Coupling of solar p-modes of high degree $l$ by joint effects of differential rotation and meridional circulation

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Abstract. Perturbational analysis of the p-mode coupling at high degree $l$, induced by the solar differential rotation [1] is extended to include, into a unified theoretical description, the effects of global meridional circulation. The predicted observational signatures of the meridional circulation in the global p-mode measurements are briefly discussed.

1. Introduction
Large-scale flows in the solar interior induce the potentially observable signatures in both the oscillation frequencies of global solar p modes, and in their velocity fields (the eigenfunctions). The most pronounced are the effects of the strongest global flow, the differential rotation, which allow its measurements from the frequency splittings. Starting from degree $l$ of about 100, the distortion of the eigenfunctions (the p-mode coupling) by the effects of differential rotation becomes observable in the p-mode measurements [2,3]. The theoretical description of this effect was first suggested by Woodard [4] and later generalized in [1]. The effect grows rapidly with degree $l$ (when frequency separation between the interacting modes of the same p-mode ridge becomes smaller), and needs to be accurately taken into account in any adequate analysis of the observational p-mode power spectra, including frequency measurements (e.g. [3]).

This work represents a direct extension to the analysis reported in [1] to include the effects of a stationary axisymmetric meridional circulation. Apart from a potential possibility of measuring the meridional flow from its effects in global p-mode data, a more immediate motivation of this study is to see what is the degree range where the effects in the observational power spectra can be safely ignored. Technically, this work represents a generalization, by applying the quasi-degenerate normal-mode perturbation technique, of the earlier approach of Woodard [5], which was based on a local analysis of the surface waves.

2. Mode coupling
Solutions to the zero-order equations (no internal flows) are designated with tilde; in operator form,

$$\rho_0 \omega^2 \ddot{\hat{u}}_l = H_0 \hat{u}_l,$$

(1)
with the displacement field
\[ \tilde{u}_l = iU(r)Y_{lm}(\theta, \phi) + V(r)\nabla_1 Y_{lm}(\theta, \phi), \] (2)
where \( \nabla_1 \) is angular part of gradient operator, and time dependence is separated as \( \exp(-i\omega t) \). Radial order \( n \) and azimuthal order \( m \) are dropped from indexing the solutions for shortness, since interaction is limited by modes of the same \( m \) and will be considered for modes of the same \( n \) only. The unperturbed eigenfunctions are assumed to be normalized as
\[ (\tilde{u}_l', \rho_0 \tilde{u}_l) = \delta_{l'l}, \] (3)
where scalar product is defined as
\[ (u_1, u_2) = \int_V u_1^\ast \cdot u_2 \, dv, \] (4)
and the integration is performed over the spherical volume occupied by the Sun.

For configuration with a slow differential rotation and meridional circulation, the equation (1) is replaced with
\[ \rho_0 \omega^2 u = (H_0 + 2\omega \delta H) u, \] (5)
where
\[ \delta H = -i\rho_0 (v \cdot \nabla) \] (6)
and \( v \) is the sum of the two axisymmetric stationary velocity fields
\[ v = v_{\text{rot}} + v_{\text{mer}} \] (7)
which are represented by their vector spherical-harmonic decomposition as
\[ v_{\text{rot}} = -\sum_{s=1,3,...} \tilde{w}_s(r) \hat{r} \times \nabla_1 Y_{s,0}(\theta, \phi), \] (8)
\[ v_{\text{mer}} = \sum_{s=1,2,...} \left[ u_s(r) \hat{r} Y_{s,0}(\theta, \phi) + v_s(r) \nabla_1 Y_{s,0}(\theta, \phi) \right]. \] (9)
Since the fluid flow is stationary, \( v_{\text{mer}} \) satisfies the mass-conservation equation
\[ \nabla \cdot (\rho_0 v_{\text{mer}}) = 0, \] (10)
with surface boundary conditions \( u_s(R) = 0 \). The perturbing operator can thus be written as
\[ \delta H = \delta H_{\text{rot}} + \delta H_{\text{mer}}. \] (11)

The matrix elements of operator \( \delta H_{\text{rot}} \) are evaluated as described in [1], and the matrix elements \( (\tilde{u}_l', \delta H_{\text{rot}} \tilde{u}_l') \) of the operator of meridional circulation \( \delta H_{\text{mer}} \)—as described in the Appendix.

We now restrict the analysis, to make it shorter, to the meridional flows which are symmetric around the equatorial plane (only the components with even \( s \) in the equation 9 differ from zero). The selection rules, specified by the matrix elements for the operator of meridional circulation, are then the same as for the differential rotation: only modes with degree \( l \) of the same parity can interact.

As in [1], we now implement a quasi-degenerate perturbational analysis, looking for a perturbed solution as
where the expansion coefficients \( c_p \) are non necessarily small. These coefficients solve the algebraic system

\[
\left( \tilde{\omega}^2_{l+2p} - \omega^2_l \right) c_p + 2\omega_l \sum_{p'=0, \pm 1, \pm 2, \ldots} \left( \delta H \delta H_{rot} \right) c_{p'} = 0, \quad p = 0, \pm 1, \pm 2, \ldots
\]  

Assuming the unperturbed eigenfrequencies to be equidistant along the p-mode ridge, we get

\[
\omega^2_l = \tilde{\omega}^2_l + 2\tilde{\omega}_l \left( \delta H_{rot} \right),
\]  

which shows that the meridional circulation does not change the frequency splittings, and

\[
2p c_p - \sum_{p'=\pm 1, \pm 2, \ldots} b_{p'} c_{p-p'} = 0, \quad p = 0, \pm 1, \pm 2, \ldots,
\]  

which are the recurrence relations between the expansion coefficients \( c_p \), with solutions

\[
c_p = \frac{1}{\pi} \int_0^\pi \cos \left( pt - \sum_{p'=1,2,\ldots}^{P} \frac{1}{p'} \text{Re}(b_{p'}) \sin(p't) \right) \exp \left( i \sum_{p'=1,2,\ldots}^{P} \frac{1}{p'} \text{Im}(b_{p'}) \cos(p't) \right) \, dt,
\]  

where

\[
b_{p'} = - \left( \frac{\partial \tilde{\omega}_l}{\partial t} \right)^{-1} \left( \delta H \delta H_{rot} \right)_{l+2p', -2p'}
\]  

is now complex-valued, with real part governed by the differential rotation, and imaginary part governed by the meridional circulation. Derivation of the solutions (14,16) to the eigenvalue problem (13) follows closely a corresponding derivation described in [1], were the analysis was limited by the effects of differential rotation. The only new feature in the derivation is that the matrix elements of \( \delta H \) are now complex-valued.

When \( \text{Im}(b_{p'}) \) are all zero (no meridional circulation), the coupling coefficients given by equation (16) are purely real. A slow meridional flow (with \( |\text{Im}(b_{p'})| \ll 1 \)) brings small corrections which are proportional to \( v^2_{\text{mer}} \) in the real part of \( c_p \) and to \( v^2_{\text{mer}} \) in its imaginary part. The effects induced in the observational power spectra are thus proportional to \( v^2_{\text{mer}} \), if we discard small spatial leaks with imaginary amplitudes which may result from some observational or instrumental distortions (e.g. from a CCD tilt).

To evaluate the expected effects of the meridional circulation, we choose zonal (\( m=0 \)) modes. For these modes, coupling by the meridional flows is stronger than for tesseral modes, and effects of the differential rotation vanish (\( \text{Re}(b_{p'}) = 0 \) when \( m = 0 \), see [1]). A simple numerical estimate for a two-cell meridional circulation (one cell in each hemisphere) with maximum surface velocity of 15 m/s shows that the value of \( |\text{Im}(c_1)| \) reaches about 0.1 for solar \( f- \) and \( p_2 \)-modes of degree \( l = 300 \) (which is an upper-degree limit of the MDI “medium-L” measurements). We thus expect that the effects of the meridional circulation are hardly detectable in the observational power spectra at \( l < 300 \). The effects can well be measurable, however, if analyzed in the correlations between amplitude spectra, as was first suggested by Woodard [5].

Since unperturbed eigenfrequencies are not purely equidistant along the p-mode ridge, the meridional circulation can contribute slightly to the frequency splittings; this contribution has been addressed recently by Roth and Stix [9]. A weak point of the analysis reported in [9] is that when addressing the effects of the meridional flows, the differential rotation has been discarded. It is a distinctive property of the quasi-degenerate perturbation analysis that the result is not necessarily linear (additive) in the perturbation. However, the reported frequency shifts are small, indicating that the effects of the meridional circulation are hardly detectable in the oscillation frequencies.
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Appendix: the matrix elements
A complete expression for the matrix elements $\langle \hat{u}_l, \delta H_{\text{mer}} \hat{u}_l \rangle$ of operator of meridional circulation can be borrowed from Lavely and Ritzwoller [6] (we note that an alternative convention $\exp(i\omega t)$ for time dependence was used in their paper, which is equivalent to changing sign of $\omega$). For modes of high degree $l$, the result can be significantly simplified with using semiclassical approximation for Wigner’s 3-j symbols [7,8]. When $s \ll l$ and $|l' - l| \ll l$, we have

$$
\left( \begin{array}{ccc} s & l' & l \\ 0 & m & -m \end{array} \right) \simeq \frac{(-1)^{l'} m}{2l^{1/2}} \frac{(s - l' + l)!}{(s + l' - l)!} P_{s-l'}^m \left( \frac{m}{l} \right),
$$

(18)

where $P_{s-l'}^m$ is associated Legendre polynomial. For the product of two 3-j symbols, which enter the expression for these matrix elements [6], we get

$$
\left( \begin{array}{ccc} s & l' & l \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} s & l' & l \\ 0 & m & -m \end{array} \right) \simeq (-1)^{(s-l'+l)/2+m} \frac{(s-l'+l-1)!(s+l'-l-1)!!}{2l(s+l'-l)!} P_{s-l'}^m \left( \frac{m}{l} \right)
$$

(19)

when $s + l' + l$ is even, and zero otherwise.

Assuming radial eigenfunctions of interacting modes (with $|l' - l| \ll l$ and the same radial order $n$) to be nearly the same, the required matrix elements are reduced to

$$
\langle \hat{u}_{l'}, \delta H_{\text{mer}} \hat{u}_l \rangle \simeq \frac{i l (l' - l)}{s + l + l' + l even} \frac{2s + 1}{4\pi} \left( \frac{s-l'+l-1}{s+l'-l-1} \right) \int_0^R \rho \rho_r v_s(r) \left[V^2 + l(l+1)V^2 \right] dr.
$$

(20)

Note that these matrix elements are purely imaginary, with zero diagonal elements. We also have $\langle \hat{u}_{l'}, \delta H_{\text{mer}} \hat{u}_l \rangle = -\langle \hat{u}_l, \delta H_{\text{mer}} \hat{u}_l \rangle$ since operator $\delta H$ is self-adjoint; this property is ensured by the relation

$$
P_{s-l'}^m(z) = (-1)^l \frac{(s-t)!}{(s+t)!} P_{s-l'}^m(z).
$$

(21)

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