Quasi-reductivity of Logically Constrained Term Rewriting Systems

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This paper considers quasi-reductivity—essentially, the property that an evaluation cannot get “stuck” due to a missing case in pattern matching—in the context of term rewriting with logical constraints.

1. INTRODUCTION

The formal framework of Logically Constrained Term Rewriting Systems (LCTRSs), introduced in [Kop and Nishida 2013], combines term rewriting with constraints and calculations over an arbitrary theory. This for instance allows users to specify rules with integers, arrays and strings, and can be used to analyze both imperative and functional programs (without higher-order variables) in a natural way.

Many methods to analyze term rewriting systems naturally extend to LCTRSs. In this paper we will study quasi-reductivity, the property that the only ground irreducible terms are constructor terms. We provide a simple method to prove quasi-reductivity: essentially, we will test that the rules do not omit any patterns.

Structure: For completeness, we will first set out the definition of LCTRSs, following [Kop and Nishida 2013]. In Section 2 we consider the definition of quasi-reductivity; in Section 3 we present three restrictions: left-linearity, constructor-soundness and left-value-freeness. The core of this work is Section 4 where we provide an algorithm to confirm quasi-reductivity for LCTRSs which satisfy these restrictions, and prove its soundness.

2. PRELIMINARIES

In this section, we briefly recall Logically Constrained Term Rewriting Systems (usually abbreviated as LCTRSs), following the definitions in [Kop and Nishida 2013].

Many-sorted terms. We introduce terms, typing, substitutions, contexts and subterms (with corresponding terminology) in the usual way for many-sorted term rewriting.

Definition 2.1. We assume given a set S of sorts and an infinite set V of variables, each variable equipped with a sort. A signature Σ is a set of function symbols f, disjoint from V, each equipped with a sort declaration [f] × × t i  => κ, with all t i and κ sorts. For readability, we often write κ instead of [f] × × t i  => κ. The set Terms(S, V) of terms over Σ and V contains any expression s such that t : ι can be derived for some sort ι, using:

\[ x : ι \vdash s : ι \hspace{1cm} t : ι \vdash s_1 : t_1 \ldots \vdash s_n : t_n \vdash f(s_1, \ldots, s_n) : κ \]

We fix Σ and V. Note that for every term s, there is a unique sort ι with t : ι.

Definition 2.2. Let t : ι. We call ι the sort of t. Let Var(s) be the set of variables occurring in s; we say that s is ground if Var(s) = ∅.

Definition 2.3. A substitution γ is a sort-preserving total mapping from V to Terms(S, V). The result sγ of applying a substitution γ to a term s with all occurrences of a variable x replaced by γ(x). The domain of this substitution, Dom(γ), is the set of variables x with γ(x) ≠ x. The notation [x_1 := s_1, \ldots, x_k := s_k] denotes a substitution γ with γ(x_i) = s_i for 1 ≤ i ≤ n, and γ(y) = y for y ∉ {x_1, \ldots, x_n}.

Definition 2.4. A context C is a term containing a typed hole □ : ι. If t : ι, we define C[t] as C with □ replaced by t. If we can write s = C[t], then t is a subterm of s.

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Logical terms. Specific to LCTRSs, we consider different kinds of symbols and terms.

Definition 2.5. We assume given:

d-signatures $\Sigma_{\text{terms}}$ and $\Sigma_{\text{theory}}$ such that $\Sigma = \Sigma_{\text{terms}} \cup \Sigma_{\text{theory}}$;

d-a mapping $I$ which assigns to each sort $i$ occurring in $\Sigma_{\text{theory}}$ a set $I_i$;

d-a mapping $J$ which assigns to each $f : \{I_1 \times \cdots \times I_n\} \Rightarrow k \in \Sigma_{\text{theory}}$ a function in $I_1 \times \cdots \times I_n \Rightarrow I_k$;

d-for all sorts $i$ occurring in $\Sigma_{\text{theory}}$ a set $Val_i \subseteq \Sigma_{\text{theory}}$ of values: function symbols $a : [] \Rightarrow i$ such that $J$ gives a bijective mapping from $Val_i$ to $I_i$.

We require that $\Sigma_{\text{terms}} \cap \Sigma_{\text{theory}} \subseteq Val = \bigcup Val_i$. The sorts occurring in $\Sigma_{\text{theory}}$ are called theory sorts, and the symbols theory symbols. Symbols in $\Sigma_{\text{theory}} \setminus Val$ are calculation symbols. A term in $\text{Terms}(\Sigma_{\text{theory}}, V)$ is called a logical term.

Definition 2.6. For ground logical terms, let $\{f(s_1, \ldots, s_n)\} := J_f([s_1], \ldots, [s_n])$. Every ground logical term $s$ corresponds to a unique value $c$ such that $[s] = [c]$; we say that $c$ is the value of $s$. A constraint is a logical term $\varphi$ of some sort bool with $I_{\text{bool}} = \mathbb{B} = \{\top, \bot\}$, the set of booleans. A constraint $\varphi$ is valid if $[\varphi] = \top$ for all substitutions $\gamma$ which map $\text{Var}(\varphi)$ to values, and satisfiable if $[\varphi] = \top$ for some substitutions $\gamma$ which map $\text{Var}(\varphi)$ to values. A substitution $\gamma$ respects $\varphi$ if $\gamma(x)$ is a value for all $x \in \text{Var}(\varphi)$ and $[\varphi] = \top$.

Formally, terms in $\text{Terms}(\Sigma_{\text{terms}}, V)$ have no special function, but we see them as the primary objects of our term rewriting systems: a reduction would typically begin and end with such terms, with calculation symbols only used in intermediate terms. Their function is to perform calculations in the underlying theory. Usually, which values are expected to occur in starting terms and end terms should be included both in $\Sigma_{\text{terms}}$ and $\Sigma_{\text{theory}}$, while values only used in constraints and calculations would only be in $\Sigma_{\text{theory}}$: true and false often fall in the latter category.

Example 2.7. Let $S = \{$int, bool$\}$, and consider the signature $\Sigma = \Sigma_{\text{terms}} \cup \Sigma_{\text{theory}}$ where $\Sigma_{\text{terms}} = \{$fact : [int] $\Rightarrow$ int$\} \cup \{$n : int $|$ n $\in$ $\mathbb{Z}$\}$ and $\Sigma_{\text{theory}} = \{$true, false : bool, $\land, \lor, \Rightarrow ;$ bool $\times$ bool $\Rightarrow$ bool, $+$, $-$, $*$ : [int $\times$ int] $\Rightarrow$ int$\} \cup \{$int $|$ n $\in$ $\mathbb{Z}$\}$. Then both int and bool are theory sorts, and the values are true, false and all symbols $n$ representing integers. For the interpretations, let $I_{\text{int}} = \mathbb{Z}$, $I_{\text{bool}} = \mathbb{B}$, and let $J$ be the evaluation function which interprets these symbols as expected.

Using infix notation, examples of logical terms are $0 = 0 + (-1)$ and $x + 3 \leq y + 42$. Both are constraints. $5 + 9$ is also a (ground) logical term, but not a constraint. Expected starting terms are for instance $\text{fact}(42)$ or $\text{fact}(\text{fact}(4))$: ground terms fully built using symbols in $\Sigma_{\text{terms}}$.

Rules and rewriting. We adapt the standard notions of rewriting (see, e.g., [Baader and Nipkow 1998]) by including constraints and adding rules to perform calculations.

Definition 2.8. A rule is a triple $\ell \rightarrow r[\varphi]$ where $\ell$ and $r$ are terms of the same sort and $\varphi$ is a constraint. Here, $\ell$ has the form $f(\ell_1, \ldots, \ell_n)$ and contains at least one symbol in $\Sigma_{\text{terms}} \setminus \Sigma_{\text{theory}}$ (so $\ell$ is not a logical term). If $\varphi = \text{true}$ with $J(\text{true}) = \top$, the rule may be denoted $\ell \rightarrow r$. Let $L\text{Var}(\ell \rightarrow r[\varphi])$ denote $\text{Var}(\varphi) \cup (\text{Var}(r) \setminus \text{Var}(\ell))$. A substitution $\gamma$ respects $\ell \rightarrow r[\varphi]$ if $\gamma(x)$ is a value for all $x \in L\text{Var}(\ell \rightarrow r[\varphi])$, and $[\varphi] = \top$.

Note that it is allowed to have $\text{Var}(r) \not\subseteq \text{Var}(\ell)$, but fresh variables in the right-hand side may only be instantiated with values. This is done to model user input or random choice, both of which would typically produce a value. Variables in the left-hand sides do not need to be instantiated with values (unless they also occur in the constraint); this is needed for instance to support a lazy evaluation strategy.
Definition 2.9. We assume given a set of rules \( \mathcal{R} \) and let \( \mathcal{R}_{\text{calc}} \) be the set 
\( \{ f(x_1, \ldots, x_n) \rightarrow y \mid y = f(\overline{x}) \} \) for \( x_1, \ldots, x_n \). The rewrite relation \( \rightarrow_\mathcal{R} \) is a binary relation on terms, defined by:
\[
C[\ell] \rightarrow_\mathcal{R} C[r] \quad \text{if} \quad \ell \rightarrow r [\varphi] \in \mathcal{R} \cup \mathcal{R}_{\text{calc}} \text{ and } \gamma \text{ respects } \ell \rightarrow r [\varphi]
\]
Here, \( C \) is an arbitrary context. A reduction step with \( \mathcal{R}_{\text{calc}} \) is called a calculation. A term is in normal form if it cannot be reduced with \( \rightarrow_\mathcal{R} \).

We will usually call the elements of \( \mathcal{R}_{\text{calc}} \) rules—or calculation rules—even though their left-hand side is a logical term.

Definition 2.10. For \( f(\ell_1, \ldots, \ell_n) \rightarrow r [\varphi] \in \mathcal{R} \) we call \( f \) a defined symbol; non-defined elements of \( \Sigma_{\text{terms}} \) and all values are constructors. Let \( \mathcal{D} \) be the set of all defined symbols, and \( \text{Cons} \) the set of all constructors. A term in \( \text{Terms}(\text{Cons},\mathcal{V}) \) is a constructor term.

Now we may define a logically constrained term rewriting system (LCTRS) as the abstract rewriting system \( \langle \text{Terms}(\Sigma,\mathcal{V}), \rightarrow_\mathcal{R} \rangle \). An LCTRS is usually given by supplying \( \Sigma, \mathcal{R} \), and also \( I \) and \( J \) if these are not clear from context.

Example 2.11. To implement an LCTRS calculating the factorial function, we use the signature \( \Sigma \) from Example 2.7 and the following rules:
\[
\mathcal{R}_{\text{fact}} = \{ \text{fact}(x) \rightarrow 1 \mid x \leq 0 \}, \text{fact}(x) \rightarrow x \ast \text{fact}(x-1) \mid \neg(x \leq 0) \}
\]

Using calculation steps, a term \( 3 - 1 \) reduces to \( 2 \) in one step (using the calculation rule \( x - y \rightarrow z \mid z = x - y \)), and \( 3 \ast (2 \ast (1 \ast 1)) \) reduces to \( 6 \) in three steps. Using also the rules in \( \mathcal{R}_{\text{fact}} \), \( \text{fact}(3) \) reduces in ten steps to \( 6 \).

Example 2.12. To implement an LCTRS calculating the sum of elements in an array, let \( \mathcal{I}_{\text{bool}} = \mathbb{B}, \mathcal{I}_{\text{int}} = \mathbb{Z}, \mathcal{I}_{\text{array(int)}} = \mathbb{Z}^* \), so \( \text{array(int)} \) is mapped to finite-length integer sequences. Let \( \Sigma_{\text{theory}} = \Sigma_{\text{int}}^{\text{theory}} \cup \{ \text{size} : [\text{array(int)}] \Rightarrow \text{int}, \text{select} : [\text{array(int)} \times \text{int}] \Rightarrow \text{int} \} \cup \{ a \mid a \in \mathbb{Z}^* \}. \) (We do not encode arrays as lists: every “array”—integer sequence—a corresponds to a unique symbol \( a \)). The interpretation \( J \) behaves on \( \Sigma_{\text{int}}^{\text{theory}} \) as usual, maps the values \( a \) to the corresponding integer sequence, and has:
\[
J_{\text{size}}(a) = k \quad \text{if } a = \langle n_0, \ldots, n_{k-1} \rangle \\
J_{\text{select}}(a,i) = n_i \quad \text{if } a = \langle n_0, \ldots, n_{k-1} \rangle \text{ with } 0 \leq i < k \\
0 \quad \text{otherwise}
\]

In addition, let:
\[
\Sigma_{\text{terms}} = \{ \text{sum} : [\text{array(int)}] \Rightarrow \text{int}, \text{sum0} : [\text{array(int)} \times \text{int}] \Rightarrow \text{int} \} \cup \\
\{ n : \text{int} \mid n \in \mathbb{Z} \} \cup \{ a \mid a \in \mathbb{Z}^* \}
\]
\[
\mathcal{R} = \left\{ \begin{array}{c}
\text{sum}(x) \rightarrow \text{sum0}(x, \text{size}(x)-1) \\
\text{sum0}(x,k) \rightarrow \text{select}(x,k) + \text{sum0}(x,k-1) \mid k \geq 0 \\
\text{sum0}(x,k) \rightarrow 0 \mid k < 0
\end{array} \right\}
\]

Note the special role of values, which are new in LCTRSs compared to older styles of constrained rewriting. Values are the representatives of the underlying theory. All values are constants (constructor symbols \( v() \) which do not take arguments), even if they represent complex structures, as seen in Example 2.12. However, not all constants are values; for instance a constant constructor \( \text{error} \in \Sigma_{\text{terms}} \) would not be a value. We will often work with signatures having infinitely many values. Note that we do not match modulo theories, e.g. we do not equate \( 0 \ast (x + y) \) with \( y + x \) for matching.

Note also the restriction on variables in a constraint being instantiated by values; for instance in Example 2.11 a term \( \text{fact}(3) \) reduces only at the inner fact.
3. QUASI-REDUCTIVITY

The most high-level definition of quasi-reductivity is likely the following.

Definition 3.1 (Quasi-reductivity). An LCTRS \((\Sigma_{\text{terms}}, \Sigma_{\text{theory}}, I, J, R)\) is quasi-reductive if for all \(s \in \text{Terms}(\Sigma, \emptyset)\) one of the following holds:

- \(s \in \text{Terms}(\text{Cons}, \emptyset)\) (we say: \(s\) is a ground constructor term);
- there is a \(t\) such that \(s \rightarrow_R t\) (we say: \(s\) reduces).

Note that \(\text{Terms}(\Sigma, \emptyset)\) is the set of ground terms. Another common definition concerns only the reduction of “basic” ground terms, but is equivalent:

Lemma 3.2. An LCTRS is quasi-reductive if and only if all terms \(f(s_1, \ldots, s_n)\) with \(f\) a defined or calculation symbol and all \(s_i \in \text{Terms}(\text{Cons}, \emptyset)\), reduce.

Proof. If the LCTRS is quasi-reductive, then each such \(f(s)\) reduces, as it is not a constructor term. If the LCTRS is not quasi-reductive, then let \(f(s)\) be a minimal ground irreducible non-constructor term. By minimality, all \(s_i\) must be constructor terms. If \(f\) is a constructor, then the whole term is a constructor term, contradiction, so \(f\) is either a defined symbol or a calculation symbol. 

4. RESTRICTIONS

For our algorithm in the next section, which proves that a given LCTRS is quasi-reductive, we will limit interest to LCTRSs which satisfy the following restrictions:

Definition 4.1 (Restrictions). An LCTRS \((\Sigma_{\text{terms}}, \Sigma_{\text{theory}}, I, J, R)\) is:

- left-linear if for all rules \(\ell \rightarrow r \ [\phi] \in R\): every variable in \(\ell\) occurs only once;
- constructor-sound if there are ground constructor terms for every sort \(i\) such that some \(f : \ldots \times i \times \ldots \Rightarrow \kappa \in D\) (so for every input sort of a defined symbol);
- left-value-free if the left-hand sides of rules do not contain any values.

Note that any LCTRS can be turned into a left-value-free one, by replacing a value \(v\) by a fresh variable and adding a constraint \(x = v\) instead. Constructor-soundness seems quite natural, with a sort representing the set of ground constructor terms of that sort. Left-linearity is probably the greatest limitation; however, note that non-left-linear systems impose syntactic equality. In a rule

\[
\text{addtoset}(x, \text{setof}(x, \text{rest})) \rightarrow \text{setof}(x, \text{rest})
\]

we can reduce \(\text{addtoset}(3 + 4, \text{setof}(3 + 4, s))\) immediately to \(\text{setof}(3 + 4, s)\). However, we cannot reduce \(\text{addtoset}(3 + 4, \text{setof}(4 + 3, s))\) with this rule. There is also no syntactic way to check for inequality. Therefore, it seems like we could better formulate this rule and its complement using constraints, or (if the sort of \(x\) has non-value constructors) by a structural check. The rule above and its complement could for instance become:

\[
\begin{align*}
\text{addtoset}(x, \text{setof}(y, \text{rest})) &\rightarrow \text{setof}(y, \text{rest}) & [x = y] \\
\text{addtoset}(x, \text{setof}(y, \text{rest})) &\rightarrow \text{setof}(y, \text{addtoset}(x, \text{rest})) & [x \neq y]
\end{align*}
\]

In this light, left-linearity also seems like a very natural restriction.

Comment: In [Falke and Kapur 2012] a similar method is introduced to prove quasi-reductivity of a different style of constrained rewriting. There, however, the systems are additionally restricted to be value-safe: the only constructors of sorts occurring in \(\Sigma_{\text{theory}}\) are values. We drop this requirement here, because it is not necessary in the definition of LCTRSs.

Constructor-soundness, arguably the most innocent of these restrictions, allows us to limit interest to certain well-behaved rules when proving quasi-reductivity:
THEOREM 4.2. A constructor-sound LCTRS with rules $\mathcal{R}$ is quasi-reductive if and only if the following conditions both hold:

- the same LCTRS restricted to constructor rules $\mathcal{R}' := \{ f(\overline{t}) \to r \mid [\varphi] \in \mathcal{R} \mid \forall i (\ell_i \in \text{Terms}(\text{Cons}, \mathcal{V})) \}$ is quasi-reductive;
- all constructor symbols with respect to $\mathcal{R}$ are also constructors w.r.t. $\mathcal{R}$.

PROOF. Suppose $\mathcal{R}'$ is quasi-reductive, and constructor terms are the same in either LCTRS. Then also $\mathcal{R}$ is quasi-reductive, as anything which reduces under $\mathcal{R}'$ also reduces under $\mathcal{R}$. Alternatively, suppose $\mathcal{R}$ is quasi-reductive.

Towards a contradiction, suppose $\mathcal{R}'$ has constructor symbols which are not constructors in $\mathcal{R}$; let $f$ be such a symbol. As $f$ is a constructor for $\mathcal{R}'$, there are no rules $f(\overline{t}) \to r \mid [\varphi] \in \mathcal{R} \cup \mathcal{R}_{\text{calc}}$ which match terms of the form $f(\overline{t'})$ with all $s_i \in \text{Terms}(\text{Cons}, \emptyset)$. Because $f$ is a defined symbol, such terms exist by constructor-soundness. As nothing matches $f(\overline{t'})$ itself, and its strict subterms are constructor terms so cannot be reduced, this term contradicts quasi-reductivity of $\mathcal{R}$!

For the first point, suppose towards a contradiction that $\mathcal{R}'$ is not quasi-reductive, yet $\mathcal{R}$ is, and the same terms are constructor terms in either. By Lemma 3.2, there is some irreducible $f(s_1, \ldots, s_n)$ with all $s_i$ constructor terms and $f$ not a constructor. As the $s_i$ are constructor terms, the rules in $\mathcal{R} \setminus \mathcal{R}'$ also cannot match! Thus, the term is also irreducible with $\rightarrow_{\mathcal{R}}$, contradiction. ∎

5. AN ALGORITHM TO PROVE QUASI-REDUCTIVITY

We now present an algorithm to confirm quasi-reductivity of a given LCTRS satisfying the restrictions from Definition 4.1. Following Theorem 4.2, we can—without loss of generality—limit interest to constructor TRSs, where the immediate arguments in the left-hand sides of rules are all constructor terms.

Main Algorithm. We assume given sequences $\ell_1, \ldots, \ell_n$ of theory sorts, $\kappa_1, \ldots, \kappa_m$ of sorts, and $x_1, \ldots, x_n$ of variables, with each $x_i : \ell_i \in \mathcal{V}$. Moreover, we assume given a set $A$ of pairs $(\overline{s'}, \varphi)$. Here, $\overline{s'}$ is a sequence $s_1, \ldots, s_m$ of constructor terms which do not contain values, such that $\vdash s_i : \kappa_i$, and $\varphi$ is a logical constraint. The $s_i$ have no overlapping variables with each other or the $x_j$; that is, a term $f(x_1, \ldots, x_n, s_1, \ldots, s_m)$ would be linear. Variables in $\overline{s'}$ and $\overline{s'}$ may occur in $\varphi$, however.

Now, for $b \in \{\text{term.value, either}\}$, 1 define the function $\text{OK}(\overline{s'}, A, b)$ as follows; this construction is well-defined by induction first on the number of function symbols occurring in $A$, second by the number of variables occurring in $A$, and third by the flag $b$ (with either $> \text{term.value}$). Only symbols in the terms $s_i$ are counted for the first induction hypothesis, so not those in the constraints.

- if $m = 0$: let $\{y_1, \ldots, y_k\} = (\bigcup_{(i, \varphi) \in A} \text{Var}(\varphi)) \setminus \{x_1, \ldots, x_n\}$;  
- if $\exists y_1 \ldots y_k (\forall (i, \varphi) \in A \varphi)$ is valid, then $\text{true}$  
- else $\text{false}$  
   Note that if $A = \emptyset$, this returns $\text{false}$.
- if $m > 0$ and $b = \text{either}$, then consider $\kappa_1$. If $\kappa_1$ does not occur in $\Sigma_{\text{theory}}$, the result is:  
   $$\text{OK}(\overline{s'}, A, \text{term})$$  
   If $\kappa_1$ occurs in $\Sigma_{\text{theory}}$ and all constructors with output sort $\kappa_1$ are values, then the result is:  
   $$\text{OK}(\overline{s'}, A, \text{value})$$

1This parameter indicates what constructor instantiations we should consider for $s_1$.  

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If \( \kappa_1 \) occurs in \( \Sigma_{\text{theory}} \) but there are also non-value constructors of sort \( \kappa_1 \), then let \( V := \{ ((s', \varphi) \in A \mid s_1 \text{ is a variable}) \} \) and \( T := \{ ((s', \varphi) \in A \mid s_1 \text{ is not a variable in } \text{Var}(\varphi)) \} \). Note that \( V \) and \( T \) overlap in cases where \( s_1 \) is a variable not occurring in \( \varphi \). The result of the function is:

\[
\text{OK}(\vec{s}, V, \text{value}) \land \text{OK}(\vec{s}, T, \text{term})
\]

In all cases, the recursive calls are defined, by the decrease in the third argument (and in the last case possibly also in the first and second argument).

- if \( m > 0 \) and \( b = \text{value} \), then we assume that \( \kappa_1 \) occurs in \( \Sigma_{\text{theory}} \) and for all \( ((\vec{s'}, \varphi) \in A \) the first term, \( s_1 \), is a variable (if not, we might define the function result as \text{false} , but this cannot occur in the algorithm). Then let \( x_{n+1} \) be a fresh variable of sort \( \kappa_1 \), and let \( A' := \{ ((s_2, \ldots, s_m), \varphi[s_1 := x_{n+1}] \mid ((s_1, \ldots, s_m), \varphi) \in A) \}; \) the result is:

\[
\text{OK}(x_1, \ldots, x_{n+1}), A', \text{either}
\]

Note that \( A' \) has equally many function symbols as and fewer variables than \( A \), and that we indeed have suitable sort sequences \( (\iota_1, \ldots, \iota_n, \kappa_1) \) for the variables and \( \kappa_2, \ldots, \kappa_m \) for the term sequences);

- if \( m > 0 \) and \( b = \text{term} \) and for all \( ((\vec{s'}, \varphi) \in A \) the first term, \( s_1 \), is a variable, then we assume (like we did in the previous case) that never \( s_1 \in \varphi \), and let \( A' := \{ ((s_2, \ldots, s_m), \varphi) \mid ((s_1, \ldots, s_m), \varphi) \in A \}; \) the result is:

\[
\text{OK}(\vec{s'}, A', \text{either})
\]

\( A' \) has at most as many function symbols as and fewer variables than \( A \).

- if \( m > 0 \) and \( b = \text{term} \) and there is some \( ((\vec{s'}, \varphi) \in A \) where \( s_1 \) is not a variable, then let \( f_1, \ldots, f_k \) be all non-value constructors with output sort \( \kappa_1 \) and let \( A_1, \ldots, A_k \) be defined as follows: \( A_i := \{ ((\vec{s'}, \varphi) \in A \mid s_1 \text{ is a variable or has the form } f_i(\vec{t}) \} \). Now, for all \( i \) if \( f_i \) has sort declaration \( [\mu_1 \times \cdots \times \mu_p] \Rightarrow \kappa_1 \), then we consider the new sort sequence \( \vec{s'} \) with \( \kappa' = (\mu_1, \ldots, \mu_p, \kappa_2, \ldots, \kappa_m) \); for every \( ((\vec{s'}, \varphi) \in A), \) we define:

\[
\text{OK}(\vec{s'}, B_1, \text{either}) \land \cdots \land \text{OK}(\vec{s'}, B_k, \text{either})
\]

Correctness of this algorithm is proved using the following technical result.

**Lemma 5.1.** For any suitable \( n, m, \vec{s'}, \vec{s}, b \) and \( A \) such that \( \text{OK}(\vec{s}, A, b) = \text{true} \), we have, for any sequence \( (s_1, \ldots, s_n) \) of values and any sequence \( (t_1, \ldots, t_m) \) of ground constructor terms: if one of the following conditions holds,

- \( b = \text{either or } m = 0 \)
- \( b = \text{value and } t_1 \text{ is a value} \)
- \( b = \text{term and } t_1 \text{ is not a value, and for all } ((u_1, \ldots, u_n), \varphi) \in A: u_1 \notin \text{Var}(\varphi) \)

then there is some \( ((u_1, \ldots, u_m), \varphi) \in A \) and a substitution \( \gamma \) with \( \gamma(x_i) = s_i \) such that:

- each \( t_i = u_i \)
- \( \varphi \gamma \) is a valid ground logical constraint
PROOF. By induction on the derivation of $\text{OK}(\vec{x}, A, b) = \text{true}$. Let values $(s_1, \ldots, s_n)$ and ground constructor terms $(t_1, \ldots, t_m)$ which satisfy the conditions be given.

If $m = 0$, then let $\psi := \bigvee_{((i, \varphi) \in A} \varphi$. By definition of $\text{OK}$, $\exists y_1 \ldots y_k(\psi)$ is valid and $\text{Var}(\psi) \subseteq \{x_1, \ldots, x_n, y_1, \ldots, y_k\}$. That is, there are values $v_1, \ldots, v_k$ such that for all values $u_1, \ldots, u_n$ the ground constraint $\psi|_{\vec{y} := \vec{v}}, \vec{x} := \vec{u}}$ is valid. In particular, we can take $\vec{s}$ for $\vec{u}$. Define $\gamma := [y_1 := v_1, \ldots, y_k := v_k], x_1 := s_1, \ldots, x_n := s_n$. Then $\gamma \psi$ is valid, and since it is ground, some clause in the disjunction must be valid; so some $(((i, \varphi) \in A$ where $\varphi\gamma$ is a valid ground constraint. This is what the lemma requires.

If $m > 0$ and $b = \text{either}$ and $\kappa_1$ does not occur in $\Sigma_{\text{theory}}$, then $\text{OK}(\vec{x}, A, \text{term})$ holds. Since there are no values of sort $\kappa_1$, the term $t_1$ is not a value; for the same reason, variables of sort $\kappa_1$ cannot occur in any constraint $\varphi$. Thus, the conditions for the induction hypothesis are satisfied; we find a suitable $\gamma$ and $((\vec{u}, \varphi)$. If $m > 0$ and $b = \text{either}$ and $\kappa_1$ does occur in $\Sigma_{\text{theory}}$, and all constructors with output sort $\kappa_1$ are values, then $\text{OK}(\vec{x}, A, \text{value})$ holds; moreover, $t_1$ is necessarily a value, so we can again apply the induction hypothesis.

If $m > 0$ and $b = \text{either}$ and there are both values and other constructors with output sort $\kappa_1$, then both $\text{OK}(\vec{x}, T, \text{term})$ and $\text{OK}(\vec{x}, \text{V}, \text{value})$ must hold. If $t_1$ is a value, then the conditions to apply the induction hypothesis with $V$ are satisfied; if not, the conditions to apply it with $T$ are satisfied! Since both $T$ and $V$ are subsets of $A$, this results in a suitable element and substitution.

If $m > 0$ and $b = \text{term}$, then we may assume that $t_1$ is a value. Let $x_{n+1}$ be a fresh variable of sort $\kappa_1$, and $A' = \{((u_2, \ldots, u_m), \varphi[u_1 := x_{n+1}] | (\vec{u}, \varphi) \in A\}$. Applying the induction hypothesis with $n + 1, m - 1, (\vec{u}, u_1), (u_2, \ldots, u_m), (x_1, \ldots, x_n, x_{n+1}), A'$, either and $(s_1, \ldots, s_n, t_1)$ and $(u_2, \ldots, u_m, t_{n+1})$, gives an element $(\vec{u}, \varphi) \in A'$ and $\gamma$ such that $\gamma |_{x_i} = s_i$ for $1 \leq i \leq n$ and $\gamma |_{x_{n+1}} = t_1$ and each $t_i = u_i \gamma$, $\varphi\gamma$ is a valid ground logical constraint. Now, $(\vec{u}, \varphi)$ can be written as $(\vec{w}, \vec{u}, \varphi)[[w_1 := \gamma(x_{n+1})]]$ for some $(\vec{w}, \varphi) \in A$. So let $\delta$ be the substitution $\gamma \cup [w_1 := \gamma(x_{n+1})]$. Noting that by linearity $w_1$ cannot occur in the other $w_i$, and that $\varphi\delta = \varphi\gamma$ because $x_{n+1}$ does not occur in $\varphi\gamma$, each further $\delta(x_i) = \gamma(x_i) = s_i$ and $t_1 = \gamma(x_{n+1}) = \delta(w_1) = w_1 \delta$ and $t_i = u_i \gamma = w_i \delta$ for larger $i$.

If $m > 0$ and $b = \text{term}$, then we can assume that $t_1$ is not a value. If all $(\vec{u}, \varphi) \in A$ have a variable for $u_1$, then we use the induction hypothesis and find a suitable element $((u_2, \ldots, u_m), \varphi) \in A'$ and substitution $\gamma$ with $\gamma |_{x_i} = s_i$ for all $1 \leq i \leq n$, such that each $t_i = u_i \gamma$ ($i > 1$) and $\varphi\gamma$ is a valid ground logical constraint. Choose $\delta := \gamma \cup [u_1 := t_1]$ (this is safe by linearity). The same requirements are satisfied, and also $t_1 = u_1 \delta!$

Finally, suppose $m > 0$ and $b = \text{term}$ and $A$ has some element $(\vec{u}, \varphi)$ where $u_1$ is not a variable; by assumption it is also not a value. By the conditions, we may assume that always $u_1 \notin \text{Var}(\varphi)$. Let $f_1, \ldots, f_k$ be all constructors in $\Sigma_{\text{terms}} \setminus \text{Val}$ with output sort $\kappa_1$. Since also $t_1$ is not a value, but is a ground constructor term, it can only have the form $f_\kappa(w_1, \ldots, w_k)$ for some $p_{\kappa, \vec{u}}$. Observing that $\text{OK}(\vec{x}, B_p, \text{either})$ must hold, we use the induction hypothesis, for $x_i, \vec{x}$ and $(w_1, \ldots, w_k, t_2, \ldots, t_m)$, and find both a suitable tuple $((q_1, \ldots, q_k, w_2, \ldots, u_m), \varphi) \in B_p$ and a substitution $\gamma$ which respects $\varphi$, maps $\vec{x}$ to $\vec{s}$ and has $q_i \gamma = w_i$ and $u_i \gamma = t_i$ for all $i, j$.

By definition of $B_p$, we have $((u_1, \ldots, u_m), \varphi) \in A$ for some $u_1$ which is either a variable (in which case all $q_i$ are fresh variables), or $u_1 = f_p(q_1, \ldots, q_k)$. In the case of a variable, $u_1$ cannot occur in any of the other $u_i$ by the linearity requirement, nor in $\varphi$ by the conditions. Thus, we can safely assume that $u_1$ does not occur in the domain of $\gamma$, and choose $\delta := \gamma \cup [u_1 := f(w_1, \ldots, w_k)]$. Then $\varphi\delta = \varphi\gamma$ is still a valid ground constraint, each $s_i = x_i \gamma = x_i \delta$ and for $i > 1$ also $t_i = u_i \gamma = u_i \delta$. Finally, $t_1 = f_p(w_1, \ldots, w_k) = u_1 \delta$ as required. In the alternative case that $u_1 = f_p(\delta_1, \ldots, q_k)$, we observe that $\gamma$ already suffices: each $s_i = x_i \gamma$, for $i > 1$ we have $t_i = u_i \gamma$ and $t_1 = f_p(w_1, \ldots, w_k) = f_p(q_1, \ldots, q_k) = u_1 \gamma$. □
With this, we can easily reach our main result:

**Theorem 5.2.** A left-linear and left-value-free constructor-sound LCTRS with rules \( R \) is quasi-reductive if for all defined and calculation symbols \( f \): \( \text{OK}(((), A_f, \text{either}) \text{ holds,} \)

where \( A_f := \{(\ell, \varphi) \mid \ell \rightarrow r \in R \cup R_{\text{calc}} \land \ell = f(\ell)\} \).

**Proof.** By Lemma 3.2 it suffices to prove that all terms of the form \( f(s_1, \ldots, s_n) \) can be reduced, where \( f \) is a defined or calculation symbol and all \( s_i \) are constructor terms. This holds if there is a rule \( f(\ell_1, \ldots, \ell_n) \rightarrow r \in R \cup R_{\text{calc}} \) and a substitution \( \gamma \) such that each \( s_i = \ell_i \gamma \) and \( \varphi \gamma \) is a satisfiable constraint. By Lemma 5.1 (which we can apply because left-hand sides of rules are linear and value-free), that is exactly the case if \( \text{OK}(((), A_f, \text{either}) = \text{true}! \)

6. **Conclusions**

In this paper, we have given an algorithm to prove quasi-reductivity of LCTRSs, whose core idea is to identify missing cases in the rules. Although we needed to impose certain restrictions to use this algorithm, these restrictions seem very reasonable.

Although we have not proved so here, we believe that our method is not only sound, but also complete for the class of left-linear, left-value-free constructor-sound LCTRSs. We intend to explore this in future work.

The method presented in this paper has been fully implemented in our tool Ctrl, which is available at

[http://cl-informatik.uibk.ac.at/software/ctrl/](http://cl-informatik.uibk.ac.at/software/ctrl/)

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