Geometry of Moduli Spaces of Flat Bundles on Punctured Surfaces.

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Abstract

For a Riemann surface with one puncture we consider moduli spaces of flat connections such that the monodromy transformation around the puncture belongs to a given conjugacy class with the property that a product of its distinct eigenvalues is not equal to 1 unless we take all of them. We prove that these moduli spaces are smooth and their natural closures are normal with rational singularities.

1 Introduction

Let \( X \) be a Riemann surface of the genus \( g > 0 \) with one puncture. We consider the moduli space of flat \( GL(n, \mathbb{C}) \)-bundles such that the monodromy transformation around the puncture belongs to a given conjugacy class \( \mathcal{C} \in SL(n, \mathbb{C}) \). We further assume that the class \( \mathcal{C} \) has property P, meaning that for the set of its eigenvalues \( (\lambda_1, \lambda_2, ..., \lambda_n) \) we have \( \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m} \neq 1 \) for any \( i_1 < i_2 < \ldots < i_m \), \( 1 \leq m < n \). We carry out all the proofs for the genus 1 and later show that all the result easily generalize for higher genera.

Due to the well-known correspondence between flat bundles on \( X \) and representations of its fundamental group, the problem (for \( g = 1 \)) is reduced to the consideration of moduli spaces of pairs of matrices \( (B, D) \) from \( GL(n, \mathbb{C}) \) such that \( BDB^{-1}D^{-1} \in \mathcal{C} \). The main result (presented in Theorem 3.2 and Proposition 3.4) is

**Theorem 1.1** Let \( X \) be a Riemann surface with one puncture and \( \mathcal{C} \) - a conjugacy class in \( SL(n, \mathbb{C}) \) with property P.

(a) The moduli space \( \mathcal{M}_\mathcal{C} \) of flat \( gl(n, \mathbb{C}) \)-connections over \( X \) with the monodromy transformations around the puncture in \( \mathcal{C} \) is smooth.

(b) Let \( \overline{\mathcal{C}} \) is the closure of \( \mathcal{C} \) in \( SL(n, \mathbb{C}) \). The variety \( \mathcal{M}_{\overline{\mathcal{C}}} \) (defined in (a) by changing \( \mathcal{C} \) to \( \overline{\mathcal{C}} \)) is normal with rational singularities.

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We extend the definition of property P to the case of orthogonal or symplectic groups to establish similar results.

We also show that if \( BDB^{-1}D^{-1} \) has property P, then the pair \( (B, D) \) algebraically generates the whole group.

The moduli spaces in question are of great importance by a number of reasons. A theorem of Mehta-Seshadri \([7]\) identifies two moduli spaces: the space of unitary representations of fundamental group and the space of parabolic bundles on \( X \). When \( C = \exp(2\pi \sqrt{-1}d/n)Id \) is (the class of) a central element, the space \( \mathcal{M}_C \) is a smooth Kähler manifold and appears in algebraic geometry as the space of holomorphic vector bundles on the closed surface of rank \( n \), degree \( d \) and fixed determinant \([1]\). Also those moduli spaces appear in topological and quantum field theories; they are related to Jones-Witten invariants (see \([2]\) for details). They are closely related to Yang-Mills theory and geometric quantization. It is necessary to mention the results due to Simpson \([13]\), which provide natural correspondence between \( D_X \)-modules, Higgs bundles and local systems on \( X \) (with extra conditions). This allows one to identify moduli spaces of those objects with the moduli spaces we consider in the present paper.

I express deep gratitude to Jean-Luc Brylinski for helpful and valuable advices. He kindly guided me and generously shared his knowledge.

2 Common stabilizer of two matrices

We say that a matrix \( C \) or a conjugacy class \( C \) in \( SL(n, \mathbb{C}) \) with the set of eigenvalues \( \{\lambda_1, \ldots, \lambda_n\} \) has property P if the following condition holds. For any \( m < n \) distinct numbers \( 1 \leq i_1 \leq i_2 \leq \ldots \leq i_m \leq n \) the product \( \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_m} \) is not equal to 1. One notices that the set of matrices with property P is Zariski open in \( SL(n, \mathbb{C}) \).

**Theorem 2.1** Let \( B, D \in GL(n, \mathbb{C}) \) be such that \([B, D]\) satisfies property P. Then the common stabilizer of \( B \) and \( D \) consists of scalar matrices only.

**Proof.** Let \( K \in GL(n, \mathbb{C}) \) be non-central matrix commuting with both \( B \) and \( D \). Let \( \lambda \) be an eigenvalue of \( K \) and let \( W \subset \mathbb{C}^n \) be the kernel of \( K - \lambda.Id \). It follows that both \( B \) and \( D \) stabilize \( W \). Hence the product of eigenvalues of \([B, D]\) which correspond to \( W \) is equal to 1. This means that \( W = \mathbb{C}^n \) and \( K \) is scalar. \( \Box \)

We will denote by \( SL(n, \mathbb{C})^2 \) or \( GL(n, \mathbb{C})^2 \) the Cartesian product of two copies of \( SL(n, \mathbb{C}) \) or \( GL(n, \mathbb{C}) \) respectively. Also denote by

\[
\kappa = [\cdot, \cdot] : GL(n, \mathbb{C})^2 \rightarrow SL(n, \mathbb{C})
\]

the commutator map.
Remark. For \( n > 2 \) the author can prove the following statement converse to the above Theorem. Let \( \mathcal{C} \) be a conjugacy class in \( SL(n, \mathbb{C}) \). If for any \( (B, D) \in GL(n, \mathbb{C})^2 \) the condition \([B, D] \in \mathcal{C}\) implies \( \dim Z(B, D) = 1 \), then \( \mathcal{C} \) has property \( P \). We do not include the proof since it is long and computational and we will not use it in the present paper.

The situation, however, is different for \( n = 2 \). If we take as \( \mathcal{C} \) the class of \(
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\)

then for all \( \{(B, D) \in GL(2, \mathbb{C})^2; [B, D] \in \mathcal{C}\} \) the common stabilizer of \( B \) and \( D \) denoted by \( Z(B, D) \) is the center of \( GL(n, \mathbb{C}) \), i.e. \( \dim Z(B, D) = 1 \). So, when \( n = 2 \), if \( B \) and \( D \) do not commute, then their common stabilizer is the center of \( GL(n, \mathbb{C}) \).

One can think of another interpretation of property \( P \). Let \( V = \mathbb{C}^n \) be the tautological representation space of \( SL(n, \mathbb{C}) \). The spaces \( \wedge^i V \) are also naturally representation spaces for \( SL(n, \mathbb{C}) \). Property \( P \) for the matrix \( C \) means that \( C \) doesn’t stabilize any \( \neq 0 \) vector in \( \wedge^i V \) for \( 0 < i < n \).

The next proposition was first proven in [12].

**Proposition 2.2** The map \( \kappa \) is onto.

For instance, the commutator of two matrices from \( SL(n, \mathbb{C}) \)

\[
\begin{pmatrix}
0 & e_1 & 0 & \ldots & 0 \\
0 & 0 & e_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & e_{n-1} \\
en & 0 & 0 & \ldots & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 & 0 & \ldots & f_n \\
f_1 & 0 & 0 & \ldots & 0 \\
f_2 & 0 & 0 & \ldots & 0 \\
0 & f_3 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots 
\end{pmatrix}
\]

can be conjugate to any semisimple element of the group. Also for any unipotent element \( U, U^{-1} \) is conjugate to \( U \), hence \( U^2 \) is also in the image of \( \kappa \). Any unipotent element is a square of another unipotent element. This proves the proposition for unipotent elements.

### 3 Moduli spaces of flat bundles

Let us fix a conjugacy class \( \mathcal{C} \) in \( SL(n, \mathbb{C}) \). Further we consider the variety of pairs and the moduli spaces of pairs of matrices with their commutator in \( \mathcal{C} \). We denote \( X_{\mathcal{C}} = \{(B, D) \in GL(n, \mathbb{C})^2; [B, D] \in \mathcal{C}\} \). It is well-known that if \( \mathcal{C} = 1 \) then the variety of commuting pairs is irreducible ([8]), but is not a smooth variety.

**Lemma 3.1** The orbit of every element \( (B, D) \), such that \([B, D] \in \mathcal{C}\) is closed in \( X_{\mathcal{C}} \) if the class \( \mathcal{C} \) satisfies property \( P \).
Proof. All orbits in the closure of $X_C$ have the same dimension when $C$ has property P. (Because all the conjugacy classes in the closure of $C$ still have property P, and whenever $[B, D]$ has property P the stabilizer $Z(B, D)$ coincides with the center of the group.) This implies that orbits are closed in $X_C$. ⊓⊔

Actually, we have proved a stronger result, namely that the orbit of $(B, D)$ is closed in $GL(n, \mathbb{C})^2$ if property P holds for $[B, D]$.

We consider the moduli space

$$M_C = \{(B, D) \in GL(n, \mathbb{C})^2; [B, D] \in C\}/SL(n, \mathbb{C}),$$

where factoring occurs by the adjoint action of the special linear group. In order to define $M_C$ properly one takes the quotient in the sense of the GIT ([9]). In the case of an affine variety $X$ there is a natural definition of the quotient variety $Y = X/G$. The algebra of the regular functions $O(Y)$ is just the subalgebra of $O(X)$ of $G$-invariant regular functions on $X$. But when all orbits are closed, each point of the quotient $M_C$ correspond to an orbit in $X_C$.

Using the above lemma we see that when $C$ is semisimple with property P, the variety $X_C$ is closed affine, the quotient $X_C/SL(n, \mathbb{C})$ exists and its points correspond exactly to orbits. It is well-known that $\bar{C} \setminus C$ is the union of finite number of conjugacy classes $C_i$. So we have the corresponding finite set of closed subvarieties $M_i$ in $M_{\bar{C}}$. We define now the algebraic variety $M_C$ which is the complement of the union $\cup_i M_i$ in $M_{\bar{C}}$.

Example. Here we consider the simple but important case $n = 2$. We have five types of conjugacy classes in $SL(2, \mathbb{C})$ (here we mention just a representative of each): $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (dim($I$) = 0), $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (dim($-I$) = 0), $R_2 = \text{the class of} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, (dim($R_2$) = 2), $R_e = \text{the class of} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, (dim($R_e$) = 2), $R_\lambda = \text{the class of} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda^2 \neq 1$, (dim($R_\lambda$) = 2). One has

$$GL(2, \mathbb{C})^2 = X_I \cup X_-I \cup X_{R_2} \cup X_{R_e} \cup \bigcup_{\lambda^2 \neq 1} X_{R_\lambda}.$$  

Each $X_C$ except for $X_I$ is a connected smooth variety. Their dimensions are 6, 5, 7, 7, 7 respectively. The varieties $X_C$ corresponding to semisimple classes are closed. Of course, $X_I$ and $X_-I$ lie in the closure of $X_{R_2}$ and $X_{R_e}$ respectively. Also, they both are limit varieties of $X_{R_\lambda}$ for $\lambda \to \pm 1$, so they both lie in the
Before we prove this assertion, we exhibit an auxiliary result.

**LEMMA 3.3** If $[B, D] = A$ and $\dim Z(B, D) = 1$ then the differential map $ds : T_{(B, D)}GL(n, \mathbb{C})^2 \to T_{A} SL(n, \mathbb{C})$ is surjective$^1$.

**Proof.** We identify the first tangent space with $gl(n, \mathbb{C})^2$ and the second one with $sl(n, \mathbb{C})$ via the corresponding left multiplications. Computations show that with these identifications $ds$ sends $(x, y) \in gl(n, \mathbb{C})^2$ to $DB(D^{-1}x D - x + y - B^{-1}yB)B^{-1}D^{-1}$. So, it is enough to show that $R(AdD - 1) + R(AdB - 1) = sl(n, \mathbb{C})$, where $R(L)$ denotes the range of a linear operator $L$. With respect to the bilinear form $Tr(XY)$ one has $R(AdB - 1) = Ker(AdB - 1)^\perp$. We notice that $Ker(AdB - 1) = z(B)$ - the centralizer of $B$ in the Lie algebra. Now the condition of the lemma implies that

$$R(AdD - 1) + R(AdB - 1) = Ker(AdB - 1)^\perp + Ker(AdD - 1)^\perp =$$

$$(z(B) \cap z(D))^\perp = z(B, D)^\perp = sl(n, \mathbb{C}). \quad \Box$$

**Proof of the theorem.** Let us consider the morphism of algebraic varieties $\kappa : X_\mathcal{C} \to \bar{C}$. We notice that every conjugacy class in the closure of a conjugacy class with property $P$ also satisfies property $P$. To see that $\kappa$ is actually a smooth morphism, one notices that smoothness is preserved by base extensions. Let $U$ be the open set of matrices with property $P$ and $U^{(2)} \subset GL(n, \mathbb{C})^2$ its preimage under $\kappa$. We saw in Lemma 3.3 that $\kappa : U^{(2)} \to U$ is smooth. For a class $\mathcal{C}$ with property $P$ we have the inclusion $\bar{C} \hookrightarrow U$. Now we make the base change $U^{(2)} \times_U \bar{C} \to \bar{C}$.

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$^1$Jean-Luc Brylinski noticed that it should be true in every characteristic.
We conclude that $\kappa : X_{\bar{C}} \to \bar{C}$ is a smooth morphism. Now we invoke the theorem of Kraft and Procesi ([5]) which tells us that $\bar{C}$ is normal, Cohen-Macaulay with rational singularities. As a consequence we obtain the fact that $X_{\bar{C}}$ has rational singularities. Now we use the theorem of Boutot [4], which implies that $M_{\bar{C}}$ has also rational singularities. In particular, $M_{\bar{C}}$ is normal and Cohen-Macaulay. 

Actually, we have proved that every connected component of $X_{\bar{C}}$ (and $M_{\bar{C}}$) is normal. But it seems likely that those varieties are connected. As far as we know this is an open problem.

It is very possible that whenever a class $C$ in the image of $\kappa$ is not the identity, we can find a pair $(B, D) \in X_{C}$ with trivial $Z(B,D)$. If it would be so, one could reprove the results of Shoda and Ree (Propositions 2.2 and 5.2) in a nice algebraic way as follows. Let $X_I \subset G \times G$ be the variety of commuting matrices. The above result on the surjectivity of $d\kappa$ implies that the image of $\kappa$ restricted to $(G \times G) \setminus X_I$ is open in $G \setminus \{I\}$. We know from previous explicit constructions that every semisimple or unipotent element is in the image of $\kappa$. One needs only to remark that each conjugacy class in $G \setminus \{I\}$ is either unipotent or contains in its closure a semisimple element $\neq I$.

**Proposition 3.4** If $C$ has property $P$ then $M_{\bar{C}}$ is a smooth algebraic variety.

**Proof.** The statement follows from the above theorem, because the fibers of the map $\kappa : X_{\bar{C}} \to \bar{C}$ are smooth varieties.

By a procedure similar to the one we described above, it is possible to define the space $M_C$ for any conjugacy class $C$. (First, we define $M_{\bar{C}}$ and then using its natural stratification we throw out irrelevant pieces.) The following lemma which was pointed out to me by J.-L. Brylinski calculates the dimension of the tangent space to $M_C$ at the class $(B,D)$. A point $(B,D) \in X_{C}$ is called a general point if $X_{C}$ is smooth at $(B,D)$ and all stabilizers of elements in some neighbourhood of $(B,D)$ in $X_{C}$ form a smooth group bundle.

**Lemma 3.5** Let $C$ be a conjugacy class in $SL(n,\mathbb{C})$. For a general point $(B,D) \in X_{C}$ one has $\dim(T_{(B,D)}M_{C}) = \dim C + 2\dim Z(B,D)$.

**Proof.** The loops $a$ and $b$ generate freely the fundamental group of the elliptic curve with one puncture $\Sigma \setminus \{O\}$. The corresponding monodromy transformations $B$ and $D$ define the local system $V$ on $\Sigma$ of the dimension $n$. Let us define a manifold $X$ as consisting of all conjugacy classes of homomorphisms $\rho : \pi_1(\Sigma \setminus \{O\}) \to SL(n,\mathbb{C})$ such that the image of $\rho$ is Zariski dense in $SL(n,\mathbb{C})$. (So that $M_{\bar{C}}$ is a subspace of $X$.) The tangent space to $X$ in the class of $\rho$ identifies, by a well-known theorem of A. Weil, with the group cohomology $H^1(\pi_1(\Sigma \setminus \{O\}), g)$, where $g = sl(n,\mathbb{C})$ is a $\pi_1(\Sigma \setminus \{O\})$-module via
the adjoint action followed by $\rho$. For any $A \in SL(n, \mathbb{C})$ we identify the tangent space $T_A SL(n, \mathbb{C})$ to $g$ via the action of the left translation by $A$. The tangent space to the conjugacy class $C_A$ of $A$ is the subspace of $g$ given as the range of $Ad(A) - 1 : g \to g$. If $\Gamma$ is the cyclic subgroup generated by $\gamma = aba^{-1}b^{-1}$ then the cohomology group $H^1(\Gamma, g)$ is the cokernel of the map $Ad(\Gamma) - 1$. So the tangent space $T_{[\rho]}M_\mathbb{C}$ identifies with the subspace $\text{Ker}(H^1(\pi_1(\Sigma \setminus \{O\}), g) \to H^1(\Gamma, g))$ of $T_{[\rho]}X = H^1(\pi_1(\Sigma \setminus \{O\}), g)$. Thus,

$$T_{(B,D)}M_\mathbb{C} = \text{Ker}[H^1(\Sigma \setminus \{O\}, End(V)) \to H^1(D^*, End(V))],$$

(3.1)

where $D^*$ is a small disk around the puncture. (It is the same as $\text{Ker}[H^1(\pi_1(\Sigma \setminus \{O\}), End(V)) \to H^1(\mathbb{Z}, End(V))]$, because $\pi_1(D^*) = \mathbb{Z}$.) The Euler characteristic of the punctured elliptic curve is $\chi(\Sigma \setminus \{O\}) = -1$ and whence,

$$\dim[H^0(\Sigma \setminus \{O\}, End(V))] - \dim[H^1(\Sigma \setminus \{O\}, End(V))] = n^2\chi(\Sigma \setminus \{O\}) = -n^2.$$

But $H^0(\Sigma \setminus \{O\}, End(V)) = End(V)^{B,D} = Z(B, D)$ in $GL(n, \mathbb{C})$. So, $\dim H^1(\Sigma \setminus \{O\}, End(V)) = n^2 + \dim Z(B, D)$. Also we notice that the map in the equation (3.1) is onto due to the exact sequence\(^1\)

$$H^1(\Sigma \setminus \{O\}, End(V)) \to H^1(D^*, End(V)) \to H^2(\Sigma \setminus \{O\}, End(V)) \to H^2(\Sigma \setminus \{0\}, End(V)) \to 0.$$

The group $H^2(\Sigma \setminus \{0\}, End(V))$ is 0 for dimension reasons. Besides, one sees that the group $H^2(\Sigma \setminus \{0\}, End(V))$ is dual to $H^0(\Sigma \setminus \{0\}, End(V))$, because the local system $End(V)$ is self-dual. The group $H^0(\Sigma \setminus \{0\}, End(V))$ is the group of matrices which commute with $B$ and $D$. It follows that the image of the linear map $H^1(\Sigma \setminus \{0\}, End(V)) \to H^1(D^*, End(V))$ has dimension equal to

$$\dim H^1(D^*, End(V)) = \dim Z(B, D) = \dim Z([B, D]) - \dim Z(B, D).$$

Therefore the dimension of the kernel of this same map is equal to

$$\dim H^1(\Sigma \setminus \{0\}, End(V)) - \dim Z([B, D]) + \dim Z(B, D)$$

which is equal to

$$n^2 + \dim Z(B, D) - \dim Z([B, D]) + \dim Z(B, D) = \dim(\mathcal{C}) + 2 \dim Z(B, D).$$

Here we used the fact that $n^2 = \dim GL(n, \mathbb{C}) = \dim(\mathcal{C}) + \dim Z(g), g \in \mathcal{C}$.  

\(^1\)The fact that $T_{(B,D)}M_\mathbb{C} = \text{Im}(H^1(\Sigma \setminus \{0\}, End(V)) \to H^1(\Sigma \setminus \{0\}, End(V)))$ was also used in [3].
COROLLARY 3.6 \( \dim(X_C) = n^2 + \dim C + \dim Z(B, D) \), where \((B, D)\) is a generic element of \(X_C\).

4 On subalgebras generated by pairs

Let \(p\) be an integer, \(p > 1\), and let \(G^p = G \times \ldots \times G\), where \(G\) is a reductive algebraic group (over \(\mathbb{C}\)) with Lie algebra \(g\). Let \(x = (x_1, \ldots, x_p) \in G^p\), \(G.x\) - its orbit, and \(A(x)\) - the algebraic subgroup of \(G\) generated by the set \(\{x_1, \ldots, x_p\}\). (So \(A(x)\) is the Zariski closure of the subgroup of \(G\) in the abstract sense generated by the set \(\{x_1, \ldots, x_p\}\).) Let also \(\pi : G^p \to G^p/G\) stand for the quotient morphism. Following Richardson \[11\] we call \(x\) a semisimple \(p\)-tuple if \(A(x)\) is linearly reductive. (Since we are in characteristic zero this it is equivalent to reductive.) We cite from \[11\] the following

THEOREM 4.1 The orbit \(G.x\) is closed if and only if \(x\) is semisimple.

This allows us to apply our knowledge to the situation of the group \(GL(n, \mathbb{C})\) and \(p = 2\).

LEMMA 4.2 If \(C\) satisfies property \(P\) and \([B, D] \in C\) then the algebraic subgroup \(A(B, D)\) of \(GL(n, \mathbb{C})\) is reductive.

\(\Box\)

PROPOSITION 4.3 If \(C\) satisfies property \(P\) and \([B, D] \in C\), then \(A(B, D) = GL(n, \mathbb{C})\).

\(\Box\)

5 Symplectic and orthogonal groups

Here we will formulate property \(P\) for orthogonal and symplectic groups. It turns out that many of the results that we proved remain valid for other semisimple algebraic groups. Let \(G\) be either \(SO(2n, \mathbb{C})\), \(SO(2n + 1, \mathbb{C})\), or \(Sp(2n, \mathbb{C})\). The group \(G\) naturally acts on \(\mathbb{C}^{2n(+1)}\) preserving the bilinear form \((,\)\). It is well-known that if \(\lambda\) is an eigenvalue of \(A \in G\) then \(\lambda^{-1}\) is an eigenvalue too of the same multiplicity and partition.

Let \(\lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \ldots, \lambda_n^{\pm 1}\) (and 1 in the case of \(SO(2n + 1, \mathbb{C})\)) be the set of eigenvalues of a conjugacy class \(C \subset G\). We say that \(C\) has property \(P\) if no
product of its eigenvalues of the form $\prod_{j \in S} \lambda_j^{e_j}$ is equal to one, where $S$ is non-empty subset of $\{1, 2, ..., n\}$ and $e_j = \pm 1$. For an element $A \in \mathcal{C}$ this is equivalent to the condition that for any isotropic subspace $V$ preserved by $A$ the product of eigenvalues of $A$ in $V$ is not equal to 1.

Remark. One can formulate property $P$ for an element $C$ for any group $G$ as follows. Let $C = C_s C_u$ be a Jordan decomposition into a product of commuting unipotent and semisimple elements and let $C'$ be the standard representation space of $G$. Consider $V = \oplus_{j=0}^n \wedge^j C'$, which is naturally a representation space of $G$ too. Now we say that $C$ (or its conjugacy class in $G$) has property $P$ if the following two stable subspaces of $V$ have the same dimension: $\dim(V^{C_s}) = \dim(V^T)$, where $T$ is a maximal torus of $G$.

Suppose that an element $K \in G$ belongs to the common stabilizer of elements $B$ and $D$. Also we make an assumption that $K^2 \neq I$ - the identity matrix. We will show that we may always find an isotropic subspace $V$ preserved by both $B$ and $D$. It means in turn that the product of eigenvalues of $[B, D]$ in $V$ is equal to 1. At first, we consider the case when $K$ has an eigenvalue $\lambda \neq \pm 1$. Here we may take as $V$ the kernel of $K - \lambda \text{Id}$. It remains to assume that $K$ has only $\pm 1$ as the set of eigenvalues. The condition that $K^2 \neq I$ implies that there exists a vector $y \in \mathbb{C}^{2n(1)}$ such that $y$ belongs to the generalized eigenspace of 1 or $-1$ and $Ky \neq \pm y$. It is equivalent to the condition that there is an eigenvalue $\mu$ of $K$ (which is, of course, 1 or $-1$), and a $K$-irreducible and $K$-invariant subspace $Y$ such that $\dim Y > 1$. Let us consider the subspace $W = \text{Ker}(K - \mu \text{Id})$. Define the subspace $V \subset W$ consisting of all $v \in W$ such that the equation $K^{-1}x = \mu x + v$ has a solution. One notices that $V$ is the intersection of $\text{Ker}(K - \mu)$ with $\text{Im}(K^{-1} - \mu)$. Then for $v \in V$ we have

$$\langle v, v \rangle = \langle v, K^{-1}x - \mu x \rangle = \langle Kv - \mu v, x \rangle = 0.$$ 

Moreover, if $B$ commutes with $K$, then $V$ is also $B$-invariant. The fact that $V \neq 0$ follows from the choice of $\mu$.

Now we will simply rewrite some of the statements which we proved before for the group $SL(n, \mathbb{C})$ and which continue to be true in the case when $G = \text{SO}(2n, \mathbb{C})$, $\text{SO}(2n + 1, \mathbb{C})$, or $\text{Sp}(2n, \mathbb{C})$. We just saw that

**Theorem 5.1** Let $B, D \in G$ be such that $[B, D]$ satisfies property $P$. Then the common stabilizer of $B$ and $D$ is finite of exponent 2.

Let $G^2$ stand for the cartesian product of two copies of $G$ and let $\kappa = [\cdot] : G^2 \to G$ be the commutator map. The next proposition is quoted from [10]:

**Proposition 5.2** The map $\kappa$ is onto.

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1The idea of this nice remark is due to Ranee Brylinski.
Analogously to Lemma 2.3 one may prove the following

**LEMMA 5.3** For \( n > 2 \) and a conjugacy class \( C \subset G \) the following two conditions are equivalent:

(i) \( C \) has property \( P \).

(ii) For any \((B, D) \in G^2\) such that \([B, D] \in C\) we have \( \dim Z(B, D) = 0 \).

Also we have

**LEMMA 5.4** If \( C \) has property \( P \) and \([B, D] \in C\) then the orbit of the element \((B, D) \in G^2\) is closed.

Just as in section 3, we may define the space \( X_C \) and the moduli space \( M_C \).

**PROPOSITION 5.5**

(a) \( \dim(M_C) = \dim C + 2 \dim Z(B, D) \) and \( \dim X_C = \dim(G) + \dim C + \dim Z(B, D) \), for a generic element \((B, D) \in X_C\).

(b) If \( C \) has property \( P \), then \( M_C \) has at worst quotient singularities.

Unfortunately, we cannot say anything about the normality of \( M_\overline{C} \), because the closures of conjugacy classes in \( G \) are not always normal (cf. [5]).

The result from the section 4 about the algebraic subgroup \( A(B, D) \) generated by a pair \((B, D) \in G^2\) still generalizes:

**PROPOSITION 5.6** If the commutator \([B, D]\) belongs to a conjugacy class \( C \) with property \( P \), then \( A(B, D) = G \).

**Remark.** We saw that the questions of normality of variety \( M_\overline{C} \) and rationality of its singularities depend only on the geometry of \( \overline{C} \) if the class \( C \) has property \( P \). In the case of orthogonal or symplectic group there occur different phenomena which are described by Kraft and Procesi in [3]. Using their results one can easily prove the analog of the part (b) of the above theorem for the group \( SO(n, \mathbb{C}) \) or \( Sp(2n, \mathbb{C}) \) according to whenever \( \overline{C} \) is normal.

6 Generalizations for \( p \)-tuples.

Here we briefly discuss how results of the paper can be extended to the case of \( p \)-tuples.

Let us consider the operation \( \kappa := [,] : G^p \to G \),

\[[A_1, A_2, \ldots, A_p] = A_1A_2\cdots A_pA_1^{-1}A_2^{-1}\cdots A_p^{-1} \]

It is clear what property \( P \) means in this case. It is still true that if \([A_1, A_2, \ldots, A_p] \in C \) has property \( P \), then the stabilizer \( Z(x) \) (where \( x := (A_1, A_2, \ldots A_p) \)) is the center of \( G \) in the case \( G = \text{SL}(n, \mathbb{C}) \) and is finite of exponent 2 when \( G \) is orthogonal or symplectic group. Taking \( A_3 = A_4 = \cdots = A_p = 1 \) we see that this fact actually characterizes the class \( C \) and that
the map \( \kappa \) is onto. If \( [A_1, A_2, \ldots, A_p] \) has property \( P \), then its orbit is closed.

As before, we can define the spaces \( X_C \) and \( M_C \). They are of dimensions \((p - 1) \dim(G) + \dim(C) + \dim(Z(x))\) and \((p - 2) \dim(G) + \dim(C) + 2 \dim(Z(x))\). When \( G = GL(n, \mathbb{C}) \) and the class \( C \) has property \( P \), both \( X_C \) and \( M_C \) are smooth varieties. Notice, that in the application to Riemann surfaces \( p \) is even and so is the dimension of \( M_C \).

In the case \( G = SL(n, \mathbb{C}) \) the same method as in section 3 proves the normality of \( M_\bar{C} \). We also deduce that if \( x \) has property \( P \), then the set \( \{A_1, A_2, \ldots, A_p\} \) generates the whole group \( G \).

**Remark.** Let \( \bar{X} \) be a compact oriented surface of genus \( g > 0 \), let \( s_1, \ldots, s_k \in \bar{X} \) and let \( X = \bar{X}\backslash\{s_1, \ldots, s_k\} \) be the punctured surface. We consider local systems of rank \( n \) on \( X \). Assume that \( a_1, a_2, \ldots, a_{2p} \) are loops in \( X \) generating the fundamental group \( \pi_1(\bar{X}, x_0) \). Also let \( \gamma_i \) be a loop which comes from \( x_0 \), goes once around \( s_i \) counterclockwise and then goes back to \( x_0 \). One can take them in such a way that \( \gamma_1 \gamma_2 \cdots \gamma_k = [a_1, a_2, \ldots, a_{2p}] \). Let us have \( k \) matrices \( C_1, \ldots, C_k \) defined up to simultaneous conjugation, such that \( C_i \) is the monodromy transformations of a local system \( V \) on \( X \) corresponding to \( \gamma_i \). So we have the corresponding matrix equation: \( C_1 C_2 \cdots C_k = [A_1, A_2, \ldots, A_{2p}] \). Here \( A_1, \ldots, A_{2p} \) are the monodromy transformations corresponding to the generators of the fundamental group of the compactified curve. (The local system \( V \) is completely determined by the \( 2p + k - 1 \) matrices \( C_1, C_2, \ldots, C_k, A_1, A_2, \ldots, A_{2p} \).)

Under the assumption that \( \det(C_1 \cdots C_k) = 1 \) Proposition 2.2 (and 5.2) shows that we always can find a solution of this equation. When the genus is zero this is not always the case (see [14] for details).

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