ON POLYNOMIALS ORTHOGONAL TO ALL POWERS OF A CHEBYSHEV POLYNOMIAL ON A SEGMENT.

F. PAKOVICH

1. INTRODUCTION.

In the recent series of papers [7]-[9] of M. Briskin, J.-P. Francoise and Y. Yomdin the following “polynomial moment problem” was proposed as an infinitesimal version of the center problem for the Abel differential equation in the complex domain: for a complex polynomial $P(z)$ and distinct $a, b \in \mathbb{C}$ such that $P(a) = P(b)$ to describe polynomials $q(z)$ such that

$$\int_a^b P^i(z)q(z)dz = 0$$

for all integer non-negative $i$.

The following “composition condition” imposed on $P(z)$ and $Q(z) = \int q(z)dz$ is sufficient for polynomials $P(z), q(z)$ to satisfy (1): there exist polynomials $\tilde{P}(z), \tilde{Q}(z), W(z)$ such that

$$P(z) = \tilde{P}(W(z)), \quad Q(z) = \tilde{Q}(W(z)), \quad \text{and} \quad W(a) = W(b).$$

Indeed, the sufficiency of condition (2) is a direct corollary of the Cauchy theorem since after the change of variable $z \rightarrow W(z)$ the new way of integration is closed.

It was suggested in the papers cited above (“the composition conjecture”) that condition (2) is actually equivalent to condition (1). This conjecture was verified in several special cases. In particular, when $a, b$ are not critical points of $P(z)$ ([7]), when $P(z)$ is indecomposable ([9]), and in some other special cases ([1]-[5], [11], [9]). Nevertheless, in general the composition conjecture is not true.

A class of counterexamples to the composition conjecture was constructed in [9]. The simplest of them has the following form:

$$P(z) = T_6(z), \quad q(z) = T'_2(z) + T'_2(z), \quad a = -\sqrt{3}/2, \quad b = \sqrt{3}/2,$$

where $T_n(z) = \cos(n \arccos z)$ is the $n$-th Chebyshev polynomial. Indeed, since $T_2(\sqrt{3}/2) = T_2(-\sqrt{3}/2)$ it follows from the equality $T_6(z) = T_3(T_2(z))$ that (1) is satisfied for $P(z) = T_6(z)$ and $q_1(z) = T'_2(z)$. Similarly, from $T_6(z) = T_2(T_3(z))$ and $T_3(\sqrt{3}/2) = T_3(-\sqrt{3}/2)$ one concludes that (1) holds for $P(z) = T_6(z)$ and $q_2(z) = T'_3(z)$. Therefore, by linearity, condition (1) is satisfied also for $P(z) = T_6(z)$ and $q(z) = q_1(z) + q_2(z)$. Nevertheless, for $P(z) = T_6(z)$ and $Q(z) = T_3(z)$ condition (2) does not hold.

More generally, it was shown in [7] that any polynomial “double decomposition” $A(B(z)) = C(D(z))$ such that $B(a) = B(b), D(a) = D(b)$ supplies counterexamples to the composition conjecture whenever $\deg B(z), \deg D(z)$ are coprime. Note that double decompositions with $\deg A(z) = \deg D(z), \deg B(z) = \deg C(z)$ and $\deg B(z), \deg D(z)$ coprime are described explicitly by Ritt’s theory of factorization.
of polynomials. They are equivalent either to decompositions with \(A(z) = z^n R^m(z)\), \(B(z) = z^m\), \(C(z) = z^m\), \(D(z) = z^n R^m(z)\) for a polynomial \(R(z)\) and \((n,m) = 1\) or to decompositions with \(A(z) = T_n(z)\), \(B(z) = T_n(z)\), \(C(z) = T_n(z)\), \(D(z) = T_m(z)\) for Chebyshev polynomials \(T_n(z)\), \(T_m(z)\) and \((n,m) = 1\) (see [11], [12]).

In this paper we give a solution of the polynomial moment problem (1) in the case when \(P(z)\) is a Chebyshev polynomial \(T_n(z)\). Denote by \(V(T_n, a, b)\) the vector space over \(\mathbb{C}\) consisting of polynomials \(q(z)\) satisfying (1) for \(P(z) = T_n(z)\). Note that any polynomial \(T_n(z)\) such that \(T_d(a) = T_d(b)\) for \(d = \text{GCD}(n,m)\) is contained in \(V(T_n, a, b)\) since \(T_n(z) = T_{n/d}(T_d(z))\) and \(T_m(z) = T_{m/d}(T_d(z))\).

**Theorem.** For any \(n \in \mathbb{N}\) and \(a, b \in \mathbb{C}\) such that \(T_n(a) = T_n(b)\) polynomials \(T'_m(z)\) such that \(T_d(a) = T_d(b)\) for \(d = \text{GCD}(n,m)\) form a basis of \(V(T_n, a, b)\).

The theorem implies that for Chebyshev polynomials the following weakened version of the composition conjecture is true: if \(q(z)\) satisfies condition (1) with \(P(z) = T_n(z)\) then \(\int q(z)dz\) can be represented as a sum of polynomials for which condition (2) is hold. Moreover, actually the number of terms in such a representation can be reduced to two.

**Corollary.** Let \(q(z) \in V(T_n, a, b)\). Then there exist divisors \(d_1, d_2\) of \(n\) such that \(\int q(z)dz = A(T_{d_1}(z)) + B(T_{d_2}(z))\) for some polynomials \(A(z), B(z)\) and the equalities \(T_{d_1}(a) = T_{d_2}(b), T_{d_1}(a) = T_{d_2}(b)\) hold.

For instance, if a polynomial \(q(z)\) belongs to \(V(T_6, -\sqrt{3}/2, \sqrt{3}/2)\) then the polynomial \(\int q(z)dz\) can be represented as \(\int q(z)dz = A(T_2(z)) + B(T_2(z))\) for some polynomials \(A(z), B(z)\). Note that such a representation in general is not unique in contrast to the one for \(q(z)\) providing by the theorem.

2. **Proofs.**

2.1. **Reduction.** First of all, we establish that the theorem can be reduced to the following statement: if \(q(z) = Q'(z)\) is contained in \(V(T_n, a, b)\) then

\[ T_d(a) = T_d(b) \quad \text{for} \quad d = \text{GCD}(n, \deg Q). \]  

(3)

Indeed, assuming that this statement is true the theorem can be deduced as follows. For \(q(z) \in V(T_n, a, b)\) set \(m_0 = \deg Q(z)\) and define \(c_0 \in \mathbb{C}\) by the condition that the degree of \(Q_1(z) = Q(z) - c_0 T_{m_0}(z)\) is strictly less than \(m_0\). Since for \(d_0 = \text{GCD}(n, m_0)\) the equalities

\[ T_n(z) = T_{n/d_0}(T_{d_0}(z)), \quad T_{m_0}(z) = T_{m_0/d_0}(T_{d_0}(z)) \]

hold it follows from \(T_{d_0}(a) = T_{d_0}(b)\) that \(T_{m_0}(z) \in V(T_n, a, b)\). Therefore, by linearity, \(Q_1(z) \in V(T_n, a, b)\). If \(\deg Q_1(z) = m_1\) then, similarly, for some \(c_{m_1} \in \mathbb{C}\) we have \(Q_1(z) = c_{m_1} T_{m_1}(z) + Q_2(z)\), where \(Q_2(z) \in V(T_n, a, b)\) and \(\deg Q_2(z) < m_1\).

Continuing in the same way and observing that \(m_{i+1} < m_i\) we eventually arrive to the representation

\[ \int q(z)dz = \sum_{i=0}^{k} c_i T_{m_i}(z), \quad c_i \in \mathbb{C}, \]  

(4)

such that \(T_{d_i}(a) = T_{d_i}(b)\) for \(d_i = \text{GCD}(n, m_i)\). Since polynomials of different degrees are linearly independent over \(\mathbb{C}\) we conclude that the polynomials \(T'_{m_i}(z)\) such that \(T_d(a) = T_d(b)\) for \(d = \text{GCD}(n, m)\) form a basis of the vector space \(V(T_n, a, b)\).
2.2. Proof of the theorem for non-singular $a, b$. By 2.1 it is enough to show that condition (1) with $P(z) = T_n(z)$, $q(z) = Q'(z)$ implies condition (3). On the other hand, it is known (see [3] or [8]) that for any polynomial $P(z)$ such that $a, b$ are not critical points of $P(z)$ the conditions (1) and (2) are equivalent. Therefore, it is enough to prove that (2) with $P(z) = T_n(z)$ implies (3).

Suppose that (2) holds and set $w = \deg W(z)$. Since by Engstrom’s theorem (see e.g. [12], Th. 5) for any double decomposition $A(B(z)) = C(D(z))$ we have

$$[C(B, D) : C(D)] = \deg D / \gcd(D, \deg D),$$

it follows from the equality

$$T_n(z) = \tilde{P}(W(z)) = T_{n/w}(T_w(z))$$

that $\mathbb{C}(W) = \mathbb{C}(T_w)$. Therefore, since $W(z), T_w(z)$ are polynomials, there exists a linear function $\sigma(z)$ such that $W(z) = \sigma(T_w(z))$ and, hence, $W(a) = W(b)$ yields $T_w(a) = T_w(b)$. Since $w$ is a divisor of $d = \gcd(n, \deg Q)$ the decomposition $T_d(z) = T_{d/w}(T_w(z))$ holds and, therefore, $T_w(a) = T_w(b)$ implies $T_d(a) = T_d(b)$.

2.3. Necessary condition for $P(z), q(z)$ to satisfy (1). To investigate the case when at least one from the points $a, b$ is a critical point of $T_n(z)$ we will use a condition, obtained in [3], formula (6) and in a more general situation in [3], Theorem 3.9, which is necessary for polynomials $P(z), q(z)$ to satisfy (1). To formulate this condition let us introduce the following notation. Say that a domain $U \subset \mathbb{C}$ is admissible with respect to the polynomial $P(z)$ if $U$ is simply connected and contains no critical values of $P(z)$. By the monodromy theorem, in such a domain there exist $n = \deg P(z)$ single-valued branches of $P^{-1}(z)$. Let $U$ be an admissible with respect to $P(z)$ domain such that its boundary $\partial U$ contains the point $z_0 = P(a) = P(b)$. Denote by $p_{u_1}^{-1}(z), p_{u_2}^{-1}(z), ..., p_{u_{d_u}}^{-1}(z)$ (resp. $p_{v_1}^{-1}(z), p_{v_2}^{-1}(z), ..., p_{v_{d_v}}^{-1}(z)$) the branches of $P^{-1}(z)$ in $U$ which map points close to $z_0$ to points close to the point $a$ (resp. $b$). In particular, the number $d_u$ (resp. $d_v$) equals the multiplicity of the point $a$ (resp. $b$) with respect to $P(z)$. In this notation a necessary condition for $P(z), q(z)$ to satisfy (1) has the following form: if $P(z), q(z) = Q'(z)$ satisfy (1) then in any admissible with respect to $P(z)$ domain $U$ such that $z_0 \in \partial U$ the equality

$$d_b \sum_{s=1}^{d_b} Q(p_{u_s}^{-1}(z)) \equiv d_a \sum_{s=1}^{d_a} Q(p_{v_s}^{-1}(z))$$

holds.

2.4. Monodromy of $T_n(z)$. To make condition (5) useful we must examine the monodromy group of $T_n(z)$. It follows from $T_n(\cos \phi) = \cos(n\phi)$, $n \geq 1$, that finite critical values of polynomial $T_n(z)$ are $\pm 1$ and that the preimages of $\pm 1$ are $\cos(\pi j/n)$, $j = 0, 1, ..., n$. To visualize the monodromy group of $T_n(z)$ consider the preimage $\lambda = P^{-1}[-1, 1]$ of the segment $[-1, 1]$ under the map $P(z) : \mathbb{C} \to \mathbb{C}$. It is convenient to consider $\lambda$ as a bicolor graph $\Omega$ embedded into the Riemann sphere. By definition, white (resp. black) vertices of $\lambda$ are preimages of the point 1 (resp. $-1$) and edges of $\lambda$ are preimages of the interval $(-1, 1)$. Since the multiplicity of each critical point of $T_n(z)$ equals 2, the graph $\lambda$ is a “chain-tree” and, as a point set in $\mathbb{C}$, coincides with the segment $[-1, 1]$ (see fig. 1).

Let us fix an admissible with respect to $T_n(z)$ domain $U$ such that $U$ is unbounded and contains the interval $(-1, 1)$. Any branch $T_{n,j}^{-1}(z), 0 \leq j \leq n - 1$, of $T_n^{-1}(z)$ in
Suppose at first that \( z \) is such that branches of \( T_{n,j}^{-1}(\infty) \) follow from \( T_{n}^{-1}(z) \) such a choice of the numeration implies that \( z \) is inverse to \( T_{n}^{-1}(z) \) where the analytic continuation of the functional element \( \{U,T_{n,j}^{-1}(z)\}, 0 \leq j \leq n-1 \), along a clockwise oriented loop around \( 1 \) (resp. \(-1, \infty\)) is the functional element \( \{U,T_{n,\pi_{11}}^{-1}(z)\} \) (resp. \( \{U,T_{n,\pi_{-11}}^{-1}(z)\} \)). The tree \( \lambda \) represents the monodromy group of \( T_{n,j}^{-1}(z) \) in the following sense: the edges of \( \lambda \) are identified with branches of \( T_{n,j}^{-1}(z) \) and the permutation \( \pi_{1} \) (resp. \( \pi_{-1} \)) is identified with the permutation arising under clockwise rotation of edges of \( \lambda \) around white (resp. black) vertices.

In order to fix a convenient numeration of branches of \( T_{n,j}^{-1}(z) \) in \( U \) consider an auxiliary domain \( U_{\infty} = U \cap B \), where \( B \) is a disc with the center at the infinity. Since at least one from \( z_{j}^{\pm} \) denotes a fixed branch of the algebraic function which is inverse to \( z_{n} \) in \( U_{\infty} \), then each branch of \( T_{n,j}^{-1}(z) \) can be represented in \( U_{\infty} \) by the convergent series

\[
\phi_{j}(z) = \sum_{k=-\infty}^{1} t_{k}z_{n}^{jk}z_{n}^{\pm k}, \quad t_{k} \in \mathbb{C}, \quad \varepsilon_{n} = \exp(2\pi i / n),
\]

for certain \( j, 0 \leq j \leq n-1 \). Now we fix a numeration of branches of \( T_{n,j}^{-1}(z) \) in \( U \) as follows: the branch \( T_{n,j}^{-1}(z) \), \( 0 \leq j \leq n-1 \), is the analytic continuation of \( \phi_{j}(z) \) from \( U_{\infty} \cap U \) to \( U \) and the branch \( z_{n}^{\pm} \) is defined by the condition that \( T_{n,j}^{-1}(z) \) maps the interval \((-1, 1)\) onto the interval \((\cos(\pi / n), 1)\) defined by the condition that \( T_{n,j}^{-1}(z) \) maps the interval \((-1, 1)\) onto the interval \((\cos(\pi / n), 1)\). Since the result of the analytic continuation of the functional element \( \{U_{\infty}, \varepsilon_{n}^{j}z_{n}^{\pm}\} \), \( 0 \leq j \leq n-1 \), along a clockwise oriented loop around \( \infty \) is the functional element \( \{U_{\infty}, \varepsilon_{n}^{j+1}z_{n}^{\pm}\} \), such a choice of the numeration implies that \( \pi_{\infty} = (012...n-1) \). Furthermore, it follows from \( \pi_{\infty}\pi_{-1}\pi_{1} = 1 \), taking into account the combinatorics of \( \lambda \), that the numeration of edges of \( \lambda \) coincides with the one indicated on the figure 1 that is \( \pi_{-1} = (0 n-1)(1 n-2)(2 n-3) \ldots \) and \( \pi_{1} = (1 n-1)(2 n-2)(3 n-3) \ldots \).

2.5. Proof of the theorem for singular \( a,b \). Again, it is enough to establish that (3) holds. Let \( Q(z) \in V(T_{n}, a, b) \) with \( \deg Q(z) = m \). Since at least one from points \( a, b \) is a critical point of \( T_{n}(z) \) the number \( z_{0} = P(a) = P(b) \) equals \( \pm 1 \). Suppose at first that \( z_{0} = 1 \). Then \( a = \cos(2j_{1} \pi / n), b = \cos(2j_{2} \pi / n) \) for certain \( j_{1}, j_{2}, 0 \leq j_{1}, j_{2} \leq n/2 \), and condition (5) has the following form:

\[
Q(T_{n,j_{1}}^{-1}(z)) + Q(T_{n,n-j_{1}}^{-1}(z)) = Q(T_{n,j_{2}}^{-1}(z)) + Q(T_{n,n-j_{2}}^{-1}(z)),
\]

(7)

1\(^1\)Note that any polynomial with two finite critical values can be represented by an appropriate bicolored plane tree and vice versa; it is a very particular case of the Grothendieck correspondence between Belyi functions and graphs embedded into compact Riemann surfaces (see e.g. [4]).
where \( T_{n,i}^{-1}(z) \) is represented in \( U_\infty \) by the series (6). Since \( t_1 \neq 0 \), the comparison of the leading coefficients of the corresponding (7) Puiseux expansions gives
\[
\varepsilon_n^{j_1m} + \varepsilon_n^{(n-j_1)m} = \varepsilon_n^{j_2m} + \varepsilon_n^{(n-j_2)m}.
\]

Therefore, the number \( \varepsilon_n^{2m} \), where \( d = \text{GCD}(n, m) \), is a root of the polynomial
\[
f(z) = \varepsilon_n^{j_1d} + z^{(n-j_1)d} - z^{j_2d} - z^{(n-j_2)d}.
\]

Since \( \varepsilon_n^{2m} \) is a primitive \( n \)-th root of unity and the coefficients of \( f(z) \) are integers, this fact implies that \( n \)-th cyclotomic polynomial \( \Phi_n(z) \) divides \( f(z) \) in the ring \( \mathbb{Z}[z] \) and, therefore, that the primitive \( n \)-th root of unity \( \varepsilon_n \) also is a root of \( f(z) \).

Hence,
\[
\varepsilon_n^{j_1d} + \varepsilon_n^{-j_1d} = \varepsilon_n^{j_2d} + \varepsilon_n^{-j_2d}.
\]

Since
\[
a = \cos(2j_1\pi/n) = \frac{1}{2}(\varepsilon_n^{j_1} + \varepsilon_n^{-j_1}), \quad b = \cos(2j_2\pi/n) = \frac{1}{2}(\varepsilon_n^{j_2} + \varepsilon_n^{-j_2}),
\]
it follows now from
\[
T_d(\varepsilon_n^{j_1d} + \varepsilon_n^{-j_1d} + 1) = \frac{1}{2}(\varepsilon_n^{j_1} + \varepsilon_n^{-j_1}) + \frac{1}{2}(\varepsilon_n^{j_2} + \varepsilon_n^{-j_2})
\]
that \( T_d(a) = T_d(b) \).

Similarly, if \( z_0 = -1 \), assuming that \( a = \cos((2j_1+1)^\pi/n), b = \cos((2j_2+1)^\pi/n) \) for certain \( j_1, j_2 \), \( 0 \leq j_1, j_2 \leq [(n - 1)/2] \), we obtain the equality
\[
T_{n,j_1}(z) + T_{n,n-j_1-1}(z) = T_{n,j_2}(z) + T_{n,n-j_2-1}(z)
\]
which implies
\[
\varepsilon_n^{j_1m} + \varepsilon_n^{(n-j_1)m} = \varepsilon_n^{j_2m} + \varepsilon_n^{(n-j_2)m}
\]

and
\[
\varepsilon_n^{j_1d} + \varepsilon_n^{-(j_1+1)d} = \varepsilon_n^{j_2d} + \varepsilon_n^{-(j_2+1)d}.
\]

It yields that
\[
\varepsilon_n^{2j_1d} + \varepsilon_n^{-(2j_1+1)d} = \varepsilon_n^{2j_2d} + \varepsilon_n^{-(2j_2+1)d},
\]
where \( \varepsilon_n = \exp(2\pi i/2n) \), and, multiplying the last equality by \( \varepsilon_n^{2m} \), we get
\[
\varepsilon_n^{(2j_1+1)d} + \varepsilon_n^{-(2j_1+1)d} = \varepsilon_n^{(2j_2+1)d} + \varepsilon_n^{-(2j_2+1)d}.
\]

Since
\[
a = \frac{1}{2}(\varepsilon_n^{2j_1+1} + \varepsilon_n^{-(2j_1+1)}), \quad b = \frac{1}{2}(\varepsilon_n^{2j_2+1} + \varepsilon_n^{-(2j_2+1)}),
\]
we conclude as above that \( T_d(a) = T_d(b) \).

2.6. Proof of the corollary. Suppose \( q(z) \in V(T_n, a, b) \). Then, by the theorem, \( \int q(z)dz \) can be represented by the sum (4). We will prove the corollary by induction on the number of non-zero terms in this representation. Since for each \( i, 0 \leq i \leq k \), in (4) we have \( T_{m_i}(z) = T_{m_i/d_i}(T_{d_i}(z)) \) with \( d_i = \text{GCD}(n, m_i) \) and \( T_{d_i}(a) = T_{d_i}(b) \), the corollary is true for \( k = 0, 1 \).

Suppose now that \( k > 1 \). By the inductive assumption there exist \( r, s \) such that
\[
\sum_{i=0}^{k-1} c_i T_{m_i}(z) = A(T_r(z)) + B(T_s(z))
\]
and \( T_r(a) = T_r(b), T_s(a) = T_s(b) \). Choose \( v, w \in \mathbb{C} \) such that
\[
a = \frac{1}{2}(v + \frac{1}{v}), \quad b = \frac{1}{2}(w + \frac{1}{w}).
\]
Then, by (8), $T_r(a) = T_r(b)$ implies that
\[
\frac{1}{2}(v^r + \frac{1}{v^r}) = \frac{1}{2}(w^r + \frac{1}{w^r}).
\]
In its turn, the last equality holds if and only if $v^r = w^{\mu_1 r}$, where $\mu_1 = \pm 1$. Similarly, the equalities $T_s(a) = T_s(b)$, $T_{m_k}(a) = T_{m_k}(b)$ yield that $v^s = w^{\mu_2 s}$, $v^{m_k} = w^{\mu_3 m_k}$, where $\mu_2, \mu_3 = \pm 1$.

Suppose $\mu_1 = \mu_2 = \mu$. Then $v^r = w^{\mu r}$, $v^s = w^{\mu s}$ and an easy reasoning involving roots of unity shows that $d = \gcd(r, s) > 1$ and $v^d = w^{\mu d}$. Therefore, by (8), $T_d(a) = T_d(b)$ and we can represent $\int q(z)dz$ as
\[
\int q(z)dz = C(T_d(z)) + c_k T_{m_k}(z),
\]
where $C(z) = A(T_{r/d}(z)) + B(T_{s/d}(z))$.

If $\mu_1 = -\mu_2$ then either $\mu_1 = \mu_3$ or $\mu_2 = \mu_3$ and we conclude as above that either
\[
\int q(z)dz = E(T_e(z)) + B(T_s(z)),
\]
where $E(z) = A(T_{r/e}(z)) + c_k T_{m_k/e}(z)$, $e = \gcd(r, m_k)$, and $T_e(a) = T_e(b)$ or
\[
\int q(z)dz = A(T_r(z)) + F(T_f(z)),
\]
where $F(z) = B(T_{s/f}(z)) + c_k T_{m_k/f}(z)$, $f = \gcd(s, m_k)$, and $T_f(a) = T_f(b)$.

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