Clique minors in graphs with a forbidden subgraph

Matija Bucić

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joint work with Jacob Fox and Benny Sudakov
What can we say about a graph with no $H$-minor?
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Hadwiger’s Conjecture

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- Wagner, 1937: $t = 5$ case is equivalent to the Four Color Theorem.
- Robertson-Seymour-Thomas, 1993: true for $t = 6$. 
**Theorem (Kostochka; Thomason, 1980s)**

*Every graph with no $K_t$-minor has average degree $O(t\sqrt{\log t})$.***
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- Delcourt, Postle: no $K_t$ minor $\implies$ $\chi(G) \leq O(t \log \log t)$ also
  no $K_t$ minor, $K_r$-free $\implies$ $\chi(G) \leq O(t)$
Conjecture (Hadwiger, 1943)

Every graph $G$ has a clique minor of size $\chi(G)$. 

Corollary: Any $n$-vertex graph $G$ has a clique minor of size $\frac{n}{\alpha(G)}$.

Duchet, Meyniel 1982: true for $n/\left(2\alpha(G) - 1\right)$.

Fox 2010: first improvement in the constant factor.

Balogh, Kostochka 2011: currently best bound $0.513 \frac{n}{\alpha(G)}$. 

Matija Bucić (IAS and Princeton) Clique minors in graphs with a forbidden subgraph Oberwolfach, January 2022
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Theorem (Kuhn and Osthus; Krivelevich and Sudakov)

If $G$ does not have a bipartite graph $F$ as a subgraph then it has a clique minor of size $(n/\alpha(G))^{1+c}$ for some $c = c(F) > 0$. 

True for average degree instead of Hall ratio as well.

Question (Dvořák and Yepremyan)
Do we get a similar improvement over what Hadwiger's conjecture implies for $F$-free graphs for any forbidden graph $F$?
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What about non-bipartite forbidden graphs?

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- We have $c(K_s) = \frac{1}{10(s-2)}$ which is tight up to an absolute constant factor.
- Simpler proof for $F = K_3$ with $c = \frac{1}{4}$.
- Simpler proof for $F$ bipartite as well.
Given a graph $G$ we wish to construct a denser minor.
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1. Color every vertex red with probability $p$ and blue with probability $1 - p$.
2. Every blue vertex chooses one among its red neighbors uniformly at random (provided it exists).

Gives rise to vertex disjoint stars with red centers and blue leaves. We obtain a random minor $M$ by contracting the stars.
A random approach for generating minors

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![Graph $G$ example](image)
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Oberwolfach, January 2022
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Analysis

- # of vertices of $\mathcal{M}$
Analysis

- # of vertices of $\mathcal{M} = # \text{ of red vertices}$

A fixed edge of $G$ gives rise to an edge of $\mathcal{M}$ with probability $(1-p)^2$. We need to control the number of parallel edges.

A 3-path $vxyu$ is activated if:

- $v, u$ are red, happens with probability $\Omega(1/d^2)$;
- $x, y$ are blue, happens with probability $p^2$;
- $x$ chose $v$ and $y$ chose $u$, happens with probability $\approx (1/pd)^2$.

We want to choose a large family $\mathcal{P}$ of 3-paths with not too many 3-paths between same pairs of vertices.
Analysis

- # of vertices of $\mathcal{M} = $ # of red vertices $\approx np$
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- # of vertices of $M = # \text{ of red vertices} \approx np$
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![Diagram of a graph with vertices v, x, y, u and edges illustrating the activated conditions]
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![Diagram of a 3-path](attachment:image.png)
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- A 3-path $vxyu$ is *activated* if:
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# Analysis

- \# of vertices of \( M = \# \) of red vertices \( \approx np \)

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- The number of vertices of $\mathcal{M} = \# \text{ of red vertices} \approx np$

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  - \( x, y \) are blue, happens with probability \( p^2 \)
  - \( x \) chose \( v \) and \( y \) chose \( u \), happens with probability \( (1 - p)^2 \)
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- We want to choose a large family \( \mathcal{P} \) of 3-paths with not too many 3-paths between same pairs of vertices.
Example: triangle-free case

Let $G$ be a triangle-free, $d$-regular graph with $\alpha = \alpha(G)$ and $p = 1/\sqrt{d}$. 
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• Let $G$ be a triangle-free, $d$-regular graph with $\alpha = \alpha(G)$ and $p = 1/\sqrt{d}$.

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![Diagram](image-url)
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- We may assume $d = O(t\sqrt{\log t})$
- We may assume $G$ is $\rho$-independent set expanding with $\rho := \Omega(n/\alpha)$
- For all $v$ we choose a family $\mathcal{P}_v$ of 3-paths starting in $v$

```
\begin{tikzpicture}
  \node[shape=circle,draw=black] (v) at (0,0) {$v$};
  \node[shape=circle,draw=black] (Nv) at (1,1) {$N(v)$};
  \node[shape=circle,draw=black] (N2v) at (2,1) {$N_2(v)$};
  \node[shape=circle,draw=black] (N3v) at (3,1) {$N_3(v)$};
  \node[shape=circle,draw=black] (u) at (4,0) {$u$};

  \draw[-] (v) -- (Nv);
  \draw[-] (v) -- (N2v);
  \draw[-] (v) -- (N3v);

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  \node[below=0.5cm] at (v) {$d$};
  \node[below=0.5cm] at (Nv) {$\geq \rho d$};
\end{tikzpicture}
```
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\[ |\mathcal{P}_v| \geq \rho^2 d \]
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- Let $G$ be a triangle-free, $d$-regular graph with $\alpha = \alpha(G)$ and $p = 1/\sqrt{d}$.
- Goal: show $G$ has a $K_t$-minor with $t = (n/\alpha)^{4/3-o(1)}$.
- We may assume $d = O(t\sqrt{\log t})$
- We may assume $G$ is $\rho$-independent set expanding with $\rho := \Omega(n/\alpha)$
- $\forall v$ we choose a family $\mathcal{P}_v$ of 3-paths starting in $v$ of size $|\mathcal{P}_v| \geq d\rho^2$:
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- For all $v$ we choose a family $\mathcal{P}_v$ of 3-paths starting in $v$ of size $|\mathcal{P}_v| \geq d\rho^2$.
- We partition $\mathcal{P}_v$ according to other endpoint $u$ into families $\mathcal{P}_{vu}$.
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Expected degree of $v$ in $M$ is $\geq \Omega(|P_v|/d^2) \geq \Omega(\rho^2/d)$

Expected average degree in $M$ is at least

$$\Omega \left( \frac{n \cdot \rho^2 / d}{np} \right)$$
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\[ \forall v \text{ we choose a family } \mathcal{P}_v \text{ of 3-paths starting in } v \text{ of size } |\mathcal{P}_v| \geq d \rho^2: \]

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Expected degree of $v$ in $\mathcal{M}$ is $\geq \Omega(|\mathcal{P}_v|/d^2) \geq \Omega(\rho^2/d)$

Expected average degree in $\mathcal{M}$ is at least

\[ \Omega\left( \frac{n \cdot \rho^2/d}{np} \right) = \Omega\left( \frac{\rho^2}{pd} \right) = \Omega\left( \frac{\rho^2}{\sqrt{d}} \right) \]
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$$\Omega\left(\frac{n \cdot \rho^2/d}{np}\right) = \Omega\left(\frac{\rho^2}{pd}\right) = \Omega\left(\frac{\rho^2}{\sqrt{d}}\right) \geq \rho^{4/3+o(1)}$$
Open problems

Question

What is the max. $c(F)$ s.t. any $F$-free graph has a clique minor of size $(n/\alpha)^{1+c(F)-o(1)}$?
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For which graphs $F$ does $G$ being $F$-free implies $G$ has a clique minor of size $(\chi(G))^{1+c}$ for some $c = c(F) > 0$?
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Open problems

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Question

For which graphs $F$ does $G$ being $F$-free implies $G$ has a clique minor of size $(\chi(G))^{1+c}$ for some $c = c(F) > 0$?

- True if $F$ is bipartite
- Not true if $F$ contains a triangle.
