An analytic semigroup generated by a fractional differential operator

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Abstract

We study a space-fractional diffusion problem, where the non-local diffusion flux is modeled by the Caputo derivative. We derive the unique existence of regular solutions to this problem by means of the semigroup theory. We show that the operator defined as divergence of the Caputo derivative is a generator of analytic semigroup.

1 Introduction

In the paper we consider the following initial-boundary value problem

\[
\begin{align*}
u_t - \frac{\partial}{\partial x} D^\alpha u &= f & \text{in } (0,1) \times (0,T), \\
u_x(0,t) = 0, \quad u(1,t) = 0 & \text{for } t \in (0,T), \\
u(x,0) = u_0(x) & \text{in } (0,1),
\end{align*}
\]

where, for \( \alpha \in (0,1) \), operator \( D^\alpha \) denotes the fractional Caputo derivative with respect to spatial variable and \( f \) and \( u_0 \) are given. The motivation for studying (1) originates from modelling of sub-surface water motion. In paper [17] the author considered the model of infiltration of water into heterogeneous soils. Due to the presence of media heterogeneity, the author proposed representing the hydraulic flux in terms of fractional derivative. The similar approach for modelling the non-locality in space was presented in [16]. This paper provides the overview of one-phase Stefan problems which exhibits anomalous behaviour. The author introduced the model where the diffusive operator is in the divergence form and the flux is the Caputo derivative of the temperature. In the present paper we consider the simplified problem (1). We would like to emphasise that from the modelling point of view, it is important that (1) is a balance law.

Our goal is to study the basic solvability problem for equation (1), by the semigroup approach. At this moment we would like to present a broader context. It is worth to mention here, that in the paper [11] the authors studied an array of fractional

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diffusion equations, for a variety of boundary conditions, but they constructed only the $C_0$-semigroup. Besides, we note that the problem (11) was not explicitly studied in [1]. Another paper that presents the probabilistic point of view on space-fractional problems is [3], where the authors consider equations with time-fractional Caputo derivative and non-local space operators. A completely different approach for solving (1) with zero Dirichlet boundary conditions is employed in [11], where the authors obtained the viscosity solutions. Further discussion was made in [2] and [12] where the authors compare the problems with diffusive flux modeled by the Caputo and the Riemann-Liouville derivative and carry a numerical analysis.

In the present paper we will present the results concerning solvability of (11) by means of the semigroup theory. We will describe the domain of $\frac{\partial}{\partial x} D^\alpha$ in terms of Sobolev spaces and as final result we will construct an analytic semigroup. Precisely, we will show that the operator $\frac{\partial}{\partial x} D^\alpha$, considered on the domain

$$D_\alpha := \{ u \in H^{1+\alpha}(0,1) : u_x \in 0H^\alpha(0,1), u(1) = 0 \}$$

generates an analytic semigroup. Here, by $H^\alpha(0,1)$ we mean the fractional Sobolev space (see [9, definition 9.1]) and the subspace $0H^\alpha(0,1)$ will be introduced in Proposition 3. We would like to emphasise that, developing the theory of analytic semigroups for the systems of type (11) seems to be especially important if we notice that the operator $\frac{\partial}{\partial x} D^\alpha$ is not self-adjoint, thus we can not expect that its eigenfunctions generate the basis of any natural Hilbert space.

The paper is organized as follows. In chapter 2 we give preliminary results concerning fractional operators. Chapter 3 is devoted to the proof of the main result. Namely, we will show that $\frac{\partial}{\partial x} D^\alpha$ is a generator of $C_0$-semigroup of contractions which can be extended to the analytic semigroup. In chapter 4, we will present the basic applications of this result to solvability of (11). At last, for the sake of completeness, in Appendix we collect the general results from calculus that we used in the paper.

## 2 Preliminaries from fractional calculus

Before we will establish the results, we will introduce definitions of fractional operators and recall some of their properties. For more comprehensive studies on general properties of fractional operators we refer to a standard literature [7], [15].

**Definition 1.** Let $L, \alpha > 0$. For $f \in L^1(0, L)$ we introduce the fractional integral $I^\alpha$ by the formula

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - p)^{\alpha-1} f(p) dp.$$  

If $\alpha \in (0, 1)$ and $f$ is regular enough we may define the Riemann-Liouville fractional derivative

$$\partial^\alpha f(x) = \frac{\partial}{\partial x} I^{1-\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_0^x (x - p)^{-\alpha} f(p) dp$$
and the Caputo fractional derivative

\[ D^\alpha f(x) = \frac{\partial}{\partial x} \left( I^{1-\alpha} [f(x) - f(0)] \right) = \frac{1}{\Gamma(1 - \alpha)} \int_0^x (x - p)^{-\alpha} [f(p) - f(0)] dp. \]

We note that if \( f \in AC[0,T] \), then \( D^\alpha f \) may be equivalently written in the form

\[ D^\alpha f(x) = I^{1-\alpha} f'(x) = \frac{1}{\Gamma(1 - \alpha)} \int_0^x (x - p)^{-\alpha} f'(p) dp. \]

We will also make use of the formal adjoint operators, that is

\[ I^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^L (p - x)^{\alpha - 1} f(p) dp, \]

\[ \partial^\alpha f(x) = -\frac{\partial}{\partial x} \left( I^{-\alpha} f \right)(x) \text{ and } D^\alpha f(x) = -\frac{\partial}{\partial x} I^{-\alpha} [f(x) - f(L)]. \]

It is clear that for absolutely continuous functions the foregoing fractional differential operators are well defined. Further discussion on the correctness of these definitions is performed in [7] and [15]. In Proposition 3 we will introduce the definition of the domain for the Riemann-Liouville fractional derivative in terms of Sobolev spaces which was established in [5]. Before we will pass to that result, we will present a simple proposition which gives us the formula for the superposition of \( \partial^\alpha \) and \( D^\alpha \).

**Proposition 1.** [8, Proposition 6.5] Let \( L > 0 \). For \( \alpha, \beta \in (0,1) \) such that \( \alpha + \beta \leq 1 \) and \( u \in AC[0,L] \) we have \( \partial^\alpha D^\beta u = D^{\alpha+\beta}u \).

The next proposition provides the energy estimate for the Riemann-Liouville fractional derivative.

**Proposition 2.** [8, Proposition 6.10] If \( w \in AC[0,L] \) then for any \( \alpha \in (0,1) \) the following equality holds,

\[
\int_0^L \partial^\alpha w(x) \cdot w(x) dx = \frac{\alpha}{4} \int_0^L \int_0^L \frac{|w(x) - w(p)|^2}{|x - p|^{1+\alpha}} dp dx
\]

\[ + \frac{1}{2\Gamma(1 - \alpha)} \int_0^L [(L - x)^{-\alpha} + x^{-\alpha}] |w(x)|^2 dx. \]

Hence, there exist a positive constant \( c \) which depends only on \( \alpha, L \), such that

\[ \int_0^L \partial^\alpha w(x) \cdot w(x) dx \geq c \| w \|_{H^\alpha(0,L)}^2 \]

(2)

and in particular

\[ \int_0^L \partial^\alpha w(x) \cdot w(x) dx \geq \frac{L^{-\alpha}}{2\Gamma(1 - \alpha)} \int_0^L |w(x)|^2 dx. \]
Now, we will introduce the characterization of the domain of the Riemann- Liouville derivative in the $L^2$-framework. It will be essential for further considerations. At first let us define the following functional spaces

$$0H^\alpha(0,1) = \begin{cases} H^\alpha(0,1) \\ \{ u \in H^\frac{1}{2}(0,1) : \int_0^1 \frac{|u(x)|^2}{x} \, dx < \infty \} \quad \text{for } \alpha = \frac{1}{2}, \\ \{ u \in H^\alpha(0,1) : u(0) = 0 \} \quad \text{for } \alpha \in \left( \frac{1}{2}, 1 \right) \end{cases}$$

and

$$0H^\alpha(0,1) = \begin{cases} H^\alpha(0,1) \\ \{ u \in H^\frac{1}{2}(0,1) : \int_0^1 \frac{|u(x)|^2}{1-x} \, dx < \infty \} \quad \text{for } \alpha = \frac{1}{2}, \\ \{ u \in H^\alpha(0,1) : u(1) = 0 \} \quad \text{for } \alpha \in \left( \frac{1}{2}, 1 \right). \end{cases}$$

We set $\|u\|_{0H^\alpha(0,1)} = \|u\|_{0H^\alpha(0,1)} = \|u\|_{0H^\alpha(0,1)}$ for $\alpha \neq \frac{1}{2}$ and

$$\|u\|_{0H^\frac{1}{2}(0,1)} = \left( \|u\|^2_{H^\frac{1}{2}(0,1)} + \int_0^1 \frac{|u(x)|^2}{x} \, dx \right)^\frac{1}{2},$$

$$\|u\|_{0H^\frac{1}{2}(0,1)} = \left( \|u\|^2_{H^\frac{1}{2}(0,1)} + \int_0^1 \frac{|u(x)|^2}{1-x} \, dx \right)^\frac{1}{2}.$$

The following Proposition is the extended version of [5, Theorem 2.1] which can be found in the Appendix of [5].

**Proposition 3.** For $\alpha \in [0, 1]$ the operators $I^\alpha : L^2(0,1) \to 0H^\alpha(0,1)$ and $\partial^\alpha : 0H^\alpha(0,1) \to L^2(0,1)$ are isomorphism and the following inequalities hold

$$c_\alpha^{-1}\|u\|_{0H^\alpha(0,1)} \leq \|\partial^\alpha u\|_{L^2(0,1)} \leq c_\alpha\|u\|_{0H^\alpha(0,1)} \quad \text{for } u \in 0H^\alpha(0,1),$$

$$c_\alpha^{-1}\|I^\alpha f\|_{0H^\alpha(0,1)} \leq \|f\|_{L^2(0,1)} \leq c_\alpha\|I^\alpha f\|_{0H^\alpha(0,1)} \quad \text{for } f \in L^2(0,1).$$

Analogously, by the change of variables $x \mapsto 1-x$, we obtain that the operators $I^\alpha : L^2(0,1) \to 0H^\alpha(0,1)$ and $\partial^\alpha : 0H^\alpha(0,1) \to L^2(0,1)$ are isomorphism and there hold the inequalities

$$c_\alpha^{-1}\|u\|_{0H^\alpha(0,1)} \leq \|\partial^- u\|_{L^2(0,1)} \leq c_\alpha\|u\|_{0H^\alpha(0,1)} \quad \text{for } u \in 0H^\alpha(0,1),$$

$$c_\alpha^{-1}\|I^\alpha f\|_{0H^\alpha(0,1)} \leq \|f\|_{L^2(0,1)} \leq c_\alpha\|I^\alpha f\|_{0H^\alpha(0,1)} \quad \text{for } f \in L^2(0,1).$$

Here $c_\alpha$ denotes a positive constant dependent on $\alpha$.

**Corollary 1.** For $\alpha, \beta \in (0, 1)$ such that $0 < \alpha + \beta \leq 1$ we have

$$I^\beta : 0H^\alpha(0,1) \to 0H^{\alpha+\beta}(0,1).$$
Proof. It is an easy consequence of Proposition 3. If \( u \in 0^H(0, 1) \), then there exists \( w \in L^2(0, 1) \) such that \( u = I^\alpha w \). Then,

\[
I^\beta u = I^\beta I^\alpha w = I^{\alpha + \beta} w \in 0^H(0, 1).
\]

We finish this section with two propositions which provide us an extension of \( I^\alpha \) and \( \partial^\alpha \) into wider functional spaces. The similar reasoning to the one carried in Proposition 4 may be found in [4, Lemma 5].

**Proposition 4.** For \( \alpha \in (0, \frac{1}{2}) \) the operators \( I^\alpha \) and \( I_-^\alpha \) can be extended to bounded and linear operators from \( H^{-\alpha}(0, 1) := (H^0(0, 1))' \) to \( L^2(0, 1) \).

**Proof.** We will prove the claim only for \( I^\alpha \) while the proof for \( I_-^\alpha \) is analogous. By the Fubini theorem for \( u, v \in L^2(0, 1) \) we obtain

\[
(I^\alpha u, v) = (u, I^\alpha_+ v).
\]

Thus, using Proposition 3 we may estimate,

\[
| (I^\alpha u, v) | \leq \| I^\alpha_+ v \|_{H^\alpha(0, 1)} \| u \|_{(H^\alpha(0, 1))'} \leq c_\alpha \| v \|_{L^2(0, 1)} \| u \|_{H^{-\alpha}(0, 1)},
\]

where we used the fact that for \( \alpha < \frac{1}{2} \) we have \( (H^0(0, 1))' = (H^\alpha(0, 1))' \). The last inequality finishes the proof.

**Proposition 5.** For \( \alpha \in (0, \frac{1}{2}) \) the operators \( \partial^\alpha \) and \( \partial_-^\alpha \) can be extended to bounded and linear operators from \( L^2(0, 1) \) to \( H^{-\alpha}(0, 1) \).

**Proof.** As in the previous proposition, we will prove the statement only for \( \partial^\alpha \), because for \( \partial_-^\alpha \) the proof is analogous. Let us assume that \( f, v \in H^\alpha(0, 1) \). (We recall that for \( \alpha \in (0, \frac{1}{2}) \) the space \( H^\alpha(0, 1) \) coincides with \( 0^H(0, 1) \) and \( 0^H(0, 1) \)). Then, from Proposition 3 there exist \( g \in L^2(0, 1) \) such that \( \partial^\alpha f = g \) and \( w \in H^\alpha(0, 1) \) such that \( v = I^\alpha_+ w \). Thus, we have

\[
(\partial^\alpha f, v) = (g, I^\alpha_+ w) = (I^\alpha g, w) = (f, \partial^\alpha_- v).
\]

Making use of Proposition 3 one more time, we may estimate

\[
| (\partial^\alpha f, v) | \leq \| f \|_{L^2(0, 1)} \| \partial^\alpha_- v \|_{L^2(0, 1)} \leq c_\alpha \| f \|_{L^2(0, 1)} \| v \|_{H^\alpha(0, 1)},
\]

Thus \( \partial^\alpha f \in (H^\alpha(0, 1))' \) which coincides with \( H^{-\alpha}(0, 1) \) for \( \alpha \in (0, \frac{1}{2}) \) and the proof is finished.
3 Operator $\frac{\partial}{\partial x} D^\alpha$ as a generator of an analytic semigroup

In this section we will proceed as follows. Firstly, we will characterize the domain of $\frac{\partial}{\partial x} D^\alpha$ in $L^2(0,1)$. Then, we will show that $\frac{\partial}{\partial x} D^\alpha$ generates a $C_0$-semigroup of contractions. Finally, we will prove that, by appropriate estimate of the resolvent operator, this semigroup may be extended to an analytic semigroup on a sector of complex plane.

We may note that, just by definition $\frac{\partial}{\partial x} D^\alpha u = \frac{\partial}{\partial x} I^{1-\alpha} u_x = \partial^\alpha u_x$, whenever one of sides of the identity is meaningful. By Proposition 3 the domain of $\partial^\alpha$ in $L^2(0,1)$ coincides with $0H^\alpha(0,1)$. Thus, we may consider the domain of $\frac{\partial}{\partial x} D^\alpha$ as $\{u \in H^{1+\alpha}(0,1) : u_x \in 0H^\alpha(0,1)\}$. Taking into account the boundary condition in (1) we finally define the domain of $\frac{\partial}{\partial x} D^\alpha$ as

$$D\left(\frac{\partial}{\partial x} D^\alpha\right) \equiv \mathcal{D}_\alpha := \{u \in H^{1+\alpha}(0,1) : u_x \in 0H^\alpha(0,1), \ u(1) = 0\}. \quad (4)$$

We may equip $\mathcal{D}_\alpha$ with the graph norm

$$\|f\|_{\mathcal{D}_\alpha} = \|f\|_{H^{1+\alpha}(0,1)} \text{ for } \alpha \in (0,1) \setminus \{\frac{1}{2}\}$$

and

$$\|f\|_{\mathcal{D}_\alpha} = \left(\|f\|_{H^{1+\alpha}(0,1)}^2 + \int_0^1 \frac{|u_x(x)|^2}{x} dx\right)^{\frac{1}{2}} \text{ for } \alpha = \frac{1}{2}.$$

**Theorem 1.** Operator $\frac{\partial}{\partial x} D^\alpha : \mathcal{D}_\alpha \subseteq L^2(0,1) \to L^2(0,1)$ generates a $C_0$-semigroup of contractions.

**Proof.** We will prove Theorem 1 by applying the Lumer-Philips theorem [13, Ch. 1, Theorem 4.3]. Firstly, we see that $\frac{\partial}{\partial x} D^\alpha$ is densely defined. In order to satisfy assumptions of Lumer-Philips theorem we need to show in addition that $-\frac{\partial}{\partial x} D^\alpha$ is accretive and that $R(I - \frac{\partial}{\partial x} D^\alpha) = L^2(0,1)$. In order to prove that $-\frac{\partial}{\partial x} D^\alpha$ is accretive we take $u \in \mathcal{D}_\alpha$, using integration by parts and Proposition 1 we get

$$\text{Re} \left( -\frac{\partial}{\partial x} D^\alpha u, u \right) = -\text{Re} \int_0^1 (\frac{\partial}{\partial x} D^\alpha u)(x) \cdot \overline{u(x)} dx$$

$$= \int_0^1 D^\alpha \text{Re} u(x) \cdot \frac{\partial}{\partial x} \text{Re} u(x) dx + \int_0^1 D^\alpha \text{Im} u(x) \cdot \frac{\partial}{\partial x} \text{Im} u(x) dx$$

$$= \int_0^1 D^\alpha \text{Re} u(x) \cdot \partial^{1-\alpha} D^\alpha \text{Re} u(x) dx + \int_0^1 D^\alpha \text{Im} u(x) \cdot \partial^{1-\alpha} D^\alpha \text{Im} u(x) dx.$$  

Since $u_x \in 0H^\alpha(0,1)$, then from Corollary 1 we know that $D^\alpha u = I^{1-\alpha} u_x \in 0H^{1}(0,1)$. We may apply inequality (2) with $w = D^\alpha \text{Re} u$ and $w = D^\alpha \text{Im} u$ to obtain

$$\text{Re} \left( -\frac{\partial}{\partial x} D^\alpha u, u \right) \geq c_\alpha \|D^\alpha u\|^2_{H^{1+\alpha}(0,1)} \geq c_\alpha \|\partial^{1-\alpha} D^\alpha u\|^2_{L^2(0,1)}$$
\[ c_\alpha \left\| D^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)}, \]  

where in the second inequality we used Proposition 3 and the equality follows from Proposition 1. Here \( c_\alpha > 0 \) denotes a generic constant dependent on \( \alpha \).

Now, we would like to show that \( R(I - \frac{\partial}{\partial x} D^\alpha) = L^2(0,1) \). In fact, we are able to show something more. We will state the result in the next lemma.

**Lemma 1.** For every \( \lambda \in \mathbb{C} \) belonging to the sector

\[ \vartheta_\alpha := \{ z \in \mathbb{C} : |\arg z| \leq \frac{\pi(\alpha + 1)}{2} \} \cup \{0\} \]  

there holds

\[ R(\lambda I - \frac{\partial}{\partial x} D^\alpha) = L^2(0,1). \]

**Proof.** To prove the lemma we fix \( \lambda \in \mathbb{C} \) belonging to \( \vartheta_\alpha \). We must prove that there exists \( u \in \mathcal{D}_\alpha \) such that

\[ \lambda u - \frac{\partial}{\partial x} D^\alpha u = g. \]  

We would like to calculate the solution directly. To that end, we will firstly solve equation (7) with the arbitrary boundary condition \( u(0) = u_0 \in \mathbb{C} \). Then, we will choose \( u_0 \) which will guarantee the zero condition at the other endpoint of the interval. We note that if we search for a solution in \( \{ f \in H^{1+\alpha}(0,1) : f_x \in \partial H^\alpha(0,1) \} \), then equation (7) is equivalent to

\[ u = u_0 + \lambda I^{\alpha+1} u - I^{\alpha+1} g. \]  

Indeed, if we apply \( I^\alpha \) to both sides of (7), recall that \( \frac{\partial}{\partial x} D^\alpha u = \partial^\alpha u_x \) and assume that \( u_x \in \partial H^\alpha(0,1) \), we obtain

\[ u_x = \lambda I^\alpha u - I^\alpha g. \]

Integrating this equality we arrive at (8). On the other hand, if we assume that \( u \in L^2(0,1) \) solves (8), then by Proposition 3 it automatically belongs to \( \{ f \in H^{1+\alpha}(0,1) : f_x \in \partial H^\alpha(0,1) \} \) and to obtain (7) it is enough to apply \( \partial^\alpha \frac{\partial}{\partial x} \) to (8).

Thus, we are going to solve (8). We apply to (8) the operator \( I^{\alpha+1} \) and we obtain

\[ u(x) = u_0 - (I^{\alpha+1} g)(x) + \lambda(I^{\alpha+1} u_0)(x) + \lambda^2(I^{2(\alpha+1)} u)(x) - \lambda(I^{2(\alpha+1)} g)(x). \]

Iterating this procedure \( n \) times we arrive at

\[ u(x) = u_0 \sum_{k=0}^{n} (\lambda I^{k(\alpha+1)+1}) - \sum_{k=0}^{n} \lambda^k (I^{(k+1)(\alpha+1)+1} g)(x) + \lambda^{n+1}(I^{(n+1)(\alpha+1)+1} u)(x). \]  

We will show, that the last expression tends to zero as \( n \to \infty \). Indeed, we may note that, since we search for the solutions in \( H^{1+\alpha}(0,1) \subseteq L^\infty(0,1) \) and due to the presence of the \( \Gamma \)-function in the denominator we have

\[ \left| \lambda^n (I^{n(\alpha+1)} u)(x) \right| \leq \|u\|_{L^\infty(0,1)} \frac{|\lambda|^n}{\Gamma((\alpha+1)n+1)} \leq \frac{\|u\|_{L^\infty(0,1)} |\lambda|^n}{\Gamma((\alpha+1)n+1)} \to 0 \text{ as } n \to \infty. \]
for each $\lambda \in \mathbb{C}$ uniformly with respect to $x \in [0, 1]$. Thus, passing to the limit with $n$ in (9) we obtain the formula

$$u(x) = u_0 \sum_{k=0}^{\infty} (\lambda I^k(\alpha+1)) - \sum_{k=0}^{\infty} \lambda^k(I^{(k+1)}(\alpha+1)g)(x). \quad (10)$$

We will show that both series in (10) are uniformly convergent. We may estimate

$$\left| \lambda^k(I^{(\alpha+1)}(k+1)g)(x) \right| \leq \frac{|\lambda|^k}{\Gamma((\alpha+1)(k+1))} \int_0^x (x-s)^{(\alpha+1)(k+1)-1} |g(s)| \, ds$$

$$\leq \|g\|_{L^2(0,1)} \frac{|\lambda|^k}{\Gamma((\alpha+1)(k+1))} \left( \int_0^x (x-s)^{2(\alpha+1)(k+1)-2} \, ds \right)^{\frac{1}{2}}$$

$$\leq \|g\|_{L^2(0,1)} \frac{|\lambda|^k}{\Gamma((\alpha+1)(k+1))} x^{(\alpha+1)(k+1)-\frac{1}{2}} \frac{x^{(\alpha+1)(k+1)+1}}{\sqrt{2(\alpha+1)(k+1) - 1}}.$$ 

If we denote the last expression by $a_k$ we may calculate

$$\frac{a_{k+1}}{a_k} = \frac{|\lambda| x^{(\alpha+1)k+1} \sqrt{2(\alpha+1)(k+1) - 1} \Gamma((\alpha+1)(k+1))}{\sqrt{2(\alpha+1)(k+2) - 1} \Gamma((\alpha+1)(k+1) + (\alpha+1))}$$

and so

$$\frac{a_{k+1}}{a_k} \leq |\lambda| x^{(\alpha+1)k+1} \frac{B((\alpha+1), (k+1)(\alpha+1))}{\Gamma((\alpha+1))} \to 0 \text{ as } k \to \infty$$

for every $\lambda \in \mathbb{C}$ and uniformly with respect to $x \in [0, 1]$ and from comparison criterion and the d’Alembert criterion the series is uniformly convergent. With the first series in (10) we may deal even more easily. Now, we would like to compute the sum of the series. By direct calculations, we see that

$$\sum_{k=0}^{\infty} \lambda^k(I^{(\alpha+1)k}) = E_{\alpha+1}(\lambda x^{\alpha+1}), \quad (11)$$

where by $E_{\alpha+1}$ we denote the Mittag-Leffler function. For the definition of Mittag-Leffler function we refer to Proposition 9 from the Appendix. To calculate the sum of the second series, we apply the definition of fractional integral (see Definition 1)

$$\sum_{k=0}^{\infty} \lambda^k(I^{(\alpha+1)(k+1)}g)(x) = \sum_{k=0}^{\infty} \lambda^k \int_0^x g(s) \frac{(x-s)^{(\alpha+1)k+\alpha}}{\Gamma((\alpha+1)k + \alpha + 1)} \, ds.$$ 

In order to interchange the order of integration and summation, we will firstly consider the finite sum and then we will pass to the limit,

$$\sum_{k=0}^{\infty} \lambda^k \int_0^x g(s) \frac{(x-s)^{(\alpha+1)k+\alpha}}{\Gamma((\alpha+1)k + \alpha + 1)} \, ds = \lim_{n \to \infty} \sum_{k=0}^{n} \lambda^k \int_0^x g(s) \frac{(x-s)^{(\alpha+1)k+\alpha}}{\Gamma((\alpha+1)k + \alpha + 1)} \, ds$$

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\[
\lim_{n \to \infty} \int_0^x g(s) \sum_{k=0}^n \lambda^k \frac{(x-s)^{(\alpha+1)k+\alpha}}{\Gamma((\alpha+1)k + \alpha + 1)} \, ds.
\]

We would like to apply the Lebesgue dominated convergence theorem, thus we need to indicate the majorant. We may estimate as follows

\[
\left| g(s) \sum_{k=0}^n \lambda^k \frac{(x-s)^{(\alpha+1)k+\alpha}}{\Gamma((\alpha+1)k + \alpha + 1)} \right| \leq |g(s)| \sum_{k=0}^\infty \frac{\left| \lambda \right|^k}{\Gamma((\alpha+1)k + (\alpha + 1))} = |g(s)| E_{\alpha+1,\alpha+1}(\left| \lambda \right|)
\]

and the last function is integrable because \( g \in L^2(0,1) \). Hence, applying the Lebesgue dominated convergence theorem we arrive at

\[
\sum_{k=0}^\infty \lambda^k (I^{(\alpha+1)(k+1)}g)(x) = g \ast x^\alpha \sum_{k=0}^\infty \frac{(\lambda x^\alpha)^k}{\Gamma((\alpha+1)k + (\alpha + 1))}.
\]

Here and in whole paper by \( \ast \) we denote the convolution on \((0, \infty)\), i.e. \( f \ast g = \int_0^x f(p)g(x-p) \, dp \). Finally, using this result together with (11) in (10) we obtain that the function \( u \) defined by the formula

\[
u(x) = u_0 E_{\alpha+1} (\lambda x^\alpha) - g \ast x^\alpha E_{\alpha+1,\alpha+1} (\lambda x^\alpha)
\]

is a solution to (7) with a boundary condition \( u(0) = u_0 \). It remains to solve equation (7) with the zero condition in the endpoint of the interval. For this purpose, we take \( x = 1 \) in (12) and we obtain

\[
u(1) = u_0 E_{\alpha+1} (\lambda) - (g \ast y^\alpha E_{\alpha+1,\alpha+1} (\lambda y^\alpha))(1)
\]

and we may calculate \( u_0 \)

\[
u_0 = (E_{\alpha+1} (\lambda))^{-1} (g \ast y^\alpha E_{\alpha+1,\alpha+1} (\lambda y^\alpha))(1).
\]

\( u_0 \) is well defined because, taking \( \nu = \alpha + 1, \mu = 1 \) in Proposition 4 from Appendix, we obtain that \( E_{\alpha+1} (\lambda) > 0 \) for \( \lambda \) belonging to the sector \( \vartheta \). Placing this \( u_0 \) in the formula (12) we obtain the solution for (8) which belongs to \( D_{\alpha} \)

\[
u(x) = (E_{\alpha+1} (\lambda))^{-1} (g \ast y^\alpha E_{\alpha+1,\alpha+1} (\lambda y^\alpha))(1) E_{\alpha+1} (\lambda x^\alpha) - g \ast x^\alpha E_{\alpha+1,\alpha+1} (\lambda x^\alpha).
\]

That way we proved the lemma.

Remark 1. We note that the above theorem as well as other results from this paper are valid for the space interval \([0, L]\) for every fixed \( 0 < L < \infty \) and \( L = 1 \) was chosen just to simplify the notation.
It remains to prove that the semigroup generated by \( \frac{\partial}{\partial x} D^\alpha \) can be extended to an analytic semigroup on a sector of complex plane. Before we will prove that result, we need to formulate two auxiliary lemmas. A similar reasoning to the one carried in Lemma 2 may be found in [6, Lemma 6].

**Lemma 2.** The formulas \( \left\| D^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)} \) and \( \left\| u \right\|_{H^{\frac{1+\alpha}{2}}(0,1)} \) define equivalent norms on \( D_\alpha \).

**Proof.** Firstly, we will show that there exists a positive constant \( c \) such that
\[
\left\| D^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)} \leq c \left\| u \right\|_{H^{\frac{1+\alpha}{2}}(0,1)}.
\]
Using Proposition 4 we may write
\[
\left\| D^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)} = \left\| I^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)} \leq c \left\| u \right\|_{H^{\frac{\alpha+1}{2}}(0,1)}.
\]
Due to Remark 12.8. [9] we know that \( \frac{\partial}{\partial x} \) is a bounded and linear operator from \( H^s(0,1) \) to \( H^{s-1}(0,1) \) for \( s \in [0,1] \setminus \{ \frac{1}{2} \} \) thus
\[
\left\| D^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)} \leq c \left\| u \right\|_{H^{\frac{\alpha+1}{2}}(0,1)}.
\]
To show the opposite inequality we notice that since \( u \in D_\alpha \) we have
\[
u(x) = -\int_0^1 u_x(s)\,ds = -I^{-\frac{1+\alpha}{2}} I^{-\frac{1-\alpha}{2}} u_x
\]
and by Proposition 3 we may estimate
\[
\left\| u \right\|_{H^{\frac{\alpha+1}{2}}(0,1)} = \left\| I^{-\frac{1+\alpha}{2}} I^{-\frac{1-\alpha}{2}} u_x \right\|_{H^0(0,1)} \leq c \left\| I^{-\frac{1-\alpha}{2}} u_x \right\|_{L^2(0,1)}.
\]
Applying Proposition 4 and Proposition 5 we may estimate further
\[
\left\| u \right\|_{H^{\frac{\alpha+1}{2}}(0,1)} \leq c \left\| u_x \right\|_{H^{\frac{\alpha-1}{2}}(0,1)} = c \left\| \frac{1+\alpha}{2} I^{-\frac{1-\alpha}{2}} u_x \right\|_{H^\frac{\alpha-1}{2}(0,1)} \leq c \left\| I^{-\frac{1-\alpha}{2}} u_x \right\|_{L^2(0,1)},
\]
which finishes the proof.

**Lemma 3.** For \( u \in D_\alpha \) we have
\[
\text{Re}(-\frac{\partial}{\partial x} D^\alpha u, u) \geq c_\alpha \left\| u \right\|_{H^{\frac{1+\alpha}{2}}(0,1)}^2 \quad \text{(13)}
\]
and
\[
\left| (-\frac{\partial}{\partial x} D^\alpha u, u) \right| \leq b_\alpha \left\| u \right\|_{H^{\frac{1+\alpha}{2}}(0,1)}^2, \quad \text{(14)}
\]
where \( c_\alpha, b_\alpha \) are positive constant which depends only on \( \alpha \).
Proof. We have already obtained in (5)

\[ \Re \left( -\frac{\partial}{\partial x} D^\alpha u, u \right) \geq c_\alpha \left\| D^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)}^2. \]

Hence, in order to prove (13) it is enough to apply the norm equivalence from Lemma 2. To show (14), we firstly notice that since \( u \in \mathcal{D}_\alpha \), we know that \( u_x \in \alpha H^\alpha(0,1) \) and from Corollary 1 we infer that \( D^\alpha u = I^{1-\alpha} u_x \in \alpha H^1(0,1) \). Applying Proposition 1 in the first and third identity and \( (D^\alpha u)(0) = 0 \) in the second one, we may write

\[ \frac{\partial}{\partial x} D^\alpha u = \partial^{\frac{1+\alpha}{2}} D^{\frac{1-\alpha}{2}} D^\alpha u = \partial^{\frac{1-\alpha}{2}} \partial^{\frac{1+\alpha}{2}} D^\alpha u = \partial^{\frac{1-\alpha}{2}} D^{\frac{1+\alpha}{2}} u = \frac{\partial}{\partial x} I^{\frac{1-\alpha}{2}} D^{\frac{1+\alpha}{2}} u. \]

Integrating by parts, in view of \( \overline{u}(1) = 0 \) we obtain

\[ \left| \left( -\frac{\partial}{\partial x} D^\alpha u, u \right) \right| = \left| \int_0^1 \frac{\partial}{\partial x} I^{\frac{1-\alpha}{2}} D^{\frac{1+\alpha}{2}} u \cdot \overline{u} dx \right| = \left| \int_0^1 I^{\frac{1-\alpha}{2}} D^{\frac{1+\alpha}{2}} u \cdot \overline{u} dx \right|. \]

Thus, by the identity 3 we get

\[ \left| \left( -\frac{\partial}{\partial x} D^\alpha u, u \right) \right| = \left| \int_0^1 D^{\frac{1+\alpha}{2}} u \cdot D^{\frac{1-\alpha}{2}} \overline{u} dx \right| \leq \left\| D^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)} \left\| D^{\frac{1-\alpha}{2}} u \right\|_{L^2(0,1)}. \tag{15} \]

Since \( u(1) = 0 \), applying Proposition 3 we obtain that

\[ \left\| D^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)} = \left\| \partial^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)} \leq b_\alpha \| u \|_{H^{\frac{1+\alpha}{2}}(0,1)} = b_\alpha \| u \|_{H^{\frac{1-\alpha}{2}}(0,1)}, \]

where by \( b_\alpha \) we denote a positive constant dependent on \( \alpha \). Making use of this estimate and the norm equivalence from Lemma 2 in (15) we obtain the estimate (14).

Finally, we are ready to prove the main theorem.

Theorem 2. The operator \( \frac{\partial}{\partial x} D^\alpha : \mathcal{D}_\alpha \subset L^2(0,1) \to L^2(0,1) \) generates an analytic semigroup.

Proof. We will give the proof of analyticity following the proof of [13, Ch. 7, Theorem 2.7], where the elliptic operators are studied.

At first, we notice that since \( L^2(0,1) \) is a Hilbert space, the numerical range of \( -\frac{\partial}{\partial x} D^\alpha \) equals

\[ S(-\frac{\partial}{\partial x} D^\alpha) = \left\{ (u, -\frac{\partial}{\partial x} D^\alpha u) : u \in \mathcal{D}_\alpha, \| u \|_{L^2(0,1)} = 1 \right\}. \]

We note that by [13] zero does not belong to \( S(-\frac{\partial}{\partial x} D^\alpha) \). Let us denote \( z = (u, -\frac{\partial}{\partial x} D^\alpha u) \). Then, in view of [13] and [14], we obtain that

\[ \left| \tan(\arg z) \right| = \left| \frac{\text{Im} z}{\text{Re} z} \right| \leq \frac{b_\alpha}{c_\alpha}, \]

where by \( b_\alpha \) and \( c_\alpha \) we denote positive constants dependent on \( \alpha \).
which implies
\[ S\left(-\frac{\partial}{\partial x} D^\alpha\right) \subseteq \left\{ \lambda \in \mathbb{C} : |\arg \lambda| \leq \arctan \left( \frac{b_\alpha}{c_\alpha} \right) \right\} \]
and \( \arctan \left( \frac{b_\alpha}{c_\alpha} \right) < \frac{\pi}{2} \). We may choose \( \nu \) such that \( \arctan \left( \frac{b_\alpha}{c_\alpha} \right) < \nu < \frac{\pi}{2} \) and denote \( \Sigma_\nu := \{ \lambda : |\arg \lambda| > \nu \} \). Then, \( \Sigma_\nu \subseteq \mathbb{C} \setminus S\left(-\frac{\partial}{\partial x} D^\alpha\right) \) and there exists a positive constant \( c_\nu \) such that
\[ d(\lambda, S\left(-\frac{\partial}{\partial x} D^\alpha\right)) \geq c_\nu |\lambda| \text{ for all } \lambda \in \Sigma_\nu. \]

By Theorem 1 we know that \((\infty, 0] \subseteq \rho\left(-\frac{\partial}{\partial x} D^\alpha\right)\), which implies that
\[ \Sigma_\nu \cap \rho\left(-\frac{\partial}{\partial x} D^\alpha\right) \neq \emptyset. \]

We may apply Proposition 7 from Appendix to the operator \(-\frac{\partial}{\partial x} D^\alpha\) to obtain that the spectrum of \(-\frac{\partial}{\partial x} D^\alpha\) is contained in \(\mathbb{C} \setminus \Sigma_\nu\), which means that \( \Sigma_\nu \subseteq \rho\left(-\frac{\partial}{\partial x} D^\alpha\right) \) and
\[ \left\| \left( \lambda I - \left(-\frac{\partial}{\partial x} D^\alpha\right) \right)^{-1} \right\| \leq \frac{1}{d(\lambda, S\left(-\frac{\partial}{\partial x} D^\alpha\right))} \leq \frac{1}{c_\nu |\lambda|} \text{ for all } \lambda \in \Sigma_\nu. \]

Thus, the set \( \{ \lambda \in \mathbb{C} : |\arg \lambda| < \pi - \nu \} \subseteq \rho\left(\frac{\partial}{\partial x} D^\alpha\right) \) and
\[ \left\| \left( \lambda I - \frac{\partial}{\partial x} D^\alpha \right)^{-1} \right\| \leq \frac{1}{c_\nu |\lambda|} \text{ for every } \lambda \in \mathbb{C} : |\arg \lambda| < \pi - \nu. \]

Making use of [13, Ch. 2, Theorem 5.2.] we obtain that the semigroup generated by \(\frac{\partial}{\partial x} D^\alpha\) can be extended to the analytic semigroup on the sector \( |\arg \lambda| < \pi - \nu. \)

That way we proved the theorem. \(\blacksquare\)

4 Applications

In this section we will present a simple application of obtained results. We will investigate the solvability of the following parabolic-type problem
\[
\begin{cases}
  u_t - \frac{\partial}{\partial x} D^\alpha u = 0 & \text{in } (0, 1) \times (0, T), \\
  u_x(0, t) = 0, \quad u(1, t) = 0 & \text{for } t \in (0, T), \\
  u(x, 0) = u_0(x) & \text{in } (0, 1).
\end{cases}
\tag{16}
\]

We will formulate the result in the theorem.

**Theorem 3.** If we assume that \( u_0 \in L^2(0, 1) \), then there exists exactly one solution to (16) which belongs to \( C([0, T]; L^2(0, 1)) \cap C^1([0, T]; D_\alpha) \cap C^1((0, T]; L^2(0, 1)) \). What
is more, there exists a positive constant $c$, such that the following estimate holds for every $t \in (0, T]$

$$\|u(t, \cdot)\|_{L^2(0,1)} + t \|u_t(t, \cdot)\|_{L^2(0,1)} + t \left\| \frac{\partial}{\partial x} D^\alpha u(t, \cdot) \right\|_{L^2(0,1)} \leq c \|u_0\|_{L^2(0,1)}.$$  

Moreover, $u \in C^\infty((0, T]; L^2(0,1))$ and for every $t \in (0, T]$, for very $k \in \mathbb{N}$ we have $u(t, \cdot) \in D((\frac{\partial}{\partial x} D^\alpha)^k) \subseteq H^{(\alpha+1)k}(0,1)$. The last inclusion implies that for every $t \in (0, T] u(t, \cdot) \in C^\infty(0,1)$.

**Proof.** Since, we know that the operator $\frac{\partial}{\partial x} D^\alpha$ generates an analytic semigroup, we may apply to (16) the general semigroup theory. Then, the above result follows from [18, Theorem 3.4] and [10, Proposition 2.1.1].

**Remark 2.** One may consider the problem (16) with nonzero right-hand-side. Then the solution is obtained by the variational of constant formula. We may also increase the regularity of solution to (16) assuming higher regularity of the initial condition in a standard way (see for instance [18, chapter 3.2.]).

5 Appendix

We collect here, the results from the general calculus which were used in the paper. At first, we would like to cite the result, concerning distribution of zeros of Mittag-Leffler function from [14].

**Proposition 6.** [14, Theorem 4.2.1] Let us recall the definition of Mittag-Leffler functions

$$E_{\nu,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu+n\nu)}, \text{ with } \mu, \nu \in \mathbb{R}, \nu > 0 \text{ and } E_{\nu}(z) := E_{\nu,1}(z).$$

If we suppose that either

$$\nu < 1, \mu \in [1,1+\nu] \text{ or } \nu \in (1,2), \mu \in [\nu-1,1] \cup [\nu,2],$$

then all roots of the function $E_{\nu,\mu}$ lie outside the angle

$$|\arg z| \leq \frac{\pi \nu}{2}.$$  

At last we quote here Theorem from [13].

**Proposition 7.** [13, Ch.1, Theorem 3.9.] Let $X$ be a Banach space. For a linear operator $A$ in $X$ we define its numerical range $S(A)$ as

$$S(A) = \{(x^*, Ax) : x \in D(A), \|x\| = 1, x^* \in X^*, \|x^*\| = 1, \langle x^*, x \rangle = 1\}$$

Let us assume that $A$ is closed, linear and densely defined in $X$. We denote by $\Sigma := \mathbb{C} \setminus S(A)$. If $\lambda \in \Sigma$ then $\lambda I - A$ is injective and has closed range. Moreover,
if $\Sigma_0 \subseteq \Sigma$ is such that $\Sigma_0 \cap \rho(A) \neq \emptyset$ then the spectrum of $A$ is contained in $\mathbb{C} \setminus \Sigma_0$ and
\[
\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{d(\lambda, S(A))} \text{ for every } \lambda \in \Sigma_0,
\]
where $d(\lambda, S(A))$ is a distance of $\lambda$ from $S(A)$.

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