1. Introduction

In [FF] B. Feigin and E. Frenkel introduced semiinfinite analogues of the classical Bernstein-Gelfand-Gelfand (BGG) resolutions of integrable simple modules over affine Lie algebras. These resolutions are two sided complexes consisting of direct sums of so-called Wakimoto modules suitable for computation of semi-infinite homology of infinitely twisted Borel subalgebras in affine Lie algebras (see e.g. [F] for the definition of the Lie algebra semiinfinite homology). Feigin and Frenkel also suggested to consider Wakimoto modules as direct limits of so-called twisted Verma modules. In [Ar3] it was shown that the semi-infinite BGG resolution itself can be constructed as a direct limit of twisted BGG resolutions of the integrable simple module and that twisted BGG resolutions are obtained from the classical BGG resolution with the help of the twisting functors.

1.1. The present paper is devoted to the analogue of this construction for affine quantum groups. Given a simply connected root datum \((X, Y, \ldots)\) of the affine type \((I, \cdot)\) we consider an associative algebra \(U\) over the field of rational functions \(Q(v)\) introduced by Drinfeld and Jimbo and called the affine quantum group. This algebra is a quantization of the universal enveloping algebra of the affine Lie algebra \(\hat{g}\) corresponding to \((I, \cdot)\). The algebra \(U\) has a natural triangular decomposition \(U = U^- \otimes U^0 \otimes U^+\) as a vector space, where \(U^-\) (resp. \(U^0\), resp. \(U^+\)) denotes the quantum analogue of the universal enveloping algebra for the standard negative nilpotent subalgebra (resp. for the Cartan subalgebra, resp. for the standard positive nilpotent subalgebra) in \(\hat{g}\). The infinitely twisted nilpotent subalgebra in \(\hat{g}\) has a natural quantum analogue \(U^{\hat{\infty}-}\) in \(U\) as well. Integrable simple modules \(L(\lambda)\) also have nice quantizations over \(U\). We preserve the notation \(L(\lambda)\) for these \(U\)-modules. The aim of this paper is to calculate semiinfinite homology (introduced in the associative algebra case in [Ar1], [Ar2]) of \(U^{\hat{\infty}-}\) with coefficients in \(L(\lambda)\).

1.2. Let us describe briefly the structure of the paper. In the second section we recall standard combinatorial definitions and constructions concerning affine root systems and affine Weyl groups. In particular we define the length function on the affine Weyl group, the twisted length function on \(W\) with the twist \(w \in W\), the semi-infinite length function, the Bruhat order on \(W\) and the semi-infinite Bruhat order.

---

Partially supported by Soros foundation.
In the third section we collect standard facts about affine quantum groups following mainly [L1] and [BeK]. We recall the construction of the automorphisms $T_w$, $w \in W$, of the affine quantum group $U$ generating the action of the affine braid group $B$ corresponding to $W$. We discuss $\theta_y$-twisted quantum nilpotent subalgebras $T_{\theta_y}(U^-)$ in the affine quantum group $U$ for an element of the transvection lattice $\theta_y \in T \subset W$ and their triangular decompositions following [Be], [BeK].

In the fourth and the fifth sections we collect necessary definitions and facts about semi-infinite cohomology of a graded associative algebra $A$ equipped with a triangular decomposition datum (see 4.1) following [Ar1] and [Ar2]. In particular we investigate the behaviour of semiinfinite homology and cohomology with respect to the change of the algebra $A$ (see Propositions 5.1.8, 5.1.9, 5.2.6, 5.2.7).

In the sixth section we consider semiregular modules over the affine quantum group $U$ with respect to various subalgebras in $U$ and endomorphism algebras of these modules. We prove that the endomorphism algebra of the semiregular $U$-module with respect to the subalgebra $U_{\theta_y} := U^- \cap T_{\theta_y}^{-1}(U^+)$ contains $U$ as a subalgebra (see Theorem 6.3.2). Thus we obtain semiregular $U$-bimodules $S_{U_{\theta_y}}^U$ enumerated by elements of the transvection lattice. Like in the affine Lie algebra case we define twisting functors with the help of these bimodules.

In the seventh section we recall quantum BGG resolutions $B^\bullet(\lambda)$ of the simple $U$-modules $L(\lambda)$ following [M]. Like in [Ar3] for affine Lie algebras, we define quantum twisted BGG resolutions as images of the complexes $B^\bullet(\lambda)$ under the twisting functors. Using quantum twisted BGG resolutions we show that the semiinfinite homology spaces of the algebras $T_{\theta_y}(U^-)$ with coefficients in $L(\lambda)$ are the same as in the affine Lie algebra case (see Lemma 7.5.3) and have a natural base enumerated by elements of $W$ graded by $\theta_y^{-1}$-twisted length.

In the eighth section we show that the semiinfinite homology spaces of the algebras $T_{\theta_y}(U^-)$ ($x \in Y'' + \subset T \subset W$ is a generic element, and $m$ tends to infinity) with coefficients in $L(\lambda)$ form a projective system with the limit equal to the semiinfinite homology of the infinitely twisted quantum affine nilpotent algebra $U^{\frac{\infty}{2}}$ with coefficients in $L(\lambda)$. In particular we prove that the latter semiinfinite homology spaces are the same as in the affine Lie algebra case and have a natural base enumerated by elements of $W$ graded by the semi-infinite length function (see Theorem 8.2.2 and Corollary 8.2.4). Unfortunately we do not know the construction of the projective system on the level of quantum twisted BGG resolutions that would provide a quantum analogue of the semi-infinite BGG resolution.

2. Notations and combinatorics of affine Weyl groups

In this section we recall basic terminology and notations concerning root systems, affine Weyl groups following mainly [L1].
2.1. Root data. A Cartan datum is a pair \((I, \cdot)\) consisting of a finite set \(I\) and a symmetric bilinear form \(\nu, \nu' \mapsto \nu \cdot \nu'\) on the free abelian group \(\mathbb{Z}[I]\) with values in \(\mathbb{Z}\) such that \(i \cdot i \in \{2, 4, 6, \ldots\}\) for any \(i \in I\) and the number \(2i \cdot j/i \cdot i \in \{0, -1, -2, \ldots\}\) for any \(i \neq j \in I\).

A Cartan datum \((I, \cdot)\) is said to be of finite type if the symmetric matrix \((i \cdot j)\) indexed by \(I \times I\) is positive definite. A Cartan datum \((I, \cdot)\) is said to be of affine type if it is irreducible and the symmetric matrix \((i \cdot j)\) indexed by \(I \times I\) is positive semi-definite but not positive definite.

2.1.1. A root datum of type \((I, \cdot)\) consists, by definition, of two finitely generated abelian groups \(Y, X\) with a perfect bilinear pairing \((\cdot, \cdot) : Y \times X \to \mathbb{Z}\) and a pair of imbeddings \(I \subset X (i \mapsto i')\) and \(I \subset Y (i \mapsto i)\) such that \(\langle i, j' \rangle = 2i \cdot j/i \cdot i\) for all \(i, j \in I\). We say that the root datum \((Y, X, \ldots)\) is simply connected if \(Y = \mathbb{Z}[I]\) with the obvious imbedding \(I \to Y; X = \text{Hom}(Y, \mathbb{Z})\) with the obvious pairing \((\cdot, \cdot)\); \(Y \times X \to \mathbb{Z}\) and with the imbedding \(I \to X\) defined by \(\langle i, j' \rangle = 2i \cdot j/i \cdot i\).

2.1.2. We fix a simply connected root datum \((X, Y, \ldots)\) of the affine type \((I, \cdot)\). We set \(d_i = i \cdot i/2\). We suppose that the chosen affine Cartan datum is untwisted, i.e., there exists \(i_0 \in I\) such that \(d_{i_0} = 1\). Let \(\mathcal{T} := I \setminus \{i_0\}\). It is known that \((\mathcal{T}, \cdot)\) is a Cartan datum of the finite type. Moreover the chosen root datum \((X, Y, \ldots)\) restricts to a root datum \((\overline{X}, \overline{Y}, \ldots)\) of the finite type \((\mathcal{T}, \cdot)\). Let \(D = \max_i d_i\), we have \(D \in \{1, 2, 3\}\) and for each \(i, d_i\) is equal either to 1 or \(D\). We define \(d_i'\) by \(d_i d_i = D\) for all \(i \in I\). There are uniquely defined strictly positive integers \(r_i, r_i'\), \(i \in I\), such that

(i) \(\sum_i r_i \langle i, j' \rangle = 0\) for all \(j\) and \(r_{i_0} = 1\),

(ii) \(\sum_j r_j' \langle i, j' \rangle = 0\) for all \(i\) and \(r_{i_0}' = 1\).

The dual Coxeter number corresponding to the Cartan datum \((I, \cdot)\) is defined as follows:

\(h' := \sum_{i \in I \setminus \{i_0\}} r_i'\).

Let \(\{\omega_i | i \in I\}\) be the basis of \(X\) dual to \(I \subset Y\). Consider an element \(c := \sum_i r_i i \in Y\). Then we have \(\langle c, j' \rangle = 0\) for all \(j \in I\) and \(\sum_i r_i' i' = 0\).

2.2. Affine Weyl groups. For \(i \in I\) we define reflections

\(s_i : Y \to Y, s_i(y) := y - \langle y, i'i \rangle i,\) and \(s_i : X \to X, s_i(x) := x - \langle i, x \rangle i'.\)

The subgroup \(W \subset \text{Aut} Y\) generated by the reflections \(s_i, i \in I\), is called the affine Weyl group. We identify \(W\) with the subgroup in \(\text{Aut}(X)\) generated by \(s_i, i \in I\). The subgroup \(\overline{W} \subset W\) generated by the reflections \(s_i, i \in \mathcal{T}\), is called the (finite) Weyl group corresponding to the Cartan datum \((\mathcal{T}, \cdot)\).
2.2.1. Consider the set $R_{re}$ (resp. $\overline{R}$) of elements of $Y$ of the form $w(i)$ for some $i \in I$ and $w \in W$ (resp. of the form $w(i)$ for some $i \in I$ and $w \in \overline{W}$). It is called the set of real affine roots (resp. the finite root system). Let $R'$ (resp. $\overline{R}'$) be the set of vectors of $X$ of the form $\omega(i')$ for some $i \in I$ (resp. for some $i \in \overline{I}$ and some $\omega \in \overline{W}$). The assignment $i \mapsto i'$ extends uniquely to a map

$$\alpha \mapsto \alpha', \; R_{re} \longrightarrow R', \text{ such that } \omega(y)' = \omega(y), \; \omega \in W, \; y \in Y.$$ 

The map restricts to bijection of $\overline{R}$ to $\overline{R}'$.

There is a unique function $\alpha \mapsto d_\alpha$ on $R_{re}$ such that it is $W$-invariant and $d_i$ is defined in 2.1.2. We define $\hat{d}_\alpha$ by $d_\alpha \hat{d}_\alpha = D$ for all $\alpha \in R_{re}$. Then we have

(i) $R_{re} = \{\alpha + \hat{d}_\alpha mc | \alpha \in \overline{R}, m \in \mathbb{Z}\}$, $R_{re} + c = R_{re}$;

(ii) $R' = \overline{R}'$, $(\alpha + \hat{d}_\alpha mc)' = \alpha'$ for all $\alpha \in \overline{R}$, $m \in \mathbb{Z}$.

Consider also the set of imaginary affine roots $R_{im} := \{mc|m \in \mathbb{Z} \setminus \{0\}\}$. The affine root system $R$ of the type $(I, \cdot)$ is defined as a union of $R_{re}$ and $R_{im}$. Note that $W$ acts trivially on $R_{im}$.

2.2.2. For $\alpha \in R_{re}$ we denote by $s_\alpha$ the element of $W$ given by the reflection in $Y$ (resp. in $X$)

$$s_\alpha(y) = y - \langle y, \alpha' \rangle \alpha \quad \text{(resp. } s_\alpha(x) = x - \langle \alpha, x \rangle \alpha').$$

For any $\alpha \in \overline{R}$ and $m \in \mathbb{Z}$ we set $s_{\alpha,m} = s_h \in W$, where $h = \alpha + \hat{d}_\alpha mc$. Let $Y' \subset X$ be a free abelian group generated by the set $\{i'|i \in \overline{I}\}$ and let $Y'' \subset Y'$ be a free abelian group generated by the set $\{\hat{d}_i i'|i \in \overline{I}\}$. For $z \in Y''$ consider a transvection $\theta_z : X \longrightarrow X$ given by $\theta_z(x) = x + \langle c, x \rangle z$.

2.2.3. **Lemma:** For $\alpha \in \overline{R}_{re}$ and $m \in \mathbb{Z}$ we have $s_{\alpha,0} \circ s_{\alpha,m} = \theta_{\hat{d}_\alpha mc}$.

In particular $\theta_z \in W$ for any $z \in Y''$. Consider the map of the sets $\theta : Y'' \longrightarrow W$, $z \mapsto \theta_z$, and denote its image by $T \subset W$. Then it is known that the map $\theta$ is an injective homomorphism of groups, $T$ is a normal subgroup in $W$ and $W$ is a semidirect product of $T$ and $\overline{W}$. Note also that $\theta_{\hat{d}_\alpha mc}$ acts on $Y$ by

$$\theta_{\hat{d}_\alpha mc}(y) = y - \langle y, \alpha' \rangle \hat{d}_\alpha mc.$$ 

2.2.4. As usual define the weight $\rho \in X$ by $\rho(i) = 1$ for all $i \in I$. Consider the dot action of $W$ on $X$:

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \; w \in W, \; \lambda \in X.$$ 

Then the dot action of the Weyl group preserves the sets $X_k := \{\lambda \in X | \langle c, \lambda \rangle = k\}$. From now on we denote the set of dominant weights at the level $k$ by $X^+_k$:

$$X^+_k := \{\lambda \in X_k | \langle i, \lambda \rangle \geq 0, \; i \in I\}.$$
2.2.5. Recall that the length of an element of the affine Weyl group $w$ is defined as follows:

$$\ell_t(w) := \sharp\{\alpha \in R^+|w^{-1}(\alpha) \in R^-\}.$$ 

**Remark:** The length of $w \in W$ is equal also to the minimal possible length of expression of $w$ via the generators $s_i$, $i \in I$ (the length of a reduced expression).

2.2.6. For $w \in W$ consider a finite set $R_w^- := \{\alpha \in R^-|w(\alpha) \in R^+\}$, $R_w^+ := -R_w^-$. Let $w_1$ and $w_2$ be elements of the Weyl group such that $\ell(w_1) + \ell(w_2) = \ell(w_2w_1)$. Then $R_{w_2w_1}^\pm = R_{w_2}^\pm \cup R_{w_1}^\pm$. Thus $R_{w_2}^\pm \cap w_1(R_{w_1}^\pm) = \emptyset$, where the set $-S \subset R$ consists of elements opposite to the ones of $S \subset R$. Recall that we have $\sum_{\alpha \in R_w^+} \alpha = \rho - w^{-1}(\rho)$.

2.2.7. *Extended affine Weyl group and affine braid group.* We will need some extension of $W$. Let $\Omega$ be the group of automorphisms of $(W, I)$ whose restriction to $T$ is a conjugation by some element of $W$. Then it is known that $\Omega$ is a finite group in correspondence with a certain group of automorphisms of the Dynkin graph of $(I, \cdot)$ (see e. g. [BeK]). Then the *extended* affine Weyl group $\tilde{W}$ is defined as a semidirect product of $W$ and $\Omega$. It is known that the decomposition of $W$ into the semidirect product of $T$ and $\tilde{W}$ can be extended to a decomposition of $W$ into a semidirect product of $\mathbb{X}$ and $\tilde{W}$.

The length function on $W$ extends to the one on $\tilde{W}$ by setting $\ell_t(\tau w) = \ell_t(w)$ for $\tau \in \Omega$.

The *braid group* $B$ associated to $W$ is the group on generators $T_w$, $w \in W$ with the relation $T_wT_w' = T_{ww'}$ if $\ell(t(ww')) = \ell(w) + \ell(w')$. The *extended braid group* $\tilde{B}$ is associated to $\tilde{W}$ in a similar way (see [Be]).

2.3. **Convex order on positive roots.** Let $Y^+$ (resp. $Y^-$) be the set of all elements in $Y$ such that all their coefficients with respect to the basis $I \subset Y$ are nonnegative (resp. nonpositive). Define the set of positive (resp. negative) roots by $R^+ = R \cap Y^+$, (resp. by $R^- = R \cap Y^-$). We have $R^+ = R^+_c = \{\rho + \alpha, \alpha \in \mathbb{T}_I, m > 0\} \cup \mathbb{T}^+$ and $R^- = R^-_c = \{|m| \in \mathbb{Z}_{>0}\}$.

The height of a positive root $\alpha = \sum_{i \in I} b_i i$ is defined as follows: $ht \alpha = \sum_{i \in I} b_i$. We extend the height function on the whole $Y$ by linearity.

2.3.1. Let $w = s_{i_1} \ldots s_{i_k}$ be the reduced expression of an element of the affine Weyl group $w \in W$. Then it is known that the set $\{i_k, s_{i_k}(i_{k-1}), \ldots, s_{i_1}s_{i_{k-1}} \ldots s_{i_2}(i_1)\}$ coincides with the set $\{\alpha \in R^+|w(\alpha) \in R^-\}$.

Recall the following construction of the set $R^+_c$ (see [Be], [Pa]). Fix an element $\theta_x \in T \subset W$ such that $\langle i, x \rangle > 0$ for all $i \in I$ and fix a reduced expression of $\theta_x$ via the generators of the affine Weyl group $\theta_x = s_{j_1} \ldots s_{j_d}$. Then in particular the opposite element of $\theta_x$ in $W$ has a reduced expression $\theta_{-x} = s_{j_d} \ldots s_{j_1}$. Let $(p_k)_{k \in \mathbb{Z}}$ be the sequence of integers such that $p_k = j_{(k \mod(d))}$. Define the positive roots $\beta_k$ by

$$\beta_k = s_{p_0}s_{p_1} \ldots s_{p_{k-1}}(i_{p_k}) \text{ for } k \leq 0, \text{ and } \beta_k = s_{p_1}s_{p_2} \ldots s_{p_{k-1}}(i_{p_k}) \text{ for } k > 0.$$
2.3.2. **Lemma:**

(i) The roots $\beta_k$ are distinct and the set $\{\beta_k\}_{k \in \mathbb{Z}}$ coincides with $R^+_\text{re}$;

(ii) each subsection $s_{p_k} \ldots s_{p_l}$ for $k < l$ is reduced.

A total order on the set of positive roots is called *convex* if for any $\alpha \in R^+$ and $\beta \in R^+$ we have $\alpha < \alpha + \beta < \beta$ in the order if $\alpha + \beta \in R^+$.

2.3.3. **Lemma:** The order on $R^+$ defined by

$$\beta_0 < \beta_{-1} < \beta_{-2} < \ldots < rc < \ldots < sc < \ldots < \beta_2 < \beta_1$$

is convex. Here $R^+_\text{re}$ is identified with the set $\{\beta_k\}_{k \in \mathbb{Z}}$ using the previous Lemma, and $r < s$.

2.4. **Semi-infinite Bruhat order on the Weyl group.** We say that $w'$ follows $w$ in the Weyl group if there exist a reduced expression of $w'$ and $p \in \{1, \ldots, \ell tw'\}$ such that $w' = s_{i_1} \ldots s_{i_{tw'}}$, $w = s_{i_1} \ldots s_{i_{p-1}} s_{i_p+1} \ldots s_{i_{tw'}}$ and $\ell tw = \ell tw' - 1$.

Recall that the usual Bruhat order on the Weyl group is the partial order on $W$ generated by the relation “$w'$ follows $w$ in $W$”. It is denoted by $\geq$.

2.4.1. **Lemma:** The relation $\{\text{there exists } \lambda_0 \in Y''^+, \langle i, \lambda_0 \rangle > 0 \text{ for every } i \in I, \text{ such that for any } \lambda \in Y''^+ \text{ we have } \theta_\lambda \theta_{\lambda_0} w' \geq \theta_\lambda \theta_{\lambda_0} w\}$ is a partial order on $W$.

2.4.2. **Definition:** We call this partial order the semi-infinite Bruhat order on $W$ and denote it by $\geq \infty$.

2.4.3. **Remark:** The semi-infinite Bruhat order defined above in fact coincides with the partial order on the affine Weyl group defined in [L2], Section 3, in terms of combinatorics of *alcoves* in $Y' \otimes \mathbb{R}$. However we will not need the comparison statement. Further details on the partial order can be found in [L2].

2.4.4. Denote the set $\{\alpha \in R|\alpha = \beta + \hat{d}_\beta mc, \beta \in \overline{R}^+, m \in \mathbb{Z}\}$ (resp. the set $\{\alpha \in R|\alpha = \beta + \hat{d}_\beta mc, \beta \in \overline{R}^-, m \in \mathbb{Z}\}$) by $R^\infty_\text{re}^+$ (resp. by $R^\infty_\text{re}^-$). By definition we set $R^\infty_\text{re}^+ := R^\infty_\text{re}^+ \sqcup R^+_\text{im}$ and $R^\infty_\text{re}^- := R^\infty_\text{re}^- \sqcup R^-\text{im}$. Following [FF] we introduce the semi-infinite length function on the affine Weyl group as follows:

$$\ell tw(w) := \sharp\{\alpha \in R^\infty_\text{re}^+ \cap R^+|w(\alpha) \in R^-\} - \sharp\{\alpha \in R^\infty_\text{re}^- \cap R^+|w(\alpha) \in R^-\}.$$
2.4.5. **Lemma:** For every \( w_1, w_2 \in W \) there exists \( \mu_0 \in -Y^{n+} \), \( \langle i, \mu_0 \rangle < 0 \) for every \( i \in \mathcal{I} \), such that for every \( \mu \in -Y^{n+} \) we have
\[
\ell t(\theta_{-\mu} \theta_{-\mu_0} w_1) - \ell t(\theta_{-\mu} \theta_{-\mu_0} w_2) = \ell t_{\mathcal{I}}^{\infty}(w_1) - \ell t_{\mathcal{I}}^{\infty}(w_2).
\]
In particular for every \( \mu \in -Y^{n+} \) we have \( \ell t_{\mu}^{\mu_0}(w) = \ell t_{\mathcal{I}}^{\infty}(w) \). \( \square \)

2.4.6. **Corollary:** For every \( x \in Y^{n+} \), \( \langle i, x \rangle > 0 \) for every \( i \in \mathcal{I} \), for every \( w \in W \) there exists \( m_0 \in \mathbb{N} \) such that for every \( m > m_0 \) we have \( \ell m_{\mu_0}(w) = \ell t_{\mathcal{I}}^{\infty}(w) \).

**Proof.** Fix \( \mu_0 = \mu_0(w) \) from the previous Lemma. There exists \( m_0 \in \mathbb{N} \) such that for every \( m > m_0 \) we have \( \mu := mx - \mu_0 \in -Y^{n+} \). Thus by the previous Lemma we obtain
\[
\ell m_{\mu_0}(w) = \ell m_{-\mu}(w) = \ell m_{\mu_0}(w) = \ell t_{\mathcal{I}}^{\infty}(w). \quad \square
\]
By 2.4.4 and the previous Corollary for \( \mu \in Y^{n+} \) we have \( \ell t_{\mathcal{I}}^{\infty}(\mu \mu) = \ell t(\mu \mu) \).

### 3. Affine Quantum Groups.

In this section we present some facts about triangular decompositions of various subalgebras in affine quantum groups following mainly [L1] and [BeK].

3.1. Assume that a root datum \((Y, X, \ldots)\) of the affine type \((I, \cdot)\) is given. We consider the associative \(Q(v)\) algebra \(\tilde{U}\) with 1 defined by generators
\[
E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad K_{\mu} \quad (\mu \in Y)
\]
and the following relations:
\[
K_0 = 1, \quad K_{\mu}K_{\mu'} = K_{\mu+\mu'} \text{ for all } \mu, \mu' \in Y;
K_{\mu}E_i = v^{(\mu, i')}E_iK_{\mu} \text{ for all } i \in I, \mu \in Y; \quad K_{\mu}F_i = v^{-(\mu, i')}F_iK_{\mu} \text{ for all } i \in I, \mu \in Y;
\]
\[
E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_{-i}}{v_i - v_i^{-1}}.
\]

Here for \( \nu = \sum_i \nu_i i \in Y \) we set \( \tilde{K}_{\nu} := \prod_i K_{(i-1)/2\nu_i} \) and \( v_i := v^{i/2} \).

Thus \(\tilde{U}\) is \(Y\)-graded. We denote the \(Y\)-grading of a homogenous element \(u \in \tilde{U}\) by \(\deg^Y u \in Y\). We define the quantum adjoint action of \(E_i\) and \(F_i\) on a homogenous element \(u \in \tilde{U}\) as follows:
\[
\text{Ad}_{E_i}^q(u) := E_iu - v^{(\deg^Y u, i')} uE_i \quad \text{and} \quad \text{Ad}_{F_i}^q(u) := F_iu - v^{-(\deg^Y u, i')} uF_i.
\]
3.1.1. **Definition:** The associative algebra obtained from $\tilde{U}$ by taking quotient by the ideal generated by *quantum Serre relations*

$$\text{Ad}_{E_i}^q - (i,j') + 1 (E_j) \quad \text{and} \quad \text{Ad}_{F_i}^q - (i,j') + 1 (F_j) \quad \text{for all} \quad i,j \in I$$

is called the *affine quantum group* $\tilde{U}$.

We denote by $\tilde{U}^-$ (resp. by $\tilde{U}^+$, resp. by $\tilde{U}^0$) the subalgebra in $\tilde{U}$ generated by $F_i$ for all $i \in I$ (resp. by $E_i$ for all $i \in I$, resp. by all $K_\mu$, $\mu \in Y$). It is known that the multiplication in $\tilde{U}$ provides the isomorphisms of vector spaces $\tilde{U}^- \otimes \tilde{U}^0 \otimes \tilde{U}^+ \to \tilde{U}$ and $\tilde{U}^+ \otimes \tilde{U}^0 \otimes \tilde{U}^- \to \tilde{U}$ (see e.g. [BeK]). By definition set $\tilde{U}^0 := \tilde{U}^0 \otimes \tilde{U}^+$ and $\tilde{U} \leq 0 := \tilde{U}^0 \otimes \tilde{U}^-$. Evidently $\tilde{U}^0$ and $\tilde{U} \leq 0$ are subalgebras in $\tilde{U}$.

3.1.2. We introduce the Q-algebra antiautomorphism $\kappa$ of $\tilde{U}$ defined by:

$$\kappa(E_i) = F_i, \quad \kappa(F_i) = E_i, \quad \kappa(K_\mu) = K_{-\mu}, \quad \kappa(v) = v^{-1}, \quad i \in I, \quad \nu \in Y.$$

3.2. **Convex PBW bases in $\tilde{U}$**. Now we recall the definition of the analogues of Lusztig generators of $\tilde{U}^+$ introduced in the affine Cartan datum case by Beck (see [Be]). First we construct the real root generators.

3.2.1. **Lemma:** (see e.g. [BeK], 1.5) The formulas

$$T_i E_i = -F_i K_i, \quad T_i(E_j) = \frac{(-1)^{(i,j')}}{(i,j')}! (\text{Ad}_{E_i}^q)^{-(i,j')}(E_j) \quad \text{if} \quad i \neq j, \quad T_i K_\nu = K_{s_i \nu}, \quad i, j \in I, \quad \nu \in Y,$$

define an action of the affine braid group $B$ on $\tilde{U}$ by automorphisms that can be extended to the action of $\tilde{B}$. \hfill $\Box$

Fix a total convex order on $R^+$ constructed starting from a positive transvection $\theta_x \in T$ (see 2.3.1). For each $\beta_k \in R^+_\text{re}$ define the root vector $E_{\beta_k} \in \tilde{U}_{\beta_k}$ as follows:

$$E_{\beta_k} = T_{i_0}^{-1} \ldots T_{i_k}^{-1} (E_{i_k}) \quad \text{if} \quad k \leq 0, \quad E_{\beta_k} = T_{i_1} \ldots T_{i_k} (F_{i_k}) \quad \text{if} \quad k > 0.$$

We will need also negative real root generators. For $\beta_k \in R^+_\text{re}$ define the root vector $F_{\beta_k} \in \tilde{U}_{\beta_k}$ by

$$F_{\beta_k} = T_{i_0}^{-1} \ldots T_{i_{k+1}}^{-1} (F_{i_k}) \quad \text{if} \quad k \leq 0, \quad F_{\beta_k} = T_{i_1} \ldots T_{i_k} (E_{i_k}) \quad \text{if} \quad k > 0.$$

3.2.2. Next we construct the imaginary root generators. For $i \in I$ and $m > 0$ set

$$\psi^{(i)}_m := \tilde{K}_i^{-1} [E_i, E_{m-i}], \quad \psi^{(i)}_{-m} := \kappa(\psi^{(i)}_m), \quad \psi^{(i)}_0 = \frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v_i - v_i^\prime}.$$
3.2.3. For \( k > 0 \) and \( i \in \mathcal{T} \) define imaginary root vectors \( E_{\kappa c}^{(i)} \) by the following functional equation:

\[
    \exp \left( (v_i - v_i^{-1}) \sum_{k=1}^{\infty} E_{\kappa c}^{(i)k} t^k \right) = 1 + (v_i - v_i^{-1}) \sum_{k=1}^{\infty} \psi^{(i)k} t^k.
\]

3.2.4. **Remark:** As defined \( E_{\beta_k} \in \widetilde{\mathbb{U}}^{\geq} \) and \( F_{\beta_k} \in \widetilde{\mathbb{U}}^{\leq} \) for \( \beta_k \in R_+^\kappa \). Set \( \dot{E}_\beta := K_{-\alpha} E_\beta \) if \( \beta = -\alpha + k\lambda , \) and \( \alpha \in \mathcal{R}^- \), otherwise set \( \dot{E}_\beta := E_\beta \) (still \( \beta \in R_+^\kappa \)). The negative real root generators \( \dot{F}_\beta \) are obtained from \( F_\beta \) in a similar way.

Consider a set \( \mathcal{R}^+ := \left( R_{\text{re}}^{\infty} \cap R^+ \right) \cup \bigcup_{\kappa \neq 1} R^+ \cup \ldots \cup \bigcup_{\kappa \neq 1} R^+ \cup \left( R_{\text{re}}^{\infty} \cap R^+ \right) \). Denote the obvious projection \( \mathcal{R}^+ \to R^+ \) by \( \pi \). Fix an arbitrary total order on \( \mathcal{R}^+ \) such that \( \pi \) would become a map of the totally ordered sets. Note that \( \mathcal{R}^+ \) enumerates the positive root vectors in \( \widetilde{\mathbb{U}}^+ \) constructed above. The sets \( \mathcal{R}^-, \mathcal{R}_{\text{re}}^{\infty} \) and \( \mathcal{R}_{\text{re}}^{\infty} \) are defined in a similar way.

Abusing notation we write \( \langle \alpha, \beta \rangle \) for \( \langle \pi(\alpha), \pi(\beta) \rangle \), where \( \alpha, \beta \in \mathcal{R}^+ \).

3.2.5. **Lemma:** (see [Be]) The elements \( \dot{E}_\beta, \beta \in \mathcal{R}^+ \), belong to \( \widetilde{\mathbb{U}}^+ \) and generate it. Let \( Z_{\mathcal{R}^+} \) be the set of all \( Z_{\geq 0} \)-valued functions on \( \mathcal{R}^+ \) with finite support. For \( (a_\beta) \in Z_{\mathcal{R}^+} \)

consider a monomial \( M(a_\beta) \in \widetilde{\mathbb{U}}^+ \) as follows: \( M(a_\beta) := \prod_{\beta \in \mathcal{R}^+} \dot{E}_\beta^{a_\beta} \) (taken in the chosen convex order on the positive roots). Then such monomials form a base of \( \widetilde{\mathbb{U}}^+ \) over \( \mathbb{Q}(v) \).

By analogy with the Lie algebra case we call the base \( \{ M(a_\beta) \} \) the convex PBW base of \( \widetilde{\mathbb{U}}^+ \). The main purpose of introducing convex PBW bases is the following statement.

3.2.6. **Theorem:** (see [Be]) Let \( \alpha, \beta \in \mathcal{R}^+ \) be such that \( \beta > \alpha \) in the total convex order on \( \mathcal{R}^+ \). Then we have

\[
    \dot{E}_\beta \dot{E}_\alpha - v^{(\alpha, \beta)} \dot{E}_\beta \dot{E}_\alpha = \sum_{\alpha < \gamma_1 < \ldots < \gamma_n < \beta} a_\gamma \dot{E}_{\gamma_1}^{b_1} \ldots \dot{E}_{\gamma_n}^{b_n},
\]

where \( \gamma \) denotes the vector \( (\gamma_1, \ldots, \gamma_n) \), \( \gamma_\beta \in \mathcal{R}^+ \), and the coefficients \( a_\gamma \in \mathbb{Q}[v, v^{-1}] \).

A similar statement holds for \( \widetilde{\mathbb{U}}^- \).

3.2.7. **Filtration on \( \widetilde{\mathbb{U}}^+ \).** Using the PBW type base we define a filtration on \( \widetilde{\mathbb{U}}^+ \) like in [DCK]. The notion of a \( S \)-filtration for a totally ordered set \( S \) was introduced there. We view \( Z_{\mathcal{R}^+} \) as a totally ordered semigroup with the usual lexicographical order. Introduce a \( Z_{\mathcal{R}^+} \)-filtration on \( \widetilde{\mathbb{U}}^+ \) by letting \( F^s \widetilde{\mathbb{U}}^+ \) be the span of the monomials

\[
    M(a_\beta) \in \widetilde{\mathbb{U}}^+, \quad (a_\beta) = (\ldots, 0, \ldots, 0, a_\beta_m, \ldots, a_\beta_n, 0 \ldots, 0, \ldots) \in Z_{\mathcal{R}^+} \text{ such that } (a_\beta) < s \text{ in the total order.} \]
3.2.8. **Proposition:** (see [BeK]) The associated $\mathbb{Z}_{\geq 0}^{\tilde{R}^+}$-graded algebra $\text{gr}\tilde{U}^+$ of the $\mathbb{Z}_{\geq 0}^{\tilde{R}^+}$-filtered algebra $\tilde{U}^+$ is an algebra over $\mathbb{Q}(v)$ on generators $E_\alpha$, $\alpha \in \hat{R}^+$, and relations $E_\alpha E_\beta = v^{(\alpha,\beta)} E_\beta E_\alpha$ for $\beta < \alpha$ in the total convex order. \qed

3.3. **Subalgebras in $\tilde{U}^+$.** Fix a positive root $\beta_k \in \hat{R}^+$. We denote the set $\{\alpha|\alpha \in \hat{R}^+, \alpha < \beta_k\}$ (resp. $\{\alpha|\alpha \in \hat{R}^+, \alpha \geq \beta_k\}$) by $\hat{R}^+_{<\beta_k}$ (resp. $\hat{R}^+_{\geq \beta_k}$). Let $\tilde{U}^+_{<\beta_k}$ (resp. $\tilde{U}^+_{\geq \beta_k}$) be the subalgebra in $\tilde{U}^+$ generated by the root elements $\{E_\alpha|\alpha \in \hat{R}^+_{<\beta_k}\}$ (resp. by the root elements $\{E_\alpha|\alpha \in \hat{R}^+_{\geq \beta_k}\}$). The corresponding subalgebras $\tilde{U}^-_{<\beta_k}$ and $\tilde{U}^-_{\geq \beta_k}$ in $\tilde{U}^-$ are defined in a similar way. Then we have the following statement.

3.3.1. **Lemma:** The monomials $M_{(a)} := \prod_{\beta \in \hat{R}^+_{<\beta_k}} E_\beta$ (resp. the monomials $M_{(a)} := \prod_{\beta \in \hat{R}^+_{\geq \beta_k}} E_\beta$) taken in the chosen convex order on the positive roots form a base of $\tilde{U}^+_{<\beta_k}$ (resp. of $\tilde{U}^+_{\geq \beta_k}$) over $\mathbb{Q}(v)$. The algebra $\tilde{U}^+$ admits a decomposition $\tilde{U}^+ \twoheadrightarrow \tilde{U}^+_{<\beta_k} \otimes \tilde{U}^+_{\geq \beta_k}$ as a $\mathbb{Q}(v)$-vector space.

**Proof.** The Lemma follows immediately from Lemma 3.2.5 and Theorem 3.2.6. \qed

In particular consider the positive roots $\beta_{md} = s_{p_1} \ldots s_{p_{md-1}}(i_{p_{md}})$ and $\beta_{\bar{md}} = s_{p_0} s_{p_1} \ldots s_{p_{md+1}}(i_{p_{md}})$ (see 2.3.1). Then we have

$\hat{R}^+_{\beta_{md}} = \{\alpha \in \hat{R}^+|\theta_{-mx}(\pi(\alpha)) \in R^-\}$ and $\hat{R}^+_{<\beta_{md-1}} = \{\alpha \in \hat{R}^+|\theta_{mx}(\pi(\alpha)) \in R^-\}$.

We denote the subalgebra $\tilde{U}^+_{<\beta_{md-1}}$ (resp. the subalgebra $\tilde{U}^-_{<\beta_{md-1}}$) by $\tilde{U}^+_{\beta_{mx}}$ (resp. by $\tilde{U}^-_{\beta_{mx}}$). Consider the antiautomorphism $T_{\theta_{mx}}^{-1} = (T_{p_1}^{-1} \ldots T_{p_l}^{-1})^{m}$ of the algebra $\tilde{U}$.

3.3.2. **Lemma:** (see [Be], 3.10, Proposition 2) We have $T_{\theta_{l}}(\tilde{E}_{mc}^{(i)}) = \tilde{E}_{mc}^{(i)}$ for all $i \in I$ and $m \in \mathbb{Z} \setminus \{0\}$. \qed

3.3.3. **Lemma:** We have $T_{\theta_{mx}}^{-1}(\tilde{U}^+_{\geq \beta_{md}}) = \tilde{U}^-_{\beta_{mx}}$ and $T_{\theta_{mx}}^{-1}(\tilde{U}^-_{\geq \beta_{md}}) = \tilde{U}^+_{\beta_{mx}}$. In particular $\tilde{U}^-_{\beta_{mx}} = \tilde{U}^- \cap T_{\theta_{mx}}^{-1}(\tilde{U}^+)$ and $\tilde{U}^+_{\beta_{mx}} = \tilde{U}^+ \cap T_{\theta_{mx}}^{-1}(\tilde{U}^-)$.

**Proof.** The first two statements are proved by the following calculation. Let $\tilde{E}_{\beta_l}$ (resp. $\tilde{F}_{\beta_l}$), $0 < l \leq md$, be one of the root generators of $\tilde{U}^+_{\beta_{md}}$ (resp. of $U^-_{\geq \beta_{md}}$). Then, forgetting about the factor from $\tilde{U}^0$, we have

$T_{\theta_{mx}}^{-1}(E_{\beta_l}) = T_{p_0}^{-1} \ldots T_{p_l}(T_{p_1} \ldots T_{p_l}(F_{p_l})) = T_{p_0}^{-1} \ldots T_{p_{md+1}}^{-1}(F_{p_{md+1}}) = F_{-\beta_{l-1}}$.

A similar calculation holds for $F_{\beta_l}$. Thus we see that the subalgebras $T_{\theta_{mx}}^{-1}(\tilde{U}^+_{\geq \beta_{md}})$ and $\tilde{U}^-_{\beta_{mx}}$ (resp. $T_{\theta_{mx}}^{-1}(\tilde{U}^-_{\geq \beta_{md}})$ and $\tilde{U}^+_{\beta_{mx}}$) have the same generators. The last two statements follow from the previous considerations. \qed
3.3.4. **Lemma**: We have \( \tilde{U}_{\geq \beta - m \Delta - 1}^+ = \tilde{U}^+ \cap T_{\theta_{mx}}^{-1}(\tilde{U}^+) \).

**Proof.** The fact that the real root generators of the two subalgebras coincide is proved the same way as in the previous Lemma. The imaginary root generators are invariant under the automorphism \( T_{\theta_{mx}}^{-1} \) by Lemma 3.3.2. Now use Lemma 3.2.5.

In particular we obtain a triangular decomposition of the algebra \( T_{\theta_{mx}}^{-1}(\tilde{U}^+) \) (resp. \( T_{\theta_{mx}}^{-1}(\tilde{U}^-) \)):

\[
T_{\theta_{mx}}^{-1}(\tilde{U}^+) \rightarrow \tilde{U}_{\geq \beta - m \Delta - 1}^+ \otimes \tilde{U}_{\theta_{mx}}^- = T_{\theta_{mx}}^{-1}(\tilde{U}^+) \cap U^+ \otimes \tilde{U}_{\theta_{mx}}^-
\]

(resp. \( T_{\theta_{mx}}^{-1}(\tilde{U}^-) \rightarrow \tilde{U}_{\leq \beta - m \Delta}^+ \otimes \tilde{U}_{\theta_{mx}}^- = T_{\theta_{mx}}^{-1}(\tilde{U}^-) \cap U^- \otimes \tilde{U}_{\theta_{mx}}^+ \)).

as a \( \mathbb{Q}(v) \)-vector space.

Consider the \( \mathbb{Z}^{R_0^+}_{\geq 0} \)-filtration on the algebra \( \tilde{U}_{\theta_{mx}}^+ \) obtained by restriction from the one on \( \tilde{U}^+ \). Note that by definition nontrivial graded quotient spaces are enumerated in fact by the totally lexicographically ordered finitely generated semigroup \( \mathbb{Z}^{R_0^+}_{\geq 0} \subset \mathbb{Z}^{R_0^+}_{\geq 0} \).

3.3.5. **Lemma**: The associated \( \mathbb{Z}^{R_0^+}_{\geq 0} \)-graded algebra \( \text{gr} \tilde{U}_{\theta_{mx}}^+ \) of the \( \mathbb{Z}^{R_0^+}_{\geq 0} \)-filtered algebra \( \tilde{U}_{\theta_{mx}}^+ \) is an algebra over \( \mathbb{Q}(v) \) on generators \( E_\alpha, \alpha \in R_{\theta_{mx}} \) and relations \( E_\alpha E_\beta = v^{(\alpha, \beta')} E_\beta E_\alpha \) for \( \beta < \alpha \) in the total convex order.

4. It is easily checked that the element \( \bar{Z} := \prod_{i \in I} K_i^\vee \) is central in \( \tilde{U} \). To simplify the exposition we add a certain root of the element \( \bar{Z} \) to the algebra \( \tilde{U} \). Namely we set \( U := \tilde{U}[Z] \) where \( Z \) is defined as a central element such that \( Z^{Dh} = \bar{Z} \). Again we have a triangular decomposition of the algebra \( U \): \( U = U^- \otimes U^0 \otimes U^+ \) as a \( \mathbb{Q}(v) \)-vector space where as before \( U^+ \) (resp. \( U^- \), resp. \( U^0 \)) denotes the subalgebra in \( U \) generated by \( E_i, \ i \in I \) (resp. by \( F_i, \ i \in I \), resp. by \( K_\mu, \ \mu \in Y \), and \( Z \)).

Fix the level \( k \in \mathbb{Z} \). We define the algebra \( U_k \) as the quotient algebra of \( U \) by the relation \( Z = v^k \).

Note that \( U^\pm = \tilde{U}^\pm \). All the statements about convex bases in \( \tilde{U} \) from the previous subsection hold for \( U \).

4. **Semiinfinite cohomology of associative algebras.**

In this section we present some definitions and statements concerning semiinfinite cohomology of associative algebras following mainly [Ar2], sections 2 and 3.

4.1. Suppose we have a graded associative algebra \( A = \bigoplus_{n \in \mathbb{Z}} A_n \) over a field \( k \). Let \( B \) and \( N \) be graded subalgebras in \( A \) satisfying the following conditions:
(i) \(N\) is positively graded;
(ii) \(N_0 = \mathbf{k}\);
(iii) \(\dim N_n < \infty\) for any \(n \in \mathbb{N}\);
(iv) \(B\) is negatively graded;
(v) the multiplication in \(A\) defines the isomorphisms of graded vector spaces
\[
B \otimes N \rightarrow A \quad \text{and} \quad N \otimes B \rightarrow A.
\]
In particular \(N\) is naturally augmented. We denote the augmentation ideal \(\mathbf{k} \oplus N_n\) by \(\overline{N}\).

4.1.1. The category of left graded \(A\)-modules with morphisms that preserve gradings is denoted by \(A\text{-mod}\). We define the functor of the grading shift
\[
A\text{-mod} \rightarrow A\text{-mod} : M \mapsto M(i), \ M(i)_m := M_{i+m}, i \in \mathbb{Z}.
\]
The space \(\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A\text{-mod}}(M_1, M_2(i))\) is denoted by \(\text{Hom}_A(M_1, M_2)\).

4.1.2. We fix a left \(B\)-augmentation on \(A\) provided by the isomorphism of graded left \(B\)-modules \(B \cong A \otimes_N \mathbf{k}\) where \(\mathbf{k} := N/\overline{N}\) is the trivial \(N\)-module. The \(A\)-module \(A \otimes_N \mathbf{k}\) is denoted by \(\overline{B}\).

4.1.3. We introduce certain subcategories in the category of complexes \(Kom(A\text{-mod})\). For \(M^\bullet \in Kom(A\text{-mod})\) the support of \(M\) is defined as follows:
\[
\text{supp} M^\bullet := \{(p, q) \in \mathbb{Z}^2 | M_p^q \neq 0\}.
\]
For \(s_1, s_2, t_1, t_2 \in \mathbb{Z}, \ s_1, s_2 > 0\), the set \(\{(p, q) \in \mathbb{Z}^2 | s_1q + p \geq t_1, \ s_2q - p \geq t_2\}\) (resp. the set \(\{(p, q) \in \mathbb{Z}^2 | s_1q + p \leq t_1, \ s_2q - p \leq t_2\}\)) is denoted by \(X^\uparrow(s_1, s_2, t_1, t_2)\) (resp. by \(X^\downarrow(s_1, s_2, t_1, t_2)\)).

Let \(\mathcal{C}^\uparrow(A)\) (resp. \(\mathcal{C}^\downarrow(A)\)) be the full subcategory in \(Kom(A\text{-mod})\) consisting of complexes \(M^\bullet\) that satisfy the following condition:

\(\text{(U)}\) there exist \(s_1, s_2, t_1, t_2 \in \mathbb{Z}, \ s_1, s_2 > 0\), such that \(\text{supp} M^\bullet \subset X^\uparrow(s_1, s_2, t_1, t_2)\) (resp. \(\text{(D)}\) there exist \(s_1, s_2, t_1, t_2 \in \mathbb{Z}, \ s_1, s_2 > 0\), such that \(\text{supp} M^\bullet \subset X^\downarrow(s_1, s_2, t_1, t_2)\)).

4.2. Relative bar resolutions. The standard bar resolution \(\overline{\text{Bar}}^\bullet(A, B, M) \in Kom(A\text{-mod})\) of an \(A\)-module \(M\) with respect to the subalgebra \(B\) is by definition the following complex of \(A\)-modules
\[
\overline{\text{Bar}}^{-n}(A, B, M) := A \otimes_B \ldots \otimes_B A \otimes_B M \ (n + 1 \text{ times}),
\]
with the standard bar differential. Consider the subspace \(\overline{\text{Bar}}(A, B, M)\) in \(\overline{\text{Bar}}^\bullet(A, B, M)\) as follows:
\[
(\overline{\text{Bar}})^{-n}(A, B, M) := \{a_0 \otimes \ldots \otimes a_n \otimes v \in \overline{\text{Bar}}^{-n}(A, B, M) | \exists s \in \{1, \ldots, n\} : a_s \in B\}.
\]
Evidently it is a submodule in \( \widehat{\text{Bar}}^\bullet(A, B, M) \) preserved by the differential.

The quotient complex \( \text{Bar}^\bullet(A, B, M) := \widehat{\text{Bar}}^\bullet(A, B, M) / \overline{\text{Bar}}^\bullet(A, B, M) \) is called the restricted bar resolution of the \( A \)-module \( M \) with respect to the subalgebra \( B \). For a complex of \( A \)-modules \( M^\bullet \in C^\bullet(A) \) its relative restricted bar resolution is defined as a total complex of the bicomplex \( \text{Bar}^\bullet(A, B, M^\bullet) \).

Let \( M \in C^\bullet(A) \). Then it is known that the restricted relative bar resolution of \( M \) also belongs to \( C^\bullet(A) \) and is quasiisomorphic to \( M \) (see [Ar2], Lemma 2.3.4).

4.3. Relative cobar DG-algebra. Here we present a construction of a canonical DG-algebra representing \( \text{RHom}^\bullet_A(B, B) \). Consider the relative restricted bar resolution \( \text{Bar}^\bullet(A, B, B) \) and the complex of graded vector spaces

\[
D^\bullet(A, B) := \text{Hom}^\bullet_A(\text{Bar}^\bullet(A, B, B), B).
\]

Clearly \( D^\bullet(A, B) \cong \text{Hom}^\bullet_k(\bigoplus_{n \geq 0} N^m_n, B) \) as a vector space, and \( D(A, B) \) belongs to \( C^\bullet(\text{Vect}) \).

4.3.1. We introduce a structure of a DG-algebra on \( D^\bullet(A, B) \). First we define a DG-algebra structure on \( \text{Hom}^\bullet_A(\text{Bar}^\bullet(A, B, B), B) \). Note that by Schapiro lemma

\[
\text{Hom}^\bullet_A(\text{Bar}^{-m}(A, B, B), B) = \text{Hom}^\bullet_B(A \otimes_B \cdots \otimes_B A \otimes_B B, B) \quad (m \text{ times}).
\]

Let \( f \in \text{Hom}^m_A(\text{Bar}^\bullet(A, B, B), B) \), \( g \in \text{Hom}^n_A(\text{Bar}^\bullet(A, B, B), B) \), i.e.

\[
f : \bigotimes_m A \otimes_B \cdots \otimes_B A \otimes_B B \rightarrow B, \quad g : \bigotimes_n A \otimes_B \cdots \otimes_B A \otimes_B B \rightarrow B.
\]

both \( f \) and \( g \) are \( B \)-linear. By definition set \( f \cdot g : \bigotimes_{m+n} A \otimes_B \cdots \otimes_B A \otimes_B B \rightarrow B : (f \cdot g)(a_1 \otimes \cdots \otimes a_{m+n} \otimes b) := f(a_1 \otimes \cdots \otimes a_n \otimes g(a_{n+1} \otimes \cdots \otimes a_{m+n} \otimes b)). \)

Then the multiplication equips \( \text{Hom}^\bullet_A(\text{Bar}^\bullet(A, B, B), B) \) with a DG-algebra structure and the subcomplex

\[
D^\bullet(A, B) := \{ f \in \text{Hom}^\bullet_A(\text{Bar}^\bullet(A, B, B), B) \mid f \equiv 0 \text{ on } \overline{\text{Bar}}^\bullet(A, B, B) \subset \text{Bar}^\bullet(A, B, B) \}
\]

is a DG-subalgebra in \( \text{Hom}^\bullet_A(\text{Bar}^\bullet(A, B, B), B) \) (see [Ar2], Lemma 2.4.2).

4.3.2. The algebra \( D^\bullet(A, B) \) has a kind of a triangular decomposition. First its zero grading component \( D^0(A, B) = \text{Hom}_A(A \otimes_B B, B) = \text{Hom}_B(B, B) \cong B^{\text{opp}} \) as an algebra (yet the inclusion \( B^{\text{opp}} \hookrightarrow D^\bullet(A, B) \) is not a morphism of DG-algebras — the differential in \( D^\bullet(A, B) \) does not preserve \( B^{\text{opp}} \).
Next consider the induction functor $\text{Ind}^A_N : \mathcal{N}\text{-mod} \to A\text{-mod}$. Then the canonical map

$$\text{Hom}^*_N(\text{Bar}^*(N, k, k), k) \to \text{Hom}^*_A(\text{Ind}^A_N(\text{Bar}^*(N, k, k)), \text{Ind}^A_N(k)) \cong \text{Hom}^*_A(\text{Bar}^*(A, B, B), B) = D^*(A, B)$$

is an inclusion of DG-algebras. Denote its image by $D^*(N, k)$. Then we have $D^*(A, B) = D^*(N, k) \otimes B^{\text{opp}}$ as a graded vector space.

4.3.3. We will need another restriction on the algebra $A$ as follows.

(vi) Both maps $D^*(N, k) \otimes B^{\text{opp}} \to D^*(A, B)$ and $B^{\text{opp}} \otimes D^*(N, k) \to D^*(A, B)$ provided by the multiplication in $D^*(A, B)$ are isomorphisms of vector spaces.

4.3.4. Free algebras over $B$. Fix a bimodule $M$ over the algebra $B$. Denote by $T_B(M)$ the free algebra over $B$ generated by $M$. By definition $T_B(M)$ is the algebra on generators space equal to $M$ and with relations as follows:

$$(m_1 \otimes \ldots \otimes m_k) \cdot (m_{k+1} \otimes \ldots \otimes m_{k+l}) = m_1 \otimes \ldots \otimes m_{k+l},$$

$$b \cdot m = bm, \ m \cdot b = mb; \ m, m_1, \ldots, m_{k+l}, bm, mb \in M, b \in B.$$

Thus we have $T_B(M) = \bigoplus_{n \geq 0} M \otimes_B \ldots \otimes_B M$ as a vector space. Consider the $B$-bimodules $A/B$ and $M := \text{Hom}_B(A/B, B)$. The latter one is provided by two $B$-module structures on the space of homomorphisms between the left $B$-modules: the left $B$-module structure is given by the right $B$-module structure in $A/B$ and the right $B$-module structure is given by the right $B$-multiplication in $B$. Then evidently we have an isomorphism of algebras $D^*(A, B) \cong T_B(M)$.

4.4. Bar duality functors. Next we construct canonical $D^*(A, B)$-DG-modules representing $\text{RHom}^*_A(B, *)^l$ and $* \otimes_A B$. Let $M^* \in \mathcal{C}^+(A)$, $M'^* \in \mathcal{C}^+(A^{\text{opp}})$. By definition set

$$D^+(A, B, M^*) := \text{Hom}^*_A(\text{Bar}^*(A, B, B), M^*), \ D^+(A, B, M'^*) := M'^* \otimes_A \text{Bar}^*(A, B, B).$$

Evidently the vector space $D^+(A, B, M^*)$ (resp. $D^+(A, B, M'^*)$) belongs to $\mathcal{C}^+(\text{Vect})$ (resp. to $\mathcal{C}^+(\text{Vect})$). Similarly to [1.3.1] we define the right action of $D^*(A, B)$ on $D^+(A, B, M^*)$ (resp. the left action of $D^*(A, B)$ on $D^+(A, B, M'^*)$) (see [Ar2], 2.4.4).

Then the multiplication equips $\text{Hom}^*_A(\text{Bar}^+(A, B, B), M^*)$ (resp. $M'^* \otimes_A \text{Bar}^+(A, B, B)$) with a structure of the right DG-module over $D^*(A, B)$ (resp. with a structure of the left DG-module over $D^*(A, B)$). Note also that $D^+(A, B, M^*) \subset \text{Hom}^*_A(\text{Bar}^+(A, B, B), M^*)$ is a DG-submodule and $D^+(A, B, M'^*)$ is a quotient DG-module of $M'^* \otimes_A \text{Bar}^+(A, B, B)$ (see [Ar2], Lemma 2.4.5).
4.4.1. Denote the category of right (resp. left) DG-modules
$$M^\bullet = \bigoplus_{p,q \in \mathbb{Z}} M^q_p, \quad d : M^q_p \to M^{q+1}_p,$$
over $D^\bullet(A,B)$, with morphisms being morphisms of DG-modules that preserve also the second grading, by $D^\bullet(A,B)$-mod (resp. by $D^\bullet(A,B)^{opp}$-mod). The subcategory in $D^\bullet(A,B)$-mod (resp. in $D^\bullet(A,B)^{opp}$-mod) that consists of DG-modules satisfying the condition (D) (resp. (U)) is denoted by $\mathcal{C}^\bullet(D^\bullet(A,B))$ (resp. by $\mathcal{C}^\bullet(D^\bullet(A,B)^{opp})$).

The localizations of $\mathcal{C}^\bullet(A)$, $\mathcal{C}^\bullet(D^\bullet(A,B))$, $\mathcal{C}^\bullet(D^\bullet(A,B)^{opp})$ by the class of quasiisomorphisms are denoted by $\mathcal{D}^\downarrow(A)$, $\mathcal{D}^\downarrow(A^{opp})$, $\mathcal{D}^\downarrow(D^\bullet(A,B))$, and $\mathcal{D}^\downarrow(D^\bullet(A,B)^{opp})$ respectively.

By 4.4 D$^\downarrow$ and $D^{\uparrow}$ define the functors
$$D^\downarrow_A : \mathcal{C}^\downarrow(A) \to \mathcal{C}^\downarrow(D^\bullet(A,B)) \quad \text{and} \quad D^{\uparrow}_A : \mathcal{C}^{\uparrow}(A^{opp}) \to \mathcal{C}^{\uparrow}(D^\bullet(A,B)^{opp}).$$

4.4.2. **Theorem:** (see [Ar2], Theorem 2.4.7)

(i) The functor $D^\downarrow_A$ is well defined as a functor from $\mathcal{D}^\downarrow(A)$ to $\mathcal{D}^\downarrow(D^\bullet(A,B))$;

(ii) $D^\downarrow_A : \mathcal{D}^\downarrow(A) \to \mathcal{D}^\downarrow(D^\bullet(A,B))$ is an equivalence of triangulated categories;

(iii) the functor $D^{\uparrow}_A$ is well defined as a functor from $\mathcal{D}^{\uparrow}(A^{opp})$ to $\mathcal{D}^{\uparrow}(D^\bullet(A,B)^{opp})$;

(iv) $D^{\uparrow}_A : \mathcal{D}^{\uparrow}(A^{opp}) \to \mathcal{D}^{\uparrow}(D^\bullet(A,B)^{opp})$ is an equivalence of triangulated categories.

----------------------------------------

4.5. **Construction of the algebra** $A^\sharp$. Now we are going to introduce an algebra $A^\sharp$ such that $A^\sharp = B^{opp} \otimes N^{opp}$ as a vector space and $D^\bullet(A,B)^{opp} \cong D^\bullet(A^\sharp,B^{opp})$.

Recall that the DG-algebra $D^\bullet(A,B)$ is a tensor product of the DG-subalgebra $D^\bullet(N,k)$ and the subalgebra (not a DG-subalgebra) $B^{opp}$. The isomorphism of vector spaces is provided by the condition (vi) from 1.3.3. Recall also that the algebra $D^\bullet(N,k)$ is isomorphic to the tensor algebra over the graded vector space $\overline{N}^*$ and the differential in it is generated by the map $\overline{N}^* \to \overline{N}^* \otimes \overline{N}^*$ dual to the multiplication map in $N$ and extended to the whole algebra $T(\overline{N}^*)$ by the Leibnitz rule.

To define a DG-algebra $C^\vee$ such that $C^\vee$ is a tensor product of its DG-subalgebra $T(\overline{N}^*)$ with a differential given by the map dual to the multiplication in $N$ and a (not DG-) subalgebra $B^{opp}$ one has to specify the following additional data:

- a linear map $B \otimes \overline{N}^* \to \overline{N}^* \otimes B$ generating the multiplication in $C^\vee$;
- a linear map $B \to \overline{N}^* \otimes B$ providing a component of the differential in $C^\vee$,

satisfying certain constraints (that provide the associativity constraint and the Leibnitz rule in the DG-algebra $C^\vee$).
On the other hand to define an algebra $C$ such that $C$ is a tensor product of two its subalgebras $B$ and $N$ as a vector space and the conditions 4.1 are satisfied one has to specify the following data (additional to the algebra structures on $B$ and $N$):

- a linear map $\overline{N} \otimes B \rightarrow B \oplus B \otimes \overline{N}$ providing the multiplication in $C$

satisfying certain constraints (that provide the associativity constraint in the algebra $C$).

4.5.1. **Proposition:** The construction of the dual algebra $C :\rightarrow C^\vee := D^\bullet(C, B)$ provides a one to one correspondence between the two described types of data, i. e. for every DG-algebra $C^\vee = T(\overline{N}^\ast) \otimes B^{\text{opp}}$ as a vector space there exists an algebra $C = B \otimes N$ as a vector space such that the DG-algebras $C^\vee$ and $D^\bullet(C, B)$ are isomorphic.

4.5.2. **Lemma:** The DG-algebras $D^\bullet(N, k)^{\text{opp}}$ and $D^\bullet(N^{\text{opp}}, k)$ are isomorphic to each other.

Thus we have a triangular decomposition for the DG-algebra $D^\bullet(A, B)^{\text{opp}}$ as follows:

$$D^\bullet(A, B)^{\text{opp}} = B \otimes D^\bullet(N^{\text{opp}}, k) = D^\bullet(N^{\text{opp}}, k) \otimes B.$$ 

4.5.3. **Corollary:** There exists an associative algebra $A^{\sharp}$ such that $A^{\sharp}$ contains two subalgebras $B^{\text{opp}}$ and $N^{\text{opp}}$ satisfying the conditions 4.1 for $A^{\sharp}$, $B^{\text{opp}}$ and $N^{\text{opp}}$ such that the DG-algebra $D^\bullet(A, B)^{\text{opp}} \cong D^\bullet(A^{\sharp}, B^{\text{opp}})$.

4.5.4. **Corollary:** The functor $D^\downarrow_{A^{\sharp}}$ provides an equivalence of triangulated categories $D^\downarrow(A^{\sharp}) \cong D^\downarrow(D^\bullet(A, B)^{\text{opp}})$.

4.6. **Definitions of semiinfinite cohomology of associative algeb ras.** Consider a left graded $N$-module $N^\ast := \bigoplus_{n \geq 0} \text{Hom}_k(N_n, k)$. The action of $N$ on the space is defined as follows.

$$f : N \rightarrow k, \ n \in N, \ \text{then} \ (n \cdot f)(n') := f(n'n).$$

Consider also the left $A$-module $S^A_N := \text{Ind}_N^A(N^\ast) = A \otimes_N N^\ast$. Evidently $S^A_N \cong B \otimes N^\ast$ as a $B$-module and by $[4.4] S^A_N \cong N^\ast \otimes B$ as a $N$-module.

There is another left $A$-module with these two properties. $S^A_N := \text{Hom}_B(A, B)$ with the left action of $A$ defined as follows.

$$f : A \rightarrow B, \ a \in A, \ \text{then} \ (a \cdot f)(a') := f(a'a).$$
4.6.1. \textbf{Lemma}: The $A$-modules $S^A_N$ and $S^A_N'$ are isomorphic.

\textbf{Proof}. Fix the decomposition $A = B \otimes N$ provided by the multiplication in $A$. For $f \in N^*$ define $f_2 : A \rightarrow B$, $f_2(b \otimes n) := f(n)b$. Define the pairing

$$S^A_N \times A \rightarrow B, \ f \otimes a \times a' \mapsto f_2(a'a).$$

One checks directly the correctness of the definition. Now the condition (vi) from 4.3.3 provides that the defined map is an isomorphism of vector spaces. Thus $S^A_N \cong \text{Coind}^A_B B$. The functors of induction from $N$ to $A$ and of coinduction from $B$ to $A$ provide the natural inclusions of algebras

$$N_{opp} \hookrightarrow \text{Hom}_A(S^A_N, S^A_N) \text{ and } B_{opp} \hookrightarrow \text{Hom}_A(S^A_N, S^A_N).$$

Note that $\text{Hom}_A(S^A_N, S^A_N) = \text{Hom}_k(N^*, B)$ as a vector space. Below we construct an inclusion of associative algebras $A^2 \subset \text{End}_A(S^A_N)$ such that the subalgebras $B_{opp}$ and $N_{opp}$ in $\text{End}_A(S^A_N)$ are exactly the ones provided by the triangular decomposition of $A^2$.

4.6.2. Consider the $D^\bullet(A, B)$-DG-module $D^\dagger_A(S^A_N)$. By Theorem 4.4.2 we have an isomorphism of associative algebras

$$\text{End}_A(S^A_N) \cong \text{Ext}^0_{D^\bullet(A, B)}(D^\dagger_A(S^A_N), D^\dagger_A(S^A_N)).$$

Here Ext functors are taken in the derived category of right $D^\bullet(A, B)$-DG-modules, that is, in the category of left $D^\bullet(A, B)^{opp}$-DG-modules. Consider also the left $D^\bullet(A, B)^{opp}$-DG-module $D^\dagger_{A^2}(A^2)$, where $A^2$ denotes the regular $A^2$-module.

4.6.3. \textbf{Lemma}: $D^\dagger_{A^2}(A^2) \cong D^\dagger_A(S^A_N)$ in the derived category of right $D^\bullet(A, B)$-DG-modules. \hfill \square

4.6.4. \textbf{Proposition}: (see [Ar2], Lemma 3.4.5) The functor $D^\dagger_{A^2}$ provides an inclusion of associative algebras

$$A^2 \hookrightarrow \text{Ext}^0_{D^\bullet(A, B)^{opp}}(D^\dagger_{A^2}(A^2), D^\dagger_{A^2}(A^2)) = \text{End}_A(S^A_N). \hfill \square$$

Thus $S^A_N$ becomes a $A - A^2$-bimodule.

4.6.5. \textit{Continuous Hom description of $A^2$}. We introduce a topology on $A$ (resp. on a graded $A$-module $M$) defined by the filtration $F^m A := \bigoplus_{n \leq m} A_n$ (resp. by the filtration $F^m M := \bigoplus_{n \leq m} M_n$). In particular the multiplication in $A$ and the action of $A$ on $M$ are given by continuous maps. Denote the space of continuous linear maps between two graded $A$-modules $M$ and $M'$ equipped with this topology by $\text{Hom}^{cont}(M, M')$. Thus we have

$$\text{Hom}^{cont}(M, M') = \bigoplus_{n > 0} \text{Hom}(M_n, M') \oplus \prod_{n \geq 0} \text{Hom}(M_n, M').$$

For left graded $A$-modules $M$ and $M'$ consider the space of continuous morphisms

$$\text{Hom}_A^{cont}(M, M') := \{ f \in \text{Hom}^{cont}(M, M') | f(am) = af(m) \text{ for } a \in A, m \in M \}. $$
In particular we have
\[
\text{Hom}_A^\text{cont}(S^A_N, S^A_N) = \text{Hom}_B^\text{cont}(S^A_N, B) = \text{Hom}^\text{cont}(N^*, B) = \bigoplus_{n \leq 0} \text{Hom}((N^*)_n, B) = \bigoplus_{n \geq 0} ((N_n)^*, B) = N \otimes B.
\]

It is easily checked that the images of the inclusions \(B^{\text{opp}} \subset \text{End}_A(S^A_N)\) and \(N^{\text{opp}} \subset \text{End}_A(S^A_N)\) belong to the space of continuous endomorphisms. Thus we obtain the following statement.

4.6.6. **Lemma:** \(A^\sharp = \text{Hom}_A^\text{cont}(S^A_N, S^A_N)\). \(\square\)

Now we give a definition of associative algebra semiinfinite cohomology and compare it with the one presented in [Ar1].

4.6.7. **Definition:** Let \(M\bullet \in \mathcal{C}^\uparrow(A^{\text{opp}})\), \(M^\sharp\bullet \in \mathcal{C}^\downarrow(A^\sharp)\). Then set
\[
\text{Ext}^{\pm \bullet}(M^\sharp\bullet, M\bullet) := \text{Ext}^\bullet_{(A,B)^{\text{opp}}}(D^A_{\text{op}}(M^\sharp\bullet), D^A_{\text{op}}(M\bullet)).
\]

Note that by definition the semiinfinite Ext functor maps complexes exact by either of the variables to zero. We will need also the semiinfinite Tor functor.

4.6.8. **Definition:** Let \(M\bullet \in \mathcal{C}^\downarrow(A)\), \(M^\sharp\bullet \in \mathcal{C}^\downarrow(A^\sharp)\). Then set
\[
\text{Tor}^{\pm \bullet}_{A^\sharp}(M^\sharp\bullet, M\bullet) := \text{Tor}^\bullet_{(A,B)}(D^A_{\text{op}}(M^\sharp\bullet), D^A_{\text{op}}(M\bullet)).
\]

Note that the definition can be rewritten as follows:
\[
\text{Tor}^{\pm \bullet}_{A^\sharp}(M^\sharp\bullet, M\bullet) := \left(\text{Ext}^{\pm \bullet}_{A^\sharp}(M^\sharp\bullet, M^\bullet^\ast)\right)^\ast.
\]

Here \(M^\bullet^\ast\) denotes the right \(A\)-module dual to \(M^\bullet\).

The definition of semiinfinite cohomology in [Ar1] used the following standard complex:
\[
\mathcal{C}^{\pm \bullet}(M^\sharp\bullet, M\bullet) := \text{Hom}_{A^\sharp}\left(\text{Bar}^\ast(A^\sharp, N^{\text{opp}}, M^\sharp\bullet), \text{Bar}^\ast(A^{\text{opp}}, B^{\text{opp}}, M\bullet) \otimes_A S^A_N\right).
\]

4.6.9. **Theorem:** (see [Ar1], Theorem 3.6.2) Let \(M\bullet \in \mathcal{C}^\uparrow(A^{\text{opp}})\), \(M^\sharp\bullet \in \mathcal{C}^\downarrow(A^\sharp)\). Then
\[
\text{Ext}^{\pm \bullet}_{A^\sharp}(M^\sharp\bullet, M\bullet) = H^\bullet(\mathcal{C}^{\pm \bullet}(M^\sharp\bullet, M\bullet)). \quad \square
\]

We present also a similar statement for the semiinfinite Tor functor. Let \(M\bullet \in \mathcal{C}^\downarrow(A)\), \(M^\sharp\bullet \in \mathcal{C}^\downarrow(A^\sharp)\). We define the standard complex for the computation of the semiinfinite Tor functor by
\[
\mathcal{C}^{\pm \bullet}_{A^\sharp}(M^\sharp\bullet, M\bullet) := \left(\mathcal{C}^{\pm \bullet}_{A^\sharp}(M^\sharp\bullet, M^\bullet^\ast)\right)^\ast.
\]

4.6.10. **Corollary:** Let \(M\bullet \in \mathcal{C}^\downarrow(A)\), \(M^\sharp\bullet \in \mathcal{C}^\downarrow(A^\sharp)\). Then
\[
\text{Tor}^{\pm \bullet}_{A^\sharp}(M^\sharp\bullet, M\bullet) = H^\bullet(\mathcal{C}^{\pm \bullet}_{A^\sharp}(M^\sharp\bullet, M\bullet)). \quad \square
4.7. Choice of resolutions. Recall that an object \( M \in A \text{-mod} \) is called \textit{injective} (resp. \textit{projective}) relative to the subalgebra \( N \) if for every complex of \( A \)-modules \( X^\bullet \) such that \( X^\bullet \) is homotopic to zero as a complex of \( N \)-modules we have \( H^\bullet(\text{Hom}_A^\bullet(X^\bullet, M)) = 0 \) (resp. \( H^\bullet(\text{Hom}_A^\bullet(M, X^\bullet)) = 0 \)). An object \( M \in A \text{-mod} \) is called \textit{semijective} if it is both \( N \)-projective and \( A \)-injective relative to \( N \). Here we recall that one can use semijective resolutions to calculate semiinfinite cohomology.

4.7.1. Lemma: (see [Ar1], Lemma 3.4.1) The following facts hold for \( L^\bullet \in C^\downarrow(\mathcal{A}^\sharp) \), \( M^\bullet \in C^\uparrow(\mathcal{A}^{\text{opp}}) \):

(i) if \( M^\bullet \) is \( N \)-projective, then we have
\[
\text{Ext}^\infty_{\mathcal{A}^\sharp}(L^\bullet, M^\bullet) = H^\bullet(\text{Hom}_{\mathcal{A}^\sharp}(\text{Bar}^\bullet(\mathcal{A}^\sharp, N, L^\bullet), M^\bullet \otimes_A S_N^A));
\]

(ii) if \( L^\bullet \) is \( B \)-projective, then we have
\[
\text{Ext}^\infty_{\mathcal{A}^\sharp}(L^\bullet, M^\bullet) = H^\bullet(\text{Hom}_{\mathcal{A}^\sharp}(L^\bullet, \text{Bar}^\bullet(\mathcal{A}, B, M^\bullet) \otimes_A S_N^A));
\]

(iii) if \( M^\bullet \) is semijective then we have
\[
\text{Ext}^\infty_{\mathcal{A}^\sharp}(L^\bullet, M^\bullet) = H^\bullet(\text{Hom}_{\mathcal{A}^\sharp}(L^\bullet, M^\bullet \otimes_A S_N^A)). \tag*{\Box}
\]

We present also the analogue of Lemma 4.7.1(iii) for semiinfinite Tor functors to be used later. We call \( M^\bullet \in C^\downarrow(\mathcal{A}) \) \textit{co-semijective} if it is both \( N \)-injective and \( A \)-projective relative to \( N \).

4.7.2. Corollary: Suppose \( M^\bullet \in C^\downarrow(\mathcal{A}) \) is co-semijective and \( L^\bullet \in C^\downarrow(\mathcal{A}^\sharp) \). Then we have
\[
\text{Tor}^A_{\mathcal{D}^\downarrow}(L^\bullet, M^\bullet) = H^\bullet(\text{Hom}_{\mathcal{A}^\sharp}(S_N^A, M^\bullet) \otimes_{\mathcal{A}^\sharp} L^\bullet). \tag*{\Box}
\]

4.7.3. Proposition: (see [Ar1], Theorem 3.5) Semiinfinite Ext functor is well defined on the corresponding derived categories:
\[
\text{Ext}^\infty_{\mathcal{A}^\sharp} : \mathcal{D}^\downarrow(\mathcal{A}^\sharp) \times \mathcal{D}^\uparrow(\mathcal{A}^{\text{opp}}) \rightarrow \mathcal{D}(\text{Vect}). \tag*{\Box}
\]

A similar fact holds evidently for semiinfinite Tor functors.

5. Functorial properties of semiinfinite cohomology

Now we describe the behaviour of semiinfinite cohomology with respect to a change of rings in spirit of [Ar3], 6.2.
5.1. **Negative extension case.** Suppose we have an inclusion of algebras $A \subset A'$ such that both $A$ and $A'$ have triangular decompositions $A = B \otimes N$ and $A' = B' \otimes N$ as vector spaces satisfying the conditions (i)-(v) from 4.1 and (vi) from 4.3.3. Suppose also that $B$ is a subalgebra in $B'$ (and note that positive parts of the triangular decompositions coincide). Denote the modules over the corresponding algebras provided by the $B$-augmentation (resp. by the $B'$-augmentation) by $B_A$ and $B'_{A'}$ respectively. We have a natural functor of induction $\text{Ind}_{A'}^A : A\text{-mod} \to A'\text{-mod}$ mapping $B_A$ to $B'_{A'}$. Consider corresponding morphism of functors
\[
\text{Ind}_{A'}^A : \text{RHom}^*(B_A, B_A) \to \text{RHom}^*(B'_{A'}, B'_{A'}). 
\]
It is represented by the morphism of DG-algebras $D^*(A, B) \to D^*(A', B')$ constructed as follows.

5.1.1. **Lemma:** We have $\text{Bar}^*(A', B', B'_{A'}) \cong \text{Ind}_{A'}^A \text{Bar}^*(A, B, B_A)$.

5.1.2. **Corollary:** The natural map
\[
D^*(A, B, B) = \text{Hom}^*(\text{Bar}^*(A, B, B_A), B_A) \to \text{Hom}^*(\text{Bar}^*(A', B', B'_{A'}), B'_{A'}) = D^*(A', B, B_A) 
\]
provided by the functor $\text{Ind}_{A'}^A$ is a morphism of DG-algebras.

5.1.3. Suppose we have a right $A'$-module $M$. By the previous Lemma we have an isomorphism of $D^*(A, B)$-DG-modules
\[
D^+_A(M) = M \otimes_{A'} \text{Bar}^*(A', B', B'_{A'}) \cong M \otimes_A \text{Ind}_{A'}^A \text{Bar}^*(A, B, B_A) \cong M = D_A^+(M).
\]

5.1.4. Next we construct the morphism of the $\ddagger$-algebras. Consider again the functor of induction
\[
\text{Ind}_{A'}^A : A\text{-mod} \to A'\text{-mod}, \; M \mapsto A' \otimes_A M. 
\]
Then we have $S_{A'} = \text{Ind}_{A'}^A N^* = \text{Ind}_{A'}^A \text{Ind}_{A}^A S_{N}^A = \text{Ind}_{A'}^A S_{N}^A$. Thus we obtain an inclusion of algebras $\text{End}_{A}(S_{N}^A) \hookrightarrow \text{End}_{A'}(S_{N}^A)$.

5.1.5. **Lemma:** Let $X \in \mathcal{C}^\ddagger(A)$. Then we have a canonical isomorphism of $A'$-modules
\[
\text{Coind}_{B'}^A \circ \text{Ind}_{B'}^A X^* \cong \text{Ind}_{A'}^A \circ \text{Coind}_{B'}^A X^*. 
\]

**Proof.** By 4.3.3(vi) we have $\text{Coind}_{B'}^A X^* \mapsto (\text{Coind}_{B}^A B_A) \otimes_B X^* \mapsto S_{N}^A \otimes_B X^*$. Evidently, $\text{Ind}_{A'}^A S_{N}^A = S_{N}^A$. Thus we obtain $\text{Ind}_{A'}^A \circ \text{Coind}_{B'}^A X^* \mapsto S_{N}^A \otimes_B X^*$. On the other hand we have
\[
\text{Coind}_{B'}^A \circ \text{Ind}_{B'}^A X^* \mapsto \text{Coind}_{B'}^A B_{A'} \otimes_{B'} \text{Ind}_{B}^A X^* = S_{N}^A \otimes_B X^*. 
\]
Taking the composition of the constructed isomorphisms we obtain the required one.
5.1.6. **Lemma:** The map constructed in 5.1.4 provides an inclusion of algebras $A^\sharp \hookrightarrow A'^\sharp$ well defined with respect to the triangular decompositions.

**Proof.** The inclusion $N^{opp} \subset \text{End}_A(S_N^A)$ (resp. $N^{opp} \subset \text{End}_{A'}(S_N^{A'})$) is provided by the realisations of the semiregular modules as coinduced ones from the coregular $N$-module. Thus the constructed map $\text{End}_A(S_N^A) \to \text{End}_{A'}(S_N^{A'})$ identifies the images of the algebra $N^{opp}$.

Next by Lemma 5.1.3 we have the isomorphism of functors $C^+(B) \to C^+(A')$:

$$\text{Coind}_{B'}^A \circ \text{Ind}_B^A \cong \text{Ind}_A^B \circ \text{Coind}_{B'}^A.$$  

Thus the images of two inclusions

$$B^{opp} \hookrightarrow \text{End}_A(S_N^A) \hookrightarrow \text{End}_{A'}(S_N^{A'})$$

coincide. Now recall that as a vector space the subalgebra $A^\sharp \subset \text{End}_A(S_N^A)$ is described as $N^{opp} \otimes B^{opp}$. \hfill \Box

Note also that the inclusions $A \subset A'$ and $A^\sharp \subset A'^\sharp$ provide the same morphism of DG-algebras $D^\bullet(A, B) \to D^\bullet(A^\sharp, B)^{opp} \to D^\bullet(A'^\sharp, B')^{opp} = D^\bullet(A', B')$.

5.1.7. Suppose we have an $A'^\sharp$-module $M'^\sharp$. By Lemma 5.1.1 we have an isomorphism of $D^\bullet(A, B)$-DG-modules

$$D^\dagger_{A'^\sharp}(M'^\sharp) = \text{Hom}_{A'^\sharp}(\text{Bar}^\bullet(A'^\sharp, B', B_A'), M'^\sharp) \cong \text{Hom}_{A'^\sharp}(\text{Ind}_{A'^\sharp}^A \text{Bar}^\bullet(A^\sharp, B, B_A'), M'^\sharp)$$

$$\cong \text{Hom}_{A'^\sharp}(\text{Bar}^\bullet(A^\sharp, B, B_A'), M'^\sharp) = D^\dagger_{A^\sharp}(M'^\sharp).$$

5.1.8. **Proposition:** There exists a natural (functorial) morphism

$$\text{Ext}^\infty_{A'}^\bullet(M^\sharp, M) \to \text{Ext}^\infty_{A'}^\bullet(M'^\sharp, M).$$

**Proof.** For any two $D^\bullet(A', B')$-DG-modules $X^\bullet$ and $Y^\bullet$ the morphism of the DG-algebras $D^\bullet(A, B) \to D^\bullet(A', B')$ obtained from the morphism of functors $\text{RHom}_{A'}^\bullet(\text{Ind}_{A'}^*(\ast), \text{Ind}_A^*(\ast))$ provides a natural restriction map

$$\text{RHom}^\bullet_{D^\bullet(A', B')}(X^\bullet, Y^\bullet) \to \text{RHom}^\bullet_{D^\bullet(A, B)}(X^\bullet, Y^\bullet).$$

In particular we have a natural morphism

$$\text{RHom}^\bullet_{D^\bullet(A', B')}(D^\dagger_{A'^\sharp}(M'^\sharp), D^\dagger_{A'}^\bullet(M)) \to \text{RHom}^\bullet_{D^\bullet(A, B)}(D^\dagger_{A'^\sharp}(M'^\sharp), D^\dagger_A^\bullet(M)).$$

Now the required morphism is obtained by the composition of the constructed one with the isomorphisms described in 5.1.3 and 5.1.7. \hfill \Box

Similarly we obtain the following statement.
5.1.9. **Proposition:** Suppose we have an inclusion of algebras \( A \subset A' \) such that both algebras have triangular decompositions \( A = B \otimes N \) and \( A' = B' \otimes N' \) as vector spaces and the inclusion preserves the triangular decompositions, i.e. \( B \subset B' \). Let \( M \in \mathcal{C}(A') \) and \( M^\mathfrak{z} \in \mathcal{C}(A'^\mathfrak{z}) \). Then there exists a natural (functorial) morphism

\[
\text{Tor}^A_{\mathfrak{z}+ \bullet} (M^\mathfrak{z}, M) \longrightarrow \text{Tor}^{A'}_{\mathfrak{z}+ \bullet} (M^\mathfrak{z}, M). \]

Note that by the definition of the semiinfinite Tor functor we have \( \text{Tor}^A_{\mathfrak{z}+ \bullet} (M^\mathfrak{z}, S^A_N) = M^\mathfrak{z} \) for every \( A^\mathfrak{z} \)-module \( M^\mathfrak{z} \).

5.1.10. **Lemma:** Let \( M^\mathfrak{z} \in \mathcal{C}(A'^\mathfrak{z}) \). Then the canonical map

\[
\text{Tor}^A_{\mathfrak{z}+ \bullet} (M^\mathfrak{z}, S^A_N) \longrightarrow \text{Tor}^{A'}_{\mathfrak{z}+ \bullet} (M^\mathfrak{z}, S^A_N)
\]

coincides with the obvious one \( \text{Ind}^{B'_{opp}}_{B_{opp}} \text{Res}^{A'^\mathfrak{z}}_A M^\mathfrak{z} \longrightarrow M^\mathfrak{z} \).

5.2. **Positive extension case.** Suppose we have an inclusion of algebras \( A \subset A' \) such that both \( A \) and \( A' \) have triangular decompositions \( A = B \otimes N \) and \( A' = B' \otimes N' \) as vector spaces satisfying the conditions (i)-(v) from 4.1 and (vi) from 4.3.3. Suppose also that \( N \) is a subalgebra in \( N' \) (and note that negative parts of the triangular decompositions coincide). Denote the modules over the corresponding algebras provided by the \( B \)-augmentations by \( B_A \) and \( B_A' \) respectively. We have a natural functor of restriction \( \text{Res}^A_{A'} : A'\text{-mod} \longrightarrow A\text{-mod} \) mapping \( B_A' \) to \( B_A \). Consider corresponding morphism of functors

\[
\text{Res}^A_{A'} : \text{RHom}^\bullet_A (B_A', B_A') \longrightarrow \text{RHom}^\bullet_A (B_A', B_A).
\]

It is represented by the morphism of DG-algebras \( D^\bullet (A', B) \longrightarrow D^\bullet (A, B) \) constucted as follows.

Consider the morphism of bar resolutions \( \text{Bar}^\bullet (A, B, B_A) \longrightarrow \text{Bar}^\bullet (A', B, B_A') \) provided by the inclusion of algebras.

5.2.1. **Lemma:** The composition of the described map with the restriction morphism provides a morphism of DG-lagebras

\[
D^\bullet (A', B) = \text{Hom}^\bullet_{A'} (\text{Bar}^\bullet (A', B, B_A'), B_A') \longrightarrow \text{Hom}^\bullet_A (\text{Bar}^\bullet (A', B, B_A'), B_A) \longrightarrow \text{Hom}^\bullet_A (\text{Bar}^\bullet (A, B, B_A), B_A) = D^\bullet (A, B). \]

Suppose we have a right \( A' \)-module \( M \). Consider the morphism of functors

\[
\text{Res}^A_{A'} : M_L \otimes_A B_A \longrightarrow M_L \otimes_{A'} B_A'.
\]

It is represented by a morphism of DG-modules over \( D^\bullet (A', B) \) constucted similarly to the one in Lemma [5.2.1]: \( D^\mathfrak{l}_A (M) \longrightarrow D^\mathfrak{l}_A' (M) \).
5.2.2. Consider also a the functor of coinduction

\[ \text{Coind}^A_{A'} : \textbf{A-mod} \rightarrow \textbf{A'}-\text{mod}, \ M \mapsto \text{Hom}_A(A', M). \]

Then we have \( S_{N'}^A = \text{Coind}^A_{B'} B = \text{Coind}^A_{B} \text{Coind}^A_{B} B = \text{Coind}^A_{B'} S_{N}^A \). Thus we obtain an inclusion of algebras \( \text{End}_A(S_{N}^A) \hookrightarrow \text{End}_A(S_{N'}^A) \).

5.2.3. **Lemma:** Let \( X^\bullet \in \mathcal{C}^+(N) \). Then we have a canonical isomorphism of \( A'-\text{modules} \)

\[ \text{Ind}^{A'}_N \circ \text{Coind}^{N'}_N X^\bullet \cong \text{Coind}^{A'}_A \circ \text{Ind}^{A}_N X^\bullet. \]

**Proof.** Suppose \( X, Y \in A-\text{mod} \) and \( \text{Hom}_A(X, Y) \in \mathcal{C}^+(\text{Vect}) \). Then we have a canonical isomorphism \( (X \otimes_A Y^*)^* \cong (\text{Hom}_A(X, Y))^* \). Using this isomorphism we obtain

\[ \text{Ind}^{A'}_N \circ \text{Coind}^{N'}_N X^\bullet \underbrace{\longrightarrow (\text{Coind}^{A'}_N \left( \text{Coind}^{N'}_N X^\bullet \right)^* )^*} \rightarrow \underbrace{(\text{Coind}^{A}^N \text{Ind}^{N'}_N (X^*)^*)^*}, \]

\[ \text{Coind}^{A'}_N \text{Ind}^{A}_N X^\bullet \underbrace{\longrightarrow (\text{Ind}^{A'}_N \left( \text{Ind}^{A}_N X^\bullet \right)^* )^*} \rightarrow \underbrace{(\text{Ind}^{A'}_N \text{Coind}^{A}_N (X^*)^*)^*}. \]

Now change all the gradings to the opposite ones and use Lemma 5.1.3. \( \square \)

5.2.4. **Lemma:** The constructed map provides an inclusion of algebras \( A^\sharp \hookrightarrow A'^\sharp \) well defined with respect to the triangular decompositions.

**Proof.** The inclusion \( B^{\text{opp}} \subset \text{End}_A(S_{N}^A) \) (resp. \( B^{\text{opp}} \subset \text{End}_{A'}(S_{N'}^A) \)) is provided by the realisations of the semiregular modules as coinduced ones from the regular \( B \)-module. Thus the constructed map \( \text{End}_A(S_{N}^A) \rightarrow \text{End}_{A'}(S_{N'}^A) \) identifies the images of the algebra \( B^{\text{opp}} \).

Next by Lemma 5.2.3 we have the isomorphism of functors \( C^+(N) \rightarrow C^+(A') \):

\[ \text{Ind}^{A'}_N \circ \text{Coind}^{N'}_N \cong \text{Coind}^{A'}_A \circ \text{Ind}^{A}_N. \]

Thus the images of two inclusions

\[ N^{\text{opp}} \hookrightarrow \text{End}_A(S_{N}^A) \hookrightarrow \text{End}_{A'}(S_{N'}^A) \text{ and } N^{\text{opp}} \hookrightarrow N'^{\text{opp}} \hookrightarrow \text{End}_{A'}(S_{N'}^A) \]

coincide. Now recall that as a vector space the subalgebra \( A^\sharp \subset \text{End}_A(S_{N}^A) \) is described as \( N^{\text{opp}} \otimes B^{\text{opp}} \).

Note also that the inclusions \( A \subset A' \) and \( A^\sharp \subset A'^\sharp \) provide the same morphism of DG-algebras \( D^\bullet(A^\sharp, B)^{\text{opp}} = D^\bullet(A', B) \rightarrow D^\bullet(A, B) = D^\bullet(A^\sharp, B)^{\text{opp}}. \)

5.2.5. Suppose we have an \( A'^\sharp \)-module \( M'^\sharp \). Consider the morphism of functors

\[ \text{Res}^{A'}_{A^\sharp} : \text{RHom}^\bullet_{A'^\sharp} (B_{A'^\sharp}, M'^\sharp) \rightarrow \text{RHom}^\bullet_{A^\sharp} (B_{A^\sharp}, M^\sharp). \]

It is represented by a morphism of DG-modules over \( D^\bullet(A', B)^{\text{opp}} \) constructed similarly to the one in Lemma 5.2.1: \( D^\bullet_{A^\sharp} (M^\sharp) \rightarrow D^\bullet_{A'} (M^\sharp). \)
5.2.6. **Proposition:** There exists a natural (functorial) morphism
\[ \text{Ext}_{A^\sharp}^\bullet(M^\sharp, M) \to \text{Ext}_{A^\sharp}^\bullet(M^\sharp, M). \]

**Proof.** For any two $D^\bullet(A, B)$-DG-modules $X^\bullet$ and $Y^\bullet$ the morphism of the DG-algebras $D^\bullet(A', B) \to D^\bullet(A, B)$ obtained from the morphism of functors $\text{RHom}_A^\bullet(\ast, \ast) \to \text{RHom}_A^\bullet(\text{Res}_A^{A'}(\ast), \text{Res}_A^{A'}(\ast))$ provides a natural restriction map
\[ \text{RHom}_{D^\bullet(A, B)}^\bullet(X^\bullet, Y^\bullet) \to \text{RHom}_{D^\bullet(A', B)}^\bullet(X^\bullet, Y^\bullet). \]

In particular we have a natural morphism
\[ \text{RHom}_{D^\bullet(A, B)}^\bullet(D^\sharp_{A'}(M^\sharp), D^\sharp_A(M)) \to \text{RHom}_{D^\bullet(A', B)}^\bullet(D^\sharp_{A'}(M^\sharp), D^\sharp_A(M)). \]

Now the required morphism is obtained by the composition of the constructed one with the maps described in 5.2.1 and 5.2.3. □

Similarly we obtain the following statement.

5.2.7. **Proposition:** Suppose we have an inclusion of associative algebras $A \subset A'$ such that both algebras have triangular decompositions $A = B \otimes N$ and $A' = B' \otimes N'$ as vector spaces and the inclusion preserves the triangular decompositions, i.e. $N \subset N'$. Let $M \in C^\bullet(A')$ and $M^\sharp \in C^\bullet(A'^\sharp)$. Then there exists a natural (functorial) morphism
\[ \text{Tor}_{\frac{\infty}{2} + \bullet}^A(M^\sharp, M) \to \text{Tor}_{\frac{\infty}{2} + \bullet}^{A'}(M^\sharp, M). \]

5.2.8. **Lemma:** Let $M^\sharp \in C^\bullet(A'^\sharp)$. Then the canonical map
\[ \text{Tor}_{\frac{\infty}{2} + \bullet}^A(M^\sharp, S_{N'}^{A'}) \to \text{Tor}_{\frac{\infty}{2} + \bullet}^{A'}(M^\sharp, S_{N'}^{A'}) \]
coincides with the obvious one $M^\sharp \to \text{Coind}_{N'_{\text{opp}}}^{N_{\text{opp}}} \text{Res}_{N'_{\text{opp}}}^{A'} M^\sharp$. □

5.3. **Semiinfinite induction functor.** Suppose we have an inclusion of associative algebras $A \subset A'$ such that, as before, both algebras have triangular decompositions $A = B \otimes N$ and $A' = B' \otimes N'$ and the inclusion preserves them — $B \subset B'$ and $N \subset N'$, but the inclusion is neither negative nor positive. Suppose also for simplicity that $B'$ (resp. $N'$) is free both as a right and as a left $B$- (resp. $N$-)module, $B' = B \otimes V^-$ and $N' = N \otimes V^+$; the spaces of generators are graded with finite dimensional grading components. Then it is still possible to construct a morphism of $\sharp$-algebras as follows.

5.3.1. **Lemma:** We have a canonical isomorphism of $A'^\sharp$-$A$-bimodules
\[ S_{N'}^{A'} \cong \text{Tor}_{\frac{\infty}{2} + 0}^A(S_{N'}^{A'}, S_N^A). \]

Note that $A^\sharp$ acts on the right hand side bimodule via the second factor. Thus we have a morphism of algebras $A^\sharp \to \text{End}_A(S_{N'}^{A'})$. A simple calculation shows that the image in fact belongs to the space of continuous endomorphisms. By [1.6.3] we obtain an inclusion of algebras $A^\sharp \to A'^\sharp$. It is easily checked that the pair of algebras $(A^\sharp, A'^\sharp)$ satisfies the conditions [5.3.]. Consider the $A'$-$A$-bimodule $\text{Hom}_{A'}^\text{cont}(S_N^A, S_{N'}^{A'})$. [end of document]
5.3.2. **Lemma:** \( \text{Hom}^\text{cont}_{A^\sharp}(S_N^A, S_N^A') \xrightarrow{\sim} V^+ \otimes V^- \otimes A \) as a right \( A \)-module. 

Consider the *semiinfinite induction* functor

\[
\text{Ind}^\infty_2 : A\text{-mod} \rightarrow A'\text{-mod}, \ M \mapsto \text{Hom}^\text{cont}_{A^\sharp}(S_N^A, S_N^A') \otimes_A M.
\]

By the previous Lemma it is exact and well defined on corresponding derived categories:

\[
\text{Ind}^\infty_2 : D(A\text{-mod}) \rightarrow D(A'\text{-mod}).
\]

Evidently for \( M \in A\text{-mod} \) we have \( \text{Ind}^\infty_2(M) = V^+ \otimes V^- \otimes M \) as a vector space.

6. \( \sharp \)-**algebras for affine quantum groups**

In this section we calculate the endomorphism algebras of semiregular modules over affine quantum groups at a fixed level with respect to various subalgebras. From now on we fix the base field \( k = \mathbb{Q}(v) \).

6.1. Consider the affine quantum group \( U_k \) at a fixed level \( k \in \mathbb{Z} \) (see 3.4). Recall that \( U \) is a \( Y \)-graded algebra. The \( Y \)-grading on \( U_k \) is induced by the one on \( U \). We define a \( \mathbb{Z} \)-grading of a homogenous element \( u \in U_k \) by \( \deg u := \text{ht}(\deg^Y u) \).

The triangular decomposition of \( U_k \) is provided by the one of \( U \). Namely, in terms of the previous section, we have \( A = U^0_k = U^- \otimes U^0_k \) and \( N = U^+ \) where \( U^0_k \) denotes the quotient algebra of \( U^0_k \) by the relation \( Z = v^k \). Evidently the triple \( (U_k, U^0_k, U^+) \) satisfies the conditions (i)-(v) from 4.1.

Consider the semiregular module \( S_{U_k} \) over \( U_k \) with respect to \( U^+ \).

6.1.1. **Lemma:** The algebra \( U_k \) satisfies also the condition (vi) from 4.3.3. 

6.2. It is known (see e. g. [V], [Ar2]) that the \( \sharp \)-algebra for the universal enveloping algebra of the affine Lie algebra at a fixed level \( k \) is equal to the universal enveloping algebra of the same Lie algebra at the level \(-2h^\vee - k\) where \( h^\vee \) denotes the dual Coxeter number (see 2.1.2). Here we present a similar statement for the affine quantum groups.

6.2.1. Consider the DG-algebra \( D^*(U_k, U^0_k) \). Then the multiplication in \( D^*(U_k, U^0_k) \) provides the isomorphisms of graded vector spaces

\[
D^*(U^+, k) \otimes U^0_k \xrightarrow{\sim} D^*(U_k, U^0_k) \quad \text{and} \quad U^0_k \otimes D^*(U^+, k) \xrightarrow{\sim} D^*(U_k, U^0_k).
\]

Recall also that \( D^*(U^+, k) \) is isomorphic to the tensor algebra of the graded vector space \( \mathcal{U}^{++} \). We are going to find the DG-algebra \( D^*(U, U^0)^{\text{opp}} \) explicitly.

Consider the map \( s : \widehat{U} \rightarrow \widehat{U}^{\text{opp}} \) defined by

\[
s(E_i) = -E_i, \quad s(F_i) = -F_i, \quad s(K_\mu) = K_{-\mu}, \quad i \in I, \ \mu \in Y.
\]
6.2.2. Lemma: The map $s$ is well defined on $\widetilde{U}$ and provides an isomorphism of algebras $U_k \longrightarrow U_{opp}$. \hfill $\Box$

In particular $s$ provides isomorphisms of algebras

$$s_{U^{\leq 0}} : U_k^{\leq 0} \longrightarrow U_{opp}^{\leq 0} \text{ and } s_{U^+} : U^+ \longrightarrow U^{+opp}.$$  

Note that $s$ has nothing to do with the antipode map being a part of the Hopf algebra structure on $\widetilde{U}$.

We denote the corresponding map of dual spaces by $s_{U^+}^* : U^{++} \longrightarrow U^{+}$. Consider the morphism of algebras $D(s_{U^+}) : T(U^{++}) \longrightarrow T(U^{+})^{opp}$ as follows:

$$D(s_{U^+})(u_1 \otimes \ldots \otimes u_m) := s_{U^*}^*(u_m) \otimes \ldots \otimes s_{U^*}^*(u_1), \ u_1, \ldots, u_m \in U^+.$$

6.2.3. Lemma: The map $D(s_{U^+})$ preserves the differentials and defines an isomorphism of DG-algebras $D^*(U^+, k) \longrightarrow D^*(U^{+}, k)^{opp}$. \hfill $\Box$

Now we use the antipode map in $U$ and $U_k$ to perform a similar construction for these algebras.

6.2.4. Antipode maps in DG-algebras. We prefer to work in a more general setup. Suppose we have an associative algebra $A$ with two its subalgebras $B$ and $N$ like in [4.3] with a triangular decomposition satisfying the conditions (i)-(vi). Suppose also there exists an antiautomorphism $s_A : A \longrightarrow A^{opp}$ preserving the triangular decomposition, i. e. $s_A(B) \subset B$ and $s_A(N) \subset N$, for simplicity we set $s_A^2 = Id$. By [1.3.4] the algebra $D^*(A, B)$ is isomorphic to the algebra $T_B(M)$ for the $B$-bimodule $M = \text{Hom}_{B\text{-left}}(A/B, B)$.

6.2.5. Lemma: The map $D(s_A) : \text{Hom}_{B\text{-left}}(A/B, B) \longrightarrow \text{Hom}_{B\text{-right}}(A/B, B)$ given by

$$D(s_A)(f)(a) = s_B(f(s_A(a))), \ f : A/B \longrightarrow B, \ a \in A,$$

provides a correctly defined isomorphism of $B$-bimodules. Here the left $B$-module structure on the space $\text{Hom}_{B\text{-right}}(A/B, B)$ is provided by the left $B$-multiplication in $A/B$ twisted by $s_A$ and the right $B$-module structure on it is provided by the left $B$-multiplication in $B$ twisted also by $s_B$.

**Proof.** First we check that $D(s_A)(f)$ belongs to the described space of homomorphisms:

$$D(s_A)(f)(ab) = s_B(f(s_A(ab))) = s_B(f(s_B(b)s_A(a)))$$

$$= s_B(s_B(b)f(s_A(a))) = s_B(f(s_A(a)))s_B^2(b) = D(s_A)(f)(a)b.$$

Next we check that $D(s_A)$ is a morphism of the left $B$-modules:

$$D(s_A)(b \cdot f)(a) = s_B(b \cdot f(s_A(a))) = s_B(f(s_A(a)b))$$

$$= s_B(f(s_A(s_B(b)a))) = D(s_A)(f)(s_B(b)a) = (b \cdot D(s_A)(f))(a).$$
Finally we check that $D(s_A)$ is a morphism of right $B$-modules:

$$D(s_A)(f \cdot b)(a) = s_B(f \cdot b(s_A(a))) = s_B(f(s_A(a))b) = s_B(b)s_B(f(s_A(a))) = (D(s_A)(f)\cdot b)(a).$$

Consider the $B^{\text{opp}}$-bimodule $M_{\text{opp}} := \text{Hom}_{B^{\text{opp}}}(A^{\text{opp}}/B^{\text{opp}}, B^{\text{opp}})$ with the left $B^{\text{opp}}$-module structure provided by the right $B^{\text{opp}}$-multiplication in $A^{\text{opp}}/B^{\text{opp}}$ (i.e. by the left $B$-multiplication in $A/B$) and the right $B^{\text{opp}}$-module structure provided by the right $B^{\text{opp}}$-multiplication in $B^{\text{opp}}$ (i.e. by the left $B$-multiplication in $B$). Consider as before the $B^{\text{opp}}$-free algebra $T_{B^{\text{opp}}}(M_{\text{opp}})$ generated by $M_{\text{opp}}$.

We extend the morphism $D(s_A)$ to the following map

$$D(s_A) : T_B(M) \to T_{B^{\text{opp}}}(M_{\text{opp}}), \ b \mapsto s_B(b);$$

$$(*) \quad f_1 \otimes \ldots \otimes f_m \mapsto D(s_A)(f_m) \otimes \ldots \otimes D(s_A)(f_1).$$

6.2.6. Lemma: The map $D(s_A)$ provides a correctly defined isomorphism of associative algebras $T_B(M)_{\text{opp}}$ and $T_{B^{\text{opp}}}(M_{\text{opp}})$.

Proof. Note that $\text{Hom}_{B^{\text{opp}}}(A/B, B)$ is isomorphic to the $B$-bimodule obtained from $M_{\text{opp}}$ by composing the $B$-actions with the antipode map $s_B$. Thus by Lemma 6.2.5 we have a vector space map $M \to M_{\text{opp}}$. Any homomorphism of vector spaces $V_1 \to V_2$ provides an antihomomorphism of tensor algebras (over the base field) $T(V_1) \to T(V_2)$ given by the formula (*). We are to check that the map is well defined with respect to the $B$-action. The necessary calculation looks as follows:

$$D(s_A)((f_1 \cdot b) \otimes f_2) = D(s_A)(f_2 \otimes D(s_A)((f_1 \cdot b)) = D(s_A)(f_2) \otimes D(s_A)(f_1) \cdot s_B(b)$$

$$= D(s_A)(f_2) \otimes s_B(b)^{\text{opp}} D(s_A)(f_1) = D(s_A)(f_2)^{\text{opp}} s_B(b) \otimes D(s_A)(f_1)$$

$$= s_B(b) \cdot D(s_A)(f_2) \otimes D(s_A)(f_1) = D(s_A)(b \cdot f_2) \otimes D(s_A)(f_1) = D(s_A)(f_1 \otimes b \cdot f_2).$$

Here $\cdot^{\text{opp}}$ denotes the left and right actions of $B^{\text{opp}}$ on $M_{\text{opp}}$ to distinguish from $\cdot$ denoting the same actions considered as the right and left actions of $B$ respectively.

Now note by 4.3.4 the associative algebra $T_{B^{\text{opp}}}(M_{\text{opp}})$ is isomorphic to the algebra $D^\bullet(A^{\text{opp}}, B^{\text{opp}})$. Thus we have the isomorphism of algebras (with differentials forgotten) $D(s_A) : D^\bullet(A, B)_{\text{opp}} \to D^\bullet(A^{\text{opp}}, B^{\text{opp}})$.

6.2.7. Corollary: We have the isomorphisms of associative algebras

(i) $D^\bullet(U, U^{\leq 0})_{\text{opp}} \to D^\bullet(U, U^{\leq 0})$;

(ii) $D^\bullet(U_k, U_k^{\leq 0})_{\text{opp}} \to D^\bullet(U_{-k}, U_{-k}^{\leq 0})$.

Again, these isomorphisms do not necessarily preserve the differentials in the DG-algebras. Note that the associative algebra structure in $D^\bullet(U_k, U_k^{\leq 0})$ does not depend on the level $k$, and it is only the differential that "remembers" the level.
6.2.8. Proposition: The map $D(s_U)$ provides an isomorphism of the DG-algebras $D^\bullet(U_k, U_k^{\leq 0})_{\text{opp}} \cong D^\bullet(U_{-2h^\vee-k}, U_{-2h^\vee-k}^{\leq 0})$.

Proof. Since the associative algebras $D^\bullet(U_k, U_k^{\leq 0})$ do not depend on $k$, we can identify them and think of the whole picture as of a family of differentials $\{d_k\}$ on the graded algebra, say, $D^\bullet(U_0, U_0^{\leq 0})$. We are to check that the differential defined by $D(s_{U_0}) \circ d_k \circ D(s_{U_0})^{-1}$ coincides with $d_{-2h^\vee-k}$. We fix the isomorphism of vector spaces

$$D^\bullet(U_0, U_0^{\leq 0}) \cong U_0^{\leq 0} \otimes D^\bullet(U^+, k) = U_0^{\leq 0} \otimes T(U^{++}).$$

Recall that $T(U^{++})$ with the differential generated by the map dual to the multiplication map in $U^+$ is a DG-subalgebra in all the DG-algebras $D^\bullet(U_k, U_k^{\leq 0})$. By Lemma 6.2.3, the restriction of the differential $D(s_{U_0}) \circ d_k \circ D(s_{U_0})^{-1}$ to $T(U^{++})$ coincides with the described one. On the other hand the associative algebra $D^\bullet(U_0, U_0^{\leq 0})$ is generated by $U_0^{\leq 0}$ and $T(U^{++})$ thus it is sufficient to check that the components $U_0^{\leq 0} \rightarrow U_0^{\leq 0} \otimes U^{++}$ of $d_{-2h^\vee-k}$ and $D(s_{U_0}) \circ d_k \circ D(s_{U_0})^{-1}$ coincide. In fact, using the Leibnitz rule, we can restrict the differentials to the space of generators of $U_0^{\leq 0}$ or, since $U_0^{\leq 0} \otimes U^{++}$ is strictly negatively graded and the differentials preserve the gradings, we have to check only that

$$d_{-2h^\vee-k}(F_i) = D(s_{U_0}) \circ d_k \circ D(s_{U_0})^{-1}(F_i) \text{ for } i \in I.$$

We omit the direct calculation that looks completely like the one in [Ar2], Corollary 4.4.2, in the case of affine Lie algebras.

6.2.9. Corollary: There exists an inclusion of associative algebras $U_{-2h^\vee-k} \subset \text{End}_{U_k}(s_{U_k^+})$.

In particular $S_{U_k}$ becomes a $U_k^{-} U_{-2h^\vee-k}$-bimodule.

6.3. Fix a positive weight $x \in Y^{\vee_+}$, $(i, x) > 0$ for every $i \in I$, and the transvection element $\theta_{mx} \in T \subset W$. Recall that we have defined the subalgebra $U_{\theta_{mx}}^+ \subset U^+$. Note that repeating word by word the considerations from the previous subsection we obtain a proof of the following statement.

6.3.1. Corollary: There exists an inclusion of associative algebras $U_{-2h^\vee-k} \subset \text{End}_{U_k}(s_{T_{\theta_{mx}}}^+(U^-))$.

Consider the semiregular $U_k$-module $S_{U_k}^{U_k} := \text{Coind}_{U_k^{\theta_{mx}}}^+ U_k^+ \text{ Ind}_{U_k^{\theta_{mx}}}^+$ with respect to the subalgebra $U_{\theta_{mx}}^+$. Here we prove that like in the affine Lie algebra case there exists an associative algebra inclusion $U_k \hookrightarrow \text{End}_{U_k}(S_{U_k}^{U_k})$. Unfortunately unlike in the Lie algebra case we cannot write down the second action of $U_k$ on the semiregular module explicitly. Instead of that we prove the following statement.
6.3.2. **Theorem:** There exists a natural isomorphism in the derived category of left $U_k$-modules

$$S^U_k \overset{L}{\otimes} U_{-2h^\vee-k} S^U_{T_{\theta mx}^{-1}(U^-)} \rightarrow S^U_k [\ell t(\theta mx)].$$

Here $[\cdot]$ denotes the shift in the derived category and the left $U_k$-module structure on the left hand side of the isomorphism is provided by the natural left $U_k$-module structure on $S^U_k$.

**Remark:** In other words we have

$$\text{Tor}_{U_k} \neq \ell t(\theta mx) (S^U_k, S^U_{T_{\theta mx}^{-1}(U^+)}) = 0 \quad \text{and} \quad \text{Tor}_{U_k} \ell t(\theta mx) (S^U_k, S^U_{T_{\theta mx}^{-1}(U^+)}) = S^U_k$$

as a left $U_k$-module. Now note that by Corollary 6.3.1 the left hand side of the latter equivalence carries the natural structure of a $U_k$-bimodule.

6.4. **Proof of Theorem 6.3.2.** The statement of the Theorem will be derived from Lemmas 6.4.1–6.4.11. Consider the natural triangular decomposition of the algebra $T_{\theta mx}^{-1}(U^-)$ provided by its inclusion into $U$.

6.4.1. **Lemma:** The algebra $T_{\theta mx}^{-1}(U^-)^g$ is isomorphic to $T_{\theta mx}^{-1}(U^-)$.

**Proof.** By 5.3 the algebra $T_{\theta mx}^{-1}(U^-)$ is a subalgebra in $U_k^g$ generated by $U_k^+$ and $T_{\theta mx}^{-1}(U^-) \cap U^-$. On the other hand we know by Theorem 6.3.2 that $U_k^g \cong U_{-2h^\vee-k}$. Finally, the subalgebra in $U_{-2h^\vee-k}$ generated by $U_k^+$ and $T_{\theta mx}^{-1}(U^-) \cap U^-$ is evidently isomorphic to $T_{\theta mx}^{-1}(U^-)$. □

The following statement is a particular case of Lemma 5.3.1.

6.4.2. **Lemma:** There exists a canonical isomorphism of $U_k$-$T_{\theta mx}^{-1}(U^-)$-bimodules

$$S^U_k \overset{\cong}{\rightarrow} \text{Ind}_{T_{\theta mx}^{-1}(U^-)}^\infty U_k S^U_{T_{\theta mx}^{-1}(U^-)}.$$ □

6.4.3. **Lemma:**

$$\left(\text{Ind}_{T_{\theta mx}^{-1}(U^-)}^\infty U_k S^U_{T_{\theta mx}^{-1}(U^-)}\right)^L \otimes_{T_{\theta mx}^{-1}(U^-)} T_{\theta mx}^{-1}(U^-)^* \overset{\cong}{\rightarrow} \text{Ind}_{T_{\theta mx}^{-1}(U^-)}^\infty U_k \left(S^U_{T_{\theta mx}^{-1}(U^-)} \otimes_{T_{\theta mx}^{-1}(U^-)} T_{\theta mx}^{-1}(U^-)^*\right).$$

**Proof.** The statement follows from the exactness of the semiinfinite induction functor. □
6.4.4. Recall that a nonnegatively graded algebra \( A = \bigoplus_{n \geq 0} A_n \), with \( A_0 = k \), is called \textit{quadratic} if it is generated by the space \( A_1 \), \( \dim A_1 < \infty \), and its relations ideal in \( T(A_1) \) is generated by a space \( J \subset A_1 \otimes A_1 \). The \textit{quadratic dual algebra} \( A^! \) for the quadratic algebra \( A \) is by definition the algebra on generators \( A_1^* \), \( \dim A_1^* < \infty \), and its relations ideal in \( T(A_1^*) \) is generated by a space \( J^\perp \subset A_1^* \otimes A_1^* \). The \textit{quadratic dual algebra} \( A^! \) for the quadratic algebra \( A \) is by definition the algebra on generators \( A_1^* \) and quadratic relations \( J^\perp \subset A_1^* \otimes A_1^* \). The \textit{quadratic dual algebra} \( A^! \) for the quadratic algebra \( A \) is by definition the algebra on generators \( A_1^* \) and quadratic relations \( J^\perp \subset A_1^* \otimes A_1^* \).

Note that the algebra \( A^! \) for the Koszul algebra \( A \) is equal to \( \operatorname{Ext}^\bullet_A(k, k) \) since the Koszul complex provides a \( A \)-free resolution of the trivial module. Evidently \( A \) is Koszul iff \( A^! \) is so.

By definition the \textit{twisted exterior algebra} \( \Lambda_q \) is given by the generators \( E^*_\beta, \beta \in R_{\theta}^+ \), and relations
\[
E^*_\alpha E^*_\beta + E^*_\beta E^*_\alpha = 0 \quad \text{if} \quad \beta > \alpha;
\]
\[
E^2_\alpha = 0 \quad \text{for} \quad \alpha \in R_{\theta}^+.
\]

6.4.5. \textbf{Lemma:} \( \text{gr} \ U_{\theta mx}^+ \) is a quadratic Koszul algebra with the quadratic dual algebra equal to \( \Lambda_q \).

Recall that a finite dimensional algebra \( A \) is called \textit{strongly Frobenius} if the coregular \( A \)-bimodule \( A^* \) is isomorphic to the regular \( A \)-bimodule.

6.4.6. \textbf{Lemma:} Suppose that a finite dimensional quadratic Koszul algebra \( A \) is strongly Frobenius. Then it is also co-Koszul.

\textbf{Proof.} The isomorphism \( A^* \cong A \) identifies the complexes \( C_A^* \) and \( K_A^{**} \).

Note that the algebra \( \Lambda_q \) is strongly Frobenius with the required bimodules’ isomorphism given by the pairing \( \Lambda_q \times \Lambda_q \longrightarrow k \) that is defined as a projection of the product in the algebra on the top grading component.

6.4.7. \textbf{Corollary:} \( \Lambda_q \) is both Koszul and co-Koszul.

\textbf{Proof.} follows immediately from Lemmas 6.4.5 and 6.4.6.

6.4.8. \textbf{Lemma:} \( \text{Tor}^\text{gr} \ U_{\theta mx}^+(k, U_{\theta mx}^{+*}) = 0 \), \( \text{Tor}^\text{gr} \ U_{\theta mx}^+(k, U_{\theta mx}^{+*}) = k \).

\textbf{Proof.} We calculate the Tor spaces using the Koszul resolution of \( k \). We have
\[
K^{**}_{gr U_{\theta mx}^+} \otimes_{gr U_{\theta mx}^+} \text{gr} U_{\theta mx}^{+*} = \Lambda_q^* \otimes \text{gr} U_{\theta mx}^{+*} = C_A^{**}[\ell(t(\theta_{mx})].
\]

Now use the previous Lemma.

A similar statement holds evidently for \( \text{gr} U_{\theta mx}^- \).
6.4.9. Lemma:

(i) $\text{Tor}^\bullet_{\bar{\theta}_{mx}}(k, U_{\bar{\theta}_{mx}}^{\pm}) = \delta_{\bullet, \ell(t(\theta_{mx}))} k$;

(ii) for any filtered graded $U_{\bar{\theta}_{mx}}^{\pm}$-module $M = \bigcup F^n M$, $\dim F^n M/F^{n-1} M < \infty$,

$$\text{Tor}^\bullet_{\bar{\theta}_{mx}}(M, U_{\bar{\theta}_{mx}}^{\pm}) = \delta_{\bullet, \ell(t(\theta_{mx}))} M;$$

(iii) $\text{Tor}^\bullet_{\bar{\theta}_{mx}}(U_{\bar{\theta}_{mx}}^{\pm}, U_{\bar{\theta}_{mx}}^{\pm}) = \delta_{\bullet, \ell(t(\theta_{mx}))} U_{\bar{\theta}_{mx}}^{\pm}$.

Proof. We prove the statements of the Lemma for the algebra $U_{\bar{\theta}_{mx}}^{+}$. The case of the algebra $U_{\bar{\theta}_{mx}}^{-}$ is treated similarly.

The well known spectral sequence

$$\text{Tor}^\bullet(U_{\bar{\theta}_{mx}}^{+} \otimes U_{\bar{\theta}_{mx}}^{\pm}, k) \Rightarrow \text{Tor}^\bullet(U_{\bar{\theta}_{mx}}^{+}, U_{\bar{\theta}_{mx}}^{\pm})$$

shows that $k \otimes_{U_{\bar{\theta}_{mx}}^{+}} U_{\bar{\theta}_{mx}}^{\pm}$ itself is isomorphic to $U_{\bar{\theta}_{mx}}^{+}[\ell(t(\theta_{mx}))]$. Here $(\bullet)$ denotes the second grading on Tor spaces obtained from the $Z^{R+}_{\bar{\theta}_{mx}}$-grading. Thus the first statement of the Lemma is proved.

To prove the second one consider the standard complex for the computation of Tor spaces as follows:

$$M \otimes U_{\bar{\theta}_{mx}}^{+} \text{Bar}^\bullet(U_{\bar{\theta}_{mx}}^{+}, k, U_{\bar{\theta}_{mx}}^{\pm}) = M \otimes T(U_{\bar{\theta}_{mx}}^{+}) \otimes U_{\bar{\theta}_{mx}}^{\pm}.$$

Note that $M = \lim \to F^n M$ and $M \otimes T(U_{\bar{\theta}_{mx}}^{+}) \otimes U_{\bar{\theta}_{mx}}^{\pm} = \lim \to F^n M \otimes T(U_{\bar{\theta}_{mx}}^{+}) \otimes U_{\bar{\theta}_{mx}}^{\pm}$. Since limit functor is exact we have

$$\text{Tor}^\bullet(U_{\bar{\theta}_{mx}}^{\pm}, U_{\bar{\theta}_{mx}}^{\pm}) = \lim \to \text{Tor}^\bullet(F^n M, U_{\bar{\theta}_{mx}}^{\pm}) = \delta_{\bullet, \ell(t(\theta_{mx}))} \lim \to F^n M = \delta_{\bullet, \ell(t(\theta_{mx}))} M.$$

The third statement is an immediate consequence of the second one since the filtration

$$F^n U_{\bar{\theta}_{mx}}^{\pm} = \bigoplus_{k \geq -n} (U_{\bar{\theta}_{mx}}^{\pm})_k$$

satisfies the condition in (ii).

6.4.10. Lemma: $S_{U_{\bar{\theta}_{mx}}^{\pm}}^{R_{\bar{\theta}_{mx}}(U^{-})}(U^{-}) \otimes_{T_{\bar{\theta}_{mx}}(U^{-})} T_{\bar{\theta}_{mx}}^{-1}(U^{-})^*[\ell(t(\theta_{mx}))]$ in the derived category of left graded $T_{\bar{\theta}_{mx}}^{-1}(U^{-})$-modules.

Proof. First we construct the required morphism in the derived category of left graded $T_{\bar{\theta}_{mx}}^{-1}(U^{-})$-modules.
By Shapiro Lemma we have

$$\text{RHom}_{U_{\theta_{mx}}}^\bullet \left( S_{U_{\theta_{mx}}}^{T_{\theta_{mx}}^{-1}}(U^-)^L \otimes_{T_{\theta_{mx}}^{-1}(U^-)} T_{\theta_{mx}}^{-1}(U^-)^*, T_{\theta_{mx}}^{-1}(U^-)^* \right)$$

$$= \text{RHom}_{K}^\bullet \left( S_{U_{\theta_{mx}}}^{T_{\theta_{mx}}^{-1}}(U^-)^L \right)$$

$$= \left( T_{\theta_{mx}}^{-1}(U^-)^L \otimes_{T_{\theta_{mx}}^{-1}(U^-)} T_{\theta_{mx}}^{-1}(U^-)^* \right)^* = \left( U_{\theta_{mx}}^{+*} \otimes_{U_{\theta_{mx}}} U_{\theta_{mx}}^{+*} \otimes_{T_{\theta_{mx}}(U^-) \cap U^-} \right)^*$$

By Lemma 6.4.11 we have $T_{\theta_{mx}}^{-1}(U^-) = U_{\theta_{mx}}^+ \otimes T_{\theta_{mx}}^{-1}(U^-) \cap U^-$. Thus by the previous Lemma we obtain

$$\text{RHom}_{U_{\theta_{mx}}}^\bullet \left( S_{U_{\theta_{mx}}}^{T_{\theta_{mx}}^{-1}}(U^-)^L \otimes_{T_{\theta_{mx}}^{-1}(U^-)} T_{\theta_{mx}}^{-1}(U^-)^*, T_{\theta_{mx}}^{-1}(U^-)^* \right)$$

In particular the canonical element in

$$\text{RHom}_{U_{\theta_{mx}}}^\bullet \left( S_{U_{\theta_{mx}}}^{T_{\theta_{mx}}^{-1}}(U^-)^L \otimes_{T_{\theta_{mx}}^{-1}(U^-)} T_{\theta_{mx}}^{-1}(U^-)^*, T_{\theta_{mx}}^{-1}(U^-)^* \right)$$

that corresponds to $1 \in T_{\theta_{mx}}^{-1}(U^-)[-\ell t(\theta_{mx})]$ provides the required morphism in the derived category of left graded $T_{\theta_{mx}}^{-1}(U^-)$-modules.

Finally we prove that this element is an isomorphism in the derived category. Since the morphism in the derived category is already constructed, we need only to check that it is a quasiisomorphism on the level of complexes of vector spaces. The calculation almost repeats the previous one:

$$S_{U_{\theta_{mx}}}^{T_{\theta_{mx}}^{-1}}(U^-)^L \otimes_{T_{\theta_{mx}}^{-1}(U^-)} T_{\theta_{mx}}^{-1}(U^-)^* \otimes_{U_{\theta_{mx}}^{+*}} U_{\theta_{mx}}^{+*} \otimes_{T_{\theta_{mx}}^{-1}(U^-)} T_{\theta_{mx}}^{-1}(U^-)^*$$

$$\otimes_{U_{\theta_{mx}}^{+*}} \left( U_{\theta_{mx}}^{+*} \otimes_{U_{\theta_{mx}}} \left( U_{\theta_{mx}}^{+*} \otimes_{U_{\theta_{mx}}} \left( U_{\theta_{mx}}^{+*} \otimes_{T_{\theta_{mx}}^{-1}(U^-)} T_{\theta_{mx}}^{-1}(U^-)^* \right) \right) \right)$$

$\text{Ind}_{T_{\theta_{mx}}^{-1}(U^-)^*}^{U_{\theta_{mx}}} \text{Coind}_{U_{\theta_{mx}}}^{U_{\theta_{mx}}^+} U_{\theta_{mx}}^{+*} = S_{U_{\theta_{mx}}}^{U_{\theta_{mx}}^+}$. 

By Lemma 6.3.11 the semiinfinite induction functor is well defined on corresponding derived categories, thus it takes the quasiisomorphism from Lemma 6.4.10 to a quasiisomorphism. Theorem 6.3.2 is proved.
7. Quantum twisted Verma modules and semiinfinite homology of the algebra \( T_{\theta mx} (U^-) \).

In this section we show that the semiinfinite homology space of the trivial module over \( T_{\theta mx} (U^-) \) has a base enumerated by elements of the affine Weyl group graded by the \( \theta_{mx}^{-1} \)-twisted length function on \( W \) (see 2.4.4) just like in the affine Lie algebra case (see [Ar3]).

7.1. Categories of \( U \)-modules. Consider the category \( \mathcal{M} \) of \( X \times \mathbb{Z} \)-graded \( U \)-modules \( M = \bigoplus_{\lambda \in X, t \in \mathbb{Z}} M_{\lambda, t} \) such that

(i) for every \( i \in I \) the standard generators \( E_i : M_{\lambda, t} \to M_{\lambda+i, t+1}, \ F_i : M_{\lambda, t} \to M_{\lambda-i, t-1} \);
(ii) every \( K_\mu \in U^0 \) acts on \( M_{\lambda, t} \) by scalar \( v^{(\mu, \lambda)} \).

Morphisms in \( \mathcal{M} \) are morphisms of \( U \)-modules that preserve \( X \times \mathbb{Z} \)-gradings.

Fix a nonnegative integer \( k \in \mathbb{Z}_{\geq 0} \). Consider a full subcategory \( \mathcal{M}_k \) in \( \mathcal{M} \) of modules \( M \) such that \( Z \) acts by the scalar \( v^k \) on \( M \).

7.1.1. We define the character of a \( U \)-module \( M \in \mathcal{M} \) such that \( \dim M_{\lambda, t} < \infty \) for all \( \lambda \in X, t \in \mathbb{Z} \) by \( \chi_M := \sum_{\lambda \in X, t \in \mathbb{Z}} \dim M_{\lambda, t} e^\lambda q^t \). Here \( q \) is a formal variable and \( e^\lambda \) is a formal expression.

7.1.2. As usual let \( \mathcal{O} \) denote the category of \( X \)-graded \( U \)-modules \( M = \bigoplus_{\lambda \in X} M_{\lambda} \) such that

(i) for every \( i \in I \) the standard generators \( E_i : M_{\lambda} \to M_{\lambda+i} \), \( F_i : M_{\lambda} \to M_{\lambda-i} \);
(ii) every \( K_\mu \in U^0 \) acts on \( M_{\lambda} \) by scalar \( v^{(\mu, \lambda)} \);
(iii) \( \dim M_{\lambda} < \infty \) for all \( \lambda \in X \);
(iv) there exist \( \lambda_1, \ldots, \lambda_n \in h^* \) such that \( M_\mu \neq 0 \) only when \( \mu \in \lambda_1 + R^- \cup \ldots \cup \lambda_n + R^- \).

7.2. Affine BGG resolution. Recall that the subalgebra \( U^+ \) is normal in \( U^\geq 0 \), that is, the left and the right ideals in \( U^\geq 0 \) generated by \( U^+ \) coincide, and the quotient algebra of \( U^\geq 0 \) by any of these ideals denoted by \( U^\geq 0 / U^+ \) equals to \( U^0 \). For the one dimensional \( U^0 \)-module \( k(\lambda), \lambda \in X \), generated by \( v_\lambda \) (such that \( K_\mu \cdot v_\lambda = v^{(\mu, \lambda)} v_\lambda \)) the (quantum) Verma module \( U_k \otimes U^\geq 0 k(\lambda) \) is denoted by \( M(\lambda) \). Evidently \( M(\lambda) \) is free as a \( U^- \)-module and belongs to \( \mathcal{O} \) and to \( \mathcal{M}_k \) if we put the \( \mathbb{Z} \)-grading of the highest weight vector \( v_\lambda \in M(\lambda) \) equal to zero.

For a fixed positive integral level \( k \) choose a dominant weight \( \lambda \in X_k^+ \). Consider the unique simple quotient module of \( M(\lambda) \) denoted by \( L(\lambda) \). Then there exists a left resolution of \( L(\lambda) \) that consists of direct sums of Verma modules as follows.
7.2.1. **Theorem:** (see e.g. [M], Theorem 3.3) There exists a resolution $B^\bullet(\lambda)$ of $L(\lambda)$ in $O$ and in $\mathcal{M}_k$ of the form

$$\ldots \rightarrow \bigoplus_{w \in W, \ell(t(w))=m} M(w, \lambda)(- \text{ht}(\lambda - w \cdot \lambda)) \rightarrow \ldots$$

$$\rightarrow \bigoplus_{w \in W, \ell(t(w))=1} M(w, \lambda)(- \text{ht}(\lambda - w \cdot \lambda)) \rightarrow M(\lambda) \rightarrow L(\lambda) \rightarrow 0.$$  

Here as usual $\langle \cdot \rangle$ denotes the shift of the $\mathbb{Z}$-grading in $\mathcal{M}$.

Like in the affine Lie algebra case it is known that the component of the differential $d_{w_1,w_2} : B^{-k}(\lambda) \rightarrow B^{-k+1}(\lambda)$, $M(w_1, \lambda) \rightarrow M(w_2, \lambda)$, is nonzero iff $w_1 \geq w_2$ in the Bruhat order on $W$ (see [M], Theorem 3.2).

7.3. **Twisting functors.** We introduce the functors of the twist of the $X$-gradings. As before, fix a positive weight $x \in Y^+$ and the transvection element $\theta_{mx} \in T \subset W$. Recall that we have defined the associative algebra automorphism

$$T_{\theta_{mx}} : U \xrightarrow{\sim} U, \ U_\alpha \rightarrow U_{\theta_{mx}(\alpha)}.$$  

We define $T_{\theta_{mx}} : \mathcal{M}_k \rightarrow \mathcal{M}_k$ as follows. For $M \in \mathcal{M}_k$ set

$$T_{\theta_{mx}}(M) = \bigoplus_{\lambda \in \mathcal{X}, t \in \mathbb{Z}} T_{\theta_{mx}}(M)_{\lambda,t}, \ T_{\theta_{mx}}(M)_{\lambda,t} := M_{\theta_{mx}(\lambda), \lambda, t + \text{ht}(\theta_{mx}(\lambda) - \lambda)},$$

$$e_\alpha \in U_\alpha, \ v_\lambda \in T_{\theta_{mx}}(M)_{\lambda,t} \text{ then } e_\alpha \cdot v_\lambda := T_{\theta_{mx}}(e_\alpha)(v_\lambda) \in M_{\theta_{mx}(\lambda + \alpha), \lambda + t + \text{ht}(\theta_{mx}(\lambda + \alpha) - \lambda),}$$

$$= M_{\theta_{mx}(\lambda + \alpha), \lambda + t + \text{ht}(\theta_{mx}(\lambda + \alpha) - \lambda + \alpha),} = T_{\theta_{mx}}(M)_{\lambda + \alpha, \lambda + t + \text{ht}(\alpha)},$$

Evidently $T_{\theta_{mx}}$ is an equivalence of categories.

7.3.1. Consider the semiregular $U_k$-module $S_{U_{\theta_{mx}}}^{U_k} = \text{Coind}_{U_{\theta_{mx}}}^{U_k} U_{\theta_{mx}}$, with respect to $U_{\theta_{mx}}$. Then by Lemma 3.2 there exists an inclusion of algebras $U_k \hookrightarrow \text{End}_{U_k}(S_{U_{\theta_{mx}}}^{U_k})$. Consider a functor

$$S_{mx} : \mathcal{M}_k \rightarrow U_k\text{-mod}, \ S_{mx}(M) := \left(\text{Hom}_{U_k}(M, S_{U_{\theta_{mx}}}^{U_k})\right)^{\ast}.$$  

The left $U_k$-module structure on $S_{mx}(M)$ is provided by the right $U_k$-multiplication in $S_{U_{\theta_{mx}}}^{U_k}$.

7.3.2. **Lemma:** $S_{mx}$ defines a functor $\mathcal{M}_k \rightarrow \mathcal{M}_k$.  

7.3.3. **Definition:** For $x \in Y^+$ the functor of twist by $\theta_{mx} \in T \subset W$

$$\Phi_{mx} := T_{\theta_{mx}} \circ S_{mx}, \ \Phi_{mx} : \mathcal{M}_k \rightarrow \mathcal{M}_k.$$  

Let us describe the image of a Verma module under $\Phi_{mx}$.
7.3.4. **Lemma:** \( \text{ch} \Phi_{mx}(M(\lambda)) = \text{ch} M(\theta_{mx} \cdot \lambda)(-\text{ht}(\lambda - \theta_{mx} \cdot \lambda)). \quad \square \)

In particular the highest weight vector \( 1 \otimes v_\lambda \in \Phi_{mx}(M(\lambda)) \) has the weight \( \theta_{mx} \cdot \lambda. \)

7.3.5. **Definition:** We call the \( U_k \)-module \( M_{\theta_{mx}}(\theta_{mx} \cdot \lambda) := \Phi_{mx}(M(\lambda))(\text{ht}(\lambda - \theta_{mx} \cdot \lambda)) \) the **quantum twisted Verma module** of the weight \( \theta_{mx} \cdot \lambda. \)

7.3.6. **Lemma:** When restricted to \( T_{\theta_{mx}}(U^-) \) the quantum twisted Verma module \( M_{\theta_{mx}}(\lambda) \) is isomorphic to the semiregular \( T_{\theta_{mx}}(U^-) \)-module \( S_{U_{\theta_{mx}}}^{T_{\theta_{mx}}(U^-)} \).

**Proof.** We prove first that the right \( U^- \)-modules \( \text{Res}^{U_k}_{U^-} \left( \text{Hom}^{\text{cont}}_{U_k}(M(\lambda), S_{U_{\theta_{mx}}}^{U_k}) \right) \) and \( \text{Coind}_{U_{\theta_{mx}}}^{U^-} U_{\theta_{mx}} \) are isomorphic to each other. On the level of vector spaces the calculation looks as follows.

\[
(*) \quad \text{Hom}^{\text{cont}}_{U_k}(M(\lambda), S_{U_{\theta_{mx}}}^{U_k}) = \text{Hom}^{\text{cont}}_{U_k}(M(\lambda), \text{Coind}_{U_{\theta_{mx}}}^{U^-} U_{\theta_{mx}}) \\
= \text{Hom}^{\text{cont}}_{U^-}(M(\lambda), \text{Coind}_{U_{\theta_{mx}}}^{U^-} U_{\theta_{mx}}) = \text{Hom}^{\text{cont}}_{U^-}(U^-, \text{Coind}_{U_{\theta_{mx}}}^{U^-} U_{\theta_{mx}}) \\
= \text{Coind}_{U_{\theta_{mx}}}^{U^-} U_{\theta_{mx}}.
\]

Now note that the right \( U^- \)-module structure on \( \text{Hom}^{\text{cont}}_{U_k}(M(\lambda), S_{U_{\theta_{mx}}}^{U_k}) \) coincides with the one obtained from the \( U^- \)-bimodule \( \text{Coind}_{U_{\theta_{mx}}}^{U^-} U_{\theta_{mx}} \) (see Lemma 6.4.1) with the help of the functor \( \text{Coind}_{U_k}^{U^-} \). This means that the equality \( (*) \) provides the required isomorphism of the right \( U^- \)-modules.

Applying the functor \( T_{\theta_{mx}} \circ * \) to both sides of the equality \( (*) \) we obtain the statement of the Lemma. \( \square \)

Next we apply the functor \( \Phi_{mx} \) to the complex \( B^*(\lambda) \). By definition the obtained complex \( \Phi_{mx}(B^*(\lambda)) \) consists of direct sums of quantum twisted Verma modules. On the other hand by Lemma 3.3.1 quantum Verma modules are \( U_{\theta_{mx}}^- \)-free, thus up to a \( X \times Z \)-grading shift cohomology spaces of \( \Phi_{mx}(B^*(\lambda)) \) coincide with \( \left( \text{Ext}^*_{U_{\theta_{mx}}}(L(\lambda), U_{\theta_{mx}}^-) \right)^* \).

7.3.7. **Proposition:** \( \text{Ext}^*_{U_{\theta_{mx}}}(L(\lambda), U_{\theta_{mx}}^-)^* = \delta_{*,-\text{ht}(\theta_{mx})} L(\lambda). \)

7.4. **Proof of Proposition 7.3.7.** Note first that \( T_{\theta_{mx}}(L(\lambda)) = L(\lambda) \) since the character of \( L(\lambda) \) is \( W \)-invariant and an integrable simple module over \( U \) is completely determined by its character. Thus up to a \( X \times Z \)-grading shift we have

\[
\text{Ext}^*_{U_{\theta_{mx}}}(L(\lambda), U_{\theta_{mx}}^-)^* = \text{Ext}^*_{T_{\theta_{mx}}(U_{\theta_{mx}}^-)}(L(\lambda), T_{\theta_{mx}}(U_{\theta_{mx}}^-))^*.
\]
7.4.1. **Lemma:** We have a canonical isomorphism of functors in $U_{\theta_{mx}}^+\text{-mod}$:

$$\text{Ext}^\bullet_{U_{\theta_{mx}}^-} (\ast, U_{\theta_{mx}}^-)^* \xrightarrow{\cong} \text{Tor}^\bullet_{U_{\theta_{mx}}^-} (\ast, U_{\theta_{mx}}^-).$$

**Proof.** It is sufficient to construct the isomorphism on the level of Hom and $\otimes$. Let $M \in U_{\theta_{mx}}^-\text{-mod}$ and $f \in \text{Hom}_{U_{\theta_{mx}}^-} (M, U_{\theta_{mx}}^-)$. Then the morphism

$$M \otimes_{U_{\theta_{mx}}} U_{\theta_{mx}}^- \otimes_{U_{\theta_{mx}}} U_{\theta_{mx}}^- \xrightarrow{f \otimes \text{Id}} U_{\theta_{mx}}^- \otimes_{U_{\theta_{mx}}} U_{\theta_{mx}}^- \xrightarrow{1} k$$

provides a pairing $(f, m \otimes \phi), m \in M, \phi \in U_{\theta_{mx}}^-$. One checks directly that the pairing is perfect for $M = U_{\theta_{mx}}^-$. Taking a free resolution of $M \in U_{\theta_{mx}}^-\text{-mod}$ we obtain the statement for an arbitrary $M$. □

By Lemma 6.4.9(ii) for a graded $T_{\theta_{mx}}(U_{\theta_{mx}}^-)$-module $M$ with a positive filtration by finite dimensional $T_{\theta_{mx}}(U_{\theta_{mx}}^-)$-submodules we have $\text{Tor}^\bullet_{T_{\theta_{mx}}(U_{\theta_{mx}}^-)} (M, T_{\theta_{mx}}(U_{\theta_{mx}}^-)) = \delta_{\bullet, \ell t(\theta_{mx})} M$.

Note that the module $L(\lambda) \in \mathcal{O}$, thus in particular, like the proof or Lemma 6.4.9(iii), the filtration obtained from the grading satisfies the condition in 6.4.9(ii) and by the previous Lemma we have

$$\text{Ext}^\bullet_{U_{\theta_{mx}}^-} (L(\lambda), U_{\theta_{mx}}^-)^* = \delta_{\bullet, \ell t(\theta_{mx})} L(\lambda).$$

Now make a $X \times \mathbb{Z}$-grading shift. Proposition 7.3.7 is proved. □

7.5. **Corollary:** There exists a complex $\mathcal{B}_{\theta_{mx}}^\bullet (\lambda)$ in $\mathcal{M}_k$ of the form

$$\ldots \rightarrow \bigoplus_{v \in W, \ell t(v) = m} M_{\theta_{mx}} (\theta_{mx}v \cdot \lambda) \langle - \text{ht}(\lambda - \theta_{mx}v \cdot \lambda) \rangle \rightarrow \ldots$$

$$\rightarrow \bigoplus_{v \in W, \ell t(v) = 1} M_{\theta_{mx}} (\theta_{mx}v \cdot \lambda) \langle - \text{ht}(\lambda - \theta_{mx}v \cdot \lambda) \rangle$$

$$\rightarrow M_{\theta_{mx}} (\theta_{mx} \cdot \lambda) \langle - \text{ht}(\lambda - \theta_{mx} \cdot \lambda) \rangle \rightarrow 0$$

such that $H^{\# - \ell t(\theta_{mx})} (\mathcal{B}_{\theta_{mx}}^\bullet (\lambda)) = 0, H^{-\ell t(\theta_{mx})} (\mathcal{B}_{\theta_{mx}}^\bullet (\lambda)) = L(\lambda)$. Here as usual $\langle \cdot \rangle$ denotes the shift of the $\mathbb{Z}$-grading. □

Recall that in 2.4.4 we have defined the twisted length function on the affine Weyl group with the twist $w \in W$ by $\ell t^w(u) = \ell t(w^{-1}u) - \ell t(w^{-1})$. 
7.5.1. **Corollary:** The complex $\tilde{B}_{\theta_{mx}}(\lambda)[-\ell t(\theta_{mx})]$ can be rewritten as follows:

\[
\ldots \rightarrow \bigoplus_{v \in W, \ell t_{\theta_{mx}}(v) = n} M_{\theta_{mx}}(v \cdot \lambda)(- \text{ht}(\lambda - v \cdot \lambda)) \rightarrow \ldots
\]

\[
\rightarrow \bigoplus_{v \in W, \ell t_{\theta_{mx}}(v) = 0} M_{\theta_{mx}}(v \cdot \lambda)(- \text{ht}(\lambda - v \cdot \lambda)) \rightarrow \ldots
\]

\[
\rightarrow \bigoplus_{v \in W, \ell t_{\theta_{mx}}(v) = -\ell t(\theta_{mx}) + 1} M_{\theta_{mx}}(v \cdot \lambda)(- \text{ht}(\lambda - v \cdot \lambda)) \rightarrow M_{\theta_{mx}}(\theta_{mx} \cdot \lambda)(- \text{ht}(\lambda - \theta_{mx} \cdot \lambda)) \rightarrow 0. \]

7.5.2. **Definition:** We call the complex $\tilde{B}^*_{\theta_{mx}}(\lambda)[-\ell t(\theta_{mx})] =: B^*_{\theta_{mx}}(\lambda)$ the quantum twisted BGG resolution of the module $L(\lambda)$ with the twist $mx$.

We conclude this section with the answer for the semiinfinite homology of $L(\lambda)$ over the algebra $T_{\theta_{mx}}(U^-)$. Note that $U^0$ acts naturally on the semiinfinite Tor spaces over $T_{\theta_{mx}}(U^-)$ because $T_{\theta_{mx}}(U^-)$ is a normal subalgebra in $T_{\theta_{mx}}(U \leq 0)$ with the quotient algebra equal to $U^0$.

7.5.3. **Lemma:** We have an equality of $U^0$-modules

\[
\text{Tor}_{\frac{T_{\theta_{mx}}(U^-)}{U^0}}(k, L(\lambda)) = \bigoplus_{v \in W, \ell t_{\theta_{mx}}(v) = n} k(v \cdot \lambda).
\]

**Proof.** The statement follows immediately from Corollary 7.5.1, Lemma 7.3.4 and Corollary 4.7.2.

8. **Limit procedure and semiinfinite homology of the infinitely twisted nilpotent affine quantum group**

Let $\mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$ be the simple Lie algebra corresponding to the root datum $(\bar{X}, \bar{Y}, \ldots)$ of the finite type $I$, let $\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}]$ be the affine Kac-Moody algebra corresponding to the root datum $(X, Y, \ldots)$ of the type $I$. Consider the infinitely twisted nilpotent subalgebra in $\hat{\mathfrak{g}}$

\[
\mathfrak{n}^{\tilde{\mathfrak{g}}} := \mathfrak{g}^- \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}] = \bigoplus_{a \in R^{\tilde{\mathfrak{g}}}} \hat{\mathfrak{g}}_a.
\]

In [FF] Feigin and Frenkel constructed semi-infinite BGG resolutions of integrable simple modules $L(\lambda)$ over $\hat{\mathfrak{g}}$. These complexes consist of direct sums of so called Wakimoto modules and provide semijective resolutions of $L(\lambda)$ over $\mathfrak{n}^{\tilde{\mathfrak{g}}}$. The semiinfinite homology spaces of $\mathfrak{n}^{\tilde{\mathfrak{g}}}$ with coefficients in $L(\lambda)$ were found this way.
In this section we consider the quantum analogue of the infinitely twisted nilpotent algebra $n_\infty^\theta$ and calculate its semifinite homology with coefficients in integrable simple modules $L(\lambda)$ over $U$. Unfortunately we lack the construction of morphisms between quantum twisted BGG resolutions that would form a porjective system of complexes like in the affine Lie algebra case (see [Ar3], section 6). Instead of it using the results of the fifth section we prove that the semifinite Tor spaces themselves form a porjective system with the limit equal to the semiinfinite homology spaces of the infinitely twisted nilpotent affine quantum group.

8.1. **Infinitely twisted nilpotent affine quantum group.** Consider a $k$-vector subspace $U_\infty^\theta^-$ in $U$ spanned by the set of PBW monomials as follows:

\[
\left\{ \prod_{\beta \in \mathbb{R}_\infty^\theta \cap \mathbb{R}^+} \tilde{E}_\beta^{a_\beta} \prod_{\beta \in \mathbb{R}_\infty^\theta \cap \mathbb{R}^-} \tilde{F}_\beta^{b_\beta} \mid a_\beta, b_\beta \geq 0 \right\}.
\]

8.1.1. **Lemma:** $U_\infty^\theta^-$ is a subalgebra in $U$.

**Proof.** It is sufficient to check that the product $\tilde{F}_\beta \tilde{E}_\alpha$ of any generators $\tilde{E}_\alpha$, $\alpha \in \mathbb{R}_\infty^\theta \cap \mathbb{R}^+$, and $\tilde{F}_\beta$, $\beta \in \mathbb{R}_\infty^\theta \cap \mathbb{R}^-$, belongs to $U_\infty^\theta^-$. Note that

\[
\mathbb{R}_\infty^\theta^+ \cap \mathbb{R}^- = \bigcup_m \left( (\theta_{mx}(\mathbb{R}^-) \cap \mathbb{R}^+) \right) \quad \text{and} \quad \mathbb{R}_\infty^\theta^- \cap \mathbb{R}^- = \bigcap_m \left( (\theta_{mx}(\mathbb{R}^-) \cap \mathbb{R}^-) \right).
\]

On the level of vector spaces we have

\[
U_{\infty}^\theta^- \cap U^+ = \bigcup_m \left( T_{\theta_{mx}}(U^-) \cap U^+ \right), \quad U_{\infty}^\theta^- \cap U^- = \bigcap_m \left( T_{\theta_{mx}}(U^-) \cap U^- \right),
\]

and

\[
U_{\infty}^\theta^- = U_{\infty}^\theta^- \cap U^+ \otimes U_{\infty}^\theta^- \cap U^-.
\]

There exists $m_0 \geq 0$ such that for any integer $m > m_0$ both $\tilde{E}_\alpha$ and $\tilde{F}_\beta$ belong to $T_{\theta_{mx}}(U^-)$. Decomposing $\tilde{F}_\beta \tilde{E}_\alpha$ into a sum of tensor product monomials in $T_{\theta_{mx}}(U^-) = T_{\theta_{mx}}(U^-) \cap U^+ \otimes T_{\theta_{mx}}(U^-) \cap U^-$ we obtain the statement of the Lemma.

8.1.2. **Lemma:** The algebra $U_{\infty}^{\theta^-, \sharp}$ is isomorphic to $U_{\infty}^{\theta^-}$.

**Proof.** By [3] we have the inclusion of algebras $U_{\infty}^{\theta^-, \sharp} \subset U_{\infty}^{\sharp}$ and the image of $U_{\infty}^{\theta^-}$ can be described as the subalgebra in $U_{\infty}^{\sharp}$ generated by $U_{\infty}^{\theta^-, \sharp} \cap U^-$ and $U_{\infty}^{\theta^-, \sharp} \cap U^+$. By Theorem 6.3.2 the algebra $U^\flat$ is isomorphic to $U_{-2\gamma^- - k}$. But the subalgebra generated by $U_{\infty}^{\theta^-} \cap U^-$ and $U_{\infty}^{\theta^-} \cap U^+$ does not depend on the central character and coincides with $U_{\infty}^{\theta^-}$.

8.2. **Limit procedure.** Fix integers $m_2 > m_1 \geq 0$. Consider the algebra $T_{\theta_{mx}}(U^-) \cap T_{\theta_{mx}}(U^-)$ and two inclusions

\[
i_{m_1} : T_{\theta_{mx}}(U^-) \cap T_{\theta_{mx}}(U^-) \hookrightarrow T_{\theta_{mx}}(U^-)
\]

and

\[
i_{m_2} : T_{\theta_{mx}}(U^-) \cap T_{\theta_{mx}}(U^-) \hookrightarrow T_{\theta_{mx}}(U^-).
\]
Thus $T_{\theta_m}(U^-) \cap T_{\theta_m}(U)\cap U^+$ has a natural triangular decomposition

$T_{\theta_m}(U^-) \cap T_{\theta_m}(U^-) = U^+ \cap T_{\theta_m}(U^-) \cap U^+$.

Note that by the same arguments as in Lemma 1.1.2 the algebra $(T_{\theta_m}(U^-) \cap T_{\theta_m}(U^-))^2$ coincides with $T_{\theta_m}(U^-) \cap T_{\theta_m}(U^-)$. Next, $i_m$ is a negative extension of algebras and $i_m$ is a positive one. By Propositions 5.2.7 and 5.1.9 we obtain canonical morphisms of semiinfinite Tor spaces:

$\text{Tor}_{T_{\theta_m}}(U^-) (k, L(\lambda)) \longrightarrow \text{Tor}_{T_{\theta_m}}(U^-) (k, L(\lambda)) \longrightarrow \text{Tor}_{T_{\theta_m}}(U^-) (k, L(\lambda))$.

Denote the composition map by $p_{m_2,m_1}$.

8.2.1. Proposition: $\text{Tor}_{T_{\theta_m}}(U^-) (k, L(\lambda)) = \lim \text{Tor}_{T_{\theta_m}}(U^-) (k, L(\lambda))$.

Proof. First note that by definition of the semiinfinite Tor functor the statement is equivalent to the following one:

$\text{Ext}_{U^+} (k, L(\lambda))^* = \lim \text{Ext}_{T_{\theta_m}}(U^-) (k, L(\lambda))^*$.

Choose a semijective resolution $M^\bullet$ of $L(\lambda)^*$ in $C(U)$ such that with the differential forgotten it is equal to a $U$-module of the form $(S^U_{U^+})^* \otimes V = \text{Cind}_{U^+}^{U} \text{Ind}_U^V$. By Proposition 4.7.3 we have

$\text{Ext}_{U^+} (k, L(\lambda))^* = \text{Ext}_{T_{\theta_m}}(U^-) (k, M^*)$ and $\text{Ext}_{T_{\theta_m}}(U^-) (k, L(\lambda))^* = \text{Ext}_{T_{\theta_m}}(U^-) (k, M^*)$.

By Lemma 7.7(iii) $\text{Ext}_{U^+} (k, M^*)$ (resp. $\text{Ext}_{T_{\theta_m}}(U^-) (k, M^*)$) are the cohomology spaces of the complex

$\text{Hom}_{U^+} (k, M^* \otimes U^+) - S^U_{U^+} - U^+ \cap U^+) \cap U^+) = \text{Hom}_{T_{\theta_m}}(U^-) (k, M^* \otimes T_{\theta_m}(U^-) - S^U_{T_{\theta_m}(U^-) - \cap U^+) \cap U^+)$.\n
Next note that

$\text{Hom}_{U^+} (k, (S^U_{U^+})^* \otimes U^+) - S^U_{U^+} - U^+) = U^+ \cap U^+ \otimes (U^+ \cap U^-)^*$ and

$\text{Hom}_{T_{\theta_m}}(U^-) (k, (S^U_{U^+})^* \otimes T_{\theta_m}(U^-) - S^U_{T_{\theta_m}(U^-) - \cap U^+) \cap U^+) = T_{\theta_m}(U^+) \cap U^+ \otimes (T_{\theta_m}(U^+) \cap U^-)^*$.

Note also that the canonical semiinfinite Ext morphisms here coincide with the obvious ones. In particular we have

$\text{Ext}_{U^+} (k, (S^U_{U^+})^*) = \lim \text{Ext}_{T_{\theta_m}}(U^-) (k, (S^U_{U^+})^*)$.

Thus the Proposition is proved since the $\lim$ functor is exact. □

From Lemma 7.3.3 we know that the space $\text{Tor}_{T_{\theta_m}}(U^-) (k, L(\lambda))$ has a base enumerated by the elements of the affine Weyl group $w \in W$ such that $\ell(w) = n$. Next recall that by Corollary 2.4.6 for every $w \in W$ there exists $m_0 \in \mathbb{N}$ such that for every $m > m_0$ we
have $\ell t^{m-1}(w) = \ell t^w(w)$. Fix $w \in W$ and choose $m > m_0 = m_0(w)$. Let $n = \ell t^w(w)$.

Consider the map
\[
p_{m+1,m} : \text{Tor}_{\frac{m+1}{2}+n} \left( \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, L(\lambda) \right) \right) \rightarrow \text{Tor}_{\frac{m}{2}+n} \left( \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, L(\lambda) \right) \right)
\]
\[
\rightarrow \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, L(\lambda) \right).
\]

Denote the first (resp. the second) map in the composition by $p_{m+1,m}$ (resp. by $p_{m+1,m}^+$). Denote the base vector corresponding to $w$ in $\text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, L(\lambda) \right)$ (resp. in $\text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, L(\lambda) \right)$) by $a_{m+1}^w$ (resp. by $a_m^w$).

8.2.2. **Theorem:** $p_{m+1,m}(a_{m+1}^w) = ca_m^w$ and $c \neq 0$ for $m >> 0$.

**Proof.** Since the map $p_{m+1,m}$ preserves the natural $X$-gradings on the semi-infinite Tor spaces, we have to prove only that $p_{m+1,m}(a_{m+1}^w) \neq 0$ for $m >> 0$. Note that $B^*_{\theta_{mx}}(\lambda)$ (resp. $B^*_{\theta_{m+1}}(\lambda)$) is a semi-injective resolution of $L(\lambda)$ not only over $T_{\theta_{mx}}(U^-)$ (resp. over $T_{\theta_{m+1}}(U^-)$) but also over $T_{\theta_{mx}}(U^-) \cap T_{\theta_{m+1}}(U^-)$. Thus by Proposition 4.7.3 we have
\[
\text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{mx}}(\lambda) \right) \rightarrow \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{m+1}}(\lambda) \right).
\]

Consider the morphisms of the complexes
\[
p_{m+1,m}^+ : \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{mx}}(\lambda) \right) \rightarrow \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{m+1}}(\lambda) \right)
\]
and
\[
\text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{mx}}(\lambda) \right) \rightarrow \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{m+1}}(\lambda) \right),
\]
the differentials in the complexes obtained from the ones in $B^*_{\theta_{m+1}}(\lambda)$ and $B^*_{\theta_{mx}}(\lambda)$ respectively. Note that the differentials in
\[
\text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{mx}}(\lambda) \right)
\]
and
\[
\text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{m+1}}(\lambda) \right)
\]
vanish. By Lemma 5.1.10 up to a $X$-grading shifts the map
\[
p_{m+1,m}^+ : \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{mx}}(\lambda) \right) \rightarrow \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{m+1}}(\lambda) \right)
\]

 coincide with the canonical map
\[
\bigoplus_{v \in W, \ell(t_{\theta_{mx}}(u)) = n} \text{Ind}_{\left( T_{\theta_{mx}}(U^-) \right) \cap U^{-1}} \mathbf{k} \rightarrow \bigoplus_{v \in W, \ell(t_{\theta_{m+1}}(u)) = n} \mathbf{k}.
\]

In particular it is surjective. By Lemma 5.2.8 up to a $X$-grading shifts the map
\[
p_{m+1,m}^+ : \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{mx}}(\lambda) \right) \rightarrow \text{Tor}_{\frac{m}{2}+n} \left( \mathbf{k}, B^*_{\theta_{m+1}}(\lambda) \right)
\]
coincides with the canonical map
\[ \bigoplus_{v \in W} k \rightarrow \bigoplus_{v \in W, \ell^1_{\Theta_{(m+1)x}}(v) = n} \text{Coind}_{U_{\Theta_{mx}}^+} \rightarrow \bigoplus_{v \in W, \ell^1_{\Theta_{mx}}(v) = n} \text{Coind}_{U_{\Theta_{mx}}^+} k. \]

Our next aim is to prove that \( p_{m+1,m}(a_{m+1}^w) \) defines a nonzero cohomology class. Denote the component of the differential \( M_{\Theta_{mx}}(v \cdot \lambda) \rightarrow M_{\Theta_{mx}}(w \cdot \lambda) \) in \( B_{\Theta_{mx}}^+(w \cdot \lambda) \) by \( d_{v,w}. \)

8.2.3. **Lemma:** For every \( w \in W \) there exists \( m_1 \in \mathbb{N} \) such that for every \( m > m_1 \) the set \( \text{Prev}(w, \lambda, m) := \{ v \in W | \ell^{1}_{\Theta_{mx}}(v) = \ell^{1}_{\Theta_{(m+1)x}}(w) - 1, d_{v,w} \neq 0 \} \) (resp. the set \( \text{Next}(w, \lambda, m) := \{ v \in W | \ell^{1}_{\Theta_{(m+1)x}}(v) = \ell^{1}_{\Theta_{mx}}(w) + 1, d_{w,v} \neq 0 \} \) coincides with the set \( \{ v \in W | \ell^1_{\Theta_{mx}}(v) = \ell^1_{\Theta_{(m+1)x}}(w) - 1, v \leq \frac{\infty}{2} \ w \} \) (resp. \( \{ v \in W | \ell^1_{\Theta_{(m+1)x}}(v) = \ell^1_{\Theta_{mx}}(w) + 1, w \leq \frac{\infty}{2} \ v \} \). □

It is known that for any \( w \in W \) both sets from the previous Lemma are finite. We suppose that \( m > m_1(w). \) Let
\[
ht^- := \max_{v \in \text{Prev}(w, \lambda, m)} |ht(w \cdot \lambda - v \cdot \lambda)|, \quad ht^+ := \max_{v \in \text{Next}(w, \lambda, m)} |ht(w \cdot \lambda - v \cdot \lambda)|.
\]

Note that the space \( \text{Coind}_{U_{\Theta_{mx}}^+} k \) has a base of dual vectors to PBW monomials with generators \( \hat{E}_\alpha, \alpha \in R_{\Theta_{(m+1)x}}^+ \setminus R_{\Theta_{mx}}^+ \). Choosing \( m \) big enough we obtain \( ht(\alpha) > ht^- \) for every \( \alpha \in R_{\Theta_{(m+1)x}}^+ \setminus R_{\Theta_{mx}}^+ \). Thus for \( m \) big enough the vector \( p_{m+1,m}^{-1}(a_{m+1}^w) \) provides a nontrivial class in the cohomology of the complex \( \text{Tor}_{\frac{\infty}{2}+0} T_{\Theta_{mx}}(U^-) \cap T_{\Theta_{(m+1)x}}(U^-) (k, B_{\Theta_{(m+1)x}}(\lambda)) \).

It is proved similarly that \( a_{m+1}^w \) belongs to
\[
\text{Im}(\text{Tor}_{\frac{\infty}{2}+n} T_{\Theta_{mx}}(U^-) \cap T_{\Theta_{(m+1)x}}(U^-) (k, L(\lambda))) \rightarrow \text{Tor}_{\frac{\infty}{2}+n} T_{\Theta_{mx}}(U^-) (k, L(\lambda)).
\]

Now note that the grading component in \( \text{Coind}_{U_{\Theta_{mx}}^+} k(w \cdot \lambda) \) of the weight \( w \cdot \lambda \) is one dimensional. Thus \( p_{m+1,m} \) necessarily takes \( a_{m+1}^w \) to a nonzero vector \( ca_{m+1}^w \). □

8.2.4. **Corollary:** We have an equality of \( U^0 \)-modules
\[
\text{Tor}_{\frac{\infty}{2}+n} (k, L(\lambda)) = \bigoplus_{v \in W, \ell^1_{\Theta_{mx}}(v) = n} k(v \cdot \lambda).
\]

**Proof.** The statement follows immediately from Lemma 7.5.3, Proposition 8.2.1 and Theorem 8.2.2. □

**References**

[Ar1] S. Arkhipov. *Semiinfinite cohomology of quantum groups.* Preprint q-alg/9601026 (1996), 1-24.

[Ar2] S. Arkhipov. *Semiinfinite cohomology of associative algebras and bar duality.*
Preprint q-alg/9602013 (1996), 1-21.

[Ar3] S. Arkhipov. A new construction of the semi-infinite BGG resolution. Preprint q-alg/9605043 (1996), 1-29.

[Be] J. Beck. Convex bases of PBW type for quantum affine algebras, Comm. Math. Phys. 165 (1994), 193-199.

[BeK] J. Beck, V. Kac. Finite dimensional representations of quantum affine algebras at roots of unity. Preprint (1996), 1-32.

[BGG] I. I. Bernstein, I. M. Gelfand, S. I. Gelfand. Differential operators on the principal affine space and investigation of $\mathfrak{g}$-modules, in Proceedings of Summer School on Lie groups of Bolyai János Math. Soc., Heidelberg, New York, 1975.

[DCKP] C. De Concini, V. Kac, C. Procesi. Some remarkable degenerations of quantum groups. Comm. Math. Phys. 157 (1993), p.405-427.

[F] B. Feigin. Semi-infinite cohomology of Kac-Moody and Virasoro Lie algebras. Usp. Mat. Nauk 33, no.2 (1984), 195-196 (in Russian).

[FF] B. Feigin, E. Frenkel. Affine Kac-Moody algebras and semi-infinite flag manifolds. Comm. Math. Phys. 128 (1990), 617-639.

[K] V. Kac. Infinite dimensional Lie algebras. Birkhäuser, Boston (1984).

[L1] G. Lusztig. Introduction to quantum groups. (Progress in Mathematics 110), Boston etc. 1993 (Birkhäuser).

[L2] G. Lusztig. Hecke algebras and Jantzen’s generic decomposition patterns. Adv. in Math. Vol.37, No.2 (1980), 121-164.

[M] F. Malikov. Quantum groups: singular vectors and BGG resolution. KURIMS preprint (1991), 1-21.

[P] P. Papi. Convex orderings in affine root systems. To appear in Journal of Algebra.

[Pr] S. Priddy. Koszul resolutions. Trans. AMS 152, no.1 (1970), 39-60.

A. Voronov. Semi-infinite homological algebra. Invent. Math. 113, (1993), 103-146.

INDEPENDENT UNIVERSITY OF MOSCOW, PERVEROMAJSKAYA ST. 16-18, MOSCOW 105037, RUSSIA

E-mail address: hippie@ium.ips.ras.ru