TRANSVERSE FUNDAMENTAL GROUP AND PROJECTED EMBEDDINGS

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ABSTRACT. For a generic degree $d$ smooth map $f: N^n \to M^n$ we introduce its “transverse fundamental group” $\pi(f)$, which reduces to $\pi_1(M)$ in the case where $f$ is a covering, and in general admits a monodromy homomorphism $\pi(f) \to S|d|$; nevertheless, we show that $\pi(f)$ can be non-trivial already for rather simple degree 1 maps $S^n \to S^n$.

We apply $\pi(f)$ to the problem of lifting $f$ to an embedding $N \hookrightarrow M \times \mathbb{R}^2$: for such a lift to exist, the monodromy $\pi(f) \to S|d|$ must factor through the group of concordance classes of $|d|$-component string links. At least if $|d| < 7$, this requires $\pi(f)$ to be torsion-free.

1. Introduction

A generic $C^\infty$ map $f: N \to M$ is called a $k$-projected embedding, or a $k$-prem if there exists a $g: N \to \mathbb{R}^k$ such that $f \times g: N \to M \times \mathbb{R}^k$ is a smooth embedding (cf. [3], [2]).

A necessary condition for a generic smooth map $f: N^n \to M^m$ to be a $k$-prem is the existence of a $\mathbb{Z}/2$-equivariant map from the double point set $\Delta_f = \{ (x, y) \in N \times N \mid f(x) = f(y), x \neq y \}$ (endowed with the restriction of the factor exchanging involution on $N \times N$) to the sphere $S^{k-1}$ (endowed with the antipodal involution $x \mapsto -x$). Indeed, given a $g: N \to \mathbb{R}^k$ as above, we define $\varphi: \Delta_f \to S^{k-1}$ by $\varphi(x, y) = \frac{g(x) - g(y)}{|g(x) - g(y)|}$; clearly, $\varphi(y, x) = -\varphi(x, y)$. One can show that when $4n - 3m \leq k$ and $m + k \geq \frac{3(n+1)}{2}$, this necessary condition is also sufficient [16].

Using this result, it is not hard to see that every generic smooth map $f$ between orientable smooth $n$-manifolds is an $n$-prem for all even $n > 2$; indeed, an equivariant map $\Delta_f \to S^{n-1}$ exists for all even $n$ (including $n = 2$) as observed in the proof of Theorem 1.4 below.

Problem 1.1. Is every generic smooth map between orientable surfaces a 2-prem?

It is known that the answer is affirmative in the following cases: for maps of any 2-manifold into $\mathbb{R}^2$ [20]; for maps of $S^2$ into any orientable 2-manifold (Yamamoto–Akhmetiev [15]); and for maps $S^1 \times S^1 \to S^1 \times S^1$ [14].

Petersen proved that the answer is also affirmative for all regular coverings of degree $< 60$ [19]. This is established as follows: a group of order $< 60$ is solvable; a solvable covering is a composition of abelian coverings; an abelian covering over a compact polyhedron with free abelian $H_1$ is induced from a covering over a torus $S^1 \times \ldots \times S^1$; a covering over a torus is equivalent by a change of coordinates to a product of coverings

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over $S^1$; a product of coverings over $S^1$ is a composition of coverings over the torus induced from coverings over $S^1$; and a covering over $S^1$ is obviously a 2-prem.

Petersen also proved that if a composition of two coverings is a 2-prem, then each of them is a 2-prem [19]. Let us note that the regular covering corresponding to the kernel of the monodromy homomorphism of a covering $p$ factors through $p$. Consequently, every covering with solvable monodromy group (hence in particular every covering of degree $< 5$) between orientable surfaces is a 2-prem.

Let us also note that a covering over a connected sum of tori is a 2-prem if it is induced from a covering of the wedge of these tori; indeed, any covering induced from a covering over a wedge of tori is a 2-prem (see Theorem 2.12 below and the subsequent remarks).

Finally, we should note that one motivation of Problem 1.1 is that its affirmative solution would yield an affirmative answer to the following

**Problem 1.2.** [5], [15] Does every inverse limit of orientable 2-manifolds embed in $\mathbb{R}^4$?

P. M. Akhmetiev proved that an inverse limit of stably parallelizable $n$-manifolds embeds in $\mathbb{R}^{2n}$ for $n \neq 1, 2, 3, 7$ [1] (see [15] for an explicit proof). It is well-known that the $p$-adic solenoid, which is an inverse limit of copies of $S^1$, does not embed in the plane. Akhmetiev also constructed inverse limits of 3- and 7-dimensional parallelizable manifolds that do not embed in $\mathbb{R}^6$, resp. $\mathbb{R}^{14}$ (see [18]).

### 1.1. Content of the paper

It is clear from Theorem 1.4 below that Problem 1.1 is a typical problem of four-dimensional topology\(^1\) in that there is no lack of potential counterexamples (such as 5-fold coverings and regular 60-fold coverings, not to mention generic approximations of various branched coverings) but an obvious lack of invariants/obstructions capable of detecting actual counterexamples.

The present note develops one approach to constructing such an obstruction in the case of generic maps other than coverings. The “transverse fundamental group” $\pi(f)$ of a generic smooth map $f: M \to N$ between manifolds of the same dimension is introduced in §2. In the case where $f$ is a covering, $\pi(f)$ specializes to $\pi_1(M)$ and so gives nothing new. On the other hand, we compute, for instance, that $\pi(f)$ contains an infinite cyclic subgroup for a certain fold map $f: S^2 \to S^2$, which is a generic $C^0$-approximation of the suspension of the double covering $S^1 \to S^1$. This and other examples are studied in §3.

The following is a special case of Corollary 2.15.

**Theorem 1.3.** Let $f$ be a generic smooth map of degree $< 7$ between orientable 2-manifolds. If $\pi(f)$ contains torsion, then $f$ is not a 2-prem.

The author does not know if $\pi(f)$ can contain torsion when $\pi_1(M)$ is torsion-free; generally speaking, nothing seems to preclude from this.

\(^1\)Keeping in mind, say, the 4-dimensional PL Poincaré conjecture, the Andrews–Curtis conjecture, the problem of PL embeddability of acyclic and contractible 2-polyhedra in $\mathbb{R}^4$, etc.
From the viewpoint of algebraic topology, the elements of $\pi(f)$ are analogous, or rather dual, to spherical classes in the 2-homology of a 4-manifold (see Remark 2.5). Even though the technique enabling us to show that $\pi(f)$ is well-defined was originally developed in the course of a study of projected embeddings [15], the present paper was written with hope that $\pi(f)$ may also find other applications.

1.2. A motivation: The double point obstruction

One may look at the following straightforward obstruction to 1.1. Let us consider, more generally, a generic smooth map $f: N^n \to M^n$. Take a generic lift $\bar{f}: N \to M \times \mathbb{R}^n$ of $f$ and pick some basepoint $b \in N$. Each (necessarily isolated) double point $z = \bar{f}(x) = \bar{f}(y)$ of $\bar{f}$ has a sign $\varepsilon_z = \pm 1$ determined by comparing the orientations of the two sheets of $N$ with the orientation of $M \times \mathbb{R}^n$. Let us pick a path $p_x$ joining $b$ and $x$ and a path $p_y$ joining $y$ and $b$. Then $f(p_x p_y)$ is an $f(b)$-based loop in $M$. The class $g_z \in G := \pi_1(M, f(b))$ of this path is well defined up to multiplication on both sides by elements of $H := f_*(\pi_1(N))$. Let $\bar{\theta}(\bar{f})$ be the algebraic sum

$$\sum_z \varepsilon_z Hg_z H \in \mathbb{Z}[H\backslash G/H]$$

of the resulting double cosets. If $\bar{f}'$ is another generic lift of $f$, a generic homotopy between $\bar{f}$ and $\bar{f}'$ over $f$ yields an oriented bordism between the set of double points of $\bar{f}$ and that of $\bar{f}'$. The critical levels of this bordism consist of cancellations/introductions of pairs $(z, z')$ such that $\varepsilon_z = -\varepsilon_{z'}$ and $g_z = g_{z'}$; and (unless $f$ is a covering) of births/deaths of individual double points $z$ such that $g_z \in H$. Hence $\theta(f) := \bar{\theta}(\bar{f}) = \bar{\theta}(\bar{f}')$ is well defined. Obviously, if $f$ is an $n$-prem, $\theta(f) = 0$.

**Theorem 1.4.** If $n$ is even, $\theta(f) = 0$ for every generic smooth $f: N^n \to M^n$.

*Proof.* Since the dimensions of $N$ and $M$ have the same parity, $\Delta_f/T$ is orientable, where $T$ is the factor changing involution on $N \times N$ (see e.g. [15; Akhmetiev’s Lemma (preceding Example 5)] or the beginning of §3). If $\lambda$ is the line bundle associated with the double covering $\Delta_f \to \Delta_f/T$, its Euler class $\varepsilon(\lambda)$ is an element of order two in the cohomology group $H^1(\Delta_f/T; \mathbb{Z}_T)$ with local coefficients (see e.g. [17; §2]). Since $\Delta_f/T$ is orientable, whereas the coefficients $\mathbb{Z}_T^{2n}$ are constant when $n$ is even, $H^n(\Delta_f/T; \mathbb{Z}_T^{2n})$ is free abelian, and therefore $\varepsilon(\lambda)^n = 0$.

This yields an equivariant oriented null-bordism $W$ of the oriented 0-manifold $\Delta_f$ in $\Delta_f$ (see e.g. [15; Lemma 7] or [17; §3, subsections “Geometric definition of $\varepsilon(f)$” and “Cohomological sectional category”]). Without loss of generality $W$ has no components without boundary. By the definition of $\Delta_f$, we have $fpT(a) = f(y) = f(x) = fp(a)$ for each $a = (x, y) \in \Delta_f$, where $p$ projects $N \times N$ onto the first factor. Since $W \subset \Delta_f$, it follows that $fpT|_{J} = fp|_{J}$ for each component $J$ of $W$.

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2This immediately implies the existence of an equivariant map $\Delta_f \to S^{n-1}$ (see e.g. [17; Alternative proof of Theorem 3.2]).
Let \( c = (x, y) \) and \( d = (x', y') \) be the endpoints of a compact component \( J \) corresponding to double points \( z = \bar{f}(x) = \bar{f}(y) \) and \( z' = \bar{f}(x') = \bar{f}(y') \) of \( \bar{f} \), so that \( \varepsilon_z = -\varepsilon_{z'} \). Let \( p_x \) be a path joining \( b = x = p(c) \) and \( p_y \) a path joining \( y = pT(c) \) to \( b \). Then \( p_x \) followed by \( p|_J \) is a path \( p_x' \) joining \( b = x = p(d) \) and the inverse of \( pT|_J \) followed by \( p_y \) is a path \( p_y' \) joining \( y = pT(d) \) to \( b \). Now \( f(pT|_J) = f(p|_J) \) implies that \( f(p_x'p_y') \) is homotopic to \( f(p_xp_y) \), whence \( g_x = g_z \).

Similarly if \( J \) is noncompact and so has only one endpoint \( a = (x, y) \) corresponding to a double point \( z = \bar{f}(x) = \bar{f}(y) \), then \( g_z \in H \).

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2. IN SEARCH OF NON-2-PREMS

2.1. Transverse fundamental group

**Definition 2.1** (Pullback). If \( L, M \) and \( N \) are smooth manifolds and \( L \xrightarrow{g} M \xleftarrow{f} N \) are smooth maps, we say that \( g \) is transverse to \( f \) and write \( g \cap f \) if \( f \times g : N \times L \to M \times M \) is transverse to \( \Delta_M \). In this case \( P := (f \times g)^{-1}(\Delta_M) \) is a smooth submanifold of \( N \times L \), and consequently the composition of the inclusion \( P \hookrightarrow N \times L \) and the projection \( N \times L \to N \) is a smooth map; if additionally \( f \) is generic, then so is this composition. This composition is called the pullback (or the "base change map") of \( f \) along \( g \) and is denoted \( g^*f \), and its domain \( P \) (also known as the pullback of the diagram \( L \xrightarrow{g} M \xleftarrow{f} N \)) may be denoted \( (g^*f)^{-1}(L) \). Note that if \( g \) is an embedding, then so is \( f^*g \), which therefore performs a homeomorphism between \( (g^*f)^{-1}(L) = (f^*g)^{-1}(N) \) and \( f^{-1}(g(L)) \).

**Definition 2.2** (Coherent homotopy). Let \( f : N \to M \) be a generic smooth map between closed oriented connected \( n \)-manifolds, \( n \geq 1 \), and let \( b \in M \) be its regular value (in particular, \( b \) is a value of \( f \), i.e. \( b \in f(N) \).) Consider \( f \)-transverse based loops \( l_0, l_1 : (S^1, pt) \to (M, b) \). A based homotopy \( h : (S^1 \times I, pt \times I) \to (M, b) \) between \( l_0 \) and \( l_1 \) will be called \( (b, f) \)-coherent if it is \( f \)-transverse and every connected component of the pullback \( (h^*f)^{-1}(S^1 \times I) \) that intersects \( (h^*f)^{-1}(pt \times I) \) is an annulus with one boundary component in \( (l_0^*f)^{-1}(S^1) \) and another in \( (l_1^*f)^{-1}(S^1) \). Note that some individual levels \( h_t : S^1 \to M \) of a \( (b, f) \)-coherent homotopy may be non-\( f \)-transverse, and the number of components in \( (h_t^*f)^{-1}(S^1) \) may vary depending on \( t \).

**Definition 2.3** (\( \pi(f) \): The case of unfolded basepoint). Suppose first the cardinality \( |f^{-1}(b)| \) equals the absolute value \( |\deg f| \) (so in particular \( \deg f \neq 0 \), since we are assuming that \( b \in f(N) \)). The set \( \pi(f, b) \) of \( b \)-based \( f \)-transverse loops in \( M \) up to \( (b, f) \)-coherent homotopy is clearly a group with respect to the usual product (i.e. the concatenation) of loops and the usual inverse of a loop.

**Example 2.4.** If the generic map \( f : N \to M \) is a covering, \( \pi(f, b) \simeq \pi_1(M) \) since coverings enjoy the covering homotopy property.
On the other hand, there exists, for instance, a fold map \( f : S^2 \to S^2 \) with four fold curves and with \( |f^{-1}(b)| = 1 \) such that \( \pi(f) \neq 1 \) (see Examples 3.5, 3.9).

**Remark 2.5.** Note that the Pontryagin construction identifies \( f \)-transverse framed loops in \( M \) with stable maps of the mapping cylinder of \( f \) into \( S^{n-1} \) that are transverse to \( pt \in S^{n-1} \). Of course, homotopies between such maps, transverse to \( pt \), are identified with arbitrary \( f \)-transverse framed bordisms, which are not necessarily homotopies (not to mention coherent homotopies). Thus from the viewpoint of algebraic topology, the question of existence of a coherent homotopy is a question of representability of a generalized cohomology class by a genus zero cocycle extending a given representation on the boundary. (By a cocycle we mean a pseudo-comanifold, i.e. an embedded mock bundle with codimension two singularities — see [4].)

**Definition 2.6** \((\pi(f))\): The general case. Without loss of generality we may assume that \( d := \deg(f) \geq 0 \). Let \( j : I \to M \) be an \( f \)-transverse path, and let \( J = (j^*f)^{-1}(I) \). Since \( M \) is oriented, \( \Delta_M \) is co-oriented in \( M \times M \), hence \( J \) is co-oriented in \( I \times N \). Since \( I \) and \( N \) are oriented, \( J \) is oriented.

A component \( C \) of \( J \) is called a positive (negative) arc if \( j^*f|_C : (C, \partial C) \to (I, \partial I) \) has degree +1 (resp. -1). Else \( C \) could be a circle or an arc with both endpoints mapping onto the same endpoint of \( I \), with \( (j^*f)(C) \neq I \). Note that the signs of the arcs reverse (along with the sign of \( \deg(f) \)) when the orientation of \( M \) or \( N \) is reversed; but remain unchanged when the orientation of \( I \) is reversed.

**Lemma 2.7.** [15; §2, proof of Observation 2] Let \( f : N \to M \) be a generic smooth map between closed oriented connected \( n \)-manifolds, \( n \geq 1 \), with \( \deg(f) \geq 0 \). Then there exists an \( f \)-transverse path \( \ell : I \to M \) such that \( (\ell^*f)^{-1}(I) \) contains no negative arcs.

Without loss of generality \( a := \ell(0) \) and \( b := \ell(1) \) are \( f \)-regular values. (In fact, since any \( f \)-transverse path \( \ell_+ \) containing \( \ell \) is again such that \( (\ell_+^*f)^{-1}(I) \) contains no negative arcs, \( a \) and \( b \) could have been any \( f \)-regular values given in advance.) Let \( L = (\ell^*f)^{-1}(I) \), and let \( D \) be a bijection between \([d] := \{0, 1, \ldots, d - 1\} \) and the set of endpoints in \( (\ell^*f)^{-1}(0) \) of the \( d \) positive arcs in \( L \).

Let \( j : (I, \partial I) \to (M, b) \) be any \( f \)-transverse loop. Then the product \( \hat{j} \) of the paths \( \ell \), \( j \) and the inverse path \( \bar{\ell} \) (defined by \( \bar{\ell}(t) = \ell(1 - t) \)) is again such that \( \hat{J} := (\hat{j}^*f)^{-1}(I) \) contains no negative arcs; moreover, each positive arc of \( L \) is contained in a unique positive arc of \( J \).

We define \( \pi(f, \ell) \) to be the set of all \( f \)-transverse \( b \)-based loops up to \( b \)-based homotopy \( j_\ell \) such that the \( a \)-based homotopy \( \hat{j}_\ell \) is \((a, f)\)-coherent. Clearly this is a group with respect to the usual product of loops and the usual inverse of a loop.

Furthermore, assigning to an endpoint of a positive arc in \( \hat{J} \) the other endpoint of this arc, we get a bijection \( h_{j, D} : [d] \to [d] \). If \([j'] = [j] \in \pi(f, \ell) \), clearly \( h_{j', D} = h_{j, D} \). Hence \([j] \mapsto h_{j, D} \) defines a homomorphism \( \varphi_{f, \ell, D} : \pi(f, \ell) \to S_d \).
Let us consider an embedding of $f$ into the pullback $C \times \{j\} \subset \mathbb{C} \times \{j\}$ to itself for $j = 0, 1$. If moreover $g$ sends each $((e^{2\pi i/k})^n, j)$ to itself for $j = 0, 1$, then $g$ is a pure string link. Thus the group $C_k$ of concordance classes of pure string links of multiplicity $k$ is the kernel of the projection $T_k \to S_k$.

By a string link of multiplicity $k$ we mean an embedding $g: (\mathbb{Z}/k) \times I \to \mathbb{C} \times I$ sending $(\mathbb{Z}/k) \times \{j\} \subset \mathbb{C} \times \{j\}$ to itself for $j = 0, 1$. If moreover $g$ sends each $((e^{2\pi i/k})^n, j)$ to itself for $j = 0, 1$, then $g$ is a pure string link. Thus the group $C_k$ of concordance classes of pure string links of multiplicity $k$ is the kernel of the projection $T_k \to S_k$.

Proposition 2.11 is analogous to the “only if” implication in
Theorem 2.12 (Hansen [10], [11]). A d-fold covering \( f : N \rightarrow M \) is a 2-prem if and only if the monodromy \( \pi_1(M) \rightarrow S_d \) factors through the projection \( B_d \rightarrow S_d \), where \( B_d \) denotes the braid group on \( d \) strands.

The “if” implication can be proved as follows (compare [6; statement of Theorem 2]). Let \( D \) be an open disk in \( M \); then \( \pi_1(M \setminus D) \) is the free group \( \langle x_1, y_1, \ldots, x_g, y_g \rangle \), where \( g \) is the genus of \( M \), and the inclusion \( M \setminus D \hookrightarrow M \) induces a homomorphism \( \pi_1(M \setminus D) \rightarrow \pi_1(M) \) whose kernel is the normal closure of \([x_1, y_1] \ldots [x_g, y_g]\). Let \( \varphi : \pi_1(M) \rightarrow B_d \) be the given homomorphism, and let \( r : M \setminus D \rightarrow W \) be a deformation retraction onto a wedge of \( g \) copies of \( S^1 \). Then the braids \( \varphi(x_1), \varphi(y_1), \ldots, \varphi(x_g), \varphi(y_g) \) combine to yield the desired lift \( \tilde{f}_0 : f^{-1}(W) \rightarrow W \times \mathbb{R}^2 \) of the restriction of \( f \) over \( W \), and its pullback \( r^*(\tilde{f}_0) \) is the desired lift of the restriction of \( f \) over \( M \setminus D \). Now over \( \partial D \) the latter partial lift restricts to the braid \( \varphi([x_1, y_1] \ldots [x_g, y_g]) \), which is trivial, since \( \varphi \) is a homomorphism; hence the lift extends over \( D \).

The “only if” part of 2.12 along with Petersen’s results discussed in §1 have the following group-theoretic consequence: every homomorphism \( G \rightarrow S_d \), where \( G \) is a finitely generated free abelian group, factors through \( B_d \). In particular, since \( B_d \rightarrow S_d \) factors through \( T_d \), we get

Corollary 2.13. Let \( f \) be a generic smooth map between compact connected oriented \( n \)-manifolds, \( n \geq 2 \). If the monodromy \( \varphi_f \) factors through a free product of finitely generated free abelian groups then it also factors through \( T_{|\deg(f)|} \).

Another immediate thing to note about 2.11 and 2.12 is that every \( f : N \rightarrow M \) factors into the composition of the embedding \( \Gamma_f : N \hookrightarrow N \times M \) and the projection \( N \times M \rightarrow M \).

Hence

Proposition 2.14. (a) The monodromy \( \pi_1(M) \rightarrow S_d \) of every d-fold covering \( N \overset{f}{\rightarrow} M \) between surfaces factors through the group \( B_d(N) \) of braids in \( N \times I \).

(b) The monodromy \( \pi(f) \rightarrow S_{|\deg(f)|} \) of every generic map \( N \overset{f}{\rightarrow} M \) between orientable surfaces factors through the group \( T_{|\deg(f)|}(N) \) of concordance classes of string links in \( N \times I \).

Le Dimet showed that the natural map \( B_d \rightarrow T_d \) is injective, i.e. concordant braids are isotopic (cf. [12; p. 312]). Indeed, the Artin representation \( B_d \rightarrow \text{Aut}(F_d) \) is injective and agrees with the representation \( T_d \rightarrow \text{Aut}(F_d/\gamma_n) \), which is well-defined for each \( n \) by the Stallings Theorem on the lower central series \( \gamma_n \) (see [9; §1]). But \( \bigcap \ker[\text{Aut}(F_d) \rightarrow \text{Aut}(F_d/\gamma_n)] = 1 \) since \( \bigcap \gamma_n = 1 \) in \( F_d \).

On the other hand, \( B_d \) and \( T_d \) have a common quotient, the homotopy braid group \( HB_d \) (see [7], where the difference between \( B_d \) and \( HB_d \) is explained). Indeed, every string link is link homotopic to a braid (see [8]) and concordance implies link homotopy by a well-known result of Giffen and Goldsmith. The latter also follows from the injectivity of \( HB_d \rightarrow \text{Aut}(F_d/\mu_0) \) [8], where \( \mu_0 \) is the product of the commutator subgroups of the normal closures of the generators of \( F_d \), which contains \( \gamma_{d+1} \).
Similarly to Artin’s combing $P_d \simeq (\ldots (F_1 \ltimes F_2) \ltimes \ldots) \ltimes F_{d-1}$ of the pure braid group, the kernel $HP_d$ of the projection $HB_d \to S_d$ admits the combing $HP_d \simeq (\ldots (F_1/\mu_0 \ltimes F_2/\mu_0) \ltimes \ldots) \ltimes F_{d-1}/\mu_0$ [7], [8]. Hence $HP_d$ is torsion-free. Using this, Humphries proved that $HB_d$ is torsion-free for $d < 7$; in fact he showed that $\alpha \in HB_d$ has infinite order if its image in $S_d$ has order divisible by 2, 3, or 5 [13].

**Corollary 2.15.** Let $f$ be a generic smooth map between compact connected oriented $n$-manifolds, $n \geq 2$. If $\pi(f)$ contains an element $\alpha$ of finite order whose monodromy $\varphi_f(\alpha) \in S_{|\deg(f)|}$ is of order divisible by 2, 3 or 5, then $f$ is not a 2-prem.

Note that for the hypothesis to hold, the group $HT_{|\deg(f)|}(N)$ of link homotopy classes of string links in $N \times I$ must contain torsion, by Proposition 2.14(b).

Taking into account the Yamamoto–Akhmetiev Theorem [15], we have

**Corollary 2.16.** If $f$ is a generic smooth map from $S^2$ to a closed orientable surface then $\pi(f)$ contains no torsion with monodromy of order divisible by 2, 3 or 5.

An interesting question is whether already some lower central series quotient of $HB_d$ is torsion free. For instance, the abelianization of $HB_3$ is not: the braid $[\sigma_{12}, \sigma_{23}]$ projects nontrivially to $S_3$, but it is easy to check that the three strands of the pure braid $[\sigma_{12}, \sigma_{23}]^3$ are pairwise unlinked. This means, in particular, that it would not be a good idea to simplify the definition of $(b, f)$-coherent homotopy into “link map bordism”, i.e. to allow the positive components of the preimage to change by pairwise disjoint bordisms.

**Remark 2.17.** It was shown by Habegger and Lin [9; §1] that

(i) the image of the group $C_d$ of concordance classes of pure string links in $\text{Aut}(F_d/\gamma_n)$ is the subgroup $\text{Aut}_0(F_d/\gamma_n)$ that depends on the chosen set $\{x_1, \ldots, x_n\}$ of free generators of $F_d$ and consists of those automorphisms that send the coset $\bar{x}_i$ of each $x_i$ to a conjugate of $\bar{x}_i$ and fix the product $\bar{x}_1 \cdots \bar{x}_d$;

(ii) $\text{Aut}_0(F_d/\gamma_2) = 1$, and each $\text{Aut}_0(F_d/\gamma_{n+1})$ is a central extension of $\text{Aut}_0(F_d/\gamma_n)$ by a free abelian group, which they denote $K_{n-1}$; in particular, for $n > 2$, $\text{Aut}_0(F_d/\gamma_n)$ is torsion-free and nilpotent of class $n - 2$.

It follows easily from these that

(i') the image of $T_d$ in $\text{Aut}(F_d/\gamma_n)$ is the subgroup $\text{Aut}_1(F_d/\gamma_n)$ consisting of those automorphisms that send the coset $\bar{x}_i$ of each $x_i$ to a conjugate of some $\bar{x}_j$ and fix the product $\bar{x}_1 \cdots \bar{x}_d$;

(ii') $\text{Aut}_1(F_d/\gamma_2) \simeq S_d$, and for $n > 1$, each $\text{Aut}_1(F_d/\gamma_{n+1})$ is a central extension of $\text{Aut}_1(F_d/\gamma_n)$ by the same free abelian group $K_{n-1}$.

Thus the homomorphism $T_d \to S_d$ factors through the limit of the inverse sequence $\cdots \to \text{Aut}_1(F_d/\gamma_3) \to \text{Aut}_1(F_d/\gamma_2) \simeq S_d$. If some term of this inverse sequence or the inverse limit is torsion-free (for each $d$) — or if $HB_d$ is torsion-free for all $d$ — then the restriction on the order of the monodromy is superfluous in Corollary 2.15.
3. SOME COMPUTATIONS OF $\pi(f)$

3.1. A certain fold map $S^n \to S^n$ of degree $d$

Let $f$ be the degree $d$ map $f: S^n \to S^n$ defined by picking $d+1$ disjoint $n$-disks in $S^n$ and sending each of them homeomorphically to its own exterior in $S^n$. Let $b$ be a point with $|f^{-1}(b)| = d$ (i.e. a point in the interior of one of the disks).

It is easy to see that $f$ lifts to an embedding $f \times g: S^n \to S^n \times \mathbb{R}^2$. Namely, fix an embedding of $pt \ast [d+1]$ (the cone over $[d+1] = \{0, \ldots, d\}$) into $\mathbb{R}^2$, and let $g: S^n \to pt \ast [d+1] \subset \mathbb{R}^2$ send the interior of the $i$th $n$-disks into $pt \ast \{i\} \setminus pt \ast \emptyset$, and the exterior of the disks into $pt \ast \emptyset$.

Example 3.1 (the case $d = 2$). Let $\alpha \in \pi(f, b)$ be the class of a loop intersecting each disk along its diameter (compare Example 6 in [15]). Then $\alpha$ is nontrivial since it is easily seen to have a nontrivial monodromy $\varphi_f(\alpha) \in S_2$. On the other hand, by Proposition 2.11 the monodromy map $\varphi_f$ lifts to $\hat{\varphi}_f: \pi(f, b) \to T_2$. Since $T_2 \to S_2$ factors through $HB_2$, we conclude that the image of $\hat{\varphi}_f(\alpha)$ in $HB_2 \simeq \mathbb{Z}$ is nontrivial. Hence $\alpha$ has infinite order, and the composition $\mathbb{Z} \simeq \langle \alpha \rangle \subset \pi(f, b) \to HB_2 \simeq \mathbb{Z}$ is an isomorphism. Thus $\pi(f)$ contains a direct summand isomorphic to $\mathbb{Z}.

Example 3.2 (the case $d = 2$, $n > 2$). We will now show that if $n > 2$ (and still $d = 2$), then $\pi(f)$ is isomorphic to $\mathbb{Z}$.

Let $A$, $B$ and $C$ denote the 3 disks, with $b \in B$. A loop representing an element of $\pi(f, b)$ gives rise to a word in the alphabet $\{A, B, C\}$ (starting and ending with the letter $B$), which encodes the sequence of disks intersected by the loop. If two loops give rise to the same word, then (using that $n \geq 3$) they represent the same element of $\pi(f, b)$. Furthermore, it is easy to see\(^3\) that $XX = X$ and $XYX = X$ in $\pi(f, b)$ for any $X, Y \in \{A, B, C\}$.

Let $F$ be the free monoid (=semi-group with 1) on the alphabet $A, B, C$ (where the product of words is given by concatenation), and let $BFB$ be the submonoid of $F$ consisting of all words of the form $BwB$, where $w \in F$. Let $G$ be the quotient of $BFB$ by the relations $XYX = X = XX$, where $X, Y \in \{A, B, C\}$. Then $G$ is a group with unit $B$ and with the inverse given by $BX_1 \cdots X_n B \mapsto BX_n \cdots X_1 B$. Indeed, $BX_n \cdots X_1 XB_1 \cdots X_n B = BX_n \cdots X_1 X_n B = \cdots = B$.

In fact, $G$ is nothing but the group of simplicial loops in the triangle $\partial \Delta^2$ (with vertices $A$, $B$, $C$) under the relation of simplicial homotopy. Thus $G \simeq \mathbb{Z}$, with $n \in \mathbb{Z}$ corresponding to the class of $B(ABC)^n B$ (where $(ABC)^{-1} = CBA$). By construction, we have an epimorphism $G \to \pi(f, b)$ sending a generator onto $\alpha$. Hence $\pi(f) \simeq \mathbb{Z}.$

Example 3.3 ($n = 2$, $d = 2$). In the case $n = 2$, $d = 2$, $\pi(f)$ is larger than $\mathbb{Z}$, for similarly to Example 3.6 below it can be shown that there are loops giving rise to the words $B$, $BAB$ and $BCB$ yet representing elements of infinite order in $\pi(f)/\mathbb{Z}$.

\(^3\)The author is grateful to P. M. Akhmetiev for pointing out these relations.
Example 3.4 \((n > 2, d > 2)\). In the case \(d > 2, n > 2\), the same considerations as in \(3.2\) show that \(\pi(f)\) is a quotient of \(\pi_1((\Delta^{d+1})^{(1)})\) (that is, of the free group on \(\frac{d(d-1)}{2}\) letters), and admits an epimorphism onto the homotopy braid group \(HB_d\).

3.2. Fold maps of geometric degree 1 with embedded spherical folds

Let \(f: S^n \to S^n\) be a generic fold map that embeds its fold surface \(\Sigma_f := \{x \in S^n \mid \ker df_x \not= 0\}\) and is such that \(|f^{-1}(b)| = 1\); in particular, \(f\) has degree \(\pm 1\). (Fold map means that every point of \(\Sigma_f\) is a fold point, rather than a point of a higher singularity type.) In the case \(n > 2\), let us additionally assume that each component \(S_i\) of \(\Sigma_f\) is a sphere.

The following notation will be used. If \(S_i\) is a component of \(\Sigma_f\), let \(B_i\) be the \(n\)-ball bounded by \(S_i\) in \(S^n \setminus f^{-1}(b)\). On the other hand, let \(D_i\) be the \(n\)-ball bounded by \(f(S_i)\) in \(S^n \setminus b\). Each \(f(B_i)\) is connected, hence coincides with some \(D_{\rho(i)}\), таким что \(D_i \subset D_{\rho(i)}\). Note that \(\rho\rho = \rho\); in other words if \(j\) is of the form \(\rho(i)\), then \(f(B_j) = D_j\). In particular, in this case \(S_j\) is outer, i.e. \(f\) sends a neighborhood of \(S_j\) in \(B_j\) into \(D_j\). Accordingly, we call an \(S_i\) inner if \(f\) sends such a neighborhood into the closure of \(S^2 \setminus D_i\).

Example 3.5 (a two-dimensional example). There exists a generic fold map \(f: S^2 \to S^2\) such that \(f\) embeds \(\Sigma_f\) and \(|f^{-1}(b)| = 1\), yet \(\pi(f, b)\) is non-trivial.

Namely, \(f\) is the unique (up to reparametrization) map such that \(\Sigma_f\) is the union of four curves \(S_1, S_2, S_3 = S_{\rho(1)}\) and \(S_4 = S_{\rho(2)}\) such that \(D_1 \cap D_2 = \emptyset\) and \(D_1 \cup D_2 \subset D_3 \subset D_4\).

Let \(l: (S^1, pt) \to (S^2, b)\) be a loop intersecting \(D_3\) by a diameter that separates \(D_1\) from \(D_2\) within \(D_3\), and intersecting \(D_4\) by some diameter (which contains the former diameter). Then the pullback \((l^*f)^{-1}(S^1)\) consists of three components \(P_0, P_1\) and \(P_2\), with \(P_0\) containing \((l^*f)^{-1}(pt)\) and with \((f^*l)(P_1)\) and \((f^*l)(P_2)\) contained respectively in \(B_1\) and \(B_2\). By a homotopy \(h\) from \(l\) to an \(\tilde{l}\) with values in \(S^2 \setminus (D_1 \cup D_2)\) one cannot eliminate either \(P_1\) or \(P_2\); that is, neither \(P_1\) nor \(P_2\) bounds a disk in \((h^*f)^{-1}(S^1 \times I)\). On the other hand, any such homotopy \(h\) with values not only in \(S^2 \setminus (D_1 \cup D_2)\) joins either \(P_1\) or \(P_2\) to \(P_0\); that is, either \(P_1\) or \(P_2\) (or both) belongs to the component \(Q\) of \((h^*f)^{-1}(S^1 \times I)\) containing \(P_0\). Hence \(Q\) has at least two boundary components in \((l^*f)^{-1}(S^1)\), and so \(h\) is not coherent. Thus one cannot eliminate either \(P_1\) or \(P_2\) by a coherent homotopy. So there exists no coherent null-homotopy of \(l\).

Similar considerations show that no power of \(l\) is trivial in \(\pi(f, b)\).

Example 3.6 (a mild generalization). Let us generalize Example 3.5 to show that if \((\text{under the hypothesis of \(\S 3.2\)}\) \(\text{there exists a pair \((i, j)\) such that } D_i \subset f(B_j)\) and \(D_j \subset f(B_i)\) (for brevity, we shall call such a pair \((i, j)\) linked) and additionally \(B_1 \cap B_2 = \emptyset\), then \(\pi(f, b)\) is non-trivial as long as \(n = 2\).

Indeed, up to renumbering we may assume that \((i, j) = (1, 2)\). Since \((1, 2)\) is linked, \(\rho(1) \neq 1\). Then, in particular, \(S_{\rho(1)}\) is outer. Hence if \(S_1\) is also outer, then there exists an \(i\) such that \(B_1 \supset B_i \supset B_{\rho(1)}\) and \(S_i\) is inner. Then \(\rho(i) = \rho(1)\) and \((i, 2)\) is a linked pair. Thus without loss of generality we may assume that \(S_1\) is inner; similarly for \(S_2\).
The construction of Example 3.5 will apply here once we show that there is a loop \( l : (S^1, pt) \to (S^2 \setminus (D_1 \cup D_2), b) \) such that the component \( L^x \) of \( L := (l^*f)^{-1}(S^1) \) that contains \( (l^*f)^{-1}(pt) \) represents, via \( f^*l \), a nontrivial element of \( H_1(S^2 \setminus (B_1 \cup B_2)) \). Let \( B \) be the union of all disks bounded by the collection of circles \( f^{-1}(f(S_1 \cup S_2)) \) in the complement to \( x := f^{-1}(b) \). Let \( S^+_1 \) be a pushoff of \( S_1 \) into the complement of \( B_1 \), and let \( y \) be a point of \( S^+_1 \). Since \( S_1 \) is inner, \( y \notin B \); and since \( S^2 \setminus B \) is connected, \( x \) and \( y \) can be joined by a path \( p \) in \( S^2 \setminus B \). By Lemma 2.10 there exists a loop \( l_0 : (S^1, pt) \to (S^2, b) \) such that \( x_0 := (l_0^*f)^{-1}(pt) = (f^*l_0)^{-1}(x) \) and \( y_0 := (f^*l_0)^{-1}(y) \) are singletons and lie in the same component \( L^x_0 \) of \( L_0 := (l_0^*f)^{-1}(S^1) \). Moreover, by the proof of Lemma 2.10 (found in [15]) we may assume that \( l_0 \) has values in a neighborhood of \( f(p(I)) \), hence in \( S^2 \setminus (D_1 \cup D_2) \). Then \( f^*l_0 \) sends \( L^x_0 \) into \( S^2 \setminus (B_1 \cup B_2) \). Now amend \( l_0 \) by cutting it open at \( f(y) \) and inserting a loop circling around \( f(S^+_1) \). Then the resulting loop \( l_1 : (S^1, pt) \to (S^2 \setminus (D_1 \cup D_2), b) \) is such that the component \( L^x_1 \) of \( L_1 := (l_1^*f)^{-1}(S^1) \) that contains \( x_1 := (l_1^*f)^{-1}(pt) \) differs from \( L^x_0 \) by a loop circling around \( S^+_1 \). Thus \( f^*l_0|_{L^x_0} \) and \( f^*l_1|_{L^x_1} \) represent distinct elements of \( H_1(S^2 \setminus (B_1 \cup B_2)) \); so at least one of them is non-trivial.

**Example 3.7** (a higher-dimensional proposition). We shall show that the phenomenon exhibited in Example 3.5 does not occur in higher dimensions; more specifically, that (under the hypothesis of §3.2) for \( n > 2 \), a \( b \)-based \( f \)-transverse loop crossing each \( S_i \) at most twice represents the trivial element of \( \pi(f, b) \).

Indeed, let \( l : (S^1, pt) \to (S^n, b) \) be such a loop. Since \( n > 2 \), it may be assumed to be embedded. (This assumption will not be essentially used, but allows to simplify notation.) Since \( l \) is \( f \)-transverse, it is transverse to the codimension one submanifold \( f(\Sigma_i) \). Up to a renumbering of \( S_i \)’s, we may assume that \( D_1 \) meets \( l(S^1) \) and is innermost among all the balls \( D_i \) in \( S^n \setminus b \) that meet \( l(S^1) \). Write \( A = l(S^1) \cap D_1 \); by our hypothesis, it is an arc. Then \( \partial A \) bounds an arc \( A' \) in \( f(S_1) \) so that \( A' \) meets \( l(S^1) \) only in \( \partial A' = \partial A \). The circle \( A \cup A' \) bounds a 2-disk \( D \) in \( D_1 \), meeting \( \partial D_1 \) only in \( A' \). Without loss of generality \( D \) meets \( l(S^1) \) only in \( A \). Since \( n > 2 \), we may assume that \( D_1 \) is disjoint from all \( D_i \) that lie in the interior of \( D_1 \). Let us homotop \( l \) across \( D_1 \) from \( A \) to \( A' \), to a loop \( l_1 \) such that \( l_1(S^1) \cap D_1 \) has fewer components than \( l(S^1) \cap D_1 \). Proceeding inductively, we obtain a pointed homotopy \( h_t, t \in [0, \ldots, N] \), from \( l_0 := l \) through loops \( l_1, l_2, \ldots, \) to a loop \( l_N \) disjoint from every \( D_i \). Since \( n > 2 \), the latter is pointed null-homotopic with values in the complement to all \( D_i \)’s and so represents the trivial element of \( \pi(f) \).

Let us show that the constructed null-homotopy of \( l \) is coherent. Let \( L_i = (l^*_i f)^{-1}(S^1) \).

If \( S_{i+1} \) is outer, then \( L_{i+1} \) is obtained from \( L_i \) by removing one component, disjoint from \( (l^*_i f)^{-1}(pt) \). Else (i.e. if \( S_{i+1} \) is inner) \( L_{i+1} \) is obtained from \( L_i \) by splitting one of the components into two. (Here we are using our hypothesis, implying by induction that \( l_i \) crosses \( S_{i+1} \) just twice.) The component of \( L_i \) being split may contain \( (l^*_i f)^{-1}(pt) \), in which case of the two resulting components \( P, Q \) of \( L_{i+1} \) one (say, \( P \)) would contain \( (l^*_i f)^{-1}(pt) \). Then for the constructed null-homotopy to be coherent, the other component \( Q \) has to be glued up by a disk in \( (h^*_{[i+1,N]} f)^{-1}(S^1 \times I) \), where \( h_{[i,j]} : S^1 \times I \to N \).
denotes the interval from $l_i$ to $l_j$ in the constructed null-homotopy $h_t$. Indeed, by construction, $Q$ will be glued up by a disk already in $(h_{i+1,\rho(i)}^*)^{-1}(S^1 \times I)$.

**Example 3.8** (a two-dimensional proposition). Let us show that the proposition in Example 3.7 remains valid in dimension two under the additional hypothesis that there are no linked pairs $(i, j)$.

Indeed, let us examine the argument of 3.7. It contains only two essential applications of the condition $n > 2$: to conclude that $D$ is disjoint from all $D_j$'s that lie in the interior of $D_1$ and to coherently null-homotop $l_N$. Now if $n = 2$ and $D$ meets some $D_j$, then it follows from our assumption of $(1, j)$ being unlinked that $D$ contains $f(B_j)$. (For if it doesn’t contain, then taking into account our assumption that $D_1$ is innermost among all $D_i$’s meeting $l(S^1)$, the only possibility is that $f(B_j)$ contains $D_1$. However $D_j \subset D_1 \subset f(B_1)$, so $(1, j)$ is linked.) Thus $D$ contains each $D_j$ together with its $f(B_j)$, which implies that $f^{-1}(D)$ is homeomorphic to a disjoint union of disks, each bounded by a component of $f^{-1}(\partial D)$. Thus the homotopy of $l$ along $D$ will be coherent. Similarly the null-homotopy of $l_N$ will be coherent.

![Figure 1](image-url)  

**Fig. 1.** A map $f: S^n \to S^n$ of geometric degree 1 with nontrivial $\pi(f)$.

**Example 3.9** (a higher-dimensional example). In fact, $\pi(f; b)$ need not be trivial for $n > 2$ (under the hypothesis of §3.2) as shown by the map depicted in Figure 1. The six thick colored (or grayscale, depending on the reader’s medium) ellipses depict the folds; the thick black curve is the loop $l$ in question; and the thin curves illustrate the pullback $l^* f$. The arrows mark those of the two components of $(l^* f)^{-1}(S^1)$ that passes through $(l^* f)^{-1}(pt)$.  


The curve \( l(S^1) \) meets each of the three \( n \)-balls \( D_i \) bounded by the images of the three inner spheres of folds \( S_i \), \( i = 1, 2, 3 \), in two arcs \( J_i, J_i' \). A key feature of this picture is that the marked component has a fold over one endpoint of each of the six arcs, and the other (unmarked) component has folds over their opposite endpoints. Thus if any of the six arcs is eliminated as in Example 3.7, this would result in the marked component being joined to the unmarked one, whence the eliminating homotopy would fail to be coherent (cf. Example 3.5).

Moreover, no preliminary tampering with the six arcs by a coherent homotopy of \( l \) is going to help. Indeed, if \( H : D^2 \to S^n \) is a coherent null-homotopy of \( l \), then the pullback \( H^* f \) of \( f \) has a fold curve over each \( H^{-1}(S_i) \). Each \( H^{-1}(D_i) \) is a codimension zero submanifold in \( D^2 \), whose boundary contains the two arcs \( l^{-1}(J_i) \) and \( l^{-1}(J_i') \). If \( j \neq i \), then these arcs alternate with \( l^{-1}(J_j) \) and \( l^{-1}(J_j') \) with respect to the cyclic order on \( S^1 \), whereas \( H^{-1}(D_i) \) and \( H^{-1}(D_j) \) are disjoint. Hence either \( l^{-1}(J_i) \) and \( l^{-1}(J_i') \) are contained in different components of \( H^{-1}(D_i) \), or \( l^{-1}(J_j) \) and \( l^{-1}(J_j') \) are contained in different components of \( H^{-1}(D_j) \) (or both assertions hold). By symmetry, we may assume the former. Since the component of \( H^{-1}(D_i) \) containing \( l^{-1}(J_i) \) does not contain \( l^{-1}(J_i') \), its boundary component containing the arc \( l^{-1}(J_j) \) otherwise contains only points of \( H^{-1}(S_i) \), which therefore must constitute an arc. Thus the two endpoints of the arc \( l^{-1}(J_i) \) belong to the same component of \( H^{-1}(S_i) \). Hence the fold curve of \( H^* f \) over this component constitutes a path in \( (H^* f)^{-1}(D^2) \) starting on the marked component of \( (l^* f)^{-1}(S^1) \) and ending on the other (unmarked) component. Therefore the latter two components are joined into one in the null-homotopy, which is therefore non-coherent. Thus \( l \) is not coherently null-homotopic.

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