An alternative approach to determining average distance in a class of scale-free modular networks

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Abstract. Various real-life networks of current interest are simultaneously scale-free and modular. Here we study analytically the average distance in a class of deterministically growing scale-free modular networks. By virtue of the recursive relations derived from the self-similar structure of the networks, we compute rigorously this important quantity, obtaining an explicit closed-form solution, which recovers the previous result and is corroborated by extensive numerical calculations. The obtained exact expression shows that the average distance scales logarithmically with the number of nodes in the networks, indicating an existence of small-world behavior. We present that this small-world phenomenon comes from the peculiar architecture of the network family.

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1. Introduction

Average distance is one of the most important measurements characterizing complex networks, which is a subject attracting a lot of interest in the recent physics literature [1, 2, 3, 4]. Extensive empirical studies showed that many, perhaps most, real networks exhibit remarkable small-world phenomenon [5], with their average distance grows as a function of network order (i.e., number of nodes in a network), or slowly [1, 2]. As a fundamental topological property, average distance is closely related to other structural characteristics, such as degree distribution [6, 7], centrality [8], fractality [9, 10, 11], symmetry [12], and so forth. All these features together play significant roles in characterizing and understanding the complexity of networks. Moreover, average distance is relevant to various dynamical processes occurring on complex networks, including epidemic spreading [5], target search [13], synchronization [14], random walks [15, 16, 17], and many more.

In addition to the small-world behavior, other two prominent properties that seem to be common to real networks, especially biological and social networks, are scale-free feature [18] and modular structure [19, 20, 21]. The former implies that the networks obey a power-law degree distribution as \( P(k) \sim k^{-\gamma} \) with \( 2 < \gamma \leq 3 \), while the latter means that the networks can be divided into groups (modules), within which nodes are more tightly connected with each other than with nodes outside. In order to describe simultaneously the two striking properties, Ravasz and Barabási (RB) presented a famous model [21], mimicking scale-free modular networks. Many topological properties of and dynamical processes on the RB model have been investigated in much detail, including degree distribution [21], clustering coefficient [21, 22], betweenness centrality distribution [22], community structure [23], random walks [24, 25], among others. Particularly, by mapping the networks onto a Potts model in one-dimensional lattices, Noh proved that the RB model is small-world [22].

In this paper, we study the average distance in the RB model by using an alternative approach very different from the previous one [22]. Our computation method is based on the particular deterministic construction of the RB model. Concretely, making use of the self-similar structure of the scale-free modular networks, we establish some recursion relations, from which we further derive the exactly analytical solution to the average distance. Our obtained rigorous expression is compatible with the previous formula. We show that the RB model is small-world. We also show that the small-world behavior is a natural result of the scale-free and modular architecture of the networks under consideration.

2. The modular scale-free networks

We first introduce the RB model for the scale-free modular networks, which are built in an iterative way [20, 21]. Let \( H_g \) stand for the network model after \( g \) (\( g \geq 1 \)) iterations (i.e., number of generations). Initially \( (g = 1) \), the model is composed \( m \) (\( m \geq 3 \)) nodes
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A sketch of a network $H_3$ for the case of $m = 4$. The filled squares and circles represent the hub node and peripheral nodes, respectively. Linked by $m(m - 1)/2$ edges forming a complete graph, among which a node (e.g., the central node in figure 1) is called hub (or root) node, and the other $m - 1$ nodes are named peripheral nodes. At the second generation ($g = 2$), $m - 1$ replicas of $H_1$ are created with the $m - 1$ peripheral nodes of each copy being connected to the root of the original $H_1$. In this way, we obtained $H_2$, the hub and peripheral nodes of which are the hub of the original $H_1$ and the $(m - 1)^2$ peripheral nodes in the $m - 1$ duplicates of $H_1$, respectively. Suppose one has $H_{g-1}$, the next generation network $H_g$ can be obtained by adding $M - 1$ copies of $H_{g-1}$ to the primal $H_{g-1}$, with all peripheral nodes of the replicas being linked to the hub of the original $H_{g-1}$ unit. The hub of the original $H_{g-1}$ and the peripheral nodes of the $m - 1$ copies of $H_{g-1}$ form the hub node and peripheral nodes of $H_g$, respectively. Repeating indefinitely the two steps of replication and connection, one obtains the scale-free modular networks. Figure 1 illustrates a network $H_3$ for the particular case of $m = 4$.

Many interesting quantities of the model can be determined explicitly [21, 22]. In $H_g$, the network order, denoted by $N_g$ is $N_g = m^g$; the degree $K_h(g) = \frac{m-1}{m-2}[(m-1)^g - 1]$ of the hub node is the largest among all nodes; the number of peripheral nodes, forming a set $P_g$, is $P_g = (m - 1)^g$; and the average degree is approximately equal to a constant $2(m - 1)(3m - 2)/m$ in the limit of infinite $g$, showing that the networks are sparse.

The model under consideration is in fact an extension of the one proposed in [26] and studied in much detail in [27, 28, 29]. It presents some typical features observed in a variety of real-world systems [21, 22]. Its degree distribution follows a power-law scaling $P(k) \sim k^{-\gamma}$ with a general exponent $\gamma = 1 + \ln m / \ln(m - 1)$ belonging to the interval $(2, 2.585)$. Its average clustering coefficient tends to a large constant dependent on $m$;
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and its average distance grows logarithmically with the network order, both of which show that the model is small-world. In addition, the betweenness distribution $P_B$ of nodes also obeys the power-law behavior $P_B \sim B^{-2}$ with the exponent regardless of the parameter $m$. Particularly, the whole class of the networks shows a remarkable modular structure. These peculiar structural properties make the networks unique within the category of complex networks.

3. Explicit formula for average distance

As shown in the introduction section, average distance is closely related to many topological properties of and various dynamical processes on complex networks. In what follows, we will derive analytically the average distance of the scale-free modular networks by applying an alternative method completely different from that in [22]. We represent all the shortest path lengths of network $H_g$ as a matrix in which the entry $d_{ij}(g)$ is the distance between nodes $i$ and $j$ that is the length of a shortest path joining $i$ and $j$. A measure of the typical separation between two nodes in $H_g$ is given by the average distance $d_g$ defined as the mean of distances over all pairs of nodes:

$$d_g = \frac{D_g}{N_g(N_g - 1)/2},$$

where

$$D_g = \sum_{i \in H_g, j \in H_g, i \neq j} d_{ij}(g)$$

denotes the sum of the distances between two nodes over all couples. Notice that in Eq. (2), for a pair of nodes $i$ and $j$ ($i \neq j$), we only count $d_{ij}(g)$ or $d_{ji}(g)$, not both.

We continue by exhibiting the procedure of determining the total distance and present the recurrence formula, which allows us to obtain $D_{g+1}$ of the $g + 1$ generation from $D_g$ of the $g$ generation. The studied network $H_g$ has a self-similar structure that allows one to calculate $D_g$ analytically. By construction (see figure 2), network $H_{g+1}$ is obtained by joining $m$ copies of $H_g$ that are labeled as $H_g^{(1)}, H_g^{(2)}, \cdots, H_g^{(m)}$. Using this self-similar property, the total distance $D_{g+1}$ satisfies the recursion relation

$$D_{g+1} = m D_g + \Delta_g,$$

where $\Delta_g$ is the sum over all shortest path length whose endpoints are not in the same $H_g^{(\rho)}$ branch. The paths that contribute to $\Delta_g$ must all go through the hub node $X$, where the $m$ copies of $H_g$ are connected. Hence, to determine $D_g$, all that is left is to calculate $\Delta_g$. The analytic expression for $\Delta_g$, referred to as the crossing path length [30, 31], can be derived as below.

Let $\Delta_g^{(a,\beta)}$ be the sum of the lengths of all shortest paths whose endpoints are in $H_g^{(a)}$ and $H_g^{(\beta)}$, respectively. According to whether the two branches are one link long or two links long, we split the crossing paths $\Delta_g^{(a,\beta)}$ into two categories: the first category composes of crossing paths $\Delta_g^{(1,\theta)}$ ($2 \leq \theta \leq m$), while the second category consists of crossing paths $\Delta_g^{(\theta_1,\theta_2)}$ with $2 \leq \theta_1 \leq m$, $2 \leq \theta_2 \leq m$, and $\theta_1 \neq \theta_2$. It is easy to see that
The numbers of the two categories of crossing paths are $m - 1$ and $(m - 1)(m - 2)/2$, respectively. Moreover, any two crossing paths in the same category have the same length. Thus, the total sum $\Delta_g$ is given by

$$\Delta_g = (m - 1)\Delta_g^{(1,2)} + \frac{(m - 1)(m - 2)}{2}\Delta_g^{(2,3)}.$$  (4)

Having $\Delta_g$ in terms of the quantities of $\Delta_g^{(1,2)}$ and $\Delta_g^{(2,3)}$, the next step is to explicitly determine the two quantities.

To calculate the crossing distance $\Delta_g^{(1,2)}$ and $\Delta_g^{(2,3)}$, we give the following notation. For an arbitrary node $v$ in network $H_g$, let $f_v(g)$ be the smallest value of the shortest path length from $v$ to any of the $(m - 1)^g$ peripheral nodes belonging to $P_g$, and the sum of $f_v(g)$ for all nodes in $H_g$ is denoted by $F_g$. Analogously, in $H_g$ let $h_v(g)$ denote the distance from a node $v$ to the hub node $X$, and let $M_g$ stand for the total distance between all nodes in $H_g$ and the hub node $X$ in $H_g$, including $X$ itself. By definition, $F_{g+1}$ can be given by the sum

$$F_{g+1} = \sum_{v \in H_g^{(1)}} f_v(g + 1) + \sum_{\eta = 2}^{m} \sum_{v \in H_g^{(\eta)}} f_v(g + 1)$$

$$= \sum_{v \in H_g} [h_v(g) + 1] + (m - 1) \sum_{v \in H_g} f_v(g)$$

$$= (m - 1)F_g + N_g + M_g,$$  (5)
An alternative approach to determining average distance in a class of scale-free modular networks and $M_{g+1}$ can be written recursively as

$$M_{g+1} = \sum_{v \in H_g^{(1)}} h_v(g + 1) + \sum_{v \in H_g^{(2)}} h_v(g + 1)$$

$$= \sum_{v \in H_g} h_v(g) + (m - 1) \sum_{v \in H_g} [f_v(g) + 1]$$

$$= M_g + (m - 1)(F_g + N_g). \quad (6)$$

Using $N_g = m^g$, and considering $F_1 = 1$ and $M_1 = m - 1$, the simultaneous equations (5) and (6) can be solved inductively to obtain:

$$F_g = m^g - 2 \left(2^g - 1\right)m - 2g + 2 \quad (7)$$

and

$$M_g = m^g - 2(m - 1)(m + 2g - 2). \quad (8)$$

With above obtained results, we can determine $\Delta_{g}^{(1,2)}$ and $\Delta_{g}^{(2,3)}$, which can be expressed in terms of these explicitly determined quantities. By definition, $\Delta_{g}^{(1,2)}$ is given by the sum

$$\Delta_{g}^{(1,2)} = \sum_{u \in H_g^{(1)}, v \in H_g^{(2)}} d_{uv}(g + 1)$$

$$= \sum_{u \in H_g^{(1)}, v \in H_g^{(2)}} \left[h_u(g) + 1 + f_v(g)\right]$$

$$= \sum_{v \in H_g^{(2)}} \sum_{u \in H_g^{(1)}} h_u(g) + \sum_{u \in H_g^{(1)}} \sum_{v \in H_g^{(2)}} [1 + f_v(g)]$$

$$= N_g M_g + (N_g)^2 + N_g F_g. \quad (9)$$

Inserting Eqs. (7) and (8) into (9), we have

$$\Delta_{g}^{(1,2)} = 2 m^{2g-4} \left[m^2 + 2(g - 2)m - 2g + 4\right]. \quad (10)$$

Proceeding similarly,

$$\Delta_{g}^{(2,3)} = \sum_{u \in H_g^{(2)}, v \in H_g^{(3)}} d_{uv}(g + 1)$$

$$= 2 \left[(N_g)^2 + N_g F_g\right]$$

$$= 2 m^{2g-4} \left[m^2 + (2g - 3)m - 2g + 4\right]. \quad (11)$$

Substituting Eqs. (10) and (11) into (4), we get

$$\Delta_g = m^{2g-3}(m - 1)^2(m + 2g - 2). \quad (12)$$

Substituting Eq. (12) into (2) and using the initial value $D_1 = m(m - 1)/2$, we can obtain the exact expression for the total distance

$$D_g = \frac{1}{2} m^{g-2} \left[4m + m^2 - m^3 + 2m^{g+2} - 4g m^g + (4g - 6) m^{g+1}\right]. \quad (13)$$
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Figure 3. Average distance $d_g$ versus network order $N_g$ on a semi-logarithmic scale. The solid lines are guides to the eye.

The expression provided by Eq. (13) is consistent with the result previously obtained [22]. Then the analytic expression for average distance can be obtained as

$$d_g = \frac{4m + m^2 - m^3 + 2m^{g+2} - 4m^{g} + (4g - 6)m^{g+1}}{m^2(m^{g} - 1)}. \quad (14)$$

We have also checked our rigorous result provided by Eq. (14) against numerical calculations for different $m$ and various $g$. In all the cases we obtain a complete agreement between our theoretical formula and the results of numerical investigation, see figure 3.

We continue to express the average distance $d_g$ as a function of network order $N_g$, in order to obtain the scaling between these two quantities. Recalling that $N_g = m^g$, we have $g = \log_{m} N_g$. Hence Eq. (14) can be rewritten as

$$d_g = \frac{4m + m^2 - m^3 + 4(m - 1)N_g \log_{m} N_g + (2m^2 - 6m)N_g}{m^2(N_g - 1)}$$

$$\quad = \frac{4(m - 1)N_g \ln N_g + (2m^2 - 6m)N_g \ln m + (4m + m^2 - m^3) \ln m}{m^2 \ln m (N_g - 1)}. \quad (15)$$

In the infinite network order limit, i.e., $N_g \rightarrow \infty$

$$d_g = \frac{4(m - 1)}{m^2 \ln m} \ln N_g + 2 - \frac{6}{m}. \quad (16)$$

Thus, for large networks, the leading behavior of average distance grows logarithmically with increasing network order.

The above observed small-world phenomenon that the leading behavior of average distance is a logarithmic function of network order can be accounted for by the following heuristic arguments based on the peculiar architecture of the networks. At first sight, this family of modular networks is not a very compact system, since in these networks, nodes with large degrees are not directly linked to one another, but connected to those...
nodes with small degree. However, this network family is made up of many small densely interconnected clusters, which combine to form larger but less compact groups connected by nodes with relatively high degrees. For node pairs in a small group, their shortest path length is very small because of the high cohesiveness of small modules. For the length of shortest paths between two nodes belonging to different large groups, it seems long because the groups that the nodes lie at are not adjacent to each other. But this is not the fact. By construction, although the relatively large groups are not directly adjacent, they are joined by some large nodes, which are connected to each other by a layer of intermediate small-degree nodes (see figure 1), such as the peripheral nodes or locally peripheral nodes [22]. Thus, different from conventional random scale-free networks, especially assortative networks [32], in the studied scale-free modular networks, although large-degree nodes are not connected to one another, they play the role of bridges linking different modules together, which is the main reason why the average distance of the networks is small.

It deserves to be mentioned that, although the studied modular scale-free networks display small-world behavior, the logarithmic scaling of average distance with respect to network order is different from the sublogarithmic scaling for conventional non-modular stochastic scale-free networks with degree distribution exponent $\gamma < 3$, in which the average distance $d(N)$ behaves as a double logarithmic scaling with network order $N$, namely, $d(N) \sim \ln \ln N$ [6, 7]. Thus, despite that the degree distribution exponent of the modular scale-free networks is smaller than 3, their average distance is larger than that of their random counterparts with the same network order. The root of this difference may also lie with the modular structure, particularly the indirect connection of large nodes, as addressed above. The genuine reasons for this dissimilarity need further studies in the future.

4. Conclusions

The determination and analysis of average distance is important to understand the complexity of and dynamic processes on complex networks, which has been a subject of considerable interest within the physics community. In this paper, we investigated analytically the average distance in a class of deterministically growing networks with scale-free behavior and modular structure, which exist simultaneously in a plethora of real-life networks, such as social and biological networks. Based on the self-similar structure of the networks, we derived the closed-form expression for the average distance. The obtained exact solution shows that for very large networks, they are small-world with their average distance increasing as a logarithmic function of network order. We confirmed the rigorous solution by using extensive numerical simulations. We also showed that the small-world behavior lies with the inherent modularity and scale-free property of the networks.
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