ON $L^p$-LIOUVILLE PROPERTY FOR SMOOTH METRIC MEASURE SPACES

JIA-YONG WU AND PENG WU

Abstract. In this short paper we study $L^p$-Liouville property with $0 < p < 1$ for nonnegative $f$-subharmonic functions on a complete noncompact smooth metric measure space $(M, g, e^{-f}dv)$ with $\text{Ric}_f^m$ bounded below for $0 < m \leq \infty$. We prove a sharp $L^p_f$-Liouville theorem when $0 < m < \infty$. We also prove an $L^p_f$-Liouville theorem when $\text{Ric}_f \geq 0$ and $|f(x)| \leq \delta(n) \ln r(x)$.

1. Introduction

This is a sequel to our previous work [16, 17]. For Riemannian manifolds, sharp $L^p$-Liouville theorems with $p > 1$, $p = 1$, and $0 < p < 1$ for subharmonic functions were proved by Yau [18], Li [8], Li and Schoen [9], respectively. Recently Pigola, Rimoldi, and Setti [12] proved a sharp $L^p_f$-Liouville theorem with $p > 1$ for $f$-subharmonic functions on smooth metric measure spaces. In [16, 17], the authors proved a sharp $L^1_f$-Liouville theorem on smooth metric measure spaces with $(\infty)$-Bakry-Émery Ricci curvature $\text{Ric}_f^{\infty} = \text{Ric} + \nabla^2 f \geq 0$ by using the $f$-heat kernel estimates on smooth metric measure spaces. For the sake of completeness, in this paper we continue to study $L^p_f$-Liouville theorem with $0 < p < 1$ on smooth metric measure spaces.

Recall a smooth metric measure space $(\mathbb{M}^n, g, e^{-f}dv_g)$ is an $n$-dimensional complete Riemannian manifold $(\mathbb{M}, g)$ together with a weighted volume element $e^{-f}dv_g$ for some $f \in C^\infty(\mathbb{M})$. The associated $m$-Bakry-Émery Ricci curvature [11] is defined as

$$\text{Ric}_f^m = \text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df,$$

for $m \in \mathbb{R} \cup \{\pm \infty\}$. When $0 < m < \infty$, the Bochner formula for $m$-Bakry-Émery Ricci curvature can be considered as the Bochner formula for the Ricci curvature of an $(n+m)$-dimensional manifold, therefore many analytic and geometric properties for manifolds with Ricci curvature bounded below can be extended to smooth metric measure spaces with $m$-Bakry-Émery Ricci curvature bounded below, see for example [2, 10, 13] for details. When $m = \infty$, one denotes

$$\text{Ric}_f = \text{Ric}_f^{\infty} = \text{Ric} + \nabla^2 f.$$

In particular, if $\text{Ric}_f = \lambda g$ for some $\lambda \in \mathbb{R}$, then $(\mathbb{M}, g, f)$ is a gradient Ricci soliton.

Date: September 15, 2014.

2000 Mathematics Subject Classification. Primary 53C44; Secondary 58J35.

Key words and phrases. Bakry-Émery Ricci curvature; smooth metric measure space; Liouville theorem.
The $f$-Laplacian $\Delta_f$ on $(M, g, e^{-f} dv)$, which is self-adjoint with respect to the weighted volume element, is defined by

$$\Delta_f = \Delta - \nabla f \cdot \nabla.$$ 

A function $u$ is called $f$-harmonic if $\Delta_f u = 0$, and $f$-subharmonic if $\Delta_f u \geq 0$. The weighted $L^p$-norm (or $L^p_f$-norm) is defined as

$$\|u\|_p = \left( \int_M |u|^p e^{-f} dv \right)^{1/p}$$ 

for any $0 < p < \infty$. We say that $u$ is $L^p_f$-integrable, i.e. $u \in L^p_f$, if $\|u\|_p < \infty$.

Recall the sharp $L^p$-Liouville theorem with $0 < p < 1$ for Riemannian manifolds proved by Li and Schoen [9],

**Theorem 1.1** (Li and Schoen [9]). Let $(M^n, g)$ be an $n$-dimensional complete noncompact Riemannian manifold. There exists a constant $\delta(n) > 0$ depending only on $n$ such that, if

$$\text{Ric} \geq -\delta(n) r^{-2}(x),$$

as $r(x) \to \infty$, then any nonnegative $L^p$-integrable subharmonic function with $0 < p < 1$ must be identically zero.

Li and Schoen [9] constructed an explicit example showing that the curvature assumption in Theorem 1.1 is sharp.

Motivated by Theorem 1.1 we first prove a sharp $L^p_f$-Liouville theorem with $0 < p < 1$ for smooth metric measure spaces with $\text{Ric}^m_f$ bounded below,

**Theorem 1.2.** Let $(M^n, g, e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. For any $0 < m < \infty$, there exists a constant $\delta(n, m) > 0$ depending only on $n$ and $m$ such that, if

$$\text{Ric}^m_f \geq -\delta(n, m) r^{-2}(x),$$

as $r(x) \to \infty$, then any nonnegative $L^p_f$-integrable $f$-subharmonic function with $0 < p < 1$ must be identically zero.

In Section 4 we show, by constructing an explicit example, that Theorem 1.2 is sharp, in fact in our example $\text{Ric}^m_f \approx -a r^{-2}(x)$ for some $a$ large enough.

For smooth metric measure spaces with $\text{Ric}_f$ bounded below, the first author [15] proved an $L^p_f$-Liouville theorem with $0 < p < 1$ when $f$ is bounded and $\text{Ric}_f \geq -\delta(n) r^{-2}(x)$ as $r(x) \to \infty$. In this paper we prove

**Theorem 1.3.** Let $(M^n, g, e^{-f} dv)$ be a complete noncompact smooth metric measure spaces with $\text{Ric}_f \geq 0$. Then there exists a constant $\delta(n)$ depending only on $n$ such that, if

$$|f(x)| \leq \delta(n) \ln r(x),$$

as $r(x) \to \infty$, then any nonnegative $L^p_f$-integrable $f$-subharmonic function with $0 < p < 1$ must be identically zero.
We expect Theorem 1.3 to be sharp too.

This rest of the paper is organized as follows. In Section 2, we prove Theorem 1.2 following the argument of Li and Schoen. In Section 3, we prove Theorem 1.3 by combining an f-mean value inequality \[16, 17\] and an argument of Yau \[19\]. In Section 4, we construct an explicit example to illustrate the sharpness of Theorem 1.2.

**Acknowledgement.** Part of the work was done when the first author was visiting the Department of Mathematics at Cornell University, he greatly thanks Professor Xiaodong Cao for his help and the department for their hospitality. The first author was partially supported by NSFC (11101267, 11271132) and the China Scholarship Council (201208310431). The second author was partially supported by an AMS-Simons postdoctoral travel grant.

**2. Proof of Theorem 1.2**

Proof of Theorem 1.2. Following the weighted Laplacian comparison theorem (for example \[3\]), the weighted volume comparison theorem, and an argument similar to that in the proof of Theorem 2.1 in Li and Schoen \[9\] (see also Theorem 5.3 in \[15\]), we get a mean value inequality for \(f\)-subharmonic functions, if \(\text{Ric}^p \geq -(n + m - 1)K(x, 5R)\), then

\[
\sup_{B_x(\beta R/2)} u^p \leq \frac{e^{c(1+\sqrt{K(x,5R)^R})}}{V_f(B_x(R))} \int_{B_x(R)} u^p e^{-f} dv
\]

for nonnegative \(f\)-subharmonic function \(u\) on \(B_x(5R)\), where \(c\) depends only on \(n\), \(m\) and \(p\).

We will show that \(u\) must vanish at infinity if \(u \in L^p_f\) by proving that the growth rate of the \(f\)-volume is large enough under the assumption using the weighted volume comparison theorem, and by the maximum principle \(u\) must be identically zero.

Let \(x \in M\) and \(\gamma\) be a minimal geodesic from \(o\) to \(x\), \(\gamma(0) = o\) and \(\gamma(T) = x\). Define \(t_i \in [0, T]\), \(0 \leq i \leq k\), such that

\[
t_0 = 0, \quad t_1 = 1 + \beta, \quad \ldots, \quad t_i = 2 \sum_{j=0}^{i} \beta^j - 1 - \beta^i,
\]

for some \(\beta > 1\) to be determined later, and \(k\) to be the number such that \(t_k < T\) and \(t_{k+1} \geq T\). Denote \(x_i = \gamma(t_i)\), so they satisfy

\[
r(x_i, x_{i+1}) = \beta^i + \beta^{i+1}, \quad r(o, x_i) = t_i \quad \text{and} \quad r(x_k, x) < \beta^k + \beta^{k+1}.
\]

Moreover, the geodesic balls \(B_{x_i}(\beta^i)\) cover \(\gamma([0, 2\sum_{j=0}^{k} \beta^j - 1])\) and their interiors are disjoint.

We claim

\[
V_f(B_{x_k}(\beta^k)) \geq C \left(\frac{\beta^{n+m}}{(\beta + 2)^{n+m} - \beta^{n+m}}\right)^k V_f(B_o(1))
\]

for a fixed

\[
\beta > \frac{2}{\frac{2}{n+m} - 1} > 1.
\]
Proof of the claim. For each $1 \leq i \leq k$, by the relative comparison theorem (see (4.10) in [14]), we have
\[
V_f(B_{x_i}(\beta^i)) \geq D_1 \left[ V_f(B_{x_i}(\beta^i + 2\beta^{i-1})) - V_f(B_{x_i}(\beta^i)) \right]
\geq D_1 V_f(B_{x_i-1}(\beta^{i-1})),
\]
where
\[
D_i = \int_0^{\beta^i} \frac{\sqrt{K(x_i,\beta^i + 2\beta^{i-1})} \sinh^{n+m-1} t \, dt}{(\beta^i + 2\beta^{i-1}) \sqrt{K(x_i,\beta^i + 2\beta^{i-1})} \sinh^{n+m-1} t \, dt},
\]
since
\[
\text{Ric}^m \geq -(n + m - 1)K(x_i,\beta^i + 2\beta^{i-1})
\]
on $B_{x_i}(\beta^i + 2\beta^{i-1})$. Therefore,
\[
(2.3) \quad V_f(B_{x_k}(\beta^k)) \geq V_f(B_{x_1}(1)) \prod_{i=1}^{k-1} D_i.
\]
Since $r(o, x_i) = 2 \sum_{j=0}^{i-1} \beta^j - 1 - \beta^i$, the curvature assumption implies that
\[
\sqrt{K(x_i,\beta^i + 2\beta^{i-1})} \leq \sqrt{\delta(n,m)} \left( \frac{2}{\beta^i - 1} \right)^{i-2} \frac{\beta - 1}{2\beta^{i-1} - \beta - 1}
\]
for sufficiently large $i$. Hence
\[
\beta^i \sqrt{K(x_i,\beta^i + 2\beta^{i-1})} \leq \sqrt{\delta(n,m)} \frac{(\beta - 1)^2}{2\beta^{i-1} - \beta - 1}
\]
which can be made arbitrarily small for a fixed $\beta > 2(2^{1/(n+m)} - 1)^{-1} > 1$ by choosing $\delta(n,m)$ to be sufficiently small. So we get
\[
D_i \approx \frac{(\beta^i)^{n+m}}{(\beta^i + 2\beta^{i-1})^{n+m} - (\beta^i)^{n+m}}
= \frac{\beta^{n+m}}{(\beta + 2)^{n+m} - \beta^{n+m}}
\]
by simply approximating $\sinh t$ by $t$. Plugging into (2.3) we proved the claim.

Next we estimate $V_f(B_x(\beta^{k+1}))$, which will be divided into two cases.  
Case 1: $r(x, x_k) \leq \beta^k(\beta - 1)$. So $B_{x_k}(\beta^k) \subset B_x(\beta^{k+1})$, and
\[
V_f(B_{x_k}(\beta^k)) \leq V_f(B_x(\beta^{k+1})).
\]
By (2.2), we get
\[
V_f(B_x(\beta^{k+1})) \geq C \left( \frac{\beta^{n+m}}{(\beta + 2)^{n+m} - \beta^{n+m}} \right)^k V_f(B_{x_1}(1)).
\]
Case 2: $r(x, x_k) > \beta^k(\beta - 1)$. So $B_{x_k}(\beta^k) \subset B_x(r(x, x_k) + \beta^k) \setminus B_{x_k}(r(x, x_k) - \beta^k)$.
By the relative comparison theorem, we have
\[
V_f(B_x(\beta^k)) \geq D \left[ V_f(B_x(r(x, x_k) + \beta^k)) - V_f(B_x(r(x, x_k) - \beta^k)) \right]
\geq D V_f(B_{x_k}(\beta^k)),
\]
where
\[
D = \frac{\int_0^{\beta_k} \sqrt{K(x,r(x,x_k)+\beta_k)} \sinh^{n+m-1} t dt}{\int (r(x,x_k)+\beta_k)^\delta (x,r(x,x_k)+\beta_k) \sinh^{n+m-1} t dt}.
\]

Since
\[
(r(x,x_k)+\beta_k) \sqrt{K(x,r(x,x_k)+\beta_k)} \leq (\beta_k+2) \sqrt{K(x,r(x,x_k)+\beta_k)} \leq \frac{\sqrt{\delta(n,m)}}{2} \beta(\beta-1)
\]
can be made sufficiently small, so we get
\[
D \approx \frac{\beta^{n+m}}{\beta(2)^{n+m} - \beta^{n+m}}.
\]
Combining with (2.2) we get
\[
V_f(B_x(\beta^{k+1})) \geq C \frac{\beta^{n+m}}{\beta(2)^{n+m} - \beta^{n+m}} \beta^{k+1} V_f(B_o(1)) \geq \tilde{C} \left( \frac{\beta^{n+m}}{\beta(2)^{n+m} - \beta^{n+m}} \right)^k V_f(B_o(1)),
\]
where \(\tilde{C}\) depends only on \(n\) and \(\beta\).

Therefore, we have
\[
V_f(B_x(\beta^{k+1})) \geq C \left( \frac{\beta^{n+m}}{\beta(2)^{n+m} - \beta^{n+m}} \right)^k V_f(B_o(1)).
\]

Next let \(r(x) \to \infty\), then \(k \to \infty\). By the choice of \(\beta\),
\[
\frac{\beta^{n+m}}{\beta(2)^{n+m} - \beta^{n+m}} > 1,
\]
so the right hand side of (2.4) approaches to infinity. Let \(R = \beta^{k+1}\), by the assumption, \(R \sqrt{K(x,5R)}\) is bounded from above. Therefore by (2.1), we have
\[
u^p(x) \leq CV_f^{-1}(B_x(R)),
\]
where \(C\) depends on \(n, m, p\) and \(\|u\|_p\). Therefore \(u(x) \to 0\) as \(r(x) \to \infty\), and Theorem 1.2 follows from the maximum principle. \(\square\)

3. Proof of Theorem 1.3

First recall the \(f\)-volume comparison theorem proved by Wei and Wylie [14],

**Theorem 3.1**. Let \((M^n, g, e^{-f} dv)\) be an \(n\)-dimensional complete noncompact smooth metric measure space. If \(\text{Ric}_f \geq 0\), then for any \(x \in B_o(R)\),
\[
\frac{V_f(B_x(R_1,R_2))}{V_f(B_x(r_1,r_2))} \leq e^{4A \frac{R_2^n - R_1^n}{r_2^n - r_1^n}}
\]
for any \(0 < r_1 < r_2\), \(0 < R_1 < R_2 < R\), \(r_1 \leq R_1\), \(r_2 \leq R_2 < R\), where \(B_x(R_1,R_2) = B_x(R_2) \backslash B_x(R_1)\), and \(A(R) = \sup_{x \in B_o(R)} |f|(x)\).

Applying Theorem 3.1, we get the following \(f\)-volume growth estimate.
Proposition 3.2. Let $(M, g, e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space with $\text{Ric}_f \geq 0$. If there exists a sufficiently small constant $\delta(n) > 0$ depending only on $n$ such that

$$|f(x)| \leq \delta(n) \ln r(x),$$

as $r(x) \to \infty$. Then there exists a constant $C > 0$ such that for $r$ sufficiently large,

$$V_f(B_o(r)) \geq C r^{1-\delta(n)}.$$

Proof. Let $x \in M$ be a point with $d(x, o) = r \geq 2$. Choosing $R_2 = r + 1$, $R_1 = r - 1$, $r_2 = r - 1$ and $r_1 = 0$, by Theorem 3.1 we have

$$\frac{V_f(B_x(r+1)) - V_f(B_x(r-1))}{V_f(B_x(r-1))} \leq c(n)(r+1)^{\delta(n)} \frac{(r+1)^n - (r-1)^n}{(r-1)^n}$$

$$\leq \frac{C(n)}{r^{1-\delta(n)}}.$$

Since $B_o(1) \subset B_x(r+1) \setminus B_x(r-1)$ and $B_x(r-1) \subset B_o(2r-1)$, therefore we have

$$V_f(B_o(2r-1)) \geq V_f(B_x(r-1))$$

$$\geq \frac{V_f(B_o(1))}{C(n)} r^{-1-\delta(n)}.$$

□

Next recall a weighted mean value inequality on complete smooth metric measure spaces proved by the authors [17].

Lemma 3.3. Let $(M, g, e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space with $\text{Ric}_f \geq 0$. Let $u$ be a smooth positive subsolution to the $f$-heat equation in $B_o(R)$. For $0 < p < \infty$, there exist constants $c_1(n, p)$ and $c_2(n, p)$ such that,

$$\sup_{B_o(R/2)} u^p \leq \frac{c_1 e^{c_2 A}}{V_f(B_o(R))} \int_{B_o(R)} u^p e^{-f} dv.$$

Now we apply Lemma 3.3 and Proposition 3.2 to prove Theorem 1.3.

Proof of Theorem 1.3. Under the assumption, by Lemma 3.3 we have the following mean value inequality

$$\sup_{B_o(R/2)} u^p \leq \frac{c_1 R^2 \delta(n)}{V_f(B_o(R))} \int_{B_o(R)} u^p e^{-f} dv$$

where constant $c_1$ and $c_2$ depend on $n$ and $p$. By Proposition 3.2 we have

$$\sup_{B_o(R/2)} u^p \leq \frac{C}{R^{1-c_3 \delta(n)}},$$

where constant $c_3$ depends on $n$ and $p$, and $C$ depends on $n$, $p$ and the $L^p_f$-norm of $u$. Taking $R \to \infty$, then $u(x) \to 0$ as long as $\delta(n) < c_3^{-1}$, and Theorem 1.3 follows. □
4. An example

In this section, we construct an example to illustrate the sharpness of Theorem 1.2.

Consider the Euclidean space \((\mathbb{R}^2, g_0)\). Let the potential function be \(f = a \ln r\) when \(r \geq 2\) for some constant \(a > 0\), and extend it smoothly to \(r < 2\). The \(f\)-Laplacian is given by

\[
\Delta_f u = u_{rr} + \left(\frac{1}{r} - f'\right) u_r + \frac{1}{r^2} u_{\theta\theta}.
\]

It is a direct computation that, when \(r \geq 2\)

\[
\text{Ric}_f^{m} = f'' - \frac{1}{m} f'^2 = -a \left(1 + \frac{a}{m}\right) \frac{1}{r^2},
\]

\[
\text{Ric}_f^{m_1} = \frac{1}{r} f' = \frac{a}{r^2},
\]

\[
\text{Ric}_f^{m_2} = 0,
\]

that is

\[
\text{Ric}_f^{m} \geq -a \left(1 + \frac{a}{m}\right) r^{-2}.
\]

It is easy to verify that \(u(r) = r^a\) is an \(f\)-harmonic function with

\[
\|u\|_p^p = \int_2^\infty \int_0^{2\pi} |u|^p r e^{-f} d\theta dr + \int_0^2 \int_0^{2\pi} |u|^p r e^{-f} d\theta dr
\]

\[
= \int_2^\infty \int_0^{2\pi} r^{ap} r e^{-a \ln r} d\theta dr + C
\]

\[
= 2\pi \int_2^\infty r^{ap-a+1} dr + C.
\]

For any \(0 < p < 1\), if \(a > 2/(1-p)\), then the integral is finite and \(u\) is an \(L^p_f\)-integrable \(f\)-harmonic function on \((\mathbb{R}^2, g_0, r^{-a} dv)\).

References

1. D. Bakry, M. Emery, *Diffusion hypercontractivitives*, in Séminaire de Probabilités XIX, 1983/1984, in: Lecture Notes in Math., vol. 1123, Springer-Verlag, Berlin, 1985, 177-206.
2. D. Bakry, Z.-M. Qian, *Some new results on eigenvectors via dimension, diameter and Ricci curvature*, Adv. Math. 155 (2000), 98–153.
3. D. Bakry, Z.-M. Qian, *Volume comparison theorems without Jacobi fields*, in Current Trends in Potential Theory, 115-122. Theta Ser. Adv. Math., 4. Theta, Bucharest (2005).
4. K. Brighton, *A Liouville-type theorem for smooth metric measure spaces*, J. Geome. Anal. 23 (2013), 562–570.
5. J. Cheeger, M. Gromov, M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom. 17 (1982), 15–53.
6. R. E. Greene, H. Wu, *Integrals of subharmonic functions on manifolds of nonnegative curvature*, Invent. Math. 27 (1974), 265–298.
7. R. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geom. 2 (1995), 7–136, International Press.
8. P. Li, *Uniqueness of \(L^1\) solutions for the Laplace equation and the heat equation on Riemannian manifolds*, J. Differential Geom. 20 (1984), 447–457.
9. P. Li, R. Schoen, *\(L^p\) and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Math. 153 (1984), 279–301.
10. X.-D. Li, *Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds*, J. Math. Pure. Appl. 84 (2005), 1295-1361.
11. J. Lott, *Some geometric properties of the Bakry-Émery-Ricci tensor*, Comment. Math. Helv. 78 (2003), 865–883.
12. S. Pigola, M. Rimoldi, A. G. Setti, *Remarks on non-compact gradient Ricci solitons*, Math. Z. 268 (2011), 777–790.
13. G. Wei, W. Wylie, *Comparison geometry for the smooth metric measure spaces*, Proceedings of the 4th International Congress of Chinese Mathematicians, 2 (2007), 191–202.
14. G. Wei, W. Wylie, *Comparison geometry for the Bakry-Émery Ricci tensor*, J. Differential Geom. 83 (2009), 377–405.
15. J.-Y. Wu, *Lp-Liouville theorems on complete smooth metric measure spaces*, Bull. Sci. Math. 138 (2014), 510–539.
16. J.-Y. Wu, P. Wu, *Heat kernel on smooth metric measure spaces with nonnegative curvature*, arxiv:math.DG/1401.6155.
17. J.-Y. Wu, P. Wu, *Heat kernel on smooth metric measure spaces and applications*, arXiv:math.DG/1406.5801.
18. S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm Pure Appl Math. 28 (1975), 201–228.
19. S.-T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. 25 (1976), 659–670.

Department of Mathematics, Shanghai Maritime University, Haigang Avenue 1550, Shanghai 201306, P. R. China
E-mail address: jywu81@yahoo.com

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA
E-mail address: wupenguin@math.cornell.edu