On the Spin-Orbital Structure of the “Zero” Magnetization Spin-Precessing Modes of the Superfluid $^3$He $- B$

G. Kharadze and N. Suramlishvili

Andronikashvili Institute of Physics, Georgian Academy of Sciences, 6 Tamarashvili St.,
380077, Tbilisi, Georgia

The spin-orbital configurations of the coherently precessing spin modes characterized by a small value of the magnetization ($M \ll M_0 = \chi H_0$) is considered. Various regimes are analysed depending on the transverse rf-field strength.

PACS: 67.57.Lm

In the superfluid phases of liquid $^3$He the unusual spin-precessing modes can be exited at the magnitude $M = |\vec{M}|$ of the magnetization essentially different from its equilibrium value $M_0 = \chi H_0$ (here $\chi$ is the magnetic susceptibility and $H_0$ denotes the strength of an applied static magnetic field). Such a possibility is realized, in particular, for $^3$He $- B$ at $M = M_0/2$ (or $M = 2M_0$), as has been pointed out in Ref.[1] and demonstrated experimentally [2-4] (for the details see, also, Ref.[5]). These homogeneously precessing spin-modes are stabilized at the local minima of time-averaged dipole-dipole potential $U_D$ and are characterized by the specific (non-Leggett) orbital configurations of the order parameter.

For the superfluid $B$-phase

$$U_D = \frac{2}{15} \chi_B \left( \frac{\Omega_B}{g} \right)^2 \left( \text{Tr} \hat{R} - \frac{1}{2} \right)^2,$$

where $\Omega_B$ is the longitudinal NMR frequency, $g$-the gyromagnetic ratio for $^3$He nuclei and the orthogonal matrix $\hat{R}$ describes 3D relative rotations of the spin and orbital degrees of freedom. By introducing the triples of Euler angles ($\alpha_S$, $\beta_S$, $\gamma_S$) and ($\alpha_L$, $\beta_L$, $\gamma_L$), which parametrize rotations in the spin and orbital spaces, respectively, it can be shown that $U_D = U_D(s_Z, l_Z, \alpha, \gamma)$, where $s_Z = \cos \beta_S$ ($l_Z = \cos \beta_L$) is the projection of the Cooper pair...
spin quantization axis (orbital momentum quantization axis) along the direction of the applied static magnetic field $\vec{H}_0$, $\alpha = \alpha_S - \alpha_L$ and $\gamma = \gamma_S - \gamma_L$. Measuring the energy density in units of $\chi H_0^2$ and noticing that $U_D$ is a periodic function of $\alpha$ and $\gamma$, we have:

$$\frac{U_D}{\chi B H_0^2} = \varepsilon f(s_Z, l_Z, \alpha, \gamma) = \varepsilon \sum_{kl} f_{kl}(s_Z, l_Z) e^{i(k\alpha + l\gamma)}$$ (2)

with $\varepsilon \propto (\Omega_B/\omega_0)^2$ (here $\omega_0 = gH_0$).

In the case of a strong magnetic field ($\omega_0 \gg \Omega_B$) the angular variables $\alpha$ and $\gamma$ perform rapid rotations within a long time scale $1/\Omega_D$ and the result of the time-averaging procedure of Eq.(2) essentially depends on the possible presence of a slow combination $\phi_{kl} = k\alpha + l\gamma$. Since in the strong-field case ($\varepsilon \ll 1$) $\dot{\alpha} \simeq -\omega_0$ and $\dot{\gamma} \simeq gM/\chi$, the phase $\phi_{kl}$ turns out to be a slow variable at the resonance condition $M/M_0 = k/l$. Addressing the explicit expressions of the Fourier coefficients $f_{kl}$ for $^3$He -- $B$ (see Ref.[6]) it is concluded that, along with the conventional resonance at $M = M_0$ with $k = l = \pm 1, \pm 2$, the two other (unconventional) resonances are realized at $M = M_0/2$ ($M = 2M_0$) with $2k = l$ ($k = 2l$). This happens in the case of $l_Z \neq 1$ (non-Leggett orbital configuration) where $f_{12} = f_{21} \neq 0$. In particular, at $M = M_0/2$ the time-averaged dipole-dipole potential

$$\bar{f} = 1 + 2s_Z^2 l_Z^2 + (1 - s_Z^2)(1 - l_Z^2) + \frac{2}{3} \sqrt{1 - s_Z^2} \sqrt{1 - l_Z^2} (1 + s_Z)(1 + l_Z) \cos \phi_{12}$$ (3)

and the half-magnetization (HM) spin-precessing mode is trapped at the local minima of Eq.(3) [1].

In Refs.[2,3], along with the above-mentioned HM spin-precessing mode, another unusual spin-precessing state with $M \ll M_0$ has been observed experimentally. This “zero” magnetization mode is not within the category of the resonances with a slow phase. Instead, the “zero” magnetization spin-precessing mode can be stabilized at the balance of the dissipative energy losses and the transverse rf-field energy pumping, as has been discussed in Ref.[7]. The experimentally realized “zero” magnetization spin-precessing mode seems to be characterized by the Cooper pairs orbital configuration close to $l_Z = 0,$
although in Ref.[7] the case of the Leggett configuration \((l_Z = 1)\) was adopted. The computer simulation was used to resolve this controversy (see, e.g., [8]). Here we apply an analytical approach to the same question.

In order to construct the stationary spin-precessing state with \(M \ll M_0\) we address the equations describing the evolution of the spin density \(\vec{S} = \vec{M}/g\). In the strong magnetic field case the spin dynamics is governed by the two pairs of canonically conjugate variables \((S_Z, \alpha)\) and \((S, \gamma)\) subject to a set of the equations

\[
\dot{\alpha} = -1 + \frac{\varepsilon \partial f_D}{\partial S_Z} + \frac{S_Z}{S_Z} \sqrt{S^2 - S_Z^2} h_\perp \cos \theta - \varepsilon \kappa \frac{S^2}{\sqrt{S^2 - S_Z^2}} \left( \frac{\partial f_D}{\partial \alpha} - s_Z \frac{\partial f_D}{\partial \gamma} \right), \quad (4)
\]

\[
\dot{\gamma} = S + \frac{\varepsilon \partial f_D}{\partial S} - \frac{S}{\sqrt{S^2 - S_Z^2}} h_\perp \cos \theta - \varepsilon \kappa \frac{S^2}{\sqrt{S^2 - S_Z^2}} \left( \frac{\partial f_D}{\partial \gamma} - s_Z \frac{\partial f_D}{\partial \alpha} \right), \quad (5)
\]

\[
\dot{S}_Z = -\varepsilon \frac{\partial f_D}{\partial \alpha} - \sqrt{S^2 - S_Z^2} h_\perp \sin \theta + \varepsilon^2 \kappa (S^2 - S_Z^2) \left( \frac{\partial f_D}{\partial S_Z} \right)^2, \quad (6)
\]

\[
\dot{S} = -\varepsilon \frac{\partial f_D}{\partial \gamma}. \quad (7)
\]

This equations are put into a dimensionless form with the time measured is the units of \(1/\omega_0\) and the spin density - in the units of \(S_0 = M_0/g\). The presence of the transverse rf field \(\vec{H}_\perp(t) = H_\perp 0 (\hat{x} \cos \phi(t) + \hat{y} \sin \phi(t))\) is taken into account \((h_\perp = H_\perp 0 / H_0, \theta = \alpha - \phi)\). The dissipation in the spin dynamics is characterized by a phenomenological parameter \(\kappa\) (as in Ref.[9]).

Now we focus on the case of the spin precession in the regime with \(S \ll 1\). In this situation \(\dot{\gamma} \ll 1\) but if \(\sqrt{\varepsilon} \ll S \ll 1\) the angular variable will change faster then \(S_Z\) and \(S\). Then, in order to describe the evolution of the spin system in \(\varepsilon\)-approximation (the Van der Pol picture), we have to use the time-averaged dipole-dipole potential \(\bar{f}_D\) with respect to both angular variables \(\alpha\) and \(\gamma\) independently. In this non-resonance case

\[
\bar{f} = f_{00} = 1 + 2s_Z^2 l_Z^2 + (1 - s_Z^2)(1 - l_Z^2), \quad (8)
\]
and the disipationless dynamics of $\alpha$ and $\gamma$ is governed by the equations

$$\dot{\alpha} = -1 + \varepsilon \frac{\partial f_{00}}{\partial S_Z} + \frac{S_Z}{\sqrt{S^2 - S_Z^2}} h_{\perp} \cos \theta,$$

$$\dot{\gamma} = S + \varepsilon \frac{\partial f_{00}}{\partial S} - \frac{S}{\sqrt{S^2 - S_Z^2}} h_{\perp} \cos \theta,$$

with $S_Z$ and $S$ being constants.

The stationary solutions of Eqs.(9) and (10) are found according to the conditions

$$\frac{\partial f}{\partial S_Z} = \frac{\partial f}{\partial S} = 0,$$  \hspace{1cm} (11)

where the free energy

$$f = \varepsilon f + \frac{1}{2}(S - \omega \gamma)^2 + (\omega_\alpha - 1)S_Z - \sqrt{S^2 - S_Z^2} h_{\perp} \cos \theta$$  \hspace{1cm} (12)

with $\omega_\alpha = -\dot{\alpha}_{\mathrm{st}}$ and $\omega_\gamma = \dot{\gamma}_{\mathrm{st}}$.

In what follows we consider the case where the first (dipole-dipole) term in Eq.(12) dominates over the rest part of the free energy. Than it is concluded that the stationarity conditions (11) are reduced to the equation

$$\frac{\partial f_{00}}{\partial \beta_s} = 2\sqrt{1 - s_Z^2} s_Z (1 - 3l_Z^2) = 0.$$  \hspace{1cm} (13)

In the situation where $l_Z$ is free to adjust to the minimal value of $f_{00}$, Eq.(13) is to be supplemented by an analogous condition

$$\frac{\partial f_{00}}{\partial \beta_L} = 2\sqrt{1 - l_Z^2} l_Z (1 - 3s_Z^2) = 0.$$  \hspace{1cm} (14)

From Eqs.(13) and (14) it is readily concluded that the minimal value of $f_{00}(= 1)$ is realized at the following spin-orbital configurations:

$$a) \quad s_Z = \pm 1, \quad l_Z = 0,$$

$$b) \quad s_Z = 0, \quad l_Z = \pm 1.$$  \hspace{1cm} (15) \hspace{1cm} (16)
The case with \( s_Z = l_Z = 1 \) corresponds to the maximal value of \( f_{00} \).

The experimentally observed “zero” magnetization spin-precessing mode is developed near the spin-orbital configuration \( s_Z = 1 \) and \( l_Z = 0 \) \([2,3]\) and we have to verify whether it corresponds to the dynamical regime with \( \sqrt{\varepsilon} < S \ll 1 \). For this purpose the set of equations for \( S \) and \( S_Z \) with dissipative terms is to be addressed. Using the results of Ref.\([6]\) and putting \( \beta \approx 1 - \frac{1}{2}\beta_S^2 \), for the case \( l_Z = 0 \) the following set of equations is obtained:

\[
\dot{S} = 4\kappa\varepsilon^2 \left( 1 - \frac{\beta_S^2}{2S} \right),
\]

\[
\dot{\beta}_S = -\frac{\kappa\varepsilon^2}{S\beta_S} \left( \frac{77}{9} + \frac{\beta_S}{S} \right) + h_\perp \sin \theta,
\]

with the stationary solution

\[
S = a \left( \frac{\varepsilon^2}{h_\perp} \right)^{2/3}.
\]

where \( a = \frac{1}{2} \left( \frac{100\kappa}{9\sin \theta} \right)^{2/3} \). It can be verified that this solution is locally stable. Since in Eq.(19) the coefficient \( a \) is of the order of unity, the initially imposed condition on the value of \( S \) (\( \sqrt{\varepsilon} \ll S \ll 1 \)) will be fulfilled for \( h_\perp \) being within the limits

\[
\varepsilon^2 \ll h_\perp \ll \varepsilon^{5/4}.
\]

For the case with \( \varepsilon \approx 10^{-3} \), which corresponds to the situation realized experimentally, the value of \( h_\perp = H_\perp/H_0 \approx 10^{-5} \) fits Eq.(20). This consideration shows that the “zero” magnetization spin-precessing mode is realized at the spin-orbital configuration \( s_Z \approx 1, l_Z \approx 0 \), in accordance with experimental observations.

Now we turn to the question of stability of the “zero” magnetization spin-precessing mode near the orbital state \( l_Z = 1 \), which realizes the maximum value of the time-averaged dipole-dipole potential (8). Again addressing the set of Eqs. for \( S \) and \( s_Z \) (see Ref.\([6]\)) this time for the case of a spin-orbital configuration with \( s_Z \approx 1 \) and \( l_Z \approx 1 \), we obtain that for \( \sqrt{\varepsilon} \ll S \ll 1 \)
\[
\dot{S} = \frac{160}{9} \kappa \varepsilon^2, \quad (21)
\]
\[
\dot{\beta}_S = -\frac{\kappa \varepsilon^2}{S \beta_S} \left( \frac{77}{9} + \frac{\beta_S}{S} \right) + h_\perp \sin \theta. \quad (22)
\]

It can be easily seen that this set of Eqs. has no stationary solution.

It is interesting to analyze another possible regime of “zero” magnetization spin-precessing state with \( S \simeq \sqrt{\varepsilon} \). In this case \( \dot{\gamma} \simeq \sqrt{\varepsilon} \) and the rate of the time evolution of angular variable \( \gamma \) is still faster than the temporal variations of \( S \) and \( S_Z \). On the other hand, \( \dot{s}_Z = (\dot{S}_Z - s_Z \dot{S})/S \simeq \sqrt{\varepsilon} \) and the only fast variable upon which depends the dipole-dipole potential is \( \alpha \). In the considered situation

\[
f_D = \sum_k f_k(s_Z, l_Z, \gamma) e^{i k \alpha}, \quad (23)
\]

where the time-averaged value

\[
f_0 = 1 + 2s_Z^2 l_Z^2 + 4s_Z l_Z \sqrt{1 - s_Z^2} \sqrt{1 - l_Z^2} \cos \gamma + (1 - s_Z^2)(1 - l_Z^2)(1 + \cos 2\gamma). \quad (24)
\]

After having minimized \( f_0 \) with respect to the slow variable \( \gamma \) it can be shown that \( f_0(s_Z, l_Z) \) is highly degenerate with respect to the spin and orbital variables: the minimum of \( f_0(s_Z, l_Z) \) is realized within the circle \( s_Z^2 + l_Z^2 \leq 1 \). Considering again the orbital state \( l_Z = 0 \) with the stationary value \( \gamma_{st} = \pi/2 \) it can be shown that the time evolution of \( s_Z \) is governed by an equation

\[
\dot{s}_Z = \frac{4 \kappa \varepsilon^2}{S(1 - s_Z^2)} (s_Z^4 - s_Z^3 s_Z^2 + s_Z^2 + 4) - h_\perp \sqrt{1 - s_Z^2} \sin \theta. \quad (25)
\]

From Eq.(25) it follow that near \( s_Z = \pm 1 \) the stationary value of \( \beta_S \) is given by

\[
\beta_S \simeq 2 \left( \frac{2 \kappa \varepsilon^2}{3S h_\perp \sin \theta} \right)^{1/3}. \quad (26)
\]

It is easy to show that the solution (26) for the case \( s_Z \simeq -1 \) is locally stable (in contrast to the case with \( s_Z \simeq 1 \)). This spin-orbital configuration \( (s_Z \simeq -1, l_Z \simeq 0) \) is
one of the stable spin-precessing states at $S \simeq \sqrt{\varepsilon}$. The phase diagram of the “zero” magnetization in the degeneracy domain $s_Z^2 + l_Z^2 \leq 1$ will be presented in a separate publication.

We are indebted to E.Sonin for his valuable comments.

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