DUALITIES IN COMPARISON THEOREMS AND
BUNDLE-VALUED GENERALIZED HARMONIC FORMS ON
NONCOMPACT MANIFOLDS

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ABSTRACT. We observe, utilize dualities in differential equations and differential inequalities (see Theorem 2.1), dualities between comparison theorems in differential equations (see Theorems E and 2.3), and obtain dualities in “swapping” comparison theorems in differential equations. These dualities generate comparison theorems on differential equations of mixed types I and II (see Theorems 2.4 and 2.5) and lead to comparison theorems in Riemannian geometry (see Theorems 2.6 and 2.8) with analytic, geometric, P.D.E.’s and physical applications. In particular, we prove Hessian comparison theorems (see Theorems 3.1 - 3.5) and Laplacian comparison theorems (see Theorems 2.6, 2.7, 3.1 - 3.5) under varied radial Ricci curvature, radial curvature, Ricci curvature and sectional curvature assumptions, generalizing and extending the work of Han-Li-Ren-Wei ([24]), and Wei ([38]). We also extend the notion of function or differential form growth to bundle-valued differential form growth of various types and discuss their interrelationship (see Theorem 5.4). These provide tools in extending the notion, integrability and decomposition of generalized harmonic forms to those of bundle-valued generalized harmonic forms, introducing Condition W for bundle-valued differential forms, and proving duality theorem and unity theorem, generalizing the work of Andreotti and Vesentini [3] and Wei [39]. We then apply Hessian and Laplacian comparison theorems to obtain comparison theorems in mean curvature, generalized sharp Caffarelli-Kohn-Nirenberg type inequalities on Riemannian manifolds, embedding theorem for weighted Sobolev spaces of functions on manifolds, geometric differential-integral inequalities, generalized sharp Hardy type inequalities on Riemannian manifolds, monotonicity formulas and vanishing theorems for differential forms of degree k with values in vector bundles, such as F-Yang Mills fields (when F is the identity map, they are Yang-Mills fields), generalized Yang-Mills-Born-Infeld fields on manifolds, Liouville type theorems for F-harmonic maps (when F(t) = 1 \( \frac{1}{p} \) \( \frac{1}{2} \) \( p \) \( 2 \), p > 1, they become p-harmonic maps or harmonic maps if \( p = 2 \)), and Dirichlet problems on starlike domains for vector bundle valued differential 1-forms and F-harmonic maps (see Theorems 4.1 - 4.7, 5.1 - 5.3, 1.1 - 1.3, 1.1 - 1.3, 4.1 - 4.7, 4.1 - 4.7, 4.1 - 4.7, generalizing the work of Caffarelli-Kohn-Nirenberg ([7]) and Costa ([14]), in which M = \( \mathbb{R}^n \) and its radial curvature \( K(r) = 0 \), the work of Wei and Li [11], Chen-Li-Wei [12, 13], Dong and Wei [17], Wei [38], Karcher and Wood [27], etc. The boundary value problem for bundle-valued differential 1-forms is in contrast to the Dirichlet problem for p-harmonic maps to which the solution is due to Hamilton [25] for the case \( p = 2 \) and \( \text{Riem}^n \leq 0 \), and Wei [38] for 1 < p < \( \infty \).

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1. Introduction

Duality is a special type of symmetry that involves with “polar opposites” and their dynamical interplays. The two (“Yin” and “Yang”, studied by a legendary sage Lao Tzu in his book Tao Te Ching and in the Book of Changes or I-Ching) are not merely opposites. The more we learn about human striving, the more we see they are supplementary, complementary, integrative and inextricably bound together. Duality is very elegant, yet powerful and has a long and distinguished history going back thousands of years. It is a natural and precious phenomenon that permeates or occurs in practically all branches of mathematics, physics, engineering, logic, psychology, real life, food science, social sciences, natural sciences, medical sciences such as alternative or energy medicine, acupuncture, meditation, qigong, physical therapy, nutrition therapy, immunotherapy, etc.

Fundamentally, duality gives two different points of view of looking at the same object. In the study of comparison theorems (in differential equations, differential geometry, differential - integral inequalities), harmonic forms, mean curvature, etc, we find many objects that have two different points of view and in principle they are all dualities. Whereas, in physics electricity and magnetism are dual objects, Hodge theory of harmonic forms is motivated in part by Maxwell’s equations of unifying magnetism with electricity in a physics world. In a complex line bundle \( L_C \) with structure group \( U(1) \) over a 4-dimensional Lorentzian manifold, Maxwell’s equations are the precise equations for the curvature 2-form \( \omega \) of a connection on \( L_C \) being both closed \((d\omega = 0)\) and co-closed \((d^*\omega = 0)\), and hence the curvature form \( \omega \) is harmonic \((\Delta \omega := -(dd^* + d^*d)\omega = 0)\). Whereas in topology, duality can broadly distinguish between contravariant functors such as cohomology, \( K \)-theory, or more general bundle theory and covariant functor such as homology or homotopy. A de Rham cohomology class involves with interplaying two objects - closed forms (coming from the source of one exterior differential operator \( d \)) and exact forms (coming from the target of its preceding operator \( d \)). Whereas in operator theory, Hodge Laplacian involves with utilizing dual objects - exterior differential operator \( d \) and codifferential operator \( d^* \), harmonic forms are privileged representatives in a de Rham cohomology class picked out by the Hodge Laplacian. From the view point of calculus of variations, harmonic forms are precisely the minimal \( L^2 \) forms within their respective topological (cohomology) classes. Whereas, many things embrace dualities Yin and Yang according to Lao Tzu, their dynamical interplay is viewed as the essence of all natural phenomena, human affairs, and Chinese medicine, and has strong impacts, interactions and connections with unity. Harmonic forms, motivated in part by Maxwell’s equations, coming out of physics, generalize harmonic functions in the study of complex analysis, partial differential equations, potential theory, and have important connections with several complex variables, Lie group representation theory and algebraic geometry through the use of Kähler metric and Bochner technique.

In the first part of this paper, we begin with studying comparison theorems in differential equations and in differential geometry and the transitions between these two fields from the viewpoint of dualities.

We observe and describe the duality between two types of differential equations, the second-order, linear Jacobi type equation (2.1) and the first-order, nonlinear Riccati type equation (2.2), and the duality between the initial condition in (2.1)
and the asymptotic condition in (2.2) (see Theorem 2.1). Moreover, to each type of differential equation, there corresponds a comparison theorem on its supersolutions and subsolutions with appropriate initial or asymptotic condition (see Theorems E and Theorem 2.2). The duality between these two types of differential equations (2.1) and (2.2) leads to the duality between two corresponding comparison theorems in differential equations of the same type. This, in turn gives rise to new duality in *swapping* differential equations of dual type and hence generates the following Comparison Theorems in Differential Equations of Mixed Types I and II with appropriate initial and asymptotic conditions:

Denote $AC(0,t)$ the set of absolutely continuous real-valued function on an open interval $(0,t)$, i.e.,

$$AC(0,t) = \{ f : (0,t) \to \mathbb{R} \mid f \text{ is absolutely continuous}, \text{ where } (0,t) \subset (0,\infty) \}.$$

**Theorem 2.3.** *(Comparing Differential Equations of Mixed Type I)* Let $G_i$ be real-valued functions defined on $(0,t_i) \subset (0,\infty)$, $i = 1,2$ satisfying

(2.15) \[ G_2 \leq G_1 \]
on $(0,t_1) \cap (0,t_2)$. Let $g_1 \in AC(0,t_1)$ be a solution of

(2.24) \[
\begin{cases}
  g_1' + \frac{g_1^2}{\kappa_1} + \kappa_1 G_1 \leq 0 \quad \text{a.e. in } (0,t_1) \\
  g_1(t) = \frac{\kappa_1}{t} + O(1) \quad \text{as } t \to 0^+
\end{cases}
\]

and $f_2 \in C([0,t_2]) \cap C^1(0,t_2)$ with $f_2 \in AC(0,t_2)$ be a positive solution of

(2.14) \[
\begin{cases}
  f_2'' + G_2 f_2 \geq 0 \quad \text{a.e. in } (0,t_2) \\
  f_2(0) = 0, f_2'(0) = \kappa_2.
\end{cases}
\]

with constants $\kappa_i$ satisfying

(2.16) \[ 0 < \kappa_1 \leq \kappa_2. \]

Then \( t_1 \leq t_2 \) and

(2.30) \[ g_1 \leq \frac{\kappa_2 f_2'}{f_2} \quad \text{on } (0,t_1). \]

**Theorem 2.4.** *(Comparing Differential Equations of Mixed Type II)* Let functions $G_i : (0,t_i) \subset (0,\infty) \to \mathbb{R}$, $i = 1,2$ satisfy (2.15). Let $f_1 \in C([0,t_1]) \cap C^1(0,t_1)$ with $f_1' \in AC(0,t_1)$ be a positive solution of

(2.13) \[
\begin{cases}
  f_1'' + G_1 f_1 \leq 0 \quad \text{a.e. in } (0,t_1) \\
  f_1(0) = 0, f_1'(0) = \kappa_1
\end{cases}
\]

and $g_2 \in AC(0,t_2)$ be a solution of

(2.25) \[
\begin{cases}
  g_2' + \frac{g_2^2}{\kappa_2} + \kappa_2 G_2 \geq 0 \quad \text{a.e. in } (0,t_2) \\
  g_2(t) = \frac{\kappa_2}{t^2} + O(1) \quad \text{as } t \to 0^+.
\end{cases}
\]

Assume (2.16). Then $t_1 \leq t_2$ and

(2.32) \[ \frac{\kappa_1 f_1'}{f_1} \leq g_2 \quad \text{on } (0,t_1). \]
When the radial Ricci curvature of a manifold is bounded below by a function \((n-1)G_1\) (cf. (2.34)), we show via Weitzenböck formula (Theorem F), this is a disguised supersolution of a Riccati type equation (2.35). Similarly, when the radial curvature of a manifold is bounded above by a function \(\tilde{G}_2\) (cf. (2.60)) (resp. bounded below by a function \(G_1\) (cf. (2.35)), we show this is a disguised subsolution (resp. supersolution) of a Riccati type equation (2.28) (resp. (2.27)) in which \(G_2 = \tilde{G}_2\) and \(t_2 = t_2\) derived from (2.60) (resp. (2.65)). Thus, we are ready to utilize dualities in comparison theorems in differential equations of mixed types to generate comparison theorems in Riemannian geometry and provide simple and direct proofs. In particular, we obtain Laplacian Comparison Theorem (2.5) when the radial Ricci curvature is bounded below as in (2.34), and Hessian and Laplacian Comparison Theorem (2.5) when the radial curvature is bounded above as in (2.60) (resp. bounded below as in (2.59)). Throughout this paper we fix a point \(x_0\) in an \(n\)-dimensional manifold \(M\). Let \(r\) be the distance function on \(M\) relative to \(x_0\), \(D(x_0) = M \setminus (\text{Cut}(x_0) \cup \{x_0\})\), \(B_t(x_0) = \{x \in M : r(x) < t\}\), and a punctured geodesic ball \(\overset{\circ}{B}_t(x_0) = B_t(x_0) \setminus \{x_0\}\).

**Theorem 2.5. (Laplacian Comparison Theorem)** Let functions \(G_i : (0, t_i) \subset (0, \infty) \rightarrow \mathbb{R}, i = 1, 2\) satisfy (2.15) on \((0, t_1) \cap (0, t_2)\). Assume

\[
(n - 1)G_1(r) \leq \text{Ric}_{\text{rad}}(r)
\]

on \(\overset{\circ}{B}_t(x_0) \subset D(x_0)\), and let \(f_2 \in C([0, t_2]) \cap C^1(0, t_2)\) with \(f'_2 \in AC(0, t_2)\) be a positive solution of

\[
\begin{align*}
\left\{ \begin{array}{l}
f''_2 + G_2 f_2 \geq 0 \quad \text{a. e. in } (0, t_2) \\
f_2(0) = 0, f'_2(0) = n - 1.
\end{array} \right.
\end{align*}
\]

Then \(t_1 \leq t_2\) and

\[
\Delta r \leq (n - 1)\frac{f''_2}{f'_2}(r)
\]

holds in \(\overset{\circ}{B}_t(x_0)\). If in addition, (2.34) occurs in \(D(x_0)\), then (2.36) holds pointwise on \(D(x_0)\) and weakly on \(M\).

**Theorem 2.5. (Hessian and Laplacian Comparison Theorems)** Let

\[
G_1(r) \leq K(r)
\]

on \(\overset{\circ}{B}_t(x_0) \subset D(x_0)\) (resp.

\[
K(r) \leq \tilde{G}_2(r)
\]

on \(\overset{\circ}{B}_{t_2}(x_0) \subset D(x_0)\), and let \(f_2 \in C([0, t_2]) \cap C^1(0, t_2)\) with \(f'_2 \in AC(0, t_2)\) be a positive solution of

\[
\begin{align*}
\left\{ \begin{array}{l}
f''_2 + G_2 f_2 \geq 0 \quad \text{a. e. in } (0, t_2) \\
f_2(0) = 0, f'_2(0) = \kappa_2,
\end{array} \right.
\end{align*}
\]

where \(G_i : (0, t_i) \rightarrow \mathbb{R}\) satisfy

\[
G_2 \leq G_1
\]
on \((0, t_1) \cap (0, t_2)\) and \(1 \leq \kappa_2\).
(resp. \( f_1 \in C([0, \tilde{t}_1]) \cap C^1(0, \tilde{t}_1) \) with \( f'_1 \in AC(0, \tilde{t}_1) \) be a positive solution of
\[
\begin{cases}
\tilde{G}_1f_1'' + \tilde{G}_1f_1 \leq 0 & \text{on } (0, \tilde{t}_1) \\
f_1(0) = 0, f'_1(0) = \kappa_1,
\end{cases}
\]
where \( \tilde{G}_1 : (0, \tilde{t}_1) \to \mathbb{R} \) satisfy
\[
\tilde{G}_2 \leq \tilde{G}_1
\]
on \((0, \tilde{t}_1) \cap (0, \tilde{t}_2) \) and \( 0 < \kappa_1 \leq 1 \).
Then \( \tilde{t}_1 \leq \tilde{t}_2 \).
\[
\text{Hess } r \leq \frac{\kappa_1 f_1''}{f_1} (g - dr \otimes dr) \quad \text{and} \quad \Delta r \leq (n - 1) \frac{\kappa_1 f_1'}{f_1}(r)
\]
on \( \tilde{B}_{\tilde{t}_1}(x_0) \) (resp. \( \tilde{t}_1 \leq \tilde{t}_2 \)).

This strengthens main theorems in [24, Theorem 4.1], and [38, Theorem D] by weakening the hypotheses on the domain and regularity of \( G_i \) (resp. \( \tilde{G}_i \)) from a smooth function on \( \mathbb{R}^+ \cup \{0\} \) to an arbitrary real-valued function on \((0, t_1)\) (resp. \((0, \tilde{t}_1)\)), (not necessarily continuous) and the hypotheses on the domain of \( K(r) \) from \( D(x_0) \) to \( \tilde{B}_{\tilde{t}_1}(x_0) \) (resp. \( \tilde{B}_{\tilde{t}_2}(x_0) \)). We also give direct and simple proofs with new applications.

In applying Theorem 2.5 under the radial Ricci curvature assumption, \( G_1 = -(n - 1) \frac{A(A - 1)}{r^2} \leq \text{Ric}^M_{\text{rad}} \), \( A \geq 1 \), or equivalently \( (2.41) \), we need \( (2.35) \) with appropriate \( G_2 \) for comparison. But in Theorem 2.6 we do not need to assume \( (2.35) \) as in Theorem 2.5. Instead, utilizing \( G_1 \), we find \( (2.44) \) as a companion system for comparison by duality, and estimate \( \frac{\tilde{B}_2}{f_2} \) (see Theorem 2.5 and its proof). Similarly, without assuming \( (2.35) \), we have

**Theorem 2.7.** (1) If
\[
(n - 1) \frac{B_1(1 - B_1)}{c + r} \leq \text{Ric}^M_{\text{rad}}(r), \quad \text{where} \quad 0 \leq B_1 \leq 1
\]
on \( \tilde{B}_{\tilde{t}_1}(x_0) \subset D(x_0) \), then
\[
\Delta r \leq (n - 1) \frac{1 + \sqrt{1 + 4B_1(1 - B_1)}}{2r}
\]
in \( \tilde{B}_{\tilde{t}_1}(x_0) \). If in addition, \( (2.51) \) occurs on \( D(x_0) \), then \( (2.52) \) holds pointwise on \( D(x_0) \) and weakly on \( M \).

(2) Equivalently, if
\[
(n - 1) \frac{B_1(1 - B_1)}{c + r} \leq \text{Ric}^M_{\text{rad}}(r), \quad 0 \leq B_1 \leq 1
\]
on \( \tilde{B}_{\tilde{t}_1}(x_0) \subset D(x_0) \), where \( c \geq 0 \), then \( (2.52) \) holds in \( \tilde{B}_{\tilde{t}_1}(x_0) \). If in addition \( (2.49) \) occurs on \( D(x_0) \), then \( (2.52) \) holds pointwise on \( D(x_0) \) and weakly on \( M \).

As applications of Theorem 2.8 we have Theorems 3.1 and 3.2 under the negative lower bound of the radial curvature assumption \( (3.1) \) \( - \frac{A(A - 1)}{r^2} \leq K(r) \), \( A \geq 1 \) (or equivalently \( (3.3) \) ), and under the negative upper bound of the radial curvature assumption \( (5.5) \) \( K(r) \leq - \frac{A(A - 1)}{r^2} \), \( A \geq 1 \) (or equivalently \( (5.7) \) ) respectively.
Corollary 3.1. (1) If the radial curvature $K$ satisfies
\[ -\frac{A(A - 1)}{r^2} \leq K(r) \leq -\frac{A_1(A_1 - 1)}{r^2} \text{ on } M\setminus\{x_0\} \] where $A \geq A_1 \geq 1$,
then we have
\[ \frac{A_1}{r} \left( g - dr \otimes dr \right) \leq \text{Hess} r \leq \frac{A_1(A_1 - 1)}{(A_1 - 1)} \left( g - dr \otimes dr \right) \text{ in the sense of quadratic forms,} \]
\[ (n - 1)\frac{A_1}{r} \leq \Delta r \leq (n - 1)\frac{A_1(A_1 - 1)}{(A_1 - 1)} \text{ pointwise on } M\setminus\{x_0\} \]
and
\[ \Delta r \leq (n - 1)\frac{A_1(A_1 - 1)}{(A_1 - 1)} \text{ weakly on } M. \]

(2) Equivalently, if $K$ satisfies
\[ -\frac{A(A - 1)}{(c + r)^2} \leq K(r) \leq -\frac{A_1(A_1 - 1)}{(c + r)^2}, \quad A \geq A_1 \geq 1, \]
on $M\setminus\{x_0\}$ where $c \geq 0$, then 3.15 holds.

Corollary 3.1 is equivalent to the following Theorem A in which (1) is due to Han-Li-Ren-Wei (24), and extends the work of Greene and Wu (21, p.38-39) from asymptotical estimates (off a geodesic ball $B_{(a-1)}(x_0)$) near infinity to pointwise estimates on $M\setminus\{x_0\}$.

Theorem A. (1)(24) Let the radial curvature $K$ satisfy
\[ -\frac{A}{r^2} \leq K(r) \leq -\frac{A_1}{r^2} \text{ where } 0 \leq A_1 \leq A \]
on $M\setminus\{x_0\}$. Then
\[ \frac{1 + \sqrt{1 + 4A_1}}{2r} \left( g - dr \otimes dr \right) \leq \text{Hess}(r) \leq \frac{1 + \sqrt{1 + 4A}}{2r} \left( g - dr \otimes dr \right) \text{ on } M\setminus\{x_0\}, \]
\[ (n - 1)\frac{1 + \sqrt{1 + 4A_1}}{2r} \leq \Delta r \leq (n - 1)\frac{1 + \sqrt{1 + 4A}}{2r} \text{ pointwise on } M\setminus\{x_0\}, \] and
\[ \Delta r \leq (n - 1)\frac{1 + \sqrt{1 + 4A}}{2r} \text{ weakly on } M. \]

(2) Equivalently, if
\[ -\frac{A}{(c + r)^2} \leq K(r) \leq -\frac{A_1}{(c + r)^2} \text{ where } 0 \leq A_1 \leq A \]
on $M\setminus\{x_0\}$, then 3.18 - 3.20 hold.

As further applications of Theorem 2.8, we also obtain the following Theorems 3.3 and 3.4 under positive lower and upper bound on radial curvature assumptions respectively.

Theorem 3.3. (1) If
\[ \frac{B_1(1-B_1)}{r} \leq K(r) \text{ where } 0 \leq B_1 \leq 1 \]
on $B_{1(x_0)} \subset D(x_0)$, then Hess $r$ and $\Delta r$ satisfy
\[ \text{Hess } r \leq \frac{1 + \sqrt{1 + 4B_1(1-B_1)}}{2r} \left( g - dr \otimes dr \right) \] and
\[ \Delta r \leq (n - 1)\frac{1 + \sqrt{1 + 4B_1(1-B_1)}}{2r} \]
on $\tilde{B}_1(x_0)$ respectively. If in addition (3.25) occurs on $D(x_0)$, then (3.26) holds pointwise on $D(x_0)$ and (3.27) holds weakly on $M$.

(2) Equivalently, if
\[
B_1(1-B_1) \leq K(r), \quad 0 \leq B_1 \leq 1
\]
on $\tilde{B}_{1i}(x_0) \subset D(x_0)$, where, $c \geq 0$, then (3.20) and (3.21) hold on $\tilde{B}_i(x_0)$. If in addition (3.28) occurs on $D(x_0)$, then (3.26) holds pointwise on $D(x_0)$ and (3.27) holds weakly on $M$.

**Theorem 3.4.** (1) If
\[
K(r) \leq \frac{B(1-B)}{r^2} \quad \text{where} \quad 0 \leq B \leq 1
\]
on $\tilde{B}_1(x_0) \subset D(x_0)$, then $Hess r$ and $\Delta r$ satisfy
\[
\frac{|B-\frac{1}{r}|}{r^2} + \frac{g - dr \otimes dr}{(n-1)} \leq Hess r \quad \text{and}
\]
\[
(n-1)\frac{|B-\frac{1}{r}|}{r^2} \leq \Delta r
\]
on $\tilde{B}_1(x_0)$ respectively.

(2) Equivalently, if
\[
K(r) \leq \frac{B(1-B)}{(c+r)^2}, \quad 0 \leq B \leq 1
\]
on $\tilde{B}_1(x_0) \subset D(x_0)$, where $c \geq 0$, then (3.36) holds on $\tilde{B}_1(x_0)$.

**Theorem 3.5.** (1) If the radial curvature $K$ satisfies
\[
\frac{B_1}{r^2} \leq K(r) \leq \frac{B}{r^2} \quad \text{where} \quad 0 \leq B_1 \leq B \leq \frac{1}{4}
\]
on $\tilde{B}_r(x_0) \subset D(x_0)$, then
\[
\frac{1+\sqrt{1+4B_1}}{2r} \left( g - dr \otimes dr \right) \leq Hess r \leq \frac{1+\sqrt{1+4B_1}}{2r} \left( g - dr \otimes dr \right) \quad \text{and}
\]
\[
(n-1)\frac{1+\sqrt{1+4B_1}}{2r} \leq \Delta r \leq (n-1)\frac{1+\sqrt{1+4B_1}}{2r}
\]
hold pointwise on $\tilde{B}_r(x_0)$. If in addition (3.33) occurs on $D(x_0)$, then
\[
\Delta r \leq (n-1)\frac{1+\sqrt{1+4B_1}}{2r}
\]
holds weakly on $M$.

(2) Equivalently, if $K$ satisfies
\[
\frac{B_1}{(c+r)^2} \leq K(r) \leq \frac{B}{(c+r)^2}, \quad 0 \leq B_1 \leq B \leq \frac{1}{4}
\]
on $\tilde{B}_r(x_0) \subset D(x_0)$, where $c \geq 0$, then (3.40) holds on $\tilde{B}_r(x_0)$. If in addition, (3.33) occurs on $D(x_0)$, then (3.47) holds weakly on $M$.

The case $K(r) \leq \frac{B}{r^2}, B \leq \frac{1}{4}$ is due to Han-Li-Ren-Wei [23]. Theorem 3.5 is equivalent to the following

**Corollary 3.5.** (1) Let the radial curvature $K$ satisfy
\[
\frac{B_1(1-B_1)}{r^2} \leq K(r) \leq \frac{B(1-B)}{r^2} \quad \text{where} \quad 0 \leq B, B_1 \leq 1
\]
on $\tilde{B}_r(x_0) \subset D(x_0)$. Then
Corollary 3.7. If the radial curvature $K$ satisfies
\[
-\frac{A}{r^2} \leq K(r) \leq \frac{B}{r^2},
\]
(resp. $-\frac{A}{(c+r)^2} \leq K(r) \leq \frac{B}{(c+r)^2}$, $0 \leq A$, $0 \leq B \leq \frac{1}{4}$

on $M \setminus \{x_0\}$, where $c > 0$, then
\[
1 + \frac{\sqrt{1 - 4B}}{2r} \left( g - dr \otimes dr \right) \leq \text{Hess}(r) \leq \frac{1 + \sqrt{1 + 4A}}{2r} \left( g - dr \otimes dr \right)
\]
and
\[
(n - 1) \frac{1 + \sqrt{1 - 4B}}{2r} \leq \Delta r \leq (n - 1) \frac{1 + \sqrt{1 + 4A}}{2r}
\]
hold on $M \setminus \{x_0\}$.

As immediate applications of Comparison Theorems in Differential Geometry (Theorems 3.1-3.5), we obtain Mean Curvature Comparison Theorems (see Theorems 4.1).

We then utilize dualities to study harmonic forms and their decomposition, integrability and growth. It is well-known that on a compact Riemannian manifold, a smooth differential form $\omega$ is harmonic if and only if it is closed and co-closed. That is,
\[
\Delta \omega = 0 \quad \text{if and only if} \quad d \omega = 0 \quad \text{and} \quad d^* \omega = 0.
\]

On complete noncompact Riemannian manifolds, although (6.1) holds for smooth $\omega$ with compact support, it does not hold in general. Simple examples include, in $\mathbb{R}^n$, a closed, non-co-closed, harmonic form $\omega_1 = x_1 dx_1$, a non-closed, co-closed, harmonic form $\omega_2 = x_n dx_1$, and a non-closed, non-co-closed, harmonic form $\omega_3 = (x_1 + x_n)dx_1$, or $\omega_4 = x_1 x_n dx_1 + x_n dx_n$ (see [39]). However, it is proved in [3].

Theorem B. (A. Andreotti and E. Vesentini [3]) On a complete noncompact Riemannian manifold $M$, (6.1) holds for every smooth $L^2$ differential form $\omega$.

It is interesting to explore any possible generalizations of Theorem B, in particular, to discuss whether or not (6.1) holds for $L^q$ differential form $\omega$, where $q \neq 2$. Some study of this generalization can be found in [43] p.663, Proposition 1], its counter-examples are given by D. Alexandru-Rugina (see [1] p. 81, Remarque 3) on the hyperbolic space $H^n_m$, $m \geq 3$, and relevant remarks of Pigola-Rigoli-Setti are discussed in [31] p.260, Remark B.8].
In [39], we introduce and add Condition W (see (6.3), where \( \Omega = \omega \) and \( A^k(\xi) = A^k \)) to the above work, so that the counter-examples provided in [11] cannot happen, the conclusion of the Proposition in [43] still holds, and works in a more general setting with geometric and physical applications. We extend the differential form in \( L^2 \) space in Theorem B in several ways. To this end, recall in extending functions in \( L^2 \) space (resp. in \( L^q \) space), we use direct simple new methods. Let \( A^k \) has the same value of \( q \), or equivalently \( p \)-imbalanced growth (see (5.1)). We also introduce the notion of \( p \)-balanced growth for functions and differential forms on complete noncompact Riemannian manifold \( M \) in [36]. Namely, a function or a differential form \( f \) has \( p \)-balanced growth (or, simply, is \( p \)-balanced) if \( f \) has one of the following: \( p \)-finite, \( p \)-mild, \( p \)-obtuse, \( p \)-moderate, or \( p \)-small growth for \( q = 2 \) (resp. for the same value of \( q \)), and their counter-parts \( p \)-infinite, \( p \)-severe, \( p \)-acute, \( p \)-immoderate, and \( p \)-large" growth [40] (see Definition 5.1). We also introduce the notion of \( p \)-imbalanced growth on \( M \) and \( p \)-moderate growth for functions and differential forms.

**Theorem C.** (Unity Theorem) ([39], Theorem 4.1) If a differential form \( \omega \in A^k \) has 2-balanced growth, for \( q = 2 \), or for \( 1 < q(\neq 2) < 3 \) with \( \omega \) satisfying Condition W, then the following six statements: (i) \( \langle \omega, \Delta \omega \rangle \geq 0 \). (ii) \( \Delta \omega = 0 \). (iii) \( d\omega = d^* \omega = 0 \). (iv) \( \langle \star \omega, \Delta \star \omega \rangle \geq 0 \). (v) \( \Delta \star \omega = 0 \). (vi) \( d \star \omega = d^* \star \omega = 0 \). are equivalent.

In the second part of the paper, we extend the notion of function growth and differential form growth (see [36], [39]) to bundle-valued differential form growth of various types, and prove their interrelationship. In particular, we show

**Theorem 5.4.** For a given \( q \in \mathbb{R} \), a function, or differential form or bundle-valued differential form \( f \) is

\[
\begin{align*}
\text{p - moderate} & \iff \text{p - small} & \Rightarrow \text{p - mild} & \Rightarrow \text{p - obtuse} & \Rightarrow \text{p - large} & \iff \text{p - immoderate.} \\
\text{p - acute} & \Rightarrow \text{p - severe} & \Rightarrow \text{p - large} & \iff \text{p - immoderate.}
\end{align*}
\]

Hence, for a given \( q \in \mathbb{R} \), \( f \) is

\[
\begin{align*}
\text{p - balanced} & \Rightarrow \text{either p - finite} & \text{or p - obtuse} \\
\text{p - imbalanced} & \Rightarrow \text{both p - infinite and p - immoderate.}
\end{align*}
\]

If in addition, \( \int_{B(x,r)} |f|^q dv \) is convex in \( r \), then the following four types of growth are all equivalent: \( f \) is \( p \)-mild, \( p \)-obtuse, \( p \)-moderate, \( p \)-small (resp. \( p \)-severe, \( p \)-acute, \( p \)-immoderate, \( p \)-large), i.e., \( f \) is

\[
\begin{align*}
(5.2) & \iff (5.3) & \iff (5.4) & \iff (5.6) \; \text{for the same value of} \; q \in \mathbb{R}.
\end{align*}
\]

In particular, we have
Corollary 5.1. Every $L^q$ function or differential form or bundle-valued differential form $f$ on $M$ has $p$-balanced growth, $p \geq 0$, and in fact, has $p$-finite, $p$-mild, $p$-obtuse, $p$-moderate, and $p$-small growth, $p \geq 0$, for the same value of $q$.

We also introduce Condition W for bundle-valued differential forms in $A^k(\xi)$ and extend the unity theorem from generalized harmonic forms in $A^k$ to generalized harmonic forms in $A^k(\xi)$ with values in a vector bundle, where $\xi : E \to M$ be a smooth Riemannian vector bundle over $(M, g)$, i.e., a vector bundle such that at each fiber is equipped with a positive inner product $\langle \cdot , \cdot \rangle_E$. Set $A^k(\xi) = \Gamma(A^kT^*M \otimes E)$ the space of smooth $k$-forms on $M$ with values in the vector bundle $\xi : E \to M$. Let $d^\nabla : A^k(\xi) \to A^{k+1}(\xi)$ relative to the connection $\nabla^E$ be the exterior differential operator, $\Delta^\nabla : A^{k+1}(\xi) \to A^k(\xi)$ relative to the connection $\nabla^E$ be the codifferential operator, and $\Delta^\nabla = -(d^\nabla \delta^\nabla + \delta^\nabla d^\nabla) : A^k(\xi) \to A^k(\xi)$. Note that if $E$ is the trivial bundle $M \times \mathbb{R}$ equipped with the canonical metric, then $A^k$ is isometric to $A^k(\xi)$, $d = d^\nabla$ (as in (6.2)), $d^* = \delta^\nabla$ (as in (6.3)), and $\Delta^\nabla : A^k(\xi) \to A^k(\xi)$ coincides with $\Delta = -(dd^* + d^*d) : A^k \to A^k$. Denote by the same notations $\langle \cdot , \cdot \rangle, | \cdot |$, and $\star : A^k(\xi) \to A^{n-k}(\xi)$, the inner product, the norm induced in fibers of various tensor bundles by the metric of $M$, and the linear operator which assigns to each $k$-form in $A^k(\xi)$ on $M$ an $(n-k)$-form in $A^{n-k}(\xi)$ and which satisfies $\star \star = (-1)^{nk+k+1}$ respectively.

Theorem 6.2. (Unity Theorem) If a bundle-valued differential $k$-form $\Omega \in A^k(\xi)$ has $2$-balanced growth, for $q = 2$, or for $1 < q(\neq 2) < 3$ with $\Omega$ satisfying Condition W (6.5), then the following six statements are equivalent.
(i) $\langle \Omega, \Delta^\nabla \Omega \rangle \geq 0$ (i.e., $\Omega$ is a generalized bundle-valued harmonic form).
(ii) $\Delta^\nabla \Omega = 0$ (i.e., $\Omega$ is a bundle-valued harmonic form).
(iii) $d^\nabla \Omega = \delta^\nabla \Omega = 0$ (i.e., $\Omega$ is closed and co-closed).
(iv) $\langle \Omega, \Delta^\nabla \star \Omega \rangle \geq 0$ (i.e., $\Omega$ is a generalized bundle-valued harmonic form).
(v) $\Delta^\nabla \star \Omega = 0$ (i.e., $\star \Omega$ is a bundle-valued harmonic form).
(vi) $d^\nabla \star \Omega = \delta^\nabla \star \Omega = 0$ (i.e., $\star \Omega$ is closed and co-closed).

This generalizes the work of Andreotti and Vesentini [3] for the case $\Omega \in A^k$ and $\Omega$ is $L^2$, Theorem C ([39, Theorem 4.1]), and leads to its immediate extension.

Corollary 6.1. Let $\Omega \in A^k(\xi)$ be in $L^2$, or in $L^q, 1 < q(\neq 2) < 3$ satisfying Condition W (see (6.5)). Then $\Omega$ is harmonic if and only if $\Omega$ is closed and co-closed.

In [41], using an analog of Bochner’s method or “$B^2 - 4AC \leq 0$” method, Wei and Li derive a geometric differential-integral inequality on a manifold (see Theorem 7.1) and give the first application of the Hessian comparison theorems to generalized sharp Caffarelli-Kohn-Nirenberg type inequalities on Riemannian manifolds (see Theorem 7.2). The case $M = \mathbb{R}^n$ is due to L. Caffarelli et al. [7] and is sharp due to Costa [14]. Furthermore, Wei and Li [41] Theorems 1 and 2, Corollaries 1.1-1.5] also gave the first application of the Hessian comparison theorems to generalized Hardy type inequalities on Riemannian manifolds. The case $M = \mathbb{R}^n$ is sharp (see [30] [26]).

In the third part of the paper, using and extending the work in [41] and [38], we apply Laplacian Comparison Theorems 2.6 and 2.7 based on Theorem 4.5 to study geometric differential-integral inequalities on manifolds. In particular, we obtain
Theorem 7.3. (Generalized sharp Caffarelli-Kohn-Nirenberg type inequalities under radial Ricci curvature assumption) Let $M$ be an $n$-manifold with a pole such that the radial Ricci curvature $\text{Ric}^\text{rad}_M(r)$ of $M$ satisfies one of the five conditions in (7.6) where $0 \leq B_1 \leq 1 \leq A$ are constants. Then for every $u \in W^{1,2}_0(M \setminus \{x_0\})$ and $a, b \in \mathbb{R}$, (7.23) holds in which the constant $C$ is given by (7.6).

When $A = 1$, the result is sharp and we recapture theorems of Wei and Li (see [11] Theorems 3 and 4) or Theorem 7.2. Analogously, we apply Hessian Comparison Theorems 3.1, 3.2, 3.3, and 3.4 based on Theorem 2.8 to study geometric differential-integral inequalities. In particular, we obtain the following. When $B_1 = 0$, we recapture theorems of Wei and Li (see [11] Theorems 3 and 4) or Theorem 7.2. The result is sharp when $M = \mathbb{R}^n$.

Theorem 7.4. (Generalized sharp Caffarelli-Kohn-Nirenberg type inequalities under radial curvature assumption) Let $M$ be an $n$-manifold with a pole and the radial curvature $K(r) \leq 0$ of $M$ satisfy one of the twelve conditions in (7.7) where $0 \leq B_1 \leq 1 \leq A$ are constants. Then for every $u \in W^{1,2}_0(M \setminus \{x_0\})$ and $a, b \in \mathbb{R}$, (7.23) holds in which the constant $C$ is given by (7.8).

Hessian comparison theorems lead to embedding theorems for weighted Sobolev spaces of functions on Riemannian manifolds (see Theorem 7.5) and various geometric integral inequalities on manifolds (see (7.13), (7.15), (7.17), (7.19), (7.21), (7.23), (7.25) in Theorem 7.6). Concurrently, Laplacian comparison Theorems lead to the following theorem.

Theorem 7.7. (Generalized sharp Hardy type inequality) Let $M$ be an $n$-manifold with a pole satisfying

$$\text{Ric}^\text{rad}_M(r) \geq -(n-1)\frac{A(A-1)}{p^2} \text{ where } A \geq 1.\ (7.27)$$

Then for every $u \in W^{1,p}_0(M)$, $\frac{u}{|x|} \in L^p(M)$ with $p > (n-1)A + 1$, we have

$$\left(\frac{p-1-(n-1)A}{p}\right)^p \int_M |u|^p \, dv \leq \int_M |\nabla u|^p \, dv. \ (7.28)$$

The case $A = 1$ is sharp and is due to Chen-Li-Wei (see [13] Theorem 5). Furthermore, the assumption $\frac{u}{|x|} \in L^p(M)$ cannot be dropped, or a counter-example is constructed in [13] in Section 5. Moreover, this theorem does not require $u \in W^{1,p}_0(M \setminus \{x_0\})$ and hence even for the case $p = 2$, the result is stronger than Hardy’s inequality (7.25), obtained by setting $a = 1$ and $b = 0$ in Theorems 7.3 and 7.4 (see Theorems 7.6, 7.7, 7.8). More recently, through the study of comparison theorems in Finsler geometry, some generalized Hardy type inequalities and generalized Caffarelli-Kohn-Nirenberg type inequalities on Riemannian manifolds have been extended to Finsler manifolds by Wei and Wu (see [12]).

In the forth part of the paper, following the framework in [18], employing and augmenting the work of Dong and Wei [17], Wei [38], we apply comparison theorems to the study of differential forms of degree $k$ with values in vector bundles. In particular, we obtain a monotonicity formula for vector bundle valued differential $k$-forms, if curvature satisfies one of the seven conditions in (8.12) (see Theorem 8.1).

Whereas a “microscopic” approach to monotonicity formulae leads to celebrated blow-up techniques due to de-Giorgi (15) and Fleming (19), and regularity theory in geometric measure theory (see [18] 4, 24, 32, 25, 29). Examining a macroscopic viewpoint of the above monotonicity formula, we derive vanishing theorems for...
vector bundle valued $k$-forms (see Theorem 9.1). This macroscopic viewpoint also leads to the study of global rigidity phenomena in locally conformal flat manifolds (see Dong-Lin-Wei [16]).

In [17], Dong and Wei introduced $F$-Yang-Mills fields (When $F$ is the identity map, they are Yang-Mills fields) (see also [20]). As an application of Theorem 9.1, we obtain Vanishing Theorem for $F$-Yang-Mills fields (see Theorem 9.2).

A natural link to unity (see [37]) leads to Liouville theorems for $F$-harmonic maps (When $F(t) = \frac{1}{p}(2t)^{\frac{2}{p}}$, $p > 1$, they become $p$-harmonic maps, or harmonic maps if $p = 2$.) (see Theorem 10.1).

In contrast to the work of Chang-Chen-Wei ([8]) on Liouville properties for a $p$-harmonic morphism or a $p$-harmonic function on a manifold that supports a weighted Poincaré inequality, we have Liouville theorems for $p$-harmonic maps (see Theorem 10.2).

The techniques can be further applied to generalized Yang-Mills-Born-Infeld fields on manifolds. In particular we have vanishing theorems for generalized Yang-Mills-Born-Infeld fields (with the plus sign) on Riemannian manifolds (see Theorem 11.2).

In [35], we solve the Dirichlet problem for $p$-harmonic maps to which the solution is due to Hamilton [23] in the case $p = 2$ and $\text{Riem}^N \leq 0$:

**Theorem D.** ([35]) Let $M$ be a compact Riemannian $n$-manifold with boundary $\partial M$ and $N$ be a compact Riemannian manifold with a contractible universal cover $\tilde{N}$. Assume that $N$ has no non-trivial $p$-minimizing tangent map of $R^\ell$ for $\ell \leq n$. Then any $u \in \text{Lip}(\partial M, N) \cap C^0(M, N)$ of finite $p$-energy can be deformed to a $p$-harmonic map $u_0 \in C^1(M, N)$ minimizing $p$-energy in the homotopic class with $u_0|_{\partial M} = u|_{\partial M}$, where $1 < p < \infty$. In particular, every $u \in C^1(M, N)$ can be deformed to a $C^{1,\alpha}$ $p$-harmonic map $u_0$ in $M \setminus \partial M$ minimizing $p$-energy in the homotopic class with Hölder continuous $u_0|_{\partial M} = u|_{\partial M}$.

Whereas the domain of the solution $u$ of the above Dirichlet problem is an arbitrary compact Riemannian manifold (with boundary), we discuss a Dirichlet problem for which the target of its solution is an arbitrary Riemannian manifold.

In the fifth part of the paper, we apply the comparison theorems to the study of Dirichlet problems on starlike domains for vector bundle valued differential 1-forms, augmenting the work of Dong and Wei ([17]), Wei [38] and generalizing and refining the work of Karcher and Wood ([27]) on harmonic maps on disc domains with constant boundary value (see Theorems 12.1 and 12.2).

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2. DUALITIES IN COMPARISON THEORY

We begin with a fundamental duality in comparison theory - the duality between the second-order linear Jacobi type equation with the initial conditions at 0 and the first-order nonlinear Riccati type equation with the asymptotic condition at 0:

2.1. Dualities between Differential Equations and Differential Inequalities.
Theorem 2.1. Let $G(t)$ be a real-valued function defined on $(0, \tau) \subset (0, \infty)$, $f \in C([0, \tau]) \cap C^1(0, \tau)$, $f > 0$ in $(0, \tau)$, $f' \in AC(0, \tau)$, $\kappa > 0$ be a constant and $g \in AC(0, \tau)$. Then

$$
\begin{aligned}
\begin{cases}
\frac{f''(t)}{f(t)} + G(t) f(t) = 0 \ (\text{resp.} \leq 0, \geq 0) \ &\in (0, \tau) \\
f(0) = 0, f'(0) = \kappa 
\end{cases}
\end{aligned}
$$

is dual to (or if and only if)

$$
\begin{aligned}
\begin{cases}
g'(t) + \frac{4G(t)^2}{\kappa} + \kappa G(t) = 0 \ (\text{resp.} \leq 0, \geq 0) \ &\in (0, \tau) \\
g(t) = \frac{\kappa}{t} + O(1) \ &\text{as} \ t \to 0^+.
\end{cases}
\end{aligned}
$$

Furthermore, (2.1) is transformed into (2.2), via the transformer

$$
g = \frac{\kappa f'}{f}
$$
in $(0, \tau)$. Conversely, (2.1) is reversed from (2.2) and solution (resp. supersolution, subsolution) $f(t)$ of (2.1) enjoys

$$
\frac{f'(t)}{f(t)} = \frac{g(t)}{\kappa}
$$
in $(0, \tau)$, via the reverser

$$
f(t) = \kappa t \exp \left( \int_0^t \frac{g(s)}{\kappa} - \frac{1}{s} ds \right)
$$
in $(0, \tau)$.

Proof. ($\Rightarrow$) Since $f > 0$ in $(0, \tau)$, (2.3) is well-defined in $(0, \tau)$ with $g \in AC(0, \tau)$.

It follows from (2.3), the quotient law and (2.1) that

$$
g' = \frac{\kappa f''(t) - \kappa (f'(t))^2}{f^2}
$$

$$
= \left(\text{resp.} \leq, \geq\right) -\frac{\kappa G f^2}{f^2} - \frac{\left(\frac{\kappa f'}{f}\right)^2}{\kappa}
$$

$$
= \left(\text{resp.} \leq, \geq\right) -\kappa G - \frac{g^2}{\kappa} \ &\text{a.e. in} \ (0, \tau).
$$

Thus, we obtain the first-order nonlinear differential equation (resp. differential inequalities )

$$
g' + \frac{g^2}{\kappa} + \kappa G = 0 \ (\text{resp.} \leq 0, \geq 0)
$$
a.e. in $(0, \tau)$, with the initial condition at $0$ in (2.1) being converted to the asymptotic condition at $0$ in (2.2). Indeed,

$$
g(t) = \kappa \frac{f'(t)}{f(t)} = \kappa \frac{\kappa + O(t)}{kt + O(t^2)} = \kappa \frac{1 + O(t)}{t} = \kappa \frac{1 + O(t)}{t + O(1)}
$$

$$
= \kappa \left( 1 + O(t) \right) + O(t) \ &\text{as} \ t \to 0^+
$$
Let $f = \kappa g$. Then $f$ satisfies (2.11) if and only if $\tilde{f}$ satisfies (2.12). Let $g = \frac{f}{\kappa}$. Then $g$ satisfies (2.12) if and only if $\tilde{g}$ satisfies (2.2). Furthermore, by Theorem 2.1, $\tilde{f}$ satisfies (2.1) if and only if $\tilde{g}$ satisfies (2.2). Consequently, “(2.11) is dual to (2.12)” is equivalent to “(2.1) is dual to (2.2)” and we have shown (3).

Proof. (1) Choose $\kappa = 1$ in Theorem 2.1 (2) and (3) Let $\tilde{f} = \kappa f$. Then $f$ satisfies (2.11) if and only if $\tilde{f}$ satisfies (2.1). Let $g = \frac{f}{\kappa}$. Then $g$ satisfies (2.12) if and only if $\tilde{g}$ satisfies (2.2). Furthermore, by Theorem 2.1, $\tilde{f}$ satisfies (2.1) if and only if $\tilde{g}$ satisfies (2.2). Consequently, “(2.11) is dual to (2.12)” is equivalent to “(2.1) is dual to (2.2)” and we have shown (3).
2.2. Dualities between Comparison Theorems in Differential Equations.

To each differential equation (2.1) (resp. (2.2)) there is associated Comparison Theorem E (resp. Theorem 2.2) between its supersolution and subsolution with appropriate initial condition (resp. asymptotic condition). We state and prove these “opposites” as well as “unity” in a general form.

**Theorem E. (Sturm-Liouville Type Comparison Theorem)** Let \( G_i : (0, t_i) \subset (0, \infty) \to \mathbb{R} \) be functions satisfying (2.13) on \((0, t_1) \cap (0, t_2)\), and \( f_i \in C([0, t_i]) \cap C^1(0, t_i) \) with \( f_i' \in AC(0, t_i) \) for \( i = 1, 2 \) be solutions of the problems

\[
\begin{align*}
\begin{cases}
   f''_1 + G_1 f_1 & \leq 0 \quad \text{a.e. in } (0, t_1) \\
   f_1(0) = 0, f_1'(0) = \kappa_1
   
   
   \end{cases}
\end{align*}
\]

(2.13)

\[
\begin{align*}
\begin{cases}
   f''_2 + G_2 f_2 & \geq 0 \quad \text{a.e. in } (0, t_2) \\
   f_2(0) = 0, f_2'(0) = \kappa_2
   
   \end{cases}
\end{align*}
\]

(2.14)

with

\[
G_2 \leq G_1
\]

(2.15)

on \((0, t_1) \cap (0, t_2)\) and

\[
0 < \kappa_1 \leq \kappa_2,
\]

(2.16)

where \( f_i > 0 \) on \((0, t_i), i = 1, 2 \). Then

\[
f_2 > 0 \text{ on } (0, t_1), \quad \frac{f_2'}{f_1} \leq \frac{f_2'}{f_2} \text{ in } (0, t_1) \quad \text{and} \quad f_1 \leq f_2 \text{ in } (0, t_1).
\]

(2.17)

**Proof.** By (2.13), (2.14) and (2.15),

\[
(f_2 f_1 - f_1' f_2)'(0) = 0
\]

(2.18)

and

\[
(f_2 f_1 - f_1' f_2)' = f_2'' f_1 - f_1'' f_2 \geq (G_1 - G_2) f_1 f_2 \geq 0
\]

a.e. in \((0, t_1) \cap (0, t_2)\) hold. Whence \( f_2 f_1 - f_1' f_2 \geq 0 \) and

\[
\frac{f_2'}{f_1} \leq \frac{f_2'}{f_2}
\]

(2.19)

in \((0, t_1) \cap (0, t_2)\). Integrating (2.20) from \( \epsilon > 0 \) to \( r(\min\{t_1, t_2\}) \), and passing \( \epsilon \) to 0 from the right, one has

\[
\ln f_1(t)|_{t=\epsilon}^r \leq \ln f_2(t)|_{t=\epsilon}^r \quad \text{and}
\]

\[
f_1(r) = \lim_{\epsilon \to 0^+} f_1(\epsilon) \leq \lim_{\epsilon \to 0^+} \frac{f_1(\epsilon)}{f_2(\epsilon)} f_2(r) = \frac{\kappa_1}{\kappa_2} f_2(r) \leq f_2(r) \text{ in } [0, t_1) \cap [0, t_2).
\]

By the continuity of \( f_1 \) and \( f_2 \),

\[
f_1 \leq f_2
\]

(2.21)

on \([0, t_1] \cap [0, t_2]\). Let \( t_2 = \sup \{ t : f_2 > 0 \text{ on } (0, t) \} \). We claim \( t_1 < t_2 \). Otherwise, \( t_2 < t_1 \) would lead to, via (2.21) \( 0 < f_1(t_2) \leq f_2(t_2) \), contradicting by continuity \( f_2(t_2) = 0 \). \( \square \)
Corollary 2.2. (1) Let $G_i : (0, t_i) \subset (0, \infty) \to \mathbb{R}$ be functions satisfying (2.15) on $(0, t_1) \cap (0, t_2)$, and let $f_i \in C([0, t_i]) \cap C^1(0, t_i)$, with $f'_i \in AC(0, t_i)$ be positive solutions of the problems

\begin{equation}
\begin{cases}
  f''_1 + G_1 f_1 \leq 0 \quad \text{a.e. in } (0, t_1) \\
  f_1(0) = 0, f'_1(0) = 1
\end{cases}
\end{equation}

(2.22)

\begin{equation}
\begin{cases}
  f''_2 + G_2 f_2 \geq 0 \quad \text{a.e. in } (0, t_2) \\
  f_2(0) = 0, f'_2(0) = 1
\end{cases}
\end{equation}

(2.23)

Then (2.17) holds.

(2) Theorem E is equivalent to (1).

Proof. (1) Choose $\kappa_1 = \kappa_2 = 1$ in Theorem E. (2) Let $\tilde{f}_i = \kappa_i f_i$. Then $\tilde{f}_1$ satisfies (2.22) if and only if $\tilde{f}_1$ satisfies (2.13), $f_2$ satisfies (2.23) if and only if $\tilde{f}_2$ satisfies (2.14). Furthermore, (2.20) holds in $(0, t_1) \cap (0, t_2)$ if and only if $\frac{\tilde{f}_1}{f_1} \leq \frac{f'_1}{\tilde{f}_2}$ in $(0, t_1) \cap (0, t_2)$. These imply that (2.21) holds on $[0, t_1] \cap [0, t_2]$ if and only if $\tilde{f}_1 \leq \tilde{f}_2$ in $[0, t_1] \cap [0, t_2]$. It follows that (2.17) holds. Consequently, Theorem E is equivalent to (1).

Theorem 2.2. (Comparison Theorem for supersolutions and subsolutions of Riccati type equations) Let functions $G_i : (0, t_i) \subset (0, \infty) \to \mathbb{R}$ satisfy (2.15) on $(0, t_1) \cap (0, t_2)$, and let $g_i \in AC(0, t_i), i = 1, 2$ be solutions of

\begin{equation}
\begin{cases}
  g'_1 + \frac{g^2_1}{\kappa_1} + \kappa_1 G_1 \leq 0 \quad \text{a.e. in } (0, t_1) \\
  g_1(t) = \frac{1}{t^\kappa} + O(1) \quad \text{as } t \to 0^+
\end{cases}
\end{equation}

(2.24)

\begin{equation}
\begin{cases}
  g'_2 + \frac{g^2_2}{\kappa_2} + \kappa_2 G_2 \geq 0 \quad \text{a.e. in } (0, t_2) \\
  g_2(t) = \frac{1}{t^\kappa} + O(1) \quad \text{as } t \to 0^+
\end{cases}
\end{equation}

(2.25)

with constants $\kappa_i$ satisfying (2.16). Then in $(0, t_1)$,

\begin{equation}
  g_1 \leq g_2.
\end{equation}

(2.26)

Proof. Proceeding as in the proof of Theorem 2.1 we use (2.5) in which $f = f_i$, $g = g_i$ and $\kappa = \kappa_i, i = 1, 2$, then the systems (2.24) and (2.25) are transformed into (2.13) and (2.14) respectively, satisfying (2.15) on $(0, t_1) \cap (0, t_2)$, and enjoying (2.3) (in which $f = f_i$, $g = g_i$ and $\kappa = \kappa_i$), $f_i > 0$ on $(0, t_i)$, $f_i \in C^0([0, t_i]) \cap C^1(0, t_i)$ and $f'_i \in AC(0, t_i), i = 1, 2$. It follows from Theorem E that (2.17) holds. By (2.3), $\frac{g_1}{\kappa_1} \leq \frac{g_2}{\kappa_2}$. Consequently, via (2.16) we obtain the desired (2.26).

Corollary 2.3. (1) Let functions $G_i : (0, t_i) \subset (0, \infty) \to \mathbb{R}$ satisfy (2.15) on $(0, t_1) \cap (0, t_2)$, and $g_i \in AC(0, t_i), i = 1, 2$ be solutions of

\begin{equation}
\begin{cases}
  g'_1 + g_1^2 + G_1 \leq 0 \quad \text{a.e. in } (0, t_1) \\
  g_1(t) = \frac{1}{t} + O(1) \quad \text{as } t \to 0^+
\end{cases}
\end{equation}

(2.27)

\begin{equation}
\begin{cases}
  g'_2 + g_2^2 + G_2 \geq 0 \quad \text{a.e. in } (0, t_2) \\
  g_2(t) = \frac{1}{t} + O(1) \quad \text{as } t \to 0^+
\end{cases}
\end{equation}

(2.28)
Then (2.26) holds on $(0, t_1)$.

(2) Theorem 2.2 is equivalent to (1).

Proof. (1) Choose $\kappa_1 = \kappa_2 = 1$ in Theorem 2.2. (2) Enough to show that (1) $\Rightarrow$ Theorem 2.2. Let $\tilde{g}_1 = \frac{g_1}{\kappa_1}$. Then $\tilde{g}_1$ satisfies (2.27) if and only if $g_1$ satisfies (2.24) and $\tilde{g}_2$ satisfies (2.28) if and only if $g_2$ satisfies (2.25). It follows from (1) that $\tilde{g}_1 \leq \tilde{g}_2$. Hence,

$$\frac{g_1}{\kappa_1} = \tilde{g}_1 \leq \tilde{g}_2 = \frac{g_2}{\kappa_2}$$

in $(0, t_1)$. Consequently, using (2.16) yields the desired (2.26) on $(0, t_1)$.

\[ \square \]

2.3. Dualities in “Swapping” Comparison Theorems in Differential Equations. We note the duality between Jacobi type equation (2.1) and Riccati type equation (2.2) leads to the duality between the above two types of corresponding comparison theorems in differential equations - Theorem E and Theorem 2.2. This duality between two comparison theorems in differential equations of the same type, in turn gives rise to the duality in swapping differential equations of different types and hence generates Comparison Theorems on Differential Equations of Mixed Types I and II. Whereas, mixed type I concerns with comparing supersolutions of a Riccati type equation with subsolutions of a Jacobi type equation, mixed type II concerns with comparing supersolutions of a Jacobi type equation with subsolutions of a Riccati type equation.

2.4. Comparison Theorems on Differential Equations of Mixed Types.

Theorem 2.3. (Comparing Differential Equations of Mixed Type I) Let functions $G_i : (0, t_i) \subset (0, \infty) \to \mathbb{R}$ satisfy (2.15) on $(0, t_1) \cap (0, t_2)$. Let $g_1 \in AC(0, t_1)$ be a solution of (2.24) and $f_2 \in C([0, t_2]) \cap C^1(0, t_2)$ with $f_2' \in AC(0, t_2)$ be a positive solution of (2.23) with constants $\kappa_i$ satisfying (2.16). Then $t_1 \leq t_2$ and

$$g_1 \leq \frac{\kappa_2 f_2'}{f_2}$$

on $(0, t_1)$.

Proof. Proceed as in the proof of Theorem 2.1 via (2.3) in which $g = g_2$, $f = f_2$, and $\kappa = \kappa_2$, the system (2.14) is transformed into (2.25). Applying Theorem 2.2 to (2.24) and (2.25), we have (2.26), and hence (2.30), via (2.3) on $(0, t_1)$.

Corollary 2.4. (1) Let $G_i : (0, t_i) \subset (0, \infty) \to \mathbb{R}$, $i = 1, 2$ satisfy (2.15) on $(0, t_1) \cap (0, t_2)$, $g_1 \in AC(0, t_1)$ be a solution of (2.27), and $f_2 \in C([0, t_2]) \cap C^1(0, t_2)$ with $f_2' \in AC(0, t_2)$ be a positive solution of (2.23). Then $t_1 \leq t_2$ and

$$g_1 \leq \frac{f_2'}{f_2}$$

on $(0, t_1)$.

(2) Theorem 2.3 is equivalent to (1).
Proof. (1) Choose $\kappa_1 = \kappa_2 = 1$ in Theorem 2.3. (2) Enough to show that $(1) \Rightarrow$ Theorem 2.3. Let $\tilde{g}_1 = \frac{\tilde{f}}{\kappa_1}$ and $\tilde{f}_2 = \kappa_2 f_2$. Then $\tilde{g}_1$ satisfies (2.27) if and only if $g_1$ satisfies (2.24) and $\tilde{f}_2$ satisfies (2.23) if and only if $f_2$ satisfies (2.14). It follows from (1) that $\tilde{g}_1 \leq \frac{\tilde{f}_2'}{f_2}$. Hence, on $(0, t_1)$,
\[ g_1 = \tilde{g}_1 \leq \frac{f_2'}{f_2} = \frac{f_2'}{f_2} \]
Consequently, using (2.16) yields $g_1 \leq \kappa_1 \frac{f_1'}{f_1} \leq \kappa_2 \frac{f_2'}{f_2}$, the desired (2.30) on $(0, t_1)$.

Theorem 2.4. (Comparing Differential Equations of Mixed Type II) Let functions $G_i : (0, t_i) \subset (0, \infty) \rightarrow \mathbb{R}$, $i = 1, 2$ satisfy (2.15) on $(0, t_1) \cap (0, t_2)$. Let $f_1 \in C^1([0, t_1]) \cap C^1(0, t_1)$ with $f_1' \in AC(0, t_1)$ be a positive solution of (2.13) and $g_2 \in AC(0, t_2)$ be a solution of (2.25) with constants $\kappa_i$ satisfying (2.16). Then $t_1 \leq t_2$ and
\[ \frac{\kappa_1 f_1'}{f_1} \leq g_2 \]
on $(0, t_1)$.

Proof. Proceed as in the proof of Theorem 2.1 via (2.3) in which $g = g_1$, $f = f_1$, and $\kappa = \kappa_1$, the system (2.13) is transformed into (2.24). Applying Theorem 2.2 to (2.24) and (2.25), we have (2.30), and hence (2.32), via (2.3) on $(0, t_1)$.

Similarly, we have

Corollary 2.5. (1) Let $G_i : (0, t_i) \subset (0, \infty) \rightarrow \mathbb{R}$, $1 \leq i \leq 2$ satisfy (2.15) on $(0, t_1) \cap (0, t_2)$. Let $f_1 \in C^1([0, t_2]) \cap C^1(0, t_1)$ with $f_1' \in AC(0, t_1)$ be a positive solution of (2.22) and $g_2 \in AC(0, t_2)$ be a solution of (2.28). Then
\[ \frac{f_1'}{f_1} \leq g_2 \]
on $(0, t_1)$.

(2) Theorem 2.4 is equivalent to (1).

2.5. Applications in Comparison Theorems in Riemannian Geometry. When the radial Ricci curvature $\text{Ric}^M_{\text{rad}}$ of a manifold $M$ has a lower bound, it can be viewed as a supersolution of Riccati type equation in disguise. Similarly, when the radial curvature $K$ of a manifold has an upper bound (resp. lower bound), it can be viewed as a subsolution (resp. supersolution) of Riccati type equation in disguise. Thus, utilizing dualities in comparison theorems in differential equations of mixed types, we generate comparison theorems in Riemannian geometry and provide simple and direct proofs.

Let $M$ be an $n$-dimensional Riemannian manifold. $M$ is said to be a manifold with a pole $x_0$, if $D(x_0) = M \setminus \{x_0\}$. We recall the radial vector field $\partial$ on $D(x_0)$ is the unit vector field such that for any $x \in D(x_0)$, $\partial(x)$ is the unit vector tangent to the unique geodesic $\gamma$ joining $x_0$ to $x$ and pointing away from $x_0$. A radial plane is a plane $\pi$ which contains $\partial(x)$ in the tangent space $T_xM$. By the radial curvature $K$ of a manifold, we mean the restriction of the sectional curvature function to all
the radial planes. We define $K(t)$ to be the radial curvature of $M$ at $x$ such that $r(x) = t$. We also study

**Definition 2.1.** The radial Ricci curvature of a manifold is the restriction of the Ricci curvature function to all the radial vector fields. Denoted $\text{Ric}^\text{rad}_M(r)$, the radial Ricci curvature of $M$ at $x$ such that $\text{dist}(x_0, x) = r$.

Let a tensor $g - dr \otimes dr = 0$ on the radial direction, and be just the metric tensor $g$ on the orthogonal complement $\partial^\perp$. At $x \in M$, the Hessian of $r$, denoted by $\text{Hess} r$ is a quadratic form on $T_x M$ given by $\text{Hess} r(v, w) = \langle \nabla_v dr, w \rangle$ for $v, w \in T_x M$. Here $\nabla_v$ is the covariant derivative in $v$ direction, $\nabla r$ is the gradient vector field of $r$, and hence is dual to the differential $dr$ of $r$. Thus, $\text{Hess} r(\nabla_r, \nabla_r) = 0$. The Laplacian of $r$, is defined to be $\Delta r = \text{trace}(\text{Hess} r)$. We say $\Delta r \leq f(r)$ holds weakly on $M$, if for every $0 \leq \phi(r) \in C^\infty_0(M)$, $\int_M \phi(r)\Delta r \, dv \leq \int_M \phi(r)f(r) \, dv$. We recall

**Lemma A.** [24, Lemma 9.1] (see [31, 11]) If $\Delta r \leq f(r)$ holds pointwise in $D(x_0)$, where $f \in C^0(0, \infty)$, then $\Delta r \leq f(r)$ holds weakly on $M$.

2.6. Under radial Ricci curvature assumptions.

**Theorem 2.5. (Laplacian Comparison Theorem)** Let functions $G_i : (0, t_i) \subset (0, \infty) \to \mathbb{R}$, $i = 1, 2$ satisfy (2.34) on $(0, t_1) \cap (0, t_2)$. Assume

$$(n - 1)G_i(r) \leq \text{Ric}^\text{rad}_M(r)$$

on $B_{t_i}(x_0) \subset D(x_0)$, and let $f_2 \in C([0, t_2]) \cap C^1(0, t_2)$ with $f_2 \in AC(0, t_2)$ be a positive solution of

$$
\begin{cases}
  f''_2 + G_2 f_2 \geq 0 \quad \text{a.e. in } (0, t_2) \\
  f_2(0) = 0, f_2(0) = n - 1.
\end{cases}
$$

Then $t_1 \leq t_2$ and

$$(2.36) \quad \Delta r \leq (n - 1)\frac{f'_i}{f_2}(r)$$

in $B_{t_i}(x_0)$. If in addition, (2.34) occurs in $D(x_0)$, then (2.27) holds pointwise on $D(x_0)$ and weakly on $M$.

**Proof.** For a bilinear form $A$, we write $|A|^2 = \text{trace}(AA^t)$. Recall

**Theorem F. (Weitzenböck formula)** For every function $f \in C^3(M)$,

$$(2.37) \quad \frac{1}{2} \Delta |\nabla f|^2 = |\text{Hess} f|^2 + \langle \nabla \Delta f, \nabla f \rangle + \text{Ric}(\nabla f, \nabla f)$$

Substituting the distance function $r(x)$ for $f(x)$ into Weitzenböck formula (2.37), we have inside the cut locus of $x_0 (|\nabla r| = 1)$, via Gauss lemma

$$0 = |\text{Hess} r|^2 + \frac{\partial}{\partial r} \Delta r + \text{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})$$

Since $\text{Hess}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 0$, $|\text{Hess} r|^2 \geq \frac{(\Delta r)^2}{n - 1}$, $\text{Ric}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = \text{Ric}^\text{rad}_M(r)$ and (2.34) occurs, it follows that

$$(2.38) \quad \begin{cases}
  \frac{\partial}{\partial r}(\Delta r) + \frac{(\Delta r)^2}{n - 1} + (n - 1)G_1(r) \leq 0 \quad \text{on } B_{t_i}(x_0) \\
  \Delta r = \frac{n - 1}{r} + O(1) \quad \text{as } t \to 0^+.
\end{cases}$$
Let \( g_1 = \Delta r, \kappa_1 = n - 1 = \kappa_2 \). Then \( 2.38 \) becomes \( 2.24 \) and \( 2.35 \) becomes \( 2.14 \). Via \( 2.24 \) and \( 2.33 \) with \( G \) satisfying \( 2.15 \) on \( (0, t_1) \cap (0, t_2) \) and \( \kappa_1 \) satisfying \( 2.16 \). Applying Theorem 2.3 we have \( 2.30 \) holds on \( (0, t_1) \), i.e., \( 2.36 \) holds in \( B_t(x_0) \). If in addition, \( 2.34 \) occurs in \( D(x_0) \), then \( 2.36 \) holds pointwise on \( D(x_0) \). By Lemma A or using Green’s Identity and a double limiting argument (see [24, Lemma 9.1], [31], [41]), \( 2.36 \) holds weakly on \( M \).

In applying Theorem 2.5, when a specific Ricci curvature lower bound assumption is given, we do not need to assume \( f_2 \) in \( 2.35 \) for comparison. Instead, we find \( f_2 \) in \( 2.44 \) and estimate \( f_2'f_2 \) in \( 2.49 \) and obtain:

**Theorem 2.6.** (1) If
\[
-(n-1)\frac{A(A-1)}{r^2} \leq \text{Ric}^M_{\text{rad}}(r) \quad \text{where} \quad A \geq 1
\]
on \( \bar{B}_t(x_0) \subset D(x_0) \), then
\[
\Delta r \leq (n-1)\frac{A}{r}
\]
on \( \bar{B}_t(x_0) \). If in addition, \( 2.39 \) occurs on \( D(x_0) \), then \( 2.40 \) holds pointwise on \( D(x_0) \) and weakly on \( M \).

(2) Equivalently, if
\[
-(n-1)\frac{A(A-1)}{(c+r)^2} \leq \text{Ric}^M_{\text{rad}}(r) \quad \text{where} \quad A \geq 1
\]
on \( \bar{B}_t(x_0) \subset D(x_0) \), where \( c \geq 0 \), then \( 2.40 \) holds on \( \bar{B}_t(x_0) \). If in addition \( 2.41 \) occurs on \( D(x_0) \), then \( 2.40 \) holds pointwise on \( D(x_0) \) and weakly on \( M \).

**Proof.** (2) \( \Rightarrow \) (1) Apparently (1) is a special case \( c = 0 \) in (2). (1) \( \Rightarrow \) (2) If \( 2.39 \) occurs, then \( 2.39 \) takes place, since \( \frac{A(A-1)}{(c+r)^2} \subset \frac{A(A-1)}{r^2} \). We then apply (1). Now we prove (2). First assume \( 2.41 \) occurs for \( c > 0 \). Choose \( \phi = (n-1)r^A \), where \( A \geq 1 \). Then \( \phi' = (n-1)Ar^{A-1} \). Hence, for \( r > 0 \)
\[
\phi' = \frac{A}{r} \quad \text{and} \quad \phi'' = (n-1)A(A-1)r^{A-2}.
\]
That is, for \( r > 0 \), \( \phi \) is a solution of the following differential equation:
\[
\phi'' + \frac{A(A-1)}{r^2} \phi = 0.
\]

Let \( f_2 \) be a positive solution of
\[
\begin{cases}
  f_2'' + G_2f_2 = 0 \quad \text{a.e. in} \quad (0, t_2) \\
  f_2(0) = 0, f_2'(0) = \kappa_2,
\end{cases}
\]
where \( G_2 = \frac{A(A-1)}{(c+r)^2} \), \( \kappa_2 = n - 1 \), and
\[
t_2 = \sup \{ r: f_2 > 0 \quad \text{on} \quad (0, r) \}.
\]
We note \( t_2 = \infty \). This can be seen by comparing the solution \( f_2 \) of (2.44) with the solution \( \tilde{f}_2(r) = (n-1)r \) of the following
\[
\begin{cases} 
\tilde{f}_2'' + 0 \cdot \tilde{f}_2 = 0, \\
\tilde{f}_2(0) = 0, \tilde{f}_2'(0) = n - 1, 
\end{cases}
\]
and applying a standard Sturm comparison theorem. Furthermore,
\[
(\phi' \tilde{f}_2 - \tilde{f}_2' \phi)(0) = 0. \tag{2.46}
\]
In view of (2.43), (2.44), (2.45), and (2.46), and \( f_2 \phi (\frac{A(A-1)}{r^2} - \frac{A(A-1)}{(c+r)^2}) \geq 0 \), for \( r \in (0, \infty) \)
\[
(\phi' f_2 - f_2' \phi)' = \phi'' f_2 - f_2'' \phi \geq 0 \tag{2.47}
\]
The monotonicity then implies that
\[
\phi' f_2 \geq f_2' \phi \tag{2.48}
\]
for \( r \in (0, \infty) \) which in turn via (2.42) yields
\[
\frac{f_2'}{f_2} \leq \frac{\phi'}{\phi} = \frac{A}{r} \tag{2.49}
\]
on \((0, \infty)\). Applying comparison Theorem 2.2 in which \( G_1(r) = \frac{A(A-1)}{(c+r)^2} \geq \frac{A(A-1)}{(c+r)^2} = G_2(c > 0) \), where \( G_1 \) is defined in \((0, t_1)\), \( G_2 \) is defined on \((0, \infty)\) and \( \frac{f_2'}{f_2} \) is estimated as in (2.49), we have shown (2.40) holds on \( \tilde{B}_{t_1}(x_0) \) under (2.41) for every \( c > 0 \). Now we prove (1) by passing \( c \to 0 \) in (2.41) in the following way: For every \( x \in \tilde{B}_{t_1}(x_0) \), there exist sequences \( \{x_i\} \) in a radial geodesic ray in \( \tilde{B}_{t_1}(x_0) \) and \( \{c_i > 0\} \) such that \( x_i \to x, c_i \to 0 \), and \( r(x_i) + c_i = r(x) \). It follows that for every \( c_i > 0 \),
\[
\text{Ric}^M_\text{rad}(r(x_i)) \geq -(n-1)\frac{A(A-1)}{(r(x_i) + c_i)^2} \Rightarrow \Delta r(x_i) \leq (n-1)\frac{A}{r(x)}(x_i),
\]
and hence, as \( i \to \infty \),
\[
\text{Ric}^M_\text{rad}(r(x)) \geq -(n-1)\frac{A(A-1)}{(r(x))} \Rightarrow \Delta r(x) \leq (n-1)\frac{A}{r(x)}. \tag{2.50}
\]
This proves that (2.40) holds on \( \tilde{B}_{t_1}(x_0) \) under (2.41) for \( c \geq 0 \), or under (2.39) on \( \tilde{B}_{t_1}(x_0) \). If in addition (2.39) or (2.41) occurs on \( D(x_0) \), then applying Lemma A (Lemma 9.1 in [24]), or a double limiting argument (see [41, Sect. 3]), we obtain the desired (2.40) pointwise on \( D(x_0) \), and weakly on \( M \). This proves (1) \( \Rightarrow \) (2), and (1).

**Theorem 2.7.** (1) If
\[
(n-1)\frac{B_1(1-B_1)}{r^2} \leq \text{Ric}^M_\text{rad}(r), \quad \text{where} \quad 0 \leq B_1 \leq 1
\]
on \( \tilde{B}_{t_1}(x_0) \subset D(x_0) \), then
\[
\Delta r \leq (n-1)\frac{1 + \sqrt{1 + 4B_1(1-B_1)}}{2r} \tag{2.52}
\]
Corollary 2.7. If the radial curvature $K$ satisfies

$$-\frac{A(A-1)}{r^2} \leq K(r) \quad (\text{or equivalently, } -\frac{A(A-1)}{(c+r)^2} \leq K(r), \ c \geq 0) \quad \text{where} \quad 1 \leq A$$

on $\bar{B}_t(x_0) \subset D(x_0)$, then (2.54) holds on $\bar{B}_t(x_0)$, and if in addition (2.57) occurs on $D(x_0)$, then (2.54) holds pointwise on $D(x_0)$ and weakly on $M$.

**Proof.** (2) $\Rightarrow$ (1) Obviously, (1) is a special case $c = 0$ in (2). To prove (1) $\Rightarrow$ (2), enough to show (2.53) holds under (2.54), for any $c > 0$. Choose $\phi_1 = r^\alpha$, where $\alpha = \frac{1+\sqrt{1+4B_1(1-B_1)}}{2}$. Then $\phi_1'' = \alpha(\alpha - 1)r^{\alpha - 2}$, i.e., for $r > 0$,

$$\frac{\phi_1''}{\phi_1} = \frac{\alpha}{r} = \frac{1+\sqrt{1+4B_1(1-B_1)}}{2r}$$

Let $f_2 > 0$ satisfy (2.41), where $G_2 = -\frac{B_1(1-B_1)}{(c+r)^2}$, $\kappa_2 = n - 1$ and $t_2$ is as in (2.45). Then $t_2 = \infty$, by applying the standard Sturm comparison theorem as in the proof of Theorem 2.6. Similarly, we have (2.46) and (2.48) hold for $r \in (0, \infty)$ in which $\phi_1 = \phi_1$, since $f_2\phi_1\left(\frac{B_1(1-B_1)}{r^2}\right) + G_2 \geq 0$, for $r \in (0, \infty)$. It follows from (2.52) that

$$\frac{f_2'}{f_2} \leq \frac{\phi_1'}{\phi_1} = \frac{1+\sqrt{1+4B_1(1-B_1)}}{2r}$$

on $(0, \infty)$. Applying comparison Theorem 2.6 in which $G_1(G_1 : (0,t_1) \rightarrow \mathbb{R}) = \frac{B_1(1-B_1)}{(c+r)^2} \geq -\frac{B_1(1-B_1)}{(c+r)^2} = G_2(G_2 : (0,t_2) \rightarrow \mathbb{R}), c > 0$ on $(0,t_1) \cap (0,t_2), \alpha = \frac{1+\sqrt{1+4B_1(1-B_1)}}{2r} \text{ and } f_2' \leq \frac{1+\sqrt{1+4B_1(1-B_1)}}{2r}$. We have shown (2.54) holds on $\bar{B}_t(x_0)$ under (2.53) for any $c > 0$. Proceeding the limiting argument as in the proof of Theorem 2.6 and (2.50), we prove similarly (1) $\Rightarrow$ (2), and (1).

**Corollary 2.6.** If the radial curvature $K$ satisfies

$$-\frac{A(A-1)}{r^2} \leq K(r) \quad (\text{or equivalently, } -\frac{A(A-1)}{(c+r)^2} \leq K(r), \ c \geq 0) \quad \text{where} \quad 1 \leq A$$

on $\bar{B}_t(x_0) \subset D(x_0)$, then (2.40) holds on $\bar{B}_t(x_0)$, and if in addition (2.57) occurs on $D(x_0)$, then (2.53) holds pointwise on $D(x_0)$ and weakly on $M$.

**Corollary 2.7.** If the radial curvature $K$ satisfies

$$\frac{B_1(1-B_1)}{r^2} \leq K(r) \quad (\text{or equivalently, } \frac{B_1(1-B_1)}{(c+r)^2} \leq K(r), \ c \geq 0) \quad \text{where} \quad 0 \leq B_1 \leq 1$$

on $\bar{B}_t(x_0) \subset D(x_0)$, then (2.52) holds on $\bar{B}_t(x_0)$, and if in addition (2.58) occurs on $D(x_0)$, then (2.52) holds pointwise on $D(x_0)$ and weakly on $M$. 


2.7. **Under the radial curvature assumptions.** The following Theorem strengthens main theorems in [24, Theorem 4.1], and [38, Theorem D]. We also give direct and simple proofs with applications from the viewpoint of dualities.

**Theorem 2.8.** *(Hessian and Laplacian Comparison Theorems)* Let

(2.59) \[ G_1(r) \leq K(r) \]

on \( B_{\tilde{t}_1}(x_0) \subset D(x_0) \) (resp.

(2.60) \[ K(r) \leq \tilde{G}_2(r) \]

on \( \tilde{B}_{\tilde{t}_2}(x_0) \subset D(x_0) \), and let \( f_2 \in C([0,t_2]) \cap C^1(0,t_2) \) with \( f'_2 \in AC(0,t_2) \) be a positive solution of (2.14), where \( G_i : (0,t_i) \to \mathbb{R} \) satisfy

(2.61)
\[
\begin{cases}
  f''_1 + \tilde{G}_1 f_1 \leq 0 \text{ on } (0,\tilde{t}_1) \\
  f_1(0) = 0, f'_1(0) = \kappa_1,
\end{cases}
\]

where \( \tilde{G}_i : (0,\tilde{t}_i) \to \mathbb{R} \) satisfy

(2.62) \[ \tilde{G}_2 \leq \tilde{G}_1 \]

on \( (0,\tilde{t}_1) \cap (0,\tilde{t}_2), \text{ where } 0 < \kappa_1 \leq 1 \).

Then \( t_1 \leq t_2 \).

(2.63) \[ \text{Hess } r \leq \frac{\kappa_2 f'_2}{f_2} (g - dr \otimes dr) \quad \text{and} \quad \Delta r \leq (n-1) \frac{\kappa_2 f'_2}{f_2} (r) \]

on \( B_{\tilde{t}_1}(x_0) \) (resp. \( \tilde{t}_1 \leq \tilde{t}_2 \)),

(2.64) \[ \frac{\kappa_1 f'_1}{f_1} (g - dr \otimes dr) \leq \text{Hess } r \quad \text{and} \quad (n-1) \frac{\kappa_1 f'_1}{f_1} (r) \leq \Delta r \]

on \( \tilde{B}_{\tilde{t}_2}(x_0) \) in the sense of quadratic forms. If in addition (2.59) occurs on \( D(x_0) \), then the second part of (2.63) holds pointwise on \( D(x_0) \) and weakly in \( M \).

**Proof.** Following the notation in [24]. Let \( \gamma \) be the unit speed geodesic curve joining \( x_0 = \gamma(0) \) to \( x = \gamma(t_0) \), and \( V \) be a parallel vector field along \( \gamma(t) \), for \( 0 \leq t \leq t_0 \).

A direct computation shows that

(2.65) \[ \frac{d}{dt} \langle \nabla_V \nabla r, V \rangle + \langle \nabla_V \nabla r, \nabla_V \nabla r \rangle = - \langle R(V, \nabla r) \nabla r, V \rangle \leq -G_1 \]

(see [24, p.174, (4.5)]). Define

\[ \lambda_{\text{max}}(x) = \max_{\{v \in T_x(M) \setminus \{0\}, v \bot \nabla r(x)\}} \frac{\text{Hess } r(v,v)}{(v,v)} \).

Select a unit vector \( v \) at \( x = \gamma(t_0) \) such that

\[ \langle \nabla_V \nabla r, v \rangle := \text{Hess } r(v,v) = \lambda_{\text{max}} \circ \gamma(t_0) \).

Then

\[ \langle \nabla_v \nabla r, \nabla_v \nabla r \rangle = \lambda_{\text{max}}^2 \circ \gamma(t_0) \).
Let $g_1 = \lambda_{\max} \circ \gamma$, and let the parallel vector field $V$ along $\gamma$ satisfying $V(t_0) = v$. Then the function $\operatorname{Hess} r(V, V) - g_1(t) \leq 0$, attains its maximum value 0 at $t = t_0$, and at this point
\[
\frac{d}{dt} \bigg|_{t=t_0} \operatorname{Hess} r(V, V) = \frac{d}{dt} \bigg|_{t=t_0} g_1(t).
\]
It follows from (2.65) that $g_1$ satisfies (2.27) (see [24, p.175]). By assumption, $f_2$ is a positive solution of (2.14), with $\kappa_2 \geq 1$, and (2.15) holds on $(0, t_1)$. Applying Theorem 2.3 in which $g_1 = \lambda_{\max} \circ \gamma$, we have
\[
\operatorname{Hess} r(w, w) \leq \operatorname{Hess} r(v, v) = g_1 \leq \frac{\kappa_2 f_2^2}{f_2}
\]
on $\tilde{B}_{t_1}(x_0)$ for any unit vector $w \perp \nabla r(x)$ at $x$. Taking the trace, we obtain the desired (2.66) on $\tilde{B}_{t_1}(x_0)$. If in addition (2.59) occurs on $D(x_0)$, then the second part of (2.63) holds weakly in $M$ by Lemma A.

Similarly, let $\tilde{V}$ be a parallel vector field along $\gamma(t)$, for $0 \leq t \leq t_0$. By the radial curvature assumption (2.66),
\[
\frac{d}{dt} (\nabla \tilde{V} \cdot \nabla r, \tilde{V}) + (\nabla \tilde{V} \cdot \nabla r, \nabla \tilde{V} \cdot \nabla r) = - (R(\tilde{V}, \nabla r) \nabla r, \tilde{V}) \geq - \tilde{G}_2.
\]
Define
\[
\lambda_{\min}(x) = \min_{\{x \in T_x(M) \setminus \{0\}, v \perp \nabla r(x)\}} \frac{\operatorname{Hess} r(v, v)}{(v, v)}.
\]
Select a unit vector $\tilde{v}$ at $x = \gamma(t_0)$ such that
\[
(\nabla \tilde{v} \cdot \nabla r, \tilde{v}) := \operatorname{Hess} r(\tilde{v}, \tilde{v}) = \lambda_{\min} \circ \gamma(t_0).
\]
Then
\[
(\nabla \tilde{v} \cdot \nabla r, \nabla \tilde{v} \cdot \nabla r) = \lambda_{\min}^2 \circ \gamma(t_0).
\]
Let $g_2 = \lambda_{\min} \circ \gamma$, and let the parallel vector field $\tilde{V}$ along $\gamma$ satisfying $\tilde{V}(t_0) = \tilde{v}$. Then the function $\operatorname{Hess} r(V, \tilde{V}) - g_2(t) \geq 0$, attains its minimum value 0 at $t = t_0$, and at this point
\[
\frac{d}{dt} \bigg|_{t=t_0} \operatorname{Hess} r(\tilde{V}, \tilde{V}) = \frac{d}{dt} \bigg|_{t=t_0} g_2(t).
\]
It follows from (2.66) that $g_2$ satisfies
\[
\left\{ \begin{array}{ll}
g_2' + g_2^2 + \tilde{G}_2 & \geq 0 \quad \text{a. e. in } (0, \tilde{t}_2) \\
g_2(t) = \frac{1}{t} + O(1) & \text{as } t \to 0^+,
\end{array} \right.
\]
We note $f_1$ is a positive solution of (2.61), where $0 < \kappa_1 \leq 1$. (see [24, p.176]) and (2.02) holds on $(0, \tilde{t}_1) \cap (0, \tilde{t}_2)$. Applying Theorem 2.4 in which $G_1 = \tilde{G}_1, t_1 = \tilde{t}_1, G_2 = \tilde{G}_2, t_2 = \tilde{t}_2$ and $\kappa_2 = 1$, we have
\[
\frac{\kappa_1 f_1'}{f_1} \leq g_2 = \operatorname{Hess} r(\tilde{v}, \tilde{v}) \leq \operatorname{Hess} r(w, w)
\]
on $\tilde{B}_{\tilde{t}_1}(x_0)$ for any unit vector $w \perp \nabla r(x)$ at $x$. Taking the trace, we obtain the desired (2.64) on $\tilde{B}_{\tilde{t}_1}(x_0)$.
\[\square\]
Remark 2.1. If $f_1 \in C([0,\tilde{t}_1]) \cap C^1(0,\tilde{t}_1)$ with $f_1' \in AC(0,\tilde{t}_1)$ is a positive solution of
\begin{align*}
\begin{cases}
  f_1'' + G_1f_1 = 0 & \text{on } (0, \tilde{t}_1) \\
  f_1(0) = 0, f_1'(0) = \kappa_1
\end{cases}
\end{align*}
then obviously $f_1$ is a positive solution of (2.61). Hence, we are ready for applying Theorem 2.8. The advantage of (2.68) over (2.61) is that a solution $f_1$ of (2.68) will make the subsequent (3.11), the monotonicity of $\phi f_1 - f_1' \phi$, and the estimate (3.41) work.

Corollary 2.8. (1) Let the radial curvature $K$ satisfy (2.59) (resp. (2.60)), and let $f_2 \in C([0,t_2]) \cap C^1(0,t_2)$ with $f_2' \in AC(0,t_2)$ (resp. $f_1 \in C([0,\tilde{t}_1]) \cap C^1(0,\tilde{t}_1)$ with $f_1' \in AC(0,\tilde{t}_1)$) be a positive solution of (2.23) (resp. (2.61), $\kappa_1 = 1$). Assume (2.15) occurs on $\langle 0,t_1 \rangle \cap (0,t_2)$ (resp. (2.62) occurs on $\langle 0,\tilde{t}_1 \rangle \cap (0,\tilde{t}_2)$). Then (2.63) holds on $\tilde{B}_{t_1}(x_0)$ for $\kappa_2 = 1$ (resp. (2.64) holds on $\tilde{B}_{\tilde{t}_1}(x_0)$ for $\kappa_1 = 1$). If in addition (2.59) occurs on $D(x_0)$, then the second part of (2.63)
(2) Theorem 2.8 is equivalent to (1).

2.8. Under the sectional curvature and Ricci curvature assumptions.

Corollary 2.9. Denote $\text{Ric}^M$ the Ricci curvature of $M$ and $\text{Sec}^M$ the sectional curvature of $M$. If we replace “$\text{Ric}_\text{rad}^M$” in Theorems 2.5, 2.6 and 2.7 by “$\text{Ric}^M$”, the results remain to be true. Likewise, if we replace “$K$ or $K(r)$” in Theorems 2.8, 3.1, 3.2, 3.3, 3.4, 3.5, and Corollaries 2.6 - 2.8, 3.1 - 3.5 by “$\text{Sec}^M$”, the results remain to be true.

Proof. This follows at once from Definition 2.1, the definition of the radial curvature, and the above Theorems and Corollaries stated in Corollary 2.9. □

3. CURVATURE IN COMPARISON THEOREMS

Hessian and Laplacian Comparison Theorem 2.8 has many geometric applications under curvature assumptions. Let $A, A_1, B, B_1$ be constants and $K(r)$ be the radial curvature of $M$.

Theorem 3.1. (1) If
\begin{align*}
(3.1) \quad - \frac{A(A - 1)}{r^2} \leq K(r) \quad \text{where } A \geq 1
\end{align*}
on $\tilde{B}_{t_1}(x_0) \subset D(x_0)$, then
\begin{align*}
(3.2) \quad \text{Hess} \, r \leq \frac{A}{r} \left( g - dr \otimes dr \right) \quad \text{in the sense of quadratic forms}
\end{align*}
and
\begin{align*}
(3.3) \quad \Delta r \leq (n - 1) \frac{A}{r}
\end{align*}
hold on \( \bar{B}_1(x_0) \). If in addition (3.2) occurs on \( D(x_0) \), then (3.2) and (3.3) hold pointwise on \( D(x_0) \), and (3.3) holds weakly on \( M \).

(2) Equivalently, if

\[
- \frac{A(A - 1)}{(c + r)^2} \leq K(r), \quad A \geq 1
\]

on \( \bar{B}_1(x_0) \subset D(x_0) \), where \( c \geq 0 \), then (3.2) and (3.3) hold on \( \bar{B}_1(x_0) \). If in addition (3.4) occurs on \( D(x_0) \), then (3.2) holds pointwise on \( D(x_0) \), and weakly on \( M \).

Proof of Theorem 3.2

(2) \( \Rightarrow \) (1) (1) is the special case \( c = 0 \) in (2). (1) \( \Rightarrow \) (2) If (3.4) occurs, then (3.1) takes place, since \( - \frac{A(A - 1)}{(c + r)^2}, \infty \subset \left[ - \frac{A(A - 1)}{r}, \infty \right) \). Then apply (1). Now we prove (2). First assume (3.4) occurs for \( c > 0 \). Proceed as in the proof of Theorem 2.6 choose \( \phi = r^A \), where \( A \geq 1 \). Then \( \phi' = A r^{A - 1} \). Hence, for \( r > 0 \), we have (2.43) and (2.44). Let \( f_2 \) be a positive solution of (2.44), where \( G_2 = - \frac{A(A - 1)}{(c + r)^2}, \kappa_2 = 1 \) and \( t_2 \) be as in (2.45). We note \( t_2 = \infty \), by a standard Sturm comparison theorem. Furthermore, (2.49) \( (\phi' f_2 - f_2') (0) = 0 \) holds. In view of (2.43), (2.44) and (2.45), we have (2.47) \( (\phi' f_2 - f_2')' = f_2 \phi (\frac{A(A - 1)}{(c + r)^2} - \frac{A(A - 1)}{(c + r)^2}) \geq 0 \), for \( r \in (0, \infty) \). The monotonicity then implies that \( \phi' f_2 \geq f_2' \phi \) for \( r \in (0, \infty) \) which in turn via (2.42) yields

\[
\frac{f_2'}{f_2} = \phi' = \frac{A}{r}
\]

on \( (0, \infty) \). Applying comparison Theorem 2.8 in which \( G_1 = - \frac{A(A - 1)}{(c + r)^2} \geq - \frac{A(A - 1)}{(c + r)^2} = G_2 \) \( (c > 0 \), \( \kappa_2 = 1 \), \( f_2' \leq \frac{A}{r} \), and taking the trace, we have shown on \( \bar{B}_1(x_0) \), (2.63) holds, i.e., (3.2) and (3.3) hold under (3.4) for every \( c > 0 \). Proceeding the limiting argument as in (2.50) and applying Lemma A, we prove similarly (1) \( \Rightarrow \) (2), and (1).

Theorem 3.2. (1) If

\[
K(r) \leq - \frac{A_1(A_1 - 1)}{r^2} \quad \text{on } M \setminus \{x_0\} \quad \text{where} \quad A_1 \geq 1,
\]

then Hess \( r \) and \( \Delta r \) satisfy

\[
\frac{A_1}{r} \left( g - dr \otimes dr \right) \leq \text{Hess } r \quad \text{and} \quad (n - 1) \frac{A_1}{r} \leq \Delta r
\]

on \( M \setminus \{x_0\} \), respectively.

(2) Equivalently, if

\[
K(r) \leq - \frac{A_1(A_1 - 1)}{(c + r)^2}, \quad A_1 \geq 1
\]

on \( M \setminus \{x_0\} \) where \( c \geq 0 \), then (3.3) holds.

Proof of Theorem 3.2

(2) \( \Rightarrow \) (1) (1) is the special case \( c = 0 \) in (2). (1) \( \Rightarrow \) (2) Enough to show (2) holds for the case \( c > 0 \) in (3.7). Choose \( \phi = (c + r)^A_1 \), where \( A_1 \geq 1 \). Then \( \phi' = A_1 (c + r)^{A_1 - 1} \). Hence, for \( r > 0 \)

\[
\frac{\phi'}{\phi} = \frac{A_1}{c + r}
\]
and \( \phi'' = A_1 (A_1 - 1)(c + r)^{A_1 - 2} \). That is, for \( r > 0 \), \( \phi \) is a solution of the equation:

\[
(3.9) \quad \phi'' + \frac{-A_1(A_1 - 1)}{(c + r)^2} \phi = 0.
\]

Let \( f_1 \) be a positive solution of

\[
(2.68) \quad \begin{cases}
  f'' + \tilde{G_1} f_1 = 0 & \text{on } (0, \tilde{t}_1) \\
  f_1(0) = 0, f'_1(0) = \kappa_1
\end{cases}
\]

where \( \tilde{G_1} = \frac{-A_1(A_1 - 1)}{(c + r)^2} \), \( \kappa_1 = 1 \) and

\[
(3.10) \quad \tilde{t}_1 = \sup \{ r : f_1 > 0 \text{ on } (0, r) \}
\]

We note \( \tilde{t}_1 = \infty \), by the same standard Sturm comparison theorem. In view of (3.9), (2.68) and (3.10), for \( r \in (0, \infty) \)

\[
(3.11) \quad (\phi' f_1 - f'_1 \phi)' = f_1 \phi \left( \frac{A_1(A_1 - 1)}{(c + r)^2} - \frac{A_1(A_1 - 1)}{(c + r)^2} \right)
\]

\[
= 0.
\]

Since \( (\phi' f_1 - f'_1 \phi)(0) = -c^{A_1} \leq 0 \), the monotonicity then implies that \( \phi' f_1 \leq f'_1 \phi \) on \( (0, \infty) \) which in turn via (3.8) yields

\[
(3.12) \quad \frac{A_1}{c + r} = \frac{\phi'}{\phi} \leq \frac{f'_1}{f_1}
\]

on \( (0, \infty) \). Applying Remark 2.1 and Theorem 2.6 in which \( \tilde{G_1}(r) = \frac{-A_1(A_1 - 1)}{(c + r)^2} = \tilde{G_2}(r)(c > 0), \kappa_1 = 1, \frac{f'_1}{f_1} \geq \frac{A_1}{c + r} \) and \( \tilde{t}_1 = \infty \) we have shown via (2.64) that

\[
(3.6) \quad \frac{A_1}{c + r} \left( g - \text{dr} \otimes \text{dr} \right) \leq \text{Hess } r \quad \text{and} \quad (n - 1) \frac{A_1}{c + r} \leq \Delta r
\]

on \( M \setminus \{ x_0 \} \), under (3.7) for every \( c > 0 \). Now proceed analogously to the proof of Theorem 2.6: For every \( x \in M \setminus \{ x_0 \} \), there exist sequences \( \{ x_i \} \) in a radial geodesic ray in \( M \setminus \{ x_0 \} \) and \( \{ c_i > 0 \} \) such that \( x_i \to x, c_i \to 0 \), and \( r(x_i) + c_i = r(x) \). It follows that for every \( c_i > 0 \),

\[
(3.13) \quad K(r)(x_i) \leq -\frac{A_1(A_1 - 1)}{(c_i + r(x_i))^2} \quad \Rightarrow \quad \Delta r(x_i) \geq (n - 1) \frac{A_1}{c_i + r(x_i)},
\]

and hence, as \( i \to \infty \),

\[
K(r)(x) \leq -\frac{A_1(A_1 - 1)}{(r(x))^2} \quad \Rightarrow \quad \Delta r(x) \geq (n - 1) \frac{A_1}{r(x)},
\]

and similarly, \( \text{Hess } r(x) \geq \frac{A_1}{r(x)} \left( g - \text{dr} \otimes \text{dr} \right)(x) \) for every \( x \in M \setminus \{ x_0 \} \). This proves (1) \( \Rightarrow \) (2) and (1).

\[\Box\]

**Corollary 3.1.** (1) If the radial curvature \( K \) satisfies

\[
(3.14) \quad -\frac{A(A - 1)}{r^2} \leq K(r) \leq \frac{-A_1(A_1 - 1)}{r^2} \quad \text{where} \quad A \geq A_1 \geq 1
\]
on $M \backslash \{x_0\}$, then
\begin{equation}
\frac{A_1}{r} \left( g - dr \otimes dr \right) \leq \text{Hess } r \leq \frac{A}{r} \left( g - dr \otimes dr \right) \text{ in the sense of quadratic forms,}
\end{equation}
\begin{equation}
(n - 1) \frac{A_1}{r} \leq \Delta r \leq (n - 1) \frac{A}{r} \quad \text{pointwise on } M \backslash \{x_0\} \quad \text{and}
\end{equation}
\begin{equation}
\Delta r \leq (n - 1) \frac{A}{r} \quad \text{weakly on } M.
\end{equation}

(2) Equivalently, if $K$ satisfies
\begin{equation}
- \frac{A(A - 1)}{(c + r)^2} \leq K(r) \leq - \frac{A_1(A_1 - 1)}{(c + r)^2}, \quad A \geq A_1 \geq 1
\end{equation}
on $M \backslash \{x_0\}$, where $c \geq 0$, then (3.15) holds.

Proof. This follows at once from Theorems 3.1 and 3.2. \qed

Theorem A. (1) Let the radial curvature $K$ satisfy
\begin{equation}
- \frac{A}{r^2} \leq K(r) \leq - \frac{A_1}{r^2} \quad \text{where } 0 \leq A_1 \leq A
\end{equation}
on $M \backslash \{x_0\}$. Then
\begin{equation}
\frac{1 + \sqrt{1 + 4A_1}}{2r} \left( g - dr \otimes dr \right) \leq \text{Hess}(r) \leq \frac{1 + \sqrt{1 + 4A}}{2r} \left( g - dr \otimes dr \right) \quad \text{on } M \backslash \{x_0\},
\end{equation}
\begin{equation}
(n - 1) \frac{1 + \sqrt{1 + 4A_1}}{2r} \leq \Delta r \leq (n - 1) \frac{1 + \sqrt{1 + 4A}}{2r} \quad \text{pointwise on } M \backslash \{x_0\}, \text{ and}
\end{equation}
\begin{equation}
\Delta r \leq (n - 1) \frac{1 + \sqrt{1 + 4A}}{2r} \quad \text{weakly on } M.
\end{equation}

(2) Equivalently, if
\begin{equation}
- \frac{A}{(c + r)^2} \leq K(r) \leq - \frac{A_1}{(c + r)^2} \quad \text{where } 0 \leq A_1 \leq A
\end{equation}
on $M \backslash \{x_0\}$, then (3.15) - (3.20) hold.

Proof of Theorem A. Let $A(A - 1) = a^2$ in (3.1) (resp. $A_1(A_1 - 1) = a_1^2$ in (3.20)). Then
\begin{equation}
A = \frac{1 + \sqrt{1 + 4a^2}}{2r} \quad (\text{resp. } A_1 = \frac{1 + \sqrt{1 + 4a_1^2}}{2r}) \geq 1. \quad \text{Hence, } \frac{A}{r} = \frac{1 + \sqrt{1 + 4a^2}}{2r} \quad (\text{resp. } A_1 = \frac{1 + \sqrt{1 + 4a_1^2}}{2r}).
\end{equation}
Substitute these into Theorem 3.1 (resp. Theorem 3.2) so that this Theorem is rephrased in terms of $a^2$ (resp. $a_1^2$). Replacing $a^2$ (resp. $a_1^2$) in this rephrase by $A$ (resp. $A_1$), we transform Corollary 3.1 into Theorem A. \qed

Corollary 3.2. (1) If the radial curvature $K$ satisfies
\begin{equation}
K(r) = - \frac{A(A - 1)}{r^2}, \quad \text{where } 1 \leq A \quad (\text{resp. } K(r) = - \frac{A}{r^2}, \quad \text{where } 0 \leq A)
\end{equation}
on $M\setminus\{x_0\}$, then

$$\text{Hess } r = \frac{A}{r} \quad \text{(resp. Hess } r = \frac{1 + \sqrt{1 + 4A}}{2r}(g - dr \otimes dr)) \quad \text{and}$$

$$\Delta r = (n - 1)\frac{A}{r} \quad \text{(resp. } \Delta r = (n - 1)\frac{1 + \sqrt{1 + 4A}}{2r})$$

on $M\setminus\{x_0\}$.

(2) Equivalently, if $K$ satisfies

$$K(r) = -\frac{A(A - 1)}{(c + r)^2}, \quad \text{where } 1 \leq A \quad \text{\text{(resp. } K(r) = -\frac{A}{(c + r)^2}, \quad \text{where } 0 \leq A)$$

on $M\setminus\{x_0\}$, and $c \geq 0$, then (3.23) holds on $M\setminus\{x_0\}$.

Proof. This follows at once from combining Theorems 3.1 and 3.2 in which $A_1 = A$.

Theorem 3.3. (1) If

$$\frac{B_1(1 - B_1)}{r^2} \leq K(r) \quad \text{where } 0 \leq B_1 \leq 1$$

on $\tilde{B}_1(x_0) \subset D(x_0)$, then Hess $r$ and $\Delta r$ satisfy

$$\text{Hess } r \leq \frac{1 + \sqrt{1 + 4B_1(1 - B_1)}}{2r}(g - dr \otimes dr) \quad \text{and}$$

$$\Delta r \leq (n - 1)\frac{1 + \sqrt{1 + 4B_1(1 - B_1)}}{2r}$$

on $\tilde{B}_1(x_0) \subset D(x_0)$ respectively. If in addition (3.25) occurs on $D(x_0)$, then (3.26) holds pointwise on $D(x_0)$ and (3.27) holds weakly on $M$.

(2) Equivalently, if

$$\frac{B_1(1 - B_1)}{(c + r)^2} \leq K(r), \quad 0 \leq B_1 \leq 1$$

on $\tilde{B}_1(x_0) \subset D(x_0)$, where $c \geq 0$, then (3.26) and (3.27) hold on $\tilde{B}_1(x_0)$. If in addition (3.28) occurs on $D(x_0)$, then (3.26) holds pointwise on $D(x_0)$ and (3.27) holds weakly on $M$.

Proof. (2) ⇒ (1) (1) is the special case $c = 0$ in (2). To prove (1) ⇒ (2), enough to show that on $\tilde{B}_1(x_0)$, (3.26) and (3.27) hold under (3.28), on $\tilde{B}_1(x_0)$ for any $c > 0$, since we can combine this with (1). Proceed as in the proof of Theorem 2.4 choose $\phi_1 = r^\alpha$, where $\alpha = \frac{1 + \sqrt{1 + 4B_1(1 - B_1)}}{2}, \quad 0 \leq B_1 \leq 1$. Then $\alpha \geq 1$, $\phi'_1 = \alpha \phi_1^{\alpha - 1}$ and $(2\alpha - 1)^2 = 1 + 4B_1(1 - B_1)$, i.e. $\alpha(\alpha - 1) = B_1(1 - B_1)$. Hence, for $r > 0$, we have $\phi''_1 = \phi_1^{\alpha - 2}$, and $\phi''_2 = \alpha(\alpha - 1)^{\alpha - 2}$, i.e. (2.54) and (2.55) hold. Let $f_2$ be a positive solution of (2.44), in which $G_2 = -\frac{B_1(1 - B_1)}{(c + r)^2}$, $\kappa_2 = 1$ and $t_2$ be as in (2.45). Then by the standard comparison theorem as before, $t_2 = \infty$.

Furthermore, we have (2.40), (2.44) and (2.48) hold on $\tilde{B}_1(x_0)$ in which $\phi = \phi_1$, since $f_2\phi_1 (\frac{B_1(1 - B_1)}{r^2} + G_2) \geq 0$, for $r \in (0, \infty)$. It follows from (2.54) that (2.56), i.e., $\frac{f_2'}{f_2} \leq \frac{\phi'}{\phi_1} = \frac{1 + \sqrt{1 + 4B_1(1 - B_1)}}{2r}$ holds on $(0, \infty)$. 
Applying comparison Theorem 2.8, in which \( G_1 = \frac{B_1(1-B_1)}{(c+r)^2} \geq \frac{B_1(1-B_1)}{(c+r)^2} = G_2 \) \((c > 0), \kappa_2 = 1, \beta_2 \leq \frac{1+2B_1(1-B_1)}{2r},\) we have shown that on \( \hat{D}_{B_1}(x_0),\) \((2.63)\) holds, i.e., \((3.26)\) and \((3.27)\) hold under \((3.28)\) for any \(c > 0.\) This proves \((1) \Rightarrow (2).\) This also proves \((1),\) since \((3.25)\) occurs on \( \hat{D}_{B_1}(x_0)\) implies that \((3.28)\) happens on \( \hat{D}_{B_1}(x_0)\) for any \(c > 0.\) If in addition \((3.25)\) occurs on \( D(x_0),\) then \((3.26)\) holds pointwise on \( D(x_0)\) and \((3.27)\) holds weakly on \( M\) by Lemma A \((24, \text{Lemma 9.1}),\) or a double limiting argument (see [41, Sect. 3]).

**Remark 3.1.** In proving Theorem 3.3, we choose auxiliary function \( \phi_1 = r^\alpha,\) where \( \alpha = \frac{1+\sqrt{1+4B_1(1-B_1)}}{2}.\) We may choose, for example a different auxiliary function \( \phi_1 = r^\alpha_1+1,\) where \( \alpha_1 = \frac{1+\sqrt{1+4B_1(1-B_1)}}{2},\) and obtain estimates

\[
\text{Hess } r \leq \frac{|B_1 - \frac{1}{2}| + \frac{3}{2}}{r} \left( g - dr \otimes dr \right) \quad \text{and}
\]

\[
\Delta r \leq (n-1) \frac{|B_1 - \frac{1}{2}| + \frac{3}{2}}{r} \quad \text{pointwise on } \hat{D}_{B_1}(x_0) \subset D(x_0)
\]

If in addition \((3.29)\) occurs on \( D(x_0),\) then

\[
\Delta r \leq (n-1) \frac{|B_1 - \frac{1}{2}| + \frac{3}{2}}{r} \quad \text{weakly on } M.
\]

However, the upper bound estimate \(\frac{1+\sqrt{1+4B_1(1-B_1)}}{2r}\) obtained in \((3.26)\) and \((3.27)\) by selecting \( \phi_1 = r^\alpha\) is better than the upper bound estimate \(\frac{|B_1 - \frac{1}{2}| + \frac{3}{2}}{r}\) obtained in \((3.29)\) and \((3.30)\) by selecting \( \phi_1 = r^\alpha_1+1.\) Indeed, \(\frac{1+\sqrt{1+4B_1(1-B_1)}}{2r} \leq \frac{B_1+1}{r} = \frac{B_1 - \frac{1}{2} + \frac{3}{2}}{r} \leq \frac{|B_1 - \frac{1}{2} + \frac{3}{2}|}{r}.\) For the completeness or comparison, we include the following derivation for the estimates \((3.29)\) and \((3.30).\)

Note \((2\alpha_1 - 1) = 2|B_1 - 1|.\) Analogously, \( \phi_1 = r^\alpha_1+1\) implies \( \phi_1' = (\alpha_1 + 1)r^{\alpha_1}.\) For \( r > 0,\)

\[
\left(\phi_1'\right)' = \frac{\alpha_1 + 1}{r} \frac{|B_1 - \frac{1}{2}| + \frac{3}{2}}{r}
\]

and \( \phi_1'' = (\alpha_1 + 1)\alpha_1 r^{\alpha_1-1},\) i.e.

\[
\phi_1'' = \frac{-(\alpha_1 + 1)\alpha_1}{r^2} \phi_1 = 0.
\]

Let \( f_2 \) be a positive solution of \((2.44),\) where \( G_2 = \frac{-(\alpha_1 + 1)\alpha_1}{(c+r)^2}, \kappa_2 = 1\) and let \( t_2 \) be as in \((2.45).\) Then \( t_2 = \infty,\) \((\phi_1' f_2 - f_2^2 \phi_1)(0) = 0,\) and for \( r \in (0, \infty)\)

\[
(\phi_1' f_2 - f_2^2 \phi_1)' = f_2 \phi_1 \left( \frac{\alpha_1(\alpha_1 + 1)}{r^2} + G_2 \right) \geq 0.
\]

Monotonicity implies

\[
\frac{f_2'}{f_2} \leq \frac{\phi_1'}{\phi_1} = \frac{|B_1 - \frac{1}{2}| + \frac{3}{2}}{r} \quad \text{on } (0, \infty)
\]

Applying Theorem 2.8, in which \( G_1 = \frac{B_1(1-B_1)}{(c+r)^2} \geq 0 \geq \frac{-(\alpha_1 + 1)\alpha_1}{(c+r)^2} = G_2, \kappa_2 = 1, \frac{f_2'}{f_2} = \frac{|B_1 - \frac{1}{2} + \frac{3}{2}|}{r},\) and taking the trace, we obtain the estimates \((3.29)\). Applying Lemma A, if in addition, \((3.25)\) occurs on \( D(x_0),\) \((3.30)\) follows.
Corollary 3.3. If the radial curvature $K$ satisfies
\begin{equation}
\frac{B_1}{r^2} \leq K(r) \quad \text{(resp. } \frac{B_1}{(c + r)^2} \leq K(r), c \geq 0), \text{ where } 0 \leq B_1 \leq \frac{1}{4},
\end{equation}
on $\tilde{B}_1(x_0) \subset D(x_0)$, then
\begin{equation}
\text{Hess } r \leq \frac{1 + \sqrt{1 + 4B_1}}{2r} \left( g - dr \otimes dr \right) \quad \text{and } \Delta r \leq (n - 1) \frac{1 + \sqrt{1 + 4B_1}}{2r}
\end{equation}
holds pointwise on $\tilde{B}_1(x_0)$. If in addition, (3.33) holds in $D(x_0)$, then the second part of (3.34) holds weakly on $M$.

Proof. This follows immediately from substituting $B_1(1 - B_1)$ in Theorem 3.3 for $B_1$, and $B_1(1 - B_1) = -(B_1 - \frac{1}{2})^2 + \frac{1}{4} \leq \frac{1}{4}$, $B_1(1 - B_1) \geq 0$ if $0 \leq B_1 \leq 1$. \hfill \Box

Theorem 3.4. (1) If
\begin{equation}
K(r) \leq \frac{B(1 - B)}{r^2}, \quad 0 \leq B \leq 1
\end{equation}
on $\tilde{B}_1(x_0) \subset D(x_0)$, then Hess $r$ and $\Delta r$ satisfy
\begin{equation}
\frac{|B - \frac{1}{2}| + \frac{1}{2}}{r} \left( g - dr \otimes dr \right) \leq \text{Hess } r \quad \text{and}
(n - 1) \frac{|B - \frac{1}{2}| + \frac{1}{2}}{r} \leq \Delta r
\end{equation}
on $\tilde{B}_1(x_0)$ respectively.
(2) Equivalently, if
\begin{equation}
K(r) \leq \frac{B(1 - B)}{(c + r)^2}, \quad 0 \leq B \leq 1
\end{equation}
on $\tilde{B}_1(x_0) \subset D(x_0)$, where $c \geq 0$, then (3.36) holds on $\tilde{B}_1(x_0)$.

Proof. (2) $\Rightarrow$ (1) (1) is the special case $c = 0$ in (2). (1) $\Rightarrow$ (2) If (3.37) occurs, then (3.36) occurs, since $(-\infty, \frac{B(1 - B)}{(c + r)^2}] \subset (-\infty, \frac{B(1 - B)}{r^2}]$. We then apply (1). To prove (2), we first assume (3.37) occurs for $c > 0$. Choose $\beta = \frac{1 + \sqrt{1 - 4B(1 - B)}}{2}$, with $0 \leq B \leq 1$, and $\phi_2 = r^\beta$. Then $2\beta - 1 = \sqrt{(2B - 1)^2}$, i.e.
\begin{equation}
\beta = |B - \frac{1}{2}| + \frac{1}{2} = \max\{B, 1 - B\} = \begin{cases} B & \text{if } \frac{1}{2} \leq B \\ 1 - B & \text{if } 0 \leq B < \frac{1}{2}, \end{cases}
\end{equation}
$\beta(\beta - 1) = B(B - 1)$ and $\phi_2' = \beta r^{\beta - 1}$ for $r > 0$. Hence,
\begin{equation}
\frac{\phi_2'}{\phi_2} = \frac{\beta}{r}
\end{equation}
fors $r > 0$, and for $0 \leq B \leq 1$, $\phi_2'' = -B(1 - B)r^{\beta - 2}$, i.e.
\begin{equation}
\phi''_2 + \frac{B(1 - B)}{r^2} \phi_2 = 0.
\end{equation}
Let \( f_1 \) be a positive solution of (2.68), where \( \widetilde{G}_1 = \frac{B(1-B)}{(c+r)^2} \), \( \kappa_1 = 1 \) and let \( \tilde{t}_1 \) be as in (3.10). Then

\[
(\phi'_2 f_1 - f'_1 \phi_2)(0) = 0 \quad \text{if} \quad \beta = 1
\]

\[
\lim_{r \to 0^+} (\phi'_2 f_1 - f'_1 \phi_2)(r) = \lim_{r \to 0^+} \frac{\beta}{1-\beta} f'_1(r) r^\beta = 0 \quad \text{if} \quad \beta \neq 1.
\]

by l’Hospital’s Rule. Moreover, in view of (3.40), (2.68) and (3.10), for \( r \in (0, \tilde{t}_1) \)

\[
(\phi'_2 f_1 - f'_1 \phi_2)' = \phi_2 f_1 \left( \frac{-B(1-B)}{r^2} + \frac{B(1-B)}{(c+r)^2} \right)
\]

\[
\leq 0
\]

Monotonicity implies that

\[
(3.41) \quad \frac{\beta}{r} = \frac{\phi'_2}{\phi_2} \leq \frac{f'_1}{f_1} \quad \text{on} \quad (0, \tilde{t}_1).
\]

Integrating (3.41) over \([\epsilon_1, \tilde{t}_1 - \epsilon_2] \subset (0, \tilde{t}_1)\), and passing \( \epsilon_2 \to 0 \), we have

\[
0 < C(\epsilon_1) r^\beta \leq f_1 \quad \text{on} \quad [\epsilon_1, \tilde{t}_1],
\]

where \( C(\epsilon_1) > 0 \) is a constant depending on \( \epsilon_1 \). Thus \( f_1 > 0 \) on \((0, \tilde{t}_1)\). We claim \( \tilde{t}_1 = \infty \). Otherwise there would exist a \( \delta > 0 \) such that \( \tilde{t}_1 + \delta < \infty \) at which \( f_1 > 0 \) by the continuity, and would lead to, via (3.10) a contradiction

\[
\tilde{t}_1 < \tilde{t}_1 + \delta \leq \tilde{t}_1.
\]

Applying Theorem 2.8 in which \( \widetilde{G}_1 = \frac{B(1-B)}{(c+r)^2} \geq \frac{B(1-B)}{(c+r)^2} = \widetilde{G}_2 \), \( c > 0 \), \( \kappa_1 = 1 \), \( \kappa_2 = \frac{\beta}{r} \), \( \tilde{t}_1 = t_1, \tilde{t}_2 = t_2 \), we have shown on \( \tilde{B}_{t_2}(x_0) \), (3.36) holds under (3.37) for every \( c > 0 \). To prove (1), we proceed analogously to the proof of Theorem 3.2: For every \( x \in \tilde{B}_{t_2}(x_0) \), there exist sequences \( \{x_i\} \) in a radial geodesic ray in \( \tilde{B}_{t_2}(x_0) \) and \( \{c_i > 0\} \) such that \( x_i \to x, c_i \to 0 \), and \( r(x_i) + c_i = r(x) \). It follows that for every \( c_i > 0 \),

\[
K(r)(x_i) \leq \frac{B(1-B)}{(c_i + r(x_i))^2} \quad \Rightarrow \quad \Delta r(x_i) \geq (n-1) \left| \frac{B - \frac{1}{2}}{c_i + r(x_i)} + \frac{1}{2} \right|,
\]

(3.42)

and consequently, as \( i \to \infty \),

\[
K(r)(x) \leq \frac{B(1-B)}{r^2}(x) \quad \Rightarrow \quad \Delta r(x) \geq (n-1) \left| \frac{B - \frac{1}{2}}{r(x)} + \frac{1}{2} \right|,
\]

and (3.36) holds on \( \tilde{B}_{t_2}(x_0) \).

\[ \square \]

**Corollary 3.4.** If the radial curvature \( K \) satisfies

\[
K(r) \leq \frac{B}{r^2} \quad \text{(resp.} \ K(r) \leq \frac{B}{(c+r)^2} ; \ c \geq 0 \text{)}, \quad \text{where} \ 0 \leq B \leq \frac{1}{4}
\]

\[
(3.43)
\]
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on $\tilde{B}_{12}(x_0) \subset D(x_0)$, then

$$\frac{1 + \sqrt{1 - 4B}}{2r} \leq \text{Hess} r \left( g - dr \otimes dr \right)$$

and

$$\frac{(n-1)(1 + \sqrt{1 - 4B})}{2r} \leq \Delta r$$

(3.44)

hold pointwise on $\tilde{B}_{12}(x_0)$.

Proof. Let $B(1 - B) = b^2$ in (3.35). Then $B = \frac{1 + \sqrt{1 - 4b^2}}{2}$, and by completing the square, we have

$$b^2 = B(1 - B) = -(B - \frac{1}{2})^2 + \frac{1}{4} \leq \frac{1}{4}.$$

Hence, $|B - \frac{1}{2}| + \frac{1}{2} = \frac{1 + \sqrt{1 - 4b^2}}{2}$. Substitute these into Theorem 3.4, so that this Theorem is rephrased in terms of $b^2$. We then replace $b^2$ in this rephrased Theorem by $B$ and obtain the desired. \qed

Theorem 3.5. If the radial curvature $K$ satisfies

$$\frac{B_1}{r^2} \leq K(r) \leq \frac{B}{r^2} \quad \text{where} \quad 0 \leq B_1 \leq B \leq \frac{1}{4}$$

on $\tilde{B}_{r}(x_0) \subset D(x_0)$, then

$$\frac{1 + \sqrt{1 - 4B}}{2r} \leq \text{Hess} r \left( g - dr \otimes dr \right) \leq \frac{1 + \sqrt{1 + 4B_1}}{2r} \left( g - dr \otimes dr \right)$$

and

$$(n-1)\frac{1 + \sqrt{1 - 4B}}{2r} \leq \Delta r \leq (n-1)\frac{1 + \sqrt{1 + 4B_1}}{2r}$$

hold pointwise on $\tilde{B}_{r}(x_0)$. If in addition (3.33) occurs on $D(x_0)$, then

$$\Delta r \leq (n-1)\frac{1 + \sqrt{1 + 4B_1}}{2r}$$

(3.47)

holds weakly on $M$.

(2) Equivalently, if $K$ satisfies

$$\frac{B_1}{(c + r)^2} \leq K(r) \leq \frac{B}{(c + r)^2}, \quad 0 \leq B_1 \leq B \leq \frac{1}{4}$$

on $\tilde{B}_{r}(x_0) \subset D(x_0)$, where $c \geq 0$, then (3.46) holds on $\tilde{B}_{r}(x_0)$. If in addition, (3.33) occurs on $D(x_0)$, then (3.47) holds weakly on $M$.

Proof. This follows at once from Corollaries 3.1 and 3.4. \qed

Corollary 3.5. (1) Let the radial curvature $K$ satisfy

$$\frac{B_1(1 - B_1)}{r^2} \leq K(r) \leq \frac{B(1 - B)}{r^2} \quad , \quad 0 \leq B, B_1 \leq 1$$

(3.49)
on $B_{r}(x_{0}) \subset D(x_{0})$. Then

\begin{equation}
\frac{|B - \frac{1}{2}| + \frac{1}{2}}{r} \left( g - dr \otimes dr \right) \leq \text{Hess} r \leq \frac{1 + \sqrt{1 + 4B_{1}(1 - B_{1})}}{2r} \left( g - dr \otimes dr \right)
\end{equation}

and

\begin{equation}
(n - 1) \frac{|B - \frac{1}{2}| + \frac{1}{2}}{r} \leq \Delta r \leq (n - 1) \frac{1 + \sqrt{1 + 4B_{1}(1 - B_{1})}}{2r}
\end{equation}

hold pointwise on $B_{r}(x_{0})$. If in addition, \(3.25\) occurs on $D(x_{0})$, then

\begin{equation}
\Delta r \leq (n - 1) \frac{1 + \sqrt{1 + 4B_{1}(1 - B_{1})}}{2r}
\end{equation}

holds weakly on $M$.

(2) Equivalently, if $K$ satisfies

\begin{equation}
\frac{B_{1}(1 - B)}{(c + r)^{2}} \leq K(r) \leq \frac{B(1 - B)}{(c + r)^{2}}, \quad 0 \leq B, B_{1} \leq 1
\end{equation}

on $B_{r}(x_{0})$, where $c \geq 0$, then \(3.30\) holds on $B_{r}(x_{0})$. If in addition, \(3.25\) occurs on $D(x_{0})$, then \(3.27\) holds weakly on $M$.

**Corollary 3.6.** If the radial curvature $K$ satisfies

\begin{equation}
K(r) = 0 \text{ (resp. } \leq 0, \geq 0)\text{ on } M \{x_{0}\},
\end{equation}

then

\begin{equation}
\text{Hess} r = (\text{resp. } \text{Hess} r \geq, \text{Hess} r \leq) \frac{1}{r} \left( g - dr \otimes dr \right)
\end{equation}

and

\begin{equation}
\Delta r = (\text{resp. } \Delta r \geq, \Delta r \leq) (n - 1) \frac{1}{r}
\end{equation}

on $M \{x_{0}\}$.

**Proof.** This follows at once from Corollary 3.5 in which $B_{1} = B = 1$, or Theorem 3.5 in which $B_{1} = B = 0$, or Corollary 3.2 in which $A = 1$ (resp. $A = 0$).

**Corollary 3.7.** If the radial curvature $K$ satisfies

\begin{equation}
-\frac{A}{r^{2}} \leq K(r) \leq \frac{B}{r^{2}}\text{ (resp. } -\frac{A}{(c + r)^{2}} \leq K(r) \leq \frac{B}{(c + r)^{2}}, \quad 0 \leq A, 0 \leq B \leq \frac{1}{4})\text{ on } M \{x_{0}\},
\end{equation}

where $c > 0$, then

\begin{equation}
\frac{1 + \sqrt{1 - 4B}}{2r} \left( g - dr \otimes dr \right) \leq \text{Hess}(r) \leq \frac{1 + \sqrt{1 + 4A}}{2r} \left( g - dr \otimes dr \right)
\end{equation}

and

\begin{equation}
(n - 1) \frac{1 + \sqrt{1 - 4B}}{2r} \leq \Delta r \leq (n - 1) \frac{1 + \sqrt{1 + 4A}}{2r}
\end{equation}

hold on $M \{x_{0}\}$.

**Proof.** This follows at once from Theorems A and Corollary 3.4.
4. Geometric Applications in Mean Curvature

The following is an immediate geometric application:

**Theorem 4.1.** (Mean Curvature Comparison Theorems) Let $H(r)$ be the mean curvature of the geodesic sphere $\partial B_r(x_0)$ of radius $r$ centered at $x_0$ in $M$ with respect to the unit outward normal. Then

\begin{align*}
(4.1) \quad H(r) &\leq \begin{cases} 
A \frac{1}{r}, & \text{if } \text{Ric}^M_{\text{rad}}(r) \text{ satisfies } (2.39) \text{ or } (2.41), \\
1 + \sqrt{1 + 4B_1(1 - B_1)} \frac{1}{2r}, & \text{if } \text{Ric}^M_{\text{rad}}(r) \text{ satisfies } (2.51) \text{ or } (2.53);
\end{cases} \\
(4.2) \quad H(r) &\leq \begin{cases} 
A \frac{1}{r}, & \text{if } K(r) \text{ satisfies } (3.5) \text{ or } (3.7), \\
1 + \sqrt{1 + 4B_1(1 - B_1)} \frac{1}{2r}, & \text{if } K(r) \text{ satisfies } (3.25) \text{ or } (3.28), \\
1 + \sqrt{1 + 4B_1} \frac{1}{2r}, & \text{if } K(r) \text{ satisfies } (3.43).
\end{cases}
\end{align*}

\begin{align*}
(4.3) \quad H(r) &\geq \begin{cases} 
A \frac{1}{r}, & \text{if } K(r) \text{ satisfies } (3.5) \text{ or } (3.7), \\
1 + \sqrt{1 - 4B} \frac{1}{2r}, & \text{if } K(r) \text{ satisfies } (3.43); \\
\frac{|B - 1| + \frac{1}{2}}{r}, & \text{if } K(r) \text{ satisfies } (3.22) \text{ or } (3.24),
\end{cases}
\end{align*}

\begin{align*}
(4.4) \quad &\begin{cases} 
\frac{A_1}{r} \leq H(r) \leq A \frac{1}{r}, & \text{if } K(r) \text{satisfies } (3.14) \text{ or } (3.16), \\
1 + \sqrt{1 + 4A_1} \frac{1}{2r}, \leq H(r) \leq (n - 1) + \sqrt{1 + 4A} \frac{1}{2r}, & \text{if } K(r) \text{satisfies } (3.17) \text{ or } (3.21), \\
1 + \sqrt{1 - 4B} \frac{1}{2r}, \leq H(r) \leq 1 + \sqrt{1 + 4B_1} \frac{1}{2r}, & \text{if } K(r) \text{satisfies } (3.45) \text{ or } (3.48), \\
\frac{|B - 1| + \frac{1}{2}}{r} \leq H(r) \leq 1 + \sqrt{1 + 4B_1(1 - B_1)} \frac{1}{2r}, & \text{if } K(r) \text{satisfies } (3.39) \text{ or } (3.51), \\
1 + \sqrt{1 - 4B} \frac{1}{2r} \leq H(r) \leq \frac{1}{r} + \sqrt{1 + 4A} \frac{1}{2r}, & \text{if } K(r) \text{satisfies } (3.54);
\end{cases}
\end{align*}

\begin{align*}
(4.5) \quad H(r) &\begin{cases} 
= A \frac{1}{r}, & \text{resp. } 1 + \sqrt{1 + 4A} \frac{1}{2r}, & \text{if } K(r) \text{satisfies } (3.22) \text{ or } (3.24), \\
= (\text{resp. } \geq, \leq) \frac{1}{r}, & \text{if } K(r) \text{satisfies } (3.52);
\end{cases}
\end{align*}

where the corresponding geodesic spheres $\partial B_r(x_0)$ are as in Theorems 3.1-3.5, Theorem A, and Corollaries 3.1-3.7.
Proof. By Gauss lemma

\begin{align*}
\frac{1}{n-1} \Delta r := \frac{1}{n-1} \text{trace}(\text{Hess } r) \\
= \frac{1}{n-1} \left( \text{the trace of the second fundamental form of } \partial B_r(x_0) \text{ in } M \right) \\
:= H(r)
\end{align*}

(see [35 (3.28)]). The results follow from Theorems 2.6-2.7, 3.1-3.5, Theorem A, and Corollaries [31, 37]. □

5. The Growth of Bundle-Valued Differential Forms and Their Interrelationship

Let \((M, g)\) be a smooth Riemannian manifold. Let \(\xi : E \to M\) be a smooth Riemannian vector bundle over \((M, g)\), i.e., a vector bundle such that at each fiber is equipped with a positive inner product \(\langle \cdot, \cdot \rangle_E\). Set \(A^k(\xi) = \Gamma(\Lambda^k T^* M \otimes E)\) the space of smooth \(k\)-forms on \(M\) with values in the vector bundle \(\xi: E \to M\). For two forms \(\Omega, \Omega' \in A^k(\xi)\), the induced inner product \(\langle \Omega, \Omega' \rangle\) is defined as in (6.4). For \(\Omega \in A^k(\xi)\), set \(|\Omega|^2 = \langle \Omega, \Omega \rangle\). Then \(|\Omega|^q = \langle \Omega, \Omega \rangle^{\frac{q}{2}}\) and we are ready to make the following

Definition 5.1. For a given \(q \in \mathbb{R}\), a function or a differential form or a bundle-valued differential form \(f\) has \(p\)-finite growth (or, simply, is \(p\)-finite) if there exists \(x_0 \in M\) such that

\begin{align*}
\liminf_{r \to \infty} \frac{1}{r^p} \int_{B(x_0; r)} |f|^q \, dv < \infty, \tag{5.1}
\end{align*}

and has \(p\)-infinite growth (or, simply, is \(p\)-infinite) otherwise.

For a given \(q \in \mathbb{R}\), a function or a differential form or a bundle-valued differential form \(f\) has \(p\)-mild growth (or, simply, is \(p\)-mild) if there exist \(x_0 \in M\), and a strictly increasing sequence of \(\{r_j\}_{j=0}^\infty\) going to infinity, such that for every \(l_0 > 0\), we have

\begin{align*}
\sum_{j=l_0}^\infty \left( \frac{(r_{j+1}-r_j)^p}{\int_{B(x_0; r_{j+1}) \setminus B(x_0; r_j)} |f|^q \, dv} \right)^{\frac{1}{p-1}} = \infty, \tag{5.2}
\end{align*}

and has \(p\)-severe growth (or, simply, is \(p\)-severe) otherwise.

For a given \(q \in \mathbb{R}\), a function or a differential form or a bundle-valued differential form \(f\) has \(p\)-obtuse growth (or, simply, is \(p\)-obtuse) if there exists \(x_0 \in M\) such that for every \(a > 0\), we have

\begin{align*}
\int_a^\infty \left( \frac{1}{\int_{B(x_0; r)} |f|^q \, ds} \right)^{\frac{1}{p-1}} \, dr = \infty, \tag{5.3}
\end{align*}

and has \(p\)-acute growth (or, simply, is \(p\)-acute) otherwise.

For a given \(q \in \mathbb{R}\), a function or a differential form or a bundle-valued differential form \(f\) has \(p\)-moderate growth (or, simply, is \(p\)-moderate) if there exist \(x_0 \in M\), and \(\psi(r) \in F\), such that

\begin{align*}
\limsup_{r \to \infty} \frac{1}{r^p \psi^{p-1}(r)} \int_{B(x_0; r)} |f|^q \, dv < \infty, \tag{5.4}
\end{align*}
and has $p$-immoderate growth (or, simply, is $p$-immoderate) otherwise, where

\[(5.5) \quad \mathcal{F} = \{ \psi : [a, \infty) \to (0, \infty) \mid \int_a^\infty \frac{dr}{r \psi(r)} = \infty \text{ for some } a \geq 0 \} . \]

(Notice that the functions in $\mathcal{F}$ are not necessarily monotone.)

For a given $q \in \mathbb{R}$, a function or a differential form or a bundle-valued differential form $f$ has $p$-small growth (or, simply, is $p$-small) if there exists $x_0 \in M$, such that for every $a > 0$, we have

\[(5.6) \quad \int_a^\infty \left( \frac{r}{\int_{B(x_0;r)} |f|^q dv} \right)^{-\frac{1}{p'}} \, dr = \infty , \]

and has $p$-large growth (or, simply, is $p$-large) otherwise.

**Definition 5.2.** For a given $q \in \mathbb{R}$, a function or a differential form or a bundle-valued differential form $f$ has $p$-balanced growth (or, simply, is $p$-balanced) if $f$ has one of the following: $p$-finite, $p$-mild, $p$-obtuse, $p$-moderate, or $p$-small growth, and has $p$-imbalanced growth (or, simply, is $p$-imbalanced) otherwise.

The above definitions of “$p$-balanced, $p$-finite, $p$-mild, $p$-obtuse, $p$-moderate, or $p$-small” and their counter-parts “$p$-imbalanced, $p$-infinite, $p$-severe, $p$-acute, $p$-immoderate, $p$-large” growth depend on $q$, and $q$ will be specified in the context in which the definition is used.

**Theorem 5.1.** For a given $q \in \mathbb{R}$, $f$ is $p$-moderate (resp. $p$-immoderate) if and only if $f$ is $p$-small (resp. $p$-large).

**Proof.** $(\Rightarrow)$ Since $f$ has $p$-moderate growth, we may assume, by the definition of the limit superior of functions, there exists a $\psi \in \mathcal{F}$ (as in $(5.5)$), and a constant $K' > 0$ such that $\frac{1}{r \psi(r)} \int_{B(x_0;r)} |f|^q \, dv \leq K'$ for $r > \ell_0$. This implies that

\[(5.7) \quad \frac{r}{\int_{B(x_0;r)} |f|^q \, dv} \geq \frac{1}{K' r^{p-1} \psi^{-1}(r)} \quad \text{for } r > \ell_0 . \]

Taking both sides to the power $\frac{1}{p'}$ and integrating, we have

\[(5.8) \quad \int_a^\infty \left( \frac{r}{\int_{B(x_0;r)} |f|^q \, dv} \right)^{-\frac{1}{p'}} \, dr \geq \frac{1}{(K')^{-\frac{1}{p}}} \int_a^\infty \frac{1}{r \psi(r)} \, dr \quad \text{for } r > \ell_0 . \]

If $f$ were $p$-large, $(5.8)$ would lead to $\int_a^\infty \frac{1}{r \psi(r)} \, dr < \infty$, contradicting $(5.5)$.

$(\Leftarrow)$ If $\int_{B(x_0;r)} |f|^q \, dv = 0$ for $r > a > 0$, then we are done. Or, we let $\psi(r) = (\frac{1}{r^p} \int_{B(x_0;r)} |f|^q \, dv)^{-\frac{1}{p'}}$, $r > a$. Then

\[(5.9) \quad \frac{1}{r \psi(r)} = \left( \frac{r}{\int_{B(x_0;r)} |f|^q \, dv} \right)^{-\frac{1}{p'}} . \]

Integrating $(5.9)$ from $a$ to $\infty$, we have

\[\int_a^\infty \frac{1}{r \psi(r)} \, dr = \int_a^\infty \left( \frac{r}{\int_{B(x_0;r)} |f|^q \, dv} \right)^{-\frac{1}{p'}} \, dr = \infty , \]

by the assumption of $f$ being $p$-small. Hence $\psi(r) \in \mathcal{F}$. Suppose the contrary, $f$ were $p$-immoderate, i.e. $(5.4)$ were not true. Then we would have, via $(5.9)$
Proof. For a strictly increasing sequence \( \{r_j\} \) with \( r_{j+1} = 2r_j \), we obtain

(5.10)

\[
\sum_{j=0}^{\ell} \left( \frac{(r_{j+1} - r_j)^p}{\int_{B(x_0, r_{j+1}) \setminus B(x_0, r_j)} |f|^q \, dv} \right)^{\frac{1}{p-1}} \geq \sum_{j=0}^{\ell} \left( \frac{r_{j+1} - r_j}{\int_{B(x_0, r_{j+1})} |f|^q \, dv} \right)^{\frac{1}{p-1}} \cdot (r_{j+1} - r_j)^{p-1} = \sum_{j=0}^{\ell} \left( \frac{r_{j+1} - r_j}{\int_{B(x_0, r_{j+1})} |f|^q \, dv} \right)^{\frac{1}{p-1}} \cdot \frac{1}{2} (r_{j+2} - r_{j+1}) \\
\geq \sum_{j=0}^{\ell} \frac{\frac{1}{2} r_{j+2}}{2^{p-1}} \int_{r_{j+1}}^{r_{j+2}} \left( \frac{r_{j+1} - r_j}{\int_{B(x_0, r_{j+1})} |f|^q \, dv} \right)^{\frac{1}{p-1}} \, dr
\]

where we have applied the Mean Value Theorem for integrals to the forth step.

Now suppose contrary, \( f \) were \( p \)-severe. Letting \( \ell \to \infty \) in (5.10) would imply

\[
\infty \geq \sum_{j=0}^{\ell} \left( \frac{(r_{j+1} - r_j)^p}{\int_{B(x_0, r_{j+1}) \setminus B(x_0, r_j)} |f|^q \, dv} \right)^{\frac{1}{p-1}} \geq \frac{1}{2^{p-1}} \int_{r_0}^{\infty} \left( \frac{r}{\int_{B(x_0, r)} |f|^q \, dv} \right)^{\frac{1}{p-1}} \, dr,
\]

contradicting the hypothesis that \( f \) has a \( p \)-small growth. \( \square \)

Theorem 5.3. For a given \( q \in \mathbb{R} \), if \( f \) has \( p \)-small growth then \( f \) has \( p \)-mild growth.

Proof. (i) \( \forall 0 < s < t \), by the Hölder inequality, one gets:

\[
t - s = \int_s^t \left( \frac{d}{dr} \int_{B(x_0, r)} (f^2 + \epsilon)^{\frac{q}{2}} \, dv \right)^{\frac{1}{q}} \left( \frac{1}{\int_{B(x_0, r)} (f^2 + \epsilon)^{\frac{q}{2}} \, dv} \right)^{\frac{1}{q}} \, dr
\]

\[
\leq \left( \int_s^t \frac{d}{dr} \int_{B(x_0, r)} (f^2 + \epsilon)^{\frac{q}{2}} \, dv \, dr \right)^{\frac{1}{q}} \left( \int_s^t \left( \frac{1}{\int_{B(x_0, r)} (f^2 + \epsilon)^{\frac{q}{2}} \, dv} \right)^{\frac{1}{p-1}} \, dr \right)^{\frac{1}{p}}.
\]

Taking this to the power of \( \frac{p}{p-1} \) on both sides, we have

(5.11)

\[
\left( \frac{(t-s)^p}{\int_{B(x_0, t) \setminus B(x_0, s)} (f^2 + \epsilon)^{\frac{q}{2}} \, dv} \right)^{\frac{1}{p-1}} \leq \int_s^t \left( \frac{1}{\int_{B(x_0, r)} (f^2 + \epsilon)^{\frac{q}{2}} \, dv} \right)^{\frac{1}{p-1}} \, dr.
\]
Hence, for every strictly increasing sequence of \( \{r_j\} \) and every \( r_{i_0} > a \), setting \( s = r_i, t = r_{i+1} \) and summing over \( i \) from \( \ell_0 \) to \( \ell \) and letting \( \ell \to \infty \), one gets:

\[
\sum_{i=\ell_0}^{\infty} \left( \frac{(r_{i+1} - r_i)^p}{\int_{B(x_0, r_i) \setminus B(x_0, s)} |f|^{\frac{q}{2}} \, dv} \right)^\frac{\alpha}{\beta} \leq \int_a^\infty \left( \frac{1}{\int_{B(x_0, r) \setminus B(x_0, a)} |f|^{\frac{q}{2}} \, dv} \right)^\frac{1}{\alpha} \, dr.
\]

Let \( \epsilon \to 0 \), we get the desired first assertion by the coarea formula and the Dominated Convergence Theorem. The proof of the second assertion follows from Theorem 5.1 and Theorem 5.2.

(ii) By the convexity assumption, comparing the slope of the tangent and the slope of the chord at two distinct points (analogous to [22]), we have

\[
\frac{d}{dr} \int_{B(x_0, r)} |f|^q \, dv \geq \frac{\int_{B(x_0, r)} |f|^q \, dv - \int_{B(x_0, a)} |f|^q \, dv}{r - a} \quad \text{a.e.}.
\]

Hence, \( \int_a^\infty \left( \frac{1}{\int_{B(x_0, r) \setminus B(x_0, a)} |f|^{\frac{q}{2}} \, dv} \right)^\frac{1}{\alpha} \, dr \leq \int_a^\infty \left( \frac{r}{\int_{B(x_0, r) \setminus B(x_0, a)} |f|^{\frac{q}{2}} \, dv} \right)^\frac{1}{\alpha} \, dr < \infty \). The assertion follows from the coarea formula. \( \square \)

**Proof of Theorem 5.4.** This follows at once from Theorems 5.1, 5.2 and 5.3. \( \square \)

**Definition 5.3.** A function or differential form or bundle-valued differential form \( f \) on \( M \) is said to belong to \( L^p(M) \) (or, simply, is \( L^p \)), denoted by \( f \in L^p(M) \) if \( \int_M |f|^p \, dv < \infty \).

**Corollary 5.1.** Every \( L^q \) function or differential form or bundle-valued differential form \( f \) on \( M \) has \( p \)-balanced growth, \( p \geq 0 \), and in fact, has \( p \)-finite, \( p \)-mild, \( p \)-obtuse, \( p \)-obtuse, \( p \)-balanced growth, \( p \)-finite, \( p \)-small growth, \( p \geq 0 \), for the same value of \( q \)

**Proof.** Since \( f \) is \( L^q \) on \( M \), by Definition 5.3, we may assume \( \int_M |f|^q \, dv = K \) for some constant \( K < \infty \). It follows from Definition 5.1 that every \( L^q \) function or differential form or bundle-valued differential form \( f \) on \( M \) is both \( p \)-finite and \( p \)-small, \( p \geq 0 \). Indeed,

\[
\liminf_{r \to \infty} \frac{1}{r^p} \int_{B(x_0, r)} |f|^q \, dv \leq \liminf_{r \to \infty} \frac{1}{r^p} K = 0 < \infty
\]

and

\[
\int_a^\infty \left( \frac{r}{\int_{B(x_0, r) \setminus B(x_0, a)} |f|^{\frac{q}{2}} \, dv} \right)^\frac{1}{\alpha} \, dr \geq \int_a^\infty \left( \frac{r}{K} \right)^\frac{1}{\alpha} \, dr = \infty.
\]

Now the assertion follows from Definition 5.2 and Theorem 5.4. \( \square \)

6. **Generalized harmonic forms with values in vector bundle, decomposition, integrability and growth**

By Stokes’ theorem, a smooth differential form \( \omega \) on a compact Riemannian manifold is harmonic if and only if it is closed and co-closed. That is,

\[
\Delta \omega = 0 \quad \text{if and only if} \quad d \omega = 0 \quad \text{and} \quad d^* \omega = 0.
\]
The exterior differential operator $d^\nabla : A^k(\xi) \to A^{k+1}(\xi)$ relative to the connection $\nabla^E$ is given by
\[
d^\nabla \sigma (X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \nabla^E_{X_i} (\sigma(X_1, \ldots, X_{k+1}))
\]
(6.2)
\[+ \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}),
\]
where the symbols covered by $\widehat{}$ are omitted. Since the Levi-Civita connection on $TM$ is torsion-free, we also have
\[
(d^\nabla \sigma)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1}(\nabla X_i, \sigma)(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1}).
\]
(6.3)

For two forms $\Omega, \Omega' \in A^k(\xi)$, the induced inner product is defined as follows:
\[
\langle \Omega, \Omega' \rangle = \sum_{i_1 < \cdots < i_k} \langle \Omega(e_{i_1}, \ldots, e_{i_k}), \Omega'(e_{i_1}, \ldots, e_{i_k}) \rangle_E
\]
(6.4)
\[= \frac{1}{k!} \sum_{i_1, \ldots, i_k} \langle \Omega(e_{i_1}, \ldots, e_{i_k}), \Omega'(e_{i_1}, \ldots, e_{i_k}) \rangle_E,
\]
where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame field on $(M, g)$. For $\Omega \in A^k(\xi)$, set $|\Omega|^2 = \langle \Omega, \Omega \rangle$ defined as in (6.4).

**Definition 6.1.** A bundle-valued differential form $\Omega \in A^k(\xi)$ on $M$ is said to satisfy Condition $W$ if
\[
|\langle d(|\Omega|^2) \wedge \Omega, d^\nabla \Omega \rangle| \leq 2|\Omega|^2|d^\nabla \Omega|^2
\]
\[|\langle d(| \ast \Omega|^2) \wedge \ast \Omega, d^\nabla \ast \Omega \rangle| \leq 2| \ast \Omega|^2|d^\nabla \ast \Omega|^2.
\]
(6.5)

Relative to the Riemannian structures of $E$ and $TM$, the codifferential operator $\delta^\nabla : A^k(\xi) \to A^{k-1}(\xi)$ is characterized as the adjoint of $d$ via the formula
\[
\int_M \langle d^\nabla \sigma, \rho \rangle dv_g = \int_M \langle \sigma, \delta^\nabla \rho \rangle dv_g,
\]
where $\sigma \in A^{k-1}(\xi), \rho \in A^k(\xi)$, one of which has compact support, and $dv_g$ is the volume element associated with the metric $g$ on $TM$. Then
\[
(\delta^\nabla \rho)(X_1, \ldots, X_{k-1}) = -\sum_i (\nabla_{e_i} \rho)(e_i, X_1, \ldots, X_{k-1}).
\]
(6.6)

Let $\Delta^\nabla : A^k(\xi) \to A^k(\xi)$ be given by
\[
\Delta^\nabla = -(d^\nabla \delta^\nabla + \delta^\nabla d^\nabla).
\]
(6.7)

**Definition 6.2.** $\Omega \in A^k(\xi)$ is said to be a harmonic form with values in the vector bundle $\xi : E \to M$, if $\Delta^\nabla \Omega = 0$, and a generalized harmonic form with values in the vector bundle $\xi : E \to M$, if $\langle \Omega, \Delta^\nabla \Omega \rangle \geq 0$.

**Theorem 6.1.** (Duality Theorem) (i) $\Omega \in A^{k}(\xi)$ is harmonic if and only if $\ast \Omega \in A^{n-k}(\xi)$ is harmonic. (ii) $\Omega \in A^k(\xi)$ has 2-balanced growth on $M$, for $q = 2$, or for $1 < q(\neq 2) < 3$ with $\Omega$ satisfying Condition $W$ (see (6.5)), if and only if $\ast \Omega \in A^{n-k}(\xi)$ has 2-balanced growth on $M$, for $q = 2$, or for $1 < q(\neq 2) < 3$ with
\(* \Omega \) satisfying Condition W (see (6.3)). (iii) \(* \Omega \) is a solution of \( \langle * \Omega, \Delta * \Omega \rangle \geq 0 \) on \( M \) if and only if \(* \Omega \) is closed and co-closed.

**Theorem 7.1.** (Unity Theorem) If a bundle-valued differential k-form \( \Omega \in A^k(\xi) \) has 2-balanced growth, for \( q = 2 \), or for \( 1 < q(\neq 2) < 3 \) with \( \Omega \) satisfying Condition W, then the following six statements (i) \( \langle \Omega, \Delta \Omega \rangle \geq 0 \), (ii) \( \Delta \Omega = 0 \), (iii) \( d^\nabla \Omega = \delta^\nabla \Omega = 0 \), (iv) \( \langle * \Omega, \Delta \nabla * \Omega \rangle \geq 0 \), (v) \( \Delta \nabla * \Omega = 0 \), (vi) \( d^\nabla * \Omega = \delta^\nabla * \Omega = 0 \) are equivalent.

**Proof.** (i) \( \iff (ii) \iff (iii) \) It is obvious. (i) \( \implies (iii) \) Choose a smooth cut-off function \( \psi(x) \) as in [35, (3.1)], i.e. for any \( x_0 \in M \) and any pair of positive numbers \( s, t \) with \( s < t \), a rotationally symmetric Lipschitz continuous nonnegative function \( \psi(x) = \psi(x; s, t) \) satisfies \( \psi \equiv 1 \) on \( B(s) \), \( \psi \equiv 0 \) off \( B(t) \), and \( |\nabla \psi| \leq C_1 \), \( a.e. \) on \( M \), where \( C_1 > 0 \) is a constant (independent of \( x_0, s, t \)). Let \( 1 < q < \infty \), to be determined later. For any constant \( \epsilon > 0 \), we compute

\[
0 \leq \int_{B(t)} \langle \psi^2(|\Omega|^2 + \epsilon)^{\frac{q-2}{2}} \Omega, \Delta \nabla \Omega \rangle \ dv
\]

and obtain

\[
\left( \int_{B(t)} \psi^2(|\Omega|^2 + \epsilon)^{\frac{q-2}{2}} \right)^2 \left( \int_{B(t) \setminus B(s)} (|\Omega|^2 + \epsilon)^{\frac{q}{4}} \ dv \right)^2 \\
\leq \left( \frac{2\sqrt{C_1}}{1 - q^{-2}} \right)^2 \left( \frac{1}{t-s} \right)^2 \left( \int_{B(t) \setminus B(s)} (|\Omega|^2 + \epsilon)^{\frac{q}{4}} \ dv \right)^2 \\
\cdot \left( \int_{B(t) \setminus B(s)} \psi^2(|\Omega|^2 + \epsilon)^{\frac{q-2}{2}} \left( |\nabla \Omega|^2 + |\delta \nabla \Omega|^2 \right) \ dv \right).
\]

Proceeding as in [39], we obtain the desired (iii). Hence, (i) \( \iff (ii) \iff (iii) \). On the other hand, applying Duality Theorem 6.2. we prove (ii) \( \iff (v) \), and (iv) \( \iff (v) \iff (vi) \). Consequently, (i) through (vi) are all equivalent. \( \square \)

**Corollary 6.1.** Every \( L^2 \) bundle-valued harmonic k-form \( \Omega \in A^k(\xi) \) is closed and co-closed and every \( L^q, 1 < q(\neq 2) < 3 \) bundle-valued harmonic k-form \( \Omega \in A^k(\xi) \) satisfying Condition W is closed and co-closed.

7. **Applications in Geometric Differential-Integral Inequalities on Manifolds**

The derived Laplacian comparison Theorems 2.6 - 2.7 and Hessian comparison Theorems 5.1 - 5.4 have many applications. In particular, they shed light on geometric differential-integral inequalities on Riemannian manifolds. Using an analog of Bochner’s Method or “\( B^2 - 4AC \leq 0 \)” Method, Wei and Li [41] proved the following theorem.

**Theorem 7.1.** ([41]) For every \( u \in W^{1,2}_0(M \setminus \{x_0\}) \), and every \( a, b \in \mathbb{R} \), the following inequality holds on a manifold \( M \) with a pole \( x_0 \):

\[
\frac{1}{2} \left\{ \int_M \frac{|u|^2}{r^2} \right\} \left( \int_M \frac{r^2}{|u|^2} \right)^{\frac{1}{2}} \leq \left( \int_M \frac{|u|^2}{r^2} \right)^{\frac{1}{2}} \left( \int_M \frac{|\nabla u|^2}{r^2} \right)^{\frac{1}{2}},
\]

where \( dv \) is the volume element of \( M \).
7.1. Generalized sharp Caffarelli-Kohn-Nirenberg type inequalities on Riemannian manifolds.

**Definition 7.1.** \( u : M \to \mathbb{R} \) is said to belong to \( W^{1,p}_0(M) \) (or, simply, is \( W^{1,p} \)), denoted by \( u \in W^{1,p}_0(M) \) if there exists a sequence \( \{u_i\} \) in \( C_0^\infty(M) \) such that

\[
(\int_M |u - u_i|^p + |\nabla(u - u_i)|^p \, dv)^{\frac{1}{p}} \to 0, \quad \text{as } i \to \infty.
\]

Applying Corollary 3.6 to Theorem 7.1, Wei and Li [41] proved the following theorem.

**Theorem 7.2.** (3.6) Let \( M \) be an \( n \)-dimensional complete Riemannian manifold with a pole such that the radial curvature \( K(r) \) of \( M \) satisfies one of the following three conditions:

(1) \( K(r) \geq 0 \) and \( n \leq a + b + 1 \),

(2) \( K(r) \leq 0 \) and \( a + b + 1 \leq n \),

(3) \( K(r) = 0 \) and \( a, b \in \mathbb{R} \) are any constants.

Then for every \( u \in W^{1,2}_0(M \setminus \{x_0\}) \), and \( a, b \in \mathbb{R} \)

\[
C \int_M \frac{|u|^2}{r^{a+b+1}} \, dv \leq \left( \int_M \frac{|u|^2}{r^{2a}} \, dv \right)^{\frac{1}{2}} \left( \int_M \frac{|\nabla u|^2}{r^{2b}} \, dv \right)^{\frac{1}{2}},
\]

where the constant \( C \) is given by

\[
C = C(a, b) = \begin{cases} 
\frac{-n-(a+b+1)}{n-(a+b+1)} , & \text{if } K(r) \text{ satisfies (1)}, \\
\frac{n-(a+b+1)}{2} , & \text{if } K(r) \text{ satisfies (2)}, \\
\frac{-n-(a+b+1)}{2} , & \text{if } K(r) \text{ satisfies (3)}.
\end{cases}
\]

The case \( M = \mathbb{R}^n \) is due to Caffarelli et al. [7] and Costa [14].

**Theorem 7.3.** (Generalized sharp Caffarelli-Kohn-Nirenberg type inequalities under radial Ricci curvature assumptions) Let \( M \) be an \( n \)-dimensional complete Riemannian manifold with a pole such that the radial Ricci curvature \( \text{Ric}^M_{\text{rad}} \) of \( M \) satisfies one of the following five conditions:

(1) \( \text{Ric}^M_{\text{rad}}(r) \geq -(n-1) \frac{A(A-1)}{r^2} \) and \( n \leq \frac{a+b+A}{A} \),

(2) \( \text{Ric}^M_{\text{rad}}(r) \geq -(n-1) \frac{A(A-1)}{(c+r)^2} \) and \( n \leq \frac{a+b+A}{A} \),

(3) \( \text{Ric}^M_{\text{rad}}(r) \geq (n-1) \frac{B_1(1-B_1)}{r^2} \) and \( n \leq \frac{2a+2b+1 + \sqrt{1+4B_1(1-B_1)}}{1+\sqrt{1+4B_1(1-B_1)}} \),

(4) \( \text{Ric}^M_{\text{rad}}(r) \geq (n-1) \frac{B_1(1-B_1)}{(c+r)^2} \) and \( n \leq \frac{2a+2b+1 + \sqrt{1+4B_1(1-B_1)}}{1+\sqrt{1+4B_1(1-B_1)}} \),

(5) \( \text{Ric}^M_{\text{rad}}(r) \geq 0 \) and \( n \leq a+b+1 \),
where \( 0 \leq B_1 \leq 1 \leq A \) are constants. Then for every \( u \in W^{1,2}_0(M \setminus \{x_0\}) \), and \( a, b \in \mathbb{R} \),

\[
C \int_M \frac{|u|^2}{r^{a+b+1}} dv \leq \left( \int_M \frac{|u|^2}{r^{2a}} dv \right)^{\frac{1}{2}} \left( \int_M \frac{|
abla u|^2}{r^{2b}} dv \right)^{\frac{1}{2}},
\]

where

\[
C = C(a, b, A, B_1) = \begin{cases} 
\frac{a+b-(n-1)A}{2} & \text{if } \text{Ric}^M_{\text{rad}} \text{ satisfies (i) or (ii)}, \\
\frac{2a+2b-(n-1)(1+\sqrt{1+4B_1(1-B_1)})}{4} & \text{if } \text{Ric}^M_{\text{rad}} \text{ satisfies (iii) or (iv)}, \\
\frac{a+b+1-n}{2} & \text{if } \text{Ric}^M_{\text{rad}} \text{ satisfies (v)}. 
\end{cases}
\]

**Proof.** This follows from Theorem 7.6 and Theorem 7.7 that we can apply (7.40) and (2.52) to simplify the integral inequality (7.1), based on Theorem 7.1 and obtain the desired. \( \square \)

**Theorem 7.4.** (Generalized sharp Caffarelli-Kohn-Nirenberg type inequalities under radial curvature assumptions) Let \( M \) be an \( n \)-dimensional complete Riemannian manifold with a pole such that the radial curvature \( K(r) \) of \( M \) satisfies one of the following twelve conditions:
where the constant \( C \) is given by

\[
C = C(a, b, B) = \begin{cases} 
\frac{a+b-(n-1)A}{2}, & \text{if } K(r) \text{ satisfies (i) or (ii)}, \\
\frac{a+b-(n-1)A}{2^2}, & \text{if } K(r) \text{ satisfies (iii) or (iv)}, \\
\frac{|a+b-(n-1)A|}{2}, & \text{if } K(r) \text{ satisfies (v) or (vi)}, \\
\frac{2a+2b-(n-1)}{4(1+\sqrt{1+4B_1(1-B_1)})}, & \text{if } K(r) \text{ satisfies (vii) or (viii)}, \\
\frac{(n-1)(B+\frac{1}{2}+4)\cdot a-b}{2a+2b-(n-1)(1+\sqrt{1+4B_1(1-B_1)})}, & \text{if } K(r) \text{ satisfies (ix) or (x)}, \\
\end{cases}
\]
Proof. This follows from Theorems 3.1, 3.2, Corollary 3.2, and Theorems 3.3, 3.4, 3.2, and Corollary 3.2, and Theorems 3.3, 3.4, 3.5 that we can apply (3.3), (3.6), (3.23), (3.27), and (3.36) to simplify the integral inequality (7.1), based on Theorem 7.1. □

7.2. Embedding theorems for weighted Sobolev spaces of functions. Hessian comparison theorems via these geometric inequalities lead to embedding theorems for weighted Sobolev spaces of functions on Riemannian manifolds. Let $M$ be manifold as in Theorem 7.3, or Theorem 7.4, or Theorem 7.2. Following Costa’s notation [14], or [41], we let $D^{1,2}_0(M)$ denote the completion of $C^\infty_0(M \setminus \{x_0\})$ with respect to the norm

$$
||u||_{D^{1,2}_0(M)} := \left( \int_M \frac{|\nabla u|^2}{r^{2\gamma}} dv \right)^{\frac{1}{2}}.
$$

$L^2_\gamma(M)$ denote the completion of $C^\infty_0(M \setminus \{x_0\})$ with respect to the norm

$$
||u||_{L^2_\gamma(M)} := \left( \int_M \frac{|u|^2}{r^{2\gamma}} dv \right)^{\frac{1}{2}}
$$

and $H^{1}_{\alpha,\beta}(M)$ denote the completion of $C^\infty_0(M \setminus \{x_0\})$ with respect to the Sobolev norm

$$
||u||_{H^{1}_{\alpha,\beta}(M)} := \left( \int_M \left( \frac{|u|^2}{r^{2a}} + \frac{|\nabla u|^2}{r^{2b}} \right) dv \right)^{\frac{1}{2}}.
$$

If $\gamma = 0$ in (7.9), we simply write $D^{1,2}_0(M) = D^{1,2}(M)$. Applying arithmetic-mean-geometric-mean inequality to Theorems 7.3 - 7.4 and 7.2, we have

**Theorem 7.5.** (Embedding Theorem) Let $M$ be a manifold of radial curvature $K$ satisfying one of the five conditions in (7.5), or one of the twelve conditions in (7.7), or one of the three conditions in (7.2). Then the following continuous embeddings hold

$$
H^{1}_{\alpha,\beta}(M) \subset L^{\frac{2}{2a+1}}(M) \quad \text{and} \quad H^{1}_{\beta,\alpha}(M) \subset L^{\frac{2}{2b+1}}(M).
$$

7.3. Geometric differential-integral inequalities on manifolds. As a consequence of embedding theorem, we have geometric differential-integral inequalities on manifolds:

**Theorem 7.6.** Let $M$ be a manifold of radial curvature $K$ satisfying one of the five conditions in (7.5), or one of the twelve conditions in (7.7). Then we have the following seven geometric differential-integral inequalities:

i) For any $u \in H^{1}_{\beta+1,\beta}(M)$,

$$
C_1 \int_M \frac{|u|^2}{r^{2(b+1)}} dv \leq \int_M |\nabla u|^2 r^{2b} dv,
$$

where $C_1$ is a constant.
where

\[ (7.14) \]

\[
C_1 = \begin{cases} 
\left( \frac{(n-1)A-1}{2} - b \right)^2, & \text{if } \text{Ric}^M(r) \text{ satisfies } (7.5) \text{ (i) or } (7.5) \text{ (ii)} \\
\left( \frac{4b+2-(n-1)(1+\sqrt{1+4A})}{4} \right)^2, & \text{if } K(r) \text{ satisfies any of } (7.5) \text{ (i) - (vi)}, \\
\left( \frac{4b+2-(n-1)(1+\sqrt{1+4B_1(1-B_1)})}{4} \right)^2, & \text{if } K(r) \text{ satisfies } (7.5) \text{ (vii) or } (7.5) \text{ (viii)}, \\
\left( \frac{2b+2-n}{2} \right)^2, & \text{if } \text{Ric}^M(r) \text{ satisfies } (7.5) \text{ (iii) or } (7.5) \text{ (iv)} \\
\left( \frac{n-1|B-\frac{1}{2}| + \frac{a-2}{2} - 2b}{2} \right)^2, & \text{if } K(r) \text{ satisfies } (7.5) \text{ (ix) or } (7.5) \text{ (x)}, \\
\end{cases}
\]

in which \( a = b + 1 \) for (7.5) and (7.7);

\( \text{ii}) \) For any \( u \in H^1_{a,a+1}(M) \),

\[ (7.15) \]

\[
C_2 \left( \int_M \frac{|u|^2}{r^{2(a+1)}} dv \right)^2 \leq \left( \int_M \frac{|u|^2}{r^{2a}} dv \right) \left( \int_M \frac{|\nabla u|^2}{r^{2(a+1)}} dv \right),
\]

where

\[ (7.16) \]

\[
C_2 = \begin{cases} 
\left( \frac{(n-1)A-1}{2} - a \right)^2, & \text{if } \text{Ric}^M(r) \text{ satisfies } (7.5) \text{ (i) or } (7.5) \text{ (ii)} \\
\left( \frac{4a+2-(n-1)(1+\sqrt{1+4A})}{4} \right)^2, & \text{if } K(r) \text{ satisfies any of } (7.5) \text{ (i) - (vi)}, \\
\left( \frac{4a+2-(n-1)(1+\sqrt{1+4B_1(1-B_1)})}{4} \right)^2, & \text{if } K(r) \text{ satisfies } (7.5) \text{ (vii) or } (7.5) \text{ (viii)}, \\
\left( \frac{2a+2-n}{2} \right)^2, & \text{if } \text{Ric}^M(r) \text{ satisfies } (7.5) \text{ (iii) or } (7.5) \text{ (iv)} \\
\left( \frac{n-1|B-\frac{1}{2}| + \frac{a-2}{2} - 2a}{2} \right)^2, & \text{if } K(r) \text{ satisfies } (7.5) \text{ (ix) or } (7.5) \text{ (x)}, \\
\end{cases}
\]

in which \( b = a + 1 \) for (7.5) and (7.7);

\( \text{iii}) \) If \( u \in H^1_{a,b+1}(M) \) then \( u \in L^2(M) \) and

\[ (7.17) \]

\[
C_3 \left( \int_M |u|^2 dv \right)^2 \leq \left( \int_M r^{2(b+1)} |u|^2 dv \right) \left( \int_M \frac{|\nabla u|^2}{r^{2b}} dv \right),
\]
where
\[ C_3 = \begin{cases} 
\frac{(n-1)A+1}{2}^2, & \text{if } \operatorname{Ric}^M(r) \text{ satisfies } (7.20)(i) \text{ or } (7.20)(ii) \\
\frac{2+(n-1)(1+\sqrt{1+4A})}{4}^2, & \text{if } K(r) \text{ satisfies any of } (7.7)(i) - (vi), \\
\frac{2+(n-1)(1+\sqrt{1+4B_1(1-B_1)})}{4}^2, & \text{if } \operatorname{Ric}^M(r) \text{ satisfies } (7.20)(vii) \text{ or } (7.20)(viii), \\
\frac{n^2}{4}, & \text{if } \operatorname{Ric}^M(r) \text{ satisfies } (7.20)(iii) \text{ or } (7.20)(iv), \\
\frac{(n-1)|B-\frac{1}{2}|+\frac{n-3}{2}}{2}^2, & \text{if } K(r) \text{ satisfies } (7.7)(ix) \text{ or } (7.7)(x), \\
\frac{(n-1)|B-\frac{1}{2}|+\frac{n-3}{2}}{2}^2, & \text{if } K(r) \text{ satisfies } (7.7)(xi) \text{ or } (7.7)(xii), 
\end{cases} \]

in which \( a = -b - 1 \) for (7.3) and (7.4);

iv) If \( u \in H^1_{0,1}(M) \), then \( u \in L^2_1(M) \) and

\[ C_4 \left( \int_M \frac{|u|^2}{r^2} \, dv \right)^2 \leq \left( \int_M |u|^2 \, dv \right) \left( \int_M \frac{|
abla u|^2}{r^2} \, dv \right), \]

where
\[ C_4 = \begin{cases} 
\frac{(n-1)A-1}{2}^2, & \text{if } \operatorname{Ric}^M(r) \text{ satisfies } (7.20)(i) \text{ or } (7.20)(ii) \\
\frac{2-(n-1)(1+\sqrt{1+4A})}{4}^2, & \text{if } K(r) \text{ satisfies any of } (7.7)(i) - (vi), \\
\frac{(n-1)(1+\sqrt{1+4B_1(1-B_1)})-2}{4}^2, & \text{if } \operatorname{Ric}^M(r) \text{ satisfies } (7.20)(vii) \text{ or } (7.20)(viii), \\
\frac{(n-2)^2}{2}, & \text{if } \operatorname{Ric}^M(r) \text{ satisfies } (7.20)(iii) \text{ or } (7.20)(iv), \\
\frac{(n-1)|B-\frac{1}{2}|+\frac{n-3}{2}}{2}^2, & \text{if } K(r) \text{ satisfies } (7.7)(ix) \text{ or } (7.7)(x), \\
\frac{(n-1)|B-\frac{1}{2}|+\frac{n-3}{2}}{2}^2, & \text{if } K(r) \text{ satisfies } (7.7)(xi) \text{ or } (7.7)(xii), 
\end{cases} \]

in which \( a = 0 \), \( b = 1 \) for (7.5) and (7.7);

v) If \( u \in H^1_{-1,1}(M) \), then \( u \in L^2_{1.5}(M) \) and

\[ C_5 \left( \int_M \frac{|u|^2}{r^2} \, dv \right)^2 \leq \left( \int_M |u|^2 \, dv \right) \left( \int_M \frac{|
abla u|^2}{r^2} \, dv \right), \]
Remark 7.1. The case $M = \mathbb{R}^n$ is due to Costa \cite{14}.

Proof. We make special choices in Theorems 7.3 and 7.4 as follows:

i) Let $a = b + 1$;

ii) Let $b = a + 1$;

iii) Let $a = -b - 1$;

where

(7.22)

\[
C_5 = \begin{cases}
\left(\frac{(n-1)A}{2}\right)^2, & \text{if } \text{Ric}^M_{\text{rad}}(r) \text{ satisfies (7.5)(i) or (7.5)(ii)} \\
\left(\frac{(n-1)(1+\sqrt{1+4A})}{4}\right)^2, & \text{if } K(r) \text{ satisfies any of (7.4)(i) - (vi)} \\
\left(\frac{(n-1)(1+\sqrt{1+4B_1(1-B_1)})}{4}\right)^2, & \text{if } K(r) \text{ satisfies (7.5)(vi) or (7.5)(vii)} \\
\left(\frac{(n-1)(B-B_1)(B_1-1)}{2}\right)^2, & \text{if } K(r) \text{ satisfies (7.7)(ix) or (7.7)(x)} \\
\left(\frac{(n-1)^2}{2}\right)^2, & \text{if } K(r) \text{ satisfies (7.7)(ii) or (7.7)(iii)} \\
\end{cases}
\]

in which $a = -1, b = 1$ for (7.5) and (7.7).

vi) If $u \in H^1(M) = H^1_{0,0}(M)$, then $u \in L^2(M)$ and

(7.23)

\[
C_6 \left(\int_M |u|^2 \, dv\right)^2 \leq \left(\int_M |\nabla u|^2 \, dv\right) \left(\int_M |\nabla u|^2 \, dv\right),
\]

where

(7.24)

\[
C_6 = C_5,
\]

in which $a = 0, b = 0$ for (7.5) and (7.7).

vii) For any $u \in D^{1,2}(M)$,

(7.25)

\[
C_7 \int_M \frac{|u|^2 \, dv}{|u|^2} \leq \int_M |\nabla u|^2 \, dv,
\]

where

(7.26)

\[
C_7 = \begin{cases}
\left(\frac{(n-1)A-1}{2}\right)^2, & \text{if } \text{Ric}^M_{\text{rad}}(r) \text{ satisfies (7.5)(i) or (7.5)(ii)} \\
\left(\frac{2-(n-1)(1+\sqrt{1+4A})}{4}\right)^2, & \text{if } K(r) \text{ satisfies any of (7.7)(i) - (vi)} \\
\left(\frac{(n-1)(1+\sqrt{1+4B_1(1-B_1))}}{4}\right)^2, & \text{if } K(r) \text{ satisfies (7.5)(vi) or (7.5)(vii)} \\
\left(\frac{(n-1)(1+\sqrt{1+4B_1})}{4}\right)^2, & \text{if } K(r) \text{ satisfies (7.7)(ix) or (7.7)(x)} \\
\left(\frac{(n-1)2}{2}\right)^2, & \text{if } K(r) \text{ satisfies (7.7)(ii) or (7.7)(iii)} \\
\end{cases}
\]

in which $a = 1, b = 0$ for (7.5) and (7.7).
Lemma 7.1. Let $a = 0, b = 1$; 
\[ \text{v)} \quad \text{Let } a = -1, b = 1; \]
\[ \text{vi)} \quad \text{Let } a = 0, b = 0; \]
\[ \text{vii)} \quad \text{Let } a = 1, b = 0. \]

7.4. Generalized sharp Hardy type inequalities on Riemannian manifolds.
Laplacian comparison Theorems further lead to

Theorem 7.7. (Generalized Sharp Hardy Type Inequality) Let $M$ be an $n$-manifold with a pole satisfying

\begin{equation}
(2.39) \quad \text{Ric}_{\text{rad}}(r) \geq -(n-1) \frac{A(A-1)}{r^2} \quad \text{where } \ A \geq 1.
\end{equation}

Then for every $u \in W^{1,p}_0(M), \ \frac{\partial}{r^p} \in L^p(M)$ with $p > (n-1)A + 1$, we have

\begin{equation}
(7.27) \quad \left( \frac{p-1-(n-1)A}{p} \right) \int_M |u|^p \, dv \leq \int_M |\nabla u|^p \, dv.
\end{equation}

We will apply a double limiting argument to the following

Lemma 7.1. (Geometric Differential-Integral Inequality) (see [41] (1.3), [13] (3), [42] (8.1)) Let $u \in C^1_0(M)$ and $\partial B_\delta(x_0)$ be the $C^1$ boundary of the geodesic ball $B_\delta(x_0)$ centered at $x_0$ with radius $\delta > 0$. Let $V$ be an open set with smooth boundary $\partial V$ such that $V \subset \subset M$, and $u = 0$ off $V$. We choose a sufficiently small $\delta > 0$ so that $\partial V \cap \partial B_\delta(x_0) = \emptyset$. Then for every $\epsilon > 0$ and $p > 1$, we have

\begin{equation}
(7.28) \quad \left| -\int_{V \cap \partial B_\delta(x_0)} \frac{1}{r^p + \epsilon} |u|^p \, dS + \int_{M \setminus B_\delta(x_0)} \frac{(r^p + \epsilon)(r^{\Delta r} + 1) - pr^p}{(pr^p + \epsilon)^2} |u|^p \, dv \right| \\
\leq p \left( \int_{M \setminus B_\delta(x_0)} \left| \frac{|u|^{p-1} r}{r^p + \epsilon} \right| \, dv \right)^{\frac{p-1}{p}} \left( \int_{M \setminus B_\delta(x_0)} |\nabla u|^p \, dv \right)^{\frac{1}{p}},
\end{equation}

where $dS$ and $dv$ are the volume element of $\partial B_\delta(x_0)$ and $M$ respectively.

Proof of Theorem 7.7. Applying the Laplacian comparison Theorem 7.6 under the assumption $\text{Ric}_{\text{rad}} \geq -(n-1) \frac{A(A-1)}{r^2}$, one has $r^{\Delta r} + 1 \leq (n-1)A + 1 < p$. By the triangle inequality, (7.28) implies

\begin{equation}
(7.29) \quad \int_{M \setminus B_\delta(x_0)} \frac{r^{-(n-1)A-1} - (n-1)A+1}{(r^p + \epsilon)^2} |u|^p \, dv \\
\leq p \left( \int_{M \setminus B_\delta(x_0)} \left| \frac{|u|^{p-1} r}{r^p + \epsilon} \right| \, dv \right)^{\frac{p-1}{p}} \left( \int_{M \setminus B_\delta(x_0)} |\nabla u|^p \, dv \right)^{\frac{1}{p}} + \left| \int_{\partial B_\delta(x_0)} \frac{r}{r^p + \epsilon} |u|^p \, dS \right|.
\end{equation}

For sufficiently small $\delta > 0$, one has

\begin{equation}
(7.30) \quad \int_{\partial B_\delta(x_0)} \frac{r}{r^p + \epsilon} |u|^p \, dS = 0 \quad \text{if } x_0 \notin V
\end{equation}

and

\begin{equation}
(7.31) \quad \left| \int_{\partial B_\delta(x_0)} \frac{r}{r^p + \epsilon} |u|^p \, dS \right| \to 0 \quad \text{as } \delta \to 0 \quad \text{if } x_0 \in V,
\end{equation}

\[ \text{iv)} \quad \text{Let } a = 0, b = 1; \]
\[ \text{v)} \quad \text{Let } a = -1, b = 1; \]
\[ \text{vi)} \quad \text{Let } a = 0, b = 0; \]
\[ \text{vii)} \quad \text{Let } a = 1, b = 0. \]
Lemma 7.2. (see Theorem 3.3 in [42]) \( L^p(M), 1 \leq p \leq \infty \) is complete, i.e., every Cauchy sequence \( \{u_i\} \) in \( L^p(M) \) converges (This means that if for every \( \epsilon > 0 \), there exists \( N \) such that \( (\int_M |u_i - u_j|^p dv)^\frac{1}{p} < \epsilon \), when \( i > N \) and \( j > N \), then there exists a unique function \( u \in L^p(M) \), such that \( (\int_M |u_i - u|^p dv)^\frac{1}{p} \to 0 \), as \( i \to \infty \).

Lemma 7.3. (see Theorem 3.4 in [42]) If \( \{u_i\} \) is a Cauchy sequence in \( L^p(M), 1 \leq p \leq \infty \), then there exists a subsequence \( \{u_{i_k}\} \) and a nonnegative function \( U \) in \( L^p(M) \) such that

1. \( |u_{i_k}| \leq U \) almost everywhere in \( M \).
2. \( \lim_{k \to \infty} u_{i_k} = u \) almost everywhere in \( M \).

In view of Lemma 7.2, there exists a limiting function \( f(x) \in L^p(M) \) satisfying...
\[
\int_M |f(x)|^p \, dv = \lim_{i \to \infty} \int_M \frac{|u_i(x)|^p}{r^p} \, dv \leq \left( \frac{p}{p - (n-1)A - 1} \right)^p \lim_{i \to \infty} \int_M |\nabla u_i|^p \, dv
\]
\[
\leq \left( \frac{p}{p - (n-1)A - 1} \right)^p \int_M |\nabla u|^p \, dv.
\]

On the other hand, since \( \frac{1}{r^p} \) is bounded in \( M \setminus B_\epsilon(x_0) \), where \( B_\epsilon(x_0) \) is the open geodesic ball of radius \( \epsilon > 0 \), centered at \( x_0 \), and the pointwise convergence in Lemma 7.3 (2), we have for every \( \epsilon > 0 \),

\[
\int_{M \setminus B_\epsilon(x_0)} |f(x)|^p \, dv = \lim_{i \to \infty} \int_{M \setminus B_\epsilon(x_0)} \frac{|u_i(x)|^p}{r^p} \, dv = \int_{M \setminus B_\epsilon(x_0)} \frac{|u|^p}{r^p} \, dv = \int_M \chi_{M \setminus B_\epsilon(x_0)} \frac{|u|^p}{r^p} \, dv,
\]

where \( \chi_{M \setminus B_\epsilon(x_0)} \) is the characteristic function on \( M \setminus B_\epsilon(x_0) \). As \( \epsilon \to 0 \), monotone convergence theorem and (7.38) imply that

\[
\int_M |f(x)|^p \, dv = \lim_{i \to \infty} \int_M \frac{|u_i|^p}{r^p} \, dv = \int_M \frac{|u|^p}{r^p} \, dv.
\]

Substituting (7.39) into (7.37) we obtain the desired (7.27) for \( u \in W^{1,p}_0(M) \). □

**Remark 7.2.** Theorem 7.7. is sharp and recaptures a theorem of Chen-Li-Wei (see [12]), when \( A = 1 \). A double limiting argument and techniques in [42] are employed in proving Theorem 7.7.

### 8. Monotonicity Formulae

Let \( F : [0, \infty) \to [0, \infty) \) be a \( C^2 \) function such that \( F' > 0 \) on \([0, \infty)\), and \( F(0) = 0 \).

**Definition 8.1.** The \( F \)-degree \( d_F \) is defined to be

\[
d_F = \sup_{t \geq 0} \frac{tF'(t)}{F(t)}.
\]

Let \( \omega \) be a smooth \( k \)-forms on a smooth \( n \)-dimensional Riemannian manifold \( M \) with values in the vector bundle \( \xi : E \to M \). At each fiber of \( E \) is equipped with a positive inner product \( \langle \, , \rangle_E \). Set \( |\omega|^2 = \langle \omega, \omega \rangle_E \). The efficiency functional given by

\[
\mathcal{E}_{F,g}(\omega) = \int_M F\left( \frac{|\omega|^2}{2} \right) \, dv_g.
\]

The stress-energy tensor \( S_{F,\omega} \) associated with the \( \mathcal{E}_{F,g} \)-energy functional is defined as follows (see [6, 5, 4, 28, 17]):

\[
S_{F,\omega}(X, Y) = F\left( \frac{|\omega|^2}{2} \right) g(X, Y) - F'(\frac{|\omega|^2}{2}) \langle i_X \omega, i_Y \omega \rangle,
\]

where \( i_X \omega \) is the interior multiplication by the vector field \( X \).
Definition 8.2. \( \omega \in A^k(\xi) \ (k \geq 1) \) is said to satisfy an \( F \)-conservation law if \( S_{F,\omega} \) is divergence free, i.e. the \((0,1)\)-type tensor field \( \text{div} \, S_{F,\omega} \) vanishes identically

\[
\text{div} \, S_{F,\omega} \equiv 0.
\]

Let \( \flat \) denote the bundle isomorphism that identifies the vector field \( X \) with the differential one-form \( X^\flat \), and let \( \nabla \) be the Riemannian connection of \( M \). Then the covariant derivative \( \nabla X^\flat \) of \( X^\flat \) is a \((0,2)\)-type tensor, given by

\[
\nabla X^\flat(Y,Z) = \langle \nabla_Z X, Y \rangle, \quad \forall \, X, Y \in \Gamma(M).
\]

If \( X \) is conservative, then

\[
X = \nabla f, \quad X^\flat = df \quad \text{and} \quad \nabla X^\flat = \text{Hess}(f).
\]

for some scalar potential \( f \) (see [11], p. 1527). A direct computation yields (see, e.g., [17])

\[
\text{div}(i_X S_\omega) = \langle S_{F,\omega}, \nabla X^\flat \rangle + (\text{div} \, S_{F,\omega})(X), \quad \forall \, X \in \Gamma(M).
\]

It follows from the divergence theorem that, if \( \omega \) satisfies an \( F \)-conservation law, we have for every bounded domain \( D \) in \( M \) with \( C^1 \) boundary \( \partial D \),

\[
\int_{\partial D} S_{F,\omega}(X,\nu)ds_g = \int_D \langle S_{F,\omega}, \nabla X^\flat \rangle dv_g,
\]

where \( \nu \) is unit outward normal vector field along \( \partial D \) with \((n-1)\)-dimensional volume element \( ds_g \). When we choose scalar potential \( f(x) = \frac{1}{2}r^2(x) \), curvature via (8.6) and our Hessian comparison theorems will influence the behavior of the \( F \)-stress energy \( S_{F,\omega} \) and the underlying criticality \( \omega \) with the help from the following.

Lemma 8.1. ([17]) Let \( M \) be a complete manifold with a pole \( x_0 \). Assume that there exist two positive functions \( h_1(r) \) and \( h_2(r) \) such that

\[
h_1(r)[g - dr \otimes dr] \leq \text{Hess}(r) \leq h_2(r)[g - dr \otimes dr]
\]

on \( M \setminus \{x_0\} \). If \( h_2(r) \) satisfies

\[
rh_2(r) \geq 1,
\]

then

\[
\langle S_{F,\omega}, \nabla X^\flat \rangle \geq (1 + (n-1)rh_1(r) - 2kd_F rh_2(r)) \frac{F(|\omega|^2)}{2},
\]

where \( X = r \nabla r \).

Theorem 8.1. Let \((M,g)\) be an \( n \)-dimensional complete Riemannian manifold with a pole \( x_0 \). Let \( \xi : E \to M \) be a Riemannian vector bundle on \( M \) and \( \omega \in A^k(\xi) \). Assume that the radial curvature \( K(r) \) of \( M \) satisfies one of the following seven conditions:
(8.12) (8.12) holds with $1 + (n - 1)A_1 - 2kd_F A > 0$;

(ii) (8.17) holds with $1 + (n - 1) \frac{1 + \sqrt{1 + 4A_1}}{2} - kd_F (1 + \sqrt{1 + 4A}) > 0$;

(iii) (8.19) holds with $1 + (n - 1) \frac{1 + \sqrt{1 - 4B}}{2} - kd_F (1 + \sqrt{1 + 4B_1 (1 - B_1)}) > 0$;

(iv) (8.19) holds with $1 + (n - 1) \frac{1 + \sqrt{1 - 4B}}{2} - kd_F (1 + \sqrt{1 + 4B_1}) > 0$;

(v) $- \alpha^2 \leq K(r) \leq -\beta^2$ with $\alpha > 0, \beta > 0$ and $(n - 1) \beta - 2kd_F \geq 0$;

(vi) $K(r) = 0$ with $n - 2kd_F > 0$;

(vii) $-\frac{A}{(1 + r^2)^{1+\epsilon}} \leq K(r) \leq \frac{B}{(1 + r^2)^{1+\epsilon}}$ with $\epsilon > 0, A \geq 0, 0 < B < 2c$ and $n - (n - 1)\frac{B}{2c} - 2ke^{\frac{n}{kd_F}} A > 0$.

If $\omega$ satisfies an $F$-conservation law, then

(8.13) $\frac{1}{\rho_1^2} \int_{B_{\rho_1}(x_0)} F\left(\frac{|\omega|^2}{2}\right)dv \leq \frac{1}{\rho_2^2} \int_{B_{\rho_2}(x_0)} F\left(\frac{|\omega|^2}{2}\right)dv$

for any $0 < \rho_1 \leq \rho_2$, where

(8.14) $\lambda = \begin{cases} 1 + (n - 1)A_1 - 2kd_F A, & \text{if } K(r) \text{ satisfies (i)}, \\ 1 + (n - 1)\frac{1 + \sqrt{1 + 4A_1}}{2} - kd_F (1 + \sqrt{1 + 4A}), & \text{if } K(r) \text{ satisfies (ii)}, \\ 1 + (n - 1)\frac{1 + \sqrt{1 - 4B}}{2} - kd_F (1 + \sqrt{1 + 4B_1 (1 - B_1)}), & \text{if } K(r) \text{ satisfies (iii)}, \\ 1 + (n - 1)\frac{1 + \sqrt{1 - 4B}}{2} - kd_F (1 + \sqrt{1 + 4B_1}), & \text{if } K(r) \text{ satisfies (iv)}, \\ n - 2ke^{\frac{n}{kd_F}} A, & \text{if } K(r) \text{ satisfies (v)}, \\ n - 2kd_F, & \text{if } K(r) \text{ satisfies (vi)}, \\ n - (n - 1)\frac{B}{2c} - 2ke^{\frac{n}{kd_F}} A, & \text{if } K(r) \text{ satisfies (vii)}. \end{cases}$

Proof. Choose a smooth conservative vector field $X = \nabla (\frac{1}{2} r^2)$ on $M$. If $K(r)$ satisfies (8.12), then by Corollary 3.1, Theorem A, Corollary 3.5 and Theorem 3.3, Lemma 8.1 holds. Hence Lemma 8.1 is applicable and by (8.11), we have on $B_\rho(x_0) \setminus \{x_0\}$, for every $\rho > 0$,

(8.15) $\langle S_{F,\omega}, \nabla X^\flat \rangle \geq \lambda F\left(\frac{|\omega|^2}{2}\right)$,

where $\lambda$ is as in (8.14). Thus, by the continuity of $\langle S_{F,\omega}, \nabla X^\flat \rangle$ and $F\left(\frac{|\omega|^2}{2}\right)$, and (8.3), we have for every $\rho > 0$, (8.13) holds on $B_\rho(x_0)$ and

(8.16) $\rho F\left(\frac{|\omega|^2}{2}\right) \geq S_{F,\omega}(X, \frac{\partial}{\partial r})$ on $\partial B_\rho(x_0)$.

It follows from (8.3), (8.15) and (8.16) that

(8.17) $\rho \int_{\partial B_\rho(x_0)} F\left(\frac{|\omega|^2}{2}\right)ds \geq \lambda \int_{B_\rho(x_0)} F\left(\frac{|\omega|^2}{2}\right)dv$. 
Hence we get from (8.17) the following

\[(8.18)\]
\[
\int_{\partial B(\rho)} \rho(x_0) F(\frac{|\omega|^2}{2}) \, ds \geq \frac{\lambda}{\rho} \int_{B(\rho)} \rho(x_0) F(\frac{|\omega|^2}{2}) \, dv
\]

The coarea formula implies that

\[
\frac{d}{d \rho} \int_{B(\rho)} \rho(x_0) F(\frac{|\omega|^2}{2}) \, dv = \int_{\partial B(\rho)} \rho(x_0) F(\frac{|\omega|^2}{2}) \, ds
\]

Thus, we have

\[(8.19)\]
\[
\frac{d}{d \rho} \int_{B(\rho)} F(\frac{|\omega|^2}{2}) \, dv \geq \frac{\lambda}{\rho} \int_{B(\rho)} \rho(x_0) F(\frac{|\omega|^2}{2}) \, dv
\]

for a.e. \( \rho > 0 \). By integration (8.19) over \([\rho_1, \rho_2]\), we have

\[
\ln \int_{B(\rho_2)} F(\frac{|\omega|^2}{2}) \, dv - \ln \int_{B(\rho_1)} F(\frac{|\omega|^2}{2}) \, dv \geq \ln \rho_2 - \ln \rho_1.
\]

This proves (8.13).

9. **Vanishing Theorems**

In this section we list some results that are immediate applications of the monotonicity formulae in the last section.

9.1. **Vanishing theorems for vector bundle valued \( k \)-forms.**

**Theorem 9.1.** Suppose the radial curvature \( K(r) \) of \( M \) satisfies the condition (8.12). If \( \omega \in A^k(\xi) \) satisfies an \( F \)-conservation law (8.4) and

\[(9.1)\]
\[
\int_{B(\rho)} F(\frac{|\omega|^2}{2}) \, dv = o(\rho^\lambda) \quad \text{as } \rho \to \infty
\]

where \( \lambda \) is given by (8.14) depending on the curvature \( K(r) \), then \( F(\frac{|\omega|^2}{2}) \equiv 0 \), and hence \( \omega \equiv 0 \). In particular, if \( \omega \) has finite \( E_{F,g} \)-energy, then \( \omega \equiv 0 \).

**Proof.** By Theorem 8.1, the monotonicity formula (8.13) holds. Let \( \lambda_2 \to \infty \) in (8.13). Then (9.1) implies that \( F(\frac{|\omega|^2}{2}) \equiv 0 \), and hence \( \omega \equiv 0 \). \( \square \)

9.2. **Applications in \( F \)-Yang-Mills fields.** Let \( R^\nabla \) be an \( F \)-Yang-Mills field, associated with an \( F \)-Yang-Mills connection \( \nabla \) on the adjoint bundle \( Ad(P) \) of a principle \( G \)-bundle over a manifold \( M \). Then \( R^\nabla \) can be viewed as a 2-form with values in the adjoint bundle over \( M \), and by [17 Theorem 3.1], \( \omega = R^\nabla \) satisfies an \( F \)-conservation law. We have the following vanishing theorem for \( F \)-Yang-Mills fields:

**Theorem 9.2.** Suppose the radial curvature \( K(r) \) of \( M \) satisfies the condition (8.12), in which \( k = 2 \). Assume \( F \)-Yang-Mills field \( R^\nabla \) satisfies the following growth condition

\[(9.2)\]
\[
\int_{B(\rho)} F(\frac{|R^\nabla|^2}{2}) \, dv = o(\rho^\lambda) \quad \text{as } \rho \to \infty,
\]

where \( \lambda \) satisfies the condition (8.14) for \( k = 2 \). Then \( R^\nabla \equiv 0 \) on \( M \). In particular, every \( F \)-Yang-Mills field \( R^\nabla \) with finite \( F \)-Yang-Mills energy vanishes on \( M \).
Proof. This follows at once from Theorem 9.1 in which \( k = 2 \) and \( \omega = R^\nabla \).

This theorem becomes the following vanishing theorem for \( p \)-Yang-Mills fields, when \( F(t) = \frac{1}{p}(2t)^{\frac{p}{2}} \), \( p > 1 \) (Hence \( dF = \frac{t}{4} \) by Definition 8.1):

**Theorem 9.3.** Suppose the radial curvature \( K(r) \) of \( M \) satisfies one of the following seven conditions:

\[(9.3)\]

(i) \((9.11)\) holds with \( 1 + (n - 1)A_1 - 2pA > 0; \)

(ii) \((9.17)\) holds with \( 1 + (n - 1)\frac{1 + \sqrt{1 + 4A_1}}{2} - p(1 + \sqrt{1 + 4A}) > 0; \)

(iii) \((9.41)\) holds with \( 1 + (n - 1)((B - \frac{1}{2}) + \frac{1}{2}) - p(1 + \sqrt{1 + 4B_1(1 - B_1)}) > 0; \)

(iv) \((9.43)\) holds with \( 1 + (n - 1)\frac{1 + \sqrt{1 - 4B}}{2} - p(1 + \sqrt{1 + 4B_1}) > 0; \)

(v) \(- \alpha^2 \leq K(r) \leq -\beta^2 \) with \( \alpha > 0, \beta > 0 \) and \((n - 1)\beta - 2p\alpha \geq 0; \)

(vi) \( K(r) = 0 \) with \( n - 2p > 0; \)

(vii) \(- \frac{A}{(1 + r^2)^{1+\epsilon}} \leq K(r) \leq B - \frac{(1 + r^2)^{1+\epsilon}}{n - (n - 1)} \) with \( \epsilon > 0, A \geq 0, 0 < B < 2\epsilon \) and \( n - 2p > 0. \)

Then every \( p \)-Yang-Mills field \( R^\nabla \) with the following growth condition vanishes:

\[(9.4)\]

\[
\frac{1}{p} \int_{B_r(x_0)} |R^\nabla|^p \, dv = o(\rho^\lambda) \text{ as } \rho \to \infty
\]

where

\[(9.5)\]

\[
\lambda = \begin{cases} 
1 + (n - 1)A_1 - 2pA, & \text{if } K(r) \text{ satisfies (i)}, \\
1 + (n - 1)\frac{1 + \sqrt{1 + 4A_1}}{2} - p(1 + \sqrt{1 + 4A}), & \text{if } K(r) \text{ satisfies (ii)}, \\
1 + (n - 1)((B - \frac{1}{2}) + \frac{1}{2}) - p(1 + \sqrt{1 + 4B_1(1 - B_1)}), & \text{if } K(r) \text{ satisfies (iii)}, \\
1 + (n - 1)\frac{1 + \sqrt{1 - 4B}}{2} - p - p\sqrt{1 + 4B_1}, & \text{if } K(r) \text{ satisfies (iv)}, \\
\frac{n - 2p\alpha}{\beta}, & \text{if } K(r) \text{ satisfies (v)}, \\
n - 2p, & \text{if } K(r) \text{ satisfies (vi)}, \\
\frac{n - (n - 1)\beta}{2p}, & \text{if } K(r) \text{ satisfies (vii)}. 
\end{cases}
\]

In particular, every \( p \)-Yang-Mills field \( R^\nabla \) with finite \( YM_p \)-energy vanishes on \( M \).

**Corollary 9.1.** Let \( M, N, K(r), \lambda, \) and the growth condition \((9.4)\) be as in Theorem 9.3 in which \( p = 2 \). Then every Yang-Mills field \( R^\nabla \equiv 0 \) on \( M \).

10. **Liouville Type Theorems**

10.1. **Liouville type theorems for \( F \)-harmonic maps.** Let \( u : M \to N \) be an \( F \)-harmonic map. Then its differential \( du \) can be viewed as a 1-form with values in the induced bundle \( u^{-1}TN \). Since \( \omega = du \) satisfies an \( F \)-conservation law \((8.3)\), we obtain the following Liouville-type
Theorem 10.1. Let $N$ be a Riemannian manifold. Suppose the radial curvature $K(r)$ of $M$ satisfies one of the following seven conditions:

(i) $(3.14)$ holds with $1 + (n - 1)A_1 - pA > 0$;

(ii) $(3.17)$ holds with $1 + (n - 1)\frac{1 + \sqrt{1 + 4A_1}}{2} - p\frac{1 + \sqrt{1 + 4A}}{2} > 0$;

(iii) $(3.43)$ holds with $1 + (n - 1)(|B - \frac{1}{2}| + \frac{1}{2}) - p\frac{1 + \sqrt{1 + 4B_1(1 - B_1)}}{2} > 0$;

(iv) $-\alpha^2 \leq K(r) \leq -\beta^2$ with $\alpha > 0, \beta > 0$ and $(n - 1)\beta - p\alpha \geq 0$;

(v) $K(r) = 0$ with $n - pk > 0$;

(vii) $\frac{A}{(1 + r^2)^{1+\epsilon}} \leq K(r) \leq \frac{B}{(1 + r^2)^{1+\epsilon}}$ with $\epsilon > 0, A \geq 0, 0 < B < 2\epsilon$ and $n - (n - 1)\frac{B}{2\epsilon} - pe^{\frac{A}{\epsilon}} > 0$.

Then every $p$-harmonic map $u : M \to N$ with the following $p$-energy growth condition is a constant:

$$\frac{1}{p} \int_{B_r(x_0)} |du|^p \, dv = o(\rho^{\lambda}) \quad \text{as } \rho \to \infty,$$

where

$$\lambda = \begin{cases} 
1 + (n - 1)A_1 - pA, & \text{if } K(r) \text{ satisfies (i)}, \\
1 + (n - 1)\frac{1 + \sqrt{1 + 4A_1}}{2} - p\frac{1 + \sqrt{1 + 4A}}{2}, & \text{if } K(r) \text{ satisfies (ii)}, \\
1 + (n - 1)(|B - \frac{1}{2}| + \frac{1}{2}) - p\frac{1 + \sqrt{1 + 4B_1(1 - B_1)}}{2}, & \text{if } K(r) \text{ satisfies (iii)}, \\
1 + (n - 1)\frac{1 + \sqrt{1 - 4B_1}}{2} - p\frac{1 + \sqrt{1 - 4B}}{2}, & \text{if } K(r) \text{ satisfies (iv)}, \\
n - p\frac{A}{\epsilon}, & \text{if } K(r) \text{ satisfies (v)}, \\
n - p, & \text{if } K(r) \text{ satisfies (vi)}, \\
n - (n - 1)\frac{B}{2\epsilon} - pe^{\frac{A}{\epsilon}}, & \text{if } K(r) \text{ satisfies (vii)}. 
\end{cases}$$

In particular, every $p$-harmonic map $u : M \to N$ with finite $p$-energy is a constant.

Proof. This follows immediately from Theorem 10.1 in which $F(t) = \frac{t^p}{p} (2t)^\frac{p}{2}$ and $dF = \frac{F'}{F}$. \qed
Corollary 10.1. Let \( M, N, K(r), \lambda \) and the growth condition (10.3) be as in Theorem 10.2, in which \( p = 2 \). Then every harmonic map \( u : M \rightarrow N \) is a constant.

11. Generalized Yang-Mills-Born-Infeld fields (with the plus sign)
on Manifolds

In [11], Sibner-Sibner-Yang consider a variational problem which is a generalization of the (scalar valued) Born-Infeld model and at the same time a quasilinear generalization of the Yang-Mills theory. This motivates the study of Yang-Mills-Born-Infeld solutions with no finite-energy solution except the trivial solution on \( \mathbb{R}^4 \). In [17], Dong and Wei introduced the following notions:

Definition 11.1. The generalized Yang-Mills-Born-Infeld energy functional with the plus sign on a manifold \( M \) is the mapping \( \mathcal{YM}_{BI}^+ : C \rightarrow \mathbb{R}^+ \) given by

\[
\mathcal{YM}_{BI}^+(\nabla) = \int_M \sqrt{1 + ||R^\nabla||^2 - 1} \ dv.
\]

The generalized Yang-Mills-Born-Infeld energy functional with the negative sign on a manifold \( M \) is the mapping \( \mathcal{YM}_{BI}^- : C \rightarrow \mathbb{R}^+ \) given by

\[
\mathcal{YM}_{BI}^-(\nabla) = \int_M (1 - \sqrt{1 - ||R^\nabla||^2}) dv.
\]

The associate curvature form \( R^\nabla \) of a critical connection \( \nabla \) of \( \mathcal{YM}_{BI}^+ \) (resp. \( \mathcal{YM}_{BI}^- \)) is called a generalized Yang-Mills-Born-Infeld field with the plus sign (resp. with the minus sign) on a manifold.

Theorem 11.1 (See [17]). Every generalized Yang-Mills-Born-Infeld field (with the plus sign or with the minus sign) on a manifold satisfies an \( F \)-conservation law.

Theorem 11.2. Let the radial curvature \( K(r) \) of \( M \) satisfy one of seven conditions in [8,12] in which \( k = 2 \) and \( d_F = 1 \). Let \( R^\nabla \) be a generalized Yang-Mills-Born-Infeld field with the plus sign on \( M \). If \( R^\nabla \) satisfies the following growth condition

\[
\int_{B_\rho(x_0)} \sqrt{1 + ||R^\nabla||^2 - 1} \ dv = o(\rho^{\lambda}) \quad \text{as} \quad \rho \rightarrow \infty,
\]

where

\[
\lambda = \begin{cases} 
1 + (n-1)A_1 - 4A, & \text{if } K(r) \text{ satisfies (i)}, \\
1 + (n-1)\frac{1 + \sqrt{1 + 4A} - 2\sqrt{1 + 4A}}{2}, & \text{if } K(r) \text{ satisfies (ii)}, \\
1 + (n-1)(|B - \frac{1}{2}| + \frac{1}{2}) - 2(1 + \sqrt{1 + 4B(1 - B_1)}), & \text{if } K(r) \text{ satisfies (iii)}, \\
1 + (n-1)\frac{1 + \sqrt{1 + 4B(1 - B_1)} - 2(1 + \sqrt{1 + 4B_1})}{2}, & \text{if } K(r) \text{ satisfies (iv)}, \\
n - 4\frac{\sigma}{B}, & \text{if } K(r) \text{ satisfies (v)}, \\
n - 4, & \text{if } K(r) \text{ satisfies (vi)}, \\
n - (n-1)\frac{B}{2} - 4e^\frac{\phi}{A}, & \text{if } K(r) \text{ satisfies (vii)},
\end{cases}
\]

then its curvature \( R^\nabla \equiv 0 \). In particular, if \( R^\nabla \) has finite \( \mathcal{YM}_{BI}^+ \)-energy, then \( R^\nabla \equiv 0 \).

Proof. By applying Theorem 11.1 and \( F(t) = \sqrt{1 + 2t} - 1 \) to Theorem 8.1 in which \( d_F = 1 \), and \( k = 2 \), for \( R^\nabla \in A^2(AdP) \), the result follows immediately. \( \square \)
12. Dirichlet Boundary Value Problems

We recall $F$-lower degree $l_F$ is defined to be

$$(12.1) \quad l_F = \inf_{t \geq 0} \frac{tF'(t)}{F(t)}.$$ 

A bounded domain $D \subset M$ with $C^1$ boundary is called starlike (relative to $x_0$) if there exists an inner point $x_0 \in D$ such that

$$(12.2) \quad \langle \frac{\partial}{\partial r_{x_0}}, \nu \rangle |_{\partial D} \geq 0,$$

where $\nu$ is the unit outer normal to $\partial D$, and for any $x \in D \setminus \{x_0\} \cup \partial D$, $\frac{\partial}{\partial r_{x_0}}(x)$ is the unit vector field tangent to the unique geodesic emanating from $x_0$ to $x$.

It is obvious that any disc or convex domain is starlike.

**Theorem 12.1.** Let $D$ be a bounded starlike domain (relative to $x_0$) with $C^1$ boundary in a complete Riemannian $n$-manifold $M$. Let $\xi : E \to M$ be a Riemannian vector bundle on $M$ and $\omega \in A^1(\xi)$. Assume that the radial curvature $K(r)$ of $M$ satisfies one of the following seven conditions:

$$(12.3) \quad \begin{align*}
(i) & \quad (3.13) \text{ holds with } 1 + (n-1)A_1 - 2d_F A > 0; \\
(ii) & \quad (3.17) \text{ holds with } 1 + (n-1)\frac{1+\sqrt{1+4A_1}}{2} - d_F(1+\sqrt{1+4A_1}) > 0; \\
(iii) & \quad (3.49) \text{ holds with } 1 + (n-1)(|B - \frac{1}{2}| + \frac{1}{2}) - d_F(1+\sqrt{1+4B_1(1-B_1)}) > 0; \\
(iv) & \quad (3.45) \text{ holds with } 1 + (n-1)\frac{1+\sqrt{1-4B}}{2} - d_F(1+\sqrt{1+4B_1}) > 0; \\
(v) & \quad -\alpha^2 \leq K(r) \leq -\beta^2 \quad \text{ with } \alpha > 0, \beta > 0 \quad \text{ and } \quad (n-1)\beta - 2\alpha d_F \geq 0; \\
vii & \quad K(r) = 0 \quad \text{ with } \quad n - 2d_F > 0; \\
viii & \quad \frac{A}{(1+r^2)^{1+\epsilon}} \leq K(r) \leq \frac{B}{(1+r^2)^{1+\epsilon}} \quad \text{ with } \quad \epsilon > 0, A \geq 0, 0 < B < 2\epsilon \quad \text{ and } \quad n - (n-1)\frac{B}{2\epsilon} - 2\epsilon \frac{A}{\epsilon} d_F > 0.
\end{align*}$$

Assume that $l_F \geq \frac{1}{2}$. If $\omega \in A^1(\xi)$ satisfies $F$-conservation law and annihilates any tangent vector $\eta$ of $\partial D$, then $\omega$ vanishes on $D$.

**Proof.** By the assumption, there exists a point $x_0 \in D$ such that the distance function $r_{x_0}$ relative to $x_0$ satisfies (12.2). Take $X = r\nabla r$, where $r = r_{x_0}$. From the proof of Theorem 8.1, we know that

$$(8.15) \quad \langle S_{F,\omega}, \nabla X^\flat \rangle \geq \lambda F\left(\frac{||\omega||^2}{2}\right)$$
in $D$, where $\lambda$ is a positive constant given by

$$
\lambda = \begin{cases} 
1 + (n-1)A_1 - 2d_F A, & \text{if } K(r) \text{ satisfies (i)}, \\
1 + (n-1)\frac{1 + \sqrt{1 + 4A}}{2} - d_F (1 + \sqrt{1 + 4A}), & \text{if } K(r) \text{ satisfies (ii)}, \\
1 + (n-1)\frac{1 + \sqrt{1 + 4B_1(1 - B_1)}}{2} - d_F (1 + \sqrt{1 + 4B_1(1 - B_1)}), & \text{if } K(r) \text{ satisfies (iii)}, \\
1 + (n-1)\frac{1 + \sqrt{1 + 4B_1(1 - B_1)}}{2} - d_F (1 + \sqrt{1 + 4B_1(1 - B_1)}), & \text{if } K(r) \text{ satisfies (iv)}, \\
n - 2\frac{\partial}{\partial r}, & \text{if } K(r) \text{ satisfies (v)}, \\
n - 2d_F, & \text{if } K(r) \text{ satisfies (vi)}, \\
n - (n-1)\frac{B}{2} - 2e^{\frac{\phi}{r_d}} d_F, & \text{if } K(r) \text{ satisfies (vii)}. 
\end{cases}
$$

Since $\omega \in A^1(\xi)$ annihilates any tangent vector $\eta$ of $\partial D$, we easily derive via \((8.3)\) and \((12.4)\), the following inequality on $\partial D$

$$
S_{F,\omega}(X, \nu) = rS_{F,\omega}(\frac{\partial}{\partial r}, \nu)
= r \left( F(\frac{|\omega|^2}{2})(\frac{\partial}{\partial r}, \nu) - F'(\frac{|\omega|^2}{2})(\omega(\frac{\partial}{\partial r}), \omega(\nu)) \right)
= r(\frac{\partial}{\partial r}, \nu) \left( F(\frac{|\omega|^2}{2}) - 2F'(\frac{|\omega|^2}{2})|\omega|^2 \right)
\leq r(\frac{\partial}{\partial r}, \nu) F(\frac{|\omega|^2}{2})(1 - 2d_F) \leq 0.
$$

From \((8.3)\), \((12.3)\) and \((12.5)\), we have

$$
0 \leq \int_D \lambda F(\frac{|\omega|^2}{2})dv \leq 0,
$$

which implies that $\omega \equiv 0$. \hfill $\square$

**Theorem 12.2.** (Dirichlet problems for $F$-harmonic maps) Let $M$, $D$, and $\xi$ be as in Theorem 12.1. Assume that the radial curvature $K(r)$ of $M$ satisfies one of the following seven conditions:

$$
(12.6)
$$

(i) \((8.3)\) holds with $1 + (n-1)A_1 - 2d_F A > 0$;

(ii) \((8.11)\) holds with $1 + (n-1)\frac{1 + \sqrt{1 + 4A_1}}{2} - d_F (1 + \sqrt{1 + 4A}) > 0$;

(iii) $-\alpha^2 \leq K(r) \leq -\beta^2$ with $\alpha > 0, \beta > 0$ and $(n-1)\beta - 2d_F \alpha \geq 0$;

(iv) $K(r) = 0$ with $n - 2d_F > 0$;

(v) $-\frac{A}{(1 + r^2)^{1+\epsilon}} \leq K(r) \leq \frac{B}{(1 + r^2)^{1+\epsilon}}$ with $\epsilon > 0, A \geq 0, 0 < B < 2\epsilon$, and

$$
n - (n-1)\frac{B}{2\epsilon} - 2e^{\frac{\phi}{r_d}} d_F > 0;$$

(vi) \((8.49)\) holds with $1 + (n-1)(|B - \frac{1}{2} + \frac{1}{2}) - d_F (1 + \sqrt{1 + 4B_1(1 - B_1)}) > 0$;

(vii) \((8.43)\) holds with $1 + (n-1)\frac{1 + \sqrt{1 - 4B}}{2} - d_F (1 + \sqrt{1 + 4B_1(1 - B_1)}) > 0$.

Let $u : \overline{D} \to N$ be an $F$-harmonic map with $l_F \geq \frac{1}{2}$ into an arbitrary Riemannian manifold $N$. If $u|_{\partial D}$ is constant, then $u|_{D}$ is constant.
Proof. Take $\omega = du$. Then $\omega|_{\partial D} = 0$. Hence $\omega$ satisfies an $F$-conservation law and annihilates any tangent vector $\eta$ of $\partial D$. The assertion follows at once from Theorem 12.1 and [17, Theorem 6.1]. □

Corollary 12.1. Suppose $M$ and $D$ satisfy the same assumptions of Theorem 12.2. Let $u : \overline{\Omega} \to N$ be a $p$-harmonic map ($p \geq 1$) into an arbitrary Riemannian manifold $N$. If $u|_{\partial D}$ is constant, then $u|_{D}$ is constant.

Proof. For a $p$-harmonic map $u$, we have $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$. Obviously $dF = lF = \frac{p}{2}$. Take $\omega = du$. This corollary follows immediately from Theorem 12.1 or Theorem 12.2. □

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