Thermalization in a quasi-one-dimensional ultracold bosonic gas

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Abstract. We study the collisional processes that can lead to thermalization in one-dimensional (1D) systems. For two-body collisions, excitations of transverse modes are the prerequisite for energy exchange and thermalization. At very low temperatures, excitations of transverse modes are exponentially suppressed, thermalization by two-body collisions stops and the system should become integrable. In quantum mechanics, virtual excitations of higher radial modes are possible. These virtually excited radial modes give rise to effective three-body velocity-changing collisions, which lead to thermalization. We show that these three-body elastic interactions are suppressed by pairwise quantum correlations when approaching the strongly correlated regime. If the relative momentum $k$ is small compared with the two-body coupling constant $c$, the three-particle scattering state is suppressed by a factor of $(k/c)^{12}$, which is proportional to $\gamma^{-12}$, that is, to the square of the three-body correlation function at zero distance in the limit of the Lieb–Liniger parameter $\gamma \gg 1$. This demonstrates that in 1D quantum systems, it is not the freeze-out of two-body collisions but the strong quantum correlations that ensure absence of thermalization on experimentally relevant time scales.

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1. Introduction

One-dimensional (1D) systems \([1, 2]\) are a model to study the fundamental processes of dynamics and (de)coherence in interacting many-body quantum systems. Ultracold atoms in strongly elongated traps with \(\omega_r \gg \omega_z\) (\(\omega_r\) and \(\omega_z\) being the frequencies of the radial and longitudinal confinement, respectively) offer the possibility to implement 1D quantum physics if both the temperature \(T\) and chemical potential \(\mu\) are small compared with the energy scale given by the transverse confinement:

\[
\mu < \hbar \omega_t, \quad k_B T < \hbar \omega_t. \tag{1}
\]

One-dimensional systems of ultracold atoms were implemented in both optical lattices \([3]\) and atom chips \([4]\). In the limit of zero temperature, they are a realization of the Lieb–Liniger model \([5]\) of spinless bosons with contact (point-like) interaction, a prime example of an integrable system.

An important parameter characterizing a 1D system of bosons with point-like interactions described by the (3D) s-wave scattering length \(a_s\) is the Lieb–Liniger parameter \([5]\)

\[
\gamma = \frac{2a_s}{n_{1D}l_r^2}, \tag{2}
\]

where \(m\) is the mass of the bosonic atom, \(n_{1D}\) the linear density of the atoms in the 1D trap characterized by the transverse confinement frequency \(\omega_t\), and \(l_r\) is the fundamental length scale of the localization of an atom in the transversal direction given by

\[
l_r = \sqrt{\frac{\hbar}{m\omega_t}}. \tag{3}
\]

The limit \(\gamma \ll 1\) corresponds to a weakly interacting regime, whereas \(\gamma \gg 1\) signifies strongly interacting, strongly correlated (Tonks–Girardeau) regime.

In an integrable system \([6, 7]\), the number of integrals of motion equals exactly the number of degrees of freedom. Thus such a system always ‘remembers’ its initial state in the course of its dynamical evolution, and thermalization does not occur. In an integrable system, the finite spread of initial energy may lead only to relaxation towards the generalized Gibbs (or fully constrained thermodynamic) ensemble \([8]\). Strictly speaking, there is no thermalization in any

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Because we consider identical bosons the main quantum number of both the radial confinement Hamiltonian, 

\[ \hat{H}_{3D} = \int d^3r \left[ \hat{\psi}^\dagger(r) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \hat{H}^{(r)} \right) \hat{\psi}(r) + \frac{2\pi \hbar^2 \alpha_s}{m} \hat{\psi}^\dagger(r) \hat{\psi}(r) \hat{\psi}(r) \hat{\psi}(r) \right], \]

(4)

\[ \hat{H}^{(r)} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{ma_r^2}{2}(x^2 + y^2). \]

(5)

Thereby the field operators \( \hat{\psi}(r) \) are assumed to vanish for \( x^2 + y^2 \to \infty \) and to be periodic along \( z \) with the period \( L \). For the solutions to equation (5) we make the ansatz:

\[ \hat{\psi}(r) = \sum_{n, \ell, k} \hat{a}_{n, \ell, k} \phi_{n, \ell}(x, y) \frac{\exp(ikz)}{\sqrt{L}}, \]

(6)

where \( L \) is the quantization length and the atomic annihilation and creation operators \( \hat{a}_{n, \ell, k} \) and \( \hat{a}_{n, \ell, k}^\dagger \) obey the standard bosonic commutation rules. \( \phi_{n, \ell}(x, y) \) is the normalized eigenfunction of both the radial confinement Hamiltonian,

\[ \hat{H}^{(r)} \phi_{n, \ell}(x, y) = (n + 1)\hbar\omega_r \phi_{n, \ell}(x, y) \]

and the \( z \)-projection of the orbital momentum,

\[ -i [x(\partial / \partial y) - y(\partial / \partial x)] \phi_{n, \ell}(x, y) = \ell \phi_{n, \ell}(x, y). \]

Because we consider identical bosons the main quantum number \( n = 0, 1, 2, \ldots \), and the \( z \)-projection quantum number \( \ell \) of the orbital momentum is restricted by

\[ |\ell| = \text{mod}(n, 2), \text{mod}(n, 2) + 2, \ldots, n - 2, n \]

and thus has the same parity as the main quantum number.
2. Two-body collisions

We first look at collisions of two identical bosonic atoms that are initially in the transverse ground state of the radial confinement. If the collision is restricted to 1D, that is, both atoms remain after the collision in the transverse ground state, then there can be no energy exchange and consequently no thermalization. For such two-body collisions to contribute to energy exchange and thermalization, they have to lead to a change in transverse excitation. By symmetry $\Delta n_1 + \Delta n_2$ must be even. For atoms in the transverse ground state ($n_1 = n_2 = 0$) $\Delta n_1 = \Delta n_2$ band, following the above considerations, their orbital-momentum quantum numbers after collision are restricted to $-\ell$ and $+\ell$. The rate of populating the radially excited modes by pairwise atomic collisions $\Gamma_{2b}$ can then be estimated for a non-degenerate Bose gas, using Fermi’s golden rule. For $k_b T < \hbar \omega_r$, this rate is

$$\Gamma_{2b} \approx \frac{2\sqrt{2} \hbar n_{1D} \alpha_s^2}{m l_t^3} e^{-\frac{2\hbar \omega_r}{\hbar \omega}} = 2\sqrt{2} \omega_r \xi e^{-\frac{2\hbar \omega_r}{\hbar \omega}}. \quad (7)$$

The dimensionless quantity

$$\xi = n_{1D} \alpha_s^2 / l_t$$

combines two dimensionless parameters characterizing a 1D system. $n_{1D} \alpha_s \propto \mu / \hbar \omega_r$ is a measure of how much the interaction energy (the chemical potential $\mu$) is smaller than the energy scale given by the transverse confinement. $\alpha_s / l_t$ characterizes the relation between the transverse confinement and the strength of the contact interaction. Its importance can be seen when looking at how the effective 1D coupling constant $g_{1D}$ of pairwise interacting bosonic atoms in a waveguide changes with confinement due to virtual excitation of the radial modes. Following Olshanii [16], $g_{1D} = 2\hbar \omega_r \alpha_s / [1 - C' \alpha_s / (\sqrt{2} l_t)]$, $C' \approx 1.46$ and increases as $\alpha_s / l_t$ grows. This points to the ratio $\alpha_s / l_t \ll 1$ as the measure of how much the 1D approximation is violated. In a general case, we have to change in all the following expressions $\alpha_s$ to $\alpha_s / [1 - C' \alpha_s / (\sqrt{2} l_t)]$.

Equation (7) also has a transparent physical interpretation: the rate $\Gamma_{2b}$ is related to the 3D atomic density ($\sim n_{1D} / l_t^2$), times the s-wave scattering cross-section ($\sim \alpha_s^2$), times the exponential Boltzmann factor for the fraction of atoms fast enough to scatter into higher radial modes, times the corresponding velocity of the collision ($\sim \hbar / (ml_t)$).

Looking at the scaling of equation (7), one immediately sees that the collision rate leading to thermalization ($\Gamma_{2b}$) rapidly diminishes when the temperature approaches $T \sim \hbar \omega_r$ and is suppressed by more than a factor of 50 ($e^{-4}$) for $T = \frac{1}{2} \hbar \omega_r$. Estimating the numbers for recent experiments [12] in $^{87}$Rb: $\alpha_s = 5.3$ nm, $n_{1D} = 50 \mu m^{-1}$, $\omega/(2\pi) = 3 \text{ kHz}$, $T = 30 \text{ nK}$ ($\xi \approx 0.007$), one obtains a collision rate of $\Gamma_{2b} \sim 0.02 \text{ s}^{-1}$. This is at least one order of magnitude too small for two-body collisions to be responsible for the thermalization required in the evaporative cooling process leading to these low temperatures.

3. Three-body collisions

If the kinetic energy of the collision is less than $2\hbar \omega_r$, then the radial modes can be excited only virtually. Such processes contribute to the system dynamics in the second and higher orders of perturbation theory.

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The simplest case is when after the collision the radial motion state is \(|\{n_1', \ell_1'\}, \{n_2', \ell_2'\}\rangle = |\{0, 0\}, \{2p, 0\}\rangle\). Then only one more collision is enough to de-excite the radial mode and bring the system back on the energy shell (see figure 1(a)). Such a process yields an effective three-body collision already in the second order of perturbation theory.

In contrast, processes involving a virtual excitation to \(|\{n_1', -\ell\}, \{n_2', +\ell\}\rangle, \ell \neq 0\), shown in figure 1(b), contribute only in the third order, and thus will be neglected.

### 3.1. Perturbative approach

We will now calculate the matrix elements for the process shown in figure 1(a), which leads to effective three-body collisions. In our perturbation calculation, the small parameter is \(n_{1D}\alpha_s\), that is, the mean field interaction \(\mu\) is much smaller than the energy scale \(\hbar \omega_r\) connected to the transverse confinement. In addition, to avoid complications related to the confinement-induced resonance in 1D scattering [16], we assume \(\alpha_s \ll \ell\). We can then rewrite the Hamiltonian (4) as

\[
\hat{H}_{3D} = \sum_{n,\ell,k} \left( \frac{\hbar^2 k^2}{2m} + n\hbar \omega_r \right) \hat{a}_{[n,\ell]k}^\dagger \hat{a}_{[n,\ell]k} + \frac{2\pi \hbar^2 \alpha_s f_{0,0,0,0}^{0,0,0,0}}{mL} \sum_{k,k',q} \hat{a}_{[0,0]k-q}^\dagger \hat{a}_{[0,0]k+q}^\dagger \hat{a}_{[0,0]k} \hat{a}_{[0,0]k}^\dagger + \frac{4\pi \hbar^2 \alpha_s}{mL} \sum_{k,k',q} \sum_{p=1}^{\infty} f_{0,0,0,0}^{2p,0,0,0} \hat{a}_{[2p,0]k-q}^\dagger \hat{a}_{[0,0]k+q}^\dagger \hat{a}_{[0,0]k} \hat{a}_{[0,0]k}^\dagger + \text{h.c.} \right) + \hat{R},
\]

where all the terms irrelevant to the process under discussion (figure 1(a)) are gathered in \(\hat{R}\), and the radial matrix element \(f_{0,0,0,0}^{2p,0,0,0}\) is given by

\[
f_{0,0,0,0}^{2p,0,0,0} = \int dx \int dy \int dx' \int dy' \varphi_{n=2p,\ell=0}^*(x, y) \varphi_{0,0}^*(x', y') \delta(x - x') \delta(y - y') \times \varphi_{0,0}(x', y') \varphi_{0,0}(x, y), \quad p = 0, 1, 2, 3, \ldots.
\]
It connects to two atoms in the ground state of the incoming channel, to one atom remaining in the same state, and the other being excited to a state with zero orbital-momentum quantum number and even main quantum number \( n = 2p \), \( p = 0, 1, 2, \ldots \) (remember that \( n \) and \( \ell \) are required to have the same parity).

To evaluate equation (9), we recall that the normalized radial wave functions \( \varphi_{n, \ell}(x, y) \) of interest are real and can be expressed through Laguerre polynomials \( L_p \)

\[
\varphi_{n=2p, \ell=0}(x, y) = (\pi l_r^2)^{\frac{1}{2}} \exp \left( -\frac{x^2 + y^2}{2l_r^2} \right) L_p \left( \frac{\sqrt{x^2 + y^2}}{l_r} \right).
\]  

(10)

Since \( L_p(0) = 1 \), we obtain [17]

\[
\int dx \int dy \int dx' \int dy' \varphi_{n=2p, \ell=0}(x-x', y-y') \varphi_{0,0}(x+x', y+y') \delta(x-x') \delta(y-y') \varphi_{0,0}(x, y) = \frac{1}{2\pi l_r^2}.
\]

(11)

independently of \( p \). Then we can easily obtain the necessary matrix element as \( f_{0,0,0,0}^{2p,0,0} = C_{2p,0,0,0}^{2p,0,0,0} / 2\pi l_r^2 \), where the coefficient \( C_{2p,0,0,0}^{2p,0,0,0} \) is defined by the expansion

\[
\varphi_{2p,0}(x-x', y-y') \varphi_{0,0}(x+x', y+y') = \sum_n \sum_\ell C_{2p-n, \ell, n, -\ell}^{2p,0,0,0} \varphi_{2p-n, \ell}(x, y) \varphi_{n, -\ell}(x', y').
\]

(12)

Comparing the coefficients in front of \( (x^2 + y^2)^p \) in the left- and right-hand sides of equation (12), we obtain \( C_{2p,0,0,0}^{2p,0,0,0} = 2^{-p} \) and

\[
f_{0,0,0,0}^{2p,0,0} = \frac{1}{2p+1\pi l_r^2}.
\]

(13)

In our consideration, we are only interested in the case where the collision energy of the two atoms is always much smaller than \( \hbar \omega_k \). Then, using the matrix element (13) and adiabatically eliminating the radially excited mode operators, we obtain from the original Hamiltonian equation (4) an effective 1D Hamiltonian:

\[
\hat{H}_{1D} = \sum_k \frac{\hbar^2 k^2}{2m} \hat{a}_k^\dagger \hat{a}_k + \frac{\hbar \omega_k \alpha_k}{L} \sum_{k,k',q} \hat{a}_{k+q}^\dagger \hat{a}_{k'-q} \hat{a}_{k'} - \frac{\xi \hbar \omega_k \alpha_k^2}{2L^2} \sum_{k,k',q} \hat{a}_{k+q}^\dagger \hat{a}_{k'}^\dagger \hat{a}_{k} \hat{a}_{k'},
\]

(14)

where we write for simplicity \( \hat{a}_k = \hat{a}_{(0,0)k} \) and the numerical constant \( \xi \) is given by

\[
\xi = 4 \sum_{p=1}^\infty 1/(4^p p) = 4 \ln(4/3) \approx 1.15.
\]

(15)

Note that the relative contribution of virtual states with excitation energy higher than \( 2\hbar \omega_k \) given by \( (\xi - 1)/\xi \) is remarkably small. The summation in the last term of equation (14) is taken over all the kinetic momenta obeying the conservation law

\[
k_1 + k_2 + k_3 = k_1 + k_2 + k_3.
\]

Introducing the field operator \( \hat{\psi}(z) = L^{-1/2} \sum_k \hat{a}_k \exp(ikz) \), we can rewrite equation (14) as

\[
\hat{H}_{1D} = \int dz \left( \frac{\hbar^2}{2m} \frac{\partial^2 \hat{\psi}}{\partial z^2} + \hbar \omega_k \alpha_k \hat{\psi} \hat{\psi} - \frac{\xi}{2} \hbar \omega_k \alpha_k^2 \hat{\psi} \hat{\psi} \hat{\psi} \hat{\psi} \right).
\]

(16)

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The first and second terms in equation (14) or (16) correspond to the Lieb–Liniger model. The third (cubic) term stems from the effective three-body collisions mediated by virtually excited states\(^5\). This third (cubic) term in equation (16) violates the integrability in the 1D system.

The fact that the effective three-body interactions are dominated by virtual excitations of the lowest even-parity excited state may seem surprising, since the correct calculation of the effective two-body 1D coupling constant requires taking into account the infinite number of states [16]. However, in the latter case one deals with the calculation of the two-body wave function, which has in 1D a 1/z singularity (stemming from the 3D boundary condition at \(r \to 0\) that provides the correct asymptotic form of the scattered s-wave). The removal of this singularity yields the regular part of the two-body wave function and, through this regular part, the scattering amplitude and, hence, the effective coupling in 1D. On the other hand, if one tries to obtain the effective 1D interaction by adiabatic elimination of all the excited states, one gets a divergent series \(\sum_{n=0}^{\infty} n^{-1/2}\) in the expression for the effective 1D coupling constant. The aforementioned regularization of the wave functions formally corresponds to the renormalization of this divergent series via substituting it by a finite expression \(\lim_{r \to \infty} (\sum_{n=0}^{s} n^{-1/2} - \int_0^s \text{d}v v^{-1/2})\) [16]. In our case, processes related to three-body collisions do not give rise to additional singularities in the many-body wave function, and no additional regularization is needed. The sum over all excited states thus converges. The convergence is rapid enough to ensure fair estimation of the whole sum by its first term.

Accurate calculation of the effective three-body interaction potential \(U_{3b}\) yields

\[
U_{3b}(z_1, z_2, z_3) = -\frac{\hbar}{2} \omega_0 \alpha_s^2 \left[\mathcal{Y}(z_1, z_2; z_3) + \mathcal{Y}(z_1, z_3; z_2) + \mathcal{Y}(z_2, z_3; z_1)\right],
\]

where

\[
\mathcal{Y}(z_1, z_2; z_3) = [\delta(z_3 - z_1) + \delta(z_3 - z_2)] \sum_{p=1}^{\infty} \frac{\sqrt{p}}{2^{p-1} p l_r} \exp \left(-\frac{\sqrt{2p|z_1 - z_2|}}{l_r}\right).
\]

Obviously, the sum in equation (18) converges and gives a sharp-peaked function rapidly (exponentially) decreasing at distances much larger than \(l_r\). Since equation (1) holds, we consider scattering events with transferred momenta much less than \(\hbar/l_r\). In this case we can use approximation \(\mathcal{Y}(z_1, z_2; z_3) \approx \xi [\delta(z_3 - z_1) + \delta(z_3 - z_2)]\delta(z_1 - z_2)\). Then, by taking the matrix element of the effective interaction \(U_{3b}\), and dividing it by 3! (the number of permutations of three identical particles), we obtain the last term in the second-quantized Hamiltonian (14).

Before continuing we want to point out similarities with other recent works: (i) the mechanism discussed here is to a certain extent similar to the virtual association of atoms with a molecular dimer [18]. In our discussion here, virtual excitation of radial modes during a two-atom collision temporarily localizes the interatomic distance on the length scale \(\sim l_r\). Scattering a third atom on such a transient structure of finite size and mass 2m leads to an effective three-body collision. In [18], collisions of a third atom bring ‘virtual’ dimers, enhanced in size by a Feshbach resonance, down to the energy shell, thus bringing about ‘quantum chemistry’ in 1D. (ii) In a similar way, effective three-body interactions between polar molecules emerge, due to virtual transitions to an off-resonant internal state [19].

\(^5\) Note that in our previous work [14], the coefficient in front of this cubic term was estimated a factor 4 too large.
3.2. Variational approach

The cubic term in equation (16) is negative and thus supports no bound ground state. Therefore we have to consider it as a first correction term to the purely pairwise interaction energy in the effective 1D Hamiltonian. Considering the mean-field limit of equation (16), \( \hat{\psi} \approx \sqrt{n_{1D}} \exp(i\theta) \), one obtains the energy density (per unit length)

\[
\mathcal{E}_{\text{pert}} = \frac{\hbar^2}{2m} \left[ \left( \frac{\partial \sqrt{n_{1D}}}{\partial z} \right)^2 + n_{1D} \left( \frac{\partial \theta}{\partial z} \right)^2 \right] + \hbar \omega_1 \alpha_s n_{1D}^2 - \frac{\xi}{2} \hbar \omega_1 \alpha_s^2 n_{1D}^3, \tag{19}
\]

this expansion is correct in the limit of the small linear density \( n_{1D} \alpha_s \ll 1 \). If one neglects the contribution of the radial levels with the main quantum number larger than 2 by setting \( \xi \approx 1 \), one sees that equation (19) is the expansion up to the cubic term of the energy density obtained by the variational method of Salasnich et al [20]

\[
\mathcal{E}_{\text{var}} = \frac{\hbar^2}{2m} \left[ \left( \frac{\partial \sqrt{n_{1D}}}{\partial z} \right)^2 + n_{1D} \left( \frac{\partial \theta}{\partial z} \right)^2 \right] + n_{1D} \left( \frac{\hbar^2}{2m \sigma^2} + \frac{1}{2} m \omega_1^2 \sigma^2 \right) \left( \frac{\hbar^2}{m \sigma^2} \right) + \frac{\hbar^2 \alpha_s n_{1D}}{m \sigma^2}, \tag{20}
\]

Here one assumes the wave function of the transversal atomic motion to be \( \alpha \exp[-(x^2 + y^2)/(2\sigma^2)] \), with \( \sigma^2 = \frac{\hbar}{m \omega_1 \sqrt{1 + 2n_{1D} \alpha_s}} \), which minimizes \( \mathcal{E}_{\text{var}} \).

4. Calculations of the collision rates

We now turn to the collision rates for these effective three-body collisions. We start with calculating the rate \( \Gamma_{k_1k_2k_3} \) for the decay of a specific state \( |k_1, k_2, k_3\rangle \equiv a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3}^\dagger |\text{vac}\rangle \) with \(|\text{vac}\rangle \) being the vacuum state of the atomic field wherein atoms are absent (should not be confused with the vacuum of elementary excitations) due to three-body collisions. The final states of the decay are \( |k_1', k_2', k_3'\rangle \equiv a_{k_1'}^\dagger a_{k_2'}^\dagger a_{k_3'}^\dagger |\text{vac}\rangle \). To make the calculation simple, we assume that the 1D bosonic gas is strongly non-degenerate \( (k_B T \text{ much higher than the chemical potential}) \) and weakly interacting. This enables us to neglect the probability of double (and higher) occupation of any \( k \)-mode, therefore assuming all involved atomic momenta to be different, and to assume that the elementary excitations coincide with the atomic plane waves with the free-particle (quadratic) dispersion law. Then Fermi’s golden rule yields

\[
\Gamma_{k_1k_2k_3} = \frac{2\pi}{\hbar} L^2 \int \frac{dk_1'}{2\pi} \int \frac{dk_2'}{2\pi} \delta \left[ \frac{\hbar^2}{2m} \left( \sum_j k_j'^2 - \sum_j k_j^2 \right) \right] \left( \frac{\xi \hbar \omega_1 \alpha_s^2}{2L^2} \right) \times \left| \langle k_1', k_2', k_3' | \sum_{i,j} \hat{a}_{q_i}^\dagger \hat{a}_{q_j}^\dagger \hat{a}_{q_i} \hat{a}_{q_j} |k_1, k_2, k_3\rangle \right|^2, \tag{21}
\]
To account for the condition \( k'_1 + k'_2 + k'_3 = k_1 + k_2 + k_3 \) we add an additional integration over \( k'_1 \) with the necessary delta-function:

\[
\Gamma_{k_1 k_2 k_3} = \frac{L^2}{2\pi \hbar} \int \int \int d k'_1 d k'_2 d k'_3 \delta \left[ \frac{\hbar^2}{2m} \left( \sum_j k''_j - 3 \sum_j k'_j \right) \right] \delta \left( \sum_j k'_j - 3 k_3 \right) \times \left( \frac{\xi \hbar \omega_0 \alpha^2}{2L^2} \right)^2 \left| \langle k'_1, k'_2, k'_3 | \hat{\alpha}^\dagger_{q_1} \hat{\alpha}^\dagger_{q_2} \hat{\alpha}^\dagger_{q_3} \hat{\alpha}_{q_1} \hat{\alpha}_{q_2} \hat{\alpha}_{q_3} | k_1, k_2, k_3 \rangle \right|^2.
\]  

(22)

Since the atoms are indistinguishable, we need to integrate over the volume \( V' \) in the \( k' \)-space that corresponds to a unique ordering of the variables, e.g. \( k'_1 > k'_2 > k'_3 \). The integral over \( V' \) of a function fully symmetric over permutations of \( k'_1, k'_2, k'_3 \) amounts to 1/3! of the integral over the whole \( k' \)-space. If all the involved momenta are different (as is the case for a system far from degeneracy), all possible ways of ordering three bosonic creation operators with lower indices \( k'_1, k'_2, k'_3 \) and three bosonic annihilation operators with lower indices \( k_1, k_2, k_3 \) finally yield

\[
\langle k'_1, k'_2, k'_3 | \sum_{\{q'_j\}} \hat{\alpha}^\dagger_{q'_1} \hat{\alpha}^\dagger_{q'_2} \hat{\alpha}^\dagger_{q'_3} \hat{\alpha}_{q'_1} \hat{\alpha}_{q'_2} \hat{\alpha}_{q'_3} | k_1, k_2, k_3 \rangle = (3!)^3.
\]  

(23)

To evaluate the integral in equation (23), we perform an orthogonal transformation from \( k'_1, k'_2, k'_3 \) to Jacobi coordinates (here we deal with the 1D analogue of the hyperspherical coordinates, which are used in the three-body problem [21]) in the wavenumber space:

\[
k'_1 = \frac{1}{\sqrt{3}}(k'_1 + k'_2 + k'_3), \quad k'_{12} = \frac{1}{\sqrt{2}}(k'_1 - k'_2), \quad k'_{321} = \sqrt{\frac{2}{3}} \left( k'_3 - \frac{k'_1 + k'_2}{2} \right).
\]  

(24)

Further, we introduce the hyperangle \( \chi' \) via

\[
k'_{12} = \tilde{k}' \sin \chi', \quad k'_{321} = \tilde{k}' \cos \chi',
\]  

(25)

and express equation (23) as

\[
\Gamma_{k_1 k_2 k_3} = \frac{(3!)^3 \xi^2 \omega_0^2 \alpha^4}{4\pi \hbar L^2} \int_{-\infty}^{\infty} d k'_1 \int_{-\pi}^{\pi} d \chi' \int_{0}^{\infty} d \tilde{k}' \left( \delta(\sqrt{3}k'_c - k_1 - k_2 - k_3) \delta(k''_1 + \tilde{k}'_1 - k_1^2 - k_2^2 - k_3^2) \right) = 3! C_{3b} \frac{\omega_0 \alpha^4}{L^2 l_s^2},
\]  

(26)

and accurate evaluation of the integral yields (cf [14])

\[
C_{3b} = 3\sqrt{3}\xi^2 \approx 6.88.
\]  

(27)

To calculate the three-body collision rate \( \Gamma_{3b} \) per atom, we need to multiply \( \Gamma_{k_1 k_2 k_3} \) by the product of populations \( N f_{k_j} \) of the states \( | k_j \rangle = \alpha_k^\dagger \mid \text{vac} \rangle \), \( j = 1, 2, 3 \) (the occupation probabilities are normalized to unity, \( \int_{-\infty}^{\infty} d k \int f_k = 1 \)), integrate over the whole \( k_1, k_2, k_3 \)-space and divide by 3! to take into account the indistinguishability of the bosons. Then we obtain the number of three-body collisions per unit time in the whole system. This number should be divided by \( N \) to obtain the rate per atom

\[
\Gamma_{3b} = C_{3b} \frac{\hbar}{m} \left( \frac{n_{1D}}{l_s^2} \right)^2 \alpha^4 = C_{3b} \xi^2 \omega_0.
\]  

(28)
The result of equation (28) may seem counterintuitive at first: the collision rate is independent of 
temperature, and it is proportional to the dimensional parameter $\xi^2$ and the radial confinement 
$\omega_r$.

The physics behind the first observation is related to the fact that the collision kinetic energy is small compared to the virtual excitation energy. This was one of our assumptions in deriving the effective three-body collisions and is required by the condition ($k_B T < \hbar \omega_r$) to be fulfilled when building a 1D system (equation (1)). Consequently, the composite matrix element of the second-order process should not depend in leading order on the velocities (energies) of the colliding particles and hence on temperature (see equation (9)). In addition, the phase space volume for the scattered particles is independent of the incoming momenta $k_1, k_2$ and $k_3$.

The other terms can be motivated by the following basic physics considerations: since effective three-body elastic scattering is the dominant process, the scattering rate must be 
proportional to the 3D density squared: $(n_{1D}/l_1^2)^2$. Furthermore, the scattering rate contains the square of the matrix element corresponding to the diagram in figure 1(a), where each vertex is 
proportional to $\alpha_s$; therefore this rate is proportional to $\alpha_s^4$. The factor $\hbar/m$ provides the correct dimensionality (s$^{-1}$).

We can now compare the scattering rates for the two-body collisions $\Gamma_{2b}$ (equation (7)) or effective three-body collisions $\Gamma_{3b}$ (equation (28)) and evaluate their contributions to 
thermalization and the breakdown of integrability in 1D systems. For $k_B T < \hbar \omega_r$ we find a simple scaling:

$$\frac{\Gamma_{3b}}{\Gamma_{2b}} = \frac{C_{3b}}{C_{2b}} \xi \frac{e^{2\hbar \omega_r/k_B T}}{2\sqrt{2}} \xi e^{2\hbar \omega_r/k_B T} \approx 2.43 \xi e^{2\hbar \omega_r/k_B T}. \quad (29)$$

The relative importance of two-body collisions and the effective three-body collisions mediated 
by virtual excitations is determined by the dimensionless quantity $\xi e^{2\hbar \omega_r/k_B T}$. For large $\xi$ 
and small temperatures ($k_B T \ll \hbar \omega_r$), the three-body scattering rate due to virtual excitations 
dominates and can lead to thermalization even when the thermalization processes due to two-body collisions are frozen out. For example, in the typical atom chip experiment, [11]–[13] the three-body rate $\Gamma_{3b}$ dominates above the two-body collisions at $k_B T \lesssim 1/\hbar \omega_c$. A detailed comparison of the two rates $\Gamma_{2b}$ and $\Gamma_{3b}$ and their relation to typical experimental parameters is given in figure 2. The scattering rate due to virtual excitations of the radial modes can dominate 
over real excitations for typical parameters of the recent experiment [12].

The above calculation was for a non-degenerate ultracold gas. In a degenerate gas we need 
to consider the Bogoliubov-type spectrum of elementary excitations [22], which is phononic in 
the long-wavelength regime, as well as the relation between atoms and elementary excitations 
via the Bogoliubov transformations and bosonic amplification of scattering to modes, which are 
initially occupied.

Taking into account all these factors, we find the rate of damping of a fast particle in a 
quasi-condensate (see also [24])

$$\Gamma_{k_0}^{\text{damp}} = \frac{9\sqrt{3}\xi^2 \omega_c \xi^2}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \frac{dk'}{k'} \frac{d\chi'}{\pi} \frac{dk''}{k''} \delta(k'' - k_0/\sqrt{3}) \delta(n_{k'} + n_{k''} + n_{k'} - n_k). \quad (30)$$

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Figure 2. Ratio between the scattering rates for the two routes to thermalization in quasi-1D systems: $\Gamma_{3b}$ for the effective three-body collisions and $\Gamma_{2b}$ for two-body collisions leading to excited transverse states. The points represent the ratios evaluated for various sets of experimental parameters from [11] (points), [12] (crosses), [13] (triangle) and [23] (diamonds). In these experiments, the parameter $\zeta$ was often close to 0.007 (the ratio $\Gamma_{3b}/\Gamma_{2b}$ for $\zeta = 0.007$ exactly is shown by the solid curve). For comparison, we plot also $\Gamma_{2b}/\Gamma_{3b}$ for $\zeta = 0.002$ (dashed curve) and 0.02 (dot-dashed curve). Units on the axes are dimensionless.

Here the momenta of the scattered elementary excitations are defined by the expressions reciprocal to equation (24, 25):

$$
\begin{align*}
  k_1' &= \frac{k_c'}{\sqrt{3}} + \sqrt{\frac{2}{3}} \tilde{k}' \cos(\chi' - 2\pi/3), \\
  k_2' &= \frac{k_c'}{\sqrt{3}} + \sqrt{\frac{2}{3}} \tilde{k}' \cos(\chi' + 2\pi/3), \\
  k_3' &= \frac{k_c'}{\sqrt{3}} + \sqrt{\frac{2}{3}} \tilde{k}' \cos \chi'.
\end{align*}
$$

(31)

The energy of a mode with the momentum $\hbar k$ is $\varepsilon_k = \hbar^2 \eta_k/(2m)$ with

$$
\eta_k = \sqrt{k^2(k^2 + 8n_{1D}a_s/l_r^2)}.
$$

The static structure factor of a quasi-condensate is

$$
S_k = k^2/\eta_k.
$$

In equilibrium, the population of the elementary mode with the momentum $\hbar k$ is given by the Bose–Einstein statistics with the mean occupation number for the mode with the momentum $\hbar k$

$$
n_k = \frac{1}{\exp[\varepsilon_k/(k_B T)] - 1}.
$$

The initial kinetic energy $\hbar^2 k_0^2/(2m)$ of the fast atom is assumed here to be large compared with both the mean-field interaction energy per particle in the quasi-condensate and the temperature:

$$
k_0^2 \gg n_{1D}a_s/l_r^2, \quad k_0^2 \gg mk_B T/\hbar^2.
$$

(32)

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Under condition (32), one of the scattered particles is always fast, and one of the three structure factors appearing in equation (30) is always very close to 1 (and the corresponding occupation number is close to 0). In most scattering events the other two particles are also fast and, hence, $S_{k_j} \approx 1$ and $n_{k_j} \approx 0$ for all three particles, $j = 1, 2, 3$. Only for the scattering events with small transferred momentum, two of the structure factors are significantly less than 1 and/or the corresponding populations approach the high-temperature limit $k_B T / \varepsilon_{k_j}$. However, in the practically interesting case where $k_B T \sim \bar{\hbar} \omega r n_{1D}$ the contribution of scattering events with small transferred momentum is relatively small, and

$$\Gamma_{k_0} \approx \frac{9 \sqrt{3}}{2} \xi^2 \omega \xi^2.$$ (33)

The result of equation (33) for a fast atom in a quasi-condensate is also obtained by Tan et al [24].

5. Calculations of the thermalization rates

We now turn to quantify thermalization in tightly confined bosons in a quasi-1D geometry by both two-body collisions and the effective interaction (14). We again for simplicity consider a non-degenerate, weakly interacting (the Lieb–Liniger parameter $\gamma = 2 \alpha_s / (n_{1D} l_\Xi^2)$ being much less than 1) gas of bosonic atoms. The assumptions of weak interaction and non-degeneracy enable us to express the three-particle distribution function through the product of single-particle distribution functions $f_k$. In contrast, calculation of relaxation via three-body collisions of low-energy excited states in the stronger interacting regime and especially for $\gamma \geq 1$ requires that we consider the (strong) quantum correlations in the quasi-1D bosonic system. We will discuss the effects of correlations on scattering rate $\Gamma_{3b}$ and on thermalization in section 6.

We start by writing the Boltzmann equation with a three-body collision integral [25], taking into account the indistinguishability of the particles:

$$\frac{d}{dt} f_k = \Gamma_{3b} \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dk'' \int_{-\pi}^{\pi} \frac{d\gamma}{2\pi} \left( f_{K_0} f_{K_{-1}} f_{K_{+1}} - f_k f_{k'} f_{k''} \right),$$ (34)

with

$$K_s = \frac{k + k' + k''}{3} + \sqrt{\frac{2}{3}} \bar{k} \cos(\gamma + 2s\pi/3), \quad s = 0, \pm 1,$$ (35)

$$\bar{k} = \sqrt{\frac{k^2 + k'^2 + k''^2 - (k + k' + k'')^2}{3}}.$$ (36)

Equation (34) can be easily understood: after integration over $k'$ and $k''$ the loss term in equation (34) is simply $-\Gamma_{3b} (f)_k$, which is the elastic three-body collision rate per atom. On the other hand, the three-atom state $|k, k', k''\rangle$ is populated by elastic three-body collisions from those states $|K_0, K_{-1}, K_{+1}\rangle$, which have the same center-of-mass momentum: $K_0 + K_{-1} + K_{+1} = k + k' + k''$. Since the kinetic energy of the relative motion, $\hbar^2 \bar{k}^2/(2m)$, is conserved,
Figure 3. Dependence of the rate $\Gamma^{3b}_{[4]}$ of thermalization induced by effective three-body collisions in a weakly interacting, quasi-1D $^{87}$Rb gas on the radial trapping frequency for the linear densities $n_{1D}$ from 80 to 30 $\mu$m$^{-1}$ (from top to bottom) with the step 10 $\mu$m$^{-1}$ (dot-dashed curve).

The states $|K_0, K_{-1}, K_{+1}\rangle$ (from where the state $|k, k', k''\rangle$ can be populated from) can be fully parameterized by the hyperangle $\gamma$.

We now use the following ansatz for the perturbed momentum distribution

$$f_k(t) = \frac{n_{1D}}{\sqrt{\pi}k_{th}} \exp(-k^2/k_{th}^2)[1 + \epsilon_4(t)H_4(k/k_{th})], \quad (37)$$

to solve equation (34). Thereby $k_{th} = \sqrt{2mk_B T/\hbar}$ and $H_4$ is the Hermite polynomial of the fourth order. This ansatz is the simplest nontrivial perturbation that retains $\int dk k f_k = 0$. We then proceed to linearize equation (34) with respect to the perturbation amplitude $\epsilon_4(t)$ and obtain an exponential solution $\epsilon_4(t) = \epsilon_4(0)\exp(-\Gamma^{3b}_{[4]} t)$ with

$$\Gamma^{3b}_{[4]} = C_{[4]} \frac{\hbar}{m} \left( \frac{n_{1D}}{l_f^2} \right)^2 \alpha_4^2 = C_{[4]} \omega_4 \xi^2 \quad \text{with the numerical constant} \quad C_{[4]} = \frac{8}{27} C_{3b} = \frac{8\xi^2}{3\sqrt{3}} \approx 2.04. \quad (38)$$

To estimate the validity of our ansatz, we note that using a higher-order Hermite polynomial $H_n$ in equation (37) leaving the functional dependence on the parameters of the system unchanged leads only to a minor modification of the numerical prefactor. For example, for $n = 5$ and 6, the thermalization rates are given by $10^{27} C_{3b} \omega_4 \xi^2$ and $34^{81} C_{3b} \omega_4 \xi^2$, respectively. It is interesting to note that in these 1D systems the thermalization rate due to the effective three-body collisions ($\Gamma^{3b}_{[4]}$) is about a factor of 3 smaller than the collision rate ($\Gamma_{3b}$). This suggests that in 1D systems thermalization requires also about three collisions, similar to 3D [26]. Figure 3 shows numerical values of $\Gamma^{3b}_{[4]}$ as a function of the 1D density of $^{87}$Rb atoms and the radial trapping frequency.

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For comparison we calculate numerically the thermalization rate \( \Gamma_{2b}^{2b} \) for two-body collisions involving the real transitions between the ground and excited radial states. We follow hereby the same ansatz and perturb the velocity distribution of atoms in the ground and excited states as given by equation \((37)\), the Boltzmannian distribution of overall populations between the levels being kept intact. In the parameter range of interest, we find numerically

\[
\Gamma_{2b}^{2b} \approx (0.33 \pm 0.03) \Gamma_{2b}, \text{i.e.}
\]

\[
\Gamma_{2b}^{2b} \approx 0.93 \omega_r \zeta e^{- \frac{2b \omega_r}{\bar{\hbar} \omega_r}}. \tag{40}
\]

The ratio of the thermalization rates for the two-body and three-body processes is therefore very close to the respective ratio of the collision rates, shown in figure 2.

It is interesting to note that we find for both processes that thermalization in 1D needs about three collisions capable of distributing energy. This is very close to the 2.7 collisions required for thermalization in 3D \([26]\).

For the typical parameters of an ultracold \( ^{87} \text{Rb} \) gas on an atom chip \([12]\) \((\omega_r \approx 2\pi \times 3 \text{ kHz}, n_{1D} \approx 50 \mu \text{m}^{-1})\), we obtain \( \Gamma_{3b}^{3b} \approx 2 \text{s}^{-1} \). This thermalization rate is temperature independent and much larger than the one calculated from the simple two-body collisions with the energy sufficient to excite radial modes \( \Gamma_{2b}^{2b} \approx 3 \times 10^{-3} \text{s}^{-1} \) at the lowest temperatures measured \((30 \text{nK})\). The estimated \( \Gamma_{3b}^{3b} \) is consistent with the time needed for evaporative cooling of an \( ^{87} \text{Rb} \) gas on an atom chip well below \( \hbar \omega_r \) \([11,12]\).

6. Suppression of thermalization by atomic correlations

The thermalization rate \( \Gamma_{3b}^{3b} \) given by equation \((38)\) was calculated for a weakly interacting, non-degenerate gas. Calculation of the thermalization rate \( \Gamma_{3b}^{3b, G} \) in a general case requires that we take into account additional physics. Firstly, we need to consider the effects of quantum degeneracy and, secondly, the fact that the dispersion relations for the elementary excitations in a 1D quantum (degenerate) system may differ significantly compared with a free particle, especially for phonon-like excitations. These effects, together with the bosonic amplification of the scattering into thermally populated modes, tend to accelerate thermalization. A third observation is that the three-body rates require three particles to be close to the same location. Quantum mechanically this is characterized by the third-order correlation function \( g_3(0) \). A full consideration of the above competing effects will require extended numerical analysis of many particular cases and beyond the scope of the present paper. We will give here physical arguments of what to expect.

We start by pointing out that the form of the secondary-quantized Hamiltonian equation \((16)\) allows us to give a simple estimate of the ratio of these two rates

\[
\Gamma_{3b}^{3b, G} / \Gamma_{3b}^{3b} = \varrho \left( g_3(0) / 6 \right)^2.
\]

Here \( \varrho \) is a phase-space factor accounting for the dispersion law of elementary excitation, which changes from free-particle-like to phonon-like. But it changes the thermalization rate less dramatically than the second factor associated with the local three-body correlation function

\[
g_3(0) = \langle \hat{\psi}_{1D}(z) \hat{\psi}_{1D}^3(z) \rangle / n_{1D}^3,
\]

and will become the dominating factor when approaching the strongly correlated regime \((\gamma > 1)\). For a non-degenerate weakly interacting Bose gas \( g_3(0) = 3! = 6 \). For a degenerate 1D
Bose gas, \( g_3(0) \) has been recently calculated for the whole range of atomic repulsion strengths \( 0 < \gamma < \infty \) by Cheianov et al.\[27\]. In the zero-temperature limit, \( g_3(0) \) rapidly decreases from 1 to \( 16\pi^6/(15\gamma^6) \) as \( \gamma \) grows from 0 to values \( \gg 1 \).

We now turn to the above conjecture on suppression of the thermalization by atomic correlations. A detailed calculation can be found in [15]; here we sketch the basic physics argument. To look at the correlations, we start by considering \( N \) identical bosons in 1D configuration with the Hamiltonian (16), which, after rescaling of units, takes the form

\[
\hat{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + 2c \sum_{j>j'} \delta(z_j - z_{j'}) + \sum_{j>j'>j''} U_{3b}(z_j - z_{j'}, z_j - z_{j''}).
\] (41)

\( c = 2\alpha_s/\ell^2 \) is the strength of interaction of two atoms in the tight waveguide with ground state size \( \ell \). \( U_{3b} \) is obtained by adiabatic elimination of transverse modes virtually excited by the 3D short-range pairwise atomic interaction [14]. The explicit form of \( U_{3b} \) is given by equation (17) within a numerical prefactor, \( U_{3b} = [\hbar^2/(2m)]U_{3b} \).

We follow now our detailed calculations in [15] and estimate the three-body scattering amplitude in the presence of the delta-functional pairwise repulsive interactions. The stronger the pairwise interparticle repulsion, the smaller the probability of a close encounter of three particles, which will result in a suppression of the three-body scattering amplitude. The simplest case is to analyze the Hamiltonian (41) for \( N = 3 \) particles. For that purpose we express it in hyperspherical coordinates \( R, \chi \) defined as \[21\]

\[
Z_c = \frac{z_1 + z_2 + z_3}{\sqrt{3}}, \quad R \sin \chi = \frac{z_1 - z_2}{\sqrt{2}}, \quad R \cos \chi = \sqrt{\frac{2}{3}}(z_3 - \frac{z_1 + z_2}{2})
\] (42)

and obtain for the Hamiltonian

\[
\hat{H} = -\frac{\partial^2}{\partial Z_c^2} - \frac{1}{R} \frac{\partial}{\partial R} R \frac{\partial}{\partial R} - \frac{1}{R^2} \frac{\partial^2}{\partial \chi^2} + \sqrt{\frac{2}{3}}c \sum_{\nu=-2}^{3} \delta(\chi - \nu \pi/3) + U_{3b}(R, \chi).
\] (43)

The corresponding Schrödinger equation for the three-particle wave function is

\[
\hat{H} \Psi(z_1, z_2, z_3) = (k_1^2 + k_2^2 + k_3^2) \Psi(z_1, z_2, z_3).
\]

The wavenumbers \( k_j \) are defined from the set of transcendental equations [5], provided that the periodic boundary conditions are set on the interval of length \( L \). By setting \( L \to \infty \), we obtain a continuous spectrum, where \( k_j \)'s are real for repulsive interaction \( (c > 0) \). We can now separate the center-of-mass motion and describe the relative motion in hyperspherical coordinates. This leads to the ansatz:

\[
\Psi(z_1, z_2, z_3) = \exp[i(k_1 + k_2 + k_3)(z_1 + z_2 + z_3)/3] \psi(R, \chi),
\]

where the kinetic energy of the relative motion is given by

\[
k^2 = \frac{1}{4}[(k_1 - k_2)^2 + (k_2 - k_3)^2 + (k_3 - k_1)^2].
\]

In the adiabatic hyperspherical approximation [30], which holds in the long-wavelength limit

\[
k \ll c
\] (44)

and neglects coupling of different scattering channels as well as accumulation of phase shifts of the scattered wave due to non-adiabatic effects we obtain

\[
\psi(R, \chi) = F_0(R) B_0(R, \chi).
\] (45)
The hyperangular part \( B_0(R, \chi) \) of this wave function is the eigenfunction of the Hamiltonian (43) with fixed \( R \), corresponding to the lowest eigenvalue \( \lambda_0(R) \), which is the smallest positive root of the transcendental equation

\[
\lambda(R) \tan[\pi \lambda(R)/6] = cr/\sqrt{2}.
\]  

Using the regular hexagon symmetry group \([31]\) of the Hamiltonian (43), we write the hyperangular part of equation (45) as

\[
B_0(R, \chi) = \tilde{B}_0(R, \chi) + \tilde{B}_0(R, \chi - 2\pi/3) + \tilde{B}_0(R, \chi + 2\pi/3),
\]  

where

\[
\tilde{B}_0(R, \chi) = \begin{cases} \cos[\lambda_0(R)(\pi/6 - |\chi|)], & |\chi| \leq \pi/3, \\ 0, & \text{otherwise}. \end{cases}
\]  

After integrating out the hyperangular variable, the Schrödinger equation in the adiabatic hyperspherical approximation reduces to

\[
-\frac{1}{R} \frac{d}{dR} R \frac{d}{dR} F_0 + \left[ \frac{\lambda_0^2(R)}{R^2} + \tilde{U}_{00}(R) \right] F_0 = k^2 F_0,
\]  

where

\[
\tilde{U}_{00}(R) = \int_0^{\pi/3} d\chi \frac{\tilde{B}_0^2(R, \chi)U_{3b}(R, \chi)}{\int_0^{\pi/3} d\chi \tilde{B}_0^2(R, \chi)}
\]  

with the boundary conditions requiring \( F_0 \) to be finite for both \( R = 0 \) and \( R \to \infty \), for the ‘partial wave’ corresponding to the lowest eigenvalue \( \lambda_0(R) \), whose asymptotic expressions are

\[
\lambda_0(R) \approx \begin{cases} \sqrt{3\sqrt{2}cR}, & cR \ll 1, \\ \frac{3 - 18\sqrt{2}}{\pi cR}, & cR \gg 1. \end{cases}
\]  

We can solve now equation (49) analytically in two regions, \( cR \ll 1 \) and \( cR \gg 1 \), with \( \lambda_0(R) \) approximated by equation (51), and tailoring the solutions by quasi-classical expressions for \( F_0(R) \) in the intermediate hyper-radius range. The scattering amplitude \( \tilde{f}_0 \) (for its definition in planar geometry see [32]–[35]) can then be obtained from the asymptotic form of the wave function at \( R \to \infty \)

\[
F_0(R) \approx J_3(kR) - i\tilde{f}_0 H_3^{(1)}(kR),
\]  

where \( H_3^{(1)}(z) = J_3(z) + iY_3(z) \) is the Hankel function of the first kind and \( J_3(z) \) and \( Y_3(z) \) are the third-order Bessel functions. The behavior of \( F_0 \) at \( R \to 0 \) is defined by the details of the potential \( U_{00}(R) \), but the result can be finally expressed via the effective vertex of the three-body elastic collisions, thus yielding

\[
\tilde{f}_0 = \frac{6(\alpha_\lambda/l_\lambda)^2 \xi}{2\pi \Omega (c/k)^6 - i\pi},
\]  

where \( \Omega \approx 1 \) is a numerical constant and \( \xi \approx 1.15 \) is defined by equation (15). We use thereby the fact that \( U_{00}(R) \) differs significantly from zero on the length scale \( l_\lambda \), over which a virtually excited particle can propagate, and \( cl_\lambda \ll 1 \).
From equation (53), we conclude that the three-body scattering amplitude decreases in proportion to \((k/c)\) as \(k/c \to 0\), i.e. when the interaction is strong enough to induce significant atomic correlations. The three-body scattering rate in a 1D system of bosons in the case of strong pairwise interaction is suppressed by a factor \(\sim (k/c)^{12}\). By averaging over collision momenta in a moderately excited strongly interacting state, we obtain the scattering rate suppression factor \(\sim (k/c)^{12} \sim \gamma^{-12}\).

We can now compare our result with the zero-distance three-particle correlation function \(g_3(0)\). In the strong interaction limit \(\gamma \gg 1\) we have \(g_3(0) \propto \gamma^{-6}\) [27, 36] and this gives a direct physical motivation for our original conjecture [14] that the pairwise interactions and the quantum correlations induced by them in a strongly interacting 1D bosonic system suppress the three-body elastic scattering rate and, hence, thermalization, by a factor \(\propto g_3^2(0)\). In other words, strong quantum correlations extend the time scale, on which a quasi-1D system approaching the Tonks–Girardeau regime can be considered as approximately integrable.

Moreover, we can corroborate this conjecture by observing that the thermalization rate is proportional to the square of the matrix element of the transition operator that is proportional to \(g_3^2(0)\). Since the energies of the products of the elastic three-body process are low (about \(k_B T\)), we may assume that the correlations in the initial and final states are the same, and the transition matrix element can be regarded as proportional to \(g_3(0)\) that yields again the \(g_3^2(0)\) scaling of the rate. In contrast to this, the inelastic three-body processes are accompanied by a large energy release, and after an inelastic collision, the newly formed dimer molecule and the fast atom leave the system almost immediately. Therefore the inelastic three-body relaxation rate in a 1D ultracold Bose gas is proportional to the first power of \(g_3(0)\) [37].

We now compare the calculated thermalization rates to the quantum Newton’s cradle experiment [10] where a lower boundary for the damping time of the 1D motion towards a Gaussian profile was estimated. For three different Lieb–Liniger parameters \(\gamma = 1.4, 3.2\) and 18, Kinoshita et al estimate lower bounds to the thermalization time from the consistency of the experimental momentum distributions with the experimentally observed heating during the time interval of 0.5, 0.5 and 1.0 s probed. They find one sigma lower limits of 2.6, 25 and 13 s, respectively. In their experiment, the motion of two groups of \(^{87}\)Rb atoms was excited at a relative velocity equal to 4 recoil velocities, which is far above the width of the ground-state velocity distribution. We therefore cannot expect the collision rate to be suppressed in proportion to \(g_3^2(0)\), as described above for slow collisions. Instead, we have to apply the estimates for damping of a fast particle discussed in the end of section 4. In a strongly correlated system one has to take the particle correlations into account and equation (33) has to be multiplied by a factor \(g_2(0)\) denoting the two-particle correlation function at zero distance (Tan et al [24]):

\[
\Gamma_{k_0}^{\text{damp}} \approx \frac{9\sqrt{3}}{2} \xi^2 \omega_k \xi^2 g_2(0).
\]  \hspace{1cm} (54)

Substituting the experimental parameters of [10] and taking the values for \(g_2(0)\) from [38], we obtain \(\Gamma_{k_0}^{\text{damp}} \approx 15, 1.7\) and \(7 \times 10^{-3}\) s\(^{-1}\) for \(\gamma = 1.4, 3.2\) and 18, respectively. To compare these calculated damping rates to the experiment in [10], one has to consider that (i) the two colliding clouds overlap only for a very short time during each oscillation and (ii) that the thermalization rate is a factor of 3 longer (section 5). Taking this into account we estimate the respective thermalization times of 2.6, 35 and \(>1000\) s. These rates are consistent with the experimental findings of [10]. For a more detailed comparison, one would need longer time
scale experiments with lower intrinsic heating and more detailed calculations of the dynamics including the damping due to three-body collisions discussed here and in [24].

7. Conclusion

A radially confined atomic gas is never perfectly 1D and radial motion can be excited either in reality or virtually even if equation (1) holds. This possibility leads to effective three-body collisions, which arise in the second order of perturbation theory and can be associated with virtual excitation of radial modes. These processes lead to thermalization even when two-body collisions are frozen out at $k_B T \ll \hbar \omega$ and provide a mechanism to break integrability in 1D systems. In other words, the freeze-out of the radial modes is only a necessary, but not sufficient condition for integrability in 1D systems. Our estimations of the relaxation rates for weakly interacting quasi-1D Bose gases are consistent with recent experimental observations for weakly interacting quasi-condensates [11, 12].

These effective three-body collisions can be suppressed by quantum correlations caused by strong pairwise repulsions. If they dominate, as in a strongly correlated 1D Tonks–Girardeau gas, they suppress the influence of the integrability-breaking interaction term. The thermalization rate decreases in proportion to $g^2(0)$ as the system enters the regime of strong correlations ($\gamma \gg 1$), and the system (remaining non-integrable in the strict sense) behaves as (almost) integrable on time scales short compared with the inverse thermalization rate.

The effective three-body collisions and their suppression by quantum correlations should be accessible in experiments looking at the damping of fast, particle-like excitations in systems with $\gamma$ varying in a broad range of values from $< 1$ to $\sim 10$.

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