Decorated Young Tableaux and the Poissonized Robinson-Schensted Process

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Abstract

We introduce an object called a decorated Young tableau that slightly extends the notion of a standard Young tableau. This object can equivalently be viewed as a continuous time trajectory of Young diagrams or as a non-intersecting line ensemble. The Robinson-Schensted correspondence can be extended to a bijection between finite configurations of points in the plane to pairs of decorated Young tableaux of the same shape. By applying this to a Poisson point process in the plane, we get a random pair of decorated Young tableaux, which we can think of as a Young diagram valued stochastic process in continuous time. By using properties of Young tableaux, we show that the finite dimensional distributions of this random object are a Schur process as introduced by Okounkov and Reshetikhin in [11]. We also show that when viewed as a non-intersecting line ensemble, this has the same law as certain Poisson walkers conditioned not to intersect. Finally, we relate this model to previous work by Borodin and Olshanski in [4] and Johansson in [7]. This allows us to compute asymptotic properties of our model, for example convergence to the Airy 2 process under the correct scaling.

1 Introduction

The Poissonized Plancherel measure is a one parameter family of measures on Young diagrams. For fixed $\theta$, this is a mixture of the classical Plancherel measures by Poisson weights. This mixture has nice properties that make it amenable to analysis, see for instance [3] and [9]. One way this measure is obtained is to take a unit rate Poisson point process in the square $[0, \theta] \times [0, \theta]$, then interpret the collection of points as a permutation, and finally apply the Robinson-Schensted (RS) correspondence. The RS correspondence gives a pair of Young tableaux of the same shape. The law of the shape of the Young tableaux constructed in this way has the Poissonized Plancherel measure. Other than the shape, the information inside the tableaux themselves are discarded in this construction. This construction has many nice properties: for example, by the geometric construction of the RS correspondence due to Viennot (see for example [12] for details), this shows that the maximum number of Poisson points an up-right path can pass through has the distribution of the length of the first row of the Poissonized Plancherel measure. One can use this to tackle problems like the longest increasing subsequence problem.

In this article, we extend the above construction slightly in order to keep the information in the Young tableaux that are generated by the RS algorithm; we do not discard the information in the tableaux. As a result, we get a slightly richer random object which we call the Poissonized Robinson-Schensted process. This object can be interpreted in several ways. If one views the object as a continuous time Young diagram valued stochastic process, then its fixed time marginals are exactly the Poissonized Plancherel measure. Moreover, the joint distribution at several times form a Schur process as defined in [11]. The model is defined in Section 2 and its distribution is characterized in Section 3.

We also show that the process itself is a special case of stochastic dynamics related to Plancherel measure studied in [4]. These connections allow us to immediately see asymptotics for the model, in particular it converges to the Airy-2 line ensemble under the correct scaling. This connection is discussed in Section 4.

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It is also possible to obtain the Poissonized RS process as a limit of a discrete time Young diagram process in a natural way. Instead of starting with a Poisson point process, one instead starts with a point process on a lattice so that the number of points at each site has a geometric distribution. This model was first considered by Johansson in Section 5 of [7], in particular see his Theorem 5.1. Under the right scaling, this model converges back to the model we have here. The details of the connection between his model and the model considered here are also discussed in Section 5.

1.1 Notation and Background

We very briefly go over the definitions/notations used here. For more details, see [13] or [12].

We denote by $\mathcal{Y}$ the set of Young diagrams. We think of a Young diagram $\lambda \in \mathcal{Y}$ as a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ where $\lambda_i$ are weakly decreasing and with finitely many non-zero entries. We can equivalently think of each $\lambda \subset \mathbb{N}^2$ as a collection of stacked unit boxes by $(i, j) \in \lambda \iff j \leq \lambda_i$. We denote by $|\lambda| = \sum_{i=1}^{n} \lambda_i$ the total number of boxes, or equivalently the sum of the row lengths. We will sometimes also consider skew tableau, which are the collection of boxes one gets from the difference of two Young diagrams $\lambda \setminus \mu$.

A standard Young tableau $T$ can be thought of as Young diagram $\lambda$ whose boxes have been filled with the numbers $1, 2, \ldots, |\lambda|$, so that the numbers are increasing in any row and in any column. We call the diagram $\lambda$ in this case the shape of the tableau, and denote this by $\text{sh}(T)$. We denote by $T(i, j)$ the entry written in the box at location $i, j$. We will also use the notation $\dim(\lambda)$ to denote the number of standard Young tableau of shape $\lambda$. This is called the “dimension” since this is also the dimension of the irreducible representations of the symmetric group $S(|\lambda|)$ associated with $\lambda$.

In the above notation the Poissonized Plancherel Measure of parameter $\theta$ is:

$$P_\theta(\lambda) = e^{-\theta^2} \left( \frac{\theta^{|\lambda|}}{|\lambda|!} \right)^2$$

The Robinson-Schensted (RS) correspondence is a bijection from the symmetric group $S_n$ to pairs of standard Young tableaux of the same shape of size $|\text{sh}(T)| = n$ (See [13] Section 7.11 for details on this bijection). We will sometimes refer to this here as the “ordinary” RS correspondence, not to diminish the importance of this, but to avoid confusion with a closely related map we introduce called the “decorated RS correspondence”.

We will also make reference to the Schur symmetric functions $s_\lambda(x_1, \ldots)$, and the skew Schur symmetric functions $s_{\lambda/\mu}(x_1, \ldots)$ as they appear in [14] or [12]. A specialization is a homomorphism from symmetric functions to complex numbers. We denote by $f(\rho)$ the image of the function $f$ under the specialization $\rho$. We denote by $\rho_t$ the Plancherel specialization (also known as exponential or “pure gamma” specialization) that has $h_n(\rho_t) = \frac{\theta^n}{n!}$ for each $n \in \mathbb{N}$. This is a Schur positive specialization, in the sense that $s_\lambda(\rho_t) \geq 0$ always, and moreover there is an explicit formula for $s_\lambda(\rho_t)$ in terms of the number of Young tableaux of shape $\lambda$:

$$s_\lambda(\rho_t) = \frac{\dim(\lambda)t^{\lambda}|}{|\lambda|!}$$

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2 Decorated Young Tableaux

Definition 2.1. A decorated Young tableau is a pair $\tilde{T} = (T, (t_1, \ldots, t_{|\text{sh}(T)|}))$ where $T$ is a standard Young tableau and $0 \leq t_1 < \ldots < t_{|\text{sh}(T)|}$ is an increasing list of non-negative numbers whose length is equal to the size of the tableau. We refer to the list $(t_1, \ldots, t_{|\text{sh}(T)|})$ as the decorations of the tableau. We represent this graphically when drawing the tableau by recording the number $t_{T(i,j)}$ in the box $(i,j)$.
Example 2.2. The decorated Young Tableau:

\[
\begin{bmatrix}
1 & 2 & 4 \\
3 & 5 \\
6
\end{bmatrix},
\begin{bmatrix}
0.02, 0.03, 0.05, 0.07, 0.11, 0.13
\end{bmatrix}
\]

is represented as:

\[
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 \\
6 & 0.02 & 0.03 & 0.07 \\
& 0.05 & 0.11 \\
& & 0.13
\end{array}
\]

Remark 2.3. Since the decorations are always sorted, we see that from the above diagram one could recover the entire decorated tableau without the labels “1”, “2” written in the tableau. In other words, one could equally well think of a decorated Young tableau as a map \( T : \text{sh}(T) \rightarrow \mathbb{R}_+ \), so that \( T \) is increasing in each column and in each row. Having \( T = (T, (t_1, \ldots, t_{\text{sh}(T)})) \) will be slightly more convenient for our explanations here, and particularly to relate the model to previous work.

Definition 2.4. A decorated Young tableau can also be thought of as a trajectory of Young diagrams evolving in continuous time. The Young diagram process of the decorated Young tableau \( T = (T, (t_1, \ldots, t_{\text{sh}(T)})) \) is a map \( \lambda_T : \mathbb{R}_+ \rightarrow \mathbb{Y} \) defined by:

\[ \lambda_T(t) = \{(i, j) : t_{T(i, j)} \leq t\} \in \mathbb{Y} \]

One can also think about this as follows: the process starts with \( \lambda(0) = \emptyset \), and then it gradually adds boxes one by one. The decoration \( t_{T(i, j)} \) is the time at which the box \((i, j)\) is added. The fact that \( T \) is a standard Young tableau ensures that \( \lambda(t) \) is indeed a Young diagram at every time \( t \). Notice that the Young diagram process for a decorated Young tableau is always increasing \( \lambda(t_1) \subset \lambda(t_2) \) whenever \( t_1 \leq t_2 \), and it can only increase by at most one box at a time. This holds since \( \lambda_T(t) \in \mathbb{Y} \) is a Young diagram. It is clear that one can recover the Young diagram process from the non-intersecting line ensemble by \( \lambda_T(t) = M_T(i; t) + i \).

Definition 2.5. A decorated Young tableau can also be thought of as an ensemble of non-intersecting lines. The non-intersecting line ensemble of the decorated Young tableau \( T = (T, (t_1, \ldots, t_{\text{sh}(\lambda)})) \) is a map \( M_T : \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{Z} \) defined by:

\[ M_T(i; t) = \lambda_T(t) - i = |\{j : t_{T(i, j)} \leq t\}| - i \]

where \( \lambda_T(t) = (\lambda_T(1), \ldots) \) is the Young diagram process of \( T \). The lines \( M_T(i; t) \) are non-intersecting in the sense that \( M_T(i; t) < M_T(j; t) \) for \( i < j \) and for every \( t \in \mathbb{R}_+ \). This holds since \( \lambda_T(t) \in \mathbb{Y} \) is a Young diagram.

Remark 2.6. The map from Young diagrams to collection of integers by \( \lambda \rightarrow \{\lambda_i - i\}_{i=1}^{\infty} \) is a well known map with mathematical significance, see for instance [2] for a survey. This is sometimes presented as the map \( \lambda \rightarrow \{\lambda_i - i + \frac{1}{2}\}_{i=1}^{\infty} \), where the target is now half integers. This presentation sometimes makes the resulting calculations much nicer. In this work they do not play a big role, so we will omit the \( \frac{1}{2} \) that some other authors use.

2.1 Robinson–Schensted Correspondence

Definition 2.7. Fix a parameter \( \theta \in \mathbb{R}_+ \) and let \( C_n^\theta \) be the set of \( n \) point configurations in the square \([0, \theta] \times [0, \theta] \subset \mathbb{R}_+^2 \) so that no two points lie in the same horizontal line and no two points lie in the same
vertical line. Every configuration of points \( \Pi \in \mathcal{C}_n^\theta \) has an
associated permutation \( \sigma \in S_n \) by the following prescription. Suppose that
\( 0 \leq r_1 < \ldots < r_n \leq \theta \) and \( 0 \leq \ell_1 < \ldots < \ell_n \leq \theta \) are respectively the
sorted lists of \( x \) and \( y \) coordinates of the points form \( \Pi \). Then find the unique permutation \( \sigma \in S_n \) so
that \( \Pi = \{(r_1, \ell_\sigma(k))\}_{k=1}^n \). Equivalently, if we are given the list of points \( \Pi = \{(x_i, y_i)\}_{i=1}^n \) sorted so that
\( 0 \leq x_1 < \ldots < x_n \leq \theta \) then \( \sigma \) is the permutation so that \( 0 \leq y_{\sigma^{-1}(1)} < y_{\sigma^{-1}(2)} < \ldots < y_{\sigma^{-1}(n)} \leq \theta \)

**Definition 2.8.** Let:
\[
\mathcal{T}_n^\theta = \{(L, (\ell_1, \ldots, \ell_n)), (R, (r_1, \ldots, r_n)) : \text{sh}(L) = \text{sh}(R), 0 \leq \ell_1 < \ldots < \ell_n \leq \theta, 0 \leq r_1 < \ldots < r_n \leq \theta\}
\]
be the set of pairs of decorated Young tableaux of the same shape and of size \( n \), whose decorations lie in the
interval \([0, \theta]\).

The decorated Robinson–Schensted (RS) correspondence is a bijection \( dRS : \mathcal{T}_n^\theta \to \mathcal{C}_n^\theta \) from pairs
of decorated Young tableaux in \( \mathcal{T}_n^\theta \) to configuration of points in \( \mathcal{C}_n^\theta \) defined as follows:

Given a pair of decorated Tableau of size \( n \), \( (L, (\ell_1, \ldots, \ell_n)), (R, (r_1, \ldots, r_n)) \), use the ordinary RS bijection and the pair of Young tableaux \( (L, R) \) to get a permutation \( \sigma \in S_n \). Then define:
\[
dRS ((L, (\ell_1, \ldots, \ell_n)), (R, (r_1, \ldots, r_n))) = \{(r_1, \ell_{\sigma(1)}), (r_2, \ell_{\sigma(2)}), \ldots, (r_n, \ell_{\sigma(n)})\}
\]
Going the other way, the inverse \( dRS^{-1} : \mathcal{C}_n^\theta \to \mathcal{T}_n^\theta \) is described as follows. Given a configuration of
points from \( \mathcal{C}_n^\theta \), first take the permutation \( \sigma \) associated with the configuration as described in Definition 2.7. Then use the ordinary RS bijection to find a pair of standard Young tableaux \( (L, R) \) corresponding to this
permutation. Define:
\[
dRS^{-1} \{(x_1, y_1), \ldots, (x_n, y_n)\} = (L, (y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)})), (R, (x_1, \ldots, x_n))
\]
Since the ordinary RS algorithm is a bijection from pairs of standard Young diagrams of size \( n \) to permutations
in \( S_n \), and since the decorations can be recovered from the coordinates of the points and vice versa
as described above, the decorated RS algorithm is indeed a bijection as the name suggests.

For convenience we will later on use the notations \( \mathcal{C}^\theta = \cup_{n \in \mathbb{N}} \mathcal{C}_n^\theta \) and \( \mathcal{T}^\theta = \cup_{n \in \mathbb{N}} \mathcal{T}_n^\theta \).

**Remark 2.9.** With the viewpoint as in Remark 2.3, one can equivalently construct the decorated RS bijection by starting with the list of points in \( \Pi \) in “two line notation” \( (x_1, x_2 \ldots x_n) \), where the points \( \{(x_i, y_i)\}_{i=1}^n \) are sorted by \( x \)-coordinate, and then apply the RS insertion algorithm on these points to build up the Young tables \( \tilde{L} \) and \( \tilde{R} \). The same rules for insertion in the ordinary RS apply; the only difference is that the
entries and comparisons the algorithm makes are between real numbers instead of natural numbers.

Each of the individual decorated tableaux from a pair \( (\tilde{L}, \tilde{R}) \in \mathcal{T}_n \) have an associated Young diagram
process as defined in Definition 2.4 and an associated non-intersecting line ensemble as defined in Definition 2.5. Since both \( L \) and \( R \) are the same shape, and since the decoration are all in the range \([0, \theta]\), the Young

diagram processes and the non-intersecting line ensembles will agree at all times \( t \geq \theta \). That is to say
\( \lambda_L(t) = \lambda_{\tilde{R}}(t) \) and \( M_{\tilde{L}}(\cdot; t) = M_{\tilde{R}}(\cdot; t) \) for \( t \geq \theta \). For this reason, it will be more convenient to do a change of coordinates on the time axis so that the Young diagram process and non-intersecting line ensemble are defined on \([-\theta, \theta]\), and the meeting of the left and right tableau happen at \( t = 0 \). The following definition makes this precise.

**Definition 2.10.** We define the Young diagram process \( \lambda_{\tilde{L}, \tilde{R}} : [-\theta, \theta] \to \mathbb{Y} \) of the pair \( (\tilde{L}, \tilde{R}) \) \( \in \mathcal{T}_n \) by:
\[
\lambda_{\tilde{L}, \tilde{R}}(t) = \begin{cases} 
\lambda_{\tilde{L}}(\theta + t) & t \leq 0 \\
\lambda_{\tilde{R}}(\theta - t) & t \geq 0
\end{cases}
\]
Notice that this is well defined at \( t = 0 \) since \( \lambda_{\tilde{L}}(\theta) = \text{sh}(L) = \text{sh}(R) = \lambda_{\tilde{R}}(\theta) \). In this way \( \lambda_{\tilde{L}, \tilde{R}} \) is an
increasing sequence of Young diagrams when \( t < 0 \) and is a decreasing when \( t > 0 \).

Similarly, we define the non-intersecting line ensemble \( M_{\tilde{L}, \tilde{R}} : \mathbb{Z} \times [-\theta, \theta] \to \mathbb{Z} \) of the pair \( (\tilde{L}, \tilde{R}) \) \( \in \mathcal{T}_n \) by:
\[
M_{\tilde{L}, \tilde{R}}(i; t) = \begin{cases} 
M_{\tilde{L}}(i; \theta + t) & t \leq 0 \\
M_{\tilde{R}}(i; \theta - t) & t \geq 0
\end{cases}
\]
Again, each of the lines are well defined at \( t = 0 \) because \( M_{\tilde{L}}(i; \theta) = M_{\tilde{R}}(i; \theta) \).

4
3 The Poissonized Robinson-Schensted Process

3.1 Definition

Definition 3.1. Fix a parameter \( \theta \in \mathbb{R}^+ \). A rate 1 Poisson point process in \([0, \theta] \times [0, \theta]\) is a probability measure on the set of configurations \( \mathcal{C}^\theta \). By applying the decorated RS correspondence this induces a probability measure on \( \mathcal{T}^\theta \). We will refer to the resulting random pair of tableaux \((\tilde{L}, \tilde{R}) \in \mathcal{T}^\theta \) as the Poissonized RS tableaux, we refer to the resulting random Young diagram process as the Poissonized RS process and the resulting random non-intersecting line ensemble as the Poissonized RS line ensemble. A realization of the Poissonized RS line ensemble for the case \( \theta = 40 \) is displayed in Figure 1.

The main results of this article are to characterize the law of the Poissonized RS process. Both the Young diagram process and the non-crossing line ensemble of this object have natural descriptions. In this section we describe the laws of these. For the remainder of the section, denote by \((\tilde{L}, \tilde{R}) = ((L, (\ell_1, \ldots, \ell_{\text{sh}L})), (R, (r_1, \ldots, r_{\text{sh}R})))\) a Poissonized RS random variable. The Young diagram process \(\lambda_{\tilde{L}, \tilde{R}}\) is a \(\mathcal{Y}\) valued stochastic process, and \(M_{\tilde{L}, \tilde{R}}\) is a random non-intersecting line ensemble.

Figure 1: A realization of the non-intersecting line ensemble for the Poissonized RS process in the case \( \theta = 40 \) created from \( \approx 1600 \) points in the plane. Only the lines that are non-constant are shown. This can be simulated very efficiently because the tableaux are created by exactly one execution of the decorated Robinson-Schensted algorithm.
3.2 Law of the Young diagram process

**Theorem 3.2.** Fix \( \theta > 0 \) and times \( t_1 < \ldots < t_n \in [-\theta, 0] \) and \( s_1 < \ldots < s_m \in [0, \theta] \). Suppose we are given an increasing list of Young diagrams \( \lambda^{(1)} \subset \ldots \subset \lambda^{(n)} \), a decreasing list of Young diagrams \( \mu^{(1)} \supset \ldots \supset \mu^{(m)} \) and a Young diagram \( \nu \) with \( \nu \supset \lambda^{(n)} \) and \( \nu \supset \mu^{(1)} \). To simplify the presentation we will use the convention \( \lambda^{(0)} = \emptyset, \lambda^{(n+1)} = \nu, \mu^{(0)} = \nu, \mu^{(m+1)} = \emptyset \) and \( t_0 = -\theta, t_{n+1} = 0, s_0 = 0, s_{m+1} = \theta \). The Poissonized RS process \( \lambda_{L,R} \) has the following finite dimensional distribution:

\[
\mathbb{P} \left( \bigcap_{i=1}^{n} \left\{ \lambda_{L,R}(t_i) = \lambda^{(i)} \right\} \cap \left\{ \lambda_{L,R}(0) = \nu \right\} \cap \bigcap_{j=1}^{m} \left\{ \lambda_{L,R}(s_j) = \mu^{(j)} \right\} \right) = e^{-\theta^2} \left( \prod_{i=0}^{n} \dim(\lambda^{(i+1)}/\lambda^{(i)}) \frac{(t_{i+1} - t_i)\lambda^{(i+1)}/\lambda^{(i)}}{\lambda^{(i+1)}/\lambda^{(i)}!} \right) \cdot \left( \prod_{j=0}^{m} \dim(\mu^{(j+1)}/\mu^{(j)}) \frac{(s_{j+1} - s_j)\mu^{(j)}/\mu^{(j+1)}}{\mu^{(j)}/\mu^{(j+1)}!} \right)
\]

Where \( \dim(\lambda/\mu) \) is the number of standard Young tableau of skew shape \( \lambda/\mu \).

The proof of Theorem 3.2 is deferred to Subsection 3.3 and is proven in a number of lemmas.

**Remark 3.3.** The conclusion of the theorem can be rewritten in a very algebraically satisfying way in terms of Schur functions specialized by the Plancherel specialization, \( \rho(t) \) (see Subsection 1.1 for our notations).

Using the identity from Equation 1 the result of Theorem 3.2 can be rewritten:

\[
\mathbb{P} \left( \bigcap_{i=1}^{n} \left\{ \lambda_{L,R}(t_i) = \lambda^{(i)} \right\} \cap \left\{ \lambda_{L,R}(0) = \nu \right\} \cap \bigcap_{j=1}^{m} \left\{ \lambda_{L,R}(s_j) = \mu^{(j)} \right\} \right) = e^{-\theta^2} \left( \prod_{i=0}^{n} s_{\lambda^{(i+1)}}(\rho_{t_{i+1} - t_i}) \right) \cdot \left( \prod_{j=0}^{m} s_{\mu^{(j)}}(\rho_{s_{j+1} - s_j}) \right)
\]

In the literature (see for instance [11] or [2] for a survey) this type of distribution arising from specializations on sequence of Young diagrams is known as a Schur Process. The particular Schur process that appears here has a very simple “staircase” diagram, illustrated here in the case \( n = m = 2 \):

![Diagram](https://via.placeholder.com/150)

In Section 4, we will further see that the Poissonized RS process \( \lambda_{L,R} \) is the same as a particular instance model introduced in [4], which is itself a special case of dynamics studied in [5].

**Corollary 3.4.** At any fixed time \( t \), the Young diagram \( \lambda_{L,R}(t) \) has the law of the Poissonized Plancherel measure with parameter \( \sqrt{\theta|\theta-t|} \).

**Proof.** Suppose first that \( t \leq 0 \) with the case \( t \geq 0 \) being analogous. By Theorem 3.2 we have the two time probability distribution of \( \lambda_{L,R} \) at time \( t \) and time 0 is:

\[\mathbb{P} \left( \left\{ \lambda_{L,R}(t) = \lambda \right\} \cap \left\{ \lambda_{L,R}(0) = \nu \right\} \right) = e^{-\theta^2} s_{\lambda}(\rho_{t}) s_{\nu}(\rho_{\theta}) \]
Summing over $\nu \in \mathcal{Y}$ and employing the Cauchy identity $\sum_\nu s_{\nu/\lambda}(\rho)s_{\nu}(\rho') = H(\rho; \rho')s_\lambda(\rho')$, and using $H(\rho; \rho') = \exp(ab)$ for the exponential specialization, we have:

$$P\left(\lambda_{L,R}(t) = \lambda\right) = e^{-\theta^2} s_\lambda(\rho t + \theta) \sum_{\nu \in \mathcal{Y}} s_{\nu/\lambda}(\rho_0 - t) s_\nu(\rho_0 - \theta)$$

$$= e^{-\theta^2} H(\rho - t; \rho_0) s_\lambda(\rho_0)$$

$$= e^{-\theta^2} e^{t |\theta|} s_\lambda(\rho_0)$$

$$= e^{-\left(\sqrt{\theta(\theta - |t|)}\right)^2} \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2$$

□

The Poissonized Plancherel measure and its asymptotics are well studied, see for example [3] or [9]. The analysis lets us see that, for any fixed $t$, the points of the line ensemble $L_{L,R}(\cdot; t)$ form a determinantal point process whose kernel is the discrete Bessel kernel. We can also use these results to write some asymptotics for the Poissonized RS line ensemble, for instance the following:

**Corollary 3.5.** Let $(L_{\theta}, R_{\theta})$ be the Poissonized RS tableaux of parameter $\theta$. For fixed $\tau \in (-1, 1)$, the top line $M_{L_{\theta}, R_{\theta}}(1; \cdot)$ of the line ensemble at some fixed time $t$ satisfies the following law of large numbers type behavior:

$$\lim_{\theta \to \infty} \frac{M_{L_{\theta}, R_{\theta}}(1; \tau) - \sqrt{1 - |\tau|}}{\theta} = 2\sqrt{1 - |\tau|} \text{ a.s.}$$

The fluctuations are of the Tracy-Widom type:

$$\lim_{\theta \to \infty} P\left(\frac{M_{L_{\theta}, R_{\theta}}(1; \tau) - \sqrt{1 - |\tau|}}{\theta^{1/3}(1 - |\tau|)^{1/6}} \leq s\right) = F(s)$$

where $F(s)$ is the GUE Tracy Widom distribution.

### 3.3 The non-intersecting line ensemble

**Definition 3.6.** Fix a parameter $\theta > 0$ and an initial location $x \in \mathbb{Z}$. Let $P_L : [0, \theta] \to \mathbb{Z}$ and $P_R : [0, \theta] \to \mathbb{Z}$ be two independent rate 1 Poisson jump processes with initial condition $P_L(0) = P_R(0) = x$. Define $P : [-\theta, \theta] \to \mathbb{Z}$ by:

$$P(t) = \begin{cases} P_L(\theta + t) & t < 0 \\ P_R(\theta - t) & t \geq 0 \end{cases}$$

A Poisson arch on $[-\theta, \theta]$ with initial location $x$ is the stochastic process $\{A(t)\}_{t \in [-\theta, \theta]}$ whose probability distribution is the conditional probability distribution of the process $\{P(t)\}_{t \in [-\theta, \theta]}$ conditioned on the event that $\{P_L(\theta) = P_R(\theta)\}$. This has $A(-\theta) = A(\theta) = x$ and the conditioning ensures that $A(t)$ is actually continuous at $t = 0$.

The Poissonized RS line ensemble, $M_{L_{\theta}, R_{\theta}}(\cdot; \cdot)$, has a simple description in terms of Poisson arches which are conditioned not to intersect:

**Theorem 3.7.** Fix $\theta > 0$ and times $-\theta < t_1 < \ldots < t_n < \theta$. For any $N \in \mathbb{N}$, consider a non-intersecting line ensemble $A : \{1, 2, \ldots, N\} \times [-\theta, \theta] \to \mathbb{Z}$, so that $\{A(i; \cdot)\}_{i = 1}^N$ is a collection of $N$ Poisson arches on $[-\theta, \theta]$ with the initial condition $A_i(\theta) = A_i(\theta) = 1 - i$ which are conditioned not to intersect i.e. $A(i; t) < A(j; t)$ for all $i > j$. Then the joint probability distributions of the line ensemble $A$ has the same conditional distribution.
as top $N$ lines of the non-intersecting line ensemble $M_{L,R}$, conditioned on the event that all of the other lines $M_{L,R}(k;\cdot)$ for $k > N$ do not move at all. To be precise, for fixed target points $\{x_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq N}$ we have:

$$\Pr \left( \bigcap_{i=1}^{n} \bigcap_{j=1}^{N} \{A(j; t_i) = x_{i,j}\} \right) = \Pr \left( \bigcap_{i=1}^{n} \bigcap_{j=1}^{N} \{M_{L,R}(j; t_i) = x_{i,j}\} \bigg| M_{L,R}(k;\cdot) \equiv 1 - k \forall k > N \right)$$

The proof of this goes through the Karlin-MacGregor theorem and is deferred to Section 3.5.

3.4 Proof of Theorem 3.2

We prove this theorem by splitting it into several lemmas. The idea behind these is to separate the activity of the left and right tableaux as much as possible, and also to separate out the behavior of the shape of the diagrams from the sizes of the diagrams.

**Lemma 3.8.** Let $N = |\lambda_{L,R}(0)|$. Then $N$ has the distribution of a Poisson random variable, $N \sim \text{Poisson}(\theta^2)$. Moreover, conditioned on the event $\{N = n\}$, the decorations $(\ell_1, \ldots, \ell_n)$ and $(r_1, \ldots, r_n)$ of $(L, R)$ are independent. Still conditioned on $\{N = n\}$, the permutation $\sigma \in S_n$ associated with $(L, R)$ via the RS correspondence is uniformly distributed in $S_n$ and is independent of both sets of decorations.

**Proof.** From the construction of the Poissonized RS process, $|\lambda_{L,R}(0)|$ is the number of points in the square $[0, \theta] \times [0, \theta]$ of a rate 1 Poisson point process in the plane. Hence $N \sim \text{Poisson}(\theta^2)$ is clear. Conditioned on $\{N = n\}$ the points of the rate 1 Poisson process in question are uniformly distributed in the square $[0, \theta] \times [0, \theta]$ (This is a general property of Poisson point processes). Consequently, each point has an $x$-coordinate and $y$-coordinate which are uniformly distributed in $[0, \theta]$ and are independent of all the other coordinates. Hence, since the decorations $(\ell_1, \ldots, \ell_n), (r_1, \ldots, r_n)$ consist of the sorted $y$-coordinates and $x$-coordinates respectively, these decorations are independent of each other. (More specifically: they have the distribution of the order statistics for a sample of $n$ uniformly distributed points in $[0, \theta]$.)

To see that the permutation $\sigma \in S_n$ associated with this points is uniformly distributed in $S_n$ first notice that this is the same as the permutation associated with the points of the Poisson Process. Then notice that if $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ are independent drawings of the order statistics for a sample of $n$ uniformly distributed points in $[0, \theta]$ and $\pi \in S_n$ is drawn uniformly at random and independently of everything else, then the points $\{(a_i, b_{\pi(i)})\}_{i=1}^{n}$ is a sample of $n$ points chosen uniformly from $[0, \theta] \times [0, \theta]$. This construction of the $n$ uniform points shows that the permutation $\sigma$ is uniformly distributed and independent of both the $x$ and $y$ coordinates. \qed

**Corollary 3.9.** For any Young diagram $\nu$, we have:

$$\Pr \left( \lambda_{L,R}(0) = \nu \right) = \left( \frac{\dim(\nu)^2}{|\nu|!} \right) \Pr (N = |\nu|)$$

**Proof.** By construction, the Young diagram $\lambda_{L,R}(0)$ is the common shape of the tableaux $(L, R)$. Hence $\{\lambda_{L,R}(0) = \nu\} = \{N = |\nu|\} \cap \{\text{sh}(L) = \nu\}$. Notice that $\text{sh}(L)$ depends only on the associated permutation $\sigma$, whose conditional distribution is known here to be uniform in $S_n$ by Lemma 3.8. Since the RS correspondence is a bijection, we have only to count the number of pairs of tableaux of shape $\nu$. This gives:

$$\Pr \left( \lambda_{L,R}(0) = \nu \right) = \Pr \left( \text{sh}(L) = \nu \big| N = |\nu| \right) \Pr (N = |\nu|) = \left( \frac{\dim(\nu)^2}{|\nu|!} \right) \Pr (N = |\nu|)$$

\qed
Lemma 3.10. Consider the law of the process conditioned on the event \(\{\lambda_{L,R}(0) = \nu\}\). The conditional probability that \(\lambda_{L,R}\) has the correct sizes at times \(t_1, \ldots, t_n \in [-\theta, 0]\) is given by:

\[
P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \mid \lambda_{L,R}(0) = \nu \right) = \frac{\prod_{i=0}^{n} P_{\theta(t_{i+1} - t_i)} \left( |\lambda(i+1)| - |\lambda(i)| \right)}{P(N = |\nu|)}
\]

where \(P_r(k) = e^{-r} \frac{r^k}{k!}\) is the Poisson probability mass function.

An analogous formula holds for \(P\left( \bigcap_{i=1}^{m} \{ |\lambda_{L,R}(s_i)| = |\mu(i)| \} \mid \lambda_{L,R}(0) = \nu \right)\) by the same principles.

Moreover, we have the following type of conditional independence at times \(-\theta < t_1 < \ldots < t_n < 0\) and at times \(0 < s_1 < \ldots < s_m < \theta\):

\[
P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \cap \bigcap_{i=1}^{m} \{ |\lambda_{L,R}(s_i)| = |\mu(i)| \} \mid \lambda_{L,R}(0) = \nu \right)
\]

\[
= P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \mid \lambda_{L,R}(0) = \nu \right) \cdot P\left( \bigcap_{i=1}^{m} \{ |\lambda_{L,R}(s_i)| = |\mu(i)| \} \mid \lambda_{L,R}(0) = \nu \right)
\]

Proof. As in the previous lemma, we have \(\{\lambda_{L,R}(0) = \nu\} = \{N = |\nu|\} \cap \{\text{sh}(L) = \nu\}\). Then consider:

\[
P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \cap \{\lambda_{L,R}(0) = \nu\} \right)
\]

\[
= P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \cap \{N = |\nu|\} \cap \{\text{sh}(L) = \nu\} \right)
\]

\[
= P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \cap \{\text{sh}(L) = \nu\} \mid N = |\nu| \right) P(N = |\nu|)
\]

But, conditioned on \(N = |\nu|\), the event \(\bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \}\) and \(\{\text{sh}(L) = \nu\}\) are independent. The former depends only on the decorations \((t_1, \ldots, t_n)\) by the definition of \(\lambda_{L,R}\) and since \(t_i's < 0\) while the later depends only on the associated permutation \(\sigma\), and these are conditionally independent by Lemma 3.8.

Hence:

\[
P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \cap \{\text{sh}(L) = \nu\} \mid N = |\nu| \right) P(N = |\nu|)
\]

\[
= P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \mid N = |\nu| \right) P\left( \text{sh}(L) = \nu \mid N = |\nu| \right) P(N = |\nu|)
\]

\[
= \frac{P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \cap \{N = |\nu|\} \right) P\left( \lambda_{L,R}(0) = \nu \right)}{P(N = |\nu|)}
\]

\[
= P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \cap \{N = |\nu|\} \right) P\left( \lambda_{L,R}(0) = \nu \right)
\]

\[
= P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \mid N = |\nu| \right) P\left( \lambda_{L,R}(0) = \nu \right)
\]

Putting this together, we have:

\[
P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \mid \lambda_{L,R}(0) = \nu \right) = P\left( \bigcap_{i=1}^{n} \{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \} \mid N = |\nu| \right)
\]
Now, from the definition of the Young diagram process, we have for $-\theta < t < 0$, that $|\lambda_{L,R}(t)| = |\{i : \ell_i < t + \theta\}|$ and we see that the event $\cap_{i=0}^{n+1} \left\{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \right\} \cap \{ N = \nu \}$ depends only on counting the number of decorations from $(\ell_1, \ldots, \ell_{|\nu|})$ in the appropriate regions:

$$\bigcap_{i=1}^{n} \left\{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \right\} \cap \{ N = \nu \} = \bigcap_{i=0}^{n+1} \left\{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \right\}$$

$$= \bigcap_{i=0}^{n+1} \left\{ |j : t_i + \theta < \ell_j < t_{i+1} + \theta| = |\lambda(i+1)| - |\lambda(i)| \right\}$$

Finally, from the construction, we notice that the random variable $|\{ j : t_i + \theta < \ell_j < t_{i+1} + \theta| \}$ counts the number of points of the Poisson point process in the region $[t_i + \theta, t_{i+1} + \theta] \times [0, \theta]$. Consequently these random variables are independent for different values of $i$ and are distributed according to $|\{ j : t_i + \theta < \ell_j < t_{i+1} + \theta| \sim \text{Poisson}(\theta(t_{i+1} - t_i))$

This observation, together with the preceding display, gives the desired first result of the lemma.

To see the second result about the conditional independence at times $-\theta < t_1 < \ldots < t_n < 0$ and at times $0 < s_1 < \ldots < s_m < \theta$, we repeat the arguments above and notice that times $-\theta < t_1 < \ldots < t_n < 0$ depend only on the decorations $(\ell_1, \ldots, \ell_n)$ (because of $|\lambda_{L,R}(t)| = |\{i : \ell_i < t + \theta\}|$) while the times $0 < s_1 < \ldots < s_m < \theta$ depend only on the decorations $(r_1, \ldots, r_m)$ (because of $|\lambda_{L,R}(s)| = |\{i : r_i < \theta - s\}|$). These decorations are conditionally independent when conditioned on $\{ N = \nu \}$ by Lemma 3.8 and the desired independence result follows.

**Lemma 3.11.** Let $C_0 = \{ \lambda_{L,R}(0) = \nu \}$. We have:

$$\mathbb{P} \left( \bigcap_{i=0}^{n+1} \left\{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \right\} \cap C_0 \right) = \frac{\prod_{i=1}^{n+1} \dim(\lambda(i+1)/\lambda(i))}{\dim(\nu)}$$

An analogous formula holds for $\mathbb{P} \left( \bigcap_{i=0}^{m+1} \left\{ |\lambda_{L,R}(s_i)| = |\mu(i)| \right\} \cap C_0 \right)$ by the same principles.

Moreover, we have the following type of conditional independence at times $-\theta < t_1 < \ldots < t_n < 0$ and at times $0 < s_1 < \ldots < s_m < \theta$:

$$\mathbb{P} \left( \bigcap_{i=0}^{n+1} \left\{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \right\} \cap \bigcap_{i=0}^{m+1} \left\{ |\lambda_{L,R}(s_i)| = |\mu(i)| \right\} \cap C_0 \right)$$

$$= \mathbb{P} \left( \bigcap_{i=0}^{m+1} \left\{ |\lambda_{L,R}(s_i)| = |\mu(i)| \right\} \cap \bigcap_{i=0}^{m+1} \left\{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \right\} \cap C_0 \right)$$

**Proof.** For a standard Young tableau $T$, $a < b \in \mathbb{N}$ we will denote by $\text{sh}(T_{a,b})$ the skew Young diagram which consist of the boxes of $T$ which are labeled with an number $i$ so that $a \leq i \leq b$ and the empty young diagram in the case $b < a$. With this notation, we now notice that that $\{ \lambda_{L,R}(t) = \lambda \} \iff \{ \text{sh}(L_1,|\lambda_{L,R}(t_i)|) = \lambda \}$. By the same token:

$$\bigcap_{i=1}^{n+1} \left\{ |\lambda_{L,R}(t_i)| = |\lambda(i)| \right\} = \bigcap_{i=0}^{n} \left\{ \text{sh}(L_1,|\lambda_{L,R}(t_i)|) = \lambda(i) \right\}$$

$$= \bigcap_{i=0}^{n} \left\{ \text{sh}(L,|\lambda_{L,R}(t_i)|+1^{-1},|\lambda_{L,R}(t_{i+1})|) = \lambda(i+1)/\lambda(i) \right\}$$

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Now consider, using the above and facts from the proof of Lemma 3.10

\[
P \left( \bigcap_{i=0}^{n+1} \left\{ \lambda_L, \tilde{R}(t_i) = \lambda(i) \right\} \right) \cap \bigcap_{i=0}^{n+1} \left\{ |\lambda_L, \tilde{R}(t_i)| = |\lambda(i)| \right\} \cap C_0 \]

\[
= P \left( \bigcap_{i=0}^{n+1} \left\{ \text{sh}(L|\lambda_L, \tilde{R}(t_i)|+1, |\lambda_L, \tilde{R}(t_{i+1})|) = \lambda(i+1)/\lambda(i) \right\} \right) \cap \bigcap_{i=0}^{n+1} \left\{ |\lambda_L, \tilde{R}(t_i)| = |\lambda(i)| \right\} \cap C_0 \]

We now notice that the event in question depends only on the Young tableau \(L\), which is entirely determined by the associated permutation \(\sigma\) (via the RS algorithm). Consequently, the conditioning on \(\bigcap_{i=0}^{n} \left\{ |\lambda_L, \tilde{R}(t_i)| = |\lambda(i)| \right\}\), which depends only on the decorations \((\ell_1, \ldots, \ell_{|\lambda|})\), has no effect here since \(\sigma\) and these decorations are conditionally independent by Lemma 3.8. Removing this conditioning on \(\bigcap_{i=0}^{n} \left\{ |\lambda_L, \tilde{R}(t_i)| = |\lambda(i)| \right\}\), we remain with:

\[
P \left( \bigcap_{i=0}^{n+1} \left\{ \lambda_L, \tilde{R}(t_i) = \lambda(i) \right\} \right) \cap \bigcap_{i=0}^{n+1} \left\{ |\lambda_L, \tilde{R}(t_i)| = |\lambda(i)| \right\} \cap C_0 \]

With this conditioning, since \(\sigma\) is uniformly distributed in \(S_{|\nu|}\), and because the RS algorithm is a bijection, the Young Tableau \(L\) is uniformly distributed among the set of all Young tableau of shape \(\nu\). Hence it suffices to count the number of tableau of shape \(\nu\) with the correct intermediate shapes i.e. the tableau \(L\) that have \(\text{sh}(L|\lambda(i)|+1, |\lambda(i+1)|) = \lambda(i+1)/\lambda(i)\) for each \(i\). Since each \(L|\lambda(i)|+1, |\lambda(i+1)|\) must itself be a standard Young tableau of skew shape \(\lambda(i+1)/\lambda(i)\), this is:

\[
\left\{ \text{S.Y.T. } L : \text{sh}(L) = \nu, \text{ sh} \left( L|\lambda(i)|+1, |\lambda(i+1)| \right) = \lambda(i+1)/\lambda(i) \forall 1 \leq i \leq n \right\} \]

\[
= \dim(\lambda_1/\emptyset) \dim(\lambda_2/\lambda_1) \cdots \dim(\lambda_n/\lambda_{n-1}) \dim(\nu/\lambda_n). \]

Dividing by \(\dim(\nu)\), the total number of tableau of shape \(\nu\), gives the desired probability and completes the first result of the lemma.

To see the second result about the conditional independence at times \(-\theta < t_1 < \ldots < t_m < \theta\) and at times \(0 < s_1 < \ldots < s_m < \theta\), we repeat arguments analogous to the above to the point where we have:

\[
P \left( \bigcap_{i=0}^{n+1} \left\{ \lambda_L, \tilde{R}(t_i) = \lambda(i) \right\} \cap \bigcap_{i=0}^{m+1} \left\{ \lambda_L, \tilde{R}(s_i) = \mu(i) \right\} \right) \cap \bigcap_{i=0}^{n+1} \left\{ |\lambda_L, \tilde{R}(t_i)| = |\lambda(i)| \right\} \cap \bigcap_{i=0}^{m+1} \left\{ |\lambda_L, \tilde{R}(s_i)| = |\mu(i)| \right\} \cap C_0 \]

\[
= P \left( A_L \cap A_R \right) \cap C_0 \]

\[
A_L := \bigcup_{i=0}^{n+1} \left\{ \text{sh}(L|\lambda(i)|+1, |\lambda(i+1)|) = \lambda(i+1)/\lambda(i) \right\} \]

\[
A_R := \bigcup_{i=0}^{m+1} \left\{ \text{sh}(R|\mu(i)|+1, |\mu(i+1)|) = \mu(i+1)/\mu(i) \right\} \]

Now the event \(A_L\) depends only on the left tableau \(L\) while the event \(A_R\) depends only on the right tableau \(R\). Hence \(A_L\) and \(A_R\) are conditionally independent when conditioned on the event \(C_0\) by Lemma 3.8 we get the desired independence result.

\[\square\]

**Proof.** (Of Theorem 3.2). This is a simple matter of conditioning on the right things and applying the lemmas until we reach an explicit formula. To make it a bit easier to read, let \(C_0 = \left\{ \lambda_L, \tilde{R}(0) = \nu \right\}\). We have:
Proof. (Of Theorem 3.7) First, by an application of the Karlin-MacGregor theorem and the Jacobi-Trudi identity, we compute the distribution of the Poisson arches in terms of Schur functions. Then, by Theorem 3.2, the right hand side is computed to be the same expression. For convenience of notation, divide the times into two parts, times \( -\theta < t_1 < \ldots < t_n < 0 \) and \( 0 < s_1 < \ldots < s_m < \theta \), and put \( t_0 = -\theta, t_{n+1} = 0, s_0 = 0, s_{m+1} = \theta \). Set target points \( \{x_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq N}, \{z_j\}_{1 \leq j \leq N} \) and \( \{y_{i,j}\}_{1 \leq i \leq m, 1 \leq j \leq N} \) and consider the event \( \left\{ \cap_{i=1}^{N} \left( \cap_{j=1}^{N} \{ A(j; t_i) = x_{i,j} \} \right) \cap \cap_{i=1}^{m} \left( \cap_{j=1}^{N} \{ A(j; s_i) = y_{i,j} \} \right) \right\} \). To each of the fixed time “slices”, we Young diagram \( \{\lambda^{(i)}\}_{1 \leq i \leq n}, \{\mu^{(i)}\}_{1 \leq i \leq m} \) and \( \nu \) by prescribing the length of the rows:

\[
\lambda_i \begin{cases} k \quad &1 \leq k \leq N \\ 0 \quad &k > N \end{cases}, \quad \nu_k \begin{cases} k \quad &1 \leq k \leq N \\ 0 \quad &k > N \end{cases}, \quad \mu_k \begin{cases} k \quad &1 \leq k \leq N \\ 0 \quad &k > N \end{cases}
\]

Reuse the same conventions as from Theorem 3.2. \( \lambda^{(0)} = \emptyset, \lambda^{(n+1)} = \nu, \mu^{(0)} = \nu, \mu^{(m+1)} = \emptyset \) and \( t_0 = -\theta, t_{n+1} = 0, s_0 = 0, s_{m+1} = \theta \). Notice that by the definitions, \( \lambda^{(i)} \) and \( \mu^{(j)} \) are always Young diagrams with at most \( N \) non-empty rows. Moreover, the admissible target points are exactly in bijection with the space \( \mathcal{Y}(N) \) of Young diagrams with at most \( N \) non-empty rows.
By application of the Karlin-MacGregor theorem [10], we know that:

\[
P \left( \bigcap_{i=1}^{n} \bigcap_{j=1}^{N} \{ A(j; t_i) = x_{i,j} \} \cap \bigcap_{i=1}^{m} \bigcap_{j=1}^{N} \{ A(j; s_i) = y_{i,j} \} \right)
= Z_{t_1, \ldots, t_n, s_1, \ldots, s_m}^{-1} \prod_{i=1}^{n+1} \det \left( W_{t_i-t_{i-1}}^{+}(x_{i-1,a}, x_{i,b}) \right)_{1 \leq a, b \leq N} \cdot \prod_{i=1}^{m+1} \det \left( W_{s_i-s_{i-1}}^{-}(y_{i-1,a}, y_{i,b}) \right)_{1 \leq a, b \leq N}
\]

Here the weights \( W_t^+ \) and \( W_s^- \) are Poisson weights for an increasing/decreasing Poisson process:

\[
W_t^+(x, y) = \frac{e^{(y-x)}}{(y-x)!} 1_{y>x} \quad W_t^-(x, y) = \frac{e^{(x-y)}}{(x-y)!} 1_{x>y}
\]

(We can safely ignore the factor of \( e^{-t} \) that appears in the transition probabilities as long as we are consistent with this convention when we compute the normalizing constant \( Z_{t_1, \ldots, s_m} \).) We will now use some elementary facts from the theory of symmetric functions to simplify the result (see again [13] or [12]). Firstly, we use the following identity for the complete homogenous symmetric functions, specialized to the exponential elementaries from the theory of symmetric functions to simplify the result (see [13] or [12]).

Firstly, we use the following identity for the complete homogenous symmetric functions, specialized to the exponential parameter \( t \), namely:

\[
h_n(\rho_t) = \frac{t^n}{n!}
\]

With this in hand, we notice that \( W_t^+(x, y) = h_{y-x}(\rho_t) \) and \( W_t^-(x, y) = h_{x-y}(\rho_t) \). Hence by the Jacobi-Trudi identity (see again [13] or [12]) we have:

\[
\det \left( W_t^+(x_{i-1,a}, x_{i,b}) \right)_{1 \leq a, b \leq N} = \det \left( h_{\lambda_i(\rho_t)} \right)_{1 \leq a, b \leq N} = s_{\lambda(\rho_t)} = s_{\lambda(i)/\lambda(i-1)}(\rho_t)
\]

Similarly, we have:

\[
\det \left( W_t^+(x_{i-1,a}, x_{i,b}) \right)_{1 \leq a, b \leq N} = s_{\mu(i)/\mu(i-1)}(\rho_t)
\]

Thus:

\[
P \left( \bigcap_{i=1}^{n} \bigcap_{j=1}^{N} \{ A(j; t_i) = x_{i,j} \} \cap \bigcap_{i=1}^{m} \bigcap_{j=1}^{N} \{ A(j; s_i) = y_{i,j} \} \right)
= Z_{t_1, \ldots, t_n, s_1, \ldots, s_m}^{-1} \prod_{i=0}^{n} s_{\lambda(i+1)/\lambda(i)}(\rho_{t_i+1-t_i}) \cdot \prod_{i=0}^{m} s_{\mu(i)/\mu(i+1)}(\rho_{s_i+1-s_i})
\]

We now recognize from the statement of Theorem 3.2 that this is exactly this is the probability of the Young diagram process \( \lambda_{L,R} \) passing through the Young diagrams \( \{ \lambda(i) \}_{1 \leq i \leq n} \) and \( \{ \mu(j) \}_{1 \leq j \leq m} \) at the appropriate times except for the constant factor of \( Z_{t_1, \ldots, t_n, s_1, \ldots, s_m} \exp \left(-\theta^2 \right) \). By the construction of the non-intersecting line ensemble in terms of the Young diagram process, this is exactly the same as the first \( N \) lines of the non-intersecting line ensemble hitting the targets \( \{ x_{i,j} \} \) and \( \{ y_{i,j} \} \) at the appropriate times and, since these Young diagrams have at most \( N \) non-empty rows, the remaining rows must be trivial:

\[
\left( \prod_{i=0}^{n} s_{\lambda(i+1)/\lambda(i)}(\rho_{t_i+1-t_i}) \right) \cdot \left( \prod_{i=0}^{m} s_{\mu(i)/\mu(i+1)}(\rho_{s_i+1-s_i}) \right)
\]

\[
P \left( \bigcap_{i=1}^{n} \{ \lambda_{L,R}(t_i) = \lambda(i) \} \cap \{ \lambda_{L,R}(0) = \nu \} \cap \bigcap_{j=1}^{m} \{ \lambda_{L,R}(s_j) = \mu(j) \} \right)
= P \left( \bigcap_{i=1}^{n} \left\{ M_{L,R}(j; t_i) = x_{i,j} \right\} \cap \bigcap_{i=1}^{m} \left\{ M_{L,R}(j; s_i) = y_{i,j} \right\} \cap \left\{ M_{L,R}(k; \cdot) = 1-k \forall k > N \right\} \right)
\]

\[
\text{Box} \]

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The constant $Z_{t_1,\ldots,t_n \ldots,s_m}$ can be calculated as a sum over all possible paths the non-crossing arches can take. By our above calculation, this is the following sum over all possible sequences of Young diagrams \(\{\alpha(i)\}_{i=1}^{n+1} \subset \mathbb{Y}(N), \{\beta(i)\}_{i=1}^{m+1} \subset \mathbb{Y}(N), \nu = \alpha(n+1) = \beta(m)\), which have at most \(N\) non-empty rows:

$$Z_{t_1,\ldots,t_n \ldots,s_m} = \sum_{\{\alpha(\cdot)\}, \{\beta(\cdot)\}} \left( \prod_{i=0}^{n+1} s_{\alpha(i)/\alpha(i-1)}(\rho_{t_{i+1}-t_i}) \right) \left( \prod_{i=0}^{m+1} s_{\beta(i)/\beta(i-1)}(\rho_{s_{i+1}-s_i}) \right)$$

Again, by Theorem 3.2, except up to a constant factor \(\exp(-\theta^2)\), this can be interpreted as a probability for the Young diagram process \(\tilde{L},\tilde{R}\) or the line ensemble \(M_{\tilde{L},\tilde{R}}\). Because we sum over all possibilities for the first \(N\) rows, we remain only with the probability that the Young diagram process \(\lambda_{\tilde{L},\tilde{R}}\) never has more than \(N\) non-trivial rows, or equivalently that all the line ensemble remains still for all \(k > N\):

$$Z_{t_1,\ldots,t_n \ldots,s_m} \propto \sum_{\{\alpha(\cdot)\}, \{\beta(\cdot)\}} P \left( \bigcap_{i=1}^{n} \{\lambda_{\tilde{L},\tilde{R}}(t_i) = \alpha(i)\} \cap \{\lambda_{\tilde{L},\tilde{R}}(0) = \nu\} \bigcap_{j=1}^{m} \{\lambda_{\tilde{L},\tilde{R}}(s_j) = \beta(j)\} \right)$$

$$= P \left( \lambda_{\tilde{L},\tilde{R}}(\cdot) \text{ has at most } N \text{ non-empty rows} \right)$$

$$= P \left( M_{\tilde{L},\tilde{R}}(k; \cdot) = 1 - k \text{ } \forall k > N \right)$$

Combining the two calculations, we see that the two factors of \(\exp(-\theta^2)\) cancel and we remain with:

$$P \left( \bigcap_{i=1}^{n} \left( \bigcap_{j=1}^{N} \{ A(j; t_i) = x_{i,j}\} \right) \cap \bigcap_{i=1}^{m} \left( \bigcap_{j=1}^{N} \{ A(j; s_i) = y_{i,j}\} \right) \right)$$

$$= P \left( \bigcap_{i=1}^{n} \left( \bigcap_{j=1}^{N} \{ M_{\tilde{L},\tilde{R}}(j; t_i) = x_{i,j}\} \right) \cap \bigcap_{i=1}^{m} \left( \bigcap_{j=1}^{N} \{ M_{\tilde{L},\tilde{R}}(j; s_i) = y_{i,j}\} \right) \cap \{ M_{\tilde{L},\tilde{R}}(k; \cdot) = 1 - k \text{ } \forall k > N \} \right)$$

$$= P \left( \bigcap_{i=1}^{n} \left( \bigcap_{j=1}^{N} \{ M_{\tilde{L},\tilde{R}}(j; t_i) = x_{i,j}\} \right) \cap \bigcap_{i=1}^{m} \left( \bigcap_{j=1}^{N} \{ M_{\tilde{L},\tilde{R}}(j; s_i) = y_{i,j}\} \right) \bigg| M_{\tilde{L},\tilde{R}}(k; \cdot) = 1 - k \text{ } \forall k > N \right)$$

### 4 Relationship to Stochastic Dynamics on Partitions

In this section we show that the Poissonized RS process can be understood as a special case of certain stochastic dynamics on partitions introduced by Borodin and Olshanski in [1].

**Theorem 4.1.** Let \((u(t), v(t)), t \in [-\theta, \theta]\) be the parametric curve in \(\mathbb{R}^2_+\) given by:

$$u(t) = \begin{cases} \theta & t \leq 0 \\ \theta - |t| & t \geq 0 \end{cases}, \quad v(t) = \begin{cases} \theta - |t| & t \leq 0 \\ \theta & t \geq 0 \end{cases}$$

Denote by \(\square(u(t), v(t))\) the rectangular region \([0, u(t)] \times [0, v(t)] \subset \mathbb{R}^2\). For any point configuration \(\Pi \in \mathcal{C}\), let \(\pi(t)\) be the permutation associated with the point configuration \(\Pi \cap \square(u(t), v(t))\) (as in Definition 2.7) and let \(\lambda_\Pi(t) = sh(RS(\pi(t)))\) be the shape of the Young tableau that one gets by applying the ordinary RS bijection to the permutation \(\pi(t)\).

If \((\tilde{L}, \tilde{R}) = dRS(\Pi)\) is the decorated Young tableau that one gets by applying the decorated RS bijection to the configuration \(\Pi\), then the Young diagram process of \((\tilde{L}, \tilde{R})\) is exactly \(\lambda_\Pi(t)\):

$$\lambda_{\tilde{L},\tilde{R}}(t) = \lambda_{\Pi}(t) \quad \forall t \in [-\theta, \theta]$$

**Proof.** For concreteness, let us suppose there are \(n\) points in the configuration \(\Pi\) and label them \((x_i, y_i)\) sorted in ascending order of \(x\)-coordinate. Also label \(\sigma = \pi(0) \in S_n\) be the permutation associated to the
configuration $\Pi$ and let $(\tilde{L}, \tilde{R}) = (L, (y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}), R, (x_1, \ldots, x_n))$ be the output of the decorated RS correspondence.

We prove first the case $t \in [0, \theta]$, then use a symmetry property of the RS algorithm to deduce the result for $t \in [-\theta, 0]$. Fix a $k$ with $1 \leq k \leq n$ and let us restrict our attention to times $t \in [0, \theta]$ for which $x_{k-1} < \theta - t \leq x_k$. By the definition of the Young diagram process, we have then by this choice of $t$ that:

$$\lambda_{\tilde{L}, \tilde{R}}(t) = \{(i, j) : x_{R(i,j)} \leq \theta - t\} = \{(i, j) : R(i,j) \leq k\}$$

Now, by the definition of the decorated RS correspondence, the pair of tableaux $(L, R)$ correspond to the permutation $\sigma$ when one applies the ordinary RS algorithm. Since $\tilde{R}$ is the recording tableau here, the set \{(i, j) : R(i,j) \leq k\} is exactly the shape of the tableaux in the RS algorithm after $k$ steps of the algorithm. At this point the algorithm has used only comparisons between the numbers $\sigma(1), \sigma(2), \ldots, \sigma(k)$; it has not seen any other numbers yet.

On the other hand, we have $\square (u(t), v(t)) = \{(x_i, y_i) \in \Pi : x_i < \theta - t\} = \{(x_i, y_i)\}_{i=1}^k$ by the choice $x_{k-1} < \theta - t \leq x_k$ and since $x_i$ are sorted. So $\lambda_{u,v}(t) = \text{sh} (RSK(\pi(t)))$ is the shape outputted by the RS algorithm after it has worked on the permutation $\pi(t) \in S_k$ using comparisons between the numbers $\pi(t)(1), \pi(t)(2), \ldots, \pi(t)(k)$.

But we now notice that for $1 \leq i, j \leq k$ that $\pi(t)(i) \leq \pi(t)(j)$ if and only if $\sigma(i) < \sigma(j)$ since they both happen if and only if $y_i < y_j$. Hence, in computing $\lambda_{u,v}(t)$, the RS algorithm makes the exact same comparisons as the first $k$ steps of the RS algorithm on the list $\sigma(1), \sigma(2), \ldots, \sigma(k)$. For this reason, $\lambda_{\Pi}(t) = \{(i, j) : R(i,j) \leq k\}$. Hence $\lambda_{\tilde{L}, \tilde{R}}(t) = \lambda_{\Pi}(t)$, as desired. Since this works for any choice of $k$, this covers all of $t \in [0, \theta]$. To handle $t \in [-\theta, 0]$ consider as follows. Let $\Pi^T$ be the reflection of the point configuration $\Pi$ about the line $x = y$, in other words swapping the $x$ and $y$ coordinates of every point. Then the permutation associated with $\Pi^T$ is $\sigma^{-1}$. Using the remarkable fact of the RS correspondence $RS(\sigma) = (L, R) \iff RS(\sigma^{-1}) = (R, L)$ we will have from the definition that $dRS(\Pi^T) = (\tilde{R}, \tilde{L}) = (R, (x_1, \ldots, x_n), L, (y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}))$.

Using the result for $t \in [0, \theta]$ applied to the configuration $\Pi$, we have $\lambda_{\tilde{R}, \tilde{L}}(t) = \lambda_{R, R}(t)$ for all $t \in [0, \theta]$. Now, $\lambda_{\tilde{R}, \tilde{L}}(t) = \lambda_{\tilde{L}, \tilde{R}}(-t)$ follows from the definition of the Young diagram process, we have then by this choice of $t$ that:

$$\lambda_{\tilde{L}, \tilde{R}}(t) = \{(i, j) : x_{R(i,j)} \leq \theta - t\} = \{(i, j) : R(i,j) \leq k\}$$

It is also true that $\lambda_{\Pi^T}(t) = \lambda_{\Pi}(t)$; this follows from the fact that $(\square (u(t), v(t)))^T = \square (u(-t), v(-t))$ as regions in the plane $\mathbb{R}^2$ and so the permutations at time $t$ will have $\sigma_{\Pi^T}(t) = (\sigma_{\Pi^{-1}}(-t))^{-1}$. Since the RS correspondence assigns the same shape to the inverse permutation have $\lambda_{\Pi^T}(t) = \lambda_{\Pi}(t)$. Hence we conclude $\lambda_{\tilde{L}, \tilde{R}}(t) = \lambda_{\Pi}(t)$ for $t \in [-\theta, 0]$ too.

**Remark 4.2.** This is exactly the same construction of the random trajectories $\lambda_{\Pi}(u(t), v(t))$ from Theorem 2.3 in [4]. Note that in this work, the curve $(u(t), v(t))$ was going the other way going “clockwise” around the outside edge of the box $[0, \theta] \times [0, \theta]$ rather than “counterclockwise” as we have here. This difference just arises from the convention of putting the recording tableau as the right tableau when applying the RS algorithm and makes no practical difference. Since our construction is a special case of the stochastic dynamics constructed from this paper, we can use the scaling limit results to compute the limiting behavior of the Poissonized RS tableaux. The only obstruction is that one has to do some change of time coordinate to translate to what is called “interior time” in [4] along the curve so that $s = \frac{1}{2} (\ln u - \ln v)$. In the below corollary, we record the scaling limit for the topmost line of the ensemble. It is also possible to get a convergence result to the multi-line Airy 2 line ensemble.

**Corollary 4.3.** Let $(\tilde{L}_\theta, \tilde{R}_\theta)$ be the Poissonized Robinson-Schensted tableaux of parameter $\theta$. If we scale around the point $\tau = 0$, there is convergence to the Airy 2 process on the time scale $\theta^{2/3}$, namely:

$$M_{\tilde{L}_\theta, \tilde{R}_\theta}(1; 2\theta^{2/3} |\tau|) - (2\theta - 2\theta^{2/3} |\tau|) + \tau^2 \Rightarrow A_2(\tau)$$

Where $A_2(\cdot)$ is the Airy 2 process.

**Proof.** We first do a change of variables in the parameter $\theta$ by $\alpha = \theta^2$ so that the curves we consider have area $u_\alpha(0)v_\alpha(0) = \alpha$ at time 0. We will use $s$ to denote “interior time” along the curve $(u_\alpha, v_\alpha)$ constructed in
Theorem 4.1 (see Remark 1.4 in [3] for an explanation of the interior time). This is a change of time variable $s = s(t)$ given by:

$$s(t) = \frac{1}{2} \left( \ln u_\alpha(t) - \ln v_\alpha(t) \right) = \frac{1}{2} \text{sgn} \left( t \right) \ln \left( 1 - \frac{|t|}{\sqrt{\alpha}} \right)$$

Notice by Taylor expansion now that $s(2\alpha^{1/3}\tau) = -\tau \alpha^{-1/6} + o(\alpha^{-1/6})$ as $\alpha \to \infty$. By application of Theorem 4.4. from [3], (see also Corollary 4.6. for the simplification of Airy line ensemble to the Airy process) we have that as $\alpha \to \infty$:

$$M_{L_\alpha, R_\alpha} \left( 1; 2\alpha^{1/3}\tau \right) - 2\sqrt{u_\alpha(2\alpha^{1/3}\tau)v_\alpha(2\alpha^{1/3}\tau)} \alpha^{1/6} \Rightarrow \mathcal{A}_2(\tau)$$

Doing a Taylor expansion now for $2\sqrt{u_\alpha(t)v_\alpha(t)} = 2\sqrt{\alpha} \sqrt{1 - \frac{|t|}{\sqrt{\alpha}}}$ gives:

$$2\sqrt{u_\alpha(2\alpha^{1/3}\tau)v_\alpha(2\alpha^{1/3}\tau)} = 2\sqrt{\alpha} - 2\alpha^{1/3}|\tau| - \alpha^{1/6}\tau^2 + o(\alpha^{1/6})$$

Putting this into the above convergence to $\mathcal{A}_2$ gives the desired result. \hfill \Box

**Remark 4.4.** The scaling that is needed for the convergence of the top line to the Airy 2 process here is exactly the same as the scaling that appears for a family of non-crossing Brownian bridges to converge to the Airy 2 process, see [3]. This is not entirely surprising in light of Theorem 3.2, which shows that $M_{L_\alpha, R_\alpha}$ is related to a family of non-crossing Poisson arches, and it might be expected that non-crossing Poisson arches have the same scaling limit as non-crossing Brownian bridges.

## 5 A discrete limit

The Poissonized RS process can be realized as the limit of a discrete model created from geometric random variables in a certain scaling limit. This discrete model is a special case of the corner growth model studied in Section 5 of [7]. We will present the precise construction of the model here, rather than simply citing [7] in order to present it in a way that makes the connection to the Poissonized RS tableaux more transparent. We also present a different argument yielding the distribution of the model here, again to highlight the connection to the Poissonized RS tableaux. Our proof is very different than the proof from [7]; it has a much more probabilistic flavor closer to the proof of Theorem 4.2.

One difference between the discrete model and the Poissonized RS process is due to the possibility of multiple points with the same x-coordinate or y-coordinate. (These events happen with probability 0 for the Poisson point process.) To deal with this we must use Robinson-Schensted-Knuth (RSK) correspondence, which generalizes the RS correspondence to a bijection from generalized permutations to semistandard Young tableau (SSYT). See Section 7.11 in [13] for a reference on the RSK correspondence.

### 5.1 Geometric Robinson-Schensted-Knuth Process

**Definition 5.1.** Fix a parameter $\theta \in \mathbb{R}_+$ and an integer $k \in \mathbb{N}$. Let $L^{\theta,k} = \{\theta/k, 2\theta/k \ldots \theta\}$ be a discretization of the interval $[0, \theta]$ with $k$ points. Let $T$ be any semistandard Young tableau (SSYT) whose entries do not exceed $k$. The $L^{\theta,k}$-discretized Young diagram process for $T$ is a Young diagram valued map $\lambda^{\theta,k}_T : \mathbb{R}_+ \to \mathcal{Y}$ defined by:

$$\lambda^{\theta,k}_T(t) = \left\{ (i, j) : T(i, j) \leq t \right\}$$

If we are given two such SSYT $L, R$ so that $sh(L) = sh(R)$, we can define the $L^{\theta,k}$-discretized Young diagram process $\lambda^{\theta,k}_{L,R} : [-\theta, \theta] \to \mathcal{Y}$ for the pair $(L, R)$ by:

$$\lambda^{\theta,k}_{L,R}(t) = \begin{cases} \lambda^{\theta,k}_L(\theta + t) & t \leq 0 \\ \lambda^{\theta,k}_R(\theta - t) & t \geq 0 \end{cases}$$
would be represented graphically as: (compare with Example 2.2)

\[
\begin{array}{cccc}
\frac{1}{\theta/k} & \frac{2}{2\theta/k} & \frac{2}{2\theta/k} \\
\frac{3}{3\theta/k} & \frac{4}{4\theta/k} & \\
\end{array}
\]

In other words, the decorations are proportional to the entries in the Young tableau by a constant of proportionality $\theta/k$. This scaling of the $L^{\theta,k}$-discretized Young diagram process, despite being a very simple proportionality, will however be important to have convergence to the earlier studied Poissonized RS model in a limit as $k \to \infty$.

**Definition 5.3.** Let $C_n^{\theta,k}$ be the set of $n$ point configurations on the lattice $L^{\theta,k} \times L^{\theta,k} \subset \mathbb{R}^2$, where we allow the possibility of multiple points to sit at each site. Since there are only $k^2$ possible locations for the points, one can think of elements of $C_n^{\theta,k}$ in a natural way as $N$-valued $k \times k$ matrices, whose entries sum to $n$:

\[
C_n^{\theta,k} = \left\{ \left( \pi_{a,b} \right)_{a,b=1}^k : \sum_{a,b=1}^k \pi_{a,b} = n \right\}
\]

Let $T_n^{\theta,k}$ be the set of pairs of semistandard tableaux of size $n$ and of the same shape whose entries do not exceed $k$. That is:

\[
T_n^{\theta,k} = \{(L, R) : \text{sh}(L) = \text{sh}(R), |L| = |R| = n, L(a, b) \leq k, R(a, b) \leq k \forall a, b\}
\]

The Robinson-Schensted-Knuth (RSK) correspondence is a bijection between $N$-valued matrices (or equivalently generalized permutations) and pairs of semistandard tableaux of the same shape. Thinking of $C_n^{\theta,k}$ as $N$-valued matrices, we see more precisely that the RSK correspondence is a bijection between $C_n^{\theta,k}$ and $T_n^{\theta,k}$. Composing this with the definition of the $L^{\theta,k}$-discretized Young diagram process we have a bijection, which we call the $L^{\theta,k}$-discretized RSK bijection between configurations in $C_n^{\theta,k}$ and $L^{\theta,k}$-discretized Young diagram processes $\lambda_{L,R}^{\theta,k} : [-\theta, \theta] \to \mathcal{Y}$.

We will also use the shorthand $C_n^{\theta,k} = \bigcup_n C_n^{\theta,k}$ and $T_n^{\theta,k} = \bigcup_n T_n^{\theta,k}$.

**Definition 5.4.** Let $\{\xi_{i,j} : 1 \leq i \leq k, 1 \leq j \leq k\}$ be an iid collection of Geometric random variables with parameter $\theta^2/k^2$. To be precise, each $\xi$ has the following probability mass function:

\[
P(\xi = x) = \left(1 - \frac{\theta^2}{k^2}\right) \left(\frac{\theta^2}{k^2}\right)^x
\]

This gives a probability measure on the set of point configurations $C_n^{\theta,k}$ by placing exactly $\xi_{i,j}$ points at the location $(i \theta/k, j \theta/k)$. By applying the $L^{\theta,k}$-discretized RSK bijection, this induces a probability measure on $L^{\theta,k}$-discretized Young diagram processes. We refer to the resulting pair of random semistandard tableaux $(L, R)$ as the $L^{\theta,k}$-Geometric RSK tableaux, and we refer to the Young diagram process $\lambda_{L,R}^{\theta,k}$ as the $L^{\theta,k}$-Geometric RSK process.

**Remark 5.5.** It is possible to construct similar models where the parameter of the geometric random variable used to place particles differs from site to site. For our purposes, however, we will stick to this simple case where all are equal to make the construction and the convergence to the Poissonized RS process as clear as possible. This type of setup with geometric random variables is sometimes referred to as the corner growth model with geometric weights. See Section 5 of [2] for a more general treatment.

With this setup, we have the following very close analogue of Theorem 3.2 for the $L^{\theta,k}$-Geometric RSK process, which characterizes the law of this random object.
Theorem 5.6. Fix \( \theta > 0 \), \( k \in \mathbb{N} \) and times \( t_1 < \ldots < t_n \in [-\theta,0] \) and \( s_1 < \ldots < s_m \in [0,\theta] \). Suppose we are given an increasing list of Young diagrams \( \lambda^{(1)} \subset \ldots \subset \lambda^{(n)} \), a decreasing list of Young diagrams \( \mu^{(1)} \supset \ldots \supset \mu^{(m)} \) and a Young diagram \( \nu \) with \( \nu \supset \lambda^{(n)} \) and \( \nu \supset \mu^{(1)} \). To simplify the presentation we will use the convention \( \lambda^{(0)} = \emptyset \), \( \lambda^{(n+1)} = \nu \), \( \mu^{(0)} = \nu_f \mu^{(m+1)} = \emptyset \) and \( t_0 = -\theta \), \( t_{n+1} = 0 \), \( s_0 = 0 \), \( s_{m+1} = \theta \). The geometric RSK process \( \lambda^{\theta,k}_{L,R} \) has the following finite dimensional distribution:

\[
\mathbb{P} \left( \bigcap_{i=1}^{n} \left\{ \lambda^{\theta,k}_{L,R}(t_i) = \lambda^{(i)} \right\} \bigcap \left\{ \lambda^{\theta,k}_{L,R}(0) = \nu \right\} \bigcap \left\{ \lambda^{\theta,k}_{L,R}(s_j) = \mu^{(j)} \right\} \right) = \left( 1 - \frac{\theta^2}{k^2} \right)^{k^2} \left( \frac{\theta^2}{k^2} \right)^{|\nu|} \prod_{i=0}^{n} \text{Dim}_{\lambda^{\theta,k}_{L,R}(t_i+1),t_i} \left( \frac{\lambda^{(i+1)}}{\lambda^{(i)}} \right) \cdot \prod_{i=0}^{m} \text{Dim}_{\lambda^{\theta,k}_{L,R}(s_i+1),s_i} \left( \frac{\mu^{(i)}}{\mu^{(i+1)}} \right)
\]

Where \( \text{Dim}_k(\lambda/\mu) \) is the number of SSYT of skew shape \( \lambda/\mu \) and whose entries do not exceed \( k \) and \( \mathcal{L}^{\theta,k}(x,y) = |\mathcal{L}^{\theta,k} \cap (x,y)| \) is the number of discretization points from \( \mathcal{L}^{\theta,k} \) in the interval \( (x,y) \).

Remark 5.7. The above theorem is purely combinatorial in terms of \( \text{Dim}_k(\lambda/\mu) \), which is the enumerating semistandard Young tableaux. As was the case for Theorem 3.2, this can be written in a very nice way using Schur functions and specializations. Let \( \sigma_{x,y}^{\theta,k} \) to be the specialization that specializes the first \( \mathcal{L}^{\theta,k}_{(x,y)} \) variables to \( \theta/k \) and the rest to zero. Namely:

\[
f(\sigma_{x,y}^{\theta,k}) = f \left( \frac{\theta}{k}, \frac{\theta}{k}, \ldots, \frac{\theta}{k}, 0, 0, \ldots \right)
\]

This specialization differs by a constant factor from the so called "principle specialization", see Section 7.8 of [134]. It is an example of a "finite length" specialization as defined in Section 2.2.1 in [1]. One has the identity that:

\[
\text{s}_{x,y}^{\theta,k}(\lambda/\mu) = \left( \frac{\theta}{k} \right)^{|\lambda/\mu|} \text{Dim}_{\mathcal{L}^{\theta,k}_{(x,y)}}(\lambda/\mu)
\]

Plugging this into the above theorem, after some very nice telescoping cancellations, we can rewrite the probability as a chain of Schur functions:

\[
\mathbb{P} \left( \bigcap_{i=1}^{n} \left\{ \lambda^{\theta,k}_{L,R}(t_i) = \lambda^{(i)} \right\} \bigcap \left\{ \lambda^{\theta,k}_{L,R}(0) = \nu \right\} \bigcap \left\{ \lambda^{\theta,k}_{L,R}(s_j) = \mu^{(j)} \right\} \right) = \left( 1 - \frac{\theta^2}{k^2} \right)^{k^2} \left( \frac{\theta^2}{k^2} \right)^{|\nu|} \prod_{i=0}^{n} \text{Dim}_{\sigma_{x,y}^{\theta,k}(t_i+1),t_i,\lambda^{(i+1)},\lambda^{(i)}} \left( \sigma_{t_i+1,t_i}^{\theta,k} \right) \cdot \prod_{i=0}^{m} \text{Dim}_{\sigma_{s_i+1,s_i}^{\theta,k},\mu^{(i+1)},\mu^{(i)}} \left( \sigma_{s_i+1,s_i}^{\theta,k} \right)
\]

Remark 5.8. For fixed \( \theta \), one might notice that the normalizing prefactor of the Geometric RSK process \( (1 - \theta^2/k^2)^{k^2} \) converges as \( k \to \infty \) to \( e^{-\theta^2} \), the normalizing prefactor of the Poissonized RS process. Even more remarkably, the specializations \( \sigma_{x,y}^{\theta,k} \) that appear in Remark 5.7 converge to the specialization \( \rho_{y-x} \) that appear in Remark 3.3 in the sense that for any symmetric function \( f \), one has that:

\[
\lim_{N \to \infty} f(\sigma_{x,y}^{\theta,N}) = f(\rho_{y-x})
\]

One can verify this convergence by checking the effect of the specialization on the basis \( p_\lambda \) of power sum symmetric functions. These have:

\[
p_\lambda(\rho_{y-x}) = \begin{cases} (y-x)^n & \lambda = (1,1,\ldots,1) \\ 0 & \text{otherwise} \end{cases}
\]
And for $\sigma^\theta_N$, we have:

$$p_\lambda(\sigma^\theta_N) = \left( L^\theta_N(x,y) \right)^\ell(\lambda) \cdot \left( \frac{\theta}{k} \right)^{|\lambda|}$$

(Here $\ell(\lambda)$ is the number of rows of $\lambda$.) Using the bound, $\left| (y-x)k/\theta \right| \leq L^\theta_N(x,y) \leq \left| (y-x)k/\theta \right|$, we know that $L^\theta_N(x,y)$ differs from $(y-x)k/\theta$ by no more than one. Since $\ell(\lambda) \leq |\lambda|$ holds for any Young diagram, the above converges to 0 as $k \to \infty$ unless $\ell(\lambda) = |\lambda|$. This only happens if $\lambda = (1,1,\ldots,1)$ is a single vertical column and in this case we get exactly a limit of $(y-x)^n$ as $k \to \infty$, which agrees with $p_\lambda(\rho_{y-x})$. See section 7.8 of [13] for these formulas. This convergence of finite length specializations to Plancherel specializations is also mentioned in Section 2.2.1. of [1].

This observations shows us that the finite dimensional distributions of the Geometric RSK process converge to the finite dimensional distributions of the Poissonized RS process in the limit $k \to \infty$. One might have expected this convergence since the point process of geometric points from which the Geometric RSK process is built converges in the limit $k \to \infty$ to a Poisson point process of rate 1 in the square $[0, \theta] \times [0, \theta]$, from which the Poissonized RS process is built. However, since the decorated RS correspondence can be very sensitive to moving points even very slightly, it is not apriori clear that convergence of point processes in general always leads to convergence at the level of Young diagram processes.

### 5.2 Proof of Theorem 5.6

The proof follows by similar methods to the proof of Theorem 3.2. We prove some intermediate results which are the analogues of Lemma 3.8, Corollary 3.9 and Lemma 3.10.

**Lemma 5.9.** Let $N = \left| \lambda^\theta_{L,R}(0) \right|$. Then $N$ has the distribution of the sum of $k^2$ i.i.d. geometric random variable of parameter $\theta^2/k^2$. Moreover, conditioned on the event $\{N = n\}$ the pair $(L,R)$ is uniformly distributed in the set $T_n^{k,\theta}$.

**Proof.** This is analogous to Lemma 3.10. In this case, $N = \sum_{i,j=1}^k \xi_{i,j}$ is the sum of geometric random variables.

The fact that all elements of $T_n^{k,\theta}$ are equally likely in the conditioning $\{N = n\}$ is because of the following remarkable fact about geometric distributions. For a collection of iid geometric random variables, the probability of any configuration $\bigcap_{i,j=1}^k \{ \xi_{i,j} = x_{i,j} \}$ depends only on the sum $\sum_{i,j=1}^k x_{i,j}$. Indeed, when $p$ is the parameter for the geometric random variables, the probability is:

$$\mathbb{P} \left( \bigcap_{i,j=1}^k \{ \xi_{i,j} = x_{i,j} \} \right) = p^\sum_{i,j=1}^k x_{i,j} \cdot (1-p)^{k^2}$$

Since this depends only on the sum, and not any other detail of the $x_{i,j}$, when one conditions on the sum, all the configurations are equally likely. Since the RSK is a bijection, it pushes forward the uniform distribution on $C_n^{k,\theta}$ to a uniform distribution on $T_n^{k,\theta}$ as desired. \hfill \Box

**Remark 5.10.** This remarkable fact about geometric random variables is the analogue of the fact that the points of a Poisson point process are uniformly distributed when one conditions on the total number of points. This was a cornerstone of Lemma 3.8. This special property of geometric random variables is what makes this distribution so amenable to analysis: see Lemma 2.2. in the seminal paper by Johansson [8] where this exact property is used.

**Corollary 5.11.** For any Young diagram $\nu$, we have:

$$\mathbb{P} \left( \lambda^\theta_{L,R}(0) = \nu \right) = \left( \frac{\text{Dim}_k(\nu)^2}{k^2 + |\nu| - 1} \right) \mathbb{P} (N = |\nu|)$$

$$= \left( 1 - \frac{\theta^2}{k^2} \right)^{k^2} \left( \frac{\theta^2}{k^2} \right)^{|\nu|} \text{Dim}_k(\nu)^2$$
Proof. This is analogous to the proof of Corollary 3.9. The only difference is that \( T^\theta,k_n \) contains pairs of semi-standard with entries no larger than \( k \), of which we are interested in the number of pairs of shape \( \nu \). This is exactly what \( \text{Dim}_k(\nu) \) enumerates.

\[
|C^\theta,k_n| = |T^\theta,n| = \binom{k^2 + |\nu| - 1}{k^2}
\]

is the number of elements in \( T^\theta,k_n \), so it appears as a normalizing factor. In this case, since \( N \) has the distribution of the sum of \( k^2 \) geometric random variables, we can simplify using the probability mass function:

\[
P(N = x) = \binom{k^2 + x - 1}{k^2} \left( \frac{\theta^2}{k^2} \right)^x \left( 1 - \frac{\theta^2}{k^2} \right)^{k^2}
\]

Lemma 5.12. We have:

\[
P\left( \bigcap_{i=0}^{n+1} \left\{ \lambda_{L,R}^\theta(t_i) = \lambda^{(i)} \right\} \bigg| \lambda_{L,R}^\theta(0) = \nu \right) = \frac{\prod_{i=1}^{n+1} \text{Dim}_{(t_{i+1}, t_i)}(\lambda^{(i+1)} / \lambda^{(i)})}{\text{Dim}_k(\nu)}
\]

An analogous formula holds for \( P \left( \bigcap_{i=0}^{m+1} \left\{ \lambda_{L,R}^\theta(s_i) = \mu^{(i)} \right\} \bigg| \lambda_{L,R}^\theta(0) = \nu \right) \).

Moreover, we have the same type of conditional independence as from Lemma 3.11 between times \( t > 0 \) and times \( t < 0 \) when we condition on the event \( \{ \lambda_{L,R}^\theta(0) = \nu \} \).

Proof. This is the analogue of Lemma 3.11. The proof proceeds in the same way with the important observation that, when conditioned on \( \{ \lambda_{L,R}^\theta(0) = \nu \} \), the pair of SSYT \( (L, R) \) is uniformly chosen from the set of pairs of shape \( \nu \) from \( T^\theta,k_n \). This set is the Cartesian product of the set of all such SSYT of shape \( \nu \) with itself. Hence, the two SSYT are independent and are both uniformly distributed among the set of all SSYT of shape \( \nu \) in this conditioning. For this reason, it suffices to count the number of SSYT of shape \( \nu \) with the correct intermediate shapes at times \( t_1, \ldots, t_n \). The counting of these SSYT then follows by the same type of argument as Lemma 3.11. Each intermediate SSYT of shape \( \mu^{i+1}/\mu^i \) must be filled with entries from the interval \( \mathcal{L}_{(t_i, t_{i+1})}^\theta \cap (t_i, t_{i+1}] \) in order for the resulting SSYT to have the correct subshapes. Since we are only interested in the number of such SSYT, counting those with entries between \( 1 \) and \( L_{(t_{i+1}, t_i)}^\theta \) will do. This is precisely what \( \text{Dim}_{\mathcal{L}_{(t_{i+1}, t_i)}^\theta}(\lambda^{(i+1)} / \lambda^{(i)}) \) enumerates.

Remark 5.13. In the proof of Theorem 3.2 there were additional lemmas needed to separate the dependence of the decorations and the entries appearing in the Young diagrams. As explained in Remark 5.2, the discrete Geometric RSK tableaux case is simpler in this respect because the decorations are proportional to the entries in the tableaux by a factor of \( \theta/k \).

Proof. (Of Theorem 5.6) Exactly as in the proof of Theorem 3.2, the proof follows by conditioning and combining the lemmas.
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