Renormalization of circle diffeomorphisms with a break-type singularity

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Abstract
Let $f$ be an orientation-preserving circle diffeomorphism with an irrational rotation number and with a break point $\xi_0$, that is, its derivative $f'$ has a jump discontinuity at this point. Suppose that $f'$ satisfies a certain Zygmund condition dependent on a parameter $\gamma > 0$. We prove that the renormalizations of $f$ are approximated by Möbius transformations in $C^1$-topology if $\gamma \in (0, 1]$ and in $C^2$-topology if $\gamma \in (1, +\infty)$. Moreover, it is shown that, in case of $\gamma \in (1, +\infty)$ the coefficients of Möbius transformations get asymptotically linearly dependent. Further, consider two circle diffeomorphisms with a break point, with the same size of the break and satisfying Zygmund condition with $\gamma \in (1, +\infty)$. We prove that, under a certain technical condition on rotation numbers, the renormalizations of these diffeomorphisms approach each other in $C^2$-topology.

Keywords: circle diffeomorphism, break point, renormalization, Möbius transformations, convergence, rotation number
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\textsuperscript{*} This paper is dedicated to Professors Akhtam Abdurakhmanovich Dzhalilov and Konstantin Mikhailovich Khanin on the occasion of their 60th birthdays.
1. Introduction

One of the most studied classes of dynamical systems are orientation-preserving diffeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Poincaré (1885) noticed that the orbit structure of orientation-preserving diffeomorphism $f$ is determined by some irrational mod 1, called the rotation number of $f$ and denoted $\rho = \rho(f)$, in the following sense: for any $\xi \in S^1$, the mapping $f^j(\xi) \to j\rho \mod 1$, $j \in \mathbb{Z}$, is orientation-preserving. Denjoy proved that if $f$ is an orientation-preserving $C^1$-diffeomorphism of the circle with irrational rotation number $\rho$ and $\log f'$ has bounded variation, then the orbit $\{f^j(\xi)\}_{j \in \mathbb{Z}}$ is dense and the mapping $f^j(\xi) \to j\rho \mod 1$ can therefore be extended by continuity to a homeomorphism $h$ of $S^1$, which conjugates $f$ to the rigid rotation $f_\rho : \xi \to \xi + \rho \mod 1$.

In this context, a natural question to ask is under what condition can one obtain the smoothness of conjugacy $h$. The first local results, that is the results requiring the closeness of diffeomorphism to the rigid rotation, were obtained by Arnold [2] and Moser [25]. Next, Herman [8] obtained a first global result (i.e. not requiring the closeness of diffeomorphism to the rigid rotation) asserting the regularity of conjugacy of the circle diffeomorphism. Further, his result was developed by Yoccoz [32], Stark [26], Khanin and Sinai [15], Katnelson and Ornstein [10, 11], and Khanin and Teplinsky [18]. They have shown that if $f$ is $C^3$ or $C^{3+\nu}$, and $\rho$ satisfies a certain Diophantine condition, then the conjugacy will be at least $C^1$. Notice that the renormalization approach used in [15] and [26] is more natural to Herman’s theory. In this approach, the regularity of the conjugacy statement can be obtained by using the convergence of renormalizations of sufficiently smooth circle diffeomorphisms. In fact, the renormalizations of a smooth circle diffeomorphism converge exponentially fast to a family of linear maps with slope 1. Such a convergence together with the condition of rotation number of Diophantine type imply the regularity of conjugacy.

A natural generalization of diffeomorphisms of the circle are diffeomorphisms with break points, i.e. those circle diffeomorphisms which are smooth everywhere with the exception of finitely many points at which their derivatives have jump discontinuities. Circle diffeomorphisms with breaks were investigated by Herman [8] in the piecewise-linear (PL) case. The studies of more general (non PL) circle diffeomorphisms with a break started with the works of Khanin and Vul [19, 29] at the beginning of the 1990s. It turns out that, the renormalizations of circle diffeomorphisms with break points are rather different from those of smooth diffeomorphisms. Indeed, the renormalizations of such a circle diffeomorphism converge exponentially fast to a two-parameter family of Möbius transformations (see theorem 3.1) with very non-trivial dynamics. In the sequel we emphasize the importance of Khanin and Vul’s [29] result in the context of renormalization conjecture and rigidity theory. Renormalization conjecture is the convergence of renormalizations of two break-equivalent (for the definition see [5]) circle diffeomorphisms and rigidity is the phenomenon of smooth conjugacy between any two maps which a priori are only topologically equivalent.

Since the renormalization of a circle diffeomorphism with a break converge to a two-parameter family of Möbius transformations, it is convenient to study the action of the renormalization operator on a space of pairs (for a particular size of a break) in $\mathbb{R}^2$ which correspond to Möbius pairs. The investigations of those Möbius transformations in [12, 17, 20, 28] showed that the renormalization operator possesses strong hyperbolic properties in a certain domain of that space, which are analogous to those predicted by Lanford [21] in case of critical rotations.

The hyperbolicity of the renormalization operator, the approximations of renormalizations by Möbius transformations and the analysis of coefficients of these transformations give the exponential convergence of renormalizations of two circle diffeomorphisms with a break in $C^2$-topology, under a certain technical condition on rotation numbers. This result was proved by
Khanin and Khmelev [12] for quadratic rotation numbers and for half bounded rotation numbers by Khanin and Teplinsky [17]. Recently, the renormalization conjecture has been proven by Khanin and Kocić [13] for all irrational rotation numbers. The main idea of their work is to introduce a notion of renormalization strings and analyze their asymptotic properties. The renormalization conjecture for the real-analytic critical circle maps was proved by de Faria and de Melo [23] for rotation numbers of bounded type and extended by Yampolsky [31] to cover all irrational rotation numbers. For the maps with lower smoothness, that is for $C^3$ critical maps with bounded type of rotation numbers the conjecture was proved by Guarino and de Melo [6] and for $C^4$ critical maps with any irrational rotation numbers by Guarino et al [7].

In addition, Khanin and Vul’s result plays an important role to show ‘K-regularity’ of renormalizations which is developed in [14] and [17]. ‘K-regularity’ and the convergence of renormalizations together with a certain condition on rotation number ensure $C^1$-rigidity of circle maps with singularities. The rigidity problem for circle diffeomorphisms with breaks with quadratic rotation numbers was solved by Khanin and Khmelev [12] and for circle diffeomorphisms with breaks with half bounded rotation numbers by Khanin and Teplinsky [17]. Recently, this problem was completely solved by Khanin et al [14] for almost all irrational rotation numbers.

For the critical real-analytic circle maps with bounded types of rotation numbers, the rigidity problem was solved by de Faria and de Melo [23] and their result was extended to all irrational rotation numbers by Yampolsky [31].

Recently, the essential results also have been obtained in this direction. Guarino and de Melo [6] have solved the rigidity problem for $C^3$ critical circle maps with the bounded type of rotation numbers, and Guarino et al [7] have solved this problem for $C^4$ critical maps with almost all irrational rotation numbers.

The purpose of this work is to study the renormalizations for a wider class of circle maps with breaks. In this work we consider a class of circle diffeomorphisms with breaks satisfying a certain Zygmund condition depending on a parameter $\gamma > 0$. The class of such diffeomorphisms is wider than the class of $C^{2+\nu}$ diffeomorphisms. In our first main result, theorem 3.3, we show that, if $\gamma \in (0, 1]$ then the renormalizations of the diffeomorphisms from our considered class approach Möbius transformations with the rate of $O(n^{-\gamma})$ in $C^1$-topology. In the case of $\gamma \in (1, +\infty)$ this class is a subset of $C^2$, therefore one can investigate the renormalizations in $C^2$-topology. In the second main result, theorem 3.4, we show that, if $\gamma \in (1, +\infty)$ then the renormalizations approach Möbius transformations with the rate of $O(n^{-\gamma})$ in $C^1$-topology and the second derivatives approach each other with the rate of $O(n^{-(\gamma-1)})$ in $C^0$-topology. Moreover, it is also shown that, the coefficients of Möbius transformations get asymptotically linearly dependent.

Further, consider two circle diffeomorphisms with a break satisfying the Zygmund condition with $\gamma \in (1, +\infty)$. In this case, theorem 3.4 allows us to investigate the convergence of renormalizations of these two circle diffeomorphisms. In theorem 3.5 we prove that, for a certain class of rotation numbers, the renormalizations of two circle diffeomorphisms satisfying the Zygmund condition with $\gamma \in (1, +\infty)$ and with the same size (see below) of the break approach each other with the rate of $O(n^{-\gamma})$ in $C^1$-topology and the second derivatives approach each other with the rate of $O(n^{-(\gamma-1)})$ in $C^0$-topology.

At the end of this introduction, let us emphasize the importance and limitations of our results. Since the convergence rates in the main theorems are given in explicit form, we believe that these results will have applications in the regularity problem of conjugation between two circle maps with breaks. Of course, for $\gamma \in (0, 1/2]$ it is difficult to expect the regularity of conjugacy. Because in this case the second derivative of circle diffeomorphisms can be very...
‘bad’ (see [33]). However, in the case of $\gamma > 1/2$ the situation gets better, that is, in this case, due to theorem 4.2 in section 2, $f'$ is absolute continuous on $S^1 \setminus \{\xi_0\}$ and $f'^n \in L^p(S^1)$ for every $p > 1$. Such smooth diffeomorphisms are known as the class of Katznelson and Ornstein (KO class) in the theory of circle maps. Katznelson and Ornstein [11] proved that diffeomorphisms from KO class with the bounded type of irrational rotation numbers are absolute continuously conjugated with rigid rotation. In our upcoming paper, for $\gamma > 1/2$, we have generalized Katznelson and Ornstein’s result for smooth diffeomorphisms with unbounded type irrational rotation numbers using the techniques of the proof of theorems 3.3–3.5 gives an opportunity to study the absolute continuity of the conjugation between two circle diffeomorphisms with breaks, however, from this theorem it is difficult to expect $C^1$-rigidity because the rate of convergence of renormalizations is sub-exponential. In order to obtain $C^1$-rigidity, it is very important that the rate of convergence of renormalizations is exponential. This point is known from previous works, e.g. in [16, 22].

2. Renormalizations of circle diffeomorphisms with a break

Let $f : S^1 \to S^1$ be a circle diffeomorphism with a single break point $\xi_0$ i.e. $f$ satisfies the following conditions:

(i) $f \in C^1([\xi_0, \xi_0 + 1])$;
(ii) $\inf_{\xi \neq \xi_0} f'(\xi) > 0$;
(iii) $f$ has one-sided derivatives $f'((\xi_0 \pm 0) > 0$ and

$$c := \frac{f'(\xi_0 - 0)}{f'(\xi_0 + 0)} \neq 1.$$ 

The number $c$ is called the size of break of $f$ at $\xi_0$. Assume that the rotation number $\rho$ is irrational and we use continued fraction expansion of the rotation number

$$\rho = 1/(k_1 + 1/(k_2 + ...)) := [k_1, k_2, ..., k_n, ...].$$

The sequence of positive integers $(k_n)$ with $n \geq 1$ are called partial quotients and it is infinite if and only if $\rho$ is irrational. Every irrational $\rho$ defines uniquely the sequence of partial quotients. Conversely, every infinite sequence of partial quotients defines uniquely an irrational number $\rho$ as the limit of the sequence of rational convergents $p_n/q_n = [k_1, k_2, ..., k_n]$. The coprime numbers $p_n$ and $q_n$ satisfy the recurrence relations $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$ for $n \geq 1$, where, for convenience we set $p_0 = 0$, $q_0 = 1$ and $p_{-1} = 1$, $q_{-1} = 0$. Taking the break point $\xi_0 \in S^1$, we define the $n$th fundamental segment $I^n_{\xi_0} := I^n_{\xi_0}(\xi_0)$ as the circle arc $[\xi_0, f^n(\xi_0)]$ if $n$ is even and $[f^n(\xi_0), \xi_0]$ if $n$ is odd. The union of two consequent fundamental segments $I^{n-1}_{\xi_0}$, $I^n_{\xi_0}$ is called the $n$th renormalization neighborhood of $\xi_0$ and we denote it by $\mathcal{V}_n$. A certain number of images of fundamental segments $I^n_{\xi_0}$ and $I^{n-1}_{\xi_0}$, under iterates of $f$, cover whole the circle without overlapping beyond the endpoints and form the $n$th dynamical partition of the circle

$$\mathbb{P}_n := \mathbb{P}_n(\xi_0, f) = \{I^n_j := f^j(I^n_{\xi_0}), \ 0 \leq j < q_{n-1}\} \cup \{I^{n-1}_i := f^i(I^{n-1}_{\xi_0}), \ 0 \leq i < q_n\}.$$

On $\mathcal{V}_n$ we define the Poincaré map $\pi_n = (f^{q_n}, f^{q_n-1}) : \mathcal{V}_n \to \mathcal{V}_n$ as follows

$$\pi_n(\xi) = \begin{cases} f^{q_n}(\xi), & \text{if } \xi \in I^{n-1}_{\xi_0}, \\ f^{q_n-1}(\xi), & \text{if } \xi \in I^n_{\xi_0}. \end{cases}$$

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The main idea of the renormalization method is to study the behaviour of $\pi_n$ as $n \to \infty$. For this, rescaling the coordinates are usually used. Let $A_n : R \to S^1$ be an affine covering map such that $A_n([-1,0]) = I_0^n$, with $A_n(0) = \xi_{0}$ and $A_n(-1) = f^n(\xi_{0})$. Define $a_n \in R$ to be a positive number such that $A_{n-1}(a_n) = f^n(\xi_{0})$. Obviously, $A_{n-1} : [0,a_n] \to I_0^n$ and $A_{n-1} : [-1,0] \to I_0^{n-1}$. A pair of functions $(f_n,g_n) : [-1,a_n] \to [-1,a_n]$ defined by $(f_n,g_n) = A_n^{-1} \circ \pi_n \circ A_{n-1}$, is called the $n$th renormalization of $f$ with respect to $\xi_{0}$, where $A_{n-1}^{-1}$ is the inverse branch that maps $V_i$ onto $[-1,a_n]$. The sequence of renormalizations $(f_n,g_n)$ can also be defined by the action of renormalization operator $R$ on a space of commuting pairs (see [13, 17]). A commuting pair $(F, G)$ consists of two real-valued, continuous and strictly increasing functions $F$ and $G$, with $F(0) > 0$ and $G(0) < 0$, defined on $[G(0),0]$ and $[0,F(0)]$ respectively, and commuting at zero, i.e. $F(G(0)) = G(F(0))$. If $G(0) = -1$, the commuting pair is called normalized. For a commuting pair $(F, G)$ with $G(0) \neq 0$, we define its normalization $(F, G) = (\nu^{-1} \circ F \circ \nu, \nu^{-1} \circ G \circ \nu)$, where $\nu(z) = -G(0)z$. A commuting pair is called non-degenerate if $F(0) > 0$. For a normalized, non-degenerate pair $(F, G)$, we define its height $k(F,G) \in N_0 \cup \{\infty\}$ as $k$ if $F^{k}(-1) \leq 0 < F^{k+1}(0)$ and $k(F,G) = \infty$ if no such $k$ exists (note that in this case the map $F$ has a fixed point). A commuting pair is called renormalizable if its height is finite and nonzero. On the set of renormalizable commuting pairs we define a renormalization operator $R(F,G) := (F^{k} \circ G,F)$. A commuting pair $(F, G)$ is called infinitely-renormalizable if $R^{n}(F,G)$ is renormalizable for all $n \in N_0$. It is clear, if the rotation number of $f$ is irrational, then $R(f_n,g_n) = (f_{n+1},g_{n+1})$ for all $n \in N_0$, where $f_0 := L \mid (-1,0]$ is the restriction of a lift $L$ of $f$ to $[-1,0]$ satisfying $L(0) \in (0,1]$ and $g_0 : x \to x - 1$ on $[0, L(0)]$.

### 3. Statements of the results

#### 3.1. Obtained results

In this paragraph, we state the results of Khanin and Vul [29] and Khanin and Teplinsky [17], asserting the approximations of $(f_n,g_n)$ with Möbius transformations and the convergence of renormalizations of two circle maps, respectively. Define two Möbius transformations $F_n := F_{a_n,b_n,m_n}$ and $G_n := G_{a_n,b_n,m_n}$, as follows

$$F_n(z) = \frac{a_n + (a_n + b_n m_n)z}{1 + (1 - m_n)z}, \quad \quad G_n(z) = \frac{-a_n c_n + (c_n - b_n m_n)z}{a_n c_n + (m_n - c_n)z}$$

where

$$c_n = c^{(-1)^n}, \quad b_n = \frac{|n| - |q_n|}{|n|^{q_n-1}}, \quad m_n = \exp \left( (-1)^n \sum_{i=0}^{q_n-1} \int_{f_i}^{f_{i+1}} \frac{f''(x)}{2f'(x)} \, dx \right).$$

The following theorem was proved by Khanin and Vul [29].

**Theorem 3.1.** Let $f$ be a $C^{2+\nu}(S^1 \setminus \{\xi_0\})$, $\nu > 0$ diffeomorphism with a break point $\xi_0$ and with irrational rotation number. There exist constants $C > 0$ and $0 < \lambda = \lambda(f) < 1$ such that

$$\|f_n - F_n\|_{C^2([-1,0])} \leq C \lambda^n, \quad \|g_n - G_n\|_{C^2([-1,0])} \leq C \lambda^n$$

and

$$\|g''_n - G''_n\|_{C^1([-a_n,0])} \leq \frac{C \lambda^n}{a_n}. $$
Moreover,
\[ |a_n + b_n m_n - c_n| \leq C a_n \lambda^n. \]

**Remark.**
- Note that, the constant \( \lambda \) in theorem 3.1 can be chosen independently of \( f \), (i.e. universal) depending on \( c \) and \( \nu \) (see [13]).
- Recently, Begmatov et al. [3] have studied the renormalizations of circle diffeomorphisms with breaks satisfying KO conditions. They have proved that the renormalizations of a circle diffeomorphism with a break, satisfying KO conditions are approximated by Möbius transformations in \( C^{1+\epsilon}\)-topology, that is, \( f \) is approximated in \( C^{1}\)-topology and \( f'' \) is approximated in \( L_1\)-topology. It is interesting to note that, the arguments in [3] use considerations from the theory of martingals.
- Cunha and Smania [4] have studied Rauzy-Veech renormalizations of \( C^{2+\nu}\)-circle diffeomorphisms with several break points. The main idea of their work is to consider the piecewise-smooth circle diffeomorphisms as generalized interval exchange transformations. They have showed that Rauzy-Veech renormalizations of \( C^{2+\nu}\)-generalized interval exchange maps satisfying certain combinatorial conditions are approximated by Möbius transformations in \( C^2\)-topology.

**Half-bounded rotation numbers.** Let \( f \) be a circle diffeomorphism with a break of size \( c \) and rotation number \( \rho \). Denote by \( M_o \) and \( M_e \) the class of all irrational rotation numbers \( \rho = [k_1, k_2, ...] \), for which the subsequence of partial quotients \( k_n \) with odd and with even indices \( n \) respectively are bounded, i.e.
\[ M_o = \{ \rho : (\exists C > 0) (\forall m \in \mathbb{N}) k_{2m-1} \leq C \}, \quad M_e = \{ \rho : (\exists C > 0) (\forall m \in \mathbb{N}) k_{2m} \leq C \}. \]

Note that, the sets of rotation numbers \( M_o \) and \( M_e \) were defined by Khanin and Teplinsky in [17] and the following theorem was proved there.

**Theorem 3.2.** Let \( f \) and \( \tilde{f} \) be two \( C^{2+\nu}\)-smooth circle diffeomorphisms with breaks of the same size \( c \) and the same rotation number \( \rho \in M_e \) in case of \( c > 1 \), or \( \rho \in M_o \) in case of \( 0 < c < 1 \). There exist constants \( C = C(f, \tilde{f}) > 0 \) and \( \mu \in (0, 1) \) such that
\[ \| f^n - \tilde{f}^n \|_{C^1([-1, 0])} \leq C \mu^n. \]

**3.2. Main results**

In this paragraph we state our main theorems. Define a new class of circle diffeomorphisms with breaks as follows. Consider the function \( Z_\gamma : [0, 1) \to (0, +\infty) \)
\[ Z_\gamma(x) = \frac{1}{(\log \frac{1}{x})^\gamma}, \quad x \in (0, 1) \]
and \( Z_\gamma(0) = 0 \), where \( \gamma > 0 \). Let \( f \) be a circle diffeomorphism with break point \( \xi_0 \). Denote by \( \Delta^2 f''(\xi, \tau) \) the second symmetric difference of \( f'' \), that is
\[ \Delta^2 f''(\xi, \tau) = f''(\xi + \tau) + f''(\xi - \tau) - 2f''(\xi) \]
where \( \xi \in S^1 \setminus \{ \xi_0 \} \) and \( \tau \in [0, \frac{1}{2}] \). Suppose that there exists a constant \( C > 0 \) such that
\[ \|\Delta^2 f'(\xi, \tau)\|_{L^\infty(B^1)} \leq C \tau Z_\gamma(\tau). \]  

(2)

Note that the class of real functions satisfying (2) with \( Z_\gamma(\tau) \) replaced by 1, is called the Zygmund class (see [33, p 43]). The Zygmund class was applied to the theory of circle homeomorphisms for the first time by Jun Hu and Sullivan [9, 27]. They extended the classical Denjoy’s theorem to this class. Note that if \( f' \) satisfies (2) then it is not necessarily of bounded variation and vice versa (see [24, 33]).

In this work we study the class of circle diffeomorphisms \( f \) with break point \( \xi_0 \), whose derivatives \( f' \) have bounded variation and satisfy the inequality (2). We denote this class by \( D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\}) \). Let \( f \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\}) \) and its rotation number is irrational. Define two Möbius transformations similarly as in (1) as follows

\[
\tilde{F}_n(z) = \frac{a_n + (a_n + b_n \tilde{m}_n)z}{1 + (1 - m_n)z}, \quad \tilde{G}_n(z) = \frac{-a_n \tilde{m}_n + (\tilde{m}_n - b_n)z}{a_n \tilde{m}_n + (1 - m_n)z}
\]

(3)

where

\[
\tilde{m}_n = \exp\left( \sum_{i=0}^{q_n-1} f'(\xi_i - f'(\xi_{i+q_n-1}) \right), \quad \tilde{m}_n = \exp\left( \sum_{j=0}^{q_n-1} f'(\xi_{i+q_n} - f'(\xi_i) \right)
\]

and the points \( \xi_i, \xi_{i+q_n-1} \) and \( \xi_i, \xi_{i+q_n} \) are endpoints of the intervals \( I_{i-1}^n, I_i^n \) respectively. Our first main result is the following.

**Theorem 3.3.** Let \( f \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\}) \) and \( \gamma \in (0, 1] \). Suppose the rotation number of \( f \) is irrational. There exists a constant \( C = C(f) > 0 \) and a natural number \( N_0 = N_0(f) \) such that

\[
\|f_n - \tilde{F}_n\|_{C^1([-1, 0])} \leq C \frac{n^\gamma}{n}, \quad \|g_n - \tilde{G}_n\|_{C^1([0, a_n])} \leq C \frac{n^\gamma}{n}
\]

(4)

for all \( n \geq N_0 \).

Note that the class \( D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\}) \) will be ‘better’ when \( \gamma \) increases. This gives more opportunities to better understand the behavior of \( (f_n, g_n) \). Further, we consider the case \( \gamma > 1 \). In this case, because of theorem 4.3 in section 2, \( f \in C^2(S^1 \setminus \{\xi_0\}) \) hence \( (f_n, g_n) \in C^2([-1, a_n] \setminus \{0\}) \). Our second result is the following.

**Theorem 3.4.** Let \( f \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\}) \) and \( \gamma > 1 \). Suppose the rotation number of \( f \) is irrational. There exists a constant \( C = C(f) > 0 \) and a natural number \( N_0 = N_0(f) \) such that

\[
\|f_n - F_n\|_{C^1([-1, 0])} \leq C \frac{n^\gamma}{n^{\gamma-1}}, \quad \|g_n - G_n\|_{C^1([0, a_n])} \leq C \frac{n^{\gamma-1}}{a_n n^{\gamma-1}}
\]

(5)

\[
\|f''_n - F''_n\|_{C^0([-1, 0])} \leq C \frac{a_n}{n^{\gamma-1}}, \quad \|g''_n - G''_n\|_{C^0([0, a_n])} \leq C \frac{a_n}{n^{\gamma-1}}
\]

(6)

and

\[
|a_n + b_n m_n - c_n| \leq C a_n \frac{n^{\gamma-1}}{n^{\gamma}}
\]

(7)

for all \( n \geq N_0 \), where \( F_n \) and \( G_n \) are defined in (1).
Further, consider two circle diffeomorphisms \( f, \tilde{f} \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\}) \) for \( \gamma > 1 \). Our third result is the following.

**Theorem 3.5.** Let \( f, \tilde{f} \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\}) \) and \( \gamma > 1 \). Assume that \( f \) and \( \tilde{f} \) have the same break size \( c \) and the same rotation number \( \rho \in M_e \) in the case of \( c > 1 \), or \( \rho \in M_o \) in the case of \( 0 < c < 1 \). There exists a constant \( C = C(f, \tilde{f}) > 0 \) and a natural number \( N_0 = N_0(f, \tilde{f}) \) such that

\[
\|f_n - \tilde{f}_n\|_{C^\gamma([-1,0])} \leq \frac{C}{n^\gamma}, \quad \|f_n'' - \tilde{f}_n''\|_{C^0([-1,0])} \leq \frac{C}{n^{\gamma-1}},
\]

for all \( n \geq N_0 \).

**Remark.**

- It is obvious that the considered classes of diffeomorphisms in our main theorems are wider than the class of \( C^{\gamma+\nu}(S^1 \setminus \{\xi_0\}) \) diffeomorphisms in theorems 3.1 and 3.2, however the rates of approximations of our theorems are not exponential.
- The Zygmund condition is quite natural in the context of cross-ration distortion (CRD). The relations between Zygmund conditions and CRD estimates have been studied in [24] (chapter IV) for \( \gamma = 1 \). Since ratio distortion (RD) is a partial case of CRD, the approaches in [24] work very well to estimate RD for the considered Zygmund class. On the other hand, the renormalizations \((f_n, g_n)\) can be represented by RD, therefore in the proofs of theorems 3.3 and 3.4 we use RD methods rather than CRD methods.
- The proof of theorem 3.5 follows closely that of [13, 17].

The structure of paper is as follows. In section 4, we provide brief facts about Zygmund functions and following Khanin and Vul [29] we define a relative coordinate of an interval. Then we obtain some estimates for the distortion of intervals. Moreover, we provide relations between distortion and relative coordinates of intervals. In section 5, we get estimates for the ratio of \( f^{\nu} \)-distortion of intervals i.e. distortion of intervals with respect to \( f^{\nu} \) for the different \( \gamma \)s. In section 6, we compare the relative coordinates with Möbius transformations. Finally, in section 7 we prove the main theorems.

### 4. Ratio distortions and Zygmund condition

#### 4.1. Notes on Zygmund functions

In this paragraph we study the functions satisfying inequality (2). Let \( I = [a, b] \) be an interval with length less than 1. Consider a continuous function \( K : I \to \mathbb{R} \) satisfying the inequality (2) on \( I \) i.e.

\[
\|\Delta^2 K(\cdot, \tau)\|_{L^\infty([a,b])} \leq C \tau Z_\gamma(\tau),
\]

where \( \tau \in [0, |I|/2] \). The following theorems will be used in the proof of main theorems.

**Theorem 4.1.** Let \( K : I \to \mathbb{R} \) be continuous and satisfy the inequality (8) on \( I \). If \( \gamma \in (0, 1) \) then

\[
\omega(\delta, K) = O\left(\delta (\log \frac{1}{\delta})^{1-\gamma}\right).
\]
If $\gamma = 1$ then
\[
\omega(\delta, K) = O\left(\delta(\log \log \frac{1}{\delta})\right)
\]
where $\omega(\cdot, K)$ is the modulus of continuity of $K$.

Proof. The proof is similar to the proof of theorem 3.4 in [33].

The following theorem was proved by Weiss and Zygmund in [30]. This theorem will be used in the proof of next theorem.

Theorem 4.2. Let $K : \mathbb{R} \to \mathbb{R}$ be $2\pi$-periodic and satisfies (8) for some $\gamma \in \left(\frac{1}{2}, 1\right]$. Then $K$ is absolute continuous and $K \in L^p[0, 2\pi]$ for every $p > 1$.

In this theorem the assumption $\gamma \in \left(\frac{1}{2}, 1\right]$ is crucial. The theorem is false for $\gamma \in \left(0, \frac{1}{2}\right]$ but almost nowhere differentiable (see e.g. [33]). Next consider the case $\gamma > 1$. To state the next theorem we define the following function
\[
S_\gamma(x) = \sum_{n=1}^{\infty} Z_\gamma(x 2^{-n}) \quad \text{where } x \in (0, 1) \quad \text{and } \gamma > 1.
\]
(9)

It is clear that $S_\gamma$ is continuous and $\lim_{x \to 0} S_\gamma(x) = 0$.

Theorem 4.3. Let $K : I \to \mathbb{R}$ be continuous and satisfies (8) for some $\gamma > 1$. Then $K \in C^1(I)$ and the modulus of continuity of $K'$ is $S_\gamma$.

Proof. We give only the sketch of proof; the details will be left to the reader. According to theorem 4.2 the function $K$ is at least absolute continuous on $I$ in the case of $\gamma > 1$, Hence $K'$ exists almost everywhere and $K$ is an indefinite integral of $K'$. To prove the theorem we take any points $\xi, \eta \in I$ that are Lebesgue points of $K'$ and using the same manner as in the proof of theorem 3.4 in [33], one can show that
\[
|K'(\xi) - K'(\eta)| \leq C \cdot S_\gamma(|\xi - \eta|).
\]
Thus, it can be continuously extended to whole interval $I$.

4.2. The distortion of interval and relative coordinate

In this paragraph we introduce the distortion of interval $I = [a, b]$ with respect to continuous and monotone function $K : I \to \mathbb{R}$. We obtain some estimates for the distortion of intervals. After that following Vul and Khanin [29], we define the relative coordinate of intervals and provide relations between distortion and relative coordinate. These estimates and relations will be used in the proofs of main theorems. The distortion of the interval $I$ with respect to $K$ is
\[
D(I; K) = \frac{|K(I)|}{|I|}.
\]
The distortion is multiplicative with respect to composition. Henceforth, take any $x \in [a, b]$ and consider the distortions
\[
D_a(x) := D([a, x]; K) \quad \text{and} \quad D_b(x) := D([x, b]; K).
\]
(10)
Below we study the distortions $D_a(x)$ and $D_b(x)$ as the functions of $x \in [a, b]$. Consider the following function $\Omega : (0, 1) \times (0, +\infty) \to \mathbb{R}$,
\[
\Omega(\delta, \gamma) = \begin{cases} 
\delta(\log \frac{1}{2})^{1-\gamma} & \text{if } (\delta, \gamma) \in (0,1) \times (0,1); \\
\delta(\log \log \frac{1}{2}) & \text{if } (\delta, \gamma) \in (0,1) \times \{1\}; \\
\delta & \text{if } (\delta, \gamma) \in (0,1) \times (1, +\infty).
\end{cases}
\]

In fact the function \( \Omega(\delta, \gamma) \) is the modulus of continuity of the functions satisfying relation (8) for the different cases of \( \gamma \). Denote by \( D^{1+\frac{Z_1}{2}}(I) \) the class of diffeomorphisms \( K \) whose derivatives \( K' \) satisfy the inequality (8) on \( I \). The following lemmas will be used in the proof of main theorems.

**Lemma 4.4.** Let \( K \in D^{1+\frac{Z_1}{2}}(I) \) and \( \gamma \in (0, +\infty) \). We have
\[
\frac{D_a(x)}{D_b(x)} - 1 = \frac{K'(a) - K'(b)}{2K'(b)} + \mathcal{O}\left(|I| \cdot Z_\gamma(|I|) + |K'(a) - K'(b)| \cdot \Omega(|I|, \gamma)\right).
\]

**Proof.** Since \( K' \) satisfies (8), one can easily show that
\[
D_a(x) = \frac{K'(x) + K'(a)}{2} + \mathcal{O}\left(|I| \cdot Z_\gamma(|I|)\right), \quad D_b(x) = \frac{K'(b) + K'(x)}{2} + \mathcal{O}\left(|I| \cdot Z_\gamma(|I|)\right).
\]

These relations imply
\[
\frac{D_a(x)}{D_b(x)} - 1 = \frac{K'(a) - K'(b)}{K'(b) + K'(x)} + \mathcal{O}\left(|I| \cdot Z_\gamma(|I|)\right).
\]

It is obvious
\[
\frac{K'(a) - K'(b)}{K'(b) + K'(x)} = \frac{K'(a) - K'(b)}{2K'(b)} + |K'(a) - K'(b)| \cdot \mathcal{O}\left(|K'(x) - K'(b)|\right).
\]

Hence, the claim of lemma follows from theorems 4.1 and 4.3. \( \Box \)

Note that due to theorem 4.3 the function \( K' \) is differentiable in the case of \( \gamma > 1 \). Hence we have the following

**Corollary 4.5.** Let \( K \in D^{1+\frac{Z_1}{2}}(I) \) for some \( \gamma \in (1, +\infty) \). We have
\[
\frac{D_a(x)}{D_b(x)} - 1 = - \int_a^b \frac{K''(y)}{2K'(y)} dy + \mathcal{O}\left(|I| \cdot Z_\gamma(|I|) + |K'(a) - K'(b)| \cdot \Omega(|I|, \gamma)\right).
\]

The relative coordinate \( z \) defined as \( x \mapsto \frac{b-x}{b-a} \) plays an important role in the proof of main theorems.

**Lemma 4.6.** Let \( K \in D^{1+\frac{Z_1}{2}}(I) \) and \( \gamma \in (0, +\infty) \). We have
\[
(x - a)(b - x) \left( \frac{D_a(x) - D_a(x)}{b - a} \right) = \frac{1}{2} \left( zK'(a) + (1 - z)K'(b) - K'(x) \right) + \mathcal{O}\left(|I| \cdot Z_\gamma(|I|)\right).
\]

**Proof.** By differentiating \( D_a \) and \( D_b \) we obtain
\[
D_a'(x) = \frac{K'(x) - D_a(x)}{x - a} \quad \text{and} \quad D_b'(x) = \frac{D_a(x) - K'(x)}{b - x}.
\]
Applying the same arguments as in the proof of lemma 4.4, one can show
\[ \mathcal{D}_b'(x) = \frac{1}{2} \frac{\kappa'(x) - \kappa'(a)}{x - a} + \frac{1}{1 - z} O \left( \mathcal{Z}_z(|I|) \right), \quad \mathcal{D}_b''(x) = \frac{1}{2} \frac{\kappa''(b) - \kappa'(x)}{b - x} + \frac{1}{z} O \left( \mathcal{Z}_z(|I|) \right). \]

Hence
\[ (x - a)(b - x) \left( \frac{\mathcal{D}_b'(x) - \mathcal{D}_b'(z)}{b - a} \right) = \frac{1}{2} \left( z\kappa'(a) + (1 - z)\kappa'(b) - \kappa'(x) \right) + O \left( |I| \cdot \mathcal{Z}_z(|I|) \right). \]

Lemma 4.6 is proved. □

Next we estimate the expression in the right-hand side of equation (11). For this we define the following function $T_\gamma : [0, 1/2] \times (0, 1) \to \mathbb{R}$ as
\[ T_\gamma(s, t) = s \int_s^1 \frac{\mathcal{Z}_\gamma(xt)dx}{x} + \int_0^s \mathcal{Z}_\gamma(xt)dx \quad \text{if} \quad s \in (0, 1/2] \]
and $T_\gamma(0, t) = 0$, for any $t \in (0, 1)$.

**Lemma 4.7.** Let $\mathcal{K} \in D^{1+\gamma}(\mathbb{R})$ and $\gamma \in (0, +\infty)$. There exists a constant $C > 0$ such that
\[ |\kappa'(a) + (1 - z)\kappa'(b) - \kappa'(x)| \leq C \begin{cases} |I| \cdot T_\gamma(z, |I|) & \text{if} \quad z \in [0, \frac{1}{2}]; \\ |I| \cdot T_\gamma(1 - z, |I|) & \text{if} \quad z \in (\frac{1}{2}, 1]. \end{cases} \]

**Proof.** Let us consider the function $\kappa(z) := \kappa'(b + z(a - b))$, $z \in [0, 1]$. It is clear that, to prove the lemma we have to estimate $|1 - z|\kappa(0) + z\kappa(1) - \kappa(z)|$. Since $\mathcal{K}'$ satisfies (8) we have
\[ |\frac{1}{2}\kappa'(\xi + \tau) + \frac{1}{2}\kappa'(\xi - \tau) - \kappa'(\xi)| \leq C \cdot |\tau| \mathcal{Z}_\gamma(|\tau||I|), \tag{12} \]
for all $\xi + \tau, \xi - \tau \in [0, 1]$. Denote by $\mathcal{D}_\ell, \ell = 0, 1, 2, \ldots$ the dyadic partition of $[0, 1]$. We take any $z \in (0, 1)$ and fix it. Denote by $J_\ell = [a_\ell, b_\ell)$ the dyadic interval of $\mathcal{D}_\ell$ which contains $z$. Define the following quantities
\[ t_\ell = \frac{b_\ell - z}{b_\ell - a_\ell} \kappa(a_\ell) + \frac{z - a_\ell}{b_\ell - a_\ell} \kappa(b_\ell) - \kappa(z), \quad t_\ell = \frac{1}{2} \kappa(a_\ell) + \frac{1}{2} \kappa(b_\ell) - \kappa(\frac{a_\ell + b_\ell}{2}). \]

After an easy computation we get
\[ |t_{\ell+1} - t_\ell| = \frac{\text{dist}(z, \partial J_\ell)}{|J_{\ell+1}|} t_\ell. \]

This implies
\[ |t_0| \leq \sum_{\ell=0}^{\infty} \frac{\text{dist}(z, \partial J_\ell)}{|J_{\ell+1}|} |t_\ell|. \]

There are two possibilities for $z$; either $z \in (0, \frac{1}{2})$ or $z \in (\frac{1}{2}, 1)$. Consider the first case. Let $m$ be the biggest natural number such that $z \leq 2^{-m}$. It is easy to verify that...
\[ \text{dist}(z, \partial(J_\ell)) \leq \begin{cases} z, & \text{if } \ell \leq m; \\ |J_\ell|, & \text{if } \ell > m. \end{cases} \]

Similarly, in the second case we define the biggest natural number \( p \) such that \( 1 - z \leq 2^{-p} \). One can easily verify that

\[ \text{dist}(z, \partial(J_\ell)) \leq \begin{cases} 1 - z, & \text{if } \ell \leq p; \\ |J_\ell|, & \text{if } \ell > p. \end{cases} \]

Consequently

\[ |\tau_0| \leq \begin{cases} \sum_{\ell=0}^{m} 2^{\ell+1}|t_\ell| + 2 \sum_{\ell=m+1}^{\infty} |t_\ell|, & \text{if } z \in (0, \frac{1}{2}]; \\ (1 - z) \sum_{\ell=0}^{m} 2^{\ell+1}|t_\ell| + 2 \sum_{\ell=p+1}^{\infty} |t_\ell|, & \text{if } z \in (\frac{1}{2}, 1). \end{cases} \]

Using inequality (12) we get

\[ |\tau_0| \leq C \begin{cases} |I| \left( \sum_{\ell=0}^{m} Z_\gamma(2^{-\ell-1}|I|) + \sum_{\ell=m+1}^{\infty} 2^{-\ell} Z_\gamma(2^{-\ell-1}|I|) \right), & \text{if } z \in (0, \frac{1}{2}]; \\ |I| \left( (1 - z) \sum_{\ell=0}^{m} Z_\gamma(2^{-\ell-1}|I|) + \sum_{\ell=p+1}^{\infty} 2^{-\ell} Z_\gamma(2^{-\ell-1}|I|) \right), & \text{if } z \in (\frac{1}{2}, 1). \end{cases} \]

It is obvious that if \( z \in (0, \frac{1}{2}] \) then

\[ \sum_{\ell=0}^{m} Z_\gamma(2^{-\ell-1}|I|) + \sum_{\ell=m+1}^{\infty} 2^{-\ell} Z_\gamma(2^{-\ell-1}|I|) \leq C \cdot T_\gamma(z, |I|). \]

Similarly, if \( z \in (\frac{1}{2}, 1) \) then

\[ (1 - z) \sum_{\ell=0}^{m} Z_\gamma(2^{-\ell-1}|I|) + \sum_{\ell=p+1}^{\infty} 2^{-\ell} Z_\gamma(2^{-\ell-1}|I|) \leq C \cdot T_\gamma(1 - z, |I|). \]

Thus

\[ |\tau_0| \leq C \begin{cases} |I| \cdot T_\gamma(z, |I|), & \text{if } z \in (0, \frac{1}{2}]; \\ |I| \cdot T_\gamma(1 - z, |I|), & \text{if } z \in (\frac{1}{2}, 1). \end{cases} \]

On the other hand

\[ |(1 - z)\kappa(0) + z\kappa(1) - \kappa(z)| = |\tau_0|. \]

Hence, we have proved lemma 4.7 for \( z \in (0, 1) \). For \( z \in \{0, 1\} \) the claim of lemma is obvious. \( \square \)

It is easy to see that the function \( T_\gamma(z, |I|) \) is an increasing function of \( z \) on \([0, \frac{1}{2}]\). Hence the function \( T_\gamma(1 - z, |I|) \) is a decreasing function of \( z \) on \([\frac{1}{2}, 1]\). Therefore \( T_\gamma(z, |I|) \leq T_\gamma(\frac{1}{2}, |I|) \) for all \( z \in [0, \frac{1}{2}] \) and \( T_\gamma(1 - z, |I|) \leq T_\gamma(\frac{1}{2}, |I|) \) for all \( z \in [\frac{1}{2}, 1] \). Moreover, if the length of interval \( I \) is sufficiently small then it can be easily shown that
Thus, lemmas 4.6 and 4.7 imply the following.

**Corollary 4.8.** Let \( K \in D^{1+Z}_\gamma(I) \) and \( \gamma \in (0, +\infty) \). If the length of interval \( I \) is sufficiently small then we have

\[
(x - a)(b - x) | \frac{D_b'(x) - D_a'(x)}{b - a} | = O \left( |I| Z_\gamma(|I|) \right).
\]

Next, consider the subcase \( \gamma \in (1, +\infty) \). Due to theorem 4.3, \( K' \) is differentiable, therefore \( D_b'' \) and \( D_a'' \) are differentiable also.

**Lemma 4.9.** Let \( K \in D^{1+Z}_\gamma(I) \) and \( \gamma \in (1, +\infty) \). There exists a constant \( C > 0 \) such that

\[
| (x - a)(b - x) \left( D_b''(x) - D_a''(x) \right) | \leq C \cdot |I| S_\gamma(|I|),
\]

\[
| D_b'(x) - D_a'(x) | \leq C \cdot S_\gamma(|I|)
\]

where function \( S_\gamma \) is defined in (9).

**Proof.** The proof lemma follows from easy computations and theorem 4.3. \( \square \)

### 5. Estimates for the ratio of \( f^n \)-distortions

In this section we first define relative coordinates on the intervals of dynamical partition \( P_n \) and the ratio of \( f^n \)-distortions i.e. distortions of intervals with respect to \( f^n \). Then we describe the ratio of \( f^n \)-distortions by initial relative coordinates and provide estimates for these descriptions and their derivatives. Note that the relative coordinates on intervals of dynamical partition \( P_n \) were introduced and very well investigated by Sinai and Khanin in [15]. Here and in the next sections we always assume the rotation number is irrational. We introduce the relative coordinates \( z_i : I_i^{n-1} \rightarrow [0, 1] \) for all \( 0 \leq i \leq q_n \) and \( z_j : I_j^n \rightarrow [0, 1] \) for all \( 0 \leq j \leq q_{n-1} \), by the formulae respectively:

\[
z_i = \frac{\xi_i - x}{\xi_i - \xi_{i+q_n-1}}, \quad x \in I_i^{n-1} \quad \text{and} \quad z_j = \frac{\xi_j + q_n - y}{\xi_j + q_n - \xi_j}, \quad y \in I_j^n.
\]

Here and below, we discuss only the case where \( n \) is even; the case where \( n \) is odd is obtained by reversing the orientation of \( S^1 \). To formulate the next lemmas, we introduce some further notations

\[
\tilde{T}_n(x) = \log \frac{D([\xi_{0}; f^n])}{D([\xi_{0}; f^n])} + \log \bar{m}_n, \quad x \in I_0^{n-1}.
\]

\[
\tilde{T}_n(x) = \log \frac{D([\xi_{0}; f^n])}{D([\xi_{0}; f^n])} + \log \bar{m}_n, \quad x \in I_0^n.
\]

To abbreviate the notations, let us denote

\[
\alpha_i := \xi_{i+q_n-1}, \quad \beta_i := \xi_i \quad \text{and} \quad \gamma_i := f^i(x) \in I_i^{n-1}, \quad 0 \leq i \leq q_n.
\]
\[
\bar{\alpha}_j := \xi_j, \quad \bar{\beta}_j := \xi_j + q_j, \quad \text{and} \quad y_j := f^j(y) \in I^n, \quad 0 \leq j \leq q_n - 1
\]

and

\[
\tilde{T}_n(z_0) := \tilde{T}_n(\beta_0 + z_0(\alpha_0 - \beta_0)), \quad \tilde{T}_n(\bar{z}_0) := \tilde{T}_n(\bar{\beta}_0 + \bar{z}_0(\bar{\alpha}_0 - \bar{\beta}_0)).
\]

Below we estimate \(\tilde{T}_n\) and \(\hat{T}_n\). These estimates will be used in the next section to approximate relative coordinates with Möbius transformations.

**Lemma 5.1.** Let \(f \in D^{1+\mathcal{Z}_{\gamma}(S^1 \setminus \{\xi_0\})}\) and \(\gamma \in (0, +\infty)\). There exists a constant \(C > 0\) such that

\[
\max_{z_0 \in [0, 1]} |\tilde{T}_n(z_0)| \leq \frac{C}{n^\gamma}, \quad \max_{z_0 \in [0, 1]} |\hat{T}_n(z_0)| \leq \frac{C}{n^\gamma}
\]

for all \(n \geq 1\).

**Proof.** We prove only the first inequality; the second inequality can be proved analogously. Since the ratio distortion is multiplicative with respect to composition, we have

\[
\tilde{T}_n(z_0) = \sum_{i=0}^{q_n-1} \log \frac{D_{\alpha_i}(x_i)}{D_{\beta_i}(x_i)} + \log \tilde{m}_n.
\]

On the other hand, by lemma 4.4 we get

\[
\sum_{i=0}^{q_n-1} \log \frac{D_{\alpha_i}(x_i)}{D_{\beta_i}(x_i)} = -\log \tilde{m}_n + \mathcal{O}\left(\mathcal{Z}_{\gamma}(d_n-1) + \Omega(d_n-1, \gamma)\right)
\]

where \(d_n = \|f^n - \text{Id}\|_{C^1}\). It is well known that \(d_n = \mathcal{O}(\lambda^n)\) (see [18]) and this implies

\[
\mathcal{Z}_{\gamma}(d_n-1) + \Omega(d_n-1, \gamma) = \mathcal{O}\left(\frac{1}{n^\gamma}\right).
\]

Thus

\[
\max_{z_0 \in [0, 1]} |\tilde{T}_n(z_0)| \leq \frac{C}{n^\gamma}.
\]

Lemma 5.1 is proved.

The following estimates will be used in the next section to approximate relative coordinates with Möbius transformations in \(C^1\)-topology.

**Lemma 5.2.** Let \(f \in D^{1+\mathcal{Z}_{\gamma}(S^1 \setminus \{\xi_0\})}\) and \(\gamma \in (0, +\infty)\). There exists a constant \(C > 0\) and a natural number \(N_0 = N_0(f)\) such that

\[
\max_{z_0 \in [0, 1]} |\tilde{T}_n(z_0)| \leq \frac{C}{n^\gamma}, \quad \max_{z_0 \in [0, 1]} \left|\tilde{T}_n(z_0)\frac{d\tilde{T}_n(z_0)}{dz_0}\right| \leq \frac{C}{n^\gamma}
\]

for all \(n \geq N_0\).

**Proof.** We prove the first inequality, the second one can be proved analogously. Setting

\[
\Psi(x_i) := \log \frac{D_{\alpha_i}(x_i)}{D_{\beta_i}(x_i)}
\]

(17)
rewrite $\tilde{Y}_n$ as follows
\[ \tilde{Y}_n(z_0) = \sum_{i=0}^{q_n-1} \Psi(x_i) + \log \tilde{m}_n. \] (18)

It is clear
\[ \frac{d\tilde{Y}_n(z_0)}{dz_0} = (\alpha_0 - \beta_0) \cdot \frac{d\tilde{Y}_n(x)}{dx} \quad \text{and} \quad \frac{d\Psi(x_i)}{dx} = \frac{d\Psi(x_i)}{dx} \cdot (f'(x))'. \] (19)

Using Finzi’s inequality (see [1]) we obtain
\[ e^{-\nu} \leq \frac{(f'(x))' (\alpha_0 - \beta_0)}{(\alpha_i - \beta_i)} \leq e^{\nu} \quad \text{and} \quad e^{-2\nu} \leq \frac{z_0 (1 - z_0)}{z_i (1 - z_i)} \leq e^{2\nu}. \] (20)

Relations (18)–(20) imply
\[ \left| 20 (1 - z_0) \frac{d\tilde{Y}_n(z_0)}{dz_0} \right| \leq e^{\nu} \left| \sum_{i=0}^{q_n-1} z_i (1 - z_i) (\alpha_i - \beta_i) \frac{d\Psi(x_i)}{dx_i} \right|. \] (21)

By differentiating (17) we get
\[ \frac{d\Psi(x_i)}{dx_i} = \left( \frac{1}{\mathcal{D}_{\alpha_i}(x_i)} - \frac{1}{\mathcal{D}_{\beta_i}(x_i)} \right) \cdot \frac{d\mathcal{D}_{\alpha_i}(x_i)}{dx_i} + \frac{1}{\mathcal{D}_{\beta_i}(x_i)} \left( \frac{d\mathcal{D}_{\alpha_i}(x_i)}{dx_i} - \frac{d\mathcal{D}_{\beta_i}(x_i)}{dx_i} \right). \] (22)

Utilizing the mean value theorem and theorems 4.1 and 4.3 we get
\[ \left| \frac{1}{\mathcal{D}_{\alpha_i}(x_i)} - \frac{1}{\mathcal{D}_{\beta_i}(x_i)} \right| = \left| \frac{1}{f'(\tilde{\alpha}_i)} - \frac{1}{f'(\tilde{\beta}_i)} \right| \leq C \cdot \Omega(d_{n-1}, \gamma) \] (23)
for any $\gamma \in (0, +\infty)$, where $\tilde{\alpha}_i \in [\alpha_i, x_i]$ and $\tilde{\beta}_i \in [x_i, \beta_i]$. Using (23), we estimate $|\frac{d\Psi(x_i)}{dx_i}|$ as
\[ \left| \frac{d\Psi(x_i)}{dx_i} \right| \leq C \cdot \Omega(d_{n-1}, \gamma) \left| \frac{d\mathcal{D}_{\alpha_i}(x_i)}{dx_i} \right| + \frac{1}{\inf f''(\xi)} \left| \frac{d\mathcal{D}_{\alpha_i}(x_i)}{dx_i} - \frac{d\mathcal{D}_{\beta_i}(x_i)}{dx_i} \right|. \] (24)

Applying this inequality to the right-hand side of (21) we get
\[ \left| 20 (1 - z_0) \frac{d\tilde{Y}_n(z_0)}{dz_0} \right| \leq C e^{\nu} \Omega(d_{n-1}, \gamma) \sum_{i=0}^{q_n-1} z_i (1 - z_i) |\alpha_i - \beta_i| \left| \frac{d\mathcal{D}_{\alpha_i}(x_i)}{dx_i} \right| \]
\[ + \frac{e^{3\nu}}{\inf f''(\xi)} \sum_{i=0}^{q_n-1} z_i (1 - z_i) |\alpha_i - \beta_i| \left| \frac{d\mathcal{D}_{\alpha_i}(x_i)}{dx_i} - \frac{d\mathcal{D}_{\beta_i}(x_i)}{dx_i} \right| := A_n + B_n. \] (25)

Next we estimate $A_n$ and $B_n$. To estimate $A_n$, we use the obvious equality
\[ z_i (1 - z_i) |\alpha_i - \beta_i| \left| \frac{d\mathcal{D}_{\alpha_i}(x_i)}{dx_i} \right| = z_i |f'(x_i) - f'(\tilde{\alpha}_i)| \]
and get $A_n = O\left(\Omega(d_{n-1, \gamma})\right)$, since $f'$ has bounded variation and the system of intervals $\{[x_i, q_i], 0 < i < q_i\}$ do not intersect. Next we estimate $B_n$. By the definition of $z_i$

$$z_i(1 - z_i)|\alpha_i - \beta_i| \left| \frac{d\mathbf{D}_\alpha(x_i)}{dx_i} - \frac{d\mathbf{D}_\beta(x_i)}{dx_i} \right| = \left| (x_i - \alpha_i)(\beta_i - x_i) \right| \frac{d\mathbf{D}_\alpha(x_i)}{dx_i} - \frac{d\mathbf{D}_\beta(x_i)}{dx_i}.$$

Due to corollary 4.8 we have

$$\frac{(x_i - \alpha_i)(\beta_i - x_i)}{\beta_i - \alpha_i} \left| \frac{d\mathbf{D}_\alpha(x_i)}{dx_i} - \frac{d\mathbf{D}_\beta(x_i)}{dx_i} \right| = O\left(|f_i^{n-1}| \Omega(\gamma)|f_i^{n-1}|\right)$$

for sufficiently large $n$. Hence $B_n = O\left(\Omega(d_{n-1})\right)$ and $A_n + B_n = O\left(\frac{1}{n^r}\right)$ for sufficiently large $n$. Lemma 5.2 is proved.

In the next two lemmas we estimate the first and second derivatives of $\tilde{T}_n$ and $\tilde{\gamma}_n$ in the case of $\gamma \in (1, +\infty)$. These estimates will be used in the next section to approximate relative coordinates with M"obius transformations in $C^2$-topology for $\gamma > 1$. Note that, in this case due to theorem 4.3, $f'$ is differentiable on $[\xi_0, \xi_0 + 1]$ and the modulus of continuity of $f''$ is $S_\gamma$.

**Lemma 5.3.** Let $f \in D^{1+Z_\gamma}(S^1 \setminus \{\xi_0\})$ and $\gamma \in (1, +\infty)$. There exists a constant $C > 0$ such that

$$\max_{\mathcal{Z}_\gamma} \left| \frac{d\tilde{T}_n(z_0)}{dz_0} \right| \leq \frac{C}{n^{\gamma - 1}}, \quad \max_{\mathcal{Z}_\gamma} \left| \frac{d\tilde{\gamma}_n(z_0)}{dz_0} \right| \leq \frac{C}{n^{\gamma - 1}}$$

for all $n > 1$.

**Proof.** We prove only the first inequality, the second one can be proved analogously. The same manner as in proof of lemma 5.2 one can show that

$$\left| \frac{d\tilde{T}_n(z_0)}{dz_0} \right| \leq e^n \left| \sum_{i=0}^{q_i-1} (\alpha_i - \beta_i) \frac{d\mathbf{D}_\alpha(x_i)}{dx_i} \right| \leq Ce^n \Omega(d_{n-1, \gamma}) \left| \sum_{i=0}^{q_i-1} |\alpha_i - \beta_i| \left| \frac{d\mathbf{D}_\alpha(x_i)}{dx_i} \right| \right|$$

$$+ \frac{e^n}{\inf_{\mathcal{S}_\gamma} f'(\xi)} \sum_{i=0}^{q_i-1} |\alpha_i - \beta_i| \left| \frac{d\mathbf{D}_\alpha(x_i)}{dx_i} - \frac{d\mathbf{D}_\beta(x_i)}{dx_i} \right| := C_n + D_n.$$

Since $f'$ is differentiable, it can be easily shown that

$$|\alpha_i - \beta_i| \left| \frac{d\mathbf{D}_\alpha(x_i)}{dx_i} \right| = |\alpha_i - \beta_i| \left| f'(x_i) - \mathbf{D}_\alpha(x_i) \right|$$

$$= |\alpha_i - \beta_i| \left| \int_{\alpha_i}^{x_i} f''(y) (y - \alpha_i)^2 dy \right| = O\left(|\alpha_i - \beta_i|\right).$$

This implies $C_n = O\left(\Omega(d_{n-1, \gamma})\right)$. According to lemma 4.9 we have

$$\left| \frac{d\mathbf{D}_\alpha(x_i)}{dx_i} - \frac{d\mathbf{D}_\beta(x_i)}{dx_i} \right| \leq C S_\gamma(|f_i^{n-1}|).$$
This implies \( D_n = O \left( S_{\gamma}(d_{n-1}) \right) \). It is obvious that
\[
S_{\gamma}(d_{n-1}) \leq \frac{1}{\gamma - 1} \left( \log \frac{1}{d_{n-1}} \right)^{1-\gamma}.
\]

As before, using the relation \( d_n = O(\lambda^n) \) we obtain
\[
\left( \log \frac{1}{d_{n-1}} \right)^{1-\gamma} + \Omega(d_{n-1}, \gamma) = O\left( \frac{1}{n^{\gamma-1}} \right).
\]

Hence, \( C_n + D_n = O\left( \frac{1}{n^{\gamma-1}} \right) \). Lemma 5.3 is proved. \( \square \)

**Lemma 5.4.** Let \( f \in D^{1+\varphi}(S^1 \setminus \{z_0\}) \) and \( \gamma \in (1, +\infty) \). There exists a constant \( C > 0 \) and a natural number \( N_0 = N_0(f) \) such that
\[
\max_{x \in [0,1]} \left| z_0(1 - z_0) \frac{d^2 \tilde{T}_n(z_0)}{d_n^2} \right| \leq \frac{C}{n^{\gamma-1}}, \quad \max_{x \in [0,1]} \left| z_0(1 - z_0) \frac{d^2 \tilde{T}_n(z_0)}{d_0^2} \right| \leq \frac{C}{n^{\gamma-1}}
\]
for all \( n \geq N_0 \).

**Proof.** As above we prove only the first inequality, the second one can be proved analogously. Differentiating \( \frac{d \tilde{T}_n(z_0)}{d_n} \) we find
\[
z_0(1 - z_0) \frac{d^2 \tilde{T}_n(z_0)}{d_n^2} = z_0(1 - z_0)(\alpha_0 - \beta_0)^2 \sum_{i=0}^{q_n-1} \frac{d^2 \Psi(x_i)}{d x_i^2} : \frac{\mathrm{d} x_i}{\mathrm{d} x}
\]
\[
+ z_0(1 - z_0)(\alpha_0 - \beta_0)^2 \sum_{i=0}^{q_n-1} \frac{d \Psi(x_i)}{d x_i} \cdot \frac{d^2 x_i}{d x^2} := E_n + F_n.
\]

Next we estimate \( E_n \) and \( F_n \). Using (20) we get
\[
z_0(1 - z_0)(\alpha_0 - \beta_0)^2 \left( \frac{\mathrm{d} x_i}{\mathrm{d} x} \right)^2 \leq e^{\nu z_i}(1 - z_i)(\alpha_i - \beta_i)^2
\]
for any \( 0 \leq i < q_n \). By differentiating (22) we obtain
\[
\frac{d^2 \Psi(x_i)}{d x_i^2} = -\frac{1}{\mathcal{D}_{\alpha_i}(x_i)} \cdot \left( \frac{\mathrm{d} \mathcal{D}_{\alpha_i}(x_i)}{\mathrm{d} x_i} \right)^2 + \frac{1}{\mathcal{D}_{\beta_i}(x_i)} \cdot \frac{d^2 \mathcal{D}_{\beta_i}(x_i)}{d x_i^2}
\]
\[
+ \frac{1}{\mathcal{D}_{\beta_i}(x_i)} \cdot \left( \frac{\mathrm{d} \mathcal{D}_{\beta_i}(x_i)}{\mathrm{d} x_i} \right)^2 - \frac{1}{\mathcal{D}_{\beta_i}(x_i)} \cdot \frac{d^2 \mathcal{D}_{\beta_i}(x_i)}{d x_i^2}
\]
\[
= \left( \frac{1}{\mathcal{D}_{\alpha_i}(x_i)} - \frac{1}{\mathcal{D}_{\beta_i}(x_i)} \right) \cdot \left( \frac{\mathrm{d} \mathcal{D}_{\alpha_i}(x_i)}{\mathrm{d} x_i} \right)^2 + \left( \frac{\mathrm{d} \mathcal{D}_{\beta_i}(x_i)}{\mathrm{d} x_i} \right)^2 - \left( \frac{\mathrm{d} \mathcal{D}_{\beta_i}(x_i)}{\mathrm{d} x_i} \right)^2 \right) \cdot \frac{1}{\mathcal{D}_{\beta_i}(x_i)}
\]
\[
+ \left( \frac{1}{\mathcal{D}_{\alpha_i}(x_i)} \right) \cdot \frac{d^2 \mathcal{D}_{\alpha_i}(x_i)}{d x_i^2} + \left( \frac{d^2 \mathcal{D}_{\alpha_i}(x_i)}{d x_i^2} - \frac{d^2 \mathcal{D}_{\beta_i}(x_i)}{d x_i^2} \right) \cdot \frac{1}{\mathcal{D}_{\beta_i}(x_i)} := E_n^{(1)} + E_n^{(2)} + E_n^{(3)} + E_n^{(4)}
\]
(28)
We multiply each $E^{(s)}_n$, $s = 1, 2, 3, 4$ to the right-hand side of (27) and estimate them separately. Relations (23) and (26) imply
\[ e^{4\nu} \sum_{i=0}^{q_s-1} z_i (1 - z_i) (\alpha_i - \beta_i)^2 |E^{(1)}_n| \leq C d_{n-1} \Omega(d_{n-1}, \gamma). \] (29)

From the second inequality of lemma 4.9 follows
\[ e^{4\nu} \sum_{i=0}^{q_s-1} z_i (1 - z_i) (\alpha_i - \beta_i)^2 |E^{(2)}_n| \leq C d_{n-1} \mathcal{S}_s(d_{n-1}). \] (30)

Using theorem 4.3 and the obvious equality
\[ \frac{d^2 \mathcal{D}_{\alpha_i}(x_i)}{dx_i^2} = 2 \int_{\alpha_i}^{\alpha_i} \frac{f''(x_i) - f''(y))}{(x_i - \alpha_i)^3} dy \]
we obtain
\[ |(1 - z_i)(\alpha_i - \beta_i) \frac{d^2 \mathcal{D}_{\alpha_i}(x_i)}{dx_i^2}| \leq 2 \int_{\alpha_i}^{\alpha_i} \frac{f''(x_i) - f''(y))}{(x_i - \alpha_i)^2} dy \leq C \mathcal{S}_s(|{\mathcal{J}}|^{n-1}). \]

This and inequality (23) imply
\[ e^{4\nu} \sum_{i=0}^{q_s-1} z_i (1 - z_i) (\alpha_i - \beta_i)^2 |E^{(3)}_n| \leq C \mathcal{S}_s(d_{n-1}) \Omega(d_{n-1}, \gamma). \] (31)

From the first inequality of lemma 4.9 follows
\[ e^{4\nu} \sum_{i=0}^{q_s-1} z_i (1 - z_i) (\alpha_i - \beta_i)^2 |E^{(4)}_n| \]
\[ \leq C \sum_{i=0}^{q_s-1} (\alpha_i - \beta_i)(\beta_i - x_i) \left| \frac{d^2 \mathcal{D}_{\alpha_i}(x_i)}{dx_i^2} - \frac{d^2 \mathcal{D}_{\beta_i}(x_i)}{dx_i^2} \right| \leq C \mathcal{S}_s(d_{n-1}). \] (32)

Hence the relations (27)–(32) imply the estimate $|E_n| = O \left( \frac{1}{n^{1-\gamma}} \right)$. We move on to estimate $F_n$. First we prove the following inequalities
\[ e^{-2\nu} \inf_{x \in \mathbb{R}^3} \left| \frac{f''(x)}{f'(x)} \right| \cdot |\alpha_i - \beta_i| \leq (\alpha_0 - \beta_0)^2 \left| \frac{d^2 x_i}{dx_i^2} \right| \leq e^{2\nu} \sup_{x \in \mathbb{R}^3} \left| \frac{f''(x)}{f'(x)} \right| \cdot |\alpha_i - \beta_i| \] (33)
for any $0 \leq i < q_n$. Consider the function
\[ \mathcal{H}_i(x) = \sum_{j=0}^{i-1} (f_j(x))' \]
where $x \in \mathcal{J}^{n-1}_0$ and $0 \leq i < q_n$. Using Finzi’s inequality it can be easily shown that
\[ e^{-\nu} \leq \frac{\mathcal{H}_i(x)}{\mathcal{H}_i(y)} \leq e^\nu \]  
\[ \text{for any } x, y \in \mathbb{I}_0^{-i} \text{ and } 0 \leq i < q_n. \]

On the other hand
\[ \int_{\mathbb{I}_{i-1}} \mathcal{H}_i(x) dx = \sum_{j=0}^{i-1} |f_j^{n-1}|. \]

This and inequality (34) imply
\[ e^{-\nu} \frac{1}{|f_0^{n-1}|} \sum_{j=0}^{i-1} |f_j^{n-1}| \leq \mathcal{H}_i(x) \leq e^\nu \frac{1}{|f_0^{n-1}|} \sum_{j=0}^{i-1} |f_j^{n-1}|. \]

We find
\[ \frac{d^2 x_i}{dx^2} = (f'(x))' \sum_{j=0}^{i-1} \frac{f''(x_j)}{f'(x_j)} (f'(x))^j. \]

Inequalities (20) and (35) imply
\[ (\alpha_0 - \beta_0)^2 \left| \frac{d^2 x_i}{dx^2} \right| \leq \sup_{x \in \mathbb{B}} \frac{f''(x)}{f'(x)} \left| \alpha_0 - \beta_0 \right| (f'(x))^j \mathcal{H}_i(x) \leq e^{2\nu} \sup_{x \in \mathbb{B}} \left| \frac{f''(x)}{f'(x)} \right| \cdot |\alpha_i - \beta_i|. \]

This proves the right-hand side of inequalities (33), the proof of the left-hand side is similar. Now we continue estimating \( F_n \). Using (33) we get
\[ |F_n| \leq C \sum_{i=0}^{q_n-1} z_i (1 - z_i) |\alpha_i - \beta_i| \left| \frac{d\Psi(x_i)}{dx_i} \right|. \]

This and inequalities (22)–(25) imply \( |F_n| \leq C (A_n + B_n) \). As we have shown in the proof of lemma 5.2, \( A_n + B_n = \mathcal{O}\left(\frac{1}{n^\gamma} \right) \) for sufficiently large \( n \). Finally \( |E_n| + |F_n| = \mathcal{O}(\frac{1}{n^\gamma}) \) for sufficiently large \( n \). Lemma 5.4 is therefore completely proved.

6. Approximations of the relative coordinates

In this section we show that the relative coordinates \( z_{q_n} (z_0) \) and \( \tilde{z}_{q_{n-1}} (\tilde{z}_0) \) are approximated by Möbius transformations. Define a Möbius transformation \( \mathcal{M}_T \) as
\[ \mathcal{M}_T(z) = \frac{z^T}{1 + z(T - 1)}. \]

**Lemma 6.1.** Let \( f \in \mathcal{D}^{1+Z_\gamma} (\{z_0\} \setminus \{\xi_0\}) \) and \( \gamma \in (0, 1] \). There exists a constant \( C > 0 \) and a natural number \( N_0 = N_0(f) \) such that
\[ \|z_{q_n} - \mathcal{M}_{\bar{m}_n}\|_{C^1([0,1])} \leq \frac{C}{n^\gamma}, \quad \|\tilde{z}_{q_{n-1}} - \mathcal{M}_{\bar{m}_n}\|_{C^1([0,1])} \leq \frac{C}{n^\gamma}, \]
for all \( n \geq N_0 \).
Proof. We prove the first inequality. After simple computations we get

\[
\frac{1 - z_{q_n}}{z_{q_n}}, \quad \frac{z_0}{1 - z_0} = \frac{\mathcal{D}(\{\xi_{q_n}, \ldots, \xi\}; f_{q_n})}{\mathcal{D}(\{x, \xi_0\}; f_{q_0})}.
\]

On the other hand, the relation (13) implies

\[
\frac{1 - z_{q_n}}{z_{q_n}}, \quad \frac{z_0}{1 - z_0} = \frac{1}{\tilde{m}_n} \exp(\tilde{\Upsilon}_n(z_0)).
\]

Therefore, the last two relations imply

\[
\frac{1 - z_{q_n}}{z_{q_n}}, \quad \frac{z_0}{1 - z_0} = \frac{1}{\tilde{m}_n} \exp(\tilde{\Upsilon}_n(z_0)).
\]

Solving for \(z_{q_n}\), we get

\[
z_{q_n}(z_0) = \frac{z_0 \tilde{m}_n}{(1 - z_0) \exp(\tilde{\Upsilon}_n(z_0)) + z_0 \tilde{m}_n}.
\] (37)

Using lemma 5.1 we get

\[
\max_{z_0 \in [0,1]} |z_{q_n}(z_0) - \mathcal{M}_{\tilde{m}_n}(z_0)| \leq \frac{C}{n^\gamma}.
\] (38)

for all \(n \geq 1\). By differentiating (37) we obtain

\[
z'_{q_n}(z_0) = \left(1 - z_0 (1 - z_0) \tilde{\Upsilon}'_n(z_0) \frac{\tilde{m}_n \exp(\tilde{\Upsilon}_n(z_0))}{(1 - z_0) \exp(\tilde{\Upsilon}_n(z_0)) + z_0 \tilde{m}_n}^2.
\] (39)

Utilizing lemma 5.2 we get

\[
\max_{z_0 \in [0,1]} |z'_{q_n}(z_0) - \mathcal{M}'_{\tilde{m}_n}(z_0)| \leq \frac{C}{n^\gamma}.
\] (40)

for all \(n \geq N_0\). Inequalities (38) and (40) prove the first inequality of lemma 6.1. The proof of the second inequality is similar. \(\square\)

Next consider the case \(\gamma > 1\).

Lemma 6.2. Let \(f \in D^{1+Z_2}(\mathbb{S}^1 \setminus \{\xi_0\})\) and \(\gamma > 1\). There exists a constant \(C > 0\) and natural number \(N_0 = N_0(f)\) such that

\[
\|z_{q_n} - \mathcal{M}_{\tilde{m}_n}\|_{C^1([0,1])} \leq \frac{C}{n^\gamma}, \quad \|z'_{q_{n-1}} - \mathcal{M}'_{\tilde{m}_n}\|_{C^1([0,1])} \leq \frac{C}{n^\gamma}.
\] (41)

and

\[
\|z''_{q_n} - \mathcal{M}''_{\tilde{m}_n}\|_{C^1([0,1])} \leq \frac{C}{n^{\gamma-1}}, \quad \|z''_{q_{n-1}} - \mathcal{M}''_{\tilde{m}_n}\|_{C^1([0,1])} \leq \frac{C}{n^{\gamma-1}}.
\] (42)

for all \(n \geq N_0\).
Proof. To prove the lemma we first show
\[
\tilde{m}_n = m_n + O(\lambda^n) \quad \text{and} \quad \tilde{m}_n = \frac{c_n}{m_n} + O(\lambda^n).
\]
(43)

Due to theorem 4.3, \( f' \) is differentiable. This implies
\[
\left| \log \tilde{m}_n - \log m_n \right| \leq \frac{1}{2} \sum_{i=0}^{q-1} \int_{\xi_i}^{\xi_{i+1}} \frac{f''(x)}{f'(x)} \frac{f''(x)}{f'(x)} \, dx.
\]

From theorem 4.1 follows
\[
\int_{\xi_{i-1}}^{\xi_i} \frac{f''(x)}{f'(x)} \, dx \leq \frac{\Omega(d_{n-1}, \gamma)}{\inf_{\lambda \geq 1} |f'(x)|^2} \int_{\xi_{i-1}}^{\xi_i} |f''(x)| \, dx.
\]

It is clear that \( \Omega(d_{n-1}, \gamma) = O(\lambda^n) \) in case of \( \gamma > 1 \). Hence \( \tilde{m}_n = m_n + O(\lambda^n) \). Similarly it can be shown
\[
\tilde{m}_n = \exp \left( (-1)^n \sum_{j=0}^{q-1} \int_{\xi_j}^{\xi_{j+1}} \frac{f''(x)}{2f'(x)} \, dx \right) + O(\lambda^n).
\]

On the other hand
\[
m_n \cdot \exp \left( (-1)^n \sum_{j=0}^{q-1} \int_{\xi_j}^{\xi_{j+1}} \frac{f''(x)}{2f'(x)} \, dx \right) = \exp \left( (-1)^n \int_{\xi_{i-1}}^{\xi_i} \frac{f''(x)}{2f'(x)} \, dx \right) = c_n.
\]

The last two relations imply \( \tilde{m}_n = c_n m_n^{-1} + O(\lambda^n) \).

Now, relations (37), (39), (43) and lemmas 5.1 and 5.2 imply the proof of the first inequality of (41). The proof of the second one is similar. To prove the first inequality of (42), we find the explicit forms of \( z''_0 \) as follows
\[
z''_0(z_0) = \frac{\tilde{m}_n \exp(\tilde{\Upsilon}_n(z_0)) \left( \tilde{\Upsilon}'_n(z_0) \left( 2z_0 - z_0(1 - z_0) \tilde{\Upsilon}'_n(z_0) \right) - z_0(1 - z_0) \tilde{\Upsilon}''_n(z_0) \right)}{\left( 1 - z_0 \exp(\tilde{\Upsilon}_n(z_0)) + z_0 \tilde{m}_n \right)^2}
\]
\[
- \frac{2\tilde{m}_n \exp(\tilde{\Upsilon}_n(z_0)) \left( 1 - z_0(1 - z_0) \tilde{\Upsilon}'_n(z_0) \right) \left( \tilde{m}_n - \exp(\tilde{\Upsilon}_n(z_0)) + (1 - z_0) \tilde{\Upsilon}'_n(z_0) \right)}{\left( 1 - z_0 \exp(\tilde{\Upsilon}_n(z_0)) + z_0 \tilde{m}_n \right)^3}.
\]

Using relation (43) together with lemmas 5.1–5.4 we obtain the first inequality of (42). The proof of the second inequality of (42) is similar. Lemma 6.2 is therefore completely proved.

7. Proofs of main theorems

The structure of proofs are as follows: first we introduce a new renormalized coordinate \( \varsigma \) on \( \mathcal{V}_L = L_0^L \cup L_0^{-1} \) and then express the functions \( f_n \) and \( g_n \) through the renormalized coordinate. Then, using relations between new renormalized coordinate \( \varsigma \) and relative coordinates \( z_0, \tilde{z}_0 \), etc.
and applying lemmas 6.1, 6.2 we obtain the proof of theorem 3.3. Using a homological equation and theorems 3.3, 4.1 and 4.3 we get the proof of theorem 3.4. To prove theorem 3.5 we use the same strategy as developed in [13] and [17].

7.1. Proof of theorem 3.3

Renormalized coordinate \( z \) on \( V_n \) is defined by \( x \mapsto \frac{x - \xi_0}{\xi_n - \xi_{n-1}} \). It is clear that for any \( x \in [\xi_0, \xi_n] \) there exists a unique relative coordinate \( z_0 \in [0, 1] \) and a unique renormalized coordinate \( z \in [-1, 0] \) which corresponds to \( x \). Similarly, for any \( y \in [\xi_0, \xi_n] \) there exists a unique \( \tilde{z}_0 \in [0, 1] \) and a unique \( \tilde{z} \in [0, an] \) which corresponds to \( y \). Using definitions of relative coordinate and renormalized coordinate, one can show that \( z_0 = -z \) and \( \tilde{z}_0 = 1 - \frac{z}{an} \). Denote by \((-b_n)\) the renormalized coordinate of point \( \xi_n + q_n \). A direct calculation shows that

\[
\begin{align*}
    f_n(z) &= an - (an + b_n)z_{qn}(-z), \\
    g_n(z) &= -bn - (1 - bn)\tilde{z}_{qn} \left(1 - \frac{z}{an}\right).
\end{align*}
\]  

(44)

On the other hand, it can be easily shown that the functions \( \tilde{F}_n, \tilde{G}_n \) can be represented by Möbius transformations \( M_{\tilde{m}_n}, M_{\tilde{m}_n} \) respectively, as follows

\[
\begin{align*}
    \tilde{F}_n(z) &= an - (an + b_n)M_{\tilde{m}_n}(-z), \\
    \tilde{G}_n(z) &= -bn - (1 - bn)M_{\tilde{m}_n} \left(1 - \frac{z}{an}\right).
\end{align*}
\]  

(45)

Relations (44) and (45) imply

\[
\begin{align*}
    f_n(z) - \tilde{F}_n(z) &= -(an + b_n) \left(z_{qn}(-z) - M_{\tilde{m}_n}(-z)\right), \quad z \in [-1, 0], \\
    g_n(z) - \tilde{G}_n(z) &= -(1 - bn) \left(\tilde{z}_{qn} \left(1 - \frac{z}{an}\right) - M_{\tilde{m}_n} \left(1 - \frac{z}{an}\right)\right), \quad z \in [0, an].
\end{align*}
\]  

(46)

(47)

By differentiating (46) and (47) we obtain

\[
\begin{align*}
    f'_n(z) - \tilde{F}'_n(z) &= (an + b_n) \left(z'_{qn}(-z) - M'_{\tilde{m}_n}(-z)\right), \quad z \in [-1, 0], \\
    g'_n(z) - \tilde{G}'_n(z) &= \frac{1 - bn}{an} \left(z'_{qn} \left(1 - \frac{z}{an}\right) - M'_{\tilde{m}_n} \left(1 - \frac{z}{an}\right)\right), \quad z \in [0, an].
\end{align*}
\]  

(48)

(49)

Using Denjoy’s inequality (see [1]) and properties of dynamical partition one can obtain

\[
a_n + b_n \leq e^e, \quad \frac{1 - bn}{an} \leq e^e \quad \text{and} \quad 0 < 1 - bn < 1.
\]  

(50)

The last relations together with (46)-(49) and lemma 6.1 imply

\[
\|f_n - \tilde{F}_n\|_{C([-1,0])} \leq \frac{C}{n^2}, \quad \|g_n - \tilde{G}_n\|_{C([0,an])} \leq \frac{C}{n^2}
\]

for all \( n \geq N_0 \). Theorem 3.3 is proved.

7.2. Proof of theorem 3.4

From lemma 5.1 and relations (43), (46)–(50) directly follow the inequalities in (5) of theorem 3.4. To prove the inequalities in (6) we use theorem 4.3. According to this theorem, \( f' \) is
differentiable, hence \( z'_{q_n} \) and \( \hat{z}'_{q_{n-1}} \) are also differentiable. By differentiating (48) and (49) we obtain

\[
f''_n(z) - \tilde{F}''_n(z) = -(a_n + b_n) \left( z''_{q_n}(-z) - \mathcal{M}''_{\tilde{m}_n}(-z) \right), \quad z \in [-1, 0]
\]

\[
g''_n(z) - \tilde{G}''_n(z) = -\frac{1 - b_n}{a_n^2} \left( z''_{q_{n-1}} \left( 1 - \frac{z}{a_n} \right) - \mathcal{M}''_{\tilde{m}_n} \left( 1 - \frac{z}{a_n} \right) \right), \quad z \in [0, a_n].
\]

These, together with relations (43) and (50) and the second assertion of lemma 6.2, imply

\[
\|f''_n - F''_n\|_{C^r([-1,0])} \leq \frac{C}{n^{\gamma - 1}}, \quad \|g''_n - G''_n\|_{C^r([0,a_n])} \leq \frac{C}{a_n n^{\gamma - 1}}
\]

for all \( n \geq N_0 \). The inequalities in (6) of theorem 3.4 are proved.

To prove the inequality (7) of theorem 3.4 we use the obvious homological equation

\[
g_{n+1}(z) = -\frac{1}{a_n} f_n(-a_n z).
\]

This equation implies

\[
g'_{n+1}(0) = f'_n(0), \quad g_{n+1}(a_n) = -\frac{1}{a_n} f_n(-a_n a_{n+1}). \tag{51}
\]

Using (44) and the explicit forms of \( z'_{q_n}, \hat{z}'_{q_{n-1}} \) one can rewrite (51) as

\[
\left( 1 - b_{n+1} \right) \frac{\exp(\tilde{\gamma}_{n+1}(1))}{\tilde{m}_{n+1}} = \frac{(a_n + b_n) \tilde{m}_n}{\exp(T_n(0))}, \tag{52}
\]

\[
-b_{n+1} = -1 + \frac{(a_n + b_n) a_{n+1} \tilde{m}_n}{(1 - a_n a_{n+1}) \exp(T_n(a_n a_{n+1})) + a_n a_{n+1} \tilde{m}_n}. \tag{53}
\]

From the last equation we find \( (1 - b_{n+1})/a_{n+1} \) and then putting this expression in (52) we obtain

\[
\exp(\tilde{\gamma}_{n+1}(1) + \tilde{\gamma}_n(0)) = (1 - a_n a_{n+1}) \exp(\tilde{\gamma}_n(a_n a_{n+1})) + a_n a_{n+1} \tilde{m}_n.
\]

Using this and (53) we get

\[
a_n \frac{b_{n+1}}{c_{n+1}} = \frac{a_{n+1}}{c_{n+1}} \left( \frac{1}{c_{n+1}} a_n \exp(\tilde{\gamma}_n(a_n a_{n+1})) - b_n \tilde{m}_n \right) + \exp(\tilde{\gamma}_n(a_n a_{n+1})) - 1.
\]

Utilizing lemma 5.1 and the relation (43) one can show that

\[
a_{n+1} + b_{n+1} a_{n+1} c_{n+1} = c_{n+1} a_{n+1} (c_n - a_n - b_n m_n) + O \left( \frac{1}{n^1} \right) \tag{54}
\]

for all \( n \geq N_0 \). Set \( r_{n+1} := a_{n+1} + b_{n+1} a_{n+1} c_{n+1} \) and \( a_{n+2} r_{n+2} := 1 \). Iterating the relation (54) we get
\[ r_{n+1} = r_1 \prod_{i=2}^{n+1} (-a_i c_i) + O\left( \sum_{j=3}^{n+2} \prod_{i=j}^{n+2} (-a_i c_i) (i-2)^{\gamma} \right). \]

It is clear
\[ \left| \prod_{i=j}^{n+1} (-a_i c_i) \right| \leq \max \left\{ c, \frac{1}{c} \right\} a_{n+1} \cdot \frac{|f_0|}{|H_0^{-1}|}. \]

From the well-known fact \( \frac{|f_0|}{|H_0^{-1}|} = O(\lambda^{n-j+1}) \) (see [1]) it follows that
\[ |r_{n+1}| = O\left( a_{n+1} \sum_{i=1}^{n} \frac{\lambda^{n-i}}{i^\gamma} \right). \]

On the other hand
\[ \sum_{i=1}^{n} \frac{\lambda^{n-i}}{i^\gamma} = \sum_{i=1}^{(n/2)} \frac{\lambda^{n-i}}{i^\gamma} + \sum_{i=(n/2)+1}^{n} \frac{\lambda^{n-i}}{i^\gamma} = O\left( \left( \left\lfloor \frac{n}{2} \right\rfloor \lambda^{\left\lfloor \frac{n}{2} \right\rfloor} \right) + \frac{1}{(\frac{n}{2})^\gamma} \right) = O\left( \frac{1}{n^\gamma} \right) \]

where \([\cdot]\) is an integer part of a given number. Hence
\[ |r_{n+1}| = O\left( \frac{a_{n+1}}{n^\gamma} \right). \]

Theorem 3.4 is completely proved.

### 7.3. Proof of theorem 3.5

In this paragraph we consider a circle diffeomorphism \( f \) with a break of size \( c \) and rotation number \( \rho \) such that either \( c > 1 \) and \( \rho \in M_c \), or \( c < 1 \) and \( \rho \in M_o \). We recall some notations and notions from [13] and [17].

The rotation number of a commuting pair. For a normalized commuting pair \((F, G)\) we define its rotation number \( \rho(F, G) \in [0, 1] \), by substituting its consecutive heights for partial quotients in the continued fractions expansion \( \rho(F, G) = [k_1, k_2, ...] \), where \( k_n \) is the height of \( R^{n-1}(F, G) \).

The canonical lift. For a normalized commuting pair \((F, G)\) with a positive height, consider a real function determined by the form
\[
H_{F,G}(w) = \begin{cases} 
m + H_{F,G}^{(1)}(w), & w \in [m - 1, m + \phi(F^{-1}(0))], 
m + 1 + H_{F,G}^{(2)}(w), & w \in [m + \phi(F^{-1}(0)), m], 
\end{cases} \quad m \in \mathbb{Z},
\]

where \( H_{F,G}^{(1)} = \phi \circ F \circ \phi^{-1} \), \( H_{F,G}^{(2)} = \phi \circ G \circ F \circ \phi^{-1} \) and \( \phi : [-1, F(0)] \rightarrow \mathbb{R} \) is the Möbius transformation that maps \((-1, 0, F(0))\) into \((-1, 0, 1)\). One can easily check that \( H_{F,G} \) is continuous on \( \mathbb{R} \) and satisfies the equivalence \( H_{F,G}(w + 1) = H_{F,G}(w) + 1 \), thus it is a lift for a certain circle homeomorphism. We call \( H_{F,G} \) the canonical lift for the commuting pair \((F, G)\).

We will denote the circle homeomorphism generated by the canonical lift by the same symbol \( H_{F,G} \). It is easy to see \( \rho(H_{F,G}) = \rho(F, G) \). A commuting pair is called a commuting pair with a break, if \( c = \sqrt{\frac{\tau_0(F)(0)}{\tau_0(G)(0)}} \neq 1 \) and we refer to \( c \) as the size of the break. A circle homeomorphism \( H_{F,G} \) generated by a commuting pair \((F, G)\) with a break of size \( c \) has, generally speaking, two break points rather then one, but they belong to the same orbit and the product
of their sizes is equal to \( c \). Notice that, the renormalization operator \( R \) acting on the commuting pair \((F,G)\) with a break of size \( c \), switches the size \( c \) to \( 1/c \), therefore, sometime we write \( R_c \) instead of \( R \).

**Absorbing area.** Consider the following two parameter families of Möbius transformations

\[
F_{a_n,v_n,c_n}(z) = \frac{a_n + c_n z}{1 - v_n z}, \quad G_{a_n,v_n,c_n}(z) = \frac{-c_n + z}{c_n - \frac{c_n - 1 - v_n}{a_n} z}
\]  

(56)

where \( v_n = m_n - 1 \). In the investigations of renormalizations of commuting pairs, it is convenient to consider these Möbius transformations instead of \( F_n, G_n \). Estimates (5)–(7) imply

\[
\|f_n - F_{a_n,v_n,c_n}\|_{C^1([-1,0])} \leq \frac{C}{n^\gamma}, \quad \|f_n'' - F_{a_n,v_n,c_n}''\|_{C^0([-1,0])} \leq \frac{C}{n^\gamma-1},
\]

\[
\|g_n - G_{a_n,v_n,c_n}\|_{C^1([0,a_n])} \leq \frac{C}{n^\gamma}, \quad \|g_n'' - G_{a_n,v_n,c_n}''\|_{C^0([0,a_n])} \leq \frac{C}{a_n n^{\gamma-1}},
\]

(57)

for some \( C > 0 \). For every particular \( c \) we identify a point \((a,v)\) on the \( \mathbb{R}^2 \) plane with the corresponding pair of Möbius transformations \((F_{a,v,c},G_{a,v,c})\). Following [13] and [17] we define

\[
D_c = \left\{ (a,v) : \frac{1}{2} \leq \frac{v}{c-1} < 1, \quad \frac{c(c-v-1)}{v} \leq a \leq c \right\},
\]

(58)

\[
\bar{D}_c = D_c \cup \left\{ (a,v) : a > \frac{(c-1)^2}{4v} \right\} \text{ if } c > 1, \quad \text{and } \quad \tilde{D}_c = D_c \cap \left\{ (a,v) : v > \frac{a(c-1)^2}{4} + c - 1 \right\}
\]

if \( c < 1 \). These sets play an important role in the investigations of the renormalization of commuting pairs of Möbius transformations. It was shown in [17] that these sets are absorbing areas for the dynamics of the renormalization operator on a space of infinitely many renormalizable commuting pairs of Möbius transformations, i.e. each infinitely renormalizable commuting pair of Möbius transformations falls inside these sets, under action of \( R^2 = R_1/ \circ R_1 \) and stays there forever afterwards. As was shown in [17], the set \( \gamma_{p,c} \) of points in \( \{(a,v) : 0 < a \leq c, \ a + v - c + 1 > 0\} \supset \tilde{D}_c \) with the same irrational rotation number \( \rho \in (0,1) \) is a continuous graph of the form \( a = \gamma_{p,c}(v), v > -1 \).

**Estimates of \((a_n,v_n)\).** Next we give some general estimates for the coefficients of Möbius transformations in (56).

**Proposition 7.1.** For sufficiently large \( n \), the following statements hold.

1. There exists a constant \( \varepsilon = \varepsilon(f) > 0 \) such that \( a_n \in (\varepsilon,c_n - \varepsilon) \) and \( \frac{c_n}{c_n - 1} \in (\varepsilon, 1 - \varepsilon) \);
2. The point \((a_n,v_n)\) belongs to \( O(n^{-\gamma}) \)-neighborhood of the set \( D_c \).

**Proof.** Since \( \gamma > 1, f \) is continuously twice differentiable (except the break point). Moreover, the set of rotation numbers is the same as in [17]. Therefore the proofs of the statements (1) and (2) are exactly similar to the proofs of propositions 5 and 6 in [17], respectively. The only one difference is, in place of the terms \( O(\lambda^n) \) appear the terms \( O(n^{-\gamma}) \), since \( f_n' \) is approximated with \( F_n' \) with this rate due to theorem 3.4. Notice that, in the proof, the second derivative approximation of \( f_n \) is not used.

The following set plays an important role in the investigations of renormalizations of commuting pairs of Möbius transformations (see [17]).

\[
\Phi_\varepsilon^c = \left\{ (a,v) : \varepsilon < a < c - \varepsilon, \ v < \frac{v}{c-1} < 1 - \varepsilon, \ v + a - c + 1 > \varepsilon \right\}, \ \varepsilon > 0.
\]
Proposition 7.1 implies that for any $f$ satisfying our assumptions for this section, there exists a natural number $n_0 = n_0(f)$ such that, the projection $(a_n, v_n)$ of a renormalization $(f_n, g_n)$ belongs to $\Phi_{\varepsilon_n}$.

Estimates of canonical lifts of the renormalizations. To abbreviate the notations we denote by $H_{\varepsilon_n}(a_n, v_n)$ the canonical lift of $(F_{a_n, v_n}, G_{a_n, v_n})$ and set $H_{\varepsilon_n}(w, a_n, v_n) := \phi \circ F_{a_n, v_n} \circ \phi^{-1}(w)$, $H_{\varepsilon_n}^{(2)}(w, a_n, v_n) := \phi \circ G_{a_n, v_n} \circ F_{a_n, v_n} \circ \phi^{-1}(w)$.

**Proposition 7.2.** There exists a constant $h_{\varepsilon_n, \varepsilon_n} > 0$ such that, for any $(a_n, v_n) \in \Phi_{\varepsilon_n}$ one has

$$\frac{\partial H_{\varepsilon_n}^{(i)}(w; a_n, v_n)}{\partial a_n} \geq h_{\varepsilon_n, \varepsilon_n} \text{ for } i \in \{1, 2\}.$$

**Proof.** We prove by contradiction. Assume there exist points $(a_n, v_n)$ such that

$$\frac{\partial H_{\varepsilon_n}^{(i)}(w; a_n, v_n)}{\partial a_n} < h_{\varepsilon_n, \varepsilon_n} \text{ for } i \in \{1, 2\}.$$

Since $f_n(0) = F_{a_n, v_n}(0)$, the function $\phi$ is the same for $(f_n, g_n)$ and $(F_{a_n, v_n}, G_{a_n, v_n})$, by its definition

$$\phi(z) = \frac{(a_n + 1)z}{2a_n + (a_n - 1)z}.$$

Let $\alpha_n = \min \left\{ \phi(F_{\alpha_n, \varepsilon_n}^{-1}(0)), \phi(f_n^{-1}(0)) \right\}$ and $\beta_n = \max \left\{ \phi(F_{\alpha_n, \varepsilon_n}^{-1}(0)), \phi(f_n^{-1}(0)) \right\}$.

Using the mean value theorem, inequality (57) and proposition 7.1 we get

$$\left| H_{\varepsilon_n}(w) - H_{\varepsilon_n}^{(i)}(w; a_n, v_n) \right| = \left| \phi \circ f_{\alpha_n, \varepsilon_n} \circ \phi^{-1}(w) - \phi \circ F_{\alpha_n, \varepsilon_n} \circ \phi^{-1}(w) \right|$$

$$= \phi'(\zeta) |f_{\alpha_n, \varepsilon_n} \circ \phi^{-1}(w) - F_{\alpha_n, \varepsilon_n} \circ \phi^{-1}(w)| \leq C h_{\varepsilon_n, \varepsilon_n}.$$

on the interval $[-1, \alpha_n]$, where $\zeta$ is a point between $f_{\alpha_n} \circ \phi^{-1}(w)$ and $F_{\alpha_n, \varepsilon_n} \circ \phi^{-1}(w)$. Similarly, we have

$$\left| H_{\varepsilon_n}(w) - H_{\varepsilon_n}^{(i)}(w; a_n, v_n) \right| \leq \frac{C}{n^\gamma}.$$

on the interval $[\beta_n, 0]$. Notice that the length of the interval $[\alpha_n, \beta_n]$ is less than $\varepsilon_n$. Since the functions $H_{\varepsilon_n}$ and $H_{\varepsilon_n}^{(i)}(w; a_n, v_n)$ are continuous and increasing, their closeness over the interval $[\alpha_n, \beta_n]$ follows from their closeness at its endpoints.

**A projector operator.** Note that the rotation number $\rho(a_n, v_n)$ of the M"obius transformation $(a_n, v_n)$ does not necessarily equal to the rotation number $\rho_n := \rho_n(f_n, g_n)$ of $(f_n, g_n)$. Therefore, we define the projection operator $P$ from the space of all commuting pairs $(f_n, g_n)$ with a well-defined rotation number to $D_{\varepsilon_n}$, as $P(f_n, g_n) = (a_n^*, v_n^*)$, where $a_n^* = \gamma_{\rho_n, \varepsilon_n}(v_n)$, $v_n^* = v_n$ if $(\gamma_{\rho, \varepsilon_n}(y), v_n) \in D_{\varepsilon_n}$, otherwise let $(a_n^*, v_n^*)$ be the closest to $(\gamma_{\rho, \varepsilon_n}(v_n), v_n)$ the intersection point of the curve $a = \gamma_{\rho, \varepsilon_n}(v)$ with the boundary of $D_{\varepsilon_n}$. Since this projection is determined uniquely by $f_n$ and rotation number $\rho_n$, we write $P f_n = (a_n^*, v_n^*)$. 

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Proposition 7.4. There exists $C = C(f) > 0$ such that for sufficiently large $n$, one has
\[ |\gamma_{\rho_n, c_n}(v_n) - a_n| \leq \frac{C}{n^\gamma}. \]

**Proof.** We assume that $a_n \geq \gamma_{\rho_n, c_n}(v_n).$ For the opposite case the proof is similar. Proposition 7.3 implies
\[ H_{c_n}(w; a_n, v_n) \geq H_{c_n}(w; a_n, v_n) - \frac{C}{n^\gamma} \]
for all $w \in [-1, 0].$ On the other hand, since $(a_n, v_n) \in \Phi_{c_n}^+$ for sufficiently large $n,$ at least half of the segment $[\gamma_{\rho_n, c_n}(v_n), a_n] \times \{v_n\}$ lies inside $\Phi_{c_n}^2.$ This and proposition 7.2 imply
\[ H_{c_n}(w; a_n, v_n) \geq H_{c_n}(w; \gamma_{\rho_n, c_n}(v_n), v_n) + \frac{h_{c_n, \gamma_n}(a_n - \gamma_{\rho_n, c_n}(v_n))}{2} - \frac{C}{n^\gamma} \]
for all $w \in [-1, 0].$ The last inequality implies $a_n - \gamma_{\rho_n, c_n}(v_n) \leq \frac{2C}{n^\gamma},$ where $h = \min\{h_{c_n, z}, h_{c_n, z}^\gamma\}.$ Otherwise by (59) and (60) there would be $H_{c_n}(w; a_n, v_n) > H_{c_n}(w; \gamma_{\rho_n, c_n}(v_n), v_n)$ for all $w \in [-1, 0],$ and therefore $\rho_n > \rho(\gamma_{\rho_n, c_n}(v_n), v_n),$ while, in fact they are equal. □

Proposition 7.5. There exists $C = C(f) > 0$ such that for sufficiently large $n,$ one has
\[ |a_n^* - a_n| \leq \frac{C}{n^\gamma} \quad \text{and} \quad |v_n^* - v_n| \leq \frac{C}{n^\gamma}. \]

**Proof.** The proof of this proposition follows from the definition of $\mathcal{P}$ and propositions 7.1 and 7.4.

An alternative coordinate system in $D_c.$ On the set of commuting pairs $(F_{a,v,c}, G_{a,v,c})$ in $D_c$ we define the new alternative coordinate as
\[ (x, y) = \pi_c(a, v) = \left( av, \frac{v + 1 - c}{ca} \right) \]
which was introduced in [17]. Notice, that the coordinates $x$ and $y$ can be considered as the independent parameters characterizing the nonlinearity of $F_{a,v,c}$ and $G_{a,v,c}.$ Indeed, if we normalize $F_{a,v,c}$ and $G_{a,v,c}$ by the linear change of variable $z = at$ and $z = -t$ respectively, one can obtain the linear-fractional functions
\[ M_{x,c} : t \rightarrow \frac{1 + ct}{1 - xt} \quad \text{and} \quad M_{y,c} : t \rightarrow \frac{1 + ct}{1 - yt}. \]
The following lemma is proved in [17] and below we use this lemma.

**Lemma 7.6.** For any $(x, y) \in D_c$ we have $R_c(x, y) = (x', y')$ with $y' = x.$

The closeness of renormalization and projection operators. On the set of commuting pairs in $D_c,$ we define two metrics: the standard metric
\[ d((a, v), (\tilde{a}, \tilde{v})) = |a - \tilde{a}| + |v - \tilde{v}| \]
and the metric
\[d_\varepsilon((x, y), (\tilde{x}, \tilde{y})) = |x - \tilde{x}| + |y - \tilde{y}|\]

where the parameters \((a, v), (x, y) = \pi_c(a, v)\) and \((\tilde{a}, \tilde{v}), (\tilde{x}, \tilde{y}) = \pi_c(\tilde{a}, \tilde{v})\) correspond to the pairs \((F, G)\) and \((\tilde{F}, \tilde{G})\) respectively. One can easily show that there exists a constant \(K = K(\varepsilon, c) > 1\), such that for any two commuting pairs \((a, v)\) and \((\tilde{a}, \tilde{v})\) in \(D_c \cap \Phi^c\) and with corresponding coordinate \((x,y)\) and \((\tilde{x}, \tilde{y})\), we have

\[K^{-1}d((a, v), (\tilde{a}, \tilde{v})) \leq d_\varepsilon((x, y), (\tilde{x}, \tilde{y})) \leq Kd((a, v), (\tilde{a}, \tilde{v})).\]  \hspace{1cm} (61)

**Proposition 7.7.** There exists a constant \(C = C(f) > 0\) such that

\[d_{n+1}(\mathcal{P}f_{n+1}, \mathcal{R}_c, \mathcal{P}f_n) \leq \frac{C}{n^\gamma}\]

for sufficiently large \(n\).

**Proof.** Notice that for sufficiently large \(n\), both \(a_n, v_n\) and \(a_n^*, v_n^*\) lie within \(\Phi^c\), for some \(\varepsilon = \varepsilon(f) > 0\), thus due to propositions 7.1 and 7.5 the values \(a_n, c_n - a_n, a_n^*, c_n - a_n^*\) and \(v_n/(c_n - 1)\) are positive, bounded and separated from zero by constants independent on \(n\). Denote \((\tilde{a}_{n+1}, \tilde{v}_{n+1}) = \mathcal{R}_c(a_n, v_n), (x_n, y_n) = \pi_c(a_n, v_n), (\tilde{x}_{n+1}, \tilde{y}_{n+1}) = \pi_c(\tilde{a}_{n+1}, \tilde{v}_{n+1})\) and \((x_n^*, y_n^*) = \pi_c(a_n^*, v_n^*)\). It follows from (7) that \(\tilde{y}_{n+1} = c_n + 1 + \varepsilon_n(a_n, a_{n+1}) - 1 + \mathcal{O}(\frac{1}{n^\gamma})\), hence \(\tilde{x}_{n+1} = x_n + \mathcal{O}(\frac{1}{n^\gamma})\). By lemma 7.6 and proposition 7.5 we have \(\tilde{y}_{n+1} = x_n^* = x_n + \mathcal{O}(\frac{1}{n^\gamma})\), hence \(\tilde{x}_{n+1} = y_{n+1} + \mathcal{O}(\frac{1}{n^\gamma})\). On the other hand, again by proposition 7.5 one can get \(\tilde{x}_{n+1} = y_{n+1} + \mathcal{O}(\frac{1}{n^\gamma})\). So, we have \(\tilde{y}_{n+1} = y_{n+1} + \mathcal{O}(\frac{1}{n^\gamma})\).

Now, by exactly the same way as in proof of proposition 6.1 in [13] and also proposition 8 in [17], one can show that \(d((\tilde{a}_{n+1}, \tilde{v}_{n+1}), (a_{n+1}, v_{n+1}^*)) \leq C|\tilde{y}_{n+1} - y_{n+1}^*|\) for some constant \(C > 0\). So, the proof of proposition follows from (61).

**Hyperbolicity of renormalization operator.** On the set of renormalizable commuting pairs in \(D_c\), with the same irrational rotation number, the renormalization operator \(\mathcal{R}\) is a Lipschitz and the two-step renormalization operator \(\mathcal{R}^2 = \mathcal{R}_{1/c} \circ \mathcal{R}_c\) has strong hyperbolic property.

**Theorem 7.8 ([17]).** For every \(0 < c \neq 1\), there exist constants \(B = B(c) > 0\) and \(\beta = \beta(c) \in (0, 1)\) such that for any two points \((a, v)\) and \((\tilde{a}, \tilde{v})\) in \(D_c \setminus \{(0, c - 1)\}\), with the same irrational rotation number, and corresponding coordinates \((x, y)\) and \((\tilde{x}, \tilde{y})\), respectively, we have

\[d_{1/c}(\mathcal{R}_c(x, y), \mathcal{R}_c(\tilde{x}, \tilde{y})) \leq Bd_{1/c}(x, y), (\tilde{x}, \tilde{y})\]

and

\[d_{c}(\mathcal{R}_{1/c} \circ \mathcal{R}_c(x, y), \mathcal{R}_{1/c} \circ \mathcal{R}_c(\tilde{x}, \tilde{y})) \leq \beta d_c(x, y), (\tilde{x}, \tilde{y})\].

The following lemma will be used in the proof of theorem 3.5.

**Lemma 7.9 ([17]).** There exists a constant \(A = A(\varepsilon, c) > 0\) such that

\[\|F_{a, \varepsilon} - F_{\tilde{a}, \varepsilon}\|_c \leq A d((a, v), (\tilde{a}, \tilde{v}))\]

for any \((a, v), (\tilde{a}, \tilde{v}) \in \Phi^c\).
Convergence of renormalizations. Let \( \hat{f} \) be a circle diffeomorphism with a break of the same size \( c \), and the same (half bounded) rotation number \( \rho \) as \( f \). Let \((\hat{f}_n, \bar{f}_n)\) be the \( n \)th renormalization of \( \hat{f} \). Due to theorem 3.4 and lemma 7.9 we have

\[
\|f_n - \hat{f}_n\|_{C^s([-1,0])} \leq C \left( \frac{1}{n^\gamma} + d_c(\mathcal{P}f_n, \mathcal{P}\hat{f}_n) \right)
\]

(62)

and

\[
\|f_n'' - \hat{f}_n''\|_{C^s([-1,0])} \leq C \left( \frac{1}{n^{\gamma-1}} + d_c(\mathcal{P}f_n, \mathcal{P}\hat{f}_n) \right).
\]

(63)

Using the triangle inequality, we get

\[
d_c(\mathcal{P}f_n, \mathcal{P}\hat{f}_n) \leq d_c(\mathcal{P}f_n, \mathcal{R}_{c_{n-1}} \mathcal{P}f_{n-1})
\]

\[
+ d_c(\mathcal{R}_{c_{n-1}} \mathcal{P}f_{n-1}, \mathcal{R}_{c_{n-1}} \mathcal{P}\hat{f}_{n-1}) + d_c(\mathcal{R}_{c_{n-1}} \mathcal{P}\hat{f}_{n-1}, \mathcal{P}\bar{f}_n).
\]

Proposition 7.7 and theorem 7.8 imply

\[
d_c(\mathcal{P}f_n, \mathcal{P}\hat{f}_n) \leq \frac{C}{n^\gamma} + d_c(\mathcal{P}f_{n-2}, \mathcal{P}\bar{f}_{n-2})
\]

\[
+ d_c(\mathcal{R}_{c_{n-2}} \circ \mathcal{R}_{c_{n-1}} \mathcal{P}f_{n-2}, \mathcal{R}_{c_{n-2}} \circ \mathcal{R}_{c_{n-1}} \mathcal{P}\bar{f}_{n-2}) + d_c(\mathcal{R}_{c_{n-2}} \circ \mathcal{R}_{c_{n-1}} \mathcal{P}\bar{f}_{n-2}, \mathcal{R}_{c_{n-1}} \mathcal{P}\bar{f}_{n-1})
\]

\[
\leq \frac{C}{n^\gamma} + \beta d_c(\mathcal{P}f_{n-2}, \mathcal{P}\bar{f}_{n-2}).
\]

By iteration of the last inequality, we get

\[
d_c(\mathcal{P}f_n, \mathcal{P}\hat{f}_n) \leq C \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor -1} \frac{\beta^k}{(n - 2k)^\gamma} + \beta^{\lfloor \frac{n}{2} \rfloor} \max \left\{ d_c(\mathcal{P}f_0, \mathcal{P}\hat{f}_0), d_c(\mathcal{P}f_1, \mathcal{P}\hat{f}_1) \right\}. \]

It is easy to show that

\[
\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor -1} \frac{\beta^k}{(n - 2k)^\gamma} = O\left( \frac{1}{n^\gamma} \right).
\]

Hence

\[
d_c(\mathcal{P}f_n, \mathcal{P}\hat{f}_n) \leq C \left( \frac{1}{n^\gamma} \right).
\]

Thus, the last inequality and inequalities (62) and (63) imply the statement of theorem 3.5.

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