On the exponential local-global principle for meromorphic functions and algebraic functions

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Abstract. We prove the rank one case of Skolem’s Conjecture on the exponential local-global principle for algebraic functions and discuss its analog for meromorphic functions.

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1. Introduction and results. Let \( K \) be a number field and \( S \) a finite set of places of \( K \) containing all the archimedean places. Denote by \( \mathcal{O}_S := \{ \alpha \in K : |\alpha|_v \leq 1 \text{ for places } v \notin S \} \) the ring of \( S \)-integers and by \( \mathcal{O}_S^* := \{ \alpha \in K : |\alpha|_v = 1 \text{ for places } v \notin S \} \) the group of \( S \)-units. Let \( \lambda_1, \ldots, \lambda_m \) be non-zero elements in \( \mathcal{O}_S \) and \( \alpha_1, \ldots, \alpha_m \) in \( \mathcal{O}_S^* \). For every \( n \in \mathbb{Z} \), we consider the following power sum with respect to \( \lambda_i \) and \( \alpha_i \),

\[
A(n) := \lambda_1 \alpha_1^n + \cdots + \lambda_m \alpha_m^n \in \mathcal{O}_S.
\]

The following conjecture was suggested by Skolem in [6].

Conjecture 1. (Exponential local-global principle) Assume that for every non-zero ideal \( \mathfrak{a} \) of the ring \( \mathcal{O}_S \), there exists \( k \in \mathbb{Z} \) such that \( A(k) \in \mathfrak{a} \). Then there exists \( n \in \mathbb{Z} \) such that \( A(n) = 0 \).

Recently, Bartolome et al. [1] proved this conjecture for the case when the rank of the multiplicative group generated by \( \alpha_1, \ldots, \alpha_m \) is one. Their proof relies on the gcd theorem of Corvaja and Zannier [2] and a celebrated inequality of Baker (see the first two contributions in [10]). We refer to [1] for the statements of these theorems and also a survey of results related to Conjecture 1.

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Due to some interesting analogy between Diophantine approximation and Nevanlinna theory, we are also interested in the corresponding statements for complex meromorphic functions. Denote by $\mathcal{R}$ the ring of entire functions and by $\mathcal{R}^*$ its group of units (i.e. entire functions without zeros). Also, $\mathcal{M}$ is the field of functions that are meromorphic in the complex plane. Occasionally we write $U_n$ for the group of $n$-th roots of unity.

Let $\lambda_1, \ldots, \lambda_m$ be non-zero elements in $\mathcal{R}$ and $f_1, \ldots, f_m$ in $\mathcal{R}^*$. For every $n \in \mathbb{Z}$, we define the power sum with respect to $\lambda_i$ and $f_i$ by

$$B(n) := \lambda_1 f_1^n + \cdots + \lambda_m f_m^n \in \mathcal{R}.$$ 

It is tempting to ask the following question which seems to be the analogue of Conjecture 1 in the complex case.

**Question 2.** Assume that for every non-zero ideal $a$ of the ring $\mathcal{R}$, there exists $k \in \mathbb{Z}$ such that $B(k) \in a$. Then there exists $n \in \mathbb{Z}$ such that $B(n) = 0$.

However, this question might be too naive since it has an easy answer due to the following reason: The assumption implies that for each $z_0 \in \mathbb{C}$ there exists $k \in \mathbb{Z}$ such that $B(k)(z_0) = 0$. Therefore, there exists $n \in \mathbb{Z}$ such that $B(n)$ has uncountably many zeros which is impossible as $B(n)$ is an entire function. This observation brings out the difference between the complex case and the arithmetic case, and also indicates that the assumption should be loosened up. Instead of considering all ideals in $\mathcal{R}$, we will restrict ourselves to more specific conditions.

Let more generally $f_i, \lambda_i \in \mathcal{M}^*$. If the multiplicative group generated by the $f_i$ has rank 1, it is a finitely generated group of rank 1, so it is a direct product of its (finite) torsion subgroup and an infinite cyclic group. But a torsion element in $\mathcal{M}^*$ can only be a function that is identically equal to a root of unity. Choosing a generator $f$ of the infinite cyclic part, we can write

$$f_i = \epsilon_i f^r_i$$

with $\epsilon_i$ a root of unity and $r_i \in \mathbb{Z}$. This form will be the most convenient to formulate our results.

Basic notation, definitions, and results will be collected in Section 2.1.

**Theorem 3.** Let $f$ be a non-constant meromorphic function and $\lambda_1, \ldots, \lambda_m$ meromorphic functions such that there exists $\varrho < 1$ with

$$T_{\lambda_i}(r) \leq \frac{\varrho}{m + \tilde{m}} T_f(r) + S_f(r)$$

for $i = 1, \ldots, m$, where $\tilde{m}$ is the number of $\lambda_i$ that are not entire. Fix an $m$-tuple of integers $(r_1, \ldots, r_m)$ and an $m$-tuple of roots of unity $(\epsilon_1, \ldots, \epsilon_m)$ and let

$$B(n) := \lambda_1(z)(\epsilon_1 f^{r_1}(z))^n + \cdots + \lambda_m(z)(\epsilon_m f^{r_m}(z))^n.$$ 

Let $N = \max\{r_1, \ldots, r_m\} - \min\{r_1, \ldots, r_m\}$. Fix an integer $e > \frac{2}{1-\varrho}$ such that $\epsilon_i^e = 1$ for all $i = 1, 2, \ldots, m$. Let $a$ be a positive integer such that

$$p^{1+\ord_a a - \ord_p e} > N + \frac{2}{1-\varrho}.$$