AUTOMORPHISM-INVARIANT MODULES SATISFY THE EXCHANGE PROPERTY

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Abstract. Warfield proved that every injective module has the exchange property. This was generalized by Fuchs who showed that quasi-injective modules satisfy the exchange property. We extend this further and prove that a module invariant under automorphisms of its injective hull satisfies the exchange property. We also show that automorphism-invariant modules are clean and that directly-finite automorphism-invariant modules satisfy the internal cancellation and hence the cancellation property.

1. Introduction.

A module $M$ which is invariant under automorphisms of its injective hull is called an automorphism-invariant module. This class of modules was first studied by Dickson and Fuller in [4] for the particular case of finite-dimensional algebras over fields $\mathbb{F}$ with more than two elements. Clearly, every quasi-injective and hence injective module is automorphism-invariant. Recently, it has been shown in [5] that a module $M$ is automorphism-invariant if and only if every monomorphism from a submodule of $M$ extends to an endomorphism of $M$. And it has been shown in [9] that any automorphism-invariant module $M$ is quasi-injective provided that $\text{End}(M)$ has no homomorphic image isomorphic to the field of two elements $\mathbb{F}_2$, thus extending the results obtained by Dickson and Fuller in [4]. However, Remark [2] below shows that an automorphism-invariant module does not need to be quasi-injective if we do not assume this additional hypothesis. For more details on automorphism-invariant modules, see [5, 9, 13, 18, 19, 20].

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On the other hand, it was shown by Fuchs [7] that every quasi-injective module satisfies the exchange property, thus extending a previous result of Warfield for injective modules [21]. Recall that the notion of exchange property for modules was introduced by Crawley and Jónsson [3]. A right $R$-module $M$ is said to satisfy the exchange property if for every right $R$-module $A$ and any two direct sum decompositions $A = M' \oplus N = \bigoplus_{i \in I} A_i$ with $M' \cong M$, there exist submodules $B_i$ of $A_i$ such that $A = M' \oplus \bigoplus_{i \in I} B_i$. If this hold only for $|I| < \infty$, then $M$ is said to satisfy the finite exchange property. Crawley and Jónsson raised the question whether the finite exchange property always implies the full exchange property but this question is still open.

A ring $R$ is called an exchange ring if the module $R_R$ (or $R_R$) satisfies the (finite) exchange property. Goodearl [8] and Nicholson [15] provided several equivalent characterizations for a ring to be an exchange ring. Warfield [22] showed that exchange rings are left-right symmetric and that a module $M$ has the finite exchange property if and only if $\text{End}(M)$ is an exchange ring.

The goal of this paper is to show that, besides Remark 2 shows that an automorphism-invariant module $M$ does not need to be quasi-injective in general, it shares several important decomposition properties with quasi-injective modules. Namely, the endomorphism ring of an automorphism-invariant module $M$ is always a von Neumann regular ring modulo its Jacobson radical $J$, idempotents lift modulo $J$ and $J$ consists of those endomorphism of $M$ which have essential kernel (see Proposition 1). As a consequence, we show in Theorem 3 that any automorphism-invariant module satisfies the exchange property.

A module $M$ is said to have the cancellation property if whenever $M \oplus A \cong M \oplus B$, then $A \cong B$. A module $M$ is said to have the internal cancellation property if whenever $M = A_1 \oplus B_1 \cong A_2 \oplus B_2$ with $A_1 \cong A_2$, then $B_1 \cong B_2$. A module with the cancellation property always satisfies the internal cancellation property but the converse need not be true, in general. Fuchs [7] had shown that if $M$ is a module with the finite exchange property, then $M$ has the cancellation property if and only if $M$ has the internal cancellation property.

A module $A$ is said to have the substitution property if for every module $M$ with decompositions $M = A_1 \oplus H = A_2 \oplus K$ with $A_1 \cong A_2 \cong A$, there exists a submodule $C$ of $M$ (necessarily $\cong A$) such that $M = C \oplus H = C \oplus K$.

A module $M$ is called directly-finite (or Dedekind-finite) if $M$ is not isomorphic to a proper summand of itself. A ring $R$ is called directly-finite if $xy = 1$ implies $yx = 1$ for any $x, y \in R$. It is well-known
that a module $M$ is directly-finite if and only if its endomorphism ring $\text{End}(M)$ is directly-finite.

In general, we have the following hierarchy:

substitution $\implies$ cancellation $\implies$ internal cancellation $\implies$ directly-finite

In this paper, we show in Corollary 8 that for an automorphism-invariant module the above four notions are equivalent.

Throughout this paper, $R$ will always denote an associative ring with identity element and modules will be right unital. $J(R)$ will denote the Jacobson radical of the ring $R$. We will use the notation $N \subseteq_e M$ to stress that $N$ is an essential submodule of $M$. We refer to [1] and [14] for any undefined notion arising in the text.

**Results.**

It was proved by Faith and Utumi [6] that if $M$ is a quasi-injective module and $R = \text{End}(M)$, then $J(R)$ consists of all endomorphisms of $M$ having essential kernel and $R/J(R)$ is a von Neumann regular ring. Later, Osofsky [17] proved that $R/J(R)$ is right self-injective too.

Let us fix some notation that we will follow along this paper. Let $M$ be a module and $E = E(M)$, its injective hull. Call $R = \text{End}(M)$, $S = \text{End}(E)$ and $J(S)$, the Jacobson radical of $S$. Let us denote by $\varphi : R \to S/J(S)$ the ring homomorphism which assigns to every $r \in \text{End}(M)$ the element $s + J(S)$, where $s : E \to E$ is an extension of $r$ (see e.g. [9]). Set $\Delta = \{r \in R : \text{Ker}(r) \subseteq_e M\}$. As $J(S)$ consists of all endomorphisms $s \in S$ having essential kernel, we get that $\text{Ker}(\varphi) = \Delta$ and thus, $\varphi$ factors through an injective ring homomorphism $\psi : R/\Delta \to S/J(S)$.

Let $N$ be a submodule of a module $M$. A submodule $L$ of $M$ is called a complement of $N$ in $M$ if it is maximal with respect to the condition $N \cap L = 0$. Using Zorn’s Lemma it is very easy to check that any submodule $N$ of $M$ has a (not necessarily unique) complement $L$ and $N \oplus L \subseteq_e M$ (see e.g. [11 Proposition 5.21]).

In our first proposition, we extend the above mentioned result of Faith and Utumi to automorphism-invariant modules.

**Proposition 1.** Let $M$ be an automorphism-invariant module. Then $\Delta = J(R)$ is the Jacobson radical of $R$, $R/J(R)$ is a von Neumann regular ring and idempotents lift modulo $J(R)$.

**Proof.** Let $r \in R$ and let $s \in E$ be an extension of $R$. Call $K = \text{Ker}(r)$. Let $L$ be a complement of $K$ in $M$. Then $K \oplus L \subseteq_e M$ and thus,


$E = E(K) \oplus E(L)$. Set $g \in S$ as $g|_{E(K)} = 0$ and $g|_{E(L)} = s|_{E(L)}$. Then $(g - s)|_{K \oplus E(L)} = 0$ and therefore, $g - s \in J(S)$. Therefore, $1 - (g - s)$ is an automorphism of $E$. Since $M$ is automorphism-invariant, $(1 - (g - s))(M) \subseteq M$. This means that $(g - s)(M) \subseteq M$. Now as $s$ is an extension of $r \in R$, we have $s(M) \subseteq M$. This yields $g(M) \subseteq M$.

As $L \cap \text{Ker}(g) = 0$, $g|_{E(L)}$ is a monomorphism. Let $E' = \text{Im}(g) = \text{Im}(g|_{E(L)})$. Then $E' \cong E(L)$ is injective. Moreover, as $g|_{E(L)} : E(L) \to E'$ is an isomorphism, there exists a homomorphism $h : E' \to E(L)$ such that $h \circ g \circ u = u \circ 1_{E(L)}$ and thus, $u \circ h \circ g = u \circ \pi$, where $u : E(L) \to E$ and $\pi : E \to E(L)$ are the inclusion and projection associated to the decomposition $E = E(K) \oplus E(L)$. As $L$ is essential in $E$, $g(L)$ is essential in $E'$ and thus, $N = M \cap g(L)$ is also essential in $E'$, because $M$ is essential in $E$. As $M$ is automorphism-invariant, this means that the monomorphism $h|_N : N \to L \subseteq M$ extends to an endomorphism $t : M \to M$ (see [5]). Let $t' : E \to E$ be an extension of $t$. As $N$ is essential in $E'$, $g^{-1}(N)$ is essential in $E$ and thus, $N' = (K \oplus L) \cap g^{-1}(N)$ is also essential in $E$. Let us choose an element $x \in N'$. By definition of $N'$, we get that $g(x) \in N$. Moreover, we can write $x = k + l$ with $k \in K$ and $l \in L$. Therefore, $g(l) = g(k) + g(l) = g(x) \in N \subseteq M$. Thus, $t' \circ g(x) = t' \circ g(l) = t \circ g(l) = h \circ g(l) = l$. We have shown that $t' \circ g|_{N'} = u \circ \pi|_{N'}$ and, as $N'$ is essential in $E$, this means that $t' \circ g + J(S) = u \circ \pi + J(S)$. Therefore, $s \circ t' \circ s + J(S) = g \circ t' \circ g + J(S) = g \circ u \circ \pi + J(S) = g + J(S) = s + J(S)$. But then, we deduce that $\psi((r \circ t \circ r) + \Delta) = (s \circ t' \circ s) + J(S) = s + J(S) = \psi(r + \Delta)$ and, as $\psi$ is injective, we get that $(r \circ t \circ r) + \Delta = r + \Delta$. This shows that $R/\Delta$ is von Neumann regular.

Since $R/\Delta$ is von Neumann regular, $J(R/\Delta) = 0$. This gives $J(R) \subseteq \Delta$. Now let $a \in \Delta$. Since $\text{Ker}(a) \cap \text{Ker}(1 - a) = 0$ and $\text{Ker}(a) \subseteq M$, $\text{Ker}(1 - a) = 0$. Hence $(1 - a)$ is an isomorphism from $M$ to $(1 - a)(M)$. Since $M$ is automorphism-invariant, $M$ satisfies the $(C_2)$ property (see [5]), that is, submodules isomorphic to a direct summand of $M$ are direct summands. Therefore, $(1 - a)(M)$ is a direct summand of $M$. However, $(1 - a)(M) \subseteq M$ since $\text{Ker}(a) \subseteq (1 - a)(M)$. Thus $(1 - a)(M) = M$ and therefore, $1 - a$ is a unit in $R$. This means that $a \in J(R)$ and therefore $\Delta \subseteq J(R)$. Hence $J(R) = \Delta$.

This shows that $R/J(R)$ is a von Neumann regular ring. Now, we proceed to show that idempotents lift modulo $J(R)$. Let $e' + J(R)$ be an idempotent in $R/J(R)$. Let $f' + J(S) = \psi(e' + J(R))$. Then $f' + J(S)$ is an idempotent in $S/J(S)$. Since idempotents lift modulo $J(S)$, there exists an idempotent $f$ in $S$ such that $f' = f + j$ with $j \in J(S)$. Now, $1 - j$ is a unit in $S$, and so $M$ is invariant under $1 - j$. 
and hence \( j(M) \subseteq M \). And thus, \( f(M) \subseteq f'(M) + j(M) \subseteq M \). This means that \( e = f|_M \) belongs to \( R = \text{End}(M) \) and it is an idempotent since so is \( f \). By construction, \( \psi(e + J(R)) = f + J(S) = f' + J(S) = \psi(e' + J(R)) \). And, as \( \psi \) is an injective homomorphism, we deduce that \( e + J(R) = e' + J(R) \). This shows that idempotents lift modulo \( J(R) \).

\[ \square \]

Remark 2. Note that in the above proposition unlike quasi-injective modules, \( R/J(R) \) need to be right self-injective. For example, let \( R \) be the ring of all eventually constant sequences \( (x_n)_{n \in \mathbb{N}} \) of elements in \( \mathbb{F}_2 \). Then \( E(R_R) = \prod_{n \in \mathbb{N}} \mathbb{F}_2 \), and it has only one automorphism, namely the identity automorphism. Thus, \( R_R \) is automorphism-invariant but it is not self-injective. Also, as \( R \) is von Neumann regular, \( J(R) = 0 \). This means \( R/J(R) \) is not self-injective.

As a consequence of the above proposition, we are ready to generalize the results of Warfield and Fuchs.

**Theorem 3.** An automorphism-invariant module satisfies the exchange property.

**Proof.** Let \( M \) be an automorphism-invariant module and \( R = \text{End}(M) \). By the above proposition, \( R/J(R) \) is a von Neumann regular ring and idempotents lift modulo \( J(R) \). Such rings are called semiregular rings (or f-semiperfect rings). Nicholson in [15, Proposition 1.6] proved that every semiregular ring is an exchange ring. Hence \( R \) is an exchange ring. This proves that \( M \) has the finite exchange property.

Now, we know that \( M = P \oplus Q \) where \( Q \) is quasi-injective and \( P \) is a square-free module [5, Theorem 3]. Since a direct summand of a module with the finite exchange property also has the finite exchange property, \( P \) has the finite exchange property. But for a square-free module, the finite exchange property implies the full exchange property [16, Theorem 9]. So \( P \) has the full exchange property. We already know that every quasi-injective module has the full exchange property, so \( Q \) has the full exchange property. Since a direct sum of two modules with the full exchange property also has the full exchange property, it follows that \( M \) has the full exchange property. \[ \square \]

We would like to thank Professor Yiqiang Zhou for kindly pointing out the next corollary. Recall that a ring \( R \) is called a clean ring if each element \( a \in R \) can be expressed as \( a = e + u \) where \( e \) is an idempotent in \( R \) and \( u \) is a unit in \( R \). A module \( M \) is called a clean module if \( \text{End}(M) \) is a clean ring. It was shown in [2] that continuous modules are clean.
Corollary 4. Automorphism-invariant modules are clean.

Proof. Let $M$ be an automorphism-invariant module. We have $M = P \oplus Q$ where $Q$ is quasi-injective and $P$ is a square-free module [5, Theorem 3]. By [2], $Q$ is a clean module. By the above theorem, $\text{End}(P)$ is an exchange ring. We know that the idempotents in $\text{End}(P)/J(\text{End}(P))$ are central (see [14]). Thus $\text{End}(P)/J(\text{End}(P))$ is an exchange ring with all idempotents central. Hence $\text{End}(P)/J(\text{End}(P))$ is a clean ring. Since idempotents lift modulo $J(\text{End}(P))$ by the Proposition 1, it follows that $\text{End}(P)$ is a clean ring. Thus $P$ is a clean module and hence $M$ is clean. \(\square\)

We recall that a module $M$ is called indecomposable if its only direct summands are 0, and $M$. And a decomposition of a module $M$ as a direct sum of indecomposable modules, say $M = \oplus_{i \in I} M_i$, is said to complement (maximal) direct summands provided that for any (resp. maximal) direct summand $N$ of $M$ there exists a subset $I' \subseteq I$ such that $M = N \oplus (\oplus_{i \in I'} M_i)$ (see [1, §12]). In particular, it is shown in [1, 12.4] that if $M = \oplus_{i \in I} M_i$ is a decomposition that complements (maximal) direct summands, then this decomposition is unique up to isomorphisms in the sense of the Krull-Remak-Schmidt Theorem. On the other hand, it has been shown in [14, Theorem 2.25] that a decomposition of a module $M = \oplus_{i \in I} M_i$, satisfying that $\text{End}(M_i)$ is a local ring for every $i \in I$, complements maximal direct summands if and only if $M$ is an exchange module. We can then show.

Corollary 5. Let $M$ be an automorphism-invariant module. If $M$ is a direct sum of indecomposable modules, then this decomposition complements direct summands.

Proof. We know that $\text{End}(M)/J(\text{End}(M))$ is von Neumann regular and therefore, so is $\text{End}(M_i)/J(\text{End}(M_i))$, for every $i \in I$. And, as $M_i$ is indecomposable, this means that $\text{End}(M_i)$ is local. The result now follows from the previous comments. \(\square\)

Next, we state a useful lemma for directly-finite automorphism-invariant modules.

Lemma 6. Let $M$ be an automorphism-invariant module. If $M$ is directly-finite, then so is $E = E(M)$.

Proof. Let $s, t \in S$ such that $t \circ s = 1_E$. This means that $s : E \rightarrow E$ is a monomorphism. Call $N = M \cap (s^{-1}(M))$. Then $N$ is an essential submodule of $M$ and $s|_{N} : N \rightarrow M$ is a monomorphism. Therefore, it extends to an endomorphism of $r : M \rightarrow M$ by [5].
Let $s' : E \to E$ be an extension of $r$. Then $s'|_N = s'|_N$ and thus, $s + J(S) = s' + J(S) = \psi(r + J(R))$. Moreover, as $N$ is essential in $M$ and $r|_N = s|_M$, we get that $r$ is also a monomorphism. As $R/J(R)$ is von Neumann regular and idempotents lift modulo $J(R)$, there exists an idempotent $e \in R$ such that $(Re + J(R))/J(R) = (Rr + J(R))/J(R)$. But this means that, in particular, $r = r'e + j$ for some $r' \in R$ and $j \in J(R)$. As $J(R)$ consists of $r \in \text{End}(M)$ with essential kernel by Proposition 1, we deduce that $K = \text{Ker}(j)$ is essential in $M$. Moreover, $r(m) = r'e(m) + j(m) = r'e(m)$ for every $m \in K$. And, as $r$ is injective, this means that $K \cap \text{Im}(1 - e) = 0$. But, $K$ being essential in $M$, this implies that $1 - e = 0$ and thus, $e = 1_M$. Therefore, we deduce that $(Rr + J(R))/J(R) = R/J(R)$ and thus, there exists an element $r'' \in R$ such that $1 - r'' \circ r \in J(R)$ and thus, $r'' \circ r$ is an automorphism. As we are assuming that $M$ is directly finite, this means that $r$ must be an automorphism and thus, $s + J(S) = \psi(r + J(R))$ is also an automorphism. Therefore, $s$ is an automorphism and, as we are assuming that $t \circ s = 1_S$, we get that $s \circ t = 1_S$ too. This shows $S$ is directly-finite and consequently, $E$ is directly-finite. \(\square\)

Recall that a ring $R$ is called unit-regular if, for every element $x \in R$, there exists a unit $u \in R$ such that $x = xux$. We can now prove.

**Theorem 7.** Let $M$ be an automorphism-invariant module. If $M$ is directly-finite, then $R/J(R)$ is unit-regular.

**Proof.** We are going to adapt the proof of Proposition 1. Let $r \in R$ and let us construct $s, g \in S$ as in Proposition 1. We know that, if we call $E' = \text{Im}(g)$, $E' \cong E(L)$ is an injective submodule of $E$. So there exists a submodule $E''$ of $E$ such that $E = E' \oplus E''$. We have that $E$ is directly finite by the above lemma. Since a directly-finite injective module satisfies the internal cancellation property, we have $E'' \cong E(K)$. Let $\varphi : E'' \to E(K)$ be an isomorphism. Define $t : E \to E$ as follows: $t|_{E''} = \varphi$ whereas $t|_{E'} = h$. Clearly, $t \circ g = u \circ \pi$, where $u : E(L) \to E$ and $\pi : E \to E(L)$ are the inclusion and projection associated to the decomposition $E = E(K) \oplus E(L)$. Moreover, $t$ is clearly an automorphism and this implies that $t(M) \subseteq M$. Call $t' : M \to M$ the restriction of $t$ to $M$. Then $t'$ is a monomorphism and, as $R/J(R)$ is von Neumann regular and idempotents lift modulo $J(R)$, there exists an idempotent $e \in R$ such that $(Rt' + J(R))/J(R) = (Re + J(R))/J(R)$. Once again as in the proof of Lemma 6, we get that $t'$ must be an automorphism. Finally,

$$\psi((r \circ t' \circ r) + J(R)) = (s \circ t' \circ s) + J(S) = (g \circ t' \circ g) + J(S) = g + J(S) = \psi(r + J(R))$$
and, as $\psi$ is injective, we get that $(r \circ t' \circ r) + J(R) = r + J(R)$. As $t' \in R$ is an automorphism, this shows that $R/J(R)$ is unit-regular. 

The example given in Remark 2 shows that if $M$ is a directly-finite automorphism-invariant module, even then $R/J(R)$ need not be self-injective.

As a consequence of the above theorem, we have the following

**Corollary 8.** Let $M$ an automorphism-invariant module. Then the following are equivalent;

(i) $M$ is directly finite.
(ii) $M$ has the internal cancellation property.
(iii) $M$ has the cancellation property.
(iv) $M$ has the substitution property.

*Proof.* (i) $\implies$ (ii). If $M$ is a directly-finite automorphism-invariant module, then by the above theorem $R/J(R)$ is unit-regular and therefore, by [12], $R$ is a ring with the internal cancellation. Now it follows from Guralnick and Lanski [10], that $M$ has the internal cancellation property. This shows (i) implies (ii).

(ii) $\implies$ (iii). Since we have already shown that an automorphism-invariant module satisfies the exchange property, (ii) implies (iii), by Fuchs [7].

(iii) $\implies$ (i). This implication holds for any module.

Thus we have established the equivalence of (i), (ii) and (iii). For a module with the finite exchange property, the equivalence of (ii) and (iv) follows from [23]. This completes the proof. 

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