ON CELLULAR RATIONAL APPROXIMATIONS TO $\zeta(5)$

FRANCIS BROWN AND WADIM ZUDILIN

Abstract. We analyse a certain family of cellular integrals, which are period integrals on the moduli space $\mathcal{M}_{0,8}$ of curves of genus zero with eight marked points, which give rise to simultaneous rational approximations to $\zeta(3)$ and $\zeta(5)$. By exploiting the action of a large symmetry group on these integrals, we construct infinitely many effective rational approximations $p/q$ to $\zeta(5)$ satisfying

$$0 < \left| \zeta(5) - \frac{p}{q} \right| < \frac{1}{q^{0.86}}.$$

1. Introduction

Following Apéry’s legendary proof [1] of the irrationality of $\zeta(3)$, the traditional strategy for approaching the irrationality of $\zeta(5)$ is to try to construct linear forms in $1$ and $\zeta(5)$ with rational coefficients and good arithmetic properties. At the time of writing, it has not succeeded despite many years of effort. In this paper we propose a different method for tackling this problem. It involves constructing small linear forms in a larger set of multiple zeta values, which, after setting the unwanted numbers to zero, leads to the approximations described in the abstract.

The starting point for this method is the recent work [6] of one of the authors which revisits irrationality proofs from a new geometric perspective. It reproduces Apéry’s irrationality proofs for $\zeta(2)$ and $\zeta(3)$, via Beukers’ famous re-interpretation [4] as Euler-type double and triple integrals, and produces natural families of multiple integrals of higher order which are linear forms in a controllable set of multiple zeta values. Whilst a literal generalisation of Apéry’s approach for higher weight zeta values remains a tough challenge, the machinery in [6] gives us hope to seek alternative approaches.

In this manuscript we investigate just one particular family of such 5-fold integrals, which was disguised as the family $8\pi^5$ and highlighted in [6, Example 7.5],

$$I(\mathbf{a}) = I(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$$

$$= \int \cdots \int_{0 < t_1 < t_2 < \cdots < t_5 < 1} \frac{t_1^{a_1} (t_2 - t_1)^{a_2} (t_3 - t_2)^{a_3} (t_4 - t_3)^{a_4} (t_5 - t_4)^{a_5} (1 - t_5)^{a_6}}{(t_3 - t_1)^{b_{24}} t_3^{b_{14}} (1 - t_4)^{b_{57}} (t_4 - t_2)^{b_{35}} (t_5 - t_2)^{b_{36}}} \times \frac{dt_1 dt_2 dt_3 dt_4 dt_5}{(t_3 - t_1) t_3 (1 - t_4) (t_4 - t_2) (t_5 - t_2)},$$

(1)

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where
\[ b_{24} = a_2 + a_3 + a_6 - a_7 - a_8, \quad b_{14} = a_4 + a_7 + a_8 - a_2 - a_6, \]
\[ b_{57} = a_4 + a_5 + a_8 - a_2 - a_3, \quad b_{35} = a_2 + a_3 - a_8, \quad b_{36} = a_8 \]  
(2)
and all the parameters \( a_1, \ldots, a_8 \) are assumed to be integers. According to [6, Sects. 5.1 and 5.2] the integral \( I(\bf{a}) \) converges if and only if the following seventeen linear forms in the \( a_i \) are non-negative:

\[ a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_1 + a_5 - a_3, a_3 + a_6 - a_8, \]
\[ a_4 + a_5 + a_7 + a_8 - a_2 - a_3 - a_6, a_7 + a_8 - a_6, a_4 + a_8 - a_2, a_2 + a_3 + a_6 - a_4 - a_8, \]
\[ a_1 + a_8 - a_3, a_1 + a_2 - a_4, a_4 + a_5 - a_2, a_4 + a_7 + 2a_8 - a_2 - a_3 - a_6. \]
(3)

General results about periods of moduli spaces [5] imply \textit{a priori} that the family \([\Pi]\) of integrals is a \( \mathbb{Q} \)-linear combination of multiple zeta values of weight \( \leq 5 \), namely: \( 1, \zeta(2), \zeta(3), \zeta(4), \zeta(5) \) and \( \zeta(3)\zeta(2) \). The cellular nature of this integral (more precisely, Poincaré duality) suggests that the term of subleading weight \( \zeta(4) \) vanishes, since it is dual to the non-existent ‘\( \zeta(1) \)’. Cohomological considerations furthermore imply that the coefficients of the two terms \( \zeta(5) \) and \( \zeta(2)\zeta(3) \) always occur in the same proportion, i.e., there is a single period in leading weight, namely \( \zeta(5) + 2\zeta(3)\zeta(2) \).

The main interest of this family is that, additionally, the coefficient of \( \zeta(3) \) always vanishes [6, Sect. 10.2.4]. Therefore, as hinted at in [6], the decomposition of \( I = I(\bf{a}) \) into a \( \mathbb{Q} \)-linear combination of zeta values takes the very special form:

\[ I = Q \cdot (2\zeta(5) + 4\zeta(3)\zeta(2)) - 4\hat{P} \cdot \zeta(2) - 2P \]  
(4)
for some \( Q, P, \hat{P} \in \mathbb{Q} \) (in fact, \( Q \in \mathbb{Z} \) since it may be expressed as a 5-fold residue of the integrand). The fact that these linear forms in \( \zeta(5) + 2\zeta(3)\zeta(2), \zeta(2) \) and 1 are very small follows from bounds for the integrand along the domain of integration. From this one may deduce that at least one of the two numbers \( \{\zeta(2), \zeta(5) + 2\zeta(3)\zeta(2)\} \) is irrational, but since this is already known for \( \zeta(2) \), one cannot deduce any new irrationality result from \( I \). In this regard, the number \( \zeta(2) \) is usually viewed as ‘parasitic’.

However, another hint from [6] suggests that the linear forms

\[ I' = I'(\bf{a}) = Q\zeta(5) - P, \]

obtained by ‘setting \( \zeta(2) \) to zero’, are also reasonably small. \textit{A priori} this operation does not make sense, but can be justified either by cohomological arguments, or by passing to motivic versions of the integral \( I \) and motivic zeta values, for which it does.

The fact that the \( I' \) are small implies \textit{a posteriori} that the linear forms

\[ I'' = I''(\bf{a}) = Q\zeta(3) - \hat{P} \]

are small as well, since \( I = 2I' + 4I''\zeta(2) \). Thus the original cellular integral \([\Pi]\) is, in disguise, a pair of simultaneous approximations \( I', I'' \) to \( \zeta(5) \) and \( \zeta(3) \). It is our principal goal here to quantify these hints from [6] as well as to analyse the arithmetic properties of the coefficients \( Q = Q(\bf{a}), P = P(\bf{a}) \) and \( \hat{P} = \hat{P}(\bf{a}) \) of the simultaneous
ON CELLULAR RATIONAL APPROXIMATIONS TO $\zeta(5)$

rational approximations to $\zeta(5)$ and $\zeta(3)$. As we will see below, there is a large transformation group $\mathcal{G}$ (of order $7! = 5040$) acting on normalised versions of the integrals $I(a)$ as well as on $I'(a), I''(a)$ and on the coefficients $Q(a), P(a), \hat{P}(a)$; this group allows us to sharply compute a denominator $D = D(a) \in \mathbb{Z}$ for which $DI' \in \mathbb{Z}[\zeta(5)]$. Such groups famously appear, and prove themselves to be arithmetically useful, in constructions of rational approximations to $\zeta(2), \zeta(3)$ and $\zeta(4)$ (and other mathematical constants); see [13,16,17,19,24]. Our ‘group structure for $\zeta(5)$’ shares similarities with its predecessors but also features interesting novelties which we will try to highlight in due course.

One outcome of our construction and analysis is the following result, for which we need to recall a related concept of effective rational approximations to a real number $\alpha$, as explained by Nesterenko in his paper [14]. A family of linear forms $q_n\alpha - p_n$, with $p_n, q_n \in \mathbb{Q}$, is called effective if it is given by the solution to a linear difference equation with polynomial coefficients (also known as an Apéry-type recursion). Equivalently, its generating function satisfies a (Picard–Fuchs) differential equation of geometric origin. We say that a family of rational approximations to $\alpha$ is effective if it can be written in the form $p_n/q_n$, where $q_n\alpha - p_n$ is effective. Such effective rational approximations $p_n/q_n$ are distinguished from (ineffective!) solutions to

$$0 < \left| \frac{\alpha - p}{q} \right| < \frac{1}{q}$$

in integers $p, q$ whose existence follows from squeezing the rational number in question via $r/q \leq \alpha \leq (r+1)/q$, with $r = \lfloor qa \rfloor$.

**Theorem 1.** There are infinitely many effective rational approximations $p/q$, with $p, q \in \mathbb{Z}$, to $\zeta(5)$ such that

$$0 < \left| \zeta(5) - \frac{p}{q} \right| < \frac{1}{q^{0.86}}.$$  

Note that this result does not imply the (expected!) irrationality of $\zeta(5)$ which would follow if the upper bound in the inequality were of the form $1/q^{1+\varepsilon}$ for some $\varepsilon > 0$. The ‘worthiness’ exponent $0.86$ is nevertheless best possible when compared to any other known constructions of effective rational approximations to $\zeta(5)$ (see the introduction in [14] for a related comparison in the case of Catalan’s constant). The result and analysis in this paper also gives us confidence that further exploration of the cellular integrals from [6] will produce new arithmetic surprises.

A more general context for this subject is the study of Mellin transforms

$$\int_{\gamma} f_1^{a_1} \cdots f_n^{a_n} \omega$$

where $f_1, \ldots, f_n : X \to \mathbb{G}_m$ are morphisms from an algebraic variety $X$ defined over $\mathbb{Q}$ to the multiplicative group $\mathbb{G}_m$, the $a_i \in \mathbb{C}$ are complex parameters, $\gamma \subset X(\mathbb{C})$ is a (locally finite) chain of integration, and $\omega$ is a differential form of degree equal to

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1Be aware that passing from $p_n/q_n$ to $p/q$ with $p, q \in \mathbb{Z}$ may involve multiplying both $p_n$ and $q_n$ by the same rational factor: rescaling is implicit in the definition.
the dimension of $\gamma$. In the present note, $X = \mathcal{M}_{0,n}$, the moduli space of Riemann spheres with $n = 8$ marked points, the $f_1, \ldots, f_8$ are cross-ratios, the $a_i$ are integers, and $\omega$ is the 5-form on the second line of (1). The combinatorial, arithmetic and analytic structures that we have unearthed in this particular case may point to the existence of general theorems for Mellin transforms in algebraic geometry. We hope that our results may serve as inspiration for future research along these lines.

Since our main task is to emphasise the ideas and methods behind the analysis of the integrals (1), we have tried to stay non-technical in our exposition as far as possible, and details of proofs which are either obtainable by finite calculation, or have been verified by computer computation, are omitted. Each section presents different mathematical features and structures underlying the integrals (1) and ends with some suggestions for generalisation. We leave it to the reader to judge whether our narrative style is sufficiently clear, self-contained and reader-friendly.

2. Totally symmetric case

We first focus our attention on the ‘totally symmetric’ case when all the parameters $a_1, \ldots, a_8$ are equal,

$$a_1 = \cdots = a_8 = n.$$ 

This also means that all the exponents in (1) including $b_{24}, b_{14}, b_{57}, b_{35}, b_{36}$ are equal to $n$:

$$I_n = I(n, \ldots, n) = \int_{0 < t_1 < t_2 < \cdots < t_5 < 1} \cdots \int \frac{(t_1(t_2 - t_1)(t_3 - t_2)(t_4 - t_3)(t_5 - t_4)(1 - t_5))}{(t_3 - t_1)t_3(1 - t_4)(t_4 - t_2)(t_5 - t_2)}^n dt_1 dt_2 dt_3 dt_4 dt_5 \times \frac{1}{(t_3 - t_1)t_3(1 - t_4)(t_4 - t_2)(t_5 - t_2)}.$$ 

It is the simplest possible choice of the parameters; as we will witness later, the transformation group $\mathcal{G}$ acts trivially in this case.

The integrals $I_n$ are effectively computed (up to $n = 10$) using Panzer’s HyperInt [15]. We find out that we indeed have

$$I_n = Q_n \cdot (2\zeta(5) + 4\zeta(3)\zeta(2)) - 4\hat{P}_n \cdot \zeta(2) - 2P_n$$  \hspace{1cm} (5)

for this range; more specifically,

$$Q_0 = 1, \ Q_1 = 21, \ Q_2 = 2989, \quad \hat{P}_0 = 0, \ \hat{P}_1 = \frac{101}{4}, \ \hat{P}_2 = \frac{344923}{96}, \quad P_0 = 0, \ P_1 = \frac{87}{4}, \ P_2 = \frac{1190161}{384}.$$
for \( n = 0, 1, 2 \). Then Koutschan’s HolonomicFunctions\[11\] produces a third order
Apéry-type recursion for the integrals \( I_n \):

\[
2(2n + 1)(41218n^3 - 48459n^2 + 20010n - 2871)(n + 1)^5Q_{n+1}
- (97604224n^9 + 178061760n^8 + 72005308n^7 - 48634688n^6 - 39076836n^5
+ 2622730n^4 + 758106n^3 + 920112n^2 - 543402n - 120582)Q_n
- 2n(3874492n^8 - 2617900n^7 - 3144314n^6 + 2947148n^5 + 647130n^4 - 1182926n^3
+ 115771n^2 + 170716n - 44541)Q_{n-1}
+ n(41218n^3 + 75195n^2 + 46746n + 9898)(n - 1)^5Q_{n-2} = 0,
\]

which is also satisfied by the rational coefficients \( Q_n, \hat{P}_n, P_n \). This already proves
the decomposition (5), so that

\[
I_n = 2I_n' + 4I_n'' \zeta(2) \quad \text{with} \quad I_n' = Q_n\zeta(5) - P_n \quad \text{and} \quad I_n'' = Q_n\zeta(3) - \hat{P}_n.
\]

The characteristic polynomial of the recurrence equation is

\[
4\lambda^3 - 2368\lambda^2 - 188\lambda + 1.
\]

If

\[
\lambda_1 = 0.00500378\ldots, \quad \lambda_2 = -0.08438431\ldots \quad \text{and} \quad \lambda_3 = 592.07938053\ldots
\]
denote its roots (ordered according to their absolute value), then a standard localisation
procedure leads to the asymptotics

\[
\lim_{n \to \infty} \frac{\log |I_n|}{n} = \log |\lambda_1| = -5.29756135\ldots,
\]

\[
\lim_{n \to \infty} \frac{\log |I_n'|}{n} = \lim_{n \to \infty} \frac{\log |I_n''|}{n} = \log |\lambda_3| = 6.38364071\ldots.
\]

Finally, based on an extensive computation of the rational coefficients \( Q_n, \hat{P}_n, P_n \) we
observe experimentally that

\[
Q_n, \ d_n^2d_{2n}\hat{P}_n, \ d_n^5P_n \in \mathbb{Z} \quad \text{for} \quad n = 0, 1, 2, \ldots,
\]

where \( d_n \) denotes the least common multiple of \( 1, 2, \ldots, n \). As we show later,

\[
Q_n = \sum_{k_1=0}^{n} \binom{n + k_1}{n} \binom{n}{k_1} \sum_{k_2=0}^{n-k_1} \binom{n + k_2}{n} \binom{n}{k_2} \sum_{k_3=0}^{n-k_2} \binom{n + k_3}{n} \binom{n}{k_3}
\]

implying in particular that the coefficients \( Q_n \) are integral. The validity of this
formula can be independently established by verifying, again on the basis of \[11\],
that the double sum on the right-hand side satisfies the above recursion.

The approximating forms are similar in spirit to the ones for \( 1, \zeta(2), \zeta(3) \) constructed in\[25\] Section 2] (although their characteristic polynomials have two roots
of the same size). A comment in loc. cit. helps one to identify our approximations with those constructed in \textit{[22] Theorem 1}: 
\[ Q_n = \frac{(-1)^{n+1} q_n}{\binom{2n}{n}}, \quad \hat{P}_n = \frac{(-1)^{n+1} \hat{p}_n}{\binom{2n}{n}}, \quad P_n = \frac{(-1)^{n+1} p_n}{\binom{2n}{n}} \quad \text{for } n = 0, 1, 2, \ldots \]
(in the notation of \textit{[22]}). The identities above may be proven using the recursions and initial data.

How good are these effective rational approximations in the totally symmetric case? To answer this question, we consider the following measure of ‘worthiness’. Assume, more generally, that we have constructed some sequence of effective rational approximations 
\[ q_n \zeta(5) - p_n \in \mathbb{Z} \zeta(5) + \mathbb{Z} \]
such that 
\[ c_0 = \lim_{n \to \infty} \frac{\log |q_n \zeta(5) - p_n|}{n} \quad \text{and} \quad c_1 = \lim_{n \to \infty} \frac{\log |q_n|}{n} > c_0. \]
Then for any choice of \( \varepsilon > 0 \) and all \( n \) sufficiently large,
\[ \left| \zeta(5) - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{c_1 - c_0}} \]
with \( \gamma = (c_1 - c_0)/c_1 \). We call this exponent \( \gamma \) the \textit{worthiness} of the approximations. When \( \gamma > 1 \), the inequalities imply that \( \zeta(5) \) is irrational; in that case one can also conclude that the irrationality exponent of \( \zeta(5) \) is at most \( \gamma/(\gamma - 1) \).

In our particular situation here we find that \( c_0 = \log |\lambda_2| + 5 \), and \( c_1 = \log |\lambda_3| + 5 \), implying that the worthiness exponent is \( \gamma = 0.77795976 \ldots \): in other words, with the choice \( q_n = d_n^5 Q_n, p_n = d_n^5 P_n \) we have
\[ \left| \zeta(5) - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{7/9}} \]
for all \( n \) sufficiently large. The group structure for \( \zeta(5) \) will enable us to improve this exponent considerably.

3. \textbf{THE ASSOCIATED GENERALISED CELLULAR INTEGRALS}

From \textit{[6]} we know that the subgroup of automorphisms of \( \mathcal{M}_{0,8} \) (which is the symmetric group permuting the 8 marked points) preserving the domain of integration in \textit{[1]} is a dihedral group of order 16. It is generated by a cyclic rotation
\[ \sigma: (t_1, t_2, t_3, t_4, t_5) \mapsto \left( 1 - \frac{t_1}{t_2}, 1 - \frac{t_1}{t_3}, 1 - \frac{t_1}{t_4}, 1 - \frac{t_1}{t_5}, 1 - t_1 \right) \]
(of order 8) and a reflection
\[ \tau: (t_1, t_2, t_3, t_4, t_5) \mapsto \left( t_1, \frac{t_1}{t_5}, \frac{t_1}{t_4}, \frac{t_1}{t_3}, \frac{t_1}{t_2} \right). \]
The group however does not preserve the form of the integrand.

Applying to the integral \( I(a) = I(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \) the fifth power of \( \sigma \) and passing from simplicial to cubical coordinates
\[ t_1 = x_1 x_2 x_3 x_4 x_5, \quad t_2 = x_2 x_3 x_4 x_5, \quad t_3 = x_3 x_4 x_5, \quad t_4 = x_4 x_5, \quad t_5 = x_5, \]
we arrive at the integral
\[
I(a) = \int \cdots \int_{[0,1]^5} x_1^{a_4} (1 - x_1)^{a_2} x_2^{a_4 + a_5} (1 - x_2)^{a_6} x_3^{a_2 + a_4 + a_6 + a_8} (1 - x_3)^{a_1 + a_5 + a_3} \\
\times x_4^{a_1 + a_2} (1 - x_4)^{a_5} x_5^{a_2} (1 - x_5)^{a_1} \\
\times x_2 x_3 x_4 \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 \\
\times (1 - x_1 x_2) (1 - x_2 x_3) (1 - x_3 x_4) (1 - x_4 x_5). \tag{8}
\]
In this representation, the involution \(x_j \mapsto x_{6-j}\) for \(j = 1, 2, 3, 4, 5\) does not change the form of the integral but acts on the set of exponents as follows:
\[
i_1: a \mapsto (a_5, a_4, a_3, a_2, a_1, a_7, a_6, a_4 + a_7 + a_8 - a_2 - a_6). \tag{9}
\]
One shows that it generates a subgroup of order two of the dihedral group alluded to earlier, which preserves the original form of the integrand.

A substitution
\[
x = (x_1, x_2, x_3, x_4, x_5) = \left(1 - \frac{y_1}{1 - y_1 y_2}, 1 - y_1 y_2, y_3, 1 - y_4 y_5, \frac{1 - y_5}{1 - y_4 y_5}\right)
\]
transforms the 8-parameter integral \(I(a)\) into a subfamily of the following 12-parameter integrals which have only two rational factors in their denominator:
\[
J = J(p; q) = J(p_0, p_1, p_2, p_3, p_4, p_5, p_6; q_1, q_2, q_3, q_4, q_5)
\]
\[
= \int \cdots \int_{[0,1]^5} y_1^{p_1} (1 - y_1)^{q_1} y_2^{p_2} (1 - y_2)^{q_2} y_3^{p_3+1} (1 - y_3)^{q_3} y_4^{p_4} (1 - y_4)^{q_4} \\
\times y_5^{p_5} (1 - y_5)^{q_5} \, dy_1 \cdots dy_5. \tag{10}
\]
An integral of this form with 12 parameters reduces to \(I(a)\) if and only if the parameters satisfy the constraints:
\[
p_1 = p_0 + q_4 + q_5 - q_1 - q_3, \quad p_3 = p_0 + q_4 + q_5 - q_3, \quad p_5 = p_0 + q_4 - q_3, \quad p_6 = p_0 + q_4 + q_5 - q_1 - q_2. \tag{11}
\]
In this case, the parameters are related to the \(a_1, \ldots, a_8\) as follows:
\[
p_0 = a_5 + a_6 - a_8, \quad p_1 = a_2 + a_3 + a_6 - a_4 - a_8, \quad p_2 = a_6, \quad p_3 = a_2 + a_3 + a_6 - a_8, \quad p_4 = a_7, \quad p_5 = a_3 + a_6 - a_8, \quad p_6 = a_1 + a_2 + a_6 - a_4 - a_8,
\]
\[
q_1 = a_4, \quad q_2 = a_5, \quad q_3 = a_1 + a_5 - a_3, \quad q_4 = a_1, \quad q_5 = a_2.
\]
and, conversely, by
\[
a_1 = q_4, \quad a_2 = q_5, \quad a_3 = q_2 + q_4 - q_3, \quad a_4 = q_1,
\]
\[
a_5 = q_2, \quad a_6 = p_2, \quad a_7 = p_4, \quad a_8 = p_2 + q_2 - p_0.
\]
The change of variables \(y_j \mapsto y_{6-j}\) for \(j = 1, 2, 3, 4, 5\) shows that the involution \(i_1\) naturally lifts to the 12-parameter family as follows:
\[
i_1: (p; q) \mapsto (p_6, p_5, p_4, p_3, p_2, p_1, p_0; q_5, q_4, q_3, q_2, q_1). \tag{12}
\]
We know that not on such a motive are necessarily of the form $1, \zeta, I\omega$ which we shall deduce later by different methods. We have general tools to compute the de Rham realisation $M$ have not been rigorously established: to this end, it would be very interesting to have tools to compute the de Rham realisation $M_{dR}$ via computer. This would have the considerable benefit of providing contiguity relations for the forms $\omega_a$, and hence direct access to the asymptotics and arithmetic of the forms $I(a)$ which we shall deduce later by different methods.

Computations of periods suggest that the motives in question, for a fixed family $I(an)$, are of rank 3, of the form $\text{gr}_W M = \mathbb{Q}(0) \oplus \mathbb{Q}(-2) \oplus \mathbb{Q}(-5)$. The periods of such a motive are necessarily of the form $1, \zeta(2), \zeta_5 = \zeta(5) + \lambda \zeta(3) \zeta(2)$, for a fixed $\lambda \in \mathbb{Q}$, which — in the case of the family of cellular integrals under consideration — is $\lambda = 2$. Therefore, a period matrix for $M$ looks like

$$
\begin{pmatrix}
1 & \zeta(2) \\
0 & (2\pi i)^2 \\
0 & (2\pi i)^2 \zeta(3) \\
\end{pmatrix}
$$

where the columns correspond to a basis $\omega_0, \omega_2, \omega_5$ of $M_{dR} \cong \mathbb{Q}_{dR}(0) \oplus \mathbb{Q}_{dR}(-2) \oplus \mathbb{Q}_{dR}(-5)$ and the first two rows $\gamma_1, \gamma_2$ correspond to real (i.e., invariant under the action of real Frobenius) homology classes $M^p_\beta$, and the last row to an imaginary (real Frobenius anti-invariant) homology class $\gamma_3$. The integration domain in the integral $I(a)$ corresponds to the homology class $\gamma_1$, which is represented by the real simplex $0 \leq t_1 \leq \cdots \leq t_5 \leq 1$. Since the de Rham class of the integrand $\omega_a$ is a $\mathbb{Q}$-linear combination of $\omega_0, \omega_1, \omega_2$, the general integral $I(a)$ is a $\mathbb{Q}$-linear combination of the entries in the first row of this period matrix. Our linear forms in $1, \zeta(5)$ are obtained by replacing the homology cycle $\gamma_1$ with a particular linear

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\end{pmatrix}
$$

where the columns correspond to a basis $\omega_0, \omega_2, \omega_5$ of $M_{dR} \cong \mathbb{Q}_{dR}(0) \oplus \mathbb{Q}_{dR}(-2) \oplus \mathbb{Q}_{dR}(-5)$ and the first two rows $\gamma_1, \gamma_2$ correspond to real (i.e., invariant under the action of real Frobenius) homology classes $M^p_\beta$, and the last row to an imaginary (real Frobenius anti-invariant) homology class $\gamma_3$. The integration domain in the integral $I(a)$ corresponds to the homology class $\gamma_1$, which is represented by the real simplex $0 \leq t_1 \leq \cdots \leq t_5 \leq 1$. Since the de Rham class of the integrand $\omega_a$ is a $\mathbb{Q}$-linear combination of $\omega_0, \omega_1, \omega_2$, the general integral $I(a)$ is a $\mathbb{Q}$-linear combination of the entries in the first row of this period matrix. Our linear forms in $1, \zeta(5)$ are obtained by replacing the homology cycle $\gamma_1$ with a particular linear
This follows from the fact that the motive \( M(13) \). It necessarily also forces the term \( \zeta(2) \zeta(3) \) in the top row and right hand column to drop out. This is because the extension group \( \text{Ext}^1(Q(-5), Q(0)) \) in the category of mixed Tate motives over \( \mathbb{Z} \) is one-dimensional, with period \( \zeta(5) \).

Stated in an equivalent but different way: there is a change of basis of the Betti homology such that the period matrix takes the form

\[
\begin{pmatrix}
1 & 0 & \zeta(5) \\
0 & (2\pi i)^2 & (2\pi i)^2 \zeta(3) \\
0 & 0 & (2\pi i)^5
\end{pmatrix}
\]

This follows from the fact that the motive \( M \) contains at most two non-trivial extensions: between \( Q(0) \) and \( Q(-5) \), giving a possible period \( \zeta(5) \); and between \( Q(-2) \) and \( Q(-5) \), giving a possible period \( (2\pi i)^2 \zeta(3) \). Any extension between \( Q(0) \) and \( Q(-2) \) necessarily splits, which explains the shape of the matrix \( (14) \), and more precisely the zero in the top row and middle column.

In the rest of this note, the word ‘motivic’ means any property of the integrals \( I(\alpha) \) which holds on the level of the object \( M \), and hence on the entire period matrix. For example, the involutive symmetry \( i_1 \) is clearly motivic. It would be very interesting to prove that the entire group \( \mathfrak{G} \) we shall associate to \( \zeta(5) \) is also motivic.

5. Barnes-type representation of the integrals, and asymptotics

The internal integrals over \( y_1, y_2 \), and over \( y_4, y_5 \) in \( J(p; q) \) can be recognised as \( 3F_2 \)-hypergeometric functions using the identity

\[
3F_{2} \left( \frac{\alpha_{0}, \alpha_{1}, \alpha_{2}}{\beta_{1}, \beta_{2}} \bigg| z \right) = \frac{\Gamma(\beta_{1})\Gamma(\beta_{2})}{\Gamma(\alpha_{1})\Gamma(\alpha_{2})\Gamma(\beta_{1}-\alpha_{1})\Gamma(\beta_{2}-\alpha_{2})} \times \int_{[0,1]^2} x_{1}^{\alpha_{1}-1}(1-x_{1})^{\beta_{1}-\alpha_{1}-1}x_{2}^{\alpha_{2}-1}(1-x_{2})^{\beta_{2}-\alpha_{2}-1} (1-zx_{1}x_{2})^{\alpha_{0}} \, dx_{1} \, dx_{2}
\]

and observing that

\[
1 - y_{3}(1 - y_{1}y_{2}) = \left( 1 + \frac{y_{3}}{1 - y_{3}} y_{1}y_{2} \right) (1 - y_{3}).
\]

We deduce that

\[
J(p; q) = \frac{p_{1}!q_{1}!p_{2}!q_{2}!p_{4}!q_{4}!p_{5}!q_{5}!}{(p_{1} + q_{1} + 1)! (p_{2} + q_{2} + 1)! (p_{4} + q_{4} + 1)! (p_{5} + q_{5} + 1)!} \\
\times \int_{0}^{1} 3F_{2} \left( \frac{p_{0} + 1, p_{1} + 1, p_{2} + 1}{p_{1} + q_{1} + 2, p_{2} + q_{2} + 2} \bigg| \frac{-y_{3}}{1 - y_{3}} \right) \\
\times 3F_{2} \left( \frac{p_{4} + 1, p_{5} + 1, p_{6} + 1}{p_{4} + q_{4} + 2, p_{5} + q_{5} + 2} \bigg| \frac{-y_{3}}{1 - y_{3}} \right) \frac{y_{3}^{p_{0} + 1} \, dy_{3}}{(1 - y_{3})^{p_{0} + p_{6} - q_{5} + 2}}
\]
which, after changing the variable $z = y_3/(1 - y_3)$, becomes

$$J(p, q) = \frac{p_1!q_1!p_2!q_2!p_4!q_4!p_5!q_5!}{(p_1 + q_1 + 1)!(p_2 + q_2 + 1)!(p_4 + q_4 + 1)!(p_5 + q_5 + 1)!}$$

$$\times \int_0^\infty 3F_2 \left( \begin{array}{c} p_0 + 1, p_1 + 1, p_2 + 1 \\ p_1 + q_1 + 2, p_2 + q_2 + 2 \end{array} \right) \left| -z \right|$$

$$\times 3F_2 \left( \begin{array}{c} p_4 + 1, p_5 + 1, p_6 + 1 \\ p_4 + q_4 + 2, p_5 + q_5 + 2 \end{array} \right) \left| -z \right| \frac{z^{p_3 + 1} \, dz}{(1 + z)^{p_3 + q_1 - p_0 - p_6 + 1}}.$$  

(15)

Applying now the Barnes integral representation

$$3F_2 \left( \begin{array}{c} \alpha_0, \alpha_1, \alpha_2 \\ \beta_1, \beta_2 \end{array} \right) = \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} \frac{\Gamma(\alpha_1 + s)\Gamma(\alpha_2 + s)\Gamma(\alpha_3 + s)\Gamma(-s)}{\Gamma(\beta_1 + s)\Gamma(\beta_2 + s)} z^s \, ds,$$

where the vertical line $\text{Re} \, s = -c$ separates the poles of $\Gamma(-s)$ from those of $\Gamma(\alpha_j + s)$ for $j = 1, 2, 3$, and the Eulerian integral

$$\int_0^\infty \frac{z^{\alpha - 1}}{(1 + z)^{\alpha + \beta}} \, dz = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

where $\text{Re} \, \alpha, \text{Re} \, \beta > 0$, we obtain

$$J(p, q) = \frac{q_1!q_2!q_4!q_5!}{p_0!p_6!(p_3 + q_3 - p_0 - p_6)!}$$

$$\times \frac{1}{2\pi i} \int_{-c_1-i\infty}^{-c_1+i\infty} \frac{\Gamma(p_0 + 1 + s)\Gamma(p_1 + 1 + s)\Gamma(p_2 + 1 + s)\Gamma(-s)}{\Gamma(p_1 + q_1 + 2 + s)\Gamma(p_2 + q_2 + 2 + s)} \, ds$$

$$\times \frac{1}{2\pi i} \int_{-c_2-i\infty}^{-c_2+i\infty} \frac{\Gamma(p_4 + 1 + t)\Gamma(p_5 + 1 + t)\Gamma(p_0 + 1 + t)\Gamma(-t)}{\Gamma(p_4 + q_4 + 2 + t)\Gamma(p_5 + q_5 + 2 + t)} \, dt$$

$$\times \frac{1}{2\pi i} \int_{-c_3-i\infty}^{-c_3+i\infty} \frac{\Gamma(p_3 + 2 + s + t)\Gamma(q_3 - p_0 - p_6 - 1 - s - t)}{\Gamma(p_3 + 2 + s)\Gamma(q_3 - p_0 - p_6 - 1 - s - t)} \, ds,$$

(16)

where the real numbers $c_1, c_2$ satisfy $0 < c_1 < 1 + p_0^* = 1 + \min\{p_0, p_1, p_2\}, 0 < c_2 < 1 + p_0^* = 1 + \min\{p_4, p_5, p_6\}$ and $c_1 + c_2 > 1 + p_0 + p_6 - q_3$.

Expanding this Barnes-type double integral into a $\mathbb{Q}$-linear form in single and double zeta values we find that

$$J(p, q) = 2Q(p, q)(\zeta(5) + 2\zeta(2)\zeta(3)) + \cdots,$$

where the extra terms encode a $\mathbb{Q}$-linear combination of zeta values of weight strictly less than 5. Furthermore, the leading coefficient has the following explicit double-sum binomial expression:

$$Q(p, q) = (-1)^{p_0 + p_1 + \cdots + p_6} \sum_{k_1, k_2 \in \mathbb{Z}} \binom{k_1}{p_0} \binom{k_2}{p_6} \binom{k_1 + k_2 + q_3 - p_0 - p_6}{p_3 + q_3 - p_0 - p_6}$$

$$\times \binom{q_1}{k_1 - p_1} \binom{q_2}{k_1 - p_2} \binom{q_4}{k_2 - p_4} \binom{q_5}{k_2 - p_5}.$$  

(17)
In the totally symmetric case (when all \( p_j = q_k = n \), except for \( p_3 = 2n \)) this gives the explicit expression \( (7) \) displayed in Section 2.

**Remark 1.** Performing the decomposition of \( J(p, q) \) into a linear form in multiple zeta values is a difficult technical task. One potential way of doing so is to find an appropriate collection of contiguity relations for the integral and perform a related multiple induction in the spirit of [16,17]. There is an alternative approach hinted at in [5], which forms the basis of the HyperInt algorithm of [15]: in principle it should be possible to perform the integration steps over the ring of integers, inverting only those primes which are necessary for taking primitives and performing logarithmic Taylor expansions of hyperlogarithms. A strategy we have executed here makes use of the integral (16) and is more in line with [13, 24, 26], combined with Beukers’ original technology in [4]. Explicitly, one writes the integrand in (16) as a product of a rational function and of reciprocals of sines; decomposes the rational part into a sum of partial fractions and makes relevant shifts of the vertical integration paths. The original integral ultimately becomes a \( \mathbb{Q} \)-linear combination of the integrals

\[
f^{(s_1, s_2)}_{k_1, k_2} = \frac{1}{(2\pi i)^2} \int_{1/3-i\infty}^{1/3+i\infty} \int_{1/3-i\infty}^{1/3+i\infty} \frac{1}{(t_1 + k_1)^{s_1} (t_2 + k_2)^{s_2}} dt_1 \ dt_2 \nonumber
\]

for \( k_1, k_2 \in \mathbb{Z}_{\geq 0} \) and \( s_1, s_2 \in \{1, 2\} \). The very same integrals can then be cast in the form

\[
I^{(s_1, s_2)}_{k_1, k_2} = \frac{1}{\Gamma(s_1) \Gamma(s_2)} \int_{[0,1]^2} u^{k_1} v^{k_2} f(u, v) (\log u)^{s_1-1} (\log v)^{s_2-1} \frac{du}{u} \ \frac{dv}{v},
\]

where

\[
f(u, v) = \frac{1}{(2\pi i)^2} \int_{1/3-i\infty}^{1/3+i\infty} \int_{1/3-i\infty}^{1/3+i\infty} u^{t_1} v^{t_2} \frac{\pi}{\sin \pi t_1} \frac{\pi}{\sin \pi t_2} \frac{\pi}{\sin \pi (t_1 + t_2)} dt_1 \ dt_2
\]

for \( 0 < u, v < 1 \). The latter double integral can be explicitly computed via a careful residue analysis:

\[
f(u, v) = \begin{cases} 
\frac{uv \log u}{(1-u)(1-v)} - \frac{u \log(u/v)}{(1-u/v)(1-v)} & \text{if } 0 < u < v < 1, \\
\frac{uv \log u}{(1-u)(1-v)} - \frac{u \log(u/v)}{(1-u/v)(1-v/u)} & \text{if } 0 < v < u < 1.
\end{cases}
\]

We skip the details of the remaining computation and only mention that, for the 8-parameter subfamily subject to (11) translated to the form \( I(a) \), we indeed find that the latter decomposes as (4), and we have the inclusion (28) below for the linear form \( I'(a) \). A sharp arithmetic analysis for the companion linear form \( I''(a) \in \mathbb{Z} \zeta(3) + \mathbb{Q} \) is somewhat harder to obtain via the above techniques but, fortunately, these linear forms have a different expression which we discuss in Section 6.
We now turn our attention to the growth of the quantities associated with $J(p; q)$. To this end, we can adapt standard techniques for computing asymptotics of Barnes-type integrals (and the coefficients in their decomposition) to the integral in \([16]\) and expression \([17]\) (see, e.g., \([21]\) Sect. 2] and \([27]\) Sect. 5)). This allows us to compute the asymptotics of $J(pn; qn)$ as $n \to \infty$ (hence of $I(\mathcal{A}n)$ as well), but also of all other integrals and sums that appear in the decomposition of these 5-fold integrals. The result of the asymptotic analysis is as follows. Consider a suitable solution $(x, y)$ of the system of equations $F_1 = F_2 = 0$, where

$$
F_1(x, y) = x(p_1 + q_1 - x)(p_2 + q_2 - x)(x + y + q_3 - p_0 - p_6) - (x - p_0)(x - p_1)(x - p_2)(x + y - p_3),
$$

$$
F_2(x, y) = (x + y + q_3 - p_0 - p_6)(p_4 + q_4 - y)(p_5 + q_5 - y)y - (x + y - p_3)(y - p_4)(y - p_5)(y - p_6),
$$

then the limit of $|J(pn; qn)|^{1/n}$ as $n \to \infty$ is equal to

$$
\frac{|x - p_0|^p |x - p_1|^p |x - p_2|^p |x + y - p_3|^p}{|p_1 + q_1 - x|^{p_1 + q_1} |p_2 + q_2 - x|^{p_1 + q_2} |x + y + q_3 - p_0 - p_6|^{p_0 + p_6 - q_3}} \times \frac{|y - p_4|^p |y - p_5|^p |y - p_6|^p}{|p_4 + q_4 - y|^{p_4 + q_4} |p_5 + q_5 - y|^{p_5 + q_5}} \times \frac{q_1 q_2 q_4 q_5}{p_0^p (p_3 + q_3 - p_0 - p_6)^{p_3 + q_3 - p_0 - p_6} p_6^p}.
$$

If one substitutes other (‘non-degenerate’) solutions $(x, y)$ into the latter expression, one obtains asymptotics related to different choices of homology cycles. This recipe for determining asymptotics may be computed in practice: one can eliminate the variable $y$ by computing the resultant $F(x)$ of the polynomials $F_1$ and $F_2$ with respect to the variable $y$; for any root $x_0$ of $F(x) = 0$, the related value of $y$ is found by solving the linear equation $F_1(x_0, y) = 0$.

When the 12 parameters $p_0, \ldots, p_6, q_1, \ldots, q_5$ are independent, the resultant $F(x)$ has generic degree 9 in $x$. When the parameters are subject to

$$
p_3 + q_3 = p_0 + q_4 + q_5, \quad p_3 + q_3 = p_6 + q_1 + q_2,
$$

which are consistent with \([11]\), then the degree of $F(x)$ drops (generically) to 5. In addition, the system $F_1 = F_2 = 0$ features ‘trivial’ solutions $x = 0, y = p_3$ and $x = p_3, y = 0$ when the following two conditions are satisfied:

$$
p_3 \in \{p_1 + q_1, p_2 + q_2\} \quad \text{and} \quad p_3 \in \{p_4 + q_4, p_5 + q_5\}.
$$

These are automatically met under the constraints \([11]\) which imply that

$$
p_3 = p_1 + q_1 \quad \text{and} \quad p_3 = p_5 + q_5.
$$

Thus, under conditions \([18]\), \([19]\), which determine when $J(p; q)$ reduces to $I(\mathcal{A})$, the asymptotics are completely determined by the cubic polynomial

$$
\frac{F(x)}{x(p_3 - x)},
$$

(20)
This justifies heuristically the appearance of just three (presumably $\mathbb{Q}$-linearly independent) periods, namely $\zeta(5) + 2\zeta(3)\zeta(2)$, $\zeta(2)$ and 1, in the decomposition of the integrals $I(a)$, as well as the rank of the motive $M$ in Section 4.

**Remark 2.** The above analysis reveals three real asymptotic quantities

$$\lambda_1 = \lim_{n \to \infty} I(an)^{1/n}, \quad \lambda_2 = \lim_{n \to \infty} I''(an)^{1/n}, \quad \lambda_3 = \lim_{n \to \infty} Q(an)^{1/n}. \quad (21)$$

The inequality $|\lambda_1| < |\lambda_2|$, which can be checked numerically for any particular admissible choice of $a = (a_1, \ldots, a_8)$, implies that

$$\lim_{n \to \infty} I'(an)^{1/n} = \lim_{n \to \infty} (Q(an)\zeta(5) - P(an))^{1/n} = \lambda_2,$$

and implies in particular the nonvanishing of our linear forms $I'(an) \in \mathbb{Q} + \mathbb{Z}\zeta(5)$. A practical (though technically challenging!) task for existing creative telescoping realisations is writing down explicitly a (third order Apéry-type) recursion for the integrals $I(an) = J(pn; qn)$, which is then automatically valid for their coefficients $Q(an), P(an)$, $\hat{P}(an)$ and linear forms $I'(an)$. The numbers $(21)$ are precisely the roots of the characteristic polynomial of the recurrence equation (compare with the situation in Section 2), and the nonvanishing can alternatively be established by showing that the first three entries of $Q(an), P(an), \hat{P}(an)$ generate three linearly independent solutions (over $\mathbb{C}$).

Finally notice that any automorphism of the 12- or 8-parameter (sub)family $J(p; q)$ should respect the corresponding asymptotics of $J(pn; qn)^{1/n}$ as $n \to \infty$. See Section 8 for the results of an analysis along these lines.

### 6. Descent to $\zeta(3)$

In this part we assume that conditions $(18)$ and $(19)$ are satisfied, so that we indeed have the decomposition $(1)$ for the induced integral $I(a)$ and its version $J(p; q)$. Notice that formula $(17)$ is nothing but an iterated residue of the integrand in $(19)$. Our computation mentioned in Remark 1 reveals that $I''(a)$ can be read off from

$$\frac{1}{(2\pi i)^2} \int_{[0,1]^3} \int_{|y_4| = |y_5| = \varepsilon} y_1^{p_1}(1 - y_1)^{q_1} y_2^{p_2}(1 - y_2)^{q_2} y_3^{p_3+1}(1 - y_3)^{q_3} y_4^{p_4}(1 - y_4)^{q_4} \times y_5^{p_5}(1 - y_5)^{q_5} dy_1 \cdots dy_5$$

for $\varepsilon < 1/4$; one can also choose $|y_1| = |y_2| = \varepsilon$ instead. Using

$$\frac{1}{(1 - y_3(1 - y_4y_5))^{p_6+1}} = \frac{1}{(y_3y_4y_5)^{p_6+1}} \left(1 + \frac{1 - y_3}{y_3y_4y_5}\right)^{-(p_6+1)}$$

$$= \frac{1}{(y_3y_4y_5)^{p_6+1}} \sum_{k \geq p_6} \binom{k}{p_6} \left(-\frac{1 - y_3}{y_3y_4y_5}\right)^{k-p_6}$$

$$= \sum_{k \in \mathbb{Z}} (-1)^{k+p_6} \binom{k}{p_6} (y_3y_4y_5)^{-k-1}(1 - y_3)^{-p_6+k},$$
so that
\[
\frac{y_3^{p_3+1}(1 - y_3)^{q_3}y_4^{p_4}(1 - y_4)^{q_4}y_5^{p_5}(1 - y_5)^{q_5}}{(1 - y_3(1 - y_4 y_5))^{p_6+1}} = (-1)^{p_6} \sum_{k \in \mathbb{Z}} (-1)^k \left( \frac{k}{p_6} \right) y_3^{p_3-k}(1 - y_3)^{q_3-k} y_4^{p_4-k}(1 - y_4)^{q_4-k} y_5^{p_5-k}(1 - y_5)^{q_5-k},
\]
and applying the following formula with \( y = y_4 \) and \( y = y_5 \):
\[
\frac{1}{2\pi i} \oint_{|y| = \varepsilon} \frac{(1 - y)^q}{y^{k-p+1}} dy = \frac{1}{2\pi i} \oint_{|y| = \varepsilon} \sum_{m=0}^q \left( \frac{q}{m} \right) (-1)^m y^m \frac{dy}{y^{k-p+1}} = (-1)^{k-p} \left( \frac{q}{k-p} \right),
\]
we find out that
\[
I''(a) = (-1)^{p_4+p_5+p_6} \sum_{k \in \mathbb{Z}} (-1)^k \left( \frac{k}{p_6} \right) \left( \frac{q_4}{k-p_4} \right) \left( \frac{q_5}{k-p_5} \right)
\times \int_{[0,1]^3} y_1^{p_1}(1 - y_1)^{q_1} y_2^{p_2}(1 - y_2)^{q_2} y_3^{p_3}(1 - y_3)^{q_3-k} dy_1 dy_2 dy_3
\times \int_{[0,1]^3} (1 - y_3(1 - y_1 y_2))^{p_0+1}
\]
\[
= (-1)^{p_4+p_5+p_6} \sum_{k \in \mathbb{Z}} (-1)^k \left( \frac{k}{p_6} \right) \left( \frac{q_4}{k-p_4} \right) \left( \frac{q_5}{k-p_5} \right)
\times J_3(p_0, p_1, p_2, p_3 - k; q_1, q_2, q_3 - p_6 + k), \tag{22}
\]
where
\[
J_3 = J_3(p_0, p_1, p_2, p_3; q_1, q_2, q_3)
\]
\[
= \int_{[0,1]^3} y_1^{p_1}(1 - y_1)^{q_1} y_2^{p_2}(1 - y_2)^{q_2} y_3^{p_3}(1 - y_3)^{q_3} dy_1 dy_2 dy_3
\times \int_{[0,1]^3} (1 - y_3(1 - y_1 y_2))^{p_0+1}
\]
is the generalised Beukers integral \([4]\). As shown in \([17]\), the condition
\[
p_3 + q_3 = q_1 + q_2 \tag{24}
\]
on its parameters implies that \( J_3 = 2A\zeta(3) - B \); furthermore, by the results in \([24]\) we have
\[
A = A(p_0, p_1, p_2, p_3; q_1, q_2, q_3)
\]
\[
= (-1)^{p_0+p_1+p_2+p_3} \sum_{k \in \mathbb{Z}} \left( \frac{k}{p_0} \right) \left( \frac{k + q_3 - p_0}{p_3 + q_3 - p_0} \right) \left( \frac{q_1}{k-p_1} \right) \left( \frac{q_2}{k-p_2} \right),
\]
which combined with \((22)\) leads to another proof of formula \((17)\).

There are several other important consequences of the explicit reduction of \( I''(a) \) to \( J_3 \) given in \((22)\). The formula provides us with access to the arithmetic of \( I''(a) \) (for example, it ‘explains’ the appearance of the \( d_{2n} \) factor in \((9)\)). It also means that any \( \mathbb{Q} \)-linear relations between the integrals \( I(a) \) deduced from manipulating the integrand (for example, contiguity relations) also hold for \( I'(a), I''(a) \) and the rational coefficients \( Q(a), \hat{P}(a), P(a) \).
It is important to point out that the integrals (23), subject to (24), are equivalent to generalised cellular integrals on $\mathcal{M}_{0,n}$ [6]. The formulae described in this section are indicative of a general recursive structure between cellular integrals on $\mathcal{M}_{0,n}$ for different $n$, which should reflect the recursive structure of their boundary strata.

7. Group structure for $\zeta(5)$

In addition to the automorphism $i_1$ recorded in Section 3, the families $I(a)$ and $J(p; q)$ admit ‘hypergeometric’ transformations. These relate integrals within a given family up to a rational prefactor which may be expressed as a quotient of factorials. Such extended automorphisms play an important role in extracting arithmetic information about the coefficients $Q, \hat{P}, P$ in the decomposition (4) of $I(a)$.

First observe that the $_3F_2$-hypergeometric functions in representation (15) of $J(p; q)$ are symmetric with respect to permutations of their top parameters. Such permutations however affect the factor $p_1! q_1! p_2! q_1! q_2! \cdot$. A simple way to describe these permutations is through manipulations of the internal double integral over $y_1, y_2$ in (10):

$$
\int_{[0,1]^2} \frac{y_1^{p_1} (1 - y_1)^{q_1} y_2^{p_2} (1 - y_2)^{q_2}}{(1 - y_3(1 - y_1 y_2))^{p_0+1}} \, dy_1 \, dy_2
$$

$$
= \frac{1}{(1 - y_3)^{p_0+1} (p_1 + q_1 + 1)! (p_2 + q_2 + 1)!} \, _3F_2 \left( \begin{array}{c} p_0 + 1, p_1 + 1, p_2 + 1 \\ p_1 + q_1 + 2, p_2 + q_2 + 2 \end{array} \right| -y_3 \right).
$$

The symmetry of the parameters $p_1 + 1$ and $p_2 + 1$ in the $_3F_2$-representation implies that

$$
\int_{[0,1]^2} \frac{y_1^{p_1} (1 - y_1)^{q_1} y_2^{p_2} (1 - y_2)^{q_2}}{(1 - y_3(1 - y_1 y_2))^{p_0+1}} \, dy_1 \, dy_2
$$

$$
= \frac{q_1! q_2!}{(p_1 - p_2 + q_1)! (p_2 - p_1 + q_2)!} \, \int_{[0,1]^2} \frac{x_1^{p_0} (1 - x_1)^{p_1 - p_0 + q_1} x_2^{p_2} (1 - x_2)^{p_2 - p_1 + q_2}}{(1 - y_3(1 - y_1 y_2))^{p_0+1}} \, dy_1 \, dy_2;
$$

the symmetry of the parameters $p_0 + 1$ and $p_1 + 1$ leads to

$$
\int_{[0,1]^2} \frac{y_1^{p_1} (1 - y_1)^{q_1} y_2^{p_2} (1 - y_2)^{q_2}}{(1 - y_3(1 - y_1 y_2))^{p_0+1}} \, dy_1 \, dy_2
$$

$$
= (1 - y_3)^{p_1 - p_0} \frac{p_1! q_1!}{p_0! (p_1 - p_0 + q_1)!} \, \int_{[0,1]^2} \frac{x_1^{p_0} (1 - x_1)^{p_1 - p_0 + q_1} x_2^{p_2} (1 - x_2)^{q_2}}{(1 - y_3(1 - y_1 y_2))^{p_1+1}} \, dy_1 \, dy_2.
$$

Inserting these findings in the 5-fold integral (10) we see that the quantity

$$
\frac{J(p; q)}{p_1! p_2! q_1! q_2!}
$$

is invariant under the group (of order 6) generated by two involutions

$$
p_{12}: (p; q) \mapsto (p_0, p_2, p_1, p_3, p_4, p_5, p_6; p_1 + q_1 - p_2, p_2 + q_2 - p_1, q_3, q_4, q_5).
$$
and

\[ p_{01} : (p, q) \mapsto (p_1, p_0, p_2, p_3, p_4, p_5, p_6; p_1 + q_1 - p_0, q_2, p_1 + q_3 - p_0, q_4, q_5). \]

These transformations respect both conditions \([18]\) and \([19]\). With the reflection \([12]\) we also get another group of order 6 generated by two involutions

\[ p_{45} = i_1 p_{12} i_1 : (p, q) \mapsto (p_0, p_1, p_2, p_3, p_4, p_5, p_6; q_1, q_2, p_4 + q_4 - p_5, p_5 + q_5 - p_4) \]

and

\[ p_{50} = i_1 p_{01} i_1 : (p, q) \mapsto (p_0, p_1, p_2, p_3, p_4, p_5, q_1, q_2, p_5 + q_3 - p_0, q_4, q_5 + q_5 - p_4). \]

All these transformations are in the group \((i_1, h_{01}, h_{12})\) of order \(2 \times 3! \times 3! = 72\); they respect both \([18]\) and \([19]\) and keep the quantity

\[ \frac{J(p, q)}{p_1! p_2! p_4! p_5! q_1! q_2! q_4 q_5!} \]

invariant. This group can be regarded as a ‘trivial’ hypergeometric group as it only takes into account trivial symmetries of the underlying hypergeometric representation. It acts on both the 12-parameter family \(J(p, q)\) and 8-parameter family \(I(a)\).

There are more transformations available for the 8-parameter family \(I(a)\), although they can be observed more easily on the corresponding subfamily of \(J(p, q)\). Conditions \([18]\) imply that both integrals

\[ \frac{1}{2\pi i} \int_{-c_1 - i\infty}^{c_1 + i\infty} \frac{\Gamma(p_0 + 1 + s)\Gamma(p_1 + 1 + s)\Gamma(p_2 + 1 + s)\Gamma(p_3 + 2 + t + s)\times \Gamma(q_3 - p_0 - p_6 - 1 - t - s)\Gamma(-s)}{\Gamma(p_1 + q_1 + 2 + s)\Gamma(p_2 + q_2 + 2 + s)} \, ds \]

and

\[ \frac{1}{2\pi i} \int_{-c_2 - i\infty}^{c_2 + i\infty} \frac{\Gamma(p_4 + 1 + t)\Gamma(p_5 + 1 + t)\Gamma(p_6 + 1 + t)\Gamma(p_3 + 2 + s + t)\times \Gamma(q_3 - p_0 - p_6 - 1 - s - t)\Gamma(-t)}{\Gamma(p_4 + q_4 + 2 + t)\Gamma(p_5 + q_5 + 2 + t)} \, dt \]

in the double integration in \([16]\) are subject to Bailey’s transformation \([2\] \S 6.3, eq. (2)] (see also \([18] \text{eq. (4.7.1.3)}\)) and therefore can be cast as \(\gamma F_6\) very well-poised hypergeometric series:

\[ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(b + t)\Gamma(c + t)\Gamma(d + t)\Gamma(1 + a - e - f + t)\times \Gamma(1 + a - b - c - d - t)\Gamma(-t) \, dt}{\Gamma(1 + a - e + t)\Gamma(1 + a - f + t)\times \Gamma(1 + a - b - c)\Gamma(1 + a - e - f)\Gamma(1 + a - c - d)\Gamma(1 + a - b - d)\times \Gamma(1 + a - b - c - d)\Gamma(1 + a - e - f)\times \gamma F_6 \left( \begin{array}{cccccc} a, & 1 + \frac{1}{2}a, & b, & c, & d, & e, & f \\ 1, & 1 + a - b, & 1 + a - c, & 1 + a - d, & 1 + a - e, & 1 + a - f \end{array} \right| 1 \right). \]  

Note that it is this transformation which plays a crucial role in a hypergeometric interpretation \([24]\) of the Rhin–Viola group \([17]\) for the linear forms in 1 and \(\zeta(3)\).
If we choose

\[ a = p_2 + q_1 + q_2 + 2, \ b = p_0 + 1, \ c = p_3 + t + 2, \ d = p_2 + 1, \ e = p_2 - p_1 + q_2 + 1, \ f = q_1 + 1 \]

in (25), and then apply the same identity with \( d \) and \( e \) interchanged — this does not affect the \( \zeta_F \) series — we find out that

\[
\frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \Gamma(p_0 + 1 + s) \Gamma(p_1 + 1 + s) \Gamma(p_2 + 1 + s) \times \Gamma(p_3 + 2 + t + s) \Gamma(q_1 - p_0 - p_6 - 1 - t - s) \Gamma(-s) \frac{ds}{\Gamma(p_1 + q_1 + 1 + s) \Gamma(p_2 + q_2 + 2 + s)}
\]

\[
= \frac{p_1! p_2! (q_1 + q_2 - p_0)! \Gamma(q_1 + q_2 - q_1 - p_3 - t)}{q_2! (p_1 + q_1 - p_0)! (p_2 + q_2 - p_1)! \Gamma(p_1 + q_1 - p_3 - t)} \times \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \Gamma(p_0 + 1 + s) \Gamma(q_2 + 1 + s) \Gamma(p_2 + q_2 - p_1 + 1 + s) \Gamma(p_3 + 2 + t + s) \Gamma(p_1 - p_0 + q_1 - p_3 - 1 - t - s) \Gamma(-s) \frac{ds}{\Gamma(q_1 + q_2 + 2 + s) \Gamma(p_2 + q_2 + 2 + s)}
\]

Assuming now (19), that is, using \( p_1 + q_1 = p_3 \), and substituting into the double integral expression for \( J(p; q) \) we obtain

\[
J(p; q) = \frac{p_1! p_2! q_1! q_4! q_5!}{p_0! p_6! (p_3 - p_0)! (p_2 + q_2 - p_1)!} \times \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} ds \cdot \frac{\Gamma(p_0 + 1 + s) \Gamma(q_2 + 1 + s) \Gamma(p_2 + q_2 - p_1 + 1 + s) \Gamma(-s)}{\Gamma(q_1 + q_2 + 2 + s) \Gamma(p_2 + q_2 + 2 + s)}
\]

\[
\times \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt \cdot \frac{\Gamma(p_4 + 1 + t) \Gamma(p_5 + 1 + t) \Gamma(p_6 + 1 + t) \Gamma(-p_0 - 1 - t - s)}{\Gamma(p_4 + q_4 + 2 + t) \Gamma(p_5 + q_5 + 2 + t)}
\]

\[
\times \Gamma(p_3 + 2 + s + t) \Gamma(-p_0 - r - 1 - t - s)
\]

(after the shift \( t \mapsto t + r \) with \( r = q_1 + q_2 - p_3 = q_3 - p_6 \))

\[
= \frac{p_1! p_2! q_1! q_4! q_5!}{p_0! p_6! (p_3 - p_0)! (p_2 + q_2 - p_1)!} \times \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} ds \cdot \frac{\Gamma(p_0 + 1 + s) \Gamma(q_2 + 1 + s) \Gamma(p_2 + q_2 - p_1 + 1 + s) \Gamma(-s)}{\Gamma(q_1 + q_2 + 2 + s) \Gamma(p_2 + q_2 + 2 + s)}
\]

\[
\times \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt \cdot \frac{\Gamma(p_4 + r + 1 + t) \Gamma(p_5 + r + 1 + t) \Gamma(p_6 + r + 1 + t) \Gamma(-t)}{\Gamma(p_4 + q_4 + r + 2 + t) \Gamma(p_5 + q_5 + r + 2 + t)}
\]

\[
\times \Gamma(p_3 + r + 2 + s + t) \Gamma(-p_0 - r - 1 - t - s)
\]

\[
= \frac{p_1! p_2! q_1! q_4! q_5!}{p_0! p_6! (p_3 - p_0)! (p_2 + q_2 - p_1)!} \cdot \frac{p_0! p'_6! (q_1' + q_2' - p'_1)!}{q_1' q_2' q_4' q_5'} \cdot J(p'; q')
\]

\[
= \frac{p_2! q_3!}{p_6! (p_2 + q_2 - p_1)!} \cdot J(p'; q'),
\]
where

\((p': q') = (p_0, q_2, p_2 + q_2 - p_1, p_3 + r, p_4 + r, p_5 + r, p_6 + r; q_1, p_1, q_3 - r, q_4, q_5)\)

\[= (p_0, q_2, p_2 + q_2 - p_1, q_1 + q_2, p_4 - p_6 + q_3, p_5 - p_6 + q_3, q_1, p_1, p_6, q_4, q_5).\]

The identity can be interpreted as the invariance of

\[J(p; q)\]

under the involution \(h': (p; q) \mapsto (p'; q').\) A different manipulation with Bailey’s transformation \([25],\) more in line with the normalisation in \([24,\) Sect. 4], leads to a different hypergeometric involution

\[h: (p; q) \mapsto (q_2, p_1, p_2 - p_0 + q_2, p_3 - p_0 + q_2, p_4 - p_0 + q_2, p_5 - p_0 + q_2, p_6 - p_0 + q_2; q_1 - p_0 + q_2, p_0, q_3, q_4, q_5)\]

which keeps the quantity

\[J(p; q)\]

invariant.

The group \(\mathcal{G}\) generated by \(i_1, p_{01}, p_{12}\) and \(h\) (or by \(h'\) instead of the latter) acts perfectly on the 8-parameter set \((p; q)\) subject to the constraints \([18], [19].\) Because the 8-parameter set is in the one-to-one correspondence with the original parameter set \(a = (a_1, \ldots, a_8),\) the action of \(\mathcal{G}\) can be translated into one on the parameters \(a.\) We obtain

\[i_1: a \mapsto (a_5, a_4, a_3, a_2, a_1, a_7, a_6, -a_2 + a_4 - a_6 + a_7 + a_8),\]

\[p_{01}: a \mapsto (a_1, a_2, -a_2 + a_4 + a_5, a_2 + a_5 - a_5, a_5, a_6, a_7, -a_2 - a_3 + a_4 + a_5 + a_8),\]

\[p_{12}: a \mapsto (a_1, a_2, -a_2 + a_4 + a_8, a_2 + a_4 - a_8, -a_2 - a_3 + a_4 + a_5 + a_8,\]

\[a_2 + a_3 - a_4 + a_6 - a_8, a_7, a_8),\]

\[h: a \mapsto (a_1, a_2, a_3 + a_6 - a_8, a_4 - a_6 + a_8, a_5 + a_6 - a_8, a_8, -a_6 + a_7 + a_8, a_6),\]

\[h': a \mapsto (a_1, a_2, a_3 + a_4 + a_2 + a_3 - a_4 + a_6 - a_8,\]

\[a_2 - a_3 + a_4 + a_5 + a_8, -a_2 - a_3 + a_4 + a_5 + a_6 + a_7 + a_8, a_8).\]

It can be checked that the group is a permutation group on the 28 elements

\[h_i = a_i \ \text{for} \ i = 1, \ldots, 8,\]

\[h_9 = a_1 + a_2 - a_4, \ h_{10} = a_1 + a_5 - a_3, \ h_{11} = a_1 + a_8 - a_3,\]

\[h_{12} = a_2 + a_3 - a_5, \ h_{13} = a_2 + a_3 - a_8, \ h_{14} = a_3 + a_6 - a_8,\]

\[h_{15} = a_3 + a_4 - a_1, \ h_{16} = a_4 + a_5 - a_2, \ h_{17} = a_4 + a_8 - a_6,\]

\[h_{18} = a_4 + a_8 - a_2, \ h_{19} = a_5 + a_6 - a_8, \ h_{20} = a_7 + a_8 - a_6,\]

\[h_{21} = a_1 + a_2 + a_6 - a_4 - a_8, \ h_{22} = a_1 + a_7 + a_8 - a_3 - a_6,\]

\[h_{23} = a_2 + a_3 + a_6 - a_4 - a_8, \ h_{24} = a_2 + a_3 + a_6 - a_7 - a_8,\]

\[h_{25} = a_4 + a_5 + a_8 - a_2 - a_3, \ h_{26} = a_4 + a_7 + a_8 - a_2 - a_6,\]

\[h_{27} = a_4 + a_7 + 2a_8 - a_2 - a_3 - a_6, \ h_{28} = a_4 + a_5 + a_7 + a_8 - a_2 - a_3 - a_6.\]
The hyperplanes $h_j = 0$ for $j = 1, \ldots, 28$ are precisely orbits of $a_i = 0$ for $i = 1, \ldots, 8$ under the action of $\mathfrak{G}$. There is one additional hyperplane

$$a_1 + a_5 + a_7 + a_8 - a_2 - a_3 = 0$$

which is also preserved by this action. The analysis above implies that the quantity

$$\frac{I(\mathbf{a})}{\prod_{i \in \mathcal{F}} h_i!}, \quad \text{where } \mathcal{F} = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 14, 16, 18, 20, 23, 27, 28\}, \quad (27)$$

is invariant under $\mathfrak{G}$.

**Remark 3.** Notice that the convergence conditions on the integral $I(\mathbf{a})$ in the introduction, namely, the non-negativity of the seventeen quantities (3), are precisely $h_i \geq 0$ for $i \in \mathcal{F}$.

This invariance can alternatively be stated for the quantity

$$J(\mathbf{p}; \mathbf{q}) = \frac{p_1! p_2! p_4! p_5! q_1! q_2! q_3! q_4! q_5! (p_4 - p_0 + q_2)! (p_2 - p_0 + q_4)!}{\times (p_2 - p_0 + q_3)! (p_4 - p_0 + q_3)! (p_2 + p_4 - p_0 - p_6 + q_3)!} \times (q_2 - q_3 + q_4)! (q_2 - q_3 + q_4)! (q_4 - q_1 + q_3)!$$

provided that the parameters $(\mathbf{p}; \mathbf{q})$ are subject to the conditions (18), (19). Furthermore, the 28-element multiset $\mathbf{h}$ naturally splits into two multisets

$$\mathbf{h'} = \mathfrak{G} \mathbf{a}_1 = \{h_1, h_3, h_5, h_6, h_7, h_8, h_9, h_{10}, h_{11}, h_{14}, h_{16}, h_{18}, h_{19}, h_{20}, h_{21}, h_{22}, h_{23}, h_{25}, h_{26}, h_{27}, h_{28}\}$$

of size 21 and

$$\mathbf{h''} = \mathfrak{G} \mathbf{a}_2 = \{h_2, h_4, h_{12}, h_{13}, h_{15}, h_{17}, h_{24}\}$$

of size 7, where $\mathbf{h'} \cap \mathbf{h''} = \emptyset$. Each of these sets is individually acted upon by the above symmetry group $\mathfrak{G}$. Also notice that

$$\{h_i : i \in \mathcal{F}\} \cap \mathbf{h'} = \{h_1, h_3, h_5, h_6, h_7, h_9, h_{10}, h_{11}, h_{14}, h_{16}, h_{18}, h_{20}, h_{23}, h_{27}, h_{28}\}$$

and

$$\{h_i : i \in \mathcal{F}\} \cap \mathbf{h''} = \{h_2, h_4\}$$

have size 15 and 2, respectively (while their complements in $\mathbf{h'}$ and $\mathbf{h''}$ have size 6 and 5).

**Remark 4.** It is a good moment to bring some related numerology to the reader’s attention. Firstly, we recognise the sizes 28, 21 and 7 of the multisets $\mathbf{h}$, $\mathbf{h'}$ and $\mathbf{h''}$ as being $\binom{7}{2}$, $\binom{7}{3}$ and $\binom{7}{1}$, respectively. Secondly, the order of the group $\mathfrak{G}$ is $7! = 5040$, which is the order of the symmetric group $\Sigma_7$ on 7 letters. The precise explanation for this will be made apparent in the next sections, as we shall show that indeed $\mathfrak{G}$ is naturally isomorphic to $\Sigma_7$. Note that the group $\Sigma_7$ is also isomorphic to the Weyl group $W(A_6)$ of the root system $A_6$, which is consistent with the known hypergeometric groups for rational approximations to $\zeta(2)$ (as in [16]), to $\zeta(3)$ (as in [17]) and to $\zeta(4)$ (as in [13]): they are equal to $|W(A_6)| = 120$, $|W(D_5)| = 1920$ and $|W(E_6)| = 51840$, respectively. Precisely how this pattern of symmetry groups extends to more general cellular integrals (as we expect) remains a mystery.
Now recall that by the results in Section 5 the group $\mathcal{G}$ acts not only on the integrals $I(a)$ but also on their parts $I'(a) \in \mathbb{Q}(\zeta(5) + \mathbb{Q})$ and $I'(a) \in \mathbb{Q}(\zeta(3) + \mathbb{Q})$, where $I(a) = 2I'(a) + 4I''(a)\zeta(2)$. As discussed in Remark 4 it is in principle possible to prove, with considerable effort, the experimental observation:

$$d_{m_1}d_{m_2}d_{m_3}d_{m_4}d_{m_5}I'(a \cdot n) \in \mathbb{Z}\zeta(5) + \mathbb{Z},$$

(28)

where $m_1 \geq m_2 \geq m_3 \geq m_4 \geq m_5$ are five consecutive maxima of the corresponding 28-element multiset $h_1, \ldots, h_{28}$ and $d_N$ denotes the least common multiple of $1, 2, \ldots, N$. Because the latter multiset is invariant under $\mathcal{G}$, we conclude from (28) that

$$d_{m_1}d_{m_2}d_{m_3}d_{m_4}d_{m_5}I'(ga \cdot n) \in \mathbb{Z}\zeta(5) + \mathbb{Z} \quad \text{for any } g \in \mathcal{G};$$

in particular,

$$\prod_{i \in F} (gh_i)! \cdot d_{m_1}d_{m_2}d_{m_3}d_{m_4}d_{m_5}I'(a \cdot n)$$

$$= d_{m_1}d_{m_2}d_{m_3}d_{m_4}d_{m_5}I'(ga \cdot n) \in \mathbb{Z}\zeta(5) + \mathbb{Z} \quad \text{for any } g \in \mathcal{G}.$$

This means that if we take

$$\nu_p = \max_{g \in \mathcal{G}} \text{ord}_p \left( \prod_{i \in F} (gh_i)! \right)$$

and consider the quantity

$$\Phi_n = \Phi_n(a) = \prod_{p > \sqrt{m_1 n}} p^{\nu_p},$$

then

$$\Phi_n^{-1} \cdot d_{m_1}d_{m_2}d_{m_3}d_{m_4}d_{m_5}I'(a \cdot n) \in \mathbb{Z}\zeta(5) + \mathbb{Z}. \quad (30)$$

Furthermore, we can compute the limit of $|I'(a \cdot n)|^{1/n}$ as $n \to \infty$ using the asymptotics of $J(p; q)$ discussed in Section 5.

**Remark 5.** The above choice of $m_1, \ldots, m_5$ may be not optimal. It is plausible to expect that, for an appropriate value of $\ell \in \{1, \ldots, 5\}$, one can take it as follows: $m_1 \geq \cdots \geq m_{\ell}$ are successive maxima of the 21-element multiset $h'$ and $m_{\ell+1} \geq \cdots \geq m_5$ are successive maxima of the 7-element multiset $h''$. Notice that for any such $\ell$, the set $\{m_1, \ldots, m_5\}$ is preserved under the action of $\mathcal{G}$.

8. Automorphisms of the asymptotic polynomials

Consider the discriminant of the cubic polynomial defined in (20), viewed as a polynomial function of the parameters $a_1, \ldots, a_8$. Upon dividing out linear factors of the form $a_2^2, a_3^2, (a_2 + a_3 - a_4 - a_5 - a_8)^2$, one is left with an irreducible polynomial $D \in \mathbb{Q}[a_0, \ldots, a_8]$ which is homogenous of degree 16. We wish to compute the subgroup $\text{Aut}_D \leq \text{GL}_8(\mathbb{Z})$ of linear automorphisms acting on the parameters $a_0, \ldots, a_8$ which preserves the equation $D = 0$.

To this end, we consider hyperplanes $H$, defined by a homogeneous linear form in the parameters $a_1, \ldots, a_8$ with the property that the restriction of the discriminant $D|_H$ to $H$ factorizes into a product $D_6D_5^2$, where $D_6, D_5$ are irreducible of degree
ON CELLULAR RATIONAL APPROXIMATIONS TO $\zeta(5)$

6 and 5. We find that the hyperplanes of this form are the 28 hyperplanes $h_i = 0$ with $h_1, \ldots, h_{28}$ listed in [26], as well as the exceptional hyperplane

$$a_5 + a_7 + a_8 + a_1 - a_2 - a_3 = 0.$$ 

Since the action of $\text{Aut}_D$ is linear, it permutes this set of 29 hyperplanes. In order to understand this action, we consider the restriction $D|_{H_i \cap H_j}$ to all pairs of such hyperplanes $H_i, H_j$. We find that for certain pairs $H_i, H_j$ (which we call of the ‘first kind’), we have a factorisation of the form

$$D|_{H_i \cap H_j} = (P_1 P_1' P_3 P_3')^2$$

where $P_1, P_1'$ are linear and $P_3, P_3'$ are of degree 3. For other pairs, another type of factorisation may also occur, whereby $D|_{H_i \cap H_j}$ has 5 linear factors of multiplicity two, and a single irreducible component of degree 6. A third type of factorisation occurs if and only if one of $H_i, H_j$ is the exceptional hyperplane.

The action of $\text{Aut}_D$ thus preserves the zero loci of the set of linear forms $h_1, \ldots, h_{28}$ together with the data of which pairs $(h_i, h_j)$ are of the first kind. This data may be encoded by a graph $G$, with one vertex for every hyperplane $H_i = V(h_i)$, and with an edge between vertices $H_i$ and $H_j$ if and only if $(h_i, h_j)$ are of the first kind. It is depicted in Figure 1, where the vertex $H_i$ is denoted by $i$.

![Figure 1. The graph $G$.](image-url)

One finds that this graph has 168 edges and that each vertex has degree exactly 12. Furthermore, one can easily check by computer that its automorphism group is of order $8!$. On observing that $168 = 3 \binom{8}{3}$ and $28 = \binom{8}{2}$, one is led to suspect that $G$ may be isomorphic to the highly symmetric graph $G_8$ defined as follows:

Let $K_n$ denote the complete graph with $n \geq 2$ vertices. Define a new graph $G_n$ whose vertices are given by the set of edges $\{i, j\}$ of $K_n$, with an edge between $\{i, j\}$ and $\{k, \ell\}$ if and only if $\{i, j, k, \ell\}$ is a set with 3 elements. Equivalently, the graph
$G_n$ may be constructed by placing a single vertex along every edge of the complete graph $K_n$, and connecting every pair of such vertices which lie on the same face of $K_n$. This construction is illustrated in Figure 2.

![Figure 2](image)

**Figure 2.** The complete graph $K_4$ on 4 vertices labelled $s_0, \ldots, s_3$ is depicted on the left as a tetrahedron. The two front faces have been subdivided into triangles. Subdividing all faces leads to the graph $G_4$ with 6 vertices and 12 edges shown on the right.

Once one suspects that $G$ might be isomorphic to $G_8$, it takes only a little detective work to find an isomorphism. Indeed, if we label the vertices of $G_8$ by $s_0, s_1, \ldots, s_7$ we find that an explicit isomorphism $G \cong G_8$ is given by the following table:

|     | $s_0$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| $s_0$ | 15    | 2     | 4     | 12    | 13    | 17    | 24    |
| $s_1$ | 1     | 9     | 10    | 11    | 21    | 22    |
| $s_2$ | 3     | 16    | 18    | 14    | 26    |
| $s_3$ | 5     | 8     | 23    | 20    |
| $s_4$ |       | 25    | 19    | 28    |
| $s_5$ |       |       | 6     | 27    |
| $s_6$ |       |       |       | 7     |

The table means, for example, that the vertex of $G$ assigned to hyperplane $h_1$ corresponds to the vertex $\{s_1, s_2\}$ of $G_8$, and the vertex assigned to $h_{27}$ corresponds to $\{s_5, s_7\}$.

From this it is clear that the automorphism group $\text{Aut}(G) \cong \text{Aut}(G_8)$ is the symmetric group of order $8!$ acting via permutations on $S = \{s_0, \ldots, s_7\}$. Furthermore, we discover that the hypergeometric group $\mathcal{G}$ is precisely the subgroup of order $7!$ which fixes the vertex $s_0$. In fact, we find with a little more calculation that

$$\mathcal{G} = \text{Aut}(G) \cap \text{GL}_8(\mathbb{Z})^+$$

where $\text{GL}_8(\mathbb{Z})^+$ denotes the subgroup of linear transformations $g \in \text{GL}_8(\mathbb{Z})$ which are positive. This means the following: a linear form $\ell(a_1, \ldots, a_8)$ is called positive if in the symmetric case $a_1 = a_2 = \cdots = a_8 = n$, where $n > 0$, it satisfies $\ell(n, \ldots, n) > 0$. An element $g$ is positive if it sends a positive defining equation...
of \( h_i \) to a positive linear form. The above calculation implies that the group \( G \) is largest possible and that we have indeed exhausted all symmetries of \( \Omega \).

Nevertheless, there exist elements in \( \text{Aut}_D \) which are not positive. For example, by considering a permutation which does not fix \( s_0 \), we are led to the automorphism

\[(a_1, a_2, a_4) \mapsto (-a_2, -a_1, a_4 - a_1 - a_2),\]

which fixes all other \( a_i \), preserves the exceptional hyperplane, acts upon the set of hyperplanes \( h_i = 0 \), for \( 1 \leq i \leq 28 \), and preserves the discriminant locus \( D = 0 \). It is very unclear whether the existence of this curious automorphism has any implications for the arithmetic of linear forms in \( \zeta(5) \). In any case, we check that this non-positive automorphism, together with the group \( G \), generates the full group of symmetries of the graph \( G \) of order 8!.

9. Duality

A general duality was proven for \emph{totally symmetric} cellular integrals in [6]. A natural definition of duality for generalised cellular integrals is as follows.

Let \( N \geq 5 \) and let \( \pi \) be a convergent permutation on \( \{1, \ldots, N\} \) in the sense of [6]. In that paper, the generalised rational cellular function was defined by

\[ f_{\pi}(a, b) = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} \frac{(z_{i} - z_{i+1})^{a_{i,i+1}}}{(z_{\pi_{i}} - z_{\pi_{i+1}})^{b_{\pi_{i},\pi_{i+1}}}} \]

where the multi-indices \( a, b \) are subject to homogeneity equations

\[ a_{i-1,i} + a_{i,i+1} = b_{\pi_{i-1},\pi_{i}} + b_{\pi_{i},\pi_{i+1}} \]

for all \( i, j \) such that \( \pi_{j} = i \), and where indices are taken modulo \( N \). The associated generalised cellular integral is defined, when it converges, by

\[ I_{\pi}(a, b) = \int_{0 \leq t_1 \leq \cdots \leq t_{N-3} \leq 1} f_{\pi}(a, b) \omega_{\pi} \quad (31) \]

where we set \( z_1 = 0, z_2 = t_1, \ldots, z_{N-2} = t_{N-3}, z_{N-1} = 1 \) and \( z_N = \infty \), and \( \omega_{\pi} \) is the cellular integrand defined in \textit{loc. cit.}. In the symmetric case, when all parameters \( a \) and \( b \) are equal to a non-negative integer \( n \), then one retrieves the ‘basic’ (or totally symmetric) cellular integrals:

\[ I_{\pi}(n) = \int_{0 \leq t_1 \leq \cdots \leq t_{N-3} \leq 1} (f_{\pi})^{n} \omega_{\pi} \]

where \( f_{\pi} \) is defined from \( f_{\pi}(a, b) \) by setting all parameters \( a, b \) equal to 1.

The dual configuration was defined in \textit{loc. cit.} to be the (equivalence class) of the inverse permutation \( \pi^{-1} \). It is convergent if and only if \( \pi \) is. It was shown in the same paper using Poincaré–Verdier duality that the family of basic cellular integrals \( I_{\pi^{-1}}(n) \) are dual to the \( I_{\pi}(n) \) and satisfy the dual recurrence relation. It would be interesting to prove a similar result for generalised cellular integrals.
Here, then, is a definition of duality in the setting of generalised cellular integrals. First of all define the rational function dual to \( f_{x}(a, b) \) by

\[
f_{x}^{\vee}(a, b) = \prod_{i \in \mathbb{Z}/N\mathbb{Z}} \frac{(z_i - z_{i+1})^{b_{x,i} \pi_{i+1}}}{(z_{\pi_{i}} - z_{\pi_{i+1}})^{a_{i,i+1}}}.\]

It automatically satisfies the same homogeneity equations. A candidate for the dual of the generalised cellular integral \((31)\) is therefore

\[
I_{\pi}(a, b)^{\vee} = \int \cdots \int f_{x}^{\vee}(a, b) \omega_{\pi^{-1}}
\]

which, by construction, is a family of generalised cellular integrals for \( \pi^{-1} \). In the totally symmetric case, this reduces to the duality on basic cellular integrals.

Let us apply this to the permutation \( \pi: (1, 2, 3, 4, 5, 6, 7, 8) \mapsto (8, 2, 4, 1, 7, 5, 3, 6) \) which represents the configuration called \( s_{8} \pi_{8} \) in \([6]\). The associated generalised cellular integral is precisely that defined in the introduction upon setting

\[
a_{12} = a_{1}, \quad a_{23} = a_{2}, \quad \ldots, \quad a_{67} = a_{6}, \quad a_{18} = a_{7}, \quad a_{78} = a_{5} + a_{1} - a_{3}
\]

\[
b_{68} = a_{5} + a_{6} - a_{8}, \quad b_{17} = a_{1} + a_{2} + a_{6} - a_{4} - a_{8}, \quad b_{28} = a_{1} + a_{7} + a_{8} - a_{3} - a_{6}
\]

where the remaining expressions for the \( b_{ij} \) were given in \([2]\). These equations represent a solution to the homogeneity conditions stated above. The inverse permutation is \( \pi^{-1}: (1, 2, 3, 4, 5, 6, 7, 8) \mapsto (4, 2, 7, 3, 6, 8, 5, 1) \). To bring the resulting family of integrals \((32)\) into more convenient form, we apply dihedral symmetries (more precisely, \( \sigma^{2} \) followed by the involution \( i_{1} \)) to obtain:

\[
\int \cdots \int \frac{t_{1}^{-a_{2}+a_{4}-a_{6}+a_{7}+a_{8}}(t_{2} - t_{1})^{a_{1}+a_{2}-a_{4}+a_{6}-a_{8}}(t_{3} - t_{2})^{-a_{2}+a_{3}+a_{4}+a_{5}+a_{8}} \times (t_{4} - t_{3})^{a_{2}+a_{3}-a_{5}+a_{6}-a_{8}} dt_{1} \cdots dt_{5}}{(1 - t_{1})^{a_{2}+1}(1 - t_{2})^{a_{1}-a_{3}+a_{5}+1}(t_{5} - t_{2})^{a_{6}+1}(t_{5} - t_{3})^{a_{5}+1}t_{4}^{a_{4}+1}t_{4}^{a_{3}+1}},
\]

which are generalised cellular integrals for the configuration \( s_{8} \pi_{8} = (8, 2, 5, 1, 6, 4, 7, 3) \).

The expected duality between \((1)\) and \((33)\) should have many consequences including a duality between the corresponding contiguity relations. In particular, for families of the form \( I(an) \) with \( a = (a_{1}, \ldots, a_{8}) \) fixed, the coefficients of zeta values in the two integrals should satisfy recurrence relations with respect to \( n \) which are dual to each other. Secondly, their asymptotics should be reciprocal. Thirdly, the group of symmetries of both integrals should be related to each other. The last two properties are discussed in Section \([10]\) below.

The dual family \((33)\) admits a hypergeometric interpretation. Indeed, it was shown in \([6]\) that a subfamily of \( s_{8} \pi_{8} \) is equivalent to the very-well-poised hypergeometric family of \((3)\) (see also \([7, 9, 12]\) for further work and discussion) which gives linear forms in \( 1, \zeta(3), \zeta(5) \) but depends on fewer parameters. In fact, as we shall now explain, this type of result may be extended to the fully general family \((33)\).
To this end, consider the very-well-poised hypergeometric series

$$F_k(b) = F_k(b; b_0, \ldots, b_k)$$

$$= \frac{(b_0 + 1)! \prod_{j=1}^{k} b_j!}{\prod_{j=1}^{k} (b_0 - b_j + 1)!} \cdot k + 2 \Gamma \left( b_0 + 2, \frac{1}{2} b_0 + 2, b_1 + 1, \ldots, b_k + 1 \right) (-1)^{k+1}$$

$$= \sum_{\mu=0}^{\infty} \frac{(b_0 + 2 \mu + 2)}{\mu!} \frac{\Gamma(b_0 + \mu + 2) \prod_{j=1}^{k} \Gamma(b_j + \mu + 1)}{\Gamma(b_0 - b_j + \mu + 2)} (-1)^{(k+1)\mu},$$

(34)

where \(b_0, b_1, \ldots, b_k\) are non-negative integers satisfying \(b_0 \geq 2b_i\) for \(1 \leq i \leq k\), and \(k \geq 4\). These series are known to represent \(Q\)-linear forms in 1 and zeta values \(\zeta(i)\) where \(2 \leq i \leq k - 2\) and \(i \equiv k \mod 2\) by [23, Sect. 3].

The case for general \(k\) will be discussed elsewhere, but here we shall only focus on the case \(k = 7\) which represents linear forms in \(Q + Q\zeta(3) + Q\zeta(5)\). Consider the arithmetic renormalisation of \(\tilde{F}_7(b)\) defined by:

$$F_7(b) = \frac{(b_0 - b_1 - b_6)! (b_0 - b_1 - b_7)! (b_0 - b_2 - b_7)! \times (b_0 - b_3 - b_5)! (b_0 - b_4 - b_5)! (b_0 - b_4 - b_6)!}{b_2! b_3!} \cdot \tilde{F}_7(b)$$

(35)

This series is intrinsically linked with the integral \(I(a)\) via results in the literature connecting very-well-poised series [34] with multiple integrals (known as integrals of Sorokin type and of Vasilyev type). In more detail, by generating an appropriate change of variables out of automorphisms of \(\mathcal{M}_{0,k}\) for \(4 \leq k \leq 8\) akin to [6, proof of Proposition 7.2], and combining with [20, Theorem 2] and [23, Theorem 3] we can identify the 8-parameter families \(F_7(b)\) with \(\tilde{\tau}_8\) via

$$F_7(b) = \int_{0}^{1} \cdots \int_{0}^{1} \frac{t_1^{b_0 - b_2 - b_7} (t_2 - t_1)^{b_0 - b_1 - b_6} (t_3 - t_2)^{b_0 - b_4 - b_5} (t_4 - t_3)^{b_5} \times (t_5 - t_4)^{b_0 - b_3 - b_5} (1 - t_5)^{b_0 - b_4 - b_6} dt_1 \cdots dt_5}{(1 - t_1)^{b_0 - b_6 - b_7 + 1} (1 - t_2)^{b_0 - b_1 - b_4 + 1} (t_5 - t_2)^{b_0 - b_5 - b_6 + 1} \times (t_5 - t_3)^{b_0 - b_4 - b_1 + 1} t_3^{b_0 - b_2 - b_3 + 1} t_4^{b_0 - b_2 - b_3 + 1}}.$$

Using the following formulae (which will follow naturally from the symmetric parametrisation considered in the following section)

$$b_0 = a_2 + a_3 + a_4, \quad b_1 = -a_1 + a_3 + a_4, \quad b_2 = a_2, \quad b_3 = a_4, \quad b_4 = a_2 + a_3 - a_5, \quad b_5 = a_2 + a_3 - a_8, \quad b_6 = a_4 - a_6 + a_8, \quad b_7 = a_2 + a_3 + a_6 - a_7 - a_8$$

we see that \(F_7(b)\) coincides with [33]. This concludes the interpretation of the family of cellular integrals [1] as being dual to very-well-poised hypergeometric series, and points to a general theory which we plan to discuss elsewhere.
10. Symmetric parameters and their duals

Based on the graph-theoretic interpretation of the group $\mathcal{G}$ and the table in Section \[\text{[7]}\] we are emboldened to introduce new parameters $s_0, s_1, \ldots, s_7$:

$$
\begin{align*}
a_1 &= s_1 + s_2, & a_2 &= s_0 - s_2, & a_3 &= s_2 + s_3, & a_4 &= s_0 - s_3, \\
a_5 &= s_3 + s_4, & a_6 &= s_5 + s_6, & a_7 &= s_6 + s_7, & a_8 &= s_3 + s_5.
\end{align*}
$$

We find that with this parametrization, the group $\mathcal{G}$ acts on $\mathbf{a}$ via the group $\mathcal{G}_7$ which permutes $s_1, \ldots, s_7$ and fixes $s_0$. The inverse transformation

$$
\begin{align*}
s_0 &= \frac{1}{2}(a_2 + a_3 + a_4), & s_1 &= a_1 + \frac{1}{2}(a_2 - a_3 - a_4), & s_2 &= \frac{1}{2}(a_3 + a_4 - a_2), \\
s_3 &= \frac{1}{2}(a_2 + a_3 - a_4), & s_4 &= a_5 + \frac{1}{2}(a_4 - a_2 - a_3), & s_5 &= a_8 + \frac{1}{2}(a_4 - a_2 - a_3), \\
s_6 &= a_6 - a_8 + \frac{1}{2}(a_2 + a_3 - a_4), & s_7 &= a_7 + a_8 - a_6 + \frac{1}{2}(a_4 - a_2 - a_3)
\end{align*}
$$

shows that, in general, the symmetric parameters $s_i$ are half-integers.

With these new parameters, we find that the multiset $\mathbf{h}$ of 28 hyperplanes $h_1, \ldots, h_{28}$ is precisely given by the union of two multisets

$$
\mathbf{h}' = \{s_i + s_j : 1 \leq i < j \leq 7\} \quad \text{and} \quad \mathbf{h}'' = \{s_0 - s_i : 1 \leq i \leq 7\}
$$

corresponding to the two orbits under the action of $\mathcal{G} \cong \Sigma_7$. Furthermore, if we combine all entries of $\mathbf{h}$ into a symmetric 7 × 7 matrix

$$
H = (H_{ij})_{1 \leq i,j \leq 7}, \quad \text{where} \quad H_{ij} = \begin{cases} 
  s_0 - s_i & \text{if } i = j, \\
  s_i + s_j & \text{if } i \neq j,
\end{cases}
$$

then this action is identified with simultaneous row-column permutations of $H$.

Armed with the symmetric parameters $s_i$, we are now in a position to verify that the cellular integrals \[\text{[1]}\] have isomorphic symmetry groups to their duals \[\text{[33]}\], and have reciprocal asymptotics.

It is known in general that the very-well-poised hypergeometric series $\tilde{\mathbb{F}}_k(b)$ in \[\text{[34]}\] is invariant under any permutation of the parameters $b_1, \ldots, b_k$. This symmetry group can be interpreted as an action of a permutation group on $k$ elements acting by simultaneous row-column permutations of the $k \times k$ symmetric matrix

$$
B = (B_{ij})_{1 \leq i,j \leq k}, \quad \text{where} \quad B_{ij} = \begin{cases} 
  b_i & \text{if } i = j, \\
  b_0 - b_i - b_j & \text{if } i \neq j.
\end{cases}
$$

In the case of interest, namely $k = 7$, we may identify this 7 × 7 matrix $B$ with \[\text{[36]}\], and thus identify the symmetry group $\mathcal{G}$ for \[\text{[1]}\] with that of its dual by setting

$$
\mathbf{b} = (2s_0; s_0 - s_1, s_0 - s_2, \ldots, s_0 - s_7).
$$

This change of variables is equivalent to the equations for $b_0, \ldots, b_7$ in terms of $a_1, \ldots, a_8$ stated at the very end of Section \[\text{[9]}\]. Thus the expected duality between \[\text{[1]}\] and \[\text{[33]}\] is indeed verified on the level of symmetry groups.
Returning now to the original normalisation (35) of well-poised hypergeometric series \( F_7(b) \), the above can equivalently be expressed by saying that

\[
F_7(b) \cdot \prod_{i \in F} h_i!
\]

is invariant under \( \mathfrak{S} \), for the 17-element index set \( F \) defined in eq. (27).

This change in normalisation enables us to compare the asymptotics of (1) with that of its dual. More precisely, it is known that there are three asymptotic quantities \( \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3 \in \mathbb{R} \) assigned to the behaviour of \( F_7(bn) \) (see, e.g., [21, 23]) as \( n \to \infty \), which are related to the real numbers (21). With appropriate labels (for example, setting \( \hat{\lambda}_3 = \lim_{n \to \infty} \frac{F_7(bn)}{n} \)) they satisfy

\[
\hat{\lambda}_1 = \lambda_1^{-1}, \quad \hat{\lambda}_2 = \lambda_2^{-1}, \quad \hat{\lambda}_3 = \lambda_3^{-1}
\]

for any fixed choice of \( a \).

The above analysis gives good reasons to expect that the symmetries and duality between \( 8\pi^\vee \) and \( 8\pi_8 \) (namely, of cellular integrals (1) and (33)) extends to general \( k \)-fold cellular integrals. This is bolstered by an analysis of the (known!) cases \( k = 4, 5 \), which correspond to (self-dual) rational approximations to \( \zeta(2) \) and \( \zeta(3) \).

11. Proof of Theorem II

Let us look at a concrete example \( a = (8, 16, 10, 15, 12, 16, 18, 13) \) corresponding to the following matrix (36):

\[
\begin{pmatrix}
17 & 8 & 9 & 10 & 11 & 12 & 13 \\
8 & 16 & 10 & 11 & 12 & 13 & 14 \\
9 & 10 & 15 & 12 & 13 & 14 & 15 \\
10 & 11 & 12 & 14 & 14 & 15 & 16 \\
11 & 12 & 13 & 14 & 13 & 16 & 17 \\
12 & 13 & 14 & 15 & 16 & 12 & 18 \\
13 & 14 & 15 & 16 & 17 & 18 & 11
\end{pmatrix}
\]

The ordered version of the corresponding 28-multiset \( h \) is

\[
\{8, 9, 10, 11, 11, 11, 11, 12, 12, 12, 12, 12, 13, 13, 13, 13, 13, 14, 14, 14, 14, 15, 15, 15, 16, 16, 16, 16, 17, 17, 18\},
\]

so that \( m_1 = 18, m_2 = m_3 = 17 \) and \( m_4 = m_5 = 16 \). (If the refined estimate on the denominators were known, as discussed in Remark 5, then in view of

\[
h' = \{8, 9, 10, 11, 11, 12, 12, 12, 13, 13, 13, 13, 14, 14, 14, 15, 15, 16, 16, 16, 17, 17, 18\},
\]

the only way that it could coincide with the above selection \( m = \{18, 17, 17, 16, 16\} \) would be if \( \ell = 3 \) or \( 4 \) in the notation of that remark. If the true value of \( \ell \) in this case were to differ from 3 and 4, there would be a small potential gain for the worthiness of our approximations to \( \zeta(5) \).) The orbit of the vector \( a \) under the
group $\mathcal{G}$ is of full order 5040; by computation, the gain in the denominator coming from the factorial prefactors in the action of $\mathcal{G}$ is
$$\lim_{n \to \infty} \frac{\log \Phi_n}{n} = 34.39425186 \ldots .$$

For the asymptotics, we find that
$$\lim_{n \to \infty} \frac{\log |I(a_n)|}{n} = -66.05784567 \ldots,$$
$$\lim_{n \to \infty} \frac{\log |I'(a_n)|}{n} = \lim_{n \to \infty} \frac{\log |Q_n \zeta(5) - P_n|}{n} = -31.55296934 \ldots,$$
$$\lim_{n \to \infty} \frac{\log Q_n}{n} = 85.08768883 \ldots.$$

Because
$$-31.55296934 \ldots + 18 + 17 + 16 + 16 - 34.39425186 \ldots > 0,$$
we cannot make any conclusions about the irrationality of $\zeta(5)$. However, in the notation
$$C_0 = \lim_{n \to \infty} \frac{\log |Q_n \zeta(5) - P_n|}{n}, \quad C_1 = \lim_{n \to \infty} \frac{\log Q_n}{n} > C_0,$$
$$C_2 = m_1 + \cdots + m_5 - \lim_{n \to \infty} \frac{\log \Phi_n}{n},$$
we can apply the calculation from Section 2 with $c_0 = C_0 + C_2, c_1 = C_1 + C_2$ to conclude that the worthiness of our approximations is
$$\gamma(a) = \frac{C_1 - C_0}{C_1 + C_2} = 0.86597135 \ldots .$$

This establishes the result in Theorem 1.

12. (In)Conclusive Comments

There are still some mysteries remaining behind the true arithmetic of the integrals (1). We have observed numerically that the exponents $\nu_p$ in (29) extracted from the group $\mathcal{G}$ action are not always optimal (although any losses are insignificant). For example, with the choice $a = (15, 20, 16, 14, 17, 16, 20)$ (which leads to a slightly worse worthiness exponent $\gamma = 0.85163139 \ldots$) we obtain $\nu_p = 3$ for the primes $p$ satisfying $\frac{1}{19} < \{n/p\} < \frac{1}{18}$; experimentally, we find (for $n$ up to 40) that the inclusions (30) remain valid if one replaces this exponent by 4 in the definition of $\Phi_n(a)$. This additional gain will increase the worthiness in this case (very moderately) to 0.85665016\ldots . Such arithmetic losses\footnote{No losses are detected for our choice $a = (8, 16, 10, 15, 12, 16, 18, 13)$ in Section 11.} do not mean that there exists in reality a larger group of transformations acting on the integrals $I(a)$; as explained in Section 8, the group $\mathcal{G}$ is exhaustive. Nevertheless, we expect the extra savings to come from different integral representations, each possessing their own arithmetic. Such a phenomenon has been observed for the simultaneous rational approximations to $\zeta(2)$ and $\zeta(3)$ in [26]. But what are those other representations? The machinery
in \([6]\) allows one to generate a very large number of alternative forms of the integral \([1]\), which are slightly more involved but which potentially have arithmetic of their own. For example, a change of variables brings the integral to the form

\[
I(a) = \int \cdots \int_{[0,1]^5} \frac{x_1^{a_1}(1-x_1)^{a_1}x_2^{a_2}(1-x_2)^{-a_2-a_3+a_4-a_6+a_7+2a_8}x_3^{a_3-a_4+a_8}}{(1-x_2x_3x_4x_5)^{a_1-a_3}} (1-x_1x_2)^{a_1-a_3-a_6+a_7+a_8} (1-x_2x_3)^{a_8} \times (1-x_3x_4)^{-a_2-a_3+a_4+a_6+a_8} (1-x_4x_5)^{a_5+a_6-a_8} \times \frac{x_2x_3x_4 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5}{(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_5)},
\]

which is similar to \([8]\) except for the extra factor \((1-x_2x_3x_4x_5)^{a_1-a_3}\) in the denominator. Clearly, such representations \([37]\) are available for every \(I(\mathfrak{g}a)\), where \(\mathfrak{g} \in \mathfrak{G}\), and they do not degenerate to the form \([8]\) if the exponent of \(1-x_2x_3x_4x_5\) is nonzero.

The central idea of this note was to present a new approach to the elimination of parasitic terms amongst small linear forms in zeta values by viewing them as simultaneous approximations. The price to pay was that one must pass to the subleading asymptotic. The example we have studied here is by no means isolated, and extends to the case of 7-fold and higher-dimensional cellular integrals. By way of example, consider the cellular integral corresponding to \((10, 2, 4, 1, 6, 3, 8, 5, 9, 7)\) on \(\mathcal{M}_{0,10}\), referred to as ‘vanishing in the middle’ in \([6, \text{Sect. 10.2.6, Example (4)}]\). Experimentally, it underlies a motive of rank 4 with semi-simple pieces \(\mathbb{Q}(0), \mathbb{Q}(-2), \mathbb{Q}(-5), \mathbb{Q}(-7)\). The totally symmetric version of this family is given by the integrals

\[
I_n = \int \cdots \int_{0<t_1<\cdots<t_7<1} \frac{(t_2-t_1)(t_3-t_2)(t_4-t_3)(t_5-t_4)(t_6-t_5)(t_7-t_6)}{(t_1-t_3)t_3t_5(t_5-t_2)(t_7-t_2)(t_7-t_4)(1-t_4)(1-t_6)} \mathrm{d}t_1 \cdots \mathrm{d}t_7 \times \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_7}{(t_3-t_4)t_3t_5(t_5-t_2)(t_7-t_2)(t_7-t_4)(1-t_4)(1-t_6)}.
\]

On the surface, they give linear forms in \(1, \zeta(2), \zeta(5), \zeta(7)\) where \(\zeta_5, \zeta_7\) are multiple zeta values of weights 5 and 7, respectively. Upon closer scrutiny, we may write them as a combination of two linear forms:

\[
I_n = I'_n + I''_n \zeta(2),
\]

where \(I''_n\) is a linear form in \(1, \zeta(3), \zeta(5)\) and \(I'_n\) is a linear form in \(1, \zeta(5), \zeta(7)\) obtained by setting \(\zeta(2) = 0\) in \(I_n\). For example, the cases \(n = 0, 1, 2\) give

\[
I_0 = \frac{75}{4} \zeta(7) - 9 \zeta(5) \zeta(2),
\]

\[
I_1 = \left(61 \cdot \frac{75}{4} \zeta(7) - 300 \cdot 3 \zeta(5) - 220\right) - \left(61 \cdot 9 \zeta(5) - 300 \cdot 2 \zeta(3) + 152\right) \zeta(2),
\]

\[
I_2 = \left(52921 \cdot \frac{75}{4} \zeta(7) - 261153 \cdot 3 \zeta(5) - \frac{6021219}{32} \right) - \left(52921 \cdot 9 \zeta(5) - 261153 \cdot 2 \zeta(3) + \frac{535857}{4}\right) \zeta(2),
\]
where the linear forms $I'_n$ and $I''_n$ are also small (but not as small as $I_n$ itself). Thus, a similar analysis to the one undertaken in this note might possibly lead to a result of the form ‘at least one of $\zeta(5)$ and $\zeta(7)$ is irrational’.

The idea of using the subleading asymptotics of integrals to remove parasitic periods should prove fruitful in other contexts where it may take a more subtle form than simply setting $\zeta(2) = 0$. There are several technical limitations at the moment to execute this strategy in higher weights. For example, calculating higher weight integrals for small values of the parameters does not seem practical with current tools. A possible way to overcome these difficulties is to create an entirely new theoretical machinery for the arithmetic and asymptotics of the integrals in question based on the underlying algebraic geometry and contiguity relations. This seems to be a good challenge already for the family $8\pi^8$ analysed in this note.

From a broader perspective, this note is possibly the first systematic exploration of a cellular family of integrals which goes beyond the previously known cases for $\zeta(2)$ and $\zeta(3)$. In the process, we have uncovered several inter-related mathematical structures which include: duality relations, the existence of large symmetry groups, and a recursive structure between cellular integrals of different weights via iterated residues. More subtle features include the role of the symmetric parameters $s_i$, the combinatorial structures underlying the asymptotic behaviour of the integrals [1], and the arithmetic of their denominators. We expect that all these structures point to a general theory for cellular integrals, which combines arithmetic, geometric, combinatorial and analytic data. Several other promising ideas for a unifying approach to irrationality proofs may be found in [8, 10]. It would seem that the interplay between these different mathematical structures, which arguably gives the subject its appeal, lies at the heart of the difficulty of finding new irrationality proofs.

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All Souls College, University of Oxford, Oxford, OX1 4AL, UK
*URL:* [https://www.maths.ox.ac.uk/people/francis.brown](https://www.maths.ox.ac.uk/people/francis.brown)

Department of Mathematics, IMAPP, Radboud University, PO Box 9010, 6500 GL Nijmegen, Netherlands
*URL:* [https://www.math.ru.nl/~wzudilin/](https://www.math.ru.nl/~wzudilin/)