Wess-Zumino terms for AdS D-branes

Machiko Hatsuda*† and Kiyoshi Kamimura*

*Theory Division, High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki, 305-0801, Japan
†Urawa University, Saitama 336-0974, Japan
e-mail: mhatsuda@post.kek.jp

*Department of Physics, Toho University, Funabashi, 274-8510, Japan
e-mail: kamimura@ph.sci.toho-u.ac.jp

Abstract

We show that Wess-Zumino terms for D-p branes with p > 0 in the Anti-de Sitter (AdS) space are given in terms of “left-invariant” currents on the super-AdS group or the “expanded” super-AdS group. As a result there is no topological extension of the super-AdS algebra. In the flat limit the global Lorentz rotational charges of the AdS space turn out to be brane charges of the supertranslation algebra representing the BPS mass. We also show that a D-instanton is described by the $GL(1)$ degree of freedom in the Roiban-Siegel formalism based on the $GL(4|4)/[Sp(4) \times GL(1)]^2$ coset.

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1 Introduction

Dynamics of supersymmetric extended objects play essential roles in nonperturbative aspects of superstring theories. A supersymmetric $p$-brane is described by an action with a Wess-Zumino (WZ) term. In contrast to that geometrical interpretation of flat $p$-branes are discussed well in [1] and [2], the one for AdS $p$-branes has not been clarified. The WZ term for a flat $p$-brane is characterized by the nontrivial element of the $(p + 2)$-th Chevalley-Eilenberg cohomology of the super-translation group [1], and the superalgebra contains topological charges originated from the WZ term [2]. The superalgebra gives a BPS condition with the BPS mass determined by the topological charge.

Chevalley and Eilenberg formulated cohomology theory of Lie algebras [3]: let $C^q(G, P)$ be a vector space of $q$-th cochain for a representation $P$ of a Lie algebra $G$ over a field of characteristic 0. In our case $c \in C^q(G, \mathbb{R})$ is a left-invariant (LI) differential $q$ form on the group $G$ mapping to real space $\mathbb{R}$, and then the cohomology group is defined as $H^q(G, \mathbb{R}) = Z^q(G, \mathbb{R})/B^q(G, \mathbb{R})$ with $Z^q(G, \mathbb{R}) = \{ c \in C^q(G, \mathbb{R}) \mid dc = 0 \}$ and $B^q(G, \mathbb{R}) = \{ dc \mid c \in C^{q-1}(G, \mathbb{R}) \}$. A super-$p$-brane in a flat space is described by the $(p + 1)$-form WZ term $c$, and is classified by the nontrivial class of the $(p + 2)$-th CE cohomology of $G =$ the “supertranslation” group

$$dr = 0, \quad r = dc, \quad r \in C^{p+2}(G, \mathbb{R}), \quad c \notin C^{p+1}(G, \mathbb{R}) \quad .$$

The WZ term is not manifestly superinvariant but pseudo-invariant so the supercharge is modified, and this modification contributes to topological terms in the flat superalgebra.

Recently brane dynamics in AdS spaces have been examined widely following the early works by Metsaev and Tseytlin [4, 5]. In these references the WZ terms of $p$-branes for $p = 1, 3$ are given as a $(p + 2)$-dimensional integration of the closed $(p + 2)$-forms. On the other hand it was shown that in AdS spaces the WZ terms for $p$-branes can be written in terms of the left-invariant (LI) currents for $p = 1$ cases [6, 7, 8, 9], and for a $p = 0$ case [10]. These WZ terms are manifestly superinvariant so there is no modification of the supercharges, therefore no topological charge in the superalgebras appears. It was also shown that the topological brane charges in a flat superalgebra can be traced back into the super-AdS algebra through the limiting procedure. For a BPS 0-brane in AdS$_2 \times \mathrm{S}^2$ space the origin of the 0-brane charge is the Lorentz rotation charge of the AdS algebra [10]. To clarify these issues for more general cases, in this paper we focus on $Dp$-branes for $p = -1, 1, 3$ in the AdS$_5 \times \mathrm{S}^5$ space.

The closed 5-form for an AdS D3-brane is obtained in the reference [5]. It is not simple enough to find the explicit local form of the WZ term to examine the superalgebra
and the brane charges. In this paper we use “expansion” technique. It was originally introduced in [8, 11] as a generalization of the Inönü-Wigner (IW) contraction [12], and further developed in [13]. The IW contraction relates the super-AdS algebra with the supertranslation algebra, while the generalized IW contraction is useful to relate the WZ term in the AdS space and the one in the flat space [8]. This generalization is a contraction in which the next to leading terms in the expansion parameter, inverse of the AdS radius, are kept. It was also shown that the expansion truncated at certain order gives another closed algebra [11]. In this paper we will show that this “expansion” procedure without truncation is useful to analyze the WZ term.

The organization of this paper is the following. In the next section we will review the Roiban-Siegel formulation [7] which makes these computation possible. In the section 3 after reviewing the closed 3-form and its 2-form WZ term for an AdS superstring, we examine the ones for an AdS D-string. By taking care of GL(1) degree of freedom carefully, the field strength for an AdS D-instanton is obtained. In the section 4, we first examine the closed 5-form and the 4-form WZ term for an bosonic D3-brane. We will show that it can not be written in terms of LI currents of AdS$_5$ algebra. We analyze it by using the “expansion” procedure. It turns out that the 4-form potential can be written in terms of the “expanded” currents of the AdS$_5$ LI currents. Next we examine the ones for an super D3-brane. In the section 5 the flat limit is discussed. We will discuss that there is no topological extension of the super-AdS algebra by examining the super-variation of the WZ term concretely. We will also discuss the origin of the topological terms of the flat superalgebra in the super-AdS algebra.

2 Notations

The non-linear realization of the super AdS$_5 \times$S$^5$ group has been discussed in [4, 5] using

\[
\frac{PSU(2, 2|4)}{SO(5, 1) \otimes SO(6)}
\]

(2.1)

In an alternative formulation of it Roiban and Siegel used the coset [7]

\[ G = \frac{GL(4|4)}{[Sp(4) \otimes GL(1)]^2} \sim \frac{PSU(2, 2|4)}{SO(5, 1) \otimes SO(6)} \]  \hspace{1cm} (2.2)

It requires a Wick rotation for the matrix valued coordinate of $G$ to be the real coordinates, and introduces scaling factors of $[GL(1)]^2$. This formulation made computations of the equations of motion and the symmetry algebras much easier [14]. We follow the notations of [7] and [14] in this paper.
An element of the coset $Z_M^A$ is transformed under a global $GL(4|4)$ group element $g$ with the indices $M = m, \bar{m}$, $(m, \bar{m} = 1, \cdots, 4)$ as
\[
Z_M^A \rightarrow g_M^N Z_N^B h_B^A
\]
where $h$ is a local $[Sp(4) \otimes GL(1)]^2$ group element with $A = a, \bar{a}$, $(a, \bar{a} = 1, \cdots, 4)$. The LI 1-form current is given by
\[
J_A^B = Z_A^M dZ_M^B
\]
and transforms as
\[
J \rightarrow h^{-1} J h + h^{-1} d h.
\]
The bosonic components are decomposed as
\[
J^{ab} = J^{(ab)} + J^{(ab)} - \frac{1}{4} C^{ab} \text{tr} J \equiv \langle J \rangle + (J) - \frac{1}{4} C \text{ tr} J
\]
\[
J^{\bar{a} \bar{b}} = J^{(\bar{a} \bar{b})} + J^{(\bar{a} \bar{b})} - \frac{1}{4} C^{\bar{a} \bar{b}} \text{tr} J \equiv \langle J \rangle + (\bar{J}) - \frac{1}{4} C \text{ tr} \bar{J}
\]
The index $\langle ab \rangle$ denotes antisymmetric traceless index which can be expanded by $(C_{\gamma a}^{\alpha})^{ab}$, and the index $(ab)$ denotes symmetric index expanded by $(C_{\gamma a, \beta}^{\alpha})^{ab}$ with the 5-dimensional gamma matrices $\gamma_{\alpha}$, $(\alpha = 0, \cdots, 4)$. The $Sp(4)$ invariant antisymmetric metric $C$, denoted by “\( \Omega \)” in [7], is the charge conjugation matrix, with $C^{-1} = C^T = -C$, in this paper. Trace is taken as $\text{tr}M = M^{a}_a = M^{ab}C_{ba}$. The fermionic components of $J$ in (2.4) will be denoted by small characters;
\[
J^{\bar{a} \bar{b}} = j^{\bar{a} \bar{b}}, \quad J^{a b} = \bar{j}^{a \bar{b}}.
\]
These LI 1-forms satisfy the Maurer-Cartan (MC) equations, $-dJ_A^B = J_A^C J_C^B$, more explicitly
\[
\begin{cases}
-d\langle J \rangle = \langle J \rangle \langle J \rangle + \langle J \rangle \langle J \rangle + \langle j \bar{j} \rangle = \{ \langle J \rangle, \langle J \rangle \} + \langle j \bar{j} \rangle \\
-d\langle J \rangle = \langle J \rangle \langle J \rangle + \langle J \rangle \langle J \rangle + \langle j \bar{j} \rangle = \langle J \rangle^2 + \langle J \rangle^2 + \langle j \bar{j} \rangle \\
-dj = \langle J \rangle j + \langle J \rangle j + j \langle J \rangle + j \langle J \rangle - \frac{1}{2} (\text{Str} J) j \\
-d\bar{j} = \langle \bar{J} \rangle \bar{j} + \langle \bar{J} \rangle \bar{j} + \bar{j} \langle J \rangle + \bar{j} \langle J \rangle + \frac{1}{2} (\text{Str} J) \bar{j}
\end{cases}
\]
and similar expressions for barred sectors. It is noted that we use a notation in which the exterior derivative “$d$” commutes with both Grassman even and odd components of “$Z_M^A$.”
3 2-form WZ terms

3.1 Superstring

The $\kappa$-symmetry of a superstring requires the 2-form WZ term. It was shown in [7] that the WZ term for an AdS-string is given as

$$L_{WZ,AdS,F_1} = B^{NS}_{[2]} = B_{[2]}^-$$  \hspace{1cm} (3.1)

with

$$B_{[2]}^\pm = \frac{1}{2} \left\{ E^{1/2} J^{ab} J_{ab} \pm E^{-1/2} J^{\bar{a}b} J_{\bar{a}b} \right\} = \frac{1}{2} \text{tr} \left\{ E^{1/2} jj \pm E^{-1/2} \bar{jj} \right\}$$  \hspace{1cm} (3.2)

and it has the correct flat limit. The invariance under the local $GL(1)$ scaling transformations requires introducing $E = \text{Sdet} Z_M^A$ factors. In the gauge $E = 1$ the coset $G = GL(4|4)/[Sp(4) \otimes GL(1)]^2$ reduces into $PSL(4|4)/[Sp(4)]^2$. The closed three form is given by

$$dB^{NS}_{[2]} = H^{NS}_{[3]} \quad , \quad H^{NS}_{[3]} = H_{[3]}^+$$  \hspace{1cm} (3.3)

with

$$H_{[3]}^\pm = \langle J \rangle^{ab} \left\{ E^{1/2} \langle J_a \bar{J}_b \rangle \pm E^{-1/2} \langle J^\bar{a} J_b \rangle \right\} - \langle J \rangle^{\bar{a}b} \left\{ E^{1/2} \langle J^a \bar{J}_b \rangle \pm E^{-1/2} \langle J^a J_\bar{b} \rangle \right\} = -\text{tr} \left[ \langle J \rangle \left\{ E^{1/2} \langle jj \rangle \pm E^{-1/2} \langle \bar{jj} \rangle \right\} - \langle J \rangle \left\{ E^{1/2} \langle jj \rangle \pm E^{-1/2} \langle \bar{jj} \rangle \right\} \right]$$  \hspace{1cm} (3.4)

satisfying

$$d(B_{[2]}^\pm) = H_{[3]}^\pm$$  \hspace{1cm} (3.5)

Then the 3-form field strengths in the super-AdS space are trivial elements of the 3-rd CE cohomology of $G$,

$$dH_{[3]} = 0 , \quad H_{[3]} = dB_{[2]} , \quad H_{[3]} \in C^3(G, \mathbb{R}) , \quad B_{[2]} \in C^2(G, \mathbb{R})$$  \hspace{1cm} (3.6)

Since these left invariant 2-forms are manifestly global superinvariant, the supercharges do not contain additional shift from the WZ action. So there is no topological extension of the superalgebra for a AdS-superstring.

3.2 D-string and D-instanton

The WZ term for Dp-brane is given as the integral of the closed and invariant $(p+2)$ -form,

$$dL_{WZ} = e^F \mathcal{R} = e^{dA} \hat{\mathcal{R}} , \quad \mathcal{F} = dA - B^{NS}_{[2]} , \quad \hat{\mathcal{R}} = e^{-B^{NS}_{[2]}} \mathcal{R}$$  \hspace{1cm} (3.7)
where \( \hat{R} \) satisfies \( d\hat{R} = 0 \) while \( dR = H^{NS}[3]R \). We comment that the Dirac-Born-Infeld \( U(1) \) gauge field \( A \) and \( B^{NS}_{[2]} \) are separately superinvariant, \( \delta_e A = \delta_e B^{NS}_{[2]} = 0 \), in our formulation. This is possible by the existence of total derivative bilinear term, \( d(\delta d\bar{\delta}) \), in \( B^{NS}_{[2]} \) which can not be introduced in the conventional 2-form \( \int H^{NS}_{[3]} \) as discussed in \( \text{[3]} \).

Recently it is observed that \( PSU(2,2|4) \) is extended to \( SU(2,2|4) \) by an external automorphism \( U(1)_B \) under the circumstance of the existence of BPS multiplets in the \( \mathcal{N} = 4 \) SYM \( \text{[15]} \). In our description the \( E = 1 \) gauge fixing of \( GL(1) \) plays the similar role, so we also keep the \( GL(1) \) field unfixed. In the reference \( \text{[7]} \) the auxiliary \( GL(1) \) field, \( E \), is introduced to preserve the local \( GL(1) \) invariance of the action \( \text{(3.1)} \) and \( \text{(3.2)} \), and the resultant three-form field strength includes \( E \) \( \text{(3.3)} \) and \( \text{(3.4)} \). For the R/R two-form we instead break the local \( GL(1) \) invariance to take into account the BPS effect as observed above. The 3-form field strength is determined in such a way that it does not depend on \( E \), \( \hat{C}^{[2]} = \hat{B}^{[2]+} \), \( B^{NS}_{[2]} = \hat{B}^{[2]-} \), \( \hat{B}^{[2]} = \frac{1}{2} \text{tr} \{ jj \pm \bar{j} \bar{j} \} \) \( \text{(3.8)} \) (3.9)

Then the exterior derivative of the R/R two-form becomes
\[
\hat{R}^{[3]} \equiv d\hat{C}^{[2]} = \hat{H}^{[3]-} + \frac{1}{2} \text{Str} J \hat{B}^{[2]-} \ .
\]

On the other hand it is also written as
\[
\hat{R}^{[3]} = R^{[3]} - B^{NS}_{[2]} \ R^{[1]} \ , \quad \begin{cases} R^{[3]} = \hat{H}^{[3]-} \\ R^{[1]} = -\frac{1}{2} \text{Str} J \end{cases} \ (3.11)
\]

from \( \text{(3.7)} \). If we identify \( R^{[1]} \) as the field strength of a D-instanton, then the WZ term is given as
\[
\mathcal{L}_{WZ, AdS, \text{D instanton}} = C^{[0]} , \quad C^{[0]} = \frac{1}{2} \text{ln} E \ (3.12)
\]

where \( \text{Str} J = -d\ln E = 0 \) is used. With this degree of freedom, the WZ term for an AdS-D-string is given by
\[
\mathcal{L}_{WZ, AdS, \text{D1}} = \hat{C}^{[2]} + dA \ C^{[0]} \ . \quad (3.13)
\]

The WZ terms for AdS D-string and AdS D-instanton are written in terms of the LI 1-form currents or \( E = S \text{det} Z_M^A \) which are manifestly superinvariant. So there is no contribution to the supercharges from these WZ terms, then no topological term is produced in the AdS superalgebra.
4 4-form WZ term

In the AdS$_5$ space the closed 5-form has a contribution from the 5-form RR flux \[5\]. At first we will concentrate on this bosonic 5-form RR flux effect and then we will extend it to the supersymmetric case.

4.1 Bosonic D3-brane

We begin with the bosonic AdS$_5$ background where $z_m^a$ is an element of the bosonic coset $G = GL(4)/[Sp(4) \otimes GL(1)]$ and the LI current is $j_a^b = z_a^m dz_m^b$. These currents are basis of the $q$-forms on $G$. They satisfy the following MC equations

\[
\begin{cases}
-d(j) = \{ (j), (j) \} \\
-d(j) = (j)^2 + (\langle j \rangle)^2
\end{cases}
\] (4.1)

Using them the closed and invariant “bosonic” 5-form is

\[ \hat{R}_5 = \text{tr} \langle j \rangle^5. \] (4.2)

The corresponding 4-form potential

\[ \hat{R}_4 = d\hat{C}_4, \; \hat{C}_4 = \int \hat{R}_5 \] (4.3)

can be evaluated explicitly as follows. We take the local Lorentz $Sp(4)$ and the $GL(1)$ gauge choice as a “unitary gauge”

\[ z_m^a = \frac{1}{\sqrt{1-x^2}} 1_m^b \{ 1 + x^\alpha \gamma_\alpha \}_b^a, \; z_a^m = \frac{1}{\sqrt{1-x^2}} \{ 1 - x^\alpha \gamma_\alpha \}_a^b 1_b^m, \] (4.4)

where $x^\alpha (\alpha = 0, \cdots, 4)$ are the AdS$_5$ coordinates. The LI currents, $j_a^b = z_a^m dz_m^b$, are

\[ \langle j_{\text{unitary}} \rangle_a^b = \frac{dx^\alpha}{1-x^2} (\gamma_\alpha)_a^b, \; \langle j_{\text{unitary}} \rangle_a^b = -\frac{x^\alpha dx^\beta}{1-x^2} (\gamma_{\alpha\beta})_a^b, \; \text{tr} j_{\text{unitary}} = d \{ \ln(1-x^2) \} \] (4.5)

and they satisfy the MC equations \[16\]. Using the Cartan’s homotopy formula \[16\], where $x_t = tx$, $k_t x_t \equiv 0$ and $k_t dx_t \equiv x_t$, the “bosonic” 4-form potential is given as

\[ \hat{C}_4 = \int_0^1 dt k_t R_5(x_t) \]
\[ = \int_0^1 dt \frac{5t^4}{(1-t^2x^2)^5} x^\alpha dx^\beta dx^\gamma dx^\delta x^\epsilon \varepsilon_{\alpha\beta\gamma\delta\epsilon} \]
\[ = f(x) x^\alpha dx^\beta dx^\gamma dx^\delta x^\epsilon \varepsilon_{\alpha\beta\gamma\delta\epsilon} \] (4.6)
with

\[ f(x) = \int_0^1 dt \frac{5t^4}{(1-t^2x^2)^5} \]

\[ = \frac{5}{128|x|^5(1-|x|^2)^4} \left[ -3|x|(1+|x|^6) + 11|x|^3(1+|x|^2) + 3(1-|x|^2)^4 \arctanh|x| \right] \]

\[ = \frac{1}{(1-x^2)^4} \left[ 1 - \frac{3}{7}x^2 + o(x^4) \right] \equiv \frac{1}{(1-x^2)^4}[1 + \alpha(x^2)] . \quad (4.7) \]

The relation (4.3) is checked by the following relation

\[ f(x^2) + 2 \frac{2}{5}x^2 \frac{d}{dx^2}f(x^2) = \frac{1}{(1-x^2)^3} . \quad (4.8) \]

In terms of LI currents (4.5) the 4-form is written as

\[ \hat{C}_{[4]} = -(1 + \alpha(x^2)) \text{ tr } (j_{\text{unitary}})(j_{\text{unitary}})^3 . \quad (4.9) \]

The 4-form potential \( \hat{C}_{[4]} \) in (4.9) cannot be written only in terms of LI currents, but it has explicit dependence on \( x \) through “\( \alpha(x^2) \)”. Therefore we conclude that

\[ d\hat{R}_{[5]} = 0, \quad \hat{R}_{[5]} = d\hat{C}_{[4]}, \quad \hat{R}_{[5]} \in C^5(G, \mathbb{R}), \quad \hat{C}_{[4]} \notin C^4(G, \mathbb{R}) , \quad (4.10) \]

and the bosonic 5-form belongs to a nontrivial class of the 5-th CE cohomology group for \( G = \{ GL(4)/[Sp(4) \otimes GL(1)] \} \).

Now we analyze the 5-form and 4-form by “expansion” procedure. This procedure realizes systematic computation not only for the leading term in the flat limit but also for all order terms. We begin with a rescaling

\[ x^\alpha \to sx^\alpha , \quad (4.11) \]

then LI currents are expanded as

\[ j_{\text{unitary}}(x) \to j_{\text{unitary}}(sx) = \sum_{n=1}^\infty s^n j_{\text{unitary},n}(x) . \quad (4.12) \]

The MC equations (4.11) is satisfied at each power in \( s \), and they are expanded as

\[ \begin{cases} 
-d\langle j_{\text{unitary},1} \rangle &= 0 \\
-d\langle j_{\text{unitary},2} \rangle &= \langle j_{\text{unitary},1} \rangle^2 \\
-d\langle j_{\text{unitary},3} \rangle &= \{ \langle j_{\text{unitary},2} \rangle, \langle j_{\text{unitary},1} \rangle \} \\
&
\end{cases} \quad (4.13) \]

From these MC equations we read off the structure constant, \([T_a, T_b] = f_{abc}T_c \leftrightarrow dJ^c = -\frac{1}{2}f_{ab}^\epsilon J^\epsilon J^b\), and recognize it as the structure constant of the “expanded” algebra. It is important that the expanded currents are no more left-invariant, since the AdS transformations depend on \( s \) as \( \delta_{\text{AdS}} = \delta_0 + s\delta_1 + \cdots \). The global Lorentz and scaling transformations
include $\delta_0$ only while the AdS translations includes $\delta_0$ and $\delta_1, \cdots$ in the unitary gauge. Expanded currents transform by themselves only under $\delta_0 x = \Lambda x$ as $j_n \to h^{-1}(-\Lambda)j_nh(-\Lambda)$.

The closed 5-form and the 4-form potential are also expanded in the power of $s$ as

$$\hat{R}_5 = \sum_{n=5}^{\infty} s^n \hat{R}_{5,n}, \quad \hat{C}_4 = \sum_{n=5}^{\infty} s^n \hat{C}_{4,n}, \quad \hat{R}_{5,n} = d\hat{C}_{4,n}.$$  \hspace{1cm} (4.14)

A clue of the analysis is that “expanded” 4-form potentials $\hat{C}_{4,n}$ can be written in terms of the “expanded” currents $j_n$’s. At the lowest $s^0$ order $\hat{R}_{5,5}$ and $\hat{C}_{4,5}$ satisfying $d\hat{R}_{5,5} = 0$ and $\hat{R}_{5,5} = d\hat{C}_{4,5}$ are

$$\hat{R}_{5,5} = \text{tr} (j_{\text{unitary},1})^5, \quad \hat{C}_{4,5} = -\text{tr} (j_{\text{unitary},2}) (j_{\text{unitary},1})^3.$$  \hspace{1cm} (4.15)

This relation can be generalized for all $n > 5$ as $\hat{R}_{5,n} = d\hat{C}_{4,n}$, where both $\hat{R}_{5,n}$ and $\hat{C}_{4,n}$ are written in terms of expanded currents. Therefore it leads to that the expanded 5-forms belong to trivial class of the 5-th CE cohomology group on $G' = \{\text{“expanded”}\ GL(4)/[Sp(4) \otimes GL(1)]\}$:

$$d\hat{R}_{5,n} = 0, \quad \hat{R}_{5,n} = d\hat{C}_{4,n}, \quad \hat{R}_{5,n} \in C^5(G', \mathbb{R}) \quad \hat{C}_{4,n} \in C^4(G', \mathbb{R}).$$  \hspace{1cm} (4.16)

In general gauge, we introduce full 16 variables of the $GL(4)$ matrix $z_{m}^a$. The $Sp(4)$ and $GL(1)$ gauge invariance removes $10 + 1$ degrees of freedom leaving 5 physical coordinates. For example we parameterize it as

$$z_m^a = z_m^a e^{\phi} e^{\varphi_{[\alpha\beta]} \gamma_{\alpha\beta}}, \quad z_a^m = e^{-\phi} e^{-\varphi_{[\alpha\beta]} \gamma_{\alpha\beta}} z_a^m$$  \hspace{1cm} (4.17)

with $z_m^a$ and $z_a^m$ in (4.3), then the LI currents contain $\phi$ and $\varphi_{\alpha\beta}$ dependent terms. The following rescaling

$$\varphi_{[\alpha\beta]} \to \varphi_{[\alpha\beta]}, \quad \phi \to \phi, \quad x^\alpha \to sx^\alpha$$  \hspace{1cm} (4.18)

leads to the expansion of the LI currents as

$$\langle j \rangle \to \langle j_1 \rangle, \langle j_3 \rangle, \langle j_5 \rangle, \cdots$$

$$\langle j \rangle \to \langle j_0 \rangle, \langle j_2 \rangle, \langle j_4 \rangle, \cdots$$  \hspace{1cm} (4.19)

$$\text{tr } j \to \text{tr } j_0, \text{tr } j_2, \text{tr } j_4, \cdots.$$

They satisfy the following “expanded” MC equations:

$$\begin{cases}
-d(j_0) = (j_0)^2 \\
-d(j_1) = \{(j_0), (j_1)\} \\
-d(j_2) = \{(j_0), (j_2)\} + (j_1)^2 \\
& \vdots
\end{cases}$$  \hspace{1cm} (4.20)
The first relation represents the Lorentz algebra $Sp(4)$, and all expanded currents except $(j_0)$ are transformed as bi-spinors under it. Both the expanded 5-forms $\hat{\mathcal{R}}_{[5],n}$ and the expanded 4-forms $\hat{\mathcal{C}}_{[4],n}$ do not contain $(j_0)$ to guarantee the local Lorentz invariant 4-forms. In this gauge the IW contraction is easily seen by the truncation of the expanded MC equations, because of the manifestation of the subgroup, $(j_0)$. The IW contraction “AdS $\rightarrow$ Lorentz” corresponds to the truncation of the (4.20) at the first level in such a way that the first equation is kept. The IW contraction “AdS $\rightarrow$ Poincaré” corresponds to the truncation at the second level. The property of (4.16) is also the same in this gauge.

4.2 Super D3-brane

Now we will extend the bosonic analysis to the supersymmetric case. We first find a closed and invariant 5-form in terms of the LI one forms. The leading term includes supersymmetric version of the $\langle J \rangle^5$ term (4.2). The five form $\hat{\mathcal{R}}_{[5]}$ is obtained as

$$\hat{\mathcal{R}}_{[5]} = -\text{tr} \left[ \frac{16}{15} \left( \langle J \rangle^5 - \langle \bar{J} \rangle^5 \right) + 8 \left( \frac{1}{3} \langle J \rangle^3 \langle Y \rangle + \langle J \rangle^2 j_{\frac{1}{2}} \langle J \rangle j_{\frac{1}{2}} - \langle \bar{J} \rangle^2 j_{\frac{1}{2}} \langle J \rangle j_{\frac{1}{2}} - \frac{1}{3} \langle \bar{J} \rangle^3 \langle \bar{Y} \rangle \right) + 2 \left( \langle J \rangle \langle Y \rangle^2 - 2 \langle J \rangle j \langle \bar{Y} \rangle j + 2 \langle \bar{J} \rangle \bar{j} \langle Y \rangle j - \langle \bar{J} \rangle \langle \bar{Y} \rangle^2 \right) \right] , \quad (4.21)$$

with

$$Y^{ab} = j^{a\bar{c}} j^b_{\bar{c}} , \quad \bar{Y}^{\bar{a}\bar{b}} = \bar{j}^{\bar{a}c} j_{\bar{b}c} . \quad (4.22)$$

The closure of the 5-form $\hat{\mathcal{R}}_{[5]}$ is shown by using the MC equations (2.8). It is not only closed but also manifestly invariant under the local $h$ transformations (2.5).

It is important that (4.21) can be rewritten as

$$\hat{\mathcal{R}}_{[5]} = \mathcal{R}_{[5]} - B_{[2]}^{NS} \mathcal{R}_{[3]} \quad (4.23)$$

$$\mathcal{R}_{[5]} = -\text{tr} \left[ \frac{16}{15} \left( \langle J \rangle^5 - \langle \bar{J} \rangle^5 \right) + 8 \left( \frac{1}{3} \langle J \rangle^3 \langle Y \rangle + \langle J \rangle^2 j_{\frac{1}{2}} \langle J \rangle j_{\frac{1}{2}} - \langle \bar{J} \rangle^2 j_{\frac{1}{2}} \langle J \rangle j_{\frac{1}{2}} - \frac{1}{3} \langle \bar{J} \rangle^3 \langle \bar{Y} \rangle \right) \right] \quad (4.24)$$

with $\mathcal{R}_{[3]}$ given in (3.10). We take the $E = 1$ gauge, i.e. D-instanton free background, so $\hat{\mathcal{R}}_{[3]} = \mathcal{R}_{[3]}$. In order to write the last line of (4.21) as $-B_{[2]}^{NS} \mathcal{R}_{[3]}$ we use an identity

$$\text{tr} \left[ \langle J \rangle \langle Y \rangle^2 - 2 \langle J \rangle j \langle \bar{Y} \rangle j \right] = \frac{1}{4} \left\{ \text{tr} \left[ \langle J \rangle \langle jj \rangle \right] \text{tr} jj + \text{tr} \left[ \langle J \rangle \langle jj \rangle \right] \text{tr} jj \right\} . \quad (4.25)$$
The last line of (4.21) is combined with $dA \mathcal{R}_3$ to give $\mathcal{F} \mathcal{R}_3$ in (4.26)

$$d\mathcal{L}_{WZ,AdS,D3} = \hat{\mathcal{R}}_5 + dA\mathcal{R}_3 = \mathcal{R}_5 + \mathcal{F}\mathcal{R}_3$$

and is consistent with the result in [3].

The “super” local 4-form potential shares the same property as the “bosonic” one; it can be written “almost” in terms of LI currents but not completely. The $\hat{\mathcal{R}}_5$ belongs to the nontrivial class of the CE cohomology of $\mathcal{G}$

$$d\hat{\mathcal{R}}_5 = 0, \quad \hat{\mathcal{R}}_5 = d\hat{C}_4, \quad \hat{\mathcal{R}}_5 \in C^5(\mathcal{G}, \mathbb{R}), \quad \hat{\mathcal{C}}_4 \notin C^4(\mathcal{G}, \mathbb{R}) \quad . \quad (4.27)$$

Now we perform the “expansion” analysis for the supersymmetric case to examine the properties in (4.16). A coset parametrization is, for example, taken as

$$Z = \begin{pmatrix} 1 + x & \theta \\ \bar{\theta} & 1 + \bar{x} \end{pmatrix} \quad (4.28)$$

where $x$ and $\bar{x}$ have $4 \times 4$ components now. $\theta$ and $\bar{\theta}$ are $4 \times 4$ fermionic matrices. Rescale the coordinates as

$$x \rightarrow sx, \quad \theta \rightarrow s^{1/2}\theta \quad (4.29)$$

and similar for barred variables, then it follows the expansion of the LI currents as

$$\langle J \rangle \rightarrow \langle J_1 \rangle, \quad \langle J_2 \rangle, \quad \langle J_3 \rangle, \cdots$$
$$\langle J \rangle \rightarrow (J_1), \quad (J_2), \quad (J_3), \cdots \quad (4.30)$$
$$\langle J \rangle \rightarrow (J_1), \quad (J_2), \quad (J_3), \cdots$$
$$j \rightarrow j_{\frac{1}{2}}, \quad j_{\frac{3}{2}}, \quad j_{\frac{5}{2}}, \cdots \quad (4.31)$$

The expanded MC equations for the super-AdS$_5 \times $S$^5$ are given as follows:

$$\begin{aligned}
-dj_{\frac{1}{2}} & = 0 \\
-d\langle J_1 \rangle & = \langle Y_1 \rangle \\
-d(J_1) & = \langle Y_1 \rangle \\
-dj_{\frac{3}{2}} & = \langle (J_1) + (J_1) \rangle j_{\frac{3}{2}} + j_{\frac{3}{2}} ((\bar{J}_1) + (\bar{J}_1)) - \frac{1}{3} (\text{Str} J_1)j_{\frac{3}{2}} \\
-d\langle J_2 \rangle & = \{ (J_1), (J_1) \} + \langle Y_2 \rangle \\
-d(J_2) & = \langle J_1 \rangle^2 + (J_1)^2 + \langle Y_2 \rangle \\
\end{aligned} \quad (4.32)$$
where

\[ Y_n = \sum_{r<n} j_{n-r} j_r, \quad \begin{cases} n & \text{is a positive integer,} \\ r & \text{is a positive half integer} \end{cases} \tag{4.33} \]

satisfy following MC equations

\[ -dY_n = \sum_{m<n} [J_{n-m}, Y_m]. \tag{4.34} \]

The invariant closed 5-form \( \mathcal{R}_{[5]} \) is expanded as follows. It starts from the \( s^3 \) order terms,

\[ \mathcal{R}_{[5],3} = -\text{tr} \left[ 2 \langle J_1 \rangle (Y_1)^2 - \langle J_1 \rangle j_{\frac{1}{2}} \bar{Y}_1 \bar{j}_{\frac{1}{2}} + \langle J_1 \rangle \bar{j}_{\frac{1}{2}} (Y_1) j_{\frac{1}{2}} - 2 \langle J_1 \rangle (\bar{Y}_1)^2 \right] \tag{4.35} \]

The corresponding 4-form potential satisfying \( \mathcal{R}_{[5],3} = d\mathcal{C}_{[4],3} \) can be found using the MC equations \( 1 \leq r \leq 2 \),

\[ \mathcal{C}_{[4],3} = -\text{tr} \left[ \frac{1}{3} \left\{ 2 \langle J_1 \rangle \{(J_1), (Y_1)\} + (\langle J_1 \rangle + \langle J_1 \rangle)) j_{\frac{1}{2}} (\langle J_1 \rangle + \langle J_1 \rangle) \bar{j}_{\frac{1}{2}} - 12 \langle J_1 \rangle j_{\frac{1}{2}} \langle J_1 \rangle \bar{j}_{\frac{1}{2}} \\
- W_2 (\langle Y_1 \rangle + \langle Y_1 \rangle)) \right\} - \{\text{barred terms}\} \tag{4.36} \]

where 4-form potentials always contain total derivative term ambiguity. It is also convenient to define

\[ W_n = j_{n-\frac{3}{2}} \bar{j}_{\frac{1}{2}} - j_{\frac{1}{2}} \bar{j}_{n-\frac{1}{2}} \tag{4.37} \]

for example \( W_2 = j_{\frac{3}{2}} j_{\frac{3}{2}} - j_{\frac{1}{2}} j_{\frac{1}{2}} \) and it satisfies

\[ -dW_2 = \{J_1, Y_1\} + 2 j_{\frac{1}{2}} J_1 \bar{j}_{\frac{1}{2}}. \tag{4.38} \]

At \( s^4 \) order the 5-form and the 4-form, satisfying \( d\mathcal{R}_{[5],4} = 0 \) and \( \mathcal{R}_{[5],4} = d\mathcal{C}_{[4],4} \), are

\[ \mathcal{R}_{[5],4} = -\text{tr} \left[ \left\{ \frac{8}{3} \langle J_1 \rangle^3 (Y_1) + 8 \langle J_1 \rangle^2 j_{\frac{1}{2}} (J_1) \bar{j}_{\frac{1}{2}} + 4 \langle J_1 \rangle (Y_1) (Y_2) + 2 \langle J_2 \rangle (Y_1)^2 \\
- 4 \left( \langle J_2 \rangle j_{\frac{1}{2}} \langle Y_1 \rangle \bar{j}_{\frac{1}{2}} + \langle J_1 \rangle j_{\frac{1}{2}} \langle Y_2 \rangle \bar{j}_{\frac{1}{2}} + j_{\frac{1}{2}} (Y_1) \bar{j}_{\frac{1}{2}} + j_{\frac{3}{2}} (Y_1) \bar{j}_{\frac{3}{2}} \right) \right\} - \{\text{barred terms}\} \right] \tag{4.39} \]

\[ \mathcal{C}_{[4],4} = -\text{tr} \left[ \left\{ \frac{1}{3} \left\{ 2 \langle J_1 \rangle^3 (J_1) + \frac{3}{2} \langle J_1 \rangle \{(J_1), (Y_2)\} + \frac{5}{2} \langle J_1 \rangle \{(J_2), (Y_1)\} + 2 \langle J_2 \rangle \{(J_1), (Y_1)\} \\
- \frac{1}{2} \langle Y_1 \rangle \{(J_2), (J_2)\} - 3 J_2 j_{\frac{1}{2}} \bar{J}_1 \bar{j}_{\frac{3}{2}} - 12 \langle J_2 \rangle j_{\frac{1}{2}} \langle J_1 \rangle \bar{j}_{\frac{1}{2}} - J_1 \left( j_{\frac{3}{2}} \bar{J}_1 \bar{j}_{\frac{3}{2}} + j_{\frac{3}{2}} \bar{J}_1 \bar{j}_{\frac{3}{2}} \right) \\
+ 6 \langle J_1 \rangle (j_{\frac{1}{2}} \langle J_1 \rangle) \bar{j}_{\frac{1}{2}} + j_{\frac{3}{2}} \langle J_1 \rangle \bar{j}_{\frac{3}{2}} - \frac{1}{2} j_{\frac{1}{2}} W_2 + \frac{1}{4} (W_3 Y_1 + W_2 Y_2) \right\} - \{\text{barred terms}\} \right] \tag{4.40} \]
In this way once we find the closed 5-form \( \hat{\mathcal{R}}_5 \), it is expanded as

\[
\hat{\mathcal{R}}_5 = \sum_{n=3}^{\infty} s^n \hat{\mathcal{R}}_5[n] , \quad d\hat{\mathcal{R}}_5[n] = 0 .
\] (4.41)

Each \( \hat{\mathcal{R}}_5[n] \) can be expressed as the exact form by the “expanded” MC equations (4.32) as

\[
\hat{\mathcal{R}}_5[n] = d\hat{\mathcal{C}}_4[n] .
\] (4.42)

Then the 4-form potential is given as

\[
\hat{\mathcal{C}}_4 = \sum_{n=3}^{\infty} s^n \hat{\mathcal{C}}_4[n] , \quad d\hat{\mathcal{C}}_4 = \hat{\mathcal{R}}_5 .
\] (4.43)

It is important that \( \hat{\mathcal{C}}_4 \) can not be written in terms of LI currents \( J \)'s, but in terms of “expanded” currents “\( J_n \)'s. From these facts we may deduce that the expanded 5-forms belong to trivial class of the 5-th CE cohomology group on \( G' = \{ \text{the “expanded” } \ GL(4) \times [Sp(4) \otimes GL(1)]^2 \} : \)

\[
d\hat{\mathcal{R}}_5[n] = 0, \quad \hat{\mathcal{R}}_5[n] = d\hat{\mathcal{C}}_4[n], \quad \hat{\mathcal{R}}_5[n] \in C^5(G', \mathbb{R}) , \quad \hat{\mathcal{C}}_4[n] \in C^4(G', \mathbb{R}) .
\] (4.44)

As a result the WZ term for an AdS D3-brane, which is integration of (4.26), is given as

\[
\mathcal{L}_{WZ,AdS,D3} = \hat{\mathcal{C}}_4 + dA \hat{\mathcal{C}}_2
\] (4.45)

with \( \hat{\mathcal{C}}_2 \) in (3.8).

## 5 Flat Limit

The super-AdS algebra is reduced into the supertranslation algebra by the IW contraction. In the rescaling of the coset coordinates (4.29) the parameter \( s \) plays the role of the inverse of the AdS radius, then the flat limit corresponds to \( s \to 0 \) limit.

We will see how the WZ term of the D3 brane in the super-AdS\( _5 \times S^5 \) background goes to one in the flat space. In order to get the correct flat limit of the WZ term, the closed \((p+2)\)-form \( \hat{\mathcal{R}}_{p+2} \) must be reconstructed in such a way that the super-AdS invariance is replaced by the supertranslation invariance. The flat limit of the DBI U(1) potential \( A \) must be taken suitably there.

The super-AdS variation mixes up different order in \( s \) as

\[
\delta_\epsilon,AdS = \delta_\epsilon,0 + s \delta_\epsilon,1
\]

\[
\delta_\epsilon,0 \theta^\dot{m}^\dot{b} = \epsilon^m \bar{n} \delta_\dot{b}^\dot{a}, \quad \delta_\epsilon,0 x^mb = \epsilon^m \bar{n} \bar{\theta}^b, \quad \delta_\epsilon,1 \theta^\dot{m}^\dot{b} = \epsilon^m \bar{n} \bar{x}^\dot{b}.
\] (5.1)
in the gauge \([128]\). Under the flat limit the super-AdS variation reduces to the super-translation \(\delta_{\epsilon, \text{AdS}} \rightarrow \delta_{\epsilon, \text{flat}} = \delta_{\epsilon, 0}\). Then expanded currents satisfies the following transformation rules

\[
\delta_{\epsilon, \text{AdS}} J = (\delta_{\epsilon, 0} + s \delta_{\epsilon, 1}) \sum s^n J_n \rightarrow \begin{cases}
\delta_{\epsilon, 0} j_2 = 0 \\
\delta_{\epsilon, 0} j_3 + \delta_{\epsilon, 1} j_2 = 0 \\
\vdots \\
\delta_{\epsilon, 0} J_1 = 0 \\
\delta_{\epsilon, 0} J_2 + \delta_{\epsilon, 1} J_1 = 0 \\
\vdots
\end{cases}
\] (5.2)

The lowest order terms in \(s\) of invariant forms are invariant under \(\delta_{\epsilon, 0}\). The following relations are also useful

\[
\begin{cases}
\delta_{\epsilon, 0} \hat{R}_{[5]3} = 0 \\
\delta_{\epsilon, 0} \hat{R}_{[5]4} + \delta_{\epsilon, 1} \hat{R}_{[5]3} = 0 \\
\vdots \\
\delta_{\epsilon, 0} R_{[3]2} = 0 \\
\delta_{\epsilon, 0} R_{[3]3} + \delta_{\epsilon, 1} R_{[3]2} = 0
\end{cases}
\] (5.3)

and

\[
\begin{cases}
\delta_{\epsilon, 0} B_{[2], 1}^{\text{NS}} = 0 \\
\delta_{\epsilon, 0} B_{[2], 2}^{\text{NS}} + \delta_{\epsilon, 1} B_{[2], 1}^{\text{NS}} = 0 \\
\vdots
\end{cases}
\] (5.4)

because \(B_{[2]}\) is bilinear in fermionic currents \(j\)’s.

In flat space the kinetic term of the D3-brane action is rescaled as \(\mathcal{L}_{D3} \rightarrow s^4 \mathcal{L}_{D3}\), so the WZ term should also be rescaled as \(\mathcal{L}_{WZ,D3} \rightarrow s^4 \mathcal{L}_{WZ,D3}\). Since exterior derivative \(d\) does not change the order of the expansion \(n\), the \(d\mathcal{L}_{WZ,\text{flat},D3}\) is given by

\[
d\mathcal{L}_{WZ,\text{flat},D3} = \lim_{s \to 0} \frac{1}{s^4} d\mathcal{L}_{WZ,\text{AdS},D3}
= \lim_{s \to 0} \frac{1}{s^4} \left( \hat{R}_{[5]} + dA \ R_{[3]} \right)
= \lim_{s \to 0} \frac{1}{s^4} \left( R_{[5]} + \mathcal{F} \ R_{[3]} \right)
= \lim_{s \to 0} \frac{1}{s} \mathcal{F}_{[2], 1} R_{[3], 2} + \left( R_{[5], 4} + \mathcal{F}_{[2], 1} R_{[3], 3} + \mathcal{F}_{[2], 2} R_{[3], 2} \right). \tag{5.5}
\]

There appear \(s^3\) terms which are singular in the flat limit. The singular term is absent if

\[
\mathcal{F}_{[2], 1} = dA_1 - B_{[2], 1}^{\text{NS}} = 0 \tag{5.6}
\]

for \(A = \sum_{n \geq 1} s^n A_n\). Since \(B_{[2], 1}^{\text{NS}}\) is an exact form it is absorbed in the DBI field \(A\) \([8]\), so that \((5.6)\) is realized. Now the DBI U(1) field strength is

\[
\mathcal{F} = \mathcal{F}_{[2], 2} = dA_2 - B_{[2], 2}^{\text{NS}} . \tag{5.7}
\]
$F$ is superinvariant but $A_2$ and $B_{[2],[2]}^{NS}$ are not separately superinvariant in the flat limit, contrasting to the super-AdS case in our formulation (3.7).

$\hat{\mathcal{R}}_{[5],4}$ is not supertranslation invariant from (5.3) and (5.4)

$$
\delta_{e,0} \hat{R}_{[5],4} = -\delta_{e,1} \hat{R}_{[5],3}
$$

$$
= (\delta_{e,1} B_{[2],[1]}^{NS}) \hat{R}_{[3],2} + B_{[2],[1]}^{NS} \left( \delta_{e,1} \hat{R}_{[3],2} \right)
$$

$$
= -\delta_{e,0} \left( B_{[2],[2]}^{NS} \hat{R}_{[3],2} + B_{[2],[1]}^{NS} \hat{R}_{[3],3} \right),
$$

(5.8)

but $R_{[5],4}$ is supertranslation invariant

$$
\delta_{e,0} R_{[5],4} = \delta_{e,0} \left[ \hat{R}_{[5],4} - \left( B_{[2],[2]}^{NS} \hat{R}_{[3],2} + B_{[2],[1]}^{NS} \hat{R}_{[3],3} \right) \right] = 0.
$$

(5.9)

The closed 5-form $R_{[5],4}$ in a flat space is consistent with known results [17, 18]. The WZ term for a D3-brane becomes

$$
L_{WZ, flat, D3} = \hat{C}_{[4],4} + dA_2 \hat{\gamma}_{[2],2},
$$

(5.10)

which is consistent with the flat expression [18] up to total derivative terms.

The MC equations of the supertranslation algebra is obtained by truncating the "expanded" MC equations (4.32) preserving the first two equations. The LI currents of the supertranslation algebra are only $\langle J_1 \rangle$, $j_{1/2}$, $\langle \bar{J}_1 \rangle$, $\bar{j}_{1/2}$. Other expanded currents are not supertranslation invariant, and they are rewritten in terms of the above currents with the non-constant coefficients $x$’s and $\theta$’s. The first term of the WZ term, $\hat{C}_{[4],4}$ in (4.40), contributes to the modification of the supercharges producing the topological D3-brane charge in a flat space which involves the D3-brane volume $\langle dx \rangle^3$. Its coefficient in the term is formally $\langle J_1 \rangle$ but it is not LI currents of the supertranslation algebra. After gauging away the symmetric part of $dx$ from $\langle J_1 \rangle$, it is written as $\langle J_1 \rangle \to \theta \bar{j}_{1/2}$;

$$
L_{WZ,AdS,D3} \sim \text{tr} \left[ \langle J_1 \rangle^3 \langle J_1 \rangle \right] \to L_{WZ,flat,D3} \sim \text{tr} \left[ \langle J_1 \rangle^3 \theta \bar{j}_{1/2} \right],
$$

(5.11)

Under the supertransformation, $\delta_{e,0}$, $L_{WZ,AdS,D3}$ is invariant from (5.2), but $L_{WZ,flat,D3}$ is not invariant any more. There is no possibility of modification of the supercharge involving $dx^3$ by $L_{WZ,AdS,D3}$. On the other hand there arises contribution to the supercharge by $L_{WZ,flat,D3}$, and we know that taking anticommutator of the supercharges gives the D3-brane charge involving $dx^3$. Since the D3-brane charge must involve $\langle dx \rangle^3$ in both flat space and curved space, it is enough to conclude that there is no D3-brane charge in the AdS superalgebra. In this paper we did not consider other types of the general topological terms which vanish when $\theta \to 0$ [19].
6 Discussion and conclusions

We have obtained concrete expression of the WZ terms for Dp-branes with $p = -1, 1, 3$ in the $\text{AdS}_5 \times S^5$ space. D-instanton is described by the auxiliary degree of freedom of $GL(1)$ in the Roiban-Siegel formalism. We have shown that the WZ term for a D3-brane in the AdS space can not be written in terms of the LI currents but in terms of the “expanded” currents. This “expansion” procedure can be generalized as the prescription of obtaining an expression of a $(p+1)$-form WZ term for a $p$-brane in AdS spaces.

Flat limit of the WZ term for a D3-brane is also examined. Since the super-AdS transformation rules depend on the expansion parameter, the supertranslation invariance and the closure of the WZ term must be examined carefully in the flat limit. The degree of freedom of the DBI field plays essential role to absorb the divergent term in the flat limit.

The “would-be” topological term, $dx^3$, in the flat WZ term can be traced back into the AdS WZ term as $\text{tr}(J_1)^3(J_1)$. This term keeps the left-invariance $(\delta_{\nu,0})$ in the AdS space, so the corresponding term does not modify the supercharges in the AdS space. There is no topological extension of the super-AdS algebra. The indices of the “would-be” topological term, $\langle J \rangle_3^{(ab)}$, has the same indices as the Lorentz rotation generator $M_{(ab)}$, so it can be absorbed into the Lorentz rotation generator analogous to [10].

For a fundamental superstring (F1) case the AdS-WZ term is given by $\mathcal{L}_{WZ, \text{AdS}, F1} = B^{NS}_{[2]}$ as (3.1). In flat space the action of the F1-brane is rescaled as $L_{F1} \rightarrow s^2 L_{F1}$; $\mathcal{L}_{WZ, \text{flat}, F1} = \lim_{s \rightarrow 0} \frac{1}{s^2} \mathcal{L}_{WZ, \text{AdS}, F1}$, so the singular term $B^{NS}_{[2],1}$ must be subtracted in the limiting procedure. The non-singular term $B^{NS}_{[2],2}$ corresponds to the familiar WZ term for a flat superstring. Since $B^{NS}_{[2],1}$ is a total derivative term, it may be also subtracted from the WZ term for a superstring in the AdS space

$$\tilde{L}_{WZ, \text{AdS}, F1} = B^{NS}_{[2]} - B^{NS}_{[2],1} = \int_0^1 dt H^{NS}_{[3]}(t)$$

which corresponds to the one given by Metsaev and Tseytlin [4]. As shown in (5.4) $B^{NS}_{[2],1}$ is not super-AdS invariant but pseudoinvariant, $\tilde{L}_{WZ, \text{AdS}, F1}$ is also pseudoinvariant under the super-AdS transformation. In this case we have a string charge in the anticommutator of supercharges which corresponds to the one calculated in [3].

\footnote{We thank Peeters and Zamaklar for suggesting the following consideration. In our notation the}
containing the string charge satisfies the Jacobi identity. This formulation manifests the topological term in the flat limit. In the AdS space it seems that both \( \mathcal{L}_{WZ, AdS,F1} \) in (3.1) and \( \tilde{\mathcal{L}}_{WZ, AdS,F1} \) in (6.2) are allowed.

The point is that WZ terms for branes in the flat space can not be superinvariant resulting the brane charges in their superalgebras, but the ones in the AdS space can be “superinvariant” resulting no brane charges. Careful flat limiting leads to the correct flat BPS condition realized as the topological extension of the superalgebra.

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Subtracted total derivative term is \( B_{[2,1]}^{NS} = \frac{1}{2} \text{tr} (d\bar{d}d\theta - d\bar{d}d\bar{\theta}) \). Under the super-AdS transformation in this gauge the WZ term becomes total derivative term as

\[
\delta_\epsilon \tilde{\mathcal{L}}_{WZ, AdS,F1} = d \text{ tr} (d\delta_\epsilon \theta \bar{\theta} - d\delta_\epsilon \bar{\theta} \theta) \tag{6.3}
\]

where \( dx \) is preferred than bare \( x \) at the time-component surface term. We denote the global GL(4|4) generators as \( G_{MN} = Z_M A^A_{\Pi_N} \) with canonical conjugates \( \Pi^A_M \). These charges get total derivative term contribution from \( \tilde{G}_{m\bar{n}} = G_{m\bar{n}} + \int (\partial_\sigma x_m^a \bar{\theta})_{n\bar{a}}, \quad \tilde{G}_{\bar{m}n} = \tilde{G}_{\bar{m}n} - \int (\partial_\sigma \bar{x}_{\bar{m}}^\bar{a}) \theta_{n\bar{a}}, \tag{6.4}
\]

then anticommutator of \( \tilde{G}_{m\bar{n}} \)'s produces the string charge as

\[
\{ \tilde{G}_{m\bar{n}}, \tilde{G}_{i\bar{k}} \} = \Omega_{m\bar{n}} \int \partial_\sigma (x_m^a x_{\bar{i}a}) \tag{6.5}
\]

which corresponds to the string charge in [8].
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