Photon Stars

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Abstract

We discuss numerical solutions of Einstein’s field equation describing static, spherically symmetric conglomerations of a photon gas. These equations imply a back reaction of the metric on the energy density of the photon gas according to Tolman’s equation. The 3-fold of solutions corresponds to a class of physically different solutions which is parameterized by only two quantities, e.g. mass and surface temperature. The energy density is typically concentrated on a shell because the center contains a repelling singularity, which can, however, not be reached by timelike or null geodesics. The physical relevance of these solutions is completely open, although their existence may raise some doubts w.r. to the stability of black holes.

1 Introduction

The starting point of this investigation was the discussion of the Carnot-Bekenstein-process in the environment of a Schwarzschild black hole [HUS]. There it was assumed that the black hole is surrounded by a cloud of radiation with a local temperature according to Tolman’s equation [TOL]

\[ T(r) = \frac{T_\infty}{g_{00}(r)} \]  

where \( T_\infty \) is the usual Hawking temperature at \( r = \infty \). This equation also follows for the equilibrium distribution of photons in relativistic kinetic theory [NEU] or from elementary thermodynamical gedanken experiments [HUS]. These yield a modified formula for the efficiency of the Carnot-Bekenstein-process

\[ \eta = 1 - \frac{T_2 \sqrt{g_{00}(2)}}{T_1 \sqrt{g_{00}(1)}} \]  

between two heat reservoirs at different height. In the equilibrium we have \( \eta = 0 \) which implies \([\text{I}]\). All this holds as long as the back reaction of the radiation to the metric can be neglected.

But, since \( g_{00}(r) = 1 - \frac{\mathcal{R}}{r} \), at the Schwarzschild radius \( r = \mathcal{R} \) the temperature and the energy density diverges and cannot longer be regarded as a mere perturbation. Rather one would have to solve the field equation anew, this time allowing
for the energy-stress tensor of the photon gas to act as a source of gravitation. It is not at all clear whether the resulting metric will be a modified black hole in some sense. So we have the following problem which may be considered independently of the original motivation:

Calculate the static, spherically symmetric metric of Einstein’s field equations with the energy-stress tensor of a perfect fluid

\[ T_{ab} = (\rho + P)u_a u_b + P g_{ab} \]  

consisting of photons, i.e.

\[ \rho = 3P. \]  

In section 2 the corresponding field equations are transformed into an autonomous two-dimensional system of differential equations which is discussed in section 3. In section 4 we investigate the metric of a photon star for \( r \to 0 \) and study its global properties by means of numerical solutions. Physical characteristics like radius, mass and temperature are defined in section 5. Section 6 contains concluding remarks.

2 Transformation of the field equations

With except of eq. \( \text{(4)} \) the problem stated above is just the well-known problem of constructing the interior solution of a star. We may choose coordinates \((t, r, \theta, \phi)\) such that the metric is given by

\[ ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2 \]  

where \( f \) and \( h \) are unknown functions and \( d\Omega^2 \) is the surface element of a unit sphere. With respect to these coordinates the field equations boil down to a system of three coupled differential equations (cf. \([\text{WAL}]\ 6.2.3–6.2.5)"

\[ \frac{8\pi G}{c^2} \rho = \frac{h'}{r h^2} + \frac{h - 1}{hr^2} \]  

\[ \frac{8\pi G}{3c^2} \rho = \frac{f'}{rf h} - \frac{h - 1}{hr^2} \]  

\[ \frac{16\pi G}{3c^2} \rho = \frac{f'}{rf h} - \frac{h'}{hr^2} + \frac{1}{\sqrt{fh}} \frac{dr}{\sqrt{fh}} \left( \frac{f'}{\sqrt{fh}} \right) \]  

We recall that Tolman’s equation \([\text{TOL}]\) was derived under the same assumptions we made, except spherical symmetry. Hence we may adopt \([\text{TOL}]\) or, equivalently,

\[ \rho(r) = \frac{\rho_1}{f^2(r)}, \]  

since

\[ \rho = \sigma T^4 \]  

from local statistical mechanics, where \( \sigma \) is the Stefan-Boltzmann constant. We write

\[ \rho_1 = \frac{c^4}{8\pi G C} \]  

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and obtain from (6) and (7)
\[
\frac{C}{f^2} = \frac{h'}{rh^2} + \frac{h - 1}{hr^2} \quad (12)
\]
\[
\frac{C}{3f^2} = \frac{f'}{rfh} - \frac{h - 1}{hr^2} \quad (13)
\]
Eq. (8) is identically satisfied, which confirms Tolman’s result. Otherwise the system of differential equations would be overdetermined.

The set of solutions \( S \) of the system (12), (13) can be parameterized by 3 parameters, e. g. \( C \) and two initial values for \( f \) and \( h \). But \( S \) is invariant under the 2-parameter group of scale transformations
\[
r \mapsto \lambda r, \quad f \mapsto \mu f, \quad h \mapsto h, \quad C \mapsto \frac{\lambda^2}{\mu^2} C
\]
(14)
Thus there exists only a 1-parameter family of solutions looking qualitatively different.

An equivalent second order equation is obtained by eliminating \( h \) using
\[
h = \frac{3f(f + rf')}{Cr^2 + 3f^2}
\]
and inserting the derivative of (15) into (12):
\[
f'' = \frac{6r f^2 + 6r^2 f f' - 6f^3 f' + 2r^3 f'^2}{Cr^3 f + 3rf^3}
\]
(16)
For \( C = 0 \) we have the well-known equations which lead to a 2-parameter family of Schwarzschild metrics. For \( C \neq 0 \) we may scale every solution such that it becomes a solution with
\[C = 1.\]
(17)
The remaining subgroup of (14) with \( \lambda = \mu \) may be used to simplify the differential equation (16). We perform the transformation
\[
s = \ln \frac{r}{r_0}, \quad x = \frac{f(r)}{r}, \quad y = f'(r) + x,
\]
(18)
where \( r_0 > 0 \) is arbitrary. The resulting reduced equations read
\[
\frac{dx}{ds} = y - 2x
\]
(19)
\[
\frac{dy}{ds} = \frac{y(2y - 3x(x^2 - 1))}{x(3x^2 + 1)}.
\]
(20)
Note that this system is autonomous. The resulting symmetry \( s \mapsto s + s_0 \) reflects the scale invariance of (16) w.r. to the subgroup \( \lambda = \mu \). Even if we cannot solve it exactly, it seems to be better accessible for intuition and developing approximation schemes. Any two different solutions of (14), (20) correspond to different similarity classes of \( S \).
3 Discussion of the reduced equations

We calculate some typical solutions of (19), (20) numerically and display them as curves in the $x - y$-plane.

![Figure 1: A selection of numerical solution curves of the reduced equation (19), (20) together with the parabolic Schwarzschild approximation](image)

It is obvious that

$$x_0 = \sqrt{\frac{3}{7}}, \quad y_0 = 2\sqrt{\frac{3}{7}} \quad (21)$$

is a stable stationary point of (19), (20) which is an attractor of the whole open quadrant $x > 0, y > 0$. It follows by inverting the transformation (18) that

$$f(r) = \sqrt{\frac{7}{3}}r, \quad h(r) = \frac{7}{4} \quad (22)$$

is an exact solution of (19), (20) which is asymptotically approached by any other solution for $r \to \infty$. It follows that the space-time of photon stars is not asymptotically flat. The other exact solution of (19), (20)

$$y = 0, \quad x = ae^{-2s} \quad (23)$$

yields

$$f(r) = \frac{a}{r}, \quad h(r) = 0 \quad (24)$$

and is hence unphysical. A typical solution curve of (19), (20) starts from $x = +\infty$, $y = 0$, $s = -\infty$ and runs close to the solution (23) until it reaches small values of $x$. Then according to the "$x$" in the denominator of (20), $\frac{dy}{ds}$ increases rapidly and the curve is turned up towards the $y$-axis. Then it describes a parabolic-like bow and approaches the stationary point by a clockwise vortex.

It is instructive to draw the general Schwarzschild solutions ($C = 0$)

$$f_S(r) = a - \frac{b}{r} \quad (25)$$
into the $x - y$-diagram. They are given by the family of parabolas

$$x = y - \frac{b}{a^2}y^2,$$

which approximate the solution curves of (19), (20) having the same vertex at

$$x_1 = \frac{a^2}{4b}, \quad y_1 = \frac{a^2}{2b}.\quad (27)$$

In this way, for each solution $\langle f(r), h(r) \rangle$ of (12), (13) we can define a unique SCHWARZSCHILD approximation $\langle f_S(r), h_S(r) \rangle$.

4 Properties of the metric

We now turn to the discussion of the solutions of (16) for $f(r)$ which yield $h(r)$ by (15).

To study the behaviour for smaller $r$ we expand $f$ into a Laurent series and insert this series into (16). It turns out that the series starts with

$$f(r) = \frac{A}{r} + B + \cdots$$

in accordance with a SCHWARZSCHILD solution for $C = 0$. The coefficients $A, B$ are left undetermined since they represent the two initial values for (16). The next terms are uniquely determined by (16). It is straightforward to calculate the first, say, 20 terms by using a computer algebra software like MATHEMATICA. Here we only note down the first nonvanishing extra terms for $f$ and $h^{-1}$:

$$f(r) = \frac{A}{r} + B + \frac{CB}{15A^2} + \mathcal{O}(r^5),\quad (29)$$

$$h^{-1}(r) = \frac{A}{Br} + 1 - \frac{C}{15A^2}r^4 + \mathcal{O}(r^5).\quad (30)$$

For small $r$, $f$ and $h$ are also approximated by SCHWARZSCHILD solutions, but unlike the approximation discussed above, we have to choose $A > 0$ in order to obtain a positive solution $f(r) > 0$. $f$ cannot change its sign in a continuous way, since $f(r_0) = 0$ implies $f''(r_0) = \infty$ by (16). (According to the scale invariance (14), $f(r)$ may be multiplied by $-1$, but this gives no physically different solution.) Thus we may state that for $r \to 0$ the metric looks like that of a SCHWARZSCHILD black hole with negative mass, independent of $C$. For the geodesic motion close to $r = 0$ we may thus adopt the effective potential of SCHWARZSCHILD theory ([WAL] 6.3.15) (with $M \mapsto -M$):

$$V = \frac{1}{2} \kappa + \kappa \frac{M}{r} + \frac{L^2}{2r^2} + \frac{ML^2}{r^3}\quad (31)$$

where

$$\kappa = \begin{cases} 1 & \text{(timelike geodesics)} \\ 0 & \text{(null geodesics)} \end{cases}\quad (32)$$

It follows that $r = 0$ can never be reached by particles or photons due to the infinite high potential barrier. Although curvature blows up for $r \to 0$, as in the
SCHWARZSCHILD case, the nature of the singularity is less harmful. We conjecture that it cannot be regarded as a "naked singularity" in whatever technical sense (see [EAR] for details of the various definitions) and that Cauchy surfaces still exist in the spacetime given by [12], [13], if the line \( r = 0 \) is excluded from the spacetime manifold.

From a computational point of view, the singularity of \( f(r) \) at \( r = 0 \) suggests to transform (16) into a differential equation for

\[
F(r) := rf(r)
\]

and to re-transform to \( f(r) \) after a numerical solution for \( F \) has been obtained. We used the \texttt{NDSolve}-command of MATHEMATICA to produce the following numerical solutions:

![Numerical solutions](image)

Figure 2: A typical numerical solution \( f(r), h(r) \) of the system (12), (13). \( h(r) \) has its maximum at \( r = r_0 \). The corresponding SCHWARZSCHILD solution \( f_S(r) \) with \( f_S(r_0) = 0 \) is also displayed.

A typical solution is shown in Fig. 2. Recall that for the SCHWARZSCHILD metric \( h(r) \) diverges at \( r = R \) and \( f(R) = 0 \). For the solution of Fig. 2 \( h(r) \) has a relatively sharp maximum at \( r_0 \) and \( f(r_0) \) is becoming small. For \( r > r_0 \), \( f \) and \( h \) are comparable with their SCHWARZSCHILD approximations \( f_S \) and \( h_S \). If \( r \) is not too large. For \( r < r_0 \), \( f \) remains small within some shell \( r_1 < r < r_0 \) and diverges for \( r \to 0 \) according to (29). By (11) this means that the energy density is concentrated within that shell and \( r_0 \) may be viewed as the "radius of the photon star".

Other solutions with larger values of \( C \) show a more diffuse cloud of photons and a less sharp maximum of \( h \), see Fig. 3 and 4. These solutions are all scaled to the same value of \( r_0 \).
Figure 3: Numerical solutions $h(r)$ for different $C$. The solutions are scaled such that they obtain their maximum at the same value $r_0 = 1$.

Figure 4: Numerical solutions $f(r)$ for different $C$ and the same scaling as in Fig. 3.

5 Physical parameters of a photon star

We have seen that the set of solutions $S$ may be characterized by 3 parameters, e.g. $A, B$ and $C$ in (29). From the analogy with the Schwarzschild case ($C = 0$) we expect that only a 2-parameter family represents physically different spacetimes. In the Schwarzschild case, one parameter is set to 1 by the choice of the units, and the remaining parameter $\mathcal{R}$ distinguishes between black holes of different mass. More specifically, one postulates that the velocity of light, expressed by $\frac{dr}{dt}$ approaches 1 for $r \to \infty$. The metric then obtains the form

$$f_S(r) = 1 - \frac{\mathcal{R}}{r}, \quad h_S = f_S^{-1}. \quad (34)$$
In the case of the photon star, we cannot proceed in the same way, since the metric will not be asymptotically flat. But instead we may postulate the "gauge condition" that the Schwarzschild approximation of $\langle f, h \rangle$, defined in section 3, should obey condition (34). If this is not the case, one has to perform a suitable scale transformation (14). In this way we obtain a two-fold of physically different solutions.

We now consider physical parameters characterizing this two-fold of solutions. One could be the radius $r_0$ of the photon star defined above.

In analogy to the Schwarzschild theory ([WAL] 6.2.7) we introduce the (gravitational) mass function

$$m(r) := \frac{r}{2}(1 - h^{-1}(r)) c^2 G.$$  

In the domain where $f(r) \approx f_S(r)$ this is the "would-be-mass" of an equivalent black hole. In the domain $r < r_0$ the interpretation of (35) is not so obvious. As to be expected from the above discussion of the metric for $r \to 0$, it turns out that $m(0) < 0$. A typical mass function is shown in Fig. 5, where also the "proper mass"

$$m_p(r) = \frac{4\pi}{c^2} \int_0^r \rho(r') h(r')^{1/2} dr'$$

and the difference $m_p - m$ is displayed.

![Figure 5](image-url)

**Figure 5:** A typical numerical solution of the mass functions $m(r), m_p(r)$ and the difference $m_p(r) - m(r)$.

It may be, as in this case, that the majority of the photons are "hidden" by the apparent negative mass in the center with respect to gravitation. Nevertheless, we could use

$$m_0 = m(r_0)$$

as a further physical parameter characterizing a photon star. Generally, by (35) and $h(r_0) > \frac{4}{7} = \lim_{r \to \infty} h(r)$

$$\frac{3}{7} M_S(r_0) < m_0 < M_S(r_0),$$

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where
\[ M_S(r_0) := \frac{r_0c^2}{2G}. \] (39)

If \( h(r_0) \gg 1 \) we have \( m_0 \approx M_S(r_0) \), as in the Schwarzschild case.

As another physical parameter we consider the "surface temperature"
\[ T_0 := T(r_0) = \left( \frac{\rho(r_0)}{\sigma} \right)^{1/4}. \] (40)

If we have only a two-fold of physically different solutions, as we claimed above, \( T_0 \) should be a function of \( r_0 \) and \( m_0 \). Indeed, (12) together with \( h'(r_0) = 0 \) shows that \( f(r_0) \) is a function of \( r_0 \) and \( h(r_0) \), hence of \( r_0 \) and \( m_0 \). Then, by (40) and (3), also \( T_0 \) depends only on \( r_0 \) and \( m_0 \). Since \( \sigma \) depends on \( \tilde{h} \), the result can be conveniently expressed by using Planck units, indicated by a subscript P:
\[ \frac{T_0}{T_P} = \frac{15^{1/4}}{(2\pi)^{3/4}} \left[ \frac{m_0}{M_P} \left( \frac{L_P}{r_0} \right)^3 \right]^{1/4}. \] (41)

To give a numerical example, if we take \( m_0 \) as the mass of the sun, \( r_0 \) as the corresponding Schwarzschild radius, we obtain \( T_0 \approx 4 \cdot 10^{12} \text{K} \). This would correspond to a very hard gamma radiation with a wavelength \( \lambda \approx 10^{-15} \text{m} \). The Hawking temperature of this example is \( T_H \approx 10^{-8} \text{K} \), since \( T_H \sim m^{-1} \) whereas \( T_0 \sim m_0^{-1/2} \) for \( m \approx M_S(r_0) \).

6 Conclusion

It is difficult to assess the physical relevance of our findings. But one point seems to be clear: The global character of the solutions \( \langle f, h \rangle \) is completely different from the Schwarzschild approximations \( \langle f_S, h_S \rangle \), no matter how small \( C \) is. So the class \( S \) of solutions of (12), (13) does not depend continuously on \( C \) in any reasonable sense. Any small amount of radiation will destroy the event horizon, at least if an equilibrium is approached. If this would also be the case in simulations of the birth of "black holes" by collapsing matter, then they would never be born, and perhaps won’t exist at all.
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