Link Invariants of Finite Type and Perturbation Theory

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Abstract

The Vassiliev-Gusarov link invariants of finite type are known to be closely related to perturbation theory for Chern-Simons theory. In order to clarify the perturbative nature of such link invariants, we introduce an algebra $V_\infty$ containing elements $g_i$ satisfying the usual braid group relations and elements $a_i$ satisfying $g_i - g_i^{-1} = \epsilon a_i$, where $\epsilon$ is a formal variable that may be regarded as measuring the failure of $g_i^2$ to equal 1. Topologically, the elements $a_i$ signify crossings. We show that a large class of link invariants of finite type are in one-to-one correspondence with homogeneous Markov traces on $V_\infty$. We sketch a possible application of link invariants of finite type to a manifestly diffeomorphism-invariant perturbation theory for quantum gravity in the loop representation.

1 Introduction

The manner in which the braid group $B_n$ takes the place of the symmetric group $S_n$ in the representation theory of quantum groups is by now well known. Recall that the braid group $B_n$ has a presentation with generators $s_i$, $1 \leq i < n$, and relations

$$s_i s_j = \begin{cases} s_j s_i & |i - j| > 1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. & \end{cases}$$

The symmetric group $S_n$ is the quotient of $B_n$ by the further relations $s_i^2 = 1$. If $G$ is a semisimple Lie group, then the corresponding quantized enveloping algebra $U_q g$ is a deformation of the universal enveloping algebra $U g$ as Hopf algebras. Given any representation of the group $G$, there is a natural action of the group algebra $CS_n$ as intertwining operators on the representation $E^{\otimes n}$. Similarly, given any representation $E$ of $U_q g$ there is a natural action of $CB_n$ as intertwining operators on $E^{\otimes n}$.

Naively, one might be led to hope that $CB_n$ is a kind of deformation of $CS_n$. While this is not true according to the standard definition - after all, $CB_n$ is infinite-dimensional while $CS_n$ is finite-dimensional - we show here that some sense can be
made of this idea. Roughly speaking, one can form an algebra $V_n$ over $\mathbb{C}[\epsilon]$, where $\epsilon$ is a formal variable, by adjoining to $\mathbb{C}B_n$ elements $a_i$, $1 \leq i < n$, such that

$$s_i - s_i^{-1} = \epsilon a_i.$$ 

The parameter $\epsilon$ should be thought of as measuring the failure for $s_i^2$ to equal 1. Setting $\epsilon$ equal to any nonzero constant gives the algebra $\mathbb{C}B_n$, while setting $\epsilon = 0$ gives an algebra containing $\mathbb{C}S_n$. More generally, for any $d \geq 0$, the quotient algebra $V_n/\langle \epsilon^{d+1} \rangle$ should be regarded as a $d$th-order perturbative approximation to $\mathbb{C}B_n$.

These quotients are closely related to the theory of link invariants of finite type, as developed by Vassiliev, Gusarov, Birman, Lin, Bar-Natan, and others [3, 5, 10, 13, 19]. The basic idea here is that one can canonically extend an invariant $L$ of oriented links to an invariant of generalized links admitting nice self-intersections (transverse double points) by means of the rule

$$L(L_+) - L(L_-) = \epsilon L(L_\times),$$

Here $L_+$ is as in Figure 1, $L_-$ is as in Figure 2, and $L_\times$ is as in Figure 3, with the strands oriented so as to be pointing downwards.

Those invariants vanishing on all generalized links with more than $d$ self-intersections are said to be of degree $d$. The space of link invariants of degree $d$ can be regarded as a $d$th-order approximation to the dual of the space with basis given by isotopy classes of links. If one takes one of the $\mathbb{C}(q)$-valued link invariants derived from quantum
group representations by the procedure of Turaev [18], sets \( q = \exp(\epsilon) \) to obtain a formal power series in \( \epsilon \), and takes the coefficient of \( \epsilon^d \), one obtains a link invariant of degree \( d \). More generally, we show that there is a one-to-one correspondence between a large class of link invariants of degree \( d \) and Markov traces \( \tau: V_\infty \to \mathbb{C}[\epsilon] \) that are “homogeneous of degree \( d \)” in a certain sense. Such Markov traces may be thought of as defined on \( V_\infty / \langle \epsilon^{d+1} \rangle \).

The connection between link invariants of finite type and physical perturbation theory is presently clearest in the context of Chern-Simons theory with semisimple gauge group \( G \). Here the action is given by

\[
S = \frac{k}{4\pi} \int_{S^3} \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),
\]

where \( A \) is a \( G \)-connection and the level \( k \geq 0 \) is an integer. From the work of Witten [20] and others it is clear that, for example, the Jones polynomial \( V_L(q) \) may be obtained from the vacuum expectation values of Wilson loops in Chern-Simons theory with \( G = SU(2) \) and \( q = \exp(2\pi i/(k + 2)) \). The coefficient of the \( \epsilon^d \) term of the Jones polynomial, an invariant of finite degree, should thus be calculable by a \( d \)th-order perturbation expansion in Chern-Simons theory. This has been pursued by Cotta-Ramusino et al, Smolin, and others [8, 15]. In particular, the relation to knot invariants of finite type has been studied by Bar-Natan [3], who dealt with an arbitrary classical Lie group, and by abstraction obtained a general combinatorial scheme for constructing knot invariants of finite type.

Link invariants of finite type may also be expected to play a role in a novel perturbation theory for 4-dimensional quantum gravity. In the loop representation of quantum gravity developed by Rovelli and Smolin [14], states are described by linear combinations of isotopy classes of framed unoriented links (or tangles), possibly admitting self-intersections. We briefly comment on the relation between Chern-Simons perturbation theory, link invariants of finite type, and perturbative quantum gravity in the final section of this paper.

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2 The Vassiliev Algebra

Let the generalized braid monoid, $GB_n$, denote the monoid with generators $g_i, g_i^{-1}, a_i$, $1 \leq i < n$, and relations

\[
\begin{align*}
[g_i, g_j] &= [a_i, a_j] = [a_i, g_j] = 0 & |i - j| > 1, \\
g_i g_i^{-1} &= g_i^{-1} g_i = 1, \\
a_i g_i &= g_i a_i, \\
g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, \\
a_{i+1} g_i^{-1} &= a_{i+1} g_i^{-1}, \\
\end{align*}
\]

In pictures of generalized braids, $g_i$ represents a right-handed crossing and $g_i^{-1}$ represents a left-handed crossing of the $i$th and $(i + 1)$st strands, as in Figures 1 and 2. The element $a_i$ represents an intersection of the $i$th and $(i + 1)$ strands, as in Figure 3. The relations above express topological facts about generalized braids, which admit intersections as well as crossings; the reader is strongly encouraged to draw these relations. The generalized braid monoid appears in the work of Kaufmann [11], as well as in the work of Brügmann, Gambini and Pullin [7, 9] on quantum gravity.

Let $CGB_n$ denote the monoid algebra of the generalized braid monoid. We define the Vassiliev algebra, $V_n$, to be the quotient of $CGB_n \otimes C[\epsilon]$ by the ideal generated by the elements

\[g_i - g_i^{-1} = \epsilon a_i.\]

It is clear that there is a homomorphism

\[v: CB_n \rightarrow V_n\]

given by $s_i \mapsto g_i$. The basic properties of this homomorphism are as follows.

**Lemma 1.** Let $C(\epsilon)$ denote the algebra of Laurent polynomials in $\epsilon$. Then

\[v \otimes 1: CB_n \otimes C(\epsilon) \rightarrow V_n \otimes C[\epsilon] \rightarrow C(\epsilon)\]

is an isomorphism.

Proof - The inverse is given by $g_i \mapsto s_i$, $a_i \mapsto \epsilon^{-1}(s_i - s_i^{-1})$. \hfill \square

**Corollary 1.** Let $j: V_n \rightarrow V_n / \langle \epsilon - x \rangle$ denote the quotient map. The map

\[j \circ v: CB_n \rightarrow V_n / \langle \epsilon - x \rangle\]

is an isomorphism if $x \in C$ is nonzero, while if $x = 0$ it factors through $C S_n$. 

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Proof - The composite $j \circ v$ is an isomorphism for $x \neq 0$ by Lemma 1, while if $x = 0$, $s_i^2 = 1$ in $V_n/\langle \varepsilon \rangle$, so $j \circ v$ factors through $\mathbf{C} S_n$. □

**Corollary 2.** The homomorphism $v: \mathbf{C} B_n \to V_n$ is one-to-one.

Proof - This is immediate from Lemma 1. □

We conclude this section with a word on the universal role the Vassiliev algebra plays in the context of braided tensor categories. It is clear that, given any object $E$ in a $\mathbf{C} [\varepsilon]$-linear strict braided monoidal category, if $R \equiv R^{-1} \mod \varepsilon$, where $R: E \otimes E \to E \otimes E$ is the braiding, then there is a canonical action of $V_n$ as endomorphisms of $E^\otimes n$. In particular, this applies to the category of quantum group representations, where we write $q = \exp(\varepsilon)$.

### 3 Link Invariants of Finite Type

By a link invariant we will always mean an ambient isotopy invariant of oriented links. It is easy to see that any $\mathbf{C}$-valued link invariant $L$ uniquely extends to a $\mathbf{C}(\varepsilon)$-valued invariant of generalized links admitting transverse double points, which we also call $L$, by means of the rule

$$L(L_+) - L(L_-) = \varepsilon L(L_\times),$$

where $L_+$, $L_-$, and $L_\times$ denote link diagrams with a right-handed crossing, a left-handed crossing, and an intersection, respectively, at a given point, the rest of the diagrams being the same. We define a $\mathbf{C}$-valued link invariant to be of degree $d$ if it vanishes on all generalized links with $d + 1$ or more self-intersections. A link invariant of degree $d$ for some $d$ is said to be of finite type.

For all $n$ there are algebra inclusions $V_n \hookrightarrow V_{n+1}$ and $\mathbf{C} B_n \hookrightarrow \mathbf{C} B_{n+1}$. Let $V_\infty$ and $\mathbf{C} B_\infty$ denote the inductive limits of the algebras $V_n$ and $\mathbf{C} B_n$, respectively, and let

$$v: \mathbf{C} B_\infty \to V_\infty$$

denote the inductive limit of the maps $v: \mathbf{C} B_n \to V_n$. We define a Markov trace on $V_\infty$ to be a $\mathbf{C}[\varepsilon]$-linear map $\tau: V_\infty \to E$, where $E$ is some $\mathbf{C}[\varepsilon]$-module, satisfying

$$\tau(xy) = \tau(yx)$$

for all $x, y \in V_\infty$, and for some fixed $z \in \mathbf{C}$

$$\tau(g_n^{\pm 1} x) = z \tau(x)$$

for all $x \in V_n \subset V_\infty$. A similar definition is standard for Markov traces $\text{tr}: \mathbf{C} B_\infty \to \mathbf{C}$. We say that a Markov trace $\tau: V_\infty \to \mathbf{C}[\varepsilon]$ is homogeneous of degree $d$ if for every $x \in \mathbf{C} B_\infty$, $\tau(v(x))$ is homogeneous of degree $d$ as a polynomial in $\varepsilon$. 

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For any braid $x \in B_n$, let $\hat{x}$ denote its closure. Of course, given $x \in B_\infty$, $\hat{x}$ depends on a choice of $n$ such that $x \in B_n$. Given a link $L$, let $L \cup \circ$ denote the distant union of $L$ with the unknot.

**Theorem 1.** There is a one-to-one correspondence between $C$-valued link invariants $L$ of degree $d$ such that for some $z \neq 0$ and all links $L$,

$$L(L \cup \circ) = z^{-1}L(L),$$

and Markov traces $\tau: V_\infty \to C[\epsilon]$ that are homogeneous of degree $d$. The invariant $L$ determines the trace $\tau$, and conversely, by the property that

$$\tau(v(x)) = \epsilon^d z^{n-1}L(\hat{x})$$

for $x \in B_n$.

Proof - Let $E$ be a vector space, and suppose $L$ is a $E$-valued link invariant satisfying equation (1). Then by Markov’s theorem \[4\] there is a Markov trace $\text{tr}: CB_\infty \to E$, given by

$$\text{tr}(x) = z^{n-1}L(\hat{x})$$

for all $x \in B_n$. Note in particular that $\text{tr}$ is well-defined because if $y \in CB_{n+1}$ is the image of $x \in CB_n$ under the inclusion $CB_n \hookrightarrow CB_{n+1}$, then

$$\text{tr}(y) = z^nL(\hat{y}) = z^nL(\hat{x} \cup \circ) = z^{n-1}L(\hat{x}) = \text{tr}(x).$$

Moreover $\text{tr}$ is a trace because $\hat{x}\hat{y} = \hat{y}\hat{x}$ for all $x, y \in B_n$, and $\text{tr}$ has the Markov property:

$$\text{tr}(s_n^{\pm 1}x) = z^nL((s_n^{\pm 1}x)^\ast) = z^nL(\hat{x}) = z\text{tr}(x).$$

In fact, Markov’s theorem implies that equation (3) gives a one-to-one correspondence between $E$-valued link invariants satisfying equation (1) and Markov traces $\text{tr}: CB_\infty \to E$.

Now let $L$ be a $C$-valued link invariant of degree $d$ satisfying equation (1). Then there is a unique Markov trace $\text{tr}: CB_\infty \to C$ given by equation (3). We claim that there exists a Markov trace $\tau_0: V_\infty \to C(\epsilon)$ such that

$$\epsilon^d\text{tr} = \tau_0v.$$ 

To see this, note that by Lemma 1 there is a Markov trace $\tilde{\tau}_0: V_\infty \otimes_{C[\epsilon]} C(\epsilon) \to C(\epsilon)$ such that

$$\epsilon^d(\text{tr} \otimes 1) = \tilde{\tau}_0(v \otimes 1)$$

as maps from $CB_\infty \otimes C(\epsilon)$ to $C(\epsilon)$. Let $\tau_0$ denote the restriction of $\tilde{\tau}_0$ to $V_\infty$, which may be regarded as a subalgebra of $V_\infty \otimes_{C[\epsilon]} C(\epsilon)$. It follows that $\epsilon^d\text{tr} = \tau_0v$, as desired.
Note that $\tau_0(x) = ce^{d-\ell}$ for some $c \in \mathbb{C}$ if $x \in V_\infty$ is a product of $\ell$ elements of the form $a_i$ and arbitrarily many of the form $g_i$. Moreover, if $\ell > d$ then $\tau_0(x)$ vanishes, since $L$ is of degree $d$. It follows that

$$\tau_0(x) = \sum_{i=0}^{d} c_i \epsilon^i,$$

with $c_i \in \mathbb{C}$, if $x$ is a product of elements of the form $a_i$ and $g_i$. It follows that $\tau_0$ factors through a map $\tau: V_\infty \to \mathbb{C}[\epsilon]$. It is easy to check that $\tau_0$, hence $\tau$, is a Markov trace that is homogeneous of degree $k$. Moreover, equation (2) holds by construction.

Conversely, suppose that $\tau: V_\infty \to \mathbb{C}[\epsilon]$ is a Markov trace homogeneous of degree $d$. We may define a Markov trace $\text{tr}: \mathcal{C}B_\infty \to \mathbb{C}$ by

$$\tau v = e^d \text{tr}.$$ 

Associated to this trace there is a link invariant $\mathcal{L}$ satisfying equation [1], given by equation (3). We claim that $\mathcal{L}$ is an invariant of degree $d$. For this, we need an analog of Alexander’s theorem for generalized links. Given an element $x \in \mathcal{G}B_n$, we may form a generalized link $\hat{x}$, the closure of $x$, in a manner analogous to the usual closure of a braid.

**Lemma 2.** For every generalized link $L$, for some $n$ there is an element $x \in \mathcal{G}B_n$ such that $\hat{x}$ is ambient isotopic to $L$.

Proof - We omit the proof, as it is similar to the usual proof of Alexander’s theorem [4], but would be quite long with all the details included. \qed

Now let $L$ be a generalized link with $\ell$ self-intersections, and let $x_0 \in \mathcal{G}B_n$ be such that $\hat{x}_0$ is ambient isotopic to $L$. Let $x_1$ be the image of $x_0$ in $V_\infty$. Then

$$x_1 = y_1 a_{i_1} y_2 a_{i_2} \cdots a_{i_{\ell}} y_{\ell+1}$$

where the elements $y_i$ are products of elements $g_j, g_j^{-1}$. Define $x \in \mathcal{C}B_\infty$ by

$$x = y_1 (g_{i_1} - g_j^{-1}) y_2 \cdots (g_{i_{\ell}} - g_{i_{\ell}}^{-1}) y_{\ell+1}.$$ 

Note that

$$e^d \text{tr}(x) = \tau(v(x)) = e^\ell \tau(x_1).$$ 

Since $\text{tr}(x) \in \mathbb{C}$ and $\tau(x_1) \in \mathbb{C}[\epsilon]$, so if $\ell > d$ we must have $\text{tr}(x) = 0$. By construction,

$$\mathcal{L}(L) = z^{1-n} \text{tr}(x),$$

so it follows that $\mathcal{L}(L) = 0$.

Now let us show that the invariant $\mathcal{L}$ uniquely determines the trace $\tau$, and vice versa, by equation (3). Since every link is the closure of some braid, $\mathcal{L}$ is determined
by $\tau$. Conversely, $L$ determines $\tau$ on the image of $v$, since $v$ is one-to-one by Corollary 2. It then follows by the $C[\epsilon]$-linearity of $\tau$ that $\tau$ is determined on all of $V_\infty$, since $\epsilon a_i = s_i - s_i^{-1}$.  

The reader may be puzzled by the fact that every link invariant of degree $d$ is obviously of degree $d + 1$, while a normalized Markov trace $\tau : V_\infty \to C[\epsilon]$ that is homogeneous of degree $d$ is definitely not homogeneous of degree $d + 1$. The point is that $\epsilon \tau$ will be a normalized Markov trace homogeneous of degree $d + 1$.

Note that to reconstruct the link invariant coming from a normalized Markov trace $\tau : V_\infty \to C[\epsilon]$ that is homogeneous of degree $d$, it suffices to know the composite of $\tau$ with the quotient map $C[\epsilon] \to C[\epsilon]/\langle \epsilon^{d+1} \rangle$, which may be regarded as a Markov trace $\tau : V_\infty / \langle \epsilon^{d+1} \rangle \to C[\epsilon]/\langle \epsilon^{d+1} \rangle$.

As an illustration of the theorem, let $tr_0 : CB_\infty \to C(q)$ be one of the traces obtained from quantum group representations by Turaev’s procedure. Writing $q = \exp(\epsilon)$, we regard $tr_0$ as having values in $C[\epsilon]$. For some invertible $z \in C[\epsilon]$, $tr_0(s_n^\pm x) = z tr_0(x)$ for all $x \in CB_n$. Since $z \not\in C$, the theorem above is not directly applicable. However, we may define a new Markov trace $tr : CB_\infty \to [\epsilon]$ with $tr(s_n^\pm x) = tr(x)$ for all $x \in CB_n$ by setting $tr(x) = z^{m-n} tr_0(x)$, for $x \in B_n$, where $m$ denotes the number of components of $\hat{x}$. There is a unique Markov trace $\tau : V_\infty \to C[\epsilon]$ such that $\tau v = tr$.

Moreover, we may write $\tau = \sum_{d=0}^{\infty} \tau_d$ where $\tau_d : V_\infty \to C[\epsilon]$ is a Markov trace homogeneous of degree $d$, with $\tau_d(g_n^\pm x) = \tau_d(x)$ for all $x \in V_n$. By the theorem, each such trace gives a link invariant $L_d$ of degree $d$, such that $L_d(\hat{x}) = \epsilon^{-d} \tau_d(v(x))$ for $x \in B_n$.

It is also important that the space of Markov traces $\tau : V_\infty \to C[\epsilon]$ that are homogeneous of degree $d$ is finite-dimensional. This may be seen by graph-theoretic reasoning along the lines of Birman and Lin [3], Bar-Natan [4], and Stanford [17]. Suppose, for example, that we fix the value of $z$. Then if $d = 0$, $\tau$ is determined by its value on 1. If $d = 1$, $\tau$ is determined by its value on 1, $a_1$, and $a_1 s_1$. If $d = 2$, $\tau$ is determined by its value on 1, $a_1$, $a_1 s_1$, $a_1^2$, $a_1^2 s_1$, $a_1 a_2$, $a_1 a_2 s_1 s_2$, $a_1 a_3$, $a_1 a_3 s_1$, and $a_1 a_3 s_1 s_3$.  

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4 Connections to Physics

The Vassiliev algebra formalism makes clear that link invariants of finite type arise from a sort of “topological perturbation theory.” We have pointedly used the symbol $\epsilon$ in our paper, instead of the suggestive letter $h$ (common in the quantum group literature), because the physical interpretation of this sort of perturbation theory is an interesting issue.

In $SU(n)$ Chern-Simons theory one makes the identification

$$\epsilon = \frac{2\pi i}{k+n}.$$  

The limit $\epsilon \to 0$ thus corresponds to $k \to \infty$, which may regarded either as a weak coupling limit or as a classical limit. In the weak coupling approach \[20\] we may write $A = A_0 + k^{1/2} B$, where $A_0$ is a flat connection, and obtain

$$S = kI + \frac{1}{8\pi} \int \epsilon^{abc} \text{tr}(B_a D_b B_c + \frac{2}{3} k^{-\frac{1}{2}} B_a [B_b, B_c]),$$

where $I$ is the Chern-Simons invariant of $A_0$ and $D$ denotes the covariant derivative with respect to $A_0$. Here we see that the $k \to \infty$ limit is closely related to a deformation of the Lie algebra $\mathfrak{su}(n)$ to an abelian Lie algebra by scaling the bracket. Alternatively, since $k$ appears where one would expect a factor of $\hbar^{-1}$ in the partition function

$$Z = \int \mathcal{D}A \exp \left( \frac{ik}{4\pi} \int \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \right),$$

one may also regard $k \to \infty$ as a classical limit. This is consistent with the weak coupling interpretation, of course, since the classical solutions of Chern-Simons theory are flat connections.

In quantum gravity the two obvious limits to consider are the classical limit $\hbar \to 0$ and the $G \to 0$ limit, where $G$ is Newton’s gravitational constant. The latter has been considered in the loop representation of Euclidean quantum gravity by Smolin \[16\], with the aim of developing a new perturbation theory for quantum gravity which is manifestly diffeomorphism-invariant at every order, as opposed to perturbation about a flat background spacetime. In this approach an $SU(2)$ connection is a key dynamical variable \[1\], and Smolin shows that the role of $G$ is to scale the Lie bracket in $\mathfrak{su}(2)$. The analogy with the $k \to \infty$ limit of Chern-Simons theory is no accident, since states of quantum gravity may be obtained from Wilson loops in $SU(2)$ Chern-Simons theory \[12\]. In this connection, the extension of the Jones polynomial to generalized links by Brügmann, Gambini and Pullin \[7\] is quite intriguing.

It is thus natural to hope that link invariants of finite type will play an important role in the $G \to 0$ limit of quantum gravity, or similar perturbation expansions in more general tangle field theories \[2\]. Roughly, we may expect that the true physical Hilbert space $\mathcal{H}$ is an inverse limit of spaces $\mathcal{H}_d$, where two states in $\mathcal{H}$ are identified in $\mathcal{H}_d$ if they cannot be distinguished by link (or tangle) invariants of degree $d$. 

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