PATH-DEPENDENT SHRINKING TARGETS IN GENERIC AFFINE ITERATED FUNCTION SYSTEMS

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ABSTRACT. We calculate the Hausdorff dimension of path-dependent shrinking target sets in generic affine iterated function systems. Here, by a path-dependent shrinking target set, we mean a set of points whose orbits infinitely often hit small balls with a fixed generic centre and with radius decreasing and dependent on the point itself. It turns out that the Hausdorff dimension of such a set is given by the zero point of a certain limsup pressure function. The result generalizes the work of Koivusalo and Ramírez, and Bárány and Troscheit, as well as that of Hill and Velani.

1. Background

In analogy with the classical metric theory of Diophantine approximation, Hill and Velani [HV95] initiated the investigation of the shrinking target problem. Consider a transformation $T$ on a metric space $(X,d)$. For any fixed point $z_0 \in X$, and any decreasing sequence $(r_n)$ such that $r_n \to 0$ as $n \to \infty$, the shrinking target problem is to study the size, in terms of Hausdorff dimension, of the set

$$\mathcal{S}(z_0, (r_n)) := \{x \in X : d(T^n x, z_0) < r_n, \text{ for infinitely many } n\},$$

which is in fact the set of points whose orbits, under the action of $T$, hit infinitely often the shrinking targets, i.e., the balls $B(z_0, r_n)$.

In [HV95], Hill and Velani calculated the Hausdorff dimension of $\mathcal{S}(z_0, (r_n))$ when $T$ is an expanding rational map of the Riemann sphere and $X$ is its Julia set. Later, in [HV97], with the same setting of [HV95], they studied a variation of $\mathcal{S}(z_0, (r_n))$ where they let the radius $r_n$ depend on the point $x$. More precisely, for a given Hölder continuous function $\psi$, satisfying $\psi(x) \geq \log|T'(x)|$ for all $x$ in the Julia set, they proved that for any $z_0$ in the Julia set, the Hausdorff dimension of the set

$$\left\{x \in X : T^n x \in B(y, \exp \left\{-\sum_{i=0}^{n-1} \psi(T^i x)\right\}, \text{ for infinitely many } (y, n) \text{ with } y \in T^{-n}(z_0)\right\},$$

is given by the zero point of a pressure function $s \mapsto P(T, -s\psi)$. Because of the dependence of the radius on the path of $x$, let us call this latter set a path-dependent shrinking target set. Such path-dependent shrinking target set was also studied by Urbański [U02], who proved that the result of Hill and Velani [HV97] also holds for the conformal iterated function systems. Recently, variations of path-dependent shrinking target sets have received much

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attention. In particular, for any \( z_0 \in X \), the Hausdorff dimension of the set

\[
\mathcal{H}(z_0, \psi) := \left\{ x \in X : T^n x \in B(z_0, \exp \left\{ - \sum_{i=0}^{n-1} \psi(T^i x) \right\}) \right\}
\]

has been proved to be the zero of the pressure function \( s \mapsto P(-s(-\log |T'| + \psi)) \) for \( \beta \)-transformation by Bugeaud and Wang [BW14], for the Gauss map by Li, Wang, Wu, and Xu [LWWX14], and for countable Markov maps by Reeve [R11].

In this paper, we study path-dependent shrinking target sets in simple non-conformal dynamical systems, namely, on self-affine sets. Versions of the non-path-dependent case have been covered in [KR18, BT].

Let \( \{f_1, \ldots, f_N\} \) be a collection of affine contractions on \( \mathbb{R}^d \) with strong separation condition. It is well-known that such a collection, known as an affine iterated function system, always gives rise to a self-affine set \( \Lambda \), which is invariant under the action of the maps. In this situation, an expanding map \( E \) on \( \Lambda \) also exists, with \( f_i \) as its local inverses. Letting \( \psi : \Lambda \to \mathbb{R} \) be a Hölder continuous function, the starting point to the set-up in the current article is the path-dependent shrinking target set

\[
\left\{ x \in \Lambda : E^n x \in B(z_0, \exp \left\{ - \sum_{i=0}^{n-1} \psi(E^i x) \right\}) \right\}.
\]

However, we will not study this exact set, and indeed, believe that its general solution is out of reach of current research, as the symbolic description of geometric balls is a very involved problem in general. A ball centred at \( z_0 \) might intersect many cylinders coded by a finite word on the alphabet \( \{1, \ldots, N\} \) and there is no easy way to determine which ones do. In a sense, geometric balls are incompatible with the dynamical system, making the above formulation of path-dependent shrinking target set slightly unnatural from a dynamics perspective. However, it should be pointed out as a sensible question from geometry point of view, and indeed, for a special class of self-affine sets known as Bedford-McMullen carpets, the non-path-dependent version of this geometric shrinking target set has been studied [BR]. The structure of Bedford-McMullen carpets allows for a straightforward translation between geometric and symbolic languages, a tool which is not available in the general case.

To circumvent this geometric difficulty, we turn our attention to the investigation of a symbolic version of the path-dependent shrinking target sets. Then, by projecting the symbolic space to \( \Lambda \), we can calculate for generic translations of the affine maps \( \{f_1, \ldots, f_N\} \), the Hausdorff dimension of this symbolically induced path-dependent shrinking target set on \( \Lambda \). We will give details of the model underneath in Section 2.

There is also a dynamical difficulty in the dimension theory of non-conformal iterated function systems. That is, the contractions in affine iterated function systems usually do not have multiplicativity, and hence even determining the sizes of the cylinders is difficult. In the literature there are several workarounds, and in particular a lot of modern theory relies on various weak quasi-multiplicativity conditions, which can be shown to be generic. Inspired by an idea of Bárány and Troscheit [BT], we suggest a novel approach to dimension estimation based on writing the space modularly, see Section 5. We believe this new technique to be of independent interest.
2. Preliminaries and the statement of results

2.1. Symbolic space. Denote \( A = \{1, \ldots, N\} \), \( \Sigma := A^N \) and let \( \sigma \) be the left shift operator on \( \Sigma \). Then the pair \((\Sigma, \sigma)\) is a dynamical system called the full shift dynamics on the alphabet \( A \) of \( N \) symbols. Let \( \Sigma_n := A^n \) be the set of words of length \( n \). We denote infinite words by bold letters \( i, j, a \) and so on, and finite words by \( i, j, a \) and so on. The set \( \Sigma_* := \bigcup_{n=1}^{\infty} \Sigma_n \) is the collection of all finite words. For any \( i = i_1 \cdots i_n \in \Sigma_n \), denote by \([i]\) the cylinder corresponding to \( i \), i.e.,

\[
[i] := \{ j = (j_1, j_2, \ldots) \in \Sigma : j_1 = i_1, \ldots, j_n = i_n \}.
\]

The length \( n \) of \( i \in \Sigma_n \) is denoted by \(|i|\). For \( i \in \Sigma \), denote by \( i_n := i_1 i_2 \cdots i_n \) the finite word composed of the first \( n \) symbols of \( i \), and by \( i^n_m := i_{m+1} i_{m+2} \cdots i_n \) the finite word composed of the symbols between the positions \( m \) and \( n \). Such a finite word \( i^n_m \) is called a subword of \( i \). For convenience, for a positive real number \( \ell \), we write \( i_{\lfloor \ell \rfloor} \) for \( i_{\lfloor \ell \rfloor} \), where \( \lfloor \cdot \rfloor \) denotes the integer part. One can easily interpret the similar symbols \( i_{\lfloor n \rfloor}, i^n_m \) and \( i_{\lfloor \ell \rfloor} \) for a finite word \( i \in \Sigma_* \).

2.2. Symbolic shrinking targets. We will investigate a variation of the shrinking target problem for \((\Sigma, \sigma)\). A length sequence function is a function \( \ell : \Sigma \to (\mathbb{R}^+)^N \) defined as \( i \mapsto \ell(i) = (\ell_n(i))_{n \geq 1} \). The value \( \ell(i) \), which depends on an infinite word \( i \), will stand for the lengths (sizes) of the shrinking targets. The center of the shrinking targets will be an infinite word \( j \in \Sigma \). Then for any \( j, i \in \Sigma \), we define a sequence of finite words, i.e., a family of targets:

\[
j(i, \ell) := \{ j|_{\ell_n(i)} \}_{n \geq 1}.
\]

For a length sequence function \( \ell \) and \( j \in \Sigma \), define the following symbolic path-dependent shrinking target set

\[
R(j, \ell) = \{ i \in \Sigma | \sigma^n(i) \in [j|_{\ell_n(i)}] \text{ for infinitely many } n \}.
\]

2.3. Iterated function systems and shrinking targets. Let \( \{f_1, \ldots, f_N\} \) be a collection of affine contractions, that is, let \( T_1, \ldots, T_N \) be linear contractions and \( v_1, \ldots, v_N \in \mathbb{R}^d \), and let \( f_i(x) = T_i(x) + v_i \) for \( i = 1, \ldots, N \). This is called an affine iterated function system. For \( i = (i_1 i_2 \cdots i_n) \in \Sigma_* \), we denote

\[
T_i := T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n},
\]

and similarly for \( f_i \). Recall that by a classical theorem of Hutchinson [H81], an iterated function system defines a unique, non-empty, compact, invariant set \( \Lambda \) such that

\[
\Lambda = \bigcup_{i=1}^{N} f_i(\Lambda).
\]

Assume throughout that this affine iterated function system under consideration satisfies the strong separation condition, which means that for its invariant set \( \Lambda \), the images \( f_i(\Lambda) \) are disjoint. In particular, then the mapping

\[
\pi : \Sigma \to \Lambda, \pi(i) = \lim_{n} f_{i_n}(0)
\]

is a bijection, and each \( x \in \Lambda \) corresponds to exactly one infinite sequence \( i = \pi^{-1}(x) = x_1 x_2 \ldots \).

The symbols \( x_k \) are called the digits of \( x \). Further, there is an expanding map \( E \) on \( \Lambda \) with \( f_i \) being its local inverses, given by \( E : \Lambda \to \Lambda, E(x) = f_{x_1}^{-1}(x) \), where \( x_1 \) is the first digit of \( \pi^{-1}(x) \). The map \( E \) is conjugate to the associated symbolic shift dynamics. That is we have the following commutative diagram:
Hence, the symbolic shrinking target sets $R(j, \ell)$ defined above have geometric interpretation as subsets of $\Lambda$, as

$$\pi \circ R(j, \ell) = \{ x \in \Lambda \mid E^n(x) \in \pi[j|\ell_n(\pi^{-1}(x))] \text{ for infinitely many } n \}.$$ 

Generally speaking, the target sets $\pi[j|\ell_n(i)]$ do not have a nice geometric meaning, but for example, when there is a rectangle $C$ such that $\bigcup_{i=1}^m f_i(C) \subseteq C$ and the union is disjoint, the target sets can be taken to be rectangles.

### 2.4. Additive and sub-additive potentials.

A potential is a function $\phi : \Sigma \to \mathbb{R}$. Together with a potential one considers its Birkhoff sums, for $i \in \Sigma$,

$$S_n \phi(i) = \sum_{i=0}^{n-1} \phi(\sigma^i i).$$

A special class of potentials are piecewise constant potentials whose values depend only on the first symbol, $\phi(i) = \phi(i_1)$. For piecewise constant potentials, their Birkhoff sums also depend only on the first finitely many symbols: for $i \in \Sigma$,

$$S_n \phi(i) = S_n \phi(i_1, \ldots, i_n).$$

The notion of potential was generalized to sub-additive potentials (the ‘usual’ potentials are sometimes called additive potentials, to distinguish them from sub-additive ones). A sub-additive potential is a family of functions $\phi_n : \Sigma \to \mathbb{R}$; $n = 1, 2, \ldots$ satisfying for $i \in \Sigma$

$$\phi_{m+n}(i) \leq \phi_m(i) + \phi_n(\sigma^m i).$$

Clearly, for $\phi_n = S_n \phi$ the inequality (3) is automatically satisfied and is an equality, hence the sub-additive potential is indeed a generalization of the additive potential or, more precisely, a generalization of its family of Birkhoff sums.

The piecewise constant potentials have their analogue among sub-additive potentials: potentials such that for every $n$, $\phi_n$ depends only on the first $n$ symbols. In this case, (3) takes the following ‘concatenating’ form: for an element $i \in \Sigma$, $i = (i_1, i_2, i_3, \ldots)$,

$$\phi_{m+n}(i_1, \ldots, i_{m+n}) \leq \phi_m(i_1, \ldots, i_m) + \phi_n(i_{m+1}, \ldots, i_{m+n}).$$

Note that a function $\psi : \Sigma_* \to \mathbb{R}$ induces a family of functions $\phi_n$ defined on $\Sigma$, and depending only on the first $n$ symbols in the following natural way:

$$\phi_n(i) = \psi(i), \quad \forall i \in \Sigma_*.$$ 

Thus, we usually study functions defined on $\Sigma_*$, and a potential $\psi : \Sigma_* \to \mathbb{R}$ is sub-additive if the induced family $\phi_n$ is sub-additive.
In this paper any mention of 'potential' will always mean a sub-additive potential.

2.5. Weakly quasi-additive singular value potentials. To a linear map \( T : \mathbb{R}^d \to \mathbb{R}^d \) and a parameter \( s > 0 \) we can associate a singular value function

\[
\phi^s(T) = \alpha_1 \alpha_2 \cdots \alpha_{\lfloor s \rfloor} \alpha_{\lfloor s \rfloor + 1}^{s - \lfloor s \rfloor},
\]

where \( \alpha_i \) are the singular values of \( T \) in descending order. The singular value function was first introduced in affine dimension theory by Falconer \([\text{F88}]\). For a fixed collection \( T_1, \ldots, T_N \) of linear maps on \( \mathbb{R}^d \), we write \( \phi^s(i) \) for \( \phi^s(T_i) \).

When investigating affine iterated function systems, one needs to work with the thermodynamic formalism for the sub-additive potential \( \log \phi^s \) defined on some matrix cocycle (generated by contracting maps, hence \( \log \phi^s \) is strictly negative). Fortunately, this potential quite often has better properties than mere sub-additivity. In particular, there is an open set (in the parameter space) of matrix cocycles for which this potential \( \psi = \log \phi^s \) is actually quasi-additive: there exists a constant \( Q \) such that for any two finite words \( i, j \in \Sigma \) we have

\[
\psi(ij) \geq Q + \psi(i) + \psi(j).
\]

This property is used in many papers including \([\text{KR18}]\), we will not present an exhaustive list.

However, when looking for a property that would be satisfied by \( \log \phi^s \) for a generic matrix cocycle, we only find something much weaker: weak quasi-additivity.

**Definition 2.1** (weak quasi-additivity). A potential \( \psi \) is weakly quasi-additive when there exist constants \( Q, K \) such that for any two finite words \( i, j \in \Sigma \) we have some \( k \in \Sigma, |k| \leq K \) such that

\[
\psi(ikj) \geq Q + \psi(i) + \psi(j).
\]

**Remark.** We emphasize at this point that the property of weak quasi-additivity is in fact not a property of \( \psi \) alone, but of \( \psi \) and the matrix cocycle generated by \( T_1, \ldots, T_N \) together. However, since in the context of this article an underlying predefined collection of linear maps is considered fixed, we use slightly imprecise language and call the potential \( \psi \) weakly quasi-additive.

The weak quasi-additivity condition is noticeably weaker than quasi-additivity. In \([\text{BT}]\) Bárány and Troscheit proposed a very interesting approach to handling thermodynamic formalism for weakly quasi-additive potentials, which in some sense allows us to reduce the weakly quasi-additive situation to quasi-additive one. In the arguments below we are building on their proof idea.

2.6. Statement of the main theorem. We are now ready to start formulating our main theorem, which concerns the Hausdorff dimension of a path-dependent shrinking target set. The motivation for setting up the problem in this way is from Hill and Velani \([\text{HV97}]\), Bugeaud and Wang \([\text{BW14}]\), Li, Wang, Wu, and Xu \([\text{LWWX14}]\), Reeve \([\text{R11}]\), et al, who have treated in the conformal setting the problem of path-dependent shrinking targets, where the target balls are given by a Hölder continuous potential.

We need some assumptions on the length sequence function \( \ell \).

**Definition 2.2** (Assumptions on \( \ell \)). Let \( \ell : \Sigma \to (\mathbb{R}^+)^\mathbb{N} \) be a function defined as

\[
\ell(i) = (\ell_n(i))_{n \geq 1}, \quad \forall i \in \Sigma.
\]
Assume that $\ell_n(i)$ depends only on the first $n$ symbols of $i$. Then, for a finite word $i$, we can define an associated function $\underline{\ell}: \Sigma \to (\mathbb{R}^+)$ by $\underline{\ell}(i) := \ell(i|a)$ for any $a \in [i]$. We assume that $\underline{\ell}$ is approximately additive on finite words, in the sense that there exists a constant $\kappa > 0$ such that for any $i, a \in \Sigma$,

$$
|\underline{\ell}(ia) - \underline{\ell}(i) - \underline{\ell}(a)| \leq \kappa.
$$

We also assume that

$$
\ell_n(i) \to \infty \quad \text{for every } i \in \Sigma.
$$

**Remark.** We can easily find a function $\ell$ satisfying the assumptions in Definition 2.2. Let $\psi$ be a potential on $\Sigma$ which depends only on the first symbol, and let

$$
\ell_n(i) = \sum_{i=0}^{n-1} \psi \circ \sigma^i(i).
$$

Then $\ell_n$ only depends on the first $n$ symbols and $\underline{\ell}$ is approximately additive on finite words, with $\kappa = 0$.

**Definition 2.3.** For $j \in \Sigma$, define the following limsup pressure function

$$
P^*(s, j) = \limsup_n \frac{1}{n} \log \sum_{|i|=n} \phi^s(i) \underline{\ell}(i).
$$

When the limit exists, we denote the pressure by $P(s, j)$.

For the path-dependent shrinking target set defined as in (2), we prove the following theorem.

**Theorem 2.4.** Let $\{f_1, \ldots, f_N\}$ with $f_i = T_i + a_i$ be an affine iterated function system satisfying the strong separation condition. Suppose that $\|T_i\| < \frac{1}{2}$ for all $1 \leq i \leq N$ and that $\log \phi^s$ is weakly quasi-additive for all $s \in [0, d]$. Let $\ell$ be as in Definition 2.2. Let $\mu$ be any ergodic measure on $(\Sigma, \sigma)$.

Then for $\mu$-almost every choice of $j$, the limit defining the pressure (7) exists and is independent of the choice of $j$. Further, for these $j$, for Lebesgue almost all $a_1, \ldots, a_N \in \mathbb{R}^d$, the Hausdorff dimension of the path-dependent shrinking target set $R^*(j, \ell)$ is given by $\min(s_0, d)$, where $s_0$ is the unique value for which $P(s_0, j) = 0$.

Moreover, for any fixed $j \in \Sigma$, the Hausdorff dimension of the path-dependent shrinking target set $R^*(j, \ell)$ is given by $\min(s_0, d)$, where $s_0$ is the unique value for which $P^*(s_0, j) = 0$ for the limsup pressure $P^*$.

**Remark.** We remark that our Theorem 2.4 is new even in the conformal case. In fact, when we project the targets $j(i, n)$ to balls in $\Lambda$, the radius of the balls not only depend on $i (\pi(i))$, but also on $j$, or more precisely the Lyapunov exponent at $j$. This situation has not been investigated in [HV97, BW14, LWWX14, R11]. If we consider an IFS with constant Lyapunov exponents, then our result recovers the Hausdorff dimension of the set $\mathcal{H}(z_0, \psi)$ from Section 1 in the case of finite IFS. We also underline that our Theorem 2.4 is a natural way of generalizing the Hausdorff dimension result of $\mathcal{H}(z_0, \psi)$ in the non-conformal case.

The article is organized as follows. In Section 3 we give a simple proof for the upper bound of the Hausdorff dimension and other preliminary observations. In Section 5 we describe the general framework of studying the dynamics of the IFS modularly, and in particular various forms of the pressures for weakly multiplicative potentials. This method is likely.
to be applicable more widely than just in the context of shrinking targets. In Section 6 we specialize to singular value potential relevant in the shrinking target problem. In Section 7 we apply these to define a Cantor set and a mass distribution that are used to prove the lower bound of the Hausdorff dimension in the case of the \( \mu \)-typical \( j \). In Section 8 we explain how to modify the proofs of Sections 6 and 7 so that they can be applied in the fixed target case.

3. Preliminaries and the upper bound

In this section, we first prove some preliminary technical lemmas. Then, we go on prove that the zero point of the limsup pressure defined above always gives an upper bound to the Hausdorff dimension of the shrinking target set \( R^*(j, \ell) \).

We have the following two lemmas on the properties of the length function \( \ell \). Recall its definition in Definition 2.2.

**Lemma 3.1.** If for all \( n \geq 1 \) there exists \( i \) such that \( \ell_n(i) < \kappa \), then there exists \( i \) such that for infinitely many \( n \)'s we have \( \ell_n(i) < 3\kappa \).

**Proof.** We first assert that for all \( i \) and for all \( n \in \mathbb{N} \), we have \( \ell_n(i) > -\kappa \). Otherwise, if there exists some \( i \in \Sigma \), and some \( n \in \mathbb{N} \), such that \( \ell_n(i) = \ell_n(i) < -\kappa \), then by (5),

\[
\ell_{k_n}(\ell_0) = \ell_{k_n}(\ell_0) = \ell_{k_n}(\ell_0) \leq \ell_{k_n}(\ell_0) + (k - 1)(\ell_{k_n}(\ell_0) + \kappa) \to -\infty \quad (k \to \infty),
\]

which contradicts with our assumption (6).

Then, we assert that if for some \( i \in \Sigma \), and some \( n \in \mathbb{N} \), \( \ell_n(i) < \kappa \), then for all \( m < n \) we have \( \ell_m(i) < 3\kappa \). Otherwise, if for some \( m < n \), \( \ell_m(i) \geq 3\kappa \). Then, by (5), and the first assertion, we have

\[
\ell_n(i) \geq \ell_m(i) + \ell_{n-m}(\sigma^m i) - \kappa \geq 3\kappa - \kappa - \kappa = \kappa,
\]

which is a contradiction.

Now, by assumption, for all \( n \in \mathbb{N} \), there exists \( i_n \) such that \( \ell_n(i_n) < \kappa \). Since we have only \( N \) choices for the first symbol for the infinite sequence \( (i_n)_{n \geq 1} \), we can find a symbol \( a_1 \) which appears as the first symbol in infinitely many infinite words \( i_n \). Hence, by the second assertion, we have \( \ell_1(a_1) < 3\kappa \). Similarly, let \( a_2 \) be the symbol such that infinitely many infinite words \( i_n \) begin with \( a_1 a_2 \). Then \( \ell_2(a_1 a_2) < 3\kappa \). Go on this process, we will then obtain an infinite word \( i = a_1 a_2 \ldots \) such that \( \ell_n(i) < 3\kappa \) for all \( n \geq 1 \).

**Lemma 3.2.** There exist some \( L_{\max} > L_{\min} > 0 \) and \( \kappa' > 0 \), such that we have the simple estimates

\[
nL_{\min} - \kappa' \leq \ell_n(i) \leq nL_{\max} + \kappa'.
\]

**Proof.** By assumption (6) and Lemma 3.1, there exists \( n_0 \in \mathbb{N} \), such that \( \ell_{n_0}(i) > \kappa \) for all \( i \in \Sigma \). Denote by \( \kappa_{\min}, \kappa_{\max} \) the minimum and maximum of \( \{\ell_n(i) : i \in \Sigma\} \). Denote by \( \kappa'_{\min}, \kappa'_{\max} \) the minimum and maximum of \( \{\ell_k(i) : k \in [0, n_0 - 1], i \in \Sigma\} \). Then, one can easily check that for every \( i \in \Sigma \) and for every \( n \) we have

\[
\frac{n}{n_0} (\kappa_{\min} - \kappa) + \kappa'_{\min} \leq \ell_n(i) \leq \frac{n}{n_0} (\kappa_{\max} + \kappa) + \kappa'_{\max},
\]

which gives (8) with \( L_{\min} = (\kappa_{\min} - \kappa)/n_0, L_{\max} = (\kappa_{\max} + \kappa)/n_0 \) and \( \kappa' = \max(\kappa_{\max}, \kappa_{\min} - \kappa') \).
We will prove that the limsup pressure \( s \mapsto P^*(s,\mathbf{j}) \) has a unique zero. The proof relies on the following lemma.

**Lemma 3.3.** Let \((T_1, \ldots, T_N)\) be linear maps in \( \mathbb{R}^d \). Denote
\[
\alpha_- := \max_{1 \leq j \leq N} \{ \| T_j^{-1} \| \}, \quad \alpha_+ := \max_{1 \leq j \leq N} \{ \| T_j \| \}.
\]
Let \( i \in \Sigma_n \) be a finite word. Let \( 0 \leq t < s \leq d \). Then
\[
(\alpha_-)^{-n(s-t)} \leq \frac{\phi^s(T_i)}{\phi^t(T_i)} \leq (\alpha_+)^{n(s-t)}.
\]
In particular,
\[
(\alpha_-)^{-ns} \leq \phi^s(T_i) \leq (\alpha_+)^{ns}.
\]

**Proof.** Both are straightforward consequences of the definition, by the facts that \( \alpha_1(TU) \leq \alpha_1(T)\alpha_1(U) \) and \( \alpha_d(TU) \geq \alpha_d(T)\alpha_d(U) \) for any two linear maps \( T \) and \( U \). \( \square \)

**Lemma 3.4.** Fix \( \mathbf{j} \in \Sigma \). There is a unique \( s_0 \) such that the limsup pressure \( P^*(s_0,\mathbf{j}) = 0 \).

**Proof.** By Lemma 3.2, there exist \( L_{\max} > L_{\min} > 0 \) and \( \kappa' > 0 \) satisfying
\[
L_{\min} |i| - \kappa' \leq \ell(i) \leq L_{\max} |i| + \kappa', \quad \forall i \in \Sigma, \text{ with } |i| \to \infty.
\]
Thus, by Lemma 3.3, for any \( \mathbf{j} \in \Sigma \) and for any \( k \in \mathbb{N} \), we have
\[
\alpha_-(\kappa'+L_{\max}k) \delta \leq \frac{\sum_{|i|=k} \phi^{s+\delta}(ij|\ell(i))}{\sum_{|i|=k} \phi^{s}(ij|\ell(i))} \leq \alpha_+(\kappa'+L_{\min}k) \delta, \quad \forall s, \delta > 0.
\]
Then, it follows that \( P^*(s,\mathbf{j}) \) is continuous and strictly decreasing in \( s \). Further, \( P^*(0,\mathbf{j}) > 0 \) and \( P^*(s,\mathbf{j}) \to -\infty \) as \( s \to \infty \). Therefore, a unique zero always exists. \( \square \)

Now, we are ready to give the upper bound of the Hausdorff dimension.

**Lemma 3.5.** For every \( \mathbf{j} \in \Sigma \) and every \( \ell \) as in Definition 2.2, the Hausdorff dimension of \( R^*(\pi(\mathbf{j}), \ell) \) is bounded from above by \( \min\{s_0, d\} \), where \( s_0 \) is the unique real number satisfying \( P^*(s_0,\mathbf{j}) = 0 \).

**Proof.** This is a standard affine covering argument. We provide the details of the proof for the convenience of the reader.

The Hausdorff dimension is always bounded from above by \( d \). For the upper bound \( s_0 \), let \( s > s_0 \) be arbitrary. Notice that \( R^*(\pi(\mathbf{j})) \) is a limsup set, so that for all \( n \) it is covered by
\[
\bigcup_{|i|=n} \pi[ij|\ell(i)].
\]
By the definition of the singular value function, each of the cylinders \( \pi[ij|\ell(i)] \) can be covered by \( c\phi^{s}(T_{ij|\ell(i)}) \alpha_{|s|+1}(ij|\ell(i))^{-s} \) cubes of sidelength \( \alpha_{|s|+1}(ij|\ell(i)) \), where \( c \) is an absolute constant. Hence,
\[
\mathcal{H}^s(R^*(\mathbf{j}, \ell)) \leq \lim_{n \to \infty} c \sum_{|i|=n} \phi^{s}(T_{ij|\ell(i)}).
\]
Since \( s > s_0 \), we have \( \sum_{|i|=n} \phi^{s}(T_{ij|\ell(i)}) < \infty \). Therefore, (9) implies that \( \mathcal{H}^s(R^*(\mathbf{j}, \ell)) = 0 \), completing the proof of the lemma. \( \square \)
4. Glossary of pressures and partial sums

In the following three sections we will define and compare pressure functionals on many different spaces and for many different potentials. We list here all the notation with a reference to the definition in the text, for the reader's convenience.

4.1. Pressures defined from general potential $\psi$.

- $P_{\text{mod}}(\psi) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{i \in A^n} e^{\psi(i)}$ (10)
- $P_{\text{full}}(\Sigma, \psi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{|i|=n} e^{\psi(i)}$ (11)
- $P_{\text{full}}(\Sigma^R, \tilde{\psi}) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{a_1, \ldots, a_n \in A^R} e^{\tilde{\psi}(a_1 \ldots a_n)}$ with $\tilde{\psi}$ extended from $\psi$ by additivity. (12)

4.2. Partial sums for pressures from general potential $\psi$.

- $S_{\text{mod}}(\Sigma, \psi, n) = \sum_{i \in A^n} e^{\psi(i)}$ (14)
- $S_{\text{full}}(\Sigma, \psi, n) := \sum_{i \in A^n} e^{\psi(i)}$ (20)
- $S_{\text{full}}(\Sigma^R, \tilde{\psi}, n) = \sum_{a_1, \ldots, a_n \in A^R} e^{\tilde{\psi}(a_1 \ldots a_n)}$. (13)

4.3. Pressures from the potentials corresponding to shrinking targets.

- $P(s, j) = P_{\text{full}}(\psi_{s,j}) = P_{\text{full}}(\Sigma, \psi_{s,j})$, where $\psi_{s,j} : i \mapsto \log \phi^s(\ell(i))$ (24)
- $P^*(s, j) = P_{\text{full}}^*(\psi_{s,j}) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{|i|=n} e^{\psi_{s,j}(i)}$ (23)
- $P_{\text{mod}}^*(\psi_{s,\partial}) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{|i|=n} \phi^s(i \partial(i))$, where $\psi_{s,\partial} : i \mapsto \log \phi^s(i \partial(i)), \partial(i) = cj_{\ell(c)}$ (27)
- $P_{\text{mod}}^*(s, j) := P_{\text{mod}}^*(\psi_{s,\partial}) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{k=0}^{K+R-1} \sum_{i \in \mathcal{U}_R^{\partial+k}} \phi^s(i \partial(i))$ (29)
- $P^*_{\ell}(s, j) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{n \leq \ell(i) \leq n + H} \phi^s(i \ell(i)) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{n \leq \ell(i) \leq n + H} \phi^s(i \partial(i))$ (37)
- $\tilde{P}^*_{\ell}(s, j) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{n \leq \ell(i) \leq n + H} \phi^s(i)e^{Z^*_{\ell}(i)} = Z^* + \lim_{n \to \infty} \frac{1}{n} \log \sum_{n \leq \ell(i) \leq n + H} \phi^s(i)$. (38)

4.4. Partial sums for shrinking target pressures.

- $S_{\text{full}}(\psi_{s,j}, n) = \sum_{|i|=n} e^{\psi_{s,j}(i)}$ (28)
- $S_{\text{full}}(\psi_{s,\partial}, n) = \sum_{|i|=n} e^{\psi_{s,\partial}(i)}$ (28)
- $S_{\text{mod}}(\psi_{s,\partial}, n) = \sum_{i \in A^n} \phi^s(i \partial(i))$ (30)
- $S_{\text{full}}(\Sigma_{n,m}, \psi) = \sum_{r_1, \ldots, r_m \in \Sigma_{n,m}} e^{\psi(r_1 + \ldots + r_m)}$, where $\psi(b) = \log \psi^a(b) + Z(j)_{\ell(b)}$. (31)

5. $R$-modular symbolic spaces and pressure

To estimate the Hausdorff dimension of the shrinking target set in Sections 6, we need consider the pressure formulas for the singular value potential $\log \phi^s$. However, in the interest of more general applicability, in this section we will define and investigate $R$-modular symbolic spaces and pressure on them for a general weakly quasi-additive potential $\psi$. Hence, let $\psi$ now be a fixed, weakly quasi-additive potential.
Recall that $\psi$ being weakly quasi-additive means that there are constants $Q,K$ such that for all $i,j \in \Sigma_*$ we can find $k \in \Sigma_*$, $|k| \leq K$ such that

$$\psi(ikj) \geq Q + \psi(i) + \psi(j).$$

Let

$$L = \sup_{i \in \Sigma_*} |\psi(i)|/|i|.$$

**Definition 5.1 (R-modular words).** Fix a large positive integer $R$. We say that a word $i \in \Sigma_*$ is $R$-modular if it can be presented in the form

$$i = r_1k_1r_2\ldots k_{n-1}r_n,$$

where all the words $r_m$ have length $R$ and all the words $k_m$ are connecting words of length $|k_m| \leq K$ for weak quasi-additivity:

$$\psi(r_1k_1\ldots r_{m+1}) \geq Q + \psi(r_1k_1\ldots r_m) + \psi(r_{m+1}).$$

The set of $R$-modular words is denoted by $\mathcal{M}_R$. The set of $R$-modular words of length $n$ is denoted by $\mathcal{M}_R^n$.

**Definition 5.2 (R-modular extension).** Let $i_1 \in \Sigma_*$. We say that $i_2 \in \Sigma_*$ is an $R$-modular extension of $i_1$ if for some $n \in \mathbb{N}$

$$i_2 = i_1k_0r_1k_1r_2\ldots k_{n-1}r_n,$$

where for every $m$ we have $|r_m| = R$, and all the words $k_m$ are connecting words for weak quasi-additivity:

$$\psi(i_1k_0r_1k_1\ldots r_{m+1}) \geq Q + \psi(i_1k_0r_1k_1\ldots r_m) + \psi(r_{m+1}).$$

The definition of $R$-modular extension is recursive: modular extension of a modular extension is a modular extension. Also, the $R$-modular words can be thought of as $R$-modular extensions of the empty word (with $k_0$ also chosen as the empty word).

We want to apply thermodynamic formalism on $R$-modular spaces, thus we need to define an $R$-modular pressure.

**Definition 5.3 (R-modular pressure).** Given a potential $\psi$ defined on $\Sigma_*$, define the $R$-modular pressure as the limit

$$(10)\quad P_{\text{mod}}(\psi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{i \in \mathcal{M}_R^n} e^{\psi(i)}.$$

**Remark.** For a word of length $n$, the lengths of its $R$-modular extensions do not start before $n + R$, and sometimes even $n + R + K$. This is the reason for the sum over $k = 0,\ldots,R + K - 1$ in the definition of $R$-modular pressure.

We would like to compare the $R$-modular pressure to the ordinary pressure of the potential $\psi$ on $\Sigma$. In what follows, we shall have to vary the space on which we consider the pressure, and hence we use for the most part the following slightly cumbersome notation.

**Definition 5.4 (Full pressure).** Given a potential $\psi$ defined on $\Sigma_*$, define the full pressure corresponding to $\psi$ to be

$$(11)\quad P_{\text{full}}(\Sigma,\psi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{|i|=n} e^{\psi(i)}.$$
It is immediate from the definitions that

\[ P_{\text{mod}}(\psi) \leq P_{\text{full}}(\Sigma, \psi). \]

One of our main goals in this subsection will be showing that the opposite inequality, or at least something close to it, also holds.

Up until now we worked with the symbolic space \( \Sigma \) defined on an alphabet \( A \) of size \( N \), but from here on we will also need to consider the alphabet \( A^R \) and the symbolic space \( \Sigma^R \) built over it. Given \( \psi \) defined on \( \Sigma^* \), there is a natural corresponding potential \( \tilde{\psi} \) defined on \( A^R \) by the formula

\[ \tilde{\psi}(a) = \psi(a), \]

and on longer words in the alphabet \( A^R \), extend \( \tilde{\psi} \) by additivity: for \( a_1, \ldots, a_n \in A^R \), let

\[ \tilde{\psi}(a_1a_2\ldots a_n) = \tilde{\psi}(a_1) + \cdots + \tilde{\psi}(a_n). \]

Hence \( \tilde{\psi} \) coincides with \( \psi \) on the words of length \( R \), but not on the words of length \( 2R \), \( 3R \), and so on.

For a word \( a_1\ldots a_n \in (A^R)^n \), we have an associated \( R \)-modular word with blocks \( r_1 = a_1, \ldots, r_n = a_n \). Conversely, for an \( R \)-modular word \( i \) with blocks \( r_1, \ldots, r_n \), we can associate the word \( r_1r_2\ldots r_n \in (A^R)^n \). Unfortunately, neither of these associations gives us a well-defined map. Fortunately, these multivalued maps obtained in the above way are not too-multi-valued, as the following lemma shows.

**Lemma 5.5.** Given a word \( a_1\ldots a_n \in (A^R)^n \), there are at most \( K_0^{-1} \) \( R \)-modular words with blocks \( r_1 = a_1, \ldots, r_n = a_n \), where \( K_0 := 1 + N + \cdots + N^K \). These words have length between \( Rn \) and \( Rn + K(n - 1) \).

Given an \( R \)-modular word \( i \) of length \( m \), it can be divided into \( R \)-blocks with legal connecting words in at most \( (K + 1)^{m/R} \) ways. The number of \( R \)-blocks in these representations is between \( m/(R + K) \) and \( m/R \).

**Proof.** For the first claim: to know an \( R \)-modular word with prescribed blocks \( r_1, \ldots, r_n \), we need still to choose the connecting parts \( c_1, \ldots, c_{n-1} \), and each of them can be chosen in no more than \( K_0 := 1 + N + \cdots + N^K \) ways. Thus, for a given word \( a_1\ldots a_n \in (A^R)^n \) we get no more than \( K_0^{-1} \) \( R \)-modular words built with these blocks. Further, the obtained \( R \)-modular words have length between \( Rn \) and \( Rn + K(n - 1) \).

In the opposite direction, given an \( R \)-modular word of length \( m \), there might be many ways in which it can be divided into blocks. Basically, we need to mark the beginning of each \( R \)-block, and only then we have the full information. The distance between the consecutive beginnings of the \( R \)-blocks varies between \( R \) and \( R + K \), that is no more than \( m/R \) times that we need to make a choice and we have at most \( K + 1 \) possibilities each time. Thus, for a given word of length \( m \) it can be presented as an \( R \)-modular word in at most \( (K + 1)^{m/R} \) ways, and the number of \( R \)-blocks in these representations is between \( m/(R + K) \) and \( m/R \). \( \square \)

We need one more notion of pressure, defined as follows:

\[ (12) \quad P_{\text{full}}(\Sigma^R, \tilde{\psi}) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{a_1, \ldots, a_n \in A^R} e^{\tilde{\psi}(a_1\ldots a_n)}. \]

We will compare \( P_{\text{full}}(\Sigma^R, \tilde{\psi}) \) with the pressure \( P_{\text{mod}}(\Sigma, \psi) \). More precisely, we will compare the \( n \)-level approximation to \( P_{\text{full}}(\Sigma^R, \tilde{\psi}) \) with the \( Rn \)-level approximation to \( P_{\text{mod}}(\Sigma, \psi) \).
Denote
\begin{equation}
S_{\text{full}}(\Sigma^R, \tilde{\psi}, n) = \sum_{a_1, \ldots, a_n \in A^R} e^{\tilde{\psi}(a_1 \ldots a_n)}
\end{equation}
and
\begin{equation}
S_{\text{mod}}(\Sigma, \psi, n) = \sum_{i \in \mathcal{L}^n} e^{\psi(i)}.
\end{equation}

**Proposition 5.6.** Let $n, R \in \mathbb{N}$. There exist $c_1, c_2$ depending on $\psi, Q, K$ but not on $R$ such that
\[
\left| \frac{1}{Rn} \log S_{\text{full}}(\Sigma^R, \tilde{\psi}, n) - \frac{1}{Rn} \log \sum_{m=0}^{R+K-1} S_{\text{mod}}(\Sigma, \psi, Rn + m) \right| \leq \frac{c_1}{n} + \frac{c_2}{R}.
\]

**Proof.** Given an $R$-modular word $i$ built on $R$-blocks $r_1, \ldots, r_n$, we have
\begin{equation}
Q(n-1) \leq \psi(i) - \sum_{k=1}^{n} \tilde{\psi}(r_k) \leq LK(n-1).
\end{equation}

We use this chain of inequalities together with Lemma 5.5 to compare $S_{\text{full}}(\Sigma^R, \tilde{\psi}, n)$ and $S_{\text{mod}}(\Sigma, \psi, n)$ to each other. There are two immediate consequences of (15), which we will state next.

The first consequence of (15) follows like this: by Lemma 5.5, for every summand $e^{\tilde{\psi}(a_1 \ldots a_n)}$ appearing in $S_{\text{full}}(\Sigma^R, \tilde{\psi}, n)$, we have at most $K_0^{n-1}$ corresponding summands $e^{\psi(i)}$ belonging to some $S_{\text{mod}}(\Sigma, \psi, m)$ with $m \in [Rn, Rn + K(n-1)]$. Each of these summands is bounded from above by $e^{\tilde{\psi}(r_1) + \ldots + \tilde{\psi}(r_n)} \cdot e^{KL(n-1)}$. Thus,
\begin{equation}
\sum_{m=0}^{Rn+K(n-1)} S_{\text{mod}}(\Sigma, \psi, m) \geq K_0^{n-1} e^{KL(n-1)} S_{\text{full}}(\Sigma^R, \tilde{\psi}, n) =: C_1^{n-1} S_{\text{full}}(\Sigma^R, \tilde{\psi}, n).
\end{equation}

The second consequence is deduced as follows: by Lemma 5.5, for every summand $e^{\psi(i)}$ appearing in $S_{\text{mod}}(\Sigma, \psi, n)$ we have at most $(K+1)^{n/R}$ corresponding summands $e^{\tilde{\psi}(a_1 \ldots a_n)}$ belonging to some $S_{\text{full}}(\Sigma^R, \tilde{\psi}, m)$ with $m \in [n/(R+K), 1+n/R]$. By (15), each of these summands is at most $Q^{-n/R} e^{\psi(i)}$. Thus,
\begin{equation}
\sum_{m=n/(R+K)}^{1+Rn} S_{\text{full}}(\Sigma^R, \tilde{\psi}, m) \leq (K+1)^{n/R} Q^{-n/R} S_{\text{mod}}(\Sigma, \psi, n) =: C_2^{n/R} S_{\text{mod}}(\Sigma, \psi, n).
\end{equation}

That is, if $S_{\text{full}}(\Sigma^R, \tilde{\psi}, n)$ is large then there must be some $m_0 \in [Rn, Rn + K(n-1)]$ such that $S_{\text{mod}}(\Sigma, \psi, m_0)$ is large, and the same in the other direction.

To deduce the statement of the proposition from (16) and (17), we will need an intermediate fact. This is the following inequality, which we will next prove for any $m_0 \geq Rn$:
\begin{equation}
S_{\text{mod}}(\Sigma, \psi, m_0) \leq C_3^{m_0-Rn} \sum_{k=0}^{R+K-1} S_{\text{mod}}(\Sigma, \psi, Rn + k), \quad \text{with} \quad C_3 = Ne^L.
\end{equation}

Indeed, for each $i \in A^{m_0}$, and for each $k = 0, \ldots, m_0 - Rn$, there is $i' = i|Rn+k \in A^{Rn+k}$ which is modular such that
\begin{equation}
e^{\psi(i)} \leq e^{\psi(i')+(m_0-Rn-k)L} \leq e^{(m_0-Rn)L} e^{\psi(i')}.
\end{equation}
and there are at most $N^{m_0-Rn-k}$ such words $i$ corresponding to one $i'$. This implies

$$S_{\text{mod}}(\Sigma, \psi, m_0) = \sum_{i \in \mathcal{M}_R^{m_0}} e^{\psi(i)} \leq \sum_{k=0}^{R+K-1} N^{m_0-Rn-k} \cdot e^{(m_0-Rn)L} \sum_{i \in \mathcal{M}_R^{Rn+k}} e^{\psi(i)}.$$  

Hence,

$$S_{\text{mod}}(\Sigma, \psi, m_0) \leq N^{m_0-Rn} \cdot e^{(m_0-Rn)L} \sum_{k=0}^{R+K-1} S_{\text{mod}}(\Sigma, \psi, Rn+k),$$

which proves (18).

We will now combine these estimations to prove the claim. First, note that (18) implies

$$\sum_{m=Rn}^{Rn+K(n-1)} S_{\text{mod}}(\Sigma, \psi, m) \leq \sum_{m=Rn}^{Rn+K(n-1)} S_{\text{mod}}(\Sigma, \psi, m) \cdot \left(1 + \sum_{r=R+K}^{Rn+K(n-1)} C_3^{m-Rn}\right).$$

Substituting to (16), we get

$$C_1^{n-1} S_{\text{full}}(\Sigma^R, \tilde{\psi}, n) \leq \left(1 + \sum_{r=R+K}^{K(n-1)} C_3^r\right) \sum_{m=Rn}^{Rn+K(n-1)} S_{\text{mod}}(\Sigma, \psi, m).$$

Taking logarithms, we have

$$\log S_{\text{full}}(\Sigma^R, \tilde{\psi}, n) - \log \sum_{m=Rn}^{Rn+K(n-1)} S_{\text{mod}}(\Sigma, \psi, m) \leq \log \left(1 + \sum_{r=R+K}^{K(n-1)} C_3^r\right) - (n - 1) \log C_1.$$  

This implies one of the inequalities needed to conclude the proposition.

In the opposite direction, we want to carry on from (17). To this end, note that $\tilde{\psi}$ is an additive potential and constant on the first level cylinders. Thus, for any $m \in \mathbb{N}$, we have

$$S_{\text{full}}(\Sigma^R, \tilde{\psi}, m) = S_{\text{full}}(\Sigma^R, \tilde{\psi}, 1)^m,$$

which implies

$$\sum_{m=[n/(R+K)]}^{[n/R]} S_{\text{full}}(\Sigma^R, \tilde{\psi}, m) = \sum_{m=[n/(R+K)]}^{[n/R]} S_{\text{full}}(\Sigma^R, \tilde{\psi}, 1)^m.$$  

Hence,

$$\sum_{m=[n/(R+K)]}^{[n/R]} S_{\text{full}}(\Sigma^R, \tilde{\psi}, m) = \sum_{m=[n/(R+K)]}^{[n/R]} S_{\text{full}}(\Sigma^R, \tilde{\psi}, 1)^m \cdot S_{\text{full}}(\Sigma^R, \tilde{\psi}, [n/R]).$$

$$\leq \left(\frac{nK}{R(R+K)} + 1\right) \cdot \max \left\{1, \frac{1}{S_{\text{full}}(\Sigma^R, \tilde{\psi}, 1)}\right\} \cdot S_{\text{full}}(\Sigma^R, \tilde{\psi}, [n/R]).$$

Observe that

$$S_{\text{full}}(\Sigma^R, \tilde{\psi}, 1) \geq N^R e^{-R} \sup_{i \in A} |\psi(i)| = C_4^R.$$  

Then, combining this with (17), we obtain

$$\left(\frac{nK}{R(R+K)} + 1\right) C_4^{nK/(R+K)} \cdot S_{\text{full}}(\Sigma^R, \tilde{\psi}, [n/R]) \geq C_2^{n/R} S_{\text{mod}}(\Sigma, \psi, n).$$
Therefore,
\[
\sum_{m=0}^{R+K-1} \left( \frac{(Rn + m)K}{R(R + K)} + 1 \right) C_4^{(Rn + m)K/(R + K)} \cdot S_{full}(\Sigma^R, \tilde{\psi}, [(Rn + m)/R]) \\
\geq \sum_{m=0}^{R+K-1} C_2^{(Rn + m)/R} S_{mod}(\Sigma, \psi, Rn + m).
\]
This implies
\[
(R + K) \left( \frac{(Rn + R + K - 1)K}{R(R + K)} + 1 \right) C_4^{(Rn + R + K - 1)K/(R + K)} \cdot S_{full}(\Sigma^R, \tilde{\psi}, [(Rn + R + K - 1)/R]) \\
\geq C_2^n \sum_{m=0}^{R+K-1} S_{mod}(\Sigma, \psi, Rn + m).
\]
Taking logarithms, we have
\[
\log S_{full}(\Sigma^R, \tilde{\psi}, [(Rn + R + K - 1)/R]) - \log \sum_{m=0}^{R+K-1} S_{mod}(\Sigma, \psi, Rn + m) \\
\geq n \log C_2 - \log \left( n + 1 + \frac{K(K - 1)}{R} + R + K \right) - \left( \frac{(Rn - 1)K}{R + K} + K \right) \log C_4.
\]
Finally, remark that when \( R \geq K \),
\[
S_{full}(\Sigma^R, \tilde{\psi}, [(Rn + R + K - 1)/R]) \\
\leq S_{full}(\Sigma^R, \tilde{\psi}, n + 1) \\
= S_{full}(\Sigma^R, \tilde{\psi}, 1) \cdot S_{full}(\Sigma^R, \tilde{\psi}, n) \\
\leq N^R e^R \sup_{i \in A} |\psi(i)| \cdot S_{full}(\Sigma^R, \tilde{\psi}, n).
\]
This finishes the proof of the second inequality needed for the claim, and hence the proof of the proposition.

Let us now look at
\[(20)\] 
\[S_{full}(\Sigma, \psi, n) := \sum_{i \in A^n} e^{\psi(i)}.
\]
We want to compare it with \( S_{mod}(\Sigma, \psi, n) \), which comparison we will achieve via \( S_{full}(\Sigma^R, \tilde{\psi}, n) \) and Proposition 5.6.

**Corollary 5.7.** There exist \( c_3, c_4 \) depending on \( \psi, Q, K \) but not on \( R \) such that
\[
\left| \frac{1}{n} \log S_{full}(\Sigma, \psi, n) - \frac{1}{n} \log \sum_{m=0}^{R+K-1} S_{mod}(\Sigma, \psi, n + m) \right| \leq \frac{c_3}{n} + \frac{c_4}{R}.
\]

**Proof.** Let \( i \in \Sigma^* \) be a word of length \( Rn \). We can divide \( i \) into \( n \) words of length \( R \)
\[
i = r_1 r_2 \ldots r_n
\]
and we see that
\[
\psi(i) \leq \sum_{k=1}^{n} \tilde{\psi}(r_k).
\]
Thus,
\[
S_{\text{full}}(\Sigma, \psi, Rn) \leq S_{\text{full}}(\Sigma^R, \bar{\psi}, n).
\]

We have already mentioned the trivial fact that \(S_{\text{full}}(\Sigma, \psi, n) \geq S_{\text{mod}}(\Sigma, \psi, n)\). Thus, the claim is a corollary of Proposition 5.6. \(\square\)

To summarize the results of this subsection: The \(R\)-modular sequences, on which the potential is quasi-additive, carry (for large \(R\)) almost the full pressure. The \(R\)-modular space of modular sequences is not invariant under the shift map, but it is closely related to \(\Sigma^R\). This relation is going to be sufficient to obtain regularity properties needed to estimate the dimension-like characteristics of sets constructed with use of the modular words and modular extensions. We will look at this in the context of shrinking targets in the next section.

### 6. \(R\)-modular shrinking target pressure

In this section we will apply the \(R\)-modular pressure analysis of the last section to the problem of shrinking targets. We consider a fixed affine iterated function system satisfying the strong separation condition. Throughout this section, we assume that the length sequence \(\ell\) satisfies Definition 2.2, and that \(\log \phi^s\) is weakly quasi-additive for all \(s \in [0, d]\).

Consider the singular value function \(\phi^s\) defined in Section 2, and assume that \(\log \phi^s\) is weakly quasi-additive with constants \(Q_s, K_s\). We will prove that these constants can be chosen uniformly. In fact, as we have seen in Lemma 3.3, the function \(s \mapsto \log \phi^s(i)\) is decreasing and we can define two constants \(0 < L_1 < L_2\) (choose \(L_1 = \log \alpha_+\) and \(L_2 = \log \alpha_-\) in the notation of the lemma) such that
\[
-L_2 \leq \frac{\log \phi^{s_2}(i) - \log \phi^{s_1}(i)}{|i(s_2 - s_1)|} \leq -L_1
\]
holds for every \(s_1 \neq s_2\) and for every \(i \in \Sigma^*_s\). This implies the continuity of \(s \mapsto \phi^s\), and hence we have uniform constants \(Q, K\) for all \(s \in [0, d]\), for the weak-quasi-additivity (4), that is there exist constants \(Q, K\) such that for any two finite words \(i, j \in \Sigma^*_s\) we have some \(k \in \Sigma^*_s\) with \(|k| \leq K\) satisfying
\[
\log \phi^s(ikj) \geq Q + \log \phi^s(i) + \log \phi^s(j).
\]

#### 6.1. Shrinking target pressure

Recall the definition of the limsup shrinking target pressure:
\[
P^\ast(s, j) = \limsup_n \frac{1}{n} \log \sum_{|i| = n} \phi^s(ij)_{\ell(i)}
\]
and if the limit exists, denote it by \(P(s, j)\). Given \(s\) and \(j\), denote \(\psi_{s,j} : i \mapsto \log \phi^s(ij)_{\ell(i)}\). Then, in the notation of the previous section,
\[
P(s, j) = P_{\text{full}}(\Sigma, \psi_{s,j}).
\]

In this subsection we have all the sums over the whole space \(\Sigma\) (or the whole \(R\)-modular space), so to simplify notation we will leave out \(\Sigma\) from the notation for the most part. That is, we denote
\[
P(s, j) = P_{\text{full}}(\psi_{s,j}) \quad \text{and} \quad P^\ast(s, j) = P^\ast_{\text{full}}(\psi_{s,j})
\]
for the pressure and limsup pressure, respectively. Set the notation
\[(26) \quad Z(j) = \lim_{n \to \infty} \frac{1}{n} \log \phi^s(j|_n)\]
for future use. We will show now that this limit exists for almost every \(j\).

**Lemma 6.1.** Let \(j \in \Sigma\) and let \(\mu\) be an ergodic measure on \((\Sigma, \sigma)\). Then, for \(\mu\)-almost every \(j \in \Sigma\) the limit \(Z(j) = \lim_{n \to \infty} \frac{1}{n} \log \phi^s(j|_n)\) exists and takes a common value independent of \(j\).

**Proof.** Define
\[X(m, n) = \log \phi^s(\sigma^m(j)|_{n-m}).\]
Then applying Kingman’s sub-additive ergodic theorem to \(X(m, n)\), we obtain the existence and almost everywhere uniqueness of
\[Z(j) = \lim_{n \to \infty} \frac{1}{n} X(0, n).\]

From now on, fix \(j\) to be such that \(Z(j)\) exists. By Lemma 6.1 this is a generic property.

The first thing we need to do is to modify the pressure in such a way that it works well on modular words. Given \(i \in \Sigma^*\), we denote \(
\partial(i) := c(i|_\ell(i))\), where \(c = c(i, j)\) is a connecting word of length at most \(K\) such that
\[\phi^s(ic|_\ell(i)) \geq e^{Q'} \phi^s(i)\phi^s(j|_\ell(i)).\]
If there are more than one choices for \(c\), take the smallest in lexicographic order. We note that by Definition 2.2 and Lemmas 3.1 and 3.2, \(j|_\ell(i)\) and \(j|_\ell(i)\) differ by at most \(2\kappa + KL_{\max}\) letters at the end, and the singular value functions on two words that differ by just few letters on one end are almost the same. Hence, the same \(c\) satisfies also
\[\phi^s(ic|_\ell(i)) \geq Q' \phi^s(i)\phi^s(j|_\ell(i)).\]
for some uniform constant \(Q'\). We can then define \(\psi_{s, \partial} : i \mapsto \log \phi^s(i\partial(i))\), and
\[(27) \quad P^*_{\text{full}}(\psi_{s, \partial}) := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{|i| = n} \phi^s(i\partial(i)) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{|i| = n} \phi^s(i)\phi^s(j|_\ell(i)).\]
Denote by
\[(28) \quad S_{\text{full}}(\psi_{s, j}, n) = \sum_{|i| = n} e^{\psi_{s, j}(i)} \quad \text{and} \quad S_{\text{full}}(\psi_{s, \partial}, n) = \sum_{|i| = n} e^{\psi_{s, \partial}(i)}\]
the sums at level \(n\) in the definitions of \(P^*_{\text{full}}(\psi_{s,j})\) and \(P^*_{\text{full}}(\psi_{s,\partial})\).

The following lemma shows that the difference between \(\psi_{s,j}\) and \(\psi_{s,\partial}\) is asymptotically unimportant.

**Lemma 6.2.**
\[P^*_{\text{full}}(\psi_{s,j}) = P^*_{\text{full}}(\psi_{s,\partial}).\]

**Proof.** For every \(i \in \Sigma^*, |i| = n\) we have
\[\phi^s(i\partial(i)) \geq c\phi^s(i|_\ell(i)).\]
Hence,
\[cS_{\text{full}}(\psi_{s,j}, n) \leq S_{\text{full}}(\psi_{s,\partial}, n).\]
we have proved an analogous result for a general $\psi_{s,j}$, $n+k = 0, \ldots, K$. Therefore,

$$S_{\text{full}}(\psi_{s,j}, n+k) \leq \sum_{k=0}^{K} S_{\text{full}}(\psi_{s,j}, n+k),$$

and we are done. \hfill \square

After defining the 'multiplicative' version of the pressure $P^*_\text{full}(\psi_{s,j})$, we can create a modular version of it. Fix some positive integer $R$ and remember the definition of $R$-modular words from the previous section. We recall that $\psi_{s,j}(i) = \log \phi^s(i \partial(i))$ and define

$$(29) \quad P^*_\text{mod}(s,j) := P^*_\text{mod}(\psi_{s,j}) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{\mathcal{A}_R^{n+k}} \phi^s(i \partial(i)).$$

Denote

$$(30) \quad S_{\text{mod}}(\psi_{s,j}, n) = \sum_{i \in \mathcal{A}_R^{n}} \phi^s(i \partial(i)).$$

The main result of this subsection is the following result.

**Proposition 6.3.** There exist $c_5$ and $c_6$ depending on $Q$ and $K$, but not on $R$, such that

$$\left| \frac{1}{n} \log S_{\text{full}}(\psi_{s,j}, n) - \frac{1}{n} \log \sum_{m=0}^{R+K-1} S_{\text{mod}}(\psi_{s,j}, n+m) \right| \leq \frac{c_5}{n} + \frac{c_6}{R}.$$ 

**Proof.** In Corollary 5.7 we have proved an analogous result for a general $\psi$. To apply Corollary 5.7 in the shrinking target context, the proof will only require a minimal modification. Let $i$ be a word of length $n$ which gives us a summand $\phi^s(i \partial(i))$ appearing in $S_{\text{full}}(\psi_{s,j}, n)$. We divide $i$ into blocks of size $R$ and construct an $R$-modular word $i'$ out of them. Then, for some uniform constant $c > 0$, we have all of the following:

i) $|i'| \leq |i| \cdot (1 + c/R)$,

ii) $\phi^s(i') \geq \phi^s(i) \cdot e^{-c|ii|R}$,

iii) $\ell(i') \leq \ell(i) + c|ii|R$, and hence $\phi^s(\partial(i')) \geq \phi^s(\partial(i)) \cdot e^{-c|ii|R}$,

iv) the map $i \mapsto i'$ is at most $e^{c|ii|R}$-to-1.

Indeed, i) and iv) are consequences of Lemma 5.5, and ii) is the exact counterpart of (19) for the potential $\log \phi^s$. The property iii) follows by combining the approximate additivity property (5) with Lemma 3.2, since $i'$ is $i$ with $|ii|/R$ inserted additional subwords each of which has length at most $K$.

The points i)-iv) imply the assertion. \hfill \square

6.2. **Shrinking target pressure on abstract modular space $\Sigma^R$.** Fix a large positive integer $R$. Let $a \in \Sigma_*$ be a finite word. In this subsection our goal is to describe the set of $R$-modular extensions of $a$ in the language of alphabet $A^R$. In this subsection, the alphabet varies, so we use a more careful notation again, writing

$$P(s,j) = P_{\text{full}}(\Sigma, \psi_{s,j})$$

and so forth.

Let $s_0$ be the solution of the equation

$$P^*(s_0,j) = P^*_{\text{full}}(\Sigma, \psi_{s_0,j}) = 0.$$
As above, we assume that \( \log \phi^{s_0} \) is weakly quasi-additive with constants \( Q, K \) (see (4)). We assume approximate additivity of \( \ell \) as in Definition 2.2. We also assume that \( j \) is such that the limit \( Z(j) \) from (26) exists.

Consider a word \( r_1 r_2 \ldots r_m \in (A^R)^m \). We can find an \( R \)-modular extension of \( a \) of the form \( a k_0 r_1 k_1 r_2 \ldots k_{m-1} r_m \). In fact, as discussed above, generally more than one such extension exists as we might have some freedom in choosing the \( k_i \)'s. To make sure that everything is well-defined, out of these extensions let us choose one, for example let it be the extension for which \( k_0 \) is the first in lexicographical order of all possible \( k_0 \)'s, then \( k_1 \) is first in lexicographical order of all possible \( k_1 \)'s under the condition that \( k_0 \) is already chosen, and so on. We denote the resulting \( R \)-modular extension by \( \pi_a(r_1 \ldots r_m) \).

This gives us a well-defined map \( \pi_a : \Sigma^R_a \to \Sigma_\ast \). Moreover, \( \pi_a(r_1 r_2 \ldots r_{m+1}) \) is an \( R \)-modular extension of \( \pi_a(r_1 r_2 \ldots r_m) \), so \( \pi_a \) preserves the cylinder structure of \( \Sigma_\ast \). We also have that
\[
R \leq |\pi_a(r_1 r_2 \ldots r_{m+1})| - |\pi_a(r_1 r_2 \ldots r_m)| \leq R + K.
\]
Hence for every \( r \in \Sigma^R \) and every \( n > 0 \) there exists a smallest \( z(a, r, n) \) such that
\[
n \leq |\pi_a(r_1 r_2 \ldots r_{z(a, r, n)})| \leq n + R + K - 1,
\]
where \( r_1 r_2 \ldots r_{z(a, r, n)} \) is an initial segment of \( r \). Naturally, for every \( r \) we have
\[
\frac{n}{R + K} \leq z(a, r, n) \leq 1 + \frac{n}{R}.
\]
We will denote
\[
\Sigma^R_{a, n} := \{r_1 r_2 \ldots r_{z(a, r, n)} ; r \in \Sigma^R \}.
\]
The sets of corresponding cylinders \([r_1 r_2 \ldots r_{z(a, r, n)}] \) form a disjoint cover of \( \Sigma^R \).

Let us now define an additive potential \( \Psi \) on \( \Sigma^R \), constant on first level cylinders: for \( b \in A^R \) we set
\[
\Psi(b) := \log \phi^{s_0}(b) + Z(j)\ell(b).
\]
Denote
\[
S_{\text{full}}(\Sigma_{a, n}^R, \Psi) := \sum_{r_1 \ldots r_m \in \Sigma_{a, n}^R} e^{\Psi(r_1) + \ldots + \Psi(r_m)}.
\]

**Proposition 6.4.** One can find \( c_7, c_8, c_9 \), independent of \( a \) and \( R \), such that
\[
e^{-c_7 n/R - c_8 c_9 n} \leq S_{\text{full}}(\Sigma_{a, n}^R, \Psi) \leq e^{c_7 n/R + c_8 + c_9 n}
\]
Moreover, we can make \( c_9 \) arbitrarily small while keeping \( c_7 \) fixed.

**Proof.** We want to compare the pressure sum for \( \Psi \) to the pressure sum for \( \psi_{s_0, \partial} \), and then make use of the choice of \( s_0 \). The sum with which we are comparing \( S_{\text{full}}(\Sigma_{a, n}^R, \Psi) \) is the sum
\[
S_{\text{full}}(\Sigma, \psi_{s_0, \partial}, n) := \sum_{|i| = n} \phi^{s_0}(i \partial(i)),
\]
where \( \psi_{s_0, \partial}(i) \) is the notation for the potential \( \log \phi^{s_0}(i \partial(i)) \) as above.

Every word \( i \in \Sigma_\ast \) with \( |i| = h R =: n \) can be divided into subwords of length \( R \) as \( i = r_1 r_2 \ldots r_k \), where, \( r_1, r_2, \ldots, r_k \in A^R \). Then
\[
\log \phi^{s_0}(i) = \log \phi^{s_0}(r_1) + \ldots + \log \phi^{s_0}(r_k) + O(n/R)
\]
and
\[
\ell(i) = \ell(r_1) + \ldots + \ell(r_k) + O(n/R).
\]
Moreover, we know from Lemma 3.1 that $\ell(i) < L_{\max} n + \kappa'$ and hence

$$|\log \phi s_0(\delta(i)) - Z(j)\ell(i)| = o(n).$$

Comparing $S_{\text{full}}(\Sigma_{a,n}, \Psi)$ to $S_{\text{full}}(\Sigma, \psi_{s_0,\delta}, n)$ term by term, we obtain from the above estimates that

$$|S_{\text{full}}(\Sigma_{a,n}, \Psi) - S_{\text{full}}(\Sigma, \psi_{s_0,\delta}, n)| = O(n/R) + o(n).$$

Recall from Lemma 6.2 that $P^*_{\text{full}}(\psi_{s,j}) = P^*_{\text{full}}(\psi_{s,j})$. By our assumptions, $P^*_{\text{full}}(\psi_{s_0,j}) = 0$ and so

$$|S_{\text{full}}(\Sigma, \psi_{s_0,\delta}, n)| = o(n).$$

Here, we will need that the limsup in the definition of $P^*(s,j)$ is actually a limit. Fortunately, the limit exists by the assumption that the limit of $Z(j)$ exists. In fact, by sub-additivity, we have

$$P^*(s,j) \leq P_{\text{full}}(\Sigma, \phi^s) + Z(j)$$

and in the opposite direction, by weak quasi-additivity (4), we have

$$P^*(s,j) = P^*_{\text{full}}(\Sigma, \psi_{s,j}) \geq P_{\text{full}}(\Sigma, \phi^s) + Z(j).$$

Combining (33) and (34) gives

$$|\log S_{\text{full}}(\Sigma_{a,n}, \Psi) - 0| = O(n/R) + o(n),$$

as desired. \hfill \square

We finish this subsection by defining a measure on $\Sigma^R$, which we will utilize in the next subsection when looking for a measure supported on $R(j, \ell)$ that will give the lower bound for the Hausdorff dimension of $R^*(j, \ell)$.

**Definition 6.5** (Measure on $\Sigma^R$). Recall the potential $\Psi$ satisfying for $b \in A^R$

$$\Psi(b) = \log \phi_{s_0}(b) + Z(j)\ell(b),$$

and extended to $(A^R)^n$ by additivity. Then, for each $b \in A^R$, set

$$\nu_R[b] := \frac{\exp(\Psi(b))}{w_R},$$

where

$$w_R = \sum_{b_1 \in A^R} \exp(\Psi(b_1))$$

is a normalizing factor. The measure $\nu_R$ extends to a measure on $\Sigma^R$ in the natural way by taking for $b_1, \ldots, b_m \in A^R$

$$\nu_R[b_1, \ldots, b_m] = \nu_R[b_1] \cdots \nu_R[b_m],$$

and hence to a measure on $\Sigma^R$ by the Carathéodory extension theorem.

**Lemma 6.6.** There exists a uniform constant $c_{10}$ such that we have a bound for the normalizing factor, valid for every $R$:

$$c_{10}^{-1} \leq w_R \leq c_{10}.$$
**Proof.** As \( v_R \) is a probability measure and \( \Sigma_{a,n}^R \) gives a disjoint cover of \( \Sigma^R \), for every \( n \) we have
\[
\sum_{r_1 \ldots r_k \in \Sigma_{a,n}^R} v_R[r_1 \ldots r_k] = 1.
\]
As \( n/R \geq k \geq n/(R + K) \), we get
\[
\log S_{\text{full}}(\Sigma_{a,n}, \nu) = w_R \cdot (n/R \cdot (1 + O(1/R))),
\]
and the assertion follows from Proposition 6.4, after we choose \( n \) large enough.

\[\Box\]

7. Lower bound for Hausdorff dimension

We can finally begin our study of the path-dependent shrinking target set \( R(j, \ell) \). We fix \( s < \min\{s_0, d\} \) for the time being. The proof strategy is as follows. We will fix some large \( R \) and then find a Cantor subset \( W_R \subset R(j, \ell) \), constructed with the help of \( R \)-modular extensions. On the subset \( W_R \) we will then distribute a measure \( \mu \) such that for any cylinder \([i]\) we have

\[
\mu([i]) \leq c\phi^s(i)
\]

for some constant \( c > 0 \). The proof of the lower bound is then finished by applying the following well-known theorem of Falconer and Solomyak. The theorem is stated here in an altered form (for closed subsets \( A \subset \Sigma \)), but it follows from the same proofs line by line.

**Theorem 7.1** (Lemma 3.1 of [F88], Proposition 3.1 of [S98]). Consider an affine iterated function system \( \{f_1, \ldots, f_N\} \) with \( f_i = T_i + a_i \), and its corresponding sequence space \( \Sigma \) as in Subsection 2.3. Assume that \( \|T_i\| < \frac{1}{2} \) for all \( i = 1, \ldots, N \). Let \( \mu \) be a finite measure supported on a closed subset \( A \subset \Sigma \) such that for all \( q \in A, n \in \mathbb{N} \), we have \( \mu(q|_n) \leq c\phi(q|_n) \). Then

\[
\int \int |x - y|^{-s} d\pi_* \mu d\pi_* \mu < \infty,
\]

and in particular \( \dim_H \pi(A) \geq s \), for Lebesgue almost all choices of \( (a_1, \ldots, a_N) \).

7.1. Construction of the Cantor subset and the corresponding mass distribution. Let us begin with the construction of \( W_R \). We fix some fast increasing sequence \( (n_i) \), satisfying \( n_1 \gg R \) and \( n_{i+1}/n_i \to \infty \). Recall the notation

\[
\Sigma_{a,n_1}^R = \{r_1 \ldots r_{x(a,r,n_1)} | r \in \Sigma^R\},
\]

and the definition of \( \pi_a \) from Subsection 6.2.

For the first step of the construction, we define

\[
W_{R,1} := \bigcup_{b_1 \in \Sigma_{\phi,n_1}^R} \{\pi_\phi(b_1) \partial(\pi_\phi(b_1))\}.
\]

Recall the notation

\[
\Sigma_{\phi,n_1}^R = \{r_1 \ldots r_{\phi(x,r,n_1)} | r \in \Sigma^R\}
\]

and in particular, the union in the definition of \( W_{R,1} \) is taken over finite words of the form \((A^R)^m\) with \( m \) varying. The set \( W_{R,1} \) is a collection of finite words, each of the form \( a_1 \partial(a) \).

The second step:

\[
W_{R,2} := \bigcup_{a_1 \in W_{R,1}} \bigcup_{b_2 \in \Sigma_{a_1,n_2}^R} \{\pi_{a_1}(b_2) \partial(\pi_{a_1}(b_2))\}.
\]
That is, $W_{R,2}$ consists of words of the form $a\delta(a)$, where $a$ are $R$-modular extensions of the words from $W_{R,1}$, such that their length is as close to $n_2$ as possible. And so we carry on:

$$W_{R,k} := \bigcup_{a_{k-1} \in W_{R,k-1}} \bigcup_{b_k \in \Sigma_{a_{k-1},n_k}} \{\pi_{a_{k-1}}(b_k)\partial(\pi_{a_{k-1}}(b_k))\}.$$ 

Finally we define

$$W_R = \bigcap_{k=1}^\infty W_{R,k}.$$ 

Every infinite word in $W_R$ is described by a sequence of finite words $(b_1, b_2, \ldots)$, each $b_k$ belonging to the set of $R$-modular extensions of some word determined by the previous $b_1, \ldots, b_{k-1}$. As we can see, for every point in $W_R$ there are infinitely many times when the initial segment of this word are of the form $a\delta(a)$, hence indeed $W_R \subset R(j, \ell)$.

One important observation: for an infinite word in $W_R$ the sequence $(b_1, b_2, \ldots)$ is in general not uniquely defined. In fact, even in the first step it can happen that two different $b_1 \in \Sigma^R$ produce the same $a_1 \in W_{R,1}$ and the same for the other $b_k$ in the sequence. This technicality is important now, as we start distributing a measure on $W_R$.

The construction of the measure $\mu$ is based on the measure $\nu_R$ from Definition 6.5, and goes as follows. First, on cylinders from $W_{R,1}$ we distribute the measure

$$\mu_{R,1} := \sum_{b_1 \in \Sigma_{\varnothing,n_1}^R} (\pi_{\varnothing})_* (\nu_R|b_1).$$

That is, for each cylinder $[\pi_{\varnothing}(b_1)\partial(\pi_{\varnothing}(b_1))]$ from $W_{R,1}$, we assign the mass $\nu_R(b_1)$. Note that if there is $b'_1 \in \Sigma_{\varnothing,n_1}^R$ such that

$$\pi_{\varnothing}(b_1)\partial(\pi_{\varnothing}(b_1)) = \pi_{\varnothing}(b'_1)\partial(\pi_{\varnothing}(b'_1))$$

then the mass of $[\pi_{\varnothing}(b_1)\partial(\pi_{\varnothing}(b_1))]$ will be the sum.

Next, for each $b_1 \in \Sigma_{\varnothing,n_1}^R$ corresponding to $a_1 = \pi_{\varnothing}(b_1)\partial(\pi_{\varnothing}(b_1)) \in W_{R,1}$, we subdivide the measure according to:

$$\mu_{R,2}[a_1] := \mu_{R,1}[a_1] \sum_{b_2 \in \Sigma_{a_1,n_2}^R} (\pi_{a_1})_* (\nu_R|b_2).$$

That is, for each sub-cylinder $[\pi_{a_1}(b_2)\partial(\pi_{a_1}(b_2))]$ of $[a_1]$, we assign the mass $\mu_{R,1}(a_1) \cdot \nu_R(b_2)$, with multiplicity if there is repetition. Notice that if repetition is disregarded,

$$\mu_{R,2}[\pi_{a_1}(b_2)\partial(\pi_{a_1}(b_2))] = \nu_R(b_1)\nu_R(b_2).$$

In general, assume that some $a_1 \in W_{R,1}, \ldots, a_{k-1} \in W_{R,k-1}$ and the corresponding $b_1 \in \Sigma_{\varnothing,n_1}^R$, $\ldots, b_{k-1} \in \Sigma_{a_{k-2},n_{k-1}}^R$ have been inductively chosen. Then, we define $\mu_{R,k}$ on the cylinders of $W_{R,k}$, by setting for each $a_{k-1} \in W_{R,k-1}$,

$$\mu_{R,k}[a_{k-1}] = \mu_{R,1}[a_1] \cdots \mu_{R,k-1}[a_{k-1}] \sum_{b_k \in \Sigma_{a_{k-1},n_k}^R} (\pi_{a_{k-1}})_* (\nu_R|b_k).$$

That is, each sub-cylinder $[\pi_{a_{k-1}}(b_k)\partial(\pi_{a_{k-1}}(b_k))]$ of $[a_{k-1}]$ gets assigned the weight

$$\mu_{R,1}[a_1] \cdots \mu_{R,k-1}[a_{k-1}] \nu_R(b_k)$$

with multiplicity if there is repetition. Notice that, again, ignoring repetition would lead to the simple product formula

$$\mu_{R,k}[\pi_{a_{k-1}}(b_k)\partial(\pi_{a_{k-1}}(b_k))] = \nu_R(b_1) \cdots \nu_R(b_{k-1})\nu_R(b_k).$$

(36)
Finally, we take \( \mu_R \) as the weak limit of \( \mu_{R,k} \).

Let \( w \in W_R \) and let \( b_1, b_2, \ldots \) be one of its generating symbolic sequence. Let \( a_1(b_1) = \varphi(b_1) \partial(\varphi(b_1)), a_2(b_1, b_2) = \varphi_a(b_2) \partial(\varphi_a(b_2)), \ldots, a_k(b_1, b_2, \ldots, b_k) \) be as above. For any \( n \) we want to compare \( \mu_R([w_{n+1}]) \) with \( \phi^{s_0}(w_{n+1}) \). Let us start by looking at the part of the measure \( \mu_R \) coming from the sequence \( b_1, b_2, \ldots \), that is, let us for the time being follow the simplified formula (36) above where repetition is ignored, and let us denote the consequent product measure by \( \tilde{\mu} \). In that case, for \( |a_{r-1}| < n \leq |a_r| \),

\[
\tilde{\mu}([w_{n+1}]) := \nu_R([b_1]) \cdot \nu_R([b_2]) \cdot \ldots \cdot \nu_R([b_{r-1}]) \cdot \nu_R([\sigma^{|a_{r-1}|}(w_{n+1})]).
\]

**Lemma 7.2.** Let \( w \in W_R, n \in \mathbb{N} \). There exist universal constants \( c'_1, c'_8, c'_9 \) independent of \( n, R, w \) such that

\[
\frac{\tilde{\mu}([w_{n+1}])}{\phi^{s_0}(w_{n+1})} \leq e^{c'_1 n/R + c'_8 + c'_9 n}.
\]

Moreover, \( c'_9 \) can be chosen arbitrarily small keeping \( c'_1 \) constant and increasing \( c'_8 \).

**Proof.** Consider first \( n \leq |a_1| \). Let \( b_1 = r_1 r_2 \ldots r_m, \) with \( r_k \in A^R \). The beginning of the sequence \( w \) is \( \varphi(r_1 r_2 \ldots r_m) \partial(\varphi(r_1 r_2 \ldots r_m)) \), with \( |\varphi(r_1 r_2 \ldots r_m)| \approx n_1 \).

For \( n < n_1 \) we have

\[
\tilde{\mu}([w_{n+1}]) = \exp\left(\sum_{i=1}^{k} \Psi(r_i) + O(k)\right),
\]

where \( n/(R + K) \leq k \leq n/R \). We also have

\[
\phi^{s_0}(w_{n+1}) = \prod_{i=1}^{k} \phi^{s_0}(r_i) \cdot e^{-O(k)}.
\]

By (8), we have

\[
\ell(i) \geq L_{\min} |i| - \kappa, \quad \forall i \in \Sigma_*.
\]

Hence, \( \ell(r_i) \geq L_{\min} R - \kappa' \). Recalling that \( \Psi(r_i) = \log \phi^{s_0}(r_i) + Z \ell(r_i) \), we see that for \( n < n_1 \),

\[
\frac{\tilde{\mu}([w_{n+1}])}{\phi^{s_0}(w_{n+1})} \leq e^{O(n/R) + L_{\min} Z n}.
\]

Therefore, the assertion follows with a safe margin (we can choose \( c_6 = c_7 = 0 \) and also we have an additional exponentially decreasing factor).

This safe margin is immediately used to deal with the case \( n_1 < n \leq |a_1| \). In this range \( \tilde{\mu}([w_{n+1}]) \) stays constant while \( \phi^{s_0}(w_{n+1}) \) constantly decreases, so we only need to check the situation for \( n = |a_1| \). There, we have, up to constants,

\[
\phi^{s_0}(w_{|a_1|}) = \phi^{s_0}(w_{|a_1|}) \cdot \phi^{s_0}(j|\ell(w_{|a_1|})).
\]

However, for some \( c > 0 \),

\[
\tilde{\mu}([w_{|a_1|}]) = e^{cn} \prod_{i=1}^{m} e^{\Psi(r_i)} = \phi^{s_0}(w_{|a_1|}) \cdot e^{Z(j)\ell(w_{|a_1|})} \cdot e^{O(n/R)},
\]

hence

\[
\frac{\tilde{\mu}([w_{|a_1|}])}{\phi^{s_0}(w_{|a_1|})} \leq e^{O(n/R)} \cdot \frac{e^{Z(j)\ell(w_{|a_1|})}}{\phi^{s_0}(j|\ell(w_{|a_1|}))},
\]

and the last factor is sub-exponential.
The proof then follows by induction. If we know the assertion holds for \( n = |a_{r-1}| \), we write \( b_r = r_1 \ldots r_m \), and use the formulas

\[
\phi^{s_0}(w|n) = \phi^{s_0}(w|a_{r-1}) \cdot \prod_{i=1}^{k} \phi^{s_0}(r_i) \cdot e^{O(k)}
\]

and

\[
\tilde{\mu}([w|n]) = \tilde{\mu}([w|a_{r-1}]) \cdot \exp(\sum_{i=1}^{k} \psi(r_i) + O(k)).
\]

Then, almost identical calculations as above give us the assertion for all \( |a_{r-1}| < n \leq |a_r| \). □

Lemma 7.2 has the following corollary, which finishes the proof of the lower bound as explained at the beginning of the section.

**Corollary 7.3.** For a measure \( \mu_R \) defined using a large enough \( R \) in the definition of the \( R \)-modular space, there is a constant \( C > 0 \) such that for all \( w \in W_R, n \in \mathbb{N} \),

\[
\mu_R([w|n]) \leq C \phi^t(w|n).
\]

**Proof.** We only need to estimate the impact of the overlaps on the measure \( \mu_R \). To this end, we need to estimate in how many possible ways a given \( R \)-modular word (or a given \( R \)-modular extension) can be obtained. This is a calculation we have already done in Lemma 5.5: the representation of a word as an \( R \)-modular word (or \( R \)-modular extension) is uniquely determined by marking the beginnings of the \( R \)-blocks, thus a word of length \( n \) has at most \((K+1)^n/R \) possible representations. Thus, for every \( w \in W_R \) and every \( n \) there are at most \((K+1)^n/R \) words \( w_i \in W_R \) such that

\[
\mu_R[w|n] = \sum_{i=1}^{(K+1)^n/R} \tilde{\mu}[w_i|n].
\]

Together with Lemma 7.2, this implies

\[
\mu_R[w|n] \leq \exp(cn/R + c' + c''n) \phi^s(w|n),
\]

where we can choose \( c'' \) arbitrarily small by taking \( c' \) larger. We have, for all \( \min\{s_0, d\} > t > s \), that \( \phi^s(w|n) \leq a^{-n(t-s)} \phi^t(w|n) \). Choosing \( R \) so large and \( c'' \) so small that

\[
a^{-n(t-s)} \exp(c/R + c'') < 1,
\]

we obtain the claim. □

**8. General j and ℓ-modular spaces**

In order to work with a general center point \( j \) we will need to complicate our approach a bit more.

Denote

\[
Z^* = \limsup_{n \to \infty} \frac{1}{n} \log \phi^{s_0}(j),
\]

where \( s_0 \) is the zero of the pressure

\[
P^*(s,j) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{|i|=n} \phi^s(i|\ell(i)) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{|i|=n} \phi^s(i\overline{\ell}(i)).
\]

Contrary to the situation in the previous subsections, this pressure is indeed obtained as a limsup, as when \( Z \) is not a limit, \( P \) need not to be a limit neither. We will first introduce some
new auxiliary pressures, and then use them to distribute a measure with an upper bound for the concentration, and hence a lower bound for the Hausdorff dimension, as before.

8.1. Pressures. Let $H$ be such a number that for every word $i \in \Sigma^*$ and every symbol $a \in \{1, \ldots, N\}$ we have $\ell(i) \leq \ell(ia) + H$. By Lemma 3.2, we can take $H = L_{\text{max}} + \kappa'$. In analogue to the modular pressure, we define the $\ell$-pressure by the following formula:

$$P^\ell(s, j) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{n \leq \ell(i) \leq n + H} \phi^s(i) \ell(i).$$

We also define another version of $\ell$-pressure in the following way:

$$\bar{P}^\ell(s, j) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{n \leq \ell(i) \leq n + H} \phi^s(i) e^{\tilde{Z}^\ell(i)} = \tilde{Z}^* + \lim_{n \to \infty} \frac{1}{n} \log \sum_{n \leq \ell(i) \leq n + H} \phi^s(i).$$

We note that the partial sums of the pressure $\bar{P}^\ell$ are sub-additive, hence the limit indeed exists by the usual Fekete Sub-additive Lemma argument (see [PU10, Lemma 2.4.3]).

It is absolutely clear that, as

$$\phi^s(\delta(i)) \leq e^{\tilde{Z}^\ell(i) + o(\ell(i))},$$

we have $P^\ell(s, j) \leq \bar{P}^\ell(s, j)$. On the other hand, we have some sequence $(m_k)$ for which

$$\phi^s(\delta(i)) \geq e^{\tilde{Z}^\ell(i) - o(\ell(i))}$$

whenever $\ell(i) \in [m_k, m_k + H]$, and this implies $P^\ell(s, j) \geq \bar{P}^\ell(s, j)$. Thus,

$$P^\ell(s, j) = \bar{P}^\ell(s, j).$$

The $\ell$-pressure $P^\ell(s, j)$ does not have much in common with $P^*(s, j)$, except for one property: it has the same zero.

Proposition 8.1. We have

$$P^\ell(s_0, j) = 0$$

Proof. There exists a sequence of times $(m_k)$ such that

$$S := \sum_{m_k \leq \ell(i) \leq m_k + H} \phi^{s_0}(i \delta(i)) \geq e^{m_k P^\ell(s_0, j) - o(m_k)}.$$

Every word $i$ appearing in this sum has length between $m_k/M_{\text{max}}$ and $m_k/M_{\text{min}}$, where $M_{\text{max}}$ and $M_{\text{min}}$ exist due to the almost additivity of $\ell$ and the consequence of Lemma 3.2. Present each of these words in the form

$$i = i_1 i_2$$

with $|i_1| = m_k/M_{\text{max}}$. Then, there exists a constant $C_5$, such that

$$S \leq C_5 \sum_{|i_1| = m_k/M_{\text{max}}} \sum_{\ell(i_2) \in [m_k - \ell(i_1), m_k + H - \ell(i_1)]} \phi^{s_0}(i_1 i_2 \delta(i_1 i_2)).$$

Note that for some constant $C_6$

$$\phi^{s_0}(i_1 i_2 \delta(i_1 i_2)) \leq C_6 \cdot \phi^{s_0}(i_1 \delta(i_1)) \cdot \phi^{s_0}(i_2 \delta(i_2)),$$

hence

$$S \leq C_7 \cdot \sum_{i_1} \left( \phi^{s_0}(i_1 \delta(i_1)) \sum_{i_2} \phi^{s_0}(i_2 \delta(i_2)) \right),$$

with some constant $C_7$. 

On the other hand, we have
\[\sum_{\ell(i_2) \in [m_k, \ell(i_1) + H - \ell(i_1)]} \phi^{s_0}(i_2 \partial(i_2)) \leq e^{(m_k - \ell(i_1))P^*_\ell(s_0, j) + o(m_k - \ell(i_1))}\]
and
\[\sum_{|i_1| = m_k M_{\max}} \phi^{s_0}(i_1 \partial(i_1)) \leq e^{o(m_k / M_{\max})}.
\]
As \(m_k - \ell(i_1) \leq m_k (1 - \frac{M_{\min}}{M_{\max}})\), we have
\[e^{m_k P^*_\ell(s_0, j) - o(m_k)} < W < e^{m_k (1 - \frac{M_{\min}}{M_{\max}}) P^*_\ell(s_0, j) + o(m_k)},\]
and hence
\[P^*_\ell(s_0, j) \leq 0.\]
To get the other inequality we do a similar argument. Let now \((m_k)\) be a sequence of times when
\[S_k := \sum_{|i| = m_k} \phi^{s_0}(i \partial(i)) > e^{-o(m_k)}.
\]
We have
\[\sum_{m_k M_{\min} \leq \ell(i) \leq m_k M_{\max} + H} \phi^{s_0}(i \partial(i)) \leq e^{m_k M_{\min} P^*_\ell(s_0, j) + o(m_k M_{\min})}.
\]
Presenting every word \(i\) of length \(m_k\) as
\[i = i_1i_2\]
with \(\ell(i_1) \in [m_k M_{\min}, m_k M_{\max} + H]\), we can write \(S_k\) as a double sum
\[S_k = \sum_{\ell(i_1) \in [m_k M_{\min}, m_k M_{\max} + H]} \sum_{|i_2| = m_k - |i_1|} \phi^{s_0}(i_1i_2 \partial(i_1i_2)).\]
Applying (39) and the inequalities
\[\sum_{|i_2| = m_k - |i_1|} \phi^{s_0}(i_2 \partial(i_1)) < e^{o(m_k)}\]
and
\[\sum_{\ell(i_1) \in [m_k M_{\min}, m_k M_{\max} + H]} \phi^{s_0}(i_1 \partial(i_1)) < e^{m_k M_{\min} P^*_\ell(s_0, j) + o(m_k M_{\min})}\]
we get
\[P^*_\ell(s_0, j) \geq 0.\]

8.2. \(\ell\)-modular spaces. Let \((n_k)\) be a sequence such that
\[Z^* = \lim_{k \to \infty} \frac{1}{n_k} \log \phi^{s_0}(j_{n_k}).\]
By choosing a further subsequence we can freely assume that \(n_{k+1} \gg n_k\).
For any infinite sequence \(i \in \Sigma\), let \(m(i, n_k)\) be the first time \(m\) for which \(\ell(i_m) \geq n_k\). Let us denote \(A_k := \{i_{m} : i \in \Sigma\}\). The cylinders \([a]\) with \(a \in A_k\) form a disjoint cover of \(\Sigma\), and hence every sequence \(i \in \Sigma\) can be uniquely presented as an infinite concatenation of elements \(a_m \in A_k: i = a_1a_2 \ldots\). This gives us a natural bijection between \(A_k^\infty\) and \(\Sigma\). In the following, \(A_k\) will play the same role as \(A^R\) did in the previous sections. Denote by \(A^*_k\) the finite words in the alphabet \(A_k\).
We say that a word is $A_k$-modular if it can be presented in the form
$$w = a_1k_1a_2\ldots k_{m-1}a_m$$
where each $a_r \in A_k$ and each $k_r$ with $|k_r| \leq K$ is a connecting word coming from the weak quasi-multiplicativity of $\phi^{s_0}$:
$$\phi^{s_0}(a_1k_1\ldots a_{r+1}) \geq Q \phi^{s_0}(a_1k_1\ldots a_r) \phi^{s_0}(a_{r+1}).$$
We define the $A_k$-modular extensions analogously.

We construct a probability vector on $A_k$: for any $a \in A_k$ we write
$$p(a) = \frac{\phi^{s_0}(a) \cdot e^{Z^*\ell(a)}}{\sum_{b \in A_k} \phi^{s_0}(b) \cdot e^{Z^*\ell(b)}}.$$ 
This measure gives us a Bernoulli measure $\nu$ on $(A_k^\infty, \sigma_{A_k})$, where $\sigma_{A_k} : A_k^\infty \to A_k^\infty$ is the shift map: for a word $i_1i_2i_3\ldots \in A_k^\infty$ with each $i_j \in A_k$, the shift is defined to be $\sigma_{A_k}(i_1i_2i_3\ldots) = (i_2i_3\ldots)$.

**Lemma 8.2.** There exists a constant $c'_{10}$ not depending on $k$ such that
$$(c'_{10})^{-1} < \sum_{b \in A_k} \phi^{s_0}(b) \cdot e^{Z^*\ell(b)} < c'_{10}.$$

**Proof.** This is the direct analogue of Lemma 6.6 and the proof is almost identical to the one presented in Section 6. One needs to construct the modular version of $\hat{P}_\ell^*$-pressure, show that its partial sums are almost equal to both the partial sums of the usual pressure on $(A_k^\infty, \sigma_{A_k})$ and the partial sums of $\hat{P}_\ell^*$, and then apply the convergence of $\hat{P}_\ell^*$ and Proposition 8.1. We omit the preparation of the construction of a measure by defining for each $d_k \in A_k$
$$\nu_k(d_k) = \frac{\exp(\Psi_k(d_k))}{\sum_{a \in A_k} \exp(\Psi_k(a))},$$
where
$$\Psi_k(a) = \log \phi^{s_0}(a) + Z^*n_k.$$

**8.3. Construction of the measure.** Finally, we need to repeat (with changes) the construction of the measure $\mu$ from Section 7. Fix $k$ for the time being.

**Step 1:** construction of the Cantor set $W_k$. The Cantor set $W_k$ will consist of the infinite words of the form
$$d_k \partial(d_k)d_{k+1}\partial(d_k\partial(d_k)d_{k+1})d_{k+2}\ldots,$$
where:
- $d_k \in A_k$,
- each $d_{k+n}$ gives an $A_k$-modular extension of the previous part of the word (thus, $d_k \partial(d_k)d_{k+1}$ is an $A_k$-modular extension of $d_k\partial(d_k)$ and so on),
- each $d_{k+n}$, giving an $A_k$-modular extension, is of the form $k_0a_1k_1a_2\ldots a_m$, with $a_i \in A_k$. The words $(a_1, \ldots, a_m)$ are not repeated: for any $(d_k, \ldots, d_{k+n-1})$ and any sequence $(a_1, \ldots, a_m)$ which could potentially give us some $A_k$-modular extensions of the form $k_0a_1k_1a_2\ldots a_m$, $a_i \in A_k$ we only choose one collection of $k_0, \ldots, k_{m-1}$ (say, the first in lexicographical order) and discard the other possibilities. Also, if we can do the extension for $(a_1, \ldots, a_m)$ then we do not take any extensions for $(a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+r})$. 

That is, for every choice of \((d_k, \ldots, d_{k+n-1})\) we have a projection \(\pi_{k+n}(d_{k+n}) = (a_1, \ldots, a_m)\) from the set of possible \(d_{k+n}\)'s into \(A^*_k\) and it is a bijection, and the cylinders of the projected words \(\pi_{k+n}(d_{k+n})\) form a disjoint cover of \(A^\infty_k\).

**Step 2: symbolic structure on \(W_k\).** As seen above, for every sequence \((d_k, d_{k+1}, \ldots)\) describing a point in \(W_k\) each of the words \(d_{k+n}\) can be projected to \(A^*_k\). Combining these projections we can define

\[
\pi(d_k, d_{k+1}, \ldots) := (d_k, \pi_{k+1}(d_{k+1}, \pi_{k+2}(d_{k+2}), \ldots))
\]

acting from \(W_k\) to \(A^\infty_k\). As the projections \(\pi_{k+n}\) were bijective, so is \(\pi\). As the cylinders of the projected words \(\pi_{k+n}(d_{k+n})\) form a disjoint cover of \(A^\infty_k\), \(\pi\) is onto. Thus, we can identify \(W_k\) with \(A^\infty_k\).

**Step 3: distribution of the measure.** We define \(\mu_k\) as a Bernoulli measure on \(A^\infty_k\), with

\[
\mu_k([d_k]) := \nu_k(d_k).
\]

**Step 4: why does this measure work for the concentration calculation?** For each \(i \in W_k\) and \(n \in \mathbb{N}\), we need to estimate the ratio \(\mu_k([i]_n)!) / \phi^s_0([i]_n)\). Like in Lemma 7.2, it will be bounded from above by some \(e^{c_n^\ell + c_n^s + c_n^s_9}\), with \(c_n^\ell\) arbitrarily small and \(c_n^s_9\) fixed. At times \(\ell([i]_n) = m_{k+n}\) we get the estimation from the definition of \(Z^*\) and \(s_0\). At other times we have even lower ratio because \(Z^*\) is a limsup.

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**References**

[BJKR] B. Bárány, T. Jordan, A. Käenmäki, M. Rams, Birkhoff and Lyapunov spectra on planar self-affine sets, Int. Math. Res. Not., to appear.

[BR] B. Bárány, M. Rams, Shrinking targets on Bedford-McMullen carpets, Proc. Lond. Math. Soc. (3) 117 (2018), no. 5, 951–995.

[BT] B. Bárány, S. Troscheit, Dynamically defined subsets of generic self-affine sets, Available online at https://arxiv.org/pdf/2109.06727.pdf.

[BW14] Y. Bugeaud, B. Wang, Distribution of full cylinders and the Diophantine properties of the orbits in \(\beta\)-expansions, J. Fractal Geom. 1 (2014), 221–241.

[F88] K. Falconer, The Hausdorff dimension of self-affine fractals, Math. Proc. Cambridge Philos. Soc. 103(2) (1988), 339-350.

[HV95] R. Hill, S. Velani, The ergodic theory of shrinking targets, Invent. Math. 119 (1995), 175–198.

[HV97] R. Hill, S. Velani, Metric Diophantine approximation in Julia sets of expanding rational maps, Inst. Hautes Etudes Sci. Publ. Math. 85 (1997), 193-216.

[H81] J. E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J., 30(5), 713–747 (1981).

[KR14] A. Käenmäki, H. Reeve, Multifractal analysis of Birkhoff averages for typical infinitely generated self-affine sets, J. Fractal Geom. 1 (2014), no. 1, 83–152.

[KR18] H. Koivusalo, F. A. Ramírez. Recurrence to shrinking targets on typical self-affine fractals, Proc. Edinb. Math. Soc. (2) 61 (2018), no. 2, 387–400.

[LWWX14] B. Li, B. Wang, J. Wu, J. Xu, The shrinking target problem in the dynamical system of continued fractions, Proc. Lond. Math. Soc. (3) 108 (2014), 159–186.

[PR17] T. Persson, M. Rams, On shrinking targets for piecewise expanding interval maps, Ergod. Th. & Dynam. Sys., 37, (2017), 646–663.
[R11] H. Revee, *Shrinking targets for countable Markov maps*, arXiv: 1107.4736 (2011).

[PU10] F. Przytycki, M. Urbański, Conformal fractals: ergodic theory methods. London Mathematical Society Lecture Note Series, 371. Cambridge University Press, Cambridge, 2010.

[S98] B. Solomyak, *Measure and dimension for some fractal families*. Math. Proc. Cambridge Philos. Soc., 124(3), 531–546 (1998).

[MS19] I. D. Morris and P. Shmerkin, *On equality of Hausdorff and affinity dimensions, via self-affine measures on positive subsystems*, Trans. Amer. Math. Soc. 371 (2019), 1547-1582.

[U02] M. Urbański, *Diophantine analysis of conformal iterated function systems*, Monatsh. Math. 137(4), 325-340 (2002).

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