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Implicit Riesz wavelets based-method for solving singular fractional integro-differential equations with applications to hematopoietic stem cell modeling

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Riesz wavelets in $L_2(\mathbb{R})$ have been proven as a useful tool in the context of both pure and numerical analysis in many applications, due to their well prevailing and recognized theory and its natural properties such as sparsity and stability which lead to a well-conditioned scheme. In this paper, an effective and accurate technique based on Riesz wavelets is presented for solving weakly singular type of fractional order integro-differential equations with applications to solve system of fractional order model that describe the dynamics of uninfected, infected and free virus carried out by cytotoxic T lymphocytes (CTL).

The Riesz wavelet in this work is constructed via the smoothed pseudo-splines refizable functions. The advantage of using such wavelets, in the context of fractional and integro-differential equations, lies on the simple structure of the reduced systems and in the powerfulness of obtaining approximated solutions for such equations that have weakly singular kernels. The proposed method shows a good performance and high accuracy orders.

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1. Introduction

Fractional differential modeling is widely used in many areas of applications in physics, engineering and many other major sciences. Recently, Atangana et al used fractional calculus in image processing [1], modeling the dynamics of novel coronavirus (COVID-19) [2], finance [3], dynamical systems [4] and many other applications and developments can also be found in [5–8]. Other interesting applications can be found in [9–16] and references therein. It is important to mention that an extensive development of fractional calculus has been achieved and well established to describe many applications that using integro-differential equations (see for example [18–21,17]). Note that, the kernel of many of these integro-differential equations is singular with a fractional order and hence obtaining the exact solutions is challenging and sometimes even impossible. Therefore, several methods involving integro-differential operators have been proposed to solve fractional differential equations. For a complete picture we refer to Refs. (see [22–25]).

Wavlets and their generalizations appear in a variety of advanced applications in filter banks analysis, in image processing and image recognition, transmission and storage [26,28,30,27,29]. This is largely due to the fact that wavelets have the right structure to capture the sparsity in “physical” images, perfect mathematical properties such as its multi-scale structure, sparsity, smoothness, compactly supported, and high vanishing moments. For example, the FBI center is using wavelets in their fingerprint database system for image reconstruction, see Fig. 1. Note that, a function $g \in L_2(\mathbb{R})$ is said to have vanishing moments of order $m$ if it is orthogonal to the polynomials $x^d$ for all $s = 0, 1, 2, \ldots, m – 1$. This property is connected to the multi-scale systems and its sparsity [26].

Wavelet expansions have been successful in developing many numerical algorithms for investigating and solving various types of fractional, differential and integral equations (see for example [31–45,47–51,46]). Motivated by the above contributions and properties, that are essential to develop efficient algorithms for the numerical solutions of a given fractional integro-differential equations (FIDEs), the main goal of the proposed work is to develop an efficient algorithm based on Riesz wavelets using the collocation method to solve fractional order of integro-differential equations with weakly singular kernels.
Before we proceed further, let us recall some definitions and notations needed for this paper. A function \( \phi \in L_2(\mathbb{R}) \) is called ren- 
fine if it satisfies the following equation 

\[
\phi(x) = \sum_{k \in \mathbb{Z}} a[k] \phi(2x - k),
\]

where \( a[k] \in \ell_2(\mathbb{Z}) \) is a finitely supported sequence and is called the refinement mask of \( \phi \). The corresponding wavelet function is defined by 

\[
\psi(x) = \sum_{k \in \mathbb{Z}} b[k] \phi(2x - k),
\]

where \( b[k] \in \ell_2(\mathbb{Z}) \) is finitely supported sequence and is called the high pass filter of \( \psi \).

In this paper, for \( f \in L_1(\mathbb{R}) \) (which can be extended to \( L_2(\mathbb{R}) \)), we use the following Fourier transform defined by 

\[
\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x) dx.
\]

The Fourier series of the sequence \( a \) is defined by 

\[
\hat{a}(\xi) = \sum_{k \in \mathbb{Z}} a[k] e^{-ik\xi}, \quad \xi \in \mathbb{R}.
\]

Pseudo-splines have attracted many researcher due to their significant contribution to both numerical computations and analysis. The constructions of pseudo-splines tracked back to the well-known work produced by Daubechies et al. in [52,53]. It is a family of refinable functions with compact support and have extensive flexibility in wavelets and applications. Pseudo-splines known as a generalization of many well-known refinable functions such as the B-splines, interpolate and orthonormal refinable functions [28]. We refer the reader to [52–58] and references therein for more details.

2. Riesz wavelets via smoothed pseudo-splines

We use the smoothed pseudo-splines introduced in [58] to construct Riesz wavelets and use it to apply our numerical scheme for solving different types of FIDEs. Pseudo-splines of order \((p, q)\) of type I and II, \( \psi_{(p,q)}, k = 1, 2 \), are defined in terms of their refinement masks, where

\[
|\hat{a}_{(p,q)}(\xi)|^2 = \sum_{m=0}^{q} \left( \frac{p+q}{m} \right) (\cos (\xi/2))^{2(p+q-m)} \sin^{2m}(\xi/2),
\]

and

\[
|\hat{a}_{(p,q)}(\xi)|^2 = |\hat{a}_{(p,q)}(\xi)|^2.
\]

Note that, the refinement mask of the pseudo-splines of type I with order \((p, q)\) is obtained using Fejér-Riesz Theorem. Using the Fourier transform, the refinable pseudo-spline function generated using the above refinement masks is defined by

\[
\psi_{(p,q)}(\xi) = \prod_{m=1}^{\infty} |\hat{a}_{(p,q)}(-m)|/2^m, \quad k = 1, 2.
\]

They are two types of smoothed pseudo-splines defined by its refinable masks. For \( r \geq p \), we have the smoothed pseudo-splines of type I \((k = 1)\), and II \((k = 2)\) with order \((r, p, q)\), such that 

\[
\psi_{(r,p,q)}(\xi) = \psi_{(p,q)} * \chi_{1-(r-p)}(\xi), \quad k = 1, 2,
\]

where

\[
\chi_{1-(r-p)}(\xi) = \chi_{1-(r-p)} * \cdots * \chi_{1-(r-p)} \quad \text{for } (r-p) \text{ times},
\]

and for \( r \geq 2p\),

\[
\psi_{(r,p,q)}(\xi) = \sum_{m=0}^{q} \left( \frac{p+q}{m} \right) (\cos (\xi/2))^{2q+(r-p)} \sin^{2m}(\xi/2).
\]

Riesz wavelets have been studied extensively in the literature, for example, see [59–62] and other references.

**Definition 2.1.** We say that the set \( \mathcal{M}(\psi^\ell) = \{ \psi_{j,k}^{\ell} = 2^{\ell/2} \psi_{j}^{\ell}(2^j - k), \ell = 1, \ldots, N \} \) is an \( L_2(\mathbb{R}) \) generate a Riesz wavelet for \( L_2(\mathbb{R}) \) if for any finitely supported sequence \( [n_{j,k}^\ell, \ell = 1, \ldots, N; j, k \in \mathbb{Z}] \) there exist positive constants \( C \) and \( C \) such that

\[
C \sum_{j=1}^{N} \sum_{k \in \mathbb{Z}} |n_{j,k}^\ell|^2 \leq \| \sum_{\ell=1}^{N} \sum_{j=1}^{N} \sum_{k \in \mathbb{Z}} n_{j,k}^\ell \psi_{j,k}^\ell \| \leq C \sum_{j=1}^{N} \sum_{k \in \mathbb{Z}} |n_{j,k}^\ell|^2, \quad \forall g \in L_2(\mathbb{R}),
\]

where

\[
\|g\|^2 = \langle g, g \rangle, \quad \text{and } \langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} dx.
\]

If \( \mathcal{M} \) defined in Definition 2.1 defined in Definition generates a Riesz wavelet for \( L_2(\mathbb{R}) \), then we can conclude the following expansion for any function \( f \in L_2(\mathbb{R}) \) such that

\[
f = \sum_{j=1}^{N} \sum_{k \in \mathbb{Z}} \langle f, \psi_{j,k}^\ell \rangle \psi_{j,k}^\ell.
\]

**Eq. (2.3)** can be truncated by

\[
\mathcal{M}_f = \sum_{\ell=1}^{N} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \langle f, \psi_{j,k}^\ell \rangle \psi_{j,k}^\ell.
\]

3. Riesz wavelets construction

In this section, we present the construction of Riesz wavelet systems using smoothed pseudo-splines that will be used to solve some examples of FIDEs. Dong in [54] and Bin in [62] for B-splines, and pseudo-splines proved that the system \( \mathcal{M} \) defined in Definition 2.1 forms a Riesz wavelet for \( L_2(\mathbb{R}) \) where the corresponding wavelet function \( \psi \) is defined by

\[
\psi(y) = \sum_{k} 2(-1)^{k-1} \hat{a}(1-k) \varphi(2y - k).
\]
For smoothed pseudo-splines $s\psi_{(r,p,q)}$, $k = 1.2$, Chuang in [58] proved the same result for the wavelet function $s\psi_{(r,p,q)}$, $k = 1.2$ defined as

$$k_{0} \hat{\psi}_{(r,p,q)}(\xi) = e^{-\frac{i\pi}{r}} k_{0} \hat{\phi}_{(p,q)}(2^{-1}\xi + \pi) k_{0} \hat{\phi}_{(p,q)}(2^{-1}\xi).$$

In this case, it was also shown in [62], for a given sequence of integer numbers $M$, we have the following representation for the smoothed pseudo-splines refinable masks given by

$$|\hat{a}_{(r,p,q)}(\xi)|^2 = 2 \hat{a}_{(r,p,q)}(\xi) = 2^{-2p}(1 + e^{-i\pi})2\hat{M}_{1}(\xi),$$

and that

$$\int_{\mathbb{R}} \phi(x)\phi(x-k)dx = \int_{\mathbb{R}} \psi(x)\psi(x-k)dx = \delta_{k},$$

$$\int_{\mathbb{R}} \phi(x)\psi(x-k)dx = \int_{\mathbb{R}} \psi(x)\phi(x-k)dx = 0,$$

where $\delta_{k}$ is the Kronecker delta function, defined by $\delta_{k} = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases}$ The corresponding refinable and wavelet functions are given by

$$k_{0} \psi_{(r,p,q)}(2\xi) = k_{0} a_{(p,q)}(\xi) \phi_{(p,q)}(\xi),$$

$$k_{0} \psi_{(r,p,q)}(2\xi) = k_{0} b_{(p,q)}(\xi) \phi_{(p,q)}(\xi).$$

Now we present some examples of Riesz wavelets via some smoothed pseudo-splines of both types and different order.

**Example 3.1.** For $(r, p, q) = (6, 2, 1)$, we have the following refinable masks:

$$\hat{a}_{(6,2,1)}(\xi) = \frac{1}{2}(\sqrt{3} + 1)e^{-\frac{i\pi}{6}} (1 + (\sqrt{3} - 2)e^{i\pi}) \cos\left(\frac{\xi}{2}\right),$$

$$\hat{a}_{(6,2,1)}(\xi) = \frac{1}{64} \left(e^{-\frac{i\pi}{6}} + e^{i\pi}\right)^{6} \left(1 - \frac{1}{2}(e^{-\frac{i\pi}{6}} - e^{i\pi})^{2}\right).$$

$$\hat{b}_{(6,2,1)}(\xi) = e^{-i\pi} \hat{a}_{(6,2,1)}(\xi + \pi),$$

$$\hat{b}_{(6,2,1)}(\xi) = e^{-i\pi} \frac{1}{2}\hat{a}_{(6,2,1)}(\xi + \pi),$$

where $\hat{a}_{(6,2,1)}(\xi)$ is obtained using Fejér-Riesz factorization theorem, so

$$|\hat{a}_{(6,2,1)}(\xi)|^2 = 2 \hat{a}_{(6,2,1)}(\xi).$$

Then, $M(1 \hat{\psi}_{(6,2,1)})$ and $M(2 \hat{\psi}_{(6,2,1)})$ form Riesz wavelet systems for $L_{2}(\mathbb{R})$. Note that the vanishing moments for both systems is equal to 6. On how to plot these generators, we recommend Han’s book [27] for a complete analysis of graphing such wavelets. Fig. 2 shows the graphs of the smoothed refinable functions and their corresponding Riesz wavelets.

**Example 3.2.** For $(r, p, q) = (9, 3, 2)$, we have the following refinable masks:

$$\hat{a}_{(9,3,2)}(\xi) = 0.0428071 \cos(\xi) - 0.162854 \cos(2\xi) - 0.293955 \cos(3\xi) + 0.146444 \cos(4\xi) + 0.629857 \cos(5\xi) + 0.483202 \cos(6\xi) + 0.121291 \cos(7\xi) + i(-0.0428071 \sin(\xi) + 0.293955 \sin(3\xi) - 0.146444 \sin(4\xi) - 0.629857 \sin(5\xi) - 0.483202 \sin(6\xi) - 0.121291 \sin(7\xi) + 0.325708 \sin(\xi) \cos(\xi) + 0.0) + 0.033212.$$

$$\hat{b}_{(9,3,2)}(\xi) = \frac{1}{4} \cos^{10}\left(\frac{\xi}{2}\right)(-156 \cos(\xi) + 33 \cos(2\xi) + 127).$$

Here we found $\hat{a}_{(9,3,2)}(\xi)$ numerically, see Fig. 3. Note that

$$|\hat{a}_{(9,3,2)}(\xi)|^2 \approx 2 \hat{a}_{(9,3,2)}(\xi).$$

Then, $M(1 \hat{\psi}_{(9,3,2)})$ and $M(2 \hat{\psi}_{(9,3,2)})$ form Riesz wavelet systems for $L_{2}(\mathbb{R})$. Fig. 4 shows the graphs of the smoothed refinable functions and its corresponding Riesz wavelets.

**4. Description of the proposed method**

In this paper, we consider several types of FIDEs integro-differential equation in the sense of Caputo fractional operator. Let us first start with some basic definitions and notation preliminaries of the fractional calculus needed in this present work.

**Definition 4.1.** For a real function $u(t)$ where $t, \alpha > 0$, and $n \in \mathbb{N}$ we have the following fractional operators of order $\alpha$, namely:
The Caputo's fractional derivative,
\[ D_\alpha^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u^{(n)}(x)}{(t-x)^{\alpha+1}} \ dx & \text{if } n-1 < \alpha \leq n \\ \frac{d^n u(t)}{dt^n} & \text{if } \alpha = n \end{cases} \]

The Atangana-Baleanu fractional derivative sense,
\[ \text{ABC} D_\alpha^\alpha u(t) = B(\alpha) \int_0^t u'(y) \left( \frac{\alpha}{1-\alpha} (y-\alpha)^\alpha \right) dy, \]
where \[ B(\alpha) \] is a normalization function such that \[ B(0) = B(1) = 0. \]

The Riemann-Liouville fractional derivative,
\[ D_\alpha^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(x)}{(t-x)^{\alpha+1}} \ dx & \text{if } n-1 < \alpha \leq n \\ \frac{d^n u(t)}{dt^n} & \text{if } \alpha = n \end{cases} \]

The Reimann-Liouville fractional integral operator (FIO),
\[ \mathcal{I}_\alpha^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(x)}{(t-x)^{1-\alpha}} \ dx, \text{ and } n-1 < \alpha \leq n. \]

Note that,
\[ D_\alpha^\alpha u(t) = \mathcal{I}_\alpha^{1-\alpha} \left( \frac{d^n u(t)}{dt^n} \right). \]

and for \( t > 0, \)
\[ \mathcal{I}_\alpha^\alpha T_\alpha^\alpha u(\cdot) = u(\cdot) - \sum_{k=0}^{n-1} u^{(k)}(0^+) \frac{t^k}{k!}. \]

We apply the collocation method based on Riesz wavelets generated using some smoothed pseudo-splines refinable functions. First, we consider the following linear FIDE with weakly singular kernel type,
\[ D_\alpha^\alpha u(x) = f(x) + \beta_1 \int_0^x \frac{u(\mu)}{(x-\mu)^\theta} \ d\mu + \beta_2 \int_0^1 K(x, \mu) u(\mu) \ d\mu, \] (4.1)
such that \( u^{(m)}(0) = 0 \) for all \( m = 1, 2, \ldots, n \) where \( n-1 < \alpha \leq n. \)
\( \beta_1, \beta_2 \) are real numbers, \( 0 < \theta < 1, f, \) and \( K \) are known functions, and \( u \) is the known function that needs to be numerically determined.

The scheme works as per the following steps:

1. We create an ansatz for the function that needs to be approximated by choosing a suitable collocation points based on the support of the Riesz wavelets and the domain of the function.
2. Use $\nu_{\alpha}U(\mu)$ as a truncated representation of unknown function $u^{(n)}(\mu)$, where
\[
\nu_{\alpha}U(\mu) = \sum_{\ell=1}^{N} \sum_{j=1}^{M-1} \sum_{k=1}^{L-1} \nu_{jk}\psi_{j}^{k}. \tag{4.2}
\]
3. Plug $\nu_{\alpha}U(\mu)$ in Eq. (4.1).
4. Plug the chosen collocation points to generate a linear system.
5. Solve the resulting system to get the coefficients $\nu_{jk}$. 

Following the steps above, we have
\[
\frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{\nu_{\alpha}U(\mu)}{(x-\mu)^{n+1-\alpha}} d\mu = f(x) + \beta_1 \int_0^x \frac{\nu_{\alpha}U(\mu)}{(x-\mu)^n} d\mu + \beta_2 \int_0^1 \mathcal{K}(x, \mu) \nu_{\alpha}U(\mu) d\mu
\]

With a few algebra and after plugging the interpolating points, the system of the form $\mathcal{J} \mathbf{c} = \mathbf{F}$.

\[
\mathcal{J} = \begin{pmatrix}
\vdots & \ldots & \vdots \\
\sigma_{jk}^{1} & \ldots & \sigma_{jk}^{N} \\
\end{pmatrix},
\]

where
\[
\sigma_{jk}^{i} = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{\psi_{j}^{k}(\mu)}{(x-\mu)^{n+1-\alpha}} d\mu - \beta_1 \int_0^x \frac{\nu_{\alpha}U(\mu)}{(x-\mu)^n} d\mu - \beta_2 \int_0^1 \mathcal{K}(x, \mu) \nu_{\alpha}U(\mu) d\mu
\]

$x^\nu \nu_{\alpha}U(\mu)$ will generate the Riesz wavelets operational matrix of fractional integration of order $n$. This generate a linear system of the form $\mathcal{J} \mathbf{c} = \mathbf{F}$.

Next, we study the hematopoietic stem cell (HSC) model based on fractional order derivatives. Recent research results proved the usefulness of using the stem cells for medical and treatment purposes. For example, for it was used for treating type 1 diabetes patients, cancer cells, strengthen immune system to fight several infections and many more. Most recently, the UAE has used stem cells for the treatment of Covid-19 to assess the effectiveness in fighting the virus. Therefore, handling modeling that describe the dynamics of such cells is crucial.

In this work, we consider the fractional model of HSC introduced in [63] and propose a generalization of the model based on Atangana-Baleanu fractional derivative as follows:
\[
\begin{align*}
\frac{ABC_0^\alpha D_\gamma^\alpha x(t)}{\alpha} &= \eta - \lambda x(t) - \delta x(t) y(t) \tag{4.3} \\
\frac{ABC_0^\alpha D_\gamma^\alpha y(t)}{\alpha} &= \delta x(t) y(t) - a y(t) - b(t) z(t) \tag{4.4}
\end{align*}
\]
Accordingly, there will be 5 corresponding ODEs with non-negative initial conditions related to specific needs. To solve the system of Eqs. (4.3)–(4.7), we consider the truncated expansion given in (4.2) to approximate each variable. Hence, we have

\[
\epsilon_a \int_a^t \sum_{\ell=1}^N \sum_{j:k=M-1}^k \zeta(x)_{j,k}^\ell(y) \left( -\frac{\alpha}{1-\alpha} (y-\alpha)^a \right) dy
\]

\[
= \eta - \lambda x(t) - \delta x(t) y(t)
\]

\[
\epsilon_a \int_a^t \sum_{\ell=1}^N \sum_{j:k=M-1}^k \zeta(y)_{j,k}^\ell(y) \left( -\frac{\alpha}{1-\alpha} (y-\alpha)^a \right) dy
\]

\[
= \delta x(t) y(t) - a y(t) - b y(t) z(t)
\]
Fig. 8. The graphs of the exact solution (black), approximated solution (red) and the error function (blue) for Example 5.2 based on $\mathcal{M}(\phi_{3,3})$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

\[
\epsilon_{\alpha} \int_{0}^{1} \sum_{\ell=1}^{N} \sum_{j \in M-1} \sum_{k \in \mathbb{Z}} \xi (w)^{j,k} \psi_{\alpha}^{\nu/2}(y) \left( -\frac{\alpha}{1-\alpha} (y-a)^{\alpha} \right) dy = -\sigma \partial h(t) + c \nu(t)x(t)y(t) - q(t)w(t)y(t) - \frac{\alpha}{1-\alpha} (y-a)^{\alpha} dy
\]

where $\epsilon_{\alpha} = \frac{\alpha}{1-\alpha}$. Then, the same procedure by the collocation methods will be applied to solve the CTL cells fractional order system.

5. Numerical applications

In this section, we use the Riesz systems presented in Section 3 to demonstrate and show the efficiency of the proposed method. We deal with the following examples.

Example 5.1. Consider the following CTL fractional model

\[
\frac{\partial}{\partial t}^{\alpha} x(t) = 0.2 - 0.5x(t) - 0.15x(t) y(t)
\]

subject to $x(0) = 1$, $y(0) = 2$, $w(0) = 3$, $z(0) = 4$, $h(0) = 5$. The graph of the variables $x(t)$ and $w(t)$ are shown in Fig. 5.

Example 5.2. Consider the following FDE

\[
\frac{\partial}{\partial t}^{\alpha} u(x) = f(x) + \beta_{1} \int_{0}^{x} u(\mu)(x - \mu)^{-1/2} d\mu + \beta_{2} \int_{0}^{x} (x - \mu) u(\mu) d\mu,
\]

such that $u(0) = 0$,

\[
f(x) = -\frac{1}{105} 8(6x + 7)x^{5/2} + \frac{2(12x + 11)x^{7/4}}{11(14)} - \frac{7x}{38} + \frac{3}{20},
\]

and $\alpha = \frac{1}{4}$, $\beta_{1} = \frac{1}{2}$, and $\beta_{2} = \frac{1}{4}$. The exact solution of this FDE is $u(x) = x^{3} + x^{5}$. By applying the algorithm in Section 4, we obtain numerical results and comparison for the exact and approximated solutions using different Riesz wavelet systems shown in Table 2 and Figs. 6, 7, 8 and 9. The graphs of the numerical solutions using
Fig. 10. Some illustrations for the exact and approximated solutions of Example 5.3 based on $M(\psi_{(a,2,1)})$ and $M(\psi_{(a,3,1)})$, respectively using different values of $\alpha$: $\alpha_1 = 0.25$ (red), $\alpha_2 = 0.20$ (blue), $\alpha_3 = 0.15$ (green). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 11. The graphs of the exact (black) and approximated (red) solutions with the error (blue) function of Example 5.2 based on $M(\psi_{(a,2,1)})$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 12. The graphs of the exact (black) and approximated (red) solutions with the error (blue) function Example 5.3 based on $M(\psi_{(a,2,1)})$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
Fig. 13. The graphs of the exact (black) and approximated (red) solutions with the error (blue) function of Example 5.3 based on $M(\psi_{(0,1,2)})$. [For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.]

Fig. 14. The graphs of the exact (black) and approximated (red) solutions with the error (blue) function of Example 5.3 based on $M(\psi_{(0,3,2)})$. [For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.]

Fig. 15. Some illustrations for the exact and approximated solutions for Example 5.3 based on $M(\psi_{(0,3,1)})$ and $M(\psi_{(0,3,2)})$ for different values of $\alpha : \alpha_1 = 0.15$ (red), $\alpha_2 = 0.20$ (blue), $\alpha_3 = 0.25$ (green). [For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.]

Different values of $\alpha$, $\alpha_1 = 0.25$, $\alpha_2 = 0.20$, $\alpha_3 = 0.15$, are depicted in Fig. 10.

Example 5.3. Consider the following FDE

$$D_\alpha^\nu u(x) = f(x) + \beta_1 \int_0^x u(\mu)(x - \mu)^{-\frac{1}{\nu}} d\mu$$

such that $u(0) = 0$,

$$f(x) = \frac{1}{15} (5 - 4x)x^{3/2} + \frac{(40x - 37)x^{17/20}}{371^{17/20}} - \frac{1}{7}(e - 3)e^x.$$
and $\alpha = \frac{17}{23}, \beta_1 = \frac{1}{2}, \text{ and } \beta_2 = \frac{1}{4}$. The exact solution of this FDE is $u(x) = x^2 - x$. By applying the algorithm in Section 4, we obtain numerical results and comparison between the exact solution and different Riesz wavelet systems shown in Table 3 and Figs. 11, 12, 13, and 14. The graphs of the numerical solutions using different values of $\alpha, \alpha_1 = 0.15, \alpha_2 = 0.20, \alpha_3 = 0.25$ are depicted in Fig. 15.

6. Conclusion

In this framework, the collocation method based on Riesz wavelets has been applied to numerically solve fractional order type of integro-differential equations with singular kernel type. The Riesz wavelets constructed via the smoothed pseudo-splines of type I and II and using different orders. These examples of Riesz wavelets have high vanishing moments orders and this effectively raised the approximation order of the numerical solution for the FIDE being handled. We showed some illustrations for the numerical solutions of two examples of FIDEs with singular kernel, which is of a great interest in applications. The proposed algorithm showed high order of accuracy and excellent agreement with the exact solutions. It turns out that increasing the vanishing moments of the underlying wavelet function resulting in improving the accuracy orders of the numerical solutions.

The utilized numerical algorithm can be expected to solve various types of FDEs including the nonlinear case that will be considered in the future work. Furthermore, we intended to study this important case on constructing appropriate higher dimensional smoothed pseudo-splines Riesz wavelets in order to solve problems in higher dimensions of fractional integro-differential equations with singular kernels.

### Table 2
Comparison results between the exact solution and its numerical approximation among different Riesz wavelet systems.

| $I^2\frac{\partial u}{\partial t}(x)$ of Example 5.2 | $x$ | Exact | $M(1,\Psi_{\alpha/2,1})$ | $M(2,\Psi_{\alpha/2,1})$ | $M(1,\Psi_{\alpha/2,2})$ | $M(2,\Psi_{\alpha/2,2})$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.011 | 0.011495692 | 0.011234666 | 0.011022234 | 0.0110009453 |
| 0.2 | 0.048 | 0.0472883645 | 0.0488432663 | 0.0479902808 | 0.0480001234 |
| 0.3 | 0.117 | 0.1134776453 | 0.1172437542 | 0.1168565323 | 0.1169993596 |
| 0.4 | 0.224 | 0.2247735578 | 0.2243426649 | 0.2239335362 | 0.2239235402 |
| 0.5 | 0.375 | 0.3746344427 | 0.3746344420 | 0.3741303598 | 0.3749543209 |
| 0.6 | 0.576 | 0.5775543674 | 0.5765664722 | 0.5773138140 | 0.5770803432 |
| 0.7 | 0.833 | 0.8371922233 | 0.8329465683 | 0.8331737266 | 0.8332900112 |
| 0.8 | 1.152 | 1.1586717244 | 1.1538873666 | 1.1526717244 | 1.1523244546 |
| 0.9 | 1.539 | 1.5493378608 | 1.5437764378 | 1.5593371623 | 1.5290332111 |
| 1.0 | 2.000 | 1.9890026011 | 2.0123456784 | 1.9973564864 | 2.0021311073 |

### Table 3
Comparison results between the exact solution and its numerical approximation among different Riesz wavelet systems.

| $I^2\frac{\partial u}{\partial t}(x)$ of Example 5.3 | $x$ | Exact | $M(1,\Psi_{\alpha/2,1})$ | $M(2,\Psi_{\alpha/2,1})$ | $M(1,\Psi_{\alpha/2,2})$ | $M(2,\Psi_{\alpha/2,2})$ |
|---|---|---|---|---|---|---|
| 0.1 | –0.09 | –0.08914006921 | –0.0894526322 | –0.09000000537 | –0.0899997888 |
| 0.2 | –0.16 | –0.15807764520 | –0.1597304857 | –0.16000000832 | –0.1599995726 |
| 0.3 | –0.21 | –0.20798023629 | –0.2097230810 | –0.21000000887 | –0.2099994112 |
| 0.4 | –0.24 | –0.23904935042 | –0.2393907984 | –0.24000000701 | –0.2399992852 |
| 0.5 | –0.25 | –0.25129409425 | –0.2501397390 | –0.25000000725 | –0.2499991939 |
| 0.6 | –0.24 | –0.23881395932 | –0.2395154647 | –0.2399999662 | –0.2399991377 |
| 0.7 | –0.21 | –0.20744015139 | –0.2096346005 | –0.2099998898 | –0.2099991164 |
| 0.8 | –0.16 | –0.15733757741 | –0.1595806080 | –0.1599997999 | –0.1599991301 |
| 0.9 | –0.09 | –0.08836742432 | –0.0892052360 | –0.0899996959 | –0.0900000078 |
| 1.0 | 0.000 | –0.00021355936 | 0.00006223241 | 0.000000042231 | 0.000000007349 |

### Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### CRediT authorship contribution statement

**Mutaz Mohammad:** Conceptualization, Methodology, Visualization, Investigation, Supervision, Validation, Writing - review & editing. **Alexander Trounev:** Software, Writing - original draft.

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