Spin-valley collective modes of the electron liquid in graphene

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We develop the theory of collective modes supported by a Fermi liquid of electrons in pristine graphene. Under reasonable assumptions regarding the electron-electron interaction, all the modes but the plasmon are over-damped. In addition to the SU(2) symmetric spin mode, these include also the valley imbalance modes obeying a U(1) symmetry, and a U(2) symmetric valley spin imbalance mode. We derive the interactions and diffusion constants characterizing the over-damped modes. The corresponding relaxation rates set fundamental constraints on graphene valley- and spintronics applications.

An extremely long electron mean free path1 combined with a fairly strong electron-electron interaction2 makes graphene an interesting platform for investigating Fermi liquid effects in solids. The strength of interaction may be tuned by varying the electron density with respect to the neutrality point in graphene and by changing the dielectric environment encapsulating the graphene sheet. A hallmark of Fermi liquid behavior is an emergence of the collective modes associated with the symmetries of the system. The charge density mode, the plasmon, is the easiest to couple to, and can be probed in various spectroscopic experiments3. Plasmons are protected from Landau damping by their high propagation velocity, leading to fairly narrow spectroscopic lines. The spectra of plasmons have been calculated for and measured in various settings 34.

More recently, hydrodynamic electron flow has attracted much attention, occurring in the regime where the electron-electron mean free path is much smaller than other collision scales ℓe-e ≪ ℓ. In the absence of disorder, a uniform electric current is protected by charge conservation and translation invariance, but a non-uniform electron flow is associated with size affects which arise from the electron viscosity. To find it, one needs to solve the two-dimensional Fermi liquid kinetic equation 56.7 for the quasiparticle distribution function. A variety of hydrodynamic size effects have been predicted and addressed experimentally in electric DC transport 711.

A fairly weak spin-orbit interaction in carbon leaves electron spin in graphene approximately conserved. Furthermore, the two Dirac points in graphene’s electron spectrum are located far from each other in the Brillouin zone. As the result, electron scattering between the valleys associated with the Dirac points is suppressed. The approximate conservation of the spin and valley indices of a quasiparticle raises the question of the existence of sound modes in these channels. It has also given rise to a host of proposals for “spintronics” and “valleytronics” graphene applications 1213 exploiting in various ways spin or valley currents. The valleytronics proposals are specific for multi-valley materials, while the spintronics ones are a part of a broader semiconductor physics literature 1415. In the analysis of spin- and valley-current conservation, the majority of theory works consider the effect of static disorder scattering non-interacting electrons at the Fermi level, see e.g. Refs. 16 and 17. A notable exception is Ref. 17 which evaluated in the Born approximation the spin relaxation rate due to electron-electron collisions in a two-dimensional electron gas, partially spin-polarized by a magnetic field. Later this theory was applied to analyze the measurement of spin diffusion in the absence of polarizing magnetic field 18. We note in passing that the spin diffusion in the SU(2)-symmetric neutral three-dimensional Fermi liquid was considered in the context of the low-temperature He-3 properties 1920, and for ungapped graphene in 2D 21.

In this work, we study dynamics of the neutral modes supported by the electron Fermi liquid in graphene. The neutral modes include a SU(2)-symmetric spin mode, U(1)-symmetric modes of the inter-valley coherence and imbalance, as well as U(2) symmetric inter-valley spin modes. We identify and estimate the relative strength of the microscopic interactions which determine the values of parameters in a phenomenological Fermi liquid theory for these modes. Under reasonable assumptions, all of the neutral modes are overdamped, and there is no neutral zero- or first-sound. The spread and decay of the spin polarization density and of the inter-valley coherence and imbalance densities are thus characterized by their diffusion constants. To find the diffusion constants, we evaluate the corresponding transport relaxation rates from the linearized collision integral which accounts for the electron-electron scattering and solve the linearized kinetic equations. The obtained rates depend on the temperature, electron density, and gaps at the Dirac points. These gaps may appear due to deformation of the graphene lattice, for example in graphene encapsulated in hexagonal boron nitride 22, and have a profound effect on electron backscattering due to the
Berry flux redistribution across the Brillouin zone. Indeed, backscattering of electrons in graphene may only occur in the presence of such gaps [23, 24].

A realistic level of the electron density induced by electrostatic gating corresponds to fairly small Fermi wave-vectors $k_F \ll |\mathbf{K}|$ (measured from the respective Dirac points). Therefore, we adopt the description of the electron system in terms of slowly-varying in space Fermi fields [25, 26].

$$
\Psi_{\sigma}(\mathbf{r}) = \left( u_{kA}(\mathbf{r}) u_{kB}(\mathbf{r}) u_{K'a}(\mathbf{r}) - u_{K'A}(\mathbf{r}) \right) \cdot \chi_{\sigma} \psi_\sigma(\mathbf{r}) \tag{1}
$$

where $u_{k\Sigma}(\mathbf{r})$ are the Bloch wavefunctions concentrated near the $\Sigma = A, B$ sub-lattice sites. For definiteness, we consider the Fermi level above the Dirac point. We may then perform a projection onto the conduction band $\tilde{\psi}_{\sigma} = \sum_{\zeta} \chi_{\zeta\Sigma} \hat{c}_{\zeta\sigma}$ where the sub-lattice pseudo-spinors $\chi_{\zeta\Sigma}$ are eigenvectors of the Dirac Hamiltonian,

$$
(\mathbf{v} \cdot \zeta \Delta \Sigma) \chi_{\zeta\Sigma} = \sqrt{v^2 k^2 + \Delta^2} \chi_{\zeta\Sigma} \tag{2}
$$

and $\Sigma_a$ are Pauli matrices in sub-lattice space [27]. In terms of the upper-band operators, the low-energy Hamiltonian is

$$
\hat{H} = \sum_{k\zeta\sigma} \left[ \sqrt{v^2 k^2 + \Delta^2} - (\Delta + E_F) \right] c_{\zeta\Sigma}^\dagger c_{\zeta\Sigma} + \hat{H}_{\text{int}} \tag{3}
$$

where $\zeta$ and $\sigma$ are respectively the valley and spin indices, $E_F$ is the Fermi level measured from the gap edge, and the interaction Hamiltonian is

$$
\hat{H}_{\text{int}} = \frac{1}{2} \sum_{k,k',q} \sum_{\zeta_1 \zeta_2} g_{\zeta_1\zeta_2} c_{\zeta_1\sigma_1}^\dagger c_{\zeta_2\sigma_2} (k, k', q) \times : \hat{c}_{\zeta_1\sigma_1}^\dagger (k+q) \hat{c}_{\zeta_2\sigma_2} (k') : - (\Delta + E_F) \tag{4}
$$

The valley charge and spin symmetries constrain the short range interactions to be of the form [28]

$$
g(p, p', q) = g^v_{p,p',q} \sigma \cdot \tau + g^m_{p,p',q} \tau \cdot \sigma + g^z_{p,p',q} \tau \cdot \tau \cdot \sigma \tag{5}
$$

where $\tau^i$ and $\sigma^i$ are Pauli matrices in valley and spin space, respectively, and all of the functions $g^\alpha$ are short ranged except for $g^v$ which includes thelong range part of the Coulomb interaction, $V(q)$. The six functions, $g^v_{p,p',q}$, are the inputs of our theory. These, in turn, can be expressed in terms of the interaction constants $g_{zz}, g_{\perp\perp}, g_{00}$ of an unprojected Hamiltonian [26] and screened Coulomb potential $V(q)$, combined with the matrix elements of the projection operator constructed from eigenspinors $\chi_{k\zeta}$. The latter contribute to the dependence of $g(p, p', q)$ on the respective momenta [28].

The space-group symmetry of the graphene lattice constrains the form of the interaction Hamiltonian. Allowing for the presence of a long-range density-density interaction, and neglecting the overlap of the Bloch functions on the $A$ and $B$ sub-lattices, the interaction Hamiltonian in terms of the unprojected operators $\psi_{k\zeta}$ in Eq. (1) takes the form [27] (cf. [28])

$$
\hat{H}_{\text{int}} = \frac{1}{2} \sum_{k, r, r'} V(|\mathbf{r} - \mathbf{r}'|) : \psi_{k\zeta}^\dagger \psi_{k\zeta} \psi_{k'\zeta}^\dagger \psi_{k'\zeta} : \tag{6}
$$

with only $g_{00}, g_{zz},$ and $g_{\perp\perp} = g_{xx} = g_{yy} = g_{yy}$ non-zero in the second term. Projecting onto the upper bands reproduces the $U(2) \times U(2)$ symmetric form of the interaction Eq. (5). The interaction functions in Eq. (5) can then be expressed in terms of the interaction constants $g_{zz}, g_{\perp\perp}, g_{00},$ and the function $V(q)$ combined with matrix elements of the eigenspinors $\chi_{k\zeta}$ [28].

One may estimate the interaction parameters here in terms of the matrix elements of the screened Coulomb interaction $V(q)$. To the lowest order one finds

$$
g_{zz} \sim V(|\mathbf{K} - \mathbf{K}'|), \quad g_{\perp\perp} \sim V(|\mathbf{b}|), \quad g_{00} \sim \frac{g_{zz} \Delta}{v_F |\mathbf{K} - \mathbf{K}'|}. \tag{7}
$$

Like the Dirac gap $\Delta$, the constant $g_{00}$ is non-zero only if the lattice $C_6$ symmetry is broken. From the hierarchy of scales $q_{TF}, k_F \ll |\mathbf{K} - \mathbf{K}'|$ we then have

$$
V(q \sim k_F) \gg g_{zz} > g_{\perp\perp} > g_{00} \tag{8}
$$

(9)

(9)
For compactness, it is useful to define the arrays \( \hat{X}^\nu, \rho^\nu, f^\nu \), with multi-index \( \mu = (\alpha, \beta) \):

\[
\hat{X}^{\alpha \beta} = \hat{\rho}^{\alpha \beta}; \quad \rho^{00} = n, \quad \rho^{0i} = s_i, \quad \rho^{i0} = Y_i, \quad \rho^{ij} = M^i_j;
\]

\[
f^{\mu 0} = f_{a0}, \quad f^{0i} = f_{si}, \quad f^{i=1,2} = f_{m0}, \quad f^{3i} = f_{mz}.
\]

Here we have introduced the Landau functions \( f_i(\mathbf{p} \cdot \mathbf{p}') \) for each channel. This allows us to write the Landau quasi-particle energy matrix as

\[
\hat{\varepsilon}(\mathbf{p}, r) = \xi(\mathbf{k}) \hat{X}^{00} + \varphi(\mathbf{r}) \hat{X}^{\mu} + \sum_{\mu'} \sum_{\mu} \hat{X}^{\mu} f^{\mu}(\mathbf{p} \cdot \mathbf{p}') \hat{\rho}^{\mu}(\mathbf{r}, r'),
\]

where \( \xi(\mathbf{k}) \) is the free particle excitation energy and \( \varphi(\mathbf{r}) \) is a generalized potential conjugate to \( \rho^\mu \) (for the density channel this includes the self-consistent Vlasov field).

We may obtain linearized equations of motion for each of the collective coordinates from the Landau-Silin kinetic equation

\[
\frac{\partial \hat{\rho}^\mu}{\partial t} + \frac{1}{2} \left( \frac{\partial \xi}{\partial \mathbf{r}} \cdot \frac{\partial \hat{\rho}^\mu}{\partial \mathbf{r}} \right) - \frac{1}{2} \left( \frac{\partial \xi}{\partial \mathbf{r}} \cdot \frac{\partial \hat{\rho}^\mu}{\partial \mathbf{p}} \right) + \delta \hat{\rho}^\mu(\mathbf{k}, \mathbf{r}) = \hat{I}[\hat{\rho}^\mu] (11)
\]

where \( \hat{I}[\hat{\rho}^\mu] \) is the collision integral. We introduce the linearized deviation \( \hat{\rho}^\mu(\mathbf{k}, \mathbf{r}) = n_F(\hat{\varepsilon}(\mathbf{k})) + \delta \hat{\rho}^\mu(\mathbf{k}, \mathbf{r}) \), where we have defined the self-consistently determined local equilibrium energy \( \hat{\varepsilon}(\mathbf{k}) = \xi(n_F(\hat{\varepsilon}(\mathbf{k}))) \) via Eq. (10). Expanding Eq. (11) to linear order in \( \delta \rho^\mu \), one may take traces of the equation multiplied by the matrices \( \hat{X}^\mu \), as in Eq. (9), to obtain

\[
\frac{\partial \delta \rho^\mu (\mathbf{k}, \mathbf{r})}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \delta \rho^\mu (\mathbf{k}, \mathbf{r}) + \frac{\partial n}{\partial \xi} \mathbf{v} \cdot \mathbf{F}^\mu = \frac{1}{G_s G_v^2} \text{Tr} \hat{X}^\mu [\hat{I}][\hat{\rho}] (12)
\]

with generalized forces \( \mathbf{F}^\mu = -\nabla \varphi^\mu \), linearized deviations in each channel

\[
\delta \rho^\mu (\mathbf{r}, \mathbf{p}) = \rho^\mu (\mathbf{r}, \mathbf{p}) - n_F(\hat{\varepsilon}(\mathbf{k})) \delta \rho^0 (0, 0) (13)
\]

and deviation from local equilibrium \( \delta \rho^0 (0, 0) \)

\[
\delta \rho^\mu (\mathbf{k}, \mathbf{r}) = \delta \rho^\mu (\mathbf{k}, \mathbf{r}) - \frac{\partial n}{\partial \xi} \mathbf{v} F^\mu = \frac{1}{G_s G_v^2} \sum_{\mathbf{p}} f^\mu (\mathbf{k} \cdot \mathbf{p}) \delta \rho^\mu (\mathbf{p}, \mathbf{r}) (14)
\]

Note that while the Berry connection contributes to the full kinetic equation \( [30] \), it does not enter the linearized equations of motion as long as we are interested in long-wavelength responses \( [31] \).

At low temperatures \( T \ll T_F \) the derivative of the Fermi function is sharply peaked at the Fermi level, pinning energies to the Fermi surface. We therefore reparametrize the linearized deviations and Fermi liquid functions

\[
\delta \rho^\mu (\mathbf{p}, \mathbf{r}) \equiv - \frac{\partial n}{\partial \xi} \mathbf{v} \rho^\mu (\phi, \mathbf{r}) (15)
\]

in terms of the angular coordinate \( \phi \mod 2\pi \) of \( \mathbf{p} \) on the Fermi surface.

We may now consider the collisionless limit of the kinetic equation Eq. (12), to determine whether undamped zero-sound modes exist. The charge-channel in this case gives rise to the usual zero-temperature 2D plasmon mode, which has been thoroughly studied \( [11, 32–34] \), and thus we will focus here solely on the charge-neutral modes. Using the parametrization along with the real-space and time Fourier transforms in Eq. (12), we find for the collisionless (\( \hat{I}[\hat{\rho}] \to 0 \)) limit:

\[
\omega \rho^\mu (\phi) - \nu_F q \cos \phi [\rho^\mu (\phi) + \frac{G_s G_v^2}{\nu_F} \oint d\varphi f^\mu (\phi - \varphi) \rho^\mu (\varphi)] = \nu_F \mathbf{F}^\mu \cos \phi (16)
\]

where \( \nu_F \) is the Fermi velocity, \( \nu_F \) the Fermi momentum, and \( \phi \) and \( \chi \) are the angles which the vectors \( \mathbf{p} \) and \( \mathbf{F}^\mu \) respectively make with \( \mathbf{q} \).

We may estimate the Landau-Fermi liquid functions \( f^\mu (\mathbf{p} \cdot \mathbf{p}') \) within the Hartree-Fock approximation \( [28] \). Due to the symmetrization of the interaction Hamiltonian one finds

\[
f^\mu (\mathbf{p} \cdot \mathbf{p}') \approx 2 \delta^\mu_{\mathbf{p} \cdot \mathbf{p}', \mathbf{q} \to 0} (17)
\]

Given the hierarchy of energy scales Eq. (8), the leading contribution to the Fermi-Liquid interaction functions comes from the long ranged-function \( V(q) \),

\[
f^\mu (\theta) \approx -\frac{1}{2} V \left[ k_F \sin \frac{\theta}{2} \right] \times \left[ \cos^2 \frac{\theta}{2} + \frac{\Delta^2}{(\Delta + E_F)^2} \sin^2 \frac{\theta}{2} \right] (18)
\]

where \( \theta = \phi - \varphi' \) is the angle between \( \mathbf{p} \) and \( \mathbf{p}' \). We note that \( f^\mu (\theta) < 0 \) at all \( \theta \) for all neutral modes identified in Eq. (9). Accounting for the smaller interaction constants identified in Eq. (7) does not change this conclusion. At \( F^\mu = 0 \), we may bring Eq. (16) to the form

\[
\tilde{\tau}^\mu (\phi) = \frac{G_s G_v^2}{\nu_F} \int d\varphi f^\mu (\phi - \varphi) \tilde{\tau}^\mu (\varphi) s / (s - \cos \phi) (19)
\]

with \( s \equiv \omega / (\nu_F q) \) and \( \tilde{\tau}^\mu = (s - \cos \phi) \rho^\mu \). Following the arguments of \( [33] \) we do not expect real-valued solutions with \( s > 1 \) for \( f^\mu < 0 \) (one may show that for constant interactions there are no solutions at all when \( -1/2 < F^\mu_0 < 0 \)). Therefore we infer that all neutral modes are overdamped \( [35] \), albeit this does not exclude the possibility of a non-trivial response in the time domain \( [30] \).
The absence of propagating neutral modes leads us to conclude that at a finite temperature $T$ spreading of an initially-localized perturbation in these channels is ultimately controlled by diffusion. In fact, as we will see, neutral first sound modes are not supported and the finite temperature behavior in these channels is generically diffusive. Next, we evaluate the corresponding diffusion coefficients. For that, we need to find transport scattering times $\tau^\mu_{tr}$. These are defined by the linearized collision integral in Eq. (12). For temperatures below the Bloch-Gruneisen temperature $T \ll T_{BG}$, electron-phonon scattering can be neglected as it will contribute at order $T^4$ [24]. Thus in a clean system, at low temperatures, the collision integral will be dominated by electron-electron collision which will be seen to contribute at order $T^2$.

The evaluation of the linearized collision integrals is greatly simplified by a choice of basis for each channel in which the linearized deviation of the density matrix is diagonal. Because of the SU(2) spin and $U(1)$ valley symmetries we need only consider the kinetic equations for $s^\pm, Y_s, M_s^z$, and for $Y_z, M_z^+$ we therefore consider the collision integral in $\sigma_z\tau_z$ and $\sigma_z\tau_x$ bases, respectively. In each of the two, the linearized collision-integral matrix is diagonal and can be written in a familiar form

$$I[p_i, \alpha] = -\frac{1}{T} \sum_{\beta,\gamma} \sum_{J} (2\pi)^2 \delta(\sum J' P_J)$$

$$\times 2\pi \delta(\sum J) n_i n_j (1-n_i)(1-n_j) \times W_{ij;\alpha;\gamma;\beta}(\bar{\nu}_{i\alpha} + \bar{\nu}_{j\beta} - \bar{\nu}_{i\gamma} - \bar{\nu}_{j\delta}) \quad (20)$$

in terms of the reparametrization $\bar{\nu}_{i\alpha}(p_i, r) \equiv -\langle \partial n/\partial \epsilon \rangle \bar{\nu}_{i\alpha}(p_i)$ where $\bar{\nu}_{i\alpha} = \bar{\nu}_{i0}(p_i)$ is the $\alpha = \sigma, \zeta$ component of the diagonal deviation, defined analogously to Eq. (14), in the chosen basis. Here we have used the short-hand $\sum J h_J$ to denote sums of the form $h_i + h_j - h_i - h_j$, $n_i = n_F(p_i)$ is the Fermi function at momentum $p_i$, and $W$ is the square of the amplitude for two particles in states $\alpha, \beta$ to scatter into states $\gamma, \delta$, which may be written in terms of the two particle $t$ matrix as $W_{ij;\alpha;\beta;\gamma;\delta} = |\langle \alpha \beta | t | i \gamma j \delta \rangle|^2$.

The trace operation on the right-hand of Eq. (12) is then simply the weighted sum $\text{tr}(X^\mu) = \sum_{\alpha} \lambda^\alpha_{\mu} T_{\alpha}$, where $\lambda^\alpha_{\mu}$ is the eigenvalue corresponding to eigenvector $|\alpha\rangle$ of matrix $X^\mu$. Upon performing the trace the collision integral for each channel can be separated into two parts: those involving scattering of particles with the same quantum numbers in that channel and those involving scattering of particles with different quantum numbers (e.g. same spin or opposite spin respectively for the case of the spin mode, cf. Eq. (22)). The former has the same structure as the collision integral for the charge channel and is known in 2D to give rise to a transport scattering time which goes as $T^4$ for low temperatures [25]. The latter term on the other hand will be seen to scale as $T^2 \ln T$ and comprises the dominant contribution to relaxation of currents in each channel,

$$I[p_i] = -\frac{1}{T} \sum_{p,p',p'} (2\pi)^2 \delta(\sum J' P_J)$$

$$\times 2\pi \delta(\sum J) n_i n_j (1-n_i)(1-n_j) \times W_{ij} \left( \bar{\nu}_{i\mu} - \bar{\nu}_{j\mu} - \bar{\nu}_{i\mu} + \bar{\nu}_{j\mu} \right) \quad (21)$$

with scattering probabilities

$$W_{ij;\alpha;\beta;\gamma;\delta} = 2(w^D_{ij;\alpha;\beta;\gamma;\delta} + w^D_{ij;\alpha;\beta;\gamma;\delta}) \quad (22)$$

where $W^D_{ij;\alpha;\beta;\gamma;\delta}$ is the scattering probability for two distinguishable particles with spins $\sigma, \sigma'$ and valley indices $\zeta, \zeta'$ and the superscript $x$ indicates the choice of the $\tau^x$ eigen-basis in valley space. These may in turn be written in terms of components of the $t$-matrix. In the first Born approximation, this can be expressed in terms of the functions in Eq. (3) as

$$W_{ij;\alpha;\beta;\gamma;\delta} = 4 |g_d - g_s + g_vz - g_mz|^2$$

$$W_{ij;\alpha;\beta;\gamma;\delta} = 4 |g_d - g_s - g_vz + g_mz|^2$$

$$W_{ij;\alpha;\beta;\gamma;\delta} = 4 |g_d - g_s - g_vz - g_mz|^2$$

$$W_{ij;\alpha;\beta;\gamma;\delta} = 4 |g_d - g_s + g_vz + g_mz|^2$$

$$W_{ij;\alpha;\beta;\gamma;\delta} = 4 |g_d - g_s - g_vz + g_mz|^2$$

To the lowest order in $T/T_F$, the Fermi functions restrict the summation over momenta in Eq. (21) to the states close to the Fermi surface. One may then transform [29] the summation to integration over the energies $\epsilon, \epsilon'$ of the incoming particles, the energy transferred in a collision $\omega$, and the scattering angle $\theta_{sc}$. Due to the constraints on $\epsilon, \epsilon'$ and conservation laws, the incoming particles collide almost head-on, or their momenta are almost collinear to each other. To evaluate the transport relaxation times $\tau^\mu_{tr}$, we use $\nu^\mu(\phi) \propto \cos \phi$ to arrive at

$$1\tau^\mu_{tr} = \frac{4v_F T^2}{v_F^2} \int d\Sigma \int_0^{\pi - \theta_{sc}} \frac{d\theta_{sc}}{\sin \theta_{sc}} (1 - \cos \theta_{sc})$$

$$\times \left( W^\mu_{\text{collinear}}(\theta_{sc}) + W^\mu_{\text{head-on}}(\theta_{sc}) \right) \quad (24)$$
where \( \theta_e \sim T/E_F \) cuts off the logarithmic divergence \([40]\) at \( \theta_{sc} = \pi \), and we have defined the dimensionless energy integration measure

\[
ds = \frac{1}{4\pi^2} d\Omega d\epsilon_{q} (1 - n_{\epsilon})(1 - n_{\epsilon'}) \tag{25}
\]
in terms of dimensionless variables \( u = \epsilon/T, \ w = \omega/T, \ \epsilon_i, \epsilon_f = \epsilon \pm \omega/2, \ \epsilon_j, \epsilon_{j'} = \epsilon' \mp \omega/2 \). In the case where \( q_{TF} \ll 2k_F \) there is also a logarithmic contribution due to collinear scattering by the Coulomb potential, which is cut off by the Thomas-Fermi wavenumber \( q_{TF} \). In graphene, unlike more conventional Fermi liquids, \( q_{TF} = G_s G_{V0} k_F/v_F \) can be greater than \( 2k_F \) due to the degeneracy factors and strong effective coupling \( \alpha = e^2/\kappa v_F \). In this complementary regime, the collinear scattering logarithm is absent and the dominant matrix element for backscattering will be screened. We present here explicit expressions for the former case, \( q_{TF} \ll 2k_F \), but one may straightforwardly perform the analogous calculations in the latter case and the qualitative results remain the same. Which regime is realized experimentally will depend on the background dielectric constant and doping.

Thus, performing the integration in Eq. \([24]\), with \( q_{TF} \ll 2k_F \), we find

\[
\frac{1}{\tau''_m(T)} = \frac{4\pi}{3} (E_F + \Delta) \alpha^2 T^2 \frac{(1 + E_F/\Delta)^2}{1 + E_F/2\Delta} \times \left[ \left( \frac{\Delta}{\Delta + E_F} \right)^4 \ln \frac{\sqrt{E_F(E_F + 2\Delta)}}{T} \right. \\
+ \ln \frac{\sqrt{E_F(E_F + 2\Delta)}}{\max(v_F q_{TF}, T)} \right] \tag{26}
\]
in the leading logarithmic approximation, applicable at \( T \ll E_F \Delta \). In the evaluation of the scattering probabilities entering Eq. \([24]\), we used the Born approximation and the Thomas-Fermi screened Coulomb potential, cf. Eqs. \([5]\) and \([23]\). The first logarithmic contribution in Eq. \(26\) comes from the backscattering amplitude \( (\theta_{sc} = \pi) \), as long as it remains finite. Notably, backscattering is suppressed in graphene at \( \Delta \ll E_F \) due to the presence of Berry phase in the electron eigenfunctions \([23, 24]\). Therefore, in the relativistic limit \( \Delta \to 0 \), Eq. \(26\) is replaced by

\[
\frac{1}{\tau''_m(T)} = \frac{4T^2}{3\pi E_F} \left\{ \frac{N_\mu}{g_\mu} \left( \frac{E_F}{v_F} \right)^2 \ln \frac{E_F}{T} \right. \\
+ (2\pi\alpha)^2 E_F \ln \frac{E_F}{\max(v_F q_{TF}, T)} \right\} \tag{27}
\]

Here \( g_\mu = g_{\perp,\perp} \) for \( \mu = v, m, v \parallel, m \parallel \), and \( g_\mu = g_{zz} \) for \( \mu = s \), and the numerical factor \( N_\mu = 8 \) for \( \mu = v, m, v \parallel, m \parallel \), and \( N_\mu = 10 \) for \( \mu = v, m \parallel \), and \( N_\mu = 4 \) for \( \mu = s \). We note that the logarithmic terms associated with the backscattering in Eq. \(27\) generically are smaller than those present at \( \Delta \sim E_F \), cf. Eqs. \([7, 5]\) and \([26]\). The diffusion constant for each of the channels is \( D_\mu = v_F^2 \tau''_m/2 \) (consistent with the Born approximation, here we dispensed with the Fermi liquid correction \([6, 29]\) to the Fermi velocity). The diffusion regime settles in at times \( \tau \gtrsim \tau_m(T) \).

The relaxation rates for the neutral modes can be experimentally probed through non-local resistance measurements using the spin- and valley- Hall effects \([40, 41, 42]\). Currently experimental measurements of spin and valley diffusion below the Bloch-Griessen temperature have obtained diffusion constants corresponding to mean free paths of the order 0.1 \( \mu \). \([42, 43]\). Calculations performed in the limit \( q_{TF} \ll k_F \) give significantly longer mean free paths, indicating either that impurities play the dominant role, or the system is in the regime \( k_F \ll q_{TF} \). Nonetheless, experimental works on electron hydrodynamics suggest that it could be possible to reach the regime where electron-electron effects dominate the response of the neutral modes \([4, 11]\), as considered here. Thus, the predicted relaxation rates and their temperature dependence – should be measurable, either via the same types of experiments as have been previously used to measure spin and valley diffusion, or through other methods \([46]\).

The generalization of Eq. \([24]\) to higher angular harmonics \( m \geq 2 \) of the distribution function \( \nu(\phi) \) is presented in \([28]\). The relaxation rate \( 1/\tau_2(T) \) of the \( m = 2 \) harmonic is similar to \( 1/\tau_m(T) \) of Eqs. \([26]\) and \([27]\) and is likely lower than \( 1/\tau_m(T) \) in the respective limits, as \( 1/\tau_2(T) \) lacks the logarithmic enhancement of the backscattering contribution.

The hierarchy \( \tau_n \lesssim \tau_2 \) for all of the graphene Fermi liquid neutral modes excludes the possibility of a hydrodynamic sound mode, contrary to the case of the density mode in a conventional neutral Fermi liquid (for the density mode, \( 1/\tau_n \equiv 0 \) by translation invariance). Combined with the discussion below Eq. \([19]\), we thus find that both zero and first sound are absent in all neutral channels.

Relaxation of the higher-\( m \) moments of the distribution function can be measured in magnetic focusing experiments \([47]\). With the increase of \( m \), the role of the forward-scattering contribution (which gave rise to the second term in Eqs. \([26]\) and \([27]\)) strengthens. At \( m \gg 2k_F/q_{TF} \) and sufficiently low temperatures the small-angle scattering involving the screened Coulomb potential \( V(0) = 1/(G_s G_{V0}[1 + F_0^2]) \) dominates the relaxation rate \( 1/\tau_m \sim (T^2/v_F^2) \ln m \). The asymptotic large-\( m \) relaxation of spin modes can then be accessed in focusing experiments utilizing spin-polarized leads in a setup similar to that of Ref. \([47]\).

Throughout this work we disregarded the trigonal warping of the electron spectrum in graphene. Warping does not destroy the used \( U(2) \times U(2) \) spin-valley symmetries. The modification of the spectrum, however,
cuts off the logarithmic singularity of backscattering in Eq. \(24\) and introduces anisotropy of the diffusion coefficient due to the dependence of the Fermi velocity on the direction of the electron wavevector. For similar reasons, the calculations presented herein may be extended to twisted bilayer graphene, which also possesses an internal \(U(2) \times U(2)\) symmetry [38] [29], and thus at a generic filling we also expect similar qualitative behavior such as the absence of neutral zero or first sound, and a transport scattering time for the neutral modes which scales as \((W/T)^2 \ln W/T\) where \(W\) is the bandwidth of the nearly-flat bands. As the flat-band limit is approached one must consider both the valence and conduction bands together and modes associated with the inter-band transitions will appear which could exhibit different behaviors given the enlarged symmetry \(U(4) \times U(4)\) for the chiral case) [48] [49] of the flat-band limit.

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**Supplemental Material**

**Appendix A: Symmetry imposed form of the interaction functions**

**A.1. Symmetry Analysis of short ranged interactions**

Ignoring out of plane behavior the point group of gapped graphene is $C_{3v}$. In the absence of spin-orbit coupling, which is generally weak in graphene, the local symmetry is enlarged to $C_{3v} \oplus SU(2)$. In the valley-sublattice basis with spinor

$$
\Psi_k = \left( \begin{array}{c} \psi_{KA}(k) \\ \psi_{KB}(k) \\ \psi_{K'B}(k) \\ -\psi_{K'A}(k) \end{array} \right)
$$

we can define the representative symmetry operators

$$
C_3 : e^{i\frac{2\pi}{3} \hat{S}_3}, \quad \sigma_V : \hat{\Sigma}_2 \hat{\tau}_2, \quad \Theta : \sigma_V K, \quad T : e^{\frac{2\pi i}{\tau}}
$$
describing respectively, $C_3$ rotation, mirror plane, time reversal, and lattice translation. Here $\Sigma$, $\sigma$, and $\tau$ denote sub-lattice, spin, and valley Pauli matrices respectively. Doing so we classify the bilinears of matrices which form invariants under the symmetry group. We note that $s_x$, $s_y$, and $\tau_x$, $\tau_y$ form doublets under $C_3$ and $T$ respectively. We may now tabulate which matrices are even or odd under $\sigma_V$ and $\theta$ in Table S1. Combining all this we find that

| $\sigma_V$ even | $\theta$ even | $\sigma_V$ odd | $\theta$ odd |
|-----------------|-------------|----------------|-------------|
| $\Sigma_y \tau_y$ | $\Sigma_x \tau_x$, $\Sigma_z \tau_z$, $\Sigma_x \tau_z$, $\Sigma_z \tau_x$ |
| $\Sigma_y \tau_x$, $\Sigma_y \tau_z$, $\Sigma_x \tau_y$, $\Sigma_z \tau_y$ |

TABLE S1. Transformation properties of short range interaction vertices, grouped by whether they are even or odd under time reversal $\theta$ and mirror plane $\sigma_V$.

the symmetry allowed short ranged interactions are

$$I^1 \cdot I^2, \, \Sigma^1 \cdot \Sigma^2, \, \tau^1 \cdot \tau^2, \, \Sigma^1 \cdot \Sigma^2 \cdot \tau^1 \cdot \tau^2, \, \Sigma^1 \cdot \Sigma^2 \cdot \tau^1 \cdot \tau^2, \, \tau^1 \cdot \tau^2, \, \Sigma^1 \cdot \Sigma^2 \cdot \tau^1 \cdot \tau^2, \, \Sigma^1 \cdot \tau^2, \, \Sigma^1 \cdot \tau^2, \, \Sigma^1 \cdot \tau^2.$$

(S3)

where we use the superscript $1, 2$ to indicates that the vertex is for particle $1, 2$, respectively. We can estimate the interaction constants for these channels by inserting the Dirac cone ansatz for the creation operator

$$\Psi = u_{KA} \psi_{KA} + u_{KB} \psi_{KB} + u_{K'B} \psi_{K'B} + u_{K'A} \psi_{K'A}$$

(S4)

into the Coulomb interaction. Because of the relations

$$u_{KA} = u_{K'A}^*, \, u_{KB} = u_{K'B}^*,$$

(S5)

the vertices which are odd under time reversal vanish. The remaining vertices are

$$I^1 \cdot I^2, \, \Sigma^1 \cdot \Sigma^1 \cdot \tau^1 \cdot \tau^1, \, \Sigma^1 \cdot \Sigma^1 \cdot \tau^1 \cdot \tau^1, \, \Sigma^1 \cdot \Sigma^1 \cdot \tau^1 \cdot \tau^1, \, \Sigma^1 \cdot \tau^1 \cdot \tau^1,$$

(S6)

to which we assign, respectively, the coupling constants

$$g_{00}, \, g_{z+}, \, g_{z-}, \, g_{zz}, \, \tilde{g}_{00}.$$

(S7)

Neglecting the overlap of the $A$ and $B$ sub-lattice Bloch wavefunctions we can further discard the $g_{z\pm}$, $g_{zz}$ terms we arrive at

$$\hat{H}_{int}^\psi = \frac{1}{2} \sum_{\mathbf{r}, \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') : \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \psi^\dagger(\mathbf{r}') \psi(\mathbf{r}') : + \frac{1}{2} \sum_{\mathbf{r}} \sum_{\sigma} \sum_{\alpha \beta} [g_{0\beta} : \psi^\dagger(\mathbf{r}) \Sigma^\sigma \cdot \tau^\beta \psi(\mathbf{r}) \psi^\dagger(\mathbf{r}') \Sigma^\sigma \tau^\beta \psi(\mathbf{r}') : + \tilde{g}_{00} : \psi^\dagger(\mathbf{r}) \Sigma^z r^z \psi(\mathbf{r}) \psi^\dagger(\mathbf{r}') \psi(\mathbf{r}') : ] \right).$$

(S8)

### A.2. Interaction functions in terms of short ranged interaction constants

Writing the Fermionic annihilation operator

$$\hat{\Psi}_\sigma(\mathbf{r}) = (u_{KA}(\mathbf{r}) \, u_{KB}(\mathbf{r}) \, u_{K'B}(\mathbf{r}) \, -u_{K'A}(\mathbf{r}) \cdot \tilde{\psi}^\sigma(\mathbf{r}))$$

(S9)

the upper band projected operator has the form

$$\tilde{\Psi}_{k\sigma} = \sum_{\mathbf{k}} \chi_{k\sigma} \mathbf{c}_{k\sigma}$$

where

$$\chi_{k+} = \begin{pmatrix} \sqrt{\frac{1}{2}} (1 + \frac{\Delta}{E_k}) \\ \sqrt{\frac{1}{2}} (1 - \frac{\Delta}{E_k}) \end{pmatrix}, \, \chi_{k-} = - \begin{pmatrix} e^{-i\phi_k} \sqrt{\frac{1}{2}} (1 - \frac{\Delta}{E_k}) \\ \sqrt{\frac{1}{2}} (1 + \frac{\Delta}{E_k}) \end{pmatrix}.$$ 

(S10)

By plugging the upper band operator into Eq. (6), the interaction functions in Eq. (5) can then be expressed in terms of the interaction constants $g_{zz}, \, g_{\perp \perp}, \, \tilde{g}_{00}$ and the function $V(\mathbf{q})$ combined with matrix elements of the eigispinors $\chi_{k\sigma}$. Using the shorthand

$$F_{ij';jj'} = F(p_i, p_j; p_{i'}, p_{j'}) \equiv F(p + q/2, p' - q/2; p - q/2, p' + q/2)$$

(S11)
we can write

\[ g_{\mathbf{p}, \mathbf{p}'}_q = \frac{1}{2} \left( V_{\mathbf{q}}^{L_{i'i'}L_{j'j'}} - \frac{1}{4} V_{\mathbf{p}-\mathbf{p}'}^{L_{i'i'}L_{j'j'}} + g_{zz} \left( N_{ii'}N_{jj'} - \frac{1}{4} N_{ij'}N_{j'i'} \right) \right) + \frac{1}{2} g_{\mathbf{00}}^{Q_{ii'jj'}} - \frac{1}{2} g_{\mathbf{11}}^{Q_{ij'j'i'}} \]

\[ g_{\mathbf{p}, \mathbf{p}'}^2 = -\frac{1}{4} \left( \frac{1}{4} V_{\mathbf{p}-\mathbf{p}'}^{L_{i'i'}L_{j'j'}} + \frac{1}{2} g_{zz} N_{ij'}N_{j'i'} + \frac{1}{4} g_{\mathbf{00}}^{Q_{ij'j'i'}} - \frac{1}{2} g_{\mathbf{11}}^{Q_{ij'j'i'}} \right) \]

\[ g_{\mathbf{p}, \mathbf{p}'}^v = \frac{1}{2} \left( \frac{1}{4} V_{\mathbf{p}-\mathbf{p}'}^{L_{i'i'}L_{j'j'}} + \frac{1}{2} g_{zz} N_{ij'}N_{j'i'} + \frac{1}{4} g_{\mathbf{00}}^{Q_{ij'j'i'}} - \frac{1}{2} g_{\mathbf{11}}^{Q_{ij'j'i'}} \right) \]

\[ g_{\mathbf{p}, \mathbf{p}'}^m = -\frac{1}{8} \left( V_{\mathbf{p}-\mathbf{p}'}^{L_{i'i'}L_{j'j'}} + \frac{1}{2} g_{zz} N_{ij'}N_{j'i'} + \frac{1}{2} g_{\mathbf{00}}^{Q_{ij'j'i'}} \right) \]

where

\[ Q_{ii'jj'}^{\mathbf{1}} = L_{ij'}^{\mathbf{1} A} + N_{ij'}^{\mathbf{1} A}, \quad Q_{ii'jj'}^{\mathbf{0}} = L_{ij'}^{\mathbf{0} A} + N_{ij'}^{\mathbf{0} A} \]

\[ L_{i'i'}^{\mathbf{1}} = \chi_{kk}^{\mathbf{1} A} + \chi_{kk}^{\mathbf{1} B}, \quad N_{i'i'}^{\mathbf{1}} = \chi_{kk}^{\mathbf{1} A} + \chi_{kk}^{\mathbf{1} B}, \quad L_{i'i'}^{\mathbf{0}} = \chi_{kk}^{\mathbf{0} A} + \chi_{kk}^{\mathbf{0} B}, \quad N_{i'i'}^{\mathbf{0}} = \chi_{kk}^{\mathbf{0} A} + \chi_{kk}^{\mathbf{0} B} \]

Of particular note are the limits of the coherence factors \( L, N, L, N \),

\[ L_{i'i'} = N_{i'i'} = 1, L_{i'i'} = N_{i'i'} = \frac{\Delta}{E_k} \]

where the latter relation is responsible for the suppression of backscattering in the relativistic, \( \Delta \to 0 \) limit.

**Appendix B: Low energy interactions from Coulomb**

We can obtain estimates for the interactions strengths occurring in Eq. 6 by looking at the matrix elements of the Coulomb interaction. In particular we aim to obtain the lowest harmonic of the Coulomb interaction contributing to each interaction strength above.

We begin by writing the interaction Hamiltonian for the physical electrons in the bands nearest the Fermi surface.

\[ \hat{H}_{\text{int}} = \frac{1}{2} \sum_{\lambda, \lambda', \sigma} \int d\mathbf{r} d\mathbf{r}' V(|\mathbf{r} - \mathbf{r}'|) : c_{\sigma \lambda}^{\dagger}(\mathbf{r}) c_{\sigma \lambda}(\mathbf{r}) c_{\sigma' \lambda'}^{\dagger}(\mathbf{r}') c_{\sigma' \lambda'}(\mathbf{r}') : \]

where \( \sigma \) denotes spin, \( \lambda = \pm 1 \) denotes the sublattice A and B states respectively and \( V(|\mathbf{R}|) \) is the Coulomb interaction. The operators may be expanded in terms of Bloch states

\[ c_{\sigma \lambda}(\mathbf{r}) = \int_{\text{BZ}} \frac{d\mathbf{p}}{(2\pi)^3} \psi_{\lambda \mathbf{p}}(\mathbf{r}) c_{\sigma \lambda}(\mathbf{p}) \]

with wavefunction

\[ \psi_{\lambda \mathbf{p}}(\mathbf{r}) = e^{i\mathbf{p} \cdot \mathbf{r}} u_{\lambda \mathbf{p}}(\mathbf{r}). \]

Using the Bloch basis and the Fourier transform of the Coulomb interaction we can rewrite Eq. \( \text{(S1)} \)

\[ \hat{H}_{\text{int}} = \frac{1}{2} \sum_{\lambda, \lambda', \sigma, \sigma'} \int_{\text{BZ}} \left[ \prod_i p_i (2\pi)^3 \right] \int_{\mathbb{R}^2} \frac{d\mathbf{q}}{(2\pi)^2} : c_{\sigma \lambda}(\mathbf{p}_1) c_{\sigma \lambda}(\mathbf{p}_1') c_{\sigma' \lambda'}(\mathbf{p}_2) c_{\sigma' \lambda'}(\mathbf{p}_2') : \]

\[ \times \int_{\mathbb{R}^2} d\mathbf{r} d\mathbf{r}' e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \psi_{\lambda \mathbf{p}_1}(\mathbf{r}) \psi_{\lambda \mathbf{p}_1}(\mathbf{r}) \psi_{\lambda' \mathbf{p}_2}(\mathbf{r}') \psi_{\lambda' \mathbf{p}_2}(\mathbf{r}'). \]
We can rewrite the integral of \( q \) as integral over the Brillouin zone and a sum over all reciprocal lattice vectors.

\[
\int_{\mathbb{R}^2} \frac{dq}{(2\pi)^2} f(q) = \sum_G \int_{BZ} \frac{dq}{(2\pi)^2} f(q + G)
\]  
(S5)

where we have introduced the shorthand

\[
\sum_G f(G) \equiv \sum_{n_1,n_2} f(n_1b_1 + n_2b_2)
\]  
(S6)

with \( b_i \) the primitive reciprocal lattice vectors. Similarly we can rewrite the integration over all space in terms of integration over the unit cell

\[
\int_{\mathbb{R}^2} df(r) = \sum_R \int_{uc} df(r + R)
\]  
(S7)

where we have analogously defined the shorthand

\[
\sum_R f(R) \equiv \sum_{n_1,n_2} f(n_1a_1 + n_2a_2)
\]  
(S8)

with \( a_i \) the primitive translation vectors of the lattice. Using both these relations, the interaction hamiltonian becomes

\[
\hat{H}_{int} = \frac{1}{2} \sum_{\lambda\lambda',\sigma\sigma'} \sum_G \int_{BZ} \left[ \prod_i \frac{P_i}{(2\pi)^2} \right] \int_{BZ} \frac{dq}{(2\pi)^2} \hat{c}^\dagger_{\sigma\lambda}(p_1) c_{\sigma\lambda}(p'_1) \hat{c}^\dagger_{\sigma'\lambda'}(p_2) c_{\sigma'\lambda'}(p'_2) : \\
\times \sum_{R,R'} e^{i(q \cdot R - R') - i(p_1 \cdot p'_1 - p_2 \cdot p'_2)} R' \int_{BZ} dr' e^{i(q + G \cdot r - r') \psi^\ast_{\lambda p_1}(r) \psi_{\lambda' p_2}(r') \psi^\ast_{\lambda' p_2}(r') \psi_{\lambda p_1}(r')}
\]  
(S9)

where we have used \( \exp(i\mathbf{G} \cdot \mathbf{R}) = 1 \), and the periodicity properties of the Bloch wavefunctions. The sums over \( R, R' \) can be performed to obtained

\[
(2\pi)^2 \delta_{BZ}(p_1 - p'_1 - q)(2\pi)^2 \delta_{BZ}(p_2 - p'_2 + q)
\]  
(S10)

where we \( \delta_{BZ} \) is to be resolved mode the Brillouin zone, as the integrand is periodic in all \( p_i \). We then obtain

\[
\hat{H}_{int} = \frac{1}{2} \sum_{\lambda\lambda',\sigma\sigma'} \sum_G \int_{BZ} \frac{dp}{(2\pi)^2} \frac{dp'}{(2\pi)^2} \int_{BZ} \frac{dq}{(2\pi)^2} \hat{c}^\dagger_{\sigma\lambda}(p + q) c_{\sigma\lambda}(p) \hat{c}^\dagger_{\sigma'\lambda'}(p' - q) c_{\sigma'\lambda'}(p') : \\
\times V(|q + G|) \int_{BZ} dr dr' e^{i(q + G \cdot r - r') \psi^\ast_{\lambda p_1+q}(r) \psi_{\lambda p_1}(r) \psi^\ast_{\lambda' p_2}(r') \psi_{\lambda' p_2}(r')}
\]  
(S11)

and in terms of the Bloch factors

\[
\hat{H}_{int} = \frac{1}{2} \sum_{\lambda\lambda',\sigma\sigma'} \sum_G \int_{BZ} \frac{dp}{(2\pi)^2} \frac{dp'}{(2\pi)^2} \int_{BZ} \frac{dq}{(2\pi)^2} \hat{c}^\dagger_{\sigma\lambda}(p + q) c_{\sigma\lambda}(p) \hat{c}^\dagger_{\sigma'\lambda'}(p' - q) c_{\sigma'\lambda'}(p') : \\
\times V(|q + G|) \int_{BZ} dr dr' e^{iG \cdot (r - r') \psi^\ast_{\lambda p_1+q}(r) u_{\lambda p_1}(r) \psi^\ast_{\lambda' p_2}(r') u_{\lambda' p_2}(r')}
\]  
(S12)

We may thus write the interaction as

\[
\hat{H}_{int} = \frac{1}{2} \sum_{\lambda\lambda',\sigma\sigma'} \int_{BZ} \frac{dp}{(2\pi)^2} \frac{dp'}{(2\pi)^2} \int_{BZ} \frac{dq}{(2\pi)^2} g_{\lambda\lambda'}(p,p',q) : \hat{c}^\dagger_{\sigma\lambda}(p + q) c_{\sigma\lambda}(p) \hat{c}^\dagger_{\sigma'\lambda'}(p' - q) c_{\sigma'\lambda'}(p') :
\]  
(S13)

where we have defined

\[
g_{\lambda\lambda'}(p,p',q) \equiv V(|q + G|) \int_{BZ} dr dr' e^{iG \cdot (r - r') \psi^\ast_{\lambda p_1+q}(r) u_{\lambda p_1}(r) \psi^\ast_{\lambda' p_2}(r') u_{\lambda' p_2}(r')}.
\]  
(S14)
Now we make the Dirac cone approximation. We restrict the momenta of each of the electron operators to be in the vicinity of the $\mathbf{K}$ or $\mathbf{K}'$ point. Each of the Bloch factors will be evaluated at the corresponding point. Introducing the notation
\[ \int \frac{d\mathbf{p}}{(2\pi)^2} \ldots \] (S15)
for integration over momenta $\mathbf{p} \ll |\mathbf{K} - \mathbf{K}'|$ and the operators
\[ c_{\sigma\lambda\zeta}(\mathbf{p}) = c_{\sigma\lambda}(\zeta\mathbf{K} + \mathbf{p}) \] (S16)
with $\zeta = \pm 1$ indexing the valley degree of freedom, we may approximate the interaction Hamiltonian as
\[ \hat{H}_{\text{int}} \approx \frac{1}{2} \sum_{\lambda\lambda',\sigma\sigma',\zeta\zeta',\zeta} \int \frac{d\mathbf{p}}{(2\pi)^2} \frac{d\mathbf{p}'}{(2\pi)^2} \int \frac{dq}{(2\pi)^2} g_{\lambda\lambda';\zeta\zeta'}(q) : c_{\sigma\lambda\zeta}^\dagger(\mathbf{p} + \mathbf{q}) c_{\sigma\lambda\zeta}^\dagger(\mathbf{p}' - \mathbf{q}) c_{\sigma'\lambda'\zeta'}(\mathbf{p}') c_{\sigma'\lambda'\zeta'}(\mathbf{p}') : \] (S17)
with
\[ g_{\lambda\lambda';\zeta\zeta'}(q) = \sum_{\mathbf{G}} V(|\mathbf{q} + \mathbf{G} + (\zeta_1 - \zeta_1')\mathbf{K}|) \delta_{\zeta_1 - \zeta_1'} \delta_{\zeta_2 - \zeta_2} \int_{\text{uc}} d\mathbf{r} d\mathbf{r}' e^{i\mathbf{G} \cdot (\mathbf{r} - \mathbf{r}')} u_{\lambda\zeta}^*(\mathbf{r}) u_{\lambda\zeta'}(\mathbf{r}) u_{\lambda\zeta'}(\mathbf{r}') u_{\lambda\zeta}^*(\mathbf{r}'). \] (S18)
There are two types of terms, $\zeta_1 = \zeta_1'$ and $\zeta_2 = \zeta_2'$. We therefore write
\[ \hat{H}_{\text{int}} \approx \frac{1}{2} \sum_{\lambda\lambda',\sigma\sigma',\zeta'} \int \frac{d\mathbf{p}}{(2\pi)^2} \frac{d\mathbf{p}'}{(2\pi)^2} \int \frac{dq}{(2\pi)^2} g_{\lambda\lambda';\zeta\zeta'}^{\text{intra}}(q) : c_{\sigma\lambda\zeta}^\dagger(\mathbf{p} + \mathbf{q}) c_{\sigma\lambda\zeta}^\dagger(\mathbf{p}') : \]
\[ + \frac{1}{2} \sum_{\lambda\lambda',\sigma\sigma'} \int \frac{d\mathbf{p}}{(2\pi)^2} \frac{d\mathbf{p}'}{(2\pi)^2} \int \frac{dq}{(2\pi)^2} g_{\lambda\lambda';\zeta\zeta'}^{\text{inter}}(q) : c_{\sigma\lambda\zeta}^\dagger(\mathbf{p} + \mathbf{q}) c_{\sigma\lambda,-\zeta}(\mathbf{p}) c_{\sigma\lambda,-\zeta'}(\mathbf{p}') : \] (S19)
with
\[ g_{\lambda\lambda';\zeta\zeta'}^{\text{intra}}(q) = \sum_{\mathbf{G}} V(|\mathbf{q} + \mathbf{G}|) \int_{\text{uc}} d\mathbf{r} d\mathbf{r}' e^{i\mathbf{G} \cdot (\mathbf{r} - \mathbf{r}')} |u_{\lambda\zeta}(\mathbf{r})|^2 |u_{\lambda\zeta}(\mathbf{r}')|^2 \] (S20)
where we have used that fact that $u_- = u_+^*$ $\Rightarrow |u_+|^2 = |u_-|^2$ and
\[ g_{\lambda\lambda';\zeta\zeta'}^{\text{inter}}(q) = \sum_{\mathbf{G}} V(|\mathbf{q} + \mathbf{G} + 2\zeta\mathbf{K}|) \int_{\text{uc}} d\mathbf{r} d\mathbf{r}' e^{i\mathbf{G} \cdot (\mathbf{r} - \mathbf{r}')} u_{\lambda\zeta}(\mathbf{r}) u_{\lambda,-\zeta}(\mathbf{r}) u_{\lambda,-\zeta'}(\mathbf{r}') u_{\lambda\zeta}(\mathbf{r}'). \] (S21)
We then see that $g_{\text{intra}}$ corresponds to $g_{00}, g_{00}, g_{zz}$ and $g_{\text{inter}}$ to $g_{11}$. From this we can immediately deduce that the smallest harmonic of the potential contributing to $g_{11}$ is $V(|\mathbf{K} - \mathbf{K}'|)$. For $g_{\text{intra}}$ we rewrite
\[ |u_\lambda|^2 = \frac{1}{2} \left( |u_A|^2 + |u_B|^2 \right) + \frac{\lambda}{2} \left( |u_A|^2 - |u_B|^2 \right) \] (S22)
giving
\[ g_{\lambda\lambda'}^{\text{intra}}(q) = \frac{1}{4} \sum_{\mathbf{G}} V(|\mathbf{q} + \mathbf{G}|) \left\{ \left| \int_{\text{uc}} d\mathbf{r} d\mathbf{r}' e^{i\mathbf{G} \cdot \mathbf{r}'} \left( |u_{A+}(\mathbf{r})|^2 + |u_{B+}(\mathbf{r})|^2 \right) \right|^2 \right. \]
\[ + \lambda' \left| \int_{\text{uc}} d\mathbf{r} d\mathbf{r}' e^{i\mathbf{G} \cdot \mathbf{r}'} \left( |u_{A+}(\mathbf{r})|^2 - |u_{B+}(\mathbf{r})|^2 \right) \right|^2 \]
\[ + \lambda \left| \int_{\text{uc}} d\mathbf{r} d\mathbf{r}' e^{i\mathbf{G} \cdot (\mathbf{r} - \mathbf{r}') \left( |u_{A+}(\mathbf{r})|^2 + |u_{B+}(\mathbf{r}')|^2 \right) \left( |u_{A+}(\mathbf{r})|^2 - |u_{B+}(\mathbf{r})|^2 \right) \right. \right. \]
\[ + \lambda' \left| \int_{\text{uc}} d\mathbf{r} d\mathbf{r}' e^{i\mathbf{G} \cdot (\mathbf{r} - \mathbf{r}') \left( |u_{A+}(\mathbf{r})|^2 + |u_{B+}(\mathbf{r}')|^2 \right) \left( |u_{A+}(\mathbf{r})|^2 - |u_{B+}(\mathbf{r})|^2 \right) \right. \]. (S23)
The first and second terms can be identified with $g_{00}$ and $g_{zz}$ respectively, while the third and fourth are $g_{00}$. It is clear that in the presence of sublattice symmetry the third and fourth terms must vanish as they are odd under
the exchange of $A$ and $B$. Using the fact that the Bloch factors are normalized to 1 we see that lowest order contribution to $g_{00}$ is

$$g_{00} = V(q) + \text{higher harmonics}$$

(S24)

while the $\mathbf{G} = 0$ contributions for the other terms all vanish since

$$\int_{uc} d\mathbf{r} \left( |u_{A+}(\mathbf{r})|^2 - |u_{B+}(\mathbf{r})|^2 \right) = 1 - 1 = 0.$$  

(S25)

The lowest contributing harmonics are

$$g_{00} \approx V(q)$$

$$g_{\perp \perp} \approx \frac{1}{2} V \left( \frac{4\pi}{3a} \right) \sum_i \left( \int_{uc} d\mathbf{r} \cos(\mathbf{b}_i \cdot \mathbf{r}) \left( |u_{A+}(\mathbf{r})|^2 + |u_{B+}(\mathbf{r})|^2 \right) \right)^2$$

$$+ \left( \int_{uc} d\mathbf{r} \sin(\mathbf{b}_i \cdot \mathbf{r}) \left( |u_{A+}(\mathbf{r})|^2 + |u_{B+}(\mathbf{r})|^2 \right) \right)^2$$

$$g_{zz} \approx \frac{1}{2} V \left( \frac{4\pi}{3a} \right) \sum_{i=1,2,3} \left( \int_{uc} d\mathbf{r} \cos(\mathbf{b}_i \cdot \mathbf{r}) \left( |u_{A+}(\mathbf{r})|^2 - |u_{B+}(\mathbf{r})|^2 \right) \right)^2$$

$$+ \left( \int_{uc} d\mathbf{r} \sin(\mathbf{b}_i \cdot \mathbf{r}) \left( |u_{A+}(\mathbf{r})|^2 - |u_{B+}(\mathbf{r})|^2 \right) \right)^2$$

(S26)

$$\tilde{g}_{00} \approx \sum_{i=1,2,3} V \left( \frac{4\pi}{3a} \right) \int_{uc} d\mathbf{r} d\mathbf{r} \cos(\mathbf{b}_i \cdot (\mathbf{r} - \mathbf{r}')) \left( |u_{A+}(\mathbf{r})|^2 |u_{A+}(\mathbf{r}')|^2 - |u_{B+}(\mathbf{r})|^2 |u_{B+}(\mathbf{r}')|^2 \right)$$

where $\mathbf{b}_{1,2}$ are the primitive reciprocal lattice vectors and $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2$. In what follows we therefore, set $g_{\perp \perp}, g_{zz}, \tilde{g}_{00}$ to constants and replace $g_{00}$ with the long range part of the Coulomb interaction.

**Appendix C: Relaxation rates for arbitrary angular harmonic**

With illustrate here the evaluation of Eq. (21). At low temperatures, the particles are restricted to the Fermi surface and we can write the scattering rates as $W^\mu(\phi - \phi', \theta)$, where we have defined $\phi, \phi'$ as the angles of $\mathbf{p}_i + \mathbf{p}_j$ and $\mathbf{p}_i + \mathbf{p}_j'$ respectively and where $\theta = \sin^{-1}(q/2k_F)$ is the scattering angle. We also reparametrize the linearized deviations in terms of angular harmonics on the Fermi surface

$$\delta \bar{\rho}^\mu(\mathbf{p}_i, \mathbf{r}) \equiv -\frac{\partial n_i}{\partial \bar{\rho}} \sum_m e^{im\phi_i} \eta^\mu_m(\mathbf{r}).$$

(S1)

Plugging this form into Eq. (21) allows us to straightforwardly obtain the angular harmonics of the collision integral

$$I^\mu_m[\eta^\mu] \equiv v_F^{-1} \sum_{\mathbf{p}} e^{-im\phi} I^\mu_0[\mathbf{p}] = -\frac{1}{v_F T} \int \frac{d^2 p_i d^2 p_j d^2 p_i' d^2 p_j'}{(2\pi)^8} \delta(\sum_j (\mathbf{p}_i') \delta(\sum_j (\mathbf{p}_j')} \eta^\mu_{i,m'} e^{im'\phi_i} - \eta^\mu_{j,m'} e^{im'\phi_j'} + \eta^\mu_{i,m'} e^{im'\phi_j'}$$

(S2)

where

$$W^d_+ = W^{\uparrow \uparrow:++} + 2W^{\uparrow \downarrow:++} + 2W^{\downarrow \downarrow:++} + 2W^{\downarrow \uparrow:++}$$

(S3)
are the rates due to scattering of particles with the same quantum number in the associated channel. These are important only for the even modes and the charge channel.

We follow here the methodology of Ref. 38, wherein the integral is evaluated for the + terms. First we rewrite the delta functions

$$
\delta(\epsilon_i + \epsilon_j - \epsilon_i' - \epsilon_j') = \int d\omega \delta(\epsilon_i - \epsilon_i' - \omega) \delta(\epsilon_j - \epsilon_j' + \omega)
$$

(S4)

and

$$
\delta(p_i + p_j - p_i' - p_j') = \int d^2 q \delta(p_i - p_i' - q) \delta(p_j - p_j' + q).
$$

(S5)

Changing variables to

$$
p_i, p_i' = p + \frac{1}{2}, \quad p_j, p_j' = p' - \frac{1}{2},
$$

we can resolve the delta functions to find $l = l' = q$. We rewrite the momentum integrals in polar coordinates

$$
\int d^2 p = \int dp \int d\phi \int d\delta.
$$

(S7)

This transforms the collision integral to

$$
I_{m, n}^\mu \sim \frac{1}{(2\pi)^3} \sum_{m} \eta_{m}^\mu \int dp dp' dq \int d\phi d\delta
$$

$$
\times \delta(\epsilon_i - \epsilon_i' - \omega) \delta(\epsilon_j - \epsilon_j' + \omega) n_i n_j (1 - n_i')(1 - n_j')
$$

$$
\times e^{-im\phi} W^\mu_l (\phi - \phi', \theta) \left( e^{im\phi_i} - e^{im\phi_i'} e^{im\phi_j} - e^{im\phi_j'} \right).
$$

(S8)

The difference of energies appearing in the delta functions can be written

$$
\epsilon_i - \epsilon_i' = \epsilon(p + q/2) - \epsilon(p - q/2)
$$

$$
= \sqrt{v^2 p^2 + \frac{v^2 q^2}{4} + v^2 qp \cos(\phi - \phi_q) + \Delta^2} - \sqrt{v^2 p^2 + \frac{v^2 q^2}{4} - v^2 qp \cos(\phi - \phi_q) + \Delta^2}
$$

$$
= \epsilon \left[ \sqrt{1 + \frac{v^2 qp}{\epsilon^2} \cos(\phi - \phi_q)} - \sqrt{1 - \frac{v^2 qp}{\epsilon^2} \cos(\phi - \phi_q)} \right] \equiv \delta \epsilon,
$$

(S9)

where we have defined

$$
\epsilon = \sqrt{v^2 p^2 + \Delta^2 + \frac{v^2}{q^2} 4}.
$$

(S10)

The delta function may then be written

$$
\delta(\delta \epsilon - \omega).
$$

(S11)

Projected to the Fermi surface, the convexity of the square-root implies that

$$
\delta \epsilon(\phi) = 0 \implies \cos(\phi - \phi_q) = 0.
$$

(S12)

This allows us to approximate the delta-functions

$$
\delta(\delta \epsilon - \omega) \approx \frac{1}{|\partial \delta \epsilon / \partial \cos(\phi - \phi_q)|} \delta(\cos(\phi - \phi_q)) \approx \sum_{\chi = \pm} \frac{\mu}{v^2 qp |\sin(\phi - \phi_q)|} \delta(\phi - \phi_q - \chi \frac{\pi}{2})
$$

$$
\delta(\delta \epsilon' - \omega) \approx \frac{1}{|\partial \delta \epsilon' / \partial \cos(\phi' - \phi_q)|} \delta(\cos(\phi' - \phi_q)) \approx \sum_{\chi' = \pm} \frac{\mu}{v^2 qp |\sin(\phi' - \phi_q)|} \delta(\phi' - \phi_q - \chi' \frac{\pi}{2})
$$

(S13)
The Fermi surface value of $p$ can be obtained from

$$p^2 = \frac{\mu^2 - \Delta^2 - v^2 q^2}{v^2} = p_F^2 \left(1 - \frac{q^2}{4p_F^2}\right),$$

(S14)

This along with $v_F = \frac{\sqrt{\mu^2 - \Delta^2}}{\mu}$ allows us to simplify Eq. (S13) to

$$\delta(\delta \epsilon - \omega) \approx \sum_{\chi = \pm} \frac{1}{v_F q \sqrt{1 - (q/2p_F)^2}} \delta(\phi - \phi_q - \chi \frac{\pi}{2})$$

$$\delta(\delta \epsilon' - \omega) \approx \sum_{\chi' = \pm} \frac{1}{v_F q \sqrt{1 - (q/2p_F)^2}} \delta(\phi' - \phi_q - \chi' \frac{\pi}{2}).$$

(S15)

Putting this back in

$$I^\mu_{m,-}[\eta^\mu] = -\frac{1}{\nu_F v_F^2 T (2\pi)^5} \sum_{m'} \eta^\mu_{m'} \int dpdp' d\omega \frac{dq}{q(1 - (q/2p_F)^2)}$$

$$\times \int \frac{dq}{q(1 - (q/2p_F)^2)} \delta(\phi - \phi_q - \chi \frac{\pi}{2})$$

$$\delta(\phi' - \phi_q - \chi' \frac{\pi}{2})$$

$$\times e^{-im\phi} W^\mu((\chi - \chi') \frac{\pi}{2}, \theta) \left(e^{im'\phi_i} - e^{im'\phi_{i'}} - e^{im'\phi_j} + e^{im'\phi_{j'}}\right).$$

(S16)

We now change integration variables to

$$\epsilon = \sqrt{v^2 p^2 + \frac{q^2}{4} + \Delta^2}, \quad \epsilon' = \sqrt{v^2 p'^2 + \frac{q^2}{4} + \Delta^2}$$

(S17)

rescale the energy variables by the temperature

$$\epsilon = \mu + uT, \quad \epsilon' = \mu + u'T, \quad \omega = \omega T$$

(S18)

and change variables for $q$ to

$$\sin(\theta) = \frac{q}{2k_F} \Rightarrow \frac{dq}{q(1 - (q/2p_F)^2)} = \frac{d\theta |\cos \theta|}{\sin \theta \cos \theta} = \frac{d\theta}{\sin \theta \cos \theta}.$$  

(S19)

Then to leading order this renders the collision integral

$$I^\mu_{m,-}[\eta^\mu] = -\sum_{m'} \eta^\mu_{m'} \frac{v_F^2}{v_F^2} \int d\Sigma$$

$$\times \int_0^{\pi/2} d\theta \frac{\sin \theta \cos \theta}{\sin \theta \cos \theta} \sum_{\chi, \chi'} \int \frac{dq}{2\pi} e^{-im\phi} W^\mu((\chi - \chi') \frac{\pi}{2}, \theta) \left(e^{im'\phi_i} - e^{im'\phi_{i'}} - e^{im'\phi_j} + e^{im'\phi_{j'}}\right).$$

(S20)

where for compactness we have defined the integration measure

$$d\Sigma = \frac{1}{4\pi^2} dudu' dwn_i n_j (1 - n_i')(1 - n_{i'}).$$

(S21)

We now turn to re-expressing the angles $\phi_\alpha$ in terms of $\phi_q$ and $\theta$. In general, we have

$$p_i - p_{i'} = q = p_j - p_{j'}$$

(S22)

So

$$\sin(|\phi_i - \phi_{i'}|) = \frac{q}{2k_F} = \sin \theta = \sin(|\phi_j - \phi_{j'}|)$$

(S23)

meaning

$$|\phi_i - \phi_{i'}| = |\phi_j - \phi_{j'}| = \theta.$$  

(S24)
with the sign determined by the relative direction of $\mathbf{p}$ and $\mathbf{q}$. Since $\phi = \phi_q + \chi \frac{\pi}{2}$,

$$\begin{cases} \phi_i - \phi_i' < 0, & \chi = 1 \\ \phi_i - \phi_i' > 0, & \chi = -1 \end{cases} \tag{S25}$$

and similarly since $\phi' = \phi_q + \chi' \frac{\pi}{2}$,

$$\begin{cases} \phi_j - \phi_j' > 0, & \chi' = 1 \\ \phi_j - \phi_j' < 0, & \chi' = -1 \end{cases} \tag{S26}$$

Thus,

$$\frac{\phi_i - \phi_i'}{2} = -\chi \theta \tag{S27}$$

$$\frac{\phi_j - \phi_j'}{2} = \chi' \theta. \tag{S28}$$

Now for the sums of angles

$$\tan \phi = \frac{\sin \phi_i + \sin \phi_i'}{\cos \phi_i + \cos \phi_i'} = \tan \frac{\phi_i + \phi_i'}{2} \implies \phi = \frac{\phi_i + \phi_i'}{2} \tag{S29}$$

$$\tan \phi' = \frac{\sin \phi_j + \sin \phi_j'}{\cos \phi_j + \cos \phi_j'} = \tan \frac{\phi_j + \phi_j'}{2} \implies \phi' = \frac{\phi_j + \phi_j'}{2} \tag{S30}.$$

Combining Eqs. (S28) and (S30) we have

$$\phi_i = \phi - \chi \theta = \phi_q + \chi \left(\frac{\pi}{2} - \theta\right) \tag{S31}$$

$$\phi_i' = \phi + \chi \theta = \phi_q + \chi \left(\frac{\pi}{2} + \theta\right) \tag{S32}$$

$$\phi_j = \phi' + \chi' \theta = \phi_q + \chi' \left(\frac{\pi}{2} + \theta\right) \tag{S33}$$

$$\phi_j' = \phi' - \chi' \theta = \phi_q + \chi' \left(\frac{\pi}{2} - \theta\right). \tag{S34}$$
We consider the two cases, $\chi = \chi'$ and $\chi = -\chi'$ corresponding to the left and right hand diagrams in Fig. S1. When $\chi = \chi'$,

$$\phi = \phi' = \phi_q + \chi \pi/2$$  \hspace{1cm} (S35)

and we have collinear scattering

$$\phi_i = \phi_j = \phi_q + \chi(\frac{\pi}{2} - \theta), \quad \phi_j = \phi'_i = \phi_q + \chi(\frac{\pi}{2} + \theta).$$  \hspace{1cm} (S36)

In the other case

$$\phi = \phi_q + \chi \frac{\pi}{2} = \phi_q - \chi \frac{\pi}{2} + \pi = \phi' + \pi$$  \hspace{1cm} (S37)

which describes head on scattering

$$\phi_i = \phi_j + \pi = \phi_q + \chi(\frac{\pi}{2} - \theta), \quad \phi_{j'} = \phi_{j'} + \pi = \phi_q + \chi(\frac{\pi}{2} + \theta).$$  \hspace{1cm} (S38)

We then have

$$I^{\mu: For. [\eta^\mu]}_{m,-} = -2 \sum_{m'} \eta_{m'}^{\mu} \frac{\nu_F T^2}{v_F} \int d\Sigma \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta \cos \theta} W_{m}^{\mu}(0,\theta) \sum_{\chi} \int_0^{\frac{\pi}{2}} \frac{d\phi_q}{2\pi} \times e^{-i m (\phi_q + \chi \pi/2 - \chi \theta)} \left( e^{i m' (\phi_q + \chi \pi/2 - \chi \theta)} - e^{i m' (\phi_q + \chi \pi/2 + \chi \theta)} \right)$$

$$= -2 \eta_{m}^{\mu} \frac{\nu_F T^2}{v_F} \int d\Sigma \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta \cos \theta} W_{m}^{\mu}(0,\theta) \sum_{\chi} \left( 1 - e^{2 i m \chi \theta} \right)$$

$$= -4 \eta_{m}^{\mu} \frac{\nu_F T^2}{v_F} \int d\Sigma \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta \cos \theta} W_{m}^{\mu}(0,\theta) (1 - \cos 2m\theta)$$  \hspace{1cm} (S39)

and

$$I^{\mu: H. O. [\eta^\mu]}_{m,-} = -2 \sum_{m'} \eta_{m'}^{\mu} \frac{\nu_F T^2}{v_F} \int d\Sigma \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta \cos \theta} \sum_{\chi} W_{m}^{\mu}(\pi,\theta) e^{-i m (\phi_q + \chi \pi/2 - \chi \theta)} \times \left( e^{i m' (\phi_q + \chi \pi/2 - \chi \theta)} - e^{i m' (\phi_q + \chi \pi/2 + \chi \theta)} \right) \left( e^{i m' (\phi_q + \chi \pi/2 - \chi \theta)} + e^{i m' (\phi_q + \chi \pi/2 + \chi \theta)} \right)$$

$$= -2 \eta_{m}^{\mu} \frac{\nu_F T^2}{v_F} \int d\Sigma \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta \cos \theta} W_{m}^{\mu}(\pi,\theta) \sum_{\chi} \frac{1}{2} \left( 1 - e^{i m \pi} \right) \left( 1 - e^{2 i m \chi \theta} \right)$$

$$= -4 \eta_{m}^{\mu} \frac{\nu_F T^2}{v_F} \int d\Sigma \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta \cos \theta} W_{m}^{\mu}(\pi,\theta) \frac{1}{2} \left( 1 - e^{i m \pi} \right) \left( 1 - \cos 2m\theta \right)$$  \hspace{1cm} (S40)

Combining the above two expressions, we can identify the contributions to the scattering rate at order $T^2$

$$\frac{1}{\tau^{\mu}_m} = \frac{8 \nu_F T^2}{v_F^2} \int d\Sigma \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta \cos \theta} \left( W_{m}^{\mu}(0,\theta) + \frac{1 - e^{im\pi}}{2} W_{m}^{\mu}(\pi,\theta) \right) \sin^2 m\theta.$$  \hspace{1cm} (S41)

Separating into even and odd terms

$$\frac{1}{\tau^{\mu}_{m,\text{even}}} = \frac{8 \nu_F T^2}{v_F^2} \int d\Sigma \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta \cos \theta} \sin^2 m\theta \left( W_{m}^{\mu}(0,\theta) + W_{m}^{\mu}(\pi,\theta) \right) \sim T^2 \log m$$  \hspace{1cm} (S42)

and

$$\frac{1}{\tau^{\mu}_{m,\text{odd}}} = \frac{8 \nu_F T^2}{v_F^2} \int d\Sigma \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta \cos \theta} \sin^2 m\theta \left( W_{m}^{\mu}(0,\theta) + W_{m}^{\mu}(\pi,\theta) \right)$$  \hspace{1cm} (S43)

where we have restored the usual contribution from the $+$ channel for the even modes. Note that Eq. (S43) is logarithmically divergent at $\theta \rightarrow \frac{\pi}{2}$, corresponding to $q = 2p_F$, if $W(0, \pi/2) + W(\pi, \pi/2)$ is finite. From Fig. S1 one can see that then that the divergence is due to backscattering, $\mathbf{p}_i \rightarrow -\mathbf{p}_i$ and $\mathbf{p}_j \rightarrow -\mathbf{p}_j$. 
These can be put into a more conventional form by writing in terms of the scattering angle $\theta_{sc} = 2\theta$

$$\frac{1}{\tau_{m,\text{even}}} = \frac{4\nu_F T^2}{v_F^2} \int d\Sigma \int_{0}^{\pi} \frac{d\theta_{sc}}{\sin \theta_{sc}} (1 - \cos m\theta_{sc}) \left(W_{-,\text{Collinear}}(\theta_{sc}) + W_{+,\text{Head On}}(\theta_{sc})\right)$$  \hspace{1cm} (S44)

and

$$\frac{1}{\tau_{m,\text{odd}}} = \frac{4\nu_F T^2}{v_F^2} \int d\Sigma \int_{0}^{\pi} \frac{d\theta_{sc}}{\sin \theta_{sc}} (1 - \cos m\theta_{sc}) \left(W_{-,\text{Collinear}}(\theta_{sc}) + W_{+,\text{Head On}}(\theta_{sc})\right)$$ \hspace{1cm} (S45)

where the subscript $sc$ on the rates $W$ is just to show that we are now considering them as functions of $\theta_{sc}$. For the odd terms, the logarithmic divergence at $\theta \rightarrow \frac{\pi}{2}$ should then be the dominant contribution to the relaxation rate. To leading log order we can resolve the divergence, but simplify modifying the integration to include the cutoff on scattering angle due to $\omega$. However, in the presence of the Coulomb interaction, when $q_{TF} \ll 2k_F$ one must also consider the singular contribution coming from forward scattering

$$W_{-}^\mu(\theta \rightarrow 0) \propto \frac{1}{(q + q_{TF})^2} \propto \frac{1}{(\frac{q_{TF}}{2k_F} + \sin \theta)^2}$$ \hspace{1cm} (S46)

leading to a logarithmic contribution from $\theta \rightarrow 0$.

### C.1. Coulomb forward scattering

The above discussion neglects the contribution arising from the long-ranged part of the Coulomb interaction. In two dimensions, the phase space for collinear collisions is logarithmically divergent \[21, 24, 50, 51\]. This being the case, the scattering rate for the unscreened Coulomb interaction, gives a divergent contribution to the transport scattering rate, despite the usual $1 - \cos \theta_{sc}$ suppression of forward scattering contributions. Physically this divergence is cut off by the screening wavevector. In the case where $q_{TF} \gg 2k_F$ the Coulomb interaction is effectively shortrange and one need not consider the forward scattering contribution, as it is effectively suppressed by the transport form factor. However, in the limit $q_{TF} \ll 2k_F$ the region of the integral $\sin \theta_{sc} \sim q_{TF}/2k_F$ leads to a $T^2 \log \frac{1}{q_{TF}}$ contribution to the transport scattering rate.

One may see that the the Coulomb term enters all the distinguishable rates. Its leading log behavior is given by the integral

$$\frac{1}{\tau_{1,c}} \approx 2^6 (\frac{2\pi e^2}{k^2})^2 \nu_F T^2 \int d\Sigma \int_{0}^{\pi} d\theta \frac{\tan \theta}{(q_{TF} + 2k_F \sin \theta)^2}$$ \hspace{1cm} (S47)

where $\kappa$ is the background dielectric constant and the leading prefactor $2^6 = 8 \times 2 \times 2 \times 2$ comes from the prefactor of the relaxation rate, the factor of 2 for distinguishable particles, the sum over forward and head-on collisions, and finally the two different eigenvalue-changing channels. Defining the effective fine-structure constant

$$\alpha = \frac{e^2}{\kappa v}$$ \hspace{1cm} (S48)

we can rewrite

$$\frac{1}{\tau_{1,c}} \approx 2^6 \cdot 4\pi^2 \alpha^2 \frac{\nu_F T^2 v^2}{v_F^2} \int d\Sigma \int_{0}^{\pi} d\theta \frac{\tan \theta}{(q_{TF} + 2k_F \sin \theta)^2}$$ \hspace{1cm} (S49)

Letting $x = \sin \theta$

$$\frac{1}{\tau_{1,c}} \approx 2^6 \cdot \pi^2 \alpha^2 \frac{\nu_F T^2 v^2}{v_F^2} \int d\Sigma \int_{0}^{1} \frac{dx}{1 - x^2} \frac{1}{x(x + a)^2}$$ \hspace{1cm} (S50)

where we have defined $a = q_{TF}/2k_F \ll 1$. Extracting the logarithmic divergence due to the Coulomb potential we have

$$\frac{1}{\tau_{1,c}} \approx 2^6 \cdot \pi^2 \alpha^2 \frac{\nu_F T^2 v^2}{k_F^2 v_F^2} \int d\Sigma \int_{0}^{1} \frac{dx}{(x + a)^2}$$ \hspace{1cm} (S51)
Changing variables to \( y = x + a \)

\[
\frac{1}{\tau_{1,c}^\mu} \approx 2^6 \cdot \pi^2 \alpha^2 \nu_F T^2 v^2 \int d\Sigma \int_{a_c+a}^1 dy \frac{y-a}{y^2} = 2^6 \cdot \pi^2 \alpha^2 \nu_F T^2 v^2 \int d\Sigma \ln \frac{\sqrt{\mu^2 - \Delta^2}}{\max(vq_{TF}, T)}
\]  

(S52)

where we have used

\[
\sqrt{\mu^2 - \Delta^2} = v k_F.
\]  

(S53)

Performing the integral over \( d\Sigma \), we then have

\[
\frac{1}{\tau_{1,c}^\mu} \approx \frac{2^6 \pi^2 \alpha^2}{6} \frac{\nu_F T^2 v^4}{v_F^2 v^2 k_F^2} \ln \frac{\sqrt{\mu^2 - \Delta^2}}{\max(vq_{TF}, T)}
\]  

(S54)

Making use of

\[
v_F = \frac{v^2 k_F}{\mu}, \quad \nu_F = \frac{\mu}{2\pi v^2}
\]  

(S55)

we have

\[
\frac{v^4}{v_F} \nu_F = \frac{\mu^3}{2\pi v^2 k_F^2} = \frac{\mu}{2\pi} \frac{\mu^2}{\mu^2 - \Delta^2}
\]  

(S56)

and arrive at

\[
\frac{1}{\tau_{1,c}^\mu} \approx 2^4 \pi^2 \alpha^2 \frac{\mu^2 T^2}{3 (\mu^2 - \Delta^2)^2} \ln \frac{\sqrt{\mu^2 - \Delta^2}}{\max(vq_{TF}, T)}
\]  

(S57)