Strichartz estimates and low-regularity solutions to multiplicative stochastic NLS

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Abstract

We study Strichartz estimates with very rough potentials, the spatial white noise on the 2 dimensional torus being of particular interest. Applications include solving the multiplicative stochastic NLS with general integer powers in a low-regularity setting, moreover we obtain global well-posedness in both the energy and strong regime.

1 Introduction

This work is devoted to proving Strichartz estimates leading to low-regularity local-in-time and high regularity global-in-time well-posedness of defocussing NLS (nonlinear Schrödinger equations) with very rough potentials $\xi$, so

$$i\partial_t u - \Delta u = u \cdot \xi - u|u|^{2n} \text{ on } \mathbb{T}^2,$$

$$u(0) = u_0,$$

(1.1)

where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the 2 dimensional torus and general integer power nonlinearity $n \in \mathbb{N}$. Our chief interest is the case where $\xi$ is spatial white noise, which is a distribution whose regularity is only $C^{-1-\epsilon}$ for $\epsilon > 0$, see (3.8) for its precise definition and the appendix for a reminder of the definition of the Hölder-Besov spaces $C^\alpha$.

In the case of the white noise potential there turns out to be a peculiarity in the form of renormalisation, which means that in order to make sense of (1.1) one is required to shift by an infinite correction term, formally $\infty \cdot u$. This can be interpreted as an infinite phase shift in the PDE, since $e^{ict} u$ solves the equation with an additional mass $c$. This kind of renormalisation is now well known in the theory of singular SPDEs which has seen a rapid growth in recent years following the introduction of the theory of Regularity Structures by Hairer [23], the theory of Paracontrolled Distributions by Gubinelli, Perkowski, and Imkeller [18] and others.

The approach we follow in this paper is to put the potential $\xi$ into the definition of the operator, i.e. we try to define the operator

$$H = \Delta + \xi$$

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as a self-adjoint and semi-bounded operator on $L^2(\mathbb{T}^d)$. This was first done by Allez and Chouk in [2], where the operator together with its domain were constructed in 2d with the white noise potential—hereafter called the **Anderson Hamiltonian**—using Paracontrolled Distributions. A similar approach was used in [21] to construct the operator and its domain in 3d with an eye also on solving PDEs like (1.1). In Section 3 we recall the main ideas of [21] since the results are integral to the current work. The domain of the Anderson Hamiltonian was also constructed by Labbé using Regularity Structures [26] with Dirichlet boundary conditions and by Mouzard [28] on compact surfaces.

The equation (1.1) with white noise potential in 2d was solved, but not shown to be well-posed, by Debussche and Weber [13] in the cubic case and by Visciglia and Tzvetkov [35], [34] for other powers and on the whole space with a sub-cubic power in the nonlinearity by Debussche and Martin in [12] which was then generalised in [11] to higher powers. In [21] global well-posedness (GWP) in the cubic case was proved in the domain of the Anderson Hamiltonian in 2d, whereas in 3d one gets a blow-up alternative when starting in the domain analogously to the case of classical $H^2$ solutions in [9]. Furthermore, in [21] global existence in the energy space in 2d was shown, but not well-posedness. Achieving GWP for energy solutions to (1.1) is one of the results of this paper, see Theorem 1.3.

The (nonlinear) Schrödinger equation (1.1) with a (random) potential has certain physical interpretations, see [16] and the references therein. In this paper we consider the white noise potential but the same results hold for a large class of potentials, see Remark 3.7.

Some potentials of interest are actually **critical** in the sense of scaling, like the Dirac Delta in 2d (see the monograph [1]) or the potential $|\cdot|^{-2}$ treated in [8]. Our method does not apply in these cases, but in the aforementioned examples the analysis depends in a crucial way on the structure of the potential.

Stochastic NLS of the form (1.1) but with different noises (e.g., white in time coloured in space) have also been considered, see [10], [6], [14] to name but a few. Other stochastic dispersive PDEs which have been studied in recent years include stochastic NLS with additive space-time noise [31], [15] and nonlinear stochastic wave equations with additive space-time noise in [20] and [19]. Let us also mention [17], where the theory of **Rough Paths**—the precursor to both Regularity Structures and Paracontrolled Distributions—is used to solve the deterministic low-regularity KdV equation and which showcases nicely how tools from singular SPDEs can be applied to non-stochastic PDE problems.

We state the main (shortened) results of the paper relating to the multiplicative stochastic NLS. $H$ “$\Delta + \xi$” is the Anderson Hamiltonian whose exact definition and properties are recalled in Section 3.

**Theorem 1.1 [2d Anderson Strichartz Estimates]** Let $r \geq 4$, then we have for any $\delta > 0$

$$\|e^{-itH}u\|_{L^r_t(\mathbb{T}^d \times [0,1])} \lesssim \|u\|_{H^{(r-3)(1+\delta)/r}}$$

(1.2)
Theorem 1.2 [2d low regularity local well-posedness] Let \( n \in \mathbb{N} \), then the SPDE
\[
(i \partial_t - H)u = -u|u|^{2n} \quad \text{on } \mathbb{T}^2
\]
\[u(0) = u_0\]
is locally well-posed (LWP) in \( \mathcal{H}^s \) for \( s \in (1 - \frac{1}{n}, 1) \) up to a time \( T \sim (1 + \|u_0\|_{\mathcal{H}^s})^{-K} \) for some \( K > 0 \) depending on \( n \) polynomially.

Theorem 1.3 [2d GWP for energy solutions] Let \( n \in \mathbb{N} \), then the SPDE
\[
(i \partial_t - H)u = -u|u|^{2n} \quad \text{on } \mathbb{T}^2
\]
\[u(0) = u_0\]
is globally well-posed (GWP) in the energy space, \( \mathcal{D} (\sqrt{-H}) \), whose definition is recalled in Theorem 3.1.

Theorem 1.4 [2d GWP for strong solutions] Let \( n \in \mathbb{N} \), then PDE
\[
(i \partial_t - H)u = -u|u|^{2n} \quad \text{on } \mathbb{T}^2
\]
\[u(0) = u_0\]
is globally well-posed(GWP) in the domain \( \mathcal{D}(H) \) of the operator \( H \), see Section 3.

Remark 1.5 In a previous version of the paper, we claimed that the bound (1.2) holds with only arbitrarily small \( \delta > 0 \) loss in regularity but the proof contained an error. In the meantime, together with Mouzard [29], we proved a version of the theorem on smooth surfaces, here we now just give a simpler proof which uses the heavier machinery of Bourgain’s periodic Strichartz estimates [4], [5].

It is still an open interesting question whether the loss in derivatives can be reduced on the torus or manifolds with special geometries. For general compact surfaces, the results from [29] can likely not be improved since it is as good as [30], which is optimal for general surfaces, with an \( \varepsilon \) loss.

The paper is organised as follows: In Section 2 we recall the well-known Strichartz estimates on the whole space and how their counterparts on the torus differ. Section 3 is meant to recapitulate the construction of the Anderson Hamiltonian and its domain following [21]. In Section 4 we prove the Strichartz estimates for the Anderson Hamiltonian on \( \mathbb{T}^2 \) i.e. Theorem 1.1. Then in Section 5 we utilise these bounds to prove well-posedness of the multiplicative stochastic NLS in three different regimes i.e. Theorems 1.2, 1.3 and 1.4.

Notations and conventions
The spaces we work in are \( L^p \)-spaces, for \( p \in [1, \infty] \), meaning the usual \( p \)-integrable
Lebesgue functions; $\mathcal{H}^\alpha, W^{\alpha,p}$ spaces, with $\alpha \in \mathbb{R}$, $p \in [1, \infty]$ the usual Sobolev potential spaces with $\mathcal{H}^\alpha = W^{\alpha,2} = B^\alpha_{2,2}$; and $B^\alpha_{r,q}$, the Besov spaces, whose definition is recalled in the appendix and which cover $\mathcal{H}^\alpha$ and $C^\alpha$—so called Hölder-Besov spaces—as special cases. Also we write
\[
\|f\|_X := \|f\|_{X(T^2)} \text{ and } \|f(t)\|_{Y_{t,[0,T]}} := \|f\|_{Y([0,T])},
\]
where $X$ is one of the function spaces above on the torus $T^2$ $Y$ is a function space in the time variable, usually $C[0,T], L^p[0,T]$ for $1 \leq p \leq \infty$ and $T > 0$.

We write, as is quite common,
\[
a \lesssim b
\]
to mean $a \leq Cb$ for a constant $C > 0$ independent of $a, b$ and their arguments. Also we write
\[
a \sim b \iff a \lesssim b \text{ and } b \lesssim a.
\]
For the sake of brevity we also allow every constant to depend exponentially on the relevant noise norm $\|\Xi\|_{\mathcal{X}^\alpha}$, see Definition 3.5 for the exact definition of the norms; This can be written schematically as
\[
\lesssim \iff \lessim \Xi,
\]
this comes with the tacit understanding that everything is continuous with respect to this norm. Another convention is that if we write something like
\[
\|F(u)\|_X \lesssim \|u\|_{\mathcal{H}^{\alpha+\varepsilon}} \text{ for } \varepsilon > 0,
\]
we of course mean
\[
\|F(u)\|_X \leq C_\varepsilon \|u\|_{\mathcal{H}^{\alpha+\varepsilon}} \text{ with } C_\varepsilon \to \infty \text{ as } \varepsilon \to 0.
\]

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Although there was an error in the previous version, together with Antoine Mouzard in [29] we fixed the proof of the Strichartz estimate and generalised many of the original results to the setting of compact surfaces, see Remark 1.5 for a comparison.

2 Classical Strichartz estimates on the torus

We start by recalling the well-known Strichartz estimates for Schrödinger equations on $\mathbb{R}^d$. 

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Theorem 2.1 [Strichartz on $\mathbb{R}^d$, Theorem 2.3 in [33]] Let $d \geq 1$ and $(p, q)$ be a Strichartz pair, i.e.

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2} \text{ and } (d, p, q) \neq (2, 2, \infty),$$

we also take $(r', s')$ to be a dual Strichartz pair, which means that they are Hölder duals of a Strichartz pair $(r, s)$, explicitly

$$\frac{2}{r'} + \frac{d}{s'} = \frac{d + 4}{2},$$

then the following are true

i. $\|e^{it\Delta}u\|_{L^p_t L^q(\mathbb{R}^d)} \lesssim \|u\|_{L^2(\mathbb{R}^d)}$ “homogeneous Strichartz estimate”

ii. $\left| \int_{\mathbb{R}} e^{-it\Delta} F(t) dt \right|_{L^2(\mathbb{R}^d)} \lesssim ||F||_{L'^r(\mathbb{R})L'^s(\mathbb{R}^d)}$ “dual homogeneous Strichartz estimate”

iii. $\left| \int_{t' < t} e^{i(t-t')\Delta} F(t') dt' \right|_{L^p_t L^q(\mathbb{R}^d)} \lesssim ||F||_{L'^r(\mathbb{R})L'^s(\mathbb{R}^d)}$ “inhomogeneous Strichartz estimates”.

Next we cite some, by now, classical Strichartz estimates on the torus and how they differ from those on the whole space. Moreover we sketch how they allow to solve NLS in spaces below $H^{\frac{d}{2} + \epsilon}$, which is an algebra.

The first results we state are the Strichartz estimates proved by Burq-Gerard-Tzvetkov in [7]. They hold on general manifolds, i.e. not only the torus. The results are not optimal for the torus but we nonetheless cite them because the methods we use are strongly inspired by this paper.

Theorem 2.2 [Strichartz estimates on compact manifolds, [7] Theorem 1] Let

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$ 

We have on the finite time interval $[0, 1]$

$$\|e^{-it\Delta}u\|_{L^p_t[0,1]L^q(M)} \lesssim ||u||_{H^{\frac{d}{2}}(M)} \quad (2.1)$$

and

$$\left| \int_0^t e^{-i(t-s)\Delta} f(s) ds \right|_{L^p_t[0,1]L^q(M)} \lesssim \int_0^1 ||f(s)||_{H^{\frac{d}{2}}(M)} ds.$$

Note that, as opposed to the whole space, one has a loss of $\frac{1}{p}$ derivatives. Together with Mouzard, we proved an analogous Strichartz estimate on smooth surfaces using a microlocal approach as in [30], see also [28] for the construction of the operator $H$ on a smooth surface.
Theorem 2.3 [Anderson Strichartz estimates on smooth surfaces, [29]] Let $M$ be a smooth compact surface and
\[ \frac{2}{p} + \frac{d}{q} = 1. \]
We have on the finite time interval $[0, 1]$
\[ \|e^{-itH}u\|_{L^p_t L^q} \lesssim \|u\|_{H^{\frac{1}{2} + \epsilon}} \]
and
\[ \left| \int_0^t e^{-i(t-s)H} f(s) ds \right|_{L^p_t L^q} \lesssim \int_0^1 \|f(s)\|_{H^{\frac{1}{2} + \epsilon}} ds \]
for any $\epsilon > 0$.

The next result we cite is an almost sharp Strichartz estimate on the torus due to Bourgain and Demeter in [5] which was refined in [25] by Killip and Visan whose version we cite because it is more amenable to our situation. The result is stated for functions which are localised in frequency but the corresponding Sobolev bound is immediate.

Theorem 2.4 [Sharp Strichartz estimate, Theorem 1.2 [25], [5]] Let $d \geq 1$ and $p \geq \frac{2(d+2)}{d}$, then, for any $\epsilon > 0$ we have
\[ \|e^{-it\Delta} P \subseteq N f \|_{L^p_t L^p(\mathbb{T}^d)} \lesssim N^{\frac{4}{d} - \frac{d+2}{p} + \epsilon} \|f\|_{L^2(\mathbb{T}^d)}. \]
For $d = 2$ this means $p \geq 4$.

This will allow us to give a comparatively simple proof of Theorem 1.1 which is very similar to Theorem 2.3. In order to prove this, we first state the following simple result.

Proposition 2.5 [Inhomogeneous short time bound]
Let $I = [t_0, t_1]$ be a subinterval of $[0, 1]$ and $4 \leq p < \infty$ and let $f \in L^\infty_{[0,1]} H^{1 - \frac{4}{p} + \epsilon}$ for $\epsilon > 0$, then
\[ \left\| \int_{t_0}^t e^{-i(t-s+t_0)\Delta} f(s) ds \right\|_{L^p_t L^p} \lesssim \int_I \|f(s)\|_{H^{1 - \frac{4}{p} + \epsilon}} ds \] (2.2)
\[ \lesssim |I| \|f\|_{L^\infty_I H^{1 - \frac{4}{p} + \epsilon}}. \] (2.3)

Proof The bound follows in the usual way that one obtains inhomogeneous estimates from the linear Strichartz estimate Theorem 2.4.

Furthermore, we give a quick sketch about why these kinds of estimates are needed for solving NLS.
Take for simplicity the cubic NLS on the two-dimensional torus.

\[
i\partial_t u - \Delta u = -u|u|^2 \text{on } T^2
\]
\[
 u(0) = u_0.
\]

The Duhamel formula reads

\[
u(t) = e^{-it\Delta} u_0 - i \int_0^t e^{-i(t-s)\Delta} |u|^2 u(s) ds. \tag{2.4}
\]

Since for \( u_0 \in \mathcal{H}^\sigma \) with \( \sigma \in \mathbb{R} \) we have

\[
e^{-it\Delta} u_0 \in C_t \mathcal{H}^\sigma
\]

with the continuity in time following from Stone’s theorem, it is natural to try to solve (2.4) in a space like \( C_t \mathcal{H}^\sigma \) for, say, \( \sigma \geq 0 \). Now, since the unitary group \( e^{-it\Delta} \) has no smoothing properties, the \textit{“best”} possible way to bound the nonlinear expression in (2.4) is

\[
\left| \int_0^t e^{-i(t-s)\Delta} |u|^2 u(s) ds \right|_{C_t(0,T) \mathcal{H}^\sigma} \lesssim \int_0^T \| |u|^2 u(s) \|_{\mathcal{H}^\sigma} ds \tag{2.5}
\]

\[
\lesssim \int_0^T \| u(s) \|_{L^\infty}^2 \| u(s) \|_{\mathcal{H}^\sigma} ds, \tag{2.6}
\]

where the second inequality follows from the \textit{“tame”} estimate (see Lemma A.7). In the case that \( \sigma > \frac{d}{2} (=1 \text{ in } 2d) \) the \( L^\infty \) norm is controlled by the \( \mathcal{H}^\sigma \) norm so it is easy to close the fixed point argument. But even in the case of \( \mathcal{H}^1 \), which is natural as it is the \textit{“energy space”}, one is not able to close the contraction argument without additional input.

The key observation to make – and to see where the Strichartz estimates enter – is that in (2.6) we can apply Hölder’s inequality in time to obtain

\[
\int_0^T \| u(s) \|_{L^\infty}^2 \| u(s) \|_{\mathcal{H}^\sigma} ds \lesssim \| u \|_{L^2_{t} L^\infty_{x}}^2 \| u \|_{L^\infty_{[0,T]} \mathcal{H}^\sigma}
\]

and note that we need not control the \( L^\infty_{t} L^\infty_{x} \) norm of the solution but it suffices to control the \( L^2_{t} L^\infty_{x} \) norm.

So, if we were able to control the \( L^p_{t} L^\infty_{x} \) norm of the right-hand side of (2.4) by the \( C_t \mathcal{H}^\sigma \) norm of \( u \) we would be able to get a contraction to get a solution to (2.4) for a short time interval. Of course, bounding the \( L^\infty \) norm directly is hopeless but recall the Sobolev embedding in \( d \)-dimensions

\[
W^{\frac{d}{2} + \varepsilon, q} \hookrightarrow L^\infty \text{for any } q \in (1, \infty) \text{ and } \varepsilon > 0.
\]
This is the point where the Strichartz estimates come in, since, for example by Theorem 2.4, one gets the bound
\[
\|e^{-it\Delta}u_0\|_{L^p_t[0,T]L^\infty} \lesssim \|e^{-it\Delta}u_0\|_{L^p_t[0,T]W^{\frac{2}{\sigma}+\varepsilon,p}} \lesssim T \|u_0\|_{\mathcal{H}^{1-\frac{2}{\sigma}+\varepsilon}}
\]
for \(4 \leq p < \infty\) for the linear evolution. If we take \(\varepsilon > 0\) small then the regularity exponent \(1 - \frac{2}{p} + \varepsilon \leq 1\) so this is strictly better than what we would get from estimating the \(L^\infty\) norm by the \(\mathcal{H}^{1+\varepsilon}\) norm.

For the nonlinear term we get similarly (assuming for simplicity \(T \leq 1\))
\[
\left| \int_0^t e^{-i(t-s)\Delta}|u|^2u(s)ds \right|_{L^p_t[0,T]L^\infty} \lesssim \left| \int_0^t e^{-i(t-s)\Delta}|u|^2u(s)ds \right|_{L^p_t[0,T]W^{\frac{2}{\sigma}+\varepsilon,p}} \lesssim \int_0^T \|u|^2u(s)\|_{\mathcal{H}^{1-\frac{2}{\sigma}+\varepsilon}} \lesssim T \|u\|_{L^3[0,T]H^\sigma}^3
\]
where \(1 - \frac{2}{p} + \varepsilon < \tilde{\sigma} < 1\) can be computed explicitly using the fractional Leibniz rule, see Lemma [A.6].

Clearly these bounds can be sharpened in different ways but the important thing is that Strichartz estimates lead to local-in-time well-posedness for some range of \(\sigma \leq 1\).

3 The Anderson Hamiltonian in 2 dimensions

The main aim of this work is to establish Strichartz estimates for the Anderson Hamiltonian which is formally given by
\[
H = \Delta + \xi(x) - \infty,
\]
where \(\xi(x)\) is spatial white noise, see [3,8]. This operator was initially studied by Allez and Chouk on \(T^2\) in [2] and later by Gubinelli, Ugurcan and the author in [21] on \(T^2\) and \(T^3\) using the theory of \textit{Paracontrolled Distributions} which was introduced in [18]. The operator was also studied by Labbé in [26] using the theory of \textit{Regularity Structures} introduced in [23] also including boundary conditions and on surfaces by Mouzard in [28]. Naively one might think that it is simply a suitably well-behaved perturbation of the Laplacian in which case Theorem 6 in [7] would more or less directly apply. However, it was shown that the domain of \(H\) in both \(2d\) and \(3d\) can be quite explicitly determined and we even have (almost surely)
\[
\mathcal{D}(H) \cap H^2 = \{0\},
\]
so it is tricky to directly compare the operators \(H\) and \(\Delta\).
We briefly recall some of the main ideas from [21] in the 2d setting and slightly reformulate it. An observation made in [2] was that a function $u$ is in the domain of $H$ if
\begin{equation}
 u - (u \prec (1 - \Delta)^{-1} \xi + B_\Xi(u)) \in \mathcal{H}^2, \tag{3.1}
\end{equation}
see the appendix for the definition and properties of paraproducts. By the paraproduct estimates (Lemma A.1) and the regularity of the noise, the term $u \prec (1 - \Delta)^{-1} \xi$ is no better than $\mathcal{H}^{1-\varepsilon}$. The “lower order” correction term $B_\Xi$ is also worse than $\mathcal{H}^2$ (in fact it is $\mathcal{H}^2 - \varepsilon$). This for example rules out that $u$ is regular, rather it fixes its regularity at $\mathcal{H}^{1-\varepsilon}$.

One of the chief innovations in [21] as opposed to [2] was that by observing that the statement (3.1) is equivalent to
\begin{equation}
 u - P_{>N}(u \prec (1 - \Delta)^{-1} \xi + B_\Xi(u)) \in \mathcal{H}^2, \tag{3.2}
\end{equation}
where $P_{>N} = \mathcal{F}^{-1}\Pi_{>N}\mathcal{F}$, for any $N > 0$ cuts out the low frequencies. By choosing $N$ large enough depending on the $X^a$ norm of $\Xi$ (see Definition 3.5) it was shown that the map
\begin{align*}
 \Phi(u) &:= u - P_{>N}(u \prec (1 - \Delta)^{-1} \xi + B_\Xi(u)) \\
 \mathcal{D}(H) &\mapsto \mathcal{H}^2
\end{align*}
which sends a paracontrolled function to its remainder admits an inverse which we call $\Gamma$ and we rename $\Phi$ as $\Gamma^{-1}$.

In the following we use the short-hand notation
\begin{equation}
 u = \Gamma u^\sharp = P_{>N}(\Gamma u^\sharp \prec (1 - \Delta)^{-1} \xi + B_\Xi(\Gamma u^\sharp)) + u^\sharp, \tag{3.3}
\end{equation}
where the term $B_\Xi$ is explicitly given by
\begin{equation}
 B_\Xi(u) := (1 - \Delta)^{-1}(\Delta u \prec X + 2\nabla u \prec \nabla X + \xi \prec u - u \prec (\xi \diamond X)),
\end{equation}
with
\begin{equation}
 X = (1 - \Delta)^{-1}\xi \text{ and } \xi \diamond X = \Xi_2 \text{ is the second component of } \Xi, \text{ see (3.10).}
\end{equation}

Moreover, in the new coordinates, $u^\sharp$, the operator $H$ is given by
\begin{equation}
 H\Gamma u^\sharp = (\Delta - 1)u^\sharp + u^\sharp \circ \xi + P_{<N}(\Gamma u^\sharp \prec \xi + \Gamma u^\sharp \succ \xi) + P_{>N}(-B_\Xi(\Gamma u^\sharp) - \Gamma u^\sharp \prec X + C(\nabla^2 u^\sharp, X, \xi) + B_\Xi(\Gamma u^\sharp) \circ \xi), \tag{3.4}
\end{equation}
recalling the convention “$\succ = +\circ$” It is also natural to consider the operator $H$ conjugated by $\Gamma$, i.e.
\begin{equation}
 \mathcal{H}^2 := \Gamma^{-1}H\Gamma, \tag{3.5}
\end{equation}
which can be expressed as
\[
H^\sharp u^\sharp = H\Gamma u^\sharp - P_{>N}(H\Gamma u^\sharp < X + B_\Xi(H\Gamma u^\sharp))
\]
\[
= (\Delta - 1)u^\sharp + u^\sharp \circ \xi + P_{<N}(\Gamma u^\sharp < \xi + \Gamma u^\sharp > \xi) + P_{>N}(-B_\Xi(\Gamma u^\sharp) - \Gamma u^\sharp < X + \Gamma u^\sharp \geq \Xi_2 + C(\Gamma u^\sharp, X, \xi) + B_\Xi(\Gamma u^\sharp) \circ \xi) - P_{>N}(H\Gamma u^\sharp < X + B_\Xi(H\Gamma u^\sharp)).
\]
(3.5)

We remark that while $H$ was shown to be self-adjoint on $L^2$, the conjugated operator $H^\sharp$ is not and in particular the map $\Gamma$ is not unitary.

We quote some results from [21] and [29], this result assumes that we have shifted the operator by a constant. We tacitly assume $-H$ to be positive as opposed to just being semi-bounded which is achieved by adding a constant depending on the noise as in [21].

**Theorem 3.1** [Proposition 2.27, Lemma 2.33, Lemma 2.34 [21], Proposition 1.14 [29]] We have, writing again $u = \Gamma u^\sharp$,

i. $\|u\|_{D(H)} = \|Hu\|_{L^2} \sim \|u^\sharp\|_{H^2}$

ii. $\|u\|_{D(\sqrt{-H})} = \|\sqrt{-H}u\|_{L^2} = -(u, Hu)_{L^2}^{1/2} \sim \|u^\sharp\|_{H^1}$

iii. $D(H) \hookrightarrow L^\infty$ and $D(\sqrt{-H}) \hookrightarrow L^p$ for any $2 \leq p < \infty$

iv. $\|u^\sharp\|_{H^s} \sim \|u\|_{H^s} \sim \|H^{\sharp/2}u\|_{L^2}$ for $s \in (-1, 1)$.

The following proposition quantifies the idea that the transformed operator $H^\sharp := \Gamma^{-1} H\Gamma$ is a lower-order perturbation of the Laplacian.

**Proposition 3.2** Take $u^\sharp \in H^2$, then the following holds for any $\varepsilon, \kappa > 0$ with $1+\varepsilon+\kappa \leq 2$

\[
\|(H^\sharp - \Delta)u^\sharp\|_{H^\kappa} \lesssim \|u^\sharp\|_{H^{1+\varepsilon+\kappa}}.
\]

**Proof** This essentially follows by noting that in terms of regularity the worst term to bound in (3.5) is $u^\sharp \circ \xi$ which is bounded by (see Lemma A.1)

\[
\|u^\sharp \circ \xi\|_{H^\kappa} \lesssim \|u^\sharp\|_{H^{1+\varepsilon+\kappa}} \|\xi\|_{C^{1-\varepsilon}}.
\]

The other terms are bounded similarly by $H^s$ norms of $u^\sharp$ with $s < 1$ multiplied by Hölder norms of objects related to $\xi$ which appear in the $X^\alpha$ norm, see Definition 3.5. This result is also proved in Proposition 2.1 in [29].

We collect all relevant results about the map $\Gamma$.

**Lemma 3.3**
i. $\Gamma : \mathcal{H}^s \to \mathcal{H}^s$ is bounded and invertible for any $s \in [0,1)$

ii. $\Gamma : L^p \to L^p$ is bounded and invertible for any $p \in [2, \infty]$

iii. $\Gamma : \mathcal{H}^1 \to D(\sqrt{-H})$ is bounded and invertible.

iv. $\Gamma : \mathcal{H}^2 \to D(H)$ is bounded and invertible.

v. One has the bounds

$$\|(\Gamma - 1)v\|_{\mathcal{H}^{1-\varepsilon-s}} \lesssim \|v\|_{\mathcal{H}^{-s}} \quad \text{for } s \in (0,1-\varepsilon) \text{ and } \varepsilon > 0$$

and

$$\|\Gamma v - v - v \prec X\|_{\mathcal{H}^{1+s-\varepsilon}} \lesssim \|v\|_{\mathcal{H}^s} \quad \text{for } \sigma \in (0,1-\varepsilon) \text{ and } \varepsilon > 0$$

where both bounds remain true if we replace the Sobolev spaces $\mathcal{H}^s$ by Besov-Hölder spaces $C^s$.

**Proof** Everything but ii. and v. was proved in Section 2.1.1 of [21]. The cases $p = 2, \infty$ were also already proved. For a different $p$ we note that the result follows by interpolation.

To prove v., one simply observes that in the case (3.6) the dominant term is $u \prec X$ which has precisely that bound and in the latter case (3.7) one just has to consider how $B_{\Xi}(v)$ is bounded for $v \in \mathcal{H}^s$ and use the paraproduct estimates from the appendix. □

Lastly we prove a statement about the “sharpened” group, which is the transformation of the unitary group $e^{itH}$

$$e^{itH^\sharp} := \Gamma^{-1} e^{itH} \Gamma.$$  

It is clear that one still has the group property (even though the unitarity is lost) since

$$e^{itH^\sharp} e^{isH^\sharp} = \Gamma^{-1} e^{itH} \Gamma^{-1} e^{isH} \Gamma = \Gamma^{-1} e^{i(t+s)H} \Gamma = e^{i(t+s)H^\sharp}.$$  

and one has the bounds for all times $t \in \mathbb{R}$

$$\|e^{itH} u\|_{L^2} \lesssim \|u\|_{L^2}$$

$$\|e^{itH} u\|_{D(H)} \lesssim \|u\|_{D(H)}.$$  

$$\left\|H^\frac{s}{2} e^{itH} u\right\|_{L^2} \lesssim \left\|H^\frac{s}{2} u\right\|_{L^2} \quad \text{for } s \in \mathbb{R}.$$  

We have the analogous results for the transformed group.

**Lemma 3.4** For $s \in [0,2]$ we get the following at any time $t \in \mathbb{R}$

$$\|e^{itH^\sharp} v\|_{\mathcal{H}^s} \lesssim \|v\|_{\mathcal{H}^s}$$
We finish this section by recalling the definition of the enhanced noise space $X^\alpha$ and the fact that smooth regularisations of the white noise $\xi$ which can be defined as a random series on $T^2$

$$\xi(\omega) = \sum_{n \in \mathbb{Z}^2} g_n (\omega) e_n$$

with $e_n$ the Fourier basis and $g_n = \overline{g_{-n}}$ i.i.d complex Gaussians

(3.8)

converges in the $X^\alpha$ topology, see e.g. [2] for details.

**Definition 3.5** Let $\alpha = 1 + \kappa'$ for $0 < \kappa' \ll 1$, then we define the metric space

$$X^\alpha := \{(g, (1 - \Delta)^{-1} g \circ g - a) : g \in C^\infty(T^2), a \in \mathbb{R}\} | C^{-\alpha} \times C^{2-2\alpha}$$

which we call the enhanced noise space.

For a smooth regularisation $\xi_\varepsilon = \xi * \rho_\varepsilon$ with smooth standard mollifier $\rho_\varepsilon$ and we set

$$\Xi_\varepsilon^2 := (1 - \Delta)^{-1} \xi_\varepsilon \circ \xi_\varepsilon - c_\varepsilon$$

(3.9)

for $c_\varepsilon = \mathbb{E}((1 - \Delta)^{-1} \xi_\varepsilon \circ \xi_\varepsilon) \sim \log (\frac{1}{\varepsilon})$ a diverging constant.

**Lemma 3.6** The lift of the regularised noise, $(\xi_\varepsilon, \Xi_\varepsilon^2)$ converges to a limit $(\xi, \Xi^2)$ in $X^\alpha$ in probability.

In particular, from now on, we can use that the limit objects pathwise have the following regularities

$$\xi \in C^{-1-\kappa'}, \xi \diamond (1 - \Delta)^{-1} \xi = \Xi^2 \in C^{-2\kappa'} \quad \text{for } 0 < \kappa' \ll 1 \text{ small.}$$

(3.10)

**Remark 3.7** Clearly a generic element in $C^{-\alpha}$ will not have a lift in $X^\alpha$, however, note that any potential in $C^{1+\kappa}$ does. Also, by using the Besov embedding, Lemma A.4, we can see that $V \in L^2 \hookrightarrow C^{-1-\kappa}$ and $(1 - \Delta)^{-1} V \in H^2 \hookrightarrow C^{1-\kappa}$ so $(V \circ (1 - \Delta)^{-1} V) \in \mathcal{H}^{1-\kappa} \hookrightarrow C^{-2\kappa}$ meaning that $L^2$ is canonically contained in the space $X^\alpha$ so in principle all our results would be valid for $L^2$ potentials $V$, and in principle one can push this further but this would lead to a higher loss in regularity in the Strichartz estimate. In a recent paper [24], Huang and Sogge proved Strichartz estimates like (2.1) for $-\Delta + V$ for $V \in L^{1+\delta}$ so our result would not immediately imply theirs although in their paper it is not clear whether $V$ may be chosen as a distribution. It would be interesting to see if both approaches could be combined.

4 Strichartz estimates for the Anderson Hamiltonian

In this section we prove Theorem 1.1.
Proposition 4.1 We have the following identity for a regular function, say \( v \in H^2 \), and at any time \( t \in \mathbb{R} \):

\[
(e^{-itH^z} - e^{-it\Delta})v = -i \int_0^t e^{-i(t-s)\Delta}((H^z - \Delta)(e^{-isH^z}v))ds,
\]

(4.1)

moreover, fixing some \( t_0 \in \mathbb{R} \), we get the related result

\[
(e^{-i(t-t_0)H^z} - e^{-i(t-t_0)\Delta})v = -i \int_{t_0}^t e^{-i(t-s)\Delta}((H^z - \Delta)(e^{-i(s-t_0)H^z}v))ds,
\]

(4.2)

where we recall, from Proposition 3.2, that there is a cancellation between \( H^z \) and the Laplacian.

Moreover, on the interval \([t_0, t_1]\) with \(|t_0 - t_1| \leq 1\), we have for any small \( \delta > 0 \) the bound

\[
\|e^{-i(t-t_0)H^z} - e^{-i(t-t_0)\Delta})v\|_{L^\infty_t[H^\sigma]} \lesssim |t_1 - t_0|\|v\|_{H^{\sigma+1+i\delta}} \quad \text{for } \sigma \in [0, 1 - \delta).
\]

(4.3)

Also, for \( r \geq 4 \) we have

\[
\|e^{-i(t-t_0)H^z} - e^{-i(t-t_0)\Delta})v\|_{L^r_t[H^\sigma]} \lesssim \int_{t_0}^{t_1} \|e^{-i(s-t_0)H^z}v\|_{H^{\sigma+2 - \frac{4}{r} + \delta}} \lesssim |t_1 - t_0|\|v\|_{H^{\sigma+2 - \frac{4}{r} + \delta}}
\]

(4.4)

for \( \sigma \geq 0 \) s.t. \( \sigma + 2 - \frac{4}{r} + \delta \leq 2 \).

Proof To prove (4.1), note that the l.h.s. solves a PDE. Set \( v_1(t) = e^{-it\Delta}v \), \( v_2(t) = e^{-itH^z}v \) and \( \bar{v} = v_1 - v_2 \). Then

\[
(i\partial_t - \Delta)v_1 = 0 \quad \quad v_1(0) = v
\]

\[
(i\partial_t - \Delta)v_2 = (H^z - \Delta)v_2 \quad \quad v_2(0) = v
\]

\[
(i\partial_t - \Delta)\bar{v} = -(H^z - \Delta)v_2 \quad \quad \bar{v}(0) = 0
\]

From this we deduce that the mild formulation for \( \bar{v} \) reads

\[
\bar{v}(t) = -i \int_0^t e^{-i(t-s)\Delta}(H^z - \Delta)(v_2(s))ds
\]

which is (4.1). To prove (4.2), we proceed as above, with the difference that we replace \( t \) by \( t - t_0 \) and do a change of variables in the integral.
Theorem 4.2 \[2-D Anderson Strichartz\] Let \( r \geq 4, \sigma \geq 0, \delta > 0 \) s.t. \( \sigma + (1 + \delta) \left( 1 - \frac{3}{r} \right) < 1 \). Then we have on a finite time interval \([0, T]\), \( T \leq 1\) the following bound
\[
\|e^{-itH^2}v\|_{L^r_t[0,T]W^\sigma,r} \lesssim \|v\|_{H^{\sigma+(1+\delta)\left(1-\frac{3}{r}\right)}}
\]
and
\[
\left| \int_0^t e^{-i(s-r)H^2}f(s)ds \right|_{L^r_t[0,T]W^\sigma,r} \lesssim \int_0^T \|f(s)\|_{H^{\sigma+(1+\delta)\left(1-\frac{3}{r}\right)}}ds
\]

**Proof** We start by proving (4.6) with \( \sigma = 0 \) and \( r = 4 \). The general case follows by interpolation. By Proposition 4.1 and the Strichartz estimates in Theorem 2.4 we can write, setting \( v_N = P_{\leq N}v, I := [t_0, t_1] \) a subinterval of length \( \sim \frac{1}{N} \) and \( \delta > 0 \)
\[
P_{\leq N}e^{-itH^2}v_N = P_{\leq N}e^{-i(t-t_0)H^2}e^{-it_0H^1}v_N = e^{-i(t-t_0)\Delta}P_{\leq N}e^{-it_0H^2}v_N + P_{\leq N}(e^{-i(t-t_0)H^2} - e^{-i(t-t_0)\Delta})e^{-it_0H^1}v_N
\]
\[
= e^{-i(t-t_0)\Delta}P_{\leq N}e^{-it_0H^2}v_N - i \int_{t_0}^t P_{\leq N}e^{-i(s-t)\Delta}(H^2 - \Delta)(e^{-i(s-t_0)H^1}v_N)ds.
\]
Now we decompose the time interval into slices \( I_j = [t_j^0, t_j^1] \) s.t. \( \cup_j I_j = [0, T] \) with \( |I_j| \sim \frac{1}{N} \)
\[
\|P_{\leq N}e^{-itH^2}v_N\|_{L^r_t[0,T]}^4 \lesssim \sum_{I_j=[t_j^0,t_j^1]} \|P_{\leq N}e^{-itH^2}v_N\|_{L^4_{t,j}L^4}^4 + \|P_{\leq N}(e^{-i(t-t_0)H^2} - e^{-i(t-t_0)\Delta})e^{-it_0H^1}v_N\|_{L^4_{t,j}L^4}^4 + \sum_{I_j=[t_j^0,t_j^1]} \|v_N\|_{H^5}^4 + (*)
\]
\[
\lesssim \sum_{I_j=[t_j^0,t_j^1]} \|v_N\|_{H^5}^4 + (*)
\]
\[
\lesssim N^{1+4\delta} \|v_N\|_{L^2}^4 + \sum_{I_j=[t_j^0,t_j^1]} (*)
\]
Here we have used (4.2) in each subinterval and applied the triangle inequality from the first to the second line. In the next step we have used the short-time bound from Proposition \[2.5\] and lastly Lemma \[3.4\] and the fact that there are \(\sim N\) summands allow us to conclude.

Next, we treat the perturbative part which we called \((\star)\)

\[
\|P_{\leq N}(e^{-i(t-t_0)H^{\sharp}} - e^{-i(t-t_0)\Delta})e^{-it_0H^{\sharp}}v_N\|_{L^4_t L^4_x}^4 = \left| \int_{t_0}^t e^{-i(t-s)\Delta} P_{\leq N}(H^{\sharp} - \Delta)(e^{-i(s-t_0)H^{\sharp}}v_N)ds \right|_{L^4_t L^4_x}^4
\]

\[
\lesssim \left( \int_{I_j} \|P_{\leq N}(H^{\sharp} - \Delta)(e^{-i(s-t_0)H^{\sharp}}v_N)\|_{H^{4\delta}} ds \right)^4
\]

\[
\lesssim \left( \int_{I_j} \|e^{-i(s-t_0)H^{\sharp}}v_N\|_{H^{1+\delta}} ds \right)^4
\]

\[
\lesssim \left( \int_{I_j} \|v_N\|_{H^{1+\delta}} ds \right)^4
\]

\[
\lesssim |I_j|^4 N^4 \|v_N\|_{H^{4\delta}}^4
\]

\[
\lesssim N^{4\delta} \|v_N\|_{L^2}^4
\]

having used the second bound in Proposition \[2.5\] to get to the second line and thereafter Proposition \[3.2\] Lemma \[3.4\] and Bernstein’s inequality, Lemma \[A.3\].

Thus we can conclude

\[
\|P_{\leq N}e^{-itH^{\sharp}}v_N\|_{L^4_t[0,T] L^4_x}^4 \lesssim N^{1+4\delta} \|v_N\|_{L^2}^4
\]

hence

\[
\|P_{\leq N}e^{-itH^{\sharp}} P_{\leq N}v\|_{L^4_t[0,T] L^4_x} \lesssim N^{4+\delta} \|P_{\leq N}v\|_{L^2}
\]

for any \(\delta > 0\), which directly implies the result. The case \(\sigma > 0\) is analogous since on this level it basically corresponds to multiplying with a power of \(N\) and for general \(r \geq 4\) we interpolate between the bounds

\[
\|P_{\leq N}e^{-itH^{\sharp}} P_{\leq N}v\|_{L^4_t[0,T] L^4_x} \lesssim \|P_{\leq N}v\|_{H^{4+\delta}}
\]

and

\[
\|P_{\leq N}e^{-itH^{\sharp}} P_{\leq N}v\|_{L^\infty_t L^\infty_x} \lesssim \|P_{\leq N}v\|_{H^{1+\delta}}
\]

where the latter is simply the trivial bound obtained from the Sobolev embedding \(H^{1+\delta} \hookrightarrow L^\infty\).

The inhomogeneous Strichartz estimate \[4.7\] follows from the first in the usual way. \(\square\)
5 Well-posedness of multiplicative stochastic NLS

5.1 Low-regularity solutions

We turn our attention to “low-regularity” solutions to the stochastic NLS

\[(i \partial_t - H)u = -u|u|^{2n} \]
\[u(0) = u_0 \in \mathcal{H}^\sigma \]

for some \(\sigma < 1\), which is formally

\[(i \partial_t - \Delta)u = u \cdot \xi + \infty u - u|u|^{2n}.\]

In [21] this PDE was studied in the “high regularity” regime, meaning \(u_0 \in D(H)\) or \(D(\sqrt{-H})\). Now we employ the Strichartz estimates to solve it in spaces of lower regularity. In particular now we solve it in a space that does not depend on the realisation of the noise \(\xi\).

Since this result is analogous to the result from [29] on general smooth surfaces where only the cubic case was considered, we only sketch the proof and point out where the value of \(n\) enters.

**Theorem 5.1 [LWP below energy space]** For \(\sigma \in (1 - \frac{1}{n}, 1)\) (5.1) is LWP in \(\mathcal{H}^\sigma\). More precisely, there exists a short time \(T > 0\) which is of size \(T \sim (1 + \|u_0\|_{\mathcal{H}^\sigma})^{-K}\) for some \(K\) depending on \(n\) s.t. there exists a unique solution to

\[u(t) = e^{-i t H}u_0 - i \int_0^t e^{-i(t-s)H} u|u|^{2n}(s)ds\]

in the space \(C_{[0,T]}\mathcal{H}^\sigma \cap L^{2n}_{[0,T]} W^{\frac{1}{n} + \kappa, 2n}_{\infty}\) for \(\kappa > 0\) sufficiently small where the solution in this norm depends continuously on the initial data in \(\mathcal{H}^\sigma\).

**Proof** By applying \(\Gamma^{-1}\) to both sides and renaming both \(\Gamma^{-1}u_0 = u^\sharp_0\) and \(\Gamma^{-1}u = u^\sharp\) this becomes

\[u^\sharp(t) = e^{-i H^2} u^\sharp_0 - i \int_0^t e^{-i(t-s)H^2} \Gamma^{-1}(\Gamma u^\sharp)|\Gamma u^\sharp|^{2n}(s)ds.\]

We want to show that this equation has a solution for a short time by setting up a fixed point argument in the space

\[C_T\mathcal{H}^\sigma \cap L^{2n+\kappa}_{[0,T]} W^{\frac{1}{n} + \kappa, \frac{2n}{1 - \kappa}},\]

where \(T > 0\) is chosen later and \(\kappa > 0\) is small enough that

\[\frac{1}{n} + \kappa + (1 + \kappa)\left(1 - \frac{3(1 - \kappa)}{2n}\right) \leq \sigma\]
Then we bound, using several times Lemma 3.3 and the Strichartz estimate Theorem 4.2

\[ \|u^2\|_{L^{2n+\kappa}_{[0,T]}} \lesssim \|u^0\|_{\mathcal{H}^\sigma} + \int_0^T \|\Gamma^{-1}(\Gamma u^2|\Gamma u^2|^{2n})\|_{\mathcal{H}^\sigma} d\tau \]
\[ \lesssim \|u^0\|_{\mathcal{H}^\sigma} + \int_0^T \|\Gamma u^2|\Gamma u^2|^{2n}\|_{L^\infty} \|u^2\|_{L^{\infty}_{[0,T],\mathcal{H}^\sigma}} d\tau \]
\[ \lesssim \|u^0\|_{\mathcal{H}^\sigma} + \|u^2\|_{L^{2n}_{[0,T]}}^{2n} \|u^2\|_{L^{\infty}_{[0,T],\mathcal{H}^\sigma}} d\tau \]
\[ \lesssim \|u^0\|_{\mathcal{H}^\sigma} + T^{\kappa} \|u^2\|_{L^{2n+\kappa}_{[0,T]}} \|u^2\|_{L^{\infty}_{[0,T],\mathcal{H}^\sigma}} d\tau . \]

For the other term we bound

\[ \|u^2\|_{L^\infty_{[0,T],\mathcal{H}^\sigma}} \lesssim \|u^0\|_{\mathcal{H}^\sigma} + \int_0^T \|\Gamma^{-1}(\Gamma u^2|\Gamma u^2|^{2n})\|_{\mathcal{H}^\sigma} d\tau \]
\[ \lesssim \|u^0\|_{\mathcal{H}^\sigma} + T^{\kappa} \|u^2\|_{L^{2n+\kappa}_{[0,T]}} \|u^2\|_{L^{\infty}_{[0,T],\mathcal{H}^\sigma}} d\tau . \]

From here we can get a contraction for small times in the usual way. Thus we solve the sharpened equation and by applying \( \Gamma \) we get a solution to the original equation. Observe that \( u^2 \in W^{1,2} \) implies that \( (\Gamma - 1) \in W^{1,2} \) and thus \( u \in W^{1,2} \).

\[ \Box \]

**Remark 5.2** The fact that \( s < 1 \) as opposed to \( s \geq 1 \) makes the bound for the term \( \Gamma^{-1}(u|u|^2) \) easier since the paraproducts and other correction terms are actually more regular than \( u \) and \( u^2 \).

**Remark 5.3** If we used the Strichartz estimates proved in [29] which are

\[ \|e^{-itH}v\|_{L^p_{[0,1]}} \lesssim \|v\|_{\mathcal{H}^{1/2}} \quad \text{for } \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \text{ and } \delta > 0 \text{ small}, \]

we would get exactly the same condition \( \sigma > 1 - \frac{1}{n} \) for the wellposedness with the only difference that the function spaces in the contraction would be a bit different i.e. one would use \( L^r_{[0,T]}L^s \) spaces with different parameters \( r, s \).

### 5.2 Global well-posedness in the energy space

Now we turn to proving global in time well-posedness of

\[
(i\partial_t - H)u = -u|u|^{2n} \quad \text{on } \mathbb{T}^2 \]
\[ u(0) = u_0 \quad \text{(5.3)} \]
in the energy space. We recall that this is the space of functions

\[ v \in L^2 : \| \Gamma^{-1} v \|_{H^1} \sim \| v \|_{D(\sqrt{-H})} = |(u, Hu)|^{\frac{1}{2}} < \infty, \]

see Lemma 3.3 and Lemma B.1. This is similar to the argument in [30] for NLS on 2d compact manifolds.

**Proof** [of Theorem 1.3] The main point is that the energy

\[ E(u)(t) := -\frac{1}{2} (u(t), Hu(t)) + \frac{1}{2n} \int |u(t)|^{2n} \]  

is conserved in time which controls the \( L^\infty_{[0,T]} D(\sqrt{H}) \) norm for all times since

\[
\| u(t) \|_{D(\sqrt{H})}^2 = 2E(u)(t) - \frac{1}{n} \int |u(t)|^{2n} \\
\leq 2E(u)(t) = 2E(u)(0) \\
= \| u_0 \|_{D(\sqrt{H})}^2 + \frac{1}{n} \| u_0 \|_{L^{2n}}^2 =: C(E(u_0)),
\]

where in the last line we write the suggestive notation \( C(E(u_0)) \) for a constant depending polynomially on \( \| u_0 \|_{D(\sqrt{H})} \) since the \( L^{2n} \) norm is also controlled by the \( D(\sqrt{H}) \) norm by Theorem 3.1. We use the notation \( C(E(u_0)) \) to emphasise the dependence of the constant on the conserved energy.

This observation apriori just gives global in time existence as was observed in [21] but not uniqueness or continuous dependence on the data. This is analogous to the classical setting, where energy methods are not enough to get well-posedness in the energy space but Strichartz estimates are needed, as noted for example in [30].

Since we now have access to Strichartz estimates with loss as in the manifold setting of [30], we can actually prove global in time well-posedness giving a better result than in [21]. Note that from Theorem 1.2 and Section 5.1 we have for \( \sigma = 1 - \frac{1}{2n} \) that \( (5.3) \) is locally well-posed in \( H^\sigma \). So for \( u_0 \in D(\sqrt{H}) \subset H^\sigma \) we get a unique solution (with continuous dependence) up to time \( T \sim (1 + \| u_0 \|_{H^\sigma})^{-K} \). Clearly this is also true if we choose the smaller time \( T' \sim (1 + \| u_0 \|_{D(\sqrt{H})})^{-K} \).

Then we may restart the fixed point procedure with initial condition \( u_1 := u(T') \) for which

\[ \| u_1 \|_{D(\sqrt{H})} \sim \| u_0 \|_{D(\sqrt{H})} \]

by energy conservation. Thus we get a unique solution up to time \( 2T' \) which depends continuously on the initial data \( u_0 \) in \( H^\sigma \) and can of course iterate this as often as we want.
Thus we are almost finished, it remains to show that the solution map is even continuous in $D\left(\sqrt{H}\right)$.

One way of getting this is by using the fact that for a sequence of initial data

$$D\left(\sqrt{H}\right) \supset u_0^{(m)} \rightarrow u_0 \quad D\left(\sqrt{H}\right)$$

(5.6)

we have the global existence/uniqueness/continuity in a weaker norm of the solutions $u^{(m)}(t)$ to (5.3) with initial data $u_0^{(m)}$. We have also by the energy conservation that the bound

$$\|u^{(m)}(t)\|_{D(\sqrt{H})} \leq \|u_0^{(m)}\|_{D(\sqrt{H})} + \|u_0^{(m)}\|^n_{L^{\infty}} \leq C(E(u_0))$$

holds uniformly in $m$ and $t$. Thus (for any subsequence) we can extract a weakly (w.r.t. the Hilbert space $D(\sqrt{H})$) converging subsequence for a dense set of times. Since the energies converge because of

$$E(u^{(m)}(t)) = E(u_0^{(m)}) \rightarrow E(u_0) = E(u(t))$$

and thus the $D\left(\sqrt{H}\right)$ norms (because of the Sobolev embedding, Theorem 3.1 also the second term in the energy converges) we have even strong convergence (recall that in a Hilbert space weak convergence and convergence of norms implies strong convergence) for a subsequence. Since we already have uniqueness in $H^\sigma$, we have a unique weak (and thus strong) limit for a dense set of times. Finally, to extend the result to all times and get continuity in time, we note that strong convergence in $D\left(\sqrt{H}\right)$ is enough to pass to the limit in the mild formulation

$$u(t) = e^{-itH}u_0 - i \int_0^t e^{-i(t-s)H}u|u|^{2n-2}(s)ds ,$$

(5.7)

which tells us that the limit solves the equation and is continuous in time due to Stone’s theorem.

5.3 Global well-posedness of strong solutions and growth of norms

We consider the equation

$$(i\partial_t - H)u = -u|u|^{2n}$$

$$u(0) = u_0$$

(5.8)

for general $n$ and $u_0 \in D(H)$ which we call the strong regime. This is equivalent to solving

$$(i\partial_t - H^2)u^\sharp = -\Gamma^{-1}(\Gamma u^\sharp |\Gamma u^\sharp|^{2n})$$

(5.9)

with initial data $u^\sharp(0) = \Gamma^{-1}u_0 \in H^2$. 

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The first order conserved energy is given by
\[
E(u(t)) := -\frac{1}{2}(u, Hu) + \frac{1}{2n+2} \int |u|^{2n+2} = E(u_0)
\]
which controls the norm \(\|u\|_{L^\infty_0[T]} D(\sqrt{H}) \sim \|u^2\|_{L^\infty_0[T]} H^1\).

We recall from [21] that in this regime we have local-in-time well-posedness from a contraction argument in the mild formulation (5.7) following on the other hand from the formal norm equivalence \(\|u\|_{L^\infty_0[T]} D(H) \sim \|\partial_t u\|_{L^\infty_0[T]} L^2\), see Lemma 5.6 and the observation that (see [21] for details)
\[
\left\| \int_0^t e^{-i(t-s)H} u|u|^{2n-2}(s) ds \right\|_{L^\infty_0[T] D(H)} \lesssim \|\partial_t u\|_{L^\infty_0[T]} D(H) \lesssim \|\partial_t u\|_{L^\infty_0[T]} D(H)\|u\|^{2n-2}_{L^\infty_0[T]} L^\infty
\]
and the embedding \(D(H) \hookrightarrow L^\infty\), since one can not apply \(H\) to a nonlinear term but one may integrate by parts in the time integral to instead.

**Remark 5.4** In [21], Remark 3.9, it was wrongly claimed that global-in-time well-posedness follows in a similar way for general powers. Local well-posedness, however, can be proved in the same way for general powers.

The main result of this section is the following, which says that we can extend these solutions to all times but we also get a bound on the growth of Sobolev bounds as in [32]

**Theorem 5.5** Let \(T > 0\) and \(\kappa > 0\) small. We can find an almost conserved energy \(E^{(1)}(u)\) for which one has
\[
\left| E^{(1)}(u(t)) - \frac{1}{2} \int |\partial_t u(t)|^2 \right| \lesssim C(u_0)(\|\partial_t u(t)\|_{L^2}^6 + 1) \quad \forall t \in [0, T]
\]
and which has growth
\[
\sup_{0 \leq t \leq T} |E^{(1)}(u(t))| \lesssim T^{8+\kappa}.
\]

This is based on the approach from [32] and is a comparable result to what was shown in [35] and [34] but uses the paracontrolled approach from [21] and the Strichartz estimates as in [29]. Moreover, the growth of the norm is not shown in those papers.

For the second order energy that should control the norm \(\|u\|_{L^\infty_0[T]} D(H) \sim \|u^2\|_{L^\infty_0[T]} H^2\) or equivalently \(\|\partial_t u\|_{L^\infty_0[T]} L^2 \sim \|\partial_t u^2\|_{L^\infty_0[T]} L^2\). We collect these norm equivalences in a small lemma.
Lemma 5.6 Let $u$ be a solution of (5.8) and $\Gamma^{-1}u = u^\sharp$ solve (5.9). Then we have that the following bounds hold

\begin{align}
\|u\|_{L_0^\infty T^1_1}^{D(H)} & \lesssim \|u^\sharp\|_{L_0^\infty T^1_1}^{D(H)} \lesssim \|u\|_{L_0^\infty T^1_1}^{D(H)} \tag{5.10} \\
\|\partial_t u\|_{L_0^\infty T^1_1}^{L^2} & \lesssim \|\partial_t u^\sharp\|_{L_0^\infty T^1_1}^{L^2} \lesssim \|\partial_t u\|_{L_0^\infty T^1_1}^{L^2} \tag{5.11} \\
& \lesssim \|u\|_{L_0^\infty T^1_1}^{D(H)} + C(E(u_0)) \tag{5.12} \\
\|Hu\|_{L_0^\infty T^1_1}^{H^{-\alpha}} & \lesssim \|u^\sharp\|_{L_0^\infty T^1_1}^{H^{-\alpha}} \lesssim \|\partial_t u\|_{L_0^\infty T^1_1}^{H^{-\alpha}} + C(E(u_0)) \text{ for } \alpha \in (0, 1) \tag{5.13} \\
\|\partial_t u\|_{L_0^\infty T^1_1}^{H^{-\alpha}} & \lesssim \|\partial_t u^\sharp\|_{L_0^\infty T^1_1}^{H^{-\alpha}} \lesssim \|Hu\|_{L_0^\infty T^1_1}^{H^{-\alpha}} + C(E(u_0)) \text{ for } \alpha \in (0, 1) \tag{5.14} \\
\|u^\sharp\|_{L_0^\infty T^1_1}^{H^{-\gamma}} & \lesssim C(E(u_0))\|u^\sharp\|_{L_0^\infty T^1_1}^{H^{-\gamma}} \text{ for } \gamma \in [1, 2]. \tag{5.15}
\end{align}

Moreover, the difference

\[ \partial_t u - \partial_t u^\sharp = (\Gamma - 1)\partial_t u^\sharp \tag{5.16} \]

satisfies the bound

\[ \| (\Gamma - 1)\partial_t u^\sharp \|_{H^{-\alpha}} \lesssim \|\partial_t u^\sharp\|_{H^{-1+\kappa}} \text{ for } \alpha < 1 \text{ and } \kappa < 1 - \alpha \tag{5.17} \]

similarly we have

\[ \| (\Gamma - 1)u^\sharp\|_{L_0^\infty T^1_1}^{C^{1-2\kappa}} \lesssim \|u^\sharp\|_{L_0^\infty T^1_1}^{C^{-\kappa}} \lesssim \|u^\sharp\|_{L_0^\infty T^1_1}^{H^1} \lesssim C(E(u_0)) \text{ for } \kappa > 0. \tag{5.18} \]

**Proof** The first two equivalences follow from the properties of the $\Gamma$ map from Lemma 5.3. To get (5.12), we use the equation to get the bound

\[ \|\partial_t u\|_{L_0^\infty T^1_1}^{L^2} \leq \|Hu\|_{L_0^\infty T^1_1}^{L^2} + \|u\|_{L_0^\infty T^1_1}^{L^{2n+1}} \leq \|Hu\|_{L_0^\infty T^1_1}^{L^2} + E^{2n+1} (u_0) \]

where the last step follows from the embedding $\mathcal{D} \left( \sqrt{-H} \right) \hookrightarrow L^2 (2n+1)$, see Lemma 3.3 and the energy conservation (5.5). The two bounds (5.13), (5.14) follow as above combined with Theorem 3.1 and (5.15) follows from $H^{-\gamma}$ interpolation and the energy bounding $\|u^\sharp\|_{L_0^\infty T^1_1}^{H^{-\gamma}}$. Finally, (5.17) and (5.18) follow from Lemma 3.3.

We next use the Strichartz estimate, Theorem 4.2

\[ \|e^{itH^s}v\|_{L_0^{4} W^{\alpha, 4}} \lesssim \|v\|_{H^{\alpha + 4} + \kappa} \text{ for } \kappa > 0 \text{ and } \alpha + 1/4 + \kappa \leq 2 \]

and obtain some bounds as a consequence.
Lemma 5.7 Let $u$ be a solution of (5.8) and $\Gamma^{-1} u = u^2$ solve (5.9). Then we have

\[
\|u\|_{L_{[0,1]}^{4}C^{1/4 - \kappa}} \lesssim C(E(u_0)) \quad \text{for any } \kappa > 0 \quad (5.19)
\]

\[
\|u|^m u\|_{L_{[0,1]}^{4}C^{\alpha}} \lesssim C(E(u_0))\|u\|_{L_{[0,1]}^{4}C^{\alpha + \kappa}} \quad \text{for } m \in \mathbb{N} \text{ and } \alpha + \kappa < 1 \quad (5.20)
\]

in particular

\[
\|u|^m u\|_{L_{[0,1]}^{4}C^{1/4 - \kappa}} \lesssim C(E(u_0)) \quad \text{for any } \kappa > 0 \quad (5.21)
\]

and

\[
\|\partial_t u|^m u\|_{L_{[0,1]}^{4}H^{-\alpha}} \lesssim C(E(u_0))\|\partial_t u\|_{L_{[0,1]}^{\infty}H^{-\alpha}} \quad \text{for } \alpha \in \left[0, \frac{1}{4}\right) \quad (5.22)
\]

\[
\|u^2\|_{L_{[0,1]}^{4}W^{1/4 - \kappa,4}} \lesssim C(E(u_0))(1 + \|\partial_t u\|_{L_{[0,1]}^{\infty}L^2}) \quad \text{for } \kappa > 0. \quad (5.23)
\]

Proof Using the Sobolev embedding $W^{\beta + \frac{1}{2} + \varepsilon, 4} \hookrightarrow C^\beta$ for $\varepsilon > 0$ and the Strichartz estimate, we get

\[
\|u\|_{L_{[0,1]}^{4}C^{1/4 - \kappa}} \lesssim \|u\|_{L_{[0,1]}^{4}W^{1/4 - \kappa, 4}}
\]

\[
\lesssim \|u_0\|_{H^{1/2 - \kappa}} + \int_0^1 \|u|u|^{2n}(s)\|_{H^{1/2 - \kappa}} ds
\]

\[
\lesssim \|u_0\|_{H^{1/2 - \kappa}} + \int_0^1 \|u(s)\|_{H^{1/2 - \kappa}}ds
\]

\[
\lesssim \|u_0\|_{D(\sqrt{-\Pi})} + \|u\|_{L_{[0,1]}^{\infty}D(\sqrt{-\Pi})}^{2n+1}
\]

\[
\lesssim C(E(u_0))
\]

where $1 \gg \kappa \gg \tilde{\kappa} > 0$ are small enough and can be determined using the fractional Leibnitz inequality, Lemma 5.6. This proves (5.19).

For (5.20), we use the Sobolev embedding, for every $\kappa > 0$ there exists $p \gg 1$ s.t. $W^{\alpha + \kappa, p} \hookrightarrow C^\alpha$ and fractional Leibnitz, Lemma 5.6.

\[
\|u|^m u\|_{L_{[0,1]}^{4}C^{\alpha}} \lesssim \|u|^m u\|_{L_{[0,1]}^{4}W^{\alpha + \kappa, p}} \lesssim \|u\|_{L_{[0,1]}^{4}L^{\infty}L^{mp}}\|u\|_{L_{[0,1]}^{4}C^{\alpha + \kappa}} \lesssim C(E(u_0))\|u\|_{L_{[0,1]}^{4}C^{\alpha + \kappa}}
\]

giving (5.20), so in particular $\|u|^m u\|_{L_{[0,1]}^{4}C^{1/4 - \kappa}} \lesssim C(E(u_0))$ as claimed. This also implies immediately the bound (5.22) by the Leibnitz rule for functions/distributions.

Finally, we want to prove (5.23) which would be obvious in the classical case, but needs an additional argument since this is the first time we want to take more than one derivative in the Strichartz estimate. We observe that for general functions $f \in C^1_{[0,T]} L^2$ we have by
integrating by parts in time
\[ \int_0^t e^{-i(t-s)\mathcal{H}^2} f(s) \, ds = \frac{1}{i} e^{-it\mathcal{H}^2} \int_0^t (\mathcal{H}^2)^{-1} \left( \frac{d}{ds} e^{is\mathcal{H}^2} \right) f(s) \, ds \]
\[ = \frac{1}{i} \int_0^t e^{-i(t-s)\mathcal{H}^2} (\mathcal{H}^2)^{-1} \frac{d}{ds} f(s) \, ds + \frac{1}{i} (\mathcal{H}^2)^{-1} (f(t) - e^{-it\mathcal{H}^2} f(0)). \]

Then, using the mild formulation for \( u^\sharp \) (set \( \Gamma^{-1}((\Gamma u^\sharp)|\Gamma u^\sharp|^{2n}) = f \) for readability)
\[ u^\sharp(t) = e^{-it\mathcal{H}^2} u_0^\sharp + i \int_0^t e^{-i(t-s)\mathcal{H}^2} f(s) \, ds \]
\[ = e^{-it\mathcal{H}^2} u_0^\sharp + \int_0^t e^{-i(t-s)\mathcal{H}^2} (\mathcal{H}^2)^{-1} \frac{d}{ds} f(s) \, ds - (\mathcal{H}^2)^{-1}(f(t) - e^{it\mathcal{H}^2} f(0)) \]
so we get the bound from the Strichartz estimate Theorem 4.2 and Lemma 3.3
\[
\| u^\sharp \|_{L^4_{[0,1]} W^{\frac{1}{2}-\kappa,4}} \lesssim \| u_0^\sharp \|_{\mathcal{H}^2} + \| (\mathcal{H}^2)^{-1} \Gamma \partial_t f \|_{L^4_{[0,1]} \mathcal{H}^2} + \| (\mathcal{H}^2)^{-1} f \|_{L^4_{[0,1]} W^{\frac{1}{2}-\kappa,4}} + \| e^{it\mathcal{H}^2} (\mathcal{H}^2)^{-1} \Gamma^{-1} f(0) \|_{L^4_{[0,1]} W^{\frac{1}{2}-\kappa,4}}
\]
\[
\lesssim \| u_0^\sharp \|_{\mathcal{H}^2} + C(E(u_0)) \| \partial_t u \|_{L^2_{[0,1]} L^2} + \| u |u|^{2n} \|_{L^2_{[0,1]} \mathcal{H}^{\frac{1}{2}-\kappa}} + \| u_0 |u_0|^{2n} \|_{L^2}
\]
\[
\lesssim C(E(u_0)) \| \partial_t u \|_{L^2_{[0,1]} L^2} + C(E(u_0))
\]
\[
\lesssim C(E(u_0))(1 + \| \partial_t u \|_{L^2_{[0,1]} L^2})
\]
finishing the proof. \( \square \)

The general approach, following [32], is to make a simple initial ansatz \( E^0 = \frac{1}{2} \int |\partial_t u|^2 \) for the almost conserved energy \( E^{(1)} \) and then compute its time derivative and try to write
\[ \frac{d}{dt} E^0 = \frac{d}{dt} A + O((E^0)^{1-}), \]
so terms on the rhs should either be total time derivatives or be “lower order” i.e. can be bounded by sublinear terms of \( E^0 \). Usually this will result from interpolating between the conserved energy \( E(u) \) which is equivalent to the \( \mathcal{H}^1 \) norm of \( u^\sharp \) and the almost conserved energy \( E^0 \) which is comparable to the \( \mathcal{H}^2 \) norm of \( u^\sharp \) or equivalently the \( L^2 \) norm of \( \partial_t u \), i.e. we will apply 5.15.

**Proof** [of Theorem 5.5] We make an ansatz for the almost conserved energy
\[ E^0(\partial_t u) = \frac{1}{2} \text{Re} \int |\partial_t u|^2 = \frac{1}{2} (\partial_t u, \partial_t u) \]
then we make the straightforward computation which is taking a time derivative and inserting the equation
\[
\frac{d}{dt}E^0(\partial_t u) = \text{Re} \int \overline{\partial_t u} \partial_t^2 u
\]
\[
= -\text{Re} i \int \overline{\partial_t u} \partial_t (Hu - |u|^{2n} u)
\]
\[
= -\text{Re} i \int \overline{\partial_t u} \partial_t (|u|^{2n} u)
\]
\[
= -\text{Re} i \int \overline{\partial_t u} (\partial_t |u|^{2n} u + u \partial_t |u|^{2n-2})
\]
\[
= -\text{Re} i \int \overline{\partial_t u} |u|^{2n-2},
\]
having used also the self-adjointness of \(H\) in the second line and the realness of the first expression in the penultimate line. Further we insert the equation again and obtain
\[
\frac{d}{dt}E^0(\partial_t u) = \text{Re} \int (Hu + |u|^{2n} u) \partial_t |u|^{2n-2}
\]
\[
= \text{Re} \int H u \partial_t |u|^{2n-2} + \text{Re} \int |u|^{2n+2} \partial_t |u|^{2n-2}
\]
\[
= \text{Re} \int \overline{H u} u \partial_t |u|^{2n-2} + (n-1) \int |u|^{4n-2} \partial_t |u|^{2n-2}
\]
\[
= \text{Re} \int \overline{H u} u \partial_t |u|^{2n-2} + \frac{n-1}{2n} \frac{d}{dt} \int |u|^{4n}
\]
\[
=: (I) + \frac{n-1}{2n} \frac{d}{dt} \int |u|^{4n}
\]
(5.24)
having inserted \(\Gamma u^\sharp = u\) and the definition of the real part for the first term.

Now, the main point is that, up to more regular terms, the term \(\overline{H u} u \partial_t |u|^{2n-2}\) in \((I)\) is comparable to \(\Delta |u|^{2n-2}\) and the \(\partial_t |\Gamma u^\sharp|^{2n-2}\) should be replaced with \(\partial_t |u^\sharp|^{2n-2}\) and then one can proceed similarly to \(32\).

In other words, we aim to prove the bound
\[
|\text{Re} \int \overline{H u} u \partial_t |u|^{2n-2} - \text{Re} \int \overline{\Delta u^\sharp u} \partial_t |u^\sharp|^{2n-2}| \lesssim \left( \|\partial_t u\|_{L^\infty_{[0,T]}L^2}^\gamma + 1 \right)
\]
(5.25)
for some \(0 < \gamma < 2\) since the second
To begin with, we want to replace the time derivative \(\partial_t |\Gamma u^\sharp|^{2n-2}\) by \(\partial_t |u^\sharp|^{2n-2}\) in \((I)\). We
bound the difference as follows
\[
(\pi Hu, (\partial_t |\Gamma u^2|^{2n-2} - \partial_t |u^2|^{2n-2})) = (\pi Hu, (\Gamma - 1)\partial_t u^2 \pi |u|^{2n-4}) + (\pi Hu, \partial_t u(\pi |u|^{2n-2} - |u^2|^{2n-2}))
\]
\[
= (Hu, |u|^{2n-2}(\Gamma - 1)\partial_t u^2) + (Hu, \partial_t u(\Gamma - 1)u^2 (P^{(2n-3)}(u, u^2))) + (Hu, (\Gamma - 1)\pi^2 \tilde{P}^{(2n-3)}(u, u^2)),
\]  
(5.26)
where we have written
\[
|u|^{2n-2} - |u^2|^{2n-2} = (\Gamma - 1)u^2 P^{(2n-3)}(u, u^2) + (\Gamma - 1)\pi^2 \tilde{P}^{(2n-3)}(u, u^2)
\]
for some polynomials $P^{(2n-3)}$, $\tilde{P}^{(2n-3)}$ of degree $2n - 3$ in $u, \pi, u^2$ and $\pi^2$. Now we start by
bounding the first term in (5.26) integrated out in time on an interval $[0, T]$ for some $T \leq 1$
\[
|| (Hu, |u|^{2n-2}(\Gamma - 1)\partial_t u^2)||_{L^4_{[0, T]}} \lesssim ||| u|^{2n-2} Hu||_{L^4_{[0, T]} H^{\frac{1}{2}+\delta}} ||| (\Gamma - 1)\partial_t u^2||_{L^4_{[0, T]} H^{\frac{1}{2}+\delta}}
\]
\[
\lesssim ||| u|^{2n-2}||_{L^4_{[0, T]}} C(\delta) \frac{1}{2} ||| Hu||_{L^4_{[0, T]} H^{\frac{1}{2}+\delta}} ||| (\Gamma - 1)\partial_t u^2||_{L^4_{[0, T]} H^{\frac{1}{2}+\delta}}
\]
\[
\lesssim C(E(u_0)) \left( || u^2 ||_{L^4_{[0, T]} H^{\frac{1}{2}+\delta}} + ||| \partial_t u^2||_{L^4_{[0, T]} H^{\frac{1}{2}+\delta}} + 1 \right)
\]
\[
\lesssim C(E(u_0)) \left( || u^2 ||_{L^4_{[0, T]} H^{\frac{1}{2}+\delta}} + ||| \partial_t u^2||_{L^4_{[0, T]} H^{\frac{1}{2}+\delta}} + 1 \right)
\]
having used Lemmas 5.6 and 5.7. Next we bound the second term in (5.26), the third being
completely analogous, this term is worse in terms of regularity and we split off the worst
terms using paraproducts, see the appendix for the definition and recall the convention
$\lesssim := \prec \circ \prec$ we have
\[
(Hu, \partial_t u(\Gamma - 1)u^2 (P^{(2n-3)}(u, u^2))) = (Hu, (\partial_t u(\Gamma - 1)u^2 \lesssim (P^{(2n-3)}(u, u^2))) +
\]
\[
+ (Hu, (\partial_t u(\Gamma - 1)u^2 \succ (P^{(2n-3)}(u, u^2)))
\]
\[
= (Hu, (\partial_t u(\Gamma - 1)u^2 \lesssim (P^{(2n-3)}(u, u^2))) +
\]
\[
+ (Hu, (\partial_t u \leq (\Gamma - 1)u^2 \succ (P^{(2n-3)}(u, u^2))))
\]
\[
+ (Hu, (\partial_t u \succ (\Gamma - 1)u^2 \succ (P^{(2n-3)}(u, u^2))))
\]
which we bound as

\[ ||...||_{L^4_{[0,T]}} \lesssim \|Hu\|_{L^\infty_{[0,T]}\mathcal{H}^{-\frac{3}{4}}} \|\partial_t u\|_{L^\infty_{[0,T]}\mathcal{H}^{-\frac{1}{2}}} ||(\Gamma - 1)u^2||_{L^\infty_{[0,T]}\mathcal{H}^{\frac{1}{2}}} + ||P^{(2n-3)}(u, u^2)||_{L^4_{[0,T]}C^{\frac{1}{2}} + \frac{1}{2}} + \]

\[ + \|(Hu, (\partial_t u \gtrsim (\Gamma - 1)u^2))\|_{C^{\frac{1}{2}}} + \]

\[ + \|(Hu, (\partial_t u \preceq (\Gamma - 1)u^2))\|_{C^{\frac{1}{2}}} \]

\[ \lesssim C(E(u_0)) \left( ||\partial_t u||_{L^\infty_{[0,T]}L^2}^{\frac{7}{4} + \frac{2}{5}} + 1 \right) + \]

\[ + \|(\Gamma - 1)u^2||_{L^\infty_{[0,T]}C^{1-\delta}} ||\partial_t u||_{L^\infty_{[0,T]}L^2} ||P^{(2n-3)}(u, u^2)||_{L^4_{[0,T]}L^\infty} \|Hu\|_{L^\infty_{[0,T]}\mathcal{H}^{\frac{1}{2}}} + \]

\[ + \|(Hu, (\partial_t u \gtrsim (\Gamma - 1)u^2))\|_{C^{\frac{1}{2}}} \]

\[ \lesssim C(E(u_0)) \left( ||\partial_t u||_{L^\infty_{[0,T]}L^2}^{\frac{7}{4} + \frac{2}{5}} + 1 \right) + ||(*)||_{L^4_{[0,T]}}. \]

Naively, the term (\textasteriskcentered) gives us a bound no better than \( C(u_0)\|\partial_t u\|_{L^\infty_{[0,T]}L^2} \|Hu\|_{L^\infty_{[0,T]}L^2} \sim \|\partial_t u\|_{L^\infty_{[0,T]}L^2}^2 \) which would be enough if we wanted to apply Gronwall, but we are able to deal with it separately to get an improved bound, namely with only powers of \( ||\partial_t u||_{L^\infty L^2} \) strictly smaller than 2 appearing.

By applying Corollary B.3 to this term, we may bound it as

\[ ||(*)||_{L^4_{[0,T]}} \lesssim \|Hu\|_{L^\infty_{[0,T]}\mathcal{H}^{-\frac{3}{4}}} ||\partial_t u||_{L^\infty_{[0,T]}\mathcal{H}^{-\frac{1}{2}}} ||u - u^2||_{L^\infty_{[0,T]}\mathcal{H}^{\frac{1}{2}}} + ||P^{(2n-3)}(u, u^2)||_{L^4_{[0,T]}C^{\frac{1}{2}}} \]

\[ \lesssim C(E(u_0)) \left( ||\partial_t u||_{L^\infty_{[0,T]}L^2}^{\frac{7}{4} + \frac{2}{5} + 2\delta} + 1 \right) \]

This means in (5.24) the term (I) we can replaced Re \( \overline{HTu^2 \Gamma u^2 \partial_i |u|^2} \) by Re \( \overline{HTu^2 \Gamma u^2 \partial_i |u|^2} \) up to an error \( \sim \left( ||\partial_t u||_{L^\infty_{[0,T]}L^2}^{\frac{7}{4} + \frac{2}{5} + 2\delta} + 1 \right) \).

Next, we want to replace Re \( \overline{HTu^2 \Gamma u^2 \partial_i |u|^2} \) by Re \( \overline{\Delta u^2 \Gamma u^2 \partial_i |u|^2} \) in (I). In order to do this, we rewrite it as

\[ \operatorname{Re} \int \overline{HTu^2 \Gamma u^2 \partial_i |u|^2} - \operatorname{Re} \int \overline{\Delta u^2 \Gamma u^2 \partial_i |u|^2} = \]

\[ = \operatorname{Re} \int (\Gamma - \Delta)u^2 \Gamma u^2 \partial_i |u|^2 - \operatorname{Re} \int \overline{\Delta u^2 (\Gamma - 1)u^2 \partial_i |u|^2} \]
now we bound the first of these terms as, using Lemmas 3.3, 5.6 and 5.7

\[
\left\| \text{Re} \int (\hat{H}\Gamma - \Delta)u^2 \Gamma u^2 \partial_t |u^2|^{2n-2} \right\|_{L^4_{[0,T]}} \lesssim \|u\|_{L^4_{[0,T]} H^{s+\eta}} \|\Delta u^2\|_{L^4_{[0,T]} L^2} + \|\text{Re} u^2 \Gamma u^2 \partial_t |u^2|^{2n-2}\|_{L^4_{[0,T]}}
\]

\[
\lesssim \|u\|_{L^4_{[0,T]} H^{s+\eta}} \|\Delta u^2\|_{L^4_{[0,T]} H^{s+\eta}} + \|\text{Re} u^2 \Gamma u^2 \partial_t |u^2|^{2n-2}\|_{L^4_{[0,T]}}
\]

\[
\lesssim C(E(u)) \|\text{Re} u^2 \Gamma u^2 \partial_t |u^2|^{2n-2}\|_{L^4_{[0,T]}} + \|\text{Re} u^2 \Gamma u^2 \partial_t |u^2|^{2n-2}\|_{L^4_{[0,T]}}
\]

\[
\lesssim C(E(u)) \left( 1 + \|\partial_t u\|_{L^2} + \|u\|_{L^2} \right).
\]

and similarly the second one

\[
\left\| \left( \Delta u^2, (\Gamma - 1)u^2 \partial_t |u^2|^{2n-2} \right) \right\|_{L^4_{[0,T]}} \lesssim \left\| (\Delta u^2, (\Gamma - 1)u^2 \partial_t |u^2|^{2n-2}) \right\|_{L^4_{[0,T]}}
\]

\[
\lesssim \|\Delta u^2\|_{L^4_{[0,T]}} + \|\Gamma - 1\|_{L^4_{[0,T]}} + \|\partial_t u^2\|_{L^4_{[0,T]}}
\]

\[
\lesssim C(E(u)) \|\partial_t u\|_{L^4_{[0,T]}} + \|\partial_t u\|_{L^4_{[0,T]}}
\]

\[
\lesssim C(E(u)) \left( 1 + \|\partial_t u\|_{L^2} + \|u\|_{L^2} \right).
\]

So we have successfully replaced the term \((I)\) by the term \(\text{Re} \int \Delta u^2 \Gamma u^2 \partial_t |u^2|^{2n-2}\), which we deal with as in [32]. Indeed, by the Leibnitz rule

\[
\text{Re} \int \Delta u^2 \partial_t |u^2|^{2n-2} = 2 \left( \int \Delta u^2 u^2 + u^2 \Delta u^2 \right) \partial_t |u^2|^{2n-2}
\]

\[
= 2(n - 1) \int \Delta |u^2|^2 - 2|\nabla u^2|^2 \partial_t |u^2|^2 |u^2|^{2n-4}
\]

and the first term we rewrite as

\[
\int \Delta |u^2|^2 \partial_t |u^2|^2 |u^2|^{2n-4} = \int -\nabla |u^2|^2 \partial_t \nabla |u^2|^2 |u^2|^{2n-4} - \int \nabla |u^2|^2 \partial_t |u^2|^2 \nabla |u^2|^{2n-4}
\]

\[
= -\frac{d}{dt} \left( \int \frac{1}{2} |\nabla |u^2|^2|^2 |u^2|^{2n-4} \right) + \frac{(2 - n)}{2} \int |\nabla |u^2|^2|^2 \partial_t |u^2|^2 |u^2|^{2n-4}
\]

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where the first term will be included in the energy and the second one we bound integrated in time as

\[
\left\| \int |\nabla|u^2|^2 \partial_t |u^2|^2 |u^2|^{2n-4} \right\|_{L^2\left[0,T\right]} \leq \left\| \nabla|u^2|^2 \right\|_{L^2\left[0,T\right]} \left\| \partial_t |u^2|^2 \right\|_{L^2\left[0,T\right]} \left\| u^2 \right\|_{L^\infty\left[0,T\right]}^{2n-1} L^\frac{2}{3} \\
\leq C(E(u_0)) \left\| \partial_t u \right\|_{L^2\left[0,T\right]} \left\| \nabla|u^2|^2 \right\|_{L^2\left[0,T\right]} \left\| |u^2|^2 \right\|_{L^\frac{4}{3} L^{1-\delta}} \\
\leq C(E(u_0)) \left\| \partial_t u \right\|_{L^2\left[0,T\right]} \left\| |u^2|^2 \right\|_{L^2\left[0,T\right]} \left\| u^2 \right\|_{L^\frac{4}{3} W^{1+\kappa,4}} \\
\leq C(E(u_0)) \left\| \partial_t u \right\|_{L^2\left[0,T\right]} \left\| |u^2|^2 \right\|_{L^2\left[0,T\right]} \left\| u^2 \right\|_{L^\frac{4}{3} W^{\frac{1}{2}+\kappa,4}} \left\| u^2 \right\|_{L^\frac{4}{3} W^{\frac{1}{2}-\kappa,4}} \\
\leq C(E(u_0)) \left( \left\| \partial_t u \right\|_{L^\infty L^2} + 1 \right)
\]

for small \( 1 \gg \delta, \kappa(\delta), \bar{\kappa}(\kappa) > 0 \).

In total, we define (as usual, \( u = \Gamma u^\delta \))

\[
E^{(1)}(u)(t) := \frac{1}{2} \int |\partial_t u(t)|^2 - \frac{(n-1)}{2} \int |\nabla|u^\delta(t)|^2|^2 |u^\delta(t)|^{2n-2} - \frac{n-1}{2n} \int |u(t)|^{4n}
\]

for which we have for any \( t \in [0,T], \kappa > 0 \)

\[
\left| E^{(1)}(u)(t) - \frac{1}{2} \int |\partial_t u(t)|^2 \right| \leq C(E(u_0))(\left\| |u^2|^2(t) \right\|_{L^2+\kappa} + 1) \leq C(E(u_0))(\left\| u^2(t) \right\|_{H^{1-\kappa}} + 1) \leq C(E(u_0))(\left\| u^2(t) \right\|_{H^{\kappa}} + 1)
\]

meaning that, up to lower order terms, we can treat \( E^{(1)}(u)(t) \) as controlling \( \left\| \partial_t u \right\|_{L_\infty L^2} \).

Collecting all the previous computations, we have for \( s < t \) with \( |s-t| \leq 1 \) since the Strichartz estimates only hold in short intervals, we have

\[
\left| \int |\partial_t u(t)|^2 - \int |\partial_t u(s)|^2 \right| \leq |E^{(1)}(u)(t) - E^{(1)}(u)(s)| + C(E(u_0)) \left\| \partial_t u \right\|_{L^\infty L^2}^{4n} \left[ t - s \right]^{rac{1}{4}} \left( 1 + \left\| \partial_t u \right\|_{L^\infty L^2}^{rac{7}{2}+\kappa} \right) \left( 1 + \left\| \partial_t u \right\|_{L^\infty L^2}^{rac{7}{2}+\kappa} \right).
\]

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We also get
\[
\sup_{\tau \in [s,t]} \int |\partial_t u(\tau)|^2 \leq \int |\partial_t u(s)|^2 + C(E(u_0)) \left( |t-s|^{\frac{1}{2}} + 1 + \| \partial_t u \|_{L^2_{[s,t]}}^{2+ \kappa} \right) + \| \partial_t u \|_{L^2_{[s,t]}}^{4\kappa}
\]
\[
\leq |\partial_t u(s)|^2 + \left. \left( 1 + \| \partial_t u \|_{L^2_{[s,t]}}^{2+ \kappa} \right) + C(E(u_0)) \| \partial_t u \|_{L^2_{[s,t]}}^{4\kappa} \right)
\]
for \(|t-s| = \left( \frac{1}{1+C(E(u_0))} \right)^2 =: \tau_0 < 1\).

To conclude, for a general \(T > 0\) we fix \(N = \lceil \tau_0 T \rceil\) and thus we have
\[
\sup_{\tau \in [0,T]} \int |\partial_t u(\tau)|^2 = \max_{i=1,...,N} \sup_{\tau \in \left[ \frac{i}{N} T, \frac{i+1}{N} T \right]} \int |\partial_t u(\tau)|^2
\]
\[
\leq \int |\partial_t u_0|^2 + N \left( 1 + \| \partial_t u \|_{L^2_{[0,T]}}^{2+\kappa} + C(E(u_0)) \| \partial_t u \|_{L^2_{[0,T]}}^{4\kappa} \right)
\]
\[
\leq \int |\partial_t u_0|^2 + C(E(u_0)) N^{1+\hat{\kappa}} + C N^{8+\hat{\kappa}} + \frac{1}{2} \| \partial_t u \|_{L^2_{[0,T]}}^{4\kappa}
\]
for \(\hat{\kappa}(\kappa) > 0\) as small as we want. This finally implies
\[
\sup_{\tau \in [0,T]} \int |\partial_t u(\tau)|^2 \lesssim E(1)(u_0) + C(E(u_0)) + T^{8+\hat{\kappa}}
\]
so we have polynomial growth as claimed. \(\square\)

A Paracontrolled Distributions and Besov spaces

We collect some elementary results about paraproducts, see [18], [2], [3] for more details. We work on the \(d\)–dimensional torus \(\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d\) in the current paper always \(d = 2\).

The Sobolev space \(\mathcal{H}^\alpha(\mathbb{T}^d)\) with index \(\alpha \in \mathbb{R}\) is defined as
\[
\mathcal{H}^\alpha(\mathbb{T}^d) := \left\{ u \in \mathcal{S}'(\mathbb{T}^d) : \left\| (1-\Delta)^{\frac{\alpha}{2}} u \right\|_{L^2} < \infty \right\}.
\]

Next, we recall the definition of Littlewood-Paley blocks. We denote by \(\chi\) and \(\rho\) two non-negative smooth and compactly supported radial functions \(\mathbb{R}^d \to \mathbb{C}\) such that

i. The support of \(\chi\) is contained in a ball and the support of \(\rho\) is contained in an annulus \(\{ x \in \mathbb{R}^d : a \leq |x| \leq b \}\).

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ii. For all $\xi \in \mathbb{R}^d$, $\chi(\xi) + \sum_{j \geq 0} \rho(2^{-j} \xi) = 1$;

iii. For $j \geq 1$, $\chi(\cdot) \rho(2^{-j} \cdot) = 0$ and $\rho(2^{-j} \cdot) \rho(2^{-i} \cdot) = 0$ for $|i - j| > 1$.

The Littlewood-Paley blocks $(\Delta_j)_{j \geq -1}$ associated to $f \in S'(\mathbb{T}^d)$ are defined by

$$\Delta_{-1} f := \mathcal{F}^{-1} \chi \mathcal{F} f \quad \text{and} \quad \Delta_j f := \mathcal{F}^{-1} \rho(2^{-j} \cdot) \mathcal{F} f \quad \text{for} \ j \geq 0.$$  

We also set, for $f \in S'(\mathbb{T}^d)$ and $j \geq -1$

$$S_j f := \sum_{i=-1}^{j-1} \Delta_i f.$$  

Then the Besov space with parameters $p, q \in [1, \infty)$, $\alpha \in \mathbb{R}$ can now be defined as

$$B_{p,q}^\alpha(\mathbb{T}^d) := \{ u \in S'(\mathbb{T}^d) : \|u\|_{B_{p,q}^\alpha} < \infty\},$$

where the norm is defined as

$$\|u\|_{B_{p,q}^\alpha} := \left( \sum_{k \geq -1} \left( 2^{\alpha k} \|\Delta_k u\|_{L^p} \right)^q \right)^{\frac{1}{q}},$$

with the obvious modification for $q = \infty$. We also define the Besov-H"older spaces

$$C^\alpha := B_{\infty,\infty}^\alpha$$

which for $\alpha \in (0,1)$ agree with the usual Hölder spaces $C^\alpha$.

Using this notation, we can formally decompose the product $f \cdot g$ of two distributions $f$ and $g$ as

$$f \cdot g = f \prec g + f \circ g + f \succ g,$$

where

$$f \prec g := \sum_{j \geq -1} S_{j-1} f \Delta_j g \quad \text{and} \quad f \succ g := \sum_{j \geq -1} \Delta_j f S_{j-1} g$$

are referred to as the paraproducts, whereas

$$f \circ g := \sum_{j \geq -1} \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g$$

is called the resonant product. An important point is that the paraproduct terms are always well defined whatever the regularity of $f$ and $g$. The resonant product, on the other hand, is a priori only well defined if the sum of their regularities is positive. We collect some results.
Lemma A.1 [cf. Theorem 3.17 [27]] Let \( \alpha, \alpha_1, \alpha_2 \in \mathbb{R} \) and \( p, p_1, p_2, q \in [1, \infty] \) be such that
\[
\alpha_1 \neq 0 \quad \alpha = (\alpha_1 \wedge 0) + \alpha_2 \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.
\]
Then we have the bound
\[
\| f \prec g \|_{B_{p,q}^{\alpha}} \lesssim \| f \|_{B_{p_1,\infty}^{\alpha_1}} \| g \|_{B_{p_2,q}^{\alpha_2}}
\]
and in the case where \( \alpha_1 + \alpha_2 > 0 \) we have the bound
\[
\| f \circ g \|_{B_{p,q}^{\alpha_1+\alpha_2}} \lesssim \| f \|_{B_{p_1,\infty}^{\alpha_1}} \| g \|_{B_{p_2,q}^{\alpha_2}}.
\]

Remark A.2 For the majority of the paper we care only about the case where \( p = p_2 = q = 2 \) and \( p_1 = \infty \).

Lemma A.3 [Bernstein’s inequality] Let \( A \) be an annulus and \( B \) be a ball. For any \( k \in \mathbb{N}, \lambda > 0, \) and \( 1 \leq p \leq q \leq \infty \) we have
1. if \( u \in L^p(\mathbb{T}^d) \) is such that \( \text{supp}(\mathcal{F}u) \subset \lambda B \) then
\[
\max_{\mu \in \mathbb{N}^d: |\mu| = k} \| \partial^\mu u \|_{L^q} \lesssim_k \lambda^{k+d\left(\frac{1}{p} - \frac{1}{q}\right)} \| u \|_{L^p}
\]
2. if \( u \in L^p(\mathbb{T}^d) \) is such that \( \text{supp}(\mathcal{F}u) \subset \lambda A \) then
\[
\lambda^k \| u \|_{L^p} \lesssim_k \max_{\mu \in \mathbb{N}^d: |\mu| = k} \| \partial^\mu u \|_{L^p}.
\]

Lemma A.4 [Besov embedding] Let \( \alpha < \beta \in \mathbb{R}, q_1 \leq q_2, \) and \( p > r \in [1, \infty] \) be such that
\[
\beta = \alpha + d \left( \frac{1}{r} - \frac{1}{p} \right),
\]
then we have the following bound
\[
\| f \|_{B_{p,q}^\alpha(\mathbb{T}^d)} \lesssim \| f \|_{B_{r,q_1}^\beta(\mathbb{T}^d)}.
\]

Proposition A.5 [Commutator Lemma, Proposition 4.3 in [2]]
Given \( \alpha \in (0,1), \beta, \gamma \in \mathbb{R} \) such that \( \beta + \gamma < 0 \) and \( \alpha + \beta + \gamma > 0 \), the following trilinear operator \( C \) defined for any smooth functions \( f, g, h \) by
\[
C(f, g, h) := (f \prec g) \circ h - f(g \circ h)
\]
can be extended continuously to the product space \( \mathcal{H}^{\alpha} \times C^\beta \times C^\gamma \). Moreover, we have the following bound
\[
\| C(f, g, h) \|_{\mathcal{H}^{\alpha+\beta+\gamma-\delta}} \lesssim \| f \|_{\mathcal{H}^{\alpha}} \| g \|_{C^\beta} \| h \|_{C^\gamma}
\]
for all \( f \in \mathcal{H}^{\alpha}, g \in C^\beta \) and \( h \in C^\gamma \), and every \( \delta > 0 \).
Lemma A.6 [Fractional Leibniz, 22] Let $1 < p < \infty$ and $p_1, p_2, p'_1, p'_2$ such that

$$
\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p'_1} + \frac{1}{p'_2} = \frac{1}{p}.
$$

Then for any $s, \alpha \geq 0$ there exists a constant s.t.

$$
\|\langle \nabla \rangle^s (fg)\|_{L^p} \leq C\|\langle \nabla \rangle^{s+\alpha} f\|_{L^{p_2}} \|\langle \nabla \rangle^{-\alpha} g\|_{L^{p_1}} + C\|\langle \nabla \rangle^{-\alpha} f\|_{L^{p'_2}} \|\langle \nabla \rangle^{s+\alpha} g\|_{L^{p'_1}}.
$$

Lemma A.7 [Tame estimate, Corollary 2.86 in 3] For any $s > 0$ and $(p, r) \in [1, \infty]^2$, the space $B^s_{p,r} \cap L^\infty$ is an algebra and the bound

$$
\|u \cdot v\|_{B^s_{p,q}} \lesssim \|u\|_{B^s_{p,q}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{B^s_{p,q}}
$$

holds.

B Almost adjointness lemmas for paraproducts

Lemma B.1 [cf appendix of 21] We have

$$
D(f, g, h) := (f, g \circ h) - (f \ast g, h)
$$

defined on smooth functions, extends to a bounded map

$$
B^\alpha_{p_1 \infty} \times B^\beta_{p_2} \times B^\gamma_{p_3} \to \mathbb{R}
$$

for $1 = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$ and $\alpha + \beta + \gamma = 0$ and $\beta + \gamma < 0$.

Proof We use the orthogonality and self-adjointness of Littlewood-Paley blocks to get

$$
D(f, g, h) = \sum_{i \sim j \sim k} (\Delta_i f, \Delta_j g \Delta_k h) - \sum_{i \leq j \sim k} (\Delta_i f \Delta_j g, \Delta_k h)
$$

$$
= \sum_{i \sim j \sim k} (\Delta_i f \Delta_j g, \Delta_k h) - \sum_{i \leq j \sim k} (\Delta_i f \Delta_j g, \Delta_k h)
$$

$$
= \sum_{i \sim j \sim k} (\Delta_i f \Delta_j g, \Delta_k h) - \sum_{i \leq j \sim k} (\Delta_i f \Delta_j g, \Delta_k h)
$$

$$
= \sum_{i \geq j \sim k} (\Delta_i f \Delta_j g, \Delta_k h)
$$

$$
\leq \sum_{i \geq j \sim k} \|\Delta_i f\|_{L^{p_1}} \|\Delta_j g\|_{L^{p_2}} \|\Delta_k h\|_{L^{p_3}}
$$
by Hölder’s inequality. Now using $\alpha + \beta + \gamma = 0$ and $\alpha > 0$ and $\beta + \gamma < 0$ we have

$$D(f, g, h) \leq \sum_{i \leq j < k} 2^{i(\alpha+\beta+\gamma)} \| \Delta_i f \|_{L^p_1} \| \Delta_j g \|_{L^p_2} \| \Delta_k h \|_{L^p_3}$$

$$\lesssim \left\| 2^\alpha \| \Delta_i f \|_{L^p_1} \right\|_{L^\infty} \sum_{i \leq j < k} 2^{i(\beta+\gamma)} \| \Delta_j g \|_{L^p_2} \| \Delta_k h \|_{L^p_3}$$

$$\lesssim \left\| f \right\|_{B^\alpha_{p_1} \infty} \sum_{j < k} 2^{j\beta} \| \Delta_j g \|_{L^p_2} 2^{k\gamma} \| \Delta_k h \|_{L^p_3}$$

$$\lesssim \left\| \Delta_j g \right\|_{B^\beta_{p_2} \infty} \| \Delta_k h \|_{B^\gamma_{p_3} 2} \lesssim \| f \|_{B^\alpha_{p_1} \infty} \| g \|_{B^\beta_{p_2} \infty} \| h \|_{B^\gamma_{p_3} 2}$$

having bounded $\sum_{i \geq j} 2^{-i\sigma} \sim 2^{-j\sigma}$ for $\sigma > 0$ to remove the sum in $i$.

This concludes the proof. \qed

Next we have a similar bound in spirit involving iterated para- and resonant products. We define the triple resonant product

$$\Pi(g, z, h) := \sum_{j \sim k \sim l} \Delta_j g \Delta_k z \Delta_l h. \quad (B.1)$$

Lemma B.2 The map

$$D^{(2)}(f, g, h, z) := (f, \Pi(g, z, h)) - ((f \prec g) \prec z, h)$$

defined for smooth functions, extends to a bounded map

$$B^\alpha_{p_1} \infty \times B^\beta_{p_2} \infty \times B^\gamma_{p_3} 2 \times B^\delta_{p_4} 2 \to \mathbb{R}$$

for $1 = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}$ and $\alpha + \gamma + \delta \geq 0$, $\beta + \gamma + \delta > 0$ and $\gamma + \delta < 0$

Proof We have (using the product property $\text{supp} \hat{\Delta_i \cdot \Delta_j} \subset \bigcup_{k \sim i, j} \text{supp} \hat{\Delta_k}$ and the ortho-
\[ D^{(2)}(f, g, h, z) = \sum_{j \sim k \sim l} (\Delta_i f, \Delta_j g, \Delta_k z, \Delta_l h) - \sum_{i \leq j \leq k \leq l} (\Delta_i f, \Delta_j g, \Delta_k z, \Delta_l h) \]

\[ = \sum_{j \sim k \sim l} (\Delta_i f, \Delta_j g, \Delta_k z, \Delta_l h) - \sum_{i \leq j \leq l \leq k} (\Delta_i f, \Delta_j g, \Delta_k z, \Delta_l h) \]

\[ = \sum_{i \leq j \leq l \leq k} (\Delta_i f, \Delta_j g, \Delta_k z, \Delta_l h) \]

\[ \leq \sum_{i \geq j \sim k \sim l} \| \Delta_i f \|_{L^p} \| \Delta_j g \|_{L^p} \| \Delta_k z \|_{L^p} \| \Delta_l h \|_{L^p} \]

using Hölder’s inequality in the final step. Then we bound, using \( \alpha + \gamma + \delta \geq 0, \beta \geq 0 \) and \( \gamma + \delta < 0 \)

\[ D^{(2)}(f, g, h, z) \leq \sum_{i \geq j \sim k \sim l} 2^{i(\alpha + \gamma + \delta)} \| \Delta_i f \|_{L^p} 2^{j\beta} \| \Delta_j g \|_{L^p} 2^{k\gamma} \| \Delta_k z \|_{L^p} 2^{l\delta} \| \Delta_l h \|_{L^p} \]

\[ \leq \| f \|_{B^\alpha_{p_1} \infty} \sum_{j \sim k \sim l} 2^{j(\beta + \gamma + \delta)} \| \Delta_j g \|_{L^p} \| \Delta_k z \|_{L^p} \| \Delta_l h \|_{L^p} \]

\[ \leq \| f \|_{B^\alpha_{p_1} \infty} \sum_{j \sim k \sim l} 2^{j\beta} \| \Delta_j g \|_{L^p} 2^{k\gamma} \| \Delta_k z \|_{L^p} 2^{l\delta} \| \Delta_l h \|_{L^p} \]

\[ \leq \| f \|_{B^\alpha_{p_1} \infty} \| g \|_{B^\beta_{p_2} \infty} \| z \|_{B^\gamma_{p_3} \infty} \| h \|_{B^\delta_{p_4} \infty} \]

having used the geometric series \( \sum_{i \geq j} 2^{-i\sigma} \sim 2^{-j\sigma} \) for \( \sigma > 0 \) to get rid of the sum in \( i \) and Hölder’s inequality for the other sums.

This result is different from the previous one, since the first lemma has a resonant product that is ill-defined by itself but by duality can be defined. In this second result, it is rather the converse, that an ill-defined dual pairing can be defined by using that one term has a paraproduct structure and then defining it by duality.
Corollary B.3 We may bound

\[ |((f \prec g) \prec z, h)| \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta} \|z\|_{H^\gamma} \|h\|_{H^\delta} \] (B.2)

for \( \alpha + \gamma + \delta > 0, \beta + \gamma + \delta > 0 \) and \( \gamma + \delta < 0 \).

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