VALUATIONS AND METRICS ON PARTIALLY ORDERED SETS

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Abstract. We extend the definitions of upper and lower valuations on partially ordered sets, and consider the metrics they induce, in particular the metrics available (or not) based on the logarithms of such valuations. Motivating applications in computational linguistics and computational biology are indicated.

1. Introduction

This expository note is motivated by our answer, given herein as Propositions 7 and 8, to the following question: let \( P = (P, \leq) \) be a poset with an upper or lower valuation \( v(x) : P \to \mathbb{R}^+ \); then is \( \ell(x) = \log v(x) \) necessarily an upper or lower valuation? (These terms are defined below.)

The question arises from the common practice in information systems (see e.g. \cite{2}) of using measures of “semantic similarity” in large taxonomic vocabularies such as WordNet\(^1\) (in computational linguistics) or the Gene Ontology\(^2\) (in computational biology) \cite{5}. Such similarity measures are based on a quantification of information content as \( I(x) = -\log p(x) \), where \( p(x) \) is a kind of cumulative probability defined on a poset \( P \) representing the hierarchical structure of the taxonomy. As such, \( p(x) \) has the form stated in Proposition 4 and is often a lower valuation.

This question also arises from Example\(^3\) that deals with valuations on the \( \land \)-semilattice \( \mathcal{L} \) of finite subgroups \( X \) of a given group \( G \): both \( c(X) = |X| \) and \( v(X) = \log c(X) \) are lower valuations on \( \mathcal{L} \). Notwithstanding this example, the logarithm of a (positive) lower valuation need not be a lower valuation. On the other hand, the logarithm of a positive upper valuation is always an upper valuation.

By focusing on this question we bring together some results (some of which are only implicit in \cite{6}, the primary predecessor of this work) concerning the practical differences between upper and lower valuations defined on partially ordered sets; and we describe the metrics they induce. We extend the previous definitions of upper and lower valuations to allow for antitone maps (instead of requiring that valuation be isotone). The symmetry introduced by this extension allows us to consider the composition \( \log(K \cdot v(x) + A) \) where \( v(x) \) is an upper valuation or a lower valuation, \( K \in \mathbb{R}/\{0\}, A \in \mathbb{R}, \) and \( K \cdot v(x) + A > 0 \).

Distance formulas involving \( I(x) = -\log p(x) \) appear in the literature. We note that such a formula introduced by Jiang and Conrath \cite{3} does not, in general, define a metric on a partially ordered set. (Under the tacit assumption that the poset is a tree, however, it does yield a metric.)

While the literature on lattice valuations extends back to Wilcox and Smiley (1939) \cite{9,10} and Birkhoff (1940) \cite{1}, the literature on general poset valuations is quite thin: we are only aware of \cite{6,7}.

\[^1\]http://wordnet.princeton.edu
\[^2\]http://geneontology.org

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2. Preliminaries and notation

In the sequel $P = (P, \leq)$ always denotes a partially ordered set. The greatest lower bound or meet of two elements $x, y \in P$ need not exist, but if it does it is denoted $x \land y$. An ordered set $P$ in which $x \land y$ always exists is $\land$-semilattice. The least upper bound or join of two elements $x, y \in P$ need not exist, but if it does it is denoted $x \lor y$. An ordered set $P$ in which $x \lor y$ always exists is $\lor$-semilattice. If $P$ is both a $\land$-semilattice and $\lor$-semilattice then $P$ is a lattice.

We write $a < c$ if $c$ covers $a$ ($a \leq b \leq c$ and $a \neq c$ implies $a = b$ or $b = c$). The notation $\{a, b\} < \{c, d\}$ means both $c$ and $d$ cover both $a$ and $b$, etc. Given a subset $S \subseteq P$, $\min(S) \subseteq S$ and $\max(S) \subseteq S$ denote the minimal and maximal elements of $S$ respectively. If $P$ has a unique minimal or maximal element, it is denoted by 0 or 1, respectively. If $0 \in P$ and $1 \in P$ then $P$ is said to be a bounded.

Given an element $x \in P$, $F_x = \{x' \in P : x \leq x'\}$ is the principal filter generated by $x$. Given an element $x \in P$, $I_x = \{x' \in P : x' \leq x\}$ is the principal ideal generated by $x$.

We shall call a finite $\lor$-semilattice $L$ in which $I_x \cap I_y = \emptyset$ for all $x, y \in L$ a tree.

If $S$ and $T$ are nonempty subsets of a multiplicative group $G = (G, \cdot, e)$, then $ST = \{st : s \in S \text{ and } t \in T\}$. The index of a subgroup $S \subseteq G$ is $[G : S]$, and if $G$ is finite $[G : S] = |G|/|S|$. If $S$ and $T$ are subgroups of $G$ the smallest subgroup containing both $S$ and $T$ is denoted $S \lor T$.

3. Valuations and metrics

Let $P$ be a poset. A function $f : P \to \mathbb{R}$ is isotone if $x \leq y$ implies $f(x) \leq f(y)$ and strictly isotone if $x < y$ implies $f(x) < f(y)$. It is antitone if $x \leq y$ implies $f(x) \geq f(y)$ and strictly antitone if $x < y$ implies $f(x) > f(y)$. Assuming $f$ is monotone (that is, either isotone or antitone) we use the notation

$$f^-(x, y) = \begin{cases} \sup\{f(z) : z \in I_x \cap I_y\}, & \text{if } f \text{ is isotone}, \\ \inf\{f(z) : z \in I_x \cap I_y\}, & \text{if } f \text{ is antitone}, \end{cases}$$

$$f^+(x, y) = \begin{cases} \inf\{f(z) : z \in F_x \cap F_y\}, & \text{if } f \text{ is isotone}, \\ \sup\{f(z) : z \in F_x \cap F_y\}, & \text{if } f \text{ is antitone}. \end{cases}$$

Note that $I_x \cap I_y$ or $F_x \cap F_y$ may be empty. We use the convention $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

**Definition 1.** Let $P$ be a poset. An isotone (antitone) function $v : P \to \mathbb{R}$ is a lower valuation if for all $x, y \in P$, $I_x \cap I_y \neq \emptyset$ ($F_x \cap F_y \neq \emptyset$) and

$$v(x) + v(y) \leq v^-(x, y) + v^+(x, y).$$

An isotone (antitone) function $v : P \to \mathbb{R}$ is an upper valuation if for all $x, y \in P$, $F_x \cap F_y \neq \emptyset$ ($I_x \cap I_y \neq \emptyset$) and

$$v^-(x, y) + v^+(x, y) \leq v(x) + v(y).$$

In the sequel we shall assume, as part of the definition of $v : P \to \mathbb{R}$ being an upper or lower valuation, the associated condition on filters or ideals in $P$. Definition 1 generalizes the definitions given by Monjardet [6], Leclerc [4]. A benefit of considering both isotone and antitone valuations may be seen in Proposition 8.

**Definition 2 (Monjardet [6]).** Let $P$ be a poset with 0. An isotone function $v : P \to \mathbb{R}$ is a lower valuation if for all $x, y, z \in P$ with $x \leq z$, $y \leq z$,

$$v(x) + v(y) \leq v^-(x, y) + v(z).$$
Let $P$ be a poset with 1. An isotone function $v : P \to \mathbb{R}$ is an upper valuation if for all $x, y, z \in P$ with $z \leq x, z \leq y$,

$$v^+(x, y) + v(z) \leq v(x) + v(y). \quad (6)$$

**Definition 3** (Leclerc [4]). Let $L = (L, \leq, \land)$ be a $\land$-semilattice. A strictly isotone function $v : L \to \mathbb{R}$ is a lower valuation if and only if it satisfies the following property whenever $x \lor y$ exists:

$$v(x) + v(y) \leq v(x \lor y) + v(x \land y). \quad (7)$$

Let $L = (L, \leq, \lor)$ be a $\lor$-semilattice. A strictly isotone function $v : L \to \mathbb{R}$ is an upper valuation if and only if it satisfies the following property whenever $x \land y$ exists:

$$v(x \lor y) + v(x \land y) \leq v(x) + v(y). \quad (8)$$

**Proposition 1.** Let $P$ be a finite poset equipped with a valuation $v(x) : P \to \mathbb{R}$ having the properties listed (by row) in Table 1. Then the corresponding formula for $d_v(x, y)$ defines a metric on $P$.

Proof. Suppose that $v$ is a strictly isotone lower valuation; the other cases are similar. We verify the triangle inequality. Fix $x, y, z \in P$. The inequality $v^-(x, y) + v^-(y, z) \leq v^-(x, z) + v(y)$, which we now establish, implies $d_v(x, z) \leq d_v(x, y) + d_v(y, z)$. Let

$$\alpha \in \{ p \in I_x \cap I_y : v(p) = v^-(x, y) \}, \quad (9)$$

$$\beta \in \{ p \in I_y \cap I_z : v(p) = v^-(y, z) \}. \quad (10)$$

(These sets are nonempty by the hypothesis that $P$ is finite.) Since $v$ is an isotone lower valuation, we have

$$v^-(x, y) + v^-(y, z) = v(\alpha) + v(\beta) \leq v^- (\alpha, \beta) + v(y),$$

and since $\alpha < x, \beta < z$, it follows that $v^- (\alpha, \beta) < v^- (x, z)$. \qed

If the valuation is merely isotone (or antitone), then the corresponding $d_v(x, y)$ is a quasimetric, which is defined by relaxing the metric condition `$d(x, y) = 0 \Rightarrow x = y$'. Note that if $P$ is not finite, then $d_v(x, y)$ is a quasimetric but it need not be a metric.

### 3.1. Bounds on $d_v(x, y)$

Before turning to examples we note the following bounds, and a condition for their universal attainment.

**Proposition 2.** Suppose $v : P \to \mathbb{R}$ is either an upper or lower valuation. Then

$$d_v(x, y) \leq v^+(x, y) - v^-(x, y) \quad \text{if $v$ is isotone,}$$

$$d_v(x, y) \leq v^-(x, y) - v^+(x, y) \quad \text{if $v$ is antitone.} \quad (11)$$

In either case, equality holds for all $x, y \in P$ if and only if $v$ is both an upper and lower valuation.
Proposition 3. Let \( L = (L, \leq, \lor) \) be a finite \( \lor \)-semilattice with \( 0 \in L \), and \( v : L \to \mathbb{R} \) be a strictly isotone upper or lower valuation. Equality can not hold in \((12)\) for all \( x, y \in L \) unless \( L \) is a modular lattice.

Proof. Let \( z_0 \in \{ z \in I_x \cap I_y : v(z) = v^- (x, y) \} \neq \emptyset \). Then \( z_0 \) is a lower bound of \( x \) and \( y \). Let \( c \) be any other lower bound of \( x \) and \( y \). Then \( c \lor z_0 \) is a lower bound of both \( x \) and \( y \) which implies \( v(c \lor z_0) \leq v(z_0) \). But since \( v \) is isotone \( v(z_0) \leq v(c \lor z_0) \). Strict isotonicity of \( v \) implies \( z_0 = c \lor z_0 \), hence \( z_0 \geq c \) and \( z_0 \) is the greatest lower bound of \( x \) and \( y \). Therefore \( L \) is a lattice. Now assume that equality holds in \((11)\) for all \( x, y \in L \). By Proposition \( \square \) \( v \) is both an upper and lower valuation, hence \( v \) is a valuation on \( L \) (meaning that \( v(x) + v(y) = v(x \lor y) + v(x \land y) \) for all \( x, y \in L \)). It is well known that the existence of a strictly isotone valuation on a lattice \( L \) implies that \( L \) is a modular lattice \( \square \).

3.2. Examples and discussion. On a finite \( \land \)-semilattice \( v_\ast (x) = |I_x| \) is an isotone lower valuation; on a finite \( \lor \)-semilattice \( v^\ast (x) = |F_x| \) is an antitone lower valuation. More generally we have the following, in which cardinality is replaced by a sum over a nonnegative weighting function:

Proposition 4. Let \( P \) be a finite \( \land \)-semilattice, \( t(x) : P \to [0, \infty) \) a non-negative weighting function. Then the map \( v_\ast : P \to [0, \infty) \) defined by

\[
(12) \quad v_\ast (x) = \sum_{x' \leq x} t(x')
\]

is an isotone lower valuation. If \( t(x) \) is strictly positive then \( v_\ast (x) \) is strictly isotone.

Proof. Since \( P \) is a \( \land \)-semilattice, \( v^- (x, y) = v_\ast (x \land y) \); it is sufficient to establish that for all \( x, y, z \in P \) such that \( x \leq z, y \leq z \), that \( v_\ast (x) + v_\ast (y) \leq v_\ast (z) + v_\ast (x \land y) \). Fix \( x, y \in P \) and let \( J_x \) and \( J_y \) denote the disjoint sets \( J_x = I_x \cap (I_x \land y)^c \), \( J_y = I_y \cap (I_x \land y)^c \). For any \( z \in P \) such that \( x \leq z, y \leq z \), we have the disjoint union and inclusion:

\[
(13) \quad J_x \cup J_y \cup I_x \land y = I_x \cup I_y \subseteq I_z.
\]

Then

\[
(14) \quad v_\ast (x) + v_\ast (y) - v_\ast (x \land y) = \sum_{w \leq x} t(w) + \sum_{w \leq y} t(w) - \sum_{w \leq x \land y} t(w)
\]

\[
(15) \quad = \sum_{w \in J_x} t(w) + \sum_{w \in J_y} t(w) + \sum_{w \in I_x \land y} t(w) \leq v(z). \quad \square
\]

The proof of the following proposition is similar and is omitted:

Proposition 5. Let \( P \) be a finite \( \lor \)-semilattice, \( t(x) : P \to [0, \infty) \) a non-negative weighting function. Then the map \( v^\ast : P \to [0, \infty) \) defined by

\[
(16) \quad v^\ast (x) = \sum_{x' \leq x} t(x')
\]

is an antitone lower valuation. If \( t(x) \) is strictly positive then \( v_\ast (x) \) is strictly antitone.
If \( t(x) \) is the indicator function of any subset \( K \subseteq P \), where \( P \) is a finite \( \lor \)-semilattice, then \( v^*(x) \) as given by (16), is a lower valuation and \( \kappa(x) = A - v^*(x) \) is an upper valuation for any \( A \in \mathbb{R} \). Letting \( K \) denote the meet-irreducible elements of \( P \), \( K(x) = \{ k \in K : x \leq k \} \), and \( A = |K| \), yields the upper valuation \( \kappa(x) = |K/K(x)| \) given in [4]. (The use of meet-irreducible elements is not necessary for defining the upper valuation \( \kappa(x) \) given in [4]; we may replace \( K \) by any subset of \( P \) and obtain an upper valuation.)

If \( P \) is a poset that is not a \( \land \)-semilattice, then \( v_+(x) = |I_x| \) need not be a lower valuation. For example, \( v_+(x) = |I_x| \) is not a lower valuation on the poset defined by the covering relations: \( 0 < \{ a, b, c \} < \{ d, e \} < 1 \). Similarly if \( P \) is a poset that is not a \( \lor \)-semilattice then \( v^*(x) = |F_x| \) need not be a lower valuation.

To extend this counterexample, we consider sufficient conditions for \( v_+(x) = |I_x| \) and \( v^*(x) = |F_x| \) to be lower valuations: let \( \mathcal{P}^* \) denote the collection of finite bounded partially ordered sets, which includes all finite lattices. A measure of the degree to which a poset \( P \in \mathcal{P}^* \) deviates from being a \( \land \)-semilattice or \( \lor \)-semilattice (which are equivalent for \( P \in \mathcal{P}^* \)) is given by the functions \( \Delta_\land, \Delta_\lor : \mathcal{P}^* \to \mathbb{N}_0 \), defined by

\[
\Delta_\land(P) = \max_{x,y \in P} D_\land(x,y), \quad D_\land(x,y) = |I_x \land I_y| - \max\{|I_x| : z \in \max(I_x \land I_y)\},
\]

\[
\Delta_\lor(P) = \max_{x,y \in P} D_\lor(x,y), \quad D_\lor(x,y) = |F_x \lor F_y| - \min\{|F_x| : z \in \min(F_x \lor F_y)\}.
\]

**Proposition 6.** Suppose \( P \in \mathcal{P}^* \). Then \( P \) is a lattice if and only if \( \Delta_\land(P) = 0 \) or \( \Delta_\lor(P) = 0 \). If \( \Delta_\land(P) \leq 1 \) then \( v_+(x) = |I_x| \) is a lower valuation on \( P \). If \( \Delta_\lor(P) \leq 1 \) then \( v^*(x) = |F_x| \) is a lower valuation on \( P \).

**Proof.** The first assertion follows directly from the definitions. For the second, assume \( \Delta_\land(P) \leq 1 \). Accordingly, \( v_+(x,y) = \max\{|I_z| : z \in \max(I_x \land I_y)\} \geq |I_x \land I_y| - 1 \).

We also have

\[
v_+(x) = 1 + |I_x/(I_x \land I_y)| + |I_y/(I_x \land I_y)| + |I_x \land I_y|,
\]

\[
v_+(y) = |I_x/(I_x \land I_y)| + |I_y|,
\]

so that \( v_+(x) \) satisfies (3), and is therefore a lower valuation. The case \( \Delta_\lor(P) \leq 1 \) is similar.

Our next example leads back to the question on logarithms:

**Example 1.** Let \( G = (G, \cdot, e) \) be a multiplicative group and \( \mathcal{L} = (\mathcal{L}, \subseteq) \) be the collection of finite subgroups of \( G \), partially ordered by inclusion. Then \( \mathcal{L} \) is a \( \land \)-semilattice in which \( X \land Y = X \cap Y \). The maps \( c(X) = |X| \) and \( v(X) = \log |X| \) are both lower valuations on \( \mathcal{L} \), the latter inducing the so-called finite subgroup metric:

\[
d_v(X,Y) = \log \frac{|X||Y|}{(|X \cap Y|^2}.
\]

If \( G \) is abelian, then \( \mathcal{L} \) is a lattice (but \( \mathcal{L} \) is not necessarily a complete lattice) and \( v(X) = \log |X| \) is an upper valuation as well.

**Proof.** Whether or not \( XY \) is a subgroup of \( G \), the product formula \([8\text{, p. 14}]\) states that

\[
|X||Y| = |XY||X \cap Y|.
\]

Let \( m = |X : X \cap Y| \) and \( n = |Y : X \cap Y| \). Then (24) implies \( |XY| = mn|X \cap Y| \), and since \( m + n \leq mn + 1 \) for all \( m, n \in \mathbb{Z}^+ \) it follows that \( |X| + |Y| \leq |X \cap Y| + |XY| \). Hence if \( X \lor Y \in \mathcal{L} \), then \( |X| + |Y| \leq |X \land Y| + |X \lor Y| \) and \( c(X) \) is a lower valuation. The fact
Proposition 8. \( v(X) = \log |X| \) is a lower valuation follows from (24) and \( XY \subseteq X \lor Y \). If \( G \) is abelian then \( |X \lor Y| = |XY| \) and \( v(X) \) is also an upper valuation. If \( G \) is an infinite abelian group then \( X, Y \in \mathcal{L} \Rightarrow X \land Y \in \mathcal{L} \), \( X \lor Y \in \mathcal{L} \) (\( X \lor Y \) is finite) so that \( \mathcal{L} \) is a lattice, but the the join over an arbitrary number of finite subgroups need not be finite so \( \mathcal{L} \) need not be a complete lattice.

4. Composition with logarithms

Suppose \( v(x) : P \to \mathbb{R} \) is either an upper valuation or lower valuation, either isotone or antitone. Observe that

\[
\ell'(x) = K \cdot v(x) + A, \quad K \in \mathbb{R}/\{0\}, \quad A \in \mathbb{R},
\]

is also an upper or lower valuation, and if \( K < 0 \), upper and lower are interchanged, as well as isotone and antitone.

Proposition 7. Suppose \( u : P \to \mathbb{R}^+ \) is a strictly positive isotone (antitone) upper valuation. Then \( \ell(x) = \log u(x) \) is an isotone (antitone) upper valuation. On the other hand, if \( v(x) : P \to \mathbb{R}^+ \) is a strictly positive isotone lower valuation then \( \ell'(x) = \log v(x) \) need not be an upper valuation or a lower valuation.

Proof. Let \( x, y \in P \), and let \( a = u^+(x,y) \), \( b = u(x) \), \( c = u(y) \), \( d = u^-(x,y) \). We treat the case that \( u(x) \) is isotone. By hypothesis

\[
a + d \leq b + c, \quad a, b, c, d > 0,
\]

\[
d \leq \min\{b,c\} \leq \max\{b,c\} \leq a.
\]

Since \( \ell(x) = \log u(x) \) is isotone, \( \ell^+(x,y) = \log u^+(x,y) \) and \( \ell^-(x,y) = \log u^-(x,y) \). The function \( \ell(x) \) is an upper valuation because it satisfies (4), that is,

\[
\ell^+(x,y) + \ell^-(x,y) = \log a + \log d \leq \log b + \log c = \ell(x) + \ell(y),
\]

or equivalently, \( ad \leq bc \). Indeed, let \( d = \min\{b,c\} - X, a = \max\{b,c\} + Y \), where \( X, Y \geq 0 \). Note that \( Y \leq X \) follows from (26). Since \( bc = \min\{b,c\} \cdot \max\{b,c\} \), we have

\[
ad = bc + Y \cdot \min\{b,c\} - X \cdot \max\{b,c\} - XY
\]

\[
\leq bc + X (\min\{b,c\} - \max\{b,c\}) - XY \leq bc.
\]

The case that \( u(x) \) is antitone may be treated similarly. If \( u(x) \) is antitone, then instead of (27) we have \( a \leq \min\{b,c\} \leq \max\{b,c\} \leq d \), while (26) still holds.

Finally, as a counterexample, consider the lower valuation \( v(x) = \sum_{x' \leq x} t(x') \) defined on the Boolean lattice \( M_2 \) with covering relations \( 0 < \langle p, q \rangle < 1 \), where \( t : M_2 \to \mathbb{R}^+ \) is a discrete probability distribution (\( v(1) = 1 \)). Then \( \log v(x) \) need not be an upper or lower valuation (depending on \( t(x) \)).

Combining Proposition 7 with the observation preceding its statement yields the following:

Proposition 8. Suppose \( u : P \to \mathbb{R} \) is an isotone (antitone) upper valuation. Then \( L(x) = \log(K \cdot u(x) + A) \) is an isotone (antitone) upper valuation for any \( K > 0 \) and \( A > -\min_{x \in P} K \cdot u(x) \). Suppose \( v : P \to \mathbb{R} \) is an isotone (antitone) lower valuation. Then \( L'(x) = -\log(K \cdot v(x) + A) \) is an isotone (antitone) lower valuation for any \( K < 0 \) and \( A > \max_{x \in P} |K| \cdot v(x) \).

While the valuations \( L(x) \) and \( L'(x) \) of Proposition 8 are available for defining metrics on \( P \), we note that the formula of Jiang and Conrath [3]

\[
\text{dist}_{IC}(x,y) = I(x) + I(y) - 2I^+(x,y), \quad I(x) = -\log p(x),
\]

(30)
in which \( p(x) \) is a cumulative probability of the form \( \left( \frac{x}{12} \right) \), is not necessarily a metric defined on a general poset. As a counterexample, consider the poset defined by the covering relations: \( \{ z_1, z_2 \} \prec a, \{ z_1, z_3 \} \prec b, \{ z_2, z_3 \} \prec c, \{ a, b, c \} \prec 1 \) with discrete probability distribution \( t(x) \) and cumulative probability \( p(x) = \sum_{x' \leq x} t(x) \). Then \( \text{dist}_{JC}(x, y) \) need not be a metric: depending on \( t(x) \), it can happen that

\[
\text{dist}_{JC}(z_1, z_2) + \text{dist}_{JC}(z_2, z_3) \leq \text{dist}_{JC}(z_1, z_3).
\]

A sufficient condition for \( \text{dist}_{JC}(x, y) \) to be a metric is that the poset be a tree.

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