Rationality and Brauer group of a moduli space of framed bundles

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Abstract

We prove that the moduli spaces of framed bundles over a smooth projective curve are rational. We compute the Brauer group of these moduli spaces to be zero under some assumption on the stability parameter.

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1 Introduction

Let \(X\) be a compact connected Riemann surface of genus \(g\), with \(g \geq 2\). A framed bundle on \(X\) is a pair of the form \((E, \varphi)\), where \(E\) is a vector bundle on \(X\), and

\[ \varphi : E_{x_0} \longrightarrow \mathbb{C}^r \]

is a non-zero \(\mathbb{C}\)-linear homomorphism, where \(r\) is the rank of \(E\). The notion of a (semi)stable vector bundle extends to that for a framed bundle. But the (semi)stability condition depends on a parameter \(\tau \in \mathbb{R}_{>0}\). Fix a positive integer \(r\), and also fix a holomorphic line bundle \(L\) over \(X\). Also, fix a positive number \(\tau \in \mathbb{R}\). Let \(M^\tau_L(r)\) be the moduli space of \(\tau\)-stable framed bundles of rank \(r\) and determinant \(L\).

In [BGM], we investigated the geometric structure of the variety \(M^\tau_L(r)\). The following theorem was proved in [BGM]:

Assume that \(\tau \in (0, \frac{1}{(r-1)(r-1)})\). Then the isomorphism class of the Riemann surface \(X\) is uniquely determined by the isomorphism class of the variety \(M^\tau_L(r)\).

Our aim here is to investigate the rationality properties of the variety \(M^\tau_L(r)\). We prove the following (see Theorem 2.3 and Corollary 3.2):

The variety \(M^\tau_L(r)\) is rational.

If \(\tau \in (0, \frac{1}{(r-1)(r-1)})\), then

\[ \text{Br}(M^\tau_L(r)) = 0, \]

where \(\text{Br}(M^\tau_L(r))\) is the Brauer group of \(M^\tau_L(r)\).

The rationality of \(M^\tau_L(r)\) is proved by showing that \(M^\tau_L(r)\) is birational to the total space of a vector bundle over the moduli space of stable vector bundles \(E\) on \(X\) together with a line in the fiber of \(E\) over a fixed point. The rationality of these moduli spaces can also be derived from [Ho2]
by taking $D$ in Example 6.9 to be the point $x_0$; we thank N. Hoffmann for pointing this out. The Brauer group of $\mathcal{M}^r_L(r)$ is computed by considering the morphism to the usual moduli space that forgets the framing.

2 Rationality of moduli space

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$. Fix a holomorphic line bundle $L$ over $X$, and take an integer $r > 0$. Fix a point $x_0 \in X$. A framed coherent sheaf over $X$ is a pair of the form $(E, \varphi)$, where $E$ is a coherent sheaf on $X$ of rank $r$, and $\varphi : E_{x_0} \to \mathbb{C}^r$ is a non-zero $\mathbb{C}$–linear homomorphism. Let $\tau > 0$ be a real number. A framed coherent sheaf is called $\tau$–stable (respectively, $\tau$–semistable) if for all proper subsheaves $E' \subset E$, we have

$$\deg E' - \varepsilon(E', \varphi) \tau < \frac{\deg E - \tau}{\text{rk } E}$$

(respectively, $\deg E' - \varepsilon(E', \varphi) \tau \leq \frac{\deg E - \tau}{\text{rk } E}$), where

$$\varepsilon(E', \varphi) = \begin{cases} 1 & \text{if } \varphi|_{E'_{x_0}} \neq 0, \\ 0 & \text{if } \varphi|_{E'_{x_0}} = 0. \end{cases}$$

A framed bundle is a framed coherent sheaf $(E, \varphi)$ such that $E$ is locally free.

We remark that the framed coherent sheaves considered here are special cases of the objects considered in [HL], and hence from [HL] we conclude that the moduli space $\mathcal{M}^r_L(r)$ of $\tau$–stable framed bundles of rank $r$ and determinant $L$ is a smooth quasi–projective variety.

Let $(E, \varphi)$ be a $\tau$–semistable framed coherent sheaf. We note that if $\tau < 1$, then $E$ is necessarily torsion–free, because a torsion subsheaf of $E$ will contradict $\tau$–semistability, hence in this case $E$ is locally free. But if $\tau$ is large, then $E$ can have torsion. In particular, the natural compactification of $\mathcal{M}^r_L(r)$ using $\tau$–semistable framed coherent sheaves could have points which are not framed bundles.

Lemma 2.1. There is a dense Zariski open subset

$$\mathcal{M}^r_L(r)^0 \subset \mathcal{M}^r_L(r)$$

corresponding to pairs $(E, \varphi)$ such that $E$ is a stable vector bundle of rank $r$, and $\varphi$ is an isomorphism.

The moduli space $\mathcal{M}^r_L(r)$ is irreducible.

Proof. From the openness of the stability condition it follows immediately that the locus of framed bundles $(E, \varphi)$ such that $E$ is not stable is a closed subset of the moduli space $\mathcal{M}^r_L(r)$ (see [Ma, p. 635, Theorem 2.8(B)] for the openness of the stability condition). It is easy to check that the locus of framed bundles $(E, \varphi)$ such that $\varphi$ is not an isomorphism is a closed subset of $\mathcal{M}^r_L(r)$. Therefore, $\mathcal{M}^r_L(r)^0$ is a Zariski open subset of $\mathcal{M}^r_L(r)$.

We will now show that this open subset $\mathcal{M}^r_L(r)^0$ is dense. Let $(E, \varphi)$ be a $\tau$–stable framed bundle. The moduli stack of stable vector bundles is dense in the moduli stack of coherent sheaves, and both stacks are irreducible (see, for instance, [Ho, Appendix]). Therefore we can construct a family $\{E_t\}_{t \in T}$ of vector bundles parametrized by an irreducible smooth curve $T$ with a base point $0 \in T$ such that the following two conditions hold:
1. \( E_0 \cong E \), and
2. the vector bundle \( E_t \) is stable for all \( t \neq 0 \).

Shrinking \( T \) if necessary (by taking a nonempty Zariski open subset of \( T \)), we get a family of frames \( \{ \varphi_t \}_{t \in T} \) such that \( \varphi_0 \) is the given frame \( \varphi \), and \( \varphi_t : E_t |_{x_0} \to \mathbb{C}^r \) is an isomorphism for all \( t \neq 0 \). Since \( E_t \) is stable, and \( \varphi_t \) is an isomorphism, it is easy to check that \((E_t, \varphi_t)\) is \( \tau \)-stable. Therefore, \( \mathcal{M}_\tau^r(r)^0 \) is dense in \( \mathcal{M}_\tau^r(r) \).

To prove that \( \mathcal{M}_\tau^r(r)^0 \) is irreducible, first note that \( \mathcal{M}_\tau^r(r)^0 \) is irreducible because the moduli stack of stable vector bundles of fixed rank and determinant is irreducible. Since \( \mathcal{M}_\tau^r(r)^0 \subset \mathcal{M}_\tau^r(r) \) is dense, it follows that \( \mathcal{M}_\tau^r(r) \) is irreducible. \( \square \)

Let \( \mathcal{N}_P \) be the moduli space of pairs of the form \((E, \ell)\), where \( E \) is a stable vector bundle on \( X \) of rank \( r \) with determinant \( L \), and \( \ell \subset E_{x_0} \) is a line. Consider \( \mathcal{M}_\tau^r(r)^0 \) defined in (2.2). Let
\[
\beta : \mathcal{M}_\tau^r(r)^0 \to \mathcal{N}_P
\]
be the morphism defined by \((E, \varphi) \mapsto (E, \varphi^{-1}(C \cdot e_1))\), where the standard basis of \( \mathbb{C}^r \) is denoted by \( \{e_1, \ldots, e_r\} \).

**Proposition 2.2.** The variety \( \mathcal{M}_\tau^r(r)^0 \) is birational to the total space of a vector bundle over \( \mathcal{N}_P \).

**Proof.** We will first construct a tautological vector bundle over \( \mathcal{N}_P \). Let \( \mathcal{N}_L(r) \) be the moduli space of stable vector bundles on \( X \) of rank \( r \) and determinant \( L \). Consider the projection
\[
f : \mathcal{N}_P \to \mathcal{N}_L(r)
\]
defined by \((E, \ell) \mapsto E \). Let \( P_{\text{PGL}} \to \mathcal{N}_L(r) \) be the principal \( \text{PGL}(r, \mathbb{C}) \)-bundle corresponding to \( f \); the fiber of \( P_{\text{PGL}} \) over any \( E \in \mathcal{N}_L(r) \) is the space of all linear isomorphisms from \( P(\mathbb{C}^r) \) (the space of lines in \( \mathbb{C}^r \)) to \( P(E_{x_0}) \) (the space of lines in \( E_{x_0} \)); since the automorphism group of \( E \) is the nonzero scalar multiplications (recall that \( E \) is stable), the projective space \( P(E_{x_0}) \) is canonically defined by the point \( E \) of \( \mathcal{N}_L(r) \). Let
\[
Q \subset \text{PGL}(r, \mathbb{C})
\]
be the maximal parabolic subgroup that fixes the point of \( P(\mathbb{C}^r) \) representing the line \( \mathbb{C} \cdot e_1 \). The principal \( \text{PGL}(r, \mathbb{C}) \)-bundle
\[
f^* P_{\text{PGL}} \to \mathcal{N}_P
\]
has a tautological reduction of structure group
\[
\tilde{E}_Q \subset f^* P_{\text{PGL}}
\]
to the parabolic subgroup \( Q \); the fiber of \( \tilde{E}_Q \) over any point \((E, \ell) \in \mathcal{N}_P \) is the space of all linear isomorphisms
\[
\rho : P(\mathbb{C}^r) \to P(E_{x_0})
\]
such that \( \rho(\mathbb{C} \cdot e_1) = \ell \). The standard action of \( \text{GL}(r, \mathbb{C}) \) on \( \mathbb{C}^r \) defines an action of \( Q \) on \( (\mathbb{C} \cdot e_1)^* \otimes \mathbb{C}^r \). Let
\[
W := f^* P_{\text{PGL}}((\mathbb{C} \cdot e_1)^* \otimes \mathbb{C}^r) \to \mathcal{N}_P
\]
be the vector bundle over \( \mathcal{N}_P \) associated to the principal \( \text{PGL}(r, \mathbb{C}) \)-bundle \( f^* \mathcal{P} \) for the above \( \text{PGL}(r, \mathbb{C}) \)-module \( (\mathbb{C} \cdot e_1)^* \otimes \mathbb{C}^{r'} \). The action of \( Q \) on \( (\mathbb{C} \cdot e_1)^* \otimes \mathbb{C}^{r'} \) fixes

\[
\text{Id}_{\mathbb{C} \cdot e_1} \in (\mathbb{C} \cdot e_1)^* \otimes \mathbb{C}^{r'} = \text{Hom}(\mathbb{C} \cdot e_1, \mathbb{C}^{r'}). 
\]

Therefore, the element \( \text{Id}_{\mathbb{C} \cdot e_1} \) defines a nonzero section

\[
\sigma \in H^0(\mathcal{N}_P, W),
\]

where \( W \) is the vector bundle in (2.5). Note that the fiber of \( W \) over \( (E, \ell) \) is \( \ell^* \otimes E_{x_0} \), and the evaluation of \( \sigma \) at \( (E, \ell) \) is \( \text{Id}_\ell \).

The projective bundle \( P(W) \to \mathcal{N}_P \) parametrizing lines in \( W \) is identified with the pullback \( f^* \mathcal{N}_P \) of the projective bundle \( \mathcal{N}_P \) to the total space of \( \mathcal{N}_P \), where \( f \) is constructed in (2.4). The tautological section \( \mathcal{N}_P \to f^* \mathcal{N}_P \) of the projection \( f^* \mathcal{N}_P \to \mathcal{N}_P \) coincides with the section given by \( \sigma \) in (2.6).

Let \( U \subset \mathcal{N}_P \) be some nonempty Zariski open subset such that there exists

\[
V \subset W|_U,
\]

a direct summand of the line subbundle of \( W|_U \) generated by \( \sigma \). Consider the vector bundle

\[
\mathcal{W} := V^* \otimes \mathbb{C}^{r'} \to U.
\]

The total space of \( \mathcal{W} \) will also be denoted by \( \mathcal{W} \). Consider the map \( \beta \) defined in (2.3). Let

\[
\gamma : \mathcal{M}_L^0(r)^0 \supset \beta^{-1}(U) \to \mathcal{W}
\]

be the morphism that sends any \( y := (E, \varphi) \in \beta^{-1}(U) \) to the homomorphism

\[
V_{\beta(y)} \to \mathbb{C}^{r'}
\]

defined by \( v \mapsto \varphi(v)/\lambda \), where \( \lambda \in \mathbb{C}^* - \{0\} \) satisfies the identity

\[
\varphi(\sigma(\beta(y))) = \lambda \cdot e_1.
\]

The morphism \( \gamma \) is clearly birational. q.e.d.

**Theorem 2.3.** The moduli space \( \mathcal{M}_L^0(r) \) is rational.

**Proof.** Since any vector bundle is Zariski locally trivial, the total space of a vector bundle of rank \( n \) over \( \mathcal{N}_P \) is birational to \( \mathcal{N}_P \times \mathbb{A}^n \). Therefore, from Proposition 2.2 we conclude that \( \mathcal{M}_L^0(r)^0 \) is birational to \( \mathcal{N}_P \times \mathbb{A}^n \), where \( n = \dim \mathcal{M}_L^0(r)^0 - \dim \mathcal{N}_P \).

The variety \( \mathcal{N}_P \) is known to be rational [BY, p. 472, Theorem 6.2]. Hence \( \mathcal{N}_P \times \mathbb{A}^n \) is rational, implying that \( \mathcal{M}_L^0(r)^0 \) is rational. Now from Lemma 2.1 we infer that \( \mathcal{M}_L^0(r) \) is rational. q.e.d.

3 Brauer group of moduli of framed bundles

We quickly recall the definition of Brauer group of a variety \( Z \). Using the natural isomorphism \( \mathbb{C}^r \otimes \mathbb{C}^{r'} \to \mathbb{C}^{r+r'} \), we have a homomorphism \( \text{PGL}(r, \mathbb{C}) \times \text{PGL}(r', \mathbb{C}) \to \text{PGL}(rr', \mathbb{C}) \). So a principal \( \text{PGL}(r, \mathbb{C}) \)-bundle \( \mathbb{P} \) and a principal \( \text{PGL}(r', \mathbb{C}) \)-bundle \( \mathbb{P}' \) on \( Z \) together produce a principal \( \text{PGL}(rr', \mathbb{C}) \)-bundle on \( Z \), which we will denote by \( \mathbb{P} \otimes \mathbb{P}' \). The two principal bundles \( \mathbb{P} \) and \( \mathbb{P}' \) are called equivalent if there are vector bundles \( V \) and \( V' \) on \( Z \) such that the principal
bundle $\mathbb{P} \otimes \mathbb{P}(V)$ is isomorphic to $\mathbb{P}' \otimes \mathbb{P}(V')$. The equivalence classes form a group which is called the Brauer group of $Z$. The addition operation is defined by the tensor product, and the inverse is defined to be the dual projective bundle. The Brauer group of $Z$ will be denoted by $\text{Br}(Z)$.

As before, fix $r$ and $L$. Define
\[
\tau(r) := \frac{1}{(r-1)!(r-1)}.
\]
Henceforth, we assume that $\tau \in \left(0, \tau(r)\right)$, where $\tau$ is the parameter in the definition of a (semi)stable framed bundle. As before, let $M^r_L(r)$ be the moduli space of $\tau$–stable framed bundles of rank $r$ and determinant $L$.

Let $N_L(r)$ be the moduli space of semistable vector bundles on $X$ of rank $r$ and determinant $L$. As in the previous section, the moduli space of stable vector bundles on $X$ of rank $r$ and determinant $L$ will be denoted by $N^r_L(r)$.

If $E$ is a stable vector bundle of rank $r$ and determinant $L$, then for any nonzero homomorphism $\varphi : E_{x_0} \longrightarrow \mathbb{C}^r$, the framed bundle $(E, \varphi)$ is $\tau$–stable (see [BGM, Lemma 1.2(ii)]). Also, if $(E, \varphi)$ is any $\tau$–stable framed bundle, then $E$ is semistable [BGM, Lemma 1.2(i)]. Therefore, we have a morphism
\[
\delta : M^r_L(r) \longrightarrow N_L(r)
\]
defined by $(E, \varphi) \longrightarrow E$. Define
\[
M^r_L(r)' := \delta^{-1}(N_L(r)) \subset M^r_L(r),
\]
where $\delta$ is the morphism in (3.1). From the openness of the stability condition (mentioned in the proof of Lemma 2.1) it follows that $M^r_L(r)'$ is a Zariski open subset of $M^r_L(r)$.

**Lemma 3.1.** The Brauer group of the variety $M^r_L(r)'$ vanishes.

**Proof.** We noted above that $(E, \varphi)$ is $\tau$–stable if $E$ is stable. Therefore, the morphism
\[
\delta_1 := \delta|_{M^r_L(r)'} : M^r_L(r)' \longrightarrow N_L(r)
\]
defines a projective bundle over $N_L(r)$, where $\delta$ is constructed in (3.1); for notational convenience, this projective bundle $M^r_L(r)'$ will be denoted by $\mathcal{P}$. The homomorphism
\[
\delta_1^* : \text{Br}(N_L(r)) \longrightarrow \text{Br}(\mathcal{P})
\]
is surjective, and the kernel of $\delta_1^*$ is generated by the Brauer class
\[
\text{cl}(\mathcal{P}) \in \text{Br}(N_L(r))
\]
of the projective bundle $\mathcal{P}$ (see [Ga, p. 193]). In other words, we have an exact sequence
\[
\mathbb{Z} \cdot \text{cl}(\mathcal{P}) \longrightarrow \text{Br}(N_L(r)) \xrightarrow{\delta_1^*} \text{Br}(M^r_L(r)') \longrightarrow 0.
\]
Let
\[ \mathbb{P} := \mathcal{N}_L(r) \times \mathcal{P}(\mathbb{C}^r) \to \mathcal{N}_L(r) \]
be the trivial projective bundle over $\mathcal{N}_L(r)$. Consider the projective bundle
\[ f : \mathcal{N}_P \to \mathcal{N}_L(r) \]
in (2.4). Let
\[ (\mathcal{N}_P)^* \to \mathcal{N}_L(r) \]
be the dual projective bundle; so the fiber of $(\mathcal{N}_P)^*$ over any point $z \in \mathcal{N}_L(r)$ is the space of all hyperplanes in the fiber of $\mathcal{N}_P$ over $z$. It is easy to see that
\[ \mathcal{P} = (\mathcal{N}_P)^* \otimes \mathbb{P} \quad (3.4) \]
(the tensor product of two projective bundles was defined at the beginning of this section).

Since $\mathbb{P}$ is a trivial projective bundle, from (3.4) it follows that
\[ \text{cl}(\mathcal{P}) = \text{cl}((\mathcal{N}_P)^*) = -\text{cl}(\mathcal{N}_P) \in \text{Br}(\mathcal{N}_L(r)) \, . \]
But the Brauer group $\text{Br}(\mathcal{N}_L(r))$ is generated by $\text{cl}(\mathcal{N}_P)$ [BBGN, Proposition 1.2(a)]. Hence $\text{cl}(\mathcal{P})$ generates $\text{Br}(\mathcal{N}_L(r))$. Now from (3.3) we conclude that $\text{Br}(\mathcal{M}_L(r)') = 0$. 

**Corollary 3.2.** The Brauer group of the moduli space $\mathcal{M}_L(r)$ vanishes.

**Proof.** Since $\mathcal{M}_L(r)'$ is a nonempty Zariski open subset of $\mathcal{M}_L(r)$, the homomorphism
\[ \text{Br}(\mathcal{M}_L(r)) \to \text{Br}(\mathcal{M}_L(r)') \]
induced by the inclusion $\mathcal{M}_L(r)' \hookrightarrow \mathcal{M}_L(r)$ is injective. Therefore, from Lemma 3.1 it follows that $\text{Br}(\mathcal{M}_L(r)) = 0$. 

**q.e.d.**

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