FINITE TIME BLOW-UP FOR A WAVE EQUATION WITH DYNAMIC BOUNDARY CONDITION AT CRITICAL AND HIGH ENERGY LEVELS IN CONTROL SYSTEMS

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ABSTRACT. We study the initial boundary value problem of linear homogeneous wave equation with dynamic boundary condition. We aim to prove the finite time blow-up of the solution at critical energy level or high energy level with the nonlinear damping term on boundary in control systems.

1. Introduction

In this paper, we mainly discuss the initial boundary value problem of linear homogeneous wave equation with dynamic boundary condition

(1.1) \( u_{tt} - \Delta u = 0 \) in \((0, \infty) \times \Omega\),

(1.2) \( u(x,t) = 0 \) on \([0, \infty) \times \Gamma_0\),

(1.3) \( \frac{\partial u}{\partial \nu} = -Q(u_t) + f(u) \) on \([0, \infty) \times \Gamma_1\),

(1.4) \( u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \) on \(\Omega\),

where \( \Omega \subset \mathbb{R}^n, n \geq 1 \) is a regular, bounded and connected domain with boundary \( \partial \Omega = \Gamma_0 \cup \Gamma_1, \ \Gamma_0 \cap \Gamma_1 = \emptyset \), where \( \Gamma_0 \) and \( \Gamma_1 \) are measurable over \( \partial \Omega \) endowed with \((n-1)\)-dimensional Lebesgue measures \( \lambda_{n-1}(\Gamma_0) \) and \( \lambda_{n-1}(\Gamma_1) \); \( \Delta \) is the Laplacian operator with respect to the \( x \); \( Q(u_t) = |u_t|^{m-2}u_t, \ f(u) = |u|^{p-2}u, \ m \geq 2, \ p \geq 2 \). These properties of \( \Omega, \ \Gamma_0 \) and \( \Gamma_1 \) will be assumed throughout the paper. The initial data are \( u_0 \in H^1(\Omega) \) and \( u_1 \in L^2(\Omega) \), with the compatibility condition \( u_0 = 0 \) on \( \Gamma_0 \). We always assume that \( \lambda_{n-1}(\Gamma_0) > 0 \) and \( \lambda_{n-1}(\Gamma_1) > 0 \) throughout the paper.

For the wave equation with nonlinear dynamic boundary condition like problem (1.1)-(1.4), arising in the physical models and control systems, there have been many papers dealing with the existence and blow-up of the solution. In [4]-[7], [12]-[18] and [33], the global existence and decay properties of the solution of the problem (1.1)-(1.4) were proved for arbitrarily large initial data when \( f \equiv 0 \) or \( f(x,u)u \leq 0 \) if \( Q \) and \( f \) were under some special assumptions. When \( f(x,u)u \geq 0 \), which means \( f \) is a source term, the situation is quite different. If \( f(x,u) = |u|^{p-2}u, \ p > 2 \) and \( Q \equiv 0 \), when the \((n-1)\)-dimensional Lebesgue measure \( \lambda_{n-1}(\Gamma_0) \) and \( \lambda_{n-1}(\Gamma_1) \) are assumed to be positive, the authors of [18] obtained the finite time blow-up when the initial
energy is negative for problem (1.1)-(1.4). For the same problem above, Levine and Smith [11] proved the global existence of the solution when initial data \( u_0 \) and \( u_1 \) are very small. Zhang et al. [30] considered the Kirchhoff equation with dynamic boundary condition. They obtained the energy decay and blow-up of a solution with negative and small positive initial energy. Recently, some authors have studied the viscoelastic wave equation with boundary damping and source terms. In [9], Lee et al. proved the global existence and exponential growth solution. In [16], they proved the blow-up result of the solution of the problem (1.1)-(1.4) when \( p > m \) and showed that the solution of problem (1.1)-(1.4) globally exists in time for arbitrary initial data when \( 2 \leq p \leq m \), in opposition with the finite time blow-up occurring when \( m = 2 < p \). In [2] they proved more general existence and stability results by using a natural tool for the problem–monotone operator theory that contrast with the Schauder fixed point arguments used in [25]. For the same problem with \( p > m \), Zhang and Hu [29] obtained the nonexistence of the solution under the energy level \( E(0) < d \) when the initial data are in the unstable set. As well known, in the frame of potential well theory, the variational arguments are usually taken by considering different levels of the initial data, see its applications to differential kinds of model equations in [21], [27] and [28], which are very different from the other aspects of the studies on the wave equations [3], [10], [11], [22], [23], [26] and [31]. However, no results were obtained about the finite time blow-up of the solution for problem (1.1)-(1.4) when \( Q(u_t) = |u_t|^{m-2}u_t \) and \( f(u) = |u|^{p-2}u \), \( p > m \) at critical energy level \( E(0) = d \) or high energy level \( E(0) > d \). The main purpose of this paper is to get the finite time blow-up result of the solution of the problem (1.1)-(1.4) when \( Q(u_t) = |u_t|^{m-2}u_t \) and \( f(u) = |u|^{p-2}u \), \( p > m \) for the critical initial data and arbitrarily large initial data. We mainly adapt the method introduced by Vitillaro in [24] to study the solution of problem (1.1)-(1.4) when \( Q(u_t) = |u_t|^{m-2}u_t \) and \( f(u) = |u|^{p-2}u \), \( p > m \) at critical energy level and the convexity method introduced by Gazzola and Squassina in [8] to get the finite time blow-up of the solution at high energy level.

In Section 2, we introduce some basic setup, notations and some known results of the solution to problem (1.1)-(1.4). In section 3, we prove the blow-up result of the solution when \( E(0) = d \). Finally in Section 4, we obtain the blow-up result when \( E(0) > d \).

2. Set up and notations

First we denote

\[
\| \cdot \| = L^2(\Omega), \quad \| \cdot \|_q = L^q(\Omega), \quad \| \cdot \|_{q, \Gamma_1} = L^q(\Gamma_1), \quad 1 \leq q \leq \infty,
\]

and

\[
H^1_{0, \Gamma_0}(\Omega) = \{ u \in H^1(\Omega) \mid u|_{\Gamma_0} = 0 \},
\]

\((u|_{\Gamma_0} \text{ is in the sense of trace}).\) We can endow \( H^1_{0, \Gamma_0}(\Omega) \) the equivalent norm \( \| u \|_{p, \Gamma_1} = \| \nabla u \| \) because of the Poincaré inequality (see [17]) and the fact that \( \lambda_{n-1}(\Gamma_0) > 0 \). We also define some useful functionals

\[
J(u) = \frac{1}{2} \| \nabla u \|^2 - \frac{1}{p} \| u \|_{p, \Gamma_1}^p,
\]

\[
I(u) = \| \nabla u \|^2 - \| u \|_{p, \Gamma_1}^p,
\]
(2.4) \[ E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 - \frac{1}{p} \|u\|^{p}_{p,\Gamma_1}. \]

All these functionals are defined on \( H^{1}_{\Gamma_1}(\Omega) \). According to the definition of \( E(t) \), we have

\[ E(0) = \frac{1}{2} \|u_1\|^2 + \frac{1}{2} \|\nabla u_0\|^2 - \frac{1}{p} \|u_0\|^{p}_{p,\Gamma_1}. \]

We also use the trace-Sobolev embedding \( H^{1}_{\Gamma_1}(\Omega) \hookrightarrow L^p(\Gamma_1) \) for \( 2 \leq p < r \) introduced in [1], where

\[ r = \begin{cases} \frac{2(n-1)}{n-2}, & \text{if } n \geq 3; \\ +\infty, & \text{if } n = 1,2. \end{cases} \]

We also have the embedding inequality

\[ \|u\|^{p}_{p,\Gamma_1} \leq C_* \|\nabla u\|, \]

where \( C_* \) is the embedding constant. Then we introduce the unstable set \( V \) defined by

\[ V = \{ (u_0, u_1) \in H^{1}_{\Gamma_1}(\Omega) \times L^2(\Omega) \mid I(0) < 0, 0 < E(0) = d \}, \]

where \( d \) is the mountain pass level, characterized as

\[ d = \inf_{u \in H^{1}_{\Gamma_1}(\Omega), u|_{\Gamma_1} \neq 0} \left( \sup_{\lambda > 0} J(\lambda u) \right). \]

We define

\[ \lambda_1 := C_*^{-\frac{1}{p}}. \]

It has been proved in [25] that \( \left( \frac{1}{2} - \frac{1}{p} \right) \lambda_1^2 \) is the potential well depth, that is

\[ d = \left( \frac{1}{2} - \frac{1}{p} \right) \lambda_1^2. \]

In [25], the author have proved the local and global existence of the solution for problem (1.1)-(1.4). Now we introduce some results in [25] as follows:

**Theorem 2.1. (Local existence of the solution)** Suppose that \( m > 1, 2 \leq p < r \) and \( m > \frac{r}{r+1-p} \). Then, given initial data \( u_0 \in H^1_{\Gamma_1}(\Omega) \) and \( u_1 \in L^2(\Omega) \), there is a \( T > 0 \) and a weak solution \( u \) of the problem (1.1)-(1.4) on \((0, T) \times \Omega \) such that \( u \in C([0, T]; H^1_{\Gamma_1}(\Omega) \cap C^1([0, T]; L^2(\Omega))) \) , \( u_t \in L^m((0, T) \times \Gamma_1) \),

\[ E(t) + \int_0^t \|u_\tau(\tau)\|^{m}_{m,\Gamma_1} \, d\tau = E(s), \]

holds for \( 0 \leq s \leq t \leq T \).

### 3. Finite time blow-up for critical initial energy \( E(0) = d \)

In this section, we mainly show the finite time blow-up of the solution when initial data are at critical level. In order to prove the finite time blow-up of the solution, we first prove some basic lemmas.
Lemma 3.1. (Invariant manifolds) We suppose that \( m > 1, \ 2 \leq p < r \) and \( m > \frac{r-1}{r+1-p} \). Let
\[
V' = \{ (u_0, u_1) \in H^1_t(\Omega) \times L^2(\Omega) \mid \|\nabla u_0\| > \lambda_1, 0 < E(0) = d \},
\]
then we have \( V = V' \).

Proof. First we show that \( V \subset V' \). Let \( (u_0, u_1) \in V \), then we have \( I(0) < 0 \) (i.e. \( \|\nabla u_0\|^2 < \|u_0\|_{p,G_1}^p \)). By using the embedding inequality (2.6), we can obtain that
\[
\|\nabla u_0\|^2 < \|u_0\|_{p,G_1}^p \leq C_2 \|\nabla u_0\|^p.
\]
Hence we have \( \|\nabla u_0\| > C_2^{-\frac{p}{p-2}} = \lambda_1 \). So we get that \( V \subset V' \). Then we show that \( V' \subset V \). Let \( (u_0, u_1) \in V' \), then we have \( \|\nabla u_0\| > \lambda_1 \) and \( E(0) = d \). Supposing by contradiction that \( I(0) \geq 0 \), we have
\[
\|\nabla u_0\|^2 \geq \|u_0\|_{p,G_1}^p.
\]
Combining
\[
E(0) = \frac{1}{2} \|u_1\|^2 + \frac{1}{2} \|\nabla u_0\|^2 - \frac{1}{p} \|u_0\|_{p,G_1}^p,
\]
we can get
\[
\frac{1}{2} \|\nabla u_0\|^2 - \frac{1}{p} \|u_0\|_{p,G_1}^p \leq d.
\]
Then we obtain that
\[
d \geq \left( \frac{1}{2} - \frac{1}{p} \right) \|\nabla u_0\|^2.
\]
Since \( \|\nabla u_0\| > \lambda_1 \), it follows that
\[
d > \left( \frac{1}{2} - \frac{1}{p} \right) \lambda_1^2 = d,
\]
which leads to a contradiction. This completes the proof. \( \square \)

Lemma 3.2. (Invariant manifolds and boundness) Suppose that \( m > 1, \ 2 \leq p < r \) and \( m > \frac{r-1}{r+1-p} \). Let \( (u_0, u_1) \in V \) and \( u(x, t) \) be the weak solution of the problem (1.1)-(1.4) on \([0, T_{max}]\). Then \( (u(t, \cdot), u_t(t, \cdot)) \) remains inside \( V \) for any \([0, T_{max}]\). Furthermore, we have
\[
\|\nabla u(t)\|^2 < \|u(t)\|_{p,G_1}^p, \quad t \in [0, t_{max}),
\|u(t)\|_{p,G_1} > C_1 \lambda_1, \quad t \in [0, t_{max}),
\|\nabla u(t)\| > \lambda_1, \quad t \in [0, t_{max}).
\]

Proof. From \( I(0) < 0 \) and the continuity of \( I(u) \) respecting to \( t \), it follows that there exists a sufficiently small \( t_1 > 0 \), such that \( I(u) < 0 \) for \( 0 < t < t_1 \). Combining (2.11), we set that \( d_1 = E(t_1) \), then we have
\[
0 < d_1 = d - \int_0^{t_1} \|u_\tau(\tau)\|_{m,G_1}^m d\tau < d.
\]
We choose \( t = t_1 \) as the initial time, by the same proceeding in Theorem 2.3 \([12]\), we can get that \( (u(t, \cdot), u_t(t, \cdot)) \) remains inside \( V \) for any \( t \in [0, T_{max}] \). (Here we have already known the invariance when \( E(0) < d \), so by selecting \( t_1 \) as the initial time, we can have \( E(t_1) < d \) again, then we can use the invariance conclusion for
$E(t_1) < d$). Furthermore, noting $I(u) < 0$ for all $t \in [0, T_{max})$, according to the definition of $I(u)$, we have

$$\|\nabla u(t)\|^2 < \|u(t)\|_{p, \Gamma_1}^p, \quad t \in [0, T_{max}).\tag{3.1}$$

Moreover, according to Lemma 3.1, we can obtain that

$$\|\nabla u(t)\| > \lambda_1, \quad t \in [0, T_{max}).\tag{3.2}$$

Then, by using (2.9) and (3.1), we have

$$\|u(t)\|_{p, \Gamma_1} > C_\lambda \lambda_1, \quad t \in [0, T_{max}).\tag{3.3}$$

This completes the proof.

By the similar method in [24], we can prove that in the manifold $V' = \{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times \mathcal{L}^2(\Omega) \mid \|\nabla u_0\| > \lambda_1, 0 < E(0) = d\}$, there is a constant $\lambda_2$ between $\|\nabla u(t)\|$ and $\lambda_1$, i.e., there is a $\lambda_2 > \lambda_1$ such that $\|\nabla u(t)\| \geq \lambda_2 > \lambda_1$. This will be given by the following lemma. And this lemma will be used to prove the finite time blow-up for the critical case $E(0) = d$.

**Lemma 3.3.** Suppose that $m > 1$, $2 \leq p < r$ and $m > \frac{r}{r+1-p}$. Let $(u_0, u_1) \in V$ and $u(x, t)$ be the weak solution of the problem (1.1)-(1.4) on $[0, T_{max})$. There is a $\lambda_2 > \lambda_1$ such that $\|\nabla u(t)\| \geq \lambda_2 > \lambda_1$.

**Proof.** According to Lemma 3.2, we have $I(u) < 0$ for all $t \in [0, T_{max})$. By (2.6), we have

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 - \frac{1}{p} \|u\|_{p, \Gamma_1}^p \geq \frac{1}{2} \|\nabla u\|^2 - \frac{1}{p} \|u\|_{p, \Gamma_1}^p \tag{3.4}$$

where $g(\lambda) = \frac{1}{2} \lambda^2 - \frac{1}{p} C_p \lambda^p$ for $\lambda \geq 0$. It is easy to see that $g$ takes its maximum at $\lambda = \lambda_1$, with $g(\lambda_1) = d$, being strictly decreasing for $\lambda \geq \lambda_1$, and $g(\lambda) \to -\infty$ as $\lambda \to \infty$. Combining the fact that $E(t)$ is decreasing when $t \in [0, T_{max})$ and $E(0) = d$, we can continue to argue as follows. By the continuity of $\|\nabla u(t)\|$, there are only two possibilities:

(a) there is a $t_0 \geq 0$ such that $E(t_0) < d$ and $\|\nabla u(t_0)\| > \lambda_1$;

(b) there is an $\varepsilon_0 > 0$ such that $E(t) = d$ on $[0, \varepsilon_0)$.

In the first case, we choose $t_0$ as the initial time. Due to the fact that $E(t)$ and $g(\lambda)$ are both decreasing and continuous, there exists a $\lambda_2 > \lambda_1$ such that $E(t_0) = g(\lambda_2)$. We now claim that

$$\|\nabla u(t_0)\| \geq \lambda_2.$$

Suppose for contradiction that $\|\nabla u(t_0)\| < \lambda_2$ for some $t \in [0, T_{max})$. By using (3.4) and the fact that $g(\lambda)$ is a decreasing function, we have

$$E(t_0) \geq g(\|\nabla u(t_0)\|) > g(\lambda_2) = E(t_0),$$

which leads to a contradiction. We can also obtain that

$$\frac{1}{2} \|\nabla u\|^2 > \frac{1}{2} \lambda_2^2 \geq \frac{1}{2} \lambda_1^2.$$
We then choose \( \lambda_0 = \frac{1}{2} \lambda_1^2 \) and \( E_1 = (1 - \frac{p}{r}) \lambda_0 \). By doing the same process in Theorem 2.3 in [24], we can conclude the proof for the first case.

In the second case, for \( t \in [0, \varepsilon_0) \), we have \( E(t) = d \). By using (2.11), we have

\[
\int_0^t \|u_\tau(\tau)\|_{m, \Gamma_1}^m d\tau = 0 \quad t \in [0, \varepsilon_0).
\]

Due to the fact that \( \|u_t(t)\|_{m, \Gamma_1}^m \geq 0 \), we have \( u_t = 0 \) and \( u(t) = u_0 \) on \([0, \varepsilon_0)\)

Suppose for contradiction that \( \|\nabla u_0\| < \lambda_2 \) for \( t \in [0, \varepsilon_0) \). We can obtain that

\[
d = E(0) = \frac{1}{2} \|\nabla u_0\|^2 - \frac{1}{p} \|u_0\|_{p, \Gamma_1}^p \geq g(\|\nabla u_0\|) > g(\lambda_2) = E(0) = d,
\]

which leads to a contradiction. This completes the proof. \( \Box \)

**Theorem 3.4.** (Finite time blow-up of solutions for \( E(0) = d \)) Assume that \( 1 < m < p, m > \frac{p}{r + 1 - p}, 2 \leq p < r \). If \( (u_0, u_1) \in V' \), then the solution of problem (1.1)-(1.4) blows up in finite time.

**Proof.** Arguing by contradiction, we assume that there exists a global weak solution of problem (1.1)-(1.4). We set \( H(t) = d - E(t) \). By Theorem 2.1, \( E(t) \) is decreasing about \( t \). So \( H(t) \) is an increasing function, then we have

\[
(3.5) \quad H(t) \geq H(0) = d - E(0) = 0, \quad t \geq 0.
\]

Next, by using the definition of \( E(t) \), we have

\[
(3.6) \quad H(t) \leq d - \frac{1}{2} \|\nabla u(t)\|^2 + \frac{1}{p} \|u(t)\|_{p, \Gamma_1}^p, \quad t \geq 0.
\]

By using (3.2) and (2.10), we can obtain that

\[
d - \frac{1}{2} \|\nabla u(t)\|^2 \leq d - \frac{1}{2} \lambda_1^2 = - \frac{1}{p} \lambda_1^2 < 0.
\]

Combining (3.6), we have

\[
H(t) - \frac{1}{p} \|u(t)\|_{p, \Gamma_1}^p \leq d - \frac{1}{2} \|\nabla u(t)\|^2 < 0.
\]

Then (3.5) tells

\[
(3.7) \quad H(0) \leq H(t) < \frac{1}{p} \|u(t)\|_{p, \Gamma_1}^p, \quad t \geq 0.
\]

Next, using the definition of \( E(t) \) and (3.6), we have

\[
\begin{align*}
\frac{d}{dt}(u, u_t) & = \|u_t\|^2 - \|\nabla u(t)\|^2 + \|u(t)\|_{p, \Gamma_1}^p - \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \\
& = 2\|u_t\|^2 + \left( 1 - \frac{2}{p} \right) \|u(t)\|_{p, \Gamma_1}^p - 2E(t) - \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \\
& = 2\|u_t\|^2 + \left( 1 - \frac{2}{p} \right) \|u(t)\|_{p, \Gamma_1}^p - 2d + 2H(t) - \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma.
\end{align*}
\]
Then, According to Lemma 3.3, we choose a $\lambda_2$, such that $\|\nabla u(t)\| \geq \lambda_2 > \lambda_1$, then by using (3.3), we have

$$\frac{d}{dt}(u, u_t) \geq 2\|u_t\|^2 + \left(1 - \frac{2}{p} - 2d(C_2\lambda_2)^{-p}\right)\|u(t)\|_{p, \Gamma_1}^p$$

$$+ 2H(t) - \int_{\Gamma_1} |u_t|^{m-2}u_t u d\sigma$$

(3.8)

$$= 2\|u_t\|^2 + C_1\|u(t)\|_{p, \Gamma_1}^p + 2H(t) - \int_{\Gamma_1} |u_t|^{m-2}u_t u d\sigma,$$

where $C_1 = 1 - \frac{2}{p} - 2d(C_2\lambda_2)^{-p} > 0$. To obtain $\frac{d}{dt}(u, u_t)$, we first estimate the last term in (3.8). By Hölder’s inequality, we obtain

$$\left|\int_{\Gamma_1} |u_t|^{m-2}u_t u d\sigma\right| \leq \|u_t\|_{m, \Gamma_1}^{m-1}\|u\|_{p, \Gamma_1} = \|u_t\|_{m, \Gamma_1}^{m-1}\|u\|_{p, \Gamma_1}^{1-\frac{m}{p}}\|u\|_{p, \Gamma_1}^{\frac{m}{p}},$$

in which $\frac{1}{p} + \frac{1}{m} = 1$, and then, by (3.7), Hölder’s inequality, Young inequality and the fact that $H'(t) = \|u_t\|_{m, \Gamma_1}^m$, we obtain that

$$\left|\int_{\Gamma_1} |u_t|^{m-2}u_t u d\sigma\right| \leq \|u\|_{p, \Gamma_1}^{1-\frac{m}{p}}\|u\|_{p, \Gamma_1}^{\frac{m}{p}}\|u_t\|_{m, \Gamma_1}^{m-1}$$

$$\leq C_2H_{\frac{1}{p} - \frac{1}{m}}(t)\|u\|_{p, \Gamma_1}^{\frac{m}{p}}\|u_t\|_{m, \Gamma_1}^{m-1}$$

$$\leq C_3(\varepsilon^m\|u\|_{p, \Gamma_1}^p + \varepsilon^{-m'}\|u_t\|_{m, \Gamma_1}^m)H^{-\alpha}(t)$$

$$\leq C_3(\varepsilon^m\|u\|_{p, \Gamma_1}^p + \varepsilon^{-m'}H'(t)H^{-\alpha}(t)),$$

for any $\varepsilon > 0$, where $\alpha = \frac{1}{m} - \frac{1}{p} > 0$ and $\frac{1}{m} + \frac{1}{p} = 1$, and we denote $C_1, C_2, ..., $ as suitable positive constants. Let $0 < \alpha < \tilde{\alpha}$, by (3.5) and (3.10), we have

$$\left|\int_{\Gamma_1} |u_t|^{m-2}u_t u d\sigma\right| \leq C_3(\varepsilon^mH^{-\tilde{\alpha}}(0)\|u\|_{p, \Gamma_1}^p + \varepsilon^{-m'}H'(t)H^{-\alpha}(t)H^{\tilde{\alpha}}(0)).$$

Now we introduce an auxiliary function

$$Z(t) = H^{1-\alpha}(t) + \delta \int_{\Omega} uu_t dx,$$

where $\delta$ is a small positive constant which will be decided later. By (3.8) and (3.11), we have

$$Z'(t) \geq (1 - \alpha)H^{-\alpha}(t)H'(t)$$

$$+ \delta \left(2\|u_t\|^2 + C_1\|u\|_{p, \Gamma_1}^p + 2H(t) - \int_{\Gamma_1} |u_t|^{m-2}u_t u d\sigma\right)$$

(3.12)

$$\geq (1 - \alpha - \delta C_3\varepsilon^{-m'}H^{\alpha-\tilde{\alpha}}(0))H^{-\alpha}(t)H'(t)$$

$$+ \delta(C_1 - C_3\varepsilon^mH^{-\tilde{\alpha}}(0))\|u\|_{p, \Gamma_1}^p + 2\delta\|u_t\|^2 + 2\delta H(t).$$

Let $\delta < (1 - \alpha)C_3^{-1}\varepsilon^{-m'}H^{\alpha-\tilde{\alpha}}(0)$, so the first term on the right hand side of (3.12) is positive. Moreover, if we choose $\varepsilon$ sufficiently small, we can obtain that

$$C_1 - C_3\varepsilon^mH^{-\tilde{\alpha}}(0) \geq \frac{1}{2}C_1,$$

further,

$$Z'(t) \geq \frac{1}{2}C_3\delta\|u\|_{p, \Gamma_1}^p + 2\delta\|u_t\|^2 + 2\delta H(t) \geq C_4\delta(\|u\|_{p, \Gamma_1}^p + \|u_t\|^2 + H(t)).$$

(3.13)
Letting $\delta$ sufficiently small, we have $Z'(0) > 0$. And noting that $H(t)$ is an increasing function, we have $Z(t) \geq Z(0)$ for $t \geq 0$. Now we set $r = \frac{1}{1 - \alpha}$, since $\alpha < \bar{\alpha} < 1$, it is evident that $1 < r < \bar{r} := \frac{1}{1 - \bar{\alpha}}$. According to the following inequality

$$|a + b|^r \leq 2^{r-1}(|a|^r + |b|^r) \quad \text{for } r \geq 1,$$

Young inequality and Cauchy-Schwarz inequality, we have

$$Z(t) \leq 2^{r-1}(H(t) + \delta^r \|u_t\|^r \|v\|^r) \leq C_4(H(t) + \|u_t\|^2 + \|u\|^{\frac{1}{2-\alpha}}).$$

Now by choosing $\alpha$ sufficiently small, we have

$$\|u\|^{\frac{1}{2-\alpha}} \leq 1 + \|u\|^2.$$  \hfill (3.15)

Using Poincaré inequality and combining (3.1), (3.14) and (3.15), we have

$$Z'(t) \geq C_7 Z(t).$$

Solving (3.17), we can obtain that there exist positive constants $C_8$ and $C_9$, such that

$$Z(t) \geq \frac{1}{C_8 t + C_9}.$$  \hfill (3.17)

Then we have

$$\lim_{t \to C_8} Z(t) = \infty.$$  \hfill (3.18)

So $Z(t)$ is not global. This completes the proof. \hfill \Box

4. Blow up for high initial energy $E(0) > d$ when $Q(u_t) = 0$

In this section, we mainly discuss the problem (1.1)-(1.4) without damping term, i.e., $Q(u_t) = 0$. We mainly adapt the convexity method introduced in [8].

**Lemma 4.1.** Let $\gamma > 0$, $T > 0$ and $h(t)$ be a Lipschitzian function over $[0, T)$. Assume that $h(0) \geq 0$ and $h'(t) + \gamma h(t) > 0$ for all $t \in (0, T)$. Then $h(t) > 0$ for all $t \in (0, T)$.

**Theorem 4.2.** (Finite time blow-up of solutions for $E(0) > d$.) Assume that $Q(u_t) = 0$, $1 < m < p$, $m > \frac{r}{r+1-p}$, $2 \leq p < r$. If

$$I(0) < 0,$$

$$E(0) > d,$$

$$\int_\Omega u_0 u_1 dx \geq 0,$$

$$\|u_0\|^2 > \frac{2p}{p-2} E(0),$$

then the solution of problem (1.1)-(1.4) blows up in finite time.
Proof. We will prove the result by two steps.

**Step I.** We first show that

\[ I(u(t)) < 0 \text{ and } \|u\|^2 > \frac{2p}{p-2} E(0), \quad t \geq 0. \]

Arguing by contradiction, we suppose by the continuity of \( I(t) \) that there exists a first time \( t_0 > 0 \) such that \( I(u(t_0)) = 0 \). Then we consider the \( L(t) \) function defined by

\[ L(t) := \|u\|^2. \]

We have

\[ L'(t) = 2 \int_{\Omega} uu_t \, dx. \]

From the definition of \( I(u) \), we obtain that

\[ L''(t) = 2\|u_t\|^2 - 2I(u). \]

Noticing that \( I(u) \leq 0, \quad t \in (0, t_0] \).

As \( L''(t) \geq 0 \) and \( L'(0) = \int_{\Omega} u_0 u_1 \, dx \geq 0 \) holds for all \( t \in (0, t_0] \), by Lemma 4.1, we can obtain that \( L'(t) > 0 \) for all \( t \in (0, t_0) \). So we can know that \( L(t) \) is strictly increasing on \( (0, t_0) \). Thus,

\[ L(t) \geq \|u_0\|^2 > \frac{2p}{p-2} E(0), \quad t \in (0, t_0]. \]

As a consequence, we have

\[ L(t_0) > \frac{2p}{p-2} E(0). \]

On the other hand, combining the fact that \( E(t) \) is a decreasing function, we have

\[ E(t_0) \leq E(t) < E(0), \quad t \in (0, t_0]. \]

According to the assumption, when \( I(t_0) = 0 \), we have

\[ \|\nabla u(t_0)\|^2 \leq \frac{2p}{p-2} E(0). \]

From \( \|u\|_{H_{\Gamma_0}^{1} (\Omega)} = \|\nabla u\| \) for \( u \in H_{\Gamma_0}^{1} (\Omega) \), we have

\[ L(t_0) = \|u(t_0)\|^2 \leq \|u(t_0)\|^2_{H_{\Gamma_0}^{1} (\Omega)} = \|\nabla u(t_0)\|^2 \leq \frac{2p}{p-2} E(0) \]

which leads to a contradiction. Thus we have proved that

\[ I(u) < 0, \quad t \in [0, T_{\text{max}}). \]

By the discussion above, we see that \( L(t) \) is strictly increasing on \( [0, T_{\text{max}}) \) provided \( I(u) < 0 \), which implies

\[ \|\nabla u\|^2 \geq L(t) > \frac{2p}{p-2} E(0), \quad t \in [0, T_{\text{max}}). \]
**Step II.** Now we prove the finite time blow-up of the solution. As discussed above, we assume that \( u \) is global first. Then for any \( t > 0 \), we use Cauchy-Schwarz inequality
\[
\|u_t\| \geq \int_{\Omega} u_t \, dx
\]
to get
\[
L(t)L''(t) - \frac{p+2}{4}(L'(t))^2 = 2\|u\|^2 (\|u_t\|^2 - I(u)) - \frac{p+2}{4} (2 \int_{\Omega} uu_t \, dx)^2
\]
\[
\geq 2\|u\|^2 (\|u_t\|^2 - I(u)) - (p+2)\|u\|^2 \|u_t\|^2
\]
\[
\geq \|u\|^2 \xi(t),
\]
where
\[
\xi(t) = -2pE(0) + (p-2)\|\nabla u\|^2.
\]
According to (4.2), we have
\[
\|\nabla u\|^2 > \frac{2p}{p-2} E(0).
\]
So we can obtain that
\[
2pE(0) < (p-2)\|\nabla u\|^2.
\]
Then we have \( \xi(t) > 0 \). Further we have
\[
L(t)L''(t) - \frac{p+2}{4}(L'(t))^2 \geq 0.
\]
Thus
\[
(L(t)^{-\alpha})'' = \frac{-\alpha}{L^{n+2}(t)} (L(t)L''(t) - (1+\alpha)(L'(t))^2) < 0,
\]
where \( \alpha = \frac{p+2}{4} > 0 \). Hence, it proves that \( L^{-\alpha} \) reaches 0 in finite time, then there exists a \( T \in [0, \infty) \) such that
\[
\lim_{t\to T} L(t) = \infty.
\]
Finally we prove that \( L(t) \) is not global. This completes the proof. \( \square \)

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