(\ell, i-1) \) is not an rc-graph. Hence by the second part of Lemma 3.1,\( R(\ell, i-1) \) is an rc-graph, then (4.3) holds.\( \)

4.3.1. **Proof of (3).** Let \( R(\ell, i) \) be an rc-graph and (3.3) holds. \( R(\ell, i) \) is an rc-graph by the second part of Lemma 3.1. Moreover, \( l(w(R(\ell, i)) = l(w(R(\ell, i-1)) + 1 \) and

\[
(4.2) \quad w(R(\ell, i)) = w(R(\ell, i-1)t_{cd}, \quad w(T(\ell, i)) = w(T(\ell, i-1))t_{cd},
\]

which immediately implies that (3.5) holds.

4.3.2. **After insertion, (3.2) holds.** If \( i = 1 \), so that the insertion step corresponds to the first letter \( k_1 = \ell \), condition (3.2) is vacuous. Otherwise, we will show that there exist \( j \), such that insertion into \((\ell, j)\) is allowed and

\[
(4.3) \quad w(R(\ell, i-1)t_{c'd'})((d')) > w(R(\ell, i-1))(f)
\]

where \( c', d' \) are the strands passing place \((\ell, j)\) and \((ef)\) is the entry of box \( i \) of \( T(\ell, i-1) \). This will be enough to prove (3.2). Indeed if \( R(\ell, i) \) is defined by adding a crossing of strand \( c, d \) in place \((\ell, j_0)\), then \( j_0 \geq j \) and therefore (4.3) holds for \( c', d' \) substituted by \( c, d \). Hence \( E(w(R_{\geq \ell}), T(\ell, i)) \) is row and column strict.

To show that \( j \), satisfying (4.3), exists, consider sequence \( c_0 \) with different \( c_0 \): if strand \( e \in R(\ell, i-1) \) intersects vertically another strand \( e' \) in row \( \ell \), that is \( e' \cap e = \ell \), then set \( c_0 = e' \) otherwise set \( c_0 = e \) (notice that \( c' < e \leq \ell \)). Since \( c_0 \leq \ell \) and \( c_k \geq r \) for large \( k \), there exists \( k \) with \( c_k \leq r < c_k+1 \). Let \( c' = c_k \) and \( d' = c_{k+1} \). Then strands \( c' \) and \( d' \) pass next to each other in row \( \ell \) at a place \((\ell, j)\) and insertion into \((\ell, j)\) is allowed. Moreover, since strand \( c_0 \) is either strand \( e \) or it intersects strand \( e \) horizontally in row \( \ell \), the following calculation proves (4.3)

\[
\quad w(R(\ell-1, i)t_{c'd'})((d')) = w(R(\ell, i-1))(c') \geq w(R(\ell, i-1))(e) > w(R(\ell, i-1))(f).
\]

4.4. **Proof of (4).** Recall that \( R' \) is constructed out of \( R(\ell, i-1) \) by adding a crossing of strands \( a, b \) to guarantee \( R' \) coincides with \( S_{i_+} P(\ell + 1) \) below row \( \ell \).

Let us show that \( R' \) is not an rc-graph if and only if strands \( a \) and \( b \) intersect in row \( \ell \) as shown in Figure 3. Indeed, if \( a, b \) intersect in row \( \ell \), they intersect twice in \( R' \), so \( R' \) is not an rc-graph. Conversely, if they do not intersect in row \( \ell \) of \( R' \), then they do not intersect in \( R(\ell, i-1) \) (If they intersect below row \( \ell \) then \( S_{i_+} P(\ell + 1) \) is not an rc-graph.) Hence by the second part of Lemma 3.1 \( R' \) is an rc-graph.

Since \( w(R') = w(R(\ell, i-1))t_{ab} \), we can immediately conclude that if \( R' \) is an rc-graph, then \( (w(R_{\geq \ell}), R', T') \) is an \( r \)-Bruhat package.
To finish the proof of (4), it remains to prove that if \( E(w(R_{\geq i}), T') \) is not column or row strict then the second crossing from Figure 5 must occur. We will do it separately for cases (2.5) and (2.6).

4.4.1. Case (2.5). We start with preliminaries, which we also use in the proof of (5).

**Lemma 4.1.** If \( u \in S_\infty \) is an \( r \)-semi-shuffle, \( u' = ut_{ab} \) with \( a \leq r < b \) and \( l(u') = l(u) + 1 \), then \( u' \) is an \( r \)-semi-shuffle.

**Proof.** We must show that if \( r < b' < b'' \) then \( ut_{ab}(b') < ut_{ab}(b'') \). If \( b' \neq b \) and \( b'' \neq b \), then \( ut_{ab}(b') = u(b') < u(b'') = ut_{ab}(b'') \), since \( u \) is an \( r \)-semi-shuffle.

If \( b' = b \), then \( ut_{ab}(b') = u(a) < u(b) = ut_{ab}(b'') \), since \( l(ut_{ab}) = l(u) + 1 \) and \( u \) is an \( r \)-semi-shuffle.

If \( b'' = b \), then \( ut_{ab}(b') = u(b') < u(a) = ut_{ab}(b'') \), where \( u(b') < u(a) \), since otherwise \( a < b' < b \) and \( u(a) < u(b') < u(b) \), which contradicts (4.4).

Assume \( T \) is a tableau of transpositions (possibly partially filled). If \((a_k b_k)\) are the entries of \( T \), let \( B(T) \) be the tableau of the same shape with the entries \( b_k \).

**Lemma 4.2.** Let \( w \) be an \( r \)-semi-shuffle and \( T \) be an \( r \)-Bruhat chain of \( w \). Then \( E(w, T) \) is row and column strict if and only if \( B(T) \) is row strict.

**Proof.** Assume \( u \in S_\infty \) is an \( r \)-semi-shuffle and \( l(ut_{ab}) = l(u) + 1 \) for \( a \leq r < b \). Let \( b' > r \) then it is easy to see by Lemma 4.1 that

\[
(4.4) \quad u(b') < ut_{ab}(b) \quad \text{if and only if} \quad b' < b.
\]

Clearly, (4.4) implies that rows of \( E(w, T) \) strictly increase from left to right if and only if the same holds for \( B(T) \).

Let us show that if \( B(T) \) is row strict, then \( E(w, T) \) is column strict. Denote by \( e_k \) the entry of box \( k \) of \( E(w, T) \). Let box \( i' \) be directly above box \( i \) in \( sh(T) \). Consider boxes \( i \) through \( i' \) in the diagram \( sh(T) \), as shown in Figure 7.

**Figure 7.** Boxes \( i \) through \( i' \) of \( T \).

To show \( E(w, T) \) is column strict, it is enough to show \( e_{i'} < e_i \), for any \( i \) not in the top row. If \( B(T) \) is row-strict, then \( b_j \neq b_i \) for \( i < \tilde{i} < i' \). If \( b_{i'} < b_i \), then

\[
e_i = \text{ww}_{\tilde{i}}(T)(b_i) = \text{ww}_{\tilde{i}}(T)(b_i) > \text{ww}_{\tilde{i}}(T)(b_{i'}) = e_{i'},
\]

since by Lemma 4.1, \( \text{ww}_{\tilde{i}}(T) \) is an \( r \)-semi-shuffle. If \( b_{i'} = b_i \), then

\[
e_i = \text{ww}_{\tilde{i}}(T)(b_i) = \text{ww}_{\tilde{i}-1}(T)(b_i) = \text{ww}_{\tilde{i}}(T)(a_i) > \text{ww}_{\tilde{i}}(T)(b_i) = e_{i'},
\]

since \( l(\text{ww}_{\tilde{i}-1}(T)) + 1 = l(\text{ww}_{\tilde{i}}(T)) \).

Conversely, assume \( E(w, T) \) is row and column strict. To show \( B(T) \) is row strict, it is enough to show \( b_j \leq b_i \) for any \( j < \tilde{i} \), such that box \( i \) is not in the top row. Assume for a moment \( b_j \neq b_i \) for any \( j \leq \tilde{i} \). Then

\[
\text{ww}_{\tilde{i}}(T)(b_i) = \text{ww}_{\tilde{i}}(T)(b_i) = e_i > e_{i'} = \text{ww}_{\tilde{i}}(T)(b_{i'}),
\]
since \( E(w, T) \) is row and column strict. Thus, since \( w w' (T) \) is an \( r \)-semi-shuffle, we conclude \( b_r < b_1 \).

Otherwise, if \( b_i = b_1 \) for some \( j \leq i \leq i' \), to show that \( i = i' \), use induction on \( i \).

If \( i \) is the first box in its row, then \( b_r = b_1 \), as \( i' = j \). Otherwise, assume the box underneath box \( i \) contains \( b_1 \). By induction \( \bar{b} \geq b_r \). On the other hand, we know that 
\( \bar{b} < b_1 \), if \( \bar{b} \neq i \). Hence, if \( b_1 = b_1 \), then box \( i \) must be underneath box \( \bar{i} \).

To finish the proof of (4) in case (2.3), we will prove that if \( R' \) is an \( r \)-graph, then \( E(w(R_{=\ell}), T') \) is row and column strict, or by Lemma 4.2 it is enough to show \( B(T') \) is row strict.

Let \( b_k \) be the entries of \( B(T') \), so that \( b_1 = b \). Since boxes 1 through \( i - 1 \) of \( B(T') \) and \( B(T(\ell, i - 1)) \) coincide, \( B(T') \) can fail to be row strict if box \( i - 1 \) is in the same row as box \( i \) and \( b_{i - 1} \geq b_i = b \), or if there is box \( i_1 \) underneath box \( i \), such that \( b = b_i > b_{i_1} \). Let us show both cases are impossible. This will finish the proof of (4) in case (2.3).

If \((\ell f)\) is the entry of box \( i_1 - 1 \) of \( T(\ell + 1) \), then by construction \( b_{i_1 - 1} \leq \bar{f} \). At the same time if box \( i_1 - 1 \) is in the same row as box \( i \), then \( \bar{f} < b_i = b \), since \( B(T(\ell + 1)) \) is row strict. So \( b_{i_1 - 1} < b \), whenever box \( i - 1 \) is in the same row as box \( i \).

It remains to show that if box \( i_1 \) is the box underneath box \( i \) in \( B(T') \), then \( b \leq b_r \). We will prove it by induction on \( i \). We will prove the induction step when step \((\ell + 1, i_1)\) is an insertion. If it is a rectification, the proof is almost identical.

Denote temporarily \( \bar{R} = R(\ell, i_1 - 1) \). We will prove there exists a place in \( \bar{R} \) in row \( \ell + 1 \) shown in Figure 4 with \( d \geq b \). Then we will be guaranteed \( b_{i_1} \geq b \).

Look at how strand \( b \) passes row \( \ell + 1 \) in \( \bar{R} \). If it passes it vertically, then it intersects certain strand \( a' \) with \( a' \leq r \) (since \( w(\bar{R}) \) is an \( r \)-semi-shuffle). Then consider the sequence \( c_k \) for \( \bar{R} \) as defined in (4.1), with \( c_0 = a' \). By the same argument as in the proof of (3) we can find a place \((\ell, j)\), shown in Figure 4, to the right of the place where strand \( b \) passes row \( \ell + 1 \). Hence \( d > b \).

If strand \( b \) does not pass row \( \ell \) vertically, look again at the sequence \( c_k \) for \( \bar{R} \) with \( c_0 = \ell \). Strand \( b \) is an element of this sequence, let \( b = c_k \). Let us show that

\[
(4.5) \quad c_{k - 1} \leq r \quad \text{or} \quad c_{k - 1} = b - 1.
\]

Indeed, if \( c_{k - 1} > r \), then if there exist \( b' \) with \( c_{k - 1} < b' < b \), then strand \( b' \) must intersect either strand \( c_k \) or strand \( b \), which is impossible, since \( w(\bar{R}) \) is an \( r \)-semi-shuffle. Hence (4.5) holds.

If \( c_{k - 1} \leq r \), then strands \( c = c_k \) and \( d = b \) pass next to each other in row \( \ell + 1 \) at a place \((\ell, j)\), so insertion into \((\ell, j)\) is allowed. It implies \( b_r \geq b \).

It remains to consider the case when \( c_{k - 1} = b - 1 \). Let \( \bar{b}_k \) denote the entries of \( B(T(\ell + 1)) \). We will prove that

\[
(4.6) \quad i_1 \neq 1 \text{ and } c_{k - 1} = \bar{b}_{i - 1} = b - 1.
\]

If (4.6) holds, then, since step \((\ell, i_1)\) is an insertion, \( i_1 \) is not the first box in row \( \ell \) and box \( i_1 - 1 \) is in the same row as box \( i_1 \). Hence by induction assumption

\[
\bar{b} - 1 = \bar{b}_{i - 1} \leq b_{i - 1} < b_{i_1}.
\]

Therefore, since \( b > \bar{b}_{i - 1} = b - 1 \), we conclude \( b = b_1 \leq b_{i_1} \). 

It remain to show that if \( c_{k-1} = b \) then (4.4) holds. Since \( w(\bar{R}) \) is an \( r \)-semi-shuffle and strands \( b \) and \( b-1 \) pass next to each other in row \( \ell \) of \( \bar{R} \),
\[
w(\bar{R}_{\geq \ell+1})(b) - 1 = w(\bar{R}_{\geq \ell+1})(b-1).
\]

\( R(\ell, i-1)_{\geq \ell+1} \) is constructed out of \( \bar{R}_{\geq \ell+1} \) by adding some crossings. It is not difficult to see that if (4.7) fails, none of these crossings involve strands \( b \) or \( b-1 \). So
\[
w(R(\ell, i-1)_{\geq \ell+1})(b) - 1 = w(R(\ell, i-1)_{\geq \ell+1})(b-1).
\]
But it is impossible by (3.1), since \( a < b-1 < b \)
\[
w(R(\ell, i-1)_{\geq \ell+1})(a) < w(R(\ell, i-1)_{\geq \ell+1})(b-1) < w(R(\ell, i-1)_{\geq \ell+1})(b)
\]
while \( l(w(R(\ell, i-1)_{\geq \ell+1})_{ab}) = l(w(R(\ell, i-1)_{\geq \ell+1})) + 1 \).

4.4.2. Case (2.4). As before, let \((ab)\) and \((ef)\) be the entries of boxes \( i \) and \( i-1 \) of \( T' \). Assume \((\ell, i-1)\) is a rectification (the argument below can be easily modified to provide a proof in the case step \((\ell, i-1)\) is an insertion). Assume \((\bar{e}, \bar{f})\) is the entry of box \( i+1 \) of \( T(\ell+1) \).

Start with the case when box \( i \) is not in the first column of \( sh(T') \). Then let us show that if \( R' \) is an rc-graph, then \( E(w(R_{\geq \ell}), T') \) is row and column strict. It is obvious if \((ef) \neq (\bar{e}, \bar{f})\), since in this case the entry of box \( i-1 \) of \( E(w(R_{\geq \ell}), T') \) is smaller than the value of box \( i+1 \) of \( E(w(R_{\geq \ell}), T(\ell+1)) \).

If \((ef) = (\bar{e}, \bar{f})\) and \( E(w(R_{\geq \ell}), T') \) is not row and column strict, then, strands \( b \) and \( f \) intersect in row \( \ell \) in \( R' \), such that \( f \oplus b = \ell \). But then \( f \oplus a \) in \( R(\ell, i-1) \), which is impossible since \( a < f \).

In the case when \( i \) is in the first column and \( R' \) is an rc-graph, we will show one of the following holds:

(4.7) \[ w(R(\ell, i-1))(f) > w(R(\ell, i-1))(a), \]
(4.8) \[ a = e \text{ and } a \oplus f = \ell \text{ in } R(\ell, i-1), \]

If (4.7) holds, then \( E(w(R_{\geq \ell}), T') \) is row and column strict. If (4.8) holds, then \( b \oplus f = \ell \text{ in } R' \) as shown in the second picture of Figure 3. So, it remain to prove (4.7) or (4.8) hold in the case when \( i \) is in the first column and \( R' \) is an rc-graph.

Since \( E(w(R_{\geq \ell+1}), T(\ell+1)) \) is a row and column strict tableau, we know
\[
w(S_{i_{k-1}}(P(\ell+1)))(\bar{f}) > w(S_{i_{k-1}}(P(\ell+1)))(b) = w(S_{i_{k-1}}(P(\ell+1)))(a).
\]

Moreover, removing crossings from row \( \ell \) of \( R(\ell, i-1) \) produces \( S_{i_{k-1}}(P(\ell+1)) \).

In the case \((ef) = (\bar{e}, \bar{f})\), the inequality (4.9) implies (4.7), unless \( a \oplus f = \ell \) in \( R(\ell, i-1) \). But it is not difficult to see that if \((ef) = (\bar{e}, \bar{f})\), strands \( a \) and \( f \) cannot intersect.

Otherwise, if \((ef) \neq (\bar{e}, \bar{f})\), we will show during the proof of (5) that a crossing of strands \( \bar{e}, \bar{f} \) has been removed during step \((\ell, i-1)\) from place \((\ell, \bar{j})\) and then another crossing has been inserted to the left of \((\ell, \bar{j})\). Assume \( R' \) is the intermediate rc-graph in step \((\ell, i-1)\) constructed by removing a crossing from \( R(\ell, i-2) \). Consider sequence \( c_k \) defined by (3.1) for \( R' \). Let \( c_0 = a \), if \( a \) does not intersect row \( \ell \) vertically, otherwise set \( c_0 = a' \) with \( a' \oplus a = \ell \). Then \( f \) is an element of the sequence \( c_k \). Let \( \bar{f} = c_k \). Clearly, there exist a place \((\ell, \bar{j})\), where insertion is allowed with strands \( c_j \) and \( c_{j+1} \) passing through \((\ell, j)\) with \( 0 \leq \bar{k} < \bar{k} \). Choose such place with the largest possible \( j \), let it be \((\ell, j_1)\), then we define \( R(\ell, i-1) \) and \( T(\ell, i-1) \) is such a way that \( e = c_{\bar{k}} \) and \( f = c_{\bar{k}+1} \). If \( \bar{k} = 0 \) and \( c_0 = a \), (4.8) holds, otherwise (4.7) must be satisfied.
4.5. **Proof of (5).** Assume that a crossing at place $(\ell, j_0)$ in $R'$ has been removed to produce $R''$. It is not difficult to see that $R''$ is an rc-graph, $(w(R_{\geq \ell}), R'', T'')$ is an r-Bruhat package and $E(w(R_{\geq \ell}), T')$ is row and column strict. We need to show that there exist a place where insertion is allowed to the left of $(\ell, j_0)$ and after $R(\ell, i)$ and $T(\ell, i)$ are defined, (3.4) and (3.3) are satisfied.

4.5.1. **Case (2.4).** As in the proof of (3), we can use sequence $c_k$ for $R''$ to show that a place, where insertion is allowed, to the left of place $(\ell, j_0)$ exists. Moreover, the rightmost place $(\ell, j_1)$ where insertion is allowed looks like the first picture in Figure 6.

After inserting crossing $(\ell, j_1)$ into $R''$ to defining $R(\ell, i)$ and $(cd)$ into box $i$ of $T''$ to define $T(\ell, i-1)$, it is easy to see $R(\ell, i)$ is an rc-graph and (3.5) is satisfied. By Lemma 4.2 to show (3.4) holds, it is enough to show $B(T, i)$ row strict. Notice that $B(T, i)$ differs from $B(T, i-1)$ only in box $i$. So we just have to check that the entry box $i$ is still greater than the entry of the box to the left of box $i$ and not greater than the entry of the box below box $i$. This can be done by an argument, which is almost identical to the argument used in Section 4.4.1.

4.5.2. **Case (2.4).** Recall that $(ab)$ is the entry of box $i$ of $T''$, $(eg)$ is the entry of box $i - 1$ of $T''$. Consider sequence $c_k$ for $R''$, defined by (3.3), which starts with $c_0 = \ell$ and ends with $c_k = b$. Then there exists $k$ between 0 and $k - 1$, such that strand $c_k$ and $c_{k + 1}$ pass next to each other in a place where insertion is allowed. Let the rightmost place to the left of $(\ell, j_0)$, where insertion is allowed, be $(\ell, j_1)$.

Consider the case when box $i$ is in the first column of $T''$. Then using sequence $c_k$, it is easy to see that place $(\ell, j_1)$ looks like the first pictures from Figure 6. (We used this in Section 4.4.2.) Therefore, as for the insertion step, $R(\ell, i)$ is an rc-graph and (3.3) holds. Moreover (3.4) holds, since strand $g$ passes row $\ell$ to the right of place $(\ell, j_1)$.

Otherwise, if box $i$ is in the first row, but not the first element of this row, then strand $g$ is either an element of the sequence $c_k$ or intersects one of the strand $c_k$ in row $\ell$. So place $(\ell, j_1)$ could look like the first picture of Figure 6 and strand $g$ passes to the left of this place. Or, it could look like the second picture of Figure 6.

If it is the first picture, then, as before, $R''$ is an rc-graph, (3.3) holds, while (3.4) holds, since strand $g$ passes row $\ell$ to the left of place $(\ell, j_1)$.

If it is the second picture, it is easy to see that $R''$ is an rc-graph and that (3.4) holds, while to prove (3.3), we must show

$$l(w(R(\ell, i))) = l(w(R(\ell, i))t_{eg}t_{cd}) + 2 = l(w(R(\ell, i))t_{eg}) + 1. \quad (4.10)$$

To prove the first equality of (4.10), notice $t_{eg}t_{cd} = t_{gd}t_{eg}$, hence

$$l(w(R(\ell, i))t_{eg}t_{cd}) = l(w(R(\ell, i)t_{gd}t_{eg}) = l(w(R'')t_{eg}) = l(w(R'')) - 1 = l(w(R(\ell, i))) - 2.$$

For the second equality, notice that $e < g$ and $w(R(\ell, i))(e) > w(R(\ell, i))(g)$, thus

$$l(w(R(\ell, i))t_{eg}) < l(w(R(\ell, i))).$$

At the same time, $e < d$ and $w(R(\ell, i))t_{eg}(e) > w(R(\ell, i))t_{eg}(d)$, hence

$$l(w(R(\ell, i))t_{eg}) < l(w(R(\ell, i))t_{eg}).$$

This proves the second part of (4.10).
5. Inverse Insertion Algorithm

Given an rc-graph $U$ and a tableau of transposition $T$ of $\nu(\lambda, r)$, inverse insertion algorithm defines rc-graphs $U \to T$ and $Y(R, T)$, given that $T$ is an $r$-Brihat chain of $w = w(U)w(T)^{-1}$, $E(w, T)$ is row and column strict, and $w, \nu(\lambda, r)$ satisfy (2.5) or (2.6). This section describes the inverse algorithm.

5.1. Sequence of inverse steps. Inverse insertion algorithm performs the same steps as insertion algorithm but in the opposite order. Each step will be either an inverse row-to-to step, an inverse insertion or an inverse rectification.

Each step $(\ell, i)$ with $1 \leq i \leq m_\ell$ constructs rc-graph $R(\ell, i - 1)$ with no crossings above row $\ell$ and tableau of transposition $T(\ell, i - 1)$ filled up to $i - 1$. Each step $(\ell, 0)$ defines integer $m_{\ell+1}$, an rc-graph $R(\ell + 1, m_{\ell+1})$ with no crossings above row $\ell + 1$ and a tableau of transpositions $T(\ell + 1, m_{\ell+1})$. Conditions (3.4), (3.5) always hold.

5.2. Start of the algorithm. Set $m_1 = |T|$, $R(1, m_1) = U$ and $T(1, m_1) = T$.

5.3. Inverse insertion. Consider step $(\ell, i)$ with $i > 0$. We need to construct $R(\ell, i - 1)$ and $T(\ell, i - 1)$. Let $(cd)$ be the entry of box $i$ of $T(\ell, i)$. By Lemma 3.1, strand $c$ and $d$ intersect in $R(\ell, i)$ at some place $(t_0, j_0)$. If $\ell = t_0$ define $T''$ by removing the entry of box $i$ from $T(\ell, i)$. Define $R''$ by removing the crossing of strands $c$ and $d$ from $R(\ell, i)$ from place $(\ell, j_0)$. We say that insertion into place $(\ell, j)$ is allowed if strands $a, b$ pass this place as shown in Figure 8. If there are no places $(\ell, j)$, where insertion is allowed, with $j > j_0$, this step is an inverse insertion, which sets $R(\ell, i - 1) = R''$ and $T(\ell, i - 1) = T''$.

5.4. Inverse rectification. All steps $(\ell, i)$ with $i > 0$, which are not inverse insertions are inverse rectifications.

Adopt the notation from previous section. If $t_0 \neq \ell$, define $T(\ell, i - 1)$ by emptying box $i$ of $T(\ell, i)$ and define $R(\ell, i - 1)$ by removing the crossing of strands $c$ and $d$ and move on to the next step, except for one case. Namely, if $(ef)$ is the entry of $i - 1$ of $T(\ell, i)$, $c = e$ and $f \equiv d = \ell$. In this case define $R''$ by removing the crossing of $b$ and $f$ and define $T''$ by emptying box $i$ of $T(\ell, i)$ and placing $(ed) = (cd)$ in box $i - 1$. If $t_0 = \ell$, define $R'$ and $T'$ as it was done in the previous section.

Once $R''$ and $T''$ are constructed, we say that insertion into places in row $\ell$ shown in Figure 8 are allowed. Find the leftmost place where insertion is allowed

\[
\begin{array}{c}
\text{Figure 8. Place where insertion is allowed}
\end{array}
\]

$b \overline{a}$ with $a \leq r < b$,

and $T''$ to produce $T'$. If this place looks like the first picture of Figure 8, add $(ab)$ to box $i$ of $T''$ to construct $T'$, otherwise insert $(ed)$ and $(eg)$ into boxes $i - 1$ and $i$ of $T''$ to produce $T'$. 

\[
\begin{array}{c}
\text{Figure 9. Places, where insertions are allowed during rectification.}
\end{array}
\]
Once $R'$ and $T'$ are constructed, let $(ab)$ be the entry of box $i$ of $T'$. Then it can be shown that strands $a$ and $b$ intersect below row $\ell$. Remove this crossing to produce $R(\ell, i - 1)$ and construct $T(\ell, i - 1)$ by emptying box $i$ of $T'$.

5.5. **Inverse row-to-row steps.** Steps $(\ell, 0)$ are inverse row-to-row steps. They define $m_\ell$ to be the number of inverse rectifications $(\ell, i_1), \ldots, (\ell, i_{m_\ell+1})$ for row $\ell$. Also each step $(\ell, 0)$ sets $R(\ell+1, m_\ell+1) = R(\ell, m_\ell)$.

The shape of $T(\ell+1, m_\ell+1)$ is the subdiagram of $sh(T(\ell, m_\ell))$ consisting of boxes $(i_1, \ldots, i_{m_\ell+1})$. By construction, this will be a Young diagram. The entry $(a_k b_k)$ of box $k$ of $T(\ell+1, m_\ell+1)$ is determined by

$$w(R(\ell, i_{k-1}) \geq \ell+1) = w(R(\ell, i_k) \geq \ell+1) t_a b_k.$$

5.6. **End of the inverse algorithm.** Set $U \rightarrow T = R(r, 0)$. We will define $Y(U, T)$ by presenting its word. Set word$_{r+1}$ to be empty. Define word$_\ell$ of length $m_\ell$ by adding letters $\ell$ to word$_{\ell+1}$ as follows. If $(\ell, i_1), \ldots, (\ell, i_{m_\ell+1})$ are the rectification steps for row $\ell$, then set letter $i_k$ of word$_\ell$ to be the same as letter $k$ of word$_{\ell+1}$, and set all the other letters of word$_\ell$ to be equal to $\ell$. Finally set $\text{word}(Y(U, T)) = \text{word}_1$.

6. **Examples**

6.1. **Example of rc-graphs** $S_j(P)$. Assume $R$ and $T$ are given in Figure 10. Define $w = w(R) w(T)^{-1} = (2, 1, 4, 3, 5, 6, \ldots)$. Then $P = (w, R, T)$ is an $r$-Bruhat package.

![Figure 10. Re-graph $R$ and tableau of transpositions $T$.](image)

Then sequence $S_j(P)$ is given in Figure 11. In each graph $S_j(P)$ the circled crossing needs to be removed to construct $S_{j-1}(P)$. Since $\text{word}(T) = (14)(23)(25)(15)$, $S_3$ is constructed out of $S_4$ by removing the crossing of strands 1 and 5, $S_2$ out of $S_3$ by removing the crossing of strands 2 and 5, and so on.
6.2. Example of insertion algorithm in case (2,5). From now on we draw only crossings of rc-graphs without drawing strands, as it was done in Figure 1. It makes it easier to see how rc-graphs change during the algorithm. At the same time, as usual, we assume each rc-graph extends infinitely to the right and to the bottom and the part of each rc-graph, which is not shown, has no crossings.

Assume the rc-graphs $R$ and $Y$ are given in Figure 12, so that $r = 3$, $w(R) = (1, 4, 3, 2, 5, 6, \ldots)$ is a 3-semi-shuffle and $w(Y) = (1, 4, 5, 2, 3, 6, \ldots) = v((2, 2), 3)$ is a 3-shuffle. We will illustrate all the steps of the algorithm for $R \leftarrow Y$.

![Figure 11. Re-graph $S_4(P)$ through $S_0(P)$.](image)

![Figure 12. Rc-graph $R$ and $Y$.](image)

Figures 13-16 show rc-graphs $R(\ell, i)$ and tableaux of transposition $T(\ell, i)$. Steps (3, 0), (2, 0) and (1, 0) are row-to-row steps. Steps (3, 1), (2, 1), (2, 3) and (1, 2) are insertions. Steps (2, 2), (1, 1), (1, 3) and (1, 4) are rectifications. We circle all crossings of $R(\ell, i)$ with $i > 0$, which are removed or added by the current step. We also show by an arrow how crossing move during rectifications.

Let us also recall that each row-to-row step $(\ell, 0)$ constructs the sequence of rc-graphs $S_j(P(\ell + 1))$ and then sets $R(\ell, 0) = S_0(P(\ell + 1))$. We omit the details of this construction and refer to Section 14 for an example of such construction. Also, after each row-to-row step $w(R(\ell, 0)) = w(R_{\geq \ell})$, but rc-graphs $R(\ell, 0)$ and $R_{\geq \ell}$ could be different. For example, see step (1, 0) in Figure 14.
6.3. Example of insertion algorithm in case (2.6). Let us now present an example in case shape of $Y$ is a hook. Let $R$ and $Y$ be shown in Figure 17. In particular, $w(R) = (1, 2, 4, 6, 3, 5, 7, 8, \ldots)$ and $w(Y) = (1, 3, 5, 2, 4, 6, 7, \ldots)$, both are shuffles, but $w(R)$ has descent at 4, while $w(Y)$ has descent at 3, so case (2.5) does not apply.

Figures 18-20 contain the results of all the steps of the algorithm. Steps (3,0), (2,0) and (1,0) are row-to-row steps, steps (3,1), (1,1) and (1,3) are insertions, while (2,1) and (1,2) are rectifications. Notice that step (1,2) is the only step, where the second situation of Figure 5 occurs.
6.4. **Another example in case (2.6).** The last example is for rc-graphs $R$ and $Y$ defined in Figure 21. In this case $w(R) = (1, 2, 5, 4, 6, 3, 7, 8, \ldots)$ and $w(Y) = (1, 4, 2, 3, 5, 6, \ldots) = v((2, 0), 2)$, the shape of $Y$ is a row, while $w(R)$ is a permutation with two descents.

The steps of the algorithm are shown in Figures 22-23. Steps (2, 0) and (1, 0) are row-to-row steps, steps (2, 1) and (2, 1) are insertions, while steps (1, 1) and (1, 2) are rectifications. Notice that step (1, 2) is the only step where the second case of Figure 6 occurs.
GENERALIZATION OF SCHENSTED INSERTION ALGORITHM.

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | + | + |   |   |
| 2 | + | + |   |   |
| 3 | + | + |   |   |

$R \begin{array}{cccc} 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$Y \begin{array}{cccc} 2 & + & + \\ 3 & + & + \end{array}$

Figure 21. Rc-graph $R$ and $Y$.

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 |   |   |   |   |
| 2 |   |   |   |   |
| 3 |   |   |   |   |

$R(2,0) \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$T(2,0) \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$R(2,1) \begin{array}{cccc} 1 & 2 & 3 | 4 \\ 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$T(2,1) \begin{array}{cccc} 1 & 2 | 3 \\ 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$R(2,2) \begin{array}{cccc} 1 & 2 & 3 \\ 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$T(2,2) \begin{array}{cccc} 1 & 2 & 3 \\ 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

Figure 22. Steps (2,0), (2,1) and (2,2).

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 |   |   |   |   |
| 2 |   |   |   |   |
| 3 |   |   |   |   |

$R(1,0) \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & \cdot & \cdot & + \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$T(1,0) \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & \cdot & \cdot & + \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$R(1,1) \begin{array}{cccc} 1 & 2 & 3 | 4 \\ 1 & \cdot & \cdot & + \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$T(1,1) \begin{array}{cccc} 1 & 2 | 3 \\ 1 & \cdot & \cdot & + \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$R(1,2) \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & \cdot & \cdot & + \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

$T(1,2) \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & \cdot & \cdot & + \\ 2 & \cdot & \cdot & + \\ 3 & \cdot & \cdot & + \end{array}$

Figure 23. Steps (1,0), (1,1) and final step (1,2).

REFERENCES

[1] N. Bergeron, S. Billey. RC-graphs and Schubert polynomials, Experimental Math., 2 (1993), 257-269.

[2] N. Bergeron, F. Sottile. Schubert polynomials, the Bruhat order, and the geometry of flag manifolds. Duke Math. J. 95 (1998), no. 2, 373–423.

[3] S. Billey, W. Jockusch, R. Stanley. Some combinatorial properties of Schubert polynomials. J. Algebraic Combin. 2 (1993), no. 4, 345–374.

[4] S. Fomin, A. N. Kirillov. Yang-Baxter equation, symmetric functions, and Schubert polynomials, Discrete Mathematics 153, (1996) 123-143.

[5] S. Fomin, R. Stanley, Schubert polynomials and the nil-Coxeter algebra. Adv. Math. 103 (1994), no. 2, 196-207.

[6] W. Fulton. Young tableaux. London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997

[7] A. Knutson Descent-cycling in Schubert calculus. arXiv:math.CO/0009112

[8] M. Kogan. RC-graphs and a generalized Littlewood-Richardson rule. Internat. Math. Res. Notices 2001, no. 15, 765–782

[9] M. Kogan, A. Kumar. A proof of Pieri’s formula using generalized Schensted insertion algorithm for rc-graphs. arXiv:math.CO/0010109

[10] A. Kumar. A generalized insertion algorithm for rc-graphs to prove Pieri’s column multiplication rule. Unpublished note.

[11] A.Lascoux, M. P. Schutzenberger. Schubert polynomials and the Littlewood-Richardson Rule, Let. Math. Phys. 10 (1985) 111-124.

[12] I. G. Macdonald Symmetric functions and Hall polynomials. Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
[13] I. G. Macdonald, *Notes on Schubert Polynomials*, Publications du LACIM 6, Universite du Quebec a Montreal (1991)
[14] L. Manivel. *Fonctions symetriques, polynomes de Schubert et lieus de degenerescence*. Cours Specialiss, 3. Societ Mathematique de France, Paris, 1998.
[15] C. Schensted. *Longest increasing and decreasing subsequences*. Canad. J. Math. 13 1961 179–191.
[16] F. Sottile. *Pieri’s formula for flag manifolds and Schubert polynomials*. Ann. Inst. Fourier (Grenoble) 46 (1996), no. 1, 89–110.

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