Stochastic maximum principle for weighted mean-field system

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Abstract. We study the optimal control problem for a weighted mean-field system. A new feature of the control problem is that the coefficients depend on the state process as well as its weighted measure and the control variable. By applying variational technique, we establish a stochastic maximum principle. As an application, we investigate the optimal premium policy of an insurance firm for asset–liability management problem.

Keyword. McKean-Vlasov equation, stochastic maximum principle.

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1 Introduction

Optimal control problems have been studied extensively since the pioneering work of Pontryagin [19], where a maximum principle is obtained by using spike variation. Kushner ([10], [11]) investigated the applicability of the maximum principle to the design of controllers for stochastic system, i.e. stochastic maximum principle (SMP) for optimal control. He showed that for system with additive white noise a maximum principle of the form developed by Pontryagin is valid when diffusion coefficient does not depend on the control variable.

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When the diffusion coefficient contains a control variable, Bensoussan (2, 3) studied such a case. The maximum principle he obtained are local conditions, and his method depends heavily on the control domain being convex. Peng (18) broke through this difficulty in 1990. In that paper, first and second order variational inequalities are introduced, when the control domain need not to be convex, and the diffusion coefficient contains the control variable.

In recent years, stochastic optimal control problems for the mean-field stochastic differential equations (SDEs) have attracted an increasing attention. The mean-field SDEs can trace their roots to the McKean–Vlasov model, which was first introduced by Kac (13) and McKean (17) to study physical systems with a large number of interacting particles. Lasry and Lions (5) extended the applications of the mean-field models to economics and finance. It is not until Buckdahn et al. (4) and Buckdahn et al. (5) established the theory of the mean-field BSDEs that the SMP for the optimal control system of mean-field type has become a popular topic. For example, Li (15) studied SMP for mean-field controls, with the domain of the control convex; Buckdahn et al. (6) obtained the related SMP for a class of general stochastic control problem with McKean-Vlasov dynamics, in which the coefficients depend nonlinearly on both the state process as well as its distribution; Acciaio et al. (1) studied mean-field stochastic control problems where the cost functional and the state dynamics depend on the joint distribution of the controlled state and the control process. Strongly inspired by Buckdahn et al. (4), Buckdahn et al. (7) investigated a generalized mean-field SMP for the optimal control problem where the coefficients not only depend on the distribution of $(X, v)$ but also with partial information. Interested readers may refer to Shen and Siu (20), Djehiche et al. (8), Guo and Xiong (9), Zhang et al. (23), Lakhdari et al. (16), Zhang (22) and Wang and Wu (21) for various versions of the SMPs for the mean-field models.

As we mentioned above, one of the motivations of the study of mean-field SDE is from mathematical finance. The state process is usually the valuation of a certain asset, such as the stock price, by a typical investor which is affected by this investor as well as the others who are interested in this asset, in a collective manner. So far, this collective interaction is
represented by the average measure of the whole system. By a propagation of chaos limit, it is described by the distribution of the state process. However, in the real-world situations, the opinion of the investors about the asset should be weighted accord to their wealth levels. This motivates our study of optimal control with weighted mean-field interaction.

In this paper, we are interested in the SMP for a stochastic control problem with McKean-Vlasov dynamics:

$$\begin{cases}
    dX_t = b(X_t, \mu_t^{X,A}, u_t)dt + \sigma(X_t, \mu_t^{X,A}, u_t)dW_t \\
    dA_t = A_t \left( \alpha(X_t, \mu_t^{X,A}, u_t)dt + \beta(X_t, \mu_t^{X,A}, u_t)dW_t \right) \quad t \in [0, T] \\
    X_0 = x, \quad A_0 = a,
\end{cases}$$

(1.1)

where $W_t$ is an $m$-dimension Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In the above, $(b, \sigma) : \mathbb{R}^d \times \mathcal{M}_F(\mathbb{R}^d) \times U \to \mathbb{R}^d \times \mathbb{R}^{d \times m}$ and $(\alpha, \beta) : \mathbb{R}^d \times \mathcal{M}_F(\mathbb{R}^d) \times U \to \mathbb{R} \times \mathbb{R}^{1 \times m}$ are continuous mappings, $u \in L^2_{\mathbb{F}}([0, T]; U)$ is a control process, $U \subset \mathbb{R}^k$ is a convex domain for the control, and $\mu_t^{X,A} \in \mathcal{M}_F(\mathbb{R}^d)$ is the weighted measure given by

$$\left\langle \mu_t^{X,A}, f \right\rangle = \mathbb{E}(A_t f(X_t)), \quad \forall f \in C^b_b(\mathbb{R}^d).$$

(1.2)

The equation of the form (1.1) without the control has been studied by Kurtz and Xiong [12] as the limit of a system of interacting weighted particles. The main difficulty in obtaining the uniqueness of the solution to this equation as well as the interacting system related to it is the non-Lipschitz property of the coefficients such as $(a, x, \mu) \mapsto a\alpha(x, \mu, u)$ because $a$ is unbounded. A stopping time technique was used in [12]. However, for stochastic control problem, the adjoint equation will be a backward stochastic differential equation (BSDE), and hence, the stopping argument will not be convenient. In this article, we overcome this difficulty by estimating $X_t$ and $A_t$ using suitably chosen different norms.

The rest of this paper is organized as follows: In the next section, we formulate the problem and present the stochastic maximum principle for our stochastic control problem.
In Section 3, as an application, we give an example about the optimal premium policy for an insurance firm for asset–liability management problem. In Sections 4 and 5, we give detailed proofs about the existence and uniqueness for the solution of the weighted mean-field system and about the stochastic maximum principle, respectively.

2 Stochastic Maximum Principle

We recall that we consider the stochastic control problem with the state equation (1.1). Let $U[0, T]$ be the collection of all $U$-valued $\mathcal{F}_t$-adapted processes $u_t$ satisfying

$$E \int_0^T |u_t|^2 dt < \infty.$$ 

$U[0, T]$ is called the set of admissible controls. In order not to overcomplicate the already notational heavy presentation of this paper, in what follows we shall assume all processes are 1-dimensional (i.e., $d = k = m = 1$). We should note that the higher dimensional cases can be argued along the same lines without substantial difficulties, except for even heavier notations.

We will take the cost function $J(u)$ as

$$J(u) = E \left( \int_0^T f(X_t, A_t, \mu_t^{X,A}, u_t) dt + \Phi(X_T, A_T) \right),$$

where $f : \mathbb{R} \times \mathbb{R} \times \mathcal{M}_F(\mathbb{R}) \times U \to \mathbb{R}$, $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Here we have dropped the superscript $u$ of $X, A$ for simplicity of notation.

Our goal is to find a control to minimize the cost functional over $U$, namely, an admissible control $u \in U$ is said to be optimal if

$$J(u) = \min_{v \in U} J(v).$$

(2.2)
In this section, we consider the necessary condition for the optimal control of the problem (2.2), also known as stochastic maximum principle, with the convexity assumption on the control set $U$.

For weighted mean-field type SDEs and BSDEs, we have to introduce some notations. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a copy of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}'(\cdot) = \int_{\Omega'} \mathcal{F}'(\cdot) d\mathbb{P}'$, $\xi'$ is a random variable defined on $(\Omega', \mathcal{F}', \mathbb{P}')$. For $\mu_1, \mu_2 \in M_F(\mathbb{R})$, the Wasserstein metric is defined by

$$\rho(\mu_1, \mu_2) = \sup\{|\langle \mu_1, f \rangle - \langle \mu_2, f \rangle| : f \in \mathbb{B}_1\},$$

where $\mathbb{B}_1 = \{f : |f(x) - f(y)| \leq |x - y|, |f(x)| \leq 1, \forall x, y \in \mathbb{R}\}$, $\langle \mu, f \rangle$ stands for the integral of the function $f$ with respect to the measure $\mu$.

**Definition 2.1.** Suppose $f : M_F(\mathbb{R}) \to \mathbb{R}$. We say that $f \in C^1(M_F(\mathbb{R}))$ if there exists $h(\mu; \cdot) \in C_b(\mathbb{R})$ such that

$$f(\mu + \epsilon \nu) - f(\mu) = \langle \nu, h(\mu; \cdot) \rangle \epsilon + o(\epsilon).$$

We denote $h(\mu; x) = f_{\mu}(\mu; x)$.

Next, we make the following standard assumptions.

**Hypothesis 2.1.** The coefficients $b, \sigma, \alpha, \beta, f, \Phi$ are measurable in all variables. Furthermore

(1) $b, \sigma, f \in C^1(\mathbb{R} \times M_F(\mathbb{R}) \times U)$ with bounded partial derivatives;

(2) $\alpha, \beta \in C^1(\mathbb{R} \times M_F(\mathbb{R}) \times U)$ are bounded and with bounded partial derivatives;

(3) $\varphi_{\mu}(x, \mu, u, x')$ is differentiable in $x'$ with bounded derivative for $\varphi = b, \sigma, \alpha, \beta$. We denote the partial derivative of $\varphi_{\mu}(x, \mu, u; x')$ with respect to $x'$ by $\varphi_{\mu,1}(x, \mu, u; x')$;

(4) $\Phi(x, a)$ has bounded partial derivatives in $x, a$. 
Theorem 2.1. For $u \in \mathcal{U}$ being fixed, the weighted mean-field SDE (1.1) has a unique solution.

The main result of this paper is to prove the following SMP. More precisely, we define Hamiltonian

$$H(X, A, \mu, u, P, Q, p, q) := A(p\alpha(X, \mu, u) + q\beta(X, \mu, u)) + Pb(X, \mu, u) + Q\sigma(X, \mu, u) - f(X, A, \mu, u).$$

The adjoint processes $(p_t, q_t), (P_t, Q_t)$ are governed by the following equations

$$
\begin{align*}
\frac{dp_t}{dt} & = -\left\{ p_t\alpha(\theta_t) + q_t\beta(\theta_t) - f_a(\kappa_t) \right\}dt \\
& - \mathbb{E}\left\{ \left( P_t' b_\mu + Q_t' \sigma_\mu + A_t' (p_t' \alpha_\mu + q_t' \beta_\mu) \right) (\theta_t'; X_t) - f_\mu(\kappa_t'; X_t) \right\} dt + q_t dW_t,
\end{align*}
$$

(2.3)

and

$$
\begin{align*}
\frac{dP_t}{dt} & = -\left\{ P_t b_x(\theta_t) + Q_t \sigma_x(\theta_t) + A_t (p_t \alpha_x(\theta_t) + q_t \beta_x(\theta_t)) - f_x(\kappa_t) \right\}dt \\
& - \mathbb{E}\left\{ A_t \left( (P_t' b_\mu + Q_t' \sigma_\mu + A_t' (p_t' \alpha_\mu + q_t' \beta_\mu)) (\theta_t'; X_t) - f_\mu(\kappa_t'; X_t) \right) \right\} dt \\
& + Q_t dW_t,
\end{align*}
$$

(2.4)

where $\kappa_t = (X_t, A_t, \mu_t, u_t), \theta_t = (X_t, \mu_t, u_t), \varphi_x$ stands for the partial derivative of $\varphi$ with respect to $x$. The same convention also apply to variables $a$ and $u$. Note that the subscript $t$, such as $X_t$, stands for the time parameter of a stochastic process.

We are now ready to state the main theorem of the paper.

Theorem 2.2. (Stochastic Maximum Principle) Suppose that Hypothesis 2.1 holds. Let $(X_t, u_t)$ be an optional solution of control problem (2.2). Then there are two pairs of $\mathcal{F}_t$-
adapted processes \((p_t, q_t)\) and \((P_t, Q_t)\) there satisfy the equations (2.3) and (2.4) such that

\[
H_u(X_t, A_t, \mu_t, u_t, P_t, Q_t, p_t, q_t) = 0.
\]

3 An asset liability management problem

As an application of Theorem 2.2 in this section, we consider the optimal premium policy of an insurance firm for asset liability management problem.

Let \(u\) be the premium strategy of the firm; \(X_t\) be the cash flow; \(l_t \equiv l_t^{X,A,u}\) be the liability process. Suppose that \(a_t, b_t, c_t\) are deterministic and uniformly bounded on \([0, T]\). Suppose that the liability process \(l_t\) satisfies

\[
dl_t = -\left(a_t \mathbb{E}(A_t X_t) + b_t u_t\right) dt - c_t dW_t,
\]

where \(c_t\) denotes the volatility rate, and \(A_t\) is the weighted process satisfying

\[
dA_t = A_t \alpha_t dt + A_t \beta_t dW_t \tag{3.1}
\]

with initial value \(a_0\) and \(\alpha_t, \beta_t\) are deterministic and bounded on \([0, T]\). Suppose that the firm has no liability at time 0, and only invests in a money account due to certain market regulations. Accordingly, the insurance firm only invests in a money account with compounded interest rate \(\rho_t\), and hence its cash-balance process \(X_t\) is

\[
X_t = e^{\int_0^t \rho_s ds} \left( X_0 - \int_0^t e^{-\int_0^r \rho_s dr} dl_t \right), \quad X_0 = x_0
\]

where \(x_0 \geq 0\) represents the initial reserve. According to Itô’s formula, we see that the cash
flow satisfies a mean-field SDE

\[
\begin{align*}
    dX_t &= (\rho_t X_t + a_t \mathbb{E}(A_t X_t) + b_t u_t) \, dt + c_t \, dW_t, \\
    X_0 &= x_0.
\end{align*}
\] (3.2)

Let \( U \subset \mathbb{R} \) be convex. For any \( u \in U \), equation (3.2) admits a unique adapted cash flow \( X_t \).

The insurance firm hopes to drive its cash-balance process to \( c_0 \) in average at the terminal time \( T \) to meet some regulatory requirement, i.e. \( \mathbb{E}X_T = c_0 \). In order to reconcile the contradiction between the liquidity and profitability of insurance, we introduce a performance functional of the firm

\[
J(u) = \frac{1}{2} \mathbb{E} \left[ \int_0^T (LX_t^2 + MA_t^2 + Nu_t^2) \, dt + R(X_T - \mathbb{E}(X_T))^2 \right],
\] (3.3)

with terminal constraint \( \mathbb{E}(X_T) = c_0 \), for some given \( c_0 \in \mathbb{R} \), where \( L, M, N, R \) are positive constants.

Then the asset liability management problem is to find an admissible premium policy \( u \in U \) such that \( J(u) = \inf_{v \in U} J[v] \). Applying Lagrange multiplier technique, we define a cost functional for any \( \lambda \in \mathbb{R} \),

\[
J_\lambda(u) = \frac{1}{2} \mathbb{E} \left[ \int_0^T (LX_t^2 + MA_t^2 + Nu_t^2) \, dt + R(X_T - c_0)^2 + 2\lambda(X_T - c_0) \right]. \] (3.4)

For every \( \lambda \in \mathbb{R} \), we find \( u^\lambda \) to minimize \( J_\lambda(u) \). In this situation, by Thorem 2.2 we write the Hamiltonian function

\[
H(X, A, \mu, u, P, Q, p, q) = A(\alpha_t p + \beta_t q) + P \left( \rho_t X + a_t \int x \mu(dx) + b_t u \right) + c_t Q - \frac{1}{2} (LX^2 + MA^2 + Nu^2),
\]
and the adjoint equations

\[
\begin{align*}
dp_t &= -(p_t \alpha_t + q_t \beta_t - M A_t + a_t X_t \mathbb{E}(P_t)) \, dt + q_t \, dW_t, \\
pt &= 0,
\end{align*}
\]

and

\[
\begin{align*}
dP_t &= -(\rho_t P_t - L X_t + a_t A_t \mathbb{E}(P_t)) \, dt + Q_t \, dW_t, \\
P_T &= -R(X_T - c_0) - \lambda.
\end{align*}
\]  

From Theorem 2.2 we have

\[ Nu_t = b_t P_t. \]  

Hence, the related feedback control system takes the form

\[
\begin{align*}
dX_t &= (\rho_t X_t + a_t \mathbb{E}(A_t X_t) + N^{-1} b_t^2 P_t) \, dt + c_t \, dW_t, \\
\dot{A}_t &= \alpha_t \, dt + \beta_t \, dW_t, \\
\dot{P}_t &= - (\rho_t P_t - L X_t + a_t A_t \mathbb{E}(P_t)) \, dt + Q_t \, dW_t, \\
X_0 &= x_0, \quad A_0 = a, \quad P_T = -R(X_T - c_0) - \lambda, \quad t \in [0, T].
\end{align*}
\]  

Here we have to deal with a fully coupled mean-field forward–backward SDE. In order to solve this equation we set

\[ P_t = \varphi_t X_t + \varphi_t \mathbb{E}X_t + \chi_t \mathbb{E}(A_t X_t) + \psi_t, \]  

where \( \varphi \) is a deterministic differentiable function with terminal \(-R, \varphi_t, \chi_t, \) and \( \psi_t \) are stochas-
tic processes. We write

\[
\begin{align*}
    d\varphi_t &= \Gamma_t dt + \Lambda_t dW_t, \\
    d\chi_t &= \Pi_t dt + \Theta_t dW_t, \\
    d\psi_t &= \Upsilon_t dt + \Sigma_t dW_t, \\
    \varphi_T &= \chi_T = 0, \quad \psi_T = Rc_0 - \lambda.
\end{align*}
\] (3.10)

Differentiating on \( P_t \) in (3.9) and comparing with (3.6), we get

\[
\begin{align*}
    -\rho_t P_t + LX_t - a_t A_t \mathbb{E}(P_t) \\
    &= \varphi_t \rho_t X_t + \varphi_t a_t \mathbb{E}(A_t X_t) + \varphi_t b_t u_t \\
    + \varphi_t \rho_t \mathbb{E}(X_t) + \varphi_t a_t \mathbb{E}(A_t X_t) + \varphi_t b_t \mathbb{E}(u_t) \\
    + \chi_t (\rho_t + \alpha_t + a_t \mathbb{E}(A_t)) \mathbb{E}(A_t X_t) + \chi_t b_t \mathbb{E}(A_t u_t) + \beta_t c_t \chi_t \mathbb{E}(A_t) \\
    + \Upsilon_t + X_t \frac{d}{dt} \varphi_t + \Gamma_t \mathbb{E}(X_t) + \Pi_t \mathbb{E}(A_t X_t)
\end{align*}
\] (3.11)

From (3.7) and by comparing the coefficients of \( X_t, \mathbb{E}[X_t], \mathbb{E}[A_t X_t] \), in the first equation of (3.11), we obtain

\[
\begin{align*}
    \frac{d}{dt} \varphi_t + 2\rho_t \varphi_t + N^{-1} b_t^2 \varphi_t^2 - L &= 0, \\
    \varphi_T &= -R.
\end{align*}
\] (3.12)
and

\[
\begin{aligned}
\text{for } t \leq T,
\begin{aligned}
\xi_t^1 &= \rho_t + N^{-1}b_t^1 \varphi_t + N^{-1}b_t^2 \mathbb{E}(\varphi_t), \\
\xi_t^2 &= a_t + N^{-1}b_t^2 \mathbb{E}(\chi_t), \\
\eta_t^1 &= N^{-1}b_t^2 \mathbb{E}(A_t \varphi_t), \\
\eta_t^2 &= \rho_t + a_t \mathbb{E}(A_t) + \alpha_t + N^{-1}b_t^2 \varphi_t + N^{-1}b_t^2 \mathbb{E}(A_t \chi_t), \\
\end{aligned}
\end{aligned}
\]

Hence,

\[
\begin{aligned}
\frac{\partial}{\partial t} \tau &= \xi_t^1 + \eta_t^1 + \lambda = 0, \\
\frac{\partial}{\partial t} \tau_T, \frac{\partial}{\partial a_t} \tau_T &= \xi_t^2 + \eta_t^2 + \lambda = 0, \\
\frac{\partial}{\partial a_t} \tau_T &= \lambda g_t^1 + \tilde{g}_t^1, \\
\frac{\partial}{\partial \sigma_t} \tau_T &= \lambda g_t^2 + \tilde{g}_t^2,
\end{aligned}
\]

with \(\varphi_t, \varphi_t, \chi_t, \psi_t\), given by Riccati equation (3.12) and the linear BSDE (3.13).

Note that \(\psi_T = R\sigma_0 - \lambda\) implies that \(\mathbb{E}(\psi_t), \mathbb{E}(A_t \psi_t)\) are linear functions of \(\lambda\). We write

\[
\mathbb{E}(\psi_t) = \lambda g_t^1 + \tilde{g}_t^1 \quad \text{and} \quad \mathbb{E}(A_t \psi_t) = \lambda g_t^2 + \tilde{g}_t^2,
\]

where \(g_t^i, \tilde{g}_t^i\) \((i = 1, 2)\) are determined by equations (3.1) and (3.13) explicitly.

To obtain the optimal premium policy, we must identify the value of \(\lambda\). To his end, plugging (3.14) into (3.2), and set

\[
\begin{aligned}
\xi_t^1 &= \rho_t + N^{-1}b_t^1 \varphi_t + N^{-1}b_t^2 \mathbb{E}(\varphi_t), \\
\xi_t^2 &= a_t + N^{-1}b_t^2 \mathbb{E}(\chi_t), \\
\eta_t^1 &= N^{-1}b_t^2 \mathbb{E}(A_t \varphi_t), \\
\eta_t^2 &= \rho_t + a_t \mathbb{E}(A_t) + \alpha_t + N^{-1}b_t^2 \varphi_t + N^{-1}b_t^2 \mathbb{E}(A_t \chi_t), \\
\end{aligned}
\]
we obtain linear ODEs:

\[
\begin{align*}
&d\mathbb{E}(X_t) = (\xi_1^t \mathbb{E}(X_t) + \xi_2^t \mathbb{E}(A_tX_t) + N^{-1}b_2^t \mathbb{E}(\psi_t))dt \\
&d\mathbb{E}(A_tX_t) = (\eta_1^t \mathbb{E}(X_t) + \eta_2^t \mathbb{E}(A_tX_t) + N^{-1}b_1^t \mathbb{E}(A_t\psi_t) + c_t \beta_t \mathbb{E}(A_t))dt
\end{align*}
\]  

\[ (3.15) \]

Since \( \xi_i^t, \eta_i^t, i = 1, 2 \) are continuous on an interval \([0, T]\) and can be determined by equation (3.1), (3.2) and (3.13), equations (3.15) have a unique explicit solution

\[
M_t = M_0 \exp\left(\int_0^t F_s^* ds + \int_0^t N_s(\lambda)e^{(\int_0^t F_r^* dr)}ds \right) := \lambda(h_1^t, h_2^t)^* + (\tilde{h}_1^t, \tilde{h}_2^t)^*.
\]

where \( M_t = (\mathbb{E}(X_t), \mathbb{E}(A_tX_t))^*, M_0 = (x_0, a_0x_0)^*, F_t = (\zeta_1^t, \zeta_2^t) \) with \( \zeta_1^t = (\xi_1^t, \eta_1^t)^*, \zeta_2^t = (\xi_2^t, \eta_2^t)^* \) and

\[
N_s(\lambda) = \lambda N^{-1}b_1^t (g_1^t, g_2^t)^* + N^{-1}b_2^t (\tilde{g}_1^t, \tilde{g}_2^t) + Nb_2^{-2}c_t \beta_t \mathbb{E}(A_t)\]

\( A^* \) denotes the transpose of the matrix \( A \). Finally we can identify \( \lambda \) by \( c_0 = \mathbb{E}X_T = \lambda h_1_T + \tilde{h}_1_T \).

**Remark 3.1.** The model studied above is inspired by Wang and Wu \[21\], where the liability process depend on the average cash flow. In our example we consider a more realistic model to allow the liability process to depend on the average of weighted cash flow with a terminal constraint.

### 4 The proof of the existence and uniqueness

In this section, we present the proof of Theorem 2.1. To this end, we need the following lemma.
Lemma 4.1. Under \((2)\) of Hypothesis \(2.1\) \(\forall p \geq 1\), we have

\[
\sup_{t \leq T} \mathbb{E}(A_t^p) < \infty.
\]

Proof: By Itô's formula, it is easy to show that

\[
A_t = a \exp \left( \int_0^t \tilde{\alpha}(X_s, \mu_s, u_s) ds + \int_0^t \beta(X_s, \mu_s, u_s) dW_s \right) \tag{4.1}
\]

where \(\tilde{\alpha} = \alpha - \frac{1}{2} \beta^2\). Then,

\[
\mathbb{E}(A_t^p) = a^p \mathbb{E} \exp \left( \int_0^t p \tilde{\alpha}(X_s, \mu_s, u_s) ds + \int_0^t p \beta(X_s, \mu_s, u_s) dW_s \right)
\]

\[
\leq a^p \mathbb{E} \exp \left( \int_0^t \left( p\tilde{\alpha} + \frac{p^2}{2} \beta^2 \right) (X_s, \mu_s, u_s) ds \right)
\]

\[
\leq K,
\]

where \(\mathbb{E}\) is the expectation with respect to an equivalent probability measure given by Girsanov's formula. \(\square\)

Now, we are ready to present

Proof of Theorem \(2.1\): We denote \(\tilde{A}_t = \ln A_t\). For the existence, we take a Picard sequence with

\[
X_t^{n+1} = x + \int_0^t b(X_s^n, \mu_s^{X^n, A^n}, u_s) ds + \int_0^t \sigma(X_s^n, \mu_s^{X^n, A^n}, u_s) dW_s
\]

and

\[
\tilde{A}_t^{n+1} = \ln a + \int_0^t \tilde{\alpha}(X_s^n, \mu_s^{X^n, A^n}, u_s) ds + \int_0^t \beta(X_s^n, \mu_s^{X^n, A^n}, u_s) dW_s.
\]

Then,

\[
\mathbb{E} \sup_{s \leq t} |X_s^{n+1} - X_s^n|^2 \leq K \int_0^t \left( \mathbb{E}|X_s^n - X_s^{n-1}|^2 + \rho(\mu_s^{X^n, A^n}, \mu_s^{X^{n-1}, A^{n-1}})^2 \right) ds.
\]
As

\[ \rho(\mu^n_s, A^n_s, \mu^{n-1}_s, A^{n-1}_s) = \sup_{f \in B_1} |E(A^n_s f(X^n_s) - A^{n-1}_s f(X^{n-1}_s))| \]
\[ \leq E|A^n_s - A^{n-1}_s| + E(A^{n-1}_s |X^n_s - X^{n-1}_s|) \]
\[ \leq E|A^n_s - A^{n-1}_s| + K (E|X^n_s - X^{n-1}_s|^2)^{1/2}, \]

we can continue with

\[ E \sup_{s \leq t} |X^{n+1}_s - X^n_s|^2 \leq K \int_0^t \left( E|X^n_s - X^{n-1}_s|^2 + (E|A^n_s - A^{n-1}_s|)^2 \right) ds. \]

Similarly, we have

\[ E \sup_{s \leq t} |\tilde{A}^{n+1}_s - \tilde{A}^n_s|^2 \leq K \int_0^t \left( E|X^n_s - X^{n-1}_s|^2 + (E|A^n_s - A^{n-1}_s|)^2 \right) ds. \]

Then,

\[ \left( E \sup_{s \leq t} |A^{n+1}_s - A^n_s|^2 \right)^2 \leq \left( E \sup_{s \leq t} (A^{n+1}_s + A^n_s)|\tilde{A}^{n+1}_s - \tilde{A}^n_s| \right)^2 \]
\[ \leq K E \sup_{s \leq t} |\tilde{A}^{n+1}_s - \tilde{A}^n_s|^2 \]
\[ \leq K \int_0^t \left( E|X^n_s - X^{n-1}_s|^2 + (E|A^n_s - A^{n-1}_s|)^2 \right) ds. \]

Let

\[ f^n(t) \equiv E \sup_{s \leq t} |X^{n+1}_s - X^n_s|^2 \]
\[ \quad + \left( E \sup_{s \leq t} |A^{n+1}_s - A^n_s| \right)^2. \]

Then,

\[ f^n(t) \leq K \int_0^t f^{n-1}(s) ds. \] (4.2)

By iterating, we see that

\[ f^n(T) \leq K' \frac{(KT)^n}{n!}. \]
which is summarable. Hence, there exists process \((X, A)\) such that

\[
\mathbb{E} \sup_{s \leq T} |X^n_s - X_s|^2 + \left( \mathbb{E} \sup_{s \leq t} |A^n_s - A_s| \right)^2 \to 0.
\]

It is then easy to show that \((X, A)\) is a solution to (1.1).

To prove the uniqueness, we take two solutions and denote the difference of them as \((Y, B)\). Let

\[
g(t) \equiv \mathbb{E} \sup_{s \leq t} |Y_s|^2 + \left( \mathbb{E} \sup_{s \leq t} |B_s| \right)^2.
\]

Similar to (4.2), we get

\[
g(t) \leq K \int_0^t g(s) ds
\]

and hence \(g = 0\). This yields the uniqueness. \(\square\)

5 The proof of the stochastic maximum principle

Suppose that \(u_t\) is an optimal control and \(v_t\) is such that \(v_t + u_t \in \mathcal{U}\). Then, \(u^\epsilon_t \equiv u_t + \epsilon v_t \in \mathcal{U}\). Hence, \(J(u) \leq J(u^\epsilon)\) for all \(\epsilon \in [0, 1]\). Let \((X^\epsilon, A^\epsilon)\) be the state with control \(u^\epsilon\). Then,

\[
\begin{align*}
dX^\epsilon_t &= b(X^\epsilon_t, \mu^\epsilon_t, u^\epsilon_t) dt + \sigma(X^\epsilon_t, \mu^\epsilon_t, u^\epsilon_t) dW_t \\
dA^\epsilon_t &= A^\epsilon_t (\alpha(X^\epsilon_t, \mu^\epsilon_t, u^\epsilon_t) dt + \beta(X^\epsilon_t, \mu^\epsilon_t, u^\epsilon_t) dW_t) \\
X^\epsilon_0 &= x, A^\epsilon_0 = a.
\end{align*}
\]

Define \(Y^\epsilon_t = X^\epsilon_t - X_t\). Then,

\[
dY^\epsilon_t = (b(X^\epsilon_t, \mu^\epsilon_t, u^\epsilon_t) - b(X_t, \mu_t, u_t)) dt + (\sigma(X^\epsilon_t, \mu^\epsilon_t, u^\epsilon_t) - \sigma(X_t, \mu_t, u_t)) dW_t.
\]

Let \(\theta^\epsilon_t\) be between \((X^\epsilon_t, \mu^\epsilon_t, u^\epsilon_t)\) and \((X_t, \mu_t, u_t)\) such that

\[
b(X^\epsilon_t, \mu^\epsilon_t, u^\epsilon_t) - b(X_t, \mu_t, u_t) = b_x(\theta^\epsilon_t) Y^\epsilon_t + \langle \mu^\epsilon_t - \mu_t, b_{\mu}(\theta^\epsilon_t) \rangle + \epsilon b_u(\theta^\epsilon_t) v_t. \tag{5.1}
\]
Note that
\[ | \langle \mu^\varepsilon_t - \mu_t, b^\varepsilon_t(\theta_t^\varepsilon, \cdot) \rangle | = |\mathbb{E} (A_t^\varepsilon b^\varepsilon_t(\theta_t^\varepsilon, X^\varepsilon_t) - A_t b^\varepsilon_t(\theta_t^\varepsilon, X_t))| \]
\[ \leq K\mathbb{E}|A_t^\varepsilon - A_t| + K\mathbb{E} (A_t|Y^\varepsilon_t|). \tag{5.2} \]

We first give the following lemmas.

**Lemma 5.1.** Under the Hypothesis 2.1 on the coefficients we have,
\[ \mathbb{E}|X^\varepsilon_t - X_t|^2 + (\mathbb{E}|A^\varepsilon_t - A_t|)^2 \leq K\varepsilon^2. \tag{5.3} \]

**Proof:** By (5.1, 5.2), we obtain
\[ |b(X^\varepsilon_t, \mu^\varepsilon_t, u^\varepsilon_t) - b(X_t, \mu_t, u_t)|^2 \leq K|Y^\varepsilon_t|^2 + K(\mathbb{E}|A^\varepsilon_t - A_t|)^2 + K\mathbb{E}(|Y^\varepsilon_t|^2) + K\varepsilon^2. \]

The same estimate holds for \( \sigma \). Then,
\[ \mathbb{E}(|Y_s^\varepsilon|^2) \leq K\mathbb{E} \left( \int_0^t \left| b(X^\varepsilon_s, \mu^\varepsilon_s, u^\varepsilon_s) - b(X_s, \mu_s, u_s) \right| ds \right)^2 \]
\[ + K\mathbb{E} \int_0^t \left| \sigma(X^\varepsilon_s, \mu^\varepsilon_s, u^\varepsilon_s) - \sigma(X_s, \mu_s, u_s) \right|^2 ds \]
\[ \leq K \int_0^t (\mathbb{E}(|Y_s^\varepsilon|^2) + (\mathbb{E}|A^\varepsilon_s - A_s|)^2 + \varepsilon^2) ds. \tag{5.4} \]

On the other hand, we denote \( \tilde{A}_t = \ln A_t \). Then,
\[ (\mathbb{E}|A^\varepsilon_t - A_t|)^2 = \mathbb{E} \left| \tilde{\zeta}_t^\varepsilon \left( \int_0^t (\tilde{\alpha}(X^\varepsilon_s, \mu^\varepsilon_s, u^\varepsilon_s) - \tilde{\alpha}(X_s, \mu_s, u_s)) ds + \int_0^t (\tilde{\beta}(X^\varepsilon_s, \mu^\varepsilon_s, u^\varepsilon_s) - \tilde{\beta}(X_s, \mu_s, u_s)) dW_s \right) \right|^2 \]
\[ \leq K \int_0^t (\mathbb{E}(|Y_s^\varepsilon|^2) + (\mathbb{E}|A^\varepsilon_s - A_s|)^2 + \varepsilon^2) ds, \tag{5.5} \]
where $\zeta^\epsilon_t$ is between $A^\epsilon_t$ and $A_t$. Denote the left hand side of (5.3) by $f(t)$. Adding inequalities (5.4, 5.5), we obtain
\[ f(t) \leq K \int_0^t f(s)ds + K \epsilon^2. \]
The desired conclusion follows from Gronwall’s inequality.

Next, we hope to prove that
\[ \lim_{\epsilon \to 0} \epsilon^{-1}(X^\epsilon_t - X_t) = Y_t \text{ and } \lim_{\epsilon \to 0} \epsilon^{-1}(A^\epsilon_t - A_t) = B_t, \]
where $Y_t$ and $B_t$ are two processes. We first taking derivative formally to guess the form of $Y_t$ and $B_t$. In fact, we will show that
\[
\begin{align*}
   dY_t &= \left( b_x(X_t, \mu_t, u_t)Y_t + b_u(X_t, \mu_t, u_t)v_t ight. \\
   &\quad + \mathbb{E}' \left( B'_x b_\mu(X_t, \mu_t, u_t; X'_t) + A'_x b_{\mu,1}(X_t, \mu_t, u_t; X'_t)Y'_t \right) \bigg) dt \\
   &\quad + \left( \sigma_x(X_t, \mu_t, u_t)Y_t + \sigma_u(X_t, \mu_t, u_t)v_t ight. \\
   &\quad + \mathbb{E}' \left( B'_x \sigma_\mu(X_t, \mu_t, u_t; X'_t) + A'_x \sigma_{\mu,1}(X_t, \mu_t, u_t; X'_t)Y'_t \right) \bigg) dW_t,
\end{align*}
\]
(5.6) and
\[
\begin{align*}
   d\tilde{B}_t &= \left( \tilde{\alpha}_x(X_t, \mu_t, u_t)Y_t + \tilde{\alpha}_u(X_t, \mu_t, u_t)v_t ight. \\
   &\quad + \mathbb{E}' \left( B'_x \tilde{\alpha}_\mu(X_t, \mu_t, u_t; X'_t) + A'_x \tilde{\alpha}_{\mu,1}(X_t, \mu_t, u_t; X'_t)Y'_t \right) \bigg) dt \\
   &\quad + \left( \beta_x(X_t, \mu_t, u_t)Y_t + \beta_u(X_t, \mu_t, u_t)v_t ight. \\
   &\quad + \mathbb{E}' \left( B'_x \beta_\mu(X_t, \mu_t, u_t; X'_t) + A'_x \beta_{\mu,1}(X_t, \mu_t, u_t; X'_t)Y'_t \right) \bigg) dW_t,
\end{align*}
\]
(5.7) where $B_t = A_t \tilde{B}_t$. 

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Lemma 5.2. Under Hypothesis 2.1 we have,

\[ \lim_{\epsilon \to 0} \epsilon^{-1}(X_t^\epsilon - X_t) = Y_t \text{ and } \lim_{\epsilon \to 0} \epsilon^{-1}(A_t^\epsilon - A_t) = B_t. \]

Proof: Let

\[ Z_t^\epsilon = \epsilon^{-1}(X_t^\epsilon - X_t) - Y_t \text{ and } C_t^\epsilon = \epsilon^{-1}(A_t^\epsilon - A_t) - B_t. \]

Then

\[ dZ_t^\epsilon = b'(t)dt + \sigma'(t)dW_t, \]

where

\[
b'(t) = \epsilon^{-1}(b(X_t^\epsilon, \mu_t^\epsilon, u_t^\epsilon) - b(X_t, \mu_t, u_t)) - (b_x(X_t, \mu_t, u_t)Y_t + b_u(X_t, \mu_t, u_t)\nu_t) \\
- \mathbb{E}' \left( \frac{\partial}{\partial \mu}(B_t^\epsilon b_{\mu}(X_t, \mu_t, u_t; X_t^\epsilon)) + A_t^\epsilon b_{\mu,1}(X_t, \mu_t, u_t; X_t^\epsilon)Y_t^\epsilon \right) \\
+ b_u(\theta_t^\epsilon)\nu_t - b_u(X_t, \mu_t, u_t)\nu_t \\
= b_x(\theta_t^\epsilon)Y_t^\epsilon - b_x(X_t, \mu_t, u_t)Y_t + \epsilon^{-1}(\mu_x^\epsilon - \mu_x, \nu(\theta_t^\epsilon, \cdot)) \\
- \mathbb{E}' \left( \frac{\partial}{\partial \mu}(B_t^\epsilon b_{\mu}(X_t, \mu_t, u_t; X_t^\epsilon)) + A_t^\epsilon b_{\mu,1}(X_t, \mu_t, u_t; X_t^\epsilon)Y_t^\epsilon \right) \\
+ b_u(\theta_t^\epsilon)\nu_t - b_u(X_t, \mu_t, u_t)\nu_t.
\]

and \( \sigma'(t) \) is given similarly.

Note that

\[
b'(t) = b_x(\theta_t^\epsilon)Y_t^\epsilon - b_x(X_t, \mu_t, u_t)Y_t + \epsilon^{-1}(\mu_x^\epsilon - \mu_x, \nu(\theta_t^\epsilon, \cdot)) \\
- \mathbb{E}' \left( \frac{\partial}{\partial \mu}(B_t^\epsilon b_{\mu}(X_t, \mu_t, u_t; X_t^\epsilon)) + A_t^\epsilon b_{\mu,1}(X_t, \mu_t, u_t; X_t^\epsilon)Y_t^\epsilon \right) \\
+ b_u(\theta_t^\epsilon)\nu_t - b_u(X_t, \mu_t, u_t)\nu_t.
\]
where
\[
\delta b_x(t) = b_x(\theta_t^\varepsilon) - b_x(X_t, \mu_t, u_t) \to 0.
\]

Similar estimate holds for \(\sigma_\varepsilon(t)\). Then,
\[
\mathbb{E}|Z_t^\varepsilon|^2 \leq K \int_0^t \left( \mathbb{E}|Z_s^\varepsilon|^2 + (\mathbb{E}|C_s^\varepsilon|)^2 + \mathbb{E}\delta_s^\varepsilon \right) ds,
\]
where \(\mathbb{E}\delta_s^\varepsilon \to 0\).

On the other hand, we denote
\[
\tilde{C}_t^\varepsilon = \varepsilon^{-1}(\tilde{A}_t^\varepsilon - A_t) - \tilde{B}_t.
\]
Then
\[
C_t^\varepsilon = A_t \tilde{C}_t^\varepsilon + A_t \varepsilon^{-1} \left( e^{\varepsilon(\tilde{B}_t + \tilde{C}_t^\varepsilon)} - 1 - \varepsilon(\tilde{B}_t + \tilde{C}_t^\varepsilon) \right) = A_t \tilde{C}_t^\varepsilon + \delta_t^\varepsilon,
\]
and \(\mathbb{E}|\delta_t^\varepsilon| \to 0\).

Similar to above, we have
\[
\mathbb{E}|\tilde{C}_t^\varepsilon|^2 \leq K \int_0^t \left( \mathbb{E}|Z_s^\varepsilon|^2 + (\mathbb{E}|C_s^\varepsilon|)^2 + \mathbb{E}\delta_s^\varepsilon \right) ds.
\]
Thus,
\[
(\mathbb{E}|C_t^\varepsilon|)^2 \leq K \mathbb{E}|\tilde{C}_t^\varepsilon|^2 + (\mathbb{E}|\delta_t^\varepsilon|)^2 \leq K \int_0^t \left( \mathbb{E}|Z_s^\varepsilon|^2 + (\mathbb{E}|C_s^\varepsilon|)^2 + \mathbb{E}\delta_s^\varepsilon \right) ds + (\mathbb{E}|\delta_t^\varepsilon|)^2.
\]
The conclusion follows from (5.8, 5.9). \(\square\)

Denote \(\kappa_t = (X_t, A_t, \mu_t, u_t)\) and note that as \(\varepsilon \to 0\),
\[
0 \leq \varepsilon^{-1} (J(u^\varepsilon) - J(u))
\]
Recall “adjoint processes” \((p, q), (P, Q)\) statistics equations \((2.3), (2.4)\), which we rewrite as \((2.3), (2.4)\) as

\[
dP_t = g_t dt + q_t dW_t, P_T = -\Phi_a(X_T, A_T), \quad dP_t = G_t dt + Q_t dW_t, P_T = -\Phi_x(X_T, A_T),
\]

where

\[
g_t = - \left\{ p_t \beta_t(\theta_t) + q_t \alpha_t(\theta_t) - f_x(\kappa_t) \right\} - \mathbb{E}' \left\{ (P_t' b_{\mu} + Q_t' \sigma_{\mu})(\theta' t; X_t) + A_t'(p_t' \alpha_{\mu} + q_t' \beta_{\mu})(\theta' t; X_t) - f_{\mu}(\kappa' t; X_t) \right\},
\]

\[
G_t = - \left\{ P_t b_x(\theta_t) + Q_t x(\theta_t) + A_t(p_t \alpha_x(\theta_t) + q_t \beta_x(\theta_t)) - f_x(\kappa_t) \right\} - \mathbb{E}' \left\{ A_t(P_t' b_{\mu 1}(\theta' t; X_t) + Q_t' \sigma_{\mu 1}(\theta' t; X_t)) \right\} - \mathbb{E}' \left\{ A_t'(A_t'(p_t' \alpha_{\mu 1} + q_t' \beta_{\mu 1})(\theta t; X t) - f_{\mu 1}(\kappa' t; X t)) \right\},
\]

where \(\theta_t = (X_t, \mu_t, u_t)\). Applying Itô’s formula of \(p_t B_t\), we get

\[
\frac{d(p_t B_t)}{p_t dB_t + B_t dp_t + d\langle B, p \rangle_t}.
\]

To continue, we calculate

\[
\begin{align*}
\frac{dB_t}{} & = A_t d\tilde{B}_t + \tilde{B}_t dA_t + d\left\langle A, \tilde{B} \right\rangle_t \\
& = A_t (\tilde{\alpha}_x(\theta_t) Y_t + \mathbb{E}'(B_t' \tilde{\alpha}_{\mu}(\theta t; X_t') + A_t' \tilde{\alpha}_{\mu 1}(\theta t; X_t') Y_t') + \tilde{\alpha}_u(\theta_t) v_t) dt \\
& + A_t (\tilde{\beta}_x(\theta_t) Y_t + \mathbb{E}'(B_t' \beta_{\mu}(\theta t; X_t') + A_t' \beta_{\mu 1}(\theta t; X_t') Y_t') + \beta_u(\theta_t) v_t) dW_t \\
& + \tilde{B}_t A_t (\alpha(\theta_t) dt + \beta(\theta_t) dW_t) \\
& + (\tilde{\beta}_x(\theta_t) Y_t + \mathbb{E}'(B_t' \beta_{\mu}(\theta t; X_t') + A_t' \beta_{\mu 1}(\theta t; X_t') Y_t') + \beta_u(\theta_t) v_t) A_t \beta(\theta_t) dt.
\end{align*}
\]
Thus, taking integration and expectation, we get

\[
\begin{align*}
&= (A_t\alpha_x(\theta_t)Y_t + A_t\alpha_u(\theta_t)v_t + B_t\alpha(\theta_t)) \ dt \\
&+ A_t\mathbb{E}'(B'_t\alpha_\mu(\theta_t; X'_t) + A'_tY'_t\alpha_{\mu,1}(\theta_t; X'_t)) \ dt \\
&+ (A_t (\beta_x(\theta_t)Y_t + \beta_u(\theta_t)v_t) + B_t\beta(\theta_t)) \ dW_t \\
&+ A_t\mathbb{E}'(B'_t\beta_\mu(\theta_t; X'_t) + A'_t\beta_{\mu,1}(\theta_t; X'_t)Y'_t) \ dW_t
\end{align*}
\]

Thus,

\[
d(p_tB_t) = p_t (A_t\alpha_x(\theta_t)Y_t + A_t\alpha_u(\theta_t)v_t + B_t\alpha(\theta_t)) \ dt \\
+ p_tA_t\mathbb{E}'(B'_t\alpha_\mu(\theta_t; X'_t) + A'_tY'_t\alpha_{\mu,1}(\theta_t; X'_t)) \ dt + B_tg_t \ dt \\
+ q_t (A_t (\beta_x(\theta_t)Y_t + \beta_u(\theta_t)v_t) + B_t\beta(\theta_t)) \ dt \\
+ q_tA_t\mathbb{E}'(B'_t\beta_\mu(\theta_t; X'_t) + A'_t\beta_{\mu,1}(\theta_t; X'_t)Y'_t) \ dt \\
+ (\cdots) \ dW_t.
\]

Taking integration and expectation, we get

\[
-\mathbb{E}(\Phi_u(X_T, A_T)B_T) \tag{5.13}
\]

\[
= \mathbb{E} \int_0^T p_t (A_t\alpha_x(\theta_t)Y_t + A_t\alpha_u(\theta_t)v_t + B_t\alpha(\theta_t)) \ dt \\
+ \mathbb{E} \int_0^T (p_tA_t\mathbb{E}'(B'_t\alpha_\mu(\theta_t; X'_t) + A'_tY'_t\alpha_{\mu,1}(\theta_t; X'_t)) + B_tg_t) \ dt \\
+ \mathbb{E} \int_0^T q_t (A_t (\beta_x(\theta_t)Y_t + \beta_u(\theta_t)v_t) + B_t\beta(\theta_t)) \ dt \\
+ \mathbb{E} \int_0^T q_tA_t\mathbb{E}'(B'_t\beta_\mu(\theta_t; X'_t) + A'_t\beta_{\mu,1}(\theta_t; X'_t)Y'_t) \ dt.
\]

Similarly, applying Itô’s formula to \(P_tY_t\), we get

\[
d(P_tY_t) = P_t dY_t + Y_t dP_t + d \langle Y, P \rangle_t
\]

\[
= P_t (b_x(\theta_t)Y_t + \mathbb{E}'(B'_t\beta_\mu(\theta_t; X'_t) + A'_t\beta_{\mu,1}(\theta_t; X'_t)Y'_t) + b_u(\theta_t)v_t) \ dt
\]
Taking integration and expectation, we get

$$-\mathbb{E}(\Phi_x(X_T, A_T)Y_T)$$

(5.14)

$$= \mathbb{E} \int_0^T \left( P_t (b_x(\theta_t)Y_t + b_u(\theta_t)v_t) + Q_t (\sigma_x(\theta_t)Y_t + \sigma_u(\theta_t)v_t) + Y_tG_t \right) dt$$

$$+ \mathbb{E} \int_0^T \mathbb{E}' \left( B_t'(P_t b_\mu(\theta_t; X'_t) + Q_t \sigma_\mu(\theta_t; X'_t)) \right) dt$$

Combining (5.10, 5.14), we get

$$0 \leq \mathbb{E} \int_0^T \left( f_x(\kappa_t)Y_t + f_\alpha(\kappa_t)B_t + \mathbb{E}' \left( B_t' f_\mu(\kappa_t; X'_t) + A_t' f_\mu,1(\kappa_t; X'_t)Y'_t \right) + f_u(\kappa_t)v_t \right) dt$$

$$+ \mathbb{E} \int_0^T \left( -P_t (b_x(\theta_t)Y_t + b_u(\theta_t)v_t) - Q_t (\sigma_x(\theta_t)Y_t + \sigma_u(\theta_t)v_t) - Y_tG_t \right) dt$$

$$+ \mathbb{E} \int_0^T \mathbb{E}' \left( - B_t'(P_t b_\mu(\theta_t; X'_t) + Q_t \sigma_\mu(\theta_t; X'_t)) \right) dt$$

$$+ \mathbb{E} \int_0^T \mathbb{E}' \left( - A_t' Y'_t(P_t b_\mu,1(\theta_t; X'_t) + Q_t \sigma_\mu,1(\theta_t; X'_t)) \right) dt$$

$$+ \mathbb{E} \int_0^T -p_t (A_t\alpha_x(\theta_t)Y_t + A_t\alpha_u(\theta_t)v_t + B_t\alpha(\theta_t)) dt$$

$$+ \mathbb{E} \int_0^T \left( -p_t A_t\mathbb{E}' \left( B_t'\alpha_\mu(\theta_t; X'_t) + A_t'Y'_t\alpha_{\mu,1}(\theta_t; X'_t) \right) - B_t g_t \right) dt$$

$$+ \mathbb{E} \int_0^T -q_t (A_t(\beta_x(\theta_t)Y_t + \beta_u(\theta_t)v_t) - B_t \beta(\theta_t)) dt$$
\[ + \mathbb{E} \int_0^T -q_t A_t E' \left( B'_t \beta_\mu (\theta_t ; X'_t) + A'_t \beta_{1,\mu} (\theta_t ; X'_t) Y'_t \right) dt \]
\[ \leq - \mathbb{E} \int_0^T v_t (P_t b_u (\theta_t) + Q_t \sigma_u (\theta_t) + A_t (p_t \alpha_u (\theta_t) + q_t \beta_u (\theta_t)) - f_u (\kappa_t)) dt \]

By the define of Hamilton \( H(X, A, \mu, u, P, Q, p, q) \), we then have
\[ 0 \leq - \mathbb{E} \int_0^T v_t H_u (X_t, A_t, \mu_t, u_t, P_t, Q_t, p_t, q_t) dt. \]
This implies
\[ H_u (X_t, A_t, \mu_t, u_t, P_t, Q_t, p_t, q_t) = 0, \]
and hence, finished the proof.

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