Polynomial constants of motion for Calogero-type systems in three dimensions

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Abstract

We give an explicit and concise formula for higher-degree polynomial first integrals of a family of Calogero-type Hamiltonian systems in dimension three. These first integrals, together with the already known ones, prove the maximal superintegrability of the systems.

1 Introduction

In the paper [1] the system with Hamiltonian

\[ H_0 = \frac{1}{2} \left( p_z^2 + \frac{1}{r^2} p_\varphi^2 + p_r^2 \right) + \frac{F(\psi)}{r^2} \]  

is studied. This family of systems include many well-known integrable systems with three degrees of freedom, as well as the rational Calogero system, the Wolfe's system and the Tremblay-Turbiner-Winternitz system without the harmonic oscillator term. The Hamiltonian (1) admits the following four constants of motion

\[ H_1 = \frac{1}{2} p_z^2, \quad H_2 = \frac{1}{2} p_\varphi^2 + F(\psi), \]

\[ H_3 = \frac{1}{2} \left[ (rp_z - zp_r)^2 + \left( 1 + \frac{z^2}{r^2} \right) p_\varphi^2 \right] + \left( 1 + \frac{z^2}{r^2} \right) F(\psi), \]

\[ H_4 = \frac{1}{2} \left( zp_r^2 + \frac{z}{r^2} p_\varphi^2 - rp_r p_z \right) + \frac{z}{r^2} F(\psi), \]
but, since the rank of the Jacobian of \((H_0, \ldots, H_4)\) is only four, these integrals establish only the quasi-maximal superintegrability of the system. In order to prove the maximal superintegrability, a further integral is needed.

It is clear that the maximal superintegrability of the Hamiltonian \((1)\) depends only on the maximal superintegrability of the two-dimensional Hamiltonian

\[ H = H_0 - H_1 = \frac{1}{2} \left( p_r^2 + \frac{1}{r^2} p_\psi^2 \right) + \frac{F(\psi)}{r^2} \]

i.e., on the existence of an extra constant of the motion besides the angular component

\[ H_2 = \frac{1}{2} p_\psi^2 + F(\psi) \]

In the preprint [2] (first version) an extra independent integral was found by direct computation for the particular case

\[ F(\psi) = \frac{k_1}{\cos^2 \psi} + \frac{k_2}{\sin^2 \psi} \]

and for the pairs of equivalent potentials

\[ F(\psi) = \frac{k_1}{\cos^2 \lambda \psi} \quad \text{and} \quad F(\psi) = \frac{k}{\sin^2 \lambda \psi} \quad (2) \]

in the cases \(\lambda = 2, 3, 4, 5, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}\). Moreover in [1] was conjectured that the general expression for the case

\[ F(\psi) = \frac{k}{\sin^2 \lambda \psi}, \quad \lambda = 2n + 1, \]

is given, up to \((-1)^n\), by the quite complicated explicit formula (here slightly simplified)

\[ \sum_{\sigma=0}^{n} \sum_{i=0}^{2\sigma+1} \frac{p_r^l p_\psi^{l-i}}{\lambda^{l-i}} (-2F)^{n-\sigma} \binom{\lambda}{i} \left[ \binom{\lambda - i}{l-i} \right] \left( \frac{1}{\lambda} \frac{d}{d\psi} \right)^{l-i} \cos \lambda \psi \quad (3) \]

where \(l = 2\sigma + 1, \left( \binom{a}{b} \right) = \frac{a!}{b! (a-b)!} \) denotes the Newton binomial symbol and \([a]\) the integer part of \(a\). The expression (3) is indeed a first integral of \((1)\) functionally independent from \(H_0, H_1, H_2\) and \(H_4\). The proof can be done by expanding the Poisson brackets of (3) with \((1)\) and verifying that all coefficients of each monomial in \(p_r\) and \(p_\psi\) are identically zero. The functional independence is proved in the same way as for the expression (11) below.

The system of Hamiltonian

\[ p_r^2 + \frac{1}{r^2} p_\psi^2 + \alpha r^2 + \frac{\beta}{r^2 \cos^2 k \psi} + \frac{\gamma}{r^2 \sin^2 k \psi}, \quad (4) \]

with \(\alpha, \beta, \gamma\) real and \(k\) rational, that includes \((1)\) as a subcase, has been intensively studied in [8] and [9] and proved to be maximally superintegrable in [4]. By a different approach, in the following we provide a compact expression for the additional functionally independent polynomial first integral in the cases \(\alpha = \beta = 0\) and \(k\) integer, improving and generalizing to even integers the result obtained with the formula (3).
2 A simple formula for the extra integral

We consider the Hamiltonian

\[ H = \frac{1}{2} p_r^2 + \frac{1}{r} \left( \frac{1}{2} p_\psi^2 + F(\psi) \right). \]  \tag{5}

The Hamiltonian vector field associated with \( H \) is

\[ X_H = p_r \frac{\partial}{\partial r} + \frac{L}{r^3} \frac{\partial}{\partial p_r} + \frac{1}{r^2} \mathcal{X}_L, \] \tag{6}

where \( L \) is the angular part of \( H \)

\[ L = \frac{1}{2} p_\psi^2 + F(\psi) \] \tag{7}

and \( \mathcal{X}_L \) is the associated Hamiltonian vector field

\[ \mathcal{X}_L = p_\psi \frac{\partial}{\partial \psi} - \dot{\dot{F}} \frac{\partial}{\partial p_\psi}. \] \tag{8}

After recalling that, for any two differential operators \( A \) and \( B \), if their commutator \([A, B]\) commutes with \( B \) then

\[ AB^\lambda = \lambda [A, B] B^{\lambda - 1} + B^\lambda A = \lambda B^{\lambda - 1} [A, B] + B^\lambda A, \] \tag{9}

we show that this property can be applied to \( X_H \) and to the operator

\[ U = p_r + \frac{\mu}{r} \mathcal{X}_L, \quad (\mu \in \mathbb{R}). \] \tag{10}

**Lemma 1.** We have

\[
[X_L, L] = 0, \\
\left[ \frac{2L}{r^3} \frac{\partial}{\partial p_r}, U \right] = \frac{2L}{r^3}, \\
\left[ p_r \frac{\partial}{\partial r}, U \right] = -\frac{\mu}{r^2} p_r \mathcal{X}_L.
\]

Therefore, \([[X_H, U], U] = 0.\]

**Proof.** Since \( X_L \) is the Hamiltonian vector field associated to \( L \), the identity \([X_L, L] = \{L, L\} = 0\) follows, where \( \{\cdot, \cdot\} \) is the standard Poisson bracket. The last two commutators are obtained by straightforward evaluations. Hence,

\[
[[X_H, U], U] = \left[ \frac{2L}{r^3} - \frac{\mu}{r^2} p_r X_L, p_r + \frac{\mu}{r} \mathcal{X}_L \right] = 0.
\]

\[ \square \]

**Lemma 2.** The square of the differential operator \( X_L \) is

\[ X_L^2 = p_\psi^2 \frac{\partial^2}{\partial \psi^2} - \dot{\dot{F}} \frac{\partial}{\partial \psi} + \dot{\dot{F}}^2 \frac{\partial^2}{\partial p_\psi^2} - p_\psi \left( \ddot{F} + 2 \dot{\dot{F}} \frac{\partial}{\partial \psi} \right) \frac{\partial}{\partial p_\psi} \]

and when applied to a function of \( \psi \) only it coincides with the operator

\[ p_\psi^2 \frac{\partial^2}{\partial \psi^2} - \dot{\dot{F}} \frac{\partial}{\partial \psi}. \]
Proof. It follows from a straightforward calculation.

Theorem 3. The function

\[ I_\lambda = \mathcal{U}^\lambda G(\psi) = \left( p_r + \frac{\mu}{r} \mathcal{X}_L \right)^\lambda G(\psi) \]

(11)
is a constant of the motion of the Hamiltonian (5),

\[ H = \frac{1}{2} p_r^2 + \frac{1}{r^2} \left( \frac{1}{2} p_\psi^2 + F(\psi) \right), \]

if and only if the following conditions are satisfied

\[ \lambda \mu = 1, \quad F(\psi) = \frac{k}{\sin^2(\lambda \psi + \psi_0)}, \quad G(\psi) = \cos(\lambda \psi + \psi_0). \]

(12)

Proof.

Let \( G(\psi) \) and \( F(\psi) \) be arbitrary functions. By Lemma 1 we can apply formula (9) to \( \mathcal{X}_H \) and \( \mathcal{U} \), and by Lemma 2 we get

\[ \mathcal{X}_H \mathcal{U}^\lambda = \mathcal{X} \mathcal{U}^{\lambda-1} \mathcal{X}_H \mathcal{U} + \mathcal{U}^\lambda \mathcal{X}_H \]

\[ = \mathcal{X} \mathcal{U}^{\lambda-1} \left[ p_r \partial_r + \frac{2L}{r^3} \partial_{p_r} + \frac{1}{r^2} \mathcal{X}_L, p_r + \frac{\mu}{r} \mathcal{X}_L \right] + \mathcal{U}^\lambda \mathcal{X}_H \]

\[ = \mathcal{X} \mathcal{U}^{\lambda-1} \left( -\frac{\lambda \mu}{r^2} p_r \mathcal{X}_L + \frac{\lambda}{r^3} (p_\psi^2 + 2F) + \frac{1}{r^2} p_r \mathcal{X}_L + \frac{\mu}{r^3} \mathcal{X}_L^2 \right) + \mathcal{U}^\lambda \left( p_r \partial_r + \frac{2L}{r^3} \partial_{p_r} \right) \]

\[ = \mathcal{U}^{\lambda-1} \left( \frac{1 - \lambda \mu}{r^2} p_r \mathcal{X}_L + \frac{1}{r^3} (\lambda + \mu \partial_\psi^2)p_\psi^2 + \frac{1}{r^3} (2\lambda F - \mu \dot{F} \partial_\psi) \right) + \frac{\mu}{r^3} \mathcal{U}^{\lambda-1} \left( \dot{G} \partial_\psi^2 - p_\psi (\ddot{G} + 2\ddot{F} \partial_\psi) \partial_{p_r} + \mathcal{U}^\lambda \left( p_r \partial_r + \frac{2L}{r^3} \partial_{p_r} \right) \right). \]

The evaluation of the operator \( \mathcal{X}_H \mathcal{U}^\lambda \) on the function \( G(\psi) \) results in the polynomial in the momenta function

\[ \mathcal{U}^{\lambda-1} \left( \frac{1 - \lambda \mu}{r^2} p_r \mathcal{X}_L + \frac{1}{r^3} (\lambda + \mu \partial_\psi^2)p_\psi^2 + \frac{1}{r^3} (2\lambda F - \mu \dot{F} \partial_\psi) \right) G. \]

(13)

The operator \( \mathcal{U} \), acting on a polynomial function of the momenta, increases by one the degree of the latter in \( p_r \). Therefore, the expression \( \text{(13)} \) vanishes if and only if

\[ \frac{1 - \lambda \mu}{r^2} G p_r p_\psi + \frac{1}{r^3} (\lambda G + \mu \dot{G}) p_\psi^2 + \frac{1}{r^3} (2\lambda F G - \mu \dot{F} \dot{G}) = 0, \]

that is, when \( \dot{G} \neq 0 \), if and only if the three following equations are satisfied

\[ \begin{cases} \lambda \mu = 1, \\ \mu \dot{G} + \lambda G = 0, \\ 2\lambda F G - \mu \dot{F} \dot{G} = 0. \end{cases} \]
The general solution of $\ddot{G} + \lambda^2 G = 0$ is clearly, up to a multiplicative constant, $G(\psi) = \cos(\lambda \psi + \psi_0)$ and substituting in the last equation we find the form of $F(\psi)$ given in the thesis. The case $\ddot{G} = 0$ leads to $G = \text{const.}$ and either $\lambda = 0$, $G = 0$ or $F = 0$, conditions all corresponding to trivial solutions.

Remark. 1. The constant of motion (11) is well defined in the Euclidean space only when $\lambda$ is a positive integer. Theorem 3 states that the Hamiltonian $H$ (5) is essentially the only one (up to a phase shift) that admits a constant of motion of the form (11). In particular a constant of motion for rational values of $\lambda$ takes necessarily a different form. Although there are strong evidences that a constant of motion for rational values of $\lambda$ can be obtained in a similar way, the exact form of the generating differential operator still remains an open problem.

Remark. 2. Making the change of coordinates $\phi = \lambda \psi$, $p_\phi = \lambda p_\psi$ and setting $k = \lambda^2 \tilde{k}$, the Hamiltonian $H$ (5), with $F$ given by (12), takes the form

$$\frac{1}{2} p_r^2 + \frac{\lambda^2}{r^2} \left( \frac{1}{2} p_\phi^2 + \frac{\tilde{k}}{\sin^2 \phi} \right)$$

in which the parameter $\lambda$ take the role of a coupling constant. Obviously, all the proprieties of $H$ hold for this Hamiltonian also.

Corollary. 4. Let $F(\psi)$ and $G(\psi)$ be defined as in (12).

i) The function

$$X_L^\nu \left( p_r + \frac{1}{\lambda r} X_L \right)^\lambda G(\psi)$$

is a constant of motion for the Hamiltonian (5),

$$H = \frac{1}{2} p_r^2 + \frac{1}{r^2} \left( \frac{1}{2} p_\phi^2 + F(\psi) \right),$$

for any $\nu \in \mathbb{N}$.

ii) For any $\lambda \in \mathbb{N}$, the function

$$\left( p_r + \frac{p_\phi}{\lambda r} \frac{\partial}{\partial \psi} \right)^\lambda G(\psi)$$

(15)

is a constant of motion for the geodesic part

$$H_g = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right)$$

of the Hamiltonian $H$.

Proof. i) It follows immediately from Theorem 3 and from the fact that the operator $X_L$ commute with the operator $X_H$, as a consequence of Lemma 1.

ii) It is well-known that, given a polynomial constant of motion for a natural Hamiltonian $H_g + V$, the part of highest degree in the momenta is a constant
of motion for $H_g$. The function (15) is clearly the part of highest degree in the momenta of the function

$$\left( p_r + \frac{1}{\lambda r}(p_\psi \partial_\phi - \hat{F} \partial_{p_\psi}) \right)^\lambda G(\psi),$$
given by Theorem 3.

The following theorem states the maximal superintegrability of the Hamiltonian $H$ (5) with $F$ given as in (12) and, as a consequence, of $H_0$ (1) under the same hypothesis.

**Theorem 5.** The three functions $H$, $L$, $I_{\lambda}$ - under the assumptions (12) - are functionally independent and hence the five functions $H_0, \ldots, H_3$ and $I_{\lambda}$ are also functionally independent.

**Proof.** We give a direct check of the independence of the three integrals of motion $H$, $L$ and $I_{\lambda}$. Their $3 \times 4$ Jacobian matrix w.r.t. the variables $(p_r, p_\psi, r, \psi)$ is

$$\begin{bmatrix}
  p_r & r^{-2}p_\psi & -2r^{-3}L & r^{-2}\hat{F} \\
  0 & p_\psi & 0 & \hat{F} \\
  \partial_{p_r}I_{\lambda} & \partial_{p_\psi}I_{\lambda} & \partial_rI_{\lambda} & \partial_\psi I_{\lambda}
\end{bmatrix},$$

the minor obtained by deleting the fourth column is

$$p_\psi(p_r \partial_r I_{\lambda} + \frac{2L}{r^3} \partial_{p_r}I_{\lambda}),$$

which is a polynomial in the momenta. Since $I_{\lambda}$ is a polynomial of degree $\lambda$ in $p_r$, the highest term in $p_r$ in the second addendum in the brackets is of degree $\lambda - 1$, while

$$p_r \partial_r I_{\lambda} = p_r \partial_r \left( \sum_{i=0}^{\lambda} \binom{\lambda}{i} p_r^{\lambda-i} \left( \frac{1}{\lambda r} \lambda L \right)^i \right) G$$

has not a $(\lambda+1)$th-degree term in $p_r$ (being the coefficient of $p_r^\lambda$ in $I_{\lambda}$ independent of $r$), but it contains a nonzero term of degree $\lambda$

$$p_r \lambda p_r^{\lambda-1} \frac{1}{\lambda r} \lambda L(G) = p_r^\lambda \frac{1}{\lambda} \lambda L(G).$$

Hence, being the minor not identically zero, the functions are functionally independent (up to a closed singular set). The functional independence of the five polynomials $H_0, \ldots, H_3$ and $I_{\lambda}$, with $F$ and $G$ given by (12), follows immediately.

**Remark. 3.** A comparison between (11) and the old result (3) has been done for the moment only through the explicit computation of the two expressions, in a large number of cases. The two expressions always coincide. Hence the constant of motion given by (11) reasonably corresponds with the conjectured one. Recently, different expressions have been obtained for higher-order polynomial first integrals of the Hamiltonian of Theorem 3. In these expressions, the degree of the first integral, while depending on $\lambda$, is greater than $\lambda$ itself, see for example [4], [6].
The existence of the $\lambda$th-order first integral (11) for $F = k \sin^2 \lambda \psi$ can be understood in connection with the $2\lambda$th-order dihedral symmetry of the function $F$: hexagonal for $\lambda = 3$ (i.e., the Calogero case), octagonal for $\lambda = 4$ and so on.

The potentials (4) with rational coefficients $k = \frac{p}{q}$, represented in polar coordinates on the Euclidean plane, if $\beta \neq \gamma$, have period $q\pi$, if $q$ is even, or $2q\pi$ otherwise. Therefore, they are not single-valued for $q > 2$ but still show, at least formally, dihedral symmetry of order $p$ or $2p$ respectively. If $\beta = \gamma$, dihedral symmetries and periods are the same as above with $2k$ instead of $k$. Indeed, by substituting $2 \cos k \psi = 1 + \cos 2k \psi$ and $2 \sin^2 h \psi = 1 - \cos 2k \psi$ in (4), one obtains for the angular part of the potential

$$2 \frac{(\beta + \gamma) + (\gamma - \beta) \cos 2k \psi}{\sin^2 2k \psi},$$

the function has the symmetries and the period of $\cos 2k \psi$, the same as $\sin^2 k \psi$, unless $\beta = \gamma$ when the numerator is a constant and the denominator only determines period and symmetries.

For $\alpha = 0$, the Hamiltonian (4) has been studied in and a necessary condition is obtained for its maximal superintegrability, namely, that $k$ must be rational. It follows that for non-rational $k$, (4) is not maximally superintegrable.

For non-rational values of $k$, (4) in $E^2$ becomes aperiodic, infinitely-many valued and loses the dihedral symmetry, confirming the connection between the existence of an extra first-integral and the invariance under dihedral symmetry groups.

It is an open problem if other systems with dihedral symmetry admit corresponding higher-order first integrals, and under what conditions. Some attempts have been done to the analysis of integrability and superintegrability in connection with discrete symmetries in three-dimensional manifolds [7], [5].

While the Hamiltonian (5), with $F(\psi)$ given by (12), admits a dihedral symmetry group of order $2\lambda$, the corresponding first integral $I_\lambda$ (11) admits instead a dihedral symmetry group of order $\lambda$. Indeed,

**Proposition. 6.** Let $\psi_h = \psi + \frac{h}{\lambda} \pi$ with $h$ integer. Then,

$$I_\lambda(\psi_h) = (-1)^h I_\lambda(\psi).$$

**Proof.** The transformation clearly leaves unchanged both the Hamiltonian and the operator $U$, while $\cos(\lambda \psi_h) = (-1)^h \cos(\lambda \psi)$. The statement follows immediately from Theorem 3. □

**Remark. 4.** In [1] and [3] the superintegrability of three and $n$-body systems on a line and on $m$-dimensional manifolds is deduced from the superintegrability of one-particle systems in three and $mn$-dimensional Euclidean spaces. By this approach, equivalence classes of $n$-body systems are determined by finite rotations of the same one-particle system in the $mn$-dimensional space, in such a way that all equivalent $n$-body systems are described by the same system of differential equations in $E_{mn}$ in which angular phases only change. It is a $\pi/6$ phase which realizes the equivalence between the Calogero and the Wolfes systems (2).
3 Conclusion and open problems

We give an explicit and compact expression for the $\lambda$th-degree polynomial in the momenta first integral of (5) with

$$F = \frac{k}{\sin^2 \lambda \psi},$$

for each positive integer $\lambda$. This polynomial, together with other four already known quadratic in the momenta first integrals, makes the Hamiltonian system (1) maximally superintegrable. The system considered here is a particular case of the more general natural Hamiltonian whose potential is given by (4) that has recently proven to be maximally superintegrable not only for integer but also for rational values of the parameter $k$. It seems not impossible to generalize the formula (11) to include the whole potential (4). The problems of finding such a concise expression also for rationals parameters and of explicitly proving the correspondence between (3) and (11) remain open.

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