Low-Energy Pion-Nucleon Interaction and the Sigma-Term

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Abstract

A dispersion framework, appropriate for discussing the low-energy pion-nucleon interaction, is reviewed. Sensitivity of the isoscalar D-amplitude at the unphysical Cheng-Dashen point to input from different phase shift analyses and low-energy experiments is discussed.

INTRODUCTION

In the limit of vanishing u- and d-quark masses the QCD Hamiltonian has chiral SU(2) × SU(2) symmetry. The symmetry group would be SU(3) × SU(3), if also the strange quark is treated as massless. The QCD vacuum does not, however, have the same symmetry, and the symmetry is said to be hidden, or spontaneously broken. As a consequence, the Goldstone theorem dictates that massless pseudoscalar mesons appear. They acquire a mass through the small, but nonzero, quark masses. The quark masses also shift the mass of the proton. The proton matrix element of the light quark mass term in the Hamiltonian, the sigma-term, is a measure of the explicit chiral symmetry breaking.

THE SIGMA TERM

The sigma-term is defined as

\[ \sigma = \frac{\hat{m}}{2m} \langle p | \bar{u}u + \bar{d}d | p \rangle, \]  

where \( \hat{m} = \frac{1}{2}(m_u + m_d) \), \( m \) is the proton mass and \( |p\rangle \) denotes the physical one-proton state normalized as \( \langle p' | p \rangle = (2\pi)^3 2p_0 \delta(p' - p) \). Algebraically \( \sigma \) can be written in the form

\[ \sigma \approx \frac{\hat{m}}{2m} \frac{\langle p | \bar{u}u + \bar{d}d - 2\bar{s}s | p \rangle}{1 - y}, \]

where the parameter \( y \), the strange quark content of the proton, is defined as

\[ y = \frac{2\langle p | \bar{s}s | p \rangle}{\langle p | \bar{u}u + \bar{d}d | p \rangle}. \]

The combination \( \sigma(1 - y) \) can be calculated in Chiral Perturbation Theory (CHPT), and with \( m_s/\hat{m} \approx 25 \) an estimate can be given in different orders of the quark mass expansion

\[ \sigma(1 - y) \approx \begin{cases} 26 \text{ MeV} & \mathcal{O}(m_q) \\ 35 \pm 5 \text{ MeV} & \mathcal{O}(m_q^{3/2}) \\ 36 \pm 7 \text{ MeV} & \mathcal{O}(m_q^2). \end{cases} \]

The corrections to the leading order calculation reflect the meson cloud effects. The result of the order \( m_q^{3/2} \) is from Ref. [1,2] and the \( m_q^2 \) result from Ref. [3]. In the OZI rule limit \( (y = 0) \) a value for \( \sigma \) follows to a given order of the quark mass expansion.

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A low-energy theorem of chiral symmetry allows for another determination of the $\sigma$-term. Namely, the pion-nucleon scattering amplitude at a particular unphysical (but on-shell) point, so called the Cheng-Dashen point, can be related to the nucleon scalar form factor up to corrections of order $\mu^4 \log \mu^2$ \cite{4}

$$\Sigma \equiv F_\pi^2 \bar{D}^+(\nu = 0, t = 2\mu^2) = \sigma(2\mu^2) + \Delta_R. \quad (5)$$

Here $F_\pi = 92.4$ MeV is the pion decay constant, $\bar{D}^+$ denotes the isoscalar combination of the pion-nucleon amplitudes $D = A + \nu B$, where the pseudovector Born term has been subtracted, i.e. $\bar{D} = D - D_{pv}$, $\nu = (s - u)/4m$ in terms of the Mandelstam variables and $\mu$ is the pion mass. The scalar form factor $\sigma(t)$ is defined by

$$\bar{u}(p')\sigma(t)u(p) = (p'\mid \hat{m}(\bar{u}u + \bar{d}d)\mid p) \quad ; \quad t = (p' - p)^2. \quad (6)$$

The correction term, $\Delta_R$, is formally of order $\mu^4$, and to one loop in chiral perturbation theory \cite{5} $\Delta_R = 0.35$ MeV. An estimate for the upper limit of $\Delta_R$ including all terms of the order $m_q^2$ is 2 MeV \cite{6} (it is the upper limit, because the sign of the contribution from the scalar meson is unknown; the conservative choice is taken in \cite{6}). Also, to this order in the quark mass expansion no chiral logarithm is found. In view of this result, the approximation $\Sigma \simeq \sigma(2\mu^2)$ is good enough for the present level of accuracy. In addition to fixing the value of $\Sigma$ from pion-nucleon information, one still has to consider the $t$-dependence of the scalar form factor to get an estimate for $\sigma (= \sigma(t = 0))$, so the quantity of interest is

$$\Delta_\sigma = \sigma(2\mu^2) - \sigma(0). \quad (7)$$

The first calculation was published in Ref. \cite{7} and later more refined results were given in [1,5]. CHPT to one loop gives \cite{5} $\Delta_\sigma \simeq 5$ MeV, which, however, corresponds to the treatment of the pion-pion scattering at tree level. In \cite{8} $\Delta_\sigma$ was calculated with a dispersion relation with $\pi\pi$ input consistent with CHPT. The result is

$$\Delta_\sigma = 15.2 \pm 0.4 \text{ MeV}, \quad (8)$$

which is also obtained by Bernard et al. \cite{9} in a calculation which includes some of the contributions of order $q^4$.

Using Eqs. 5 and 7 the $\sigma$ can be calculated once $\Sigma$ is fixed from pion-nucleon information, namely

$$\sigma = \Sigma - (\Delta_\sigma + \Delta_R), \quad (9)$$

i.e. the difference between $\Sigma$ and $\sigma$ is about 15 MeV with relatively small uncertainty in comparison with the current uncertainties in $\Sigma$ and the CHPT result for $\sigma(1 - y)$.

**DETERMINATION OF $\Sigma$**

Some time ago Koch used \cite{10} hyperbolic dispersion relations to determine $\Sigma$ and got the result

$$\Sigma = 64 \pm 8 \text{ MeV}, \quad (10)$$

where the uncertainty reflects the internal consistency of the method. To be able to estimate the uncertainty due to the experimental errors of the low-energy data, a dispersion method was proposed in Ref. \cite{11}. That approach involves six forward dispersion relations, the standard ones for $D^\pm$ and $\bar{D}^\pm$ and, in addition, two dispersion relations for the newly defined amplitudes

$$E^\pm(\omega) = \frac{\partial}{\partial t} (A^\pm + \omega B^\pm)|_{t=0}, \quad (11)$$
where $\omega$ is the pion total energy. The point here is that in the low-energy range the 2 $s$-waves and 4 $p$-waves matter, and the $d$- and $f$-waves can be treated as corrections (to be taken from an existing phase shift analysis; KA85). The six dispersion relations will give six energy dependent functions, the partial wave amplitudes ($l = 0, 1$). The method is a low-energy approach; it makes sense only to the extent that $d$- and higher partial waves can be treated as corrections. In Ref. [12] it was assumed that the $d$-waves were accurate to 30%, and the error estimate there involved variations of the $d$-waves by that amount around the KA85 values [13]. In addition to the low-energy $l \geq 2$ waves, input is needed for the higher energies. (The “high” energy piece starts here at the pion momentum $k_0 = 185$ MeV/c corresponding to a laboratory kinetic energy of 92 MeV.) In this range the imaginary parts of the invariant amplitudes $D^\pm$, $B^\pm$ and $E^\pm$ are needed, and they are constructed from partial wave solutions of the Karlsruhe group (KH80, KA84 and KA85) and, also, from solutions FA93, SM95 [14] and SM97 of the VPI group. For the asymptotic behaviour of the amplitudes Karlsruhe forms [15] have been adopted.

The six dispersion relations fully fix the amplitudes up to two subtraction constants both of which have their origin in the $A^+$ amplitude. The subtraction constant in the dispersion relation for $D^+$ is proportional to the $s$-wave scattering length $a_{0+}^0$, and the subtraction constant for the $E^+$ amplitude to $a_{1+}^1$. Once these two threshold parameters are known, the amplitude is fully fixed. Experimental data at low energies (below the momentum $k_0$) will be needed to determine $a_{0+}^0$ and $a_{1+}^1$. At the lowest energies there are results from differential cross section measurements which are used as input in the $\chi^2$ search. For each data set the contribution to the total $\chi^2$ is

$$\chi^2 \equiv \left( \frac{z - 1}{\Delta z} \right)^2 + \sum_{\text{DATA}} \left( \frac{z \sigma^*(\theta) - \sigma(\theta)}{\Delta \sigma(\theta)} \right)^2,$$

where $\sigma^*(\theta)$ is the experimental differential cross section with error $\Delta \sigma(\theta)$. The computed cross section is denoted by $\sigma(\theta)$, and it contains the full observable cross section constructed from the solutions for the hadronic amplitudes from the dispersion relations, and the electromagnetic corrections according to the formalism of Tromborg et al. [16]. Also, the $P_{33}$ splitting between the $\pi^- p$ and $\pi^+ p$ channels has been taken into account. For each data set there is one parameter $z$ which takes care of the normalization of that set, i.e. in the present analysis there is a contribution to the $\chi^2$ from the normalization. However, it does not depend on the number of data points in the data set. The experimental normalization uncertainty has been used for $\Delta z$, if given in the experimental paper. The iterative procedure for solving the coupled dispersion relations has then the following steps:

1. Take initial values for $(a_{0+}^+, a_{1+}^+)$, and set $\delta_l \equiv 0$ for $l \leq 1$ and $k_{\text{lab}} < k_0$.
2. Construction of the imaginary parts of the six invariant amplitudes for all momenta.
3. Solution of the dispersion relations.
4. Partial waves from the real parts of the invariant amplitudes at low energy.
5. Back to step 2 until the phase shifts stabilize.

Usually only a few iterations are needed to get smooth partial wave amplitudes.

It is practical to write the subthreshold expansion at the origin of the $(\nu, t)$ plane

$$\bar{D}^+ = d_{00}^+ + d_{10}^+ \nu^2 + d_{01}^+ t + d_{20}^+ \nu^4 + d_{11}^+ \nu^2 t + ..., \quad (13)$$

where two of the constants can be directly related to the dispersion relations

$$d_{00}^+ = \bar{D}^+(0), \quad d_{01}^+ = \bar{E}^+(0). \quad (14)$$
The expression for $\Sigma$ then gets the form
\[
\Sigma = F_\pi^2 (d_{00}^+ + 2 \mu_\pi^2 d_{01}^+) + \Delta_D = \Sigma_d + \Delta_D,
\] (15)
where the curvature term $\Delta_D$ is dominated by the $\pi\pi$ cut giving [8]
\[
\Delta_D = 11.9 \pm 0.6 \text{ MeV.}
\] (16)
However, there is also the left-hand cut contribution, so a new $\pi N$ analysis could modify this number slightly. The term linear in $t$ is a relatively sensitive quantity as can be seen from the numbers for the solutions A and B in [12]
\[
\begin{align*}
\Sigma_d &= (-91.3 + 138.8) \text{ MeV (solution A)} \\
\Sigma_d &= (-94.5 + 144.2) \text{ MeV (solution B)}
\end{align*}
\] (17)
where the first figure corresponds to the $d_{00}^+$ contribution and the second the $2d_{01}^+$ piece.

**RESULTS AND DISCUSSION**

The problem with the sigma values used to be that the value for $y$, the strange quark content, tended to be large in view of what could reasonably be expected for the strange quark contribution for the proton mass. Then it turned out [8,12], however, that the $t$-dependence of the scalar form factor was underestimated in the previous analyses, and that the new value $\Delta_\sigma \approx 15$ MeV led to a more reasonable result, about 130 MeV of the proton mass coming from the strange quark piece [12].

This analysis relied heavily on the Karlsruhe amplitudes, which included data from the 70’s and earlier. So only a very limited amount of information from the meson factories was incorporated. Especially at low energies new measurements have been performed for different observables. New data point to some changes in the Karlsruhe amplitudes, but there is, at present, no general agreement of the new amplitudes, because even the new data contain conflicts. Also, the discussion of the value of the $\pi N$ coupling constant has continued with increasing vigour. In [12] it was not possible to check the dependence of the values of $\Sigma$ on the pion-nucleon coupling, because one was bound to use the coupling strength corresponding to the input amplitudes, i.e. the Karlsruhe value $g_\pi^2/4\pi = 14.28$; $f^2 = 0.079$. The VPI group has, however, started to incorporate fixed-$t$ constraints to the phase shift analysis, which gives a possibility to make some checks of the influence of the value of the coupling. Their value for the $\pi N$ coupling is $f_\pi^2 = 0.076$ [14]. Exploratory searches were reported in [17,18] for the VPI amplitudes FA93 and SM95. Definite values for $\Sigma$ cannot be given, because the curvature term $\Delta_D$ has not been estimated with these amplitudes, but the linear part $\Sigma_d$ gets values of about 3 MeV larger than the ones resulting from the use of the Karlsruhe amplitudes when the same low-energy data sets have been used as input. Another matter is that the favoured data sets would be different from the ones which are consistent with the Karlsruhe amplitudes. In any case, the effect of the $\pi N$ coupling seems to be relatively unimportant in comparison with the error bar due to other uncertainties. The reason could be that the $\Sigma$ involves the $D^+$-amplitude where the Born term has been subtracted and, therefore, the large contribution directly proportional to $f^2$ is removed from the amplitude. In [18] some issues in the selection of the low-energy data was investigated for the case of VPI-SM95 [14] input amplitudes. The trend seems to be towards slightly larger values of the $\Sigma_d$, but the effect is typically about half of the estimated total uncertainty, so the overall picture does not change essentially.

New low-energy data has also appeared [19,20]. The differential cross section data of Ref. [19] are slightly too high in energy to be suitable for the present analysis. Ref. [20] gives the first analyzing power results at low energy, and with more data to come [21] an important...
constraint on the low-energy analysis will be obtained. A careful discussion of the $\pi^+p$ data base has recently appeared in [22].

Another piece of new information has appeared quite recently, namely, a measurement of the backward elastic $\pi^-p$ cross section [23] as a function of energy. With the KH80 input a good fit can be found, and the result is $\Sigma_d = (-86.0 + 145.8) \text{ MeV}$. This result is quite sensitive to the amplitudes and the $P_{33}$ splitting of $\pi^-p$ and $\pi^+p$. Even the $d$-waves start to matter, because the VPI-SM97 $d$-waves differ up to a factor of two from the KA85 results. No attempt was made to try to estimate the error bar due to the errors in data. A nice feature is, however, that the resulting scattering lengths are agreeing quite well with the results from pionic hydrogen experiment. The search gives

$$a_{\pi^-p} = 0.0882 \, \mu^{-1}$$
$$a_{0^+} = 0.0922 \, \mu^{-1}$$

(18)

to be compared with the experimental results from pionic hydrogen [24,25]

$$a_{\pi^-p} = 0.0883 \pm 0.0008 \, \mu^{-1}$$
$$a_{0^+} = 0.0920 \pm 0.0042 \, \mu^{-1}.$$  

(19)

Another approach for the sigma-term discussion is gradually becoming more and more useful. Namely, already now lattice calculations can make statements about the value of sigma [26,27]. Using the Feynman-Hellmann theorem

$$\sigma = \bar{m} \frac{\partial m}{\partial \bar{m}}$$

(20)

the sigma can be calculated from the quark mass dependence of the proton mass. Of course, there are the difficulties related to the question of dynamical quarks which until now cannot be treated. Also, the quark mass values in actual calculations are relatively large, but gradual improvement can be expected here. The current numbers for $\sigma$ are 40-60 MeV [26], and 50 MeV [27] where the latter calculation cites a very small error (3 MeV).

CONCLUSIONS

In this workshop views were expressed that the modern $\pi N$ data base is essentially internally consistent. In the strongly constrained low-energy analysis discussed here the picture is not quite that clear. Also, there seems to be data sets which simply have to be “thrown away” in all analyses. In my opinion, one should aim at understanding from the experimental point of view why there is so much trouble with the $\pi^+p$ data. A good example is the disagreement of the differential cross section results with the integrated cross section measurements [28,29].

Another set of problems is opened by the possibility of isospin violation which has not been addressed here. The sensitivity of the extrapolation to the Cheng-Dashen point may call for an improvement of the precision in this respect.

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