A VANISHING THEOREM FOR CO-HIGGS BUNDLES ON THE MODULI SPACE OF BUNDLES

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Abstract. We consider smooth moduli spaces of semistable vector bundles of fixed rank and determinant on a compact Riemann surface \( X \) of genus at least 3. The choice of a Poincaré bundle for such a moduli space \( M \) induces an isomorphism between \( X \) and a component of the moduli space of semistable sheaves over \( M \). We prove that \( \dim H^0(M, \text{End}(\mathcal{E}) \otimes T M) = 1 \) for any vector bundle \( \mathcal{E} \) on \( M \) coming from this component. Furthermore, there are no nonzero integrable co-Higgs fields on \( \mathcal{E} \).

1. Introduction

Higgs bundles have been mostly studied on Riemann surfaces although moduli spaces have been constructed for arbitrary smooth projective varieties [Si]. On Riemann surfaces, Higgs bundles moduli spaces are only nonempty in positive genus, where the canonical line bundle has sections. One way to extend the theory of Higgs bundles to genus 0, without introducing a parabolic structure, is to consider co-Higgs bundles as in [Ra2]. In general, a co-Higgs bundle on a variety \( X \) is a holomorphic bundle \( E \) with a co-Higgs field \( \phi : E \to E \otimes TX \) for which \( \phi \wedge \phi \) vanishes in \( H^0(X, \text{End}(E) \otimes \wedge^2 TX) \). This vanishing is the analogue of the integrability condition in [Si]. Co-Higgs bundles arose originally in generalized complex geometry [Hi3], as the limit of generalized holomorphic bundles as a generalized complex structure becomes ordinary complex.

The pattern continues in higher dimension, with Higgs bundles and co-Higgs bundles existing largely as general-type and Fano phenomena, respectively (see [Ra3, Co, B1, B2, BBGL]). Families of integrable co-Higgs bundles on Fano surfaces, namely \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \), have been constructed in [Ra3, VC]. Another natural variety to consider is \( M_\xi \), the moduli space of semistable bundles of rank \( r \) and fixed determinant \( \xi \) on a compact Riemann surface \( X \) of genus \( g \) with \( g \geq 2 \). This variety is known to be Fano and has \( \text{Pic}(M_\xi) \cong \mathbb{Z} \). It is an irreducible normal projective variety, and it is smooth whenever \( \text{gcd}(r, \text{deg}(\xi)) = 1 \).

One immediate example of an integrable co-Higgs bundle on \( M_\xi \) is the one with \( E = \mathcal{O}_{M_\xi} \oplus TM_\xi \) and the co-Higgs field \( \phi \) that maps \( TM_\xi \) to \( \mathcal{O}_{M_\xi} \otimes TM_\xi \) via the identity and acts as zero on the component \( \mathcal{O}_{M_\xi} \subset E \). This co-Higgs field is nilpotent of order 2, and so in particular \( \phi \wedge \phi = 0 \). This is the so-called canonical co-Higgs bundle, which can be defined on any variety [Ra1, Hi3]. A sufficient condition for \((E, \phi)\) to be slope stable in the sense of Hitchin [Hi1] is that \( TM_\xi \) should be slope stable as a bundle [Mi, Ra1]. This has been conjectural for some time. It has been known for \( r = 2 \) since [Hw] (see [Iy] for \( r > 2 \)). It is natural to ask if there are examples of integrable co-Higgs bundles on \( M_\xi \) where \( E \) is stable.

2010 Mathematics Subject Classification. 14H60, 14D20, 14D21.

Key words and phrases. Co-Higgs bundle, integrability, moduli space, Poincaré bundle.
as a vector bundle, as the stability of $E$ makes the stability of $(E, \phi)$ automatic. Under the assumption that $g \geq 3$, we show that if $(E, \phi)$ is integrable, and $E$ is a stable bundle from the component of the moduli space of sheaves on $M_\xi$ isomorphic to $X$, then $\phi = 0$.

More precisely, we prove that the total space of the tangent bundle $TX$ is a component of the moduli space of semistable co-Higgs bundles on $M_\xi$, while the moduli space of integrable co-Higgs bundles sits inside $TX$ as the image of the zero section $X \rightarrow TX$.

This work is inspired in part by a conversation between E. Witten and the second named author concerning branched covers of moduli spaces of bundles, with reference to spectral covers constructed using Higgs bundles with poles [FW]. We observe that, according to Theorem 3.1, co-Higgs bundles $(E, \phi)$ with $E$ in the given component of the space of stable sheaves on $M_\xi$ do not generate interesting spectral covers — they are just the zero section in the total space of $TM_\xi$. In the canonical co-Higgs example, the spectral cover is the first-order neighborhood of the zero section of $TM_\xi$ and can potentially be perturbed by deforming $(E, \phi)$ in such a way that $\phi$ is no longer nilpotent but still integrable. We leave these speculations as inspiration for future work.

2. A class of co-Higgs bundles

2.1. Poincaré bundle and co-Higgs fields. Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 3$. The holomorphic tangent and cotangent bundles of $X$ will be denoted by $TX$ and $K_X$ respectively. Fix an integer $r \geq 2$ and also a holomorphic line bundle $\xi \rightarrow X$ such that degree($\xi$) is coprime to $r$. Let $M_\xi$ denote the moduli space of stable holomorphic vector bundles $E$ on $X$ such that rank($E$) = $r$ and $\bigwedge^r E = \xi$. This $M_\xi$ is an irreducible smooth projective variety of dimension $(r^2 - 1)(g - 1)$. The Picard group of $M_\xi$ is isomorphic to $\mathbb{Z}$. Hence the notion of (semi)stability of vector bundles over $M_\xi$ does not depend on the choice of polarization on $M_\xi$.

A Poincaré vector bundle on $X \times M_\xi$ is a holomorphic vector bundle

$$ E \rightarrow X \times M_\xi $$

such that for every point $E \in M_\xi$, the restriction $E|_{X \times \{E\}}$ lies in the isomorphism class of vector bundles on $X$ corresponding to $E$. There are Poincaré vector bundles on $X \times M_\xi$; two such vector bundles differ by tensoring with a line bundle pulled back from $M_\xi$. Fix a Poincaré vector bundle $E$ on $X \times M_\xi$ as in (2.1). For any point $x \in X$, the vector bundle $E|_{\{x\} \times M_\xi}$ on $M_\xi$ will be denoted by $E_x$. It is known that $E_x$ is stable [LN, p. 174, Proposition 2.1]. If $x$ and $y$ are two distinct points of $X$, then $E_x$ is not isomorphic to $E_y$ [LN, p. 174, Theorem]. On the other hand, the infinitesimal deformation map

$$ T_x X \rightarrow H^1(M_\xi, \text{End}(E_x)) $$

is an isomorphism [NR2] p. 392, Theorem 2]. Therefore, $X$ is a connected component of the moduli space of stable sheaves on $M_\xi$ with numerical type that of $E_x$. It may be noted that all sheaves on $M_\xi$ lying in this component of the moduli space are actually locally free.

Fix a point $x \in X$. We will construct a family of co-Higgs fields on $E_x$ parametrized by the tangent line $T_x X \subset TX$. 
The fiber $\mathcal{O}_X(x)_x$ over $x$ of the line bundle $\mathcal{O}_X(x)$ is identified with $T_xX$ using the Poincaré adjunction formula. More precisely, for any holomorphic coordinate $z$ on $X$ defined around $x$ with $z(x) = 0$, the evaluation at $x$ of the section $\frac{\partial}{\partial z}$ of $TX \otimes \mathcal{O}_X(-x)$ is independent of the choice of the function $z$. The isomorphism between $\mathcal{O}_X(x)$ and $T_xX$ is given by this element of $(TX \otimes \mathcal{O}_X(-x))_x = T_xX \otimes \mathcal{O}_X(-x)_x$. Let

$$p : X \times M_\xi \to X \quad \text{and} \quad q : X \times M_\xi \to M_\xi$$

be the natural projections. Consider the following short exact sequence of sheaves on $z$:

$$0 \to \text{End}(\mathcal{E}) \to \text{End}(\mathcal{E}) \otimes p^* \mathcal{O}_X(x) \to \text{End}(\mathcal{E})_x \otimes \mathcal{O}_X(x)_x = \text{End}(\mathcal{E})_x \otimes T_xX \to 0,$$

where $\text{End}(\mathcal{E})_x = \text{End}(\mathcal{E}_x)$ is supported on $\{x\} \times M_\xi$, and $T_xX$ (respectively, $\mathcal{O}_X(x)_x$) denotes the trivial line bundle on $\{x\} \times M_\xi$ with fiber $T_xX$ (respectively, $\mathcal{O}_X(x)_x$). Let

$$R^0q_*(\text{End}(\mathcal{E})_x \otimes T_xX) = \text{End}(\mathcal{E})_x \otimes T_xX \xrightarrow{\gamma'} R^1q_*(\text{End}(\mathcal{E}))$$

be the homomorphism in the long exact sequence of direct images associated to (2.2) for the projection $q$.

We have $\text{End}(\mathcal{E}) = \text{ad}(\mathcal{E}) \oplus \mathcal{O}_{X \times M_\xi}$, where $\text{ad}(\mathcal{E}) \subset \text{End}(\mathcal{E})$ is the subbundle of corank one defined by the sheaf of trace zero endomorphisms, and the homomorphism $\mathcal{O}_{X \times M_\xi} \hookrightarrow \text{End}(\mathcal{E})$ is given by the scalar multiplications of $\mathcal{E}$. Therefore, we have

$$R^1q_*(\text{End}(\mathcal{E})) = R^1q_*(\text{ad}(\mathcal{E})) \oplus R^1q_*\mathcal{O}_{X \times M_\xi}.$$  

(2.4)

On the other hand, we have

$$R^1q_*(\text{ad}(\mathcal{E})) = TM_\xi,$$

because the fiber of $TM_\xi$ over any vector bundle $V \in M_\xi$ is $H^1(X, \text{ad}(V))$. Therefore, from (2.4) we get a surjective homomorphism

$$R^1q_*(\text{End}(\mathcal{E})) \twoheadrightarrow TM_\xi \to 0.$$  

(2.5)

Let $\text{End}(\mathcal{E})_x \otimes T_xX \xrightarrow{\gamma'} R^1q_*(\text{End}(\mathcal{E})) \hookrightarrow TM_\xi$ be the composition of the homomorphism $\gamma'$ in (2.3) with the homomorphism in (2.5). This composition produces a homomorphism

$$\gamma : T_xX \to H^0(\mathcal{M}_\xi, \text{End}(\mathcal{E})_x^* \otimes TM_\xi) = H^0(M_\xi, \mathcal{E}_x \otimes TM_\xi).$$

(2.6)

In other words, $\gamma(v)$ is a co-Higgs field on $\mathcal{E}_x$ for all $v \in T_xX$.

2.2. Non-integrability.

**Proposition 2.1.**

1. The homomorphism $\gamma$ in (2.6) is injective.
2. For any nonzero vector $v \in T_xX$, the co-Higgs field $\gamma(v)$ is not integrable.

**Proof.** The dimension of $T_xX$ in (2.6) is one. Hence if the homomorphism $\gamma$ from $T_xX$ is not injective, then we have $\gamma = 0$. Therefore, the first statement in the proposition follows from the second statement. We will prove the second statement.

Lemma 2.2 says that there are vector bundles $E \in M_\xi$ on $X$ such that

$$\dim H^0(X, \text{End}(E) \otimes \mathcal{O}_X(x)) = 1.$$  

(2.7)
Fix such a vector bundle \( E \in M_\xi \). Consider the natural short exact sequence
\[
0 \to \text{End}(E) \to \text{End}(E) \otimes \mathcal{O}_X(x) \to \text{End}(E)_x \otimes \mathcal{O}_X(x)_x = \text{End}(E)_x \otimes T_x X \to 0;
\] (2.8)
note that it coincides with the restriction of the exact sequence in (2.2) to \( X \times \{E\} \subset X \times M_\xi \).

Let
\[
\begin{align*}
0 \to H^0(X, \text{End}(E)) \to H^0(X, \text{End}(E) \otimes \mathcal{O}_X(x)) \\
\to \text{End}(E)_x \otimes T_x X \to H^1(X, \text{End}(E)) = T_E M_\xi \oplus H^1(X, \mathcal{O}_X)
\end{align*}
\] (2.9)
be the exact sequence of cohomologies associated to (2.7). Since \( E \) is stable, we have \( H^0(X, \text{End}(E)) = \mathbb{C} \cdot \text{Id}_E \). Therefore, from (2.7) it follows that the homomorphism \( b \) in (2.9) is injective. Hence \( s = 0 \), which implies that \( h \) in (2.9) is injective.

The exact sequence in (2.8) is a direct sum of the following two short exact sequences
\[
0 \to \mathcal{O}_X \to \mathcal{O}_X(x) \to \mathcal{O}_X(x)_x = T_x X \to 0
\]
and
\[
0 \to \text{ad}(E) \to \text{ad}(E) \otimes \mathcal{O}_X(x) \to \text{ad}(E)_x \otimes \mathcal{O}_X(x)_x = \text{ad}(E)_x \otimes T_x X \to 0,
\]
where \( \text{ad}(E) \subset \text{End}(E) \) is the subbundle of co-rank one defined by the sheaf of trace-zero endomorphisms. Let
\[
h' : \text{ad}(E)_x \otimes T_x X \to H^1(X, \text{ad}(E)) = T_E M_\xi
\]
be the homomorphism in the long exact sequence of cohomologies associated to the second short exact sequence. Since \( h \) (2.9) is injective, we conclude that \( h' \) is also injective. Therefore, the homomorphism of second exterior products induced by \( h' \)
\[
\bigwedge^2 h' : \bigwedge^2 \text{ad}(E)_x \otimes (T_x X)^{\otimes 2} = \bigwedge^2 \text{ad}(E)_x \otimes T_x X \to \bigwedge^2 H^1(X, \text{ad}(E)) = \bigwedge^2 T_E M_\xi
\]
is also injective.

Consider the Lie bracket homomorphism \( \bigwedge^2 \text{End}(E_x) \to \text{End}(E_x) \), defined by \( A \wedge B \mapsto \frac{1}{2}(AB - BA) \). Let
\[
\eta : \text{End}(E_x) = \text{End}(E_x)^* \to (\bigwedge^2 \text{End}(E_x))^* = \bigwedge^2 \text{End}(E_x)
\]
be the dual of this Lie bracket homomorphism. Let
\[
\eta' : \text{End}(E_x) \to \bigwedge^2 \text{ad}(E_x)
\]
be the composition of \( \eta \) with the projection \( \bigwedge^2 \text{End}(E_x) \to \bigwedge^2 \text{ad}(E_x) \) induced by the natural projection \( \text{End}(E_x) \to \text{ad}(E_x) \).

For any \( v \in T_x X \), the element
\[
(\gamma(v) \bigwedge \gamma(v))(E) \in \text{End}(E_x) \otimes (\bigwedge^2 T_E M_\xi)
\]
\[
= \text{End}(E_x)^* \otimes (\bigwedge^2 T_E M_\xi) = \text{Hom}(\text{End}(E_x), \bigwedge^2 T_E M_\xi)
\]
coincides with the homomorphism \( \text{End}(E_x) \to \bigwedge^2 T_E M_\xi \) defined by
\[
w \mapsto (\bigwedge^2 h')(\eta'(w) \otimes v^{\otimes 2}), \ w \in \text{End}(E_x) .\]
Now from the injectivity of $\wedge^2 h'$ it follows immediately that
\[(\gamma(v) \wedge \gamma(v))(E) \neq 0\]
if $v \neq 0$. Therefore, the co-Higgs field $\gamma(v)$ is not integrable for all $v \neq 0$. \qed

**Lemma 2.2.** Fix a point $x \in X$. There is a nonempty Zariski open subset $U_x \subset M_\xi$ such that for all $E \in U_x$,
\[
\dim H^0(X, \text{End}(E) \otimes \mathcal{O}_X(x)) = 1.
\]

**Proof.** Write
\[
\text{degree}(\xi) = d + rm_0,
\]
where $d$ and $m_0$ are integers with $1 \leq d < r$. We will first construct a vector bundle $V_r$ on $X$ of rank $r$ and degree $d_0$
\[
\dim H^0(X, \text{End}(V_r) \otimes \mathcal{O}_X(x)) = 1.
\]
The locus in $\text{Pic}^{g-1}(X)$ of line bundles $L$ with $H^0(X, L) \neq 0$ is the theta divisor. Therefore, for a general line bundle $L$ on $X$ with degree($L$) = $g - 1$, we have $H^0(X, L) = 0$. Given that $g \geq 3$, this implies that for a general line bundle $L$ in $\text{Pic}^1(X)$ or $\text{Pic}^2(X)$, we have $H^0(X, L) = 0$. Consequently, there are holomorphic line bundles
\[
\{L_1, \ldots, L_d, L_{d+1}, \ldots, L_r\}
\]
on $X$ such that

1. degree($L_i$) = 1 for all $1 \leq i \leq d$,
2. degree($L_i$) = 0 for all $d + 1 \leq i \leq r$
3. $H^0(X, \text{Hom}(L_i, L_j) \otimes \mathcal{O}_X(x)) = 0$ for all $(i, j) \in \{1, \ldots, r\} \times \{1, \ldots, r\}$ with $i \neq j$.

We will now inductively construct holomorphic vector bundles $V_i$ of rank $i$, $0 \leq i \leq r$.

Set $V_0 = 0$ and $V_1 = L_1$. For each $2 \leq i \leq r$, the vector bundle $V_i$ fits in a short exact sequence of holomorphic vector bundles
\[
0 \rightarrow V_{i-1} \rightarrow V_i \rightarrow L_i \rightarrow 0
\]
such that the corresponding extension class
\[
\mu_i \in H^1(X, \text{Hom}(L_i, V_{i-1}))
\]
satisfies the following condition: consider the exact sequence
\[
0 \rightarrow \text{Hom}(L_i, V_{i-2}) \rightarrow \text{Hom}(L_i, V_{i-1}) \rightarrow \text{Hom}(L_i, L_{i-1}) \rightarrow 0;
\]
it produces a surjective homomorphism
\[
h_i : H^1(X, \text{Hom}(L_i, V_{i-1})) \rightarrow H^1(X, \text{Hom}(L_i, L_{i-1})) \rightarrow 0.
\]
The condition that $\mu_i$ is required to satisfy states as
\[
h_i(\mu_i) \neq 0;
\]
note that $H^1(X, \text{Hom}(L_i, L_{i-1})) \neq 0$ because by Riemann Roch,
\[
\chi(\text{Hom}(L_i, L_{i-1})) = \text{degree}(\text{Hom}(L_i, L_{i-1})) - g + 1 \leq 1 - g + 1 < 0.
\]
It should be clarified that the above conditions do not determine $V_i$ uniquely. We take \( \{V_i\}_{i=0} \) to be a collection of vector bundles satisfying the above conditions.

We will prove that (2.11) holds.

To prove (2.11), consider the filtration
\[
0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r \quad (2.14)
\]
of \( V_r \) by holomorphic subbundles obtained from (2.12). Take any
\[
T \in H^0(X, \text{End}(V_r) \otimes \mathcal{O}_X(x)) .
\]
We will first prove that \( T \) preserves the filtration in (2.14), meaning
\[
T(V_i) \subset V_i \otimes \mathcal{O}_X(x) \quad (2.15)
\]
for all \( i \).

Fix any \( i \in \{1, \cdots, r\} \), and consider the short exact sequence
\[
0 \rightarrow \text{Hom}(V_{k+1}/V_k, L_j) \otimes \mathcal{O}_X(x) \rightarrow \text{Hom}(V_{k+1}, L_j) \otimes \mathcal{O}_X(x) \rightarrow \text{Hom}(V_k, L_j) \otimes \mathcal{O}_X(x) \rightarrow 0 \quad (2.16)
\]
obtained from (2.12) by tensoring with \( \mathcal{O}_X(x) \), where \( j > i \) and \( k < i \). Since
\[
H^0(X, \text{Hom}(V_{k+1}/V_k, L_j) \otimes \mathcal{O}_X(x)) = H^0(X, \text{Hom}(L_{k+1}, L_j) \otimes \mathcal{O}_X(x)) = 0
\]
(see the third condition on \( \{L_i\}_{i=1}^r \)), from the long exact sequence of cohomologies associated to (2.16) we conclude that
\[
H^0(X, \text{Hom}(V_{k+1}, L_j) \otimes \mathcal{O}_X(x)) = 0
\]
if \( H^0(X, \text{Hom}(V_k, L_j) \otimes \mathcal{O}_X(x)) = 0 \). Hence using induction on \( k \) it follows that
\[
H^0(X, \text{Hom}(V_i, V_j/V_{j-1}) \otimes \mathcal{O}_X(x)) = H^0(X, \text{Hom}(V_i, L_j) \otimes \mathcal{O}_X(x)) = 0 \quad (2.17)
\]
if \( j > i \). Now from the exact sequence obtained from (2.12) by tensoring with \( \mathcal{O}_X(x) \)
\[
0 \rightarrow \text{Hom}(V_i, V_j/V_{j-1}) \otimes \mathcal{O}_X(x) \rightarrow \text{Hom}(V_i, V_i/V_{j-1}) \otimes \mathcal{O}_X(x) \rightarrow \text{Hom}(V_i, V_i/V_j) \otimes \mathcal{O}_X(x) \rightarrow 0,
\]
where \( j > i \), it follows that
\[
H^0(X, \text{Hom}(V_i, V_i/V_{j-1}) \otimes \mathcal{O}_X(x)) = 0
\]
if \( H^0(X, \text{Hom}(V_i, V_i/V_j) \otimes \mathcal{O}_X(x)) = 0 \), because (2.17) holds. Therefore, using induction on \( j \) it follows that
\[
H^0(X, \text{Hom}(V_i, V_i/V_i) \otimes \mathcal{O}_X(x)) = 0,
\]
which means that (2.15) holds.

Since (2.15) holds, the homomorphism \( T \) induces a homomorphism
\[
T_i : L_i := V_i/V_{i-1} \rightarrow V_i/V_{i-1} \otimes \mathcal{O}_X(x) = L_i \otimes \mathcal{O}_X(x)
\]
for each \( i \). Now, \( H^0(X, \mathcal{O}_X(x)) = \mathbb{C} \) (for this it is enough that \( g \geq 1 \)), and hence it follows that
\[
T_i = \lambda_i \cdot \text{Id}_{L_i}, \quad (2.18)
\]
where \( \lambda_i \in \mathbb{C} \).
As $H^0(X, \text{Hom}(V_i / V_{i-1}, V_j / V_{j-1}) \otimes \mathcal{O}_X(x)) = H^0(X, \text{Hom}(L_i, L_j) \otimes \mathcal{O}_X(x)) = 0$ if $j < i$, it follows that there is no nonzero homomorphism $S: V_r \to V_r \otimes \mathcal{O}_X(x)$ over $X$ which is nilpotent with respect to the filtration in (2.14), meaning $S(V_i) \subset V_{i-1} \otimes \mathcal{O}_X(x)$ for all $i \geq 1$.

If $\lambda_1 = \cdots = \lambda_r$ (constructed in (2.18)), then $T - \lambda_1 \cdot \text{Id}_{V_r}$ is nilpotent with respect to the filtration in (2.14). From the above observation that there are no nonzero nilpotent homomorphisms it would then follow that $T = \lambda_1 \cdot \text{Id}_{V_r}$. Therefore, to prove (2.11) it suffices to show that

$$\lambda_1 = \cdots = \lambda_r. \quad (2.19)$$

Assume that (2.19) fails. Let $k$ be the smallest integer such that $\lambda_k \neq \lambda_1$. Now consider the restriction

$$T' := T|_{V_k} : V_k \to V_k \otimes \mathcal{O}_X(x).$$

Let $L \subset V_k$ be the line subbundle generated by kernel$(T' - \lambda_k \cdot \text{Id}_{V_k})$. The restriction to $L$ of the projection $V_k \to V_k / V_{k-1} = L_k$ is an isomorphism. So $L$ provides a splitting of the short exact sequence obtained by setting $i = k$ in (2.12). But this contradicts the assumption that the extension class $\mu_k$ is nonzero (see (2.13)). Therefore, we conclude that (2.19) holds. As noted before, this proves (2.11).

Let $\mathcal{M}$ be the moduli stack of vector bundles $W$ on $X$ such that rank$(W) = r$ and degree$(W) = d$ (see (2.10)). Let $\mathcal{C} \subset \mathcal{M}$ be the locus of all $W$ such that

$$\dim H^0(X, \text{End}(W) \otimes \mathcal{O}_X(x)) = 1.$$ 

This $\mathcal{C}$ is open by semi-continuity and it is nonempty because $V_n \in \mathcal{C}$. Let $\mathcal{S} \subset \mathcal{M}$ be the stable locus which is also nonempty and open [Ma, p. 635, Theorem 2.8(B)]. Finally, $\mathcal{C}$ and $\mathcal{S}$ intersect because $\mathcal{M}$ is irreducible [BL, p. 396, Proposition 3.4], [BL, p. 394, Proposition 2.6(e)] (see also [DS]).

Let $F$ be a stable vector bundle on $X$ of rank $r$ and degree $d$ such that

$$\dim H^0(X, \text{End}(F) \otimes \mathcal{O}_X(x)) = 1.$$ 

There is a holomorphic line bundle $L$ of degree $m_0$ (see (2.10)) such that $F \otimes L \in M_\xi$. Since $\text{End}(F) = \text{End}(F \otimes L)$, semi-continuity ensures the existence of $U_x$ in the statement of the lemma.

3. Computation of co-Higgs fields

As before, take any point $x \in X$.

**Theorem 3.1.** The homomorphism $\gamma$ in (2.6) is an isomorphism.

**Proof.** The homomorphism $\gamma$ is injective by Proposition (2.1). We will prove that

$$\dim H^0(M_\xi, \text{End}(\mathcal{E}_x) \otimes TM_\xi) = 1 \quad (3.1)$$

which would prove that $\gamma$ is surjective.

Let

$$P := P(\mathcal{E}_x) \xrightarrow{\phi} M_\xi$$
be the projective bundle of relative dimension $r - 1$ that parametrizes all the hyperplanes in the fibers of the vector bundle $\mathcal{E}_x$. Let

$$T_\phi \longrightarrow \mathbb{P}$$

be the relative tangent bundle for the projection $\phi$, so $T_\phi$ is the kernel of the differential $d\phi : T\mathbb{P} \longrightarrow \phi^*TM_\xi$ of $\phi$.

**Lemma 3.2.** There is a natural isomorphism

$$H^0(M_\xi, \text{End}(\mathcal{E}_x) \otimes TM_\xi) \sim \longrightarrow H^0(\mathbb{P}, T_\phi \otimes \phi^*TM_\xi).$$

*Proof.* We have $\phi_* T_\phi = \text{ad}(\mathcal{E}_x)$, so by the projection formula,

$$\phi_* (T_\phi \otimes \phi^*TM_\xi) = \text{ad}(\mathcal{E}_x) \otimes TM_\xi.$$ 

This implies that $H^0(\mathbb{P}, T_\phi \otimes \phi^*TM_\xi) = H^0(M_\xi, \text{ad}(\mathcal{E}_x) \otimes TM_\xi)$. But

$$H^0(M_\xi, \text{End}(\mathcal{E}_x) \otimes TM_\xi) = H^0(M_\xi, \text{ad}(\mathcal{E}_x) \otimes TM_\xi) \oplus H^0(M_\xi, TM_\xi),$$

and $H^0(M_\xi, TM_\xi) = 0$ [NR2, p. 391, Theorem 1(a)], [Hi2, p. 110, Theorem 6.2]. Therefore, the lemma follows. □

Let $\mathcal{N}$ denote the moduli space of semistable vector bundles $E$ on $X$ of rank $r$ and $\bigwedge^r E = \xi \otimes \mathcal{O}_X(-x)$.

For any point $(E, H) \in \mathbb{P}$, we have a vector bundle $V$ on $X$ that fits in the short exact sequence

$$0 \longrightarrow V \longrightarrow E \longrightarrow E_x/H \longrightarrow 0 .$$  (3.2)

Note that $\bigwedge^r V = \xi \otimes \mathcal{O}_X(-x)$, however $V$ is not semistable in general. Nevertheless, there is a nonempty Zariski open subset

$$\mathcal{U} \subset \mathbb{P}$$

satisfying the following four conditions:

1. the codimension of the complement $\mathcal{U}^c \subset \mathbb{P}$ is at least three,
2. for every $(E, H) \in \mathcal{U}$, the corresponding vector bundle $V$ constructed in (3.2) is semistable,
3. for the resulting map

$$\psi : \mathcal{U} \longrightarrow \mathcal{N},$$

the image $\mathcal{Y} := \psi(\mathcal{U})$ is Zariski open in $\mathcal{N}$ with the codimension of the complement being $\mathcal{Y}^c \subset \mathcal{N}$ at least three, and
4. the map $\psi$ is a projective fibration over $\mathcal{Y}$.

(See [NR2].)

Consider the differential $d\psi : T\mathcal{U} \longrightarrow \psi^*\mathcal{N}$ of $\psi$ in (3.3). The kernel

$$T_\psi := \text{kernel}(d\psi) \subset T\mathcal{U}$$

is the relative tangent bundle on $\mathcal{U}$.

Let $\tilde{T}_x := \mathcal{U} \times T_x X \longrightarrow \mathcal{U}$ be the trivial line bundle on $\mathcal{U}$ with fiber $T_x X$. On $\mathcal{U}$, we have

$$T_\phi = \tilde{T}_x \otimes T_\psi^*$$  (3.4)
Next we have
\[ R^0\psi_*(T^*_\psi \otimes \phi^*TM_\xi) = \mathcal{O}_Y \]
(see [Bi, p. 266, Lemma 3.1]). So,
\[ H^0(U, T^*_\psi \otimes \phi^*TM_\xi) = \mathbb{C} . \]
Now combining Lemma 3.2 and (3.4) it follows that
\[ H^0(M_\xi, \text{End}(E_x) \otimes TM_\xi) = T_xX . \]
This proves (3.1).

Remark 3.3. The assumption that \( g \geq 3 \) was used in the proofs of Proposition 2.1 and Theorem 3.1; more precisely, this assumption is used in the proofs of Lemma 2.2 and Lemma 3.2. Well-known is the fact that a holomorphic vector bundle on \( \mathbb{C}P^1 \) decomposes into a direct sum of holomorphic line bundles [Gr]; hence, there are no strictly stable vector bundles on \( \mathbb{C}P^1 \) of rank larger than 1. From Atiyah’s classification of vector bundles on an elliptic curve \( Y \) [At], we know that the moduli space \( M_\xi \) for \( Y \) is a single point. What remains is the case of \( g = 2 \). When \( r = g = 2 \), the moduli space \( M_\xi \) has an explicit description [NR1]. However it is not clear whether Proposition 2.1 and Theorem 3.1 hold in this special case.

Acknowledgements

We thank the referee for helpful comments. The first author acknowledges support of a J. C. Bose Fellowship. The second author acknowledges the support of a New Faculty Recruitment Grant from the University of Saskatchewan.

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