Some comparisons of Blanchfield pairings and cohomology pairings of knots

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Abstract

We study some comparison between a bilinear cohomology pairing in local coefficients and the Blanchfield pairing of a knot. We show that the former pairing is an \( S \)-equivalent invariant, and give a criterion to a relation between the two pairings. We also observe that the pairings of some knots are equivalent, and that the pairings of other knots are not equivalent.

Keywords

Cup product, knot, Blanchfield pairing, infinite covering, quandle

1 Introduction

1.1 Motivation and background

The interaction between cup products and intersection forms is a basic method to essentially analyse a \( C^\infty \)-manifold \( Y \) in the history (e.g., the classification theorem of simply connected manifolds). As a typical example, as seen in the Poincaré duality with trivial coefficients, non-degeneracy of the intersection form can be shown from the view of the cup product. Here, it is worth noting that the interactions are implicitly reflected on algebraic futures of the coefficients \( \mathbb{Z} \) and \( \mathbb{Z}/p \).

However, once the (co)homology groups are investigated with local coefficients \( A \), such an interaction has many unknown aspects with ambiguity and differences. Actually, graded commutativity and the Krull dimension of \( A \) appear as obstructions: for example, we can perceive such a difference even from some dualities of infinite cyclic covering spaces \( \tilde{Y} \) of closed manifolds; while Blanchfield duality \([\text{Bla}]\) on the homology of \( \tilde{Y} \) is defined from some Bockstein operator to realize hermitian intersection forms over \( \mathbb{Z}[t^\pm 1] \), Milnor duality \([\text{MI}]\) is anti-hermitianly constructed with regard to the cup products of \( \tilde{Y} \) over fields and requires some assumptions; moreover, there are not so many descriptions to explicitly connect the two dualities (cf. Theorem 1.2; however, partial connections on signatures can be seen in \([\text{Ke2}, \text{MP}]\)).

1.2 Settings: cohomology pairings and the Blanchfield duality

In this paper, we focus on the case \( Y \) is a complement \( S^3 \setminus K \) of a knot \( K \), and study a relation between cohomology pairings and the Blanchfield duality. The former pairing is constructed from the abelianization \( \text{Ab} : \pi_1(Y) \to \mathbb{Z} = \langle t \rangle \), as follows. Choosing a Seifert surface \( \Sigma \), we regard it as a relative homology 2-class in \( H_2(Y, \partial Y; \mathbb{Z}) \). Set up \( \mathbb{Z}[t^\pm 1] \)-modules \( M \) and \( M' \), and a sesquilinear\(^2\) bilinear function \( \psi : M \times M' \to A \) for some \( \mathbb{Z}[t^\pm 1] \)-module \( A \). Then, we can define the pairing as a bilinear form

\[
Q_\psi : H^1(Y, \partial Y; M) \otimes H^1(Y, \partial Y; M') \longrightarrow H^2(Y, \partial Y; M \otimes M') \stackrel{\bullet \otimes \Sigma}{\longrightarrow} M \otimes M' \stackrel{\psi}{\longrightarrow} A. \tag{1}
\]

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\(^2\)A bilinear map \( \psi : M \times M' \to A \) over \( \mathbb{Z} \) is said to be \textit{sesquilinear}, if \( \psi(tx, y) = t\psi(x, y) = \psi(x, t^{-1}y) \) holds for any \( x \in M, y \in M' \).
Here we regard $M$ and $M'$ as the local coefficient modules of $Y$ via $\text{Ab}$, and the first map $\sim$ is the cup-product, and the second is the cap-product with $\Sigma$. Though this pairing $Q_\psi$ seems speculative and uncomputable from definitions, the author [No2] has developed a diagrammatic computation of the $Q_\psi$ (see also [2]).

On the other hand, we roughly review the Blanchfield pairing $\text{Bl}a$ of a knot $K$ with Alexander polynomial $\Delta$. The first homology $H_1(S^3 \setminus K; \mathbb{Z}[t^\pm 1])$ with local coefficients is called the Alexander module, i.e., the first homology of the covering space $\tilde{Y}$. Then, from the view of intersection forms in $\tilde{Y}$, the Blanchfield pairing is defined as a sesquilinear form

$$\text{Bl}_K : H_1(S^3 \setminus K; \mathbb{Z}[t^\pm 1]) \otimes \mathbb{Z}[t^\pm 1] \rightarrow \mathbb{Z}[t^\pm 1]/(\Delta),$$

(3 reviews the formulation $\text{Bl}$). This $\text{Bl}_K$ is known to be non-singular, hermitian and sesquilinear (see $\text{Bl}$, $\text{Ka}$, $\text{Hil}$); further, it is a complete invariant of “the $S$-equivalences” in knots (see $\text{T2}$, $\text{NS}$ for details). More precisely, two knots $K$ and $K'$ are $S$-equivalent if and only if the associated pairings $\text{Bl}_K$ and $\text{Bl}_{K'}$ are isomorphic as a bilinear form.

### 1.3 Main results

Thus, it is natural to ask whether $Q_\psi$ is invariant under $S$-equivalence. The main result is as follows:

**Theorem 1.1.** If two knots $K$ and $K'$ are $S$-equivalent, then the cohomology pairings $Q_\psi$ and $Q'_\psi$ are equal up to bilinear isomorphisms.

We put the proof in Section [3]. In conclusion, this theorem implies that the cohomology pairing $Q_\psi$ can be described from the $\text{Bl}_K$ in principle.

The main purpose of this paper is a study of such a description. For this, we point out that it is reasonable to suppose $M = \Lambda/(\Delta)$, since there is a $\mathbb{Z}[t^\pm 1]$-module isomorphism

$$\kappa : H_1(S^3 \setminus K; \mathbb{Z}[t^\pm 1]) \cong H^1(S^3 \setminus K, \partial(S^3 \setminus K); M),$$

which is explicitly described in Section [3]. Let $\bar{\cdot} : \mathbb{Z}[t^\pm 1] \rightarrow \mathbb{Z}[t^\pm 1]$ be the involution defined by $\bar{t} = t^{-1}$ The following theorem asserts that a constant multiple of the Blanchfield pairing can be recovered from some bilinear form $Q_\psi$ in some cases (we put the proof in $\text{§3}$):

**Theorem 1.2.** Let $K$ be a knot with Alexander polynomial $\Delta$, and let $M$ be the quotient module $\mathbb{Z}[t^\pm 1]/(\Delta)$. Define $\psi_0 : M \otimes M \rightarrow \mathbb{Z}[t^\pm 1]/(\Delta)$ by $\psi_0(x, y) = \bar{xy}$.

Then, there is a constant $\alpha_K \in \mathbb{Z}[t^\pm 1]/(\Delta)$ with $\bar{\alpha_K} = \alpha_K$ such that the following equality holds as bilinear forms:

$$Q_{\psi_0}(\kappa(x), \kappa(y)) = \alpha_K \frac{1 + t}{1 - t} \cdot \text{Bl}_K(x, y) \in \mathbb{Z}[t^\pm 1]/(\Delta),$$

for any $x, y \in H_1(S^3 \setminus K; \mathbb{Z}[t^\pm 1])$. Here $\kappa$ is written in $\text{[3]}$.

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3In many cases (see $\text{Ka}$, $\text{T2}$, $\text{Hil}$), the image is described as the injective module $Q(t)/\mathbb{Z}[t^\pm 1]$. However, such as $\text{T4}$, the pairing factors through the inclusion $\mathbb{Z}[t^\pm 1]/(\Delta) \hookrightarrow Q(t)/\mathbb{Z}[t^\pm 1]$ that sends $[f]$ to $[f/\Delta]$.  

2
In summary, it is fair to state that, this theorem gives a cohomological approach to $\text{Bl}_K$ in the sense 2, and an obstruction $\alpha_K$ from the approach, in contrast with the previous works $\text{Bl}a, \text{FP}, \text{MP}, \text{Ke}2$ as homological approach. However, it is a future problem to ask a relation between the Milnor pairing and our pairing $Q_\psi$.

We give some remarks on this theorem: We note that $1 - t$ is invertible in $\mathbb{Z}[t^{\pm 1}]/(\Delta)$ because of $\Delta(1) = \pm 1$. Furthermore, the constant multiple of $(1 + t)(1 - t)^{-1}$ is a key to connect the hermitian pairing $\text{Bl}_K$ with the anti-one $Q_{\psi_0}$. This theorem implies that, on the assumption, if $\alpha_K$ and $\Delta(-1) \in \mathbb{Z}$ are not zero-divisors in $\Lambda/(\Delta)$, then the Blanchfield pairing $\text{Bl}_K$ can be completely recovered from the pairing $Q_{\psi_0}$ (see §3 for such examples); however, conversely, if either of $\alpha_K$ and $\Delta(-1) \in \mathbb{Z}$ is zero-divisor, all information of $\text{Bl}_K$ cannot be recovered from the pairing $Q_{\psi_0}$; Section 4 observes some cases where the cup products $Q_\psi$ lose many information of $\text{Bl}_K$, according to complexity of the Alexander module $H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])$. In addition, in Section 6 we will see that, for even the torus knot, the problem of the recovery is not so easy.

Finally, in Appendix A we will see that “the cocycle knot-invariants [CJKLS] from Alexander quandles” also turn to be topologically recovered from the Blanchfield pairing (see Theorem A.1 for the details);

This paper is organized as follows. Section 2 reviews the computation in [No2], and Section 3 gives the proof of Theorem 1.2. Sections 4–6 give some computations of the pairings.

**Conventional notation.** Every knot $K$ is understood to be smooth, oriented, and embedded in the 3-sphere $S^3$ as a circle. We regard the complement $S^3 \setminus K$ as the 3-manifold which is obtained from $S^3$ by removing an open tubular neighborhood of $K$. In ordinary papers on the Blanchfield pairing, by $\Lambda$ we mean the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$, with involution $\bar{t} = t^{-1}$. Moreover, we denote a $\Lambda$ module by $M$; furthermore, let $M^{\text{op}}$ be $M$ with the $\Lambda$ module structure by $\lambda \cdot m := \bar{\lambda}m$, where $\lambda \in \Lambda, m \in M^{\text{op}} = M$.

### 2 Review; diagrammatic computation of the cohomology pairing.

This section strictly describes diagrammatic computation of the cohomology pairing, and gives the proof of Theorem 1.2. We will need some knowledge of quandles before proceeding.

Throughout this section, we fix two $\mathbb{Z}[t^{\pm 1}]$-modules $X$ and $A$. Further, define a binary operation on $X$ by

$$\triangleright : X \times X \longrightarrow X; \quad (a,b) \longmapsto t(a - b) + b.$$ (4)

The pair $(X, \triangleright)$ is called an Alexander quandle [CJKLS, Joy, No1].

We review colorings. Pick a knot $K \subset S^3$ with orientation and an oriented knot diagram $D$ of $K$. A map $C : \{\text{arcs of } D\} \rightarrow X$ over $f$ is an $X$-coloring if it satisfies $C(\alpha) \triangleright C(\beta) = C(\gamma)$ at each crossing of $D$ illustrated as Figure 1. Let $\text{Col}_X(D)$ denote the set of all $X$-colorings. By definition, this $\text{Col}_X(D)$ canonically injects into the product $X^{\text{Arc}(D)}$, where $\text{Arc}(D)$ is the number of arcs of $D$. Therefore, $\text{Col}_X(D)$ serves as a $\Lambda$-submodule of $X^{\text{Arc}(D)}$. Furthermore,
the diagonal submodule $X_{\text{diag}} \subset M^{\lambda(D)}$ is contained in $\text{Col}_X(D)$, and is a direct summand of $\text{Col}_X(D)$. Thus, we denote by $\text{Col}_X^{\text{red}}(D)$ another direct summand, i.e., $\text{Col}_X(D) \cong X_{\text{diag}} \oplus \text{Col}_X^{\text{red}}(D)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{positive_negative_crossings.png}
\caption{Positive and negative crossings, where arcs are assigned by $A$.}
\end{figure}

Furthermore, one introduces a sesquilinear form on the $\Lambda$-module $\text{Col}_X(D)$ as follows. Take another Alexander quandle $X'$. Let $\psi : X \times X' \rightarrow A$ be a sesquilinear map over $\mathbb{Z}$. Define a map

$$Q_{D,\psi} : \text{Col}_X(D) \times \text{Col}_{X'}(D) \rightarrow A; \quad (C, C') \mapsto \sum_{\tau} \epsilon_{\tau} \cdot \psi(C(\alpha_{\tau}) - C(\beta_{\tau}), C'(\beta_{\tau})(1 - t^{-1})), \quad (5)$$

where $\tau$ runs over all the crossings of $D$, and the symbols $\alpha_{\tau}$, $\beta_{\tau}$ are the arcs and $\epsilon_{\tau} \in \{\pm 1\}$ is the sign of $\tau$ according to Figure 1. The sum in (5) is sometimes called a weight sum. Then, the sesquilinear form $Q_{D,\psi}$ is topologically detected by the following sense:

**Theorem 2.1** (A special case of [No2, Theorem 2.2]). Let $E_K = S^3 \setminus K$. Let $M$ be $X$ and $M'$ be $X'$ as above. Then, there are $\Lambda$-module isomorphisms

$$\text{Col}_X(D) \cong H^1(E_K, \partial E_K; M) \oplus M, \quad \text{Col}_X^{\text{red}}(D) \cong H^1(E_K, \partial E_K; M).$$

Furthermore, on the isomorphisms, the restriction of $Q_{\psi,D}$ on $\text{Col}_X^{\text{red}}(D) \times \text{Col}_X^{\text{red}}(D)$ is equal to the bilinear cohomology pairing $Q_{\psi}$ in (1).

To summarize, the point is that, given a diagram $D$, we can diagrammatically compute the form $Q_{\psi,D}$ by definitions, and that we need no description of the Seifert surface $\Sigma$; in a comparison, there are approaches to the signature of knots without using Seifert surfaces [Ke2, MP].

### 2.1 Proof of Theorem 1.1

To prove the invariance of the cohomology pairing (1) under $S$-equivalence, we review the $S$-equivalence of knots. While there are several definitions of the $S$-equivalences (see [T2, Lic]), this paper uses the definition in the sense of [NS]. Two knots $K$ and $K'$ are $S$-equivalent if they are related by a finite sequence of the (double delta) local moves shown in Figure 2. Furthermore, Trotter [T2] showed that $K$ and $K'$ are $S$-equivalent if and only if the associated Blanchfield pairings are isomorphic as bilinear forms.

Since the following lemma is elementary, we omit writing details of the proofs.

**Lemma 2.2.** Consider an $X$-coloring and an $X'$-coloring of the eight arcs illustrated in the figure below, where the alphabets are elements in $X$ or $X'$. Then, the weight sum with respect to the four crossings is $\psi((1 - t)(a - b), c' - d') \in A$. 

![Positive and negative crossings](positive_negative_crossings.png)
Figure 2: A double delta move with 24 arcs

Proof of Theorem 1.1. Suppose that two knots $K$ and $K'$ are $S$-equivalent. Let $D$ and $D'$ be diagrams of $K$ and $K'$, respectively. By the results mentioned above, we may assume that the difference between $D$ and $D'$ is only a double delta move. Given an $X$-coloring $C$ of $D$, consider the assignment $C_1$ of $D'$ such that $C_1(\beta_i) = C(\alpha_i)$. We can easily check that $C_1$ is an $X$-coloring of $D'$, and the correspondence $C \mapsto C_1$ gives rise to a $\Lambda$-isomorphism $\lambda : \text{Col}_X(D) \rightarrow \text{Col}_X(D')$. Here, we can easily check that this $\lambda$ preserves the direct sum decomposition $\text{Col}_X(D^\bullet) = H^1(E_K^\bullet; \partial E_K^\bullet; M) \oplus M$. Furthermore, we can easily verify from Lemma 2.2 the equality $Q_{\psi,D}(C, C') = Q_{\psi,D'}(\lambda(C), \lambda(C'))$ for any colorings $C, C'$. In the sequel, the associated bilinear forms $Q_\psi$ and $Q'_\psi$ are equivalent; hence, so are the corresponding cohomology pairings by Theorem 2.1 as required.

3 Proof of Theorem 1.2

The end of this section is devoted to proving Theorem 1.2, which gives a trial to recover the Blanchfield pairing from $Q_\psi$. The reader, who has mainly an interest in examples of computation, may read only Proposition 3.2 and skip this section. In this section, given a matrix $V$, we denote the transposed matrix by $V'$.

3.1 The Blanchfield pairing from cup product

We first recall the calculation of the Blanchfield pairing in terms of homology [FP, Ke1]. Choose a Seifert surface $F$ of $K$ whose genus is $g$, where we may assume the existence of a bouquet of circles $W \subset F$ such that $W$ is a deformation retract of $F$ and the inclusion $F \subset S^3$ is isotopic to the standard embedding $W \subset F$. Then, we have the Seifert form $\alpha : H_1(F; \mathbb{Z}) \otimes H_1(F; \mathbb{Z}) \rightarrow \mathbb{Z}$; see [Lic, Chapter 6] for the definition. The matrix presentation is commonly written by $V \in \text{Mat}(2g \times 2g; \mathbb{Z})$, and is called the Seifert matrix.

Theorem 3.1 (Ke1). See also [FP]4. The first homology $H_1(E_K; \Lambda)$ is isomorphic to the

4Strictly speaking, the notation of $t$ is that in [Ke1, FP]. However, if we replace $t$ by $t^{-1}$, the notations are equal.
Furthermore, we consider the cochain complex \( C \) here the differential maps \( c \) Let 

Next, in cohomological terminology, we will reformulate the Blanchfield pairing: Considering the exact sequence \( 0 \to \Lambda \xrightarrow{\Delta} H \to \Lambda/\langle \Delta \rangle \to 0 \), we have the Bockstein map \[
\beta : H^i(E_K, \partial E_K; \Lambda/\langle \Delta \rangle) \to H^{i+1}(E_K, \partial E_K; \Lambda). \tag{6}
\]

Here, it is worth noticing that this map with \( i = 1 \) is an isomorphism, since \( H^2(E_K, \partial E_K; \Lambda) \cong H_1(E_K; \Lambda) \) is annihilated by the Alexander polynomial \( \Delta \); see \cite[Theorem 6.17]{Lic}. We define the cohomological Blanchfield pairing to be the bilinear map 

\[
\text{cBl} : H^1(E_K, \partial E_K; \Lambda/\langle \Delta \rangle)^2 \to \Lambda/\langle \Delta \rangle
\]

by setting \( \text{cBl}(u, v) = \langle \beta(u) \prec v, [E_K, \partial E_K] \rangle \). Consider the following kernel:

\[
\ker(tV - V')_{\Lambda/\Delta} := \{ w \in (\Lambda/\langle \Delta \rangle)^2g \mid (tV - V')w = 0 \in (\Lambda/\langle \Delta \rangle)^2g \}. 
\]

Furthermore, we introduce two maps

\[
\begin{align*}
\psi_l : \Delta \Lambda^{\text{op}} \otimes \Lambda & \to \mathbb{Z}[t^{\pm 1}]/\langle \Delta \rangle; \quad \Delta x \otimes y \mapsto \bar{x} y, \\
\psi_r : \Lambda^{\text{op}} \otimes \Delta \Lambda & \to \mathbb{Z}[t^{\pm 1}]/\langle \Delta \rangle; \quad z \otimes \Delta w \mapsto \bar{z} w.
\end{align*}
\]

**Proposition 3.2.** Choose a section \( s : (\Lambda/\langle \Delta \rangle)^{2g} \to \Lambda^{2g} \). The cohomology \( H^1(E_K, \partial E_K; \Lambda/\langle \Delta \rangle) \)

is isomorphic to \( \ker(tV - V')_{\Lambda/\Delta} \). Furthermore, the cohomological Blanchfield pairing is isomorphic to the bilinear form

\[
\ker(A)_{\Lambda/\Delta} \times \ker(A)_{\Lambda/\Delta} \to \Lambda/\langle \Delta \rangle; \quad (v, w) \mapsto (1 - t)\psi_l((tV - V')s(v), w).
\]

Here, \( \psi_l : (\Delta \Lambda^{\text{op}} \otimes \Lambda)^{2g} \to \Lambda/\langle \Delta \rangle \) is the direct sum of \( (7) \).

To prove Proposition 3.2 we review from \cite{T1} the relative cellular chain complex of \((E_K, \partial E_K)\) with local coefficients \( R \). Here, we let \( R \) be either \( \Lambda \) or \( \Lambda/\langle \Delta \rangle \). According to \cite[Proposition 4.1]{T1}, the complex is isomorphic to

\[
C_* : 0 \to R \xrightarrow{\partial_3} R^{2g} \oplus R^2 \xrightarrow{\partial_2} R^{2g} \oplus R \xrightarrow{\partial_1} R \to 0. \tag{8}
\]

Here the differential maps \( \partial_3 \) have matrix presentations

\[
\partial_3 = (0, 0, \cdots, 0, 1 - t), \quad \partial_2 = \begin{pmatrix} tV - V' & 0 & 0 \\ 0 & 1 - t & 0 \end{pmatrix}, \quad \partial_1 = (0, \cdots, 0, 1 - t)'.
\]

Furthermore, we consider the cochain complex \( C^* : \text{Hom}(C_*; R) \). Pick a 2-cochain and a 1-cochain of the forms

\[
c^2 = (f_1, f_2) \in \text{Hom}(R^{2g}, R) \oplus \text{Hom}(R^2, R), \quad c^1 = (g_1, g_2) \in \text{Hom}(R^{2g}, R) \oplus \text{Hom}(R, R)
\]

Let \( c^1 \prec c^2 \) be \((1 - t)f_1 \cdot g_1' \in \text{Hom}(R, R) = C^3\). Then, it is shown \cite{T1} that the map \( H^1 \otimes H^2 \to H^3 \) induced by \( \prec : C^1 \otimes C^2 \to C^3 \) coincides with the natural cup product.
Proof of Proposition 3.2. Notice that $\det(tV - V') = \Delta$. By the presentation (8), we have a canonical isomorphism $H_1(E_K, \partial E_K; \Lambda/\Delta) \cong \text{Ker}(tV - V')_{\Lambda/\Delta}$, and can identify the Bockstein map $\beta : H^1(E_K, \partial E_K; \Lambda/\Delta) \to H^2(E_K, \partial E_K; \Lambda)$ with the mapping $v \mapsto (tV - V')s(v)$. Therefore, by the above formula of the cup product, we readily see $\text{cBl}(u, v) = (1-t)\psi_l((tV - V')s(u), v)$ as required.

Next, we will see a corollary. In general, it is easier to quantitatively compute kernels rather than cokernels. Using adjugate matrices, consider the linear map

$$\kappa : \Lambda^{2g}/(tV - V')\Lambda^{2g} \to \text{Ker}(tV - V')_{\Lambda/\Delta}; \quad v \mapsto \text{adj}(tV - V')v.$$ 

This map is an isomorphism. Indeed, if we choose a section $s : \Lambda^{2g}/(tV - V')\Lambda^{2g} \to \text{Ker}(tV - V')_{\Lambda/\Delta}$, the inverse mapping is defined by the map $w \mapsto (tV - V')s(w)/\Delta$. In summary, from Theorem 3.1 and Proposition 3.2, we immediately have

**Corollary 3.3.** The map $\kappa$ gives the isomorphism

$$\kappa : H_1(E_K, \partial E_K; \Lambda) = \Lambda^{2g}/(tV - V') \cong \text{Ker}(tV - V')_{\Lambda/\Delta} = H^1(E_K, \partial E_K; \Lambda/\Delta)$$

such that, for any $x, y \in H_1(E_K, \partial E_K; \Lambda)$,

$$\text{Bl}_K(x, y) = \text{cBl}_K(\kappa(x), \kappa(y)) \in \Lambda/\Delta.$$ 

### 3.2 Three Bockstein maps

We further need three Bockstein maps and their properties. We focus on the case $M = A = \Lambda/\Delta).$ Consider exact sequences

$$0 \to \Lambda \xrightarrow{\Delta} \Lambda \to \Lambda/\Delta \to 0,$$

$$0 \to \Lambda^{\text{op}} \xrightarrow{\Delta} \Lambda^{\text{op}} \to \Lambda^{\text{op}}/\overline{(\Delta)} \to 0. \quad (10)$$

The tensor products over $\mathbb{Z}$ canonically give rise to an exact sequence

$$0 \to (\overline{(\Delta^{\text{op}} \otimes \Lambda)}) \oplus (\Lambda^{\text{op}} \otimes \Delta^{\text{op}}) \to \Lambda^{\text{op}} \otimes \Lambda \to \Lambda^{\text{op}}/\overline{(\Delta)} \otimes \Lambda/\Delta \to 0.$$

Noticing $\overline{\Delta} = \Delta$, we have

$$0 \to \overline{(\Delta^{\text{op}} \otimes \Lambda)} \oplus (\Lambda^{\text{op}} \otimes \Delta^{\text{op}}) \to \Lambda^{\text{op}} \otimes \Lambda \to \Lambda^{\text{op}}/\overline{(\Delta)} \otimes \Lambda/\Delta \to 0 \quad (11)$$

Denote by $\gamma$ the associated Bockstein map. Then, using (7), they induce

$$(\psi_l \oplus \psi_r)_* : H^3(E_K, \partial E_K; \overline{(\Delta^{\text{op}} \otimes \Lambda^{\text{op}})} \oplus (\Lambda^{\text{op}} \otimes \Delta^{\text{op}})) \to H^3(E_K, \partial E_K; \Lambda/\Delta).$$

Moreover, let us define the following composite homomorphisms:

$$\Upsilon : H^2(E_K, \partial E_K; \Lambda^{\text{op}}/(\Delta) \otimes \Lambda/(\Delta)) \xrightarrow{(\psi_l \oplus \psi_r)_* \circ \gamma} H^3(E_K, \partial E_K; \Lambda/\Delta) \to A, \quad \Upsilon : H^2(E_K, \partial E_K; \Lambda^{\text{op}}/(\Delta) \otimes \Lambda/(\Delta)) \xrightarrow{(\psi_l \oplus \psi_r)_* \circ \gamma} H^3(E_K, \partial E_K; \Lambda/\Delta) \to A. \quad (12)$$
\[ \Phi : H^2(E_K, \partial E_K; \Lambda/(\Delta) \otimes \Lambda^\text{op}/(\Delta)) \xrightarrow{\bullet \cap \nu} \Lambda^\text{op}/(\Delta) \otimes \Lambda/(\Delta) \xrightarrow{\psi_0} A, \]

where \( \bullet \cap [E_K, \partial E_K] \) is the cap-product with the relative fundamental 3-class in \( H_3(E_K, \partial E_K; \mathbb{Z}) \).

In addition, we will explain a Leibniz rule of Bockstein maps, and show Lemma 3.4 below. Let \( \alpha, \beta, \gamma \) be the associated Bockstein maps with (12), (10), (11), respectively. Let \( C \) be the first term in (11), and \( \nu : (\Delta \Lambda^\text{op} \otimes \Lambda) \oplus (\Lambda^\text{op} \otimes \Delta \Lambda^\text{op}) \to C \) be the projection. Then, it follows from [3] Proposition in p. 451 that

\[
\gamma(u \sim v) = \nu_*(\alpha(u) \sim v + (-1)^{\text{dim}(u)}(u \sim \beta(v))), \tag{13}
\]

for \( u \in H^*(X; \Lambda/(\Delta)), v \in H^*(X; \Lambda^\text{op}/(\Delta)) \).

**Lemma 3.4.** Let \( \alpha, \beta, \gamma \) be as above. Take \( u, v \in H^1(E_K, \partial E_K; \Lambda/\Delta) \). Then,

\[
\langle \gamma(u \sim v), [E_K, \partial E_K] \rangle = \frac{1 + t}{1 - t} \langle \beta u \sim v, [E_K, \partial E_K] \rangle. \tag{14}
\]

*Proof.* From the definitions of \( \psi_r \) and \( \psi_l \), we notice that

\[
\psi_r(u, (tV - V') \cdot s(v)) = \psi_l((tV - V') \cdot s(u), v). \tag{15}
\]

By (13), the left hand side in (14) is formulated

\[
\psi_l((tV - V')s(u), v) - \psi_r(u, (tV - V') \cdot s(v)),
\]

which is computed as

\[
= \psi_l((tV - V')s(u), v) - \psi_l((t^{-1}V' - V) \cdot s(u), v)
\]

\[
= \psi_l((tV - V')s(u), s(v)) + \psi_l((V - t^{-1}V') \cdot s(u), v)
\]

\[
= \psi_l((tV + V - V' - t^{-1}V')s(u), v)
\]

\[
= (1 + t)\psi_l((tV - V')s(u), v).
\]

Since \( \langle \beta u \sim v, [E_K, \partial E_K] \rangle \) is \( (1 - t)\psi_l((tV - V')s(u), s(v)) \) by Proposition 3.2, we have the desired equality. \( \square \)

### 3.3 Final discussion

To prove Theorem 1.2, we need a lemma:

**Lemma 3.5.** There is a constant \( \alpha_K \) such that \( \Phi = \alpha_K \Upsilon \), where \( \Phi \) and \( \Upsilon \) are given in (12).

*Proof.* First, we notice that, if the tensor is defined over \( \Lambda \), the coefficient \( \Lambda^\text{op}/(\Delta) \otimes \Lambda/(\Delta) \) is a trivial coefficient, and is additively isomorphic to \( \Lambda/(\Delta) \). Thus, we have an additive isomorphism

\[
H^2(E_K, \partial E_K; \Lambda^\text{op}/(\Delta) \otimes \Lambda/(\Delta)) \cong H^2(E_K, \partial E_K; \mathbb{Z}) \otimes \Lambda/(\Delta) \cong \Lambda/(\Delta).
\]

By the definition of \( Q_\psi \) (see (11)), \( Q_\psi \) factors through this second homology. Thus, the maps \( \Phi \) and \( \Upsilon \) are regarded as multiplications of \( P \) and \( Q \) for some \( P, Q \in \Lambda/(\Delta) \), respectively. Here, we notice that \( P \) is invertible in \( \Lambda/(\Delta) \), since, if not so, the Blanchfield pairing is not non-singular. Hence, defining \( \alpha_K \) by \( P^{-1}Q \), we have \( \Phi = \alpha_K \Upsilon \) as required. \( \square \)
Proof of Theorem 1.2. Recall the definition of $\Upsilon$ in (12), the left hand side in (14) is equal to $\Upsilon(u \sim v)$. If $u = \kappa(x), v = \kappa(y)$, the right hand side in (14) equals $\frac{1+t}{1-t} \cdot \text{Bl}_K(x, y)$ by Corollary 3.3. On the other hand, by the definition of $\Phi$ in (12), $\Phi(u \sim v) = Q_{\psi_0}(u, v)$. By Lemma 3.5, $\Phi = \alpha_K \Upsilon$ for some $\alpha_K \in \Lambda/(\Delta)$; we have

$$Q_{\psi_0}(\kappa(x), \kappa(y)) = Q_{\psi_0}(u, v) = \Phi(u \sim v) = \alpha_K \Upsilon(u \sim v) = \alpha_K \frac{1+t}{1-t} \cdot \text{Bl}_K(x, y) \in \mathbb{Z}[[t^{\pm 1}]/(\Delta),$$

which is the required equality. It remains to show $\overline{\alpha_K} = \alpha_K$. Indeed, since $\text{Bl}_K$ is hermitian and $Q_{\psi_0}$ is anti-hermitian, $\alpha_K$ must satisfy $\overline{\alpha_K} = \alpha_K$. 

\section{Computation I: small knots and some non-fibered knots}

From this section, we will give the resulting computations of the Blanchfield pairings and the other pairing $Q_{\psi_0}$ for some knots; recall the definition $Q_{\psi_0}$ in Theorem 2.1. Here, the former pairing is easily computed by Proposition 3.2 in terms of Seifert matrices, and the latter is computed from Theorem 2.1 in terms of $X$-colorings. Here, we use data of the Seifert matrices from KnotInfo [CL].

We give a list of the resulting computations of all knots of crossing number < 8; see Table 1. Here, if $\alpha_K \neq 1$ in the table, it is not hard to verify that $\alpha_K$ is not invertible and not an zero divisor in $\Lambda/\Delta$.

| Knot type | Alexander polynomial $\Delta$ | $\alpha_K \in \Lambda/(\Delta)$ |
|-----------|-----------------------------|-----------------------------|
| 3,1       | $t^2 - t + 1$               | 1                           |
| 4,1       | $t^2 - 3t + 1$              | 1                           |
| 5,1       | $t^4 - t^3 + t^2 - t + 1$   | $t^{-1} + 2 + t$            |
| 5,2       | $2t^2 - 3t + 2$             | 1                           |
| 5,3       | $2t^2 - 5t + 2$             | 1                           |
| 6,1       | $t^4 - 3t^3 + 3t^2 - 3t + 1$| $3t^{-1} - 7 + 3t$          |
| 6,3       | $t^4 - 3t^3 + 5t^2 - 3t + 1$| $t + t^{-1}$               |
| 7,1       | $t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$ | $3t^{-2} - 2t^{-1} + 4 - 2t + 3t^2$ |
| 7,2       | $3t^2 - 5t + 3$             | $2t^{-1} - 3 + 2t$          |
| 7,3       | $2t^4 - 3t^3 + 3t^2 - 3t + 2$| $(-3 + 2t)(-2 + 3t^{-1})$   |
| 7,4       | $4t^2 - 7t + 4$             | 1                           |
| 7,5       | $2t^4 - 4t^3 + 5t^2 - 4t + 2$| $2(2t^{-1} - 3 + 2t)$       |
| 7,6       | $t^4 - 5t^3 + 7t^2 - 5t + 1$| $t^{-1} - 5 + t$            |
| 7,7       | $t^4 - 5t^3 + 9t^2 - 5t + 1$| $t^{-1} - 4 + t$            |

Table 1: The constants $\alpha_K$ for knots of crossing number < 8.

Similarly, we can compute $Q_{\psi_0}$ for knots of higher crossing number. In our experience in the computation, when the Alexander polynomial $\Delta$ is divisible by a square of a polynomial and $K$ is not fibered, the constant $\alpha_K$ might be a non-trivial divisor of $\Delta$ and not a unit. As examples, we give a table: Thus, the Blanchfield pairings of such knots can not be recovered from $Q_{\psi_0}$'s. As these tables imply, it might be a difficult problem to give a formula to determine $\alpha_K$ for every knot $K$. 


Knot type | Alexander polynomial $\Delta$ | $\alpha_K \in \Lambda/(\Delta)$
---|---|---
$8_{20}$ | $(t^2 - t + 1)^2$ | $t - 1 + t^{-1}$
$11_{73}$ | $(t^2 - t + 1)^2$ | 0
$12_60169$ | $(2t^2 - 3t + 2)^2$ | $4(-2 + t)(2t^{-1} - 3 + 2t)$
$12_60057$ | $(t^2 - t + 1)^2$ | $(t + t^{-1})(t - 1 + t^{-1})$
$12_{n087}$ | $(2t^2 - 3t + 2)^2$ | $2t - 3 + 2t^{-1}$
$12_{n0279}$ | $(t^2 - 3t + 1)^2$ | $(t - 3 + t^{-1})(3t - 7 + 3t^{-1})$

Table 2: The constants $\alpha_K$ for some non-fibered knots.

5 Computation II: Pretzel knots

We will focus on the Pretzel knot as in Figure 3. Take odd numbers $p, q, r \in \mathbb{Z}$ such that $p = 2\ell + 1, q = 2m + 1, r = 2n + 1$. Then, the Alexander polynomial is known to be $\Delta = \frac{1}{4}((pq + qr + rp)(t^2 - t + 1) + t^2 + t + 1)$. By observing the discriminant, $\Delta$ cannot be any square of some polynomial. The purpose of this section is to show the following:

**Theorem 5.1.** If $K$ is the Pretzel knot $P(p, q, r)$, then $K$ satisfies the assumption of Theorem 1.2 and $\alpha_K = 1$.

In other words, $\text{Bl}_K$ is completely recovered from the cohomology pairing. Since this theorem immediately follows from a comparison between Propositions 5.2 and 5.3 below, we will show the propositions.

Figure 3: The Pretzel knot $P(p, q, r)$ and the $T_{m,n}$-torus link with labeled arcs. Here, the boxes in the left hand side mean $p$, $q$, $q$-twists, respectively.

For the purpose, one computes the Blanchfield pairing $\text{Bl}_K$. According to [Lic, Example 6.9], we can choose the Seifert matrix of the form $V = \frac{1}{2} \left( \begin{array}{cc} p + q & q + 1 \\ q - 1 & q + r \end{array} \right)$.

**Proposition 5.2.** The kernel $\text{Ker}_{\Lambda/\Delta}(tV - V') \subset (\Lambda/\Delta)^2$ is generated by two elements

$((1 + m + n)(t - 1), t + mt - m), \quad w = (mt - 1 - m, (1 + \ell + m)(t - 1))$.

Furthermore, we have

$$
\begin{pmatrix}
\text{cBl}_K(v, v) & \text{cBl}_K(v, w) \\
\text{cBl}_K(w, v) & \text{cBl}_K(w, w)
\end{pmatrix} = (1 - t^{-1}) \begin{pmatrix}
(1-t)(1+m+n) & (-1-m+mt) \\
(-m+t+mt) & (1-t)(1+m+\ell)
\end{pmatrix}.
$$
Thanks to Proposition 3.2, the proof can be easily obtained by the help of a computer program of Mathematica; we omit the details.

On the other hand, we will compute the cohomology pairing $Q_{\psi_0}$.

**Proposition 5.3.** Let $X$ and $A$ be $\Lambda/\Delta$ in usual. Consider the submodule, $K$, of $(\Lambda/\Delta)^2$ independently generated by two elements

$$v = ((1 + m + n)(t - 1), t + mt - m), \quad w = (mt - 1 - m, (1 + \ell + m)(t - 1)) \in (\Lambda/\Delta)^2.$$  

Then, there exists a $\Lambda$-isomorphism $\theta : K \cong \text{Col}_X^\text{red}(D)$ such that

$$\begin{pmatrix}
Q_{\psi_0}(\theta(v), \theta(v)) & Q_{\psi_0}(\theta(v), \theta(w)) \\
Q_{\psi_0}(\theta(w), \theta(v)) & Q_{\psi_0}(\theta(w), \theta(w))
\end{pmatrix} = (1 + t^{-1})
\begin{pmatrix}
(1 - t)(1 + m + n) & (-1 - m + mt) \\
(-m + t + mt) & (1 - t)(1 + m + \ell)
\end{pmatrix}.$$  

**Proof.** For simplicity, we suppose that all of $p, q, r$ is positive. Since the proofs of other cases can be done in the same way, we omit considering the other cases.

We first notice the following lemma, which can be obtained by definitions:

**Lemma 5.4.** Consider the $(2,2)$-tangle with $(2N + 1)$-twist, and the labels of the arcs in Figure 4. Choose $a_1, a_2, x_1, x_2, y_1, y_2 \in X$. For $j \in \{1, 2\}$, the assignment $C_j$

$$\alpha_k \mapsto a_j + k(1 - t)(y_j - x_j) + x_j, \quad \beta_k \mapsto a_j + k(1 - t)(y_j - x_j) + y_j$$

defines an $X$-coloring. Moreover, the weight sum with respect to the $2N + 1$ crossings is

$$Q_{\psi_0}(C_1, C_2) = -((x_1 - y_1)(a_2 + y_2) + N(x_1 - y_1)(x_2 - y_2))(1 - t^{-1}).$$  

(16)

![Figure 4: The $(2,2)$-tangle as a $(2N + 1)$-twist. Here the box means a twist.](image)

Given an $X$-coloring $C$ of $P(p, q, r)$, we put $a, x, y \in X$ such that $C(\alpha^u) = a, C(\beta^u) = a + x, C(\gamma^u) = a + y$. By Lemma 5.4, we have the simultaneous equations

$$C(\alpha_b) = (\ell + 1)(1 - t)x + a = m(y - x)(1 - t) + y + a,$$

$$C(\beta_b) = (1 + m)(y - x)(-1 + t) + x + a = -ny(1 - t) + a,$$

$$C(\gamma_b) = -(1 + n)y(-1 + t) + y + a = \ell(1 - t)x + x + a \in \Lambda/\Delta.$$

Then, by the help of a computer program of Mathematica, we have two solutions

$$\begin{cases}
x_1 = (1 + m + n)(t - 1), \\
y_1 = t + mt - m,
\end{cases} \quad \text{or} \quad \begin{cases}
x_2 = mt - 1 - m, \\
y_2 = (1 + \ell + m)(t - 1).
\end{cases}$$
Furthermore, it can be verified that every solutions of \((x, y)\) is a linear sum of the two solutions. Hence, we have the desired isomorphism \(\theta : \mathcal{K} \cong \text{Col}_X^\text{red}(D)\).

For \(i \in \{1, 2\}\), let \(\mathcal{C}_i \) be the \(X\)-coloring associated with the solution \((x_i, y_i)\). Thanks to Lemma 5.4 again, given two \(X\)-colorings \(\mathcal{C}_i \) and \(\mathcal{C}'_j\), the sum \(Q_{\psi_0}(\mathcal{C}_i, \mathcal{C}'_j)\) is equal to

\[-(\bar{x}_i(a_j + x_j) + \ell \bar{x}_i x_j + (\bar{x}_i - \bar{y}_i)(a_j + y_j) + m(\bar{x}_i - \bar{y}_i)(x_j - y_j) + \bar{y}_ia_j + n\bar{y}_i x_j)(1 - t^{-1})\]

Using Mathematica for the computation modulo \(\Delta\), we can obtain the desired equality in the 2 \(\times\) 2-matrix. \(\square\)

6 Computation III: the torus knot

We will compute the cohomology pairing of the \((m, n)\)-torus knot \(T_{m,n}\). Here note the known fact that the Alexander module is isomorphic to \(\mathbb{Z}[t^{\pm 1}]/(\Delta)\), where the Alexander polynomial \(\Delta\) is \((t^{mn} - 1)(t - 1)/(t^{m-1})(t^{m-1})\); see [Rol].

**Proposition 6.1.** Fix \((n, m, a, b) \in \mathbb{Z}^4\) with \(an + bm = 1\), and let \(K = T_{m,n}\). Then,

\[Q_{\psi_0}(y_1, y_2) = \frac{nm(1 - t^{-1})}{(1 - t^{bm})(1 - t^{an})} \cdot \bar{y}_1 y_2 \in \mathbb{Z}[t^{\pm 1}]/(\Delta),\]

for \(y_1, y_2 \in H^1(S^3 \setminus T_{m,n}; \mathbb{Z}[t^{\pm 1}]/(\Delta)) \cong \mathbb{Z}[t^{\pm 1}]/(\Delta)\).

Remark that the coefficient in (17) lies in \(\mathbb{Z}[t^{\pm 1}]/(\Delta)\) because of l’Hôpital’s rule in \(\mathbb{Z}[t^{\pm 1}]\).

**Proof.** Let \(X = A = \mathbb{Z}[t^{\pm 1}]/(\Delta)\). By Theorem 5.2 we have \(\text{Col}_X(D) \cong H_1(S^3 \setminus K; A) \oplus X \cong X^2\); it is enough to compute the bilinear form \(Q_{\psi}\) with \(\psi(y, z) = \bar{y}z\).

To this end, let us start by examining \(\text{Col}_X(D)\) in more details. Let \(\alpha_1, \ldots, \alpha_m\) be the arcs depicted in Figure 3. Because of the shape of \(D\), every coloring in \(\text{Col}_X(D)\) is characterized by colors of these \(m\) arcs. Hence, we can view \(\text{Col}_X(D)\) as a submodule of \(X^m\). In addition, for \(k \in \{1, 2\}\), consider elements of the forms

\[\bar{y}_k = (\delta_k, y_k + \delta_k, \frac{1 - t^{2an}}{1 - t^{an}} y_k + \delta_k, \ldots, \frac{1 - t^{an(m-1)}}{1 - t^{an}} y_k + \delta_k) \in X^m,\]

for some \(y_k, \delta_k \in X\). From this view, we can easily see that these elements (18) define \(X\)-colorings. Further, the first and second components imply that these elements in (18) give a basis of \(\text{Col}_X(D) \cong X^2\). Let \(\tau_{i,j}\) be the \(j\)-th crossing point on the arc \(\alpha_i\). Then, concerning the \(X\)-coloring \(C\) arising from (18), the colors around \(\tau_{i,j}\) as Figure 4 are formulated as

\[\left(\mathcal{C}(\alpha_{\tau_{i,j}}), \mathcal{C}(\beta_{\tau_{i,j}})\right) = \left(\frac{1 - t^{bmj}}{1 - t^{bm}} t^{an(i,j)} y_k + \frac{1 - t^{an(i-1)}}{1 - t^{an}} y_k + \delta_k, \frac{1 - t^{an(i-1)}}{1 - t^{an}} y_k + \delta_k\right) \in A^2.\]

Accordingly, we now deal with the 2-form \(Q_{\psi}(\bar{y}_1, \bar{y}_2)\). By definition, compute it as

\[\sum_{i \leq m, j \leq n-1} \psi_0 \left( t^{an(i-2)} \frac{1 - t^{bmj}}{1 - t^{bm}} y_1, \frac{1 - t^{an(i-1)}}{1 - t^{an}} y_2 + \delta_2 \right) \left(1 - t^{-1}\right) \]
\[
= \psi_0 \left( \sum_{j=1}^{n-1} \frac{1 - t^{bmj}}{1 - t^{bm}} y_1, \left( \sum_{i=1}^{m} \frac{t^{an} - t^{an(2-i)}}{1 - t^{an}} \right) y_2(1 - t^{-1}) \right) \\
= \psi_0 \left( \frac{1 - t^{bm(n-1)}}{(1 - t^{bm})^2} - (n - 1)(1 - t^{bm}) \right) y_1, \frac{m(1 - t^{an}) + 1 - t^{-amn}}{(1 - t^{an})^2} t^{an} \cdot y_2(1 - t^{-1}) \\
= \psi_0 \left( \frac{n}{1 - t^{bm}} \cdot y_1, \frac{m}{1 - t^{an}} \cdot y_2(1 - t^{-1}) \right) = \frac{nm(1 - t^{-1})}{(1 - t^{bm})(1 - t^{an})} \cdot \bar{y}_1 y_2 \in \mathbb{Z}[t^{\pm 1}]/(\Delta),
\]

where the first equality is obtained by the \(t\)-invariance of \(\psi\) and the equality \(\sum_{i=1}^{m} t^{ani} = 0\), and an elementary computation lifted in \(\mathbb{Z}[t^{\pm 1}]\) can imply the third equality by noting \(t^{mn} = 1 \in X\). \(\square\)

Finally, we will give a comparison with the Blanchfield pairing of \(T_{m,n}\). The pairing has not since been computed; the essential reason is the Seifert genus is \((n - 1)(m - 1)/2\), i.e., it seems impossible to compute the pairing from Theorem [3.2] using the Seifert matrix. Furthermore, it is a subtle problem whether \(\text{Bl}_{T_{m,n}}\) can be recovered from \(Q_{\psi_0}\) or not. In fact, the coefficients \(nm/(1 - t^{bm})(1 - t^{an})\) are not units in \(\Lambda/(\Delta)\) in many cases; for example, we can easily verify that, if \(m\) is even, the coefficient is divisible by \((1 + t)^2\), and that if \((m, n) = (3, 7)\), \(\Delta(2) = 7 \cdot 337\) and the coefficient modulo \(t + 2\) is \(2^3 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 17\). Meanwhile, the coefficient has some important information of \(\text{Bl}_{T_{m,n}}\): for example, Matsumoto [Mat], Kearton [Ke2], and Litherland [Lit] independently use other technical formula to compute all the local signatures of \(T_{m,n}\), and the signatures can be recovered from the form of the coefficient.

A Universality of Alexander quandle cocycle invariants.

The paper [CJKLS] constructed a knot invariant from quandle cocycles. However, the invariant was defined in a combinatorial way without topological meanings. This appendix reviews the invariant, and gives a topological meaning in the Alexander case, as an application of Theorems [1.2] and [2.1].

For the purpose, we first briefly review the invariant, supposing that the reader has read Sections 2–4. Let \(X\) be a \(\mathbb{Z}[t^{\pm 1}]\)-module, which is regarded as an Alexander quandle. Further, given an abelian group \(A\), we suppose a map \(\phi : X^2 \to A\) satisfying the equality

\[
\phi(x, z) - \phi(y, z) - \phi(x \triangleleft y, z) + \phi(x \triangleleft z, y \triangleleft z) = 0,
\]

\[
\phi(x, x) = 0,
\]

for any \(x, y, z \in X\). Such a map \(\phi\) is called a quandle 2-cocycle [CJKLS]. Then, in analogy of [2] let us define a map \(\mathcal{I}_\phi : \text{Col}_X(D) \to A\):

\[
C \mapsto \sum_{\tau} \epsilon_\tau \cdot \phi(C(\alpha_\tau), C(\beta_\tau)),
\]

where \(\tau\) ranges over all the crossings of \(D\), and the symbols \(\alpha_\tau, \beta_\tau\) are the arcs and \(\epsilon_\tau \in \{\pm 1\}\) is the sign of \(\tau\) according to Figure [1]. Then, it is known [CJKLS] that, thanks to (19), the map
\(\mathcal{I}_\Phi\) is independent of the choice of \(D\) by \((19)\); then, \(\mathcal{I}_\Phi\) is called the quandle cocycle invariant. For example, given an additive homomorphism \(\psi : X^2 \to A\) satisfying \(\psi(tx, ty) = \psi(x, y)\) for any \(x, y \in X\), we can easily verify that the map \(\phi_{\psi}\) defined by \(\phi_{\psi}(x, y) = \psi(x - y, y - yt^{-1})\) satisfies \((19)\), and, by definitions, that the associated invariant \(\mathcal{I}_\Phi\) is equal to the restricted 2-form \(Q_\psi \circ \triangle\), where \(\triangle\) is the diagonal map \(\text{Col}_X(D) \to \text{Col}_X(D)^2\) and \(Q_\psi\) with \(X = M = M'\) is defined in \((1)\).

Similar to Theorem 1.2 we will show the S-equivalence and a universality of the quandle 2-cocycle invariants.

**Theorem A.1.** If two knots \(K\) and \(K'\) are S-equivalent, then for every Alexander quandle \(X\) and every quandle 2-cocycle \(\phi : X^2 \to A\), the associated cocycle invariants \(\mathcal{I}_\Phi\) and \(\mathcal{I}_\Phi'\) are equivalent.

Furthermore, for a knot \(K\), there is an Alexander quandle \(X_0\) and a bilinear map \(\psi_{X_0} : X_0 \times X_0 \to X_0\) such that, for any Alexander quandle \(X\), any quandle 2-cocycle \(\phi : X^2 \to A\) and any \(X\)-coloring \(C\) of \(K\), there are an \(X_0\)-coloring \(C_0\) and an additive homomorphism \(\mathcal{P}_\phi : X_0 \to A\) such that \(\mathcal{P}_\phi(Q_{\psi_{X_0}}(C_0, C_0)) = \mathcal{I}_\Phi(C)\).

In conclusion, this theorem implies that every 2-cocycle invariants from Alexander quandles can be described from the Blanchfield pairing \(\text{Bl}_K\) in principle. While the cocycle invariants are diagrammatically defined, this theorem implies a topological interpretation of the 2-cocycle invariants in the sense of cohomology pairings \(Q_\psi\).

**Proof of Theorem A.1.** As the first step, we claim that, for any bilinear function \(\psi : X^2 \to A\) satisfying \(\psi(x, y) = \psi(ty, x)\), the associated bilinear form \(Q_{\phi_{\psi}}\) is an S-equivalent knot invariant. Notice that \(\psi\) satisfies \(\psi(x, y) = \psi(ty, x) = \psi(tx, ty)\). Therefore, in the same way as the proof of Theorem 1.1, the claim can be easily shown.

We will deal with any quandle 2-cocycle \(\phi : X^2 \to A\). According to \cite[Theorem 17.3]{Joy} concerning “Abelianization of the knot quandle”, for any Alexander quandle \(X\), there is a functorial \(\mathbb{Z}[t^{\pm 1}]-\text{module isomorphism}\)

\[
\text{Col}_X(D) \cong \text{Hom}_{\mathbb{Z}[t^{\pm 1}]-\text{mod}}(H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}]), X) \oplus X.
\]

In other words, \(\text{Col}_X(D)\) is representable by the Alexander module \(H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])\). Hence, by the universality, we may assume \(X = H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])\) hereafter. In particular, \((1-t)X = X\).

Further, consider “the quandle second homology \(H_2^Q(X; \mathbb{Z})\)” defined in \cite{CJKLS}, which is isomorphic to the quotient \(\mathbb{Z}\)-module

\[
G_X := X \otimes_{\mathbb{Z}} X/\{x \otimes y - (ty) \otimes x \mid x, y \in X\}.
\]

This result is essentially due to Clauwens \cite{Cla} (see also \cite{No1} §5 or \cite{BKMNP} for the proof). Further, consider a homomorphism \(\psi_{\text{univ}} : X \otimes X \to G_X\) which sends \((x, y)\) to \([x \otimes y]\). Then, it is known (see, e.g., \cite{CJKLS, No1}), every cocycle invariant \(\triangle \circ Q_\psi\) factors through the homology \(H_2^Q(X; \mathbb{Z})\). Precisely, there is an additive homomorphism \(P_\psi : G_X \to A\) such that
\(Q_{\psi_{\text{uni}}} \circ \triangle = P_{\psi}(Q_{\psi} \circ \triangle)\) for any link \(L\). By the discussion in the first paragraph, \(Q_{\psi_{\text{uni}}}\) is \(S\)-equivalent; hence, so is \(Q_{\psi}\) for any quandle 2-cocycle \(\psi\).

To show the latter part, let \(X_0\) be \(H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])\) as above. Since \(X_0\) is finitely generated over \(\Lambda\), we can make \(X_0\) into a \(\Lambda\)-algebra. Then, we let \(\psi_{X_0} : X_0 \times X_0 \rightarrow X_0\) send \((x, y) \mapsto xy\), which factors through \(G_{X_0}\) via \(\psi_{\text{uni}}\). Therefore, by the discussion in the previous paragraph, any cocycle invariant \(I_\Phi\) is derived from \(I_{\Phi_{X_0}} = (Q_{\psi_{\text{uni}}} \circ \triangle)\). This means the desired statement.

\[\square\]

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