Iterative solution of a nonlinear static beam equation

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Abstract The paper deals with a boundary value problem for the nonlinear integro-differential equation
\[ u''''(x) - m(\int_0^l u'^2 \, dx) u''(x) = f(x, u, u'), \quad m(z) \geq \alpha > 0, \quad 0 \leq z < \infty, \]
modelling the static state of the Kirchhoff beam. The problem is reduced to a nonlinear integral equation which is solved using the Picard iteration method. The convergence of the iteration process is established and the error estimate is obtained.

Keywords: Kirchhoff type beam equation, Picard iteration method, error estimate.

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1. Statement of the Problem

Let us consider the nonlinear beam equation
\[ u'''(x) - m \left( \int_0^l u'^2 \, dx \right) u''(x) = f(x, u(x), u'(x)), \quad x \in (0, l), \]
with the conditions
\[ u(0) = u(l) = 0, \quad u''(0) = u''(l) = 0. \]

Here \( u = u(x) \) is the displacement function of length \( l \) of the beam subjected to the action of a force given by the function \( f(x, u, u') \), the function \( m(z) \),
\[ m(z) \geq \alpha > 0, \quad 0 \leq z < \infty, \]
describes the type of a relation between stress and strain. Namely, if the function \( m(z) \) is linear, this means that this relation is consistent with Hooke’s linear law, while otherwise we deal with material nonlinearities.

Equation [1] is the stationary problem associated with the equation
\[ u_{tt} + u_{xxxx} - m \left( \int_0^l u'^2 \, dx \right) u_{xx} = f(x, t, u, u_x), \]
\[ m(z) \geq \text{const} > 0, \]
which for the case where \( m(z) = m_0 + m_1 z, \quad m_0, m_1 > 0 \), and \( f(x, t, u, u_x) = 0 \), was proposed by Woinowsky-Krieger [11] as a model of deflection of an extensible dynamic beam with hinged ends. The nonlinear term \( \int_0^l u'^2 \, dx \) was for first time used by Kirchhoff [3] who generalized D’Alembert’s classical linear model. Therefore [1] is frequently called a Kirchhoff type equation for a static beam.
The problem of construction of numerical algorithms and estimation of their accuracy for equations of type (1) is investigated in [1], [5], [8] and [9]. In [4], the existence of a solution of problem (1), (2) is proved when the right-hand part of equation is written in the form \( q(x)f(x, u, u') \), where \( f \in C([0, l] \times [0, \infty) \times \mathbb{R}) \) is a nonnegative function and \( q \in C[0, l] \) is a positive function.

In the present paper, in order to obtain an approximate solution of the problem (1), (2), an approach is used, which differs from those applied in the above-mentioned references. It consists in reducing the problem (1), (2) by means of Green’s function to a nonlinear integral equation, to solve which we use the iterative process. The condition for the convergence of the method is established and its accuracy is estimated.

The Green’s function method with a further iteration procedure has been applied by us previously also to a nonlinear problem for the axially symmetric Timoshenko plate [6].

2. Assumptions

Let us assume that besides (3) the function \( m(z) \) also satisfies the Lipschitz condition
\[
|m(z_1) - m(z_2)| \leq l_1|z_2 - z_1|, \quad 0 \leq z_1, z_2 < \infty, \quad l_1 = \text{const} > 0.
\]
Suppose that \( f(x, u, v) \in L_2 ((0, l), \mathbb{R}, \mathbb{R}) \) and, additionally, that the inequalities
\[
|f(x, u, v)| \leq \sigma_1(x) + \sigma_2(x)|u| + \sigma_3(x)|v|, \quad (4)
\]
\[
|f(x, u_2, v_2) - f(x, u_1, v_1)| \leq l_2(x)|u_2 - u_1| + l_3(x)|v_2 - v_1|, \quad (5)
\]
where
\[
0 < x < l, \quad u, v, u_i, v_i \in \mathbb{R}, i = 1, 2, \quad \sigma_1(x) \in L_2(0, l), \quad \sigma_i(x), l_i(x) \in L_{\infty}(0, l), \quad i = 2, 3,
\]
\[
\sigma_1(x) \geq \text{const} > 0, \quad \sigma_i(x) \geq 0, \quad l_i(x) > 0, \quad i = 2, 3,
\]
are fulfilled.

We impose one more restriction on the beam length \( l \) and the parameters \( \alpha \) and \( \sigma_2(x), \sigma_3(x) \) from the conditions (3) and (4), (5) in the form
\[
\omega = \alpha + \left( \frac{\pi}{l} \right)^2 - \frac{l}{\pi} \left( \frac{2}{\pi} \left\| \sigma_2(x) \right\|_{\infty} + \left\| \sigma_3(x) \right\|_{\infty} \right) > 0. \quad (6)
\]
Let us assume that there exists a solution of the problem (1), (2) and \( u \in W^{2,2}_0(0, l) \) [2].

3. The Method

We will need the Green function for the problem
\[
v''''(x) - av''(x) = \psi(x),
\]
\[
0 < x < l, \quad a = \text{const} > 0,
\]
\[
v(0) = v(l) = 0, \quad v''(0) = v''(l) = 0. \quad (7)
\]
In order to obtain this function, we split problem (7) into two problems
\[
w''(x) - aw(x) = \psi(x),
\]
\[
w(0) = w(l) = 0.
\]
and

\[ v''(x) = w(x), \]
\[ v(0) = v(l) = 0. \]

Calculations convince us that

\[ w(x) = -\frac{1}{\sqrt{a \sinh(\sqrt{a}l)}} \left( \int_0^x \cosh(\sqrt{a}(x-l)) \cosh(\sqrt{a}\xi) \psi(\xi) d\xi + \int_x^l \cosh(\sqrt{a}x) \cosh(\sqrt{a}(\xi - l)) \psi(\xi) d\xi \right), \]
\[ v(x) = \frac{1}{l} \left( \int_0^x (x-l) w(\xi) d\xi + \int_x^l x(\xi - l) w(\xi) d\xi \right). \]

Substituting the first of these formulas into the second and performing integration by parts, we obtain

\[ v(x) = \frac{1}{a} \left( \int_0^x (k_1(l - x)\xi + k_2 \sinh(\sqrt{a}(x-l)) \sinh(\sqrt{a}\xi) \psi(\xi)) d\xi + \int_x^l (k_1 x(\xi - l) + k_2 \sinh(\sqrt{a}x) \sinh(\sqrt{a}(\xi - l)) \psi(\xi))) d\xi \right), \]
\[ k_1 = \frac{1}{l}, \quad k_2 = \frac{1}{\sqrt{a \sinh(\sqrt{a}l)}}. \]

The application of (7) to problem (1), (2) makes it possible to replace the latter problem by the integral equation

\[ u(x) = \int_0^l G(x, \xi) f(\xi, u(\xi), u'(\xi)) d\xi, \quad 0 < x < l, \quad (8) \]

where

\[ G(x, \xi) = \frac{1}{\tau} \begin{cases} \frac{1}{l} (x - l)\xi + \frac{1}{\sqrt{\tau} \sinh(\sqrt{\tau} l)} \sinh(\sqrt{\tau}(x-l)) \sinh(\sqrt{\tau}\xi), & 0 < \xi \leq x < l, \\ \frac{1}{l} x(\xi - l) + \frac{1}{\sqrt{\tau} \sinh(\sqrt{\tau} l)} \sinh(\sqrt{\tau}x) \sinh(\sqrt{\tau}(\xi - l)), & 0 < x \leq \xi < l, \end{cases} \]
\[ \tau = m \left( \int_0^l u^2(x) \, dx \right). \]

The equation (8) is solved by the method of the Picard iterations. After choosing a function \( u_0(x), \) \( 0 \leq x \leq l, \) which together with its second derivative vanish for \( x = 0 \) and \( x = l, \) we find subsequent approximations by the formula

\[ u_{k+1}(x) = \int_0^l G_k(x, \xi) f(\xi, u_k(\xi), u'_k(\xi)) d\xi, \quad 0 < x < l, \quad k = 0, 1, \ldots \quad (9) \]

where

\[ G_k(x, \xi) = \frac{1}{\tau_k} \begin{cases} \frac{1}{l} (x - l)\xi + \frac{1}{\sqrt{\tau_k} \sinh(\sqrt{\tau_k} l)} \sinh(\sqrt{\tau_k}(x-l)) \sinh(\sqrt{\tau_k}\xi), & 0 < \xi \leq x < l, \\ \frac{1}{l} x(\xi - l) + \frac{1}{\sqrt{\tau_k} \sinh(\sqrt{\tau_k} l)} \sinh(\sqrt{\tau_k}x) \sinh(\sqrt{\tau_k}(\xi - l)), & 0 < x \leq \xi < l, \end{cases} \]
\[ \tau_k = m \left( \int_0^l u_k^2(x) \, dx \right), \]
and \( u_k(x) \) is the \( k \)th approximation of the solution of equation (8).

4. The Equation for the Method Error

Our aim is to estimate the error of the method, by which we understand the difference between the approximate and exact solutions

\[ \delta u_k(x) = u_k(x) - u(x), \quad k = 0, 1, \cdots. \]  

For this, it is advisable to use not formula (9), but the system of equalities

\[ u''''_{k+1}(x) - m \left( \int_0^l u_k^2(x) \, dx \right) u''_{k+1}(x) = f(x, u_k(x), u'_k(x)), \]  

\[ u_k(0) = u_k(l) = 0, \quad u''_k(0) = u''_k(l) = 0, \]  

which follows from (9).

If we subtract the respective equalities in (1) and (2) from (11) and (12), then we get

\[ \delta u''''_k(x) - \frac{1}{2} \left[ m \left( \int_0^l u_{k-1}^2(x) \, dx \right) + m \left( \int_0^l u'^2(x) \, dx \right) \right] \delta u''_k(x) + \]

\[ + \left[ m \left( \int_0^l u_{k-1}^2(x) \, dx \right) - m \left( \int_0^l u'^2(x) \, dx \right) \right] \left( u''_k(x) + u''(x) \right) = \]

\[ = f(x, u_{k-1}(x), u'_{k-1}(x)) - f(x, u(x), u'(x)), \]

\[ \delta u_k(0) = \delta u_k(l) = 0, \quad \delta u''(0) = \delta u''(l) = 0, \quad k = 1, 2, \cdots. \]

We will come back to (13), (14) to estimate the error of method (9). In meantime we have to derive several a priori estimates.

5. Auxiliary Inequalities

Let

\[ \|u(x)\|_p = \left( \int_0^l \left( \frac{d^p u}{dx^p}(x) \right)^2 \, dx \right)^{1/2}, \quad p = 0, 1, 2, \quad \|u(x)\| = \|u(x)\|_0. \]  

The symbol \((\cdot, \cdot)\) is understood as a scalar product in \( L_2(0, l) \).

**Lemma 1.** The following estimates are true

\[ \|u(x)\| \leq \frac{l}{\pi} \|u(x)\|_1, \quad \|u(x)\|_1 \leq \frac{l}{\pi} \|u(x)\|_2, \]

respectively for \( u(x) \in W^{1,2}_0(0, l) \) and \( u(x) \in W^{2,2}(0, l) \cap W^{1,2}_0(0, l) \).
Proof. Indeed, the first estimate of (16) is Friedrich’s inequality (see, e.g. [7], p. 192). Applying this inequality and taking into account that

\[ \|u(x)\|^2_1 = u(x)u'(x)|_0^l - (u(x), u''(x)) = -(u(x), u''(x)) \leq \|u(x)\| \|u(x)\|_2 \]

we get the second inequality of (16).

Lemma 2. The inequality

\[ \|f(x, u(x), u'(x))\| \leq \|\sigma_1(x)\| + \left( \frac{l}{\pi} \|\sigma_2(x)\|_\infty + \|\sigma_3(x)\|_\infty \right) \|u(x)\|_1 \]  

(17)

is fulfilled for \(u(x) \in W^{1,2}_0(0, l)\).

Proof. By (4) we write

\[ \|f(x, u(x), u'(x))\| \leq \|\sigma_1(x)\| + \|\sigma_2(x)\|_\infty \|u(x)\| + \|\sigma_3(x)\|_\infty \|u'(x)\| \]

Recall also (16). The result is (17).

Lemma 3. For the solution of problem (1), (2) we have the inequality

\[ \|u(x)\|_1 \leq c_1, \]  

(18)

where

\[ c_1 = \frac{l}{\omega \pi} \|\sigma_1(x)\|. \]  

(19)

Proof. We multiply equation (1) by \(u(x)\) and integrate the resulting equality with respect to \(x\) from 0 to \(l\). Using (2), we get

\[ \|u(x)\|^2_2 + m(\|u(x)\|^1_1)\|u(x)\|^2_1 = (f(x, u(x), u'(x)), u(x)). \]

By (16) and (3) we obtain

\[ \left( \alpha + \left( \frac{\pi}{l} \right)^2 \right) \|u(x)\|^2_1 \leq \frac{l}{\pi} \|f(x, u(x), u'(x))\| \|u(x)\|_1. \]

Therefore by (17),

\[ \left( \alpha + \left( \frac{\pi}{l} \right)^2 - \left( \frac{l}{\pi} \right)^2 \|\sigma_2(x)\|_\infty - \frac{l}{\pi} \|\sigma_3(x)\|_\infty \right) \|u(x)\|_1 \leq \frac{l}{\pi} \|\sigma_1(x)\|. \]

From this relation and (6) follows (18).

Lemma 4. Suppose where given some numbers \(v_k \geq 0, k = 0, 1, \cdots\), for which the inequality

\[ v_k \leq av_{k-1} + b, \quad k = 1, 2, \cdots, \]  

(20)

where \(0 \leq a < 1, b > 0\), holds. Then we have the following uniform estimate with respect to the index \(k\)

\[ v_k \leq \frac{b}{1-a} + a \max \left( 0, v_0 - \frac{b}{1-a} \right), \quad k = 1, 2, \cdots. \]  

(21)
Proof. By virtue of (20), by the method of mathematical induction we have \( v_k \leq a^k v_0 + (a^{k-1} + a^{k-2} + \cdots + 1)b \), \( k = 1, 2, \cdots \), which implies
\[
v_k \leq a^k v_0 + \frac{1 - a^k}{1 - a} b = \frac{b}{1 - a} + a^k \left( v_0 - \frac{b}{1 - a} \right). \tag{22}
\]
Let us denote \( \nu_k = a^k \left( v_0 - \frac{b}{1 - a} \right) \) and consider two cases \( v_0 \leq \frac{b}{1 - a} \) and \( v_0 > \frac{b}{1 - a} \).

In the first case \( \nu_k \leq 0 \) and by virtue of (22) \( v_k \leq \frac{b}{1 - a} \), \( k = 1, 2, \cdots \). In the second case \( \nu_k > 0 \), \( \max \nu_k = \nu_1 = a(v_0 - \frac{b}{1 - a}) \), which, by virtue of (22) yields \( v_k = \frac{b}{1 - a} + a(v_0 - \frac{b}{1 - a}) \), \( k = 1, 2, \cdots \). From this conclusions the validity of estimate (21) follows.

Lemma 5. Approximations of iteration method (9) satisfy the inequality
\[
\|u_k(x)\|_1 \leq c_2, \ k = 1, 2, \cdots, \tag{23}
\]
where
\[
c_2 = \begin{cases} c_1, & \text{if } \|\sigma_2(x)\|_\infty + \|\sigma_3(x)\|_\infty = 0, \\ c_1 + c_0 \max(0, \|u_0(x)\|_1 - c_1), & \text{if } \|\sigma_2(x)\|_\infty + \|\sigma_3(x)\|_\infty \neq 0, \end{cases}
\]
\[
c_0 = \left( 1 + \omega \left( \left( \frac{1}{\pi} \right)^2 \|\sigma_2(x)\|_\infty + \frac{l}{\pi} \|\sigma_3(x)\|_\infty \right) \right)^{-1}.
\]

Proof. Replace \( k \) by the index \( k - 1 \) in equation (11), multiply the resulting relation by \( u_k(x) \) and integrate over \( x \) from 0 to \( l \). Taking (12) into account, we get
\[
\|u_k(x)\|_1^2 + m(\|u_{k-1}(x)\|_1^2) \|u_k(x)\|_1^2 = \left( f(x, u_{k-1}(x), u'_{k-1}(x)), u_k(x) \right), \ k = 1, 2, \cdots.
\]
Applying (3) and (15), we have
\[
\left( \alpha + \left( \frac{\pi}{l} \right)^2 \right) \|u_k(x)\|_1^2 \leq \frac{l}{\pi} \|f(x, u_{k-1}(x), u'_{k-1}(x))\| \|u_k(x)\|_1,
\]
which implies
\[
\left( \alpha + \left( \frac{\pi}{l} \right)^2 \right) \|u_k(x)\|_1 \leq \frac{l}{\pi} \|f(x, u_{k-1}(x), u'_{k-1}(x))\|.
\]
Hence, using (17), we conclude that
\[
\|u_k(x)\|_1 \leq \frac{1}{\alpha + \left( \frac{\pi}{l} \right)^2} \frac{l}{\pi} \left( \|\sigma_1(x)\| + \left( \|\sigma_2(x)\|_\infty \frac{l}{\pi} + \|\sigma_3(x)\|_\infty \right) \|u_{k-1}(x)\|_1 \right).
\]
This relation is an inequality of type (20), where
\[
v_k = \|u_k(x)\|_1, \quad a = \frac{1}{\alpha + \left( \frac{\pi}{l} \right)^2} \frac{l}{\pi} \left( \|\sigma_2(x)\|_\infty \frac{l}{\pi} + \|\sigma_3(x)\|_\infty \right), \quad b = \frac{1}{\alpha + \left( \frac{\pi}{l} \right)^2} \frac{l}{\pi} \|\sigma_1(x)\|.
\]
Let us apply (6), (19) to these formulas and carry out some calculations. As a result, for \( \|\sigma_2(x)\|_\infty + \|\sigma_3(x)\|_\infty = 0 \) we obtain \( a = 0 \) and \( \frac{b}{1 - a} = c_1 \), while for \( \|\sigma_2(x)\|_\infty + \|\sigma_3(x)\|_\infty \neq 0 \) we obtain \( a \) and \( b \) from (21) and (23), respectively.
Taking (10), (19) and (16) into consideration we come to the following result where

\[\|\sigma(x)\|_\infty \neq 0 \] we have \(a = c_0\) and \(b = c_1\). By considering these two cases with estimate (21) we get convinced that (23) is valid.

By Lemma 3 and Lemma 5 it will be natural to require that the initial approximation \(u_0(x)\) in (9) satisfy the condition

\[\|u_0(x)\|_1 \leq c_1.\]  

(25)

Then, by virtue of (24) and (23), we have \(\|u_k(x)\|_1 \leq c_1\), which, with (19) taken into account, implies

\[\|u_k(x)\|_1 \leq \frac{l}{\omega\pi}\|\sigma_1(x)\|, \quad k = 0, 1, \ldots.\]  

(26)

6. Convergence of the Method

Multiplying (13) by \(\delta u_k(x)\), integrating the resulting equality with respect to \(x\) from 0 to \(l\) and using (14), we come to the relation

\[\|\delta u_k(x)\|_2^2 + \frac{1}{2} \left( (m (\|u_{k-1}(x)\|_1^2) + m (\|u(x)\|_1^2) ) \|\delta u_k(x)\|_1^2 + \right.\]

\[+ (m (\|u_{k-1}(x)\|_1^2) - m (\|u(x)\|_1^2) ) (u_k(x) + u'(x), \delta u_k(x)) \right) =\]

\[= \left( f(x, u_{k-1}(x)), u_k(x) - f(x, u(x), u'(x)), \delta u_k(x) \right), \quad k = 1, 2, \ldots.

Applying (3)-(5) and (16) we first obtain

\[\|\delta u_k(x)\|_2^2 + \frac{1}{2} \left( \frac{l}{\omega\pi} \|\sigma_1(x)\| \right) (\|u_{k-1}(x)\|_1 + \|u(x)\|_1) \|\delta u_{k-1}(x)\|_1 + \frac{l}{\pi} (\|l_2(x)\|_\infty + \|l_3(x)\|_\infty) \|u_k(x)\|_1 \leq\]

\[\leq \frac{1}{2} \frac{l_1}{\omega\pi} \prod_{p=0}^1 \|u_{k-p}(x)\|_1 + \|u(x)\|_1 + \frac{l}{\pi} (\|l_2(x)\|_\infty + \|l_3(x)\|_\infty) \|u_k(x)\|_1,\]

and after that, by virtue of (18) and (26) we have

\[\|\delta u_k(x)\|_1 \leq \left( \alpha + \left( \frac{\pi}{l} \right)^2 \right)^{-1} \left( \frac{1}{2} \frac{l_1}{\omega\pi} \prod_{p=0}^1 \|u_{k-p}(x)\|_1 + \|u_1(x)\|_1 \right) +\]

\[+ \left( \frac{l}{\pi} \right)^2 (\|l_2(x)\|_\infty + \|l_3(x)\|_\infty) \|\delta u_{k-1}\|_1 \leq q \|\delta u_{k-1}\|_1, \quad k = 1, 2, \ldots,\]

where

\[q = \left( \alpha + \left( \frac{\pi}{l} \right)^2 \right)^{-1} \left( 2c_2 l_1 + \|l_2(x)\|_\infty \left( \frac{l}{\pi} \right)^2 + \|l_3(x)\|_\infty \frac{l}{\pi} \right).\]

Taking (10), (19) and (16) into consideration we come to the following result
Theorem 1. Let assumptions (3)-(6) and (25) are fulfilled. Suppose besides
\[ q = \frac{1}{\alpha + \left(\frac{\pi}{l}\right)^2} \left( \frac{l}{\pi} \right)^2 \left( 2l_1 \left( \frac{\|\sigma_1(x)\|}{\omega} \right)^2 + \|l_2(x)\|_{\infty} + \frac{\pi}{l} \|l_3(x)\|_{\infty} \right) < 1. \]
Then the approximations of the iteration method (9) converge to exact solution of problem (1), (2) and for the error the following estimate
\[ \|u_k(x) - u(x)\|_p \leq \left( \frac{l}{\pi} \right)^{1-p} q^k \|u_0(x) - u(x)\|_1, \quad k = 1, 2, \ldots, \quad p = 0, 1, \]
is true.

7. Numerical Experiment

The theoretical results about the convergence of approximations of iteration method (9) to exact solution \( u(x) \) of problem (1), (2) is confirmed. For illustration, the results of numerical computations of one of the test problems are given below.

We consider a special case, where \( m(z) = m_0 + m_1 \cdot z, \quad m_0, m_1 > 0, \quad m_0 = 1, \quad m_1 = \frac{1}{3} \), the beam length \( l = 1 \), exact solution \( u(x) = x(x-1)(x^2-x-1) \), i.e. \( u(x) = x^4 - 2x^3 + x \), the right-hand side
\[ f(x, u(x), u'(x)) = \frac{1}{35} \left( 43.5u''(x) - 348x^3u'(x) - 1566u(x) + 696x^6 - 3132x^3 + 2088x + 796.5 \right). \]

We carried out five, seven and nine iterations. To compute the integrals on \([0, 1]\) we divided the interval into \( n = 10, 20 \) parts (\( h = 0.1, \ 0.05, \) respectively) and used the square formula of trapezoid. The error in the \( k \)-iteration is defined as
\[ \text{error} \ k = \max_{i=0,1,\ldots,n} \{|u_k(x_i) - u(x_i)|\}, \quad x_i = ih, \quad k = 1, 2, \ldots, 9. \]

Numerical values for the errors are calculated (see Table 1).

The function \( u_0(x) = 0 \) is taken as the initial approximation. In case of five, seven and nine iterations for \( n = 10, 20 \) the exact and approximate solutions are graphically illustrated (Figs. 1-6).

**Remark.** In the figures the green line color denotes the exact solution graph, yellow is the first approximation, red – the second, blue – the third, pink – the fourth, golden – the fifth, brown – the sixth, purple – the seventh, orange – the eighth and black – the ninth.

| \( n \) | error 1 | error 2 | error 3 | error 4 | error 5 | error 7 | error 9 |
|---|---|---|---|---|---|---|---|
| 10 | 0.43203 | 0.16734 | 0.06405 | 0.02473 | 0.00953 | 0.00142 | 0.00021 |
| 20 | 0.43328 | 0.16715 | 0.06365 | 0.02446 | 0.00938 | 0.00138 | 0.00020 |

Table 1
Figure 1. Iteration = 5, n = 10

Figure 2. Iteration = 5, n = 20

Figure 3. Iteration = 7, n = 10

Figure 4. Iteration = 7, n = 20
The numerical experiments clearly show the convergence of iteration approximate solutions to the exact solution of the problem. The error decreases with the growth of the parameters $n$ and $k$.

References

[1] C. Bernardi and M.I.M. Copetti, Finite element discretization of a thermoelastic beam. *Archive Ouverte HAL-UPMC*, 29/05/2013, 23pp.

[2] S. Fučík and A. Kufner, Nonlinear differential equations. Studies in Applied Mechanics, 2. *Elsevier Scientific Publishing Company, Amsterdam-Oxford-New York*, 1980.

[3] G. Kirchhoff, Vorlesungen über mathematische physik, I. Mechanik. *Teubner, Leipzig*, 1876.

[4] T.F. Ma, Positive solutions for a nonlocal fourth order equation of Kirchhoff type. *Discrete Contin. Dyn. Syst.* 2007, 694–703.

[5] J. Peradze, A numerical algorithm for a Kirchhoff-type nonlinear static beam. *J. Appl. Math.* **2009**, Art.ID 818269, 12pp.

[6] J. Peradze, On an iteration method of finding a solution of a nonlinear equilibrium problem for the Timoshenko plate. *ZAMM Z. Angew. Math. Mech.* **91** (2011), no. 12, 993 –1001.

[7] K.Rektorys, Variational methods in mathematics, science and engineering. *Springer Science & Business Media*, 2012.
[8] H. Temimi, A.R. Ansari and A.M. Siddiqui, An approximate solution for the static beam problem and nonlinear integro-differential equations. *Comput. Math. Appl.* **62** (2011), no. 8, 3132–3139.

[9] S.Y. Tsai, Numerical computation for nonlinear beam problems. *M.S. thesis, National Sun Yat-Sen University, Kaohsiung, Taiwan*, 2005.

[10] F. Wang and Y. An, Existence and multiplicity of solutions for a fourth-order elliptic equation. *Bound. Value Probl.* **2012**, 2012:6, 9 pp.

[11] S.Woinowski-Krieger, The effect of an axial force on the vibration of hinged bars. *J. Appl. Mech.* **17** (1950), 35–36.