ABSTRACT

Bosonic end perturbative calculations for quantum mechanical anyon systems require a regularization. I regularize by adding a specific δ-function potential to the Hamiltonian. The reliability of this regularization procedure is verified by comparing its results for the 2-anyon in harmonic potential system with the known exact solutions. I then use the δ-function regularized bosonic end perturbation theory to test some recent conjectures concerning the unknown portion of the many-anyon spectra.

Submitted to: Physics Letters B
In 2+1 dimensions the triviality of the rotation group SO(2) allows the existence of anyons: particles with “anomalous” spin and statistics\cite{1}. It has been conjectured\cite{2} that anyons might play an important role in the understanding of some planar condensed matter phenomena, most notably the fractional quantum Hall effect, and this has motivated numerous recent investigations of quantum mechanical anyon systems and of field theory realizations of anomalous quantum statistics.

One would like to develop a description of anyons as complete and intuitive as the ones available for bosons and fermions, but thus far this program has had only partial success. In particular, the anyon quantum mechanics is not completely understood. Even for the simple cases of $N$ identical anyons in an external magnetic field and/or harmonic potential the complete set of eigensolutions is known\cite{1} only for $N = 2$. For systems of more than 2 anyons just a particular class of eigensolutions has been found\cite{3,4}. In order to obtain some information concerning the “missing eigensolutions” various approximation techniques have been used: “bosonic end perturbation theory”\cite{4−9}, “fermionic end perturbation theory”\cite{4,10}, and numerical methods\cite{11,12}.

The results presented in this letter are in the framework of the bosonic end perturbation theory. This technique concerns the study of anyons with small statistical parameter $\nu$ (here defined following the convention of Refs.[7,9]), by using a perturbative expansion in $\nu$. The unperturbed wave functions are bosonic\cite{*}; in fact in the limit $\nu \to 0$ anyons acquire bosonic statistics\cite{1}. Due to the known\cite{1−7} non-analyticity of the limit $\nu \to 0$, in applying this perturbative approach some divergencies appear, and therefore regular-

\* In fermionic end perturbation theory $\nu \simeq 1$ and the unperturbed wave functions are fermionic.
ization procedures are necessary\textsuperscript{[4−9]}. For example, the magnetic gauge\textsuperscript{[1,7]} Hamiltonian which describes the relative motion\textsuperscript{*} of 2 anyons in a common harmonic well is given by

\[
H_2 = -\frac{1}{r} \partial_r (r \partial_r) - \frac{1}{r^2} \partial^2_\phi + r^2 - \frac{2i\nu}{r^2} \partial_\phi + \frac{\nu^2}{r^2} ,
\] (1)

and logarithmic divergencies in the bosonic end perturbative analysis originate from some of the matrix elements of the $\nu^2/r^2$ term, like

\[
< \Omega_0 \left| \frac{\nu^2}{r^2} \right| \Omega_0 > = 2\nu^2 \int_0^\infty dr \frac{\exp(-r^2)}{r} \sim \infty ,
\] (2)

where $|\Omega_0 > \equiv (e^{-r^2/2})/\pi^{1/2}$ is the unperturbed (i.e. bosonic) ground state.

Many of the results presently available on the unknown portion of the anyon spectra, have been obtained using bosonic end perturbation theory in the analysis of few anyon systems. However, as already emphasized in Ref.[13], the regularization procedures used in these calculations require rather arbitrary manipulations.

In this letter I calculate for arbitrary $N$ and to second order in $\nu$ some of the eigenenergies of the $N$-anyon in harmonic potential system. This analysis extends some of the results obtained in the literature on the anyon spectra, and could also be useful in the search of new exact solutions to the $N$-anyon problem by leading to some “educated ansaetze”. Moreover, in my calculations I use a new regularization procedure, based on the introduction of a repulsive $\delta$-function potential, which has a simpler physical interpretation than the ones used in the literature.

\textsuperscript{*} The center of mass motion is simply a free motion and it is essentially irrelevant for the discussion in this letter.
Let me start by observing that, because the anyonic wave functions vanish* at the points of overlap, the addition of a repulsive \( \delta \)-function potential to the Hamiltonian \( H_N \) of a quantum mechanical \( N \)-anyon system has no physical consequences (i.e. the exact eigensolutions are unaffected by it), but it can be used to implement in the bosonic end perturbation theory the hard core boundary condition\(^{[15]}\). I can therefore apply small-\( \nu \) perturbation theory, rather than to the original Hamiltonian \( H_N \), to the equivalent Hamiltonian \( H_N^\delta \), given by**

\[
H_N^\delta \equiv H_N + 2\pi |\nu| \sum_{m<n} \delta^{(2)}(r_{mn}) ,
\]

where \( r_{mn} \) is the relative position of the \( m \)-th and \( n \)-th anyon in the system.

In the following we will see that the \( \delta \)-function potential added in (3) eliminates the divergencies of bosonic end perturbation theory. This is not surprising in light of the results obtained in some recent investigations of field theory realizations of fractional quantum statistics in which the addition of a quartic contact term, the field theory analog of a quantum mechanical \( \delta \)-function potential, has been shown to eliminate some divergencies\(^{[13,17,18]}\).

As a first test of the reliability of this “\( \delta \)-function regularization”, it is convenient to discuss its application to the 2-anyon in harmonic potential problem; the regularized

---

* The possibility of anyons with wave functions which do not vanish at the points of overlap has also been considered in the literature (for example this subject is discussed in Ref.\(^{[14]}\)), but in this letter only the conventional “non-colliding” anyons are considered.
** Note that in Ref.\(^{[16]}\) an Hamiltonian of the type \( H_N^\delta \) was already considered, as a result of an analysis of an induced anyon magnetic moment.
Hamiltonian is

\[
H_2^\delta \equiv -\frac{1}{r} \partial_r (r \partial_r) - \frac{1}{r^2} \partial^2_\phi + r^2 - \frac{2i\nu}{r^2} \partial_\phi + 2\pi |\nu| \delta^{(2)}(r) + \frac{\nu^2}{r^2} = H_2 + 2\pi |\nu| \delta^{(2)}(r) .
\]  

(4)

Although the \(H_2^\delta\)-eigenproblem is equivalent to the \(H_2\)-eigenproblem, \(H_2^\delta\) is more suitable for perturbation theory; in fact, the added \(\delta\)-function potential leads to divergencies which exactly cancel those introduced by the \(\nu^2/r^2\) term, rendering finite the results of bosonic end perturbation theory. Let me illustrate this mechanism by verifying that the first and second order eigenenergies and the first order eigenfunctions obtained with the \(\delta\)-function regularized bosonic end perturbation theory are finite and in agreement with the known eigensolutions of the 2-anyon in harmonic potential problem, which are given by

\[
E_{n,l,\nu}^{exact} = (4n + 2|l + \nu| + 2) ,
\]

(5a)

\[
|\Psi_{n,l,\nu}^{exact} > = N_{n,l}^\nu r^{l+\nu} e^{-\frac{r^2}{2} + il\phi} L_{n}^{l+\nu}(r^2) ,
\]

(5b)

where the \(L_n^x\) are Laguerre polynomials, and the \(N_{n,l}^\nu\) are normalization constants.

My analysis is limited to the states with \(l = 0\). For the states with \(l \neq 0\) no divergence is present to begin with\(^4-7\), and the consistency of the \(\delta\)-function regularization can be verified in complete analogy with the corresponding results obtained for the other regularization procedures. [N.B. The \(\delta\)-function potential does not contribute to the matrix elements involving unperturbed states with \(l \neq 0\), because these states vanish for \(r = 0\).]

Concerning the first order energies, one easily finds

\[
E_{n,0,\nu}^{(1)} = <\Psi_{n,0}^{(0)}| - \frac{2i\nu}{r^2} \partial_\phi + 2\pi |\nu| \delta^{(2)}(r)|\Psi_{n,0}^{(0)} >= 2|\nu| ,
\]

(6)

where the unperturbed eigenfunctions are

\[
|\Psi_{n,l}^{(0)} > \equiv \left( \frac{n!}{\pi (n + l)!} \right)^{1/2} r^{l+\nu} e^{-\frac{r^2}{2} + il\phi} L_{n}^{l+\nu}(r^2) = |\Psi_{n,l,0}^{exact} >
\]

(7)
The result (6) is clearly in agreement with Eq.(5a).

The first order eigenfunctions are given by

$$|\Psi_{n,0,\nu}^{(1)}> = \sum_{m,l,\not= n,0} \frac{<\Psi_{m,l}^{(0)}|\Psi_{n,0}^{(0)}>}{E_{n,0}^{(0)} - E_{m,l}^{(0)}} |\Psi_{m,l}^{(0)}>$$

$$= - \frac{\nu}{2\sqrt{\pi}} \sum_{m\not= n} \frac{L_m^0(r^2)}{m-n} e^{-\frac{r^2}{2}}.$$  (8)

Using properties of the Laguerre polynomials one can verify (with some algebra) that the result (8) is in agreement with Eq.(5b). For example one obtains ($\gamma$ is the Euler constant)

$$|\Psi_{2,0,\nu}^{(1)}> = \frac{\nu}{\sqrt{\pi}} e^{-\frac{r^2}{2}} \left[ \frac{3}{2} - r^2 + \frac{1}{4}(2\gamma - 3 + 4ln(r))L_2^0(r^2) \right],$$  (9)

which is in perfect agreement with the first order term in the expansion in $\nu$ of $|\Psi_{2,0,\nu}^{exact}>$.

From (4) one sees that the second order energies are given by:

$$E_{n,0,\nu}^{(2)} = <\Psi_{n,0,\nu}^{(0)}|\nu^2 r^2 |\Psi_{n,0}^{(0)}> + <\Psi_{n,0,\nu}^{(0)}|\frac{2i\nu}{r} \partial_\phi + 2\pi |\nu|\delta^{(2)}(r)|\Psi_{n,0,\nu}^{(1)}> \equiv E_{n,0,\nu}^{(2,a)} + E_{n,0,\nu}^{(2,b)}.$$  (10)

Both $E_{n,0,\nu}^{(2,a)}$ and $E_{n,0,\nu}^{(2,b)}$ are divergent, but the final result is finite. In order to illustrate the details of the cancellation of the infinities, it is useful to follow a definite calculation; for example for $E_{2,0,\nu}^{(2)}$ the contributions are:

$$E_{2,0,\nu}^{(2,a)} = <\Psi_{2,0,\nu}^{(0)}|\frac{\nu^2}{r^2} |\Psi_{2,0}^{(0)}> = \nu^2 \int_0^\infty \int_0^{2\pi} dr d\phi \frac{\exp(-r^2)}{\pi r} [L_2^0(r^2)]^2$$

$$= \nu^2 \lim_{\epsilon \to 0} \int_\epsilon^\infty dr \nu^2 \frac{\exp(-r^2)}{r} [L_2^0(r^2)]^2 = \nu^2 \lim_{\epsilon \to 0} \left[ -2\ln(\epsilon) - \gamma - \frac{3}{2} \right],$$  (11a)

$$E_{2,0,\nu}^{(2,b)} = <\Psi_{2,0,\nu}^{(0)}| - \frac{2i\nu}{r^2} \partial_\phi + 2\pi |\nu|\delta^{(2)}(r)|\Psi_{2,0,\nu}^{(1)}> = <\Psi_{2,0,\nu}^{(0)}| 2\pi |\nu|\delta^{(2)}(r)|\Psi_{2,0,\nu}^{(1)}>$$

$$= 2\nu^2 \int_{-\infty}^\infty \int_{-\infty}^\infty dr_x dr_y \delta^{(2)}(r) e^{-r^2} L_2^0(r^2) \left[ \frac{3}{2} - r^2 + \frac{1}{4}(2\gamma - 3 + 4ln(r))L_2^0(r^2) \right]$$

$$= \nu^2 \lim_{\epsilon \to 0} \left[ 2\ln(\epsilon) + \frac{3}{2} + \gamma \right].$$  (11b)
Note that, in order to see the cancellation of infinities and evaluate the left-over finite result, I introduced a cut-off $\epsilon$, which will be ultimately removed by taking the limit $\epsilon \to 0$. In general a similar cut-off must be introduced in all the divergent matrix elements of $r^{-2}$ and $\delta^{(2)}(r)$ by using

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dr_x \, dr_y \, \frac{1}{r^2} \, f(r_x, r_y) = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \int_{0}^{2\pi} r \, dr \, d\phi \, \frac{1}{r^2} \, f(r \cos \phi, r \sin \phi) \ , \quad (12a)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dr_x \, dr_y \, \delta^{(2)}(r) \, f(r_x, r_y) = \lim_{\epsilon \to 0} f\left(\frac{\epsilon}{\sqrt{2}}, \frac{\epsilon}{\sqrt{2}}\right) . \quad (12b)$$

From Eqs.(10) and (11) one concludes that $E^{(2)}_{2,0,\nu} = 0$, and this is in agreement with Eq.(5a), which indicates that the 2-anyon in harmonic potential spectra is linear in $\nu$.

This completes my test of the reliability of the $\delta$-function regularization. I am now ready to calculate perturbatively from the bosonic end to second order in the statistical parameter $\nu$ and for arbitrary $N$ the eigenenergies of some $N$-anyon in harmonic potential states.

The $\delta$-function regularized Hamiltonian which describes the relative motion of $N$ identical anyons in an harmonic potential is given by

$$H_N^\delta = H^{(0)} + H_L^{(1)} + H_\delta^{(1)} + H^{(2)} \quad (13)$$

where $H_0$ is the relative motion Hamiltonian for $N$ bosons in an harmonic potential, and

$$H_L^{(1)} \equiv \frac{\nu}{2} \sum_{m \neq n} \frac{1}{|z_n - z_m|^2} L_{n,m} \ , \quad (14a)$$

$$H_\delta^{(1)} \equiv 2\pi|\nu| \sum_{m < n} \delta^{(2)}(z_m - z_n) \ , \quad (14b)$$

$$H_2 = \frac{\nu^2}{4} \sum_{m \neq n, n \neq k} \left( \frac{1}{(z_n - z_m)(z_n^* - z_k^*)} + h.c. \right) . \quad (14c)$$
The operators $L_{n,m}$ are given by

$$L_{n,m} \equiv (z_n - z_m) \left( \frac{\partial}{\partial z_n} - \frac{\partial}{\partial z_m} \right) - (z_n^* - z_m^*) \left( \frac{\partial}{\partial z_n^*} - \frac{\partial}{\partial z_m^*} \right), \quad (15)$$

and I am using the conventional notation $z_n \equiv x_n + iy_n$, $z_n^* \equiv x_n - iy_n$.

The three $N$-anyon states whose energies I shall evaluate perturbatively are the ones which correspond, in the limit $\nu \to 0$, to the following bosonic states

$$|\Omega\rangle \equiv \frac{1}{\sqrt{\pi^{N-1}}} \exp \left[ \sum_{n=1}^{N-1} |u_n(\{z_i\})|^2 \right], \quad (16a)$$

$$|+\rangle \equiv \frac{1}{\sqrt{2(N-1)\pi^{N-1}}} \left( \sum_{n=1}^{N-1} u_n^2(\{z_i\}) \right) \exp \left[ \sum_{n=1}^{N-1} |u_n(\{z_i\})|^2 \right], \quad (16b)$$

$$|-\rangle \equiv |+\rangle^*, \quad (16c)$$

where $u_n(\{z_i\}) \equiv (z_1 + z_2 + ... + z_n - n\ z_{n+1})/\sqrt{n(n+1)}$.

$|\Omega\rangle$ is the $N$-boson groundstate (and the corresponding perturbative results give the approximate $N$-anyon groundstate energy for small $\nu$), and the states $|\pm\rangle$ are in the first excited bosonic energy level and have angular momentum $\pm 2$.

It is easy to verify that $E_{\Omega}(1,2)(\nu) = E_{\Omega}(1,2)(-\nu)$, $E_{+}(1)(\nu) = -E_{-}(1)(-\nu)$, and $E_{+}(2)(\nu) = E_{-}(2)(-\nu)$; therefore, I can limit the calculations to the case $\nu > 0$ without any loss of generality.

The first order energies can be easily calculated, they are given by

$$E^{(1)}_{\Omega} = <\Omega|H^{(1)}_L + H^{(1)}_\delta|\Omega> = \frac{N(N-1)}{2} \nu, \quad (17a)$$

$$E^{(1)}_{\pm} = <\pm|H^{(1)}_L + H^{(1)}_\delta|\pm> = \frac{N(N-2)}{2} \nu \pm \frac{N}{2} \nu. \quad (17b)$$

Concerning the evaluation of the second order energies, let us start by noticing that from (13) and (14) it follows that
\[ E_{\Psi(0)}^{(2)} = \langle \Psi(0) | H^{(2)} | \Psi(0) \rangle + E_{\Psi(0), L}^{(2)} + E_{\Psi(0), \delta}^{(2)}, \quad (18a) \]

\[ E_{\Psi(0), L}^{(2)} = \sum_{m > \not\in D} \frac{\langle \Psi(0) | H^{(1)}_L + H^{(1)}_\delta | m \rangle \langle m | H^{(1)}_L | \Psi(0) \rangle}{E^{(0)} - E^{(0)}_m}, \quad (18b) \]

\[ E_{\Psi(0), \delta}^{(2)} = \sum_{m > \not\in D} \frac{\langle \Psi(0) | H^{(1)}_L + H^{(1)}_\delta | m \rangle \langle m | H^{(1)}_\delta | \Psi(0) \rangle}{E^{(0)} - E^{(0)}_m}, \quad (18c) \]

where \( D \) is the space of bosonic states with energy \( E^{(0)} \) (\( \dim D \) is the “degeneracy” of \( |\Psi(0)\rangle \)). Using the symmetries of \( H^{(2)} \) and of the unperturbed wave functions (16), one easily obtains

\[ < \Omega | H^{(2)} | \Omega > = \lim_{\epsilon \to 0} \left[ \frac{\nu^2}{4} N(N - 1) \left( 2(N - 2) \ln \left( \frac{4}{3} \right) - \gamma - \ln(\epsilon) \right) \right], \quad (19a) \]

\[ < \pm | H^{(2)} | \pm > = \lim_{\epsilon \to 0} \left[ \frac{\nu^2}{8} N \left( 9 - 4N + 4(N + 1)(N - 2) \ln \left( \frac{4}{3} \right) - 2(N - 2)(\gamma + \ln(\epsilon)) \right) \right], \quad (19b) \]

where I introduced the cut-off \( \epsilon \) using (12).

For the states (16) the evaluation of \( E_{\Psi(0), L}^{(2)} \) and \( E_{\Psi(0), \delta}^{(2)} \) (which is usually possible only numerically) is relevantly simplified by the following results

\[ \frac{1}{|z_1 - z_2|^2} L_{12} |\pm > = \frac{1}{4} \left( \left[ C_{12}, H^{(0)} \right] - \pi \delta^{(2)} \left( \frac{z_1 - z_2}{\sqrt{2}} \right) + 1 \right) L_{12} |\pm >, \]

\[ < m | \pi \delta^{(2)} \left( \frac{z_1 - z_2}{\sqrt{2}} \right) |\Omega(\pm) > = < m | \left[ C_{12}, H^{(0)} \right] |\Omega(\pm) >, \quad (20) \]

\[ H^{(0)} L_{12} |\pm > = E_{\pm}^{(0)} L_{12} |\pm >; \quad L_{12} |\Omega > = 0, \]

where \( C_{mn} \equiv \ln \left( \frac{|z_m - z_n|^2}{2} + \gamma - 1 \right) / 2 \). Using the properties (20) (and introducing again the cut-off \( \epsilon \) using (12)) one finds

\[ E_{\Omega, L}^{(2)} = 0, \quad (21a) \]

\[ E_{\Omega, \delta}^{(2)} = \lim_{\epsilon \to 0} \left[ \frac{\nu^2}{4} N(N - 1) \left( \ln(\epsilon) + \gamma - 2(N - 2) \ln \left( \frac{4}{3} \right) \right) \right], \quad (21b) \]
\[ E_{\pm, L}^{(2)} = \pm \frac{3}{8} \nu^2 N(N - 2) \ln\left(\frac{3}{4}\right) + \frac{\nu^2}{16} N \left(5N - 12 - 18(N - 2) \ln\left(\frac{4}{3}\right)\right), \]  \hspace{1cm} (21c)

\[ E_{\pm, \delta}^{(2)} = \pm \frac{3}{8} \nu^2 N(N - 2) \ln\left(\frac{3}{4}\right) + \frac{\nu^2}{16} N \lim_{\epsilon \to 0} \left[3N - 6 + 4(N - 2)(\gamma + \ln(\epsilon)) + (38N - 44 - 8N^2) \ln\left(\frac{4}{3}\right)\right]. \]  \hspace{1cm} (21d)

From (17), (18), (19), and (21) one obtains the following final results:

\[ E_{\Omega}^{(1)} + E_{\Omega}^{(2)} = \frac{N(N - 1)}{2} \nu, \]  \hspace{1cm} (22a)

\[ E_{+}^{(1)} + E_{+}^{(2)} = \frac{N(N - 1)}{2} \nu, \]  \hspace{1cm} (22b)

\[ E_{-}^{(1)} + E_{-}^{(2)} = \frac{N(N - 3)}{2} \nu + \ln\left(\frac{4}{3}\right) \frac{3N(N - 2)}{2} \nu^2. \]  \hspace{1cm} (22c)

\( E_{\Omega} \) and \( E_{+} \) are among the exactly known eigenenergies of the \( N \)-anyon in harmonic potential problem (see Ref.[4]), and they are in perfect agreement with (22a) and (22b). \( E_{-} \) is not known exactly, but, for the special cases \( N = 3 \) and \( N = 4 \), \( E_{-}^{(1)} + E_{-}^{(2)} \) has been calculated in Refs.[8,9], using the regularization method I, and in Refs.[11,12], using numerical methods; Eq.(22c), for the corresponding values of \( N \), is in agreement with those results.

In conclusion, the \( \delta \)-function regularized bosonic end perturbation theory appears to be a reliable tool for the investigation of anyon quantum mechanics. I have shown explicitly for various anyon states that its results are in agreement with the exact analytic eigensolutions, when such eigensolutions are known, and with the numerical analysis of the few anyon spectra presented in Refs.[11,12].

This bosonic end perturbative approach may be preferable to the ones previously introduced in the literature because of the more physically intuitive regularization procedure.
Moreover, in some instances properties of the matrix elements of the \( \delta \)-function regularized Hamiltonian can be used to simplify the calculations.

Indeed, by exploiting properties of some matrix elements of the \( \delta \)-function regularized Hamiltonian for \( N \) anyons in an harmonic potential, I obtained the first direct evidence (Eq.(22c)) of the fact that there are many-anyon eigenenergies with nonlinear dependence on the statistical parameter \( \nu \). [N.B. All the presently exactly known anyon eigenenergies\[1,3,4\] depend linearly on \( \nu \), and the existence of eigenenergies with nonlinear dependence on \( \nu \) had previously been shown only for some 3 and 4 anyon systems\[4,8-12\].]

The presence of a factor \( \ln(\frac{4}{3}) \) in Eq.(22c) supports the conjecture made in Refs.[8,9] that such factors should characterize a part of the general \( N \)-anyon spectra. A further investigation of the role and the physical meaning, if there is any, of this factor \( \ln(\frac{4}{3}) \) would be very interesting; in particular, as it has already been suggested\[19\] for other “noticeable factors” which have appeared in the study of anyons, they might be related to presently unidentified symmetries of the problem. Eq.(22c) is also consistent with the ansatz proposed in Refs.[8,9] for the unknown portion of the spectra, and this should encourage attempts to use that ansatz in the search for new exact eigensolutions.

I thank S.-Y. Pi for many useful discussions on this subject, R. Jackiw for suggesting that the results of Refs.[8,9] might be related to the ones of Ref.[13], and O. Bergman and G. Lozano for conversations concerning Ref.[13]. I also thank J. Negele for hospitality. I acknowledge support from the Istituto Nazionale di Fisica Nucleare, Frascati, Italy. Part of the research reported in this paper was done at the Department of Physics, Boston University.
Note Added

After this letter had been submitted for publication, the Center for Theoretical Physics received a preprint of Manuel and Tarrach\cite{20} in which, in a different calculation, an analogous $\delta$-function regularization is proposed. The combination of my results with the ones of Ref.\cite{20} suggests that this regularization procedure might be useful in the study of several problems.
References

1  J. M. Leinaas and J. Myrheim, Nuovo Cimento B 37 (1977) 1; G. A. Goldin, R.
    Menikoff, and D. H. Sharp, J. Math. Phys. 21 (1980) 650; 22 (1981) 1664; F. Wilczek,
    Phys. Rev. Lett. 48 (1982) 1144; 49 (1982) 957 . Also see: F. Wilczek, Fractional
    Statistics and Anyon Superconductivity, (World Scientific, 1990).

2  R. B. Laughlin, Phys. Rev. Lett. 50 (1983) 1395; B. I. Halperin, Phys. Rev. Lett.
    52 (1984) 1583.

3  Y. S. Wu, Phys. Rev. Lett. 53 (1984) 111; G. V. Dunne, A. Lerda, and C. A.
    Trugenberger, Mod. Phys. Lett. A 6 (1991) 2819; G. V. Dunne, A. Lerda, S. Sciuto,
    and C. A. Trugenberger, Nucl. Phys. B 370 (1992) 601; K. H. Cho and C. Rim, Ann.
    Phys. 213 (1992) 295; Chihong Chou, Phys. Lett. A 155 (1991) 245; S. Mashkevich,
    Int. J. Mod. Phys. A 7 (1992) 7931.

4  C. Chou, Phys. Rev. D 44 (1991) 2533, D 45 (1992) 1433(E).

5  A. Comtet, J. McCabe, and S. Ouvry, Phys. Lett. B 260 (1991) 372; J. McCabe and
    S. Ouvry, Phys. Lett. B 260 (1991) 113; A. Dasnieres de Veigy and S. Ouvry, Phys.
    Lett. B 291 (1992) 130; Nucl. Phys. B 388 (1992) 715.

6  D. Sen, Nucl. Phys. B 360 (1991) 397.

7  G. Amelino-Camelia, Phys. Lett. B 286 (1992) 97.

8  C. Chou, L. Hua, and G. Amelino-Camelia, Phys. Lett. 286 (1992) 329.

9  G. Amelino-Camelia, Phys. Lett. 299 (1992) 83.

10 M. Sporre, J. J. M. Verbaarschot, and I. Zahed, Nucl. Phys. B 389 (1993) 645.

11 M. Sporre, J. J. M. Verbaarschot, and I. Zahed, Phys. Rev. Lett. 67 (1991) 1813;
12 M. V. N. Murthy, J. Law, M. Brack, and R. K. Bhaduri, Phys. Rev. Lett. 67 (1991) 1817.

13 O. Bergman and G. Lozano, MIT preprint MIT-CTP-2182, January 1993.

14 M. Bourdeau and R.D. Sorkin, Phys. Rev. D45 (1992) 687.

15 More on the physical interpretation of the $\delta$-function regularization, together with a detailed comparison to the regularization procedures previously used in the literature, will be included in a forthcoming longer publication: G. Amelino-Camelia, in preparation.

16 I.I. Kogan, Phys. Lett. B 262 (1991) 83; I.I. Kogan and G.W. Semenoff, Nucl. Phys. B 368 (1992) 718.

17 G. Lozano, Phys. Lett. B 283 (1992) 70.

18 M.A. Valle Basagoiti, Phys. Lett. B306 (1993) 307; R. Emparan and M.A. Valle Basagoiti, Universidad del Pais Vasco preprint EHU-FT-93-5, April 1993.

19 A. Cappelli, C.A. Trugenberger, and G.R. Zemba, Nucl. Phys. B396 (1993) 465.

20 C. Manuel and R. Tarrach, Universitat de Barcelona preprint UB-ECM-PF-19/93, September 1993.