VARIATION OF THE CANONICAL HEIGHT FOR A FAMILY OF POLYNOMIALS

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Abstract. A theorem of Tate asserts that, for an elliptic surface $E \to X$ defined over a number field $k$, and a section $P : X \to E$, there exists a divisor $D = D(E, P) \in \text{Pic}(X) \otimes \mathbb{Q}$ such that
\[
\hat{h}_{E_t}(P_t) = h_D(t) + O(1),
\]
where $\hat{h}_{E_t}$ is the Néron-Tate height on the fibre above $t$. We prove the analogous statement for a one-parameter family of polynomial dynamical systems. Moreover, we compare, at each place of $k$, the local canonical height with the local contribution to $h_D$, and show that the difference is analytic near the support of $D$, a result which is analogous to results of Silverman in the elliptic surface context.

1. Introduction

Let $k$ be a number field, $X$ a smooth, projective curve over $k$, and $E$ an elliptic curve over the function field $K = k(X)$ with associated Néron-Tate height $\hat{h}_E$. If $E$ has good reduction at $t \in X(k)$, then the fibre $E_t$ is an elliptic curve over $k$ with an associated Néron-Tate height $\hat{h}_{E_t}$. Given a point $P \in E(K)$, it is natural to ask how the height $\hat{h}_{E_t}(P_t)$ varies as a function of the parameter. If $h$ is a height on $X$ with respect to a divisor of degree 1, then a result of Silverman [11] shows that
\[
\hat{h}_{E_t}(P_t) = \hat{h}_E(P)t + o(h(t)),
\]
where $o(h(t))/h(t) \to 0$ as $h(t) \to \infty$. This was improved by Tate [16], who showed that, for some divisor $D \in \text{Pic}(X) \otimes \mathbb{Q}$, of degree $\hat{h}_E(P)$, we have
\[
\hat{h}_{E_t}(P_t) = h_D(t) + O(1).
\]
In particular, if $X = \mathbb{P}^1$, then the error term in (1) can be replaced with something bounded by an absolute constant (depending on $E$ and $P$), while in general Silverman’s bound is improved to $O(h(t)^{\frac{1}{2}})$.

Now, let $f \in K(z)$ be a rational function, and $P \in \mathbb{P}^1(K)$. There is, associated to $f$, a canonical height $\hat{h}_f : \mathbb{P}^1(K) \to \mathbb{R}$ determined uniquely by the properties
\[
\hat{h}_f(f(P)) = \deg(f)\hat{h}_f(P) \quad \text{and} \quad \hat{h}_f(P) = h(P) + O(1),
\]
and similarly to each specialization $f_t$ at which $f$ has good reduction. The analogue of (1) holds again here; Call and Silverman [4, Theorem 4.1] have shown that
\[
\hat{h}_{f_t}(P_t) = \hat{h}_f(P)t + o(h(t)).
\]

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It is natural to ask if the analogue of Tate’s theorem holds in this context. We show that it does, when \( f \) is a polynomial.

**Theorem 1.** Let \( k, X, \) and \( K \) be as above, let \( f \in K[z] \), and let \( P \in \mathbb{P}^1(K) \). Then there is a divisor \( D = D(f, P) \in \text{Pic}(X) \otimes \mathbb{Q} \) of degree \( \hat{h}_f(P) \) such that

\[
\hat{h}_f(P_t) = h_D(t) + O(1),
\]

as \( t \in X(K) \) varies, where the implied constant depends only on \( f \) and \( P \).

The divisor \( D(f, P) \) is not hard to define: identifying elements of \( K \) with morphisms \( X \to \mathbb{P}^1 \), and associating to these the usual pull-back maps from \( \text{Pic}(\mathbb{P}^1) \) to \( \text{Pic}(X) \), we may take

\[
D(f, P) = \lim_{N \to \infty} d^{-N} f^N(P)^*(\infty).
\]

One immediate application of Theorem 1 is that it allows one to count points on the base for which \( \hat{h}_f(P_t) \) is less than a given bound. It follows from the result of Call and Silverman that for any \( B \) and \( d \), the quantity

\[
N_{f,P}(B, D) = \# \{ t \in X(K) : |h(t) : k| \leq d \text{ and } \hat{h}_f(P_t) \leq B \}
\]

is finite, so long as \( \hat{h}_f(P) \neq 0 \), but nothing stronger than finiteness follows from (2).

In the case \( X = \mathbb{P}^1 \), Theorem 1 combined with a result of Schanuel allows one to deduce that

\[
N_{f,P}(B, d) \ll e^{2Bd/\hat{h}_f(P)},
\]

where the implied constants depend on \( k, d, f, \) and \( P \).

Theorem 1 also leads to an improved error term in (2), using an observation due to Lang.

**Corollary 2.** Let \( k, X, K, f, \) and \( P \) be as above. If \( h \) is any height on \( X \) relative to a divisor of degree 1, then for \( t \in X(K) \) we have

\[
\hat{h}_f(P_t) = \hat{h}_f(P)h(t) + O\left(h(t)^{\frac{3}{2}}\right),
\]

as \( h(t) \to \infty \), where the implied constant depends only on \( f \) and \( P \). If \( X = \mathbb{P}^1 \), then we have the further improvement

\[
\hat{h}_f(P_t) = \hat{h}_f(P)h(t) + O(1).
\]

It is perhaps somewhat surprising that an analogue of Tate’s theorem can be derived in this context. The proof of Tate’s result relies heavily on both the Néron model and group structure of elliptic curves. Neither of those tools are available in the context of dynamics. Call and Silverman introduced a notion of weak Néron models, which one might hope would help in this context, but Hsia has shown, over local fields, that these sometimes fail exist. Indeed, in the present context, the situation is somewhat more dire. If a given rational function \( f(z) \in K(z) \) admits a weak Néron model at every place, then by Theorem 3.1 of [5], the multipliers of the periodic cycles are integral at every place, and hence constant. In other words, if \( \mathcal{M}_d \) is the moduli space of rational functions of degree \( d \), and one considers the map \( F : X \to \mathcal{M}_d \) the generic fibre of which is \( f \), and \( \Lambda_N : \mathcal{M}_d \to \mathbb{A}^m \) is the map taking a rational function to the symmetric functions in the multipliers of its points of period dividing \( N \), we have that \( \Lambda_N \circ F \) is constant. A result of McMullen shows that the map \( \Lambda_N \) is finite-to-one, for \( N \) large enough, except on Lattès maps,
and so we have shown that for \( f \) to admit a weak Néron model at every place, \( f \) must either be isotrivial, or a family of Lattès maps (i.e., a family coming from an elliptic surface, and hence to which Tate’s result applies). In light of this, it would be particularly interesting if one could extend Theorem 1 to apply to all rational functions. If such a result could be shown, this would give a proof of Tate’s theorem which makes no fundamental use of the Néron model or the group structure of an elliptic curve, via the machinery of Lattès maps.

Tate’s results in [16] are not the end of the story for the variation of canonical heights on elliptic surfaces. Silverman [12, 13, 14] showed that the difference \( \hat{h}_{E_t}(P_t) - h_D(t) \), in addition to being bounded, varies quite regularly as a function of \( t \), breaking up into a finite sum of well-behaved functions at various places of \( k \). For example, if

\[
E_t: y^2 = x^3 + t^2(1 - t^2)x \quad \text{and} \quad P_t = (t^2, t^2),
\]

then the first result of [12] shows that there is a real-analytic function \( F(x) \) defined on a neighbourhood of 0, such that \( F(0) = 0 \) and, for all \( t \in \mathbb{Z} \) sufficiently large,

\[
\hat{h}_{E_t}(P_t) = \hat{h}_E(P)h(t) + \frac{1}{4}\log 2 + F\left(\frac{1}{t^2}\right).
\]

In the present context, we may also derive results analogous to those of [12, 13, 14].

For example, let \( k = \mathbb{Q}, X = \mathbb{P}^1, f_t(z) = z^2 + t, \) and \( P_t = 0 \). One can show that for \( t \in \mathbb{Q} \) in a (real) neighbourhood of infinity,

\[
\hat{h}_{f_t}(P_t) = \hat{h}_f(P)h(t) + \frac{1}{4t} - \frac{1}{8t^2} + \frac{5}{24t^3} - \frac{5}{16t^4} - \frac{17}{40t^5} - \frac{29}{48t^6} + \cdots
\]

(where in this case \( \hat{h}_f(P) = \frac{1}{2} \), and \( h \) is the usual Weil height on \( \mathbb{P}^1 \)). More generally, we derive the following result for quadratic polynomials over \( \mathbb{Q}(t) \).

**Theorem 3.** Let \( f_t(z) = z^2 + t, \) and let \( P_t \in \mathbb{Z}[t] \) be a monic polynomial. Then there exists a function \( F(z) \in \mathbb{Q}[z] \), convergent in a (real) neighbourhood of 0 and satisfying \( F(0) = 0 \), such that for all \( t \in \mathbb{Q} \) with \( |t| \) sufficiently large,

\[
\hat{h}_{f_t}(P_t) = \hat{h}_f(P)h(t) + F\left(\frac{1}{t}\right).
\]

Theorem 3 is essentially a special case of a more general theorem, which is analogous to the results of Silverman [12, 13, 14]. Roughly speaking, the theorem below says that the difference between \( \hat{h}_{f_t}(P_t) \) and \( h_D(t) \) is given by a sum of real-analytic functions, so long as \( t \) is close enough to \( \text{Supp}(D) \), on some prescribed set of places of \( k \). The statement of the result is somewhat more involved, however, since the analytic functions depend on which point in \( \text{Supp}(D) \) is approached by \( t \) at each place. It should be noted that, since the points in \( \text{Supp}(D) \) need not be \( k \)-rational, the following theorem assumes that we have fixed an extension of each valuation on \( k \) to a valuation on \( \overline{k} \). It should also be noted that the height \( h_D \) below is a particular height function, although it will be clear from the proof that one can adjust the terms involved to accommodate any suitably well-behaved height.

**Theorem 4.** Let \( k, X, f, \) and \( P \) be as above. Then there exists a finite set of places \( S \subseteq M_k \), containing all infinite places; for each pair \( \beta \in \text{Supp}(D) \) and \( v \in S \) a neighbourhood \( U_{\beta,v} \subseteq X(\mathbb{F}_v) \) of \( \beta \); and for each pair \( \beta \in \text{Supp}(D) \) and \( v \in S \)
archimedean, a function \( F_{\beta,v} : U_{\beta,v} \to \mathbb{R} \) which is real-analytic, with \( F_{\beta,v}(\beta) = 0 \), such that for any \( \phi : S \to \text{Supp}(D) \) there exists a \( C(\phi) \in \mathbb{R} \) such that
\[
\hat{h}_{f_t}(P_t) = h_D(t) + C(\phi) + \sum_{v|\infty} F_{\phi(v),v}(t)
\]
for any \( t \in X(k) \) satisfying \( t \in U_{\phi(v),v} \) for all \( v \in S \). In particular,
\[
\hat{h}_{f_t}(P_t) = h_D(t) + C(\phi) + o(1),
\]
where \( o(1) \to 0 \) as \( t \to \phi(v) \in \text{Supp}(D) \) in the \( v \)-adic topology, simultaneously for all \( v \in S \).

Remark. We will, in fact, prove something stronger. It turns out that our maps \( F_{\beta,v} \) are of the form
\[
F_{\beta,v}(t) = \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \log |\tilde{F}_\beta(t)|_v
\]
for some \( \tilde{F}_\beta \in \tilde{O}_{\beta,X} \), where \( O_{\beta,X} \) is the local ring of \( X/k \) at \( \beta \), and \( \tilde{O}_{\beta,X} \) its completion in the local topology. This \emph{a priori} formal function \( \tilde{F}_\beta \) turns out to be \( v \)-adic analytic at \( \beta \) for all \( v \in M_k \). This is noticeably stronger than the statement of Theorem 4, as it shows that the real-analytic functions \( F_{\beta,v} \) arise from more fundamental analytic functions which depend only on the \( \beta \in \text{Supp}(D) \). It also shows that the power series defining the functions \( F_{\beta,v} \) have coefficients in some finite extension of \( k \). Similarly, the constants \( C(\phi) \) turn out to have the form
\[
C(\phi) = d^{-N}(d-1)^{-1} \sum_{v\in S} \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \log |c_{\phi(v)}|_v,
\]
for some \( N \geq 0 \), and some values \( c_\beta \in k^* \) indexed by \( \beta \in \text{Supp}(D) \), which are \( v \)-units for any \( v \notin S \). In particular, it follows from the product formula that \( C(\phi) = 0 \) if \( \phi \) is constant.

Remark. The proof of Theorem 4 is easily modified to give a similar result for points \( t \in X(\overline{k}) \), and we present that (somewhat more complicated) statement below. Indeed, since the proof of this theorem turns out to be purely local, we could replace the \( k \)-rational points on \( X \) with the points rational over the adele ring \( A_\mathbb{R} \), and similarly for Theorem 4.

Before proceeding, we consider a slightly more revealing example of Theorem 4. If \( f_t(z) = z^2 + t \) and \( P_t = 7t + t^{-1} \), then our definition above gives \( D(f,P) = (0) + (\infty) \). Let \( v_\infty \) and \( v_7 \), respectively, denote the archimedean and \( 7 \)-adic valuations on \( \mathbb{Q} \), and let \( \phi : S = \{ v_\infty, v_7 \} \to \text{Supp}(D) = \{ 0, \infty \} \). From the proof of Theorem 4 we see that for \( t \in \mathbb{P}^1(\mathbb{Q}) \), we have
\[
\hat{h}_{f_t}(P_t) = h_D(t) + \log |\phi(v_\infty)|_\infty + \log |c_{\phi(v_7)}|_7 + o(1)
\]
where \( o(1) \to 0 \) as \( t \to \phi(v) \) in the \( v \)-adic topologies. It turns out, in this case, that \( c_\infty = 7 \) and \( c_0 = 1 \). In particular, as \( t \to \infty \) at the archimedean place, and \( t \to 0 \) at the \( 7 \)-adic place, we have
\[
\hat{h}_{f_t}(P_t) = h_D(t) + \log 7 + o(1).
\]
In contrast, as \( t \to \infty \) in both topologies, we have
\[
\hat{h}_{f_t}(P_t) = h_D(t) + o(1).
\]
2. Local and Global Heights

To begin, we will set down some notation and preliminary results. Most of the terminology is standard, and can be found, for example, in [4], [7], and [15], but we will recall the basic notation here. First of all, let \( L \) be a field, and let \( v \) be a valuation on \( L \). Then for any polynomial \( f(z) \in L[z] \) of degree \( d \geq 2 \), we define a local canonical height

\[
\hat{\lambda}_{f,v}(z) = \lim_{N \to \infty} d^{-N} \max \{0, \log |f^N(z)|_v\}.
\]

It is perhaps not immediately clear that this limit exists for all \( z \in L \). If \( f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0 \), with \( a_i \in L \) and \( a_d \neq 0 \), let

\[
\mathcal{B}_v(f) = \{ z : |f^N(z)|_v \to \infty \text{ as } N \to \infty \}
\]
denote the \( v \)-adic basin of infinity. Furthermore, let the symbol \( (2d)_v \) denote \( 2d \) if \( v \) is archimedean, and \( 1 \) otherwise, let

\[
\Lambda_v = \max \left\{ \max_{0 \leq i < d} \left\{ \left| \frac{a_i}{a_d} \right|_v^{1/(d-i)} \right\} , |a_d|_v^{2/(d-1)} , 1 \right\},
\]

and let

\[
\mathcal{B}_v^0(f) = \{ z : |z|_v > (2d)_v \Lambda_v \}.
\]

Note that it is perhaps more natural to replace \( |a_d|_v^{-2/(d-1)} \), in the above definition, with \( |a_d|_v^{-1/(d-1)} \), but the more restrictive bound is critical later.

Roughly speaking, \( \mathcal{B}_v^0(f) \) will play the rôle played in the elliptic curve context by \( \mathcal{E}_v^0 \), the identity component of the fibre of the Néron model above \( v \). In other words, \( \mathcal{B}_v^0(f) \) is some domain on which the local heights are particularly well-behaved. The following elementary results describe the behaviour of local heights in \( \mathcal{B}_v^0(f) \); similar results appear in [2] and [6].

**Lemma 5.** For all \( z \in L \), the limit defining \( \hat{\lambda}_{f,v}(z) \geq 0 \) exists, and \( \hat{\lambda}_{f,v}(z) > 0 \) if and only if \( z \in \mathcal{B}_v(f) \). Furthermore, both \( \mathcal{B}_v(f) \) and \( \mathcal{B}_v^0(f) \) are closed under \( f \), and

\[
\mathcal{B}_v(f) = \{ z : f^N(z) \in \mathcal{B}_v^0(f) \text{ for some } N \geq 0 \}. \]

Finally, for all \( z \in \mathcal{B}_v^0(f) \) and all \( N \), we have

\[
c_1 \leq \frac{1}{d^N} \log |f^N(z)|_v \leq \left( \frac{1 - d^{-N}}{d - 1} \log |a_d|_v + \log |z|_v \right) \leq c_2,
\]

where \( c_1 = \log \frac{1}{2} \) and \( c_2 = \log \frac{3}{2} \) if \( v \) is archimedean, and \( c_1 = c_2 = 0 \) otherwise. In particular,

\[
c_1 \leq \hat{\lambda}_{f,v}(z) - \left( \frac{1}{d - 1} \log |a_d|_v + \log |z|_v \right) \leq c_2.
\]

**Proof.** Let \( d = \deg(f) \). First, we note that for \( z \in \mathcal{B}_v^0(f) \), we have by hypothesis,

\[
(2d)_v |a_i|_v |z|_v^i \leq (2d)_v^{(d-i)} |a_i|_v |z|_v^i < |a_d|_v |z|_v^d.
\]

If \( v \) is non-archimedean, this implies

\[
|f(z)|_v = |a_d|_v |z|_v^d \geq |z|_v,
\]
whereupon \( f(z) \in B^0_v(z) \). On the other hand, if \( v \) is archimedean, we have

\[
|f(z)|_v = \left| \sum a_i z^i \right|_v \geq |a_d|_v |z|^d - d \max |a_i z|_v^i \geq \frac{1}{2} |a_d|_v |z|^d
\]

by (6). It follows again that \( |f(z)|_v \geq |z|^\lambda_{f,v} \), and hence \( f(z) \in B^0_v(f) \). Thus, in either case, \( B^0_v(f) \) is closed under \( f \).

Now, if \( v \) is non-archimedean, then (7) implies

\[
|f^N(z)|_v = |a_d|_v |z|^d \quad \text{by induction, for all } z \in B^0_v(f).
\]

Since \(|z|_v > 1\), we obtain

\[
\lambda_{f,v}(z) = \lim_{N \to \infty} d^{-N} \log |a_d|_v |z|^d
\]

for all \( z \in B^0_v(f) \), and so by induction,

\[
\left( \frac{1}{d} |a_d|_v \right)^{d^{-N}} |z|^{d^N} \leq |f^N(z)|_v \leq \left( \frac{3}{2} |a_d|_v \right)^{d^{-N}} |z|^{d^N}.
\]

Taking logarithms and limits yields

\[
\frac{1}{d-1} \log \frac{1}{2} \leq \lambda_{f,v}(z) - \left( \frac{1}{d-1} \log |a_d|_v + \log |z|_v \right) \leq \frac{1}{d-1} \log \frac{3}{2},
\]

which, in the worst case \( d = 2 \), is what was claimed.

Now, if \( z \not\in B_v(f) \), then \( |f^N(z)|_v \) is bounded as \( N \to \infty \), and so \( \lambda_{f,v}(z) = 0 \). On the other hand, if \( z \in B_v(f) \) then there is some \( N \) with \( |f^N(z)|_v > (2d)_v \lambda_v \), and so we have both \( f^N(z) \in B^0_v(f) \), and \( \lambda_{f,v}(z) > 0 \).

We now recall the definition of various height functions. Throughout, \( k \) will denote some number field, and \( M_k \) will be the standard set of places on \( k \). We will adopt the convention that the valuation \( | \cdot |_v \) for each \( v \in M_k \), has been extended in some way to \( \mathbb{F} \). For each \( v \in M_k \), we define a local (naïve) height on \( \mathbb{P}^1 \) by

\[
\lambda_v(x) = \max \{0, \log |x|_v\}.
\]

The global (naïve) height on \( \mathbb{P}^1(k) \) is defined by

\[
h(x) = \sum_{v \in M_k} \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \lambda_v(x).
\]

It is easy enough to see that this can be extended to \( \mathbb{F} \) by defining

\[
h(x) = \sum_{v \in M_k} \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \left( \frac{1}{[L : k]} \sum_{\sigma \in \text{Gal}(L/k)} \lambda_v(x^\sigma) \right),
\]

where \( L \supset k \) is any Galois extension containing \( x \). It is, of course, necessary to check that this definition does not depend on the particular Galois extension chosen, but it does not. We define the canonical height with respect to \( f \in k[z] \) by

\[
h_f(x) = \sum_{v \in M_k} \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \lambda_{f,v}(x),
\]
and similarly for \( \overline{k} \).

We define heights in function fields similarly, although we work over the algebraic closure of the constant field (so that valuations on \( K = k(X) \) are the same as valuations on \( K \otimes \overline{k} \)). Also, we will denote the valuation corresponding to \( \beta \in X(\overline{k}) \) by \( \text{ord}_\beta \) to avoid confusion with valuations on the constant field \( k \). For any \( z \in K \) and \( \beta \in X(\overline{k}) \), we define

\[
\lambda_\beta(z) = \max\{0, -\text{ord}_\beta(z)\},
\]

so that \( \lambda_\beta(z) \) is the order of the pole of \( z \) at \( \beta \), if there is one, and 0 otherwise. For \( f \in K[z] \) we define \( \hat{\lambda}_{f, \beta} \) as in [5], and set

\[
\hat{h}_f(z) = \sum_{\beta \in X(\overline{k})} \hat{\lambda}_{f, \beta}(z).
\]

At this point we can define our divisor \( D = D(f, P) \in \text{Pic}(X) \otimes \mathbb{Q} \), which will simply be

\[
D(f, P) = \sum_{\beta \in X(\overline{k})} \hat{\lambda}_{f, \beta}(\beta).
\]

This clearly has degree \( \hat{h}_f(P) \), and is equivalent to the definition of \( D(f, P) \) given in the introduction.

In addition to the above heights on \( \mathbb{P}^1 \), we will define Néron functions, and heights relative to divisors on \( X \). Since we want to claim that the difference \( \hat{\lambda}_{f, v}(P_t) - \lambda_{D, v}(t) \) is real-analytic in certain neighbourhoods, we need to be fairly specific as to how we define these local heights. Let \( D = D(f, P) \) be as defined above, for a particular \( f \in K[z] \) and \( P \in K \). If it should happen that \( D = 0 \), then we will simply define \( \lambda_{D, v}(x) = 0 \) for all \( v \in M_k \) and \( x \in X(\overline{k}_v) \). To deal with the case \( D \neq 0 \), we will employ the following simple lemma.

**Lemma 6.** With \( f \) and \( P \) as above, suppose that \( D = D(f, P) \neq 0 \). Then there is an \( N \) and a morphism \( g : X \to \mathbb{P}^1 \) (defined over \( k \)) such that \( d^N(d - 1)D = g^*(\infty) \).

**Proof.** First we must show that there is an \( N \) with \( d^N(d - 1)D \in \text{Div}(X) \), under the hypothesis that \( D > 0 \). For each \( \beta \in \text{Supp}(D) \), we have \( \hat{\lambda}_{f, \beta}(P) > 0 \), and so by Lemma [5] there is an \( N \) such that \( f^N(P) \in \mathcal{B}^N_\beta(f) \). For this value of \( N \), we have

\[
d^N(d - 1)\hat{\lambda}_{f, \beta}(P) = (d - 1)\hat{\lambda}_{f, \beta}(f^N(P)) = (d - 1)\log|f^N(P)|_\beta + \log|a_d|_\beta \in \mathbb{Z}.
\]

If we choose \( N \) large enough that \( f^N(P) \in \mathcal{B}^N_\beta(f) \) for all \( \beta \in \text{Supp}(D) \), we have

\[
d^N(d - 1)D = \sum_{\beta \in X(\overline{k})} d^N(d - 1)\hat{\lambda}_{f, \beta}(P)(\beta) \in \text{Div}(X).
\]

Now, since \( D > 0 \), we may choose \( N \) to be large enough so that

\[
d^N(d - 1) \deg(D) \geq 2g(X)
\]

which ensures, by the Riemann-Roch theorem, that there is a morphism \( g : X \to \mathbb{P}^1 \) such that \( d^N(d - 1)D = g^*(\infty) \).

Thus in the case \( D \neq 0 \), we may choose \( N \) and \( g \) as in Lemma [5] and set for each \( v \in M_k \)

\[
\lambda_{D, v}(t) = d^{-N}(d - 1) \max\{0, \log|g(t)|_v\}.
\]
The global height is defined by
\[ h_D(t) = \sum_{v \in M_k} \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \lambda_{D,v}(t) \]
for \( t \not\in \text{Supp}(D) \), and \( h_D(t) = 0 \) for \( t \in \text{Supp}(D) \) (and similarly for points \( t \in X(\overline{K}) \)). By linearity and functoriality of heights (see, e.g., [7]), the global height \( h_D : X(\overline{K}) \to \mathbb{R} \) differs from any other height relative to \( D \) by at most a bounded amount. Everything below transfers over to any suitably well-behaved choice of local heights.

To keep track of bounds which depend on places \( v \in M_k \), we will use Weil’s notion of an \( M_k \)-divisor [4, p. 29]. A (multiplicative) \( M_k \)-divisor is a function \( c : M_k \to \mathbb{R}^+ \) such that \( c(v) = 1 \) for all but finitely many \( v \in M_k \), and such that for each non-archimedean \( v \), \( c(v) = |\alpha|_v \) for some \( \alpha \in k^* \). It is clear that the \( M_k \)-divisors form a group under pointwise multiplication, and that the pointwise maximum or minimum of two \( M_k \)-divisors is again an \( M_k \)-divisor.

Additionally, given a place \( v \in M_k \), a point \( \beta \in X(\overline{K}) \), and a function \( u_\beta \in \overline{k}(X) \), vanishing only at \( \beta \), we will set
\[
\mathbb{D}_v(\beta; \varepsilon) = \left\{ t \in X(\overline{K}_v) : |u_\beta(t)|_v^{1/\text{ord}_\beta(u_\beta)} < \varepsilon \right\}
\]
and
\[
\mathbb{D}_v(\beta; \delta, \varepsilon) = \left\{ t \in X(\overline{K}_v) : \delta < |u_\beta(t)|_v^{1/\text{ord}_\beta(u_\beta)} < \varepsilon \right\}.
\]
Note that these sets depend on the choice of \( u_\beta \), but for a different choice the set will agree at all but finitely many places.

3. Analytic properties

The proofs of Theorem 1 and Theorem 4 are, not surprisingly, largely analytic in nature. Here we lay some of the groundwork. In this section, we will typically agree at all but finitely many places.

Throughout, \( X/L \) will be a smooth projective curve, and \( K = L(X) \) its function field. For each \( \beta \in X(\overline{L}) \), we fix a uniformizer \( w_\beta \in K \), and let \( K_\beta \) denote the completion of \( K \) with respect to \( \text{ord}_\beta \). As usual, \( \mathcal{O}_{\beta,X} \subseteq K \) will denote the subring consisting of elements regular at \( \beta \), and similarly for \( \hat{\mathcal{O}}_{\beta,X} \subseteq K_\beta \). Note that there is a natural isomorphism \( \hat{\mathcal{O}}_{\beta,X} \cong L[w_\beta] \), and we will associate elements of \( \hat{\mathcal{O}}_{\beta,X} \) with their series representations, and similarly for \( K_\beta \). The element \( g \in \hat{\mathcal{O}}_{\beta,X} \) is \( v \)-adic analytic if the corresponding series converges on the disk \( \mathbb{D}_v(\beta; \varepsilon) \), for some \( \varepsilon > 0 \), and an element in \( K_\beta \) is analytic if it is the quotient of two analytic elements (note that such a function might have a pole at \( \beta \)). Similarly, if \( v \in M_L \) is non-archimedean, then \( \mathcal{O}_{v,L} \) will denote the ring of \( v \)-adic integers, defined by \( \{ x \in L : |x|_v \leq 1 \} \).

For the remainder of the section, we will fix \( f \in K_\beta[z] \) and \( P \in K_\beta \), such that \( P \in \mathcal{B}_{\text{ord}_\beta}^0(f) \), and set \( 0 < m = -\text{ord}_\beta(P) \). The next lemma is (in the case where \( L \) is a number field) a well-known adelic version of the implicit function theorem, which we will use to translate the problem into one of pure analysis. Given a formal power series \( F \in L[[w]] \), we will say that \( \varepsilon > 0 \) is a \( (v \text{-adic}) \) radius of convergence.
for $F$ if the sum $F(w)$ converges $v$-absolutely for $|w|_v < \varepsilon$. We will say that the $M_L$-divisor $\epsilon$ is a \textit{global radius of convergence} for $F$ if $\epsilon(v)$ is a $v$-adic radius of convergence for $F$, for each $v \in M_L$. Given a Laurent series $F \in L((w))$, we will say that $\varepsilon$ or $\epsilon$ is a radius of convergence for $F$ if it is for $w^{m}F(w)$, where $m$ is the order of the pole of $F$ at $w = 0$.

\textbf{Lemma 7.} Let $g \in \mathcal{O}_{\beta,X}$. Then $g$ is analytic at every place, i.e., there is a (global) radius of convergence for the image of $g$ in $L[[w_\beta]]$.

\textbf{Proof.} If $v$ is an archimedean place, this is simply the implicit function theorem. There is a non-archimedean version of the implicit function theorem, and we may apply this at finitely many of the non-archimedean places, but for the conclusion when $L$ is a number field, we need to know that the radius of convergence at $v$ is 1 for all but finitely many $v$.

Let $R = L[[w_\beta]]$, and let $p = w_\beta R$. If $U \subseteq X$ is an affine subscheme of $X$, containing $\beta$, then $U$ defines a 0-dimensional affine scheme over $R$. Suppose that

$$U = \text{Spec}(R[X_1,\ldots,X_s]/(F_1,\ldots,F_s)).$$

Then, as usual, we may use Newton’s Method to lift the point $\beta \in U(R/p)$ to a point in $U(R)$. In other words, if $F(X)$ denotes the vector

$$\langle F_1(X_1,\ldots,X_s),\ldots,F_s(X_1,\ldots,X_s) \rangle,$$

and $J(F)$ denotes the Jacobian matrix of this system, we let $X_0 = \beta$, and take

$$X_{n+1} = X_n - J(F)(X_n)^{-1}F(X_n).$$

Note that $J(F)(X_n)$ is invertible for each $n$, since

$$\det(J(F)(X_n)) = \det(J(F)(\beta)) \not\equiv 0 \pmod{p}.$$ It is simple enough to show that $F(X_n) \in p^{2^n}$, for each $n$, and so by the completeness of $R$, this sequence of points converges to a limit $Y = (Y_1,\ldots,Y_s)$ in $R^s$. This vector satisfies $Y \equiv \beta \pmod{p}$ or, viewing the entries as functions of $w_\beta$, $Y(0) = \beta$. The tuple $Y$ is also the unique element of $R^s$ with this property. Thus, the series $Y_1,\ldots,Y_s$, within their radius of convergence, define the coordinate functions $y_1,\ldots,y_s$ on $U$.

Now, let $S$ be a finite set of places such that for $v \notin S$, the coefficients of all of equations defining $U$ are $v$-integral, and $\det(J(F))(\beta)$ is a $v$-unit. Then it is easy to check, by induction, that $X_n$ has $v$-integral coefficients, and so

$$Y_i \in \mathcal{O}_{v,L}[[w_\beta]] \subseteq L[[w_\beta]].$$

Now, if we represent $g$ in these coordinates

$$g(y_1,\ldots,y_s) = \frac{G_1(y_1,\ldots,y_s)}{G_2(y_1,\ldots,y_s)},$$

we see that $g(Y_1,\ldots,Y_s)$ has $v$-integral coefficients, so long as $G_2(Y_1(0),\ldots,Y_s(0))$ is a $v$-unit. If we enlarge $S$ to contain all places for which this fails, we have $g \in \mathcal{O}_{v,L}[[w_\beta]]$ for all $v \notin S$, and so $g(w)$ converges for any $|w|_v < 1$. \hfill $\Box$

\textbf{Remark.} What we have in fact proven, and we will make use of this below, is that there is a finite set $S \subseteq M_L$, containing all archimedean places, such that for $v \notin S$, the series for $g$ in $L[[w_\beta]]$ has $v$-integral coefficients, a result essentially due to Eisenstein.
The following statement is essentially a continuity claim, which states that a function on a curve can take \( v \)-adically large values only at points \( v \)-adically close to its poles.

**Lemma 8.** Let \( Z \subseteq X(L) \) be a finite set, and for each \( \alpha \in Z \) let \( c_\alpha \) be an \( M_L \)-divisor. Then for any rational function \( g \) on \( X \) having no poles on \( X \setminus Z \), there is an \( M_L \)-divisor \( \mathfrak{d} \) such that for any \( t \in X(L) \) and any \( \nu \in M_L \), \( |g(t)|_\nu > \mathfrak{d}(\nu) \) implies \( |w_\alpha(t)|_\nu < c_\alpha(\nu) \) for some \( \alpha \in Z \).

**Proof.** The conclusion of the lemma only gets weaker as \( Z \) gets larger, and so we will assume that \( Z \) is exactly the set of poles of \( g \). The curve \( X \) is smooth and projective, and so by Lemma 2.2 of [1], for any functions \( f_1, \ldots, f_n \in \mathcal{T}(X) \) with no common zero, there is an \( M_L \)-divisor \( \mathfrak{c} \) with

\[
\sup_{1 \leq i \leq n} |f_i(t)|_\nu \geq \mathfrak{c}(\nu)
\]

for all \( \nu \in M_L \) and all \( t \in X(\mathbb{Z}) \). Let \( n_\alpha \) be the order of the pole of \( g \) at \( \alpha \), and let \( f_1 = 1/g, f_\alpha = 1/(w_\alpha^n g) \), for each \( \alpha \in Z \). The zeros of \( f_1 \) are contained in \( Z \), but \( \alpha \) is not a zero of \( f_\alpha \), and so the lemma applies to this collection of functions. Let \( \mathfrak{c} \) be the \( M_L \)-divisor with the above property, and choose

\[
\mathfrak{d}(\nu) = \max_{\alpha \in Z} \{ c_\alpha(\nu)^{-n_\alpha}, 1 \} \mathfrak{c}(\nu)^{-1}.
\]

Note that \( |g(t)|_\nu > \mathfrak{d}(\nu) \) immediately implies \( |f_1(t)|_\nu < \mathfrak{c}(\nu) \), and so for each \( \nu \in M_L \), there is some \( \alpha \in Z \) with \( |f_\alpha(t)|_\nu \geq \mathfrak{c}(\nu) \). For that particular \( \alpha \) and \( \nu \), then, we have

\[
|w_\alpha(t)|_\nu^{-n_\alpha} = |g(t)f_\alpha(t)|_\nu > \mathfrak{d}(\nu)\mathfrak{c}(\nu) = \max_{\alpha' \in Z} \{ c_{\alpha'}(\nu)^{-n_{\alpha'}}, 1 \} \geq c_\alpha(\nu)^{-n_\alpha}.
\]

Since \( n_\alpha \geq 1 \), the result is proven. \( \square \)

Our next lemma produces a formal object which, in certain isotrivial cases (take \( X = \mathbb{P}^1 \), \( f \in L[z] \), and \( P_t = t \)), corresponds to the Böttcher coordinate. Note that the lemma is slightly ambiguous, since there may be several \( d^N \)th roots of an element of \( K_\beta \), but a choice of roots is made in the proof.

**Lemma 9.** With the notation above, there exists a \( \mathcal{G}_\beta \in K_\beta \) such that

\[
\left( f^N(P) a_d^{-(d^N-1)/(d-1)} \right)^{1/d^N} \rightarrow \mathcal{G}_\beta
\]

in the \( m \)-adic topology, as \( N \to \infty \). Furthermore, there is a finite set \( S \subseteq M_L \) such that for any \( \nu \not\in S \), the image of \( \mathcal{G}_\beta \) in \( L[[w]] \) lies in the subring \( \mathcal{O}_{v,L}[[w]] \) consisting of power series with coefficients integral at \( v \).

**Proof.** Since \( P \in \mathcal{B}_\beta^0(f) \), we have for all \( N \),

\[
|f^N(P)|_m = |P|^{(d^N-1)/(d-1)},
\]

by Lemma 5 If \( m = \hat{\lambda}_{f,\beta}(P) = -\text{ord}_{\beta}(P) \), and \( w = w_\beta \) is a uniformizer at \( \beta \), let

\[
\xi_N = f^N(P) w^{md^N} a_d^{-(d^N-1)/(d-1)} \in \mathcal{O}_{\beta,X}^N.
\]

Note that, for all \( Q \in \mathcal{B}_\beta^0(f) \), we have

\[
|f(Q) - a_dQ^d|_m = |a_{d-1}Q^{d-1} + \cdots + a_1Q + a_0|_m < |a_dQ^d|_m.
\]
and so the leading term of the series $f(Q)$ agrees with the leading term of $a_d Q^d$. Thus if $P = a w^{-m} + O(w^{1-m})$ and $a_d = \gamma w^n + O(w^{1+n})$, say, we have

$$f^N(P) = \gamma(d^N-1)/(d-1) \alpha^d w^n + O(w^{1+n})$$

for $q = n(d^N - 1)/(d-1) - ma^N$, and hence

$$\xi_N = \alpha^d + O(w).$$

Thus the polynomial $\Phi(X) = X^d - \xi_N$ has a simple root, modulo $m$, at $X = \alpha$, and so by Hensel’s Lemma (a special case of the argument used in Lemma 7), there is some $G_N \in \hat{O}_{\beta,N}$ such that $G_N^d = \xi_N$, and $G_N \equiv \alpha \pmod{m}$. We wish to show that the sequence $G_N$ has a limit in $\hat{O}_{\beta,N}$.

Suppose that we set $B = \max_{0 \leq i < d} |a_i/a_d|_m$, and take $Q \in B_0^d(f)$. Then certainly

$$|f(Q) - a_d Q^d|_m = |a_d|_m \left| i=0 \sum_{i=0}^{d-1} \frac{a_i}{a_d} Q^i \right|_m \leq B |a_d|_m |Q|_m^{d-1}$$

(recalling that $Q \in B_0^d(f)$ implies $|Q|_m > 1$). Since $B_0^d(f)$ is closed under $f$, then, we have

$$|f^{N+1}(P) - a_d f^N(P)|_m \leq B |a_d|_m |f^N(P)|_m^{d-1} = B |a_d|_m |P|_m^{dN(d-1)},$$

by Lemma 5. It follows that

$$|\xi_{N+1} - \xi_N|_m = |w|_m^{mdN+1} |a_d|_m^{-(dN+1)/(d-1)} |f^{N+1}(P) - a_d f^N(P)|_m^{d-1} \leq B |P|_m^{-dN} |a_d|_m^{-dN/(d-1)} B |a_d|_m^{dN} |P|_m^{dN-1} \leq B |P|_m^{-dN/2}.$$
By the triangle inequality, and the generous estimate $N < d^N$, we have for all $M \geq N$,

$$|G_M - G_N|_m \leq \sum_{i=N}^{M-1} B |P|^d_i^{-d/2} < B \sum_{i=N}^{M-1} |P|^{-i/2} < B \int_{N-1}^{\infty} |P|^{-s/2} ds = \frac{B |P|^{-N-1/2}}{\frac{1}{2} \log |P|_m}.$$  

Since $P \in B^d_\beta(F)$, we have $B < |P|_m^d$, and it follows that the sequence of $G_N$ is Cauchy. Since $\hat{O}_{\beta,X}$ is complete with respect to the $m$-adic metric, there is a limit $G_\infty \in \hat{O}_{\beta,X}$ of this sequence. We can simply take $G_\beta = w^{-m}G_\infty$.

It remains to show that, in the case where $L$ is a global field, we may find a finite set of places $S \subseteq M_L$ such that for $v \not\in S$, we have $G_\beta \in \hat{O}_{v,L}(w)$. Invoking Lemma 7, choose a finite set of primes $S \subseteq M_L$ such that for $v \not\in S$, the series for $P$ and each $a_i$ have coefficients in $\hat{O}_{v,L}$, and such that $\alpha$ (the lead coefficient of the series for $P$), $\beta$ (the lead coefficient of $a_d$), and $d$ are units in $\hat{O}_{v,L}$. If $v \not\in S$, it follows from the fact that $a_d \in \hat{O}_{v,L}(w)$ and $\beta \in \hat{O}_{v,L}^*$ that $a_d^{-1} \in \hat{O}_{v,L}(w)$. It is now clear that $\xi_n \in \hat{O}_{v,L}[w]$ for all $N$, and the leading coefficient of $\xi_N$ is $\alpha^dN \in \hat{O}_{v,L}^*$. Now, the Hensel’s Lemma construction of $G_N$ is as follows. For $\Phi_N(X) = X^d - \xi_N$, we let $X_0 = \alpha$, and then

$$X_{n+1} = X_n - \frac{\Phi(X_n)}{\Phi'(X_n)}.$$  

Hensel’s Lemma shows that the $X_n$ converge m-adically, and $G_N$ is their limit. Since $X_0 = \alpha \in \hat{O}_{v,L}[w]$, suppose that $X_n \in \hat{O}_{v,L}[w]$. Then $\Phi'(X_n) = d^N X_n^{d^N-1}$ has the leading term $d^N \alpha^{d^N-1} \in \hat{O}_{v,L}^*$, and hence $\Phi'(X_n)$ is a unit in $\hat{O}_{v,L}[w]$. In other words, $X_n \in \hat{O}_{v,L}[w]$ implies $X_{n+1} \in \hat{O}_{v,L}[w]$. But $\hat{O}_{v,L}[w] \subseteq L[w]$ is closed in the m-adic topology, and so $G_N = \lim_{n \to \infty} X_n \in \hat{O}_{v,L}[w]$. Similarly, $G_\infty = \lim_{N \to \infty} G_N$ has coefficients in $\hat{O}_{v,L}$, and hence so too does $G_\beta$.  

At this point, the power series $G_\beta$ is simply a formal limit. It is easy enough to see, if $L = \mathbb{C}$, say, that this sort of formal convergence of functions in the local ring $\hat{O}_{\beta,X}$ neither implies, nor is implied by, uniform convergence as functions in a neighbourhood of $\beta$. On the other hand, it is easy to show that if a sequence should converge both uniformly and formally, then the two limits must be identical, since a derivative of a uniform limit is the limit of the derivatives. Before showing that $G_\beta$ does, in fact, define a smooth function at $\beta$, we will prove a version of the Schwarz Lemma.

**Lemma 10** (Schwarz Lemma). Let $v \in M_L$, suppose that $\varepsilon > 0$, and if $v$ is non-archimedean, that there is an $\alpha \in \mathbb{L}^*$ with $\varepsilon = |\alpha|_v$. If the series $g \in \mathbb{L}[w]$ converges uniformly on $U = \{w \in \mathbb{T}_v : |w|_v < \varepsilon\}$, and $|g(w)|_v \leq B$ for all $w \in U$, then

$$|g(w)|_v \leq \frac{|w|^{\ord(g)} \cdot B}{\varepsilon^{\ord(g)}}$$

for $w \in U$.

**Proof.** There is nothing to prove if $g(0) \neq 0$, so suppose that $n = \ord(g) \geq 1$. We have $g(w) = w^n h(w)$ for some $h$ with $h(0) \neq 0$, and $h$ analytic on $U$. For any
for each $0 < r < \varepsilon$ we have, by the maximum modulus principle (for the non-archimedean maximum principle, see [9, p. 318]),

$$\max_{|w|_v \leq r} |h(w)|_v = \max_{|w|_v = r} \left| \frac{g(w)}{w^n} \right|_v \leq r^{-n} B.$$

Since this is true for all $r < \varepsilon$, we in fact have $|h(w)|_v \leq \varepsilon^{-n} B$ for all $w \in U$, and hence $|g(w)|_v \leq |w|^n \varepsilon^{-n} B$.

\[\square\]

Lemma 11. Maintaining the notation above, there is an $M_L$-divisor $\varepsilon$ such that the a priori formal power series $G_\beta$ defines a $v$-adic analytic function on $\mathbb{D}_v(\beta; 0, \varepsilon(v))$ for each $v \in M_L$. Furthermore, for any $0 < \delta_1 < \delta_2 < \varepsilon(v)$, we have

$$\left(f^N(P)a_d^{-(dN-1)/(d-1)}\right)^{1/d^N} \in G_\beta$$

uniformly on $\mathbb{D}_v(\beta; \delta_1, \delta_2)$.

Proof. Since $P \in B^0_\beta(f)$, we have

$$-(d - i) \text{ord}_\beta(P) > -\text{ord}_\beta(a_i/a_d),$$

for each $0 \leq i < d$, and so we may choose a disk $\mathbb{D}_v(\beta; 0, \varepsilon)$ on which

$$|P_t|_v > 2^v \left| \frac{a_i(t)}{a_d(t)} \right|^{1/(d-i)}_v,$$

where as usual $2^v = 2$ if $v$ is archimedean, and 1 otherwise. For all but finitely many places, this disk can be chosen to have radius one, since we can choose a finite set of places outside of which $P_t^{d-i}a_d(t)/a_i(t)$ is given by a power series with integral coefficients, a leading coefficient which is a unit, and a pole at zero. Proceeding similarly with $2^v|a_d(t)|^{-2/(d-1)}_v$ and the constant function $2^v$, we see that we may choose an $M_L$-divisor $\varepsilon$ such that $P_t \in B^0_\beta(f)$ for all $t \in \mathbb{D}_v(\beta; 0, \varepsilon(v))$.

Also note that since $\text{ord}_\beta(P) = -m$, we may assume without loss of generality that the function $P w_\beta^m$ is analytic and bounded on the disk of radius $\varepsilon(v)$, say

$$|P_t w_\beta(t)^m|_v \leq b(v),$$

where $b$ is an $M_L$-divisor. In particular, if we set

$$\xi_N = w_\beta^{md^N} f^N(P)a_d^{-(dN-1)/(d-1)} \in O^*_\beta X$$

as above, then for any given place $v \in M_L$, $\xi_N$ extends to an analytic function with no zeros on $\mathbb{D}_v(\beta; \varepsilon(v))$, and hence (since $\xi_N(\beta) = \alpha^d$) there is an analytic function $G_N$ with $G_N(\beta) = \alpha$ and $G_N^N = \xi_N$. It is not hard to see that $G_N$ is defined by the formal power series $G_N$ in the previous proof. If $\varepsilon(v) = \frac{3}{2}$ for $v$ archimedean, and I otherwise, we have for $t \in \mathbb{D}_v(\beta; \varepsilon(v))$

$$|G_N(t)|_v = \left| \frac{\xi_N(t)}{w_\beta(t)^m f^N(P_t)}a_d^{-(dN-1)/(d-1)} \right|_v$$

$$\leq \left| w_\beta(t)^m \right|_v \left| f^N(P_t) \right|_v^{1/(d-1)} \left| a_d(t) \right|_v^{-(dN-1)/(dN-1)}$$

$$\leq b(v) \left| \frac{c(v)}{1-d^N} \right|^{(1-d^{-N})/(d-1)}_v \left| a_d(t)^{-1} \right|_v^{(1-d^{-N})/(d-1)}$$

$$\leq b(v) c(v)^{1/(d-1)} \max\{1, |a_d(t)^{-1}|_v\}^{1/(d-1)}.$$
First, we treat the case \( \operatorname{ord}_\beta(a_d) \leq 0 \). In this case, we have for some \( M_k \)-divisor \( \mathfrak{d}_v \),
\[
|G_N(t)|_v \leq \mathfrak{d}_v \quad \text{for all } t \in U,
\]
and hence
\[
|G_M(t) - G_N(t)|_v \leq 2_v \mathfrak{d}_v.
\]
Now, by the Schwarz Lemma, we have
\[
|G_M(t) - G_N(t)|_v \leq \frac{|w_\beta(t)|^{|\operatorname{ord}_\beta(G_N - G_M)|}}{\epsilon(v)^{|\operatorname{ord}_\beta(G_N - G_M)|}} \frac{2_v \mathfrak{d}_v}{\epsilon(v)^{|\operatorname{ord}_\beta(G_N - G_M)|}}
\]
for all \( t \in \mathbb{D}_v(\beta; \epsilon(v)) \). In particular, if \( \delta_2 < \epsilon(v) \), we have
\[
|G_M(t) - G_N(t)|_v \leq 2_v \mathfrak{d}_v \left( \frac{\delta_2}{\epsilon(v)} \right)^{|\operatorname{ord}_\beta(G_N - G_M)|}
\]
on the disk \( \mathbb{D}_v(\beta; \delta_2) \). We’ve seen that \( \operatorname{ord}_\beta(G_N - G_M) \to \infty \) as \( \min\{N, M\} \to \infty \), and so the sequence \( G_N \) is uniformly Cauchy on this disk. In particular, we have \( G_N \to G_\infty \) uniformly on this domain.

Now, consider the case where \( a_d(0) = 0 \). Shrinking \( \epsilon \) if necessary, we may assume that \( |a_d(t)^{-1}|_v \geq 1 \) for all \( t \in \mathbb{D}_v(\beta; \epsilon(v)) \), and so for some \( M_k \)-divisor \( \mathfrak{d} \), we have
\[
|G_M(t) - G_N(t)|_v \leq 2_v \mathfrak{d}_v |a_d(t)|_v^{-1/(d-1)}
\]
on \( \mathbb{D}_v(\beta; \epsilon(v)) \). Applying the Schwarz Lemma to \( a_d(G_N - G_M)^{(d-1)} \), we find that
\[
|G_M(t) - G_N(t)|_v \leq 2_v \mathfrak{d}_v |w_\beta(t)|^{|\operatorname{ord}_\beta(G_N - G_M)|} + \frac{1}{\operatorname{ord}_\beta(a_d)} |a_d(t)|_v^{-1/(d-1)}.
\]
Supposing that \( w_\beta^{\operatorname{ord}_\beta(a_d)} a_d^{-1} \) is bounded on \( \mathbb{D}_v(\beta; \epsilon(v)) \) by \( f(v) \), we have for all \( t \) with \( |w_\beta(t)|_v < \delta_2 \),
\[
|G_M(t) - G_N(t)|_v \leq 2_v \mathfrak{d}_v \left( \frac{\delta_2}{\epsilon(v)} \right)^{|\operatorname{ord}_\beta(G_N - G_M)|} f(v)^{-\frac{1}{d-1}}.
\]
Again we see that \( G_M - G_N \to 0 \) uniformly as \( \min\{N, M\} \to \infty \), and so \( G_N \to G_\infty \) uniformly on \( \mathbb{D}_v(\beta; \delta_2) \).

Since \( w_\beta \) is analytic on the annulus \( \mathbb{D}_v(\beta; \delta_1, \delta_2) \), for any \( 0 < \delta_1 < \delta_2 \), and since \( \mathcal{G}_\beta = w_\beta^{-m} G_\infty \), we see that \( \mathcal{G}_\beta \) is analytic on this annulus, and is the uniform limit of
\[
w_\beta^{-m} G_N = \left( f^N(P) a_d^{-(dN-1)/(d-1)} \right)^{1/dN}.
\]
Since \( \mathcal{G}_\beta \) is analytic on any annulus of this form, it is analytic on all of \( \mathbb{D}_v(\beta; 0, \epsilon(v)) \).

4. Proof of Theorem 1

First of all, we dispatch the somewhat pathological case where \( D = D(f, P) = 0 \). Note that Theorem 1 says nothing at all in this case, since \( D \) is of empty support.

**Lemma 12.** Theorem 1 holds in the case where \( D(f, P) = 0 \).

*Proof.* One possible case in which \( D(f, P) = 0 \) is the case in which \( P \) is preperiodic for \( f \), that is, the case where \( f^m(P) = f^n(P) \) for some \( m > n \geq 0 \). In this case, \( f^m(P_t) = f^n(P_t) \) for all \( t \in X(\kbar) \), and so the set \( \{ f^N(P_t) : N \geq 0 \} \) is finite, for each \( t \in X(\kbar) \). It follows immediately that
\[
\hat{h}_{f_t}(P_t) = \lim_{N \to \infty} d^{-N} h(f^N_t(P_t)) = 0
\]
identically on \( X(\kbar) \). The inequality (3), in this case, is trivial.
Suppose that $P$ is not preperiodic for $f$, but that $D(f, P) = 0$, and hence $\hat{h}_f(P) = 0$. By a theorem of Benedetto \[^{[3]}\], the polynomial $f$ is (affine) isotrivial. Thus, there exists an affine transformation

$$\psi(z) = \alpha z + \beta$$

with $\alpha \neq 0$, and $\alpha, \beta \in \overline{K}$, such that $\psi \circ f \circ \psi^{-1} \in \overline{k}[z]$. In other words, there is a dominant morphism $\phi : Y \to X$ defined over $\overline{K}$, $\alpha, \beta \in \overline{k}(Y)$, and $g \in \overline{k}[z]$ such that

$$f_{\phi(s)}(z) = \psi_s^{-1} \circ g \circ \psi_s(z).$$

If we let $Q = \psi(P \circ \phi) \in \overline{K}(Y)$, and fix any $\gamma \in Y(\overline{K})$, then

$$\text{ord}_\gamma(g^N(Q)) = \text{ord}_\gamma(\alpha(f^N(P) \circ \phi) + \beta)$$

$$\geq \min \left\{ \text{ord}_\gamma(\alpha) + \text{ord}_\gamma((f^N(P) \circ \phi), \text{ord}_\gamma(\beta) \right\}$$

$$= \min \left\{ \text{ord}_\gamma(\alpha) + \text{ord}_\gamma(\phi(f^N(P)) + \text{ord}_\gamma(\beta) \right\}$$

with equality in \((9)\) if the two terms in the minimum are distinct (here $e(\phi)$ is the ramification index of $\phi$ at $\gamma$). If $\hat{\lambda}_{f, \phi(\gamma)}(P) > 0$, then $\text{ord}_\gamma(f^N(P))$ decreases without bound as $N \to \infty$. It follows that for $N$ sufficiently large we have

$$\text{ord}_\gamma(g^N(Q)) = \text{ord}_\gamma(\alpha) + e(\phi) \text{ord}_\gamma(f^N(P)),$$

and hence

$$\hat{\lambda}_{g, \gamma}(Q) = e(\phi) \hat{\lambda}_{f, \phi(\gamma)}(P).$$

On the other hand, if $\hat{\lambda}_{f, \phi(\gamma)}(P) = 0$, then $\text{ord}_\gamma(f^N(P))$ is bounded as $N \to \infty$, and so $\text{ord}_\gamma(g^N(Q))$ is bounded as well; it follows that $\hat{\lambda}_{g, \gamma}(Q) = 0$. In other words, we have shown that

$$D(g, Q) = \sum_{\gamma \in Y(\overline{K})} \hat{\lambda}_{g, \gamma}(Q)(\gamma) = \sum_{\beta \in X(\overline{K})} \hat{\lambda}_{f, \beta}(P) \left( \sum_{\gamma \in \phi^{-1}(\beta)} e(\phi)(\gamma) \right) = \phi^*D(f, P).$$

This is true in general, but in particular $D(f, P) = 0$ implies $D(g, Q) = 0$. It is easy to see that if $g \in \overline{k}[z]$, then $D(g, Q) = Q^*(\infty)$, and so $D(g, Q) = 0$ implies that $Q$ is constant.

Now, for each fixed $s \in Y(\overline{K})$, $\psi_s^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$ is a morphism of degree 1, and so

$$h \left( f_{\phi(s)}^N(P_{\phi(s)}) \right) = h \left( \psi_s^{-1} \circ g^N(Q) \right) = h(g^N(Q)) + O(1),$$

where the implied constant depends on $s$, but not on $N$. Dividing by $D^N$ and letting $N \to \infty$, we have $\hat{h}_f(P_t) = \hat{h}_g(Q)$ for all $t \in X(\overline{K})$, since $\phi$ was dominant, and so $\hat{h}_f(P_t)$ is constant. Since $h_D = 0$, \([3]\) holds.

We now prove a lemma which contains most of the content of Theorems\[^{[1]}\] and \[^{[4]}\]. We set up the notation as above, with $L$ a field, $v \in M_L$ some valuation, $X/L$ a smooth and projective curve, $f(z) \in L(X)[z]$, and $P \in L(X)$. Furthermore, in light of Lemma \[^{[12]}\] we will suppose that $D(f, P) \neq 0$, whereupon

$$\hat{h}_f(P) = \deg(D(f, P)) > 0.$$

**Lemma 13.** There is an $M_L$-divisor $b$ such that

$$\left| \hat{\lambda}_{f, v}(P_t) - \lambda_{D, v}(t) \right| \leq \log b(v)$$
for all \( t \in X(\overline{T}_v) \) and all \( v \in M_L \) (in particular, the difference vanishes identically at all but finitely many places). Furthermore there is an integer \( N \) such that for each \( \beta \in \text{Supp}(D) \) there is a germ \( E_{\beta} \in \hat{O}_{\beta,X} \), and an \( M_L \)-divisor \( \epsilon \) such that \( E_{\beta} \) is \( v \)-adic analytic on \( D_v(\beta; \epsilon(v)) \), and

\[
\hat{\lambda}_{f,v}(P_t) - \lambda_{D,v}(t) = \frac{1}{d^N(d-1)} \log |E_{\beta}(t)|_v,
\]
on \( D_v(\beta; \epsilon(v)) \).

Proof. Let \( N \) be the integer chosen in Lemma 11 which we may take as the least non-negative integer with the property that

\[
d^N(d-1) \deg(D(f,P)) \geq 2g(X)
\]
and \( f^N(P) \in B^0_{\beta}(f) \) for all \( \beta \in \text{Supp}(D) \). We have

\[
\hat{\lambda}_{f,v}(f^N(P_t)) = d^N \hat{\lambda}_{f,v}(P_t)
\]
and (by definition)

\[
\lambda_{D(f,f^N(P)),v}(t) = d^N \lambda_{D(f,P),v}(t),
\]
and so the general case clearly follows from the special case where \( N = 0 \). Consequently, we will suppose throughout that \( N = 0 \).

To begin, set \( Z \subseteq X(\overline{L}) \) to be a finite set of points containing all of the poles of \( P \) and of the \( a_i \), and at each \( \beta \in Z \) we fix a uniformizer \( w_{\beta} \in L(X) \). Recall that, for each \( \beta \), we have an inclusion \( L(X) \hookrightarrow L((w_{\beta})) \), and we will associate functions with their images (their Laurent series).

Choose a \( \beta \in \text{Supp}(D) \). By Lemma 11 there is an \( M_L \)-divisor \( \epsilon_{\beta} \) such that the formal limit \( G_{\beta} \in L((w_{\beta})) \) of \( (f^N(P) a_d^{-\left(d^N-1)/(d-1)\right)} \) defines a \( v \)-adic analytic function on \( D_v(\beta; 0, \epsilon_{\beta}(v)) \) with a pole of order \( m = - \text{ord}_{\beta}(P) \) at \( \beta \). For simplicity, and since \( \beta \) is fixed, we will drop the subscripts.

Now, as in the proof of Lemma 11 we may suppose that \( \epsilon(v) \) is small enough that \( P_t \in B^0_{\beta}(f_t) \) for all \( t \in D_v(\beta; 0, \epsilon(v)) \). By the definition of \( B^0_{\beta}(f_t) \), and the fact that this set is closed under \( f_t \), this implies \( |f_t^N(P_t)|_v > 1 \) for all \( N \), and so in particular,

\[
\hat{\lambda}_{f,v}(P_t) = \lim_{N \to \infty} d^{-N} \log |f_t^N(P_t)|_v.
\]
for \( t \in D_v(\beta; 0, \epsilon(v)) \). On the other hand, for each \( t \in D_v(\beta; 0, \epsilon(v)) \) we have by Lemma 11

\[
\log |G(t)|_v = \log \lim_{N \to \infty} \left| f_t^N(P_t) a_d(t)^{-\left(d^N-1)/(d-1)\right)} \right|_v^{1/d^N}
= \lim_{N \to \infty} d^{-N} \log |f_t^N(P_t)|_v - \frac{1}{d-1} \log |a_d(t)|_v
\]
\[
= \hat{\lambda}_{f,v}(P_t) - \frac{1}{d-1} \log |a_d(t)|_v.
\]
Since \( P \in B^0_{\beta}(f) \),

\[
(d-1)\hat{\lambda}_{f,\beta}(P) = (d-1) \log |P|_\beta + \log |a_d|_\beta,
\]
and so the function \( g \) defining the local heights has a pole of this order at \( \beta \). Shrinking \( \epsilon \) again, if necessary, we suppose that \( \epsilon \) is also a radius of convergence.
for the series defining \( g \), and that \( |g(t)|_v \geq 1 \) for all \( t \in \mathbb{D}_v(\beta; 0, \epsilon(v)) \). Thus, for \( t \) in this domain,

\[
\lambda_{D,v}(t) = (d - 1)^{-1} \log |g(t)|_v,
\]

whereupon

\[
(10) \quad \hat{\lambda}_{f,v}(P_t) - \lambda_{D,v}(t) = \log |G(t)|_v + \frac{1}{d - 1} \log |a_d(t)|_v - \frac{1}{d - 1} \log |g(t)|_v
\]

\[
= \frac{1}{d - 1} \log |G(t)^{d - 1}a_d(t)/g(t)|_v
\]

Now, if \( E_\beta = G^{d - 1}a_d/g \in K_\beta \), then \( \text{ord}_\beta(E_\beta) = 0 \), and hence \( E_\beta = 0 \) has a power series representation of the form

\[
E_\beta = b_0 + b_1w + b_2w^2 + \cdots
\]

with \( b_i \in \mathbb{L} \) and \( b_0 \neq 0 \). Since \( G, a_d, \) and \( g \) are \( v \)-adic analytic functions on \( \mathbb{D}_v(\beta; 0, \epsilon(v)) \), and \( g \) has no zeros in this region, \( E_\beta \) is also \( v \)-adic analytic. This is the function \( E_\beta \) in the statement of the lemma.

Note that, by Lemmas 3 and 4 there is a finite set \( S \subseteq M_L \) of places (containing all infinite places) such that for \( v \notin S \), we have \( G, a_d, g \in \mathcal{O}_{v,L}[[w]] \), and the leading coefficients of \( G, a_d, \) and \( g \) are in \( \mathcal{O}_{v,L}^* \). It follows that for \( v \notin S \), we have \( E_\beta \in \mathcal{O}_{v,L}[[w]] \), and \( E_\beta(\beta) = b_0 \in \mathcal{O}_{v,L}^* \). Now, if \( v \notin S \), if \( \epsilon(v) = 1 \), and if \( t \in \mathbb{D}_v(\beta; 0, \epsilon(v)) \), then

\[
\hat{\lambda}_{f,v}(P_t) - \lambda_{D,v}(t) = (d - 1)^{-1} \log |b_0 + w(t)(b_1 + b_2w(t) + \cdots)|_v
\]

\[
= (d - 1)^{-1} \log |b_0|_v = 0.
\]

At the other finitely many places, we may shrink \( \epsilon(v) \) to ensure that \( E_\beta \) has no zeros on \( \mathbb{D}_v(\beta; 2\epsilon(v))_v \). This ensures that \( \log |E_\beta(t)|_v \) is bounded above and below for \( t \in \mathbb{D}_v(\beta; \epsilon(v)) \), and hence so is \( \hat{\lambda}_{f,v}(P_t) - \lambda_{D,v}(t) \). So, we may choose an \( M_L \)-divisor \( \delta_\beta \) such that \( t \in \mathbb{D}_v(\beta; \epsilon_\beta(v)) \) implies

\[
\left| \hat{\lambda}_{f,v}(P_t) - \lambda_{D,v}(t) \right| = (d - 1)^{-1} \log |E_\beta(t)|_v \leq \log \delta_\beta(v).
\]

We construct \( \epsilon_\beta \) and \( \delta_\beta \) in this way, for each \( \beta \in \mathcal{D}(D) \), and set

\[
\delta_1 = \max_{\beta \in \text{Supp}(D)} \delta_\beta.
\]

Now let \( \beta \in Z \setminus \text{Supp}(D) \), and again choose \( \epsilon = \epsilon_\beta \) which is a global radius of convergence for all of the series representing \( P, a_i, \) and \( g \) in \( \mathbb{L}((w_\beta)) \). As above, we choose a set of places \( S \subseteq M_L \) large enough that for any \( v \notin S, P, \) the \( a_i, \) and \( g \) are in \( \mathcal{O}_{v,L}((w_\beta)) \). We will enlarge \( S \), if necessary, to contain the finitely many places \( v \in M_L \) with \( \epsilon(v) \neq 1 \). Since \( f^N(P) \) is a polynomial in \( P \) and the \( a_i \), it is clear that each of these is defined by a series in \( \mathcal{O}_{v,L}((w)) \), for \( v \notin S \), convergent within the same radius \( \epsilon \). Now, for each \( N \), let \( m_N = -\text{ord}_\beta(f^N(P)) \).

First, consider \( v \notin S \), so that \( \epsilon_\beta(v) = 1 \). Then for \( t \in \mathbb{D}_v(\beta; 0, \epsilon(v)) \), we have

\[
|w(t)^m f^N_t(P_t)|_v \leq 1,
\]

since \( w^m f^N(P) \) is regular at \( \beta \), and defined by a series in \( w \) with coefficients which are integral at \( v \). Now let \( \delta > 0 \) be a real number. For \( t \in \mathbb{D}_v(\beta; \delta, \epsilon(v)) \), we have

\[
d^{-N} \log |f^N_t(P_t)| = d^{-N} \left( \log |w(t)^{-m_N}|_v + \log |w(t)^m f^N_t(P_t)|_v \right)
\]

\[
\leq d^{-N} m_N \log \delta^{-1}.
\]
Since \( m \in N \) have \( \hat{\lambda} \), and so \( \hat{\lambda} \) whenever \( \lambda \in B \). On the other hand, \( \Phi \) is a real polynomial with non-negative coefficients, and so is non-decreasing on positive values. Thus, for all \( \beta \in B \),

\[
|w(t)^m f^N(P_t)|_v \leq \max_{|w(t)|_v = \epsilon(v)} |w(t)^m f^N(P_t)|_v = \epsilon_\beta(v)^m |f^N(P_t)|_v.
\]

The same follows for non-archimedean valuations by the non-archimedean maximum modulus principle [9, p. 318], given that our definition of an \( M_L \)-divisor required that \( \epsilon(v) = |\alpha|_v \) for some \( \alpha \in L^* \). Fixing \( v \) for the moment, define

\[
\|j\| = \max_{|w(t)|_v = \epsilon(v)} |j(t)|_v,
\]

for any \( j \in L(X) \) for which this maximum exists, and define \( \Phi(X) \in \mathbb{R}[X] \) by

\[
\Phi(X) = \sum_{i=0}^d |a_i| X^i.
\]

Note that the triangle inequality gives

\[
\|f^{N+1}(P)\| \leq \Phi(\|f^N(P)\|)
\]

On the other hand, \( \Phi \) is a real polynomial with non-negative coefficients, and so is non-decreasing on positive values. Thus, for all \( N \), \( \|f^N(P)\| \leq \Phi^N(\|P\|) \) and so, by Lemma [5] there is a \( B_r \) (which depends on \( \|P\| \)) such that

\[
\|f^N(P)\| \leq \Phi^N(\|P\|) \leq B_r^{dN}
\]

for all \( N \). If we suppose that \( t \in D_v(\beta; \delta, \epsilon(v)) \), then we have

\[
|f^N(P_t)|_v = |w(t)^{-mN}|_v |w(t)^m f^N(P_t)|_v \leq \left( \frac{\epsilon_\beta(v)}{\delta} \right)^m |f^N(P)|_v \leq \left( \frac{\epsilon_\beta(v)}{\delta} \right)^m B_r^{dN}.
\]

Since \( m_N \) is bounded as \( N \to \infty \), taking logarithms and limits gives

\[
\hat{\lambda}_{f,\epsilon}(P_t) \leq \log B_v,
\]

whenever \( t \in D_v(\beta; \delta, \epsilon(v)) \). As \( \delta \) was arbitrary, and \( B_r \) did not depend on \( \delta \), we have \( \hat{\lambda}_{f,\epsilon}(z_t) \leq \log B_v \) for all \( t \in D_v(\beta; 0, \epsilon(v)) \). Let \( n_\beta \) be the \( M_L \)-divisor with value \( B_v \) at each of these places, and 1 everywhere else, so that for any \( v \in M_L \),

\[
\hat{\lambda}_{f,\epsilon}(P_t) \leq \log n(v)
\]

whenever \( t \in D_v(\beta; 0, \epsilon(v)) \).

On the other hand, since the poles of the function \( g \) are all contained in \( \text{Supp}(D) \), there is, by Lemma [8] an \( M_L \)-divisor \( m_1 \) such that \( |g(t)|_v \leq m_1(v) \) whenever
whenever $t \in \mathbb{D}_v(\beta'; e_{\beta'}(v))$, for some $\beta' \in \text{Supp}(D)$, in which case we have

$$\left| \hat{\lambda}_{f_t,v}(z_t) - \lambda_{D,v}(t) \right| \leq \log d_1(v),$$

or else $\lambda_{D,v}(t) \leq \log m_1(v)$. In the latter case, if $t \in \mathbb{D}_v(\beta; 0, e_{\beta}(v))$, then we have shown that $\hat{\lambda}_{f_t,v}(P_t) \leq \log n_\beta(v)$. Combining these, we have

$$\left| \hat{\lambda}_{f_t,v}(z_t) - \lambda_{D,v}(t) \right| \leq \log m_1(v) + \log n_\beta(v).$$

Constructing $e_\beta$ and $n_\beta$ as above, for each $\beta \in Z \setminus \text{Supp}(D)$, and letting

$$d_2(v) = \max_{\beta \in Z \setminus \text{Supp}(D)} \{ d_1(v), m_1(v)n_\beta(v) \},$$

then, we have

$$\left| \hat{\lambda}_{f_t,v}(z_t) - \lambda_{D,v}(t) \right| \leq \log d_2(v)$$

whenever $t \in \mathbb{D}_v(\beta; e_{\beta}(v))$, for some $\beta \in Z$.

For each $\beta \in Z$ we have chosen an $M_L$-divisor $e_{\beta}$. Since none of the functions $P$, $a_i$, and $g$ have any poles outside of $Z$, we employ Lemma 8 to construct an $M_L$-divisor $m_2$ such that for each $v \in S$, the condition $|j(t)|_v > m_2(v)$ for any of the functions $j \in \{ P, a_i, g \}$, implies $|w_{\beta}(t)|_v < e_{\beta}(v)$ for some $\beta \in Z$. For any place $v$, let $Y_v \subseteq X(\overline{\mathbb{Q}}_v)$ be the set of points $t$ such that

$$\max\{|P_t|_v, |a_i(t)|_v, |g(t)|_v\} \leq m_2(v).$$

Then we have, as above, $M_L$-divisors $m_3$ and $m_4$ such that for $t \in Y_v$,

$$\hat{\lambda}_{f_t,v}(P_t) \leq \log m_3(v) \quad \text{and} \quad \lambda_{D,v}(t) \leq \log m_4(v).$$

That is,

$$\left| \hat{\lambda}_{f_t,v}(P_t) - \lambda_{D,v}(t) \right| \leq \log d_3(v),$$

for $d_3 = m_3 m_4$. On the other hand, if $t \notin Y_v$, then $|w_{\beta}(t)|_v < e_{\beta}(v)$ for some $\beta \in Z$. By the previous two arguments, we have

$$\left| \hat{\lambda}_{f_t,v}(P_t) - \lambda_{D,v}(t) \right| \leq \log d_i(v)$$

for $i = 1$ or $2$. Letting $d_4 = \max_{1 \leq i \leq 3} d_i$, pointwise, we have

$$\left| \hat{\lambda}_{f_t,v}(P_t) - \lambda_{D,v}(t) \right| \leq \log d_4(v)$$

for all $t \in X(\overline{\mathbb{Q}}_v)$.

Theorem 1 is an immediate consequence of the above lemma.

**Proof of Theorem 1.** Let $k$, $X$, $K$, $f$, and $P$ be as in the statement of Theorem 1. Then, if $b$ be the $M_k$-divisor prescribed by Lemma 13 and $L/k$ is any finite Galois extension, we have for any $t \in X(L)$,

$$\hat{h}_f(P_t) = \sum_{v \in M_k} \left[ k_v : Q_v \right] \left( \frac{1}{[L : k]} \sum_{\sigma \in \text{Gal}(L/k)} \hat{\lambda}_{f_{\sigma},v}(P_{t^\sigma}) \right)$$

and

$$h_D(t) = \sum_{v \in M_k} \left[ k_v : Q_v \right] \left( \frac{1}{[L : k]} \sum_{\sigma \in \text{Gal}(L/k)} \lambda_{D,v}(t^\sigma) \right),$$

where $[k_v : Q_v]$ is the degree of the extension of $k_v$ over $Q_v$, and $\hat{\lambda}_{f_{\sigma},v}(P_{t^\sigma})$ and $\lambda_{D,v}(t^\sigma)$ are the canonical heights of $f_{\sigma}(P_{t^\sigma})$ and $D_{t^\sigma}$ in $X(L_{\sigma})$. The theorem follows from the definitions and the properties of canonical heights.
and hence
\[
\left| \hat{h}_{f_i}(P_t) - \hat{h}_D(t) \right| = \left| \sum_{v \in \mathcal{M}_k} \frac{[k_v : Q_v]}{[k : Q]} \sum_{\sigma \in \text{Gal}(L/k)} \frac{\hat{\lambda}_{f_i,v}(P_{t^\sigma}) - \lambda_{D,v}(t^\sigma)}{[L : k]} \right|
\]
\[
\leq \sum_{v \in \mathcal{M}_k} \frac{[k_v : Q_v]}{[k : Q]} \sum_{\sigma \in \text{Gal}(L/k)} \frac{\hat{\lambda}_{f_i,v}(P_{t^\sigma}) - \lambda_{D,v}(t^\sigma)}{[L : k]}
\]
\[
\leq \sum_{v \in \mathcal{M}_k} \frac{[k_v : Q_v]}{[k : Q]} \sum_{\sigma \in \text{Gal}(L/k)} \log b(v) \frac{[L : k]}{[L : k]}
\]
\[
= \sum_{v \in \mathcal{M}_k} \frac{[k_v : Q_v]}{[k : Q]} \log b(v).
\]
This final sum is finite, and independent of both \( t \) and \( L \). Since \( L/k \) was arbitrary, we have
\[
\hat{h}_{f_i}(P_t) = h_D(t) + O(1),
\]
for \( t \in X(\overline{k}) \). \( \square \)

To prove the corollary, we use an argument due to Lang.

Proof of Corollary 4. Let \( \eta \in \text{Pic}(X) \) have degree 1, and let \( h_\eta \) be a height relative to this divisor. By the linearity of heights,
\[
h_D(f,P) - \hat{h}_f(P)h_\eta = h_D(f,P)_{|\hat{h}_f(P)\eta} + O(1).
\]

Note that the divisor \( D(f,P) - \hat{h}_f(P)\eta \) has degree 0. In general, by [7, Proposition 5.4, p. 115], we have
\[
h_D(f,P)_{|\hat{h}_f(P)\eta} = O(h_\eta^{1/2}).
\]
If \( X = \mathbb{P}^1 \), then \( D(f,P) - \hat{h}_f(P)\eta \) is linearly equivalent to the zero divisor, and so
\[
h_D(f,P)(t) - \hat{h}_f(P)h_\eta(t) = O(1).
\]
\( \square \)

5. PROOF OF THEOREMS 4 AND 4

We will begin by proving Theorem 4. Theorem 3 follows by clarifying some of the details of this proof in the special case where \( f_i(z) = z^2 + t \), and \( P_t \in \mathbb{Z}[t] \).

Proof of Theorem 4. Let \( k, X, f, \) and \( P \) be as in the theorem. By Lemma 13 there is a finite set of places \( S \subseteq M_k \), containing all infinite places, such that for \( v \not\in S \), we have
\[
\hat{\lambda}_{f_i,v}(P_t) = \lambda_{D,v}(t),
\]
for all \( t \in X(\overline{k}) \). Again by Lemma 13 we may choose, for each \( \beta \in \text{Supp}(D) \), a germ \( E_\beta \in \hat{\mathcal{O}}_{\beta,X} \), defined over \( k \), and an \( M_k \)-divisor \( e_\beta \) such that
\[
\hat{\lambda}_{f_i,v}(P_t) - \lambda_{D,v}(t) = \frac{1}{d^N(d-1)} \log |E_\beta(t)|_v,
\]
for all \( t \in D_v(\beta; e(v)) \). We will enlarge \( S \), if necessary, to ensure that \( E_\beta(\beta) \) is an \( S \)-unit for all \( \beta \in \text{Supp}(D) \). Let \( \tilde{E}_\beta(t) = E_\beta(t)/E_\beta(\beta) \), so that \( \tilde{E}_\beta \) is \( v \)-adic analytic.
on $D_v(\beta; \epsilon(v))$, and $\tilde{E}_{\beta}(\beta) = 1$. Since $1^{dN(d-1)} \equiv \tilde{E}_{\beta} \pmod{m}$, for $m$ the maximal ideal of $\mathcal{O}_{\beta,X}$, we see (by Hensel’s Lemma) that there is a $\tilde{F}_\beta \in \mathcal{O}_{\beta,X}$ such that

\[ \tilde{F}_\beta(t) = 1 + O(m) \quad \text{and} \quad \tilde{F}_\beta^{dN(d-1)} = E_{\beta}. \]

The germ $\tilde{F}_\beta$ defines a $v$-adic analytic function on $D_v(\beta; \epsilon'(v))$, for some $M_k$-divisor $\epsilon'$ and, shrinking $\epsilon'(v)$ at finitely many places if necessary, we may assume that $|\tilde{F}_\beta(t)|_v = 1$ for all $t \in D_v(\beta; \epsilon'(v))$, whenever $v \in S$ is non-archimedean.

If $\phi : S \rightarrow \text{Supp}(D)$, and $t \in X(k)$ satisfies $t \in D_v(\phi(v); \epsilon'(v))$ for each $v \in S$, then we have

\[
\hat{h}_{f_t}(P_t) - h_D(t) = \sum_{v \in M_k} \left[ \frac{k_v : Q_v}{[k : Q]} \left( \hat{\lambda}_{f_t,v}(P_t) - \lambda_{D,v}(t) \right) \right] = \sum_{v \in S} \left[ \frac{k_v : Q_v}{[k : Q]} \left( \hat{\lambda}_{f_t,v}(P_t) - \lambda_{D,v}(t) \right) \right]
\]

\[ = \sum_{v \in S} \left[ \frac{k_v : Q_v}{[k : Q]} \frac{1}{dN(d-1)} \log |E_{\phi(v)}(t)|_v \right] = \sum_{v \in S} \left[ \frac{k_v : Q_v}{[k : Q]} \log |\tilde{F}_{\phi(v)}(t)|_v + \frac{1}{dN(d-1)} \log |E_{\phi(v)}(\phi(v))|_v \right] = \sum_{v \mid \infty} \left[ \frac{k_v : Q_v}{[k : Q]} \log |\tilde{F}_{\phi(v)}(t)|_v \right] + C(\phi),
\]

where

\[ C(\phi) = \frac{1}{dN(d-1)} \sum_{v \in S} \left[ \frac{k_v : Q_v}{[k : Q]} \log |E_{\phi(v)}(\phi(v))|_v \right]. \]

Since $\tilde{F}_\beta$ is $v$-adic analytic, for each $v \mid \infty$, and $\tilde{F}_\beta(t) = 1$, we may shrink $\epsilon'(v)$ again to ensure that $\tilde{F}_\beta(t) \neq 0$ for $t \in D_v(\beta; \epsilon'(v))$. It follows that

\[ F_{\beta,v}(t) = \left[ \frac{k_v : Q_v}{[k : Q]} \right] \log |\tilde{F}_\beta(t)|_v \]

is real analytic on $D_v(\beta; \epsilon'(v)) \subseteq X(\mathcal{O}_v)$. Since $\tilde{F}_\beta(\beta) = 1$, we have $F_{\beta,v}(\beta) = 0$.

If $k_v = \mathbb{R}$, then since

\[ \tilde{F}_\beta = 1 + c_1w + c_2w^2 + \cdots, \]

the function

\[ F_{\beta,v} = \left[ \frac{k_v : Q_v}{[k : Q]} \right] \left( c_1w + \left( c^2 - \frac{c_1^2}{2} \right)w^2 + \left( c_3 - c_1c_2 + \frac{c_1^3}{3} \right)w^3 + \cdots \right) \]

is given by a power series in $w$ with coefficients in $k$. Similarly, if $k_v = \mathbb{C}$, and the disk $|w|_v < \epsilon'(v)$ is identified with a disk in $\mathbb{R}^2$ by $w = x + iy$, then $\tilde{F}_{\beta,v} \in k[x,y]$.

The result is easily extended to a theorem quantified over $X(\mathcal{O}_v)$. Let $L/k$ be a Galois extension, and let $\phi : S \times \text{Gal}(L/k) \rightarrow \text{Supp}(D)$. For any $t \in X(L)$ satisfying $t^\sigma \in D_v(\phi(v,\sigma); \epsilon'(v))$, for all $v \in S$ and $\sigma \in \text{Gal}(L/k)$, we have

\[
\hat{h}_{f_t}(P_t) - h_D(t) = \frac{1}{[L : Q]} \sum_{v \mid \infty} F_{\phi(v,\sigma)}(t^\sigma) + C(\phi),
\]
where
\[
C(\phi) = \frac{1}{d^N(d-1)} \sum_{\sigma \in \text{Gal}(L/k)} \sum_{v \in S} \left[ \frac{k_v : Q_v}{[L : Q]} \right] \log |E_{\phi(v, \sigma)}(\phi(v, \sigma))| 
\]
by the same argument as above.

\[\square\]

Remark. It should be pointed out that, in the statement of Theorem 4, it is entirely possible that for certain \(\phi : S \to \text{Supp}(D)\), there will be no \(t \in X(k)\) satisfying \(t \in D_v(\phi(v); c'(v))\) for all \(v \in S\). In particular, suppose that \(k = \mathbb{Q}\), \(X = \mathbb{P}^1\), and that \(D = (i) + (-i)\), for \(i^2 = -1\) (this \(D\) arises, for example, when \(f(z) = z^2 + t\) and \(P_t = t^3(t^2 + 1)^{-1}\)). Then for \(\varepsilon\) small enough, there is no \(t \in \mathbb{P}^1(\mathbb{Q})\) with \(t \in D_\infty(i; \varepsilon)\). In this case, Theorem 4 is vacuously true, but becomes non-trivial after a finite extension.

Proof of Theorem 3. Let \(f_t(z) = z^2 + t\), and let \(P \in \mathbb{Z}[t]\) be a polynomial of degree at least one, and leading coefficient \(\alpha\).

Note that \(\text{ord}_\beta(P), \text{ord}_\beta(t) \geq 0\) for all \(\beta \neq \infty = [1 : 0] \in \mathbb{P}^1\). It follows that \(\hat{\lambda}_{f_t,\beta}(P) = 0\) for all \(\beta \neq \infty\). On the other hand, \(\text{ord}_\infty(P) = \text{deg}(P) \geq 1 > \frac{1}{2} = \frac{1}{2} \text{ord}_\infty(t)\), and consequently, \(P \in B^0_\infty(f)\). It follows at once that
\[
D(f, P) = \hat{\lambda}_{f,\infty}(P)(\infty) = \text{deg}(P)(\infty),
\]
and so \(h_D\) can be taken to be \(\text{deg}(P)h_t\), for \(h\) the usual Weil height on \(\mathbb{P}^1\). Similarly, the Néron functions can be taken to be
\[
\lambda_{D,v}(x) = \max\{0, \log |x^{\text{deg}(P)}|_v\}.
\]

Now, for each non-archimedean \(v \in M_k\), if \(|t|_v \leq 1\), then \(|f_t^N(P_t)|_v \leq 1\) for all \(N\), since \(P\) and \(f\) have integral coefficients. In this case, we have
\[
\hat{\lambda}_{f_t,v}(P_t) = \lambda_{D,v}(t).
\]
If, on the other hand, \(|t|_v > \max\{1, |\alpha|_v^{-1}\}\) then we have (since the coefficients of \(P\) are integral)
\[
|P_t|_v = |\alpha|_v |t|_v^{\text{deg}(P)} \geq |t|_v,
\]
whence \(P_t \in B^0_v(f_t)\). It follows from Lemma 5 that
\[
\hat{\lambda}_{f_t,v}(P_t) = \log |\alpha t^{\text{deg}(P)}|_v = \lambda_{D,v}(t) + \log |\alpha|_v
\]
in this case. Thus, if we consider only \(t \in X(\mathbb{Q})\) such that \(|t|_v > \max\{1, |\alpha|_v^{-1}\}\) for all \(v \in M_k\) with \(|\alpha|_v \neq 1\) it follows that
\[
\hat{h}_{f_t}(P_t) - h_D(t) = \hat{\lambda}_{f_t,\infty}(P_t) - \lambda_{D,\infty}(t) - \log |\alpha|,
\]
by the product formula. Now, just as in the proof of Theorem 4, we see that (taking \(u(t) = t^{-1}\) as a uniformizer)
\[
\hat{\lambda}_{f,\infty}(P_t) - \lambda_{D,\infty}(t) = \log |G_\infty(t)^{-1} t^{-\text{deg}(P)}|,
\]
where \(G_\infty(t) \in \mathbb{Q}[t^{-1}]\) is analytic in a punctured neighbourhood of \(\infty\), and
\[
G_\infty(t) = \alpha t^{\text{deg}(P)} + O(t^{\text{deg}(P)-1})
\]
by construction, the leading coefficient being that of $P$. We have
\[ G_\infty(t)^{\deg(P)} = \alpha + O(t^{-1}), \]
and so if we choose a neighbourhood small enough that $G_\infty(t)^{\deg(P)} \neq 0$, for all $t$ in the neighbourhood, we have
\[ \log |G_\infty(t)^{\deg(P)}| = \log |\alpha| + F(t^{-1}), \]
for some $F(x) \in \mathbb{Q}[x]$ with $F(0) = 0$. Thus, for all $t \in X(\mathbb{Q})$ in this real neighbourhood of $\infty$, and such that $|t|_v > \max\{1, |\alpha|_v^{-1}\}$ for all $v \in M^0_\mathbb{Q}$ with $|\alpha|_v \neq 1$, we have
\[ \hat{h}_{\phi_t}(P_t) - h_D(t) = F \left( \frac{1}{t} \right). \]
Note that the second condition is satisfied vacuously by all $t \in X(\mathbb{Q})$ if, for example, $P$ is monic.

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