On the geometry of a class of $N$-qubit entanglement monotones

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A family of $N$-qubit entanglement monotones invariant under stochastic local operations and classical communication (SLOCC) is defined. This class of entanglement monotones includes the well-known examples of the concurrence, the three-tangle, and some of the four, five and $N$ qubit SLOCC invariants introduced recently. The construction of these invariants is based on bipartite partitions of the Hilbert space in the form $\mathbb{C}^{2^n} \simeq \mathbb{C}^N \otimes \mathbb{C}^l$ with $L = 2^{N-l} \geq l = 2^n$. Such partitions can be given a nice geometrical interpretation in terms of Grassmannians $Gr(L, l)$ of $l$-planes in $\mathbb{C}^L$ that can be realized as the zero locus of quadratic polynomials in the complex projective space of suitable dimension via the Plücker embedding. The invariants are neatly expressed in terms of the Plücker coordinates of the Grassmannians.

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I. INTRODUCTION

Since the advent of quantum information science \cite{1} which regards entanglement as a resource it has become of fundamental importance to characterize different classes of entanglement via the use of suitable entanglement measures. Though there are a number of very useful and spectacular results \cite{2, 3, 4, 5, 6} for quantifying the amount of entanglement present in pure and mixed states of multipartite systems, the subject is still at its infancy. For pure states for example we know that it is unlikely that the complete classification of $N$-qubit states will ever be found\cite{7} due to formidable computational difficulties. Under such conditions it seems reasonable to try to find a characteristic subclass of $N$-qubit entanglement that can be described in a unified way. In this paper we attempt a modest step towards the identification of such a class which provides a way of understanding $N$-qubit entanglement in geometric terms.

The use of geometric ideas in understanding entanglement has already been used in a number of papers \cite{8, 9, 10, 11, 12}. In particular it was observed \cite{10, 11} that two and a special class of three-qubit entangled states can be described by certain maps that are entanglement sensitive. These maps enable a geometric description of entanglement in terms of fiber bundles. Fiber bundles are spaces which locally look like the product of two spaces the base and the fiber globally, however they can exhibit a nontrivial twisted structure. In this picture this twisting of the bundle accounts for some portion of quantum entanglement. For two qubits these ideas were elaborated \cite{13} using the correspondence between fibre bundles and the language of gauge fields. The essence of this approach was to provide a description of entanglement by regarding the local unitary (LU) transformations corresponding to a fixed subsystem as gauge degrees of freedom. In our recent paper \cite{13} we have generalized this approach to describe the interesting geometry of three-qubit entanglement. For this purpose we have taken into account the more general class of transformations corresponding to stochastic local operations and classical communication (SLOCC). Using twistor methods we have shown that the relevant fibration in this case is a one over the Grassmannian $Gr(4, 2)$ of two-planes in $\mathbb{C}^4$ with the gauge group being the SLOCC transformations of an arbitrarily chosen qubit i.e. $GL(2, \mathbb{C})$. For every three-qubit state we have associated a pair of planes in $\mathbb{C}^4$, or equivalently a pair of lines in the complex projective space $\mathbb{C}P^3$. In this picture entanglement can be described by the intersection properties of a pair of lines in $\mathbb{C}P^3$. Unlike the one in \cite{11} this method turned out to be capable of characterizing geometrically all the entanglement classes introduced in Ref. \cite{15}. For example the two inequivalent classes of genuine three party entanglement namely the GHZ and W classes correspond to the geometric situation of a pair of nonintersecting lines or lines intersecting in a point respectively.

The aim of the present paper is to generalize these geometric ideas for multiqubit systems. We will see that for an interesting subclass of $N$ qubit entanglement such a generalization can indeed be done. The starting point of our investigations is a recent paper of Emary \cite{16} introducing a class of entanglement monotones based on bipartite partitions of multiqubit systems. By reformulating and generalizing the results of Ref. \cite{16} we are naturally led to a class of SLOCC entanglement monotones giving back the well-known examples of the concurrence \cite{6}, 3-tangle \cite{17} and some of the four \cite{7}, five \cite{18} and $N$-qubit \cite{8} invariants introduced recently. Moreover, these invariants can be rewritten in a nice and instructive form of geometric significance. In fact these invariants are the natural ones associated to higher dimensional Grassmannians of $l$-planes that can be embedded in a complex projective space of suitable dimension. This observation leads us to the interesting possibility of understanding entanglement in terms of the intersection properties of projective subspaces of a complex projective space of suitable dimension. This approach being interesting and useful in its own right also shows a nice connection with twistor theory \cite{14, 19}. 


II. A BIPARTITE CLASS OF ENTANGLEMENT MONOTONES

As a starting point we reformulate the results of Ref. [10] in a geometric fashion convenient for our purposes. Let us consider an arbitrary $N$-qubit pure normalized state $|\Psi\rangle \in \mathbb{C}^{2^N}$

\[ |\Psi\rangle = \sum_{i_1, i_2, \ldots, i_N = 0}^{1} C_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle \quad (1) \]

where the states $|i_1 i_2 \ldots i_N\rangle \equiv |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle$ correspond to the computational base of our $N$-qubit state. Let us single out the computational base of our $N$-qubit state. 

Let us first consider single qubits where the states $|i_1 i_2 \ldots i_N\rangle$ correspond to the computational base of our $N$-qubit state. 

Let us now construct the $L \times L$ matrix $Z_{aa}, a = 0, 1, \ldots, L - 1, a = 0, 1, \ldots, l - 1$ of $2^N = L \times L$ complex entries to be just $C_{i_1 i_2 \ldots i_N}$ arranged according to this partition. This means that the first $N - n$ terms of the binary string $i_1 i_2 \ldots i_N$ written in decimal form are represented by $\alpha$ (rows) while the remaining $n$ terms in decimal form are represented by the letter $a$ (columns). Since according to our assumption $N - n \geq n$ the matrix $Z_{aa}$ is of rectangular shape with the number of rows is greater or equal to the number of columns.

Let us assume now that the columns $Z_{a0}, Z_{a1}, \ldots, Z_{aL-1} \equiv Z_0, Z_1, \ldots, Z_{l-1}$ considered as unnormalized vectors in $\mathbb{C}^L$ are linearly independent. Then the matrix $(Z'Z)_{ab}$ (the reduced density matrix of the last $n$ qubits) is of maximal rank. Hence the assumption of linear independence for all bipartite partitions is equivalent to the one that $|\Psi\rangle$ reinterpreted as the state of a bipartite system in $\mathbb{C}^{N-n} \otimes \mathbb{C}^n$ for all $N - n \geq n$ is totally entangled [21].

Our unnormalized linearly independent vectors $Z_0, Z_1, \ldots, Z_{l-1}$ span an $l$-plane in $\mathbb{C}^L$. The set of $l$-planes in $\mathbb{C}^L$ forms an $L - l \times l$ dimensional complex manifold the Grassmannian $Gr(L, l)$. There are a number of ways to introduce complex coordinates for this manifold. First the entries of the $L \times l$ matrix define the so-called homogeneous or Stiefel coordinates. Their number is greater than the (complex) dimension of the manifold. This redundancy in the homogeneous coordinates has its origin in the fact that any linear combination of the vectors $Z_a, a = 0, 1, \ldots, l - 1$ spans the same $l$-plane. Equivalently, the transformation $Z \mapsto ZS$ where $S \in GL(l, \mathbb{C})$ (the set of invertible $l \times l$ matrices with complex entries) can be regarded as a gauge degree of freedom. It merely amounts to a redefinition of the vectors spanning the $l$-plane in question. It can be shown [21] that $S(L, l)$ the set of complex $L \times l$ matrices $Z_{aa}$ of full rank forms a fiber bundle over the Grassmannian $Gr(L, l)$ with gauge group, i.e. we have $Gr(L, l) = S(L, l)/GL(l, \mathbb{C})$.

Another way of defining homogeneous coordinates for $Gr(L, l)$ is to use the so-called Plücker coordinates. By definition the Plücker coordinate $P_{\alpha_0 \alpha_1 \ldots \alpha_{l-1}}$ of the $l$-plane defined by $Z$ is just the maximal minor of $Z_{aa}$ formed by using the rows singled out by the $l$ fixed values $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$. It is obvious that if we make the transformation $Z \mapsto ZS$ with $S \in GL(l, \mathbb{C})$ the Plücker coordinates transform as $P_{\alpha_0 \alpha_1 \ldots \alpha_{l-1}} \mapsto Det(S)P_{\alpha_0 \alpha_1 \ldots \alpha_{l-1}}$.

The number of such coordinates is $\binom{l}{2}$ which is greater than the dimension of the Grassmannian $Gr(L, l)$, this means that the Plücker coordinates are not independent. They are subject to special relations called the Plücker relations.

In order to illustrate these abstract concepts let us consider the example of a three-qubit system $N = 3$. The state of the system can then be written in the form

\[ |\Psi\rangle = \sum_{i_1, i_2, i_3 = 0}^{1} C_{i_1 i_2 i_3} |i_1 i_2 i_3\rangle. \quad (2) \]

Let us chose $n = 1$ corresponding to the last qubit, then we have $L = 4$ ($\alpha = 0, 1, 2, 3$) and $l = 2$ ($a = 0, 1$) hence

\[ Z_0 \equiv \begin{pmatrix} Z_{00} \\ Z_{10} \\ Z_{20} \\ Z_{30} \end{pmatrix} = \begin{pmatrix} C_{00} \\ C_{01} \\ C_{10} \\ C_{11} \end{pmatrix}. \quad (3) \]

\[ Z_1 \equiv \begin{pmatrix} Z_{01} \\ Z_{11} \\ Z_{21} \\ Z_{31} \end{pmatrix} = \begin{pmatrix} C_{01} \\ C_{11} \\ C_{10} \\ C_{11} \end{pmatrix}. \quad (4) \]

Now the Plücker coordinates are the maximal minors of the $4 \times 2$ matrix $Z_{aa}$ formed by the columns above. Let us chose arbitrary two values $\alpha_0 = \alpha$ and $\alpha_1 = \beta$ then the Plücker coordinates are

\[ P_{\alpha \beta} = Z_{\alpha \beta} - Z_{\beta \alpha}Z_{\alpha 1}. \quad (5) \]

The number of such coordinates is 6 which is greater than the complex dimension of $Gr(4, 2)$ which is 4.

Clearly under a $GL(2, \mathbb{C})$ transformation

\[ \begin{pmatrix} Z_{00} & Z_{01} \\ Z_{10} & Z_{11} \\ Z_{20} & Z_{21} \\ Z_{30} & Z_{31} \end{pmatrix} \mapsto \begin{pmatrix} Z_{00} & Z_{01} \\ Z_{10} & Z_{11} \\ Z_{20} & Z_{21} \\ Z_{30} & Z_{31} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (6) \]

these coordinates transform as

\[ P_{\alpha \beta} \mapsto (AD - BC)P_{\alpha \beta}. \quad (7) \]

Hence although the number of Plücker coordinates is greater then the complex dimension of $Gr(4, 2)$ we see from the equation above that these coordinates are defined merely projectively i.e. up to a nonzero complex
number hence only their ratios count as coordinates. The number of ratios is 5, moreover one can check that the quadratic Plücker relation
\[ P_{01}P_{23} - P_{02}P_{13} + P_{03}P_{12} = 0 \]  
holds which reduces the number of independent complex coordinates to 4 the complex dimension of $Gr(4,2)$.

As we have seen the Plücker coordinates are defined up to a common scalar factor. Since these coordinates are defined merely projectively we should be able to embed $Gr(L,l)$ into the complex projective space $CP^D$ with $D = \binom{l}{1} - 1$. Such embedding really exists it is the Plücker embedding
\[ Gr(L,l) \hookrightarrow CP^\binom{l}{1} = P\left( \bigwedge^1 L \right) \]  
associating to the vectors $Z_a$, $a = 0, \ldots, l - 1$ spanning the $l$-plane in question the separable $l$-vector $Z_0 \wedge Z_1 \wedge \cdots \wedge Z_{l-1}$ in the $l$-fold antisymmetric tensor product of $C^2$ with itself. For the three qubit case the Plücker embedding associates to the $2$-plane determined by the vectors $Z_0$ and $Z_1$ the separable bivector $Z_0 \wedge Z_1$. Writing out this bivector as an antisymmetric matrix we get Eq. (5). Hence we can alternatively regard the Plücker coordinates as separable $l$-vectors or as totally antisymmetric matrices with $l$ indices, satisfying additional constraints (Plücker relations). In the language of $l$-vectors the transformation property of Plücker coordinates is
\[ Z_0 \wedge \cdots \wedge Z_{l-1} \mapsto (\text{det} S) Z_0 \wedge \cdots \wedge Z_{l-1}, \]  
where $S \in GL(l, C)$ is the usual $l \times l$ matrix acting on our $L \times l$ matrix $Z$. Clearly Eq. (4) is just a special case of (10).

After illustrating our use of Plücker coordinates let us use them to express the entanglement monotones of Ref. [10] in a simpler form. For this following [10] let us introduce the operator $dx_\alpha$, which assigns to vectors $\{Z\} \equiv \{Z_0, \ldots, Z_{l-1}\}$ their $\alpha$th component, i.e. $dx_\alpha(Z_a) = Z_{\alpha a}$, and combines them in the wedge product defined as
\[ \bigwedge_{\alpha=0}^{l-1} dx_\alpha(\{Z\}) = \det(dx_\alpha(Z_0))_{a,b=0,\ldots,l-1}. \]  
Clearly this quantity is just the maximal minor of $Z$ labelled by the rows $\alpha_a, a = 0, \ldots, l - 1$, i.e the Plücker coordinate $P_{\alpha_1,\ldots,\alpha_{l-1}}$. In this notation the entanglement monotones
\[ D_n^{(k_1,\ldots,k_n)} = l^2 \left( \sum_{\alpha_0 < \cdots < \alpha_{l-1} = 0} \left| \bigwedge_{\alpha=0}^{l-1} dx_\alpha(\{Z\}) \right| \right)^{2/l} \]  
of [10] take the following instructive form
\[ D_n^{(k_1,\ldots,k_n)} = \frac{l^2}{l!} \left( \sum_{\alpha_0,\ldots,\alpha_{l-1} = 0} \left| P_{\alpha_1,\ldots,\alpha_{l-1}} \right|^2 \right)^{2/l}. \]  
Notice that here we have introduced the general notation $(k_1,\ldots,k_n)$ of [10] to identify the location of the $n$ qubits. In our simplified case $(k_1,\ldots,k_n) = (N-n+1,\ldots,N)$ i.e. we have placed the $n$ qubits to the end of the $N$ qubit string. Clearly our considerations can be repeated for any partition with $n$ qubit locations labelled as $(k_1,\ldots,k_n)$ and a suitable adjustment for the definition of the Plücker coordinates for this case. It should be obvious that for each such partition with fixed $L$ and $l$ we have a different bundle of the form $S(L,l)/GL(l,C)$. For a given $n$ we have $\binom{N}{n}$ such entanglement monotones associated with these bundles, except for $n = N/2$ when we have half of this number. The important property of the quantities $D_n^{(k_1,\ldots,k_n)}$ is that they are invariant under local unitary (LU) transformations of the qubits [10]. Moreover, writing $P_{\alpha_1,\ldots,\alpha_{l-1}} = (Z_0 \wedge \cdots \wedge Z_{l-1})_{\alpha_1,\ldots,\alpha_{l-1}}$ and using Eq. (5) we see that they are also invariant under the more general transformations of $U(l)$ acting on the $l$ qubit Hilbert subspace. Note that the quantities $D_n^{(k_1,\ldots,k_n)}$ are not necessarily independent.

### III. SLOCC ENTANGLEMENT MONOTONES

In order to motivate our generalization of the LU entanglement monotones to SLOCC entanglement monotones we turn once again to the three-qubit case. Let us single out the last qubit to be the one characterizing the partition. Then we can write the antisymmetric matrix of Plücker coordinates in the form $P = Z_0 \wedge Z_1$ i.e. as a separable bivector (see Eqs. (4) and (5)). Then we have $l = 2$ and $L = 4$ and the entanglement monotone $D_1^{(3)}$ can be written in the form
\[ D_1^{(3)} = 2 \sum_{\alpha,\beta=0}^3 |P_{\alpha,\beta}|^2 = 4\det \begin{pmatrix} (Z_0|Z_0) & (Z_0|Z_1) \\ (Z_1|Z_0) & (Z_1|Z_1) \end{pmatrix}, \]  
where $(Z_a|Z_b) = \sum_{\alpha=0}^3 Z_{\alpha a}Z_{\alpha b}$, with the overbar denotes complex conjugation. As it is well-known [17][22][23] $D_1^{(3)} = \tau_{123} = 4\det \rho_3 = 2(1 - \text{Tr}\rho_3^2)$ which is the linear entropy of the third qubit. Repeating the same construction with the first and then the second qubit one gets the monotones $D_1^{(1)}$ and $D_1^{(2)}$ related to the linear entropies of these qubits. The quantity $Q_1 = \frac{1}{2} (D_1^{(1)} + D_1^{(2)} + D_1^{(3)})$ is the permutation invariant used in [22] and [23].

Let us now introduce a bilinear form $g : C^4 \times C^4 \rightarrow C$ such that for two vectors $A, B \in C^4$ we have
\[ (A, B) \mapsto g(A, B) = A^\alpha B^\beta = A_\alpha B^\alpha \]  

where
\[ g_{\alpha\beta} = g_{i_1 i_2, j_1 j_2} = \varepsilon_{i_1 j_1} \varepsilon_{i_2 j_2}, \]  
(16)
\[ \text{or explicitly} \]
\[ g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]  
(17)

where \( \alpha, \beta = 0, 1, 2, 3 \) and summation for repeated indices is understood. Clearly \( \overrightarrow{A} \cdot \overrightarrow{B} = -\langle A | B \rangle \) where the right hand side is expressed via the spin flip operation of \( \mathcal{L} \) i.e. \( |B\rangle = \sigma_2 \otimes \sigma_2 |B\rangle \).

Let us now define the quantity similar to the one in \( \mathcal{L} \)
\[ E_1^{(3)} = 2|P_{\alpha\beta} P^{\alpha\beta}| = 4 \left| \text{Det} \left( \begin{array}{cccc} Z_0 & Z_0 & Z_0 & Z_1 \\ Z_1 & Z_0 & Z_1 & Z_1 \end{array} \right) \right|. \]  
(18)
Notice the crucial changes we have made, namely we have taken the modulus of the sum, and the sum was understood with respect to the metric \( \mathcal{L} \). Since \( M \cdot M^t = \varepsilon \) with \( M \in SL(2, C) \) this sum with respect to \( g \) is invariant under \( SL(2, C) \times SL(2, C) \) i.e. of determinant one SLOCC transformations acting on the first and second qubits respectively. Moreover, the \( \mathcal{L} \) transformation property shows that the Plücker coordinates are invariant under the remaining \( SL(2, C) \) transformation of the third qubit. Hence \( E_1^{(3)} \) is an \( SL(2, C)^{\otimes 3} \) invariant which can be shown \( \mathcal{L} \) to be the three-tangle \( \tau_{123} \) \( \mathcal{L} \) which is also an entanglement monotone \( \mathcal{L} \). Moreover, it is easy to check that the invariants \( E_1^{(1)} \) and \( E_1^{(2)} \) defined similarly are equal to \( E_1^{(3)} \) reflecting the permutation invariance of the three-tangle.

Having gained some insight into the structure of three qubit invariants now we turn to our generalization of SLOCC entanglement monotones. (In the following by SLOCC transformations we mean the group \( SL(2, C)^{\otimes N} \).) The monotones we wish to propose are of the form
\[ E_n^{(k_1, \ldots, k_n)} \equiv \frac{l^2}{l!} |P_{a_0 \ldots a_{l-1}} P^{a_0 \ldots a_{l-1}} |^{2/l}, \]  
(19)
where summation is now understood with respect to the \( SL(2, C)^{\otimes (N-n)} \) invariant bilinear form with matrix
\[ g_{\alpha\beta} = \varepsilon_{i_0 j_0} \otimes \cdots \otimes \varepsilon_{i_{N-n-1} j_{N-n-1}}. \]  
(20)
Hence the matrix of \( g \) is just the \( N-n \)-fold tensor product of the fundamental \( SL(2, C) \) invariant tensor \( \varepsilon \). An alternative formula using the \( l \) linearly independent vectors spanning the \( l \)-plane in question is
\[ E_n^{(k_1)} \equiv l^2 \left| \text{Det} \left( \begin{array}{cccc} Z_0 & Z_0 & \cdots & Z_0 & Z_{l-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ Z_{l-1} & Z_0 & \cdots & Z_{l-1} & Z_{l-1} \end{array} \right) \right|^{2/l}. \]  
(21)
In the following we adopt the convention of regarding the bilinear form \( g \) to be fundamental, i.e. we consider the pair \( (C^L, g) \) meaning that \( C^L \) is equipped with the extra structure defined by \( g \). Notice that for \( N-n \) even the matrix \( g \) is symmetric and for \( N-n \) odd it is antisymmetric. For \( N-n \) odd, \( g \) defines a simplectic structure on \( C^L \).

The \( SL(2, C)^{\otimes (N-n)} \) invariance of the quantities \( E_n^{(k_1)} \) \( (\{k_1, k_2, \ldots, k_n \}) \) follows from the invariance of the bilinear form, and the \( SL(2, C)^{\otimes n} \) invariance follows from the \( \mathcal{L} \) transformation formula of the Plücker coordinates used for the subgroup \( SL(2, C)^{\otimes n} \subset SL(l, C) \). Hence the \( E_n^{(k_1)} \) are invariant under the full group of determinant one SLOCC transformations i.e. \( SL(2, C)^{\otimes N} \).

The other important property of the quantities \( E_n^{(k_1)} \) is that they are entanglement monotones, meaning that on average they are non-increasing under the action of any local protocol. Now any local protocol can be decomposed into POVM (positive operator valued measures) acting on a single qubit. Since any POVM can be further be decomposed into a sequence of two-outcome POVMs, it is enough to demonstrate the non-increasing property of the \( E_n^{(k_1)} \) under two outcome POVMs. The proof that this property is indeed satisfied is simply a slightly modified rerun of the standard arguments that can be found in \( \mathcal{L} \) \( \mathcal{L} \) \( \mathcal{L} \). The choice of the power \( 2/l \) in the \( \mathcal{L} \) definition makes \( E_n^{(k_1)} \) transform under local POVMs in the same way as the concurrence-squared and the three-tangle do \( \mathcal{L} \).

IV. EXAMPLES

A. Two and three qubits

As our first example it is easy to show that in the case of two qubits \((N = 2, n = 1, L = l = 2)\) our entanglement monotones give back the usual definition of the concurrence squared. Indeed in this case \( Z_{a 0} = C_{a 0}, (\alpha, a = 0, 1) \), hence we have a \( 2 \times 2 \) matrix \( Z = C \) with linearly independent columns. Two linearly independent vectors in \( C^2 \) define the trivial Grassmannian \( Gr(2, 2) \) which is just a point. For the monotone \( E_1^{(2)} \) we have the formula
\[ E_1^{(2)} = 4 |\text{Det}(Z_a \cdot Z_b)| = 4 |\text{Det}(Z^t g Z)| = 4 |\text{Det}C|^2, \]  
(22)
which is just the concurrence squared. Clearly \( E_1^{(1)} = E_1^{(1)} \).

For the three qubit case we have already shown that \( E_1^{(1)} = E_1^{(2)} = E_1^{(3)} = \tau_{123} \) with \( \tau_{123} \) being the three-tangle. Moreover, from Eq. \( \mathcal{L} \) we see that \( E_1^{(3)} \) is just four times the magnitude of the discriminant of \( \text{Det}(x C_{i_1 i_2} + y C_{i_1 i_3}) = 0 \), i.e. a binary form of de-
gree two in the complex variables $x$ and $y$. According to the method of Schläfi [21] this discriminant is just the hyperdeterminant $D(C)$ of $C_{i_1,i_2,i_3}$.

B. Four qubits

As our first nontrivial example let us consider an arbitrary four-qubit state

$$|\Psi\rangle = \sum_{i_1,i_2,i_3,i_4=0}^1 C_{i_1 i_2 i_3 i_4}|i_1 i_2 i_3 i_4\rangle.$$  (23)

Let us first consider the partition $N-n=3,n=1$. In this case $L=8$ and $l=2$, hence for each four-qubit state totally entangled for this partition we have a 2-plane in $\mathbb{C}^8$. Geometrically a four qubit state of this kind determines a point in the Grassmannian $Gr(8,2)$, or equivalently a line in $\mathbb{C}P^7$. Moreover the Grassmannian $Gr(8,2)$ as a manifold of complex dimension 12 can be embedded in $\mathbb{C}P^{27}$ via the Plücker embedding. In this case we have a $8 \times 2$ matrix $Z_{ao}$ with $a=0,1 \ldots 7$ and $a=0,1$ consisting of the two columns

$$Z_0 = \begin{pmatrix} C_{0000} & C_0 \\ C_{0010} & C_2 \\ C_{0100} & C_4 \\ C_{0110} & C_6 \\ C_{1000} & C_8 \\ C_{1010} & C_6 \\ C_{1100} & C_{12} \\ C_{1110} & C_{14} \end{pmatrix}$$  (24)

where for later use we also have written out explicitly the four qubit amplitudes using also decimal labeling. The bilinear form on $\mathbb{C}^8$ is antisymmetric with the explicit form

$$g = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$  (26)

Due to the antisymmetry of $g$ we have $Z_0 \cdot Z_0 = Z_1 \cdot Z_1 = 0$, hence for the entanglement monotone $E_4^{(4)}$ we have the formula

$$E_4^{(4)} = 4\text{Det} \begin{pmatrix} 0 & Z_0 \cdot Z_1 \\ Z_0 \cdot Z_1 & 0 \end{pmatrix} = 4|Z_0 \cdot Z_1|^2.$$  (27)

We can also write this using the decimal labeling of the four qubit amplitudes as

$$E_4^{(4)} = 4|C_0 C_15 - C_2 C_13 - C_4 C_11 + C_6 C_9 - C_8 C_7 + C_10 C_5 + C_{12} C_3 - C_{14} C_{11}|^2.$$  (28)

Hence $E_4^{(4)} = 4|H|^2$ where $H$ is the $SL(2,\mathbb{C})^{\otimes 4}$ invariant introduced in [7]. Calculating the invariants $E_1^{(1)}, E_1^{(2)}$ and $E_1^{(3)}$ by choosing the reduced qubits to be the first second and respectively the third a similar calculation shows that they are all equal to $E_2^{(3)}$ in accordance with the permutation invariance of $H$ [7]. Later when we look at this invariant in a more general context we will give a simple proof of this fact.

Let us now calculate the invariant $E_2^{(34)}$. In this case we have $N-n=2$ and $n=2$, hence $L=l=4$. In this case we have four vectors in $\mathbb{C}^4$ hence the Grassmannian $Gr(4,4)$ being a point is again trivial. One then shows that

$$Z_{oa} = \begin{pmatrix} C_0 & C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 & C_7 \\ C_8 & C_9 & C_{10} & C_{11} \\ C_{12} & C_{13} & C_{14} & C_{15} \end{pmatrix}.$$  (29)

Hence similar to the two-qubit case we have merely one Plücker coordinate which is just the determinant of the matrix above, then we have

$$E_2^{(34)} = 16|\text{Det}(Z_o - Z_b)|^{1/2} = 16|\text{Det}Z|,$$  (30)

hence $E_2^{(34)} = 16|\text{Det}|L|$ where $L$ is the $SL(2,\mathbb{C})^{\otimes 4}$ invariant introduced in [7]. We can calculate two more invariants of this kind, namely $E_2^{(24)}$ and $E_2^{(14)}$ (the remaining ones are not independent). A calculation shows that $E_2^{(24)} = 16|\text{Det}|L$ and $E_2^{(14)} = 16|\text{Det}|N$ in the notation of [7]. The $SL(2,\mathbb{C})^{\otimes 4}$ invariants $L$, $M$ and $N$ are still not independent due to the relation $L + M + N = 0$. Notice also that the same invariants arise from the ones of Emary, namely $D_2^{(34)}, D_2^{(24)}$ and $D_2^{(14)}$ due to the fact that in this very special case the number of Plücker coordinates is merely one so the sums in [18] and [19] contain merely one term (the sum of magnitudes in this case equals the magnitude of the sum). Moreover, since the Hilbert series for the algebra of $SL(2,\mathbb{C})^{\otimes 4}$ invariants is known [7] it follows that the invariants $E_1^{(4)}, E_2^{(34)}$ and $E_2^{(24)}$ are algebraically independent. Moreover there are
four invariants of degrees 2, 4, 4, 6 generating freely the algebra of SLOCC invariants of a four qubit system. Our monotones already reproducing three of such fundamental invariants.

C. Five qubits

For a five qubit state

\[ |\Psi\rangle = \sum_{i_1,i_2,i_3,i_4,i_5=0}^1 C_{i_1,i_2,i_3,i_4,i_5} |i_1i_2i_3i_4i_5\rangle. \]  

(31)

first we consider the partition \( N = 5, n = 1 \). In this case we have \( L = 16 \) and \( l = 2 \) so we have 2-planes in \( \mathbb{C}^{16} \). The set of such 2-planes is the Grassmannian \( Gr(16,2) \). Alternatively one can think of this space as the one parametrizing the set of lines in \( \mathbb{C}^{15} \). Now a five qubit state is characterized by the pair of vectors \( Z_0 \) and \( Z_1 \) forming the 16 x 2 matrix \( Z_{\alpha\alpha} (\alpha = 0, 1, \ldots, 15, a = 0, 1) \).

Now the invariant \( E_1^{(5)} \) has the same form as Eq. (18) where now \( g = \varepsilon \otimes \varepsilon \otimes \varepsilon \). Written out explicitly we see that the quantities \( Z_0 \cdot Z_0 \) and \( Z_1 \cdot Z_1 \) have the same structure as the one appearing in Eq. (28). Indeed it is known that the invariant \( H \) of degree two responsible for this structure defines a quadratic binary form in the variables \( x \) and \( y \). The discriminant of this form defines an invariant of degree 4 [18]. This discriminant is precisely of the form we are already familiar from the definition of the three-tangle via the use of Cayley’s hyperdeterminant. We can define four other invariants \( E_1^{(1)}, E_1^{(2)}, E_1^{(3)} \) and \( E_1^{(4)} \) similarly. One can show [18] that the invariants \( E_1^{(j)} \) with \( j = 1, \ldots, 5 \) are algebraically independent.

Let us now consider the partition \( N = 5, n = 2 \). In this case \( Z_{\alpha\alpha} \) is a 8 x 4 matrix. Since \( N - n = 3 \) is odd \( g \) is antisymmetric, hence \( Z_a \cdot Z_b = -Z_b \cdot Z_a \). Hence the invariant \( E_2^{(45)} \) has the form

\[ E_2^{(45)} = 16|\text{Det}(Z_a \cdot Z_b)|^{1/2}. \]  

(32)

Since the determinant of an even dimensional antisymmetric matrix can always be written as a square (the Pfaffian) we can write this as

\[ E_2^{(45)} = 16|Z_0 \cdot Z_1 Z_2 \cdot Z_3 - Z_0 \cdot Z_2 Z_1 \cdot Z_3 + Z_0 \cdot Z_3 Z_1 \cdot Z_2|. \]  

(33)

Notice that there are 10 entanglement monotones of this kind based on a partition of the form \( 5 = 3 \oplus 2 \). However, these invariants cannot be independent from the ones \( E_1^{(j)} (j = 1, \ldots, 5) \) due to the results of [18] showing that the number of algebraically independent fourth order invariants is five.

D. The \( N \) qubit invariants of Wong and Christensen

In a paper Wong and Christensen have introduced a potential entanglement measure calling it the \( N \)-tangle [3]. In our notation they are just the invariants \( E_1^{(N)} \) based on the partition \( N = N - 1 \oplus 1 \) corresponding to Grassmannians \( Gr(2^{N-1},1) \) of 2-planes in \( \mathbb{C}^{2^{N-1}} \). In [3] it was observed that for \( N \) even these invariants can be written as a square of the pure state concurrence [2]. This structure is indeed exhibited by our two and four-qubit invariants [22] and [27]. This result easily follows from the observation that the matrix \( g \) of Eq. (20) in this case is antisymmetric. Since the pure state concurrence is a permutation invariant we conclude that the invariants \( E_1^{(N)} \) for \( N \) even are also permutation invariants. For the four qubit case we recover the well-known permutation invariance of \( H \) of Ref. [2].

We also see that the invariants \( E_n^{(k)} \) arising from the partition \( N = N - n \oplus n \) can always be written as a square of another invariant when \( N - n \) is odd. This again follows from the antisymmetry of \( g \) and the fact that the determinant of an even dimensional antisymmetric matrix can be represented as a square of the Pfaffian. The simplest example of a Pfaffian is the combination (the Plücker relation) appearing in Eq. (33).

V. CONCLUSIONS

In this paper we have introduced a class of \( N \)-qubit entanglement monotones based on bipartite decompositions \( N = N - n \oplus n \) of the Hilbert space \( \mathcal{H} \simeq \mathbb{C}^{2^N} \). This decomposition has naturally led us to the use of Grassmannians \( Gr(L,l) \) of \( l \)-planes in \( \mathbb{C}^{L} \) where \( L = 2^{N-n} \geq l = 2^n \) as the natural structure characterizing the geometry of a subclass of \( N \)-qubit entanglement. Our construction of such monotones was based on the paper of Emary [16]. The new monotones unlike the ones in [16] are SLOCC invariants, i.e. invariant under stochastic operations and classical communication. We have shown how the well-known invariants such as the concurrence, three-tangle, \( N \)-tangle and some of the four and five qubit invariants introduced recently can be obtained as special cases.

There are a lot of interesting possibilities left to be explored. The most important is of course to see what is the physical meaning of our monotones \( E_n^{(k)} \), for what kind of states we have \( E_n^{(k)} = 0 \) etc. Moreover, an interesting development would be the extension of the approach initiated in [14] of characterizing different SLOCC classes of entanglement via studying the intersection properties of \( l - 1 \)-planes in \( \mathbb{C}P^{L-1} \). A geometric approach of this kind would establish interesting links between the theory of entanglement and twistor theory [14,19]. Such interesting questions will be addressed in a future publication.
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[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[2] C. H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A53, 2046 (1996).
[3] A. Wong and N. Christensen, Phys. Rev. A63, 044301 (2001).
[4] A. Peres, Phys. Rev. Lett. 77 1413 (1996).
[5] M. Horodecki, P. Horodecki and R. Horodecki, Physics Letters A223, 1 (1996).
[6] W. K. Wootters, Phys. Rev. Lett. 80 2245 (1998).
[7] J-G. Luque and J-Y. Thibon, Phys. Rev. A67 042303 (2003).
[8] I. Bengtsson, J. Brännlund and K. Życzkowski, Int. J. Mod. Phys. A17 4675 (2002).
[9] D. C. Brody and L. P. Hughston, Journal of Geometry and Physics 38 19 (2001).
[10] R. Mosseri and R. Dandoloff, J. Phys. A34 10243 (2001).
[11] B. A. Bernevig and H. D. Chen, J. Phys. A36 8325 (2003).
[12] A. Miyake, Phys. Rev. A67, 012108 (2003).
[13] P. Lévay, J. Phys. A37 1821 (2004).
[14] P. Lévay, Phys. Rev. A71 012334 (2005).
[15] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A62, 062314 (2000).
[16] C. Emary, J. Phys. A37 8293 (2004).
[17] V. Coffman, J Kundu and W. K. Wootters, Phys. Rev. A61 052306 (2000).
[18] J-G Luque and J-Y Thibon, quant-ph/0506058.
[19] R. S. Ward, R. O. Wells jr., Twistor geometry and field theory, Cambridge monographs mathematical physics (1990).
[20] G. Ghirardi, L. Marinatto and T. Weber, Journal of statistical Physics 108 49 (2002).
[21] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky Discriminants, resultants and multidimensional determinants, Birkhäuser Boston 1994.
[22] D. A. Meyer and N. R. Wallach, J. Math. Phys. 43 4273 (2001).
[23] G. K. Brennen, Quantum Inf. Comput. 3 619 (2003).
[24] F. Verstraete, J. Dehaene and B. De Moor, Phys. Rev. A68 012103 (2003).