On bifurcation of cusps
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Abstract Let $f = \mathbb{R} \times \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ be an analytic mapping having a critical point at the origin. There is the corresponding one-parameter family of mappings $f_t = f(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$.

There will be presented effective algebraic methods of computing the number of cusps of $f_t$, where $0 < |t| \ll 1$, emanating from the origin and having a positive/negative cusp degree.

1 Introduction

Mappings between surfaces are a natural object of study in the theory of singularities. Whitney [27] proved that critical points of such a generic mapping are folds and cusps. There are several results concerning relations between the topology of surfaces and the topology of the critical locus of a mapping (see [17], [26], [27]). Singularities of map germs of the plane into the plane were studied in [9], [10], [12], [19], [20], [23].

Let $f_t$, where $t \in \mathbb{R}$, be an analytic family of plane-to-plane mappings with $f_0$ having a critical point at the origin. Under some natural assumptions there is a finite family of cusp points of $f_t$ bifurcating from the origin. There are important results [9, Theorem 3.1], [20, Proposition 7.1] concerning the parity of the number of those points.

In this paper we show how to compute the number of cusps of $f_t$ which are represented by germs having either positive or negative local topological degree (see Theorem 6.8).

The paper is organized as follows. In Sections 2 and 3 we collect some useful facts. The curve in $\mathbb{R} \times \mathbb{R}^2$ consisting of points $(t, x)$, where $x$ is a cusp point of $f_t$, is defined by three analytic equations, so that it is not a complete intersection. In Section 4 we show how to adopt in this case some more general techniques from [21] concerning curves in $\mathbb{R}^n$ defined by $m$ equations, where $m \geq n$.

In Sections 5 and 6 we prove the main result. In Section 7 we present examples computed by a computer. We have implemented our algorithm...
with the help of SINGULAR [7]. We have also used a computer program written by Łęcki [16].

2 Mappings between surfaces

Let \((M, \partial M)\) and \((N, \partial N)\) be compact oriented connected surfaces, and let \(f : M \to N\) be a smooth mapping such that \(f^{-1}(\partial N) = \partial M\). Assume that

(i) every point in \(M\) is either a fold point, a cusp point or a regular point, and there is only a finite number of cusps which all belong to \(M \setminus \partial M\),

(ii) the 1-dimensional manifold consisting of fold points is transverse to \(\partial M\), so that \(f|\partial M : \partial M \to \partial N\) is locally stable, i.e. its critical points are non-degenerate.

We shall write \(M^−\) for the closure in \(M\) of the set of regular points at which \(f\) does reverse the orientation.

If \(p \in M \setminus \partial M\) is a cusp point, we define \(\mu(p)\) to be the local topological degree of the germ \(f : (M, p) \to (N, f(p))\). Put

\[
\text{cusp deg } (f) = \sum \mu(p),
\]

where \(p\) runs through the set of all cusp points of \(f\).

Fukuda and Ishikawa [9] have generalized the result by Quine [22] concerning surfaces without boundary, proving

**Theorem 2.1.** Let \(M, N\) and \(f\) be as above and \(\partial M \neq \emptyset\). Then

\[
\text{cusp deg } (f) = 2\chi(M^−) + (\deg f|\partial M)\chi(N) - \chi(M) - \#C(f|\partial M)/2,
\]

where \(C(f|\partial M)\) is the set of critical points of \(f|\partial M\).

In fact, in [9] there is a stronger assumption that both \(f : M \to N\) and \(f|\partial M : \partial M \to \partial N\) are \(C^\infty\)–stable mappings. However, if \(f\) satisfies (i), (ii) then there exists its \(C^\infty\)–stable perturbation \(\tilde{f}\), which is arbitrary close to \(f\) in \(C^\infty\)–Whitney topology, such that all corresponding numbers associated to \(f\) and \(\tilde{f}\) which appear in the above theorem stay the same.

Let \(f = (f_1, f_2) : U \to \mathbb{R}^2\), where \(U \subset \mathbb{R}^2\) is open, be a smooth mapping.

Set \(J = \partial(f_1, f_2)/\partial(x_1, x_2)\), \(F_i = \partial(f_i, J)/\partial(x_1, x_2), i = 1, 2\). Applying the same arguments as in the proof of [15, Proposition 2, p. 815] one gets
Proposition 2.2. The set of all common solutions in $U$ of the system of equations $J = F_1 = F_2 = \partial(F_1, J)/\partial(x_1, x_2) = \partial(F_2, J)/\partial(x_1, x_2) = 0$ is empty if and only if the set of critical points of $f$ consists of either fold or cusp points.

If that is the case then the set of cusp points is discrete and equals $\{J = F_1 = F_2 = 0\}$.

3 Families of germs

In this section we recall some useful facts concerning 1-parameter families of real analytic germs.

For $r > 0$, let $D^n(r) = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$, and $S^{n-1}(r) = \partial D^n(r)$. We shall write $(t, x) = (t, x_1, \ldots, x_n) \in \mathbb{R} \times \mathbb{R}^n$. Assume $J(t, x) : \mathbb{R} \times \mathbb{R}^n, 0 \to \mathbb{R}$ is an analytic function defined in a neighbourhood of the origin having a critical point at $0$. We shall write $L_0 = \{x \in S^{n-1}(r) \mid J(0, x) = 0\}$,

$M^-_t = \{x \in D^n(r) \mid J(t, x) \leq 0\}$,

where $0 < |t| \ll r \ll 1$.

Let $f : \mathbb{R} \times \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ be an analytic mapping. Put $f_t(x) = f(t, x)$. Suppose that there exists a small $r > 0$ such that $f_0^{-1}(0) \cap D^n(r) = \{0\}$. For $0 < \delta \ll r$, put $\tilde{S}^{n-1}(\delta) = f_t^{-1}(S^{n-1}(\delta)) \cap D^n(r)$ and $\tilde{D}^n_t(\delta) = f_t^{-1}(D^n(\delta)) \cap D^n(r)$. We shall write

$\tilde{L}_0 = \{x \in \tilde{S}^{n-1}_0(\delta) \mid J(0, x) = 0\}$,

$\tilde{M}^-_t = \{x \in \tilde{D}^n_t(\delta) \mid J(t, x) \leq 0\}$,

where $0 < |t| \ll \delta \ll 1$.

Lemma 3.1. We have $\chi(\tilde{M}^-_t) = \chi(M^-_t)$ and $\chi(\tilde{L}_0) = \chi(L_0)$.

Proof. There exist small positive $\delta_1 < \delta_2$, $r_1 < r_2$ and $t_0$, such that for $0 < |t| < t_0$ we have

$\{x \in \tilde{D}^n_t(\delta_1) \mid J(t, x) \leq 0\} \subset \{x \in D(r_1) \mid J(t, x) \leq 0\}$

$\subset \{x \in \tilde{D}^n_t(\delta_2) \mid J(t, x) \leq 0\} \subset \{x \in D(r_2) \mid J(t, x) \leq 0\}$,
and inclusions
\[ \{ x \in \tilde{D}_t^n(\delta_1) \mid J(t, x) \leq 0 \} \subset \{ x \in \tilde{D}_t^n(\delta_2) \mid J(t, x) \leq 0 \}, \]
\[ \{ x \in D(r_1) \mid J(t, x) \leq 0 \} \subset \{ x \in D(r_2) \mid J(t, x) \leq 0 \} \]
induce isomorphisms of corresponding homology groups. Then
\[ \chi(\tilde{M}^-) = \chi(\{ x \in \tilde{D}_t^n(\delta_1) \mid J(t, x) \leq 0 \}) \]
\[ = \chi(\{ x \in D(r_2) \mid J(t, x) \leq 0 \}) = \chi(M^-). \]
The proof of the second assertion is similar. □

Define a mapping \( d_0 : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) by
\[ d_0(x) = \left( \frac{\partial J}{\partial x_1}(0, x), \ldots, \frac{\partial J}{\partial x_n}(0, x) \right), \]
and mappings \( d_1, d_2 : \mathbb{R} \times \mathbb{R}^n, 0 \to \mathbb{R} \times \mathbb{R}^n, 0 \), by
\[ d_1(t, x) = \left( \frac{\partial J}{\partial t}(t, x), \frac{\partial J}{\partial x_1}(t, x), \ldots, \frac{\partial J}{\partial x_n}(t, x) \right), \]
\[ d_2(t, x) = \left( J(t, x), \frac{\partial J}{\partial x_1}(t, x), \ldots, \frac{\partial J}{\partial x_n}(t, x) \right). \]
Applying directly results by Fukui [11] and Khimshiasvili [13, 14] we get

**Theorem 3.2.** Suppose that the origin is isolated in \( d_0^{-1}(0), d_1^{-1}(0) \) and \( d_2^{-1}(0) \), so that the local topological degrees \( \deg_0(d_0), \deg_0(d_1) \) and \( \deg_0(d_2) \) are defined.

Then both \( J(0, x) \) and \( J(t, x) \) have an isolated critical point at the origin.
If \( 0 \neq t \) is sufficiently close to zero then
\[ \chi(\tilde{M}^-) = \chi(M^-) = 1 - (\deg_0(d_0) + \deg_0(d_1) + \text{sign}(t) \cdot \deg_0(d_2))/2, \]
\[ \chi(\tilde{L}_0) = \chi(L_0) = 2 \cdot (1 - \deg_0(d_0)). \]
In particular, if \( n = 2 \) then \( \tilde{L}_0 \) is finite and \( \#\tilde{L}_0 = 2 \cdot (1 - \deg_0(d_0)). \)

It is proper to add that there exists an efficient computer program which may compute the local topological degree (see [16]).
4 Number of half–branches

In this section we shall show how to adopt some techniques developed in [21, 24, 25] so as to compute the number of half–branches of an analytic set of dimension $\leq 1$ emanating from a singular point.

Let $O_n = \mathbb{R}\{t, x_1, \ldots, x_{n-1}\}$ denote the ring of germs at the origin of real analytic functions. If $I$ is an ideal in $O_n$, let $V(I) \subset \mathbb{R}^n$ denote the germ of zeros of $I$ near the origin, and let $V_C(I) \subset \mathbb{C}^n$ denote the germ of complex zeros of $I$.

Remark 4.1. If $I$ is proper then $\dim \mathbb{R} O_n/I < \infty$ if and only if $V_C(I) = \{0\}$.

Let $w_1, \ldots, w_m \in O_n$, where $m \geq n - 1$, be germs vanishing at the origin. We shall write $\langle w_1, \ldots, w_m \rangle$ for the ideal in $O_n$ generated by $w_1, \ldots, w_m$.

Let $W \subset O_n$ denote the ideal generated by $w_1, \ldots, w_m$ and all $(n - 1) \times (n - 1)$–minors of the Jacobian matrix $[\partial w_i/\partial x_j]$. The ideal $W$ is proper if and only if the rank of this matrix at the origin is $\leq n - 2$.

If $V(W) = \{0\}$ then by the implicit function theorem the germ $V(w_1, \ldots, w_m)$ is of dimension $\leq 1$, so that this set is locally an union of a finite family of half-branches emanating from the origin. We shall say that $V(w_1, \ldots, w_m)$ is a curve having an algebraically isolated singularity at the origin if $W$ is proper and $\dim \mathbb{R} O_n/W < \infty$.

From now on we shall assume that $m = n = 3$ Let $M(3, 3)$ denote the space of all $3 \times 3$–matrices with coefficients in $\mathbb{R}$. By [21, Theorem 3.8] and comments in [21, p. 1012] we have

Theorem 4.2. Assume that $V(w_1, w_2, w_3)$ is a curve having an algebraically isolated singularity. There exists a proper algebraic subset $\Sigma \subset M(3, 3)$ such that for every non-singular matrix $[a_{ij}] \in M(3, 3) \setminus \Sigma$ and $g_s = a_{1s}w_1 + a_{2s}w_2 + a_{3s}w_3$, where $1 \leq s \leq 3$, the set $V(g_1, g_2)$ is a curve having an algebraically isolated singularity at the origin and $V(w_1, w_2, w_3) = V(g_1, g_2, g_3) \subset V(g_1, g_2)$.

In particular, if $V(w_1, w_2)$ is a curve having an algebraically isolated singularity then one may take $g_s = w_s$.

If that is the case and $J_p = \langle g_1, g_2, g_3^p \rangle$, where $p = 1, 2$, then $J_2 \subset J_1$ and $\dim \mathbb{R} (J_1/J_2) < \infty$.

From now on we shall assume that

(1) $\dim \mathbb{R} O_3/(t, g_1, g_2) < \infty$. 

5
As \( \dim_\mathbb{R}(J_1/J_2) < \infty \) and \( g_3(0) = 0 \), then by the Nakayama lemma \( \xi = \min\{s \mid t^s \cdot g_3 \in J_2 \} \) is finite. (In \cite{25} there are presented effective methods for computing this number.) Let \( k > \xi \) be an even positive integer.

Now we shall adopt to our case some arguments presented in \cite{25} pp. 529-531]. There are germs \( h_1, h_2, h_3 \in \mathcal{O}_3 \) such that

\[
t^\xi g_3 = h_1g_1 + h_2g_2 + h_3g_3^2.
\]

Let \( Y_C = V_C(g_1, g_2) \setminus V_C(g_3) \). By \( \mathbb{I} \), the germ \( t^k \) does not vanish at points in \( V_C(g_1, g_2) \setminus \{0\} \). If \( (t, x_1, x_2) = (t, x) \in Y_C \) lies sufficiently close to the origin then \( |h_3(t, x)| < M \) for some \( M > 0 \), \( g_1(t, x) = g_2(t, x) = 0 \) and \( g_3(t, x) \neq 0 \). Hence

\[
|g_3(t, x)| \geq |t|^\xi/M > |t|^k.
\]

Then the origin is isolated in both \( V_C(g_3 \pm t^k, g_1, g_2) \).

Take \( (t, x) \in V(g_1, g_2) \setminus \{0\} \) near the origin. By \( \mathbb{I} \), \( t \neq 0 \). If \( g_3(t, x) \neq 0 \) then \( g_3(t, x) \pm t^k \) has the same sign as \( g_3(t, x) \). If \( g_3(t, x) = 0 \) then \( g_3(t, x) + t^k > 0 \) and \( g_3(t, x) - t^k < 0 \). Write \( b_+ \) (resp. \( b_- \), \( b_0 \)) for the number of half-branches of \( V(g_1, g_2) \) on which \( g_3 \) is positive (resp. \( g_3 \) is negative, \( g_3 \) vanishes). Put

\[
H_\pm = \left( \frac{\partial (g_3 \pm t^k, g_1, g_2)}{\partial (t, x_1, x_2)}, g_1, g_2 \right): \mathbb{R}^3, \mathbf{0} \to \mathbb{R}^3, \mathbf{0}.
\]

By \cite{24} Theorem 3.1] or \cite{25} Theorem 2.3], the origin is isolated in both \( H_\pm^{-1}(0) \) and

\[
b_+ + b_0 - b_- = 2 \deg_0(H_+), \\
b_+ - b_0 - b_- = 2 \deg_0(H_-).
\]

**Theorem 4.3.** If \( \dim_\mathbb{R}\mathcal{O}_3/\langle t, g_1, g_2 \rangle < \infty \) then the number \( b_0 \) of half-branches of \( V(w_1, w_2, w_3) \) emanating from the origin equals \( \deg_0(H_+) - \deg_0(H_-) \).

**Proof.** As the matrix \( [a_{ij}] \) is non-singular, then \( V(w_1, w_2, w_3) = V(g_1, g_2, g_3) \). Of course, \( b_0 \) equals the number of half-branches of \( V(g_1, g_2, g_3) \). Moreover,

\[
b_0 = \frac{1}{2}( (b_+ + b_0 - b_1) - (b_+ - b_0 - b_1)) = \deg_0(H_+) - \deg_0(H_-). \quad \Box
\]

Now we shall explain how to compute the number of half-branches of \( V(w_1, w_2, w_3) \) in the region where \( t > 0 \).
Proposition 4.4. Put \( g'_i(t, x) = g_i(t^2, x) \). Then \( \dim_{\mathbb{R}} \mathcal{O}_3/\langle t, g'_1, g'_2 \rangle < \infty \) and \( V(g'_1, g'_2) \) has an isolated singularity at the origin.

Proof. By (1), as \( V_{\mathbb{C}}(t, g_1, g_2) = \{0\} \) then \( V_{\mathbb{C}}(t, g'_1, g'_2) = \{0\} \). By Remark 4.1 \( \dim_{\mathbb{R}} \mathcal{O}_3/\langle t, g'_1, g'_2 \rangle < \infty \). We have

\[
\frac{\partial (g'_i, g'_j)}{\partial (t, x_p)}(t, x) = 2t \frac{\partial (g_i, g_j)}{\partial (t, x_p)}(t^2, x), \quad \frac{\partial (g'_i, g'_j)}{\partial (x_1, x_2)}(t, x) = \frac{\partial (g_i, g_j)}{\partial (x_1, x_2)}(t^2, x),
\]

and then \( V(g'_1, g'_2) \) is a curve having an algebraically isolated singularity at the origin. \( \square \)

Remark 4.5. Let \( J'_p = \langle g'_1, g'_2, (g'_3)^p \rangle \). Put \( \xi' = \min\{s \mid t^s \cdot g'_3 \subset J'_2\} \). Of course, \( \xi' \leq 2 \cdot \xi \).

Applying the same methods as above, one may compute the number \( b'_0 \) of half-branches of \( V(g'_1, g'_2, g'_3) \). Obviously \( b'_0/2 \) equals the number of half-branches of \( V(w_1, w_2, w_3) \) lying in the region where \( t > 0 \).

Other methods of computing the number of half-branches were presented in [1, 2, 3, 4, 5, 6, 7, 8, 18].

According to Khimshiashvili [13, 14], if a germ \( f : \mathbb{R}^2, 0 \to \mathbb{R}, 0 \) has an isolated critical point at the origin then the number of real half-branches in \( f^{-1}(0) \) equals \( 2 \cdot (1 - \deg_0(\nabla f)) \), where \( \nabla f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0 \) is the gradient of \( f \).

5 Mappings between curves

In this section we give sufficient conditions for a mapping between some smooth plane curves to have only non-degenerate critical points.

Let \( f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) be a smooth mapping. Put \( g = f_1^2 + f_2^2 \). Assume that \( \delta^2 > 0 \) is a regular value of \( g \) and \( P = g^{-1}(\delta^2) \) is non-empty, so that \( P \) is a smooth curve. Obviously, \( P = f^{-1}(S^1(\delta)) \) and \( f|P : P \to S^1(\delta) \) is a smooth mapping between 1-dimensional manifolds.

At any \( p \in P \) the gradient \( \nabla g(p) = (\partial g/\partial x_1(p), \partial g/\partial x_2(p)) \) is a non-zero vector perpendicular to \( P \), and the vector \( T(p) = (-\partial g/\partial x_2(p), \partial g/\partial x_1(p)) \) obtained by rotating \( \nabla g(p) \) counterclockwise by an angle of \( \pi/2 \) is tangent to \( P \). This way \( T : P \to \mathbb{R}^2 \) is a non-vanishing tangent vector field along \( P \).
Take \( p \in P \). There exists a smooth mapping \( x(t) = (x_1(t), x_2(t)) : \mathbb{R} \to P \) such that \( x(0) = p \) and \( x'(t) = T(x(t)) \). Hence

\[
(2) \quad x_1'(t) = -2 \cdot \left( f_1 \frac{\partial f_1}{\partial x_2} + f_2 \frac{\partial f_2}{\partial x_2} \right) \bigg|_{(x(t))},
\]

\[
x_2'(t) = 2 \cdot \left( f_1 \frac{\partial f_1}{\partial x_1} + f_2 \frac{\partial f_2}{\partial x_1} \right) \bigg|_{(x(t))}.
\]

As \( g(x(t)) = \delta^2 \), then \( f(x(t)) = (\delta \cos \theta(t), \delta \sin \theta(t)) \) for some smooth function \( \theta : \mathbb{R}, 0 \to \mathbb{R} \). Of course, \( (\delta \cos \theta(0), \delta \sin \theta(0)) = f(x(0)) = f(p) \).

Applying the complex numbers notation we may write

\[
(3) \quad \delta \cdot e^{i\theta} = f_1(x(t)) + i f_2(x(t)), \text{ where } i = \sqrt{-1}.
\]

Put \( J = \partial(f_1, f_2)/\partial(x_1, x_2) \) and \( F_j = \partial(f_j, J)/\partial(x_1, x_2) \), where \( j = 1, 2 \).

**Lemma 5.1.** A point \( p \in P \) is a critical point of \( f|P : P \to S^1(\delta) \) if and only if \( J(p) = 0 \).

**Proof.** By \( (2) \), the derivative of the equation \( (3) \) equals

\[
i \delta \theta' \cdot e^{i\theta} = \left( \frac{\partial f_1}{\partial x_1} x_1' + \frac{\partial f_1}{\partial x_2} x_2' \right) + i \cdot \left( \frac{\partial f_2}{\partial x_1} x_1' + \frac{\partial f_2}{\partial x_2} x_2' \right)
\]

\[
= 2i(f_1 + i f_2) \cdot J = 2i \delta \cdot e^{i\theta} \cdot J.
\]

So \( p \in P \) is a critical point of \( f|P \) if and only if \( \theta'(0) = 0 \), i.e. if \( J(p) = 0 \). \( \square \)

**Lemma 5.2.** Suppose that \( p \in P \) is a critical point of \( f|P : P \to S^1(\delta) \). Then

\[
\text{sign} (\theta''(0)) = \text{sign} (f_1 \cdot F_1 + f_2 \cdot F_2)|_p.
\]

In particular, a point \( p \in P \) is a non-degenerate critical point of \( f|P : P \to S^1(\delta) \) if and only if \( J(p) = 0 \) and \( (f_1 \cdot F_1 + f_2 \cdot F_2)|_p \neq 0 \).

**Proof.** Since \( \theta'(0) = 0 \) and \( J(p) = 0 \), after computing the second derivative of \( (3) \) the same way as above one gets

\[
i \delta \theta'' \cdot e^{i\theta}|_0 = 2i \delta \cdot e^{i\theta} \cdot \left( \frac{\partial J}{\partial x_1} x_1' + \frac{\partial J}{\partial x_2} x_2' \right)|_0
\]

\[
= 4i \delta \cdot e^{i\theta(0)} \cdot (f_1 \cdot F_1 + f_2 \cdot F_2)|_p. \quad \square
\]
Lemma 5.3. Let \( f = (f_1, f_2) : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0 \) be an analytic mapping such that \( J(0) = 0 \), and the origin is isolated in both \( f^{-1}(0) \) and \( \nabla J^{-1}(0) \).

If \( 0 < \delta \ll r \ll 1 \) then \( \tilde{S}^1(\delta) = D(r) \cap f^{-1}(S^1(\delta)) \) is diffeomorphic to a circle, \( \tilde{D}^2(\delta) = D(r) \cap f^{-1}(D^2(\delta)) \) is diffeomorphic to a disc, and \( f : \tilde{S}^1(\delta) \to S^1(\delta) \) has only non-degenerate critical points. Moreover the one-dimensional set \( J^{-1}(0) \) consisting of critical points of \( f \) is transverse to \( \tilde{S}^1(\delta) \).

Proof. If the origin is isolated in \( J^{-1}(0) \) then \( f|\mathbb{R}^2 \setminus \{0\} \) is a submersion near the origin, and so \( f : \tilde{S}^1(\delta) \to S^1(\delta) \) has no critical points.

In the other case, \( J^{-1}(0) \setminus \{0\} \) is locally a finite union of analytic half-branches emanating from the origin. Let \( B \) be one of them. The gradient \( \nabla J(p) \) is a non-zero vector perpendicular to \( T_p B \) at any \( p \in B \).

The origin is isolated in \( f^{-1}(0) \). By the curve selection lemma one may assume that \( (f_1^2 + f_2^2)|B \) has no critical points, so that \( \nabla J \) and

\[
\nabla(f_1^2 + f_2^2) = \left( 2f_1 \frac{\partial f_1}{\partial x_1} + 2f_2 \frac{\partial f_2}{\partial x_1}, 2f_1 \frac{\partial f_1}{\partial x_2} + 2f_2 \frac{\partial f_2}{\partial x_2} \right)
\]

are linearly independent along \( B \). Then

\[
0 \neq \nabla J \times \nabla(f_1^2 + f_2^2) = 2f_1 \frac{\partial(J, f_1)}{\partial(x_1, x_2)} + 2f_2 \frac{\partial(J, f_2)}{\partial(x_1, x_2)} = -2(f_1 \cdot F_1 + f_2 \cdot F_2)
\]

along \( B \). By previous lemmas, \( f : \tilde{S}^1(\delta) \to S^1(\delta) \) has only non-degenerate critical points. Other assertions are rather obvious. \( \square \)

6 Families of self-maps of \( \mathbb{R}^2 \)

In this section we investigate 1-parameter families of plane-to-plane analytic mappings.

Let \( f = (f_1, f_2) : \mathbb{R} \times \mathbb{R}^2, 0 \to \mathbb{R}^2, 0 \) be an analytic function defined in a neighbourhood of the origin. We shall write \( f_t(x_1, x_2) = f(t, x_1, x_2) \) for \( t \) near zero. Define three germs \( \mathbb{R} \times \mathbb{R}^2, 0 \to \mathbb{R} \) by

\[
J = \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}, \quad F_i = \frac{\partial(f_i, J)}{\partial(x_1, x_2)}.
\]

Put \( J_t(x_1, x_2) = J(t, x_1, x_2) \).
From now on we shall also assume that

\[ \dim_{\mathbb{R}} \mathcal{O}_3/\langle t, f_1, f_2 \rangle < \infty, \quad \dim_{\mathbb{R}} \mathcal{O}_3/\langle t, F_1, F_2 \rangle < \infty, \]

\[ J(0) = 0, \quad \dim_{\mathbb{R}} \mathcal{O}_3/\langle t, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \rangle < \infty, \]

i.e. the origin is isolated in both \( \{0\} \times \mathbb{C}^2 \cap V_C(f_1, f_2) \), \( \{0\} \times \mathbb{C}^2 \cap V_C(F_1, F_2) \), and \( J_0 \) has an algebraically isolated critical point at the origin.

**Lemma 6.1.** Let \( Q = \mathcal{O}_3/\langle t, J, F_1, F_2 \rangle \). Then \( \dim_{\mathbb{R}} Q < \infty \), i.e. the origin is isolated in \( \{0\} \times \mathbb{C}^2 \cap V_C(J, F_1, F_2) \).

**Proof.** Of course \( \langle t, F_1, F_2 \rangle \subset \langle t, J, F_1, F_2 \rangle \). Then \( \dim_{\mathbb{R}} Q \leq \dim_{\mathbb{R}} \mathcal{O}_3/\langle t, F_1, F_2 \rangle < \infty \). □

We shall write \( g = f_1^2 + f_2^2 \) and \( g_t(x_1, x_2) = g(t, x_1, x_2) \). There exists a small \( r_0 > 0 \) such that \( f_0^{-1}(0) \cap D^2(r_0) = \{0\} \). For \( |t| \ll \delta \ll r_0 \), put \( \tilde{S}_t^1(\delta) = f_t^{-1}(S^1(\delta)) \cap D^2(r_0) \) and \( \tilde{D}_t^2(\delta) = f_t^{-1}(D^2(\delta)) \cap D^2(r_0) \). If \( \delta^2 \) is a regular value of \( g_0 D^2(r_0) \), then it is also a regular value of \( g_t | D^2(r_0) \). If that is the case then \( \tilde{S}_t^1(\delta) \) is diffeomorphic to \( \tilde{S}_t^1(\delta) \simeq S^1(1) \). By the same argument, \( \tilde{D}_t^2(\delta) \) is diffeomorphic to \( \tilde{D}_t^2(\delta) \simeq D^2(1) \).

By Lemmas \[5.2\], \[5.3\] we get

**Lemma 6.2.** Critical points of \( f_0 : \tilde{S}_t^1(\delta) \to S^1(\delta) \) are non-degenerate, and \( C(f_0 | \tilde{S}_t^1(\delta)) = \tilde{S}_t^1(\delta) \cap \{J_0 = 0\} \).

For \( t \) near zero, critical points of \( f_t : \tilde{S}_t^1(\delta) \to S^1(\delta) \) are non-degenerate too, and the number of critical points \( \#C(f_t | \tilde{S}_t^1(\delta)) \) equals \( \#(\tilde{S}_t^1(\delta) \cap \{J_0 = 0\}) \). Moreover the set of critical points of \( f_t \), i.e. \( J_t^{-1}(0) \), is transverse to \( \tilde{S}_t^1(\delta) \). □

Let \( I \) denote the ideal in the ring \( \mathcal{O}_3 \) generated by \( J, F_1, F_2 \), and let \( V(I) \subset \mathbb{R} \times \mathbb{R}^2 \) denote a representative of the germ of zeros of \( I \) near the origin. By Lemma \[6.1\] there exists \( 0 < \delta \ll 1 \) such that \( \{0\} \times \tilde{D}_t^2(\delta) \cap V(I) = \{0\} \), and \( \{t\} \times \tilde{S}_t^1(\delta) \cap V(I) = \emptyset \) for \( t \) sufficiently close to zero. Put \( \Sigma_t = \{x \in \tilde{D}_t^2(\delta) | (t, x) \in V(I)\} \). Hence \( \Sigma_0 = \{0\} \) and \( \Sigma_t \) is contained in the interior of \( \tilde{D}_t^2(\delta) \).

Let \( I' \) denote the ideal in \( \mathcal{O}_3 \) generated by germs \( J, F_1, F_2, \partial(F_1, J)/\partial(x_1, x_2) \) and \( \partial(F_2, J)/\partial(x_1, x_2) \). Suppose that \( V(I') = \{0\} \). Hence \( \{t\} \times \tilde{D}^2(\delta) \cap V(I') \) is empty for \( 0 \neq t \) close to zero. By Proposition \[2.2\] one gets
Lemma 6.3. Suppose that $0 < \delta \ll 1$ and $0 \neq t$ is sufficiently close to zero. Then the set of critical points of $f_t : \tilde{D}^2_t(\delta) \rightarrow D^2(\delta)$ consists of fold points, and a finite family $\Sigma_t$ of cusp points. □

Remark 6.4. By [9, Theorem 3.1], if $0 \neq t$ is sufficiently close to zero then

$$\# \Sigma_t \leq \dim R Q \text{ and } \# \Sigma_t = \dim R Q \mod 2.$$  

For $t \neq 0$ we shall write $\Sigma_t^\pm = \{x \in \Sigma_t \mid \mu_t(x) = \pm 1\}$, where $\mu_t(x)$ is the local topological degree of $f_t$ at $x$. Put $\text{cusp deg}(f_t) = \sum_{x \in \Sigma_t} \mu_t(x) = \# \Sigma_t^+ - \# \Sigma_t^-$. By Lemmas 5.3, 6.2, 6.3 and Theorem 2.1 we get

Proposition 6.5. Suppose that $0 < \delta \ll 1$, and $0 \neq t$ is sufficiently close to zero. Then

(i) the pair $(\tilde{D}^2_t(\delta), \tilde{S}^1_t(\delta))$ is diffeomorphic to $(D^2(1), S^1(1))$, and $f_t : \tilde{D}^2_t(\delta) \rightarrow D^2(\delta)$ is such a mapping that $f_t^{-1}(S^1(\delta)) = \tilde{S}^1_t(\delta)$.

(ii) every point in $\tilde{D}^2_t(\delta)$ is either a fold point, a cusp point or a regular point, and there is a finite family of cusps which all belong to $\tilde{D}^2_t(\delta) \setminus \tilde{S}^1_t(\delta)$.

(iii) $f_t|_{\tilde{S}^1_t(\delta)} : \tilde{S}^1_t(\delta) \rightarrow S^1(\delta)$ is locally stable, and the set of critical points of $f_t$, i.e. $J_t^{-1}(0)$, is transverse to $S^1_t(\delta)$.

(iv) $\text{cusp deg}(f_t) = 2\chi(\tilde{M}_t^-) + \text{deg}(f_t|_{\tilde{S}^1_t(\delta)}) - 1 - \#C(f_t|_{\tilde{S}^1_t(\delta)})/2$

$$= 2\chi(\tilde{M}_t^-) + \text{deg}_0(f_0) - \#C(f_0|_{\tilde{S}^1_0(\delta)})/2 - 1,$$

where $\tilde{M}_t^- = \{x \in \tilde{D}^2_t(\delta) \mid J_t(x) \leq 0\}$. □

Let $d_1, d_2 : \mathbb{R} \times \mathbb{R}^2, 0 \rightarrow \mathbb{R} \times \mathbb{R}^2, 0$ be defined as in Section 3.

Theorem 6.6. Let $f = (f_1, f_2) : \mathbb{R} \times \mathbb{R}^2, 0 \rightarrow \mathbb{R}^2, 0$ be an analytic function defined in a neighbourhood of the origin such that [11] holds. Suppose that the origin is isolated in $V(I')$, $d_1^{-1}(0)$ and $d_2^{-1}(0)$.

Then there exists $r > 0$ such that the set of critical points of $f_t : D^2(r) \rightarrow \mathbb{R}^2$, where $0 \neq t$ is sufficiently close to zero, consists of fold points, and a finite family $\Sigma_t$ of cusp points. Moreover, the origin is isolated in $f_0^{-1}(0)$ and

$$\text{cusp deg}(f_t) = \text{deg}_0(f_0) - \text{deg}_0(d_1) - \text{sign}(t) \cdot \text{deg}_0(d_2).$$
Proof. For any small $\delta > 0$ there is $r > 0$ such that $D^2(r) \subset \tilde{D}_0^2(\delta) \setminus \tilde{S}_0^1(\delta)$, so that also $D^2(r) \subset \tilde{D}_0^2(\delta) \setminus \tilde{S}_1^1(\delta)$ if $|t|$ is small.

By Lemma 6.3 the set of critical points of $f|\tilde{D}_i^2(\delta)$ consists of fold points, and a finite family $\Sigma_i$ of cusp points. Because $\Sigma_0 = \{0\}$ then $\Sigma_i$ is the set of cusp points of $f|\tilde{D}_i^2(\delta)$.

By (4), the germ $d_0 = \nabla J_0 : \mathbb{R}^2, 0 \rightarrow \mathbb{R}^2, 0$ has an isolated zero at the origin. By Theorem 3.2 and Lemma 6.2,

$$\#(f|\tilde{S}_1^1(\delta)) = \#(\tilde{S}_0^1(\delta) \cap \{J_0 = 0\}) = 2 \cdot (1 - \deg_0(d_0)),$$

for $0 \neq t$ sufficiently close to zero. Our assertion is then a consequence of Proposition 6.5 and Theorem 3.2 $\square$

Put $J' = J(t^2, x_1, x_2)$, $F'_i = F_i(t^2, x_1, x_2)$.

**Lemma 6.7.** Suppose that $V(I') = \{0\}$. Then $\dim V(J, F_1, F_2) \leq 1$ and $\dim V(J', F'_1, F'_2) \leq 1$.

Moreover, if $\dim_\mathbb{R} \mathcal{O}_3/I' < \infty$ then $V(J', F'_1, F'_2)$, as well as $V(J, F_1, F_2)$, is a curve having an algebraically isolated singularity.

**Proof.** We have

$$\{0\} = V(I') = V(J, F_1, F_2) \cap V\left(\frac{\partial(F_1, J)}{\partial(x_1, x_2)}, \frac{\partial(F_2, J)}{\partial(x_1, x_2)}\right),$$

so by the implicit function theorem $\dim V(J, F_1, F_2) \leq 1$. Of course, $(t, x_1, x_2) \in V(J', F'_1, F'_2)$ if and only if $(t^2, x_1, x_2) \in V(J, F_1, F_2)$. Hence $\dim V(J', F'_1, F'_2) \leq 1$ too.

The ideal

$$K = \left\langle J', F'_1, F'_2, \frac{\partial(F'_1, J')}{\partial(x_1, x_2)}, \frac{\partial(F'_2, J')}{\partial(x_1, x_2)} \right\rangle \subseteq \mathcal{O}_3$$

is contained in the ideal $L$ generated by $J', F'_1, F'_2$ and all $2 \times 2$-minors of the derivative matrix of $(J', F'_1, F'_2)$.

As $\dim_\mathbb{R} \mathcal{O}_3/I' < \infty$, by the local Nullstellensatz, the origin is isolated in the set of complex zeros of $I'$. Since

$$\frac{\partial(F'_1, J')}{\partial(x_1, x_2)}(t, x_1, x_2) = \frac{\partial(F_1, J)}{\partial(x_1, x_2)}(t^2, x_1, x_2),$$

the origin is isolated in the set of complex zeros of $K$. Hence $\dim_\mathbb{R} \mathcal{O}_3/L \leq \dim_\mathbb{R} \mathcal{O}_3/K < \infty$, and then $V(J', F'_1, F'_2)$ is a curve having an algebraically
isolated singularity at the origin. The proof of the last assertion is similar. □

Suppose that the origin is isolated in \( V(I') \). Let \( b_0 \) (resp. \( b_0' \)) be the number of half branches in \( V(J, F_1, F_2) \) (resp. \( V(J', F_1', F_2') \)) emanating from the origin.

By Lemma 6.1 no half-branch is contained in \( \{0\} \times \mathbb{R}^2 \). Then by the curve selection lemma the family of half-branches is a finite union of graphs of continuous functions \( t \mapsto x_i(t) \in \mathbb{R}^2 \), where \( t \) belongs either to \((-\epsilon, 0]\) or to \([0, \epsilon)\), \( 0 < \epsilon \ll 1 \), \( x_i(0) = 0 \), \( 1 \leq i \leq b_0 \) (resp. \( 1 \leq i \leq b_0' \)), and those graphs meet only at the origin.

Hence, if \( 0 < t \ll 1 \) then

\[
b_0 = \#\Sigma_t + \#\Sigma_{-t} = \#\Sigma_t^+ + \#\Sigma_{-t}^-,
\]

\[
b_0'/2 = \#\Sigma_t = \#\Sigma_t^+ + \#\Sigma_{-t}^-.
\]

By Theorem 6.6 we have

\[
\deg_0(f_0) - \deg_0(d_1) - \deg_0(d_2) = \#\Sigma_t^+ - \#\Sigma_{-t}^-,
\]

\[
\deg_0(f_0) - \deg_0(d_1) + \deg_0(d_2) = \#\Sigma_{-t}^+ - \#\Sigma_{-t}^-.
\]

Then we have

**Theorem 6.8.** Suppose that assumptions of Theorem 6.6 hold. Then numbers \( \#\Sigma_{\pm t}^\pm \), where \( t > 0 \) is small, are determined by \( b_0, b_0', \deg_0(f_0), \deg_0(d_1), \deg_0(d_2) \).

Moreover, if \( \dim \mathcal{O}_3/I' < \infty \) then \( V(J, F_1, F_2) \) and \( V(J', F_1', F_2') \) are curves having an algebraically isolated singularity at the origin. In that case one may apply Theorem 4.3 so as to compute \( b_0 \) and \( b_0' \). In particular, if \( \dim \mathbb{R}\mathcal{O}_3/I'' < \infty \), where

\[
I'' = \left\langle F_1, F_2, \frac{\partial(F_1, F_1)}{\partial(t, x_1)}, \frac{\partial(F_1, F_1)}{\partial(t, x_2)}, \frac{\partial(F_1, F_1)}{\partial(x_1, x_2)} \right\rangle,
\]

then \( V(F_1, F_2) \) is a curve having an algebraically isolated singularity at the origin. In that case one may take \( g_1 = F_1, g_2 = F_2, g_3 = J \).

### 7 Examples

Examples presented in this section were calculated with the help of SINGULAR [7] and the computer program written by Andrzej Łęcki [16].
Example 1. Let \( f = (f_1, f_2) = (x_1^3 + x_2^2 + t x_1, x_1 x_2) \). Since \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle t, f_1, f_2 \rangle = 5 \), \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle t, F_1, F_2 \rangle = 7 \), \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle t, \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2} \rangle = 2 \), then (1) holds. Moreover, \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle t, \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2} \rangle = 8 \), \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle t, \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2} \rangle = 1 \), and \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle J, \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2} \rangle = 3 \). Then the origin is isolated in \( V(I') \), \( d_1^{-1}(0) \) and \( d_2^{-1}(0) \). Using the computer program by Leckl one may compute \( \deg_0(f_0) = -1 \), \( \deg_0(d_1) = +1 \) and \( \deg_0(d_2) = -1 \). By Theorem 6.6 cusp \( \deg(f_t) = \text{sign}(t) - 2 \) for \( 0 \neq t \) sufficiently close to zero.

By Lemma 6.7 the set \( V(J, F_1, F_2) \), as well as \( V(J', F_1', F_2') \), is a curve having an algebraically isolated singularity. Hence we may apply techniques presented in Section 4 so as to compute the number of half-branches of those curves.

One may verify that \( \dim_{\mathbb{R}} \mathcal{O}_3 / I'' = 8 \), so that \( V(F_1, F_2) \) is a curve with an algebraically isolated singularity at the origin.

Put \( J_p = \langle F_1, F_2, J_p \rangle \), where \( p = 1, 2 \). In that case \( \xi = 2 \), and so \( k = 4 \).

As \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle t, F_1, F_2 \rangle < \infty \), then (1) holds. Set

\[
H_+ = \left( \frac{\partial (J + t^4, F_1, F_2)}{\partial (t, x_1, x_2)}, F_1, F_2 \right) : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0.
\]

One may compute \( \deg_0(H_+) = +2 \), \( \deg_0(H_-) = -2 \). By Theorem 4.3, \( V(J, F_1, F_2) \) is an union of four half-branches emanating from the origin, i.e. \( b_0 = 4 \).

Now we shall apply the same techniques so as to compute the number of half-branches of \( V(J', F_1', F_2') \). By Proposition 4.4 \( V(F_1', F_2') \) is a curve with an algebraically isolated singularity at the origin. Put \( J_p' = \langle F_1', F_2', (J')^p \rangle \), where \( p = 1, 2 \). By Remark 4.5 \( \xi' \leq 4 \) and so one may take \( k = 6 \). Let

\[
H'_+ = \left( \frac{\partial (J' + t^6, F_1', F_2')}{\partial (t, x_1, x_2)}, F_1', F_2' \right) : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0.
\]

One may compute \( \deg_0(H'_+) = +1 \), \( \deg_0(H'_-) = -1 \). Then \( V(J', F_1', F_2') \) is an union of two half-branches emanating from the origin, i.e. \( b'_0/2 = 1 \). Hence, if \( 0 < t \ll 1 \) then \( \#\Sigma^+_t = 0 \), \( \#\Sigma^-_t = 1 \), \( \#\Sigma^+_{-t} = 0 \) and \( \#\Sigma^-_{-t} = 3 \).

Example 2. Let \( f = (f_1, f_2) = (x_1^4 + x_1^2 + x_1^2 x_2^2 + t x_1, x_1 x_2 + t x_2) \). In that case \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle t, f_1, f_2 \rangle = 8 \), \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle t, F_1, F_2 \rangle = 24 \), \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle t, \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2} \rangle = 9 \), \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle t, \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2} \rangle = 33 \), \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle J, \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2} \rangle = 3 \), and \( \dim_{\mathbb{R}} \mathcal{O}_3 / \langle J, \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2} \rangle = 12 \). Then the origin is isolated in \( V(I') \), \( d_1^{-1}(0) \) and \( d_2^{-1}(0) \). One may compute \( \deg_0(f_0) = 0 \), \( \deg_0(d_1) = +1 \) and \( \deg_0(d_2) = 0 \). By Theorem 6.6 cusp \( \deg(f_t) = -1 \) for \( 0 \neq t \) sufficiently close to zero, i.e. \( \#\Sigma^+_t - \#\Sigma^-_t = -1 \).
As \(|\text{dim}_\mathbb{R} \mathcal{O}_\delta/I''| = 45\) then \(V(F_1, F_2)\) is a curve having an isolated singularity at the origin. Let \(J_p\) be defined the same way as in the previous example. One may verify that \(\xi = 2\), and so \(k = 4\). Put

\[ H_\pm = \left( \frac{\partial (J \pm t^4, F_1, F_2)}{\partial (t, x_1, x_2)}, F_1, F_2 \right) : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0. \]

One may compute \(\deg_0(H_+) = 0\), \(\deg_0(H_-) = -2\). Then \(V(J, F_1, F_2)\) is an union of two half-branches emanating from the origin, i.e. \(b_0 = 2\).

Because \(f_t(x_1, x_2) = f_{-t}(-x_1, -x_2)\), then \(b_0'/2 = 1\) and \(#\Sigma^+_t = \#\Sigma^-_t\), \(#\Sigma^+_t = \#\Sigma^-_t\). So in this case there is no need to compute \(\deg_0(H'_\pm)\). Hence, if \(t > 0\) then \(#\Sigma^+_t = \#\Sigma^-_t = 0\) and \(#\Sigma^+_t = \#\Sigma^-_t = 1\).

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17