Uniqueness of the electrostatic solution in Schwarzschild space

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In this Brief Report we give the proof that the solution of any static test charge distribution in Schwarzschild space is unique. In order to give the proof we derive the first Green’s identity written with \( p \)-forms on (pseudo) Riemannian manifolds. Moreover, the proof of uniqueness can be shown for either any purely electric or purely magnetic field configuration. The spacetime geometry is not crucial for the proof.

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I. INTRODUCTION

In 1971, Cohen and Wald [1] presented the electrostatic field of a point test charge in the Schwarzschild background by using a multipole expansion. For the same problem, Linet found the solution in algebraic form [2]. Recently, Molnár generalized the solution for any test charge distribution in Schwarzschild space with boundary values [3]. He gave the solution not only in algebraic form but also in terms of a multipole expansion. Therefore the question arises whether the solution of any test charge distribution held at rest near a Schwarzschild black hole is unique.

The Brief Report is organized as follows: In Sec. II we derive Green’s first identity written with \( p \)-forms on (pseudo) Riemannian manifolds. Holmann and Rummler gave in their book a version for a pair of \( p \)-forms on Riemannian manifolds [4]. However, our version is more general because it is valid for a pair of two \( p \)-forms and for pseudo-Riemannian manifolds too. The Schwarzschild spacetime, for example, is pseudo-Riemannian. In Sec. III the uniqueness of the electrostatic solution is derived with the help of Green’s first identity. In Sec. IV we replace the spherical symmetry by the axial one and show that, for the purely electric or purely magnetic solutions, the proof can be completed along the same lines.

II. THE FIRST GREEN’S IDENTITY

Let \((M, g)\) be an \(n\)-dimensional oriented (pseudo) Riemannian manifold and let \(D\) be a region of \(M\) with smooth boundary such that \(\bar{D}\) is compact. For the two \(p\)-forms \(u, v \in \bigwedge^p(M)\) we consider the combination

\[
\int_{\partial D} u \wedge * dv = \int_{\bar{D}} d(u \wedge * dv) = \int_{\bar{D}} du \wedge * dv + (-1)^p \int_{\bar{D}} u \wedge d * dv. \tag{2}
\]

Now, we write the term \(d \ast dv\) in Eq. (2) in a different form. For every form \(\omega \in \bigwedge^k(M)\)

\[
\ast \ast \omega = (-1)^{(n-k)} \text{sgn}(g) \omega. \tag{3}
\]

This gives us

\[
(-1)^{(k-n)} \text{sgn}(g) \ast \ast \omega = \omega. \tag{4}
\]

With \(\omega = d \ast dv\) and \(k = n - p\)

\[
d \ast dv = (-1)^{(n-p)(n-p-n)} \text{sgn}(g) \ast \ast d \ast dv = (-1)^{p(n-p-n)} \text{sgn}(g) \ast \ast d \ast dv. \tag{5}
\]

The codifferential \(\delta : \bigwedge^q(M) \to \bigwedge^{q-1}(M)\) is defined by

\[
\delta := \text{sgn}(g) (-1)^{nq+n} \ast d \ast. \tag{6}
\]

We solve Eq. (6) for \(\ast d\ast\)

\[
\ast d\ast = (-1)^{-nq-n} \text{sgn}(g) \delta. \tag{7}
\]

Setting Eq. (7) in Eq. (5) we obtain for \(d \ast dv\) (\(q = p + 1\))

\[
d \ast dv = (-1)^p \ast \delta dv. \tag{8}
\]

We set Eq. (8) in Eq. (2)

\[
\int_{\partial D} u \wedge d \ast dv = \int_{D} du \wedge \ast dv + \int_{D} u \wedge \delta dv. \tag{9}
\]

Since for two \(p\)-forms \(\alpha, \beta\)

\[
\alpha \wedge \ast \beta = \beta \wedge \ast \alpha,
\]

Eq. (9) becomes

\[
\int_{\partial D} u \wedge \ast dv = \int_{D} du \wedge \ast dv + \int_{D} \delta dv \wedge \ast u. \tag{10}
\]
Next, we consider
\[ \delta v \wedge u \in \bigwedge_{n-1} (M) \, . \]

Using Stokes’ theorem and the anti-Leibniz rule again we have
\[ \int_{\partial D} \delta v \wedge u = \int_D d\delta v \wedge u + (-1)^{p-1} \int_D \delta v \wedge d \ast u \, . \quad (11) \]

Now we add Eqs. (11) and (10)
\[ \int_{\partial D} (u \wedge dv + \delta v \wedge u) = \int_D du \wedge dv + \int_D \Box v \wedge u + (-1)^{p-1} \int_D \delta v \wedge d \ast u \, . \]

This is Green’s first identity written with \( p \)-forms on a (pseudo) Riemannian manifold, where \( \Box := d \circ \delta + \delta \circ d \) is the Laplace-Beltrami operator.

One can easily derive the second Green’s identity with the help of Eq. (12). If we write down Eq. (12) again with \( u \) and \( v \) interchanged, and subtract it from Eq. (12), we have
\[ \int_{\partial D} (u \wedge dv - v \wedge du + \delta v \wedge u - \delta u \wedge v) \]
\[ = \int_D (\Box v \wedge u - \Box u \wedge v) + (-1)^{p-1} \int_D \delta v \wedge d \ast u \, . \quad (13) \]

Note that \( dv \wedge du = \delta v \wedge du \). By definition it follows that
\[ \delta v \wedge d \ast u = \text{sgn}(g) (-1)^{p+n} \ast d \ast v \wedge d \ast u \]
\[ = \text{sgn}(g) (-1)^{p+n} \ast d \ast u \wedge d \ast v = \delta u \wedge d \ast v \, . \quad (14) \]

With the help of Eq. (14) the second term on the right-hand side of Eq. (13) cancels and we find
\[ \int_{\partial D} (u \wedge dv - v \wedge du + \delta v \wedge u - \delta u \wedge v) \]
\[ = \int_D (\Box v \wedge u - \Box u \wedge v) \, . \quad (15) \]

This is Green’s second identity which was already derived by Molnár [3].

### III. UNIQUENESS OF THE ELECTROSTATIC SOLUTION

We write Maxwell’s equations with the exterior calculus
\[ dF^{(1)} = 0 \, , \quad \delta F^{(1)} = 4\pi J \, . \quad (16) \]

By Poincaré’s lemma we can introduce a potential form \( A^{(1)} = A^{(1)}_\mu dx^\mu \) with \( F^{(1)} = dA^{(1)} \). Then, the inhomogeneous Maxwell equations become
\[ \delta dA^{(1)} = 4\pi J \, . \quad (17) \]

The gauge freedom permits us to choose a special gauge condition for \( A^{(1)} \). We require the Lorenz condition \( \delta A^{(1)} = 0 \). Thus, Eq. (17) may be written in the form
\[ \Box A^{(1)} = 4\pi J \, . \quad (18) \]

Now, suppose that there exists another solution \( A^{(2)} \) satisfying the Lorenz condition, Eq. (18), and the same boundary conditions. Let \( A \in \bigwedge_1 (M) \)
\[ A := A^{(2)} - A^{(1)} \, . \quad (19) \]

Then
\[ \Box A = 0 \, , \quad \delta A = 0 \, , \quad A|_{\partial D} = 0 \, . \quad (20) \]

We set \( A = u = v \) in Green’s first identity (12)
\[ \int_{\partial D} (A \wedge dA + \delta A \wedge A) = \int_D dA \wedge dA + \int_D \Box A \wedge A + (-1)^{p-1} \int_D \delta A \wedge d \ast A \, . \quad (21) \]

With the specified properties of \( A \), this reduces to
\[ \int_D dA \wedge dA = 0 \, . \quad (22) \]

In the following we write \( F = dA \). \( F \in \bigwedge_2 (M) \) denotes, like \( A \), the difference of two solutions
\[ F = F^{(2)} - F^{(1)} \, . \quad (23) \]

Thus, Eq. (22) becomes
\[ \int_D F \wedge dF = 0 \, . \quad (24) \]

One can show that
\[ F \wedge dF = (F, F) \eta \, , \quad (25) \]

where
\[ (F, F) = \frac{1}{2} F_{\mu \nu} g^{\mu \alpha} g^{\nu \beta} F_{\alpha \beta} \quad (26) \]

is the scalar product induced in \( \bigwedge_2 (M) \) and \( \eta \in \bigwedge_4 (M) \) is the volume element. The uniqueness of solutions of Eq. (22) can only be shown for particular cases where the scalar product is semidefinite.

Now, consider the electrostatic potential \( \Phi^{(1)} \) of a static test charge distribution in the Schwarzschild background [2, 3] and suppose that there is another solution \( \Phi^{(2)} \). Let \( \Phi \) be the difference between \( \Phi^{(2)} \) and \( \Phi^{(1)} \)
\[ \Phi := \Phi^{(2)} - \Phi^{(1)} \, . \quad (27) \]

Then, the vector potential is
\[ A_\mu = \Phi \delta^0_\mu \quad (28) \]
and for the Schwarzschild metric we can write
\[ g^\mu\nu = 0 \quad \text{for} \quad \mu \neq \nu . \] (29)
Thus, Eq. (20) becomes
\[ (F, F) = g^{00} g^{i\dot{i}} (\Phi_i) = \text{definite} . \] (30)
Since the scalar product (20) is definite for this special case, we can conclude with Eq. (24) that
\[ \Phi_j = 0 \quad \text{for all} \quad i = 1, 2, 3 . \] (31)
Consequently, inside $D$, $\Phi$ is constant. For Dirichlet boundary conditions (26), $\Phi = 0$ on $\partial D$ so that, inside $D$, $\Phi(1) = \Phi(2)$ and the solution is unique.

So far we restricted our analysis to the static case. However, the proof of uniqueness can be straightforwardly generalized to the case of any purely electric or purely magnetic field configuration, because one can infer with the scalar product (26) that all solutions are unique for which $F_{00}$ or $F_{ij}$ (i. e. $E_i$ or $B_k$) vanish and the scalar product of $E$ or $B$ is definite.

IV. STATIONARY AND AXISYMMETRIC SYSTEMS

We call a system stationary and axisymmetric when all physical quantities, including the metric tensor components $g_{\mu\nu}$, which describe the system are independent of time $t$ and of a toroidal angle $\varphi$. We choose a coordinate system $(x^\mu)$ with $x^0 = ct$ ($c = 1$), $x^1 = \varphi$, and $x^2, x^3$ some poloidal coordinates. Assuming in addition that all physical quantities are invariant to the simultaneous inversion of $t$ and $\varphi$ — which is reasonable for any rotating equilibrium — the most general line element $ds$ can be represented as follows [8]:
\[ (ds)^2 = g_{rs} dx^r dx^s + g_{ab} dx^a dx^b , \]
\[ g_{ab} = 0 \quad \text{for} \quad a \neq b , \] (32)

Thus, we can write
\[ (F, F) = A_{0,0} A_{0,0} g^{ab} g^{00} + A_{1,0} A_{1,0} g^{ab} g^{11} + 2A_{0,0} A_{1,0} g^{ab} g^{01} . \] (37)

We see again that uniqueness is only obtained in general if either $A_0$ or $A_1$ are zero, i. e., if either the electrostatic field or the magnetic field vanishes. This is also true for a diagonal metric (Schwarzschild, Minkowski, $g^{01} = 0$). In other words, the symmetry of the spacetime geometry is not crucial for the proof.

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