We review warped compactifications of superstring theory with some attention to the limit in which these resemble “bottom-up” phenomenological models. In addition to some discussion of the original Klebanov-Witten and Klebanov-Strassler set-ups, we also touch on various generalizations of the geometry that have been considered. Various other systems with a holographic duality are also briefly reviewed. The point of this latter exploration is to illustrate how far beyond the standard $AdS_5 \times S^5$ set-up things have moved over the years.
1 Introduction

Strongly coupled extensions to the standard model of particle physics have a long history, beginning with ideas that became known as technicolor [1, 2]. Mostly, these ideas are now known as composite Higgs. Various elaborations on this scenario were made to address various phenomenological problems with the original proposal. For instance, walking technicolor was introduced to address the scale of fermion masses [3, 4, 5, 6, 7, 8, 9]. However, the space of possible theories is vast once extensions are included to address issues of flavor. See [10] for just one example. Furthermore, it has proven difficult to make reliable predictions because the theories are beyond the reach of perturbation theory. As a result, various lattice studies have been conducted in the last several years, using large scale numerical simulations. (See for example [11, 12] and references therein.) For each theory several years must be devoted to developing the codes, exploring the parameter space and extracting physics results with increasing levels of sophistication. Even after all of this, there are significant limitations coming from the difficulty of simulating light states, of signal-to-noise when fermion-disconnected diagrams are involved, and the effect of excited states in the regime where this signal is strongest. It should also be noted that the theories that are actually simulated are ones that are very QCD-like, so that theories of for instance extended technicolor where chiral gauge theories might play a role are never addressed.

However, an alternative approach to studying strongly coupled gauge theories has been available since the late 1990s, namely the holographic correspondence [13]. Here, a weakly coupled gravitational alternative can be utilized, albeit in a larger number of spacetime dimensions. A key component of many of these dualities is the use of anti-deSitter (AdS) space in the extra-dimensional theory. In the case of gauge theories with four-dimensional spacetimes (4d), the AdS space is five-dimensional (AdS$^5$). When this gravitational theory comes to us from string theory, there is in addition another five dimensions, and the complete geometry is AdS$^5 \times X_5$, at least in some regime of the “radial” coordinate (variously $r$, $z$, $\tau$ or $y$). Typically $X_5$ is compact. The original example has $X_5 = S^5$, the five-dimensional sphere. However, in the case of Type IIB string theory this leads to a maximally supersymmetric gauge theory in 4d, $\mathcal{N} = 4$ super-Yang-Mills, with no matter and no possibility for a complex representation. This leads one to wonder what other choices of $X_5$ could be made, how much supersymmetry they would have, and what the dual gauge theory happens to be. In this review we discuss some of the investigations in this direction. Furthermore, there are modifications to the geometry or string theory set-up that can lead to interesting matter representations. The initial efforts in this direction involve adding “probe branes,” i.e., neglecting back-reaction on the geometry, but still having objects that can carry “flavor” quantum numbers.

There are a couple of features of the original AdS/CFT formulation that need modification when it comes to realistic particle phenomenology. As mentioned, the Maldacena system has too much supersymmetry. We need $\mathcal{N} = 1$ SUSY in order to have a chiral gauge theory like the Standard Model. Second, we have to move away from a conformal system. Again, the Standard Model has many mass scales and is not conformal. In particular, we need to accommodate phenomena such as confinement, spontaneous symmetry breaking and
chiral condensates.

These matters are addressed to a great extent with compactifications on the deformed or resolved conifold, which has $T^{1,1}$ for its base instead of $S^5$. The five-dimensional manifold $T^{1,1}$ is topologically equivalent to $S^3 \times S^2$. In the case of the deformed conifold, the $S^3$ remains finite at the tip of the cone ($r \to 0$) whereas the $S^2$ shrinks to zero size. For the resolved conifold the situation is reversed. These correspond to an IR cutoff in the theory. In the Klebanov-Strassler construction \[14\] this is dual to confinement in the gauge theory, complete with a tower of Seiberg dualities.

In this review we also take the opportunity to briefly discuss some other versions of holography that are of interest. One is holographic cosmology, which makes predictions that differ somewhat from conventional cosmology \[15\]. It is remarkable that aspects of cosmology could be described by studying a 3d field theory, say, using lattice techniques. We also present some aspects of the D1-D5 system \[16\], and related theories of 6d supergravity formulated on $AdS_3 \times S^3$. These give rise to 2d CFTs, about which a lot can be said due to the large symmetry algebra.

## 2 The conifold

The conifold is described in terms of four complex coordinates satisfying

$$z_1 z_2 = z_3 z_4 \quad (2.1)$$

or equivalently

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0 \quad (2.2)$$

The transformation between $z_i$ and $w_i$ is straightforward to work out and has been tabulated many times in the literature. It is given by

$$z_1 = w_1 + iw_2, \quad z_2 = w_1 - iw_2$$
$$z_3 = -w_3 + iw_4, \quad z_4 = w_3 + iw_4 \quad (2.3)$$

Four complex coordinates subject to one complex constraint is equivalent to eight real coordinates subject to two real constraints, leaving a six-dimensional “surface” $Y$. Given the form of the above equations, which allows for a rescaling of all complex coordinates, $z_i \to e^\lambda z_i$, there is one overall scale, which will be our radial coordinate $r$, and five compact coordinates, which are angles $\psi, \theta_1, \theta_2, \phi_1, \phi_2$. The geometry has the structure of a cone, $Y = R \times X_5$.

Then it is found that the solution to Eq. (2.1) is

$$z_1 = e^{3/2} e^{i\theta_1/2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}$$
$$z_2 = e^{3/2} e^{i\theta_1/2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2}$$
$$z_3 = e^{3/2} e^{i\theta_2/2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2}$$
$$z_4 = e^{3/2} e^{i\theta_2/2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \quad (2.4)$$
The conifold is a non-compact Calabi-Yau (CY) manifold. In string theory, a warping of
the spacetime is obtained by placing a stack of D3-branes at the conical singularity \( r \to 0 \).
In some of the more advanced models, the singularity may be resolved or deformed to make
it non-singular. In the gauge theory this translates into infrared (IR) dynamics associated
with a cutoff scale. In terms of the IR dynamics, there is a special role for fluxes through
the cycles of the CY, as will be seen below. Indeed, this can break conformal invariance and
lead to confinement at low energies.

The compact manifold is described by:

\[
 ds^2_{T^{1,1}} = \frac{1}{6} \sum_{i=1}^{2} (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} (d\psi + \sum_{i=1}^{2} \cos \theta_i d\phi_i)^2 
\]  

(2.5)

This is one of the manifolds found by Romans, related to preserving some supersymmetry
under compactification \[17\].

The presence of the D3 branes warps the space. So on top of the conifold we have warp
factors \( H(r) \): 

\[
 ds^2 = H(r)^{-1/2} \eta_{\mu \nu} dx^\mu dx^\nu + H(r)^{1/2} (dt^2 + r^2 ds^2_{T^{1,1}}) 
\]  

(2.6)

where

\[
 H(r) = 1 + \frac{L^4}{r^4} 
\]  

(2.7)

Here, when comparison is made to the string theory, one finds \( L^4 = 4\pi g_s N\alpha'^2 \) where \( g_s \)
is the string coupling and \( \alpha' \) is the Regge slope parameter. It is amusing that this gives a power-law
warp factor, whereas in Randall-Sundrum type discussions one often finds an exponential
warp factor. This is a reflection of the general coordinate reparameterization invariance of
general relativity: we can make a change of coordinates which drastically changes the way
the space looks.

The geometry (2.6) is the basic geometry of the Klebanov-Witten (KW) construction \[18\]. In that case, the dual gauge theory is superconformal, reflecting the conical singularity,
but with reduced supersymmetry, reflecting the \( T^{1,1} \) compact space.

3 Probe branes

There are low energy phenomena that we would like to insert into these super-Yang-Mills
(SYM) theories. In particular, we would like to have the mesons and baryons of quantum
chromodynamics (QCD). For this, we need fields in the fundamental representation of
the color group \( SU(N_c) \), and which also carry flavor quantum numbers under \( SU(N_f) \). In order
to accomplish this, a traditional pathway has been to introduce “probe” D7 branes. The
reason why the probe approximation is used is that the backreaction of D7 branes on the
geometry is difficult to account for, though there have been steps in that direction \[19\].
Levi and Ouyang have studied the mesons in the Klebanov-Witten with D7 probe brane scenario [20]. They work from an embedding based on the coordinates (2.4). I.e., fixing one of these defines a 7 + 1 dimensional hypersurface which defines the worldvolume of the D7 brane. This is then substituted into the Dirac-Born-Infeld (DBI) action and expanded around fluctuations in this embedding to define the “mesons” of the theory.

Our interest in this topic began with [21, 22]. In those papers we were looking for a string theoretic basis for warped extra dimension models of the Randall-Sundrum type [23]. One of the issues in these models is stabilizing the hierarchy. We believed that there is a string theoretic version of this, based on the work of Giddings, Kachru and Polchinski [24]. By contrast, effective field theoretic approaches to the problem of stabilizing the extra dimension were initiated by Goldberger and Wise in [25].

The D7 is embedded into the spacetime by the equation

\[
z_1 = r^{3/2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{\frac{1}{2}(\psi - \phi_1 - \phi_2)} = \mu > 0
\]  

(3.1)

Here, \(\mu\) is an adjustable (real) parameter of the embedding which is related to how far towards the tip of the conifold the D7 branes extend. This leads to \(r\) and \(\psi\) being functions of the other coordinates:

\[r = r_0(\theta_i), \quad \psi = \psi_0(\phi_i)\]

(3.2)

In particular,

\[
r^{3/2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} = \mu, \quad \psi_0 = \phi_1 + \phi_2
\]

(3.3)

These equations are important in the analysis of fluctuations of the D7 branes, which lead to modes of excitation in the effective theory. The fluctuations of the embedding are parameterized by

\[r = r_0(1 + \chi), \quad \psi = \psi_0 + 3\eta\]

(3.4)

Here, \(\chi\) and \(\eta\) represent the fluctuations and are functions of the D7 worldvolume coordinates \(\xi^a, a = 0, 1, \ldots, 7:\)

\[\chi = \chi(\xi^a), \quad \eta = \eta(\xi^a)\]

(3.5)

We will identify \(\xi^a\) with the 8 coordinates \(x^\mu\) (4d spacetime), \(\theta_i\) and \(\phi_i\).

The embedding metric can be obtained in a couple of ways. One is through the formula

\[G_{ab} = g_{MN} \frac{\partial X^M}{\partial \xi^a} \frac{\partial X^N}{\partial \xi^b}\]

(3.6)

where \(X^M = X^M(\xi)\) are the 10d coordinates describing the embedding of the probe D7 brane. An alternative is to start with the 10d metric (2.6) and substitute in the expressions

\[
dr = \frac{\partial r_0}{\partial \theta_i} d\theta_i + r_0 \left( \frac{\partial \chi}{\partial x^\mu} dx^\mu + \frac{\partial \chi}{\partial \theta_i} d\theta_i + \frac{\partial \chi}{\partial \phi_i} d\phi_i \right)
\]

\[
d\psi = d\phi_1 + d\phi_2 + 3 \left( \frac{\partial \eta}{\partial x^\mu} dx^\mu + \frac{\partial \eta}{\partial \theta_i} d\theta_i + \frac{\partial \eta}{\partial \phi_i} d\phi_i \right)
\]

(3.7)
This can then be substituted into the 10d metric (2.6) above to obtain an equation for the
8d metric of the D7 brane embedding.

Working in the near-horizon limit of (2.6) \( r \to 0 \), and substituting the expressions (3.7),
we obtain the 8d metric including fluctuations:

\[
\begin{align*}
\text{ds}_8^2 &= \frac{r_0^2}{L^2}(1 + \chi)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r_0^2}(1 + \chi)^{-2} dr^2 + ds_{T^1,1}^2
\end{align*}
\]

Here, \( ds_{T^1,1}^2 \) takes into account the expression for \( d\psi \) in (3.7) above, and \( dr \) should also use
the corresponding expression from (3.7). The nonzero elements of the 8d metric are then
found to be:

\[

G_{\mu\nu} = \frac{r_0^2}{L^2}(1 + \chi)^2 \eta_{\mu\nu}, \\
G_{\theta_i\theta_i} = \frac{L^2}{r_0^2}(1 + \chi)^{-2} \left( \frac{\partial r_0}{\partial \theta_i} + \frac{\partial \chi}{\partial \theta_i} \right)^2 + \frac{1}{6} + \left( \frac{\partial \eta}{\partial \theta_i} \right)^2, \\
G_{\theta_1\theta_2} = G_{\theta_2\theta_1} = \frac{L^2}{r_0^2}(1 + \chi)^{-2} \left( \frac{\partial r_0}{\partial \theta_1} + \frac{\partial \chi}{\partial \theta_1} \right) \left( \frac{\partial r_0}{\partial \theta_2} + \frac{\partial \chi}{\partial \theta_2} \right) + \frac{\partial \eta}{\partial \theta_1} \frac{\partial \eta}{\partial \theta_2}
\]

This is then substituted into the DBI effective action:

\[
S_{\text{DBI}} = -\tau_7 \int d^4 x \ d^2 \theta \ d^2 \phi \ \sqrt{\det[G + \varphi^*(B) + 2\pi \alpha' F]} \tag{3.10}
\]

where \( \varphi^*(B) \) is the pullback of the antisymmetric tensor (which will not be important for
our purposes), \( F \) is the \( U(1) \) field strength associated with the charge of the D7 brane and
\( \tau_7 = (2\pi)^{-7} \alpha'^4 g_s^{-1} \) is the brane tension.

By a sequence of redefinitions and approximations, angular coordinates can be integrated
out and one obtains for an effective, quadratic action:

\[
S(\chi) \approx -2\pi^2 L^{-5} \tau_7 \int d^4 x \int_R^\infty dr \left\{ \frac{r}{L} \eta^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + \frac{r^5}{L^5} (\partial_r \chi)^2 - \frac{15}{4L^2} \frac{r^3}{L^3} \chi^2 \right\} \tag{3.11}
\]

Here, \( L \) is the \( AdS_5 \) radius. A similar equation holds for \( \eta \). This is the action of a conformally
coupled scalar in \( AdS_5 \). Thus we see that an effective five-dimensional theory can be derived
from D7 probe branes embedded into the KW geometry and that it has a standard form.

4 Klebanov-Witten duality

Klebanov and Witten showed the duality of the conifold compactified string theory to a
superconformal gauge theory [18]. Here we review some key aspects of that duality.

For the conifold, we can describe it in terms of a quaternion:

\[
Z = \frac{1}{\sqrt{2}} \sum_{n=1}^{4} w_n \sigma^n \tag{4.1}
\]
Here, $\sigma^4$ is the unit matrix multiplied by $i = \sqrt{-1}$. Then the conifold equation can be described by:

$$\det Z = 0 \quad (4.2)$$

This can be related to the dual gauge theory by writing $Z$ as:

$$Z = \begin{pmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_1 & A_2 B_2 \end{pmatrix} \quad (4.3)$$

Then the superpotential is given by

$$W = \lambda \det Z \quad (4.4)$$

which is a function of the chiral superfields $A_i$ and $B_i$. Here, $\lambda$ is a parameter. In supergravity, one of the conditions for a supersymmetric ground state is $W = 0$. Hence we recover the conifold equation from this condition. In actuality, if $A_i$ and $B_i$ are all commuting variables, then it is identically true that

$$\det Z = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1 = 0 \quad (4.5)$$

Thus the parameterization of $Z$ in terms of these variables automatically generates the conifold condition $\det Z = 0$. However, if $A_i$ and $B_i$ are matrices, as in the $SU(N) \otimes SU(N)$ gauge theory (where they are bi-fundamentals), then

$$W = \lambda \text{Tr} (A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) \quad (4.6)$$

does not vanish identically. In this case we have a nontrivial superpotential and the vacuum conditions correspond to the conifold. In particular, if on the moduli space $A_i$ and $B_i$ are all diagonal matrices, then the condition $W = 0$ is satisfied.

The parameterization in terms of $Z$ also reveals symmetry. We note that (4.2) allows for the transformation

$$Z \rightarrow U Z V^\dagger \quad (4.7)$$

where $U$ and $V$ are $SU(2)$ group matrices. This is because

$$\det U = \det V = 1 \quad (4.8)$$

So, there is an $SU(2) \otimes SU(2)$ isometry for the conifold. Also, one can rephase $Z$:

$$Z \rightarrow e^{i\phi} Z \quad (4.9)$$

where $\phi$ is a real parameter. Thus there is an additional $U(1)$ isometry. Of course one can also see these isometries from the equation (2.2). One simply notes that $SO(4) \simeq SU(2) \otimes SU(2)$. Obviously equation (2.2) is invariant under a four-dimensional rotation. Similarly, the $z$s can all be simultaneously rephased by $e^{i\varphi}$.

\footnote{This particular condition is related to the supersymmetric variation of the gravitino field.}
5 Klebanov-Strassler

The Klebanov-Strassler (KS) construction [14] has been reviewed previously in Strassler [26] and Gwyn and Knauf [27]. For details beyond what is found in this brief section, see those references.

Recall that in the Klebanov-Witten construction the dual gauge theory has gauge group $G = SU(N) \otimes SU(N)$. By contrast, KS has a gauge group $G = SU(N + M) \otimes SU(N)$ which undergoes a cascade of Seiberg duality transformations [28] as one descends in energy. Because of this, the theory confines in the IR. The consequent chiral symmetry breaking resolves the IR singularity of the KW construction.

On the brane side, Klebanov-Strassler corresponds to $N$ D3 branes and $M$ wrapped D5 branes. This takes advantage of the fact that $T^{1,1}$ is topologically equivalent to $S^3 \times S^2$. Thus there is an $S^2$ to wrap, leaving effectively branes with three spatial dimensions (fractional D3 branes). Often it is said that the D5 branes wrap a two-cycle.

5.1 Geometry

In terms of the geometry, the KS case has a simple modification of the conifold equation: the sum is nonzero. That is,

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = \epsilon^2$$

(5.1)

with $\epsilon$ some nonzero number. This means that there is a minimal radius allowed. Thus the tip of the conifold is rounded off and is no longer singular. This space is referred to as the deformed conifold. Because warp factors will ultimately be applied to this geometry, KS is built on the warped deformed conifold.

One of the nice features about the KS construction is that it makes explicit what was suspected intuitively all along: the radial coordinate on the gravity side maps to the RG scale on the gauge theory side. This occurs through a duality cascade and is a result of the fractional D3 branes.

It is interesting that the two principle modifications of the conifold can be related to each other. In the deformed conifold the size of the 3-cycle is described by the complex structure modulus. On the other hand in the resolved conifold the size of the 2-cycle corresponds to the Kähler modulus. These are mapped to each other in the topological string context [29]. In that case the large $N$ Chern-Simons theory on $S^3$ is shown to be dual to the topological string on the $S^2$ blow-up of the conifold. This suggests a geometric transition may also be possible between the deformed and resolved geometries in the full string theory. It would be interesting to see what this does to the dual gauge theory.

The resolved conifold is described as follows. Translating the notation in Elituv [30] to the one used here,

\[
\begin{pmatrix}
  z_1 & -z_3 \\
 -iz_4 & i\bar{z}_2
\end{pmatrix}
\begin{pmatrix}
  \eta_1 \\
  \eta_2
\end{pmatrix}
\]

(5.2)
for some real parameters $\eta_1$ and $\eta_2$. It can be seen that this is quite different from (5.1), so it is a little surprising that there may be a duality here.

KS has the additional features beyond those of KW: fractional D3 branes, duality cascades, IR resolution of the singularity and chiral symmetry breaking. These are all captured in the modified geometry, which we now describe.

The KS metric is conveniently described in terms of 1-forms:

$$ds_{10}^2 = A^2(\tau)\eta_{\mu\nu}dx^\mu dx^\nu + B^2(\tau)d\tau^2 + C^2(\tau)(g^5)^2 + D^2(\tau)[(g^3)^2 + (g^4)^2] + E^2(\tau)[(g^1)^2 + (g^2)^2]$$ (5.3)

The one-forms in the above expressions can be related to the original KW coordinates:

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}}, \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}}$$
$$g^3 = \frac{e^1 + e^3}{\sqrt{2}}, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}}$$
$$g^5 = e^5$$ (5.4)

where

$$e^1 = -\sin \theta_1 d\phi_1, \quad e^2 = d\theta_1$$
$$e^3 = \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2$$
$$e^4 = \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2$$
$$e^5 = d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2$$ (5.5)

Based on various considerations about the allowed fluxes in the theory, this is further specialized to [14]:

$$ds_{10}^2 = h^{-1/2}(\tau)\eta_{\mu\nu}dx^\mu dx^\nu + h^{1/2}(\tau)ds_6^2$$ (5.6)
$$ds_6^2 = \frac{1}{2}e^{4/3}K(\tau)\left[\frac{1}{3K^3(\tau)}(d\tau^2 + (g^5)^2) + \cosh^2(\tau/2)[(g^3)^2 + (g^4)^2]
+ \sinh^2(\tau/2)[(g^1)^2 + (g^2)^2]\right]$$ (5.7)
$$K(\tau) = \frac{\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh \tau}$$ (5.8)

The warp factor $h(\tau)$ is obtained from solving the supergravity equations of motion, yielding:

$$h(\tau) = \alpha^2 \frac{2^{2/3}}{4} \int_{\tau}^{\infty} dx \frac{x \cosh x - 1}{\sinh^2 x}(\sinh(2x) - 2x)^{1/3}$$ (5.9)

Obviously this is far more intricate than KW, reflecting that this is a supergravity dual of a theory with confinement and spontaneous chiral symmetry breaking (gauge condensation). It is therefore more realistic as a starting point for phenomenology.
It is interesting to compare KS to KW in detail. For instance, (5.6) versus (2.6) looks quite similar. But the 6d part in (5.7) looks quite a bit more complicated. Also, the function \( H(r) \) in (2.7) is quite simple, just reflecting a stack of D3 branes at the origin \( r = 0 \), whereas the corresponding function (5.9) in KS is rather involved, reflecting wrapped D5 branes at all different values of the radial coordinate \( \tau \). In the dual gauge theory this latter complicated \( \tau \) dependence is exhibited by the Seiberg duality cascade.

In KS, we find the relation \( r^3 \sim \epsilon^2 e^\tau \). Based on this, we could choose to embed a probe D7 brane in a way that is analogous to \( z_1 = \mu \) in KW, namely:

\[
\epsilon e^{\tau/2} e^{\frac{1}{2} (\psi - \phi_1 - \phi_2)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} = \mu
\]  

(5.10)

There is a question however: is it a supersymmetric embedding? To answer this question requires a detailed investigation that we will not address in this review.

## 5.2 Forms and couplings

The background field strength \( F_3 \) is governed by the number of D5 branes \( M \) according to [14]:

\[
F_3 = M \omega_3
\]

\[
\omega_3 = \frac{1}{2} d\psi \wedge (\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2)
\]

\[
+ \frac{1}{2} d\phi_1 \wedge d\phi_2 \wedge (\cos \theta_1 \sin \theta_2 d\theta_2 + \sin \theta_1 \cos \theta_2 d\theta_1)
\]  

(5.11)

The gauge couplings in the two factors of the gauge group are related by

\[
\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim \frac{1}{g_s} \left( \int B_2 - \frac{1}{2} \right)
\]  

(5.12)

where \( g_s \) is the string coupling and \( B_2 \) is the NS-NS 2-form. The latter is given by

\[
B_2 = 3 g_s M \omega_2 \ln \left( \frac{r}{r_0} \right)
\]

\[
\omega_2 = \frac{1}{2} (\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2)
\]  

(5.13)

\( r_0 \) is a reference scale. Thus we see that the difference in gauge couplings depends on the radial coordinate \( r \), corresponding under the duality to a renormalization group scale. This is one of the interesting things about holography: in a theory with a running coupling, the extra dimension corresponds to the renormalization group scale. Here it is important to have an example like KS, where the gauge theory is not conformal.

In the KW and KS constructions there are two gauge couplings, \( g_1 \) and \( g_2 \), corresponding to the two factors of the gauge group. In KW, they can take any values without destroying the conformal invariance. In KS we have to keep in mind the duality cascade. What do
these couplings correspond in the dual SUGRA? Klebanov and Strassler discuss this and we mention a few details here because the story is interesting.

For instance in the $M = 0$ scenario (no fractional D3 branes, i.e., just KW), we have the formulae:

$$\frac{1}{g_1^2} + \frac{1}{g_2^2} \sim e^{-\phi} \quad (5.14)$$

and

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim e^{-\phi} \left[ \int_{S^2} B_2 - \frac{1}{2} \right] \quad (5.15)$$

equivalent to (5.12) above. In the T-dual Type IIA picture, it is related to the positions of NS5 branes [14]:

$$\frac{1}{g_1^2} = l_6 - a g_s, \quad \frac{1}{g_2^2} = a g_s \quad (5.16)$$

Here, $l_6$ is the size of the compact dimension upon which things are T-dualized, and $a$ is the separation of the two NS5 branes.

In the KW case, $\int_{S^2} B_2$ is normalized to that its period is 1. By contrast in KS with $M \neq 0$ we have

$$\int_{S^2} B_2 \sim M e^{\phi} \ln(r/r_0) \quad (5.17)$$
as can be seen from (5.13) above, so that a logarithmic running occurs for the gauge couplings as we follow the duality cascade down to small $r$.

6 Other generalizations of the conifold

In [30], Elituv explores both Kähler and non-Kähler 3-complex-dimensional manifolds (locally $\mathbb{C}^3$) that are generalizations of the various constructions such as Klebanov-Witten, Klebanov-Strassler, etc. He begins with the forms $dr, d\psi, d\theta_i, d\phi_i$ and imposes that it is a complex manifold by computing $d\Omega = 0$ where $\Omega$ is a rather general $(3, 0)$ form. Then, the Kähler condition is tackled using $d(e^{2\phi} J)$ where $J$ is a rather general $(1, 1)$ form and $\phi$ is the dilaton. From this he derives conditions on the various coefficient functions, including some differential equations that they should satisfy. This opens up the possibility of many new manifolds that could serve as the basis for holographic duals to some $\mathcal{N} = 1$ supersymmetric gauge theory.

6.1 Conditions for supersymmetry

To establish that a conifold-like theory has $\mathcal{N} = 1$ supersymmetry, Elituv explains that we need to check three things:
1. The complex manifold has a (3,0) form $\Omega$. We need to then require that the manifold be closed, $d\Omega = 0$. This imposes constraints on the components $\Omega_{abc}$. Solving these constraints then tells us something about the (3,0) form. But the manifold must be consistent with the existence of a solution. So it also tells us something about the manifold.

2. We construct the 3-form $G_3 = F_3 - ie^{i\phi}H_3$. Here, $F_3$ is the field strength associated with a 2-form gauge potential and $H_3$ is given by $H_3 = dB_2$ where $B_2$ is the usual 2-form associated with string theory. $\phi$ is the dilaton. Next we demand that the the 3-form $G_3$ is a (2,1) form. That is, all of the non-(2,1) form components need to vanish. This requires that we specify the complex basis of the manifold, and again tells things about both the form and the manifold.

3. Lastly, we have require that the (2,1) form is primitive. This means that $J \wedge G_3 = 0$, where

$$J = ig_{a\bar{a}}dz^a \wedge d\bar{z}^{\bar{a}}$$

is the Kähler form.

### 6.2 A more general metric

As elucidated in Elituv, the more general metric in the case of deformed and resolved conifolds can be described by:

$$ds^2 = G_1 dr^2 + G_2 (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2$$
$$+ G_3 (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + G_4 (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2)$$
$$+ G_5 \cos \psi (d\theta_1 d\theta_2 - \sin \theta_1 \sin \theta_2 d\phi_1 d\phi_2)$$
$$+ G_6 \sin \psi (\sin \theta_1 d\phi_1 d\theta_2 + \sin \theta_2 d\phi_2 d\theta_1)$$

The functions $G_1, \ldots, G_6$ then need to be specified in either case. Furthermore, this form allows for various generalizations. Elituv calls this a “non-Kähler resolved deformed conifold.”

Elituv reports that the deformed conifold corresponds to

$$G_1 = \gamma + (r^2 \gamma' - \gamma) \left(1 - \frac{\mu^4}{r^4}\right)$$
$$G_2 = \frac{1}{4} \left[ \gamma - (r^2 \gamma' - \gamma) \left(1 - \frac{\mu^4}{r^4}\right) \right]$$
$$G_3 = G_4 = \frac{\gamma}{4}, \quad G_6 = \frac{\mu^2 \gamma}{2r^2}$$

Here $\mu^2$ is a deformation parameter related to $\epsilon^2$ above. $\gamma$ is some function of the radial coordinate $r$. $\gamma' = d\gamma/dr(r^2)$. 

11
Note that for D5 branes, the natural gauge potential that couples to them is a 6-form \( C_6 \). The corresponding field strength is \( F_7 = dC_6 \). The Hodge dual of this is \( F_3 = * F_7 \). This is the field above that appears in \( G_3 \).

In the next subsection, we describe how Elituv is able to construct the metric, the (1,1) form \( J \) and the (3,0) form \( \Omega \) from a basis of complex forms \( E_1, E_2 \) and \( E_3 \). This involves both the functions \( G_1, ..., G_4 \) and some free parameters \( \alpha_1, ..., \alpha_4 \) and \( \beta_1, ..., \beta_4 \). He can then study the conditions on these quantities if the manifold is Kähler, \( d(e^{2\phi}J) = 0 \), and/or complex, \( d\Omega = 0 \). Having started with a rather general set-up, he can construct manifolds that are more general than the ones appearing in the original KW and KS formulations. It is an interesting question how these modifications would be reflected in the dual gauge theory.

### 6.3 Complex geometry

The conifold has the nice feature that it illustrates a system describable by complex geometry. This is already apparent from the fact that we utilize four complex coordinates \( z_i \) satisfying \( (2.1) \). This is similar to how we often describe a sphere in terms of embedding coordinates in a higher dimensional space, subject to a constraint equation. Here we begin with four complex dimensions, equivalent to eight real dimensions. But we impose one constraint equation, which happens to be complex. This reduces us to a three-complex-dimensional subspace, equivalent to a six-real-dimensional subspace. As in the case of a sphere, we begin with a flat metric in the higher dimensional space:

\[
ds^2 = \sum_{i=1}^{4} d\bar{z}_i dz_i
\]

To get to the metric written in terms of the Gaussian coordinates \( r, \psi, \theta_1, \phi_1, \phi_2 \) we substitute in the solutions \( (2.4) \) to the constraint equation \( (2.1) \).

In the theory of the conifold, one deals with the left-invariant Maurer-Cartan forms:

\[
\begin{align*}
\sigma_1 &= \cos \psi_1 d\theta_1 + \sin \psi_1 \sin \theta_1 d\phi_1 \\
\sigma_2 &= -\sin \psi_1 d\theta_1 + \cos \psi_1 \sin \theta_1 d\phi_1 \\
\sigma_3 &= d\psi_1 + \cos \theta_1 d\phi_1 \\
\Sigma_1 &= \cos \psi_2 d\theta_2 + \sin \psi_2 \sin \theta_2 d\phi_2 \\
\Sigma_2 &= -\sin \psi_2 d\theta_2 + \cos \psi_2 \sin \theta_2 d\phi_2 \\
\Sigma_3 &= d\psi_2 + \cos \theta_2 d\phi_2
\end{align*}
\]

where \( \psi_1 = \psi_2 = \frac{1}{2} \psi \). Using these, one can build up the various structures that arise. These are discussed for instance in Section 3.1 of \[30\].

Note that \( \sigma_1 \) and \( \sigma_2 \) can be obtained as 2d rotations of the more basic forms \( d\theta_1 \) and \( \sin \theta_1 d\phi_1 \) that appear in the metric of \( S^2 \):

\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix} =
\begin{pmatrix}
\cos \psi_1 & \sin \psi_1 \\
-\sin \psi_1 & \cos \psi_1
\end{pmatrix}
\begin{pmatrix}
d\theta_1 \\
\sin \theta_1 d\phi_1
\end{pmatrix}
\]

(6.6)
A similar relation holds for $\Sigma_1$ and $\Sigma_2$.

Elituv relates these left-invariant Maurer-Cartan forms back to his rather general metric after various manipulations and assumptions. Thus these forms can be viewed as building blocks for constructing modifications of the original conifold geometry.

Elituv uses these forms to compose other forms that are particularly useful for describing his geometry. For instance, he has:

\begin{align*}
e_1 &= \sqrt{G_1}dr \\
e_2 &= \sqrt{G_2}(\sigma_3 + \Sigma_3) \\
e_3 &= \sqrt{G_3}(\alpha_1 \sigma_1 + \beta_3 \Sigma_1) \\
e_4 &= \sqrt{G_4}(\alpha_2 \sigma_2 - \beta_4 \Sigma_2) \\
e_5 &= \sqrt{G_3}(\beta_1 \sigma_1 + \alpha_3 \Sigma_1) \\
e_6 &= \sqrt{G_4}(-\beta_2 \sigma_2 + \alpha_4 \Sigma_2)
\end{align*}

(6.7)

From these he then constructs complex forms:

\begin{align*}
E_1 &= e^{-\phi}(e_2 + ie_1), \quad E_2 = e^{-\phi}(e_3 + ie_4), \quad E_3 = e^{-\phi}(e_5 + ie_6)
\end{align*}

(6.8)

These can then be used to form a 6d metric.

\begin{equation}
e^{-2\phi}ds_6^2 = E_1 \otimes \bar{E}_1 + E_2 \otimes \bar{E}_2 + E_3 \otimes \bar{E}_3
\end{equation}

(6.9)

Elituv also uses these in the construction of the (1,1) form:

\begin{equation}
J = -\frac{i}{2}(E_1 \wedge \bar{E}_1 + E_2 \wedge \bar{E}_2 + E_3 \wedge \bar{E}_3)
\end{equation}

(6.10)

Likewise, he obtains the (3,0) form from these:

\begin{equation}
\Omega = e^{3\phi}E_1 \wedge E_2 \wedge E_3
\end{equation}

(6.11)

Thus we have a general prescription for how to build up the relevant structures in the complex geometry, based on the left-invariant Maurer-Cartan forms.

The conifold is a noncompact Calabi-Yau manifold. This means it has $SU(3)$ holonomy since it has 3 complex dimensions. The result is $\mathcal{N} = 1$ supersymmetry in the 4d low energy theory since the manifold has a single Killing spinor. Now one might ask: if it is noncompact, how does one get 4d physics? One might think that one has to completely compactify the 6 extra dimensions of superstring theory in order to get 4d physics. So, how does it work? Part of the answer is that we have D3 branes and wrapped D5 branes (on the $\sim S^2$ part of $T^{1,1}$).

So of course the worldvolume theory on these branes is 4d. But it is more than that, as we can see from the effective 5d picture derived in [21]. There we see that we have effectively a Randall-Sundrum type setup with a UV brane and an IR brane. It is well-known that this will produce an effective 4d theory at low energy. Evidence of the extra 5th dimension begins to appear at the TeV scale through KK excitations. Thus there is a compactification of 5 dimensions on $T^{1,1}$, and the remaining extra dimension is dealt with in a Randall-Sundrum sort of way.
7 Lattice theory

The most straightforward lattice fermion is the Wilson fermion, which has an action

\[ S = \sum_x \{ \bar{\psi} \gamma_\mu D_\mu \psi + \frac{1}{2} ra \bar{\psi} D^2 \psi + m \bar{\psi} \psi \} \]  

(7.1)

The role of the Wilson “mass term” \( \bar{\psi} D^2 \psi \) is to lift spectral doublers. \( a \) is the lattice spacing and \( r \) is a dimensionless constant. The factor of \( \frac{1}{2} \) is conventional. Even with \( m = 0 \) chiral symmetry is violated explicitly, due to the \( \bar{\psi} D^2 \psi \) term. This causes problems for preservation of global symmetries in supersymmetric theories, as we now detail.

In two-component notation, the part of the SUSY action for fermions coming from the superpotential is given by\(^2\)

\[ -\frac{1}{2} W_{ij} \chi_i \chi_j - \frac{1}{2} \bar{W}_{ij} \bar{\chi}_i \bar{\chi}_j \]  

(7.2)

Here, \( W_{ij} = \partial^2 W / \partial \phi_i \partial \phi_j \) is the second derivative of the superpotential. The example that we will refer to is the one for the KW and KS theories:

\[ W = \lambda \epsilon_{ij} \epsilon_{kl} \text{Tr} (A_i B_k A_j B_l) \]  

(7.3)

Thus for instance the SUSY Lagrangian will contain a term

\[ -\lambda \epsilon_{ij} \epsilon_{kl} \text{Tr} (\chi^\alpha_{A_i} B_k \chi_{A_j A} B_l) \]  

(7.4)

where \( \alpha = 1, 2 \) is the spinor index on the two-component fermions, \( \chi_{A_i} \) being the fermionic partner of the scalar \( A_i \).

In order to understand the chiral symmetry, and how the Wilson “mass term” will impact it, it is useful to go over to a four-component notation. For this purpose the fermions are grouped into Majorana fermions

\[ \Psi_{M_A i} = \begin{pmatrix} \chi_{A_i} \\ \bar{\chi}_{A_i} \end{pmatrix}, \quad \Psi_{M_B i} = \begin{pmatrix} \chi_{B_i} \\ \bar{\chi}_{B_i} \end{pmatrix} \]  

(7.5)

The superpotential (7.3) has a global \( SU(2)_A \times SU(2)_B \times U(1) \) symmetry. The two \( SU(2) \)s are obvious, and the \( U(1) \) acts on the superfields according to \( A_i \to e^{i \alpha} A_i, B_i \to e^{-i \alpha} B_i \).

Focusing on the \( \sigma^3 \) part of the first \( SU(2) \), we have

\[ \chi_{A1} \to e^{i \alpha} \chi_{A1}, \quad \chi_{A2} \to e^{-i \alpha} \chi_{A2}, \quad \bar{\chi}_{A1} \to e^{-i \alpha} \bar{\chi}_{A1}, \quad \bar{\chi}_{A2} \to e^{i \alpha} \bar{\chi}_{A2} \]  

(7.6)

In terms of the Majorana fermions, this is

\[ \Psi_{MA1} \to e^{-i \alpha \gamma_5} \Psi_{MA1}, \quad \Psi_{MA2} \to e^{i \alpha \gamma_5} \Psi_{MA2} \]  

(7.7)

\(^2\)Often, the index on the fermion and other fields would be raised, to make the Kähler geometric interpretation more clear. However, we will not use this notation here in our relatively brief discussion.
The Wilson “mass term”
\[
\bar{\Psi} M_{A1} D^2 \Psi_{M1} + \bar{\Psi} M_{A2} D^2 \Psi_{M2} = \Psi^T M_{A1} CD^2 \Psi_{M1} + \Psi^T M_{A2} CD^2 \Psi_{M2}
\] (7.8)
badly violates this symmetry. However, from the last expression, it can be seen that an SO(2) subgroup of the SU(2) is preserved:
\[
\begin{pmatrix}
\Psi_{M1} \\
\Psi_{M2}
\end{pmatrix} \rightarrow \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
\Psi_{M1} \\
\Psi_{M2}
\end{pmatrix}
\] (7.9)
Importantly, this symmetry is restrictive on the necessary counterterms.
Thus we see that trying to formulate either the KW or the KS theories on the lattice using the Wilson discretization for fermions will violate the SU(2) \times SU(2) \times U(1) flavor symmetry of these models. Because it is broken by the regulator, we do not expect it to be restored in the IR without tuning counterterms. The number of such counterterms will be large and they would have to be fine-tuned nonperturbatively, which would require many simulations. This is an example of the types of challenges that are faced when trying to study these theories from first principles using numerical methods.

8 The D1-D5 system
Although the phenomenological implications are unclear\(^3\) the D1-D5 system on \(T^4\) is of particular interest in the field of holography, such as the recent work [31]. It is supposed to be dual to the supersymmetric gauge theory on \(AdS_3 \times S^3 \times T^4\). Indeed, we have previously studied an approach that may realize a latticization of this system [32]. More generally, this setup is just one example of a whole class of 2d conformal theories that are dual to 6d supergravity formulated on \(AdS_3 \times S^3\). Early stages of such an analysis have been started by de Boer [16], but much more work remains to be done. One can ask questions such as: have we learned anything from the bootstrap studies that might shed light on these types of dualities? How much is really known about these 6d supergravities? Are there more elegant formulations of them, analogous to the Kähler U(1) superspace for N=1 sugra in 4d? (See [33] and references therein.)

Six-dimensional Calabi-Yau manifolds \(K\) are an important compactification target within string theoretic studies, because of the nice properties of these spaces. For this reason there is also some attention on compactification of F-theory (which is 12d) on \(AdS_3 \times S^3 \times K\) and M-theory (which is 11d) on \(AdS_3 \times S^2 \times K\). The common element is \(AdS_3\), which means that these theories will have dual 2d CFTs. Thus they generally fall into the same class as the D1-D5 based theories.

In [31], a connection is made to twisted supersymmetry. This is especially challenging on the gravity side of the duality, since it requires supergravity to be twisted. Nevertheless, this

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\(^3\)Since the dual CFT is two-dimensional, it may be that the D1-D5 system and its close relatives could have applications in condensed matter systems close to a second order critical point. This could include static properties of 2+1 dimensional systems or dynamical properties of 1+1 dimensional systems.
is an interesting direction in light of how many of the supersymmetric lattices are formulated in a twisted framework.

In [34] supersymmetry is broken in systems analogous to D1-D5. This is accomplished through supergravity in geometries that are topologically $AdS_3 \times S^3$. In this particular situation, the $S^3$ is replaced by a squashed sphere. These surfaces preserve an $SO(2) \otimes SO(2)$ isometry group of the original $SO(4)$ isometry group of $S^3$. These calculations rely heavily on exceptional field theory.

For the D1-D5 system with D5s wrapped on the four-dimensional manifold K3, the metric is given by [16]

$$ds^2 = \frac{U^2}{\ell^2} (-dt^2 + (dx^5)^2) + \frac{\ell^2}{U^2} dU^2$$

$$+ \ell^2 d\Omega_5^2 + \sqrt{Q_1 v Q_5} ds_{K3}^2 \quad (8.1)$$

Here, $Q_1$ is the number of D1 branes and $Q_5$ is the number of D5 branes. The $d\Omega_5^2$ is associated with an $S^3$. Note that $U$ is a sort of radial coordinate. The parameter $\ell$ is a length scale. It is given by $\ell^2 = g_6 \sqrt{N}$, where $N = Q_1 Q_5$ and $g_6$ is the six-dimensional gauge coupling, associated with reducing the ten-dimensional theory using compactification on K3.

$AdS_3$ can be expressed through the metric

$$ds^2 = \ell^2 (-\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2) \quad (8.2)$$

Here, $\ell$ is the AdS radius (inverse of AdS curvature). The masses of primary fields on $AdS_3$ are given by [16]:

$$m^2 \ell^2 = (h + \bar{h})(h + \bar{h} - 2) \quad (8.3)$$

where $h$ and $\bar{h}$ are the conformal weights at the boundary. This is typical of the AdS/CFT correspondence. We see that there is a relationship between the mass of the field in anti-de-Sitter space and the conformal weights in the CFT at the boundary. A similar relation holds in more phenomenological applications that relate $AdS_5$ to 4d effective field theory in terms of Kaluza-Klein decomposition.

Associated with this geometry are Virasoro operators. Those corresponding to the $SL(2, \mathbb{R})$ subalgebra of the 2d conformal group are given by:

$$L_0 = i \partial_u$$

$$L_{-1} = i e^{-iu} \left( \coth(2\rho) \partial_u - \frac{1}{\sinh(2\rho)} \partial_v + \frac{i}{2} \partial_{\rho} - \frac{i}{2} s \coth \rho \right)$$

$$L_1 = i e^{iu} \left( \coth(2\rho) \partial_u - \frac{1}{\sinh(2\rho)} \partial_v - \frac{i}{2} \partial_{\rho} + \frac{i}{2} s \coth \rho \right) \quad (8.4)$$

where $u = \tau + \phi$ and $v = \tau - \phi$. There is also a barred set of generators $\bar{L}_0$, $\bar{L}_{-1}$ and $\bar{L}_1$ obtained through $u \leftrightarrow v$ and $s \rightarrow -s$. Thus the full subalgebra is $SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R})$. In a Virasoro algebra, there is an infinite number of generators $L_n$ with $n \in \mathbb{Z}$. 16
We will now make some comments that will make the role of the parameter $s$ clear.

In the conformal field theory, one seeks primary fields in order to develop the modules associated with the Virasoro algebra. These should satisfy

$$
L_1 \psi = \bar{L}_1 \psi = 0 \\
L_0 \psi = h \psi, \quad \bar{L}_0 \psi = \bar{h} \psi
$$

(8.5)

where $h$ and $\bar{h}$ are the conformal weights. ($L_{-1}$ and $\bar{L}_{-1}$ behave as raising operators, generating other parts of the spectrum.) In order for such primary fields to exist, one finds that $s = h - \bar{h}$, which is a sort of “spin.” The primary fields then take the form [16]:

$$
\psi \sim e^{-i(h+\bar{h})\tau - is\phi} (\cosh \rho)^{h+\bar{h}}
$$

(8.6)

From the last equation we can see why $s$ is called “spin,” seeing how it is dual to the angular variable $\phi$. On the other hand, $h + \bar{h}$ is the total conformal dimension of the field.

What one finds from applying the corresponding Casimir operator of $SL(2, \mathbb{R})$ built from $L_1$, $L_{-1}$ and $L_0$ is that the primary field $\psi$ of spin $s$ satisfies the differential equation [16]:

$$
[2h(h-1) + 2\bar{h}(\bar{h}-1)]\psi = \ell^2 \Box \psi + s^2 \coth^2 \rho \psi
$$

(8.7)

This makes clear where the connection between the mass of the field in the supergravity and the conformal weights $h$ and $\bar{h}$ of the boundary CFT comes from. Note that there is a contribution coming from the spin $s$ which calls for a nontrivial $\rho$ dependent value of $\Box \psi$. We also see some of the special features of 2d CFTs as opposed to 4d CFTs, where both $h$ and $\bar{h}$ play a role, including through the spin $s$, which is their difference. This is closely related to the fact that in 2d one can have right-movers and left-movers, which allows for a sort of chirality even for bosonic fields. Of course this is consistent with the fact that in 2d we can perform bosonization of fermionic modes.

To compute the “Laplacian” (or more accurately a D’Alembertian) associated with $AdS_3$ in the coordinate system above, one has to work out $\nabla^2 \psi = \nabla^\mu \nabla_\mu \psi$ for a scalar field $\psi$ in this nontrivial geometry. Because of the curvature of $AdS_3$, this entails working out the Christoffel symbols so that the covariant derivative can be computed. By straightforward calculations we find

\[
\begin{align*}
\Gamma^\tau_{\tau\tau} &= 0, \quad \Gamma^\tau_{\tau\rho} = \Gamma^\tau_{\rho\tau} = \tanh \rho, \quad \Gamma^\tau_{\tau\phi} = \Gamma^\tau_{\phi\tau} = 0 \\
\Gamma^\rho_{\rho\rho} &= 0, \quad \Gamma^\rho_{\rho\phi} = \Gamma^\rho_{\phi\rho} = 0, \quad \Gamma^\rho_{\rho\phi} = 0 \\
\Gamma^\rho_{\tau\rho} &= \sinh \rho \cosh \rho, \quad \Gamma^\rho_{\rho\tau} = \Gamma^\rho_{\rho\tau} = 0, \quad \Gamma^\rho_{\tau\phi} = \Gamma^\rho_{\phi\tau} = 0 \\
\Gamma^\phi_{\phi\rho} &= -\sinh \rho \cosh \rho, \quad \Gamma^\phi_{\rho\rho} = 0, \quad \Gamma^\phi_{\rho\phi} = 0, \quad \Gamma^\phi_{\rho\phi} = 0 \\
\Gamma^\phi_{\tau\tau} &= 0, \quad \Gamma^\phi_{\rho\phi} = 0, \quad \Gamma^\phi_{\phi\phi} = 0 \\
\Gamma^\phi_{\tau\phi} &= \Gamma^\phi_{\phi\tau} = 0, \quad \Gamma^\phi_{\rho\phi} = 0, \quad \Gamma^\phi_{\phi\phi} = 0 \\
\Gamma^\phi_{\rho\phi} &= \Gamma^\phi_{\phi\rho} = \coth \rho
\end{align*}
\]
Next, the D’Alembertian of a scalar field is given by
\[
\nabla_\mu \nabla^\mu \psi = g^{\mu\nu} (\partial_\mu \partial_\nu \psi - \Gamma^\sigma_{\mu\nu} \partial_\sigma \psi)
= g^{\tau\tau} (\partial_\tau^2 \psi - \Gamma^\sigma_{\tau\tau} \partial_\sigma \psi)
+ g^{\rho\rho} (\partial_\rho^2 \psi - \Gamma^\sigma_{\rho\rho} \partial_\sigma \psi)
+ g^{\phi\phi} (\partial_\phi^2 \psi - \Gamma^\sigma_{\phi\phi} \partial_\sigma \psi)
= -\frac{1}{\ell^2} \frac{1}{\cosh^2 \rho} \partial_\tau^2 \psi + \frac{1}{\ell^2} \partial_\rho^2 \psi + \frac{1}{\ell^2} \frac{1}{\sinh^2 \rho} \partial_\phi^2 \psi
+ \frac{1}{\ell^2} (\tanh \rho + \coth \rho) \partial_\rho \psi
\]

(8.9)

We can then apply this to the primary field (8.6). We find:
\[
\partial_\tau^2 \psi = -(h + \bar{h})^2 \psi
\]
\[
\partial_\rho^2 \psi = -s^2 \psi
\]
\[
\partial_\phi \psi = -(h + \bar{h}) \tanh \rho \cdot \psi
\]
\[
\partial_\phi^2 \psi = -(h + \bar{h}) \text{sech}^2 \rho \cdot \psi + (h + \bar{h})^2 \tanh^2 \rho \cdot \psi
\]

(8.10)

Then using these one finds after some hyperbolic trigonometric identities that
\[
\Box \psi = \nabla_\mu \nabla^\mu \psi
= [(h + \bar{h})^2 - 2(h + \bar{h}) - s^2 \text{csch}^2 \rho] \frac{\psi}{\ell^2}
\]

(8.11)

Now we perform the manipulation
\[
(h + \bar{h})^2 - 2(h + \bar{h}) = 2h(h - 1) + 2\bar{h}(\bar{h} - 1) - s^2
\]

(8.12)

to obtain
\[
\Box \psi = [2h(h - 1) + 2\bar{h}(\bar{h} - 1) - s^2 \coth^2 \rho] \frac{\psi}{\ell^2}
\]

(8.13)

It can be seen that this agrees with (8.7) above.

The D1-D5 system can be described by 6d supergravity on \( AdS_3 \times S^3 \), or the string version thereof. Thus the low energy excitations of the 2d CFT associated with D1-D5 can be extracted from a mode analysis of the 6d gravity compactified on \( S^3 \). For this, it is important to understand representations of the isometry group \( SO(4) \) of \( S^3 \) and how they correspond to spherical harmonics on that space. Supersymmetry is of a big help here.

To get to a 6d SUGRA from string theory, one must compactify on a 4d manifold. Two possibilities are considered in de Boer [16]: \( T^4 \) and \( K3 \). Let’s start with \( K3 \). For heterotic or Type I, this gives a \((0,1)\) theory. For Type IIA, this gives a \((1,1)\) theory. For Type IIB this gives \((0,2)\) theory. In contrast, on \( T^4 \), Types IIA and IIB give a \((2,2)\) theory. Here the notation indicates left-handed and right-handed supersymmetries. This is because in 2d
or 10d, we can have Weyl-Majorana spinors. Thus there are more possibilities than in 4d. These are denoted by \((n_L, n_R)\).

AdS\(_3\) \times S^3\) has isometry group \(SO(2, 2) \times SO(4)\) where the \(SO(2, 2)\) is associated with the AdS\(_3\) and the \(SO(4)\) is associated with \(S^3\). These can be seen by the coordinate descriptions that embed these spaces in one higher dimension. AdS\(_3\) can be described by the hyperbolic equation

\[
X_1^2 + X_2^2 - X_3^2 - X_4^2 = R^2
\]

which has an obvious \(SO(2, 2)\) symmetry. \(S^3\) of course can be described by

\[
X_1^2 + X_2^2 + X_3^2 + X_4^2 = R^2
\]

which has an \(SO(4)\) symmetry. In terms of covering groups,

\[
SO(2, 2) \simeq SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R})
\]

and

\[
SO(4) \simeq SU(2) \otimes SU(2)
\]

These are associated with left-handed and right-handed fields, each of which fall into representations of \(SL(2, \mathbb{R}) \otimes SU(2)\). These are actually embedded into larger groups \(G_L\) and \(G_R\), where

\[
G_L \supset SL(2, \mathbb{R}) \otimes SU(2)
\]

and similarly for \(G_R\). The related supergroup is \(SU(1, 1|2)\). Thus we are particularly interested in the representation theory of this supergroup, since we are dealing with a super-CFT.

There are some important analogies between the D1-D5 system and \(\mathcal{N} = 4\) super-Yang-Mills obtained through compactification of superstring theory on AdS\(_5\) \times S\(^5\). In the latter case, only short multiplets of the superconformal algebra are involved due to the 32 supersymmetries (4 Majorana supercharges with 4 components each and 4 additional superconformal fermionic generators that are also Majorana spinors with 4 components each) and the fact that the representations must have at most spin 2 corresponding to the graviton. In the case of 6d supergravity with 8 or 16 supersymmetries, corresponding to \((1,0)\), \((1,1)\) and \((2,0)\), again only short multiplets are involved. As shown by de Boer, this means that the KK spectrum and hence states in the 2d CFT can be computed completely using representation theory.

As usual with 2d CFTs with a large amount of supersymmetry, quite beautiful structures arise once all charges are taken into account. One has \(L_1\), \(L_0\) and \(L_{-1}\) defining an \(SL(2, \mathbb{R})\) algebra. Then one has \(J_+\), \(J_0\) and \(J_-\) defining an \(SU(2)\) algebra. Finally, one has \(Q^+\) and \(Q^-\) that are supersymmetry generators taking states back and forth from the bosonic side and fermionic side. Altogether this rounds out \(SU(2|1, 1)\). One has this both in the left movers and right movers in the case of \((1,1)\) supersymmetry. That gives an overall
$SU(2|1,1) \otimes SU(2|1,1)$ superalgebra. Or, in the case of $(0,2)$ supersymmetry both $SU(2|1,1)$s are in the right-moving sector.

We are interested in the CFT of the D1-D5 theory with the D5 branes wrapped on a four-dimensional compact manifold. A particular example is when the D5 branes are wrapped on the well-known manifold $K3$. In this case it has been conjectured that the CFT is a deformation of a supersymmetric $\sigma$ model \cite{35,36}. Furthermore, it has been conjectured that the target space of this deformed $\sigma$ model is $K3^N/S_N$. Here, $K3^N$ represents the product space built from $N$ copies of $K3$. Also, $S_N$ is the permutation group of order $N$. E.g., $S_3$ would contain the group operations that give the 6 permutations of $(1,2,3)$:

\begin{align}
&(1,2,3), \quad (2,3,1), \quad (3,1,2) \\
&(3,2,1), \quad (2,1,3), \quad (1,3,2)
\end{align}

(8.19)

In the first line we give the 3 even permutations and in the second line we give the 3 odd permutations. For large $N$, $S_N$ contains a very large number of elements. The notation $K3^N/S_N$ indicates that we have an equivalence between the points of the $K3^N$ space if they are related by the permutation group $S_N$. For example, if $(x_1,x_2,x_3)$ is a point in $K3 \times K3 \times K3$, with each $x_i$ a four-component position vector, then we would have the equivalence under the permutation $(2,1,3)$ of

$$(x_1,x_2,x_3) \simeq (x_2,x_1,x_3)$$

(8.20)

This is an orbifold because it is a quotient space built using a discrete group, $S_N$ (or $S_3$ in the specific example just given).

In the analysis of the CFT, a key quantity is the Poincare polynomial \cite{16}:

$$P_{t,\bar{t}} = \text{Tr} \left( t^{J_0} \bar{t}^{\bar{J}_0} \right)$$

(8.21)

Here, $J_0$ and $\bar{J}_0$ are the generators in the Cartan subalgebra of the $SU(2) \otimes SU(2)$ internal symmetry. Also, the trace is only over the chiral primaries of the CFT. The quantities $t$ and $\bar{t}$ are independent complex numbers. For instance, in counting states, de Boer was particularly interested in arbitrary $t$ but $\bar{t} = -1$, and the case of the $K3$ manifold: $P_{t,-1}(K3^N/S_N)$.

It turns out that this Poincare polynomial is directly related to the manifold of the target space in an interesting way:

$$P_{t,\bar{t}} = \sum_{p,q} h_{p,q} t^p \bar{t}^q$$

(8.22)

Here, $h_{p,q}$ are the Betti numbers of the manifold. Thus there is an intimate connection between homology and this polynomial.

There is also an interesting connection here to orbifold CFT. This is where we start with a CFT on a space $X$, and then perform an orbifold of the target space,

$$X \rightarrow X^n/Z_n$$

(8.23)
Or, if the CFT is $M$, then the orbifold is $M^n/Z_n$. It turns out that if we understand the representations of the conformal algebra on $X$, then there is a straightforward relation to representations on $X^n/Z_n$.

The untwisted sector of the orbifold CFT has

$$h' = nh, \quad q' = nq$$

(8.24)

Here, $h$ is the conformal weight of the CFT $M$ and $q$ is the charge under the internal symmetry. The primed values correspond to the same quantities in the orbifold CFT $M^n/Z_n$.

For the twisted sector labeled by $m$ (a non-negative integer), we have

$$h' = \frac{h + m}{n} + \frac{c}{24} \left( n^2 - 1 \right), \quad q' = q$$

(8.25)

where $c$ is the central charge of the CFT $M$.

Related to all of this is the generating function for the Poincare polynomial [16]:

$$\sum_{N \geq 0} Q^N P_{t,\bar{t}}(M^N/S_N) = \prod_{m=1}^{\infty} \prod_{p,q} (1 + (-1)^{p+q+1} Q^m t^{p+\frac{c}{2}(m-1)} \bar{t}^{q+\frac{c}{2}(m-1)} (-1)^{p+q+1} h_{p,q})$$

(8.26)

One should think about $Q$ as being a small complex number. Then one can take the $N$th derivative w.r.t. $Q$ and set $Q \to 0$ after differentiating. On the l.h.s. one then obviously ends up with the quantity of interest, $P_{t,\bar{t}}(M^N/S_N)$. Then on the r.h.s., one obtains an expression for this in terms of the Betti numbers.

Through various calculations along these lines, de Boer is able to count states, such as $67131N - 244053$ states of the form

$$| q, h \rangle_L \otimes | q', h' \rangle_R = | 0, 2 \rangle_L \otimes | q', q'/2 \rangle_R$$

(8.27)

Thus we can learn various numerological data about orbifold CFTs. One could imagine tabulating such things almost ad infinitum. de Boer’s motivation was to resolve certain quandries about missing states noticed by Vafa [37].

9 de Sitter holography

Holographic ideas can also be applied to de Sitter spacetime and hence cosmology. This is described for instance in Skenderis et al. [15]. (See also references 4-8 of that paper.) This particular approach has been coined “holographic cosmology.” It can be thought of as a holographic framework to address some of the shortcomings of conventional inflationary theory. Additionally, it is motivated by the fact that quantum gravity appears to be holographic in nature and there is little doubt that quantum gravity also plays a role in the earliest stages of the universe.

This scenario has a number of interesting features:

1. The dual theory to the cosmological universe is a 3d quantum field theory (QFT).
2. This QFT is located at future infinity.

3. The partition function, in the presence of sources for gauge invariant operators, is the wavefunction of the universe.

4. Cosmic evolution is mapped to (inverse) RG flow in the QFT.

5. It provides a model for a non-geometric early universe.

6. The fields in the QFT are taken to be in the adjoint representation. Thus 3d super-Yang-Mills falls into the class of theories to be considered, and may have a special cosmology.

Standard $\Lambda$CDM has six adjustable parameters. Skendaris et al. holographic cosmology addresses two of them: the tilt $n_s$ and the scalar perturbation power spectrum normalization. In the conventional cosmology these appear through the power spectrum:

$$\Delta^2_R(q) = \Delta^2_0(q_*) \left( \frac{q}{q_*} \right)^{n_* - 1}$$

(9.1)

In the holographic cosmology there is a different prediction for the power spectrum, which is given by

$$\Delta^2_R(q) = \frac{\Delta^2_0(q_*)}{1 + (gq_*/q) \ln |q/\beta gq_*| + O(gq_*/q)^2}$$

(9.2)

In the discussion of these authors, one finds that they can no longer trust their formula for small $\ell$ (the multipole moment of the CMB), because the two-loop term becomes competitive with the one-loop term. So really to handle the small $\ell$ behavior one needs to do a numerical simulation and compute the two-point function of the energy momentum tensor at small $q$.

The analytic continuation (14) in their paper raises the question of how this is related to an imaginary time correlator. Of course to carry out a computation using lattice field theory, one has to derive the energy momentum tensor. This can be nontrivial. For instance, in the formulation of $\mathcal{N} = 4$ 3d super-Yang-Mills on the lattice using the Blau-Thompson twist, it is a significant challenge to find the combination of operators and renormalization constants that really give a conserved energy-momentum tensor. This is something that we hope to complete in the future. Then we would have a supersymmetric holographic cosmology.

In [38] it is shown how the standard cosmological problems that motivate inflation are resolved in the context of holographic cosmology. More specifically, they look at the horizon problem, the flatness problem and the problem of relics (such as GUT monopoles).

For instance, for the horizon problem, points on the surface of last scattering are correlated through correlations in the dual QFT assuming they run deep enough into the IR. While these super-renormalizable theories are perturbatively IR divergent, it is believed that they are non-perturbatively IR finite. This then corresponds to a resolution of the cosmic initial singularity. This is certainly an interesting possibility. As is well-known, the presence of singularities in classical gravity has long intrigued physicists. It has always been hoped that a theory of quantum gravity would somehow resolve them. In the present context, this is accomplished through the dual QFT description, exploiting holography.
10 SUSY warped extra dimension

Now we briefly discuss some aspects of the low-energy effective field theory of formulating physics on a slice of AdS\(_5\). The leading terms of the action are of course

\[
S = - \int d^4x dy \sqrt{-g} \left\{ \text{Tr} F^{MN} F_{MN} + D^M \phi^\dagger D_M \phi + \bar{\Psi} \Gamma^M D_M \Psi \right\}
\]  

(10.1)

That is, a gauge action, a scalar action and a fermion action. The indices will be chosen as \(M = 0, 1, \ldots, 3\) or 5, where \(M = 5\) denotes the extra dimension; i.e., corresponding to the coordinate \(x^5 = y\) in equation (10.2) below. The metric appears implicitly in a few places, such as \(F^{MN} = g^{MP} g^{NQ} F_{PQ}, D^M = g^{MN} D_N\) and \(\Gamma^M = e_A^M \Gamma_A^\beta\). Here, \(\{\Gamma^A, \Gamma^B\} = -2 \eta^{AB}\) with \(\eta^{AB} = \text{diag}(-1, 1, 1, 1, 1)\). As usual, the frame is related to the metric, such as \(g^{MN} = e_A^M e_B^N \eta^{AB}\).

So far, our description is general to any five-dimensional (5d) spacetime. We now want to specify the Randall-Sundrum geometry, which is a slice of 5d anti-deSitter (AdS\(_5\)) space. There are several coordinate systems that we could choose from. Here we highlight one of the more common coordinate systems. The 5d metric is given by

\[
ds^2 = e^{-2k|y|} (-dt^2 + dx^2) + dy^2
\]  

(10.2)

where \(k\) is related to the inverse of the AdS radius \(L\) and is generally of order the 4d Planck mass. The extra dimension \(y\) is taken to be an orbifold, \(S^1/Z_2\). That is, one begins with a circle, \(y \simeq y + 2\pi R\). Then a further identification is made: \(y \simeq -y\). This results in two fixed points \(y = 0\) and \(y = \pi R\). Points \(y \in (\pi R, 2\pi R)\) are identified with points \(y \in (0, \pi R)\) since on \(S^1/Z_2\) we have \(y \simeq -y \simeq 2\pi R - y\).

Next one performs a KK decomposition on the space \(S^1/Z_2\). To complete this process, one must know the orbifold action on the fields. In particular, the fermions are subject to even or odd conditions depending on their chirality.

There is another coordinate system where the extra dimension is denoted as \(z\), with metric

\[
ds^2 = \frac{L^2}{z^2} (-dt^2 + dx^2 + dz^2)
\]  

(10.3)

We wish to relate this to the metric (10.2) above. By comparing these two we can figure out the change of coordinates. For instance matching the \(dt^2\) terms we see that

\[
e^{-2k|y|} = \frac{L^2}{z^2}
\]  

(10.4)

Taking the \(y > 0\) branch, we have that

\[
z = L e^{ky}, \quad \frac{dz}{z} = k \, dy
\]  

(10.5)
\[ L^2 \frac{dz^2}{z^2} = (kL)^2 dy^2 \]  

(10.6)

Finally, to match the two metrics, we have

\[ kL = 1 \]  

(10.7)

so that as stated, \( k \) is the inverse of the AdS\(_5\) radius \( L \). It will turn out that (10.3) is convenient for generalizations of AdS\(_5\) that are inspired by developments in string theory.

## 11 Deformed AdS

In [22] we considered a deformation of AdS\(_5\) that was inspired by some supergravity constructions that broke supersymmetry [39]. The idea was that by doing so we would be able to have supersymmetry breaking arise from the background geometry, but now in a Randall-Sundrum type phenomenological theory.

The geometry is

\[ ds^2 = A^2(z)(-dt^2 + dx^2 + dz^2) \]

\[ A^2(z) = \frac{1}{(kz)^2} \left[ 1 - \epsilon \left( \frac{z}{z_1} \right)^4 \right] \]  

(11.1)

with \( z_0 \leq z \leq z_1 \). \( z_0 \) is the position of the UV brane in the extra dimension. \( z_1 \) is the position of the IR brane. If \( \epsilon = 0 \) we would have a slice of AdS\(_5\). The presence of the term with coefficient \( \epsilon \) is supposed to be dual to dynamical supersymmetry breaking in the supersymmetric gauge theory.

The fermion equation of motion (free particle limit) after the rescaling \( \hat{\Psi} = A^2\Psi \) is

\[ (\delta^\mu_\alpha \gamma^\alpha \partial_\mu + \gamma_5 \partial_z + cA)\hat{\Psi} = 0 \]  

(11.2)

In the slice of AdS, the scalar mass is given by:

\[ M_\Phi^2 = ak^2 + 2bk^2 z[\delta(z-z_0) - \delta(z-z_1)] \]  

(11.3)

Thus the parameters \( a \) and \( b \) are key to the construction. The equation of motion for the scalars is:

\[ \partial_\mu \partial^\mu \Phi + A^{-3} \partial_5 (A^3 \partial^5 \Phi) - ak^2 A^2 \Phi = 0 \]  

(11.4)

This equation is solved by separation of variables:

\[ \Phi(x, z) = \sum_{n=0}^{\infty} \phi_n(x) f_n(z) \]  

(11.5)
The functions $\tilde{f}_n(z)$ are known as the “profiles” of the Kaluza-Klein (KK) modes. In the supersymmetric limit, which corresponds to $\epsilon \to 0$ in the deformed metric, zeromodes are produced when the modified Neumann conditions

$$\left. \left( \tilde{f}'_n - bk^2 z A^2 \tilde{f}_n \right) \right|_{z_0, z_1} = 0$$

(11.6)

are imposed. This also require that $b$ and $a$ be related by

$$b = 2 \pm \sqrt{4 + a}$$

(11.7)

In the $\epsilon \neq 0$ case, deviation from this produces masses for the scalars, leading to a supersymmetry violating splitting in the chiral multiplets.

## 12 Conclusions

We have seen that there are many interesting examples of holography beyond the standard $AdS_5 \times S^5$ construction. In some cases they have been brought quite close to phenomenology. This can occur through reducing the amount of supersymmetry or eliminating conformal invariance. It can also consist of a bottom-up approach that is inspired by string theory and supergravity. The D1-D5 system also shows that holographic duality for 2d CFTs and $AdS_3$ can be rich in representation theory, generating functions, Virasoro operators and other nice mathematical features. We described some of the difficulties of formulating these theories in terms of a lattice theory that can be simulated. However, we believe that with consistent effort progress can also be made in that direction. Then one would be able to compute features of quantum gravity by simulating a QFT. A interesting example of this occurs in holographic cosmology, which we briefly described.

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