A lecture note on scale invariance vs conformal invariance

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Abstract

This is a lecture note on scale invariance vs conformal invariance, on which I gave lectures at Taiwan Central University for the 5th Taiwan School on Strings and Fields.

In this lecture note, we discuss the distinction and possible equivalence between scale invariance and conformal invariance in relativistic quantum field theories. As of January 2013, our consensus is that there is no known example of scale invariant but non-conformal field theories in $d = 4$ under the assumptions of (1) unitarity, (2) Poincaré invariance (causality), (3) discrete spectrum in scaling dimension, (4) existence of scale current and (5) unbroken scale invariance. We have a perturbative proof based on the higher dimensional analogue of Zamolodchikov’s $c$-theorem, but the non-perturbative proof is yet to come. We give a complementary holographic argument to support the claim.

We also try to make this lecture note a good reference for examples of scale invariance without conformal invariance. We have tried to collect as many interesting examples as possible. I appreciate your comments very much.\footnote{Reach me at nakayama@theory.caltech.edu}
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1 Introduction

In elementary school, we learn a rectangle is not a square. In graduate school, we learn (?) scale invariance is conformal invariance. In the era of AdS/CFT, everybody touts conformal invariance, so without much reflection we are somehow accustomed to the “belief” that scale invariant quantum field theories show conformal invariance. I confess I was, too. But, hey, has it been proved? Are you sure your beloved $\mathcal{N}=4$ supersymmetric Yang-Mills theory is really conformal invariant?

Of course scale invariance does not imply conformal invariance at least at the level of the superficial mathematical definition. Otherwise, we do not need two different names for the identical concept. However, our nature may be more beautiful than we naively expect. Our ultimate goal of the lectures is to understand the mysterious symmetry enhancement in quantum field theories and gravitational systems.

Indeed, this may happen. In general relativity, for instance. there is a famous theorem by Israel [1] that states all axisymmetric static black holes must be spherically symmetric in $d=4$ (i.e. Schwartzshild black hole solution) as vacuum solutions of Einstein’s equation. This is a symmetry enhancement from the axisymmetry to the spherical symmetry due to the dynamics of the classical gravitational system. Presumably it has a deep quantum gravitational origin. The black hole has no hair, and any classical probe cannot distinguish the microscopic degrees of freedom. We expect that the symmetry enhancement does not occur with no good reason.

Scale invariance is ubiquitous in our nature. We can easily find them in our daily lives. The coastline, snowflakes, lightening, and stock charts, all show scale invariance or fractal structure. My favorite one is a vegetable called roman cauliflower (a.k.a broccoli romanesque). The (discrete) scale invariance here is realized as a self-similarity: if we look at the same system closer or further away, it looks similar. Does our society show a self-similar organization structure? The repetitive structure begs the question: is there any fundamental component in such a self-similar or scale invariant object? Or, is the self-similar structure itself the fundamental organizing principle?

The most notable application of the scale invariance in theoretical physics is the renormalization group flow. One of the central dogmas in the 20th century physics was Wilson’s renormalization group (see e.g. [2]). In a plain word, his idea of the renormalization group is a successive application of scale transformation and coarse graining. If we are interested in the long range universal physics, we can then integrate out the short-range degrees of freedom that might show non-universal dynamics. Afterwards we can talk about the effective theory of the long range degrees of freedom by keeping only relevant degrees of freedom of the theory and focusing on the relevant parameters. All the detailed short range information is judiciously encoded in the process of the renormalization of the relevant parameters.

Schematically, renormalization group transformation is realized as the path integral form:

$$e^{-S_{\text{eff}}[\bar{\Phi};g(\Lambda)]} = \int_{\Lambda}^{\Lambda_0} D\Phi e^{-S_0[\Phi;g(\Lambda_0)]}$$

where $\Lambda$ is the renormalization scale. We integrate out the “high energy degrees of freedom” from the scale between $\Lambda_0$ and $\Lambda$ contained in a field $\Phi$ by keeping only the low energy mode below $\Lambda$. We expect that when we take $\Lambda$ sufficiently small, there are only a few universal parameters that will characterise the system. The success of the idea of the renormalization group transformation explains why we can understand our world with sufficient accuracy even without knowing the most

\footnote{If you are unfamiliar with the vegetable, please search online. It is good with pasta, in particular with oil-based source.}
detailed elementary microscopic physics (e.g. string theory?). Without exaggeration, this is how our standard model works in elementary particle physics, how Einstein’s general relativity works in gravitational physics, how the hydrodynamics works in such a way that our plane flies, and how the idea of universality of the statistical mechanics work in condensed matter physics.

From Wilson’s renormalization group viewpoint, it may be natural that we would expect a scale invariant fixed point in the infrared limit. The intuition is that if we encounter any non-trivial energy scale, we can simply integrate out the corresponding degrees of freedom. Eventually, there should be no scale at all! As a consequence, the universality class of the long range behavior in quantum field theories or many body systems is characterized by the fixed point of the renormalization group flow. This idea has a great success in particular in \( d = 2 \), where the universality combined with conformal invariance made it possible to classify the critical phenomena.

What is conformal invariance? We will describe the mathematical definition of the conformal invariance momentarily. An intuitive idea can be guessed from the root: “con-formal” comes from a Latin word \( \text{conformalis} \) “having the same shape”. It is the transformation that leaves the size of the angle between corresponding curves unchanged. Such transformations are more general than the scale transformation. See fig 1.

Therefore there is no a priori reason why a given scale invariant system must show conformal invariance. Nevertheless, as I mentioned at the most beginning, there should be a reason why theoretical physicists (our smart colleagues) have not made a clear distinction in our everyday research. I believe it is because we have an empirical knowledge that almost all scale invariant quantum field theories that we know show conformal invariance so there is no point in emphasizing it in our textbooks. Do we have to talk about unicorns or dragons in zoology lectures (even though they might exist in principle due to our limited knowledge)?

But it is still a mystery: why does scale invariance have to accompany conformal invariance? The aim of our lectures is to uncover the puzzle behind the enhancement of conformal invariance from scale invariance. As we will see, the underlying reason must be related to a deep property of the renormalization group flow. In particular, the notion of irreversibility of the renormalization group flow and counting degrees of freedom will play a crucial role in our discussions.

The idea of irreversibility of the renormalization group flow can be understood in an intuitive way: as mentioned, the renormalization group flow is accompanied with a kind of coarse graining. We lose information along the flow. It is very counterintuitive if the renormalization group flow shows cyclic or chaotic behavior (although it was envisaged by the pioneers \[3\][4]). See fig 2 for illustration. The field theory understanding of this coarse graining is supported by the so-called “c-theorem” that dictates there exists a function that monotonically decreases along the renormalization group flow. Roughly speaking, this function counts the degrees of freedom at a given energy scale. If such a function exists, the cyclic or chaotic behavior in renormalization group flow cannot occur. In relativistic quantum field theories in \( d = 2 \), there is a proof that such a function indeed exists and the renormalization group flow is irreversible \[3\].

As we will see, scale invariant but non-conformal field theories are intimately connected with the possibility of such a cyclic or chaotic behavior in the renormalization group flow. At least this is the case within perturbation theory in \( d = 4 \) as first emphasized in \[4\]. There is a clear tension between them. The above mentioned proof in \( d = 2 \) implies that scale invariance must be enhanced to conformal invariance in \( d = 2 \). In these lectures, we would like to report the current status of the situations in higher dimensions. Unfortunately, as of January 2013, there is no compelling non-perturbative proof in higher dimensions with no counterexamples reported so far under some reasonable assumptions.
Figure 1: We see a graphical distinction between scale invariance and conformal invariance in $d = 2$. Our perception is approximately invariant under scale transformation but not invariant under conformal transformation. Do you think conformal transformation keeps the “same shape”?

To look for a non-perturbative evidence, we will study the holographic argument. This is the second aim of our lectures. Holographic principle is by far the most profound but conjectural principle that connects non-gravitational quantum physics and the corresponding (quantum) gravity. It has a beautiful concrete realization known as AdS/CFT correspondence, and we have culminating evidence that it is true. Our idea is to explore the hidden side of the quantum field theories from the analysis in gravitational backgrounds. According to the AdS/CFT correspondence, the classical gravity will describe a certain strongly coupled limit of the dual quantum field theories, and it is expected that it provides non-perturbative understanding of them.

As I mentioned in the black hole example, gravitational systems show their own symmetry enhancement mechanism. We conjecture that the consistency of the quantum gravity is encoded in the consistency of the renormalization group flow through the holographic equivalence, and vice versa. The $c$-function associated with the renormalization group flow can be viewed as “entropy” of the gravitational system, which should be monotonically decreasing along the evolution. Along the same
Figure 2: We show artificially generated examples of (possible?) renormalization group flow. The left hand side contains UV fixed point as well as IR fixed point. The right hand side shows cyclic behavior with UV fixed point.

... line of reasoning, we will argue that it is reasonable that scale invariant holographic configurations show further enhanced conformal invariance.

At the same time, we would like to ask some pertinent questions in quantum gravity. What would be the fundamental mechanism to exclude seemingly pathological geometries from quantum gravity such as superluminal propagation of information, closed time-like curves, and so on? We believe that the consistency of the renormalization group flow e.g. absence of the limit cycle or chaotic behavior would give a hint to understand the fundamental aspects of quantum geometry and quantum gravity through the holography. Our earlier attempt to discuss the issue from the holographic approach is summarized in a review paper [7].

There was a debate whether scale invariance without conformal invariance is possible or not in $d = 4$ over the last couple of years, but I’m happy to announce that we converge to the point our holographic argument predicts [8][9].

1.1 Ancient History of Conformal Field Theory

The following quote is taken from A. A. Migdal’s historic remark [11]. It is my great pleasure to express my gratitude to Prof. Migdal for his kind permission to quote the great anecdote in our lecture note. I vividly recall how I got more attracted to the subject when I first read the quote.

... Conformal Field Theory was the next step of development of the idea of anomalous dimensions, based on remarkable observation that one-dimensional scale invariance in local Euclidean Field Theory necessarily leads to a wider symmetry, with 15 parameters in our four dimensions (including
translations and rotations). The generators of conformal symmetry are related to various conserved currents

\[ D_\nu = x^\mu \Theta_{\mu \nu}, \quad \partial^\mu D_\mu = 0; \]  
(1.2)

for dilatations

\[ K_{\mu \nu} = (x^2 \delta^\lambda_{\mu} - 2x_{\mu}x^\lambda)\Theta_{\lambda \nu}, \quad \partial^\nu K_{\mu \nu} = 0; \]  
(1.3)

for special conformal transformations, etc.

Here the conserved symmetric stress-energy tensor \( \Theta_{\mu \nu} \) is in addition traceless in case there is scale invariance (no massive fields), which leads to conservation of both of these currents simultaneously.

This invariance leads to very specific predictions such as explicit form of 3-point correlation functions and vanishing two-point correlations between fields with different dimensions.

It happened so, that there was an International Conference in Dubna, the main topic of which was so-called scale symmetry, promoted with great fanfare by the Bogoliubov School of Thought. This scale symmetry was mostly a political slogan, good for dissertations and career moves but not for any practical applications in a world of Physics.

After the Plenary Session devoted to the Scale Symmetry, one of the Western Physicists asked the speaker: “What is the difference between Scale Symmetry and Conformal Symmetry” Apparently, the rumors about new symmetry were already spread, so this was what KGB used to call “provocative question”.

The speaker hesitated, but the Chairman of the Session, great mathematician N.N. Bogoliubov took the microphone and said literally the following: “There is no mathematical difference, but when some young people want to use a fancy word they call it Conformal Symmetry”.

Obviously, his ignorant lieutenants misinformed him, and he did not bother to look up for himself what was the Conformal Symmetry.

I could not stand it any longer. I raised the hand to give everybody brief introduction to Conformal Symmetry (naturally, nobody invited any of us, suckers, to speak at such an important International Conference, but I was allowed into the audience). Vigilant Organizers of the Conference ignored my raised hand, the break was quickly announced, so that my indignant cry: “15 parameters!” went apparently unnoticed. (By the way, somebody told me recently that he heard that cry and wondered for years what could that mean, until he learned conformal symmetry).

Here is an interesting part. I came home and said to my father: “Look, what a fool N.N is really is” then I told him the story of 15 parameters. My father laughed with me; surely he knew what Conformal Symmetry was about; then he said something remarkable: “You know, Sasha, there are two kinds of intellect. The first kind helps you to say smart things. But the second kind helps you to do smart things. N.N used to have great intellect of the first kind, but later he switched to the second kind. Do you think he cares about parameters of Conformal Group? He is involved in Big Science, where political truth is more important than scientific truth. You would do yourself some good by borrowing the second kind of intellect from N.N.”

I wish I would know how to follow this wise advice! Fortunately, in my new life there are lawyers around to zip my mouth when the smart thing would be to keep it shut.
I respect both Prof. Migdal and Prof. Bogoliubov and we will see that the story around scale invariance and conformal invariance shows various twists and turns and it will eventually reveal beautiful structures of quantum field theories and the space-time.

1.2 Plan of lectures

In Lecture 1, we would like to begin with the definition of scale invariance and conformal invariance, and give criteria to distinguish between the two. Then we show various examples of scale invariant systems with or without conformal invariance. In Lecture 2, we try to give a field theoretic argument why scale invariant phenomena typically show enhanced conformal invariance under several assumptions. Unfortunately, the argument known in the literature is not complete in non-perturbative regimes, so we hope interested readers will contribute to the subject in the future. In Lecture 3, we give holographic discussions on the distinction and show possible equivalence between scale invariance and conformal invariance from the properties of the space-time dynamics.

- As broader audience in mind, let us fix our notation for the space-time dimensionality. When we say $d$-dimension, it means either $d$-dimensional Euclidean (statistical) system, or relativistic $d$-dimensional space-time which is given by $d - 1$ dimensional space and 1 dimensional time. $\mu = 0, 1, 2 \cdots d - 1$ refers to Lorentz index with the Minkowski metric $\eta_{\mu \nu} = (- + + \cdots)$. We also work in the Euclidean signature field theories with no particular mentioning of Wick rotation $x_d = ix_0$. In the Euclidean signature, we use $\mu = 1, 2 \cdots d$ with the Euclidean metric $\delta_{\mu \nu} = (+ + + \cdots)$. The antisymmetrization of the tensor indices is represented by $[IJ]$, and the symmetrization is represented by $(IJ)$. Only when we discuss field theories in $d = 2$, we will use the complex coordinate which will be explained in section 3.1.

- In Lecture 3, we add the extra holographic coordinate and use $M = 0, 1, \cdots d, r$ or $M = 0, 1, \cdots d, z$ referring to $d + 1$-dimensional space-time coordinate. We also work in the Euclidean signature as above.

- We use the natural unit: $\hbar = c = 1$. In Lecture 3, we further assume the Planck constant $\kappa_d = 1$.

- Otherwise stated, the summation convention of Einstein is used (e.g. $A_\mu B^\mu \equiv \sum_\mu A_\mu B^\mu$). The coordinate indices are raised and lowered with an appropriate metric $g_{\mu \nu}$. Such a “metric” may or may not be naturally available when tensor indices take the value in more abstract spaces such as “coupling constant spaces”, and we will be explicit about raising and lowering indices then.

- Our metric and curvature convention is same as that of Wald’s textbook \[12\]. See Appendix A.1 for more about our conventions.

- In most of the sections dealing with quantum interacting field theories, we assume fields appearing in various formulae are all finitely renormalized. Although we do give some basic explanations of the renormalization procedure in the lectures, it is beyond our scope to perform the explicit

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3I should add that K. Wilson later recalls at the Nobel lecture that he was inspired by the most mysterious part of the textbook by Bogoliubov and Shirkov on the leading order summation of logs (i.e. “renormalization group”) for his later study of the subject. A. Migdal and A. Polyakov were among the first who understood that $3d$ phase transitions and $4d$ relativistic theory were the same theory in different space dimension by analytic continuation.

4With this regards, we will not discuss some subtle aspects of the global conformal transformation in Minkowski space-time. See section 2.8.
renormalization program, and we refer to textbooks (e.g. [13], [14], [15], [16]). Unfortunately, at a certain point, we have to go beyond the textbook treatment because the distinction between scale invariance and conformal invariance is so subtle. We hope interested readers will find reference provided throughout the lectures useful and fill the gap if necessary.

• When we talk about quantum gauge theories, the gauge fixing procedure will be always implicit. After the gauge fixing, we have to add various terms both in the action and the energy-momentum tensor. However, all these terms (that could violate scale invariance or conformal invariance) are BRST trivial, so the physical discussions will not be affected.

• We do not cover various techniques in conformal field theories developed in particular in \( d = 2 \). We refer to [17], [18] and references therein. We also refer to the lecture note [19] for a complementary approach in higher dimensional conformal field theories.

• We try to avoid spinors and supersymmetry as much as possible. In some advanced sections, we assume some basic knowledge. For supersymmetry, we refer to textbooks [20], [21] and a lecture note [22]. (But in reality, I’m more afraid that most of the high-energy-theory-oriented readers know what the \( R \)-current is, while they don’t know what the virial current is.) In various symbolic formulae such as a field variation \( \delta S / \delta \phi \), we implicitly pretend as if \( \phi \) were bosonic and suitable modifications are necessary for anti-commuting fermionic fields.

• The discussion on the holographic approach in Lecture 3 is relatively independent. Lecture 1 and Lecture 2 are self-contained within field theory arguments. Those who are not interested in holography can skip Lecture 3 entirely. On the other hand, although understanding of Lecture 3 requires some basic facts presented in Lecture 1 and Lecture 2, one may directly start with Lecture 3 to grasp the holographic approach. In the latter case, we also recommend [24] for reference.

• A few exercises are scattered throughout the lecture note for some random reasons. They are boxed such as

(Exercise) Find as many typos and sign errors in the lecture note as possible⁴

⁴Don’t forget to report them!

Most of them are easy, and they are more or less irrelevant for the main discussions of the lectures. However, I think some of them are fun problems to solve.

• If you find the reference list incomplete, please let me know. In particular, I’m more on the high energy physics side, so if you find anything missing from the statistical mechanics side, I welcome a lot.

• Minor revisions will be updated at https://sites.google.com/site/scalevsconformal/

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⁴It is my pleasure to recall that I first encountered the mentioning of distinction between scale invariance and conformal invariance in the lecture note [24] when I was a grad student, which refers to the original paper [23].


2 Lecture 1

In this lecture, we begin with the mathematical distinction between scale invariance and conformal invariance, and discuss the criteria to distinguish them in quantum field theories. We then show various examples of scale invariant field theories that often (but not always) show conformal invariance. In the examples we will discuss, I tried to collect all examples of scale invariant but non-conformal relativistic field theories as far as I have recognized.

2.1 Conformal algebra as maximal bosonic space-time symmetry

In special relativity, we postulate the Poincaré algebra as the most fundamental symmetry of our space-time:

\[
\begin{align*}
[i J^\mu, J^\rho] &= \eta^{\mu\nu} J^\nu - \eta^{\mu\rho} J^\nu - \eta^{\nu\sigma} J^\rho + \eta^{\nu\nu} J^\mu \\
[i P^\mu, J^\rho] &= \eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho \\
[P^\mu, P^\nu] &= 0 .
\end{align*}
\]  

(2.1)

In quantum mechanics, they are realized by Hermitian operators acting on a given Hilbert space. The representation of Poincaré algebra in terms of particles will naturally lead to the formalism of the quantum field theory \[15\].

For a massless scale invariant theory, one can augment this Poincaré algebra by adding the dilatation operator \(D\) as

\[
\begin{align*}
[P^\mu, D] &= i P^\mu \\
[J^\mu, D] &= 0 .
\end{align*}
\]  

(2.2)

We use the terminology “dilatation” and “scale transformation” interchangeably in our lecture note. The representation theory naturally leads to the notion of unparticles \[24\][25]. The theory of unparticles sometimes relies on a delicate difference between scale invariance and conformal invariance (see e.g. \[26\]).

The generalization of the Coleman-Mandula theorem \[27\][28] asserts (for \(d \geq 3\)) that the maximally enhanced bosonic symmetry of the space-time for massless particles is obtained by adding the special conformal transformation \(K^\mu\):

\[
\begin{align*}
[K^\mu, D] &= -i K^\mu \\
[P^\mu, K^\nu] &= 2i \eta^{\mu\nu} D + 2i J^{\mu\nu} \\
[K^\mu, K^\nu] &= 0 \\
[J^{\rho\sigma}, K^\mu] &= i \eta^{\rho\sigma} K^\mu - i \eta^{\mu\sigma} K^\rho .
\end{align*}
\]  

(2.3)

As is clear from the group theory structure above, the conformal invariance demands scale invariance from the closure of the algebra in (2.3) but the converse is not necessarily true: scale invariance does not always imply conformal invariance. However, in many field theory examples as we will show in section 2.4, we typically observe the emergence of the full conformal invariance rather than the

\[6\] With this assertion, we have to be careful about the assumption in the Haag-Lopuszanski-Sohnius theorem. In particular, the analyticity assumption of S-matrix can be violated with infrared divergence in interacting conformal field theories. For example, as noted in Weinberg’s textbook, the validity of the Haag-Lopuszanski-Sohnius theorem (even the Coleman-Mandula theorem) for the Banks-Zaks fixed point has not been proved \[21\].

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scale invariance alone. The aim of these lectures is to uncover deep dynamical reasons behind the enhancement of the symmetry from scale invariance to conformal invariance.

(Exercise) Count the number of generators for the conformal group in \( d = 4 \) to show it is 15.

2.2 Space-time symmetry and energy-momentum tensor

In section 2.1, we introduced the symmetry of a given quantum system as algebra of conserved charges that act on the Hilbert space (or S-matrix in the Haag-Lopuszanski-Sohnius theorem). In quantum field theories, we postulate that these symmetries are realized by conserved currents. Obviously, if a current \( j_\mu \) is conserved: \( \partial_\mu j_\mu = 0 \), one can construct the conserved charge

\[
Q = \int d^{d-1}x j_0 .
\]  

(2.4)

Strictly speaking, this is not necessary, but this assumption (Noether assumption) covers most of interesting examples we will discuss in this lecture note. The assumption implies that we will exclusively consider local quantum field theories. We will not discuss, for instance, conformal or scale invariant string (field) theories (or more generally membrane field theories if any).

With the Noether assumption, the translational invariance means that the theory possesses a conserved energy-momentum tensor:

\[
\partial_\mu T_{\mu\nu} = 0
\]  

(2.5)

The Lorentz invariance further demands that the energy momentum tensor can be chosen to be symmetric (known as the Belinfante prescription):

\[
T_{\mu\nu} = T_{\nu\mu}
\]  

(2.6)

so that the Lorentz current \( J_\mu^{M\nu} = x^\mu T_\mu^{\nu} \) is conserved.

The scale invariance \( (x^\mu \to \lambda x^\mu) \) requires that

\[
T_\mu^{\nu} = \partial_\mu J_\nu
\]  

(2.7)

so that \( D_\mu = x^\rho T_{\mu\rho} - J_\mu \) is the conserved scale current (dilatation current). As we will see, roughly speaking, the first term generates the space-time dilatation while the second term generates the additional scaling of fields to preserve the total scale invariance of the theory. The current \( J_\mu \) is known as the virial current. The word “virial” is derived from Latin “vis” meaning “power” or “energy”. According to Wikipedia, it was Clausius who first used the name with the definition \( \sum x^i p_i \), which reminds us of the virial current for a free scalar field theory we will describe in section 2.4.1. Probably it refers to the “internal” degrees of freedom responsible for the scale transformation like those in “molecules”.

\[8\]The reason why they discussed the symmetry of the S-matrix rather than the symmetry of the Hilbert space is based on the hypothesis that the Hilbert space may not be good observables in relativistically interacting systems. According to the purists at the time, only S-matrix was observable. We will not be so pedantic about it.

\[9\]Actually, we do not have to assume that the energy-momentum tensor is symmetric: We do not need Lorentz invariance for scale invariance.
The special conformal invariance is a symmetry under
\[ x^\mu \rightarrow x^\mu + v^\mu x^2 \frac{1}{1 + 2v^\mu x_\mu + v^2 x^2} . \]  
(2.8)

It requires that the energy-momentum tensor is traceless:
\[ T^\mu_\mu = 0 \]  
(2.9)

so that we can construct the special conformal current \( K^{(\rho)}_\mu = [\rho_\nu x^2 - 2x_\nu (\rho_\sigma x^\sigma)] T^{\nu}_\mu \) .

In the literature, it is often claimed that the inversion and translation generates the full conformal transformation. This is true because \( K_\mu = IP_\mu I \) with \( I \) generating space-time inversion \( x^\mu \rightarrow x^\mu x^2 \), but the converse may not hold. Invariance under the conformal algebra does not imply invariance under the inversion (see e.g. [30][31] for a related comment). The point is that inversion is a disconnected component of the conformal group and it is only an outer automorphism. We can see it explicitly if we recall that the action of inversion on the cylinder \( S^{d-1} \times \mathbb{R}^1 \) is given by the time reversal (on \( \mathbb{R}^1 \)) in the radial quantization of conformal field theories. We refer e.g. to [17][18] for the radial quantization in \( d = 2 \). The similar construction is possible in any \( d \geq 2 \) [32]. The time-reversal may or may not be a symmetry of the theory on \( S^{d-1} \times \mathbb{R}^1 \).

Energy-momentum tensor is not unique: one can improve it without spoiling the conservation law (2.5):
\[ T_{\mu\nu} \rightarrow T_{\mu\nu} + \partial^\rho B_{\mu\nu\rho} \quad B_{\mu\nu\rho} = -B_{\rho\mu\nu} . \]  
(2.10)

Belinfante showed [33] that by using the ambiguity, one can always make it symmetric when the theory is Poincaré invariant (see also [34]) by explicitly constructing \( B_{\mu\nu\rho} \) from the spin current
\[ B_{\mu\nu\rho} = \frac{1}{2} (s_{\nu\rho\mu} + s_{\mu\rho\nu} + s_{\mu\nu\rho}) \]  
(2.11)

where the Lorentz current is given by \( j^M_\rho = x^{[\mu} T_\rho^{\nu]} + s^{\mu\nu}_{\rho} \) with possibly non-symmetric energy-momentum tensor \( \tilde{T}_{\mu\nu} \).

The non-uniqueness of the energy-momentum tensor has an important consequence in conformal invariance. Suppose the energy-momentum tensor is given by
\[ T^\mu_\mu = \partial^\mu \partial^\nu L_{\mu\nu} \quad (d \geq 3) \]
\[ T^\mu_\mu = \partial^\mu \partial_\mu L \quad (d = 2) \]  
(2.12)

with certain local operators \( L_{\mu\nu} \) and \( L \), then by using this ambiguity, one can define the improved energy-momentum tensor (see e.g. [29][35][23])
\[ \Theta_{\mu\nu} = T_{\mu\nu} + \frac{1}{d-2} \left( \partial_\mu \partial_\alpha L^\alpha_{\nu} + \partial_\nu \partial_\alpha L^\alpha_{\mu} - \Box L_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta L^\alpha_{\beta} \right) \]
\[ + \frac{1}{(d-2)(d-1)} \left( \eta_{\mu\nu} \Box L^\alpha_{\alpha} - \partial_\mu \partial_\nu L^\alpha_{\alpha} \right) \]  
(2.13)

for \( d \geq 3 \), and
\[ \Theta_{\mu\nu} = T_{\mu\nu} + \eta_{\mu\nu} \Box L^\alpha_{\alpha} - \partial_\mu \partial_\nu L^\alpha_{\alpha} \]  
(2.14)

\(^9\)Note that unlike scale invariance, we have to assume that the energy-momentum tensor is symmetric here. We need the Lorentz invariance for the special conformal invariance to exist as can be seen from the algebra [2.3].
for \( d = 2 \). The improved energy-momentum tensor is traceless (as well as symmetric and conserved). Thus the precise condition for the conformal invariance is (2.12). The traceless energy-momentum tensor may not be unique because we can still add \( \partial^\rho \partial^\sigma \Upsilon_{\mu\rho\nu\sigma} \) with \( \Upsilon_{\mu\rho\nu\sigma} \) possessing the symmetry of Weyl tensor (symmetry properties of Riemann tensor plus traceless condition. See appendix A.1). When there is such a possibility, a different choice will give a different Weyl invariant theory in the curved background as we will describe more about the Weyl transformation in the next section.\(^{10}\) If we allow more than two derivative modifications of the energy-momentum tensor, it is possible to introduce further higher derivative improvement terms in \( d > 4 \), but we will not discuss them in the lecture note. As we will see in section 2.2.3, the unitarity demands that the only allowed important term in unitary theory in \( d \geq 3 \) is from \( L_{\mu\nu} = \eta_{\mu\nu} L \) with a dimension \( d - 2 \) scalar operator \( L \) if we demand the energy-momentum tensor has the canonical dimension \( d \).

### 2.2.1 Curved background

So far, we have discussed the space-time symmetry of the flat Minkowski (or Euclidean) space-time. The above mentioned properties of the energy-momentum tensor are more succinctly derived in the curved background. The important fact is that the energy-momentum tensor is a source of gravity through the coupling to the metric \( g_{\mu\nu} \) in general relativity. Suppose we can couple our Poincaré invariant quantum field theory to a general covariant gravitational theory (not necessarily in a unique fashion). Let us conventionally define the energy-momentum tensor for matter fields \( \Phi \) as

\[
T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}}, \quad T^{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}},
\]

where \( S = \int d^d x \sqrt{|g|} \mathcal{L}(\Phi, g_{\mu\nu}) \) is the matter action with the action density \( \mathcal{L}(\Phi, g_{\mu\nu}) \).\(^{12}\) The diffeomorphism invariance of the action \( S \) automatically gives the conservation \( D^\mu T_{\mu\nu} = 0 \) with the covariant derivative \( D^\mu \), and the symmetry \( T_{\mu\nu} = T_{\nu\mu} \). Of course, when the theory contains a spinor, we have to consider the spin connection with vielbein \( e^a_\mu \) as fundamental gravitational degrees of freedom (rather than the metric \( g_{\mu\nu} = e^a_\mu e^a_\nu \) itself), but the generalization is obvious.

If the action is scale invariant or constant Weyl invariant, i.e. \( g_{\mu\nu} \rightarrow e^{2\tilde{\sigma}} g_{\mu\nu} \), where \( \tilde{\sigma} \) is a space-time independent constant, then the action density is scale invariant up to a total derivative term \( \delta \mathcal{L} = -\tilde{\sigma} D^\mu J_\mu \), so

\[
T^\mu_\mu = \frac{2}{\sqrt{|g|}} g^{\mu\nu} \frac{\delta S}{\delta g^\mu_\nu} = D^\mu J_\mu.
\]

This is the origin of the virial current.

On the other hand, if the action is Weyl invariant, i.e. \( g_{\mu\nu} \rightarrow e^{2\sigma(x)} g_{\mu\nu} \), where \( \sigma(x) \) is space-time dependent arbitrary scalar function, then the action density itself must be invariant, so the energy-momentum tensor is traceless.

\[
T^\mu_\mu = \frac{2}{\sqrt{|g|}} g^{\mu\nu} \frac{\delta S}{\delta g^\mu_\nu} = 0.
\]

\(^{10}\)Fortunately, the unitarity of the operator dimension restrict the possibilities of improvement, and we will not encounter such inequivalent Weyl invariant theories in \( d \geq 3 \) with the assumption of unitarity.

\(^{11}\)Another common definition of the energy-momentum tensor is the Noether energy-momentum tensor: \( T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial^\mu \Phi)} \partial^\mu \Phi - \eta_{\mu\nu} \mathcal{L} \) for the classical two-derivative action \( S = \int d^d x \mathcal{L}(\Phi, \partial_\mu \Phi) \). It is related to the energy-momentum tensor in general relativity (2.15) by improvement we discussed in the last section.

\(^{12}\)Due to our convention, the action density is minus of the Lagrangian density in the Lorentzian signature.
We recall that conformal killing vectors of the \(d\)-dimensional \((d > 2)\) Minkowski space-time are generated by the conformal algebra \(so(2, d)\) (see e.g. [17] [18]). They are precisely translation, Lorentz rotation, dilatation, and special conformal transformation. The conformal killing vector induces the diffeomorphism

\[
ds^2 = \Omega(\tilde{x})\eta_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu = \eta_{\mu\nu}dx^\mu dx^\nu
\]

that makes the metric invariant up to overall Weyl factor \(\Omega\). When the theory is Weyl invariant, if we restrict ourselves to the Minkowski space-time, the above argument implies that the theory under study is automatically conformal invariant [16]. Since the energy-momentum tensor may not be unique for a given conformal field theory, the converse may not be true: we might be able to couple a conformal field theory to gravity in a non-Weyl invariant way. We will see a free scalar example in section 2.4.1.

To close this section, let us briefly discuss the interpretation of the improvement of the energy-momentum tensor. If we add the curved space action

\[
S_{\text{imp}} = \int d^dx \sqrt{|g|} (RL + R_{\mu\nu}L^{\mu\nu} + R_{\mu\nu\rho\sigma}L^{\mu\nu\rho\sigma}) ,
\]

the energy-momentum tensor defined in (2.17) obtains extra contributions such as \((\partial_\mu \partial_\nu - \eta_{\mu\nu}\Box)L\) in the flat space-time limit \(g_{\mu\nu} = \eta_{\mu\nu}\). This is the origin of the improvement terms and ambiguities of the energy-momentum tensor from the curved space-time viewpoint. In a certain situation, by choosing appropriate terms in (2.19), one may be able to construct the traceless energy-momentum tensor, and then the theory is Weyl invariant.

### 2.2.2 Consequence of conformal invariance

- The conformal Ward-Takahashi identity constrains the forms of correlation functions. For a review, we refer to [17] [18] for \(d = 2\). In higher dimensions, the discussion in [37] would be the most comprehensive one except that they assume invariance under inversion and CP transformation. Two-point functions of primary operators are diagonal with respect to their conformal dimensions. For instance, scalar two-point functions are given by

\[
\langle O_1(x_1)O_2(x_2) \rangle = \frac{c_{12}\delta_{\Delta_1\Delta_2}}{(x_1 - x_2)^{2\Delta_1}}. \tag{2.20}
\]

Three-point functions of scalar primary operators are uniquely fixed [38] [39] [40]

\[
\langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle = \frac{c_{123}}{(x_1 - x_2)^{-\Delta_1+\Delta_2+\Delta_3}(x_2 - x_3)^{-\Delta_1+\Delta_2+\Delta_3}(x_1 - x_3)^{-\Delta_1+\Delta_2+\Delta_3}}. \tag{2.21}
\]

- Due to the unitarity constraint, the conformal dimension of primary operators are bounded [11] (see [12] for a pedagogical review). For instance in \(d = 4\), the conformal dimension of primary operators with \((j_1, j_2)\) Lorentz spin must satisfy

\[
\Delta \geq j_1 + j_2 + 2 - \delta_{j_1,j_2,0} . \tag{2.22}
\]

\(^{13}d = 2\) gives the infinite dimensional Virasoro algebra.
The four-point functions of conformal field theories satisfy the beautiful bootstrap equations. Let us consider the simplest one \(x_{ij} = x_i - x_j\) for a shorter notation) with four identical scalar operators with conformal dimension \(\Delta\):

\[
\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle = \frac{g(u, v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}} \tag{2.23}
\]

with \(u = \frac{x_{12}^2 x_{23}^2}{x_{13} x_{24}}\), \(v = \frac{x_{12}^2 x_{24}^2}{x_{14} x_{23}}\). From the operator product expansion (OPE) \(^{13}\) (see also \(^3\) for the earlier application of the OPE in conformal field theories)

\[
O(x)O(0) = \sum_i C_{OOi} \mathcal{C}(x_\mu, \partial_\mu) O^i(x) \tag{2.24}
\]

where \(\mathcal{C}(x_\mu, \partial_\mu)\) gives the sub-leading non-primary operator contributions fixed by conformal invariance, we see that the four-point function is determined

\[
g(u, v) = \sum_i (C_{OOi})^2 g^{(l)}_\Delta(u, v) \tag{2.25}
\]

where \(g^{(l)}_\Delta(u, v)\) are explicitly known conformal blocks with spin \(l\) and conformal dimension \(\Delta\). On the other hand, the four-point function must satisfy the crossing symmetry

\[
g(u, v) = (u/v)^\Delta g(v, u) \tag{2.26}
\]

With the unitarity constraint, it gives interesting constraints on the spectrum \(\Delta\) and the OPE coefficients \(C_{OOi}\) of the theory. In \(d = 2\), the program was carried out in the seminal work \(^{16}\) thanks to the infinite dimensional extra constraint from the Virasoro symmetry.

### 2.2.3 Consequence of scale invariance alone

- The scale invariance constrains the form of correlation functions, but more weakly than in the conformal invariant case. Two-point functions may not be diagonal with respect to scaling dimensions.

\[
\langle O_1(x_1)O_2(x_2) \rangle = \frac{c_{12}}{(x_1 - x_2)^{\Delta_1 + \Delta_2}} \tag{2.27}
\]

Three-point function takes the less restrictive form (see e.g. \(^3\)):

\[
\langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle = \sum_{\delta_1, \delta_2} c_{123}^{\delta_1, \delta_2} (x_1 - x_2)^{\delta_1} (x_2 - x_3)^{\delta_2} (x_3 - x_1)^{\Delta_{123} - \delta_1 - \delta_2} \tag{2.28}
\]

with \(\Delta_{123} = \Delta_1 + \Delta_2 + \Delta_3\). Note that scaling dimension may not be diagonalizable. Then the form of correlation function will be more complicated.

- Assuming the diagonalizability of the scaling dimension, one can show similar but weaker conditions on scaling dimensions \(^{17}\):

\[
\Delta \geq j_1 + j_2 + 1 \tag{2.29}
\]

\(^{14}\)Historically, there is another “bootstrap equation” that is the Schwinger-Dyson-like self-consistent equations with anomalous dimensions (see e.g. \(^{40}\) for its demonstration in conformal field theories). It is also known as the skeleton expansion. We would like to thank S. Rychkov and A. Migdal for the reference.
• The OPE factorization and the crossing symmetry are not a privilege of conformal field theories. They must be satisfied by the scale invariant field theories as well. However, there are more complications because (1) the two-point functions of higher spin operators are not uniquely fixed by the symmetry, (2) the two-point functions may not be diagonalized with respect to the scaling dimension and (3) therefore OPE can be much more complicated. Thus, at this point, the usage of the bootstrap technique was not fully appreciated in the literature. Of course, the correlation functions must satisfy these conditions.

2.3 Weyl Anomaly

In section 2.5, we learned that local conformal field theories require $T^\mu_\mu = 0$ in the flat space-time. In curved background, however, the conformal field theory breaks Weyl invariance due to the so-called Weyl anomaly [48][49] (see e.g. [50][51] for historical reviews). In $d=2$, it is proportional to the scalar curvature $R$ as

$$\langle T^\mu_\mu \rangle = + \frac{1}{2\pi} \frac{c}{12} R .$$

The constant number $c$ is known as the central charge. It is because in $d=2$, the conformal symmetry is enhanced to the infinite dimensional symmetry known as Virasoro symmetry, and $c$ coincides with the center of the Virasoro algebra. As we will see in Lecture 2, the central charge $c$ has a physical interpretation of counting massless degrees of freedom at the conformal fixed point.

In $d=4$, the most generic possibility of the Weyl anomaly from naive dimensional analysis (without any other background sources) is

$$\langle T^\mu_\mu \rangle = c (\text{Weyl})^2 - a \text{Euler} + b R^2 + \tilde{b} \Box R + d \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{\alpha\beta} R_{\alpha\beta\rho\sigma} ,$$

where Euler $= R^2_{\mu\nu\rho\sigma} - 4 R^2_{\mu\nu} + R^2 C\text{Weyl}^2 = R^2_{\mu\nu\rho\sigma} - 2 R^2_{\mu\nu} + \frac{1}{3} R^2$, $\Box R = D^\mu D_\mu R$ and Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ is an antisymmetric tensor (rather than tensor density). The term $\tilde{b} \Box R$ can be removed by adding a local counterterm $\tilde{b} R^2$ to the effective action, so it is not an anomaly in a conventional sense. In addition, for conformal field theories, we can show $b = 0$ due to the Wess-Zumino consistency condition as we will discuss momentarily. The Pontryagin term $\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{\alpha\beta} R_{\alpha\beta\rho\sigma}$ is consistent, but it breaks the CP transformation [52]. There is no known unitary field theory model that gives the Pontryagin term as Weyl anomaly (the self-dual two-form gauge field theory in the Euclidean signature may be an exception [53]).

All anomalies must satisfy the Wess-Zumino consistency condition. Or more precisely, we can add local non-gauge invariant counterterms so that the effective action satisfies it [54] (see also [52][54] for the discussions of such counterterms in the context of Weyl anomaly: the upshot is we can use them so that the Weyl anomaly is given by diffeomorphism invariant terms). The Weyl transformation is Abelian, and the Wess-Zumino condition for the Weyl variation $\delta \sigma(x) = -2 \int d^4x \sqrt{|g|} \sigma(x) g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)}$ is simply (see e.g. [57][55][58][59])

$$[\delta \sigma(x), \delta \tilde{\sigma}(x')] S_{\text{eff}} [g_{\mu\nu}] = 0 ,$$

and the first order variation is given by the Weyl anomaly $\delta \sigma(x) S_{\text{eff}} [g_{\mu\nu}] = - \int d^4x \sqrt{|g|} \sigma(x) \sqrt{|g|} T^\mu_\mu(x)$ if the theory is conformal invariant in the flat space-time. It is trivial to see that Weyl$^2$ and Pontryagin

\footnote{The factor $-1/2\pi$ is due to our different normalization than the most common convention in the string theory literature. We conventionally normalize $c = 1$ for a free scalar in $d = 2$. There seems no standard convention for the Weyl anomaly in higher dimensions.}
term in the Weyl anomaly satisfy this condition because they are Weyl invariant by themselves. The Euler term non-trivially satisfies the condition after partial integration. However, we can check $R^2$ term does not satisfies the condition due to the term proportional to $\sigma \partial^\mu \sigma' - \sigma' \partial^\mu \sigma$, so $b$ must vanish for Weyl invariant field theories.

For free field theories, the Weyl anomaly can be read from the Schwinger-De Wit computation of the one-loop determinant (see e.g. [60] for a review). Let us consider the Laplace-type differential operator $\Delta$ (e.g. $\Delta = -\Box + \xi R$ acting on a scalar) with eigenvalues $\Delta \phi_n = \lambda_n \phi_n$. We introduce the Schwinger-De Wit heat kernel

$$F(x,y,\rho) = \sum_n e^{-\rho \lambda_n} \phi_n(x) \phi_n(y) .$$

When we expand the diagonal heat kernel

$$F(x,x,\rho) = \sum_m \rho^{m-\frac{d}{2}} \int d^d x \sqrt{g} \phi_m(x) ,$$

we can obtain the regularized one-loop determinant

$$S_{\text{eff}}[g_{\mu\nu}] = \log \text{Det} \Delta - \frac{1}{2} \int d^d x \sqrt{g} \phi_m(x) .$$

Since the Schwinger-De Wit heat kernel computes the one-loop logarithmic divergence that gives renormalization of the corresponding terms in gravity in even dimension (gravitational beta function), we have the formula

$$\langle T^\mu_{\mu} \rangle = b_{d/2}(x,x) .$$

For example, we obtain the Weyl anomaly for a non-minimally coupled scalar in $d = 4$ from (2.33). Note that with this definition of the Weyl anomaly, $bR^2$ term is non-zero if we consider the non-conformal scalar with $\xi \neq \frac{1}{6}$. In (2.33), $\zeta$-function regularization is assumed and a different regularization gives a different coefficient in the $\Box R$ term (see e.g. [61]).

Alternatively, as is the case with all the other anomalies, we may regard the Weyl anomaly as the non-invariance of the path integral measure under the Weyl transformation. After carefully choosing the path integral variables to preserve the diffeomorphism invariance, the free field computation of the anomalous Jacobian gives the same result. See e.g. [62] for a review of the path integral approach.

For reference, we give the free field values for the Weyl anomaly in $d = 4$. The Euler term $a$ for a real scalar, a Dirac fermion, and a real vector is given by $\frac{1}{90(8\pi)^2}$, $\frac{11}{90(8\pi)^2}$ and $\frac{62}{90(8\pi)^2}$ with our normalization. The Weyl$^2$ term $c$ for a real scalar, a Dirac fermion, and a real vector is $\frac{1}{30(8\pi)^2}$, $\frac{6}{30(8\pi)^2}$ and $\frac{12}{30(8\pi)^2}$ with our normalization. It is not immediately obvious which combination of $a$ and $c$ will count the degrees of freedom in $d = 4$ compared with the situation in $d = 2$, where there is no other choice but to use $c$. Note that the number $a$ and $c$ are quite different from the naive “number of helicities” that usually appear in thermal properties of non-interacting massless particles. We will revisit the problem in section 3.2.

Although we will not discuss it further in this lecture, the Weyl anomaly can be generalized to any even dimensions. See [63] for the complete classification. In odd dimensions, we can convince ourselves that there is no Weyl anomaly from the naive dimensional counting. However, there is a more subtle anomaly in contact terms that may be inconsistent with the conformal invariance [64].

16 If the original theory is not conformal invariant in the flat space-time, we should regard the Weyl anomaly here as the additional violation due to the curved background [50].
2.4 Examples

In this section, we will present several examples of scale invariant field theories that may or may not show conformal invariance in various dimensions.

2.4.1 Free theories

A free massless scalar theory in \(d\) dimension has the minimally coupled action

\[
S = \frac{1}{2} \int d^d x \sqrt{|g|} (\partial^\mu \phi \partial_\mu \phi) . \tag{2.37}
\]

The (canonical) energy-momentum tensor

\[
T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}} |_{g_{\mu\nu} = \eta_{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - \frac{\eta_{\mu\nu}}{2} (\partial_\rho \phi)^2 . \tag{2.38}
\]

The trace can be computed as

\[
T^\mu_\mu = \frac{2 - d}{2} (\partial_\mu \phi)^2 = \frac{2 - d}{4} (\Box \phi^2) . \tag{2.39}
\]

In the last line, we have used the equation of motion (EOM). In classical field theories, there is nothing wrong with the usage of equations of motion in deriving conserved currents. Even in quantum mechanics, the equations of motion hold as an operator identity (as long as there is no anomaly) in a suitably renormalized sense.

The free massless scalar is obviously scale invariant. The virial current is given by

\[
J_\mu = \frac{2 - d}{2} \phi \partial_\mu \phi . \tag{2.40}
\]

Moreover, it is conformal invariant in any dimension because \(T_{\mu\nu} = \partial^\mu \partial^\nu L_{\mu\nu}\) with

\[
L_{\mu\nu} = \frac{2 - d}{4} \eta_{\mu\nu} \phi^2 . \tag{2.41}
\]

Indeed, one can improve the curved space action by adding \(\frac{1}{2} \int d^d x \sqrt{|g|} \frac{d-2}{12} R \phi^2\) so that the theory is manifestly Weyl invariant, and the energy-momentum tensor is traceless. The improved action (e.g. \(\frac{1}{2} \int d^4 x \sqrt{|g|} (\Box \phi^2 + \frac{R}{6} \phi^2)\) in \(d = 4\)) is known as conformal scalar action.

Although we can improve the energy-momentum tensor as we wish, there can be a conflict with other symmetries. For instance, a free massless scalar theory can possess the shift symmetry \(\phi \rightarrow \phi + c\). A physically relevant situation is when the massless scalar is given by a Nambu-Goldstone boson. In such a case, it is unnatural to improve the energy-momentum tensor because \(\frac{1}{2} \int d^d x \sqrt{|g|} \frac{d-2}{12} R \phi^2\) term will be incompatible with the shift symmetry. Indeed, the shift symmetry does not commute with the scale transformation or special conformal transformation.

(Exercise) Show free massless Dirac fermion is conformal invariant in any space-time dimension.

---

\[\text{Note:} \text{One exceptional subtlety may be that it is possible that the symmetry algebra may only close up to the equations of motion (on-shell symmetry rather than off-shell symmetry). Correspondingly, we have a so-called zilch symmetry whose variation is proportional to the equations of motion, which does not have the corresponding Noether current. They are related to field redefinition ambiguities.}\]
Another interesting example is free $U(1)$ Maxwell theory in $d$ dimension\cite{65,66}.

\[
S = \int d^d x \sqrt{|g|} \frac{1}{4} F_{\mu \nu} F^{\mu \nu} .
\]

(2.42)

Canonical gauge invariant energy-momentum tensor $T_{\mu \nu} = 2 \frac{2}{\sqrt{|g|}} \delta S |_{g_{\mu \nu} = \eta_{\mu \nu}}$ can be computed as

\[
T_{\mu \nu} = F_{\mu \rho} F_{\nu}^{\rho} - \frac{\eta_{\mu \nu}}{4} (F_{\rho \sigma}^2).
\]

(2.43)

Its trace does not vanish when $d \neq 4$:

\[
T^\mu_{\mu} = \frac{4 - d}{4} (F_{\rho \sigma})^2 = \frac{4 - d}{8} \partial_\mu [A_\rho F^{\mu \rho}] ,
\]

(2.44)

but it is a divergence of a current by using the free Maxwell equation. The massless free vector field is scale invariant with the virial current

\[
J_\mu = \frac{4 - d}{8} A^\nu F_{\mu \nu} .
\]

(2.45)

When $d = 4$, it is well-known that Maxwell theory is conformal invariant. However in other dimensions, we cannot improve the energy-momentum tensor so that it is traceless. Therefore Maxwell theory in $d \neq 4$ is scale invariant, but not conformal invariant.

One peculiar feature of the scale invariance of the free Maxwell theories in $d \neq 4$ is that the scale current $D_\mu = x^\nu T_{\mu \nu} - J_\mu$ is not gauge invariant due to the gauge non-invariance of the virial current (2.43). This is related to the fact that the scale dimension of the vector potential is different from the geometric dimension of 1-form. Because of this fact, strictly speaking, the Noether assumption is violated. Nevertheless the scale charge $D = \int d^{d-1} x D_0$ is obviously gauge invariant (after partial integration of the gauge parameter), and all the correlation functions scale as they should.

One may note that in $d = 3$, something special happens. A free massless vector is dual to a free massless scalar $\phi$ in $d = 3$ by dualizing $F_{\mu \nu} = \epsilon_{\mu \nu \rho} \partial \phi$ with the dual action $\int d^3 x \partial^\mu \phi \partial_\mu \phi$, so one may reformulate it with scalar fields, and we can see that the virial current is then given by $J_\mu \sim \partial_\mu (\phi^2)$. Note that the dual scalar must accompany the gauged shift symmetry, so the theory cannot be Weyl invariant (because would-be improvement term is not gauge invariant). It is still embedded in a conformal field theory\cite{65,66}.

(Exercise) Compute the three-point function of $F_{\mu \nu}^2$ in $d \neq 4$ and show it is not conformal. The result can be found e.g. in\cite{66}.

In the above discussions, we have been careless about the gauge fixing, but the conclusion does not change by the gauge fixing procedure. It is interesting to note, however, the introduction of the BRST charge together with the hidden “conformal generator” will generate infinite dimensional graded algebra\cite{66}.

Generic massless vector field theories without gauge invariance (thus without unitarity) are scale invariant but not conformal invariant in any dimension as emphasized by Riva and Cardy\cite{67}.

\[
S = \int d^d x \left( \frac{1}{4} (\partial_\mu v_\nu - \partial_\nu v_\mu)^2 - \frac{\alpha}{2} (\partial^\mu v_\mu)^2 \right) .
\]

(2.46)

with

\[
T^\mu_\mu = \left( 2 - \frac{d}{2} \right) \left( \partial_\mu v_\nu \partial^\mu v^\nu - \partial_\nu v_\nu \partial^\nu v^\mu \right) - \alpha \left( (2 - d) v_\mu \partial^\nu v^\nu - \frac{d}{2} (\partial^\mu v_\mu)^2 \right) .
\]

(2.47)
This can be improved to be traceless only when $\alpha = d - 4$ (see e.g. [66]). In the Euclidean signature, this model is regarded as a theory of elasticity [68], where $v_\mu$ is the displacement vector. The model can also be regarded as a free field theory describing the theory of perception [69][70][71].

### 2.4.2 Interacting theories

In $d = 4$, we can argue that classically all unitary renormalizable scale invariant actions are conformal invariant [35]. The statement can be verified by explicitly writing down all the interactions. The quantum corrections due to the renormalization and necessity of cut-off hiddenly introduces the scale, so the scale invariance can be broken. The effect of the renormalization plays a crucial role in discussing scale invariance and conformal invariance. In particular, the trace of the energy-momentum tensor obtains a quantum correction

$$T_\mu^\mu = \beta^I O_I + \beta^a (\partial_\mu J_a^\mu) ,$$

(2.48)

where $\beta^I = \frac{dg^I}{d\log g}$ is the beta function for the coupling constant $g^I$ and tells how much our coupling constant runs along the renormalization group flow. The second term $\beta^a$ (the so-called vector beta function) may be unfamiliar, and we will discuss more in section 2.6. For now, we just say it is the beta function that removes the divergence of the background space-time dependent current coupling $\int d^4x (N^a) \partial_\mu J_a^\mu$ with a space-time dependent background field $N^a$. The general expression (2.48) for the violation of the trace of the energy-momentum tensor by quantum corrections is known as the trace identity and we have more detailed discussions in section 2.5 and 2.6.

As discussed in 2.2, the energy-momentum tensor is not unique. In (2.48), we have assumed that we can use the ambiguities to remove $\Box O$ term with a scalar operator $O$ (preferably dimension 2 but not necessary). In interacting field theories, the term $\Box O$ can be renormalized and it may mix with the other terms, so this assumption is more non-trivial than we naively think, but we leave the problem set a side for now (see e.g. [72][73] for reference), and come back to the point when necessary. We also note that the expression for the trace of the energy-momentum tensor (2.48) is only up to the usage of the equations of motion, and we will discuss the consequence of the equations of motion in quantum field theories later in section 2.5 in relation to the renormalization group equation.

As an example, let us consider massless many flavor QCD [74] (a.k.a Banks-Zaks model [75]) of $SU(N_c)$ gauge group with $N_f$ pairs of Dirac fermions in fundamental representation. The two-loop beta functions are given by

$$\beta(g) = -\frac{g^3}{48\pi^2} (11N_c - 2N_f) - \frac{g^5}{(16\pi^2)^2} \left( \frac{34}{3} N_c^2 - \frac{1}{2} N_f \left( \frac{16}{3} + \frac{20}{3} N_c \right) \right)$$

$$\beta^a = 0 .$$

(2.49)

The absence of the vector beta functions is essentially because there is no parity-even non-conserved gauge invariant vector operator with dimension 3. Note that when $J_a^\mu$ is conserved, it does not give any contribution in (2.48). We also do not have any good candidate for the virial current $J_\mu$ in perturbation theory with the same reasons for the absence of $\beta^a \partial J_a^\mu$. As a consequence, the requirement of scale invariance reduces to $\beta(g) = 0$ and it automatically implies conformal invariance. Indeed for $N_f \sim \frac{11N_c}{2}$, we can find a perturbative conformal fixed point [77].
It is not our main scope to discuss the details of renormalization and compute beta functions (in particular at higher loops), but let us present how the trace of the energy-momentum tensor can appear and why it is related to the beta functions in dimensional regularization at one-loop level. As can be inferred from section 2.4.1 with the $U(1)$ Maxwell field theory example, in $d = 4 - \epsilon$ dimension, the trace of the energy-momentum tensor in massless QCD is given by

$$T^\mu_\mu = \frac{\epsilon}{4g_0^2} \text{Tr}(F^{(0)}_{\rho\sigma})^2$$

(2.50)

up to terms that vanish with the equation of motion. If we naively take $\epsilon \to 0$, this vanishes, but we have to renormalized the bare field strength operator $\text{Tr}(F^{(0)}_{\mu\nu})^2$ and bare gauge coupling constant $g_0$. Indeed, both of them contain $\epsilon^{-1}$ pole in dimensional regularization, and it can result in the cancellation in (2.50). Since the renormalization necessary here is the same as that for the bare Lagrangian density, we conclude that in $\epsilon \to 0$ limit

$$T^\mu_\mu = -\frac{\beta(g)}{2g^3} \text{Tr}([F_{\rho\sigma}]^2),$$

(2.51)

where $\text{Tr}([F_{\rho\sigma}]^2)$ is the renormalized finite operator. Note that the $\epsilon^{-1}$ poles are related to the beta function in the standard dimensional regularization. Although the heuristic argument here is essentially correct for massless QCD, we have to be more careful about the renormalization of the composite operator in the derivation of the right hand side of (2.51), which is related to the appearance of $\beta^a(\partial_\mu J^{\mu}_a)$ in more complicated examples. See section 2.6 for further details.

At one-loop level, we do not need to be careful about the composite operator renormalization and the simple application of the background field method (see e.g. [76]) by decomposing $A^{(0)}_\mu = \bar{A}_\mu + \delta A_\mu$ and integrating out the fluctuation $\delta A_\mu$ from the one-loop determinant gives the $\epsilon^{-1}$ pole in

$$\langle\text{Tr}(F_{\rho\sigma})^2\rangle = g_0^2 \text{Tr}(\bar{F}_{\rho\sigma})^2 \frac{b_0}{(4\pi)^2} \epsilon^{-1} + \cdots$$

(2.52)

so substituting it in (2.51) gives the one-loop formula for (2.51) with $\beta(g) = -\frac{b_0 g_0^3}{(4\pi)^2}$ with $b_0 = \frac{1}{8}(11N_c - 2N_f)$. As in chiral anomaly, one-loop background field computation gives the bare trace anomaly formula.

We refer to [77],[78],[79] for further details on the operator renormalization needed beyond one-loop. See also [80] for the detailed structure of the renormalized energy-momentum tensor and trace anomaly at higher loops in QED.

The above discussion is based on the perturbative power-counting renormalization, and we do not know whether in non-perturbative regime, there can be other (possibly emergent) operators that appear in the trace of the energy-momentum tensor in massless QCD. This is a very difficult problem for many flavor massless QCD, and we do not have any good theoretical tool to investigate it while such a possibility is often neglected. Presumably lattice computer simulations will shed some lights on it. See [81] for the current status of lattice simulations of the conformal windows of massless QCD. As far as we know, what they have computed so far could not distinguish scale invariance and conformal invariance. Eventually, we hope to compute the three-point functions to see whether the conformal invariance is realized. We also would like to refer to our collaboration [82],[83] with this respect.

We emphasize that it is not absurd to imagine such a possibility. For instance, when a chiral symmetry is broken, the Nambu-Goldstone boson appears and it does have a non-zero (but a kind
of trivial) virial current as mentioned in section 2.4.1, which is indeed emergent. Similarly, in the magnetic free phase of Seiberg-duality [84], we have emergent conformal dimension two operator (due to the emergence of magnetic infrared free fields) that may appear in the trace of the energy momentum tensor as an improvement term.

We now consider more non-trivial situations in which the symmetry does not forbid the non-trivial existence of the perturbative virial current. The most general power-counting renormalizable classically scale invariant field theories in \( d = 4 \) have interactions with gauge couplings, Yukawa couplings \( y^{ab} \), and \( \phi^4 \) scalar self-interactions \( \lambda^{ijkl} \). Each interaction may have non-trivial beta functions, so the trace of the energy-momentum tensor is schematically given by

\[
T_\mu = -\frac{\beta(g)}{2g^4} \text{Tr} F^{\mu\nu} F_{\mu\nu} + (\beta y^{abi} \psi_a \psi_b) + \beta \partial_\mu J^\mu + \beta a \partial_\mu J^\mu .
\]  

We have assumed that \( \theta \) angles are not renormalized.\(^{19}\) We also assume that we fine-tune mass terms for \( \phi \) and the cosmological constant to make them vanish during the renormalization. As a further technical assumption, we assume that the energy-momentum tensor stays in the improved form (i.e. absence of \( \Box O \) term) during the renormalization [35][85].

In these theories, we have candidates for the non-trivial virial current

\[
J_\mu = q^{ij} (\phi_i \partial_\mu \phi_j) + p^{ab} (\bar{\psi}_a \gamma_\mu \psi_b)
\]

that corresponds to \( O(N_b) \) rotations for scalars (\( q^{ij} \) is anti-symmetric), and \( U(n_f) \) rotations for fermions (\( p^{ab} \) is anti-Hermitian). Depending on the details of the interactions, some of them are conserved and do not contribute to the virial current. The naive application of the equations of motion schematically gives

\[
\partial_\mu J_\mu = q^{im} \lambda^{ijkl} \phi_i \phi_j \phi_k \phi_l + (q^{im} y^{abm} + p^{ac} g^{cb}) \phi_i \psi_a \psi_b + c.c.
\]

so it may be possible to rewrite some of the terms in (2.53) as a divergence of the virial current. We note that the number of possible candidate currents for the non-trivial virial current is same as that of the redundant perturbations (see section 2.7 for more details about redundant perturbations).

A priori, if we look for a conformal invariant fixed point with \( T_\mu = 0 \), we have to solve the equations \( \beta^I = 0 \) (if \( \beta^a = 0 \) whose number is the same as that of the coupling constants \( g^I \), and we expect that we typically find a fixed point. If we relax the condition so that \( T_\mu = \partial_\mu J_\mu \), we naively expect more solutions because we have more free parameters available and it seems much easier to find a scale invariant but non-conformal invariant fixed point. Does this work in reality? Actually it does not seem so for the physical reason we will argue in the next lecture.

It was known that up to two-loops within the minimal subtraction scheme, \( \beta^a = 0 \) and there is no non-trivial solutions of \( J_\mu \) that would give scale invariant but non-conformal invariant fixed point (see also [23][36][77] for an attempt in \( d = 4 - \epsilon \) dimensions). The significantly more complicated three-loop (four-loop for gauge interaction) computation was done in [88] for diagrams that are relevant for our discussions, and they found that within the minimal subtraction scheme, there exists a non-trivial solution to the equation

\[
\beta^I O_I = \partial^\mu K_\mu ,
\]  

\(^{19}\)More precisely, one may introduce the beta function for the \( \theta \) angle as long as the contribution to the trace of the energy-momentum tensor is cancelled by the anomalous conservations of \( J^\mu \) (so that \( \mathcal{B} \) function is zero). This corresponds to introducing field rotations on chiral fermions as a part of our renormalization scheme, whose anomaly is cancelled by the flow of the \( \theta \) angle. Physically this is a redundant flow.
where $K_\mu$ has the same ansatz as (2.54).

By construction, the eigenvalue of $\beta^I$ flow is pure imaginary when it is given by the divergence of a current because the current generates $O(N_b)$ or $U(n_f)$ rotations in perturbation theory. Thus if we define the renormalization group flow by $\frac{dg^I(\mu)}{d\log \mu} = \beta^I(g(\mu))$, the non-trivial solution of (2.56) gives the cyclic renormalization group flow. It was quite surprising, and it was interpreted that it gives the first non-trivially interacting counterexample of scale but non-conformal field theories in $d = 4$.

However, in order to understand the conformal invariance, we had to compute the additional terms in the trace of the energy-momentum tensor $\beta^a \partial_\mu J^\mu_a$ independently to argue if the total trace of the energy-momentum tensor vanishes. We anticipated that this must cancel against the beta functions because it looks inconsistent with the general argument from the strong version of the $\alpha$-theorem as we will discuss in the next lecture. Soon after, the additional terms $\beta^a \partial_\mu J^\mu_a$ have been computed [10], and they exactly cancel against the $\partial_\mu K_\mu$ term computed within the same regularization scheme (see appendix of [8] for the first observation and the physical explanation).

We will revisit how and why the naive expectation that it is much easier to solve the scale invariant but non-conformal condition than to solve the conformal invariant condition is not true in Lecture 2. As for the counting goes, we realize that whenever there is a candidate for the virial current, there is a corresponding symmetry in the coupling constant space, and the beta functions are no longer independent. As a consequence, the number of free parameters does not increase as naively expected in the above considerations.

So far there is no known scale invariant but not conformal invariant unitary quantum field theories in $d = 4$. Although there is no non-perturbative proof, it is presumably true under some assumptions such as

- unitarity
- Poincaré invariance (causality)
- discrete spectrum in scaling dimension
- existence of scale current
- unbroken scale invariance

We will see in Lecture 2, these assumptions are sufficient to prove the conformal invariance in $d = 2$. We also note that thanks to the fourth assumption, we can exclude the counterexample (free Maxwell theory in $d \neq 4$) in section 2.4.1. As far as we know, there is no known counterexample of scale invariance without conformal invariance in other dimensions than $d = 2, 4$, either, with the above assumptions.

### 2.4.3 More examples

We have more examples of possible scale invariance without conformal invariance discussed in various literatures. We describe a few of them in this section.

- Non-linear sigma model for a quasi-Ricci flat manifold by Hull and Townsend [89]

  Probably this is the first physical example in which the distinction between scale invariance and conformal invariance was emphasized (see also [90] [91] [92]). They observed that in order to achieve the scale invariance of the non-linear sigma models in $d = 2$, whose action is $S =$
∫ d²xG_{IJ}(X)∂_μX^I∂^μX^J with G_{IJ}(X) being the metric for the target space X, we require that the target space metric must be quasi-Ricci flat \[93\] in one-loop approximation:

\[ R_{IJ}(G(X)) = D_I V_J(X) + D_J V_I(X) , \] (2.57)

where \( R_{IJ}(G(X)) \) is the target space Ricci-tensor and \( V_I(X) \) is a certain (non-zero) vector field in the target space. On the other hand, the conformal invariance demands that the target space metric is Ricci flat (up to possible dilaton improvement terms).

\[ R_{IJ}(G(X)) = D_I D_J \Phi(X) . \] (2.58)

Here \( \Phi(X) \) is a certain scalar in the target space, which can be removed by improvement. Note that the pull-back of the vector filed \( V_B \) (i.e. \( J_μ = ∂_μX^I V_I(X) \)) will be identified with the virial current. The condition for scale invariance is weaker than conformal invariance.

As we will see, however, under certain assumptions, scale invariance must imply conformal invariance in two-dimensional non-linear sigma models. Indeed, it is a mathematical fact that any quasi-Ricci flat manifold \( X \) must be Ricci flat when \( X \) is compact, and therefore there is no scale invariant but non-conformal field theories realized by a compact non-linear sigma model. We are delighted to know that the field theory theorem is in perfect agreement with the mathematical theorem on Riemannian geometry \[23\].

The two-dimensional black hole (Euclidean cigar geometry) is an example of conformal field theory whose target space is quasi-Ricci flat \[94\]:

\[ ds^2 = k(dr^2 + \tanh^2 r dθ^2) . \] (2.59)

The target space is non-compact, and it solves (2.58). It has an exact conformal field theory description by \( SL(2, \mathbb{R})/U(1) \) coset model at level \( k \). This type of quasi-Ricci flat space-time and its higher dimensional generalization was studied in \[25\] (but they are all conformal invariant).

A scale invariant but non-conformal quasi-Ricci flat space-time may be obtained in the linearized order around the Minkowski space-time as a vector gravitational wave. Let \( G_{IJ} = η_{IJ} + h_{IJ} e^{ikX} \) with the small fluctuation \( h_{IJ} \). The scale invariance requires (see e.g. section 3.6 of the textbook \[98\] with a slight generalization mentioned in exercise 15.12 \[77\])

\[ -k^2 h_{IJ} + k_I k^L h_{JL} + k_J k^L h_{IL} - k_I k_J (h^{L} + φ) = ik_I V_J + ik_J V_I \] (2.60)

with a constant vector \( V_I \) and a constant scalar \( φ \). By introducing a particular little group and the vector \( n_I \) such that \( n^2 = 0, n_I k^I = 1 \), we can assume \( n^I h_{IJ} = 0 \). The scale invariant condition becomes \( k^2 = 0 \) and

\[ k^L h_{IL} = -ik_I (V_L n^L) + i V_I . \] (2.61)

Clearly, when \( V_L = 0 \), we have transverse traceless tensor mode as well as dilaton scalar mode for a conformal invariant solution. However, we have additional \( d - 2 \) vector polarization for a scale invariant but non-conformal solution specified by \( V_I \) (up to gauge transformation). This is only possible because we violate the unitarity and the discreteness of the spectrum.

- Wilson-Fisher fixed point \[18\]: Is 3d Ising conformal invariant?

It is an extremely interesting and important problem to show if the critical phenomena of 3d Ising model shows conformal invariance. In \( d = 2 \), it is long known that the critical phenomenon
is described by a free Majorana fermion, which is conformal invariant. The direct proof of conformal invariance from the statistical model is, however, mathematically very hard.

In $d = 3$, the success of the bootstrap approach \cite{99,100} suggests that it must show conformal invariance. How much do we know about it? If we assume that the critical phenomenon of the 3d Ising model has the same universality class as the Landau-Ginzburg model with $\lambda \Phi^4$ potential in $4 - \epsilon$ dimension analytically continued to $\epsilon = 1$, the trace of the energy-momentum tensor is given by

$$T^\mu_\mu = [-\epsilon \lambda + \beta(\lambda, \epsilon)]\Phi^4$$

within perturbation theory (after fine-tuning $m^2\Phi^2$ term). If we employ minimal subtraction scheme, $\beta(\lambda, \epsilon)$ does not depend on $\epsilon$ and we can recycle the four-dimensional computation in the minimal subtraction scheme (see e.g. \cite{14} for explanations). As in the Banks-Zaks theory, the significant feature is there is no candidate for the virial current in perturbation theory for one-component Landau-Ginzburg model. Therefore, perturbative fixed point (Wilson-Fisher fixed point) is necessarily conformal invariant.

Unfortunately, $\epsilon$ expansion is asymptotic, and it is not obvious if there is any non-perturbative emergence of virial current (but we will have little to say about scale invariance and conformal invariance in $d = 3$ in this lecture). Since we know that at $\epsilon = 2$ the theory is conformal because it is described by a free fermion, we anticipate that there is no such a possibility, but it is extremely important to give more rigorous non-perturbative argument for it.

- The fermionic version of the Landau-Ginzburg theory is known as the Gürsoy model \cite{101} (in $d = 4$).\footnote{We define conformal invariance in $d = 4 - \epsilon$ as vanishing of the trace of the energy-momentum tensor.} The action is $S = \int d^4x \left( i\bar{\psi}\gamma^\mu \partial_\mu \psi + \lambda(\bar{\psi}\gamma_H \psi)^{4/3} \right)$. It is classically scale invariant as well as conformal invariant. In two-dimension it is known as the massless Thirring model, where interaction term is $\lambda(\bar{\psi}\gamma^\mu \psi)^2$ and it is conformal invariant quantum mechanically. In these models, there is a potential candidate for the virial current $\bar{\psi}\gamma^\mu \psi$, but in $d = 2$, it is a total derivative after bosonization, so it can be improved away in any way. We do not know much about the situations in the other dimensions.

- Scalar Riegert Theory $S = \int d^4x (\Box \phi)^2$.\footnote{I would like to thank K. Akama for the reference.} In Polchinski’s classic paper \cite{23} (see the textbook \cite{14} for the same remark), it was mentioned that a fourth derivative scalar theory is scale invariant but not conformal invariant, but we can find the Weyl invariant extension of the fourth derivative operator, which is commonly known as scalar Riegert operator\footnote{The operator was also found by Paneitz at the same time \cite{103}. As far as we are aware, the same operator appeared earlier in the work by Fradkin and Tseytlin \cite{105,106} in the context of conformal supergravity.} in $d = 4$:

$$\Delta_4 = \Box^2 + 2G_{\mu\nu}D^\mu D^\nu + \frac{1}{3}\Delta^\mu R \Delta_\mu$$\hspace{1cm}(2.63)$$

The corresponding action

$$S = \int d^4x \sqrt{|g|}\phi \Delta_4 \phi$$\hspace{1cm}(2.64)$$
is Weyl invariant (with zero Weyl weight for the scalar). Indeed, we could construct the improved energy-momentum tensor such that the theory is conformal invariance (although it is not unitary).

This is not so surprising, but probably a more surprising thing is that there is no supersymmetric Riegert operator at the non-linear level in the old minimal supergravity [107] [108]. The supersymmetric Riegert operator does exist in the new minimal supergravity [109].

In other dimensions, when \( d \) is odd, we can construct higher derivative Weyl invariant free scalar actions of order \( \Box^n \) for any \( n \) [110] [111] [112]. In even dimensions, the Weyl invariant higher derivative free scalar actions exist when \( n \leq d \). As an example, there does not exist Weyl invariant fourth (or higher) order derivative actions in \( d = 2 \). We can directly check that the energy-momentum tensor for the action \( S = \int d^d x (\Box \phi)^2 \) is given by

\[
T_{\mu \nu} = (\partial_\mu \phi \partial_\nu \Box \phi + \partial_\nu \phi \partial_\mu \Box \phi - \eta_{\mu \nu} \partial_\rho \phi \partial_\rho \Box \phi + (\Box \phi)^2/2)
\]

and see that it cannot be improved to be traceless in \( d = 2 \) (in contrast to the case \( d \geq 3 \), where it is possible).

- **Spontaneous broken case with the non-linear action** \( S = \int d^4 x \phi^4 f \left( \frac{\partial^\mu \partial_\mu \phi}{\phi^2} \right) \) [113] [65].

  This scale invariant action is not conformal invariant classically unless \( f(x) = c_0 + c_1 x \), in which case, we have just \( \phi^4 \) self interaction with the conventional kinetic term. The scale invariant vacua at \( \phi = 0 \) is singular with higher terms. When \( \phi \neq 0 \), scale invariance is spontaneously broken.

  Of course, the spontaneous breaking of scale invariance does not necessarily exclude conformal invariance (which is spontaneously broken). One example is

\[
S = \int d^4 x \phi^4 f \left( \frac{\Box \phi}{\phi^4} \right).
\]

Here the theory can be made manifestly Weyl invariant in the curved background by replacing \( \Box \) with \( \Box - \frac{1}{3} R \).

- **Liouville theory coupled with matter**

  The analogue of the above example in \( d = 2 \) is the Liouville action.

\[
S = \frac{1}{4\pi} \int d^2 x \sqrt{|g|} \left( \partial_\mu \varphi \partial_\mu \varphi + R(b + b^{-1}) \varphi + \lambda e^{-2b \varphi} \right).
\]

It is Weyl invariant and importantly, the conformal invariance is not spontaneously broken, which is a special feature of \( d = 2 \). The quantization of the Liouville theory can be performed without breaking the conformal invariance (see e.g. [114] and references therein).

Once we couple the Liouville theory to a non-linear sigma model with the specific interaction

\[
S = \frac{1}{4\pi} \int d^2 x \sqrt{|g|} \left( G_{IJ}(X) \partial_\mu X^I \partial^\mu X^J + h(X) \partial_\mu \varphi \partial^\mu \varphi + R(b + b^{-1}) \varphi + \lambda e^{-2b \varphi} \right),
\]

the theory is scale invariant but not conformal invariant classically [115]. In [116], we have shown that by considering the light-like wave form for \( h(X) \), we may be able to preserve the scale invariance without conformal invariance after quantization. The model breaks the assumption of discreteness of spectrum as well as unitarity (in the light-like wave case) to avoid Zamolodchikov-Polchinski theorem that claims scale invariance implies conformal invariance in \( d = 2 \).

\[23\]I would like to thank S. Kuzenko for the reference and related discussions.
• The above mentioned Liouville theory and Riegert theory were studied in the context of generating classical effective action (so-called Wess-Zumino action) for the Weyl anomaly. If we restrict ourselves to the A-type Weyl anomaly (Euler density term), the Wess-Zumino action is invariant under the constant Weyl transformation due to the space-time integration, but it is not invariant under the non-constant Weyl transformation. This Wess-Zumino action plays a significant role in the next section. To avoid a confusion, in flat space-time, one can always make the Riegert and Liouville action conformal invariant.

• As a simple generalization of the free Maxwell theory example discussed in section 2.4.1, in d-dimensional space-time, among various Abelian free form fields, only zero-form field (scalar), d/2-form field and d − 1 form field (dual to scalar) are conformal invariant (see e.g. [117]). In the first quantized approach, this was discussed in [118].

• Within the Lagrangian formulation in d = 4, it was mentioned in [119] [120] that the non-gauge invariant interaction terms such as \( \phi \partial_\mu \phi A_\mu \) is scale invariant but not conformal invariant. It violates unitarity.

2.4.4 Theory without action

As mentioned in section 2.3, it is possible that theories are scale or conformal invariant without having local currents. In particular, this is the case when the action principle does not exist.

In d = 4, the conformally covariant massless wave-equations (known as Bargmann-Wigner equation) must take the form [121]

\[
\partial^{\alpha} \Psi_{\alpha \beta \gamma \ldots} = 0 ,
\]

(2.69)

where \( \Psi_{\alpha \beta \gamma \ldots} \) has completely symmetric spinor indices. The CPT conjugate equation is obtained with dotted spinors. We note that except for helicity 0, 1/2, 1, there is no simple Lagrangian formulation whose equations of motion are equivalent to (2.69) with the local energy-momentum tensor. After the second quantization, the theory is conformal invariant in the sense that all the correlation functions transform in a conformally covariant manner as the helicity h field \( \Psi_{\alpha \beta \gamma \ldots} \) being conformal primaries with conformal dimension \( h + 1 \).

An interesting observation is that the massless rank-four spinor \( \Psi_{\alpha \beta \gamma \delta} \), which has helicity two, can be regarded as the linearized Weyl tensor around the Minkowski vacuum. Indeed, the linearized Einstein equation makes the other components of linearized curvature tensor vanish, and the Bianchi identity is nothing but the conformal wave equation (2.69). Therefore, the linearized Einstein gravity is on-shell conformal invariant. However, there is no conserved energy-momentum tensor, nor conformal current, so the Noether assumption is clearly violated [119] [120].

The discussion also applies to helicity 3/2 massless rank-three spinor \( \Psi_{\alpha \beta \gamma} \) with the conformal wave-equation (2.66). The theory is equivalent to the on-shell Rarita-Schwinger theory of a massless spin 3/2 particle. Combining it with the above helicity 2 wave-equation, we can conclude that the

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24 The author would like to thank I. Shapiro for the comment and discussion.
25 The vacuum Einstein equation \( R_{\mu \nu} = 0 \) is not Weyl invariant, so the Minkowski vacuum after Weyl transformation would not solve the vacuum Einstein equation. The conformal transformation here only acts on the linearized variation from the Minkowski vacuum. We also note that \( \Psi_{\alpha \beta \gamma \delta} \) has the conformal dimension 3, which may be unexpected from the non-linear Einstein gravity.
linearized supergravity around the Minkowski vacuum is on-shell superconformal invariant. Again, we do not have a conserved supercurrent supermultiplet nor a superconformal current.

Without assuming the existence of the action, or more precisely the existence of the energy-momentum tensor, it seems possible to construct unitary scale invariant field theories without conformal invariance by considering so-called generalized free field theories [86]. For instance, we start with the two-point functions of the vector operator

\[
\langle A_\mu(x)A_\nu(0) \rangle = \frac{1}{x^{2\Delta}} \left( \alpha \eta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2} \right)
\]  

(2.70)

and demand that the higher point functions are defined by the Wick contraction. One could write down the formal (possibly non-local) free action from the inverse of the two-point function (2.70).

Unless we take a very particular value for \( \alpha \) (e.g. \( \alpha = 1 \) or \( \alpha = 1/\Delta \)), the theory is not conformal invariant. Note that for a sufficiently large value of \( \Delta \), the unitarity is preserved irrespective of the conformal invariance (see section 2.2.3). Since the theory does not seem to possess the energy-momentum tensor, we do not know how to couple it to gravity, and this peculiar property will enable us to evade various assumptions in the proof of the equivalence between scale invariance and conformal field theories. Indeed, we will see that the existence of the energy-momentum tensor plays a crucial role in our argument for the equivalence.

2.5 EOM, contact terms, and Callan-Symanzik equation

In classical field theories, there is nothing subtle about using equations of motion to simplify the trace of the energy-momentum tensor. Indeed, we have shown that the trace of the improved energy-momentum tensor for a free scalar vanishes up on the usage of the equations of motion. In quantum mechanics, the use of the equations of motion introduces contact terms in correlation functions, and they play an important role in deriving the Ward-Takahashi identities and the scaling properties of the correlation functions.

To see this, let us consider the path integral of a scalar field theory with action \( S[\phi] \) as an example. By using the field redefinition

\[
\phi(x) \rightarrow (1 + \alpha(x))\phi(x)
\]  

(2.71)

with the invariance of the path integral measure within the path integral expression

\[
\langle \phi(x_1) \cdots \phi(x_n) \rangle = \int D\phi \exp(-S[\phi])
\]  

(2.72)

and taking \( \alpha(x) \rightarrow \delta(x) \) limit, we can formally obtain

\[
\left\langle \phi(x) \frac{\delta S[\phi]}{\delta \phi(x)} \phi(x_1) \cdots \phi(x_n) \right\rangle = \sum_i \left\langle \delta(x_i - x)\phi(x_i) \prod_{i \neq j} \phi(x_j) \right\rangle .
\]  

(2.73)

This means that the equations of motion \( \frac{\delta S[\phi]}{\delta \phi(x)} = 0 \) is valid up to contact terms (if the anomaly were present, the use of the equations of motion within a composite operator would be modified). In

\[^{26}\text{To some extent, this is just a peculiar coincidence in (1 + 3) dimension. In higher dimensions, not all massless equations (like Maxwell theory or linearized gravity) are conformal invariant [118].}\]

\[^{27}\text{This is non-trivial because we are interested in the composite operator insertion. Indeed, Konishi anomaly [122] of the supersymmetric gauge theories does suggest the violation here. In our discussion of the operator identity, we have to use the correct quantum operator identity.}\]
addition, if we integrate \(2.73\) over \(x\), the insertion of the equations of motion can be used to rescale the bare fields in the path integral computation of the correlation functions. For example, in a free scalar field theory, we found in section 2.4.1 that the trace of the energy-momentum tensor is a total derivative (or zero in the improved case) up to the equations of motion \( \frac{2}{3} \phi \square \phi \). This explains the canonical scaling dimension of scalar under scale transformation after inserting the trace of the energy-momentum tensor for scale transformation in the path integral.

The contact terms associated with the equations of motion play an important role in understanding the renormalization group flow. To understand the situation more clearly, we compare the renormalization group equation and the Ward-Takahashi identity for the dilatation. Within power-counting renormalization, the renormalized correlation functions must satisfy the Callan-Symanzik equation 123 124 125 with the renormalization scale \( \Lambda \):

\[
\begin{align*}
\left( \frac{\partial}{\partial \log \Lambda} + \beta^I \frac{\partial}{\partial g^I} \right) \langle \phi_{i_1} \cdots \phi_{i_n} \rangle &= \gamma_i^{j_1} \langle \phi_{j_1} \cdots \phi_{i_n} \rangle + \cdots + \gamma_i^{j_n} \langle \phi_{i_1} \cdots \phi_{j_n} \rangle ,
\end{align*}
\]

(2.74)

where \( \beta^I \) is the beta function \( \beta^I(g) = \frac{\partial \beta^I}{\partial \log X} \big|_{g_0} \) and \( \gamma_i^j \) is the anomalous dimension matrix.

We now give a formal interpretation of the trace identity (2.48) from the Callan-Symanzik equation. The response to the scale transformation \( \frac{\partial}{\partial \log \Lambda} \) would define the dilatation current through the naive Ward-Takahashi identity if the theory were scale invariant

\[
\frac{\partial}{\partial \log \Lambda} \langle \phi_{i_1} \cdots \phi_{i_n} \rangle = \int d^4x \langle (\partial^\mu D_\mu) \phi_{i_1} \cdots \phi_{i_n} \rangle = \int d^4x \langle (T_\mu^\mu - \partial^\mu J_\mu) \phi_{i_1} \cdots \phi_{i_n} \rangle = 0 .
\]

(2.75)

We recall \( D_\mu = x^\mu T_{\mu\nu} - J_\mu \) (see section 2.3).26 However, this is not generically correct because the renormalization breaks the scale invariance, so we may not have a conserved dilatation current. We rewrite the left hand side with the Callan-Symanzik equation to introduce the anomalous violation of the scaling transformation, while we retain the right hand expression because this is what we obtain from the direct definition of the energy-momentum tensor and scaling transformation:

\[
-\beta^I \frac{\partial}{\partial g^I} \langle \phi_{i_1} \cdots \phi_{i_n} \rangle + \gamma_i^{j_1} \langle \phi_{j_1} \cdots \phi_{i_n} \rangle + \cdots + \gamma_i^{j_n} \langle \phi_{i_1} \cdots \phi_{j_n} \rangle = \int d^4x \langle (T_\mu^\mu - \partial^\mu J_\mu) \phi_{i_1} \cdots \phi_{i_n} \rangle .
\]

(2.76)

Due to the integration over space-time, the total derivative term \( \partial^\mu J_\mu \) can be dropped here. This is the correct (broken) Ward-Takahashi identity for the scale transformation.

If we assume the action principle \( \frac{\partial}{\partial g^I} \langle \cdots \rangle = - \int d^4x \langle O^I(x) \cdots \rangle \), then the \( \beta \) function term in the Callan-Symanzik equation (2.76) is understood as the insertion of \( \beta^I O_I \). Similarly, we can use (2.73) to express the anomalous dimension term (2.76) by using the insertion of the equation of motion operator. Therefore, up to total derivative terms, we obtain

\[
T_\mu^\mu = \beta^I O_I + \beta^a (\partial_\mu J_a^\mu) + (d_0 + \gamma) \int \phi \frac{\delta S}{\delta \phi} ,
\]

(2.77)

\[\text{28}\text{Since we are not considering the explicit mass terms in our most of the discussions, probably it is more appropriate to call it Gell-Mann Low equation} \quad 28\text{29}\text{More precisely, one should really think of the Callan-Symanzik equation as the anomalous conservation law for the dilatation current (rather than the trace of the energy-momentum tensor) because the energy-momentum tensor (and its trace) has a further ambiguity associated with the contact terms that vanish up to equations of motion such as} \delta T_{\mu\nu} = \eta_{\mu\nu} C^{IJ} \int \phi \phi \phi \phi \text{ (do not confuse with the anomalous dimension term below). This extra freedom cancels in the Ward-Takahashi identity for dilatation. See e.g. appendix of [127]. In this lecture, we ignore this unphysical contribution.} \]
where the matrix structure of $\gamma$ acting on $\phi$ is suppressed. Here $d_0$ is the “canonical dimension” of the field $\phi$. The inclusion of $d_0$ here is rather conventional in the Callan-Symanzik equation with reference to “free” field theories. The sum $d_0 + \gamma$ gives the total scaling dimension of the field $\phi$, which has an intrinsic meaning without referring to reference free field theories. We note that the Callan-Symanzik equation says nothing about the total derivative term that drops after space-time integration such as vector beta function terms like $\beta^a(\partial_\mu J_a^\mu)$ in $T_\mu^\mu$.

Whenever $\beta^I O_I$ can be transformed into the virial current, the Callan-Symanzik equation can be further transformed as
\[
\left( \frac{\partial}{\partial \log \Lambda} + \tilde{\beta}^I \frac{\partial}{\partial g^I} \right) \langle \phi_{i_1} \cdots \phi_{i_n} \rangle = (\gamma_{i_1} j_{1} + S_{i_2} j_{2}) \langle \phi_{j_1} \cdots \phi_{i_n} \rangle + \cdots + (\gamma_{i_n} j_{n} + S_{i_1} j_{1}) \langle \phi_{i_1} \cdots \phi_{j_n} \rangle \quad (2.78)
\]
by introducing the “flavor” rotation matrix $S^i_j$ with $\beta^I O_I = \tilde{\beta}^I O_I + \partial^\mu J_\mu$ up to equations of motion because the change of the coupling constant in the virial current direction can be absorbed by the rotations of fields (or more abstractly operators).

Correspondingly, the trace of the energy-momentum tensor is rewritten as
\[
T_\mu^\mu = \tilde{\beta}^I O_I + \beta^a(\partial_\mu J_a^\mu) + (d_0 + \gamma + S) \int \phi \frac{\delta S}{\delta \phi} \quad (2.79)
\]
by using the equation of motion (operator identity). The use of the equation of motion is manifest in the last term of (2.79) so that it gives the extra wavefunction renormalization factor $S$ in the Callan-Symanzik equation (2.78). When $\tilde{\beta}^I$ vanishes by choosing a wavefunction renormalization factor $S$, the theory is indeed scale invariant. If in addition, all $\tilde{\beta}^a$ vanish in this choice of $S$, then the theory is conformal invariant. Although the a wavefunction renormalization factor $S$ introduces non-standard anti-symmetric part (rather than symmetric part) $[4]$, we may diagonalize the dilatation operator if we wish.

Unfortunately, the global Callan-Symanzik equation says nothing about the distinction between scale invariance and conformal invariance. We have to study the unintegrated trace of the energy-momentum tensor to see the distinction. We may wonder if it is possible to consider the local version of the renormalization group equation, and this is indeed one of the approaches to the scale invariance and conformal invariance as we will discuss in section 3.3.1.

2.6 Computation of trace of energy-momentum tensor

Let us summarize the problem of finding scale invariant but non-conformal field theories within power-counting renormalization of perturbatively renormalizable quantum field theories in $d = 4$. We also would like to clarify the origin of the $\beta^a(\partial_\mu J_a^\mu)$ term in the trace of the energy-momentum tensor in dimensional regularization. Our discussion here is based on the dimensional regularization with minimal subtraction, but since our final expression is renormalization group scheme covariant, any other regularization should work in principle. The scheme covariance of the cyclic renormalization group flow was discussed in [88] under the change of the coordinate transformation in the coupling constant space $g^I \rightarrow \tilde{g}^I(g)$. We will further discuss the more non-trivial scheme associated with the “gauge transformation” on the coupling constant space in the following. The discussion of this section is based on [128][22] (c.f. [10] for a concise summary).

$^{30}$Throughout the lecture note, the “flavor” symmetry with quotation mark refers to the spurious broken symmetry acting on the interaction terms. In perturbation theory, it is the symmetry of the kinetic terms (e.g. $O(N_b)$ for scalars and $U(n_f)$ for fermions), but is broken by the interaction terms such as Yukawa interactions or scalar self-interactions.
First of all, we recall that all classically scale invariant power-counting renormalizable quantum field theories have the classical energy-momentum tensor whose trace is zero up on improvement (classical Weyl invariance) in $d = 4$. To regularize the quantum field theory, we use the dimensional regularization and evaluate the trace of the energy-momentum tensor in $d = 4 - \epsilon$. The trace is proportional to the total action density up to the terms that vanish with equations of motion

$$T^\mu_\mu = \epsilon \mathcal{L} + \int \phi \frac{\delta S}{\delta \phi}, \quad (2.80)$$

We will renormalize the action density operator $\mathcal{L}$ so that it satisfies the renormalization group equation in $d = 4 - \epsilon$

$$\left(\hat{\beta}^I \frac{\partial}{\partial g^I} + \hat{\gamma} \phi \frac{\partial}{\partial \phi} - \epsilon \right) \mathcal{L} = 0, \quad (2.81)$$

where $\hat{\beta}^I = \epsilon (k g)^I + \beta^I (g)$ are beta functions in $d = 4 - \epsilon$ ($k$ is a constant that depends on the power of the coupling constants $g^I$ appearing in the action) and $\hat{\gamma} = \epsilon + \gamma$ are anomalous dimension in $d = 4 - \epsilon$. In massless QCD, there is no complication at this point, and we can simply take $\epsilon \to 0$ and rederive the equality.

In a more complicated situation, this naive limit must be modified in a subtle way. The point is that although $\hat{\beta}^I \frac{\partial}{\partial g^I} \mathcal{L}$ is a finite operator, $\frac{\partial}{\partial g^I} \mathcal{L}$ might not be. We have to expand $\frac{\partial}{\partial g^I} \mathcal{L} = \{O_I\} + N_I^a \partial_a [J^\mu_a] + M_{I k} \Box [O_k^{(2)}]$, where all $\{O\}$ are finite operators while $N_I^a$ and $M_{I k}$ can contain $\epsilon^{-1}$ and higher poles. (Note that $\int d^d x \frac{\partial}{\partial g^I} \mathcal{L}$ must be finite so the divergence appears only in derivatives). Thus, if we express the trace of energy-momentum tensor in terms of finite operators, we should obtain

$$T^\mu_\mu = \beta^I \{O_I\} + \partial^\mu [J^\mu] + \Box [O^{(2)}] + (d_0 + \gamma) \int \phi \frac{\delta S}{\delta \phi}, \quad (2.82)$$

where we have taken $\epsilon \to 0$ limit safely because all the operators are finite now.

One important point to notice is that for $[J^\mu]$ to be finite, we have to cancel the poles in $N_I^a$ and linear $\epsilon$ terms in $\hat{\beta}^I$. This means that at the leading order, we obtain $[J^\mu] = \beta^a [J^\mu_a] = g^I N_I^{a(1)} J^\mu_a$ with $N_I^{a(1)}$ is the $\epsilon^{-1}$ term in $N_I^a$. The higher terms are also constrained because of the delicate cancellations between $N_I^a$ and $\epsilon (k g)^I$. The coefficient $\beta^a$ is interpreted as the beta function for the divergence of a vector current $\partial^\mu J^\mu_a$. A similar argument applies for $[O^{(2)}]$, but it is of little relevance for our perturbative discussions. In the following, we assume $\Box [O^{(2)}]$ term is removed by improvement of the renormalized energy-momentum tensor (see e.g. [7, 73] for reference).

However, this is not the end of the story because there is an operator identity (equations of motion) to relate $\partial_\mu [J^\mu_a]$ to sum of $[O^I]$s. Therefore, the separation between $\beta^I \{O_I\}$ and $\beta^a \partial_\mu [J^\mu_a]$ is actually arbitrary. After all, the possibility of the equality

$$T^\mu_\mu = \beta^I \{O_I\} + \beta^a \partial_\mu [J^\mu_a] = \partial^\mu [K^\mu] \quad (2.83)$$

up to equations of motion, which we are looking for the scale invariant field theories, assumes the operator identity such as $\beta^I \{O_I\} = \partial^\mu [K^\mu]$ for a certain current operator $[K^\mu]$.

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31To assure this, we have to include suitable improvement terms for scalars.

32In this section, we make a careful distinction between unrenormalized operators $O$ and the finitely renormalized composite operators $[O]$. We should remember that most of the other part of the lecture note, the composite operators $O$ are finitely renormalized implicitly and they could have been written as $[O]$ as in this section.

33For technical reason, it is important that we use minimal subtraction here because $\epsilon$ only appears in the first term in $\hat{\beta}^I (g, \epsilon) = (k g)^I + \beta (g)$ and the higher $\epsilon$ terms does not appear in the beta function.
With this operator identity, the trace of the energy-momentum tensor is invariant under

\[ \beta^I \to \beta^I + (S \cdot g)^I \]
\[ \beta^a \to \beta^a + S^a, \tag{2.84} \]

where \( S \) acts on coupling constant as an element of the “flavor” symmetry generator (i.e. \( (S \cdot g)^I = S^a T^I_a g^I \) with a representation matrix \( T^I_a \) for the symmetry). Thus, the beta functions are ambiguous.

To cancel the ambiguity, it is customary to introduce the \( \mathcal{B} \) function \([128][59]\), which is defined by the full trace of the energy momentum tensor,

\[ T = \mathcal{B}^I[O_I] + (d_0 + \gamma + S)\phi \frac{\delta S}{\delta \phi} = \beta^I[O_I] + \beta^a \partial_\mu [J^\mu_a] + (d_0 + \gamma) \int \phi \frac{\delta S}{\delta \phi}. \tag{2.85} \]

We can see that \( \mathcal{B} \) function is invariant under the gauge transformation (2.84). Clearly, the conformal invariance requires the vanishing of \( \mathcal{B} \) function rather than the vanishing of beta functions.

We note the appearance of the additional equation of motion operator with \( S \) if we use \( \mathcal{B} \) function as a renormalization group flow of the coupling constant \( g^I \): \( \frac{dg^I}{d \log \mu} = \beta^I \). This changes the wavefunction renormalization factor compared with the “standard” one \( \frac{dg^I}{d \log \mu} = \beta^I \) which we started with. Actually, this could have been asked at (2.81) because the renormalization group equation itself was ambiguous as discussed around (2.78). If we had renormalized the action density operator \( \mathcal{L} \) with the usage of the additional wavefunction factor \( S \) and the corresponding \( \mathcal{B} \) function, we would not have to introduce the vector beta functions \( \beta^a \) when we rewrite the bare operator into the finite ones because the same renormalization prescription removes \( \epsilon^{-1} \) poles in \( N^I_f \). In this way, the renormalization group flow has various ambiguities if we allow the appearance of virial current operators, but they all cancel out in the final expression for the trace of the energy-momentum tensor, and the question over scale invariance vs conformal invariance is a physically well-posed one.

So far, we have not discussed how to compute the vector beta function \( \beta^a \) in practice. In general, the renormalization of the composite operator discussed above is complicated. Conceptually, it is easier to consider the space-time dependent coupling constant \( g^I(x) \), and introduce the additional counterterms \( \int d^dx N^\alpha(x) \partial_\mu J^\mu_a \) in the action. As we have mentioned \( \beta^a \) can be regarded as the beta function for \( N^\alpha(x) \).

More generally, we can consider the counterterm \( \int d^dx N^\alpha(g) \partial_\mu g^I J^\mu_a \) part of which generates \( \int d^dx N^\alpha(x) \partial_\mu J^\mu_a \) after partial integration (i.e. “symmetric part”) \([128]\). In the dimensional regularization, we may identify \( N^\alpha(g) \) here with the operator renormalization factor \( N^\alpha \) used in the computation of the virial current \( J_\mu \) in (2.82) because the functional derivative in the local Callan-Symanzik operator \( \delta^I \frac{\delta}{\delta g^I(x)} \) will act on the renormalized action to give the finite operator relation \( \delta^I \frac{\delta}{\delta g^I(x)} S|_{D_\mu g^I=0} = [O_I] + N^\alpha \partial_\mu [J^\mu_a] + M_{1k} \Box [O_k^2] \).

In this way, in the dimensional regularization with minimal subtraction, the computation of the vector beta function \( N^\alpha \) gives the virial current through \( N^\alpha \). For an explicit computation of the counterterm \( N^\alpha \), we can study the anti-symmetric wavefunction renormalization with additional momentum flow to accommodate the position dependence. We refer to the literature \([128][10]\) how to

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Footnotes:

[34] For instance in \( \phi^4 \) theory, the coupling constants \( \lambda^{ijkl} \) transforms as forth rank symmetric tensor under the \( O(N_0) \) rotation induced by the wavefunction renormalization \( S \) on \( \phi^4 \).

[35] The term that cannot be written as \( \int d^dx N^\alpha(x) \partial_\mu J^\mu_a \) (i.e. “anti-symmetric part”) is related to the extra term in the trace of the energy-momentum tensor \( \rho \) that appears when the coupling constant is position dependent (see section 3.3.1 for more details).
compute the diverging part of $N^a$ and consequently $\beta^a$. If we use the dimensional regularization with the prescription that $\gamma$ is symmetric, $g^I N^{\sigma(1)}_I J^a_\mu$ vanishes up to two loops. At three loops, there is a non-trivial contribution and we have quoted the result in section 2.4.2, where non-zero term $g^I N^{\sigma(1)}_I J^a_\mu$ in the trace of the energy-momentum tensor played a crucial role in confirming conformal invariance at three-loop order.

2.7 (Redundant) conformal perturbation theory

As a complementary but concrete approach to the discussions in the last section, we will try to understand the role of the redundant operators and the computation of the beta functions in conformal perturbation theory in this final section of Lecture 1. It will also give some general perspectives on the perturbative searches for scale invariant but non-conformal field theories.

First of all, we should recall that quantum field theories have intrinsic ambiguities due to the field redefinition. In high energy-physics, this is manifested in the invariance of the S-matrix under the field redefinition [129] [130], and in statistical physics, it is know as the invariance of the partition function [131]. Correspondingly, the deformation of the effective action that is related to total derivative terms up on using the equations of motion is the so-called redundant perturbation because it does not affect any physics. Clearly, it is of importance to tame the redundant perturbation to discuss the perturbative scale invariance without conformal invariance.

The conformal perturbation theory [132] [133] is defined by perturbing the reference conformal field theory by adding relevant or marginal perturbations $\delta S = \int d^d x g^I O_I(x)$. From the unitarity, $O_I(x)$ must be conformal primary operators of the reference conformal field theory. For technical simplicity, we focus on the situation when all $O_I(x)$ have conformal dimension $d$, but the generalization of the following argument for including slightly relevant deformations is possible.

We assume that $O_I(x)$ have the canonical normalization in the reference conformal field theory:

$$\langle O_I(x)O_J(0) \rangle_0 = \frac{\delta_{IJ}}{(x-y)^{2d}}. \quad (2.86)$$

The conformal invariance demands that the three-point functions among $O_I(x)$ must be given by

$$\langle O_I(x)O_J(y)O_K(z) \rangle_0 = \frac{C_{IJK}}{(x-y)^d (y-z)^d (z-x)^d}. \quad (2.87)$$

In these expressions, the subscript 0 means the expectation value in the reference conformal field theory. So far, it is a standard conformal perturbation theory setup. In order to allow the non-trivial existence of the virial current, we allow the appearance of the conserved current $J^a_\mu$ in the reference conformal field theory in the OPE (see e.g. [133] for a similar argument in $d = 2$)

$$O_I(x)O_J(y) = \frac{C_{IJK}}{(x-y)^d} O_K(y) + \frac{C^a_{IJJ}(x-y)_\mu}{(x-y)^{d+2}} J^a_\mu(y) + \cdots. \quad (2.88)$$

Here $C_{IJK}$ is totally symmetric while $C^a_{IJJ} = -C^a_{JJI}$ is a certain representation matrix of the “flavor symmetry” generated by $J^a_\mu$.

Before going on, let us discuss the current contribution in the OPE (2.88). From the unitarity, we require $\partial^\mu J^a_\mu = 0$ in the reference conformal field theory, so a possible addition of $\partial^\mu J^a_\mu$ in the action is a redundant perturbation in a double sense because (A) it is a total derivative, and (B) it vanishes by conservation. However, the OPE (2.88) means that some operators $O^I$ are charged under the “flavor symmetry” because we can derive the Ward-Takahashi identity

$$\langle \partial^\mu J^a_\mu(x) O^I(x_1) \cdots \rangle_0 = \delta(x-x_1) C^a_{IJJ}(O^J(x_1) \cdots)_0 \quad (2.89)$$

34
from the OPE. It follows that in the perturbed conformal field theory, we have the violation of the symmetry as
\[ \partial^\mu J^a_\mu = g^I C^a_{I J} O^J. \] (2.90)

The equation will get renormalized at the higher order, but since it is outside of our scope to develop a systematic higher order conformal perturbation theory, it will not be important.

The conformal perturbation theory begins with the formal definition
\[ \langle \ldots \rangle = \langle e^{-\int d^d x g^I(x) O_I(x) \ldots} \rangle_0 \] (2.91)
for the correlation functions of the perturbed theory as a perturbative series in \( g^I \). The right hand side is typically divergent and we need a suitable renormalization. To discuss the vector beta functions, we have promoted the coupling constant \( g^I \) to be space-time dependent as mentioned in section 2.6. Accordingly, we need more counterterms, which is suppressed here (see also section 3.3.1).

Let us compute the beta function in a conventional way.\(^{36}\) At the second order in perturbation theory, we encounter the divergence in the “vacuum diagram” by colliding \( \int d^d x g^I(x) O_I(x) \) and \( \int d^d y g^J(y) O_J(y) \) near \( x \sim y \). The divergence from the scalar three-point function
\[ \int d^d x d^d y g^I(x) O_I(x) g^J(y) O_J(y) \sim \log \mu \int d^d z C_{IJK} g^I(z) g^J(z) O_K(z) \] (2.92)
can be removed by renormalizing the coupling constant with the beta function.\(^{37}\)

As we mentioned in section 2.6, the vector beta functions could have been obtained from the divergence in \( \int d^d x g^I(\partial^\mu J^a_\mu N^a_{I J}) \) (with symmetric \( N^a_{I J} \)). At the second order in conformal perturbation theory with the above conventional prescription, however, the would-be divergent term is only
\[ \log \mu \int d^d z C_{IJK} g^I(z) \partial^\mu g^J(z) C^a_{I J} J^a_\mu \] (2.94)
which does not affect the vector beta function because \( C^a_{I J} \) is anti-symmetric. This term itself is renormalized by \( \rho_I \) term in the space-time dependent coupling constant term in the trace of the energy-momentum tensor \( \Box \) that was mentioned in footnote \(^{35}\) and we will discuss more in the next lecture, but it has nothing to do with the discussion relevant for the computation of \( B^I \) function here. The symmetric part does not appear due to the conservation \( \partial^\mu J^a_\mu = 0 \) in the reference theory. Thus the vector beta functions are zero in this prescription at this order. We therefore conclude
\[ B^I = C_{I KL} g^K g^L + \mathcal{O}(g^3), \] (2.95)
and we observe it is given by the gradient flow with the potential
\[ \tilde{c} = \frac{1}{3} C_{IJK} g^I g^J g^K + \mathcal{O}(g^4). \] (2.96)

\(^{36}\)One cautious remark is that we do not pay attention to the “Lagrangian density operator” of the reference theory, which gives an additional redundant deformation. In usual quantum field theories, we do renormalize the wavefunction to reduce the number of independent running coupling constants, but this has not been attempted here. Anyway, it will be higher order corrections than we study here.

\(^{37}\)In the following discussions of the conformal perturbation theory, \( d \)-dependent numerical factors that appear in the integration over the space-time are omitted. One may always absorb them in the normalization of \( g^I \).
so that $\partial_I \tilde{c} = C_{IJK} g^J g^K = B^I$. We have more to say about the gradient formula in Lecture 2. For a later reference, we note that the Zamolodchikov metric, which we will discuss in Lecture 2, is $\chi_{IJ} = \delta_{IJ}$, and $w_I = 0$.

The potential $\tilde{c}$ is invariant under the “flavor” symmetry transformation $\delta^a g^I = C^a_{IL} g^L$. As a consequence, we obtain

$$\delta^a g^I \cdot B^I = 0 \, ,$$

which means at the leading order in conformal perturbation theory, the renormalization group flow is orthogonal to the “flavor symmetry” transformation and the virial current must vanish.

Let us briefly discuss the ambiguities of the beta functions with this setup. The point is that we could subtract more in the scalar operator beta functions as long as we add more to the vector beta functions. We consider the counterterm

$$\log \mu \int d^d z g^I w^a_I \partial_\mu J^\mu_a$$

with $w^a_I$ of $\mathcal{O}(1)$, which is arbitrary. In contrast to the conventional counterterm (2.94), it is non-zero at the second order in perturbation theory because of the broken conservation law (2.90). Or if we stick to the reference conformal field theory, one can perturb it once more by $g^I O_I$. It gives the vector beta function

$$\tilde{\beta}^a = g^I w^a_I \, .$$

Clearly the added term (2.98) by itself is divergent and we have to cancel it. This is done by further adding the scalar operator counterterm

$$\log \mu \int d^d z g^I w^a_I g^K C^a_{KL} O_L$$

which precisely cancels with (2.98) after using the equations of motion. It gives the scalar operator beta function at the second order in addition to the original one that was needed to cancel the OPE singularity:

$$\tilde{\beta}^I = C_{IKL} g^K g^L + (g^I w^a_K C^a_{KL}) g^K \, .$$

Of course, such artificial adding and subtracting the same term up to the equation of motion does not change the physics, and this is what we called the ambiguities in the beta functions discussed in section 2.6. Although we may think the conventional computation seems more natural at this order, at higher orders in perturbation theory it becomes more non-trivial. In anyway, the most important object is the the invariant $B$ function (2.95) that appears in the total trace of the energy-momentum tensor. We can confirm that it does not change under the ambiguity.

### 2.8 Literature guides

At the end of each lecture, we put the literature guide section. This is intended to supplement materials that are not covered in the main part of the lecture note.

The conformal invariance in relativistic systems was (as far as I know) first discussed in the context of the symmetry of the Maxwell theory (with massless matters) at the very beginning of the 20th century [135][136]. We have learned that this discovery is essentially due to the fact that the dimensionality of our space-time is precisely $d = 4$. A legend says that Poincaré should have known
the special relativity before Einstein because he knew that the Maxwell equation is not invariant under the Galilei group, but is invariant under the Lorentz group. I wonder what people had imagined for the discovery of the further symmetry there.

Weyl, on the other hand, studied the space-time dilatation, and he introduced the concept of Weyl transformation \[ \Lambda_{\mu\nu} \rightarrow e^{2\sigma} \Lambda_{\mu\nu} \] His motivation was to explain the electromagnetism from geometry. He was the first one who introduced the concept of gauge invariance, and he believed that the Weyl transformation \[ \Lambda_{\mu\nu} \rightarrow e^{2\sigma} \Lambda_{\mu\nu} \] is related to the electro-magnetic gauge invariance \[ A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \sigma. \] Actually, he further noticed that this idea crucially relies on the fact that our universe is \( d = 4 \). Otherwise, the Maxwell theory is not invariant under the Weyl transformation. We refer to \[ \text{for more details on the historical development.} \]

Apart from the historical origin, I cannot help but to think that the structure of the space-time is deeply related to the (non-existence) of the scale invariance without conformal invariance. A non-trivial existence of the interacting field theories crucially depend on the space-time dimensionality. We refer to \[ \text{for an interesting application of this idea to supersymmetric theories. Of course, these are related to the renormalization group flow through our attempts to classify all the quantum field theories. A possible constraint on the classification will be the main subject of the next lecture.} \]

In this lecture, I have tried to collect the list of all the known scale invariant but non-conformal field theories. Here, I will list some more controversial ones, which are not discussed in the main part of the lecture. I welcome your suggestions.

- In this lecture note, we do not discuss a subtle aspect of global conformal invariance in Minkowski space-time. The problem is that the global conformal transformation \[ (2.8) \] can affect the causal structure by making the space-like separation into the time-like one and vice versa. We are satisfied with the infinitesimal conformal symmetry, and do not discuss the global aspects (with possible breakdowns). See \[ \text{for reference and possible resolutions.} \]

- In \[ , \] a non-unitary example of supersymmetric scale invariant but non-conformal field theories was constructed. The structure of the renormalization group flow is Jordan block type, which we never expect in unitary conformal field theories.

- An interesting but confusing example of possible scale invariant but not conformal field theories in \( d = 2 \) is the so-called time-like Liouville theory obtained by a certain “analytic continuation” of the conventional Liouville theory. It was observed in \[ \text{the two-point functions are not diagonal with respect to the operator dimension, suggesting a possible violation of conformal invariance. An alternative interpretation was presented in \[ , \] but the situation is not conclusive.} \]

- In \[ \text{it was pointed out that a certain hermitian deformation of the unitary minimal model shows the periodic structure in its S-matrix, suggesting that the renormalization group flow is cyclic. A similar idea appeared in Zamolodchikov’s work \[ \text{The result seems to be inconsistent with the c-theorem we will discuss in Lecture 2, and I would be happy to know the resolution of the puzzle.} \]

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\[ \text{The Maxwell theory has yet another mysterious symmetry “electric-magnetic duality”, which has its own theoretical impact afterwards.} \]

\[ \text{I would like to thank Prof. Hortacsu for pointing out the reference.} \]

\[ \text{My current understanding is as follows: in the perturbative regime, their renormalization group flow satisfies the gradient formula, and Zamolodchikov’s c-theorem applies. The cyclic structure necessarily brings us to the non-perturbative regime, in which something wrong could occur.} \]
3 Lecture 2

In this lecture, we would like to discuss a possible proof of equivalence between scale invariance and conformal invariance. We begin with the well-established situation in $d = 2$, and expand our surveys in higher dimensions, in which we have partial but promising results. Various attempts are reviewed in relation to the higher dimensional analogue of Zamolodchikov’s $c$-theorem.

3.1 Zamolodchikov-Polchinski theorem

In $d = 2$, we can give a rigorous argument that scale invariance is equivalent to conformal invariance under the following assumptions [5][23] (see also [151])

- unitarity
- Poincaré invariance (causality)
- discrete spectrum in scaling dimension
- existence of scale current
- unbroken scale invariance

Most interesting classes of two-dimensional quantum field theories satisfy these assumptions, but we should remember that if we violate one of them, we may construct counterexamples (see examples in section 2.4.3 and literature guides in section 2.8). One important class of exceptions is the string world-sheet theory, in which the assumption of unitarity and the discreteness of the spectrum are both violated. Thus in string perturbation theory, it is not enough to check the scale invariance, but we have to show the conformal invariance for its consistency.

Let us present the proof. In $d = 2$, it is convenient to use the complex coordinate notation $z = \sigma + i\tau$. $T = T_{zz}$ and $\Theta = T^{\mu}_{\mu}$. The conservation of the energy-momentum tensor gives

$$\bar{\partial}T + 4\partial\Theta = 0.$$  

Following Zamolodchikov [5], we introduce

$$F(|z|^2) = (2\pi)^2 z^4 \langle T(z, \bar{z})\bar{T}(0) \rangle$$

$$G(|z|^2) = (2\pi)^2 z^3 \bar{z} \langle \Theta(z, \bar{z})\bar{T}(0) \rangle$$

$$H(|z|^2) = (2\pi)^2 z^2 \bar{z}^2 \langle \Theta(z, \bar{z})\Theta(0) \rangle,$$

which only depend on $|z|$ by the combination $\log |z|\Lambda$, where $\Lambda$ is the renormalization scale.\[41\] Let us define the $c$-function

$$C = 2 \left( F - \frac{1}{2}G - \frac{3}{16}H \right).$$  

The response to the renormalization group flow is fixed by the conservation (3.1) as

$$\frac{dC}{d\log |z|^2} = -\frac{3}{4}H \leq 0.$$  

\[41\] We use the convention that all the coupling constants are dimensionless.
This is celebrated Zamolodchikov’s c-theorem: the c-function decreases along the renormalization group flow, and it agrees with the central charge at the fixed point (since $H = G = 0$ at the fixed point as we will see).

At the scale invariant fixed point, one can assume that the energy-momentum tensor shows canonical scaling behavior (see section 3.1.2 for a further discussion), so $T_{\mu\nu}$ has a canonical scaling dimension of 2, and hence $C$ is a constant. Then

$$\langle \Theta(z, \bar{z})\Theta(0) \rangle = 0 \quad (3.5)$$

which means from unitarity and causality (according to Reeh-Schlieder theorem [152]), $\Theta(z, \bar{z}) = 0$ as an operator identity. Since $\Theta$ is the trace of the energy-momentum tensor, the scale invariance implies conformal invariance in $d = 2$.

For later purposes, let us expand $\Theta$ with respect to the operators in the theory $\Theta = B^I O_I$. The c-theorem can be expressed as

$$\frac{dc}{d \log \mu} = B^I \chi_{IJ} B^J \geq 0$$

$$\chi_{IJ} = \frac{3}{2} (2\pi)^2 |z|^4 \langle O_I(z, \bar{z})O_J(0) \rangle |z| = \mu^{-1} . \quad (3.6)$$

At this point, we identify $C$ defined in (3.3) with $c(g(\mu))$ as an interpolating function between the central charges $c$ at conformal fixed points. The manifestly positive definite metric $\chi_{IJ}$ is known as Zamolodchikov’s metric. Since it is positive definite, the c-function stays constant along the renormalization group flow if and only if $B^I$ vanishes and the theory is conformal invariant.

There is a physical meaning in $c$ as counting degrees of freedom. If we quantize the conformal field theory on a cylinder with the radial quantization, the scaling dimension of the operator in $\mathbb{R}^{1,1}$ is identified with the energy spectrum on the cylinder. The modular invariance of the partition function dictates that the asymptotic density of states with a given radial energy $E$ is

$$\rho(E) \sim \exp \left(4\pi \sqrt{\frac{cE}{6}} \right) . \quad (3.7)$$

This is known as Cardy formula [153], and it tells that the central charge dictates the effective degrees of freedom of the conformal field theory. It is therefore reasonable that the central charge decreases along the renormalization group flow from our intuition that the renormalization group flow gives a coarse graining and the effective reduction of the degrees of freedom.

We have one comment on Zamolodchikov’s c-theorem. A priori, we know that the c-function (3.3) at $|z| = \mu^{-1}$ is a function of the energy scale $\mu$, but it is not immediately obvious if it is a function of the running coupling constants alone (i.e. $c(\mu) = c(g^I(\mu))$ evaluated at the energy scale $\mu$, and does not depend on the energy scale $\mu$ explicitly. An intuitive reason why the dependence is only through the running coupling constants $g^I(\mu)$ is the renormalizability. Since the renormalized two-point functions do not depend on the renormalization scale $\Lambda$, we obtain the Callan-Symanzik equation for the two-point functions, or more general correlation functions of the energy-momentum tensor (by assuming that there is no anomalous dimension for $T_{\mu\nu}$):

$$\frac{d}{d \log \Lambda} \langle T_{\mu\nu} \cdots \rangle = \left( \frac{\partial}{\partial \log \Lambda} + \beta^I \frac{\partial}{\partial g^I} \right) \langle T_{\mu\nu} \cdots \rangle = 0 . \quad (3.8)$$

In particular it applies to the above c-function constructed out of energy-momentum tensor two-point functions. On the other hand, since $c$ at $|z| = \mu^{-1}$ is a dimensionless quantity, we have the Euler
This explains the simple chain rule
\[ \frac{d}{d \log \mu} c = \beta^I \frac{\partial}{\partial g^I} \hat{c} \left( = B^I \frac{\partial}{\partial g^I} c \right) \] (3.10)
with the running coupling constants \( g^I(\mu) \). Note that when \( T_{\mu\nu} \) is a singlet under the “flavor” rotation, there is no distinction between beta function and \( B \) function here. We also know that at the fixed point, \( c(\mu) \) is a function of the running coupling constants and does not depend on the trajectory of the renormalization group flow since it is specified by the Weyl anomaly and therefore it is intrinsic to the conformal fixed point.

Indeed, the local renormalization group analysis (as we will review in section 3.3.1) tells that the \( c \)-function is actually a function of the running coupling constants alone and does not depend on the trajectory of the renormalization group flow. In particular, within the power-counting renormalization, one can show the “gradient formula” [59]:

\[ 8 \partial_I \hat{c} = (\chi_{IJ}^g + w_{[JJ]}^I)B^J + (P_I g)^J w_J , \] (3.11)
where \( w_{[JJ]} = \partial_J w_J - \partial_J w_I \), and \( P_I \) is a “flavor” rotation as we will explain in more detail later in section 3.3.1. By multiplying it with \( B^J \), we obtain (we use \( P_I B^I = 0 \) which we will prove later in section 3.3.1)

\[ \frac{d \hat{c}}{d \log \mu} = B^I \partial_I \hat{c} = B^I \chi_{IJ} B^J \] (3.12)
with the fact that \( \hat{c} \) only depends on \( \mu \) through the running coupling constants \( g^I(\mu) \). In addition, we can use a certain freedom in local renormalization group flow in order to make the Zamolodchikov metric \( \chi_{IJ} \) in (3.6). Therefore, Zamolodchikov’s \( c \)-function coincides with the \( \hat{c} \)-function that appeared in the local renormalization group analysis, and the \( c \)-function is really a function of the running coupling constants.

The flow equation (3.11) is known as the “gradient formula”. It would have been a true gradient formula if there would be no \( w_J \). See also [154] for further details on the validity of the gradient formula in general quantum field theories in \( d = 2 \).

### 3.1.1 Simple alternative derivation

Without referring to the \( c \)-theorem, there is a more direct way to derive the equivalence between scale invariance and conformal invariance in \( d = 2 \)[40]. For this purpose, we study two-point functions of the energy-momentum tensor in momentum space. The point is that the conservation and the canonical scaling of the energy-momentum tensor gives the unique structure of the two-point functions so that the trace must vanish.

The assumption of the canonical scaling dimension in position space of the energy-momentum tensor leads to the requirement that the momentum space energy-momentum tensor must show (we

\[ ^{42} \text{The argument here is close to the historically original one presented in [154].} \]
use the complex momentum $k = k_x + i k_y$ and $\bar{k} = k_x - i k_y$)

\[
\langle T(k)T(p) \rangle = c \frac{k^3}{k} \delta(k + p)
\]

\[
\langle T(k)\Theta(p) \rangle = e k^2 \log |k|^2 \delta(k + p)
\]

\[
\langle \Theta(k)\Theta(p) \rangle = h |k|^2 \log |k|^2 \delta(k + p)
\]

\[
\langle T(k)\bar{T}(p) \rangle = w |k|^2 \log |k|^2 \delta(k + p)
\]

where we have neglected the contact terms that are polynomial in $k$ and $\bar{k}$.

The conservation of the energy-momentum tensor (again up to the contact terms) requires

\[
e = 0 \ , \ h = 0 \ , \ w = 0.
\]

From the Reeh-Schlieder theorem by going back to the position space, we conclude that $\Theta(x) = 0$ as an operator identity. Thus, the scale invariance implies conformal invariance. If we had kept track of the contact terms, we could see $\langle \Theta(x)\Theta(0) \rangle$ contains the contact term proportional to $c$, which can be related to the Weyl anomaly from the second variation of the effective action with the metric.

We may attempt a similar derivation in higher dimensions [23][86]. However, we can immediately realize that the number of independent two-point functions of the energy-momentum tensor is larger than the constraint from conservation and unitarity even if we assumed the canonical scaling of the energy-momentum tensor, so we cannot derive the similar result in this way. In retrospect, we have a good reason for this: two-point functions of the energy-momentum tensor do not seem to be a good barometer to show the higher dimensional analogue of Zamolodchikov’s $c$-theorem as we will see.

3.1.2 Some technical details

In this section, we will discuss some technical details we have encountered in the derivation of the Zamolodchikov-Polchinski theorem.

- Canonical scaling of $T_{\mu\nu}$:

In Zamolodchikov’s argument, we tacitly assumed that the energy-momentum tensor has a canonical scaling dimension. This assertion can be proved in $d = 2$ with the assumption of the discreteness of scaling dimensions of operators in the theory (in particular, there is no dimension zero operator other than the identity operator) [23]. The canonical scaling of the energy-momentum tensor is violated when $T_{\mu\nu}$ is not an eigenoperator under dilatation:

\[
i[D, T_{\mu\nu}] = x^\rho \partial_\rho T_{\mu\nu} + d T_{\mu\nu} + y_a \partial^\sigma \partial^\rho \hat{Y}^a_{\mu\sigma\nu\rho},
\]

where $\hat{Y}^a_{\mu\sigma\nu\rho}$ is the complete set of tensor operators (excluding the trivial contribution from the identity operator) that have the symmetry of Riemann tensor and the scaling properties

\[
i[D, \hat{Y}^a_{\mu\sigma\nu\rho}] = x^\rho \partial_\rho \hat{Y}^a_{\mu\sigma\nu\rho} + \tilde{\gamma}_b^a \hat{Y}^b_{\mu\sigma\nu\rho}.
\]

This is still consistent with the scale invariance because the algebra of $D$ and $P_\mu$ is not affected from this mixing thanks to the space integration.

Polchinski argued that in $d = 2$, one can always improve the energy-momentum tensor so that it has a canonical scaling dimension as long as there is no dimension zero operators than the identity operator. He introduced the improved energy-momentum tensor by

\[
\Theta'_{\mu\nu} = T_{\mu\nu} + y^a (d - 2 - \tilde{\gamma})^{-1}_{ab} \partial^\sigma \partial^\rho \hat{Y}^b_{\mu\sigma\nu\rho}.
\]
The discreteness of the scaling dimensions, and the absence of dimension zero operators allows us to invert the matrix in $d = 2$, so the energy-momentum tensor with canonical scaling dimension always exists.

In other dimensions, this argument is subtle because if the scaling dimension matrix $\hat{\gamma}$ has an eigenvalue $d$, then one cannot invert the matrix. Within power counting renormalization in $d = 4$, in many non-trivial examples [72][73], one can explicitly construct the so-called new improved energy-momentum tensor that is not renormalized and finite [35], and therefore it has the canonical scaling dimension. The new improved energy-momentum tensor is defined so that its trace vanishes at the conformal fixed point. Indeed, once we assume the existence of the new improved energy-momentum tensor, the Wess-Zumino consistency condition of the local renormalization group flow (see discussions in section 3.3.1) constrains the mixing of operators in a non-trivial way, and it makes it possible to choose a basis in which energy-momentum tensor has a canonical dimension (see [85][155]).

• Reeh-Schlieder theorem:

In the discussion of the Zamolodchikov-Polchinski theorem, we have used the Reeh-Schlieder theorem. In order to justify the discussion, we have to show $O(x)|0\rangle = 0 \iff O(x) = 0$ in general quantum field theories. This is trivially true in CFT by state operator correspondence, but we cannot assume conformal invariance here. Still unitarity tells that $\langle O(x)O(0) \rangle = 0$ implies $O(x)|0\rangle = 0$ and there is a theorem in general quantum field theories that this is sufficient to conclude that $O(x) = 0$ for any local operators. Obviously this is not true for non-local operators such as charge or momentum.

The proof is not elementary [152], so we only give a sketch of the physical idea. What we would like to show boils down to the claim that in any correlation functions, the insertion of $O(x)$ is zero except for contact terms:

$$\langle 0|O_1(x_1)\cdots O(x)\cdots O_n(x_n)|0\rangle = 0 . \quad (3.18)$$

To see this, we suppose that the insertion point $x$ is space-like separated with all the other $x_i$. Then the microscopic causality demands $[O_i(x_i), O(x)] = 0$, so

$$\langle 0|O_1(x_1)\cdots O(x)\cdots O_n(x_n)|0\rangle = \langle 0|O_1(x_1)\cdots \cdots O_n(x_n)O(x)|0\rangle \quad (3.19)$$

vanishes by acting $O(x)$ on the vacuum. The correlation functions (more precisely Feynman $T^*$ function) in the other causal domains are related by analytic continuation, so they must vanish identically in any causal domain.

The argument crucially relies on causality, so if the theory does not have a notion of causality, the proof fails. A good example is the Schrödinger field theory in which the annihilation operator $\Psi(x)$, which is local, annihilates the vacuum, but it is obviously not zero identically. We also note that for non-local operators like charge or momentum, the above argument does not apply.
3.1.3 Averaged c-theorem

Zamolodchikov’s argument can be presented in a slightly different way [154]. Define

\[ c^M_{(2)} = - \int d^2 x G(\mu) \langle \Theta(x) \Theta(0) \rangle \]

\[ \Theta = B^I O_I \]

\[ \chi_{IJ} = - \frac{d}{d \log \mu} \int d^2 x G(\mu)(x) \langle O_I(x) O_J(0) \rangle \]

(3.20)

where

\[ G(\mu)(x) = 3\pi x^2 \theta(1 - \mu |x|) \, . \quad (3.21) \]

The metric \( \chi_{IJ} \) is positive definite from unitarity (no dangerous contact term will contribute because \( \frac{dG(\mu)}{d \log \mu} \) has a support only when \( O_I \) are separated). It can be easily shown that

\[ \frac{dc^M_{(2)}}{d \log \mu} = B^I \chi_{IJ} B^J \geq 0 \, . \quad (3.22) \]

We note that this \( c^M_{(2)} \)-function is equivalent to the one in section 3.1.

One can now repeat the same analysis in \( d \geq 2 \). We define

\[ c^M_{(d)} = - \int d^d x G^{(d)}(\mu) \langle \Theta(x) \Theta(0) \rangle \]

\[ \Theta = T^\mu \mu O_I \]

\[ \chi_{IJ} = - \frac{d}{d \log \mu} \int d^d x G^{(d)}(\mu)(x) \langle O_I(x) O_J(0) \rangle \]

(3.23)

where

\[ G^{(d)}(\mu)(x) = 3\pi x^d \theta(1 - \mu |x|) \, . \quad (3.24) \]

The metric is again positive definite from unitarity. It can be easily shown that

\[ \frac{dc^M_{(d)}}{d \log \mu} = B^I \chi_{IJ} B^J \geq 0 \, . \quad (3.25) \]

Can we declare the proof of c-theorem in any dimension? Does it mean scale invariance implies conformal invariance in any dimension?

A related idea was explored in [156]. The integrated \( c^M_{(d)} \)-function is known as the averaged c-function. In the later works [157] [158] [159], it was argued that the integral of the two-point functions of the trace of the energy-momentum tensor (3.1.3) is directly related to the difference of \( \tilde{b} \) coefficient in the Weyl anomaly in \( d = 4 \) [43]. It was also argued that although \( \tilde{b} \) itself is scheme dependent, the ambiguity cancels in the difference of the ultraviolet (UV) fixed point and infrared (IR) fixed point, and the integral only depends on the trajectory of the renormalization group flow. Since it depends on the trajectory, the quantity has a very different nature than Zamolodchikov’s c-function in \( d = 2 \). We have no direct way to connect the averaged c-function to the local correlation function so we cannot use the Callan-Symanzik equation to trade the \( \mu \) dependence with beta functions.

43More precisely, we had to fine-tune local counterterms (see section 3.3.1) to achieve this claim.
What is special in \(d = 2\) is the identity \[54\]

\[
\partial^\mu[(2x^\nu x^\rho x^\sigma - 2x^2 x^\nu \eta^{\rho\sigma} - x^2 x^\sigma \eta^{\rho\nu})(T_{\mu\nu}(x)T_{\rho\sigma}(0))] = -3x^2(\Theta(x)\Theta(0)). \tag{3.26}
\]

It enables us to integrate \(c^M_2\) by part to rewrite the averaged quantity \(3.20\) into the local form

\[
c^M_2(\mu) = 2\pi^2(2x_\mu x_\nu x_\rho x_\sigma - x^2 x_\mu x_\nu \eta^{\rho\sigma} - x^2 x_\sigma x_\rho \eta^{\nu\mu} - x^2 x_\rho x_\sigma \eta^{\nu\mu})(T_{\mu\nu}(x)T_{\rho\sigma}(0))|_{\mu = 1}, \tag{3.27}
\]

which is nothing but the one defined by Zamolodchikov. The application of the Callan-Symanzik equation leads to the claim that \(\mu\) dependence is only through the running coupling constants.

We will come back to this point later when we discuss the renormalization scale dependence in the proof of the higher dimensional analogue of Zamolodchikov’s \(c\)-theorem and its application to scale invariance and conformal invariance. Here we only emphasize that the crucial distinction between \(d = 2\) and \(d > 2\) in relation to the argument of this section is that the so-defined averaged \(c\)-function is not an intrinsic quantity of the fixed point, but it is a quantity of the flow. In particular, the equation \(3.25\) by itself is consistent with the cyclic renormalization group flow with \(B^I \neq 0\) because there is no reason why \(c^M_{(d)}\) should take a constant value when the theory is scale invariant. However, it is remarkable to mention that within a few orders in perturbation theory when the theory is classically conformal invariant, \(3.23\) gives the same renormalization scale dependence as that for the higher dimensional analogue of Zamolodchikov’s \(c\)-function we will discuss in the next section, which only depends on the running coupling constants at the scale \(\mu\).

### 3.2 Cardy’s conjecture (a.k.a “\(a\)-theorem”)

Cardy conjectured \[160\] that in \(d = 4\), the higher dimensional analogue of Zamolodchikov’s \(c\)-function is given by the Weyl anomaly \(a\). As discussed in section 2.3, the Weyl anomaly in \(d = 4\) is given by

\[
\langle T_{\mu\nu} \rangle = c(\text{Weyl})^2 - a\text{Euler} + \tilde{b}\square R + d\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu} \alpha\beta R_{\alpha\beta\rho\sigma}. \tag{3.28}
\]

We know \(\tilde{b}\) is not universal, and \(c\) does not show monotonicity along the renormalization group flow (see the examples below) so they cannot be the candidate. The remaining possibility is \(a\). We recall that in \(d = 2\), the scalar curvature \(R\) can be also regarded as the Euler density, and it shares the common feature with \(a\) in \(d = 4\). In general even dimensions, the Euler density (A-type Weyl anomaly in the classification of \([\mathbf{8}]\)) seems to be a good candidate. Cardy formulated it as \(a = -\frac{1}{2d-2\pi^2} \int_{S^d} d^4x\sqrt{g} (T_{\mu\nu})\) because we can see that Weyl tensor vanishes on \(S^4\) since it is conformally flat and \(\square R\) term is zero after integration (and we assumed \(d = 0\)). Note that the Euler characteristic of \(S^{2n}\) is given by

\[
\chi(S^{2n}) = \int_{S^{2n}} d^{2n}x\sqrt{g}_{(2n+1)} \frac{1}{(2\pi^2)^{n+1}} \epsilon_{\mu_1\cdots\mu_{2n}} \epsilon_{\nu_1\cdots\nu_{2n}} R^{\mu_1\nu_1\mu_2\nu_2} \cdots R^{\mu_{2n-1}\nu_{2n-1}\nu_2\mu_2} = 2.
\]

The conjecture can be stated in different versions.

- **Weak version:** \(a_{1R} \geq a_{1UV}\) between the flow of two CFTs, which is proved in \([161] [162]\) recently.

- **Strong version:** \(\frac{da(g(\mu))}{d\log \mu} \geq 0\) along the renormalization group flow.

- **Gradient formula:** \(\beta I = \chi^- I \partial_j a\) (we will make more precise about the statement).\[^44\]

\[^44\]Even in \(d = 2\), the proof of the gradient formula is much more non-trivial than that of the \(c\)-theorem discussed in section 3.1. Within the power-counting renormalization it was discussed in \([50]\). With relevant perturbations, we find more recent discussions in \([154]\).
The gradient formula suggests \[ \frac{d\beta(g(\mu))}{d\log \mu} = \beta^I \chi_{IJ} \beta^J, \] so it is also called the strongest version (with a tacit assumption \( \chi_{IJ} \) is positive definite). The gradient formula for the renormalization group flow in \( d = 4 \) was first discussed in \[ 163 \] [164].

So far, we have introduced the \( a \)-function that can be a candidate for the higher dimensional analogue of Zamolodchikov’s \( c \)-function from the Weyl anomaly. Since Weyl anomaly appears once we put our conformal field theories on a curved background, we have a natural question if we can read the Weyl anomaly coefficients \( a \) and \( c \) from correlation functions in the flat space-time. We know that in \( d = 2 \), the Weyl anomaly \( c \) is related to the two-point function of the energy-momentum tensor. The point is that the Weyl anomaly \( c \) appears in the contact terms of two-point functions of the trace of the energy-momentum tensor in \( d = 2 \), and the conservation condition relates it to the two-point functions of the energy-momentum tensor \( T_{zz} \). Therefore, we can compute the Weyl anomaly \( c \) by studying the two-point functions of \( T_{zz} \).

In \( d = 4 \), the situation is more complicated. We can show that the two-point function of the energy-momentum tensor for conformal field theories is completely fixed by the Weyl anomaly \( c \). The three-point function of the energy-momentum tensor depends on three independent numbers, which are determined by \( a \) and \( c \) (and one additional parameter). The way to read \( a \) from the three-point functions of the energy-momentum tensor has been developed in \[ 157 \] [165] (see also appendix \( \text{A.2} \)).

What else do we know about \( a \) and \( c \)? In \[ 166 \], the general bound on the value of \( a \) and \( c \) has been derived. The bound on \( c \geq 0 \) is easily obtained from the positivity of the energy-momentum tensor two-point function. The bound on \( a \) is more non-trivial: they studied the energy flux operator

\[
E(\theta) = \int d^2r^2 n^i T^i(t, r\vec{n})|_{r\to\infty} \tag{3.29}
\]

and assumed its positivity from the averaged null energy condition. Then by imposing the condition of the positivity of the energy flux operator for the state constructed out of the energy-momentum operator \( T_{\mu\nu} \), they schematically required

\[
\langle T_{\mu\nu}|E(\theta)|T_{\mu\nu} \rangle \geq 0 , \tag{3.30}
\]

which is related to \( a \) and \( c \) from the three-point functions of \( T_{\mu\nu} \). The resulting condition is

\[
\frac{31}{18} c \geq a \geq \frac{1}{3} c . \tag{3.31}
\]

Since \( c \) is bounded from below \( c \geq 0 \), \( a \) is also bounded from below \( a \geq \frac{1}{3} c \geq 0 \). The bound of \( a \) will be important in our later discussions. The bound \( a \geq 0 \) was also discussed in the paper \[ 167 \] by assuming the “quantum modified null energy-condition”.

Let us come back to the comparison between \( a \) and \( c \) under renormalization group flow. To motivate \( a \) rather than any other linear combinations of the Weyl anomaly coefficients, we will now show examples of renormalization group flow in which \( c \) increases. We consider \( SU(N_c) \) SQCD with \( N_f \) fundamental flavors within conformal window \( \frac{2}{3} N_c \leq N_f \leq 3N_c \). Supersymmetry allows the exact computation of \( a \) and \( c \) both in UV and IR (see section \( \text{3.3.2} \)).

\[
a_{\text{UV}} = \frac{1}{48} (9N^2 + 9 + 2N_f N_c) \]

\[
c_{\text{UV}} = \frac{1}{16} (3N^2 - 3 + 2N_f N_c) \tag{3.32}
\]
and
\[
a_{\text{IR}} = \frac{3}{16} \left( 2N_c^2 - 1 - 3 \frac{N_f^4}{N_f^2} \right)
\]
\[
c_{\text{IR}} = \frac{1}{16} \left( 7N_c^2 - 2 - 9 \frac{N_f^4}{N_f^2} \right).
\] (3.33)

One can see that near \( N_f \sim 3N_c, c_{\text{UV}} - c_{\text{IR}} \leq 0 \) so \( c \) cannot be monotonically decreasing along the renormalization group flow \[168\]. On the other hand, one can see that \( a_{\text{UV}} - a_{\text{IR}} \geq 0 \). By taking \( N_f \rightarrow 3N_c \) limit, we can further conclude that any other combinations of \( a + kc \) are not monotonically decreasing for \( k > 0 \).

### 3.3 Toward a proof in \( d = 4 \)

We give a review of some earlier attempts to prove Cardy’s conjecture in \( d = 4 \). They will give some insights about the renormalization group flow even though the argument is not complete.

#### 3.3.1 Local renormalization group

Local renormalization group analysis gives a very strong constraint on the renormalization group flow even in the flat space-time limit. Indeed, the analysis gives a perturbative proof of the strong version of Cardy’s conjecture \[128\][59] as well as the gradient formula. Since the argument is based on the generic consistency conditions of the effective action, the positivity of the target space metric appearing in the effective action, for instance, was not derived (because their argument works also in non-unitary field theories). Nevertheless, it was shown that perturbatively, \( a \)-theorem is true, and scale invariance implies conformal invariance from the subsequent result.

The idea of the local renormalization group is to generalize the Wess-Zumino consistency condition for the Weyl anomaly mentioned in section 2.3 not only in the non-trivial metric background but with the space-time dependent coupling constants (a.k.a Schwinger’s source theory). This is conceptually very natural because if we consider the non-trivial renormalization group flow, the Weyl transformation acts on coupling constants non-trivially, so the coupling constants must be treated space-time dependent after the Weyl transformation even if we started with a constant background. In addition to the space-time dependent coupling constants, for each current operators, we will introduce the background gauge field \( A_\mu \). If the currents are non-conserved, we further transform the coupling constants under the background gauge transformation, so that the theory is spuriously invariant. We also note that the space-time dependent source is natural in AdS/CFT correspondence as we will discuss in Lecture 3.

As a consequence, in order to properly regularized and renormalize the theory, we have to introduce various additional counterterms that are not present in the flat space-time limit, and the consistency of the renormalization group flow will give more non-trivial constraints, where the consequence will be the main subject of our analysis. Unfortunately, the entire analysis is slightly complicated partly because there are many terms, which is not essential but technical, so we will focus on the points relevant for our discussions, and leave the other aspects to the original literature \[59\] (see also appendix A.3).

Let us first revisit the operator Weyl anomaly. In addition to the scalar beta functions corresponding to background coupling constants \( g'^I \), we have to introduce the term given by the beta function for the background currents \( A_\mu^a \). Within power-counting renormalization, we have the field dependent
part of the trace of the energy-momentum tensor as

$$T^\mu_{\mu;\text{field}} = \beta^I O_I + \rho^I(g)(D_{\mu}g)^I J^\mu_\alpha + D_{\mu}(S^\alpha(g)J^\mu_\alpha)$$  \hspace{1cm} (3.34)

up to the terms that vanish upon using the equations of motion. The second term is particular to the space-time dependent coupling constants, and the third term is related to the vector beta function term discussed in section 2.4.2.

As discussed before, due to the operator identities, or in this case, due to the background gauge independence, the last term \(3.34\) can be removed

$$T^\mu_{\mu;\text{field}} = B^I O_I + P^I(g)(D_{\mu}g)^I J^\mu_\alpha$$  \hspace{1cm} (3.35)

up to the term that vanishes by equations of motion, where \(B^I = \beta^I - (Sg)^I\) and \(P^I(g)(D_{\mu}g)^I = \rho^I(D_{\mu}g)^I + (D_{\mu}S^\alpha)\). This is nothing but the partial current conservation, and \(B^I\) are the same \(B\) functions introduced in section 2.4.

Correspondingly, the Weyl variation generator is given by

$$\delta_{\sigma(x)} = - \int d^4 x \sqrt{|g|} \sigma(x) \left(2g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} - B^\mu \frac{\delta}{\delta g^\mu} - P^\mu D_{\mu}g^I \frac{\delta}{\delta A^I_\mu} \right).$$  \hspace{1cm} (3.36)

If there were no anomaly, \(\delta_{\sigma(x)} S_{\text{eff}}[g_{\mu\nu}, g^I, A^I_\mu] = - \int d^4 x \sqrt{|g|} \sigma \left(T^\mu_{\mu;\text{field}} - B^I O_I - P^I(g)(D_{\mu}g)^I J^\mu_\alpha \right) = 0,\) and this would be nothing but the trace identity. In the following, we study consistency condition on the anomalous terms in the Weyl variation.

To go further, we demand the Wess-Zumino consistency condition as in section 2.3,

$$[\delta_{\sigma(x)}, \delta_{\tilde{\sigma}(x')}S_{\text{eff}}[g_{\mu\nu}, g^I, A^I_\mu] = 0 ,$$  \hspace{1cm} (3.37)

but now the effective action \(S_{\text{eff}}[g_{\mu\nu}, g^I, A^I_\mu]\) depends not only on the background metric but also background space-time dependent coupling constants as well as background gauge fields.

In section 2.3, we wrote down all the possible first order variation of \(S_{\text{eff}}\) from the metric alone. Similarly, we should consider all the possible invariant terms (within power-counting renormalization) from \(g_{\mu\nu}, g^I\) and \(A^I_\mu\), and study the consistency equations. We only focus on three terms that are relevant for our discussions:

$$-\delta_{\sigma(x)} S_{\text{eff}}[g_{\mu\nu}, g^I, A^I_\mu] = - \int d^4 x \sqrt{|g|} \left(a \sigma \text{Euler} + \frac{1}{2} \sigma G^{\mu\nu} \chi^I_{J\mu}D_{\mu}g^I D_{\nu}g^J + \partial_\mu \sigma G^{\mu\nu} (w_I D_{\nu}g^I) + \cdots \right),$$  \hspace{1cm} (3.38)

where we recall \(G_{\mu\nu}\) is the Einstein tensor. The right hand side is regarded as the Weyl anomaly on the curved background with space-time dependent coupling constants because (3.36) gives the trace of the energy-momentum tensor from the left hand side of (3.38). A particular class of the Wess-Zumino consistency condition demands (we refer [59] and appendix A.3 for the full details)

$$8 \partial_\alpha a = \chi^I_{J\mu}B^I - \partial_J w_I B^J - \partial_I B^J w_J + (P_I g)^J w_J$$

$$B^I P_I^a = 0 .$$  \hspace{1cm} (3.39)

Here \((P_I g)^J = P_I^a T^a_{J\mu} K g^K\) with some representation matrix \(T^a\) of the “flavor symmetry”. The former equation in (3.39) comes from the term proportional to \(G^{\mu\nu} D_{\mu}g^I \sigma \partial_\nu \tilde{\sigma}\), and the latter comes from the consistency of

$$0 = \left[ \int d^4 x \sqrt{|g|} \sigma(x) \left( B^I \frac{\delta}{\delta g^I} + P^I D_{\mu}g^I \frac{\delta}{\delta A^I_\mu} \right), \int d^4 y \sqrt{|\tilde{g}|} \tilde{\sigma}(x') \left( B^I \frac{\delta}{\delta g^I} + P^I D_{\mu}g^I \frac{\delta}{\delta A^I_\mu} \right) \right]$$

$$= \int d^4 x \sqrt{|g|} (\sigma \partial_\mu \tilde{\sigma} - \tilde{\sigma} \partial_\mu \sigma) B^I P_I^a \frac{\delta}{\delta A^I_\mu} .$$  \hspace{1cm} (3.40)
Now we proceed to the physical interpretation of the Wess-Zumino consistency condition. If we define \( \tilde{a} = a + \frac{1}{8} w_I B^I \), the first line of (3.39) gives the flow equation or “gradient formula”

\[
8 \partial_I \tilde{a} = (\chi_{IJ}^g + w_{[I,J]}) B^J + (P_I g)^J w_J ,
\]

where \( w_{[I,J]} = \partial_I w_J - \partial_J w_I \).

One important consequence of the “gradient formula” is

\[
\frac{d \tilde{a}}{d \log \mu} \equiv B^I \partial_I \tilde{a} = B^I \chi_{IJ}^g B^J
\]

from \( B^I P_I^a = 0 \). This means that \( \tilde{a} \)-function would be decreasing monotonically along the renormalization group flow (defined by \( \frac{d a^I}{d \log \mu} = B^I \)) if the metric \( \chi_{IJ}^g \) is positive definite. Since we have not assume any physical requirement such as unitarity, the argument here cannot say the positivity of the metric, but in perturbation theory, we can check that this metric is positive definite.

The \( \tilde{a} \)-function of the local renormalization group flow is not unique. The flow equation itself is invariant under the dressing transformation

\[
\delta \chi_{IJ}^g = L_B C_{IJ} = B^K \partial_K C_{IJ} + C_{JK}(\partial_I B^K - (P_I g)^K) + C_{IK}(\partial_J B^K - (P_J g)^K),
\]

\[
\delta w_I = -8 \partial_I A + C_{IJ} B^J , \quad \delta \tilde{a} = B^I C_{IJ} B^J ,
\]

where \( C_{IJ} \) and \( A \) are curved space-time counterterms that can be chosen as an arbitrary tensor of coupling constants. Note that \((P_J g)^K \partial_K A = 0 \) due to the gauge invariance of \( A \).

The reason why we can add coupling constant dependent local counterterms

\[
S_{ct} = - \int d^4 x \sqrt{|g|} \left( -\frac{1}{2} G^{\mu\nu} C_{IJ}(g^I) D_\mu g^I D_\nu g^J - A(g^I) \text{Euler} \right) ,
\]

which generates the additional terms in the trace of the energy-momentum tensor so that we have the dressing transformation as in (3.43). These are related to the ambiguities in the contact terms in various correlation functions among \( T^\mu_\nu \) and \( O^I \) (see [59] for details). There are more terms we could add than (3.44) but they do not contribute to our discussions on our \( a \)-theorem.

Let us point out one important consequence of the formula (3.42). As pointed out in [65], the scale invariance demands that Osborn’s \( \tilde{a} \)-function must take a constant value. By assuming the positivity of \( \chi_{IJ}^g \), it means that \( B^I = 0 \) with the scale invariance, forbidding the cyclic behavior.

The trace identity (3.39) tells that the energy-momentum tensor is traceless in the flat space-time limit, and the theory must be conformal invariant.

The similar ambiguity existed in \( d = 2 \) [55], in which we can introduce the scheme dependent \( \tilde{a} \)-function from the local renormalization group analysis with the ambiguity as in (3.43). From this viewpoint, the main claim of Zamolodchikov is that one can choose a good counterterm \( C_{IJ} \) so that \( \chi_{IJ}^g \) agrees with Zamolodchikov metric and positive definite. Or more precisely what Zamolodchikov did is he first read the counterterms from the two-point functions and then defined the monotonically

\footnote{Indeed, we can show \( \chi_{IJ}^g \) is always positive definite at the unitary conformal fixed point when \( B^I = 0 \) (with a suitable choice of counterterms). Thus, the deviation is small as long as \( B^I \) are small in perturbation theory.}

\footnote{By using this ambiguity, one can show that the \( R^2 \) Weyl anomaly is given by \( B^I \chi_{IJ}^g B^J \), which is expected because when a theory is conformal invariant, \( R^2 \) anomaly must vanish. \( 2 \chi_{IJ}^g \) agrees with \( \chi_{IJ}^g \) in a certain order of perturbation theory (indeed \( 2 \chi_{IJ}^g = \chi_{IJ}^g \) when \( B^I = 0 \) and if we set \( S_{IJ} \) by using the further ambiguity), but they can deviate at the higher order.}
decreasing \(c\)-function by considering a particular combination to cancel the ambiguity. We may not be able to remove the anti-symmetric part \(w_{IJ}\) unless \(w_I\) is closed, but this is unimportant for the strong version of Zamolodchikov’s \(c\)-theorem which we derived in section 3.1. Also note the above ambiguity does not affect the value of the \(a\)-function at the conformal fixed point because \(\mathcal{B}^I = 0\) there. It is consistent with the fact that at the conformal fixed point the trace anomaly does not have a local counterterm.

3.3.2 \(a\)-maximization

In supersymmetric field theories, the conformal anomaly is directly related to the anomaly of the superconformal \(\mathcal{R}\)-current. From the structure of the superconformal supermultiplet [168], we can show

\[
a = \frac{9}{16 \cdot (8\pi)^2} \left(3\text{Tr}\mathcal{R}^3 - \text{Tr}\mathcal{R}\right)
\]

\[
c = \frac{9}{16 \cdot (8\pi)^2} \left(3\text{Tr}\mathcal{R}^3 - \frac{5}{3}\text{Tr}\mathcal{R}\right).
\]

The Tr here means a schematic notation to compute the triangle anomaly of \(\mathcal{R}\)-currents. By using the formula, once we can determine the superconformal \(\mathcal{R}\)-current, we may compute \(a\) and \(c\). In particular, if the conformal \(\mathcal{R}\)-symmetry is a well-defined symmetry in the UV theory, we can use the ’t Hooft anomaly matching argument [170] and evaluate (3.45) by using the free field theory computation.

To compute \(a\) (and \(c\)), all we need is to determine the superconformal \(\mathcal{R}\)-current, and there is a principle so called “\(a\)-maximization” [171]. The principle gives a three-line proof of the weak version of Cardy’s conjecture in supersymmetric field theories under some technical assumptions. The idea of \(a\)-maximization is that under all possible candidates of \(U(1)_{\mathcal{R}}\) symmetry, the superconformal one is the one that maximize the “trial \(a\)-function”: \(a_{\text{trial}} = 3\text{Tr}\mathcal{R}^3 - \text{Tr}\mathcal{R}\). Since relevant deformations generically break the flavor symmetries, the set for candidates of the \(\mathcal{R}\)-symmetry in IR is a subset of that in UV. Thus, the maximized \(a\) must satisfy \(a_{\text{UV}} \geq a_{\text{IR}}\).

While \(a\)-maximization is very powerful in practice, we have a limitation. One limitation is that we might have accidental symmetries in IR that could spoil the above argument because the set for the candidate \(\mathcal{R}\)-symmetry is no longer a subset of that for UV. However, since we have a more general proof of the \(a\)-theorem, this means that the mixing of accidental symmetry at the superconformal fixed point is somehow bounded (see [172] for discussions). We also note that the \(a\)-maximization argument relies on the existence of the superconformal fixed point, so we cannot exclude scale invariant but non-conformal supersymmetric field theories from this argument (see e.g. [173][8]). Furthermore, in principle, supersymmetric field theories can be scale invariant without any \(\mathcal{R}\)-symmetry [8]. For instance, the non-renormalizable theory with the supersymmetric action

\[
\mathcal{S} = \int d^4x d^4\theta |\Phi|^2(\Phi^2 + (\Phi^\dagger)^2) + \int d^4x d^2\theta \Phi^6 + c.c,
\]

where \(\Phi\) is a chiral superfield, is classically scale invariant, but not \(\mathcal{R}\)-symmetric, and therefore it is not superconformal invariant [8].

3.4 Proof of weak \(a\)-theorem

There have been various attempts to prove the \(a\)-theorem in \(d = 4\). Finally, Komargodski and Schwimmer gave a reasonable and ingenious physical argument for the weak version of the theorem.
Consider the renormalization group flow from CFT$_{UV}$ to CFT$_{IR}$ in $d = 4$. For technical simplicity, we assume that both are Weyl invariant for a while. We assume that the flow is induced by adding a relevant deformation $O$ with the conformal dimension $\Delta$ to the UV CFT so that under the Weyl transformation $g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$ it transforms as $O \rightarrow e^{-\Delta \sigma} O$. From unitarity, the deformation must be a conformal primary operator.

The deformed theory is no-longer Weyl invariant, but we may introduce the "dilaton" $\tau$ to compensate the violation of the Weyl invariance due to the deformation. We can always do it by dressing the Weyl transformation with $\tau$. Under the Weyl transformation, we assume that the dilaton transforms as $\tau \rightarrow \tau + \sigma$ to make the deformation spuriously Weyl invariant: $\sqrt{|g|} e^{-\Delta \sigma} \tau \rightarrow \sqrt{|g|} e^{-(\Delta-\sigma)} \tau$.

The dilaton compensated UV action is schematically given by

$$S_{UV} = S_{\text{CFT}_{UV}} + \int d^4x \sqrt{|g|} e^{-(\Delta-\Delta)^\tau} O + S_{\text{ct}} + f^2 \int d^4x \sqrt{|g|} e^{-2\tau} \left((\partial_\mu \tau)^2 + \frac{1}{6} R\right) + \tilde{S}_{\text{nu}} \ ,$$

where $S_{\text{ct}}$ is the dilaton compensated (relevant) counterterms that contain various relevant operators in the UV CFT (including cosmological constant) that will be fine-tuned during the renormalization group flow so that it will end up with the desired IR fixed point. The kinetic term for the dilaton (added by hand) is Weyl invariant by itself. Here $f$ is arbitrarily large dimensionful decay constant of the dilaton, which we could add if we wish. $\tilde{S}_{\text{nu}}$ is Weyl invariant non-universal dilaton counterterms that can again be introduced by hand.

The dilaton is very weakly coupled (or we can regard it as an external source) so it will not affect the dynamics or the properties of the IR CFT. This is equivalent to the claim that the dilaton will decouple in the IR physics so that the IR effective action has the decoupled form

$$S_{\text{eff}} = S_{\text{CFT}_{IR}} + f^2 \int d^4x \sqrt{|g|} e^{-2\tau} \left((\partial_\mu \tau)^2 + \frac{1}{6} R\right) + S_{\text{WZ}} + S_{\text{nu}} \ .$$

The dilaton decay constant $f$ and the non-universal term $S_{\text{nu}}$ can be different from those of UV, but this does not affect the following discussions. In addition, we may want to introduce counterterms that are associated with the position dependent coupling constants in relation to the local renormalization group flow discussed in section 3.3.1. As will be discussed in section 66, these do not affect the analysis in this section basically because we assume UV and IR theories are Weyl invariant and $B$ function vanishes.

The 't Hooft anomaly matching condition [170] (for Weyl invariance [174]) fixes the form of the Wess-Zumino term $S_{\text{WZ}}$

$$S_{\text{WZ}} = \int d^4x \sqrt{|g|} \left((a_{UV} - a_{IR}) \left(\tau \text{Euler} + 4G_{\mu\nu} \partial^\mu \tau \partial^\nu \tau - 4(\partial_\mu \tau)^2 \Box \tau + 2(\partial_\mu \tau)^4\right)
- (c_{UV} - c_{IR}) \tau (\text{Weyl})^2
- (\tilde{b}_{UV} - \tilde{b}_{IR}) \frac{R^2}{12}\right) \ .$$

If the theory breaks CP invariance, there is a potential addition of Pontryagin term

$$(d_{UV} - d_{IR}) \tau (\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu} a_{\alpha\beta} R_{\alpha\beta\rho\sigma})$$

but it will play no role in the following (see footnote 49). The necessity of the anomaly matching is as follows. Suppose we would like to hypothetically gauge the Weyl symmetry. We had to cancel...
the Weyl anomaly of the UV theory. We do it by adding (non-unitary) spectator Weyl invariant theory with the opposite Weyl anomaly. The consistency of the gauging suggests that the IR theory must show the same anomaly (‘t Hooft matching condition) to cancel the contribution from the added spectators. Because the anomaly of the IR theory with \( S_{\text{CFT}} \) is different than that of the original theory, it must be somehow compensated. This is precisely what Wess-Zumino terms do. Under the classical Weyl variation \( g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu} \) and \( \tau \to \tau + \sigma \), the Wess-Zumino terms give

\[
\delta_\sigma S_{\text{WZ}} = \int d^4x \sqrt{|g|} \left( (a_{\text{UV}} - a_{\text{IR}})\sigma \text{Euler} - (c_{\text{UV}} - c_{\text{IR}})\sigma \text{(Weyl)}^2 + (b_{\text{UV}} - b_{\text{IR}})\sigma \Box R \right). \tag{3.51}
\]

The Wess-Zumino term can be obtained by trial and error or by using the Wess-Zumino trick [174] (see also [175] for higher dimensional computations)\(^4\)

\[ S_{\text{WZ}} = (a_{\text{UV}} - a_{\text{IR}}) \int_0^1 dt \int d^4x \tau \sqrt{|g|} \text{Euler}(g \to ge^{-2t\tau}) \]

\[- (c_{\text{UV}} - c_{\text{IR}}) \int_0^1 dt \int d^4x \tau \sqrt{|g|} \text{Weyl}^2(g \to ge^{-2t\tau}) \tag{3.52}
\]

and it is constructed so that it cancels the Weyl anomaly at the conformal fixed point.

The non-universal terms contain all possible Weyl invariant counterterms

\[ S_{\text{mu}} = \int d^4x \sqrt{|g|} \left( \hat{a}_c \text{Euler}(\hat{g}) + \hat{c}_c (\text{Weyl}^2(\hat{g})) + \hat{b}_c \hat{R}^2 \right). \tag{3.53} \]

We define \( \hat{R} = \hat{g}^{\mu\nu}R_{\mu\nu}[\hat{g}] \) with \( \hat{g}_{\mu\nu} = e^{-2\tau}g_{\mu\nu} \), which is the Weyl compensated metric so that \( \hat{g}_{\mu\nu} \to \hat{g}_{\mu\nu} \) under the Weyl transformation. We cannot determine these terms from the symmetry alone. In UV, we can just add them by hand, and in IR, they can be generated from the renormalization group flow. This ambiguity will turn out to be irrelevant for our argument in this section.

(Exercise) Use the counterterms to rewrite the Wess-Zumino term in the Riegert form [102]. See [174] and compare.

Following the strategy of Komargodski and Schwimmer [161], we will study the scattering amplitudes of the dilatons in the flat Minkowski background \( g_{\mu\nu} = \eta_{\mu\nu} \). We focus on the two-two dilaton scattering amplitudes. In particular, we are interested in the s-channel forward scattering amplitude \((t = 0)\) in the leading order of \( s \) (see appendix A.4 for a brief review of the scattering theory). For this purpose, we can assume the on-shell dilaton condition \((\partial_\mu\tau)^2 = \Box \tau \). In other words, we introduce the canonically normalized dilaton field \( \varphi \) defined by \( e^{-\tau} = 1 + \varphi \) with \( \Box \varphi = 0 \) as on-shell condition. The error of using on-shell condition in the interaction term is suppressed by \( \frac{s}{\tau} \) compared with the leading term we are interested in. With the on-shell condition, the flat space-time IR action is given by\(^4\)

\[ S_{\text{eff}} = S_{\text{CFT}} + \int d^4x \left( f^2 e^{-2\tau}(\partial_\mu\tau)^2 - 2(a_{\text{UV}} - a_{\text{IR}})(\partial_\mu\tau)^4 \right). \tag{3.54} \]

\(^4\)Up to the non-universal term, the Wess-Zumino action itself was known in [102]. See also [176] [177] [51].

\(^5\)It is instructive to see what happened to the other trace anomaly terms. The Wess-Zumino terms for \( c(\text{Weyl})^2 \) and \( \hat{b} R^2 \) vanish by \( g_{\mu\nu} = \eta_{\mu\nu} \). Similarly, that for Pontryagin term (if any) does not affect the dilaton scattering amplitudes. The non-universal term \( S_{\text{nu}} \) does not contribute either due to the on-shell condition \((\partial_\mu\tau)^2 = \Box \tau \).
In the following, we would like to argue that the coefficient of \((\partial_{\mu} \tau)^4\) must be negative definite from causality and unitarity so that the weak version of the \(a\)-theorem \(a_{UV} \geq a_{IR}\) must hold for its consistency. A quick heuristic way to show this is the following argument: if we consider a particular non-trivial background \(\varphi = c_\mu x^\mu\) for the dilaton effective action (3.54) with small \(c_\mu/f\), then the propagation of the dilaton \(\varphi\) around the background is superluminal unless \(a_{UV} - a_{IR} \geq 0\), suggesting the violation of causality [178].

A more rigorous argument can be made by using the dispersion relation [161] [162] (see also [9]). We study the forward scattering \((t = 0\); see again appendix A.4 for a brief review of the scattering theory) of the two-two dilaton scattering in the \(s \to 0\) limit. The behavior of the forward scattering amplitude \(A(s) = A(s, t = 0)\) in the \(s \to 0\) limit is governed by the IR effective action (3.54) and

\[
A(s) \to 8(a_{UV} - a_{IR})\frac{s^2}{f^4} + \mathcal{O}(s^{\Delta_{IR} - 2}),
\]

(3.55)

where \(\Delta_{IR} > 4\) is the lowest dimension of the irrelevant deformations at the IR fixed point. Note that relevant deformations are fine-tuned to be absent (otherwise it does not flow to the fixed point we are focusing on).

\[
0 = \frac{1}{2\pi i} \oint ds \frac{A(s)}{s^3}.
\]

(3.56)

\[50\]Our metric convention is opposite to that used in [161], and we have a negative sign here.
Around the $s = 0$ pole, we have $I_1 = -\frac{8(a_{\text{UV}} - a_{\text{IR}})}{2f^4}$. Just above the cut on the real axis, by noting that $A(s) = A(-s)$ from crossing symmetry (see appendix A.4 for more details), we obtain

$$I_2 = \frac{1}{\pi} \int_{\epsilon}^{\infty} ds \frac{\text{Im} A(s)}{s^3}.$$

(3.57)

The integral is convergent both in UV and IR. Here, $\sigma(s)$ is the total cross section of $\varphi \varphi \rightarrow \text{CFT}$ from the optical theorem (A.33) so it must be manifestly positive from unitarity. Finally, the large semi circle contribution is zero by noting in UV there is no irrelevant deformation from renormalizability. Thus $a_{\text{UV}} \geq a_{\text{IR}}$.

The above discussion applies when $B$ function near the fixed point has a first order zero both in UV and IR, but we can study the case with higher order zero (which corresponds to marginally relevant/irrelevant couplings like UV gauge coupling constants), and we can still prove the convergence, so the proof is also valid [9] in more generality.

We may wonder whether the argument here suggests a possibility to define the $a$-function not only at the fixed point but also along the renormalization group flow to derive the strong version of the $a$-theorem: $\frac{da(g)}{d\log \mu} \geq 0$. One candidate [161] is $a_{\text{KS}}(\mu) = \int_{\mu}^{\infty} ds \frac{\sigma(s)}{s^2}$. As we will see in the next subsection, this behaves very similarly to Osborn’s $\tilde{a}$-function at least within perturbation theory, and since $\sigma(s) \geq 0$, it is manifestly monotonically decreasing. However, we still have to show that this is a function of the running coupling constants at the energy scale $\mu$ alone, and does not depend on the path of the renormalization group flow. Otherwise, the monotonicity along the renormalization group flow itself is not physically relevant (see also our discussions in section 3.1.3 on averaged $c$-theorem).

The discussion here is made sharper in [9]: They introduced the averaged amplitude over the semi-circle $C(\mu)$ of radius $\mu$ as

$$\tilde{a}(\mu) = -\frac{2f^4}{\pi} \int_{C(\mu)} \frac{ds}{s^3} A(s)$$

(3.58)

with the differential relation

$$\frac{d\tilde{a}(\mu)}{d\log \mu} = \frac{2f^4}{\pi \mu^2} \text{Im} A(\mu).$$

(3.59)

The optical theorem implies that $\tilde{a}$ is a monotonically decreasing function of $\mu$. By Cauchy’s theorem, $\tilde{a}(\mu)$ is same as $a_{\text{KS}}(\mu)$ up to a constant.

### 3.5 Scale vs Conformal in $d = 4$

Given the non-perturbative proof of the weak version of the $a$-theorem, can we say anything about scale invariance vs conformal invariance? Recall that in $d = 2$, the argument was essentially based on the strong version of the $c$-theorem as discussed in section 3.1, so we can imagine we have to be a little bit more creative here since the strong version of the $a$-theorem is not proved yet.

---

51 Although in perturbative examples, we can directly check it from Feynman diagrams, the validity of the usage of optical theorem may cause some suspicion because the dilaton is not physical, and the use of the unitarity may be invalid (in particular due to non-renormalizability). It would be more desirable if we had a better understanding.

52 For instance, we can always define $2a(\mu) = (a_{\text{UV}} - a_{\text{IR}}) \tanh(\log \mu) + (a_{\text{IR}} + a_{\text{UV}})$. as “$c$-function” in any renormalization group flow (without unitarity, Poincaré invariance and so on), which is monotonically decreasing by definition (one can even choose whatever number for $a_{\text{UV}} \geq a_{\text{UV}}$ here). This does not reflect the intrinsic properties of the flow, and completely useless. Needless to say, we cannot conclude anything about scale invariance and conformal invariance from this function.
Let us suppose the IR limit of the deformed theory is not Weyl invariant but only scale invariant. Can we infer any inconsistency to exclude such a possibility? Note this is a slightly more subtle problem because we allow non-Weyl invariant but conformally equivalent IR fixed point such as Nambu-Goldstone bosons, which must be, of course, consistent.

The first thing we have to notice is that the dilaton does not decouple from the IR effective action of the matter when the IR theory is not Weyl invariant [179]. The reason is that we are enforcing the Weyl invariance, so the non-zero trace of the energy-momentum tensor of the IR limit of the deformed theory with non-trivial virial current (recall $T^\mu_\mu = \partial^\mu J_\mu$) must be cancelled by the coupling between the dilaton and the matter.

\[
S_{\text{eff}} = S_{\text{SFT}_{\text{IR}}} + S_{\text{dilaton}} + \int d^4x \left( \tau \partial^\mu J_\mu + \mathcal{O}(\tau^2) O_{\text{SFT}_{\text{IR}}} \right).
\] (3.60)

The first order term $\tau \partial^\mu J_\mu$ is uniquely specified by the Noether procedure, but the $\mathcal{O}(\tau^2)$ terms are non-universal and model dependent. It is related to the Weyl transformation property of $J_\mu$ and it is not specified by the scale invariance, so we have a certain degree of freedom here. In addition, we recall that $J_\mu$ is determined up to the equations of motion of $S_{\text{SFT}_{\text{IR}}}$ with $\tau = 0$. After the inclusion of $\tau$ coupling, the “up to EOM” term contributes to the higher order contact terms so we cannot say much about what $\mathcal{O}(\tau^2)$ are from the generic argument alone. In principle, in a given theory, one can determine these terms order by order (while they may not be unique).

For instance, within the powercounting renormalization, the renormalization of the trace of the energy-momentum tensor can be specified by

\[
\delta_\sigma T^\mu_\mu = \sigma \beta^k_\eta \Box O_k^{(2)} + \mathcal{O}(\partial^\mu \sigma)
\] (3.61)

with operator $O_k^{(2)}$ whose tree-level scale dimension is 2, so at least we have to add $\tau^2 \beta^k_\eta \Box O_k^{(2)}$ to cancel the scale variation. We should stress that this effect did not include the possibility that under the non-constant Weyl transformation, $T^\mu_\mu$ could contain the additional terms proportional to $\partial_\mu \sigma$. One important remark here is that if we choose the renormalization scheme so that $T^\mu_\mu = B^I O_I$, then the consistency requires $\beta^k_\eta$ is always proportional to $B^I$ multiplied by an extra loop factor [59] so that this contribution is always small in perturbation theory.

The second thing we have to notice is that the structure of the Weyl anomaly is modified. For instance, the scale invariant but non-conformal invariant field theory introduces additional terms such as $R^2$ (see non-conformal scalar example). This may raise a puzzle in deriving the dilaton effective action: $R^2$ does not satisfy the Wess-Zumino consistency condition, so is there any $S_{\text{WZ}}[\tau]$ that reproduces the $R^2$ Weyl anomaly?

(Exercise) Show this is impossible. See trial and error method does not work. Try Wess-Zumino trick and see how it fails.

The solutions to these problems are related. Now the IR theory is modified by the external source coupling to the dilaton, we have to reconsider the Weyl anomaly of the dilaton coupled scale invariant field theories.

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[53] Here we still assume that the UV theory is conformal invariant, but one may relax the condition. In this case, it may be more convenient to introduce the external gauge field $C_\mu$ corresponding to the virial current to compensate the Weyl transformation \[54\]. Since $C_\mu$ and $\partial_\mu \tau$ transform in the same way under the Weyl transformation, we have some freedom to choose.

[54] SFT stands for scale invariant field theory, and as far as I know, it was first used in \[54\]. Do not confuse with string field theory.
There are two approaches we can take:

- The first approach is to assume that the dilaton is fully dynamical. Then the total theory is Weyl invariant by construction \([179]\). The Wess-Zumino consistency condition dictates that there is no \(R^2\) anomaly. The on-shell Wess-Zumino action is given by

\[
S_{WZ} = -2 \int d^4x (a_{UV} - \hat{a}_{IR}) (\partial_\mu \tau)^4.
\]

Here \(\hat{a}_{IR}\) may be different from \(a_{SFT_{IR}}\).

- The second approach is to assume that the dilaton is still an external source. The possible Weyl anomaly is larger than without dilaton because we can construct more non-trivial solutions to the Wess-Zumino consistency condition. Note that \(\hat{g}\) is always Weyl invariant, so for instance

\[
T_{\text{new}} = be^{-2\tau} \hat{R}^2 = b(R + 6(\Box \tau) - 6(\partial_\mu \tau)^2)^2
\]

are completely allowed trace anomaly term. The above mentioned \(R^2\) anomaly is completed to be \(\hat{R}^2\) by adding extra dilaton depending terms so that it satisfies the Wess-Zumino condition. If we assume that these have local Wess-Zumino action, there are 11 terms that could appear in the dilaton effective action (see appendix A.3 and the discussions in section 65; for instance the dressing transformation by \(C^0_{IJ}\) induces \((\Box \tau)^2\) term in the dilaton effective action and the corresponding Weyl anomaly term).

If we impose the on-shell condition, the Wess-Zumino terms that contribute to the two-two dilaton scattering all reduce to

\[
S_{WZ} = -2 \int d^4x (a_{UV} - \hat{a}_{IR}) (\partial_\mu \tau)^4.
\]

Here \(\hat{a}_{IR}\) may be different from \(a_{IR}\) which gives the pure Euler term in the Weyl anomaly. We realize that \(\hat{R}^2\) in the Weyl anomaly did not matter, but there are many other terms that could contribute to the dilaton scattering amplitudes. We will discuss relevant terms coming from the space-time dependent coupling constant counterterms later in section 65.

Presumably, this modification of the four-dilaton scattering amplitude explains the “modification” of \(\hat{a}\) from \(a\) away from the fixed point to see the monotonicity in Osborn’s argument (see \([10]\) for the similar argument).\(^{55}\) In the following argument, we do not need the precise relation, but the only crucial thing is that the dilaton scattering amplitudes must be bounded. As emphasized in \([10]\), actually we do not need the explicit form of the Wess-Zumino terms, either, for the following argument. We will identify \(\hat{a}\) with the averaged dilaton scattering amplitude \(\bar{\alpha}(\mu)\) in \((3.58)\).

In either approaches we take, the dilaton does not decouple and we have to regard the action \((3.60)\) as the effective action renormalized at a particular energy scale \(\mu\). We now argue that generically \(\hat{a}_{IR}\) is logarithmically renormalized \([173]\) suggesting its inconsistency with the strict scale invariance \([9]\).

To see this, we again compute the two-two forward dilaton scattering amplitude. From the effective action \((3.60)\), we know that the computation of the S-matrix schematically follows from the matrix element of

\[
A = \int \langle \phi\phi| TT| \phi\phi \rangle + \int \langle \phi\phi| TTTT| \phi\phi \rangle + \int \langle \phi\phi| OO| \phi\phi \rangle + \cdots.
\]

\(^{55}\)We should note that the amplitude is now scheme dependent because of the existence of trivial anomaly terms involving \(\tau\). This is rather consistent with local renormalization group analysis, in which Osborn’s \(\tilde{a}\)-function away from the conformal fixed point is not unique. We will revisit the effect in section 8 from the space-time dependent coupling constant counterterms.
The first two terms come from universal coupling to the virial current, and the last term comes from the non-universal terms in the dilaton effective action. We again emphasize that $T_{\mu}^{\mu}$ is determined up to the equation of motion, so there is an intrinsic ambiguities in defining $T_{\mu}^{\mu}$ correlation functions. In particular, away from the fixed point, the contact terms contain various ambiguities from the renormalization group prescriptions. These explain the presence of the last term in (3.64).

Whenever the dilaton scattering amplitude $A$ has a logarithmic divergence in IR, it is inconsistent with the assumption of scale invariance.\footnote{We refer to \cite{Osborn:1993yt} for the absence of scale anomaly for on-shell dilaton scattering.} The optical theorem tells that logarithmic divergence comes from the on-shell $\varphi \varphi \rightarrow \text{SFT}$ amplitudes and they must vanish. The cancellation is very unlikely unless $T = 0$ (and $O = 0$) \footnote{\cite{Osborn:1993yt}}, and this is how the scale invariance without conformal invariance may be excluded. Alternatively speaking, generically, $\hat{a}$ is monotonically decreasing along the renormalization group flow for scale invariant but non-conformal field theory, and as long as we demand that $\hat{a}$ is determined by the local running coupling constants, it must be finite and the logarithmic divergence here is inconsistent. If we can modify the IR behavior of the scale invariant but not conformal theory by adding further relevant deformations so that it will flow to a conformal invariant fixed point (which could be a trivial one), the inconsistency is more transparent: at the conformal fixed point, $\hat{a}$ or $\bar{\alpha}$ is identical to $a$ and is bounded from below (recall $a \geq 0$ as discussed in section \ref{sec:conf}) so $\hat{a}$ cannot decrease forever.

To make the above discussion more precise, we assume that $T_{\mu}^{\mu}$ is small and $O$ is negligible. Recall that $T_{\mu}^{\mu} = B^I O_I$, so in perturbation theory, the first assumption is valid. The second assumption is more tricky, but in dimensional regularization with the most generic renormalizable action (as discussed in section \ref{sec:dim}), it is true as long as we use the suitably renormalized new-improved energy-momentum tensor.

Under these assumptions, the dilaton scattering amplitude is dominated by $\langle \varphi \varphi | T T | \varphi \varphi \rangle$. Then the logarithmic divergence of the scattering amplitude is equivalent to

$$
\frac{d\hat{a}_{\text{IR}}}{d\log \mu} = \frac{d\hat{\alpha}}{d\log \mu} = B^I \chi_{IJ} B^J + O(B^4),
$$

where $T = B^I O_I$, and $\chi_{IJ} \sim \langle O_I(x) O_J(0) \rangle x^8 |_{x^8} = \mu^{-1}$. As discussed, we cannot absorb the logarithmic divergence in running coupling constants, so for the scale invariance, it must vanish, which means $B^I = 0$ and the theory is conformal invariant.

Therefore under some technical assumptions, scale invariant fixed points must be conformal invariant, and the strong $a$-theorem (or more precisely strong $\hat{a}$-theorem) must hold in perturbation theory. Beyond the perturbation theory, the proof is not complete because although implausible $TTTT$ and $OO$ term might cancel the divergence (the latter may be more dangerous because it can be only proportional to $B^I$). This is related to the open question of the positivity of the metric in the strong $a$-theorem from the local renormalization group flow analysis.

Clearly, there is a parallel between the dilaton scattering amplitude and $\hat{a}$-function discussed in the local renormalization group flow. There is a slight difference, however, beyond the leading order in $B^I$ discussed above. One advantage of the dilaton scattering amplitude is that the derivative with respect to the energy scale is always positive (if we assume the optical theorem) unlike $\hat{a}$-function of Osborn. One disadvantage, on the other hand, is that it is not obvious whether the averaged amplitude is a function of the running coupling constants. We will give a further comment on this point below.

In relation to the running coupling constant dependence on the $\hat{a}$-function as well as $\bar{\alpha}$ from the dilaton scattering amplitude, it is crucial that the dependence is not a multi-valued function.
without any monodromy. Clearly, if we allow the monodromy structure, the monotonically decreasing function along the renormalization group flow is consistent with the cyclic renormalization group flow. We have implicitly assumed this in the local renormalization group flow analysis and we can check that $\tilde{a}$-function in perturbation theory is a single-valued function on $g^I$. In any perturbation theory, we have an explicit power series expansion of $\tilde{a}$ with respect to coupling constants. We expect the multivaluedness does not happen in the dilaton scattering amplitude either because it is a physical observable at a given energy. Furthermore when there is a monodromy in the $a$-function, it is inconsistent with the further possibility to deform the theory so that it flows to a well-defined conformal fixed point (in particular the trivial fixed point by mass deformation) with a fixed value of $a$. We will continue to assume that $a$-function is a single-valued function of the running coupling constants.

### 3.5.1 Some technical comments on possible cancellation

We recall that in section 3.4, we assumed that the IR theory under study is Weyl invariant. We might ask a pedantic question if all the IR fixed points of unitary quantum field theories are Weyl invariant (say in $d = 4$). The answer to this question is no. Take massless QCD for example. In the IR limit, the chiral symmetry is spontaneously broken, and the IR theory is described by Nambu-Goldstone bosons. They possess the shift symmetry, so the natural energy-momentum tensor is not traceless (such a question was raised in \cite{156,180} in the discussion of the $a$-theorem). After all, the scale transformation as well as Weyl transformation do not commute with the shift symmetry because the improved curvature coupling $R\phi^2$ is incompatible with the shift symmetry.

We wanted to use the consequence of the $a$-theorem in the above examples. For instance, the weak $a$-theorem (if applied) gives a constraint on the possibility of symmetry breaking: too many spontaneous symmetry breaking could be inconsistent with the constraint from the $a$-theorem \cite{183} etc. Note that we can compare the Weyl anomaly of Nambu-Goldstone bosons with that of conformally coupled scalar to see the difference is only in $R^2$ term. In particular $a$ is same. From the viewpoint of counting degrees of freedom, it seems natural to count the Nambu-Goldstone boson and conformal scalar with the same unit.

We may see a possible complication in general, but we would like to focus on the simplest Nambu-Goldstone boson case to see what we would expect. The action for the Nambu-Goldstone boson has a “trivial” virial current, so the coupling with the dilaton is non-zero $\int d^4x \tau \partial^\mu J_\mu$ with $J_\mu = \phi \partial_\mu \phi$. Since we have used the equations of motion to derive $J_\mu$, we have to augment the contact terms. Also the virial current does not transform covariantly under the Weyl transformation, so we have to add higher terms. In the end, we obtain the complete coupling between the dilaton and Nambu-Goldstone boson as

$$S = \tilde{f}^2 \int d^4xe^{-2\tau}(\partial^\mu \phi \partial_\mu \phi) ,$$

which is Weyl invariant and shift-symmetric. Note that the Weyl weight of $\phi$ becomes zero by higher dilaton couplings.

The two-two dilaton scattering may obtain logarithmic divergence due to the intermediate Nambu-Goldstone boson channel. However, we can easily see that various terms cancel against the virial current exchange so that the total cross-section is zero \cite{179}, and there is no logarithmic divergence, which would lead to the same inconsistency that we encountered in the perturbative study of scale but non-conformal IR theories due to $\varphi \varphi \rightarrow$ SFT channels. A more direct way to see the cancellation
is to realize that we can perform a simple field redefinition of dilaton and Nambu-Goldstone boson so that they are completely decoupled (by going to “Cartesian coordinate” in (3.66)).

This is an example in which the cancellation of various terms can occur in (3.64) when the contribution of each term is of same order (even in unitary example). As emphasized in [9], this kind of cancellation is generically very unlikely with no good physical reason (in the example here because of the Nambu-Goldstone symmetry). In non-unitary scale but non-conformal field theory such as Riva-Cardy model, we also encounter a similar cancellation.

As we have discussed, in perturbation theory, the cancellation does not occur as long as we use the new-improved energy momentum tensor. See also general arguments in [9] how the improvement terms such as
\[ \int d^4 x \sqrt{|g|} \text{RO term in the effective action do not affect the argument for the weak } \alpha \text{-theorem.} \]
We see that the terms that can be improved away and the terms that cannot be improved away give qualitatively different contributions to the dilaton scattering amplitude. Thus we conclude that the non-zero dilaton scattering is really an obstruction not only for the Weyl invariance but also for the conformal invariance up to the improvement since it is insensitive to the improvement terms.

This types of coupling between dilaton and Nambu-Goldstone boson naturally appears in supersymmetric extension of our construction because the dilaton must accompany the \( R \)-axion from superconformal symmetry, and the supersymmetric extension of the dilaton effective action always contains such couplings (see e.g. [174]).

### 3.5.2 Space-time dependent coupling constant counterterms

One of the key ingredients in local renormalization group flow analysis is the introduction of the space-time dependent coupling constants. In particular, we recall that the local counterterms associated with the space-time dependent coupling constants introduce the ambiguity in \( \tilde{\alpha} \)-function. We would like to argue that the similar ambiguity can be introduced in the dilaton effective action as well as in the dilaton scattering amplitudes away from the conformal fixed point.

We recall that the effective action of the original theory (before the Weyl compensation) may contain additional space-time dependent coupling constant counterterms (in \( d = 4 \)) \[\begin{align*}
S_{\text{ct}} &= \int d^4 x \sqrt{|g|} \left( \frac{1}{2} C^g_{IJ}(g) D_\mu g^I D^\mu g^J G^{\mu\nu} + A(g) \text{Euler} \\
&\quad - C^a_{IJ}(g) D^2 g^I D^2 g^J - \frac{1}{4} C^c_{IJKL}(g) D_\mu g^I D^\mu g^J D_\nu g^K D^\nu g^L + \cdots \right). \tag{3.67}
\end{align*}\]
These counterterms are related to various contact terms of operator \( O^I \) and \( T_{\mu\nu} \) in the flat space-time limit (see [59] for details). Once they are at the fixed point, the counterterms are Weyl invariant since \( B^I = -d g^I/d \log \mu \), so the introduction of these counterterms does not change the dilaton effective action. However, away from the conformal fixed point, the counterterms are not Weyl invariant, and we need the dilaton compensation so that they will give extra contributions to the dilaton scattering amplitudes. Alternatively speaking, the counterterms \( \{1.67\} \) parametrize the ambiguities of the dilaton coupling beyond the leading order, which we have mentioned in \( \{1.66\} \).

Let us focus on the first two terms \( C^g_{IJ} \) and \( A \), which will be the most important ones for our discussions.\[\text{For the } C^g_{IJ} \text{ term, within the order of perturbation we are interested in, one can replace } D_\mu g^I \text{ with } D_\mu g^I + B^I \partial_\mu \tau \text{ and replace } G^{\mu\nu} \text{ with } \tilde{G}^{\mu\nu} \text{ to make it Weyl invariant.} \]
Once we impose the
\footnote{Most of the other terms will not contribute to the dilaton scattering amplitudes in flat space-time in the leading order \( \mathcal{O}(B^2) \) once we impose the on-shell condition. See the argument below.}
on-shell condition and restrict ourselves to the flat space-time (i.e. \( g_{\mu\nu} = \eta_{\mu\nu} \) and \( D^\mu g^I = 0 \)), we obtain the additional four-dilaton terms

\[
S_{ct} = \int d^4x 2(C^\rho_{IJ} B^I B^J)(\partial_\mu \tau)^4.
\] (3.68)

This is nothing but the ambiguity in defining \( \tilde{a} \)-function in local renormalization group flow (see section 3.3.1) because \( \tilde{a} \) appearing in the Wess-Zumino action is modified by the extra contribution \( B^I C^\rho_{IJ} B^J \). Correspondingly, the energy scale dependence of the dilaton scattering amplitude must be modified by various terms as in (3.43). Similarly, \( A \) term (after dilaton compensation by replacing \( A \rightarrow A - \tau B^I \partial_I A \) in the leading order) does not contribute to the dilaton scattering amplitude to the order we are interested in, which corresponds to the fact that \( A \) term gives no correction to \( \tilde{a} \) (but not in \( a ! \)) in local renormalization group flow. However \( A \) does change the form of the gradient formula. It will give the gauge transformation to \( w_I \).

The origin of these terms are the contact terms in higher order interaction terms in defining the dilaton scattering amplitudes. Although we may implicitly fix the scheme as in [9] or [155], we need not. A different renormalization group prescription gives different dilaton scattering amplitudes, and in \( d = 2 \), we have used this ambiguity to demonstrate that the metric \( \chi_{IJ} \) is manifestly positive definite.

We should realize that by choosing the counterterms, it is possible to make the \( (\partial_\mu \tau)^4 \) term positive in the dilaton effective action at a certain energy scale if we wish. Apparently, such a choice would be inconsistent with the unitarity and the optical theorem that follows. The contact terms must be chosen so that the unitarity of the dilaton scattering amplitudes must be intact (and we believe there is such a choice). At this point, one may rephrase one of the implicit assumptions of [1]. There are good choices of counterterms so that the dilaton scattering amplitudes are unitary. Essentially, this is a different way to impose the positivity of the metric in the strong \( a \)-theorem. Obviously the renormalization by itself may not know the unitarity, and we have to respect it by choosing the good counterterms.

We emphasize that at the conformal fixed point, there is no such an ambiguity, so our discussion for the weak \( a \)-theorem in section 3.4 is not affected. At the UV conformal fixed point, the space-time dependent counterterms for irrelevant perturbations are fine-tuned away because they are non-renormalizable, and those for relevant perturbations are zero because they have vanishing beta functions at the UV fixed point. Similarly, at the IR conformal fixed point, the space-time dependent counterterms for relevant perturbations are fine-tuned away because they must sit at the fixed point, and those for irrelevant perturbations are zero because they have vanishing beta functions at the IR fixed point.

As for the other terms (e.g. second line of (3.67)), Osborn [59] used the freedom to cancel various local Weyl anomaly terms with space-time dependent coupling constant. In spirit, it is close to removing \( \Box R \) term in Weyl anomaly. In this prescription, the remaining ambiguity in the dilaton scattering amplitude from the counterterm (3.67) is the above mentioned ambiguity in \( \tilde{a} \)-function from \( C^\rho_{IJ} \). If we did not use this particular prescription, then the relation between \( \tilde{a} \) and dilaton scattering amplitude is less clear. They seem to deviate at order \( \mathcal{O}(B^4) \).

There are some other subtleties in identifying the averaged dilaton scattering amplitude \( \bar{\alpha} \) and \( \tilde{a} \)-function. As mentioned at the end of section 3.3.1 \( \tilde{a} \)-function is a function of the running coupling constants alone and does not depend on the energy scale \( \mu \) explicitly, while it is not immediately obvious whether the averaged dilaton scattering amplitude \( \bar{\alpha} \) can be written as a function of the running coupling constants alone and does not depend on the energy scale in addition.
Indeed, if we truncate the computation at order $O(B^2)$ as was done in \cite{9} and use the leading order Zamolodchikov metric, the averaged dilaton scattering amplitude $\bar{\alpha}$ is nothing but the averaged $c$-function discussed in section 3.1.3. Thus it may seem more natural to identify the dilaton scattering amplitudes with the averaged $c$-function. As discussed in section 3.1.3, however, the averaged $c$-function is known to depend on the trajectory of the renormalization group flow, which is equivalent to our concern that the averaged dilaton scattering amplitude $\bar{\alpha}$ is not a function of the running coupling constants alone, but it may depend on the energy scale $\mu$ separately or on the renormalization group trajectory.

The reason why this kind of complication happens is that the renormalization group flow $\frac{dc^M_d}{d\log \mu}$ of the averaged $c$-function and that of Osborn’s $\tilde{\alpha}$-function $\frac{d\tilde{\alpha}}{d\log \mu}$ as well as that of the averaged dilaton scattering amplitude $\frac{d\bar{\alpha}}{d\log \mu}$ all coincide with one another up to the order we have computed. In particular, the “Zamolodchikov metric” $\chi_{IJ}$ appearing in the averaged $c$-function (as well as in the averaged dilaton scattering amplitude $\bar{\alpha}$) is essentially identical to the metric $\chi^g_{IJ}$ appearing in the $\tilde{\alpha}$-flow equation in many usual renormalization schemes such as dimensional regularization (at least up to two-loops). In \cite{10}, however, they pointed out that the “metric” $\chi_{IJ}$ for the averaged dilaton scattering amplitude at the leading order (which is identical to that for the averaged $c$-function in section 3.1.3) is related to $\chi^g_{IJ}$, and it can be different from that of the $\tilde{\alpha}$-function, namely $\chi^g_{IJ}$, at the third order or higher.\footnote{We recall $\chi^g_{IJ}$ is derived from $D^2g^I_D^2g^J$ term in the Weyl anomaly, so it governs the Weyl non-invariance of the contact terms in $(O_I(x)O_J(0))$. Thus, it is related to the Weyl non-invariance of $(\Theta(x)\Theta(0))$. On the other hand $\chi^g_{IJ}$ appears at higher point functions of $T_{\mu\nu}$ and $O_I$. See \cite{59} for more details.}

Of course, we have neglected the higher order corrections in the dilaton scattering amplitudes, too. The “metric” that appears in the dilaton scattering amplitude can be different from $\chi_{IJ}$ in the averaged $c$-theorem beyond the leading order. In addition, the ambiguities discussed here are not treated carefully, so it is premature to say whether or not there is a good “scheme” in which the metric in the dilaton scattering amplitude and $\chi^g_{IJ}$ really coincide to give a clue for the non-perturbative result as in $d = 2$.

Let us summarize the non-perturbative situation. Osborn’s $\tilde{\alpha}$-function is a function of the running coupling constants at the given energy scale alone, but it is not obvious if it shows monotonicity. The averaged $c$-function $c^M_d$ is by definition monotonically decreasing, but it is not (a priori) defined as a function of the running coupling constants alone and presumably it is not. Finally, the averaged dilaton scattering amplitude $\bar{\alpha}$ is monotonically decreasing once we assume the optical theorem, but it is again not (a priori) obvious if it is a function of the running coupling constants alone. In the perturbative approach, they differ at $O(B^3)$ or higher but we could not see this distinction at $O(B^2)$.\footnote{The difference in the integrated form was discussed in \cite{159} as a sum rule. The positivity of the difference is tricky due to the contact terms.}

In principle, the counterterms discussed here might modify the argument for the equivalence between scale invariance and conformal invariance, but the effect is always higher order $\sim O(B^4)$, so it does not change the perturbative conclusion in section 3.5. The perturbative proof — both from the analysis of $\tilde{\alpha}$-function \cite{10} based on the local renormalization group analysis \cite{58} and from the dilaton scattering amplitude \cite{9} — that scale invariance implies conformal invariance is therefore robust.\footnote{See also appendix of \cite{169} for the comment that Osborn’s argument essentially implies that scale invariance must be enhanced to conformal invariance in perturbation theory. The argument there is not modified either.}
3.6 Physical reason why scale implies conformal in perturbative fixed point

In the last sections, we have discussed how scale invariant but non-conformal field theories are inconsistent with the perturbative $\alpha$-theorem. We have showed that the suitably defined $\alpha$-function (whose particular realization is the dilaton scattering amplitudes) decreases at the rate dominantly proportional to the bilinear of $B$-functions in perturbation theory. In order to achieve the finite dilaton scattering amplitude, it must be accompanied with vanishing $B$ functions near the asymptotic IR region, and therefore it must show conformal invariance.

If we perform an explicit computation, however, we may realize something a little bit more about the structure of the renormalization group flow. We realize that $B$ functions in certain directions are identically zero irrespective of if the coupling constants are at the fixed point or not. In addition, within a few orders in perturbation theory, the zero-directions directly correspond to the “would-be” virial current direction that may induce scale invariance without conformal invariance by using the equations of motion. We have seen it in section 2.7 when we discussed the conformal perturbation theory.

This may sound unexpected because we are solving the weaker equation $T^\mu_\mu = \partial^\mu J_\mu$ rather than $T^\mu_\mu = 0$ by introducing extra parameters in $J_\mu$ (as many as the number of non-conserved current). Why can’t we expect more solutions generically? The argument based on the $\alpha$-theorem gives the constraint near the fixed point, but what is the origin of this zero direction during the entire renormalization group flow?

To see it in a simple example, we consider Yukawa interaction $y\phi\psi^2$ in $d = 4$. The Yukawa interaction has the one-loop $B$ function

$$T^\mu_\mu = B^I O_I = \frac{|y|^2}{16\pi^2} y(\psi\psi\phi) + c.c. + O(\phi^4).$$

(3.69)

We realize that the $B$ function is orthogonal to the direction $iy(\psi\psi\phi) + c.c.$, which can be rewritten as the divergence of the non-conserved current (which is given by $i\partial^\mu(\bar{\psi}\gamma^\mu\psi)$). In other words, the phase of the Yukawa coupling constants are not renormalized (as observed in [86][87]), and this is the reason why the “would-be” virial current direction does not appear in this one-loop computation. We can easily see that the phase of the Yukawa coupling is unphysical from the beginning because we can remove it by a field redefinition of $\psi$. Since the phase is not a parameter of the theory nor does it affect any observables of the theory, it had better not show any physical consequences in the renormalization group flow. Of course, the strong $\alpha$-theorem $\frac{\delta a}{d\log\mu} \sim \frac{1}{16\pi^2}|y|^6$ does tell that the $\tilde{a}$-function must be decreasing along the renormalization group flow, but again note that the phase of the Yukawa coupling does not appear in the $\tilde{a}$-function, either.

As discussed in section 2.7, the $B$ function that can be completely rewritten as a divergence of a virial current is related to redundant directions in the renormalization group flow. The physical reason why the scale invariant but non-conformal field theory is difficult to achieve in perturbation theory can be understood as the claim that redundant directions do not acquire $B$ functions as can be checked directly within a first few orders of perturbation theory. At higher orders, $B$ functions in the virial current direction can be non-zero, but they still possess the zero directions in the $B$ function flow, which essentially excludes the cyclic renormalization group flow.

The above observation can be made more precise when we can find the counterterm in which $w_I = 0$ in local renormalization group flow discussed in section 3.3.1[61]. In this situation, the $B$ functions must

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[61]This is shown to be possible within the first few orders in perturbation theory. The superpotential flows in holography also suggest this is the case in explicit holographic models. In general this may not be true unless $w_I$ is exact, but I do not know any explicit examples that show $w_I$ is not exact.
have the same number of zero directions from the consistency condition \( B^I = g^{IJ} \partial_J \tilde{a} \) together with the fact that \( \tilde{a} \) is a singlet under the “flavor” symmetries generated by the candidates of the virial current.

Let us label \( i, j \) as the direction corresponding to the “would-be” virial current direction, and \( M, N \) as the direction that cannot be transformed to currents. The Wess-Zumino consistency condition of the local renormalization group and the “flavor” invariance of \( \tilde{a} \) requires

\[
0 = \partial_i \tilde{a} = g_{iM} \beta^M + g_{ij} B^j. \tag{3.70}
\]

Suppose \( \beta^M = 0 \) but \( B^i \neq 0 \) so that we obtain scale invariant but non-conformal background. This cannot occur because it is inconsistent with the expected zero directions in \( B \) function flow (3.70) as long as \( g_{ij} \) is non-degenerate (as expected because \( g_{ij} \) must be isometric under the “flavor” symmetry group).

Even when \( w_I \) is not zero, we know that \( B^I \) functions are not independent because \( \partial_i \tilde{a} = 0 \) gives the constraint along the renormalization group flow, which reflects the fact that there are as many redundant perturbations in renormalization group flow as the number of candidates for the virial current. The condition \( (3.70) \) is replaced by

\[
0 = \partial_i \tilde{a} = (g_{iM} + w_{iM}) \beta^M + (g_{ij} + w_{ij}) B^j + (P_i g)^j w_j \tag{3.71}
\]

By demanding \( \beta^M = 0 \) and contracting it with \( B^i \), we obtain the same conclusion that \( B^i = 0 \).

In summary, we have two physical intuitions for non-existence of (perturbative) scale invariant but non-conformal field theories in power-counting renormalization. The first one is the \( a \)-theorem: the dilaton scattering (or Osborn’s \( \tilde{a} \)-function) must be bounded and it cannot decrease forever from non-trivial \( B \) functions of scale invariant but non-conformal field theories. The second one is that \( B \) functions in the directions that might be used to construct a non-trivial virial current are actually redundant directions in perturbation theory. Even at higher orders, we expect to keep the same number of zero directions in the \( B \) function flow as the number of redundant directions. The both intuitions are beautifully realized by the generalized “gradient formula” if true. In holographic computations, we will precisely see both obstructions if we try to construct scale invariant but non-conformal field theory duals.

### 3.7 Other dimensions

- In \( d = 1 \), due to the lack of Poincaré invariance, we cannot use the Reeh-Schlieder theorem. This is a major drawback. If we assume its validity then scale invariance implies conformal invariance [184]. Similarly, the boundary \( g \)-theorem [185], which claims that boundary entropy of the two-dimensional system is monotonically decreasing along the renormalization group flow, can be proved [186][187]. On the other hand, a quantum field theory in \( d = 1 \) is equivalent to a simple quantum mechanical system, and there are examples of cyclic renormalization group flow [188] realized in non-relativistic field theories [189] as well as the system with scale invariance without conformal invariance [184].

- In \( d = 3 \), the candidate of Zamolodchikov’s \( c \)-function is the finite part of the \( S^3 \) partition function \( F = -\log Z_{S^3|_{\text{reg}}} \) as we will elaborate a little more in the next section. This is equivalent to the finite part of the entanglement entropy of the half \( S^3 \). It is an interesting open question if there is a strong version of the \( F \)-theorem that would imply equivalence between scale invariance and conformal invariance in \( d = 3 \). We have more to say in section [3.7.1].
In even dimensions, Cardy’s conjecture has a natural generalization: the coefficient in front of the Euler density in the Weyl anomaly must be monotonically decreasing along the renormalization group flow. In $d = 6$, so far we have not been successful in using the dilaton-scattering argument to show the weak version of the $a$-theorem. A reported problem is that it is hard to show the positivity of the dilaton scattering amplitudes in $d = 6$. On the other hand, there is no counterexample of $a$-theorem reported, nor there is no known scale invariant but non-conformal invariant field theories (with gauge invariant scale current). According to Wikipedia, it is reported 6d self dual non-critical string theory is scale invariant but not conformal.

In higher dimension $d \geq 7$, it is likely that there is no interacting unitary conformal field theory, but there is no proof of it. The reason why higher dimensional free Maxwell theory cannot be conformal invariant is consistent with the fact that there is no superconformal algebra in $d \geq 7$, but we know supersymmetric Maxwell theories exist in $d \leq 10$. If it were conformal, it would be inconsistent with the non-existence of the superconformal algebra (unless it breaks supersymmetry). We note that Nahm’s classification is based on the assumption of the existence of the S-matrix, so it does not exclude the superconformal membrane field theories in higher dimension than 6.

### 3.7.1 F-theorem in $d = 3$

In section 3.3.2, we briefly discussed the relation between the problem of finding the superconformal $R$-symmetry in superconformal field theories and the $a$-theorem via the $a$-maximization principle. A similar question arises in the other dimensions. In $d = 2$, we can apply the completely similar argument: in order to determine the superconformal $R$-symmetry, we can maximize the trial $c$-function, and it gives a new perspective of the $c$-theorem in supersymmetric field theories in $d = 2$. What about the situation in $d = 3$, where we have no Weyl anomaly?

The question was answered in the paper by Jafferis. He proposed that in order to determine the superconformal $R$-symmetry of $N = 2$ superconformal field theories in $d = 3$, we can minimize the (real part of the) supersymmetric free energy on $S^3$ (hence it is known as F-theorem, $F$ referring to the free energy). Due to a subtle anomaly in the contact terms in $N = 2$ superconformal field theories in $d = 3$, the partition function acquires a non-trivial phase, but the phase is irrelevant for the discussion here. The derivation is based on the supersymmetric localization technique to compute the partition function and the holomorphic dependence on the real mass parameters of the $N = 2$ supersymmetric field theories. We refer to for the discussions.

Given the analogy with Cardy’s conjecture in even dimensions, it is natural to conjecture that the free-energy on $S^3$ should give a natural candidate for Zamolodchikov’s $c$-function in $d = 3$. Indeed, there have been various non-trivial checks if this is indeed the case. Perturbative arguments have been presented both with slightly relevant deformations as well as with marginal deformations in . Unlike the supersymmetric situation, where the localization computation is available, the regularization can be very tricky and there are some issues. However, the formal argument that the $S^3$ partition function is related to the entanglement entropy suggests that it seems a promis-

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62We believe that the little string theory with (2, 0) supersymmetry becomes superconformal in the IR limit. On the other hand, the little string theory with (1, 1) supersymmetry becomes free super Yang-Mills theory in the IR limit, which is scale invariant but not conformal invariant. Presumably, the statement refers to the latter situation. Otherwise I have no idea what it means. We refer for a review of the little string theory.

63One thing I found curious is that the topological Chern-Simons gauge theories give non-zero contributions to the $S^3$ partition function although they do not carry any dynamical degrees of freedom.
ing direction to pursue (see the following sections on the entanglement entropy). The establishment of the strong \( F \)-theorem is related to the question of equivalence between scale invariance and conformal invariance in \( d = 3 \) (see e.g. [184] [155] for perturbative argument), and the non-perturbative understanding is desirable.

The heuristic argument goes as follows. We first note that the conformal invariance and the absence of the conformal anomaly in \( d = 3 \) dictate that if we evaluate the one-point function of a conformal primary operator on \( S^3 \), it must vanish. This means that the sphere partition function does not depend on the exactly marginal deformation:

\[
\frac{\partial}{\partial g^I} Z_{S^3}^{\text{CFT}} = 0 .
\] (3.72)

Near the conformal fixed point, the one-point function does not vanish, but the deviation must be proportional to \( B^I \) function:

\[
\frac{\partial}{\partial g^I} Z_{S^3} = B^I \hat{g}_{IJ} .
\] (3.73)

We can show (e.g. [184] [155]) that in perturbation theory \( \hat{g}_{IJ} \) is equivalent Zamolodchikov metric in certain loop orders, but it can contain the anti-symmetric part at higher loop orders.

We now compute the scale dependence of the sphere partition function. Since \( T^\mu_\mu = B^I O_I \), we can derive

\[
\frac{\partial}{\partial \log \mu} Z_{S^3} = B^I \hat{g}_{IJ} B^I ,
\] (3.74)

which gives the perturbative \( F \)-theorem with the renormalizability as long as \( \hat{g}_{IJ} \) is positive definite.

On the other hand, if the theory is scale invariant (but not necessarily conformal invariant), the energy-momentum tensor is divergence of a certain current: \( T^\mu_\mu = D^\mu J_\mu \). Thus, the partition function cannot depend on the scale as long as the Virial current is well-defined. Therefore we should obtain

\[
\frac{\partial}{\partial \log \mu} Z_{S^3}^{\text{SFT}} = \beta^I \hat{g}_{IJ} B^I = 0 ,
\] (3.75)

which is possible only if \( B^I = 0 \) as long as the metric \( \hat{g}_{IJ} \) is positive definite. Therefore, we can conclude that the scale invariance must imply conformal invariance within perturbation theory in \( d = 3 \).

For a consistency check of the above argument, we would like to point out the effect of the operator identity \( B^I O_I = \beta^I O_I + \beta^a \partial_\mu J^\mu_\mu \). Since the total derivative does not contribute to the partition function after spatial integration, (3.76) can be also written as

\[
\frac{\partial}{\partial \log \mu} Z_{S^3} = \beta^I \hat{g}_{IJ} B^I .
\] (3.76)

However, the partition function is invariant under the “flavor rotation” induced by \( J_\mu^\mu \), so (3.77) tells that \( \beta^I \hat{g}_{IJ} B^I = B^I \hat{g}_{IJ} B^I \). At the scale invariant fixed point, \( B^I = 0 \) and the conformal invariance is recovered irrespective of our gauge freedom in the definition of the beta functions. The structure is in complete parallel with that in \( d = 4 \).

\[\text{64}\] Once we are away from the conformal fixed point, there is a certain ambiguity in defining the partition function. The improvement terms in the action do change the partition function itself, but it will not change the expectation value of the trace of the energy-momentum tensor. In addition, there are counterterm ambiguities that may introduce ambiguities in \( \hat{g}_{IJ} \).

\[\text{65}\] A counterexample is the three-dimensional free Maxwell field theory. The author would like to thank Z. Komargodski for pointing out the fact.

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3.7.2 In relation to entanglement entropy

It was observed that the Weyl anomaly is intimately related to the entanglement entropy of the vacuum states in relativistic field theories (see e.g. [198] for a review of the entanglement entropy in quantum field theories). The entanglement entropy is defined as follows. We first divide the space into two domains $A$ and $\bar{A}$. We compute the partial density matrix $\rho_A = \text{Tr}_{\bar{A}} \rho_{\text{tot}}$, where $\rho_{\text{tot}}$ is the density matrix of the total system, and in particular in our case it is the pure vacuum state $\rho_{\text{tot}} = |0\rangle\langle 0|$. The entanglement entropy is given by

$$S_A = -\text{Tr}_A \rho_A \log \rho_A .$$  \hfill (3.77)

It satisfies various interesting properties such as

$$S_A = S_{\bar{A}} .$$  \hfill (3.78)

In particular, the so-called strong subadditivity holds (see e.g. [199] for a review. See [200] for its original derivation):

$$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C} .$$  \hfill (3.79)

In relativistic quantum field theories on $\mathbb{R}^{d-1} \times \mathbb{R}^t$, one can argue that the (regularized) entanglement entropy has the following structure

$$S = \cdots + (-1)^{\frac{d}{2}-1} a_d \log (R_{\text{IR}}/\epsilon) + \cdots$$  \hfill (3.80)

for even dimension $d$, and

$$S = \cdots + (-1)^{\frac{d}{2}-1} a_d + \cdots$$  \hfill (3.81)

for odd dimension $d$, where $R_{\text{IR}}$ is a typical IR scale of the entangling surface that divides $A$ and $\bar{A}$, and $\epsilon$ is the UV cut-off to regularize the trace over the field theory Hilbert space. In these expressions, we have neglected both the UV diverging terms $a_2 \left( \frac{R_{\text{IR}}}{\epsilon} \right)^{d-2} + a_3 \left( \frac{R_{\text{IR}}}{\epsilon} \right)^{d-3} + \cdots$ and the finite terms $a_{d+1} \left( \frac{R_{\text{IR}}}{\epsilon} \right)^{-1} + a_{d+2} \left( \frac{R_{\text{IR}}}{\epsilon} \right)^{-2} \cdots$. The leading diverging part is known as the “area law” because it is proportional to the area of the entangling surface.

The universal contribution $a_d$ does depend on the shape of the entangling surfaces $\partial A$ as well as the background geometry, but if we take $\partial A$ to be $(d-2)$ dimensional sphere $S^{d-2}$ inside $\mathbb{R}^{d-1}$, it can be argued that in even dimension $d$, the coefficient $a_d$ coincides with the coefficients of the Euler density in the Weyl anomaly if the quantum field theory under consideration is a conformal field theory [201][202]. Thus we have an alternative way to define the $a$-function in even dimension by using the universal part of the entanglement entropy of the vacuum state divided by $S^{d-2}$.

Moreover, in $d = 2$, one may even prove the inequality $a_{\text{UV}} \geq a_{\text{IR}}$ by using the strong subadditivity condition (3.73) together with the Lorentz invariance in a clever way [203]. Here the strong subadditivity is replacing the unitarity condition in Zamolodchikov’s argument: of course the unitarity (in the sense of no-negative norm state) was crucially assumed in the derivation of the strong subadditivitiy. It is an open question if the similar argument is possible in $d = 4$ to derive the weak $a$-theorem. It is also an interesting question if there is any pathology if we consider the scale invariant but non-conformal field theory and study the properties of the entanglement entropy. In a recent paper [204],...
the author computed the entanglement entropy of the dilaton compensated effective field theories in
the flat Minkowski space-time, and argued that it is governed by the $G_{\mu\nu}\partial^\mu\tau\partial^\nu\tau$ term in the dilaton
effective action. The evaluation of the term led to the similar expression to the averaged $c$-function
$c_d^M$. 

In odd dimensions, there is no Weyl anomaly, but it is conjectured that the universal part of
the entanglement entropy can be used as a candidate of the $\alpha$-function. There are some supporting
evidence from AdS/CFT correspondence \[202\]. In $d = 3$, more detailed analysis is needed, but there
have been some attempts in this direction, suggesting that the entanglement entropy is monotonically
decreasing along the holographic renormalization group flow \[205\]. Again, it is an interesting question
to ask if the equivalence between scale invariance and conformal invariance follows from such an
argument.

3.8 Reduced symmetry

- For chiral field theories in $d = 2$, we actually have two central charges, $c$ and $\bar{c}$, which can take
different values (modulo gravitational anomaly). It is easy to generalize our argument in section
3.1 by taking CP transformation and see that both $c$ and $\bar{c}$ are monotonically decreasing with
the same rate governed by $\langle \Theta(x)\Theta(0) \rangle$ \[206\]. Thus the difference $c - \bar{c}$ is a constant along
the renormalization group flow, which is consistent with the ’t Hooft anomaly matching condition for
the gravitational anomaly. In our discussions in $d = 4$, we have not discussed the gravitational
anomaly but it is possible to incorporate the gravitational anomaly in our discussions of the
local renormalization group flow of section \[3.3.1\]. See also \[3\] for a related discussion on the
gravitational anomaly in dilaton scattering amplitudes.

- In finite temperature situations, thermodynamic $c$-theorem was proposed in \[207\]. See also \[208\]
for applications in condensed matter physics.

- It is an interesting question whether the scale invariant boundary conditions will lead to confor-
mal invariant boundary conditions. Let us consider a $d$-dimensional bulk conformal field
theory and put a $(d - 1)$-dimensional Poincaré invariant boundary at $r = 0$ in $(t,x_1,\cdots,r)$ plane. The
condition for the Poincaré invariant boundary is given by \[209\] \[210\]

$$T_{ri}(r = 0) = \partial^j \tau_{ji} ,$$

(3.82)

where $\tau_{ji}$ is the symmetric “boundary energy-momentum tensor”, where $i$ runs through $(0, \cdots d - 1)$.

The boundary scale invariance further requires

$$\tau^i_{\phantom{i}j} = \partial^i j_{\phantom{i}j} ,$$

(3.83)

where we call $j_i$ the boundary virial current. Much like in the bulk situation, if we can improve
the boundary energy-momentum tensor so that it is traceless, then the boundary is conformal
invariant.

With the assumption of the canonical scaling of the (boundary) energy-momentum tensor, we
can argue that Cardy’s condition \[211\] $T_r(r = 0) = 0$ is a necessary condition for the conformal
boundary. This is because boundary conformal invariance demands $\tau_{ij}$ is a symmetric traceless

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\[66\] Again, we should point out that the effective action term has the contact term ambiguity as discussed in section .
tensor whose conformal dimension is $d-1$. The unitarity of the boundary conformal algebra then demands it must be conserved. The Poincaré invariance further dictates that $T_\gamma(r = 0) = 0$, and Cardy’s condition follows.

The sufficiency of Cardy’s condition is more non-trivial. We believe that the scale invariant boundary condition (with some extra assumptions) implies the conformal boundary conditions as in the bulk case, but no rigorous derivation is available (in $d > 3$). In $d = 2$, the argument on the boundary $g$-theorem implies that the scale invariance without conformal invariance is inconsistent as in the bulk situation. In $d = 3$, we can show that Cardy’s condition is a sufficient condition, but it has not been proved whether this can be derived from the scale invariance alone.

Within the boundary perturbation theory, one can show that the higher dimensional analogue of the $g$-theorem should apply in boundary deformations, and therefore, the scale invariance must imply vanishing of the boundary $B$ function, resulting in the boundary conformal invariance. It would be very interesting to see if this argument can hold beyond the leading order in perturbation theory.

- The chiral scale invariance studied in [212] (see also [213]) states that the theory is invariant under the translation

$$t \rightarrow t + \epsilon_t, \quad x \rightarrow x + \epsilon_x,$$

and the chiral dilatation

$$t \rightarrow \lambda t$$

in $(1 + 1)$ dimensional local quantum field theories. Correspondingly, the theory possesses three conserved charges $H$, $P$, and $D$ with the commutation relation

$$i[D, H] = H, \quad i[D, P] = 0, \quad i[H, P] = 0.$$  

We assume that all the symmetries are linearly realized in a unitary manner.

From the Noether assumption, the translational invariance requires the existence of a conserved energy-momentum tensor

$$\partial_x T_{tx} + \partial_t T_{xx} = 0, \quad \partial_x T_{tt} + \partial_t T_{xt} = 0,$$  

which is not necessarily symmetric $T_{xt} \neq T_{tx}$ due to the lack of Lorentz invariance. The chiral scale invariance implies that the “trace” of the energy-momentum tensor must be given by the “divergence” of the “virial current”:

$$T_{xt} = \partial_t J_x + \partial_x J_t.$$  

Then the chiral dilatation current

$$D_t = t T_{tt} - J_t, \quad D_x = t T_{xt} - J_x$$

is conserved: $\partial_x D_t + \partial_t D_x = 0$.

As discussed in [212], we can always remove $J_t$ by defining the new conserved energy-momentum tensor

$$\tilde{T}_{tt} = T_{tt} + \partial_t J_t, \quad \tilde{T}_{xt} = T_{xt} - \partial_x J_t.$$
When, in addition, \( \partial_t J_x \) vanishes, the theory possesses the chiral special conformal transformation induced by the conserved current

\[
K_t = t^2 \tilde{T}_{tt}, \quad K_x = 0
\]  

(3.91)

together with the infinite tower of the chiral Virasoro symmetry \( \{ L^n_t = t^n \tilde{T}_{tt}, L^n_x = 0 \} \). The chiral special conformal transformation \( K \) with the chiral dilatation will generate the \( SL(2) \times U(1) \) subalgebra

\[
i[K, H] = D, \quad i[D, K] = -K, \quad i[K, P] = 0.
\]  

(3.92)

The vanishing of \( \partial_t J_x \) in unitary quantum field theories comes from the fact that the chiral scale invariance demands

\[
\langle J_x(x,t)J_x(0) \rangle = f(x),
\]

indicating \( \partial_t J_x(x,t)|0\rangle = 0 \) from the unitarity and translational invariance \[212\]. Furthermore, if the analogue of the Reeh-Schlieder theorem \[152\] is true, then \( \partial_t J_x(x,t)|0\rangle = 0 \) is equivalent to the vanishing of the local operator itself \( \partial_t J_x(x,t) = 0 \) (in any correlation functions): in relativistic field theories, the proof requires the microscopic causality in addition to the unitarity (see section 3.1.2). This shows that the chiral scale invariant field theories in \( (1+1) \) dimension are automatically invariant under the full chiral conformal transformation (with various technical assumptions).

- Since our primary interest in this lecture note is relativistic field theories, we have little to say about the non-relativistic scale invariance and conformal invariance. Some interesting classes of scale invariant algebra in non-relativistic systems include Schrödinger algebra \[214\][215\], Lifshitz algebra \[216\], and Galilean conformal algebra \[217\][218\] with rotation, time-translation, and space-translation in common. The Lifshitz algebra does not contain “special conformal symmetry”, so indeed we have an example of scale invariant but non-conformal field theories simply there is no way to enhance the symmetry.

Let us consider the Galilean invariant field theories with scale invariance. We can ask the question if we automatically obtain the non-relativistic conformal transformation, leading to the Schrödinger algebra \[220\]. The question is very similar to the one we have been working on in this lecture note with relativity.

We begin with a (possibly non-symmetric) conserved energy-momentum tensor (see e.g. \[214\]) from time-translation and space-translation

\[
\partial_t T^{0i} + \partial_j T^{ji} = 0
\]

\[
\partial_t T^{00} + \partial_i T^{i0} = 0.
\]  

(3.93)

The \( U(1) \) particle number conservation demands

\[
m \dot{\rho} = -\partial_i T^{0i}.
\]  

(3.94)

Then, the Galilean boost density \( G^i = tT^{0i} - mx^i \rho \) is conserved.

Suppose the energy-momentum tensor satisfies the condition

\[
2T^{00} - T^{ij} \delta_{ij} = \partial_t S + \partial_j A^j,
\]  

(3.95)

then the dilatation density \( D = tT^{00} - \frac{1}{2} x_i T^{0i} - \frac{S}{2} \) is conserved. We can always improve the energy-momentum tensor to remove \( A_j \) by \( 2T^{00} \to 2T^{00} + \partial_j A^j \). When \( S \) is a total divergence
$S = \partial_i \sigma^i$, one can further improve the energy-momentum tensor by $T^{00} \rightarrow T^{00} + \partial_j \partial_i \sigma^j$ so that the right hand side of (3.94) is zero \[220\].

When we can improve the energy-momentum tensor in this way, we are able to construct the non-relativistic special conformal density

$$K = i^2 T^{00} - tx_i T^{0i} + \frac{m}{2} x^2 \rho,$$  \hspace{1cm} (3.96)

which is conserved. We see that the structure of the symmetry enhancement is very close to the relativistic situation.

It would be interesting to prove if the non-relativistic special conformal invariance can be derived from the non-relativistic scale invariance possibly with additional assumptions. We have tried some perturbative searches in \[220\], and as far as we are aware, there are no known counterexamples. Again, it seems that the absence of the Reeh-Schlieder theorem can be a major obstruction for the proof. Another difficulty would be the absence of the analogue of Zamolodchikov’s $c$-theorem. In non-relativistic systems, it would be possible to have limit cycles in renormalization group flow, and it would make it more difficult to imagine the proof similar to the one for the relativistic field theories.

### 3.9 Literature guides

The main subject of this lecture has been the irreversibility of the renormalization group flow. Presumably, the concept of the irreversibility was not envisaged when the terminology of the renormalization group (by Stueckelberg and Petermann \[221\]) was first invented in the context of the quantum field theories with high energy physics in mind.

What do they mean by “group” in the renormalization group? Can a group transformation be irreversible? These are interesting questions, and a formal answer is that if we throw away the irrelevant parameters (as we implicitly did in our studies when we interpret the renormalization group flow in Wilson’s sense), then the transformation is only semi-group, and clearly there is a preferred direction. In the conventional field theory language, the renormalization group was just the $U(1)$ Abelian group of the scale transformation\[69\], but the gradient property makes it possible to introduce the notion of the preferred direction. In a sense, the Wess-Zumino consistency condition is the most advanced way to use the “group properties”. We should recall that we have to supplement the unitarity constraint to say anything about the irreversibility.

With this respect, it is very much similar to time. The time translation is Abelian, and from the group structure, there seems no preferred direction. However, if we throw away information along the evolution, the entropy increases, and there is a preferred future direction. We have seen that the notion of “time” (or unitarity and causality) actually played a hidden but crucial role in our discussions of the irreversibility of the renormalization group flow. For instance, our argument relies on the assumption of unitarity and causality.

From time to time, the violations of the weak version of the $a$-theorem were reported. In most cases, it turned out that either such hypothetical theories did not exist or computations of the central charge were erroneous in a subtle way. One example of the former is given by the series of AdS/CFT dual pairs with a certain Sasaki-Einstein manifold induced from the cone $z_1^4 + z_2^2 + z_3^2 + z_4^2 = 0$ \[222\]\[223\]\[224\] whose dual field theory construction can be found in \[223\]. The naive conclusion of the AdS/CFT correspondence is $a$ is increasing as $k$ decreases when $k \geq 4$, which is supposed to be a

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\[67\] At that time, because of the “Gruppen Pest”, I imagine everyone wanted to use the terminology “group”.

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relevant flow. What is happening here is that such a Sasaki-Einstein manifold does not exist due to a geometrical obstruction [222]. From the dual field theory perspective, the assumption of the existence of the non-trivial fixed point in $\alpha$-maximization was incorrect. Another example would be certain $\mathcal{N} = 1$ gauge theories obtained from the strongly coupled cousins of $\mathcal{N} = 2$ gauge theories induced by M5-branes wrapping around Riemann surfaces. Those theories do not possess the Lagrangian description, but some of the flow among them seem to violate the $\alpha$-theorem. It was interpreted in [226] that such theories should not exist, and it would be interesting to see what was wrong with them.

An example of the former was reported in [224], which was later rebutted in [227]. The problem was due to a subtlety in taking the IR limit and the assumption that the IR fixed point is described by a single superconformal field theory (rather than many weakly coupled with each other). It is important to note that after the physical proof of the $\alpha$-theorem, our gears are shifted so that we use the $\alpha$-theorem to exclude these hypothetical possibilities by seeking the flaw in the arguments rather than claiming them as counterexamples.
4 Lecture 3

In this lecture, we study the relation between scale invariance and conformal invariance from the holographic perspective. The holography is an alternative but powerful way to understand the strongly coupled quantum field theories in \( d \)-dimension by using the gravitational system in \( d + 1 \) (or higher) dimension. We discuss the holographic realization of the higher dimensional analogue of Zamolodchikov’s \( c \)-theorem and the equivalence between scale invariance and conformal invariance based on the energy-condition in general relativity. We will mention the validity and a possible violation of the energy-condition and its consequence in the holographic argument.

4.1 AdS/CFT and holography

Holography is one of the most powerful guiding principles to understand the fundamental aspects of quantum gravity \[228 \][229]. The holography dictates that \( d + 1 \) dimensional physics of the quantum gravity (referred to as “bulk” hereafter) is described by \( d \) dimensional non-gravitational physics (referred to as “boundary” hereafter), typically realized by quantum field theories. The (in)consistency of the dual field theory would yield strong constraints on the properties of quantum theories of gravity in the bulk. Conversely, we may be able to answer some unsolved problems in boundary field theories by using the holographic bulk argument.

One of the most established examples of the holography is the duality between a gravitational theory on \( d + 1 \) dimensional AdS space-time and a non-gravitational conformal field theory on \( d \) dimensional boundary of the AdS space-time (AdS/CFT correspondence \([230]\): see e.g. \([231]\) for an earlier review). AdS space-time is defined by the maximally symmetric space-time with constant negative curvature. In Poincaré coordinate, the \( d + 1 \) dimensional metric is given by

\[
ds^2 = g_{MN}dx^Mdx^N = L^2 \frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2}.
\]

It solves the Einstein equation \[68\] \( R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \) with negative cosmological constant \( \Lambda = \frac{d(d-1)}{2L^2} \).

As its name suggests, Poincaré coordinate has the manifest \( d \) dimensional Poincaré invariance acting on \( x^\mu \). We can easily see that the metric (4.1) is also invariant under the scaling transformation

\[
z \rightarrow \lambda z, \quad x^\mu \rightarrow \lambda x^\mu.
\]

Thus, the dual field theory is expected to be scale invariant. The corresponding killing vectors satisfy the commutation relation of the scaling algebra (2.2) acting on the Poincaré symmetry. It is less obvious but the metric (4.1) is invariant under the “special conformal transformation”

\[
\delta x^\mu = 2(\rho^\nu x_\nu) x^\mu - (z^2 + x^\nu x_\nu) \rho^\mu, \quad \delta z = 2(\rho^\nu x_\nu) z.
\]

The full isometry group of the AdS space-time is \( SO(2,d) \) and it agrees with the conformal group of \( d \)-dimensional quantum field theories. This leads to the first conjecture by Maldacena that the \((d + 1)\)-dimensional gravitational theory on AdS space-time describes a \( d \)-dimensional conformal field theory.

In AdS/CFT correspondence, we identify the \( SO(2,d) \) isometry of the AdS space-time with the conformal group of the dual conformal field theories. From the scaling transformation \( [1.2] \), it is

\[68\] We suppress the \( d \)-dimensional Planck constant throughout the lecture note.
We will identify the holographic direction $z$ with the direction of the renormalization group flow of the dual quantum field theory. The boundary $z \to 0$ corresponds to the UV limit of the dual field theory, and $z \to \infty$ corresponds to the IR limit of the dual field theory. For a later purpose, we note that the simple coordinate transformation (i.e. $z = e^{-Ar}A^{-1}$ with $A = L^{-1}$) makes the Poincaré metric $(4.1)$ into the warped metric

$$ds^2 = dr^2 + e^{2Ar} \eta_{\mu\nu} dx^\mu dx^\nu . \quad (4.4)$$

We will identify $Ar$ with the energy scale of the renormalization group flow $Ar \sim \log \mu$. We will assume $A$ and $L$ are both positive hereafter.

The most studied example of AdS/CFT correspondence is the duality between the type IIB string theory on $\text{AdS}_5 \times S^5$ background with Ramond-Ramond five-form flux, and the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with $SU(N)$ gauge group in $d = 4$. The underlying string theory construction allows us to identify various parameters in the both sides (e.g $N^2 = 4\pi^2 L^3$ for $d = 4$ case). In our following discussions, we do not specify the concrete realization of AdS/CFT from the string theory construction. We rather take an effective field theory approach of (quantum) gravity, and we will study the required consistency of the properties of the (quantum) gravitational system from the existence of the consistent dual quantum field theory interpretation. Of course, the following argument should apply to the string theory construction. Indeed, the validity of various assumptions such as the energy-condition or unitarity can be directly checked within the allegedly consistent string theory background.

To be more concrete, let us consider how to compute correlation functions via AdS/CFT correspondence. For this purpose, we recall that one of the most basic recipes in AdS/CFT correspondence is the Gubser-Klebanov-Polyakov-Witten (GKP-W) prescription $[232][233]$ that connects the computation of the partition function in the gauge theory side and the computation of the partition function in the gravity side. Schematically, we postulate the relation

$$\langle \exp \left( \int d^d x \phi^{(0)}(x) O(x) \right) \rangle_{\text{CFT}_d} = e^{-S_{\text{grav}}[\phi(x)|_{z=0=\phi^{(0)}(x)}]} \quad (4.5)$$

within the classical approximation of the gravity side. Obviously we should regard it as a certain hypothetical classical limit (saddle point approximation) of the quantum gravity “path integral” in the right hand side. It is not obvious such “path integral” exists, or we should do the path integral from the beginning, but we do believe that the right hand side should exist in a string theory context while the details are not well-established because of the difficulty in quantizing strings in AdS background with the Ramond-Ramond background.

Within the classical approximation, we can use the prescription $(4.3)$ for the study of correlation functions of conformal field theories from the corresponding classical equations of motion of the bulk theories. For instance, let us demonstrate it in a free massive scalar field $\phi(x, z)$ in $\text{AdS}_{d+1}$ with the minimally coupled action

$$S = \int d^{d+1}x \sqrt{|g|} \left( \partial_M \phi \partial^M \phi + m^2 \phi^2 \right). \quad (4.6)$$

The asymptotic solution ($z \to 0$) of the equation of motion gives

$$\phi \sim \phi^{(0)} \Delta_+ + \langle O \rangle z^\Delta_+ \quad (4.7)$$

with

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2}. \quad (4.8)$$
The scaling shows that the dual operator $O$ has the scaling dimension $\Delta_+$ and the coupling constant $\phi^{(0)}$ has the scaling dimension $\Delta_-$. We are able to check that the GKP-W prescription gives the scalar two-point function

$$\langle O(x)O(y) \rangle = \frac{c^\Delta_d}{(x-y)^{2\Delta_+}},$$

by functionally differentiating the GKP-W partition function \(1.5\) twice with respect to the source $\phi^{(0)}(x)$. Here $c^\Delta_d$ is a calculable number that depends on $\Delta$ and $d$, but after all it can be changed by the overall normalization of the scalar action.

We have seen that scalar fields such as $\phi(x_\mu,z)$ in the bulk correspond to scalar operators in the dual conformal field theories. The other important classes of fields that we typically encounter in the bulk is the graviton ($d+1$ dimensional metric) and gauge fields. Since the graviton couples with the conserved energy-momentum tensor and the gauge field couples with the conserved current, it is natural to postulate the duality between the $d+1$ dimensional metric fluctuation and $d$ dimensional energy-momentum tensor, and the duality between $d+1$ dimensional bulk gauge field with the $d$ dimensional conserved current. Indeed, the dimensional analysis similar to the above scalar example suggests that massless graviton must have $\Delta^+ = d$, and massless gauge field must have $\Delta^+ = d - 1$, which are precisely the values for the scaling dimension of the energy-momentum tensor and the conserved current of the dual $d$-dimensional conformal field theories.

It is important to realize that the GKP-W formalism of computing the generating function for the correlation functions by adding space-time dependent source terms (near $z = 0$) is very similar to the introduction of space-time dependent coupling constants discussed in the local renormalization group flow argued in section \[3.3.1\]. In the following sections, we will see how the consistency condition for the local renormalization group flow is related to the dynamics of the $d+1$ dimensional space-time through gravity.

In the actual computation of the GKP-W partition function, we encounter various divergence in the on-shell action due to the integration near $z \to 0$ limit (typically when the conformal dimension $\Delta$ takes an integer value). The resolution is obtained by first adding the finite cut-off at $z = \epsilon$ and add local counterterms at the boundary of the AdS space-time. They are given by the boundary fields such as boundary metric or the value of the scalar fields at the boundary and their derivatives. Note that they are local functional of the boundary fields, and they only changes the GKP-W prescription in contact terms. The structure of the holographic counterterms are very similar to the one we discussed in section \[3.3.1\] in relation to the local counterterms for the quantum effective actions with space-time dependent coupling constants. The procedure is called holographic renormalization and systematically developed in \[23,24\].

Generically, we have to impose boundary conditions to solve the second order equations of motion in gravitational theory. In the following, we will be interested in the so-called domain wall solution that interpolates two AdS space-time (with different cosmological constants). Suppose the gravitational theory under consideration admits multiple AdS vacua. We can consider the domain wall connecting two different vacua in the radial direction. By assuming that the domain wall preserves the $d$-dimensional Poincaré invariance, the metric must take the form

$$ds^2 = dr^2 + e^{2A(r)} \eta_{\mu\nu}dx^\mu dx^\nu,$$

where $A(r)$ approaches $A_{\text{UV}}$ in $r \to \infty$ and $A_{\text{IR}}$ in $r \to -\infty$ limit. It will interpolate two different AdS vacua with different cosmological constants. As we will see, the holographic interpretation of the
domain wall solution is the renormalization group flow between a UV conformal field theory described by one particular AdS vacua (with $e^{2A_{UV}}r$ as the warp factor) and an IR conformal field theory described by one particular AdS vacua (with $e^{2A_{IR}}r$ as the warp factor).

In this flow, the boundary condition is fixed both at $r \to \pm \infty$, and the solution is uniquely specified by the choice of the vacua. The details of the flow depends on the potential of the theory that determines vacua, and it may not be simple to solve the equations of motion with the fixed boundary conditions both at UV and IR. However, there is a beautiful simple realization of such a flow by using the (fake) superpotential as we will review. Such a simple flow is motivated by the Hamilton-Jacobi formalism of the flow \[236\] (see e.g. \[237\]-\[238\] for reviews) as well as the stability of the vacua in AdS space-time \[239\], and of course supersymmetry when available. In the next section, we will argue that it has a holographic interpretation as the gradient renormalization group flow.

4.2 Holographic $c$-theorem

As we mentioned, in AdS/CFT, we can regard the radial direction $r$ as renormalization scale $\log \mu$. The dynamics of the $r$ direction is governed by the gravitational equation of motion, and our task is to relate the gravitational dynamics with the renormalization group equation. In the holographic renormalization group flow \[240\]-\[241\]-\[242\]-\[243\]-\[244\], we consider the metric

$$
\text{d} s^2 = \text{d}r^2 + e^{2A(r)} \eta_{\mu\nu} \text{d}x^\mu \text{d}x^\nu ,
$$

(4.11)
special case of which corresponds to the domain wall solution discussed at the end of the previous section. When $A(r) = A_* r$, the metric describes the Poincaré patch of the AdS space-time and the dual field theory is conformal invariant (AdS/CFT correspondence).

The first observation is that without any matter, in diffeomorphism invariant gravitational theories, the requirement of the scale invariance is equivalent to the requirement of the AdS isometry. Indeed, under the scale transformation $x^\mu \to \lambda x^\mu$, the Minkowski metric $\eta_{\mu\nu} \text{d}x^\mu \text{d}x^\nu$ acquires $\lambda^2$. On the other hand, we can always assume that the scaling transformation acts on $r$ as the shift transformation $r \to r + c \log \lambda$ with a certain constant $c$. We see that the condition of the isometry under the scale transformation fixes the metric to be AdS space-time

$$
\text{d} s^2 = \text{d}r^2 + e^{2A_* r} \eta_{\mu\nu} \text{d}x^\mu \text{d}x^\nu
$$

(4.12)
so that the isometry vector fields give the desired scaling transformations of the Poincaré algebra \[2.2\]. We realize that the geometry is nothing but the AdS space-time, and we have obtained the enhanced AdS isometry $SO(2, d)$ even though we did not require the invariance under the special conformal transformation.

Once we have understood the scale invariant vacuum solutions, we would like to take a further look at the non-trivial flow solution with the metric ansatz \[2.2\]. Here, we will see the holographic analogue of Zamolodchikov’s $c$-theorem. To be concrete, our starting point is a gravitational theory described by the classical Einstein Hilbert action (with negative cosmological constant) minimally coupled with a classical matter (we will relax some of the conditions in later sections by introducing higher derivative terms or quantum corrections).

$$
\mathcal{S} = -\frac{1}{2} \int d^d x \sqrt{|g|} \left( R + \frac{d(d-1)}{L^2} \right) + \mathcal{S}_{\text{matter}} .
$$

(4.13)

\[69\] However, we emphasize the (fake) superpotential flow is not limited to supergravity.
In this setup, we define the holographic c-function \( a_d(r) \) by (hereafter prime denotes the derivative with respect to the radial direction:

\[
a_d(r) = \frac{\pi^{d/2}}{\Gamma(d/2)((A'(r))^d-1).}
\]

(4.14)

The overall normalization factor is fixed from the holographic Weyl anomaly argument we will explain later. Then by using the Einstein equation, we obtain

\[
\frac{da_d(r)}{dr} = -\frac{(d-1)\pi^{d/2}}{\Gamma(d/2)(A'(r)^d)} A''(r) = -\frac{\pi^{d/2}}{\Gamma(d/2)(A'(r))^d}[T^t_t - T^r_r] \geq 0.
\]

(4.15)

In the last equality we have assumed the null energy condition (NEC). Therefore, the holographic c-function \( a_d(r) \) is monotonically decreasing along the renormalization group flow.

The null energy condition is the condition for the energy-momentum tensor: for all null vectors \( k^M \) such that \( g_{MN}k^Mk^N = 0 \), the energy-momentum tensor satisfies \( k^Mk^NT_{MN} \geq 0 \). In the fluid frame, it requires that sum of the energy and pressure is semi-positive: \( \epsilon + p \geq 0 \). By using the Einstein equation, it constrains possible geometries through relating it with the constraints on the energy-momentum tensor: \( k^Mk^NR_{MN} \geq 0 \).

Apart from the holographic c-theorem, the requirement of the null energy-condition leads to a deep consequence in general relativity. Einstein equation itself allows many seemingly pathological solutions like worm-hole, superluminal propagation of information, classically decreasing black hole horizon, time-machine and so on, but the null energy condition forbids them (see e.g. [146][147][148][149]).

At the conformal fixed point, we can relate the above defined holographic c-function \( a_d(r) \) to the Weyl anomaly when space-time dimension of the dual field theory \( d \) is even. The idea is to replace the boundary Minkowski metric \( \eta_{\mu\nu} \) with a general weakly curved metric \( g_{\mu\nu} \) and study the expectation value of \( T_{\mu\nu} \) through the GKP-W prescription. We will not go into the detailed computation (see [250] for the original computation), but we only quote the result in \( d = 4 \). The holographic Weyl anomaly is given by

\[
\langle T^\mu_\mu \rangle = \frac{L^3}{16}(\text{Weyl}^2 - \text{Euler}).
\]

(4.16)

We see that the Einstein gravity predicts \( a = c = \frac{L^3}{16} = \frac{a_4}{16\pi^2} \) in holography. This holographic computation indeed suggests that the central charge \( a \) has a natural interpolation function, which is monotonically decreasing along the holographic renormalization group flow. However, at this stage, we cannot distinguish two different Weyl anomaly \( a \) and \( c \) in the Einstein gravity.

Let us discuss more details on the structure of the holographic c-theorem. Suppose the matter action is given by a generic non-linear sigma model (possibly with a potential):

\[
S_{\text{matter}} = \int d^{d+1}x \sqrt{|g|} (G_{IJ}(\Phi)\partial^M \Phi^I \partial_M \Phi^J + V(\Phi)).
\]

(4.17)

Then the explicit computation of the energy-momentum tensor gives\(^72\)

\[
T^t_t - T^r_r = -g^{tt}G_{IJ}\partial_t \Phi^I \partial_r \Phi^J.
\]

(4.18)

\(^70\)When \( d \) is even, the positivity of the factor in front is obvious. When \( d \) is odd, we can still argue\(^243\) that the sign of \( A'(r) \) cannot change along the renormalization group flow. We will also see it from (4.23).

\(^71\)Here, we assume the canonical (non-improved) energy-momentum tensor that corresponds to the Einstein frame of gravity. This is reasonable because we use the gravity equations in the Einstein frame.
We may regard $\partial_r \Phi^I$ as the beta function $\beta^I$ for the coupling constant corresponding to $\Phi^I$ under AdS/CFT correspondence (up to a choice of the renormalization scheme). We may also regard the target space metric $\mathcal{G}_{IJ}$ as the Zamolodchikov metric $\chi_{IJ}$ (up to a choice of the renormalization scheme and the associated constant multiplicative factor $a(r)$ we will discuss below). Indeed, at the conformal fixed point, $\mathcal{G}_{IJ}$ determines the two-point function via the GKP-W prescription.

Under the holographic renormalization group flow, the Einstein equation demands

$$\frac{da_d}{dr} = \frac{\pi^{d/2}}{\Gamma(d/2)(A'(r))^d} g^{rr} \mathcal{G}_{IJ}\partial_r \Phi^I \partial_r \Phi^J,$$

which is interpreted as

$$\frac{d\tilde{a}}{d \log \mu} = \beta^I \chi_{IJ} \beta^J.$$

This is nothing but the strong $c$-theorem (or what we called the strong $\tilde{a}$-theorem in $d = 4$) we have discussed in the previous lecture.

To be more precise, we have to make the relation between the energy scale $\log \mu$ and the radial direction $r$ because away from the scale invariant fixed point, there is some arbitrariness. The standard choice would be the so-called holographic scheme [251]:

$$\log \mu \equiv A(r), \quad \beta^I = \frac{d\Phi^I}{dA(r)},$$

in which “Zamolodchikov metric” $\chi_{IJ}$ is given by $a_d \mathcal{G}_{IJ}$. The proportional factor by $a_4 \sim N^2$ (in $d = 4$) can be well-understood in the large $N$ CFT (e.g. $\mathcal{N} = 4$ supersymmetric Yang-Mills theory), in which supergravity light operators are described by single trace operators like $\text{Tr}(F_{\mu\nu}^2)$.

We also note that coupling constant dependent Weyl anomaly can be computed by assuming (nearly) massless fields $\Phi^I$ are space-time dependent at boundaries [252][253][254]. In particular, it turns out that the Weyl anomaly with $\beta^I = 0$ in $d = 4$ is governed by the (boundary) Riegert operator $\Delta_4$ in (2.63) as $\mathcal{G}_{IJ} \Phi^I \Delta_4 \Phi^J$ (up to boundary counterterms), which is reasonable because the Wess-Zumino condition is trivially solved due to the Weyl invariance of $\Delta_4$ then. Thus, by comparing it with the result in section 3.3.1 we see that the AdS/CFT at the conformal fixed point predicts $\chi^{g}_{IJ} = 2a_4 \mathcal{G}_{IJ}$ and $\chi^{g}_{IJ} = 4a_4 \mathcal{G}_{IJ}$ [25]. The computation can be generalized in which $\beta^I$ are small but non-zero and it agrees with our discussion in section 3.3.1.

In addition, the holographic renormalization group flow shows the gradient property whenever the potential takes a “holographically renormalizable form”. Suppose the potential admits the “superpotential” $W(\Phi)$ so that

$$V(\Phi) = \mathcal{G}^{IJ} \partial_I W(\Phi) \partial_J W(\Phi) - \frac{d}{4(d-1)} W(\Phi)^2,$$

then the flow of the scalar field $\Phi^I$ turns out to be a gradient flow:

$$\partial_r \Phi^I = \mathcal{G}^{IJ} \partial_J [W],$$

$$A'(r) = \frac{1}{d-1} [W].$$

---

The proportional relation $\chi^{g}_{IJ} = \frac{1}{2} \chi^{g}_{IJ}$ at the conformal fixed point may be explained by the Wess-Zumino consistency condition of the local renormalization group.
The first formula corresponds to the gradient formula of the renormalization group. Indeed, the second formula suggests that the holographic c-function (4.14) is proportional to \(|W|^{−d+1}\) and with the holographic scheme (4.21), we can precisely reproduce the gradient formula of the field theory.

Note that for a given \(V(\Phi)\) there could be many different \(W\) that satisfies (4.22) and each gives a different flow. On the other hand, not every \(V(\Phi)\) possess the corresponding fake superpotential \(W(\Phi)\). The potential flow (4.23) has a natural interpretation as the gradient formula of the beta function. We recall that the gradient formula in the renormalization group could contain the anti-symmetric part. If there were “B-field”, then it would have an anti-symmetric part in \(G_{IJ}\), but the significance is not so clear at this point. It seems that the anti-symmetric part vanishes in holographic computation.

From the Hamilton-Jacobi formulation of the holographic renormalization group flow, the condition (4.22) is regarded as a holographic renormalizability from adding the boundary counterterms of scalar fields. In the literature it was suggested that the condition (4.22) is probably a necessary condition to guarantee a consistent stable renormalization group flow. For instance, the form of the potential (4.22) automatically guarantees the Breitenlohner-Freedman bound: \(m^2 ≥ −\frac{d^2}{4L^2}\) that assures the unitarity bound of the scaling dimensions of operators at the conformal fixed point. It seems remarkable that the consistency of the renormalization group interpretation gives a constraint on the possible potential in the bulk gravity.

Before going on, let us say a few words about the situation in the case with odd \(d\). When \(d\) is odd, we do not have (holographic) Weyl anomaly, so the question arises what is the physical interpretation of the monotonically decreasing holographic c-function (4.14). Priori, there are many physical quantities related to the number: two-point functions (or higher-point functions) of energy-momentum tensor, thermal free-energy, entanglement entropy and so on. At the level of the Einstein gravity, however, they are indistinguishable. The study of the higher order derivative corrections revealed that it is related to the Euclidean \(S^d\) partition function and the entanglement entropy. We refer to [202] for more details.

So far, we have not considered the \(\beta\) function for the vector operators. In our applications of scale invariance vs conformal invariance, it is important to realize the operator identity such as \(\beta^I O_I = −\beta^a \partial_\mu J^a_\mu\) as discussed in Lecture 1. The redundant operators in the conformal field theory are realized by gauge symmetries in the holographic renormalization group flow. Suppose we gauge the non-linear sigma-model by requiring it is invariant under the gauge transformation \(\Phi → e^{i\Lambda}\Phi\) and \(A → A + d\Lambda\) (we can easily generalize the situations with non-Abelian symmetry). Then the gauged non-linear sigma model is described by the action

\[
S_{\text{matter}} = \int d^d x \sqrt{|g|} \left( G_{IJ}(\Phi) D_M \Phi^I D^M \Phi^J + V(\Phi) \right),
\]

where \(D_M = \partial_M − A_M\) contains the gauge connection, and the kinetic term \(G_{IJ}\) and the potential \(V(\Phi)\) must be gauge invariant.

Now, the energy-momentum tensor appearing in the holographic renormalization group flow is replaced by the gauged one

\[
T^r_i - T^i_r = −g^{rr} G_{IJ} D_r \Phi^I D^r \Phi^J.
\]

\(^{73}\)Another interesting feature of the holographic computation is \(\chi^3_{IJ} = \frac{1}{2} \chi^3_{IJ} \) in a natural holographic scheme as we mentioned above. Of course, this relation can be modified by adding local counterterms, so it is not a robust prediction.

\(^{74}\)The holographic discussions on the entanglement entropy first appeared in seminal papers by Ryu and Takayanagi [256], [257]. See e.g. [258] for a review.
We can regard $D_r \Phi^I$ as $B^I$ rather than the beta function $\beta^I \sim \partial_r \Phi^I$. Indeed, as we will discuss, the arbitrary separation of $B^I O_I = \beta^I O_I + \beta^a \partial^a J_a^I$ is the corresponding gauge transformation. By substituting the energy-momentum tensor (4.23) into the holographic renormalization group flow (4.15), we interpret the holographic renormalization group flow in the gauged non-linear sigma-model

$$\frac{da_d}{dr} = \frac{\pi [d/2]}{\Gamma(d/2)(2')_d} g^{\tau \tau} G_{IJ} D_r \Phi^I D_r \Phi^J,$$

as the holographic realization of the strong $c$-theorem with respect to the $B$ function flow

$$\frac{d\tilde{a}}{d \log \mu} = B^I \chi_{IJ} B^J \quad (4.27)$$

as expected from field theory discussions of the strong $c$-theorem in the previous lecture.

It is interesting to observe that the gauge invariance of the action imposes some interesting restrictions of the holographic renormalization group flow with the operator identity $B^I O_I = \beta^I O_I + \beta^a \partial^a J_a^I$. First of all, the holographic $c$-function does not depend on the coupling constant that can be removed from the gauge transformation. This is due to the gauge invariance, and the holographic $c$-function has flat directions corresponding to the redundant perturbations. If the gradient formula holds, then this further suggests that the $B^I$ functions have as many zero directions as the gauged directions. These directions must be in contrast with exactly marginal directions that are not gauged: physics changes along the exactly marginal but non-redundant directions. One example of the exactly marginal direction is the dilaton in type IIB string theory on AdS$_5 \times S^5$ which corresponds to the coupling constant of the $\mathcal{N} = 4$ super Yang-Mills theory. On the other hand, the redundant directions appear in $\mathcal{N} = 8$ gauged supergravity in which various scalar fields are gauged under the $\mathcal{R}$-symmetry $[243]$. These gauge directions in holographic renormalization group flow precisely correspond to redundant perturbations, and the flow in that direction (if any) should be regarded as physically equivalent. Indeed, the argument here is in complete parallel with the one in section 3.6. In particular, $w_I$ is exact in holographic computation and can be gauged away, so the gradient formula does not contain the inhomogeneous terms in the holographic scheme.

### 4.3 Scale vs Conformal from holography

As we have mentioned, the scale invariance dictates that the metric must take the form

$$ds^2 = L^2 dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu \quad (4.28)$$

up to diffeomorphism invariance. It is accidentally invariant under the full AdS isometry. The holographic interpretation is obvious: only with the gravity, there is no other operators than the energy-momentum tensor in the dual field theory, and in particular there is no candidate for the virial current. Without the candidate for the virial current, the only possible way to realize scale invariance is $T^\mu_\mu = \partial^\mu J_\mu = 0$, and the theory is conformal invariant.

In order to break the conformal invariance, therefore, we need the matter sector that represents a non-trivial existence of the virial current. As discussed in $[259][260]$, the non-trivial configuration of the matter may break the conformal invariance while preserving the scale invariance. Some examples are vector condensation

$$A = \alpha \frac{dz}{z} \quad (4.29)$$
and the scalar condensation (with gauge invariance)

\[ \Phi = \gamma z^{i\alpha} \]  

in the AdS background. The gauge invariance is needed because under the scale transformation \( z \rightarrow \lambda z \), \( \Phi \) acquires the phase: \( \Phi \rightarrow \lambda^{i\alpha} \Phi \), and we need to cancel it by the gauge invariance. Without the gauge invariance, it is only invariant under the discrete scale transformation.

Clearly these configurations are invariant under the scaling transformation, but it is not invariant under the AdS isometry that corresponds to the special conformal transformation \((4.3)\). As we will discuss, such configurations are directly related to the non-zero contribution to the trace of the energy-momentum tensor of the dual field theory and gives a non-trivial existence of the virial current realized in the holographic renormalization group.

In the previous section, we have introduced the gauged non-linear sigma-model to realize the operator identity of the dual field theory, which is crucial to understand the emergence of the virial current as discussed in Lecture 1. From this perspective, the above two configurations \((4.29)\) and \((4.30)\) are mutually related by the gauge transformation. Indeed, suppose \( \Phi \) and \( A \) are related by the gauge transformation: \( \Phi \rightarrow e^{i\Lambda} \Phi \) and \( A \rightarrow A + d\Lambda \), then the configuration

\[ \Phi = \gamma z^{i\alpha} \]
\[ A = 0 \]  

which can be interpreted as \( \beta = i\alpha \) with \( \partial_\mu J_\mu^a = 0 \) in the dual field theory, is gauge equivalent to

\[ \Phi = \gamma \]
\[ A = \frac{\alpha dz}{z} \]  

which can be interpreted as \( \beta = 0 \) with \( \partial_\mu J_\mu^a \neq 0 \) in the dual field theory.

The former shows a cyclic renormalization group flow because the phase of the field \( \Phi \) (phase of the dual coupling constant) is rotating along the evolution in \( z \) direction in the holographic renormalization group flow from our identification \( \beta^I \sim \partial_\phi \Phi^I \) discussed in section \(4.3\). On the other hand the latter gives the non-zero background gauge field renormalization with the identification \( \partial^\mu j_\mu \) with \( A_z \), which is eventually related to the non-zero virial current. Of course, the gauge invariant quantity \( B^I \) that appears in the trace of the energy-momentum tensor of the dual field theory is non-zero whichever gauge one uses because of the gauge invariant identification \( B^I \sim D_r \Phi^I \).

Our central question is whether such a flow is possible in a reasonable theory of holography. We argue that there are two main obstructions. The first one is that the potential of the gauge direction is always zero from the gauge invariance, so it is unlikely that such a flow is generated from the beginning. In particular in the superpotential flow or gradient flow, the radial evolution of the field \( \Phi^I \) is uniquely specified by the gradient of the gauge invariant superpotential, and the field theory discussions in section \(3.6\) directly applies. The second one is the inconsistency with the holographic \(c\)-theorem. Suppose that the metric of the non-linear sigma-model in the holographic renormalization group flow is positive definite. Then such a scale invariant but non-conformal background gives a non-trivial flow for the warp-factor: we derived the holographic \(c\)-theorem

\[ \frac{d\alpha_d}{dr} = \frac{\pi^{d/2}}{\Gamma(d/2)(A'(r))^2} g^{rr} G_{IJ} D_r \Phi^I D_r \Phi^J, \]  

(4.33)
or its field theoretic interpretation
\[
\frac{da_d}{d\log \mu} = a_d B^I G_{IJ} B^J
\] (4.34)
but whenever \( D_r \Phi^I \neq 0 \) these are inconsistent with the scale invariance because the warp-factor does not take the scale invariant form \( e^{2A \tau} \) as long as the metric is non-degenerate. Note that the requirement of the positivity of the metric is natural because of the unitarity of the bulk theory although strictly speaking it is not guaranteed by the null energy-condition alone. We will introduce the notion of the strict null energy-condition to give a sufficient condition for the unitarity as well as the equivalence between scale invariance and conformal invariance.

In retrospect, the “counterexample” of the scale invariant but non-conformal field theory in beta function flow (see section 2.4.2) can be understood in holography as follows. We start with the manifestly conformal invariant background
\[
\Phi = \gamma \\
A = 0
\] (4.35)
and perform the gauge transformation
\[
\Phi = cz^i \alpha \\
A = -\frac{\alpha dz}{z}
\] (4.36)
so now we interpret that beta function is non-zero \( \beta = i \alpha \) and the renormalization group flow appears to be cyclic from \( \frac{da_d}{d\log \mu} = \beta^I \). However, there is an extra contribution from the beta function for the background vector fields and the total trace of the energy-momentum tensor is zero and the theory is conformal invariant as it must be. In spirit, this is close to the artificial separation we did in section 2.7.

The gauge transformation in effective \( d + 1 \) dimensional gravity may be seen as the coordinate transformation in the higher dimensional gravity. Take AdS\(_5 \times S_5\) solution in type IIB string theory with the metric
\[
ds^2 = L^2 \left( \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2} + d\gamma^2 + \cos^2 \gamma d\phi^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\theta_1^2 + \sin^2 \psi d\theta_2^2) \right)
\] (4.37)
and perform the coordinate transformation \( \varphi \to \varphi + \alpha \log z \):
\[
ds^2 = L^2 \left( \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2} + d\gamma^2 + \cos^2 \gamma (d\varphi + \frac{\alpha dz}{z})^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\theta_1^2 + \sin^2 \psi d\theta_2^2) \right)
\] (4.38)
The geometry is still manifestly scale invariant, but the isometry corresponding to the special conformal transformation is obscured. The \( \varphi \) direction was isometry so we are mixing the dilatation current with the conserved current. If we shifted the non-isometric direction, say \( \gamma \) with \( \log z \), we would have non-zero artificial virial current that is not conserved and it would induce the apparent cyclic renormalization group flow in holography. The scaling transformation must accompany the shift of \( \gamma \) as is consistent with (4.36).

Note that from the holographic perspective, there is nothing wrong with the second background (4.36), and it may be actually more mysterious why the perturbative computation of the beta functions
(say in the minimal subtraction scheme) prefers the seemingly zero renormalization for the background vector fields. There seems to be a certain perturbative mechanism to choose a gauge.

In Lecture 1 and 2, we have discussed how the use of the equations of motion in \( B^I O_I = \beta^I O_I + \beta^0 \partial_\mu J^\mu_a \) introduces the additional anti-symmetric wavefunction renormalization resulting in the anti-symmetric contributions in the anomalous dimensions. As long as we are interested in the physical spectrum of the AdS space-time, we cannot see it: the only gauge invariant object is the mass of the physical excitations that correspond to the dimensions of operators after the diagonalization. In this way, the anti-symmetric wavefunction renormalization does not play a significant role in the holographic approach, suggesting that they are simply an artifact of perturbative computations (with reference to the trivial fixed point) and are not intrinsic to the theory.

We have a small comment on the anomalous dimension of the current operator. As discussed above, we are led to the conformal invariant fixed point once we assume the scale invariance. We note that there is an operator identity \( \beta^I O_I = -\beta^0 \partial_\mu J^\mu_a \) so that \( B^I = 0 \). This means that \( J^\mu_a \) are not conserved and must acquire anomalous dimensions: otherwise they must be conserved from unitarity. The holographic realization of the anomalous dimension is the Higgs mechanism. We note that for the operator identity to be realized in our holographic setup, the charged scalar must obtain a vacuum expectation value (as in (4.35)), and then the gauge field becomes massive. Following the discussion in section 4.1, such a massive vector field corresponds to a non-conserved vector operator with the scaling dimension \( \Delta > d - 1 \). In contrast, the combination \( \beta^I O_I + \beta^0 \partial_\mu J^\mu_a \) does not acquire the anomalous dimension because it is the gauge direction.

So far, we have assumed a particular classical action for the matter sector. More generically, we argue that if the matter field satisfies the strict null energy condition, scale invariance implies conformal invariance in holography. The strict null energy condition states that the equality of the null energy condition is saturated if and only if the field contributing to the energy momentum tensor takes the trivial configuration \([261]\). More precisely, we demand that if there exists any null-vector that makes \( T_{MN} k^M k^N = 0 \), then the field configuration must be invariant under all the isometry transformation of the space-time. Note that the null energy-condition itself cannot exclude the degenerate metric for the (gauged) non-linear sigma-model but the strict null energy condition does. Supergravity analysis of the strict null energy-condition in string compactification can be found in \([260]\).

### 4.3.1 Holographic counterexample 1: null energy violation

In the last section, we used a certain energy-condition to prove the holographic \( c \)-theorem as well as the equivalence between scale invariance and conformal invariance. Without imposing the strict null energy-condition, one can construct a counterexample of scale invariant but non-conformal dual field theories in holography \([259] [260]\). Let us consider the vector field theory with the generic (gauge non-invariant) potential

\[
S = \int d^d x \sqrt{|g|} \left( -\frac{1}{2} R + \frac{1}{4} F_{MN} F^{MN} + V(A_M A^M) \right).
\]

We assume that the potential \( V(A_M A^M) \) for the vector field \( A_M \) has a non-trivial extremum (e.g. Mexican hat potential \( V(A_M A^M) = \Lambda_0 - m^2(A_M A^M) + \lambda (A_M A^M)^2 \)). We see that the theory may admit the non-trivial vector condensation solution

\[
ds^2 = L^2 \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2}
A = \alpha \frac{dz}{z}
\]

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with non-zero $\alpha$, depending on the shape of the potential. Note that the violation of the AdS isometry in the matter does not back-react to the metric, so it will keep the AdS metric.

(Exercise) Compute the matter energy-momentum tensor from (4.39) and show that the solution (4.40) only gives the contribution proportional to the metric.

As discussed in the last section, the configuration preserves the scale invariance, but does not preserve the special conformal invariance in the matter vector condensation. Therefore, we may regard it as a holographic realization of scale invariant but non-conformal field theories.

It is interesting to point out its relation to ghost condensation [262]. The above vector condensation model can be made gauge invariant by introducing the Higgs field with higher derivative kinetic terms:

$$S = \int d^{d+1}x \sqrt{|g|} \left( -\frac{1}{2} R + \frac{1}{4} F_{MN} F^{MN} + V(D_M \Phi D^M \Phi) \right).$$

(4.41)

Here the gauge invariant higher derivative kinetic term is given by e.g. $V(D_M \Phi D^M \Phi) = \Lambda_0 - m^2 (D_M \Phi D^M \Phi) + \lambda (D_M \Phi D^M \Phi)^2$. If we fix the gauge in a unitary gauge $\Phi = \text{const}$, then it is equivalent to the vector condensation model. Again this particular form of the kinetic term does not back-react to the AdS space-time even though the isometry is broken by the non-trivial field configuration.

On the other hand, if we ignore the gauge field, this is nothing but a model of the ghost condensation studied in relation to the alternative gravity with the action

$$S = \int d^{d+1}x \sqrt{|g|} \left( -\frac{1}{2} R + V(D_M \Phi D^M \Phi) \right).$$

(4.42)

Our discussion suggests that the holographic dual of the (gauged) ghost condensation in AdS space-time would be inconsistent (at least when $d = 2$) because it is incompatible with the equivalence of scale invariance and conformal invariance, which we know must be true from the field theory argument. Presumably, the unitarity is sacrificed in order to reconcile with the situation, which is suggested by the violation of the (strict) null energy condition.

As emphasized in [259], the situation can be different in dS space-time, where we allow time-like condensation $A = \frac{dt}{t}$ with the de-Sitter metric $ds^2 = L^2 \frac{dt^2 - \delta_{\mu\nu} dx^\mu dx^\nu}{t^2}$. Although we can still postulate holography like dS/CFT [263] to discuss the properties of this hypothetical dual of the scale invariant but not conformal invariant Euclidean field theory, they may not disagree with our field theory arguments. The point is that the dS/CFT does not assure the unitarity of the dual field theory, and without unitarity (more precisely reflection positivity in the Euclidean signature), it is possible to construct scale invariant but non-conformal field theories without unitarity. See some examples in section 2.4 and section 2.8. The cyclic behavior in the time-like condensation or dS/CFT is something like “time crystal” studied in [264].

4.3.2 Holographic counterexample 2: foliation preserving diffeomorphic theory of gravity

Another interesting possibility to realize the holographic dual of scale invariant but not conformal field theories is to abandon the full space-time diffeomorphism [169]. We have discussed that the scale invariance and Poincaré invariance naturally leads to the AdS metric

$$ds^2 = g_{MN} dx^M dx^N = L^2 \frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2}.$$

(4.43)
The metric has a natural foliation with respect to the \(d\)-dimensional Minkowski space-time. In order to preserve the scale invariance, we do not have to assume the full \(d+1\) dimensional diffeomorphism.

In [265][266] (see e.g. [267] for a review), they discussed a foliation preserving diffeomorphic theory of gravity. Their motivation is to improve the power-counting renormalizability of the quantum gravity by adding higher spatial curvature terms without introducing higher time derivative terms to avoid ghost, or negative norm states. We are not particularly interested in the renormalizability problem, but we can borrow their idea, and consider the foliation preserving diffeomorphic theory of gravity in the holographic radial direction rather than time directions as in the original proposal.

We write the \(d+1\) dimensional space-time metric in the form similar to the Arnowitt-Deser-Misner (ADM) decomposition [268]:

\[
ds^2 = N^2 dz^2 + g_{\mu \nu} (dx^\mu + N^\mu dz)(dx^\nu + N^\nu dz),
\]

where \(g_{\mu \nu}\) has the Lorentzian signature, and \(N\) is an analogue of the lapse function and \(N^\mu\) is that of the shift vector. We demand the theory be invariant under the foliation preserving diffeomorphism

\[
\delta x^\mu = \xi^\mu(z, x^\mu), \quad \delta z = f(z).
\]

The simplest action up to the second order derivatives would be

\[
S = \int N dz \sqrt{|g|} d^d x \left( K_{\mu \nu} K_{\mu \nu} - \lambda K^2 + R_d + \Lambda \right),
\]

where \(K_{\mu \nu} = \frac{1}{2N} \left( \partial_z g_{\mu \nu} - D_\mu N_\nu - D_\nu N_\mu \right)\), and \(K = g^{\mu \nu} K_{\mu \nu}\) with \(D_\mu\) being the covariant derivative with respect to \(d\)-dimensional metric \(g_{\mu \nu}\). \(R_d\) is the \(d\)-dimensional Ricci scalar out of \(g_{\mu \nu}\). The parameter \(\lambda\) describes the deviation from the Einstein-Hilbert action, and \(\lambda = 1\) formally corresponds to the Einstein-Hilbert action up to surface terms.

We can easily show that the theory has a solution of the Poincaré AdS-metric (4.43), but crucial point is that although the scaling isometry is a foliation preserving diffeomorphism, the special conformal isometry is not: the coordinate transformation

\[
\delta x^\mu = 2(\rho^\nu x_\nu) x^\mu - (z^2 + x^\nu x_\nu) \rho^\mu, \quad \delta z = 2(\rho^\nu x_\nu) z
\]

is not the foliation preserving form (4.45). Therefore, if the foliation diffeomorphic theory of gravity with the Poincaré AdS-metric solution has a dual field theory interpretation, it cannot possess the conformal invariance as the space-time symmetry. It still possesses the scale invariance and the \(d\)-dimensional Poincaré invariance, so it should be dual to a scale invariant but non-conformal field theory.

Furthermore, if we perform the holographic Weyl anomaly computation in \(d = 4\), which is a generalization of the one we have reported in Einstein gravity in section 4.2, we can derive

\[
T^\mu_{\text{\mu}} = \frac{L^3}{16} \left( \text{Weyl}^2 - \text{Euler} + \frac{2}{3} \frac{\lambda - 1}{4\lambda - 1} R^2 \right),
\]

and the explicit appearance of \(R^2\) term dictates that the dual field theory cannot be conformal invariant due to the Wess-Zumino consistency condition (see section 2.4). We learned that the \(R^2\) term cannot appear unless the theory violates the conformal invariance. In addition, the term is related to bilinear of \(B\) functions, so effectively, the deviation from the Einstein-Hilbert limit introduces non-trivial virial current.
At the same time, given our understanding of the importance of unitarity to show the equivalence between scale invariance and conformal invariance, it is very likely that the theory effectively violates the (strict) null energy-condition and the unitarity is lost. Again, the situation can be different in the original Horava time-like setup, in which we foliate the space-time with space-like Cauchy surface. Their setup may be consistent with the holography because as in the ghost condensation, the lack of the unitarity of the dual field theory may not be directly related to the inconsistency of the gravity dual (if any) in the Euclidean signature. It would be interesting to understand a possible (non reflection positive) scale invariant but non-conformal field theory as the dual of Horava gravity in the de-Sitter solution.

4.4 Beyond classical Einstein gravity

The holographic arguments so far have assumed the classical Einstein gravity. Some of the predictions such as $a = c$ are particular to the classical approximation to the gravity dual and it does not cover the entire space of the conformal field theories. In this section, we will discuss various attempts to introduce corrections to the Einstein gravity in holography such as higher derivative corrections and quantum anomalous corrections. We discuss these aspects within the effective $d + 1$ dimensional gravity. Ultimately, these must be embedded in the string theory to fully understand the quantum gravity, which we will leave for the future study.

4.4.1 Higher derivative corrections

In quantum gravity (like string theory) the Einstein equation is modified in two different ways. The first one is higher derivative corrections that can be derived from the local action principle (e.g. $\alpha'$ corrections in string theory). The second one is possibly non-local corrections from quantum loop effects (e.g. $g_s$ corrections in string theory) including anomalous terms.

The effects of local higher derivative corrections in holographic renormalization group flow have been studied \cite{245\cite{269} within the assumption that the gravity part of the action and the matter part of the action are separated in a minimal way. This restriction is ultimately related to the usage of the “null energy condition” in higher derivative gravity: without the separation, the notion of the energy-condition becomes very much obscured.

Let us consider the higher derivative gravity with $O(R^2)$ correction. We start with the action

$$S = -\frac{1}{2} \int d^{d+1}x \sqrt{|g|} \left( \frac{d(d-1)}{L^2} + R + b_1 L^2 R_{MNLK}^2 + b_2 L^2 R_{MN}^2 + b_3 L^2 R^2 \right) + S_{\text{matter}}. \quad (4.49)$$

The assumption of the minimal separation is that the matter action $S_{\text{matter}}$ does not contain curvature couplings. We do not expect that every gravitational theory satisfies the holographic c-theorem because the unitarity may be violated in a higher derivative gravity. Our strategy is to compute $T^t_t - T^r_r$ with the use of the higher derivative corrected equations of motion, and demand the null energy-condition. Can we still find the holographic c-function that monotonically decreases along the holographic renormalization group flow in the radial direction?

For this purpose, it is sufficient to require that $T^t_t - T^r_r$ can be written as the second order derivatives of the warp factor $A(r)$. Intuitively speaking, this eliminates the ghost mode in the fluctuations in the radial direction. If the equation of motion contains higher order in derivatives, it
would suggest the ghost mode. This gives a constraint on the parameters in higher derivative gravity

\[ b_1 + \frac{d+1}{4}b_2 + db_3 = 0 . \]  
(4.50)

Then one can define the monotonically decreasing holographic \( c \)-function

\[ a_d(r) \equiv \frac{\pi^{d/2}}{\Gamma(d/2)A'(r)^{d-1}}(1 - \hat{\lambda}L^2A'(r)^2) , \]  
(4.51)

where \( \hat{\lambda} = 2(2b_1 + db_2 + d(d+1)b_3) \) with the constraint (4.50). By using the higher derivative corrected Einstein equation with the condition (4.50) we obtain the higher derivative corrected holographic \( c \)-theorem:

\[ a_d'(r) = -\frac{\pi^{d/2}}{\Gamma(d/2)A'(r)^{d}}(T^t_t - T^r_r) \geq 0 , \]  
(4.52)

where we have assumed the null energy condition for \( T_{\mu\nu} \) that is obtained from the matter action. Note that the requirement of vanishing higher derivative terms in \( T^t_t - T^r_r \) in terms of the warp factor by using the modified Einstein equation gives the integrability condition on the holographic flow. Without the condition (4.50) we cannot find the good (or at least simple) monotonically decreasing function along the holographic renormalization group flow.

What was the physical origin of the constraint (4.50)? We have seen that there are two independent parameters that allow holographic \( c \)-theorem in \( O(R^2) \) gravity. We can check that these are the sum of the Gauss-Bonnet term (which is Euler density in \( d = 4 \)) and Weyl\( ^2 \) term with no independent \( R^2 \) term. The appearance of the Weyl\( ^2 \) term is accidental because the geometry for the holographic renormalization group flow is conformally flat (irrespective of the shape of \( A(r) \)), so the Weyl\( ^2 \) term cannot affect the renormalization group flow equation that we study at this point. The origin of the Gauss-Bonnet term is deeper. It does affect the equations of motion, but it does in such a way that there is no ghost mode along the renormalization group flow, and in addition, it assures the existence of the monotonically decreasing holographic \( c \)-function in any space-time dimension.

Actually, if we demand that there is no ghost mode not only along the radial direction but along the other directions in the geometry of the holographic renormalization group flow, the only allowed \( O(R^2) \) corrections to the Einstein gravity is the Gauss-Bonnet term. Although the Weyl\( ^2 \) term does not affect the holographic renormalization group flow, the fluctuation in the other directions contain a ghost. From the field theory viewpoint, unitarity of the theory is guaranteed by the absence of the ghost mode of the gravity and the null energy condition of the matter, so it seems reasonable that we have to assume the absence of the ghost mode in gravitational fluctuations to obtain the holographic \( c \)-theorem.

In \( d = 4 \), we can compute the value of \( A'(r) \) at the AdS fixed point, and read the Weyl anomaly from the holographic renormalization analysis with higher derivative corrections. Since it requires a certain amount of computational details, we only quote the result [270] \( a_d(r) \) defined in (4.51) agrees with the holographic Weyl anomaly \( a \) (that couples to Euler density) at the conformal fixed point, but not with \( c \) (that couples to Weyl\( ^2 \)) in \( d = 4 \). This is non-trivial because within the Einstein gravity, we always obtain \( a = c \) and we cannot make a distinction. We also note that there is no known way to construct the monotonically decreasing function “\( c(r) \)” (in contrast with the above holographic

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\(^{75}\)Strictly speaking, this explanation is rather superficial because the derivative here is the radial derivative and what is actually important is the time derivative. As discussed in [243][269], the condition is anyway necessary to avoid the ghost, so a posteriori, the argument here is justifiable.

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c-function \( a_d(r) \) that naturally interpolates \( c \) in the higher derivative corrected holography. This seems in complete agreement with the field theory result in which \( c \) is not monotonically decreasing but \( a \) is. In general even dimension, we can show that the monotonically decreasing holographic \( c \)-function \( a_d(r) \) is related to the Weyl anomaly that couples with the Euler density in even dimensions. It gives a supporting evidence for Cardy’s conjecture in higher dimensions. In particular, we should note that the strong version of the \( a \)-theorem is realized in holography. Furthermore, the dilaton degrees of freedom can be introduced in the holography discussion (see [272][273]) in connection with the proof of the weak \( a \)-theorem reviewed in section 3.4.

In odd dimensions, in particular in \( d = 3 \), higher derivative corrections again enable us to distinguish various proposals for the interpretation of the monotonically decreasing function \( a_d(r) \) along the holographic renormalization group flow. It turned out [269] that the higher derivative corrected \( a_d(r) \) corresponds to the \( S_d \) partition function and the entanglement entropy with sphere entangling surface. In reference [245][269] they generalize the construction including \( O(R^3) \) corrections and obtain the same conclusion. There are some restricted parameter regions in which the holographic renormalization group flow allows the monotonically decreasing holographic \( c \)-function \( a_d(r) \). The parameter regions are interpreted as the combination of Weyl terms that do not change the holographic renormalization group equations, and quasi topological terms that avoid the ghost modes in holographic renormalization group flows. Some other classes of higher derivative gravities (e.g. \( f (R) \) gravity) have been studied in [274][275]. In most of these examples, the decoupling between matter and the gravity sector is assumed, and it would be very interesting to see if we can generalize the discussion when the matter and gravity couple with each other through various curvature corrected terms because the notion of the energy-condition is very much obscured. Presumably, the unitarity of the total system must become important in the holographic realization of the generalized \( c \)-theorem.

Let us move on to our interest in the holographic equivalence between scale invariance and conformal invariance [276]. Once the holographic \( c \)-theorem is established, the argument in the last section can be naturally generalized. At the scale invariant fixed point, the metric must take the form of the AdS space-time. The higher derivative corrections do not affect the conclusion that the energy-momentum tensor \( T^i_j - T^i_r \) must vanish for scale invariance. We postulate that this occurs if and only if the matter shows a trivial field configuration (a.k.a strict null energy-condition with higher derivative corrections), then the conformal invariance follows.

Of course, as long as we use the same matter action, the requirement of the strict null energy-condition is no different than in the Einstein theory. As long as we postulate the separation of the matter and gravity, the leading order unitarity is governed by the same strict null energy condition, and we cannot relax it. The importance of the no-ghost mode is slightly indirect. The ghost mode in the radial direction would allow non-trivial (non-AdS) holographic renormalization group solution even if the matter saturates the null energy condition, which seems pathological, meaning that the effective matter metric responsible to the radial flow is singular.

### 4.4.2 Quantum violation of null energy condition

In the holographic argument above, the assumption of the null energy condition played a crucial role. It is interesting to observe, however, that the null energy condition can be violated quantum mechanically. Various sources of violation [277][278][279] include

- Casimir effect
• general squeezed quantum states
• Weyl anomaly induced energy-momentum tensor in curved background
• Hawking radiation
• Orientifold

It is an important question if these reported violation will invalidate the holographic $a$-theorem argument [280].

Since we do not know the general mechanism for the violation of the null energy condition, we focus on the universal violation from the Weyl anomaly, which should appear in any quantum field theories in curved backgrounds. For this purpose, we study the $\text{AdS}_4$ case, whose field theory dual is a three-dimensional conformal field theory[4] based on the discussion in [280]. We emphasize that the correction of the equation of motion from anomaly is of order $O(R^2)$, and it gives the same order effect as the one studied in the last section. The effect is clearly distinguishable, however, because the anomaly term cannot be absorbed by the local curvature correction terms discussed in the last section.

The Weyl anomaly induces the anomalous transformation for the energy-momentum tensor in $d=4$ under the Weyl rescaling $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ (see also [283][58][284] for a conformally flat case):

\[
\bar{T}_M^N = \Omega^{-4} T_M^N - 8c\Omega^{-4} \left((C_{BN}^A \log \Omega)^B_A + \frac{1}{2} R_B^C C_{BN}^A \log \Omega \right) - a \left(4R_B^C C_B^A + 2\bar{H}_N^M - \Omega^{-4}(4R_A^C C_B^M - 2H^M_N)\right)
\]

(4.53)

with

\[
H_{MN} = -R_M^A R_A^N + \frac{2}{3} R R_{MN} + \left(\frac{1}{2} R_{AB} R^{AB} - \frac{1}{4} R^2\right) g_{MN}.
\]

(4.54)

We have assumed $b' = 0$ in the Weyl anomaly by adding suitable counterterms $b/R^2$ because they are indistinguishable from the local higher derivative corrections studied in the previous section[7].

Let us begin with the AdS vacuum in which $T_{\mu\nu} \propto g_{\mu\nu}$. The anomalous Weyl transformation of the energy-momentum tensor generates the universal contribution to the energy-momentum tensor in the holographic renormalization group background

\[
(T^r_r - \bar{T}^t_t)_{\text{anom}} = 4a A''(r)(A'(r))^2 \leq 0,
\]

(4.55)

which by itself violate the null energy condition because $A''(r) \leq 0$ and $a \geq 0$. Accordingly, the equation of motion for the warp factor is modified as

\[
2A''(r) = (T^t_t - \bar{T}^r_r)_{\text{class}} - 4a A''(r)(A'(r))^2.
\]

(4.56)

Does this invalidate the holographic $c$-theorem? We claim that as long as the classical part of the energy-momentum tensor satisfies the null energy-condition, the holographic $c$-theorem is still intact in a slightly modified sense. To see this, we introduce the modified holographic $c$-function as

\[
a_3(r) \equiv \frac{\pi^{3/2}}{\Gamma(3/2)(A'(r))^2} - 4a \frac{\pi^{3/2}}{\Gamma(3/2)} \log A'(r),
\]

(4.57)

\[\text{A canonical example of AdS/CFT correspondence in } d = 3 \text{ is the so-called ABJM model [281], which is given by a quiver Chern-Simons gauge theory with } \mathcal{N} = 6 \text{ supersymmetry.}\]

\[\text{As discussed in the last section, we have to tune this parameter to avoid the ghost mode and derive the holographic } c\text{-theorem.}\]
and we observe it is monotonically decreasing

$$a_3'(r) = -\frac{\pi^{3/2}}{\Gamma(3/2)(\mathcal{A}'(r))^3}(T_t - T_r)_{\text{class}} \geq 0$$  \hspace{1cm} (4.58)

once we assume the null energy condition for the classical energy-momentum tensor.

The field theory interpretation of the logarithmic corrections to the holographic \(c\)-function seems very interesting. In [285][286], the localization computation showed that the \(S^3\) free-energy of the ABJM model contains \(\log N\) corrections, and the supergravity 1-loop computation also suggests its existence [287]. Since it was argued that classically the \(S^3\) free-energy is identified with the \(S^3\) free energy, the appearance of the logarithmic corrections in holographic \(c\)-function is very promising.

As for the question of scale invariance and conformal invariance, if we assume the strict null energy-condition for the classical energy-momentum tensor, the argument is still valid. Actually, for AdS geometry, which is required from the scale invariance alone, the anomalous part of the energy-momentum tensor gives only a trivial contribution to the null energy-condition, so the discussion is in complete parallel with the higher derivative case.

### 4.5 Reduced symmetry

The holography in a broader sense is applicable not only to Poincaré invariant quantum field theories, but also other non-gravitational systems with different space-time symmetries. In this section, we give a holographic dual approach to the case with reduced symmetry mentioned in section 3.8.

- First we would like to consider the holographic dual of the chiral version of the \(c\)-theorem. Such a left-right asymmetric CFT in \(d = 2\) is obtained by adding the gravitational Chern-Simons term. In [288], they computed the holographic renormalization group flow and showed the validity of the holographic \(c\)-theorem in dual chiral conformal field theories (with \(c - \bar{c}\) constant along the renormalization group flow).

- A generalization of the AdS/CFT correspondence with a boundary was proposed in [289][290][291], where they introduced the new boundary in the bulk space-time with the Neumann boundary condition (in contrast to the Dirichlet boundary condition at \(z \to 0\) limit of the AdS space-time).

In this setup, we consider the Poincaré-AdS metric

$$ds^2 = L^2 dz^2 + d\xi^2 + \eta_{ij} dx^i dx^j$$  \hspace{1cm} (4.59)

with the boundary at \(\xi = \xi(z)\). The holographic \(g\)-function is defined by

$$\log g(z) = \text{Arcsinh} \left( \frac{d\xi(z)}{dz} \right).$$  \hspace{1cm} (4.60)

When the boundary energy-momentum tensor satisfies the null energy-condition, one can prove the holographic \(g\)-theorem (indices \(a\) run through the boundary coordinate)

$$-\frac{d \log g(z)}{d \log z} \sim T_{ab} k^a k^b \geq 0.$$  \hspace{1cm} (4.61)

As in the bulk case, the further assumption of the strict null energy condition demands that when the boundary \(g\)-function takes a constant value due to the scale invariance, there is no non-trivial field configurations at the boundary, and it results in the boundary conformal invariance [184]. If

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we assume the gauged non-linear sigma model as our boundary matter fields, then the argument is in complete parallel with the field theory argument. In particular, (4.61) is understood as
\[ \frac{d \log g(\mu)}{d \log \mu} = B^a \chi_{ab} B^b, \] (4.62)
which we expect to hold in the boundary quantum field theories in general. The stronger gradient formula was derived in $1+1$ dimensional setup in [186] from the field theory argument. Presumably, the introduction of the boundary “superpotential” will give the holographic gradient formula.

- In section 3.8, we have discussed the enhancement of chiral scale invariance to chiral conformal invariance. A similar argument applies [213]. If we assume time and space translations and chiral scale invariance, the most generic metric is given by the warped AdS$_3$ space-time:
\[ ds^2 = \frac{1}{c} \left( \frac{b dt}{z} - c dx \right)^2 + e(\frac{dz}{z})^2 - (a + \frac{b^2}{z}) dt^2 \] (4.63)
up to coordinate transformation, which has the enhanced $SL(2) \times U(1)$ isometry. This corresponds to the enhanced chiral conformal invariance.

The matter contributions may break the chiral conformal invariance. For instance is it possible to introduce a vector condensation
\[ A = \alpha \frac{dz}{z} + \beta dx + \gamma \frac{dt}{z} \] (4.64)
but again the (strict) null energy condition will forbid such a non-trivial field configuration unless it is consistent with the $SL(2) \times U(1)$ isometry (up to the field redefinition discussed in [213]).

The chiral conformal invariance in the warped AdS$_3$ space-time has been studied in the literature (see e.g. [293][294]), in which the emergence of the chiral Virasoro symmetry is also discussed.

- Non-relativistic systems show various interesting scaling or conformal symmetries such as Schrödinger symmetry [293][296], Lifshitz symmetry [297], Galilean conformal symmetry [298] and so on. Correspondingly, gravitational dual descriptions with these symmetries have been investigated.

For our interest in relation between scale invariance and conformal invariance, let us take a look at the Schrödinger holography. The $d$ dimensional Schrödinger algebra can be realized as an isometry of $d+2$ dimensional space-time with one null direction $\zeta$ compactified.
\[ ds^2 = -2 \frac{dt^2}{z^4} + \frac{-2 dt d\zeta + dx_i^2 + dz^2}{z^2} . \] (4.65)

The non-relativistic special conformal transformation is realized by the isometry
\[ (t, \zeta, x_i, z) \rightarrow \left( \frac{t}{1 + \eta t}, \zeta - \frac{\eta x_i^2 + z^2}{2 (1 + \eta t)}, \frac{x_i}{1 + \eta t}, \frac{z}{1 + \eta t} \right) . \] (4.66)

\[ ^{78} \text{A similar argument was presented by D. Honda and M. Nakamura [292].} \]

\[ ^{79} \text{The strict null energy condition in this context is actually stronger than the requirement in our non-chiral discussions in the other sections. Although we do not know any physical counterexamples, it is interesting to see why this must be satisfied. Presumably, it is related to an extra assumption of the validity of the Reeh-Schlieder theorem in the field theory argument.} \]
It can be shown that if we try to deform the geometry so that we preserve the scale invariance and Galilei invariance acting as \((\zeta, x_i) \rightarrow (\zeta - v_ix_i + \frac{1}{2}v_i^2t, x_i - v_it)\), then there is no such a geometric deformation that breaks non-relativistic special conformal invariance.

In order to support the Schrödinger geometry, which is not Einstein, we have to introduce the matter. A typical example is a massive vector field. From scale invariance and Galilean invariance, the matter vector field takes the form

\[
A = \alpha \frac{dt}{z^2} + \gamma \frac{dz}{z}.
\]

(4.67)

The first term is compatible with the non-relativistic conformal invariance and needed for the geometry. The second term, however, is not invariant under the non-relativistic conformal invariance, and may lead to the scale invariant but non conformal field configuration. It may be possible to forbid such a configuration by introducing a certain stronger notion of the strict null energy-condition, but the validity is yet to be addressed.

### 4.6 Final Project

Let us finish the lectures with the final project, which I challenged you at the very beginning of the introduction. Do you remember?

(Projec) Prove the \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory is superconformal invariant.

First, we try to answer the question from the field theory perspective.

1. We begin with the simplest case: show that the \(U(1)\) \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory is superconformal invariant. The Abelian theory is free, so it must be easy. What is the role of the improvement terms?

2. Let us move on the non-Abelian case. Our first task is to show that beta function vanishes for the \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory. The rough argument goes like this. Use the holomorphic scheme and show that the holomorphic gauge coupling constant is not renormalized. Then consider the wavefunction renormalization of the adjoint “matter” superfields. Use the \(\mathcal{N} = 4\) supersymmetry to connect it to the gauge coupling constant and show it is not renormalized either. Then the physical beta function must vanish. See e.g. [299] for a more complete argument.

3. But we know that vanishing of the beta function is not enough to declare the (super)conformal invariance. Use the superfield structure and show that there is no vector beta functions. We can find the discussions in [300].

4. Combining them together, we learn that the \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory has vanishing \(B\) function, and it is superconformal.

Alternatively, we could begin with the holographic dual.

1. Show that the type IIB supergravity on \(\text{AdS}_5 \times \mathbb{S}_5\) has the symmetry that corresponds to the \(\mathcal{N} = 4\) superconformal algebra. In this lecture, we discussed the bosonic part of the statement. What is the role of the improvement terms?
2. Make an argument that the symmetry is not anomalous in the perturbative string theory on $\text{AdS}_5 \times S^5$. Maybe we had better use Berkovits formulation [301].

3. Prove the AdS/CFT correspondence.

If you are a firm believer of the holography, you may skip the last part of the holographic argument.

4.7 Literature guides

A precursor of the AdS/CFT correspondence [302] is a discovery of the fact that the AdS space-time has the so-called asymptotic symmetry group isomorphic to the conformal group. In particular, this asymptotic symmetry group is how the Virasoro symmetry is realized in $\text{AdS}_3$ space-time, in which the geometry itself does not possess the infinite dimensional isometry. In this lecture, we did not discuss the realization of scale invariance or conformal invariance as an asymptotic symmetry group. We have only focused on the realization by the isometry. It is an interesting question if such a realization would give a new perspective on the subject.

With this regard, the Kerr-CFT [303] (see e.g. [304] for a review) is one concrete example of realizing the (chiral) conformal algebra not as an isometry of the system, but as an asymptotic symmetry group. Since the asymptotic symmetry group does not specify whether the theory is in the vacuum state, the Kerr-CFT has its own temperature. Most of our discussion in this lecture is done in the vacuum state, and it would be interesting to see if we can generalize the argument with non-zero temperature.

In [305], they proposed an interesting attempt to derive (a special class of) AdS/CFT from a free field theory. They showed that the (singlet) correlation functions of the free scalar field theory can be computed from the restricted sector of the higher spin gauge theory. They further showed that the vacuum solution of the higher spin gauge theory is the AdS space-time. This begs the question what will happen if we consider the free Maxwell theory (rather than the scalar theory) which is not conformal invariant (unless $d = 4$). A similar construction was done in [169], and it was shown that the theory seems only invariant under the foliation preserving diffeomorphism, but the situation in $d = 4$ must be clarified.

In the very recent paper, a possible way to circumvent the holographic $c$-theorem and and a possibility to construct the holographic geometry that shows the cyclic behavior (with manifest Poincaré invariance) was proposed in [306]. It was argued that the higher dimensional null energy condition does not necessarily lead to an immediate inconsistency with the holographic $c$-theorem when the warped compactification does not allow the effective truncation to the lower $d+1$ dimension. It would be very interesting to understand the physics of the dual field theory, if any.

Finally let us point out that there are a couple of interesting but slightly different geometric constructions of the holographic dual for scale invariant but non-conformal field theories. In [307], they discussed the violation of the conformal invariance by putting the boundary theory on a curved background while preserving the scale invariance. In [308], they discussed the violation of the conformal invariance by putting the boundary theory on a non-anti-commutative background while preserving the scale invariance.
5 Conclusions

So our lectures are almost finished, but our journey still continues. In this lecture note, we have shown as many examples of scale invariant but possibly non-conformal field theories. We have tried to argue such examples are extremely rare and most probably inconsistent with some important assumptions in quantum field theories.

We hope that in the near future, the equivalence between scale invariance and conformal invariance is proved (or disproved), and the necessary condition for the claim is stated in a clear manner. On the other hand, the implication of the equivalence in holography would be very helpful to understand the detailed consistency conditions for quantum gravity.

I always feel that there is a deep space-time structure behind it. Even though our belief in this correspondence is based on the intuition of “coarse graining” in renormalization group flow, whenever we try to make the statement concrete, we had to assume the notion of “time” like unitarity, causality, energy-condition and so on, all of which are not always available in the Euclidean statistical systems. Probably there is a magic in Wick rotation and the renormalization group flow. With this regard, I always find the first chapter of the textbook by Polyakov [76] mesmerizing.

As Einstein once said, “Subtle is the Lord, but malicious He is not”. His own explanation of the meaning is “Nature hides her secret because of her essential loftiness, but not by means of ruse.” Indeed, a beautiful symmetry may be secretly hidden unless we try hard to understand it as our conformal invariance. We need to choose a good probe (e.g. energy-momentum-tensor) and respect it very carefully (in the renormalization prescription).

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure4.png}
\caption{(A) My family name is written alphabet. (B) Same but in Japanese (or Chinese) characters. (C) This is the traditional form which Sun Yat-sen must have encountered in Tokyo.}
\end{figure}

Let me offer one analogy to finish the lecture note. My family name “Nakayama” has a hidden symmetry. See fig [4] You cannot see it in alphabet. To uncover the hidden symmetry, you need to pay respect, and use a proper probe. In this case, you have to look at their Japanese or Chinese
characters (which are the same in this case). Gradually you will see the symmetry pattern, but you need one more step. You may be accustomed to writing the characters from left to right (and then top to bottom), but in traditional Japanese or Chinese, you write them from top to bottom (and right to left). Now you understand that there is a hidden axisymmetry in my family name. It literally means the middle mountain.

There is a further story to it. A great leader of Taiwan, Sun Yat-sen (1866-1925) once visited Japan. He lived close to Hibiya-Park in Tokyo. Near the park, there was a mansion whose family name tag was “Nakayama” (I don’t think they are my relatives). He immediately liked the symmetry of the name very much (of course Japanese name tag is written from top to bottom), and he decided to call himself “Zhongshan”, which is the Chinese way to read the Japanese characters for “Nakayama”. Thanks to this great leader, my family name has become very popular in Taiwan. Unfortunately, since Japanese and Chinese read the same characters in a very different way, without writing down in characters, they do not recognize they are the same. That is why I wrote my name down in Chinese characters when I gave a talk in Taiwan. The symmetry is only shared after using the proper communication tool. Anyway, it was my greatest pleasure to give these lectures in Taiwan. I wish the participants had learned something from them.

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80I would like to thank H. Nakajima for informing me of the history after my lecture.
A Useful formulae and miscellaneous topics

A.1 Weyl transformation

In this appendix, Weyl transformation properties of various tensors will be shown. The sign convention in the lecture is the same as that of Wald [12], which is $s_1 = s_2 = s_3$ in the Misner-Thorne-Wheeler convention [309]. Note, however, that the action density used in this lecture note is minus of the Lagrangian density in the Lorentzian signature.

\[ \begin{align*}
R_{\mu\nu\rho\sigma} &= \partial_\rho \Gamma_{\mu\nu}^\lambda + \Gamma_{\tau\nu}^\lambda \Gamma_{\mu\sigma}^\tau - \partial_\mu \Gamma_{\nu\sigma}^\lambda - \Gamma_{\tau\mu}^\lambda \Gamma_{\nu\sigma}^\tau \\
R_{\mu\nu} &= R_{\mu\lambda\nu}^\lambda \\
R &= g^{\mu\nu} R_{\mu\nu} \\
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} .
\end{align*} \] (A.1)

Under the finite Weyl transformation

\[ g_{\mu\nu} \rightarrow e^{2\sigma(x)} g_{\mu\nu} \] (A.2)

they transform as (see e.g. appendix of [12])

\[ \begin{align*}
R_{\mu\nu\rho\sigma}^\lambda &\rightarrow R_{\mu\nu\rho\sigma}^\lambda + \delta_\rho^\lambda (\sigma_{\mu\nu} - \sigma_{\mu\sigma} + g_{\mu\nu} \sigma_{\rho\sigma}) + g_{\mu\nu}(\sigma^\lambda_{\sigma\sigma} - \sigma^{\lambda\sigma}_{\sigma\rho} - \sigma^{\lambda\rho}_{\rho\sigma}) - (\nu \leftrightarrow \sigma) \\
R_{\mu\nu} &\rightarrow R_{\mu\nu} - g_{\mu\rho} \Box \sigma - (d-2)(\sigma_{\mu\nu} - \sigma_{\mu\sigma} + g_{\mu\nu} \sigma_{\lambda\sigma}) \\
R &\rightarrow e^{-2\sigma}(R - 2(d-1)\Box \sigma - (d-1)(d-2)\sigma_{\mu\sigma} \sigma^{\mu\sigma}) .
\end{align*} \] (A.3)

The traceless part of the Riemann-tensor is known as Weyl tensor

\[ C_{\mu\nu\rho\sigma}^\lambda = R_{\mu\nu\rho\sigma}^\lambda - \frac{1}{d-2}(\delta_\rho^\lambda R_{\mu\sigma} + g_{\mu\rho} R_{\nu\sigma}^\lambda - (\nu \leftrightarrow \sigma)) - \frac{1}{(d-2)(d-1)}(\delta_\rho^\lambda g_{\mu\sigma} - \delta_\nu^\lambda g_{\mu\sigma}) R , \] (A.4)

and it is invariant under the Weyl transformation

\[ C_{\mu\nu\rho\sigma}^\lambda \rightarrow C_{\mu\nu\rho\sigma}^\lambda . \] (A.5)

In $d = 4$, the Weyl transformation of the Euler term is given by

\[ \sqrt{g} E \rightarrow \sqrt{g} E + 4\sqrt{g} D^\mu \left( -R \sigma_{\mu\nu} + 2R_{\mu\nu} \sigma^{\mu\nu} - D_{\mu}(\sigma_{\nu\sigma}) + 2\sigma_{\mu\rho} \Box \sigma + 2\sigma_{\nu\rho} \sigma^{\mu\rho} \sigma_{\mu\nu} \right) . \] (A.6)

Note that the inhomogeneous term is a total derivative.

A.2 Energy-momentum tensor correlation functions

In this appendix, we will show the correlation functions of the energy-momentum tensor in conformal filed theories to read $\alpha$ and $\beta$ anomaly. We implicitly assume that the theory does not violate CP, so the following formulae contain no CP-violating term with Levi-Civita tensor. We also ignore the contact terms.

\[^8\text{Note that our Weyl transformation is minus that of [59].}\]
In two-dimensional conformal field theories, the two-point function and three-point function of the energy-momentum tensor are governed by the conformal invariance up to one-number, which is the central charge $c$:

$$\langle T(z)T(w) \rangle = \frac{1}{(2\pi)^2} \frac{c}{2(z-w)^4}$$

$$\langle T(z_1)T(z_2)T(z_3) \rangle = -\frac{1}{(2\pi)^3} \frac{c}{(z_1-z_2)^2 (z_2-z_3)^2 (z_3-z_1)^2} , \quad \text{(A.7)}$$

where we use the holomorphic coordinate.

In higher dimensions, the two-point function is again uniquely specified up to an overall number

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(0) \rangle = \frac{c_T}{x^{2d}} \mathcal{I}_{\mu\nu,\sigma\rho}(x) , \quad \text{(A.8)}$$

where

$$\mathcal{I}_{\mu\nu,\sigma\rho}(x) = \frac{1}{2} (I_{\mu\sigma}(x)I_{\nu\rho}(x) + I_{\mu\rho}(x)I_{\nu\sigma}(x)) - \frac{1}{d} \delta_{\mu\nu}\delta_{\sigma\rho} \quad \text{(A.9)}$$

with

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - 2 \frac{x_{\mu}x_{\nu}}{x^2} . \quad \text{(A.10)}$$

In $d = 4$ the coefficient of the two-point function $c_T$ is given by the Weyl anomaly $c$ as

$$c_T = \frac{640}{\pi^2} c . \quad \text{(A.11)}$$

The three-point function is sufficiently complicated. It was shown in \[37\] (see also \[163\]) that it is given by

$$\langle T_{\mu\nu}(x_1)T_{\sigma\rho}(x_2)T_{\alpha\beta}(x_3) \rangle = \frac{1}{x_{12} x_{13} x_{23}} \Gamma_{\mu\nu,\sigma\rho,\alpha\beta}(x_1, x_2, x_3) \quad \text{(A.12)}$$

with

$$\Gamma_{\mu\nu,\sigma\rho,\alpha\beta}(x_1, x_2, x_3) = \mathcal{E}_{\mu\nu,\rho'\sigma'} \mathcal{E}_{\sigma'\rho,\alpha'\beta'} \mathcal{E}_{\alpha'\beta,\alpha\beta'}$$

$$\left[ AI_{\nu'\sigma'}(x_{12})I_{\rho'\mu'}(x_{23})I_{\beta'\mu'}(x_{31}) + BI_{\mu'\sigma'}(x_{12})I_{\nu'\alpha'}(x_{23})X_{\rho'} X_{\beta'}(x_2-x_3)^2 + \text{perm} \right]$$

$$+ CT_{\mu\nu,\sigma\rho}(x_{12}) \left( \frac{X_{\alpha'} X_{\beta'}}{(X^3)^2} - \frac{1}{d} \delta_{\alpha\beta} \right) + \text{perm}$$

$$+ D \mathcal{E}_{\mu\nu,\rho'\sigma'} \mathcal{E}_{\sigma'\rho,\alpha'\beta'} X_{\alpha'\beta'}(x_1-x_2)^2 X_{\rho'}(x_{12}) \left( \frac{X_{\alpha'} X_{\beta'}}{(X^3)^2} - \frac{1}{d} \delta_{\alpha\beta} \right) + \text{perm}$$

$$+ E \left( \frac{X_{\mu'} X_{\nu'}}{(X^3)^2} - \frac{1}{d} \delta_{\mu\nu} \right) \left( \frac{X_{\sigma'} X_{\rho'}}{(X^3)^2} - \frac{1}{d} \delta_{\sigma\rho} \right) \left( \frac{X_{\alpha'} X_{\beta'}}{(X^3)^2} - \frac{1}{d} \delta_{\alpha\beta} \right) \quad \text{(A.13)}$$

where

$$\mathcal{E}_{\mu\nu,\sigma\rho} = \frac{1}{2} (\delta_{\mu\sigma} \delta_{\nu\rho} + \delta_{\mu\rho} \delta_{\nu\sigma}) - \frac{1}{d} \delta_{\mu\nu} \delta_{\sigma\rho} \quad \text{(A.14)}$$

\[82\]Our normalization is different from the string theory literature.
is the projection operator onto symmetric traceless tensors. We have also introduced \((i = 1, 2, 3 \text{ mod } 3)\)

\[
X^i_\mu = \frac{(x_{i+1} - x_i)_\mu}{(x_{i+1} - x_i)^2} - \frac{(x_{i+2} - x_i)_\mu}{(x_{i+2} - x_i)^2}.
\] (A.15)

The conservation of the energy-momentum tensor demands

\[
(d^2 - 4)A + (d + 2)B - 4dC - 2D = 0
\]
\[
(d - 2)(d + 4)B - 2d(d + 2)C + 8D - 4E = 0.
\] (A.16)

There are three free parameters left, two of which are related to \(a\) and \(c\) in \(d = 4\). The relation has been worked out in [37][165] as

\[
c = \frac{\pi^4}{640 \times 12} (9A - B - 10C)
\]
\[
a = \frac{\pi^4}{512 \times 90} (13A - 2B - 40C).
\] (A.17)

In terms of free fields (in \(d = 4\)), we have

\[
A = \frac{1}{\pi^6} \left( \frac{8}{27} N_0 - 16N_1 \right)
\]
\[
B = -\frac{1}{\pi^6} \left( \frac{16}{27} N_0 + 4N_{1/2} + 32N_1 \right)
\]
\[
C = -\frac{1}{\pi^6} \left( \frac{2}{27} N_0 + 2N_{1/2} + 16N_1 \right).
\] (A.18)

### A.3 Local Wess-Zumino condition

In this appendix, we will summarize the Wess-Zumino consistency condition of the local renormalization group. We follow the convention of [254] rather than that in [59]. There are a couple of sign difference there.

We begin with the most generic candidates for the Weyl anomaly that depends on the metric and space-time dependent coupling constants in \(d = 4\):

\[-\delta_\sigma S_{\text{eff}} = \int d^4x \sqrt{|g|} \left( \sigma T + \partial^\mu \sigma Z_\mu \right)\] (A.19)

where

\[
T = c_{\text{Weyl}}^2 - a_{\text{Euler}} + \frac{1}{9} b R^2
\]
\[
+ \frac{1}{3} \chi_i^g D_\mu g^g D^\mu \partial R + \frac{1}{6} \chi^f_{ij} D_\mu g^i D^\mu g^j R - \frac{1}{2} \chi^g_{ij} D_\mu g^i D_\nu g^j G^{\mu \nu} + \frac{1}{2} \chi^g_{ij} D^2 g^i D^2 g^j
\]
\[
+ \frac{1}{2} \chi^{b}_{ijk} D_\mu g^i D^\mu g^j D^2 g^K + \frac{1}{4} \chi^{c}_{ijkl} D_\mu g^i D_\nu g^j D_\sigma g^K D^{\mu \nu} g^b
\]
\[
+ \frac{1}{4} F_{\mu \nu} \zeta_{ij} D_\mu g^i D_\nu g^j.
\] (A.20)

and

\[
Z_\mu = -G_{\mu \nu} w_I D^{\nu} g^I + \frac{1}{3} \partial_\mu (q R) + \frac{1}{3} R Y_I D_\mu g^I + F_{\mu \nu} \eta_I D^{\nu} g^I
\]
\[
+ \partial_\mu (U_I D^2 g^I + \frac{1}{2} V_{ij} D_\mu g^i D^\mu g^j) + S_{ij} D_\mu g^i D^2 g^j + \frac{1}{2} T_{ij} D_\nu g^{i} D^{\nu} g^{j} D_\mu g^K.
\] (A.21)
c, a, b, \chi^c_I, \chi^b_I, \chi^a_I, \chi^b_{IK}, \chi^a_{IKL}, w_I, q, Y_I, U_I, V_{IJ}, S_{IJ} and T_{IJK} are gauge invariant tensors on the coupling constant space g^I. \kappa, \zeta_{IJ} and \eta_I are tensors that take values on the Lie algebra of the “flavor” symmetries. As discussed in the main text, we “flavor” symmetries act on this case, we can introduce various local counterterms given by 11 terms as in \[11\] terms as in \[59\], we do not consider the CP violating terms as well as anomaly for the “flavor” symmetries.

(Exercise) Generalize the following computation by including various gravitational as well as “flavor” anomalies.

The Wess-Zumino consistency condition

\[ \delta_{\sigma(x)} = - \int d^4x \sqrt{|g|} \delta \sigma(x) \left( 2g^{\mu \nu} \frac{\delta}{\delta g^{\mu \nu}} - B^I J^I - P_I D_\mu g^I J^I A_\mu^a \right) \]

with

\[ \delta_{\sigma(x)} = \frac{8}{\sqrt{g}} \partial_I a - \chi^a_{IJ} B^I = - \mathcal{L}_B w_I \]

\[ 2\chi^a_I + \chi^a_{IJ} B^J = - \mathcal{L}_B U_I \]

\[ 8b - \chi^b_{IJ} B^J B^J = - \mathcal{L}_B (2q + U_I B^I) \]

\[ - \chi^b_{IJ} + 2\chi^b_{IJ} + \Lambda_{IJ} = \mathcal{L}_B S_{IJ} \]

\[ 2(\chi^b_{IJ} + \chi^a_{IJ} + \Lambda_{IJ} + B^K (2\chi^b_{K(I,J)} - \bar{\chi}^b_{IKJ}) = \mathcal{L}_B ( \chi_{IJ} - \chi^a_{IJ} - 2U_{(IJ)} + V_{IJ} ) \]

\[ \chi^b_{IJK} - \chi^b_{K(I,J)} = \frac{1}{2} \chi^a_{IJK} - D_K B^L \chi^a_{IJK} + \chi^c_{IJK} B^L = \frac{1}{2} \mathcal{L}_B T_{IJK} + D_I D_J B^L S_{KL} \]

\[ P_I B^I = 0 \]

\[ \eta_I B^I = 0 \]

\[ \kappa P_I + \zeta_{IJ} B^J = \mathcal{L}_B \eta_I + g^I P_I \eta_I \]

where

\[ \Lambda_{IJ} = 2D_I B^K \chi^a_{KJ} + B^K \chi^b_{IJ} \]

\[ \bar{\chi}^a_{IKJ} = \chi^a_{IKJ} - \chi^b_{IKJ} \]

and the modified Lie derivative is defined as

\[ \mathcal{L}_B t_I = B^I \partial_I t_I + t_I (\partial_I B^J - (P_I g)^J) \]

for a 1-form and similarly for other tensors.

As discussed in the main text, anomaly is defined up to the addition of the local counterterms. In this case, we can introduce various local counterterms given by 11 terms as in \[T\] of \[\text{(A.20)}\].

\[ S_{ct} = - \int d^4x \sqrt{|g|} \left( C_{\text{Weyl}}^2 - AEuler + \frac{1}{9} B R^2 \right. \]

\[ + \frac{1}{3} C_i^c D_\mu g^I \partial^\mu R + \frac{1}{6} C_i^f D_\mu g^I D_\nu g^j g^I D_\nu g^I + \frac{1}{2} C_i^{a I} D^2 g^I D^2 g^I \]

\[ + \frac{1}{2} C_i^{b IJK} D_\mu g^I D_\mu g^I D_\nu g^K + \frac{1}{4} C_i^{c IJKL} D_\mu g^I D_\nu g^j D_\nu g^K D_\nu g^L \]

\[ + \frac{1}{4} F_{\mu \nu} K F^\mu \nu + \frac{1}{2} F^\mu \nu Z_{IJ} D_\mu g^I D_\nu g^J \right) \]

\[ (A.27) \]
Here $C, A, \tilde{B}, C^I, C^I_{i,j}, C^g_{i,j}, C^u_{i,j}, C^b_{i,j,k}$ and $C^e_{i,j,k,l}$ are gauge invariant tensors of coupling constant spaces $g^I$ and $K$ and $Z_{i,j}$ are tensors that take values on the Lie algebra of the “flavor” symmetries. The induced local contributions to the Weyl anomaly are

$$\delta(c, a, \tilde{b}, \chi^c_{i,j}, \chi^f_{i,j}, \chi^g_{i,j}, \chi^a_{i,j}) = \mathcal{L}_B(C, A, \tilde{B}, C^I, C^g_{i,j}, C^u_{i,j})$$

$$\delta \chi^b_{i,j,k,l} = \mathcal{L}_B C^g_{i,j,k,l} + 2D_I D_J B^{i} C^b_{i,j,k,l}$$

$$\delta \chi^c_{i,j,k,l} = \mathcal{L}_B C^g_{i,j,k,l} + D_I D_J B^{i} C^b_{i,j,k,l} + D_K D_L B^{i} C^b_{i,j,k,l}$$

$$\delta \chi^a_{i,j,k,l} = \mathcal{L}_B C^g_{i,j,k,l} + 2D_I D_J B^{i} C^c_{i,j,k,l} + 2D_K D_L B^{i} C^c_{i,j,k,l}.$$  \(\text{(A.28)}\)

The above expression is directly taken from Osborn’s paper \cite{59}. Since he did not discuss the $K$ and $Z_{i,j}$ term, it does not contain the effect of $K$ and $Z_{i,j}$ counterterms. If we introduced these counterterms we would schematically get

$$\delta \zeta_{i,j} \sim D[I P] + \mathcal{L}_B Z_{i,j}$$

$$\delta \eta_{i,j} \sim -P_I K + B^j Z_{i,j}$$

$$\delta K \sim \mathcal{L}_B K.$$  \(\text{(A.29)}\)

and so on. Presumably, other terms like $T_{i,j,k}$ are modified. An interested reader may complete the transformation rule.

As mentioned in \cite{54}, one can use the freedom to make $q, Y_I$ or $U_I$, $V_{i,j}$, $S_{i,j}$, and $T_{i,j,k}$ vanish. The remaining ambiguity for $\delta a$ and $\delta w_I$ are used in the dressing transformation of the gradient formula in section \cite{60}.

### A.4 Analytic properties of S-matrix

We need some elementary facts about analytic properties of S-matrix when we use the dilaton scattering amplitudes to derive constraints on the renormalization group flow. We briefly summarize them here. One cautious remark is that a formal textbook derivation of the following formulae on the S-matrix assume a mass gap in the spectrum, and strictly speaking, we need a careful treatment for massless theories like a dilaton coupled with the deformed conformal field theories.\cite{83}

We are interested in two-two scattering of the identical massless particles with the initial momenta $(p^1_1, p^2_1)$ to the final ones $(p^3_2, p^4_2)$. Let us introduce the conventional Mandelstam variables

$$s = -(p_1 + p_2)^2$$

$$t = -(p_1 - p_3)^2$$

$$u = -(p_1 - p_4)^2.$$  \(\text{(A.30)}\)

\(^{83}\)To some extent, the problem is alleviated since we do not consider the internal loop of massless dilatons.
They are not independent because \( s + t + u = 0 \) from the conservation of the energy-momentum. The scattering amplitude is a function of the two of them, e.g. \( A(s,t) \). We see that the forward scattering corresponds to \( t = 0 \) (or \( u = 0 \)). The scattering amplitude \( A(s,t) \) is related to the S-matrix as

\[
(f|S|i) = \delta_{f_i} + i(2\pi)^4 \delta^{(4)}(p_i - p_f) \langle f|T|i \rangle ,
\]

and we identify \( A(s,t) \) with \( \langle f|T|i \rangle \) for two-two scattering.

We recall that the S-matrix is unitary: \( SS^\dagger = S^\dagger S = 1 \), therefore the T-matrix satisfies

\[
2\text{Im} \langle i|T|i \rangle = \sum_f (2\pi)^4 \delta^{(4)}(p_i - p_f)|\langle f|T|i \rangle|^2 .
\]

Now, in our two-two scattering, Fermi’s golden rule tells that the right hand side of (A.32) is proportional to the total cross section of the two-dilaton initial states (because we summed over the final states), while the left hand side of (A.32) is the imaginary part of the forward scattering amplitude of two dilatons (because initial state and final state are identical). Thus we obtain the special case of the optical theorem:

\[
\text{Im} A(s,t = 0) = s\sigma(s) .
\]

Scattering amplitudes have some important analytic structures. In particular, the diagrammatic computations do not distinguish the exchange of the initial state and final state (up on replacing particles with anti-particles). The \( u \)-channel exchange gives

\[
A_{a+d \to b+c}(u,t,s) = A_{a+b \to c+d}(s,t,u)
\]

In the forward two-two dilaton scattering we are interested in, this gives the crossing symmetry relation \( A(s) = A(u) = A(-s) \) since \( t = 0 \) and \( u = -s \).

The two-two dilaton scattering amplitudes have a cut along the real \( s \) axis. This is due to the massless multi-particle intermediate states. If the intermediate channels were massive, \( A(s) \) would be real near \( s = 0 \) on the real \( s \) axis, and the analytic continuation of \( s \) in complex plane should satisfy the Schwarz reflection principle

\[
A(s^*) = A(s)^* .
\]

Although our dilaton scattering may have a bad IR behavior, we postulate (A.35) holds. Then, for real \( s \) we obtain

\[
A(-s + i\epsilon) = A(s - i\epsilon) = [A(s + i\epsilon)]^* ,
\]

which is the basis of the first equality (3.57) in the main text.
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