SPACE-TIME FRACTIONAL STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

JEBESSA B MIJENA AND ERKAN NANE

ABSTRACT. We consider non-linear time-fractional stochastic heat type equation
\[ \partial_t^\beta u_t(x) = -\nu(-\Delta)^{\alpha/2}u_t(x) + \mathcal{I}_{t}^{1-\beta}[\sigma(u) W(t,x)] \]
in \((d+1)\) dimensions, where \( \nu > 0, \beta \in (0, 1), \alpha \in (0, 2] \) and \( d < \min\{2, \beta^{-1}\} \alpha \), \( \partial_t^\beta \) is the Caputo fractional derivative, \( -(-\Delta)^{\alpha/2} \) is the generator of an isotropic stable process, \( \mathcal{I}_{t}^{1-\beta} \) is the fractional integral operator, \( W(t,x) \) is space-time white noise, and \( \sigma : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous. Time fractional stochastic heat type equations might be used to model phenomenon with random effects with thermal memory. We prove existence and uniqueness of mild solutions to this equation and establish conditions under which the solution is continuous. Our results extend the results in the case of parabolic stochastic partial differential equations obtained in [13; 36]. In sharp contrast to the stochastic partial differential equations studied earlier in [13; 18; 36], in some cases our results give existence of random field solutions in spatial dimensions \( d = 1, 2, 3 \). Under faster than linear growth of \( \sigma \), we show that time fractional stochastic partial differential equation has no finite energy solution. This extends the result of Foondun and Parshad [14] in the case of parabolic stochastic partial differential equations. We also establish a connection of the time fractional stochastic partial differential equations to higher order parabolic stochastic differential equations.

1. Introduction

Stochastic partial differential equations (SPDE) have been studied in mathematics, and various sciences; see, for example, Khoshnevisan [18] for a big list of references. The area of SPDEs is interesting to mathematicians as it contains big number of hard open problems. SPDE’s have been applied in many disciplines that include applied mathematics, statistical mechanics, theoretical physics, theoretical neuroscience, theory of complex chemical reactions, fluid dynamics, hydrology, and mathematical finance. In this paper we introduce new time fractional stochastic partial differential equations (TSPDE) in the sense of Walsh [36], and prove existence and uniqueness of mild solutions. We establish the conditions under which the solution is continuous. When \( \sigma \) grows faster than Lipschitz we show that no finite energy solution exists.

The study of the time-fractional diffusion equations has recently attracted a lot of attention and a typical form of the time fractional equations is \( \partial_t^\beta u = \Delta u \) where \( \partial_t^\beta \) is the Caputo fractional derivative with \( \beta \in (0, 1) \) and \( \Delta = \sum_{i=1}^{d} \partial^2_{x_i} \) is the Laplacian. These equations are related with anomalous diffusions or diffusions in...
non-homogeneous media, with random fractal structures; see, for instance, [24]. The Caputo fractional derivative $\partial^\beta_t$ first appeared in [7] is defined for $0 < \beta < 1$ by

$$\partial^\beta_t u_t(x) = \frac{1}{\Gamma(1-\beta)} \int_0^t \partial_r u_r(x) \frac{dr}{(t-r)^\beta}. \quad (1.1)$$

Its Laplace transform

$$\int_0^\infty e^{-st} \partial^\beta_t u_t(x) dt = s^\beta \tilde{u}_s(x) - s^{\beta-1} u_0(x), \quad (1.2)$$

where $\tilde{u}_s(x) = \int_0^\infty e^{-st} u_t(x) dt$ and incorporates the initial value in the same way as the first derivative.

Starting from the works [19; 29; 39; 40] much effort has been made in order to introduce a rigorous mathematical approach; see, for example, [28] for a short survey on these results. The solutions to fractional diffusion equations are strictly related with stable densities. Indeed, the stochastic solutions of time fractional diffusions can be realized through time-change by inverse stable subordinators. A couple of recent works in this field are [23; 31].

It might come natural to add just a noise term to the time fractional diffusion and study the equation

$$\partial^\beta_t u_t(x) = \Delta u_t(x) + W(t, x); \quad u_t(x)|_{t=0} = u_0(x), \quad (1.3)$$

where $W(t, x)$ is a space-time white noise with $x \in \mathbb{R}^d$.

We will make use of time fractional Duhamel’s principle [33; 35; 34] to get the correct version of (1.3). Let $G_t(x)$ be the fundamental solution of the time fractional PDE $\partial^\beta_t u = \Delta u$. The solution to the time-fractional PDE with force term $f(t, x)$

$$\partial^\beta_t u_t(x) = \Delta u_t(x) + f(t, x); \quad u_t(x)|_{t=0} = u_0(x), \quad (1.4)$$

is given by Duhamel’s principle, the influence of the external force $f(t, x)$ to the output can be count as

$$\partial^\beta_t V(\tau, t, x) = \Delta V(\tau, t, x); \quad V(\tau, \tau, x) = \partial_{1-\beta} t f(t, x)|_{t=\tau}, \quad (1.5)$$

which has solution

$$V(t, \tau, x) = \int_{\mathbb{R}^d} G_{t-\tau}(x-y) \partial_{1-\beta} \tau f(\tau, y) dy.$$ 

Hence solution to (1.4) is given by

$$u(t, x) = \int_{\mathbb{R}^d} G_t(x-y) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} G_{t-\tau}(x-y) \partial_{1-\beta} \tau f(\tau, y) dy dr. \quad (1.6)$$

Hence if we use time fractional Duhamel’s principle we will get the mild (integral) solution of (1.3) to be of the form (informally):

$$u(t, x) = \int_{\mathbb{R}^d} G_t(x-y) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} G_{t-\tau}(x-y) \partial_{1-\beta} \tau f(\tau, y) dy dr. \quad (1.6)$$

It is not clear what the fractional derivative of the space-time white noise mean.
We can remove the fractional derivative of the noise term in (1.6) in the following way. Let $\gamma > 0$, define the fractional integral by
\[
I_\gamma^t f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t - \tau)^{\gamma - 1} f(\tau) d\tau.
\]
For every $\beta \in (0, 1)$, and $g \in L^\infty(\mathbb{R}_+)$ or $g \in C(\mathbb{R}_+)
\partial^\beta \partial^\beta_t g(t) = g(t).
\]
We consider the time fractional PDE with a force given by $f(t, x) = I_1^{1-\beta} g(t, x)$, then by the Duhamel’s principle the mild solution to (1.4) will be given by
\[
u(t) = \int_{\mathbb{R}^d} G_t(x - y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y)g(r, y)dydr.
\]
Hence, the preceding discussion suggest that the correct TFSPDE is the following model problem:
\[
\partial^\beta_t u_t(x) = \Delta u_t(x) + I_1^{1-\beta} W(t, x); \quad u_t(x)|_{t=0} = u_0(x).
(1.7)
\]
When $d = 1$, the fractional integral above in equation (1.7) is defined as
\[
I_1^{1-\beta}[\sigma(u) W(t, x)] = \frac{1}{\Gamma(1-\beta)} \int_0^t (t - \tau)^{-\beta} \partial W(dr, x) \partial x,
\]
is well defined only when $0 < \beta < 1/2$. By the Duhamel’s principle, mentioned above, mild (integral) solution of (1.7) will be (informally):
\[
u_t(x) = \int_{\mathbb{R}^d} G_t(x - y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y)W(dydr).
(1.8)
\]
Next we want to give a **Physical motivation** to study time fractional SPDEs. The time-fractional SPDEs studied in this paper may arise naturally by considering the heat equation in a material with thermal memory; see [9] for more details: Let $u_t(x), e(t, x)$ and $\vec{F}(t, x)$ denote the body temperature, internal energy and flux density, respectively. Then the relations
\[
\partial_t e(t, x) = -\text{div} \vec{F}(t, x),
\]
\[
e(t, x) = \beta u_t(x), \quad \vec{F}(t, x) = -\lambda \nabla u_t(x),
(1.9)
\]
yields the classical heat equation $\beta \partial_t u = \lambda \Delta u$.

According to the law of classical heat equation, the speed of heat flow is infinite but the propagation speed can be finite because the heat flow can be disrupted by the response of the material. In a material with thermal memory Lunardi and Sinestrari [20], von Wolfersdorf [37] showed that
\[
e(t, x) = \tilde{\beta} u_t(x) + \int_0^t n(t - s)u_s(x)ds,
\]
holds with some appropriate constant $\tilde{\beta}$ and kernel $n$. In most cases we would have $n(t) = \Gamma(1 - \beta_1)^{-1}t^{-\beta_1}$. The convolution implies that the nearer past affects
the present more. If in addition the internal energy also depends on past random effects, then
\begin{equation}
 e(t, x) = \beta u_t(x) + \int_0^t n(t - s)u_s(x)ds
 + \int_0^t l(t - s)h(s, u_s(x)) \frac{\partial W(ds, x)}{\partial x},
 \tag{1.10}
\end{equation}
where $W$ is the space time white noise modeling the random effects. Take $l(t) = \Gamma(2 - \beta_2)^{-1}t^{1 - \beta_2}$, then after differentiation (1.10) gives
\begin{equation}
 \partial_t^{\beta_1} u = \text{div} \ F + \frac{1}{\Gamma(1 - \beta_2)} \int_0^t (t - s)^{-\beta_2} h(s, u_s(x)) \frac{\partial W(ds, x)}{\partial x}.
\end{equation}

In this paper we will study existence and uniqueness of mild solutions to this type of stochastic equations and its extensions:
\begin{equation}
 \partial_t^{\beta_1} u_t(x) = -\nu(-\Delta)^{\alpha/2} u_t(x) + \Gamma^{1 - \beta}|\sigma(u) W(t, x)|, \quad \nu > 0, t > 0, x \in \mathbb{R}^d; \quad \tag{1.11}
 u_t(x)|_{t=0} = u_0(x),
\end{equation}
where the initial datum $u_0$ is measurable and bounded, $-(-\Delta)^{\alpha/2}$ is the fractional Laplacian with $\alpha \in (0, 2]$, $W(t, x)$ is a space-time white noise with $x \in \mathbb{R}^d$, and $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function.

Let $G_t(x)$ be the fundamental solution of the fractional heat type equation
\begin{equation}
 \partial_t^{\beta_1} G_t(x) = -\nu(-\Delta)^{\alpha/2} G_t(x). \tag{1.12}
\end{equation}
We know that $G_t(x)$ is the density function of $X(E_t)$, where $X$ is an isotropic $\alpha$-stable Lévy process in $\mathbb{R}^d$ and $E_t$ is the first passage time of a $\beta$-stable subordinator $D = \{D_r, \ r \geq 0\}$, or the inverse stable subordinator of index $\beta$: see, for example, Bertoin [13], for more on time fractional diffusion equations, and Meerschaert and Scheffler [25] for properties of the inverse stable subordinator $E_t$.

Let $p_{X(s)}(x)$ and $f_{E_t}(s)$ be the density of $X(s)$ and $E_t$, respectively. Then the Fourier transform of $p_{X(s)}(x)$ is given by
\begin{equation}
 \hat{p}_{X(s)}(\xi) = e^{-s\xi^\alpha}, \tag{1.13}
\end{equation}
and
\begin{equation}
 f_{E_t}(x) = t^{\beta - 1} x^{-1 - 1/\beta} g_\beta(t x^{-1/\beta}), \tag{1.14}
\end{equation}
where $g_\beta(\cdot)$ is the density function of $D_1$. The function $g_\beta(u)$ (cf. Meerschaert and Straka [26]) is infinitely differentiable on the entire real line, with $g_\beta(u) = 0$ for $u \leq 0$.

By conditioning, we have
\begin{equation}
 G_t(x) = \int_0^\infty p_{X(s)}(x) f_{E_t}(s)ds. \tag{1.15}
\end{equation}

A related time-fractional SPDE was studied by Karczewska [17], Chen et al. [9], and Baeumer et al [3]. They have proved regularity of the solutions to the time-fractional parabolic type SPDEs using cylindrical Brownian motion in Banach spaces in line with the methods in [12].

In this paper we study the existence and uniqueness of the solution to (1.11) under global Lipchitz conditions on $\sigma$, using the white noise approach of [36]: We
say that an $\mathcal{F}_t$-adapted random field $\{u_t(x), t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution of (1.11) with initial value $u_0$ if the following integral equation is fulfilled

$$u_t(x) = \int_{\mathbb{R}^d} u_0(y) G_t(x-y) dy + \int_0^t \int_{\mathbb{R}^d} \sigma(u_r(y)) G_{t-r}(x-y) W(dr,dy). \quad (1.16)$$

For a comparison of the two approaches to SPDE’s see the paper by Dalang and Quer-Sardanyons [11].

We now briefly give an outline of the paper. We adapt the methods of proofs of the results in [18] with many crucial nontrivial changes. We give some preliminary results in section 2. Moment estimates for time increments and spatial increments of the solution is given in Section 3. The main result in this section is Theorem 1. In section 4, we give main results of the paper about existence and uniqueness, and the continuity of the solution to the time fractional SPDEs. The main result is Theorem 2. In Section 5, we show that under faster than linear growth of $\sigma$ there is no finite energy solution. In section 6, we discuss the equivalence of time-fractional SPDEs to higher order SPDEs. Throughout the paper, we use the letter $C$ or $c$ with or without subscripts to denote a constant whose value is not important and may vary from places to places.

## 2. Preliminaries

In this section, we give some preliminary results that will be needed in the remaining sections of the paper.

Applying the Laplace transform with respect to time variable $t$, Fourier transform with respect to space variable $x$. Laplace-Fourier transform of $G$ defined in (1.15) is given by

$$\int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t + i\xi \cdot x} G_t(x) dx dt = \int_0^\infty e^{-\lambda t} \int_0^\infty f_{E_t(s)}(s) ds \int_{\mathbb{R}^d} e^{i\xi \cdot x} p_X(s)(x) dx = \int_0^\infty e^{-\nu |\xi| \alpha} ds \int_0^\infty e^{-\lambda t} f_{E_t}(s) dt = \frac{\lambda^\beta}{\lambda^\beta + \nu |\xi| \alpha} = \lambda^{\beta-1} \frac{\lambda^{\beta-1}}{\lambda^\beta + \nu |\xi| \alpha}, \quad (2.1)$$

here we used the fact that the laplace transform $t \to \lambda$ of $f_{E_t}(u)$ is given by $\lambda^{\beta-1} e^{-\lambda t}$. Using the convention, $\sim$ to denote the Laplace transform and $*$ the Fourier transform we get

$$\tilde{G}_t^*(x) = \frac{\lambda^{\beta-1}}{\lambda^\beta + \nu |\xi| \alpha}, \quad (2.2)$$

Inverting the Laplace transform, it yields

$$G_t^*(\xi) = E_{\beta}(-\nu |\xi| \alpha t^\beta), \quad (2.3)$$

where

$$E_{\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1 + \beta k)}; \quad (2.4)$$
Note that for the last equation above. Now using equation (2.8), we get
\[ \int_0^\infty \frac{z^{d/\alpha - 1}}{(1 + \Gamma(1 - \beta)z)^2} dz \leq \int_0^\infty z^{d/\alpha - 1}(E_\beta(-z))^2 dz \]
\[ \leq \int_0^\infty \frac{z^{d/\alpha - 1}}{(1 + \Gamma(1 + \beta)^{-1}z)^2} dz . \]

**Proof.** Using Plancherel theorem and (2.3), we have
\[ \int_{R^d} |G_t(x)|^2 dx = \frac{1}{(2\pi)^d} \int_{R^d} |G_t^\alpha (\xi)|^2 d\xi = \frac{1}{(2\pi)^d} \int_{R^d} |E_\beta(-\nu |\xi|^\alpha t^\beta)|^2 d\xi \]
\[ = \frac{2\pi^{d/2}}{(2\pi)^d} \frac{1}{\sqrt{\Gamma(d/2)}} \int_0^\infty r^{d-1}(E_\beta(-\nu r^\alpha t^\beta))^2 dr. \]
\[ = \frac{(\nu t^\beta)^{-d/\alpha} 2\pi^{d/2}}{\alpha \Gamma(d/2)} \frac{1}{(2\pi)^d} \int_0^\infty z^{d/\alpha - 1}(E_\beta(-z))^2 dz . \]
We used the integration in polar coordinates for radially symmetric function in the last equation above. Now using equation (2.8) we get
\[ \int_0^\infty \frac{z^{d/\alpha - 1}}{(1 + \Gamma(1 - \beta)z)^2} dz \leq \int_0^\infty z^{d/\alpha - 1}(E_\beta(-z))^2 dz \]
\[ \leq \int_0^\infty \frac{z^{d/\alpha - 1}}{(1 + \Gamma(1 + \beta)^{-1}z)^2} dz . \]
Hence \( \int_0^\infty z^{d/\alpha - 1} (E_\beta(-z))^2 dz < \infty \) if and only if \( d < 2\alpha \). In this case the upper bound in equation (2.12) is
\[
\int_0^\infty \frac{z^{d/\alpha - 1}}{(1 + \Gamma(1 + \beta)^{-1})^{2}} dz = \frac{B(d/\alpha, 2 - d/\alpha)}{\Gamma(1 + \beta)^{-d/\alpha}},
\]
where \( B(d/\alpha, 2 - d/\alpha) \) is a Beta function.

\( \square \)

**Remark 1.** For \( d < 2\alpha \),
\[
\frac{B(d/\alpha, 2 - d/\alpha)}{\Gamma(1 + \beta)^{-d/\alpha}} z^{-\beta d/\alpha} \leq \int_0^\infty z^{d/\alpha - 1} (E_\beta(-z))^2 dz \leq \frac{B(d/\alpha, 2 - d/\alpha)}{\Gamma(1 + \beta)^{-d/\alpha}} z^{-\beta d/\alpha}.
\]

**Lemma 2.** For \( \lambda \in \mathbb{R}^d \) and \( \alpha = 2 \),
\[
\int_{\mathbb{R}^d} e^{\lambda^T x} G_s(x) dx = E_\beta(\nu | \lambda|^2 s^\beta).
\]

**Proof.** Using uniqueness of Laplace transform we can easily show \( \mathbb{E}[E_s^k] = \frac{\Gamma(1 + k)s^\beta}{\Gamma(1 + \beta k)} \) for \( k > -1 \). Therefore, using this and moment-generating function of Gaussian densities we have
\[
\int_{\mathbb{R}^d} e^{\lambda^T x} G_s(x) dx = \int_0^\infty \int_{\mathbb{R}^d} e^{\lambda^T x} \frac{e^{-|\lambda|^2 \nu}}{(4\pi \nu)^{d/2}} dx f_{E_s}(u) du
\]
\[
= \int_0^\infty e^{\nu |\lambda|^2 u} f_{E_s}(u) du
\]
\[
= \sum_{k=0}^\infty \nu^k |\lambda|^{2k} \int_0^\infty u^k f_{E_s}(u) du
\]
\[
= \sum_{k=0}^\infty \frac{\nu^k |\lambda|^{2k}}{k!} \frac{\Gamma(1 + k)s^\beta}{\Gamma(1 + \beta k)}.
\]

\( \square \)

\( W(t, x) \) is a space-time white noise with \( x \in \mathbb{R}^d \), which is assumed to be adapted with respect to a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), where \( \mathcal{F} \) is complete and the filtration \( \{ \mathcal{F}_t, t \geq 0 \} \) is right continuous. \( W(t, x) \) is generalized processes with covariance given by
\[
\mathbb{E} \left[ W(t, x) W(s, y) \right] = \delta(x - y)\delta(t - s).
\]
That is, \( W(f) \) is a random field indexed by functions \( f \in L^2((0, \infty) \times \mathbb{R}^d) \) and for all \( f, g \in L^2((0, \infty) \times \mathbb{R}^d) \), we have
\[
\mathbb{E} \left[ W(f) W(g) \right] = \int_0^\infty \int_{\mathbb{R}^d} f(t,x)g(t,x) dxdt.
\]

Hence \( W(f) \) can be represented as
\[
W(f) = \int_0^\infty \int_{\mathbb{R}^d} f(t,x) W(dxdt).
\]
Note that \( W(f) \) is \( \mathcal{F}_t \)-measurable whenever \( f \) is supported on \([0, t] \times \mathbb{R}^d\).

Next we give the definition of Walsh-Dalang Integrals that is used in equation (1.16). We use the Brownian Filtration \( \{\mathcal{F}_t\} \) and the Walsh-Dalang integrals defined as follows:
(t, x) \to \Phi_t(x) is an elementary random field when \(\exists 0 \leq a < b\) and an \(\mathcal{F}_{a}\text{-measurable} \ X \in L^2(\Omega)\) and \(\phi \in L^2(\mathbb{R}^d)\) such that
\[
\Phi_t(x) = X_{[a,b]}(t)\phi(x) \quad (t > 0, x \in \mathbb{R}^d).
\]

- If \(h = h_t(x)\) is non-random and \(\Phi\) is elementary, then
\[
\int h\Phi dW := X \int_{(a,b) \times \mathbb{R}^d} h_t(x)\phi(x)W(dxdt).
\]

- The stochastic integral is Wiener’s, and it is well-defined iff \(h_t(x)\phi(x) \in L^2([a,b] \times \mathbb{R}^d)\).

- We have Walsh isometry,
\[
\mathbb{E}\left(\left( \int h\Phi dW \right)^2 \right) = \int_0^\infty \int_{\mathbb{R}^d} dy[|h_s(y)|^2\mathbb{E}[|\Phi_s(y)|^2]].
\]

Let \(\Phi\) be a random field, and for every \(\gamma > 0\) and \(k \in [2, \infty)\) define
\[
\mathcal{N}_{\gamma,k}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}^d} \left( e^{-\gamma t}\|\Phi_t(x)\|_k \right) = \sup_{t \geq 0} \sup_{x \in \mathbb{R}^d} \left( e^{-\gamma t} \mathbb{E}[|\Phi_t(x)|^k] \right)^{1/k}.
\]

If we identify a.s.-equal random fields, then every \(\mathcal{N}_{\gamma,k}\) becomes a norm. Moreover, \(\mathcal{N}_{\gamma,k}\) and \(\mathcal{N}_{\gamma',k}\) are equivalent norms for all \(\gamma, \gamma' > 0\) and \(k \in [2, \infty)\). Finally, we note that if \(\mathcal{N}_{\gamma,k}(\Phi) < \infty\) for some \(\gamma > 0\) and \(k \in (2, \infty)\), then \(\mathcal{N}_{\gamma,2}(\Phi) < \infty\) as well, thanks to Jensen’s inequality.

**Definition 1.** We denote by \(\mathcal{L}^{\gamma,2}\) the completion of the space of all simple random fields in the norm \(\mathcal{N}_{\gamma,2}\).

Given a random field \(\Phi := \{\Phi_t(x)\}_{t \geq 0, x \in \mathbb{R}^d}\) and space-time noise \(W\), we define the [space-time] stochastic convolution \(G \ast \Phi\) to be the random field that is defined as
\[
(G \ast \Phi)_t(x) := \int_{(0,t) \times \mathbb{R}^d} G_{t-s}(y-x)\Phi_s(y)W(sdsdy),
\]
for \(t > 0\) and \(x \in \mathbb{R}^d\), and \((G \ast W)_0(x) := 0\).

We can understand the properties of \(G \ast \Phi\) for every fixed \(t > 0\) and \(x \in \mathbb{R}^d\) as follows. Define
\[
G_s^{(t,x)}(y) := G_{t-s}(y-x) \cdot 1_{(0,t)}(s) \quad \text{for all } s \geq 0 \text{ and } y \in \mathbb{R}^d.
\]

Clearly, \(G_s^{(t,x)} \in L^2(\mathbb{R}^+ \times \mathbb{R}^d)\) for \(d < \min\{2, \beta^{-1}\}a\); in fact,
\[
\int_0^\infty ds \int_{\mathbb{R}^d} [G_s^{(t,x)}(y)]^2dy = \int_0^t ds \int_{\mathbb{R}^d} [G_s(y)]^2dy = C' t^{1-\beta d/a} < \infty.
\]

This computation follows from Lemma 1. Thus, we may interpret the random variable \((G \ast \Phi)_t(x)\) as the stochastic integral \(\int G_s^{(t,x)}\Phi dW\), provided that \(\Phi\) is in \(\mathcal{L}^{\beta,2}\) for some \(\beta > 0\). Let us recall that \(\Phi \to G \ast \Phi\) is a random linear map; that is, if \(\Phi, \Psi \in \mathcal{L}^{\beta,2}\) for some \(\beta > 0\), then for all \(a, b \in \mathbb{R}\) the following holds almost surely:
\[
\int G_{t-s}(y-x)\Phi_s(y)W(dxdy) = a \int G_{t-s}(y-x)\Phi_s(y)W(dxdy) + b \int G_{t-s}(y-x)\Psi_s(y)W(dxdy)
\]
3. Estimates on the moments of the increments of the solution

In this section we prove continuity of various increments related to the solution of (1.11). We start with the next result

**Lemma 3.** Suppose \( d < \min(2, \beta^{-1}) \alpha \), and \( p \geq 2 \), then we have the following estimates for time increments of the density \( G \).

(i). For \( t \leq t' \), we have

\[
\int_0^t \int_{\mathbb{R}^d} |G_{t'}(x - y) - G_{t}(x - y)|^2 dy dr \leq \frac{C^*}{(1 - \beta d/\alpha)} (t' - t)^{1 - \beta d/\alpha},
\]

where \( C^* \) is a constant given in Lemma 1.

(ii). For \( x, x' \in \mathbb{R}^d \), we have

\[
c|x' - x|^2 \leq \int_0^t \int_{\mathbb{R}^d} |G_{t}(x - y) - G_{t'}(x' - y)|^2 dy dr
\]

\[
\leq C|x' - x|^2 \min\left\{\left(\frac{\alpha - \beta d}{\pi}\right)^{-2}\right\}.
\]

**Proof.** Using \( G_{t'}(x - \cdot) - G_{t}(x - \cdot) = e^{i\xi \cdot x} G_{t'}^*(\xi) - e^{i\xi \cdot x} G_{t}^*(\xi) \), Plancherel theorem and computation in Lemma 1 we have

\[
\int_{\mathbb{R}^d} |G_{t'}(x - y) - G_{t}(x - y)|^2 dy
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |G_{t'}^*(\xi) - G_{t}^*(\xi)|^2 d\xi
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (G_{t'}^*(\xi))^2 d\xi + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (G_{t}^*(\xi))^2 d\xi
\]

\[
- \frac{2}{(2\pi)^d} \int_{\mathbb{R}^d} G_{t'}^*(\xi) G_{t}^*(\xi) d\xi
\]

\[
= C^* (t' - r)^{-\beta d/\alpha} + C^* (t - r)^{-\beta d/\alpha} - \frac{2}{(2\pi)^d} \int_{\mathbb{R}^d} G_{t'}^*(\xi) G_{t}^*(\xi) d\xi.
\]

Using Plancharel theorem, integration in polar coordinates in \( \mathbb{R}^d \), and the fact that \( f(z) = E_{\beta}(-z) \) is decreasing (since it is completely monotonic, i.e. \((-1)^n f^{(n)}(z) \geq 0 \) for all \( z > 0, n = 0, 1, 2, 3, \ldots \)), we get

\[
\frac{2}{(2\pi)^d} \int_{\mathbb{R}^d} G_{t'}^*(\xi) G_{t}^*(\xi) d\xi
\]

\[
= \frac{2}{(2\pi)^d} \int_{\mathbb{R}^d} E_{\beta}(-\nu(t' - r)^\beta |\xi|^\alpha) E_{\beta}(-\nu(t - r)^\beta |\xi|^\alpha) d\xi
\]

\[
= \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{\nu^{\alpha/2}}{(2\pi)^d} \int_0^{\infty} s^{d-1} E_{\beta}(-\nu(t' - r)^\beta s\alpha) E_{\beta}(-\nu(t - r)^\beta s\alpha) ds
\]

\[
\leq \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{\nu^{\alpha/2}}{(2\pi)^d} \int_0^{\infty} s^{d-1} E_{\beta}(-\nu(t' - r)^\beta s\alpha) E_{\beta}(-\nu(t - r)^\beta s\alpha) ds
\]

\[
\leq \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{\nu^{\alpha/2}}{(2\pi)^d} \int_0^{\infty} s^{d-1} \left(E_{\beta}(-\nu(t' - r)^\beta s\alpha)\right)^2 ds
\]

\[
= \frac{2\pi^{d/2} \nu^{-d/\alpha} (t' - r)^{-\beta d/\alpha}}{\alpha(2\pi)^d} \int_0^{\infty} z^{d/\alpha - 1} E_{\beta}(z)^2 dz
\]
\[ = 2C^* (t' - r)^{-\beta d/\alpha}. \]

Now integrating both sides wrt to \( r \) from 0 to \( t \) we get

\[
\int_0^t \int_{\mathbb{R}^d} [G_{t' - r}(x - y) - G_{t - r}(x - y)]^2 dr dy \leq \frac{-C^* (t' - t)^{1-\beta d/\alpha}}{1 - \beta d/\alpha} + \frac{C^* (t')^{1-\beta d/\alpha}}{1 - \beta d/\alpha} + \frac{C^* t^{1-\beta d/\alpha}}{1 - \beta d/\alpha}
- 2C^* (t' - t)^{1-\beta d/\alpha} - \frac{2C^* (t')^{1-\beta d/\alpha}}{1 - \beta d/\alpha}
= \frac{C^* (t' - t)^{1-\beta d/\alpha}}{1 - \beta d/\alpha} - \frac{C^* (t')^{1-\beta d/\alpha}}{1 - \beta d/\alpha} + \frac{C^* t^{1-\beta d/\alpha}}{1 - \beta d/\alpha}
\leq \frac{C^* (t' - t)^{1-\beta d/\alpha}}{1 - \beta d/\alpha},
\]

the last inequality follows since \( t < t' \).

Now we prove (ii). Using \( G_{t' - r}(x' - \cdot) - G_{t - r}(x - \cdot) = e^{i\xi \cdot x'}G_{t' - r}(\xi) - e^{i\xi \cdot x}G_{t - r}(\xi) \) and Plancherel theorem, we have

\[
\int_0^t \int_{\mathbb{R}^d} [G_{t' - r}(x' - y) - G_{t - r}(x - y)]^2 dy dr = \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} [G_{t'}(\xi)]^2 \left| 1 - e^{i\xi (x' - x)} \right|^2 d\xi dr
\]

In the following, we divide our proof into two cases:

Case 1. If \( d + 2 < \min(2, \beta^{-1})\alpha \), then we apply the following inequality

\[
\left| 1 - e^{i\xi (x' - x)} \right|^2 = 4 \sin^2 \left( \frac{\xi \cdot (x' - x)}{2} \right) \leq |\xi|^2 |x' - x|^2
\]

(3.4)

to (3.3) to derive that

\[
\int_0^t \int_{\mathbb{R}^d} [G_{t' - r}(x' - y) - G_{t - r}(x - y)]^2 dr dy \leq \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} [G_{t'}(\xi)]^2 |\xi|^2 |x' - x|^2 d\xi dr
= \frac{|x' - x|^2 2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int_0^t \int_0^\infty u^{d+1} E_\beta(-\nu(t - r)\beta u^\alpha)^2 du dr
\leq \frac{|x' - x|^2 2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int_0^t \int_0^\infty [1 + F(1 + \beta)^{-1} \nu(t - r)\beta u^\alpha]^2 du dr
= \frac{2\pi^{d/2} B((d + 2)/\alpha, 2 - (d + 2)/\alpha) \nu^{-(d+2)/\alpha}}{(2\pi)^d \Gamma(d/2) \Gamma(1 + \beta)^{-1} \nu^{(d+2)/\alpha}} |x' - x|^2.
\]

Case 2. If \( d + 2 \geq \min(2, \beta^{-1})\alpha \), then we choose \( \varepsilon > 0 \) fixed such that \( d + 2\varepsilon < \min(2, \beta^{-1})\alpha \). [This can be done since \( d < \min(2, \beta^{-1})\alpha \).] Clearly, \( \varepsilon < 1 \). then we apply the following inequality

\[
\left| 1 - e^{i\xi (x' - x)} \right|^2 \leq 4^{1-\varepsilon} |\xi|^2 |x' - x|^{2\varepsilon}
\]

(3.5)
to (3.3) to derive that
\[
\int_0^t \int_{\mathbb{R}^d} |G_{t-r}(x') - G_{t-r}(x-y)|^2 dy dr \\
\leq \frac{4^{1-\epsilon}}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} |G_{t-r}(\xi)|^2 |\xi|^{2\epsilon} |x'|^2 d\xi dr \\
= \frac{|x' - x|^{2\epsilon} 4^{1-\epsilon} 2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int_0^t \int_0^\infty u^{d+2\epsilon-1} (E_{\beta}(-\nu(t-r)^\beta u^\alpha))^2 du dr \\
\leq \frac{|x' - x|^{2\epsilon} 4^{1-\epsilon} 2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int_0^t \int_0^\infty \left(1 + \Gamma(1 + \beta)^{-1} \nu(t-r)^\beta u^\alpha \right)^2 du dr \\
= \frac{2\pi^{d/2} 4^{1-\epsilon} B((d + 2\epsilon)/\alpha, 2 - (d + 2\epsilon)/\alpha) \mu^{-(d+2\epsilon)/\alpha}}{(2\pi)^d \Gamma(d/2) \alpha (1 - (d + 2\epsilon)/\alpha) \Gamma(1 + \beta)^{-1} (d+2\epsilon)/\alpha} E_{\beta}(-\nu(t-r)^\beta u^\alpha)^2 dx dr.
\]

To estimate lower bound we need to use $\frac{1}{3} |z| < |e^z - 1| < \frac{7}{8} |z|$ for $|z| < 1$ [27, Eq. 4.2.38]. Let $A = \{ \xi \in \mathbb{R}_+^d : |\xi| < x_0 < \frac{1}{|x' - x|} \}$.
\[
\int_0^t \int_{\mathbb{R}^d} |G_{t-r}(x') - G_{t-r}(x-y)|^2 dy dr \\
= \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} |G_{t-r}(\xi)|^2 \left| 1 - e^{i\xi(x' - x)} \right|^2 d\xi dr \\
\geq \frac{1}{4(2\pi)^d} \int_0^t \int_A |G_{t-r}(\xi)|^2 |\xi|^{2\epsilon} |x' - x|^2 d\xi dr \\
= \frac{c |x' - x|^2}{},
\]
this is true because by (2.8) $E_{\beta}(-x) < 1$ and using this we have
\[
\int_0^t \int_A |G_{t-r}(\xi)|^2 |\xi|^{2\epsilon} d\xi dr = \int_0^t \int_A (E_{\beta}(-\nu|\xi|^a(t-r)^\beta))^2 |\xi|^{2\epsilon} d\xi dr \\
\leq \int_0^t \int_A |\xi|^{2\epsilon} d\xi dr \\
\leq x_0^{2\epsilon} t |A|, \quad (3.6)
\]
where $|A|$ is $d$-dimensional volume of $A$. This completes the proof. \(\square\)

Our results in this section extend the results in [18, Chapter 5] to the time fractional stochastic heat type equations setting. Our presentation in this section follow the presentation in [18, Chapter 5] with many crucial changes. We next prove that the linear map $\Phi \to G \circ \Phi$ is a continuous map from $\mathcal{L}^{\gamma, 2}$ into itself. In particular this will show that if $\Phi \in \mathcal{L}^{\gamma, 2}$, then $\Psi(x) := (G \circ \Phi)_{\mu}(x)$ is also in $\mathcal{L}^{\gamma, 2}$, and hence the stochastic convolution $G \circ \Phi W$ is also a well-defined random field in $\mathcal{L}^{\gamma, 2}$.

**Theorem 1.** If $\Phi \in \mathcal{L}^{\gamma, 2}$ for some $\gamma > 0$, then $G \circ \Phi$ has a continuous version that is in $\mathcal{L}^{\gamma, 2}$.

We will prove this theorem in steps.

The following result extends the analogous result on SPDEs by [10; 13; 21].
Proposition 1. (A stochastic Young inequality). For all $\gamma > 0, k \in [2, \infty), d < \min\{2, \beta^{-1}\} \alpha$, and $\Phi \in L^{\gamma, 2}$,

$$\mathcal{N}_{\gamma, k}(G \circ \Phi) \leq c_0 k^{1/2} \cdot \mathcal{N}_{\gamma, k}(\Phi)$$

where $c_0 = \sqrt{4(2\gamma)^{-1-\beta d/\alpha}\Gamma(1-\beta d/\alpha)}$.

Proof. According to the BDG inequality in [18, Proposition 4.4], and Lemma 1, for all $k \geq 2$,

$$\|(G \circ \Phi)_t(x)\|_k^2 \leq 4k \int_0^t ds \int_{\mathbb{R}^d} dy [G_{t-s}(y-x)]^2 \|\Phi(y)\|_k^2$$

$$\leq 4k \mathcal{N}_{\gamma, k}(\Phi)^2 \int_0^t e^{2\gamma s} ds \int_{\mathbb{R}^d} [G_{t-s}(y-x)]^2 dy$$

$$= 4kC^* \mathcal{N}_{\gamma, k}(\Phi)^2 \int_0^t e^{2\gamma s} (t-s)^{-\beta d/\alpha} ds$$

$$= 4kC^* \mathcal{N}_{\gamma, k}(\Phi)^2 e^{2\gamma t} \int_0^t e^{-2\gamma u} u^{-\beta d/\alpha} du$$

$$\leq 4kC^* \mathcal{N}_{\gamma, k}(\Phi)^2 e^{2\gamma t} (2\gamma)^{-(1-\beta d/\alpha)}\Gamma(1-\beta d/\alpha).$$

(3.7)

Divide both sides by $\exp(2\gamma t)$, take supremum over all $(t, x)$, and then take square roots to finish. \hfill \Box

Lemma 4. There exists a finite universal constant $C_1$ such that for all $\gamma > 0, k \in [1, \infty), t > 0, \Phi \in L^{\gamma, 2}, d < \min\{2, \beta^{-1}\} \alpha$ and $x, x' \in \mathbb{R}^d$,

$$E\left(\left\|(G \circ \Phi)_t(x) - (G \circ \Phi)_t(x')\right\|^k\right) \leq (C_1 k)^{k/2} e^{\gamma kt} [\mathcal{N}_{\gamma, k}(\Phi)]^k \|x' - x\|_{\min\{\frac{\alpha - \beta d}{\alpha}, 2\}}^k.$$

Proof. We need to only consider when $\mathcal{N}_{\gamma, k}(\Phi) < \infty$, a condition that we now assumed. In that case we may apply the BDG inequality to obtain the following

$$\|(G \circ \Phi)_t(x) - (G \circ \Phi)_t(x')\|_k^2 = \left\|\int_{(0,t) \times \mathbb{R}^d} [G_{t-s}(y-x) - G_{t-s}(y-x')] \Phi_s(y) W(dsdy)\right\|$$

$$\leq c_0 k \int_0^t ds \int_{\mathbb{R}^d} dy [G_{t-s}(y-x) - G_{t-s}(y-x')]^2 \|\Phi_s(y)\|_k^2$$

Now let us observe that whenever $0 < s < t$,

$$\|\Phi_s(y)\|_k^2 \leq e^{2\gamma s} [\mathcal{N}_{\gamma, k}(\Phi)]^2 \leq e^{2\gamma t} [\mathcal{N}_{\gamma, k}(\Phi)]^2.$$ 

This and Lemma 3 gives

$$\|(G \circ \Phi)_t(x) - (G \circ \Phi)_t(x')\|_k^2 \leq c_0 k e^{2\gamma t} [\mathcal{N}_{\gamma, k}(\Phi)]^2 \int_0^t ds \int_{\mathbb{R}^d} [G_{t-s}(y-x) - G_{t-s}(y-x')]^2 dy$$

$$\leq Cc_0 k e^{2\gamma t} [\mathcal{N}_{\gamma, k}(\Phi)]^2 \|x' - x\|_{\min\{\frac{\alpha - \beta d}{\alpha}, 2\}}.$$

Raise both sides to the power of $k/2$ to finish. \hfill \Box
Lemma 5. There exists a finite universal constant $C_1$ such that for every $\gamma > 0, k \in [1, \infty], t, t' > 0, x \in \mathbb{R}^d, d < \min\{2, \beta^{-1}\} \alpha$ and $\Phi \in \mathcal{L}^{\gamma,2}$, then
\[
\mathbb{E}\left(\|(G \circledast \Phi)_t(x) - (G \circledast \Phi)_{t'}(x)\|_k^k\right) \leq (C_1 k)^{k/2} e^{\gamma k t} [N_{\gamma,k}(\Phi)]^k \cdot |t - t'|^{(1 - \beta d/\alpha)k/2}.
\]

Proof. Without loss of generality, we suppose that $0 < t < t'$ and $N_{\gamma,k}(\Phi) < \infty$. In that case, we may write
\[
(G \circledast \Phi)_t(x) - (G \circledast \Phi)_{t'}(x) = J_1 + J_2,
\]
where
\[
J_1 := \int_{(0,t) \times \mathbb{R}^d} [G_{t-s}(y-x) - G_{t'-s}(y-x)] \Phi_s(y) W(dsdy),
\]
and
\[
J_2 := \int_{(t,t') \times \mathbb{R}^d} G_{t'-s}(y-x) \Phi_s(y) W(dsdy).
\]
We apply the BDG inequality and Lemma 1 and 3 to obtain the bound
\[
\|J_1\|^2_k \leq c_0 k \int_0^t ds \int_{\mathbb{R}^d} dy [G_{t-s}(y-x) - G_{t'-s}(y-x)]^2 \|\Phi_s(y)\|_k^2
\]
and
\[
\|J_2\|^2_k \leq c_0 k \int_t^{t'} ds \int_{\mathbb{R}^d} dy [G_{t'-s}(y-x)]^2 \|\Phi_s(y)\|_k^2
\]
\[
\leq C k e^{2\gamma t} [N_{\gamma,k}(\Phi)]^2 (t'-t)\gamma, \quad \text{for } 0 < t < t'.
\]
In this way we obtain the bound
\[
\|(G \circledast \Phi)_t(x) - (G \circledast \Phi)_{t'}(x)\|_k \leq 2\|J_1\|^2_k + 2\|J_2\|^2_k
\]
\[
\leq C k e^{2\gamma t} [N_{\gamma,k}(\Phi)]^2 (t'-t)^{1 - \beta d/\alpha},
\]
where $C$ is a finite constant. Hence, we deduce the lemma after we raise both sides of the preceding display to the power $k/2$. \hfill \square

In the following, we consider
\[
U_t(x) = \int_0^t \int_{\mathbb{R}^d} G_{t-r}(x-y) W(dr dy).
\]
This is the random part of the mild solution to (1.11) when $\sigma = 1$.

Proposition 2. Suppose $d < \min\{2, \beta^{-1}\} \alpha$, and $k \geq 2$, then we have the following moment estimates for time increments and spatial increments, respectively.

(i). For $t \leq t'$, we have
\[
c_0^{-1} |t' - t|^{(1 - \beta d/\alpha)k/2} \leq \mathbb{E} \left[ |U_{t'}(x) - U_t(x)|^k \right] \leq c_0 |t' - t|^{(1 - \beta d/\alpha)k/2}.
\]

(ii). For $x, x' \in \mathbb{R}^d$, we have
\[
c|x' - x|^k \leq \mathbb{E} \left[ |U_t(x) - U_t(x')|^k \right] \leq c_0 |x - x'|^\min\left\{\left(\frac{\alpha - \beta d}{\alpha - \beta d} - .2\right) \frac{k}{2}\right\}.
\]
Proof. We prove the lower bound in (i) at first. By the Hölder’s inequality, the fact that the stochastic integrals over nonoverlapping time intervals are independent, isometry of the stochastic integrals and the proof of Lemma 3, we have that for \( k \geq 2, \ t \leq t' \)

\[
\mathbb{E} \left[ |U_t'(x) - U_t(x)|^k \right] \geq \left( \mathbb{E} \left[ |U_t'(x) - U_t(x)|^2 \right] \right)^{k/2}
\]

\[
= \left( \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} |G_{t'-r}(x-y) - G_{t-r}(x-y)|^2 W(dr) dy \right] \right)^{k/2}
\]

\[
= \left( \int_0^t \int_{\mathbb{R}^d} |G_{t'-r}(x-y) - G_{t-r}(x-y)|^2 dy dr \right)^{k/2}
\]

\[
= \left( \frac{C^*}{1 - \beta d/\alpha} \right)^{k/2} \ |t' - t|^{(1 - \beta d/\alpha) \frac{k}{2}}.
\]

(3.11)

Now, for the upper bound, applying the Burkholder’s inequality, the fact that \( |a + b|^k \leq 2^k (|a|^k + |b|^k) \), and Lemma 3 we have that

\[
\mathbb{E} \left[ |U_t'(x) - U_t(x)|^k \right] 
\leq c_2 \left( \int_0^t \int_{\mathbb{R}^d} |G_{t'-r}(x-y) - G_{t-r}(x-y)|^2 dy dr \right)^{k/2}
\]

\[+ c_2 \left( \int_0^{t'} \int_{\mathbb{R}^d} G_{t'-r}^2(x-y) dy dr \right)^{k/2} \]

\[\leq c_2 \left( \frac{C^* (t' - t)^{1 - \beta d/\alpha}}{1 - \beta d/\alpha} \right)^{k/2} + c_2 \left( \frac{C^* (t' - t)^{1 - \beta d/\alpha}}{1 - \beta d/\alpha} \right)^{k/2} \]

\[= 2c_2 \left( \frac{C^*}{1 - \beta d/\alpha} \right)^{k/2} |t' - t|^{(1 - \beta d/\alpha) \frac{k}{2}}.\]

We now prove (ii). For \( p \geq 2 \) with \( x, x' \in \mathbb{R}^d \), by the Burkholder’s inequality and Lemma 3, we have

\[
\mathbb{E} \left[ |U_t(x') - U_t(x)|^k \right] \leq c_3 \left( \int_0^t \int_{\mathbb{R}^d} |G_{t-r}(x'-y) - G_{t-r}(x-y)|^2 dy dr \right)^{k/2}
\]

\[= C |x - x'|^{\min \left\{ \left( \frac{\alpha-\beta d}{\alpha} \right)^{-\frac{k}{2}}, 2 \right\}}.\]

To prove the lower bound we use the fact that \( k \geq 2 \) and Lemma 3(ii)

\[
\mathbb{E} \left[ |U_t(x) - U_t(x')|^k \right] \geq \left( \mathbb{E} \left[ |U_t(x) - U_t(x')|^2 \right] \right)^{k/2}
\]

\[= \left( \int_0^t \int_{\mathbb{R}^d} |G_{t-r}(x'-y) - G_{t-r}(x-y)|^2 dy dr \right)^{k/2}
\]

\[\geq c |x' - x|^k.\]

\[\square\]

**PROOF OF THEOREM 1.** If \( \Phi \in \mathcal{L}^{\gamma,2} \) for some \( \gamma > 0 \), then \( G \circ \Phi \) is an adapted random field. This can be seen by considering simple random fields \( \Phi \), as
limits of adapted random fields are adapted. But the result is easy to deduce when \( \Phi \) is simple.

Lemma 4 and Lemma 5 shows that \( G \ast \Phi \) is continuous in \( L^2(\Omega) \), then Proposition 4.6 in [18] implies that \( G \ast \Phi \) is in \( L^{7/2} \). We give some details of the preceding.

Let us choose and fix \( t > 0 \) and \( x \in \mathbb{R}^d \). We need to show that \( (\tau, z) \rightarrow (G(t,x) \ast \Phi)_{\tau}(z) \) is continuous in \( L^2(\Omega) \) to show that \( G \ast \Phi \) is in \( L^{7/2} \).

Since

\[
\mathbb{E}\left( |(G(t,x) \ast \Phi)_{\tau}(z) - (G(t,x) \ast \Phi)_{\tau}(z')|^2 \right)
= \mathbb{E}\left( |(G(t,0) \ast \Phi)_{\tau}(z + x) - (G(t,0) \ast \Phi)_{\tau}(z' + x)|^2 \right),
\]

the proof of Lemma 4 ensures that the preceding tends to zero, uniformly on \((\tau, z, z') \in (0, t) \times (-n, n)^d \times (-n, n)^d \) for any \( n > 0 \) fixed, as \( |z - z'| \rightarrow 0 \). This shows uniform continuity in \( L^2(\Omega) \), in the space variable.

On the other hand, we may follow the steps in the proof of Lemma 5 to deduce that

\[
\mathbb{E}\left( |(G(t,x) \ast \Phi)_{\tau}(z) - (G(t,x) \ast \Phi)_{\tau}(z')|^2 \right)
= \int_0^{\tau'-\tau} dr \int_{\mathbb{R}^d} dy |G(t-s)(y - z - x)|^2 \mathbb{E}(\Phi_s(y)^2)
\]

for \( 0 < \tau < \tau' \). Then we get

\[
\mathbb{E}\left( |(G(t,x) \ast \Phi)_{\tau}(z) - (G(t,x) \ast \Phi)_{\tau}(z')|^2 \right)
= [N_{\gamma,2}(\Phi)]^2 \int_0^{\tau'-\tau} e^{2\gamma s} dr \int_{\mathbb{R}^d} dy |G(t-s)(y - z - x)|^2.
\]

The preceding quantity goes to zero, as \( \tau' - \tau \rightarrow 0 \), uniformly for all \( 0 < \tau < \tau' < t \), and \( z \in \mathbb{R}^d \). This proves the remaining \( L^2 \)-continuity in the time variable, and completes the proof of the fact that \( G \ast \Phi \in L^{7/2} \).

The continuity of a modification of the stochastic convolution \( G \ast \Phi \) follows from Lemmas 4 and 5 by using a suitable form of the Kolmogorov continuity theorem [18, Theorem C.6].

4. Existence and Uniqueness of Solutions to the Time Fractional SPDEs

Recall that \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) is Lipschitz continuous. This means that there exists \( \text{Lip} > 0 \) such that

\[
|\sigma(x) - \sigma(y)| \leq \text{Lip}|x - y| \quad \text{for all } x, y \in \mathbb{R}.
\]

We may assume, without loss of generality, that \( \text{Lip} \) is also greater than \( |\sigma(0)| \). Since \( |\sigma(x)| \leq |\sigma(0)| + \text{Lip}|x| \), it follows that \( |\sigma(x)| \leq \text{Lip}(1 + |x|) \) for all \( x \in \mathbb{R} \).

The next theorem establishes existence and uniqueness of mild solutions of (1.11). It is our first main result in this paper.

**Theorem 2.** Let \( d < \min\{2, \beta^{-1}\alpha\} \). If \( \sigma \) is Lipschitz continuous and \( u_0 \) is measurable and bounded, then there exists a continuous random field \( u \in \cup_{\gamma \geq 0} C^{\gamma,2} \) that solves (1.11) with initial function \( u_0 \). Moreover, \( u \) is a.s.-finite among all
random fields that satisfy the following: There exists a positive and finite constant $L$-depending only on Lip, and $\sup_{z \in \mathbb{R}^d} |u_0(z)|$-such that

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}\left( |u_t(x)|^k \right) \leq L^k \exp(Lk^{1+\alpha/(\alpha-\beta d)t}). \quad (4.2)$$

Remark 2. Theorem 2 implies that a random field solution exists when $d < \min\{2, \beta^{-1}\}$. So in the case $\alpha = 2, \beta < 1/2$, a random field solution exists when $d = 1, 2, 3$. The analogous result in the case $\beta = 1$ is only valid when $d = 1$.

Remark 3. Is the power $k^{1+\alpha/(\alpha-\beta d)}$ in Theorem 2 artificial? Is it possible to find a lower bound for the moments with the same power of $k$?

We use Picard iteration to prove Theorem 2 that is outlined in [18, Chapter 1] with crucial changes. Define $u_t^{(0)}(x) := u_0(x)$, and iteratively define $u_t^{(n+1)}$ from $u_t^{(n)}$ as follows:

$$u_t^{(n+1)}(x) := (G_t * u_0)(x) + \int_{(0,t) \times \mathbb{R}^d} G_{t-r}(x-y)\sigma(u^{(n)}(r,y))W(dr,dy)$$

$$:= (G_t * u_0)(x) + (G * \sigma(u^{(n)}))W_t(x) \quad (4.3)$$

for all $n \geq 0, t > 0$, and $x \in \mathbb{R}^d$. Moreover, we set $u_t^{(k)}(x) := u_0(x)$ for every $k \geq 1$ and $x \in \mathbb{R}^d$.

Proposition 3. The random fields $\{u_t^{(n+1)}\}_{n=0}^\infty$ are well-defined, and each is in $\cup_{\gamma \geq 0} \mathcal{L}^{\gamma,2}$. Moreover, there exist positive and finite constants $L_1$ and $L_2$-depending only on Lip, and $\sup_{z \in \mathbb{R}^d} |u_0(z)|$-such that

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}\left( |u_t^{(n)}(x)|^k \right) \leq L_1^k \exp(L_2k^{1+\alpha/(\alpha-\beta d)t}), \quad (4.4)$$

simultaneously for all $k \in [1, \infty)$, $n \geq 0$, and $t > 0$.

Proof. We prove this proposition using induction on $n$. Since $u_0$ is non-random, bounded, and measurable, it is in $\cup_{\gamma \geq 0} \mathcal{L}^{\gamma,2}$. Moreover (4.4) holds since

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}\left( |u_t^{(0)}(x)|^k \right) = \sup_{z \in \mathbb{R}^d} |u_0(z)| \leq L_1^k \exp(L_2k^{1+\alpha/(\alpha-\beta d)t}).$$

Next suppose that the proposition holds for some integer $n \geq 0$. We will show the proposition for $n + 1$.

Let $k$ denote a fixed real number $\geq 2$. Now

$$u_t^{(l+1)}(x) := (G_t * u_0)(x) + B_t^{(l)}(x), \quad \text{where :}$$

$$B_t^{(l)}(x) := \int_{(0,t) \times \mathbb{R}^d} G_{t-r}(x-y)\sigma(u^{(0)}(r,y))W(dr,dy),$$

for all $t > 0, x \in \mathbb{R}^d$, and $l \geq 0$.

It is easy to see that $|(G_t * u_0)(x)| \leq \sup_{z \in \mathbb{R}^d} |u_0(z)|$, hence for all $\gamma > 0$

$$\mathcal{N}_{\gamma,k}(G_t * u_0) \leq \sup_{z \in \mathbb{R}^d} |u_0(z)|. \quad (4.5)$$

We estimate $B^{(n)}$ using the Stochastic Young inequality in Proposition 1:
\[ N_{\gamma,k} \left( B^{(n)} \right) \leq c_{\alpha, \beta, d} \frac{\sqrt{k}}{\gamma^{1-\beta d/\alpha}} N_{\gamma,k} \left( \sigma(u^{(n)}) \right) \leq c_{\alpha, \beta, d} \cdot \text{Lip} \cdot \frac{\sqrt{k}}{\gamma^{1-\beta d/\alpha}} \left\{ 1 + N_{\gamma,k} \left( u^{(n)} \right) \right\}, \] (4.6)

where \( c_{\alpha, \beta, d} \) is the constant in Proposition 1.

Now combining equations (4.5) and (4.6) we get for all \( \gamma > 0 \)
\[ N_{\gamma,k} \left( u^{(n+1)} \right) \leq \sup_{z \in \mathbb{R}^d} |u_0(z)| + c_{\alpha, \beta, d} \cdot \text{Lip} \cdot \frac{\sqrt{k}}{\gamma^{1-\beta d/\alpha}} \left\{ 1 + N_{\gamma,k} \left( u^{(n)} \right) \right\}. \] (4.7)

We can find a constant \( L_2 > 0 \) depending only on \( \text{Lip}, \alpha, \beta, d \) such that \( \gamma := L_2 k^{\alpha/(\alpha - \beta d)} \) satisfies
\[ c_{\alpha, \beta, d} \cdot \text{Lip} \cdot \frac{\sqrt{k}}{\gamma^{1-\beta d/\alpha}} \leq \frac{1}{4}, \]
then
\[ N_{\gamma,k} \left( u^{(n+1)} \right) \leq \sup_{z \in \mathbb{R}^d} |u_0(z)| + \frac{1}{4} + \frac{1}{4} N_{\gamma,k} \left( u^{(n)} \right). \]

In order to simplify notation, let
\[ \theta = \sup_{z \in \mathbb{R}^d} |u_0(z)| + \frac{1}{4}. \]

We solve recursively to find that
\[ N_{\gamma,k} \left( u^{(n+1)} \right) \leq \theta + \frac{\theta}{4} + \frac{1}{4} N_{\gamma,k} \left( u^{(n-1)} \right) \leq \cdots \leq \sum_{j=1}^{n} \frac{\theta}{4^j} + \frac{1}{4^n+1} N_{\gamma,k} \left( u^{(0)} \right). \]

But
\[ N_{\gamma,k} \left( u^{(0)} \right) = \sup_{t \geq 0} \sup_{x \in \mathbb{R}^d} \left( e^{-\gamma t} |u_0(x)| \right) = \sup_{z \in \mathbb{R}^d} |u_0(z)|. \]

Thus, for our choice of \( \gamma \), we have
\[ N_{\gamma,k} \left( u^{(n+1)} \right) \leq 4 \theta + 4^{-n+1} \sup_{z \in \mathbb{R}^d} |u_0(z)| \leq \frac{4}{3} \theta + \sup_{z \in \mathbb{R}^d} |u_0(z)| = L_1. \]

Therefore by Theorem 1 we get \( u^{(n+1)} \in \cup_{\gamma > 0} \mathcal{L} \gamma^2 \). Moreover, the last inequality means:
\[ \mathbb{E} \left( \left| u^{(n+1)}_t (x) \right|^k \right)^{L_k^k e^{\gamma k t}} = L_k^k \exp(L_2 k^{1+\alpha/(\alpha - \beta d)} t). \]

Hence, (4.4) holds for \( n + 1 \), and this proves the induction step, and proves the proposition. \( \square \)
PROOF OF THEOREM 2. Our proof follows similar steps as in the proof of Theorem 5.5 in [18] with nontrivial crucial changes. Let’s choose and fix some \( k \in [2, \infty) \). Let us write
\[
J := u^{(n+1)}_t(x) - u^{(n)}_t(x) = \int_{(0,t) \times \mathbb{R}^d} G_{t-r}(x-y) \left[ \sigma(u^{(n)}(r, y)) - \sigma(u^{(n-1)}(r, y)) \right] W(dr dy).
\]
By proposition 3 every \( u^{(n)} \in \mathcal{L}^{\gamma,2} \) for some \( \gamma > 0 \), and hence \( J \) is a well-defined stochastic integral for every \( n \geq 0 \). By the same proposition we can also choose a continuous version of \( (t, x) \rightarrow u^{(n)}(x) \) for every \( n \).

Next we estimate \( J \). We apply the BDG inequality (Proposition 4.4 in [18]) to bound \( \|J\|_k \) as follows:
\[
\|J\|_k^2 \leq C \cdot k \int_0^t dr \int_{\mathbb{R}^d} dy [G_{t-r}(x-y)]^2 \left\| \sigma(u^{(n)}_r(y)) - \sigma(u^{(n-1)}_r(y)) \right\|_k^2
\leq C \cdot \text{Lip}^2 k \int_0^t dr \int_{\mathbb{R}^d} dy [G_{t-r}(x-y)]^2 \left\| u^{(n)}_r(y) - u^{(n-1)}_r(y) \right\|_k^2
\leq C \cdot \text{Lip}^2 k \left[ \mathcal{N}_{\gamma,k}(u^{(n)} - u^{(n-1)}) \right]^2 \int_0^t e^{2\gamma r} dr \int_{\mathbb{R}^d} dy [G_{t-r}(x-y)]^2
\]
Since \( \int_{\mathbb{R}^d} dy [G_{t-r}(x-y)]^2 = C \cdot (t-r)^{-\beta d/\alpha} \), it follows that
\[
\|J\|_k^2 \leq C \cdot \text{Lip}^2 k e^{2\gamma t} \left[ \mathcal{N}_{\gamma,k}(u^{(n)} - u^{(n-1)}) \right]^2 \int_0^t e^{-2\gamma r} dr \cdot \frac{\sqrt{k}}{\gamma^{1-\beta d/\alpha}}.
\]
The preceding holds for all \( \gamma > 0 \). We can choose \( \gamma_0 := q \cdot k^{\alpha/(\alpha - \beta d)} \), where \( q > L_2 \) depends only on \( L_2 \) and ensures that
\[
\mathcal{N}_{\gamma_0,k}(u^{(n+1)} - u^{(n)}) \leq \frac{1}{4} \mathcal{N}_{\gamma_0,k}(u^{(n)} - u^{(n-1)}).
\]
According to Proposition 3
\[
\mathcal{N}_{\gamma_0,k}(u^{(1)} - u^{(0)}) \leq \mathcal{N}_{\gamma_0,k}(u^{(1)}) + \mathcal{N}_{\gamma_0,k}(u^{(0)}).
\]
Since \( q > L_2 \), Proposition 3 implies that
\[
\mathcal{N}_{\gamma_0,k}(u^{(1)}) \lor \mathcal{N}_{\gamma_0,k}(u^{(0)}) \leq L_1.
\]
Hence we obtain the estimate
\[
\mathcal{N}_{\gamma_0,k}(u^{(n+1)} - u^{(n)}) \leq \frac{L_1}{4^n}
\]
valid for all \( n \geq 0 \) and \( k \in [2, \infty) \). From this we obtain
(1) The random field \( u := \lim_{n \to \infty} u^{(n)} \) exists, where the limit takes place almost surely and in every norm \( \mathcal{N}_{\gamma_0, k} \);
(2) the random field \( \mathbb{S} \) defined by
\[
\mathbb{S}_t(x) := \lim_{n \to \infty} \int_{(0,t) \times \mathbb{R}^d} G_{t-r}(x-y)\sigma(u^{(n)}(r,y))W(drdy),
\]
exists, where the limit takes place almost surely and in every norm \( \mathcal{N}_{\gamma_0, k} \).

Combining Lemmas 4 and 5, applied to \( \Phi := u^{(n)} \), with Proposition 4.4 we see that for every \( k \in [2, \infty) \) and \( \tau \in (0, \infty) \) there exists a finite constant \( A_{k, \tau} \) such that
\[
\mathbb{E}\left( |u_t^{(n)}(x) - u_{t'}^{(n)}(x')|^k \right) \leq A_{k, \tau} \left( |x - x'| \min\left\{ \left( \frac{\alpha - \beta d}{2} \right)^{-2}, 2 \right\} \frac{1}{k} + |t - t'| \left( \frac{\alpha - \beta d}{2} \right)^{-2} \right),
\]
simultaneously for all \( t, t' \in [0, \tau] \), \( x, x' \in \mathbb{R}^d \). The right hand side of this inequality does not depend on \( n \). Hence, using Fatou’s lemma we get
\[
\mathbb{E}\left( |u_t(x) - u_{t'}(x')|^k \right) \leq A_{k, \tau} \left( |x - x'| \min\left( \frac{\alpha - \beta d}{2} \right)^{-2} \right) \frac{1}{k} + |t - t'| \left( \frac{\alpha - \beta d}{2} \right)^{-2}.
\]

Now by a suitable form of Kolmogorov continuity theorem in [18, Theorem c.6] we get that \( u \) has a version that is continuous. Moreover, (4.2) holds with \( L = \max\{q, L_1\} \) since \( \mathcal{N}_{\gamma_0, k}(u) \leq L_1 \) for all \( k \in [2, \infty) \).

So far, we know that
\[
u_t(x) = (G_t \ast u_0)(x) + S_t(x),
\]
where the equality is understood in the sense that the \( \mathcal{N}_{\gamma_0, k} \)-norm of the difference between the two sides of that inequality is zero. Equivalently, by the use of Fubini theorem we have shown that with probability one, the identity (4.9) holds for almost every \( t > 0 \) and \( x \in \mathbb{R}^d \).

Since \( u = \lim_{n \to \infty} u^{(n)} \) and \( u^{(n)} \in \mathcal{L}^{\gamma, 2} \) for some \( \gamma > 0 \) that is independent of \( n \), we can conclude that \( u \in \mathcal{L}^{\gamma, 2} \), and hence
\[
\mathbb{S}_t(x) := \int_{(0,t) \times \mathbb{R}^d} G_{t-r}(x-y)\sigma(u(r,y))W(drdy),
\]
is well-defined for all \( t > 0 \), and \( x \in \mathbb{R}^d \). Now we apply the Fatou’s Lemma together with Proposition 1 with \( \Phi = u^{(n)} - u \) and \( \gamma_0 = qk^{\alpha/(\alpha - \beta d)} \) to get that
\[
\mathcal{N}_{\gamma_0, k}(\mathbb{S} - \bar{\mathbb{S}}) \leq C \cdot \liminf_{n \to \infty} \mathcal{N}_{\gamma_0, k}(u^{(n)} - u) = 0.
\]

Hence \( \mathbb{S} \) and \( \bar{\mathbb{S}} \) are versions of one another. Another application of Lemmas 4 and 5 shows that \( \mathbb{S} \) has a continuous version. We combine these results with (4.9) to see that the present version of \( u \) is a mild solution—in the sense of (1.16) for the right versions of the integrals—for the time fractional stochastic heat type equation (1.11). This completes the proof of existence.

Suppose \( v \in \mathcal{L}^{\gamma, 2} \) is another random field that is mild solution to (1.11) with initial function \( u_0 \). We can argue similar to the arguments above to show that if \( Q \) is sufficiently large, \( \gamma_1 = Qk^{\alpha/(\alpha - \beta d)} \), then \( \mathcal{N}_{\gamma_1, k}(u - v) \leq \frac{1}{Q} \mathcal{N}_{\gamma_1, k}(u - v) \). Hence \( u \) and \( v \) are versions of one another. The other details are omitted.

\( \square \)
5. Non-existence of solutions

In this section, we will establish the non-existence of finite energy solutions when \( \sigma \) grows faster than linear. We say that a random field \( u \) is a finite energy solution to the stochastic heat equation (1.11) when \( u \in \cup_{\gamma>0} \mathcal{L}^{\gamma,2} \) and there exists \( \rho_*>0 \) such that

\[
\int_0^\infty e^{-\rho_*t} \mathbb{E}(|u_t(x)|^2) dt < \infty \quad \text{for all} \quad x \in \mathbb{R}^d.
\]

**Remark 4.** If \( \rho \in (0, \infty) \), then

\[
\int_0^\infty e^{-\rho t} \mathbb{E}(|u_t(x)|^2) dt \leq [\mathcal{N}_{1,2}(u)]^2 \cdot \int_0^\infty e^{-(\rho-2\gamma)t} dt.
\]

Therefore if \( \rho > 2\gamma \) and \( \mathcal{N}_{1,2}(u) < \infty \), then the preceding integral is finite. By Theorem 2, when \( \sigma \) is Lipschitz-continuous function and \( u_0 \) is bounded and measurable, then there exists a finite energy solution to the time fractional stochastic heat type equation (1.11).

When \( \sigma \) is Lipschitz continuous then it is at most linear growth. The following theorem shows that if we drop the assumption of linear growth, then we do not have a finite energy solution to the time fractional stochastic heat equation (1.11). This theorem extends the result of Foondun and Parshad [14]

**Theorem 3.** Suppose \( \inf_{z \in \mathbb{R}^d} u_0(z) > 0 \) and \( \inf_{y \in \mathbb{R}^d} |\sigma(y)|/|y|^{1+\epsilon} > 0 \). Then, there is no finite-energy solution to the time fractional stochastic heat type equation (1.11).

**Proof.** We will adapt the methods in [14] with many crucial changes. Let \( c := \inf_{y \in \mathbb{R}^d} |\sigma(y)|/|y|^{1+\epsilon} \) and \( l := \inf_{x \in \mathbb{R}^d} u_0(x) \).

\[
\mathbb{E}(|u_t(x)|^2) = |(G_t * u_0)(x)|^2 + \int_0^t ds \int_{\mathbb{R}^d} dy |G_{t-s}(y-x)|^2 \mathbb{E}(\sigma^2(u_s(y)))
\]

\[
\geq l^2 + c^2 \int_{\mathbb{R}^d} |G_{t-s}(y-x)|^2 \mathbb{E}(|u_s(y)|^{2(1+\epsilon)})dy \geq l^2 + c^2 \int_{\mathbb{R}^d} |G_{t-s}(y-x)|^2 \inf_{y \in \mathbb{R}^d} \mathbb{E}(|u_s(y)|^2)^{1+\epsilon} dy,
\]

with an application of the Jensen’s inequality in the last inequality. If we let

\[
I(t) := \inf_{y \in \mathbb{R}^d} \mathbb{E}(|u_s(y)|^2) \quad (t \geq 0),
\]

it satisfies the following inequality

\[
I(t) \geq l^2 + c^2 \int_0^t \mathbb{E}(|G_{t-s}|_{L^2(\mathbb{R}^d)}^2)^{1+\epsilon} ds = l^2 + c^2 \int_0^t (t-s)^{-d\beta/\alpha} [I(s)]^{1+\epsilon} ds.
\]

Now we define the Laplace transform of \( I \) as

\[
\tilde{I}(\theta) := \int_0^\infty e^{-\theta s} I(s) ds.
\]
Under the assumption that $0 < \theta < 1$, we shall show that $\theta > 0$. For all $\theta > 0$, where $C_1 = C^*\Gamma(1 - \beta d/\alpha)$. It follows that

$$\tilde{I}(\theta) = \frac{l^2}{\theta} + c^2 C_1 \theta^{-(1 - \beta d/\alpha)} \int_0^\infty e^{-\theta s}[I(s)]^{1+\epsilon} ds.$$  

We multiply both sides by $\theta$ and use Jensen’s inequality to find that

$$\theta \tilde{I}(\theta) \geq l^2 + c^2 C_1 \theta^{-(1 - \beta d/\alpha)} \left[ \int_0^\infty \theta e^{-\theta s}[I(s)] ds \right]^{1+\epsilon}$$

for all $\theta > 0$. It follows that $\theta \tilde{I}(\theta) \geq l^2 > 0$, and hence $\tilde{I}(\theta) > 0$ for all $\theta > 0$. Moreover, $|\theta \tilde{I}(\theta)|^{1+\epsilon} \geq l^2 \theta \tilde{I}(\theta)$, and hence

$$\theta \tilde{I}(\theta) \geq l^2 + c^2 C_1 \theta^{-(1 - \beta d/\alpha)} l^{2\epsilon} \theta \tilde{I}(\theta).$$

We shall show that $\tilde{I}(\theta) = \infty$ for $0 < \theta \leq \theta_0 := (c^2 C_1 l^{2\epsilon})^{\alpha/\alpha - \beta d}$. Under the assumption that $0 < \theta \leq \theta_0$, the constant $A := c^2 C_1 \theta^{-(1 - \beta d/\alpha)} l^{2\epsilon}$ is greater than or equal to 1. With recursive argument we can show that for any positive integer $n$,

$$\theta \tilde{I}(\theta) \geq l^2 + A^n \theta \tilde{I}(\theta).$$

Since $A \geq 1, l > 0$ and $n$ is arbitrary. We see that $\theta \tilde{I}(\theta) = \infty$. Which shows that $\tilde{I}(\theta) = \infty$ for a given range of $\theta$.

Now we will show that there exists $\theta_1 > \theta_0$ such that $\tilde{I}(\theta_1) = \infty$. Recall that $\tilde{I}(\theta_0) = \infty$. That means $I(t) \geq ce^{\theta_0 t}$ for large $t$.

For $t$ large enough, we have

$$I(t) \geq l^2 + c^2 \int_0^t (t - s)^{-\beta d/\alpha}[I(s)]^{1+\epsilon} ds$$

$$\geq l^2 + c^2 t^{-\beta d/\alpha} \int_0^t [I(s)]^{1+\epsilon} ds$$

$$\geq l^2 + c^2 t^{-\beta d/\alpha} \int_{1/2}^t e^{\theta_0 (1+\epsilon) s} ds$$

$$\geq l^2 + c^2 t^{-\beta d/\alpha} e^{\theta_0 (1+\epsilon) t} \left(1 - e^{-\theta_0 (1+\epsilon) t/2}\right).$$  

Using the inequality $x/(1+x) < (1 - e^{-x})$ for $x > -1$ we get

$$I(t) \geq l^2 + c^2 t^{-\beta d/\alpha} e^{\theta_0 (1+\epsilon) t},$$

again for large enough $t$. Now if we take $\theta_1 = \theta_0 (1 + \epsilon)$, we have $\tilde{I}(\theta_1) = \infty$. Since we can repeat this process over and over again there is no minimum $\theta < \infty$ such that $\tilde{I}(\theta) = \infty$. 

Because

$$\int_0^\infty e^{-\theta s} |G_s|_L^2 ds = \int_0^\infty e^{-\theta s} C^{*} s^{-\beta d/\alpha} ds = C_1 \theta^{-(1 - \beta d/\alpha)}$$

for all $\theta > 0$, where $C_1 = C^*\Gamma(1 - \beta d/\alpha)$. It follows that

$$\tilde{I}(\theta) \geq \frac{l^2}{\theta} + c^2 C_1 \theta^{-(1 - \beta d/\alpha)} \int_0^\infty e^{-\theta s}[I(s)]^{1+\epsilon} ds.$$  

(5.4)
Now, if we assume there is a finite energy solution, we certainly have \(\tilde{I}(\theta_\ast) < \infty\) for some \(\theta_\ast > 0\). With simple algebra we can show that
\[
\tilde{I}(\theta) = \int_0^\infty e^{-\theta s} I(s) ds = \int_0^\infty e^{-(\theta - \theta_\ast + \theta_\ast) s} I(s) ds \leq \int_0^\infty e^{-\theta_\ast s} I(s) ds < \infty,
\]
for \(\theta \geq \theta_\ast\). That means \(\tilde{I}(\theta) < \infty\) for all \(\theta \geq \theta_\ast\). But this contradicts the above argument. Hence there is no finite energy solution. This completes the proof.

\[\square\]

6. Equivalence of time fractional SPDEs to higher order SPDE’s

In this section we will give a connection of time fractional SPDE’s and Higher order parabolic SPDE’s.
Allouba \([1; 2]\) considered the following SPDE \((k = 1, 2, \cdots)\):
\[
\begin{align*}
\partial_t u_t^k(x) &= \sum_{j=1}^{2^k-1} \frac{t^{j/2^k-1}}{\Gamma(j/2^k)} \Delta^j u_t^k(x) + \Delta^2 u_t^k(x) + \sigma(u_t^k) W(t, x); \\
\quad u_t^k(x)|_{t=0} &= u_0^k(x)
\end{align*}
\]
(6.1)

Where \(\sigma\) satisfy Lipschitz and linear growth conditions. Allouba \([2]\) showed that there exists a path wise unique strong (mild) solution in all space dimensions \(d = 1, 2, 3\) given by
\[
\begin{align*}
u_t^k(x) &= \int_\mathbb{R} K_{t}^{BM^d, E^\beta}(x-y) u_0^k(y) dy \\
&\quad + \int_0^t \int_\mathbb{R} K_{t-r}^{BM^d, E^\beta}(x-y) \sigma(u_s^k(y)) W(dy, dr)
\end{align*}
\]
(6.2)

where \(K_{t}^{BM^d, E^\beta}(x-y)\) is the transition density of a \(d\)-dimensional process \(B(E^\beta)\), here \(B\) is a \(d\)-dimensional Brownian motion, and \(E^\beta\) is inverse of a stable subordinator of index \(\beta = 2^{-k}\):
\[
K_{t}^{BM^d, E^\beta}(x-y) = \int_0^\infty e^{-\frac{|x-y|^2}{4s}} f_{E_t}(s) ds.
\]

Allouba \([1; 2]\) also showed that the Hölder continuity exponent (time, space) of the mild solution of (6.1) is
\[
\left( \left( \frac{2\beta^{-1} - d}{4\beta^{-1}} \right)^{-}, \left( \frac{4 - d}{2} \wedge 1 \right)^{-}\right),
\]
(where \(\beta = 2^{-k}\)) in space dimensions \(d = 1, 2, 3\).
Since Baeumer et al.\([5]\) showed that \(G_t(x-y) = K_{t}^{BM^d, E^\beta}(x-y)\) for \(\alpha = 2\) and
\[
K_{t}^{Y, E^\beta}(x-y) = \int_0^\infty p_{Y(s)}(x-y) f_{E_t}(s) ds = G_t(x-y), \quad \text{for } 0 < \alpha < 2,
\]
we obtain the next result
Theorem 4. Let $L_x = -\nu(-\Delta)^{\alpha/2}$ for $\alpha \in (0, 2]$ and suppose $\sigma$ is Lipschitz. For any $k = 1, 2, 3, \ldots$ both the higher order SPDE
\[
\partial_t u^k_t(x) = \sum_{j=1}^{2^k-1} \frac{t^{1/2^k-1}}{\Gamma(j/2^k)} L_x^j u^k_0(x) + L_x^{2^k} u^k_t(x) + \sigma(u^k) W(t, x); 
\]
and the time fractional SPDE
\[
\partial^{1/2^k} u^k_t(x) = L_x u^k_t(x) + I^{1-1/2^k}_{t} [\sigma(u^k) W(t, x)]; \quad u^k_t(x)|_{t=0} = u^k_0(x),
\]
have the same unique mild solution given by
\[
u_t(x) = \int_{\mathbb{R}^d} K^{Y,E}_t(x-y)u^k_0(y)dy + \int_0^t \int_{\mathbb{R}} K^{Y,E}_s(x-y)\sigma(u^k_s(y))W(dydr). \tag{6.5}
\]

Remark 5. Theorem 4 extends the particular case considered by Baeumer et al. [5]—the case of $\sigma \equiv 0$—to the equivalence of SPDE’s. Allouba [2, equation (1.8)] mentions equivalence of (6.1) to the equation
\[
\partial^{1/2^k} u^k_t(x) = \Delta_x u^k_t(x) + \sigma(u^k) W(t, x); \quad u^k_t(x)|_{t=0} = u^k_0(x). \tag{6.6}
\]
Our theorem 4 gives the correct equivalence.

References

[1] H. Allouba. Brownian-time Brownian motion SIEs on $\mathbb{R}^p \times \mathbb{R}^d$: Ultra Regular direct and lattice-limits solutions, and fourth order SPDEs links. DCDS-A. 33 (2013), no. 2, 413-463.
[2] H. Allouba. Time-fractional and memoryful $\Delta^{2^k}$ SIEs on $\mathbb{R}^p \times \mathbb{R}^d$: how far can we push white noise? Illinois J. Math. 57 (2013), no. 3, 50pp.
[3] B. Baeumer, M. Geissert, and M. Kovacs. Existence, uniqueness and regularity for a class of semilinear stochastic Volterra equations with multiplicative noise. Preprint.
[4] B. Baeumer and M.M. Meerschaert. Stochastic solutions for fractional Cauchy problems, Fractional Calculus Appl. Anal. (2001) 4 481–500.
[5] B. Baeumer, M.M. Meerschaert, and E. Nane. Brownian subordinators and fractional Cauchy problems, Trans. Amer. Math. Soc. 361 (2009), 3915-3930.
[6] J. Bertoin. Lévy Processes. Cambridge University Press, Cambridge (1996).
[7] M. Caputo. Linear models of dissipation whose $Q$ is almost frequency independent, Part II. Geophys. J. R. Astr. Soc. 13 (1967), 529-539.
[8] René A. Carmona and S. A. Molchanov, Parabolic Anderson problem and intermittency, Mem. Amer. Math. Soc. 108 (1994), no. 518, viii+125.
[9] Z.-Q. Chen, K.-H. Kim and P. Kim. Fractional time stochastic partial differential equations. Preprint 2014.
[10] D. Conus and D. Khoshnevisan. On the existence and position of the farthest peaks of a family of stochastic heat and wave equations, Probab. Theory Related Fields 152 (2012), no. 3-4, 681–701.
[11] Dalang, Robert C.; Quer-Sardanyons, Lluís Stochastic integrals for spde’s: a comparison. Expo. Math. 29 (2011), no. 1, 67109.
[12] Giuseppe Da Prato and Jerzy Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992.
[13] M. Foondun and D. Khoshnevisan. Intermittence and nonlinear parabolic stochastic partial differential equations, Electron. J. Probab. 14 (2009), no. 21, 548–568.
[14] M. Foondun and R. Parshad, On non-existence of global solutions to a class of stochastic heat equations, to appear in Proc. Amer. Math. Soc., 2013.
[15] N. Georgiou, M. Joseph, D. Khoshnevisan, P. Mahboubi, and S-Y. Shiu. Semi-discrete semi-linear parabolic SPDEs, 2013.
[16] H. J. Haubold, A. M. Mathai and R. K. Saxena. Review Article: Mittag-Leffler functions and their applications, Journal of Applied Mathematics. Volume 2011 (2011) Article ID 298628, 51 pages
[17] A. Karczewska. Convolution type stochastic Volterra equations, 101 pp., Lecture Notes in Nonlinear Analysis 10, Juliusz Schauder Center for Nonlinear Studies, Torun, 2007.
[18] D. Khoshnevisan. Analysis of stochastic partial differential equations. CBMS Regional Conference Series in Mathematics, 119. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2014.
[19] A.N. Kochubei, The Cauchy problem for evolution equations of fractional order, Differential Equations, 25 (1989) 967 – 974.
[20] A. Lunardi and E. Sinestrari, An inverse problem in the theory of materials with memory, Nonlin. Anal. Theory Meth. Appl. 12 (1988), 13171355,
[21] P. Mahboubi. Intermittency of the Malliavin Derivatives and Regularity of the Densities for a Stochastic Heat Equation, Ph.D. thesis, University of California, Los Angeles, 2012.
[22] A. M. Mathai and H. J. Haubold, Special functions for applied scientists. Springer, 2007.
[23] M.M. Meerschaert, E. Nane and P. Vellaisamy. Fractional Cauchy problems on bounded domains. Ann. Probab. 37 (2009), 979-1007.
[24] M.M. Meerschaert, E. Nane, and Y. Xiao. Fractal dimensions for continuous time random walk limits, Statist. Probab. Lett., 83 (2013) 10831093.
[25] M.M. Meerschaert and H.P. Scheffler. Limit theorems for continuous time random walks with infinite mean waiting times. J. Applied Probab. 41 (2004) No. 3, 623–638.
[26] M.M. Meerschaert and P. Straka. Inverse stable subordinators. Mathematical Modeling of Natural Phenomena, Vol. 8 (2013), No. 2, pp. 116.
[27] M. Abramowitz and I. A. Stegun. Handbook of mathematical functions Dover Publications, INC., New York.
[28] E. Nane, Fractional Cauchy problems on bounded domains: survey of recent results, In : Baleanu D. et al (eds.) Fractional Dynamics and Control, 185198, Springer, New York, 2012.
[29] R.R. Nigmatullin. The realization of the generalized transfer in a medium with fractal geometry. Phys. Status Solidi B. 133 (1986) 425 – 430.
[30] E. Orsingher and L. Beghin. Time-fractional telegraph equations and telegraph processes with Brownian time. *Prob. Theory Rel. Fields* **128** (2004), 141–160.

[31] E. Orsingher and L. Beghin. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab.* **37** (2009) 206 – 249.

[32] T. Simon. Comparing Fréchet and positive stable laws. *Electron. J. Probab.* **19** (2014), no. 16, 125

[33] S. Umarov and E. Saydamatov. A fractional analog of the Duhamel principle. *Fract. Calc. Appl. Anal.* **9** (2006), no. 1, 5770.

[34] S.R. Umarov, and É. M. Saidamatov. Generalization of the Duhamel principle for fractional-order differential equations. (Russian) *Dokl. Akad. Nauk* **412** (2007), no. 4, 463–465; translation in *Dokl. Math.* **75** (2007), no. 1, 9496

[35] S. Umarov. On fractional Duhamel’s principle and its applications. *J. Differential Equations* **252** (2012), no. 10, 52175234.

[36] J. B. Walsh. An Introduction to Stochastic Partial Differential Equations, *École d’été de Probabilités de Saint-Flour, XIV—1984*, Lecture Notes in Math., vol. 1180, Springer, Berlin, (1986), pp. 265–439.

[37] L. von Wolfersdorf. An identification of memory kernels in linear theory of heat equation, *Math. Meth. appl. sci.* **17** (1994) 919-932

[38] D. Wu. On the solution process for a stochastic fractional partial differential equation driven by space-time white noise. *Statist. Probab. Lett.* **81** (2011), no. 8, 11611172.

[39] W. Wyss. The fractional diffusion equations. *J. Math. Phys.* **27** (1986) 2782 – 2785.

[40] G. Zaslavsky. Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion. *Phys. D* **76** (1994) 110-122.

Department of Mathematics, Georgia College & State University, Milledgeville, GA 31061

*E-mail address:* jebessa.mijena@gcsu.edu

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849, USA

*E-mail address:* nane@auburn.edu