Elementary subgroups of relatively hyperbolic groups and bounded generation.

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Abstract

Let $G$ be a group hyperbolic relative to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$. We say that a subgroup $Q \leq G$ is hyperbolically embedded into $G$, if $G$ is hyperbolic relative to $\{H_\lambda, \lambda \in \Lambda\} \cup \{Q\}$. In this paper we obtain a characterization of hyperbolically embedded subgroups. In particular, we show that if an element $g \in G$ has infinite order and is not conjugate to an element of $H_\lambda, \lambda \in \Lambda$, then the (unique) maximal elementary subgroup contained $g$ is hyperbolically embedded into $G$. This allows to prove that if $G$ is boundedly generated, then $G$ is elementary or $H_\lambda = G$ for some $\lambda \in \Lambda$.

1 Introduction

Originally the notion of a relatively hyperbolic group was proposed by Gromov in order to generalize various examples of algebraic and geometric nature. Gromov’s idea has been elaborated by Bowditch \cite{4} in terms of the dynamics of group actions on hyperbolic spaces, and by Farb \cite{7} in terms of the geometry of Cayley graphs. Another definition of relative hyperbolicity of a group $G$ with respect to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$ is suggested in \cite{17}. In contrast to the definitions of Bowditch and Farb, this approach does not require the group $G$ and the subgroups $H_\lambda$ to be finitely generated, as well as the collection $\{H_\lambda, \lambda \in \Lambda\}$ to be finite. This is important for some applications (see \cite{13}) and, in particular, allows to include the small cancellation theory over free products developed in \cite{13} Ch. 5 within the general frameworks of the theory of relatively hyperbolic groups. On the other hand, in case the group $G$ is finitely generated our definition is equivalent to the definitions of Bowditch and Farb \cite{17}.

More precisely, let $G$ be a group, $\{H_\lambda, \lambda \in \Lambda\}$ a collection of subgroups of $G$, $X$ a subset of $G$. We say that $X$ is a relative generating set of $G$ with respect to $\{H_\lambda, \lambda \in \Lambda\}$ if $G$ is generated by the set $\left( \bigcup_{\lambda \in \Lambda} H_\lambda \right) \cup X$. (We always assume

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that $X$ is symmetrized, i.e. $X^{-1} = X$.) In this situation the group $G$ can be regarded as the quotient group of the free product

$$F = (\ast_{\lambda \in \Lambda} H_{\lambda}) \ast F(X),$$

(1)

where $F(X)$ is the free group with the basis $X$. Let $N$ denote the kernel of the natural homomorphism $F \to G$. If $N$ is a normal closure of a finite subset $\mathcal{R} \subseteq N$ in the group $F$, we say that $G$ has relative presentation

$$\langle X, H_{\lambda}, \lambda \in \Lambda \mid R = 1, \ R \in \mathcal{R} \rangle.$$  

(2)

If $\sharp X < \infty$ and $\sharp \mathcal{R} < \infty$, the relative presentation (2) is called finite and the group $G$ is called finitely presented relative to the collection of subgroups \{\(H_{\lambda}, \lambda \in \Lambda\}\).

Let $\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} (H_{\lambda} \setminus \{1\})$ (3)

(we regard $H_{\lambda}$ as subgroups of $F$ here). Given a word $W$ in the alphabet $X \cup \mathcal{H}$ such that $W$ represents 1 in $G$, there exists an expression

$$W = F \prod_{i=1}^{k} f_i^{-1} R_i f_i$$

(4)

with the equality in the group $F$, where $R_i \in \mathcal{R}$ and $f_i \in F$ for any $i$. The smallest possible number $k$ in a representation of type (4) is denoted by $\text{Area}_{rel}(W)$.

**Definition 1.1.** We say that a function $f : \mathbb{N} \to \mathbb{N}$ is a relative isoperimetric function of (2) if for any $n \in \mathbb{N}$ and any word $W$ over $X \cup \mathcal{H}$ of length $\|W\| \leq n$ representing the identity in the group $G$, we have $\text{Area}_{rel}(W) \leq f(n)$. The smallest relative isoperimetric function of (2) is called the relative Dehn function of $G$ with respect to \{\(H_{\lambda}, \lambda \in \Lambda\}\} and is denoted by $\delta_{rel} G, \{H_{\lambda}, \lambda \in \Lambda\}$ (or simply by $\delta_{rel}$ when the group $G$ and the collection of subgroups are fixed).

We note that $\delta_{rel}(n)$ is not always well–defined, i.e., it can be infinite for certain values of the argument, since the number of words of bounded relative length can be infinite. Indeed consider the group

$$G = \langle a, b \mid [a, b] = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$$

and the cyclic subgroup $H$ generated by $a$. Clearly $X = \{b^{\pm 1}\}$ is a relative generating set of $G$ with respect to $H$. It is easy to see that the word $W_n = [a^n, b]$ has length 4 as a word over $X \cup H$ for every $n$, but $\text{Area}_{rel}(W_n)$ grows linearly as $n \to \infty$. Thus we have $\delta_{rel}(4) = \infty$ in this case.

However if $\delta_{rel}$ is well–defined, it is independent of the choice of the finite relative presentation up to the following equivalence relation [17 Theorem 2.32]. Two functions $f, g : \mathbb{N} \to \mathbb{N}$ are called equivalent if there are positive constants $A, B, C$ such that $f(n) \leq Ag(Bn) + Cn$ and $g(n) \leq Af(Bn) + Cn$.  

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**Definition 1.2.** A group $G$ is hyperbolic relative to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$ if $G$ is finitely presented relative to $\{H_\lambda, \lambda \in \Lambda\}$ and the corresponding relative Dehn function is linear. In particular, a group is hyperbolic (in the ordinary non-relative sense) if and only if it is hyperbolic relative to the trivial subgroup.

The next theorem allows to regard Definition 1.2 as a generalization of previously known approaches. For details concerning Bowditch’s and Farb’s definition of relative hyperbolicity we refer the reader to [4, 7, 17].

**Theorem 1.3** ([17], Theorem 1.7). Let $G$ be a finitely generated group, $\{H_1, \ldots, H_m\}$ a collection of subgroups of $G$. Then the following conditions are equivalent.

1) $G$ is finitely presented with respect to $\{H_1, \ldots, H_m\}$ and the corresponding relative Dehn function is linear.

2) $G$ is hyperbolic with respect to the collection $\{H_1, \ldots, H_m\}$ in the sense of Farb and satisfies the Bounded Coset Penetration property (or, equivalently, $G$ is hyperbolic relative to $\{H_1, \ldots, H_m\}$ in the sense of Bowditch).

The set of groups which have a relatively hyperbolic structure includes fundamental groups of finite-volume non-compact Riemannian manifolds of pinched negative curvature, geometrically finite Kleinian groups, word hyperbolic groups, small cancellation quotients of free products, and many other examples. This paper continues the investigation initiated in [17]. It is the second article in the sequence of three and is supposed to establish a background for [18], where we use relative hyperbolicity to prove certain embedding theorems for countable groups.

**Definition 1.4.** Let $G$ be a group hyperbolic relative to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$. A subgroup $Q \leq G$ is said to be hyperbolically embedded into $G$ if $G$ is hyperbolic relative to $\{H_\lambda, \lambda \in \Lambda\} \cup \{Q\}$.

For every element $g \in G$, we denote by $|g|_{X \cup \mathcal{H}}$ its relative length that is the word length with respect to the generating set $X \cup \mathcal{H}$. Our main result is the following characterization of hyperbolically embedded subgroups.

**Theorem 1.5.** Suppose that $G$ is a group hyperbolic relative to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$ and $Q$ is a subgroup of $G$. Then $Q$ is hyperbolically embedded into $G$ if and only if the following conditions hold:

(Q1) $Q$ is generated by a finite set $Y$.

(Q2) There exist $\lambda, c \geq 0$ such that for any element $q \in Q$, we have $|q|_Y \leq \lambda |q|_{X \cup \mathcal{H}} + c$, where $|q|_Y$ is the word length of $q$ with respect to the generating set $Y$ of $Q$.

(Q3) For any $g \in G$ such that $g \notin Q$, we have $\sharp(Q \cap Q^g) < \infty$. 

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For ordinary hyperbolic groups the 'if' part of this theorem was proved in [4] (a weaker result is also obtained in [8]). Let us mention some corollaries of Theorem 1.5.

**Corollary 1.6.** If $Q$ is hyperbolically embedded into $G$, then $Q$ is a hyperbolic group.

Recall that a group is called *elementary* if it contains a cyclic subgroup of finite index. We also say that an element $g \in G$ is *parabolic* if it is conjugate to an element of $H_\lambda$ for some $\lambda \in \Lambda$. Otherwise $g$ is said to be *hyperbolic*. In Section 3 we notify that any hyperbolic element $g \in G$ of infinite order is contained in a unique maximal elementary subgroup of $G$, which is denoted by $E(g)$. Using results about cyclic subgroups of relatively hyperbolic groups proved in [17], we obtain the following corollary of Theorem 1.5.

**Corollary 1.7.** For any hyperbolic element $g \in G$ of infinite order, $E(g)$ is hyperbolically embedded into $G$.

This result can be applied to the study of boundedly generated relatively hyperbolic groups. A group $G$ is said to be *boundedly generated*, if there are elements $x_1, \ldots, x_n$ of $G$ such that for any $g \in G$ there exist integers $\alpha_1, \ldots, \alpha_n$ satisfying the equality $g = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Bounded generation is closely related to the Congruence Subgroup Property of arithmetic groups [19]. It is also interesting in connection with subgroup growth [12], unitary representations and Kazhdan Property (T) of discrete groups [2, 20]. Many lattices in semi–simple Lie groups of $\mathbb{R}$–rank at least 2 are known to be boundedly generated. For instance, Carter and Keller [5] established bounded generation for $SL_n(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a number field and $n \geq 3$ (see also [4] for an elementary proof in case $\mathcal{O} = \mathbb{Z}$ and [6, 11, 21] for other results).

In contrast if $\Gamma$ is a uniform lattice and $G$ has $\mathbb{R}$–rank 1, then $\Gamma$ is not boundedly generated. Indeed any such a group $\Gamma$ is non–elementary hyperbolic. The absence of bounded generation property for a non–elementary hyperbolic group immediately follows from the existence of infinite periodic quotients [15] (the direct proof can be found in [14]). On the other hand, the problem of whether non–uniform lattices in simple Lie groups of $\mathbb{R}$–rank 1 are boundedly generated was open until now. It is well known that any such a lattice is hyperbolic relative to maximal parabolic subgroups [7, 9]. Thus the following theorem answers this question negatively.

**Definition 1.8.** Let $G$ be a group hyperbolic relative to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$. We say that $G$ is *properly hyperbolic relative to $\{H_\lambda, \lambda \in \Lambda\}$, if each of the subgroups $H_\lambda$ is proper.

**Theorem 1.9.** Let $G$ be a non–elementary group properly hyperbolic relative to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$. Then $G$ is not boundedly generated.
2 Preliminaries

Hyperbolic spaces  Recall that a metric space $M$ is called hyperbolic (or, more precisely, $\delta$–hyperbolic) if for any geodesic triangle, each side of the triangle belongs to the union of the closed $\delta$–neighborhoods of the other two sides.

For a path $p$ in a metric space $M$, we denote by $p_-$ and $p_+$ the initial and the terminal points of $p$ respectively. All paths under consideration are assumed to be rectifiable (i.e., of finite length). The length of a path $p$ is denoted by $l(p)$.

A path $p$ in a metric space $M$ is called $(\lambda, c)$–quasi–geodesic for some $\lambda \geq 1, c \geq 0$, if for any subpath $q$ of $p$, we have

$$l(p) \leq \lambda \text{dist}_M(q_-, q_+) + c.$$ 

In our paper we will often use the following property of quasi–geodesic paths in hyperbolic spaces (see [10] or [9]).

Lemma 2.1. For any $\delta \geq 0, \lambda \geq 1, c \geq 0$, there exists $H = H(\delta, \lambda, c)$ such that for any $\delta$–hyperbolic space, any two $(\lambda, c)$–quasi–geodesic paths $p, q$ such that $p_- = q_-, p_+ = q_+$ are contained in the closed $H$–neighborhoods of each other.

Two paths $p, q$ in a metric space $M$ are called $k$–connected, if

$$\max\{\text{dist}_M(p_-, q_-), \text{dist}_M(p_+, q_+)\} \leq k.$$ 

The next lemma can easily be derived from the definition of a hyperbolic space by drawing the diagonal.

Lemma 2.2. Suppose that $p, q$ are $k$–connected geodesic paths in a $\delta$–hyperbolic space and $u$ is a point on $p$ such that

$$\min\{\text{dist}_M(u, p_-), \text{dist}_M(u, p_+)\} \geq 2\delta + k.$$ 

Then there exists a point $v$ on $q$ such that $\text{dist}_M(u, v) \leq 2\delta$.

From Lemma 2.1 and Lemma 2.2, we immediately obtain

Corollary 2.3. Suppose that $p, q$ are $k$–connected $(\lambda, c)$–quasi–geodesic paths in a $\delta$–hyperbolic space and $u$ is a point on $p$ such that

$$\min\{\text{dist}_M(u, p_-), \text{dist}_M(u, p_+)\} \geq H + 2\delta + k.$$ 

Then there exists a point $v$ on $q$ such that $\text{dist}_M(u, v) \leq 2(H + \delta)$.

The next lemma is a simplification of Lemma 10 from [15].

Lemma 2.4. Suppose that the set of all sides of a geodesic $n$–gon $P = p_1p_2\ldots p_n$ in a $\delta$–hyperbolic space is partitioned into two subsets $R$ and $S$. Let $\rho$ (respectively $\sigma$) denote the sum of lengths of sides from $R$ (respectively $S$). Assume, in addition, that $\sigma > \max\{\xi n, 10^3 \rho\}$ for some $\xi \geq 3\delta \cdot 10^4$. Then there exist two distinct sides $p_i, p_j \in S$ that contain $13\delta$–connected segments of length greater than $10^{-3}\xi$. 

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Relatively hyperbolic groups. We begin with a necessary conditions for relative Dehn functions to be well-defined.

Lemma 2.5 ([17], Theorem 1.6). Let $G$ be a group, $\{H_\lambda, \lambda \in \Lambda\}$ a collection of subgroups of $G$. Suppose that $G$ is finitely presented with respect to $\{H_\lambda, \lambda \in \Lambda\}$ and the Denh function of $G$ with respect to $\{H_\lambda, \lambda \in \Lambda\}$ is finite for all values of the argument. Then the following conditions hold.

1) For any $g \in G$, the intersection $H_\lambda^g \cap H_\mu$ is finite whenever $\lambda \neq \mu$.
2) The intersection $H_\lambda^g \cap H_\mu$ is finite for any $g \in H_\lambda$.

Let $G$ be a group generated by a (not necessarily finite) set $A$. Recall that the Cayley graph $\Gamma(G, A)$ of a group $G$ with respect to the set of generators $A$ is an oriented labelled 1–complex with the vertex set $V(\Gamma(G, A)) = G$ and the edge set $E(\Gamma(G, A)) = G \times A$. An edge $e = (g, a)$ goes from the vertex $g$ to the vertex $ga$ and has the label $\phi(e) = a$. As usual, we denote the origin and the terminus of the edge $e$, i.e., the vertices $g$ and $ga$, by $e^-$ and $e^+$ respectively.

One can regard $\Gamma(G, A)$ as a metric space assuming the length of any edge to be equal to 1 and taking the corresponding path metric. Also, it is easy to see that a word $W$ in $A$ represents 1 in $G$ if and only if some (or, equivalently, any) path in $\Gamma(G, A)$ labelled $W$ is a cycle.

Given a combinatorial path $p = e_1e_2\ldots e_k$ in the Cayley graph $\Gamma(G, A)$, where $e_1, e_2, \ldots, e_k \in E(\Gamma(G, A))$, we denote by $\phi(p)$ its label. By definition, $\phi(p) = \phi(e_1)\phi(e_2)\ldots\phi(e_k)$. We also denote by $p^- = (e_1)^-$ and $p^+ = (e_k)^+$ the origin and the terminus of $p$ respectively. A path $p$ is called irreducible if it contains no subpaths of type $ee^{-1}$ for $e \in E(\Gamma(G, A))$. The length $l(p)$ of $p$ is, by definition, the number of edges of $p$.

In the next three lemmas we suppose that $G$ is a group hyperbolic relative to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$. The next lemma follows, for example, from [17] Corollary 2.54.

Lemma 2.6. The Cayley graph $\Gamma(G, X \cup H)$ of $G$ with respect to the generating set $X \cup H$ is a hyperbolic metric space.

Lemma 2.7 ([17], Theorem 1.16). For any hyperbolic element $g \in G$ of infinite order, there exist positive constants $\lambda, c$ such that

$$|g^n|_{X \cup H} > \lambda|n| - c$$

for any $n \in \mathbb{N}$.

Lemma 2.8 ([17], Corollary 1.17). If $g \in G$ is hyperbolic and $f^{-1}g^mf = g^n$ for some $f \in G$, then $m = \pm n$.

$H_\lambda$–components. We are going to recall an auxiliary terminology introduced in [17], which plays an important role in our paper. As usual, by a cyclic word $W$ we mean the set of all cyclic shifts of $W$. A word $V$ is a subword of a cyclic word $W$ if $V$ is a subword of some cyclic shift of $W$.
For any Lemma 3.1. Let $p \in G, X$ we denote the natural metric on the Cayley graph $\Gamma(G, X \cup \mathcal{H})$. A subpath $p$ of $q$ is called an $H_\lambda$–component, if the label of $p$ is an $H_\lambda$–syllable of the the word $\phi(q)$ (respectively cyclic word $\phi(q)$).

**Definition 2.9.** Given a word $W$ (cyclic or not) in the alphabet $X \cup \mathcal{H}$, we say that a subword $V$ of $W$ is an $H_\lambda$–syllable if $V$ consists of letters from $H_\lambda \setminus \{1\}$ and is not contained in a bigger subword entirely consisting of letters from $H_\lambda \setminus \{1\}$. Let $q$ be a path (respectively cyclic path) in $\Gamma(G, X \cup \mathcal{H})$. A subpath $p$ of $q$ is called an $H_\lambda$–component, if the label of $p$ is an $H_\lambda$–syllable of the the word $\phi(q)$ (respectively cyclic word $\phi(q)$).

**Definition 2.10.** Two $H_\lambda$–components $p_1, p_2$ of a path $q$ (cyclic or not) in $\Gamma(G, X \cup \mathcal{H})$ are called connected if there exists a path $c$ in $\Gamma(G, X \cup \mathcal{H})$ that connects some vertex of $p_1$ to some vertex of $p_2$ and $\phi(c)$ is a word consisting of letters from $H_\lambda \setminus \{1\}$. In algebraic terms this means that all vertices of $p_1$ and $p_2$ belong to the same coset $gH_\lambda$ for some $g \in G$. Note that we can always assume that $c$ has length at most 1, as every element of $H_\lambda$ is included in the set of generators. An $H_\lambda$–component $p$ of a path $q$ (cyclic or not) is called isolated if no (distinct) $H_\lambda$–component of $q$ is connected to $p$.

The next lemma is a simplification of Lemma 2.27 from [17]. The subsets $\Omega_\lambda$ mentioned below are exactly the sets of all elements of $H_\lambda$ represented by $H_\lambda$–components of defining words $R \in \mathcal{R}$ in a suitably chosen finite relative presentation $\langle X, H_\lambda, \lambda \in \Lambda \mid R = 1, R \in \mathcal{R} \rangle$ of $G$.

**Lemma 2.11.** Suppose that $G$ is a group hyperbolic relative to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$. Then there exists a constant $K > 0$ and subsets $\Omega_\lambda \subseteq H_\lambda$ such that the following conditions hold.

1) The union $\Omega = \bigcup_{\lambda \in \Lambda} \Omega_\lambda$ is finite.

2) Let $q$ be a cycle in $\Gamma(G, X \cup \mathcal{H})$, $p_1, \ldots, p_k$ a set of isolated $H_\lambda$–components of $q$ for some $\lambda \in \Lambda$, $g_1, \ldots, g_k$ the elements of $G$ represented by the labels of $p_1, \ldots, p_k$ respectively. Then for any $i = 1, \ldots, k$, $g_i$ belongs to the subgroup $\langle \Omega_\lambda \rangle \leq G$ and the lengths of $g_i$ with respect to $\Omega_\lambda$ satisfy the inequality

$$\sum_{i=1}^{k} |g_i|_{\Omega_\lambda} \leq Kl(q).$$

**3 Hyperbolically embedded subgroups**

Throughout this section we fix a group $G$ hyperbolic relative to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$, a finite relative generating set $X = X^{-1}$ of $G$ with respect to $\{H_\lambda, \lambda \in \Lambda\}$, and the set $\Omega$ provided by Lemma 2.11. By $\text{dist}_{X \cup \mathcal{H}}$ we denote the natural metric on the Cayley graph $\Gamma(G, X \cup \mathcal{H})$.

**Lemma 3.1.** For any $\lambda \geq 1$, $c \geq 0$, there exists $\alpha_1 = \alpha_1(\lambda, c) > 0$ such that for any $k \geq 0$ there exists $\alpha_2 = \alpha_2(k, \lambda, c) > 0$ satisfying the following condition. Let $p, q$ be two $k$–connected $(\lambda, c)$–quasi–geodesics in $\Gamma(G, X \cup \mathcal{H})$, such that the labels of $p$ and $q$ are words in the alphabet $X$. Let $u$ be a vertex on $p$ such that

$$\min\{\text{dist}_{X \cup \mathcal{H}}(u, p_-), \text{dist}_{X \cup \mathcal{H}}(u, p_+)\} \geq \alpha_2.$$
Then there exists a vertex $v$ on $q$ such that the element $u^{-1}v$ belongs to the subgroup $\langle X \cup \Omega \rangle$ and the length of $u^{-1}v$ with respect to $X \cup \Omega$ satisfies the inequality $|u^{-1}v|_{X \cup \Omega} \leq \alpha_1$.

Proof. By Lemma 2.6, the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ is a hyperbolic metric space with respect to the metric $\text{dist}_{X \cup \mathcal{H}}$. Let $\delta$ denote the hyperbolicity constant of $\Gamma(G, X \cup \mathcal{H})$. We set

$$\alpha_2 = 5H + 6\delta + k,$$

where $H = H(\delta, \lambda, c)$ is the constant provided by Lemma 2.1.

Without loss of generality we may assume that $4(H + \delta)$ is integer. Since $\text{dist}_{X \cup \mathcal{H}}(p_-, u) \geq \alpha_2$, there exists a vertex $u_0$ on the segment $[p_-, u]$ such that

$$\text{dist}_{X \cup \mathcal{H}}(u_0, u) = 4(H + \delta)$$

(5)

Note that

$$\max\{\text{dist}_{X \cup \mathcal{H}}(p_-, u_0), \text{dist}_{X \cup \mathcal{H}}(p_+, u_0)\} \geq H + 2\delta + k.$$

Hence by Corollary 2.8 there exist points $v_0$ and $v$ on $q$ such that

$$\text{dist}_{X \cup \mathcal{H}}(u, v) \leq 2(H + \delta),$$

(6)

and

$$\text{dist}_{X \cup \mathcal{H}}(u_0, v_0) \leq 2(H + \delta).$$

(7)

Clearly we may assume that $v$ and $v_0$ are vertices of $p$. Let us consider the combinatorial loop

$$r = [u, u_0][u_0, v_0][v_0, v][u, v]^{-1},$$

where $[u_0, v_0]$ and $[u, v]$ are arbitrary geodesics in $\Gamma(G, X \cup \mathcal{H})$ and $[u, u_0], [v_0, v]$ are segments of $p^{-1}$ and $q$ (or $q^{-1}$) respectively. Obviously inequalities (5), (6) and (7) imply

$$l([v_0, v]) \leq \lambda \text{dist}_{X \cup \mathcal{H}}(v_0, v) + c \leq \lambda(\text{dist}_{X \cup \mathcal{H}}(u_0, v_0) + \text{dist}_{X \cup \mathcal{H}}(u_0, u) + \text{dist}_{X \cup \mathcal{H}}(u, v)) + c \leq 8\lambda(H + \delta) + c.$$
Therefore,

\[
l(r) \leq \lambda \text{dist}_{X \cup H}(u, u_0) + c + \text{dist}_{X \cup H}(u_0, v_0) + l([v_0, v]) + \text{dist}_{X \cup H}(v, u) \leq (12\lambda + 4)(H + \delta) + 2c.
\]

We are going to show that for any \( \lambda \in \Lambda \), any \( H_\lambda \)-component of \([u, v]\) is isolated in \( r \). Indeed assume that a certain \( H_\lambda \)-component \( t \) of \([u, v]\) is not isolated in \( r \). Since the labels \( \phi([u_0, u]) \) and \( \phi([v_0, v]) \) are words in \( X \), they contain no \( H_\lambda \)-components at all. Thus the only possibility is that there exists an \( H_\lambda \)-component \( s \) of \([u_0, v_0]\) that is connected to \( t \). This means that there exists a path \( e \) of length at most 1 in \( \Gamma(G, X \cup H) \) connecting \( s_- \) to \( t_- \) (see Fig. 1). Using (6) and (7) we obtain

\[
dist_{X \cup H}(u_0, u) \leq dist_{X \cup H}(u_0, s_-) + dist_{X \cup H}(s_-, t_-) + dist_{X \cup H}(t_-, u) \leq (dist_{X \cup H}(u_0, v_0) - 1) + 1 + (dist_{X \cup H}(v, u) - 1) < 4(H + \delta)
\]

which contradicts to (6).

Thus for any \( \lambda \in \Lambda \), any \( H_\lambda \)-component \( t \) of \([u, v]\) is isolated in \( r \). Therefore, by Lemma 2.1 we have \( t \in \langle X \cup \Omega \rangle \) and \( |t|_{X \cup \Omega} \leq KI(r) \), where \( K \) depends on \( G \) only. Combining these with (8) and (6), we obtain \( u^{-1}v \in \langle X \cup \Omega \rangle \) and

\[
|u^{-1}v|_{X \cup \Omega} \leq KI(r)l([u, v]) \leq 2K(H + \delta)((12\lambda + 4)(H + \delta) + 2c).
\]

It remains to set \( \alpha_1 \) to be equal to the right side of (9).

\[\square\]

In the next two lemmas \( Q \) is a subgroup of \( G \) satisfying conditions (Q1)–(Q3). Since \( \mathcal{Y} \prec \infty \), without loss of generality we may assume that \( Y \subseteq X \).

**Lemma 3.2.** For every \( \alpha > 0 \) there exists \( A = A(\alpha) > 0 \) such that the following holds. If \( a, b \in Q \) and \( f, g \in G \) are arbitrary elements such that \( \max\{|a|_Y, |b|_Y\} \geq A \), \( \max\{|f|_{X \cup \Omega}, |g|_{X \cup \Omega}\} \leq \alpha \), and \( a = fbg \), then \( f, g \in Q \).

**Proof.** Let \( \alpha_1 = \alpha_1(\lambda, c) \) and \( \alpha_2 = \alpha_2(\alpha, \lambda, c) \) be constants provided by Lemma 3.1 where \( \lambda \) and \( c \) are given by (Q2). We also denote by \( N \) the number of different elements of the subgroup \( \langle X \cup \Omega \rangle \) of length at most \( \alpha_1 \) with respect to the generating set \( X \cup \Omega \). Since \( \mathcal{Y}(X \cup \Omega) \prec \infty \), \( N \prec \infty \). By (Q3), there is an integer \( M > 0 \) such that for any element \( t \in \langle X \cup \Omega \rangle \) of length \( |t|_{X \cup \Omega} \leq \alpha_1 \), any element of \( Q \cap Q^t \) has length strictly less than \( M \) with respect to \( Y \) whenever \( t \notin Q \). Set

\[A = 2(\lambda \alpha_2 + c) + MN.\]

Increasing constants if necessary, we can assume that \( \lambda, c, \alpha_2, \) and \( M \) are integer. By the condition of the lemma there is a quadrangle \( rqsp^{-1} \) in \( \Gamma(G, X \cup \Omega) \).
we obtain $f$. By the choice of vertices \([t_i \in \Omega \setminus \{p, q\}] is at least \(p\) of \(G\) respectively) together with the following commutative diagram of homomorphisms.

\[ \begin{array}{ccc}
    a_0 & \cdots & a_i \\
    t_i \\
    b_i & \cdots & b_j \\
    q \\
    s \\
\end{array} \]

Figure 2:

$\mathcal{H}$, where \(r\) and \(s\) are geodesics in \(\Gamma(G, X \cup \mathcal{H})\) whose labels represent \(f\) and \(g\), and \(p, q\) are paths labelled by the shortest words in \(Y \subseteq X\) representing \(a\) and \(b\) respectively. Note that paths \(p\) and \(q\) are \((\lambda, c)\)-quasi–geodesic according to \((Q2)\). We take a vertex \(a_0\) on \(p\) such that the length of the segment \([p_-, a_0]\) of \(p\) equals \(\lambda \alpha_2 + c\). By the choice of \(A\) the length of the segment \([a_0, p_+]\) of \(p\) is at least \(\lambda \alpha_2 + c + MN\). Let \(a_0, \ldots, a_N\) denote the subsequent vertices of \([a_0, p_+]\) such that the length of the segment \([a_{k-1}, a_k]\) of \(p\) is equal to \(M\) for each \(k = 1, \ldots, N\). Note that for any \(i = 1, \ldots, N\),

\[
\min\{\text{dist}_{\mathcal{H}}(p_-, a_i), \text{dist}_{\mathcal{H}}(a_i, p_+)\} \geq \frac{\min\{l([p_-, a_i]), l([a_i, p_+])\} - c}{\lambda} \geq \alpha_2.
\]

Since \(l(r) = |f|_{\mathcal{H}}\) and \(l(s) = |g|_{\mathcal{H}}\), the paths \(p\) and \(q\) are \(\alpha\)-connected. By Lemma \(3.1\) there are vertices \(b_0, \ldots, b_N\) on \(q\) and paths \(t_0, \ldots, t_N\) such that \((t_k)_- = a_k, (t_k)_+ = b_k\), and \(\phi(t_k)\) represents an element of the subgroup \(\langle X \cup \Omega \rangle\) of length at most \(\alpha_1\) with respect to \(X \cup \Omega\) (see Fig. \(2\)). By our choice of the constant \(N\), there are two paths \(t_i, t_j\) such that \(\phi(t_j)\) and \(\phi(t_i)\) represent the same element \(t\) in \(G\). Reading the label of the cycle \(t_i [b_i, b_j] t_j^{-1} [a_i, a_j]^{-1}\), where \([b_i, b_j]\) is the segment of \(q\), gives us the equality \(t q t^{-1} = g_2\) for some \(q_1, q_2 \in Q\). By the choice of vertices \(a_0, \ldots, a_N\), we have \(|q_2|_Y = l([a_i, a_j]) \geq M\). Therefore, \(t \in Q\) according to the choice of \(M\). Since \(\phi([p_-, a_i] t [q_-, b_i]^{-1})\) represents the same element of \(G\) as \(\phi(r)\) and \(\phi([p_-, a_i]), \phi([q_-, b_i])\) represent elements of \(Q\), we obtain \(f \in Q\). Similarly \(g \in Q\).

Recall that \(G\) is generated by \(X\) relative to \(\{H_\lambda, \lambda \in \Lambda\}\), \(Q\) is generated by \(Y\), and \(Y \subseteq X\). Let \(Z = X \setminus Y\). We consider the groups

\[ F = (\ast_{\lambda \in \Lambda} H_\lambda) \ast F(Y) \ast F(Z) \]

and

\[ F_Q = (\ast_{\lambda \in \Lambda} H_\lambda) \ast Q \ast F(Z) \]

(here \(F(Y)\) and \(F(Z)\) stand for free groups freely generated by \(Y\) and \(Z\) respectively) together with the following commutative diagram of homomorphisms defined in the obvious way.
Since $G$ is hyperbolic with respect to $\{ H_\lambda, \lambda \in \Lambda \}$, there is a finite subset $R$ such that $\text{Ker } \beta = (R)^F$. Then obviously $\text{Ker } \gamma = \langle \varepsilon(R), R \in R \rangle^{F_Q}$. In particular, $G$ is finitely presented with respect to the collection $\{ H_\lambda, \lambda \in \Lambda \} \cup \{ Q \}$. To simplify our notation we denote by $Q$ alphabet $X \cup H \cup (Q \setminus \{1\})$.

For any word $T$ in the alphabet $X \cup H$ such that $T \in \text{Ker } \beta$, we denote by $\text{Area}^\text{rel}_F(T)$ the minimal possible number $k$ of factors in the decomposition

$$T = F \prod_{i=1}^k f_i^{-1} R_i f_i,$$

where $f_i \in F$ and $R_i \in R$, $i = 1, \ldots, k$. Similarly, given a word $W$ in $Q$ such that $W \in \text{Ker } \gamma$, we denote by $\text{Area}^\text{rel}_Q(W)$ the minimal possible number $k$ of factors in the decomposition

$$W = F_Q \prod_{i=1}^k g_i^{-1} \varepsilon(R_i) g_i,$$

where $g_i \in F_Q$ and $R_i \in R$, $i = 1, \ldots, k$. The next lemma is quite obvious and we leave the proof to the reader.

**Lemma 3.3.** Let $U, V, W$ be words in $Q$, $T$ a word in $X \cup H$. Suppose that $U, V \in \text{Ker } \gamma$ and $T \in \text{Ker } \beta$. Then:

(a) $\text{Area}^\text{rel}_Q(UV) \leq \text{Area}^\text{rel}_Q(U) + \text{Area}^\text{rel}_Q(V)$.

(b) $\text{Area}^\text{rel}_Q(W^{-1} VW) = \text{Area}^\text{rel}_Q(V)$.

(c) $\text{Area}^\text{rel}_Q(\varepsilon(T)) \leq \text{Area}^\text{rel}(T)$.

In the proof of Theorem 1.5 we will use the following auxiliary notion.

**Definition 3.4.** A word $W$ in $Q$ is called primitive if $W$ can not be decomposed as

$$W \equiv W_1 q_1 W_2 q_2 W_3,$$

where $q_1, q_2 \in Q \setminus \{1\}$ and the subword $W_2$ represents an element of the subgroup $Q$ in $G$.

**Lemma 3.5.** Suppose that there exists a constant $\kappa > 0$ such that for any primitive word $W$ in $Q$, $W \in \text{Ker } \gamma$, we have $\text{Area}^\text{rel}_{F_Q}(W) \leq \kappa \| W \|$. Then the relative Dehn function of $G$ with respect to $\{ H_\lambda, \lambda \in \Lambda \} \cup \{ Q \}$ is linear.
Proof. Let \( U \) be an arbitrary word in \( Q \) such that \( U \in Ker \gamma \). To prove the lemma it suffices to show that

\[
\text{Area}_{Q}^{rel}(U) \leq \kappa \|U\|. \tag{12}
\]

Let \( q(U) \) denote the number of letters from \( Q \setminus \{1\} \) that appear in \( U \). We proceed by induction on \( q(U) \). If \( q(U) = 0 \), the inequality \( \text{12} \) obviously holds since \( U \) is primitive in this case.

Now let \( q \) be non–primitive and \( q(U) > 0 \). Then \( U \equiv U_1q_1U_2q_2U_3 \), where \( q_1, q_2 \in Q \setminus \{1\} \) and \( \gamma(U_2) \in Q \). Let \( r_1, r_2 \) be letters from the alphabet \( Q \setminus \{1\} \) that represent the same elements as \( \gamma(U_2) \) and \( \gamma(q_1U_2q_2) \) in \( G \) respectively. In particular, we have \( \gamma(r_2) = G \gamma(q_1r_1q_2) \). However, the restriction of \( \gamma \) to \( Q \) is injective. Hence \( r_2 = q_1r_1q_2 \). Therefore,

\[
U =_{Q_1} (U_1q_1r_1q_2U_3)(U_3^{-1}q_2^{-1}r_1^{-1}U_2q_2U_3) =_{Q_2} (U_1r_2U_3)(U_3^{-1}q_2^{-1}r_1^{-1}U_2q_2U_3).
\]

Note that \( q(U_1r_2U_3) < q(U) \) and \( q(r_1^{-1}U_2) < q(U) \). Moreover,

\[
q(U_1r_2U_3) + q(r_1^{-1}U_2) = q(U_1) + 1 + q(U_3) + 1 + q(U_2) = q(U). \tag{13}
\]

Applying Lemma 3.3 together with inequality 13 and the inductive hypothesis, we obtain

\[
\text{Area}_{Q}^{rel}(U) \leq \text{Area}_{Q}^{rel}(U_1r_2U_3) + \text{Area}_{Q}^{rel}(r_1^{-1}U_2) \leq \kappa \|U_1r_2U_3\| + \kappa \|r_1^{-1}U_2\| \leq \kappa \|U\|.
\]

Now we are ready to prove the main theorem.

Proof of Theorem 1.5. First suppose that \( Q \) satisfies (Q1)–(Q3). We keep the notation introduces above. In particular, we assume that \( Y \) is a subset of \( X \).

Since \( G \) is hyperbolic relative to \( \{H_\lambda, \lambda \in \Lambda\} \), there exists \( L > 0 \) such that for any word \( V \) in \( X \cup H \) such that \( V \in Ker \beta \), we have \( \text{Area}_{rel}^{rel}(V) \leq Ln \). Let us take a word \( W \) in \( Q, W \in Ker \gamma \), of length \( \|W\| = n \). We want to bound \( \text{Area}_{Q}^{rel}(W) \) from above by a linear function of \( n \). Taking into account Lemma 3.3, we may assume that \( W \) is primitive. In case \( W \) contains no letters from \( Q \setminus \{1\} \), we immediately obtain

\[
\text{Area}_{Q}^{rel}(W) \leq Ln. \tag{14}
\]

In what follows we assume that at least one letter from \( Q \setminus \{1\} \) appears in \( W \). Let

\[
W \equiv W_0t_1W_1 \ldots t_lW_l,
\]

where \( W_0, W_l \in Q \setminus \{1\} \), \( t_1, \ldots, t_l \) are letters from \( Ker \gamma \), and \( W_1, \ldots, W_l \) are words in \( Q \). By induction on \( n \), we can reduce \( W \) to a primitive word in \( Q \) without \( Q \setminus \{1\} \). Then

\[
\text{Area}_{Q}^{rel}(W) \leq \text{Area}_{Q}^{rel}(W_0t_1W_1) + \sum_{i=2}^{l} \text{Area}_{Q}^{rel}(W_i) \leq Ln.
\]

Hence

\[
\text{Area}_{Q}^{rel}(W) \leq Ln.
\]
where \( t_i \in Q \setminus \{1\} \) and subwords \( W_0, \ldots, W_l \) contain no letters from \( Q \setminus \{1\} \).
For each \( t_i, i = 1, \ldots, l \), we fix a shortest word \( V_i \) in the alphabet \( Y \) such that \( V_i = Q_{t_i} \) and consider the word
\[
V = W_0 V_1 W_1 \ldots V_l W_l
\]
in the alphabet \( X \cup H \) regarded as an element of the group \( F \). Clearly \( \epsilon(V) = W \).

We set
\[
\theta = \sum_{i=1}^{l} \| V_i \|,
\]
\[
\omega = \sum_{i=1}^{l} \| W_i \|.
\]

Note that
\[
\omega < n. \quad (15)
\]

Also let \( \lambda, c \) be given by (Q2), \( \delta \) the hyperbolicity constant of \( \Gamma(G, X \cup H) \),
\( H = H(\delta, \lambda, c) \) the constant from Lemma 2.1, \( A = A(13\delta) \) the constant provided
by Lemma 3.2 and
\[
\xi = \max\{3\delta \cdot 10^4, (A + 2H) \cdot 10^3\}.
\]

There are three possibilities to consider.

a) First assume that \( \theta \leq (\lambda \xi + c)n \). Taking into account inequality (15), we obtain
\[
\| V \| = \theta + \omega < (\lambda \xi + c + 1)n.
\]
By Lemma 3.3 we have

\[ \text{Area}_Q^{rel}(W) \leq \text{Area}_Q^{rel}(V) < L(\lambda \xi + c + 1)n. \quad (16) \]

b) Further suppose that \( \theta \leq 10^3 \lambda \omega + cn \). Similarly to the previous case, we obtain

\[ \|V\| \leq 10^3 \lambda \omega + cn + \omega < (10^3 \lambda + c + 1)n \]

and

\[ \text{Area}_Q^{rel}(W) \leq \text{Area}_Q^{rel}(V) < L(10^3 \lambda + c + 1)n. \quad (17) \]

c) Finally, suppose that

\[ \theta > \max\{(\lambda \xi + c)n, 10^3 \lambda \omega + cn\}. \quad (18) \]

We consider a cycle \( q \) in the Cayley graph \( \Gamma(G, X \cup H) \) labelled \( V \). Let

\[ q = q_0 p_1 q_1 \ldots p_l q_l, \]

where \( q_i, p_i \) are subpaths labelled \( W_i \) and \( V_i \) respectively.

Consider the \( 2l + 1 \)-gon \( a_0 b_1 a_1 \ldots b_l a_l \), where \( a_i \) (respectively \( b_i \)) is a geodesic path with the same endpoints as \( p_i \) (respectively \( q_i \)). Let \( S = \{a_i, i = 1, \ldots, l\} \), \( R = \{b_i, i = 0, \ldots, l\} \). By \( \rho \) (respectively \( \sigma \)) we denote the sum of lengths of sides from \( R \) (respectively \( S \)). Obviously \( \rho \leq \omega \). Further, according to (Q2) \( p_i \) is \((\lambda, c)\)-quasi–geodesic in \( \Gamma(G, X \cup H) \) for any \( i = 1, \ldots, l \). In particular, we have

\[ l(a_i) \geq \frac{1}{\lambda}(\|V_i\| - c). \]

Hence,

\[ \sigma \geq \sum_{i=1}^{l} \frac{1}{\lambda}(\|V_i\| - c) = \frac{1}{\lambda}(\theta - cl) \geq \frac{1}{\lambda}(\theta - cn). \]

Together with (18), this yields

\[ \sigma > \max\{\xi n, 10^3 \omega\} \geq \max\{\xi n, 10^3 \rho\}. \]

By Lemma 2.4 there are two sides, say \( a_j \) and \( a_k, k > j \), having \( 13\delta \)-bounded segments of length at least \( \xi \cdot 10^{-3} \). Therefore, by Lemma 2.2, \( p_j \) and \( p_k \) have \((13\delta + 2H)\)-bounded segments \( s = [o_1, o_2], s' = [o_1', o_2'] \) of length at least \( \xi \cdot 10^{-3} - 2H \geq A \) (see Fig. 3). Set \( a = a_1^{-1}o_2, f = a_1^{-1}o_1', b = (o_1')^{-1}o_2, g = (o_2')^{-1}o_2 \). Obviously the elements \( a, b, f, g \) satisfy the requirements of Lemma 3.2. Thus \( f, g \in Q \). Let \((p_j)_+ = u, (p_k)_- = v \). Then

\[ u^{-1}v = (u^{-1}a_1)f^{-1}((o_1')^{-1}v) \in Q, \]

as labels of the segments \([u, a_1]\) and \([o_1, v]\) of \( p \) and \( q \) respectively are words in \( Y \) and hence represent elements of \( Q \).

Clearly the words

\[ W_0 t_1 \ldots W_{j-1} t_j \]

and

\[ W_0 t_1 \ldots W_{k-2} t_{k-1} W_{k-1} \]
represent \( u \) and \( v \) respectively in \( G \). Therefore, the subword
\[
W_j t_{j+1} \ldots W_{k-2} t_{k-1} W_{k-1}
\]
of \( W \) represents the element \( u^{-1}v \in Q \). However this contradicts to the assumption that \( W \) is primitive. Thus case c) is impossible for primitive \( W \).

Taking together (14), (16), and (17), we obtain
\[
\text{Area}^\text{rel}_Q(W) < \kappa n,
\]
where
\[
\kappa = \max\{L(10^3 \lambda + c + 1), L(\lambda \xi + c + 1)\}.
\]
This completes the proof of the first part of the theorem.

Now suppose that the subgroup \( Q \) is hyperbolically embedded into \( G \). By \( \Gamma(G, Q) \) we denote the Cayley graph of \( G \) with respect to \( Q \). Also let \( \Omega_Q \) denote the subset of \( Q \) given by Lemma 2.11 applied to the collection of subgroups \( \{H_\lambda, \lambda \in \Lambda\} \cup \{Q\} \).

For every nontrivial element \( q \in Q \), we fix a shortest word \( W \) over \( X \cup H \) that represents \( q \) in \( G \). Let us consider a cycle \( d = pr \) in \( \Gamma(G, Q) \), where \( p \) is an edge labelled \( q \) and \( \phi(r) = W^{-1} \). Clearly \( p \) is an isolated \( Q \)-component of \( q \) as \( r \) contains no edges labelled by elements of \( Q \). Applying Lemma 2.11 we obtain \( q \in \langle \Omega_Q \rangle \). Therefore \( \Omega_Q \) generates \( Q \). Moreover, we have
\[
|q|_{\Omega_Q} \leq K_Q l(d) \leq K_Q(\|W\| + 1) \leq K_Q(\|q|_{X \cup H} + 1),
\]
where \( K_Q \) is some constant independent of \( q \). Thus the conditions (Q1) and (Q2) hold. The fulfilment of (Q3) follows from Lemma 2.5.

Recall that a group is hyperbolic if it is finitely generated and its Cayley graph with respect to some finite generating set is a hyperbolic metric space.

**Proof of Corollary 1.6.** Let \( Y \) be a finite generating set of \( Q \). As above we assume that \( Y \) is a subset of \( X \). The inclusion \( Y \subseteq X \) defines the embedding of the Cayley graph
\[
\iota : \Gamma(Q, Y) \to \Gamma(G, X \cup H)
\]
whose restriction on \( Q = V(\Gamma(Q, Y)) \) is the identity map. If \( \Delta \) is a geodesic triangle in \( \Gamma(Q, Y) \), then \( \iota(\Delta) \) is a triangle in \( \Gamma(G, X \cup H) \) whose sides are \( (\lambda, c) \)-quasi–geodesic according to (Q2). Let \( v \) be a vertex on a side of \( \Delta \). If \( \delta \) is the hyperbolicity constant of \( \Gamma(G, X \cup H) \), then there is a vertex \( w \) on the union of the other two sides of \( \Delta \) such that
\[
|v^{-1}w|_{X \cup H} = \text{dist}_{X \cup H}(\iota(v), \iota(w)) \leq 2H + \delta,
\]
where \( H = H(\delta, \lambda, c) \) is the constant from Lemma 2.1. Therefore, by (Q2) the distance between \( v \) and \( w \) in \( \Gamma(Q, Y) \) is
\[
|v^{-1}w|_Y \leq \lambda|v^{-1}w| + c \leq \lambda(2H + \delta) + c.
\]
This shows that \( \Gamma(Q, Y) \) is \( (\lambda(2H + \delta) + c + 1) \)-hyperbolic.
4 Elementary subgroups and bounded generation

All assumptions and notation listed at the beginning of the previous section remain valid here. We begin with auxiliary results about elementary subgroups of relatively hyperbolic groups.

**Lemma 4.1.** For any hyperbolic element of infinite order \( g \in G \), there exists a constant \( C = C(g) \) such that if \( f^{-1}g^n f = g^n \) for some \( f \in G \) and some \( n \in \mathbb{N} \), then there are \( m \in \mathbb{Z} \) and \( h \in \langle X \cup \Omega \rangle \) such that \( f = h g^m \) and \( |h|_{X \cup \Omega} \leq C \).

**Proof.** Without loss of generality we may assume that \( g \in X \). Let \( \lambda, c \) be constants from Lemma 2.7 and \( \alpha_1 = \alpha_1(\lambda, c), \alpha_2 = \alpha_2(k, \lambda, c) \) be constants provided by Lemma 3.1 where \( k = |f|_{X \cup \Omega} \).

Since \(fg^n = g^n f\), we have \( f g^n t = g^n f t \) for any \( t \in \mathbb{Z} \). Consider the \( k \)-connected paths \( p, q \) in \( \Gamma(G, X \cup \mathcal{H}) \) labelled \( g^n t \) such that \( q_- = 1, q_+ = g^n t, p_- = f, p_+ = f g^n t = g^n t f \). If \( t \) is big enough, there is a vertex \( u \) on \( p \) such that \( u = f g^n j \) for some \( j \in \mathbb{Z} \) and

\[
\min\{\text{dist}_{X \cup \mathcal{H}}(p-, u), \text{dist}_{X \cup \mathcal{H}}(p+, u)\} \geq \alpha_2.
\]

By Lemma 3.1 there exists a vertex \( v \) on \( q \) such that \( |v^{-1}u|_{X \cup \Omega} \leq \alpha_1 \). Note that \( v = g^i \) for some \( i \in \mathbb{Z} \). We consider the vertex \( w = g^n j \) on \( q \). The equality \( f g^n j = g^n j f \) implies \( f = w^{-1} u = (w^{-1} v)(v^{-1} u) = g^{i-n} h \) for \( h = v^{-1} u \). \( \square \)

**Definition 4.2.** For any hyperbolic element of infinite order \( g \in G \), we set

\[
E(g) = \{ f \in G : f^{-1} g^n f = g^{\pm n} \text{ for some } n \in \mathbb{N} \}.
\]

**Theorem 4.3.** Every hyperbolic element \( g \in G \) is contained in a unique maximal elementary subgroup, namely in \( E(g) \).

**Proof.** Let

\[
E_+(g) = \{ f \in G : f^{-1} g^n f = g^n \text{ for some } n \in \mathbb{N} \}.
\]

By Lemma 4.1 \( \langle g \rangle \) has finite index in \( E_+(g) \). Clearly \( E_+(g) \) has index 2 in \( E(g) \). Hence \( |E(g) : \langle g \rangle| < \infty \). In particular, \( E(g) \) is elementary.

It remains to show that if \( E \) is another elementary subgroup containing \( g \) then \( E \leq E(g) \). Let \( s \) be an element of \( E \) such that \( \langle s \rangle \) has finite index in \( E \). Passing to a subgroup \( \langle s^i \rangle \) for some \( i \) if necessary, we may assume that \( \langle s \rangle \) is normal in \( E \). Obviously \( s^l = g^k \) for some \( k, l \in \mathbb{Z} \setminus \{0\} \). In particular, \( s \) is hyperbolic. Indeed if \( s \in H_\Lambda^a \) for some \( \Lambda \in \Lambda, a \in G \), then the cyclic subgroup \( \langle s^l \rangle \) is contained in the intersection \( H_\Lambda^a \cap H_\Lambda^a = (H_\Lambda^a g a^{-1} \cap H_\Lambda) a \) as \( s^l \) commutes with \( g \). Therefore, \( H_\Lambda g a^{-1} \cap H_\Lambda \) is infinite. By Lemma 2.4 \( aga^{-1} \in H_\Lambda \) that

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Figure 4:

contradicts to hyperbolicity of $g$. Thus $s$ is hyperbolic and for any element $t \in E$, we have $t^{-1}st = s^{\pm 1}$ by Corollary 2.8. Hence 

$$t^{-1}g^kt = t^{-1}s^lt = s^{\pm l} = g^{\pm k}.$$ 

By the definition of $E(g)$, we have $t \in E(g)$.

**Proof of Corollary 1.7.** Let us show that conditions (Q1)–(Q3) hold for $Q = E(g)$. The fulfillment of (Q1) is obvious. Let $Y$ be a finite generating set of $E(g)$ containing $g$. Since $\langle g \rangle$ has finite index in $E(g)$, there is a constant $D > 0$ such that any element $t \in E(g)$ can be represented as $t = hg^m$ for some $m \in \mathbb{Z}$, where $\max\{|h|_{X \cup H}, |h|^Y \} \leq D$. Using Lemma 2.7 we obtain 

$$|t|^Y \leq |g^m|^Y + D \leq |m| + D \leq \frac{1}{\lambda} |g^m|_{X \cup H} + c + D \leq \frac{1}{\lambda} (|t|_{X \cup H} + D) + c + D$$

for some positive $\lambda, c$. Thus (Q2) holds.

Finally if the intersection $E(g)^f \cap E(g)$ is infinite for some $f \in G$, then it contains $g^n$ for some $n \in \mathbb{Z} \setminus \{0\}$. Hence $f^{-1}g^nf = g^k$ for some $k \in \mathbb{Z} \setminus \{0\}$. By Corollary 2.8, $k = \pm n$ and thus $f \in E(g)$.

**Lemma 4.4.** For any $\lambda \in \Lambda$ there is a finite subset $F_\lambda \subseteq H_\lambda$ such that if $h \in H_\lambda \setminus F_\lambda$, $a \in G \setminus H_\lambda$, and $|a|_{X \cup H} = 1$, then $ah$ is a hyperbolic element of infinite order.

**Proof.** Set 

$$F_\lambda = \{f \in (\Omega_\lambda), |f|^\Omega_\lambda \leq 5K\},$$

where $K$ and $\Omega_\lambda$ are given by Lemma 2.11. Since $a \in G \setminus H_\lambda$ has relative length 1, we can think of $a$ is a letter from $X \cup H$. For every $m \in \mathbb{N}$, we consider a path $p_m = q_1r_1q_2r_2 \ldots q_mr_m$, where $q_i$ (respectively $r_i$) is labelled $a$ (respectively $h$), $i = 1, \ldots, m$.

Note that $r_1, \ldots, r_m$ are $H_\lambda$–components of $p_m$. First of all we are going to show that they are isolated. Indeed suppose that $r_i$ is connected to $r_j$ for some $j > i$ and $j - i$ is minimal possible. Let $s$ denote the segment $[(r_i)^+, (r_j)^-]$ of $p_m$, and let $e$ be a path of length at most 1 in $\Gamma(G, X \cup H)$ labelled by an element of $H_\lambda$ such that $e_- = (r_i)^+, e_+ = (r_j)^-$ (see Fig. 4). If $j = i + 1$, then $\phi(s) = a$. Since $\phi(s)$ and $\phi(e)$ represent the same element in $G$, we arrive
at a contradiction with the assumption \( a \notin H_\lambda \). Therefore, \( j = i + 1 + k \) for some \( k \geq 1 \). Note that the components \( r_{i+1}, \ldots, r_{i+k} \) are isolated in the cycle \( se^{-1} \). (Otherwise we can pass to another pair of connected \( H_\lambda \)-components with smaller value of \( j - i \).) By Lemma 2.11 we have \( h \in \langle \Omega_\lambda \rangle \) and
\[
k|\lambda|_{\Omega_\lambda} \leq Kl(se^{-1}) = K(2k + 2).
\]
Hence \( |h|_{\Omega_\lambda} \leq K(2 + 2/k) \leq 4K \) which contradicts to the choice of \( h \). Thus all components \( r_1, \ldots, r_m \) are isolated in \( p_m \). In particular, this means that the element \( ah \) has infinite order. Indeed if \( (ah)^n = 1 \) for some \( n \in \mathbb{N} \), then the components \( r_1 \) and \( r_{n+1} \) of \( p_{n+1} \) coincide (and thus they are connected).

Let us show that \( ah \) is hyperbolic. Indeed suppose that \( ah = b^{-1}fb \) for some \( b \in G, f \in H_\nu, \nu \in \Lambda \). We take \( m = 4|b|_{\mathcal{X} \cup \mathcal{H}} + 6 \). Let \( B \) be a shortest word in \( X \cup \mathcal{H} \) representing \( b \) in \( G \). Consider a cycle \( d = p_m q_m \) in \( \Gamma(G, X \cup \mathcal{H}) \) such that \( \phi(q_m) \equiv (B^{-1} t B)^{-1} \), where \( t \) is the letter from \( H_\nu \setminus \{1\} \) representing the same element as \( f^m \) in \( G \). Note that \( r_2, \ldots, r_{m-1} \) are components of \( d \) and any \( H_\lambda \)-component of the subpath \( q_m \) of \( d \) is connected to at most one \( H_\lambda \)-component from the set \( \{r_2, \ldots, r_m m - 1\} \). (If an \( H_\lambda \)-component of \( q_m \) is connected to \( r_i \) and \( r_j \), then \( r_i \) and \( r_j \) are connected.) As the total number of \( H_\lambda \)-components in \( q_m \) does not exceed \( l(q_m) \leq 2|b|_{X \cup \mathcal{H}} + 1 \), at least \( m - 2|b|_{X \cup \mathcal{H}} - 3 = m/2 \) components from the set \( \{r_2, \ldots, r_{m-1}\} \) are isolated. Lemma 2.11 yields
\[
\frac{1}{2} m|b|_{\Omega_\lambda} \leq Kl(d) \leq K(2m + 2|b|_{X \cup \mathcal{H}} + 1) < \frac{5}{2} mK.
\]
Dividing by \( m/2 \), we obtain \( |h|_{\Omega_\lambda} \leq 5K \). This contradicts to \( h \notin F_\lambda \). The lemma is proved.

**Corollary 4.5.** Let \( G \) be an infinite group properly hyperbolic relative to a collection of subgroups \( \{H_\lambda, \lambda \in \Lambda\} \). Then \( G \) contains a hyperbolic element of infinite order.

**Proof.** Removing trivial subgroups from the set \( \{H_\lambda, \lambda \in \Lambda\} \) if necessary, we may assume that \( H_\lambda \neq \langle 1 \rangle \) for any \( \lambda \in \Lambda \). First suppose that \( \sharp \Lambda = \infty \). Since \( G \) is defined by a finite presentation \( \mathcal{P} \) with respect to \( \{H_\lambda, \lambda \in \Lambda\} \), there is a finite subset \( \Lambda_0 \subseteq \Lambda \) such that no relators from \( \mathcal{R} \) involve letters from \( H_\lambda \) for \( \lambda \in \Lambda \setminus \Lambda_0 \). Therefore \( G = G_0 \ast \langle \ast_{\lambda \in \Lambda \setminus \Lambda_0} H_\lambda \rangle \), where \( G_0 \) is the subgroup of \( G \) generated by all \( H_\lambda, \lambda \in \Lambda_0 \). If \( \lambda_1, \lambda_2 \in \Lambda \setminus \Lambda_0, \lambda_1 \neq \lambda_2 \), then for any two nontrivial elements \( g_1 \in H_{\lambda_1}, g_2 \in H_{\lambda_2} \), the product \( g_1 g_2 \) is hyperbolic and has infinite order.

Now suppose that \( \sharp \Lambda < \infty \). If all subgroups \( H_\lambda \) are finite, then \( \Gamma(G, X \cup \mathcal{H}) \) is locally finite. As \( \Gamma(G, X \cup \mathcal{H}) \) is hyperbolic by Lemma 2.6, \( G \) is a hyperbolic group (in the ordinary non–relative sense). It is well–known that any infinite hyperbolic group contains an element of infinite order \( g \). Obviously \( g \) is hyperbolic in this case, as all parabolic elements of \( G \) have finite orders.

Finally if there is an infinite subgroup in the collection \( \{H_\lambda, \lambda \in \Lambda\} \), the desired hyperbolic element exists by Lemma 2.11. The element \( a \in G \setminus H_\lambda \) of
relative length 1 mentioned in Lemma 4.4 exists as $G$ is generated by elements of relative length 1 and $H_\lambda \neq G$.

Corollary 4.3 together with Lemma 2.7 immediately imply the following.

**Corollary 4.6.** If $G$ is infinite and properly hyperbolic relative to a collection of subgroups $\{H_\lambda, \lambda \in \Lambda\}$, then $G$ has infinite diameter with respect to the metric $\text{dist}_{X \cup H}$.

Finally let us prove Theorem 1.9.

**Proof of Theorem 1.9.** We assume that $G$ is non–elementary and is properly hyperbolic relative to $\{H_\lambda, \lambda \in \Lambda\}$. Suppose that there are elements $x_1, \ldots, x_n$ of $G$ such that for any $g \in G$ there exist integers $\alpha_1, \ldots, \alpha_n$ satisfying the equality $g = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. According to Corollary 1.7, if $x_i$ is a hyperbolic element of infinite order for some $i$, then $G$ is properly hyperbolic relative to the collection $\{H_\lambda, \lambda \in \Lambda\} \cup \{E(x_i)\}$. In case $x_i$ has finite order, $G$ is properly hyperbolic relative to the collection $\{H_\lambda, \lambda \in \Lambda\} \cup \{\langle x_i \rangle\}$ by Theorem 1.5, as any finite subgroup satisfies (Q1)–(Q3). Thus joining new subgroups to the collection $\{H_\lambda, \lambda \in \Lambda\}$ if necessary, we may assume that elements $x_1, \ldots, x_n$ are hyperbolic, i.e., for any $i = 1, \ldots, n$, $x_i = a_i^{-1} h_i a_i$, where $h_i \in H_\lambda$, for some $\lambda_i \in \Lambda$.

Then for any integers $\alpha_1, \ldots, \alpha_n$, we have

$$x_1^{\alpha_1} \ldots x_n^{\alpha_n} = a_1 h_1^{\alpha_1} a_1^{-1} \ldots a_n h_n^{\alpha_n} a_n^{-1}.$$ 

Therefore, for every element $g = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, we have

$$|g|_{X \cup H} \leq \sum_{i=1}^n (2|\alpha_i|_{X \cup H} + 1).$$

This means that $G$ has finite diameter with respect to $\text{dist}_{X \cup H}$ contradictory to Corollary 4.6. □

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