A rigorous bound on quark distributions in the nucleon

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I deduce an inequality between the helicity and the transversity distribution of a quark in a nucleon, at small energy scales. Then I establish, thanks to the positivity constraint, a rigorous bound on longitudinally polarized valence quark densities, which finds nontrivial applications to $d$-quarks. This, in turn, implies a bound for the distributions of the longitudinally polarized sea, which is probably not $SU(3)$-symmetric. Some model predictions and parametrizations of quark distributions are examined in the light of these results.

1. INTRODUCTION

The leading twist distributions of quarks and antiquarks inside the nucleon are not completely known. Indeed, the longitudinally polarized sea distributions are only poorly known [1,2,3], while the transversely polarized densities are unknown at all. In this situation it is quite important to establish rigorous inequalities, which are very useful in best fits [1,2] to data. As an example, I recall the famous Soffer inequality [4], which is a consequence of the positivity constraint. Here I deduce, as an application of that inequality, a rigorous bound concerning unpolarized and longitudinally polarized valence quark distributions.

The talk is organized as follows. First of all (sect. 2), starting from the Soffer inequality, from the Melosh-Wigner rotation [5,6,7] and from general considerations on evolution equations, I prove bound (17) for any $Q^2$-value, $-Q^2$ being the modulus square of the four-momentum of the probe (e. g., a virtual photon). Secondly (sect. 3), I consider applications to $d$-quark distributions and to the first moments of polarized sea distributions. Lastly, in sect. 4, I draw a short conclusion.

2. A RIGOROUS BOUND

2.1. The Melosh-Wigner rotation

First of all, I deduce, for very small values of $Q^2$, an inequality between the helicity and the transversity distributions, for which I adopt the notations by Mulders and Tangerman [8].

In the nucleon rest frame, let $q_0^\pm (p)$ be the probability density for a quark of momentum $p$ to have spin parallel (+) or antiparallel (-) to the proton spin. A boost parallel to the nucleon spin, and such that the nucleon momentum becomes much greater than the nucleon rest mass $M$, produces a spin dilution in the transverse momentum (t.m.) polarized density, which turns out to coincide with the longitudinally polarized t.m. distribution, i. e.,

$$g_1 (x, p_\perp) = \left[ q_0^+ (p) - q_0^- (p) \right] \cos \theta_M.$$  (1)

Here $x = (p_0 + p_3)/\sqrt{2M}$, $p_\perp = p - p_3 k$, $k$ is a unit vector in the direction of the boost, $p_3 = p \cdot k$, $p_0^2 = m^2 + p_3^2 + p_\perp^2$ and $m$ is the quark rest mass. Lastly $\theta_M$ is the Melosh-Wigner rotation angle [5,6,7], i. e.,

$$\theta_M = \arccos \left[ \frac{(m + \sqrt{2xM})^2 - p_\perp^2}{(m + \sqrt{2xM})^2 + p_\perp^2} \right].$$  (2)

On the other hand, a boost from the nucleon rest system, similar to the previous one, but in a direction perpendicular to its spin, produces a less drastic spin dilution. Indeed, in this case the distribution results in the t.m. transversity, the Melosh-Wigner rotation giving [8]

$$h_1 (x, p_\perp) = \left[ q_0^+ (p) - q_0^- (p) \right] D_\perp (\theta_M, \phi).$$  (3)

Here

$$D_\perp (\theta_M, \phi) = \cos^2 \frac{\theta_M}{2} + \sin^2 \frac{\theta_M}{2} (2 \sin^2 \phi - 1).$$  (4)
and $\phi$ is the azimuthal angle of $p_\perp$ with respect to the plane perpendicular to the nucleon spin vector. Eqs. (1) to (4) imply
\[
\frac{h_1(x)}{g_1(x)} \geq 1,
\]
where $g_1(x) = \int d^2 p_\perp g_1(x, p_\perp^2)$ and $h_1(x)$ is defined analogously. This inequality - which reduces to equality for a nonrelativistic bound state - holds at the starting point for QCD evolution[9,10,11,12], i.e., for very small values of $Q^2$ (<< 1 GeV$^2$). This is confirmed by previous calculations, based on the constituent quark model[5,6], on the bag model[13,14], on light cone formalism, and so forth[15]. The elementary process determining the evolution is described by an equation similar to (9).

2.2. The Soffer inequality

At increasing $Q^2$, $h_1$ decreases much more rapidly than $g_1$, owing to a different evolution kernel. Therefore inequality (5) no longer holds true for sufficiently large values of $Q^2$. However, as I shall show in the next subsection, relation (6), together with the Soffer inequality, i.e.,
\[
2|h_1(x)| \leq f_1(x) + g_1(x),
\]
implies an inequality which holds for any $Q^2$.

Indeed, (4) and (6) yield
\[
2|g_1(x)| \leq f_1(x) + g_1(x),
\]
which is nontrivial for negative values of $g_1(x)$; in this case one has
\[
-3g_1(x) \leq f_1(x).
\]
Since $f_1(x)$ is nonnegative, inequality (6) holds true for any value of $x$. It is interesting to compare this inequality with a result of the nonrelativistic SU(6) quark model. For a valence $d$-quark in the nucleon, this model predicts
\[
g_{1v}^d(x) = -\frac{1}{3} f_{1v}^d(x).
\]
As I have shown above, in a relativistic bound state with spinning constituents one has to take into account the Melosh-Wigner rotation, which produces a spin dilution (see also[16,17]). Therefore eq. (4) is to be replaced by
\[
0 < -g_{1v}^d(x) \leq \frac{1}{3} f_{1v}^d(x),
\]
whose upper bound is a particular case of (8).

2.3. Evolution equations

Now I shall show that, under rather general assumptions, inequality (5), if referred to valence quarks, holds true for any $Q^2 > Q_0^2$, with $Q_0^2 \leq 1$ GeV$^2$. To this end, define the function
\[
\phi(x,t) = f_{1v}(x,t) + 3g_{1v}(x,t), \quad t = \ln \frac{Q^2}{Q_0^2}.
\]
and set $\phi(x,0) > 0$. For $Q_0^2 \simeq 1$ GeV$^2$, it looks reasonable to assume, like Bourrely et al.[18], that $f_{1v}(x,t)$ and $g_{1v}(x,t)$ evolve according to the DGLAP equations at least for $Q^2 \geq Q_0^2$. Then, for $t \geq 0$, the leading order (LO) QCD evolution equation reads
\[
\frac{d}{dt} \phi(x,t) = \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dy}{y} \phi(y,t) P\left(\frac{x}{y}\right),
\]
\[
P(z) = C_F \left( 1 + \frac{z^2}{1 - z^2} \right) + .
\]
This is a consequence of the fact that at LO $f_{1v}(x,t)$ and $g_{1v}(x,t)$ have the same evolution scheme dependent and introduces a complication in the evolution of $\phi$ - are completely screened by the LO term[19] and therefore do not affect the result[20,19].

Now, if I push $Q_0^2$ down to very small values (<< 1 GeV$^2$), a nonperturbative evolution of the type described in refs.[15,21,22] - based essentially on the chiral quark model[10] - could be assumed. In this case the elementary process determining the evolution is
\[
q \rightarrow \pi q',
\]
where $q$ is a quark and $\pi$ a pion. Considerations analogous to the DGLAP equation can be done in the framework of this model. In fact the LO evolution is described by an equation similar to (12), i.e.,
\[
\frac{d}{dt} \phi(x,t) = \frac{g_{\pi}^2}{8\pi^2} \int_x^1 \frac{dy}{y} \phi(y,t) P'\left(\frac{x}{y}\right).
\]
Here $g_{\pi}$ is the pion-quark coupling constant and $P'(z)$ the splitting function for the process[14].
Since $P'(z)$ again preserves positivity, and $g_\pi$ is small enough to assure the screening of NLO effects by the LO term, the model leads to conclusions similar to those drawn in the framework of perturbative QCD.

It is important to realize that, in the case of nonperturbative evolution, the result we have found does not depend on the specific model assumed. Indeed, the same reasoning holds equally true for any evolution mechanism of the type

$$q \to Bq', \quad (16)$$

where $q$ is a quark and $B$ a boson - not necessarily a gluon or a pion -, provided the probability of such an elementary process

i) is sufficiently small,

ii) preserves positivity,

iii) is helicity independent at LO.

But the third requirement is a necessary consequence of helicity conservation and implies that $f_{1v}(x,t)$ and $g_{1v}(x,t)$ have the same LO evolution kernel. On the other hand, any realistic process of the type $(16)$ satisfies conditions i) and ii).

Therefore I conclude that inequality $(8)$ is true under quite general assumptions, and practically for any $Q^2$, since $Q_0^2$ may assume very small values. This constitutes a nontrivial bound to the parametrizations of the quark distribution functions. In fact, it implies, together with the positivity constraint,

$$-1/3f_{1v}(x) \leq g_{1v}(x) \leq f_{1v}(x), \quad (17)$$

which is stronger than the inequality

$$|g_t(x)| \leq f_t(x), \quad (18)$$

usually taken into account in the fits to data of polarized deep inelastic scattering.[1]

3. APPLICATIONS

Now I compare bound $(16)$ with models and parametrizations of quark distributions, especially as to the down flavor. For example, the predictions of some models, like the constituent quark model[24], and the Carlitz-Kaur model[17], fulfill inequality $(17)$. On the contrary, in a best fit to semi-inclusive data[23], the ratio $g_1^d(x)/f_1^d(x)$ does not satisfy this inequality for $x > 0.1$; this casts some doubts on that fit, since for sufficiently large $x$ ($x > x_0$, with $x_0 = 0.2$) the quark distributions derive their contributions essentially from valence quarks.

Integrating over $x$ from 0 to 1 all the terms which appear in the inequality $(17)$ yields

$$-n_q/3 < \Delta q_v \leq n_q, \quad (19)$$

Here $\Delta q_v$ is the first moment of $g_{1v}(x)$ and $n_q$ the valence number, $n_q = 2$ for $u$-quarks and 1 for $d$-quarks. The lower bound $(19)$ is not respected by the $d$-quark parametrizations deduced from the best fits in the literature[12]. Leader, Sidorov and Stamenov[1], who assume an SU(3)-symmetric polarized sea, find, at $Q^2 = 1$ GeV$^2$, $-\Delta d_v = 0.339 - 0.341$. On the other hand, the best fits by Bartelski and Tatur[2], who do not assume any constraint for the sea, result, at the same $Q^2$-value, in $-\Delta d_v = 0.60 - 0.68$. Therefore I conclude that, although SU(3) symmetry is probably violated by polarized sea quark distributions, a fit without contraints is quite unreliable. As I shall show in a moment, some bounds on polarized sea distributions must be taken into account.

Indeed, $(17)$, together with the well-known relations

$$\Delta u + \Delta \bar{u} - \Delta d - \Delta \bar{d} = \bar{a}_3, \quad (20)$$

$$\Delta u + \Delta \bar{u} + \Delta d + \Delta \bar{d} - 2\Delta s - 2\Delta \bar{s} = \bar{a}_8, \quad (21)$$

allows to deduce an important inequality about longitudinally polarized sea distributions. In eqs. $(20)$ and $(21)$ $\Delta q$ and $\Delta \bar{q}$ denote the first moments of the quark and antiquark polarized distributions, $\bar{a}_3 = (1.2670 \pm 0.0035) - \alpha_s/\pi$ and $\bar{a}_8 = (0.585 \pm 0.025) - \alpha_s/\pi$. Considering the splitting $\Delta q = \Delta q_v + \Delta q_s$ into valence and sea contributions, eqs. $(20)$ and $(21)$ yield

$$\Delta d_v = -0.341 \pm 0.014 + \Delta \bar{s} - \Delta \bar{d}, \quad (22)$$

having set $\Delta \bar{q} = \Delta q_v + \Delta \bar{q}$. Therefore the first inequality $(19)$ implies

$$\Delta \bar{s} - \Delta \bar{d} > 0.008 \pm 0.014, \quad (23)$$

indicating that the SU(3) flavor symmetry for the longitudinally polarized sea distributions is very unlikely, although it cannot be completely excluded. This result confirms the analysis by other authors[25].
Lastly, since \( \Delta \tilde{s} \) is negative\(^{26} \), inequality \(^{23} \) implies
\[
\Delta \tilde{d} < \Delta \tilde{s} < 0, \tag{24}
\]
i.e., a negatively polarized sea. This confirms the result that can be inferred\(^{27} \) from data of polarized deep inelastic scattering\(^{28} \). \textit{Vice versa}, inequality \(^{24} \) contradicts the interpretation in terms of a positive sea polarization of recent data of semi-inclusive deep inelastic scattering\(^{29,30} \).

4. CONCLUSION

Here I recall the main results illustrated in my talk.
1. I have proved bound \(^{17} \) for any \( Q^2 \)-value.
2. This bound is especially important in setting limits to fit parameters and in discriminating between correct and wrong models. In particular, I find that the predictions of some models agree with that bound; on the contrary, recent distributions, resulting from best fits, violate it.
3. A bound on the polarized sea distributions is deduced, which turns out to agree with some analyses\(^{27} \) of the EMC effect, also known as "spin crisis"\(^{28} \).

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