ERRATUM TO
“ON THE CUSPIDAL COHOMOLOGY OF S-ARITHMETIC
SUBGROUPS OF REDUCTIVE GROUPS OVER NUMBER FIELDS”
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Let $G$ be a reductive algebraic group defined over a number field $k$. A $k$-automorphism of $G$ is said to be of Cartan-type if, at each non-archimedean places, it differs from a Cartan involution by an inner automorphism. In [1] the following result regarding the existence of non-trivial cuspidal cohomology classes for $S$-arithmetic subgroups of $G$ is proved:

**Theorem 1.** Let $G$ be an absolutely almost simple algebraic group defined over $k$ that admits a Cartan-type automorphism. When the coefficient system is trivial, the cuspidal cohomology of $G$ over $S$ does not vanish, that is, every $S$-arithmetic subgroup of $G$ has a subgroup of finite index with non-zero cuspidal cohomology with respect to the trivial coefficient system.

This is Theorem 10.6 in [1]. The following assertion appears as Corollary 10.7:

Assume that $G$ is $k$-split and $k$ totally real or $G = \text{Res}_{k'/k} G'$ where $k'$ is a CM-field. Then the cuspidal cohomology of $G$ over $S$ with respect to the trivial coefficient system does not vanish.

It was observed by J. Rohlfs and L. Clozel independently that, in the second case where $G = \text{Res}_{k'/k} G'$ with $k'$ a CM-field, the assertion must be corrected since, to make sense, the argument implicitly uses strong extra assumptions. In fact $G'$ has to be defined over $k$ so that the complex conjugation $c'$ induced by the non-trivial element $\sigma$ in $\text{Gal}(k'/k)$ acts as a $k$-rational automorphism, and a further assumption is necessary to make sense of the argument. Therefore we have to replace Corollary 10.7 in [1] by the following statement:

**Corollary 2.** Let $k$ be a totally real number field and $G$ be an absolutely almost simple algebraic group defined over $k$. Assume that $G$ is $k$-split or $G = \text{Res}_{k'/k} G'$ where $k'$ is a CM-field with $G'$ defined over $k$ and that the complex conjugation $c'$ is of Cartan-type. Then the cuspidal cohomology of $G$ over $S$ with respect to the trivial coefficient system does not vanish.

**Proof.** When $G$ is $k$-split, the proof is given in [1]. The second case is a particular case of Theorem 1. \[\square\]

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1Strictly speaking, since $G'$ is defined over $k$, one has to extend the scalars to $k'$ before applying the restriction functor. For any $k$-algebra $R$ the group $\text{Res}_{k'/k} G'(R)$ of points of $\text{Res}_{k'/k} G'$ with value in $R$ is given by $G'(R \otimes_k k')$. 

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We now give equivalent forms of the condition used in the second case. To simplify the notation we forget for a while about $k$ and $k'$ since in the next proposition we are only interested in archimedean places.

Consider a complex almost quasi-simple group $G = G'({\mathbb C})$ where $G'$ is defined over $\mathbb R$. Let $H = G'({\mathbb R})$ be the the real Lie group of fixed points under the complex conjugation $c'$. The group $G'$ has a quasi-split inner form $G''$. Denote by $c^*$ the action of $\sigma$ on $G^*$. Consider the root system for the maximal torus $T$ in a Borel pair $(B, T)$ defined over $\mathbb R$ in $G^*$. Let $w$ be the element of maximal length in the Weyl group for $T$. Parts of the following proposition are well known (see in particular Corollary 2.9 of [7] and Lemma 3.1 of [6]) but not knowing of a complete reference for it we sketch the argument for the convenience of the reader.

**Proposition 3.** The following assertions are equivalent:

(i) The complex conjugation $c'$ is of Cartan-type.

(ii) The group $G'$ has an inner form $G''$ such that $U = G''({\mathbb R})$ is a compact Lie group.

(iii) The real Lie group $H$ has a compact Cartan subgroup.

(iv) The real Lie group $H$ admits discrete series.

(v) The product $w \circ c^*$ acts by $-1$ on the root system of $T$.

**Proof.** The group $U$ of fixed points of a Cartan involution $\theta$ of $G = G'({\mathbb C})$ is a maximal compact subgroup and $U = G''({\mathbb R})$ where $G''$ is a form of $G'$, not necessarily inner. The complex conjugation $c'$ is of Cartan-type iff $c'(g) = x\theta(g)x^{-1}$ for some $x \in G$ i.e. iff $G''$ is an inner form of $G'$ or equivalently of $G^*$. This proves the equivalence of (i) and (ii). Now consider an inner form $G''$ of $G'$. Up to isomorphism, we may assume that the complex conjugation $c''$ for $G''$ is of the form $c'' = Ad(x') \circ c'$ with $x'$ belonging to the normalizer of $T'$ a maximal torus in $G'$ defined over $\mathbb R$. In particular $x'$ is semi-simple and its centralizer $L = Z_{G'}(x')$ is a reductive group of maximal rank. Now, $(c'')^2 = 1$ implies that $\text{Ad}(x'c'(x')) = 1$ and hence the group $L$ is stable under $c'$. Let $M$ be the group of fixed points in $L$ under $c'$. Such points are also fixed by $c''$. A Cartan subgroup $C \subset M = H \cap U$ is compact if $U = G''({\mathbb R})$ is compact and hence (ii) implies (iii). Observe one may now choose $T'$ such that $C = T'({\mathbb R})$. On a compact Cartan subgroup $C \subset H$ the complex conjugation $c'$ acts by $-1$ on the root system of $T'$ and hence there is $x^* \in G$ which belongs to the normalizer of $T \subset G^*$ such that $\text{Ad}(x^*) \circ c^*$ acts as $-1$ on the root system of $T$. In particular, $w = \text{Ad}(x^*)|_T$ is the element of maximal length in the Weyl group. This shows that (iii) implies (v). The equivalence of (iii) and (iv) is a well known theorem due to Harish-Chandra ([3], Theorem 13). Finally Lemma 4 below shows that (v) implies (ii). □

**Lemma 4.** Assume $w \circ c^*$ acts by $-1$ on the root system. Then, $G^*$ has an inner form $G''$ such that $G''({\mathbb R})$ is compact.

**Proof.** Consider the complex Lie algebra $\mathfrak g = \text{Lie}(G)$. Choose a Borel pair $(B, T)$ defined over $\mathbb R$ in $G^*$. Let $\Sigma$ be the set of roots, $\Sigma^+$ the set of positive roots and $\Delta$ the set of simple roots. Denote by $\mathfrak g_\alpha$ the vector space attached to $\alpha \in \Sigma$. Following Weyl [9], Chevalley [2] and Tits [8] one may choose elements $X_{\alpha} \in \mathfrak g_\alpha$ and $H_{\alpha} \in \mathfrak h = \text{Lie}(T)$ for $\alpha \in \Sigma$, such that $\alpha(H_{\alpha}) = 2$, $[X_{\alpha}, X_{-\alpha}] = H_{\alpha}$ and $[X_{\alpha}, X_{\beta}] = N_{\alpha, \beta}X_{\alpha + \beta}$ if $\alpha + \beta \in \Sigma$ with $N_{\alpha, \beta} = -N_{-\alpha, -\beta} \in \mathbb Z$. One may moreover assume that the splitting $(B, T, \{X_\alpha\}_{\alpha \in \Delta})$ is preserved by $c^*$ i.e. $c^*(X_\alpha) = X_{c^*(\alpha)}$. Let $w$ be the element of maximal length in the Weyl group for $T$. There is an $x'' \in G$, uniquely defined modulo the center of $G$, such that the inner automorphism $\phi = \text{Ad}(x'')$ acts as $w$ on $T$ and such that $\phi(X_{\alpha}) = -X_{w^\alpha}$ for
α ∈ Σ. It is of order 2 and commutes with c∗. Now assume w ◦ c∗ acts by −1 on Σ and let:

\[ Y_\alpha = X_\alpha - X_{-\alpha} \quad Z_\alpha = i(X_\alpha + X_{-\alpha}) \quad U_\alpha = iH_\alpha. \]

The elements \(Y_\alpha\) and \(Z_\alpha\) for \(\alpha \in \Sigma^+\) together with the \(U_\alpha\) for \(\alpha \in \Delta\) build a basis for a real Lie algebra \(u\) and \(g = u + iu\). As in the proof of Theorem 6.3 in Chapter III of [4], we see that the Killing form is negative definite on \(u\). The involution \(\theta = \phi \circ c^*\) induces a Cartan involution on \(G\): its fixed point set is a compact group \(U = G''(\mathbb{R})\) with Lie algebra \(u\) and \(G''\) is the inner form of \(G^*\) defined by the cocycle \(a_1 = 1\) and \(a_\sigma = \text{Ad}(x'')\).

We observe that condition (v) in Proposition 3 may not hold when the Dynkin diagram has a non trivial automorphism of order 2. Assume that the Dynkin diagram of \(G^*\) is irreducible. Then (v) does not hold when \(G^*\) is split of type \(A_n\) with \(n \geq 2\), or \(D_n\) with \(n \geq 3\) odd, or \(E_6\) or when \(G^*\) is quasi-split but non split of type \(D_n\) with \(n \geq 4\) even. It holds in all other cases. This follows from the classification of irreducible root systems.

Nevertheless the conclusion of Corollary 2 may hold true even when condition (v) is not satisfied. In fact we have the

**Theorem 5.** Consider \(G = \text{Res}_{k'/k} G'\) where \(k'\) is a CM-field with \(G'\) absolutely almost simple. Assume that either condition (v) of Proposition 3 holds or that \(G'\) is split over \(k\) and simply connected. Then the cuspidal cohomology of \(G\) over \(S\) with respect to the trivial coefficient system does not vanish.

**Proof.** When \(G'\) is split over \(k\) and simply connected the assertion is a particular case of Theorem 4.7.1 of [5], which in turn relies on Theorem 10.6 in [11]. Now, when condition (v) in Proposition 3 holds the result follows from this Proposition and Corollary 2. \(\square\)

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