Quantum vertex $\mathbb{F}((t))$-algebras and their modules

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Abstract

This is a paper in a series to study vertex algebra-like structures arising from various algebras including quantum affine algebras and Yangians. In this paper, we develop a theory of what we call (weak) quantum vertex $\mathbb{F}((t))$-algebras with $\mathbb{F}$ a field of characteristic zero and $t$ a formal variable, and we give a conceptual construction of (weak) quantum vertex $\mathbb{F}((t))$-algebras and their modules. As an application, we associate weak quantum vertex $\mathbb{F}((t))$-algebras to quantum affine algebras, providing a solution to a problem posed by Frenkel and Jing. We also explicitly construct an example of quantum vertex $\mathbb{F}((t))$-algebras from a certain quantum $\beta\gamma$-system.

1 Introduction

In the earliest days of vertex (operator) algebra theory, Lie algebras had played an important role. In particular, an important family of vertex operator algebras (see [FLM], [FZ], [DL]) was associated to untwisted affine Lie algebras. A fundamental problem, posed in [FJ] (see also [EFK]), has been to establish a suitable theory of quantum vertex algebras so that quantum vertex algebras can be canonically associated to quantum affine algebras in the same (or similar) way that vertex operator algebras are associated to affine Lie algebras. In the past, several theories of quantum vertex algebras have been studied ([eFR], [EK], [Bo], [Li3], [Li4], [AB]), however this particular problem is still to be solved.

This is a paper in a series, starting with [Li3], to study vertex algebra-like structures arising from various algebras such as quantum affine algebras and Yangians, with an ultimate goal to solve the aforementioned problem. In the present paper, we develop a theory of (weak) quantum vertex $\mathbb{F}((t))$-algebras with $\mathbb{F}$ a field of characteristic zero and $t$ a formal variable, and we establish a general construction of weak quantum vertex $\mathbb{F}((t))$-algebras and their modules. As an application we associate weak quantum vertex $\mathbb{F}((t))$-algebras canonically to quantum affine algebras, providing a desired solution to the very problem.

The notion of weak quantum vertex $\mathbb{F}((t))$-algebra in a certain way generalizes the notion of weak quantum vertex algebra, which was introduced and studied previously in this series (see [Li3], [Li4]). A rough description of all these “quantum vertex algebras” is that they are various generalizations of ordinary vertex algebras where the locality, namely weak commutativity, is replaced by a braided locality, while the

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weak associativity is retained. A *weak quantum vertex* $\mathbb{F}((t))$-*algebra* is defined to be an $\mathbb{F}((t))$-module $V$, equipped with an $\mathbb{F}$-linear map

$$Y(\cdot, x) : V \to \text{Hom}_{\mathbb{F}((t))}(V, V((x)))$$

and equipped with a distinguished vector $1 \in V$, satisfying the conditions that

$$Y(f(t)v, x) = f(t + x)Y(v, x) \quad \text{for } f(t) \in \mathbb{F}((t)), \ v \in V,$$

$$Y(1, x)v = v, \ Y(v, x)1 \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)1 = v \quad \text{for } v \in V,$$

and that for $u, v \in V$, there exist (finitely many)

$$u^{(i)}, v^{(i)} \in V, \ f_i(x_1, x_2) \in \mathbb{F}_*(x_1, x_2) \ (i = 1, \ldots, r)$$

such that

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2)$$

$$-x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) \sum_{i=1}^{r} \iota_{t,x_2,x_1}(f_i(x_1 + t, x_2 + t))Y(v^{(i)}, x_2)Y(u^{(i)}, x_1)$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2).$$

(See Section 2 for the definitions of $\mathbb{F}_*(x_1, x_2)$ and $\iota_{t,x_2,x_1}$.) Furthermore, a quantum vertex $\mathbb{F}((t))$-algebra is a weak quantum vertex $\mathbb{F}((t))$-algebra equipped with a unitary quantum Yang-Baxter operator on $V$ with two (independent) spectral parameters, which describes the braiding and satisfies some other conditions.

In [EK], Etingof and Kazhdan developed a fundamental theory of quantum vertex operator algebras in the sense of formal deformation. The notion of (weak) quantum vertex $\mathbb{F}((t))$-algebra as well as that of (weak) quantum vertex algebra (see [Li3], [Li4]) largely reflects Etingof-Kazhdan’s notion of quantum vertex operator algebra, however there are essential differences. As the map $Y(\cdot, x)$ for a weak quantum vertex $\mathbb{F}((t))$-algebra is not $\mathbb{F}((t))$-linear (where linearity is deformed), the formal variable $t$ is not a deformation parameter, unlike the formal variable $\hbar$ in Etingof-Kazhdan’s theory. On the other hand, the braiding operator in Etingof-Kazhdan’s theory is a rational quantum Yang-Baxter operator (with one parameter), whereas the braiding operator here is more general with two parameters.

The theory of quantum vertex $\mathbb{F}((t))$-algebras is also significantly different from Anguelova and Bergvelt’s theory of $H_D$-quantum vertex algebras (see [AB]). The notion of $H_D$-quantum vertex algebra generalizes Etingof-Kazhdan’s notion of braided vertex operator algebra (see [EK]) in certain directions. In particular, the underlying space of an $H_D$-quantum vertex algebra is a topologically free $\mathbb{F}[[t]]$-module and the vertex operator map $Y(\cdot, x)$ is $\mathbb{F}[[t]]$-linear, where the variable $t$ plays the same role as $\hbar$ does in [EK]. We note that weak quantum vertex $\mathbb{F}((t))$-algebras satisfy
the same associativity for ordinary vertex algebras. Unlike (weak) quantum vertex $\mathbb{F}((t))$-algebras, general $H_D$-quantum vertex algebras do not satisfy the associativity for ordinary vertex algebras (though they do satisfy a braided associativity).

The theory of (weak) quantum vertex $\mathbb{F}((t))$-algebras is deeply rooted in \cite{Li3}. To better state the results of the present paper we review a conceptual result obtained therein. Let $W$ be an arbitrary vector space and let $\mathcal{E}(W)$ denote the space $\text{Hom}(W, W((x)))$ alternatively. The essential idea is to study the algebraic structures generated by various types of subsets of $\mathcal{E}(W)$. The most general type consists of what we called quasi compatible subsets, where a subset $U$ of $\mathcal{E}(W)$ is quasi compatible if for any finite sequence $a^{(1)}(x), \ldots, a^{(r)}(x)$ in $U$, there exists a nonzero polynomial $p(x, y)$ such that

$$
\left( \prod_{1 \leq i < j \leq r} p(x_i, x_j) \right) a_1(x_1) \cdots a_r(x_r) \in \text{Hom}(W, W((x_1, \ldots, x_r))).
$$

Furthermore, the notion of compatible subset is defined by assuming that the nonzero polynomial $p(x, y)$ is of the form $(x - y)^k$ with $k \in \mathbb{N}$. It was proved therein that any (quasi) compatible subset $U$ of $\mathcal{E}(W)$ generates what we called a nonlocal vertex algebra $(U)$ in a certain canonical way with $W$ as a (quasi) module in a certain sense. (A nonlocal vertex algebra is the same as a weak $G_1$-vertex algebra in the sense of \cite{Li1} and is also essentially the same as a field algebra in the sense of \cite{BK}.) In contrast with that vertex algebras are analogs of commutative and associative algebras, nonlocal vertex algebras are analogs of noncommutative associative algebras. It follows from this general result that nonlocal vertex algebras can be associated to a wide variety of algebras including quantum affine algebras.

In the present paper, based on \cite{Li3}, as one of our main results we prove that for any quasi compatible subset $U$ of $\mathcal{E}(W)$, the $\mathbb{F}((x))$-span $\mathbb{F}((x))|U|$ is what we call a nonlocal vertex $\mathbb{F}((t))$-algebra. (Note that $\mathcal{E}(W)$ is naturally an $\mathbb{F}((x))$-module.) The notion of nonlocal vertex $\mathbb{F}((t))$-algebra is a counterpart of the notion of nonlocal vertex algebra, where a nonlocal vertex $\mathbb{F}((t))$-algebra $V$ is a nonlocal vertex algebra over $\mathbb{F}$ and an $\mathbb{F}((t))$-module such that

$$
Y(f(t)u, x)g(t)v = f(t + x)g(t)Y(u, x)v \quad \text{for } f(t), g(t) \in \mathbb{F}((t)), \ u, v \in V.
$$

Furthermore, to deal with quantum affine algebras, we study what we call quasi $\mathcal{S}(x_1, x_2)$-local subsets of $\mathcal{E}(W)$. A subset $U$ of $\mathcal{E}(W)$ is said to be quasi $\mathcal{S}(x_1, x_2)$-local if for any $a(x), b(x) \in U$, there exist (finitely many)

$$
u^{(i)}(x) \in U, \ f_i(x_1, x_2) \in \mathbb{F}_s(x_1, x_2) \ (i = 1, \ldots, r)
$$

such that

$$
p(x_1, x_2)a(x_1)b(x_2) = p(x_1, x_2) \sum_{i=1}^r t_{x_2, x_1}(f_i(x_1, x_2)) \nu^{(i)}(x_2) \nu^{(i)}(x_1)
$$

where
for some nonzero polynomial \( p(x_1, x_2) \). We note that quasi \( \mathcal{S}(x_1, x_2) \)-local subsets are quasi compatible. Our key result is that for any quasi \( \mathcal{S}(x_1, x_2) \)-local subset \( U \) of \( \mathcal{E}(W) \), \( \mathbb{F}((x)) \langle U \rangle \) is a weak quantum vertex \( \mathbb{F}((t)) \)-algebra.

The theory of (weak) quantum vertex \( \mathbb{F}((t)) \)-algebras runs largely parallel to that of (weak) quantum vertex algebras. To construct quantum vertex \( \mathbb{F}((t)) \)-algebras from weak quantum vertex \( \mathbb{F}((t)) \)-algebras, we extend Etingof-Kazhdan’s notion of non-degeneracy for nonlocal vertex \( \mathbb{F}((t)) \)-algebras and we prove that every non-degenerate weak quantum \( \mathbb{F}((t)) \)-algebra has a (unique) canonical quantum vertex \( \mathbb{F}((t)) \)-algebra structure, just as with weak quantum vertex algebras in [Li4] (see also [EK]). We furthermore establish certain general non-degeneracy results, analogous to those obtained in [Li4].

We note that this theory of quantum vertex \( \mathbb{F}((t)) \)-algebras has a great generality and our conceptual result is applicable to many better known quantum algebras, particularly including quantum affine algebras. Take \( W \) to be a highest weight module for a quantum affine algebra and set \( \mathbb{F} = \mathbb{C} \). We show that the generating functions of the generators in Drinfeld’s realization form a quasi \( \mathcal{S}(x_1, x_2) \)-local subset \( U \) of \( \mathcal{E}(W) \). Therefore we have a weak quantum vertex \( \mathbb{C}((t)) \)-algebra \( \mathbb{C}((x)) \langle U \rangle \) with \( W \) as a canonical quasi module. To a certain extent, this solves the aforementioned problem, though we yet have to show that this weak quantum vertex \( \mathbb{C}((t)) \)-algebra is a quantum vertex \( \mathbb{C}((t)) \)-algebra, or sufficiently to show that it is non-degenerate.

In the theory of (weak) quantum vertex \( \mathbb{F}((t)) \)-algebras, an important issue is about notions of module. Notice that for a quasi \( \mathcal{S}(x_1, x_2) \)-local subset \( U \) of \( \mathcal{E}(W) \) with \( W \) a vector space as before, the weak quantum vertex \( \mathbb{F}((t)) \)-algebra \( \mathbb{F}((x)) \langle U \rangle \) has the natural module \( W \) (a vector space over \( \mathbb{F} \)) and the adjoint module \( \mathbb{F}((x)) \langle U \rangle \) (a vector space over \( \mathbb{F}((t)) \)), which are significantly different. This leads to us to two categories of modules for weak quantum vertex \( \mathbb{F}((t)) \)-algebras.

This paper is organized as follows: In Section 2, we study notions of nonlocal vertex \( \mathbb{F}((t)) \)-algebra and weak quantum vertex \( \mathbb{F}((t)) \)-algebra. In Section 3, we study notions of quantum vertex \( \mathbb{F}((t)) \)-algebra and non-degeneracy. In Section 4, we give a conceptual construction of nonlocal vertex \( \mathbb{F}((t)) \)-algebras and weak quantum vertex \( \mathbb{F}((t)) \)-algebras. In Section 5, we present two existence theorems. In Section 6, we associate weak quantum vertex \( \mathbb{C}((t)) \)-algebras to quantum affine algebras and we construct a quantum vertex \( \mathbb{C}((t)) \)-algebra from a certain quantum \( \beta\gamma \)-system.

2 Nonlocal vertex \( \mathbb{F}((t)) \)-algebras and weak quantum vertex \( \mathbb{F}((t)) \)-algebras

In this section, we define notions of nonlocal vertex \( \mathbb{F}((t)) \)-algebra and weak quantum vertex \( \mathbb{F}((t)) \)-algebra, and we study what we call type zero modules and type one modules for nonlocal vertex \( \mathbb{F}((t)) \)-algebras. We also present some basic axiomatic results.
We begin by fixing some basic notations. In addition to the standard usage of symbols $\mathbb{Z}$ and $\mathbb{C}$, we use $\mathbb{N}$ for the set of nonnegative integers. We shall use the standard formal variable notations and conventions as in [FLM] and [FHL] (cf. [LL]). Letters such as $t, x, y, z, x_0, x_1, x_2, \ldots$ stand for mutually commuting independent formal variables. We shall be working on a scalar field $\mathbb{F}$ of characteristic zero, where typical examples of $\mathbb{F}$ are $\mathbb{C}$ and the field $\mathbb{C}((t))$ of lower truncated formal Laurent series in $t$. Denote by $\mathbb{F}[[x_1, \ldots, x_r]]$ the algebra of formal Laurent series which are globally truncated with respect to all the variables. By $\mathbb{F}^*((x_1, x_2, \ldots, x_r))$ we denote the extension of the algebra $\mathbb{F}[[x_1, x_2, \ldots, x_r]]$ of formal nonnegative power series by joining the inverses of nonzero polynomials.

We recall the iota maps from [Li3], which will be used extensively. For any permutation $(i_1, i_2, \ldots, i_r)$ on $\{1, \ldots, r\}$,

$$\iota_{x_{i_1}, \ldots, x_{i_r}} : \mathbb{F}_* (x_1, x_2, \ldots, x_r) \to \mathbb{F}((x_{i_1})) \cdots ((x_{i_r}))$$

(2.1)

denotes the unique algebra embedding that extends the identity endomorphism of $\mathbb{F}[[x_1, \ldots, x_r]]$ (cf. [FHL]). Note that both $\mathbb{F}_* (x_1, x_2, \ldots, x_r)$ and $\mathbb{F}((x_{i_1})) \cdots ((x_{i_r}))$ contain $\mathbb{F}((x_1, \ldots, x_r))$ as a subalgebra. The map $\iota_{x_{i_1}, \ldots, x_{i_r}}$ preserves $\mathbb{F}((x_1, \ldots, x_r))$ element-wise and is $\mathbb{F}((x_1, \ldots, x_r))$-linear.

We recall the notion of nonlocal vertex algebra ([Li1], [Li3]; see also [K], [BK]), which is essential to this paper.

**Definition 2.1.** A nonlocal vertex algebra over $\mathbb{F}$ is a vector space $V$, equipped with a linear map $Y(\cdot, x) : V \to \text{Hom}(V, V((x))) \subset (\text{End} V)[[x, x^{-1}]]$, $v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$ (with $v_n \in \text{End} V)$ and a distinguished vector $1 \in V$, satisfying the conditions that

$$Y(1, x)v = v, \quad Y(v, x)1 \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)1 = v \quad \text{for} \quad v \in V,$$

and that for $u, v, w \in V$, there exists a nonnegative integer $l$ such that

$$(x_0 + x_2)^l Y(u, x_0 + x_2) Y(v, x_2)w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w$$

(the weak associativity).

The following two notions can be found either in [Li1] or [Li3]:

**Definition 2.2.** Let $V$ be a nonlocal vertex algebra. A $V$-module is a vector space $W$ equipped with a linear map

$$Y_W(\cdot, x) : V \to \text{Hom}(W, W((x))) \subset (\text{End} W)[[x, x^{-1}]],$$

$w \mapsto Y_W(v, x),$

and a distinguished vector $1 \in W$, satisfying the conditions that

$$Y_W(1, x)v = v, \quad Y_W(v, x)1 \in W[[x]] \quad \text{and} \quad \lim_{x \to 0} Y_W(v, x)1 = v \quad \text{for} \quad v \in V,$$

and that for $u, v, w \in W$, there exists a nonnegative integer $l$ such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2)w = (x_0 + x_2)^l Y_W(Y_W(u, x_0)v, x_2)w$$

(the weak associativity).
satisfying the conditions that
\[ Y_W(1, x) = 1_W \quad \text{ (the identity operator on } W) \]
and that for \( u, v \in V, \ w \in W \), there exists a nonnegative integer \( l \) such that
\[ (x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w. \]
The notion of quasi \( V \)-module is defined as above with the last condition replaced by a weaker condition that for \( u, v \in V, \ w \in W \), there exists a nonzero polynomial \( p(x_1, x_2) \in F[x_1, x_2] \) such that
\[ p(x_0 + x_2, x_2)Y_W(u, x_0 + x_2)Y_W(v, x_2)w = p(x_0 + x_2, x_2)Y_W(Y(u, x_0)v, x_2)w. \quad (2.2) \]

The following follows immediately from [LTW] (Lemma 2.9):

**Proposition 2.3.** Let \( V \) be a nonlocal vertex algebra. In the definition of a \( V \)-module, in the presence of other axioms weak associativity can be equivalently replaced by the condition that for \( u, v \in V, \ ) there exists \( k \in \mathbb{N} \) such that
\[ (x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))), \]
\[ x_0^k Y_W(Y(u, x_0)v, x) = ((x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2))|_{x_1=x_2+x_0}. \]

**Remark 2.4.** For \( A(x_1, x_2) \in \text{Hom}(W, W((x_1))((x_2))) \) with \( W \) a vector space over \( F \), we have been using the convention
\[ A(x_1, x_2)|_{x_1=x_0+x_2} = A(x_0 + x_2, x_2) = e_{x_0,x_2}A(x_0 + x_2, x_2). \]
Note that the substitutions \( A(x_2 + x_0, x_2), A(x_1, x_1 + x_0) \) and \( A(x_1, x_0 + x_1) \) do not exist in general. On the other hand, for \( E(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2))), \) all the substitutions \( E(x_0 + x_2, x_2), E(x_2 + x_0, x_2), E(x_1, x_1 - x_0), \) and \( E(x_1, -x_0 + x_1) \) exist, and we have
\[ (E(x_1, x_2)|_{x_2=x_1-x_0})|_{x_1=x_2+x_0} = E(x_1, x_2)|_{x_1=x_2+x_0}. \quad (2.3) \]

Let \( F \) be a field of characteristic zero as before and let \( t \) be a formal variable. Notice that as \( F((t)) \) is a field containing \( F \) as a subfield, every \( F((t)) \)-module is naturally a vector space over \( F \).

**Definition 2.5.** A nonlocal vertex \( F((t)) \)-algebra is a nonlocal vertex algebra \( V \) over \( F \), equipped with an \( F((t)) \)-module structure, such that
\[ Y(f(t)u, x)(g(t)v) = f(t + x)g(t)Y(u, x)v \quad (2.4) \]
for \( f(t), g(t) \in F((t)), \ u, v \in V \), where it is understood that
\[ f(t + x) = e^{tx} f(t) \in F((t))[[x]]. \]
A homomorphism of nonlocal vertex \( F((t)) \)-algebras is a homomorphism of nonlocal vertex algebras over \( F \), which is also \( F((t)) \)-linear.
Definition 2.6. Let $V$ be a nonlocal vertex $\mathbb{F}((t))$-algebra. A $V$-module of type zero is a module $(W,Y_W)$ for $V$ viewed as a nonlocal vertex algebra over $\mathbb{F}$, satisfying the condition that

$$Y_W(f(t)v, x)w = f(x)Y_W(v, x)w \quad \text{for } f(t) \in \mathbb{F}((t)), \; v \in V, \; w \in W.$$ (2.5)

We define a notion of quasi $V$-module of type zero in the obvious way—with the word “module” replaced by “quasi module” in the two places.

The following immediately follows from the corresponding results for nonlocal vertex algebras (see [Li3]):

Lemma 2.7. Let $V$ be a nonlocal vertex $\mathbb{F}((t))$-algebra and let $\mathcal{D}$ be the $\mathbb{F}$-linear operator on $V$ defined by $\mathcal{D}v = v \cdot 1$ for $v \in V$. Then

$$Y(v, x)1 = e^{x \mathcal{D}}v,$$ (2.6)

$$[\mathcal{D}, Y(v, x)] = Y(\mathcal{D}v, x) = \frac{d}{dx} Y(v, x),$$ (2.7)

$$e^{x \mathcal{D}}(f(t)v) = f(t + x)e^{x \mathcal{D}}v \quad \text{for } f(t) \in \mathbb{F}((t)), \; v \in V.$$ (2.8)

Furthermore, for any type-zero quasi $V$-module $(W,Y_W)$ we have

$$Y_W(\mathcal{D}v, x) = \frac{d}{dx} Y_W(v, x) \quad \text{for } v \in V.$$ (2.9)

Note that as $\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ is a derivation of $\mathbb{F}[[x_1, x_2]]$, $e^{t(\partial/\partial x_1 + \partial/\partial x_2)}$ is an algebra embedding of $\mathbb{F}[[x_1, x_2]]$ into $\mathbb{F}[[t, x_1, x_2]]$ with

$$e^{t(\partial/\partial x_1 + \partial/\partial x_2)} \mathbb{F}[x_1, x_2] \subset \mathbb{F}[t, x_1, x_2].$$

Consequently, this gives rise to an algebra embedding of $\mathbb{F}_s(x_1, x_2)$ into $\mathbb{F}_s(t, x_1, x_2)$, where for $f(x_1, x_2) \in \mathbb{F}_s(x_1, x_2),

$$e^{t(\partial/\partial x_1 + \partial/\partial x_2)} f(x_1, x_2) = f(x_1 + t, x_2 + t) \in \mathbb{F}_s(t, x_1, x_2).$$

We now define the main object of this paper.

Definition 2.8. A weak quantum vertex $\mathbb{F}((t))$-algebra is a nonlocal vertex $\mathbb{F}((t))$-algebra $V$, satisfying the condition that for any $u, v \in V$, there exist

$$u^{(i)}, v^{(i)} \in V, \; f_i(x_1, x_2) \in \mathbb{F}_s(x_1, x_2) \quad (i = 1, \ldots, r)$$

such that

$$x_0^{-1}\delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2)$$

$$-x_0^{-1}\delta \left( \frac{x_2 - x_1}{-x_0} \right) \sum_{i=1}^{r} t_{\ell, x_2, x_1}(f_i(t + x_1, t + x_2))Y(v^{(i)}, x_2)Y(u^{(i)}, x_1)$$

$$= x_2^{-1}\delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)$$ (2.10)

(the $\mathcal{S}_s(x_1, x_2)$-Jacobi identity).
In the following we study certain axiomatic aspects. For convenience we recall from [Li1] the following result (cf. [FHL]):

**Lemma 2.9.** Let $W$ be a vector space over $F$ and let

$A(x_1, x_2) \in W((x_1))((x_2)), \ B(x_1, x_2) \in W((x_2))((x_1)), \ C(x_0, x_2) \in W((x_2))((x_0))$.

Then

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) A(x_1, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) B(x_1, x_2)$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) C(x_0, x_2)$$

if and only if there exist nonnegative integers $k$ and $l$ such that

$$(x_1 - x_2)^k A(x_1, x_2) = (x_1 - x_2)^k B(x_1, x_2),$$

$$(x_0 + x_2)^l C(x_0, x_2) = (x_0 + x_2)^l A(x_0 + x_2, x_2).$$

As weak associativity holds for every nonlocal vertex algebra, in view of Lemma 2.9 we immediately have:

**Proposition 2.10.** In Definition 2.8 the $S_t(x_1, x_2)$-Jacobi identity axiom in the presence of other axioms can be equivalently replaced by $S_t(x_1, x_2)$-locality: For $u, v \in V$, there exist

$$u^{(i)}, v^{(i)} \in V, \ f_i(x_1, x_2) \in F_*(x_1, x_2) \ (i = 1, \ldots, r)$$

such that

$$Y(u, x_1)Y(v, x_2)$$

$$= \sum_{i=1}^{r} (x_1 - x_2)^k u_{t,x_2,x_1}(f_i(x_1 + t, x_2 + t))Y(v^{(i)}, x_2)Y(u^{(i)}, x_1) \quad (2.11)$$

for some nonnegative integer $k$ depending on $u$ and $v$.

We also have:

**Proposition 2.11.** Let $V$ be a nonlocal vertex $\mathbb{F}((t))$-algebra and let

$$u, \ v, \ u^{(i)}, v^{(i)} \in V, \ f_i(x_1, x_2) \in F_*(x_1, x_2) \ (i = 1, \ldots, r).$$

Then (2.11) holds for some nonnegative integer $k$ if and only if

$$Y(u, x)v = \sum_{i=1}^{r} u_{t,x}(f_i(x + t, t))e^{xP}Y(v^{(i)}, -x)u^{(i)}. \quad (2.12)$$
Proof. We follow the proof of the analogous assertion for ordinary vertex algebras in [LL]. Assume that (2.11) holds for some $k \in \mathbb{N}$. We can choose $k$ so large that

$$x^k Y(v^{(i)}, x) u^{(i)} \in V[[x]] \quad \text{for } i = 1, \ldots, r.$$ 

Let $p(x_1, x_2) \in \mathbb{F}[x_1, x_2]$ be a nonzero polynomial such that

$$p(x_1, x_2) f_i(x_1, x_2) \in \mathbb{F}[[x_1, x_2]] \quad \text{for } i = 1, \ldots, r.$$ 

Using (2.11) and the $\mathcal{D}$-properties in Lemma 2.7 we have

$$p(x_1 + t, x_2 + t)(x_1 - x_2)^k Y(u, x_1) Y(v, x_2)\textbf{1} = \sum_{i=1}^{r} (x_1 - x_2)^k p(x_1 + t, x_2 + t) t_{t,x_1} (f_i(x_1 + t, x_2 + t)) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1)\textbf{1}$$

$$= \sum_{i=1}^{r} (x_1 - x_2)^k (pf_i)(x_1 + t, x_2 + t) Y(v^{(i)}, x_2) e^{x_1 \mathcal{D}} u^{(i)}$$

$$= \sum_{i=1}^{r} (x_1 - x_2)^k (pf_i)(x_1 + t, x_2 + t) e^{x_1 \mathcal{D}} Y(v^{(i)}, x_2 - x_1) u^{(i)}$$

$$= \sum_{i=1}^{r} (pf_i)(x_1 + t, x_2 + t) e^{x_1 \mathcal{D}} [(x_1 - x_2)^k Y(v^{(i)}, -x_1 + x_2) u^{(i)}].$$

Notice that it is safe now to set $x_2 = 0$. By doing so we get

$$p(x_1 + t, t)x_1^k Y(u, x_1) v$$

$$= \sum_{i=1}^{r} x_1^k (pf_i)(x_1 + t, t) e^{x_1 \mathcal{D}} Y(v^{(i)}, -x_1) u^{(i)}$$

$$= \sum_{i=1}^{r} x_1^k p(x_1 + t, t) t_{t,x_1} (f_i(x_1 + t, t)) e^{x_1 \mathcal{D}} Y(v^{(i)}, -x_1) u^{(i)}.$$ 

By cancellation (namely, multiplying both sides by $t_{t,x_1}(x_1^{-k}/p(x_1 + t, t))$) we obtain

$$Y(u, x_1) v = \sum_{i=1}^{r} t_{t,x_1} (f_i(x_1 + t, t)) e^{x_1 \mathcal{D}} Y(v^{(i)}, -x_1) u^{(i)}.$$ 

On the other hand, assume that this skew-symmetry relation holds. By Proposition 2.3 there exists $k \in \mathbb{N}$ such that

$$(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) \in \text{Hom}(V, V((x_1, x_2))),$$

$$x_0^k Y(Y(u, x_0)v, x_2) = ((x_1 - x_2)^k Y(u, x_1) Y(v, x_2)) |_{x_1 = x_2 + x_0};$$

$$(x_1 - x_2)^k Y(v^{(i)}, x_2) Y(u^{(i)}, x_1) \in \text{Hom}(V, V((x_1, x_2))),$$

$$x_0^k Y(Y(v^{(i)}, -x_0) u^{(i)}, x_1) = ((x_1 - x_2)^k Y(v^{(i)}, x_2) Y(u^{(i)}, x_1) w) |_{x_2 = x_1 - x_0},$$
and such that
\[ t_{t,x_1,x_2}(x_1 - x_2)^k f_i(t + x_1, t + x_2) = t_{t,x_2,x_1}(x_1 - x_2)^k f_i(t + x_1, t + x_2), \]
lying in \( \mathbb{F}((t))[x_1, x_2] \) for \( i = 1, \ldots, r \) (recall Lemma 6.12). Set
\[ E(x_1, x_2) = \sum_{i=1}^{r} t_{t,x_2,x_1}(f_i(t + x_1, t + x_2))(x_1 - x_2)^{2k} Y(v^{(i)}, x_2) Y(u^{(i)}, x_1). \]
Then
\[ E(x_1, x_2) \in \text{Hom}(V, V((x_1, x_2))). \]
Using the skew-symmetry relation and the basic \( D \)-properties we get
\[
Y(Y(u, x_0) v, x_2)
= \sum_{i=1}^{r} Y(t_{t,x_0}(f_i(t + x_0, t)) e^{x_0 D} Y(v^{(i)}, -x_0) u^{(i)}, x_2)
= \sum_{i=1}^{r} t_{t,x_2,x_0}(f_i(t + x_2 + x_0, t + x_2)) e^{x_0 D} Y(v^{(i)}, -x_0) u^{(i)}, x_2)
= \sum_{i=1}^{r} t_{t,x_2,x_0}(f_i(t + x_2 + x_0, t + x_2)) Y(v^{(i)}, -x_0) u^{(i)}, x_2 + x_0).
\]
Then
\[
x_0^{2k} Y(Y(u, x_0) v, x_1 - x_0)
= \sum_{i=1}^{r} t_{t,x_1,x_0}(x_0^k f_i(t + x_1, t + x_1 - x_0)) e^{x_0 D} Y(v^{(i)}, -x_0) u^{(i)}, x_1)
= \sum_{i=1}^{r} t_{t,x_1,x_0}(x_0^k f_i(t + x_1, t + x_1 - x_0))
\cdot \left( (x_1 - x_2)^k Y(v^{(i)}, x_2) Y(u^{(i)}, x_1) \right) \bigg|_{x_2 = x_1 - x_0}
= \sum_{i=1}^{r} \left( t_{t,x_1,x_2}(x_1 - x_2)^k f_i(t + x_1, t + x_2) \right) \bigg|_{x_2 = x_1 - x_0}
\cdot \left( (x_1 - x_2)^k Y(v^{(i)}, x_2) Y(u^{(i)}, x_1) \right) \bigg|_{x_2 = x_1 - x_0}
= \left( \sum_{i=1}^{r} t_{t,x_2,x_1}(f_i(t + x_1, t + x_2))(x_1 - x_2)^{2k} Y(v^{(i)}, x_2) Y(u^{(i)}, x_1) \right) \bigg|_{x_2 = x_1 - x_0}
= E(x_1, x_2) \bigg|_{x_2 = x_1 - x_0},
\]
where we are using the basic facts from Lemmas 6.11 and 6.12. Thus
\[
\left( (x_1 - x_2)^{2k} Y(u, x_1) Y(v, x_2) \right) \bigg|_{x_2 = x_2 + x_0}
= x_0^{2k} Y(Y(u, x_0) v, x_2)
= (E(x_1, x_2) \big|_{x_2 = x_1 - x_0}) \bigg|_{x_1 = x_2 + x_0}
= E(x_1, x_2) \bigg|_{x_1 = x_2 + x_0}.
\]
It follows that
\[
(x_1 - x_2)^{2k}Y(u, x_1)Y(v, x_2) = E(x_1, x_2)
\]
\[
= \sum_{i=1}^{r} \iota_{t, x_1}(f_i(t + x_1, t + x_2))(x_1 - x_2)^{2k}Y(v^{(i)}, x_2)Y(u^{(i)}, x_1),
\]
proving (2.11). \(\square\)

As an immediate consequence we have:

**Corollary 2.12.** A nonlocal vertex \(\mathbb{F}((t))\)-algebra \(V\) is a weak quantum vertex \(\mathbb{F}((t))\)-algebra if and only if for any \(u, v \in V\), there exist \(u^{(i)}, v^{(i)} \in V\), \(f_i(x_1, x_2) \in \mathbb{F}_s(x_1, x_2)\) (\(i = 1, \ldots, r\)) such that
\[
Y(u, x)v = \sum_{i=1}^{r} \iota_{t, x}(f_i(x + t, t))e^{x\partial}Y(v^{(i)}, -x)u^{(i)}.
\]

The following result implies that if \(V\) is a weak quantum vertex \(\mathbb{F}((t))\)-algebra, for any type zero \(V\)-module \(W\), a variant of \(S_t(x_1, x_2)\)-Jacobi identity (2.10) holds:

**Proposition 2.13.** Let \(V\) be a nonlocal vertex \(\mathbb{F}((t))\)-algebra, let \((W, Y_W)\) be a type zero \(V\)-module, and let
\(u, v, u^{(i)}, v^{(i)} \in V\), \(f_i(x_1, x_2) \in \mathbb{F}_s(x_1, x_2)\) (\(i = 1, \ldots, r\)). Assume that (2.11) holds for some nonnegative integer \(k\). Then
\[
x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y_W(u, x_1)Y_W(v, x_2)
\]
\[
- \sum_{i=1}^{r} x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\iota_{x_2, x_1}(f_i(x_1, x_2))Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1)
\]
\[
= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y_W(Y(u, x_0)v, x_2).
\]  
(2.13)

**Proof.** Since \(Y_W(f(t)a, x) = f(x)Y_W(a, x)\) for \(f(t) \in \mathbb{F}((t)), a \in V\), using Proposition 2.11 and Lemma 2.7 we get
\[
Y_W(Y(u, x_0)v, x_2)
\]
\[
= \sum_{i=1}^{r} Y_W(\iota_{t, x_0}(f_i(t + x_0, t))e^{x_0\partial}Y(v^{(i)}, -x_0)u^{(i)}, x_2)
\]
\[
= \sum_{i=1}^{r} \iota_{x_2, x_0}(f_i(x_2 + x_0, x_2))Y_W(e^{x_0\partial}Y(v^{(i)}, -x_0)u^{(i)}, x_2)
\]
\[
= \sum_{i=1}^{r} \iota_{x_2, x_0}(f_i(x_2 + x_0, x_2))Y_W(Y(v^{(i)}, -x_0)u^{(i)}, x_2 + x_0).
\]

Then it follows from the second half of the proof of Proposition 2.11. \(\square\)

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Next, we study another category of modules for nonlocal vertex \( \mathbb{F}((t)) \)-algebras.

**Definition 2.14.** Let \( V \) be a nonlocal vertex \( \mathbb{F}((t)) \)-algebra. A type one (resp. quasi) \( V \)-module is an \( \mathbb{F}((t)) \)-module \( W \) which is also a (resp. quasi) module for \( V \) viewed as a nonlocal vertex algebra over \( \mathbb{F} \) such that

\[
Y_W(f(t)v,x)(g(t)w) = f(t+x)g(t)Y_W(v,x)w \tag{2.14}
\]

for \( f(t), g(t) \in \mathbb{F}((t)), v \in V, w \in W \).

We have the following simple fact:

**Lemma 2.15.** Let \( V \) be a nonlocal vertex \( \mathbb{F}((t)) \)-algebra. a) Let \( W \) be a type one quasi \( V \)-module and let \( U \) be a quasi submodule of \( W \) for \( V \) viewed as a nonlocal vertex algebra over \( \mathbb{F} \). Then \( U \) is an \( \mathbb{F}((t)) \)-submodule of \( W \). b) Let \( U \) and \( W \) be type one quasi \( V \)-modules and let \( \psi : U \to W \) be a homomorphism of quasi modules for \( V \) viewed as a nonlocal vertex algebra over \( \mathbb{F} \). Then \( \psi \) is \( \mathbb{F}((t)) \)-linear.

**Proof.** For a), by assumption, \( U \) is an \( \mathbb{F} \)-subspace of \( W \), which is closed under the action of \( V \). For \( f(t) \in \mathbb{F}((t)), w \in U \), we have

\[
f(t+x)w = f(t+x)Y_W(1,x)w = Y_W(f(t)1,x)w \in U((x)),
\]

which implies \( f(t)w \in U \). Thus \( U \) is an \( \mathbb{F}((t)) \)-submodule of \( W \).

For b), we are given that \( \psi \) is an \( \mathbb{F} \)-linear map such that

\[
\psi(Y_U(v,x)u) = Y_W(v,x)\psi(u) \quad \text{for} \quad v \in V, u \in U.
\]

For \( f(t) \in \mathbb{F}((t)), u \in U \), we have

\[
\psi(f(t+x)u) = \psi(Y_U(f(t)1,x)u) = Y_W(f(t)1,x)\psi(u) = f(t+x)\psi(u),
\]

which implies \( \psi(f(t)u) = f(t)\psi(u) \). Thus \( \psi \) is \( \mathbb{F}((t)) \)-linear. \( \square \)

The same proof (the second half) of Proposition 2.11 yields the following analog of Proposition 2.13.

**Proposition 2.16.** Let \( V \) be a nonlocal vertex \( \mathbb{F}((t)) \)-algebra, let \( (W, Y_W) \) be a type one \( V \)-module, and let

\[
u, v, u^{(i)}, v^{(i)} \in V, f_i(x_1, x_2) \in \mathbb{F}_*(x_1, x_2) \quad (i = 1, \ldots, r).
\]

Assume that \( \text{(2.11)} \) holds for some nonnegative integer \( k \). Then

\[
x_0^{-1}\delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1)Y_W(v, x_2)
\]

\[
- \sum_{i=1}^r x_0^{-1}\delta \left( \frac{x_2 - x_1}{-x_0} \right) u_{x_2,x_1}(f_i(t + x_1, t + x_2))Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1)
\]

\[
= x_2^{-1}\delta \left( \frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0)v, x_2). \tag{2.15}
\]
In the following two lemmas, we present some technical results which we shall need in later sections.

**Lemma 2.17.** Let $V$ be an $\mathbb{F}(t)$-module and a nonlocal vertex algebra over $\mathbb{F}$. Assume that there exists a subset $U$ of $V$ such that $\mathbb{F}(t)U$ generates $V$ as a nonlocal vertex algebra over $\mathbb{F}$ and such that

$$Y(f(t)u,x)g(t) = f(t+x)g(t)Y(u,x) \quad \text{for } f(t), g(t) \in \mathbb{F}(t), \ u \in U \cup \{1\}.$$ 

Then $V$ is a nonlocal vertex $\mathbb{F}(t)$-algebra.

**Proof.** Set

$$K = \{ v \in V \mid Y(f(t)v,x)g(t) = f(t+x)g(t)Y(v,x) \quad \text{for } f(t), g(t) \in \mathbb{F}(t) \}.$$ 

We must prove $V = K$. It is clear that $K$ is an $\mathbb{F}(t)$-submodule. From assumption we have $\mathbb{F}(t)U \cup \{1\} \subseteq K$, so that $K$ generates $V$ as a nonlocal vertex algebra. Now, it suffices to show that $K$ is closed. Let $u, v \in K$, $f(t), g(t) \in \mathbb{F}(t)$. For any $w \in V$, there exists $l \in \mathbb{N}$ such that

$$(x_0 + x_2)^l Y(Y(f(t)u,x_0)v, x_2)g(t)w = (x_0 + x_2)^l Y(f(t)u, x_0 + x_2)Y(v, x_2)g(t)w$$

$$(x_0 + x_2)^l Y(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2)w.$$ 

Then

$$(x_0 + x_2)^l Y(f(t + x_0)Y(u, x_0)v, x_2)g(t)w$$

$$= (x_0 + x_2)^l Y(Y(f(t)u, x_0)v, x_2)g(t)w$$

$$= (x_0 + x_2)^l Y(f(t)u, x_0 + x_2)Y(v, x_2)g(t)w$$

$$= (x_0 + x_2)^l f(t + x_0 + x_2)Y(u, x_0 + x_2)Y(v, x_2)g(t)w$$

$$= (x_0 + x_2)^l f(t + x_0 + x_2)g(t)Y(u, x_0 + x_2)Y(v, x_2)w$$

$$= (x_0 + x_2)^l f(t + x_0 + x_2)g(t)Y(Y(u, x_0)v, x_2)w.$$ 

Note that as $Y(u, x_0)v \in V((x_0))$, both expressions

$$Y(f(t + x_0)Y(u, x_0)v, x_2)g(t)w \quad \text{and} \quad f(t + x_0 + x_2)g(t)Y(Y(u, x_0)v, x_2)w$$

lie in $V((x_2))((x_0))$. It follows that

$$Y(f(t + x_0)Y(u, x_0)v, x_2)g(t)w = f(t + x_0 + x_2)g(t)Y(Y(u, x_0)v, x_2)w.$$ 

We note that this also holds with $f(t)$ replaced by its derivatives of all orders.
Next, we show \( u_m v \in K \) for \( m \in \mathbb{Z} \) by using the above information. Let \( m \in \mathbb{Z} \) be arbitrarily fixed. Choosing \( k \in \mathbb{Z} \) such that \( x_0^{m+k}Y(u, x_0)v \in V[[x_0]] \), we get
\[
Y(f(t)u_m v, x_2)g(t)w = \text{Res}_{x_0} x_0^m Y(f(t)Y(u, x_0)v, x_2)g(t)w
= \text{Res}_{x_0} x_0^m Y(e^{-x_0 f(t)}Y(u, x_0)v, x_2)g(t)w
= \text{Res}_{x_0} \sum_{n=0}^{k} (-1)^i \frac{1}{n!} x_0^{m+n} Y(f^{(n)}(t + x_0)Y(u, x_0)v, x_2)g(t)w
= \text{Res}_{x_0} \sum_{n=0}^{k} (-1)^i \frac{1}{n!} x_0^{m+n} f^{(n)}(t + x_0 + x_2)g(t)Y(Y(u, x_0)v, x_2)w
= \text{Res}_{x_0} x_0^m \left( e^{-x_0 f(t)}Y(u, x_0)v + x_2 \right) g(t)Y(Y(u, x_0)v, x_2)w
= f(t + x_2)g(t)Y(u_m v, x_2)w.
\]

Thus \( u_m v \in K \). This proves that \( K \) is closed, concluding the proof. \( \square \)

For \( F(x_1, x_2), G(x_1, x_2) \in V[[x_1^{\pm 1}, x_2^{\pm 1}]] \), we define \( F \sim_{\pm} G \) if
\[
(x_1 \pm x_2)^p F = (x_1 \pm x_2)^p G
\]
for some \( p \in \mathbb{N} \). It is clear that the defined relations “\( \sim_{\pm} \)” are equivalence relations.

Let \( U \) be a subset of a nonlocal vertex \( \mathcal{F}((t)) \)-algebra \( V \). We say \( U \) is \( \mathcal{S}_i \)-local if the \( \mathcal{S}_i \)-locality condition in Proposition 2.10 holds with \( U \) in place of \( V \).

We have (cf. [Li4], Lemma 2.7; [LTW], Proposition 2.6):

**Lemma 2.18.** Let \( V \) be a nonlocal vertex \( \mathcal{F}((t)) \)-algebra. Assume that there exists an \( \mathcal{S}_i \)-local subset \( U \) of \( V \) such that \( \mathcal{F}((t))U \) generates \( V \) as a nonlocal vertex algebra over \( \mathcal{F} \). Then \( V \) is a weak quantum vertex \( \mathcal{F}((t)) \)-algebra.

**Proof.** First we introduce a technical notion. We say that an ordered pair \( (A, B) \) of subsets of \( V \) is \( \mathcal{S}_i \)-local if for any \( a \in A, b \in B \), there exist
\[
a^{(i)} \in A, b^{(i)} \in B, f_i(x_1, x_2) \in \mathcal{F}_i(x_1, x_2) \quad (i = 1, \ldots, r)
\]
such that
\[
Y(a, x_1)Y(b, x_2) \sim - \sum_{i=1}^{r} \tau_{t, x_2, x_1}(f_i(t + x_1, t + x_2))Y(b^{(i)}, x_2)Y(a^{(i)}, x_1),
\]
or equivalently (in view of Corollary 2.12)
\[
Y(a, x)b = \sum_{i=1}^{r} \tau_{t, x}(f_i(t + x, t))e^{xP}Y(b^{(i)}, -x)a^{(i)}.
\]

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It is clear that if \((A, B)\) is \(S_t\)-local, so is \((\mathbb{F}(t))A, \mathbb{F}(t)B\). For any subset \(A\) of \(V\), we set
\[
A^{(2)} = \mathbb{F}(t)\text{-span}\{u_nv \mid u, v \in A, n \in \mathbb{Z}\} \subset V.
\]

We are going to prove that if an ordered pair \((A, P)\) of \(\mathbb{F}(t)\)-submodules of \(V\) is \(S_t\)-local, then \((A, P^{(2)})\) and \((A^{(2)}, P)\) are \(S_t\)-local. Then it follows from this and induction that \((\langle \mathbb{F}(t)U \rangle, \langle \mathbb{F}(t)U \rangle)\) is \(S_t\)-local. Therefore, \(V\) is a weak quantum vertex \(\mathbb{F}(t)\)-algebra.

We first prove that \((A, P^{(2)})\) is \(S_t\)-local. Let \(a \in A, u, v \in P\). By \(S_t\)-locality and by Proposition 2.11, there exist
\[
f_i(x_1, x_2), g_{ij}(x_1, x_2) \in \mathbb{F}_s(x_1, x_2), a^{(i)}, a^{(ij)} \in A, u^{(i)}, v^{(j)} \in P
\]
for \(1 \leq i \leq r, 1 \leq j \leq s\), such that
\[
Y(a, x_1)Y(u, x_2)v \sim - \sum_{i=1}^{r} t_{t,x_2,x_1}(f_i(t + x_1, t + x_2))Y(u^{(i)}, x_2)Y(a^{(i)}, x_1)v,
\]
\[
Y(a^{(i)}, x)v = \sum_{j=1}^{s} t_{t,x}(g_{ij}(t + x, t))e^{x_1D}Y(v^{(j)}, -x)a^{(ij)},
\]
and
\[
Y(u^{(i)}, x_2 - x_1)Y(v^{(j)}, -x_1)a^{(ij)} \sim Y(Y(u^{(i)}, x_2)v^{(j)}, -x_1)a^{(ij)}. \tag{2.16}
\]
Then using the \(D\)-bracket-derivative property (2.7) and weak associativity we get
\[
Y(a, x_1)Y(u, x_2)v
\sim- \sum_{i=1}^{r} t_{t,x_2,x_1}(f_i(t + x_1, t + x_2))Y(u^{(i)}, x_2)Y(a^{(i)}, x_1)v
\sim- \sum_{i=1}^{r} t_{t,x_2,x_1}(f_i(t + x_1, t + x_2))Y(u^{(i)}, x_2)\sum_{j=1}^{s} g_{ij}(t + x_1, t)e^{x_1D}Y(v^{(j)}, -x_1)a^{(ij)}
\sim- \sum_{i=1}^{r} \sum_{j=1}^{s} t_{t,x_2,x_1}f_i(t + x_1, t + x_2)g_{ij}(t + x_1, t)e^{x_1D}Y(u^{(i)}, x_2 - x_1)Y(v^{(j)}, -x_1)a^{(ij)}
\sim- \sum_{i=1}^{r} \sum_{j=1}^{s} t_{t,x_1,x_2}f_i(t + x_1, t + x_2)g_{ij}(t + x_1, t)e^{x_1D}Y(Y(u^{(i)}, x_2)v^{(j)}, -x_1)a^{(ij)}.
\]
That is, there exists a nonnegative integer \(k\) such that
\[
(x_1 - x_2)^kY(a, x_1)Y(u, x_2)v
= (x_1 - x_2)^k \sum_{i=1}^{r} \sum_{j=1}^{s} t_{t,x_1,x_2}f_i(t + x_1, t + x_2)g_{ij}(t + x_1, t)e^{x_1D}Y(Y(u^{(i)}, x_2)v^{(j)}, -x_1)a^{(ij)}.
\]

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As both sides involve only finitely many negative powers of \( x_2 \), multiplying both sides by \( (x_1 - x_2)^{-k} \), we obtain

\[
Y(a, x_1)Y(u, x_2)v = \sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{t, x_1, x_2} f_i(t + x_1, t + x_2)g_{ij}(t + x_1, t)e^{x_1D}Y(u^{(i)}, x_2)v^{(j)}, -x_1)a^{(ij)}. 
\]

It follows that \((A, P^{(2)})\) is \(S_t\)-local.

Next, we prove that \((A^{(2)}, P)\) is \(S_t\)-local. Let \(a, b \in A\), \(w \in P\). There exist

\[
f_i(x_1, x_2), g_{ij}(x_1, x_2) \in \mathbb{F}_*(x_1, x_2), \ a^{(ij)}, b^{(j)} \in A, \ w^{(i)}, w^{(ij)} \in P
\]

for \(1 \leq i \leq r, 1 \leq j \leq s\), such that

\[
Y(b, x_2)w = \sum_{i=1}^{r} \lambda_{t, x} f_i(t + x, t)e^{x_2D}Y(w^{(i)}, -x)b^{(i)},
\]

\[
Y(a, x_1)Y(w^{(i)}, -x_2)b^{(i)} \sim + \sum_{j=1}^{s} \lambda_{t, x_2, x_1} (g_{ij}(t + x_1, t - x_2))Y(w^{(ij)}, -x_2)Y(a^{(ij)}, x_1)b^{(i)}.
\]

Then we get

\[
Y(Y(a, x_1)b, x_2)w \sim + Y(a, x_1 + x_2)Y(b, x_2)w
\]

\[
\sim + Y(a, x_1 + x_2) \sum_{i=1}^{r} \lambda_{t, x_2} f_i(t + x_2, t)e^{x_2D}Y(w^{(i)}, -x_2)b^{(i)}
\]

\[
\sim + \sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{t, x_2, x_1} f_i(t + x_2, t)e^{x_2D}g_{ij}(t + x_1, t - x_2)Y(w^{(ij)}, -x_2)Y(a^{(ij)}, x_1)b^{(i)}
\]

\[
= \sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{t, x_2, x_1} (f_i(t + x_2, t)g_{ij}(t + x_1 + x_2, t))e^{x_2D}Y(w^{(ij)}, -x_2)Y(a^{(ij)}, x_1)b^{(i)}.
\]

By a similar reasoning we obtain

\[
Y(Y(a, x_1)b, x_2)w = \sum_{i=1}^{r} \sum_{j=1}^{s} \lambda_{t, x_2, x_1} f_i(t + x_2, t)g_{ij}(t + x_1 + x_2, t)e^{x_2D}Y(w^{(ij)}, -x_2)Y(a^{(ij)}, x_1)b^{(i)}.
\]

It follows that \((A^{(2)}, P)\) is \(S_t\)-local. Now, the proof is complete. \(\square\)
3 Quantum vertex $\mathbb{F}((t))$-algebras and non-degeneracy

In this section we formulate and study a notion of quantum vertex $\mathbb{F}((t))$-algebra and we study Etingof-Kazhdan’s notion of non-degeneracy for nonlocal vertex $\mathbb{F}((t))$-algebras. As a key result we show that every non-degenerate weak quantum vertex $\mathbb{F}((t))$-algebra has a canonical quantum vertex $\mathbb{F}((t))$-algebra structure. In this section we also present some basic results on non-degeneracy.

We begin with some basics on quantum Yang-Baxter operators. Let $H$ be a vector space over $\mathbb{F}$. The symmetric group $S_3$ naturally acts on $H^\otimes 3$ with $\sigma \in S_3$ acting as $P_\sigma$ which is defined by

$$P_\sigma(u_1 \otimes u_2 \otimes u_3) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes u_{\sigma(3)} \quad \text{for } u_1, u_2, u_3 \in H.$$

For $1 \leq i < j \leq 3$, set $P_{ij} = P_{(ij)}$ (with $(ij)$ denoting the transposition). We have

$$P_{12}P_{23}P_{12} = P_{13} = P_{23}P_{12}P_{23}.$$

Let $P$ denote the flip operator on $H \otimes H$ with $P(u \otimes v) = v \otimes u$ for $u, v \in H$. Then

$$P_{12} = P \otimes 1, \quad P_{23} = 1 \otimes P.$$

A quantum Yang-Baxter operator with two parameters on $H$ is a linear map

$$S(x_1, x_2) : H \otimes H \to H \otimes H \otimes \mathbb{F}_*(x_1, x_2),$$

satisfying the quantum Yang-Baxter equation

$$S_{12}(x_1, x_2)S_{13}(x_1, x_3)S_{23}(x_2, x_3) = S_{23}(x_2, x_3)S_{13}(x_1, x_3)S_{12}(x_1, x_2), \quad (3.1)$$

where $S_{ij}(x_i, x_j)$ are the linear maps from $H^\otimes 3 \to H^\otimes 3 \otimes \mathbb{F}_*(x_i, x_j)$, defined by $S_{12}(x, z) = S(x, z) \otimes 1$, $S_{23}(x, z) = 1 \otimes S(x, z)$, and

$$S_{13}(x, z) = P_{23}(S(x, z) \otimes 1)P_{23}.$$

Furthermore, $S(x_1, x_2)$ is said to be unitary if

$$S_{21}(x_2, x_1)S(x_1, x_2) = 1, \quad (3.2)$$

where $S_{21}(x_2, x_1) = PS(x_2, x_1)P$. Set

$$R(x_1, x_2) = S(x_1, x_2)P : \ H \otimes H \to H \otimes H \otimes \mathbb{F}_*(x_1, x_2). \quad (3.3)$$

It is known that (3.1) is equivalent to the following braided relation

$$R_{12}(x_1, x_2)R_{23}(x_1, x_3)R_{12}(x_2, x_3) = R_{23}(x_2, x_3)R_{12}(x_1, x_3)R_{23}(x_1, x_2). \quad (3.4)$$
Definition 3.1. A quantum vertex $F((t))$-algebra is a weak quantum vertex $F((t))$-algebra $V$ equipped with an $F$-linear unitary quantum Yang-Baxter operator $S(x_1, x_2)$ (with two parameters) on $V$, satisfying the conditions that
\[
S(x_1, x_2)(f(t)u \otimes g(t)v) = f(x_1)g(x_2)S(x_1, x_2)(u \otimes v) \tag{3.5}
\]
for $f(t), g(t) \in F((t))$, $u, v \in V$, and that for $u, v \in V$,
\[
(x_1 - x_2)^k Y(v, x_2)Y(u, x_1) = \sum_{i=1}^{r} (x_1 - x_2)^k u_{i,x_2}(f_i(x_1 + t, x_2 + t))Y(u^{(i)}, x_1)Y(v^{(i)}, x_2)
\]
for some nonnegative integer $k$, where $u^{(i)}, v^{(i)}, f_i (i = 1, \ldots, r)$ are given by
\[
S(x_1, x_2)(u \otimes v) = \sum_{i=1}^{r} u^{(i)} \otimes v^{(i)} \otimes f_i(x_1, x_2),
\]
and that
\[
[D \otimes 1, S(x_1, x_2)] = -\frac{\partial}{\partial x_1} S(x_1, x_2), \quad [1 \otimes D, S(x_1, x_2)] = -\frac{\partial}{\partial x_2} S(x_1, x_2), \quad \tag{3.6}
\]
\[
S(x_1, x_2)(Y(x) \otimes 1) = (Y(x) \otimes 1)S_{23}(x_1, x_2)S_{13}(x_1 + x, x_2). \quad \tag{3.7}
\]
We modify Etingof-Kazhdan’s notion of non-degeneracy (see [EK]) as follows:

**Definition 3.2.** Let $V$ be a nonlocal vertex $F((t))$-algebra. Denote by $V^{\otimes n}$ the tensor product space over $F$ and define $V^{\otimes n} \boxtimes F_*(x_1, \ldots, x_n)$ to be the quotient space of $V^{\otimes n} \otimes F_*(x_1, \ldots, x_n)$ by the relations
\[
f_1(t)u^{(1)} \cdots f_n(t)u^{(n)} \otimes f = u^{(1)} \otimes \cdots \otimes u^{(n)} \otimes f_1(x_1) \cdots f_n(x_n)f
\]
for $f \in F_*(x_1, \ldots, x_n)$, $f_i(t) \in F((t))$, $v^{(i)} \in V$ ($i = 1, \ldots, n$). We say that $V$ is non-degenerate if for every positive integer $n$, the $F$-linear map
\[
Z_n : V^{\otimes n} \boxtimes F_*(x_1, \ldots, x_n) \rightarrow V((x_1)) \cdots ((x_n))
\]
\[
(v^{(1)} \otimes \cdots \otimes v^{(n)}) \boxtimes f \mapsto t_{x,x_1,\ldots,x_n}f(t + x_1, \ldots, t + x_n)Y(v^{(1)}, x_1) \cdots Y(v^{(n)}, x_n)1
\]
is injective. (One can see that $Z_n$ is indeed well defined.)

**Remark 3.3.** Given a nonlocal vertex $F((t))$-algebra $V$, let $V^0$ be an $F$-subspace such that $V = F((t)) \otimes_F V^0$. We see that $Z_n$ is injective if and only if the restriction
\[
Z_n^0 : (V^0)^{\otimes n} \boxtimes F_*(x_1, \ldots, x_n) \rightarrow V((x_1)) \cdots ((x_n))
\]
is injective. For $g_i(x) \in F((x))$, $v^{(i)} \in V^0$ ($i = 1, \ldots, r$), we have
\[
Z_1 \left( \sum_{i=1}^{r} v^{(i)} \otimes g_i(x) \right) = \sum_{i=1}^{r} g_i(t + x)Y(v^{(i)}, x)1 = \sum_{i=1}^{r} g_i(t + x)e^{xD}v^{(i)}
\]
\[
= e^{xD} \sum_{i=1}^{r} g_i(t)v^{(i)}.
\]
From this we see that $Z_1$ is always injective.
Actually, what we need in practice are certain variations of the maps $Z_n$.

**Definition 3.4.** For each $n \geq 1$, we define an $F$-linear map

$$
\pi_n : V^\otimes n \boxtimes F_*(x_1, \ldots, x_n) \to \text{Hom}(V, V((x_1)) \cdots ((x_n)))
$$

(3.8)

by

$$
\pi_n(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f) = \iota_{t, x_1, \ldots, x_n} f(t + x_1, \ldots, t + x_n) Y(v^{(1)}, x_1) \cdots Y(v^{(n)}, x_n)
$$

for $f \in F_*(x_1, \ldots, x_n)$, $v^{(1)}, \ldots, v^{(n)} \in V$.

Noticing that

$$
\pi_n(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f)(1) = Z_n(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f),
$$

we see that the injectivity of $Z_n$ implies the injectivity of $\pi_n$.

We follow [EK] to denote by

$$
Y(x) : V \otimes V \to V((x)) \subset V[[x, x^{-1}]]
$$

the $F$-linear map defined by $Y(x)(u \otimes v) = Y(u, x)v$ for $u, v \in V$. As a common practice, $Y(x)$ is always extended:

$$
Y(x) : V \otimes V \otimes F_*(t, x_1, \ldots, x_k)((x)) \to (V \otimes F_*(t, x_1, \ldots, x_k))((x)),
$$

where

$$
Y(x)(u \otimes v \otimes f) = fY(x)(u \otimes v) = fY(u, x)v
$$

(3.9)

for $u, v \in V$, $f \in F_*(t, x_1, \ldots, x_k)((x))$, where $k$ is a positive integer.

The following, which is lifted from [EK] (Proposition 1.11), plays a very important role in the theory of quantum vertex $F((t))$-algebras:

**Theorem 3.5.** Let $V$ be a weak quantum vertex $F((t))$-algebra. Assume that $V$ is non-degenerate. Then there exists an $F$-linear map

$$
S(x_1, x_2) : V \otimes V \to V \otimes V \otimes F_*(x_1, x_2),
$$

which is uniquely determined by the condition that for $u, v \in V$,

$$
Y(v, x_2)Y(u, x_1)w \sim Y(x_1)(1 \otimes Y(x_2))(S(t + x_1, t + x_2)(u \otimes v) \otimes w)
$$

(3.10)

for all $w \in V$. Furthermore, $S(x_1, x_2)$ is a unitary quantum Yang-Baxter operator on $V$, and $V$ equipped with $S(x_1, x_2)$ is a quantum vertex $F((t))$-algebra.
Proof. First of all, with \( V \) non-degenerate, all the maps \( \pi_n \ (n \geq 1) \) are injective. Notice that

\[
Y(x_1)(1 \otimes Y(x_2))(S(t + x_1, t + x_2)(u \otimes v) \otimes w) = (\pi_2 S(x_1, x_2)(u \otimes v))(w).
\]

It follows that \( S(x_1, x_2) \) is uniquely determined by the very condition.

Let \( u, v, w \in V, f(t), g(t) \in F((t)) \). We have

\[
\begin{align*}
Y(x_1)(1 \otimes Y(x_2))S_{12}(t + x_1, t + x_2)(f(t)u \otimes g(t)v \otimes w),
\sim_\pi Y(x_2)(1 \otimes Y(x_1))(g(t)v \otimes f(t)u \otimes w)
= f(t + x_1)g(t + x_2)Y(x_2)(1 \otimes Y(x_1))(v \otimes u \otimes w)
\sim_\pi f(t + x_1)g(t + x_2)Y(x_1)(1 \otimes Y(x_2))S_{12}(t + x_1, t + x_2)(u \otimes v \otimes w)
= Y(x_1)(1 \otimes Y(x_2))f(t + x_1)g(t + x_2)S_{12}(t + x_1, t + x_2)(u \otimes v \otimes w).
\end{align*}
\]

As the first term and the last term both lie in \( V((x_1))((x_2)) \), the equivalence relation between them actually amounts to equality. With \( \pi_2 \) injective, we obtain

\[
S(x_1, x_2)(f(t)u \otimes g(t)v) = f(x_1)g(x_2)S(x_1, x_2)(u \otimes v).
\]

The quantum Yang-Baxter relation, the unitarity, and the \( D \)-bracket-derivative property (3.6) follow from the same proof of Theorem 4.8 in [Li3] with obvious modifications. It remains to prove (3.7). For \( u, v, w, a \in V \), we have

\[
\begin{align*}
\text{Res}_{x, x^n}Y(x_1)(1 \otimes Y(x_2))(S(x_1 + t, x_2 + t)(Y(u, x)v \otimes w) \otimes a)
\sim_\pi \text{Res}_{x, x^n}Y(w, x_2)Y(Y(u, x)v, x_1)a
\end{align*}
\]

for any fixed \( n \in \mathbb{Z} \). On the other hand, we have

\[
\begin{align*}
Y(w, x_2)Y(u, z)Y(v, x_1)a
\sim_\pi Y(z)(1 \otimes Y(x_2))(1 \otimes 1 \otimes Y(x_1))S_{12}(z + t, x_2 + t)(u \otimes w \otimes v \otimes a)
\sim_\pi Y(z)(1 \otimes Y(x_1))(1 \otimes 1 \otimes Y(x_2))S_{23}(x_1 + t, x_2 + t)P_{23} \cdot
\cdot \cdot S_{12}(z + t, x_2 + t)P_{23}(u \otimes v \otimes w \otimes a)
= Y(z)(1 \otimes Y(x_1))(1 \otimes 1 \otimes Y(x_2)) \cdot
\cdot \cdot S_{23}(x_1 + t, x_2 + t)S_{13}(z + t, x_2 + t)(u \otimes v \otimes w \otimes a). \tag{3.11}
\end{align*}
\]

Notice that for any \( u', v' \in V \), there exists a nonnegative integer \( k \) such that

\[
(x_1 - x_2)^k Y(u', x_1)Y(v', x_2) \in \text{Hom}(V, V((x_1, x_2))),
\]

\[
x_0^k Y(Y(u', x_0)v', x_2) = ((x_1 - x_2)^k Y(u', x_1)Y(v', x_2)) |_{x_1 = x_2 + x_0}.
\]

Using (3.11), by choosing \( k \) sufficiently large, we have

\[
\begin{align*}
x^k Y(w, x_2)Y(Y(u, x)v, x_1)a
= ((z - x_1)^k Y(w, x_2)Y(u, z)Y(v, x_1)a) |_{z = x_1 + x}
\sim_\pi x^k Y(x_1)(Y(x) \otimes 1)(1 \otimes 1 \otimes Y(x_2)) \cdot
\cdot \cdot S_{23}(x_1 + t, x_2 + t)S_{13}(x_1 + x + t, x_2 + t)(u \otimes v \otimes w \otimes a).
\end{align*}
\]

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Thus
\[ \text{Res}_x x^n Y(x_1)(1 \otimes Y(x_2)) (S(x_1 + t, x_2 + t)(Y(u, x)v \otimes w) \otimes a) \]
\[ \sim \text{Res}_x x^n Y(x_1) (Y(x) \otimes 1) (1 \otimes 1 \otimes Y(x_2)) \cdot \]
\[ \cdot S_{23}(x_1 + t, x_2 + t) S_{13}(x_1 + x + t, x_2 + t)(u \otimes v \otimes w \otimes a) \]
for any fixed \( n \in \mathbb{Z} \). As both sides are in \( V((x_1))((x_2, x)) \), we have
\[ \text{Res}_x x^n Y(x_1)(1 \otimes Y(x_2)) (S(x_1 + t, x_2 + t)(Y(u, x)v \otimes w) \otimes a) \]
\[ = \text{Res}_x x^n Y(x_1)(Y(x) \otimes 1)(1 \otimes 1 \otimes Y(x_2)) \cdot \]
\[ \cdot S_{23}(x_1 + t, x_2 + t) S_{13}(x_1 + x + t, x_2 + t)(u \otimes v \otimes w \otimes a) \]
Since \( n \) is arbitrary, we can drop off \( \text{Res}_x x^n \). Then \((\ref{eq:3.7})\) follows.

For the rest of this section we focus on non-degeneracy of nonlocal vertex \( \mathbb{F}(\!(t)\!)\)-algebras. Let \( V \) be a nonlocal vertex \( \mathbb{F}(\!(t)\!)\)-algebra. From Lemma \ref{lemma:2.15} a \( V \)-submodule of \( V \) for \( V \) viewed as a nonlocal vertex algebra over \( \mathbb{F} \) is the same as a \( V \)-submodule of \( V \) for \( V \) viewed as a nonlocal vertex \( \mathbb{F}(\!(t)\!)\)-algebra. Furthermore, a module endomorphism for \( V \) viewed as a nonlocal vertex algebra over \( \mathbb{F} \) is the same as a module endomorphism for \( V \) viewed as a nonlocal vertex \( \mathbb{F}(\!(t)\!)\)-algebra. We denote by \( V^{\text{mod}} \) the adjoint \( V \)-module.

**Proposition 3.6.** Let \( V \) be a nonlocal vertex \( \mathbb{F}(\!(t)\!)\)-algebra such that \( V \) as a \( V \)-module is irreducible with \( \text{End}_V(V^{\text{mod}}) = \mathbb{F}(\!(t)\!) \). Then \( V \) is non-degenerate.

**Proof.** We are going to use induction to show that \( Z_n \) is injective for every positive integer \( n \), following the proof of a similar result in \cite{Li4}. Recall from Remark 3.3 that \( Z_1 \) is always injective. Now, assume that \( n \geq 2 \) and \( Z_{n-1} \) is injective. Let \( U \) be the quotient space of \( V^{\otimes(n-1)} \otimes \mathbb{F}_*(x_1, \ldots, x_n) \), viewed as a vector space over \( \mathbb{F} \), by the relations
\[ f_2(t)v^{(2)} \otimes \cdots \otimes f_n(t)v^{(n)} \otimes f = v^{(2)} \otimes \cdots \otimes v^{(n)} \otimes f_2(x_2) \cdots f_n(x_n)f \]
for \( f \in \mathbb{F}_*(x_1, \ldots, x_n) \), \( f_i(t) \in \mathbb{F}(\!(t)\!) \), \( v^{(i)} \in V \ (i = 2, \ldots, n) \). Note that \( U \) is naturally an \( \mathbb{F}_*(x_1, \ldots, x_n) \)-module while \( \mathbb{F}_*(x_1, \ldots, x_n) \) is an algebra over \( \mathbb{F}(\!(x_1)\!) \). Furthermore, viewing \( V \) as an \( \mathbb{F}(\!(x_1)\!) \)-module with \( f(x_1) \) acting as \( f(t) \), we have
\[ V^{\otimes n} \otimes \mathbb{F}_*(x_1, \ldots, x_n) = V \otimes \mathbb{F}(\!(x_1)\!) U. \]
Let \( B \) be the subalgebra of the endomorphism algebra \( \text{End}_{\mathbb{F}(\!(t)\!)}(V) \) (over \( \mathbb{F}(\!(t)\!) \)) generated by \( v_n \) for \( v \in V \), \( n \in \mathbb{Z} \). Then \( V \) is an irreducible \( B \)-module with \( \text{End}_B(V) = \)
\[ F((t)). \] From [Li4] (Lemma 3.8), the kernel of \( Z_n \) is a \( B \otimes F((t)) \) submodule of \( V \otimes F((x_i)) U \) (with \( B \) acting on the first factor). By a classical fact (cf. [L2], Lemma 2.10), we have \( \ker Z_n = V \otimes F((x_i)) P \) for some submodule \( P \) of \( U \). Let \( a \in P \subset U \). There exists a nonzero polynomial \( q(x_1, \ldots, x_n) \) such that

\[
q(x_1, \ldots, x_n) a \in V^{\otimes(n-1)} \otimes F[x_1, \ldots, x_n].
\]

Write

\[
q(x_1, \ldots, x_n) a = \sum_{m \in \mathbb{N}} x_1^m a_m
\]

with \( a_m \in V^{\otimes(n-1)} \otimes F[x_2, \ldots, x_n] \). As \( 1 \otimes qa \in \ker Z_n \), we have \( a_m \in \ker Z_{n-1} \) for \( m \in \mathbb{Z} \). Then \( a_m = 0 \) for \( m \in \mathbb{Z} \), and hence \( q(x_1, \ldots, x_n) a = 0 \). Thus \( a = 0 \). This proves that \( P = 0 \), which implies that \( Z_n \) is injective. \( \square \)

**Remark 3.7.** Let \( V \) be a nonlocal vertex \( F((t)) \)-algebra and let \( \mathcal{F} = \{ F_n \}_{n \in \mathbb{Z}} \) be an increasing filtration of \( F((t)) \)-submodules of \( V \), satisfying the condition that \( \mathbf{1} \in F_0 \),

\[
u_k F_n \subset F_{m+n-k-1} \quad \text{for \( u \in F_m, \ k \in \mathbb{Z}, \ m, n \in \frac{1}{2} \mathbb{Z}. \n\]

Form the \( n \)-graded \( F((t)) \)-module

\[
\text{Gr}_\mathcal{F}(V) = \oplus_{n \in \mathbb{Z}} (F_n/F_{n-1/2}).
\]

For \( u + F_{m-1/2} \in (F_m/F_{m-1/2}), \ v + F_{n-1/2} \in (F_n/F_{n-1/2}) \) with \( m, n \in \frac{1}{2} \mathbb{Z} \), define

\[
(u + F_{m-1/2}) (v + F_{n-1/2}) = u_k v + F_{m+n-k-3/2} \in (F_{m+n-k-1}/F_{m+n-k-3/2})
\]

for \( k \in \mathbb{Z} \). It is straightforward to show that \( \text{Gr}_\mathcal{F}(V) \) is a nonlocal vertex \( F((t)) \)-algebra with \( 1 + F_{-1/2} \) as the vacuum vector (cf. [KL]).

**Proposition 3.8.** Let \( V \) be a nonlocal vertex \( F((t)) \)-algebra with an increasing filtration \( \mathcal{F} = \{ F_n \}_{n \in \mathbb{Z}} \) of \( F((t)) \)-submodules, satisfying the condition that \( F_n = 0 \) for \( n \) sufficiently negative, \( \mathbf{1} \in F_0 \), and

\[
u_k F_n \subset F_{m+n-k-1} \quad \text{for \( u \in F_m, \ k \in \mathbb{Z}, \ m, n \in \frac{1}{2} \mathbb{Z}. \n\]

Assume that \( \text{Gr}_\mathcal{F}(V) \) as a \( \text{Gr}_\mathcal{F}(V) \)-module is irreducible with \( \text{End}(\text{Gr}_\mathcal{F}(V)^{\text{mod}}) = F((t)) \). Then \( V \) as a \( V \)-module is irreducible with \( \text{End}_V(V^{\text{mod}}) = F((t)) \) and \( V \) is non-degenerate.

**Proof.** Notice that the assertion on non-degeneracy follows from the other assertions and Proposition 3.6. The irreducibility assertion follows from Proposition 2.11 of [KL]. It remains to prove \( \text{End}_V(V^{\text{mod}}) = F((t)) \). Let \( \psi \in \text{End}_V(V^{\text{mod}}) \). If \( \psi(\mathbf{1}) = 0 \), we have \( \psi = 0 \in F((t)) \) as

\[
\psi(v) = \psi(v_{-1}\mathbf{1}) = v_{-1}\psi(\mathbf{1}) = 0 \quad \text{for \( v \in V. \n\]

\[ 22 \]
Thus, $\psi(1) \neq 0$ for any nonzero $\psi \in \text{End}_V(V^{\text{mod}})$. Assume $\psi \neq 0$. Since $\psi(1) \neq 0$ and since $F_n = 0$ for $n$ sufficiently negative, there exists $m \in \frac{1}{2}\mathbb{Z}$ such that $\psi(1) \in F_m - F_{m-1/2}$. For $v \in V$, we have

$$v_n\psi(1) = \psi(v_n1) = 0 \quad \text{for } n \geq 0.$$ 

By Lemma 6.1 of [Li3], we have a $\text{Gr}_F(V)$-module endomorphism $\bar{\psi}$ of $\text{Gr}_F(V)$, sending $1 + F_{-1/2} \in F_0/F_{-1/2}$ to $\psi(1) + F_m/F_{m-1/2}$. From assumption we have $\bar{\psi} = f(t)$ for some $f(t) \in \mathbb{F}(t)$. As $\text{Gr}_F(V)$ is $\frac{1}{2}\mathbb{Z}$-graded, we must have that $m = 0$ and $\psi(1) - f(t) 1 \in F_{-1/2}$. If $\psi \neq f(t)$, with $\psi - f(t)$ in place of $\psi$ we have $(\psi - f(t))(1) \in F_0 - F_{-1/2}$, a contradiction. Thus, $\psi = f(t) \in \mathbb{F}(t))$. \hfill $\square$

**Remark 3.9.** Let $V$ be a nonlocal vertex $\mathbb{F}(t))$-algebra and let $\mathcal{E} = \{E_n\}_{n \in \mathbb{Z}}$ be an increasing filtration of $\mathbb{F}(t))$-submodules of $V$, satisfying the condition that $1 \in E_0$,

$$u_k E_n \subset E_{m+n} \quad \text{for } u \in E_m, \ m, n, k \in \mathbb{Z}.$$ 

Form the $\mathbb{Z}$-graded $\mathbb{F}(t))$-module

$$\text{Gr}_\mathcal{E}(V) = \bigoplus_{n \in \mathbb{Z}} (E_n/E_{n-1}).$$ 

For $u \in E_m$, $v \in E_n$ with $m, n \in \mathbb{Z}$ and for $k \in \mathbb{Z}$, define

$$(u + E_{m-1})_k(v + E_{n-1}) = u_k v + E_{m+n-1} \in (E_{m+n}/E_{m+n-1}).$$ 

It is straightforward to show that $\text{Gr}_\mathcal{E}(V)$ is a nonlocal vertex $\mathbb{F}(t))$-algebra with $1 + E_{-1} \in E_0/E_{-1}$ as the vacuum vector (cf. [Li4]).

The following follows from the same proof of Proposition 3.14 of [Li4] (with obvious notational modifications):

**Proposition 3.10.** Let $V$ be a nonlocal vertex $\mathbb{F}(t))$-algebra and let $\mathcal{E} = \{E_n\}_{n \in \mathbb{Z}}$ be an increasing filtration of $\mathbb{F}(t))$-submodules, satisfying the condition that $E_n = 0$ for $n$ sufficiently negative, $1 \in E_0$, and

$$u_k E_n \subset E_{m+n} \quad \text{for } u \in E_m, m, n, k \in \mathbb{Z}.$$ 

If $\text{Gr}_\mathcal{E}(V)$ is non-degenerate, then $V$ is non-degenerate.

Let $U$ be a nonlocal vertex $\mathbb{F}(t))$-algebra and let $K$ be a nonlocal vertex algebra over $\mathbb{F}$. Equip $U \otimes K$ with the $\mathbb{F}(t))$-module structure with $\mathbb{F}(t))$ acting on $U$ and also equip $U \otimes K$ with the nonlocal vertex algebra structure by tensor product over $\mathbb{F}$. It can be readily seen that $U \otimes K$ becomes a nonlocal vertex $\mathbb{F}(t))$-algebra. Note that from Borcherds’ construction of vertex algebras, $\mathbb{F}(t))$ is a vertex algebra with $1$ as the vacuum vector and with

$$Y(f(t), x) g(t) = (e^{x \frac{d}{dt}} f(t)) g(t) = f(t + x) g(t)$$

for $f(t), g(t) \in \mathbb{F}(t))$. Thus, for any nonlocal vertex algebra $V^0$ over $\mathbb{F}$, $\mathbb{F}(t)) \otimes V^0$ is a nonlocal vertex $\mathbb{F}(t))$-algebra. We have:

$$23.$$
Lemma 3.11. Let $V^0$ be a non-degenerate nonlocal vertex algebra over $F$. Then the nonlocal vertex $F((t))$-algebra $F((t)) \otimes V^0$ is non-degenerate.

Proof. From Remark 3.3, for $n \geq 1$, $Z_n$ is injective if and only if the restriction $Z_n^0 : (V^0)^n \otimes F[[x_1, \ldots, x_n]] \to (F((t)) \otimes V^0)((x_1)) \cdots ((x_n))$ is injective. Furthermore, we see that $Z_n^0$ is injective if and only if its restriction on $(V^0)^n \otimes F[[x_1, \ldots, x_n]]$ is injective. Assume that $A \in (V^0)^n \otimes F[[x_1, \ldots, x_n]]$ such that $Z_n^0(A) = 0$. By extracting the constant term in variable $t$, we see that $A$ lies in the kernel of the $Z_n$-map for $V^0$. Then it follows.

4 Conceptual construction of weak quantum vertex $F((t))$-algebras and their modules

In this section, we present a conceptual construction of nonlocal vertex $F((t))$-algebras, weak quantum vertex $F((t))$-algebras, and their quasi modules of type zero, by using quasi compatible subsets and quasi $S(x_1, x_2)$-local subsets of formal vertex operators. This construction is based on the conceptual construction in [Li3] of nonlocal vertex algebras and their quasi modules.

We begin with the conceptual construction of nonlocal vertex algebras and their (quasi) modules, established in [Li3]. Let $W$ be a vector space over $F$. Set

$$E(W) = \text{Hom}(W, W((x))) \subset \text{End}(W)[[x, x^{-1}]],$$

which contains the identity operator $1_W$ on $W$ as a special element.

Definition 4.1. A finite sequence $a_1(x), \ldots, a_r(x)$ in $E(W)$ is said to be quasi compatible if there exists a nonzero polynomial $p(x, y) \in F[x, y]$ such that

$$\left( \prod_{1 \leq i < j \leq r} p(x_i, x_j) \right) a_1(x_1) \cdots a_r(x_r) \in \text{Hom}(W, W((x_1, \ldots, x_r))).$$

(4.2)

The sequence $a_1(x), \ldots, a_r(x)$ is said to be compatible if there exists a nonnegative integer $k$ such that

$$\left( \prod_{1 \leq i < j \leq r} (x_i - x_j)^k \right) a_1(x_1) \cdots a_r(x_r) \in \text{Hom}(W, W((x_1, \ldots, x_r))).$$

(4.3)

Furthermore, a subset $T$ of $E(W)$ is said to be quasi compatible (resp. compatible) if every finite sequence in $T$ is quasi compatible (resp. compatible).
Let \((a(x),b(x))\) be a quasi compatible ordered pair in \(\mathcal{E}(W)\). By definition, there exists a nonzero polynomial \(p(x,y) \in \mathbb{F}[x,y]\) such that
\[
p(x_1,x_2)a(x_1)b(x_2) \in \text{Hom}(W,W((x_1,x_2))).
\] (4.4)
Define \(a(x)_n b(x) \in \mathcal{E}(W)\) for \(n \in \mathbb{Z}\) in terms of the generating function
\[
Y_\mathcal{E}(a(x),x_0)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n b(x)x_0^{-n-1}
\] (4.5)
by
\[
Y_\mathcal{E}(a(x),x_0)b(x) = \iota_{x,x_0} \left( \frac{1}{p(x+x_0,x)} \right) (p(x_1,x)a(x_1)b(x))|_{x_1=x+x_0}.
\] (4.6)
A quasi compatible \(\mathbb{F}\)-subspace \(U\) of \(\mathcal{E}(W)\) is said to be \(\mathcal{Y}_\mathcal{E}\)-closed if
\[
a(x)_n b(x) \in U \quad \text{for} \ a(x), b(x) \in U, \ n \in \mathbb{Z}.
\]

The following was obtained in \cite{Li3} (though the scalar field therein is \(\mathbb{C}\), it is clear that the results hold for any field of characteristic 0):

**Theorem 4.2.** Let \(W\) be a vector space over \(\mathbb{F}\) and let \(U\) be a (resp. quasi) compatible subset of \(\mathcal{E}(W)\). There exists a \(\mathcal{Y}_\mathcal{E}\)-closed (resp. quasi) compatible subspace that contains \(U\) and \(1_W\). Denote by \(\langle U \rangle\) the smallest such subspace of \(\mathcal{E}(W)\). Then \((\langle U \rangle, \mathcal{Y}_\mathcal{E}, 1_W)\) carries the structure of a nonlocal vertex algebra with \(W\) as a (resp. quasi) module where \(Y_W(\alpha(x),x_0) = \alpha(x_0)\) for \(\alpha(x) \in \langle U \rangle\).

Let \(W\) be a vector space over \(\mathbb{F}\) as before. Notice that for any \(f(x) \in \mathbb{F}((x))\), \(a(x) \in \mathcal{E}(W) = \text{Hom}(W,W((x)))\), we have \(f(x)a(x) \in \mathcal{E}(W)\). Thus, \(\mathcal{E}(W)\) is naturally an \(\mathbb{F}((x))\)-module, namely, a vector space over the field \(\mathbb{F}((x))\).

We now present our first main result of this section.

**Theorem 4.3.** Let \(W\) be a vector space over \(\mathbb{F}\) and let \(U\) be any (resp. quasi) compatible subset of \(\mathcal{E}(W)\). Let \(\langle U \rangle\) be the \(\mathcal{Y}_\mathcal{E}\)-closed (resp. quasi) compatible \(\mathbb{F}\)-subspace of \(\mathcal{E}(W)\) as in Theorem 4.2. Then \(\mathbb{F}((x))\langle U \rangle\) is a \(\mathcal{Y}_\mathcal{E}\)-closed (resp. quasi) compatible \(\mathbb{F}((x))\)-submodule of \(\mathcal{E}(W)\). Furthermore, \((\mathbb{F}((x))\langle U \rangle, \mathcal{Y}_\mathcal{E}, 1_W)\) carries the structure of a nonlocal vertex \(\mathbb{F}((t))\)-algebra, where
\[
f(t)a(x) = f(x)a(x) \quad \text{for} \ f(t) \in \mathbb{F}((t)), \ a(x) \in \mathbb{F}((x))\langle U \rangle,
\]
and \((W,Y_W)\) carries the structure of a (resp. quasi) \(\mathbb{F}((x))\langle U \rangle\)-module of type zero with \(Y_W(a(x),x_0) = a(x_0)\) for \(a(x) \in \mathbb{F}((x))\langle U \rangle\).

**Proof.** This had been essentially proved in \cite{Li3} though the notion of nonlocal vertex \(\mathbb{F}((t))\)-algebra was absent. It was proved in \cite{Li3} (Proposition 3.12) that if
Let \((a(x), b(x))\) be a quasi compatible ordered pair in \(E(W)\), then for any \(f(x), g(x) \in \mathbb{F}((x))\), \((f(x)a(x), g(x)b(x))\) is a quasi compatible ordered pair and

\[
Y_E(f(x)a(x), x_0)(g(x)b(x)) = f(x + x_0)g(x)Y_E(a(x), x_0)b(x).
\]

\[(4.7)\]

It was also proved that the \(\mathbb{F}((x))\)-span of any \(Y_E\)-closed quasi compatible \(\mathbb{F}\)-subspace of \(E(W)\) is quasi compatible and \(Y_E\)-closed. It can be readily seen from the proof that this is also true for compatible case. The rest follows from Theorem 4.2.

We continue to establish a construction of weak quantum vertex \(\mathbb{F}((t))\)-algebras.

**Definition 4.4.** Let \(W\) be a vector space over \(\mathbb{F}\) as before. A subset \(U\) of \(E(W)\) is said to be *quasi* \(S(x_1, x_2)\)-local if for any \(a(x), b(x) \in U\), there exist finitely many

\[
u^{(i)}(x), \nu^{(i)}(x) \in U, \quad f_i(x_1, x_2) \in \mathbb{F}_s(x_1, x_2) \quad (i = 1, \ldots, r)
\]

such that

\[
p(x_1, x_2)a(x_1)b(x_2) = \sum_{i=1}^{r} p(x_1, x_2)u^{(i)}_{x_2, x_1}(f_i(x_1, x_2))u^{(i)}(x_2)\nu^{(i)}(x_1)
\]

\[(4.8)\]

for some nonzero \(p(x_1, x_2) \in \mathbb{F}[x_1, x_2]\), depending on \(a(x)\) and \(b(x)\). We say that \(U\) is *\(S(x_1, x_2)\)-local* if the polynomial \(p(x_1, x_2)\) is of the form \((x_1 - x_2)^k\) with \(k \in \mathbb{N}\).

We note that a quasi \(S(x_1, x_2)\)-local subset is the same as a pseudo-local subset as defined in [Li3]. The following is straightforward to prove:

**Lemma 4.5.** The \(\mathbb{F}((x))\)-span of any (resp. quasi) \(S(x_1, x_2)\)-local subset of \(E(W)\) is (resp. quasi) \(S(x_1, x_2)\)-local.

We also have:

**Proposition 4.6.** Every (resp. quasi) \(S(x_1, x_2)\)-local subset \(U\) of \(E(W)\) is (resp. quasi) compatible. Furthermore, the \(\mathbb{F}((x))\)-submodule \(\mathbb{F}((x))\langle U \rangle\) as in Theorem 4.3 is (resp. quasi) \(S(x_1, x_2)\)-local.

**Proof.** It was proved in [Li3] (Lemma 3.2 and Proposition 3.9) that if \(U\) is quasi \(S(x_1, x_2)\)-local, \(U\) is quasi compatible and \(\langle U \rangle\) is quasi \(S(x_1, x_2)\)-local. Following the same proof with the obvious changes, we confirm the corresponding assertions without the word “quasi” in the three places. Then, by Lemma 4.5 \(\mathbb{F}((x))\langle U \rangle\) is (resp. quasi) \(S(x_1, x_2)\)-local.

Furthermore, we have:

**Proposition 4.7.** Let \(W\) be a vector space over \(\mathbb{F}\) as before and let \(V\) be a \(Y_E\)-closed quasi compatible \(\mathbb{F}((x))\)-submodule of \(E(W)\). Let

\[
a(x), b(x), \nu^{(i)}(x), \nu^{(i)}(x) \in V, \quad f_i(x_1, x_2) \in \mathbb{F}_s(x_1, x_2) \quad (i = 1, \ldots, r).
\]
Assume that there exists a nonzero polynomial \( p(x_1, x_2) \in \mathbb{F}[x_1, x_2] \) such that

\[
p(x_1, x_2)a(x_1)b(x_2) = \sum_{i=1}^{r} p(x_1, x_2)\iota_{x_2,x_1}(f_i(x_1, x_2))u^{(i)}(x_2)v^{(i)}(x_1).
\]

Then

\[
(x_1 - x_2)^k\mathcal{E}(a(x), x_1)\mathcal{E}(b(x), x_2)
= (x_1 - x_2)^k \sum_{i=1}^{r} \iota_{x_2,x_1}(f_i(x + x_1, x + x_2))\mathcal{E}(u^{(i)}(x), x_2)\mathcal{E}(v^{(i)}(x), x_1), \tag{4.9}
\]

where \( p(x_1, x_2) = (x_1 - x_2)^k q(x_1, x_2) \) with \( k \in \mathbb{N}, q(x_1, x_2) \in \mathbb{F}[x_1, x_2] \) such that \( q(x_1, x_1) \neq 0 \).

**Proof.** By Proposition 3.13 of [Li3] we have

\[
p(x + x_1, x + x_2)\mathcal{E}(a(x), x_1)\mathcal{E}(b(x), x_2)
= p(x + x_1, x + x_2) \sum_{i=1}^{r} \iota_{x_2,x_1}(f_i(x + x_1, x + x_2))\mathcal{E}(u^{(i)}(x), x_2)\mathcal{E}(v^{(i)}(x), x_1).
\]

Note that

\[
p(x + x_1, x + x_2) = (x_1 - x_2)^k q(x + x_1, x + x_2).
\]

Write \( q(x + x_1, x + x_2) = q(x, x) + x_1g + x_2h \) with \( g, h \in \mathbb{F}[x, x_1, x_2] \). As \( q(x, x) \neq 0 \), by Lemma 6.12 we have

\[
\iota_{x,x_1}(q(x + x_1, x + x_2)^{-1}) = \iota_{x_2,x_1}(q(x + x_1, x + x_2)^{-1}) \in \mathbb{F}((x))[x_1, x_2]].
\]

Then we can cancel the factor \( q(x + x_1, x + x_2) \) to obtain the desired relation. \( \square \)

As our second main result of this section we have the following refinement of Theorem 4.3

**Theorem 4.8.** Let \( W \) be a vector space over \( \mathbb{F} \) and let \( U \) be any (resp. quasi) \( S(x_1, x_2) \)-local subset of \( \mathcal{E}(W) \). Then the nonlocal vertex \( \mathbb{F}((t))\)-algebra \( \mathbb{F}((x))(U) \) which was obtained in Theorem 4.3 is a weak quantum vertex \( \mathbb{F}((t))\)-algebra with \( W \) as a type zero (resp. quasi) module.

**Proof.** Since a (resp. quasi) \( S(x_1, x_2) \)-local subset is (resp. quasi) compatible by Proposition 4.6, the assertion on module structure follows from Theorem 4.3. As for the first assertion, by Proposition 4.6 \( \mathbb{F}((x))(U) \) is (resp. quasi) \( S(x_1, x_2) \)-local. Then it follows from Proposition 4.7 that the nonlocal vertex \( \mathbb{F}((t))\)-algebra \( \mathbb{F}((x))(U) \) satisfies \( S \)-locality. In view of Proposition 2.10 \( \mathbb{F}((x))(U) \) is a weak quantum vertex \( \mathbb{F}((t))\)-algebra. \( \square \)

We end this section with the following technical result:
Proposition 4.9. Let \( W \) be a vector space over \( \mathbb{F} \), let \( V \) be a \( Y_\mathcal{E} \)-closed quasi compatible \( \mathbb{F}(x) \)-submodule of \( \mathcal{E}(W) \), and let

\[ n \in \mathbb{Z}, \; a(x), \; b(x), \; u^{(i)}(x), \; v^{(i)}(x) \in V, \; f_i(x_1, x_2) \in \mathbb{F}_+(x_1, x_2) \; (i = 1, \ldots, r), \]

\[ c^{(0)}(x), c^{(1)}(x), \ldots, c^{(s)}(x) \in V. \]

Assume

\[
(x_1 - x_2)^n a(x_1) b(x_2) - (-x_2 + x_1)^n \sum_{i=1}^r t_{x_2,x_1}(f_i(x_1, x_2)) u^{(i)}(x_2) v^{(i)}(x_1)
\]

\[
= \sum_{j=0}^s c^{(j)}(x_2) \frac{1}{j!} \left( \frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left( \frac{x_2}{x_1} \right). \tag{4.10}
\]

Then

\[
(x_1 - x_2)^n Y_\mathcal{E}(a(x), x_1) Y_\mathcal{E}(b(x), x_2)
\]

\[ -(-x_2 + x_1)^n \sum_{i=1}^r t_{x_2,x_1}(f_i(x + x_1, x + x_2)) Y_\mathcal{E}(u^{(i)}(x), x_2) Y_\mathcal{E}(v^{(i)}(x), x_1)
\]

\[
= \sum_{j=0}^s Y_\mathcal{E}(c^{(j)}(x, x_2)) \frac{1}{j!} \left( \frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left( \frac{x_2}{x_1} \right). \tag{4.11}
\]

Proof. Let \( k \) be a nonnegative integer such that \( k \geq s+1 \) and \( k+n \geq 0 \). Multiplying both sides of (4.10) by \( (x_1 - x_2)^k \) we obtain

\[
(x_1 - x_2)^{k+n} a(x_1) b(x_2) = (x_1 - x_2)^{k+n} \sum_{i=1}^r t_{x_2,x_1}(f_i(x_1, x_2)) u^{(i)}(x_2) v^{(i)}(x_1), \tag{4.12}
\]

as \( (x_1 - x_2)^k \left( \frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) = 0 \) for \( 0 \leq j \leq s \). By Proposition 4.7 we have

\[
(x_1 - x_2)^{n+k} Y_\mathcal{E}(a(x), x_1) Y_\mathcal{E}(b(x), x_2)
\]

\[
= (x_1 - x_2)^{n+k} \sum_{i=1}^r t_{x_2,x_1}(f_i(t + x_1, t + x_2)) Y_\mathcal{E}(u^{(i)}(x), x_2) Y_\mathcal{E}(v^{(i)}(x), x_1),
\]

which together with weak associativity implies (by Lemma 2.9)

\[
x_0^{-1} \delta \left( \frac{x_1 - x}{x_0} \right) Y_\mathcal{E}(a(x), x_1) Y_\mathcal{E}(b(x), x_2)
\]

\[
- x_0^{-1} \delta \left( \frac{x - x_1}{-x_0} \right) \sum_{i=1}^r t_{t,x_2,x_1}(f_i(t + x_1, t + x_2)) Y_\mathcal{E}(u^{(i)}(x), x_2) Y_\mathcal{E}(v^{(i)}(x), x_1)
\]

\[
= x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y_\mathcal{E}(Y_\mathcal{E}(a(x), x_0) b(x), x_2). \tag{4.13}
\]
With (4.12) we have
\[ Y_E(a(x), x_0)b(x) = x_0^{-n-k} \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x}{x_1} \right) ((x_1 - x)^{n+k} a(x_1) b(x)) \]
\[ = \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x}{x_0} \right) a(x_1) b(x) \]
\[ - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x - x_1}{x_0} \right) \sum_{i=1}^{r} t_{x, x_1}(f_i(x_1, x)) u^{(i)}(x) v^{(i)}(x_1), \]
from which we obtain
\[ a(x)_n b(x) = c^{(0)}(x), \quad a(x)_{n+1} b(x) = c^{(1)}(x), \ldots, \quad a(x)_{n+s} b(x) = c^{(s)}(x), \]
and \( a(x)_m b(x) = 0 \) for \( m > n + s \). Then applying \( \text{Res}_{x_0} x_0^n \) to (4.13) we obtain (4.11).

5 General existence theorems

In this section we present two existence theorems for a nonlocal vertex \( \mathbb{F}(t) \)-algebra structure and for a weak quantum vertex \( \mathbb{F}(t) \)-algebra structure. These are analogs of the existence theorem in the theory of weak quantum vertex algebras (see [Li3], [Li4]) and in the theory of vertex algebras (see [FKRW], [MP]; cf. [LL]).

We begin by reexamining Section 4 with \( \mathbb{F}(t) \) in place of \( \mathbb{F} \) as the scalar field. Let \( W \) be an \( \mathbb{F}(t) \)-module, namely a vector space over \( \mathbb{F}(t) \). By \( E(W) \) we mean the \( \mathbb{F}(t) \)-module
\[ E(W) = \text{Hom}_{\mathbb{F}(t)}(W, W((x))), \quad (5.1) \]
which is a canonical \( \mathbb{F}(t)((x)) \)-module. Let \( W_\mathbb{F} \) denote \( W \) viewed as a vector space over \( \mathbb{F} \). We see that \( E(W) \subset E(W_\mathbb{F}) \) and that every compatible subset of \( E(W) \) is also a compatible subset of \( E(W_\mathbb{F}) \).

As a convention, for \( f(t) \in \mathbb{F}(t) \) we define
\[ f(t + x) = t_{x, x} f(t + x) = e^{x \frac{d}{t}} f(t) \in \mathbb{F}((t))[[x]] \subset \mathbb{F}(t)((x)). \]

The following is immediate:

**Lemma 5.1.** Let \( W \) be an \( \mathbb{F}(t) \)-module and let \( t_1 \) be another formal variable. Then \( E(W) \) becomes an \( \mathbb{F}(t_1) \)-module with
\[ f(t_1) a(x) = f(t + x) a(x) \quad \text{for } f(t_1) \in \mathbb{F}(t_1), \ a(x) \in E(W). \quad (5.2) \]

With this we have:
Proposition 5.2. Let \( W \) be an \( \mathbb{F}(t)) \)-module and let \( U \) be a compatible subset of \( \mathcal{E}(W) \). Denote by \( (U)_{\mathbb{F}} \) the nonlocal vertex algebra over \( \mathbb{F} \) generated by \( U \). Then \( \mathbb{F}(\langle t \rangle)(U)_{\mathbb{F}} \) is a nonlocal vertex \( \mathbb{F}(\langle t \rangle) \)-algebra, and \( W \), viewed as an \( \mathbb{F}(\langle t \rangle) \)-module with \( f(t_1) \in \mathbb{F}(\langle t \rangle) \) acting as \( f(t) \), is a module of type one.

Proof. By Theorem 4.2 with \( \mathbb{F}(\langle t \rangle) \) in place of \( \mathbb{F} \), \( U \) generates a nonlocal vertex algebra \( \langle U \rangle \) over \( \mathbb{F}(\langle t \rangle) \). Furthermore, by Theorem 4.3 the span \( \mathbb{F}(\langle t \rangle)(\langle x \rangle)(\langle U \rangle) \) is also a nonlocal vertex algebra over \( \mathbb{F}(\langle t \rangle) \), satisfying the condition that

\[
Y_{\mathbb{F}}(g(x)a(x), x_0)(h(x)b(x)) = g(x + x_0)h(x)Y_{\mathbb{F}}(a(x), x_0)b(x)
\]

for \( g(x), h(x) \in \mathbb{F}(\langle t \rangle)(\langle x \rangle) \), \( a(x), b(x) \in \mathbb{F}(\langle t \rangle)(\langle x \rangle)(\langle U \rangle) \). From this we have

\[
Y_{\mathbb{F}}(f(t_1)a(x), x_0)(g(t_1)b(x)) = Y_{\mathbb{F}}(f(t + x)a(x), x_0)(g(t + x)b(x)) = f(t + x + x_0)g(t + x)Y_{\mathbb{F}}(a(x), x_0)b(x) = f(t_1 + x_0)g(t_1)Y_{\mathbb{F}}(a(x), x_0)b(x)
\]

for \( f(t_1), g(t_1) \in \mathbb{F}(\langle t \rangle) \), \( a(x), b(x) \in \mathbb{F}(\langle t \rangle)(\langle x \rangle)(\langle U \rangle) \). It follows that \( \mathbb{F}(\langle t \rangle)(\langle x \rangle)(\langle U \rangle) \) is a nonlocal vertex \( \mathbb{F}(\langle t \rangle) \)-algebra with \( \mathbb{F}(\langle t_1 \rangle)(U)_{\mathbb{F}} \) as a subalgebra. Also, by Theorem 4.2 \( W \) is a module for \( \mathbb{F}(\langle t \rangle)(\langle x \rangle)(\langle U \rangle) \) viewed as a nonlocal vertex algebra over \( \mathbb{F}(\langle t \rangle) \) with \( Y_{\mathbb{F}}(\alpha(x), x_0) = \alpha(x_0) \) for \( \alpha(x) \in \mathbb{F}(\langle t \rangle)(\langle x \rangle)(\langle U \rangle) \). For \( f(t_1) \in \mathbb{F}(\langle t \rangle) \), \( a(x) \in \mathbb{F}(\langle t \rangle)(\langle x \rangle)(\langle U \rangle) \), \( w \in W \), we have

\[
Y_{\mathbb{F}}(f(t_1)a(x), x_0)w = Y_{\mathbb{F}}(f(t + x)a(x), x_0)w = f(t + x_0)a(x_0)w = f(t_1 + x_0)Y_{\mathbb{F}}(a(x), x_0)w.
\]

Then the last assertion follows.

Definition 5.3. Let \( W \) be an \( \mathbb{F}(\langle t \rangle) \)-module. A subset \( U \) of \( \mathcal{E}(W) \) is said to be \( \mathcal{S}_t \)-local if for any \( a(x), b(x) \in U \), there exist (finitely many)

\[
u^{(i)}(x), v^{(i)}(x) \in U, f_i(x_1, x_2) \in \mathbb{F}_*(x_1, x_2) \quad (i = 1, \ldots, r)
\]

such that

\[
(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k \sum_{t = 1}^{r} \epsilon_{t, x_1, x_2} f_i(t + x_1, t + x_2) u^{(i)}(x_2)v^{(i)}(x_1)
\]

for some nonnegative integer \( k \) depending on \( a(x) \) and \( b(x) \).

With this notion we have:

Theorem 5.4. Let \( W \) be an \( \mathbb{F}(\langle t \rangle) \)-module and let \( U \) be an \( \mathcal{S}_t \)-local subset of \( \mathcal{E}(W) \). Then \( U \) is compatible. Furthermore, \( U \) is an \( \mathcal{S}_t \)-local subset of the nonlocal vertex \( \mathbb{F}(\langle t_1 \rangle) \)-algebra \( \mathbb{F}(\langle t \rangle)(U)_{\mathbb{F}} \), which was obtained in Proposition 5.2, and \( \mathbb{F}(\langle t \rangle)(U)_{\mathbb{F}} \) is a weak quantum vertex \( \mathbb{F}(\langle t \rangle) \)-algebra and \( W \), viewed as an \( \mathbb{F}(\langle t \rangle) \)-module with \( f(t_1) \in \mathbb{F}(\langle t \rangle) \) acting as \( f(t) \), is a module of type one.

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Proof. Let \( f(x_1, x_2) \in \mathbb{F}_*(x_1, x_2) \). We have \( f(t + x_1, t + x_2) \in \mathbb{F}_*(t, x_1, x_2) \) with
\[
\iota_{t,x_1} f(t + x_1, t + x_2) \in \mathbb{F}((t))((x_2))((x_1)).
\]
We can also view \( f(t + x_1, t + x_2) \) as an element of \( \mathbb{F}((t))_*(x_1, x_2) \) (with \( \mathbb{F}((t)) \) as the scalar field), which we denote by \( f_1(x_1, x_2) \), noticing that for \( q/p \in \mathbb{F}_*(x_1, x_2) \) with \( q \in \mathbb{F}[x_1, x_2] \), \( p \in \mathbb{F}[x_1, x_2] \), we have
\[
q(t + x_1, t + x_2) \in \mathbb{F}[t, x_1, x_2] \subset \mathbb{F}((t))[x_1, x_2],
\]
\[
p(t + x_1, t + x_2) \in \mathbb{F}[t, x_1, x_2] \subset \mathbb{F}((t))[x_1, x_2].
\]
With the iota-map \( \iota_{x_2,x_1}: \mathbb{F}((t))_*(x_1, x_2) \rightarrow \mathbb{F}((t))((x_2))((x_1)) \), we have
\[
\iota_{x_2,x_1} f_1(x_1, x_2) \in \mathbb{F}((t))((x_2))((x_1)).
\]
It is straightforward to show that
\[
\iota_{x_2,x_1} f_1(x_1, x_2) = \iota_{t,x_1} f(t + x_1, t + x_2).
\]
In view of this, we see that an \( \mathcal{S}_r \)-local subset of \( \mathcal{E}(W) \) is also an \( \mathcal{S}(x_1, x_2) \)-local subset with \( \mathbb{F}((t)) \) in place of \( \mathbb{F} \). By Proposition 4.6, every \( \mathcal{S}_r \)-local subset is compatible.

Let \( a(x), b(x) \in U \). There exist
\[
u^{(i)}(x), \mu^{(i)}(x) \in U, f_i(x_1, x_2) \in \mathbb{F}_*(x_1, x_2) \quad (i = 1, \ldots, r)
\]
such that (5.3) holds for some nonnegative integer \( k \). Viewing \( f_i(t + x_1, t + x_2) \) as elements of \( \mathbb{F}((t))_*(x_1, x_2) \), from Proposition 4.9 with \( \mathbb{F}((t)) \) in place of \( \mathbb{F} \), we have
\[
(x_1 - x_2)^k Y_{\mathcal{E}}(a(x), x_1) Y_{\mathcal{E}}(b(x), x_2)
= (x_1 - x_2)^k \sum_{i=1}^{r} \iota_{t,x_1} f_i(t + x_1, t + x_2) Y_{\mathcal{E}}(\mu^{(i)}(x), x_2) Y_{\mathcal{E}}(\nu^{(i)}(x), x_1)
= (x_1 - x_2)^k \sum_{i=1}^{r} \iota_{t,x_1} f_i(t_1 + x_1 + x_2) Y_{\mathcal{E}}(\mu^{(i)}(x), x_2) Y_{\mathcal{E}}(\nu^{(i)}(x), x_1).
\]
This proves that \( U \) is an \( \mathcal{S}_{t_1} \)-local subset of the nonlocal vertex \( \mathbb{F}((t_1)) \)-algebra \( \mathbb{F}((t_1))\langle U \rangle_{\mathbb{F}} \). Then by Lemma 2.18, \( \mathbb{F}((t_1))\langle U \rangle_{\mathbb{F}} \) is a weak quantum vertex \( \mathbb{F}((t_1)) \)-algebra. The last assertion on module structure has already been established in Proposition 5.2.

Now, we are ready to present our first existence theorem.

**Theorem 5.5.** Let \( V \) be an \( \mathbb{F}((t)) \)-module, \( 1 \) a vector of \( V \), \( D \) an \( \mathbb{F} \)-linear operator on \( V \), \( U \) an \( \mathbb{F}((t)) \)-submodule of \( V \),
\[
Y_0(\cdot, x) : U \rightarrow \mathcal{E}(V) = \text{Hom}_{\mathbb{F}((t))}(V, V((x)))
\]
\[
u \mapsto Y_0(\nu, x) = u(x) = \sum_{n \in \mathbb{Z}} u_n x^{-n-1},
\]

For the structure of a nonlocal vertex algebra over \( \mathbb{F} \), noticing that for vectors in the assumption. It follows from [Li3] (Theorem 6.3) that \( (V,\{,1 \} \) carries the structure of a nonlocal vertex algebra over \( \mathbb{F} \). Then it follows from the first part with \((U,Y,\cdot)\)-algebra. Furthermore, if \( U(x) \) is \( S\)-local, \( V \) is a weak quantum vertex \( \mathbb{F}(t) \)-algebra.

Proof. First consider the case with \( 1 \in U \). Then \( \mathbb{F}(t)1 \in U \). Since \( (f(t)1)_{-1}1 = f(t)1 \) for \( f(t) \in \mathbb{F}(t) \), we see that \( V \) is actually linearly spanned over \( \mathbb{F} \) by those vectors in the assumption. It follows from [Li3] (Theorem 6.3) that \( (V,Y,1) \) carries the structure of a nonlocal vertex algebra over \( \mathbb{F} \). It is clear that \( U \) generates \( V \) as a nonlocal vertex algebra over \( \mathbb{F} \) and we have

\[
Y(f(t)u,x)g(t) = Y_0(f(t)u,x)g(t) = f(t+x)g(t)Y_0(u,x) = f(t+x)g(t)Y(u,x)
\]

for \( f(t),g(t) \in \mathbb{F}(t) \), \( u \in U \). It follows from Lemma 2.17 that \( V \) is a nonlocal vertex \( \mathbb{F}(t) \)-algebra. Furthermore, if \( \{Y_0(u,x) \mid u \in U\} \) \( (= U(x) \) is \( S\)-local, it follows from Lemma 2.18 that \( V \) is a weak quantum vertex \( \mathbb{F}(t) \)-algebra.

Now, assume \( 1 \notin U \). Then \( U \cap \mathbb{F}(t)1 = 0 \). Set \( \bar{U} = U \oplus \mathbb{F}(t)1 \). Extend the map \( Y_0 \) to \( \bar{U} \) by defining \( \bar{Y}_0(f(t)1,x) = f(t+x) \) for \( f(t) \in \mathbb{F}(t) \). We have

\[
[D,Y_0(f(t)1,x)] = [D,f(t+x)] = f'(t+x) = \frac{d}{dx}Y_0(f(t)1,x),
\]

\[
\bar{Y}_0(f(t)1,x)1 = f(t+x)1 \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} \bar{Y}_0(f(t)1,x)1 = f(t)1.
\]

Noticing that for \( f(t) \in \mathbb{F}(t) \), \( a(x) \in \mathcal{E}(V) \),

\[
Y_\mathcal{E}(f(t+x),x_0)a(x) = (f(t+x_1)a(x))|_{x_1=x+x_0} = f(t+x+x_0)a(x),
\]

we get

\[
\psi_x(\bar{Y}_0(f(t)1,x_0)v) = \psi_x(f(t+x_0)v) = f(t+x+x_0)\psi_x(v) = Y_\mathcal{E}(f(t+x),x_0)\psi_x(v)
\]

for \( v \in V \). Then it follows from the first part with \((\bar{U},Y_0)\) in place of \((\bar{U},Y_0)\).
Thus θ retains all the assumptions that do not involve D. First we prove the following (cf. [4], Theorem 2.9):

**Lemma 5.6.** Let V be a nonlocal vertex \( \mathbb{F}((t)) \)-algebra and let \((W,Y_W)\) be a type one V-module. Suppose that \( e \) is a vector in W and U is an \( S_t \)-local subset of V such that

\[
Y_W(u,x)e \in W[[x]] \quad \text{for} \ u \in U.
\]

Then \( Y_W(v,x)e \in W[[x]] \) for \( v \in \mathbb{F}((t))(U)_{\mathbb{F}} \) and the map \( \theta : \mathbb{F}((t))(U)_{\mathbb{F}} \to W \), sending \( v \) to \( v_{−1}e \) for \( v \in \mathbb{F}((t))(U)_{\mathbb{F}} \), is a homomorphism of type one \( \mathbb{F}((t))(U)_{\mathbb{F}} \)-modules with \( \theta(1) = e \).

**Proof.** Set

\[
K = \{ v \in V \mid Y_W(v,x)e \in W[[x]] \}.
\]

It can be readily seen that \( K \) is an \( \mathbb{F}((t)) \)-submodule of V. We must prove \( \mathbb{F}((t))(U)_{\mathbb{F}} \subset K \). From assumption, \( \mathbb{F}((t))1 + \mathbb{F}((t))U \) is an \( S_t \)-local \( \mathbb{F}((t)) \)-submodule of \( K \). Then there exists a maximal \( S_t \)-local \( \mathbb{F}((t)) \)-submodule \( A \) of \( K \), containing \( \mathbb{F}((t))1 + \mathbb{F}((t))U \). We now prove \( \mathbb{F}((t))(U)_{\mathbb{F}} \subset A \) \((\subset K)\). As \( \{1\} \cup U \subset A \), it suffices to prove that \( A \) is closed, i.e., \( A^{(2)} \subset A \). From the proof of Lemma 2.18 \( A^{(2)} \) is \( S_t \)-local. Now we prove \( A^{(2)} \subset K \). Let \( u, v \in A \). From Proposition 2.16 there exist \( u^{(i)}, v^{(i)} \in A \), \( f_i(x_1,x_2) \in \mathbb{F}((x_1,x_2)) \) \((i = 1,\ldots,r)\) such that the Jacobi identity holds. By Lemma 6.12 for \( 1 \leq i \leq r \), the series

\[
u_{t,x_2,x_1}(f_i(t + x_1,t + x_2))
\]

involves only nonnegative powers of \( x_1 \). By applying Res_{x_1} to \( f_i \), we see that

\[
Y_W(Y(u,x_0)v,x_2)e \in W[[x_2]]((x_0))
\]

which implies \( u_m v \in K \) for \( m \in \mathbb{Z} \). Thus \( A^{(2)} \subset K \). Since \( 1 \in A \), we have \( A \subset A^{(2)} \). As \( A \) is maximal we must have \( A = A^{(2)} \), proving that \( A \) is closed. Thus we have \( \mathbb{F}((t))(U)_{\mathbb{F}} \subset A \subset K \), proving the first assertion.

By Lemma 6.1 of [4], \( \theta \) is a module homomorphism for \( \mathbb{F}((t))(U)_{\mathbb{F}} \) viewed as a nonlocal vertex algebra over \( \mathbb{F} \). Furthermore, from Lemma 2.18 \( \theta \) is \( \mathbb{F}((t)) \)-linear. Thus \( \theta \) is a homomorphism of type one \( \mathbb{F}((t))(U)_{\mathbb{F}} \)-modules. \qed

Now, we have:

**Theorem 5.7.** Let \( V,1,Y_0(\cdot,x),U(x) \) and \( \psi_x \) be given as in Theorem 5.5 and retain all the assumptions that do not involve D. In addition, assume that \( U(x) \) is \( S_t \)-local. Set \( Y(v,x) = \psi_x(v) \in \mathcal{E}(V) \) for \( v \in V \). Then \( Y(\cdot,x) \) extends the map \( Y_0(\cdot,x) \), and \((V,Y,1)\) carries the structure of a weak quantum vertex \( \mathbb{F}((t)) \)-algebra.
Proof. Recall from Lemma 5.1 the $\mathbb{F}((t_1))$-module structure on $\mathcal{E}(V)$ with $t_1$ another formal variable. For $u \in U$, $f(t_1) \in \mathbb{F}((t_1))$, we have

$$f(t_1)u(x) = f(t + x)Y_0(u, x) = Y_0(f(t)u, x) = (f(t)u)(x),$$

so that $U(x)$ is an $\mathbb{F}((t_1))$-submodule of $\mathcal{E}(V)$. Since $U(x)$ is an $S_t$-local subset of $\mathcal{E}(V)$, by Theorem 5.4, $U(x)$ is compatible and $\mathbb{F}((t_1))\langle U(x) \rangle_\theta$ is a weak quantum $\mathbb{F}((t_1))$-algebra, where $\langle U(x) \rangle_\theta$ denotes the nonlocal vertex algebra over $\mathbb{F}$, generated by $U(x)$ inside $\mathcal{E}(V)$. Set $E = \mathbb{F}((t_1))\langle U(x) \rangle_\theta$. By Theorem 5.4 $U(x)$ is an $S_t$-local $\mathbb{F}((t_1))$-submodule of $E$ and $(V, Y_V)$ is a type one $E$-module, where $V$ is viewed as an $\mathbb{F}((t_1))$-module with $f(t_1)$ acting as $f(t)$ and where $Y_V(a(x), x_0) = a(x_0)$ for $a(x) \in E$. From our assumption we have

$$Y_V(u(x), x_0)1 = u(x_0)1 \in V[[x_0]] \quad \text{for } u \in U.$$

By Lemma 5.6 there exists an $E$-module homomorphism $\phi$ from $E$ to $V$ such that $\phi(1_V) = 1$ and

$$\phi(f(t + x)a(x)) = \phi(f(t_1)a(x)) = f(t_1)\phi(a(x)) = f(t)\phi(a(x))$$

for $f(t) \in \mathbb{F}((t))$, $a(x) \in E$. For $u \in U$, $a(x) \in E$, we have

$$\phi(Y_E(u(x), x_0)a(x)) = Y_V(u(x), x_0)\phi(a(x)) = u(x_0)\phi(a(x)).$$

That is,

$$\phi(u(x)_n a(x)) = u(x)_n \phi(a(x)) \quad \text{for } u \in U, \ a(x) \in E, \ n \in \mathbb{Z}.$$ 

It follows that $\psi_x$ is an $\mathbb{F}((t_1))$-isomorphism from $V$ to $E$ with $\phi$ as the inverse. Then we have a weak quantum vertex $\mathbb{F}((t_1))$-algebra structure on $V$, transported from $E$, where $1$ ($= \phi(1_V)$) is the vacuum vector. The defined map $Y(\cdot, x)$ coincides with the transported structure, as for $v \in V$,

$$\phi Y_E(\psi_x(v), x_0)\psi_x = Y_V(\psi_x(v), x_0) = \psi_x(v) = Y(v, x_0)$$

(recall that $\phi$ is a module homomorphism). Furthermore, for $u \in U$, we have

$$\psi_x(u) = \psi_x(u_{-1}1) = u(x_{-1})1_V = u(x),$$

so that

$$Y(u, x) = \psi_x(u) = u(x) = Y_0(u, x).$$

Now, the proof is completed. \[\square\]
6 Example of quantum vertex $\mathbb{C}((t))$-algebras

In this section we first associate weak quantum $\mathbb{C}((t))$-algebras to quantum affine algebras and then we construct an example of non-degenerate quantum vertex $\mathbb{C}((t))$-algebras from a certain quantum $\beta\gamma$-system.

First, we follow [FJ] (see also [Dr]) to present the quantum affine algebras. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank $l$ of type $A$, $D$, or $E$ and let $A = (a_{ij})$ be the Cartan matrix. Let $q$ be a nonzero complex number. For $1 \leq i, j \leq l$, set

$$f_{ij}(x) = (q^{a_{ij}} x - 1)/(x - q^{a_{ij}}) \in \mathbb{C}(x).$$  \hspace{1cm} (6.1)

Furthermore, set

$$g_{ij}(x)^{\pm 1} = \tau_{x,0} f_{ij}(x)^{\pm 1} \in \mathbb{C}[[x]],$$  \hspace{1cm} (6.2)

where $\tau_{x,0} f_{ij}(x)^{\pm 1}$ are the formal Taylor series expansions of $f_{ij}(x)^{\pm 1}$ at 0. The quantum affine algebra $U_q(\mathfrak{g})$ is (isomorphic to) the associative algebra with identity 1 and with generators

$$X_{ik}^\pm, \phi_{im}, \psi_{in}, \gamma^{1/2}, \gamma^{-1/2}$$

for $1 \leq i \leq l$, $k \in \mathbb{Z}$, $m \in -\mathbb{N}$, $n \in \mathbb{N}$, where $\gamma^{\pm 1/2}$ are central elements, satisfying the relations below, written in terms of the following generating functions:

$$X_i^+(z) = \sum_{k \in \mathbb{Z}} X_{ik}^+ z^{-k}, \quad \phi_i(z) = \sum_{m \in -\mathbb{N}} \phi_{im} z^{-m}, \quad \psi_i(z) = \sum_{n \in \mathbb{N}} \psi_{in} z^{-n}.$$  \hspace{1cm} (6.4)

The relations are

$$\gamma^{1/2} \gamma^{-1/2} = \gamma^{-1/2} \gamma^{1/2} = 1,$$

$$\phi_0 \psi_0 = \psi_0 \phi_0 = 1,$$

$$[\phi_i(z), \phi_j(w)] = 0, \quad [\psi_i(z), \psi_j(w)] = 0,$$

$$\phi_i(z) \psi_j(w) \phi_i(z)^{-1} \psi_j(w)^{-1} = g_{ij}(z/w \gamma)/g_{ij}(z \gamma/w),$$

$$\phi_i(z) X_j^+(w) \phi_i(z)^{-1} = g_{ij}(z/w \gamma^{1/2}) X_j^+(w),$$

$$\psi_i(z) X_j^-(w) \psi_i(z)^{-1} = g_{ij}(w/z \gamma^{1/2}) X_j^-(w),$$

$$(z - q^{a_{ij}} w) X_i^+(z) X_j^+(w) = (q^{a_{ij}} z - w) X_j^+(w) X_i^+(z),$$

$$[X_i^+(z), X_j^-(w)] = \frac{\delta_{ij}}{q - q^{-1}} \left( \frac{z}{w \gamma} \right) \psi_i(w \gamma^{1/2}) - \delta \left( \frac{z \gamma}{w} \right) \phi_i(z \gamma^{1/2})$$

and there is one more set of relations of Serre type.

A $U_q(\mathfrak{g})$-module $W$ is said to be restricted if for any $w \in W$, $X_{ik}^+ w = 0$ and $\psi_k w = 0$ for $1 \leq i \leq l$ and for $k$ sufficiently large. We say $W$ is of level $\ell \in \mathbb{C}$ if $\gamma^{1/2}$ act on $W$ as scalars $q^{\pm \ell/4}$. (Rigorously speaking, one needs to choose a branch of log $q$.) We have (cf. [Li3], Proposition 4.9):
**Proposition 6.1.** Let $q$ and $\ell$ be complex numbers with $q \neq 0$ and let $W$ be a restricted $U_q(\mathfrak{g})$-module of level $\ell$. Set

$$U_W = \{ \phi_i(x), \psi_i(x), X_i^\pm(x) \mid 1 \leq i \leq l \}.$$  

Then $U_W$ is a quasi $S(x_1, x_2)$-local subset of $\mathcal{E}(W)$ and $\mathbb{C}(\langle x \rangle)(U_W)$ is a weak quantum vertex $\mathbb{C}(\langle t \rangle)$-algebra with $W$ as a type zero quasi module, where $\langle U_W \rangle$ denotes the nonlocal vertex algebra over $\mathbb{C}$ generated by $U_W$.

**Proof.** As $W$ is a restricted module, we have $U_W \subset \mathcal{E}(W)$, noticing that $\phi_i(x) \in \text{End}W[[x]] \subset \mathcal{E}(W)$. Note that

$$g_{ij}(z/w) = t_{w,z}(q^{a_{ij}}z - w)/(z - q^{a_{ij}}w).$$

Then

$$g_{ij}(z/w\gamma)/g_{ij}(z\gamma/w) = t_{w,z}(q^{a_{ij}}z - w\gamma)(z\gamma - q^{a_{ij}}w)/(z - q^{a_{ij}}w\gamma - w).$$

With this, from the defining relation we get

$$(z - q^{a_{ij}}w\gamma)(q^{a_{ij}}z\gamma - w)\phi_i(z)\psi_j(w) = (q^{a_{ij}}z - w\gamma)(z\gamma - q^{a_{ij}}w)\psi_j(w)\phi_i(z). \quad (6.5)$$

With

$$g_{ij}(z/w\gamma^{\pm1/2}) = t_{w,z}(q^{a_{ij}}z - w\gamma^{\pm1/2})/(z - q^{a_{ij}}w\gamma^{\pm1/2}),$$

we get

$$(z - q^{a_{ij}}w\gamma^{1/2})\phi_i(z)X_j^+(w) = (q^{a_{ij}}z - w\gamma^{1/2})X_j^+(w)\phi_i(z), \quad (6.6)$$

$$(q^{a_{ij}}z - w\gamma^{-1/2})\phi_i(z)X_j^-(w) = (z - q^{a_{ij}}w\gamma^{-1/2})X_j^-(w)\phi_i(z). \quad (6.7)$$

Similarly, we have

$$(w - q^{a_{ij}}z\gamma^{1/2})\psi_i(z)X_j^+(w) = (q^{a_{ij}}w - z\gamma^{1/2})X_j^+(w)\psi_i(z), \quad (6.8)$$

$$(q^{a_{ij}}z - w\gamma^{1/2})\psi_i(z)X_j^-(w) = (z - q^{a_{ij}}w\gamma^{1/2})X_j^-(w)\psi_i(z). \quad (6.9)$$

As $(z - x)\delta(z) = 0$, we have

$$(z - w\gamma)(z\gamma - w)X_i^+(z)X_j^-(w) = (z - w\gamma)(z\gamma - w)X_j^-(w)X_i^+(z). \quad (6.10)$$

Now, it is clear that $U_W$ is a quasi $S(x_1, x_2)$-local subset of $\mathcal{E}(W)$. The rest follows immediately from Theorem [4.3]. □

For the rest of section, we construct a quantum vertex $\mathbb{C}(\langle t \rangle)$-algebra from a quantum $\beta\gamma$-system.
Definition 6.2. Let $q$ be a nonzero complex number. Define $A_q(\beta\gamma)$ to be the associative algebra over $\mathbb{C}$ with generators $\beta_n, \gamma_n \ (n \in \mathbb{Z})$, subject to relations

$$
\beta(x_1)\beta(x_2) = \left(\frac{qx_2 - x_1}{x_2 - qx_1}\right)\beta(x_2)\beta(x_1),
$$

$$
\gamma(x_1)\gamma(x_2) = \left(\frac{qx_2 - x_1}{x_2 - qx_1}\right)\gamma(x_2)\gamma(x_1),
$$

$$
\beta(x_1)\gamma(x_2) - \left(\frac{x_2 - qx_1}{qx_2 - x_1}\right)\gamma(x_2)\beta(x_1) = x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),
$$

where $\beta(x) = \sum_{n \in \mathbb{Z}} \beta_n x^{-n-1}$ and $\gamma(x) = \sum_{n \in \mathbb{Z}} \gamma_n x^{-n-1}$.

This algebra $A_q(\beta\gamma)$ belongs to a family of algebras, known as Zamolodchikov-Faddeev algebras (see [ZZ], [F]). Notice that $A_q(\beta\gamma)$ becomes the standard $\beta\gamma$-algebra when $q = 1$, while $A_q(\beta\gamma)$ becomes a Clifford algebra when $q = -1$. For these two special cases, it is well known that a vertex algebra for $q = 1$, or a vertex superalgebra for $q = -1$ can be associated to the algebra $A_q(\beta\gamma)$ canonically. In the following, we shall mainly deal with the case with $q \neq 1$. (All the results will still hold for $q = 1$, though a different proof is needed.)

 Remark 6.3. Notice that the defining relations involve infinite sums, so that $A_q(\beta\gamma)$ is in fact a topological algebra. One can give a precise definition using the procedure in [FZ] for defining the universal enveloping algebra $U(V)$ of a vertex operator algebra $V$. However, for this paper we shall only need a category of modules for a free associative algebra. By an $A_q(\beta\gamma)$-module we mean a vector space $W$ on which the set $\{\beta_n, \gamma_n \mid n \in \mathbb{Z}\}$ acts as linear operators, satisfying the condition that for every $w \in W$,

$$
\beta_n w = \gamma_n w = 0 \quad \text{for } n \text{ sufficiently large}
$$

and all the relations in Definition 6.2 after applied to $w$ hold.

Definition 6.4. Let $q$ be a nonzero complex number as before. Define $A_{t,q}(\beta\gamma)$ to be the associative algebra over $\mathbb{C}((t))$ with generators $\beta_t(n), \gamma_t(n) \ (n \in \mathbb{Z})$, subject to relations

$$
\beta_t(x_1)\beta_t(x_2) = \left(\frac{(q - 1)t + qx_2 - x_1}{(1 - q)t + x_2 - qx_1}\right)\beta_t(x_2)\beta_t(x_1),
$$

$$
\gamma_t(x_1)\gamma_t(x_2) = \left(\frac{(q - 1)t + qx_2 - x_1}{(1 - q)t + x_2 - qx_1}\right)\gamma_t(x_2)\gamma_t(x_1),
$$

$$
\beta_t(x_1)\gamma_t(x_2) - \left(\frac{(1 - q)t + x_2 - qx_1}{(1 - q)t + qx_2 - x_1}\right)\gamma_t(x_2)\beta_t(x_1) = x_1^{-1}\delta\left(\frac{x_2}{x_1}\right),
$$

where $\beta_t(x) = \sum_{n \in \mathbb{Z}} \beta_t(n) x^{-n-1}$, $\gamma_t(x) = \sum_{n \in \mathbb{Z}} \gamma_t(n) x^{-n-1}$, and where when $q \neq 1$, the rational-function coefficients are expanded in the non-positive powers of $t$, e.g.,

$$
\frac{1}{(1-q)t + x_2 - qx_1} = \sum_{i \geq 0} (1-q)^{-i-1} t^{-i-1} (x_2 - qx_1)^i \in \mathbb{C}((t))[\llbracket x_1, x_2 \rrbracket].
$$

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By an $A_{t,q}(βγ)$-module we mean a $\mathbb{C}((t))$-module $W$ on which $β_t(n), γ_t(n)$ for $n \in \mathbb{Z}$ act as linear operators, satisfying the condition that for any $w \in W$, $β_t(n)w = γ_t(n)w = 0$ for $n$ sufficiently large and those defining relations after applied to $w$ hold. A vacuum $A_{t,q}(βγ)$-module is an $A_{t,q}(βγ)$-module $W$ equipped with a vector $w_0$ that generates $W$ such that

$$β_t(n)w_0 = γ_t(n)w_0 = 0 \quad \text{for } n \geq 0.$$  

Let $\tilde{A}$ be the free associative algebra over $\mathbb{C}((t))$ with generators $\tilde{β_t}(n), \tilde{γ_t}(n)$ for $n \in \mathbb{Z}$. Then an $A_{t,q}(βγ)$-module amounts to an $\tilde{A}$-module $W$ such that for any $w \in W$, $\tilde{β_t}(n)w = \tilde{γ_t}(n)w = 0$ for $n$ sufficiently large and such that the three defining relations after applied to $w$ hold. Define

$$\deg 1 = 0, \quad \deg \tilde{β_t}(n) = \deg \tilde{γ_t}(n) = -n - \frac{1}{2} \quad \text{for } n \in \mathbb{Z},$$

to make $\tilde{A}$ a $\frac{1}{2}\mathbb{Z}$-graded algebra, where the degree-$k$ subspace is denoted by $\tilde{A}(k)$ for $k \in \frac{1}{2}\mathbb{Z}$. We define an increasing filtration $F = \{F_k\}_{k \in \frac{1}{2}\mathbb{Z}}$ of $\tilde{A}$ by $F_k = \oplus_{p \leq k} \tilde{A}(p)$ for $k \in \frac{1}{2}\mathbb{Z}$. Clearly,

$$F_p \cdot F_k \subset F_{p+k} \quad \text{for } p, k \in \frac{1}{2}\mathbb{Z}.$$  

**Remark 6.5.** Let $B$ be the associative algebra over $\mathbb{C}$ with generators $a_n, b_n$ ($n \in \mathbb{Z}$), subject to relations

$$a_m a_n = -a_n a_m, \quad b_m b_n = -b_n b_m, \quad a_m b_n + b_n a_m = δ_{m+n+1,0}$$

for $m, n \in \mathbb{Z}$. In terms of the generating functions

$$a(x) = \sum_{n \in \mathbb{Z}} a_n x^{-n-1}, \quad b(x) = \sum_{n \in \mathbb{Z}} b_n x^{-n-1},$$

the above defining relations amount to

$$a(x_1)a(x_2) = -a(x_2)a(x_1), \quad b(x_1)b(x_2) = -b(x_2)b(x_1),$$

$$a(x_1)b(x_2) + b(x_2)a(x_1) = x_1^{-1} δ \left( \frac{x_2}{x_1} \right).$$

Let $J$ be the left ideal of $B$, generated by $a_n, b_n$ for $n \geq 0$. Set

$$V_B = B/J,$$

a (left) $B$-module, set $1 = 1 + J \in V_B$, and set

$$a = a_{-1}1, \quad b = b_{-1}1 \in V_B.$$  

It is well known (cf. [FFR]) that $V_B$ is an irreducible $B$-module. It follows that if $U$ is a nonzero $B$-module with a vector $u_0$ satisfying the condition that $U = Bu_0$ and $a_n u_0 = b_n u_0 = 0$ for $n \geq 0$, then $U$ must be isomorphic to $V_B$. It is also well known (see [FFR]) that there exists a vertex superalgebra structure on $V_B$, which is uniquely determined by the conditions that $1$ is the vacuum vector and that $Y(a, x) = a(x), Y(b, x) = b(x)$.

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Proposition 6.6. Assume \( q \neq 1 \). Let \( (W, w_0) \) be a nonzero vacuum \( A_{t,q}(\beta \gamma) \)-module. Then \( F_{-1/2}w_0 = 0 \) and \( W \) is irreducible with \( \text{End}_{A_{t,q}(\beta \gamma)}(W) = \mathbb{C}(t) \).

Proof. From definition, \( W \) is an \( \tilde{A} \)-module satisfying that \( \tilde{\beta}_t(n)w_0 = \tilde{\gamma}_t(n)w_0 = 0 \) for \( n \geq 0 \) and \( W = \tilde{A}w_0 \). We define an increasing sequence \( W[k] \) with \( k \in \frac{1}{2} \mathbb{Z} \) as follows: For \( k < 0 \), set \( W[k] = 0 \), and for \( k \in \frac{1}{2} \mathbb{N} \), let \( W[k] \) be the span of the vectors

\[
a^{(1)}(-m_1) \cdots a^{(r)}(-m_r)w_0
\]

for \( r \geq 0 \), \( a^{(1)}, \ldots, a^{(r)} \in \{ \tilde{\beta}_t, \tilde{\gamma}_t \} \), \( m_i \geq 1 \) with

\[
\deg a^{(1)}(-m_1) + \cdots + \deg a^{(r)}(-m_r) = (m_1 - 1/2) + \cdots + (m_r - 1/2) \leq k.
\]

In the following we prove that \( W[k] = F_kw_0 \) for \( k \in \frac{1}{2} \mathbb{Z} \).

From definition, we have \( W[0] = \mathbb{C}(t)w_0 \) and

\[
a(m)W[k] \subset W[k - m - 1/2] \quad \text{for } a \in \{ \tilde{\beta}_t, \tilde{\gamma}_t \}, \quad m < 0, \quad k \in \frac{1}{2} \mathbb{Z}. \tag{6.11}
\]

Next, we show that this is also true for \( m \geq 0 \). Notice that for \( a, b \in \{ \tilde{\beta}_t, \tilde{\gamma}_t \}, \quad m, n \in \mathbb{Z}, \quad w \in W \), from the defining relations in Definition 6.4 we have

\[
a(m)b(n)w = -b(n)a(m)w + \sum_{i,j \geq 0, i+j \geq 1} f_{i,j}(t)b(n+i)a(m+j)w + \lambda \delta_{m+n+1,0}w,
\]

where \( f_{i,j}(t) \in \mathbb{C}(t) \), \( \lambda = 0 \), or \( 1 \). Then using induction on \( k \), we can show that \( \text{(6.11)} \) also holds for \( m \geq 0 \), noting that \( a(m)w_0 = 0 \) for \( m \geq 0 \). It follows that \( F_kw_0 = F_kW[0] \subset W[k] \) for \( k \in \frac{1}{2} \mathbb{Z} \). From definition, we also have \( W[k] \subset F_kw_0 \).

Therefore, \( F_kw_0 = W[k] \). Consequently, \( F_kw_0 = W[k] = 0 \) for \( k < 0 \). In particular, we have \( F_{-1/2}w_0 = 0 \).

Now, we prove \( \text{End}_{A_{t,q}(\beta \gamma)}(W) = \mathbb{C}(t) \). We see that the subspaces \( W[k] \) \( (k \in \frac{1}{2} \mathbb{N}) \) form an increasing filtration of \( W \), satisfying that \( F_pW[k] \subset W[k + p] \) for \( k, p \in \frac{1}{2} \mathbb{Z} \). Form the associated graded space \( \text{Gr}_F(W) = \oplus_{k \in \frac{1}{2} \mathbb{N}} (W[k]/W[k - 1/2]) \), which is naturally an \( \tilde{A} \)-module as \( \text{Gr}_F(\tilde{A}) \simeq \tilde{A} \). Let \( \rho : \tilde{A} \to \text{End}(\text{Gr}_F(W)) \) be the corresponding algebra homomorphism. On \( \text{Gr}_F(W) \), the following relations hold:

\[
\rho(\tilde{\beta}_t(x))\rho(\tilde{\beta}_t(z)) = -\rho(\tilde{\beta}_t(z))\rho(\tilde{\beta}_t(x)), \quad \rho(\tilde{\gamma}_t(x))\rho(\tilde{\gamma}_t(z)) = -\rho(\tilde{\gamma}_t(z))\rho(\tilde{\gamma}_t(x)),
\]

\[
\rho(\tilde{\beta}_t(x))\rho(\tilde{\gamma}_t(z)) + \rho(\tilde{\gamma}_t(z))\rho(\tilde{\beta}_t(x)) = x_1^{-1} \delta \left( \frac{x_2}{x_1} \right).
\]

We see that \( \rho(\tilde{A}) \) is a homomorphism image of \( \mathbb{C}(t) \otimes B \) (where \( B \) is the algebra defined in Remark 6.5), so that \( \text{Gr}_F(W) \) is naturally a \( (\mathbb{C}(t) \otimes B) \)-module. Since \( W = \tilde{A}w_0 \), we have \( \text{Gr}_F(W) = \tilde{A}w_0 \) with \( w_0 \) identified with \( w_0 + W[-1/2] \in W[0]/W[-1/2] \). Then

\[
\text{Gr}_F(W) = (\mathbb{C}(t) \otimes B)w_0.
\]
From Remark 6.3 the $B$-submodule of $\text{Gr}_F(W)$, generated from $w_0$, is irreducible and isomorphic to $V_B$. As $V_B$ is of countable dimension over $C$, we have $\text{End}_B(V_B) = C$. Then one can show (cf. [Li2]) that $C((t)) \otimes V_B$ is an irreducible $C((t)) \otimes B$-module. It follows that $\text{Gr}_F(W) \simeq C((t)) \otimes V_B$ as a $C((t)) \otimes B$-module. Thus $\text{Gr}_F(W)$ is an irreducible $\mathcal{A}$-module.

Set
$$\Omega(W) = \{ w \in W \mid \beta_t(n) w = 0 = \gamma_t(n) w \quad \text{for } n \geq 0 \}.$$ It is known that $\{ v \in V_B \mid a_n v = b_n v = 0 \quad \text{for } n \geq 0 \} = C1$. Then
$$\{ w \in C((t)) \otimes V_B \mid a_n w = b_n w = 0 \quad \text{for } n \geq 0 \} = C((t))1.$$ Using this and the filtration $\mathcal{F}$ we obtain $\Omega(W) = C((t))w_0$. Notice that for any endomorphism $\psi$ of $W$, $\psi(w_0) \in \Omega(W)$ and $\psi(w_0)$ determines $\psi$ uniquely. Then it follows that $\text{End}_{A_{t,q}(\beta\gamma)}(W) = C((t))$.

To prove that $W$ is irreducible, let $M$ be any submodule of $W$. Then $M \cap W[k]$ with $k \in \mathbb{Z}$ form an increasing filtration of $M$ and the associated graded space $\text{Gr}_F(M)$ can be considered canonically as a subspace of $\text{Gr}_F(W)$. It is clear that $\text{Gr}_F(M)$ is an $\mathcal{A}$-submodule. As $\text{Gr}_F(W)$ is an irreducible $\mathcal{A}$-module, we must have either $\text{Gr}_F(M) = 0$ or $\text{Gr}_F(M) = \text{Gr}_F(W)$. If $\text{Gr}_F(M) = 0$, we have $M \cap W[k] = M \cap K[k-1/2]$ for all $k \in \mathbb{Z}$. Since $W[k] = 0$ for $k$ sufficiently negative, we have $M \cap W[k] = 0$ for all $k$. Thus $M = 0$. On the other hand, if $\text{Gr}_F(M) = \text{Gr}_F(W)$, we have $M \cap W[k] + W[k-1/2] = W[k]$ for all $k$. Using induction we get $W[k] \subset M$ for all $k$. Thus $M = W$. This proves that $W$ is irreducible, concluding the proof. □

The following gives the existence of a nonzero vacuum $V_{t,q}(\beta\gamma)$-module:

**Proposition 6.7.** Let $V_B$ be the vertex superalgebra as in Remark 6.3. There exists linear maps
$$\Phi^\pm(t) : V_B \to V_B \otimes C((t))$$ satisfying the condition that
$$\Phi^\pm(t)1 = 1, \quad \Phi^\pm(t)(a) = a \otimes t^\pm, \quad \Phi^\pm(t)(b) = b \otimes t^\mp,$$
$$\Phi^\pm(x_1)Y(v, x_2) = Y(\Phi^\pm(x_1 - x_2)v, x_2)\Phi^\pm(x_1) \quad \text{for } v \in V_B,$$
$$\Phi^\pm(x_1)\Phi^\pm(x_2) = \Phi^\pm(x_2)\Phi^\pm(x_1),$$
$$\Phi^+(x)\Phi^-(x) = \Phi^-(x)\Phi^+(x) = 1.$$

Furthermore, if $q \neq 1$, the assignment
$$\beta_t(x) = (1 - q)(t + x)Y(a, qx)\Phi((1 - q)t + x),$$
$$\gamma_t(x) = Y(b, qx)\Phi^{-1}((1 - q)t + x)$$
defines a vacuum $A_{t,q}(\beta\gamma)$-module structure on $V_B \otimes C((t))$. 40
Proof. It is similar to the proof of a similar result in Section 4 of [Li]. Equip $\mathbb{C}((t))$ with the vertex algebra structure with 1 as the vacuum vector and with

$$Y(f(t), x)g(t) = (e^{-x(d/dt)} f(t))g(t) = f(t)g(t) \quad \text{for } f(t), g(t) \in \mathbb{C}((t)).$$

Furthermore, equip $V_B \otimes \mathbb{C}((t))$ with the vertex superalgebra structure by tensor product over $\mathbb{C}$. We have

$$Y(a \otimes t^{\pm 1}, x) = Y(a, x) \otimes (t + x)^{\pm 1}, \quad Y(b \otimes t^{\mp 1}, x) = Y(b, x) \otimes (t + x)^{\mp 1}.$$

It is straightforward to show that the assignments

$$a(x) \mapsto Y(a \otimes t^{\pm 1}, x), \quad b(x) \mapsto Y(b \otimes t^{\mp 1}, x)$$

give two $B$-module structures on $V_B \otimes \mathbb{C}((t))$. It follows from the universal property of $V_B$ that there exist $B$-module homomorphisms $\Phi^\pm : V_B \to V_B \otimes \mathbb{C}((t))$ such that $\Phi^\pm(1) = 1 \otimes 1$. Since $a, b$ generate $V_B$ as a vertex algebra, it follows that $\Phi^\pm$ are vertex algebra homomorphisms. We have

$$\Phi^\pm(a) = \Phi^\pm(a-1)1 = \text{Res}_x x^{-1}(Y(a, x) \otimes (t + x)^{\pm 1})(1 \otimes 1) = a \otimes t^{\pm 1}, \quad \Phi^\pm(b) = b \otimes t^{\mp 1}.$$  

Write $\Phi^\pm$ as $\Phi(t)$ and $\Phi^-(t)$, indicating the dependence on $t$. Then $\Phi^\pm(t)$ meet all the requirements.

As for the last assertion, note that $\Phi((1 - q)t + x)$ makes sense as $\Phi(x)(v) \in V \otimes \mathbb{C}((x))$. We have

$$Y(a, qx_1)\Phi((1 - q)t + x_1)Y(a, qx_2)\Phi((1 - q)t + x_2) = ((1 - q)t + x_1 -qx_2)Y(a, qx_1)Y(a, qx_2)\Phi((1 - q)t + x_1)\Phi((1 - q)t + x_2) = ((q - 1)t + qx_2 - x_1)Y(a, qx_2)Y(a, qx_1)\Phi((1 - q)t + x_1)\Phi((1 - q)t + x_2) = \left(\frac{(q - 1)t + qx_2 - x_1}{(1 - q)t + x_2 - qx_1}\right)Y(a, qx_2)\Phi^-(1 - q)t + x_2)Y(a, qx_1)\Phi^-(1 - q)t + x_1),$$

$$Y(b, qx_1)\Phi^-(1 - q)t + x_1)Y(b, qx_2)\Phi^-(1 - q)t + x_2) = ((1 - q)t + x_1 -qx_2)Y(b, qx_1)Y(b, qx_2)\Phi^-(1 - q)t + x_1)\Phi^-(1 - q)t + x_2) = ((q - 1)t + qx_2 - x_1)Y(b, qx_2)Y(b, qx_1)\Phi^-(1 - q)t + x_1)\Phi^-(1 - q)t + x_2) = \left(\frac{(q - 1)t + qx_2 - x_1}{(1 - q)t + x_2 - qx_1}\right)Y(b, qx_2)\Phi^-(1 - q)t + x_2)Y(b, qx_1)\Phi^-(1 - q)t + x_1),$$

$$Y(a, qx_1)\Phi((1 - q)t + x_1)Y(b, qx_2)\Phi^-(1 - q)t + x_2) = \frac{(1 - q)t + x_2 - qx_1}{(1 - q)t + x_2 - x_1}Y(b, qx_2)\Phi^-(1 - q)t + x_2)Y(a, qx_1)\Phi((1 - q)t + x_1) = ((1 - q)t + x_1 -qx_2)^{-1}Y(a, qx_1)Y(b, qx_2)\Phi((1 - q)t + x_1)\Phi^-(1 - q)t + x_2) = ((q - 1)t + qx_2 - x_1)^{-1}Y(b, qx_2)Y(a, qx_1)\Phi^-(1 - q)t + x_2)\Phi((1 - q)t + x_1) = (1 - q)^{-1}(1 - q)t + x_2)^{-1}x_2^{-1}\delta \left(\frac{x_1}{x_2}\right) = \Phi^-(1 - q)t + x_2)\Phi((1 - q)t + x_1) = (1 - q)^{-1}(t + x_1)^{-1}x_2^{-1}\delta \left(\frac{x_1}{x_2}\right).$$
This proves that $V_B \otimes \mathbb{C}((t))$ is an $A_{t,q}(\beta \gamma)$-module with the given action. Let $M$ be the $A_{t,q}(\beta \gamma)$-submodule of $V_B \otimes \mathbb{C}((t))$, generated from $1 \otimes 1$. It is clear that $M$ is a vacuum $A_{t,q}(\beta \gamma)$-module. Now, it suffices to prove that $M = V_B \otimes \mathbb{C}((t))$. As $V_B$ is an irreducible $B$-module, we have $V_B = B \cdot 1$, so that

$$V_B \otimes \mathbb{C}((t)) = (B \otimes \mathbb{C}((t)))(1 \otimes 1).$$

Then it suffices to prove that $M$ is stable under the action of $B$. With $\Phi^\pm(x)1 = 1$, using the commutation relations (and induction), we see that $M$ is stable under the actions of $\Phi^\pm((1-q)t + x)$. Note that by definition $M$ is stable under the actions of $\beta_t(x)$ and $\gamma_t(x)$. Consequently, $M$ is stable under the actions of $Y(a, x)$ and $Y(b, x)$. Thus $M$ is stable under the action of $B$. Therefore, we have $M = V_B \otimes \mathbb{C}((t))$, proving that $V_B \otimes \mathbb{C}((t))$ is a vacuum $A_{t,q}(\beta \gamma)$-module.

We now construct a universal vacuum $A_{t,q}(\beta \gamma)$-module. First, set $\tilde{J} = \tilde{A}F_{-1/2}$ (recall Proposition 6.6), a left ideal of $\tilde{A}$. Then consider the quotient $\tilde{A}/\tilde{J}$, a (left) $\tilde{A}$-module. One sees that for any $w \in \tilde{A}/\tilde{J}$, $\beta_t(n)w = \tilde{\gamma}_t(n)w = 0$ for $n$ sufficiently large, as for any $a \in \tilde{A}$, $\beta_t(n)a$, $\tilde{\gamma}_t(n)a \in F_{-1/2}$ for $n$ sufficiently large.

**Definition 6.8.** Let $V_{t,q}(\beta \gamma)$ be the quotient of $\tilde{A}/\tilde{J}$ modulo the relations corresponding to the defining relations of $A_{t,q}(\beta \gamma)$. We set $1 = 1 + \tilde{J} \in V_{t,q}(\beta \gamma)$.

From the construction, $(V_{t,q}(\beta \gamma), 1)$ is a vacuum $A_{t,q}(\beta \gamma)$-module and it is universal in the obvious sense. It then follows from Propositions 6.6 and 6.7 that $V_{t,q}(\beta \gamma)$ is irreducible (nonzero) and that every nonzero vacuum module is isomorphic to $V_{t,q}(\beta \gamma)$.

Now we are ready to present the main result of this section:

**Theorem 6.9.** Assume $q \neq 1$. There exists a weak quantum vertex $\mathbb{C}((t))$-algebra structure on $V_{t,q}(\beta \gamma)$ with $1$ as the vacuum vector and with

$$Y(\beta_t(-1)1, x) = \beta_t(x), \quad Y(\gamma_t(-1)1, x) = \gamma_t(x).$$

Furthermore, such a weak quantum vertex $\mathbb{C}((t))$-algebra structure is unique and non-degenerate.

**Proof.** We shall apply Theorem 5.7. Set

$$U = \mathbb{C}((t))\beta_t + \mathbb{C}((t))\gamma_t \subset V_{t,q}(\beta \gamma)$$

and define

$$Y_0(f(t)\beta_t, x) = f(t + x)\beta_t(x), \quad Y_0(f(t)\gamma_t, x) = f(t + x)\gamma_t(x) \quad \text{for } f(t) \in \mathbb{C}((t)).$$

Set $U(x) = \{Y_0(u, x) | u \in U\}$. It is clear that $U(x)$ is $S_t(x_1, x_2)$-local, so $U(x)$ generates a nonlocal vertex algebra $(U(x))$ over $\mathbb{C}((t))$ inside $\mathcal{E}(V_{t,q}(\beta \gamma))$. Furthermore,
\[ \mathbb{C}(\langle t \rangle) \langle x \rangle \langle U(x) \rangle \text{ is a nonlocal vertex algebra over } \mathbb{C}(\langle t \rangle). \] By Proposition 4.9 we have

\[ Y_E(\beta_t(x), x_1)Y_E(\beta_t(x), x_2) = \frac{(q-1)(t+x) + qx_2 - x_1}{(1-q)(t+x) + x_2 - qx_1}Y_E(\beta_t(x), x_2)Y_E(\beta_t(x), x_1), \]

\[ Y_E(\gamma_t(x), x_1)Y_E(\gamma_t(x), x_2) = \frac{(q-1)(t+x) + qx_2 - x_1}{(1-q)(t+x) + x_2 - qx_1}Y_E(\gamma_t(x), x_2)Y_E(\gamma_t(x), x_1), \]

\[ Y_E(\beta_t(x), x_1)Y_E(\gamma_t(x), x_2) - \frac{(1-q)(t+x) + x_2 - qx_1}{(q-1)(t+x) + qx_2 - x_1}Y_E(\gamma_t(x), x_2)Y_E(\beta_t(x), x_1) = x_1^{-1}\delta \left( \frac{x_2}{x_1} \right). \]

Define a \( \mathbb{C}(\langle t_1 \rangle) \)-module structure on \( \mathbb{C}(\langle t \rangle) \langle x \rangle \langle U(x) \rangle \) with \( f(t_1) \in \mathbb{C}(\langle t_1 \rangle) \) acting as \( f(t+x) \). Then \( \mathbb{C}(\langle t \rangle) \langle x \rangle \langle U(x) \rangle \) is an \( A_{t_1,q}(\beta_t) \)-module with \( \beta_t(z) \) and \( \gamma_t(z) \) acting as \( Y_E(\beta_t(z), z) \) and \( Y_E(\gamma_t(z), z) \), respectively. Furthermore, the submodule generated from \( 1_{V_{t,q}(\beta_t)} \) is a vacuum \( A_{t_1,q}(\beta_t) \)-module. It follows that there is a \( \mathbb{C} \)-linear map \( \psi \) from \( V_{t,q}(\beta_t) \) to \( \mathbb{C}(\langle t \rangle) \langle x \rangle \langle U(x) \rangle \) such that

\[ \psi(1) = 1_V, \quad \psi(f(t)v) = f(t_1)\psi(v), \]

\[ \psi(\beta_t(z)v) = Y_E(\beta_t(z), z)\psi(v), \quad \psi(\gamma_t(z)v) = Y_E(\gamma_t(z), z)\psi(v) \]

for \( f(t) \in \mathbb{C}(\langle t \rangle) \), \( v \in V_{t,q}(\beta_t) \). Now the first assertion follows from Theorem 5.7.

Next, we show that \( V_{t,q}(\beta_t) \) is non-degenerate by using Proposition 3.6. Recall the \( \frac{1}{2}\mathbb{Z} \)-graded free algebra \( \hat{A} \) over \( \mathbb{C}(\langle t \rangle) \). By Proposition 4.6, \( V_{t,q}(\beta_t) \) is an irreducible \( \hat{A} \)-module with \( \text{End}_A(V_{t,q}(\beta_t)) = \mathbb{C}(\langle t \rangle) \). As \( \beta_t, \gamma_t \) generate \( V_{t,q}(\beta_t) \), it follows that \( V_{t,q}(\beta_t) \) as a \( V_{t,q}(\beta_t) \)-module is irreducible with \( \text{End}_{V_{t,q}(\beta_t)}(V_{t,q}(\beta_t)) = \mathbb{C}(\langle t \rangle) \). Now, by Proposition 3.6, \( V_{t,q}(\beta_t) \) is non-degenerate. \( \square \)

Regarding the relationship between \( A_q(\beta_t) \)-modules and the quantum vertex \( \mathbb{C}(\langle t \rangle) \)-algebra \( V_{t,q}(\beta_t) \) we have:

**Proposition 6.10.** Assume \( q \neq 1 \). Let \( W \) be an \( A_q(\beta_t) \)-module. There exists a type zero module structure for the quantum vertex \( \mathbb{C}(\langle t \rangle) \)-algebra \( V_{t,q}(\beta_t) \) with

\[ Y_W(\beta_t(-1)1, x) = \beta(x), \quad Y_W(\gamma_t(-1)1, x) = \gamma(x). \]

**Proof.** From the defining relations of \( A_q(\beta_t) \), one sees that the generating functions \( \beta(x) \) and \( \gamma(x) \) form an \( \mathcal{S}(x_1, x_2) \)-local subset of \( \mathcal{E}(W) \). Thus by Theorem 4.2, \( \{ \beta(x), \gamma(x) \} \) generates a nonlocal vertex algebra \( K \) over \( \mathbb{C} \) with \( W \) as a module. Furthermore, by Theorem 4.3, \( \mathbb{C}(\langle t \rangle)K \) is a weak quantum vertex \( \mathbb{C}(\langle t \rangle) \)-algebra with \( W \) as a type zero module, where \( f(t) \in \mathbb{C}(\langle t \rangle) \) acts on \( \mathbb{C}(\langle t \rangle)K \) as \( f(x) \). In
view of Proposition 4.9 we have

\[
Y_\xi(\beta(z), x_1)Y_\xi(\beta(z), x_2) = \frac{(1 - q)t + x_1 - qx_2}{(q - 1)t + qx_1 - x_2} Y_\xi(\beta(z), x_2)Y_\xi(\beta(z), x_1),
\]

\[
Y_\xi(\gamma(z), x_1)Y_\xi(\gamma(z), x_2) = \frac{(1 - q)t + x_1 - qx_2}{(q - 1)t + qx_1 - x_2} Y_\xi(\gamma(z), x_2)Y_\xi(\gamma(z), x_1),
\]

\[
Y_\xi(\beta(z), x_1)Y_\xi(\gamma(z), x_2) = \frac{(q - 1)t + qx_1 - x_2}{(q - 1)t + x_1 - qx_2} Y_\xi(\gamma(z), x_2)Y_\xi(\beta(z), x_1)
\]

\[= x_1^{-1}\delta\left(\frac{x_2}{x_1}\right).\]

Thus, \(\mathbb{C}(x)K\) is an \(A_{t,q}(\beta\gamma)\)-module with \(f(t) \in \mathbb{C}(x)\) acting as \(f(x)\) and with \(\beta_t(n), \gamma_t(n)\) acting as \(\beta(z)_n, \gamma(z)_n\) for \(n \in \mathbb{Z}\), respectively. We also have \(\beta(z)_n1_W = 0 = \gamma(z)_n1_W\) for \(n \geq 0\). It follows that there exists an \(A_{t,q}(\beta\gamma)\)-module homomorphism \(\pi\) from \(V_{t,q}(\beta\gamma)\) to \(\mathbb{C}(x)K\), sending \(1\) to \(1_W\). That is, \(\pi\) is a \(\mathbb{C}(x)\)-module homomorphism satisfying the condition that \(\pi(1) = 1_W\),

\[
\pi(Y(\beta_t, z)v) = Y_\xi(\beta(x), z)\pi(v), \quad \pi(Y(\gamma_t, z)v) = Y_\xi(\gamma(x), z)\pi(v)
\]

for \(v \in V_{t,q}(\beta\gamma)\). It follows that \(\pi\) is a homomorphism of nonlocal vertex \(\mathbb{C}(t)\)-algebras. Consequently, \(W\) is a module of type zero for \(V_{t,q}(\beta\gamma)\).

**Appendix**

In this Appendix we present some technical results on iota-maps, which we use in the main body of this paper.

**Lemma 6.11.** For any \(f(x_1, x_2) \in \mathbb{F}^*_x(x_1, x_2)\), we have

\[
\begin{align*}
t_{t_1, x_2}(t_{t_0}f(t + x_0, t))_{|t = t_1 + x_2} &= t_{t_1, x_2, x_0}f(t_1 + x_2 + x_0, t_1 + x_2), \\
t_{t_1, x_0}(t_{t_2}f(t + x_2 + x_0, t + x_2))_{|x_2 = x_1 - x_0} &= t_{t_1, x_1, x_0}f(t + x_1 + t + x_1 - x_0).
\end{align*}
\]

**Proof.** If \(f(x_1, x_2) \in \mathbb{F}[[x_1, x_2]]\), it is clear as iota-maps leave nonnegative power series unchanged. Now, we consider the case with \(f = 1/p\) for \(p(x_1, x_2) \in \mathbb{F}[x_1, x_2]\). We have \(t_{t_0}(1/p(t + x_0, t)) = p(t + x_0, t)^{-1}\), the inverse of \(p(t + x_0, t)\) in \(\mathbb{F}((t))(x_0)\). The substitution \(t = t_1 + x_2\) is an algebra homomorphism from \(\mathbb{F}((t))(x_0)\) into \(\mathbb{F}((t_1))(x_2))(x_0)\). Thus \((t_{t_0}(1/p(t + x_0, t)))_{|t = t_1 + x_2}\) is the inverse of polynomial \(p(t_1 + x_2 + x_0, t_1 + x_2)\) in \(\mathbb{F}((t_1))(x_2))(x_0)\). On the other hand, we know that \(t_{t_1, x_2, x_0}(1/p(t_1 + x_2 + x_0, t_1 + x_2))\) is also the inverse of \(p(t_1 + x_2 + x_0, t_1 + x_2)\) in \(\mathbb{F}((t_1))(x_2))(x_0)\). This proves the first assertion. The second assertion can be proved similarly. \(\square\)

**Lemma 6.12.** a) Let \(q(x_1, x_2) \in \mathbb{F}[x_1, x_2]\) be such that \(q(x_1, x_1) \neq 0\). Then

\[
\begin{align*}
t_{t, x_1, x_2}(1/q(t + x_1, t + x_2)) &= t_{t, x_2, x_1}(1/q(t + x_1, t + x_2)) \in \mathbb{F}((t))[x_1, x_2],
\end{align*}
\]
b) For any \( f(x_1, x_2) \in \mathbb{F}_*(x_1, x_2) \), we have

\[
\iota_{t,x_2,x_1} f(t + x_1, t + x_2) \in \mathbb{F}(t)((x_2))[x_1],
\]

and there exists \( k \in \mathbb{N} \) such that

\[
\iota_{t,x_1,x_2}(x_1 - x_2)^k f(t + x_1, t + x_2) = \iota_{t,x_2,x_1}(x_1 - x_2)^k f(t + x_1, t + x_2)
\]

with both sides lying in \( \mathbb{F}(t)[[x_1, x_2]] \), and such that

\[
\iota_{t,x_1,x_2}(x_0^k f(t + x_1, t + x_1 - x_0)) = \left( \iota_{t,x_2,x_1}(x_1 - x_2)^k f(t + x_1, t + x_2) \right)|_{x_2 = x_1 - x_0}.
\]

Proof. We have

\[
q(t + x_1, t + x_2) = q(t, t) - x_1 g(t, x_1, x_2) - x_2 h(t, x_1, x_2)
\]

for some \( g, h \in \mathbb{F}[t, x_1, x_2] \) where \( q(t, t) \neq 0 \). Then

\[
\begin{align*}
\iota_{t,x_1,x_2}(1/q(t + x_1, t + x_2)) &= \sum_{j \geq 0} \iota_{t,0}(1/q(t, t))^{j+1}(x_1 g(t, x_1, x_2) + x_2 h(t, x_1, x_2))^j \\
&= \iota_{t,x_2,x_1}(1/q(t + x_1, t + x_2)),
\end{align*}
\]

proving the first assertion. Let \( f = g/p \) with \( g \in \mathbb{F}[[x_1, x_2]], p(x_1, x_2) \in \mathbb{F}[x_1, x_2] \) (nonzero). We have \( p(x_1, x_2) = (x_1 - x_2)^k q(x_1, x_2) \) for some \( k \in \mathbb{N}, q(x_1, x_2) \in \mathbb{F}[x_1, x_2] \) with \( q(x_2, x_2) \neq 0 \). Then the second and the third assertions follow immediately. As for the last assertion, we have

\[
\iota_{t,x_1,x_0}(1/q(t + x_1, t + x_1 - x_0)) = (\iota_{t,x_2,x_1}(1/q(t + x_1, t + x_2)))|_{x_2 = x_1 - x_0},
\]

because both sides are the inverse of \( q(t + x_1, t + x_1 - x_0) \) in \( \mathbb{F}(t)((x_1))(x_0) \). Then the last assertion follows.

\[\square\]

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