Derivative expansions of real-time thermal effective actions

Maria Asprouli, Victor Galan-Gonzalez

Imperial College
Theoretical Physics Group
Blackett Laboratory
London SW7 2BZ

September 5, 2018

Abstract

In this work we use a generalised real-time path formalism with properly regularised propagators based on Le Bellac and Mabilat \footnote{Le Bellac, É., Mabilat, S.: Real-time path integral: a generalised formalism for field theories. Phys. Rev. D 63, 105006 (2001).} and calculate the effective potential and the higher order derivative terms of the effective action in the case of real scalar fields at finite temperature. We consider time-dependent fields in thermal equilibrium and concentrate on the quadratic part of the expanded effective action which has been associated with problems of non-analyticity at the zero limits of the four external momenta at finite temperature. We derive the effective potential and we explicitly show its independence of the initial time of the system when we include both paths of our time contour. We also derive the second derivative in the field term and recover the Real Time (RTF) and the Imaginary Time Formalism (ITF) and show that the divergences associated with the former are cancelled as long as we set the regulators zero in the end. Using an alternative method we write the field in its Taylor series form and we first derive RTF and ITF in the appropriate limits, check the analyticity properties in each case and do the actual time derivative expansion of the field up to second order in the end. We agree with our previous
results and discuss an interesting term which arises in this expansion. Finally we discuss the initial time-dependence of the quadratic part of the effective action before the expansion of the field as well as of the individual terms after the expansion.

1 Introduction

The interest in the amalgamation of field theory and statistical mechanics arose from the realisation that many problems encountered experimentally and theoretically in particle physics have many-body aspects. For this reason, zero-temperature quantum field theory was reformulated by generalising the usual time-ordered products of operators to the ordered products along a path in a complex time-plane [2]. The choice of the path gives rise to different formalisms but all theories should give the same physical answers. Although the various path-ordered finite temperature field theory formalisms such as Real Time Formalism (RTF) including the closed-time approach [3] and Imaginary Time Formalism (ITF) [4] should give the same physics, there has been serious discussion about their exact equivalence.

In this paper we will tackle a problem in RTF which consists in the occurrence of pathologies associated with singularities, arising in diagrams with self-energy insertions or in some effective potential calculations. This problem appears when products of delta functions with the same argument are involved and creates non-analyticities in the effective action at finite temperature, thus making it ill-defined.

The main interest for developing an effective formalism which describes the finite temperature field theory comes from the need to tackle important problems in phase transitions, which have played a crucial role in the early evolution of the universe. The significance and observable quantities of a specific transition depend on its detailed nature and its order. A reason for a well-defined effective action comes from the fact that it represents the quantum corrections which in general might be of extreme importance in defining or changing the order of a transition. For example, analytic analysis [5] suggests that the electroweak phase transition is first order because of quantum corrections from gauge bosons while non-perturbative lattice simulations of high temperature electroweak theory suggests that this is only true if the
Higgs is lighter than 70Gev [6]. The inclusion of higher derivative terms in the derivative expansion of the effective action in a first order transition is of great importance in cases such as the derivation of the rate of the sphaleron fluctuations. These configurations have been used to explain the observed baryon asymmetry of the universe [7].

Moreover, although the effective potential can give the approximate critical temperature of a given transition, it is not adequate for answering questions concerning the departure of the field from equilibrium occurring during dynamical cooling near and below the critical temperature $T_c$. The effective potential describes static properties and therefore it is not an appropriate tool for studying the dynamical behaviour of a wide class of field theory models considered in inflationary scenarios.

The standard method for estimating the quantum corrections is to first integrate out quantum fluctuations about a constant background. This gives an effective potential for $\phi$ which is then used in the equations of motion determining $\bar{\phi}(\bar{x}, \tau)$ [8]. Integrating out fluctuations about a general inhomogeneous configuration gives the full effective action which includes the higher derivative terms. In the language of quantum field theory at finite temperature the effective potential $\Gamma$ is given by $\Gamma = \beta F$, where $F$ is the minimum of the free energy at which the system lies in the case of local thermal equilibrium. If the ensemble averages of the matter fields are homogeneous and static, then the free energy is given by the finite temperature effective potential [4].

In analogy to quantum mechanics, the decay rate of an unstable configuration $\phi_f$ with energy $\mathcal{E}_f$ is given by [10]

$$\Gamma = -\frac{2}{\hbar} \text{Im}[\mathcal{E}_f] = -2\text{Im} \left[ \lim_{T \to \infty} \frac{1}{T} \ln \langle \phi_f | e^{-HT/\hbar} | \phi_f \rangle \right]$$

where $H$ is the Hamiltonian and the matrix element can be described as a functional integral

$$\langle \phi_f | e^{-HT/\hbar} | \phi_f \rangle = N \int \mathcal{D}\phi e^{-S[\phi]/\hbar}$$

in Euclidean time. Here $\phi$ is subject to the condition $\phi(T/2) = \phi(-T/2) = \phi_f$ and $S$ denotes the Euclidean action. Evaluating the functional integral
to one loop, we expand $S(\phi)$ about a solution of the equations of motion, $\tilde{\phi}$, and keeping only terms quadratic in the fluctuations $\delta\phi = \phi - \tilde{\phi}$, we obtain

$$N \int \mathcal{D}\phi e^{-S[\phi]/\hbar} \approx Ne^{-S(\tilde{\phi}/\hbar)} [\det(-\partial_\mu \partial_\mu + V''(\tilde{\phi}))]^{-\frac{1}{2}} \equiv \exp[-S_{\text{eff}}(\tilde{\phi}/\hbar)]$$

where $S_{\text{eff}}$ is the effective action. If we expand $S_{\text{eff}}$ about a constant $\phi$, i.e. in powers of momentum about a point with zero external momenta, in position space and zero temperature this reads

$$S_{\text{eff}}(\phi) = \int d^4x \left[ -V_{\text{eff}}(\phi) + \frac{1}{2} Z(\phi) \partial_\mu \phi \partial^\mu \phi + \mathcal{O}((\partial_\mu \phi)^4) \right]$$

where we have made use of $T = 0$ Lorentz properties. For constant $\phi$ only the effective potential term survives.

Although such an expansion up to the second derivative has been performed at zero temperature for scalar and Dirac field theories \cite{11}, there are difficulties arising in the equivalent expansion at finite temperature. Das and Hott \cite{12} find a non-analyticity in the two-point functions involving the temperature dependent term of the quadratic part of the effective action. In this spirit, if the derivative expansion breaks down at finite temperature, the definition of the effective potential might not be unique. This non-analyticity, in the case of the vacuum polarisation for a scalar field coupled to a classical external field, manifests itself in the difference between the order of the limits ($p_0 \to 0$, $\mathbf{p} \to 0$) and ($\mathbf{p} \to 0$, $p_0 \to 0$) of the external momenta, the first relating to the electric screening mass of the photon and the second to the plasma frequency of the particular field theory under consideration \cite{13}. This non-commutativity appears in hot QCD \cite{14}, self interacting scalars \cite{15, 16} and gauge theories with chiral fermions \cite{17}. In ITF, setting $p_\mu = 0$ first and performing the mode sum gives the same result as taking the limit $p_0 \to 0$ first and the limit $\mathbf{p} \to 0$ afterwards. In RTF extra Feynman rules have been imposed to explain this difference in the two limits \cite{18}. The problem is neither due to subtleties in the use of Feynman parametrisation at finite temperature \cite{19}, nor to the infinite number of possible extensions of $p_0$ to the imaginary axis and its analytic continuation to the complex plane \cite{16}. The
lack of analyticity and the infrared divergences occurring in the definition of the effective action at finite temperature show the need for an effective field theory formalism from which RTF and ITF rules can be derived easily. In the next chapter we will describe a method dealing with these problems. We will calculate the two-propagator contribution (bubble diagram) to the second derivative term in the effective action using Le Bellac and Mabilat’s generalised real-time path formulation with properly regularised propagators.

2 The method

In Le Bellac and Mabilat’s approach they derive Feynman rules that take explicitly into account the vertical part of the contour and recover the RTF in the case of diagrams with at least one finite external line and the ITF in the case of vacuum fluctuations. They keep the regulators of the propagators when they find problematic products of delta functions with the same argument and show that they can use RTF when no such problems arise. They claim that the contribution of the vertical part of the contour lies in the cancellation of the \( t_i \) dependent terms of the horizontal part since the whole result should be \( t_i \) and \( t_f \) independent due to the KMS condition of the propagators. We will now describe in detail this method, which we will use throughout this paper.

2.1 Outline of the method

The specific approach uses the Mills mixed representation of the propagators for a free scalar field, with \( t \) defined in the generalised time path \( C \), starting at \( t_i \) and ending at \( t_i - i\beta \) of Fig.1. The propagator is written as

\[
D_c(t, \mathbf{k}) = \int \frac{dk_0}{2\pi} e^{-ik_0t} [\theta_c(t) + n(k_0)] \rho(k_0, \mathbf{k})
\]  

(1)

where \( \theta_c(t) \) is a contour \( \theta \) function, \( n(k_0) \) is the Bose-Einstein distribution function given by

\[
n(k_0) = \frac{1}{e^{\beta k_0} - 1}
\]  

(2)
and $\rho(k_0, k)$ is the (temperature independent) two-point spectral function given by
\[
\rho(k_0, k) = 2\pi \varepsilon(k_0) \delta(k_0^2 - \omega_k^2)
\] (3)
where $\varepsilon(k_0)$ is the sign function and
\[
\omega_k^2 = k^2 + m^2
\] (4)

However, the propagators need to be regularised because eventually we want to take the Fourier transform in time by taking the limits $t_i \to -\infty, t_f \to +\infty$ (to get energy conservation) and this is ill-defined since the integrands are linear combinations of complex exponentials. For this reason we write the $\delta$ distribution in its regularised form
\[
\delta(k_0 \mp \omega_k) = \frac{1}{2i\pi} \left[ \frac{1}{k_0 + \omega_k - i\varepsilon} - \frac{1}{k_0 + \omega_k + i\varepsilon} \right]
\] (5)
and thus $\rho(k_0, k)$ in Eq. (3) can be written as
\[
\rho(k_0, k) = \frac{i}{2\omega_k} \sum_{r,s=\pm 1} \frac{rs}{k_0 - s\omega_k + i\varepsilon r}
\] (6)

The regularised propagator can be written as
\[
D_R^c(t, k) = D_R^> (t, k) \theta_c(t) + D_R^< (t, k) \theta_c(-t)
\] (7)
and obeys the KMS condition

$$D_R^>(t - i\beta, \mathbf{k}) = D_R^<(t, \mathbf{k})$$ (8)

Momentum integration in the complex \(k_0\)-plane will give for \(D_R^>(t, \mathbf{k})\), with \(t\) defined in the region \(-\beta \leq \text{Im}t \leq 0\)

$$D_R^>(t) = \frac{1}{2\omega_k} \sum_{\varepsilon = \pm} [\theta(\varepsilon) + n(\omega_k - i\varepsilon\xi\hat{\eta})] e^{-i\varepsilon\omega_k t - \xi\hat{\eta} t} - \frac{1}{2\omega_k\beta} \sum_{\eta \geq 1} X_\eta e^{-s\omega_\eta t}$$ (9)

where \(s = \text{sign}[Re(t)]\), \(\xi\) is the regulator and \(X_\eta\) is the sum

$$X_\eta = \sum_{r,s=\pm 1} \frac{rs}{i\omega_\eta - s\omega_k + i\xi r} = \frac{8\xi\omega_\eta\omega_k}{[(i\omega_\eta - \omega_k)^2 + \xi^2][(i\omega_\eta + \omega_k)^2 + \xi^2]}$$ (10)

where \(\omega_\eta = \frac{2\pi\eta}{\beta} = 2\pi\eta T\) denotes the Matsubara frequencies. This term arises from the residues of the distribution function when we integrate \(k_0\) in the complex plane. We will discuss its contribution later.

### 2.2 The bubble term

The rest of the paper will be the calculation of the bubble diagram, which is nothing else but the product of two propagators. In order to justify its relevance, we will briefly mention where it comes from. We consider the two scalar field theory described by the Lagrangian

$$\mathcal{L}[\phi, \eta] = \frac{1}{2} \partial_\mu \eta \partial^\mu \eta - \frac{1}{2} m^2 \eta^2 - \frac{1}{2} g\phi\eta^2 + \mathcal{L}_0$$ (11)

where \(\mathcal{L}_0\) denotes the free Lagrangian for \(\phi\). If we integrate out the \(\eta\)-field fluctuations and use a one-loop approximation we find that the generating functional can be expressed as

$$\mathcal{Z} = \int_C \mathcal{D}\phi e^{iS_0(\phi) + iS_{\text{eff}}(\phi)}$$ (12)

where \(S_{\text{eff}}\) is given by

$$S_{\text{eff}}(\phi) = \frac{i}{2} \text{Tr} \ln [1 - g\Delta_c(x, x')\phi(x')]$$ (13)
and $\Delta_c(x, x')$ is the propagator for the $\eta$ field. Expanding the logarithm we get

$$S'_{\text{eff}}[\phi] = \sum_{p=1}^{\infty} S^{(p)}_{\text{eff}}$$

(14)

where $p$ denotes the number of the propagators. In this expression we will concentrate on the first non-local term which is the quadratic part $S^{(2)}_{\text{eff}}$ of the expansion and is given by

$$S^{(2)}_{\text{eff}} = -ig^2 Tr(\Delta_c(x, x')\phi(x')\Delta_c(x', x)\phi(x))$$

(15)

One can permute the order of the elements inside the trace using the identity

$$\phi(x)\Delta_c(k) = \Delta_c(k + p)\phi(x)$$

(16)

which is equivalent to the Taylor expansion of Fraser [21] for moving momentum operators to the left of functions depending on $x$, when we identify $p_\mu = -i\partial_\mu$. Thus the quadratic part of $S'_{\text{eff}}[\phi]$ can be rewritten as

$$S^{(2)}_{\text{eff}} = -\frac{1}{4} \int d^4x \int d^4x' \phi(x) iB(p, \beta)\phi(x')$$

(17)

with $iB(p, \beta)$ being the bubble term given in terms of the propagators as

$$iB(p, \beta) = g^2 \int \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \Delta_c(k, m)\Delta_c(k + p, m)$$

(18)

Separating the time dependence which interests us at finite temperature, $S^{(2)}_{\text{eff}}$ is written

$$S^{(2)}_{\text{eff}} = -\frac{ig^2}{4} \int_{t_i}^{t_i - i\beta} dt \int \frac{d^3p}{(2\pi)^3} \phi(p, t)$$

$$\times \int \frac{d^3k}{(2\pi)^3} \int_{t_i}^{t_i - i\beta} dt' \Delta_c(k; t, t')\Delta_c(k + p; t', t)\phi(p, t')$$

(19)

and the fields $\phi$ are periodic over the time path $[t_i, t_i - i\beta]$. This derivative expansion of the bubble term is well established at zero temperature. At finite temperature this is not so, since we will be expanding our theory around
an ill-defined point. This can be seen for example if we look at the \( \Delta_{11} \) component of the propagator of our theory which is given by

\[
\Delta(k, m) = \frac{1}{k^2 - m^2 + i \varepsilon} - 2i\pi n(k_0)\delta(k^2 - m^2) \tag{20}
\]

Substituting the propagator in Eq. (17), the temperature-dependent real part of the quadratic thermal effective action is given by the following expression which is nonanalytic at the zero four-momentum limit [12]

\[
\text{Re}(S_{\text{eff}}^{(2)}[\phi]) = -\frac{g^2}{32\pi^2} \int d^4x \left\{ \int_0^\infty dk f(k)\phi(x) \right\} \phi(x) \tag{21}
\]

with

\[
f(k) = \frac{kn(\omega)\text{Re}(\ln R)}{\omega(-\mathbf{\nabla}^2)^{1/2}} \tag{22}
\]

where \( \omega^2 = k^2 + m^2 \) and

\[
R = \prod_{r,s=+1,-1} (\partial_0^2 - \mathbf{\nabla}^2 + 2is\omega\partial_0 + 2irk(-\mathbf{\nabla}^2)^{1/2})^r \tag{23}
\]

This result suggests that the derivative expansion breaks down at finite temperature, due to the problematic product of the two delta functions contained in the bubble. If the derivative expansion of the effective action is not rigorously possible, then the definition of an effective potential, the lowest order term in such an expansion, is not unique. This would have consequences in any kind of study concerning symmetry breaking and restoration, unless there is a formal way to overcome these pathologies and have a well defined derivative expansion.

### 2.3 Effective potential term

We use Le Bellac and Mabilat’s formulation [1] to prove explicitly the \( t_i \)-independence of the effective potential in the case of two propagators and one external time \( t_0 \) (first term of the bubble diagram) when adding both contributions from the horizontal and vertical path. Since we are interested
in time dependent fields, we will concentrate on the time integral of the bubble term in Eq. (19). This is given by

$$G_R = \int_C D^c_R(t_0 - t_1)D^c_R(t_0 - t_1)\,dt_1$$

where \(C\) will be in the region \([t_i, t_0]\) and \([t_0, t_i]\) for the horizontal path and \([t_i, t_i - i\beta]\) for the vertical one as shown in Fig. 1. Using the representation for the propagators of Eq. (9), the integrations over the two paths give \(G_R^H\) for the horizontal and \(G_R^V\) for the vertical one [1]

$$G_R^H = -\frac{A_H}{(2\omega)^2} \left( \frac{1 - e^{-\beta \omega (\epsilon_1 + \epsilon_2) e^{2i\xi\beta}}}{i\omega (\epsilon_1 + \epsilon_2) + 2\xi} \right) \left[ 1 - e^{-i\omega (\epsilon_1 + \epsilon_2)(t_0 - t_i)} e^{2\xi(t_i - t_0)} \right]$$

$$G_R^V = -\frac{A_V}{(2\omega)^2} \left( \frac{e^{\beta \omega (\epsilon_1 + \epsilon_2)} e^{-2i\xi\beta} - 1}{i\omega (\epsilon_1 + \epsilon_2) + 2\xi} \right) \left[ e^{-i\omega (\epsilon_1 + \epsilon_2)(t_0 - t_i)} e^{2\xi(t_i - t_0)} \right]$$

where

$$A_{H,V} = \sum_{\epsilon_1, \epsilon_2} \left[ \theta(\pm\epsilon_1) + n(\omega - i\epsilon_1) \right] \left[ \theta(\pm\epsilon_2) + n(\omega - i\epsilon_2) \right]$$

are \(t_i\)-independent coefficients. The KMS relation which reads as

$$\left[ \theta(\epsilon) + n(\omega - i\epsilon) \right] = e^{\epsilon\beta\omega} e^{-i\epsilon\beta} \left[ \theta(-\epsilon) + n(\omega - i\epsilon) \right]$$

gives for the coefficients \(A_H\) and \(A_V\)

$$A_H = e^{(\epsilon_1 + \epsilon_2)\beta\omega} e^{-2i\xi\beta} A_V$$

(29)

Adding the \(t_i\)-dependent terms of both paths Eq. (25) and Eq. (26) and using the KMS condition of Eq. (28), we find that they cancel. Therefore, it is essential that we add the vertical contribution to ensure the \(t_i\)-independence of our result. The remaining part of the sum is given by

$$G^R = G_R^H + G_R^V = \frac{1}{(2\omega)^2} \frac{(A_H - A_V)}{i\omega (\epsilon_1 + \epsilon_2) + 2\xi}$$

and using the definitions of \(A_H\) and \(A_V\) from Eq. (27) and identities of the \(\theta\) functions, the sum is written as

$$G_R^H + G_R^V = \frac{1}{(2\omega)^2} \left\{ \frac{(1 + 2n(\omega))i\omega + 2i\xi^2 n'(\omega)}{-\omega^2 - \xi^2} \right\} + \frac{i\beta n(\omega)(1 + n(\omega))}{2\omega^2}$$

(31)
In Eq. (31) the first term is the result of Eq. (30) for $\varepsilon_1 = \varepsilon_2$ and the second term is the result for $\varepsilon_1 + \varepsilon_2 = 0$. Taking the limit of the regulator $\xi$ to zero at the end, we have

$$G^R_H + G^R_V = \frac{i}{4\omega^2}(1 + 2n(\omega)) + \frac{i\beta n(\omega)(1 + n(\omega))}{2\omega^2}$$

(32)

The previous result agrees completely with the ITF result for the effective potential [22], which proves the consistency of our theory to this order. Now we examine the cases of taking different limits for the $t_i$ and the regulators and try to explain their physical meaning.

1. We take the limit $t_i \to -\infty$ (keeping the regulators finite) which should give us the real time formalism. Since the total sum is $t_i$-independent, the limit of $t_i \to -\infty$ is already given by Eq. (31) in which the regulator can be taken to zero since there is no need for it any more after the limit has been performed. We notice from Eq. (26) that keeping the regulators finite the vertical part vanishes in this limit, recovering thus the RTF and the total sum is being given by the $t_i$-independent contribution of the horizontal part. In the case of unregularised propagators, the vertical contribution contains a $t_i$-independent term of the form

$$\frac{\beta}{2\omega} \frac{1}{i} n(\omega)(1 + n(\omega)) 2\pi \delta(k^2 - m^2)$$

which in the regularised approach is hidden in the two horizontal parts of the contour, as seen in the last part of Eq. (31). We see that, to this order, the regularised formalism is dealing with the pathologies of the problematic delta functions, recovering RTF in the appropriate limit.

2. Now keeping $t_i$ finite, we take the zero limits of the regulators in different orders and find

$$G^R_H(\varepsilon_1 + \varepsilon_2 = 0, \xi \to 0) = G^R_H(\xi \to 0, \varepsilon_1 + \varepsilon_2 = 0) = 0$$

and the only contribution comes from the vertical part in the limit $\varepsilon_1 + \varepsilon_2 = 0, \xi \to 0$ which is the second term of Eq. (31)

$$\lim_{\xi \to 0, \varepsilon_1 + \varepsilon_2 = 0} G = \frac{i\beta n(\omega)(1 + n(\omega))}{2\omega^2}$$
This limit is one part of the full effective potential of Eq. (32) as expected since it only corresponds to the limit of equal and opposite energies \( \omega \) for the two propagators \((\varepsilon_1 + \varepsilon_2 = 0)\).

### 2.4 The second derivative term

Now the same formalism will be used for the derivation of the second derivative term of the effective action in our 1-loop case, where only time-dependent fields are considered. Because now the sum will also contain terms polynomial in \((t_i - t_0)\) as well as exponential ones, the equivalence with the RTF and ITF is less straightforward. We will show that the RTF limit can be extracted in this case without problems of divergences as long as we keep the regulators finite and take them to zero after the limit has been done. The second derivative term will look like

\[
\Gamma^{(2)}_2 = \int_C dt_1 D^c_R(t_0, t_1) (t_0 - t_1)^2 D^c_R(t_1, t_0)
\]

where we have omitted the \(\frac{1}{2} \partial^2_t \phi(t_0)\) factor of this term in the expansion. Using the definition of Eq. (1), \(\Gamma^{(2)}_2\) can be written as

\[
\Gamma^{(2)}_2 = \int_C dt_1 \int \frac{dk_0 dk_1}{(2\pi)^2} \left( \theta_c(t_0, t_1) + n(k_0) \right) \left( \theta_c(t_1, t_0) + n(k_1) \right)
\]

\[
\times (t_1 - t_0)^2 e^{-i(k_0 - k_1 - i\varepsilon)(t_0 - t_1)} \rho(k_0) \rho(k_1)
\]

where the regulator \(\varepsilon\) is used so that the limit of \(t_i \to -\infty\) can be taken without problems. In the end it will be set to zero. The other two regulators \(\varepsilon_1\) and \(\varepsilon_0\) in the delta functions of \(\rho(k_0) \rho(k_1)\) make sure that no problems appear in the equal energy (mass) case \(k_0 = k_1 = w\).

Now we first perform the \(dt_1\) integration and then “absorb” the \((t_1 - t_0)^2\) term by differentiating the result with respect to \(k_0\). If we name the time integrals over the two paths as \(I_C\), we then write

\[
\int_C dt_1 (t_1 - t_0)^2 \theta_c(t_1, t_0) e^{-i(k_0 - k_1 - i\varepsilon)(t_0 - t_1)} = i^2 \frac{\partial^2}{\partial k_0^2} I_C
\]

Substituting this formula into our general expression Eq. (34), the contributions from the different paths can now be written

\[
\Gamma^{(2)}_H = \int \frac{dk_0 dk_1}{(2\pi)^2} (n(k_0) - n(k_1)) \rho(k_0) \rho(k_1) i^2 \frac{\partial^2}{\partial k_0^2} I_H
\]
\[ \Gamma_V^{(2)} = \int \frac{dk_0 dk_1}{(2\pi)^2} [n(k_0)(n(k_1) + 1)] \rho(k_0) \rho(k_1) i^2 \frac{\partial^2}{\partial k_0^2} I_V \]  

(37)

In \( \Gamma_H^{(2)} \) the \( \theta^2 \) term vanishes due to the opposite sign of its time arguments. The \( n^2 \) term vanishes due to the cancellation between the two horizontal paths. In \( \Gamma_V^{(2)} \) one of the \( \theta \)-functions always vanishes due to the choice of \( t_0 \) on the horizontal path.

The time integrations \( I_H, I_V \) over the two paths give

\[ I_H = i \left( 1 - e^{-i(k_0-k_1-\imath \epsilon)(t_0-t_i)} \right) \frac{1}{k_0 - k_1 - i\epsilon} \]

\[ I_V = i e^{-i(k_0-k_1-\imath \epsilon)(t_0-t_i)} \times \left( 1 - e^{\beta (k_0-k_1-\imath \epsilon)} \right) \frac{1}{k_0 - k_1 - i\epsilon} \]

The analytical calculation of the different path contributions is quite complicated since it involves first and second order residues and therefore derivatives of the distribution functions. We performed the momentum integrations and then took the same limits of our variable \( \Delta t = t_i - t_0 \) and of the regulators as before to check the consistency of our method for the second derivative term.

1. We took the \( \Delta t \to -\infty \) limit keeping the regulators finite. In the total sum the \( \Delta t \)-dependence appears in terms like \( \Delta t^n e^{\epsilon \Delta t} \) and \( \Delta t^n \) (\( n = 0, 1, 2 \)). These terms could cause divergences in the \( \Delta t \to -\infty \) limit but they disappear once we include the vertical part in our calculation. Our result is independent of the order in which the regulators are taken to zero in the end and is given by a finite term coming from the horizontal part

\[ \lim_{t_i \to -\infty} \Gamma^{(2)} = \frac{i}{8} \frac{(1 + 2n(\omega))}{\omega^5} \]

This term looks like the first order term of the effective potential divided by \( \omega^2 \), as it can be seen from Eq. (32), which is sensible since it is essentially the first correction due to the second derivative.

2. Our second limit is \( t_i = t_0, \epsilon = 0 \) in order to try to recover the ITF result \( (t_i = t_0 = \text{finite and } \epsilon \text{ is not needed any more since } t_i \text{ is finite}) \).
This proved to be also independent of the order of the zero limits of the regulators. We obtained

$$\Gamma^{(2)}(t_i = t_0, \varepsilon = 0; \varepsilon_1 = 0; \varepsilon_0 = 0) =$$

$$-\frac{i}{8\omega^5} [(2n(\omega) + 1) - \beta \omega (2n(\omega)(n(\omega) + 1) + 1)$$

$$+ \beta^2 \omega^2 (2n(\omega) + 1) + \frac{4\beta^3 \omega^3 n(\omega)(n(\omega) + 1)}{3}],$$

which is consistent with the derivation of the second derivative term in the ITF formalism [22].

### 3 Alternative method

Another possible way of performing our calculation is to consider the full Taylor series of the field but do the actual expansion and study the individual terms in the end. We will generalise our method considering different energies ($\omega$ and $\Omega$) in the delta functions of Eq. (3) for each propagator of the bubble term. In this way we will check the analyticity limits of the full derivative term by taking the limits $\Omega \to \pm \omega$ ($\nabla \to 0$) and $\partial_t \to 0$ in different orders at the end of the calculation.

The expanded field can be written as

$$\phi(t_i) = \sum_{n=0}^{\infty} \frac{1}{n!} (t_1 - t_0)^n \frac{\partial^n}{\partial t^n} \phi(t) \bigg|_{t=t_0} = e^{(t_1-t_0)\partial_t} \phi(t) \bigg|_{t=t_0}$$

The full derivative term of the field inserted between the two propagators, which is the last time integral in Eq. (19), looks like

$$\Gamma^{(B)} = \int_C dt_1 D_R^e(t_0, t_1) e^{(t_1-t_0)\partial_t} \phi(t) \bigg|_{t=t_0} D_R^e(t_1, t_0)$$

This term acts as an energy-shift by $-i\partial_t$ in the exponentials of the propagators making the time-integrals over the paths $I_C$ of Eq. (33) look like

$$\int_C dt_1 \theta_\varepsilon(t_1, t_0) e^{-i(k_0-k_1-i(\varepsilon+\partial_t))(t_0-t_1)} = I_C'$$

14
Performing the time integration for the horizontal and vertical path as before, we get

\[ I'_H = i \frac{1 - e^{-i(k_0 - k_1 - i(\varepsilon + \partial_t))(t_0 - t_i)}}{k_0 - k_1 - i(\varepsilon + \partial_t)} \]

and

\[ I'_V = i e^{-i(k_0 - k_1 - i(\varepsilon + \partial_t))(t_0 - t_i)} \times \frac{1 - e^\beta(k_0 - k_1 - i(\varepsilon + \partial_t))}{k_0 - k_1 - i(\varepsilon + \partial_t)} \]

The energy integration gives us the full bubble term as a sum over the two paths written in terms of \( \Delta t = t_i - t_0 \)

\[ \Gamma = \Gamma_H + \Gamma_V \]

with

\[ \Gamma_H = \sum_{\pm \omega, \Omega} \frac{i n(\omega - i \varepsilon_0) n(\Omega - i \varepsilon_1)}{4 \omega \Omega} \times \left[ \frac{e^{\beta(\omega + \Omega - i(\varepsilon_0 + \varepsilon_1))} - 1}{A} \right] (40) \]

and

\[ \Gamma_V = \sum_{\pm \omega, \Omega} \frac{i n(w - i \varepsilon_0) n(\Omega - i \varepsilon_1)}{4 \omega \Omega} \times \left[ \frac{e^{-iA \Delta t} e^{-i \beta(\varepsilon_1 + \partial_t)} (e^{\beta A} - 1)}{A} \right] (41) \]

where

\[ A = \omega + \Omega - i(\varepsilon_1 + \varepsilon_0 - \varepsilon - \partial_t) \]

Now we can check the analyticity of our result keeping \( \Delta t \) finite. We expand the distribution functions and take the limits of our regulators to zero (we can do that since we keep \( \Delta t \) finite). If we take the limits \( \Omega \rightarrow \pm \omega \) and \( \partial_t \rightarrow 0 \), we get finite and independent of the order of the limits results. The full derivative expansion of the bubble term, therefore, is analytical in this limit and is

\[ \Gamma_H(\Omega \rightarrow \pm \omega, \partial_t \rightarrow 0) = i \frac{(2n(\omega) + 1)(1 - \cos(2\omega \Delta t))}{2\omega^3} \] (42)

for the horizontal case and

\[ \Gamma_V(\Omega \rightarrow \pm \omega, \partial_t \rightarrow 0) = i \frac{(2n(\omega) + 1)\cos(2\omega \Delta t)}{2\omega^3} + i \frac{\beta n(\omega)(n(\omega) + 1)}{\omega^2} \] (43)

for the vertical one.
1. Now we consider the limit $t_i = t_0$ in Eq. (42) and Eq. (43). This gives

$$\Gamma^{(B)}_H = 0 \quad (44)$$

and

$$\Gamma^{(B)}_V = \frac{i(2n(\omega) + 1)}{2\omega^3} + i\beta n(\omega)(1 + n(\omega)) \frac{1}{w^2} \quad (45)$$

This is exactly the result for the effective potential using the ITF formalism, as expected since it is the zeroth time and space-derivative term, when $t_i = t_0$. It also agrees with our previous result in Eq. (32) of the effective potential after we set the regulators to zero. (In Eq. (32) the result differs by a factor of $1/2$ due to the fact that we have initially considered same energies $\omega$ for the propagators and this corresponds to half of the result of the effective potential of Eq. (45) when different energies are assumed).

2. Now we take the limit $\Delta t \to -\infty$ and we will check the analyticity again. In this case only the first $\Delta t$-independent part of $\Gamma^{(B)}_H$ survives and taking the regulators to zero after the limit has been performed, we get

$$\Gamma^{(B)}(\Delta t \to -\infty) = \sum_{\pm\omega, \Omega} \frac{i(n(\omega) - n(-\Omega))}{4\omega\Omega (\omega + \Omega + i\partial_t)} \quad (46)$$

The analyticity check for Eq. (46) gives finite but different results for different orders of performing the limits ($\Omega \to \pm\omega, \partial_t \to 0$). We found that performing the time-derivative ($\partial_t \to 0$) limit first and the spatial-derivative ($\Omega \to \pm\omega$) afterwards, we had the usual effective potential term of Eq. (45)

$$\lim_{\partial_t \to 0, \Omega \to \pm\omega} \Gamma^{(B)} = \frac{i(2n(\omega) + 1)}{2\omega^3} + i\beta n(\omega)(1 + n(\omega))$$

but reversing the order of the limits gave us only the first term of our previous result

$$\lim_{\Omega \to \pm\omega, \partial_t \to 0} \Gamma^{(B)} = \frac{i(2n(\omega) + 1)}{2\omega^3}$$

We see that although we don’t have divergence problems in taking the limits in both orders, approaching the zero from the space-derivative
first seems to produce only part of the full result in agreement with the result of Evans using ITF [2]. Now we take only the spatial derivative to zero in the $\Delta t \to -\infty$ case of Eq. (46) which gives

$$\lim_{\Omega \to \pm \omega} \Gamma^{(B)} = 2i \left( \frac{2n(\omega) + 1}{\omega(4\omega^2 + \partial_t^2)} \right)$$

If we now expand our result in powers of the time-derivative $\partial_t$, we get the zeroth order term of our analyticity check and a second order term of the form

$$\frac{i(2n(\omega) + 1)}{8\omega^5}(\partial_t^2)$$

This is exactly the second order time-derivative term derived in this limit using our previous method in section 2.4. We have to note that in the case of $\Delta t \to -\infty$, there is no term linear in $\partial_t$.

Now we perform the same expansion in powers of the time-derivative up to the second order but for a general finite $\Delta t$, for both horizontal and vertical paths in Eq. (40) and Eq. (41) and take the limits ($\Omega \to \pm \omega$) to get the effective potential and the higher derivative terms. We identify the terms as follows

1. $\partial_t^0$ term

$$\Gamma_H^0 = -i \left( \frac{2n(\omega) + 1}{4\omega^3} \right) \left( e^{2i\omega\Delta t} + e^{-2i\omega\Delta t} - 2 \right)$$

$$\Gamma_V^0 = i \left( \frac{2n(\omega) + 1}{4\omega^3} \right) \left( e^{2i\omega\Delta t} + e^{-2i\omega\Delta t} \right) + i \frac{\beta n(\omega)(n(\omega) + 1)}{\omega^2}$$

2. $\partial_t^1$ term

$$\Gamma_H^1 = \left( \frac{2n(\omega) + 1}{8\omega^4} \right) \left[ (e^{2i\omega\Delta t} - e^{-2i\omega\Delta t}) - 2i\omega(\Delta t)(e^{2i\omega\Delta t} + e^{-2i\omega\Delta t}) \right]$$
\[ \Gamma_1^V = -\frac{1}{8\omega^4}[(2n(\omega) + 1)(e^{2i\omega \Delta t} - e^{-2i\omega \Delta t})
- 2\beta \omega((n(\omega) + 1)^2e^{2i\omega \Delta t} - n^2(\omega)e^{-2i\omega \Delta t}) - 4\beta^2 \omega^2 n(\omega)(n(\omega) + 1)]
+ \frac{i}{4\omega^3}[(2n(\omega) + 1)(e^{2i\omega \Delta t} + e^{-2i\omega \Delta t})
+ 4\beta \omega n(\omega)(n(\omega) + 1)](\Delta t) \] (50)

We notice the existence of a non-zero \( \partial_t \)-dependent term unlike the zero temperature case where such a term vanishes. This could be related to the loss of Lorentz invariance in the finite temperature case and could be interpreted as an energy shift. The existence of such a linear term might be of great physical importance in the study of time-dependent systems. Such a term did not exist in the expansion for the \( \Delta t \) infinite case, where only zero and second order terms in the time-derivative survived. This makes sense since in the infinite time limit any interaction with the heat bath which gives rise to such linear terms will have been damped. Mathematically this term could arise due to the shape of the time contour, which in the finite \( \Delta t \) case is non-symmetric. However this is not the case for the zero-temperature situation or the non zero temperature one in the infinite \( \Delta t \) limit where the symmetry of the contour will make any time integration of odd terms in the derivative expansion to vanish. In the \( \Delta t = 0 \) case this term is equal to

\[ \Gamma^1 = \frac{-i\beta \Gamma^0}{2} \]

where \( \Gamma^0 \) is the effective potential term given by Eq. (47) and Eq. (48) in the \( \Delta t = 0 \) limit.

3. \( \partial_t^2 \) term

\[ \Gamma^2_H = \frac{i(2n(\omega) + 1)}{16\omega^5}[(e^{2i\omega \Delta t} + e^{-2i\omega \Delta t} - 2) - 2i\omega(\Delta t)(e^{2i\omega \Delta t} - e^{-2i\omega \Delta t})
- 2\omega^2(\Delta t)^2(e^{2i\omega \Delta t} + e^{-2i\omega \Delta t})] \] (51)

\[ \Gamma^2_1 = \Gamma^2_{11} + \Gamma^2_{12} \] (52)
with

$$\Gamma_{V_1}^2 = -\frac{i}{16\omega^5}(2n(\omega) + 1)(e^{2i\omega\Delta t} + e^{-2i\omega\Delta t}) - 2\beta\omega((n(\omega) + 1)^2 e^{2i\omega\Delta t}$$

$$+ n^2(\omega)e^{-2i\omega\Delta t})$$

$$+ 2\beta^2\omega^2((n(\omega) + 1)^2 e^{2i\omega\Delta t} - n^2(\omega)e^{-2i\omega\Delta t})$$

$$+ \frac{8}{3}\beta^3\omega^3 n(\omega)(n(\omega) + 1)] \quad (53)$$

and

$$\Gamma_{V_2}^2 = -\frac{1}{8\omega^4}[(2n(\omega) + 1)(e^{2i\omega\Delta t} - e^{-2i\omega\Delta t}) - 2\beta\omega((n(\omega) + 1)^2 e^{2i\omega\Delta t}$$

$$- n^2(\omega)e^{-2i\omega\Delta t}) - 4\beta^2\omega^2 n(\omega)(n(\omega) + 1)](\Delta t)$$

$$+ i\frac{\Delta t}{8\omega^4}[(2n(\omega) + 1)(e^{2i\omega\Delta t} + e^{-2i\omega\Delta t})$$

$$+ 4\beta\omega n(\omega)(n(\omega) + 1)](\Delta t)^2 \quad (54)$$

If we take the $t_i = t_0$ limit of our second derivative term, we recover our previous derivation of the same term in section 2.4.

In our calculation we have omitted the contribution of the $X_\eta$ term of Eq. (10). This term which arises from the residue of the distribution function vanishes since it is proportional to the regulator. In the finite $\Delta t$ case the regulators are set to zero before any limit is taken while in the infinite $\Delta t$ case they are set to zero once the infinity limit has been performed. In both cases this term does not contribute.

4 The initial time dependence

In this section we will treat the initial time-dependence of our problem in a rather more formal way.

Based on Le Bellac and Mabilat’s proof of the $t_i$-independence of a regularised Green function [1], we will prove the same for our effective potential term. Our $t_i$-dependent integrals in this case are

$$\int_{t_i}^{t_0} dt_1 G_R(t_1, t_0)G_R(t_1, t_0) + \int_{t_i}^{t_{-i\beta}} dt_1 G_R(t_1, t_0)G_R(t_1, t_0) \quad (55)$$
Differentiating the first term with respect to $t_i$ we get

$$-G^<_R(t_i, t_0)G^<_R(t_i, t_0)$$

Repeating for the second term we now get

$$G^>_R(t_i - i\beta, t_0)G^>_R(t_i - i\beta, t_0)$$

Using the KMS condition for thermal equilibrium

$$G^>_R(t - i\beta) = G^<_R(t)$$

we see that these terms cancel. If we repeat the same method for the higher derivative terms of the field of the bubble case, we get a non-zero result which shows explicitly the $t_i$-dependence of these terms. The same analysis for the second derivative term gives

$$G^<_R(t_i, t_0)\left[-\beta^2 - 2i\beta(t_i - t_0)\right]G^<_R(t_i, t_0)$$

If we generalise in the case of the m-th derivative term, the $t_i$ dependence of the derivative with respect to $t_i$ will have the form

$$G^<_R(t_i, t_0)\left[\sum_{k=1}^{m} \prod_{j=0}^{k-1} \frac{(m-j)}{k!} (-i\beta)^k(t_i - t_0)^{m-k}\right]G^<_R(t_i, t_0)$$

We see that the individual terms of the expanded field are clearly $t_i$-dependent even in the case of $t_i = t_0$, where the highest order $\beta$-term survives in the previous sum. This is somehow expected since a truncated expansion of the field makes it no longer periodic. On the other hand if we have a periodic field $\phi(t)$ in equilibrium, the derivative with respect to $t_i$ discussed earlier will give

$$-G^<_R(t_i, t_0)\phi(t_i)G^<_R(t_i, t_0) + G^>_R(t_i - i\beta, t_0)\phi(t_i - i\beta)G^>_R(t_i - i\beta, t_0)$$

which is zero for periodic fields and regularised propagators obeying the KMS condition.
5 Conclusions and possible applications

We found that using this closed time path formalism we can avoid the pathologies in the RTF and derive the ITF limit as well, both in the case of the effective potential and in the second derivative correction of the bubble diagram. The fact that we can cancel the divergence in our effective action using our prescription, hence providing us with a formalism which allows us to compute quantum corrections to the effective potential is the key point of our paper. However we found that the inclusion of the vertical path and the careful treatment of the regulators are essential for the cancellations to happen.

We showed a general way to compute higher derivative terms in the bubble and derived the complete bubble term. We checked its analyticity for finite and infinite time differences $\Delta t$ and found different limits in the second case. The non zero, linearly dependent on the time derivative of the field term found in the finite $\Delta t$ case, may be related to the loss of translational invariance at finite temperature. The physical meaning of such a term and in particular its sign and whether it is complex or real may be important in the study of time-dependent systems. Gribosky and Holstein [13] do not find such a linear term in their expansion of derivatives of the field. Motivated by the use of Feynman parametrisation at zero temperature [19] they calculate the vacuum polarisation diagram using ITF but extending to continuous $p_0$ first and evaluating the mode sum afterwards. They compare their result with Dittrich’s [23] background field method of calculating the effective Langrangian at finite temperature and in both cases there is no linear term unlike our case which appears for any finite $\Delta t$.

The extension of our calculation to higher derivative terms and to space-dependent fields will give a full effective action whose importance in field theory was discussed earlier. Our calculation may be performed for higher order diagrams in the expansion of the one-loop effective action, but this is beyond the scope of this paper. We have considered a two real scalar field theory, but we could in principle use our method in different models, such as a Yukawa or a gauge theory or even consider systems with time-dependent parameters. The possibility of evaluating quantum corrections can be more directly applied to phase transitions, where they may indicate us something
about the order of the transition.

6 Acknowledgements

We would like to thank T. S. Evans and R. J. Rivers for many helpful discussions, A. Gomez Nicola for his interesting remarks and B. D. Wandelt for his help with Mathematica. This work has also been supported in part by the European Commission under the Human Capital and Mobility programme, contract number CHRX-CT94-0423.
References

[1] M. Le Bellac and H. Mabilat, Z. Phys. C75, 137, 1997
    H. Mabilat, Z. Phys. C75, 155, 1997

[2] M. Le Bellac, “Thermal Field Theory”, Cambridge University Press, 1996

[3] L. V. Keldysh, Sov. Phys. JETP 20, 1018, 1964
    J. Schwinger, J. Math. Phys. 2, 407, 1961
    R. A. Craig, Ann. Phys. 40, 416, 1966

[4] T. Matsubara, Prog. Theor. Phys. 14, 351, 1955

[5] G. Anderson and L. Hall, Phys. Rev. D 45, 2685, 1992
    M. E. Carrington, Phys. Rev. D 45, 2993, 1992

[6] K. Kajantie and J. Kapusta, Ann. Phys. C160, 477, 1985

[7] P. Arnold and L. M. Lerran, Phys. Rev. D 36, 581, 1987

[8] S. Coleman, “Aspects of Symmetry”, Cambridge University Press, 1985
    R. J. Rivers, “Path integral methods in quantum field theory”, Cambridge University Press, 1987

[9] R. Jackiw, Phys. Rev. D 9, 3320
    S. Weinberg, Phys. Rev. D 9, 3357, 1974

[10] S. Coleman, Phys. Rev. D 15, 2929, 1977
    C. Callan and S. Coleman, Phys. Rev. D 16, 1762, 1977

[11] S. Coleman and E. Weinberg, Phys. Rev. D 7, 1883, 1973

[12] A. Das and M. Hott, Phys. Rev. D 50, 6655, 1994

[13] P. S. Gribosky and B. R. Holstein, Z. Phys. C47, 205, 1990

[14] V. P. Silin, Sov. Phys. JETP 11, 1136, 1960
    D. J. Gross, R. D. Pisarski and L. G. Jaffe, Rev. Mod. Phys. 53, 43, 1981
[15] H. A. Weldon, Phys. Rev. D 28, 2007, 1983
[16] T. S. Evans, Can. J. Phys. 71, 241, 1993
[17] H. A. Weldon, Phys. Rev. D 26, 2789, 1982
[18] T. S. Evans, Z. Phys. C36, 153, 1987
   T. S. Evans, Z. Phys. C41, 333, 1988
[19] H. A. Weldon, Phys. Rev. D 47, 594, 1993
[20] R. Mills, “Propagators for Many-Particle Systems”, Gordon and Breach, 1969
[21] C. M. Fraser, Z. Phys. C28, 101, 1985
[22] T. S. Evans, “Derivative expansions of Euclidean thermal effective actions”, (in preparation)
[23] W. Dittrich, Phys. Rev. D 19, 2385, 1979