Super-polynomial accuracy of multidimensional randomized nets using the median-of-means

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Abstract

We study approximate integration of a function $f$ over $[0, 1]^s$ based on taking the median of $2^{r-1}$ integral estimates derived from independently randomized $(t, m, s)$-nets in base 2. The nets are randomized by Matousek’s random linear scramble with a digital shift. If $f$ is analytic over $[0, 1]^s$, then the probability that any one randomized net’s estimate has an error larger than $2^{-cm^2/s}$ times a quantity depending on $f$ is $O(1/\sqrt{m})$ for any $c < 3\log(2)/\pi^2 \approx 0.21$. As a result the median of the distribution of these scrambled nets has an error that is $O(n^{-c\log(n)/s})$ for $n = 2^m$ function evaluations. The sample median of $2^{r-1}$ independent draws attains this rate too, so long as $r/m^2$ is bounded away from zero as $m \to \infty$.

1 Introduction

In this paper we study a median-of-means algorithm for multidimensional randomized quasi-Monte Carlo (RQMC) sampling over $[0, 1]^s$ for $s \geq 1$. The problem in RQMC is to estimate $\mu = \int_{[0,1]^s} f(x) \, dx$. The familiar Monte Carlo estimate is the mean $\hat{\mu}$ of $f(x_i)$ for $n$ independent $x_i \sim U[0,1]^s$, with a root mean squared error (RMSE) of $O(n^{-1/2})$ when $f$ has finite variance. A quasi-Monte Carlo (QMC) estimate [21] replaces those $n$ points by deterministic points strategically chosen to more uniformly sample the unit cube $[0,1]^s$. The resulting absolute error is $O(n^{-1+\epsilon})$ for any $\epsilon > 0$ when $f$ has finite total variation in the sense of Hardy and Krause. Randomizing those points [23] in such a way that they remain digital nets provides independent unbiased estimates of $\mu$ allowing one to estimate accuracy statistically. For smooth enough $f$, the randomization also improves the RMSE to $O(n^{-3/2+\epsilon})$ [24].

The usual way to combine independent replicates of randomized digital nets is to simply take the average of the replicate estimates. The method we study here is to instead take the median estimate from $2r-1$ independent replicates when using the random linear scramble from [18].
In [26] we studied the case \( s = 1 \). The median-of-means proposal in [26] uses a \((0, m, 1)\)-net in base 2 randomized with a random linear scramble of Matousek [18] and a digital shift. For \( f \) analytic on \([0, 1]\) with integral \( \mu \) estimated by an infinite precision RQMC estimator denoted by \( \hat{\mu}_\infty \) we saw that the median of the randomization distribution of \( \hat{\mu}_\infty - \mu \) converges to \( O(n^{-c \log_2(n)}) \) for any \( c < 3 \log(2)/\pi^2 \approx 0.21 \). That same rate could be attained by the sample median of \( 2^{r-1} \) independently replicated RQMC estimates so long as \( r = \Omega(m) \) by which we mean \( m = O(r) \) as both \( r \) and \( m \) go to infinity. That paper also considered integrands whose \( \alpha \) derivative satisfied a \( \lambda \)-Hölder condition and found an error of \( O(n^{-\alpha-\lambda} + \epsilon) \) for that case. The significance of this result is that we can attain a better rate than the customary mean of replicated estimates and that rate can adapt to an unknown smoothness level of the integrand without the user having to know the smoothness level. Indeed when many integrals are computed from the same inputs we might know that they have different smoothness levels.

The previous paper was limited to \( s = 1 \), where there are many other good ways to integrate a smooth function over \([0, 1]\), as in [4]. That paper did however include a numerical result for the OTL circuit function on \([0, 1]^6 \) from [29]. There the standard deviation of a median of means estimator was superior to that of the usual mean-of-means at practically relevant sample sizes. In the present paper we consider analytic functions \( f : [0, 1]^s \rightarrow \mathbb{R} \). We find that the median value of \( \hat{\mu}_\infty - \mu \) is now \( O(n^{-c \log_2(n)/s}) \) for any \( c < 3 \log(2)/\pi^2 \). In other words, there is still superlinear convergence but with a dimension effect.

An outline of this paper is as follows. Section 2 introduces some notation as well as the integration problem and scrambling algorithms. Section 3 decomposes the RQMC error into a sum over nonzero vectors of \( s \) nonnegative integers. It is a sum of a randomly selected set of randomly signed Walsh coefficients. That section introduces some notation that we need to describe the complexity of the Walsh basis functions and then presents an upper bound on Walsh coefficients from Yoshiki [30]. Section 4 gives asymptotic properties of the median of means estimator. It bounds the probability that a Walsh coefficient contributes to the error and it shows that the probability of an integration error above \( 2^{-\lambda m^2/s + O(m \log(m))} \) is \( O(1/\sqrt{m}) \) when scrambling a \((t, m, s)\)-net in base 2. It also shows superpolynomial convergence for some finite precision estimates where the number of bits in the sample values grows faster than a certain multiple of \( m^2/s \) and the median of \( \Omega(m^2) \) independent copies is used. Section 5 looks at finite sample performance of the method and gives conditions where a median-of-means can outperform a mean-of-means for large \( s \) and feasible \( m \), despite the dimension effect. This may happen when the integrand is dominated by contributions from a small set of important variables. Section 6 has a discussion of the results focusing on two remaining challenges: adaptation to unknown smoothness, and quantifying uncertainty. The median-of-means setting makes use of techniques from analytic combinatorics that have previously seen very little use in quasi-Monte Carlo. That literature has quite different methods and notational conventions, and the results we derive with it are in an Appendix.

We close the introduction with some bibliographic remarks on median-of-
means. It is a longstanding method in theoretical computer science. See \cite{14} and \cite{16} for some old and new uses, respectively. Several uses in information based complexity are discussed in \cite{15}. Uses in quasi-Monte Carlo include \cite{26} and \cite{27} mentioned above as well as \cite{13} for some laws of large numbers, \cite{8} for some smoothness adaptive lattice rules and \cite{8} for robust RQMC estimates.

2 Notation and background

We use $\mathbb{N}$ for the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$ for integers $n \geq 2$. For $K \subset \mathbb{N}$ we use $|K|$ for its cardinality. We use $\mathcal{N} = \{K \subset \mathbb{N} \mid |K| < \infty\}$. For $K \in \mathcal{N}$ we use $[K]$ to denote the largest element of $K$ with $\lfloor \varnothing \rfloor = 0$ by convention. When $x \in \mathbb{R}$ we use $\lfloor x \rfloor$ for the smallest integer greater than or equal to $x$. The context will make it clear whether the argument to $\lfloor \cdot \rfloor$ is a real number or a set of natural numbers.

We let $\mathbf{0}$ be a vector of $m$ zeros and we set $\mathbb{N}^s_0 = \mathbb{N}^s_0 \setminus \{0\}$. We abuse notation slightly by letting $\mathbf{0}$ be either a row or a column vector as needed. For $s \in \mathbb{N}$ and $f \in L^2[0, 1]^s$ we study approximation of

$$
\mu = \int_{[0,1]^s} f(x) \, dx \quad \text{by} \quad \hat{\mu} = \frac{1}{n} \sum_{i=0}^{n-1} f(x_i)
$$

for $n \geq 1$ and $x_i \in [0, 1]^d$. We use $1:s = \{1, 2, \ldots, s\}$ for the set of input indices to $f$. When $v \subseteq 1:s$ we use $-v$ for $1:s \setminus v$.

We use a van der Corput style mapping between natural numbers and bit vectors and points in $[0, 1)$ as follows. For $i \in \mathbb{Z}_{2^m}$ we let $\bar{i} = \bar{i}[m] = (i_1, i_2, \ldots, i_m)^T \in \{0, 1\}^m$ where $i = \sum_{\ell=1}^{m} i\ell 2^{\ell-1}$. For $a = \sum_{\ell=1}^{m} a\ell 2^{-\ell} \in [0, 1)$ we let $\vec{a} = \vec{a}[E] = (a_1, a_2, \ldots, a_E)^T$. Here $E$ is the precision of $\vec{a}$ and we typically have $E \geq m$ in our use cases. For $a$ with two binary expansions we choose the one ending in infinitely many 0s. For each $\vec{a}$ there is a unique $a \in [0, 1)$. When $E < \infty$, we can have $\vec{a} = \vec{a}'$ for $a \neq a'$.

For an integer base $b \geq 2$ and vectors $\mathbf{k}, \mathbf{c} \in \mathbb{N}^n_0$ with $0 \leq c_j < b^{k_j}$, an elementary interval in base $b$ is a Cartesian product of the form

$$
\prod_{j=1}^{s} \left[ \frac{c_j}{b^{k_j}}, \frac{c_j + 1}{b^{k_j}} \right).
$$

For integers $m \geq t \geq 0$, the points $x_0, \ldots, x_{b^{m-1}} \in [0, 1)^s$ form a $(t, m, s)$-net in base $b$ if every elementary interval with $\sum_{j=1}^{s} k_j = m - t$ contains precisely $b^t$ of those points. Here, $t$ is the quality parameter of the net with smaller values being better. It is not always possible to get $t = 0$ for a given choice of $b$ and $m$ and $s$. The infinite sequence $x_i \in [0, 1)^s$ for $i \in \mathbb{N}_0$ forms a $(t, s)$-sequence in base $b$ if for all integers $m \geq t$ and $r \geq 0$, the points $x_{r b^m}, \ldots, x_{(r+1)b^m-1}$ form a $(t, m, s)$-net in base $b$. In this paper we consider $b = 2$. This includes the most widely used nets of Sobol’ \cite{28} as well as those of Niederreiter and Xing \cite{29} that have some of the best available $t$ values.

3
Base 2 digital nets of \( n = 2^m \) points are formed by setting

\[
\bar{a}_{ij} = C_j \tilde{i} \mod 2
\]

for \( 0 \leq i < 2^m \) and \( j = 1, \ldots, s \) for carefully chosen generator matrices \( C_j = C_j[E] \in \{0,1\}^{E \times m} \) where \( E \geq m \) is a precision. Our theoretical analysis emphasizes \( E = \infty \). The attained value of \( t \) is a property of the chosen generator matrices. We always assume that \( C_j \) has full rank over \( \mathbb{Z}_2 \).

Given points \( \mathbf{a}_i \in [0,1]^s \) have components \( a_{ij} \) determined by \( \bar{a}_{ij} \) from equation (1). That is, we give expressions for \( \bar{a}_{ij} \) with the understanding that \( a_{ij} \in [0,1) = \sum_{i=1}^E 2^{-i}a_{ij} \) when \( \bar{a}_{ij} = (a_{ij1}, a_{ij2}, \ldots, a_{ijE}) \).

For a base 2 digital \((t,s)\)-sequence one uses generator matrices with infinitely many rows and columns. For \( m \geq t \), the first \( n = 2^m \) points of such a sequence are a \((t,m,s)\)-net in base 2. When we consider a digital sequence we suppose that for each finite \( m \) we are working with \( C_j \) equal to the upper left \( E \times m \) submatrix of the infinite generator matrix. Note that any entries in \( \tilde{i} \) after the \( m \)'th are zero for \( i \in \mathbb{Z}_{2^m} \) so columns of \( C_j \) after the \( m \)'th do not affect \( \hat{\mu}_n \).

Given points \( \mathbf{a}_i = (a_{i1}, \ldots, a_{is}) \in [0,1]^s \) of a digital net, we define linearly scrambled points as follows. For precision \( E \geq m \) we choose random matrices \( M_j \in \{0,1\}^{E \times m} \) and random vectors \( \mathbf{D}_j \in \{0,1\}^E \) and take

\[
\hat{x}_{ij} = \bar{x}_{ij}[E] = \bar{a}_{ij} + \mathbf{D}_j = M_j C_j \tilde{i} + \mathbf{D}_j \mod 2
\]

for \( 0 \leq i < 2^m \) and \( 1 \leq j \leq s \) to define \( \mathbf{x}_i \in [0,1]^s \). From here on, arithmetic operations on bit vectors are taken modulo two unless otherwise indicated. Our estimate of \( \mu \) is now

\[
\hat{\mu} = \hat{\mu}_E = \frac{1}{n} \sum_{i=0}^{n-1} f(x_i).
\]

For \( E' < E \) we define \( \hat{\mu}_{E'} \) as above keeping only the first \( E' \) rows of \( M_j \) and the first \( E' \) entries in \( D_j \). Our reduced precision estimate \( \hat{\mu}_{E'} \) uses \( \hat{x}_{ij}[E'] = M_j(1:E',:) C_j \tilde{i} + \mathbf{D}_j(1:E') \).

**Lemma 1.** Let \( f : [0,1]^s \rightarrow \mathbb{R} \) have modulus of continuity \( \omega_f \). Let \( M_j \) and \( D_j \) for \( j = 1, \ldots, s \) be defined with infinite precision. Then

\[
|\hat{\mu}_{\infty} - \hat{\mu}_E| \leq \omega_f \left( \frac{\sqrt{s}}{2^E} \right).
\]

**Proof.** Let \( x_i[E] \) be \( x_i \) under scrambling with precision \( E \) and \( x_i[\infty] \) be \( x_i \) under scrambling in the infinite precision limit. By Lemma 1 of [20], each coordinate of \( x_i[E] \) differs from \( x_i[\infty] \) by at most \( 2^{-E} \). Therefore \( ||x_i[E] - x_i[\infty]||_2 \leq \sqrt{s}2^{-E} \) and so

\[
|\hat{\mu}_{\infty} - \hat{\mu}_E| \leq \frac{1}{n} \sum_{i=0}^{n-1} |f(x_i[E]) - f(x_i[\infty])| \leq \omega_f \left( \frac{\sqrt{s}}{2^E} \right).
\]
We will use $\omega_f(\sqrt{s})$ as shorthand for $\sup_{x \in [0,1]^s} f(x) - \inf_{x \in [0,1]^s} f(x)$.

We focus on the random linear scrambling of [18]. The matrix $M_j \in \{0, 1\}^{E \times s}$ is lower triangular with ones on the diagonal and independent $\mathbb{U}\{0, 1\}$ entries below the diagonal. The digital shift has independent $\mathbb{U}\{0, 1\}$ elements. That is

$$M_{j,\ell\ell'} = \begin{cases} 0, & 1 \leq \ell < \ell' \leq m \\ 1, & 1 \leq \ell = \ell' \leq m \\ \mathbb{U}\{0, 1\}, & \text{else}, \end{cases}$$

and $D_{j,\ell} = \mathbb{U}\{0, 1\}$ for $\ell = 1, \ldots, E$. We sketch this setting for $m = 3$ and $E = 4$ as follows:

$$M_j = \begin{pmatrix} 1 \\ u \\ u \\ u \\ u \\ u \\ u \end{pmatrix}$$ and $D_j = \begin{pmatrix} u \\ u \\ u \end{pmatrix}$, \hspace{1cm} (3)

with $u$ representing random elements. All of the uniform random variables in $M_1, \ldots, M_s$ and $D_1, \ldots, D_s$ are independent.

### 3 Error decomposition

In order to analyze the convergence rate of median-of-means, we first derive an error decomposition formula for $\hat{\mu}_\infty - \mu$ using Walsh functions. For $k \in \mathbb{N}_0$ and $x \in [0, 1)$, we define

$$\text{wal}_k(x) = (-1)^{k^T x}. \hspace{1cm} (4)$$

Because $k$ is a finite integer, only finitely many entries in $\tilde{k}$ are nonzero and so the inner product in (4) is a finite sum. For the multivariate generalization, the $k$'th dyadic Walsh function $\text{wal}_k(x)$ for $k \in \mathbb{N}_0^s$ is defined to be

$$\text{wal}_k(x) = \prod_{j=1}^s \text{wal}_{k_j}(x_j) = (-1)^{\sum_{j=1}^s k_j^T x_j}. \hspace{1cm} (5)$$

It is known that $\{\text{wal}_k(x) \mid k \in \mathbb{N}_0^s\}$ form a complete orthonormal basis of $L^2[0, 1]^s$ [6]. Therefore for $f \in L^2[0, 1]^s$

$$f(x) = \sum_{k \in \mathbb{N}_0^s} \hat{f}(k) \text{wal}_k(x), \hspace{1cm} \text{where}$$

$$\hat{f}(k) = \int_{[0,1]^s} f(x) \text{wal}_k(x) \, dx. \hspace{1cm} (7)$$

Equation (6) holds in a mean square sense.
Theorem 1. Let $f \in L^2[0,1)^s$ and let $x_i$ be defined by $[2]$ for $0 \leq i < 2^m$. Then

$$\hat{\mu}_\infty - \mu = \sum_{k \in \mathbb{N}_s^*} 1 \left\{ \sum_{j=1}^s \bar{k}_j^T M_j C_j = 0 \right\} \hat{f}(k)(-1)^{s_{i=1}^s \bar{k}_j^T \bar{b}_j}. \quad (8)$$

Proof. From equation $(7)$ we see that $\mu = \hat{f}(0)$. So by equations $(5)$ and $(6)$,

$$\hat{\mu}_\infty - \mu = \sum_{k \in \mathbb{N}_s^*} \hat{f}(k) \frac{1}{n} \sum_{i=0}^{n-1} \text{wal}_k(x_i) = \sum_{k \in \mathbb{N}_s^*} \hat{f}(k) \frac{1}{n} \sum_{i=0}^{n-1} (-1)^{s_{i=1}^s \bar{k}_j^T x_i}. \quad (2)$$

From equation $(2)$, we have

$$\frac{1}{n} \sum_{i=0}^{n-1} (-1)^{s_{i=1}^s \bar{k}_j^T x_i} = \frac{1}{n} \sum_{i=0}^{n-1} (-1)^{s_{i=1}^s \bar{k}_j^T (M_j C_j \bar{i} + \bar{b}_j)}$$

$$= (-1)^{s_{i=1}^s \bar{k}_j^T \bar{b}_j} \frac{1}{n} \sum_{i=0}^{n-1} (-1)^{s_{i=1}^s \bar{k}_j^T M_j C_j \bar{i}}$$

$$= (-1)^{s_{i=1}^s \bar{k}_j^T \bar{b}_j} 1 \left\{ \sum_{j=1}^s \bar{k}_j^T M_j C_j = 0 \right\}$$

so the conclusion follows. \qed

We need to quantify several properties of Walsh function indices $k$ and $k$. Let $k \in \mathbb{N}_0$ have binary expansion $k = \sum_{\ell=1}^\infty b_\ell 2^{\ell-1}$ for bits $b_\ell \in \{0,1\}$. First we let

$$\bar{k} := \bar{k}[\infty] = (b_1, b_2, \ldots)^T, \quad \text{and} \quad \kappa := \left\{ \ell \in \mathbb{N} \mid b_\ell = 1 \right\}. \quad (9)$$

We will study $\text{wal}_k$ using the cardinality of $\kappa$, the sum of its elements, and its last (largest) element. For $k \geq 1$, these are

$$|\kappa|, \quad \|\kappa\|_1 = \sum_{\ell \in \kappa} \ell, \quad \text{and} \quad [\kappa] = \max_{\ell \in \kappa},$$

respectively. For $k = 0$ we set $\kappa = \emptyset$ and then $\|\kappa\|_1 = \|\kappa\|_1 = [\kappa] = 0$, the last one by convention.

In the $s$ dimensional setting we need to vectorize these quantities. For $k = (k_1, \ldots, k_s) \in \mathbb{N}_s$, we define the corresponding vectors $\bar{k}_1, \ldots, \bar{k}_s$ and sets $\kappa_1, \ldots, \kappa_s$ componentwise. We need to keep track of those indices in $k$ for which $k_j > 0$. We denote the supports of $k$ and $\kappa$ as $s(k) = \{j \in 1:s \mid k_j > 0\}$ and $s(\kappa) = \{j \in 1:s \mid \kappa_j \neq \emptyset\}$ respectively. Clearly $s(k) = s(\kappa)$.

Given $k \in \mathbb{N}_0^s$ we now define the corresponding bit matrix $\bar{k} = (\bar{k}_1, \ldots, \bar{k}_s)$ along with $\kappa = (\kappa_1, \ldots, \kappa_s) \in \mathbb{N}^s$, a list of finite sets of natural numbers. We
need some componentwise quantities for $\kappa$ and some aggregate quantities. The componentwise quantities are

$$[\kappa] = ([\kappa_1], \ldots, [\kappa_s]) \in \mathbb{N}_0^s \quad \text{and} \quad |\kappa| = ([|\kappa_1|], \ldots, [|\kappa_s|]) \in \mathbb{N}_0^s.$$ 

The first two aggregate quantities are

$$\|\kappa\|_1 = \sum_{j=1}^s |\kappa_j| \quad \text{and} \quad \|\kappa\|_0 = \sum_{j=1}^s |\kappa_j|.$$ 

Note that $\|\kappa\|_0$ is the number of one bits in $\vec{k}$. We also need the sum of largest indices

$$\|\lceil \kappa \rceil\|_1 = \sum_{j=1}^s \lceil \kappa_j \rceil.$$ 

These quantities satisfy

$$\|\kappa\|_0 \leq \|\lceil \kappa \rceil\|_1 \leq \|\kappa\|_1.$$ 

Theorem 2 of [30] provides the following crucial bound on $|\hat{f}(k)|$.

**Lemma 2.** Let $f \in C^\infty([0,1]^s)$. Then

$$|\hat{f}(k)| \leq 2^{-\|\kappa\|_1 - \|\kappa\|_0} \sup_{x \in [0,1]^{|\kappa|}} \left| \int_{[0,1]^{|\kappa|} - |\kappa|} f|\kappa| \left( x \right) dx \right|$$

where

$$f|\kappa| = f(|\kappa_1|, \ldots, |\kappa_s|) = \frac{\partial \|\kappa\|_0 f}{\partial x_1^{\|\kappa_1\|} \cdots \partial x_s^{\|\kappa_s\|}}.$$ 

Yoshiki’s Theorem 2 uses a norm defined in his Theorem 1 for smoothness $\alpha \geq 2$. Our setting has $\alpha = \infty$. We take his $p = \infty$. Our $\|\kappa\|_0 + \|\kappa\|_1$ is his $\mu'_\alpha(k_v)$.

### 4 Asymptotic convergence rate

In this section we derive the super-polynomial convergence rate of median-of-means. Many parts of the analysis will be refined in the next section to derive a tighter finite sample bound.

As a first step, we want to know the probability that $\sum_{j=1}^s \bar{e}_j^T M_j C_j = 0$ when $M_j$ is generated by random linear scrambling. Recall that we have assumed that each $C_j$ is nonsingular. We let $C_j(1:q, :)$ denote the first $q \geq 0$ rows of $C_j$ and then for $q_1, \ldots, q_s \in \mathbb{N}$ we write

$$C^{(q_1, \ldots, q_s)} = \begin{bmatrix}
    C_1(1:q_1, :)
    
    \vdots
    
    C_s(1:q_s, :)
\end{bmatrix} \in \{0,1\}^{(\sum_{j=1}^s q_j) \times m}.$$
with the convention that when \( q_j = 0 \), \( C_j(1; q_j, :) \) is an empty matrix. If
\( (q_1, \ldots, q_n) = \mathbf{0} \), we define \( C(q_1, \ldots, q_n) \) to be a \( 0 \times m \) matrix and it has rank
0. We will use Row(\( C(q_1, \ldots, q_n) \)) to denote the row space of matrix \( C \) in \( \{0, 1\}^m \). For
\( v \subseteq 1:s \) we let \( 1\{v\} \in \{0, 1\}^s \) be the vector with \( v_j = 1 \) for \( j \in v \) and \( v_j = 0 \) for
\( j \notin v \).

A very important quantity that recurs in our analysis is the matrix \( C[\kappa] \).

For every \( j \) with \( k_j > 0 \), this matrix has all the rows of \( C_j \) that will be relevant
to \( \text{wal}_k(x_i) \), namely \( C_j(1: [\kappa]_j, :) \). If we remove the last relevant row of each \( C_j \)
we obtain \( C[\kappa]-1(\kappa) \).

**Lemma 3.** If \( \max_{1 \leq j \leq s} [\kappa_j] > m \), then

\[
\Pr \left( \sum_{j=1}^{s} \vec{k}_j^T M_j C_j = \mathbf{0} \right) = 2^{-m}.
\]

If \( \max_{1 \leq j \leq s} [\kappa_j] \leq m \) and \( \sum_{j \in \kappa(k)} C_j ([\kappa]_j, :) \in \text{Row}(C[\kappa]-1(\kappa)) \), then

\[
\Pr \left( \sum_{j=1}^{s} \vec{k}_j^T M_j C_j = \mathbf{0} \right) = 2^{-\text{rank}(C[\kappa]-1(\kappa))}.
\]

Otherwise

\[
\Pr \left( \sum_{j=1}^{s} \vec{k}_j^T M_j C_j = \mathbf{0} \right) = 0.
\]

**Proof.** Because of the upper triangular form for \( M_j \) (recall the sketch in equation (3)), we see that \( \vec{k}_j^T M_j \) has the same distribution as \( M_j([\kappa_j], :) \). Because
\( C_j \) is nonsingular, if \( [\kappa_j] > m \) for any \( j^* \in 1:s \), then \( \vec{k}_j^T M_j \cdot C_{j^*} \) is uniformly
distributed on the set of \( 2^m \) possible binary vectors so that

\[
\Pr \left( \sum_{j=1}^{s} \vec{k}_j^T M_j C_j = \mathbf{0} \right) = \Pr \left( \vec{k}_j^T M_j \cdot C_{j^*} + \sum_{j \in 1:s, j \neq j^*} \vec{k}_j^T M_j C_j = \mathbf{0} \right)
\]

\[
= \Pr \left( \vec{k}_{j^*}^T M_j \cdot C_{j^*} + \sum_{j \in 1:s, j \neq j^*} \vec{k}_j^T M_j C_j = \mathbf{0} \right) = 2^{-m}
\]

establishing the first claim.

Now assume that all \( [\kappa_j] \leq m \). Then

\[
\sum_{j=1}^{s} \vec{k}_j^T M_j C_j = \sum_{j \in \kappa(k)} C_j ([\kappa]_j, :) + \sum_{j \in \kappa(k)} \left( \vec{k}_j^T M_j \cdot C_j ([\kappa]_j, :) \right).
\]

Observe that \( \vec{k}_j^T M_j \cdot C_j ([\kappa]_j, :) \) is uniformly distributed on the linear span
of first \( [\kappa_j] - 1 \) rows of \( C_j \). Hence the second sum on the right is uniformly
distributed on \( \text{Row}(C^{[\kappa]-1\{s(k)\}}) \). If \( \sum_{j \in s(k)} C_j([\kappa_j],:) \in \text{Row}(C^{[\kappa]-1\{s(k)\}}) \), then

\[
\Pr \left( \sum_{j=1}^{s} k_j^T M_j C_j = 0 \right) = \frac{1}{|\text{Row}(C^{[\kappa]-1\{s(k)\}})|} = 2^{-\text{rank}(C^{[\kappa]-1\{s(k)\}})}
\]

establishing the second claim. If \( \sum_{j \in s(k)} C_j([\kappa_j],:) \notin \text{Row}(C^{[\kappa]-1\{s(k)\}}) \), then the above probability is clearly 0, establishing the final claim.

**Corollary 1.** If \( C_1, \ldots, C_s \) generate a digital \((t, m, s)\)-digital in base 2, then

\[
\Pr \left( \sum_{j=1}^{s} k_j^T M_j C_j = 0 \right) \leq 2^{-m+t+s}.
\]

**Proof.** We only need to verify that \( \text{rank}(C^{[\kappa]-1\{s(k)\}}) \geq m - t - s \) when \( \max_{1 \leq j \leq s} [\kappa_j] \leq m \) and \( \sum_{j \in s(k)} C_j([\kappa_j],:) \in \text{Row}(C^{[\kappa]-1\{s(k)\}}) \). Notice that in this case \( C^{[\kappa]} \) is rank-deficient. By the definition of \((t, m, s)\)-digital net, a rank-deficient \( C^{[\kappa]} \) must contains \( m - t \) linearly independent rows, so \( \text{rank}(C^{[\kappa]}) \geq m - t \). Hence

\[
\text{rank}(C^{[\kappa]-1\{s(k)\}}) \geq \text{rank}(C^{[\kappa]}) - |s(k)| \geq m - t - s
\]

which proves the conclusion.

**Corollary 2.** Let \( \lambda = 3 \log(2)^2/\pi^2 \approx 0.146 \). For \( j = 1, \ldots, s \) let \( C_j = C_j(m) \) be the first \( m \) columns of the generator matrices of a digital \((t, s)\)-net in base 2. Then

\[
\Pr \left( \sum_{j=1}^{s} k_j^T M_j C_j = 0 \text{ for some } k \neq 0 \text{ with } \|k\|_1 \leq \frac{\lambda m^2}{s} \right) = O \left( \frac{1}{\sqrt{m}} \right)
\]

as \( m \to \infty \).

**Proof.** From Corollary \ref{cor:rank_bound} in the appendix, we know that

\[
|\{k \in \mathbb{N}_s^* | \|k\|_1 \leq \lambda m^2/s\}| = \Theta \left( \frac{2^m}{\sqrt{m}} \right).
\]

So from the union bound on the result of Corollary \ref{cor:rank_bound}

\[
\Pr \left( \sum_{j=1}^{s} k_j^T M_j C_j = 0 \text{ for some } k \in \mathbb{N}_s^* \text{ with } \|k\|_1 \leq \frac{\lambda m^2}{s} \right) \\
\leq 2^{-m+t+s} |\{k \in \mathbb{N}_s^* | \|k\|_1 \leq \lambda m^2/s\}| \\
= O \left( \frac{1}{\sqrt{m}} \right).
\]

\(\square\)
Now we are ready to prove the main theorem that shows $|\hat{\mu}_\infty - \mu| = 2^{-\lambda m^2/s + O(m \log m)}$ with high probability. We note that for $f$ to be analytic over $[0, 1]^s$ means that it equals its infinite order Taylor expansion on some open set containing $[0, 1]^s$.

**Theorem 2.** Let $f$ be analytic over $[0, 1]^s$. Let $x_i$ be from a $(t, s)$-sequence in base 2 with a random linear scramble plus digital shift. Then there exist constants $B_1$ and $B_2$ such that for all $m \geq 2$

$$\Pr\left( |\hat{\mu}_\infty - \mu| \geq 2^{-\lambda m^2/s + B_1 \log m} \right) \leq \frac{B_2}{\sqrt{m}}.$$

**Proof.** Because $[0, 1]^s$ is compact, we can find $\epsilon > 0$ such that for all $t \in [0, 1]^s$, the Taylor expansion of $f$ centered at $t$

$$\sum_{n_1=0}^\infty \cdots \sum_{n_s=0}^\infty \frac{\partial^{n_1+\cdots+n_s}f}{\partial x_1^{n_1} \cdots \partial x_s^{n_s}}(t) \prod_{j=1}^s \frac{(x_j - t_j)^n_j}{n_j!}$$

converges absolutely in an edge-length-2$\epsilon$ box centered at $t$. It follows that

$$\left| \frac{\partial^{n_1+\cdots+n_s}f}{\partial x_1^{n_1} \cdots \partial x_s^{n_s}}(t) \prod_{j=1}^s \frac{\epsilon_{n_j}}{n_j!} \right| \rightarrow 0$$

as $n_1 + \cdots + n_s \rightarrow \infty$. There must then be a constant $A$ such that

$$\left| \frac{\partial^{n_1+\cdots+n_s}f}{\partial x_1^{n_1} \cdots \partial x_s^{n_s}}(t) \right| \leq \frac{A n!}{\epsilon^n}$$

holds for all $t \in [0, 1]^s$ where $n = n_1 + \cdots + n_s$. Lemma 2 then implies that

$$|\hat{f}(\kappa)| \leq 2^{-\|\kappa\|_1} \|\kappa\|_0 \sup_{t \in [0, 1]^s} \left| \frac{\partial^{\|\kappa\|_0}f}{\partial x_1^{\|\kappa\|_0} \cdots \partial x_s^{\|\kappa\|_0}}(t) \right| \leq A 2^{-\|\kappa\|_1} \left( \frac{1}{2\epsilon} \right)^\|\kappa\|_0 \|\kappa\|_0!.$$

Because $\epsilon$ can be chosen arbitrarily small, we assume without loss of generality that $2\epsilon < 1$.

Let $E$ be the event that no $k \in \mathbb{N}_s^*$ with $\|\kappa\|_1 \leq \lambda m^2/s$ has $\sum_{j=1}^s k_j^T M_j C_j = 0$. Corollary 2 shows that $\Pr(E) = 1 - O(1/\sqrt{m})$ and we take $B_2$ to be the implied constant in that expression. Conditionally on $E$, equation (8) becomes

$$|\hat{\mu}_\infty - \mu| = \sum_{k \in \mathbb{N}_s^* : \|\kappa\| > \lambda m^2/s} \frac{1}{\sqrt{\lambda m^2/s}} \left\{ \sum_{j=1}^s k_j^T M_j C_j = 0 \right\} \hat{f}(k)(-1)^{\sum_{j=1}^s k_j^T D_j} \leq \sum_{k \in \mathbb{N}_s^* : \|\kappa\| > \lambda m^2/s} |\hat{f}(k)| \leq A \times \sum_{k \in \mathbb{N}_s^* : \|\kappa\| > \lambda m^2/s} 2^{-\|\kappa\|_1} \|\kappa\|_0!(2\epsilon)^{\|\kappa\|_0}.$$
Now for $k = \sum_{\ell=1}^{\infty} b_{\ell} 2^{\ell-1} = \sum_{\ell \in \kappa} 2^{\ell-1}$
\[ \|\kappa\|_1 = \sum_{\ell \in \kappa} \ell \geq \sum_{\ell=1}^{\infty} \ell \geq \frac{|\kappa|^2}{2}, \]
with equality holding for $\kappa = \emptyset$. Then
\[ \|\kappa\|_1 = \sum_{j=1}^{s} \|\kappa_j\|_1 \geq \sum_{j=1}^{s} \frac{|\kappa_j|^2}{2} \geq \frac{1}{2s} \left( \sum_{j=1}^{s} |\kappa_j| \right)^2 = \frac{1}{2s} \|\kappa\|_0^2, \]
yielding $\|\kappa\|_0 \leq \sqrt{2s} \|\kappa\|_1$. Hence
\[ \sum_{\kappa \in \mathbb{N}^s: \|\kappa\|_1 > \lambda m^2/s} A 2^{-\|\kappa\|_1} \|\kappa\|_0!/(2\epsilon) \|\kappa\|_0 \]
\[ \leq \sum_{N=[\lambda m^2/s]}^{\infty} A 2^{-N} \left( \frac{1}{2\epsilon} \right)^{\sqrt{2sN}} \Gamma(\sqrt{2sN} + 1) \{ \kappa \in \mathbb{N}^s_+ : \|\kappa\|_1 = N \} \]
where $\Gamma(\cdot)$ is the Gamma function and we have also used $2\epsilon < 1$.

By Theorem 7 in the appendix,
\[ |\{ \kappa \in \mathbb{N}^s_+ : \|\kappa\|_1 = N \}| \leq \frac{B}{\sqrt{N}} \exp\left( \frac{\pi \sqrt{2N}}{3} \right) \]
holds for $B = \pi \sqrt{s}/(2\sqrt{3})$. Hence
\[ \left( \frac{1}{2\epsilon} \right)^{\sqrt{2sN}} \Gamma(\sqrt{2sN} + 1) \{ \kappa \in \mathbb{N}^s_+ : \|\kappa\|_1 = N \} \leq 2D\sqrt{N} \log(N) \]
for some constant $D$. Because $\sqrt{N + 1} \log(N + 1) - \sqrt{N} \log(N)$ converges to 0 as $N \to \infty$, we can find $N_{\rho}$ for any $\rho > 1$ such that $2D\sqrt{N} \log(N) < \rho^N$ for $N > N_{\rho}$. Let us choose $\rho = 3/2$ for simplicity. Then when $\lambda m^2/s > N_{3/2}$,
\[ \sum_{N=[\lambda m^2/s]}^{\infty} 2^{-N} 2D\sqrt{N} \log(N) \leq 2^{-[\lambda m^2/s]} 2D\sqrt{[\lambda m^2/s] \log([\lambda m^2/s])} \sum_{N=0}^{\infty} \left( \frac{3}{4} \right)^N \]
\[ \leq 2^{-\lambda m^2/s + B_1 m \log(m)} \]
for some constant $B_1$. The conclusion follows once we increase $B_1$ sufficiently to cover all $m \geq 2$ cases.

**Corollary 3.** Under the same condition as Theorem 2, if $E \geq \lambda m^2/s$ and $r = \Omega(m^2)$, then the sample median $\hat{\mu}_E^{(r)}$ of $2r - 1$ independently generated values of $\hat{\mu}_E$ satisfies
\[ E(|\hat{\mu}_E^{(r)} - \mu|^2) \leq 4^{-\lambda m^2/s + O(m \log(m))}. \]
Proof. By Lemma 1 with probability at least \(1 - B_2/\sqrt{m},\)

\[
|\hat{\mu}_E - \mu| \leq 2^{-\lambda m^2/s+B_1 m \log m} + \frac{\sqrt{s}}{2E} \sup_{x \in [0,1]^r} \|\nabla f(x)\|_2
\]

where \(B_1\) and \(B_2\) come from Theorem 2 and \(\nabla f\) is the gradient of \(f\).

In order for the sample median of \(2r - 1\) copies of \(\hat{\mu}_E\) to violate the above bound, there must be at least \(r\) copies violating the bound. Because there are \((2r-1)^r\) subsets of size \(r\), the union bound implies that the probability of \(r\) such violations is at most

\[
\binom{2r-1}{r} \left( \frac{B_3}{\sqrt{m}} \right)^r = \prod_{j=2}^r \frac{(2j-1)(2j-2)}{j(j-1)} \left( \frac{B_3}{\sqrt{m}} \right)^r < \left( \frac{4B_3}{\sqrt{m}} \right)^r.
\]

When the above described event happens, \(|\hat{\mu}_E^{(r)} - \mu|\) is still bounded by \(\sup_{x \in [0,1]^r} |f(x)|\).

Hence

\[
E(|\hat{\mu}_E^{(r)} - \mu|^2) \leq \left( 2^{-\lambda m^2/s+B_1 m \log m} + O\left( \frac{1}{2E} \right) \right)^2 + O\left( \left( \frac{4B_3}{\sqrt{m}} \right)^r \right)
\]

under our assumptions on \(E\) and \(r\). \(\square\)

5 Finite sample analysis

Although the asymptotic convergence rate of median-of-means is super-polynomial, the bound in Corollary 3 is of limited use when \(\lambda m^2/s\) is only moderately large or even smaller than \(m\). In this section, we derive results that better describe the finite sample behavior of median-of-means. In particular, we want to study under what conditions median-of-means can outperform the usual RQMC estimator (mean-of-means) in terms of mean squared error. For simplicity, we assume that the precision \(E\) is high enough that the difference between \(\hat{\mu}_\infty\) and \(\hat{\mu}_E\) is negligible in comparison to their root mean squared error.

First let us work out the variance of \(\hat{\mu}_\infty\).

Lemma 4. For \(k \in \mathbb{N}_s^*\), let \(S(k) = \langle -1 \rangle \sum_{j=1}^s \vec{k}_j^T \vec{D}_j\). Then

\[
\Pr(S(k) = 1) = \Pr(S(k) = -1) = 1/2.
\]

For distinct \(k, k' \in \mathbb{N}_s^*\), \(S(k)\) and \(S(k')\) are independent.

Proof. The proof is similar to Lemma 4 of [21] and is omitted here. \(\square\)

Theorem 3. \(E(\hat{\mu}_\infty) = \mu\) and

\[
\text{Var}(\hat{\mu}_\infty) = \sum_{k \in \mathbb{N}_s^*} \Pr\left( \sum_{j=1}^s \vec{k}_j^T \vec{M}_j \vec{C}_j = 0 \right) f(k)^2.
\]
Proof. Let $M = (M_1, M_2, \ldots, M_s)$. By equation (8) and Lemma 4,

$$E(\hat{\mu}_\infty - \mu | M) = \sum_{k \in \mathbb{N}_s^*} 1\{\sum_{j=1}^s \vec{k}_j^T M_j C_j = 0\} \hat{f}(k) E(S(k)) = 0$$

and

$$Var(\hat{\mu}_\infty - \mu | M) = \sum_{k \in \mathbb{N}_s^*} 1\{\sum_{j=1}^s \vec{k}_j^T M_j C_j = 0\} \hat{f}(k)^2 Var(S(k))$$

$$= \sum_{k \in \mathbb{N}_s^*} 1\{\sum_{j=1}^s \vec{k}_j^T M_j C_j = 0\} \hat{f}(k)^2.$$  (12)

The conclusion follows by taking expectations with respect to $M$. \hfill \Box

To illustrate when the median-of-means can outperform the mean-of-means, suppose we can find a rare event $A$ such that $E(\hat{\mu}_\infty | A^c) = \mu$ and $Var(\hat{\mu}_\infty | A^c) \ll Var(\hat{\mu}_\infty)$. Then if we generate independent copies of $\hat{\mu}_\infty$ and look at the histogram, we should see a cluster with bandwidth comparable to $\sqrt{Var(\hat{\mu}_\infty | A^c)}$ around $\mu$. Those $\hat{\mu}_\infty$ for which $A$ happens could well be far into the tails away from $\mu$. In such a setting, the sample median is robust with respect to the event $A$ and has mean square error close to $Var(\hat{\mu}_\infty | A^c)$. We make this intuition precise with the following lemma.

**Lemma 5.** Let $A$ be an event with $Pr(A) \leq \delta$ and $E(\hat{\mu}_\infty | A^c) = \mu$. Then the sample median $\hat{\mu}_\infty^{(r)}$ of $2r - 1$ independently generated values of $\hat{\mu}_\infty$ using a digital $(t, m, s)$-net satisfies

$$E((\hat{\mu}_\infty^{(r)} - \mu)^2) \leq Pr(A^c) Var(\hat{\mu}_\infty | A^c) \delta^{-1} + (8\delta)^r \Delta_n^2$$

where

$$\Delta_n = \min \left( \omega_f(\sqrt{s}), \frac{V_{HK}(f)}{2^{m-t}} \sum_{i=0}^{s-1} \binom{m-t}{i} \right),$$

$\omega_f$ gives the modulus of continuity for $f$, and $V_{HK}(f)$ is the total variation of $f$ in the sense of Hardy and Krause.

**Proof.** Conditionally on $A^c$, we can apply Markov’s inequality to get

$$Pr\left( |\hat{\mu}_\infty - \mu|^2 \geq \frac{Pr(A^c)}{\delta} Var(\hat{\mu}_\infty | A^c) \right) \leq \frac{\delta}{Pr(A^c)}.$$

Hence

$$Pr\left( |\hat{\mu}_\infty - \mu|^2 \geq \frac{Pr(A^c)}{\delta} Var(\hat{\mu}_\infty | A^c) \right) \leq \frac{\delta}{Pr(A^c)} Pr(A^c) + Pr(A) \leq 2\delta.$$
The rest of the proof is similar to that of Corollary 3. In particular,
\[
\Pr\left( |\hat{\mu}_\infty^{(r)} - \mu|^2 \geq \frac{\Pr(A^c)}{\delta} \Var(\hat{\mu}_\infty \mid A^c) \right) \leq \left(\frac{2r - 1}{r}\right)^r (2\delta)^r \leq (8\delta)^r.
\]

When the ‘bad event’ \(A\) happens, we can bound the error in two ways: first it is clear that both \(\hat{\mu}_\infty^{(r)}\) and \(\mu\) are between \(\inf_{x \in [0,1]} f(x)\) and \(\sup_{x \in [0,1]} f(x)\), so their difference is no larger than \(\omega_f(\sqrt{s}) = \sup_{x \in [0,1]} f(x) - \inf_{x \in [0,1]} f(x)\). Second, if \(f\) has finite Hardy–Krause variation, we can apply the Koksma–Hlawka inequality \(\text{[12]}\) to conclude that \(|\hat{\mu}_\infty - \mu| \leq \star\DHK(f) D_n^*(x_0, \ldots, x_{n-1})\) where \(D_n^*(\cdot)\) denotes the star discrepancy. Because this is true for all \(\hat{\mu}_\infty\), it is also true for \(\hat{\mu}_\infty^{(r)}\). Because \(x_0, \ldots, x_{n-1}\) is a \((t, m, s)\)-net regardless of the scrambling, we can apply the bound
\[
D_n^*(x_0, \ldots, x_{n-1}) \leq \frac{1}{2m-1} \sum_{i=0}^{s-1} \left(\frac{m-t}{t}\right)
\]
from Corollary 5.3 of \(\text{[6]}\). By combining the two bounds, we derive \(|\hat{\mu}_\infty^{(r)} - \mu| \leq \Delta_n\) and hence the bound on \(\mathbb{E}(\hat{\mu}_\infty^{(r)} - \mu)^2\).

The \((8\delta)^r \Delta_n^2\) term in the bound \(\text{[13]}\) is exponentially small in \(r\) if \(\delta < 1/8\). As shown in Section 3 of \(\text{[20]}\), \(\Var(\hat{\mu}_\infty)\) is in general \(\Omega(n^{-3})\) for smooth functions \(f\). Hence we only need \(r \geq C_* m\) for some \(C_* > 0\) to make \((8\delta)^r D_n^2 \ll \Var(\hat{\mu}_\infty)\). With the same computational effort, the mean-of-means has variance equal to \(\Var(\hat{\mu}_\infty)/(2r - 1)\). So heuristically, median-of-means can significantly outperform mean-of-means in terms of MSE if there exists an event \(A\) such that \(\Pr(A) \leq \delta < 1/8\) and \(\Var(\hat{\mu}_\infty \mid A^c) \ll \Var(\hat{\mu}_\infty)/m\).

Motivated by Corollary \(\text{[2]}\) one way to choose \(A\) is to specify a set of frequencies \(K \subseteq \mathbb{N}^*_s\) and let \(A = \{\sum_{j=1}^s \hat{k}_j^T M_j C_j = 0 \text{ for some } k \in K\}\). We know that \(\mathbb{E}(\hat{\mu}_\infty \mid A^c) = \mu\) because equation \(\text{[11]}\) shows that \(\hat{\mu}_\infty\) is unbiased conditionally on \(M = (M_1, \ldots, M_s)\) and \(A\) belongs to the \(\sigma\)-algebra generated by \(M\).

Moreover, as long as \(\sum_{k \in K} \Pr(\sum_{j=1}^s \hat{k}_j^T M_j C_j = 0) \leq \delta\), we know by the union bound that \(\Pr(A) \leq \delta\) as well. According to equation \(\text{[12]}\),
\[
\Pr(A^c \mid A^c) \Var(\hat{\mu}_\infty \mid A^c) = \Pr(A^c) \mathbb{E}(\Var(\hat{\mu}_\infty - \mu \mid M) \mid A^c)
\]
\[
= \sum_{k \in \mathbb{N}^*_s} \Pr(A^c) \Pr\left(\sum_{j=1}^s \hat{k}_j^T M_j C_j = 0 \mid A^c\right) \hat{f}(k)^2
\]
\[
\leq \sum_{k \in \mathbb{N}^*_s \setminus K} \Pr\left(\sum_{j=1}^s \hat{k}_j^T M_j C_j = 0\right) \hat{f}(k)^2.
\]

So in principle, if one knows all \(\hat{f}(k)^2\) and \(\Pr(\sum_{j=1}^s \hat{k}_j^T M_j C_j = 0)\), then one can find a good candidate \(A\) by solving the following combinatorial optimization
problem:

$$\max_{K \subseteq \mathbb{N}_s^*} \sum_{k \in K} \Pr \left( \sum_{j=1}^s k_j^T M_j C_j = 0 \right) \hat{f}(k)^2$$

s.t. \( \sum_{k \in K} \Pr \left( \sum_{j=1}^s k_j^T M_j C_j = 0 \right) \leq \delta. \)

In particular, if \( \text{Var}(\hat{\mu}_\infty) \) is dominated by a few \( k \) with large \( \hat{f}(k)^2 \), then we should see a significant variance reduction after we condition on the \( A^c \) specified by the above optimization problem.

To make the problem more tractable, we examine one case where our function \( f \) is effectively low-dimensional. Suppose there are a few components \( x_j \) that contribute most of the variability to \( f \). More precisely, let

$$f_1(x) = \mathbb{E}(f(x) | x_j, j \in u) - \mu$$

and $$f_2(x) = f(x) - f_1(x) - \mu.$$  

Then \( \mu, f_1, \) and \( f_2 \) are orthogonal in the \( L^2[0,1]^s \) inner product, so that \( \sigma^2(f) = \sigma^2(f_1) + \sigma^2(f_2) \). We assume that \( \sigma^2(f_1) \gg \sigma^2(f_2) \), and then \( f_1 \) captures most of the variance of \( f \).

Given such a function, it is natural to choose \( K = \{ k \in \mathbb{N}_s^* | s(k) \subseteq u \} \) because their associated \( f(k)^2 \) are relatively large. Then, Corollary 1 can be strengthened in the following way:

**Lemma 6.** For non-empty \( u \subseteq 1:s \), define \( \mathbb{N}_u \subset \mathbb{N}_s^* \) to be the set of \( k \in \mathbb{N}_s^* \) with \( s(k) = u \). Further define

$$t_u^* = m + 1 - \min_{k \in \mathbb{N}_u} \{ \| [k] \|_1 | C[[k]] \text{ not of full rank } \}. \quad \text{If } u = \emptyset, \text{ we conventionally define } t_u^* = 0. \quad \text{Then}$$

$$\Pr \left( \sum_{j=1}^s k_j^T M_j C_j = 0 \right) \leq 2^{-m + t_u^* + |s(k)|}.$$ 

**Proof.** The proof is basically the same as Corollary 1. By the definition of \( t_u^* \), a rank-deficient \( C[[k]] \) must contains \( m - t_{s(k)}^* \) linearly independent rows, so

$$\text{rank}(C[[k]] - 1(s(k))) \geq \text{rank}(C[[k]]) - |s(k)| \geq m - t_{s(k)}^* - |s(k)|$$

which proves the conclusion. \( \Box \)

**Remark 1.** To compare \( t_u^* \) with \( t \), consider for instance a Sobol’ sequence constructed by the \( s \) lowest order irreducible polynomials. As shown in Section 4.5 of [27] the order of irreducible polynomials grows roughly like \( \log(s) \) and \( t \) is consequently \( O(s \log(s)). \) The supremum of \( t_u^* \) on the other hand, grows no faster than \( |u| \log(s) \), which is potentially much smaller than \( t \).

Now we can prove the finite sample version of Corollary 2.

15
**Theorem 4.** For non-empty \( u \subseteq 1:d \), let \( t_u = \max_{v \subseteq u} t^*_v \). For \( \delta > 0 \), let \( N^*_m \) be the largest integer \( N \) satisfying

\[
2^{t^*_u + |u| \left( \frac{m + |u|}{|u| - 1} \right)} N \exp \left( \frac{\pi \sqrt{|u| N^*_m}}{3} \right) \leq \delta^m,
\]

and

\[
\sqrt{3(|u| - 1)} N \leq \frac{\pi}{2} (m - t_u).
\]

Then

\[
\Pr \left( \sum_{i=1}^{s} k^T_i M_i C_j = 0 \text{ for some } k \neq 0 \text{ with } s(k) \subseteq u, \|\kappa\|_1 \leq N^*_m + m - t_u \right) \leq \delta.
\]

**Proof.** It is shown in Section 5 of [25] that \( t^*_v + |v| \leq t^*_u + |u| \) if \( v \subseteq u \). So for \( k \neq 0 \) with \( s(k) \subseteq u \),

\[
\Pr \left( \sum_{i=1}^{s} k^T_i M_i C_j = 0 \right) \leq 2^{-m + t^*_u + |s(k)|} \leq 2^{-m + t^*_u + |u|}.
\]

(14)

Now by the definition of \( t^*_u \), \( C[\kappa] \) has full rank if \( \|\kappa\|_1 \leq m - t^*_s(k) \), so

\[
\Pr \left( \sum_{j=1}^{s} k^T_j M_j C_j = 0 \right) = 0.
\]

Let \( N^*_u = \{ k \in N^*_m \mid s(k) \subseteq u \} \). By choosing \( s = |u| \) and \( R = m - t_u \) in Corollary 6 from the appendix, we further get

\[
\left| \left\{ k \in N^*_0 \mid \|\kappa\|_1 \leq N^*_m + m - t_u, \|\kappa\|_1 > m - t_u \right\} \right| < \left( \frac{m + |u|}{|u| - 1} \right) N^*_m \exp \left( \frac{\pi \sqrt{|u| N^*_m}}{3} \right).
\]

After taking a union bound over all \( k \) in the above set, we finally get

\[
\Pr \left( \sum_{j=1}^{s} k^T_j M_j C_j = 0 \text{ for some } k \neq 0 \text{ with } s(k) \subseteq u, \|\kappa\|_1 \leq N^*_m + m - t_u \right) \leq \left( \frac{m + |u|}{|u| - 1} \right) N^*_m \exp \left( \frac{\pi \sqrt{|u| N^*_m}}{3} \right) 2^{-m + t^*_u + |u|} \leq \delta.
\]

To interpret this result, let us consider the setting of Theorem 2. For simplicity, we will replace \( f \) by \( f_1(x) = \mathbb{E}(f(x) \mid x_j, j \in u) - \mu \) and pretend that the problem is \(|u|\)-dimensional, which is a useful approximation under our assumption on \( f \). In view of Lemma 2 and equation (12), one can argue that \( \text{Var}(\hat{\mu}_\infty \mid M) \) is proportional to \( 4^{-\|\kappa\|_1} \) for the \( k \) with the smallest \( \|\kappa\|_1 \) among
those satisfying $\sum_{j \in u} \vec{k}_j^T M_j C_j = 0$. This is certainly true in the asymptotic sense, as we have shown in the proof of Theorem 2 that $\|\kappa\|_0 \leq \sqrt{2} u \|\kappa\|_1$ and the supremum norm of partial derivatives grows no faster than $\|\kappa\|_0!$. (More precisely, Lemma 2 only provides an upper bound on $f(\kappa)^2$, but section 3 of [26] shows the factor $4^{-\|\kappa\|_1}$ is in general necessary.)

By the definition of $t_u$, there exists a set of $\kappa \in \mathbb{N}_+^s$ such that $C[\kappa]$ is rank-deficient and $\|\|\kappa\|\|_1 = m - t_u + 1$. It is also true that $C[\kappa]^{-1}[s(\kappa)]$ has full rank, because otherwise $t_u$ would be even larger. Hence $\text{rank}(C[\kappa]^{-1}[s(\kappa)]) = \|\|\kappa\|\|_1 - |s(\kappa)| \leq m - t_u$ and $\text{Pr}(\sum_{j \in u} \vec{k}_j^T M_j C_j = 0) \geq 2^{-m+t_u}$ from the second case in Lemma 3. On the other hand, if we condition on the event $A$ specified by Theorem 2 the smallest $\|\kappa\|_1$ for which $\sum_{j \in u} \vec{k}_j^T M_j C_j = 0$ is possible is $N^*_m + m - t_u + 1$ and the corresponding probability is no more than $2^{-m+t_u + |u|}$ according to equation (14). So roughly speaking, $\text{Var}(\mu_\infty \mid A')$ is a factor of $4^{-N^*_m}$ smaller than $\text{Var}(\mu_\infty)$. In view of our previous criterion $\text{Var}(\mu_\infty \mid A')$ needs to be much smaller than $\text{Var}(\mu_\infty)/m$, we see that with a proper choice on the number of replicates, median-of-means can significantly outperform mean-of-means when $N^*_m \gg \log m$.

**Remark 2.** One can easily generalize the above discussion to cases where $f$ can be approximated by multiple low-dimensional functions. For instance, suppose $f$ has effective dimension $d$ in the superposition sense [3], namely $f \approx \sum_{u \subseteq 1:s, |u| \leq d} f_u$ where $f_u$ is the ANOVA term corresponding to subset $u$. We can define $t_d = \max_{u:|u|=d} t_u$ and $T_d = \max_{u:|u|=d} t_u + |u|$. By applying the above theorem to each of the $\binom{s}{d}$ size-$d$ subsets of $1:s$, we get

$$\text{Pr}\left(\sum_{j=1}^s \vec{k}_j^T M_j C_j = 0 \text{ for some } \kappa \neq 0 \text{ with } |s(\kappa)| \leq d, \|\kappa\|_1 \leq N^*_m + m - t_d\right) \leq \delta$$

where $N^*_m$ is the largest integer $N$ satisfying

$$2^{T_d}\left(\binom{s}{d}\frac{m+d}{d-1} N \exp\left(\pi \sqrt{\frac{d N}{3}}\right)\right) \leq \delta 2^m \quad \text{and} \quad \sqrt{3(d-1)N} \leq \frac{\pi}{2}(m - t_d).$$

Again when $N^*_m \gg \log m$, we expect to see median-of-means outperform mean-of-means.

### 6 Discussion

We have shown that a median-of-means strategy based on scrambled $(t, m, s)$-nets in base 2 can attain superpolynomial accuracy for integration of analytic functions on $[0, 1]^s$. The main nets we have in mind are those that arise as the first $2^n$ points of a Sobol’ sequence. The superpolynomial rate comes with a dimension effect that has lesser impact when the integrand is dominated by low dimensional ANOVA components.
We have not shown that the method adapts to lesser levels of smoothness of the integrand. That is known to hold for $s = 1$ from \[20\]. It therefore also holds for additive functions on $[0, 1]^{s}$ with a rate given by the worst smoothness of any of the summands. We do not know the extent of adaptation for more general functions.

It remains to quantify the uncertainty in the median-of-means estimate using the sample data. For the mean-of-means we can get an unbiased estimate of the variance of the combined estimate. There is a central limit theorem (CLT) by Loh \[17\] for scrambled nets as $n \to \infty$ but it only applies to nested uniform scrambling from \[23\] and is only proved for $t = 0$. There is recent work by Nakayama and Tuffin \[20\] that describes CLTs for the mean-of-means over scrambled nets as the number of replicates increases.

For the median-of-means, things are more complicated. We can use non-parametric statistical methods to get a confidence interval for the median of $\tilde{\mu}_{\infty, r}$ over all scrambles, but that is not the same quantity as $\mu = E(\tilde{\mu}_{\infty})$ and it generally depends on $m$. There are confidence intervals for the median-of-means (see e.g., \[5\]) but in our setting those would have width proportional to $\text{Var}(\tilde{\mu}_{\infty, r})^{1/2}$. That standard deviation does not decrease at a super-polynomial rate and so the confidence intervals would not reflect the increased precision that comes from using the median-of-means. The median-of-means works so well for random linear scrambling because that estimate is usually very accurate apart from outliers that raise its variance. The presence of outliers implies that the convergence to the Gaussian distribution will be slow for the mean-of-means with the random linear scrambles we study here.

This upper bound on the error has the same rate that we would get in applying a one dimensional rule with error $O(n^{-c \log(n)})$ in an $s$-fold product. However, an $s$-fold product rule allows no nontrivial sample sizes below $2^s$ which may be far too large to use and still ineffective. It is also not clear whether there would need to be $(2r - 1)^s$-fold computation in a product rule whose factors involve medians of means. Digital nets exist for sample sizes $2^m$ for $m \geq 0$ so we can get this rate along a practically usable sequence of sample sizes and benefit from a good convergence rate on the low dimensional ANOVA or other components. The situation is similar to that in \[11\] where the optimal rate under Lipschitz continuity is attained by a grid but also by sampling along a Hilbert space-filling curve.

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Appendix

Here we prove some combinatorial results that our main theorem depends on. We use some results from analytic combinatorics. Some of the standard notations used there conflict with that in quasi-Monte Carlo. For instance, both literatures study a function denoted by $f$. Rather than change their notation to avoid duplications, we proceed with the understanding that some symbols have a different meaning in this appendix than they have in the main body of the paper. The uses in the two settings are distinct.

We use $Q[N]$ to denote the coefficient of $x^N$ in the generating function $Q(x)$. We refer the reader to [7] for background on generating functions.

We use the bijection from the main body of the paper between $\mathbb{N}_0$ and the set of finite cardinality subsets of $\mathbb{N}$, denoted by $\mathcal{N}$. Recall that for $k \in \mathbb{N}_0$, we write $k = \sum_{\ell=1}^{\infty} a_{\ell} 2^{\ell-1}$ for bits $a_{\ell} \in \{0,1\}$, and we set $\kappa = \{ \ell \in \mathbb{N} \mid a_{\ell} = 1 \} \in \mathcal{N}$. Clearly the mapping between $k$ and $\kappa$ is a bijection between $\mathbb{N}_0$ and $\mathcal{N}$. We extend this mapping to a bijection between $k \in \mathbb{N}_0^s$ and $\kappa \in \mathcal{N}^s$ componentwise. Therefore, combinatorial problems about $k \in \mathbb{N}_0^s$ can be translated into equivalent problems about $\kappa \in \mathcal{N}^s$.

We will need a theorem of Meinardus [19]. We state the version from [10], using the Gamma function $\Gamma(\cdot)$ and Riemann’s zeta function $\zeta(\cdot)$. For $b_n \geq 0$ let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n = \prod_{n=1}^{\infty} (1 - z^n)^{-b_n}$$

(15)

for complex $z$ with $|z| < 1$. Meinardus’ theorem will give an asymptotic expression for $c_n$. Let

$$D(z) = \sum_{n=1}^{\infty} b_n n^{-z}, \quad z = \sigma + it \quad \text{and} \quad G(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |z| < 1.$$  

be Dirichlet and power series, respectively, for $b_n$.

**Theorem 5** (Meinardus). Let $b_n \geq 0$ for $n \geq 1$ satisfy these conditions:

1. The Dirichlet series $D(z)$ converges in the half-plane $\sigma > r > 0$ and there is a constant $C_0 \in (0,1]$ such that $D(z)$ for $z = \sigma + it$ has an analytic continuation to the half-plane $\mathcal{H} = \{ z \mid \sigma \geq -C_0 \}$ on which it is analytic except for a simple pole at $z = r$ with residue $A > 0$.

2. There is a constant $C_1 > 0$ such that $D(\sigma + it) = O(|t|^{C_1})$ as $t \to \infty$ uniformly in $\sigma \geq -C_0$.

3. There are constants $C_2 > 0$ and $\epsilon > 0$ such that $g(\tau) = G(\exp(-\tau))$ for $\tau = \delta + 2\pi i \alpha$ with $\delta > 0$ and $\alpha \in \mathbb{R}$ satisfies $\text{Re}(g(\tau)) - g(\delta) \leq -C_2 \delta^{-\epsilon}$ for $|\text{arg}(\tau)| > \pi/4$, $0 \not= |\alpha| \leq 1/2$ for small enough $\delta$.

Then as $n \to \infty$,

$$c_n \sim C(1)n^{\gamma_1} \exp\left( n^{r/(r+1)} \left( 1 + \frac{1}{r} \right) (A\Gamma(r+1)\zeta(r+1))^{1/(r+1)} \right)$$

(16)
where
\[ \gamma_1 = \frac{2D(0) - 2 - r}{2(1 + r)} \]
and
\[ C^{(1)} = e^{D'(0)} (2\pi(1 + r))^{-1/2} (A\Gamma(r + 1)\zeta(r + 1))^{\gamma_2} \]
for
\[ \gamma_2 = 1 - \frac{2D(0)}{2(1 + r)}. \]

Proof. This is the statement from [10] based on the result of [19].

Theorem 6. For dimension \( s \geq 1 \)
\[ |\{ k \in \mathbb{N}_s^* \mid \|\kappa\|_1 = N\}| \sim \frac{C}{N^{3/4}} \exp\left(\pi \sqrt{\frac{8N}{3}}\right) \]
as \( N \to \infty \) for some constant \( C \) depending on \( s \).

Proof. For \( N > 0 \), the number of solutions \( k \) in \( \mathbb{N}_s^* \) equals the number in \( \mathbb{N}_0^* \) which we study next. By using the bijection introduced above, it suffices to bound the number of \( s \)-tuples \( (\|\kappa_1\|_1, \ldots, \|\kappa_s\|_1) \) for which \( \sum_{j=1}^s \|\kappa_j\|_1 = N \).

When \( s = 1 \), this is equal to the number of ways to partition an integer \( N \) into distinct positive integers. From Note I.18 of [7] that quantity has generating function
\[ Q(x) = \prod_{n=1}^{\infty} \frac{1}{1 - x^n}. \] (17)
For general \( s \), the generating function is given by \( Q^s(x) \), the \( s \)'th power of \( Q(x) \).

Let us denote the Dirichlet series of \( Q(x) \) as \( D^s(z) \). To prove that \( Q(x) \) has coefficients \( b_n \) which satisfy the conditions of Meinardus’ theorem, we first note that \( Q(x) \) is also the generating function for the number of ways to partition an integer \( N \) into possibly repeated odd integers. This equivalence is a famous result of Euler. The paper by Bidar [2] opens with a short discussion of how Euler’s observation follows from equation (17).

Theorem 6.4 of [1] says that Meinardus’ theorem applies to the number of ways to partition an integer \( N \) into sums of elements of \( H_{k,a} = \{ n \in \mathbb{N} \mid n = a \text{ mod } k \} \). Because \( Q(x) \) corresponds to the case \( a = 1 \) and \( k = 2 \), its coefficients satisfy those conditions. Now we can apply Meinardus’ theorem to \( Q(x) \) and compare equation (16) to the actual growth rate from Note VII.24 in [7]
\[ Q[N] \sim \frac{1}{4 \times 3^{1/4} N^{3/4}} \exp\left(\pi \sqrt{\frac{N}{3}}\right). \]
Comparing to equation (16), we see that the exponent of \( N \) within the exponential is \( r/(1 + r) = 1/2 \) and that \( \gamma_1 = (2D^s(0) - 2 - r)/(2(1 + r)) = -3/4. \)
Therefore \( r = 1 \) and \( D^s(0) = 0. \)
The Dirichlet series of $Q^s(x)$ has coefficients $s_{bn}$, so it is equal to $sD^s(z)$. It is straightforward to verify that conditions of Meinardus’ theorem still hold if all coefficients are multiplied by a positive constant, so we can apply Meinardus’ theorem to $Q^s(x)$ as well. Because $sD^s(z)$ has the same pole as $D^s(z)$ and its residue is $s$ times that of $D^s(z)$, $r$ is still 1 and $A$ is changed into $sA$. Meinardus’ theorem now gives
\[ |\{ k \in \mathbb{N}_s^* | \|\kappa\|_1 = N\} | = Q^s[N] \sim \frac{C}{N^{3/4}} \exp\left(\frac{\sqrt{2N}}{3}\right) \]
for some constant $C$ depending on $s$.

**Corollary 4.** Let $\lambda = 3 \log(2)^2/\pi^2$. Then
\[ |\{ k \in \mathbb{N}_s^* | \|\kappa\|_1 \leq \frac{\lambda m^2}{s}\} | \sim C \frac{2^m}{\sqrt{m}} \]
as $m \to \infty$ for some constant $C$ depending on $s$.

**Proof.** With slight modification, Appendix B of [26] shows that
\[ Q[N] \sim \frac{1}{N^{3/4}} \exp\left(\sqrt{\beta N}\right) \]
for some $\beta > 0$ and also that
\[ \sum_{n=1}^{N} Q[n] \sim \frac{1}{\beta^{1/4} N^{1/4}} \exp\left(\sqrt{\beta N}\right). \]
Hence
\[ |\{ k \in \mathbb{N}_s^* | \|\kappa\|_1 \leq N\} | \sim \left(\frac{3}{\pi^2 s}\right)^{1/4} \frac{C}{N^{1/4}} \exp\left(\pi \sqrt{\frac{sN}{3}}\right) \]
where $C$ is the constant from Theorem 6 and $\beta = \pi^2 s/3$. The conclusion follows once we put in $N = \lfloor \lambda m^2/s \rfloor$ and notice that
\[ \exp\left(\pi \sqrt{\frac{sN}{3}}\right) \sim \exp\left(\pi \sqrt{\frac{\lambda m^2}{3}}\right) = 2^m. \]

Next we derive some finite sample bounds using techniques from [2]. Those results give bounds for finite $N$ instead of asymptotic equivalences as $N \to \infty$.

**Theorem 7.** For integers $N \geq 1$ and $s \geq 1$
\[ |\{ k \in \mathbb{N}_s^* | \|\kappa\|_1 = N\} | < \frac{\pi \sqrt{s}}{2\sqrt{3N}} \exp\left(\pi \sqrt{\frac{sN}{3}}\right). \]
Proof. Let \( Q(x) \) and \( Q^*(x) \) be the same generating functions used in Theorem 6. From Lemma 3 of [2], \( Q[n + 1] - Q[n] \geq Q[n] - Q[n - 1] \) for \( n > 3 \). Because \( Q[1] = Q[2] = 1 \), and \( Q[3] = Q[4] = 2 \), it follows that \( Q[n] \) is nondecreasing over integers \( n \geq 1 \). Because \( Q^*[n] \) is given by a convolution sum of coefficients of \( Q(x) \), we also see that \( Q^*[n] \) is nondecreasing in \( n \). Therefore for \( 0 \leq x < 1 \),

\[
Q^*(x) \geq \sum_{n=0}^{\infty} Q^*[n] x^n \geq Q^*[N] \sum_{n=0}^{\infty} x^n = Q^*[N] \frac{x^N}{1-x}.
\]

Furthermore, from the proof of Theorem 1 in Bidar [2], for \( x = e^{-u} \) and \( u > 0 \)

\[
\log(Q^*(e^{-u})) = s \log(Q(e^{-u})) < \frac{\pi^2 s}{12u}
\]

where we have used positivity of the dilogarithm function at positive real arguments to obtain this bound from Bidar’s expression.

After combining the above two inequalities, we get

\[
\log(Q^*[N]) < Nu + \frac{s\pi^2}{12u} + \log(1 - e^{-u}).
\]

Now we can set \( u = \sqrt{(s\pi^2)/(12N)} \) and apply the inequality \( 1 - e^{-u} < u \), after which the above equation becomes

\[
\log(Q^*[N]) < \pi \sqrt{\frac{sN}{3}} + \frac{1}{2} \log \left( \frac{s\pi^2}{12N} \right).
\]

The conclusion follows once we exponentiate both sides. \( \Box \)

Corollary 5. For integers \( N \geq 1 \) and \( s \geq 1 \),

\[
|\{k \in \mathbb{N}^s_+ | \|k\|_1 \leq N\}| < \exp \left( \pi \sqrt{\frac{s(N+1)}{3}} \right).
\]

Proof. Because \( \exp(\pi \sqrt{sx^3}/\sqrt{x}) \) is an increasing function over \([1, \infty)\),

\[
|\{k \in \mathbb{N}^s_+ | \|k\|_1 \leq N\}| = \sum_{n=1}^{N} |\{k \in \mathbb{N}^s_+ | \|k\|_1 = n\}|
\]

\[
< \int_{1}^{N+1} \frac{\pi \sqrt{s}}{2\sqrt{3x}} \exp \left( \pi \sqrt{\frac{sx}{3}} \right) dx
\]

\[
= \exp \left( \pi \sqrt{\frac{s(N+1)}{3}} \right) - \exp \left( \pi \sqrt{\frac{s}{3}} \right)
\]

and hence the conclusion. \( \Box \)

Corollary 6. For \( R, s, N \in \mathbb{N} \) satisfying \( R \geq 2\sqrt{3(s-1)N}/\pi \),

\[
|\{k \in \mathbb{N}^s_+ | \|k\|_1 \leq N + R, \|k\|_1 > R\}| < \left( \frac{R + s}{s - 1} \right) N \exp \left( \pi \sqrt{\frac{sN}{3}} \right).
\]
Proof. Recall that we use $\|v\|_1$ for the sum of entries in a vector. There are $(n + s - 1)$ vectors $v \in \mathbb{N}_0^s$ with $\|v\|_1 = n$. Hence

$$\{|\{k \in \mathbb{N}_0^s | \|\kappa\|_1 \leq N, |\kappa| \leq n\} | = \sum_{v \in \mathbb{N}_0^s : \|v\|_1 > R} |\{k \in \mathbb{N}_0^s | \|\kappa\|_1 \leq N, |\kappa| = v\}| \leq \sum_{n=R+1}^{N} \binom{n + s - 1}{s - 1} \binom{n + s}{s - 1} \exp\left(\frac{\pi}{3} \sqrt{s(N-n+1)}\right)$$

where the last inequality uses the bound from Corollary 5. The ratio of the summand with index $n$ to the summand with index $n + 1$ is

$$\frac{\binom{n + s - 1}{s - 1} \binom{n + s}{s - 1}}{\binom{n + s}{s - 1} \binom{n + s + 1}{s - 1}} \times \exp\left(\frac{\pi}{3} \sqrt{s(N-n+1)} - \frac{\pi}{3} \sqrt{s(N-n)}\right)$$

$$= \frac{n + 1}{n + s} \exp\left(\frac{\pi s/3}{\sqrt{s(N-n+1)} + \sqrt{s(N-n)}}\right)$$

$$> \exp\left(-\frac{s - 1}{n + 1} + \frac{\pi \sqrt{s}}{2 \sqrt{3(N-R)}}\right)$$

where the last inequality uses $n \geq R + 1$ and $(1 + x)^{-1} > \exp(-x)$ for $1 + x = (n + 1)/(n + s)$, that is $x = (s - 1)/(n + 1)$. Let $N' = N - R$. If $n > R \geq 2\sqrt{3(s-1)N'}/\pi$, then the above ratio is larger than 1 and we know that

$$\sum_{n=R+1}^{N} \binom{n + s - 1}{s - 1} \exp\left(\frac{\pi}{3} \sqrt{s(N-n+1)}\right) < \binom{R + s}{s - 1} N' \exp\left(\frac{\pi s N'}{3}\right)$$

which implies the conclusion. 

\[\square\]