Abstract

We consider the effect that gravity has when one tries to set up a constant background form field. We find that in analogy with the Melvin solution, where magnetic field lines self-gravitate to form a flux-tube, the self-gravity of the form field creates fluxbranes. Several exact solutions are found corresponding to different transverse spaces and world-volumes, a dilaton coupling is also considered.

1 Introduction

The Melvin solution in Einstein-Maxwell theory was constructed many years ago, describing what happens to a uniform magnetic field when gravity is included. Namely, the lines of magnetic flux self gravitate and form a cylindrically symmetric configuration which is classically stable. Over the years a number of generalizations have appeared, both within the context of extending to metrics which are asymptotically Melvin and changing the field content of the model. Of particular relevance is the work of Gibbons and Maeda where the Melvin flux-tube solution was extended to arbitrary dimensions, with the inclusion of a dilaton, to give fluxbranes. The Melvin solution has also found its way into the string theory literature, with it appearing as an exact background for propagating strings.

In these days of supergravity theories we find that there is another natural generalization to be done. The Maxwell two-form field strength is just one of a more general class of n-form field strengths, so it is of interest to study the effect gravity has on a homogeneous n-form background.

The solutions we find here for an F(n) form have the character of branes with a worldvolume dimension of D − n, called fluxbranes. We find exact solutions for these fluxbranes with both curved and flat worldvolume, leaving the spacetime dimension and the rank of F(n) general. The inclusion of a dilaton has also been studied. In distinction to the Melvin solution these fluxbranes are singular at their core, asymptotically however the solutions are regular.

To get a picture of what these fluxbranes are consider the case of Melvin, where we have an infinite flux-tube. We can think of this heuristically as coming from a monopole-antimonopole pair with infinite separation, and the flux joins them in the form of a tube. In the more general case we could think of the situation where we have a brane-antibrane pair of infinite separation, where these branes are

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charged under an $n$ form field strength. For example in 11D SUGRA we have an 4-form field strength giving five-branes. Separating a 5 5 pair with infinite distance gives a $D - n - 1 = 6$ fluxbrane joining them, i.e. a fluxbrane with $D - n = 7$ world-volume dimensions.

2 The Melvin solution and its generalization

In four dimensions the Melvin solution corresponds to a cylindrically symmetric flux tube, which we take to be in the $z$-direction. The metric and field strength take the following form,

$$ds_{\text{Melvin}}^2 = (1 + \rho^2)^2 \left[-dt^2 + dz^2\right] + (1 + \rho^2)^2dr^2 + \frac{r^2}{(1 + \rho^2)^2}d\phi^2,$$

$$F = \frac{1}{(1 + \rho^2)^2}dr \wedge r d\phi. \quad (2.1)$$

The core has world volume coordinates $(t, z)$ and the form $F$ lives on the angular coordinate $\phi$ and the radial coordinate $r$ ($\rho \propto Br$), so the magnetic field (of strength $B$) points in the $z$-direction. A detailed description of the structure of this spacetime can be found in [2].

Taking the Melvin case as a template we extend this to consider an $(m + 1)$ form field strength in general spacetime dimensions. In the Melvin case the form lives on a circle and a radial direction, here we take the form to have components on a compact Euclidean manifold (with volume form $\eta(m)$) and a radial direction (with coordinate $\xi$),

$$F_{\text{mag\textsuperscript{\prime} flux}} = \alpha(\xi) d\xi \wedge \eta(m), \quad (2.3)$$

Just as in the Melvin solution the Hodge dual of the form has components only in the worldvolume coordinates. There is also a dual solution where the fluxbrane carries electric rather than magnetic flux. The duality relates a magnetic flux tube for $F_{(m+1)}$ to an electric flux tube for a $D - (m + 1)$ form field strength, $G_{(D-(m+1))}$,

$$*G_{\text{elec\textsuperscript{\prime} flux}} = \alpha(\xi) d\xi \wedge \eta(m). \quad (2.4)$$

This compares to the usual case of branes charged magnetically under an $m + 1$ form field strength, $F_{\text{mag\textsuperscript{\prime} charge}} = \beta(r)\eta(m+1)$ where the world volume has $D - (m + 2)$ dimensions, a $D - (m + 3)$ brane. The metric we take has a cylindrical symmetry,

$$ds^2 = \exp(2a(\xi))\overline{ds}^2_{(L)} + \exp(2b(\xi))d\xi^2 + \exp(2c(\xi))\overline{ds}^2_{(E)} \quad (2.5)$$

where $\overline{ds}^2_{(L)}$ is an $l + 1$ dimensional Einstein manifold of (mostly positive) Lorentzian signature, $\overline{ds}^2_{(E)}$ is an $m$ dimensional Einstein manifold of Euclidean signature. There is still a coordinate freedom in the choice of $\xi$, this shall be removed later when a suitable gauge choice will be made to simplify our resulting equations.

3 Equations of motion

3.1 Curvature equations

In order to be self contained we include here the calculation of the Riemann curvature two forms which are needed to get the equations governing the fluxbranes. We use the convention that the zero index
is time-like and the curvatures of (3.3) are normalised according to
\[
\begin{align*}
\mathcal{R}^j_\ell (L) &= l \Lambda (L) \bar{g}^j_\ell (L), \\
\mathcal{R}^\alpha_\beta (E) &= (m - 1) \Lambda (E) \bar{g}^\alpha_\beta (E),
\end{align*}
\]
where the over-line refers to the metrics \( \bar{ds}^2 (L), (E) \). Our notation for indices is that; \( i, j = 0 \ldots l \) and cover the Lorentzian metric, \( \alpha, \beta = l + 2 \ldots D - 1 \) which cover the Euclidean metric, the index \( (l + 1) \) is for the radial coordinate \( \xi \). Capital Roman letters shall be used to denote general indices, \( M, N = 0 \ldots D - 1 \).

Our orthonormal basis one forms for \( ds^2 \) are denoted \( e^i = \exp(a) \bar{e}^i, \bar{e}^\xi = \exp(b) d\xi \) and \( e^\alpha = \exp(c) \bar{e}^\alpha \), where the \( \bar{e}^i, \bar{e}^\xi \) are understood to be the orthonormal bases on \( \bar{ds}^2 (L), (E) \). In this basis then we find that the curvature two forms are
\[
\begin{align*}
R^i_\ell &= \mathcal{R}^i_\ell - (a')^2 \exp(-2b) e^i \wedge \eta_{kj}, \\
R^\alpha_\beta &= \mathcal{R}^\alpha_\beta - (c')^2 \exp(-2b) e^\alpha \wedge e^\gamma \eta_{\gamma \beta}, \\
R^\xi &= \left[ a'' - a'b' - (a')^2 \right] \exp(-2b) e^\xi \wedge e^i, \\
R^\alpha_\xi &= \left[ c'' - c'b' + (c')^2 \right] \exp(-2b) e^\xi \wedge e^\alpha, \\
R^\alpha &= -a' c' \exp(-2b) e^i \wedge e^\gamma \eta_{\gamma \alpha},
\end{align*}
\]
where \( \eta_{MN} \) is the usual Minkowski metric. Using the fact that the \( \bar{ds}^2 (L), (E) \) are Einstein manifolds we find the components of the Ricci tensor,
\[
\begin{align*}
\mathcal{R}_{ij} &= -\left[ a'' - a'b' + (l + 1)(a')^2 + ma' c' \right] \exp(-2b) \eta_{ij} + l \Lambda (L) \bar{g}_{ij} \exp(-2a), \\
\mathcal{R}_{\alpha \beta} &= -\left[ c'' - c'b' + m(c')^2 + (l + 1)a' c' \right] \exp(-2b) \eta_{\alpha \beta} + (m - 1) \Lambda (E) \bar{g}_{\alpha \beta} \exp(-2c), \\
\mathcal{R}_{\xi i} &= -(l + 1)a'' - mc'' + (l + 1)a'b' + mb' c' - (l + 1)(a')^2 - m(c')^2 \exp(-2b),
\end{align*}
\]
taking care to notice that the curvature forms \( \mathcal{R} \) are defined with an orthonormal basis of \( \bar{ds}^2 (L), (E) \) not \( ds^2 \). So, for example,
\[
\begin{align*}
\mathcal{R}^\xi_\beta &= \frac{1}{2} \mathcal{R}^\xi_\beta \gamma \delta \bar{e}^\gamma \wedge \bar{e}^\delta, \\
&= \exp(-2c) \frac{1}{2} \mathcal{R}^\xi_\beta \gamma \delta e^\gamma \wedge e^\delta.
\end{align*}
\]

### 3.2 Source equations

By analogy with the Melvin solution (2.2) we require the form \( F \) to represent a magnetic flux on the brane, that is to say the Hodge dual of \( F \) is to have its components in the worldvolume directions. We are led to consider an \( (m + 1) \) form ansatz,
\[
F = f (\xi) e^\xi \wedge e^{l+2} \wedge e^{l+3} \wedge \ldots \wedge e^{D-1},
\]
which clearly satisfies the Bianchi equation because \( e^{l+2} \wedge e^{l+3} \wedge \ldots \wedge e^{D-1} \) is, up to a function of \( \xi \), the volume form on \( \bar{ds}^2 (E) \). The equation of motion \( \ast F = 0 \) is satisfied for \( f (\xi) = \kappa \exp[-(l + 1)a] \), where \( \kappa \) is a constant, representing the strength of the flux. With the form equations satisfied we are left with the gravity equations for an \( m + 1 \) form field strength,
\[
\mathcal{R}_{MN} = \frac{1}{2(m!)} \left[ F_{MN} F_{\bar{N}} - \frac{m}{(m + 1)(l + m)} F^2 g_{MN} \right].
\]
This gives two equations of motion and one constraint,

\[ a'' - a'b' + (l + 1)(a')^2 + ma'c' - l\Lambda(L) \exp(2b - 2a) = \frac{1}{2l + m} \kappa^2 \exp[2b - 2(l + 1)a] \quad (3.11) \]

\[ c'' - c'b' + m(c')^2 + (l + 1)a'c' - (m - 1)\Lambda(E) \exp(2b - 2c) = -\frac{l}{2l + m} \kappa^2 \exp[2b - 2(l + 1)a] \quad (3.12) \]

\[ \frac{1}{2} m(m - 1)(c')^2 + \frac{1}{2} l(l + 1)(a')^2 + m(l + 1)a'c'
- \frac{1}{2} m(m - 1)\Lambda(E) \exp(2b - 2c) - \frac{1}{2} l(l + 1)\Lambda(L) \exp(2b - 2a)
= \frac{1}{4} \kappa^2 \exp[2b - 2(l + 1)a] . \quad (3.13) \]

**4 Dynamical system**

As is common practice for this type of system we may use Misner variables \footnote{[7]} to rewrite the equations in a more familiar form. By picking a suitable gauge these variables put the equations into the form of the equation of motion for a particle in a potential. To see this we define the following gauge choice,

\[ b = (l + 1)a + mc, \quad (4.1) \]

and introduce a new variable,

\[ A = la + mc. \quad (4.2) \]

With these we find

\[ A'' = l^2\Lambda(L) \exp(2A) + m(m - 1)\Lambda(E) \exp \left[ \frac{2(l + 1)}{l} A - \frac{2(m + l)}{l} c \right] , \quad (4.3) \]

\[ c'' = (m - 1)\Lambda(E) \exp \left[ \frac{2(l + 1)}{l} A - \frac{2(m + l)}{l} c \right] - \frac{l}{2l + m} \kappa^2 \exp(2mc). \quad (4.4) \]

We recognise these as describing the motion of a particle in the \(A-c\) plane, with a position dependent force acting on it. It is noted that these equations may be derived from the following Lagrangian,

\[ \mathcal{L} = \frac{l + 1}{l} (A')^2 - \frac{m(l + m)}{l} (c')^2 + l(l + 1)\Lambda(L) \exp(2A)
+ m(m - 1)\Lambda(E) \exp \left[ \frac{2(l + 1)}{l} A - \frac{2(m + l)}{l} c \right] + \frac{1}{4} \kappa^2 \exp(2mc). \]

If we calculate from this the Hamiltonian, then one finds that the constraint equation restricts us to zero energy solutions of this dynamical system.

For the explicit solution of these equations the following will be useful,

\[ \frac{\partial^2 y}{\partial \xi^2} = -\alpha^2 \exp(2\beta y) \quad (4.5) \]

\[ \Rightarrow y = -\frac{1}{\beta} \ln \left[ \frac{\alpha}{c} \sqrt{\beta} \cosh[c(\xi - \xi_0)] \right] , \quad (4.6) \]

\[ \frac{\partial^2 y}{\partial \xi^2} = \alpha^2 \exp(2\beta y) \quad (4.7) \]

\[ \Rightarrow y = -\frac{1}{\beta} \ln \left[ -\frac{\alpha}{c} \sqrt{\beta} \sinh[c(\xi - \xi_0)] \right] , \quad (4.8) \]

where the range of \(\xi\) for (4.8) is \(-\infty < \xi < \xi_0\) and \(c(>0), \xi_0\) are arbitrary constants.
5 Melvin solutions

We shall start our tour of solutions by showing that the above formalism correctly reproduces what we already know, namely the Melvin solution. This case has a Minkowski world volume so \( \Lambda_{(L)} = 0 \), one also has \( \overline{d^2 s} = d\phi^2 \) so \( m = 1 \) and because \( \overline{d^2 s} \) is one dimensional then \( \Lambda_{(E)} = 0 \). Taking \( \Lambda_{(L)} = 0 \), \( \Lambda_{(E)} = 0 \) but keeping \( m \) general we find,

\[
A = c_0 (\xi - \xi_0) \quad \text{(5.1)}
\]

\[
c = -\frac{1}{m} \ln \left[ \frac{\kappa}{c_1} \sqrt{\frac{lm}{2(l+m)}} \cosh(c_1 (\xi - \xi_1)) \right] \quad \text{(5.2)}
\]

the constraint equation requires \( c_1 = c_0 \sqrt{\frac{m(l+1)}{l+m}} \), with \( c_0 (> 0) \), \( \xi_0 \) and \( \xi_1 \) being arbitrary constants.

To make connection with the familiar form of the Melvin solution we change coordinates to \( \xi = \ln(r) \) and take \( m = 1 \). Upon choosing the constants \( \xi_0 = 0 \) and \( \xi_1 = \ln \left( \frac{2}{\kappa} \sqrt{\frac{2(l+1)}{l}} \right) \) we find

\[
ds^2 = ds^2_{(L)} \left[ (\beta r)^{2c_0} + 1 \right]^{2/l} + \frac{dr}{r}^2 \left[ (\beta r)^{2c_0} + 1 \right]^{2/l} + d\phi^2 r^{2c_0} / \left[ (\beta r)^{2c_0} + 1 \right]^2 \quad \text{(5.3)}
\]

where \( \beta = \frac{5}{2} \sqrt{\frac{l}{2(l+1)}} \). We are then left with another constant, \( c_0 \), which we are free to choose. The natural choice for \( c_0 \) is determined by regularity at the origin, choosing \( c_0 = 1 \) means that as, \( \rho \to 0 \), the metric approaches Minkowski space and so is regular. Thus we have reproduced the Melvin solution \( \lambda^2 \).

It is instructive to look at the curvature invariants in the limits \( \xi \to \pm \infty \) in this well known case, as we shall be doing a similar analysis for the new solutions.

\[
c(\xi \to \infty) \to -\xi, \quad \text{(5.4)}
\]

\[
a(\xi \to \infty) \to \frac{2}{l} \xi, \quad \text{(5.5)}
\]

\[
b(\xi \to \infty) \to \frac{l+2}{l} \xi, \quad \text{(5.6)}
\]

\[
c(\xi \to -\infty) \to \xi - \exp(2[\xi - \xi_1]) + \text{const}, \quad \text{(5.7)}
\]

\[
a(\xi \to -\infty) \to \frac{1}{l} \exp(2[\xi - \xi_1]) + \text{const}, \quad \text{(5.8)}
\]

\[
b(\xi \to -\infty) \to \xi + \frac{1}{l} \exp(2[\xi - \xi_1]) + \text{const}. \quad \text{(5.9)}
\]

We see then that in this orthonormal basis the components of the curvature two forms vanish as \( \xi \to \infty \) and are constant as \( \xi \to -\infty \). As this is the orthonormal frame then all invariants made from contractions of \( (\overline{3}) \) will be finite. We shall see later that this is peculiar to the Melvin case, where \( m = 1 \), in general the space will be singular as we approach the fluxbrane core \( \xi \to -\infty \). For \( m = 1 \) there are no such curvature components as \( R_{(E)ij} \) because \( \overline{d^2 s} \) is one dimensional, so the divergence of \( (c')^2 \exp(-2b) \) in \( R_{(E)ij} \) is not a problem. For \( m > 1 \) however these curvature components are there and do diverge at the core of the fluxbrane leading to a naked curvature singularity. We shall see however that at an infinite proper distance from the core (for \( \Lambda_{(L)} \geq 0 \)) the curvatures are well behaved, giving a regular asymptotic spacetime.

As we have written out the equations in the form of a dynamical system we may visualize this solution by considering the acceleration of a particle with location in the \( x - y \) plane of \( (x, y) = (c, A) \). Fig. 4 is a plot with this in mind. Here the arrows show the direction of the acceleration that the
Figure 1: Solutions for the Ricci flat world-volume and Euclidean manifold. The core of the fluxbrane, \( \xi \to -\infty \), is at \( A, c \to -\infty \).

The particle feels, given by \((c'', A'')\) of (4.3), but we have suppressed the magnitude of the acceleration for clearer presentation.

The different trajectories are governed by the integration constants and we show trajectories which only differ in the choice of \( c_0 \). From (5.1) we see that \( c_0 \) characterizes the speed of the particle in the \( A \) direction, so a large value will tend to give a straighter path in the \( A - c \) plane. The particle starts at \( c \to -\infty \) which represents the core of the fluxbrane, moving to \( c \to \infty \) which describes the region far away from the fluxbrane. In the case of \( m = 1 \) we get the Melvin solution by choosing the \( c_0 = 1 \) trajectory, which leads to a regular core.

6 First extension, \( \Lambda_{(L)} \neq 0, \Lambda_{(E)} = 0, m \geq 1 \)

6.1 \( \Lambda_{(L)} > 0, \Lambda_{(E)} = 0, m > 1 \)

The problem in getting a general solution to (4.3) is the term involving \( \Lambda_{(E)} \) so, to get a feel for the system we are trying to solve we start by setting \( \Lambda_{(E)} = 0 \). This leads to an unconventional transverse space as \( ds^2_{(E)} \) is usually taken to be a round sphere metric. This simplification allows us to get the general solution, in a later section we shall look at the more interesting case where this restriction is dropped. In this case the equations (4.3) may be solved to give

\[
A = -\ln \left[ \frac{t \sqrt{\Lambda_{(L)}}}{c_0} \sinh(c_0(\xi - \xi_0)) \right], \tag{6.1}
\]

\[
c = -\frac{1}{m} \ln \left[ \frac{\kappa}{c_1} \sqrt{\frac{lm}{2(l + m)}} \cosh(c_1(\xi - \xi_1)) \right], \tag{6.2}
\]
and the coordinate range is $-\infty < \xi < \xi_0$, noting that $\xi = \xi_0$ is at infinite proper distance from the core of the fluxbrane at $\xi \to -\infty$. The constraint gives the relation,

$$c_1 = c_0 \sqrt{\frac{m(l+1)}{l+m}}. \quad (6.3)$$

To study the solution at infinite proper distance we expand the solution around $\xi = \xi_0$.

$$c(\xi \to \xi_0) \to \text{constant}, \quad (6.4)$$

$$a(\xi \to \xi_0) \to \frac{1}{l} \ln(1/(\xi_0 - \xi)) + \text{constant}, \quad (6.5)$$

$$b(\xi \to \xi_0) \to \frac{l + 1}{l} \ln(1/(\xi_0 - \xi)) + \text{constant}. \quad (6.6)$$

Which shows that the curvature components go to zero, (3.3). In fact, we can see why this happens by using a more appropriate coordinate system, $R = 1/(\xi_0 - \xi)^{1/l}$. As $\xi \to \xi_0$ then $R \to 0$ and the asymptotic metric becomes

$$\text{ds}^2 \sim \text{ds}^2_{(E)} + l^2 dR^2 + R^2 \text{ds}^2_{(L)}. \quad (6.7)$$

If we take $\text{ds}^2_{(L)}$ to be the deSitter metric,

$$\text{ds}^2_{(L)} = -dt^2 + l_0^2 \cosh^2(t/l_0) d\Omega_l^2, \quad (6.8)$$

then we see that asymptotically the metric, $\text{ds}^2$, is just the product of $\text{ds}^2_{(E)}$ and Minkowski space.

The dynamical systems plot for this situation is given in Fig. 2. Again the different trajectories are labelled by the integration constants, here we show only what happens as $c_0$ varies.

### 6.2 $\Lambda_{(L)} < 0$, $\Lambda_{(E)} = 0$, $m > 1$

Given the deSitter fluxbrane above then it is natural to look for anti-deSitter fluxbranes that is, fluxbranes with negative curvature. In order to get the general solution we again consider $\Lambda_{(E)} = 0$. The starting point, (4.3), now gives the solution,

$$A = -\ln \left[ \frac{l \sqrt{|\Lambda_{(L)}|}}{c_0} \cosh(c_0(\xi - \xi_0)) \right], \quad (6.9)$$

$$c = \text{as before} (5.2), \quad \text{with the same relation holding between } c_0 \text{ and } c_1, (6.3). \quad (6.10)$$

and $c$ is as before (5.2), with the same relation holding between $c_0$ and $c_1$, (6.3). The coordinate range is $-\infty < \xi < \infty$ and we note that $\xi \to \infty$ is a finite proper distance from the core, so there is the possibility that we are not covering the whole spacetime. However, by looking at limiting behaviour of the scale factors,

$$c(\xi \to \infty) \to \frac{1}{m} c_1 \xi, \quad (6.11)$$

$$a(\xi \to \infty) \to \frac{1}{l} (c_1 - c_0) \xi, \quad (6.12)$$

$$b(\xi \to \infty) \to \frac{c_1 - (l+1)c_0}{l} \xi, \quad (6.13)$$

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we see that \( b(\xi \to \infty) < 0 \) (because \((l + 1)c_0 > c_1\)) so the scalar curvatures constructed from (3.3) diverge at a finite proper distance from the core. This leads to a rather singular spacetime, with a singularity at a finite radial distance as well as one at the core. This behaviour was seen previously [5] for the case of fluxbranes generated from a Maxwell 2-form, where the worldvolume had a negative curvature.

In this case the dynamical systems plot shows a marked difference with the preceding cases. Fig. 3 indicates some sample trajectories that show how the particle doubles back on itself in the \( c - A \) plane, it is this behaviour that makes \( b(\xi \to \infty) \to -\infty \) causing the curvature to blow up.

7 Second extension, \( \Lambda(L) = 0, \Lambda(E) > 0, m \geq 1 \)

The more interesting case is where \( d_{S(E)}^2 \) is a round sphere metric, rather than a Ricci flat one. We can solve the equations if we make the ansatz that \( A(\xi) \) is proportional to \( c(\xi) \), plus a constant. This means that we don’t get the most general solution but, as we shall argue, it is in fact the only sensible one.

For the two equations of motion to agree we find that

\[
A = \frac{lm + l + m}{l + 1} c + \ln(\beta),
\]

(7.1)

\[
\beta^{(2l+2)/l} = \frac{\kappa^2}{2\Lambda(E)} \frac{lm + l + m}{(m - 1)(l + m)},
\]

(7.2)

To find the solution for this we take the approach of substituting the ansatz into the constraint equation,
Figure 3: Solutions for the negatively curved world-volume and Ricci flat Euclidean manifold. The core of the fluxbrane, $\xi \to -\infty$, takes the values $A, c \to -\infty$. The fact that $A, c$ turn around and diverge back to $-\infty$ is the reason why there is a singularity at finite proper distance from the core.

yielding

$$c = -\frac{1}{m} \ln \left[ -\kappa m \sqrt{\frac{l + 1}{l + m}(\xi - \xi_0)} \right]$$

(7.3)

with coordinate range $-\infty < \xi < \xi_0$. We note that this could, of course, have been derived using (4.3) but the constraint equation would have meant taking a singular limit of one of the constants of integration. Again we find that the curvature two form components are finite as $\xi \to \xi_0$, but blow up as the core of the fluxbrane is approached. As an example we show the Ricci scalar,

$$\mathcal{R}(\xi \to \xi_0) \to \left[ \frac{2 - 3m - l}{ml} + \frac{1}{m^2l(l + 1)} + m(m - 1)\Lambda(E) \right] (\xi_0 - \xi)^{2/m} \to 0$$

(7.4)

Although this isn’t the most general solution it is the only one that makes sense in the asymptotic region. A perturbative analysis around this solution reveals that the perturbations to the exact solution above take the form

$$\mathbf{v}'' = \frac{1}{(\xi - \xi_0)^2} \text{M}(l, m)\mathbf{v}$$

(7.5)

where $\mathbf{v} = (\delta A, \delta c)$ and $\text{M}(l, m)$ is some matrix. It is found that the product of eigenvalues, $\mathcal{E}_1 \mathcal{E}_2$, of $\text{M}(l, m)$ is

$$\mathcal{E}_1 \mathcal{E}_2 = (-m^3 - l^3 - l^2m - 4l^2m^2 - 2m^2l^3 + m^2 + 2ml + l^2)$$

(7.6)

which is always negative for $l, m > 1$. Thus we have one growing mode and one oscillatory mode. However, because of the $1/(\xi - \xi_0)^2$ the oscillating mode increases in frequency and amplitude causing
the scale factors to oscillate more and more wildly. For this reason we believe that the particular solution found above is the relevant one.

We can get more intuition for this system by looking at the acceleration field that the dynamical systems picture gives us, Fig. 4. If we place the particle off the line marking the particular solution found above then it will accelerate towards it. It is not an attractor however because of the magnitude of the acceleration, the particle oscillates about the particular solution with an ever increasing amplitude and frequency.

8 Embedding diagram

The spacetime considered above can be neatly characterized by an embedding diagram, showing how the size of the Euclidean manifold changes as a function of proper distance from the core of the fluxbrane. Fig. 5 shows such a diagram, where we recall from (2.5) that \( \exp(c) \) characterizes the size of the Euclidean manifold. The cases where \( \Lambda(E) = 0 \) are reminiscent of the embedding diagrams of 5, where the Euclidean manifold was a circle and the form field was the Maxwell two form.

When plotted like this we see clearly that the AdS brane (\( \Lambda < 0 \)) is rather singular, with it ending at a finite proper distance from the core.

9 Dilatonic fluxbranes

In supergravities one typically finds scalar fields which couple to the field strength of the form field. It is therefore of interest to see how these scalar fields modify the fluxbranes described above. To study this we consider an action of the form,
Figure 5: Here we show the embedding diagrams for the various solutions. They describe how the volume of the Euclidean manifold, \( \exp(c(\xi)) \), varies as we move away from the core. To compare between cases we plot the volume as a function of proper distance.
\[ S = \int \left[ R * 1 - \frac{1}{2} d\phi \wedge * d\phi - \frac{1}{2} \exp(\alpha\phi) F \wedge * F \right]. \quad (9.1) \]

So, we have a scalar field with an arbitrary dilaton type coupling in the Einstein frame. From this we may derive the following equations of motion,
\[ d\left[ \exp(\alpha\phi) * F \right] = 0 \quad (9.2) \]
\[ d * d\phi = \frac{1}{2} \alpha \exp(\alpha\phi) F \wedge * F \quad (9.3) \]
\[ R_{MN} = \frac{1}{2} e_M(\phi)e_N(\phi) + \frac{1}{2m!} \exp(\alpha\phi) F \wedge * F - \frac{m}{(m+1)(l+m)} F^2 g_{MN} \quad (9.4) \]

where the \( e_M \) are the vectors dual to the one forms \( e^M \). For these solutions we take it that \( \phi = \phi(\xi) \).

Using the same metric ansatz \((2.5)\) the form equation, \((9.2)\), is solved for
\[ F = \kappa \exp\left[ -\alpha\phi - (l+1)a \right] e^\xi \wedge e^{l+2} \wedge e^{l+3} \wedge ... \wedge e^{D-1}, \quad (9.5) \]
with the Bianchi equation, \(d*F=0\), being satisfied because \( e_1(E) \wedge e_2(E) \wedge ... \wedge e_{m}(E) \) is just the volume form on \( ds^2(E) \), up to a function of \( \xi \). Taking the same gauge choice as before \((4.1)\), and using the variable \( A = la + mc \quad (4.2)\) we find the following equations of motion and constraint,
\[ \phi'' = \frac{1}{2} \kappa^2 \exp(2mc - \alpha\phi), \quad (9.6) \]
\[ A'' = l^2 \Lambda(L) \exp(2A) + m(m-1)\Lambda(E) \exp\left[ \frac{2(l+1)}{l} A - \frac{2(l+m)}{l} c \right], \quad (9.7) \]
\[ c'' = (m-1)\Lambda(E) \exp\left[ \frac{2(l+1)}{l} A - \frac{2(m+l)}{l} c \right] - \frac{1}{2} \frac{l}{l+m} \kappa^2 \exp(2mc - \alpha\phi), \quad (9.8) \]
\[ 0 = \frac{l+1}{l} (A')^2 - \frac{m}{l} (l+m)(c')^2 - \frac{1}{2} (\phi')^2 - l(l+1)\Lambda(L) \exp(2A) \quad (9.9) \]
\[ -m(m-1)\Lambda(E) \exp\left[ \frac{2l+1}{l} A - \frac{2l+m}{l} c \right] - \frac{1}{2} \kappa^2 \exp(2mc - \alpha\phi) \quad (9.10) \]

Due to the similarity with the dilaton free case we shall only consider \( \Lambda(L) = 0 \). Upon introducing new variables \( Y \) and \( Z \),
\[ Y = 2mc - \alpha\phi, \quad (9.11) \]
\[ Z = 2l+1 \quad A - 2 \frac{l+m}{l} c, \quad (9.12) \]
we find,
\[ \phi'' = \frac{1}{2} \kappa^2 \exp(Y), \quad (9.13) \]
\[ Y'' = 2m(m-1)\Lambda(E) \exp(Z) - \left[ \frac{lm}{l+m} + \frac{1}{2} \alpha^2 \right] \kappa^2 \exp(Y), \quad (9.14) \]
\[ Z'' = 2(m-1)^2 \Lambda(E) \exp(Z) + \kappa^2 \exp(Y). \quad (9.15) \]

Although we have been unable to find the general solution we are able, just as before, to find a particular solution by taking the ansatz \( Z = Y + \ln \beta \). In order for \((9.13,9.14)\) to be consistent we require,
\[ \beta = \frac{\kappa^2}{2(m-1)\Lambda(E)} \left[ \frac{lm+l+m}{l+m} + \frac{1}{2} \alpha^2 \right]. \quad (9.16) \]
The solution then that satisfies the constraint equation is

$$Y = -2\ln \left[-\frac{\zeta}{\sqrt{2}}(\xi - \xi_0)\right],$$

(9.16)

$$\phi = -\frac{\alpha \kappa^2}{\zeta^2} \ln \left[-c_1(\xi - \xi_0)\right],$$

(9.17)

$$\zeta^2 = (m - 1) \left[\frac{lm + 2(l + m)}{l + m} + \frac{1}{2} \alpha^2\right] \kappa^2.$$ 

(9.18)

This solution has much the same properties as the analogous case without the dilaton. The interpretation also follows from the previous case, with us understanding the solution as the the fluxbrane which transmits the flux between a dilatonic brane-antibrane pair of infinite separation.

10 Discussion

In supergravity theories one typically requires antisymmetric form fields, it has been the aim of this paper to construct the gravitational analogue of a constant form field. The Melvin solution describes such a scenario for a two-form (Einstein-Maxwell) where one finds that the lines of magnetic flux self-gravitate into a cylindrical flux tube. What we have found is that this is part of a more general picture, where a magnetic $n$-form field strength in $D$ dimensional spacetime creates a fluxbrane of $D - n - 1$ spatial dimensions, a $(D - n - 1)$-fluxbrane. The physical origin of these objects is made clearer by thinking of these fluxbranes as being formed between a $(D - n - 2)$ brane anti-brane pair with infinite separation.

In this paper we have performed a rather brute-force method of finding solutions, however there is another, rather elegant, way of getting the dilaton-Einstein-Maxwell solution [4]. The $D$ dimensional Melvin+dilaton solution comes from an unconventional identification of points in $D + 1$ dimensional Minkowski space. When the dimensional reduction is then performed one finds that the Kaluza-Klein vector is describing a magnetic fluxbrane. The fluxbranes considered here are not expected to come from any analogous technique as dimensional reduction of a metric does not generate form fields, other than the Kaluza-Klein vector. In this special case one is also able to generate the instantons which describe the decay of this fluxbrane [4] by the nucleation of charged spherical branes. While we have not considered the stability of the generalized fluxbranes we believe that, if classically stable, they could decay by similar instanton processes.

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