ON THE LS-CATEGORY OF HOMOMORPHISMS

ALEXANDER DRANISHNIKOV AND NURSULTAN KUANYSHOV

Abstract. We prove the equality \( \text{cat}(\phi) = \text{cd}(\phi) \) for homomorphisms \( \phi : \Gamma \to \Lambda \) of a torsion free finitely generated nilpotent groups \( \Gamma \) to an arbitrary group \( \Lambda \). We construct an epimorphism \( \psi : G \to H \) between geometrically finite groups with \( \text{cat}(\psi) > \text{cd}(\psi) \).

1. Introduction

Definition 1.1. The (reduced) Lusternik-Schnirelmann category (LS-category), \( \text{cat} \) of an ANR space \( X \) is the minimal number \( k \) such that \( X \) admits an open cover by \( k + 1 \) sets \( U_0, U_1, \ldots, U_k \) such that each \( U_i \) is contractible in \( X \).

The Lusternik-Schnirelmann category is an important invariant, since it gives a lower bound on the number of critical points for a smooth real-valued function on a closed manifold [LS]. Since it is a homotopy invariant, it can be defined for discrete groups \( \Gamma \) as \( \text{cat} \Gamma = \text{cat} B\Gamma \) where \( B\Gamma = K(\Gamma, 1) \) is a classifying space. Eilenberg and Ganea [EG] proved that the LS-category of a discrete group equals its cohomological dimension, \( \text{cat}(\pi) = \text{cd}(\pi) \).

We recall that the cohomological dimension of a group \( \Gamma \) is defined as follows,

\[
\text{cd}(\Gamma) = \max \{ k \mid H^k(\Gamma, M) \neq 0 \}
\]

where the maximum is taken over all \( \mathbb{Z}\Gamma \)-modules \( M \).

Theorem 1.2 [Sch, DR]. For the cohomological dimension of a discrete group \( \Gamma \),

\[
\text{cd}(\Gamma) = \max \{ k \mid (\beta_{\Gamma})^k \neq 0 \}
\]

where \( \beta_{\Gamma} \in H^1(\Gamma, I(\Gamma)) \) is the Berstein-Schwarz class of \( \Gamma \).

The LS-category of the map \( f : X \to Y \), \( \text{cat} f \), is the minimal number \( k \) such that \( X \) admits an open cover by \( k + 1 \) open sets \( U_0, U_2, \ldots, U_k \) with nullhomotopic restrictions \( f|_{U_i} : U_i \to Y \) for all \( i \). The LS-category \( \text{cat} \phi \) of a group homomorphism \( \phi : \Gamma \to \pi \) is defined as \( \text{cat} f \) where the map \( f : B\Gamma \to B\pi \) induces the homomorphism \( \phi \) for the fundamental groups.

The cohomological dimension \( \text{cd}(\phi) \) of a group homomorphism \( \phi : \Gamma \to \pi \) was defined by Mark Grant about 10 years ago on Mathoverflow [Gr] as maximum of \( k \) such that there is a \( \pi \)-module \( M \) with the nonzero induced homomorphism \( \phi^* : H^k(\pi, M) \to H^k(\Gamma, M) \). In view of universality of the Berstein-Schwarz class [DR] for any homomorphism \( \phi : \Gamma \to \pi \)

\[
\text{cd}(\phi) = \max \{ k \mid \phi^*(\beta_{\pi})^k \neq 0 \}.
\]

This brings the inequality \( \text{cd}(\phi) \leq \text{cat} \phi \) for all homomorphisms.

In view of the equality \( \text{cd}(\Gamma) = \text{cat} \Gamma \), the following conjecture seems to be natural:

2000 Mathematics Subject Classification. Primary 55M30; Secondary 55M25, 57R65, 57R67.

Key words and phrases. Lusternik-Schnirelmann category, group homomorphism.
Conjecture 1.3. For any group homomorphism \( \phi : \Gamma \to \pi \) always
\[
\text{cat } \phi = \text{cd}(\phi).
\]

In [Sc] Jamie Scott considered this conjecture for geometrically finite groups and he proved it for monomorphisms of any groups and for homomorphisms of free and free abelian groups.

In [Gr] Tom Goodwillie gave an example of an epimorphism of an infinitely generated group \( \phi : G \to \mathbb{Z}^2 \) with \( \text{cd}(\phi) = 1 \) that disproves the conjecture.

In the first part of this paper we prove Conjecture 1.3 for finitely generated torsion free nilpotent groups. In the second part we present a finitely generated counterexample by constructing a map between aspherical manifolds \( f : M \to N \) with
\[
\text{cat } f > \text{cd}(f_\# : \pi_1(M) \to \pi_1(N)).
\]

2. Preliminaries

2.1. Nilpotent groups. The upper central series of a group \( \Gamma \) is a chain of subgroups
\[
e = Z_0 \leq Z_1 \leq \ldots \leq Z_n \leq \ldots
\]
where \( Z_1 = Z(\Gamma) \) is the center of the group, and \( Z_{i+1} \) is the preimage under the canonical epimorphism \( \Gamma \to \Gamma/Z_i \) of the center of \( \Gamma/Z_i \). A group \( \Gamma \) is nilpotent if \( Z_n = \Gamma \) for some \( n \).

The least such \( n \) is called the nilpotency class of \( \Gamma \), denoted \( \text{nil}(\Gamma) \). Note that the groups with the nilpotency class one are exactly abelian groups.

The lower central series of a group \( \Gamma \) is a chain of subgroups
\[
\Gamma = \gamma_0(\Gamma) \geq \gamma_1(\Gamma) \geq \gamma_2(\Gamma) \geq \ldots
\]
defined as \( \gamma_{i+1}(\Gamma) = [\gamma_i(\Gamma), \Gamma] \). It’s known that for nilpotent groups \( \Gamma \) the nilpotency class \( \text{nil}(\Gamma) \) equals the least \( n \) for which \( \gamma_n(\Gamma) = 1 \).

Proposition 2.1. (1) Let \( \phi : \Gamma \to \Gamma' \) be an epimorphism. Then \( \phi(Z(\Gamma)) \subset Z(\Gamma') \) and
\[
\phi(\gamma_i(\Gamma)) = \gamma_i(\Gamma') \quad \text{for all } i.
\]

(2) For any finitely generated torsion free nilpotent group \( \Gamma \), any \( z \in \Gamma \), and any \( n \in \mathbb{N} \) the condition \( z^n \in Z(\Gamma) \) implies \( z \in Z(\Gamma) \).

Proof. (1) Straightforward (see for example [B], Theorem 5.1.3).

(2) This is Mal’cev’s Theorem 1 in [Ma2]. We note that the proof there is not selfcontained. This statement follows from the fact that \( z \) and \( z^n \) have the same centralizers. The latter can be proven using Mal’cev theorem about embedding \( \Gamma \) into the group of unipotent upper triangular matrices [Ra] and the fact that \( z \) belongs to Zariski closure of \( z^n \). \( \square \)

Corollary 2.2. For any finitely generated torsion free nilpotent group \( \Gamma \) the group \( \Gamma/Z(\Gamma) \) is torsion free finitely generated nilpotent group.

Proof. The torsion free part follows from (2). The rest follows from (1). \( \square \)

We note that \( \text{nil}(\Gamma/Z(\Gamma)) < \text{nil}(\Gamma) \).

Suppose that \( G \) is a connected, simply connected nilpotent Lie group and \( \Gamma \subset G \) is a uniform lattice. Then \( G \) is the universal cover for \( \Gamma \) and \( N = G/\Gamma \) is an aspherical manifold, called a nilmanifold.
Mal’cev Theorem. Every torsion free finitely generated nilpotent group $\Gamma$ can be realized as the fundamental group of some nilmanifold.

The corresponding simply connected Lie groups $G$ is obtained as the Mal’cev completion of $\Gamma$.

2.2. Ganea-Schwarz’s approach to cat. Recall that an element of an iterated join $X_0 \ast X_1 \ast \cdots \ast X_n$ of topological spaces is a formal linear combination $t_0 x_0 + \cdots + t_n x_n$ of points $x_i \in X_i$ with $\sum t_i = 1$, $t_i \geq 0$, in which all terms of the form $0x_i$ are dropped. Given fibrations $f_i : X_i \to Y$ for $i = 0, \ldots, n$, the fiberwise join of spaces $X_0, \ldots, X_n$ is defined to be the space $X_0 \ast Y X_1 \ast Y \cdots \ast Y X_n = \{ t_0 x_0 + \cdots + t_n x_n \in X_0 \ast \cdots \ast X_n \mid f_0(x_0) = \cdots = f_n(x_n) \}$. The fiberwise join of fibrations $f_0, \ldots, f_n$ is the fibration $f_0 \ast \cdots \ast f_n : X_0 \ast Y X_1 \ast Y \cdots \ast Y X_n \to Y$ defined by taking a point $t_0 x_0 + \cdots + t_n x_n$ to $f_i(x_i)$ for any $i$ such that $t_i \neq 0$.

When $X_i = X$ and $f_i = f : X \to Y$ for all $i$ the fiberwise join of spaces is denoted by $*_Y X$ and the fiberwise join of fibrations is denoted by $*_Y f$.

For a path connected space $X$, we turn an inclusion of a point $* \to X$ into a fibration $p_0^X : G_0(X) \to X$, whose fiber is known to be the loop space $\Omega X$. The $n$-th Ganea space of $X$ is defined to be the space $G_n(X) = *_{X}^{n+1}G_0(X)$, while the $n$-th Ganea fibration $p_n^X : G_n(X) \to X$ is the fiberwise join $*_Y^{n+1} p_0^X$. Then the fiber of $p_n^X$ is $*_Y \Omega X$.

The following theorem give the Ganea-Shwarz characterization of cat [Sch], [CLOT]:

Theorem 2.3. If $X$ is a connected ANR, then $\text{cat } X \leq n$ if and only if the fibration $p_n^X : G_n(X) \to X$ admits a section.

This characterization can be extendend to maps:

Theorem 2.4. If $f : X \to Y$ is a map between connected ANRs, then $\text{cat } f \leq n$ if and only if there is a lift of $f$ with respect to the fibration $p_n^X : G_n(Y) \to Y$ admits a section.

2.3. Berstein-Schwarz cohomology class. The Berstein-Schwarz class of a discrete group $\pi$ is the first obstruction $\beta_\pi$ to a lift of $B\pi = K(\pi, 1)$ to the universal covering $E\pi$. Note that $\beta_\pi \in H^1(\pi, I(\pi))$ where $I(\pi)$ is the augmentation ideal of the group ring $\mathbb{Z}\pi$ [Be], [Sch].

Theorem 2.5 (Universality [DR], [Sch]). For any cohomology class $\alpha \in H^k(\pi, L)$ there is a homomorphism of $\pi$-modules $I(\pi)^k \to L$ such that the induced homomorphism for cohomology takes $(\beta_\pi)^k \in H^k(\pi, I(\pi)^k)$ to $\alpha$ where $I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi)$ and $(\beta_\pi)^k = \beta_\pi \sim \cdots \sim \beta_\pi$.

In the paper we use notations $H^*(\Gamma, A)$ for cohomology of a group $\Gamma$ with coefficient in $\Gamma$-module $A$. The cohomology groups of a space $X$ with the fundamental group $\Gamma$ we denote as $H^*(X; A)$. Thus, $H^*(\Gamma, A) = H^*(B\Gamma; A)$ where $B\Gamma = K(\Gamma, 1)$.

3. Reduction to epimorphisms

Lemma 3.1. Let $\pi \subset \Lambda$ be a subgroup, $j : B\pi \to B\Lambda$ be a map generated by this inclusion, and $p' : j^*E\Lambda \to B\pi$ be the pull-back of the universal covering $p_\Lambda : E\Lambda \to B\Lambda$. Then $p'$ has a lift with respect to the universal covering $p_\pi : E\pi \to B\pi$. 
Theorem 3.2. From [Sc].

Proof. Clearly, for each path component $C$ of $j^*E\Lambda$ the restriction $p'|_C : C \to B\pi$ of a covering $p'$ is a covering. Easy diagram chasing shows that $C$ is simply connected. Hence, each path component $C$ of $j^*E\Lambda$ is homeomorphic to $E\pi$ and the restriction $p'|_C : C \to B\pi$ is the universal covering. We may assume that $j^*E\Lambda$ is a CW complex. This would imply that each path component $C$ is open. Thus, the lift $j^*E\Lambda \to E\pi$ can be defined independently on each path component. \qed

Given a homomorphism $\phi : \Gamma \to \Lambda$, by $\phi' : \Gamma \to \text{im}(\phi)$ we denote the restriction of $\phi$ from the codomain to its range. The following theorem is a generalization of Theorem 6.11 from [Sc].

Theorem 3.3. For any group homomorphism $\phi : \Gamma \to \Lambda$, cat $\phi = \text{cat} \phi'$.

Proof. Clearly, cat $\phi \leq$ cat $\phi'$. We show that cat $\phi \geq$ cat $\phi'$. Let cat $\phi = k$. This means that there is a lift $f : \Gamma \to B\Lambda$ with respect to the Ganea fibration $p_k : G_k(\Lambda) \to \Lambda$. Since the path fibration $p : P_0(\Lambda) \to \pi$ is fiber-wise homotopy equivalent to the universal covering $\pi \to \pi$ for any discrete group $\pi$, the $k$-th Ganea fibration $G_k(\pi) \to \pi$ is fiberwise homotopy equivalent to the iterated fiberwise join $*_{\pi}^{k+1} E\pi$ of the universal covering. Then $f$ admits a lift with respect to

$$*_{BA}^{k+1} p_{\Lambda} : *_{BA}^{k+1} E\Lambda \to B\Lambda.$$

The map $f$ factors as $f = j \circ f'$ where $f' : \Gamma \to B\pi$, $j : B\pi \to B\Lambda$ and $\pi = \text{im}(\phi)$. Hence, the map $f'$ admits a lift to the pull-back

$$f' : \Gamma \to j^*(*_{BA}^{k+1} E\Lambda)$$

By Lemma 2.1 there is a lift $s$ of $*_{\pi}^{k+1} p_{\pi} : j^*(*_{BA}^{k+1} E\Lambda) \to B\pi$ with respect to

$$*_{\pi}^{k+1} p_{\pi} : *_{\pi}^{k+1} E\pi \to B\pi.$$

Then the composition $s \circ f'$ is a lift of $f'$ with respect to $*_{\pi}^{k+1} p_{\pi}$. By Theorem 2.4 cat $f' \leq k$ and, hence, cat $\phi' \leq k$. \qed

Theorem 3.3. For any group homomorphism $\phi : \Gamma \to \Lambda$ and $\phi' : \Gamma \to \text{im}(\phi) = \pi$ as above, $\text{cd}(\phi) = \text{cd}(\phi')$.

Proof. Clearly, $\text{cd}(\phi') \geq \text{cd}(\phi)$. We show that $\text{cd}(\phi') \leq \text{cd}(\phi)$. Let $\text{cd}(\phi') = k$. Then $\phi' : H^k(\pi, M) \to H^k(\Gamma, M)$ is not zero for some $\pi$-module $M$. Let

$$\alpha : \text{Coind}_\pi^\Lambda M = \text{Hom}_\pi(\Lambda M, M) \to M$$

denote the canonical $\pi$-homomorphism, defined for $f : \Lambda \to M$ as $\alpha(f) = f(1)$. Consider the commutative diagram

$$
\begin{array}{ccc}
H^k(\Lambda, \text{Coind}_\pi^\Lambda M) & \xrightarrow{j^*} & H^k(\pi, \text{Coind}_\pi^\Lambda M) \\
\downarrow{\phi'^*} & & \downarrow{\phi'^*} \\
H^k(\Gamma, \text{Coind}_\pi^\Lambda M) & \xrightarrow{\alpha^*} & H^k(\Gamma, M)
\end{array}
$$

where $\alpha^*$ is the coefficient homomorphism generated by $\alpha$. By Shapiro Lemma [Br] the top row through homomorphism is an isomorphism. Therefore, the homomorphism

$$\phi'^* = \phi'^* j^* : H^k(\Lambda, \text{Coind}_\pi^\Lambda M) \to H^k(\Gamma, \text{Coind}_\pi^\Lambda M)$$

is a homotopy equivalence. Hence, cat $\phi'^* \leq k$. This means that cat $\phi' \leq k$. \qed
is nonzero. Hence, $\text{cd}(\phi) \geq k$. □

Theorem 3.2 and Theorem 3.3 imply the following:

**Corollary 3.4.** Suppose that Conjecture 1.3 holds true for all epimorphisms $\phi : \Gamma \to \pi$ for some class of groups. Then it holds for all homomorphisms for groups from that class.

4. **Homomorphisms of nilpotent groups**

**Lemma 4.1.** Let $\Gamma$ be $\pi$ finitely generated, torsion free nilpotent groups. Then every epimorphism $\phi : \Gamma \to \pi$ can be realized as a locally trivial bundle of nilmanifolds with the fiber a nilmanifold.

*Proof.* We prove it by induction on $n = s + t$ where $s = \text{nil}(\Gamma)$ and $t = \text{nil}(\pi)$ are the nilpotency classes of $\Gamma$ and $\pi$. The base of induction is the case of abelian groups where any epimorphism $\phi : \mathbb{Z}^{k+\ell} \to \mathbb{Z}^k$ is the projection onto a factor, since it is a split surjection. Clearly, the projection $\phi$ can be realized as a trivial fiber bundle of tori $T^{k+\ell} \to T^k$ with the fiber a torus $T^\ell$.

Suppose that $n > 2$ and the statement of the lemma holds true for $s + t < n$. We denote by $B = Z(\Gamma) \cap \text{Ker} \phi$ and by $A = Z(\Gamma)/B$, where $Z(\Gamma)$ is the center of $\Gamma$. Note that $B$ is a direct summand in $Z(\Gamma)$ or, equivalently, $A$ is free. For the later we claim that if $z^n \in B$, then $z \in B$. Indeed, If $z^n \in \text{Ker} \phi$ then $z \in \text{Ker} \phi$, since $\pi$ is torsion free. If $z^n \in Z(\Gamma)$ then $z \in Z(\Gamma)$ by Proposition 2.1. Thus, $Z(\Gamma) \cong B \oplus A$. Let $\tilde{A}$ be the direct summand of $Z(\pi)$ that contains $\phi(Z(\Gamma)) \cong A$ as a finite index subgroup. In view of Corollary 2.2, $\pi/\tilde{A}$ is a torsion free nilpotent group.

We consider the nilpotent group $\Gamma' = \Gamma/B$ and the epimorphism $\phi' : \Gamma' \to \pi$. induced by $\phi$. In view of the principal fiber bundle $B\Gamma \to B\Gamma'$ with the fiber a torus, it suffices to show that $\phi'$ can be realized as a fiber bundle of nilmanifolds. We consider the commutative diagram

$$
\begin{array}{ccc}
\Gamma' & \longrightarrow & \Gamma'/A \\
\phi' \downarrow & & \phi' \downarrow \\
\pi & \longrightarrow & \pi/\tilde{A}.
\end{array}
$$

Note that $\Gamma'/A = \Gamma/Z(\Gamma)$ and, hence, $\phi'$ is an epimorphism with $\text{nil}(\Gamma'/A) + \text{nil}(\pi/\tilde{A}) < s + t$. The homomorphism $\phi'$ factors through the pull-back $\Lambda = \phi'^*\pi$, $\phi' = \phi' \circ \xi$ with respect to $\phi$. Since $\phi'|_A : A \to \tilde{A}$ is an embedding of a finite index subgroup, the homomorphism $\xi : \Gamma' \to \Lambda$ in the commutative diagram of short exact sequences generated by $\phi'$ and the pull-back

$$
\begin{array}{cccccc}
1 & \longrightarrow & A & \longrightarrow & \Gamma' & \longrightarrow & \Gamma/Z(\Gamma) & \longrightarrow & 1 \\
\downarrow & & \xi & \downarrow & = & & & \\
1 & \longrightarrow & \tilde{A} & \longrightarrow & \Lambda & \longrightarrow & \Gamma/Z(\Gamma) & \longrightarrow & 1 \\
\downarrow & & \phi' & \downarrow & \phi & & & \\
1 & \longrightarrow & \tilde{A} & \longrightarrow & \pi & \longrightarrow & \pi/\tilde{A} & \longrightarrow & 1
\end{array}
$$

is an embedding of a finite index subgroup.
By induction assumption applied to $\widetilde{\phi}$ there is a fiber bundle between nilmanifolds

$$\bar{f} : B(\Gamma / Z(\Gamma)) \to B(\pi / \bar{A})$$

with the fiber a nilmanifold $F$. Consider the pull-back diagram

$$\begin{array}{ccc}
BA & \to & B(\Gamma / Z(\Gamma)) \\
\downarrow f' & & \downarrow f \\
B\pi & \to & B(\pi / \bar{A}).
\end{array}$$

Let $q : M \to BA$ be a covering that corresponds to the subgroup $\xi(\Gamma') \subset \Lambda = \pi_1(BA)$. Since $\xi(\Gamma')$ is of finite index, $M$ is a closed aspherical manifold with $\pi_1(M) = \Gamma'$. Thus, the composition $f' = \bar{f}' \circ q : M \to B\pi$ realizes the homomorphism $\phi'$ as a fiber bundle. The fiber of $f'$ is homeomorphic to the total space of a fiber bundle $F' \to F$ with a finite fiber. The homotopy exact sequence of the fibration $F' \to B\Gamma' \xrightarrow{\bar{f}'} B\pi$ and the fact that $\phi'$ is surjective imply that $F'$ is connected. Hence, $F'$ is a nilmanifold. □

**Lemma 4.2.** For every locally trivial bundle of closed aspherical manifolds $f : M^m \to N^n$ with compact connected fiber $F$ the induced homomorphism

$$f^* : H^n(N; \mathbb{Z}\pi) \to H^n(M; \mathbb{Z}\pi)$$

is nonzero where $\pi = \pi_1(N)$.

**Proof.** Consider the pull-back diagram:

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{p_2} & f^* \tilde{N} \xrightarrow{f} \tilde{N} \\
\downarrow p_M & & \downarrow p_N \\
M & \xrightarrow{f} & N
\end{array}$$

where $p_M$ and $p_N$ are the universal covering maps. Since $f^*(\tilde{N})$ is a covering of $M$ and $\tilde{M}$ is the universal covering, $p_M$ factors through covering maps $p_2$ and $p_1$.

We recall [Br, Lemma 7.4] that for every left $\pi$-module $M$ there is a natural isomorphism of right modules

$$\Theta : Hom_\pi(M, \mathbb{Z}\pi) \to Hom_c(M, \mathbb{Z})$$

defined by the formula $F \mapsto f_1$ where $F : M \to \mathbb{Z}\pi$, $F(m) = \sum_{\gamma \in \pi} f_\gamma(m)\gamma$. We consider the following commutative diagrams of cochain groups:

$$\begin{array}{ccc}
Hom_\Gamma(C_*(\tilde{M}), \mathbb{Z}\pi) & \xrightarrow{p_2^*} & Hom_\pi(C_*(\tilde{N}), \mathbb{Z}\pi) \\
\downarrow \Psi & & \downarrow \Theta \\
Hom_\pi(C_*(X), \mathbb{Z}\pi) & \xrightarrow{\theta} & Hom_c(C_*(X), \mathbb{Z}) \\
(f)^* & & (f_\gamma)^* \\
Hom_\pi(C_*(\tilde{N}), \mathbb{Z}\pi) & \xrightarrow{\theta} & Hom_c(C_*(\tilde{N}), \mathbb{Z})
\end{array}$$

where $X = f^*\tilde{N}$, $\Psi = \Theta \circ \Phi$, and

$$\Phi : Hom_\Gamma(C_*(\tilde{M}), \mathbb{Z}\pi) \to Hom_\pi(C_*(X), \mathbb{Z}\pi)$$
is defined as follows. For each simplex $\sigma$ in $X$ we fix a lift $\tilde{\sigma}$ in $\tilde{M}$. We define
\[ \Phi(F)(\sigma) = F(\tilde{\sigma}) \]
for $F \in \text{Hom}_\pi(C_*(\tilde{M}), \mathbb{Z}_2)$ and simplex $\sigma$ in $X$.

**Claim 1.** $\Phi(F) \in \text{Hom}_\pi(C_*(X), \mathbb{Z}_2)$.

**Proof.** Since the fiber $F$ is connected, it follows that $f$ induces an epimorphism of the fundamental groups. Let $g \in \pi$ and let $f_*(\tilde{g}) = g$. Then $\tilde{g}\sigma$ and $\tilde{g}\tilde{\sigma}$ both cover $g\sigma$. Hence there is $\gamma \in \text{Ker} f_*$ such that $\gamma(\tilde{g}\tilde{\sigma}) = \tilde{g}\tilde{\sigma}$. Then $\Phi$ is $\pi$-equivariant:
\[ \Phi(F)(g\sigma) = F(\tilde{g}\tilde{\sigma}) = F(\tilde{g}\tilde{\sigma}) = F(\tilde{g}\tilde{\sigma}) = gF(\tilde{\sigma}) = g\Phi(F)(\sigma). \]

**Claim 2.** $\Phi \circ p_2^* = 1$.

**Proof.** $\Phi \circ p_2^*(F')(\sigma) = \Phi(p_2^*(F'))(\sigma) = (p_2^*F')(\tilde{\sigma}) = F'(p_2(\tilde{\sigma})) = F'(\sigma)$.

Since $\tilde{N}$ is contractible, the bottom row in the diagram above gives us an isomorphism of the cohomology groups $\Theta^*: H^\ast(\tilde{N}; \mathbb{Z}_2) \to H^\ast(\tilde{N}; \mathbb{Z}_2)$. The commutative diagram on the cochain level produces the commutative diagram for cohomology:
\[
\begin{array}{ccc}
H^n(M; \mathbb{Z}_2) & \xrightarrow{\Phi^*} & H^n_c(X; \mathbb{Z}_2) \\
\uparrow f^* & & \uparrow (f^*)^* \\
H^n(N; \mathbb{Z}_2) & \xrightarrow{\Theta^*} & H^n(\tilde{N}; \mathbb{Z}_2) = \mathbb{Z}.
\end{array}
\]

Since $\tilde{N}$ is contractible, $X = f^*(\tilde{N})$ is a trivial fiber bundle with the fiber $F$. Since $X \simeq \tilde{N} \times F$ admits a proper retraction onto $\tilde{N}$, we obtain that $(\tilde{f}_e)^*$ is a monomorphism. Hence, the homomorphism $f^*: \mathbb{Z} = H^n(N; \mathbb{Z}_2) \to H^n(M; \mathbb{Z}_2)$ is not trivial. □

**Theorem 4.3.** For a homomorphism $\phi: \Gamma \to \pi$ of finitely generated torsion free nilpotent groups $\text{cat} \phi = \text{cd}(\phi)$.

**Proof.** In view of Theorem 3.2 and Theorem 3.3, it suffices to prove the theorem when $\phi$ is surjective. By Lemma 4.1 there is a fiber bundle of nilmanifolds $M^m = B\Gamma \to B\pi = N^m$ with a compact fiber. By Lemma 4.2 $\text{cd}(\phi) = n$. The inequalities $\text{cd}(\phi) \leq \text{cat} \phi \leq \dim N = n$ complete the proof. □

5. Counterexample

Answering a question of M. Grant, T. Goodwillie gave an example of an epimorphism $\phi: G \to \Gamma$ satisfying the inequality $\text{cd}(\phi) < \min\{\text{cd}(G), \text{cd}(\Gamma)\}$. We present here a slightly modified version of his example. Let $\Gamma = \mathbb{Z}^2$ and $\phi: G \to \Gamma$ be the epimorphism defined by the extension of $\Gamma$ by the abelian group $I(\Gamma)^2 = I(\Gamma) \otimes \mathbb{Z}$ that corresponds to the square of the Berstein-Schwarz cohomology class $(\beta_{\Gamma})^2 \in H^2(\Gamma, I(\Gamma)^2)$. Then $\phi^*(\beta_{\Gamma})^2 = 0$. Hence, $\text{cd}(\phi) < 2$. If $\text{cat} \phi \leq 1$, then the induced map $\tilde{\phi}: BG \to B\Gamma$ can be lifted to the Ganea’s space $G_1(B\Gamma)$ which is homotopy equivalent to 1-dimensional complex $\Sigma\Gamma$, the suspension of $\Gamma$. [CLOT]. Then the epimorphism $\phi$ factors through a free group $F$ via epimorphisms $G \to F \to \Gamma$. Hence $\text{rank} F \geq 2$. Therefore, $G$ contains a free group of rank $\geq 2$. We note that $G$ is amenable as an abelian-by-abelian extension and hence $G$ cannot contain $F$. Therefore, cat $\phi \geq 2$. Thus, $\phi$ is an infinitely generated counterexample to Conjecture 1.3.
Remark 5.1. We note there are simpler examples of epimorphisms \( \phi : G \to \Gamma \) satisfying
\[
\text{cd}(\phi) < \min\{\text{cd}(G), \text{cd}(\Gamma)\}.
\]
Namely, the homomorphism \( \phi : F_n \ast \mathbb{Z}^n \to \mathbb{Z}^n \) defined by the abelianization of the first factor and by the projection onto \( \mathbb{Z}^n \to \mathbb{Z} \subset \mathbb{Z}^n \) has \( \text{cd}(\phi) = 1 \) and \( \text{cd}(F_n \ast \mathbb{Z}^n) = \text{cd} \mathbb{Z}^n = n \). Here \( F_n \) denotes the free group on \( n \) generators.

Let \( S \) denote the sphere spectrum. We recall that every stably parallelizable manifold is \( S \)-orientable [Sw]. The \( S \)-cohomotopy groups of \( X \) are exactly the stable cohomotopy groups \( \pi^*_s(X) \). Then Lemma 3.5 of [Ru] in the case of the spectrum \( S \) can be stated as follows:

Lemma 5.2. Suppose that \( f : W \to M \) is a map of degree one between closed stably parallelizable manifolds. Then the induced map \( f^* : \pi^*_s(M) \to \pi^*_s(W) \) is injective.

We note that the natural map \( [X, S^n] \to \pi^*_s(X) \) is a bijection when \( \dim X \leq 2n - 2 \) [Hu].

5.1. Bolotov’s example. Answering a question of Gromov [Gro1], D. Bolotov constructed [Bo] a closed 4-manifold \( M \) with the fundamental group \( \pi = \mathbb{Z} \ast \mathbb{Z}^3 \) whose cohomological dimension \( \text{cd}(\pi) = 3 \) such that a classifying map \( u_M : M \to B\pi \) cannot be deformed to the 2-skeleton \( B\pi^{(2)} \). His manifold is defined as the pull-back \( M = g^*(S^3 \times S^1) \) of the \( S^1 \)-bundle \( h \times 1 : S^3 \times S^1 \to S^2 \times S^1 \), where \( h \) is the Hopf fiber bundle, with respect to the collapsing map \( g : N = (S^2 \times S^1) \# T^3 \to S^2 \times S^1 \). Here \( T^3 \) is a 3-dimensional torus. The pull-back bundle is denoted by \( p : M \to N \).

Proposition 5.3. Bolotov’s manifold has the following properties:

(a) The map \( p \) induces an isomorphism of the fundamental groups.

(b) The homomorphism \( u_M^* : H^3(B\pi; A) \to H^3(M; A) \) is trivial for any \( \pi \)-module \( A \).

(c) The through map \( M \cup_1 B\pi = S^1 \vee T^3 \cong T^3 \) is essential where \( q_1 \) collapses \( S^1 \) to the wedge point, and \( q_1 \) is a map of degree one.

(d) The manifold \( M \) is stably parallelizable.

Proof. (a) This is straightforward ([Bo]). Thus, the classifying map \( u_M \) factors through \( p : M \to N \), \( u_M = q_1q_4p \) where \( q_1 : N \to (S^2 \times S^1) \vee T^3 \) collapses the connected sum to the wedge and \( q_3 : (S^2 \times S^1) \vee T^3 \to S^1 \vee T^3 \) projects \( S^2 \times S^1 \) onto the factor \( S^1 \).

(b) Let \( a \in H^3(B\pi; A) \). By the Poincare Duality for local coefficients there is 1-dimensional class \( \beta \) in \( M \) such that \( u_M^*(a) \cup \beta \neq 0 \). In view of universality of the Berstein-Schwarz class \( \beta_{\pi} \) and the fact that \( u_M \) induces an isomorphism of 1-cohomology, we may assume that \( \beta = u_M^*(\beta_{\pi}) \). Then \( 0 \neq u_M^*(a) \cup \beta = u_M^*(a \cup \beta_{\pi}) = 0 \). The last equality is due to dimensional reason.

(c) This was proven in [Bo]. Here we present a simplified proof. Let \( q = q_1q_2q_3q_4 : N \to S^3 \):
\[
N = (S^2 \times S^1) \# T^3 \cong (S^2 \times S^1) \vee T^3 \cong S^1 \vee T^3 \cong T^3 \cong S^3.
\]
Let \( a \) be a generator in \( H^3(K(\mathbb{Z}, 3)) \) and let \( a_0 \) be its restriction to \( S^3 = K(\mathbb{Z}, 3)^{(4)} \). Part (b) implies that \( p^*(a_0) = 0 \) where \( a_0 = g^*(a) \). The exact sequence of pair \( (C_p, N) \) implies \( (j^*)^{-1}(a_0) \neq \emptyset \) where \( j : N \to C_p \) is the inclusion of the codomain of \( p \) into the mapping cone. It suffices to show that \( q : N \to S^3 \) does not extend to \( C_p \). Every extension \( \psi \) of \( q \) to the 4-skeleton \( C_p^{(4)} \) can be extended to a map \( \psi : C_p \to K(\mathbb{Z}, 3) \) defining an element
\[ \alpha \in (j^*)^{-1}(\alpha_0). \] We show that such \( \psi \) cannot be deformed to \( S^3 \subset K(\mathbb{Z}, 3) \). Here we assume that \( K(\mathbb{Z}, 3)^{(5)} = S^3 \cup \nu D^5 \).

For a cohomology class \( x \) we denote by \( \bar{x} \) its mod 2 reduction. Since \( q \) is a map of degree one, \( \bar{\alpha}_0 \neq 0 \). We recall that the obstruction to retraction of \( S^3 \cup \nu D^5 \) to \( S^3 \) is the Steenrod square \( Sq^2 \bar{\alpha} \). By the naturality of primary obstructions the obstruction to deform the map 
\[ q_{ \alpha} : S^3 \xrightarrow{\psi} S^3 \] implies that \( \bar{\alpha}_0 \). Then
\[ H^3(N, \partial N; \mathbb{Z}_2) \times H^2(N, \partial N; \mathbb{Z}_2) \xrightarrow{\cup} H^5(N, \partial N; \mathbb{Z}_2) \]
implies that \( \bar{\alpha}_0 \cup u = \bar{\alpha} \cup u \). Then
\[ 0 \neq \bar{\alpha}_0 \cup u = \bar{\alpha} \cup u = x \cup u \cup u = x \cup Sq^2 u = Sq^2(x \cup u) = Sq^2(\bar{\alpha}). \]

Here we use the equality \( Sq^2 u = u \cup u \) and the Cartan formula for Steenrod squares.

(d) The manifold \( M \) is stably parallelizable as the total space of an orientable \( S^1 \)-bundle over a stably parallelizable manifold \( N \).

We recall that for every closed manifold \( M \) there is a hyperbolization \( f : W \to M \) which a map of degree one of a closed aspherical manifold which is surjective on the fundamental groups. Moreover, the map \( f \) induces an isomorphism between stable tangent bundles of \( M \) and \( W )\).

**Theorem 5.4.** There is a map of a closed aspherical \( 4 \)-manifold \( g : W \to T^3 \) onto a 3-torus that induces an epimorphism of the fundamental groups \( g_\# : \pi_1(W) \to \mathbb{Z}^3 \) such that \( \text{cat } g_\# = 3 \) and \( \text{cd} (g_\#) < 3 \).

**Proof.** Let \( M \) be the Bolotov’s example and let \( f : W \to M \) be the hyperbolization of \( M \). Let \( g = q_2 \circ u_M \circ f \). Since Bolotov’s example is stably parallelizable and the hyperbolization is a tangential map, by Lemma [5.2] we obtain that \( f^* : \pi_3^s(M) \to \pi_3^s(W) \) is injective. By dimensional reason \( [M, S^3] = \pi_3^s(M) \) and \( [W, S^3] = \pi_3^s(W) \). Thus, since by Proposition [5.3] (c) the map \( q_1 q_2 u_M : M \to S^3 \) is essential, the map \( q_1 \circ g : W \to S^3 \) is essential as well. Hence, the map \( g \) cannot be deformed to the 2-skeleton. Therefore, \( \text{cat } g > 2 \). Clearly, \( \text{cat } g = \text{cat } g_\# = 3 \).

By Proposition [5.3] (a) the homomorphism \( (q_1 \circ u_M)^* \) is trivial on 3-dimensional cohomology. Hence, so is \( g^* \). This means that \( \text{cd} (g_\#) < 3 \).

**Acknowledgments**

The first author was supported by the Simons Foundation Grant.
References

[B] H. Bechtell, The theory of groups, Addison-Wesley, 1971.
[Be] I. Berstein, On the Lusternik-Schnirelmann category of Grassmannians. Math. Proc. Camb. Philos. Soc. 79 (1976) 129-134.
[Bo] D. Bolotov, Gromov’s macroscopic dimension conjecture, AGT 6 (2006), 1669-1676.
[Br] K. Brown, Cohomology of Groups. Graduate Texts in Mathematics, 87 Springer, New York Heidelberg Berlin, 1994.
[CD] R. M. Charney, M. W. Davis, Strict hyperbolization, Topology 34 (1995), no.2, 329-350.
[CLOT] O. Cornea, G. Lupton, J. Oprea, D. Tanre, Lusternik-Schnirelmann Category, AMS, 2003.
[DJ] M. W. Davis and T. Januszkiewicz, Hyperbolization of polyhedra, J. Differential Geom. 34 (1991), no.2, 347-388.
[DR] A. Dranishnikov, Yu. Rudyak, On the Berstein-Svarc theorem in dimension 2. Math. Proc. Cambridge Philos. Soc. 146 (2009), no. 2, 407-413.
[EG] S. Eilenberg, T. Ganea, On the Lusternik-Schnirelmann Category of Abstract Groups. Annals of Mathematics, 65, (1957), 517-518.
[Gr] M. Grant, https://mathoverflow.net/questions/89178/cohomological-dimension-of-a-homomorphism
[Gro1] M. Gromov Positive curvature, macroscopic dimension, spectral gaps and higher signatures, Functional analysis on the eve of the 21st century. Vol. II, Birkhauser, Boston, MA, 1996.
[Gro2] M. Gromov, Hyperbolic groups, Essays in group theory, Math. Sci. Res. Inst. Publ., 8, Springer, New York, (1987) 75–263.
[Hu] S.-T. Hu, Homotopy theory, Acad. Press, 1959.
[LS] L. Lusternik, L. Schnirelmann, “Sur le probleme de trois geodesiques fermees sur les surfaces de genre 0”, Comptes Rendus de l’Academie des Sciences de Paris, 189: (1929) 269-271.
[Ma1] A. I. Mal’tsev, On a class of homogeneous spaces, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, vol. 13 (1949), no 3, 201-212.
[Ma2] A. I. Mal’tsev, Nilpotent groups without torsion, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, vol. 13 (1949), no 1, 9-32.
[Ra] M. S. Raghunathan, Discrete subgroups of Lie groups, Springer, 1972.
[Ru] Yu. Rudyak, On category weight and its applications, Topology 38 (1999) no. 1, 37–55.
[Sch] A. Schwarz, The genus of a fibered space. Trudy Moscov. Mat. Obsc. 10, 11 (1961 and 1962), 217-272, 99-126.
[Sc] J. Scott, On the Topological Complexity of Maps, preprint arXiv:2011.10646 2020.
[Sw] R. Switzer, Algebraic Topology, Homotopy and Homology. Springer, Berlin, 1975.

Alexander Dranishnikov, Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, FL 32611-8105, USA
Email address: dranish@math.ufl.edu

Nursultan Kuanyshov, Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, FL 32611-8105, USA
Email address: kuanyshov@math.ufl.edu