Nature of spin-charge separation

Z.Y. Weng, D.N. Sheng, and C.S. Ting

Texas Center for Superconductivity and Department of Physics
University of Houston, Houston, TX 77204-5506

Quasiparticle properties are explored in an effective theory of the $t-J$ model which includes two important components: spin-charge separation and unrenormalizable phase shift. We show that the phase shift effect indeed causes the system to be a non-Fermi liquid as conjectured by Anderson on a general ground. But this phase shift also drastically changes a conventional perception of quasiparticles in a spin-charge separation state: an injected hole will remain stable due to the confinement of spinon and holon by the phase shift field despite the background is a spinon-holon sea. True deconfinement only happens in the zero-doping limit where a bare hole will lose its integrity and decay into holon and spinon elementary excitations. The Fermi surface structure is completely different in these two cases, from a large band-structure-like one to four Fermi points in one-hole case, and we argue that the so-called underdoped regime actually corresponds to a situation in between, where the "gap-like" effect is amplified further by a microscopic phase separation at low temperature. Unique properties of the single-electron propagator in both normal and superconducting states are studied by using the equation of motion method. We also comment on some of influential ideas proposed in literature related to the Mott-Hubbard insulator and offer a unified view based on the present consistent theory.

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I. INTRODUCTION

High-$T_c$ cuprates are regarded by many as essentially a doped Mott-Hubbard insulator [1]. At half-filling such an insulator is a pure antiferromagnet with only the spin degrees of freedom not being frozen at low energy. And a metallic phase with gapless charge degrees of freedom emerge after holes are added to the filled lower Hubbard band. To characterize the doped Mott-Hubbard insulator in the metallic regime, two important ideas were originally introduced by Anderson: Spin-charge separation [2] and unrenormalizable phase shift effect [3,4]. The first one is about elementary excitations of such a system and the second one is responsible for its non-Fermi liquid behavior.

The spin-charge separation idea may be generally stated as the existence of two independent elementary excitations, charge-neutral spinon and spinless holon, which carry spin $1/2$ and charge $+e$, respectively. It can be easily visualized in a short-range resonating-valence-bond (RVB) state [5] and has become a widely used terminology in literature, often with an additional meaning attached to it. For example, a spin-charge separation may be mathematically realized in the so-called slave-particle representation [6] of the $t-J$ model

$$ c_{i\sigma} = h_{i\sigma}^\dagger f_{i\sigma} $$ (1.1)

where the no-double-occupancy constraint, reflecting the Hubbard gap in its extreme limit, is handled by an equality $h_i^\dagger h_i + \sum_{\sigma} f_{i\sigma}^\dagger f_{i\sigma} = 1$ which commutes with the Hamiltonian. Here one sees the close relation of the spin-charge separation and the constraint condition through the counting of the quantum numbers. But the spin-charge separation also acquires a new meaning here: If those holon ($h_{i\sigma}^\dagger$) and spinon ($f_{i\sigma}$) fields indeed describe elementary excitations, the hole (electron) is no longer a stable object and must decay into a holon-spinon pair once being injected into the system as shown by Fig. 1(a). This instability of a quasiparticle will be later referred to as the deconfinement, in order to distinguish it from the narrow meaning of the spin-charge separation about elementary excitations. We will see later that these two are generally not the same thing.

The second idea, the so-called "unrenormalizable phase shift" [3,4], may be described as follows. In the presence of an upper Hubbard band, adding a hole to the lower Hubbard band could change the whole Hilbert space due to the on-site Coulomb interaction: The entire spectrum of momentum $k$'s may be shifted through the phase shift effect. It leads to the orthogonality of a bare doped hole state with its true ground state such that the quasiparticle weight $Z \equiv 0$, the key criterion for a non-Fermi liquid. In general, it implies

$$ c_{i\sigma} = c_{i\sigma} e^{i\Theta_{i\sigma}} $$ (1.2)

where $c_{i\sigma}$ is related to elementary excitation fields, e.g., $h_{i\sigma}^\dagger f_{i\sigma}$ in a spin-charge separation framework. Such an expression means that in order for a bare hole created by $c_{i\sigma}$ to become low-lying elementary excitations, a many-body
phase shift $\Theta_\sigma$ must take place in the background. In momentum space, it is easy to see how such a phase shift changes the Hilbert space by shifting $k$ values. Note that $c_{k\sigma} = \sum_{k'} h_{k',k} f_{k' + k\sigma}$ where $k$ and $k'$ belong to the same set of quantized values (for example, in a 2D square sample with size $L \times L$, the momentum is quantize at $k_\alpha = \frac{2\pi \alpha}{L}$ under the periodic boundary condition with $\alpha = x$, $y$ and $n = \text{integer}$). But because of a nontrivial $\Theta_\sigma$, $c_{k\sigma}$ and $c_{k'\sigma}$ generally may no longer be described by the same set of $k$'s, or, in the same Hilbert space, which thus constitutes an essential basis for a possible non-Fermi liquid.

The one-dimensional (1D) Hubbard model serves a marvelous example in favor of the decomposition (1.2) over (1.1). The quantitative value of the phase shift was actually determined by Anderson and Ren [9] using the Bethe-Ansatz approach [10] without using the Bethe-Ansatz also reaches the same conclusion which supports (1.2) as the correct decomposition of true holon and spinon.

Another important property of the Mott-Hubbard insulator, which is well-known but has not been fully appreciated, is the bosonization of the electrons at half-filling. Namely, the fermionic nature of the electrons completely disappears and is replaced by a bosonic one. This is one of the most peculiar features of the Mott-Hubbard insulator due to the strong on-site Coulomb interaction. In fact, under the no-double-occupancy constraint, the $t-J$ model reduces to the Heisenberg model in this limit. Its ground state for any finite bipartite lattice is singlet according to Marshall [10] and the wave function is real and satisfies a trivial Marshall sign rule as opposed to a much complicated “sign problem” assoicate with the fermionic statistics in a conventional fermionic system. This bosonization is the reason behind a very successful bosonic RVB description of the antiferromagnet: The variational bosonic RVB wave functions can produce strikingly accurate ground-state energy [11, 12] as well as elementary excitation spectrum over the whole Brillouin zone (the variational bosonic RVB wave functions can produce the correct path-integral approach [9] without using the Bethe-Ansatz as well). Such a vortex-like phase shift originates from the fact that a doped hole moving on an AF spin background will always pick up sequential $+$ and $-$ signs, $(+1) \times (-1) \times (-1) \times ...$, first identified in Ref. [18]. These signs come from the Marshall signs hidden in the AF background which are scrambled by the hopping of the doped hole on its path, determined by simply counting the index $\sigma$ of each spin exchanged with the hole during its hopping [13]. The significance of such a phase string is that it represents the sole source to generate phase frustrations in the $t-J$ model (at finite doping, the only additional signs coming from the fermionic statistics of doped holes in the original slave-fermion representation are also counted in). Namely, the ground-state wave function would become real and there should be no “sign problem” only if such a phase string effect is absent (like in the zero-doping case). The phase shift field in (1.3) precisely keeps track of such a phase string effect [13] and therefore can be considered to be a general consequence of the $t-J$ model.

The decomposition (1.3) defines a unique spin-charge separation theory where the relation between the physical electron operator and the internal elementary excitations, holon and spinon, is explicitly given. The thermodynamic
properties will be obtained in terms of the energy spectra of holon and spinon fields, while the physically observable quantities will be determined based on (1.3) where the singular phase shift field with vorticities is to play a very essential role in contrast to the conventional spin-charge separation theories in the slave-particle decompositions of (1.1).

Note that the total vorticity of (1.4) is always equal to $2\pi \times \text{integer}$ due to the no-double-occupancy constraint such that the phase shift factor $e^{i\hat{\Theta}_{i\sigma}}$ is single-valued. Such a phase shift field will play different roles in different channels. For example, in the spin channel one has $S^+_i = b^\dagger_i b_i (1)^i e^{-i(\Theta_{i\sigma}^{\text{string}} - \Theta_{i\sigma}^{\text{string}})}$ where the total vorticity

$$\oint_{\Gamma} \mathbf{r} \cdot \nabla \left( \Theta_{\uparrow}^{\text{string}} - \Theta_{\downarrow}^{\text{string}} \right) = \pm 2\pi \sum_{l \in \Gamma} n^h_l. \quad (1.5)$$

It obviously vanishes at $\delta \to 0$ ($\delta$ is the doping concentration) so that the aforementioned bosonization is naturally realized. And at finite-doping, the vorticity shown in (1.3) reflects the recovered fermionic effect and is responsible for a doping-dependent incommensurate momentum structure in the dynamic spin susceptibility function which provides a unique reconciliation of neutron scattering and NMR measurements in the cuprates. On the other hand, in the singlet pairing channel, the phase shift field appearing in the local pairing operator will contribute to a vorticity

$$\oint_{\Gamma} \mathbf{r} \cdot \nabla \left( \Theta_{\uparrow}^{\text{string}} + \Theta_{\downarrow}^{\text{string}} \right) = \pm 2\pi \sum_{l \in \Gamma} \left[ \sum_{\alpha} \alpha n^b_{l\alpha} - 1 \right], \quad (1.6)$$

which decides the phase coherence of Cooper pairs. According to (1.6), besides a trivial $2\pi$ flux quantum per site, each spinon also carries a fictitious $(\pm)2\pi$ flux tube. To achieve the phase coherence or superconducting condensation, $\uparrow$ and $\downarrow$ spinons have to be paired off to remove the vorticities associated with individual spinons in (1.6), which then connects $T_c$ to a characteristic spinon energy scale, in consistency with the experimental result of cuprate superconductors and resolving the issue why $T_c$ is too high in usual RVB theories.

The purpose of the present work is to explore the consequences of the bosonization decomposition (1.3) in 2D quasiparticle channel. First of all, we show that the phase shift field indeed causes the quasiparticle weight $Z$ to vanish. Namely, this spin-charge separation state is a 2D non-Fermi liquid, a fact almost trivial in such a particular formulation. A surprising “by-product” of this phase shift field is that it also plays a role of confinement force to “glue” spinon and holon constituents together inside a quasiparticle, as illustrated in Fig. 1(b). In other words, a hole injected into this system generally does not break up into spinon-holon elementary particles, even though the background is a spinon-holon sea. Such a quasiparticle may be regarded as a spinon-holon bound state or more properly a collective mode but will generally remain incoherent due to the same phase shift field.

Due to the confinement, an equation-of-motion description of the quasiparticle excitation is constructed, in which the dominant “scattering” process is described as the “virtual” decay of the quasiparticle into holon-spinon composite. In the superconducting phase, the composite nature of the quasiparticle predicts a unique non-BCS structure for the

![FIG. 1. Schematical illustration of the quasiparticle deconfinement (a) and confinement (b) due to the phase shift field](image-url)
A true deconfinement or instability of the quasiparticle only happens in the zero-doping limit where an injected hole indeed can decay into a holon and spinon pair, which provides a consistent explanation of angle-resolved photoemission spectroscopy (ARPES) measurements. The contrast of a large band-structure-like Fermi surface in the confinement phase to the four Fermi-points in the deconfinement phase at zero-doping limit may provide a unique explanation for the ARPES experimental measurements in cuprate superconductors. In a weakly-doped regime, a “partial” deconfinement of the quasiparticle between full-blown deconfinement and confinement will be reflected in the single-electron Green’s function which may explain the “spin gap” phenomenon.

The remainder of the paper is organized as follows. In Sec. II, we first briefly review the effective spin-charge separation theory based on the decomposition (1.3). Then in Sec. II B, we show that the phase shift field leads to $Z = 0$, i.e., a non-Fermi liquid state. In Sec. II C, we demonstrate how the phase shift field causes the confinement of the holon and spinon within a quasiparticle except in the zero-doping limit. In Sec. II D, we study the single-electron propagator in both normal and superconducting states based on an equation-of-motion approach. We then discuss an underdoped case as a crossover regime from a Fermi-point structure in one-hole case with the holon-spinon deconfinement to a large Fermi-surface in the confinement case. Finally, Sec. III is devoted to discuss some of the most influential ideas proposed in literature related to the Mott-Hubbard insulator and high-$T_c$ cuprates and offer a unified view based on the present consistent theory.

II. PROPERTIES OF A QUASIPARTICLE IN SPIN-CHARGE SEPARATION STATE

A. Effective spin-charge separation theory

The decomposition (1.3) determines an effective spin-charge separation theory of the $t - J$ model in which spinon and holon fields constitute the elementary particles. Before proceeding to the discussion of the quasiparticle properties in the next subsections, we first briefly review some basic features of this theory based on Refs. [15,20].

In the operator formalism, the phase shift field $\Theta_{i\sigma}^{\text{string}}$ satisfying (1.4) can be explicitly written down in a specific gauge as follows [18]:

$$\Theta_{i\sigma}^{\text{string}} = i/2 \left[ \Phi_i^b - \sigma \Phi_i^h \right],$$  

(2.1)

where

$$\Phi_i^b = \sum_{l \neq i} \theta_i(l) \left( \sum_{\alpha} c_{\alpha l}^b - 1 \right),$$  

(2.2)

and

$$\Phi_i^h = \sum_{l \neq i} \theta_i(l) n_i^h.$$  

(2.3)

Here $\theta_i(l)$ is defined as an angle

$$\theta_i(l) = \text{Im} \ln (z_i - z_l)$$  

(2.4)

with $z_i = x_i + iy_i$ representing the complex coordinate of a lattice site $i$.

In 2D, an effective Hamiltonian based on the decomposition (1.3) after a generalized mean-field decoupling [20] in the $t - J$ model can be written down:

$$H_{\text{eff}} = H_h + H_s,$$  

(2.5)

where the holon Hamiltonian

$$H_h = -t_h \sum_{\langle ij \rangle} \left( e^{i A_{ij}^f} \right) h_i^\dagger h_j + H.c.$$  

(2.6)

and the spinon Hamiltonian
\[ H_s = - J_s \sum_{(ij)_{\sigma}} \left[ \left( e^{i\sigma A^f_{ij}} \right) b^\dagger_{i\sigma} b^\dagger_{j-\sigma} + H.c. \right] - \sum_{ij\sigma} J^h_{ij} \left( e^{i\sigma A^h_{ij}} \right) b^\dagger_{i\sigma} b_{j\sigma}, \]  
\[(2.7)\]

with \( t_h \sim t, J_s \sim J \). In the second term of \( H_s \), \( J^h_{ij} \sim \delta t \neq 0 \) only for \( i \) and \( j \) on the same sublattice sites, which originates from \( H_t \) where a phase shift occurs \[21\] in the spinon mean-field wavefunction and results in the same-sublattice hopping of spinons. The lattice gauge fields \( A^f_{ij} \) and \( A^h_{ij} \) in the specific gauge choice of \[2.2\] and \[2.3\] are given by

\[ A^f_{ij} = \frac{1}{2} \sum_{l \neq i,j} \left[ \theta_i(l) - \theta_j(l) \right] \left( \sum_{\sigma} \sigma n^h_{i\sigma} - 1 \right) \equiv A^s_{ij} - \phi^0_{ij}, \]  
\[(2.8)\]

and

\[ A^h_{ij} = \frac{1}{2} \sum_{l \neq i,j} \left[ \theta_i(l) - \theta_j(l) \right] n^h_{l\sigma}. \]  
\[(2.9)\]

In general, \( A^s_{ij} \) and \( A^h_{ij} \) can be regarded as “mutual” Chern-Simons lattice gauge fields as, for example, \( A^h_{ij} \) is determined by the density distribution of holons but only seen by spinons.

The above effective theory is based on a RVB pairing order parameter \[20\]

\[ \Delta^s = \sum_{\sigma} \left\langle e^{-i\sigma A^h_{ij}} b_{i\sigma} b_{j-\sigma} \right\rangle, \]  
\[(2.10)\]

which in zero-doping limit \( \delta \rightarrow 0 \) reduces to the well-known bosonic RVB order parameter \[13\] as \( A^h_{ij} = 0 \). And \( H_{\text{eff}} \) recovers the Schwinger-boson mean-field Hamiltonian \[13\] of the Heisenberg model. So this theory can well describe AF correlations at half-filling. At finite doping, the “mutual” Chern-Simons gauge fields, \( A^f_{ij} \) and \( A^h_{ij} \), will play important roles in shaping superconductivity, magnetic and transport properties, and some very interesting similarities with cuprate superconductors have been discussed based on this model \[21\]. In contrast to the slave-fermion approach \[21\], \( \Delta^s \) remains the only order parameter at finite doping, controlling the short-range spin-spin correlations as \( \langle S_i \cdot S_j \rangle = -1/2|\Delta^s|^2 \) for nearest-neighboring \( i \) and \( j \). It is noted that due to the presence of the RVB pairing \[2.10\], the conventional gauge fluctuations \[20,22\] are suppressed as the gauge invariance, \( h_i^\dagger b_{i\sigma} = [h_i^\dagger e^{i\theta_i}] [b_{\sigma} e^{-i\theta_i}] \), is apparently broken by \( \Delta^s \). Here spinons no longer contribute to transport and are really charge-neutral particles.

In the ground state of the uniform-phase solution (Ref. \[20\]), \( A^f_{ij} \) and \( A^h_{ij} \) both become simplified: \( A^f_{ij} \) simply describes a \( \pi \)-flux per plaquette: \( \sum_{\Box} A^f_{ij} \approx -\sum_{\Box} \phi^0_{ij} = -\pi \) since \( A^f_{ij} \) is suppressed due to the spin pairing in the ground state; \( A^h_{ij} \) describes a uniform flux \( \sum_{\Box} A^h_{ij} = \pi \delta \) due to the Bose condensation of holons. Self-consistently, \( H_h \) \[2.6\] determines a Bose-condensed ground state of holons where the \( \pi \) flux produced by \( A^h_{ij} \) merely enlarges the effective mass near the band edge by \( \sqrt{2} \). On the other hand, \( A^h_{ij} \) in \( H_s \) \[2.7\] leads to a “resonance-like” energy structure in the spinon spectrum and the corresponding dynamic spin susceptibility function at the AF vector \( (\pi, \pi) \) is illustrated in Fig. 2. Note that \( E_g \) in Fig. 2 is twice bigger than the corresponding spinon energy \( E_s \). The doping dependence of \( E_g \) is shown in the inset. Finally, the superconducting order parameter \( \Delta^SC_{ij} = \langle \sum_{\sigma} \sigma c_{i\sigma} c_{j-\sigma} \rangle \) has a finite value (for nearest-neighboring \( i \) and \( j \)) in the ground state \[20\]

\[ \Delta^SC_{ij} = \Delta^s(-1)^i h_i^\dagger e^{i(\Phi_i^+ + \Phi_j^+)} h_j^\dagger \neq 0, \]  
\[(2.11)\]

due to the Bose condensation of holons as well as the pairing of spinons which leads to \( \Delta^s \neq 0 \) and the vortex-antivortex confinement in \( e^{\mp(\Phi_i^+ + \Phi_j^+)} \). Thus the ground state is always superconducting-condensed with a pairing symmetry of d-wave-like \[21\].

Besides the above uniform ground state, possible non-uniform solutions characterized by the coexistence of the Bose condensations of holons and spinons have been also discussed in Ref. \[20\] where the spinon spectrum has only a pseudo-gap. In any case, the ground state can be regarded as a spinon-holon sea, and low-lying elementary excitations are described in terms of spinons and holons. What we are mainly interested in this paper is to answer the question how a hole (electron) as a composite of spinon and holon behaves in this spin-charge separation state. This is one of the most fundamental questions not only because it can be directly tested in an ARPES measurement, but also because it will make crucial distinction between a conventional Fermi liquid and a non-Fermi liquid. Let us begin with the question: If this spin-charge separation state is a non-Fermi liquid.
where lattice Hamiltonians (2.6) and (2.7) are invariant since the gauge fields, in which
But a bare hole state will change under the transformation (2.12) as follows:

It corresponds to a simple change of reference axis for the angle function \( \theta_i(l) \) defined in (2.4). It is easy to see that the Hamiltonians (2.6) and (2.7) are invariant since the gauge fields, in which \( \theta_i(l) \) appears, are obviously not changed: \( A^h_{ij} \rightarrow A^h_{ij} \). Both the ground state \( |\Psi_G\rangle \) as well as single-valued \( h_i^z \) and \( b_{i\sigma} \) fields are apparently independent of \( \phi \).

One can construct a “rotational” operation by making a transformation

\[
\theta_i(l) \rightarrow \theta_i(l) + \phi. 
\]

(2.12)

due to the phase shift factor \( e^{i\theta_i(l)} \) with \( P_i^s = S^z - \sigma N^h/2 - (N - 1 - \sigma)/2 \) which remains an integer for a bipartite lattice \( (S^z \) and \( N^h \) denote total spin and hole numbers, respectively, and \( N \) is the lattice size). This implies that \( c_{i\sigma} |\Psi_G\rangle \) indeed has a nontrivial “angular” momentum in contrast to \( |\Psi_G\rangle \) which carries none.

It is probably more transparent to see the origin of the angular momentum if one rewrites, for example,

\[
e^{i\hat{\Theta}_i} = \prod_{l \neq i} (z_i - z_l^\uparrow)^{1/2} \prod_{l \neq i} (z_i^\uparrow - z_l^\uparrow)^{1/2} \prod_{l \neq i} (z_i - z_l^\downarrow)^{1/2} \prod_{l \neq i} (z_i^\downarrow - z_l^\downarrow)^{1/2} F_i 
\]

(2.14)

where \( z_{i\uparrow}, z_{i\downarrow}, \) and \( z_{i\downarrow} \) denote the complex coordinates of \( \uparrow, \downarrow \) spinons, and holons, respectively. And \( F_i = \prod_{l \neq i} |z_i - z_l| \) is a constant (which is obtained by using the no-double-occupancy constraint). It is important to note that despite the fractional (“semion”) exponents of \( 1/2 \) in (2.14), it can be directly verified that the phase shift field \( e^{i\hat{\Theta}_i} \) remains single-valued under the no-double-occupancy constraint. Generally the vortex field (2.14) introduces an extra angular momentum which can be easily identified as

\[
l = S^z + \frac{N^h}{2}. 
\]

(2.15)
Here \( l \) is always an integer. Then one has

\[
\langle \Psi_G(N_c - 1)|c_{l\sigma}|\Psi_G(N_c) \rangle = 0 \tag{2.16}
\]

due to the orthogonal condition \( 28 \) as \( l \neq 0 \) \( [S^z = O(1), N^h = O(N) \) at finite doping \] for \( c_{l\sigma}|\Psi_G \). By extending the same argument, one can quickly see that the bare hole state \( c_{l\sigma}|\Psi_G(N_c) \rangle \) has no overlap not only with \( |\Psi_G(N_c - 1) \rangle \) but also with all the elementary excitations composed of simple holons and spinons with \( l = 0 \). So \( c_{l\sigma} \) is more like a creation operator of a “collective” mode whose quantum number \( l \) is different from a simple spinon-holon pair.

C. Quasiparticle: Spinon-holon confinement

The difference in symmetry between a quasiparticle and a holon-spinon pair implies that the former cannot simply decay into the latter even though they share the same quantum numbers of charge and spin. In this section, we demonstrate that generally the holon and spinon constituents will be confined by the phase shift field within a quasiparticle although the background is a spinon-holon sea.

Intuitively such a confinement is easy to understand: If the holon and spinon constituents inside a quasiparticle could move away independently by themselves, as schematically shown in Fig. 3, the vortex-phase-shift field \( e^{i\theta_{l\sigma}} \) left behind would cost a logarithmically divergent energy as to be shown below. But a quasiparticle state \( c_{l\sigma}|\Psi_G \) \) as a local excitation should only cost a finite energy relative to the ground state energy. Such a discrepancy can be reconciled only if the holon and spinon constituents no longer behave as free elementary excitations: They have to absorb the effect of the vortex-like phase shift and by doing so make themselves bound together.

Let us consider \( |\Psi'\rangle \equiv e^{i\theta_{l\sigma}}|\Psi_G \rangle \) and compute the energy cost for the vortex-like phase shift:

\[
\langle \Psi'|H_{\text{eff}}|\Psi' \rangle - \langle \Psi_G|H_{\text{eff}}|\Psi_G \rangle \tag{2.17}
\]

We first focus on the contribution from the holon part \( H_h \ [2.6] \). Define \( E_{\text{G}}^h = \langle \Psi_G|H_h|\Psi_G \rangle \). One has

\[
-t_h\langle \Psi_G|h_f^\dagger h_m e^{iA_{lm}}|\Psi_G \rangle = E^h/4N \text{ for any nearest neighbor link (lm) due to the translational symmetry. Then a straightforward manipulation leads to}
\]

\[
\langle \Psi'|H_h|\Psi' \rangle - E_{\text{G}}^h = \frac{E_{\text{G}}^h}{2N} \sum_{\langle im \rangle} \{1 - \cos [\theta_i(l) - \theta_i(m)]/2 \}. \tag{2.18}
\]

Notice that if the (lm) link (say, along \( \hat{x} \)-direction) is far away from the site \( i \), then one has \( |\theta_i(l) - \theta_i(m)| \rightarrow a|\sin \theta|/r \) where \( r \) denotes the distance between the center of the link and the site \( i \) and \( \theta \) is the azimuth angle. Then it is easy to see that the summation over those links on the r.h.s. of (2.18) will contribute as \( \int dr dr \sin^2 \theta/r^2 \propto \ln R \) (\( R \) denotes the sample size). Namely, the vortex-like phase shift will cost a logarithmically diverged energy if it is left alone. Similar logarithmically divergent energy contributed by \( H_s \ [2.7] \) can be also obtained. It should be noted that the same conclusion still holds if one replace \( H_{\text{eff}} \) by the exact \( t-J \) model in the representation of (1.3) (Ref. 15).

Hence the vortex-like phase shift field has to be absorbed by the holon and spinon fields in order to keep the quasiparticle energy finite. In the following, let us illustrate how this will happen. We first use the vortex phase \( \frac{1}{i} \tilde{\Phi}_i^b \) in (2.4) as an example. Let us write down the following identity

\[
e^{i\tilde{\Phi}_i^b} = \left( e^{\frac{i}{2} \sum_{c_h} A^f \left[ e^{iK_h} \right] } \right) e^{i\tilde{\Phi}_i^b}, \tag{2.19}
\]

in which

\[
\sum_{c_h} A^f \equiv \sum_s A^f_{m_s m_{s+1}} \tag{2.20}
\]

where \( m_0 = i, m_1, ..., m_{k_{ch}} = j \) are sequential lattice sites on an arbitrary path \( c_h \) connecting \( i \) and \( j \). And \( K_h(c_h) \equiv \frac{1}{2} \sum_s \left[ \theta_{m_s-1}(m_s) - \theta_{m_{s+1}}(m_s) \right] \left( \sum_{\alpha} \alpha n^b_{m_s \alpha} - 1 \right) \) which is a string-like field only involving spinons on the path \( c_h \). By contrast, the line summation \( \sum_{c_h} A^f \) is contributed by spinons from the whole system nonlocally. Note that, according to \( H_h \ [2.8] \), holons see the gauge field \( A^f \) in the Hamiltonian and if a holon moves from site \( i \) to \( j \) via the same path \( c_h \) it should acquire a phase factor \( e^{-i \sum_{c_h} A^f} \) which can exactly compensate the similar phase in (2.19).
In other words, if the vortex phase factor $e^{i\Phi_i^h}$ is bound to a holon to form a new composite object $\hat{h}_i^h = h_i^h e^{i\Phi_i^h}$, then there will be no more vortex effect as it moves on the path $c_s$ shown in Fig. 3, except for a phase string field $K^h(c_h)$ left on its path, and the new object should cost only a finite energy.

Similarly, for the vortex field $-\frac{F}{2} \Phi_i^h$ in (2.1) one can rewrite

$$e^{-i\Phi_i^h} = \left(e^{-i\sigma \sum_{c_s} A^h} e^{-i\sigma K^h(c_s)}\right) e^{-i\Phi_i^h},$$

in which

$$\sum_{c_s} A^h = \sum_s A^h_{m_s, m_{s+1}}$$

where the line summation runs over a sequential lattice sites on an arbitrary path $c_s$ connecting $i$ and $j'$ shown in Fig. 3. And then we can similarly see that the spinon constituent of the quasiparticle also has to be bound to a holon to form a new composite object $\check{h}_i^s = h_i^s e^{i\Phi_i^s}$, then there will be no more vortex effect as it moves on the path $c_h$ shown in Fig. 3, except for a phase string field $K^s(c_h)$ left on its path, and the new object should cost only a finite energy.

However, there is one problem in the above argument about the absorption of the vortex phase factors $e^{i\Phi_i^h}$ and $e^{i\Phi_i^s}$ by the holon and spinon constituents. Namely, these two phase factors are not single-valued except in the zero doping limit. In fact, only the total phase factor $e^{-i\Theta_{i,j'}}$ in (2.19) and (2.21) can be well-defined in the total phase shift fields together to eliminate the divergent energy while maintain single-valued. There is another way to see this. Note that $\theta_{m_{s-1}}(m_s) - \theta_{m_{s+1}}(m_s)$ describes the angle between the nearest-neighboring links $(m_{s-1}, m_s)$ and $(m_{s+1}, m_s)$, it can have an uncertainty by $\pm 2\pi \times$integer, and it is easy to see that the phase string factors $e^{iK^h(c_h)}$ and $e^{-i\sigma K^h(c_s)}$ in (2.19) and (2.21) are not well-defined by themselves as they are multi-valued except at $\delta = 0$. On the other hand, if one chooses $c_h = c_s = c$ in Fig. 3, the mathematical ambiguity is eliminated in the total phase string field

$$K_\sigma(c) \equiv K^h(c) - \sigma K^h(c)$$

$$= \sum_{s=1}^{k_s} \left[\theta_{m_{s-1}}(m_s) - \theta_{m_{s+1}}(m_s)\right] \frac{1}{2} \left(\sum_{\alpha} an_{m_s, \alpha} - 1 - \sigma n_{m_s, \sigma}\right),$$

since by using the no-double-occupancy constraint, one can show $\frac{1}{2} \left(\sum_{\alpha} an_{m_s, \alpha} - 1 - \sigma n_{m_s, \sigma}\right) = -\frac{1+\sigma}{2} + \sigma n_{m_s, \sigma}$ which is an integer such that $e^{iK_\sigma(c)}$ remains single-valued.

Physically, it is because a fermionic quasiparticle may not decay into two bosonic holon and spinon elementary excitations in 2D. The only exception is in the zero-doping limit. We have pointed out in Introduction that at half-filling the “fermionic” nature of the electrons essentially disappears and is replaced by a “bosonic” one due to the
no-double-occupancy constraint. Then it is not surprising that in the one-hole doped case which is adjacent to the half-filling, the deconfinement of holon-spinon can happen as a result of the electron “bosonization”. Indeed, in the zero-doping limit \( \Phi_i^h \) defined in the gauge (2.3) vanishes. Without \( \Phi_i^h \), the original reason for inseparable \( e^{-i\pi/2}\Phi_i^h \) and \( e^{i\pi/2}\Phi_i^h \) in \( e^{i\Theta_i} \) is no longer present: In this case, the phase shift field \( e^{i\Theta_i} \) reduces to \( e^{i\pi/2}\Phi_i^h \) which itself becomes well-defined, and can solely accompany the holon during the propagation. As for the spinon part, \( H_s \) in (2.7) reduces to the well-known SBMFT Hamiltonian with \( A^h = 0 \) and the corresponding line summation \( \sum_s A^h \) is also absent in the propagator. Without leading to the multi-value ambiguity, the quasiparticle will break into a spinon and a composite of holon-vortex phase which can propagate independently. More discussions of the one-hole problem can be found in the Sec. II D3.

How a quasiparticle behaves in a spinon-holon sea as a single entity at finite doping will be the subject of discussion in the next subsection. In the following we will make several remarks on some implications of the confinement before concluding the present subsection. First of all, we note that a quasiparticle generally remains an incoherent excitation in contrast to the coherent spinons and holons and we assume that it will not contribute significantly to either thermodynamic and dynamic properties. In the equal-time limit \( t = 0^- \), the single-electron propagator can be expressed as

\[
G_e(i,j;0^-) = i(-\sigma)^{i-j} \left\langle \left( b_{i\sigma} e^{i\pi/2} \sum_e A^h_{i\sigma} \right) \left( h_{j\sigma} e^{-i\pi/2} \sum_e A^f_{j\sigma} \right) e^{iK_s(c)} \right\rangle.
\]  

(2.24)

At finite \( t \), temporal components have to be added to the line summations, \( \sum_e A^h \) and \( \sum_e A^f \), as well as in the phase string field \( K_s(c) \) above. Even though mathematically the path \( c \) can be chosen arbitrarily in (2.24), a natural choice is for \( c \) to coincide with the real path of the quasiparticle such that the line summations can be precisely compensated by the phases picked up by the holon and spinon constituents as mentioned above. In this case, all the singular phase effect will be tracked by \( e^{iK_s(c)} \) which is nothing but the previously-identified phase string effect [18], where it has been shown that the phase string effect is nonrepairable and represents the dominant phase interference at low energy. Physically, it reflects the fermionic exchange relation between the quasiparticle under consideration and those electrons in the background. Such a phase string field accompanying the propagation of the quasiparticle is a many-body operator in terms of elementary holon and spinon fields. Even in the one hole case, such a phase string effect results in the incoherency of the quasiparticle as has been discussed in Ref. [22].

One may also see how a Fermi-surface structure is generated from the phase shift \( \Theta_{i\sigma} \) in some limits. For example, in the 1D case (where \( A_{i\sigma}^{h,f} = 0 \) [13]), since one may always define \( \theta_{m_s-1}(m_s) - \theta_{m_s+1}(m_s) = \pm \pi \), the phase string factor \( e^{iK_s(c)} \) in (2.24) can be written as \( (-\sigma)^{i-j} e^{i\sigma k_f(x_1-x_j)} e^{i\delta k_f^s(x_1-x_j)} \) which produces the 1D Fermi surface at \( k_f = \pm \pi (1 - \delta) / 2 \) (here \( \delta k_f^s \) denotes Fermi-surface fluctuations with \( \delta k_f^s = 0 \) which is crucial to the Luttinger liquid behavior [13]). In the 2D one-hole case, \( \Theta_{i\sigma} \) also leads to a “remnant” Fermi surface structure in the equal time limit while gives rise to four Fermi points \( k_0 \) at low energy as discussed in Ref. [24]. At finite doping, the doping-dependent incommensurate peaks in the dynamic spin susceptibility function has been also related to such a phase shift field [19]. In general, the Luttinger volume theorem may even be understood based on \( e^{iK_s(c)} \) as it involves the counting of the background electron numbers. Nevertheless, the precise Fermi-surface topology will not be solely determined by the phase shift field in 2D and one must take into account of the dynamic effect.

Finally, a stable but incoherent quasiparticle excitation in which a pair of holon and spinon are confined means that a photoemission experiment, in which such a quasiparticle excitation can be created through “knocking out” an electron by a photon, does not directly probe the intrinsic information of coherent elementary excitations anymore, and the energy-momentum structure of the single-electron Green’s function is no longer a basis as fundamental and useful as in the case of conventional Fermi liquid metals to understand superconductivity, spin dynamics, and transport properties in other channels.

D. Description of the quasiparticle: Equation-of-motion method

Now imagine a bare hole is injected into the ground state of \( N_e \) electrons. By symmetry, such a state should be orthogonal to the ground state of \( N_e - 1 \) electrons. Its dissolution into a holon and a spinon is also prohibited by the symmetry introduced by the phase shift field (1.3) and the latter would otherwise cost a logarithmically divergent energy if being left alone unscreened. Therefore, one has to treat a quasiparticle as an independent collective excitation in this spin-charge separation system.

Involving infinite-body holons and spinons, a quasiparticle cannot be simply described by the mean-field theory of individual holon and spinon. The previously discussed confinement is one example of non-perturbative consequences
caused by the infinite-body phase shift field. But such a confinement of the holon and spinon inside a quasiparticle will enable us to approach this problem from a different angle.

Here it may be instructive to recall how a low-lying collective mode is determined in the BCS theory. In the BCS mean-field theory, quasiparticle excitations are well defined with an energy gap. But quasiparticle excitations do not exhaust all the low-lying excitations, and there exists a collective mode in the absence of long-range Coulomb interaction, which may be also regarded as a “bound” state of a quasiparticle pair due to the residual attractive interaction. A correct way \[29\] to handle this “bound” state is to use the full BCS Hamiltonian to first write down the equation of motion for a quasiparticle pair and then apply the BCS mean-field treatment to linearize the equation to produce the gapless spectrum, which is equivalent to the RPA scheme \[30\]. Including the long-range Coulomb interaction \[29\] will turn this collective mode into the well-known plasma mode.

Similarly we can establish an equation-of-motion description of the quasiparticle as a “collective mode”, which moves on the background of the mean-field spin-charge separation state. For this purpose, let us first write down the full equation of motion of the hole operator in the Heisenberg representation: \(-i\partial_t c_{i\sigma}(t) = [H_{t-J}, c_{i\sigma}(t)]\), based on the exact \(t-J\) model, either in the decomposition \[13\] or simply in the original \(c\)-operator representation, as follows

\[
[H_t, c_{i\sigma}] = \frac{t}{2}(1+n_i^h) \sum_{l=NN(i)} c_{l\sigma}
+ t \sum_{l=NN(i)} \left(c_{l\sigma} \sigma S_i^z + c_{-\sigma} S_i^{-\sigma}\right),
\]

(2.25)

and

\[
[H_J, c_{i\sigma}] = \frac{J}{4} c_{i\sigma} \sum_{l=NN(i)} (1-n_i^h)
- \frac{J}{2} \sum_{l=NN(i)} \left(c_{i\sigma} \sigma S_i^z + c_{-\sigma} S_i^{-\sigma}\right).
\]

(2.26)

Note that the above equations hold in the restricted Hilbert space under the no-double-occupancy constraint: \(\sum_{i} c_{i\sigma}^\dagger c_{i\sigma} \leq 1\).

There are many papers in literature dealing with the \(t-J\) model in the \(c\)-operator representation, in which the no-double-occupancy \(\sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma} \leq 1\) is disregarded. As a consequence, there is only a conventional scattering between the quasiparticle and spin fluctuations as suggested by (2.25) and (2.26). This leads to a typical spin-fluctuation theory, which usually remains a Fermi liquid theory with well-defined coherent quasiparticle excitations near the Fermi surface in contrast to the \(Z=0\) conclusion obtained here. The problem with the spin fluctuation theory is that the crucial role of the no-double-occupancy constraint hidden in (2.25) and (2.26) has been completely ignored which, in combination with the RVB spin pairing, is actually the key reason resulting in a spin-charge separation state in the present effective theory of the \(t-J\) model. In such a backdrop of the holon-spinon sea, the scattering terms in (2.25) and (2.26) will actually produce a virtual “decaying” process which is fundamentally different from the usual spin-fluctuation scattering in shaping the single-electron propagator.

By using the decomposition (1.3) and the mean-field order parameter \(\Delta^s\) defined in (2.10), the high-order spin-fluctuation-scattering terms on the r.h.s. of (2.25) and (2.26) can be “reduced” to the same order of linear \(c_{i\sigma}\), and we find

\[-i\partial_t c_{i\sigma}(t) \approx \frac{t}{2}(1+\delta) \sum_{l=NN(i)} c_{l\sigma} + J(1-\delta)c_{i\sigma}\]

\[-\frac{1}{4} (B_0 \sum_{l=NN(i)} e^{i\Theta_{l\sigma}} l_i^h b_{l\sigma} e^{-i\Theta h_i} + \frac{3}{8} J \Delta^s \sum_{l=NN(i)} e^{i\Theta_{l\sigma}} l_i^h b_{-\sigma} e^{i\Theta h_i} + \ldots \]

(2.27)

where \(B_0\) is the modified (but not an independent) order parameter for the hopping term introduced in Ref. [21]. In the following we will discuss some unique quasiparticle properties based on this equation.

So in the spin-charge separation (mean-field) background, the leading order effect of the “scattering” terms correspond to the decay of the quasiparticle: The terms in the second line of (2.27) clearly indicate the tendency for the quasiparticle to break up into holon and spinon constituents. This is in contrast to the conventional scatterings between the quasiparticle and spin fluctuations, as (2.25) and (2.26) would have suggested. Generally, the quasiparticle is expected to have an intrinsic broad spectral function extended over the whole energy range.
\[ E_{\text{quasiparticle}} > E_{\text{holon}} + E_{\text{spinon}} \]  

because of the decomposition process. But the presence of the phase factor \( e^{i\hat{\Theta}} \) in these “decaying” terms of (2.27) prevents a real decay of the quasiparticle since such a vortex field would cost a logarithmically divergent energy as has been discussed before. Thus, even in the case of (2.28), the decaying of a quasiparticle remains only a virtual process which is another way to understand the confinement discussed in Sec. II C.

Without the “decaying” terms, the equation-of-motion (2.27) would become closed with an eigen spectrum in momentum-energy space (besides a constant which can be absorbed into the chemical potential):

\[ \epsilon_k = -2t_{\text{eff}}(\cos k_x + \cos k_y), \]  

with

\[ t_{\text{eff}} = \frac{t}{2}(1 + \delta). \]  

Generally the “decaying” terms do not contribute to a coherent \( k \)-dependent correction due to the nature of the holon and spinon excitations as well as the “smearing” caused by \( e^{i\hat{\Theta}} \) in (2.27). But in the ground state, which is also superconducting, the “decaying” terms in (2.27) do produce a coherent contribution due to the composite nature of the quasiparticle which will modify the solution of the equation-of-motion.

1. Ground state: a superconducting state

In the mean-field ground state, the bosonic holons are Bose condensed with \( \langle \hat{h}_i^\dagger \rangle = h_0 \sim \sqrt{\delta} \) and the superconducting order parameter \( \Delta^{SC} \neq 0 \) (see Sec. II A). The decomposition (1.3) then implies that the electron \( c \)-operator may be rewritten in two parts:

\[ c_{i\sigma} = h_0 a_{i\sigma} + c'_{i\sigma}, \]  

where \( a_{i\sigma} \equiv b_{i\sigma} e^{i\hat{\Theta}_{i\sigma}} \) and \( c'_{i\sigma} = (\hat{h}_i^\dagger \cdot) b_{i\sigma} e^{i\hat{\Theta}_{i\sigma}} \) with \( \hat{h}_i^\dagger \cdot \equiv h_i^\dagger - h_0 \). Correspondingly, a coherent term will emerge from the “decaying” terms in (2.27) which is linear in \( a^\dagger \):

\[ J\text{-scattering term in (2.27)} \rightarrow \frac{3}{8} J \sum_{l \in N N(i)} \left( \frac{\Delta^{SC}_{il}}{h_0^2} \right) \sigma h_0 a^\dagger_{l-\sigma} + \text{high order}. \]  

In obtaining the r.h.s. of the above expression, the superconducting order parameter defined in (2.11) is used. Note that the \( t \)-scattering term in (2.27) gives rise to a term \( h_0 a_{i\sigma} \) which can be absorbed by the chemical potential \( \mu \) added to the equation. Then one finds

\[ -i\partial_t a_{i\sigma} \simeq t_{\text{eff}} \sum_{l \in N N(i)} a_{l\sigma} + \mu a_{i\sigma} + \frac{3}{8} J \sum_{l \in N N(i)} \left( \frac{\Delta^{SC}_{il}}{h_0^2} \right) \sigma a^\dagger_{l-\sigma} + \text{high order} \]  

(2.33)

where the connection between \( a \) and \( c' \) has been assumed to be in high order and thus is neglected in the leading order approximation to get a closed form in linear \( a \) and \( a^\dagger \). Finally, introducing the Bogoliubov transformation in the momentum space

\[ a_{k\sigma} = u_k \gamma_{k\sigma} - \sigma v_k \gamma^\dagger_{-k-\sigma}, \]  

(2.34)

we find that (2.33) can be reduced to

\[ -i\partial_t \gamma^\dagger_{k\sigma} = E_k \gamma^\dagger_{k\sigma}, \]  

(2.35)

where \( \gamma^\dagger_{k\sigma} \) represents the creation operator of an eigenstate of quasiparticle excitations with the energy spectrum

\[ E_k = \sqrt{(\epsilon_k - \mu)^2 + |\Delta_k|^2}. \]  

(2.36)

Here \( \Delta_k \) is defined by
\[
\Delta_k = \frac{3}{4} J \sum \Gamma_q \left( \frac{\Delta_{SC}^k + \Delta_{q}^k}{k_0^2} \right),
\]
(2.37)

with \( \Gamma_q = \cos q_x + \cos q_y \). Like in the BCS theory, \( u_k^2 = \frac{1 + (\epsilon_k - \mu)/E_k}{2} \) and \( v_k^2 = \frac{1 - (\epsilon_k - \mu)/E_k}{2} \).

The large “Fermi surface” is defined by \( \epsilon_k = \mu \) and \( \Delta_k \) then represents the energy gap opened at the Fermi surface. Note that \( \Delta_k \) changes sign as
\[
\Delta_{k+Q} = -\Delta_k,
\]
(2.38)

with \( Q = (\pm \pi, \pm \pi) \), by noting \( \Gamma_{q+Q} = -\Gamma_q \) in (2.37). It means that \( \Delta_k \) has opposite signs at \( k = (\pm \pi, 0) \) and \((0, \pm \pi)\), indicating a d-wave symmetry near the Fermi surface. In fact, since the pairing order parameter \( \Delta_{SC}^k \) is d-wave-like \([20]\), \( \Delta_k \) should be always d-wave-like with node lines \( k_x = \pm k_y \) according to (2.37).

Comparing to the conventional BCS theory with d-wave order parameter, there are several distinct features in the present case. First of all, besides the d-wave quasiparticle spectrum illustrated in Fig. 4 by the “V” shape lines along the Fermi surface, there exists a discrete spinon excitation level at \( E_s \sim \delta J \) (horizontal line in Fig. 4) which leads to \( E_g = 2E_s \sim 41 \text{meV} \) (if \( J \sim 100 \text{meV} \)) magnetic peak at \( \delta \sim 0.14 \) as reviewed in Sec. II A. This latter spin collective mode is independent of the quasiparticle excitations at the mean-field level.

Secondly, even though the superconducting order parameter \( \Delta_{SC}^k \) and the energy gap \( \Delta_k \) in the quasiparticle spectrum have the same symmetry: Both are d-wave-like, they cannot be simply identified as the same quantity as in the BCS theory. For example, while \( \Delta_{SC}^k \) apparently scales with the doping concentration \( \delta \) \( (h_0 \propto \sqrt{\delta}) \) and vanishes at \( \delta \to 0 \), the gap \( \Delta_k \) defined in (2.37) is not, and can be extrapolated to a finite value in the zero doping limit where \( T_c = 0 \). It means
\[
\frac{2\Delta_k(T = 0)}{T_c} \to \infty \quad (2.39)
\]
at \( \delta \to 0 \), whereas the BCS theory predicts a constant \( \sim 4.28 \) (d-wave case \[31]\)). The result (2.39) is consistent with the ARPES measurements \[32]\.

Thirdly, the quasiparticle gains a “coherent” part \( h_0a \) which should behave similarly to the conventional quasiparticle in the BCS theory as it does not further decay at \( E_k < E_s \) (see Fig. 4). In this sense, the quasiparticle partially restores its coherence in the superconducting state. Such a coherent part will disappear as a result of vanishing superfluid density. According to (2.31) one may rewrite the single-particle propagator as
\[
G_e \simeq h_0^2 G_a + G_c',
\]
(2.40)

where \( G_a \) denotes the propagator of \( a \)-particles with omitting the crossing term between \( a \) and \( c' \) which is assumed negligible. Then \( h_0^2 G_a \) emerges as the “coherent” part of the Green’s function in superconducting state against the “normal” part \( G'_c \):

\[\text{FIG. 4. Low-lying excitations in the superconducting phase: The “V”-shape quasiparticle spectrum and the discrete spinon energy at } E_s.\]
\[ h_0^2 G_e(k, \omega) \sim h_0^2 \left( \frac{u_k^2}{\omega - E_k} + \frac{v_k^2}{\omega + E_k} \right). \]  (2.41)

Correspondingly, the total spectral function as the imaginary part of \( G_e \) in our theory can be written as

\[ A_e(k, \omega) = h_0^2 A_n(k, \omega) + A'_e(k, \omega). \]  (2.42)

So at \( h_0 \to 0 \), even though \( \Delta_k \) does not scale with \( h_0 \), the superconducting coherent part \( h_0^2 A_n \) vanishes altogether, with \( A_e \) reduces to the normal part \( A'_e \) at \( T > T_c \).

Finally, in the present case, \( A'_e(k, \omega) \) as a normal part has nothing to do with the procedure that leads to the spectrum (2.36) with a d-wave gap, which is different from the slave-boson approach where the fermionic spinons are all paired up such that even the part of the spectral function corresponding to \( A'_e \) should also look like in a d-wave pairing state. Due to the sum rule \( \int \frac{d\omega}{2\pi} A^c(k, \omega) = 1 \), the “normal” part \( A'_e(k) \) is expected to be sort of suppressed by the emergence of the “coherent” \( h_0^2 A_n \) part, but since the latter is in order of \( \delta \), \( A'_e \) should be still dominant away from the Fermi surface at small doping. It implies that even in superconducting state, a normal-state dispersion represented by the peak of \( A'_e \) may still be present as a “hump” in the total spectral function \( A_e \). Recent ARPES experiments have indeed indicated [33] the existence of a “hump” in the spectral function which clearly exhibits normal-state dispersion in the Fermi-surface portions near the areas of \((\pm \pi, 0)\) and \((0, \pm \pi)\) where the d-wave gap is maximum.

2. Normal state

In the normal state, without any coherent contribution, the scattering terms in (2.27) only give rise to the virtual process for a quasiparticle to decay into the holon-spinon pairs. Based on (2.27), the propagator can be determined according to the following standard equation of motion for the single-particle Green’s function \( G_e(i, j; t) \):

\[ \partial_t G_e(i, j; t) = \theta(t) \langle [H_{t-J}, c_{i\sigma}(t)]c_{j\sigma}(0) \rangle - \theta(-t) \langle [c_{j\sigma}(0)[H_{t-J}, c_{i\sigma}(t)] \rangle - i\delta(t)\delta_{i,j}. \]  (2.43)

If we simply neglect the scattering terms in zero-order approximation, a closed form for \( G_e \) can be obtained in momentum-energy space:

\[ G_e(k, \omega) \sim \frac{1}{\omega - (\epsilon_k - \mu)}. \]  (2.44)

Here the quasiparticle spectrum \( \epsilon_k \) [defined in (2.29)] is essentially the same as the original band-structure spectrum except for a factor of \( 2/(1+\delta) \approx 2 \) enhancement in effective mass. It is noted that in the \( t-J \) model if the hopping term described by the tight-binding model is replaced by a more realistic band-structure model, like introducing the next-nearest-neighbor hopping terms, the above conclusion about the factor-2 enhancement of the effective mass still holds, in good agreement with ARPES experiments [34]. Here the reason for the mass enhancement is quite simple: At each step of hopping, the probability is roughly one half for a hole not to change the surrounding singlet spin configuration, which in turn reduces the “bandwidth” of the quasiparticle by a factor of two.

The expression (2.44) shows a “quasiparticle” peak at \( \epsilon_k - \mu \) and defines a large “Fermi surface” as an equal-energy contour at \( \epsilon_k = \mu \). Here \( \mu \) is determined such that \( -i2 \sum_k G_e(k, t = -0) = N_e \) [35]. So the “Fermi surface” should look like similar to that of a non-interacting band-structure fermion system as long as the virtual decaying process in (2.27) does not fundamentally alter it. [As mentioned before, we do not expect such “decaying” terms to significantly modify the \( k \)-dependence of \( \epsilon_k \) since, for example, the spinon no longer has a well-defined spectrum in momentum space (see Sec. II A) and in particular, the vortex phase \( e^{i\theta} \) in (2.27) will further “smear out” the \( k \)-dependent correction, if any, from (2.27).]

So far we have not discussed the finite life-time effect of a quasiparticle due to the “decaying” terms in (2.27). Even though the true break-up of a quasiparticle is prevented by the phase shift field as discussed before, the virtual decaying process should remain a very strong effect since the phase shift field only cost a logarithmically divergent energy at a large length scale. The corresponding confinement force is rather weak and the virtual decaying process should become predominant locally to cause an intrinsic broad feature in the spectral function at high energy. Such a broad structure reflecting the decomposition in the one-hole case has been previously discussed in Ref. [22]. At finite doping, how the “decaying” effect shapes the broadening of the quasiparticle peak will be a subject to be investigated elsewhere.
The existence of a large Fermi surface, coinciding with the non-interacting band-structure one, can be attributed to the integrity of the quasiparticle due to the confinement of spinon and holon. But as pointed out in Sec. II C, such a confinement will disappear in the zero-doping limit. The Fermi surface structure will then be drastically changed.

In this limit, the single-electron propagator may be expressed in the following decomposition form

\[ G_e \approx iG_f \cdot G_b \]  \hspace{1cm} (2.45)

where

\[ G_f(i,j;t) = -i \left\langle T_i h_i^\dagger(t) e^{-i\frac{\Phi_b^h(t)}{\hbar}} h_j(0) \right\rangle, \]  \hspace{1cm} (2.46)

and

\[ G_b(i,j;t) = -i(-\sigma)^i-j \left\langle T_i b_{i\sigma}(t) e^{-i\frac{\Phi_b^h(t)}{\hbar}} b_{j\sigma}^\dagger(0) \right\rangle, \]  \hspace{1cm} (2.47)

without the multi-value problem because \( \Phi_b^h \) in (2.3) vanishes and \( e^{i\frac{\Phi_b^h}{\hbar}} \) becomes well-defined as discussed in Sec. II C. At \( \delta \to 0 \), \( H_s \) in (2.2) reduces to the SBMFT Hamiltonian with \( A^h = 0 \) and \( G_b \) becomes the conventional Schwinger-boson propagator. Such a deconfinement can also be seen from the equation of motion (2.27) by noting that \( e^{i\Phi_b^h} \rightarrow e^{i\Phi_b^{h*}} \) can be absorbed by \( h_i^\dagger \), while \( A_{k\sigma}^h = 0 \), so that the scattering term becomes a pure decaying process for the quasiparticle without any confining force. Due to such a true decaying, equation (2.27) actually describes in real time the first step towards dissolution for the quasiparticle. In particular, the large Fermi surface structure originated from the bare hopping term in (2.23) will no longer appear in the decomposition form of the electron propagator (2.45), where the residual Fermi surface (points) will solely come from the oscillating part of the phase shift field \( e^{i\delta N_i} \) in \( G_f \).

The single-electron propagator for the one-hole case has been discussed in detail in Ref. [22]. Here the large Fermi surface is gone except for four Fermi points at \( k_0 = (\pm \pi/2, \pm \pi/2) \) with the rest part looking like all “gapped”. In fact, in the convolution form of (2.45) the “quasiparticle” peak (edge) is essentially determined by the spinon spectrum \( E_k^s = 2.32 J \sqrt{1-s_k^2} \) with \( s_k = (\sin k_x + \sin k_y)/2 \) in SBMFT through the spinon propagator \( G_b \), since the holon propagator \( G_f \) is incoherent. The intrinsic broad feature of the spectral function found in Ref. [22] is due to the convolution law of (2.45) and is a direct indication of the composite nature of the quasiparticle, which is also consistent with the ARPES results as well as the earlier theoretical discussion in Ref. [36].

Note that the Fermi points \( k_0 \) coming from \( G_f \) at low energy is due to the phase shift field \( e^{i\delta N_i} \) appearing in it. In Ref. [22], this is shown in the slave-fermion formulation which is related to the present formulation through a unitary transformation [13] with \( h_i^\dagger e^{i\delta N_i} \) being replaced by a new holon operator \( f_i^\dagger \). And the \( f-\)holon will then pick up a phase string factor \( (-1)^{N_i^\downarrow} \) \( (N_i^\downarrow \text{ denotes the total number of } \downarrow \text{ spins exchanged with the holon during its propagation along the path } c \text{ connecting sites } i \text{ and } j \) at low energy which can be written as

\[ (-1)^{N_i^\downarrow} \equiv e^{\pm i \delta N_i^\downarrow} = e^{i \delta N_i^\downarrow} = e^{i N_i^\uparrow} e^{-i N_i^\downarrow} = e^{i N_i^\uparrow} e^{-i N_i^\downarrow}, \]  \hspace{1cm} (2.48)

where \( \delta N_i^\downarrow = N_i^\downarrow - \langle N_i^\downarrow \rangle \), and \( \langle n_i^\downarrow \rangle = 1/2 \) is used. On the other hand, in the equal-time \( (t \sim 0^-) \) limit, the singular oscillating part of \( e^{iK_{\sigma\nu}(v)} \) in (2.24) will also contribute to a large “remnant Fermi surface” in the momentum distribution function \( n(k) \) which can be regarded as a precursor of the large Fermi surface in the confining phase at finite doping, and is also consistent with the ARPES experiment as discussed in Ref. [23].

The above one-hole picture may have an important implication for the so-called pseudo-gap phenomenon in the underdoped region of the high-\( T_c \) cuprates. Even though the confinement will set in once the density of holes becomes finite, the “confining force” should remain weak at small doping, and one expects the virtual “decaying” process in (2.27) to contribute significantly at weakly doping to bridge a continuum evolution between the Fermi-point structure in the zero-doping limit to a full large Fermi surface at larger doping. Recall that in the one-hole case decaying into spinon-holon composite happens around \( k_0 \) at zero energy transfer, while it costs higher energy near \( (\pi, 0) \) and \( (0, \pi) \), which shouldn’t be changed much at weakly doping. In the confinement regime, the quasiparticle peak in the electron spectral function defines a quasiparticle spectrum and a large Fermi surface as discussed before. Then due to the virtual “decaying” process in the equation of motion (2.27) as shown in Fig. 1(b), the spectral function will become much broadened with its weight shifted toward higher energy like a gap-opening near those portions of the Fermi surface far away from \( k_0 \), particularly around four corners \( (\pm \pi, 0) \) and \( (0, \pm \pi) \). With the increase of doping
concentration and reduction of the decaying effect, the suppressed quasiparticle peak can be gradually recovered starting from the inner parts of the Fermi surface towards four corners \((\pm \pi, 0)\) and \((0, \pm \pi)\). Eventually, a coherent Landau quasiparticle may be even restored in the so-called overdoped regime, when the bosonic RVB ordering collapses such that the spin-charge separation disappears.

Furthermore, at small doping (underdoping), something more dramatic can happen in the model described by \((2.6)\) and \((2.7)\). In Ref. \[20\], a microscopic type of phase separation has been found in this regime which is characterized by the Bose condensation of bosonic spinon field. Since spinons are presumably condensed in hole-dilute regions \[20\], the propagator will then exhibit features looking like in an even weaker doping concentration or more “gap” like than in a uniform case, below a characteristic temperature \(T^*\) which determines this microscopic phase separation. Therefore, the “spin gap” phenomenon related to the ARPES experiments \[24\] in the underdoped cuprates may be understood as a “partial” deconfinement of holon and spinon whose effect is “amplified” through a microscopic phase separation in this weakly-doped regime. As discussed in Ref. \[20\], \(T^*\) also characterizes other “spin-gap” properties in magnetic and transport channels in this underdoping regime.

III. CONCLUSION AND DISCUSSION: A UNIFIED VIEW

In this paper, we have studied the quasiparticle properties of doped holes based on an effective spin-charge separation theory of the \(t - J\) model. The most unique result is that a quasiparticle remains stable as an independent excitation despite the existence of holon and spinon elementary excitations. The underlying physics is that in order for a doped hole to evolve into elementary excitations described by holon and spinon, the whole system has to adjust itself globally which would take infinite time under a local perturbation. Such an adjustment is characterized by a vortex-like phase shift as shown in (1.3). As a consequence of the phase shift effect, the holon and spinon constituents are found to be effectively confined which maintains the integrity of a quasiparticle except for the case in the zero doping limit. In particular, the quasiparticle weight is zero since there is no overlap between a doped hole state and the true ground state due to the symmetry difference introduced by the vortex phase shift. Such a quasiparticle is no longer a conventional Landau quasiparticle and is generally incoherent due to the virtual decaying process. Only in the superconducting state the coherence can be partially regained by the quasiparticle excitation.

The physical origin of the “unrenormalizable phase shift” is based on the fact that a hole moving on an antiferromagnetic spin background will always pick up the phase string composed of a product of + and − signs which depend on the spins exchanged with the hole during its propagation \[18\]. Such a phase string is nonrepairable at low energy and is the only source to generate phase frustrations in the \(t - J\) model. The phase shift field in (1.3) precisely keeps track of such a phase string effect \[13\] and therefore accurately describes the phase problem in the \(t - J\) model even at the mean-field level discussed in Sec. II A.

Probably the best way to summarize the present work is to compare the present self-consistent spin-charge separation theory with some fundamental concepts and ideas proposed over years in literature related to the doped Mott-Hubbard insulator.

The RVB pairing. The present theory can be regarded as one of the RVB theories \[1,7,13\], where the spin RVB pairing is the driving force behind everything from spin-charge separation to superconductivity. The key justification for this RVB theory is that it naturally recovers the bosonic RVB description at half-filling, which represents \[1,12\] the most accurate description of the antiferromagnet for both short-range and long-range AF correlations. In the metallic state at finite doping, the RVB order \(\Delta^*\) defined in (2.10) reflects a partial “fermionization” from the original pure bosonic RVB pairing due to the gauge field \(A^*_h\) determined by doped holes. But it is still physically different from a full fermionic RVB description \[14,37\]. In contrast to the fermionic RVB order parameter, \(\Delta^*\) here serves as a “super” order parameter characterizes a unified phase covering the antiferromagnetic insulating and metallic phases, normal and superconducting states altogether \[21\].

Spin-charge separation. In our theory, elementary excitations are described by charge-neutral spinon and spinless holon fields, and the ground state may be viewed as a spinon-holon sea. Different from slave-particle decompositions, spinons and holons here are all bosonic in nature and the conventional gauge symmetry is broken by the RVB ordering. But these spinons and holons in 2D still couple to each other through the mutual Chern-Simons-like gauge interactions which are crucial to \(T_c\), anomalous transport and magnetic properties. The Bose condensation of holons corresponds to the superconducting state, while the Bose condensation of spinons in the insulating phase gives rise to an AF long-range order. The spinon Bose condensation can persist into the metallic regime, leading to a pseudo-gap phase with microscopic phase separation which can coexist with superconductivity \[20\].

Bosonization. The electron c-operator expressed in terms of bosonic spinon and holon in (1.3) naturally realizes a special form of bosonization. A 2D bosonization description has been regarded by many \[1,7,18,19\] as the long-sought
technique to replace the perturbative many-body theory in dealing with a non-Fermi liquid. The 2D bosonization scheme has been usually studied, as an analog to the successful 1D version \[14\], in the momentum space where Fermi surface patches have to be assumed first \[4,17,38–40\]. In the present scheme, which is also applicable to 1D, the Fermi surface satisfying the Luttinger volume and the so-called Fermi-surface fluctuations are presumably all generated by the phase shift field in \[1,3\], which guarantees the fermionic nature of the electron. Note that the vortex-structure involved in the phase shift field in the 2D case is the main distinction from the \(\Theta\)-function in conventional bosonization proposals \[40\].

**Non-Fermi liquid.** As a consequence of the phase shift field, representing the Fermi surface “fluctuations”, the ground state is a non-Fermi liquid with the vanishing spectral weight \(Z\), consistent with the argument made by Anderson \[3\] based on a “scattering” phase-shift description. In 1D both methods are equivalent as the phase shift value in the latter can be determined quantitatively based on the exact Bethe-Ansatz solution. But in 2D, the phase shift field \(\Theta_{\text{string}}\), which is obtained by keeping track of the nonreparable phase string effect induced by the traveling holon, provides a unique many-body version with vorticities, and our model shows how a concrete 2D non-Fermi liquid system can be realized.

**Quasiparticle: Spinon-holon confinement.** In conventional spin-charge separation theories based on slave-particle schemes, a quasiparticle does not exist at all: It always breaks up and decays into spinon-holon elementary excitations. This deconfinement has been widely perceived as a logical consequence of the spin-charge separation in literature. But in the present paper, we have shown that the phase shift field actually confines the spinon-holon constituents (at least at finite doping), which means the integrity of a quasiparticle is still preserved even in a spin-charge separation state. It may be considered as a \(U(1)\) version of quark confinement, but with a twist: the stable quasiparticle as a collective mode is generally not a coherent elementary excitation since during its propagation the phase shift field also induces a nonlocal phase string on its path. In other words, the quasiparticle here is not a Landau quasiparticle anymore. The confinement and nonreparable phase string effect both reflect the fermionic nature of the quasiparticle, namely, the fermionic quasiparticle cannot simply decay into bosonic spinon and holon; and the phase string effect comes from sequential signs due to the exchange between the propagating quasiparticle and the spinon-holon background — the latter after all is composed of fermionic electrons in the original representation.

The issue of the possible confinement of spinon and holon was already raised by Laughlin \[42\] along a different line of reasoning. He has also discussed numerical and experimental evidence that the spinon-holon may be seen, more sensibly, in high-energy spectroscopy like the way quarks are seen in particle physics. In the recent SU(2) gauge theory \[43\], an attraction between spinon and holon to form a bound state due to gauge fluctuations is also assumed in order to explain the ARPES data. But in the present work the essential point is that the holons and spinons are not confined in the ground state but are only bound in quasiparticles as a kind of incoherent (many-body) excitations which have no overlap with elementary holon and spinon excitations as guaranteed by symmetry. These incoherent quasiparticle excitations should not have any significant contribution to the thermodynamic properties (at least above \(T_c\)). At short distance and high-energy, the composite nature of a quasiparticle will become dominant which may well explain the broad intrinsic structure in the spectral function observed in the ARPES experiments. The composite structure of the quasiparticle can even show up at low-energy when one navigates through different circumstances like the superconducting condensation, underdoping regime, etc., with some unique features different from the behavior expected from Landau quasiparticles.

**Fractional statistics.** Laughlin \[44\] had made a compelling argument right after the proposal of the spin-charge separation that the holon and spinon should carry fractional statistics, by making an analogy of the spin liquid state with a fractional quantum Hall state. Even though the absence of the time-reversal symmetry-breaking evidence in experiment does not support the original version of fractional statistics (anyon) theories \[44,45\], the essential characterization of fractional statistics for the singlet spin liquid state is, surprisingly, present in our theory in the form of the phase string effect as discussed in Ref. \[15\]. But no explicit time-reversal symmetry is broken in this description \[20\].

A fractional statistics may sound strange as we have been talking about “bosonization” throughout the paper. But as pointed out in Ref. \[13\], if the phase shift field \(\Theta_{\text{string}}\) is to be “absorbed” by the bosonic spinon and holon fields, then the expression \(1,3\) can be regarded as a slave-semon decomposition with the new “spinon” and “holon” fields being “bosonic” among themselves but satisfying a mutual fractional statistics between the spin and charge degrees of freedom. The origin of mutual statistics can be traced back to the nonreparable phase string effect induced by a hole in the antiferromagnetic spin background. At finite doping, the order parameter \(\Delta^*\) in \(2.10\) actually describes the RVB pairing of spinons with mutual-statistics which reduces to the bosonic RVB only in the half-filling limit. Furthermore, in the bosonic representation of \(2.6\) and \(2.7\) the lattice Chern-Simons fields, \(A^s_{ij}\) and \(A^b_{ij}\), precisely keep track of mutual statistics \[13\], which are crucial to various peculiar properties exhibited in the model.

**Gauge theory.** Based on the slave-boson decomposition, it has been shown \[26,27\] that the gauge coupling is
the most important low-energy interaction associated with spin-charge deconfinement there. The anomalous linear-
temperature resistivity \cite{25} has become the hallmark for anomalous transport phenomenon based on the scattering
between charge carriers and gauge fluctuations. In the present theory, the holons are also subject to strong random
flux fluctuations, in terms of the effective holon Hamiltonian \cite{25,27}, in a uniform normal state where it leads \cite{4}
to the linear-T resistivity in consistency with the Monte Carlo numerical calculation \cite{4}. Other anomalous transport
properties related to the cuprates may be also systematically explained in such a simple gauge model based on some
effective analytic treatment \cite{11,17}. In contrast to the slave-boson gauge theory, however, the spinon part does not
participate in the transport phenomenon due to the RVB condensation \((\Delta^s \neq 0)\) persisting over the normal state,
and besides the Chern-Simons gauge fields \(A^{f,h}_{ij}\), the conventional gauge interaction between spinons and holons is
suppressed because of it.

Furthermore, the \(\pi\)-flux phase \cite{15} and commensurate flux phase \cite{14} in the mean-field slave-boson theory, and the
recent SU(2) gauge theory \cite{43} at small doping have a very close connection with the present approach: A fermionic
spinon in the presence of \(\pi\) flux per plaquette is actually a precursor to become a bosonic one at half-filling under
the lattice and no-double-occupancy constraint. In Ref. \cite{14}, how such a statistics-transmutation occurs has been
discussed, and in fact the bosonization decomposition \(1.3\) was first obtained there based on the fermionic flux
phase. In spite of physical proximity, however, the detailed mathematical structure of gauge description \cite{50,51} for
the fermionic flux phase and the present bosonic spinon description are obviously rather different.

**Superconducting mechanism.** Anderson \cite{1} originally conjectured that the superconducting condensation may occur
once the RVB spin pairs in the insulating phase start to move like Cooper pairs in the doped case. The superconducting
condensation in the present theory indeed follows suit. But there is an important subtlety here. Since the RVB paring
order parameter \(\Delta^s\) covers the normal state as well, there must be an another factor controlling the superconducting
transition: The phase coherence. Indeed, the vortex phase \(\Phi^b\) appearing in the superconducting order parameter
\cite{2,11} is the key to ensure the phase coherence at a relatively low temperature compatible to a characteristic spin
energy \(2T_c\). In other words, the phase coherence discussed by Emery and Kivelson \cite{52} is realized by the phase shift
field in the present theory which effectively resolves the issue why \(T_c\) is too high in previous RVB theories.

Furthermore, the interlayer pair tunneling mechanism \cite{53} for superconductivity is also relevant to the present
theory from a different angle. Recall that the quasiparticle does exist in the present theory but is always incoherent
just like the blocking of a coherent single-particle interlayer hopping conjectured in Ref. \cite{53}. On the other hand, a
pair of quasiparticles in the singlet channel can recover the coherency due to the cancellation of the frustration caused
by the phase string effect. Thus if one is to construct a phenomenological theory based on the electron representation,
the superconductivity can be naturally viewed as due to an in-plane kinetic mechanism \cite{57}.

**Phase string and \(Z_2\) gauge theory.** As pointed out at the beginning of this section, our whole theory is built on the
phase string effect identified in the \(t-J\) model. Namely, the main phase frustration induced by doping is characterized
by a sequence of signs \(\prod_c \sigma_{ij}\) on a closed path \(c\) where \(\sigma_{ij} = \pm 1\) denotes the index of spins exchanged with a hole at
a link \((ij)\) during its hopping. Thus, instead of working in the vortex representation of \(1.3\) where the singular phase
string effect has been built into the wave functions with the non-singular part described by the Chern-Simons-like
lattice gauge fields \cite{13}, one may also directly construct a 2+1 \(Z_2\) gauge theroy \cite{57} to deal with the singular phase
string \(\prod_c \sigma_{ij}\). A discrete \(Z_2\) gauge theory here seems the most natural description of the phase string effect as the
sole phase frustration in the \(t-J\) model, in contrast to the conventional continuum gauge field description \cite{23,27}.

Finally, we emphasize the close connection between the antiferromagnetism and superconductivity as both occur in
a unified RVB background controlled by \(\Delta^s\). The relation between the AF insulating phase and the superconducting
phase here is much more intrinsic than in conventional approaches to the \(t-J\) model. Especially, the coexistence
of holon and spinon Bose condensations in the underdoped regime \cite{24} makes a group theory description of such a
phase, in a fashion of the SO(5) theory \cite{44}, become possible but with an important modification: Inhomogeneity
must play a crucial role here in the Bose condensed holon and spinon fields in order to describe this underdoped
regime. Detailed investigation along this line will be pursued elsewhere.

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strictly speaking, this orthogonal condition holds only exactly in the thermodynamic limit. A finite-size lattice usually breaks the continuum rotational symmetry. In the thermodynamic limit, when the length scale under consideration is much larger than the lattice constant, i.e., in the continuum limit, a rotational symmetry and angular momentum can still be meaningfully defined. Since $l$ is a thermodynamic number [$\sim O(N)$] at finite $\delta$, it will still remain so while the local discrete coordinates get coarse-grained in the continuum limit.

Note that here is an independent variable which has not previously appeared in the mean-field theory of the ground state discussed in Sec. II A where only the Lagrangian multiplier $\lambda$ is present to enforce the double-occupancy constraint at global level while the holon number is fixed $[2]$. 

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