Presymplectic representation of bi-Hamiltonian chains

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Abstract

Liouville integrable systems, which have bi-Hamiltonian representation of the Gel’fand-Zakharevich type, are considered. Bi-presymplectic representation of one-Casimir bi-Hamiltonian chains and weakly bi-presymplectic representation of multi-Casimir bi-Hamiltonian chains are constructed. The reduction procedure for Poisson and presymplectic structures is presented.

1 Introduction

The bi-Poisson formulation of finite dimensional integrable Hamiltonian systems has been systematically developed for the last two decades (see 

and the literature quoted there). It has been found that most of the known Liouville integrable finite dimensional systems have more then one Hamiltonian representation. Moreover, in the majority of known cases, both Poisson structures of a given flow are degenerated. Perhaps this is the reason why such an important property of integrable systems was discovered so late, relative to the age of classical mechanics. For such systems, related bi-Poisson (bi-Hamiltonian) commuting vector fields belong to one or more bi-Hamiltonian chains starting and terminating with Casimirs of respective Poisson structures. An important aspect of such a construction is its relation to the recently developed geometric separability theory 

. Actually, the necessary condition for the existence of separation coordinates is the reducibility of one of the Poisson structures onto a symplectic leaf of the other one. An important fact is, that the whole procedure of variables separation is almost algorithmic.

On the other hand, it is well known from the classical mechanics, that if the Poisson structure is nondegenerate, i.e. if the rank of the Poisson tensor is equal to the dimension of a phase space, then the phase space becomes a symplectic manifold with a symplectic structure being just the inverse of the Poisson structure. In such a case there exists an alternative (dual) description of Hamiltonian vector fields in the language of symplectic geometry. So, a natural question arises, whether one can construct such a dual picture in the degenerated case, when there is no natural inverse of the Poisson tensor .

A positive answer to this question is presented in next Sections of the paper. A dual presymplectic picture will be constructed for bi-Hamiltonian chains with one Casimir as well as with many Casimirs. The paper is organized as follows. In this Section we recall some elementary facts from the Poisson and presymplectic geometry. In Section 2 we introduce notions of dual pairs, compatible pairs and Poisson pairs and investigate some of their properties. In Section 3, applying the results of the previous Section, we construct a presymplectic representation of Poisson chains. In Section 4 the deformation reduction procedure for Poisson and presymplectic chains is presented. Such a reduction is crucial for separability of underlying dynamical systems. Finally, in Section 5, we illustrate the presented theory by a nontrivial example.

Given a manifold \( M \) of dim \( M = m \), a Poisson operator \( \Pi \) of corank \( r \) on \( M \) is a bivector \( \Pi \in \Lambda^2(\mathcal{M}) \) with vanishing Schouten bracket:

\[
[\Pi, \Pi]_S = 0,
\]

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whose kernel is spanned by exact one-forms

\[ \ker \Pi = \text{Sp}\{dc_i\}_{i=1,\ldots,r}. \]

The symbol \( d \) denotes the operator of exterior derivative. In a given coordinate system \((x^1,\ldots,x^m)\) on \( \mathcal{M} \) we have

\[ \Pi = \sum_{i<j}^m \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \]

while the Poisson property \( \Pi \) takes the form

\[ \sum_l (\Pi^{lj} \partial_l \Pi^{ik} + \Pi^{il} \partial_l \Pi^{kj} + \Pi^{kl} \partial_l \Pi^{ij}) = 0, \quad \partial_i := \frac{\partial}{\partial x^i}. \]

A function \( c : \mathcal{M} \to \mathbb{R} \) is called the *Casimir function* of the Poisson operator \( \Pi \) if \( \Pi dc = 0 \). A linear combination \( \Pi_\lambda = \Pi_1 - \lambda \Pi_0 \) (\( \lambda \in \mathbb{R} \)) of two Poisson operators \( \Pi_0 \) and \( \Pi_1 \) is called a *Poisson pencil* if the operator \( \Pi_\lambda \) is Poissonian for any value of the parameter \( \lambda \). In this case we say that \( \Pi_0 \) and \( \Pi_1 \) are *compatible*. A vector field \( X_F \) related to a function \( F \) through the relation

\[ X_F = \Pi dF \] (2)

is called a Hamiltonian vector field with respect to the Poisson operator \( \Pi \). It is also important to note that if \( X \) is any vector field on \( \mathcal{M} \) that is Hamiltonian with respect to \( \Pi \), then \( L_X \Pi = 0 \), where \( L_X \) is the Lie-derivative operator in the direction \( X \).

Further, a *presymplectic operator* \( \Omega \) on \( \mathcal{M} \) defines a 2-form that is closed, i.e. \( d\Omega = 0 \), degenerated in general. In the coordinate system \((x^1,\ldots,x^m)\) on \( \mathcal{M} \) we can always represent \( \Omega \) as

\[ \Omega = \sum_{i<j}^m \Omega_{ij} dx^i \wedge dx^j, \]

where the closeness condition takes the form

\[ \partial_i \Omega_{jk} + \partial_k \Omega_{ij} + \partial_j \Omega_{ki} = 0. \]

Moreover, the kernel of any presymplectic form is always an integrable distribution. A vector field \( X^F \) related to a function \( F \) by the relation

\[ \Omega X^F = dF \] (3)

is called the inverse Hamiltonian vector field with respect to the presymplectic operator \( \Omega \). Generally, if \( \Omega \) is a closed two-form and \( X \) is an arbitrary vector field then

\[ L_X \Omega = d(\Omega X). \] (4)

Hence, if \( \Omega(Y) = 0 \) for some vector field \( Y \) on \( \mathcal{M} \) then \( L_X \Omega = 0 \). Notice that contrary to the Poisson case, a linear combination of two presymplectic operators is always presymplectic.

Poission tensor \( \Pi \), considered as the mapping \( \Pi : T^* \mathcal{M} \to T \mathcal{M} \), induces a Lie bracket on the space \( C^\infty(\mathcal{M}) \) of all smooth real-valued functions on \( \mathcal{M} \)

\[ \{.,.\}_\Pi : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \quad \{F,G\}_\Pi \overset{\text{def}}{=} \langle dF, \Pi dG \rangle = \Pi(\langle dF, dG \rangle), \] (5)

(where \( \langle .,. \rangle \) is the dual map between \( T \mathcal{M} \) and \( T^* \mathcal{M} \)) which is skew-symmetric and satisfies Jacobi identity. It is called a *Poisson bracket*.

When a Poisson operator \( \Pi \) is nondegenerate, one can always define its inverse \( \Omega = \Pi^{-1} \), called a *symplectic operator*, and then equations (2) and (3) are equivalent. Moreover, any Hamiltonian vector field with respect to \( \Pi \) is simultaneously the inverse Hamiltonian with respect to \( \Omega \) and \( X_F = X^F \). Finally, symplectic operator \( \Omega \) defines the same Poisson bracket as the related Poisson operator \( \Pi \)

\[ \{F,G\}_\Omega := \Omega(X^F,X^G) = \langle \Omega X^F, X^G \rangle = \langle dF, \Pi dG \rangle = \{F,G\}_\Pi. \] (6)

The equivalence is destroyed in the case of degeneracy. First, one cannot define \( \Omega \) as the inverse of \( \Pi \). Second, for degenerated \( \Pi \) equation (2) is valid for an arbitrary function \( F \) (as in the nondegenerate case), while for degenerated \( \Omega \) and an arbitrary \( F \) there is no such vector field \( X^F \) that (3) is fulfilled. It means that equation (3) is valid only for a particular class of functions (contrary to the nondegenerate case). Finally it is not clear how to define a Poisson bracket with respect to a presymplectic form.
2 Dual Poisson-presymplectic pairs and compatible structures

In this Section we introduce basic objects important for the theory further developed and we investigate some of their properties. As the concept of dual pairs was introduced and developed for the first time in our previous paper [12], here we only recall their main properties. Let us remark that the concept of dual pairs was introduced and developed for the first time in [12], which we are going to apply to bi-Poisson chains, is a useful particular realization of the concept of Poisson brackets on presymplectic manifolds, presented by B.A.Dubrovin at al. [11].

Consider a smooth manifold $M$ of dimension $m$ equipped with a pair of antisymmetric operators $\Pi$, $\Omega$.

**Definition 1** A pair of antisymmetric tensor fields $(\Pi, \Omega)$ such that $\Pi : T^*M \to TM$, i.e. $\Pi$ is twice contravariant, and $\Omega : TM \to T^*M$, i.e. $\Omega$ is twice covariant, is called a dual pair if there exists $r$ one-forms $\alpha_i$, $i = 1, \ldots, r$ and $r$ linearly independent vector fields $Z_i$, $i = 1, \ldots, r$ such that the following conditions are satisfied:

1. $\alpha_i(Z_j) = \delta_{ij}$ for all $i, j = 1, \ldots, r$.
2. The kernel of $\Pi$ is spanned by all $\alpha_i$, $\ker(\Pi) = Sp\{\alpha_i\}_{i=1..r}$.
3. The kernel of $\Omega$ is spanned by all the vector fields $Z_i$, $\ker(\Omega) = Sp\{Z_i\}_{i=1..r}$.
4. The following partition of unity holds on $TM$

$$I = \Pi\Omega + \sum_{i=1}^r Z_i \otimes \alpha_i$$  \hspace{1cm} (7)

where $\otimes$ denotes the tensor product.

Notice that the partition of unity (7) on $T^*M$ takes the form

$$I = \Omega\Pi + \sum_{i=1}^r \alpha_i \otimes Z_i.$$  \hspace{1cm} (8)

Let us choose the basic one-forms $\alpha_i$ in such a way that $\alpha_i = dc_i$ and let us denote a foliation of $M$ given by the functions $c_i$ by $N$. This foliation consists of the leaves $N_\nu = \{x \in M : c_i(x) = \nu_i, i = 1, \ldots, r\}$, $\nu = (\nu_1, \ldots, \nu_r)$. Condition 1 of the above definition implies that the distribution $Z$ spanned by the vector fields $Z_i$ is transversal to the foliation $N$. Thus, for any $x \in M$ we have

$$T_xM = T_xN_\nu \oplus Z_x, \quad T^*_xM = T^*_xN_\nu \oplus Z^*_x$$  \hspace{1cm} (9)

where $N_\nu$ is a leaf from the foliation $N$ that passes through $x$, the symbol $\oplus$ denotes the direct sum of the vector spaces, $Z_x$ is the subspace of $T_xM$ spanned by the vectors $Z_i$ at this point, $T^*_xN_\nu$ is the annihilator of $Z_x$ and $Z^*_x$ is the annihilator of $T_xN_\nu$. Condition 2 of the above definition implies that $\text{Im}(\Pi) = T^*N$, Condition 3 means that $\text{Im}(\Omega) = T^*N$ and Condition 4 describes the degree of degeneracy of our pair.

**Definition 2** A dual pair $(\Pi, \Omega)$ is called a dual Poisson-presymplectic pair (in short: dual P-p pair) if $\Pi$ is a Poisson bivector and if $\Omega$ is a closed 2-form.

Notice that in the case when a dual P-p pair has no degeneration ($r = 0$) we get the usual Poisson-symplectic pair of mutually inverse operators, since (7) reads then as $I = \Pi\Omega$. Moreover, for a degenerated case, when $r \neq 0$, as $\Omega$ is presymplectic, then $\ker(\Omega)$ is an integrable distribution with $[Z_i, Z_j] = 0$, $i, j = 1, \ldots, r$ and for $\Pi$ Poisson, $\alpha_i$ are exact one forms generated by Casimir functions: $\alpha_i = dc_i$, $i = 1, \ldots, r$. The commutativity of $Z_i$ follows from Condition 1 of Definition 1. The following Lemma will be useful in further considerations.

**Lemma 3** Let $(\Pi, \Omega)$ be a dual P-p pair, then

$$L_{Z_i}\Pi = 0, \quad i = 1, \ldots, r.$$
Assume that \((\Pi, \Omega)\) is a dual P-p pair and
\[
\Pi dF = X_F \tag{10}
\]
is a Hamiltonian vector field with respect to \(\Pi\). Applying \(\Omega\) to both sides of (10) and using the decomposition (8) we get
\[
dF = \Omega(X_F) + \sum_{i=1}^{r} Z_i(F) dc_i, \tag{11}
\]
which reconstructs \(dF\) from \(X_F\) and \(Z_i(F)\) in the case of degenerated Poisson structure \(\Pi\). In that sense \(\Omega\) plays the role of the "inverse" of \(\Pi\).

Notice that inverse Hamiltonian vector fields with respect to \(\Omega\) are related to functions which are annihilated by \(\ker(\Omega)\), i.e. \(Z_i(F) = 0\), \(i = 1, \ldots, r\). Then, equation (11) reduces to (3) with \(\Omega(X_F) = \Omega(X_F)\). It means that \(X_F\) is not only a Hamiltonian but also inverse Hamiltonian vector field related to the same Hamiltonian function \(F\). Moreover, it is a gauge freedom for inverse Hamiltonian vector fields \(X_F\) with respect to \(\Omega\).

Indeed, applying \(\Pi\) to both sides of equation (3) and using decomposition (7) one gets
\[
X_F - X_F = \sum_{i} X_F(c_i) Z_i.
\]
It means that an inverse Hamiltonian vector field \(X_F\) is simultaneously a Hamiltonian vector field, i.e. \(X_F = X_F\), if \(X_F\) annihilates the kernel of \(\Pi\).

The definition of dual objects is not unique and questions about the 'gauge freedom' can be posed. A possible realization of such a freedom is as follows: given a dual P-p pair \((\Pi, \Omega)\) we are looking for possible deformations of \(\Omega\) to get a new presymplectic form \(\Omega'\) ensuring that \((\Pi, \Omega')\) is dual again. Another possibility is related to a gauge freedom for the operator \(\Pi\), i.e. how can we deform \(\Pi\) to a new Poisson bivector \(\Pi'\) so that \((\Pi', \Omega)\) is also the dual pair. An example of such a gauge freedom is given in the following proposition:

**Proposition 4** Let \((\Pi, \Omega)\) be a dual P-p pair as in definitions (1) and (2). Suppose that \(F_i\) are real functions on \(M\) related to vector fields \(K_i\) which are simultaneously Hamiltonian and inverse Hamiltonian with respect to \((\Pi, \Omega)\) pair
\[
dF_i = \Omega K_i, \quad K_i = \Pi dF_i, \quad i = 1, \ldots, r.
\]
Then:

(i) \[
\Omega' = \Omega + \sum_i dF_i \wedge dc_i,
\]
is a dual to \(\Pi\) presymplectic two-form, provided that
\[
\Pi(dF_i, dF_j) = 0 \text{ for all } i, j.
\]

(ii) \[
\Pi' = \Pi + \sum_i Z_i \wedge K_i
\]
is a dual to \(\Omega\) Poisson bivector, provided that
\[
\Omega(K_i, K_j) = 0 \text{ for all } i, j.
\]

Let us now turn our attention to brackets induced on the space \(C^\infty(M)\). We know that the Poisson operator \(\Pi\) turns \(C^\infty(M)\) into a Poisson algebra with the Poisson bracket (5)
\[
\{F, G\}_\Pi = \Pi(dF, dG) = \langle dF, \Pi dG \rangle.
\]
In case when \(\Omega\) is a part of a dual P-p pair we can define the above bracket through the \(\Omega\) in the following way:

\[4\]
Lemma 5 Let \( (\Pi, \Omega) \) be a dual P-p pair. Define a new bracket on \( C^\infty(M) \)

\[
\{F, G\}_\Omega := \Omega(X_F, X_G) = \langle \Omega X_F, X_G \rangle, \quad X_F = \Pi dF.
\]

Then \( \{\cdot, \cdot\}_\Omega = \{\cdot, \cdot\}_\Pi \) i.e. both brackets are identical.

The proofs of Lemma 3, Lemma 5 and Proposition 4, as well as more details on the concept of dual P-p pairs the reader can find in [12].

Now we pass to the concept of compatibility.

Definition 6 A Poisson bivector \( \Pi \) and presymplectic two-form \( \Omega \) are called a compatible P-p pair if \( \Omega_D := \Omega \Pi \Omega \) is presymplectic.

As well known (see for example [1]) if \( (\Pi, \Omega) \) is a compatible P-p pair, then the second order tensor \( \Phi = \Pi \Omega : T^2 M \to T^2 M \) has vanishing Nijenhuis torsion

\[
L_{\Phi, \tau} - \Phi L_{\tau} = 0, \quad \forall \tau \in T^0 M,
\]

and is called a hereditary operator or recursion operator. Moreover \( \Pi_D := \Pi \Omega \Pi \) is a Poisson bivector. Observe that a dual P-p pair \( (\Pi_0, \Omega_0) \) is a trivial example of a compatible pair as

\[
\Omega_0 = \Omega_0(I - \sum_i Z_i \otimes dc_i) = \Omega_0.
\]

(12)

Lemma 7 If \( \Omega \) is a presymplectic two-form compatible with a Poisson bivector \( \Pi_0 \), then the bracket

\[
\{F, G\}_\Omega := \Omega(X^0_F, X^0_G), \quad X^0_F = \Pi_0 dF
\]

is a Poisson bracket.

Proof.

\[
\{F, G\}_\Omega \equiv \langle \Omega X^0_F, X^0_G \rangle = \langle \Omega_0 dF, \Pi_0 dG \rangle = -\langle dG, \Pi_0 \Omega_0 dF \rangle = \{F, G\}_{\Pi D}
\]

and \( \Pi_D \) is Poisson. \( \blacksquare \)

Obviously, when \( \Omega = \Omega_0 \), i.e. the compatible pair is simply a dual pair, then we deal with a special case described by Lemma 6. Moreover, if \( (\Pi, \Omega_0) \) is a compatible P-p pair and \( \ker(\Omega_0) = Sp\{Z_i\}_{i=1, \ldots, r} \), then

\[
\Omega_0(L_{Z_i} \Pi) \Omega_0 = 0, \quad i = 1, \ldots, r,
\]

(13)

which follows from (11).

Theorem 8 Let \( (\Pi_0, \Omega_0) \) be a dual P-p pair, such that \( \ker \Omega_0 = Sp\{Z_i\} \) and \( \ker \Pi_0 = Sp\{dc_i\} \). Moreover, let \( \Pi \) be a Poisson bivector compatible with \( \Omega_0 \), then:

(i)

\[
\Pi_d := \Pi_0 \Omega_0 \Pi_0 = \Pi_0 \Omega_0 \Pi_0 \Pi_0 = \Pi - \sum_i X_i \wedge Z_i + \frac{1}{2} \sum_{i,j} c_{ij} Z_i \wedge Z_j,
\]

(14)

(ii)

\[
L_{Z_i} \Pi_d = 0, \quad i = 1, \ldots, r,
\]

(15)
(iii) \[ L_Z \Pi = \sum_i [Z_i, X_i] \wedge Z_i - \frac{1}{2} \sum_{i,j} Z_i(c_{ij}) Z_i \wedge Z_j, \] (16)

where \( X_i = \Pi dc_i, \ c_{ij} = \Pi (dc_i, dc_j) = < dc_i, \Pi dc_j >, \)

(iv) \( \Pi_d \) is Poisson.

**Proof.** From the definition of \( \Pi_d \) we have

\[
\Pi_d = \Pi_0 \Omega_0 \Pi \Omega_0 = (I - \sum_i Z_i \otimes dc_i) \Pi (I - \sum_j dc_j \otimes Z_j)
\]

\[
= \Pi - \sum_i X_i \wedge Z_i + \frac{1}{2} \sum_{i,j} c_{ij} Z_i \wedge Z_j.
\]

Then, from Lemma 3 and relation (13), it follows that \( L_Z \Pi_d = 0 \). Next, from (i) and (ii) immediately follows (iii). Finally we prove the property (iv). If \( X, Y \) are some vector fields, then their Schouten bracket \( [X, Y]_S = [X, Y] = L_X Y \) is a usual Lie bracket (commutator). Moreover, for arbitrary bivector \( P \) and function \( F \), the Schouten bracket fulfills the relations

\[
[X \wedge Y, P]_S = Y \wedge [X, P]_S - X \wedge [Y, P]_S, \quad [X, P]_S = L_X P \quad (17)
\]

and

\[
L_{FX} P = F L_X P - (PdF) \wedge X. \quad (18)
\]

Now, using (17) and (18), after straightforward but lengthy calculations, one finds

\[
[\Pi_d, \Pi]_S = [\Pi, \Pi]_S - 2 [\Pi, \sum_i X_i \wedge Z_i]_S + [\Pi, \sum_{i,j} c_{ij} Z_i \wedge Z_j]_S
\]

\[
+ \left[ \sum_i X_i \wedge Z_i, \sum_j X_j \wedge Z_j \right]_S - \left[ \sum_k X_k \wedge Z_k, \sum_{i,j} c_{ij} Z_i \wedge Z_j \right]_S
\]

\[
+ \frac{1}{4} \left[ \sum_{i,j} c_{ij} Z_i \wedge Z_j, \sum_{k,l} c_{kl} Z_k \wedge Z_l \right]_S
\]

\[
= \sum_{i,j,k} X_k(c_{ij}) Z_i \wedge Z_k \wedge Z_j = 0,
\]

as

\[
\sum_{i,j,k} X_k(c_{ij}) Z_i \wedge Z_j \wedge Z_k = \frac{1}{3} \sum_{i,j,k} (X_k(c_{ij}) + X_k(c_{ij}) + X_k(c_{ij})) Z_i \wedge Z_k \wedge Z_j = 0
\]

which follows from Jacobi identity. ■

As the concept of compatibility will be important in the reduction scheme for bi-Hamiltonian chains, the following Theorem will be useful in the further considerations.

**Theorem 9** Let \( (\Pi_0, \Omega_0) \) be a dual \( P-p \) pair such that \( \ker \Omega_0 = Sp\{Z_i\} \) and \( \Pi \) be a Poisson tensor compatible with \( \Pi_0 \). Then, \( \Pi \) is compatible with \( \Omega_0 \) if

\[
\Omega_0(L_{Z_i} \Pi) \Omega_0 = 0, \quad i = 1, \ldots, k. \quad (19)
\]

**Proof.** First we gather all necessary formulas important for the calculation. For any Poisson operator \( \Pi \)

\[
L_{\Pi^2} \Pi = -\Pi(d\gamma) \Pi, \quad \forall \gamma \in T^* M, \quad (20)
\]

for any presymplectic form \( \Omega \)

\[
L_X \Omega = d(\Omega X), \quad \forall X \in TM \quad (21)
\]

and for an arbitrary second order mixed rank tensor \( \Phi \)

\[
[\Phi X_1, X_2] = \Phi[X_1, X_2] + (L_{X_2} \Phi) X_1. \quad (22)
\]
For arbitrary vectors $X_1, X_2, X$ one-forms $\alpha_1, \alpha_2$, two-form $\Omega$, and function $F$ the following relations hold

\[
(X_1 \otimes X_2)(\alpha_1 \otimes \alpha_2) = \alpha_1(X_2)X_1 \otimes \alpha_2, \quad \alpha_1(X_2) = < \alpha_1, X_2 >,
\]

\[
\Pi(\alpha_1 \otimes \alpha_2) = \Pi(\alpha_1) \otimes \alpha_2, \quad \Omega(X_1 \otimes X_2) = \Omega(X_1) \otimes X_2,
\]

\[
(\alpha_1 \otimes \alpha_2)\Pi = -\alpha_1 \otimes (\Pi \alpha_2), \quad (X_1 \otimes X_2)\Omega = -X_1 \otimes (\Omega X_2),
\]

\[
L_{FX}\Omega = FL_X\Omega + dF \wedge \Omega X. \tag{23}
\]

As $\Pi_0$ and $\Pi$ are compatible so $\Pi + \lambda \Pi_0$ is Poisson, hence for $\forall \tau \in TM$ and $\gamma = \Omega_0\tau$ from (20) we have

\[
0 = L_{(\Pi + \lambda \Pi_0)\gamma}(\Pi + \lambda \Pi_0) + (\Pi + \lambda \Pi_0)d\gamma(\Pi + \lambda \Pi_0) = \lambda(L_{\Pi_0} \Pi_0 + L_{\Pi_0} \gamma \Pi + \Pi(d\gamma)\Pi_0 + \Pi_0(d\gamma)\Pi).
\]

Applying (7), (20) and (18) we find

\[
L_{\Pi_0} \gamma \Pi = -\Pi_0(L_{\Pi_0} \gamma \Omega_0)\Pi_0 - \sum_i (\Pi_0 d\alpha_i^0) \wedge Z_i,
\]

where $a_i^0 = < dc_i, \Pi \gamma >$, $L_{Z_i} \Omega_0 = 0$ and

\[
L_{\Pi_0} \gamma \Pi = L_\tau \Pi - \sum_i L_{\tau(c_i)} Z_i \Pi,
\]

hence

\[
0 = -\Pi_0(L_{\Pi_0} \gamma \Omega_0)\Pi_0 + \sum_i L_{\alpha_i^0 Z_i} \Pi_0 + L_\tau \Pi - \sum_i L_{\tau(c_i)} Z_i \Pi + \Pi(L_\tau \Omega_0)\Pi_0 + \Pi_0(L_\tau \Omega_0)\Pi.
\]

Multiplying from left and right by $\Omega_0$ and using (7), after strenuous but straightforward calculations with the application of formulas (20), (23) we arrive at the relation

\[
0 = -d(\Omega_0 \Pi_0 \Omega_0 \tau) + L_\tau(\Omega_0 \Pi \Omega_0) - \sum_i [\Omega_0(L_{Z_i} \Pi)\Omega_0] \tau \wedge dc_i - \sum_i \tau(c_i) \Omega_0(L_{Z_i} \Pi)\Omega_0.
\]

Hence, $\Omega_0 \Pi \Omega_0$ is closed if

\[
\sum_i [\Omega_0(L_{Z_i} \Pi)\Omega_0] \tau \wedge dc_i + \sum_i \tau(c_i) \Omega_0(L_{Z_i} \Pi)\Omega_0 = 0.
\]

As the last equality holds for an arbitrary vector field $\tau$, hence

\[
\Omega_0(L_{Z_i} \Pi)\Omega_0 = 0, \quad i = 1, ..., r.
\]

Definition 10 Let $(\Pi_0, \Omega_0)$ be a dual $P$-$p$ pair and $\Pi$ be a Poisson bivector. We say that $\Pi$ is compatible with the pair $(\Pi_0, \Omega_0)$ if $\Pi$ is compatible with $\Pi_0$ and $\Omega_0$.

Up to now, we have induced a Poisson bracket on $C^\infty(M)$ in various ways using not only Poisson bivectors but also dual pairs and compatible pairs. So, the question is what is the most general way of introducing a Poisson algebra on $C^\infty(M)$.

Definition 11 Assume that $\Pi$ is some bivector and $\Omega$ is a two-form. A pair $(\Pi, \Omega)$ is called a Poisson pair if $\Pi_D = \Pi \Omega \Pi$ is Poisson. Two Poisson pairs $(\Pi_1, \Omega_1)$ and $(\Pi_2, \Omega_2)$ will be called equivalent if $\Pi_1 \Omega_1 \Pi_1 = \Pi_2 \Omega_2 \Pi_2$. 

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Each compatible pair is simultaneously a Poisson pair. For a given Poisson pair \((\Pi, \Omega)\) the bracket
\[
\{F, G\}_\Pi := \Omega(\Pi dF, \Pi dG) = \langle \Omega \Pi dF, \Pi dG \rangle = \langle dF, \Pi \Omega \Pi dG \rangle = (\Pi \Omega \Pi)(dF, dG) = \{F, G\}_\Pi
\]
is a Poisson bracket. Hence, the property of closeness of \(\Omega\) is too strong for the definition of a Poisson algebra.

**Definition 12** Let \(\Pi\) be a bivector with a kernel spanned by exact one-forms. A two-form \(\Omega\) is called weakly presymplectic with respect to \(\Pi\) if it is closed on \(\text{Im} \Pi = TN\), where \(N\) is the foliation given by functions whose differentials span the kernel of \(\Pi\).

Obviously, if \((\Pi, \Omega)\) is a Poisson pair then \(\Omega\) is weakly presymplectic with respect to \(\Pi\). As we will see later, weakly presymplectic forms play an important role in bi-Hamiltonian chains and in the reduction procedure.

### 3 Presymplectic representation of Gel’fand-Zakharevich chains

Let us consider a bi-Poisson manifold \((M, \Pi_0, \Pi_1)\) of \(\dim M = m = 2n + r\) where \(\Pi_0, \Pi_1\) is a pair of compatible Poisson tensors of rank \(2n\). Moreover we assume that the Poisson pencil \(\Pi_\lambda\) admits \(r\), polynomial with respect to the pencil parameter \(\lambda\), Casimir functions of the form
\[
H^{(j)}(\lambda) = \sum_{i=0}^{n_j} H_i^{(j)} \lambda^{n_j-i}, \quad j = 1, ..., r,
\]
(24)
such that \(n_1 + ... + n_r = n\) and \(H_i^{(j)}\) are functionally independent. The collection of \(n\) bi-Hamiltonian vector fields
\[
X_i^{(j)} = \Pi_1 dH_{i-1}^{(j)} - \Pi_0 dH_i^{(j)}, \quad i = 1, ..., n_j, \quad j = 1, ..., r,
\]
(25)
constructed from Casimirs of the pencil
\[
\Pi_\lambda dH_i^{(j)}(\lambda) = 0,
\]
is called the Gel’fand-Zakharevich system of the bi-Poisson manifold \(M\). Notice that each chain starts from a Casimir of \(\Pi_0\) and terminates with a Casimir of \(\Pi_1\). Moreover all \(H_i^{(j)}\) pairwise commute with respect to both Poisson structures
\[
\{X_i^{(j)}(H_i^{(k)}), X_i^{(j)}(H_i^{(l)})\} = \{dH_i^{(k)}, \Pi_0 dH_i^{(j)}\} = \{dH_i^{(k)}, \Pi_1 dH_i^{(j)}\} = 0.
\]
(26)
\[
\Pi_\lambda(dH_i^{(j)}, dH_i^{(k)}) = 0.
\]

#### 3.1 Bi-presymplectic representation of one-Casimir chains

As in this subsection we restrict our considerations to the simplest case of \(r = 1\), i.e. to the one-Casimir case, we will use the following notation for a single bi-Hamiltonian chain
\[
X_i = \Pi_0 dH_i = \Pi_1 dH_{i-1}, \quad i = 0, ..., n + 1.
\]
(26)
The chain starts with a Casimir \(H_0\) of \(\Pi_0\) and terminates with a Casimir \(H_n\) of \(\Pi_1\).

Let \(\Omega_0\) be a dual to \(\Pi_0\) presymplectic form. The kernels of \(\Omega_0\) and \(\Pi_0\) are one dimensional: \(\ker \Omega_0 = Z\), \(\ker \Pi_0 = dH_0\) and
\[
L_Z \Omega_0 = 0, \quad L_Z \Pi_0 = 0.
\]
We assume that $\Omega_0(LZ\Pi_1)\Omega_0 = 0$, i.e. that $\Pi_1$ is compatible with the P-p pair $(\Pi_0, \Omega_0)$, so
\[
L_Z \Pi_1 = [Z, X_1] \wedge Z, \quad X_1 = \Pi_1 dH_0
\]
and
\[
\Omega_{1D} := \Omega_0 \Pi_1 \Omega_0
\]
is also presymplectic with $\ker \Omega_0 \subseteq \ker \Omega_{1D}$.

Next, we construct the following two-form
\[
\Omega_1 = \Omega_{1D} + \Omega_0 X_1 \wedge dH_0 = \Omega_{1D} + dH_1 \wedge dH_0.
\]
It is obviously a presymplectic form. Moreover, $(\Pi_0, \Omega_1)$ is a Poisson pair. Indeed,
\[
\Pi_0 \Omega_1 \Pi_0 = \Pi_0 \Omega_{1D} \Pi_0 + \Pi_0 (dH_1 \wedge dH_0) \Pi_0 = \Pi_0 \Omega_{1D} \Pi_0 = \Pi_{1D} = \Pi_1 - X_1 \wedge Z
\]
which is Poisson according to Theorem 8.

**Lemma 13** Vector fields $Y = X_n + Z(H_n)Z$ belong to $\ker \Omega_1$.

**Proof.**
\[
\Omega_1 Y = (\Omega_{1D} - dH_0 \wedge dH_1)(X_n + Z(H_n)Z) = (\Omega_0 \Pi_1 \Omega_0) X_n - Z(H_n)Z(H_1) dH_0 + Z(H_n) dH_1.
\]
On the other hand, from (11) and the fact that $H_0$ is the only Casimir function of $\Omega_0$
\[
(\Omega_0 \Pi_1 \Omega_0) X_n = \Omega_0 \Pi_1 (dH_n - Z(H_n) dH_0) = -Z(H_n) \Omega_0 X_1 = -Z(H_n) (dH_1 - Z(H_n) dH_0)
\]
\[
= -Z(H_n) dH_1 + Z(H_n) Z(H_1) dH_0.
\]

Now we are prepared to formulate the following theorem:

**Theorem 14** Bi-presymplectic representation of the bi-Poisson chain (26) takes the form
\[
\beta_i = \Omega_0 Y_i = \Omega_1 Y_{i-1}, \quad i = 0, \ldots, n + 1,
\]
where
\[
Y_i = X_i + Z(H_i)Z, \quad \beta_i = dH_i - Z(H_i) dH_0.
\]
The chain starts with a kernel vector field $Y_0 = Z$ of $\Omega_0$ and terminates with a kernel vector field $Y_n = Y = X_n + Z(H_n)Z$ of $\Omega_1$.

**Proof.**
\[
\begin{align*}
\Omega_0 Y_i & = \Omega_0 X_i, \\
\Omega_1 Y_{i-1} & = (\Omega_0 \Pi_1 \Omega_0 - dH_0 \wedge dH_1)(X_{i-1} + Z(H_{i-1})Z) = (\Omega_0 \Pi_1 \Omega_0) X_{i-1} - Z(H_{i-1}) Z(H_1) dH_0 + Z(H_{i-1}) dH_1, \\
(\Omega_0 \Pi_1 \Omega_0) X_{i-1} & = \Omega_0 \Pi_1 (dH_{i-1} - Z(H_{i-1}) dH_0) = \Omega_0 (X_{i-1} - Z(H_{i-1}) X_1) = \Omega_0 X_{i-1} - Z(H_{i-1}) \Omega_0 X_1 = \Omega_0 X_{i-1} - Z(H_{i-1}) dH_1 + Z(H_{i-1}) Z(H_1) dH_0.
\end{align*}
\]

Observe that neither $X_i$ nor $Y_i$ vector fields are inverse Hamiltonian with respect to $\Omega_0$ and $\Omega_1$. Besides $[Y_i, Y_j] \neq 0$. Introducing a presymplectic pencil
\[
\Omega_\lambda = \Omega_1 - \lambda \Omega_0
\]
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related with each other as follows

\[ \Omega(X_i, X_j) = \{ H_i, H_j \}_\Pi = 0, \quad \Omega(X_i, X_j) = \{ H_i, H_j \}_\Pi_{id} = 0. \]

The first bracket is obvious, the second one follows from the relation \( \Pi_{0} \) and the first bracket. Additionally, Poisson tensors \( \Pi_{0} \) take the form \( \Omega \) with a kernel vector field \( Y = \sum_{i=0}^{n} Y_i \lambda^{n-i} \), the bi-presymplectic chain \( (28) \) takes the form \( \Omega \) with \( \ker \Omega \), and the kernels of \( \Omega \) take the form \( \Omega := \Omega(X_i, X_j) = \{ H_i, H_j \}_\Pi_{id} = 0. \)

As a consequence \( (\Pi_{0}, \Omega_{\lambda}) \) is a Poisson pair and \( \Omega_{\lambda}(X_i, X_j) = 0. \)

### 3.2 Weakly bi-presymplectic representation of multi-Casimir chains

In this subsection we will show that bi-presymplectic representation is purely one-Casimir phenomenon. Consider the \( r \)-Casimir Gel’fand-Zakharevich chain \( (24), (25) \). Let \( \Omega_{0} \) be a dual to \( \Pi_{0} \) presymplectic form. The kernels of \( \Omega_{0} \) and \( \Pi_{0} \) are \( r \) dimensional: \( \ker \Omega_{0} = Sp\{ Z_{i}\}_{i=1, \ldots, r} \), \( \ker \Pi_{0} = Sp\{ dH_{0}^{(i)}\}_{i=1, \ldots, r} \), and

\[ L_{Z_i} \Omega_{0} = 0, \quad L_{Z_i} \Pi_{0} = 0, \quad i = 1, \ldots, r. \]

We assume that \( \Omega_{0}(L_{Z_i} \Pi_{1}) \Omega_{0} = 0 \), i.e. that \( \Pi_{1} \) is compatible with the P-p pair \( (\Pi_{0}, \Omega_{0}) \), so from involutivity of \( H_{k}^{(i)} \) the relation \( (16) \) takes the form

\[ L_{Z_i} \Pi_{1} = \sum_{k} [Z_i, X_{1}^{(k)}] \wedge Z_{k}, \quad X_{1}^{(k)} = \Pi_{1} dH_{0}^{(k)} \]

and

\[ \Omega_{1D} := \Omega_{0} \Pi_{1} \Omega_{0} \]

is also presymplectic with \( \ker \Omega_{0} \subseteq \ker \Omega_{1D} \).

Next, we construct the following two-forms

\[ \Omega_{1D} = \Omega_{1D} + \sum_{j=1}^{r} \Omega_{0} X_{1}^{(j)} \wedge dH_{0}^{(j)}, \quad \Omega_{1} = \Omega_{1D} + \sum_{j=1}^{r} dH_{1}^{(j)} \wedge dH_{0}^{(j)}, \]

related with each other as follows

\[ \Omega_{1} = \Omega_{1D} + \frac{1}{2} \sum_{k,l} A_{kl} dH_{0}^{(k)} \wedge dH_{0}^{(l)}, \quad A_{kl} = Z_{k}(H_{1}^{(l)}) - Z_{l}(H_{1}^{(k)}). \]

Obviously \( \Omega_{1} \) is presymplectic and together with \( \Pi_{0} \) forms a Poisson pair as

\[ \Pi_{0} \Omega_{1} \Pi_{0} = \Pi_{0} \Omega_{1D} \Pi_{0} = \Pi_{0} \Omega_{0} \Pi_{1} \Omega_{0} \Pi_{0} = \Pi_{1D} = \Pi_{1} - \sum_{i} X_{1}^{(i)} \wedge Z_{i} \]

is Poisson. It is also clear that \( \Omega_{1D} \) is not closed as

\[ d\Omega_{1D} = - \frac{1}{2} \sum_{k,l} dA_{kl} \wedge dH_{0}^{(k)} \wedge dH_{0}^{(l)}, \]

\[ d\Omega_{1D} = \frac{1}{2} \sum_{k,l} dA_{kl} \wedge dH_{0}^{(k)} \wedge dH_{0}^{(l)}, \]
but is weakly presymplectic with respect to $\Pi_0$

$$d\Pi_1^* (\Pi_0 \alpha_1, \Pi_0 \alpha_2, \Pi_0 \alpha_3) = 0, \quad \forall \alpha_1, \alpha_2, \alpha_3 \in T^* \mathcal{M}.$$  

Moreover, $(\Pi_0, \Pi_1)$ is a Poisson pair equivalent to $(\Pi_0, \Omega_1)$, one as $\Pi_0 \Pi_1 \Pi_0 = \Pi_0 \Omega_1 \Pi_0 = \Pi_{1d}$.

**Theorem 15** Multi-Casimir bi-Poisson chains \([24]\) have weakly bi-presymplectic representation

$$\beta_i^{(j)} = \Omega_0 Y_i^{(j)} = \Pi_1 Y_i^{(j)} - 1, \quad j = 1, \ldots, r, \quad i = 0, \ldots, n_j + 1,$$

where

$$Y_i^{(j)} = X_i^{(j)} + \sum_{k=1}^r Z_k (H_i^{(j)}) Z_k, \quad \beta_i^{(j)} = dH_i^{(j)} - \sum_{k=1}^r Z_k (H_i^{(j)}) dH_0^{(k)}.$$  

The $j$-th chain starts with a kernel vector field $Y_0^{(j)} = \Omega_0$ and terminates with a kernel vector field $Y_{n_j}^{(j)} = X_{n_j}^{(j)} + \sum_{k=1}^m Z_k (H_{n_j}^{(j)}) Z_k$ of $\Pi_1$.

**Proof.** We have

$$\Omega_0 Y_i^{(j)} = \Omega_0 X_i^{(j)}.$$  

On the other hand

$$\Pi_1 Y_i^{(j)} = (\Omega_0 \Pi_1 \Omega_0 + \sum_l \Omega_0 X_1^{(l)} \land dH_0^{(l)}) (X_i^{(j)} + \sum_k Z_k (H_i^{(j)}) Z_k)$$

$$\quad = \Omega_0 \Pi_1 \Omega_0 X_i^{(j)} + (\sum_l \Omega_0 X_1^{(l)} \land dH_0^{(l)}) X_i^{(j)}$$

$$\quad + \sum_{l,k} Z_k (H_i^{(j)}) (\Omega_0 X_1^{(l)} \land dH_0^{(l)}) Z_k.$$  

Using decomposition \([21]\) and bi-Hamiltonian chains \([24]\), one finds

$$\Omega_0 \Pi_1 \Omega_0 X_i^{(j)} = \Omega_0 X_i^{(j)} - \sum_k Z_k (H_i^{(j)}) dH_1^{(k)} + \sum_{l,k} Z_k (H_i^{(j)}) Z_l (H_1^{(k)}) dH_0^{(l)},$$

$$\sum_{l,k} Z_k (H_i^{(j)}) (\Omega_0 X_1^{(l)} \land dH_0^{(l)}) Z_k = \sum_k Z_k (H_i^{(j)}) dH_1^{(k)} - \sum_{j,k} Z_k (H_i^{(j)}) Z_l (H_1^{(k)}) dH_0^{(l)},$$

$$(\sum_l \Omega_0 X_1^{(l)} \land dH_0^{(l)}) X_i^{(j)} = - \sum_l \Omega_0 (X_1^{(l)}, X_i^{(j)}) dH_0^{(l)} = 0.$$  

The last equality follows from the fact that $\Omega_0 (X_1^{(l)}, X_i^{(j)}) = \Omega_0 (dH_1^{(l)}, dH_i^{(j)}) = 0$. Hence

$$\Pi_1 Y_i^{(j)} = \Omega_0 X_i^{(j)}.$$  

Introducing a weakly presymplectic pencil

$$\Pi_0 \lambda = \Pi_1 - \lambda \Omega_0$$

with respect to $\Pi_0$, with a kernel vector fields

$$Y^{(j)} = \sum_{i=0}^{n_j} Y_i^{(j)} \lambda^{n_j-i}, \quad j = 1, \ldots, r,$$

the weakly bi-presymplectic chains \([24]\) take the form $\Pi_0 Y^{(j)} = 0$. On the other hand, as we mentioned before, the pairs $(\Pi_0, \Omega_0)$ and $(\Pi_0, \Pi_1)$ are Poisson pairs, hence $\Omega_0$ and $\Pi_1$ define Poisson brackets. The first one is equal to that given by $\Pi_0$ while the second one is equal to that given by $\Pi_{1d}$. Moreover,

$$\Omega_0 (X_1^{(k)}, X_2^{(j)}) = \{H_1^{(k)}, H_2^{(l)}\} \Pi_0 = 0, \quad \Omega_1 (X_1^{(k)}, X_2^{(j)}) = \{H_1^{(k)}, H_2^{(l)}\} \Pi_{1d} = 0.$$  

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The first bracket is obvious, the second one follows from the relation
\[ \Omega_1 X_i^{(k)} = (\Omega_{1D} + \sum_r dH_{1}^{(r)} \wedge dH_{0}^{(r)}) X_i^{(k)} = \Omega_0 \Pi_1 \Omega_0 X_i^{(k)} \]
\[ = \Omega_0 \Pi_1 (dH_{1}^{(k)} - \sum_r Z_r (H_{1}^{(k)}) dH_{0}^{(r)}) \]
\[ = \Omega_0 X_i^{(k)} - \sum_r Z_r (H_{1}^{(k)}) \Omega_0 X_1^{(r)} \]
and the first bracket. Additionally, Poisson tensors \( \Pi_0 \) and \( \Pi_{1d} \) are compatible as
\[ [\Pi_{1d}, \Pi_0]_S = [\Pi_1 - \sum_i X_i^{(i)} \wedge Z_i, \Pi_0]_S \]
\[ = \sum_i (X_i^{(i)} \wedge [Z_i, \Pi_0]_S - Z_i \wedge [X_i^{(i)}, \Pi_0]_S) \]
\[ = \sum_i (X_i^{(i)} \wedge L_{Z_i} \Pi_0 - Z_i \wedge L_{X_i^{(i)}} \Pi_0) \]
\[ = 0. \]
As a consequence, \((\Pi_0, \Omega_\Lambda)\) is a Poisson pair and
\[ \Omega_\Lambda (X_i, X_j) = 0. \]

Now, let us consider the presymplectic pencil
\[ \Omega_\Lambda = \Omega_1 - \lambda \Omega_0. \]
As \((\Pi_0, \Omega_1)\) is a Poisson pair equivalent to the Poisson pair \((\Pi_0, \overline{\Pi}_1)\), then
\[ \Omega_\Lambda (X_i, X_j) = 0. \]
Moreover, chains \((31)\) take the form
\[ \beta_i^{(j)} = \Omega_0 Y_i^{(j)} = \Omega_1 Y_i^{(j)} - \sum_k B_{i-1, k}^{(j)} dH_{0}^{(k)}, \quad B_{i,k}^{(j)} = \sum_l A_{kl} Z_l (H_{1}^{(j)}), \]
where \( j = 1, ..., r, \quad i = 0, ..., n_j + 1. \)

4 Reduction procedure for Gel’fand-Zakharevich chains

Let us consider a \((2n + r)\)-dimensional manifold \( \mathcal{M} \) and \( 2n \)-dimensional submanifold \( \mathcal{N} \) of \( \mathcal{M} \). Then, let us fix an integrable distribution \( \mathcal{Z} \) of constant dimension \( r \) that is transversal to \( \mathcal{N} \). As mentioned in Section 2, such a case is realized by an appropriate dual P-p pair defined on \( \mathcal{M} \). Indeed, let \((\Pi_0, \Omega_0)\) be a dual P-p pair on \( \mathcal{M} \) with \( \ker \Omega_0 = \mathcal{Z} = Sp \{ Z_i \} \) and \( \ker \Pi_0 = \mathcal{Z}^* = Sp \{ dc_i \}, \ i = 1, ..., r \) where obviously \( Z_i (c_j) = \delta_{ij} \) and \( [Z_i, Z_j] = 0 \). Then, \( \mathcal{N} \) is a fixed symplectic leave of \( \Pi \) and \( \mathcal{Z} \) consists of vector fields from \( \ker \Omega_0 \) evaluated on \( \mathcal{N} \). An appropriate decomposition of tangent and cotangent bundle of \( \mathcal{M} \) is given by \((31)\).

**Definition 16** A function \( F : \mathcal{M} \to \mathbb{R} \) is called invariant with respect to distribution \( \mathcal{Z} \) if
\[ L_{Z_i} F = Z_i (F) = 0, \quad \forall Z_i \in \mathcal{Z}. \]
The set of such functions will be denoted by \( \mathcal{A} \).

**Definition 17** The Poisson tensor \( \Pi \) is called invariant with respect to the distribution \( \mathcal{Z} \) if functions that are invariant along \( \mathcal{Z} \) form a Poisson subalgebra with respect to \( \Pi \), that is
\[ L_{Z_i} \Pi (dF, dG) = 0, \quad Z_i (F) = Z_i (G) = 0. \]
We will denote this subalgebra by \( \mathcal{A}(\Pi) \).
Notice that $\Pi_0$ is in obvious way $\mathcal{Z}$-invariant as $L_\mathcal{Z} \Pi_0 = 0$, hence $\mathcal{A}(\Pi_0)$ is also a Poisson subalgebra.

**Lemma 18** If Poisson bivector $\Pi$ is compatible with a presymplectic form $\Omega_0$, then it is invariant with respect to the distribution $\mathcal{Z} = \ker \Omega_0$.

**Proof.** Assume $Z_i(F) = Z_i(G) = 0$ for all $i$. We have to show that condition (31) is fulfilled. But due to Theorem 8 it follows that

$$L_\mathcal{Z}_i \Pi(dF, dG) = <dF, (L_\mathcal{Z}_i \Pi)dG> = < dF, \left( \sum_i [Z_i, X_i] \wedge Z_i - \frac{1}{2} \sum_{i,j} Z_i(c_{ij}) Z_i \wedge Z_j \right) dG >$$

$$= \sum_i (Z_i(G)[Z_i, X_i](F) - Z_i(F)[Z_i, X_i](G))$$

$$- \frac{1}{2} \sum_{i,j} Z_i(c_{ij}) [Z_i(G)Z_i(F) - Z_j(F)Z_i(G)]$$

$$= 0.$$

The invariance of Poisson tensors given in the form (14) was proved for the first time by Vaisman [15]. As a consequence we conclude that an arbitrary Poisson bivector $\Pi_0$ compatible with a dual $P$-$p$ pair $(\Pi_0, \Omega_0)$, is reducible onto foliation given by Casimirs of $\Pi_0$ along the distribution given by $\ker \Omega_0$. Here we propose a simple constructive method of deriving the reduced operator.

**Lemma 19** Let $\Pi$ be a Poisson bivector compatible with a dual $P$-$p$ pair $(\Pi_0, \Omega_0)$ and $\pi$ a reduction of $\Pi$ onto a symplectic leaf $\mathcal{N}_\nu$ of $\Pi_0$ along the transversal distribution $\mathcal{Z} = \ker \Omega_0$. Then, $\pi$ can be constructed by a restriction of

$$\Pi_d = \Pi_0 \Omega_0 \Pi_0 \Pi_0 = \Pi - \sum_i X_i \wedge Z_i + \frac{1}{2} \sum_{i,j} c_{ij} Z_i \wedge Z_j$$

$$\pi = \Pi_d |_{\mathcal{N}_\nu}.$$  

**Proof.** From the relation (14) and the fact that for $F, G \in \mathcal{A}$

$$< dF, (- \sum_i X_i \wedge Z_i + \frac{1}{2} \sum_{i,j} c_{ij} Z_i \wedge Z_j) dG > \geq 0,$$

the Poisson operator $\Pi$ and its deformation $\Pi_d$ both act in the same way on the set $\mathcal{A}$, so that both can be used to define the same reduced operator $\pi$ on $\mathcal{N}_\nu$. But as the image of $\Pi_d$ is tangent to $\mathcal{N}_\nu$, what follows from the fact that ker $\Pi_0 \subset$ ker $\Pi_d$, and $\Pi_d$ is Poisson, then the projection of $\Pi_d$ onto $\mathcal{N}_\nu$ means simple its restriction to $\mathcal{N}_\nu$. Obviously, if ker $\Pi_d = \ker \Pi_0$, then (32) means the restriction of $\Pi_d$ to its symplectic leaf $\mathcal{N}_\nu$.  

Now we pass to the reduction of bi-Hamiltonian chains in Poisson [30] and presymplectic [33] representations onto symplectic foliation of $\Pi_0$. Let us denote the projections of $\Pi_0, \Pi_1$ onto $\mathcal{N}$ along $\mathcal{Z}$ by $\pi_0, \pi_1$ and restrictions of $(H_k^{(1)}, ..., H_k^{(n)})|_{\mathcal{N}}$ to $\mathcal{N}$ by $(h_i^{(1)}, ..., h_i^{(n)})$.

**Proposition 20** The bi-Poisson chain [(30)], when reduced to $\mathcal{N}$ takes the form

$$\pi_1 dh_i^{(j)} = \pi_0 dh_i^{(j)} - \sum_{k=1}^r \alpha_{ki}^{(j)} \pi_0 dh_i^{(k)}, \quad j = 1, ..., r, \quad i = 1, ..., n_j,$$  

where $\alpha_{ki}^{(j)} = Z_k (H_i^{(j)})|_{\mathcal{N}}$. 

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Proof.

\[ \pi_1 dH_i^{(j)} = \Pi_{1d} |_{N} dH_i^{(j)} |_{N} = (\Pi_{1d} dH_i^{(j)}) |_{N} \]

\[ = (\Pi_{1d} dH_i^{(j)}) |_{N} - \sum_{k=1}^{r} \left( Z_k(H_i^{(j)})X_1^{(k)} \right) |_{N} \]

\[ = (\Pi_0 dH_{i+1}^{(j)}) |_{N} - \sum_{k=1}^{r} \left( Z_k(H_i^{(j)}) \Pi_0 dH_1^{(k)} \right) |_{N} \]

\[ = \Pi_0 |_{N} dH_{i+1}^{(j)} |_{N} - \sum_{k=1}^{r} Z_k(H_i^{(j)}) |_{N} \Pi_0 |_{N} dH_1^{(k)} |_{N} \]

\[ = \pi_0 dH_{i+1}^{(j)} - \sum_{k=1}^{r} Z_k(H_i^{(j)}) |_{N} \pi_0 dH_1^{(k)}. \]

The second and fifth equalities are valid as in coordinates

\[ (x^i, H_0^{(j)}), \quad i = 1, ..., 2n, \quad j = 1, ..., r \]

on \( M \), the last \( r \) rows and columns of \( \Pi_0 \) and \( \Pi_{1d} \) contain zeros only. Obviously we have

\[ \pi_0 (dh_i^{(j)}, dh_k^{(l)}) = \pi_1 (dh_i^{(j)}, dh_k^{(l)}) = 0, \]

which follows from the construction of \( \pi_0 \) and \( \pi_1 \). ■

Before we pass to the reduction of presymplectic representation, observe that as \( (\Pi_0, \Omega_0), (\Pi_1, \Omega_1) \) and \( (\Pi_0, \Omega_1) \) are Poisson pairs, then their restrictions to \( N \) are closed: \( \Omega_0 |_{N} = \omega_0 = \pi^{-1}, \Omega_1 |_{N} = \Omega_1 |_{N} = \omega_1 \). Moreover, \( \pi_0 dH_i^{(j)} := K_i^{(j)} = X_i^{(j)} |_{N} \), where \( |_{N} \) means as usually a restriction, as

\[ X_i^{(j)} |_{N} = (\Pi_0 dH_i^{(j)}) |_{N} = \Pi_0 |_{N} dH_i^{(j)} |_{N} = \pi_0 dH_i^{(j)}. \]

Proposition 21 When reduced to \( N \), the weakly bi-presymplectic chain takes the form

\[ \omega_1 K_i^{(j)} = \omega_0 K_i^{(j)} = - \sum_{k} a_{ki}^{(j)} \omega_0 K_k^{(j)}, \quad j = 1, ..., r, \quad i = 1, ..., n. \]

Proof.

\[ \omega_1 K_i^{(j)} = \Pi_1 |_{N} X_i^{(j)} |_{N} = (\Pi_1 X_i^{(j)}) |_{N} = (\Pi_1 (Y_i^{(j)} - \sum_{k} Z_k(H_i^{(j)})Z_k)) |_{N} \]

\[ = (\Pi_1 Y_i^{(j)}) |_{N} - \sum_{k} \left( Z_k(H_i^{(j)})Y_1^{(k)} \right) |_{N} \]

\[ = (\Omega_0 Y_i^{(j)}) |_{N} - \sum_{k} Z_k(H_i^{(j)}) \Omega_0 Y_1^{(k)} |_{N} \]

\[ = (\Omega_0 X_i^{(j)}) |_{N} - \sum_{k} \left( Z_k(H_i^{(j)}) \Omega_0 X_1^{(k)} \right) |_{N} \]

\[ = \Omega_0 |_{N} X_i^{(j)} |_{N} - \sum_{k} Z_k(H_i^{(j)}) |_{N} \Omega_0 |_{N} X_1^{(k)} |_{N} \]

\[ = \omega_0 K_i^{(j)} - \sum_{k} a_{ki}^{(j)} \omega_0 K_k^{(j)}. \]

The second and seventh equality are valid as in the coordinates vector fields \( X_i^{(j)} \) have the last \( r \) components equal to zero. ■

Notice that

\[ \omega_1 = \Pi_1 |_{N} = \Omega_1 |_{N} = (\Omega_0 \Pi_1 \Omega_0) |_{N} = (\Omega_0 \Pi_{1d} \Omega_0) |_{N} \]

\[ = \Omega_0 |_{N} \Pi_{1d} |_{N} \Omega_0 |_{N} = \omega_0 \pi_1 \omega_0. \]
As \( \omega_1 \) is closed then \((\pi_1, \omega_0)\) is a compatible pair and \( N = \pi_1 \omega_0 \) is a recursion operator. Moreover \( \pi_1 = N \pi_0 \) hence \( \pi_0 \) and \( \pi_1 \) are compatible. Now we immediately find that reduced chains \((\mathfrak{d}_0 \pi_1, \omega_0)\) and \((\mathfrak{d}_0 \pi_0, \omega_0)\) are equivalent. As \( K_i^{(j)} = \pi_0 h_i^{(j)} \), hence \( K_i^{(j)} \) takes the form

\[
N^* d h_i^{(j)} = d h_{i+1}^{(j)} + \sum_k \alpha_k^{i} d h_k^{(j)}, \quad j = 1, \ldots, r, \quad i = 1, \ldots, n_j,
\]

where \( N^* = \omega_0 \pi_1 \) is a recursion operator for one-forms. On the other hand, multiplying \( \mathfrak{d}_0 \pi_1 \) from left by \( \omega_0 \) we arrive at \((\mathfrak{d}_0 \pi_0, \omega_0)\) again. Moreover,

\[
\omega_0 (K_i^{(j)}, K_i^{(r)}) = \pi_0 (h_i^{(j)}, d h_i^{(l)}) = 0, \quad \omega_1 (K_i^{(j)}, K_i^{(r)}) = \pi_1 (h_i^{(j)}, h_i^{(l)}) = 0.
\]

As a consequence, the distribution tangent to the foliation of \( N \) defined by \((h_1^{(1)}, \ldots, h_n^{(r)})\) is bi-Lagrangian and the \( n \)-tuple \((h_1^{(1)}, \ldots, h_n^{(r)})\) of functionally independent Hamiltonians is separable [10]. Separated coordinates are eigenvalues of the recursion operator \( N \) and canonically conjugated momenta that put the recursion operator in the diagonal form.

We conclude this section with a statement, that the existence of weakly bi-presymplectic representation of bi-Poisson chains is a sufficient condition for the separability of related Hamiltonian systems.

## 5 Example

Let us illustrate our previous considerations with a simple nontrivial example of the integrable case of the Henon-Heiles equations

\[
(q^1)_{tt} = -3(q^1)^2 - \frac{1}{2}(q^2)^2 + c, \quad (q^2)_{tt} = -q^1 q^2.
\]

The system \((37)\) can be put into a canonical Hamiltonian form with the Hamiltonian function given by

\[
H_1 = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + (q^1)^3 + \frac{1}{2} (q^2)^2 - c q^1,
\]

where \( p_1 = q_1^1, p_2 = q_2^2 \). The second constant of motion is

\[
H_2 = \frac{1}{2} p_1^2 p_2 - \frac{1}{2} (q^1)^4 + \frac{1}{16} (q^2)^4 - \frac{1}{4} c (q^2)^2.
\]

The bi-Hamiltonian chain on \( \mathcal{M} = Sp(q^1, q^2, p_1, p_2, c) \) is of the following form

\[
\Pi_0 d H_0 = 0, \quad \Pi_0 d H_1 = X_1 = \Pi_1 d H_0, \quad \Pi_0 d H_2 = X_2 = \Pi_1 d H_1, \quad 0 = \Pi_1 d H_2,
\]

where \( H_0 = c \) and the compatible Poisson bivectors are

\[
\Pi_0 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \Pi_1 = \begin{pmatrix}
0 & 0 & q^1 & q^2 & p_1 \\
0 & 0 & \frac{1}{2} q^2 & 0 & p_2 \\
-\frac{1}{2} q^2 & 0 & 0 & \frac{1}{2} p_2 & -3(q^1)^2 - \frac{1}{4} (q^2)^2 + c \\
-\frac{1}{2} q^2 & 0 & -\frac{1}{2} p_2 & 0 & -q^1 q^2 \\
-p_1 & -p_2 & 3(q^1)^2 + \frac{1}{2} (q^2)^2 - c & q^1 q^2 & 0
\end{pmatrix}.
\]
Now, dual to the canonical Poisson tensor $\Pi_0$ is a canonical presymplectic form

$$\Omega_0 = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

with a kernel vector

$$Z = (0, 0, 0, 0, 1)^T.$$  

As evidently $\Omega_0(L_2(\Pi_1)\Omega_0 = 0$, then $\Pi_1$ is compatible with the pair $(\Pi_0, \Omega_0)$, so the second presymplectic form is

$$\Omega_1 = \Omega_{1D} + dH_1 \wedge dH_0$$

$$= \begin{pmatrix}
0 & -\frac{1}{2}p_2 & -q^1 & -\frac{1}{4}q^2 & 3(q^1)^2 + \frac{1}{4}(q^2)^2 - c \\
\frac{1}{2}p_2 & 0 & -\frac{1}{2}q^2 & 0 & q^1q^2 \\
q^1 & \frac{1}{2}q^2 & 0 & 0 & p_1 \\
0 & 0 & 0 & p_2 & 0 \\
-3(q^1)^2 - \frac{1}{2}(q^2)^2 + c & -q^1q^2 & -p_1 & -p_2 & 0
\end{pmatrix} .$$

Hence, the bi-presymplectic representation of the Henon-Heiles chain takes the form

$$\Omega_0Y_0 = 0$$

$$\Omega_0Y_1 = \beta_1 = \Omega_1Y_0$$

$$\Omega_0Y_2 = \beta_2 = \Omega_1Y_1$$

$$0 = \Omega_1Y_2$$

where vector fields $Y_i$ are

$$Y_0 = Z = (0, 0, 0, 0, 1)^T$$

$$Y_1 = X_1 + Z(H_1)Z = (p_1, p_2, -3(q^1)^2 - \frac{1}{2}(q^2)^2 + c, -q^1q^2, -q_1^1)^T$$

$$Y_2 = X_1 + Z(H_2)Z = \frac{1}{2}q^2p_1, \frac{1}{2}q^2p_1 - q^1p_1, \frac{1}{2}q^2p_1 - \frac{1}{2}q^1(q^2)^2,$$

$$-\frac{1}{2}p_1p_2 - \frac{1}{4}(q^2)^3 - \frac{1}{2}(q^1)^2q^2 + \frac{1}{2}cq^2, -\frac{1}{4}(q^2)^3)^T .$$

The chain starts with a kernel vector field $Y_0$ of $\Omega_0$ and terminates with a kernel vector field $Y_2$ of $\Omega_1$. The restriction of $\Pi_0$, $\Pi_{1d}, \Omega_0$ and $\Omega_1$ to $\mathcal{N} = Sp(q^1, q^2, p_1, p_2)$ are

$$\pi_0 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix} , \quad \omega_0 = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} ,$$

$$\pi_1 = \begin{pmatrix}
0 & 0 & q^1 & \frac{1}{2}q^2 \\
0 & 0 & \frac{1}{2}q^2 & 0 \\
-q^1 & -\frac{1}{2}q^2 & 0 & \frac{1}{2}p_2 \\
-\frac{1}{2}q^2 & 0 & -\frac{1}{2}q^2 & 0
\end{pmatrix} , \quad \omega_1 = \begin{pmatrix}
0 & -\frac{1}{2}p_2 & -q^1 & -\frac{1}{4}q^2 \\
\frac{1}{2}p_2 & 0 & -\frac{1}{2}q^2 & 0 \\
q^1 & \frac{1}{2}q^2 & 0 & 0 \\
\frac{1}{2}q^2 & 0 & 0 & 0
\end{pmatrix} ,$$

with the recursion operator $N$ of the form

$$N = \pi_1\omega_0 = \begin{pmatrix}
q^1 & \frac{1}{2}q^2 & 0 & 0 \\
\frac{1}{2}q^2 & 0 & 0 & 0 \\
0 & \frac{1}{2}p_2 & q^1 & \frac{1}{4}q^2 \\
-\frac{1}{2}p_2 & 0 & \frac{1}{2}q^2 & 0
\end{pmatrix}$$

and $N^* = N^T$.  

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