SPECTRUM OF COMPACT MANIFOLDS WITH HIGH GENUS

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Abstract. In this paper we study the behavior of the spectrum of a compact, connected Riemannian manifold \((M, g)\) of dimension \(d \geq 2\), when we add an increasing number of increasingly small handles. No assumptions on any of the curvatures are needed.

Introduction

Let \((M, g)\) be a compact (connected, Riemannian) manifold (with or without boundary) of dimension \(d \geq 2\); let \(g = (g_{ij})_{i,j=1}^{d}\) be its metric. We shall find convenient to work in the category of Lipschitz manifolds, namely the coefficients of the metric \(g_{ij}, i,j = 1\ldots,d\), are bounded, measurable functions and changes of coordinates are related by bi-lipschitzian diffeomorphisms.

We introduce the spectrum of a Lipschitz manifold \((M, g)\) in Definition 1.7; if \((M, g)\) happens to be a smooth manifold, then the spectrum of \((M, g)\) is related with sequence of eigenvalues of the Laplace-Beltrami operator \(-\Delta_g\) of \((M, g)\) (with "natural" boundary conditions) in an obvious way; cf. Definition 1.8 and Remark 1.9.

In this paper we shall be interested in exhibiting topological perturbations of \(M\) which affect the spectrum of \((M, g)\); more precisely we shall add an increasing number of increasingly small handles to \(M\), call \((N_h, g^{N_h})\) the resulting manifold, and study the spectrum of \((N_h, g^{N_h})\) as \(h \to \infty\).

Our main result is Theorem 1 where we prove that, provided that the handles have a suitable girth, the spectrum of the Laplace-Beltrami operator of \((N_h, g^{N_h})\) with "natural" boundary condition (cf. Definition 1.8) converges to the spectrum of the operator induced by the quadratic form

\[
D_\infty(u) := D_M(u) + \frac{\alpha}{2^d} \int_M [u(x) - u(T(x))]^2 \text{Vol}_g(dx) + \\
+ \lambda \int_M u^2 \text{Vol}_g(dx), \quad u \in H^1(M, g); 
\]

\[(*)\]
$T : M \to M$ is an involution which is induced by the process of attaching handles to $M$. The convergence of the spectrum mentioned above means convergence of the eigenvalues and of the eigenspaces (Definition 1.4) generated by the corresponding eigenfunctions. It should also be noted that the involution $T$ induces an orthogonal decomposition of both $L^2(M,g)$ and $H^1(M,g)$ into “odd” and “even” subspaces; therefore formula (*) above implies that the process of attaching handles affects only the odd part of the spectrum; cf. the discussion after Theorem 1.

The proof of our main result relies on two general results, Theorem 2 and Theorem 3, which have a stronger analytical flavor and may have some interest in their own. By using a general result by G. Dal Maso, R. Gulliver & U. Mosco in \cite{6} (cf. Proposition 2.17 below), we prove Theorem 1 in the framework of relaxed Lipschitz manifolds (Definition 1.18), the reason being that relaxed Lipschitz manifolds are better suited for the analysis via $\Gamma$-convergence that will be used in this paper.

We notice that Theorem 1, Theorem 2 and Theorem 3 generalize some results in \cite{6, §§4,5}: There those authors considered Lipschitz manifolds-with-boundary topologically equivalent to bounded open sets of $\mathbb{R}^d$, while our results deal with compact Lipschitz manifolds, possibly without boundary.

Theorem 1 is proved in the case of $(M,g) = (S^d,g)$, the $d$-dimensional sphere of radius 1, with its standard metric of constant sectional curvature. We have made this choice so as to approach and present the problem in a hopefully intuitive way. Even though the metric of the “base” manifold $M = S^d$ is smooth, we stress that the metric of the manifold which results from attaching handles to $M$ is in general only Lipschitz; cf. Remark 2.15 and also Remark 3.3. We also stress that, adapting some arguments in \cite{6, §§2,3}, it is possible to prove Theorem 1 in the full generality of Lipschitz manifolds $(M,g)$.

Theorem 2 and Theorem 3 deal with variational limits of sequences of perturbed Dirichlet functionals. Theorem 2 may be thought of as a compactness result for Lipschitz metrics, while Theorem 3 is a compactness result for non-local perturbations of their associated Dirichlet functionals. We point out that the convergence of manifolds treated in Theorem 2 is quite different from the convergence of manifolds as studied, among others, by R.E. Greene, M. Gromov, S. Peters and H. Wu \cite{11}, \cite{10}, \cite{15} in that we need no bounds on any of the curvatures; cf. Remark 2.15.

To obtain our result of convergence of the spectrum (Theorem 1) we must consider \textit{handles of bounded thinness}, according to the terminology
of [3]; cf. Definitions 2.6, 2.11 and Remark 2.14. This assumption is necessary in that if we allow “long-thin” handles such as $[-L, L] \times S^{d-1}(\varepsilon)$ (isometrically imbedded in $M$, as $\varepsilon \downarrow 0$) then C. Anné [1] proved that the spectrum of the resulting manifold “converges” to the spectrum of $M$ plus the spectrum of $d^2/dt^2$ on $(-L, L)$ with Dirichlet boundary condition at $t = \pm L$.

With different methods (and treating the case of smooth manifolds), I. Chavel and E.A. Feldman considered in [4], among other issues, the problem of attaching one handle to $M$ and determining the size of this handle in order to have negligible perturbations of the eigenvalues of $M$ ([4, Theorem 5]). They, too, have to rule out long-thin handles and their condition involves the isoperimetric constant of the resulting manifold.

The organization of the paper is as follows. In the first section we introduce the basic definitions and notation, define relaxed Lipschitz manifolds (Definition 1.18) and make precise what we mean by “representing a Lipschitz manifold in term of a relaxed Lipschitz manifold” (Definition 1.19).

In section 2 we introduce and attach handles to $(S^d, g)$ (Definition 2.11). Then we represent the resulting manifold by means of a relaxed Lipschitz manifold-with-boundary of simpler topological type (Proposition 2.17).

Section 3 is devoted to the statements of our main result, the proof of which is carried out in Section 4 and rests on Theorem 2 and Theorem 3; because of the general nature of these two results, we state and prove them separately in the Appendix, § 5.

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1. Notation & Preliminaries.

When we define a quantity $A$ in term of other known quantities $B$, we write $A := B$.

The disjoint union of two sets $A, B$ is denoted by $A \setminus B$. 
We consider real-valued functions; if \( u \) is a function defined on some set \( X \), then \( u|_{A} \) denotes its restriction to \( A \subset X \), \( u|_{A} : A \to \mathbb{R} \).

We let \( 1_{A}(\cdot) \) denote the characteristic function of the set \( A \), namely \( 1_{A}(x) = 1 \), if \( x \in A \), and zero otherwise.

A measure \( \mu \) on a measurable space \((X, \mathcal{A})\) is a countably additive function \( \mu : \mathcal{A} \to [0, +\infty] \) defined on the \( \sigma \)-algebra \( \mathcal{A} \) which vanishes on the empty set. Notice that we allow \( \mu \) to assume the value +\( \infty \). If \( X \) is a topological space, then a Borel measure is a measure defined on the \( \sigma \)-algebra of the Borel sets \( \mathcal{B} \) of \( X \). We shall only consider Borel measures in the following and sometimes for the sake of shortness we shall call them measures.

If \( X \) is a topological space, then the closure of a set \( A \subset X \) is denoted by \( \overline{A} \); moreover \( A \subset \subset X \) means \( A \subset \overline{A} \subset X \).

Let \((X,d)\) be a (separable) metric space, and let \( B(x,\eta) \) denote the open ball of center \( x \in X \) and radius \( \eta > 0 \).

**Definition 1.1** (\( \eta \)-packing). Let \( \eta > 0 \); a \( \eta \)-packing in \((X,d)\) is a collection of points \( \{x_i\}_{i \in I} \) in \((X,d)\) such that:

\[(p_1) \ B(x_i,\eta) \cap B(x_j,\eta) = \emptyset, \text{ for every } i \neq j, i,j \in I; \]
\[(p_2) \ \bigcup_{i \in I} B(x_i,2\eta) = X. \]

We can easily construct an \( \eta \)-packing in every metric space \((X,d)\), in particular in a manifold. Indeed, let \( \eta > 0 \) be given, and let \( x_1 \) be any point in \( X \); if the ball \( B(x_1,2\eta) \) covers \( X \), then we are done. If not, there exists \( x_2 \) whose distance from \( x_1 \) is greater than or equal to \( 2\eta \). Therefore \( B(x_1,\eta) \cap B(x_2,\eta) = \emptyset \); if \( B(x_1,2\eta) \cup B(x_2,2\eta) = X \), then we are done; otherwise there is a point, which we call \( x_3 \) whose distance from both \( x_1 \) and \( x_2 \) is greater than or equal to \( 2\eta \), hence \( B(x_i,\eta) \cap B(x_3,\eta) = \emptyset, i = 1,2 \).

Using the induction we find that there exists an (at most countable) family \( \{x_i\}_{i \in I} \) which satisfies the properties (\( p_1 \)) and (\( p_2 \)) above. Also, if \( X \) is compact, then eventually a finite number of \( x_i \)'s will cover \( X \).

**Definition 1.2.** In a metric space \((X,d)\) let \( F_h : X \to [-\infty, +\infty] \) be a functional defined on \( X \), \( h \in \mathbb{N} \). We say that the sequence \((F_h)\) \( \Gamma \)-converges in \( X \) to a functional \( F : X \to [-\infty, +\infty] \) if and only if

(a) for every sequence \((x_h)\) in \( X \), converging to \( x \in X \) we have

\[ F(x) \leq \liminf_{h \uparrow +\infty} F_h(x_h); \]
(b) for every \( x \in X \) there exists a sequence \( (x_h) \) converging to \( x \) such that
\[
\limsup_{h \uparrow +\infty} F_h(x_h) \leq F(x).
\]
We refer to Dal Maso’s monograph [5] for more information about \( \Gamma \)-convergence.

**Definition 1.3.** Let \( (H_h)_h \) be a sequence of Hilbert spaces; we say that \( (H_h)_h \) is uniformly embedded in the Hilbert space \( H \) if there exists a linear, injective map
\[
I_h : H_h \to H, \quad h \in \mathbb{N}
\]
such that the metric \( \| \cdot \|_H \) (induced by the Hilbert structure of \( H \)) on \( I_h(H_h) \) is uniformly equivalent to the metric \( \| \cdot \|_{H_h} \) of \( H_h \), i.e., there exists a constant \( c_o \), possibly depending on \( H \), but not on \( h \in \mathbb{N} \), such that
\[
c_o^{-1}\|x - y\|_{H_h} \leq \|I_h(x) - I_h(y)\|_H \leq c_o\|x - y\|_{H_h}.
\]

In Theorem 1 we shall need the following definition, which is a slight generalization of the convergence of subsets in a Hilbert space as given in [13, Definition 2.7.2].

**Definition 1.4.** Let \( (H_h)_h \) be a sequence of Hilbert spaces uniformly embedded in \( H \) and let \( S_h \) be a subset of \( H_h \), \( h \in \mathbb{N} \). We say that the sequence \( (S_h)_h \) converges to a subset \( S \) of \( H \) if the following conditions are satisfied:

(i) for every subsequence \( (S_{h'})_{h'} \) of \( (S_h)_h \) and for every \( x_{h'} \in I_{h'}(S_{h'}) \) converging weakly (in \( H \)) to some \( x \in H \), we have \( x \in S_h \); (ii) for every \( x \in S \) there exists a sequence \( (x_h)_h \) converging strongly to \( x \) such that \( x_h \in I_h(S_h) \), for every \( h \in \mathbb{N} \).

In the following we let \((M,g)\) denote a Lipschitz \( d \)-dimensional manifold (with or without boundary), \( d \geq 2 \), with \( g = (g_{ij})_{i,j=1}^d \) the metric tensor of \( M \). We denote by \( \text{Vol}_g(dx) \) the canonical measure of \( g \) on \( M \) \([4, Definition 3.90]\); in local coordinates \( x = (x_1, \ldots, x_d) \) it can be written as
\[
\text{Vol}_g(dx) = \sqrt{\det g(x)} \, dx,
\]
where \( dx \) is the Lebesgue measure on \( \mathbb{R}^d \), so that
\[
\text{Vol}_g(B) = \int_B \sqrt{\det g(x)} \, dx,
\]
for every Borel set \( B \subset M \); the term \( \sqrt{\det g(x)} \) sometimes will be referred to as the local density of \( \text{Vol}_g \).
We occasionally will also use the notation $ds^2_M, \text{Vol}_M$ to denote respectively the metric of $M$ and its canonical measure.

We point out that Definition 3.90 in [9] requires the manifold to be $C^\infty$; however their definition can be extended in our Lipschitz framework.

We define $L^2(M, g)$ as the space of all functions on $M$ such that

$$\int_M f^2 \text{Vol}_g(dx) < +\infty.$$  

In the framework of Lipschitz manifolds it still makes sense to consider the Dirichlet functional on $(M, g)$ ([7])

$$\mathcal{D}_{M,g}(u) := \int_M \sum_{i,j=1}^d g^{ij} D_i u D_j u \text{Vol}_g(dx),$$

for Lipschitz functions $u$ defined on $M$; notice that $D_i u := \partial u/\partial x_i$, $i = 1, \ldots, d$, is defined $\text{Vol}_g$-almost everywhere.

**Definition 1.5.** We let $H^1(M, g)$ denote the closure of $\text{Lip}(M)$, the family of Lipschitz functions on $M$, under the norm induced by

$$\mathcal{D}_{M,g}(u) + \int_M u^2 \text{Vol}_g(dx).$$

For $\lambda > 0$, and $f \in L^2(M, g)$, we can introduce the functional $F^\lambda : L^2(M, g) \rightarrow [0, +\infty]$ defined by

$$F^\lambda(u) := \mathcal{D}_{M,g}(u) + \lambda \int_M u^2 \text{Vol}_g(dx) - 2 \int_M f u \text{Vol}_g(dx),$$

if $u \in H^1(M, g)$ and $F^\lambda(u) := +\infty$ otherwise in $L^2(M, g)$. This functional is strictly convex, coercive and lower semi-continuous in the strong topology of $L^2(M, g)$ hence, by the Direct Method in the Calculus of Variation, it has a unique minimum point $u_f$.

Via standard arguments it is possible to prove the following result.

**Proposition 1.6.** The resolvent operator $R^\lambda : L^2(M, g) \rightarrow L^2(M, g)$, which associates to every $f \in L^2(M, g)$ the minimum point $u_f$, is a compact, positive, self-adjoint operator.

Thus the resolvent operator has a sequence of proper values $(\sigma_i^\lambda)_{i \in \mathbb{N}}$ having zero as accumulation point.
**Definition 1.7.** The spectrum of \((M, g)\) is the sequence of the proper values \((\sigma_i^o)_{i \in \mathbb{N}}\) of the resolvent operator \(R^\lambda\).

If \((M, g)\) is smooth, then let \(-\Delta_g\) denote its Laplace-Beltrami operator.

**Definition 1.8.** We say that \(\alpha\) is an eigenvalue of \(-\Delta_g\) if

\[
\begin{cases}
-\Delta_g u = \alpha u, & \text{on } M \\
\text{natural boundary condition}
\end{cases}
\]

where we agree that “natural boundary condition” means homogeneous Neumann boundary condition if \(\partial M \neq \emptyset\), and no boundary condition if \(\partial M = \emptyset\).

**Remark 1.9.** Notice that if \((M, g)\) is a smooth manifold, then \(\sigma_i^o = (\lambda + \lambda_i)^{-1}\), where \((\lambda_i)_{i \in \mathbb{N}}\) is the sequence of eigenvalues of \(-\Delta_g\).

**Definition 1.10.** In what follows we shall need the *lower semi-continuous regularization* in \(L^2(M, \text{Vol}_g)\) of the functional \(D_{M,g}(\cdot)\):

\[
\tilde{D}_{M,g}(u) := \begin{cases}
D_{M,g}(u), & \text{if } u \in H^1(M, g) \\
+\infty, & \text{otherwise in } L^2(M, g).
\end{cases}
\]

**Definition 1.12.** Let \(A\) be a given relatively compact open set contained in \((M, g)\), and let \(E\) be a Borel set, \(E \subset A\). Then the \((M, g)\)-capacity of \(E\) with respect to \(A\) is defined by

\[
(M, g)\text{-cap}(E, A) := \inf \{ D_{M,g}(u) : u \in K(E, A) \},
\]

where \(K(E, A) := \{ u \in \text{Lip}_o(A) : u \geq 1 \text{ on a neighborhood of } E \}\), and \(\text{Lip}_o(A)\) denotes the family of all Lipschitz functions whose support is contained in \(A\).

**Remark 1.13.** Notice that the property of having capacity zero is unchanged if \(A\) is replaced by a larger relatively compact open subset of \(M\). Therefore we shall say that a property \(P(x)\) holds \((M, g)\)-quasi everywhere (q.e.), or for \((M, g)\)-quasi every \(x \in M\), if the set

\[
\{ x \in M : P(x) \text{ is not true} \}
\]

has \((M, g)\)-capacity zero.
Remark 1.14. Modifying suitably some arguments in [3], it is possible to show that for every $u \in H^1(M, g)$ there exists $\tilde{u} : M \to \mathbb{R}$ such that
\[
\lim_{r \downarrow 0} \frac{1}{\text{Vol}_g(B(x, r))} \int_{B(x, r)} |\tilde{u}(x) - u(y)| \text{Vol}_g(dy) = 0,
\]
and $\tilde{u} = u$ Vol$_g$-almost everywhere on $M$. The function $\tilde{u}$ is uniquely determined up to a set of $(M, g)$-capacity zero and is continuous when restricted to the complement of open sets with arbitrarily small $(M, g)$-capacity. In the following we shall identify each $u \in H^1(M, g)$ with $\tilde{u}$, so that we can say that $u$ is determined $(M, g)$-quasi everywhere.

Definition 1.15. Let $\mathcal{M}_o(M, g)$ denote the class of all Borel measures $\mu$ on $M$ which are absolutely continuous w.r.t. the $(M, g)$-capacity, i.e.,
\[ \mu(E) = 0 \quad \text{whenever } E \text{ has } (M, g)\text{-capacity zero}. \]
We introduce an equivalence relation among measures in $\mathcal{M}_o(M, g)$: We say that $\mu, \nu \in \mathcal{M}_o(M, g)$ are equivalent, $\mu \sim \nu$, if and only if
\[ \int_M u^2 d\mu = \int_M u^2 d\nu, \]
for every $u \in H^1(M, g)$. By Remark 1.14 above each function $u \in H^1(M, g)$ is determined $(M, g)$-quasi-everywhere, hence $\mu$- and $\nu$-almost everywhere. Thus the integrals above are well-defined, possibly equal to $+\infty$.

Definition 1.16. Let $E$ be a given Borel set; then
\[ \infty_E(B) := \begin{cases} 
0, & \text{if } B \cap E \text{ has } (M, g)\text{-capacity zero;} \\
+\infty, & \text{otherwise.} 
\end{cases} \]
Thus $\infty_E(\cdot) \in \mathcal{M}_o(M, g)$. A function $v \in H^1(M, g)$ belongs to $L^2(M, \infty_E(dx))$ if and only if $v = 0$ $(M, g)$-quasi everywhere on $E$.

Remark 1.17. The canonical measure Vol$_g$ belongs to $\mathcal{M}_o(M, g)$. Indeed, let $E$ be a Borel set which has capacity zero; then $(M, g)\text{-cap}(E, A) = 0$, for some relatively compact open set $A \subset M$. Therefore, by definition, for every $\varepsilon > 0$ there is $\varphi_\varepsilon \in \text{Lip}(A)$, with the support contained in $A$, such that $D_M(\varphi_\varepsilon) < \varepsilon$, and $\varphi_\varepsilon \geq 1$ on a neighborhood of $E$. Thus the
characteristic function of $E$ is less than or equal to $(\varphi_\varepsilon)^2$ on $A$, hence $\text{Vol}_g(E) \leq \int_A (\varphi_\varepsilon(x))^2 \text{Vol}_g(dx)$. By the Poincaré inequality

$$\int_A (\varphi_\varepsilon(x))^2 \text{Vol}_g(dx) \leq C \mathcal{D}_M(\varphi_\varepsilon),$$

where the constant $C$ possibly depends on $A$, we get

$$\text{Vol}_g(E) \leq \mathcal{D}_M(\varphi_\varepsilon) < \varepsilon,$$

which implies $\text{Vol}_g(E) = 0$, by the arbitrariness of $\varepsilon > 0$.

**Definition 1.18** ([6]). A relaxed Lipschitz manifold is by definition a 4-tuple $(M, ds^2_M, T, \mu)$ where $(M, ds^2_M)$ is a Lipschitz manifold, possibly with boundary; $T : M \to M$ is an isometry, $T \circ T = \text{id}_M$ and the fixed-point set of $T$, $\text{Fix}(T) = \{ x \in M : T(x) = x \}$, is a submanifold of $(M, ds^2_M)$; $\mu$ is a measure which belongs to $\mathcal{M}_o(M, g)$.

We have the following general definition (cf. Definition 1.8 in [6]).

**Definition 1.19.** Let $(\overline{M}, ds^2_M)$ be a manifold-with-boundary and $(N, ds^2_N)$ be a manifold (possibly with $\partial N = \emptyset$); let $\text{Vol}_M, \text{Vol}_N$ be the canonical measures, and $\mathcal{D}_M(\cdot), \mathcal{D}_N(\cdot)$ be the Dirichlet functionals of respectively $M$ and $N$. Let $T : \overline{M} \to \overline{M}$ be an isometry such that $T \circ T = \text{id}_{\overline{M}}$, and the fixed-point set $\text{Fix}(T)$ is a submanifold of $M$ (possibly $\text{Fix}(T) = \emptyset$). Furthermore, let $\nu$ be a Borel measure on $\overline{M}$. We say that the manifold $(N, ds^2_N)$ is represented by the relaxed manifold $(\overline{M}, ds^2_M, T, \nu)$ if there is an isometry

$$I : L^2(N, \text{Vol}_N) \to L^2(M, \text{Vol}_M)$$

and for $v \in H^1(N)$ we have

$$\mathcal{D}_N(v) = \mathcal{D}_M(u) + \int_{\overline{M}} [u(x) - u(T(x))]^2 \nu(dx),$$

with $u = I(v)$.

**Remark 1.20.** Notice that $\mathcal{D}_Y(v) < +\infty$ if and only if $\mathcal{D}_X(u) < +\infty$ and $u(\cdot) - u(T(\cdot)) \in L^2(X, \nu(dx))$. 
2. Handles

In this section we describe an explicit procedure to attach a handle to \( S^d \), and denote the resulting manifold by \((N, g^N)\); the topological type of \( N \), then, will be that of \( S^{d-1} \times S^1 \). Then we show that \((N, g^N)\) can be represented (in the sense of Definition 1.19 above) by a relaxed manifold \((M_1, g_1, T, \infty_{\partial M_1})\), where \( M_1 \) is homeomorphic to \( S^d \) with two punctures.

We point out that the handles we shall consider here are called “handles of bounded thinness” in [6, §3].

As a matter of notation let \( S^{d-1} \) be the \((d - 1)\)-sphere of radius 1, and \( S^{d-1}(t) \) be the \( d - 1 \)-sphere of radius \( t > 0 \).

Let \( 0 < \varepsilon < 1 \); let us consider the cylinder
\[
H := [-1, 1] \times S^{d-1}(\varepsilon),
\]
with its usual product metric
\[
ds^2_H = dy^2 + \varepsilon^2 ds^2_{S^{d-1}}(\omega).
\]

Let us consider the map
\[
T : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}
\]
\[
T(x) := -x
\]
Notice that \( T \) is an isometry, \( T \circ T = \text{id}_{\mathbb{R}^{d+1}} \), and the fixed-point set of \( T \) reduces to the singleton \( \{0\} \). In the following we are only concerned with the restriction of this map to \( S^d \), and in order not to have too heavy notation we still denote this restriction by \( T \). We notice that \( T \) (the perhaps familiar antipodal map on \( S^d \)) maps \( S^d \) onto itself, is an isometry, \( T \circ T = \text{id}_{S^d} \) and the fixed-point set is empty.

We shall find useful in the following to employ cylindrical coordinates so that if \( x \in \mathbb{R}^{d+1} \), then \( x = (y, r\omega) \) where \( y \in \mathbb{R}, \omega \in S^{d-1}, r \geq 0 \), and \(|x|^2 = y^2 + r^2 \); in particular,
\[
S^d = \{(y, r\omega) \in \mathbb{R}^{d+1} : y^2 + r^2 = 1\},
\]

and the metric of \( S^d \) is in these coordinates
\[
ds^2_{S^d} = (1 - r^2)^{-1} dr^2 + r^2 ds^2_{S^{d-1}}(\omega).
\]

Also, \( H = \{(y, r\omega) : y \in [-1, 1], r = \varepsilon\} \). We shall also consider the map
\[
T_H : H \to H
\]
\[
(y, \varepsilon \omega) \mapsto (-y, \varepsilon \omega)
\]
Notice that \( T_H \) is an isometry, \( T_H \circ T_H = \text{id}_H \), \( \text{Fix}(T_h) = \{0\} \times S^{d-1}(\varepsilon) \).
Definition 2.5. Let $\Theta \in [1, +\infty)$. Let $\mathcal{F}_\Theta$ be the family of all pairs $(\varepsilon, r)$ where $\varepsilon \in (0, 1)$, and $r(\cdot)$ is an increasing, Lipschitz continuous function, $r : [0, 1] \rightarrow (0, 1)$, with

$$0 < D^{-1} < \frac{dr}{dt}(t) < D, \text{ a.e. } t \in (0, 1),$$

such that

$$\max \left\{ \frac{\varepsilon}{r(0)^{1/2}}, \frac{\varepsilon}{r(1)}, \frac{D}{\sqrt{1 - (r(1))^2}} \right\} < \Theta.$$

Definition 2.6 (Handle). Let $(\varepsilon, r_1) \in \mathcal{F}_\Theta$, and let $\varphi : H \setminus \text{Fix}(T_H) \rightarrow S^d$ be the function defined by

$$\varphi : (y, \varepsilon \omega) \mapsto (\text{sgn}(y)\sqrt{1 - (r_1(|y|))^2}, \text{sgn}(y)r_1(|y|)\omega).$$

The pair $(H, \varphi)$ is by definition a handle.

Example 2.8. Let $\Theta > 1$ be given. Let $\varepsilon > 0$, and let us consider

$$r_1(t) := \beta t + \alpha(1 - t), \quad t \in [0, 1].$$

We choose $\alpha = \varepsilon \delta_o$, $\beta = \varepsilon \delta_1$ with $0 < \delta_o < \delta_1 < 1$; if $\delta_o > 1/\Theta$, then $(\varepsilon, r_1) \in \mathcal{F}_\Theta$, hence the pair $(H, \varphi)$, where $\varphi$ is as in Definition 2.6 above, is a handle.

Let

$$U := B((1, 0), \sin^{-1} r_1(1)) \vee B((-1, 0), \sin^{-1} r_1(1)),$$

$$E := B((1, 0), \sin^{-1} r_1(0)) \vee B((-1, 0), \sin^{-1} r_1(0)).$$

Notice that

$$\varphi(\text{int}(H) \setminus \text{Fix}(T_H)) = \varphi(\{(y, \varepsilon \omega) \in H : y \neq 0\}) \subset U,$$

$$\varphi(\partial H) = \varphi(\{\pm 1\} \times S^{d-1}(\varepsilon)) = \partial U,$$

$$E = U \setminus \varphi(\text{int}(H) \setminus \text{Fix}(T_H)).$$

Definition 2.11 (Attaching the handle). To attach the handle $(H, \varphi)$ to $S^d$ we consider $(S^d \setminus E) \cup H$ and identify each point $z \in H \setminus \text{Fix}(T_H)$ with $\varphi(z) \in U \setminus E$: $z \sim \varphi(z)$. Then the topological type of

$$N := ((S^d \setminus E) \cup H) / \sim.$$
is that of $S^{d-1} \times S^1(1)$. The Lipschitz metric of the resulting manifold $N$ is by definition the metric of $S^d$ on $N \setminus E$ and that of $H$ on $\text{int}(H)$:

\begin{equation}
 ds^2_N := \begin{cases}
 ds^2_S, & \text{on } S^d \setminus U, \\
 ds^2_H, & \text{on } \text{int}(H).
\end{cases}
\end{equation}

Remark 2.13. Notice that the boundary of $H$, $\partial H = \{\pm 1\} \times S^{d-1}(\varepsilon)$, is thus identified with the hypersurfaces \{(y, r\omega) \in S^d : r = r_1(1)\}.

Remark 2.14. It can be shown (cf. Remark 3.6 in [8]) that the metric $g^N = (g^N_{i,j})_{i,j}$ on the resulting manifold $N$ is uniformly bounded by the metric $(g_{i,j})_{i,j}$ on $S^d$, i.e.,

$$\Theta^{-2} \sum_{i,j} g_{i,j} \xi_i \xi_j \leq \sum_{i,j} g^N_{i,j} \xi_i \xi_j \leq \Theta^2 \sum_{i,j} g_{i,j} \xi_i \xi_j,$$

for every $\xi \in \mathbb{R}^n$.

Moreover the complement of $E$ is uniformly strongly connected, i.e., for each $u \in H^1(S^d \setminus E, g)$ there is an extension of $u$, $v = \pi u$, with $v \in H^1(S^d, g)$, and

$$\|v\|_{H^1(S^d, g)} \leq \Theta^{1/2} \|u\|_{H^1(S^d \setminus E, g)}.$$ 

The constant $\Theta$ is the same that appears in Definition 2.6.

Remark 2.15. Let $(H, \varphi)$ be the handle with the function $r_1(\cdot)$ as in Example 2.3. Then comparing (2.3), (2.1), (2.12), it is not difficult to see that the metric on $N$ is continuous (i.e., the manifold is $C^1$) if and only if $B = r_1(1) = \varepsilon$. We can also have the resulting manifold to be of class $C^{k+1}$ provided that we modify the metric of the handle near $S^{d-1}(\varepsilon) \times \{\pm 1\}$ so that the new metric is of class $C^k$ and as $t \to \pm 1$ the metric is $ds^2_H = (1 - r_1(t))^{-1/2} (dr_1/dt)^2 dt^2 + (r_1(t))^2 ds^2_{S^{d-1}}$ plus terms which vanish at $t = \pm 1$ together with all derivatives of order less than or equal to $k$.

If instead $r_1(1) \neq \varepsilon$, then the metric is only piecewise continuous: In fact, the metric has a discontinuity along the hypersurfaces $\{(y, r\omega) \in S^d : r = r_1(1), y = \pm \sqrt{1 - r^2}\}$ where $\partial H$ is attached to $S^d$, whereas the metric is smooth elsewhere on $S^d$. Therefore, if $r_1(1) \neq \varepsilon$, $N$ is a Lipschitz, but not a $C^1$, manifold.

Notice that the sectional curvature may present a distributional component along the hypersurfaces $\{(y, r\omega) \in S^d : r = r_1(1), y = \pm \sqrt{1 - r^2}\}$. 


Let $\overline{M}_1$ be the manifold-with-boundary $S^d \setminus E$, $\partial M_1 = \partial E$, whose metric $g_1 = (g_{ij,1})_{i,j}$ is equal to

\[(g_{ij,1})_{i,j} := \begin{cases} (g_{ij})_{i,j}, & \text{on } S^d \setminus U, \\ (g^H_{ij})_{i,j}, & \text{on } U \setminus E, \end{cases}\]

where $(g_{ij})_{i,j}$ denotes the metric of $S^d$, and $(g^H_{ij})_{i,j}$ is the pull-back of the metric of $H$ under the map $\varphi^{-1} : \varphi(H \setminus \text{Fix}(T_H)) \to H$. We notice that the metric in (2.16) is uniformly bounded by the metric $g = (g_{ij})_{i,j}$ of $S^d$.

We point out that the map $T$ introduced in (2.2) maps $\overline{M}_1$ onto itself, is an isometry there and $\text{Fix}(T) = \emptyset$.

We have the following result (cf. Lemma 1.11 in [6]).

**Proposition 2.17.** The manifold $(N,g^N)$ is represented (in the sense of Definition 1.19) by the relaxed manifold-with-boundary $(\overline{M}_1,g_1,T,\infty_{\partial M_1})$.

In particular, for each $v \in H^1(N,g^N)$ there exists $u \in H^1(\Omega,a)$ such that

\[(2.18) \quad D_N(v) = D_{M_1}(u) + \int_{\overline{M}_1} [u(x) - u(T(x))]^2 \, d\mu_{\partial M_1}(dx).\]

**Remark 2.19.** Notice that the right hand side in (2.18) is finite if and only if

\[(2.20) \quad u(\cdot) = u(T(\cdot)), \text{ q.e. on } \partial M_1,\]

(cf. Definition 1.16.) Roughly speaking, we may say that the representation of $N$ has been obtained by the following “cut-and-paste” procedure: We cut the handle of $N$ along $\text{Fix}(T)$ and get the manifold-with-boundary $\overline{M}_1$; notice that $\overline{M}_1$ is homeomorphic to $S^d$ with two punctures. The presence of the handle, as far as the Dirichlet functional is concerned, is then represented by the non-local term

\[\int_{\overline{M}_1} [u(x) - u(T(x))]^2 \, d\mu_{\partial M_1}(dx),\]

which, via (2.20), glues together the two components of $\partial M_1 = \partial E$ and gives back the handle.

If the measure $\mu$ appearing in the non-local term above is finite, with $(N,g^N)$ represented by $(\overline{M}_1,g_1,T,\mu)$, then by analogy with the case of the “infinite” measure, we may say that the handle is “weakly” (or “partially”) attached to $N$. 
3. The main results

Let \((\eta_h)_h\) be a sequence of positive numbers, \(\lim_{h \to +\infty} \eta_h = 0\) and let \((x_i)_{i \in I_h}\) be an \(\eta_h\)-package in \((S^d, g)\) (cf. Definition 1.1), for each \(h \in \mathbb{N}\). Let also \(\overline{x}_i := T(x_i), i \in I_h\), where \(T\) is the antipodal map on \(S^d\).

Let \((\varepsilon_h)_h\) be a sequence of positive numbers such that

\[ \lim_{h \to +\infty} \frac{\eta_h}{\varepsilon_h} = 0. \]

(3.1)

Let us consider \(H_h := [-1, 1] \times S^{d-1}(\varepsilon_h)\), whose metric is equal to \(ds^2_{H_h} = \varepsilon_h^2(dy^2 + ds^2_{S^{d-1}})\).

Using cylindrical coordinates we can assume that \(x_i = (1, 0)\) and \(\overline{x}_i = T(x_i) = (-1, 0)\), for \(i \in I_h\). Hence, for a given \(\Theta > 1\), we can define maps (similarly as in the previous § 2)

\[ \varphi_{+i}^\pm : H_h \cap \{y > 0\} \to S^d \]

\[ \varphi_{-i}^\pm(y, \varepsilon_h \omega) = (\pm \sqrt{1 - (r_{\varepsilon_h}(\pm y))^2}, \pm r_{\varepsilon_h}(\pm y) \omega), \]

with (cf. Example 2.8)

\[ r_{\varepsilon_h}(t) := (\delta_1 \varepsilon_h \eta_h) t + (\delta_0 \varepsilon_h \eta_h)(1 - t), \quad 0 < \frac{1}{\Theta} < \delta_0 < \delta_1 < 1, \quad t \in [0, 1], \]

so that \((r_{\varepsilon_h}, \varepsilon_h \eta_h) \in \mathcal{F}_\Theta\); if we define

\[ \varphi_{i,h}^\pm(\cdot) := \begin{cases} \varphi_{i,h}^+(\cdot), & \text{on } H_h \cap \{y > 0\}, \\
\varphi_{i,h}^-(\cdot), & \text{on } H_h \cap \{y < 0\}, \end{cases} \]

then the pair \((H_h, \varphi_{i,h})\) is a handle, as in Definition 2.6.

**Definition 3.2.** (cf. Definition 2.11) We let \((N_h, g^{N_h})\) denote the manifold obtained by attaching the handles \((H_h, \varphi_{i,h}), i \in I_h\), to \(S^d\).

**Remark 3.3.** As \(r_{\varepsilon_h}(1) = \eta_h \varepsilon_h \neq \varepsilon_h\), then the metric \(g^{N_h}\) is not continuous, but only piecewise continuous; cf. Remark 2.13. Thus \((N_h, g^{N_h})\) is a Lipschitz manifold, but not a \(C^1\) manifold.

**Definition 3.4.** (cf. Definition 1.5) We let \((\sigma^h_i)_{i \in \mathbb{N}}\) denote the spectrum of \((N_h, g^{N_h})\).

Our main result is the following.
Theorem 1. Let $r_h := \arcsin r_{\varepsilon h}(0)$, and assume that

$$\alpha := \begin{cases} 
\lim_{h \uparrow +\infty} \frac{r_h^{d-2}}{n_h^d} & \text{if } d \geq 3, \\
\lim_{h \uparrow +\infty} \frac{-1}{n_h^2 \log r_h} & \text{if } d = 2,
\end{cases}$$

with $0 \leq \alpha < +\infty$. Let moreover $(\sigma_i)_i$ be the sequence of the proper values of the resolvent operator $R^\lambda_\infty : L^2(S^d, g) \to L^2(S^d, g)$ corresponding to the functional

$$D_\infty(u) := D_{S^d}(u) + \frac{\alpha}{2^d} \int_{S^d} \left[ u(x) - u(T(x)) \right]^2 \text{Vol}_g(dx) + \lambda \int_{S^d} u^2 \text{Vol}_g(dx) - 2 \int_{S^d} fu \text{Vol}_g(dx),$$

for $u \in H^1(S^d, g)$; then

$$\lim_{h \uparrow +\infty} \sigma_i^h = \sigma_i, \quad \text{for each } i \in \mathbb{N}.$$  

Moreover the sequence $(L^2(N_h, g^{N_h}))_h$ is uniformly embedded in $L^2(S^d, g)$ and if $r \leq i \leq s$ are such that $\sigma_{s-1}^h < \sigma_i^h = \sigma_s^h < \sigma_{r+1}^h$, then the linear subspace spanned by \{u_r^h, \ldots, u_s^h\} in $L^2(N_h, g^{N_h})$ converges (in the sense of Definition 1.4) to the eigenspace corresponding to $\sigma_i$, where $u_r^h, \ldots, u_s^h$ are the eigenfunctions corresponding to $\sigma_i^h$.

Let $u$ be a function defined on $S^d$; following [6, §5] let us define the respectively odd and even part of $u$ by

$$u_{\text{odd}}(\cdot) := \frac{1}{2}[u(\cdot) - u(T(\cdot))]$$

$$u_{\text{even}}(\cdot) := \frac{1}{2}[u(\cdot) + u(T(\cdot))],$$

so that $u(\cdot) = u_{\text{odd}}(\cdot) + u_{\text{even}}(\cdot)$. If moreover $\Xi$ is a space of functions defined on $S^d$, then we denote by $\Xi_{\text{odd}}$ (resp. $\Xi_{\text{even}}$) the subspace of $\Xi$ consisting of all odd (resp. even) functions on $S^d$.

It can be shown that both the standard metric $g$ and canonical measure $\text{Vol}_g$ of $S^d$ are invariant under the action of the antipodal map $T$. Thus it can be proved that both $L^2(S^d, g)$ and $H^1(S^d, g)$ split into their even and
odd part as follows
\begin{enumerate}
\item \[
L^2(S^d, g) = L^2_{\text{odd}}(S^d, g) \oplus L^2_{\text{even}}(S^d, g)
\]
\item \[
H^1(S^d, g) = H^1_{\text{odd}}(S^d, g) \oplus H^1_{\text{even}}(S^d, g).
\]
\end{enumerate}
(The orthogonal decomposition is with respect to the inner product of \(L^2(S^d, g)\) in the former, and in \(H^1(S^d, g)\) in the latter.)

Thus the limit functional \(\mathcal{D}_\infty(\cdot)\) can be written as
\[
\mathcal{D}_\infty(u) = \mathcal{D}_\infty,\text{even}(u_{\text{even}}) + \mathcal{D}_\infty,\text{odd}(u_{\text{odd}}),
\]
where
\[
\mathcal{D}_\infty,\text{odd}(u_{\text{odd}}) = \mathcal{D}_{S^d}(u_{\text{odd}}) + \\
\left(\lambda + \frac{\alpha}{2d}\right) \int_{S^d} u_{\text{odd}}^2 \text{Vol}_g(dx) - 2 \int_{S^d} u_{\text{odd}} f_{\text{odd}} \text{Vol}_g(dx),
\]
and
\[
\mathcal{D}_\infty,\text{even}(u_{\text{even}}) = \mathcal{D}_{S^d}(u_{\text{even}}) + \\
\lambda \int_{S^d} u_{\text{even}}^2 \text{Vol}_g(dx) - 2 \int_{S^d} u_{\text{even}} f_{\text{even}} \text{Vol}_g(dx)
\]
Using (3.6), (3.7) above, we have that the Euler equation associated with \(\mathcal{D}_\infty(\cdot)\) can be “decoupled” into two equations, one for the odd part and the other for the even part of \(\mathcal{D}_\infty(\cdot)\) as follows:
\begin{align*}
\begin{cases}
-\Delta_g u_{\text{odd}} + \left(\lambda + \frac{\alpha}{2d}\right) u_{\text{odd}} = f_{\text{odd}}, & \text{in } S^d \\
\end{cases}
\end{align*}
and
\begin{align*}
\begin{cases}
-\Delta_g u_{\text{even}} + \lambda u_{\text{even}} = f_{\text{even}}, & \text{in } S^d \\
\end{cases}
\end{align*}

\textit{Remark 3.10.} The decoupling of the Euler equation associated with the limit functional \(\mathcal{D}_\infty(\cdot)\) into (3.8) and (3.9) implies that the sequence \((\sigma_i)_{i \in \mathbb{N}}\) of proper values of the resolvent operator \(R^\lambda_{\infty}\) splits into two sequences \((\sigma_i^{\text{odd}})_{i \in \mathbb{N}}, (\sigma_i^{\text{even}})_{i \in \mathbb{N}}\): the odd and even part of the spectrum. Thus we see that adding an increasing number of handles affects only the odd part of the spectrum with the occurrence of the Lenz shift phenomenon in (3.3).
4. Proof of the main result

The strategy to prove our main result, Theorem 1, is as follows: We introduce the relaxed manifold \((M, g, T, \infty)\) (Definition 4.3) and prove that \((M, g, T, \infty)\) represents \((N, g_N)\) (Proposition 4.9). Thus the value of the functional \(F(\cdot)\) introduced above is equal to the value of the functional \(R(\cdot)\): \(L^2(S, g) \to [0, +\infty]\), introduced in Definition 4.8. Finally we use Theorem 2, Theorem 3 and an adaptation to our framework of a derivation-type argument in [2] to conclude the proof of Theorem 1.

Let \(r = \arcsin r_\varepsilon(0), R = \arcsin r_\varepsilon(1)\) (cf. (2.9), (2.10)); define

\[
E_h := \bigcup_{i \in I_h} \left [ B(x_i, r_h) \right ]
\]

(4.1)

\[
U_h := \bigcup_{i \in I_h} \left [ B(x_i, R_h) \right ].
\]

(4.2)

**Definition 4.3.** For \(h \in \mathbb{N}\) we let

\[
M_h := S^d \setminus E_h,
\]

so that \(\partial M_h = \partial E_h\), and let \(g_h\) denote the metric on \(M_h\) which is defined as the standard metric \(g\) of \(S^d\) on \(S^d \setminus U\), and the metric of the handle \((H_h, \varphi_i, h)\) on 

\[
\left ( B(x_i, R_h) \setminus B(x_i, r_h) \right ) \vee \left ( B(\varphi_i, R_h) \setminus B(\varphi_i, r_h) \right ), i \in I_h.
\]

Finally we let \(T\) be the antipodal map (2.2).

**Remark 4.4.** Notice that one of the sectional curvatures of \(M_h\) is equal to \(1/\varepsilon_h\). In particular the sectional curvature of \(M_h\) is unbounded, as \(h \uparrow +\infty\).

**Lemma 4.5.** The sequence \((1_{M_h})_h\) converges in measure \(Vol_g\) to the constant function 1; moreover the sequence \((g_{ij, h}1_{M_h})_h\) converges in measure \(Vol_g\) to \(g_{ij}\), \(i, j = 1, \ldots, d\).

**Proof.** Let \((x_i)_{i \in I_h}\) be the \(r_h\)-packing introduced above; let moreover \((U_h)_h\) be as in (4.2) above. To prove the lemma it is sufficient to show that \(Vol_g(U_h)\) tends to zero, as \(h \uparrow +\infty\), and to this aim, we first prove an estimate on \(#(I_h)\), and then an estimate on the volume of geodesic balls. By Definition 1.1-(p1), we get that

\[
Vol_g \left ( \bigcup_{i \in I_h} B(x_i, \eta_h) \right ) = \sum_{i \in I_h} Vol_g B(x_i, \eta_h) \leq Vol_g(S^d).
\]
Thus
\[
\#(I_h) \leq \frac{\Vol_g(S^d)}{\min_{i \in I_h} \Vol_g(B(x_i, \eta_h))}.
\]
The measure $\Vol_g$ of geodesic balls $B(x_i, \eta_h)$ can be computed by means of the following formula, which is a particular case of a Bishop-type inequality:
\[
\frac{\Vol_g(B(x_i, \eta_h))}{\omega_d \eta_h^d} = \left( \frac{\sin(\eta_h/\varepsilon_h)}{(\eta_h/\varepsilon_h)} \right)^{d-1}
\]
for $i \in I_h$, where $\omega_d$ is the $d$-dimensional euclidean volume of the unit sphere. Notice that, as $\lim_{h \uparrow +\infty} \eta_h/\varepsilon_h = 0$ (cf. 3.1), the ratio at the left-hand side of (4.7) tends to 1 as $h \uparrow +\infty$. Let us estimate the volume of $U_h$:
\[
\Vol_g(U_h) \leq \sum_{i \in I_h} \Vol_g(B(x_i, R_h)).
\]
By means of (4.7) and (4.6), we have
\[
\Vol_g(U_h) \leq \zeta_h \varepsilon_h^d,
\]
where $(\zeta_h)_h$ is a bounded sequence. Thus, passing to the limit as $h \uparrow +\infty$, $\Vol_g(U_h)$ tends to zero and we get the result.

**Definition 4.8.** For $h \in \mathbb{N}$, let $\mathcal{D}_{M_h}(\cdot)$ be the Dirichlet functional on $(M_h, g_h)$, and $f \in L^2(S^d, g)$; for $\lambda > 0$, let \( R^\lambda_h : L^2(S^d(1), g) \to [0, +\infty] \)
be defined by
\[
R^\lambda_h(u) := \mathcal{D}_{M_h}(u) + \int_{M_h} \left[ u(x) - u(T(x)) \right]^2 \arrowvert_{\partial M_h} (dx) + \lambda \int_{M_h} u^2 \Vol_{g_h}(dx) - 2 \int_{M_h} fu \Vol_{g_h}(dx)
\]
if $u|_{M_h} \in H^1(M_h, g_h)$; $R_h(u) := +\infty$ otherwise in $L^2(S^d(1), g)$.

**Proposition 4.9.** (i) The manifold $(N_h, g^{N_h})$ is represented (in the sense of Definition 1.19) by $(\overline{M}_h, g_h, T, \infty_{\partial M_h})$, $h \in \mathbb{N}$.

(ii) The sequence of manifolds $((M_h, g_h))_h$ satisfies the following conditions:
(a1) There exists $\Lambda_0 > 0$ such that for every $h \in \mathbb{N}$

\[
0 \leq \Lambda_0^{-1} \sum_{i,j=1}^{d} g_{ij} \xi_i \xi_j \leq \sum_{i,j=1}^{d} g_{ij,h} \xi_i \xi_j \leq \Lambda_0 \sum_{i,j=1}^{d} g_{ij} \xi_i \xi_j,
\]

for every $x \in M_h$ and every $\xi \in \mathbb{R}^d$;

(a2) (Strong Connectivity Condition) There exist bounded linear extension operators $\pi_h : H^1(M_h, g_h) \to H^1(M, g)$ which satisfy the uniform bound

\[
\|\pi_h u\|_{H^1(M, g)} \leq c_0 \|u\|_{H^1(M_h, g_h)},
\]

for every $u \in H^1(M_h, g_h)$, and the constant $c_0$ does not depend on $h \in \mathbb{N}$.

(a3) The sequence of characteristic functions $(1_{M_h})_h$ converges in measure to the constant function equal to 1 on $S^d$.

(a4) $T(M_h) = M_h$;

(a5) the canonical measure $\text{Vol}_{g_h}(dx) = \sqrt{\det g_h} \, dx$ is $T$-invariant, i.e.,

\[
\sqrt{\det g_h}(x) = \sqrt{\det g_h(T(x))} \det \left( \frac{\partial T}{\partial x}(x) \right),
\]

for almost every $x \in M_h$;

(a6) the metric $g_h$ is $T$-invariant, i.e.,

\[
g_{ij}^h(T(x)) = \sum_{k,\ell=1}^{d} \frac{\partial T_i^j}{\partial x_k}(x) \frac{\partial T^j}{\partial x_k}(x) g_{k\ell}^h(x),
\]

for almost every $x \in M_h$.

Proof. Arguing as in Proposition 2.17 we can show part (i). Applying [6, Proposition 3.9], Definition 4.3 and Lemma 4.5, we get (ii); more precisely, from Proposition 3.9 in [6] we get (a1) and (a2); (a3) follows from Lemma 4.5; finally, by definition, $T(M_h) = M_h$, both the metric $g_h$ and the canonical measure $\text{Vol}_{g_h}(\cdot)$ are invariant under the antipodal map $T$, and moreover $\infty_{\partial M_h}(\cdot) = \infty_{\partial M_h}(T(\cdot))$, i.e., (a4), (a5), (a6) are satisfied. This proves the proposition.

Corollary 4.11. The sequence of Hilbert spaces $(L^2(N_h, g^{N_h}))_h$ is uniformly embedded into $L^2(S^d, g)$. 
Proof of Theorem \[1\]. Adapting some arguments from Proposition 5.7 in [6], the proof of the theorem follows if we show that the \( \Gamma \)-limit (in \( L^2(S^d, g) \)) of the sequence \((R^\lambda)\) is equal to the functional

\[
D_\infty(u) := D_{S^d}(u) + \frac{\alpha}{2^d} \int_{S^d} [u(x) - u(T(x))]^2 \text{Vol}_g(dx) \\
+ \lambda \int_{S^d} u^2 \text{Vol}_g(dx) - 2 \int_{S^d} fu \text{Vol}_g(dx)
\]

for \( u \in H^1(S^d, g) \). By a general result in \( \Gamma \)-convergence ([5]) this is equivalent to prove that the sequence of functionals \( G_h : L^2(S^d, g) \rightarrow [0, +\infty] \) defined by

\[
G_h(u) : D_{M_h}(u) + \int_{\partial M_h} [u(x) - u(T(x))]^2 \text{Vol}_{\partial M_h}(dx)
\]

if \( u|_{M_h} \in H^1(M_h, g_h) \), \( G_h(u) = +\infty \) otherwise in \( L^2(S^d, g) \), \( \Gamma \)-converges to the functional

\[
G(u) := D_{S^d}(u) + \frac{\alpha}{2^d} \int_{S^d} [u(x) - u(T(x))]^2 \text{Vol}_g(dx),
\]

for \( u \in H^1(S^d, g) \).

By Proposition 4.9-(ii) the sequence of manifolds \((M_h, g_h)\) satisfies the assumptions \((a_1), (a_2), (a_3), (a_4), (a_5), (a_6)\), hence we can apply Theorems 2 and 3 in §5 and get that the \( \Gamma \)-limit of \((G_h)\) is equal to

\[
\tilde{G}(u) = D_{S^d}(u) + \int_{S^d} [u(x) - u(T(x))]^2 \mu(dx),
\]

for some measure in \( \mathcal{M}_o(S^d, g) \). What is left to prove, then, is that \( \mu(dx) = \alpha 2^{-d} \text{Vol}_g(dx) \); with our choice of \( \varepsilon_h \), plus the assumption (3.5) in Theorem 1, this can be done suitably modifying a derivation-type argument as in [2]. The proof of the theorem is then complete. \( \square \)

5. Appendix

For each \( h \in \mathbb{N} \), let \((M_h, g_h)\), be a manifold-with-boundary, \( \overline{M}_h = M_h \cup \partial M_h, \partial M_h \neq \emptyset \), with \( \overline{M}_h \subset M \), where \( M \) is a manifold (with or without boundary); we assume that \( \dim M_h = d = \dim M \).
Variational compactness of Lipschitz metrics. We shall consider the following assumptions in the rest of the paper.

(A1) There exists $\Lambda_o > 0$ such that for every $h \in \mathbb{N}$

\[
0 \leq \Lambda_o^{-1} \sum_{i,j=1}^{d} g_{ij} \xi^i \xi^j \leq \sum_{i,j=1}^{d} g_{ij,h} \xi^i \xi^j \leq \Lambda_o \sum_{i,j=1}^{d} g_{ij} \xi^i \xi^j,
\]

for every $x \in M_h$ and every $\xi \in \mathbb{R}^d$;

(A2) (Strong Connectivity Condition) There exist bounded linear extension operators $\pi_h : H^1(M_h, g_h) \to H^1(M, g)$ which satisfy the uniform bound

\[
\|\pi_h u\|_{H^1(M, g)} \leq c_0 \|u\|_{H^1(M_h, g_h)},
\]

for every $u \in H^1(M_h, g_h)$.

The constant $c_0$ does not depend on $h \in \mathbb{N}$.

(A3) The sequence of characteristic functions $(1_{M_h})_h$ converges in the weak* topology of $L^\infty(M)$ to the function $b$; moreover both $b$ and $b^{-1}$ belong to $L^\infty(M)$.

Remark 5.2. 1) From (5.1) in (A1) we get the following formula relating the local densities

\[
\Lambda_o^{-d/2} \sqrt{\det g} \leq \sqrt{\det g_h} \leq \Lambda_o^{d/2} \sqrt{\det g}
\]

for every $x \in M_h$ and $h \in \mathbb{N}$. In particular sets of Vol$_{g_h}$-measure zero are also of Vol$_g$-measure zero, and conversely.

2) Let $d(\cdot, \cdot)$, $d_h(\cdot, \cdot)$ be the distances on $M_h$ associated with the metrics $g$ (restricted to $M_h$), $g_h$ respectively. Then the assumption (5.1) implies that, for every $h \in \mathbb{N}$, the metric space $(M, d)$ is equivalent to $(M_h, d_h)$.

3) Note that from (5.1) sets of $(M_h, g_h)$-capacity zero has also $(M, g)$-capacity zero. Conversely, if $Z \subset M_h$ has $(M, g)$-capacity zero, then it also has $(M_h, g_h)$-capacity zero.

We are in a position to prove the following result.
Theorem 2. Let us assume (A1), (A2), (A3), and consider the sequence $(D_{M_h})_h$, where $D_{M_h}$ is defined by

$$D_{M_h}(u) := \begin{cases} \int_M \sum_{i,j=1}^d g_h^{ij} D_i u D_j u \ dVol_{g_h}, & \text{if } u|_{M_h} \in H^1(M_h, g_h), \\ +\infty, & \text{otherwise in } L^2(M, g), \end{cases}$$

for $h \in \mathbb{N}$. Then there exists a Lipschitz metric $a = (a_{ij})_{i,j=1}^d$ such that the sequence $(D_{M_h})_{\Gamma}$-converges in $L^2(M, g)$ to the weighted Dirichlet functional

$$D_{M,a,w}(u) := \begin{cases} \int_{M_h} \sum_{i,j=1}^d a^{ij} D_i u D_j u \sqrt{\det a} \ w(x) dx, & u \in H^1(M, a), \\ +\infty, & \text{otherwise in } L^2(M, a), \end{cases}$$

with $w(x) := \sqrt{\det g/\det a(x)}$ for $x \in M$, and $(a_{ij})_{i,j=1}^d$ satisfies

$$\Lambda_o^{-1} \sum_{i,j=1}^d g_{ij} \xi^i \xi^j \leq \sum_{i,j=1}^d a_{ij} \xi^i \xi^j \leq c_o^2 \sum_{i,j=1}^d g_{ij} \xi^i \xi^j,$$

for every $x \in M$ and every $\xi \in \mathbb{R}^d$; the constants $\Lambda_o, c_o$ are those appearing respectively in (A1), (A2).

Remark 5.5. 1) Notice that because of (5.4) both $w(\cdot)$ and $w^{-1}(\cdot)$ are contained in $L^\infty(M)$.

As another consequence of (5.4) we have that if $u \in H^1(M, g)$, then $D_{M,a,w}(u) < +\infty$; also, if $v \in H^1(M, a)$, then $D_M(v) < +\infty$.

2) Using a similar argument as in Remark 5.2-3), we have that a set has $(M, a)$-capacity zero if and only if has $(M, g)$-capacity zero. In particular, $M_o(M, a)$ coincide with $M_o(M, g)$.

3) From (5.4), and similarly as in Remark 5.2-1), we have that sets of Vol$_a$-measure zero are also of Vol$_g$-measure zero, and conversely.

4) Another consequence of (5.3) and (5.4) is that

$$\lim_{h \uparrow \infty} \|u_h - u\|_{L^2(M, g)} = 0 \text{ if and only if } \lim_{h \uparrow \infty} \|u_h - u\|_{L^2(M, a)} = 0.$$

For the proof of Theorem 2 we shall need the following generalization of Theorem 2.1 by P. Marcellini & C. Sbordone [12].
Proposition 5.6. For each relatively compact open set \( \Omega \subset M \) let us consider for every \( h \in \mathbb{N} \) the following functional

\[
F_h(u, \Omega) := \begin{cases} \int_{M_h \cap \Omega} \sum_{i,j=1}^{d} g_{ij}^h \partial_i u \partial_j u \sqrt{\det g_h} \, dx, & u \in \text{Lip}(M), \\ +\infty, & \text{otherwise in } L^2(M,g). \end{cases}
\]

Then there exist a symmetric tensor \( (a_{ij})_{i,j=1}^{d} \) on \( M \) and a functional \( F(\cdot, \Omega) : L^2(\Omega,g) \to [0, +\infty] \) such that

\[
F(u, \Omega) = \int_{\Omega} \sum_{i,j=1}^{d} a_{ij} \partial_i u \partial_j u \sqrt{\det g} \, dx,
\]

with \( u \in \text{Lip}(M) \), and the tensor \( (a_{ij})_{i,j=1}^{d} \) satisfies

\[
0 \leq \sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j \leq \Lambda_0 \sum_{i,j=1}^{d} g_{ij} \xi_i \xi_j,
\]

for all \( x \in M, \xi \in \mathbb{R}^d \) and \( a_{ij} \in L^\infty(M), i,j = 1, \ldots, d. \)

During the proof of the Proposition 5.6 we shall need the following general result [14, Lemma 1.9].

Lemma 5.7. Given any cover \( (W_\ell)_{\ell \in I}, I \subseteq \mathbb{N} \), of a paracompact, differentiable manifold \( X \), and any Borel measure \( \sigma \) on \( X \), there exists a family of open sets \( (U_\ell)_{\ell \in I} \) such that

1. \( \bigcup_{\ell \in I} U_\ell = X \setminus \left[ \bigcup_{\ell \in I} \partial U_\ell \right] \);
2. \( U_\ell \cap U_k = \emptyset \), for all \( k, \ell \in I \), with \( k \neq \ell \);
3. \( U_\ell \subset \subset W_\ell \), with \( \ell \in I \);
4. \( \sigma \left[ \bigcup_{\ell \in I} \partial U_\ell \right] = 0 \).

Proof of Proposition 5.6. For each relatively compact open set \( \Omega \) in \( M \), let \( F(\cdot, \Omega) \) be the functional which is the \( \Gamma \)-limit (in \( L^2(M,g) \)) of the sequence \( (F_h)_h \). With suitable modifications in Lemmas 2.2 through 2.6 in [12] we can prove that for each \( u \in \text{Lip}(M) \) the set function \( \Omega \mapsto F(u, \Omega) \) satisfies the following properties:
• for every relatively compact open sets $\Omega' \subset \Omega \subset M$

$$0 \leq F(u, \Omega') \leq F(u, \Omega)$$

(5.8)

$$\leq F(u, \Omega') + \Lambda_0^{1+d/2} \int_{\Omega \cap \Omega'} \sum_{i,j=1}^d g^{ij} D_i u D_j u \sqrt{\det g} \, dx;$$

• for every disjoint relatively compact open sets $\Omega, \Omega'$ in $M$

$$F(u, \Omega \cup \Omega') = F(u, \Omega) + F(u, \Omega').$$

(5.9)

Using standard argument in Measure Theory we can extend the function $\Omega \mapsto F(u, \Omega)$ to a Borel measure $\tau(u, \cdot)$ on $M$ with

$$\tau(u, \cdot) = F(u, \cdot)$$
on relatively compact open sets; moreover using a Radon-Nikodym argument, similarly as in the proof [12, Lemma 2.8], we can show that $\tau(u, \cdot)$ is absolutely continuous w.r.t. $\text{Vol}_g(\cdot)$. If in the above Lemma 5.7 we let $(W_\ell)_{\ell \in I}$ be the given atlas of $M$, and $\sigma(\cdot) = \text{Vol}_g(\cdot)$ then we can apply Lemma 2.8 in [12] on each relatively compact open set

$$\Omega \cap U_\ell, \; \ell \in I,$$

and find a matrix $(\alpha_{\ell}^{ij})_{i,j=1}^d$ such that:

$$0 \leq \sum_{i,j=1}^d \alpha_{\ell}^{ij} \xi_i \xi_j \leq \Lambda_0^{1+d/2} \sqrt{\det g} \sum_{i,j=1}^d g^{ij} \xi_i \xi_j,$$

for all $x \in U_\ell \cap \Omega$, and $\xi \in \mathbb{R}^d$,

$$\alpha_{\ell}^{ij} = \alpha_{\ell}^{ji}, \; \alpha_{\ell}^{ij} \in L^\infty(M), i, j = 1, \ldots, d,$$

so that if we define

$$a_{\ell}^{ij} := \frac{\alpha_{\ell}^{ij}}{\Lambda_0^{d/2} \sqrt{\det g}}, \; i, j = 1, \ldots, d,$$

we get

$$F(u, \Omega \cap U_\ell) = \int_{\Omega \cap U_\ell} \sum_{i,j=1}^d a_{\ell}^{ij} D_i u D_j u \sqrt{\det g} \, dx,$$

(5.10)

for $u \in \text{Lip}(M)$.

As the measure $\tau(u, \cdot)$ is absolutely continuous w.r.t. $\text{Vol}_g(\cdot) = \sigma(\cdot)$, we
have from Lemma 5.7-(4) that \( \tau(u, \bigcup_{\ell \in I} \partial U_{\ell}) = 0 \); the sets \( U_{\ell}, \ell \in I \), are disjoint, hence \( \tau(u, \Omega) = \sum_{\ell \in I} \tau(u, \Omega \cap U_{\ell}) \), which implies

\[
(5.11) \quad F(u, \Omega) = \sum_{\ell \in I} F(u, \Omega \cap U_{\ell}),
\]
as \( \tau(u, \cdot) \) coincides with \( F(u, \cdot) \) on relatively compact open sets. Therefore by (5.10) and (5.11) we get

\[
F(u, \Omega) = \sum_{\ell \in I} \int_{\Omega \cap U_{\ell}} \sum_{i,j=1}^{d} a^{ij}_{\ell} D_{i}u D_{j}u \sqrt{\det g} \, dx\]

where the \((0,2)\)-tensor \( a = (a^{ij})_{i,j=1}^{d} \) is defined by

\[
a^{ij} = a^{ij}(x) := \begin{cases} 
a^{ij}_{\ell}(x), & x \in U_{\ell} \\
g^{ij}(x), & x \in \bigcup_{\ell \in I} \partial U_{\ell}, \end{cases}
\]
so that we have

\[
0 \leq \sum_{i,j=1}^{d} a^{ij}_{\ell} \xi_{i} \xi_{j} \leq \Lambda o \sum_{i,j=1}^{d} g^{ij}_{\ell} \xi_{i} \xi_{j},
\]
for every \( x \in M \) and for every \( \xi \in \mathbb{R}^{d} \), with \( a^{ij} = a^{ji}, a^{ij} \in L^{\infty}(M), i, j = 1, \ldots, d \). The proof of the proposition is thus completed.

**Proof of Theorem 2.** Let \( \phi : L^{2}(M, g) \rightarrow [0, +\infty] \) be the \( \Gamma \)-limit (in \( L^{2}(M, g) \)) of (a possible subsequence of) \( (\tilde{D}_{Mh,g_{h}}(u))_{h} \). We notice that, for each \( h \in \mathbb{N} \), the functional \( (\tilde{D}_{Mh,g_{h}}(u))_{h} \) is also the lower semi-continuous regularization of the functional \( F_{h}(\cdot, M) \) introduced in Proposition 5.6; this statement can be proven as in the Step 1 of the proof of Theorem 4.4. in [3]. Hence it is not difficult to see that \( \phi \) coincides with \( F(\cdot, M) \), the \( \Gamma \)-limit of \( (F_{h}(\cdot, M))_{h} \). The previous Proposition 5.6 then, gives us the representation formula for \( \phi \), namely,

\[
\phi(u) = \int_{M} \sum_{i,j=1}^{d} a^{ij} D_{i}u D_{j}u \sqrt{\det g} \, dx
\]
with \( u \in \text{Lip}(M) \), and the tensor \((a^{ij})_{i,j=1}^d\) satisfies
\[
0 \leq \sum_{i,j=1}^d a^{ij} \xi_i \xi_j \leq \Lambda_o \sum_{i,j=1}^d g^{ij} \xi_i \xi_j,
\]
for all \( x \in M \), and \( \xi \in \mathbb{R}^d \), and
\[
\left\{ \begin{array}{l}
a^{ij} \in L^\infty(M), \\
a^{ij} = a^{ji},
\end{array} \right.
\]
for \( i, j = 1, \ldots, d \). As in Step 2 of the proof of [6, Theorem 4.4] we have
\[
\phi(u) = \int_M \sum_{i,j=1}^d a^{ij} D_i u D_j u \sqrt{\det g} \, dx
\]
for all \( u \in H^1(M, g) \).
We now prove that the functional \( \phi \) is equal to \(+\infty\) outside \( H^1(M, g) \); more precisely, we have
\[
\|u\|_{H^1(M, g)}^2 \leq c_2^2 \phi(u) + c_2^2 \Lambda_o^{d/2} \|u\|_{L^2(M, g)}^2,
\]
and \( \phi(u) = +\infty \) for every \( u \in L^2(M, g) \setminus H^1(M, g) \).
First of all we notice that
\[
\|v\|_{L^2((M, g_h)}^2 \leq \Lambda_o^{d/2} \|v\|_{L^2(M, g)}^2.
\]
Indeed by (5.14) we have
\[
\|v\|_{L^2((M, g_h)}^2 = \int_{M_h} |v|^2 \sqrt{\det g_h} \, dx \leq \Lambda_o^{d/2} \int_{M_h} |v|^2 \sqrt{\det g} \, dx \\
\leq \int_M |v|^2 \sqrt{\det g} \, dx \leq \Lambda_o^{d/2} \|v\|_{L^2(M, g)}^2.
\]
Let \( u \in L^2(M, g) \) be such that \( \phi(u) < +\infty \). As \( \phi \) is the \( \Gamma \)-limit of \((\phi_h)_h\), there exists by definition a sequence \((u_h)_h\) of functions converging to \( u \) in \( L^2(M, g) \) with
\[
\lim_{h \uparrow +\infty} \phi_h(u_h) = \phi(u).
\]
We notice that
\[
\|u_h\|_{H^1(M, g_h)}^2 = \phi_h(u_h) + \|u_h\|_{L^2(M, g_h)}^2,
\]
and by (5.14) \( \|u_h\|_{H^1(M, g_h)}^2 \leq \phi_h(u_h) + \Lambda_o^{d/2} \|u_h\|_{L^2(M, g)}^2 \).
Hence, by our assumption on $u$ and $(u_h)_h$,
\[ \limsup_{h \uparrow +\infty} \|u_h\|_{H^1(M_h; g_h)} \leq \phi(u) + \Lambda_o^{d/2}\|u\|^2_{L^2(M; g)} < +\infty. \]
Using the strong connectivity condition (A2) (we recall that $c_o$ is independent of $h$)
\[ \|\pi_h u_h\|^2_{H^1(M, g)} \leq c_o^2 \|u_h\|^2_{H^1(M_h; g_h)} \]
(and setting $v_h := \pi_h u_h$ for shortness) we have
\[ \limsup_{h \uparrow +\infty} \|v_h\|^2_{H^1(M, g)} \leq c_o^2 \left( \phi(u) + \Lambda_o^{d/2}\|u\|^2_{L^2(M; g)} \right). \]
Therefore, up to a subsequence, $(v_h)_h$ converges to a function $v \in H^1(M, g)$ weakly in $H^1(M, g)$ and, by Rellich’s theorem, strongly in $L^2(M, g)$. Using again (5.14), applied this time to $u - u_h$ and $v - v_h$, we have
\[ \|u - u_h\|^2_{L^2(M_h; g_h)} \leq \Lambda_o^{d/2}\|u - u_h\|^2_{L^2(M; g)} \]
\[ \|v - v_h\|^2_{L^2(M_h; g_h)} \leq \Lambda_o^{d/2}\|v - v_h\|^2_{L^2(M; g)}. \]
Being $\pi_h$ an extension operator, we also have
\[ v_h = u_h \text{ (hence } u = v) \text{ on } M_h. \]
By the assumption (A3), which we recall here,
\[ \begin{cases} (1_{M_h})_h \text{ converges } w^*-L^\infty(M) \text{ to } b, \\ b, b^{-1} \in L^\infty(M) \end{cases} \]
and by Hölder’s inequality we get
\[ (5.15) \]
\[ \|u - v\|^2_{L^2(M, g)} \leq \|b^{-1}\|_{L^\infty(M)} \int_M |u - v|^2b(x)\sqrt{\det g} \, dx. \]
From this inequality we get that $u = v \text{ Vol}_g$-almost everywhere on $M$; indeed by (5.15), (A3) and (5.3), we have
\[ \int_M |u - v|^2\sqrt{\det g} \, dx \leq \|b^{-1}\|_{L^\infty(M)} \int_M |u - v|^2b(x)\sqrt{\det g} \, dx \]
\[ \leq \|b^{-1}\|_{L^\infty(M)} \lim_{h \uparrow +\infty} \int_{M_h} |u - v|^2\sqrt{\det g} \, dx \]
\[ \leq \|b^{-1}\|_{L^\infty(M)} \Lambda_o^{d/2} \lim_{h \uparrow +\infty} \int_{M_h} |u - v|^2\sqrt{\det g_h} \, dx \]
\[ = 0. \]
Therefore $u = v$ Vol$_g$-almost everywhere on $M$, hence
\[ \phi(u) < +\infty \implies u \in H^1(M,g), \]
and
\[ \|u\|_{H^1(M,g)}^2 = \|v\|_{H^1(M,g)}^2 \leq \liminf_{h \uparrow +\infty} \|v_h\|_{H^1(M,g)} \]
\[ \leq c_0^2 \left( \phi(u) + \Lambda_o d^2/2 \|u\|_{L^2(M,g)} \right). \]

The proof of the theorem will be achieved if we prove that the tensor \((a^{ij})_{i,j=1}^d\) satisfies the following ellipticity condition
\begin{equation}
(5.16) \quad c_o^{-2} \sum_{i,j=1}^d g^{ij} \xi_i \xi_j \leq \sum_{i,j=1}^d a^{ij} \xi_i \xi_j.
\end{equation}

Indeed if \((5.16)\) holds true, then it follows that \((a^{ij})_{i,j=1}^d\) is invertible and its inverse $a = (a_{ij})_{i,j=1}^d$ is symmetric, being \((a^{ij})_{i,j=1}^d\) such, and $a_{ij} \in L^\infty(M)$, $i,j = 1, \ldots, d$. Thus $(M,a)$ is a Lipschitz manifold, and from $(5.12)$ and $(5.16)$ we get
\[ \Lambda_o^{-1} \sum_{i,j=1}^d g_{ij} \xi^i \xi^j \leq \sum_{i,j=1}^d a_{ij} \xi^i \xi^j \leq c_o^2 \sum_{i,j=1}^d g_{ij} \xi^i \xi^j \]
for every $x \in M$ and every $\xi \in \mathbb{R}^d$.

Now we prove $(5.16)$ and to this aim we make use of the family $(U_\ell)_{\ell \in I}$ of open sets as in Lemma 5.7 with $X = M$, $(W_\ell)_{\ell \in I}$ is the given atlas of $M$ and $\sigma(\cdot) = \text{Vol}_g(\cdot)$. Consider, for $\ell \in I$, the open set $U_\ell$, and let $\psi \in \text{Lip}(M)$. In local coordinates, and making use of the same computation done in Step 4 in the proof of [1, Theorem 4.4], we then have
\begin{equation}
(5.17) \quad c_o^{-2} \int_{U_\ell} \left( \sum_{i,j=1}^d g^{ij} \xi_i \xi_j \right) \psi^2 \sqrt{\det g} \, dx
\end{equation}
\[ \leq \int_{U_\ell} \left( \sum_{i,j=1}^d a^{ij} \xi_i \xi_j \right) \psi^2 \sqrt{\det g} \, dx \]
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for every \( \xi \in \mathbb{R}^d \). We apply Lemma 5.7 with \( \sigma(\cdot) = \text{Vol}_g(\cdot) \) and get, for every \( \psi \in \text{Lip}(M) \),

\[
\varepsilon_0^{-2} \int_M \left( \sum_{i,j=1}^d g^{ij} \xi_i \xi_j \right) \psi^2 \sqrt{\det g} \, dx 
\leq \int_M \left( \sum_{i,j=1}^d a^{ij} \xi_i \xi_j \right) \psi^2 \sqrt{\det g} \, dx,
\]

for every \( \xi \in \mathbb{R}^d \). Therefore, up to a redefinition on a set of \( \text{Vol}_g \)-measure zero, we get (5.16). Define then \( D_{M,a,w}(\cdot) := \phi(\cdot) \), and the proof of the theorem is complete.

**Proposition 5.18.** In addition to the assumptions (A1), (A2), (A3) let us suppose that

\[
1_{M_h} \rightharpoonup b \quad \text{in measure on } M
\]

(5.19)

\[
g_{ij,h} 1_{M_h} \rightharpoonup a_{ij} \quad \text{in measure on } M, \ i, j = 1, \ldots, d.
\]

(5.20)

Then the same conclusion of Theorem 3 holds and the Lipschitz metric is equal to \( a = (a_{ij})_{i,j} \), where the coefficients \( a_{ij} \) are given by (5.20) above.

**Proof.** It follows the lines of the proof of Proposition 4.5 in [6].

**Variational limits of handles.** In this part we shall be concerned with the asymptotic behavior of sequences of functionals \( \phi_h : L^2(M, g) \rightarrow [0, +\infty] \) of the type

\[
\phi_h(u) := D_{M_h}(u) + \int_{M_h} \left[ u(x) - u(T(x)) \right]^2 \mu_h(dx),
\]

(5.21)

if \( u|_M \in H^1(M_h, g_h) \); \( \phi_h(u) := +\infty \) otherwise in \( L^2(M, g) \).

In (5.21) above, \( (\mu_h)_h \) is a sequence of measures such that \( \mu_h \in M_o(M_h, g_h) \), \( h \in \mathbb{N} \); the map \( T : M \rightarrow M \) is an isometry, \( T \circ T = \text{id}_M \), and the fixed-point set of \( T \), \( \text{Fix}(T) \), is a submanifold of \( M \).

**Theorem 3.** Assume that \( ((M_h, g_h))_h \) satisfies (A1), (A2), (A3) above and is a \( T \)-invariant sequence of Lipschitz manifolds, i.e.,

\[
(A_4) \ T(M_h) = M_h;
\]

\[
(A_5) \ \sqrt{\det g_h(x)} = \sqrt{\det g_h(T(x))} \det \left( \frac{\partial T}{\partial x}(x) \right), \ a.e. \ in \ M_h;
\]
(A_6) \ g_h^{ij}(T(x)) = \sum_{k,\ell=1}^{d} \frac{\partial T_i}{\partial x_k}(x) \frac{\partial T_j}{\partial x_{\ell}}(x) g_h^{k\ell}(x), \ \text{a.e. in } M_h.

Let us consider, for each \( h \in \mathbb{N} \), the relaxed Dirichlet functional \( \phi_h(u) \) defined on \( L^2(M, g) \) by

\[
\phi_h(u) := D_{M_h}(u) + \int_{M_h} \left[ u(x) - u(T(x)) \right]^2 \mu_h(dx),
\]

if \( u \in L^2(M, g) \) with \( u|_{M_h} \in H^1(M_h, g_h) \); \( \phi_h(u) := +\infty \) otherwise in \( L^2(M, g) \). Assume that \( \mu_h \in \mathcal{M}_o(M_h) \), and \( \mu_h(\cdot) \sim \mu_h(T^{-1}(\cdot)) \) (in the sense of Definition 1.15).

Then there exists a \( T \)-invariant Lipschitz metric \( a = (a_{ij})_{i,j} \) on \( M \), with a \( T \)-invariant canonical measure \( \text{Vol}_a \), and a Borel measure \( \mu \in \mathcal{M}_o(M) \), with \( \mu(\cdot) \sim \mu(T^{-1}(\cdot)) \), such that the sequence of functionals given by (5.21) \( \Gamma \)-converges in \( L^2(M, g) \) to the functional \( \phi(u) \) defined by

\[
(5.22) \quad \phi(u) := D_{M,a,w}(u) + \int_{M} \left[ u(x) - u(T(x)) \right]^2 \mu(dx),
\]

if \( u \in H^1(M, g) \), and \( \phi(u) := +\infty \) otherwise in \( L^2(M, g) \). The functional \( D_{M,a,w}(\cdot) \) is the weighted Dirichlet functional introduced in Theorem 2.

Remark 5.23. Using (A_1), (A_2), (A_3), the existence of a Lipschitz metric follows from Theorem 2, where it is also proved that the sequence \( (D_{M_h}(\cdot)) \) \( \Gamma \)-converges to \( D_{M,a,w}(\cdot) \).

The \( T \)-invariance of the canonical measure \( \text{Vol}_a \) and of the metric \( a = (a_{ij})_{i,j} \) are proved similarly as in [3, Lemma 5.1], by means of (A_4), (A_5), (A_6).

What is left to prove, then, is the existence of a Borel measure \( \mu \in \mathcal{M}_o(M) \), with \( \mu(\cdot) \sim \mu(T^{-1}(\cdot)) \), such that \( (\phi_h) \ \Gamma \)-converges in \( L^2(M, g) \) to (5.22).

To this aim we need the following lemma, which is a generalization to this framework of a result in the euclidean setting by G. Buttazzo, G. Dal Maso, & U. Mosco in [3, Section 4 & Appendix]; cf. also [3, Lemma 5.3].

**Lemma 5.24.** Let \( m : H^1(M, a) \to [0, +\infty] \) be such that:

1. If \( 0 \leq u \leq v \) a.e. on \( M \), then \( m(u) \leq m(v) \);
2. \( m(|u|) = m(u) \);
3. \( m(u + v) \leq m(u) + m(v) \), if \( \min\{u(x), v(x)\} = 0 \) for almost every \( x \in M \);
(4) $m(u) = \lim_{n \to +\infty} m(u_n)$, for every increasing sequence of positive
functions $(u_n)$ such that

$$(M,a)\text{-}\text{cap}\{x \in M : u_n(x) \text{ does not converge to } u(x)\} = 0;$$

(5) $m(0) = 0$; $m(tu) = t^2 m(u)$, for $t \in \mathbb{R}$; $m(u + v) + m(u - v) = 2[m(u) + m(v)]$.

Then there exists a measure $\mu \in \mathcal{M}_0(M,a)$ such that

$$m(u) = \int_M u^2 d\mu,$$

for every $u \in H^1(M,a)$.

**Proof.** It is a simple modification of Theorem 2.3 in [14], by means of [14, Theorem 2.22].

**Proof of Theorem 3.** By means of Lemma 5.24, the proof now runs parallel
to the proof of Proposition 5.2 in [6], which gives the existence of a mea-
ure $\mu \in \mathcal{M}_0(M)$, with $\mu(\cdot) \sim \mu(T^{-1}(\cdot))$, such that $(\phi_h)$ $\Gamma$-converges in
$L^2(M,g)$ to the functional given in (5.22). The theorem is so proved.

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