Critical behavior of the 3-state Potts model on Sierpinski carpet

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Abstract

We study the critical behavior of the 3-state Potts model, where the spins are located at the centers of the occupied squares of the deterministic Sierpinski carpet. A finite-size scaling analysis is performed from Monte Carlo simulations, for a Hausdorff dimension $d_f \simeq 1.8928$. The phase transition is shown to be a second order one. The maxima of the susceptibility of the order parameter follow a power law in a very reliable way, which enables us to calculate the ratio of the exponents $\gamma/\nu$. We find that the scaling corrections affect the behavior of most of the thermodynamical quantities. However, the sequence of intersection points extracted from the Binder’s cumulant provides bounds for the critical temperature. We are able to give the bounds for the exponent $1/\nu$ as well as for the ratio of the exponents $\beta/\nu$, which are compatible with the results calculated from the hyperscaling relation.

Keywords: phase transitions, fractal, Potts model, finite size scaling, Monte Carlo.

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1 Introduction

Critical phenomena on fractal structures have been first studied by Gefen \textit{et al.} \cite{1}. Since then, most of the works on this topic have been done in the framework of the Ising model. Recently, the use of cluster algorithms and histogram methods enabled two groups \cite{2, 3, 4, 5} to study the ferromagnetic Ising transition on the fractal lattices by Monte Carlo simulation much more thoroughly than before; several fractal dimensions between
1 and 3 have been investigated. These last results show that the scaling corrections can strongly affect the finite size behavior of some thermodynamical quantities, especially when the fractal dimension is lowered from 2 to 1; the critical exponents cannot always be calculated, but bounds can be provided. The available values and bounds of the exponents are consistent with the hyperscaling relation provided that the space dimension is replaced by the Hausdorff one. Furthermore, they do not agree with $\epsilon$-expansion results [6]. The critical behavior of fractals is said to be understood in the framework of weak universality [1, 4, 7]: the critical exponents do not only depend upon the symmetry of the order parameter, the interaction range, and the fractal dimension, but also upon geometrical features of the fractal structure.

The Potts model is a generalization of the Ising model obtained by varying the number of spin states from 2 to any non zero value $q$. Since the number of states $q$ is related to the symmetrical properties of the order parameter, the critical behavior of the Potts model depends upon the value of this additional variable [8]. One of the most striking results is the effect of $q$ on the order of the transition as a function of the space dimension $d$: In the case of regular translational invariant lattices, it has been shown that the ferromagnetic phase transition is a second order one if $q$ is smaller than a value $q_c(d)$ and a first order one if $q > q_c(d)$. For instance, the phase transition in the bidimensional case is first order for $q > 4$ and second order for $q \leq 4$; the value of $q_c(3)$ is smaller than 3. Moreover, a wide variety of effects occur in the presence of disorder: bond randomness can induce a second order transition from a system exhibiting a first order one [9]. The order of the transition can be changed to a second one if strong enough aperiodic fluctuations are introduced [10].

The question arises whether the values $q_c(d)$ make sense in non integer dimensions $d_f$ or not. Fractals are natural candidates to interpolate between integer space dimensions. Since translation invariance is broken in these structures, disorder is introduced: the lattice is strongly inhomogeneous. Thus, two among the three parameters driving the critical behavior of the Potts model (structured fractal disorder and dimensionality) are closely linked in the present case of fractals. The critical properties of the Potts model on fractal structures have never been studied before, excepted by Bin [11] who generalizes the Migdal-Kadanoff bond-moving renormalization scheme on the Sierpinski carpet from the Ising model [12] to the Potts one. No Monte Carlo results are available, up to now; there are two main reasons why studies of the critical behaviors on fractal structures is a difficult task:

(i) Monte Carlo simulations come up against critical slowing down. Fortunately, the use of cluster algorithms [13, 14] is very helpful in overcoming this difficulty.

(ii) The scaling corrections are expected to be large [3]. A reliable analysis, hence, requires simulations on several large lattices of the same structure. Due to the self-similar character of the underlying network, the sizes of the lattices increase as a geometrical series. We face the high C.P.U. time consuming problem in the simulations.
The purpose of this paper is to provide a thorough study of the 3-state Potts model on the deterministic Sierpinski carpets of a given fractal dimension. The order of the transition has to be carefully checked before calculating the critical temperature and exponents. The paper is organized as follows: The model and the finite size scaling theory are briefly recalled in Sec.2.1 and Sec.2.2. The simulation methods are described in Sec.2.3. The critical behavior of the 3-state Potts model on the deterministic Sierpinski carpet is studied in Sec.3.

2 Methods and theoretical background

2.1 Sierpinski carpet $SC(3,1)$ and Potts model

The lattice structures we deal with are constructed according to an iterative segmentation process: we start from a generating cell denoted $SC(3,1,1)$, which is a $3 \times 3$ square lattice where the central square is removed. The lattice $SC(3,1,k)$ associated with the $k^{th}$ iteration step is constructed by replacing each occupied square of $SC(3,1,k-1)$ by the generating cell while enlarging the side 3 times. The size of the lattice $SC(3,1,k)$ is $L = 3^k$ and the number of occupied sites $N_{occ}^L = (3^2 - 1^2)^k$. The Hausdorff dimension $d_f$, defined in such a manner that $N_{occ}^L = L^{d_f}$, is equal to $\ln(8)/\ln(3) \approx 1.892789$. The Sierpinski carpet $SC(3,1,k)$ becomes a true fractal in the mathematical sense when $k$ tends to infinity, and we will denote it $SC(3,1)$.

The Hamiltonian of the $q$-state Potts model on $SC(3,1,k)$ reads:

$$H = -J \sum_{(i,j)} \delta(\sigma_i, \sigma_j),$$

where $\sigma_i$ and $\sigma_j$ stand for the spin states on the sites $i$ and $j$ and can take the integer values $1, 2, \ldots, q$. $\delta(\sigma_i, \sigma_j)$ is the Kronecker $\delta$-function, and $J$ is the coupling constant. The sum runs over the nearest-neighbor bonds on $SC(3,1,k)$. The order parameter per site of the Potts model is defined as :

$$m_L = \frac{q\rho_L - 1}{q - 1}$$

where $\rho_L = \max(n_1, n_2, \ldots, n_q)/N_{occ}^L$ is the largest density of spin state, since $n_s$ is the number of spins in state $s$. We notice that the values of $m_L$ lie in a range between 0 and 1. We will focus our attention on the 3-state Potts model.

2.2 Finite size scaling for second order and first order phase transitions

Finite size scaling theory has been first developed by Fisher et al. [15] in the case of the second order phase transitions. According to Widom’s homogeneity hypothesis,
applying to a system with translational symmetry, the singular part of the free energy per spin of the system under a change of the unit length from 1 to $b$ is assumed to scale as

$$f(t, h, L) = b^{-d} f(tb^y_t, hb^y_h, L/b)$$  \hspace{1cm} (3)$$

where $d$ is the lattice dimension, $L$ is the original lattice size, $y_t$ and $y_h$ are eigenvalue exponents, $t = (T - T_c)/T_c$ is the reduced temperature with $T_c$ representing the critical temperature on the infinite lattice, and $h$ is the external field; $t$ and $h$ are supposed to be small, and $b$ smaller than the correlation length $\xi$. This hypothesis (Eq.3) has to be generalized in the case of fractals by restricting the choice of $b$ to a power of the size of the generating cell of the Sierpinski lattice in order to keep invariant the structure of the renormalized lattice. Also, one has to replace the dimension $d$ in Eq.3 by the Hausdorff dimension $d_f$ of the fractal; the factor $b^{-d_f}$ describes how the number of spins decreases under the change of the unit length. Moreover, since the translational symmetry is lost on Sierpinski lattices, the two point correlation function depends on the positions of the spins. Hence, the definition of the correlation length on the fractal lattices has to be adapted. A position-independent correlation function of one scalar variable $r$ can be defined by taking the average of the two point correlation function values over all the pairs of spins having direct distance equal to $r$. The position-independent correlation function behaves as $\exp(-r/\xi(T))/r^{d-2+\eta}$ in the critical region in the case of translational symmetry networks. We assume that it keeps a similar behavior on fractal lattices too. For a second order phase transition system, $\xi(T)$ diverges at the critical point. We, therefore, expect that $\xi(T)$ reaches the finite lattice size $L$ while $T$ is in the neighborhood of $T_c$. In this temperature region, we can take $b$ equal to the limiting value $L$ and it yields the zero-external field scaling equations for the fractal lattice as:

$$C_L(T) \sim L^{\alpha/\nu} T^C(tL^{1/\nu})$$ \hspace{1cm} (4)$$

$$m_L(T) \sim L^{-\beta/\nu} M(tL^{1/\nu})$$ \hspace{1cm} (5)$$

$$\chi_L(T) \sim L^{\gamma/\nu} \chi(tL^{1/\nu})$$ \hspace{1cm} (6)$$

$$\phi_L(T) \sim L^{1/\nu} T^2 \mathcal{P}(tL^{1/\nu})$$ \hspace{1cm} (7)$$

$$U_L(T) \sim TU(tL^{1/\nu})$$ \hspace{1cm} (8)$$

where $C_L(T)$ is the heat capacity per site, $m_L(T)$ the order parameter per site, $\chi_L(T)$ the zero field susceptibility of the order parameter per site, $\phi_L(T) \equiv \partial \ln\langle m_L(T) \rangle / \partial (k_BT)^{-1}$ usually called the first logarithmic derivative of the order parameter, and $U_L(T) \equiv 1 - \langle m_L^4(T) \rangle / (3 \langle m_L^2(T) \rangle^2)$ the Binder’s cumulant. The critical exponents $\alpha$, $\beta$, $\gamma$, and $\nu$ can be written as a function of the two independent eigenvalue exponents $y_t$ and $y_h$ and the dimension $d_f$. $\alpha$ is equal to $2 - d_f/y_t$, $\beta$ is equal to $(d_f - y_h)/y_t$, $\gamma$ is equal to $(2y_h - d_f)/y_t$, and $\nu$ is equal to $1/y_t$. The finite-size shifting of the rounded singularity of a given thermodynamical average $\kappa$ (for instance, $\phi$, $\chi$, and $C$) from the critical point should follow the relation:

$$T^\kappa(L) = T_c + g_{\kappa} L^{-1/\nu},$$  \hspace{1cm} (9)$$
where the $g_k$'s are some constants. Moreover, the width of the associated rounded singularity should scale as $L^{-1/\nu}$.

For a system undergoing a first order phase transition, Fisher et al. [16] and Cardy et al. [17] propose that the relevant eigenvalue exponents in Eq.(3) should reach their allowed limiting values, said $y_t = d$ and $y_h = d$, in the phenomenological studies. In the case of the Sierpinski lattice, $y_t$ and $y_h$ should take the value of the Hausdorff dimension $d_f$. The thermodynamical averages on the fractal lattice, therefore, scale as:

\[ C_L(T) \sim L^{d_f} T C(tL^{d_f}) \]  
\[ \chi_L(T) \sim L^{d_f} \chi(tL^{d_f}) \]  
\[ \phi_L(T) \sim L^{d_f} T^2 \phi(tL^{d_f}) \]

Moreover, they are characterized by $\delta$-function like singularities as the lattice size $L$ increases; as a result, the areas under the curves displaying a given average as a function of the temperature should be independent of $L$. Also, it is well known that the probability distribution of the energy is double peaked in the vicinity of the transition temperature $T_0$.

### 2.3 Simulation process

We implement the Wolff [13] and Swendsen-Wang [14] Monte Carlo cluster algorithms; both of them have been shown to largely overcome the critical slowing down. In order to save simulation time, we used alternatively the two algorithms in our studies. Let us describe the simulation process in a more precise way. An initial configuration is built up by randomly assigning one of the three spin states to each occupied site of the lattice $SC(3,1,k)$. Periodic boundary conditions will be used. A simulation temperature $T_0$ is set and enough Wolff or Swendsen-Wang Monte Carlo steps are performed in order to ensure that the configuration has been thermalized. The energy per site $\epsilon_L(T_0)$ and the order parameter per site $m_L(T_0)$ are then calculated for every Monte Carlo step. One million steps are collected to be a sample and the above procedure is started again at least ten times. The histogram method [20], applied within each sample, allows the thermodynamical averages of a physical quantity $g$ to be calculated over a range $\Delta T$ around $T_0$; we denote it $\langle g(T_0) \rangle_T$. The reliability of the calculated averages over the temperature interval must be carefully checked on. $\phi_L(T)$, $\chi_L(T)$, $C_L(T)$, and $U_L(T)$ are then obtained according to the following relations:

\[ \phi_L(T) = N^{occ}_L \left( \langle \epsilon_L(T_0) \rangle_T - \frac{\langle \epsilon_L(T_0) m_L(T_0) \rangle_T}{\langle m_L(T_0) \rangle_T} \right) \]  
\[ \chi_L(T) = N^{occ}_L \left( \frac{\langle m_L^2(T_0) \rangle_T - \langle m_L(T_0) \rangle_T^2}{k_B T} \right) \]  
\[ C_L(T) = N^{occ}_L \left( \frac{\langle \epsilon_L^2(T_0) \rangle_T - \langle \epsilon_L(T_0) \rangle_T^2}{k_B T^2} \right) \]
The statistical errors associated with the thermodynamical averages are estimated by calculating the standard deviations from the ten samples.

\[ U_L(T) = 1 - \frac{\langle m_4^L(T_0) \rangle_T}{3\langle m_2^L(T_0) \rangle_T^2}. \]  

(16)

The width of the Lorentz distribution is \( W \) and the area under the distribution curve is \( A \). The two-parameter fits \((W, A)\) of the rounded singularities are reported in Table 2. It appears clearly that the area \( A \) under the Lorentz distribution increases as \( L \) increases; as a result of this, we can conclude that the singularity is not \( \delta \)-function like as \( L \) tends to infinity. We have the first evidence that the phase transition is not a first order one.

3 Results and discussions

3.1 Studies of rounded singularities of \( \chi_L(T) \) and \( \phi_L(T) \)

First, we study the order of the phase transition of the 3-state Potts model in the case of the fractal \( SC(3, 1) \). The behavior of the finite-size rounded singularities of \( \chi_L(T) \) calculated from our simulations are shown in Fig.1 for \( L = 27, 81, 243, 729, \) and 2187. The maximum values \( \chi_L^{\text{max}} \) and their associated temperatures \( T^\chi(L) \) are reported in Table 1. The finite-size shifts of \( \chi_L^{\text{max}} \) are clearly observed; the Lorentz distribution is a better candidate to fit the rounded singularity than the Gaussian one:

\[ \chi_L(T) = \frac{2A}{\pi} \frac{W}{4(T - T^\chi(L))^2 + W^2} \]  

(17)

The width of the Lorentz distribution is \( W \) and the area under the distribution curve is \( A \). The two-parameter fits \((W, A)\) of the rounded singularities are reported in Table 2. It appears clearly that the area \( A \) under the Lorentz distribution increases as \( L \) increases; as a result of this, we can conclude that the singularity is not \( \delta \)-function like as \( L \) tends to infinity. We have the first evidence that the phase transition is not a first order one. According to the finite size scaling for second order phase transitions, the width \( W \) should scale as \( L^{-1/\nu} \) and the area \( A \) as \( L^{\gamma/\nu - 1/\nu} \). The log-log plots of \( W \) and \( A \) versus \( L \) do not appear straight lines over the whole range from \( L = 27 \) to \( L = 2187 \); they exhibit a slight curvature, which can be interpreted as a contribution of scaling corrections to the behavior of \( W \) and \( A \). The slopes calculated from two consecutive points from \( L = 27 \) to \( L = 2187 \) are \(-0.786(17), -0.701(11), -0.6243(60), -0.5524(35)\) for \( W \) and \(0.897(12), 0.9867(81), 1.0635(40), 1.1634(25)\) for \( A \). The effect of the scaling corrections in the case of fractal lattices has been pointed out and discussed in the framework of the Ising model by Monceau et al. \[5\] and Carmona et al. \[3\]. Instead of a power law, \( W \) is expected to behave as \( L^{-1/\nu} (1 + \varphi_W(L)) \) where the scaling corrections \( \varphi_w(L) \) tend to zero as \( L \) tends to infinity; usually, \( \varphi_W(L) \) is developed as \( \varphi_W(L) = A_W L^{-\omega} \), where \( A_W \) is the amplitude of the corrections and \( \omega \) an additional exponent. A similar behavior is expected for \( A \). There is no hope to calculate the scaling corrections in a reliable way from our simulations. However, the slopes calculated above enable to estimate an upper bound for \( 1/\nu \) to be 0.5524(35) and a lower bound for \( \gamma/\nu - 1/\nu \) to be 1.1634(25).

We now pay attention to the maxima \( \chi_L^{\text{max}} \) of \( \chi_L(T) \), and \( \phi_L^{\text{max}} \) of \( \phi_L(T) \), reported in Table 1. The log-log plot of \( \chi_L^{\text{max}} \) shows a straight line. The power law fits are satisfied with reliability coefficients equal to 1.0000; they yield \( \gamma/\nu = 1.6948(37) \) with a four sizes
fit from $L = 81$ to $L = 2187$ and $\gamma/\nu = 1.7013(28)$ with a three sizes fit from $L = 243$ to $L = 2187$. The scaling corrections turn out to be very weak in the case of the susceptiblility, as already shown in the case of the Ising model. The best estimate of $\gamma/\nu$ is $1.7013(28)$. Finite size scaling theory states that the slope of the line represents the exponent $\gamma/\nu$ if the phase transition is second order and that it should be the Hausdorff dimension $d_f$ if the transition is first order. The stability of the fits involving $\chi_{\text{max}}^L$ enable to conclude that $\gamma/\nu$ is significantly different from $d_f = 1.892789$ and, therefore, consistent with a second order phase transition as we have shown at the beginning of this paragraph. On the other hand, $\phi_{\text{max}}^L$ suffers from large scaling corrections since the slopes of two successive points of $\phi_{\text{max}}^L$ in a log-log plot are measured as $0.758(20)$, $0.669(11)$, $0.613(13)$, and $0.533(11)$. These large scaling corrections may be due to the deviation of the average number of nearest neighbors on the finite size fractal from that on the infinite fractal because the critical temperature is directly related to the average number of nearest neighbors in the mean filed theory and the eigenvalue exponent $y_t = 1/\nu$ describes the scaling behavior of the reduced temperature. The behavior of $\phi_{\text{max}}^L$ provides an upper bound for $1/\nu$, which is $1/\nu = y_t \leq 0.533(11)$. Combining $\gamma/\nu = 1.7013(28)$ with $1/\nu \leq 0.533(11)$, we get a lower bound for $(\gamma/\nu - 1/\nu)$, which is $1.168(11)$. These bounds are both very close to the ones we got from the fits of $W$ and $A$. The eigenvalue exponent $y_k$ can be calculated from $(\gamma/\nu + d_f)/2$ and is equal to $1.7970(15)$.

The comparison between the present value of $\gamma/\nu$ and the one associated with the Ising model for the same fractal structure ($\gamma/\nu = 1.732(4)$) enables to distinguish the Ising and the 3-state Potts model universality classes. We mention that this comparison makes sense only if the fractal structures are exactly the same because the universality is said to be weak.

### 3.2 Estimation of the critical temperature $T_c$

The infinite lattice critical temperature $T_c$ can be calculated from the finite-size shifts of the rounded singularities provided that $\nu$ is known; for a given peaked thermodynamical average $\kappa$, the equation (9) yields:

$$T_c = \frac{T^x(\ell L) - \ell^{-1/\nu} T^x(L)}{1 - \ell^{-1/\nu}}$$

where $\ell = 3$ is the size of the generating cell. In this form, the value of $T_c$ is deduced from two consecutive iteration steps of the fractal structure. An upper bound for $1/\nu$ will yield an upper bound for $T_c$ since $T^x(L)$ and $T^\phi(L)$ decrease as $L$ increases. Taking $1/\nu = 0.533(11)$, the values of $T^x(L)$ from the two highest iteration steps provide the first estimate for the upper bound of $T_c$ equal to $0.67324(1264)$. The values of $T^\phi(L)$ from the two highest iteration steps yield $0.67357(1456)$ as a second upper bound for $T_c$. The errors in the above bounds mainly come from the error propagation of $1/\nu$. We can conclude that $T_c < 0.673(12)$.

An alternative way to measure $T_c$ without knowing $\nu$ is to study the Binder’s cumulant crossing points [19]. Fig.2 shows that the cumulant curves for different lattice sizes do
not intersect at a fixed point. We call $T_{k-1,k}$ the temperature of the cumulant intersection point between the iteration steps $(k-1)$ and $k$. The measured values are: $T_{2,3} = 0.68461(54)$, $T_{3,4} = 0.67923(26)$, $T_{4,5} = 0.67594(20)$, $T_{5,6} = 0.673916(69)$, and $T_{6,7} = 0.672866(50)$. The difference $\Delta T_k = (T_{k-1,k} - T_{k,k+1})$ behaves basically as a geometrical series as a function of $k$. If we write $\Delta T_k = ar_3 r_4 \cdots r_k$ where $r_k = \Delta T_k/\Delta T_{k-1}$, $k \geq 4$, and $ar_3 = \Delta T_3$, we will find $1 > r_4 \geq r_5 \geq r_6 \geq \cdots$. This enables us to claim that the series $\{T_{k,k+1}\}$ converges as $k$ tends to infinity. The convergent value calculated from the three intersection points $T_{4,5}$, $T_{5,6}$, and $T_{6,7}$ gives a lower bound for the critical temperature: 0.671737. We finally conclude that $0.671737 < T_c < T_{6,7} = 0.672866$. This range is in agreement with the upper bound for $T_c$ extracted from Eq.3.18.

### 3.3 Simulation near $T_c$

We firstly check the occurrence of a single peak in the energy probability distributions, which confirms again that the transition is second order. A typical energy probability distribution is shown in Fig.3. We then study the behaviors of $\phi_L(T)$, $\chi_L(T)$, and $m_L(T)$ within the previously estimated temperature range for $T_c$. The following three temperatures will be considered: 0.671737, 0.672866, and 0.673241. The first two temperatures are the lower and the upper bounds extracted from the Binder’s cumulant crossing and the last one is the upper bound calculated from Eq.3.18, assuming that $1/\nu = 0.533$ and using the positions of the peaks of $\chi_L$ at $L = 729$ and $L = 2187$. The log-log plots of $m_L(T)$, $\chi_L(T)$, and $\phi_L(T)$ at the three considered temperatures are shown in Fig.4. $m_L$ shows a straight line in the log-log plot, at $T = 0.671737$, over a range of lattice sizes covering several orders of magnitude, with a slope, measured from $L = 243$, $L = 729$, and $L = 2187$, equal to $-0.0748(1)$. When the temperature increases towards the upper bound, $m_L(T)$ becomes more and more concave while $\chi_L(T)$ is more and more convex. The asymptotic slopes for $m_L(T)$ obtained from the measurements between the largest sizes $L = 729$ and $L = 2187$ are upper bounded at $T = 0.672866$ and $T = 0.673241$, respectively, by $-0.0876(2)$ and $-0.0932(2)$; and the asymptotic slopes for $\chi_L(T)$ at $T = 0.671737$, $T = 0.672866$, and $T = 0.673241$ are lower bounded by $1.60(1)$, $1.73(1)$, and $1.78(1)$. On the other hand, the curvature of $\phi_L(T)$ in a log-log plot varies in the opposite direction, that is, the curve is less convex at the higher temperature bound ($T = 0.673241$) than at the lower one ($T = 0.671737$). The asymptotic slopes for $\phi_L(T)$ at $T = 0.671737$, $T = 0.672866$, and $T = 0.673241$ are upper bounded by $0.466(13)$, $0.521(10)$, and $0.552(11)$. The scaling corrections, depending on the physical quantities, turn out to affect strongly the critical behavior of the 3-state Potts model on the fractal structure even when the sizes are large. The estimation of the ratio of critical exponents $\beta/\nu$, $\gamma/\nu$, and $1/\nu$ from the behavior of $m_L$, $\chi_L$, and $\phi_L$ in the critical region becomes a very difficult task: firstly, the value of the critical temperature cannot be provided precisely; secondly, even if we expect scaling corrections to tend to zero as $L$ tends towards infinity (whatever thermodynamic quantity is studied), they should be included in the calculation of our exponents; as already explained in Sec.3.1, there is no hope to extract scaling
corrections from our data. We point out that the scaling corrections in the critical region are stronger than that for the case of Ising model. In the case of Ising model, the slope of $\chi_L(T)$ extracted at the expected critical temperature is consistent with the value of $\gamma/\nu$ extracted from $\chi_L^{\text{max}}$.

3.4 Discussions

The consistency of our full set of results can be achieved in the following way: The behavior of the Binder cumulant shows that $T_c$ should be greater than 0.671737 and smaller than 0.682866. From the behavior of $\chi_L(T)$, the asymptotic slopes in this temperature region are lower bounded by 1.60(1), consistent with the value $\gamma/\nu = 1.7013(29)$ extracted from $\chi_L^{\text{max}}$. The asymptotic slopes for $m_L(T)$ in the region are always smaller than -0.0748(1) and, thus, yield $\beta/\nu > 0.0748(1)$, consistent with the requirement of the hyperscaling relation, said $\beta/\nu = (d_f - \gamma/\nu)/2 = 0.0957(21)$ with $\gamma/\nu$ taking the value 1.7013(29). Similarly, $1/\nu < 0.521(11)$ can be found from the behavior of $\phi_L(T)$ in this temperature region, consistent with the results in the studies of $\phi_L^{\text{max}}$. We, therefore, give the best value and bounds of the exponents and the critical temperature that we find: $\gamma/\nu = 1.7013(29)$, $\beta/\nu > 0.0748(1)$, $1/\nu < 0.521(11)$, and $0.671737 < T_c < 0.672866$.

The critical temperature and the exponents of the 3-state Potts model on a two dimensional regular system are known to take the following values [21]: $T_c = 1/\ln(1 + \sqrt{3}) \approx 0.995$, $1/\nu = 6/5$, $\beta/\nu = 2/15$, and $\gamma/\nu = 26/15$. We find that the values of $T_c$, $1/\nu$, $\beta/\nu$ and $\gamma/\nu$ for the 3-state Potts model are smaller in the case of the Sierpinski carpet we investigated than in the case of the two dimensional regular system. A similar situation occurs in the case of the Ising model on Sierpinski carpets [3] where, regardless of the structure of the fractals, those values decrease as the Hausdorff dimension decreases.

Discrepancies are observed between our simulation results and Bin’s predictions. Bin [11] found by applying the method of Migdal-Kadanoff bond-moving renormalization, that $\exp(-K) = 0.746$ (which is equivalent to $T_c = 3.4126$) and $y_K(= y_t) = 0.619$ for the 3-state Potts model on the same Sierpinski carpet we studied in this paper. There are mainly two reasons for the occurrence of discrepancies: firstly, the Migdal-Kadanoff bond-moving renormalization is an approximate method; secondly, in Bin’s case the spins were located at the vertices of the occupied squares of the Sierpinski carpet while they are located at the centers of the occupied squares in the present work; it yields a different universality class because the duality is no more held on fractal lattices.

In conclusion, we have carefully performed the first Monte Carlo studies for the 3-state Potts model on the Sierpinski carpet with $d_f \approx 1.8928$. Although the system exhibits strong scaling corrections, the consistency of our results demonstrates the validity of the finite size scaling theory for the $q$-state Potts model on these fractal lattices. Nevertheless, scaling corrections tend to increase while $q$ is passing from 2 (Ising model) to 3. We, hence, expect the raise of these difficulties in the studies of the phase transition while $q$ goes larger. In the coming future, we will be able to perform in a reliable way the studies for the Potts model with larger $q$ on fractal dimensions higher than 2.
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Figure captions

Fig.1 Rounded singularities of the susceptibility on lattice SC(3, 1, k) where k = 3, 4, 5, 6, and 7. Each segment represents the reliable temperature range of the histogram method where the simulation temperature is located in its center. The fitting by Lorentz distribution for the susceptibility on SC(3, 1, 7) is plotted in order to clearly indicate the location of the rounded singularity.

Fig.2 Binder’s cumulant crossings.

Fig.3 Probability distribution of energy per site on SC(3, 1, 7). It’s made of 10 million Monte Carlo data and separated by 500 histogram classes. The simulation temperature is taken at 0.672845.

Fig.4 Log-log plot of m_L, χ_L, and φ_L at T = 0.671737 (open square), 0.672866 (open circle), and 0.673241 (open triangle) on the Sierpinski carpet SC(3, 1, k).
Table 1: Maxima values and associated peak positions of $\chi_L$ and $\phi_L$ on the Sierpinski carpet $SC(3, 1, k)$ of size $L = 3^k$

|       | $L = 27$      | $L = 81$      | $L = 243$     | $L = 729$     | $L = 2187$    |
|-------|---------------|---------------|---------------|---------------|---------------|
| $\chi_L^{max}$ | 30.53(12)     | 195.58(59)    | 1239.4(4.4)   | 7984(19)      | 51969(105)    |
| $T\chi(L)$     | 0.73800(22)   | 0.70448(12)   | 0.689525(65)  | 0.682115(16)  | 0.678182(11)  |
| $\phi_L^{max}$ | 8.85(18)      | 20.35(13)     | 42.46(45)     | 83.23(74)     | 149.50(1.23)  |
| $T\phi(L)$     | 0.7510(17)    | 0.70995(38)   | 0.69221(22)   | 0.68357(16)   | 0.679139(23)  |

Table 2: Fit of the Lorentz distribution (Eq.(17)) of the rounded singularity of $\chi_L$ on the Sierpinski carpet $SC(3, 1, k)$ of size $L = 3^k$.

|       | $L = 27$      | $L = 81$      | $L = 243$     | $L = 729$     | $L = 2187$    |
|-------|---------------|---------------|---------------|---------------|---------------|
| $W$   | 0.0700(11)    | 0.02949(31)   | 0.01366(8)    | 0.00688(2)    | 0.00375(1)    |
| $A$   | 3.444(38)     | 9.230(74)     | 27.29(11)     | 87.78(18)     | 315.13(57)    |
3-state Potts model on SC(3,1,k)

(W,A) fitting on SC(3,1,7)

\( \chi_L(T) \)
Binder's cumulant crossings

Graph showing various lines labeled SC(3,1,n) for different values of n, indicating critical points in the system.
Energy per spin: $e_L$

$P(e_L) \Delta e_L$

SC(3,1,7)  $T=0.672845$
