Influence of MHD Peristaltic Transport for Jeffrey Fluid with Varying Temperature and Concentration through Porous Medium

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Abstract. This paper is intended for investigating the effects of heat and mass transfer on the peristaltic motion of Magnetohydrodynamic of a non-Newtonian transport of Jeffrey fluid through a cylindrical porous medium channel. The flow is investigated in a wave frame of reference moving with the velocity of the wave. Governing equations for the problem under consideration have been simplified under the assumptions of long wavelength and low Reynolds number approximation. The distribution of temperature and concentration are discussed for various parameters governing the flow with the simultaneous effects of. The analytical formulas of the velocity and temperature have been obtained in terms of Bessel function of first and second kinds. In addition, it has been illustrated graphically for significant various parameters such as, magnetic, permeability, Radiation, Reynolds number, Prandtl number, Schmidt and Soret numbers on these velocity are discussed and illustrated graphically.

Keywords: Magnetohydrodynamic (MHD), Peristaltic, Jeffrey Fluid, mass transfer, Porous medium.

1. Introduction
The influxes attracted the attention of a number of researchers because of extensive applications in the physiology and industry. The word rheumatism comes from the Greek word Peristaltic which means clamps and pressure. The drowsiness moves to a deflationary wave along the structure of the tube, which is produced physiologically from the nerve muscle properties of any soft tubular muscle. Peristaltic movements of blood in the bodies of animals or humans, many writers have considered. It is an important blood transfusion mechanism, where the arterial cross-section is gradually graded through the spread of the gradual wave. Peristaltic infusion is found in many applications, for example, vessel movements, such as transferring sensitive or corrosive liquids, putty, liquid, and harmful fluids into the nuclear industry. Non-Newtonian fluid theory has received considerable attention in recent years, because conventional viscous fluids cannot accurately describe the properties of many physiological fluids, see [1- 2]. Hayat et al. [3], checks the effect of rotation and thermophoresis on MHD peristaltic
transport of Jeffrey fluid with convective conditions and wall properties. Bhatti et al. [4], they examine theoretically nonlinear thermal radiation effects on EMHD peristaltic propulsion of non-Newtonian fluid-particle (dusty) suspensions in a planar homogenous porous channel. S Venkateswarlu et al. [6], they investigate the unsteady hydromagnetic free convective Jeffrey fluid flow in the presence of heat source. Sankad and Nagathan [7], they analyzed the importance of magnetic effect on peristaltic motion of Jeffrey fluid inside a uniform porous medium channel. Al-Khafajy and Abdulhadi [8], they analyzed a mathematical model to study the effect of wall properties and heat transfer on swallowing the food bolus through the oesophagus. Recently Sivaiah et al. [9], they studied the two-fluid peristaltic flow of a Jeffrey fluid with a Newtonian fluid in a vertical channel.

From the above we will investigating the effects of heat and mass transfer on the peristaltic motion of Magnetohydrodynamic of a non-Newtonian transport of Jeffrey fluid through a cylindrical porous medium channel.

1. **Mathematical Formulation**

Consider a peristaltic flow of an incompressible Jeffrey fluid in a coaxial uniform circular tube. The cylindrical coordinates are considered, where $R$ is along radius of the tube and $Z$ coincides with the axes of the tube as shown in figure (1).

![Figure (1): Geometry of the](image)

The geometry of wall surface is described as:

$$H(Z,t) = a + b \sin \left[ \frac{2\pi}{\lambda} (Z - ct) \right]$$

(1)

where $a$ is the average radius of the undisturbed tube, $b$ is the amplitude of the peristaltic wave, $\lambda$ is the wavelength, $c$ is the wave propagation speed, and $\tilde{t}$ is the time.

2. **Basic Equations**

The basic equations governing the non-Newtonian Jeffrey fluid are given by:

$$\nabla \vec{V} = 0$$

(2)

The momentum equation is given by:

$$\rho (\nabla \vec{V}) = \nabla \vec{F} + \mu_e \vec{j} \times B - \frac{\mu}{K^*} \nabla + \rho g \beta_T (T - T_0) + \rho g \beta_C (C - C_2)$$

(3)
The constitutive equations for an incompressible Jeffrey fluid are given by:

\[
\tau = - \frac{\partial T}{\partial t} + S, \\
S = \frac{\mu}{1+\lambda_1} (\gamma + \lambda_2 \dot{\gamma}),
\]

where \( \dot{\gamma} \) is the shear rate, \( \mu \) is the dynamic viscosity, \( \lambda_1 \) is the ratio of relaxation to retardation times, \( \lambda_2 \) is the retardation time.

### 3. Method of Solution

Let \( \vec{U} \) and \( \vec{W} \) be the respective velocity components in the radial and axial directions in the fixed frame, respectively. For the unsteady two-dimensional flow, the velocity components may be written follows:

\[
\vec{V} = (\vec{U}(\tilde{r}, z), 0, \tilde{W}(\tilde{r}, z))
\]

The equation of motion (2) – (7) and the constitutive relations (8), (9) take the form

\[
\frac{\partial \vec{u}}{\partial t} + \vec{U} \cdot \nabla \vec{u} + \vec{W} \cdot \nabla \vec{u} = - \nabla p + \frac{1}{\rho} \left( \frac{\partial T}{\partial \tilde{r}} \right) + \frac{1}{\rho} \frac{\partial \bar{S}_{\tilde{r}z}}{\partial z} - \frac{\partial \bar{S}_{\tilde{r}z}}{\partial z} - \frac{\mu}{\rho} \vec{U}
\]

\[
\frac{\partial \bar{w}}{\partial t} + \vec{U} \cdot \nabla \bar{w} + \vec{W} \cdot \nabla \bar{w} = - \frac{\partial \bar{p}}{\partial \tilde{z}} + \frac{1}{\rho_0 \tilde{r}} \left( \frac{\partial \bar{T}}{\partial \tilde{r}} \right) + \frac{1}{\rho_0 \tilde{r}} \frac{\partial \bar{S}_{\tilde{r}z}}{\partial \tilde{z}} - \frac{\partial \bar{S}_{\til{r}z}}{\partial \til{z}} + \rho \beta_T \left( T - T_0 \right) + \rho \beta_C \left( C - C_0 \right) - \frac{\sigma B_0^2 \sin^2(\alpha) \vec{W}}{k} - \vec{W}
\]

\[
\frac{\partial \bar{T}}{\partial t} + \vec{U} \cdot \nabla \bar{T} + \vec{W} \cdot \nabla \bar{T} = \frac{K}{\epsilon \rho_f} \left( \frac{\partial^2 \bar{T}}{\partial \til{r}^2} + \frac{1}{\til{r}} \frac{\partial \bar{T}}{\partial \til{r}} + \frac{\partial^2 \bar{T}}{\partial \til{z}^2} \right) - \frac{16 \sigma \int \frac{1}{\epsilon_f c_{\rho f}} \bar{R} \frac{\partial^2 \bar{T}}{\partial \til{r}^2} \bar{R} - \frac{Q}{c_{\rho f}}}{\epsilon_f c_{\rho f}} T
\]

\[
\frac{\partial \bar{C}}{\partial t} + \vec{U} \cdot \nabla \bar{C} + \vec{W} \cdot \nabla \bar{C} = D_m \left( \frac{\partial^2 \bar{C}}{\partial \til{r}^2} + \frac{1}{\til{r}} \frac{\partial \bar{C}}{\partial \til{r}} + \frac{\partial^2 \bar{C}}{\partial \til{z}^2} \right) + \frac{D_m \kappa_T}{\bar{T}} \left( \frac{\partial^2 \bar{C}}{\partial \til{r}^2} + \frac{1}{\til{r}} \frac{\partial \bar{C}}{\partial \til{r}} + \frac{\partial^2 \bar{C}}{\partial \til{z}^2} \right)
\]
In the fixed coordinates \((\bar{R}, \bar{Z})\) the flow between the two tubes is unsteady. It becomes steady in a wave frame \((r, z)\) moving with the same speed as wave in the \(Z\) - direction. The transformations between the two frames are given by:
\[
\bar{r} = R, \quad \bar{z} = Z - c\bar{t}
\]
\[
\bar{u} = U, \quad \bar{w} = W + c
\] (15)
(16)
Where \((\bar{r}, \bar{z})\) and \((U, W)\) are the velocity components in the moving and fixed frames, respectively. After using these transformations, the equations of motion are:
\[
\rho \left( \frac{\partial \bar{u}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{w}}{\partial \bar{z}} \right) = -\frac{\partial \bar{p}}{\partial \bar{r}} + \frac{1}{r} \frac{\partial}{\partial \bar{r}} \left( r F \bar{S}_{rr} \right) + \frac{\partial}{\partial \bar{z}} \left( S_{rz} \right) - \frac{S_{zz}}{r} - \frac{\mu}{k} \bar{u}
\] (17)
\[
\rho \left( \frac{\partial \bar{w}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{w}}{\partial \bar{z}} \right) = -\frac{\partial \bar{p}}{\partial \bar{z}} + \frac{1}{r} \frac{\partial}{\partial \bar{r}} \left( r F \bar{S}_{rz} \right) + \frac{\partial}{\partial \bar{z}} \left( S_{zz} \right) + \rho g \beta_T (T - T_0) + \rho g \beta_c (C - C_0) - \sigma B_0^2 \sin^2(\alpha) \bar{w} - \frac{\mu}{k} \bar{w}
\] (18)
\[
\frac{\partial \bar{r}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{r}}{\partial \bar{r}} + \bar{w} \frac{\partial \bar{r}}{\partial \bar{z}} = \frac{K}{c_p \rho} \left( \frac{\partial^2 \bar{r}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{r}}{\partial \bar{r}} + \frac{\partial^2 \bar{r}}{\partial \bar{z}^2} \right) - \frac{16 \sigma^2 \alpha^2}{3 \pi c_p \rho} \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left( e \bar{r} \frac{\partial}{\partial \bar{r}} - \frac{Q}{c_p \rho} T \right)
\] (19)
\[
\frac{\partial \bar{c}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{c}}{\partial \bar{r}} + \bar{w} \frac{\partial \bar{c}}{\partial \bar{z}} = D_m \left( \frac{\partial^2 \bar{c}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{c}}{\partial \bar{r}} + \frac{\partial^2 \bar{c}}{\partial \bar{z}^2} \right) + \frac{D_m k r}{T_m} \left( \frac{\partial^2 \bar{c}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{c}}{\partial \bar{r}} + \frac{\partial^2 \bar{c}}{\partial \bar{z}^2} \right)
\] (20)
where \(\bar{u}\) and \(\bar{w}\) are the velocity components in the \(\bar{r}\) and \(\bar{z}\) directions, respectively, \(\rho\) is the density, \(\bar{p}\) is the pressure, \(\mu\) is the viscosity.

In order to simplify the governing equations of the motion, we may introduce the following dimensionless transformations as follows:
\[
\frac{r}{a_z}, \quad Z = \frac{Z}{c}, \quad \delta = \frac{T}{a_z}, \quad u = \frac{a_z u}{\delta}, \quad w = \frac{a_z w}{\delta}, \quad p = \frac{a_z^2 p}{\mu \delta}, \quad S = \frac{a_z^2 S}{\mu \delta}, \quad \bar{r}_1 = \frac{a_z}{a_z}, \quad \bar{c} = \bar{c}, \quad \bar{u} = \frac{\bar{u} \rho}{\bar{c}}, \quad \bar{w} = \frac{\bar{w} \rho}{\bar{c}}, \quad \bar{p} = \frac{\bar{p} \rho}{\bar{c}}, \quad \bar{r} = \frac{\bar{r} \rho}{\bar{c}}, \quad \bar{T} = \frac{\bar{T} \rho}{\bar{c}}
\]
\[
R_e = \frac{\rho a_z^2}{\mu c}, \quad \tau_2 = \frac{T_2}{T_0}, \quad \phi = \frac{\bar{c}}{a_z}, \quad \theta = \frac{\bar{c}}{a_z}, \quad \varphi = \frac{\bar{c}}{a_z}, \quad R_n = \frac{\rho a_z^2 c}{\mu \delta^2}
\]
\[
S_c = \frac{\mu \bar{c}^2}{K}, \quad S_r = \frac{D_m k r (T_2 - T_1)}{a_z c_m (C_1 - C_0)}, \quad G_r = \frac{\rho g \beta_T (T_2 - T_1)}{\mu \delta^2}, \quad G_c = \frac{\rho g \beta_c (C_1 - C_0)}{\mu \delta^2}, \quad \bar{M}_1^2 = \frac{\sigma B_0^2}{\mu \sin^2(\alpha) a_z^2}
\] (22)
where \(\phi\) is the amplitude ratio, \(R_e\) the Reynolds number, \(Da\) the Darcy number, \(S_c\) the Soret number, \(R_n\) the radiation parameter, \(S_r\) the Brandt number, \(G_r\) is Thermal Grashof number, \(G_c\) is Solutinal Grashof number, \(M_1^2\) the magnetic number and \(\delta\) is the dimensionless wave number. Substituting (22) into equations (17) - (21), we have:
\[
\left( \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \right) = 0
\] (23)
\[
R_e \delta^3 \left( w \frac{\partial w}{\partial z} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial \delta}{\partial \delta} + \frac{1}{r} \frac{\partial}{\partial \delta} \left( r S_{r_r} \right) + \delta \frac{\partial}{\partial \delta} \left( S_{rz} \right) - \delta S_{zz} \frac{a_z}{K} \delta^2 u
\] (24)
\[
R_e \delta \left( u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial \delta}{\partial \delta} + \frac{1}{r} \frac{\partial}{\partial \delta} \left( r S_{r_r} \right) + \delta \frac{\partial}{\partial \delta} \left( S_{zz} \right) - \delta \left( M_1^2 + \frac{1}{Da} \right) w + Gr \theta + Gc \varphi - \left( M_1^2 + \frac{1}{Da} \right) w
\] (25)
\[ \delta \left( u \frac{\partial \varphi}{\partial r} + w \frac{\partial \varphi}{\partial z} \right) = \frac{1}{r} \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \varphi}{\partial r} + \delta^2 \frac{\partial^2 \varphi}{\partial z^2} \right) + \delta \left( \frac{1}{\delta r} \left( \frac{\partial \varphi}{\partial r} \right)^2 + \frac{1}{\delta r^2} \frac{\partial \varphi}{\partial r} \right) + \frac{4}{3} \frac{1}{\delta r^3} \left( \delta \frac{\partial \varphi}{\partial r} - \Omega \varphi \right) \]  
(26)

\[ \delta \left( u \frac{\partial \varphi}{\partial r} + w \frac{\partial \varphi}{\partial z} \right) = \frac{1}{\delta r} \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \delta^2 \frac{\partial^2 \varphi}{\partial z^2} \right) + S_r \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \delta^2 \frac{\partial^2 \varphi}{\partial z^2} \right) \]  
(27)

where

\[ S_{rr} = \frac{2 \delta}{1 + \lambda_1} \left[ 1 + \frac{c^2 \delta}{a_2} \left( u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) \right] \left( \frac{\partial u}{\partial r} \right) \]  
(28)

\[ S_{rz} = \frac{1}{1 + \lambda_1} \left[ 1 + \frac{c^2 \delta}{a_2} \left( u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) \right] \left( \frac{\partial u}{\partial z} \right) + \delta^2 \frac{\partial u}{\partial z} \]  
(29)

\[ S_{\theta \theta} = \frac{2 \delta}{1 + \lambda_1} \left[ 1 + \frac{c^2 \delta}{a_2} \left( u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) \right] \left( \frac{\partial u}{\partial r} \right) \]  
(30)

\[ S_{zz} = \frac{2 \delta}{1 + \lambda_1} \left[ 1 + \frac{c^2 \delta}{a_2} \left( u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) \right] \left( \frac{\partial u}{\partial z} \right) \]  
(31)

The related boundary conditions regarding to the dimensionless variables in the wave frame are given by:

\[ w = -1, u = 0, \vartheta = 1, \varphi = 1 \quad \text{at} \quad r = r_1 = \varepsilon \]

\[ w = -1, u = 0, \vartheta = 0, \varphi = 0 \quad \text{at} \quad r = r_2 = 1 + \Theta \sin(2\pi z) \]  
(32)

The general solution of the governing equations (23) - (27) in the general case seems to be impossible; therefore, we shall confine the analysis under the assumption of small dimensionless wave number. It follows that \( \delta \ll 1 \). In other words, we considered the long-wavelength approximation. Along to this assumption, equations (23) - (27) become:

\[ \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) = 0 \]  
(33)

\[ \frac{\partial \varphi}{\partial r} = 0 \]  
(34)

\[ \frac{\partial \varphi}{\partial z} = \frac{1}{r} \left( S_{rz} \right) + \frac{1}{r} \left( \frac{\partial S_{rz}}{\partial r} \right) = \left( M_r^2 + \frac{1}{\delta a} \right) w + G_r \vartheta + G_r \varphi - \left( M_r^2 + \frac{1}{\delta a} \right) \]  
(35)

\[ \frac{1}{r^2} \left( \frac{\partial w}{\partial r} + \frac{4}{3} \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial \varphi}{\partial r} \frac{\partial \varphi}{\partial r} - \Omega \varphi = 0 \]  
(36)

\[ \frac{1}{\delta r} \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{\delta r} \frac{\partial \varphi}{\partial r} \right) = -S_r \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{\delta r} \frac{\partial \varphi}{\partial r} \right) \]  
(37)

Where

\[ S_{rr} = S_{\theta \theta} = S_{zz} = 0, \text{and } S_{rz} = \frac{1}{1 + \lambda_1} \left( \frac{\partial w}{\partial r} \right) \]  
(38)
Replacing $S_{r2}$ from equation (38) in equation (35), we have:

$$\frac{\partial p}{\partial z} = \frac{1}{r(1+\lambda_1)} \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{1+\lambda_1} \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial r} \right) - \left( M_1^2 + \frac{1}{\partial r} \right) w + Gr\theta + Gc\varphi - \left( M_1^2 + \frac{1}{\partial r} \right) \right)$$  (39)

4. Solution of The Problem

The temperature equation (36), can be written as:

$$(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r}) - \frac{\alpha}{(R_e Pr^{\frac{1}{3}} n)} \theta = 0$$  (40)

Set $\frac{\alpha}{(R_e Pr^{\frac{1}{3}} n)}$, the equation (40) takes the form:

$$r^2 \frac{\partial^2 \theta}{\partial r^2} + r \frac{\partial \theta}{\partial r} - A r^2 \theta = 0$$  (41)

which is the modified Bessel equation of order zero.

The general solution of equation (41) is

$$\theta = c_1 J_0[\sqrt{A}r] + c_2 Y_0[\sqrt{A}r]$$  (42)

By using the boundary conditions Eq. (32), we have

$$c_1 = \frac{J_0[\sqrt{A}] Y_0[\sqrt{A}] - J_0[\sqrt{A}] Y_0[\sqrt{A}]}{J_0[\sqrt{A}] Y_0[\sqrt{A}] - J_0[\sqrt{A}] Y_0[\sqrt{A}]} \quad \text{and} \quad c_2 = \frac{J_0[\sqrt{A}] Y_0[\sqrt{A}] - J_0[\sqrt{A}] Y_0[\sqrt{A}]}{J_0[\sqrt{A}] Y_0[\sqrt{A}] - J_0[\sqrt{A}] Y_0[\sqrt{A}]}$$

The concentration equation (37), can be written as;

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) = -S_c S_r \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right)$$  (43)

The general solution of equation (43) is

$$\varphi = -ScSr \theta + c_3 \ln[r] + c_4$$  (44)

By using equation (42) and the boundary conditions given in equation (32), we have

$$c_3 = \frac{1 + (Sc r \gamma)}{\ln(1/r_2)} \quad \text{and} \quad c_4 = -c_3 \ln(r_2).$$

Equation (34) shows that $\varphi$ dependents on $z$ only. The equation (39), can be written as;

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} - (1 + \lambda_1) \left( M_1^2 + \frac{1}{\partial r} \right) w = (1 + \lambda_1) \left( \frac{\partial p}{\partial r} - Gr\theta - Gc\varphi \right) + \left( M_1^2 + \frac{1}{\partial r} \right)$$  (45)
Sets $L = -(1 + \lambda_1) \left( M_1^2 + \frac{1}{r^2} \right)$ and $B = (1 + \lambda_1) \left( \frac{\partial p}{\partial z} - Gr \theta - Gc \psi + \left( M_1^2 + \frac{1}{r^2} \right) \right)$, the equation (45) takes the form:

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + Lw = B$$

(46)

The general solution of equation (46) is

$$w = \frac{B}{L} + c_5 J_0[\sqrt{L}r] + c_6 Y_0[\sqrt{L}r]$$

(47)

By using the boundary conditions given in equation (32), we have

$$c_5 = -\frac{(\theta + L)(h_0 V[\sqrt{L}r]_0 - Y_0 V[\sqrt{L}r]_0)}{L J_0[\sqrt{L}r]_0 V[\sqrt{L}r]_0 - J_0[\sqrt{L}r]_0 V[\sqrt{L}r]_0} \quad , \quad c_6 = -\frac{(\theta + L)(-h_0 V[\sqrt{L}r]_0 + J_0 V[\sqrt{L}r]_0)}{L J_0[\sqrt{L}r]_0 V[\sqrt{L}r]_0 - J_0[\sqrt{L}r]_0 V[\sqrt{L}r]_0}$$

and $J_0$, $Y_0$ are the modified Bessel functions of the first and second kind of zero order. By using the MATHEMATICA program and the boundary conditions given in equations (32) we have a constants $c_1$, $c_2$, $c_3$, $c_4$, $c_5$ and $c_6$.

The corresponding stream functions $u = -\frac{1}{r} \frac{\partial \psi}{\partial z}$ and $w = \frac{1}{r} \frac{\partial \psi}{\partial r}$ is

$$\psi = \int r \left( c_5 J_0[\sqrt{L}r] + c_6 Y_0[\sqrt{L}r] - \left( \frac{(1+\lambda_1)\frac{\partial p}{\partial z} - Gr \theta - Gc \psi + (M_1^2 + \frac{1}{r^2})}{L} \right) \right) dr$$

(48)

The instantaneous volume flow rate $Q(z) (= 2 \int_{r_1}^{r_2} rw dr)$ is given by:

$$\frac{dp}{dz} = \frac{1}{r_1^2 - r_2^2 + \lambda_1 (r_1^2 - r_2^2)} \left( -Q(z) - \frac{2Y_1[\sqrt{L}r]_0 C_4 r_1 \lambda_1}{\sqrt{L}} - \frac{(1+\lambda_1) r_1^2}{L} \right) + \frac{Gr (1 + \lambda_1) (C_5 J_0[\sqrt{L}r] + C_6 Y_0[\sqrt{L}r]) r_1^2}{L}$$

$$\left( \frac{Gc (1 + \lambda_1) (C_5 J_0[\sqrt{L}r] + C_6 Y_0[\sqrt{L}r]) r_1^2}{L} + \frac{1}{L (Log[h] - Log[e])} Gc (1 + \lambda_1) (-Log[h] - Sr Sc Log[h] + C_5 Sc J_0[\sqrt{L}r] Log[e] - C_6 Sc Y_0[\sqrt{L}r] Log[e]) r_1^2 - 0 F_1[2, -\frac{1}{4} L r_1^2] C_5 r_1^2 - \frac{(1+\lambda_1) M_1^2 r_1^2}{L} + \frac{2Y_1[\sqrt{L}r] C_4 r_1^2}{\sqrt{L}} + \frac{(1+\lambda_1) r_1^2}{L} \right)$$

(49)

Following the analysis given by Shapiro et al. [10], the mean volume flow $q_2$ over a period is obtained as
\[ q^2 = Q(z) + \frac{1}{2} \left( 1 - \epsilon^2 + \frac{\varphi^2}{2} \right) \]

This on using Eq. (44) yields

\[
\frac{dp}{dz} = \frac{1}{r_1^2 - r_2^2 + 1 \left( r_1^2 - r_2^2 \right)} L \left( q^2 + \frac{1}{2} \left( 1 - \epsilon^2 + \frac{\varphi^2}{2} \right) - \frac{2Y_1(\sqrt{\varphi}r_1)|c_0r_1 - (1+\lambda_1)r_2^2}{\sqrt{E}} \right) + \frac{Gr(1+\lambda_1)(C_1d^2\sqrt{\varphi}r_1 + C_2Y_0(\sqrt{\varphi}r_1))r_1^2}{\sqrt{E}} + \frac{1}{L \left( \log[h] - \log[\epsilon] \right)} Gc(1+\lambda_1)(-\log[h] - SrScLog[h] + C_3SrScJ_0(\sqrt{\varphi}r_1)|Log[h] + C_4SrScY_0(\sqrt{\varphi}r_1)|Log[h] + Log[r] + SrScLog[r] - C_3SrScJ_0(\sqrt{\varphi}r_1)|Log[\epsilon] - C_4SrScY_0(\sqrt{\varphi}r_1)|Log[\epsilon]|r_1^2 - 0F1(2, -\frac{1}{4}L \varphi r_1^2)C_5r_2^2 - \frac{(1+\lambda_1)M_1^2r_2^2}{L} \right)
\]

\[ \frac{Y_1}{L} 0F1 \text{ are the modified Bessel function of the second kind and Hypergeometric regularized function, respectively.} \]

The pressure rise \( \Delta p \) and the friction force (at the wall) on the inner and outer tubes are \( F(\varphi) \) and \( F(\psi) \), respectively, in a tube of length \( L \), in their non-dimensional forms, are given by:

\[
\Delta p = \int_0^1 \left( \frac{dp}{dz} \right) dz \quad (51)
\]

\[
F(\varphi) = \int_0^1 r_2^2 \left( - \frac{dp}{dz} \right) dz \quad (52)
\]

\[
F(\psi) = \int_0^1 r_2^2 \left( - \frac{dp}{dz} \right) dz \quad (53)
\]

Substituting from equation (48) in equations (49) - (51) with \( r_1 = \epsilon, r_2 = 1 + \varphi \sin(2\pi z) \), and then evaluating the integrations by using the language of series for several values of the parameters included, using the MATHEMATICA program, and the obtained results are discussed in the next section.

**5. Numerical Results and Discussion**

In this section, the numerical and computational results are discussed for the problem of an incompressible non-Newtonian Jeffrey fluid in a tube with heat and mass transfer through the graphical illustrations. Figure 1 shows that effects of the parameters \( \varepsilon \) and \( Re \) on the temperature distribution function \( \vartheta \) is direct, means \( \vartheta \) increases with the increasing of any one of these parameters. Figure 2 shows that effects of the parameters \( Re \) and \( \psi \) on the temperature, \( Re \) increases heat while decreasing by increasing \( \psi \). Figure 3 shows that the temperature increases with both \( \beta \) and \( Pr \). Figure 4 It appears that the effects of \( Sc \) and \( \varepsilon \) parameters are on The distribution function of the concentration is reversed when \( r < 1.17622 \), that is \( \varphi \) decreases with the increasing of \( Sc \) while increasing \( \varepsilon \) and direct when \( r > 1.17622 \). Also \( \varphi < 0 \) when \( r < 1.17622 \), and \( \varphi > 0 \) when \( r > 1.17622 \). Figure 5 The change in concentration decreases by increasing
sr while increasing by $\varnothing$, when $r < 1.17622$ and the direction changes when it is greater than the value. Figure (6) the change in concentration decreases by increasing $\varnothing$ while increasing by $\varepsilon$, when $r < 1.17622$ and the direction changes when it is greater than the value. Figure (7) Shows the effects of parameters $Gr$ and $M$ on the velocity distribution function $w$ vs. $r$. It found that $w$ increases with increase $Gr$ at $r > 0.18$ while decreases with increase $M$, and decreasing with the increase of $Gr$, and $w < 0$ at $r < 0.18$ while increases with increase of $M$. Figure (8) we see that $w$ is decreasing with the increase of $M$, when $r < 0.18$ then the $Da$ decreases, and $w$ is decreases with increase of $M$ when $r > 0.18$ the $Da$ is also in conflict with the $M$. Figure (9) $w$ increases with increase of $Sr$ and $Sc$ when $r < 0.18$, and $w$ decreases with $Sr$ and $Sc$ increase when $r > 0.18$. Figure (10) shows the effects of parameters $\lambda_1$ and $Gc$ on $dp / dz$ vs. $z$. It was found that $dp / dz$ increases with increasing each $\lambda_1$ and $Gc$. Figure (11) we see that $dp / dz$ decreases with an increase of $q_2$, while increases with increasing of $M$. Figure (12) increases $dp / dz$ with increase for each $Sr$ and $Sc$. Figure (13) illustrates the effects of the parameters $\lambda_1$ and $Gc$ on the pressure rise $\Delta p$ versus $q_2$ respectively, shows that the variation of $\Delta p$ vs. $q_2$ , it is found that $\Delta p$ decreases with the increasing for each $\lambda_1$ and $Gc$. Figure (14) we see that $\Delta p$ vs. $\phi$. It is found $\Delta p$ increases with the increasing of $\lambda_1$ while decreases with an increase of $Gr$. Figure (15) shows that the variation of $F^{(1)}$ vs. $q_2$. It is found that $F^{(1)}$ increases with the increasing for each $\lambda_1$ and $Gc$. Figure (16) shows that the variation of $F^{(1)}$ vs. $\phi$. It is found $F^{(1)}$ increases with the increasing $\lambda_1$ while decreases with an increase of $Gr$, and it changes its direction when it is $\phi < 0.07$. Figure (17) shows that the variation of $F^{(0)}$ vs. $q_2$. It is found that $F^{(0)}$ increases with the increasing for each $\lambda_1$ and $Gc$. Figure (18) shows that the variation of $F^{(0)}$ vs. $\phi$. It is found $F^{(0)}$ increases with the increasing $\lambda_1$ while decreases with an increase of $Gr$, and it changes its direction when it is $\phi < 0.07$ Figure (19) we observe the increase in $pr$ and the number of valves increases gradually. Figure (20) the bracelets grow when the $Gr$ increases. Figure (21) with the increase of $\varepsilon$ the size of the rotors increases and seems very clear. Figure (22) the $\lambda_1$ size is less than the rotors and looks very clear.

![Fig. (2): The variation of temperature $\vartheta$ vs. $r$ at $\Delta = 1$, $Re = 0.9$, $Pr = 1$, $\varepsilon = 0.3$, $z = 0.1$.](image1)

![Fig. (3): The variation of temperature $\vartheta$ vs. $r$ at $\Delta = 1$, $Pr = 1$, $\varepsilon = 0.3$, $Rn = 2$, $z = 0.1$.](image2)
Fig. (4): The variation of temperature $\vartheta$ vs. $r$ at $Rn = 2, Re = 0.9, z = 0.1, \Theta = 0.3, \epsilon = 0.1, 0.3$.

Fig. (5): The variation of concentration $\varphi$ vs. $r$ at $\Omega = 1, Re = 3, Pr = 2, Rn = 0.5, z = 0.1, \Theta = 0.3, Sr = 0.3$.

Fig. (6): The variation of concentration $\varphi$ vs. $r$ at $\Omega = 1, Re = 3, Pr = 2, Rn = 0.5, z = 0.1, \Theta = 0.3, Sr = 0.3$.

Fig. (7): The variation of concentration $\varphi$ vs. $r$ at $\Omega = 1, Re = 3, Pr = 2, Rn = 0.5, z = 0.1, Sc = 0.3, Sr = 0.3$.
Fig. (8): Velocity distribution \( w \) at \( \Omega = 0.9, \phi = 0.3, \xi = \frac{\pi}{4}, z = 0.01, \epsilon = 0.2, Da = 0.9, \lambda_1 = 0.1, Re = 1, Rn = 2, Pr = 2, Gr = 1, q_2 = 0.5, Sr = 0.1, Sc = 0.5, \epsilon = 0.2.

Fig. (10): Velocity distribution \( w \) at \( \Omega = 0.9, \phi = 0.3, \xi = \frac{\pi}{4}, z = 0.01, \epsilon = 0.2, Da = 0.9, \lambda_1 = 0.1, Re = 1, Rn = 2, Pr = 2, Gr = 1, M = 1.1, q_2 = 0.5, Sr = 0.1, Sc = 0.5, \epsilon = 0.2.

Fig. (12): The variation of \( \frac{dp}{dz} \) vs. \( z \) at \( \Omega = 0.9, Re = 1, Pr = 2, Da = 0.9, \xi = \frac{\pi}{4}, z = 0.01, Rn = 2, Gr = 1, \lambda_1 = 0.1, \epsilon = 0.2, Sc = 0.5, Sr = 0.1, Gr = 2.

Fig. (13): The variation of \( \frac{dp}{dz} \) vs. \( z \) at \( \Omega = 0.9, Re = 1, Pr = 2, Da = 0.9, \xi = \frac{\pi}{4}, z = 0.01, Rn = 2, Gr = 1, \lambda_1 = 0.1, \epsilon = 0.2, Sc = 0.5, Sr = 0.1, Gr = 2. \)
Fig. (14): The variation of $\Delta$ vs $q^2$, $\Omega = 0.9$, $R_\eta = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $Da = 0.9$, $Gr = 2$, $\epsilon = 0.2$, $\xi = \frac{\pi}{4}$, $z = 0.01$.

Fig. (15): The variation of $\Delta$ vs $q^2$, $\Omega = 0.9$, $R_\eta = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $Da = 0.9$, $Gr = 1$, $\epsilon = 0.2$, $\xi = \frac{\pi}{4}$, $z = 0.01$.

Fig. (16): The variation of $F$ vs $q^2$, $\Omega = 0.9$, $R_\eta = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $Da = 0.9$, $Gr = 2$, $\epsilon = 0.2$, $\xi = \frac{\pi}{4}$, $z = 0.01$.

Fig. (17): The variation of $F$ vs $q^2$, $\Omega = 0.9$, $R_\eta = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $Da = 0.9$, $Gr = 1$, $\epsilon = 0.2$, $\xi = \frac{\pi}{4}$, $z = 0.01$.

Fig. (18): The variation of $F$ vs $q^2$, $\Omega = 0.9$, $R_\eta = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $Da = 0.9$, $Gr = 2$, $\epsilon = 0.2$, $\xi = \frac{\pi}{4}$, $z = 0.01$.

Fig. (19): The variation of $F$ vs $q^2$, $\Omega = 0.9$, $R_\eta = 2$, $Pr = 2$, $Sc = 0.5$, $Sr = 0.1$, $Da = 0.9$, $Gr = 1$, $\epsilon = 0.2$, $\xi = \frac{\pi}{4}$, $z = 0.01$. 
Fig. (20) Streamlines for $Pr$ when $\Omega = 0.9$, $\epsilon = 0.2$, $\phi = 0.2$, $\lambda_1 = 0.1$, $Re = 1$, $Rn = 2$, $q_2 = 0.5$, $Sc = 0.5$, $S_1 = 0.1$, $Gc = 2$, $G1 = 1$, $M = 1.1$, $Da = 0.9$, $\xi = \frac{\pi}{4}$, $i = \sqrt{-1}$.

Fig. (21) Streamlines for $Gr$ when $\Omega = 0.9$, $\epsilon = 0.2$, $\phi = 0.2$, $\lambda_1 = 0.1$, $Re = 1$, $Rn = 2$, $q_2 = 0.5$, $Sc = 0.5$, $S_1 = 0.1$, $Gc = 2$, $G1 = 1$, $M = 1.1$, $Da = 0.9$, $\xi = \frac{\pi}{4}$, $i = \sqrt{-1}$. 
Fig. (22) Streamlines for $\varepsilon$ when $\Omega = 0.9, Gr = 1, \phi = 0.2, \lambda_1 = 0.1, Re = 1, Rn = 2, q_2 = 0.5, Sc = 0.5, S_1 = 0.1, Gc = 2, G_1 = 1, M = 1.1, Da = 0.9, \xi = \frac{\pi}{4}, \iota = \sqrt{-1}$.

Fig. (23) Streamlines for $\lambda_1$ when $\Omega = 0.9, Gr = 1, \phi = 0.2, \varepsilon = 0.1, Re = 1, Rn = 2, q_2 = 0.5, Sc = 0.5, S_1 = 0.1, Gc = 2, G_1 = 1, M = 1.1, Da = 0.9, \xi = \frac{\pi}{4}, \iota = \sqrt{-1}$. 
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