FINITE SUMS OF COMMUTATORS

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Abstract. We show that elements of unital C*-algebras without tracial states
are finite sums of commutators. Moreover, the number of commutators in-
volved is bounded, depending only on the given C*-algebra.

1. Introduction

It was shown in [2] that in finite von Neumann algebras, elements with central
trace zero are sums of at most 10 commutators. The C*-algebra case was considered
in [1]. The main result there states that if the unit of a C*-algebra A is properly
infinite (i.e. there exist two orthogonal projections p, q ∈ A equivalent to 1), then
any hermitian element is a sum of at most five self-adjoint commutators. In this
paper we consider the more general case of unital C*-algebras A without tracial
states and improve the previous result of T. Fack. Note that if the unit of A is
properly infinite, then A has no tracial states. The converse is known to be false, at
least when A is non-simple (see [4] for further details). C*-algebras without tracial
states have several nice characterizations, such as [3]. This paper also contains
another simple proof of the latter result of [3, Lemma 1].

2. The result

Given a, b ∈ A, their commutator is [a, b] = ab − ba. A self-adjoint commutator
is just a commutator of the form [a∗, a] = a∗a − aa∗.

Theorem 1. Let A be a unital C*-algebra. Then the following properties are equiva-
 lent:
(1) A has no tracial states.
(2) There exist an integer n ≥ 2 and elements b1, b2, . . . , bn ∈ A such that

\[ \sum_{i=1}^{n} b_i^* b_i = 1 \quad \text{and} \quad \| \sum_{i=1}^{n} b_i^* b_i \| < 1. \]

(3) There exists an integer n ≥ 2 such that any element of A can be expressed as
a sum of n commutators and any positive element can be expressed as a sum
of at most n self-adjoint commutators.

Remark 2. The equivalence of (1) and (2) is just [3 Lemma 1]. As mentioned, in
this paper we give a new simple proof.
Remark 3. The integer \( n \) appearing in (2) above automatically satisfies property (3). If the unit of \( A \) is properly infinite, there exist two isometries \( v_1, v_2 \in A \) with orthogonal ranges. Let \( b_1 = (1/\sqrt{2})v_i \) for \( i = 1, 2 \). Then \( b_1b_1^* + b_2b_2^* = 1 \) and \( b_1b_1^* + b_2b_2^* \leq 1/2 \); thus the property from (2) is achieved with \( n = 2 \). Thus in a properly infinite \( C^* \)-algebra, every element is the sum of two commutators, every positive element is the sum of two self-adjoint commutators, and every self-adjoint element is the sum of four self-adjoint commutators.

Proof. The implication (3) \( \Rightarrow \) (1) is trivial. (1) \( \Rightarrow \) (2). Consider
\[
R = \left\{ \sum_{i=1}^{s} (a_i^*a_i - a_ia_i^*) \mid s \geq 1, a_i \in A \right\}
\]
the set of finite sums of self-adjoint commutators of \( A \). Note that \( R \subseteq A_{sa} \) is a real vector subspace of \( A_{sa} \). Put \( \delta = \text{dist}(1, R) \).

We show that \( \delta < 1 \). Suppose the contrary, i.e. \( \delta = 1 \). This is equivalent to
\[
\|t + x\| \geq |t|, \quad \forall x \in R, \quad \forall t \in \mathbb{R}.
\]
It follows that \( \varphi(t + x) = t \) is a real bounded functional on \( \mathbb{R}1 + R \) of norm 1. By the Hahn–Banach theorem it can be extended to a norm-1 functional on \( A_{sa} \) and furthermore to a bounded complex functional on \( A \), also denoted by \( \varphi \). Observe that \( \varphi \) is necessarily a tracial state on \( A \), which contradicts our hypothesis.

Because \( \delta < 1 \), there exist some elements \( a_1, a_2, \ldots, a_m \in A \) such that \( t_0 = \|1 - \sum_{i=1}^{m} (a_i^*a_i - a_ia_i^*)\| < 1 \). In particular we have
\[
\sum_{i=1}^{m} a_i^*a_i \leq -1 + t_0 + \sum_{i=1}^{m} a_i^*a_i.
\]
Let \( k = \| \sum_{i=1}^{m} a_i^*a_i \| \) and \( a_{m+1} = (k - \sum_{i=1}^{m} a_i^*a_i)^{1/2} \). Then we have
\[
\sum_{i=1}^{m+1} a_i^*a_i = k;
\]
but on the other hand, by (1) we have also
\[
\sum_{i=1}^{m+1} a_i^*a_i \leq -1 + t_0 + k.
\]
The required properties are now fulfilled with \( n = m + 1 \) and \( b_i = (1/\sqrt{k})a_i \).

(2) \( \Rightarrow \) (3). Suppose that \( b_1, b_2, \ldots, b_n \) are as in (2). Define \( \Phi(a) = \sum b_iab_i^* \). Then \( \Phi \) is a bounded positive map on \( A \) with norm \( \|\Phi\| = \|\sum b_i^*b_i\| < 1 \). It follows that \( Id_A - \Phi \) is invertible in the Banach algebra \( B(A) \) of bounded maps on \( A \). Let
\[
\Psi = (Id_A - \Phi)^{-1}.
\]
Note that \( \Psi = \sum_{i=0}^{\infty} \Phi^i \), thus \( \Psi \) is positive too. By definition of \( \Psi \), for any \( a \in A \) we have
\[
a = (Id_A - \Phi)(\Psi(a)) = \Psi(a) - \sum_{i=1}^{n} b_i\Psi(a)b_i^*
\]
\[
= \sum_{i=1}^{n} b_i^*b_i\Psi(a) - \sum_{i=1}^{n} b_i\Psi(a)b_i^* = \sum_{i=1}^{n} [b_i^*, b_i\Psi(a)].
\]
so $a$ is a finite sum of at most $n$ commutators. If moreover $a$ is a positive element in $A$, then

$$a = (Id_A - \Phi)(\Psi(a)) = \Psi(a) - \sum_{i=1}^{n} b_i \Psi(a) b_i^*$$

$$= \sum_{i=1}^{n} \Psi(a)^{1/2} b_i^* b_i \Psi(a)^{1/2} - \sum_{i=1}^{n} b_i \Psi(a) b_i^* = \sum_{i=1}^{n} [\Psi(a)^{1/2} b_i^*, b_i \Psi(a)^{1/2}],$$

so $a$ is a finite sum of at most $n$ self-adjoint commutators.

3. Questions

For an infinite $C^*$-algebra $A$ (in the sense that it admits no tracial states) let $\nu(A)$ be the least positive integer such that any element of $A$ is a sum of at most $\nu(A)$ commutators. In all the examples that we know of, we have $\nu(A) = 2$. We believe that it is unlikely to always be the case.

In [3] it was shown that if $A$ is an unital exact $C^*$-algebra, then there exists an integer $m$ such that $\mathbb{M}_m(A)$ is properly infinite. It follows that $\nu(\mathbb{M}_m(A)) = 2$. Then a simple computation shows that $\nu(A) \leq 2m^2$. It would be interesting to answer the inverse problem, that is: assuming $\nu(A)$ is known, estimate the least positive integer $m$ such that $\nu(\mathbb{M}_m(A)) = 2$.

References

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