LARGE DEVIATION PRINCIPLE FOR STOCHASTIC BURGERS TYPE EQUATION WITH REFLECTION

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Abstract. In this paper, we establish a large deviation principle for stochastic Burgers type equation with reflection perturbed by the small multiplicative noise. The main difficulties come from the highly non-linear coefficient and the singularity caused by the reflection. Here, we adopt a new sufficient condition for the weak convergence criteria, which is proposed by Matoussi, Sabbagh and Zhang [14].

1. Introduction. Consider the following Burgers type stochastic partial differential equation (SPDE) with reflection:

\[
\begin{cases}
\frac{du^\varepsilon(t,x)}{dt} = \frac{\partial^2 u^\varepsilon(t,x)}{\partial x^2} dt + \frac{\partial g(u^\varepsilon(t,x))}{\partial x} dt \\
+ \sqrt{\varepsilon} \sum_{j=1}^{m} \sigma_j(u^\varepsilon(t,x)) dB_j(t) + L^\varepsilon(dt, dx), \\
\quad x \in [0,1], \; \varepsilon > 0, \\
\end{cases}
\]

(1.1)

where \( B = (B_1, B_2, \cdots, B_m) \) denotes a \( m \)-dimensional Brownian motion on a complete filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) with \( \mathcal{F}_t = \sigma(B(s); 0 \leq s \leq t) \); \( u_0 \) is a deterministic function on \([0,1] \); \( L^\varepsilon(t,x) \) is a random measure which is a part of the solution pair \((u^\varepsilon, L^\varepsilon)\). The coefficients \( \sigma \) and \( g \) satisfy the following hypotheses:

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(A.1) $\sigma = (\sigma_1, \cdots, \sigma_m) : \mathbb{R} \to \mathbb{R}^m$ satisfies the Lipschitz condition, i.e., there exists $c_{\sigma} \in (0, \infty)$ satisfying that
\[ \sum_{k=1}^{m} |\sigma_k(z_1) - \sigma_k(z_2)| \leq c_\sigma |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}. \tag{1.2} \]

(A.2) $g : \mathbb{R} \to \mathbb{R}$ is a $C^1$-function and its derivative function $g'$ satisfies that
\[ |g'(z)| \leq c_g (1 + |z|), \quad \forall z \in \mathbb{R}, \tag{1.3} \]
for some constant $c_g \in (0, \infty)$.

Let $H := L^2([0, 1]; \mathbb{R})$ be the usual $L^2$-space with the norm $\| \cdot \|_H$ and inner product $\langle \cdot, \cdot \rangle$. Denote by $V$ the Sobolev space of order one, i.e., $V$ is the completion of $C_0^\infty(0, 1)$ (the space of all smooth functions with compact supports in $(0, 1)$) under the norm $\| u \|_V := \left( \int_0^1 (\frac{\partial u}{\partial x})^2 \, dx \right)^{1/2}$.

The following definition is taken from [26].

**Definition 1.1.** A pair $(u^\varepsilon, L^\varepsilon)$ is said to be a solution of Equation (1.1), if
(i) $u^\varepsilon$ is a continuous random field on $\mathbb{R} \times [0, 1]$; for any $t \geq 0$ and $x \in [0, 1]$, $u^\varepsilon(t, x)$ is $\mathcal{F}_t$-measurable and $u^\varepsilon(t, x) \geq 0$, a.s.;
(ii) $L^\varepsilon$ is a random measure on $\mathbb{R} \times [0, 1]$ such that
   (a) $E \left[ \text{Var}_{\mathcal{G}_t} (L^\varepsilon)^2 \right] < +\infty$, $\forall T \geq 0$, where $\text{Var}_{\mathcal{G}_t}(L^\varepsilon)$ denotes the total variation of $L^\varepsilon$ on $Q_T := [0, T] \times [0, 1]$;
   (b) $L^\varepsilon$ is adapted in the sense that for any bounded measurable mapping $\psi : \mathbb{R} \times [0, 1] \to \mathbb{R}$,
\[ \int_0^T \int_0^1 \psi(s, x)L^\varepsilon(ds, dx) \text{ is } \mathcal{G}_t\text{-measurable}; \]
(iii) $(u^\varepsilon, L^\varepsilon)$ solves the SPDE with reflection in the following sense: for any $t \in \mathbb{R}_+, \phi \in C_0^\infty(0, 1)$ (the space of all differentiable functions $f$, which have continuous derivatives up to order $2$ and have compact supports in $(0, 1)$),
\[ \langle u^\varepsilon(t), \phi \rangle - \int_0^t \langle u^\varepsilon(s), \phi'' \rangle \, ds + \int_0^t \langle g(u^\varepsilon(s)), \phi' \rangle \, ds = \langle u^\varepsilon(0), \phi \rangle + \sqrt{\varepsilon} \sum_{k=1}^{m} \int_0^t \langle \sigma_k(u^\varepsilon(s)), \phi \rangle \, dB_k(s) + \int_0^t \int_0^1 \phi(x)L^\varepsilon(ds, dx), \quad \text{a.s.}, \]
where $u^\varepsilon(t) : = u^\varepsilon(t, \cdot)$;
(iv) for any $T > 0$, $\int_0^T \int_0^1 u^\varepsilon(t, x) L^\varepsilon(dt, dx) = 0$, a.s.

Zhang [26] obtained the following result about the solution to (1.1).

**Theorem 1.2.** [26, Theorem 3.1] Suppose (A.1) and (A.2) hold and $u_0 \in V$ with $u_0 \geq 0$. For any $\varepsilon > 0$, the reflected SPDE (1.1) admits a unique solution $(u^\varepsilon, L^\varepsilon)$ that satisfies
\[ E \left[ \sup_{0 \leq t \leq T} \| u^\varepsilon(t) \|_V^2 + \int_0^T \| u^\varepsilon(t) \|^2_V \, dt \right] < +\infty, \quad \forall T > 0. \tag{1.4} \]

The aim of this paper is to establish a large deviation principle (LDP) for the solution $u^\varepsilon$ of (1.1) as $\varepsilon \to 0$ in the space $C([0, T]; H) \cap L^2([0, T]; V)$.

Let us first recall some literatures about the LDPs for the stochastic systems with reflections. Based on a viscosity solution approach, an LDP for the diffusions
with Lipschitz continuous oblique reflections on regular domains was obtained in [13]. Using the weak convergence approach introduced in [1, 3, 4], LDPs for obstacle problems of linear/quasi-linear SPDEs were established in [22] and [14], respectively. There are also several papers studying the large deviations for multivalued stochastic differential equations (SDEs), which in particular contain a class of SDEs with reflection in a convex domain, see, e.g., [16, 17, 18, 20]. However, these results do not apply to Equation (1.1), since the highly non-linear term \( \frac{\partial g(u^\varepsilon(t,x))}{\partial x} \) is involved, which leads to one of the main difficulties to deal with our problem. We would also like to recall that the LDP for the stochastic Burgers type equation without reflection was proved in Cardon-Weber [5].

It is well-known that the weak convergence approach (see, e.g., [1, 3, 4]) is a powerful tool to prove the large and moderate deviations for the SPDEs, see, e.g., [9, 11, 19, 27] and so on. We refer to [2] for the excellent reviews during the past decade in this field. In Liu et al. [11] and Xiong and Zhai [21], by using the weak convergence approach, the authors provided unified proofs of the LDPs for a large class of SPDEs with locally monotonic coefficients driven by Brownian motions or by Lévy noises, respectively. However, the methods in [11, 21] do not work for the stochastic Burgers type equation with reflection, since the local monotonicity of the nonlinear operator in Equation (1.1) cannot be satisfied in a straight way because of the reflection.

In this paper, we will adopt a new sufficient condition for the weak convergence criteria given by Matoussi et al. [14]. This new condition seems to be more suitable for stochastic dynamics generated by SPDEs with reflections. It is also recently successfully applied to obtain an LDP for stochastic conservation laws in [8]. As an important part of the proof, we need to obtain the global well-posedness and the continuity of the associated skeleton equation (2.3) below. This turns out to be hard because of the highly non-linear coefficient and the singularity caused by the reflection. In order to overcome these difficulties, we introduce the penalized skeleton equations and establish a number of a priori estimates for the solutions of the penalized equations. Then, we show that the limit of the penalized solutions gives rise to a solution of the skeleton equation and obtain the continuity of the solution by approximations.

Finally, we would like to give some references [6, 7, 10, 15, 22, 23, 24, 25] for the background of the reflected SPDEs and their properties of the solutions.

The rest of the paper is organized as follows. In Section 2, we study the properties of the skeleton equation through their penalizations. The LDP for stochastic Burgers type equation with reflection and its proof are given in Section 3.

2. The skeleton equation.

2.1. The skeleton equation and its properties. The Cameron-Martin space associated with the \(m\)-dimensional Brownian motion \(B = \{B_1(t), B_2(t), \ldots, B_m(t)\}_{t \in [0,T]}\) is given by

\[
\mathcal{H} := \left\{ h = (h_1, \cdots, h_m); \int_0^T \left| \dot{h}(s) \right|^2 \, ds < \infty \right\}.
\]  

(2.1)

For any \( h \in \mathcal{H} \), let

\[
\| h \|_{\mathcal{H}} := \left( \int_0^T \left| h(s) \right|^2 \, ds \right)^{\frac{1}{2}}.
\]  

(2.2)
Let \( S_N := \{ h \in \mathcal{H}; \| h \|_{\mathcal{H}} \leq N \} \). The set \( S_N \) endowed with the weak convergence topology is a Polish space. Throughout, we use this topology on \( S_N \).

For any \( h \in \mathcal{H} \), consider the following deterministic reflected PDE (the skeleton equation):

\[
\begin{aligned}
&\begin{cases}
du^h(t, x) = \frac{\partial u^h(t, x)}{\partial x} dt + \frac{\partial g(u^h(t, x))}{\partial x} dt \\
+ \sum_{j=1}^m \sigma_j (u^h(t, x)) \hat{h}_j(t) dt + L^h(dt, dx),
\end{cases} \\
&u^h(t, x) \geq 0, \quad u^h(0, x) = u_0(x) \geq 0, \quad x \in [0, 1],
\end{aligned}
\]

(2.3)

Analogously to (1.1), a pair of \((u^h, L^h)\) is called a solution of Equation (2.3), if it satisfies

(i) \( u^h \) is a continuous function on \( \mathbb{R}_+ \times [0, 1] \); for any \( t \geq 0 \) and \( x \in [0, 1] \), \( u^h(t, x) \geq 0 \);

(ii) \( L^h \) is a measure on \( \mathbb{R}_+ \times [0, 1] \) such that \( \text{Var}_{Q_T} \{ L^h \} < +\infty, \forall T \geq 0 \);

(iii) \((u^h, L^h)\) solves the parabolic partial differential equation (PDE) with reflection in the following sense: for any \( t \in \mathbb{R}_+, \phi \in C^2_0(0, 1) \),

\[
\begin{aligned}
&\langle u^h(t), \phi \rangle - \int_0^t \langle u^h(s), \phi'' \rangle ds + \int_0^t g(u^h(s)) \phi' ds \\
= &\langle u^h(t), \phi \rangle + \sum_{k=1}^m \int_0^t \sigma_k (u^h(s)) \hat{h}_j(s) \phi ds + \int_0^t \int_{\mathbb{R}} \phi \cdot \langle L^h(\cdot), L^h(\cdot) \rangle ds, \\
\end{aligned}
\]

where \( u^h(t) := u^h(t, \cdot) \);

(iv) for any \( T > 0, \int_0^T \int_{\mathbb{R}} u^h(t, x) L^h(dt, dx) = 0 \).

The objections of this section are the following two results.

**Proposition 1.** Assume that (A.1) and (A.2) hold. For any \( u_0 \in V \) with \( u_0 \geq 0 \) and \( h \in \mathcal{H} \), the reflected PDE (2.3) admits a unique solution \((u^h, L^h)\), and \( u^h \in C([0, T]; H) \cap L^2([0, T]; V) \).

**Proposition 2.** Assume that (A.1) and (A.2) hold. If the sequence \( h^{(r)} \) converges weakly to \( h \) in \( S_N \) for some \( N > 0 \), as \( r \to 0 \), then \( u^{h^{(r)}} \) converges to \( u^h \) in the space \( C([0, T]; H) \cap L^2([0, T]; V) \). Here \((u^{h^{(r)}}, L^{h^{(r)}})\) denotes the solution of (2.3) replacing \( h \) by \( h^{(r)} \).

In order to prove the above two propositions, we introduce the penalized skeleton equations and give some a priori estimates to their solutions in the next subsection.

### 2.2. Penalized equations.

For any \( n \geq 1, h \in \mathcal{H}, u_0 \in V \), consider the penalized Burgers type equation:

\[
\begin{aligned}
&\begin{cases}
\frac{\partial u_{n,h}(t, x)}{\partial t} = u_{n,h}(0, x) + \int_0^t \frac{\partial^2 u_{n,h}(s, x)}{\partial x^2} ds + \int_0^t \frac{\partial g(u_{n,h}(s, x))}{\partial x} ds \\
+ \sum_{j=1}^m \int_0^t \sigma_j (u_{n,h}(s, x)) \hat{h}_j(s) ds + n \int_0^t u_{n,h}(s, x)^{-} ds,
\end{cases} \\
u_{n,h}(0, x) = u_0(x) \geq 0, \\
u_{n,h}(t, 0) = u_{n,h}(t, 1) = 0, \quad t \geq 0,
\end{aligned}
\]

(2.4)

here \( u^- := -\min\{u, 0\} \).
According to [5], we know that there exists a unique solution \( u^{n,h} \) to Equation (2.4). In this part, we give some estimates for the sequence \( u^{n,h}(t) := u^{n,h}(t, \cdot), n \geq 1 \).

**Lemma 2.1.** Assume that (A.1) and (A.2) hold. For any \( N > 0 \), we have

\[
\sup_{n \geq 1} \sup_{h \in S_N} \left( \sup_{0 \leq t \leq T} \left\| u^{n,h}(t) \right\|^2_H + \int_0^T \| u^{n,h}(t) \|^2_W dt \right) < +\infty \tag{2.5}
\]

and

\[
\sup_{n \geq 1} \sup_{h \in S_N} \left( n \int_0^T \left\| u^{n,h}(t) \right\|^2_H dt \right) < +\infty. \tag{2.6}
\]

**Proof.** For any \( n \geq 1, h \in S_N \), according to the chain rule, we have

\[
\left\| u^{n,h}(t) \right\|^2_H = \left\| u_0 \right\|^2_H + 2 \int_0^t \int_0^1 u^{n,h}(s, x) \frac{\partial^2 u^{n,h}(s, x)}{\partial x^2} dx ds \\
+ 2 \int_0^t \int_0^1 u^{n,h}(s, x) \frac{\partial g(u^{n,h}(s, x))}{\partial x} dx ds - 2n \int_0^t \left\| u^{n,h}(s) \right\|^2_H ds \\
+ 2 \sum_{j=1}^m \int_0^t \left\langle u^{n,h}(s), \sigma_j(u^{n,h}(s)) \right\rangle \dot{h}_j(s) ds. \tag{2.7}
\]

By the proof of Lemma 2.1 in [26], we know that

\[
\int_0^t \int_0^1 u^{n,h}(s, x) \frac{\partial^2 u^{n,h}(s, x)}{\partial x^2} dx ds = - \int_0^t \left\| u^{n,h}(s) \right\|^2_W ds \tag{2.8}
\]

and

\[
\int_0^t \int_0^1 u^{n,h}(s, x) \frac{\partial g(u^{n,h}(s, x))}{\partial x} dx ds = 0. \tag{2.9}
\]

Notice that (A.1) implies that \( \sigma \) satisfies the following linear growth property: there exists \( C_\sigma \in (0, \infty) \) such that

\[
\sum_{k=1}^m |\sigma_k(z)| \leq C_\sigma (1 + |z|), \quad \forall z \in \mathbb{R}. \tag{2.10}
\]

Hence, there exists \( c_{2,1} \in (0, \infty) \) such that

\[
\left( \sum_{j=1}^m \left| \sigma_j(u^{n,h}(s, x)) \right|^2 \right)^{\frac{1}{2}} \leq c_{2,1} \left( 1 + |u^{n,h}(s, x)|^2 \right)^{\frac{1}{2}}.
\]

By the Cauchy-Schwarz inequality, we have

\[
\sum_{j=1}^m \int_0^t \left\langle u^{n,h}(s), \sigma_j(u^{n,h}(s)) \right\rangle \dot{h}_j(s) ds \\
= \int_0^t \left[ \int_0^1 u^{n,h}(s, x) \left( \sum_{j=1}^m \sigma_j(u^{n,h}(s, x)) \dot{h}_j(s) \right) dx \right] ds \\
\leq \int_0^t \int_0^1 u^{n,h}(s, x) \left( \sum_{j=1}^m \left| \sigma_j(u^{n,h}(s, x)) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^m |\dot{h}_j(s)|^2 \right)^{\frac{1}{2}} dx ds.
\]
\[ \leq c_{2,1} \int_0^t \left[ \int_0^1 u^{n,h}(s,x) \left( 1 + |u^{n,h}(s,x)|^2 \right)^{\frac{1}{2}} dx \right] |\dot{h}(s)| ds \]
\[ \leq c_{2,1} \left\{ \int_0^t \left[ \int_0^1 u^{n,h}(s,x) \left( 1 + |u^{n,h}(s,x)|^2 \right)^{\frac{1}{2}} dx \right]^2 ds \right\} \left( \int_0^t |\dot{h}(s)|^2 ds \right)^{\frac{1}{2}} \]
\[ \leq c_{2,1} N \left\{ \sup_{0 \leq s \leq t} \|u^{n,h}(s)\|_H \right\} \left( \int_0^t \left( 1 + \|u^{n,h}(s)\|_H^2 \right) ds \right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{4} \left( \sup_{0 \leq s \leq t} \|u^{n,h}(s)\|_H^2 + c_{2,1} N^2 \int_0^t \left( 1 + \|u^{n,h}(s)\|_H^2 \right) ds. \right) \quad (2.11) \]

By (2.7)-(2.11), we have
\[ \sup_{0 \leq t \leq T} \|u^{n,h}(t)\|_H^2 + \int_0^T \|u^{n,h}(t)\|_{V^1} dt + n \int_0^T \|u^{n,h}(t)\|_H^2 dt \]
\[ \leq 4 \|u_0\|_H^2 + 4c_{2,1} N^2 \int_0^T \left( 1 + \|u^{n,h}(t)\|_H^2 \right) dt. \]

This, together with Gronwall's inequality, implies (2.5) and (2.6). The proof is complete. \( \square \)

**Lemma 2.2.** Assume that (A.1) and (A.2) hold. For any \( N > 0 \), we have
\[ \sup_{n \geq 1} \sup_{h \in S_N} \left( n \left\| (u^{n,h})^- \right\|_{L^1(Q_T)} \right) < +\infty, \quad (2.12) \]

here \( \| \cdot \|_{L^1(Q_T)} \) denotes the usual \( L^1 \)-norm in \( L^1(Q_T) \).

**Proof.** For any \( n \geq 1, h \in S_N \), by the chain rule, we have
\[ \|u^{n,h}(t) - 1\|_H^2 = \|u_0 - 1\|_H^2 + 2 \int_0^t \int_0^1 (u^{n,h}(s,x) - 1) \frac{\partial^2 u^{n,h}(s,x)}{\partial x^2} dx ds \]
\[ + 2 \int_0^t \int_0^1 (u^{n,h}(s,x) - 1) \frac{\partial g(u^{n,h}(s,x))}{\partial x} dx ds \]
\[ + 2n \int_0^t \int_0^1 (u^{n,h}(s,x) - 1) u^{n,h}(s,x) dx ds \]
\[ + 2 \sum_{j=1}^m \int_0^t \langle u^{n,h}(s) - 1, \sigma_j(u^{n,h}(s)) \rangle \dot{h}_j(s) ds \quad (2.13) \]

Using the same technique in the proof of (2.11), there exists \( c_{2,2} \in (0, \infty) \) such that
\[ 2 \sum_{j=1}^m \int_0^t \langle u^{n,h}(s) - 1, \sigma_j(u^{n,h}(s)) \rangle \dot{h}_j(s) ds \]
\[ \leq \sup_{0 \leq s \leq t} \|u^{n,h}(s) - 1\|_H^2 + c_{2,2} N^2 \int_0^t \left( 1 + \|u^{n,h}(s)\|_H^2 \right) ds. \quad (2.14) \]

From the proof of Lemma 2.3 in [26], we have
\[ 2 \int_0^t \int_0^1 (u^{n,h}(s,x) - 1) \frac{\partial g(u^{n,h}(s,x))}{\partial x} dx ds = 0 \quad (2.15) \]
Assume that (A.1) and (A.2) hold. For any $\tilde{f}$ or any $f$, we have
\begin{equation}
-2n \int_0^t \int_0^1 (u^{n,h}(s,x) - 1) u^{n,h}(s,x) - dxds \geq 2n \int_0^t \int_0^1 u^{n,h}(s,x) - dxds. \tag{2.16}
\end{equation}
It follows from (2.13)-(2.16) that
\begin{align*}
2n \|(u^{n,h})^-\|_{L^1(\Omega_T)} &= 2n \int_0^t \int_0^1 u^{n,h}(s,x) - dxds \\
&\leq \|u_0 - 1\|^2_H + \sup_{0 \leq t \leq T} \|u(t) - 1\|^2_H + c_{2,2}N^2 \int_0^t \left(1 + \|u^{n,h}(s)\|^2_H\right) ds.
\end{align*}
Combining with Lemma 2.1, we obtain (2.12). The proof is complete. \hfill \square

To show the convergence of $u^{n,h}$ in $C([0,T];H) \cap L^2([0,T];V)$ uniformly over $h \in S_N$ as $n \to \infty$, we need the following estimates.

**Lemma 2.3.** Assume that (A.1) and (A.2) hold. For any $N > 0$, we have
\begin{equation}
\lim_{n \to \infty} \sup_{h \in S_N} \sup_{0 \leq t \leq T} \|u^{n,h}(t)^-\|^2_H = 0. \tag{2.17}
\end{equation}

**Proof.** For any $j \geq 1$, define $\psi_j(z)$ and $f_j(y)$ as follows (see [7]):
\begin{align*}
\psi_j(z) &= \begin{cases} 
0, & \text{if } z \leq 0; \\
2jz, & \text{if } 0 \leq z \leq \frac{1}{j}; \\
2, & \text{if } z > \frac{1}{j},
\end{cases} \\
\text{and} \\
f_j(y) &= \begin{cases} 
0, & \text{if } y \leq 0; \\
\int_y^0 dz \int_0^{z_j} \psi_j(z)dz, & \text{if } y > 0.
\end{cases}
\end{align*}
Then we have
\begin{equation}
\begin{aligned}
f_j'(y) &= \begin{cases} 
0, & \text{if } y \leq 0; \\
2y, & \text{if } 0 \leq y \leq \frac{1}{j}; \\
2y - \frac{1}{j}, & \text{if } y > \frac{1}{j}.
\end{cases}
\end{aligned} \tag{2.18}
\end{equation}
It is clear that $f_j(\cdot) \uparrow (y^-)^2$ as $j \to \infty$. For any $p \in H$, set
\begin{equation*}
R_j(p) := \left(\int_0^1 f_j(-p(x))dx\right)^2.
\end{equation*}

Then $R_j$ has the following derivative:
\begin{equation*}
R_j(p)(p_1) = -2 \int_0^1 f_j(-p(x))dx \cdot \int_0^1 f_j'(-p(x))p_1(x)dx, \quad \text{for } p_1 \in H.
\end{equation*}
Applying the chain rule, for any $n \geq 1, h \in S_N$, we have
\begin{align*}
R_j(u^{n,h}(t)) &= R_j(u_0) + \int_0^t R_j'(u^{n,h}(s)) \left(\frac{\partial^2 u^{n,h}(s)}{\partial x^2}\right) ds \\
&\quad + \int_0^t R_j'(u^{n,h}(s)) \left(\frac{\partial g(u^{n,h}(s))}{\partial x}\right) ds \\
&\quad + n \int_0^t R_j'(u^{n,h}(s)) (u^{n,h}(s)-) ds.
\end{align*}
\[ + \sum_{k=1}^{m} \int_0^t R_j' \left( u^{n,h}(s) \right) \left( \sigma_k \left( u^{n,h}(s) \right) \right) \dot{h}_k(s) ds \]
\[ = I_n^1(t) + I_n^2(t) + I_n^3(t) + I_n^4(t) + I_n^5(t). \]  

(2.19)

From the proof of Lemma 2.4 in [26], we know that \( I_n^5(t) \leq 0 \), \( I_n^2(t) = 0 \) and \( I_n^4(t) \leq 0 \). By using the fact of \( 0 \leq f_j'(-y) \leq 2y^- \) and by using the Cauchy-Schwarz inequality and the linear growth condition of \( \sigma \), we have
\[ I_n^5(t) \leq \left( \sum_{k=1}^{m} \int_0^t R_j' \left( u^{n,h}(s) \right) \left( \sigma_k \left( u^{n,h}(s) \right) \right)^2 ds \right)^{1/2} \left( \int_0^t |\dot{h}(s)|^2 ds \right)^{1/2} \]
\[ \leq 2C_{\sigma} N^2 \left[ \int_0^t \left[ f_j \left( -u^{n,h}(s,x) \right) \right] dx \right] \left( \int_0^t f_j \left( -u^{n,h}(s,x) \right) \left( 1 + |u^{n,h}(s,x)| \right) ds \right) \]
\[ \leq \frac{1}{2} \sup_{0 \leq s \leq t} \left( \int_0^1 f_j \left( -u^{n,h}(s,x) \right) dx \right)^2 \]
\[ + 2C_{\sigma}^2 N^2 \int_0^t \left( \int_0^1 f_j \left( -u^{n,h}(s,x) \right) \left( 1 + |u^{n,h}(s,x)| \right) dx \right) ds \]
\[ \leq \frac{1}{2} \sup_{0 \leq s \leq t} \left( \int_0^1 f_j \left( -u^{n,h}(s,x) \right) dx \right)^2 \]
\[ + 8C_{\sigma}^2 N^2 \int_0^t \left( \int_0^1 |u^{n,h}(s,x)| - \left( 1 + |u^{n,h}(s,x)| \right) dx \right) ds \]
\[ \leq \frac{1}{2} \sup_{0 \leq s \leq t} R_j \left( u^{n,h}(s) \right) + 16C_{\sigma}^2 N^2 \left( \int_0^1 \|u^{n,h}(s)\|_H^2 ds + \int_0^1 \|u^{n,h}(s)\|_H^4 ds \right). \]

Putting the above estimates together, it follows from (2.19) that
\[ \sup_{0 \leq t \leq T} R_j \left( u^{n,h}(t) \right) \leq 32C_{\sigma}^2 N^2 \int_0^T \|u^{n,h}(t)\|_H^4 dt + 32C_{\sigma}^2 N^2 \int_0^T \|u^{n,h}(t)\|_H^4 dt. \]

Let \( j \to +\infty \), we have
\[ \sup_{0 \leq t \leq T} \|u^{n,h}(t)\|_H^4 \leq 32C_{\sigma}^2 N^2 \int_0^T \|u^{n,h}(t)\|_H^4 dt + 32C_{\sigma}^2 N^2 \int_0^T \|u^{n,h}(t)\|_H^4 dt. \]

Applying Gronwall’s inequality, we have
\[ \sup_{0 \leq t \leq T} \|u^{n,h}(t)\|_H^4 \leq 32C_{\sigma}^2 N^2 e^{32C_{\sigma}^2 N^2 T} \int_0^T \|u^{n,h}(t)\|_H^2 dt, \]
which converges to 0, as \( n \to +\infty \), according to Lemma 2.1. The proof is complete. \( \square \)

**Lemma 2.4.** Assume that (A.1) and (A.2) hold. For any \( N > 0 \), we have
\[ \lim_{n,n^{'} \to +\infty} \sup_{h \in \mathcal{S}_N} \left( \sup_{0 \leq t \leq T} \left\| u^{n,h}(t) - u^{n^{'},h}(t) \right\|_H^2 + \int_0^T \left\| u^{n,h}(t) - u^{n^{'},h}(t) \right\|_V^2 dt \right) = 0. \]

(2.20)
Proof. Lemma 2.1 tells us that
\[
\epsilon_{2.3}(N) := \sup_{n \geq 1} \sup_{h \in S_N} \int_0^T \| u^{n,h}(t) \|^2_V dt + T < +\infty. \tag{2.21}
\]
Thus, for any \( \lambda > 0, t \in [0, T], n, n' \geq 1, \)
\[
e_{n,n'}(t) := \exp \left( -\lambda \int_0^t \left( 1 + \| u^{n,h}(s) \|^2_V + \| u^{n',h}(s) \|^2_V \right) ds \right) \geq e^{-2\epsilon_{2.3}(N)\lambda}.
\]
Hence, to prove (2.20), it is sufficient to prove that
\[
\lim_{n,n' \to +\infty} \sup_{h \in S_N} \left[ \sup_{0 \leq t \leq T} e_{n,n'}(t) \left\| u^{n,h}(t) - u^{n',h}(t) \right\|^2_H \\
+ \int_0^T e_{n,n'}(t) \left\| u^{n,h}(t) - u^{n',h}(t) \right\|^2_V dt \right] = 0.
\tag{2.22}
\]
By the chain rule, we have
\[
e_{n,n'}(t) \left\| u^{n,h}(t) - u^{n',h}(t) \right\|^2_H \\
= -\lambda \int_0^t e_{n,n'}(s) \left( 1 + \| u^{n,h}(s) \|^2_V + \| u^{n',h}(s) \|^2_V \right) \left\| u^{n,h}(s) - u^{n',h}(s) \right\|^2_H ds \\
- 2 \int_0^t e_{n,n'}(s) \| u^{n,h}(s) - u^{n',h}(s) \|^2_V ds \\
+ 2 \int_0^t e_{n,n'}(s) \left( u^{n,h}(s) - u^{n',h}(s), \frac{\partial g}{\partial x} \left( u^{n,h}(s) \right) - \frac{\partial g}{\partial x} \left( u^{n',h}(s) \right) \right) ds \\
+ 2 \int_0^t e_{n,n'}(s) \left( u^{n,h}(s) - u^{n',h}(s), n u^{n,h}(s) - n' u^{n',h}(s) \right) ds \\
+ 2 \sum_{j=1}^m \int_0^t e_{n,n'}(s) \left( u^{n,h}(s) - u^{n',h}(s), \sigma_j \left( u^{n,h}(s) \right) - \sigma_j \left( u^{n',h}(s) \right) \right) \bar{h}_j(s) ds \\
=: I_{n,n'}^1(t) + I_{n,n'}^2(t) + I_{n,n'}^3(t) + I_{n,n'}^4(t) + I_{n,n'}^5(t). \tag{2.23}
\]
By using the following inequality (see [26, (3.3)]):
\[
\left\langle u - v, \frac{\partial g(u)}{\partial x} - \frac{\partial g(v)}{\partial x} \right\rangle \leq \frac{1}{8} \| u - v \|^2_V + c_{2.4} \left( 1 + \| u \|^2_V + \| v \|^2_V \right) : \| u - v \|^2_H, \tag{2.24}
\]
for some constant \( c_{2.4} \in (0, \infty) \), we have
\[
I_{n,n'}^3(t) \leq \frac{1}{4} \int_0^t e_{n,n'}(s) \left\| u^{n,h}(s) - u^{n',h}(s) \right\|^2_V ds \\
+ 2c_{2.4} \int_0^t e_{n,n'}(s) \left( 1 + \| u^{n,h}(s) \|^2_V + \| u^{n',h}(s) \|^2_V \right) \left\| u^{n,h}(s) - u^{n',h}(s) \right\|^2_H ds.
\tag{2.25}
\]
According to (3.4) in [26], we get that for any \( t \in [0, T], \)
\[
I_{n,n'}^4(t) \leq 2 \sup_{0 \leq s \leq t} e_{n,n'}(s) \left\| u^{n',h}(s) \right\|_{L^\infty([0,1])} \left( n \left\| (u^{n,h})^+ \right\|_{L^1(Q_T)} \right) \\
+ 2 \sup_{0 \leq s \leq t} e_{n,n'}(s) \left\| u^{n,h}(s) \right\|_{L^\infty([0,1])} \left( n' \left\| (u^{n',h})^+ \right\|_{L^1(Q_T)} \right). \tag{2.26}
\]
Using the same technique in the proof of (2.11), we can obtain that
\[
I_{n,n'}^5(t) \leq \frac{1}{2} \sup_{0 \leq t \leq t} \left( e_{n,n'}(s) \left\| u^{n,h}(s) - u^{n',h}(s) \right\|_H^2 + 8c_2^N N^2 \int_0^t e_{n,n'}(s) \left\| u^{n,h}(s) - u^{n',h}(s) \right\|_H^2 ds \right).
\]
(2.27)

Putting (2.23), (2.25)-(2.27) together, we obtain that for any \( \lambda > 2c_{2,4} \),
\[
\sup_{0 \leq t \leq T} e_{n,n'}(t) \left\| u^{n,h}(t) - u^{n',h}(t) \right\|_H^2 + \int_0^T e_{n,n'}(t) \left\| u^{n,h}(t) - u^{n',h}(t) \right\|_V^2 dt \leq 16c_2^N N^2 \int_0^T e_{n,n'}(t) \left\| u^{n,h}(t) - u^{n',h}(t) \right\|_H^2 dt + 4 \sup_{0 \leq t \leq T} e_{n,n'}(t) \left\| u^{n',h}(t)^{-} \right\|_{L^\infty([0,1])} \left( n \left\| (u^{n,h})^{-} \right\|_{L^1(Q_T)} \right) + 4 \sup_{0 \leq t \leq T} e_{n,n'}(t) \left\| u^{n,h}(t)^{-} \right\|_{L^\infty([0,1])} \left( n' \left\| (u^{n',h})^{-} \right\|_{L^1(Q_T)} \right).
\]

Applying Gronwall’s inequality, we get
\[
\sup_{0 \leq t \leq T} e_{n,n'}(t) \left\| u^{n,h}(t) - u^{n',h}(t) \right\|_H^2 + \int_0^T e_{n,n'}(t) \left\| u^{n,h}(t) - u^{n',h}(t) \right\|_V^2 dt \leq 4e^{16c_2^N N^2 T} \left( \sup_{0 \leq t \leq T} e_{n}(t) \left\| u^{n',h}(t)^{-} \right\|_{L^\infty([0,1])} \right) \left( n \left\| (u^{n,h})^{-} \right\|_{L^1(Q_T)} \right) + 4e^{16c_2^N N^2 T} \left( \sup_{0 \leq t \leq T} e_{n}(t) \left\| u^{n,h}(t)^{-} \right\|_{L^\infty([0,1])} \right) \left( n' \left\| (u^{n',h})^{-} \right\|_{L^1(Q_T)} \right),
\]
(2.28)

where
\[
e_n(t) := \exp \left( -\lambda \int_0^t \left\| u^{n,h}(s) \right\|_V^2 ds \right).
\]

Using the argument in the proof of [26, Corollary 2.1], we can obtain that by Lemma 2.3,
\[
\lim_{n \to +\infty} \sup_{h \in S_n} \left( \sup_{0 \leq t \leq T} e_n(t) \left\| u^{n,h}(t)^{-} \right\|_{L^\infty([0,1])} \right) = 0.
\]
(2.29)

By (2.28), (2.29) and Lemma 2.2, we obtain (2.22). The proof is complete. \( \square \)

2.3. Proof of Proposition 1. We prove Proposition 1 by using the estimates of the penalized equation (2.4) in the previous part.

**Proof of Proposition 1. (Existence).** For any \( n \geq 1 \), let \( u^{n,h} \) be the solution of the penalized equation (2.4). According to Lemma 2.4, as \( n \to +\infty \), \( u^{n,h} \) converges in the space \( C([0,T],H) \cap L^2([0,T],V) \), whose limit is denoted by \( u^h \). We will show that \( u^h \) is the solution of Equation (2.3).

For any \( n \geq 1 \), let
\[
L^{n,h}(dx,dt) := nu^{n,h}(t,x)^{-}dxdt.
\]
(2.30)

For any \( N > 0 \), by Lemma 2.2, we have
\[
\sup_{n \geq 1} \sup_{h \in S_N} \text{Var}_{Q_T} \left( L^{n,h} \right) \leq \sup_{n \geq 1} \sup_{h \in S_N} \left( n \int_0^T \int_0^1 u^{n,h}(t,x)^{-}dxdt \right) < +\infty.
\]
(2.31)
Let $\mathcal{M}(Q_T)$ denote the Banach space of totally finite signed measures on $Q_T$, equipped with the norm of total variation. (2.31) shows that $\{L^{n,h}; n \geq 1\}$ is bounded in $\mathcal{M}(Q_T)$, hence it is relatively compact with respect to the weak-* topology in $\mathcal{M}(Q_T)$. Thus, we may assume (take a subsequence if necessary) that $L^{n,h}$ converges to some $L^h \in \mathcal{M}(Q_T)$ with respect to the weak-* topology. Moreover, it follows from (2.31) that $\text{Var}_{Q_T}(L^h) < +\infty$.

For any $\phi \in C^2_0((0, \infty) \times [0, 1], \mathbb{R})$, according to (2.4), we have

\[
\int_0^t \int_0^1 \phi(s, x)L^{n,h}(dx, ds) = \langle u^{n,h}(t), \phi(t) \rangle - \langle u^{n,h}(0), \phi(0) \rangle \\
- \int_0^t \left( \frac{\partial \phi}{\partial s}, u^{n,h}(s) \right) ds - \int_0^t \left( \frac{\partial^2 \phi}{\partial x^2}, u^{n,h}(s) \right) ds \\
+ \int_0^t \left( \frac{\partial \phi}{\partial x}, g(u^{n,h}(s)) \right) ds - \sum_{j=1}^m \int_0^t \langle \phi(s), \sigma_j(u^{n,h}(s)) \rangle \hat{h}_j(s) ds.
\]

By Theorem 3.1 in [26], we know that

\[
\lim_{n \to +\infty} \int_0^t \left( \frac{\partial \phi}{\partial x}, g(u^{n,h}(s)) \right) ds = \int_0^t \left( \frac{\partial \phi}{\partial x}, g(u^h(s)) \right) ds.
\]

Since every term of (2.32) converges as $n \to +\infty$, we know that

\[
\int_0^t \int_0^1 \phi(s, x)L^h(dx, ds)
\]

\[
= \langle u^h(t), \phi(t) \rangle - \langle u^h(0), \phi(0) \rangle - \int_0^t \left( \frac{\partial \phi}{\partial s}, u^h(s) \right) ds - \int_0^t \left( \frac{\partial^2 \phi}{\partial x^2}, u^h(s) \right) ds
\]

\[
+ \int_0^t \left( \frac{\partial \phi}{\partial x}, g(u^h(s)) \right) ds - \sum_{j=1}^m \int_0^t \langle \phi(s), \sigma_j(u^h(s)) \rangle \hat{h}_j(s) ds.
\]

Furthermore, by the proof of Theorem 3.1 in [26], we know $u^h(t, x) \geq 0$ for all $(t, x) \in [0, T] \times [0, 1]$ and $\int_0^T \int_0^1 u^h(t, x) L^h(dt, dx) = 0$ for all $T > 0$. Thus, $(u^h, L^h)$ is a solution of (2.3).

**Uniqueness.** Let $(\tilde{u}^h, \tilde{L}^h)$ be another solution of the reflected PDE (2.3). Let $\{e_k; k \geq 1\}$ be the eigenfunction basis of the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2}$ on $[0, 1]$ and $\Delta e_k = -\lambda_k e_k$, $k \geq 1$. Denote

\[
u_k^h(t) = \langle u^h(t), e_k \rangle, \quad \tilde{\nu}_k^h(t) = \langle \tilde{u}^h(t), e_k \rangle,
\]

\[
L^{h,k}(ds) = \int_0^1 e_k(x)L^h(dx, ds), \quad \tilde{L}^{h,k}(ds) = \int_0^1 e_k(x)\tilde{L}^h(dx, ds).
\]

Multiplying $e_k$ to both sides of the equations satisfied by $(u^h, L^h)$ and $(\tilde{u}^h, \tilde{L}^h)$, we have

\[
u_k^h(t) - \tilde{\nu}_k^h(t)
\]

\[
= \langle u^h(t), e_k \rangle - \langle \tilde{u}^h(t), e_k \rangle
\]

\[
= -\lambda_k \int_0^t (\nu_k^h(s) - \tilde{\nu}_k^h(s)) ds + \int_0^t \left( \frac{\partial g(u^h(s))}{\partial x} - \frac{\partial g(\tilde{u}^h(s))}{\partial x} \right) ds
\]
\[
+ \int_0^t L^{h,k}(ds) - \int_0^t \tilde{L}^{h,k}(ds) \\
+ \sum_{j=1}^m \int_0^t \langle \sigma_j (u^h(s)) - \sigma_j (\tilde{u}^h(s)), e_k \rangle \hat{h}_j(s)ds.
\]
\[(2.33)\]

For any \( \lambda > 0 \), set
\[
I(t) := \exp \left( -\lambda \int_0^t \left( 1 + \|u^h(s)\|^2_V + \|\tilde{u}^h(s)\|^2_V \right) ds \right).
\]

Applying the chain rule and adding up \( k \) from 1 to \( M \in \mathbb{N} \), it follows that
\[
\sum_{k=1}^M \left\| u^h_k(t) - \tilde{u}^h_k(t) \right\|_H^2 I(t)
= -\sum_{k=1}^M \lambda \int_0^t I(s) \left( 1 + \left\| u^h(s) \right\|_V^2 + \left\| \tilde{u}^h(s) \right\|_V^2 \right) \left\| u^h_k(s) - \tilde{u}^h_k(s) \right\|_H^2 ds
- 2 \sum_{k=1}^M \lambda \int_0^t I(s) \left\| u^h_k(s) - \tilde{u}^h_k(s) \right\|_V^2 ds
+ 2 \sum_{k=1}^M \int_0^t I(s) \left( \frac{\partial g \left( u^h(s) \right)}{\partial x} - \frac{\partial g \left( \tilde{u}^h(s) \right)}{\partial x} \right) (u^h_k(s) - \tilde{u}^h_k(s)) e_k ds
+ 2 \sum_{k=1}^M \int_0^t I(s) \left( u^h_k(s) - \tilde{u}^h_k(s) \right) L^{h,k}(ds)
- 2 \sum_{k=1}^M \int_0^t I(s) \left( u^h_k(s) - \tilde{u}^h_k(s) \right) \tilde{L}^{h,k}(ds)
+ 2 \sum_{k=1}^M \sum_{j=1}^m \int_0^t I(s) \langle \sigma_j (u^h(s)) - \sigma_j (\tilde{u}^h(s)), (u^h_k(s) - \tilde{u}^h_k(s)) e_k \rangle \hat{h}_j(s)ds.
\]
\[(2.34)\]

Letting \( M \to +\infty \), by Theorem 3.1 in [26], we know that
\[
\lim_{M \to +\infty} \sum_{k=1}^M \int_0^t I(s) \left( u^h_k(s) - \tilde{u}^h_k(s) \right) L^{h,k}(ds)
= \int_0^t \int_0^1 I(s) \left( u^h(s,x) - \tilde{u}^h(s,x) \right) L^h(dx, ds) \leq 0
\]
\[(2.35)\]

and
\[
\lim_{M \to +\infty} \sum_{k=1}^M \int_0^t I(s) \left( u^h_k(s) - \tilde{u}^h_k(s) \right) \tilde{L}^{h,k}(ds)
= \int_0^t \int_0^1 I(s) \left( u^h(s,x) - \tilde{u}^h(s,x) \right) \tilde{L}^h(dx, ds) \geq 0.
\]
\[(2.36)\]

By (2.34)-(2.36), we have
\[
I(t) \left\| u^h(t) - \tilde{u}^h(t) \right\|_H^2
\leq -\lambda \int_0^t I(s) \left( 1 + \left\| u^h(s) \right\|_V^2 + \left\| \tilde{u}^h(s) \right\|_V^2 \right) \left\| u^h(s) - \tilde{u}^h(s) \right\|_H^2 ds
\]
\[-2 \int_0^t I(s) \left\| u^h(s) - \hat{u}^h(s) \right\|^2_V ds \]
\[+ 2 \int_0^t I(s) \left\langle \frac{\partial g(u^h(s))}{\partial x} - \frac{\partial g(\hat{u}^h(s))}{\partial x}, u^h(s) - \hat{u}^h(s) \right\rangle ds \]
\[+ 2 \sum_{j=1}^m \int_0^t I(s) \left\langle \sigma_j(u^h(s)) - \sigma_j(\hat{u}^h(s)), u^h(s) - \hat{u}^h(s) \right\rangle \dot{h}_j(s) ds. \quad (2.37)\]

Using the same technique in the proof of (2.11) and choosing \(\lambda\) sufficiently large, we obtain that by (2.25) and the Lipschitz continuity of \(\sigma\),

\[I(t) \left\| u^h(t) - \hat{u}^h(t) \right\|^2_H \leq c_{2.5} N^2 \int_0^T I(s) \left\| u^h(s) - \hat{u}^h(s) \right\|^2_H ds,\]

where \(c_{2.5} \in (0, \infty)\). Applying Gronwall’s inequality, we get \(u^h = \hat{u}^h\), which implies that \(L^h = \hat{L}^h\) by (2.3). The proof of Proposition 1 is complete. \(\square\)

2.4. Proof of Proposition 2. To prove Proposition 2, we need the following continuity result for the penalized equation (2.4).

**Lemma 2.5.** Assume that (A.1) and (A.2) hold. If the sequence \(h^{(r)}\) converges weakly to \(h\) in \(S_N\) for some \(N \geq 1\), as \(r \to 0\), then for any \(n \geq 1\), we have

\[\lim_{r \to 0} \left( \sup_{0 \leq t \leq T} \left\| u^{n,h^{(r)}}(t) - u^{n,h}(t) \right\|^2_H + \int_0^T \left\| u^{n,h^{(r)}}(t) - u^{n,h}(t) \right\|^2_V dt \right) = 0. \quad (2.38)\]

**Proof.** Set \(Z^{(r)}(t) := u^{n,h^{(r)}}(t) - u^{n,h}(t)\). Since \(u^{n,h}\) is the solution of (2.4) and \(u^{n,h^{(r)}}\) is the solution of (2.4) with \(h\) replaced by \(h^{(r)}\), the chain rule implies that

\[
\left\| Z^{(r)}(t) \right\|^2_H + 2 \int_0^t \left\| Z^{(r)}(s) \right\|^2_V ds
\]
\[
= 2 \int_0^t \int_0^1 \left( \frac{\partial g(u^{n,h^{(r)}}(s,x))}{\partial x} - \frac{\partial g(u^{n,h}(s,x))}{\partial x} \right) Z^{(r)}(s,x) dx ds
\]
\[+ 2n \int_0^t \int_0^1 \left( u^{n,h^{(r)}}(s,x) - u^{n,h}(s,x) \right) Z^{(r)}(s,x) dx ds
\]
\[+ 2 \sum_{j=1}^m \int_0^t \int_0^1 \left( \sigma_j(u^{n,h^{(r)}}(s,x)) \right) \dot{h}_j^{(r)}(s) - \sigma_j(u^{n,h}(s,x)) \dot{h}_j(s) \right\rangle Z^{(r)}(s,x) dx ds
\]
\[=: I_1^{(r)}(t) + I_2^{(r)}(t) + I_3^{(r)}(t). \quad (2.39)\]

By (2.24), there exists \(c_{2.6} \in (0, \infty)\) satisfying that

\[
\left| I_1^{(r)}(t) \right| \leq \int_0^t \left\| Z^{(r)}(s) \right\|^2_V ds
\]
\[+ c_{2.6} \int_0^t \left( 1 + \left\| u^{n,h^{(r)}}(s) \right\|^2_V + \left\| u^{n,h}(s) \right\|^2_V \right) \left\| Z^{(r)}(s) \right\|^2_H ds. \quad (2.40)\]

Using the fact that for any \(y_1, y_2 \in \mathbb{R}\), \((y_1^- - y_2^-)(y_1 - y_2) \leq 0\), we have

\[I_2^{(r)}(t) \leq 0. \quad (2.41)\]
By (A1), there exists $c_{2,7} \in (0, \infty)$ satisfying that
\[
\dot{I}_3^{(r)}(t) \leq c_{2,7} \int_0^t \| Z^{(r)}(s) \|_H^2 : |\dot{h}^{(r)}(s)| \, ds + 2 \sum_{j=1}^m \int_0^t \int_0^1 \sigma_j \left( u^{n,h}(s,x) \right) \left( \dot{h}_j^{(r)}(s) - \dot{h}_j(s) \right) Z^{(r)}(s,x) \, dx \, ds. \tag{2.42}
\]

By Gronwall’s lemma and (2.39)–(2.42), we deduce that
\[
\sup_{t \in [0,T]} \| Z^{(r)}(t) \|_H^2 + \int_0^T \| Z^{(r)}(s) \|_V^2 \, ds \\
\leq e^{c_{2,7} \int_0^T \left( 1 + \| u^{n,h}(s) \|_V^2 + \| u^{n,h}(s) \|_V^2 \right) \, ds} + c_{2,7} \int_0^T \left| \dot{h}^{(r)}(s) \right| \, ds.
\]

Since $h^{(r)} \in S_N$, (2.5) implies that
\[
C(N,T) := \sup_{r \geq 0} \left( c_{2,6} \int_0^T \left( 1 + \| u^{n,h}(s) \|_V^2 + \| u^{n,h}(s) \|_V^2 \right) \, ds + c_{2,7} \int_0^T \left| \dot{h}^{(r)}(s) \right| \, ds \right) < + \infty.
\]

Thus,
\[
\sup_{t \in [0,T]} \| Z^{(r)}(t) \|_H^2 + \int_0^T \| Z^{(r)}(s) \|_V^2 \, ds \leq C(N,T)J^{(r)}, \tag{2.44}
\]
where
\[
J^{(r)} := \sup_{t \in [0,T]} \left| \int_0^t \int_0^1 \sigma_j \left( u^{n,h}(s,x) \right) \left( \dot{h}_j^{(r)}(s) - \dot{h}_j(s) \right) Z^{(r)}(s,x) \, dx \, ds \right|.
\]

Now we estimate $J^{(r)}$ by using the argument similar to that used in the proof of Lemma 3.4 of [11].

For any $q \in \mathbb{N}, k = 0, 1, 2, \ldots, 2^q - 1$, set
\[
\pi_q := t_{k+1} \equiv (k + 1) T \cdot 2^{-q} \text{ for } s \in [k T 2^{-q}, (k + 1) T 2^{-q}).
\]

Then
\[
J^{(r)} \leq 2 \sum_{i=1}^4 \bar{I}_i, \tag{2.45}
\]
where
\[
\bar{I}_1 := \sup_{t \in [0,T]} \left| \int_0^t \int_0^1 \sigma_j \left( u^{n,h}(s,x) \right) \left( \dot{h}_j^{(r)}(s) - \dot{h}_j(s) \right) \left( Z^{(r)}(s,x) - Z^{(r)}(\pi_q,x) \right) \, dx \, ds \right|,
\]
\[
\bar{I}_2 := \sup_{0 \leq t \leq T} \left| \int_0^t \int_0^1 \left( \sigma_j \left( u^{n,h}(s,x) \right) - \sigma_j \left( u^{n,h}(\pi_q,x) \right) \right) \left( \dot{h}_j^{(r)}(s) - \dot{h}_j(s) \right) Z^{(r)}(\pi_q,x) \, dx \, ds \right|.
\]
\[
\bar{I}_3 := \sup_{1 \leq k \leq 2^m} \sup_{t_{k-1} \leq t \leq t_k} \left| \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \sigma_j \left( u^{n,h}(t_j, x) \right) Z^{(r)}(t_j, x) dx \cdot \int_{t_{k-1}}^{t_k} \left( \hat{h}^{(r)}(s) - \hat{h}_j(s) \right) ds \right|
\]
\[
\bar{I}_4 := \sum_{k=1}^{2^m} \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \sigma_j \left( u^{n,h}(t_j, x) \right) Z^{(r)}(t_j, x) dx \cdot \int_{t_{k-1}}^{t_k} \left( \hat{h}^{(r)}(s) - \hat{h}_j(s) \right) ds \right|
\]

By (2.10) and Hölder's inequality, we get
\[
\bar{I}_1 \leq C_\sigma \int_{0}^{T} \left( \| u^{n,h}(s) \|_H + 1 \right) \cdot \left( \int_{0}^{T} \| u^{n,h(r)}(s) - u^{n,h(r)}(\bar{\sigma}_e) \|_H \cdot \| \hat{h}^{(r)}(s) - \hat{h}(s) \| ds \right)
\]
\[
\leq C_\sigma \sup_{s \in [0, T]} \left( \| u^{n,h}(s) \|_H + 1 \right) \cdot \left( \int_{0}^{T} \left( \int_{0}^{T} \| u^{n,h(r)}(s) - u^{n,h(r)}(\bar{\sigma}_e) \|_H \cdot \| \hat{h}^{(r)}(s) - \hat{h}(s) \| ds \right)^{\frac{1}{2}}
\]
\[
\cdot \left( \int_{0}^{T} u^{n,h(r)}(s) - u^{n,h(r)}(\bar{\sigma}_e) \|_H^2 ds \right)^{\frac{1}{2}} + \left( \int_{0}^{T} \| u^{n,h}(s) - u^{n,h}(\bar{\sigma}_e) \|_H^2 ds \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]

To estimate \( \int_{0}^{T} \left( u^{n,h}(s) - u^{n,h}(\bar{\sigma}_e) \right) \|_H^2 ds \), by (2.4), we have
\[
\left\{ \begin{aligned}
 u^{n,h}(\bar{\sigma}_e, x) - u^{n,h}(s, x) = \int_{0}^{\bar{\sigma}_e} \frac{\partial^2 u^{n,h}(t, x)}{\partial x^2} dt + \int_{0}^{\bar{\sigma}_e} \frac{\partial g(u^{n,h}(t, x))}{\partial x} dt \\
 + \sum_{j=1}^{m} \int_{0}^{\bar{\sigma}_e} \sigma_j \left( u^{n,h}(t, x) \right) \hat{h}_j(t) dt + m \int_{0}^{\bar{\sigma}_e} u^{n,h}(t, x) dt, \\
 u^{n,h}(t, 0) = u^{n,h}(t, 1) = 0, \quad t \in [s, \bar{\sigma}_e].
\end{aligned} \right.
\]

Applying the chain rule to \( \| u^{n,h}(\bar{\sigma}_e) - u^{n,h}(s) \|_H^2 \), we get
\[
\| u^{n,h}(\bar{\sigma}_e) - u^{n,h}(s) \|_H^2 = 2 \int_{0}^{\bar{\sigma}_e} \int_{0}^{1} \frac{\partial^2 u^{n,h}(t, x)}{\partial x^2} (u^{n,h}(t, x) - u^{n,h}(s, x)) dx dt
\]
\[
+ \int_{0}^{\bar{\sigma}_e} \int_{0}^{1} \frac{\partial g(u^{n,h}(t, x))}{\partial x} (u^{n,h}(t, x) - u^{n,h}(s, x)) dx dt
\]
\[
+ 2 \sum_{j=1}^{m} \int_{0}^{\bar{\sigma}_e} \int_{0}^{1} \sigma_j \left( u^{n,h}(t, x) \right) \hat{h}_j(t) (u^{n,h}(t, x) - u^{n,h}(s, x)) dx dt
\]
\[
+ 2 \int_{0}^{\bar{\sigma}_e} \int_{0}^{1} n u^{n,h}(t, x) - (u^{n,h}(t, x) - u^{n,h}(s, x)) dx dt.
\]
\[
\begin{align*}
= - \int_s^{\bar{s}} \left\| u^{n,h}(t) - u^{n,h}(s) \right\|_V^2 \, dt \\
+ \int_s^{\bar{s}} \int_0^1 \frac{\partial^2 u^{n,h}(s,x)}{\partial x^2} (u^{n,h}(t,x) - u^{n,h}(s,x)) \, dx \, dt \\
\end{align*}
\]

and

\[
\int_s^{\bar{s}} \int_0^1 \frac{\partial g (u^{n,h}(t,x))}{\partial x} u^{n,h}(t,x) \, dx \, dt = 0.
\]  

(2.49)

It is also easy to see that

\[
\int_s^{\bar{s}} \int_0^1 nu^{n,h}(t,x) - u^{n,h}(t,x) \, dx \, dt \leq 0.
\]  

(2.50)

Combining (2.47)–(2.50), we have

\[
\begin{align*}
&\left\| u^{n,h}(s) - u^{n,h}(s) \right\|_H^2 + 2 \int_s^{\bar{s}} \left\| u^{n,h}(t) - u^{n,h}(s) \right\|_V^2 \, dt \\
\leq &\ 2 \int_s^{\bar{s}} \int_0^1 \frac{\partial^2 u^{n,h}(s,x)}{\partial x^2} (u^{n,h}(t,x) - u^{n,h}(s,x)) \, dx \, dt \\
&\ - 2 \int_s^{\bar{s}} \int_0^1 \frac{\partial g (u^{n,h}(t,x))}{\partial x} u^{n,h}(s,x) \, dx \, dt \\
&\ + 2 \sum_{j=1}^m \int_s^{\bar{s}} \int_0^1 \sigma_j (u^{n,h}(t,x)) \dot{\sigma}_j (t) (u^{n,h}(t,x) - u^{n,h}(s,x)) \, dx \, dt \\
&\ - 2 \int_s^{\bar{s}} \int_0^1 nu^{n,h}(t,x) - u^{n,h}(s,x) \, dx \, dt.
\end{align*}
\]

Then, by (2.10), Assumption (A.2) and Lemma 2.1, there exist \( c_{2.8}, c_{2.9} \in (0, \infty) \) such that

\[
\begin{align*}
&\left\| u^{n,h}(s) - u^{n,h}(s) \right\|_H^2 + \int_s^{\bar{s}} \left\| u^{n,h}(t) - u^{n,h}(s) \right\|_V^2 \, dt \\
\leq &\ \left\| u^{n,h}(s) \right\|_V^2 (\bar{s} - s) + c_{2.8} \left( 1 + \sup_{t \in [0,T]} \left\| u^{n,h}(t) \right\|_H \right) (\bar{s} - s) \\
&\ + c_{2.8} \left( 1 + \sup_{t \in [0,T]} \left\| u^{n,h}(t) \right\|_H \right) (\bar{s} - s)^{1/2} \left( \int_s^{\bar{s}} \left\| u^{n,h}(t) \right\|_H^2 \, dt \right)^{1/2} \\
&\ + c_{2.8} \sup_{t \in [0,T]} \left\| u^{n,h}(t) \right\|_H \cdot n^{1/2} (\bar{s} - s)^{1/2} \left( n \int_s^{\bar{s}} \left\| u^{n,h}(t) - u^{n,h}(s) \right\|_H^2 \, dt \right)^{1/2} \\
\leq &\ T^{2-\delta} \left\| u^{n,h}(s) \right\|_V^2 + c_{2.9} 2^{-\delta/2}.
\end{align*}
\]

Note that \( c_{2.9} \) is independent of \( h \). Integrating over \([0,T]\) with respect to \( s \), and by Lemma 2.1 again, we obtain

\[
\int_0^T \left\| u^{n,h}(s) - u^{n,h}(s) \right\|_H^2 \, ds \leq c_{2.10} 2^{-\delta/2},
\]  

(2.51)
where the positive constant $c_{2,10}$ is independent of $h$. The proof of the above inequality also implies that, for any $r > 0$,
\[
\int_0^T \left\| u^{n,h}(r)(\bar{\pi}_0) - u^{n,h}(r)(s) \right\|_H^2 \, ds \leq c_{2,10} 2^{-e/2}. \tag{2.52}
\]
By (2.46)–(2.52) and Lemma 2.1,
\[
\bar{I}_1 \leq c_{2,11} 2^{-e/4}. \tag{2.53}
\]
By Assumption (A.1) and using the similar argument in the proof of (2.53), there exist $c_{2,12}, c_{2,13} \in (0, \infty)$ which are independent of $h$ and satisfy that
\[
\begin{align*}
\bar{I}_2 & \leq \sup_{t \in [0,T]} \left( \left\| u^{n,h}(r)(t) \right\|_H + \left\| u^{n,h}(t) \right\|_H \right) \left( \int_0^T \left| \dot{h}(r)(s) - \dot{h}(s) \right|^2 \, ds \right)^{1/2} \cdot \left[ \int_0^T \left\| u^{n,h}(s) - u^{n,h}(\bar{\pi}_0) \right\|_H^2 \, ds \right]^2 r \leq c_{2,12} 2^{-e/4} \tag{2.54}
\end{align*}
\]
and
\[
\begin{align*}
\bar{I}_3 & \leq c_\sigma \sup_{t \in [0,T]} \left( 1 + \left\| u^{n,h}(r)(t) \right\|_H^2 + \left\| u^{n,h}(t) \right\|_H^2 \right) \cdot \sup_{1 \leq k \leq 2^\nu} \left( \int_{t_{k-1}}^{t_k} \left| \dot{h}(r)(s) - \dot{h}(s) \right|^2 \, ds \right)^{1/2} \cdot 2^{-e/2} T^{1/2} \leq c_{2,13} 2^{-e/4}. \tag{2.55}
\end{align*}
\]
Since the sequence $\dot{h}(r)$ converges weakly to $h$ in $S_N$ as $r \to 0$, for any $a, b \in [0, T]$ with $a < b$, \( \int_a^b \dot{h}(r)(s) \, ds \) converges to \( \int_a^b \dot{h}(s) \, ds \) in \( \mathbb{R}^m \). Therefore, by the linear growth property of $\sigma$ (see (2.10)) and Lemma 2.1, we know that
\[
\begin{align*}
& \lim_{r \to 0} \left\| \sigma \left( u^{n,h}(t_k) \right) \left( \int_{t_{k-1}}^{t_k} \dot{h}(r)(s) \, ds - \int_{t_{k-1}}^{t_k} \dot{h}(s) \, ds \right) \right\|_H \\
& \leq \lim_{r \to 0} \int_{t_{k-1}}^{t_k} \dot{h}(r)(s) \, ds - \int_{t_{k-1}}^{t_k} \dot{h}(s) \, ds \cdot C_\sigma \left( 1 + \left\| u^{n,h}(t_k) \right\|_H \right) \\
& = 0.
\end{align*}
\]
Hence, for fixed $\varrho \in \mathbb{N}$,
\[
\begin{align*}
\limsup_{r \to 0} \bar{I}_4 & = \limsup_{r \to 0} \sum_{k=1}^{2^\nu} \left( 2^{-1} \sum_{j=1}^{m} \left| \sigma \left( u^{n,h}(t_k,x) \right) Z^{(r)}(t_k,x) \, dx \cdot \int_{t_{k-1}}^{t_k} \left( \dot{h}_j(r)(s) - \dot{h}_j(s) \right) \, ds \right| \right) \\
& \leq \limsup_{r \to 0} 2^{-e} \sum_{k=1}^{2^\nu} \left\| \sigma \left( u^{n,h}(t_k) \right) \left( \int_{t_{k-1}}^{t_k} \dot{h}(r)(s) \, ds - \int_{t_{k-1}}^{t_k} \dot{h}(s) \, ds \right) \right\|_H \cdot \left\| Z^{(r)}(t_k) \right\|_H \tag{2.56}
\end{align*}
\]
\[
\leq C_N \limsup_{r \to 0} \sum_{k=1}^{2^\nu} \left\| \sigma \left( u^{n,h}(t_k) \right) \left( \int_{t_{k-1}}^{t_k} \dot{h}(r)(s) \, ds - \int_{t_{k-1}}^{t_k} \dot{h}(s) \, ds \right) \right\|_H = 0.
\]
Here (2.5) has been used to get the last inequality.
Combining (2.44), (2.45), and (2.53)–(2.56), we have
\[
\limsup_{r \to 0} \left( \sup_{t \in [0,T]} \left\| Z^{(r)}(t) \right\|_H^2 + \int_0^T \left\| Z^{(r)}(t) \right\|_V^2 \, dt \right) \leq c_{2.15,2} \frac{2^{-\theta/4}}{\theta}, \quad \forall \theta \in \mathbb{N}, \quad (2.57)
\]
where \( c_{2.15} \) is independent of \( h \). The arbitrariness of \( \theta \) in (2.57) implies that
\[
\limsup_{r \to 0} \left( \sup_{t \in [0,T]} \left\| Z^{(r)}(t) \right\|_H^2 + \int_0^T \left\| Z^{(r)}(t) \right\|_V^2 \, dt \right) = 0. \quad (2.58)
\]
This completes the proof of Lemma 2.5.
\[\square\]

**Proof of Proposition 2.** For any \( T \geq 0 \), we need to prove that
\[
\lim_{r \to 0} \left( \sup_{0 \leq t \leq T} \left\| u^{h,r}(t) - u^h(t) \right\|_H^2 + \int_0^T \left\| u^{h,r}(t) - u^h(t) \right\|_V^2 \, dt \right) = 0.
\]
Since
\[
\left\| u^{h,r}(t) - u^h(t) \right\|_H^2 \leq 3 \left\| u^{n,h}(t) - u^h(t) \right\|_H^2 + 3 \left\| u^{n,r}(t) - u^{n,h}(t) \right\|_H^2 + 3 \left\| u^{n,h}(t) - u^h(t) \right\|_H^2.
\]
By Lemma 2.4, for any \( \eta > 0 \), there exists \( M > 0 \) such that for any \( n \geq M \),
\[
\sup_{0 \leq t \leq T} \left\| u^{n,h}(t) - u^h(t) \right\|_H^2 \leq \eta \quad \text{and} \quad \sup_{0 \leq t \leq T} \left\| u^{n,h}(t) - u^h(t) \right\|_H^2 \leq \frac{\eta}{9}.
\]
By Lemma 2.5, for any \( n \geq M \), there exists \( r_n > 0 \) such that for any \( r \in (0, r_n) \), we have
\[
\sup_{0 \leq t \leq T} \left\| u^{n,h}(t) - u^{n,h}(t) \right\|_H^2 \leq \frac{\eta}{9}.
\]
Hence, for any \( n \geq M \) and \( r \in (0, r_n) \), we have
\[
\sup_{0 \leq t \leq T} \left\| u^{h}(t) - u^h(t) \right\|_H^2 \leq \eta.
\]
The convergence of \( u^{h}(t) \) in \( L^2([0,T]; V) \) can be proved in the same way. The proof of Proposition 2 is complete. \[\square\]

3. **Large deviation principle for \( u^x \).** In this section, we first recall the definition of the LDP and a sufficient condition of the weak convergence criteria for the LDP given in [14]. Then, we state the main result of this paper and give its proof.

3.1. **Large deviation principle.** Let \((\Omega, \mathcal{F}, P)\) be a probability space with an increasing family \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) of the sub-\(\sigma\)-fields of \(\mathcal{F}\) satisfying the usual conditions. Let \(\mathcal{E}\) be a Polish space with the Borel \(\sigma\)-field \(\mathcal{B}(\mathcal{E})\).

**Definition 3.1. (Rate function)** A function \( I : \mathcal{E} \to [0, \infty] \) is called a rate function on \(\mathcal{E}\), if for each \( M < +\infty \), the level set \( \{x \in \mathcal{E} : I(x) \leq M\} \) is a compact subset of \(\mathcal{E}\).

**Definition 3.2. (Large deviation principle)** Let \( I \) be a rate function on \(\mathcal{E}\). A family \( \{X^x\} \) of \(\mathcal{E}\)-valued random variables is said to satisfy the large deviation principle on \(\mathcal{E}\) with rate function \( I \), if the following two conditions hold.
(a) (Large deviation upper bound) For each closed subset $F$ of $\mathcal{E}$,
\[
\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leq - \inf_{x \in F} I(x).
\]

(b) (Large deviation lower bound) For each open subset $G$ of $\mathcal{E}$,
\[
\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \geq - \inf_{x \in G} I(x).
\]

Recall the Cameron-Martin space $\mathcal{H}$ defined by (2.1). Let $\mathcal{A}$ denote the class of all $\mathcal{H}$-valued $\mathcal{F}_t$-predictable processes, and let
\[
\mathcal{A}_N := \{ \phi \in \mathcal{A}; \phi(\omega) \in S_N, \text{P-a.s.} \}.
\]

Recall the following result from Matoussi et al. [14].

**Theorem 3.3.** ([14, Theorem 3.2]) For any $\varepsilon > 0$, let $\Gamma^\varepsilon$ be a measurable mapping from $C([0,T]; \mathbb{R}^m)$ into $\mathcal{E}$. Let $X^\varepsilon := \Gamma^\varepsilon(B(\cdot))$. Suppose that there exists a measurable map $\Gamma^0 : C([0,T]; \mathbb{R}^m) \to \mathcal{E}$ such that the following two items hold:

(a) For every $N < +\infty$ and $\{h^\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_N$ and $\delta > 0$,
\[
\lim_{\varepsilon \to 0} \mathbb{P}\left( \rho(Y^\varepsilon, Z^\varepsilon) > \delta \right) = 0,
\]
where $Y^\varepsilon = \Gamma^\varepsilon(B(\cdot)) + \frac{1}{\sqrt{\varepsilon}} \int_0^t \hat{h}^\varepsilon(s) ds$, $Z^\varepsilon = \Gamma^0(\int_0^t \hat{h}^\varepsilon(s) ds)$ and $\rho(\cdot, \cdot)$ stands for the metric in the space $\mathcal{E}$.

(b) For every $N < +\infty$ and any family $\{h^\varepsilon; \varepsilon > 0\} \subset S_N$ that converges weakly to some element $h$ as $\varepsilon \to 0$, $\Gamma^0(\int_0^t \hat{h}^\varepsilon(s) ds)$ converges to $\Gamma^0(\int_0^t \hat{h}(s) ds)$ in the space $\mathcal{E}$.

Then the family $\{X^\varepsilon\}_{\varepsilon > 0}$ satisfies an LDP in $\mathcal{E}$ with the rate function $I$ given by
\[
I(g) := \inf_{\{h \in \mathcal{H}; g = \Gamma^0(\int_0^t \hat{h}(s) ds)\}} \left\{ \frac{1}{2} \|h\|_{\mathcal{H}}^2 \right\}, \quad g \in \mathcal{E}, \tag{3.1}
\]
with the convention $\inf \emptyset = +\infty$.

### 3.2. The main result

By Theorem 1, we know that for any $h \in \mathcal{H}$, (2.3) admits a unique solution $(u^h, L^h)$. Define $\Gamma^0 : \mathcal{H} \to C([0,T]; H) \cap L^2([0,T]; V)$ by
\[
\Gamma^0 \left( \int_0^t \hat{h}(s) ds \right) := u^h, \quad \text{for } h \in \mathcal{H}. \tag{3.2}
\]

The main result in this paper is the following LDP.

**Theorem 3.4.** Assume that (A.1) and (A.2) hold, and $u_0 \in V$ with $u_0 \geq 0$. Then the family $\{u^\varepsilon\}_{\varepsilon > 0}$ in (1.1) satisfy an LDP in the space $C([0,T]; H) \cap L^2([0,T]; V)$ with the rate function $I$ given by
\[
I(g) := \left\{ \begin{array}{ll}
\inf_{\{h \in \mathcal{H}; g = \Gamma^0(\int_0^t \hat{h}(s) ds)\}} \left\{ \frac{1}{2} \|h\|_{\mathcal{H}}^2 \right\}, & \text{if } g \in \Gamma^0(\mathcal{H}); \\
+\infty, & \text{otherwise}. \end{array} \right. \tag{3.3}
\]

**Proof.** To prove this theorem, it is sufficient to prove that Conditions (a) and (b) in Theorem 3.3 are satisfied. Condition (b) has been proved in Proposition 2. The verification of Condition (a) will be given in Proposition 3 below, and the proof of Theorem 3.4 is complete. \qed
The following part is devoted to verify Condition (a) in Theorem 3.3. We first give the definition of the mapping $\Gamma^\varepsilon$ and give some a priori estimates.

By the Yamada-Watanabe theorem, for any $\varepsilon > 0$, there exists $\Gamma^\varepsilon : C([0, T]; \mathbb{R}^m) \to C([0, T]; H) \cap L^2([0, T]; V)$ such that

$$
\Gamma^\varepsilon (B(\cdot)) := u^\varepsilon, 
$$

where $u^\varepsilon$ stands for the solution of Equation (1.1).

Let $\{h^\varepsilon\}_{\varepsilon > 0} \subset A_N$ be a given family of stochastic processes. By the Girsanov theorem, we know that $Y^\varepsilon := \Gamma^\varepsilon (B(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^t \dot{h}^\varepsilon (s) ds)$ is the unique solution of the following equation:

$$
\begin{cases}
    dY^\varepsilon(t, x) = \frac{\partial^2 Y^\varepsilon(t, x)}{\partial x^2} dt + \frac{\partial g(Y^\varepsilon(t, x))}{\partial x} dt + \sqrt{\varepsilon} \sum_{j=1}^{m} \sigma_j (Y^\varepsilon(t, x)) dB_j(t) \\
    \quad + \sum_{j=1}^{m} \sigma_j (Y^\varepsilon(t, x)) \dot{h}_j^\varepsilon(t) dt + L^{\varepsilon, Y}(dt, dx), \quad t > 0, x \in [0, 1],
\end{cases}
$$

$$
Y^\varepsilon(t, x) \geq 0, \quad Y^\varepsilon(0, x) = u_0(x) \geq 0, \quad t > 0, x \in [0, 1],
$$

$$
Y^\varepsilon(t, 0) = Y^\varepsilon(t, 1) = 0, \quad t \geq 0.
$$

Recall $\Gamma^0$ defined by (3.2). Then $Z^\varepsilon := \Gamma^0 (\int_0^t \dot{h}^\varepsilon(s) ds)$ solves the following equation:

$$
\begin{cases}
    dZ^\varepsilon(t, x) = \frac{\partial^2 Z^\varepsilon(t, x)}{\partial x^2} dt + \frac{\partial g(Z^\varepsilon(t, x))}{\partial x} dt \\
    \quad + \sum_{j=1}^{m} \sigma_j (Z^\varepsilon(t, x)) \dot{h}_j^\varepsilon(t) dt + L^{\varepsilon, Z}(dt, dx), \quad t > 0, x \in [0, 1],
\end{cases}
$$

$$
Z^\varepsilon(t, x) \geq 0, \quad Z^\varepsilon(0, x) = u_0(x) \geq 0, \quad t > 0, x \in [0, 1],
$$

$$
Z^\varepsilon(t, 0) = Z^\varepsilon(t, 1) = 0, \quad t \geq 0.
$$

For any $t \geq 0$, denote

$$
Y^\varepsilon(t) := Y^\varepsilon(t, \cdot), \quad Z^\varepsilon(t) := Z^\varepsilon(t, \cdot).
$$

**Lemma 3.5.** For any $\{h^\varepsilon\}_{\varepsilon > 0} \subset A_N$ with $N \geq 1$, we have

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|Y^\varepsilon(t)\|_H^2 + \int_0^T \|Y^\varepsilon(t)\|_V^2 dt \right] < +\infty
$$

and

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|Z^\varepsilon(t)\|_H^2 + \int_0^T \|Z^\varepsilon(t)\|_V^2 dt \right] < +\infty.
$$
Proof. By Itô’s formula, we have
\[ \|Y^\varepsilon(t)\|_H^2 + 2 \int_0^t \|Y^\varepsilon(s)\|_V^2 \, ds \]
\[ = \|u_0\|_H^2 + 2 \int_0^t \left\langle Y^\varepsilon(s), \frac{\partial g(Y^\varepsilon(s))}{\partial x} \right\rangle \, ds \]
\[ + 2\sqrt{\varepsilon} \sum_{j=1}^m \int_0^t \langle Y^\varepsilon(s), \sigma_j(Y^\varepsilon(s)) \rangle dB_j(s) + \varepsilon \sum_{j=1}^m \int_0^t \|\sigma_j(Y^\varepsilon(s))\|_H^2 \, ds \]
\[ + 2 \int_0^t \langle Y^\varepsilon(s), \sigma(Y^\varepsilon(s)) \rangle \hat{h}^\varepsilon(s) \, ds + 2 \int_0^t \int_0^1 Y^\varepsilon(s, x) L^\varepsilon_Y(ds, dx) \]
\[ =: \|u_0\|_H^2 + I_1^{Y,\varepsilon}(t) + I_2^{Y,\varepsilon}(t) + I_3^{Y,\varepsilon}(t) + I_4^{Y,\varepsilon}(t) + I_5^{Y,\varepsilon}(t). \]

Set \( G(z) := \int_0^z g(y) \, dy \). Then by the boundary condition of \( Y^\varepsilon \), we have
\[ I_1^{Y,\varepsilon}(t) = -2 \int_0^t ds \int_0^1 \frac{\partial Y^\varepsilon(s)}{\partial x} g(Y^\varepsilon(s, x)) \, dx \]
\[ = -2 \int_0^t ds \int_0^1 \frac{\partial G(Y^\varepsilon(s, x))}{\partial x} \, dx = 0. \quad (3.10) \]

By Burkholder-Davis-Gundy’s inequality and the Lipschitz condition of \( \sigma \), there exist \( c_{3,1}, c_{3,2} \in (0, \infty) \) satisfying that
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| I_2^{Y,\varepsilon}(t) \right| \right] \]
\[ \leq \sqrt{\varepsilon} c_{3,1} \mathbb{E} \left[ \left( \sum_{j=1}^m \int_0^T \langle Y^\varepsilon(s), \sigma_j(Y^\varepsilon(s)) \rangle^2 \, ds \right)^{\frac{1}{2}} \right] \]
\[ \leq \sqrt{\varepsilon} c_{3,1} \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \|Y^\varepsilon(t)\|_H \right) \left( \sum_{j=1}^m \int_0^T \|\sigma_j(Y^\varepsilon(s))\|_H^2 \, ds \right)^{\frac{1}{2}} \right] \]
\[ \leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|Y^\varepsilon(t)\|_H^2 \right] + \varepsilon c_{3,2} \mathbb{E} \left[ \int_0^T \left( 1 + \|Y^\varepsilon(s)\|_H^2 \right) \, ds \right]. \quad (3.11) \]

By the Lipschitz condition of \( \sigma \), there exists \( c_{3,3} \in (0, \infty) \) satisfying that
\[ I_3^{Y,\varepsilon}(t) \leq c_{3,3} \int_0^t \left( 1 + \|Y^\varepsilon(s)\|_H^2 \right) \, ds. \quad (3.12) \]

By the Cauchy-Schwarz inequality and by using the same technique in the proof of (2.11), we can obtain that
\[ I_4^{Y,\varepsilon}(t) \leq \frac{1}{4} \sup_{0 \leq s \leq t} \|Y^\varepsilon(s)\|_H^2 + c_{3,4} N^2 \int_0^t \|Y^\varepsilon(s)\|_H^2 \, ds, \quad (3.13) \]
where \( c_{3,4} \in (0, \infty) \). By (iv) in Definition 1.1, we have
\[ I_5^{Y,\varepsilon}(t) = 0. \quad (3.14) \]
Putting (3.9)-(3.14) together, we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \| Y^\varepsilon(t) \|_H^2 \right] + \mathbb{E} \left[ \int_0^T \| Y^\varepsilon(t) \|_V^2 \, dt \right] \\
\leq 2 \| u_0 \|_H^2 + 2\varepsilon T (c_{3,2} + c_{3,3}) + 2 \left( \varepsilon c_{3,2} + \varepsilon c_{3,4} + c_{3,4}N^2 \right) \mathbb{E} \left[ \int_0^T \| Y^\varepsilon(t) \|_H^2 \, dt \right].
\]

By Gronwall’s inequality, we obtain (3.7).

The proof of (3.8) is similar and is omitted here. The proof is complete. \(\square\)

**Lemma 3.6.** There exists a constant \(\lambda > 0\) such that
\[
\lim_{\varepsilon \to 0} \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \exp \left\{ -\lambda \int_0^t \left( \| Y^\varepsilon(s) \|_V^2 + \| Z^\varepsilon(s) \|_V^2 \right) ds \right\} \| Y^\varepsilon(t) - Z^\varepsilon(t) \|_H^2 \right] \right. \\
+ \mathbb{E} \left[ \int_0^T \exp \left\{ -\lambda \int_0^t \left( \| Y^\varepsilon(s) \|_V^2 + \| Z^\varepsilon(s) \|_V^2 \right) ds \right\} \| Y^\varepsilon(t) - Z^\varepsilon(t) \|_H^2 \, dt \right] \right\} = 0.
\]

**Proof.** Denote
\[
E_{Y^\varepsilon, Z^\varepsilon}(t) := \exp \left\{ -\lambda \int_0^t \left( \| Y^\varepsilon(s) \|_V^2 + \| Z^\varepsilon(s) \|_V^2 \right) ds \right\}.
\]

By Itô’s formula, we have
\[
E_{Y^\varepsilon, Z^\varepsilon}(t) \| Y^\varepsilon(t) - Z^\varepsilon(t) \|_H^2 \\
= -\lambda \int_0^t E_{Y^\varepsilon, Z^\varepsilon}(s) \left( \| Y^\varepsilon(s) \|_V^2 + \| Z^\varepsilon(s) \|_V^2 \right) \| Y^\varepsilon(s) - Z^\varepsilon(s) \|_H^2 \, ds \\
- 2 \int_0^t E_{Y^\varepsilon, Z^\varepsilon}(s) \| Y^\varepsilon(s) - Z^\varepsilon(s) \|_H^2 \, ds \\
+ 2 \int_0^t E_{Y^\varepsilon, Z^\varepsilon}(s) \left( Y^\varepsilon(s) - Z^\varepsilon(s), \frac{\partial g(Y^\varepsilon(s))}{\partial x} - \frac{\partial g(Z^\varepsilon(s))}{\partial x} \right) ds \\
+ 2\varepsilon \sum_{j=1}^m \int_0^t E_{Y^\varepsilon, Z^\varepsilon}(s) \| \sigma_j(Y^\varepsilon(s)) \|_H^2 \, ds \\
+ \varepsilon \sum_{j=1}^m \int_0^t E_{Y^\varepsilon, Z^\varepsilon}(s) \| \sigma_j(Y^\varepsilon(s)) \|_H^2 \, ds \\
+ 2 \int_0^t E_{Y^\varepsilon, Z^\varepsilon}(s) \left( Y^\varepsilon(s) - Z^\varepsilon(s), \sigma(Y^\varepsilon(s)) - \sigma(Z^\varepsilon(s)) \right) \hat{h}(s) ds \\
+ 2 \int_0^t \int_0^1 E_{Y^\varepsilon, Z^\varepsilon}(s) \left( Y^\varepsilon(s, x) - Z^\varepsilon(s, x) \right) \left( L^\varepsilon Y(ds, dx) - L^\varepsilon Z(ds, dx) \right) ds dx \\
= I_3^\varepsilon(t) + I_7^\varepsilon(t) + I_8^\varepsilon(t) + I_9^\varepsilon(t) + I_{10}^\varepsilon(t) + I_{11}^\varepsilon(t).
\]

Using the same technique in the proof of (2.25), we have
\[
I_3^\varepsilon(t) \leq \frac{1}{4} \int_0^t E_{Y^\varepsilon, Z^\varepsilon}(s) \| Y^\varepsilon(s) - Z^\varepsilon(s) \|_V^2 \, ds \\
+ c_{3,5} \int_0^t E_{Y^\varepsilon, Z^\varepsilon}(s) \left( \| Y^\varepsilon(s) \|_V^2 + \| Z^\varepsilon(s) \|_V^2 \right) \| Y^\varepsilon(s) - Z^\varepsilon(s) \|_H^2 \, ds,
\]
for some constant \(c_{3,5} \in (0, \infty)\).
By Burkholder-Davis-Gundy’s inequality and the Lipschitz condition of \( \sigma \), there exist some constants \( c_{3,6}, c_{3,7} \in (0, \infty) \) satisfying that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |I_2^e(s)| \right] \\
\leq \varepsilon c_{3,6} \mathbb{E} \left[ \sum_{j=1}^{m} \int_0^t E_{Y^e,s,s}^2 \left( \int_0^t \sigma_j(Y^e(s)) ds \right)^2 ds \right] \\
\leq \varepsilon c_{3,6} \mathbb{E} \left[ \sup_{0 \leq s \leq t} E_{Y^e,s,s}^2 \left( \int_0^t \sigma_j(Y^e(s)) ds \right)^2 |s \rangle \right] \\
\leq \frac{1}{4} \varepsilon c_{3,6} \mathbb{E} \left[ \sup_{0 \leq s \leq t} E_{Y^e,s,s}^2 \left( \int_0^t \sigma_j(Y^e(s)) ds \right)^2 \right] \\
\leq \frac{1}{4} \varepsilon c_{3,6} \mathbb{E} \left[ \sup_{0 \leq s \leq t} E_{Y^e,s,s}^2 \left( \int_0^t \sigma_j(Y^e(s)) ds \right)^2 \right] \\
\leq \frac{1}{4} \varepsilon c_{3,6} \mathbb{E} \left[ \sup_{0 \leq s \leq t} E_{Y^e,s,s}^2 \left( \int_0^t \sigma_j(Y^e(s)) ds \right)^2 \right].
\tag{3.19}
\]
By the Lipschitz condition of \( \sigma \), there exists \( c_{3,8} \in (0, \infty) \) satisfying that
\[
I_2^e(t) \leq \varepsilon c_{3,8} \int_0^t E_{Y^e,s,s} \left( 1 + \|Y^e(s)\|_H^2 \right) ds.
\tag{3.20}
\]
By the Cauchy-Schwarz inequality and the same technique used in (2.11), there exists \( c_{3,9} \in (0, \infty) \) satisfying that
\[
I_2^e(t) \leq \frac{1}{4} \sup_{0 \leq s \leq t} \left( E_{Y^e,s,s} \left( \|Y^e(s) - Z^e(s)\|_H^2 \right) \right) \\
+ c_{3,9} N^2 \int_0^t E_{Y^e,s,s} \left( Y^e(s) - Z^e(s) \right)^2 \|E_{Y^e,s,s} \|_H^2 ds.
\tag{3.21}
\]
Since \( Y^e(t) \) and \( Z^e(t) \) are nonnegative, it follows from the definition of the solution that
\[
\int_0^t \int_0^1 E_{Y^e,s,s} \left( Y^e(s,x) - Z^e(s,x) \right) L^{e,Y}(ds,dx) \leq 0
\]
and
\[
\int_0^t \int_0^1 E_{Y^e,s,s} \left( Y^e(s) - Z^e(s) \right) L^{e,Z}(ds,dx) \geq 0.
\]
The above two inequalities imply that
\[
I_2^e(t) \leq 0.
\tag{3.22}
\]
Putting (3.17)-(3.22) together, we obtain that for any \( \lambda > c_{3,5} \),
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} E_{Y^e,s,s} \left( \|Y^e(t) - Z^e(t)\|_H^2 \right) \right] \\
\leq 2 \varepsilon (c_{3,7} + c_{3,8}) \left( T + \int_0^T E_{Y^e,s,s} \left( \|Y^e(t) - Z^e(t)\|_H^2 \right) dt \right) \\
+ 2c_{3,9} N^2 \mathbb{E} \left[ \int_0^T E_{Y^e,s,s} \left( \|Y^e(t) - Z^e(t)\|_H^2 \right) dt \right]
\tag{3.23}
\]
Applying Gronwall’s inequality to (3.23), we get (3.15). The proof is complete. \( \square \)

Now we are in a position to verify Condition (a) in Theorem 3.3.
Proposition 3. For any $\delta > 0$,
\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{0 \leq t \leq T} \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_H^2 + \int_0^T \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_V^2 \, dt > \delta \right) = 0. \tag{3.24}
\]

Proof. Recall $E_{Y^\varepsilon,Z^\varepsilon}(t)$ defined in (3.16). For any $M > 0, \lambda > 0$, we have
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_H^2 + \int_0^T \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_V^2 \, dt > \delta \right)
= \mathbb{P} \left( \sup_{0 \leq t \leq T} \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_H^2 + \int_0^T \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_V^2 \, dt > \delta, \right.
\]
\[
\left. \int_0^T \|Y^\varepsilon(t)\|_V^2 + \|Z^\varepsilon(t)\|_V^2 \, dt > M \right) + \mathbb{P} \left( \sup_{0 \leq t \leq T} \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_H^2 + \int_0^T \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_V^2 \, dt > \delta, \right.
\]
\[
\left. \int_0^T \|Y^\varepsilon(t)\|_V^2 + \|Z^\varepsilon(t)\|_V^2 \, dt \leq M \right) \leq \mathbb{P} \left( \int_0^T \|Y^\varepsilon(t)\|_V^2 + \|Z^\varepsilon(t)\|_V^2 \, dt > M \right)
\]
\[
+ \mathbb{P} \left( \sup_{0 \leq t \leq T} \exp \left\{ -\lambda \int_0^t \left( \|Y^\varepsilon(s)\|_V^2 + \|Z^\varepsilon(s)\|_V^2 \right) \, ds \right\} \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_H^2 \right.
\]
\[
\left. + \int_0^T \exp \left\{ -\lambda \int_0^t \left( \|Y^\varepsilon(s)\|_V^2 + \|Z^\varepsilon(s)\|_V^2 \right) \, ds \right\} \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_V^2 \, dt \geq e^{-\lambda M} \delta \right) \leq \frac{1}{M} \mathbb{E} \left[ \int_0^T \|Y^\varepsilon(t)\|_V^2 + \|Z^\varepsilon(t)\|_V^2 \, dt \right]
\]
\[
+ \frac{e^{\lambda M}}{\delta} \mathbb{E} \left[ \sup_{0 \leq t \leq T} E_{Y^\varepsilon,Z^\varepsilon}(t) \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_H^2 + \int_0^T E_{Y^\varepsilon,Z^\varepsilon}(t) \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_V^2 \, dt \right].
\]

By Lemma 3.5, we know that for any $\eta > 0$, there exists $M > 0$ such that
\[
\frac{1}{M} \mathbb{E} \left[ \int_0^T \|Y^\varepsilon(t)\|_V^2 + \|Z^\varepsilon(t)\|_V^2 \, dt \right] \leq \frac{\eta}{2}.
\]

By Lemma 3.6, we know that for any $\eta > 0$, there exist $\lambda > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,
\[
\frac{e^{\lambda M}}{\delta} \mathbb{E} \left[ \sup_{0 \leq t \leq T} E_{Y^\varepsilon,Z^\varepsilon}(t) \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_H^2 + \int_0^T E_{Y^\varepsilon,Z^\varepsilon}(t) \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_V^2 \, dt \right] \leq \frac{\eta}{2}.
\]

Hence, we obtain that for all $\varepsilon \in (0, \varepsilon_0)$,
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_H^2 + \int_0^T \|Y^\varepsilon(t) - Z^\varepsilon(t)\|_V^2 \, dt > \delta \right) \leq \eta.
\]

The proof is complete. \qed

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