Maps on random hypergraphs and random simplicial complexes

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Abstract

Let $L$ be a simplicial complex. In this paper, we study random sub-hypergraphs and random sub-complexes of $L$. By considering the minimal complex that a sub-hypergraph can be embedded in and the maximal complex that can be embedded in a sub-hypergraph, we define some maps on the space of probability functions on sub-hypergraphs of $L$. We study the compositions of these maps as well as their actions on the space of probability functions.

Keywords. Hypergraphs, Simplicial complexes, Randomness, Probability

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1 Introduction

Random topological objects, for example, random graphs, random simplicial complexes, random hypergraphs, etc., have important applications in large-data systems in computer science and engineering. Among these random topological objects, random graphs are the simplest case. The systematic study of random graphs was started by P. Erdös and A. Rényi and E.N. Gilbert around 1960.

Let $0 \leq p \leq 1$. In 1959, P. Erdös and A. Rényi and E.N. Gilbert constructed the Erdös-Rényi model $G(n, p)$ of random graphs by choosing each pair of vertices in $V$ as an edge uniformly and independently at random with probability $p$. In 1960, thresholds for the connectivity of $G(n, p)$ were given in [15]. In recent decades, the clique complex of $G(n, p)$ was studied in [9, 21].

Random simplicial complexes are higher-dimensional generalizations of random graphs. In 2006, N. Linial and R. Meshulam constructed the Linial-Meshulam model $Y_2(n, p)$ of random 2-complexes. They take the complete graph on $V$ and choose each 2-simplex of the complete complex on $V$ uniformly and independently at random with probability $p$. The fundamental group of $Y_2(n, p)$ was studied in [3]. The homology groups of $Y_2(n, p)$ were studied in [5, 6]. The asphericity and the hyperbolicity of $Y_2(n, p)$ were studied in [7, 8].

Let $d$ be a non-negative integer. In 2009, R. Meshulam and N. Wallach generalized $Y_2(n, p)$ and constructed a model $Y_d(n, p)$ of random $d$-complexes. They take the $(d-1)$-skeleton of the complete complex on $V$, then choose each $d$-simplex of the complete complex on $V$ uniformly and independently at random with probability $p$. The homology groups of $Y_d(n, p)$ were studied in [2, 20, 24]. The cohomology of $Y_d(n, p)$ was studied in [23]. Some thresholds for the homology of $Y_d(n, p)$ were given in [26]. The eigenvalues of the Laplacian on $Y_d(n, p)$ were studied in [17]. The collapsibility property of $Y_d(n, p)$ was studied in [11, 23]. And some sub-structures of $Y_d(n, p)$ were studied in [15].

Let $0 \leq r \leq n-1$ be an integer. Let $0 \leq p_0, p_1, \ldots, p_{n-1} \leq 1$. Let $p = (p_0, p_1, \ldots, p_{n-1})$. In 2016, $G(n, p)$, $Y_2(n, p)$, $Y_d(n, p)$ and (the $r$-skeleton of) the clique complex of $G(n, p)$ were generalized universally to a multi-parameter model of random complexes with probability function $P_{n,r,p}$ by A. Costa and M. Farber. In 2017, the fundamental group of the final-generated complexes has been studied in [11]. The dimension has been studied in [12].

Hypergraphs can be obtained by deleting certain faces of simplicial complexes. In this paper, we construct the model of random hypergraphs as an generalization of random simplicial complexes. We investigate certain maps on random hypergraphs and prove Theorem 1.1. With the helps of the maps, we study the relations between random hypergraphs and random simplicial complexes and prove Theorem 1.2.

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The remaining part of this paper is organized as follows. In Section 1.1 we give an outline of the main results. In Section 2 we study the map algebra generated by the maps. In Section 3 we give some geometric characterizations of certain compositions of the maps. In Section 4 we prove Theorem 1.1 and Theorem 1.2.

1.1 An outline of the main results

Let \( n \geq 2 \) be an integer. Let \( V \) be a set of \( n \) points. The power set of \( V \), denoted as \( 2^V \), is the collection of all subsets of \( V \). We assume that the emptyset is not in \( 2^V \) if there is no extra claim. A hypergraph \( H \) on \( V \) is a subset of \( 2^V \). In particular, \( 2^V \) is called the complete hypergraph and the emptyset \( \emptyset \) is called the empty hypergraph. An element of \( H \) is called a hyperedge, and an element of a hyperedge is called a vertex. The dimension of a hyperedge is the cardinality of the hyperedge minus one. A hyperedge of dimension \( d \) is called a \( d \)-hyperedge for short. The vertex set of \( H \), denoted as \( V_H \), is the subset of \( V \) consisting of all vertices of all hyperedges in \( H \). A hypergraph \( H' \) on \( V \) is said to be a sub-hypergraph of \( H \) if \( H' \subseteq H \). An (abstract) simplicial complex \( K \) on \( V \) is a hypergraph on \( V \) such that for any \( \sigma \subseteq K \) and any nonempty \( \tau \subseteq \sigma \), \( \tau \subseteq K \). The hyperedges of \( K \) are called simplices. A simplicial complex \( K' \) is said to be a sub-complex of \( K \) if \( K' \subseteq K \). Given a sub-complex \( K' \subseteq K \), a \( d \)-clique of \( K' \) in \( K \) is a \( d \)-simplex \( \sigma \subseteq K \) such that for any \( \tau \subseteq \sigma \), \( \tau \subseteq K' \).

Let \( L \) be a finite simplicial complex. Let \( \mathcal{H}(L) \) be the collection of all sub-hypergraphs of \( L \). A random sub-hypergraph of \( L \) is a probability function on \( \mathcal{H}(L) \). Let \( D(\mathcal{H}(L)) \) be the functional space of all probability functions on \( \mathcal{H}(L) \). Let \( Map(\mathcal{H}(L)) \) be the semigroup of all self-maps on \( \mathcal{H}(L) \). An element \( T \in Map(\mathcal{H}(L)) \) induces a self-map \( DT \) on \( D(\mathcal{H}(L)) \) by

\[
DT(f)(H) = \sum_{H' = H} f(H'),
\]

for any \( f \in D(\mathcal{H}(L)) \) and any \( H \in \mathcal{H}(L) \). And a map \( F \) from \( \mathcal{H}(L) \times 2 \) to \( \mathcal{H}(L) \) induces a map \( DF \) from \( D(\mathcal{H}(L)) \times 2 \) to \( D(\mathcal{H}(L)) \) by

\[
DF(f_1, f_2)(H) = \sum_{F(H_1, H_2) = H} f_1(H_1)f_2(H_2),
\]

for any \( f_1, f_2 \in D(\mathcal{H}(L)) \) and any \( H \in \mathcal{H}(L) \). Let \( \mathcal{K}(L) \) be the collection of all sub-complexes of \( L \). Let \( Map(\mathcal{K}(L)) \) be the semigroup of all self-maps on \( \mathcal{K}(L) \). A random sub-complex of \( L \) is a probability function on \( \mathcal{K}(L) \). Let \( D(\mathcal{K}(L)) \) be the functional space of all probability functions on \( \mathcal{K}(L) \). Similar to \( \mathcal{H}(L) \), an element of \( Map(\mathcal{K}(L)) \) induces a self-map on \( D(\mathcal{K}(L)) \) and a map from \( \mathcal{H}(L) \) to \( \mathcal{K}(L) \) induces a map from \( D(\mathcal{H}(L)) \) to \( D(\mathcal{K}(L)) \). And similar to \( \mathcal{H}(L) \), a map \( F \) from \( \mathcal{K}(L) \times 2 \) to \( \mathcal{K}(L) \) induces a map \( DF \) from \( D(\mathcal{K}(L)) \times 2 \) to \( D(\mathcal{K}(L)) \).

**Definition 1.**\(^{10}^{13}\) Let \( \Delta_n \) denote the complete simplicial complex on \( n \) vertices. Let \( \Delta_n^{(r)} \) be the \( r \)-skeleton of \( \Delta_n \). An external face of a sub-complex \( Y \subseteq \Delta_n \) is a simplex \( \sigma \subseteq \Delta_n \) such that \( \sigma \notin Y \) but the boundary of \( \sigma \) is contained in \( Y \). We use \( E(Y) \) to denote the set of all external faces of \( Y \). Let \( \mathbf{p} = (p_0, p_1, \ldots, p_r) \) with \( 0 \leq p_i \leq 1 \). We consider the probability space \( \mathcal{K}(\Delta_n^{(r)}) \). The probability function is

\[
P_{n,r,p}(Y) = \prod_{\sigma \in Y, \; \dim \sigma \leq r} p_{\dim \sigma} \cdot \prod_{\sigma \notin E(Y), \; \dim \sigma \leq r} (1 - p_{\dim \sigma}).
\]

The probability function \( P_{n,r,p} \) can be obtained as follows (cf. \( \text{[22] Section 5.5]} \):

(i). We generate the 0-skeleton by choosing each vertex of \( \Delta_n \) uniformly and independently at random with probability \( p_0 \).

(ii). For each \( 0 \leq k \leq r - 2 \), suppose the \( k \)-skeleton is generated. Then we generate the \((k+1)\)-skeleton by choosing each \((k+1)\)-clique of the \( k \)-skeleton in \( \Delta_n^{(r)} \) uniformly and independently at random with probability \( p_{k+1} \).
(iii). The final-generated complexes have probability function $P_{n,r,p}$. Thus

$$\sum_{Y \subseteq \Delta^{|r|}} P_{n,r,p}(Y) = 1.$$ 

Let $p : L \to [0,1]$ be an arbitrary function. In the next definition, we generalize Definition 4 and give a model of random sub-complex in $L$.

**Definition 2** (Generalization of Definition 1). An external face of a sub-complex $Y \subseteq L$ is a simplex $\sigma \in L$ such that $\sigma \notin Y$ but the boundary of $\sigma$ is contained in $Y$. We use $E(Y)$ to denote the set of all external faces of $Y$ in $L$. We consider the probability space $\mathcal{K}(L)$. The probability function is given by

$$P_{L,p}(Y) = \prod_{\sigma \in Y} p(\sigma) \cdot \prod_{\sigma \in E(Y)} (1 - p(\sigma)).$$

In particular, suppose $\dim L = r$ and there exists $0 \leq p_0, p_1, \ldots, p_r \leq 1$ such that for each $\sigma \in L$, $p(\sigma) = p_{\dim \sigma}$. Then we denote $P_{L,p}$ as $P_{L,p}$. We have $P_{\Delta^{|r|},p} = P_{n,r,p}$.

The random complex model in Definition 2 can be generated as follows:

(i). Choose each vertex $v \in L$ independently at random with probability $p(v)$.

(ii). For each $0 \leq k \leq \dim L - 1$, suppose the $k$-skeleton is generated. Then we generate the $(k+1)$-skeleton by choosing each $(k+1)$-clique $\sigma$ of the $k$-skeleton in $L$ independently at random with probability $p(\sigma)$.

(iii). The final-generated complexes have the probability function $P_{L,p}$. Hence

$$\sum_{Y \subseteq L} P_{L,p}(Y) = 1.$$ 

In the next definition, we consider an analogue of Definition 2 and give a model of random sub-hypergraph in $L$.

**Definition 3** (Hypergraphic analogue of Definition 2). We consider the probability space $\mathcal{H}(L)$. The probability function is given by

$$P_{L,p}(H) = \prod_{\sigma \in H} p(\sigma) \cdot \prod_{\sigma \notin H} (1 - p(\sigma)).$$

In particular, suppose $\dim L = r$ and there exists $0 \leq p_0, p_1, \ldots, p_r \leq 1$ such that for each $\sigma \in L$, $p(\sigma) = p_{\dim \sigma}$. Then we denote $P_{L,p}$ as $P_{L,p}$.

The random hypergraph in Definition 3 can be generated as follows. We choose each simplex $\sigma \in L$ independently at random with probability $p(\sigma)$. We obtain a hypergraph.

The probability function of these independent trials is $P_{L,p}$. Therefore,

$$\sum_{H \subseteq L} P_{L,p}(H) = 1.$$ 

We let $H \in \mathcal{H}(L)$. We study the minimal complex $\Delta H$ that $H$ can be embedded in, the maximal complex $\delta H$ that can be embedded in $H$, and the complement hypergraph $\gamma H$ in $L$. By composing $\Delta, \delta$ and $\gamma$ iteratively, we obtain a sub-semigroup $G$ of $\text{Map}(\mathcal{H}(L))$. And $G$ induces a semi-group $DG$ of self-maps on $D(\mathcal{H}(L))$. Moreover, by composing $\Delta \gamma$ and $\delta \gamma$ iteratively, we obtain a sub-semigroup $G'$ of $\text{Map}(\mathcal{K}(L))$. And $G'$ induces a semi-group $DG'$ of self-maps on $D(\mathcal{K}(L))$. We study the map algebra acting on $D(\mathcal{H}(L))$ induced from $\Delta, \delta$ and $\gamma$, and the map algebra acting on $D(\mathcal{K}(L))$ induced from $\Delta \gamma$ and $\delta \gamma$. In
particular, we give some explicit expressions for the actions of the map algebra on $\hat{P}_{L,p}$ and $P_{L,p}$. As consequences, we give algorithms generating large sparse random hypergraphs with probability function $F_{\Delta_n,p}$, and algorithms generating large sparse random simplicial complexes with probability function $P_{\Delta_n,p}$.

Let $\sigma \in L$. The characteristic probability $\varphi_{\sigma}$ is the function

$$\varphi_{\sigma}(\sigma') = \begin{cases} 0, & \text{if } \sigma' \neq \sigma; \\ 1, & \text{if } \sigma' = \sigma. \end{cases}$$

A path $s$ in $L$ is a sequence of simplices $\sigma_1 \sigma_2 \ldots \sigma_m$ in $L$ such that the intersection of any two consecutive simplices is nonempty. We call $m$ the length of $s$. Given two simplices $\sigma, \sigma' \in L$, the distance between $\sigma$ and $\sigma'$ is

$$d(\sigma, \sigma') = \min \{ m \mid s = \sigma_1 \sigma_2 \ldots \sigma_m \text{ is a path in } L, \sigma_1 = \sigma, \sigma_m = \sigma' \}.$$ 

The diameter of $L$ is $\text{diam}L = \max_{\sigma, \sigma' \in L} d(\sigma, \sigma')$. Let $m = \max_{\sigma, \sigma' \in \text{max}(L)} d(\sigma, \sigma')$. The first main result of this paper is the next Theorem.

**Theorem 1.1** (Main Result I). Let $k$ be a non-negative integer. Let $\text{Ext} = \Delta \gamma \delta \gamma$ and $\text{Int} = \delta \gamma \Delta \gamma$. Let $f \in D(\mathcal{H}(L))$.

(a). If $k \geq \text{diam}L$, then $(D\text{Ext})^k(f) = f(\emptyset)\varphi_{\emptyset} + (1 - f(\emptyset))\varphi_{L}$.

(b). If $k \geq \text{diam}L$, then $(D\text{Int})^k(f) = (1 - f(L))\varphi_{\emptyset} + f(L)\varphi_{L}$.

(c). There exists $f \in D(\mathcal{H}(L))$ such that

$$f, (D\text{Ext})(f), (D\text{Ext})^2(f), \ldots, (D\text{Ext})^{m-1}(f), (D\text{Ext})^m(f)$$

are distinct;

(d). There exists $f \in D(\mathcal{H}(L))$ such that

$$f, (D\text{Int})(f), (D\text{Int})^2(f), \ldots, (D\text{Int})^{m-1}(f), (D\text{Int})^m(f)$$

are distinct;

(e). Let $k \geq 1$. Then for any probability function $f \in D(\mathcal{H}(L))$, the probability that $\text{Ext}^{k-1}(\gamma H) \subseteq \gamma \text{Int}^k(\gamma H)$ and the probability that $\text{Int}^k(\gamma H) \subseteq \text{Ext}^{k+1}(\gamma H)$ are 1;

(f). For any probability function $f \in D(\mathcal{H}(L))$, the probability that $\Delta H \subseteq \text{Int} \circ \text{Ext}(H)$ is greater than or equal to $\sum_{V_H \subseteq H} f(H)$ where the condition $V_H \subseteq H$ means that each vertex of $H$ is a 0-hyperedge.

Consider the spaces of probability functions

$$F(\mathcal{H}(L)) = \{ \hat{P}_{L,p} \mid p : L \longrightarrow [0,1] \},$$

$$F(\mathcal{K}(L)) = \{ P_{L,p} \mid p : L \longrightarrow [0,1] \}.$$ 

Then $F(\mathcal{H}(L))$ is a subspace of $D(\mathcal{H}(L))$ and $F(\mathcal{K}(L))$ is a subspace of $D(\mathcal{K}(L))$. Let $\cap$ and $\cup$ be the intersection and the union of hypergraphs. The second main result of this paper is the next theorem.

**Theorem 1.2** (Main Result II). The map $D\gamma$ maps $F(\mathcal{H}(L))$ to itself. The maps $D\Delta$ and $D\delta$ map $F(\mathcal{H}(L))$ to $F(\mathcal{K}(L))$. The map $D\cap$ maps $F(\mathcal{H}(L))^{\times 2}$ to $F(\mathcal{H}(L))$, and maps $F(\mathcal{K}(L))^{\times 2}$ to $F(\mathcal{K}(L))$. And the map $D\cup$ maps $F(\mathcal{H}(L))^{\times 2}$ to $F(\mathcal{H}(L))$. Precisely,

(a). $D\gamma$ sends $\hat{P}_{L,p}$ to $\hat{P}_{L,1-p}$;
(b). $D\Delta$ sends $\mathcal{P}_{L,p}$ to a random sub-simplicial complex of $\mathcal{L}$ given by

$$(D\Delta(\mathcal{P}_{L,p}))(K) = \left( \prod_{\tau \in \max(K)} p(\tau) \right) \left( \prod_{\tau \notin K} (1 - p(\tau)) \right)$$

for any $K \in \mathcal{K}(L)$;

(c). $D\delta$ sends $\mathcal{P}_{L,p}$ to a random sub-simplicial complex of $\mathcal{L}$ given by

$$(D\delta(\mathcal{P}_{L,p}))(K) = \sum_{\delta \in K, \sigma \in H} p(\sigma) \prod_{\sigma \notin H} (1 - p(\sigma))$$

for any $K \in \mathcal{K}(L)$;

(d). $D\cap$ sends the pair $(\mathcal{P}_{L,p}', \mathcal{P}_{L,p}'')$ to $\mathcal{P}_{L,r};$

(e). $D\cup$ sends the pair $(\mathcal{P}_{L,p}', \mathcal{P}_{L,p}'')$ to $\mathcal{P}_{L,r,1-(1-p')(1-p'')}$. 

2 Map algebras on hypergraphs and simplicial complexes

In this section, we study the minimal complex $\Delta H$ that $H$ can be embedded in, the maximal complex $\delta H$ that can be embedded in $H$, and the complement hypergraph $\gamma H$ in $L$. We study the map algebras of the compositions of $\Delta, \delta$ and $\gamma$ as well as the intersections and unions. We also study the restrictions of the compositions of $\Delta, \delta$ and $\gamma$ on simplicial complexes.

2.1 The map algebra on hypergraphs

We consider the maps $\Delta, \delta : \mathcal{H}(L) \rightarrow \mathcal{K}(L)$, and $\gamma : \mathcal{H}(L) \rightarrow \mathcal{H}(L)$ given by

$$\Delta H = \{ \sigma \in L | \text{there exists } \tau \in H \text{ such that } \sigma \subseteq \tau \};$$

$$\delta H = \{ \sigma \in L | \text{for any } \tau \subseteq \sigma, \tau \in H \};$$

$$\gamma H = \{ \sigma \in L | \sigma \notin H \}$$

for any $H \in \mathcal{H}(L)$. Then (i). $\gamma^2 = \text{id};$ (ii). $\Delta \delta = \delta;$ (iii). $\delta \Delta = \Delta;$ (iv). $\Delta^2 = \Delta;$ (v). $\delta^2 = \delta;$ (vi). $(\Delta \gamma \Delta \gamma)^2 = \Delta \gamma \Delta \gamma;$ (vii). $(\delta \gamma \delta \gamma)^2 = \delta \gamma \delta \gamma$. The equalities (i) - (v) are straight-forward. Let max($L$) be the set of all maximal faces of $L$. We prove (vi) and (vii).

Proof of (vi). Let $H \in \mathcal{H}(L)$. Then

$$\Delta \gamma \Delta \gamma H = \Delta \gamma \Delta \{ \sigma \in L | \sigma \notin H \}$$

$$\Delta \gamma \Delta \{ \sigma \in L | \text{there exists } \tau \subseteq \sigma \text{ such that } \tau \notin H \}$$

$$\Delta \{ \sigma \in L | \text{there does not exist any } \sigma \subseteq \tau \text{ such that } \tau \notin H \}$$

$$\Delta \{ \sigma \in L | \text{for any } \sigma \subseteq \tau, \tau \in H \}$$

$$\Delta \{ \sigma \in \text{max}(L) | \sigma \in H \}$$

$$\Delta \{ \text{max}(L) \cap H \}.$$ (2.1)

Thus

$$(\Delta \gamma \Delta \gamma)^2 H = \Delta \{ \text{max}(L) \cap \Delta \{ \text{max}(L) \cap H \} \} = \Delta \{ \text{max}(L) \cap H \}.$$ 

Since $H$ is arbitrary, we have (vi). \qed
Proof of (vii). Let $H \in \mathcal{H}(L)$. Then

\[
\delta \gamma \delta H = \delta \gamma \delta \{ \sigma \in L \mid \sigma \notin H \} = \delta \gamma \{ \sigma \in L \mid \text{for any } \tau \subseteq \sigma, \tau \notin H \} = \delta \{ \sigma \in L \mid \text{there exists } \tau \subseteq \sigma, \tau \in H \} = \{ \sigma \in L \mid \text{for any } \sigma' \subseteq \sigma, \text{there exists } \tau \subseteq \sigma', \tau \in H \} = \{ \sigma \in L \mid \text{for any vertex } v \text{ of } \sigma, \{ v \} \in H \}.
\]

Hence $\delta \gamma \delta H$ is the sub-complex of $L$ spanned by all the 0-hyperedges in $H$. And $(\delta \gamma \delta)^2 H$ is the sub-complex of $L$ spanned by all the 0-hyperedges in $\delta \gamma \delta H$. Since the 0-hyperedges of $H$ and the 0-hyperedges of $\delta \gamma \delta H$ are same, we have $(\delta \gamma \delta)^2 H = \delta \gamma \delta H$. Since $H$ is arbitrary, we have (vii).

Let $G$ be the semi-group generated by $\Delta, \delta, \gamma$ modulo the relations (i) - (vii). The multiplication of $G$ is the composition of maps. The unit of $G$ is id, the identity map on $\mathcal{H}(L)$. There are four types of elements in $G$: (1). $\gamma x_1 \gamma x_2 \ldots \gamma x_k \gamma$; (2). $x_1 \gamma x_2 \ldots \gamma x_k \gamma$; (3). $\gamma x_1 \gamma x_2 \ldots \gamma x_k$; (4). $x_1 \gamma x_2 \ldots \gamma x_k$. In (1) - (4), $k$ is a nonnegative integer, $x_i = \Delta$ or $\delta$, and there does not exist any 4 consecutive $x_i$'s that take the same value. For any $w_1, w_2 \in G$ and any $H_1, H_2 \in \mathcal{H}(L)$, let

\[
(w_1 + w_2)(H_1, H_2) = w_1(H_1) \cup w_2(H_2);
\]

\[
(w_1 \wedge w_2)(H_1, H_2) = w_1(H_1) \cap w_2(H_2).
\]

For any positive integer $t$, let

\[
G^t = \{ (\ldots (w_1 \ast w_2) \ast \ldots \ast w_t) \mid \ast = \land \lor +, w_1, w_2, \ldots, w_t \in G
\]

with any $t - 2$ brackets $\ast$ giving the order of evaluation}$\}

For any $W \in G^t$, $W$ is a map from $\mathcal{H}(L)^{t \times 1}$ to $\mathcal{H}(L)$. Some relations among $w \in G$, $+$ and $\wedge$ are:

(I). $(w_1 \wedge w_2) \wedge w_3 = w_1 \wedge (w_2 \wedge w_3)$;

(II). $(w_1 + w_2) + w_3 = w_1 + (w_2 + w_3)$;

(III). $\gamma (w_1 + w_2) = (\gamma w_1) \wedge (\gamma w_2)$, or equivalently, $\gamma (w_1 \wedge w_2) = \gamma w_1 + \gamma w_2$;

(IV). $\Delta (w_1 + w_2) = (\Delta w_1) + (\Delta w_2)$;

(V). $\delta (w_1 \wedge w_2) = \delta w_1 \wedge \delta w_2$.

Here $w_1, w_2, w_3 \in G$, and $0$ is the constant map sending $\mathcal{H}(L)$ to $\emptyset$. (I) - (III) are straightforward. Let $H, H' \in \mathcal{H}(L)$. Let $H_1 = w_1(H)$ and $H_2 = w_2(H')$. We prove (IV) and (V).

Proof of (IV). In order to prove (IV), we only need to prove that for any $H_1, H_2 \in \mathcal{H}(L)$, $\Delta (H_1 \cup H_2) = \Delta H_1 \cup \Delta H_2$. Let $\sigma \subseteq L$. Then $\sigma \in \Delta (H_1 \cup H_2)$ iff. there exists $\tau \in H_1 \cup H_2$ such that $\sigma \subseteq \tau$. This happens iff. there exists $\tau \in H_1$ such that $\sigma \subseteq \tau$ or there exists $\tau \in H_2$ such that $\sigma \subseteq \tau$. Hence $\sigma \in \Delta (H_1 \cup H_2)$ iff. $\sigma \in \Delta H_1$ or $\sigma \in \Delta H_2$, that is, $\sigma \in \Delta H_1 \cup \Delta H_2$.

Proof of (V). In order to prove (V), we only need to prove that for any $H_1, H_2 \in \mathcal{H}(L)$, $\delta (H_1 \cap H_2) = \delta H_1 \cap \delta H_2$. Let $\sigma \subseteq L$. Then $\sigma \in \delta (H_1 \cap H_2)$ iff. for any $\tau \subseteq \sigma, \tau \in H_1 \cap H_2$. This happens iff. for any $\tau \subseteq \sigma, \tau \in H_1$ and $\tau \in H_2$. That is, $\sigma \in \delta H_1 \cap \delta H_2$. ☐
Lemma 3.1. Let \( \sigma \in \mathcal{L} \), maximal face of \( \text{Ext}(H) \) and \( \delta(H_1 \cup H_2) \geq \delta H_1 \cup \delta H_2 \). Hence for any \( H, H' \in \mathcal{H}(L) \),

\[
(\Delta(w_1 \wedge w_2)) (H, H') \subseteq (\Delta w_1 \wedge \Delta w_2) (H, H'), \\
(\delta(w_1 + w_2)) (H, H') \supseteq (\delta w_1 + \delta w_2) (H, H').
\]

2.2 The map algebra on simplicial complexes

Let \( w \in G \). A subset \( S \) of \( \mathcal{H}(L) \) is called an invariant subspace of \( \mathcal{H}(L) \) if for any \( H \in S \), \( w(H) \in S \). We consider \( w \in G \) such that \( \mathcal{K}(L) \) is an invariant subspace of \( w \). The collection of all such \( w \) forms a subgroup \( G_1 \) of \( G \). Since both \( \Delta \) and \( \delta \) act on \( \mathcal{K}(L) \) identically, we take an equivalent relation \( \sim \) identifying both \( \Delta \) and \( \delta \) as the unit element of \( G_1 \). We denote the quotient group \( G_1/ \sim \) as \( G' \).

Precisely, \( G' \) can be constructed as follows. Let \( \alpha = \Delta \gamma \) and \( \beta = \delta \gamma \). Let \( G' \) be the semi-group generated by \( \alpha \) and \( \beta \) modulo the relations (i)' \( \alpha^4 = \alpha^2 \); (ii)' \( \beta^4 = \beta^2 \). The multiplication of \( G' \) is the composition of maps. The unit of \( G' \) is \( \alpha \), the identity map on \( \mathcal{K}(L) \). The elements in \( G' \) are: (1)' \( \alpha^m, \beta^n \) \( \alpha^m \beta^n \) \( \beta^n \alpha^m \) \( \alpha^m \beta^n \alpha^k \beta^n \); (2)' \( \beta^n \alpha^m \) \( \beta^n \alpha^m \beta^n \alpha^k ; (3)' \beta^n \alpha^m \beta^n \alpha^k \beta^n \); \( \beta^n \alpha^m \beta^n \alpha^k \). Here \( k \) is a nonnegative integer and \( 1 \leq m, n, i \leq 3, i = 1, 2, \ldots \). Similar to Subsection 2.1, we define + and \( \wedge \) for the elements in \( G' \). We construct \( G'' \) for all positive integer \( t \). Each element \( W^t \in G'' \) is a map from \( \mathcal{K}(L)^{\times t} \) to \( \mathcal{K}(L) \).

3 Some characterizations of the maps

Let \( \text{Ext} = \Delta \gamma \delta \gamma \) and \( \text{Int} = \delta \gamma \Delta \gamma \). In this section, we study the properties of the maps \( \text{Ext} \) and \( \text{Int} \). We give some geometric characterizations of \( \text{Ext} \) and \( \text{Int} \). In Subsection 3.1, we use paths in complexes to characterize the powers of \( \text{Ext} \) and \( \text{Int} \). In Subsection 3.2, we use neighborhoods of sub-complexes to study \( \text{Ext} \), \( \text{Int} \) and their compositions.

3.1 Powers of maps and paths

Let \( H \in \mathcal{H}(L) \). Let \( \text{Ext}(H) = \Delta \gamma \delta \gamma (H) \). Then

\[
\text{Ext}(H) = \Delta \{ \tau \in L \mid \text{there exists } \sigma \in H \text{ such that } \sigma \subseteq \tau \} \\
= \{ \tau \in L \mid \text{there exists } \tau' \subseteq \tau \text{ such that there exists } \sigma \in H \text{ with } \sigma \subseteq \tau' \}.
\]

Hence \( \text{Ext}(H) \) is the sub-complex of \( L \) obtained by extending each hyperedge \( \sigma \) of \( H \) to a maximal face of \( L \) containing \( \sigma \). We call \( \text{Ext}(H) \) the extension of \( H \). We notice that every maximal face of \( \text{Ext}(H) \) is in \( \text{max}(L) \), and

\[
\text{Ext}(H) = \Delta \{ \text{max}(L) \cap \text{Ext}(H) \} \\
= \Delta \{ \tau_1 \in \text{max}(L) \mid \text{there exists } \sigma \in H \text{ such that } \sigma \subseteq \tau_1 \}. \quad (3.1)
\]

For any \( k \geq 2 \), by an induction on \( k \) and (3.1),

\[
\text{Ext}^k(H) = \Delta \{ \tau_k \in \text{max}(L) \mid \text{there exists } \tau_1, \tau_2, \ldots, \tau_{k-1} \in \text{max}(L) \text{ and } \sigma \in H \text{ such that } \tau_i \cap \tau_{i-1} \neq \emptyset \text{ for any } 2 \leq i \leq k \text{ and } \sigma \subseteq \tau_1 \}. \quad (3.2)
\]

A path \( s = \sigma_1 \sigma_2 \ldots \sigma_m \) in \( L \) is called a broad path if for each \( 1 \leq i \leq m, \sigma_i \) is a maximal face of \( L \). For any path \( s \) in \( L \), if we extend each \( \sigma \) of \( s \) to be a maximal face \( \tau \in \text{max}(L) \) such that \( \sigma \subseteq \tau \), then we obtain a broad path \( s' \).

Lemma 3.1. Let \( \sigma, \sigma' \in \text{max}(L) \) with \( d(\sigma, \sigma') = n \). Then there exists a broad path of length \( n \) starting from \( \sigma \) and ending at \( \sigma' \).
Proof. Since \( d(\sigma, \sigma') = n \), there exists a path \( s = \tau_1 \tau_2 \ldots \tau_n \) in \( L \) such that \( \sigma = \tau_1 \) and \( \sigma' = \tau_n \). For each \( 1 \leq i \leq n \), we extend \( \tau_i \) to be a maximal face \( \tau'_i \) of \( L \). Then we obtain the broad path \( s' = \tau'_1 \ldots \tau'_n \).

Let \( \text{Int}(H) = \delta \gamma \Delta \gamma(H) \). Then with the help of (2.1),

\[
\text{Int}(H) = \delta \{ \tau \in L \mid \text{for any } \tau \subseteq \sigma, \sigma \in H \} \\
= \{ \tau \in L \mid \text{for any } \tau' \subseteq \sigma \text{ and } \tau' \subseteq \sigma, \sigma \in H \} \\
= \{ \tau \in L \mid \text{for any } \sigma \text{ with } \sigma \cap \tau = \tau' \neq \emptyset, \sigma \in H \}. 
\]

(3.3)

Hence \( \text{Int}(H) \) is the sub-complex of \( L \) consisting of all the hyperedges \( \tau \in H \) such that for any \( \sigma \in \gamma H, \sigma \cap \tau \) is empty. We call \( \text{Int}(H) \) the interior of \( H \). It follows from (3.3) that

\[
\text{Int}(H) = \{ \tau \in L \mid \text{for any } \sigma \in \gamma H, \sigma \cap \tau = \emptyset \} \\
= \gamma \{ \tau \in L \mid \text{there exists } \sigma \in \gamma H \text{ such that } \tau \cap \sigma \neq \emptyset \}. 
\]

(3.4)

For any \( k \geq 1 \), by an induction on \( k \) and (3.3),

\[
\text{Int}^k(H) = \gamma \{ \tau_k \in L \mid \text{there exists } \sigma \in \gamma H \text{ and } \tau_1, \tau_2, \ldots, \tau_{k-1} \in L \} \\
\text{such that } \tau_1 \cap \sigma \neq \emptyset \text{ and } \tau_1 \cap \tau_1 \neq \emptyset \text{ for any } 2 \leq i \leq k \}. 
\]

(3.5)

Lemma 3.2. Let \( \sigma, \sigma' \) be simplices of \( L \) with \( d(\sigma, \sigma') = n \). If \( s = \sigma_1, \ldots, \sigma_n \) is a path in \( L \) with \( \sigma_1 = \sigma, \sigma_n = \sigma' \) and \( |s| = n \), then for any \( 1 \leq i < j \leq n, \) \( d(\sigma_i, \sigma_j) = j - i \).

Proof. Suppose to the contrary, there exists \( 1 \leq i < j \leq n \) such that \( d(\sigma_i, \sigma_j) < j - i \). Let \( s_{i,j} \) be the path \( \sigma_1 \ldots, \sigma_j \) as a subset of \( s \). Then \( |s_{i,j}| = j - i + 1 \). And we can find a path \( s'_{i,j} \) starting from \( \sigma_i \) and ending at \( \sigma_j \) with \( |s'_{i,j}| < j - i + 1 \). Replacing \( s_{i,j} \) with \( s'_{i,j} \), we obtain a new path \( s' \) starting from \( \sigma \) and ending at \( \sigma' \) with \( |s'| < |s| \). This contradicts that \( s \) is the path starting from \( \sigma \) and ending at \( \sigma' \) with the minimal length.

Lemma 3.3. Let \( H \in \mathcal{H}(L) \).

(a). Suppose \( H \neq \emptyset \). Then for any \( k \geq 1 \), \( \max(L) \cap \text{Ext}^k(H) \) is the union of all the broad paths \( s = \tau_1 \ldots \tau_k \) in \( L \) such that there exists \( \sigma \in H \) with \( \sigma \subseteq \tau_1 \).

(b). Suppose \( H \neq L \). Then for any \( k \geq 1 \), \( \gamma \text{Int}^k(H) \) is the union of all the paths \( s' = \tau'_1 \ldots \tau'_k \) in \( L \) such that there exists \( \sigma' \in \gamma H \) with \( \tau'_1 \cap \sigma' \neq \emptyset \).

Proof. Assertion (a) follows from (3.2). Assertion (b) follows from (3.3).

In the next proposition, we list some properties of the powers of Ext and Int.

Proposition 3.4. Let \( H \in \mathcal{H}(L) \), \( k \) be a nonnegative integer and \( n \) be the diameter of \( L \). Let \( m = \max_{\sigma, \sigma' \in \max(L)} d(\sigma, \sigma') \).

(a). If \( H \neq \emptyset \) and \( k \geq n \), then \( \text{Ext}^k(H) = L \).

(b). If \( H \neq L \) and \( k \geq n \), then \( \text{Ext}^k(H) = \emptyset \).

(c). There exists \( H \) such that \( H \subseteq \text{Ext}(H) \subseteq \text{Ext}^2(H) \subseteq \ldots \subseteq \text{Ext}^{m-1}(H) \subseteq L \).

(d). There exists \( H \) such that \( H \supseteq \text{Int}(H) \supseteq \text{Int}^2(H) \supseteq \ldots \supseteq \text{Int}^{m-1}(H) \neq \emptyset \).

(e). Let \( k \geq 1 \). Then \( \text{Ext}^{k-1}(\gamma H) \subseteq \gamma \text{Int}^k(H) \subseteq \text{Ext}^{k+1}(\gamma H) \).

Proof. Let \( k \geq n \). Then for any simplices \( \sigma_1, \sigma_2 \in L \), there exists a path \( s \) of length \( n \) starting from \( \sigma_1 \) and ending at \( \sigma_2 \).

(a). Suppose \( H \neq \emptyset \). Then by Lemma 3.3 (a), any \( \sigma \in \max(L) \) is in \( \text{Ext}^k(H) \). Thus \( \max(L) \cap \text{Ext}^k(H) = L \). Thus \( \text{Ext}^k(H) = L \).
(b) Suppose \( H \neq L \). Then by Lemma 3.3(b), any simplex \( \sigma \in L \) is in \( \gamma \text{Int}^k(H) \). Thus \( \text{Int}^k(H) = \emptyset \).

Choose \( \sigma, \sigma' \in \text{max}(L) \) such that \( d(\sigma, \sigma') = m \). Choose a broad path \( s = \sigma_1 \ldots \sigma_m \) in \( L \) with \( \sigma_1 = \sigma \) and \( \sigma_m = \sigma' \).

(c) Let \( H = \{\sigma\} \). Then by Lemma 3.2 and Lemma 3.3(a), for any \( 0 \leq i < m - 1 \), \( \sigma_j \in \text{max}(L) \cap \text{Ext}^i(H) \) for any \( j \leq i + 1 \), and \( \sigma_j \notin \text{max}(L) \cap \text{Ext}^i(H) \) for any \( j < i + 2 \).

Hence for any \( 0 \leq i < m - 1 \), \( \text{Ext}^i(H) \subseteq \text{Ext}^{i+1}(H) \).

(d) Let \( H = \gamma\{\sigma\} \). Then by Lemma 3.2 and Lemma 3.3(b), for any \( 0 \leq i < m - 1 \), \( \sigma_j \in \text{Int}^i(H) \) for any \( j \leq i + 2 \), and \( \sigma_j \notin \text{Int}^i(H) \) for any \( j < i + 1 \). Hence for any \( 0 \leq i < m - 1 \), \( \text{Int}^i(H) \supseteq \text{Int}^{i+1}(H) \).

(e) Let \( \tau_{k-1} \in \text{Ext}^{k-1}(\gamma H) \). Then there exists \( \sigma_{k-1} \in \text{max}(L) \cap \text{Ext}^{k-1}(\gamma H) \) such that \( \tau_{k-1} \subseteq \sigma_{k-1} \). By Lemma 3.3(a), there exists a broad path \( \sigma \ldots \sigma_{k-1} \) of length \( k - 1 \) and \( \tau \in \gamma H \) where \( \sigma \) is a maximal face of \( L \) such that \( \tau \subseteq \sigma \). Consider the path \( s = \sigma \ldots \sigma_{k-1} \tau_{k-1} \) of length \( k \) in \( L \). Since \( \sigma \cap \tau = \tau \neq \emptyset \), by Lemma 3.3(b), \( \tau_{k-1} \in \gamma \text{Int}^k(H) \). Hence \( \text{Ext}^{k-1}(\gamma H) \subseteq \gamma \text{Int}^k(H) \).

Let \( \tau_k' \in \gamma \text{Int}^k(H) \). By Lemma 3.3(b), there exists a path \( s' = \tau_1' \ldots \tau_k' \) in \( L \) and \( \sigma' \in \gamma H \) such that \( \tau_i' \cap \sigma' \neq \emptyset \). Since \( \Delta \sigma' \subseteq \text{Ext}(\gamma H) \), \( \tau_i' \cap \sigma' \in \text{Ext}(\gamma H) \). Let \( \sigma_i' \) be a maximal face of \( L \) such that \( \tau_i' \subseteq \sigma_i' \), for each \( 1 \leq i < k \). Consider the broad path \( s' = \sigma_1' \ldots \sigma_k' \) of length \( k \). Then since \( \tau_i' \cap \sigma' \subseteq \sigma_i' \), by Lemma 3.3(a), \( \sigma_k' \in \text{max}(L) \cap \text{Ext}^{k+1}(\gamma H) \). Hence \( \tau_k' \in \text{Ext}^{k+1}(\gamma H) \). Hence \( \gamma \text{Int}^k(H) \subseteq \text{Ext}^{k+1}(\gamma H) \).

The next corollary follows from Proposition 3.3(e).

**Corollary 3.5.** Let \( r \) be the smallest integer such that \( \text{Ext}^r(\gamma H) = L \). Let \( t \) be the smallest integer such that \( \text{Int}^t(H) = \emptyset \). Then \( t = r - 1, r \) or \( r + 1 \).

**Proof.** By Theorem 1.1(e), for any \( k \geq 1 \), \( \text{Ext}^{k+1}(\gamma H) \neq L \) implies \( \text{Int}^k(H) \neq \emptyset \), and \( \text{Ext}^{k-1}(\gamma H) = L \) implies \( \text{Int}^k(H) = \emptyset \). Let \( k = r - 2 \) and \( k = r + 1 \) respectively, we obtain \( \text{Int}^{r-2}(H) \neq \emptyset \) and \( \text{Int}^{r+1}(H) = \emptyset \). Thus \( t = r - 1, r \) or \( r + 1 \).

The next examples show that all three cases \( t = r - 1, r \) and \( r + 1 \) in Corollary 3.5 could happen. Hence the power-estimation in the inequality Proposition 3.3(e) is tight.

![Figure 1: The 2-complex L](image)

**Example 3.6.** Let \( L \) be the 2-complex with vertices \( v_{i,j} \), \( 0 \leq i, j \leq 6, i + j \leq 6 \), given in Figure 1.

1. Let \( H \) be the 2-dimensional sub-complex consisting of all the simplices inside the triangle \( [v_{1,2}, v_{3,2}, v_{1,4}] \), including the boundary of \( [v_{1,2}, v_{3,2}, v_{1,4}] \). Then \( t = 1, r = 2 \).

2. Let \( H \) be the 2-dimensional sub-complex consisting of all the simplices inside the triangle \( [v_{1,1}, v_{4,1}, v_{1,4}] \), including the boundary of \( [v_{1,1}, v_{4,1}, v_{1,4}] \). Then \( t = 2, r = 2 \).

3. Let \( H \) be the sub-hypergraph consisting of all the hyperedges inside the triangle \( [v_{1,2}, v_{4,1}, v_{1,4}] \), excluding the boundary of \( [v_{1,2}, v_{4,1}, v_{1,4}] \). Then \( t = 2, r = 1 \).
3.2 Compositions of maps and neighborhoods

Let $v$ be a vertex of $L$. Recall that the closed star of $v$ in $L$ is the complex

$$\overline{\text{St}}(v, L) = \Delta\{\sigma \in L \mid v \in \sigma\}.$$ 

Let $\tau \in L$. The neighborhood of $\tau$ in $L$ is the complex

$$\text{Nbd}(\tau) = \bigcup_{v \in \tau} \overline{\text{St}}(v, L).$$

By a straight-forward calculation,

$$\text{Nbd}(\tau) = \bigcup_{v \in \tau} \Delta\{\sigma \in L \mid v \in \sigma\} = \Delta \bigcup_{v \in \tau} \{\sigma \in L \mid v \in \sigma\} = \Delta\{\sigma \in L \mid \text{there exists } v \in \tau \text{ such that } v \in \sigma\} = \Delta\{\sigma \in L \mid \tau \cap \sigma \neq \emptyset\}.$$

Let $H \in \mathcal{H}(L)$. The neighborhood of $H$ in $L$ is the complex $\text{Nbd}(H) = \bigcup_{\gamma \in H} \text{Nbd}(\gamma)$. The maximal sub-hypergraph in $L$ whose neighborhood is contained in $H$ is $\text{Nbd}^{-1}(H) = \bigcup_{\text{Nbd}(H') \subseteq H} H'$.

**Proposition 3.7.** Let $H \in \mathcal{H}(L)$. Then

(a). $\text{Nbd} \circ \text{Nbd}^{-1}(H) \subseteq \delta H$;

(b). $\Delta H \subseteq \text{Nbd}^{-1} \circ \text{Nbd}(H)$;

(c). $\text{Nbd}^{-1}(H) \subseteq \text{Int}(H)$ and the equality holds if for any $\sigma' \in \gamma H$, there exists $\sigma \in \gamma H$ such that $\sigma$ is maximal in $L$ and $\sigma' \subseteq \sigma$;

(d). $\text{Ext}(H) \subseteq \text{Nbd}(H)$ and the equality holds if each vertex of $H$ is a hyperedge.

**Proof.** (a). By the definition of neighborhoods, $\text{Nbd} \circ \text{Nbd}^{-1}(H) \subseteq H$. Moreover, $\text{Nbd} \circ \text{Nbd}^{-1}(H)$ is a simplicial complex. Hence $\text{Nbd} \circ \text{Nbd}^{-1}(H) \subseteq \delta H$.

(b). By the definition of neighborhoods, any maximal face of $\text{Nbd}(H)$ is a maximal face of $L$. Thus $\text{Nbd}(H)$ is completely determined by $\max(L) \cap \text{Nbd}(H)$. In order to prove (b), we only need to show $\text{Nbd}(\Delta H) = \text{Nbd}(H)$. Since $H \subseteq \Delta H$, $\text{Nbd}(H) \subseteq \text{Nbd}(\Delta H)$. Let $\sigma' \in \max(L) \cap \text{Nbd}(\Delta H)$. Then there exists $\sigma \in \Delta H$ such that $\sigma \cap \sigma' \neq \emptyset$. Moreover, there exists $\tau \in H$ such that $\sigma \subseteq \tau$. Hence $\tau \cap \sigma' \neq \emptyset$. Hence $\sigma' \in \max(L) \cap \text{Nbd}(H)$. Thus $\max(L) \cap \text{Nbd}(\Delta H) \subseteq \max(L) \cap \text{Nbd}(H)$. Thus $\text{Nbd}(\Delta H) \subseteq \text{Nbd}(H)$. Therefore, $\text{Nbd}(\Delta H) = \text{Nbd}(H)$. Consequently, $\Delta H \subseteq \text{Nbd}^{-1} \circ \text{Nbd}(H)$.

(c). By a straight-forward calculation,

$$\text{Nbd}^{-1}(H) = \{\tau \in L \mid \text{Nbd}(\tau) \subseteq H\} = \{\tau \in L \mid \text{for any } \sigma' \in \gamma H, \sigma' \notin \Delta\{\sigma \mid \tau \cap \sigma \neq \emptyset\}\} = \{\tau \in L \mid \text{for any } \sigma' \in \gamma H \text{ and any } \sigma \in L\}$$

$$= \{\tau \in L \mid \text{for any } \sigma' \in \gamma H \text{ and any } \sigma \in L, \tau \cap \sigma \neq \emptyset, \sigma' \notin \Delta\sigma\}$$

$$\subseteq \{\tau \in L \mid \text{for any } \sigma' \in \gamma H, \tau \cap \sigma' = \emptyset\} = \text{Int}(H).$$

(3.6)

Suppose in addition that for any $\sigma' \in \gamma H$, there exists $\sigma \in \gamma H$ such that $\sigma$ is maximal in $L$ and $\sigma' \subseteq \sigma$. Then the equality holds in the penultimate inequality of (3.6). Thus $\text{Nbd}^{-1}(H) = \text{Int}(H)$. 

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\( \text{Nbd}(H) = \bigcup_{\tau \in H} \Delta \{ \sigma \in L \mid \tau \cap \sigma \neq \emptyset \} \)
\[ \supseteq \bigcup_{\tau \in H} \Delta \{ \sigma \in L \mid \tau \subseteq \sigma \} \]
\[ = \bigcup_{\tau \in H} \Delta \{ \sigma \in \max(L) \mid \tau \subseteq \sigma \} \]
\[ = \text{Ext}(H). \]

Suppose in addition that each vertex of \( H \) is a hyperedge. Then the equality holds in (3.7). Hence \( \text{Nbd}(H) = \text{Ext}(H) \). □

The next corollary follows from Proposition 3.7:

**Corollary 3.8.** Let \( H \in \mathcal{H}(L) \). Then

(a) If for any \( \sigma' \in \gamma H \), there exists \( \sigma \in \gamma H \) such that \( \sigma \) is maximal in \( L \) and \( \sigma' \subseteq \sigma \), then \( \text{Ext} \circ \text{Int}(H) \subseteq \delta H \);

(b) If each vertex of \( H \) is a hyperedge, then \( \delta H \subseteq \text{Int} \circ \text{Ext}(H) \).

## 4 Map algebras on random hypergraphs and random simplicial complexes

In this section, we study the maps \( D\Delta, D\delta \) and \( D\gamma \) as well as their compositions acting on \( D(\mathcal{H}(L)) \) and \( D(\mathcal{K}(L)) \). We prove Theorem 1.1 and Theorem 1.2.

Let \( w \in G \). For any \( f \in D(\mathcal{H}(L)) \) and any \( H \in \mathcal{H}(L) \), it follows from (1.1) that

\[ Dw(f)(H) = \sum_{w(H') = H} f(H'). \]

(4.1)

Let \( t \) be a positive integer and let \( W \in G \). Suppose \( W = (\ldots (w_1 * w_2) * \ldots * w_t) \in G \) with any \((t-2)\)-brackets \( \langle \cdot \rangle \) giving the order of evaluations, \( * = \wedge \) or \( + \), and \( w_1, w_2, \ldots, w_t \in G \). Then with the help of (1.2), \( W \) induces a map

\[ DW : D(\mathcal{H}(L))^\times t \to D(\mathcal{H}(L)) \]

(4.2)

given by

\[ DW(f_1, f_2, \ldots, f_t)(H) = \sum_{(H_1, H_2, \ldots, H_t) = H} \prod_{i=1}^{t} (Dw_i(f_i)(H_i)) \]

(4.3)

for any \((f_1, \ldots, f_t) \in D(\mathcal{H}(L))^\times t \) and any \( H \in \mathcal{H}(L) \). The next lemma shows that (4.3) gives a well-defined map (1.2).

**Lemma 4.1.** For and \( t \geq 1 \) and any \((f_1, f_2, \ldots, f_t) \in D(\mathcal{H}(L))^\times t \), \( DW(f_1, f_2, \ldots, f_t) \in D(\mathcal{H}(L)) \).

**Proof.** To prove Lemma 4.1 we need to prove

\[ \sum_{H \in \mathcal{H}(L)} DW(f_1, f_2, \ldots, f_t)(H) = 1 \]

(4.4)

for any \( t \geq 1 \). Firstly, we prove (4.4) for \( t = 1 \). Since \( w \) is a self-map on \( \mathcal{H}(L) \), for any \( f_1 \in D(\mathcal{H}(L)) \),

\[ \sum_{H_1 \in \mathcal{H}(L)} Dw_1(f_1)(H_1) = \sum_{H_1 \in \mathcal{H}(L)} \sum_{w_1(H_1') = H_1} \sum_{H_1'} f_1(H_1') = \sum_{H_1' \in \mathcal{H}(L)} f_1(H_1'). \]

(4.4)
Thus (4.4) holds for $t = 1$. Secondly, we use induction on $t$ and prove (4.4) for $t \geq 2$. By (4.4) and (4.3), we have

\[
\sum_{H \in \mathcal{H}(L)} DW(f_1, f_2, \ldots, f_t)(H)
= \sum_{H_1, H_2, \ldots, H_t \in \mathcal{H}(L)} \prod_{i=1}^{t-1} \left(\sum_{H_i \in \mathcal{H}(L)} f_i(H_i')\right)
= \frac{1}{1-p}\sum_{H_1, H_2, \ldots, H_t \in \mathcal{H}(L)} \prod_{i=1}^{t-1} \sum_{H_i \in \mathcal{H}(L)} f_i(H_i')
\]

By an induction on $t$, (4.4) follows. $
$

Now we prove Theorem 1.1.

**Proof of Theorem 1.1.** Theorem 1.1 (a), (b), (c), (d) follow from Proposition 3.4 (a), (b), (c), (d) respectively. Theorem 1.1 (e) follows from Proposition 3.4 (e). Theorem 1.1 (f) follows from Corollary 2.3 (b).

Let $p$ be a function from $L$ to $[0,1]$.

**Lemma 4.2.** The map $D\Delta$ sends $\bar{P}_{L,p} \in D(\mathcal{H}(L))$ in Definition 3 to a random simplicial complex $D\Delta(\bar{P}_{L,p}) \in D(\mathcal{K}(L))$ given by

\[
D\Delta(K) = \left( \prod_{\tau \in \max(K)} p(\tau) \right) \left( \prod_{\tau \in K} (1-p(\tau)) \right)
\]

for any $K \in \mathcal{K}(L)$.

**Proof.** Let $K$ be a sub-simplicial complex of $L$. Let $\max(K)$ be the collection of all the maximal faces in $K$. Let $S = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ be any set of distinct simplices in $K$ such that $s$ is a non-negative integer and for each $\sigma_i$, $i = 1, 2, \ldots, s$, there exists $\tau \in \max(K)$ such that $\sigma_i \subseteq \tau$. Here $S$ is allowed to be the emptyset. Suppose $S$ runs over all such sets of simplices in $K$. Then

\[
H = \max(K) \cup S
\]

runs over all the sub-hypergraphs of $L$ such that $\Delta H = K$. Consequently,

\[
\text{Prob}[\Delta H = K \text{ for the random hypergraph } H \sim \bar{P}_{L,p}(H)]
= \sum_{\Delta H = K} \bar{P}_{L,p}(H)
= \sum_{\Delta H = K} \prod_{\sigma \in H} p(\sigma) \prod_{\sigma \notin H} (1-p(\sigma))
= \sum_{H = \max(K) \cup S} \prod_{\sigma \in H} p(\sigma) \prod_{\sigma \notin H} (1-p(\sigma))
= \sum_{S \subseteq K \setminus \max(K)} \prod_{\sigma \in \max(K)} p(\sigma) \prod_{\sigma \in S} p(\sigma) \prod_{\sigma \in K \setminus \max(K) \cup S} (1-p(\sigma)) \prod_{\sigma \in K} (1-p(\sigma))
= \left( \prod_{\sigma \in \max(K)} p(\sigma) \right) \left( \sum_{S \subseteq K \setminus \max(K)} \prod_{\sigma \in S} p(\sigma) \prod_{\sigma \notin S} (1-p(\sigma)) \right) \left( \prod_{\sigma \in K} (1-p(\sigma)) \right)
= \left( \prod_{\tau \in \max(K)} p(\tau) \right) \left( \prod_{\tau \notin K} (1-p(\tau)) \right).
\]
Here the last equality follows from that
\[
\sum_{S \subseteq K \setminus \max(K)} \prod_{\sigma \in S} p(\sigma) \prod_{\sigma \in K \setminus (\max(K) \cup S)} (1 - p(\sigma)) \\
= \prod_{\sigma \in K \setminus \max(K)} \left( p(\sigma) + (1 - p(\sigma)) \right) = \prod_{\sigma \in K \setminus \max(K)} 1 = 1.
\]

We obtain the lemma.

\[\square\]

**Lemma 4.3.** The map \( D\delta \) sends \( \bar{P}_{L,p} \) in Definition 3 to a random simplicial complex \( D\delta(\bar{P}_{L,p}) \in D(\mathcal{K}(L)) \) given by

\[
D\delta(K) = \sum_{\delta H = K} \prod_{\sigma \in H} p(\sigma) \prod_{\sigma \notin H} (1 - p(\sigma))
\]

for any \( K \in \mathcal{K}(L) \).

**Proof.** Let \( K \) be a sub-simplicial complex of \( L \). Then

\[
\text{Prob}[\delta H = K \text{ for the random hypergraph } H \sim \mathcal{T}_{L,p}(H)] = \sum_{\delta H = K} \prod_{\sigma \in H} p(\sigma) \prod_{\sigma \notin H} (1 - p(\sigma)).
\]

By the definition of \( D\delta \), the lemma follows.

\[\square\]

**Lemma 4.4.** The map \( D\gamma \) sends \( \bar{P}_{L,p} \) to \( \bar{P}_{L,1-p} \).

**Proof.** Let \( H \) be a hypergraph in \( L \). Let \( \tau \in L \). Then the probability that \( \tau \) is a hyperedge in \( H \) is \( p(\tau) \). Thus the probability that \( \tau \) is a hyperedge in \( \gamma H \) is \( 1 - p(\tau) \). By the construction of the random hypergraphs in Definition 3 the probability function of \( \gamma H \) is \( \bar{P}_{L,1-p} \).

\[\square\]

Let \( p', p'' \) be functions from \( L \) to \([0, 1]\).

**Lemma 4.5.** The map \( D\cap \) sends the pair \( (\bar{P}_{L,p'}, \bar{P}_{L,p''}) \) to \( \bar{P}_{L,p',p''} \). And the map \( D\cup \) sends the pair \( (\bar{P}_{L,p'}, \bar{P}_{L,p''}) \) to \( \bar{P}_{L,1-(1-p')(1-p'')} \).

**Proof.** We choose hypergraphs \( H', H'' \in \mathcal{H}(L) \) independently at random with probability functions \( \bar{P}_{L,p'} \) and \( \bar{P}_{L,p''} \) respectively. In order to prove Lemma 4.5 we need to show

(a). the random hypergraph \( H' \cap H'' \) satisfies Definition 3 with probability function \( \bar{P}_{L,p',p''} \);

(b). the random hypergraph \( H' \cup H'' \) satisfies Definition 3 with probability function \( \bar{P}_{L,1-(1-p')(1-p'')} \).

Let \( \sigma \in L \). Consider two independent trials: (1). generate \( H' \); (2). generate \( H'' \).

**Proof of (a).** \( \sigma \in H' \cap H'' \) if and only if \( \sigma \in H' \) in trial (1) and \( \sigma \in H'' \) in trial (2). Thus \( \sigma \in H' \cap H'' \) has probability \( pp' \). Letting \( \sigma \) run over \( L \), these trials of \( \sigma \)'s are independent.

**Proof of (b).** \( \sigma \notin H' \cup H'' \) if and only if \( \sigma \notin H' \) in trial (1) and \( \sigma \notin H'' \) in trial (2). Thus \( \sigma \notin H' \cup H'' \) has probability \( (1 - p')(1 - p'') \), and \( \sigma \in H' \cap H'' \) has probability \( 1 - (1 - p')(1 - p'') \). Letting \( \sigma \) run over \( L \), these trials of \( \sigma \)'s are independent.

\[\square\]

Now we prove Theorem 1.2.

**Proof of Theorem 1.2.** Theorem 1.2 (a) follows from Lemma 4.4. Theorem 1.2 (b) follows from Lemma 4.2. Theorem 1.2 (c) follows from Lemma 4.3. Theorem 1.2 (d) follows from the first assertion of Lemma 4.5. Theorem 1.2 (e) follows from the second assertion of Lemma 4.5.

\[\square\]
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