Generalised matricvariate Pearson type II- distribution

José A. Díaz-García *
Universidad Autónoma Agraria Antonio Narro
Calzada Antonio Narro 1923
25350 Buenavista, Saltillo, Coahuila, México
E-mail: jadiaz@uaaan.mx

Francisco J. Caro-Lopera
Department of Basic Sciences
Universidad de Medellín
Carrera 87 No.30-65, of. 5-103
Medellín, Colombia
E-mail: fjcaro@udem.edu.co

Abstract
This paper proposes a generalisation of the Pearson type II distribution, which shall termed Pearson Type II-Riesz distribution, based in the Kotz-Riesz distribution. Specifically, the central nonsingular matricvariate generalised Pearson type II-Riesz distribution, beta-Riesz type I distributions and the joint density of the singular values for real normed division algebras are obtained.

1 Introduction

First, recall that if \( X \) and \( U_1 \) are random matrices independently distributed as matrix multivariate normal distribution and a Whishart distribution, respectively; it is known that the random matrix \( R = L^{-1}X \), where \( L \) is any square root of \( U = LL^* = U_1 + XX^* \), has a matricvariate Pearson type II distribution. In the real case under normality, the matricvariate Pearson type II distribution (also known in the literature as matricvariate inverted \( T \) distribution) was studied in detail by [14], see also [35]. This distribution was previously studied by [27], also in the real case. For real, complex, quaternion and octonion cases this distribution was studied by [11].

In Bayesian inference, the matricvariate Pearson type II distribution is assumed as the sampling distribution; then, considering a noninformative prior distribution, the posterior distribution and marginal distributions, the posterior mean and generalised maximum likelihood estimators of the parameters involved are found. [18]. The matricvariate Pearson type II distribution appears in the frequentist approach to normal regression as the distribution of the Studentised error, see [9] and [31]. Another opportunities and potential studies concerns the role of the matricvariate Pearson type II distribution in multivariate analysis, because if the matrix \( R \) has a matricvariate Pearson type II distribution, then the matrix \( RR^* \) (or

*Corresponding author

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\( \mathbf{R}^* \mathbf{R} \) is distributed as matrix multivariate beta type I; and the distribution of the latter, in particular, plays a fundamental role in the MANOVA model, see [27, 30] and [34].

A family of distributions on symmetric cones, termed the matrix multivariate Riesz distributions, was first introduced by Hassairi [23] under the name of Riesz natural exponential family (Riesz NEF); it was based on a special case of the so-termed Riesz measure from Faraut, and Korányi [21, p. 137], going back to Riesz [36]. This Riesz distribution generalises the matrix multivariate gamma and Wishart distributions, containing them as particular cases. Subsequently, Díaz-García [7] and Díaz-García [8] proposes two versions of the Riesz distribution and two generalised of a class of type Kotz distributions. This last two generalised type Kotz distributions are termed matrix multivariate Kotz-Riesz distribution and generalise the matrix multivariate normal distribution, containing this as particular case.

In this point, we are able to propose a generalisation of the multivariate Pearson type II distribution. With this as aim, let \( \mathbf{R} = \mathbf{L}^{-1} \mathbf{X} \), where \( \mathbf{L} \) is any square root of \( \mathbf{U} = \mathbf{LL}^* = \mathbf{U}_1 + \mathbf{XX}^* \), where now we shall assume that \( \mathbf{X} \) and \( \mathbf{U}_1 \) are random matrices independently distributed as a matrix multivariate Kotz-Riesz distribution and a matrix multivariate Riesz distribution, respectively. Then, the distribution of \( \mathbf{R} \) shall be termed multivariate Pearson type II-Riesz distribution.

Although during the 90’s and 2000’s were obtained important results in theory of random matrices distributions, the past 30 years have reached a substantial development. Essentially, these advances have been archived through two approaches based on the theory of Jordan algebras and the theory of real normed division algebras. A basic source of the mathematical tools of theory of random matrices distributions under Jordan algebras can be found in Faraut, and Korányi [20]; and specifically, some works in the context of theory of random matrices distributions based on Jordan algebras are provided in Massam [32], Casalis, and Letac [4], Hassairi [23], and Hassairi et al. [24], and the references therein. Parallel results on theory of random matrices distributions based on real normed division algebras have been also developed in random matrix theory and statistics, see Gross and Richards [22], Dumitriu [13], Forrester [21], Díaz-García and Gutiérrez-Jáimez [10], Díaz-García and Gutiérrez-Jáimez [12], among others. In addition, from mathematical point of view, several basic properties of the matrix multivariate Riesz distribution under the structure theory of normal \( j \)-algebras and under theory of Vinberg algebras in place of Jordan algebras have been studied, see Ishi [25] and Boutouria and Hassir [3], respectively.

From a applied point of view, the relevance of the octonions remains unclear. An excellent review of the history, construction and many other properties of octonions is given in Baez [1], where it is stated that:

“Their relevance to geometry was quite obscure until 1925, when Élie Cartan described ‘triality’ – the symmetry between vector and spinors in 8-dimensional Euclidian space. Their potential relevance to physics was noticed in a 1934 paper by Jordan, von Neumann and Wigner on the foundations of quantum mechanics... Work along these lines continued quite slowly until the 1980s, when it was realised that the octonions explain some curious features of string theory... However, there is still no proof that the octonions are useful for understanding the real world. We can only hope that eventually this question will be settled one way or another.”

For the sake of completeness, in the present article the case of octonions is considered, but the veracity of the results obtained for this case can only be conjectured. Nonetheless, Forrester [21, Section 1.4.5, pp. 22-24] it is proved that the bi-dimensional density function
of the eigenvalue, for a Gaussian ensemble of a $2 \times 2$ octonionic matrix, is obtained from the general joint density function of the eigenvalues for the Gaussian ensemble, assuming $m = 2$ and $\beta = 8$, see Section 2. Moreover, as is established in Faraut, and Korányi \cite{20} and Sawyer \cite{37} the result obtained in this article are valid for the algebra of Albert, that is when hermitian matrices ($\mathcal{S}$) or hermitian product of matrices ($X^*X$) are $3 \times 3$ octonionic matrices.

The present article is organised as follows; basic concepts and the notation of abstract algebra and Jacobians are summarised in Section 2. Then Section 3 derives the nonsingular central matricvariate Pearson type II-Riesz and the beta type I distributions, and some basic properties are also studied. Finally, the joint densities of the singular values and eigenvalues are derived in Section 4. We emphasise that all these results are found for real normed division algebras.

2 Preliminary results

A detailed discussion of real normed division algebras may be found in \cite{1} and \cite{17}. For convenience, we shall introduce some notation, although in general we adhere to standard notation forms.

For our purposes: Let $\mathbb{F}$ be a field. An algebra $\mathfrak{A}$ over $\mathbb{F}$ is a pair $(\mathfrak{A}; m)$, where $\mathfrak{A}$ is a finite-dimensional vector space over $\mathbb{F}$ and multiplication $m : \mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{A}$ is an $\mathbb{F}$-bilinear map; that is, for all $\lambda \in \mathbb{F}$, $x, y, z \in \mathfrak{A}$,

$$m(x, \lambda y + z) = \lambda m(x; y) + m(x; z)$$
$$m(\lambda x + y; z) = \lambda m(x; z) + m(y; z).$$

Two algebras $(\mathfrak{A}; m)$ and $(\mathfrak{C}; n)$ over $\mathbb{F}$ are said to be isomorphic if there is an invertible map $\phi : \mathfrak{A} \rightarrow \mathfrak{C}$ such that for all $x, y \in \mathfrak{A}$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write $m(x; y) = xy$ for all $x, y \in \mathfrak{A}$.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is said to be

1. alternative if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in \mathfrak{A}$,
2. associative if $x(yz) = (xy)z$ for all $x, y, z \in \mathfrak{A}$,
3. commutative if $xy = yx$ for all $x, y \in \mathfrak{A}$, and
4. unital if there is a $1 \in \mathfrak{A}$ such that $x1 = x = 1x$ for all $x \in \mathfrak{A}$.

If $\mathfrak{A}$ is unital, then the identity 1 is uniquely determined.

An algebra $\mathfrak{A}$ over $\mathbb{F}$ is said to be a division algebra if $\mathfrak{A}$ is nonzero and $xy = 0_\mathfrak{A} \Rightarrow x = 0_\mathfrak{A}$ or $y = 0_\mathfrak{A}$ for all $x, y \in \mathfrak{A}$.

The term “division algebra”, comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is a division algebra if, and only if, $\mathfrak{A}$ is nonzero and for all $a, b \in \mathfrak{A}$, with $b \neq 0_\mathfrak{A}$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in \mathfrak{A}$.

In the sequel we assume $\mathbb{F} = \mathbb{R}$ and consider classes of division algebras over $\mathbb{R}$ or “real division algebras” for short.

We introduce the algebras of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. Then, if $\mathfrak{A}$ is an alternative real division algebra, then $\mathfrak{A}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

3
Let \( \mathfrak{A} \) be a real division algebra with identity 1. Then \( \mathfrak{A} \) is said to be \textit{normed} if there is an inner product \((\cdot, \cdot)\) on \( \mathfrak{A} \) such that

\[
(xy, xy) = (x, x)(y, y) \quad \text{for all } x, y \in \mathfrak{A}.
\]

If \( \mathfrak{A} \) is a \textit{real normed division algebra}, then \( \mathfrak{A} \) is isomorphic \( \mathbb{R}, \mathbb{C}, \mathfrak{H} \) or \( \mathfrak{O} \).

There are exactly four normed division algebras: real numbers \( \mathbb{R} \), complex numbers \( \mathbb{C} \), quaternions \( \mathfrak{H} \) and octonions \( \mathfrak{O} \), see [1]. We take into account that should be taken into account, \( \mathbb{R}, \mathbb{C}, \mathfrak{H} \) and \( \mathfrak{O} \) are the only normed division algebras; furthermore, they are the only alternative division algebras.

Let \( \mathfrak{A} \) be a division algebra over the real numbers. Then \( \mathfrak{A} \) has dimension either 1, 2, 4 or 8. In other branches of mathematics, the parameters \( \alpha = 2/\beta \) and \( t = \beta/4 \) are used, see [10] and [26], respectively.

Finally, observe that

- \( \mathbb{R} \) is a real commutative associative normed division algebras,
- \( \mathbb{C} \) is a commutative associative normed division algebras,
- \( \mathfrak{H} \) is an associative normed division algebras,
- \( \mathfrak{O} \) is an alternative normed division algebras.

Let \( \mathcal{L}_{m,n}^\beta \) be the set of all \( m \times n \) matrices of rank \( m \leq n \) over \( \mathfrak{A} \) with \( m \) distinct positive singular values, where \( \mathfrak{A} \) denotes a \textit{real finite-dimensional normed division algebra}. Let \( \mathfrak{A}^{m \times n} \) be the set of all \( m \times n \) matrices over \( \mathfrak{A} \). The dimension of \( \mathfrak{A}^{m \times n} \) over \( \mathbb{R} \) is \( \beta mn \). Let \( \mathfrak{A} \in \mathfrak{A}^{m \times n} \), then \( \mathfrak{A}^* = \mathfrak{A}^T \) denotes the usual conjugate transpose.

Table 1 sets out the equivalence between the same concepts in the four normed division algebras.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Real & Complex & Quaternion & Octonion & Generic notation \\
\hline
Semi-orthogonal & Semi-unitary & Semi-symplectic & Semi-exceptional type & \( \Psi_{m,n}^\beta \) \\
Orthogonal & Unitary & Symplectic & Exceptional type & \( \mathfrak{U}^\beta(m) \) \\
Symmetric & Hermitian & Quaternion & Octonion & \( \mathfrak{G}_m^\beta \) \\
\hline
\end{tabular}
\end{table}

We denote by \( \mathfrak{G}_m^\beta \) the real vector space of all \( \mathbf{S} \in \mathfrak{A}^{m \times m} \) such that \( \mathbf{S} = \mathbf{S}^* \). In addition, let \( \Psi_m^\beta \) be the \textit{cone of positive definite matrices} \( \mathbf{S} \in \mathfrak{A}^{m \times m} \). Thus, \( \Psi_m^\beta \) consist of all matrices \( \mathbf{S} = \mathbf{X}\mathbf{X}^* \), with \( \mathbf{X} \in \mathfrak{G}_m^\beta \); then \( \Psi_m^\beta \) is an open subset of \( \mathfrak{G}_m^\beta \).

Let \( \mathfrak{D}_m^\beta \) consisting of all \( \mathbf{D} \in \mathfrak{A}^{m \times m} \), \( \mathbf{D} = \text{diag}(d_1, \ldots, d_m) \). Let \( \Sigma_m^\beta (m) \) be the subgroup of all \textit{upper triangular} matrices \( \mathbf{T} \in \mathfrak{A}^{m \times m} \) such that \( t_{ij} = 0 \) for \( 1 < i < j \leq m \).

For any matrix \( \mathbf{X} \in \mathfrak{A}^{m \times n} \), \( d\mathbf{X} \) denotes the matrix of differentials \( (d\mathbf{x}_{ij}) \). Finally, we define the \textit{measure} or \textit{volume element} \( (d\mathbf{X}) \) when \( \mathbf{X} \in \mathfrak{G}_m^\beta \), \( \mathfrak{D}_m^\beta \) or \( \Psi_m^\beta \), see [10] and [12].

If \( \mathbf{X} \in \mathfrak{A}^{m \times n} \) then \( (d\mathbf{X}) \) (the Lebesgue measure in \( \mathfrak{A}^{m \times n} \)) denotes the exterior product of the \( \beta mn \) functionally independent variables

\[
(d\mathbf{X}) = \bigwedge_{i=1}^m \bigwedge_{j=1}^n dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^\beta dx_{ij}^{(k)}.
\]

If \( \mathbf{S} \in \mathfrak{G}_m^\beta \) (or \( \mathbf{S} \in \Sigma_m^\beta (m) \) with \( t_{ii} > 0 \), \( i = 1, \ldots, m \)) then \( (d\mathbf{S}) \) (the Lebesgue measure in \( \mathfrak{G}_m^\beta \) or in \( \Sigma_m^\beta (m) \)) denotes the exterior product of the exterior product of the \( m(m-1)\beta/2 + m \)
functionally independent variables,

\[
(dS) = \bigwedge_{i=1}^{m} ds_{ii}^{\beta} \bigwedge_{i<j}^{m} ds_{ij}^{(k)}.
\]

Observe, that for the Lebesgue measure \((dS)\) defined thus, it is required that \(S \in \mathcal{P}_m^n\), that is, \(S\) must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If \(A \in \mathcal{D}_{m}^\beta\) then \((dA)\) (the Lebesgue measure in \(\mathcal{D}_{m}^\beta\)) denotes the exterior product of the \(\beta m\) functionally independent variables

\[
(dA) = \bigwedge_{i=1}^{n} d\lambda_i^{(k)}.
\]

If \(H_1 \in \mathcal{V}_{m,n}^\beta\) then

\[
(H_1^* dH_1) = \bigwedge_{i=1}^{m} \bigwedge_{j=i+1}^{n} h_j^* dh_i.
\]

where \(H = (H_1^* H_2^*)^* = (h_1, \ldots, h_m, h_{m+1}, \ldots, h_n)^* \in \mathcal{U}^\beta(n)\). It can be proved that this differential form does not depend on the choice of the \(H_2\) matrix. When \(n = 1\); \(\mathcal{V}_{m,1}^\beta\) defines the unit sphere in \(\mathbb{A}^m\). This is, of course, an \((m-1)\beta\)-dimensional surface in \(\mathbb{A}^m\). When \(n = m\) and denoting \(H_1\) by \(H\), \((H^* dH^*)\) is termed the Haar measure on \(\mathcal{U}^\beta(m)\).

The surface area or volume of the Stiefel manifold \(\mathcal{V}_{m,n}^\beta\) is

\[
\text{Vol}(\mathcal{V}_{m,n}^\beta) = \int_{H_1 \in \mathcal{V}_{m,n}^\beta} (H_1^* dH_1) = \frac{2^n \pi^{m(n-\beta)/2}}{\Gamma_m^n[n\beta/2]},
\]

where \(\Gamma_m^\beta[a]\) denotes the multivariate Gamma function for the space \(\mathcal{S}_m^\beta\). This can be obtained as a particular case of the generalised gamma function of weight \(\kappa\) for the space \(\mathcal{S}_m^\beta\) with \(\kappa = (k_1, k_2, \ldots, k_m), k_1 \geq k_2 \geq \cdots \geq k_m \geq 0\), \(k_1, k_2, \ldots, k_m\) are nonnegative integers, taking \(\kappa = (0, 0, \ldots, 0)\) and which for \(\text{Re}(\kappa) \geq (m-1)\beta/2 - k_m\) is defined by, see 22,

\[
\Gamma_m^\beta[a, \kappa] = \int_{A \in \mathcal{P}_m^n} \text{etr}\{-A\}|A|^{-a-(m-1)\beta/2-1} q_{\kappa}(A)(dA) \tag{2}
\]

\[
= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a + k_i - (i-1)\beta/2]
\]

\[
= [a]_\kappa^{\beta} \Gamma_m^\beta[a],
\]

where \(\text{etr}(\cdot) = \exp(\text{tr}(\cdot))\), \(\text{det}(\cdot)\) denotes the determinant, and for \(A \in \mathcal{S}_m^\beta\)

\[
q_{\kappa}(A) = |A_m|^k \prod_{i=1}^{m-1} |A_i|^{k_i - k_{i+1}}
\]

with \(A_p = (a_{rs}), r, s = 1, 2, \ldots, p, p = 1, 2, \ldots, m\) is termed the highest weight vector, see 22. Also,

\[
\Gamma_m^\beta[a] = \int_{A \in \mathcal{P}_m^n} \text{etr}\{-A\}|A|^{-a-(m-1)\beta/2-1}(dA)
\]

\[
= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a - (i-1)\beta/2],
\]
and \( \text{Re}(a) > (m-1)\beta/2 \).

In other branches of mathematics the highest weight vector \( q_\kappa(A) \) is also termed the generalised power of \( A \) and is denoted as \( \Delta_\kappa(A) \), see [20] and [23].

Additional properties of \( q_\kappa(A) \), which are immediate consequences of the definition of \( q_\kappa(A) \) and the following property 1, are:

1. if \( \lambda_1, \ldots, \lambda_m \), are the eigenvalues of \( A \), then
   \[
   q_\kappa(A) = \prod_{i=1}^{m} \lambda_{k_i}^{\lambda_i}. \tag{5}
   \]

2. \[
   q_\kappa(A^{-1}) = q_{-\kappa}(A^{-1}) = q_{-\kappa}(A), \tag{6}
   \]

3. if \( \kappa = (p, \ldots, p) \), then
   \[
   q_\kappa(A) = |A|^p, \tag{7}
   \]
   in particular if \( p = 0 \), then \( q_\kappa(A) = 1 \).

4. if \( \tau = (t_1, t_2, \ldots, t_m) \), \( t_1 \geq t_2 \geq \cdots \geq t_m \geq 0 \), then
   \[
   q_{\kappa+\tau}(A) = q_\kappa(A)q_\tau(A), \tag{8}
   \]
   in particular if \( \tau = (p, p, \ldots, p) \), then
   \[
   q_{\kappa+\tau}(A) \equiv q_{\kappa+p}(A) = |A|^p q_\kappa(A). \tag{9}
   \]

5. Finally, for \( B \in \mathcal{A}^{m \times m} \) in such a manner that \( C = B^*B \in \mathcal{S}^\beta_m \),
   \[
   q_\kappa(BAB^*) = q_\kappa(C)q_\kappa(A) \tag{10}
   \]
   and
   \[
   q_\kappa(B^{-1}AB^*-1) = (q_\kappa(C))^{-1}q_\kappa(A). \tag{11}
   \]

**Remark 2.1.** Let \( \mathcal{P}(\mathcal{S}^\beta_m) \) denote the algebra of all polynomial functions on \( \mathcal{S}^\beta_m \), and \( \mathcal{P}_k(\mathcal{S}^\beta_m) \) the subspace of homogeneous polynomials of degree \( k \) and let \( \mathcal{P}_k(\mathcal{S}^\beta_m) \) be an irreducible subspace of \( \mathcal{P}(\mathcal{S}^\beta_m) \) such that
\[
\mathcal{P}_k(\mathcal{S}^\beta_m) = \sum_{\kappa} \bigoplus \mathcal{P}_k^{\kappa}(\mathcal{S}^\beta_m).
\]

Note that \( q_\kappa \) is a homogeneous polynomial of degree \( k \), moreover \( q_\kappa \in \mathcal{P}_k(\mathcal{S}^\beta_m) \), see [22].

In [3], \( [a]_\kappa^\beta \) denotes the generalised Pochhammer symbol of weight \( \kappa \), defined as
\[
[a]_\kappa^\beta = \prod_{i=1}^{m} (a - (i-1)\beta/2)_k, \quad \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a + k_i - (i-1)\beta/2] \Gamma^\beta_m[a] \Gamma^\beta_m[a].
\]
where \( \text{Re}(a) > (m-1)\beta/2 - k_m \) and

\[
(a)_i = a(a+1) \cdots (a+i-1),
\]
is the standard Pochhammer symbol.

An alternative definition of the generalised gamma function of weight \( \kappa \) is proposed by [29], which is defined as

\[
\Gamma_m^\beta[a, -\kappa] = \int_{A \in \mathfrak{U}_m^\beta} \text{etr}\{-A\}|A|^{a-(m-1)\beta/2-1} q_\kappa(A^{-1})(dA) \tag{12}
\]

\[
= \pi^{m(m-1)\beta/4} \prod_{i=1}^m |a - k_i - (m-i)\beta/2|
\]

\[
= \frac{(-1)^k \Gamma_m^\beta[a]}{|-a + (m-1)\beta/2 + 1|_n}, \tag{13}
\]

where \( \text{Re}(a) > (m-1)\beta/2 + k_1 \).

In addition consider the following generalised beta functions termed, \textit{generalised c-beta function}, see Faraut, and Korányi [20, p. 130] and [5],

\[
B_m^\beta[a, \kappa; b, \tau] = \int_{0 < S \in \mathcal{I}_m} |S|^{a-(m-1)\beta/2-1} q_\kappa(S)|I_m - S|^{b-(m-1)\beta/2-1} q_\tau(I_m - S)(dS)
\]

\[
= \int_{R \in \mathfrak{U}_m^\beta} |R|^{a-(m-1)\beta/2-1} q_\kappa(R)|I_m + R|^{-(a+b)} q_{\kappa + \tau}(I_m + R)(dR)
\]

\[
= \frac{\Gamma_m^\beta[a, \kappa] \Gamma_m^\beta[b, \tau]}{\Gamma_m^\beta[a + b, \kappa + \tau]},
\]

where \( \kappa = (k_1, k_2, \ldots, k_m), k_1 \geq k_2 \geq \cdots \geq k_m \geq 0, k_1, k_2, \ldots, k_m \) are nonnegative integers, \( \tau = (t_1, t_2, \ldots, t_m), t_1 \geq t_2 \geq \cdots \geq t_m \geq 0, t_1, t_2, \ldots, t_m \) are nonnegative integers, \( \text{Re}(a) > (m-1)\beta/2 - k_m \) and \( \text{Re}(b) > (m-1)\beta/2 - t_m \). Similarly is defined the \textit{generalised k-beta function} as, see [5],

\[
B_m^\beta[a, -\kappa; b, -\tau] = \int_{0 < S \in \mathcal{I}_m} |S|^{a-(m-1)\beta/2-1} q_\kappa^{-1}(S)|I_m - S|^{b-(m-1)\beta/2-1} q^{-1}_\tau(I_m - S)(dS)
\]

\[
= \int_{R \in \mathfrak{U}_m^\beta} |R|^{a-(m-1)\beta/2-1} q_\kappa^{-1}(R)|I_m + R|^{-(a+b)} q_{\kappa + \tau}(I_m + R)(dR)
\]

\[
= \frac{\Gamma_m^\beta[a, -\kappa] \Gamma_m^\beta[b, -\tau]}{\Gamma_m^\beta[a + b, -\kappa - \tau]},
\]

where \( \kappa = (k_1, k_2, \ldots, k_m), k_1 \geq k_2 \geq \cdots \geq k_m \geq 0, k_1, k_2, \ldots, k_m \) are nonnegative integers, \( \tau = (t_1, t_2, \ldots, t_m), t_1 \geq t_2 \geq \cdots \geq t_m \geq 0, t_1, t_2, \ldots, t_m \) are nonnegative integers, \( \text{Re}(a) > (m-1)\beta/2 + k_1 \) and \( \text{Re}(b) > (m-1)\beta/2 + t_1 \).

Finally, the following Jacobians involving the \( \beta \) parameter, reflects the generalised power of the algebraic technique; the can be seen as extensions of the full derived and unconnected results in the real, complex or quaternion cases, see Faraut, and Korányi [20] and Díaz-García and Gutiérrez-Jáimez [10]. These results are the base for several matrix and matric variate generalised analysis.
Proposition 2.1. Let $X$ and $Y$ be matrices of functionally independent variables, and let $Y = AXB + C$, where $A \in L^\beta_{m,m}$, $B \in L^\beta_{n,n}$ and $C \in L^\beta_{m,n}$ are constant matrices. Then

\[(dY) = |A* A|^{n\beta/2}|B*B|^{m\beta/2}(dX). \tag{14}\]

Proposition 2.2. Let $X$ and $Y$ be matrices of functionally independent variables, and let $Y = AXA^* + C$, where $A \in L^\beta_{m,m}$ and $C \in L^\beta_{m}$ are constant matrices. Then

\[(dY) = |A^* A|^{(m-1)\beta/2+1}(dX). \tag{15}\]

Proposition 2.3 (Singular Value Decomposition, SVD). Let $X \in L^\beta_{m,n}$ be matrix of functionally independent variables, such that $X = W^*DV_1$ with $V_1 \in V^\beta_{m,n}$, $W \in U^\beta(m)$ and $D = \text{diag}(d_1, \ldots, d_m) \in \mathcal{D}_m$, $d_1 > \cdots > d_m > 0$. Then

\[
(dX) = 2^{-m} \prod_{i=1}^{m} d_i^{(n-m+1)\beta-1} \prod_{i<j}^{m} (d_i^2 - d_j^2)^{\beta/2} (dD)(V_1 dV_1^*)(WdW^*), \tag{16}\]

where

\[
\varrho = \begin{cases} 
0, & \beta = 1; \\
-m, & \beta = 2; \\
-2m, & \beta = 4; \\
-4m, & \beta = 8. 
\end{cases}
\]

Proposition 2.4. Let $X \in L^\beta_{m,n}$ be matrix of functionally independent variables, and $S = XX^* \in \mathcal{P}^\beta_m$. Then

\[
(dX) = 2^{-m} |S|^{\beta(n-m+1)/2-1} (dS)(V_1 dV_1^*), \tag{17}\]

with $V_1 \in V^\beta_{m,n}$.

3 Matricvariate Pearson type II-Riesz distribution

Two versions of the matricvariate Pearson type II-Riesz distributions and the corresponding generalised beta type I distributions are obtained in this section.

A detailed discussion of Riesz distribution may be found in Hassairi [23] and Díaz-García [8]. In addition the Kotz-Riesz distribution is studied in detail in Díaz-García [6]. For your convenience, we adhere to standard notation stated in Díaz-García [6].

Theorem 3.1. Let $\kappa = (k_1,k_2,\ldots,k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $k_1,k_2,\ldots,k_m$ are nonnegative integers, and $\tau = (t_1,t_2,\ldots,t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, $t_1,t_2,\ldots,t_m$ are nonnegative integers. Also define $R \in L^\beta_{m,n}$ as

\[R = L^{-1}X,\]

1. where $L$ is any square root of $U = U_1 + XX^*$ where $U_1 \sim \mathcal{R}^{3,1}_m(\nu/2,\kappa,I_m)$, $\text{Re}(\nu/2) > (m-1)\beta/2 - k_m$; independent of $X \sim \mathcal{K}\mathcal{R}^{3,1}_{m,n}(\tau,0,I_m,I_n)$, $\text{Re}(n\beta/2) > (m-1)\beta/2 - t_m$. Then $U \sim \mathcal{R}^{3,1}_{m,n}((\nu+n)\beta/2,\kappa+\tau,I_m)$ independent of $R$ with $\text{Re}((\nu+n)\beta/2) > (m-1)\beta/2 - k_m - t_m$. Furthermore, the density of $R$ is

\[
\frac{\Gamma_m[\nu/2]\Gamma_m[\nu/2]}{\pi^{mn\beta/2} B_m[\nu/2,\kappa,\nu/2,\tau]} q_\kappa (I_m - RR^*) q_\tau (RR^*) (dR), \tag{18}\]

which is shall be termed the matricvariate Pearson type II-Riesz distribution type I, where $I_m - RR^* \in \mathcal{P}^\beta_m$.
2. where \( L \) is any square root of \( U = LL^* = U_1 + XX^* \) where \( U_1 \sim R_{m}^{\beta,11}(\nu \beta/2, \kappa, I_m) \), \( \text{Re}(\nu/2) > (m-1)\beta/2 + k_1 \); independent of \( X \sim K R_{m \times n}^{\beta,11}(\tau, 0, I_m, I_n) \), \( \text{Re}(n/2) > (m-1)\beta/2 + t_1 \). Then \( U \sim R_{m}^{\beta,11}((\nu + n)/2, \kappa + \tau, I_m) \) independent of \( R \) with \( \text{Re}((\nu + n)/2) > (m-1)\beta/2 + k_1 + t_1 \). Furthermore, the density of \( R \) is

\[
\frac{\Gamma^\beta_m[n/2]}{\pi^{mn^\beta/2} B^\beta_m[\nu \beta/2, \kappa; n \beta/2, -\kappa \beta/2, -\tau]} q_{\kappa}(I_m - RR^*)^{-1} \times q_\tau(\lambda_2)^{-1}(dR),
\]

which is shall be termed the matricvariate Pearson type II-Riesz distribution type II, where \( I_m - RR^* \in \mathbb{P}^\beta_m \).

**Proof.** 1. From Díaz-García, the joint density of \( U_1 \) and \( X \) is

\[
\alpha(U_1)^{(\nu-n-m+1)/2-1} \text{etr}\{-\beta(U_1 + XX^*)\} q_\kappa(U_1) q_\tau(XX^*) (dU_1)(dX),
\]

where the constant of proportionality given by

\[
c = \frac{\beta^\nu n!^\beta/2 + \sum_{i=1}^m k_i}{\Gamma^\beta_m[\nu \beta/2, \kappa]}. \frac{\beta^\nu n!^\beta/2 + \sum_{i=1}^m t_i}{\Gamma^\beta_m[n \beta/2, -\kappa \beta/2, -\tau]}
\]

Making the change of variable \( U_1 = (U - XX^*) \) and \( X = LR \), where \( U = LL^* \), then by \( LR \)

\[
(dU_1)(dX) = |LL^*|^{-\beta/2}(dU)(dR) = |U|^{n \beta/2}(dU)(dR),
\]

and observing that \( |U_1| = |U - XX^*| = |U - LL^* L^*| = |U||I_m - RR^*| \), the joint density of \( U \) and \( R \) is

\[
\alpha(U)^{(\nu+n-m+1)/2-1} \text{etr}\{-\beta U\} q_\kappa(U) |I_m - RR^*|^{(\nu+n-m+1)/2-1}
\]

\[
\times q_\kappa(I_m - RR^*) q_\tau(\lambda_2)^{-1} (dU)(dT).
\]

Finally, integrating over \( U \in \mathbb{P}^\beta_m \) using the density of Riesz distribution, the desired result is obtained.

2. Its proof is similar to given for item 1.

Now, note that

1. from \( LR \)

\[
\frac{\Gamma^\beta_m[n \beta/2]}{\pi^{mn \beta/2} B^\beta_m[\nu \beta/2, \kappa; n \beta/2, -\kappa \beta/2, -\tau]} = \frac{\Gamma^\beta_n[m \beta/2]}{\pi^{mn \beta/2} B^\beta_n[\nu \beta/2, \kappa; m \beta/2, -\kappa \beta/2, -\tau]}
\]

2. also,

\[
|I_m - RR^*| = |I_n - R^* R|.
\]

3. And considering the next extension defining \( q_{\kappa}(A) \) as: if \( \lambda_1, \ldots, \lambda_r \), are the non null eigenvalues of \( A \in \mathbb{G}^\beta_{m \times n} \), then

\[
q_{\kappa}(A) = \prod_{i=1}^r \lambda_i^{k_i}.
\]

then, \( q_{\kappa}(RR^*) = q_{\kappa}(R^* R) \).
Thus, the density (18) can be expressed alternatively as

$$\frac{\Gamma_n^2[m\beta/2]}{\pi^{mn/2}B_n[(n + \nu - m)/2, -\kappa; m\beta/2, \tau]} \left| I_n - \tilde{R}^*\tilde{R} \right|^{(n + m + 1)/2 - 1}$$

$$\times q_\kappa \left( I_n - \tilde{R}^*\tilde{R} \right) q_\tau \left( \tilde{R}^*\tilde{R} \right) (d\tilde{R}), \quad I_n - \tilde{R}^*\tilde{R} \in P_n^2,$$

where \( \tilde{R} \) was denoted as \( \tilde{R} \) to distinguish them and facilitate their use in the following sections.

Similarly, the density (19) can be written alternatively as

$$\frac{\Gamma_n^2[m\beta/2]}{\pi^{mn/2}B_n[(n + \nu - m)/2, -\kappa; m\beta/2, -\tau]} \left| I_n - \tilde{R}^*\tilde{R} \right|^{(n + m + 1)/2 - 1}$$

$$\times q_\kappa \left( I_n - \tilde{R}^*\tilde{R} \right)^{-1} q_\tau \left( \tilde{R}^*\tilde{R} \right)^{-1} (d\tilde{R}), \quad I_n - \tilde{R}^*\tilde{R} \in P_n^2.$$

**Corollary 3.1.** Let \( Q = (M^*)^{-1}RN^* + \mu, \tilde{R} \) as in Theorem 3.1. \( M \) and \( N \) are any square root of the constant matrices \( \Omega = MM^* \in P_n^2 \) and \( \Xi = NN^* \in P_n^2 \), respectively, and \( \mu \in C_{m,n}^2 \) is constant.

1. Then, from (18) the density of \( Q \) is

$$\propto \left| \Omega^{-1} - (Q - \mu)\Xi^{-1}(Q - \mu)^* \right|^{(n + m + 1)/2 - 1}$$

$$\times q_\kappa \left( \Omega^{-1} - (Q - \mu)\Xi^{-1}(Q - \mu)^* \right) q_\tau \left( (Q - \mu)\Xi^{-1}(Q - \mu)^* \right) (dQ),$$

with constant of proportionality

$$\frac{\Gamma_n^2[m\beta/2]}{\pi^{mn/2}B_n[(n + \nu - m)/2, -\kappa; m\beta/2, \tau]} \left| \Omega^{-1} - (Q - \mu)\Xi^{-1}(Q - \mu)^* \right|^{(n + m + 1)/2 - 1} q_\kappa q_\tau (\Omega) \Xi^{\beta}.$$

where \( \Omega^{-1} - (Q - \mu)\Xi^{-1}(Q - \mu)^* \in P_n^2 \). And from (20) the density of \( \tilde{Q} \) is

$$\propto \left| \Xi - (\tilde{Q} - \mu)^*\Omega(\tilde{Q} - \mu) \right|^{(n + m + 1)/2 - 1}$$

$$\times q_\kappa \left( \Xi - (\tilde{Q} - \mu)^*\Omega(\tilde{Q} - \mu) \right) q_\tau \left( (\tilde{Q} - \mu)^*\Omega(\tilde{Q} - \mu) \right) (d\tilde{Q}),$$

with constant of proportionality

$$\frac{\Gamma_n^2[m\beta/2]}{\pi^{mn/2}B_n[(n + \nu - m)/2, -\kappa; m\beta/2, \tau]} \left| \Xi - (\tilde{Q} - \mu)^*\Omega(\tilde{Q} - \mu) \right|^{(n + m + 1)/2 - 1} q_\kappa q_\tau (\Xi) \Xi^{\beta}.$$

where \( \Xi - (\tilde{Q} - \mu)^*\Omega(\tilde{Q} - \mu) \in P_n^2 \). This fact is denoted as

$$Q(\tilde{Q}) \sim \mathcal{P}_{m \times n}(\nu, \kappa, \tau, \mu, \Omega, \Xi).$$

2. Analogously, from (19) the density of \( Q \) is

$$\propto \left| \Omega^{-1} - (Q - \mu)\Xi^{-1}(Q - \mu)^* \right|^{(n + m + 1)/2 - 1}$$

$$\times q_\kappa \left[ (\Omega^{-1} - (Q - \mu)\Xi^{-1}(Q - \mu)^*)^{-1} \right] q_\tau \left[ ((Q - \mu)\Xi^{-1}(Q - \mu)^*)^{-1} \right] (dQ),$$
with constant of proportionality

\[
\frac{\Gamma_m^\beta [m\beta/2] |\Omega|^{(\nu+n-m+1)/\beta-2-1}}{\pi^{mn/2}B_n^\beta(3/2, -\kappa; n/2, -\tau)||\Xi|^{m/2}q_{\kappa+\tau}(\Omega)}
\]

where \( \Omega^{-1} - (Q - \mu)^* \Xi^{-1}(Q - \mu)^* \in \mathcal{F}_m^\beta \). Similarly, from Equation (27) the density of \( \tilde{Q} \) is

\[
\propto |\Xi - (\tilde{Q} - \mu)^* \Omega(\tilde{Q} - \mu)|^{(n-\nu+1)/\beta-2-1} q_n \left[ (\Xi - (\tilde{Q} - \mu)^* \Omega(\tilde{Q} - \mu))^{-1} \right] \times q_\tau \left[ ((\tilde{Q} - \mu)^* \Omega(\tilde{Q} - \mu))^{-1} \right] (d\tilde{Q}),
\]

with constant of proportionality

\[
\frac{\Gamma_n^\beta [m\beta/2] |\Omega|^{(3/2-2-1)}q_{\kappa+\tau}(\Xi)}{\pi^{mn/2}B_n^\beta(n+\nu-m)/2, -\kappa; m\beta/2, -\tau)||\Xi|^{m/2}(\nu-2m+1/\beta-2-1)}
\]

where \( \Xi - (\tilde{Q} - \mu)^* \Omega(\tilde{Q} - \mu) \in \mathcal{F}_n^\beta \). This fact is denoted as

\[
Q(\bar{Q}) \sim \mathcal{P}_{\mathbb{R}^{\beta,1}}^{m \times n}(\nu, \kappa, \tau, \Omega, \Xi).
\]

Proof. 1. The proof follows from \([14] \) and \([20] \), respectively, observing that, by Equation (14)

\[
(dR) = |MM^*|^{\beta/2}|NN^*|^{\beta/2}(dQ) = |\Omega|^{m/2}|\Xi|^{\beta/2}(dQ),
\]

\[
|I_m - \Omega(Q - \mu)\Xi^{-1}(Q - \mu)| = |\Omega||\Xi^{-1}| - (Q - \mu)\Xi^{-1}(Q - \mu)^*|,
\]

and that

\[
|I_n - \Xi^{-1}(Q - \mu)^* \Omega(Q - \mu)| = |\Xi|^{-1} \Xi - (Q - \mu)^* \Omega(Q - \mu)^*|.
\]

2. This is similar to the given to item 1. \( \square \)

Corollary 3.2. 1. Assume that \( Q \sim \mathcal{P}_{\mathbb{R}^{\beta,1}}^{m \times n}(\nu, \kappa, \tau, \mu, \Omega, \Xi) \), then

\[
Q^* \sim \mathcal{P}_{\mathbb{R}^{\beta,1}}^{n \times m}(\nu, \kappa, \tau, \mu^*, \Xi^{-1}, \Omega^{-1}).
\]

2. Suppose that \( Q \sim \mathcal{P}_{\mathbb{R}^{\beta,1}}^{m \times n}(\nu, \kappa, \tau, \mu, \Omega, \Xi) \), then

\[
Q^* \sim \mathcal{P}_{\mathbb{R}^{\beta,1}}^{n \times m}(\nu, \kappa, \tau, \mu^*, \Xi^{-1}, \Omega^{-1}).
\]

Proof. The proof follows immediately from the two expressions for the density of \( Q \) in Corollary 3.1. \( \square \)

Corollary 3.3. Let \( \kappa = (k_1, k_2, \ldots, k_m) \), \( k_1 \geq k_2 \geq \cdots \geq k_m \geq 0 \), \( k_1, k_2, \ldots, k_m \) are nonnegative integers, and \( \tau = (t_1, t_2, \ldots, t_m) \), \( t_1 \geq t_2 \geq \cdots \geq t_m \geq 0 \), \( t_1, t_2, \ldots, t_m \) are nonnegative integers. Also define \( R \in \mathbb{L}^{\beta} \) as

\[
R = Y L_{n}^{-1},
\]

1. where \( L_1 \) is any square root of \( V = L_1 L_1^* = V_1 + YY^* \) where \( V_1 \sim \mathcal{R}_n^{\beta,1}(\nu + n - m)/2, \kappa, I_m \), \( \text{Re}(\nu + n - m)/2 > (m - 1)/2 - k_m \); independent of \( Y \sim \mathcal{K}_{R^{\beta,1}}^{m \times n}(\tau, 0, I_m, I_n) \), \( \text{Re}(n/2) > (m - 1)/2 - t_m \). Then \( V \sim \mathcal{R}_n^{\beta,1}(\nu + n)/2, \kappa + \tau, I_m \) independent of \( R \), where \( \text{Re}(\nu + n)/2 > (m - 1)/2 - k_m - t_m \). Moreover, \( R \sim \mathcal{P}_{\mathbb{R}^{\beta,1}}^{m \times n}(\nu, \kappa, \tau, 0, I_m, I_n) \).
2. where \( L_1 \) is any square root of \( V = L_1L_1^* = V_1 + Y^*Y \) where \( V_1 \sim \mathcal{R}_{\nu, \kappa, 0, I_m, I_n}^{\beta, I}((\nu + n - m)\beta/2, \kappa, I_m) \), \( \Re((\nu + n - m)\beta/2) > (m - 1)\beta/2 + k_1 \); independent of \( Y \sim \mathcal{K} \mathcal{R}_{\nu, \kappa, \tau, I_n}^{\beta, I}((\nu + n - m)\beta/2, \kappa, \tau, I_m) \), \( \Re(\nu + n - m)\beta/2) > (m - 1)\beta/2 + t_1 \). Then \( V \sim \mathcal{R}_{\nu, \kappa, \tau, I_n}^{\beta, I}((\nu + n)\beta/2, \kappa + \tau, I_n) \) independent of \( R \), with \( \Re((\nu + n)\beta/2) > (m - 1)\beta/2 + k_1 + t_1 \). Moreover, \( R \sim \mathcal{P}_{\nu, \kappa, \tau, I_n}^{\beta, I}((\nu + n)\beta/2, \kappa + \tau, I_n) \).

**Proof.** The proof is a verbatim copy of the proof of Theorem 3.1. \( \square \)

Now \( c\)-beta-Riesz type I and \( k\)-beta-Riesz type I distributions are obtained, see [5]. Let \( n \geq m \) and let \( B \in \mathcal{P}_m^\beta \) defined as \( B = RR^* \) then, under the conditions of Theorem 3.1 and Corollary 3.3 we have

\[
B = L^{-1}XX^*(L^{-1})^* = L^{-1}W(L^{-1})^* = Y(V_1 + Y^*Y)^{-1}Y^*,
\]

where \( W = XX^* \). Therefore:

**Theorem 3.2.** 1. Assuming that \( R \sim \mathcal{P}_{\nu, \kappa, \tau, I_n}^{\beta, I}((\nu + n - m)\beta/2, \kappa, \tau, I_m, I_n) \), we have that \( W = XX^* \sim \mathcal{R}_{m,n}^{\beta, I}(n\beta/2, \tau, I_m) \), \( \Re(n\beta/2) > (m - 1)\beta/2 - t_m \). Moreover, the density of \( B \), such that \( I_m - B \in \mathcal{P}_m^\beta \) is

\[
\frac{|B|^{(n-m+1)\beta/2-1}}{B_m^{\nu\beta/2, -\kappa; n\beta/2, -\tau}} |I_m - B|^{\nu - m + 1, \beta/2 - 1} q_\nu(I_m - B)q_\tau(B)(dB). \tag{22}
\]

\( B \) is said to have a matrixvariate \( c\)-beta-Riesz type I distribution.

2. Suppose that \( R \sim \mathcal{P}_{\nu, \kappa, \tau, I_n}^{\beta, I}((\nu + n - m)\beta/2, \kappa, \tau, I_m, I_n) \), then \( W = XX^* \sim \mathcal{R}_{m,n}^{\beta, I}(n\beta/2, \tau, I_m) \), \( \Re(n\beta/2) > (m - 1)\beta/2 + t_1 \) and the density of \( B \), such that \( I_m - B \in \mathcal{P}_m^\beta \) is

\[
\frac{|B|^{(n-m+1)\beta/2-1}}{B_m^{\nu\beta/2, -\kappa; n\beta/2, -\tau}} |I_m - B|^\nu q_\nu(I_m - B)q_\tau(B)(dB). \tag{23}
\]

\( B \) is said to have a matrixvariate \( k\)-beta-Riesz type I distribution.

**Proof.** The proof follows from [18] by applying [1] and [17]. \( \square \)

In addition, assume that \( n < m \) and let \( \tilde{B} \in \mathcal{P}_n^\beta \) defined as \( \tilde{B} = \tilde{R}^*\tilde{R} \) then, under the conditions of Theorem 3.1 and Corollary 3.3 we have

\[
\tilde{B} = X^*(U_1 + X^*X)^{-1}X = L_{i_n}^{-1}Y^*Y(L_{i_n}^{-1})^* = L_{i_n}^{-1}W_1(L_{i_n}^{-1})^*,
\]

where \( W_1 = Y^*Y \). Hence:

**Theorem 3.3.** 1. Assuming that \( R \sim \mathcal{P}_{\nu, \kappa, \tau, I_n}^{\beta, I}((\nu + n - m)\beta/2, \kappa, \tau, I_m, I_n) \), then \( W_1 = Y^*Y \sim \mathcal{R}_{n}^{\beta, I}(m\beta/2, \tau, I_m) \), \( \Re(m\beta/2) > (m - 1)\beta/2 - t_m \) and \( \tilde{B} \) has the density

\[
\frac{|B|^{(m-n+1)\beta/2-1}}{B_m^{(m-n+1)\beta/2, -\kappa; m\beta/2, -\tau}} |I_n - \tilde{B}|^{\nu - m + 1, \beta/2 - 1} q_\nu(I_n - \tilde{B})q_\tau(\tilde{B})(d\tilde{B}). \tag{24}
\]

where \( I_n - \tilde{B} \in \mathcal{P}_n^\beta \), also, we say that \( \tilde{B} \) has a matrixvariate \( c\)-beta-Riesz type I distribution.
2. Similarly, assuming that $R \sim P_{m \times n}^{\beta/2}(\nu, \kappa, \tau, I_m, I_n)$, then, $W_1 = Y^*Y \sim R_n^{\beta/2}(m\beta/2, \tau, I_m)$, moreover, the density of $\tilde{B}$ is

$$W_1 = Y^*Y \sim R_n^{\beta/2}(m\beta/2, \tau, I_m)$$

Moreover, the density of $\tilde{B}$ is

$$|\tilde{B}|/(m + n + 1) = q_n[(I_n - \tilde{B})^{-1}]$$ (25)

where $I_n - \tilde{B} \in \mathbb{R}^n$. We say that $\tilde{B}$ has a matricvariate k-beta-Riesz type I distribution.

**Proof.** The proof is the same as that given in Theorem 3.2.

Alternatively, observe that densities (24) and (25) can be obtained from densities (22) and (23), respectively, making the following substitutions, see [34, Eq. (7), p. 455] and [38, p. 96],

$$m \to n, \quad n \to m, \quad \nu \to \nu + n - m.$$ (26)

**Corollary 3.4.** Define $C = M^*BM$, where $M$ is any square root of $\Theta = M^*M$.

1. Assuming that $B$ has the density (23), then the density of random matrix $C$ is

$$|C|(n - m + 1)\beta/2 - 1 [\Theta - C]^{(n - m + 1)\beta/2 - 1} q_n[(\Theta - C)^{-1} q_r(C) dC),$$ (27)

with constant of proportionally

$$|\Theta|^{(n + \nu - m + 1)\beta/2 - 1} \frac{B_n^{\beta/2}}{\nu^2} \frac{q_n(\Theta)}{q_r(\Theta)}$$

for $\Theta - C \in \mathbb{R}^n$.

2. Supposing that $B$ has the density (25), then the density of random matrix $C$ is

$$|C|(n - m + 1)\beta/2 - 1 [\Theta - C]^{(n - m + 1)\beta/2 - 1} q_n[(\Theta - C)^{-1} q_r(C^{-1}) dC),$$ (28)

with constant of proportionally

$$|\Theta|^{(n + \nu - m + 1)\beta/2 - 1} \frac{B_n^{\beta/2}}{\nu^2} \frac{q_n(\Theta)}{q_r(\Theta)}$$

for $\Theta - C \in \mathbb{R}^n$.

**Proof.** This immediate from (15).

Finally, densities of $\tilde{C} = M^*\tilde{B}M$, where $\tilde{B}$ has the densities (24) and (25) are obtained from (27) and (28) applying the substitutions (26), respectively.

### 4 Singular value densities

In this section, we derive the joint density of the singular values of matrices $R$ and $\tilde{R}$. Also, the joint density of the eigenvalues of $B$ and $\tilde{B}$ are obtained.
Theorem 4.1. 1. Let \( \delta_1, \ldots, \delta_m \) be the singular values of \( R \sim \mathcal{P}_{\mathcal{I}} \mathcal{I} \mathcal{R}_{m \times n}^{\beta, I}(\nu, \kappa, \tau, 0, I_m, I_n) \),

\[
1 > \delta_1 > \cdots > \delta_m > 0.
\]
Then its joint density is

\[
\alpha \prod_{i=1}^{m} \delta_i^{2i+(n-m+1)\beta-1}(1-\delta_i^{2})^{k_i+(n-m+1)\beta/2-1} \prod_{i<j}^{m} (\delta_i^2 - \delta_j^2)^j \bigwedge_{i=1}^{m} d\delta_i, \tag{29}
\]

with constant of proportionally

\[
\frac{2^m \pi^{m^2 \beta+\varrho}}{\Gamma_m^\beta [m\beta/2] B_m^\beta [\nu \beta/2; \kappa; n\beta/2, \tau]}.\]

2. Let \( \delta_1, \ldots, \delta_m \) be the singular values of \( R \sim \mathcal{P}_{\mathcal{I}} \mathcal{I} \mathcal{R}_{m \times n}^{\beta, II}(\nu, \kappa, \tau, 0, I_m, I_n) \),

\[
1 > \delta_1 > \cdots > \delta_m > 0.
\]
Then its joint density is

\[
\alpha \prod_{i=1}^{m} \delta_i^{-2i+(n-m+1)\beta-1}(1-\delta_i)^{-k_i+(n-m+1)\beta/2-1} \prod_{i<j}^{m} (\delta_i^2 - \delta_j^2)^j \bigwedge_{i=1}^{m} d\delta_i, \tag{30}
\]

with constant of proportionally

\[
\frac{2^m \pi^{m^2 \beta+\varrho}}{\Gamma_m^\beta [m\beta/2] B_m^\beta [\nu \beta/2; -\kappa; n\beta/2, -\tau]},\]

where \( \varrho \) is defined in Lemma 2.3.

Proof. The proof follows immediately from (18) and (19), respectively, first using (16) and then applying (1).

Observing that \( \delta_i = \sqrt{\text{eig}_i(\mathbb{B}^\ast \mathbb{R}^\ast)} \), where \( \text{eig}_i(\mathbb{A}) \), \( i = 1, \ldots, m \), denotes the \( i \)-th eigenvalue of \( \mathbb{A} \). Let \( \lambda_i = \text{eig}_i(\mathbb{R}^\ast) = \text{eig}_i(\mathbb{B}) \), hence, defining \( \delta_i = \sqrt{\lambda_i} \) we have

\[
\bigwedge_{i=1}^{m} d\delta_i = \bigwedge_{i=1}^{m} 2^{-m} \prod_{i=1}^{m} \lambda_i^{-1/2} d\lambda_i,
\]

then, the corresponding joint densities of \( \lambda_1, \ldots, \lambda_m, 1 > \lambda_1 > \cdots > \lambda_m > 0 \) are obtained from (29) and (30), respectively. Specifically it get,

1. Let \( \lambda_1, \ldots, \lambda_m \) be the eigenvalues of \( \mathbb{B} \) with density (22), \( 1 > \lambda_1 > \cdots > \lambda_m > 0 \).
Then its joint density is

\[
\alpha \prod_{i=1}^{m} \lambda_i^{2i+(n-m+1)\beta/2-1}(1-\lambda_i)^{k_i+(n-m+1)\beta/2-1} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^j \bigwedge_{i=1}^{m} d\lambda_i, \tag{31}
\]

with constant of proportionally

\[
\frac{\pi^{m^2 \beta+\varrho}}{\Gamma_m^\beta [m\beta/2] B_m^\beta [\nu \beta/2; \kappa; n\beta/2, \tau]}.
\]
2. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of $B$ with density \([23]\), $1 > \lambda_1 > \cdots > \lambda_m > 0$. Then its joint density is
\[
\propto \prod_{i=1}^{m} \lambda_i^{-\nu_i + (n-m+1)/2 - 1} (1 - \lambda_i)^{-\kappa_i + (n-m+1)/2 - 1} \prod_{i<j}^{m} (\lambda_i - \lambda_j)^\beta \prod_{i=1}^{m} d\lambda_i, \tag{32}
\]
with constant of proportionally
\[
\pi^{m^2 \beta + \nu}
\frac{\Gamma_{m}^{\beta} [m\beta/2, B_m^{\beta}] [\nu\beta/2, -\kappa; n\beta/2, -\tau]}.\]

Finally, the joint density of singular values of $\tilde{R}$ type I and II are obtained from \([29]\) and \([30]\) after making the substitutions \([26]\), respectively. Analogously, the joint density of eigenvalues of $\tilde{B}$ type I and II are obtained from \([31]\) and \([32]\) after making the substitutions \([26]\), respectively.

Conclusions

Undoubtedly, in any generalisation of results there is a price to be paid, and in this case the price is that of acquiring a basic understanding of some concepts of abstract algebra, which can initially be summarised as the use of notation and a basic minimum set of definitions. However, we believe that a detailed study of mathematical properties from a statistical standpoint can have a potential impact on statistical theory. Furthermore, some statistical results in the literature have been studied. For example, \([33]\) address the problem of point estimation of parameters in complex shape theory. Also, \([28]\) considered the estimation of parameters of a complex matrix multivariate normal distribution and establishes a test of hypotheses about the mean. In a quaternionic context, \([2]\) set test statistics and their corresponding asymptotic distributions for two particular hypothesis tests. As noted by the reviewer, the statistical results in \([34]\) and \([19]\) can be extended, and in fact are being extended to the case of real normed division algebras, but first they needed to study various preliminary results, including those obtained in this work.

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