ON SEMILINEAR REPRESENTATIONS OF THE INFINITE SYMMETRIC GROUP

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Abstract. In this note the smooth (i.e. with open stabilizers) linear and semilinear representations of certain permutation groups (such as infinite symmetric group or automorphism group of an infinite-dimensional vector space over a finite field) are studied. Many results here are well-known to the experts, at least in the case of linear representations of symmetric group. The presented results suggest, in particular, that an analogue of Hilbert’s Theorem 90 should hold: in the case of faithful action of the group on the base field the irreducible smooth semilinear representations are one-dimensional (and trivial in appropriate sense). Here the Hilbert’s Theorem 90 is established in the case of the permutation group of a transcendence base of a purely transcendental field extension.

Let $G$ be a permutation group, cf. § 1. In the most general setting, we are interested in the continuous $G$-actions on the discrete sets (i.e. with open stabilizers; they are called smooth in what follows). In practice, the considered $G$-sets are endowed with extra structures, e.g., of a vector $k$-space for a field $k$. In that case our primary goal is a description of the smooth representations of $G$ in the $k$-vector spaces, especially of the irreducible ones. The smooth representations form a $k$-linear Grothendieck category with $\bigoplus_{s \gg 0} W_s$ as one of many possible generators, where $W_s = W^\otimes k$ and $W = k[\Psi]$. However, the structure of $k[G]$-modules $W_s$ can be quite complicated.

For any group $G$ and any field $k$ there is a field extension $K|k$ endowed with a faithful $G$-action. Namely, as $K$ one can take the fraction field of the symmetric algebra of a faithful representation $W$ of $G$ over $k$. Then there is a natural surjection $W \otimes_k K \to K$ of $K$-semilinear representations of $G$. (The semilinear representations are defined in § 3.)

Suppose first that $G$ is pre-compact, i.e., any open subgroup of $G$ is of finite index. It is well-known that any cyclic smooth $k[G]$-module can be embedded into an arbitrary field extension $K$ of $k$ endowed with a smooth faithful $G$-action trivial on $k$. Moreover, Hilbert’s Theorem 90 asserts the triviality of smooth semilinear representations of $G$ over $K$: the functor $H^0(G, -) : V = V^G \otimes_{k[G]} K \cong \bigoplus_I K \to V^G$ is an equivalence between the category of smooth semilinear representations of $G$ over $K$ and the category of vector spaces over the fixed field $K^G$. (Here $I$ is a basis of the $K^G$-vector space $V^G$.) In particular, the symmetric algebra of any smooth faithful $k$-representation of $G$ is a generator of the category of smooth $k$-representations of $G$.

The above remarks lead to the problem of describing smooth semilinear representations over $K$ for a field $K$ endowed with a smooth faithful $G$-action.

There is a (rather wild) description of all smooth semilinear $G$-actions on a given $K$-vector space in the case when a dense subgroup of $G$ is exhausted by pre-compact subgroups, cf. Appendix A. This description is quite explicit in the case when $G$ is exhausted by open pre-compact subgroups.

Our principal examples of $G$ will be permutation groups of $\mathfrak{S}$-type defined on p 4. Typical groups of $\mathfrak{S}$-type are the group $\mathfrak{S}_\Psi$ of all permutations of a plain set $\Psi$, or the automorphism group of an infinite-dimensional projective space $\Psi$ over a finite field $\mathbb{F}_q$, or the automorphism group $GL_{\mathbb{F}_q}(\Psi)$ of an infinite-dimensional vector space $\Psi$ over a finite field $\mathbb{F}_q$, or the automorphism group $GL_{\mathbb{F}_q}(\Psi)$ of an infinite-dimensional $\mathbb{F}_q$-vector space $\Psi$.

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Such groups admit a dense subgroup exhausted by finite subgroups.

Let $G \subseteq \mathfrak{S}_\Psi$ be a group of $\mathfrak{S}$-type for a countable set $\Psi$. It is well-known (cf., e.g., [8] §6 and references therein) that (i) any smooth cyclic representation of $G$ is admissible and of finite length (cf., e.g., Lemma 5.3.1), (ii) the isomorphism classes of irreducible smooth representations of $\mathfrak{S}_\Psi$ are in one-to-one correspondence with (finite) Young diagrams. However, the approach of loc.cite is different and I was unable to find in the literature a description of all injective smooth representation of $G$. This description follows from Lemma 1.[1]

One of the purposes of this note is to examine the similar questions in the context of semilinear representations.

The results are as follows (where [1] is my principal motivation):

1. a description of injective smooth representations of a group $G$ of $\mathfrak{S}$-type (Corollary 1.[3]);
2. any smooth finitely generated linear representation of a pseudofinite group (this class of permutation groups is introduced in 1[.4]) is of finite length (Lemma 5.3.1);
3. the category of smooth $K$-semilinear representation of the group $\mathfrak{S}_\Psi$ is locally noetherian, i.e., any smooth finite $K\langle \mathfrak{S}_\Psi \rangle$-module is noetherian (notations are in 5[.6], cf. Corollary 5.3.6);
4. [an analogue of Hilbert's Theorem 90] any smooth $K$-semilinear finite-length representation of $\mathfrak{S}_\Psi$ is isomorphic to a direct sum of copies of $K$, if $K/k$ is a purely transcendental field extension with a transcendence base identified with $\Psi$, cf. Theorem 12.2 and Lemma 9.1;
5. a generalization of the well-known cyclicity of finite-dimensional semilinear representations of infinite semigroups to the cyclicity of certain smooth finite $K\langle G \rangle$-modules (Lemma 11.1);
6. the triple $(K\langle \mathfrak{S}_\Psi \rangle, K[\Psi], K)$ is an example of a triple $(A, M, P)$ consisting of an associative unital ring $A$, a noetherian $A$-module $M$ and a simple $A$-module $P$ such that (i) any quotient of $M$ by any non-zero submodule is isomorphic to a finite direct sum of copies of $P$, (ii) for any integer $N \geq 0$ there is a quotient of $M$ isomorphic to a direct sum of $N$ copies of $P$ (Lemma 10.2).

With $K$ as in [1], there are some reasons to expect an explicit description of the indecomposable injectives, cf. Conjecture 11.5. This is compatible with Theorem 12.2.

0.1. **Permutation groups and categories associated to collections of their subgroups.**

By definition, a *permutation group* is a Hausdorff topological group $G$ admitting a base of open subsets consisting of the left and right shifts of subgroups. If we denote by $B$ a collection of open subgroups such that the finite intersections of conjugates of elements of $B$ form a base of open neighbourhoods of 1 in $G$ (e.g., the set of all open subgroups of $G$), then $G$ acts faithfully on the set $\Psi := \Pi_{U \in B} G/U$, so (i) $G$ becomes a *permutation group* of $\Psi$, (ii) the shifts of the pointwise stabilizers $G_T$ of the finite subsets $T \subset \Psi$ form a base of the topology of $G$.

Clearly, $G$ is totally disconnected.

It is well-known (e.g., [7] Exposé IV, §2.4–2.5) or [1] §8.1, Example 8.15 (iii)) that the category $\text{Sm-G}$ of smooth $G$-sets and their $G$-equivariant maps is a topos. For a base $B$ of open subgroups of $G$, considered as a poset, let $\mathcal{C}_B$ be the small full subcategory of $\text{Sm-G}$ whose objects are the images of the contravariant functor $B \to \text{Sm-G}$, $U \mapsto [U] := G/U$. Thus, any morphism in $\mathcal{C}_B$ is epimorphic and $\mathcal{C}_B([U],[V]) := \text{Maps}_G(G/U,G/V) = (G/U)^V = \{ g \in G \mid gUg^{-1} \supseteq V \}/U$.

We endow $\mathcal{C}_B$ with the maximal topology, i.e. we assume that any sieve is covering. Then the sheaves of sets, groups, etc. on $\mathcal{C}_B$ are identified with the smooth $G$-sets, groups, etc.: $\mathcal{F} \mapsto \lim_{U \in B} \mathcal{F}(U)$ (this is a smooth $G$-set, since any element in it belongs to the image of some $\mathcal{F}(U)$ and the $U$-action on it is trivial by definition) and $W \mapsto [U \mapsto W_U]$.

1. **Finiteness conditions on permutation groups**

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1A representation $V$ of a permutation group $G$ is *admissible* if $\dim V^U < \infty$ for any open subgroup $U \subseteq G$.

2*Length* of an object of a concrete category is the maximal length of the chains of its subobjects. E.g., length is cardinality for the category of sets; length is dimension for the categories of projective or vector spaces.
1.1. Pseudoprofinite groups. A permutation group $G$ is called pseudoprofinite if the set $U \backslash G/V$ is finite for any pair of open subgroups $U, V \subseteq G$.

Example. A locally compact group is pseudoprofinite if and only if it is profinite, since it is a finite union of compact double cosets.

Lemma 1.1. The following conditions on a permutation group $G$ are equivalent:

1. The category of smooth finitely generated representations of $G$ over a fixed field is tensor,
2. for any commutative ring $C$ endowed with a smooth $G$-action, any associative $C$-algebra $A$ endowed with a smooth $G$-action, any smooth finitely generated $A(G)$-module $V$ and any smooth finitely generated $C(G)$-module $W$ the $A(G)$-module $V \otimes_C W$ is finitely generated,
3. restriction of any smooth finitely generated representation of $G$ to any open subgroup is again finitely generated,
4. $G$ is pseudoprofinite,
5. any open subgroup of $G$ is pseudoprofinite.

Proof. The implications \((\ref{1}) \Rightarrow \ref{1})\) and \((\ref{2}) \Rightarrow \ref{1})\) are trivial.

\((\ref{1}) \Rightarrow \ref{5})\). For any open subgroup $H$ of $G$ and any pair of open subgroups $U, V$ of $H$ the natural map $U \backslash H/V \to U \backslash G/V$ is injective. As $U \backslash G/V$ is finite, so is $U \backslash H/V$, i.e., $H$ is pseudoprofinite.

\((\ref{1}) \Rightarrow \ref{4})\). For any pair of open subgroups $U, V$ of $G$ one has the following decomposition of the representations of $G$: $k[G/U] \otimes_k k[G/V] = \bigoplus_{O \in \mathcal{G}(G/U \times G/V)} k[O]$, so $k[G/U] \otimes_k k[G/V]$ is finitely generated if and only if the set of orbits $G/(G/U \times G/V) \cong U \backslash G/V$ is finite.

\((\ref{1}) \Rightarrow \ref{2})\). Any smooth finitely generated $A(G)$-module is a quotient of a finite sum of $A[G/U_i]$ for some open subgroups $U_i$ of $G$, so tensor product of a pair of finitely generated $A(G)$- and $C(G)$-modules is a quotient of a finite sum of $A(G)$-modules $A[G/U_i] \otimes_C C[G/V_j] = A[(G/U_i) \times (V/V_j)]$ for some open subgroups $U_i, V_j$ of $G$.

\((\ref{1}) \Leftrightarrow \ref{3})\). Any smooth finitely generated representations of $G$ is a quotient of a finite sum of $k[G/V_i]$ for some open subgroups $V_i$ of $G$, so restriction to an open subgroup $U$ of $G$ of a finitely generated representation of $G$ is a quotient of a finite sum of representations $k[G/V_j]$ for some open subgroups $V_j$ of $G$. This proves \((\ref{3}) \Rightarrow \ref{4})\) if we assume \((\ref{4})\).

Lemma 1.2. Let $G$ be a pseudoprofinite group. Then for any open subgroup $U \subseteq G$ the finite group $N_G(U)/U$ acts transitively and freely on the finite set $(G/U)^U$.

Proof. For any open subgroup $V \subseteq G$ the set $(G/U)^V = \{g \in G \mid Vg \subseteq gU\}/U$ is identified with the set Maps$_G(G/V, G/U)$ (which is the semigroup $\text{End}_G(G/V)$ if $V = U$) by $[g] \mapsto ([h] \mapsto [hg])$, $\varphi \mapsto \varphi([1])$. This set is finite, since $(G/U)^V$ embeds into the finite set $V \backslash G/U$.

If $U$ is a proper subgroup of $gUg^{-1}$ for some $[g] \in (G/U)^U$ then we get a strictly increasing sequence of subgroups in $G$: $U \subseteq gUg^{-1} \subseteq g^2Ug^{-2} \subseteq g^3Ug^{-3} \subseteq \ldots$, so the elements $[g], [g^2], [g^3], \ldots$ of $(G/U)^U$ are pairwise distinct, contradicting the finiteness of $(G/U)^U$.

This means that the natural inclusion $N_G(U)/U \hookrightarrow (G/U)^U$ is bijective.

Lemma 1.3. Let $G$ be a pseudoprofinite group. Then (i) any ascending chain of open subgroups eventually stabilizes, (ii) any open subgroup is contained in a maximal proper subgroup of $G$.

Proof. (i) For any sequence of open subgroups $U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots$ the sequence of surjections $U_1 \backslash G/U_1 \to U_2 \backslash G/U_2 \to U_3 \backslash G/U_3 \to \ldots$ eventually stabilizes, since so does the descending sequence of the orders of these finite sets. In particular, the subset $U_n \backslash U_{n+1}/U_n \subseteq U_n \backslash G/U_n$ maps bijectively onto its image $U_{n+1} \backslash U_{n+1}/U_{n+1} \subseteq U_{n+1} \backslash G/U_{n+1}$ for $n \gg 0$, i.e., the set $U_n \backslash U_{n+1}/U_n$ consists of a single element, which means that $U_{n+1} = U_n$. (ii) follows from (i).
1.2. Groups of \( \mathfrak{G}\)-type. Definition. A pseudoprofinite group \( G \) is of \( \mathfrak{G}\)-type if (i) the maps \( (G/V)^V \to (G/U)^V \) are surjective for all \( V \subseteq U \) from a base \( B \) of open subgroups of \( G \), (ii) for each \( U \in B \) the natural projection \( N_U(V) \setminus (G/U)^V \to U \setminus G/U \) is injective for sufficiently small \( V \in B \).³

Examples. The following examples of groups \( G \) of \( \mathfrak{G}\)-type are constructed as the groups of all permutations of an infinite set \( \Psi \) respecting an extra structure on \( \Psi \). (So \( G \) is a closed subgroup of the group \( \mathfrak{G}_\Psi \) of all permutations of the plain set \( \Psi \).)

(1) \( \Psi \) is a plain set, i.e., \( G = \mathfrak{G}_\Psi \). As a base \( B \) of open subgroups we take the subgroups \( G_T \) for some exhausting collection of finite subsets \( T \subset \Psi \). Then (i) \( G/G_T \) is the set of all embeddings \( T \to \Psi \), (ii) \( (G/G_T)^{G_T} \) consists of the embeddings \( T \to T' \) (it is clear that \( G_T \) is transitive on \( (G/G_T)^{G_T} \)) and (iii) elements of the sets \( G\setminus G/G_T \) and \( N_{G_T}(G_T) \setminus (G/G_T)^{G_T} \) (if \( \#T' \geq \#T \)) are pairs \((T_0, \iota)\) consisting of a subset \( T_0 \subseteq T \) and an embedding \( \iota: T_0 \hookrightarrow T \).

(2) \( \Psi = A^S \) is a free \( A \)-module of infinite rank with a finite collection of marked elements for a finite commutative ring \( A \). As a base \( B \) of open subgroups we take the subgroups \( G_T \) for sufficiently big finite subsets \( T \) of the set \( S \) of generators of the \( A \)-module \( \Psi \). Then (i) \( G/G_T \) is a subset in the set of all embeddings \( T \to \Psi \), and (ii) \( (G/G_T)^{G_T} \) consists of the embeddings \( T \to \Psi^{G_T} = \sum_{t \in T} A_t \). At inducing splittable embeddings \( \sum_{t \in T} A_t \to \Psi \), i.e., whose images admit complementary free summands. Clearly, (i) \( N_{G_T}(G_T)^{G_T} \) is transitive on \( (G/G_T)^{G_T} \), (ii) the natural projection \( N_{G_T}(G_T) \setminus (G/G_T)^{G_T} \to G/G_T \) is injective, at least if \( \#T' \geq 2\#T \).

(3) The automorphism group of an infinite-dimensional projective space \( \Psi = \mathbb{P}(F^S) \) over a finite field \( F_q \). As a base \( B \) of open subgroups we take the subgroups \( G_T \) for the finite subsets \( T \subset S \). Then \( G/G_T \) consists of the embeddings \( T \to \Psi \) with \((\#T - 1)\)-dimensional projective envelope of the image and \( (G/G_T)^{G_T} \) consists of the embeddings \( T \to \Psi^{G_T} \).

2. Injectivity of trivial representations and admissibility

Lemma 2.1. Let \( G \) be a permutation group admitting a base of open subgroups \( B \) such that for any \( V \subseteq U \) in \( B \) the group \( N_G(V)/V \) is finite and the \((N_G(V)/V)\)-action on the set \((G/U)^V \supseteq (N_G(V)/U)=(N_G(V)/(U \cap N_G(V)) \) is transitive, i.e., \( (G/U)^V = (N_G(V)/U)/U \). Let \( R \) be a \( \mathbb{Q}\)-algebra and \( M \) be an \( R \)-module considered as a trivial \( \mathbb{Q}\)-module.

Then any essential extension \( E \) of the \( R[G] \)-module \( M \) is a trivial \( G \)-module. In particular, if \( M \) is an injective \( R \)-module then (when endowed with the trivial \( G \)-action) it is an injective object of the category of smooth left \( R[G] \)-modules.

Proof. Let \( E \) be an essential extension of \( M \) in the category of smooth left \( R[G] \)-modules. Any element of \( E \) spans a smooth cyclic \( R[G] \)-module. Any smooth cyclic \( R[G] \)-module is isomorphic to a quotient of a permutation module \( R[G/U] \) for an open subgroup \( U \) in a base of open subgroups. I claim that the image of \( R[G/U] \) in \( E \) has zero intersection with \( E[G] \), and in particular, with \( M \).

Indeed, suppose that the image \( \beta \in E \) of an element \( \alpha \in R[G/U] \) is fixed by \( G \).

The support of the element \( \alpha \), i.e., a finite subset in \( G/U \), is pointwise fixed by an open subgroup \( V \in B \), so the image of \( \alpha' := \sum_{g \in N_G(V)/V} g \alpha \) in \( E \) is \( \#(N_G(V)/V) \beta \). On the other hand, as the support of \( \alpha \) is contained in a (finite) \((N_G(V)/V)\)-orbit, \( \alpha' = 0 \), and thus, \( \beta = 0 \), i.e., the image of \( R[G/U] \) in \( E \) has no non-zero vectors fixed by \( G \). Therefore, \( E \) is a quotient of a sum of trivial \( G \)-modules \( R[G/U]/R[G/U] = R \), i.e., \( E \) is a trivial \( G \)-module. \( \square \)

³The condition (i) is the transitivity condition from [3] p.5, §3.1; (ii) is the bijectivity condition of [3] p.5, §3.2.

⁴By Lemma 2.2, this is automatic if \( G \) is pseudoprofinite.

⁵The groups of \( \mathfrak{G}\)-type are examples of such groups \( G \).

⁶For any ring \( A \), an \( A \)-module \( P \) and a set \( S \) we denote by \( P[S]^o \) the \( A \)-submodule of the \( A \)-module \( P[S] := \{ \text{finite formal linear combinations } \sum_i a_i[s_i] \ \text{for all } a_i \in P, s_i \in S \} \) consisting of finite formal linear combinations \( \sum_i a_i[s_i] \) for some \( a_i \in P, s_i \in S \) with \( \sum_i a_i = 0 \).
Lemma 2.2. Let $G$ be a pseudoprofinite group, $B$ be a base of open subgroups of $G$ and $k$ be a field. Suppose that the trivial representations $k$ of any $V \in B$ are injective as smooth $k[V]$-modules. Then any smooth finite $k[G]$-module $W$ is admissible, i.e., $\dim_k W^V < \infty$ for any $V \in B$.

Proof. As $H^0(V,W)$ is a direct summand of $W$ for any $V \in B$, the natural map $H^0(V,W) \to H_0(V,W)$ is injective, so $\dim_k H^0(V,W) \leq \dim_k H_0(V,W)$. The $k[G]$-module $W$ is a quotient of $\bigoplus_{i=1}^N k[G/U_i]$ for some open subgroups $U_i \subseteq G$, in particular, $H_0(V,W)$ is a quotient of $H_0(V,k[G/U_i])$, and thus, $\dim_k H_0(V,W) \leq \sum_{i=1}^N \dim_k H_0(V,k[G/U_i]) = \sum_{i=1}^N |V \setminus G/U_i| < \infty$. Combining all these inequalities, we get $\dim_k H^0(V,W) < \infty$. □

Theorem 2.3. Let $G$ be a pseudoprofinite group such that the group $N_G(V)$ acts transitively on $(G/U)^V$ for all $V \subseteq U$ from a base $B$ of open subgroups of $G$ and $k$ be a field of characteristic 0. Then any smooth finite $k[G]$-module $W$ is admissible (but there exist infinite direct sums among admissible $k[G]$-modules).

Proof. This is Lemma 22, since any open subgroup $V$ of $G$ satisfies assumptions of Lemma 2.1. □

3. Filtered representations and local length-finiteness

Lemma 3.1. Let $G$ be a group, $B$ be a $k[G]$-module for a ring $k$, $B$ be a partially ordered set. Let $\{U_\alpha\}_{\alpha \in B}$ be a partially ordered exhausting collection of $k$-submodules in $W$: $W = \bigcup_{\alpha} U_\alpha$. Let $G_\alpha$ be a subgroup of the stabilizer in $G$ of the subspace $U_\alpha$.

Then length of the $k[G]$-module $W$ does not exceed $\inf_\beta \sup_{\alpha \geq \beta} \text{length}_{k[G_\alpha]} U_\alpha$.

Proof. Suppose that $n = \sup_{\alpha \geq 0} \text{length}_{k[G_\alpha]} U_\alpha$ is finite and $0 = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n \subseteq W$ is a chain of subrepresentations of $G$ in $W$. Choose some $i_1 \in W_1 \setminus W_{-1}$ and $\beta \in B$, $\beta > 0$, such that $e_{i_1}, \ldots, e_n \in U_\beta$. Then $0 \subseteq W_0 \cap U_\beta \subseteq W_1 \cap U_\beta \subseteq \cdots \subseteq W_n \cap U_\beta \subseteq U_\beta$ is a chain of subrepresentations of $G_\beta$ in $U_\beta$ of length $n + 1$, contradicting our assumptions. □

Corollary 3.2. Let $G$ be a permutation group, $B$ be a base of open subgroups of $G$. Let $W$ be a smooth representation of $G$ over a field $k$. Then $\text{length}_{k[G]} W \leq \inf_{V \in B} \sup_{H \in B, H \subseteq V} \text{length}_{k[N_G(H)]} W^H$.

Proof. In Lemma 3.1 we take $G_H = N_G(H)$ and $U_H = W^H$. □

The following result is standard.

Lemma 3.3. Let $F|k$ be a finite extension of fields and $A,B$ be associative unital $k$-algebras. Let $M$ be a simple $A$-module and $N$ be a simple $B$-module. Suppose that $\text{End}_B(N) = k$. Then (i) $M \otimes_k N$ is a simple $A \otimes_k B$-module, (ii) $M \otimes_k F$ is an $A \otimes_k F$-module of finite length.

Proof. (i) Fix a non-zero $A \otimes_k B$-submodule in $M \otimes_k N$ and a shortest non-zero element $\alpha = \sum_{i=1}^a m_i \otimes n_i$ in it, i.e., $a \geq 1$ is minimal possible. Clearly, $\alpha$ is a generator of $M \otimes_k N$ if $a = 1$, so assume that $a > 1$. If the annihilators of $n_i$ in $B$ are not the same, say $\text{Ann}(n_i) \not\subseteq \text{Ann}(n_j)$, then $(1 \otimes \xi)\alpha$ is a shorter non-zero element for any $\xi \in \text{Ann}(n_i) \setminus \text{Ann}(n_j)$, contradicting to the minimality of $a$. There remains the case of coincident annihilators of $n_1, \ldots, n_a$. In that case $B/\text{Ann}(n_1) \sim \cdots \sim B/\text{Ann}(n_a)$ are isomorphisms. As $\text{End}_B(N) = k$, all these isomorphisms differ by a non-zero multiple, and therefore, the images of the element 1 in $B/\text{Ann}(n_1)$ under $-n_i$ differ by a non-zero multiple as well, i.e., the elements $n_1, \ldots, n_a$ are proportional, so finally, $a = 1$.

(ii) Set $D := \text{End}_A(M)$. This is a division $k$-algebra. Then $D \otimes_k F$ is a left and right artinian $F$-algebra, since it is finite-dimensional over $D$. Therefore, its quotient $M \otimes_k F$ is of finite length. □

Lemma 3.4. Let $G$ be a group of $\mathfrak{S}$-type from Example on p.4 Then for any left artinian $Q$-algebra $R$ the category of smooth left $R[G]$-modules is a locally artinian and locally noetherian Grothendieck category, i.e., admits a system of generators of finite length.

Proof. Any smooth finite $R[G]$-module is a quotient of a finite sum of $R[G]$-modules of type $R[G/U]$ for some subgroups $U$ from a fixed base $B$ of open subgroups of $G$. Therefore, it suffices to check that the $R[G]$-modules $R[G/U]$ are of finite length.
Fix some open subgroup $U \subseteq G$ and set $\Psi := G/U$. Set $H_V := N_G(V)/V$. As $G$ is a group of $\mathcal{G}$-type, $\Psi^V \cong H_V/(H_V \cap (U/V))$ is an $H_V$-orbit for all $V \in B$, so
\begin{equation}
\begin{aligned}
\text{End}_{F[H_V]}((F[\Psi^V]) \cong F[\Psi^V]_{H_V \cap (U/V)} \cong F[(H_V \cap (U/V)) \setminus \Psi^V]
\end{aligned}
\end{equation}
for any finite field extension $F[\mathbb{Q}]$. The natural projection $(H_V \cap (U/V)) \rightarrow \Psi^V \rightarrow U \setminus \Psi$ is injective, and therefore, length of the $F[H_V] \text{-module } F[\Psi^V]$ does not exceed
\[\dim_F \text{End}_{F[H_V]}((F[\Psi^V]) = \dim_F F[(H_V \cap (U/V)) \setminus \Psi^V] \leq \dim_F F[U \setminus \Psi] = \# [U \setminus \Psi].\]

For a fixed $V$, we choose $F$ so that the $F[H_V] \text{-module } F[\Psi^V]$ is a sum of absolutely simple modules. Clearly, length of the $R[G] \text{-module } R[\Psi]$ does not exceed length of the $(R \otimes F)[G] \text{-module } (R \otimes F)[\Psi]$. By Lemma 3.1 length of the $(R \otimes F)[G] \text{-module } (R \otimes F)[\Psi]$ does not exceed
\[\sup_{V \in B} \text{length}_{(R \otimes F)[H_V]}((R \otimes F)[\Psi]).\]

On the other hand, by Lemma 3.3 (i),
\[\text{length}_{(R \otimes F)[H_V]}((R \otimes F)[\Psi]) = \text{length}_{(R \otimes F)}((R \otimes F) \cdot \text{length}_{F[H_V]}((F[\Psi^V]), so finally,}
\[\text{length}_{(R \otimes F)[G]}((R \otimes F)[\Psi]) \leq \text{length}_{R \otimes F}(R \otimes F) \cdot \# [U \setminus \Psi],\]
which is finite by Lemma 3.3 (ii).

4. Coinduction and Injectives

Let $A \subseteq R$ be a pair of associative unital topological rings with a base of neighbourhood of 0 given by a collection of left ideals. The restriction functor $R \text{-mod} \rightarrow A \text{-mod}$ admits a left adjoint (induction): $\text{Hom}_A(E, W) = \text{Hom}_R(R \otimes_A E, W)$. Let $R \text{-mod}^{sm}$ be the category of left $R$-modules such that any element is annihilated by an open left ideal. Then the restriction functor induces a functor $R \text{-mod}^{sm} \rightarrow A \text{-mod}^{sm}$ admitting a right adjoint (coinduction), sending a smooth $A$-module $E$ to the smooth part of the $R$-module $\text{Hom}_A(R, E)$: $\text{Hom}_A(W, E) = \text{Hom}_R(W, \text{Hom}_A(R, E)^{sm})$. Clearly, if a smooth $A$-module $E$ is injective then $\text{Hom}_A(R, E)^{sm}$ is an injective smooth $R$-module. In particular, if restriction to $A$ of a smooth $R$-module $W$ is injective the adjunction morphism $W \rightarrow \text{Hom}_A(R, W)^{sm}$ gives an embedding into an injective smooth $R$-module.

Our only examples of such topological rings will be skew group rings $A(G)$, cf. 8 with the base of open left ideals $I_U$ indexed by a base $B$ of open subgroups $U$ of $G$, where $I_U$ is generated by elements $u - 1$ for all $u \in U$.

**Lemma 4.1.** Let $G \subseteq \mathcal{G}$ be a group of $\mathcal{G}$-type and $J \subseteq \Psi$ be a substructure of finite length. Let $R$ be an associative unital ring endowed with the trivial $G$-action and $E$ be a left $R$-module considered as a trivial $G_J$-module. Then the left $R[G_J]$-module $W$ coinduced by $E$ (i.e., the smooth part of $R[G_J](R[G], E) = \text{Maps}_{G_J}(G, E))$ is isomorphic to $\bigoplus_{\Lambda \subseteq J} E \otimes \mathbb{Z}\{\Lambda \hookrightarrow \Psi\}$.

**Proof.** As $E$ is a trivial $G_J$-module, the $G_J$-module $W = \varinjlim_{I \subseteq \Psi} \text{Maps}_{G_J}(G/G_I, E)$, where $I$ runs over the finite substructures of $\Psi$. Splitting $G/G_I$ into the disjoint union of the $G_J$-orbits $B_{\Lambda, \tau, I} := \{\sigma : I \hookrightarrow \Psi | \sigma(I) \cap J = \Lambda, \sigma \tau = id_{\Lambda}\}$ over all $\Lambda \subseteq J$ and $\tau : \Lambda \hookrightarrow I$, we get a canonical isomorphism
\[\text{Maps}_{G_J}(G/G_I, E) = \prod_{\Lambda \subseteq J} \prod_{\tau : \Lambda \hookrightarrow I} \text{Maps}_{G_J}(B_{\Lambda, \tau, I}, E) = \prod_{\Lambda \subseteq J} E = \bigoplus_{\Lambda \subseteq J} E \otimes \mathbb{Z}\{\Lambda \hookrightarrow \Psi\},\]
where to a collection $(a_{\tau : \Lambda \hookrightarrow I})$ (consisting of elements of $E$ parametrized by substructures $\Lambda \subseteq J$ and their embeddings $\tau : \Lambda \hookrightarrow I$) corresponding to $I$ we associate the function $G \ni g \mapsto a_{\tau(g)} \in E$, where $\tau(g) : (g(I) \cap J \hookrightarrow I)$ is the restriction of $g^{-1}$. The $G$-action respects this decomposition: $g : \text{Maps}_{G_J}(B_{\Lambda, \tau, I}, E) \rightarrow \text{Maps}_{G_J}(B_{\Lambda, \tau, g(I)}, E), f \mapsto [\sigma \mapsto f(\sigma g)]$. So finally,
\[W = \varinjlim_{I} \text{Maps}_{G_J}(G/G_I, E) = \bigoplus_{\Lambda \subseteq J} E \otimes \mathbb{Z}\{\Lambda \hookrightarrow \Psi\}.\]

**Lemma 4.2.** Let $G$ be a group of $\mathcal{G}$-type (p.4) and $k$ be a field of characteristic zero. Then any smooth simple $k[G]$-module $W$ can be embedded into $V_T^\Psi \otimes k$ for some $T$. 6
Proof. Any smooth simple $k[G]$-module is a quotient of $M_T^\Psi \otimes k$ for some $T$, in particular, it is a subquotient of $M_T^\Psi \otimes k$ for some $T$. If $W$ is a subquotient of $M_T^\Psi \otimes k$ for a minimal $T$ is minimal then $W$ is, in fact, embedded into $V_T^\Psi \otimes k \subseteq M_T^\Psi \otimes k$: otherwise $W$ is a subquotient of $M_{T-\{t\}} \otimes k$ for some $t \in T$.

Corollary 4.3. Let $G \subseteq S_\Psi$ be a group of $S$-type and $k$ be a field of characteristic zero. Then the indecomposable injective smooth representation of $G$ over $k$ are precisely the direct summands of $k[\{\Lambda \to \Psi\}]$ for all finite $\Lambda$.

Proof. By Lemma 3.4 this Lemma any smooth $k[G]$-module contains a simple submodule. By Lemma 1.1 this Lemma any smooth simple $k[G]$-module is embedded into $V_T^\Psi \otimes k \subseteq M_T^\Psi \otimes k$ for some $T$.

By Lemma 2.1 the trivial representation $k$ of $G$ is injective for any $J$. By adjunction, the coinduced representation of $G$ is injective as well. By Lemma 4.1 this implies that the representation $M_T^\Psi \otimes k$ is injective, so finally, any injective indecomposable representation is a direct summand of an appropriate $M_T^\Psi \otimes k$.

5. Smooth representations of infinite symmetric groups

Let $G \subseteq S_\Psi$ be a permutation group. For a subset $S \subseteq \Psi$ set $S := \Psi^G_S$. We say that (i) a subset $S \subseteq \Psi$ is $G$-closed if $S := S$, (ii) elements $x_1, \ldots, x_n \in \Psi$ are $G$-independent if \{\{x_1, \ldots, x_n\} \neq \{y_1, \ldots, y_{n-1}\} \text{ for any } y_1, \ldots, y_{n-1} \in \Psi\}. For each integer $n \geq 0$ set $M_n^\Psi := \mathbb{Z}[\{x_1, \ldots, x_n\} \in \Psi^n | x_1, \ldots, x_n \text{ are } G\text{-independent}]$, so $M_n^\Psi \subseteq (M_1^\Psi)^{\otimes n}$.

If $G$ is transitive on the set of $G$-independent $n$-tuples then $\text{Hom}_G(M_n^\Psi, M_k^\Psi)$ is a free abelian group with the basis consisting of $G$-independent $n$-tuples in $G$-independent elements $x_1, \ldots, x_n \in \Psi$ (empty if and only if $n > m$).

For any infinite $S \subseteq \mathbb{N}$, \{\{M_n^\Psi\}_{n \in S}\} is a system of generators of the category of smooth $G$-modules. As shown in Corollary 4.3. for some groups $G$ the collection \{\{M_n^\Psi\}_{n \in \mathbb{N}}\} is a system of injective co-generators of the category of smooth $G$-modules, cf. Corollary 4.3.

Let $V_n^\Psi$ be the common kernel of all $G$-morphisms $M_n^\Psi \to M_{n-1}^\Psi$. (It suffices to consider only morphisms $M_n^\Psi \to M_{n-1}^\Psi$ given by omitting one of $n$ coordinates.) Then $V_n^\Psi = M_n^\Psi \cap (V_{n-1}^\Psi)^{\otimes n} \subseteq (M_1^\Psi)^{\otimes n}$. As the group $G_{n-1}$ acts (freely) on the basis and the orbits correspond to subobjects of $T$ isomorphic to $T'$, the common kernel of all morphisms of $G$-modules $M_n^\Psi \to M_{n-1}^\Psi$ coincides with $V_n^\Psi$.

Theorem 5.1. Let $G$ be a group of $S$-type ($p[?]$) and $k$ be a field of characteristic zero. Then any smooth simple $k[G]$-module $W$ is isomorphic to a direct summand of $V_T^\Psi \otimes k$ for some $T$. The representation $V_T^\Psi \otimes k$ of $G \times \text{Aut}(T)$ is a direct sum of representations Specht($V_\lambda^\Psi \otimes k$) for all $\lambda$, where Specht($V^\Psi_\lambda$) runs over all representatives of isomorphism classes of irreducible representations of $G$ over $\mathbb{Q}$ and Specht($V^\Psi_\lambda$) := $\text{Hom}_{\text{Aut}(T)}(\text{Specht}^\Psi_\lambda, V^\Psi_T)$ are pairwise non-isomorphic and absolutely irreducible.

Proof. The module $W$ can be embedded into $M_T^\Psi \otimes k$ for some minimal $T$. Then $W$ is embedded into $V_T^\Psi \otimes k$.

Let us show that $V_T^\Psi \otimes k$ is semisimple. Otherwise, it contains an indecomposable $W' \supset W$ with simple $W'/W$. There is an embedding $\iota : W'/W \to M_n \otimes k$. As $M_n \otimes k$ is injective, $\iota$ extends to $M_T^\Psi \otimes k/W \to M_n \otimes k$, so gives rise to a non-zero morphism $M_T^\Psi \otimes k \to M_n \otimes k$, so $n \leq \text{length}(T)$. If $n < \text{length}(T)$ then $\varphi|_{V_T^\Psi \otimes k} = 0$, so $\varphi|_{W'} = 0$, which is a contradiction. Thus, $n = \text{length}(T)$, which means that ker $\varphi$ splits as a direct summand: $M_T^\Psi \otimes k = \text{ker } \varphi \oplus M'$, and therefore, assuming that $\varphi|_{W'} \neq 0$, one has $W' \subseteq W \oplus M'$. Then $W' = W \oplus \text{pr}_{M'}(W')$, contradicting to the indecomposability.

It is clear from the identity $\text{End}_{k[G]}(V_T^\Psi \otimes k) = k[\text{Aut}(T)]$ that $V_T^\Psi \otimes k$ is multiplicity-free.

\footnote{In the case $q = 1$ of the plain sets the $\text{(Specht)} \mathbb{Q}[S_T]$-modules Specht($V^\Psi_\lambda$) correspond to the partitions of $\#T$ and are absolutely irreducible.}
Lemma 5.2. Let $k$ be a field of characteristic zero. Then the representation $V_T^\Psi \otimes k$ of $G$ admits no non-zero morphisms to subquotients of $M_n^\Psi \otimes k$.

Proof. Assuming the contrary, for any finite $J$ with $\#J > \#T$ there is a non-zero morphism of $k[\text{Aut}(J)]$-modules $V_J^I \otimes k \to V_{n-1}^I \otimes k$, contradicting definition of $V_T^\Psi$. \hfill $\square$

The following results should be well-known to the experts.

Theorem 5.3. In notation of Theorem [5.7]

1. the representation $S^\lambda_k \otimes k$ of $G$ is an injective hull of $\text{Specht}_k^\lambda \otimes k$ (in particular, it is indecomposable);
2. any indecomposable injective smooth $k[G]$-module is isomorphic to $S^\lambda_k \otimes k$ for some $\lambda$;
3. the minimal essential submodule of any module $M$ of finite length coincides with its maximal semisimple submodule (the socle), which is $V_n^\Psi \otimes k$ in the case $M = M_n^\Psi \otimes k$.

6. Skew group rings and semilinear representations

Let $A$ be a (unital) associative ring, $G$ be a semigroup acting on $A$, i.e., a (unital) semigroup homomorphism $\rho : G \to \text{End}_{\text{ring}}(A)$ is given. Denote by $A(G)_\rho = A(G)$ the unital associative subring in $\text{End}_Z(A \otimes \mathbb{Z}[G])$ generated by the natural left action of $A$ and the diagonal left action of $G$ on $A \otimes \mathbb{Z}[G]$. In other words, $A(G)$ is the ring of $A$-valued measures on $G$ with finite support. This is a central $k$-algebra, where $k := A^a(G)$ is the fixed ring.

More explicitly, the elements of $A(G)$ are the finite formal sums $\sum_{i=1}^N a_i[g_i]$ for all integer $N \geq 0$, $a_i \in A$, $g_i \in G$. Addition is defined obviously; multiplication is a unique distributive one such that $(a[g])(b[h]) = ab^\rho(a)[gh]$, where we write $a^h$ for the result of applying of $h \in \text{End}_{\text{ring}}(A)$ to $a \in A$.

An additive action of $G$ on an $A$-module $V$ is called semilinear if $g(a \cdot v) = a^g \cdot gv$ for any $g \in G$, $v \in V$ and $a \in A$. Then an $A$-module endowed with an additive semilinear $G$-action is the same as an $A(G)$-module.

The principal example of $A$ will be a field.

For a field $K$ endowed with a $G$-action, a $K$-semilinear representation of $G$ is a left $K[G]$-module. We say that a $K$-semilinear representation of $G$ is non-degenerate if the action of each element of $G$ is injective. We omit the $G$-action on $K$ from notation and denote by $k := K^G$ the fixed field.

The non-degenerate $K$-semilinear representations of $G$ form an abelian tensor $k$-linear category.

Lemma 6.1. Let $A$ be a division ring endowed with a $G$-action, $V$ be a $A(G)$-module and $\chi : G \to \text{End}_{A}(V)$ be a non-degenerate $G$-action on the $A$-module $V$ (e.g., $\chi \equiv \text{id}_V$).

Set $V_\chi := \{w \in V \mid \sigma w = \chi(\sigma)w \text{ for all } \sigma \in G\}$ (e.g., $V_{\text{id}_V} = V^G$).

Then $V_\chi$ is a $A^G$-module and the natural map $A \otimes_{A^G} V_\chi \to V$ is injective.

Proof. This is well-known: Suppose that elements $w_1, \ldots, w_m \in V_\chi$ are $A^G$-linearly independent, i.e., $w_i \not\in \sum_{j \neq i} A w_j \subset V$ for all $i$. Suppose that $w = \sum_{j} \lambda_j w_j \in V_\chi$ for some $\lambda_j \in A$. Then $\sigma w - \chi(\sigma)w = \sum_{j} (\lambda_j^\sigma - \lambda_j)\chi(\sigma)w_j = 0$, and therefore, $\lambda_j^\sigma = \lambda_j$ for any $j$, i.e., $\lambda_j \in A^G$. \hfill $\square$

A $K$-semilinear representation $V$ of $G$ is called trivial, if the natural injective map $V^G \otimes_k K \to V$ of Lemma 6.1 is bijective, i.e., if $V$ is isomorphic to a direct sum of several copies of $K$ with $G$-action via $\rho$.

There is a bijection between $k$-lattices $U$ in a $K$-vector space $V$ and the trivial $K$-semilinear $G$-actions on $V$:

\{structures on $V$ of $K$-semilinear representation isomorphic to a direct sum of copies of $K$\}.

The set of isomorphism classes of non-degenerate $K$-semilinear $G$-actions on a $K$-vector space $V := U \otimes_k K$ is canonically identified with the set $H^1(G, GL_K(V))$. Namely, there is a unique $K$-semilinear $G$-action $\xi$ on $V$ identical on the $k$-lattice $U$. This gives rise to the $K$-semilinear
$G$-action on $\text{End}_K(V)$ by $f^\tau := \xi(\tau)f\xi(\tau)^{-1} \in \text{End}_K(V)$ if $f \in \text{End}_K(V)$, so the matrix of $f^\tau$ in the basis $\tau(b)$ is the result of applying $\tau$ to the matrix of $f$ in a basis $b$ of $U$. For each element $\sigma \in G$ there exists a unique element $f_\sigma \in \text{GL}_K(V)$ such that $\sigma|_U = f_\sigma|_U$, so $\sigma v = f_\sigma \xi(\sigma)v$. This implies $\tau \sigma v = f_{\tau} \xi(\tau) f_\sigma \xi(\sigma)v = f_{\tau} f_\sigma \xi(\tau \sigma)v$ for all $v \in V$, so $f_{\tau \sigma} = f_{\tau} f_\sigma$ for all $\tau, \sigma \in G$.

Now let the semigroup $G$ be totally disconnected and the homomorphism $\rho$ be continuous. (We endow any set $H \subseteq \text{Maps}(\Psi_1, \Psi_2)$ of mappings between sets $\Psi_1$ and $\Psi_2$ with topology, where a base $\{U_\alpha\}_\alpha$ of open subsets of $H$ is indexed by the mappings $\alpha$ of finite subsets $\alpha \subset \Psi_1$ to $\Psi_2$ and $U_\alpha$ is the set of all elements of $H$ with restriction $\alpha$ to $\alpha$.)

Denote by $\mathcal{C}_K(G)$ the category of smooth $K$-semilinear representations of $G$.

The category $\mathcal{C}_K(G)$ is $K$-linear abelian. Let $F|k$ be a field extension such that $k$ is algebraically closed in $F$. Let $K'$ be the field of fractions of $K \otimes_k F$. Then $G$ acts naturally on $K'$ and the functor $K' \otimes K - : \mathcal{C}_K(G) \to \mathcal{C}_{K'}(G)$ is left adjoint to the forgetful functor $\text{Res}_K^K' : \text{Hom}_{\mathcal{C}_{K'}(G)}(K' \otimes_K V, V') = \text{Hom}_{\mathcal{C}_K(G)}(K' \otimes_K V, \text{Res}_K^K' V')$.

**Lemma 6.2.** Let $K$ be a field, $G$ be a group acting faithfully on the field $K$, i.e., an injective group homomorphism $G \to \text{Aut}_{\text{field}}(K)$ is given. Let $B$ be such a system of open subgroups of $G$ that any open subgroup contains a subgroup conjugated, for some $H \in B$, to an open subgroup of finite index in $H$. Then the objects $K[G/H]$ for all $H \in B$ form a system of generators of the category $\mathcal{C}_K(G)$.

**Proof.** Let $V$ be a smooth semilinear representation of $G$. Then the stabilizer of any vector $v$ is open, i.e., the stabilizer of some vector $v'$ in the $G$-orbit of $v$ admits a subgroup commensurable with some $H \in B$. The $K$-linear envelope of the (finite) $H$-orbit of $v'$ is a smooth $K$-semilinear representation of $H$, so it is trivial, i.e., $v'$ belongs to the $K$-linear envelope of the $K^H$-vector subspace fixed by $H$. As a consequence, there is a morphism from a finite cartesian power of $K[G/H]$ to $V$, containing $v'$ (and therefore, containing $v$ as well) in the image. \qed

**Example.** Let $S$ be an infinite set of positive integers. Let $G$ be a group of $\mathfrak{S}$-type (p. 14) acting on $\Psi$. Suppose that the $G$-action on $K$ is faithful. Then the objects $K[(\Psi)_q]$ (defined before Corollary 7.2) for $N \in S$ form a system of generators of the category $\mathcal{C}_K(G)$.

Indeed, (i) the assumptions of Lemma 6.2 hold if $B$ is the set of setwise stabilizers $G_T$ of all subobjects $T \subset \Psi$ with length in $S$, (ii) $K[(\Psi)_q]$ is isomorphic to $K[G/G_T]$ for any $T$ of length $N$. \qed

### 7. Growth estimates

**Lemma 7.1.** Let $G$ be one of the groups from Examples on p. 14. Let $A$ be a commutative integral domain (or a division ring) endowed with a $G$-action. For any $A(G)$-module $M$ we define a function $d_M : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by $d_M(N) := \text{rk}_{A \otimes A_{\Psi N}} M^{G_{\Psi N}}$ for any subobject $\Psi N \subset \Psi$ of length $N$. Suppose that $M \subset A \otimes M^\Psi_n$ for some $n \geq 0$. Then $d_M$ grows as a $q$-polynomial of degree $n$ if $M \neq 0$:

$$\frac{1}{d_n(n)} ([N]_q - [n + m - 1]_q)^n \leq d_{m,n}(N) := \frac{d_m(n)}{d_m(n)d_n(n)} \leq d_M(N) \leq d_n(N) \leq [N]_q^n$$

for some $m \geq 0$, where $[s]_q$ is the cardinality of any object of length $s$ and $d_n(N) := ([N]_q - [0]_q) \cdots ([N]_q - [n - 1]_q)$.

**Proof.** As $A \otimes A_{\Psi N} M^{G_{\Psi N}} \subset A \otimes M^\Psi_n$, one has $d_M(N) \leq \text{rk}_Z(M^\Psi_n) = \#\{\Psi n \to \Psi N \} = d_n(N)$. The lower bound of $d_M(N)$ is given by the number of subobjects in $\Psi N$ with trivial intersection with a fixed finite subobject of $\Psi$ of length $m$. Indeed, any non-zero element $\alpha = \sum a_I(x_{i_1}, \ldots, x_{i_m}) \in M$ is congruent to $\sum_{\sigma \in \text{Aut}(\Psi n)} b_\sigma \tau \sigma$ for some collections $b_\sigma \in A$ not all zero and a monomorphism $\tau : \Psi n \to \Psi$ modulo monomorphisms whose images have a nontrivial intersection with a fixed finite subobject $J \subset \Psi$ (of length $m$). \qed
Let \( q \) be either 1 or a primary integer. For an integer \( s \geq 0 \), we denote by \( \binom{\Psi}{s}_q \) the set of subobjects of \( \Psi \) length \( s \) (i.e., of subsets of order \( s \)) if \( q = 1 \) and of \( s \)-dimensional subspaces if \( q > 1 \).

**Corollary 7.2.** Let \( G \) be a group of \( \mathfrak{S} \)-type (p4). Let a ring \( A \) endowed with a \( G \)-action be as in Lemma 7.1. Let \( \Xi \) be a finite subset in \( \text{Hom}_{A(G)}(A \otimes M^\Psi_n, A \otimes M^\Psi_m) \) for some integer \( n > m \geq 0 \). Then

1. any non-zero \( A(G) \)-submodule of \( A[\binom{\Psi}{m}_q] \) is essential;
2. there are no nonzero isomorphic \( A(G) \)-submodules in \( A \otimes M^\Psi_n \) and \( A \otimes M^\Psi_m \);
3. the common kernel \( V_\Xi \) of all elements of \( \Xi \) is an essential \( A(G) \)-submodule in \( A \otimes M^\Psi_n \).

**Proof.** (1) follows from the lower growth estimate of Lemma 7.1
(2) follows immediately from Lemma 7.1
(3) Suppose that there exists a nonzero submodule \( M \subseteq A \otimes M^\Psi_n \) such that \( M \cap V_\Xi = 0 \). Then restriction of some \( \xi \in \Xi \) to \( M \) is nonzero. If \( \xi|_M \) is not injective, replacing \( M \) with \( \ker \xi \cap M \), we can assume that \( \xi|_M = 0 \). In other words, we can assume that restriction to \( M \) of any \( \xi \in \Xi \) is either injective or zero. In particular, restriction to \( M \) of some \( \xi \in \Xi \) is injective, i.e. \( \xi \) embeds \( M \) into \( A \otimes M^\Psi_n \), contradicting to (2). \( \square \)

8. Noetherian properties of smooth semilinear representations of \( \mathfrak{S}_\Psi \)

**Lemma 8.1.** Let \( G \) be a group acting on a field \( K \). Let \( U \) be a subgroup of \( G \) such that \( \{ g \in G \mid gU \subseteq U \} = U \) and \( \{ U : U \cap (gUg^{-1}) \} = \infty \), unless \( g \in U \). Then \( \text{End}_{K(G)}(K[G/U]) = K^U \) is a field, so \( K[G/U] \) is indecomposable.

**Proof.** Indeed, \( \text{End}_{K(G)}(K[G/U]) = (K[G/U])^U = K^U \oplus (K[(G \setminus U)/U])^U \). As \( U(gUg^{-1}) \) consists of \( [U : U \cap (gUg^{-1})] \) classes in \( G/(gUg^{-1}) \), we see that \( (K[(G \setminus U)/U])^U = 0 \). \( \square \)

**Examples.** 1. Let \( \Psi \) be an infinite set, possibly endowed with a structure of a projective space. Let \( G \) be the group of automorphisms of \( \Psi \), respecting the structure, if any. Let \( J \) be a ‘finite substructure’ in \( \Psi \), i.e., a finite subset or a finite-dimensional subspace. Let \( U \) be the stabilizer of \( J \) in \( G \). Then \( G/U \) is identified with the set of all substructures in \( \Psi \) of the same length as \( J \).

2. By Lemma 8.1, \( K[G/U] \) is indecomposable in the following examples:

1. \( G \) is the group of projective automorphisms of an infinite projective space \( \Psi \) (i.e., either \( \Psi \) is infinite-dimensional, or \( \Psi \) is defined over an infinite field), \( U \) is the setwise stabilizer in \( G \) of a finite-dimensional subspace \( J \subseteq \Psi \). Then \( G/U \) is identified with the Grassmannian of all subspaces in \( \Psi \) of the same dimension as \( J \).

2. \( G \) is the group of permutations of an infinite set \( \Psi \), \( U \) is the stabilizer in \( G \) of a finite subset \( J \subseteq \Psi \). Then \( G/U \) is identified with the set \( \binom{\Psi}{J} \) of all subsets in \( \Psi \) of order \( \#J \).

3. \( G \) is the automorphism group of an algebraically closed extension \( F \) of a field \( k \), \( U \) is the stabilizer in \( G \) of an algebraically closed subextension \( L/k \) of finite transcendence degree. Then \( G/U \) is identified with the set of all subextensions in \( F/k \) isomorphic to \( L/k \).

**Lemma 8.2.** Let \( G \) be a group acting on a field \( K \). Let \( U \subseteq G \) be a subgroup such that an element \( g \in G \) acts identically on \( K^U \) if and only if \( g \in U \). Then there are no irreducible \( K \)-semilinear subrepresentations in \( K[G/U] \), unless \( U \) is of finite index in \( G \). If \( G \) acts faithfully on \( K \) and \( U \) is of finite index in \( G \) then \( K[G/U] \) is trivial.

**Example.** Let \( G \) be a group acting on a field \( K \); \( U \subseteq G \) be a maximal proper subgroup. Assume that \( K^U \neq K^G \). Then we are under assumptions of Lemma 8.2.

More particularly, if \( G = \mathfrak{S}_\Psi \), \( U = \mathfrak{S}_{\Psi,I} \) (so \( G/U \cong \binom{\Psi}{I} \)) and \( K^G \) is not trivial, then there are no irreducible \( K \)-semilinear subrepresentations in \( K[\binom{\Psi}{I}] \).

**Proof.** By Artin’s independence of characters theorem (applied to the one-dimensional characters \( g : (K^U)^\times \to K^\times \)), the morphism \( K[G/U] \to \prod_{(K^U)^\times} K \), given by \( \sum \beta g \mapsto (\sum \beta g f^{|I|})_{f \in (K^U)^\times} \),
is injective. Then, for any non-zero element \( \alpha \in K[G/U] \), there exists an element \( Q \in K^U \) such that the morphism \( K[G/U] \to K \), given by \( \sum b_g g \mapsto \sum b_g Q^g \), does not vanish on \( \alpha \). Then \( \alpha \) generates a subrepresentation \( V \) surjecting onto \( K \). If \( V \) is irreducible then it is isomorphic to \( K \), so \( V^G \neq 0 \). In particular, \( K[G/U][G] \neq 0 \), which can happen only if index of \( U \) in \( G \) is finite.

If \( [G : U] < \infty \) set \( U' = \cap_{g \in G/U} gUg^{-1} \). This is a normal subgroup of finite index. Then \( K[G/U] = K \otimes K^U K^{U'}[G/U'] \) and \( K^{U'}[G/U'] \cong (K^{U'})^{|G:U'|} \) is trivial by Hilbert’s theorem 90, so we get \( K[G/U] \cong K^{|G:U'|} \).

**Lemma 8.3.** Let \( G \) be a permutation group, \( A \) be an associative ring endowed with a smooth \( G \)-action and \( U \subseteq G \) be an open subgroup. Then any smooth \( A(G) \)-module is also smooth when considered as an \( A(U) \)-module. Suppose that the set \( U \setminus G/U' \) is finite for any open subgroup \( U' \subseteq G \). Then the restriction of any smooth finite \( A(G) \)-module to \( A(U) \) is a finite \( A(U) \)-module.

**Proof.** The \( A(G) \)-modules \( A[G/U'] \) for all open subgroups \( U' \) of \( G \) form a generating family of the category of smooth \( A(G) \)-modules. It suffices, thus, to check that \( A[G/U'] \) is a finite \( A(U) \)-module for all open subgroups \( U' \) of \( G \). Choose representatives \( \alpha_i \in G/U' \) of the elements of \( U \setminus G/U' \). Then \( G/U' = \coprod_i U \alpha_i \), so \( A[G/U'] \cong \bigoplus_i A[U/(U \cap \alpha_i U' \alpha_i^{-1})] \) is a finite \( A(U) \)-module. 

**Examples.** 1. The finiteness assumption of Lemma 8.3 is valid for any open subgroup \( G \) of a group of \( \mathfrak{G} \)-type \( (p, 1) \), as well as for any compact group \( G \).

2. The restriction functor splits the indecomposable generators into finite direct sums of indecomposable generators via canonical isomorphisms of \( A(G_j) \)-modules \( A[\Psi_j ] \) \( \cong \bigoplus_{j \leq j} M_{\lambda} \), where \( M_{\lambda} \) is the free \( \Lambda \)-module on the set of all subobjects of \( \Psi \) of length \( t \) and meeting \( \Lambda \) along \( \Lambda \).

**Lemma 8.4.** Let \( s \geq 0 \) be an integer and \( M \) be a quotient of the \( K[\mathfrak{G}(\Psi)] \)-module \( K[\Psi_j ] \) by a non-zero submodule \( M_0 \). Then there is a finite subset \( I \subset \Psi \) such that the \( K[\mathfrak{G}(\Psi)_I] \)-module \( M \) is isomorphic to a quotient of \( \bigoplus_{j=1}^{\mathfrak{G}(\Psi)_I} K[\Psi_j ]_{I} \).

**Proof.** Let \( \alpha = \sum_{S \subseteq J} a_S \delta[S] \in M_0 \) be a non-zero element for a finite set \( J \subset \Psi \). Fix some \( S \subseteq J \) with \( a_S \neq 0 \). Set \( I := J \setminus S \). Then the morphism of \( K[\mathfrak{G}(\Psi)_I] \)-modules \( K[\mathfrak{G}(\Psi)_I] \delta \bigoplus_{\delta \neq \lambda \subseteq J} K[\Psi_j ]_{I} \) \( \rightarrow \) \( K[\Psi_j ]_{I} \), given (i) by the inclusion on the first summand and (ii) by \( [T] \) \( \rightarrow \) \( [T \cup \lambda] \) on the summand corresponding to \( \lambda \), is surjective.

**Proposition 8.5.** Let \( A \) be an associative left noetherian ring endowed with an arbitrary \( \mathfrak{G}(\Psi) \)-action.\footnote{Examples of \( A \): a division ring endowed an \( \mathfrak{G}(\Psi) \)-action; localization of \( \mathbb{Z}[x \mid x \in \Psi \] at all non-constant indecomposable polynomials.} Then the left \( \mathfrak{G}(\Psi)_I \)-module \( A[\Psi^s ] \) is noetherian for any integer \( s \geq 0 \) and any open subgroup \( U \subseteq \mathfrak{G}(\Psi) \).

If the \( \mathfrak{G}(\Psi)_I \)-action on \( A \) is smooth then any smooth finite \( A[\mathfrak{G}(\Psi)_I] \)-module is noetherian.

**Proof.** We need to show that every \( A(\Psi^s ) \)-submodule \( M \subset A[\Psi^s ] \) is finite for all \( U = \mathfrak{G}(\Psi)_I \) with finite \( S \subset \Psi \). We proceed by induction on \( s \geq 0 \), the case \( s = 0 \) being trivial. Assume that \( s > 0 \) and the \( A(U) \)-modules \( A[\Psi^s ] \) are noetherian for all \( j < s \). Fix a subset \( I_0 \subset \Psi \setminus S \) of order \( s \).

Let \( M_0 \) be the image of \( A(\mathfrak{G}(\Psi)_I) \otimes A[\Psi^s ] \) under all \( s \)-tuples containing elements other than those of \( I_0 \). As \( A \) is noetherian and \( I_0^s \) is finite, the \( A(\Psi^{s-1}) \)-module \( M_0 \) is finite. Let the \( A(U) \)-module \( M_0 \) be generated by the images of some elements \( \alpha_1, \ldots, \alpha_N \in M \subset A[\Psi^s ] \). Then \( \alpha_1, \ldots, \alpha_N \) belong to the \( \mathfrak{G}(\Psi)_I \)-module \( A[\Psi^s ] \) for some finite subset \( I \subset \Psi \).

Let \( J \subset I \cup S \) be the complement to \( I_0 \). For each pair \( \gamma = (j, x) \), where \( 1 \leq j \leq s \) and \( x \in J \), set \( \Psi^s_{\gamma} := \{(x_1, \ldots, x_s) \in \Psi^s \mid x_j = x \} \). This is a smooth \( \mathfrak{G}(\Psi)_J \)-set. Then the set \( \Psi^s_{\gamma} \) is the union of the \( \mathfrak{G}(\Psi)_J \)-orbit \( (\Psi^s \setminus J^s)_{\gamma} \) consisting of \( s \)-tuples of pairwise distinct elements of \( \Psi \setminus J \) and of a finite union of \( \mathfrak{G}(\Psi)_J \)-orbits embeddable into \( \Psi^s_{\gamma} \). The \( \Delta_{ij} := \{(x_1, \ldots, x_s) \in \Psi^s \mid x_i = x_j \}\) are diagonals.

As (i) \( M_0 \subset \sum_{j=1}^N A\alpha_j + \sum_{\gamma \in \{1, \ldots, s\} \times J} A[\Psi^s_{\gamma}] \), (ii) \( g(M_0) \subset A[\Psi^s ] \) is determined by \( g(I_0) \), (iii) for any \( g \in U \) such that \( g(I_0) \cap J = \emptyset \) there exists \( g' \in U_j \) with \( g(I_0) = g'(I_0) \) \( (U_j \) acts transitively on \( \Psi^s_{\gamma} \)).
on the $s$-configurations in $\Psi \setminus J$, one has inclusions of $A(U_J)$-modules

$$\sum_{j=1}^{N} A(U) \alpha_j \subseteq M \subseteq \sum_{g \in U} g(M_0) \subseteq \sum_{g \in U_J} g(M_0) + \sum_{\gamma \in \{1, \ldots, s\} \times J} A[\Psi^s_{\gamma}].$$

On the other hand, $g(M_0) \subseteq g(\sum_{j=1}^{N} A\alpha_j) + \sum_{\gamma \in \{1, \ldots, s\} \times J} A[\Psi^s_{\gamma}]$ for $g \in U_J$, and therefore, the $A(U_J)$-module $M/\sum_{j=1}^{N} A(U)\alpha_j$ becomes a subquotient of the noetherian, by the induction assumption, $A(U_J)$-module $\sum_{\gamma \in \{1, \ldots, s\} \times J} A[\Psi^s_{\gamma}]$, so the $A(U_J)$-module $M/\sum_{j=1}^{N} A(U)\alpha_j$ is finite, and thus, $M$ is finite as well. \hfill \square

**Corollary 8.6.** Let $A$ be a left noetherian associative ring endowed with a smooth $S$-action. Then (i) any smooth finite left $A(\Psi_q)$-module $W$ is noetherian if considered as a left $A(U)$-module for any open subgroup $U \subseteq S$; (ii) the category of smooth $A(\Psi_q)$-modules is locally noetherian, i.e., any smooth finite left $A(\Psi_q)$-module is noetherian.

**Proof.** The module $W$ is a quotient of a finite direct sum of $A[\Psi^m]$ for some integer $m \geq 0$, while $A[\Psi^m]$ are noetherian by Proposition 8.5 \hfill \square

The next result generalizes a description of representations $k[(\Psi)]$ of $S$ from [2].

**Lemma 8.7.** In notation preceeding Corollary 7.2, let $q \geq 1$ be either 1 or a primary number, $S$ be a finite or infinite set (if $q = 1$) or an $F_q$-vector space, $G$ be the automorphism group of $S$. Let the morphism of $G$-modules $\partial^S_q : Z[(\ell)] \to Z[(S_{\ell})_q]$ be defined by $[T] \mapsto \sum_{T' \subset T}[T']$. Then $\partial^S_q \otimes Q$ is surjective if length of $S$ is $> 2s$ \footnote{If length of $S$ is $\leq 2s$ then $\partial^S_q \otimes Q$ is not surjective, since then $\dim_Q Q[(S_{s+1})_q] = \#(S_{s+1})_q < \#(S)_q = \dim_Q Q[(S)_q]$.}

the representation $Q[(S)_q]$ of $G$ is of length $\min(s, \text{length}(S) - s) + 1$ and the irreducible quotients of its composition series are absolutely irreducible and pairwise non-isomorphic.

**Proof.** Fix a subobject $T$ of $S$ of length $s$. Then any element $\varphi \in \text{Hom}_G(Z[(S)_q], Z[(\ell)_q])$ is determined by the image of $T$. As $\varphi(T)$ is fixed by the stabilizer of $T$ and the latter acts transitively on the set of all subobjects of $S$ of length $t$ and with a given length of intersection with $T$, one has $\varphi([T]) = \sum_{i=0}^{\min(t, s)} a_i [T']$ for a collection of coefficients $a_i \in Q$ if $S$ is finite and $\varphi([T]) = a \sum [T' \subset T][T']$ for a coefficient $a \in Q$ if $S$ is infinite.

Assume first that $S$ is finite. Comparing the cases $t = s$ and $t = s - 1$ and arguing by induction, we see that (i) $Q[(S)_q]$ is a direct sum of $s + 1$ pairwise non-isomorphic absolutely irreducible subrepresentation, (ii) $Q[(S)_q]$ embeds into $Q[(S_{s+1})_q]$.

Assume that length of $S$ is $> 2s$. The morphisms $\partial^S_q \otimes Q, \partial^S_q \otimes Q, \ldots, \partial^S_q \otimes Q$ are non-injective, but they cannot drop length of modules by more than 1, since their composition $\partial^S_q \otimes Q$ is non-zero. This means that their kernels are irreducible, and thus, they are surjective.

Now assume that $S$ is infinite. Clearly, the surjectivity of $\partial^S_q \otimes Q$ and the irreducibility of its kernel follows from the case of sufficiently large finite $S$. By Lemma 5.1 length of $Q[(S)_q]$ does not exceed $s + 1$, so it is precisely $s + 1$.

There are no other irreducible subrepresentation if $S$ is infinite, since $Q[(S)_q]$ is injective and indecomposable, cf. [1] Proposition 6.9. \hfill \square

**Corollary 8.8.** In notation of Lemma 8.7 let $A$ be a torsion-free commutative integral domain (or a division ring) endowed with the trivial $G$-action. Then, any $G$-submodule $M \subseteq A[(S)_q]$ with $(M \otimes \text{Frac}(A)) \cap A[(S)_q] = M$ is the kernel of $\partial^S_q \otimes Q \otimes \partial^S_q \otimes Q \otimes \partial^S_q \otimes Q$ for some $i$. In particular, $A[(S)_q]$ is of length $s + 1$ and $\partial^S_q \otimes A$ is surjective if $A$ is a skew field and length of $S$ is $> 2s$. \hfill \square
Corollary 8.9. Let $A$ be an associative ring endowed with a smooth action of a group $G$ of $\mathcal{S}$-type (p.24). Suppose that $A$ is noetherian as a left $A(U)$-module for any open subgroup $U \subseteq G$. Then (i) any sum of smooth injective left $A(G)$-modules is again injective; (ii) any smooth injective left $A(G)$-module is a sum of uniquely determined (upto non-unique isomorphism) collection of indecomposable smooth injective left $A(G)$-modules; (iii) injective hull of a smooth noetherian left $A(G)$-module is a finite sum of indecomposables.

Proof. (i) is [1] Corollary 6.50 by Corollary 8.6 (ii) is [1] Proposition 6.51, again by Corollary 8.6 (iii) is [1] Proposition 6.41.

For any $N \geq M \geq 0$, fix some subobjects $J_0 \subseteq I_0 \subseteq \Psi$ of lengths $M$ and $N$, respectively, and set $S^{(q)}_{M,N} := K^{\Psi/I_0,J_0}$. For any $Q \in S^{(q)}_{M,N}$ and any $J \subseteq I \subseteq \Psi$ of the same lengths, set $Q(J \subseteq I) = \sigma Q$ for any $\sigma \in G$ such that $\sigma(I_0) = I$ and $\sigma(J_0) = J$. Clearly, this is independent of a particular choice of $\sigma$.

If $q = 1$ then $S_{M,N} := S^{(1)}_{M,N}$ can be considered as the (purely transcendental over $k$ of transcendence degree $N$) field of rational functions $Q$ in $N$ variables over $k$ symmetric both in the first $M$ and in the remaining $N - M$ variables. For instance, $S_{N,N}$ is the field of symmetric rational functions in $N$ variables over $k$; $S_{N,N+1}$ is the field of rational functions in the $(N + 1)$-st variable over the field $S_{N,N}$ of symmetric rational functions in the first $N$ variables.

Let from now on $K$ be a purely transcendental field extension of $k$ with a transcendence base of $K|k$ identified with $\Psi$. Then $Q(J \subseteq I)$ is a rational function in elements of $I$ symmetric in elements of $J$ and in elements of $I \setminus J$.

Lemma 8.10. There is a canonical isomorphism of $k$-vector spaces

$$\text{Hom}_{K(\mathcal{S}_\Psi)}(K[\bigl(\Psi \bigr)_q], K[\bigl(\Psi \bigr)_M]) = \begin{cases} S^{(q)}_{M,N}, & \text{if } N \geq M, \\ 0, & \text{if } N < M, \end{cases}$$

for $Q : [T] \mapsto \sum_{J \subseteq T} Q(J \subseteq T)[J]$. Under this isomorphism, the composition $K[\bigl(\Psi \bigr)_N] \xrightarrow{R} K[\bigl(\Psi \bigr)_M] \xrightarrow{Q} K[\bigl(\Psi \bigr)_I]$ is given by $$(Q \circ R)(J_0 \subseteq T) := \sum_{J_0 \subseteq J \subseteq T} Q(J \subseteq T)R(J_0 \subseteq J), \quad \text{where length of } J = M.$$

Remark. $K[\bigl(\Psi \bigr)_N] \cong \prod_{k \geq 1} K^{N_k}$, where $(s_1, \ldots, s_N) \mapsto \prod_{1 \leq i < j \leq N} (s_i - s_j)ds_1 \wedge \cdots \wedge ds_N$ for any $N \geq 0$.

The representation $K[\bigl(\Psi \bigr)_N]$ is highly reducible: any finite-dimensional $k$-vector subspace $\Xi \subseteq S_{N,N}$ determines a surjective morphism $K[\bigl(\Psi \bigr)_N] \rightarrow \text{Hom}_k(\Xi, K)$, $[I] \mapsto [Q \mapsto Q(I)]$.

9. Triviality of finite-dimensional semilinear representation of $\mathcal{S}_\Psi$

The following result is analogous to [5] Proposition 5.4, but there one can take any algebraic group.

Lemma 9.1. Any finite-dimensional $K$-semilinear representation $V$ of $\mathcal{S}_\Psi$ admitting a basis pointwise fixed by an open subgroup is trivial. In particular, any smooth finite-dimensional $K$-semilinear representation of $\mathcal{S}_\Psi$ is trivial.

Proof. Set $G := \mathcal{S}_\Psi$ and let $b \subseteq V$ be a $K$-basis, pointwise fixed by an open subgroup of $G$, so $b \subseteq V_I := V^{G_I}$ for a finite subset $I \subseteq \Psi$. By Lemma 6.1 (with $\chi \equiv 1$), the multiplication maps $V_I \otimes_{K_{I'}} K = (V_I \otimes_{K_{I'}} K_I) \otimes_{K_{I'}} K = V_I \otimes_{K_{I'}} K \rightarrow V_I \otimes_{K_{I'}} K \rightarrow V$ are injective for any subset $I \subseteq \Psi$ containing $I$, where $K_I := K^{G_I}$. The composition is an isomorphism, so $V_I \otimes_{K_{I'}} K_I \rightarrow V_I$ is an isomorphism as well. In particular, $f_a = id_V$ if $\sigma \in G_I$, where $(f_a \in \text{GL}_K(V))$ is the 1-cocycle of the $G$-action in the basis $b$. Clearly, (i) $f_a$ depends only on the class $\sigma[I]$ of $\sigma$ in $G/G_I = \{\text{embeddings of } I \text{ into } \Psi\}$, (ii) $f_a \in \text{GL}_K(\bigl(V_{\psi[I]}\bigr)_{\psi[I]}(I \subseteq \psi[I]))$.

Assume that $I, \sigma(I), \tau(I)$ are disjoint, $X, Y, Z$ are the standard collections of the elementary symmetric functions in $I, \tau(I), \tau(I)$, respectively. Then the cocycle condition $f_{\sigma}f_{\tau} = f_{\tau}f_{\sigma}$ (where $f_{\sigma} \in \text{GL}_K(V_{\psi[I]}\bigl(V_{\psi[I]}\bigr))$ becomes $\Phi(X, Z) = \Phi(X, Y)\Phi(Y, Z)$ and $\Phi(Y, X) = \Phi(X, Y)^{-1}$.
where \( f_{\sigma} = \Phi(X,Z) \), etc. Specializing \( Y \) to any \( \mathbb{Z} \)-point \( Y_0 \), where \( \Phi(X,Y) \) and \( \Phi(Y,Z) \) are regular, we get \( \Phi(X,Z) = \Phi(X,Y_0)\Phi(Y_0,Z) = \Phi(X,Y_0)\Phi(Z,Y_0)^{-1} \). Then \( \Phi(X,Y_0) \) transforms \( b \) to a basis fixed by all \( \sigma \in G \) such that \( \sigma(I) \) does not meet \( I \), i.e. fixed by entire \( G \).

\[ \qed \]

## 10. Structure of \( K[\Psi] \)

**Lemma 10.1.** Let \( K \) be a field endowed with a smooth \( \mathfrak{S}_2 \)-action. For any pair of finite sets \( S,T \) one has \( \text{Ext}^1(M^\Psi_S \otimes K, M^\Psi_T \otimes K) = 0 \) in the category of smooth \( K \)-semilinear representations of \( \mathfrak{S}_2 \). In particular, for any integer \( r \), \( s \geq 0 \) the restriction morphism

\[
\rho_{s,r} : \text{Hom}_{K(\mathfrak{S}_2)}(K[\Psi_s], K[\Psi_r]) \to \text{Hom}_{K(\mathfrak{S}_2)}(V, K[\Psi_r])
\]

is surjective if \( V = K[\Psi]^0 \) or \( s = 1 \) and \( V \subseteq K[\Psi] \) (and injective if \( r > 0 \)).

**Proof.** Consider first the case of \( T = \emptyset \). Let \( 0 \to M^\Psi_S \otimes K \to E \to K \to 0 \) be an extension. Choose a section \( v \in E \) (projecting to \( 1 \) in \( K \)). Then \( v \in E^{\mathfrak{S}_2/I} \) for a finite subset \( I \subseteq \Psi \), and therefore, \( f_\sigma := \sigma v - v \in (M^\Psi_S)^{\mathfrak{S}_2/I} \subseteq M^\Psi_S[I,I] \) for any \( \sigma \in \mathfrak{S}_2 \). Then \( f_\sigma = \sum_{\xi,S} f_{\sigma,S,k} \sigma_\xi a_{\xi,\sigma} \xi[k] \). Let \( \sigma, \tau \in \mathfrak{S}_2 \) be such elements that \( |I \cup \sigma(I)| = 3|I| \). Then \( f_\sigma = \sum_{\xi,S} f_{\sigma,S,k} \sigma_\xi a_{\xi,\tau} \xi[k] \) and \( f_\tau = \sum_{\xi,S} f_{\tau,I,k} \tau_\xi a_{\xi,\sigma} \xi[k] \). From the cocycle condition \( f_{\sigma} - f_\tau = f_\sigma + f_\tau \), we see that \( a_{\sigma,\xi} = 0 \), unless \( \xi(S) \subseteq I \) or \( \xi(S) \subseteq \tau(I) \). Moreover, \( a_{\sigma,\xi} + a_{\sigma,\tau} \xi^{-1} = 0 \) if \( \xi(S) \subseteq \tau(I) \); \( a_{\tau,\xi} = a_{\sigma,\xi} \) if \( \xi(S) \subseteq I \); \( a_{\sigma,\xi} = a_{\tau,\xi}^{-1} \) if \( \xi(S) \subseteq \tau(I) \).

Then the element \( v' := v - \sum_{\xi,S} f_{\sigma,S,k} \sigma_\xi a_{\xi,\sigma} \xi[k] \) is fixed by all elements \( \sigma \in \mathfrak{S}_2 \) such that \( |I \cup \sigma(I)| = 2|I| \). As such \( \sigma \) generate the group \( \mathfrak{S}_2 \), we get \( v' \in E^{\mathfrak{S}_2/I} \), i.e., our extension is split.

Now let \( T \) be arbitrary. Let \( 0 \to M^\Psi_S \otimes K \to E \xrightarrow{p} M^\Psi_T \otimes K \to 0 \) be a smooth extension. We know already that \( 0 \to M^\Psi_S \otimes K \to p^{-1}(K[\Psi]) \xrightarrow{p} K \cdot [\Psi] \to 0 \) is a split extension of \( K(\mathfrak{S}_2,J) \)-modules for any \( \xi : T \to \Psi \), where \( J := \xi(T) \). Then we can choose \( v \in E^{\mathfrak{S}_2/I} \) with \( p(v) = [\Psi] \). Such \( v \) spans in \( E \) a \( K(\mathfrak{S}_2,J) \)-submodule identified by \( p \) with \( M^\Psi_T \otimes K \), i.e., the extension splits. Then the short exact sequence \( 0 \to K(\Psi)^0 \to K[\Psi] \to K \to 0 \) induces the surjection \( \rho_{s,r} \) with the kernel \( \text{Hom}_{K(\mathfrak{S}_2)}(K[\psi],K[\psi]) \), which is \( k \) for \( r = 0 \), and \( 0 \) for \( r > 0 \) (cf. Lemma 8.2).

**Lemma 10.2.**

1. The cokernel of any non-zero morphism \( \varphi : K[\Psi] \to K[\Psi] \) is at most \((s-1)\)-dimensional. In particular, any non-zero submodule of \( K[\Psi] \) is of finite codimension.

2. There are natural bijections between
   a. the \( K(\mathfrak{S}_2,J) \)-modules of codimension \( s \) and the \( s \)-dimensional \( k \)-vector subspaces in \( k(T) \);
   b. the isomorphism classes of \( K(\mathfrak{S}_2,J) \)-modules of codimension \( s \) and the elements of \( k(T)^\times \setminus \text{Gr}(s,k(T)) \).

3. An element \( \sum t q_t \in K[\Psi] \) is a generator if and only if \( \sum t q_t Q(t) \neq 0 \) for any \( Q \in k(T)^\times \).

**Proof.**

1. Fix a subset \( \{b_1, \ldots, b_{s-1}\} \) of \( \Psi \) of order \((s-1)\). Then, for any element \( b \in \Psi \) the set \( \{b, b_1, \ldots, b_{s-1}\} \) is sent by \( \varphi \) to a linear combination of \( b, b_1, \ldots, b_{s-1} \) with non-zero coefficients, and therefore, the image of \( b \) in the cokernel of \( \varphi \) is a linear combination of images of \( b_1, \ldots, b_{s-1} \).

2. By \( \prod \), the quotient \( K[\Psi]/M \) is finite-dimensional for any non-zero \( K(\mathfrak{S}_2,J) \)-submodule \( M \) of \( K[\Psi] \), so it is trivial by Lemma 10.1, and therefore, \( M \) is the common kernel of a finite-dimensional \( k \)-vector subspace of \( \text{Hom}_{K(\mathfrak{S}_2,J)}(K[\Psi],K) \) (identified with the field of rational functions \( k(T) \)); to a \( k \)-vector subspace \( L \) of \( \text{Hom}_{K(\mathfrak{S}_2,J)}(K[\Psi],K) \subseteq k(T) \) we associate the submodule \( \{m \in K[\Psi] : \lambda m = 0 \text{ for all } \lambda \in \mathbb{L}\} \subseteq K[\Psi] \). Clearly, the \( \text{End}_{K(\mathfrak{S}_2,J)}(K[\Psi])^\times \)-action preserves the isomorphism classes (since \( \text{End}_{K(\mathfrak{S}_2,J)}(K[\Psi]) = k(T) \) is a field). On the other hand, \( \text{Hom}_{K(\mathfrak{S}_2,J)}(K[\Psi]/M, K[\Psi]) = 0 \) and (by Lemma 10.1) \( \text{Ext}^1_{K(\mathfrak{S}_2,J)}(K[\Psi]/M, K[\Psi]) = 0 \), so the restriction morphism \( \text{End}_{K(\mathfrak{S}_2,J)}(K[\Psi]) \to \)...
Hom\(_K(\langle S \rangle, M, K[\Psi])\) is an isomorphism, i.e., any morphism between \(K(\langle S \rangle)\)-submodule \(M\) of \(K[\Psi]\) is induced by an endomorphism of \(K[\Psi]\) (identified with an element of \(k(T)\)).

(3) Any \(Q \in k(T)^{\times}\) such that \(\sum q_i Q(t) = 0\) determines a non-zero morphism \(K[\Psi] \to K\) trivial on the submodule generated by \(\alpha := \sum q_i [t]\), so \(\alpha\) is not a generator. If \(\alpha\) generate a proper submodule \(M \subset K[\Psi]\) then the quotient \(K[\Psi]/M\) is finite-dimensional, so by Lemma 9.1 it admits a quotient isomorphic to \(K\). Finally, any morphism \(K[\Psi] \to K\) is given by some \(Q \in k(T)^{\times}\).

\[\tag*{\square}\]

Lemma 10.3. The following conditions on a non-zero rational function \(q(X, Y)\) over \(k\) are equivalent

1. \(q(a, b)[a] + q(b, a)[b]\) is not a generator of \(K[\Psi]\) (in other words, \(q : K(\langle Y_2 \rangle) \to K[\Psi]\) is not surjective),
2. the cokernel of \(q : K(\langle Y_2 \rangle) \to K[\Psi]\) is isomorphic to \(K\),
3. \(q(X, Y) = (X - Y)S(Y)R(X, Y)\) for some \(S\) and a symmetric \(R\),
4. there exists some \(S(X) \neq 0\) such that \(q(a, b)S(a) + q(b, a)S(b) = 0\),
5. there exists some \(S(X)\) such that the sequence \(K(\langle Y_2 \rangle) \xrightarrow{q} K[\Psi] \xrightarrow{S} K\) is exact.

Proof. Let \(A = \{a_0, \ldots, a_s\}\) be a subset of \(\Psi\) of order \(s + 1\). Then \(A \setminus \{a_j\}\) is sent by \(q\) to \(\sum_{i \neq j} q(A \setminus \{a_i, a_j\}; a_i)\{a_i\}\), so the image contains \(s + 1\) elements \(\sum_{i \neq j} q(A \setminus \{a_i, a_j\}; a_i)\{a_i\}\) for \(0 \leq j \leq s\). These three elements span a vector space of dimension 2 or 3. The dimension is 2 if and only if \(D := \det\begin{pmatrix} q(a, b) & q(b, a) & 0 \\ 0 & q(b, c) & q(c, b) \\ q(a, c) & 0 & q(c, a) \end{pmatrix} = q(a, b)q(b, c)q(c, a) + q(b, a)q(c, b)q(a, c)\) vanishes. Let \(q(X, Y) = P(X)S(Y)\prod q\Phi_i(X, Y)^{m_i}\) be a decomposition into a product of irreducibles. \(0 \leq j \leq s\).

Then \(D = R(a)R(b)R(c)/\prod q\Phi_i(a, b)\Phi_i(b, c)\Phi_i(c, a)^{m_i} + \prod q\Phi_i(a, b)\Phi_i(b, c)\Phi_i(c, a)^{m_i}\), where \(R = PS\), vanishes if and only if \(\prod q\Phi_i(a, b)\Phi_i(b, c)\Phi_i(c, a)^{m_i}\) is a skew symmetric function, i.e., \(\prod q\Phi_i(X, Y)^{m_i}\) is a skew symmetric function. \(\tag*{\square}\)

11. Cyclicity of finitely generated smooth modules

The following result extends the existence of a cyclic vector in any finite-degree non-degenerate semilinear representation of an endomorphism of infinite order, cf., e.g., \(6\) Lemma 2.1.

Lemma 11.1. Let \(G\) be a permutation group, \(K\) be a field endowed with a smooth \(G\)-action such that any open subgroup of \(G\) contains an element inducing on \(K\) an automorphism of infinite order. Then any smooth finite \(K(G)\)-module \(W\) admits a cyclic vector.

Proof. A finite system \(S\) of generators of the \(K(G)\)-module \(W\) is fixed by an open subgroup \(U \leq G\). By \(6\) Lemma 2.1, the \(K(U)\)-module spanned by \(S\) admits a cyclic vector \(v\). Then \(v\) is a cyclic vector of the \(K(G)\)-module \(W\).

Corollary 11.2. Let \(K\) be a field endowed with a smooth faithful \(\langle S \rangle\)-action. Then any smooth simple left \(K(\langle S \rangle)\)-module is isomorphic to \(K(\langle Y_2 \rangle)/K(\langle S \rangle)\alpha\) for some \(\alpha \in K(\langle Y_2 \rangle)\).

Proof. By Lemma 6.2 any smooth simple left \(K(\langle S \rangle)\)-module is a quotient of \(K(\langle Y_2 \rangle)\) for an appropriate \(s \geq 0\) by a left \(K(\langle S \rangle)\)-submodule \(V\). By Proposition 8.6 the \(K(\langle S \rangle)\)-module \(V\) is finite, and thus, by Lemma 11.1 it is cyclic, i.e., it is generated by some \(\alpha \in K(\langle Y_2 \rangle)\).

Corollary 11.3. The \(K\)-semilinear representations of \(\langle S \rangle\) of the following 4 classes, where \(s \geq 0\) is integer, are indecomposable: (i) \(s, K(\langle Y_2 \rangle)\), where \(s \neq 1\), (ii) \(K(\langle S \rangle)\)-subrepresentations of \(K(\langle Y_2 \rangle)\), (iii) \(s, K(\langle Y_2 \rangle)^o\), where \(s \geq 2\). Then a pair of such representations consists of isomorphic ones only if they belong to the same class (i.e., to one of (i), (ii), (iii) for some \(s\)) and, in the case (ii), have the same codimension in \(K(\Psi)\).

Proof. This follows from \(\text{End}_{K(\langle S \rangle)}(K(\langle Y_2 \rangle)^o) = k\) and Lemma 8.10. It follows from Lemmas 10.3 \(\tag*{1}\), 9.1 \(\tag*{1}\) that codimension of \(V \neq 0\) in \(K[\Psi]\) is \(\dim_k \text{Ext}^1(K, V)\). \(\tag*{\square}\)
Corollary 11.4. Any subquotient of $K[Ψ]^N$ is isomorphic to $K^{m_0} \oplus \bigoplus W_i^{m_i}$ for a unique collection of isomorphism classes of $K$-semilinear subrepresentations $W_i \subseteq K[Ψ]$ and multiplicities $m_i \geq 0$.

Conjecture 11.5. For any $s \geq 0$ the indecomposable object $K[\Psi_s]$ is injective. Any indecomposable injective object is isomorphic to $K[\Psi_s]$ for some $s \geq 0$.

12. Smooth irreducible semilinear representations of $\mathcal{S}_Ψ$

Lemma 12.1. Let $Ψ' \subseteq Ψ$ be an infinite subset, $K := k(Ψ)$ and $K' = k(Ψ')$. Fix a bijection $Ψ \xrightarrow{\sim} Ψ'$. Denote by $ι$ the induced ring isomorphism $K(\mathcal{S}_Ψ) \xrightarrow{ι} K'(\mathcal{S}_{Ψ'})$, where $I := Ψ \setminus Ψ'$. It allows to consider $K'⟨\mathcal{S}_{Ψ'}⟩$-modules as $K(\mathcal{S}_Ψ)$-modules. Then any smooth finite $K(\mathcal{S}_Ψ)$-module $M$ admits a $K'⟨\mathcal{S}_{Ψ'}⟩$-submodule $M'$ and a $K(\mathcal{S}_Ψ)$-module isomorphism $M \xrightarrow{ι} M'$. In particular, any smooth simple $K(\mathcal{S}_Ψ)$-module $M$ of $K$-dimension $> 1$ admits a simple $K'⟨\mathcal{S}_{Ψ'}⟩$-submodule $M'$ of $K'$-dimension $> 1$.

Proof. By Lemma 11.1 and Proposition 8.5, $M$ is the quotient of $K[Ψ^s]$ by the $K(\mathcal{S}_Ψ)$-submodule generated by some $α_1, \ldots, α_N$. Assume that $α_1, \ldots, α_N \in k(J)[J^s]$ for a finite $J \subset Ψ$. Choose $g \in \mathcal{S}_Ψ$ such that $g(J) \subset Ψ$. Replacing $α_j$ with $gα_j$ (and $J$ with $g(J)$), we may, thus, assume that $J \subset Ψ'$. Then the $K'⟨\mathcal{S}_{Ψ'}⟩$-submodule generated by $α_1, \ldots, α_N$ is the one we are looking for. □

Theorem 12.2. Let $K = k(Ψ)$. Then any smooth simple $K(\mathcal{S}_Ψ)$-module is isomorphic to $K$.

Proof. By Lemma 12.1 any smooth simple $K(\mathcal{S}_Ψ)$-module $M$ is isomorphic to a quotient of $K[\Psi_s]$ for some $s$. Let us show by induction on $s$ that any simple subquotient of the $K(\mathcal{S}_Ψ)$-module $K[\Psi_s]$ is isomorphic to $K$, the case $s = 0$ being trivial.

By Corollary 11.2 any simple subquotient $M$ of $K[\Psi_s]$ is contained in $K[\Psi_s]/K(\mathcal{S}_Ψ)α$ for some $α \in K[\Psi_s]$. By Lemma 8.2 there are no simple $K(\mathcal{S}_Ψ)$-submodules in $K[\Psi_s]$ if $s \geq 1$, and therefore, $α \neq 0$. Let $α = \sum_{i=1}^m a_i[I_i]$ and $J := (\bigcup_{i=1}^m I_i) \setminus I_1$. By Lemma 8.3 the $K(\mathcal{S}_{Ψ'})$-module $K[\Psi_s]/K(\mathcal{S}_Ψ)α$ is isomorphic to a quotient of $\bigoplus_{i=1}^s K[\Psi_s]^{\oplus (J_i)}$. By the induction assumption, any simple subquotient of the $K(\mathcal{S}_{Ψ'})$-module $M$ is isomorphic to $K$. By Lemma 12.1 this means that $M$ itself is isomorphic to $K$. □

Appendix A. Smooth semilinear representations of groups exhausted by compact subgroups

Let $K$ be a field and $ψ : G \rightarrow \text{Aut}_{\text{field}}(K)$ be a group of field automorphisms of $K$.

Suppose that $G$ is exhausted by its compact subgroups, i.e., any compact subset of $G$ is contained in a compact subgroup.

Let $V$ be an $K$-vector space. By Hilbert Theorem 90, cf. [9], Prop.3, p.159], restriction to any compact subgroup $U \subseteq G$ of a smooth semilinear $G$-action $Θ : G \rightarrow \text{GL}_K(V)$ on $V$ is given by the action $id_{Λ_U,Θ} ⊕ ψ$ on $Λ_U,Θ ⊗ K^K V = V$ for a $K^K$-lattice $Λ_U,Θ$ in $V$. Fix a system $B$ of compact subgroups of $G$ such that (i) $B$ covers a dense subgroup in $G$, (ii) any pair of subgroups in $B$ is contained in a subgroup in $B$ (e.g., as $B$ we can take the collection of all compact subgroups).

Each smooth semilinear action of $G$ on $V$ determines a compatible system of $K^K$-lattices $Λ_U$ in $V$ for all $U \in B$ (in other words, an element of the set $\varprojlim \{K^K$-lattices $Λ_U$ in $V\}$): if $U \subseteq U'$ then $Λ_U = Λ_{U'} \otimes K^K U'$.

Suppose that $G$ is locally compact. Then we may assume that $B$ consists of open compact subgroups of $G$ and there is a bijection between (a) the smooth semilinear actions of $G$ on $V$ and (b) compatible systems of $K^K$-lattices $Λ_U$ in $V$ for all $U \in B$.

If we fix a $K^G$-lattice $Λ$ then the mapping $[g] \mapsto g(Λ ⊗ K^K U)$ identifies the set of $K^K$-lattices in $V$ with the set $\text{GL}_K(V)/\text{GL}_{K^K}(Λ \otimes K^K U)$, and therefore, the set of smooth semilinear $G$-action on
V can be identified with the set \( \lim_{U \in B} \GL_K(V) / \GL_K(\Lambda \otimes K G K U) \), and the set of isomorphism classes of smooth semilinear actions of \( G \) on \( V \) coincides with \( \GL_K(V) \setminus \lim_{U \in B} \GL_K(V) / \GL_K(\Lambda \otimes K G K U) \).

If \( G \) is not locally compact then the smooth semilinear \( G \)-actions on a given \( K \)-vector space \( V \) are described as compatible systems \((\Lambda_U)_{U \in B}\) of \( K^U \)-lattices in \( V \) such that for any vector \( v \in V \) the intersection over all \( U \in B \) of the subfields generated over \( K^U \) by the coordinates of \( v \) with respect to the lattice \( \Lambda_U \) is of finite type over \( K^U \).

**Example.** If \( G \) is countable at infinity then as \( B \) one can choose a totally ordered collection \( U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots \) of compact subgroups in \( G \) such that \( G = \bigcup_{m \geq 1} U_m \). Any compatible systems of \( K^U \)-lattices \( \Lambda_U \) can be presented (a priori, not uniquely) as a composition, infinite to the left, \( \cdots b_3 b_2 b_1 b(\Lambda) \), where \( b_i \in \GL_{K U_i}(\Lambda \otimes K G K U_i) \) (so that \( \Lambda_i := b_{i-1} \cdots b_2 b_1 b(\Lambda) \otimes K G K U_i \)).

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