LATTICE REALIZATIONS OF UNITARY MINIMAL MODULAR INVARIANT PARTITION FUNCTIONS

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Abstract

The conformal spectra of the critical dilute $A-D-E$ lattice models are studied numerically. The results strongly indicate that, in branches 1 and 2, these models provide realizations of the complete $A-D-E$ classification of unitary minimal modular invariant partition functions given by Cappelli, Itzykson and Zuber. In branches 3 and 4 the results indicate that the modular invariant partition functions factorize. Similar factorization results are also obtained for two-colour lattice models.

1 Introduction

It is well established that the critical behaviour of two-dimensional lattice models is described by conformal field theory or, to put it another way, that the continuum limit of critical lattice models provide realizations of two-dimensional conformal field theories. An important class of conformal field theories is the unitary minimal series with central charge $c < 1$. In this case a complete $A-D-E$ classification of the theories has been obtained by Cappelli, Itzykson and Zuber \cite{1}. In this paper we present compelling numerical evidence to show that the critical dilute $A-D-E$ lattice models \cite{2, 3, 4} provide realizations of this complete $A-D-E$ classification of unitary minimal conformal field theories, as conjectured by Roche \cite{3}. The layout of the paper is as follows. In Section 2 we describe the minimal conformal field theories and their $A-D-E$ classification. In Section 3 we define the critical $A-D-E$ models due to Pasquier \cite{4} and their dilute and two-colour \cite{5} generalizations. We also summarize the conjectured modular invariant partition functions for these models. Finally, in Section 4, we present the numerical results that confirm the conjectured modular invariant partition functions.

2 Conformal Field Theory

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2.1 Minimal Models

In 1984 Belavin, Polyakov and Zamolodchikov [7] introduced the minimal series of conformally invariant field theories. These models are characterized by a central charge $c < 1$ which is restricted to the discrete values

$$c = 1 - \frac{6(p - p')^2}{pp'} \quad (2.1)$$

with $p$ and $p'$ coprime positive integers. The conformal weights of the minimal series are given by the Kac formula

$$\Delta = \Delta^{(p,p')}_{r,s} = \frac{(rp' - sp)^2 - (p' - p)^2}{4pp'} \quad (2.2)$$

with

$$1 \leq r \leq p - 1, \quad 1 \leq s \leq p' - 1. \quad (2.3)$$

Moreover, Friedan, Qiu and Shenker [8] showed that if the theory is unitary, then the central charge is further restricted by $|p - p'| = 1$, and if in fact $p' - p = 1$,

$$c = 1 - \frac{6}{p'(p' - 1)}; \quad p' = 4, 5, 6, \ldots \quad (2.4)$$

The grids of conformal weights for $p' = 4, 5$ and 6 are shown in Figure 1.

2.2 A–D–E Classification of Modular Invariant Partition Functions

For a conformal field theory on a torus, modular invariance [9] implies further constraints on the theory. The requirement of modular invariance is strong enough to fix the operator content. In fact, Cappelli, Itzykson and Zuber [1] have obtained a complete classification of minimal modular invariant partition functions. Remarkably they obtain two series in one-to-one correspondence with the $A–D–E$ classical Lie algebras, one labelled by $(A, G)$ and the other by $(G, A)$, with Coxeter numbers $(p, p')$ in each case, and $p' > p$. The $A–D–E$ classification of minimal modular invariant partition functions is shown in Table 1. The Virasoro characters in this table are defined by

$$\chi_{r,s}(q) = \frac{q^{-c/24 + \Delta^{(p,p')}_{r,s}}}{Q(q)} \sum_{n=-\infty}^{\infty} \left\{ q^{n(pp' + rp' - sp)} - q^{n(p' + s)(np + r)} \right\} \quad (2.5)$$

where $q$ is the modular parameter and

$$Q(q) = \prod_{n=1}^{\infty} (1 - q^n). \quad (2.6)$$
Figure 1: Grids of conformal weights for the unitary minimal models with \( p' = 4, 5, 6 \) and \( p = p' - 1 \). The table with \( p' = 4 \), \( c = 1/2 \) is identified with the Ising model, \( p' = 5 \), \( c = 7/10 \) is identified with the tricritical Ising model and \( p' = 6 \), \( c = 4/5 \) with the tetracritical Ising model. The odd rows of the \( p' = 6 \) Kac table give the critical exponents of the 3-state Potts model.

Here we are primarily interested in the unitary minimal models with \( p' - p = 1 \). In this case the two \( A-D-E \) series correspond to

\[
\begin{align*}
(A, G) &= \begin{cases} 
(A_{p'-2}, A_{p'-1}) \\
(A_{p'-2}, D_{(p'+2)/2}) \\
(A_{10}, E_6) \\
(A_{16}, E_7) \\
(A_{28}, E_8) 
\end{cases} \\
(G, A) &= \begin{cases} 
(A_{p-1}, A_p) \\
(D_{(p+2)/2}, A_p) \\
(E_6, A_{12}) \\
(E_7, A_{18}) \\
(E_8, A_{30}) 
\end{cases}
\end{align*}
\]

with central charges

\[
c = 1 - \frac{6}{p'(p' - 1)} = 1 - \frac{6}{p(p + 1)}
\]

The Coxeter numbers and the Coxeter exponents of the classical \( A-D-E \) Lie algebras are shown in Table 2. The Dynkin diagrams are shown in Figure 2. Some members of these series are identified as follows:

\[
\begin{align*}
(A_2, A_3) &= \text{critical Ising} \quad c = 1/2 \\
(A_4, D_4) &= \text{critical 3-state Potts} \quad c = 4/5 \\
(A_3, A_4) &= \text{tricritical Ising} \quad c = 7/10 \\
(D_4, A_6) &= \text{tricritical 3-state Potts} \quad c = 6/7
\end{align*}
\]

For this reason we will refer to the \((A, G)\) series as the critical series and the \((G, A)\) series as the tricritical series. In particular, the modular invariant partition functions of the
An exhaustive list of theories with $c < 1$ representative of each universality class allowed by the $A$ classification gives all possible critical behaviours for two-dimensional statistical systems with central charges are $r, s$ series where $r, s$ are Coxeter exponents of $(A, G)$. In this series critical and tricritical 3-state Potts models are

$$
(A_4, D_4) : \quad Z = \frac{1}{2} \sum_{r=1}^{4} \left| \chi_{r,1} + \chi_{r,3} \right|^2 + 2 \left| \chi_{r,5} \right|^2 \quad (p = 5, p' = 6)
$$

$$
(D_4, A_6) : \quad Z = \frac{1}{2} \sum_{s=1}^{6} \left| \chi_{1,s} + \chi_{5,s} \right|^2 + 2 \left| \chi_{3,s} \right|^2 \quad (p = 6, p' = 7).
$$

The $A$ classification of unitary minimal conformal field theories gives an exhaustive list of theories with $c < 1$. In other words, this is a complete list of universality classes giving all possible critical behaviours for two-dimensional statistical systems with $c < 1$. A natural question to ask is whether a solvable lattice model can be found as a representative of each universality class allowed by the $A$ classification.
| $G$ | $h$  | $s$          |
|-----|------|--------------|
| $A_L$ | $L + 1$ | $1, 2, 3, \ldots, L$ |
| $D_L$ | $2L - 2$ | $L - 1, 1, 3, 5, \ldots, 2L - 3$ |
| $E_6$ | $12$ | $1, 4, 5, 7, 8, 11$ |
| $E_7$ | $18$ | $1, 5, 7, 9, 11, 13, 17$ |
| $E_8$ | $30$ | $1, 7, 11, 13, 17, 19, 23, 29$ |

Table 2: The Coxeter number $h$ and Coxeter exponents $s$ of the classical $A$–$D$–$E$ Lie algebras.

Figure 2: The Dynkin diagrams of the classical $A$–$D$–$E$ Lie algebras. The $A$–$D$–$E$ graphs classify all connected graphs whose associated adjacency matrices have eigenvalues strictly less than 2. The eigenvalues of the adjacency matrices are in fact given by $2 \cos(s\pi/h)$ where $s$ ranges over the Coxeter exponents.
3  \textit{A–D–E} Lattice Models and Their Modular Invariant Partition Functions

3.1 Pasquier’s \textit{A–D–E} Models

By a remarkable coincidence, in the same year that Belavin, Polyakov and Zamolodchikov introduced the minimal conformal field theories, Andrews, Baxter and Forrester \cite{10} solved the first infinite hierarchy of lattice models in the form of restricted solid-on-solid (RSOS) models. The spins in these models take values on the $A_L$ Dynkin diagram and are subject to the constraint that the state of adjacent spins on the square lattice must be adjacent on the $A_L$ diagram. Huse \cite{11} showed that the critical behaviour of these $L$ height RSOS models is precisely described by the unitary minimal series. Moreover, it turns out that the modular invariant partition functions of the ABF RSOS models give the $(A_{L-1}, A_L)$ series with $L = 3, 4, 5, \ldots$

The lattice realizations of this critical series of modular invariant partition functions was completed in 1987 by Pasquier \cite{5} who generalized the ABF models by constructing solvable lattice models whose states take values on the \textit{A–D–E} graphs. The $A_L$ models of Pasquier are just the critical ABF RSOS models. We note that, although the $A$ and $D$ models admit off-critical elliptic extensions, the exceptional $E$ models can only be solved at criticality. The face weights of Pasquier’s critical \textit{A–D–E} models are given by

$$W \left( \begin{array}{cc} d & c \\ a & b \end{array} \bigg| u \right) = \begin{array}{c} d \\ \hline \\ \hline c \\ \hline \\ a \\ \hline b \end{array} \sin \left( \lambda - u \right) \sin \lambda \delta_{a,c} \delta_{A_{a,b} A_{a,d}} + \sin \left( \frac{u \sqrt{S_a S_c S_b S_d}}{A_{a,b} A_{b,c}} \right) \left( \begin{array}{c} d \\ \hline \\ \hline c \\ \hline \\ a \\ \hline b \end{array} \right) (3.1)$$

where the spins $a, b, c, d$ take values on the given \textit{A–D–E} graph. The parameter $u$ is called the spectral parameter. In the branches of interest here the spectral parameter lies in the interval $0 < u < \lambda$. The adjacency matrices are given by

$$A_{a,b} = \begin{cases} 1, & a, b \text{ connected} \\ 0, & \text{otherwise}. \end{cases} (3.2)$$

The nonnegative components $S_a$ of the Perron-Frobenius eigenvector are determined by

$$\sum_b A_{a,b} S_b = 2 \cos \lambda S_a \quad (3.3)$$

where $2 \cos \lambda$ is the largest eigenvalue of the adjacency matrix and

$$\lambda = \frac{\pi}{h} \quad (3.4)$$

is called the crossing parameter. The Coxeter number $h$ is given in Table 2.

Pasquier’s \textit{A–D–E} models include some much studied models in statistical mechanics. Some prototypes are shown in Figure 3. The modular invariant partition functions of Pasquier’s critical \textit{A–D–E} models precisely realize the $(A,G)$ series of Cappelli, Itzykson and Zuber. However, for many years realizations of the $(G,A)$ series were missing.
Figure 3: Some prototype classical $A-D-E$ lattice models.

### 3.2 Dilute $A-D-E$ Models

In 1992 Warnaar, Nienhuis and Seaton and Roche independently obtained a second series of solvable lattice models whose states take values on the $A-D-E$ graphs. These lattice models are called the dilute $A-D-E$ models. The face weights of the dilute $A-D-E$ lattice models at criticality are given by

$$ W(d \ a \ c \ b | u) = \rho_1(u)\delta_{a,b,c,d} + \rho_2(u)\delta_{a,b,c}A_{a,d} + \rho_{3}(u)\delta_{a,c,d}A_{a,b} + \rho_4(u)\delta_{b,c,d}A_{a,b} + \rho_5(u)\delta_{a,b,d}A_{a,c} + \rho_6(u)\delta_{a,b}\delta_{c,d}A_{a,c} + \rho_7(u)\delta_{a,d}\delta_{c,b}A_{a,b} + \rho_8(u)\delta_{a,c}A_{a,d} + \rho_9(u)\delta_{b,d}A_{a,b}A_{b,c} $$

where, as before, the adjacency matrix is

$$ A_{a,b} = \begin{cases} 1, & a, b \text{ adjacent} \\ 0, & \text{otherwise} \end{cases} \quad (3.6) $$

and the Perron-Frobenius vector is given by

$$ \sum_b A_{a,b} S_b = 2 \cos \left( \frac{\pi}{h} \right) S_a. \quad (3.7) $$

The effective adjacency graph is given by adding a loop to each node of the $A-D-E$ graphs, that is, the spin states at adjacent sites of the lattice are either the same or adjacent on the $A-D-E$ graph. The generalized Kronecker delta is

$$ \delta_{a,b,c,...} = \begin{cases} 1, & a = b = c = \ldots \\ 0, & \text{otherwise} \end{cases} \quad (3.8) $$

and the trigonometric weight functions are

$$ \rho_1(u) = 1 + \frac{\sin u \sin(3\lambda - u)}{\sin(2\lambda) \sin(3\lambda)} $$

$$ \rho_2(u) = \rho_3(u) = \frac{\sin(3\lambda - u)}{\sin(3\lambda)} $$
\[
\begin{align*}
\rho_4(u) &= \rho_5(u) = \epsilon \frac{\sin u}{\sin(3\lambda)} \\
\rho_6(u) &= \rho_7(u) = \epsilon \frac{\sin u \sin(3\lambda - u)}{\sin(2\lambda) \sin(3\lambda)} \\
\rho_8(u) &= \frac{\sin(2\lambda - u) \sin(3\lambda - u)}{\sin(2\lambda) \sin(3\lambda)} \\
\rho_9(u) &= -\frac{\sin u \sin(\lambda - u)}{\sin(2\lambda) \sin(3\lambda)}.
\end{align*}
\] (3.9)

Here \( \epsilon = \pm 1; \) the choices \( \epsilon = 1 \) in the \( u > 0 \) branches and \( \epsilon = -1 \) in the \( u < 0 \) branches ensure positive Boltzmann weights at the isotropic points \( u = \gamma/2. \)

The dilute \( A-D-E \) models are solvable for two choices of \( \lambda \)

\[
\lambda = \begin{cases} 
\frac{(h-1)\pi}{4h}, & \text{branches 1 and 4} \\
\frac{(h+1)\pi}{4h}, & \text{branches 2 and 3}.
\end{cases}
\] (3.10)

The physical branches are summarized in Table 3. The central charges of these models in branches 1 and 2 are given by [12, 2, 1, 13]

\[
c = \begin{cases} 
1 - \frac{6}{h(h+1)}, & \text{branch 1} \\
1 - \frac{6}{h(h-1)}, & \text{branch 2}.
\end{cases}
\] (3.11)

This suggests identifying the universality classes of the first few dilute \( A-D-E \) models as follows:

branch 2: \( A_3 = \) critical Ising \quad c = 1/2
branch 1: \( A_3 = \) tricritical Ising \quad c = 7/10
branch 2: \( D_4 = \) critical 3-state Potts \quad c = 4/5
branch 1: \( D_4 = \) tricritical 3-state Potts \quad c = 6/7

(3.12)

Notice that the dilute \( A_3 \) and \( D_4 \) are not the usual Ising and 3-state Potts models, they just have the same \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) symmetries and lie in the same universality classes.

| Branch 1 | \( \lambda = \frac{\pi}{4} \left( 1 - \frac{1}{h} \right) \) | \( \gamma = 3\lambda \) | \( u \in (0, \gamma) \) | \( c = 1 - \frac{6}{h(h+1)} \) |
| Branch 2 | \( \lambda = \frac{\pi}{4} \left( 1 + \frac{1}{h} \right) \) | \( \gamma = 3\lambda \) | \( u \in (0, \gamma) \) | \( c = 1 - \frac{6}{h(h-1)} \) |
| Branch 3 | \( \lambda = \frac{\pi}{4} \left( 1 + \frac{1}{h} \right) \) | \( \gamma = 3\lambda - \pi \) | \( u \in (\gamma, 0) \) | \( c = \frac{3}{2} - \frac{6}{h(h+1)} \) |
| Branch 4 | \( \lambda = \frac{\pi}{4} \left( 1 - \frac{1}{h} \right) \) | \( \gamma = 3\lambda - \pi \) | \( u \in (\gamma, 0) \) | \( c = \frac{3}{2} - \frac{6}{h(h-1)} \) |

Table 3: Physical branches and central charges of the dilute \( A-D-E \) lattice models

The dilute \( A-D-E \) lattice models in branch 2 in fact give a second realization of the \((A, G)\) series of Cappelli, Itzykson and Zuber. More importantly, as we show in the next
section, the dilute $A-D-E$ lattice models in branch 1 precisely realize the missing $(G, A)$ series. The dilute $A-D-E$ models thus give a complete realization of all unitary minimal conformal field theories.

### 3.3 Two-Colour $A-D-E$ Models

The two-colour models, obtained by Warnaar and Nienhuis [6], are dense RSOS models built on pairs of $A-D-E$ adjacency graphs. Each site on the lattice carries two heights, one from each graph. In moving between adjacent sites, one of the heights remains constant and the other varies as permitted by its corresponding adjacency graph. If $G^1$ and $G^2$ are the adjacency matrices of the two graphs, the effective adjacency matrix of the two-colour model is therefore 

$$A^1 = G^1 \otimes I, \quad A^2 = I \otimes G^2 \quad (3.13)$$

With a slight abuse of notation we will denote such a model by $G^1 \otimes G^2$. The Perron-Frobenius vector of $A$ is the tensor product $S = S^1 \otimes S^2$ of the Perron-Frobenius vectors of the underlying graphs. In constructing the two-colour models it is assumed that the graphs $G^1$ and $G^2$ have the same largest eigenvalue and hence the same Coxeter numbers.

| Crossing param. | Inversion pt | Phys. region | Central charge |
|-----------------|--------------|--------------|----------------|
| Branch 1        | $\gamma = 3\lambda - \pi$ | $u \in (0, \gamma)$ | $c = 2 \left( 1 - \frac{6}{h(h+1)} \right)$ |
| Branch 2        | $\gamma = 3\lambda - 2\pi$ | $u \in (\gamma, 0)$ | $c = 2 \left( 1 - \frac{6}{h(h-1)} \right)$ |

Table 4: Physical branches and central charges of the two-colour $A-D-E$ lattice models

Explicitly, the state at each site is given by an ordered pair $a = (a_1, a_2)$ so that 

$$S_a = S_{a_1}^1 S_{a_2}^2$$

$$A_{a,b}^i = G_{a_i,b_i}^i \delta_{a_3,b_3}$$ \quad (3.15)

In terms of these entities, the face weights of the critical lattice model are given by

$$W \left( \begin{array}{ccc} d & c & u \\ a & b & \end{array} \right) = \sum_{i=1}^{2} \left[ \rho_1(u) A_{a,b}^i A_{b,c}^i A_{c,d}^i A_{d,a}^i + \delta_{a,c} \left( \rho_2(u) A_{a,b}^i A_{a,d}^i + \rho_3(u) A_{a,b}^i A_{a,d}^i \right) \right. \left. + \sqrt{\frac{S_{a} S_{b}}{S_{c} S_{d}}} \delta_{b,d} \left( \rho_4(u) A_{a,b}^i A_{b,c}^i A_{d,a}^i + \rho_5(u) A_{a,b}^i A_{b,c}^i \right) \right]$$ \quad (3.16)

where

$$\rho_1(u) = \frac{\sin u \sin(3\lambda - u)}{\sin \lambda \sin 3\lambda}$$

$$\rho_2(u) = \frac{\sin(\lambda - u) \sin(3\lambda - u)}{\sin \lambda \sin 3\lambda}$$
\[ \rho_3(u) = \frac{\sin(3\lambda - u)}{\sin 3\lambda} \] (3.17)
\[ \rho_4(u) = -\frac{\sin u \sin(2\lambda - u)}{\sin \lambda \sin 3\lambda} \]
\[ \rho_5(u) = -\epsilon \frac{\sin u}{\sin 3\lambda} \]

The two-colour models have two physical regimes as summarized in Table 4 in terms of the Coxeter number \( h \) of the underlying graphs. The choice of the sign factor \( \epsilon = \pm 1 \) such that \( \epsilon = 1 \) in branch 1 and \( \epsilon = -1 \) in branch 2 ensures that the Boltzmann weights are positive at the isotropic points \( u = \gamma/2 \).

### 3.4 Conjectured Modular Invariant Partition Functions

The partition function of a critical lattice model on a finite \( \ell \times \ell' \) periodic lattice or torus can be written as
\[ Z_{\ell,\ell'} \sim \exp(-\ell\ell' f) Z(q) \] (3.18)
where \( f \) is the bulk free energy and \( Z(q) \) is a universal term describing the leading finite-size corrections in the limit of \( \ell, \ell' \) large with the aspect ratio \( \delta = \ell'/\ell \) fixed. The argument \( q \) is the modular parameter. For a spatially isotropic model, it is simply related to the aspect ratio \( \delta \) by \( q = \exp(-2\pi \delta) \).

The modular invariant partition functions of the dilute \( A-D-E \) models built on the classical graph \( G \) with Coxeter number \( h \) are conjectured to be as follows:
- **Branch 1:** \((G, A_h)\)
- **Branch 2:** \((A_{h-2}, G)\)
- **Branch 3:** \((G, A_h) \times (A_2, A_3)\)
- **Branch 4:** \((A_{h-2}, G) \times (A_2, A_3)\).

Here the modular parameter is
\[ q = \exp(2\pi i\tau), \quad \tau = \frac{\ell'}{\ell} \exp[i(\pi - \theta)] \] (3.20)
and the effective angle \( \theta \) is given by
\[ \theta = \begin{cases} \frac{\pi u}{3\lambda}, & \text{branches 1 and 2} \\ \frac{\pi u}{3\lambda - \pi}, & \text{branches 3 and 4}. \end{cases} \] (3.21)

The modular invariant partition functions of the two-colour \( A-D-E \) models are conjectured to be as follows, where once again \( h \) is the Coxeter number of the underlying graphs \( G^1 \) and \( G^2 \):
- **Branch 1:** \((G^1, A_h) \times (G^2, A_h)\)
- **Branch 2:** \((A_{h-2}, G^1) \times (A_{h-2}, G^2)\).

Here the modular parameter \( q \) is as before but now the effective angle is
\[ \theta = \begin{cases} \frac{\pi u}{3\lambda - \pi}, & \text{branch 1} \\ \frac{\pi u}{3\lambda - 2\pi}, & \text{branch 2}. \end{cases} \] (3.22)
4 Numerical Results

The central charges and scaling dimensions of critical lattice models can be extracted from the finite-size corrections to the eigenvalues of the row transfer matrices

$$\langle a|T(u)|b \rangle = \prod_{j=1}^{N} W \left( \begin{array}{cc} b_j & b_{j+1} \\ a_j & a_{j+1} \end{array} \right| u \right).$$

(4.1)

Specifically, the finite-size corrections to the largest eigenvalue $\Lambda_0$ of a periodic transfer matrix with $N$ faces take the form

$$\frac{1}{N} \log \Lambda_0(u) = -f(u) + \frac{\pi c}{6N^2} \sin \theta(u) + o \left( \frac{1}{N^2} \right),$$

(4.2)

where $f$ is the free energy, $c$ is the central charge and $\theta(u)$ is the effective angle as defined in Section 3.4. At an isotropic point for a square ordered phase $u$ is fixed such that $\theta = \pi/2$. The finite-size corrections to the next-largest eigenvalues $\Lambda_n$ with $n = 1, 2, 3, \ldots$ take the form

$$\frac{1}{N} \log \Lambda_n(u) = -f(u) + \frac{2\pi}{N^2} \left[ \left( \frac{c}{12} - x_n \right) \sin \theta(u) - is_n \cos \theta(u) \right] + o \left( \frac{1}{N^2} \right),$$

(4.3)

where $x_n = \Delta + \bar{\Delta}$ and $s_n = \Delta - \bar{\Delta}$ are respectively the scaling dimension and spin. The scaling dimension takes fractional values whereas the spin is restricted to integer values.

The free energies of the dilute and two-colour models are calculated by solving the appropriate inversion relations

$$\kappa(u)\kappa(-u) = \rho(u)\rho(-u), \quad \kappa(u) = \kappa(\gamma - u)$$

(4.4)

where $f(u) = -\log \kappa(u)$ is the free energy and

$$\rho(u) = \begin{cases} \frac{\sin(2\lambda - u) \sin(3\lambda - u)}{\sin 2\lambda \sin 3\lambda}, & \text{dilute models} \\ \frac{\sin(\lambda - u) \sin(3\lambda - u)}{\sin \lambda \sin 3\lambda}, & \text{two-colour models}. \end{cases}$$

(4.5)

The free energy of the critical dilute models is given by

$$f(u) = -2 \int_{-\infty}^{\infty} \cosh(\pi - 5\lambda)x \cosh \lambda x \cosh(\gamma - u)x \cosh \gamma x \sinh(\gamma - u)x \cosh \gamma x \sinh(\gamma - u)x \cosh \gamma x \sinh \gamma x \sinh \gamma x dx$$

(4.6)

and the free energy of the critical two-colour models is given by

$$f(u) = -2 \int_{-\infty}^{\infty} \cosh(\pi - 2\lambda)x \cosh 2(\pi - 2\lambda)x \cosh(\gamma - u)x \cosh \gamma x \sinh(\gamma - u)x \cosh \gamma x \sinh(\gamma - u)x \cosh \gamma x \sinh \gamma x \sinh \gamma x dx.$$

(4.7)

In these expressions $\gamma$ is the inversion point in the appropriate branch. The dilute and two-colour models have respectively four and two critical branches. Tables 3 and 4 summarize the crossing parameters, inversion points and central charges in each of the physical branches in terms of the Coxeter numbers $h$ of the underlying graphs as given in Table 2.
Given the free energy, the central charge \( c \) is estimated by calculating a sequence of largest eigenvalues \( \Lambda_0 \) for increasing values of \( N \) and applying a suitable extrapolation scheme. Once the central charge is determined, equations (4.3) then allows estimation of the scaling dimensions by a similar procedure using the next-largest eigenvalues \( \Lambda_n \). To obtain accurate values for the central charges and scaling dimensions we need to calculate eigenvalues for \( N \) as large as possible. It is therefore convenient to prediagonalize the transfer matrices into block diagonal form using the eigenvectors of the shift operator \( \Omega = T(0) \) and, in the case of the \( A \) and \( D \) models, the reflection operator \( R \) which arises from the \( Z_2 \) symmetry of the \( A \) and \( D \) Dynkin diagrams. Taken together, these operators reduce the transfer matrices to \( 2^N \) diagonal blocks.

Once a sequence of eigenvalues for increasing \( N \) is obtained, equations (4.2) and (4.3) imply that, for large \( N \), the graph of \( \log \Lambda_n/N + f \) against \( 1/N^2 \) should approximate a line through the origin. However, since the \( o(N^{-2}) \) corrections tend to vanish fairly slowly, a parabolic fit gives better results. A simple extrapolation scheme to extract the \( 1/N \) term is to discard all but the last two eigenvalues in the sequence and take the linear coefficient of the parabola passing through these two points and the origin. We have performed this calculation to find numerically the central charges of a variety of dilute and two-colour models as well as the scaling dimensions of the dilute \( A_3, A_4, D_4 \), and two-colour \( A_4 \otimes A_4 \) models. The approximate central charges are summarized in Tables 5 to 7 and the approximate scaling dimensions are summarized in Tables 8 to 14.

The estimates of the scaling dimensions \( x_n \) allow the first few terms of the isotropic modular invariant partition functions to be determined, since these partition functions are simply

\[
Z(q) = q^{-\frac{c}{12}} \left( 1 + \sum_{n=1}^{\infty} d_n q^{x_n} \right),
\]

where \( d_n \) are the multiplicities. For the dilute models, we see that branches 1 and 2 are described by the series of partition functions in Table 1 and that the partition functions in branches 3 and 4 are the product of the critical Ising partition function \((A_2, A_3)\) with those of branches 1 and 2 respectively.

Dilute \( A_3 \) therefore has the following modular invariant partition functions in its four
Table 6: Central charges of the dilute $A$–$D$–$E$ lattice models in the $u < 0$ branches.

| Model      | Branch 3 | Branch 4 | $N_{\text{max}}$ |
|------------|----------|----------|------------------|
|            | Approx.  | Exact    | Approx.          | Exact |            |
| $A_3$      | 1.232    | $6/5$    | 1.2              | 1.000 | 1           | 12      |
| $A_4$      | 1.327    | 13/10    | 1.3              | 1.199 | 6/5         | 1.2     | 10      |
| $A_5, D_4$ | 1.377    | 19/14    | 1.357...         | 1.299 | 13/10       | 1.3     | 10      |
| $A_7, D_5$ | 1.432    | 17/12    | 1.417...         | 1.391 | 39/28       | 1.393...| 8       |
| $A_{11}, D_7, E_6$ | 1.470 | 19/13 | 1.462... | 1.453 | 16/11 | 1.455... | 8       |
| $A_{17}, D_{10}, E_7$ | 1.487 | 169/114 | 1.482... | 1.479 | 151/102 | 1.480... | 8       |

Similarly, dilute $A_4$ has the partition functions

| Model      | Branch 3 | Branch 4 | $N_{\text{max}}$ |
|------------|----------|----------|------------------|
|            | Approx.  | Exact    | Approx.          | Exact |            |
| $A_3 \otimes A_3$ | 1.423 | 7/5 | 1.4 | 0.9991 | 1 | 1 | 8 |
| $A_4 \otimes A_4$ | 1.606 | 8/5 | 1.6 | 1.3982 | 7/5 | 1.4 | 8 |
| $[A_5, D_4] \otimes [A_5, D_4]$ | 1.712 | 12/7 | 1.714... | 1.5930 | 8/5 | 1.6 | 6 |
| $[A_{11}, D_7, E_6] \otimes [A_{11}, D_7, E_6]$ | 1.914 | 25/13 | 1.923... | 1.8993 | 21/11 | 1.909... | 6 |

Table 7: Central charges of the two-colour $A$–$D$–$E$ lattice models. Here the notation $[G, G']$ means either $G$ or $G'$. The approximations match well the predictions summarized in Table 4.
Table 8: Scaling dimensions and multiplicities for the dilute $A_3$ model in the $u > 0$ branches. We expect branch 1 to correspond to the partition function labelled by $(A_3, A_4)$ and branch 2 to that labelled by $(A_2, A_3)$. These correspondences may be verified by comparison of the above approximations to the exact exponents which arise in the partition function expansions of (4.9).

|     | Branch 1 |     | Branch 2 |     |     |
|-----|----------|-----|----------|-----|-----|
|     | Approx.  | Exact | Mult.    | Approx.  | Exact | Mult. |
| 0.0749999 | 3/40   | 0.075 | 1        | 0.125999 | 1/8   | 0.125 | 1 |
| 0.200000  | 1/5    | 0.2   | 1        | 0.998457 | 1     | 1     | 1 |
| 0.874980  | 7/8    | 0.875 | 1        | 1.12489  | 9/8   | 1.125 | 2 |
| 1.07505   | 43/40  | 1.075 | 2        | 2.01084  | 2     | 2     | 2 |
| 1.20003   | 6/5    | 1.2   | 2        | 1.98828  | 2     | 2     | 2 |
| 1.19994   | 6/5    | 1.2   | 1        | 2.13060  | 17/8  | 2.125 | 2 |
| 1.87541   | 15/8   | 1.875 | 2        |         |       |       |   |

Table 9: Scaling dimensions and multiplicities for the dilute $A_3$ model in the $u < 0$ branches. Comparison with the expansions (4.10) shows branch 4 to be in excellent agreement with the partition function product $(A_2, A_3) \times (A_2, A_3)$, and branch 3 to be in reasonable agreement with $(A_3, A_4) \times (A_2, A_3)$.

|     | Branch 3 |     | Branch 4 |     |     |
|-----|----------|-----|----------|-----|-----|
|     | Approx.  | Exact | Mult.    | Approx.  | Exact | Mult. |
| 0.0778 | 3/40   | 0.075 | 1        | 0.1250  | 1/8   | 0.125 | 1 |
| 0.1270 | 1/8    | 0.125 | 1        | 0.1250  | 1/8   | 0.125 | 1 |
| 0.2033 | 1/5    | 0.2   | 1        | 0.2500  | 1/4   | 0.25  | 1 |
| 0.2401 | 1/5    | 0.2   | 1        | 1.0000  | 1     | 1     | 1 |
|       |         |       |          | 1.0000  | 1     | 1     | 1 |
|       |         |       |          | 1.1250  | 9/8   | 1.125 | 1 |
|       |         |       |          | 1.1246  | 9/8   | 1.125 | 2 |
|       |         |       |          | 1.1248  | 9/8   | 1.125 | 1 |
|       |         |       |          | 1.1210  | 9/8   | 1.125 | 2 |
|       |         |       |          | 1.2450  | 5/4   | 1.25  | 2 |
|       |         |       |          | 1.2424  | 5/4   | 1.25  | 2 |
|       |         |       |          | 1.9987  | 2     | 2     | 2 |
|       |         |       |          | 1.9964  | 2     | 2     | 2 |
|       |         |       |          | 1.9949  | 2     | 2     | 1 |
| Branch 1 | Branch 2 |
|----------|----------|
| Approx.  | Exact    | Mult. | Approx.  | Exact    | Mult. |
| 0.050000 | 1/20     | 0.05  | 1        | 0.074999 | 3/40  | 0.075 |
| 0.013333 | 2/15     | 0.133.. | 1       | 0.19997  | 1/5   | 0.2   |
| 0.250000 | 1/4      | 0.25  | 1        | 0.873958 | 7/8   | 0.875 |
| 0.799969 | 4/5      | 0.8   | 1        | 1.075250 | 43/40 | 1.075 |
| 1.050060 | 21/20    | 1.05  | 2        | 1.199460 | 6/5   | 1.2   |
| 1.049920 | 21/20    | 1.05  | 1        | 1.196920 | 6/5   | 1.2   |
| 1.133360 | 17/15    | 1.133.. | 2      |          |        |       |

Table 10: Scaling dimensions and multiplicities for the dilute $A_4$ model in the $u > 0$ branches. Branch 1 agrees well with the expansion (4.9) of the partition function ($A_4, A_5$), as does branch 2 with the expansion of ($A_3, A_4$).

| Branch 3 | Branch 4 |
|----------|----------|
| Approx.  | Exact    | Mult. | Approx.  | Exact    | Mult. |
| 0.0501   | 1/20     | 0.05  | 1        | 0.07500  | 3/40  | 0.075 |
| 0.1256   | 1/8      | 0.125 | 1        | 0.12500  | 1/8   | 0.125 |
| 0.1357   | 2/15     | 0.133.. | 1       | 0.19985  | 1/5   | 0.2   |
| 0.1918   | 7/40     | 0.175 | 1        | 0.20000  | 1/5   | 0.2   |
| 0.2498   | 1/4      | 0.25  | 1        | 0.32500  | 13/40 | 0.325 |
|          |          |       |          | 0.87506  | 7/8   | 0.875 |
|          |          |       |          | 0.99984  | 1     | 1     |
|          |          |       |          | 0.99803  | 1     | 1     |
|          |          |       |          | 1.07508  | 43/40 | 1.075 |
|          |          |       |          | 1.09886  | 43/40 | 1.075 |

Table 11: Scaling dimensions and multiplicities for the dilute $A_4$ model in the $u < 0$ branches. The data for branch 4 agree very well with the expansion (4.10) of ($A_3, A_4$) $\times$ ($A_2, A_3$), and the data for branch 3 agree reasonably with the expansion of ($A_4, A_5$) $\times$ ($A_2, A_3$).
Table 12: Scaling dimensions and multiplicities for the dilute $D_4$ model in the $u > 0$ branches. Comparison with the partition function expansions shows branch 1 to correspond to $(D_4, A_6)$, and branch 2 to $(A_4, D_4)$; these are respectively tricritical and critical 3-state Potts.

Table 13: Scaling dimensions and multiplicities for the dilute $D_4$ model in the $u < 0$ branches. Branch 4 agrees well with the expansion of $(A_4, D_4) \times (A_2, A_3)$, and branch 3 agrees reasonably with that of $(D_4, A_6) \times (A_2, A_3)$.
Our numerical estimates for the scaling dimensions of this model are summarized in Table 14. The expansions of the isotropic partition function products are

Branch 1: \((A_4, A_5) \times (A_4, A_5)\)
Branch 2: \((A_3, A_4) \times (A_3, A_4)\)

Our numerical results for the two-colour models are consistent with the conjectured modular invariant partition functions summarized in Section 1.
Table 14: Scaling dimensions and multiplicities for the two-colour $A_4 \otimes A_4$ model. These agree well with the partition function expansions of $(A_4, A_5) \times (A_4, A_5)$ in branch 1 and $(A_3, A_4) \times (A_3, A_4)$ in branch 2.

5 Conclusion

We have presented numerical evidence that the critical dilute $A-D-E$ lattice models in the $u > 0$ branches provide a realization of both the $(A, G)$ and the $(G, A)$ series of modular invariant partition functions in the classification of Cappelli, Itzykson and Zuber. In the $u < 0$ branches we have seen that the modular invariant partition functions are products of the $(A_2, A_3)$ partition function with members of the $(A, G)$ and $(G, A)$ series. Furthermore, we have seen that the modular invariant partition functions of the two-colour $A-D-E$ lattice models at criticality are squares of members of the $(A, G)$ and $(G, A)$ series.

Since these are all exactly solvable models, it would be interesting to see some exact calculations of scaling dimensions to compare with the predicted values. Indeed such calculations have been performed in the case of the dilute $A$ models in references [13, 14].

We have here considered only unitary minimal conformal field theories. It would also be interesting to see whether, by varying the crossing parameter, the dilute $A-D-E$ lattice models might provide realizations of non-unitary minimal conformal field theories.

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References

[1] A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. B280 (1987) 445; Comm. Math. Phys. 113 (1987) 1
[2] S. O. Warnaar, B. Nienhuis and K. A. Seaton, Phys. Rev. Lett. 69 (1992) 710.

[3] Ph. Roche, Phys. Lett. B4 (1992) 929.

[4] S. O. Warnaar, P. A. Pearce, K. A. Seaton and B. Nienhuis, J. Stat. Phys. 74 (1994) 469.

[5] V. Pasquier, Nucl. Phys. B28 (1987) 162; J. Phys. A 20 (1987) L1229, 5707.

[6] S. O. Warnaar and B. Nienhuis, J. Phys. A 26 (1993) 2301.

[7] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333.

[8] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575; in “Vertex Operators in Mathematics and Physics”, eds. J. Lepowsky, S. Mandelstam and I.M. Singer, Springer, 1984.

[9] J. L. Cardy, Nucl. Phys. B270 (1986) 186; Nucl. Phys. B275 (1986) 200.

[10] G. E. Andrews, R. J. Baxter and P. J. Forrester, J. Stat. Phys. 35 (1984) 193.

[11] D. A. Huse, Phys. Rev. B 30 (1984) 3908.

[12] S. O. Warnaar, M. T. Batchelor and B. Nienhuis, J. Phys. A 25 (1992) 3077.

[13] Y-K. Zhou and P. A. Pearce, Conformal Weights of Dilute A Lattice Models, in preparation (1995).

[14] D. Kim and P. A. Pearce, J. Phys. A 20 (1987) L451.

[15] H. W. J. Blöte, J. L. Cardy and M. P. Nightingale, Phys. Rev. Lett. 56 (1986) 742; I. Affleck, Phys. Rev. Lett. 56 (1986) 746.

[16] P. A. Pearce and Y-K. Zhou, Unitary Minimal Conformal Weights of Dilute A Lattice Models, to appear in the Proceedings of the International Congress of Mathematical Physics, Paris (1995).