A method to multi-attribute decision-making based on interval-valued q-rung dual hesitant linguistic Maclaurin symmetric mean operators

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Abstract
The aim of this paper is to propose a new multi-attribute decision-making (MADM) method to rank all feasible alternatives in complex decision-making scenarios and determine the optimal one. To this end, we first propose the notion of interval-valued q-rung dual hesitant linguistic sets (IVq-RDHLSs) by combining interval-valued q-rung dual hesitant fuzzy (IVq-RDHF) sets with linguistic terms set. The proposed IVq-RDHLSs utilize IVq-RDHF membership and non-membership degrees to assess linguistic terms, so that they can fully express decision-makers’ evaluation information. Additionally, some related concepts such as the operational rules, score and accuracy functions, and ranking method of IVq-RDHLSs are presented. Considering the good performance of the classical Maclaurin symmetric mean (MSM) in integrating fuzzy information, we further generalize MSM into IVq-RDHLSs to propose the interval-valued q-rung dual hesitant linguistic MSM operator, the interval-valued q-rung dual hesitant linguistic dual MSM operator, as well as their weighted forms. Afterwards, we study the applications of IVq-RDHLSs and their aggregation operators in decision-making and propose a new MADM method. Some real decision-making problems in daily life are employed to prove the rightness of the proposed method. We also attempt to demonstrate the advantages and superiorities of our proposed method through comparing with some other methods in this paper.

Keywords Multi-attribute decision-making · Interval-valued q-rung dual hesitant fuzzy sets · Interval-valued q-rung dual hesitant linguistic sets · Maclaurin symmetric mean

Introduction
The past years have witnessed the great success of the study on Pythagorean fuzzy sets (PFSs) based multi-attribute decision-making (MADM) methods [1–11]. PFSs were proposed by Prof. Yager [12] to overcome the shortcoming of intuitionistic fuzzy sets (IFSs) [13], which fail to describe decision-making situations in which the sum of membership grade (MG) and non-membership grade (NMG) is greater than one. Hence, PFSs have been regarded as an effective tool to comprehensively express decision-makers’ (DMs’) complex evaluation information. Recently, Yager [14] generalized PFSs by relaxing their constraint and proposed the notion of q-rung orthopair fuzzy sets (q-ROFSs). Due to their high efficiency in expressing fuzzy and complex DMs’ assessment information, q-ROFS-based MADM models and methods have been a new research topic in operational research. Recent publications focused on q-ROFSs based decision-making can be roughly divided into three types. The first type is q-rung orthopair fuzzy (q-ROF) aggregation operators (AOs)-based MADM methods. Liu and Wang [15] proposed basic q-ROF operational rules and simple weighted average/geometric AOs, which are the foundation of q-rung orthopair fuzzy AO theory. On the basis of Liu and Wang’s [15] pioneering works, some other powerful AOs have been proposed, such as the q-ROF Bonferroni mean [16], q-ROF Heronian mean [17], q-ROF Hamy mean...
For example, Liu and Jin [48] generalized DMs’ evaluations both quantificationally and qualitatively to the high complexity of real decision-making problems. In most practical decision-making situations, it is almost impossible for DMs to comprehensively present their evaluation opinions. In addition, some scholars investigated the operation theory of q-ROFNs and proposed novel q-ROF operational laws. For instance, Peng et al. [25] proposed exponential operations and corresponding AOs of q-ROFNs. Xing and her colleagues [26] studied the interaction operations of q-ROFNs, which take the relationship among MGs and NMGs into consideration. Besides, scholars also investigated Hamacher operations [27], Dombi operations [28], and Archimedean operations of q-ROFNs [29], respectively. The second type is information measures of q-ROFS. Du [30] introduced Minkowski-type distance measures of q-ROFSs. Peng and Liu [31] studied distance measure, similarity measure, entropy, and inclusion measure for q-ROFS, and invested their application in MADM. Liu et al. [32] discussed the cosine similarity measure and a Euclidean distance measure of q-ROFSs, and investigated their properties and applications. Peng and Dai [33] proposed multi-parametric similarity measure of q-ROFSs and applied it in the assessment of classroom teaching quality. The third type focuses on extensions of classical q-ROFSs and studies their corresponding MADM methods. For example, Liu and Liu [34, 35] employed linguistic terms to represent the MGs and NMGs to propose the linguistics q-ROFSs and investigated their applications in MADM. Li et al. [36] and Wang et al. [37] utilized q-rung orthopair fuzzy MGs and NMGs to portray linguistic variables and proposed the q-rung orthopair linguistic sets (q-ROLSs). Similarly, Wang et al. [38], Xing et al. [39], and Liu et al. [40, 41] proposed the q-rung orthopair uncertain linguistic sets and discussed their applications in MADM, respectively.

Recently, Xu et al. [42, 43] proposed the q-rung dual hesitant fuzzy sets (q-RDHFSs) and interval-valued q-RDHFSs (IVq-RDHFSs) which are powerful to deal with DMs’ high hesitancy in giving MGs and NMGs in q-ROFSs context. In addition, Xu et al. [42, 43] introduced the AOs of q-RDHFSs and IVq-RDHFSs to propose novel MADM methods. Numerical examples have proved the effectiveness advantages of these MADM approaches. However, Xu et al.’s [43] method still has shortcomings, owing to its information expression capability. In most practical decision-making situations, it is almost impossible for DMs to comprehensively provide their evaluation values by quantitative methods, due to the high complexity of real decision-making problems. As a matter of fact, more and more scientists and scholars have realized the importance and necessity of depicting DMs’ evaluations both quantificationally and qualitatively [44–47]. For example, Liu and Jin [48] generalized intuitionistic fuzzy sets to intuitionistic uncertain linguistic sets, which employ intuitionistic fuzzy sets and uncertain linguistic to express DMs’ quantitative and qualitative evaluation values, respectively. Similarly, Du et al.’s [49] combined interval-valued PFSs with linguistic terms set (LST) and proposed the interval-valued Pythagorean linguistic sets. In light of this, Xu et al.’s [43] method is flawed as it only considers DMs’ quantitative assessment information.

Based on the above analysis, the motivation of this paper is to propose a new MADM method, which overcomes the drawback of Xu et al.’s [43] method. To do so, we first propose the concept of interval-valued q-rung dual hesitant linguistic sets (IVq-RD HLSs) to portray fuzzy decision-making information. The IVq-RD HLS can be regarded as a combination of IVq-ROFS with LTS. The IVq-RD HLSs are parallel to the interval-valued dual hesitant fuzzy linguistic sets (IVDHFLSs) proposed by Wei et al. [50], but are more powerful and flexible as they have relaxer constraint, which provides DMs more freedom to fully express their evaluation opinions. In addition, IVq-RDHLSs are also stronger than q-ROLSs as they can effectively deal with DMs’ hesitancy and indeterminacy in expressing their decision information. When considering to integrate attribute values in interval-valued q-rung dual hesitant linguistic (IVq-RDHL) MADM problems, the MSM operator has deeply impressed us, due to its flexibility of reflecting the interrelationship among multiple input variables. Because of this characteristic, MSM has been extensively applied to fuse IFSs [51, 52], interval-valued IFSs [53], hesitant fuzzy sets [54], neutrosophic sets [55]. Therefore, this paper utilizes MSM to aggregate IVq-RDHFL information and proposes a series of AOs of IVq-RDHFL variables (IVq-RDHFLVs).

The rest of this paper is organized as follows. In section “Preliminaries”, we review the existing basic notions and propose the concept if IVq-RD HLSs. The operations and comparison method are also presented in section “Preliminaries”. Section “Aggregation operators for IVq-RDHL information” proposes some AOs of IVq-RDHL information and discusses their properties in detail. Section “A novel method to MADM with IVq-RDHL information” gives the main steps of a novel MADM based on the proposed AOs. Numerical examples are investigated in section “Application of the proposed method in medical equipment selection”. Conclusion remarks are provided in “Conclusions”.

123
Preliminaries

In this section, we briefly review the basic notions that will be used in the following parts. In addition, we also propose the concept of IVq-RDHLSs as well as their operations and comparison method.

**IVq-RDHFSs, IVq-RDHLSs, and their related notions**

To comprehensively express DMs’ fuzzy judgements over alternatives with high hesitancy, Xu et al. [43] introduced the IVq-RDHFSs. The remarkable feature of IVq-RDHFS is that its MG and NMG are denoted by a set of interval values, satisfying the sum of $q$th power of MG and $q$th power of NMG is less than or equal to one ($q \geq 1$). The definition of IVq-RDHFS is given as follows.

**Definition 1** [43] Let $X$ be a fixed set, and then, an interval-valued q-rung dual hesitant fuzzy set (IVq-RDHFS) $D$ defined on $X$ is expressed as:

$$D = \{ (x, h_D(x), g_D(x)) | x \in X \},$$

where $h_D(x) = \bigcup_{[r_D^{l}, r_D^{u}] \in h_D(x)} \{ [r_D^{l}, r_D^{u}] \}$ and $g_D(x) = \bigcup_{[\eta_D^{l}, \eta_D^{u}] \in g_D(x)} \{ [\eta_D^{l}, \eta_D^{u}] \}$ are two sets of some interval values in $[0, 1]$, denoting the possible MG and NMG of the element $x \in X$ to the set $D$, respectively, such that $[r_D^{l}, r_D^{u}]$, $[\eta_D^{l}, \eta_D^{u}] \subseteq [0, 1]$, and $0 \leq \left( (r_D^{l})^q + (\eta_D^{l})^q \right)^{\frac{1}{q}} \leq 1$. $q \geq 1$, where $[r_D^{l}, r_D^{u}] \in h_D(x), [\eta_D^{l}, \eta_D^{u}] \in g_D(x), (r_D^{l})^q \in h_D^+(x) = \bigcup_{[r_D^{l}, r_D^{u}] \in h_D(x)} \max \{ r_D^{l}, \} \eta_D^{l} \} \in g_D^+(x) = \bigcup_{[\eta_D^{l}, \eta_D^{u}] \in g_D(x)} \max \{ \eta_D^{l}, \} \eta_D^{l} \}$. For convenience, we call the pair $d(x) = (h_D(x), g_D(x))$ an interval-valued q-rung dual hesitant fuzzy element (IVQ-RDHFE), which can be denoted by $d = (h, g)$ for simplicity. Especially, if $r_D^{l} = r_D^{u}$ and $\eta_D^{l} = \eta_D^{u}$, then $D$ reduces to the q-rung dual hesitant fuzzy set [42]; if $q = 1$, then $D$ reduces to interval-valued dual hesitant fuzzy set [56]; if $q = 2$, then $D$ reduces to hesitant interval-valued Pythagorean fuzzy set [57].

Similar to natural language, the LTS is an effective tool proposed by Zadeh [58] to depict DMs’ fuzzy evaluations, and, in addition, to more comprehensively and accurately portray DMs’ judgements, scholars, and scientists combined LTS with fuzzy set theories to propose new hybrid tools. The typical representatives are the intuitionistic linguistic sets [59], Pythagorean fuzzy linguistic sets [60], hesitant fuzzy linguistic sets [61], q-rung orthopair linguistic sets [36, 37], etc. Similarly, we employ IVq-RDHFs to denote the possible MGs and NMGs of linguistic terms to propose the IVq-RDHLSs. To this end, we first briefly review the concept of LTS.

**Definition 2** [58] Let $S = \{ s_i | i = 1, 2, \ldots, t \}$ be a linguistic term set (LTS) with odd cardinality. Any label $s_i$ represents a possible value for a linguistic variable, and it should satisfy the following characteristics [62]: (1) the set is ordered: $s_i > s_j$ if $i > j$; (2) there is the negation operator: $\neg(s_i) = s_j$, such that $i + j = t + 1$. For example, an LTS $S$ can be defined as:

$$S = \{ s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{medium}, s_5 = \text{good}, s_6 = \text{very good}, s_7 = \text{extremely good} \}$$

In the following, we combine IVq-RDHLSs with LTS and propose the IVq-RDHLSs.

**Definition 3** Let $X$ be a fixed set and $\tilde{S}$ be a continuous LTS of $S = \{ s_i | i = 1, 2, \ldots, t \}$, and then, an interval-valued q-rung dual hesitant linguistic set (IVq-RDHLS) $D$ defined on $X$ is expressed as:

$$D = \{ (x, s_D(p), h_D(x), g_D(x)) | x \in X \},$$

where $s_D(p) \in \tilde{S}, h_D(x) = \bigcup_{[r_D^{l}, r_D^{u}] \in h_D(x)} \{ [r_D^{l}, r_D^{u}] \}$ and $g_D(x) = \bigcup_{[\eta_D^{l}, \eta_D^{u}] \in g_D(x)} \{ [\eta_D^{l}, \eta_D^{u}] \}$ are two sets of interval values in $[0, 1]$, denoting the possible MG and NMG of the element $x \in X$ to the set $D$, respectively, such that $[r_D^{l}, r_D^{u}]$, $[\eta_D^{l}, \eta_D^{u}] \subseteq [0, 1]$, and $0 \leq \left( (r_D^{l})^q + (\eta_D^{l})^q \right)^{\frac{1}{q}} \leq 1$. $q \geq 1$, where $[r_D^{l}, r_D^{u}] \in h_D(x), [\eta_D^{l}, \eta_D^{u}] \in g_D(x), (r_D^{l})^q \in h_D^+(x) = \bigcup_{[r_D^{l}, r_D^{u}] \in h_D(x)} \max \{ r_D^{l}, \} \eta_D^{l} \} \in g_D^+(x) = \bigcup_{[\eta_D^{l}, \eta_D^{u}] \in g_D(x)} \max \{ \eta_D^{l}, \} \eta_D^{l} \}$. For convenience, we call the pair $d(x) = (h_D(x), g_D(x))$ an interval-valued q-rung dual hesitant fuzzy element (IVQ-RDHFE), which can be denoted by $d = (h, g)$ for simplicity. Especially, if $r_D^{l} = r_D^{u}$ and $\eta_D^{l} = \eta_D^{u}$, then $D$ reduces to the q-rung dual hesitant fuzzy set [42]; if $q = 1$, then $D$ reduces to interval-valued dual hesitant fuzzy set [56]; if $q = 2$, then $D$ reduces to hesitant interval-valued Pythagorean fuzzy set [57].

The operational laws of IVq-RDHLVs are proposed as follows.

**Definition 4** Let $d = \langle s_0, (h, g) \rangle$, $d_1 = \langle s_0, (h_1, g_1) \rangle$, and $d_2 = \langle s_0, (h_2, g_2) \rangle$ be any three IVq-RDHVLs, and then:
1. \( d_1 \oplus d_2 = \{ s_{h_1+g_2}, \cup_{r_1 \in h_1, r_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left[ \left( (r_1^{1/2})^q + (r_2^{1/2})^q \right) / q \right], \left( (r_1^q + r_2^q) / q \right)^{1/q}, \left( (r_1^{1/2} r_2^{1/2}) / q \right), \left( (r_1^q r_2^q) / q \right)^{1/q} \right\} \}; \)

2. \( d_1 \otimes d_2 = \{ s_{h_1 \times h_2}, \cup_{r_1 \in h_1, r_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left[ \left( (r_1^{1/2})^q + (r_2^{1/2})^q \right) / q \right], \left( (r_1^q + r_2^q) / q \right)^{1/q}, \left( (r_1^{1/2} r_2^{1/2}) / q \right), \left( (r_1^q r_2^q) / q \right)^{1/q} \right\} \}; \)

3. \( \lambda d = \{ s_{\lambda h}, \cup_{r \in h, \eta \in g} \left\{ \left[ \left( (1 - (1 - (r^{1/2})^q) / q \right) \right] \right\} \left( (1 - (1 - (r^q) / q) / q \right)^{1/q}, \left( (1 - (1 - (r^{1/2})^q) / q \right), \left( (1 - (1 - (r^q) / q) / q \right)^{1/q} \right\} \}; \)

4. \( d^\lambda = \{ s_{\lambda h}, \cup_{r \in h, \eta \in g} \left\{ \left[ \left( (1 - (1 - (r^{1/2})^q) / q \right) \right] \right\} \left( (1 - (1 - (r^q) / q) / q \right)^{1/q}, \left( (1 - (1 - (r^{1/2})^q) / q \right), \left( (1 - (1 - (r^q) / q) / q \right)^{1/q} \right\} \}; \)

According to Definition 4, the following theorem can be easily obtained.

**Theorem 1** Let \( d = (s_{\theta}, (h, g)) \), \( d_1 = (s_{\theta_1}, (h_1, g_1)) \) and \( d_2 = (s_{\theta_2}, (h_2, g_2)) \) be any three IVq-RDHLVs, and then:

1. \( d_1 \oplus d_2 = d_2 \oplus d_1; \)
2. \( d_1 \otimes d_2 = d_2 \otimes d_1; \)
3. \( \lambda (d_1 \oplus d_2) = \lambda d_1 \oplus \lambda d_2 (\lambda_1, \lambda_2) \geq 0; \)
4. \( \lambda (d_1 \otimes d_2) = \lambda d_1 \otimes \lambda d_2 (\lambda_1, \lambda_2) \geq 0; \)
5. \( d_1^\lambda \otimes d_2^\lambda = (d_1 \otimes d_2)^\lambda (\lambda \geq 0); \)
6. \( d_1^\lambda \otimes d_2^\lambda = (d_1 \otimes d_2)^\lambda (\lambda_1, \lambda_2 \geq 0). \)

We present the following comparison method to rank any two IVq-RDHLVs.

**Definition 5** Let \( d = (s_{\theta}, (h, g)) \) be an IVq-RDHLV, and then, the score function of \( d \) is defined as:

\[
S(d) = \left( \frac{1 + \left( \frac{1}{\#h} \sum_{r^1 \in h} r^1 \right)^q + \left( \frac{1}{\#h} \sum_{r^u \in h} r^u \right)^q}{2} - \left( \frac{1}{\#g} \sum_{\eta^1 \in g} \eta^1 \right)^q - \left( \frac{1}{\#g} \sum_{\eta^u \in g} \eta^u \right)^q \right) \times s_{\theta}.
\]

(3)

and the accuracy function of \( d \) is defined as:

\[
p(d) = \left( \left( \frac{1}{\#h} \sum_{r^1 \in h} r^1 \right)^q + \left( \frac{1}{\#h} \sum_{r^u \in h} r^u \right)^q \right)
- \left( \frac{1}{\#g} \sum_{\eta^1 \in g} \eta^1 \right)^q - \left( \frac{1}{\#g} \sum_{\eta^u \in g} \eta^u \right)^q \times s_{\theta},
\]

(4)

where \( \#h \) and \( \#g \) represent the numbers of interval values in \( h \) and \( g \), respectively. Let \( d_1 = (s_{\theta_1}, (h_1, g_1)) \) and \( d_2 = (s_{\theta_2}, (h_2, g_2)) \) be any two IVq-RDHLVs, and then:

1. If \( S(d_1) > S(d_2) \), then \( d_1 \) is superior to \( d_2 \), denoted by \( d_1 > d_2; \)
2. If \( S(d_1) = S(d_2) \), then \( p(d_1) = p(d_2) \), then \( d_1 \) is equivalent to \( d_2 \), denoted by \( d_1 = d_2; \)
3. If \( p(d_1) > p(d_2) \), then \( d_1 \) is superior to \( d_2 \), denoted by \( d_1 > d_2. \)

**Maclaurin symmetric mean**

**Definition 6** [63] Let \( \alpha_i (i = 1, 2, \ldots, n) \) be a collection of crisp numbers, and \( k = 1, 2, \ldots, n \). If

\[
\text{MSM}^{(k)}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left( \frac{\sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^{k} \alpha_{i_j}}{C_n^k} \right)^{\frac{1}{k}}.
\]

(5)

and then, \( \text{MSM}^{(k)} \) is called the MSM, where \( (i_1, i_2, \ldots, i_k) \) traversals all the \( k \)-tuple combination of \( (1, 2, \ldots, n) \) and \( C_n^k \) is the binominal coefficient.

In addition, Qin and Liu [51] proposed the dual Maclaurin symmetric mean.

**Definition 7** [51]. Let \( \alpha_i (i = 1, 2, \ldots, n) \) be a collection of crisp numbers, and \( k = 1, 2, \ldots, n \). If

\[
\text{DMSM}^{(k)}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{k} \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( \sum_{j=1}^{k} \alpha_{i_j} \right) C_n^k \right)^{\frac{1}{k}}.
\]

(6)
then DMSM\(^{(k)}\) is called dual Maclaurin symmetric mean
(DMSM), where \((i_1, i_2, \ldots, i_k)\) traversals all the \(k\)-tuple
combination of \((1, 2, \ldots, n)\) and \(C^k_n\) is the binomial coef-
cient.

**Aggregation operators for IVq-RDHL information**

In this section, we extend MSM and DMSM to IVq-RDHL
environment and propose a family of interval-valued q-rung
dual hesitant linguistic Maclaurin symmetric mean operators.
Moreover, some desirable properties of the proposed AOs are
presented and discussed.

\[
IVq - RDHLMSM^{(k)}(d_1, d_2, \ldots, d_n) = \left\langle \frac{1}{\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} r_{i_j}^q} \cdot \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} r_{i_j}^q, \prod_{j=1}^{k} r_{i_j}^q \right\rangle
\]

Based on the operations for IVq-RDHLVs, the following
theorem can be obtained.

**Theorem 2** Let \(d_i = (s_{i0}, (h_i, g_i)) (i = 1, 2, \ldots, n)\) be a
collection of IVq-RDHLVs, and then, the aggregated value by
the IVq-RDHLMSM operator is also an IVq-RDHLV and:

\[
IVq - RDHLMSM^{(k)}(d_1, d_2, \ldots, d_n) = \left\langle \frac{1}{\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} r_{i_j}^q} \cdot \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} r_{i_j}^q, \prod_{j=1}^{k} r_{i_j}^q \right\rangle
\]

**Interval-valued q-rung dual hesitant linguistic
Maclaurin symmetric mean operator**

**Definition 8** Let \(d_i = (s_{i0}, (h_i, g_i)) (i = 1, 2, \ldots, n)\) be
a collection of IVq-RDHLVs and \(k = 1, 2, \ldots, n\), then the
interval-valued q-rung dual hesitant linguistic Maclaurin
symmetric mean (IVq-RDHLMSM) operator is defined as:

\[
\oplus_{1 \leq i_1 < \cdots < i_k \leq n} d_{i_j} = \left\langle \frac{1}{\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} r_{i_j}^q} \cdot \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} r_{i_j}^q, \prod_{j=1}^{k} r_{i_j}^q \right\rangle
\]

**Proof** Based on the operations of IVq-RDHLVs introduced
in Definition 4, we have:

\[
\oplus_{1 \leq i_1 < \cdots < i_k \leq n} d_{i_j} = \left\langle \frac{1}{\sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} r_{i_j}^q} \cdot \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} r_{i_j}^q, \prod_{j=1}^{k} r_{i_j}^q \right\rangle
\]
Furthermore:

\[ \frac{1}{C_n^k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} d_{i_1} \cdots d_{i_k} \leq \left( \frac{1}{C_n^k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} r_{i_j}^q \right)^{\frac{1}{q}} \left( 1 - \prod_{i=1}^{n} \frac{1}{\eta_{i_j}^q} \right)^{\frac{1}{q}} \left( 1 - \prod_{j=1}^{k} \left( 1 - \frac{1}{\eta_{i_j}^q} \right) \right)^{\frac{1}{q}}. \]

Hence:

\[ \left( \frac{1}{C_n^k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} d_{i_1} \cdots d_{i_k} \right)^{\frac{1}{q}} \leq \left( \frac{1}{C_n^k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} r_{i_j}^q \right)^{\frac{1}{q}} \left( 1 - \prod_{i=1}^{n} \frac{1}{\eta_{i_j}^q} \right)^{\frac{1}{q}} \left( 1 - \prod_{j=1}^{k} \left( 1 - \frac{1}{\eta_{i_j}^q} \right) \right)^{\frac{1}{q}}. \]

In the following, we discuss some desirable properties of the IVq-RDHLMSM operator.

**Theorem 3 (Idempotency)** Let \( d_i = \{s_{i_1}, (h_i, r_i)\} \) (\( i = 1, 2, \ldots, n \)) be a collection of IVq-RDHLVs, if all the IVq-RDHLVs are equal, i.e., \( d_i = d \) for all \( i \), and \( d \) only has one MG and one NMG, then:

\[ \text{IVq - RDHLMSM}^{(k)}(d_1, d_2, \ldots, d_n) = d. \]  

**Proof** Since \( d_i = d \) for \( i = 1, 2, \ldots, n \), we can get:
IVq – RDHLMSM\(^{(k)}\)(\(d_1, d_2, \ldots, d_n\)) = \left\{ s \middle| \sum_{1 \leq i_1 \cdots i_k \leq n} \frac{C_{k,n}^{i_1 \cdots i_k} }{c_{i_1}^{f_1} \cdots c_{i_k}^{f_k}} \right\} ^{1/2} \cup \{ h_i, g_i \} \in G_j \right\}

\begin{align*}
\left\{ \left[ \begin{array}{c}
1 - \prod_{1 \leq i_1 \cdots i_k \leq n} \left( 1 - \left( \prod_{j=1}^{\eta_i} r_{i,j}^q \right) \right) \\
1 - \prod_{1 \leq i_1 \cdots i_k \leq n} \left( 1 - \left( \prod_{j=1}^{\eta_i} \left( \prod_{j=1}^{\eta_i} r_{i,j}^q \right) \right) \right)
\end{array} \right] ^{1/2} \right\},
\left\{ \left[ \begin{array}{c}
1 - \prod_{1 \leq i_1 \cdots i_k \leq n} \left( 1 - \left( \prod_{j=1}^{\eta_i} r_{i,j}^q \right) \right) \\
1 - \prod_{1 \leq i_1 \cdots i_k \leq n} \left( 1 - \left( \prod_{j=1}^{\eta_i} \left( \prod_{j=1}^{\eta_i} r_{i,j}^q \right) \right) \right)
\end{array} \right] ^{1/2} \right\}
\end{align*}

\begin{align*}
= \left\{ s \middle| \sum_{1 \leq i_1 \cdots i_k \leq n} \frac{C_{k,n}^{i_1 \cdots i_k} }{c_{i_1}^{f_1} \cdots c_{i_k}^{f_k}} \right\} ^{1/2} \cup \{ h_i, g_i \} 
\end{align*}

Theorem 4 (Monotonicity) Let \( d_i = \{ s_{h_i}, (h_i, g_i) \} \) (\( i = 1, 2, \ldots, n \)) and \( d'_i = \{ s_{h'_i}, (h'_i, g'_i) \} \) (\( i = 1, 2, \ldots, n \)) be two collections of IVq-RDHLVs, if \( s_{h_i} \geq s_{h'_i}, r_i \geq r'_i \) and \( \eta_i \leq \eta'_i \) holds for all \( i = 1, 2, \ldots, n \), where \( r_i \in h_i, r'_i \in h'_i, \eta_i \in g_i \) and \( \eta'_i \in g'_i \), then:

\[
IVq – RDHLMSM^{(k)}(d_1, d_2, \ldots, d_n) \geq IVq – RDHLMSM^{(k)}(d'_1, d'_2, \ldots, d'_n).
\] (10)

Proof For easy description, we assume that:

IVq – RDHLMSM^{(k)}(d_1, d_2, \ldots, d_n) \geq IVq – RDHLMSM^{(k)}(d'_1, d'_2, \ldots, d'_n).
\[ \text{IVq} - \text{RDHLMSM}^{(k)}(d_1, d_2, \ldots, d_n) = \left\{ \mathbf{s} \left\lfloor \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\prod_{j=1}^{k} \eta_{i_j}}{c^2} \right\rfloor \right. \cdot \mathcal{U}_{i_1} \mathcal{P}_{i_2} \ldots \mathcal{P}_{i_k} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \ldots \mathcal{E}_{i_k} \right. \\
\left. \left\{ \left( 1 - \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} (\eta'_i)^{q_i \frac{1}{c^2}} \right)^{\frac{1}{c^2}} \right) \right)^{\frac{1}{c^2}} \cdot \left( 1 - \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} (\eta'^{q_i}_{i_j})^{q_i \frac{1}{c^2}} \right)^{\frac{1}{c^2}} \right) \right)^{\frac{1}{c^2}} \right\} \right\} \\
= \{s, \{[g, l], [p, q]\} \}.
\]

and

\[ \text{IVq} - \text{RDHLMSM}^{(k)}(d'_1, d'_2, \ldots, d'_n) = \left\{ \mathbf{s}' \left\lfloor \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\prod_{j=1}^{k} \eta'_{i_j}}{c^2} \right\rfloor \right. \cdot \mathcal{U}_{i_1} \mathcal{P}'_{i_2} \ldots \mathcal{P}'_{i_k} \mathcal{E}'_{i_1} \mathcal{E}'_{i_2} \ldots \mathcal{E}'_{i_k} \right. \\
\left. \left\{ \left( 1 - \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} (\eta'_{i_j})^{q_i \frac{1}{c^2}} \right)^{\frac{1}{c^2}} \right) \right)^{\frac{1}{c^2}} \cdot \left( 1 - \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} (\eta'^{q_i}_{i_j})^{q_i \frac{1}{c^2}} \right)^{\frac{1}{c^2}} \right) \right)^{\frac{1}{c^2}} \right\} \right\} \\
= \{s', \{[k, l], [m, n]\} \}.
\]
Since \( s_{hi} \geq s'_{hi} \) hold for all \( i = 1, 2, \ldots, n \), it is easy to prove that:

\[
\left( \frac{\Sigma_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \eta_{i_j}}{c_k^q} \right)^{\frac{1}{k}} \geq s' \left( \frac{\Sigma_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \eta_{i_j}}{c_k^q} \right)^{\frac{1}{k}},
\]

i.e., \( s_i \geq s'_{hi} \).

Since \( d_i \geq d'_{i} \), we can get \( r_i \geq (r_i)' \) and \( \left( \prod_{j=1}^{k} r_{i_j}^{j} \right)^{q} \geq \left( \prod_{j=1}^{k} \left( r_{i_j}^{j} \right)^{q} \right) \). Then:

\[
1 - \left( \prod_{j=1}^{k} r_{i_j}^{j} \right)^{q} \leq 1 - \left( \prod_{j=1}^{k} \left( r_{i_j}^{j} \right)^{q} \right),
\]

\[
\prod_{1 \leq i_1 < \cdots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( r_{i_j}^{j} \right)^{q} \right) \right)^{\frac{1}{c_k^q}} \leq \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( r_{i_j}^{j} \right)^{q} \right) \right)^{\frac{1}{c_k^q}}.
\]

Therefore, Theorem 4 is kept.

**Theorem 5** (Boundedness) Let \( d_i = \{s_{hi}, (h_i, g_i)\} \) \((i = 1, 2, \ldots, n)\) be a collection of IVq-RDHLVs, if

\[
d^+ = \left\{ \max_{i=1}^{n} s_{hi}, \max_{i=1}^{n} (h_i, g_i) \right\},
\]

and

\[
d^- = \left\{ \min_{i=1}^{n} s_{hi}, \min_{i=1}^{n} (h_i, g_i) \right\},
\]

then

\[
d^- \leq \text{IVq} - \text{RDHLMSM}^{(k)}(d_1, d_2, \ldots, d_n) \leq d^+.
\]  (11)

**Proof** According to Theorem 3 and Theorem 4, we have:

\[
\text{IVq} - \text{RDHLMSM}^{(k)}(d_1, d_2, \ldots, d_n) \geq \text{IVq} - \text{RDHLMSM}^{(k)}(d^-, d^-, \ldots, d^-)
\]

and

\[
\text{IVq} - \text{RDHLMSM}^{(k)}(d^+, d^+, \ldots, d^+)
\]

\[
\geq \text{IVq} - \text{RDHLMSM}^{(k)}(d_1, d_2, \ldots, d_n).
\]

In addition, both \( d^- \) and \( d^+ \) only have one MG and NMG. Therefore:

\[
\text{IVq} - \text{RDHLMSM}^{(k)}(d^-, d^-, \ldots, d^-) = d^- 
\]

\[
\text{IVq} - \text{RDHLMSM}^{(k)}(d^+, d^+, \ldots, d^+) = d^+.
\]

Therefore, we have \( d^- \leq \text{IVq} - \text{RDHLMSM}^{(k)}(d_1, d_2, \ldots, d_n) \leq d^+ \).

In the following, we discuss some special cases of the proposed IVq-RDHLMSM operator with respect to parameters \( k \) and \( q \).
Case 1 If $k = 1$, then the IVq-RDHLMSM operator reduces to the interval-valued q-rung dual hesitant linguistic average (IVq-RDHLA) operator, that is:

$$\text{IVq - RDHLMSM}^{(1)}(d_1, d_2, \ldots, d_n) = \left\langle \frac{1}{n} \sum_{i,j=1, i \neq j} \theta_{ij} \right\rangle^{1/2}, U_{r_i \in h_i, r_j \in h_j, \eta_i \in g_i, \eta_j \in g_j}$$

$$\left\{ \left\{ \left[ 1 - \prod_{i,j=1, i \neq j} \left( 1 - \left( r^l_{ij} \right)^q \right) \right]^{1/2}, \left[ 1 - \prod_{i,j=1, i \neq j} \left( 1 - \left( \eta^u_{ij} \right)^q \right) \right]^{1/2} \right\} \right\}$$

$$= \frac{1}{n} \theta_{ij}^n d_i = \text{IVq - RDHLA}(d_1, d_2, \ldots, d_n)$$

Case 2 If $k = 2$, then the IVq-RDHLMSM operator reduces to the interval-valued q-rung dual hesitant linguistic Bonferroni mean (IVq-RDHLBM) operator; that is:

$$\text{IVq - RDHLMSM}^{(2)}(d_1, d_2, \ldots, d_n) = \left\langle \frac{1}{n} \sum_{i,j=1, i \neq j} \theta_{ij} \right\rangle^{1/2}, U_{r_i \in h_i, r_j \in h_j, \eta_i \in g_i, \eta_j \in g_j}$$

$$\left\{ \left\{ \left[ 1 - \left( 1 - \prod_{i,j=1, i \neq j} \left( 1 - \left( r^l_{ij} \right)^q \right) \right) \right]^{1/2}, \left[ 1 - \left( 1 - \prod_{i,j=1, i \neq j} \left( 1 - \left( \eta^u_{ij} \right)^q \right) \right) \right]^{1/2} \right\} \right\}$$

$$= \left( \frac{1}{n(n-1)} \theta_{ij}^n, j=1, i \neq j \right) \left( d_i \otimes d_j \right) \right\rangle^{1/2} = \text{IVq - RDHLBM}^{(1,1)}(d_1, d_2, \ldots, d_n).$$

Case 3 If $k = n$, then the IVq-RDHLMSM operator reduces to the interval-valued q-rung dual hesitant linguistic geometric (IVq-DHLG) operator:

$$\text{IVq - RDHLMSM}^{(n)}(d_1, d_2, \ldots, d_n) = \left\langle s_{n, \prod_{i=1}^n \theta_i} \right\rangle^{1/n}, U_{r_i \in h_i, \eta_i \in g_i}$$

$$\left\{ \left\{ \left[ \prod_{i=1}^n r^l_i \right]^{1/n}, \left[ \prod_{i=1}^n \eta^u_i \right]^{1/n} \right\} \right\}$$

$$= \bigotimes_{i=1}^n d_i^{1/n} = \text{IVq - RDHLG}(d_1, d_2, \ldots, d_n)$$
Case 4 If \( q = 1 \), then the \( IVq-RDHLMSM \) operator reduces to interval-valued dual hesitant linguistic Maclaurin symmetric mean (IVDHLMSM) operator; that is:

\[
IVq - RDHLMSM^{(k)}(d_1, d_2, \ldots, d_n) = \left( \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{c_n^k} \right)^{1/k} \bigcup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}}
\]

\[
\left\{ \left[ \left( 1 - \prod_{j=1}^{k} \left( 1 - \frac{1}{r_{i_j}} \right)^2 \right)^{1/C_n^k} \right]^{1/2k}, \left[ \left( 1 - \prod_{j=1}^{k} \left( 1 - \frac{1}{r_{i_j}} \right)^2 \right)^{1/C_n^k} \right]^{1/2k} \right\}
\]

\[
= IVDHLMSM^{(k)}(d_1, d_2, \ldots, d_n)
\]

(15)

Case 5 If \( q = 2 \), then the \( IVq-RDHLMSM \) operator reduces to the hesitant interval-valued Pythagorean linguistic Maclaurin symmetric mean (HIVPLMSM) operator; that is:

\[
IVq - RDHLMSM^{(k)}(d_1, d_2, \ldots, d_n) = \left( \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{c_n^k} \right)^{1/k} \bigcup_{r_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}}
\]

\[
\left\{ \left[ \left( 1 - \prod_{j=1}^{k} \left( 1 - \frac{1}{r_{i_j}} \right)^{1/2} \right)^{1/C_n^k} \right]^{1/2k}, \left[ \left( 1 - \prod_{j=1}^{k} \left( 1 - \frac{1}{r_{i_j}} \right)^{1/2} \right)^{1/C_n^k} \right]^{1/2k} \right\}
\]

\[
= HIVPLMSM^{(k)}(d_1, d_2, \ldots, d_n).
\]

(16)

Interval-valued q-rung dual hesitant linguistic weighted Maclaurin symmetric mean operator

Definition 9 Let \( d_i = (s_{i_1}, (h_{i_1}, g_{i_1})) (i = 1, 2, \ldots, n) \) be a collection of \( IVq-RDHFLVs \) and \( k = 1, 2, \ldots, n \). Let \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector, such that \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \). The interval-valued q-rung dual hesitant linguistic weighted Maclaurin symmetric mean (IVq-RDHLWMSM) operator is defined as:

\[
IVq - RDHLWMSM^{(k)}(d_1, d_2, \ldots, d_n) = \left( \bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{C_n^k} \right)^{1/k} \bigotimes_{j=1}^{k} \left( w_{i_j} \right)
\]

where \( (i_1, i_2, \ldots, i_k) \) traversals all the \( k \)-tuple combination of \( (1, 2, \ldots, n) \), and \( C_n^k \) is the binomial coefficient.

(17)
Theorem 6 Let $d_i = (s_{i_1}, (h_{i_1}, g_{i_1}))(i = 1, 2, \ldots, n)$ be a collection of IVq-RDHLVs, and then, the aggregated value by the IVq-RDHLWMSM operator is still an IVq-RDHLV and:

$$
\text{IVq-RDHLWMSM}^{(k)}(d_1, d_2, \ldots, d_n) = \left\{ \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{d_i} \pi_{i} (s_{i_1}, \pi_{i_1} (h_{i_1}, g_{i_1})) \right)^{1/c_k}, \cup_{i,j} \epsilon h_{i_1}, g_{i_1} \right\}
$$

where $\pi_{i}$ is the $i$-th element of the $k$-tuple combination of $(1, 2, \ldots, n)$ and $c_k$ is the binomial coefficient.

The proof of Theorem 6 is similar to that of Theorem 2, which is omitted here. In addition, it is easy to prove that the IVq-RDHLWMSM operator has the properties of monotonicity and boundedness.

**Interval-valued q-rung dual hesitant linguistic dual Maclaurin symmetric mean operator**

**Definition 10** Let $d_i = (s_{i_1}, (h_{i_1}, g_{i_1}))(i = 1, 2, \ldots, n)$ be a collection of IVq-RDHLVs, and $k = 1, 2, \ldots, n$, and then, the interval-valued q-rung dual hesitant linguistic dual Maclaurin symmetric mean (IVq-RDHLMSM) operator:

$$
\text{IVq-RDHLMSM}^{(k)}(d_1, d_2, \ldots, d_n)
= \frac{1}{k} \left( \bigotimes_{i=1}^{k} \left( d_i \right)^{1/c_k} \right).
$$

The proof of Theorem 7 is similar to that of Theorem 2, which is omitted here. Similar to the IVq-RDHLMSM operator, the IVq-RDHLMSM operator also has the properties of idempotency, monotonicity, and boundedness.

In the followings, we are to investigate the special cases of the IVq-RDHLMSM operator with respect to the parameters $k$ and $q$. 
Case 1 If \( k = 1 \), the IVq-RDHLDMSM operator reduces to the IVq-DHLG operator, which is shown as Eq. (14).

Case 2 If \( k = 2 \), the IVq-RDHLDMSM operator reduces to the interval-valued q-rung dual hesitant linguistic geometric Bonferroni mean (IVq-RDHLGBM) operator; that is:

\[
\text{IVq} - \text{RDHLDSM}^{(2)}(d_1, d_2, \ldots, d_n) = \left\{ \frac{1}{2} \left( \sum_{i,j=1, \neq j}^{n} \left( d_i + d_j \right) \prod_{r=1}^{n} \left( r_i^q \right) \right) \right\}, \quad \bigcup_{r_i \in \mathbb{H}, n_j \in \mathbb{H}, r_j \in \mathbb{H}, n_j \in \mathbb{H}}
\]

Case 3 If \( k = n \), the IVq-RDHLDMSM operator reduces to the IVq-RDHLA operator, which is shown as Eq. (12).

Case 4 If \( q = 1 \), the IVq-RDHLDMSM operator reduces to the interval-valued dual hesitant linguistic dual Maclaurin symmetric mean (IVDHLDMSM) operator; that is:

\[
\text{IVq} - \text{RDHLDSM}^{(k)}(d_1, d_2, \ldots, d_n) = \left\{ \frac{1}{k} \left( \prod_{i=1}^{k} \left( \sum_{j=1}^{k} d_j \right) \right) \right\}, \quad \bigcup_{r_i \in \mathbb{H}, n_j \in \mathbb{H}}
\]

Case 5 If \( q = 2 \), the IVq-RDHLDMSM operator reduces to the hesitant interval-valued Pythagorean linguistic dual Maclaurin symmetric mean (HIVPLDMSM) operator; that is:

\[
\text{IVq} - \text{RDHLDSM}^{(2)}(d_1, d_2, \ldots, d_n) = \left\{ \frac{1}{2} \left( \sum_{i,j=1, \neq j}^{n} \left( d_i + d_j \right) \prod_{r=1}^{n} \left( r_i^q \right) \right) \right\}, \quad \bigcup_{r_i \in \mathbb{H}, n_j \in \mathbb{H}, r_j \in \mathbb{H}, n_j \in \mathbb{H}}
\]
Let $\text{Definition 11}$ weighted dual Maclaurin symmetric mean operator

Theorem 8 $\text{IVq-RDHLWDMSM}$ operator is defined as:

$$\text{IVq-RDHLWDMSM}^{(k)}(d_1, d_2, \ldots, d_n) = \left\{ \frac{1}{k} \left( \prod_{i=1}^{k} \left( \sum_{j=1}^{n} \theta_{ij} \right)^{1/c^k_{ij}} \right) \right\}_{\cup \mathcal{R}_{ij} \in h_{ij}, \theta_{ij} \in \mathcal{P}_{ij}}$$

$$\left\{ \left( \begin{array}{l} \left( \begin{array}{c} 1 - \prod_{i=1}^{k} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( r_{ij}^{k} \right)^{w_{ij}} \right) \right) \right) \right) \end{array} \right) \right\}_{\cup \mathcal{R}_{ij} \in h_{ij}, \theta_{ij} \in \mathcal{P}_{ij}}$$

$$\left\{ \left( \begin{array}{l} \left( \begin{array}{c} 1 - \prod_{i=1}^{k} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( r_{ij}^{k} \right)^{w_{ij}} \right) \right) \right) \end{array} \right) \right\}_{\cup \mathcal{R}_{ij} \in h_{ij}, \theta_{ij} \in \mathcal{P}_{ij}}$$

$$= \text{HIVPLDMSM}^{(k)}(d_1, d_2, \ldots, d_n).$$

The proof of Theorem 8 is similar to that of Theorem 2, which is omitted. In addition, it is easy to prove that the IVq-RDHLWDMSM operator has the properties of monotonicity and boundedness.

**Interval-valued q-rung dual hesitant linguistic weighted dual Maclaurin symmetric mean operator**

**Definition 11** Let $d_i = (s_{i0}, (h_i, g_i))(i = 1, 2, \ldots, n)$ be a collection of IVq-RDHLVs and $k = 1, 2, \ldots, n$. Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. The interval-valued q-rung dual hesitant linguistic weighted dual Maclaurin symmetric mean (IVq-RDHLWDMSM) operator is defined as:

$$\text{IVq-RDHLWDMSM}^{(k)}(d_1, d_2, \ldots, d_n) = \frac{1}{k} \left( \prod_{i=1}^{k} \left( \sum_{j=1}^{n} w_{ij} d_{ij} \right)^{1/c^k_{ij}} \right).$$ (24)

where $(i_1, i_2, \ldots, i_k)$ traversals all the $k$-tuple combination of $(1, 2, \ldots, n)$ and $C^k_n$ is the binomial coefficient.

**Theorem 8** Let $d_i = (s_{i0}, (h_i, g_i))(i = 1, 2, \ldots, n)$ be a collection of IVq-RDHLVs, and then the aggregated value by the IVq-RDHLWDMSM operator is still an IVq-RDHLV and:

$$\text{IVq-RDHLWDMSM}^{(k)}(d_1, d_2, \ldots, d_n) = \left\{ \frac{1}{k} \left( \prod_{i=1}^{k} \left( \sum_{j=1}^{n} w_{ij} d_{ij} \right)^{1/c^k_{ij}} \right) \right\}_{\cup \mathcal{R}_{ij} \in h_{ij}, \theta_{ij} \in \mathcal{P}_{ij}}$$

$$\left\{ \left( \begin{array}{l} \left( \begin{array}{c} 1 - \prod_{i=1}^{k} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( r_{ij}^{k} \right)^{w_{ij}} \right) \right) \right) \end{array} \right) \right\}_{\cup \mathcal{R}_{ij} \in h_{ij}, \theta_{ij} \in \mathcal{P}_{ij}}$$

$$\left\{ \left( \begin{array}{l} \left( \begin{array}{c} 1 - \prod_{i=1}^{k} \left( 1 - \prod_{j=1}^{n} \left( 1 - \left( r_{ij}^{k} \right)^{w_{ij}} \right) \right) \right) \end{array} \right) \right\}_{\cup \mathcal{R}_{ij} \in h_{ij}, \theta_{ij} \in \mathcal{P}_{ij}}$$

$$= \text{HIVPLDMSM}^{(k)}(d_1, d_2, \ldots, d_n).$$ (25)

The A novel method to MADM with IVq-RDHL information

In the above sections, we have demonstrated the powerfulness and robustness of the IVq-RDHLVs and their AOs. In this section, we investigate the applications of IVq-RDHLVs and their AOs in MADM. Let us consider a MADM problem in which DMs express their evaluations in the form of IVq-RDHL information. Suppose that there are $m$ alternatives that to be evaluated, which can be denoted as $\{A_1, A_2, \ldots, A_m\}$. Let $\{C_1, C_2, \ldots, C_n\}$ be a set of attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_n)^T$, such that $\sum_{i=1}^{n} w_i = 1$ and $0 \leq w_i \leq 1$. For the attribute $C_j (j = 1, 2, \ldots, n)$
of $A_i (i = 1, 2, \ldots, m)$, DMs use an IVq-RDHLV $d_{ij} = (r_{ij}^l, \{r_{ij}^u, n_{ij}^u\}, \{n_{ij}^l, r_{ij}^l\})$ to denote their evaluation value. Hence, finally, an IVq-RDHL decision matrix can be obtained, which can be denoted as $R = (d_{ij})_{m \times n}$. If we utilize the proposed IVq-RDHL aggregation operators to solve this MADM problem, the main steps are shown as follows.

**Step 1.** Standardize the original decision matrix according to the following formula:

$$d_{ij} = \begin{cases} (s_{ij}, \{[r_{ij}^l, r_{ij}^l], [n_{ij}^l, n_{ij}^l]\}) & C_i \text{ is benefit type} \\ (s_{ij}, \{[n_{ij}^u, n_{ij}^u], [r_{ij}^l, r_{ij}^l]\}) & C_i \text{ is cost type} \end{cases}$$

(26)

**Step 2.** Calculate the overall evaluation value of each alternative by the IVq-RDHLWMSM operator:

$$d_i = \text{IVq - RDHLWMSM}^{(k)}(d_{i1}, d_{i2}, \ldots, d_{in}).$$

(27)

or the IVq-RDHLWDMSM operator:

$$d_i = \text{IVq - RDHLWDMSM}^{(k)}(d_{i1}, d_{i2}, \ldots, d_{in}).$$

(28)

Hence, a set of overall evaluation values are derived.

**Step 3.** Compute scores of the overall evaluations.

**Step 4.** Rank the alternatives according to their corresponding scores.

### Application of the proposed method in medical equipment selection

The former section introduces a novel MADM method. To better illustrate the procedure of the propose method, we apply it in a real decision-making problem. Furthermore, we also conduct comparative analysis to show the advantages and superiorities of the proposed method.

**Example 1** Medical equipment is an important infrastructure for a hospital, which is directly related to the medical service content and service capacity. The choice of medical equipment is influenced by many factors, including the hospitals’ budget, whether the fund is sufficient, whether the hospital has the conditions to utilize the equipment, the degree of equipment demand, the technical evaluation of the equipment, the selection of equipment, the maintainability, etc. In this context, medical equipment selection is a problem with high dimension and high complexity. Thus, fuzzy theory is very suitable for solving the medical equipment purchasing problem. In this paper, we assume that there are four medical equipment with different brands. Let $A_i (i = 1, 2, 3, 4)$ denote the brand of equipment $i$. DMs assess each of them under attribute $C_j (j = 1, 2, 3, 4)$, where $C_1$ represents equipment quality; $C_2$ represents equipment price; $C_3$ represents after-sale service; and $C_4$ represents user evaluation, whose weight vector is $w = (0.25, 0.35, 0.20, 0.20)^T$. Based on the procurement principle, DMs usually express their evaluation information on the medical equipment and mainly focus on 5 levels {unqualified, qualified, medium, good, excellent}. To comprehensively express the evaluation information over the performance of the feasible alternatives under attributes, DMs are requested to evaluate the four alternatives $A_i (i = 1, 2, 3, 4)$ with respect to the four attributes $C_j (j = 1, 2, 3, 4)$ by IVq-RDHLVs and an IVq-RDHL decision matrix $d_{ij}$ is obtained, which is shown in Table 1.

### The decision-making procedure

**Step 1.** As all attributes are benefit type, the original decision matrix does not need to be standardized.

**Step 2.** Utilize the IVq-RDHFLWMSM operator to aggregate DMs’ assessments, so that the overall assessments of alternatives can be derived (suppose that $k = 3, q = 3$). As the aggregation results are so complicated, we omit them here.

**Step 3.** Compute the scores of the overall assessments and we can obtain:

$$S(d_1) = 0.2621, \ S(d_2) = 0.2407, \ S(d_3) = 0.3430, \ S(d_4) = 0.3662.$$ 

**Step 4.** Based on the score values, the ranking order of alternatives is derived, i.e., $A_4 \succ A_3 \succ A_1 \succ A_2$. Hence, the optimal choice is equipment is $A_4$.

In step 3, if we utilize the IVq-RDHFLWDMSM operator to aggregate attribute values (suppose that $k = 3, q = 3$), then the scores of alternatives are:

$$S(d_1) = 0.2786, \ S(d_2) = 0.2592, \ S(d_3) = 0.3695, \ S(d_4) = 0.3933.$$ 

Accordingly, the ranking order is $A_4 \succ A_3 \succ A_1 \succ A_2$ and $A_4$ is the optimal alternative.
In this subsection, we assign different values to \( k \) in the IVq-RDHLWMSM and IVq-RDHLWDMSM operators and present the decision results in Tables 2 and 3. As we can see from Tables 2 and 3, different score values are obtained with different parameter values of \( k \); however, the ranking orders are the same, i.e., \( A_4 \succ A_3 \succ A_1 \succ A_2 \) and \( A_4 \) is the optimal alternative. This characteristic demonstrates the robustness of our proposed method. In addition, it is noted that the score value of each alternative derived by the IVq-RDHLWMSM operator becomes smaller with the increase of parameter \( k \). However, in the IVq-RDHLWDMSM operator, the increase of parameter values \( k \) leads to the increase of the score value of each alternative. Therefore, the parameter \( k \) can be regarded as DMs’ attitude to performance of alternatives. If DMs are optimistic about the alternatives, they can select a smaller value of \( k \) in the IVq-RDHLWMSM operator, or a larger value of \( k \) in the IVq-RDHLWDMSM operator. If DMs are pessimistic about the alternatives, they can choose a larger value of \( k \) in the IVq-RDHLWMSM operator and a smaller value of \( k \) in the IVq-RDHLWDMSM. If DMs do not have special preference, they choose the value of \( k \) can be taken as \( k = \lfloor k \rfloor / 2 \), where \( \lfloor k \rfloor \) is the round function and \( n \) is the number of attributes.

### The influence of parameter \( q \) on the final results

To investigate the influence of parameter \( q \) on the final decision results, we assign different values of \( q \) in the IVq-RDHLWMSM and IVq-RDHLWDMSM operators, and score values of alternatives are shown as Figs. 1 and 2. As
Table 3: Score values and ranking results with different values of \( k \) in the IVq-RDHLWDM model \( (q = 3) \)

| \( k \) | Score values \( S(d_i) (i = 1, 2, 3, 4) \) | Ranking orders |
|-------|-----------------------------------|----------------|
| \( k = 1 \) | \( S(d_1) = 0.2513, S(d_2) = 0.2296, S(d_3) = 0.3254, S(d_4) = 0.3484 \) | \( A_4 > A_3 > A_1 > A_2 \) |
| \( k = 2 \) | \( S(d_1) = 0.2700, S(d_2) = 0.2500, S(d_3) = 0.3557, S(d_4) = 0.3803 \) | \( A_4 > A_3 > A_1 > A_2 \) |
| \( k = 3 \) | \( S(d_1) = 0.2786, S(d_2) = 0.2592, S(d_3) = 0.3695, S(d_4) = 0.3933 \) | \( A_4 > A_3 > A_1 > A_2 \) |
| \( k = 4 \) | \( S(d_1) = 0.2874, S(d_2) = 0.2692, S(d_3) = 0.3828, S(d_4) = 0.4057 \) | \( A_4 > A_3 > A_1 > A_2 \) |

seen from Fig. 1, different values of \( q \) lead to different score values of alternatives. Moreover, the score values of each alternative become greater with the increase of values of \( q \) in the IVq-RDHLWDM operator. However, the ranking order is always \( A_4 > A_3 > A_1 > A_2 \) and \( A_4 \) is the optimal alternative. Similarly, from Fig. 2, we find out that the increase of the value \( q \) in the IVq-RDHLWDM operator also results in the increase of score value of each alternative. Although the ranking orders are slightly different, the best alternative is always \( A_4 \). In our proposed method, how to select an appropriate value of \( q \) is an important issue. Liu [48] and his colleagues proposed a method to determine the value of \( q \) properly in q-rung orthopair fuzzy decision context, i.e., the value of \( q \) should be taken as the smallest integer that makes \( \mu^q + v^q \leq 1 \) where \( \mu \) and \( v \) denote the MG and NMG. For instance, a DM employs a q-ROFN \((0.75, 0.95)\) as his/her judgement over a certain alternative. As \( 0.75^5 + 0.95^5 = 1.0111 > 1 \) and \( 0.75^6 + 0.95^6 = 0.9131 < 1 \), the value of \( q \) can be taken as 6. In our proposed method, the value of \( q \) should be taken as the smallest integer that makes \( 0 \leq (r_{p_j}^{(u)})^q + (r_{p_j}^{(1)})^q \leq 1 \). For example, an IVq-RDHLV \((s_3, \{[0.1, 0.2], [0.3, 0.4], [0.5, 0.8], \{[0.2, 0.6], [0.7, 0.8] \})\) is employed to express a DM’s evaluation over an alternative. As \( 0.8^3 + 0.8^3 = 1.024 > 1 \) and \( 0.8^4 + 0.8^4 = 0.8192 < 1 \), then the value of \( q \) should be taken as 4.

Validation analysis

In this subsection, we utilize our proposed method and some existing MADM methods to solve practical examples and discuss their decision results to demonstrate the validity of our proposed method. These methods involve that presented by Wei et al. [50] based on interval-valued dual hesitant fuzzy linguistic weighted average (IVDHFLWA) operator, and that introduced by Wei [64] based on interval-valued dual hesitant fuzzy uncertain linguistic weighted average (IVDHFULWA) operator.

Example 2 Suppose that there is a panel with three possible logistics outsourcing service providers in the engineering material procurement \( A_i (i = 1, 2, 3) \) to select with respect to four attributes, i.e., \( C_1 \) is the external environment; \( C_2 \) is the enterprise management risk; \( C_3 \) the logistics business risk; and \( C_4 \) is the internal environment risk. The weight vector of the attributes is \( w = (0.20, 0.30, 0.30, 0.20)^T \). DMs employed the interval-valued dual hesitant fuzzy linguistic variables (IVDHFLVs) to express their evaluations of alternatives and a normalized decision matrix is listed in Table 4. We utilize our proposed method based on the IVq-RDHLWDM operator, and Wei et al.’s [50] and Wei’s [64] methods to solve Example 2.

It is noted that Wei’s [64] method is based on interval-valued dual hesitant fuzzy uncertain linguistic sets (IVDHFULSs). Actually, the interval-valued dual hesitant fuzzy linguistic set (IVDHFLS) is a special case of IVDH-
Table 4 Normalized interval-valued dual hesitant fuzzy linguistic decision matrix of Example 2

|   | A1                  | A2                  | A3                  |
|---|---------------------|---------------------|---------------------|
| C1 | (s₂, [[0.1, 0.2], [0.2, 0.3], [0.3, 0.4]]) | (s₁, [[0.7, 0.8], [0.1, 0.2]]) | (s₄, [[0.4, 0.5], [0.3, 0.4], [0.4, 0.5]]) |
| C2 | (s₁, [[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]]) | (s₄, [[0.2, 0.3], [0.5, 0.6]]) | (s₁, [[0.7, 0.8], [0.1, 0.2]]) |
| C3 | (s₄, [[0.3, 0.4], [0.4, 0.5]]) | (s₁, [[0.6, 0.8], [0.1, 0.2]]) | (s₃, [[0.2, 0.5], [0.3, 0.4]]) |
| C4 | (s₁, [[0.4, 0.7], [0.2, 0.3]]) | (s₄, [[0.3, 0.5], [0.3, 0.4]]) | (s₂, [[0.3, 0.4], [0.2, 0.4], [0.4, 0.5]]) |

Table 5 Normalized interval-valued dual hesitant fuzzy uncertain linguistic decision matrix of Example 2

|   | A1                  | A2                  | A3                  |
|---|---------------------|---------------------|---------------------|
| C1 | ([s₂, s₂], [[0.1, 0.2], [0.2, 0.3], [0.3, 0.4]]) | ([s₃, s₃], [[0.7, 0.8], [0.1, 0.2]]) | ([s₄, s₄], [[0.4, 0.5], [0.3, 0.4], [0.4, 0.5]]) |
| C2 | ([s₁, s₁], [[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]]) | ([s₄, s₄], [[0.2, 0.3], [0.5, 0.6]]) | ([s₁, s₁], [[0.7, 0.8], [0.1, 0.2]]) |
| C3 | ([s₄, s₄], [[0.3, 0.4], [0.4, 0.5]]) | ([s₁, s₁], [[0.6, 0.8], [0.1, 0.2]]) | ([s₃, s₃], [[0.2, 0.5], [0.3, 0.4]]) |
| C4 | ([s₁, s₁], [[0.4, 0.7], [0.2, 0.3]]) | ([s₄, s₄], [[0.3, 0.5], [0.3, 0.4]]) | ([s₂, s₂], [[0.3, 0.4], [0.2, 0.4], [0.4, 0.5]]) |

Table 6 Decision results of Example 2 by different methods

| Method                                      | Score values S(dᵢ)(i = 1, 2, 3) | Ranking order |
|---------------------------------------------|----------------------------------|---------------|
| Wei et al.’s [50] method based on IVDHFLWA operator | S(d₁) = 3.2517, S(d₂) = 4.0873, S(d₃) = 3.6395 | A₂ > A₃ > A₁ |
| Wei’s [64] method based on IVDHFULWA operator | S(d₁) = 6.5034, S(d₂) = 8.1747, S(d₃) = 7.2790 | A₂ > A₃ > A₁ |
| The proposed method based on IVq-RDHFLWMSM operator (q = 1, k = 1) | S(d₁) = -0.0512, S(d₂) = -0.0453, S(d₃) = -0.0616 | A₂ > A₁ > A₃ |

Advantages of the proposed method

In this subsection, we further demonstrate the advantages and superiorities of our proposed method through comparative analysis.

Its powerfulness of expressing DMs’ judgements comprehensively

Our proposed method employs IVq-RDHLSs to depict DMs’ evaluation values over the performance of alternatives. As we know, the IVq-RDHLs should satisfy the constraint that the sum of qth power of MG and qth power of NMG is less than or equal to one. In practical decision-making situations, DMs adjust the parameter value q to make DMs’ evaluation values to meet this constraint. Hence, our method can comprehensively describe DMs’ assessment values in complicated decision-making problems. To better explain this advantage, we provide the following example.

Example 3 In Example 2, DMs employ IVDHFLSs to express their decision preference. The constraint of IVDHFLSs is that the sum of MG and NMG should be not greater than one. Nevertheless, this constraint cannot be always satisfied. For example, we replace the evaluation value of attribute C₁ of A₁ with (s₂, [[[0.1, 0.2], [0.2, 0.8]], [[0.3, 0.9]])]. The other evaluation values keep unchanged. If we utilize the Wei et al.’s [50] and Wei’s [64] MADM method and our proposed method to solve Example 3, then the decision results are obtained (listed in Table 7).

As we can see from Table 7, the methods proposed by Wei et al. [50] and Wei [64] fail to deal with Example 3, whereas our proposed method can still handle this case and the ranking order is A₂ > A₃ > A₁. This is because in the revised example, the evaluation value of the attribute C₁ of alternative A₁ cannot be represented by IVDHFLSs and IVDHFULSs as 0.8 + 0.9 = 1.7 > 1. Our proposed method
Table 7 Decision results of Example 3 by different methods

| Method                                | Score values $S(d_i)(i = 1, 2, 3)$ | Ranking order |
|---------------------------------------|------------------------------------|---------------|
| Wei et al.’s [50] method based on IVDHFLWA operator | Cannot be calculated | –             |
| Wei’s [64] method based on IVDHFULWA operator | Cannot be calculated | –             |
| The proposed method based on IVQ-RDHFLWMSM operator ($q = 5, k = 1$) | $S(d_1) = 0.2177, S(d_2) = 0.3206, S(d_3) = 0.2992$ | $A_2 > A_3 > A_1$ |

Table 8 Decision results of Example 2 by our proposed method based on the IVQ-RDHFLWMSM operator with different values of $k (q = 3)$

| $k$ | Score values $S(d_i)(i = 1, 2, 3)$ | Ranking order |
|-----|------------------------------------|---------------|
| $k = 1$ | $S(d_1) = 0.0751, S(d_2) = 0.1479, S(d_3) = 0.1024$ | $A_2 > A_3 > A_1$ |
| $k = 2$ | $S(d_1) = 0.0539, S(d_2) = 0.1090, S(d_3) = 0.0667$ | $A_2 > A_3 > A_1$ |
| $k = 3$ | $S(d_1) = 0.0444, S(d_2) = 0.0852, S(d_3) = 0.0532$ | $A_2 > A_3 > A_1$ |
| $k = 4$ | $S(d_1) = 0.0362, S(d_2) = 0.0694, S(d_3) = 0.0441$ | $A_2 > A_3 > A_1$ |

can still deal with this case, and if we set $q = 5$, then $0.8^5 + 0.9^5 = 0.9182 < 1$ (The method to appropriately determine the value of $q$ is presented in section “The influence of parameter $q$ on the final results”). Hence, our proposed method can deal with more complicated decision information, give DMs more freedom to provide their evaluations, and comprehensively express DMs’ judgements.

Its ability of capturing the interrelationship among attributes

In practical decision-making problems, there exists weak or strong interrelationship among attributes. To make the final decision results more reliable, it is necessary to take such interrelationship into consideration when determining the optimal alternative. As we can see, our method and those proposed by Wei et al. [50] and Wei [64] can effectively solve Example 2. In addition, our proposed method is based on the IVQ-RDHFLWMSM operator, which has an important parameter $k$, making the decision-making procedure flexible. If we assign different values to $k$ when dealing with Example 2 by our proposed method, we can obtain the following results (listed in Table 8).

It is noted that the methods proposed by Wei et al. [50] and Wei [64] are based on simple weighted average operator. Hence, the two MADM methods take the evaluation values and the weight information of attributes into account when calculating the overall preference of DMs over alternatives. However, Wei et al.’s [50] and Wei’s [64] methods do not consider the interrelationship among attributes. That is to say, Wei et al.’s [50] and Wei’s [64] methods assume that the attributes are independent, which is not always consistent with the reality. Our method based on the IVQ-RDHFLWMSM operator can effectively deal with the correlation among attributes. For example, when we set $k = 2$, then the interrelationship between any attributes is reflected. If any three attributes are interacted, we can set $k = 3$. If there is no interrelationship among attributes, we can let $k = 1$. Hence, as in the most practical MADM problems, the attributes are usually dependent, our proposed method is more powerful and suitable than those proposed by Wei et al. [50] and Wei [64].

The ability of portraying DMs’ evaluation information both quantificationally and qualitatively

Our proposed method employs IVQ-RDHLs to denote DMs’ evaluation information over the performance of alternatives. Basically, the IVQ-RDHL is a hybrid tool that can describe fuzzy decision information from both quantitative and qualitative aspects. When we utilize IVQ-RDHLs to express decision-making information, we first provide DMs a predefined LTS, and DMs can choose proper linguistic terms to express their qualitative decision information. In addition, the IVQ-RDHFLs are employed for DMs to express the MG and NG of a linguistic term, so that their quantitative decision ideas are expressed. Hence, via the LTS and IVQ-RDHFLs, we can denote DMs’ evaluation information more accurately, so that the final decision results are more reliable. Compared with our proposed MADM method, the decision method proposed by Xu et al. [43] uses the IVQ-RDHFLs to denote the DMs’ fuzzy judgements, which only concerns the quantitative information and neglects DMs’ qualitative evaluation information. In other words, the method of Xu et al. [43] cannot fully express DMs’ evaluation values in complicated decision-making context, so that the decision results obtained are insufficient. Hence, compared with Xu et al.’s [43] method, our proposed method is more sufficient to deal with complex decision-making problems and the final ranking orders of alternatives are more reliable. We list the characteristics of all the above-mentioned MADM methods...
in Table 9 to better illustrate the advantages and superiorities of our proposed method.

### Conclusions

The IVq-RDHFSs are a good tool in describing fuzzy information. However, the main shortcoming of IVq-RDHFSs is that they only express DMs’ quantitative decision information. To overcome such drawback, we proposed the notion of IVq-RDHLs by combining IVq-RDHFSs with TLS. The IVq-RDHLs not only portray DMs’ quantitative and qualitative evaluation values, but also effectively deal with DMs’ hesitancy in MADM procedures. In addition, due to its laxer constraint, the IVq-RDHLs also give DMs great freedom to comprehensively express their judgements. To effectively handle MADM under IVq-RDHLs, we have proposed the IVq-RDHLMSM, IVq-RDHLWMSM, IVq-RDHLDMSM, and IVq-RDHLWDMSM operators. Besides, we have also introduced a method to determine the optimal alternative based on the numerical examples, the feasibility and effectiveness of our proposed method have been clearly illustrated. In addition, advantages and superiorities of the new MADM method have been presented through comparative analysis. For future research, we will investigate new AOs of IVq-RDHLs information and propose novel powerful MADM methods.

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