CHERN CLASSES OF RANK TWO GLOBALLY GENERATED VECTOR BUNDLES ON $\mathbb{P}^2$.

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ABSTRACT. We determine the Chern classes of globally generated rank two vector bundles on $\mathbb{P}^2$.

INTRODUCTION.

Vector bundles generated by global sections come up in a variety of problems in projective algebraic geometry. In this paper we consider the following question: which are the possible Chern classes of rank two globally generated vector bundles on $\mathbb{P}^2$? (Here $\mathbb{P}^2 = \mathbb{P}^2_k$ with $k$ algebraically closed, of characteristic zero.)

Clearly these Chern classes have to be positive. Naively one may think that this the only restriction. A closer inspection shows that this is not true: since we are on $\mathbb{P}^2$, the construction of rank two vector bundles starting from codimension two, locally complete intersection subschemes is subjected to the Cayley-Bacharach condition (see Section 2). So if we have an exact sequence $0 \to \mathcal{O} \to F \to \mathcal{I}_Y(c) \to 0$, with $F$ a rank two vector bundle and $Y \subset \mathbb{P}^2$ of codimension two, then $Y$ satisfies Cayley-Bacharach for $c - 3$.

Now $F$ is globally generated if and only if $\mathcal{I}_Y(c)$ is. If $Y$ is contained in a smooth curve, $T$, of degree $d$, we have $0 \to \mathcal{O}(-d + c) \to \mathcal{I}_Y(c) \to \mathcal{I}_{Y,T}(c) \to 0$ and we see that, if $c \geq d$, $\mathcal{I}_Y(c)$ is globally generated if and only if the line bundle $\mathcal{L} = \mathcal{I}_{Y,T}(c)$ on $T$ is globally generated. But there are gaps in the degrees of globally generated line bundles on a smooth plane curve of degree $d$ (it is classically known that no such bundle exists if $d \geq 3$ and $1 \leq \deg \mathcal{L} \leq d - 2$). A remarkable theorem due to Greco-Raciti and Coppens ([3], [1] and Section 3) gives the exact list of gaps.

This is another obstruction, at least if $Y$ lies on a smooth curve, $T$, of low degree with respect with $c = c_1(F)$ (in this case $F$ tends to be not stable). The problem

2010 Mathematics Subject Classification. 14F99, 14J99.

Key words and phrases. Rank two vector bundles, globally generated, projective plane.
then is to have such a curve for every vector bundle with fixed Chern classes and then, to treat the case where $T$ is not smooth. The first problem is solved in the necessarily unstable range ($\Delta(F) = c_1^2 - 4c_2 > 0$) (see Section 3). In the stable range there are no obstructions, this was already known to Le Potier (see [7]). For the second problem we use the following remark: if a line bundle $O_T(Z)$ on a smooth plane curve of degree $d$ is globally generated, then $Z$ satisfies Cayley-Bacharach for $d - 3$. Working with the minimal section of $F$ we are able to have a similar statement even if $T$ is singular (see [4]). Finally with a slight modification of Theorem 3.1 in [3] we are able to show the existence of gaps.

To state our result we need some notations. Let $c > 0$ be an integer. Let’s say that $(c, y)$ is effective if there exists a globally generated rank two vector bundle on $\mathbb{P}^2$, $F$, with $c_1(F) = c, c_2(F) = y$. It is easy to see (cf Section 1) that it must be $0 \leq y \leq c^2$ and that $(c, y)$ is effective if and only if $(c, c^2 - y)$ is. So we may assume $y \leq c^2/2$. For every integer $t$, $2 \leq t \leq c/2$, let $G_t(0) = [c(t - 1) + 1, t(c - t) - 1]$ (we use the convention that if $b < a$, then $[a, b] = \emptyset$). For every integer $t$, $4 \leq t \leq c/2$, denote by $t_0$ the integral part of $\sqrt{t - 3}$, then for every integer $a$ such that $1 \leq a \leq t_0$ define $G_t(a) = [(t - 1)(c - a) + a^2 + 1, (t - 1)(c - a + 1) - 1]$. Finally let

$$G_t = \bigcup_{a=0}^{t_0} G_t(a) \text{ and } G = \bigcup_{t=2}^{c/2} G_t.$$ 

Then we have:

**Theorem 0.1.** Let $c > 0$ be an integer. There exists a globally generated rank two vector bundle on $\mathbb{P}^2$ with Chern classes $c_1 = c, c_2 = y$ if and only if one of the following occurs:

1. $y = 0$ or $c - 1 \leq y \leq c^2/2$ and $y \notin G$
2. $y = c^2$ or $c^2/2 < y \leq c^2 - c + 1$ and $c^2 - y \notin G$.

Although quite awful to state, this result is quite natural (see Section 3). As a by-product we get (Section 4) all the possible “bi-degrees” for generically injective morphisms from $\mathbb{P}^2$ to the Grassmannian $G(1, 3)$ (or more generally to a Grassmannian of lines). We hope to come back on this topic in the future.

**Acknowledgment:** I thank L. Gruson for pointing this problem to my attention.
1. **General facts and a result of Le Potier for stable bundles.**

Let $F$ be a rank two globally generated vector bundle on $\mathbb{P}^2$ with Chern classes $c_1(F) =: c$, $c_2(F) =: y$. Since the restriction $F_L$ to a line is globally generated, we get $c \geq 0$. A general section of $F$ yields:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_Y(c) \to 0$$

where $Y \subset \mathbb{P}^2$ is a smooth set of $y$ distinct points (cf [3], 1.4) or is empty. In the first case $y > 0$, in the second case $F \simeq \mathcal{O} \oplus \mathcal{O}(c)$ and $y = 0$. In any case the Chern classes of a globally generated rank two vector bundle are positive.

Also observe ($Y \neq \emptyset$) that $\mathcal{I}_Y(c)$ is globally generated (in fact $F$ globally generated $\iff \mathcal{I}_Y(c)$ is globally generated). This implies by Bertini’s theorem that a general curve of degree $c$ containing $Y$ is smooth (hence irreducible).

Since $rk(F) + \dim(\mathbb{P}^2) = 4$, $F$ can be generated by $V \subset H^0(F)$ with $\dim V = 4$ and we get:

$$0 \to E^* \to V \otimes \mathcal{O} \to F \to 0$$

It follows that $E$ is a rank two, globally generated vector bundle with Chern classes: $c_1(E) = c$, $c_2(E) = c^2 - y$. We will say that $E$ is the *G-dual bundle* of $F$. Since a globally generated rank two bundle has positive Chern classes we get: $0 \leq y \leq c^2$, $c \geq 0$.

**Definition 1.1.** We will say that $(c, y)$ is effective if there exist a globally generated rank two vector bundle on $\mathbb{P}^2$ with $c_1 = c$ and $c_2 = y$. A non effective $(c, y)$ will also be called a gap.

**Remark 1.2.** By considering G-dual bundles we see that $(c, y)$ is effective if and only if $(c, c^2 - y)$ is effective. Hence it is enough to consider the range $0 \leq y \leq c^2/2$.

If $c = 0$, then $F \simeq 2\mathcal{O}$ and $y = 0$.

If $y = c^2$ then $c_2(E) = 0$, hence $E \simeq \mathcal{O} \oplus \mathcal{O}(c)$ and:

$$0 \to \mathcal{O}(-c) \to 3\mathcal{O} \to F \to 0$$

Such bundles exists for any $c \geq 0$. If $y = 0$, $F \simeq \mathcal{O} \oplus \mathcal{O}(c)$.
Definition 1.3. If $F$ is a rank two vector bundles on $\mathbb{P}^2$ we denote by $F_{\text{norm}}$ the unique twist of $F$ such that $-1 \leq c_1(F_{\text{norm}}) \leq 0$. The bundle $F$ is stable if $h^0(F_{\text{norm}}) = 0$.

By a result of Schwarzenberger, if $F$ is stable with $c_1(F) = c, c_2(F) = y$, then $\Delta(F) := c^2 - 4y < 0$ (and $\Delta(F) \neq -4$). Moreover there exist a stable rank two vector bundle with Chern classes $(c, y)$ if and only if $\Delta := c^2 - 4y < 0, \Delta \neq -4$.

Concerning stable bundles we have the following result of Le Potier [7]:

Proposition 1.4 (Le Potier). Let $\mathcal{M}(c_1, c_2)$ denote the moduli space of stable rank two bundles with Chern classes $c_1, c_2$ on $\mathbb{P}^2$. There exists a non empty open subset of $\mathcal{M}(c_1, c_2)$ corresponding to globally generated bundles if and only if one of the following holds:

1. $c_1 > 0$ and $\chi(c_1, c_2) \geq 4$ ($\chi(c_1, c_2) = 2 + \frac{c_1(c_1+3)}{2} - c_2$)
2. $(c_1, c_2) = (1, 1)$ or $(2, 4)$.

Using this proposition we get:

Corollary 1.5. If $c > 0$ and

$$\frac{c^2}{4} \leq y \leq \frac{3c^2}{4}$$

then $(c, y)$ is effective.

Proof. The existence condition ($\Delta < 0, \Delta \neq -4$) translates as: $y > c^2/4$, $y \neq c^2/4 + 1$. Condition (1) of [1,4] gives: $\frac{c(c+3)}{2} - 2 \geq y$, hence if $\frac{c(c+3)}{2} - 2 \geq y > \frac{c^2}{4}$ and $y \neq \frac{c^2}{4} + 1$, $(c, y)$ is effective.

Let’s show that $(c, \frac{c^2}{4})$ is effective for every $c \geq 2$ (c even). Consider:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_Y(2) \to 0$$

where $Y$ is one point. Then $F$ is globally generated with Chern classes $(2, 1)$. For every $m \geq 0$, $F(m)$ is globally generated with $c_1^2 = 4c_2$.

In the same way let’s show that $(c, \frac{c^2}{4} + 1)$ is effective for every $c \geq 2$ (c even). This time consider:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_Y(2) \to 0$$
where $Y$ is a set of two points; $F$ is globally generated with Chern classes $(2, 2)$. For every $m \geq 0$, $F(m)$ is globally generated with the desired Chern classes.

We conclude that if $\frac{c(c+3)}{2} - 2 \geq y \geq \frac{c^2}{4}$, then $(c, y)$ is effective. By duality, $(c, y)$ is effective if $\frac{3c^2}{4} \geq y \geq \frac{c(c-3)}{2} + 2$. Putting every thing together we get the result. □

Remark 1.6. Since $3c^2/4 > c^2/2$, we may, by duality, concentrate on the range $y < c^2/4$, i.e. on not stable bundles with $\Delta > 0$, that’s what we are going to do in the next section.

2. Cayley-Bacharach.

Definition 2.1. Let $Y \subset \mathbb{P}^2$ be a locally complete intersection (l.c.i.) zero-dimensional subscheme. Let $n \geq 1$ be an integer. We say that $Y$ satisfies Cayley-Bacharach for curves of degree $n$ ($CB(n)$), if any curve of degree $n$ containing a subscheme $Y' \subset Y$ of colength one (i.e. of degree $\deg Y - 1$), contains $Y$.

Remark 2.2. Since $Y$ is l.c.i. for any $p \in \text{Supp}(Y)$ there exists a unique subscheme $Y' \subset Y$ of colength one (locally) linked to $p$ in $Y$. So Def. 2.1 makes sense even if $Y$ is non reduced.

Let’s recall the following ([4]):

Proposition 2.3. Let $Y \subset \mathbb{P}^2$ be a zero-dimensional l.c.i. subscheme. There exists an exact sequence:

$$0 \to \mathcal{O} \to F \to I_Y(c) \to 0$$

with $F$ a rank two vector bundle if and only if $Y$ satisfies $CB(c - 3)$.

Actually in [4] Proposition 2.3 is proved only under the assumption that $Y$ is reduced, but it is well known that the proof works in the general case. The proposition gives conditions on the Chern classes of bundles having a section, in our case:

Lemma 2.4. Let $F$ be a globally generated rank two vector bundle on $\mathbb{P}^2$ with $c_1(F) = c$, $c_2(F) = y$, then:

$$c - 1 \leq y \leq c^2 - c + 1 \text{ or } y = c^2 \text{ or } y = 0$$
Proof. Since $F$ is globally generated a general section vanishes in codimension two or doesn’t vanish at all. In the second case $F \simeq 2\mathcal{O}$ and $y = 0$. Let’s assume, from now on, that a general section vanishes in codimension two. We have an exact sequence:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_Y(c) \to 0$$

where $Y$ is a zero-dimensional subscheme (we may assume $Y$ smooth) which satisfies Cayley-Bacharach condition for $c - 3$.

If $c - 3 \geq y - 1$, $\forall p \in Y$ there exists a curve of degree $c - 3$ containing $Y_p := Y \setminus \{p\}$ and not containing $Y$ (consider a suitable union of lines). Since $Y$ must satisfy the Cayley-Bacharach condition, it must be $y \geq c - 1$.

Let $F$ be a globally generated rank two vector bundle with $c_1(F) = c$, $c_2(F) = y$. Consider the $G$-dual bundle:

$$0 \to F^* \to 4\mathcal{O} \to E \to 0$$

then $E$ is a rank two, globally generated, vector bundle with $c_1(E) = c$, $c_2(E) = c^2 - y$. By the previous part: $c_2(E) = 0$ (i.e. $y = c^2$) or $c^2 - y = c_2(E) \geq c_1(E) - 1 = c - 1$. So $c^2 - c + 1 \geq y$. □

Remark 2.5. It is easy to check that for $0 \leq c \leq 3$, every value of $y$, $c - 1 \leq y \leq c^2 - c + 1$ is effective (take $Y \subset \mathbb{P}^2$ of maximal rank with $c - 1 \leq y \leq c^2/2$ and use Castelnuovo-Mumford’s lemma to show that $\mathcal{I}_Y(c)$ is globally generated). In fact gaps occur only for $c \geq 6$. In the sequel we will assume that $c \geq 4$.

3. The statement.

From now on we may restrict our attention to the range: $c - 1 \leq y < c^2/4$ ([1.6, 2.4]) for $c \geq 4$ (2.3). In this range $\Delta(F) = c^2 - 4y > 0$, hence $F$ is necessarily unstable (i.e. not semi-stable). In particular, if $c$ is even: $h^0(F(-\frac{c}{2})) = h^0(\mathcal{I}_Y(\frac{c}{2})) \neq 0$ (resp. $h^0(F(-\frac{c+1}{2})) = h^0(\mathcal{I}_Y(\frac{c+1}{2})) \neq 0$, if $c$ is odd). So $Y$ is forced to lie on a curve of relatively low degree. In fact something more precise can be said, for this we need the following elementary remark:

Lemma 3.1 (The trick). Let $F$ be a rank two vector bundle on $\mathbb{P}^2$ with $h^0(F) \neq 0$. If $c_2(F) < 0$, then $h^0(F(-1)) \neq 0$. 

Proof. A non-zero section of $F$ cannot vanish in codimension two (we would have $c_2 > 0$), nor can the section be nowhere non-zero ($F$ would split as $F \simeq \mathcal{O} \oplus \mathcal{O}(c)$, hence $c_2(F) = 0$). It follows that any section vanishes along a divisor. By dividing by the equation of this divisor we get $h^0(F(-1)) \neq 0$.

Actually this works also on $\mathbb{P}^n$, $n \geq 2$.

For $2 \leq t \leq c/2$ ($c \geq 4$) we define:

$$\overline{A}_t := [(t-1)(c-t+1), \ t(c-t)] = [(t-1)c - (t-1)^2, \ (t-1)c - (t^2 - c)]$$

The ranges $\overline{A}_t$ cover $[c-1, \ c^2/4]$, the interval we are interested in. From our point of view we may concentrate on the interior points of $\overline{A}_t$. Indeed if $y = ab$, with $a + b = c$, we may take $F \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$. So we define:

$$A_t = [(t-1)(c-t+1), \ t(c-t)], \ \ 2 \leq t \leq c/2$$

**Lemma 3.2.** If $y \in A_t$, and if $Y$ is the zero-locus of a section of $F$, a rank two vector bundle with Chern classes $(c, y)$, then $h^0(I_Y(t-1)) \neq 0$.

Proof. We have an exact sequence $0 \to \mathcal{O} \to F \to I_Y(c) \to 0$. Now $c_2(F(-c-t)) = (-c + t)t + y$. By our assumptions, $y < t(c-t)$, hence $c_2(F(-c + t)) < 0$. We have $c - t \geq c/2$, by repeated use of 3.1 we conclude that $h^0(F(-c + t - 1)) = h^0(I_Y(t-1)) \neq 0$.

So if $y \in A_t$, $Y$ is forced to lie on a degree $(t-1)$ curve (but not on a curve of degree $t-2$). If general principles are respected we may think that if $y \in A_t$, $Y \subset T$, where $T$ is a smooth curve of degree $t - 1$ and that $h^0(I_Y(t-2)) = 0$. If this is the case we have an exact sequence:

$$0 \to \mathcal{O}(-t+1) \to I_Y \to I_{Y,T} \to 0$$

twisting by $\mathcal{O}_T(c)$:

$$0 \to \mathcal{O}(c-t+1) \to I_Y(c) \to \mathcal{O}_T(c-Y) \to 0$$

Since $c - t + 1 > 0$ (because $c \geq 2t$), we see that: $I_Y(c)$ is globally generated if and only if $\mathcal{O}_T(c - Y)$ is globally generated. The line bundle $\mathcal{L} = \mathcal{O}_T(c - Y)$ has degree $l := c(t-1) - y$. So the question is: for which $l$ does there exists a degree $l$ line bundle on $T$ generated by global sections? This is, by its own, a quite natural
problem which, strangely enough, has been solved only recently (\cite{3, 4}). First a definition:

**Definition 3.3.** Let $C$ be a smooth irreducible curve. The Lüroth semi-group of $C$, $LS(C)$, is the semi-group of nonnegative integers which are degrees of rational functions on $C$. In other words: $LS(C) = \{ n \in \mathbb{N} \mid \exists \mathcal{L}, \text{of degree } n, \text{such that } \mathcal{L} \text{ is globally generated} \}.$

Then we have:

**Theorem 3.4 (Greco-Raciti-Coppens).** If $C$ is a smooth plane curve of degree $d \geq 3$, then

$$LS(C) = LS(d) := \mathbb{N} \setminus \bigcup_{a=1}^{n_0} [(a-1)d+1, a(d-a)-1]$$

where $n_0$ is the integral part of $\sqrt{d-2}$.

Of course $LS(1) = LS(2) = \mathbb{N}$. We observe that $LS(C)$ doesn’t depend on $C$ but only on its degree.

Going back to our problem we see that if $c(t-1) - y \notin LS(t-1)$, then $\mathcal{L} = \mathcal{O}_T(c-Y)$ can’t be globally generated and the same happens to $\mathcal{I}_{Y}(c)$. In conclusion if $c(t-1) - y \in \bigcup_{a=1}^{n_0} [(a-1)(t-1)+1, a(t-1-a)-1]$, or if $c(t-1) - y < 0$, under our assumptions, $(c, y)$ is not effective. The assumption is that the unique curve of degree $t-1$ containing $Y$ is smooth. (Observe that $\deg \mathcal{O}_T(t-1-Y) < 0$, hence $h^0(\mathcal{I}_{Y}(t-1)) = 1$.)

Our theorem says that general principles are indeed respected. In order to have a more manageable statement let’s introduce some notations:

**Definition 3.5.** Fix an integer $c \geq 4$. An integer $y \in A_t$ for some $2 \leq t \leq c/2$, will be said to be admissible if $c(t-1) - y \in LS(t-1)$. If $c(t-1) - y \notin LS(t-1)$, $y$ will be said to be non-admissible.

Observe that $y \in A_t$ is non-admissible if and only if: $y \in G_t(0) = [c(t-1) + 1, t(c-1)-1]$ (this corresponds to $c(t-1) - y < 0$), or $y \in G_t(a) = [(t-1)(c-a) + a^2 + 1, (t-1)(c-a+1)-1]$ for some $a \geq 1$ such that $a^2 + 2 \leq t-1$ (i.e. $a \leq t_0$).
In order to prove Theorem 0.1 it remains to show:

**Theorem 3.6.** For any \( c \geq 4 \) and for any \( y \in A_t \) for some \( 2 \leq t \leq c/2 \), \((c, y)\) is effective if and only if \( y \) is admissible.

The proof splits into two parts:

(1) **(Gaps)** If \( c(t - 1) - y \notin LS(t - 1) \), one has to prove that \((c, y)\) is not effective.

This is clear if \( Y \) lies on a smooth curve, \( T \), of degree \( t - 1 \), but there is no reason for this to be true and the problem is when \( T \) is singular.

(2) **(Existence)** If \( c(t - 1) - y \in LS(t - 1) \), one knows that there exists \( L \) globally generated, of degree \( c(t - 1) - y \) on a smooth curve, \( T \), of degree \( t - 1 \).

The problem is to find such an \( L \) such that \( M := \mathcal{O}_T(c) \otimes L^* \) has a section vanishing along a \( Y \) satisfying the Cayley-Bacharach condition for \((c - 3)\).

4. **The proof (gaps).**

In this section we fix an integer \( c \geq 4 \) and prove that non-admissible \( y \in A_t \), \( 2 \leq t \leq c/2 \) are gaps. For this we will assume that such a \( y \) is effective and will derive a contradiction. From \( 3.2 \) we know that \( h^0(\mathcal{I}_Y(t - 1)) \neq 0 \). The first task is to show that under our assumption (\( y \) not-admissible), \( h^0(\mathcal{I}_Y(t - 2)) = 0 \) (see \( 4.3 \)); this will imply that \( F(-c + t - 1) \) has a section vanishing in codimension two.

To begin with let’s observe that non-admissible \( y \in A_t \) may occur only when \( t \) is small with respect to \( c \).

**Lemma 4.1.** Assume \( c \geq 4 \). If \( t > \frac{2\sqrt{3}}{3}\sqrt{c - 2} \), then every \( y \in A_t \) is admissible.

**Proof.** Recall (see \( 3.3 \)) that \( y \in A_t \), \( 2 \leq t \leq c/2 \), is non admissible if and only if \( y \in G_t(a) \) for some \( a, 0 \leq a \leq t_0 \).

We have \( G_t(0) \neq \emptyset \) \( \iff \) \( t(c - t) - 1 \geq c(t - 1) + 1 \) \( \iff \) \( t \leq \sqrt{c - 2} \).

For \( a \geq 1 \), \( G_t(a) \cap A_t \neq \emptyset \) \( \Rightarrow \) \( (t - 1)(c - a) + a^2 + 1 < t(c - t) \). This is equivalent to: \( a^2 - at + t^2 - c + a + 1 < 0 \) (*). The discriminant of this equation in \( a \) is \( \Delta = -3t^2 + 4(c - a - 1) \) and we must have \( \Delta \geq 0 \), i.e. \( \frac{2\sqrt{3}}{3}\sqrt{c - 2} \geq t \).

Let’s get rid of the \( y' \)s in \( G_t(0) \):
Lemma 4.2. If $y \in A_t$ is non-admissible and effective, then $y \in G_t(a)$ for some $a$, $1 \leq a \leq \sqrt{t - 3}$.

Proof. We have to show that if $c(t - 1) - y < 0$ and $y \in A_t$, then $y$ is not effective. By 3.2 $h^0(I_Y(t - 1)) \neq 0$. If $y$ is effective then $I_Y(c)$ is globally generated and $Y$ is contained in a complete intersection of type $(t - 1, c)$, hence $\deg Y = y \leq c(t - 1)$: contradiction.

Now we show that if $y$ is non-admissible and effective, then $h^0(I_Y(t - 1)) = 1$:

Lemma 4.3. Let $c \geq 4$ and assume $y \in A_t$ for some $t$, $2 \leq t \leq c/2$. Assume furthermore that $y$ is non-admissible and effective i.e.:

$$y = (t - 1)(c - a) + \alpha, \quad a^2 + 1 \leq \alpha \leq t - 2$$

for a given $a$ such that $t - 1 \geq a^2 + 2$. Under these assumptions, $h^0(I_Y(t - 1)) = 1$.

Proof. If $h^0(I_Y(t - 2)) \neq 0$, then $y \leq c(t - 2)$ (the general $F_c \in H^0(I_Y(c))$ is integral since $I_Y(c)$ is globally generated. Moreover $t - 1 < c$ so $F_c \neq T$). It follows that:

$$y = (t - 1)(c - a) + \alpha \leq c(t - 2) = c(t - 1) - c$$

This yields $a(t - 1) \geq c + \alpha$. We have $c + \alpha \geq c + a^2 + 1$, hence:

$$0 \geq a^2 - a(t - 1) + c + 1 \quad (*)$$

The discriminant of $(*)$ (viewed as an equation in $a$) is: $\Delta = (t - 1)^2 - 4(c + 1)$. If $\Delta < 0$, $(*)$ is never satisfied and $h^0(I_Y(t - 2)) = 0$. Now $\Delta < 0 \iff (t - 1)^2 < 4(c + 1)$.

In our context $\Delta < 0 \iff t < 1 + 2\sqrt{c + 1}$. In conclusion if $t < 1 + 2\sqrt{c + 1}$ and if $y$ is non-admissible, then $h^0(I_Y(t - 2)) = 0$.

Now by 1.1 if $y$ is non-admissible, we have: $t \leq \frac{2\sqrt{3}}{3}\sqrt{c - 2}$. Since $\frac{2\sqrt{3}}{3}\sqrt{c - 2} < 1 + 2\sqrt{c + 1}$, for $c > 0$, we are done.

Since $h^0(I_Y(t - 1)) \neq 0$, $F(-c + t - 1)$ has a non-zero section, since $h^0(I_Y(t - 2)) = 0$ the section vanishes in codimension two. Hence we have:

$$0 \rightarrow O \rightarrow F(-c + t - 1) \rightarrow I_W(-c + 2t - 2) \rightarrow 0$$

where $\deg W = y - (t - 1)(c - t + 1)$. Since $-c + 2t - 2 < 0$ (because $c \geq 2t$), we get $h^0(F(-c + t - 1)) = 1 = h^0(I_Y(t - 1))$. \qed
Notations 4.4. Let $F$ be a globally generated rank two vector bundle with Chern classes $(c, y)$. A section $s \in H^0(F)$ defines $Y_s = (s)_0$. If $y \in A_t$, $h^0(\mathcal{I}_{Y_s}(t-1)) \neq 0$, moreover if $y$ is non-admissible $h^0(\mathcal{I}_{Y_s}(t-1)) = 1$ and there is a unique $T_s \in H^0(\mathcal{I}_{Y_s}(t-1))$. It follows that $F(-c + t - 1)$ has a unique section (hence vanishing in codimension two): $0 \to \mathcal{O} \to F(-c + t - 1) \to \mathcal{I}_W(-c + 2t - 2) \to 0$.

Lemma 4.5. If $y \in A_t$ is non-admissible and effective, with notations as in 4.4:

1. $Y_s$ and $W$ are bilinked on $T_s$
2. The curves $T_s$ are precisely the elements of $H^0(\mathcal{I}_W(t-1))$
3. $\mathcal{I}_W(t-1)$ is globally generated, in particular for $s \in H^0(F)$ general, $T_s$ is reduced.

Proof.

(1) (2) We have a commutative diagram:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\mathcal{O} & = & \mathcal{O} \\
\downarrow u & & \downarrow T_s \\
0 & \to & \mathcal{O}(-c + t - 1) \xrightarrow{s} F(-c + t - 1) \to \mathcal{I}_Y(t-1) \to 0 \\
\| & & \downarrow \\
0 & \to & \mathcal{O}(-c + t - 1) \xrightarrow{s} \mathcal{I}_W(-c + 2t - 2) \to \mathcal{I}_{Y,T_s}(t-1) \to 0 \\
\downarrow & & \downarrow \\
0 & = & 0 \\
\end{array}
\]

We see that $s$ corresponds to an element of $H^0(\mathcal{I}_W(t-1))$ and the quotient $\mathcal{O} \xrightarrow{s} \mathcal{I}_W(t-1)$ has support on $T_s$ and is isomorphic to $\mathcal{L}^*(-c + 2t - 2)$ where $\mathcal{L}^* \simeq \mathcal{I}_{W,T_s}$; finally $Y_s$ is a section of $\mathcal{L}(c - t + 1)$ which shows that $W$ and $Y_s$ are bilinked on $T_s$.

(3) The exact sequence $0 \to \mathcal{O}(c-t+1) \to F \to \mathcal{I}_W(t-1) \to 0$ shows that $\mathcal{I}_W(t-1)$ is globally generated, hence the general element in $H^0(\mathcal{I}_W(t-1))$ is reduced. □

Since $W$ could well be non-reduced with embedding dimension two, concerning $T$, this is the best we can hope. However, and this is the point, we may reverse the construction and start from $W$. 
Lemma 4.6. Let $W \subset \mathbb{P}^2$ be a zero-dimensional, locally complete intersection (l.c.i.) subscheme. Assume $\mathcal{I}_W(n)$ is globally generated, then if $T, T' \in H^0(\mathcal{I}_W(n))$ are sufficiently general, the complete intersection $T \cap T'$ links $W$ to a smooth subscheme $Z$ such that $W \cap Z = \emptyset$.

Proof. If $p \in \text{Supp}(W)$, denote by $W_p$ the subscheme of $W$ supported at $p$. Since $W$ is l.c.i, $\mathcal{I}_{W,p} = (f, g) \subset \mathcal{O}_p$. By assumption the map $H^0(\mathcal{I}_W(n)) \otimes \mathcal{O}_p \xrightarrow{ev} \mathcal{I}_{W,p}$ which takes $T \in H^0(\mathcal{I}_W(n))$ to its germ, $T_p$, at $p$, is surjective. Hence there exists $T$ such that $T_p = f$ (resp. $T'$ such that $T'_p = g$). It follows that in a neighborhood of $p$: $T \cap T' = W_p$. If $G$ is the Grassmannian of lines of $H^0(\mathcal{I}_W(n))$ for $(T,T') \in G$ the property $T \cap T' = W_p$ (in a neighborhood of $p$) is open (it means that the local degree at $p$ of $T \cap T'$ is minimum). We conclude that there exists a dense open subset, $U_p \subset G$, such that for $(T,T') \in U_p$, $T \cap T' = W_p$ (locally at $p$). If $\text{Supp}(W) = \{p_1, ..., p_r\}$ there exists a dense open subset $U \subset U_1 \cap ... \cap U_r$ such that if $(T,T') \in U$, then $T \cap T'$ links $W$ to $Z$ and $W \cap Z = \emptyset$.

By Bertini’s theorem the general curve $T \in H^0(\mathcal{I}_W(n))$ is smooth out of $W$. If $C \subset T$ is an irreducible component, the curves of $H^0(\mathcal{I}_W(n))$ cut on $C$, residually to $W \cap C$, a base point free linear system. By the previous part the general member, $Z_C$, of this linear system doesn’t meet $\text{Sing}(C)$ (because $Z_C \cap W = \emptyset$), it follows, by Bertini’s theorem, that $Z_C$ is smooth. So for general $T, T' \in H^0(\mathcal{I}_W(n))$, $T \cap T'$ links $W$ to a smooth subscheme, $Z$, such that $W \cap Z = \emptyset$.

Corollary 4.7. Let $y \in A_1$ be non-admissible. If $y$ is effective, with notations as in [4.6], if $T, T' \in H^0(\mathcal{I}_W(t-1))$ are sufficiently general, then $T \cap T'$ links $W$ to a smooth subscheme, $Z$, such that $W \cap Z = \emptyset$. Furthermore $\mathcal{I}_Z(c)$ is globally generated and if $S_c \in H^0(\mathcal{I}_Z(c))$ is sufficiently general, then $T \cap S_c$ links $Z$ to a smooth subscheme $Y$, where $Y$ is the zero locus of a section of $F$ and where $Z \cap Y = \emptyset$.

Proof. The first statement follows from [4.6]. From the exact sequence

$$0 \to \mathcal{O}(c-2t+2) \to F(-t+1) \to \mathcal{I}_W \to 0$$

we get by mapping cone:

$$0 \to F^*(-t+1) \to \mathcal{O}(-c) \oplus 2\mathcal{O}(-t+1) \to \mathcal{I}_Z \to 0 \quad (*)$$
which shows that $\mathcal{I}_Z(c)$ is globally generated. Since $Z$ is smooth and contained in the smooth locus of $T$ and since $\mathcal{I}_Z(c)$ is globally generated, if $C$ is an irreducible component of $T$, the curves of $H^0(\mathcal{I}_Z(c))$ cut on $C$, residually to $C \cap Z$, a base point free linear system. In particular the general member, $D$, of this linear system doesn’t meet $\text{Sing}(C)$. By Bertini’s theorem we may assume $D$ smooth. It follows that if $S_c \in H^0(\mathcal{I}_Z(c))$ is sufficiently general, $S_c \cap T$ links $Z$ to a smooth $Y$ such that $Z \cap Y = \emptyset$. By mapping cone, we see from (*) that $Y$ is the zero-locus of a section of $F$. \hfill \Box

The previous lemmas will allow us to apply the following (classical, I think) result:

**Lemma 4.8.** Let $Y, Z \subset \mathbb{P}^2$ be two zero-dimensional subschemes linked by a complete intersection, $X$, of type $(a, b)$. Assume:

1. $Y \cap Z = \emptyset$
2. $\mathcal{I}_Y(a)$ globally generated.

Then $Z$ satisfies Cayley-Bacharach for $(b - 3)$.

**Proof.** Notice that $Z$ and $Y$ are l.c.i. Now let $P$ be a curve of degree $b - 3$ containing $Z' \subset Z$ of colength one. We have to show that $P$ contains $Z$. Since $\mathcal{I}_Y(a)$ is globally generated and since $Y \cap Z = \emptyset$, there exists $F \in H^0(\mathcal{I}_Y(a))$ not passing through $p$. Now $PF$ is a degree $a + b - 3$ curve containing $X \setminus \{p\}$. Since complete intersections $(a, b)$ verify Cayley-Bacharach for $a + b - 3$ (the bundle $\mathcal{O}(a) \oplus \mathcal{O}(b)$ exists!), $PF$ passes through $p$. This implies that $P$ contains $Z$. \hfill \Box

Gathering everything together:

**Corollary 4.9.** Let $y \in A_t$ be non-admissible. If $y$ is effective, then there exists a smooth zero-dimensional subscheme $Z$ such that:

1. $Z$ lies on a pencil $\langle T, T' \rangle$ of curves of degree $t - 1$, the base locus of this pencil is zero-dimensional.
2. $\deg Z = c(t - 1) - y$
3. $Z$ satisfies Cayley-Bacharach for $t - 4$
Proof. By 4.7 there is a zero-locus of a section of $F$ which is linked by a complete intersection of type $(c, t-1)$ to a $Z$ such that $Y \cap Z = \emptyset$. Since $\mathcal{I}_Y(c)$ is globally generated, by 4.8, $Z$ satisfies Cayley-Bacharach for $t-4$. □

Now we conclude with:

**Proposition 4.10.** Let $Z \subset \mathbb{P}^2$ be a smooth zero-dimensional subscheme contained in a curve of degree $d$. Let $a \geq 1$ be an integer such that $d \geq a^2 + 2$. Assume $h^0(\mathcal{I}_Z(a-1)) = 0$. If $(a-1)d + 1 \leq \deg Z \leq a(d-a) - 1$, then $Z$ doesn’t verify Cayley-Bacharach for $d-3$.

**Remark 4.11.** This proposition is Theorem 3.1 in [3] with a slight modification: we make no assumption on the degree $d$ curve (which can be singular, even non reduced), but we assume $h^0(\mathcal{I}_Z(a-1)) = 0$ (which follows from Bezout if the degree $d$ curve is integral).

Since this proposition is a key point, and for convenience of the reader, we will prove it. We insist on the fact that the proof given is essentially the proof of Theorem 3.1 in [3].

**Notations 4.12.** We recall that if $Z \subset \mathbb{P}^2$, the numerical character of $Z$, $\chi = (n_0, ..., n_{\sigma-1})$ is a sequence of integers which encodes the Hilbert function of $Z$ (see [3]):

1. $n_0 \geq ... \geq n_{\sigma-1} \geq \sigma$ where $\sigma$ is the minimal degree of a curve containing $Z$
2. $h^1(\mathcal{I}_Z(n)) = \sum_{i=0}^{\sigma-1} [n_i - n - 1]_+ - [i - n - 1]_+ ([x]_+ = \max\{0, x\})$.
3. In particular $\deg Z = \sum_{i=0}^{\sigma-1} (n_i - i)$.

The numerical character is said to be connected if $n_i \leq n_{i+1} + 1$, for all $0 \leq i < \sigma - 1$. For those more comfortable with the Hilbert function, $H(Z, -)$ and its first difference function, $\Delta(Z, i) = H(Z, i) - H(Z, i-1)$, we recall that $\Delta(i) = i + 1$ for $i < \sigma$ while $\Delta(i) = \#\{l \mid n_l \geq i + 1\}$. It follows that the condition $n_{r-1} > n_r + 1$ is equivalent to $\Delta(n_r + 1) = \Delta(n_r)$. Also recall that for $0 \leq i < \sigma$, $n_i = \min\{t \geq i \mid \Delta(t) \leq i\}$. 
Lemma 4.13. Let $Z \subset \mathbb{P}^2$ be a smooth zero-dimensional subscheme. Let $\chi = (n_0, \ldots, n_{s-1})$ be the numerical character of $Z$. If $n_{r-1} > n_r + 1$, then $Z$ doesn’t verify Cayley-Bacharach for every $i \geq n_r - 1$.

Proof. It is enough to show that $Z$ doesn’t verify $CB(n_r - 1)$. By [2] there exists a curve, $R$, of degree $r$ such that $R \cap Z = E'$ where $\chi(E') = (n_0, \ldots, n_{r-1})$. Moreover if $E''$ is the residual of $Z$ with respect to the divisor $R$, $\chi(E'') = (m_0, \ldots, m_{s-1-r})$, with $m_i = n_{r+i} - r$. It follows that $h^1(I_{E''}(n_r - r - 1)) = 0$. This implies that given $X \subset E''$ of colength one, there exists a curve, $P$, of degree $n_r - r - 1$ passing through $X$ but not containing $E''$. The curve $RP$ has degree $n_r - 1$, passes through $Z' := E' \cup X$ but doesn’t contain $Z$ (because $R \cap Z = E'$).

Proof of Proposition 4.10.

Observe that the assumptions imply $d \geq 3$, moreover if $d = 3$, then $a = \deg Z = 1$ and the statement is clear; so we may assume $d \geq 4$.

Assume to the contrary that $Z$ satisfies $CB(d-3)$. This implies $h^1(I_Z(d-3)) \neq 0$. If $a = 1$, then $\deg Z \leq d - 2$ and necessarily $h^1(I_Z(d-3)) = 0$, so we may assume $a \geq 2$. Now if $h^1(I_Z(d-3)) \neq 0$, then $n_0 \geq d - 1$, where $\chi(Z) = (n_0, \ldots, n_{s-1})$ is the numerical character of $Z$. Since $\sigma \geq a$, $n_{a-1} \in \chi(Z)$.

We claim that $n_{a-1} < d - 2$. Indeed otherwise $n_0 \geq d - 1$ and $n_0 \geq \ldots \geq n_{a-1} \geq d - 2$ implies

$$\deg Z = \sum_{i=0}^{a-1} (n_i - i) \geq \sum_{i=0}^{a-1} (n_i - i) \geq 1 + \sum_{i=0}^{a-1} (d - 2 - i) = 1 + a(d - 2) - \frac{a(a - 1)}{2}$$

If $a \geq 1$, then $1 + a(d - 2) - \frac{a(a - 1)}{2} > a(d - a) - 1 \geq \deg Z$: contradiction.

Let’s show that $n_{a-1} \geq d - a$. Assume to the contrary $n_{a-1} < d - a$. Then there exists $k, 1 \leq k \leq a - 1$ such that $n_k \leq d - 2$ and $n_{k-1} \geq d - 1$ (indeed $n_0 \geq d - 1$ and $n_{a-1} < d - a \leq d - 2$). If $n_{k-1} \geq n_k \geq \ldots \geq n_{a-1}$ is connected, then $n_{k-1} < d - a + r$ where $a = k + r$. Hence $d - a + r > n_{k-1} \geq d - 1$, which implies $r \geq a$ which is impossible since $k \geq 1$. It follows that there is a gap in $n_{k-1} \geq n_k \geq \ldots \geq n_{a-1}$, i.e. there exists $r, k \leq r \leq a - 1$, such that $n_{r-1} > n_r + 1$. Since $d - 2 \geq n_k \geq n_r$, we conclude by Lemma 4.13 that $Z$ doesn’t satisfy $CB(d - 3)$: contradiction.

So far we have $d - a \leq n_{a-1} < d - 2$ and $n_0 \geq d - 1$. Set $n_{a-1} = d - a + r$ ($r \geq 0$). We claim that there exists $k$ such that $n_k \geq d - 1$ and $n_k \geq \ldots \geq n_{a-1} = d - a + r$.
is connected. Since \( n_0 \geq d - 1 \), this follows from 4.13, otherwise \( Z \) doesn’t verify \( CB(d - 3) \).

We have \( \chi(Z) = (n_0, ..., n_k, ..., n_{a-1}, ..., n_{\sigma-1}) \) with \( n_k \geq d - 1 \), \( n_{a-1} = d - a + r \). Since \((n_k, ..., n_{a-1})\) is connected and \( n_k \geq d - 1 \), we have \( n_i \geq d - 1 + k - i \) for \( k \leq i \leq a - 1 \). Since \( n_{a-1} = d - a + r \geq d - 1 + k - (a - 1) \), we get \( r \geq k \). It follows that:

\[
\deg Z = \sum_{i=0}^{\sigma-1} (n_i - i) = \sum_{i=0}^{k-1} (n_i - i) + \sum_{i=k}^{a-1} (n_i - i) + \sum_{i=a} (n_i - i)
\]

\[
\geq \sum_{i=0}^{k-1} (d - 1 - i) + \sum_{i=k}^{a-1} (d - 1 - 2i + k) + \sum_{i=a} (n_i - i)
\]

\[
\geq \sum_{i=0}^{k-1} (d - 1 - i) + \sum_{i=k}^{a-1} (d - 1 - 2i + k) = (+)
\]

We have:

\[
\sum_{i=k}^{a-1} (d - 1 - 2i + k) = (a - k)(d - a) \quad (*)
\]

If \( k = 0 \), we get \( \deg Z \geq a(d - a) \), a contradiction since \( \deg Z \leq a(d - a) - 1 \) by assumption. Assume \( k > 0 \). Then:

\[
\sum_{i=0}^{k-1} (d - 1 - i) = k(d - 1) - \frac{k(k - 1)}{2} = k(d - 1 - \frac{(k - 1)}{2})
\]

From (+) and (*) we get:

\[
\deg Z \geq (a - k)(d - a) + k(d - 1 - \frac{(k - 1)}{2}) = a(d - a) + k(a - 1 - \frac{(k - 1)}{2})
\]

and to conclude it is enough to check that \( a - 1 \geq (k - 1)/2 \). Since \( r \geq k \), this will follow from \( a - 1 \geq (r - 1)/2 \). If \( a < (r + 1)/2 \), then \( n_{a-1} = d - a + r > d + a - 1 \geq d \), in contradiction with \( n_{a-1} < d - 2 \). The proof is over.

We can now conclude and get the “gaps part” of 3.6:

**Corollary 4.14.** For \( c \geq 4 \) let \( y \in A_t \) for some \( t \), \( 2 \leq t \leq c/2 \). If \( y \) is non admissible, then \( y \) is a gap (i.e. \((c, y)\) is not effective).
Proof. Since $y$ is non-admissible, $y \in G_*(a)$ for some $a \geq 1$ (see 4.2), or equivalently $\deg Z = c(t - 1) - y \in [(a - 1)(t - 1) + 1, a(t - 1 - a) - 1]$ for some $a \geq 1$ such that $a^2 + 1 \leq t - 1$. In view of 4.3 it is enough to show that $Z$ cannot verify Cayley-Bacharach for $t - 4$. For this we want to apply 1.10. The only thing we have to show is $h^0(I_Z(a - 1)) = 0$. Let $P$ be a curve of degree $\sigma < a$ containing $Z$. If $P$ doesn’t have a common component with some curve of $H^0(I_Z(t - 1))$, then $\deg Z \leq \sigma(t - 1) \leq (a - 1)(t - 1)$. But this is impossible since $\deg Z \geq (a - 1)(t - 1) + 1$. On the other hand $Z$ is contained in a pencil $\langle T, T' \rangle$ of curves of degree $t - 1$ and this pencil has a base locus of dimension zero (see 4.9). So we may always find a curve in $H^0(I_Z(t - 1))$ having no common component with $P$. \hfill \Box

5. The proof (existence).

In this section we assume that $y \in A_t$ is admissible and prove that $y$ is indeed effective. Since $y$ is admissible we know by 1.11 that there exists a smooth plane curve, $T$, of degree $t - 1$ and a globally generated line bundle, $\mathcal{L}$, on $T$ of degree $z := c(t - 1) - y$.

**Lemma 5.1.** Assume $y \in A_t$ is admissible. If $T$ is a smooth plane curve of degree $t - 1$ and if $\mathcal{L}$ is a globally generated line bundle on $T$ with $\deg \mathcal{L} = c(t - 1) - y$, then $\mathcal{L}^*(c)$ is non special and globally generated.

*Proof.* We have $\deg \mathcal{L}^*(c) = y$. It is enough to check that $y \geq 2gt + 1 = (t - 2)(t - 3) + 1$. We have $y \geq (t - 1)(c - t + 1) + 1$. Since $c \geq 2t$ it follows that $y \geq (t - 1)(t + 1) + 1 = t^2$. \hfill \Box

**Lemma 5.2.** Assume $y \in A_t$ is admissible. If there exists a smooth plane curve, $T$, of degree $t - 1$, carrying a globally generated line bundle, $\mathcal{L}$, with $\deg \mathcal{L} = c(t - 1) - y$ and with $h^1(\mathcal{L}) \neq 0$, then $y$ is effective.

*Proof.* Let $Z$ be a section of $\mathcal{L}$. If $h^1(\mathcal{L}) = h^0(\mathcal{L}^*(t - 4)) \neq 0$, then $Z$ lies on a curve, $R$, of degree $t - 4$. Set $X = T \cap R$. By 5.1 $\mathcal{L}^*(c)$ is globally generated, so we may find a $s \in H^0(\mathcal{L}^*(c))$ such that $(s)_0 \cap X = \emptyset$. Set $Y = (s)_0$. We have $\mathcal{O}_T(c) \simeq \mathcal{O}_T(Z + Y)$ and $Y \cap Z = \emptyset$. So $Y$ and $Z$ are linked by a complete intersection $I = F \cap T$. Let’s prove that $Y$ satisfies $CB(c - 3)$. First observe that there exists a degree
t − 1 curve, $T'$, containing $Z$ such that $T' \cap Y = \emptyset$: indeed since $Y \cap X = \emptyset$, we just take $T' = R \cup C$ where $C$ is a suitable cubic. Now let $p \in Y$ and let $P$ be a degree $c − 3$ curve containing $Y' = Y \setminus \{p\}$. The curve $T'P$ contains $I \setminus \{p\}$ and has degree $c + t − 4$. Since the complete intersection $I$ satisfies $CB(c + t − 4)$ and since $T' \cap Y = \emptyset$, $p \in P$.

It follows that we have: $0 \to \mathcal{O} \to F \to \mathcal{I}_Y(c) \to 0$ where $F$ is a rank two vector bundle with Chern classes $(c, y)$. Since $\mathcal{I}_{Y,T}(c)$ is globally generated, $\mathcal{I}_Y(c)$ and therefore $F$ are globally generated. □

We need a lemma:

**Lemma 5.3.** For any integer $r$, $1 \leq r \leq h^0(\mathcal{O}(t − 1)) − 3$, there exists a smooth zero-dimensional subscheme, $R$, of degree $r$ such that $\mathcal{I}_R(t − 1)$ is globally generated with $h^0(\mathcal{I}_R(t − 1)) \geq 3$.

**Proof.** Take $R$ of degree $r$, of maximal rank. If $h^0(\mathcal{O}(t − 2)) \geq r$, then $h^1(\mathcal{I}_R(t − 2)) = 0$ and we conclude by Castelnuovo-Mumford’s lemma. Assume $h^0(\mathcal{O}(t − 2)) < r$ and take $R$ of maximal rank and minimally generated (i.e. all the maps $\sigma(m) : H^0(\mathcal{I}_R(m)) \otimes H^0(\mathcal{O}(1)) \to H^0(\mathcal{I}_R(m + 1))$ are of maximal rank). If $\sigma(t − 1)$ is surjective we are done, otherwise it is injective and the minimal free resolution looks like:

$$0 \to d.\mathcal{O}(-t − 1) \to b.\mathcal{O}(-t) \oplus a.\mathcal{O}(-t + 1) \to \mathcal{I}_R \to 0$$

By assumption $a \geq 3$.

Since $\text{Hom}(d − \mathcal{O}(-t − 1), b.\mathcal{O}(-t) \oplus a.\mathcal{O}(-t + 1))$ is globally generated, if $\varphi \in \text{Hom}(d.\mathcal{O}(-t − 1), b.\mathcal{O}(-t) \oplus a.\mathcal{O}(-t + 1))$ is sufficiently general, then $\text{Coker}(\varphi) \simeq \mathcal{I}_R$ with $R$ smooth of codimension two. Furthermore since $b.\mathcal{O}(1)$ is globally generated, it can be generated by $b + 2$ sections; it follows that the general morphism $f : d.\mathcal{O} \to b.\mathcal{O}(1)$ is surjective ($d = a + b − 1 \geq b + 2$). In conclusion the general morphism $\varphi = (f, g) : d.\mathcal{O}(-t − 1) \to b.\mathcal{O}(-t) \oplus a.\mathcal{O}(-t + 1)$ has $\text{Coker}(\varphi) \simeq \mathcal{I}_R$ with $R$ smooth, with the induced morphism $a.\mathcal{O}(-t + 1) \to \mathcal{I}_R$ surjective. □

**Proposition 5.4.** Let $c \geq 4$ be an integer. For every $2 \leq t \leq c/2$, every admissible $y \in A_t$ is effective.
Proof. By \([1]\) there exists a globally generated line bundle, \(L\), of degree \(l = c(t-1) - y\) on a smooth plane curve, \(T\), of degree \(t - 1\). If \(h^1(L) \neq 0\) we conclude with \(5.2\).

Assume \(h^1(L) = 0\). Then \(h^0(L) = l - g_T + 1 \geq 2\) (we may assume \(L \neq \mathcal{O}_T\), because if \(y = c(t-1)\), we are done). So \(l \geq \frac{(t-2)(t-3)}{2} + 1\). Since \((t-1)(c-t+1) + 1 \leq y \leq t(c-t) - 1\), we have:

\[
(t-1)^2 - 1 \geq l \geq \frac{(t-2)(t-3)}{2} + 1 \quad (*)
\]

It follows that:

\[
l = (t-1)^2 - r, \quad 1 \leq r \leq \frac{t(t+1)}{2} - 3 = h^0(\mathcal{O}(t-1)) - 3 \quad (**)\]

For \(r, 1 \leq r \leq h^0(\mathcal{O}(t-1)) - 3\), let \(R \subset \mathbb{P}^2\) be a general set of \(r\) points of maximal rank, with \(h^0(\mathcal{I}_R(t-1)) \geq 3\) and \(\mathcal{I}_R(t-1)\) globally generated (see \(5.3\)). It follows that \(R\) is linked by a complete intersection \(T \cap T'\) of two smooth curves of degree \(t-1\), to a set, \(Z\), of \((t-1)^2 - r = l\) points. Since \(\mathcal{I}_R(t-1)\) is globally generated, \(\mathcal{I}_{R,T}(t-1) \simeq \mathcal{O}_T(t-1-R)\) is globally generated. Since \(\mathcal{O}_T(t-1) \simeq \mathcal{O}_T(R+Z)\), we see that \(L := \mathcal{O}_T(Z)\) is globally generated. Moreover, by construction, \(h^0(\mathcal{I}_Z(t-1)) \geq 2\).

By \(5.4\), \(L^*(c)\) is globally generated so there exists \(s \in H^0(L^*(c))\) such that: \(Y := (s)_0\) satisfies \(Y \cap (T \cap T') = \emptyset\). As in the proof of \(7.2\), we see that \(Y\) satisfies \(CB(c-3)\): indeed \(T'\) is a degree \(t-1\) curve containing \(Z\) such that \(T' \cap Y = \emptyset\). Since \(\mathcal{I}_{Y,T}(c) \simeq L\) is globally generated, we conclude that \(\mathcal{I}_{Y}(c)\) is globally generated.

Proposition \(5.4\) and Corollary \(1.14\) (and Remark \(2.5\)) prove Theorem \(3.6\). It follows that the proof of Theorem \(0.1\) is complete.

6. Morphisms from \(\mathbb{P}^2\) to \(G(1, 3)\).

It is well known that finite morphisms \(\varphi : \mathbb{P}^2 \to G(1, 3)\) are in bijective correspondence with exact sequences of vector bundles on \(\mathbb{P}^2\):

\[
0 \to E^* \to 4.\mathcal{O} \to F \to 0 \quad (*)
\]

where \(F\) has rank two and is globally generated with \(c_1(F) = c > 0\). If \(\varphi\) is generically injective, then \(\varphi(\mathbb{P}^2) = S \subset G \subset \mathbb{P}^5\) (the last inclusion is given by the Plücker embedding) has degree \(c^2\) (as a surface of \(\mathbb{P}^5\)) and bidegree \((y, c^2 - y)\), \(y = c_2(F)\) (i.e. there are \(y\) lines of \(S\) through a general point of \(\mathbb{P}^3\) and \(c^2 - y\) lines of \(S\) contained in a general plane of \(\mathbb{P}^3\)). Theorem \(5.11\) gives all the possible
(c, y) (but it doesn’t tell if φ exists). Finally, by [8], if φ is an embedding then 
(c, y) ∈ {(1, 0), (1, 1), (2, 1), (2, 3)}.

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