Meromorphic Differentials with Twisted Coefficients
on Compact Riemann Surfaces

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Abstract

This note is to concern a generalization to the case of twisted coefficients of the classical theory of Abelian differentials on a compact Riemann surface. We apply the Dirichlet’s principle to a modified energy functional to show the existence of differentials with twisted coefficients of the second and third kinds under a suitable assumption on residues.

1 Main results and discussion

Let $\bar{X}$ be a compact Riemann surface. Classically, one knows that a meromorphic (Abelian) differential can be expressed as a sum of three kinds of differentials, one of which is holomorphic, the second one differentials of the second kind, i.e. all its poles having residues 0, and the last one differentials of the third kind, i.e. its poles being log-pole. A classical problem is, fixing some points in $\bar{X}$, how to construct such a differential with poles at the fixed points, provided that the sum of residues be zero. This was completely solved, e.g. by using the Dirichlet’s principle on certain modified energy functional (cf. [8]). Briefly, the results are as follows: For arbitrarily given point $p$ of $\bar{X}$ with a local coordinate $z$ around $p$ and arbitrary integer $k \geq 1$, one can find a differential $\phi$ of the second kind such that $p$ is the only pole of $\phi$ and $\phi$ has the following asymptotic behavior near $p$

$$z^{-k-1}dz;$$ (1)

for arbitrarily given two points $p_1, p_2$ of $\bar{X}$, there exists a differential $\phi$ of the third kind such that $p_1, p_2$ are the only log-poles of $\phi$ and the residues of $\phi$ are 1, $-1$ at $p_1, p_2$ respectively; the general case can be obtained by combining the above two. As mentioned above, the method is the Dirichlet’s principle; by using the Dirichlet’s principle on a certain modified energy functional, one can get a harmonic function $u$ with prescribed asymptotic behaviors at the

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given points and then $\partial u$ is the required Abelian differential; the key is the requirement that the sum of residues be zero.

In this note, we want to generalize this classical theory to the twisted case. Let $\rho : \pi_1(\overline{X}) \to GL(n, \mathbb{C})$ be a linear representation of $\pi_1(\overline{X})$, $L_\rho$ the corresponding flat vector bundle, $D$ the canonical flat connection on $L_\rho$. A Hermitian metric $h$ on $L_\rho$ can be canonically explained as a $\rho$-equivariant map from the universal covering of $\overline{X}$ into $GL(n, \mathbb{C})/U(n)$ (equivalently, the set of all positive definite Hermitian symmetric matrices, denoted by $\mathcal{P}_n$), still denoted by $h$. Then, the differential $(dh)h^{-1}$ is a one-form valued in $End(L_\rho)$. The condition that the differential $(\partial h)h^{-1}$ is holomorphic is then read as

$$\overline{\Omega}((\partial h)h^{-1}) = 0;$$

equivalently, the map (metric) $h$ is harmonic (if $\overline{X}$ is higher dimensional, $h$ is pluri-harmonic). We consider $End(L_\rho)$ as our twisted coefficient. Then, our purpose of this note is to find meromorphic one-forms with value in $End(L_\rho)$, which have prescribed singularities, similar to classical Abelian differentials.

In order to find such differentials, we assume that the representation $\rho : \pi_1(\overline{X}) \to GL(n, \mathbb{C})$ in question is semi-simple (for the precise definition, see §3). We attempt to find certain special $\rho$-equivariant harmonic map (harmonic metric) on $L_\rho$ with (possible) singularities; equivalently, this means we apply the Dirichlet’s Principle to certain modified energy functional to get some special critical points $h$ so that $(\partial h)h^{-1}$ are the desired ones. We develop the variational technique of Siegel \[8\] so that it is appropriate for the present nonlinear setup; this is one of main points of this note.

In the following, we briefly describe our main results and their proofs. Let me first show what our singularities look like. As in the classical theory, we consider two kinds of singularities. We first consider the second kind. Fix arbitrarily a point $p \in \overline{X}$ and restrict ourself to a disk $\Delta$ with center at $p$. Let $z = x + \sqrt{-1}y$ be an Euclidean complex coordinate with $z(p) = 0$. Restrict the flat bundle $L_\rho$ to $\Delta$ and fix a suitable flat basis of $L_\rho$ on $\Delta$. Then, under the fixed basis, the asymptotic behavior \[2\] at the point $p$ of the desired harmonic metrics is of the following form

$$\exp\left(\begin{array}{ccc}
\sum a_{k_1}u_{k_1} & 0 & \\
0 & \ddots & \\
0 & & \sum a_{k_n}u_{k_n}
\end{array}\right)$$

where $k_1, k_2, \cdots, k_n \in \mathbb{N}$, $a_{k_1}, \cdots, a_{k_n} \in \mathbb{R}$, and $u_k = 2\text{Re}(z^{-k})$. It is easy to see that if a harmonic metric $K$ has the above asymptotic behavior, the cor-

\[1\] for the precise definition of the asymptotic behavior of a metric at a puncture, see §4.
responding differential \((\partial K)K^{-1}\) then has the following asymptotic behavior

\[
\left( -\sum k_1a_{k_1}z^{-k_1} \quad \cdots \quad 0 \right)
\left( 0 \quad \cdots \quad -\sum k_n a_{k_n}z^{-k_n} \right) \frac{dz}{z},
\]

(3)

Our first result is then the following:

**Theorem 1** Let \(\rho : \pi_1(X) \to \text{Gl}(n, \mathbb{C})\) be a semi-simple representation (for the precise definition, see \(\S 3\)), \(p_1, p_2, \ldots, p_s\) arbitrarily given points of \(\overline{X}\); by \(X\) denote \(\overline{X} \setminus \{p_1, p_2, \ldots, p_s\}\). Let \(L_\rho\) be the corresponding flat bundle restricted to \(X\). Then, for arbitrarily given asymptotic behaviors of the form (2) at the punctures \(p_1, p_2, \ldots, p_s\), there exists a unique harmonic metric \(K\) on \(L_\rho\) with the corresponding asymptotic behaviors; hence the differential \((\partial K)K^{-1}\) is a holomorphic one-form with twisted coefficient which has asymptotic behavior of the form (3).

We now consider the singularities of the third kind. Fix arbitrarily points \(p_1, p_2, \ldots, p_s \in \overline{X}\), take a smooth curve \(\gamma\) connecting them, say, the starting point \(p_1\), the end point \(p_s\); take a small enough tube neighborhood \(\Gamma\) of \(\gamma\) so that they are simply-connected. Now, assume that under a fixed flat basis of \(L_\rho\) on \(\Gamma\), the desired differentials at each \(p_i\) have prescribed singularity of the following form

\[
\left( a_1^i \quad \cdots \quad 0 \right)
\left( \cdots \quad 0 \quad \frac{dz^i}{z^i} \right),
\]

(4)

where \(z^i\) is a local complex coordinate at \(p_i\) and \(a_1^i, \ldots, a_n^i \in \mathbb{R}\). Then, we have the following:

**Theorem 2** Let \(\rho\) as before, \(p_1, \ldots, p_s\) arbitrarily fixed points on \(\overline{X}\); by \(X\) denote \(\overline{X} \setminus \{p_1, \ldots, p_s\}\). Let \(L_\rho\) be the corresponding flat bundle restricted to \(X\). Then, for arbitrarily given asymptotic behaviors of the form (4) at the points \(p_i\) such that the \(a_j^i\) are rational numbers (actually, we can assume the ratios of \(a_j^i\) and \(a_j^j\) are rational; see \(\S 3\)) and \(\sum_{i=1}^s a_j^i = 0, j = 1, \ldots, n\), there exists a unique harmonic metric \(K\) on \(L_\rho\), the differential of which \((\partial K)K^{-1}\) is a holomorphic one-form with twisted coefficient and has asymptotic behavior of the form (4) at each point \(p_i\).

We now outline the proof of the theorems. The proofs of Theorem 1 and 2 are completely similar, so we outline only that of Theorem 1. It is clear that the harmonic metrics with prescribed singularities are always of infinite energy. So, the variational technique for the usual energy functional does not work anymore. In order to overcome this difficulty, we use a modified energy functional, which is roughly defined as follows. Let \(K\) be the set of continuous
and piece-wise differentiable metrics on $L_\rho$ which have the asymptotic behaviors mentioned above near each puncture $p_i$. For a metric $K \in \mathcal{K}$, we define its modified energy as

$$
\hat{E}(K) = \int_{X \setminus \bigcup_{i=1}^{s} \Delta^*_i} |(dK)K^{-1}|^2 + \sum_{i=1}^{s} \int_{\Delta^*_i} |(dK)K^{-1} - (dK_0)K_0^{-1}|^2,
$$

(5)

where $K_0$ is a suitably constructed metric with the asymptotic behaviors above at each puncture $p_i$, $\Delta^*_i$ is a small punctured disk around $p_i$. We remark that, in the definition of the modified energy, we use the difference of the derivatives of two maps, which applies to the only case when the target manifolds are homogeneous. Then, we will prove that one can minimize the modified energy functional $\hat{E}(K)$ in $\mathcal{K}$ and the minimizer is a (smooth) harmonic metric with prescribed asymptotic behaviors at the punctures.

In order to minimize the modified energy functional $\hat{E}$, technically, we first need to construct a suitable initial metric $K_0$, which is harmonic near each puncture $p_i$ and has not only prescribed asymptotic behaviors at the punctures but also vanishing radial derivatives on a certain circle around each puncture. We would like to point out that both the harmonic property of $K_0$ around the punctures and vanishing radial derivatives of $K_0$ on a certain circle around each puncture are very key for our proof. After this construction, we choose a minimizing sequence of $\hat{E}$ in $\mathcal{K}$. Generally, such a minimizing sequence does not necessarily converge. In order to make such a sequence to converge, we have to modify it. To this end, we first use harmonic metrics to replace continuously each metric of the minimizing sequence on $X \setminus \bigcup_i \Delta^*_i$. It is clear that the new sequence is still a minimizing one of $\hat{E}$; furthermore, using the semi-simplicity of $\rho$, we can show that the new sequence (if necessary, going to a subsequence) is actually uniformly convergent on any compact subset of $X \setminus \bigcup_i \Delta^*_i$. We continue to modify the new minimizing sequence on the remaining part. For this, we need to solve a boundary value problem for harmonic metrics with prescribed asymptotic behavior at the puncture on a punctured disk (Proposition 4). After solving such a boundary problem, we then use such a solution to replace continuously each metric of the new minimizing sequence on a greater disk than $\Delta^*_i$ around each puncture; we can show that the sequence obtained is still a minimizing one (Proposition 5). Using the previous convergence, we can finally show that the sequence obtained is uniformly convergent on $X$ and the limit lies in $\mathcal{K}$.

From the above description, it is easy to see that our proof for convergence of minimizing sequences is slightly different from that of Siegel; we use a two-step modification of minimizing sequences and the semi-simplicity of the representation $\rho$. In fact, although the argument for convergence of Siegel can be explained as the case of one-dimensional trivial representations, it, due to the nonlinearity of maps, does not however apply to the present setting.

The idea of modifying energy functional was also used by Ding in [4] to deal
with the problems of harmonic maps with infinite energy. Due to generality
of the target manifolds he considered, he can not use the difference of the
derivatives of two maps; instead, he used the integration by parts on bounded
domains of the domain manifolds and then an approximation process.

Naturally, one should ask if there exists a holomorphic one-form with
twisted coefficients but without singularity. In the case of complex coeffi-
cient, this is a well-known result; the dimension of the set of such differentials
is the genus of \( X \). In the case of twisted coefficient, this is actually a conse-
quence of Donaldson’s result \[5\]; in the case of higher dimension, this is also
true by means of Corlette’s result \[4\] and Siu’s Bochner technique. In a future
paper, we will generalize the results and the method of the present paper to
the higher dimension case.

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2 The energy functional and the equation

In this section, for convenience, we fix some notations and state some more or
less standard facts (cf. e.g. \[9, 10\]). Let \( \mathcal{P}_n \) be the set of all positive definite
hermitian symmetric matrices of order \( n \). \( GL(n, \mathbb{C}) \) acts transitively on \( \mathcal{P}_n \) by

\[
g \circ A = gA^t \bar{g}, \quad A \in \mathcal{P}_n, g \in GL(n, \mathbb{C}).
\]

Obviously, the action has the isotropic subgroup \( U(n) \) at the identity \( I_n \). Thus
\( \mathcal{P}_n \) can be identified with the coset space \( GL(n, \mathbb{C})/U(n) \), and can be uniquely
endowed an invariant metric\footnote{In terms of matrices, such an invariant metric can be defined as follows. At the identity \( I_n \), the tangent elements just are hermitian matrices; let \( A, B \) be such matrices, then the Riemannian inner product \( \langle A, B \rangle_{\mathcal{P}_n} \) is defined by \( \text{tr}(AB) \). In general, let \( H \in \mathcal{P}_n \), \( A, B \)
two tangent elements at \( H \), then the Riemannian inner product \( \langle A, B \rangle_{\mathcal{P}_n} \) is defined by \( \text{tr}(AH^{-1}BH^{-1}) \).} up to some constants. In particular, under such
a metric, the geodesics through the identity \( I_n \) are of the form \( \exp(th) \), \( t \in \mathbb{R} \),
\( h \) being a hermitian matrix.
Let $X$ be a complex manifold, $V \to X$ a flat vector bundle, $K$ a hermitian metric on $V$. For $x \in X$, the metric $K_x$ on the fiber $V_x$, after fixing a basis, can be considered as an element $H_x \in \mathcal{P}_n$, and hence a point in the coset space $GL(n, \mathbb{C})/U(n)$. Thus, after fixing a flat basis of $V$, the metric $K$ can be considered as an equivariant map from the universal covering of $X$ into $GL(n, \mathbb{C})/U(n)$ or $\mathcal{P}_n$.

From now on, we always fix a flat basis $\{v_i\}$ of $V$, unless stated otherwise. Decompose the flat connection $D = d' + d''$ into the parts of type $(1, 0)$ and $(0, 1)$. Define the differential operators $\delta'$ and $\delta''$ by setting

$$\partial < u, v >_K = < \delta' u, v >_K + < u, d'' u >_K,$$
$$\overline{\partial} < u, v >_K = < \delta'' u, v >_K + < u, d' u >_K,$$

namely, both $\delta' + d''$ and $d' + \delta''$, as connection on $V$, preserve the metric. Clearly $d''d' + d'd'' = 0$ implies $\delta'\delta'' + \delta''\delta' = 0$. Set

$$\theta_K = (d' - \delta')/2, \overline{\theta}_K = (d'' - \delta'')/2,$$
$$\partial \theta_K = (d' + \delta')/2, \overline{\partial} \theta_K = (d'' + \delta'')/2.$$

It is easy to see that $< \theta_K u, v > = < u, \overline{\theta}_K v >$ and $\partial \theta_K + \overline{\partial} \theta_K$ preserves the metric. $\theta_K$ (resp. $\overline{\theta}_K$) is a one-form of type $(1, 0)$ (resp. $(0, 1)$) valued in $\text{End}(V)$. On the other hand, one can explicitly write down $\theta_K$ in terms of the basis $\{v_i\}$ as follows. Setting $H = (H_{i\overline{j}}) = < v_i, v_j >_K$, one has then

$$\partial H_{i\overline{j}} = \partial < v_i, v_j >_K = < \delta' v_i, v_j >_K = < \delta v_i, v_j >_K = -2 < \theta_K v_i, v_j >_K.$$

Writing $\theta_K = \theta_{i\alpha} v_k \otimes v^i \otimes dz^\alpha$ ($\{v^i\}$ is the dual basis of $\{v_i\}$, $\{z^\alpha\}$ is a local coordinate of $X$), one then has

$$\partial H_{i\overline{j}} = -2\theta_{i\alpha}^{k} H_{k\overline{j}} dz^\alpha,$$

namely, $\partial H_{i\overline{j}} H^{\overline{k}\overline{j}} = -2\theta_{i\alpha}^{j} dz^\alpha$ (or invariantly, $\partial HH^{-1} = -2\theta_K$), where $(H_{k\overline{j}})$ is the inverse of $(H_{i\overline{j}})$. Thus, $\theta_K$ (resp. $\overline{\theta}_K$) can be identified with the differential of type $(1, 0)$ (resp. $(0, 1)$) of the map into $\mathcal{P}_n$ corresponding to the metric $K$, up to some constant.

Remark. $\delta' + d'' = D - 2\theta_K$ can also be regarded as a (hermitian) connection on the flat $(d''$-holomorphic) bundle $V$ with respect to $K$, so that

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3Actually, the construction of the operators $\partial_K, \overline{\partial}_K, \theta_K, \overline{\theta}_K$ comes essentially from the Cartan decomposition of the flat connection $D$ with respect to $\mathfrak{g}(n, \mathbb{C}) = \mathfrak{u}(n) + \mathfrak{p}_n$, where $\mathfrak{p}_n$ is the set of Hermitian matrices of order $n$. This, together with the fact that $\mathcal{P}_n$ is homogeneous, implies that these operators are invariant under certain sense, as will be used in various computations of the note very often. The point can be more explicitly seen if we consider $K$ as an equivariant map into $\mathcal{P}_n$, so that the connection $\partial_K + \overline{\theta}_K$ is the pull-back of the standard invariant connection of $\mathcal{P}_n$.\hfill\square
$-2\theta_K$ is just the corresponding connection form under the fixed flat basis, as will be used in the following. The similar explanation works for $d' + \delta''$ and $\overline{\theta}_K$.

Usually, one needs to choose some "nice" metrics of $V$, which furthermore satisfy some differential equations. To this end, we from now on assume that $X$ is a Kähler manifold with a Kähler metric $\omega$, and denote by $\Lambda$ the adjoint of the operation of wedging with $\omega$ on exterior forms of $X$. Set

$$D''_K = \overline{\partial}_K + \theta_K, \quad D'_K = \partial_K + \overline{\theta}_K$$

and $G_K = (D''_K)^2$; call $D''_K$ the Higgs operator and $G_K$ the pseudo-curvature. Call a metric $K$ on the flat bundle $V$ harmonic if it satisfies

$$\Lambda G_K = 0.$$  \hfill (6)

Equivalently, this can be written as

$$\Lambda (d' \delta'' + \delta'' d' - \delta' d'' - d'' \delta') = 0.$$  \hfill (6)

In the following, we will show that the metric $K$ being harmonic is equivalent to the corresponding map being a harmonic map, and hence the equation $\Lambda G_K = 0$ is a variational one.

First, let us see how $\theta_K$ and $\overline{\theta}_K$ change when the metric $K$ changes. Let $K_1, K_0$ be two metrics on $V$. One can then define an endomorphism $h$ of $V$ by setting $< u, v >_{K_1} = < hu, v >_{K_0}$. It is clear that $h$ is self-adjoint positive with respect to $K_0$. Under the fixed basis $\{v_i\}$, write $h = (h^j_i)$, i.e. $hv_i = h^j_i v_j$; also write $K_s$ as the matrix $(H_{sij})$, the inverse of which is denoted by $(h^{-1})_{ij}$, $s = 0, 1$. Then $H_{1ij} = h^k_i H_{0kj}$ and $H_{1}^{ji} = H_{0}^{jk} (h^{-1})_{kj}$, here $h^{-1}$ is the inverse of $h$. Thus

$$-2\theta_{K_1} = \partial H_1 H_1^{-1} = (\partial h H_0 + h \partial H_0) H_0^{-1} h^{-1} = \partial hh^{-1} + h \partial H_0 H_0^{-1} h^{-1} = \partial hh^{-1} - 2h \theta_{K_0} h^{-1} = (\partial h - 2h \theta_{K_0} + 2\theta_{K_0} h - 2\theta_K h) h^{-1} = \delta' hh^{-1} - 2\theta_{K_0},$$

in the last equality, $h$ is considered as a section of $\text{End}(V)$, and $\delta' h$ is the covariant derivative of $h$ (referring to the above remark). Similarly,

$$\overline{\theta}_{K_1} = -(1/2) \delta'' hh^{-1} + \overline{\theta}_{K_0}. \hfill (7)$$

The above computation is very similar to that in the Hermitian-Yang-Mills theory (cf. e.g. [11]).
For later purpose, let’s here make some simple remarks about both inner products on the bundle \( \text{End}V \) and the tangent bundle of \( P_n \). In the following arguments, we ignore integrability of integrals. For \( \text{End}V \), we always use the trace inner product \( \langle A, B \rangle = tr(A^tB) \) under the fixed flat basis; it is easy to see that when endomorphisms are from tangent elements of \( P_n \), the trace inner product coincides with the invariant inner product on \( P_n \). Fix a metric \( K \) on \( V \) (and hence \( \text{End}(V) \)) and a point \( x \in X \). The metric \( K_x \) on the fiber \( V_x \) corresponds to the matrix \( H_x \in P_n \). Let \( A, B \) be two elements in the tangent space \( T_{H_x}P_n \). Then \( AH_x^{-1}, BH_x^{-1} \) can be considered as two self-adjoint homomorphisms of \( V_x \) with respect to \( K_x \). Using an orthogonal basis of \( V_x \) with respect to \( K_x \), it is easy to show that

\[
\langle A, B \rangle_{P_n} = tr(AH_x^{-1}BH_x^{-1}) = \langle AH_x^{-1}, BH_x^{-1} \rangle_{K_x}.
\]

This together with the previous argument concerning \( \theta_K \) and \( \overline{\theta}_K \) implies

\[
\int_X \left( \langle \theta_K H, \theta_K H \rangle_{P_n, \omega} + \langle \overline{\theta}_K H, \overline{\theta}_K H \rangle_{P_n, \omega} \right)
\]

\[
= \int_X \left( \langle \theta_K, \theta_K \rangle_{K, \omega} + \langle \overline{\theta}_K, \overline{\theta}_K \rangle_{K, \omega} \right).
\]

Based on these remarks, afterwards, we often omit the subscripts of the inner products, since it is clear from the context.

Furthermore, both integrals above are independent of choice of a basis, though defined by choosing a basis; and the first integral, up to a constant, is just the energy of the map \( H \) corresponding to the metric \( K \); for simplicity, we call it the energy of the metric \( K \), denoted by \( E(K) \), i.e.,

\[
E(K) = \int_X \left( \langle \theta_K, \theta_K \rangle + \langle \overline{\theta}_K, \overline{\theta}_K \rangle \right).
\]  (8)

Thus, taking the first variation for either of both integrals with respect to \( K \), we will get the equation of harmonic maps for \( K \). As usual, we do this for the first integral and, due to \( P_n \)'s homogeneity, we can easily pass to an integral related to the bundle \( V \). Finally the Euler-Lagrange equation is just the equation of harmonic metrics \( \Lambda G_K = 0 \).

Let \( h \) be a self-adjoint (not necessarily positive) endomorphism of \( V \) with respect to \( K \). Then \( \exp(th) \) is self-adjoint positive, \( t \in \mathbb{R} \). Set \( H_t = \exp(th)H \), denote the corresponding metric by \( K_t \), here \( H = (H_{ij}) = \langle v_i, v_j \rangle_K = H_0 \). From the previous computation, one has

\[
-2\theta_{K_t} = \delta'(\exp(th)) \exp(-th) - 2\theta_K = t\delta' h - 2\theta_K + o(t).
\]
Similarly, \(-2\overline{\theta}_K = t\delta'' h - 2\overline{\theta}_K + o(t)\). Thus,
\[
\frac{d}{dt} E(K_t)_{|t=0} = \frac{d}{dt} \int_X \left( \langle \theta_K H_t, \theta_K H_t \rangle_{\mathcal{P}_n, \omega} + \langle \overline{\theta}_K H_t, \overline{\theta}_K H_t \rangle_{\mathcal{P}_n, \omega} \right)_{|t=0} \\
= -\int_X \left( \langle \theta_K H, (\delta' h) H \rangle_{\mathcal{P}_n, \omega} + \langle \overline{\theta}_K H, (\delta'' h) H \rangle_{\mathcal{P}_n, \omega} \right) \\
= -\int_X \left( \langle \theta_K, \delta' h \rangle_{K, \omega} + \langle \overline{\theta}_K, \delta'' h \rangle_{K, \omega} \right) \\
= -\int_X \left( (\delta')^* \theta_K, h \right)_{K, \omega}.
\]

Since \(\delta' + d''\) (resp. \(d' + \delta''\)) is a hermitian connection on \(\mathcal{V}\) with respect to \(d''\) (resp. \(\delta''\)) and \(K\), so one has the related Kähler identity \((\delta')^* = \sqrt{-1}[\Lambda, d'']\) (resp. \((\delta'')^* = -\sqrt{-1}[\Lambda, d']\)). Thus,
\[
\frac{d}{dt} E(K_t)_{|t=0} = -\int_X \sqrt{-1} \left( \langle \Lambda d''(\theta_K), h \rangle_{K, \omega} - \langle \Lambda d'(\overline{\theta}_K), h \rangle_{K, \omega} \right) \\
= -\int_X \sqrt{-1} < \Lambda (d''(\theta_K) - d'(\overline{\theta}_K)), h >_{K, \omega}.
\]

So, the E-L equation is
\[
\sqrt{-1} \Lambda (d''(\theta_K) - d'(\overline{\theta}_K)) = 0,
\]
which is just equivalent to the equation \(\Lambda G_K = 0\) for \(K\) being harmonic. Summing up all the above argument, we have

**Proposition 1** Let \((X, \omega)\) be a Kähler manifold, \(\mathcal{V} \to X\) a flat vector bundle. Giving a metric \(K\) on \(\mathcal{V}\) and using the above notations, one has the energy functional
\[
E(K) = \int_X |\theta_K + \overline{\theta}_K|^2,
\]
the E-L equation of which is
\[
\sqrt{-1} \Lambda (d''(\theta_K) - d'(\overline{\theta}_K)) = 0,
\]
i.e. \(\Lambda G_K = 0\).

### 3 The construction of initial metrics

Let \(X\) be a differentiable manifold. In general, we call a linear representations \(\rho : \pi_1(X) \to Gl(n, \mathbb{C})\) semi-simple if the Zariski closure in \(Gl(n, \mathbb{C})\) of the image of \(\rho\), as an algebraic subgroup, is semi-simple. Here, for convenience of the later application, we state a more geometric definition of semi-simplicity, which is motivated by Donaldson [5]: for this, we need to use a little bit knowledge about the boundary theory of symmetric spaces (cf. e.g. [1]).
Let \( \rho : \pi_1(X) \to \text{Gl}(n, \mathbb{C}) \) be a linear representation. Call \( \rho \) semi-simple if for any boundary component \( \Sigma \) of \( P_n \), there exists an image element \( \rho(\gamma), \gamma \in \pi_1(X) \) satisfying \( \Sigma \cap \rho(\gamma)(\Sigma) = \emptyset \).

From now on, we assume that \( X \) is an open Riemann surface, i.e. \( X = \overline{X} \setminus \{p_1, p, \cdots, p_s\} \) for a compact Riemann surface \( \overline{X} \), as mentioned in the §1.

Let \( \rho : \pi_1(\overline{X}) \to \text{Gl}(n, \mathbb{C}) \) be a linear semi-simple representation. Let \( L_\rho \) be the corresponding flat bundle over \( X \); also by \( L_\rho \) denote the restriction to \( X \).

We first construct an initial metric on \( L_\rho \) needed by the proof of Theorem 1. Let \( \Delta_i^* \) (resp. \( \Delta_i^*/2 \)) be the punctured disk with radius 1 (resp. \( 1/2 \)) around \( p_i \) and \( t_i = r_i e^{\sqrt{-1} \theta} \) a complex (polar) coordinate on \( \Delta_i^* \) with \( t_i(p_i) = 0 \). Fix a flat basis of \( L_\rho \) which is clearly single-value on each \( \Delta_i^* \). Under this flat basis, we construct a metric of \( L_\rho \) on \( \Delta_i^* \) as follows:

\[
H_i = \exp \left( \begin{array}{ccc} \sum a_{k_1} u_{k_1} & 0 & \ldots \\ 0 & \ddots & 0 \\ \sum a_{k_n} u_{k_n} & & \sum a_{k_n} u_{k_n} \end{array} \right),
\]

where \( k_1, k_2, \ldots, k_n \in \mathbb{N} \), \( a_{k_1}, a_{k_2}, \cdots, a_{k_n} \in \mathbb{R} \), and

\[
u_k = \text{Re} \left( \frac{1}{t_i^k} + 4^{k} t_i^k \right).
\]

It is clear that the \( H_i \) is a harmonic metric on \( \Delta_i^* \) and

\[
\frac{\partial H_i}{\partial r_i} H_i^{-1} = 0, \quad \text{on } r_i = \frac{1}{2}.
\]

Extending the \( H_i \)'s to \( X \), we obtain the required initial metric of \( L_\rho \), denoted by \( K_0 \).

We now turn to the construction of the initial metrics needed in the proof of Theorem 2. As in §1, connect \( p_1, \cdots, p_s \) in a smooth curve \( \gamma \) and take two small enough neighborhoods \( \Gamma \subset \Gamma' \) of \( \gamma \) so that they are simply-connected. Take a compatible (with the complex structure of \( X \)) complex coordinate \( z \) on \( \Gamma' \) so that \( \Gamma \) and \( \Gamma' \) are two disks; without loss of generality, assume \( z(p_1) = 0 \). Put \( \Gamma' \) on the complex plane and take the reflections of \( p_2, \cdots, p_s \) denote by \( p'_2, \cdots, p'_s \) respectively. (If necessary, we can shrink \( \Gamma' \) so that the reflection points do not lie in \( \Gamma' \).) Denote the coordinates of \( p_2, p'_2, \ldots, p_s, p'_s \) by \( \xi_2, \xi'_2, \cdots, \xi_s, \xi'_s \) respectively. Construct a meromorphic function on \( \Gamma' \) as follows

\[
g(z) = z^{l_2 + \cdots + l_s} \prod_{i=2}^{s} (z - \xi_i)^{l_i} \cdot \prod_{i=2}^{s} (z - \xi'_i)^{l_i},
\]

where \( l_2, \cdots, l_s \in \mathbb{N} \); take the real part of the multiple-value function \( \log g(z) \), which is single-value, denoted by \( u_{l_2 \cdots l_s} \). A simple argument shows that the
radial derivatives of \( u_{l_2 \cdots l_s} \) along \( \partial \Gamma \) vanish. Then, under a fixed flat basis of \( L_{\rho} \) on \( \Gamma' \), our initial metric on \( L_{\rho} \) over \( \Gamma' \) is taken as the following

\[ H = \exp \begin{pmatrix} a_1 u_{l_1 2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n u_{l_n 2} \end{pmatrix}, \] (11)

where \( a_1, \cdots, a_n \in \mathbb{R} \). It is clear that \( H \) is a harmonic metric on \( \Gamma' \setminus p_1, p_2, \cdots, p_s \) and

\[ \frac{\partial H}{\partial r} H^{-1} = 0, \quad \text{on} \quad \partial \Gamma, \]

where \( r \) is the radial coordinate of \( z \). Now, we can extend \( H \) to \( X \) to get a desired initial metric on \( L_{\rho} \) with prescribed singularities, also denoted by \( K_0 \).

It is also clear that If a harmonic metric \( K \) has the above asymptotic behavior at the points \( p_i \), the corresponding differential \( (\partial K) K^{-1} \), under the fixed flat basis, has then asymptotic behavior of the following form at the points \( p_i \)

\[ \begin{pmatrix} b_1 & 0 \\ \vdots & \ddots \\ 0 & b_n \end{pmatrix} \cdot \frac{dz}{z}, \] (12)

where \( b_1, \cdots, b_n \in \mathbb{R} \).

4 The modified energy functional and minimizing sequences

In this section, we develop a variational technique to show the existence of a harmonic metric on \( L_{\rho} \) with the prescribed asymptotic behaviors at the punctures \( p_i \), as described in §3; since the proofs of Theorem 1 and 2 are the same, our discussion here is restricted to Theorem 1. In order to use the method of minimizing sequences to get such a harmonic metric, we need to modify the usual energy functional \( E(K) \). Due to conformal invariance, without loss of generality, we can take a special Riemannian metric \( \omega \) on \( X \) which is Euclidean on each \( \Delta_i^* \), i.e., \( \omega|_{\Delta_i^*} = \sqrt{-1} dt_i \wedge d\bar{t}_i \).

Call a (continuous and piece-wise differentiable) metric \( K (= hK_0 \) under a fixed flat basis) on \( L_{\rho} \) (or \( L_{\rho}|_{\cup_{i=1}^s \Delta_i^*} \)) admissible relative to \( K_0 \) if

1) The integral \( \sum_{i=1}^s \int_{\Delta_i^{*2}} (| \theta_K - \theta_{K_0} |^2 + | \bar{\theta}_K - \bar{\theta}_{K_0} |^2 ) \) exists; and

2) \( K \) and \( K_0 \) are mutually bounded (namely, if writing \( K = hK_0 \), the eigenvalues of \( h \) are uniformly far away from both 0 and \( \infty \)).
We remark that when one considers the metrics on $L_\rho$ as equivariant maps from the universal covering $\tilde{X}$ of $X$ into $\mathcal{P}_n$, the condition 2) is equivalent to

2)’ $K$ and $K_0$ have uniform bounded distance near the punctures under the invariant metric of $\mathcal{P}_n$.

The equivalence of 2) and 2)’ will be used very often in the following discussion. If two metrics $K_1, K_2$ satisfy the property 2) or 2)’, we say they have the same asymptotic behavior at the punctures.

Denote the set of admissible metrics $K$ on $L_\rho$ by $\mathcal{K}$. We then define the modified energy functional as

$$\hat{E}(K) = \int_{X \setminus \bigcup_{i=1}^s \Delta_{i/2}^*} |\theta_K + \bar{\theta}_K|^2 + \sum_{i=1}^s \int_{\Delta_{i/2}^*} \left( |\theta_K - \theta_{K_0}|^2 + |\bar{\theta}_K - \bar{\theta}_{K_0}|^2 \right),$$

for admissible hermitian metrics $K \in \mathcal{K}$ on $L_\rho$. For the modified energy functional, we have

**Proposition 2** Any admissible metric $K$ on $L_\rho$, if it minimizes the modified energy functional $\hat{E}$, is a harmonic metric (and hence smooth).

**Proof.** One only needs to use the same computation as in §2 (taking the first variation of $\hat{E}$ to obtain the E-L equation) and remarks that $K_0$ is harmonic on $\bigcup_{i=1}^s \Delta_i^*$ and has vanishing normal derivative on $\partial \Delta_{i/2}$. Mainly, one needs to consider variational domains containing (part of) $\partial \Delta_{i/2}$, then the condition of vanishing normal derivative applies; since if a variational domain lies completely in the interior of $X \setminus \bigcup_{i=1}^s \Delta_i^* \cup \bigcup_{i=1}^s \Delta_{i/2}^*$, the minimizer $K$ of $\hat{E}$ is naturally harmonic in such a domain using the harmonicity of $K_0$ on each $\Delta_i^*$ and the usual first variation computation.

Suppose that $D$ is a compact sub-domain containing (part of) $\partial \Delta_{i/2}$. Consider a one-parameter variation $K^t$ of $K$ with $K^0 = K$ and $K^t|_{X \setminus D} \equiv \hat{K}$; correspondingly, $K^t = h^t K$ with $h^0 = \text{id}$ and $h^t|_{X \setminus D} \equiv \text{id}$; furthermore, we can even assume that $h^t = e^{th}$ with $h$ being hermitian and $h|_{X \setminus D} \equiv 0$. For
sake of simplicity, by $X^c$ denote $X \setminus \bigcup_{i=1}^{s} \Delta_{i/2}^*$. Compute at $t = 0$

$$\frac{d}{dt}_{|t=0} \hat{E}(K_i) = \int_{X^c} \frac{d}{dt} \left| \theta_{K^t} + \overline{\theta}_{K^t} \right|^2 + \sum_{i=1}^{s} \int_{\Delta_{i/2}^*} \frac{d}{dt} \left( \left| \theta_{K^t} - \theta_{K_0} \right|^2 + \left| \overline{\theta}_{K^t} - \overline{\theta}_{K_0} \right|^2 \right)$$

$$= \int_{X^c \cap D} < \theta_K + \overline{\theta}_K, \delta' h + \delta'' h > + \sum_{i=1}^{s} \int_{\Delta_{i/2}^* \cap D} < \theta_K - \theta_{K_0} + \overline{\theta}_K - \overline{\theta}_{K_0}, \delta' h + \delta'' h >$$

$$= \int_{D} < \theta_K + \overline{\theta}_K, \delta' h + \delta'' h > - \sum_{i=1}^{s} \int_{\Delta_{i/2}^* \cap D} < \theta_{K_0} + \overline{\theta}_{K_0}, \delta' h + \delta'' h >$$

$$= \int_{D} < \theta_K + \overline{\theta}_K, \delta' h + \delta'' h > + \sum_{i=1}^{s} \int_{\Delta_{i/2}^* \cap D} d < \theta_{K_0} + \overline{\theta}_{K_0}, h > +$$

$$+ \sum_{i=1}^{s} \int_{\Delta_{i/2}^* \cap D} \Lambda(d'' \theta_{K_0} - d' \overline{\theta}_{K_0}), h >$$

$$= \int_{D} < \theta_K + \overline{\theta}_K, \delta' h + \delta'' h >,$$

in the last equality, we use the Stokes' formula, the harmonic property of $K_0$ on each $\Delta_{i}^*$, vanishing normal derivative of $K_0$ on $\partial \Delta_{i/2}$ with respect to $\Delta_{i/2}^*$, and $h$ being vanishing on $\partial D$.

Finally, the minimizing property of $K$ for $\hat{E}$ implies that $\frac{d}{dt}_{|t=0} \hat{E}(K_i) = 0$. On the other hand, since $h$ can be chosen arbitrarily with $h|_{X \setminus D} \equiv 0$, so, under the weak sense,

$$\Lambda(d'' \theta_K - d' \overline{\theta}_K) = 0,$$

namely, $K$ is weakly harmonic on $D$, and hence harmonic. □

Since $\hat{E}(K) \geq 0$ for any $K \in \mathcal{K}$, so the greatest lower bound of $\hat{E}$ on $\mathcal{K}$ is a nonnegative number, denoted by $\mu$. Therefore, there exists a sequence of admissible metrics $\{K_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} \hat{E}(K_n) = \mu$. Call such a sequence a minimizing sequence of $\hat{E}$ in $\mathcal{K}$. In general, it is not clear if such a minimizing sequence is convergent and (if so) the limit is an admissible harmonic metric. We shall however show that it is possible to prove that minimizing sequences constructed in a special way are convergent and the corresponding limits are admissible and harmonic, so that the greatest lower bound $\mu$ is actually attained as the minimum value of $\hat{E}$ for an admissible metric which has the same behavior as $H_i$ at each $p_i$. Namely, we will show

**Main Theorem** There exists an admissible metric $K \in \mathcal{K}$ with $\hat{E}(K) = \mu$, and hence $K$ is harmonic.
In order to prove the main theorem, we here recall some estimates for harmonic maps into non-positive curved manifolds, which are presently standard and also apply very well to the equivariant setting. We write these as the following

**Proposition 3** Let $M$ be (a domain of) a Riemannian manifold, $N$ a non-positive curved manifold.

1) (S-Y. Cheng [2]) Suppose that $u: M \to N$ is a harmonic map with finite energy $E(u; M)$. Then on any compact subset $M'$ of $M$, one has the following estimate on energy density of $u$

$$ e(u) \leq C(E(u), M', \dim M). $$

2) (Schoen-Yau [7]) Let $u_1, u_2$ be two harmonic maps from $M$ into $N$. Then the square of the distance function $d^2_N(u_1, u_2)$ is subharmonic.

We now show the following technical tool.

**Proposition 4** Let $K_1$ be an admissible metric with $K_1 = h_1 K_0$ under the fixed flat basis, let $K_0$ be the set of positive self-adjoint operators $h$ on $L^2_{\rho | \Delta^*_i}$ with respect to $K_0$ satisfying that $h|_{r_i=1} = h_1|_{r_i=1}$ and $hK_0$ is admissible. Then the following functional

$$ G_i(h) = \int_{\Delta^*_i} \left( |(\delta' h) h^{-1}|^2 + |(\delta'' h) h^{-1}|^2 \right) $$

has a critical point $h$ on $K_0$, satisfying

$$ G_i(h) \leq G_i(h_1). $$

Equivalently, there exists an admissible harmonic metric $K (= hK_0)$ on $L^2_{\rho | \Delta^*_i}$, which satisfies $K|_{r_i=1} = K_1|_{r_i=1}$ and

$$ \hat{E}_i(K) \leq \hat{E}_i(K_1), $$

where $\hat{E}_i(K) = \int_{\Delta^*_i} (|\theta_K - \theta_{K_0}|^2 + |\overline{\theta}_K - \overline{\theta}_{K_0}|^2)$.

Furthermore, when considering $K$ and $K_0$ as equivariant maps into $\mathcal{P}_n$, the maximum of the distance function $d_{\mathcal{P}_n}(K(x), K_0(x))$ on $\Delta^*_i$ is attained on $\{r_i = 1\}$.

**Remarks.** 1) If $h$ is a function (e.g. the representation is one-dimensional), just by setting $u = \log h$, the problem is reduced to the usual Dirichlet problem for harmonic functions on a disk. But the present situation is different, since the covariant derivatives $\delta'h, \delta''h$ are defined using the connections $\delta' + d'', \delta'' + d'$ which are not defined at the puncture $p_i$. In order to overcome this difficulty, we use bounded exhausted domains with the fixed outside circle and minimize.
the functional at each such domain; finally we show the obtained sequence of minimizers converges to the required minimizer of the functional. 2) In the proof of Prop. 4, we only use the harmonic property of \( K_0 \) on each \( \Delta^*_i \).

**Proof of Proposition 4.** Take bounded exhausted domains \( \{ \Delta_i^* \setminus \Delta^*_i, \frac{1}{k} \} \) of \( \Delta^*_i \), where \( \Delta^*_i, \frac{1}{k} \) represents the puncture disk \( \{ t_i \in \mathbb{C} \mid 0 < |t_i| < 1/k \} \). Minimizing the functional on \( K_0, k = \{ h \in K_0 \mid h \vert_{r_i = \frac{1}{k}} = h_1 \vert_{r_i = \frac{1}{k}} \} \)

\[
G_{i, \frac{1}{k}}(h) = \int_{\Delta^*_i \setminus \Delta^*_i, \frac{1}{k}} \left( |(\delta h)h^{-1}|^2 + |(\delta'' h)h^{-1}|^2 \right),
\]

one gets a (unique) minimizer, denoted by \( h_k \). This is equivalent to minimizing the functional

\[
\hat{E}_{i, \frac{1}{k}}(K) = \int_{\Delta^*_i \setminus \Delta^*_i, \frac{1}{k}} \left( |\theta_K - \theta_{K_0}|^2 + |\vec{\theta}_K - \vec{\theta}_{K_0}|^2 \right)
\]

on the set \( \{ K = hK_0 \mid h \in K_0, k \} \), which is just a boundary value problem for (equivariant) harmonic maps; since the target space \( \mathcal{P}_n \) is of non-positive sectional curvature, this can always be solved uniquely by Hamilton (the equivariant case by Schoen). We remark that the solution is a (unique) minimizer of both \( \hat{E}_{i, \frac{1}{k}}(K) \) and \( E_{i, \frac{1}{k}}(K) = \int_{\Delta^*_i \setminus \Delta^*_i, \frac{1}{k}} (|\theta_K|^2 + |\vec{\theta}_K|^2) \) under the corresponding boundary condition, this is a direct consequence of the Stokes formula and the harmonicity of \( K_0 \).

We now show that the sequence \( \{ h_k \} \) (if necessary, going to a subsequence) converges uniformly on any compact subset of \( \Delta^*_i \) to a critical point \( h \) of \( G_i \). The uniform convergence of \( h_k \) on compact subsets can be easily seen: Since \( h_k \) minimizes \( G_{i, \frac{1}{k}} \), they have uniform gradient estimate in \( k \) on any compact subset and hence \( h_k \)'s are uniform bounded. The Arzela-Ascoli’s Theorem then applies. Actually one can further show \( C^1 \)-convergence of \( h_k \). Furthermore, since \( G_{i, \frac{1}{k}}(h_k) \leq G_{i, \frac{1}{k}}(h_1) \leq G_i(h_1) \), so one has

\[
G_i(h) \leq G_i(h_1).
\]

Next, we need to show that that \( h \in K_0 \), namely, \( K = hK_0 \) and \( K_0 \) are mutually bounded; equivalently, this says that when considering them as equivariant harmonic map into \( \mathcal{P}_n \), the distance function \( d_{\mathcal{P}_n}(K, K_0) \) is bounded on \( \Delta^*_i \). This can be obtained by using 2) of Proposition 3 and Maximum Principle.

---

4 More precisely, one should put the functional on a closed convex subset of certain Hilbert manifold.
Applying Proposition 3, 2) to \( d_{P_n}(h_k K_0, K_0) \) on \( \Delta^*_i \setminus \Delta^*_{i, \frac{1}{k}} \), one has that for all \( k > 1 \), the distance functions \( d_{P_n}(h_k K_0, K_0) \) have \( \max_{\Delta^*_i} d_{P_n}(K_1, K_0) \) as an upper bound. As \( k \) goes to \( \infty \), we obtain on \( \Delta^*_i \)

\[
d_{P_n}(K, K_0) \leq \max_{\Delta^*_i} d_{P_n}(K_1, K_0).
\]

We note that from the above argument, it is not very clear if \( h \) (resp. \( K \)) is a minimizer of \( G_i \) (resp. \( \hat{E}_i \)) on \( K_0 \); but from the following lemma, we will easily see that this is actually the case.

In order to prove the final assertion of Proposition 4, we first state and prove the following

**Lemma 1** Let \( K, K' \) be two harmonic metrics on \( L_{\rho |\Delta^*_i} \) with the same boundary value \( K_1 |_{r_i = 1} \) and satisfy that \( K, K' \) are mutually bounded with \( K_0 \). Then \( K \equiv K' \) on \( \Delta^*_i \).

*Proof of Lemma 1.* The proof is obtained by again using Proposition 3, 2) and the fact that on the half cylinder there exists no nonconstant nonnegative bounded subharmonic function which takes value zero on the boundary. \( \square \)

**Continuation of Proof of Proposition 4.** Similar to the argument in the beginning of the proof, we minimize the functional

\[
E'_{i, \frac{1}{k}}(K) = \int_{\Delta^*_i \setminus \Delta^*_{i, \frac{1}{k}}} (|\theta_K|^2 + |\bar{\theta}_K|^2)
\]

on the set \( \{ K \in K_0 \mid K |_{r_i = \frac{1}{k}} = K_0 |_{r_i = \frac{1}{k}} \} \). We obtain harmonic metrics \( h'_k K_0 \) on \( \Delta^*_i \setminus \Delta^*_{i, \frac{1}{k}} \). Again by Maximum Principle, \( \max_{\Delta^*_i \setminus \Delta^*_{i, \frac{1}{k}}} d_{P_n}(h'_k K_0, K_0) = \max_{r_i = 1} d_{P_n}(K_1, K_0) \). So, one can easily show that these harmonic metrics converge to a harmonic metric \( K' = h' K_0 \) on \( \Delta^*_i \) and

\[
\max_{\Delta^*_i} d_{P_n}(K', K_0) = \max_{r_i = 1} d_{P_n}(K_1, K_0).
\]

Thus, by Lemma 1, we have \( K \equiv K' \) and hence

\[
\max_{\Delta^*_i} d_{P_n}(K, K_0) = \max_{r_i = 1} d_{P_n}(K_1, K_0).
\]

This finishes the proof of Proposition 4. \( \square \)

Let \( K_1 \in \mathcal{K} \) be an admissible metric on \( L_{\rho} \). Restricting \( K_1 \) to each \( \Delta^*_i \) and using the solution of Proposition 4 corresponding to \( K_1 |_{\Delta^*_i} \) to replace \( K_1 |_{\Delta^*_i} \), we obtain a new admissible metric on \( L_{\rho} \), denoted by \( \hat{K} \). Then, we have the following
Proposition 5

\[ \hat{E}(K) \leq \hat{E}(K_1). \]

Proof. By means of the definition of \( K \) and Proposition 4, we first have

\[
\int_{X \cup i \Delta^*_i} |\theta_K + \overline{\theta}_K|^2 + \sum_i \int_{\Delta^*_i} (|\theta_K - \theta_{K_0}|^2 + |\theta_{K_0} - \overline{\theta}_K|^2) \leq \int_{X \cup i \Delta^*_i} |\theta_{K_1} + \overline{\theta}_{K_1}|^2 + \sum_i \int_{\Delta^*_i} (|\theta_{K_1} - \theta_{K_0}|^2 + |\theta_{K_1} - \overline{\theta}_{K_0}|^2).
\]

Note that \( K \equiv K_1 \) on \( X \cup i \Delta^*_i \). The left-hand side of the above inequality can be written as

\[
\hat{E}(K) - \sum_i \int_{\Delta^*_i \setminus \Delta^*_i/2} |\theta_K + \overline{\theta}_K|^2 + \sum_i \int_{\Delta^*_i \setminus \Delta^*_i/2} (|\theta_K - \theta_{K_0}|^2 + |\theta_{K_0} - \overline{\theta}_K|^2);
\]

similarly, the right-hand side is

\[
\hat{E}(K_1) - \sum_i \int_{\Delta^*_i \setminus \Delta^*_i/2} |\theta_{K_1} + \overline{\theta}_{K_1}|^2 + \sum_i \int_{\Delta^*_i \setminus \Delta^*_i/2} (|\theta_{K_1} - \theta_{K_0}|^2 + |\theta_{K_1} - \overline{\theta}_{K_0}|^2).
\]

So, in order to prove the Proposition, it suffices to prove the following

\[
- \sum_i \int_{\Delta^*_i \setminus \Delta^*_i/2} |\theta_K + \overline{\theta}_K|^2 + \sum_i \int_{\Delta^*_i \setminus \Delta^*_i/2} (|\theta_K - \theta_{K_0}|^2 + |\theta_{K_0} - \overline{\theta}_K|^2) = - \sum_i \int_{\Delta^*_i \setminus \Delta^*_i/2} |\theta_{K_1} + \overline{\theta}_{K_1}|^2 + \sum_i \int_{\Delta^*_i \setminus \Delta^*_i/2} (|\theta_{K_1} - \theta_{K_0}|^2 + |\theta_{K_1} - \overline{\theta}_{K_0}|^2);
\]

this is equivalent to show

\[
\sum_i \int_{\Delta^*_i \setminus \Delta^*_i/2} < \theta_K + \overline{\theta}_K, \theta_{K_0} + \overline{\theta}_{K_0}> = \sum_i \int_{\Delta^*_i \setminus \Delta^*_i/2} < \theta_{K_1} + \overline{\theta}_{K_1}, \theta_{K_0} + \overline{\theta}_{K_0}>,
\]

which can be easily obtained by using the Stokes’ formula together with the facts that both \( K \) and \( K_1 \) have the same boundary value on each \( \partial \Delta_i \) and that \( K_0 \) is harmonic on each \( \Delta_i^* \) and has vanishing normal derivative on \( \partial \Delta_{i/2} \). □

Now, we can turn to the proof of Main Theorem.

The proof of Main Theorem. Let \( \{K_n\}_n^\infty \) be a minimizing sequence of \( \hat{E} \) in \( K \), i.e., \( \lim_{n \to \infty} \hat{E}(K_n) = \mu \). Without loss of generality, we can assume that each metric \( K_n \) is harmonic on \( X \cup i \Delta^*_i \); this can be done briefly as follows: on \( X \cup i \Delta^*_i \), we replace \( K_n \) by a (unique) harmonic metric which is obtained by solving the Dirichlet’s boundary problem for equivariant harmonic maps.
with the boundary value being $K_{n|\cup_i\partial\Delta_{i/2}}$. Since this replacement does not increase energy on $X\setminus\cup_i\Delta_{i/2}^*$, so the new sequence is still a minimizing sequence.

We now show that the minimizing sequence $\{K_n\}$ (if necessary, going to a subsequence) is uniformly convergent on $X\setminus\cup_i\Delta_{i/2}^*$ in the sense of $C^1$. The key is to show the $C^0$-convergence; to this end, we use an idea due to Donaldson [5]. From now on, we consider each $K_n$ as an equivariant map from the universal covering of $X\setminus\cup_i\Delta_{i/2}^*$ into $\mathcal{P}_m$. Since $\{K_n\}$ is a minimizing sequence for $\tilde{E}$, the usual energy of $K_n$ on $X\setminus\cup_i\Delta_{i/2}^*$ is uniform bounded in $n$. By means of Proposition 3, 1), the energy density $e(K_n)$ on $X\setminus\cup_i\Delta_{i}^*$ has uniform bound in $n$, namely

$$e(K_n) \leq C, \text{ on } X\setminus\cup_i\Delta_{i}^*,$$

where $C > 0$ is independent of $n$.

Take a point $\tilde{p}$ in the universal covering of $X\setminus\cup_i\Delta_{i}^*$, say, the projection of which lies in $\cup_i\partial\Delta_i$, denoted by $p$. We would like to show that the sequence $\{K_n(\tilde{p})\}_{n=1}^\infty$ in $\mathcal{P}_m$ (if necessary, going to a subsequence) converges. If NOT, we may assume that $\{K_n(\tilde{p})\}$ (if necessary, going to a subsequence) converges to a point in a certain boundary component, say $\Sigma$, of $\mathcal{P}_m$. By the semi-simplicity of the representation $\rho : \pi_1(X) \to \text{Gl}(m, \mathbb{C})$ (cf. §3), there exists an element $\gamma \in \pi_1(X)$ satisfying $\rho(\gamma)\Sigma \cap \Sigma = \emptyset$. So, we have

$$\lim_{n \to \infty} d_{\mathcal{P}_m}(K_n(\tilde{p}), \rho(\gamma)K_n(\tilde{p})) = \lim_{n \to \infty} d_{\mathcal{P}_m}(K_n(\tilde{p}), K_n(\gamma\tilde{p})) = \infty.$$

On the other hand, letting $c(t), t \in [0, 1]$ be a differentiable curve in the universal covering of $X\setminus\cup_i\Delta_{i}^*$ connecting the points $\tilde{p}$ and $\gamma(\tilde{p})$, we then have

$$d_{\mathcal{P}_m}(K_n(\tilde{p}), K_n(\gamma\tilde{p})) \leq \text{Length of the curve } K_n(c(t)).$$

By means of the uniform boundedness of energy density $e(K_n)$ in $n$ on $X\setminus\cup_i\Delta_{i}^*$, we know that the length of the curves $K_n(c(t))$ are actually uniformly bounded in $n$. Thus, we derive a contradiction; namely, the sequence $\{K_n(\tilde{p})\}$ in $\mathcal{P}_m$ (if necessary, going to a subsequence) converges.

Using the convergence of $\{K_n(\tilde{p})\}$ and the fact that $e(K_n)$ are uniformly bounded on $X\setminus\cup_i\Delta_{i}^*$, we easily show that $K_n$ (if necessary, going to a subsequence) is $C^1$-convergent on the compact subset $X\setminus\cup_i\Delta_{i}^*$.

Summing the above all up, we can assume that the minimizing sequence $\{K_n\}$ in question satisfies that 1) on $X\setminus\cup_i\Delta_{i/2}^*$, each $K_n$ is harmonic; 2) on $X\setminus\cup_i\Delta_{i}^*$, $\{K_n\}$ uniformly converges in the sense of $C^1$.

Now, using Proposition 4 and 5, we construct a new minimizing sequence from the above $\{K_n\}$. For each $K_n$, we restrict $K_n$ to $\cup_i\Delta_{i}^*$ and consider this restriction as the $K_1$ in Proposition 4 to get the corresponding $K$ in
Proposition 4; we then use this $K$ to replace $K_n$ on $\cup_i \Delta^*_i$ to get a new metric in $K$, denoted by $K'_n$. Proposition 5 tells us $\tilde{E}(K'_n) \leq \tilde{E}(K_n)$ so that the new sequence $\{K'_n\}$ is still a minimizing sequence. Note that $K'_n \equiv K_n$ on $X \setminus \cup_i \Delta^*_i$. Since $\{K_n\}$ (hence $\{K'_n\}$) converges on $\cup_i \partial \Delta_i$, using the same technique as in the proof of Proposition 4, we easily prove that $\{K'_n\}$ converges on $\cup_i \Delta^*_i$; especially, the limit is admissible. Thus, $\{K'_n\}$ converges on $X$ to an admissible metric, denoted by $K$, and $K$ minimizes the modified energy functional $\tilde{E}$. By Proposition 2, we know that $K$ is harmonic on $X$. □

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