ADE bundles over surfaces with ADE singularities

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Abstract

Given a compact complex surface $X$ with an ADE singularity and $p_g = 0$, we construct ADE bundles over $X$ and its minimal resolution $Y$. Furthermore, we describe their minuscule representation bundles in terms of configurations of (reducible) $(-1)$-curves in $Y$.

1 Introduction

It has long been known that there are deep connections between Lie theory and the geometry of surfaces. A famous example is an amazing connection between Lie groups of type $E_n$ and del Pezzo surfaces $X$ of degree $9 - n$ for $1 \leq n \leq 8$. The root lattice of $E_n$ can be identified with $K_X^\perp$, the orthogonal complement to $K_X$ in $Pic(X)$. Furthermore, all the lines in $X$ form a representation of $E_n$. Using the configuration of these lines, we can construct an $E_n$ Lie algebra bundle over $X$ [15]. If we restrict it to the anti-canonical curve in $X$, which is an elliptic curve $\Sigma$, then we obtain an isomorphism between the moduli space of degree $9 - n$ del Pezzo surfaces which contain $\Sigma$ and the moduli space of $E_n$-bundles over $\Sigma$. This work is motivated from string/F-theory duality, and it has been studied extensively by Friedman-Morgan-Witten [8][9][10], Donagi [3][4][5][7], Leung-Zhang [14][15][16] and others [6][13][17][18].

In this paper, we study the relationships between simply-laced, or ADE, Lie theory and rational singularities of surfaces. Suppose

$$\pi : Y \to X$$

is the minimal resolution of a compact complex surface $X$ with a rational singularity. Then the dual graph of the exceptional divisor $\sum_{i=1}^n C_i$ in $Y$ is an ADE Dynkin diagram. From this we have an ADE root system $\Phi := \{\alpha = \sum a_i [C_i] | \alpha^2 = -2\}$ and we can construct an ADE Lie algebra bundle over $Y$:

$$E^g_0 := O_Y^{\oplus n} \oplus \bigoplus_{\alpha \in \Phi} O_Y(\alpha)$$

Even though this bundle can not descend to $X$, we show that it can be deformed to one which can descend to $X$ provided that $p_g(X) = 0$. 

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Theorem 1 (Proposition 3, Proposition 4, Theorem 5 and Lemma 10)

Assume $Y$ is the minimal resolution of a surface $X$ with a rational singularity at $p$ of type $g$ and $C = \sum_{i=1}^n C_i$ is the exceptional divisor. If $p_g(X) = 0$, then

(i) given any $(\varphi_{C_i})_{i=1}^n \in \Omega^{0,1}(Y, \bigoplus_{i=1}^n O(C_i))$ with $\partial \varphi_{C_i} = 0$ for every $i$, it can be extended to $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha \in \Phi^+} O(\alpha))$ such that $\partial \varphi := \overline{\partial} g + \text{ad}(\varphi)$ is a holomorphic structure on $E^g_\varphi$. We denote this new holomorphic bundle as $E^g_\varphi$.

(ii) Such a $\overline{\partial} g$ is compatible with the Lie algebra structure.

(iii) $E^g_\varphi$ is trivial on $C_i$ if and only if $[\varphi_{C_i}, C_i] \neq 0 \in H^1(C_i, O(C_i)) \cong \mathbb{C}$.

(iv) There exists $[\varphi_{C_i}] \in H^1(Y, O(C_i))$ such that $[\varphi_{C_i}, C_i] \neq 0$.

(v) Such a $E^g_\varphi$ can descend to $X$ if and only if $[\varphi_{C_i}, C_i] \neq 0$ for every $i$.

Remark 2

Infinitesimal deformations of holomorphic bundle structures on $E^g_\varphi$ are parametrized by $H^1(Y, End(E^g_\varphi))$, and those which also preserve the Lie algebra structure are parametrized by $H^1(Y, \text{ad}(E^g_\varphi)) = H^1(Y, E^g_\varphi)$, since $g$ is semisimple. If $p_g(X) = q(X) = 0$, e.g. rational surface, then for any $\alpha \in \Phi^-$, $H^1(Y, O(\alpha)) = 0$. Hence $H^1(Y, E^g_\varphi) = H^1(Y, \bigoplus_{\alpha \in \Phi^+} O(\alpha))$.

This generalizes the work of Friedman-Morgan [8], in which they considered $E_n$ bundles over generalized del Pezzo surfaces. In this paper, we will also describe the minuscule representation bundles of these Lie algebra bundles in terms of $(-1)$-curves in $Y$.

Here is an outline of our results. We first study $(-1)$-curves in $Y$ which are (possibly reducible) rational curves with self intersection $-1$. If there exists a $(-1)$-curve $C_0$ in $X$ passing through $p$ with minuscule multiplicity $C_k$ (Definition 13), then $(-1)$-curves $l$'s in $Y$ with $\pi(l) = C_0$ form the minuscule representation $V$ of $g$ corresponding to $C_k$ (Proposition 21). When $V$ is the standard representation of $g$, the configuration of these $(-1)$-curves determines a symmetric tensor $f$ on $V$ such that $g$ is the space of infinitesimal symmetries of $(V, f)$. We consider the bundle

$$\mathcal{L}_0^{(g, V)} := \bigoplus_{l \in (-1)-\text{curve}, \pi(l) = C_0} O_Y(l)$$

over $Y$ constructed from these $(-1)$-curves $l$'s. This bundle can not descend to $X$ as it is not trivial over each $C_i$.

Theorem 3 (Theorem 22 and Theorem 24)

For the bundle $\mathcal{L}_0^{(g, V)}$ with the corresponding minuscule representation $\rho : g \to \text{End}(V)$,

(i) there exists $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha \in \Phi^+} O(\alpha))$ such that $\overline{\partial} g := \overline{\partial} \rho + \rho(\varphi)$ is a holomorphic structure on $\mathcal{L}_0^{(g, V)}$. We denote this new holomorphic bundle as $\mathcal{L}_\varphi^{(g, V)}$.

1Here $V$ is the lowest weight representation with lowest weight dual to $-C_k$, i.e. $V$ is dual to the highest weight representation with highest weight dual to $C_k$.

2Unless specify otherwise, $C_i$ always refers to an irreducible component of $C$, i.e. $i \neq 0$.  


(ii) $\mathcal{L}_x^{(g,V)}$ is trivial on $C_i$ if and only if $[\varphi_{C_i}|C_i] \neq 0 \in H^1(Y,O_{C_i}(C_i))$.

(iii) When $V$ is the standard representation of $\mathfrak{g}$, there exists a holomorphic fiberwise symmetric multi-linear form

$$ f : \bigotimes^r \mathcal{L}_x^{(g,V)} \to O_Y(D) $$

with $r = 0, 2, 3, 4$ when $\mathfrak{g} = A_n, D_n, E_6, E_7$ respectively such that $E_x^g \cong \text{aut}_0(\mathcal{L}_x^{(g,V)}, f)$.

When $V$ is a minuscule representation of $\mathfrak{g}$, there exists a unique holomorphic structure on $\mathfrak{L}_0^{(g,V)} := \bigoplus_i O(l)$ such that the action of $E_x^g$ on this bundle is holomorphic and it can descend to $X$ as well.

**Example 4** When we blowup 2 distinct points, we have a surface $Y$ with 2 $(-1)$-curves $l_1$ and $l_2$ as exceptional curves. $\mathfrak{L}_0 := O_Y(l_1) \oplus O_Y(l_2)$ is a $\mathbb{C}^2$-bundle and the bundle $\zeta_0^{A_1}$ of its symmetries is a $sl(2)$- or $A_1$-bundle over $Y$.

When the 2 points become infinitesimally close, then $C_1 = l_2 - l_1$ is effective, namely a $(-2)$-curve in $Y$. If we blow down $C_1$ in $Y$, we get a surface $X$ with an $A_1$ singularity. $\mathfrak{L}_0$ cannot descend to $X$ as $\mathfrak{L}_0|_{C_1} \cong O_{P^1}(-1) \oplus O_{P^1}(1)$. Using the Euler sequence $0 \to O_{P^1}(-1) \to O_{P^1}^\oplus 2 \to O_{P^1}(1) \to 0$, we deform $\mathfrak{L}_0|_{C_1}$ to become trivial and using $p_g = 0$ to lift this deformation to $Y$. The resulting bundles $\mathfrak{L}_x$ and $\zeta_0^{A_1}$ do descend to $X$.

For every ADE case with $V$ the standard representation, we have $\mathfrak{L}_0^{(g,V)}|_{C_1} \cong O_{P^1}^{\oplus m} + (O_{P^1}(1) + O_{P^1}(-1))^{\oplus n}$. For $A_n$ cases, our arguments are similar to the above $A_1$ case. For $D_n$ cases, further arguments are needed as the pairs of $O_{P^1}(\pm 1)$ in $\mathfrak{L}_0^{(D_n, \mathbb{C}^{2n})}|_{C_1}$ are in different locations comparing with the $A_n$ cases, and we also need to check the holomorphic structure $\mathcal{J}_x$ on $\mathfrak{L}_0^{(D_n, \mathbb{C}^{2n})}$ preserves the natural quadratic form $q$. For the $E_6$ (resp. $E_7$) case, since the cubic form $c$ (resp. quartic form $t$) is more complicated than the quadratic form $q$ in $D_n$ cases, the calculations are more involved. The $E_8$ case is rather different and we handle it by reductions to $A_7$ and $D_7$ cases.

The organization of this paper is as follows. Section 2 gives the construction of ADE Lie algebra bundles over $Y$ directly. In section 3, we review the definition of minuscule representations and construct all minuscule representations using $(-1)$-curves in $Y$. Using these, we construct the Lie algebra bundles and minuscule representation bundles which can descend to $X$ in $A_n, D_n$ and $E_n$ ($n \neq 8$) cases separately in section 4, 5 and 6. The proofs of the main theorems in this paper are given in section 7.

Notations: for a holomorphic bundle $(E_0, \overline{\partial}_0)$, if we construct a new holomorphic structure $\mathcal{J}_x$ on $E_0$, then we denote the resulting bundle as $E_x$.

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2 \hspace{1em} ADE Lie algebra bundles

2.1 \hspace{1em} ADE singularities

A rational singularity \( p \) in a surface \( X \) can be described locally as a quotient singularity \( \mathbb{C}^2/\Gamma \) with \( \Gamma \) a finite subgroup of \( SL(2, \mathbb{C}) \). It is also called a Kleinian singularity or ADE singularity \(^2\). We can write \( \mathbb{C}^2/\Gamma \) as zeros of a polynomial \( F(X, Y, Z) \) in \( \mathbb{C}^3 \), where \( F(X, Y, Z) = X^n + YZ, X^{n+1} + XY^2 + Z^2, X^4 + Y^3 + Z^2, X^3Y + Y^3 + Z^2 \) or \( X^5 + Y^3 + Z^2 \) and the corresponding singularity is called of type \( A_n, D_n, E_6, E_7 \) or \( E_8 \) respectively. The reason is if we consider the minimal resolution \( Y \) of \( X \), then every irreducible component of the exceptional divisor \( C = \sum_{i=1}^n C_i \) is a smooth rational curve with normal bundle \( O_{\mathbb{P}^1}(-2) \), i.e. a \((-2)\)-curve, and the dual graph of the exceptional divisor is an ADE Dynkin diagram. The corresponding roots in the Dynkin diagrams are labelled as follows:

![Figure 1. The root system of A_n](image)

![Figure 2. The root system of D_n](image)

![Figure 3. The root system of E_n](image)

There is a natural decomposition

\[
H^2(Y, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus \Lambda,
\]

where \( \Lambda = \{ \sum a_i[C_i] | a_i \in \mathbb{Z} \} \). The set \( \Phi := \{ \alpha \in \Lambda | \alpha^2 = -2 \} \) is a simply-laced (i.e. ADE) root system of a simple Lie algebra \( g \) and \( \Delta = \{ [C_i] \} \) is a base of \( \Phi \). For any \( \alpha \in \Phi \), there exists a unique divisor \( D = \sum a_iC_i \) with \( \alpha = [D] \), and we define a line bundle \( O(\alpha) := O(D) \) over \( Y \).
2.2 Lie algebra bundles

We define a Lie algebra bundle of type $\mathfrak{g}$ over $Y$ as follows:

$$\mathcal{E}_0^\mathfrak{g} := O^\oplus_n \oplus_{\alpha \in \Phi} O(\alpha).$$

For every open chart $U$ of $Y$, we take $x^U_\alpha$ to be a nonvanishing holomorphic section of $O_U(\alpha)$ and $h^U_i$ ($i = 1, \ldots, n$) nonvanishing holomorphic sections of $O^\oplus_n$. Define a Lie algebra structure $[\cdot, \cdot]$ on $\mathcal{E}_0^\mathfrak{g}$ such that $\{x^U_\alpha\}$'s, $\{h^U_i\}$'s is the Chevalley basis [12], i.e.

(a) $[h^U_i, h^U_j] = 0$, $1 \leq i, j \leq n$.

(b) $[h^U_i, x^U_\alpha] = \langle \alpha, C_i \rangle x^U_\alpha$, $1 \leq i \leq n, \alpha \in \Phi$.

(c) $[x^U_i, x^U_\alpha] = h^U_\alpha$ is a $\mathbb{Z}$-linear combination of $h^U_i$.

(d) If $\alpha, \beta$ are independent roots, and $\beta - r\alpha, \ldots, \beta + qa\alpha$ is the $\alpha$-string through $\beta$, then $[x^U_\alpha, x^U_\beta] = 0$ if $q = 0$, otherwise $[x^U_\alpha, x^U_\beta] = \pm(r + 1)x^U_{\alpha + \beta}$.

Since $\mathfrak{g}$ is simply-laced, all its roots have the same length, we have any $\alpha$-string through $\beta$ is of length at most 2. So (d) can be written as $[x^U_\alpha, x^U_\beta] = n_{\alpha, \beta}x^U_{\alpha + \beta}$, where $n_{\alpha, \beta} = \pm 1$ if $\alpha + \beta \in \Phi$, otherwise $n_{\alpha, \beta} = 0$. From the Jacobi identity, we have for any $\alpha, \beta, \gamma \in \Phi$, $n_{\alpha, \beta}n_{\alpha + \beta, \gamma} + n_{\beta, \gamma}n_{\beta + \gamma, \alpha} + n_{\gamma, \alpha}n_{\gamma + \alpha, \beta} = 0$.

This Lie algebra structure is compatible with different trivializations of $\mathcal{E}_0^\mathfrak{g}$ [13].

By Friedman-Morgan [8], a bundle over $Y$ can descend to $X$ if and only if its restriction to each irreducible component $C_i$ of the exceptional divisor is trivial. But $\mathcal{E}_0^\mathfrak{g}|_{C_i}$ is not trivial as $O([C_i]|_{C_i}) \cong O_{\mathbb{P}^1}(-2)$. We will construct a new holomorphic structure on $\mathcal{E}_0^\mathfrak{g}$, which preserves the Lie algebra structure and therefore the resulting bundle $\mathcal{E}_0^\mathfrak{g}$ can descend to $X$.

As we have fixed a base $\Delta$ of $\Phi$, we have a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots.

**Definition 5** Given any $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha \in \Phi} O(\alpha))$, we define $\overline{\varphi} : \Omega^{0,0}(Y, \mathcal{E}_0^\mathfrak{g}) \to \Omega^{0,1}(Y, \mathcal{E}_0^\mathfrak{g})$ by

$$\overline{\partial}_\varphi := \overline{\partial}_0 + ad(\varphi) := \overline{\partial}_0 + \sum_{\alpha \in \Phi^+} \overline{ad}(\varphi_\alpha),$$

where $\overline{\partial}_0$ is the standard holomorphic structure of $\mathcal{E}_0^\mathfrak{g}$. More explicitly, if we write $\varphi_\alpha = c^U_\alpha x^U_\alpha$ locally for some one form $c^U_\alpha$, then $ad(\varphi_\alpha) = c^U_\alpha \overline{ad}(x^U_\alpha)$.

**Proposition 6** $\overline{\varphi}$ is compatible with the Lie algebra structure, i.e. $\overline{\varphi}[\cdot, \cdot] = 0$.

**Proof.** This follows directly from the Jacobi identity. ■

For $\overline{\varphi}$ to define a holomorphic structure, we need

$$0 = \overline{\partial}_\varphi := \sum_{\alpha \in \Phi^+} \overline{\partial}_0 c^U_\alpha + \sum_{\beta + \gamma = \alpha} (n_{\beta, \gamma}c^U_\beta c^U_\gamma)ad(x^U_\alpha),$$

that is $\overline{\partial}_0 \varphi_\alpha + \sum_{\beta + \gamma = \alpha} (n_{\beta, \gamma}\varphi_\beta \varphi_\gamma) = 0$ for any $\alpha \in \Phi^+$. Explicitly:

$$\begin{cases}
\overline{\partial}_0 \varphi_{C_i} = 0 & i = 1, 2, \ldots, n \\
\overline{\partial}_0 \varphi_{C_i + C_j} = nc_{i, C_j} \varphi_{C_i} \varphi_{C_j} & \text{if } C_i + C_j \in \Phi^+ \\
\vdots
\end{cases}$$
Proposition 7 Given any $(\varphi_{C_i})_{i=1}^n \in \Omega^{0,1}(Y, \bigoplus_{i=1}^n O(C_i))$ with $\overline{\partial}_\varphi \varphi_{C_i} = 0$ for every $i$, it can be extended to $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha \in \Phi^+} O(\alpha))$ such that $\overline{\partial}_\varphi^2 = 0$. Namely we have a holomorphic vector bundle $E_{\varphi}^0$ over $Y$.

To prove this proposition, we need the following lemma. For any $\alpha = \sum_{i=1}^n a_i C_i \in \Phi^+$, we define $ht(\alpha) := \sum_{i=1}^n a_i$.

Lemma 8 For any $\alpha \in \Phi^+$, $H^2(Y, O(\alpha)) = 0$.

Proof. If $ht(\alpha) = 1$, i.e. $\alpha = C_i$, $H^2(Y, O(C_i)) = 0$ follows from the long exact sequence associated to $0 \to O_Y \to O_Y(C_i) \to O_{C_i}(C_i) \to 0$ and $p_2 = 0$.

By induction, suppose the lemma is true for every $\beta$ with $ht(\beta) = m$. Given any $\alpha$ with $ht(\alpha) = m+1$, by Lemma 4 in §10.2 of [13], there exists some $C_i$ such that $\alpha \cdot C_i = -1$, i.e. $\beta := \alpha - C_i \in \Phi^+$ with $ht(\beta) = m$. Using the long exact sequence associated to $0 \to O_Y(\beta) \to O_Y(\alpha) \to O_{C_i}(\alpha) \to 0$, $O_{C_i}(\alpha) \cong O_Y(-1)$ and $H^2(Y, O(\beta)) = 0$ by induction, we have $H^2(Y, O(\alpha)) = 0$. ■

Proof. (of Proposition 7) We solve the equations $\overline{\partial}_\varphi \varphi_\alpha = \sum_{\beta + \gamma = \alpha} n_{\beta, \gamma} \varphi_\beta \varphi_\gamma$ for $\varphi_\alpha \in \Omega^{0,1}(Y, O(\alpha))$ inductively on $ht(\alpha)$.

For $ht(\alpha) = 2$, i.e. $\alpha = C_i + C_j$ with $C_i \cdot C_j = 1$, since $[\varphi_{C_i}, \varphi_{C_j}] \in H^2(Y, O(C_i + C_j)) = 0$, we can find $\varphi_{C_i + C_j}$ satisfying $\overline{\partial}_\varphi \varphi_{C_i + C_j} = \pm \varphi_{C_i} \varphi_{C_j}$.

Suppose we have solved the equations for all $\varphi_\beta$’s with $ht(\beta) \leq m$. For

$\overline{\partial}_\varphi \varphi_\alpha = \sum_{\beta + \gamma = \alpha} n_{\beta, \gamma} \varphi_\beta \varphi_\gamma$

with $ht(\alpha) = m+1$, we have $ht(\beta), ht(\gamma) \leq m$. Using $\overline{\partial}_\varphi \sum_{\beta + \gamma = \alpha} n_{\beta, \gamma} \varphi_\beta \varphi_\gamma = \sum_{\delta + \lambda + \mu = \alpha} (n_{\delta, \lambda} n_{\delta, \mu} + n_{\lambda, \mu} n_{\lambda, \delta} + n_{\mu, \delta} n_{\mu, \lambda}) \varphi_\delta \varphi_\lambda \varphi_\mu = 0$, we can solve for $\varphi_\alpha$. ■

Denote $\Psi_Y \triangleq \{ \varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha \in \Phi^+} O(\alpha)) | \overline{\partial}_\varphi^2 = 0 \}$, and

$\Psi_X \triangleq \{ \varphi \in \Psi_Y | [\varphi_{C_i}|_{C_i}] \neq 0 \text{ for } i = 1, 2, \cdots, n \}$.

Theorem 9 $E_{\varphi}^0$ is trivial on $C_i$ if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(Y, O_{C_i}(C_i))$.

Proof. We will discuss the ADE cases separately in §4, §5, §6 and the proof will be completed in §7. ■

The next lemma says that given any $C_i$, there always exists $\varphi_{C_i} \in \Omega^{0,1}(Y, O(C_i))$ such that $0 \neq [\varphi_{C_i}|_{C_i}] \in H^1(Y, O_{C_i}(C_i)) \cong \mathbb{C}$.

Lemma 10 For any $C_i$ in $Y$, the restriction homomorphism $H^1(Y, O_Y(C_i)) \rightarrow H^1(Y, O_{C_i}(C_i))$ is surjective.

Proof. The above restriction homomorphism is part of a long exact sequence induced by $0 \to O_Y \to O_Y(C_i) \to O_{C_i}(C_i) \to 0$. The lemma follows directly from $p_2(Y) = 0$. ■
3 Minuscule representations and \((-1)\)-curves

3.1 Standard representations

For \(ADE\) Lie algebras, \(A_n = sl(n+1)\) is the space of tracefree endomorphisms of \(\mathbb{C}^{n+1}\) and \(D_n = o(2n)\) is the space of infinitesimal automorphisms of \(\mathbb{C}^{2n}\) which preserve a non-degenerate quadratic form \(q\) on \(\mathbb{C}^{2n}\). In fact, \(E_6\) (resp. \(E_7\)) is the space of infinitesimal automorphisms of \(\mathbb{C}^{27}\) (resp. \(\mathbb{C}^{56}\)) which preserve a particular cubic form \(c\) on \(\mathbb{C}^{27}\) (resp. quartic form \(t\) on \(\mathbb{C}^{56}\)) \[1\]. We call the above representation the **standard representation** of \(g\), i.e.

| \(g\)       | standard representation |
|-------------|-------------------------|
| \(A_n = sl(n+1)\) | \(\mathbb{C}^{n+1}\)   |
| \(D_n = o(2n)\)   | \(\mathbb{C}^{2n}\)   |
| \(E_6\)           | \(\mathbb{C}^{27}\)   |
| \(E_7\)           | \(\mathbb{C}^{56}\)   |

Note all these standard representations are the fundamental representations corresponding to the left nodes (i.e. \(C_1\)) in the corresponding Dynkin diagrams (Figure 1, 2 and 3) and they are minuscule representations.

3.2 Minuscule representations

**Definition 11** A minuscule (resp. quasi-minuscule) representation of a semi-simple Lie algebra is an irreducible representation such that the Weyl group acts transitively on all the weights (resp. non-zero weights).

Minuscule representations are always fundamental representations and quasi-minuscule representations are either minuscule or adjoint representations.

| \(g\)       | Minuscule representations                           |
|-------------|-----------------------------------------------------|
| \(A_n = sl(n+1)\) | \(\wedge^k \mathbb{C}^{n+1}\) for \(k = 1, 2, \ldots, n\) |
| \(D_n = o(2n)\)   | \(\mathbb{C}^{2n}, \quad S^+, \quad S^-\)           |
| \(E_6\)           | \(\mathbb{C}^{27}, \quad \mathbb{C}^{27}\)         |
| \(E_7\)           | \(\mathbb{C}^{56}\)                                 |

Note \(E_8\) has no minuscule representation.

3.3 Configurations of \((-1)\)-curves

In this subsection, we describe \((-1)\)-curves in \(X\) and \(Y\).

**Definition 12** A \((-1)\)-curve in a surface \(Y\) is a genus zero (possibly reducible) curve \(l\) in \(Y\) with \(l \cdot l = -1\).

**Remark 13** The genus zero condition can be replaced by \(l \cdot K_Y = -1\) by the genus formula, where \(K_Y\) is the canonical divisor of \(Y\).
Let $C_0$ be a curve in $X$ passing through $p$.

**Definition 14** (i) $C_0$ is called a $(-1)$-curve in $X$ if there exists a $(-1)$-curve $l$ in $Y$ such that $\pi(l) = C_0$, or equivalently the strict transform of $C_0$ is a $(-1)$-curve $C_0$ in $Y$. (ii) The multiplicity of $C_0$ at $p$ is defined to be $\sum a_i[\mathcal{C}_i] \in \Lambda$, where $a_i = C_0 \cdot \mathcal{C}_i$.

Recall from Lie theory, any irreducible representation of a simple Lie algebra is determined by its lowest weight. The fundamental representations\(^3\) are those irreducible representations whose lowest weight is dual to the negative of some base root. If $C_0 \subset X$ has multiplicity $C_k$ at $p$ whose dual weight determines a minuscule representation $V$, then we use $C_0^{\text{\(k\)}}$ to denote $C_0$. The construction of such $X$'s and $C_0$'s can be found in appendix.

**Definition 15** (i) We call $C_0$ has minuscule multiplicity $C_k \in \Lambda$ at $p$ if $C_0$ has multiplicity $C_k$ and the dual weight of $-C_k$ determines a minuscule representation $V$. (ii) In this case, we denote $I^{(g,V)} = \{ l : (-1)\text{-curve in } Y| \pi(l) = C_0 \}$.

If there is no ambiguity, we will simply write $I^{(g,V)}$ as $I$. Note that $I \subset C_0^k + \Lambda_{\geq 0}$, where $\Lambda_{\geq 0} = \{ \sum a_i[\mathcal{C}_i] : a_i \geq 0 \}$.

**Lemma 16** In the above situation, the cardinality of $I$ is given by $|I| = \dim V$.

**Proof.** By the genus formula and every $\mathcal{C}_i \cong \mathbb{P}^1$ being a $(-2)$-curve, we have $\mathcal{C}_i \cdot K_Y = 0$. Since $C_0^k \cdot K_Y = -1$, each $(-1)$-curve has the form $l = C_0^k + \sum a_i \mathcal{C}_i$ with $a_i$'s non-negative integers. From $l \cdot l = -1$, we can determine $\{ a_i \}$'s for $l$ to be a $(-1)$-curve by direct computations. ■

**Remark 17** The intersection product is negative definite on the sublattice of $\text{Pic}(X)$ generated by $C_0^{k_1}, C_1, \cdots , C_n$ and we use its negative as an inner product.

**Lemma 18** In the above situation, for any $l \in I$, $\alpha \in \Phi$, we have $l \cdot \alpha \leq 1$.

**Proof.** We claim that for any $v \in C_0^k + \Lambda$, we have $v \cdot v \leq -1$. We prove the claim by direct computations. In $(\mathbb{A}_n,\wedge^k \mathbb{C}^{n+1})$ case:

\[
(C_0^k + \sum a_i \mathcal{C}_i)^2 = -1 + 2a_k - (a_k^2 + (a_1 - a_2)^2 + \cdots + (a_{k-1} - a_k)^2) - (a_k - a_{k+1})^2 + \cdots + a_n^2 \leq -1.
\]

The other cases can be proven similarly.

Since $l \cdot l + \alpha, l - \alpha \in C_0^k + \Lambda$ by assumptions, we have $l \cdot l = -1 \geq (l + \alpha) \cdot (l - \alpha)$, hence $l \cdot \alpha \leq 1$. Also $l \cdot l = -1 \geq (l - \alpha) \cdot (l - \alpha)$, hence $l \cdot \alpha \geq -1$. ■

**Lemma 19** In the above situation, for any $l \in I$ which is not $C_0^k$, there exists $C_i$ such that $l \cdot \mathcal{C}_i = -1$.

**Proof.** From $l = C_0^k + \sum a_i \mathcal{C}_i \neq C_0^k$ ($a_i \geq 0$), we have $a_k \geq 1$. From $l \cdot l = -1$, we have $(\sum a_i \mathcal{C}_i)^2 = -2a_k$. If there does not exist such an $i$ with $l \cdot \mathcal{C}_i = -1$, then by Lemma 18, $l \cdot \mathcal{C}_i \geq 0$ for every $i$, $l \cdot (\sum a_i \mathcal{C}_i) \geq 0$. But $l \cdot (\sum a_i \mathcal{C}_i) = a_k + (\sum a_i \mathcal{C}_i)^2 = -a_k \leq -1$ leads to a contradiction. ■

\(^3\)The usual definition for fundamental representations uses highest weight. But in this paper, we will use lowest weight for simplicity of notations.
Lemma 20 In the above situation, for any \( l, l' \in I \), \( H^2(Y, O(l - l')) = 0 \).

Proof. Firstly, we prove \( H^2(Y, O(C_k^0 - l)) = 0 \) for any \( l = C_k^0 + \sum a_i C_i \in I \) inductively on \( \text{ht}(l) := \sum a_i \). If \( \text{ht}(l) = 0 \), i.e. \( l \in C_k^0 \), the claim follows from Lemma 17. Suppose the claim is true for any \( l' \in I \) with \( \text{ht}(l') \leq m - 1 \). Then for any \( l \in I \) with \( \text{ht}(l) = m \), by Lemma 17 there exists \( i \) such that \( l \cdot C_i = -1 \). This implies \( (l - C_i) \in I \) with \( \text{ht}(l - C_i) = m - 1 \) and therefore \( H^2(Y, O(C_k^0 - (l - C_i))) = 0 \) by induction hypothesis. Using the long exact sequence induced from

\[
0 \to O_Y(C_k^0 - l) \to O_Y(C_k^0 - (l - C_i)) \to O_{C_i}(C_k^0 - (l - C_i)) \to 0
\]

and \( O_{C_i}(C_k^0 - (l - C_i)) \cong O_{P^1}(-1) \) or \( O_{P^1} \), we have the claim.

If \( H^2(Y, O(l - l')) \neq 0 \), then there exists a section \( s \in H^0(Y, K_Y(l' - l)) \) by Serre duality. Since there exists a nonzero section \( t \in H^0(Y, O(l - C_k^0)) \), we have \( st \in H^0(Y, K_Y(l' - C_k^0)) \cong H^2(Y, O(C_k^0 - l')) = 0 \), which is a contradiction. \( \blacksquare \)

3.4 Minuscule representations from \((-1\text{-}curves)

Recall from the ADE root system \( \Phi \), we can recover the corresponding Lie algebra \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \). As before, we use \( \{x_\alpha, s, h_i\}'s \) to denote its Chevalley basis. If \( C_0 \) has minuscule multiplicity \( C_k \), we denote

\[
V_0 := \mathcal{C}' = \bigoplus_{l \in l} \mathbb{C}(v_l),
\]

where \( v_l \) is the base vector of \( V_0 \) generated by \( l \). Then we define a bilinear map \([,] : \mathfrak{g} \otimes V_0 \to V_0 \) (possibly up to \( \pm \) signs) as follows:

\[
[x, v_l] = \begin{cases} 
\langle x, l \rangle v_l & \text{if } x \in \mathfrak{h} \\
\pm v_{l+\alpha} & \text{if } x = x_\alpha, \ l + \alpha \in I \\
0 & \text{if } x = x_\alpha, \ l + \alpha \notin I
\end{cases}
\]

Proposition 21 The signs in the above bilinear map \( \mathfrak{g} \otimes V_0 \to V_0 \) can be chosen so that it defines an action of \( \mathfrak{g} \) on \( V_0 \). Moreover, \( V_0 \) is isomorphic to the minuscule representation \( V \).

Proof. For the first part, similar to \( \mathfrak{le} \), we use Lemma 17 to show \([x, y, v_l] = [x, [y, v_l]] - [y, [x, v_l]]\).

For the second part, since \([x_\alpha, v_{C_k}] = 0 \) for any \( \alpha \in \Phi^- \), \( v_{C_k} \) is the lowest weight vector of \( V_0 \) with weight corresponding to \(-C_k \). Also we know the fundamental representation \( V \) corresponding to \(-C_k \) has the same dimension with \( V_0 \) by Lemma 17. Hence \( V_0 \) is isomorphic to the minuscule representation \( V \). \( \blacksquare \)

Here we show how to determine the signs. Take any \( l \in I \), \( v_l \) is a weight vector of the above action. For \( x = x_\alpha \) and \( v_l \) with weight \( w \), we define \([x, v_l] = n_{\alpha, w} v_{l+\alpha} \), where \( n_{\alpha, w} = \pm 1 \) if \( l + \alpha \in I \), otherwise \( n_{\alpha, w} = 0 \). By \([x, y, v_l] = [x, [y, v_l]] - [y, [x, v_l]]\), we have \( n_{\alpha, \beta} n_{\alpha+\beta, w} - n_{\beta, w} n_{\alpha, \beta+w} + n_{\alpha, w} n_{\beta, \alpha+w} = 0 \).
Remark 22 Recall for any \( l = c^k_0 + \sum_i a_i C_i \in \mathcal{I} \), we define \( \text{ht}(l) := \sum a_i \). Using this, we can define a filtered structure for \( \mathcal{I} : \mathcal{I} = \mathcal{I}_0 \supset \mathcal{I}_1 \supset \cdots \supset \mathcal{I}_m \), where \( m = \max_{l \in \mathcal{I}} \text{ht}(l) \), \( \mathcal{I}_i = \{ l \in \mathcal{I} | \text{ht}(l) \leq m-i \} \) and \( \mathcal{I}_{i+1} \backslash \mathcal{I}_i = \{ l \in \mathcal{I} | \text{ht}(l) = m-i \} \).

This \( \text{ht}(l) \) also enables us to define a partial order of \( \mathcal{I} \). Say \( | \mathcal{I} | = N \), we denote \( l_N := c^k_0 \) since it is the only element with \( \text{ht} = 0 \). Similarly, \( l_{N-1} := c^k_0 + C_k \).

Of course, there are some ambiguity of this ordering, if so, we will just make a choice to order these \((-1)\)-curves.

3.5 Bundles from \((-1)\)-curves

The geometry of \((-1)\)-curves in \( Y \) can be used to construct representation bundles of \( \mathcal{E}^g_\phi \) for every minuscule representation of \( g \). The proofs of theorems in this subsection will be given in §7.

When \( C_0 \subset X \) has minuscule multiplicity \( C_k \) at \( p \) with the corresponding minuscule representation \( V \), we define\(^4\)

\[
\mathcal{L}^{(g,V)}_0 := \bigoplus_{l \in \mathcal{I}(g,V)} O(l).
\]

\( \mathcal{L}^{(g,V)}_0 \) has a natural filtration \( F^* : \mathcal{L}^{(g,V)}_0 = F^0 \mathcal{L} \supset F^1 \mathcal{L} \supset \cdots \supset F^m \mathcal{L} \), induced from the filtered structure on \( \mathcal{I} \), namely \( F^i \mathcal{L}^{(g,V)}_0 = \bigoplus_{l \in \mathcal{I}_i} O(l) \).

\( \mathcal{L}^{(g,V)}_0 \) can not descend to \( X \) as \( O_{C_k}(C_0^k) \cong O_{p^1}(1) \) (because \( C_k \cdot C_0^k = 1 \) by the definition of the minuscule multiplicity). For any \( C_i \) and any \( l \in \mathcal{I} \), we have \( O_{C_i}(l) \cong O_{p^1}(\pm 1) \) or \( O_{p^1} \) by Lemma \( \text{[3]} \). For every fixed \( C_i \), if there is a \( l \in \mathcal{I} \) such that \( O_{C_i}(l) \cong O_{p^1}(1) \), then \( (l + C_i)^2 = -1 = (l + C_i) \cdot K_Y \), i.e. \( l + C_i \in \mathcal{I} \), also \( O_{C_i}(l + C_i) \cong O_{p^1}(-1) \). That means among the direct summands of \( \mathcal{L}^{(g,V)}_0|_{C_i}, O_{p^1}(1) \) and \( O_{p^1}(-1) \) occur in pairs, and each pair is given by two \((-1)\)-curves in \( I \) whose difference is \( C_i \). This gives us a chance to deform \( \mathcal{L}^{(g,V)}_0 \) to get another bundle which can descend to \( X \).

Theorem 23 If there exists a \((-1)\)-curve \( C_0 \) in \( X \) with minuscule multiplicity \( C_k \) at \( p \) and \( \rho : g \rightarrow \text{End}(V) \) is the corresponding representation, then \( \mathcal{L}^{(g,V)}_\phi := \bigoplus_{l \in \mathcal{I}} O(l), \overline{\partial}_\phi := \overline{\partial}_0 + \rho(\phi) \) with \( \phi \in \Psi_Y \) is a holomorphic bundle over \( Y \) which preserves the filtration on \( \mathcal{L}^{(g,V)}_0 \) and it is a holomorphic representation bundle of \( \mathcal{E}^g_\phi \). Moreover, \( \mathcal{L}^{(g,V)}_\phi \) is trivial on \( C_i \) if and only if \( [\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(Y, O_{C_i}(C_i)) \).

For \( C_k \) with \( k = 1 \), the corresponding minuscule representation \( V \) is the standard representation of \( g \). When \( g = A_n \), it is simply \( sl(n + 1) = \text{aut}_0(V) \). When \( g = D_n \) (resp. \( E_6 \) and \( E_7 \)), there exists a quadratic (resp. cubic and quartic) form \( f \) on \( V \) such that \( g = \text{aut}(V,f) \). The next theorem tells us that we can globalize this construction over \( Y \) to recover the Lie algebra bundle \( \mathcal{E}^g_\phi \) over \( Y \). But this does not work for \( \mathcal{E}^g_{E_8} \) as \( E_8 \) has no standard representation.

\(^4\)When \( X \) is a del Pezzo surface, we use lines in \( X \) to construct bundles [FM]. So here we use \((-1)\)-curves in \( X \) to construct bundles.
Theorem 24  Under the same assumptions as in theorem 23 with $k = 1$, there exists a holomorphic fiberwise symmetric multi-linear form

$$f : \bigotimes^r \Omega^{(g,V)} \rightarrow O_Y(D)$$

with $r = 0, 2, 3, 4$ when $g = A_n, D_n, E_6, E_7$ respectively such that $E^{g}_\varphi \cong \text{auto}(\Omega^{(g,V)}, f)$. It is obvious that $E^{g}_\varphi$ does not depend on the existence of the $(-1)$-curve $C_0$, for the minuscule representation bundles, we have the following results.

Theorem 25  There exists a divisor $B$ in $Y$ and an integer $k$, such that the bundle $L_{\varphi}^{g(V)} := S^k \Omega^{(g,V)} \otimes O(-B)$ with $\varphi \in \Psi_X$ can descend to $X$ and does not depend on the existence of $C_0$.

3.6 Outline of Proofs for $g \neq E_8$

When $g \neq E_8$, there exists a natural symmetric tensor $f$ on its standard representation $V$ such that $g = \text{aut}_V(V, f)$. The set $I^{(g,V)}$ of $(-1)$-curves has cardinality $N = \dim V$. Given $\eta := (\eta_{i,j})_{N \times N}$ with $\eta_{i,j} \in \mathcal{O}^{0,1}(Y, O(l_i - l_j))$ for every $l_i \neq l_j \in I^{(g,V)}$, we consider the operator $\overline{\partial}_{\eta} := \overline{\partial}_0 + \eta$ on $\mathcal{L}^{(g,V)}_0 := \bigoplus_{(i,j) \in I^{(g,V)}} O_Y(l)$. We will look for $\eta$ which satisfy:

1. (filtration) $\eta_{i,j} = 0$ for $i > j$ for the partial ordering introduced in §3.4.
2. (holomorphic structure) $(\overline{\partial}_0 + \eta)^2 = 0$.
3. (Lie algebra structure) $\overline{\partial}_\eta f = 0$.
4. (descendent) For every $C_k$, if $l_i - l_j = C_k$, then $0 \neq [\eta_{i,j}|_{C_k}] \in H^1(Y, O_{C_k}(C_k))$.

Remark 26  Property (2) implies that we can define a new holomorphic structure on $\mathcal{L}_0^{(g,V)}$. Properties (1) and (3) require that for any $\eta_{i,j} \neq 0$, $\eta_{i,j} \in \mathcal{O}^{0,1}(Y, O(\alpha))$ for some $\alpha \in \Phi^+$. We will show that if $\eta$ satisfies (1), (2) and (3), then (4) is equivalent to $\mathcal{L}_0^{(g,V)}$ being trivial on every $C_k$, i.e. $\mathcal{L}_0^{(g,V)}$ can descend to $X$.

Denote

$$\Xi^0_V \triangleq \{ \eta = (\eta_{i,j})_{N \times N} | \eta \text{ satisfies (1), (2) and (3)} \},$$

and

$$\Xi^0_X \triangleq \{ \eta \in \Xi^0_V | \eta \text{ satisfies (4)} \},$$

then each $\eta$ in $\Xi^0_V$ determines a filtered holomorphic bundle $\mathcal{L}_\eta^{(g,V)}$ over $Y$ together with a holomorphic tensor $f$ on it. It can descend to $X$ if $\eta \in \Xi^0_X$.

Since $g = \text{aut}(V, f)$, for any $\eta \in \Xi^0_V$, we have a holomorphic Lie algebra bundle $\mathfrak{g}_\eta := \text{aut}(\mathcal{L}_\eta^{(g,V)}, f)$ over $Y$ of type $g$, and $\mathcal{L}_\eta^{(g,V)}$ is automatically a representation bundle of $\mathfrak{g}_\eta$. Furthermore, if $\eta \in \Xi^0_X$, then $\mathfrak{g}_\eta$ can descend to $X$.

For a general minuscule representation of $g$, given any $\eta \in \Xi^0_V$, we show that there exists a unique holomorphic structure on $\mathcal{L}_\eta^{(g,V)}$, such that the action of $\mathfrak{g}_\eta$ on the new holomorphic bundle $\mathcal{L}_\eta^{(g,V)}$ is holomorphic. Furthermore, if $\eta \in \Xi^0_X$, then $\mathcal{L}_\eta^{(g,V)}$ can descend to $X$. 

11
4 $A_n$ case

We recall that $A_n = sl(n + 1, \mathbb{C}) = aut_0(\mathbb{C}^{n+1})$ (where $aut_0$ means tracefree endomorphisms). The standard representation of $A_n$ is $\mathbb{C}^{n+1}$ and minuscule representations of $A_n$ are $\wedge^k \mathbb{C}^{n+1}$, $k = 1, 2, \ldots, n$.

4.1 $A_n$ standard representation bundle $\mathcal{L}_0^{(A_n, \mathbb{C}^{n+1})}$

We consider a surface $X$ with an $A_n$ singularity $p$ and a $(-1)$-curve $C_0$ passing through $p$ with multiplicity $C_1$, then $I^{(A_n, \mathbb{C}^{n+1})} = \{ C_0^l + \sum_{i=1}^k C_i \mid 0 \leq k \leq n \}$ has cardinality $n + 1$. We order these $(-1)$-curves: $l_k = C_0^l + \sum_{i=1}^{n+1-k} C_i$ for $1 \leq k \leq n + 1$. For any $i \neq j \in I$, $i \cdot j = 0$. Fix any $C_i$, we have

$$l_k \cdot C_i = \begin{cases} 1, & k = n + 2 - i \\ -1, & k = n + 1 - i \\ 0, & \text{otherwise}. \end{cases}$$

Define $\mathcal{L}_0^{(A_n, \mathbb{C}^{n+1})} := \bigoplus_{l \in \mathcal{I}} O(l)$ over $Y$, for simplicity, we write it as $\mathcal{L}_0^{A_n}$. $\mathcal{L}_0^{A_n}$ can not descend to $X$, since for any $C_i$,

$$\mathcal{L}_0^{A_n}|_{C_i} \cong O^{\oplus(n-1)} \oplus O^{l_1}(1) \oplus O^{l_1}(-1).$$

Our aim is to find a new holomorphic structure on $\mathcal{L}_0^{A_n}$ such that the resulting bundle can descend to $X$. First, we define $\overline{\partial}_\eta : \Omega^{0,0}(Y, \mathcal{L}_0^{A_n}) \rightarrow \Omega^{0,1}(Y, \mathcal{L}_0^{A_n})$ on $\mathcal{L}_0^{A_n} = \bigoplus_{k=1}^{n+1} O(l_k)$ as follows:

$$\overline{\partial}_\eta = \begin{pmatrix} \overline{\partial} & \eta_{1,2} & \cdots & \eta_{1,n+1} \\ 0 & \overline{\partial} & \cdots & \eta_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\partial} \end{pmatrix}$$

where $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for any $j > i$. When $j > i$, $l_i - l_j \in \Lambda$ is a positive root because of $l_i \cdot l_j = 0$ and our ordering of $l_k$'s.

The integrability condition $\overline{\partial}_\eta = 0$ is equivalent to, for $i = 1, 2, \ldots, n$,

$$\begin{cases} \overline{\partial} \eta_{i,i+1} = 0, \\ \overline{\partial} \eta_{i,j} = - \sum_{m=i+1}^{j-1} \eta_{i,m} \eta_{m,j}, \quad j \geq i + 2, \end{cases}$$

Note $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j)) = \Omega^{0,1}(Y, O(\alpha))$ for some $\alpha \in \Phi^+$. From

$$\sum_{m=i+1}^{j-1} [\eta_{i,m} \eta_{m,j}] \in H^2(Y, O(l_i - l_j)) = 0,$$

we can find $\eta_{i,j}$, such that $\overline{\partial} \eta_{i,j} = - \sum_{m=i+1}^{j-1} \eta_{i,m} \eta_{m,j}$. That is
Proposition 27 Given any $\eta_{i+1} \in \Omega^{0,1}(Y, O(l_i-l_{i+1}))$ with $\overline{\partial}_i \eta_{i+1} = 0$ for $i = 1, 2, \cdots, n$, there exists $\eta_j \in \Omega^{0,1}(Y, O(l_i-l_j))$ for every $j > i$ such that $\overline{\partial}_j \eta$ defines a holomorphic structure on $\mathcal{L}_n^\mathcal{A}_n$, i.e. $\overline{\partial}_j \eta = 0$.

We want to prove that there exists $\eta \in \mathcal{L}_n^\mathcal{A}_n$ such that $\mathcal{L}_n^\mathcal{A}_n$ can descend to $X$, i.e. $\mathcal{L}_n^\mathcal{A}_n|_C$ is trivial for every $C_i$. To prove this, we will construct $n + 1$ holomorphic sections of $\mathcal{L}_n^\mathcal{A}_n|_C$, which are linearly independent everywhere on $C_i$. The following lemma will be needed for all the ADE cases.

Lemma 28 Consider a vector bundle $L := \bigoplus_{i=1}^N O(l_i)$, $\overline{\partial}_L = \overline{\partial}_0 + (\eta_{i,j})_{N \times N}$ over $Y$ with $\eta_{i,j} = 0$ whenever $i \geq j$. Suppose $C$ is a smooth (-2)-curve in $Y$ with $H^1(C, O_C(l_i)) = 0$ for every $i = 1, 2, \cdots, N$, then for any fixed $i$ and any $s_i \in H^0(C, O_C(l_i))$, the following equation for $s_1, s_2, \cdots, s_i$ has a solution,

\[
\left( \begin{array}{cccc} \overline{\partial} & \eta_{1,2} | C & \eta_{1,3} | C & \cdots & \eta_{1,N} | C \\ 0 & \overline{\partial} & \eta_{2,3} | C & \cdots & \eta_{2,N} | C \\ 0 & 0 & \overline{\partial} & \cdots & \eta_{3,N} | C \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \overline{\partial} \end{array} \right) \left( \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ \vdots \\ \vdots \\ s_i \end{array} \right) = 0.
\]

Proof. The above equation is equivalent to:

\[
\overline{\partial} s_i = 0, \quad \text{ (1)}
\]

\[
\eta_{i-1,i} s_i + \overline{\partial} s_{i-1} = 0, \quad \text{ (2)}
\]

\[
\vdots
\]

\[
\eta_{1,i} s_1 + \cdots + \eta_{1,2} s_2 + \overline{\partial} s_1 = 0. \quad \text{ (i)}
\]

Equation (1) is automatic as $s_i \in H^0(C, O_C(l_i))$. For equation (2), since $\overline{\partial} \eta_{i-1,i} = 0$ and $\overline{\partial} s_i = 0$, we have $[\eta_{i-1,i} s_i] \in H^1(C, O_C(l_{i-1})) = 0$, hence we can find $s_{i-1}$ satisfying $\overline{\partial} s_{i-1} = -\eta_{i-1,i} s_i$.

Inductively, suppose we have found $s_1, \cdots, s_{j-1}$ for the first $(i-j)$ equations, then for the $(i-j+1)$-th equation: $\eta_{j,i} s_i + \cdots + \eta_{j+1,j+1} s_{j+1} + \overline{\partial} s_j = 0$, we have

\[
\eta_{j,i} s_i + \cdots + \eta_{j+1,j+1} s_{j+1} \in \Omega^{0,1}(C, O_C(l_j)).
\]

From $\overline{\partial}_L = 0$, we have

\[
\overline{\partial} \eta_{k,m} = -(\eta_{k,k+1} \cdot \eta_{k+1,m} + \eta_{k,k+2} \cdot \eta_{k+2,m} + \cdots + \eta_{k,m-1} \cdot \eta_{m-1,m}).
\]

Then

\[
\overline{\partial} (s_i) = -(\eta_{m,m+1} s_{m+1} + \cdots + \eta_{m,i} s_i)
\]

implies

\[
\overline{\partial} (\eta_{j,i} s_i + \cdots + \eta_{j+1,j+1} s_{j+1}) = 0.
\]

Therefore $[\eta_{j,i} s_i + \cdots + \eta_{j+1,j+1} s_{j+1}] \in H^1(C, O_C(l_j)) = 0$, hence we can find $s_j$ such that $\overline{\partial} s_j = -(\eta_{j,i} s_i + \cdots + \eta_{j+1,j+1} s_{j+1})$. $\blacksquare$
Let us recall a standard result which says that the only non-trivial extension of \( O_{P^1}(1) \) by \( O_{P^1}(-1) \) is the trivial bundle. We will give an explicit construction of this trivialization as we will need a generalization of it later.

**Lemma 29** For an exact sequence over \( \mathbb{P}^1 : 0 \to O_{P^1}(-1) \to E \to O_{P^1}(1) \to 0 \), the bundle \( E \) is determined by the extension class \([\varphi] \in \text{Ext}^1_{\mathbb{P}^1}(O(1),O(-1)) \cong \mathbb{C} \) up to a scalar multiple. If \([\varphi] \neq 0\), \( E \) is trivial, namely there exists two holomorphic sections for \( E \) which are linearly independent at every point in \( \mathbb{P}^1 \).

**Proof.** With respect to the (topological) splitting \( E = O_{P^1}(-1) \oplus O_{P^1}(1) \), the holomorphic structure on \( E \) is given by

\[
\overline{\partial}_E = \begin{pmatrix} \overline{\partial} & \varphi \\ 0 & \overline{\partial} \end{pmatrix}
\]

with \( \varphi \in \text{Ext}^1_{\mathbb{P}^1}(O(1),O(-1)) \). Let \( t_1, t_2 \) be a base of \( H^0(\mathbb{P}^1,O(1)) \cong \mathbb{C}^2 \). Since \([\varphi t_1] \in H^1(\mathbb{P}^1,O(-1)) = 0 \), we can find \( u_1, u_2 \in \Omega^0(\mathbb{P}^1,O(-1)) \), such that

\[
\begin{pmatrix} \overline{\partial} & \varphi \\ 0 & \overline{\partial} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ t_1 \end{pmatrix} = 0,
\]

i.e. \( s_1 = (u_1, t_1)^t \) and \( s_2 = (u_2, t_2)^t \) are two holomorphic sections of \( E \). Explicitly, we can take \( s_1 = (\frac{1}{1+|l|^2}, \frac{z}{1+|l|^2})^t \) and \( s_2 = (\frac{1}{1+|l|^2}, 1)^t \) in the coordinate chart \( \mathbb{C} \subset \mathbb{P}^1 \). It can be checked that \( s_1 \) and \( s_2 \) are linearly independent over \( \mathbb{P}^1 \).

From the above lemma, we have the following result.

**Lemma 30** Under the same assumption as in Lemma 29, Suppose \( \Sigma|_C \cong O_{P^1}^{\oplus m} \oplus (O_{P^1}(1) \oplus O_{P^1}(-1))^{\oplus n} \) with each pair of \( O_{P^1}(\pm 1) \) corresponding to two \((-1)\)-curves \( l_i \) and \( l_{i+1} \) with \( l_i - l_{i+1} = C \). Then \( \Sigma|_C \) is trivial if and only if \([\eta_{i,i+1}] \neq 0 \) for every \( \eta_{i,i+1} \in \Omega^{0,1}(Y,O(C)) \).

**Proof.** For simplicity, we assume \( m = n = 1 \) and \( O_C(l_1) \cong O_{P^1}, O_C(l_2) \cong O_{P^1}(-1), O_C(l_3) \cong O_{P^1}(1) \) with \( l_2 - l_3 = C \). If \([\eta_{2,3}] \neq 0 \), by Lemma 29 and Lemma 24, there exists two holomorphic sections for \( \Sigma|_C \), which are linearly independent at every point in \( C \): \( s_1 = (x_1, u_1, t_1)^t \) and \( s_2 = (x_2, u_2, t_2)^t \) with \( u_1, t_1, u_2, t_2 \) given in the proof of Lemma 24. By \( H^0(Y, O_C(l_1)) \cong H^0(\mathbb{P}^1, O) \cong \mathbb{C} \), there exists one holomorphic section for \( \Sigma|_C \) which is nowhere zero on \( C \): \( s_3 = (x_3, 0, 0)^t \). These \( s_1, s_2, s_3 \) give a trivialization of \( \Sigma|_C \). If \([\eta_{2,3}] = 0 \), then \( \Sigma|_C \) is an extension of \( O_{P^1}(1) \oplus O_{P^1}(-1) \) by \( O_{P^1} \) and there is no such nontrivial extension.

**Proposition 31** The bundle \( \Lambda_n \) over \( Y \) with \( \eta \in \Xi_n \) can descend to \( X \) if and only if \( 0 \neq [\eta_{n+1-i,n+2-i}] \in H^1(Y, O_C|_i) \) for every \( i \), i.e. \( \eta \in \Xi_{n+1}^n \).

**Proof.** Restricting \( \Lambda_n \) to \( C_i \), the corresponding line bundle summands are

\[
O_{C_i}(l_k) \cong \begin{cases} O_{P^1}(1), & k = n + 2 - i \\
O_{P^1}(-1), & k = n + 1 - i \\
O_{P^1}, & \text{otherwise.} \end{cases}
\]

By Lemma 30 and our assumption, we have the proposition.
4.2 \(A_n\) Lie algebra bundle \(\zeta^{A_n}_\eta\)

As \(A_n = sl(n+1, \mathbb{C}) = aut(\mathbb{C}^{n+1})\), \(\zeta^{A_n}_\eta := aut(\mathcal{L}^{A_n}_\eta)\) (\(\eta \in \Xi^{A_n}_X\)) is an \(A_n\) Lie algebra bundle over \(Y\) which can descend to \(X\). This \(\zeta^{A_n}_\eta\) does not depend on the existence of \(C_0\). And \(\mathcal{L}^{A_n}_\eta\) is automatically a representation bundle of \(\zeta^{A_n}_\eta\).

4.3 \(A_n\) minuscule representation bundle \(\mathcal{L}^{(A_n, \wedge^k \mathbb{C}^{n+1})}_\eta\)

Consider a surface \(X\) with an \(A_n\) singularity \(p\) and a \((-1)\)-curve \(C_0\) passing through \(p\) with multiplicity \(C_k\). By Proposition [10], \(I^{(A_n, \wedge^k \mathbb{C}^{n+1})}\) has cardinality \(\binom{n}{k+1}\). Define \(\mathcal{L}^{(A_n, \wedge^k \mathbb{C}^{n+1})}_0 := \bigoplus_{l \in I} O(l)\) over \(Y\).

**Lemma 32** \(\mathcal{L}^{(A_n, \wedge^k \mathbb{C}^{n+1})}_0 = (\wedge^k \mathcal{L}^{A_n}_\eta)(C_0^k - kC_0 - \sum_{j=1}^{k-1}(k-j)C_j)\).

**Proof.** The bundles on both sides have the same rank, so we only need to check that every line bundle summand in the right-hand side is \(O_Y(l)\) for \(l\) a \((-1)\)-curve in \(I^{(A_n, \wedge^k \mathbb{C}^{n+1})}\). For any \(k\) distinct elements \(l_i\) in \(I^{(A_n, \mathbb{C}^{n+1})}\), we denote \(l = l_{i_1} + l_{i_2} + \cdots + l_{i_k} + C_0 - (l_1 + l_2 + \cdots + l_k)\), then \(O_Y(l)\) is a summand in the right-hand side. Since the intersection number of any two distinct \((-1)\)-curves in \(I^{(A_n, \mathbb{C}^{n+1})}\) is zero, we have \(l^2 = l \cdot K_Y = -1\), i.e. \(l \in I^{(A_n, \wedge^k \mathbb{C}^{n+1})}\).

From the above lemma and direct computations, for any \(C_i\),

\[\mathcal{L}^{(A_n, \wedge^k \mathbb{C}^{n+1})}_0|_{C_i} \cong O_{P^1}^\oplus((k-i)C_i) \oplus (O_{P^1}(1) \oplus O_{P^1}(-1))^\oplus(n-i)\].

**Proposition 33** Fix any \(\eta \in \Xi^{A_n}_X\), there exists a unique holomorphic structure on \(\mathcal{L}^{(A_n, \wedge^k \mathbb{C}^{n+1})}_0\) such that the action of \(\zeta^{A_n}_\eta\) on the resulting bundle \(\mathcal{L}^{(A_n, \wedge^k \mathbb{C}^{n+1})}_\eta\) is holomorphic. Furthermore, if \(\eta \in \Xi^{A_n}_X\), then \(\mathcal{L}^{(A_n, \wedge^k \mathbb{C}^{n+1})}_\eta\) can descend to \(X\).

**Proof.** As the action of \(\zeta^{A_n}_\eta\) on \(\mathcal{L}^{A_n}_\eta\) is holomorphic, \(\zeta^{A_n}_\eta\) acts on \(\mathcal{L}^{(A_n, \wedge^k \mathbb{C}^{n+1})}_\eta := (\wedge^k \mathcal{L}^{A_n}_\eta)(C_0^k - kC_0 - \sum_{j=1}^{k-1}(k-j)C_j)\) holomorphically. The last assertion follows from Proposition [77] and the fact that \(O(C_0^k - kC_0 - \sum_{j=1}^{k-1}(k-j)C_j)|_{C_i}\) is trivial for every \(C_i\).

5 \(D_n\) case

We recall that \(D_n = o(2n, \mathbb{C}) = aut(\mathbb{C}^{2n}, q)\) for a non-degenerate quadratic form \(q\) on the standard representation \(\mathbb{C}^{2n}\). The other minuscule representations are \(S^+\) and \(S^-\) and the adjoint representation is \(\wedge^2 \mathbb{C}^{2n}\).

5.1 \(D_n\) standard representation bundle \(\mathcal{L}^{(D_n, \mathbb{C}^{2n})}_\eta\)

We consider a surface \(X\) with a \(D_n\) singularity \(p\) and a \((-1)\)-curve \(C_0\) passing through \(p\) with multiplicity \(C_k\), then \(I^{(D_n, \mathbb{C}^{2n})} = I_1 \cup I_2\) with \(I_1 = \{C_0 + \sum_{i=1}^{k}C_i | 0 \leq k \leq n-1\}\) and \(I_2 = \{F - l | l \in I_1\}\), where \(F = 2C_1 + 2C_1 + \cdots + \)
We order these \((-1)\)-curves: \(l_k = F - C_0^1 - \sum_{i=1}^{k-1} C_i \) and \\
l_{2n-k+1} = C_0^1 + \sum_{i=1}^{k-1} C_i \) for \(1 \leq k \leq n\).

For any \(l_i \neq l_j \in I\), we have \(l_i \cdot l_j = 0\) or \(1\). Given any \(l_i \in I\), there exists a unique \(l_j \in I\) such that \(l_i \cdot l_j = 1\). In this case, \(l_i + l_j = F\).

Define \(L_0^{(D_n, C^{2n})} := \bigoplus_{l \in I} O(l) \) over \(Y\), for simplicity, we write it as \(L_0^{D_n}\). If we ignore \(C_n\), then we recover the \(A_{n-1}\) case as in the last section. They are related by the following.

**Lemma 34** \(L_0^{D_n} = L_0^{A_{n-1}} \oplus (L_0^{A_{n-1}})^*(F)\).

**Proof.** Since \(A_{n-1}\) is a Lie subalgebra of \(D_n\), we can decompose the representation of \(D_n\) as sum of irreducible representations of \(A_{n-1}\). By the branching rule, we have \(2n = n + n\), that is \(C^{2n} = C^n \oplus (C^n)^* \) with \(C^{2n} \) and \(C^n\) the standard representations of \(D_n\) and \(A_{n-1}\) respectively. For \(I^{(D_n, C^{2n})} = I_1 \cup I_2\), \(I_1\) forms the standard representation \(C^n\) of \(A_{n-1}\), and \(I_2\) forms the \((C^n)^*\).

From the above lemma and direct computations, for any \(C_i\),

\[
\mathfrak{g}_D^{D_n}|_{C_i} \cong O_{\mathbb{C}^2}^{2n-k+4} \oplus (O_{\mathbb{C}^2}^{2n-1} \oplus O_{\mathbb{C}^2}^{2n-1})(-1)^{\otimes 2}.
\]

Similar to \((A_n, C^{n+1})\) case, we define \(\overline{\partial}_\eta : \Omega^0(Y, L_0^{D_n}) \to \Omega^0(Y, L_0^{D_n})\) on \(L_0^{D_n} = \bigoplus_{k=1}^{2n} O(l_k)\) by \(\overline{\partial}_\eta := \overline{\partial}_\eta + (\eta, j)_{2n \times 2n}\), where \(\eta, j \in \Omega^0(Y, O(l_i - l_j))\) for any \(j > i\), otherwise \(\eta, j = 0\).

By Lemma [20] and arguments similar to the proof of Proposition [27] for the \(A_n\) case, given any \(\eta, i+1\) with \(\overline{\partial}_\eta, i+1 = 0\) for every \(i\), there exists \(\eta, j \in \Omega^0(Y, O(l_i - l_j))\) for every \(j > i\) such that \(\overline{\partial}_\eta = 0\).

From the configuration of these \(2n\) \((-1)\)-curves, we can define a quadratic form \(q\) on the vector space \(V_0 = \mathbb{C}^{L} = \bigoplus_{l \in I} \mathbb{C}(v_l)\) spanned by these \((-1)\)-curves,

\[
q : V_0 \otimes V_0 \to \mathbb{C}, q(v_{l_i}, v_{l_j}) = l_i \cdot l_j.
\]

The \(D_n\) Lie algebra is the space of infinitesimal automorphism of \(q\), i.e. \(D_n = \text{aut}(V_0, q)\).

Correspondingly, we have a fiberwise quadratic form \(q\) on the bundle \(L_0^{D_n}\):

\[
q : L_0^{D_n} \otimes L_0^{D_n} \to O(F).
\]

**Proposition 35** There exists \(\eta\) with \(\overline{\partial}_\eta^2 = 0\) such that \(\overline{\partial}_\eta q = 0\).

**Proof.** \(\overline{\partial}_\eta q = 0\) if and only if \(q(\overline{\partial}_\eta s_i, s_j) + q(s_i, \overline{\partial}_\eta s_j) = 0\) for any \(s_i \in H^0(Y, O(l_i))\) and \(s_j \in H^0(Y, O(l_j))\). From the definition of \(q\), this is equivalent to \(\eta, 2n-1, i + \eta, 2n-1, i - j = 0\), i.e. \(\eta, i = -\eta, 2n-1, j, 2n-1, j\) for any \(j > i\). From \(l_i + l_{2n-1} = 2l_i + l_{2n-1} = F\), we have

\[
\eta, i \in \Omega^0(Y, O(l_i - l_j)) = \Omega^0(Y, O(l_{2n+1-1} = l_{2n+1-1}) \oplus \eta, 2n-1, j, 2n-1, j.
\]

We construct \(\eta\) which satisfies \(\overline{\partial}_\eta^2 = 0\) with \(\eta, i = -\eta, 2n+1, j, 2n+1, j\) inductively on \(j - i\). For \(j - i = 1\), we can always take \(\eta, i+1 = -\eta, 2n-1, i, 2n-1, i\). Note we have \(\eta, n+1 = 0\). For \(j - i = 2\), we have

\[
\overline{\partial}_\eta \eta, i+2 = -\eta, i+1, \eta, i+1, i+2.
\]
Lemma 36

Assume \( l_{i+1}, l_{i+2} \cdots l_{i+2k} \) satisfy \( l_j + C = -1 \) and \( l_{i+k+j} = l_j - C \) for \( j = 1, 2, \cdots k \). If \( \eta_{i+p,i+q} = 0 \) for \( 2 \leq p \leq k, k + 1 \leq q \leq 2k - 1 \) and \( q - p \leq k - 1 \), i.e. the corresponding submatrix of \( \partial \) given by \( l_{i+1}, l_{i+2}, \cdots l_{i+2k} \) looks like

\[
\begin{pmatrix}
\ast & \eta_{i+1,i+k+1} & \eta_{i+1,i+k+2} & \cdots & \eta_{i+1,i+2k} \\
0 & \eta_{i+2,i+k+2} & \cdots & \eta_{i+2,i+2k} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \eta_{i+k,i+2k} \\
0_{k \times k} & \ast & \eta_{i+1,i+3} & 0 & \eta_{i+2,i+4} & \eta_{i+3,i+4} & \cdots \\
0_{2 \times 2} & \partial & \eta_{i+1,i+2} & 0 & \partial & \eta_{i+3,i+4} & \cdots
\end{pmatrix}
\]

with \( \eta_{i+1,i+k+1}, \eta_{i+2,i+k+2} \cdots \eta_{i+k,i+2k} \) in \( \Omega^{0,1}(Y, O(C)) \). Suppose \( [\eta_{i+1,i+k+1}] \), \( [\eta_{i+2,i+k+2}] \), \( \cdots [\eta_{i+k,i+2k}] \) are nonzero, we can construct \( 2k \) holomorphic sections of \( \Sigma^C \) which are linearly independent at every point in \( C \).

Proof. In order to keep our notations simpler, we assume \( k = 2 \). The above matrix given by \( l_{i+1}, l_{i+2}, l_{i+3}, l_{i+4} \) has the form

\[
\begin{pmatrix}
\bar{\partial} & \eta_{i+1,i+3} & \ast & \eta_{i+1,i+2} \\
0 & 0 & \eta_{i+2,i+4} & \eta_{i+3,i+4} \\
0_{2 \times 2} & \partial & \eta_{i+1,i+2} & 0
\end{pmatrix}
\]

From \( H^0(Y, O_C(l_{i+4})) \cong H^0(\mathbb{P}^1, O(1)) \cong \mathbb{C}^2 \) and \( [\eta_{i+2,i+4}] \neq 0 \), there exist two holomorphic sections of \( \Sigma^C \) which are linearly independent at every point in \( C \): \( s_1 = (y_1, u_1, x_1, t_1)^t \) and \( s_2 = (y_2, u_2, x_2, t_2)^t \) with \( u_1, t_1, u_2, t_2 \) given in Lemma 28. Similarly, from \( H^0(Y, O_C(l_{i+3})) \cong \mathbb{C}^2 \) and \( [\eta_{i+1,i+3}] \neq 0 \), we also have two holomorphic sections of \( \Sigma^C \) which are linearly independent at every point in \( C \): \( s_3 = (y_3, 0, x_3, 0)^t \) and \( s_4 = (y_4, 0, x_4, 0)^t \). If there exist \( a_1, a_2, a_3, a_4 \) such that \( a_1 s_1 + a_2 s_2 + a_3 s_3 + a_4 s_4 \) \neq 0 at some point in \( C \), then we have \( a_1 t_1 + a_2 t_2 = 0 \) and \( a_1 u_1 + a_2 u_2 = 0 \) at some point, which is impossible by the explicit formulas for \( u_1, t_1, u_2, t_2 \) in Lemma 29. Hence we have the lemma.

\[\square\]
Lemma 37 Under the same assumption as in Lemma 28, we assume $L|_C \cong O(\mathbb{P}^n) \oplus (O(\mathbb{P}^n \oplus O(-1))^\oplus n$ with each pair of $O(\mathbb{P}^n(\pm 1)$ and the corresponding holomorphic structure as in Lemma 26. Then $L|_C$ is trivial if and only if $[\eta_i, j|_C] \neq 0$ for any $\eta_i, j \in \Omega^{0,1}(Y, O(C))$.

Proof. Same arguments as in the proof of Lemma 26 and Lemma 30. ■

Proposition 38 The bundle $L_{0}^{n+1}$ over $Y$ with $\eta \in \Xi_{Y}$ can descend to $X$ if and only if for every $C_k$ and $\eta, j \in \Omega^{0,1}(Y, O(C))$, $[\eta, j|_C] \neq 0$, i.e. $\eta \in \Xi_{X}^{D_{n}}$.

Proof. Restricting $L_{0}^{n+1}$ to $C_i \ (1 \leq i \leq n - 1)$, the line bundle summands are

$$O_{C_i}(l_j) \cong \begin{cases} O_{\mathbb{P}^1}(1), & j = i + 1 \text{ or } 2n - i \\ O_{\mathbb{P}^1}(-1), & j = i \text{ or } 2n - i + 1 \\ O_{\mathbb{P}^1}, & \text{otherwise.} \end{cases}$$

By Lemma 34, $L_{n+1}^{D_{n}}$ is trivial if and only if $[\eta_{i+1}, j_i|_C] / [\eta_{2n-i-2n+1-i}, j_i|_C]$ are not zeros. For $C_n$, the pairs of $O_{\mathbb{P}^1}(\pm 1)$ in $L_{0}^{n+1}|_{C_n}$ are given by $\{(l_{n-1}, l_{n+1})\}$ and $\{(l_{n}, l_{n+2})\}$. By Lemma 37 and $\eta, n+1 = 0$ (Proposition 35), $L_{0}^{n+1}|_{C_n}$ is trivial if and only if $[\eta_{i-1, n+1} C_n], [\eta_{n+1, n+2} C_n]$ are not zeros. In fact, this $L_{0}^{n+1}$ is just an extension of $L_{0}^{A_{n-1}}$ by $(L_{0}^{A_{n-1}}, F)$ for some $\eta \in \Xi_{X}^{A_{n-1}}$ with $\eta \subset \eta'$. ■

5.2 $D_{n}$ Lie algebra bundle $\zeta_{n}^{D_{n}}$

Note that $\zeta_{n}^{D_{n}} = aut(L_{0}^{D_{n}})$ is a $D_{n}$ Lie algebra bundle over $Y$. In order for $\zeta_{n}^{D_{n}}$ to descend to $X$ as a Lie algebra bundle, we need to show that $q_{C_i} : L_{n}^{D_{n}}|_{C_i} \otimes L_{n}^{D_{n}}|_{C_i} \rightarrow O_{C_i}(F)$ is a constant map for every $C_i$. This follows from the fact that both $L_{0}^{D_{n}}$ and $O(F)$ are trivial on all $C_i$'s and $\overline{\partial} q = 0$. From the construction, $L_{0}^{D_{n}}$ is a representation bundle of $\zeta_{n}^{D_{n}}$.

5.3 $D_{n}$ spinor representation bundles $L_{0}^{D_{n}, S^{+}}$

We will only deal with $S^{+}$, as $S^{-}$ case is analogous. Consider a surface $X$ with a $D_{n}$ singularity $p$ and a $(1)$-curve $C_0$ passing through $p$ with multiplicity $n$.

By Proposition 16 $|I^{D_{n}, S^{+}}| = 2^{n-1}$. Define $L_{0}^{D_{n}, S^{+}} := \bigoplus_{i \in I} O(l) \overline{\partial} Y$.

Lemma 39 $L_{0}^{D_{n}, S^{+}} = \bigoplus_{m=0}^{n} \lambda^{2m}(L_{0}^{D_{n}, S^{+}}(mF + C_0))$.

Proof. First we check that every line bundle summand in the right-hand side is $O_{Y}(l)$ for a $(1)$-curve $l$ in $I^{D_{n}, S^{+}}$. For any $l \in I^{(A_{n-1}, C^m)}$, we have $l \cdot C_0 = 0$, $l \cdot F = 0$ and $F \cdot C_0 = 1$. For any $2m$ distinct elements $l_{i_{1}}$'s in $I^{(A_{n-1}, C^m)}$, we denote $l = -(l_{i_{1}} + \cdots + l_{i_{2m}}) + mF + C_0$, then $O_{Y}(l)$ is a summand in the right-hand side. Since $l^2 = -1$ and $l \cdot K_Y = -1$, $l \in I^{D_{n}, S^{+}}$. Also the rank of these two bundles are the same which is $2^{n-1} = \binom{n}{0} + \frac{\binom{n}{2}}{2\cdot 2} + \cdots + \binom{n}{2^{\lfloor n/2 \rfloor}}$.

Hence we have the lemma. ■
From the above lemma and direct computations, for any \(C_i\),
\[
\mathcal{L}_0^{(D_n,S^+)}|_{C_i} \cong O_{\mathbb{P}^{2n-2}} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus 2n-3}.
\]

The \(D_n\) Lie algebra bundle \(\mathcal{L}_0^{D_n}\) has a natural fiberwise action on \(\mathcal{L}_0^{(D_n,S^+)}\),
\[
\rho : \mathcal{L}_0^{D_n} \otimes \mathcal{L}_0^{(D_n,S^+)} \rightarrow \mathcal{L}_0^{(D_n,S^+)},
\]
which can be described easily using the reduction to \(A_{n-1}\) (with the node \(C_n\) being removed): recall
\[
\mathcal{L}_0^{D_n} = (\wedge^2 \mathcal{O}_{\mathbb{P}^{n-1}}(-F)) \oplus ((\mathcal{O}_{\mathbb{P}^{n-1}})^* \otimes \mathcal{O}_{(0)}^{A_{n-1}}) \oplus ((\wedge^2 \mathcal{O}_{\mathbb{P}^{n-1}})^*(F)),
\]
\[
\mathcal{L}_0^{(D_n,S^+)} = \bigoplus_{m=0}^{[\mathcal{L}_0^{D_n}]} \wedge^{2m} (\mathcal{O}_{\mathbb{P}^{n-1}})^*(mF)\]
and \(\rho\) is given by interior and exterior multiplications for \(\wedge \mathcal{L}_0^{A_{n-1}}\).

**Proposition 40** Fix any \(\eta \in \mathbb{P}^{D_n}\), there exists a unique holomorphic structure on \(\mathcal{L}_0^{(D_n,S^+)}\) such that the action of \(\mathcal{L}_0^{D_n}\) on the resulting bundle \(\mathcal{L}_0^{(D_n,S^+)}\) is holomorphic. Furthermore, if \(\eta \in \mathbb{P}^{D_n}\), then \(\mathcal{L}_0^{(D_n,S^+)}\) can descend to \(X\).

**Proof.** First, we recall the holomorphic structure on \(\mathcal{L}_0^{D_n}\). In \(I(D_n,\mathbb{C}^n) = I_1 \cup I_2\) with \(I_1 = \{l_i = C_0^i + \sum_{m=1}^{2n-1} C_m|n+1 \leq i \leq 2n\}\) and \(I_2 = \{F - l_i|l_i \in I_1\}\), let \(S_i, S_i^*\) and \(f\) be local holomorphic sections of \(O(l_i), O(F - l_i)\) and \(O(-F)\) respectively. By Proposition 35, we have
\[
\bar{\partial}_{\mathcal{L}_0^{D_n}} s_i^* = \sum_{p=1}^{n-1} \eta_{p,i} s_p^*
\]
and
\[
\bar{\partial}_{\mathcal{L}_0^{D_n}} s_i = \sum_{p=1}^{n} \eta_{p,2n+1-i} s_p^* - \sum_{p=i+1}^{n} \eta_{i,p} s_p.
\]

Back to \((\mathcal{L}_0^{D_n})^*\), we define \(s_i_1 \cdots i_{2m} := s_{i_1} \wedge \cdots \wedge s_{i_{2m}} \otimes f^m \in \Gamma(\wedge^{2m} \mathcal{L}_0^{A_{n-1}}(-mF))\) where \(i_j \in \{1,2, \cdots n\}\) and define \(\bar{\partial}_{\mathcal{L}_0^{(D_n,S^+)}}\) as follows:
\[
\bar{\partial}_{\mathcal{L}_0^{(D_n,S^+)}} s_{i_1} \cdots i_{2m} = \sum_{p,q} (-1)^{p+q} \eta_{p,2n+1-i_1} s_{i_1} \cdots \hat{s}_{i_p} \cdots \hat{s}_{i_q} \cdots s_{i_{2m}} - \sum_{p} \sum_{k \neq i_p} \eta_{i_p,k} s_{i_1} \cdots \hat{s}_{i_p} \cdots k_{i_{2m}},
\]
where \(\hat{s}_{i_j}\) means deleting the \(i_j\) component. We verify \(\bar{\partial}_{\mathcal{L}_0^{D_n}} = 0\) by direct computations. 

\(\text{Footnote:}\) For simplicity, we omit the \(C_0^n\) factor.
We claim that $\overline{\partial}_S$ is the unique holomorphic structure such that the action of $\zeta^{D_n}$ on $\left(\mathcal{L}_0^{(D_n, S^+)}\right)^*$ is holomorphic, i.e.

$$\overline{\partial}_{\zeta_0^{D_n}}(g) \cdot x + g \cdot (\overline{\partial}_S x) = \overline{\partial}_S(g \cdot x)$$

for any $g \in \Gamma(\zeta_0^{D_n})$ and $x \in \Gamma\left(\left(\mathcal{L}_0^{S^+}\right)^*\right)$.

We prove the above claim by induction on $m$. When $m = 0$, $x = s_0 \in \Gamma(\wedge^0 \mathcal{L}_0^{A_n-1})$, by direct computations, (*) holds for any $g \in \Gamma(\zeta_0^{D_n})$ if and only if $\overline{\partial}_S s_0 = 0$ and $\overline{\partial}_S s_{ij} = -\eta_{i,2n+1-j} s_0 - \sum_{p=i+1}^{n} \eta_{i,p} s_{pj} - \sum_{p=j+1}^{n} \eta_{j,p} s_{ip}$ for any $s_{ij} \in \Gamma(\wedge^2 \mathcal{L}_0^{A_n-1})$. When $m = 2$, from the above formula for $\overline{\partial}_S s_{ij}$, we can get the formula for $\overline{\partial}_S s_{ijkl}$. Repeat this process inductively, we can get the above formula for $\overline{\partial}_S s_{i120n}$. Hence we have the first part of this proposition.

For the second part, we will rewrite $\overline{\partial}_S$ in matrix form. Firstly, we have

$$\overline{\partial}_{\zeta_0^{(D_n, S^2n)}} = \left( \frac{\overline{\partial}\left(\mathcal{L}_0^{A_n-1}\right)^*(F)}{0} \right) \left( \begin{array}{c} B \\ \frac{\overline{\partial}\left(\mathcal{L}_0^{A_n-1}\right)^*(-1)}{0} \end{array} \right)$$

with $\eta' \subset \eta$ and the upper right block $B$ has the following shape

$$B = \left( \begin{array}{cccc} : & : & \cdots & \\ \beta & * & \cdots & \\ 0 & -\beta & \cdots \end{array} \right),$$

for $[\beta] \in H^1(Y, O(C_n))$.

In particular, we have an exact sequence of holomorphic bundles:

$$0 \to (\mathcal{L}_0^{A_n-1})^*(F) \to \mathcal{L}_0^{D_n} \to \mathcal{L}_\eta^{A_n-1} \to 0. \quad (\Delta)$$

By tensoring $(\Delta)$ with $\mathcal{L}_\eta^{A_n-1}(-F)$, we obtain a bundle $S_1$ as follows,

$$0 \to O_Y \to S_1 \to \wedge^2 \mathcal{L}_\eta^{A_n-1}(-F) \to 0,$$

with the induced holomorphic structure given by

$$\overline{\partial}_{S_1} = \left( \begin{array}{c} \overline{\partial}\left(\mathcal{L}_0^{A_n-1}\right)^*(F) \\ 0 \end{array} \right) \left( \begin{array}{c} B_1 \\ \frac{\overline{\partial}\left(\mathcal{L}_0^{A_n-1}\right)^*(-1)}{0} \end{array} \right) = \left( \begin{array}{c} \overline{\partial}\left(\mathcal{L}_0^{A_n-1}\right)^*(F) \\ 0 \end{array} \right) \left( \begin{array}{c} \pm \beta \cdots \\ \frac{\overline{\partial}\left(\mathcal{L}_0^{A_n-1}\right)^*(-1)}{0} \end{array} \right).$$

The occurrence of $\pm \beta$ in that location is because $l_{n+1} + l_{n+2}$ with $l_{n+1}, l_{n+2} \in I^{(D_n, S^2n)}$ is the largest element in $I^{(A_n-1, S^2n)}$ and $F - l_{n+1} - l_{n+2} = C_n$ because $F = 2C_0 + 2C_1 + \cdots 2C_{n-2} + C_{n-1} + C_n$.

Similarly, we have an extension bundle

$$0 \to \wedge^2 \mathcal{L}_\eta^{A_n-1}(-F) \to S_2 \to \wedge^4 \mathcal{L}_\eta^{A_n-1}(-2F) \to 0,$$
with
\[ \overline{\partial}_{S_2} = \begin{pmatrix} \partial_{\wedge^2 \mathbb{P}^{n-1}}(-F) & B_2 \\ 0 & \partial_{\wedge^4 \mathbb{P}^{n-1}}(-2F) \end{pmatrix}, \]
where
\[ B_2 = \begin{pmatrix} \pm \beta & \pm \beta \\ 0 & \pm \beta \\ \vdots & \vdots \\ 0 & \cdots \pm \beta \end{pmatrix}, \]
for \([\beta] \in H^1(Y, \text{Hom}(O(l_i + l_j + l_{n+1} + l_{n+2} - 2F), O(l_i + l_j - F))) = H^1(Y, O(C_n))\] with \(i, j \in \{n + 3, n + 4, \ldots, 2n\}\). And the number of \(\pm \beta\)'s is \(\binom{n-2}{2}\).

Inductively, we obtain \(\overline{\partial}(\Sigma^{D_n, s+}_\eta)^*\) as above which has the shape that satisfies Lemma 37.

\[ \overline{\partial}(\Sigma^{D_n, s+}_\eta)^* = \begin{pmatrix} \overline{\partial}_{\wedge^0 \mathbb{P}^{n-1}} & B_1 & \cdots & \cdots \\ 0 & \overline{\partial}_{\wedge^2 \mathbb{P}^{n-1}}(-F) & B_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]

The number of \(\pm \beta \in \Omega^1(Y, O(C_n))\) in \(\overline{\partial}_{\Sigma}\) is \(\binom{n-2}{0} + \binom{n-2}{2} + \cdots + \binom{n-2}{2\left\lfloor \frac{n-2}{2} \right\rfloor} = 2^{n-3}\).

To prove that \(\Sigma^{s+}_\eta\) can descend to \(X\) when \(\eta \in \Xi^{D_n}_{\eta}\), we need to show \(\Sigma^{s+}_\eta\) is trivial for every \(C_\eta\). When \(i \neq n\), this follows from the fact that \(\Sigma^{A_n-1}_{\eta'}\) is trivial (Proposition 31) and \(\text{Ext}^1_{\eta'}(O, O) = 0\). When \(i = n\), this follows from Lemma 37 and \(\beta = \eta_{n-1, n+1} \in \Omega^{\wedge 1}(Y, O(C_n))\) with \(|\eta_{n-1, n+1}|_{c_n} \neq 0\).

6 \( E_n \) case

6.1 \( E_6 \) case

We recall that \( E_6 = \text{aut}(\mathbb{C}^{27}, c) \) for a non-degenerate cubic form \(c\) on the standard representation \(\mathbb{C}^{27}\). The other minuscule representation is \(\mathbb{C}^{27}\).

We consider a surface \(X\) with an \(E_6\) singularity \(p\) and a \((-1)\)-curve \(C_0\) passing through \(p\) with multiplicity \(C_1\). By Proposition 13, \(I^{(E_6, \mathbb{C}^{27})}\) has cardinality 27. For any two distinct \((-1)\)-curves \(l_i\) and \(l_j\) in \(I\), we have \(l_i \cdot l_j = 0\) or 1.

Define \(\Sigma^{E_6}_{0, \mathbb{C}^{27}} := \bigoplus_{l \in I} O(l)\) over \(Y\), for simplicity, we write it as \(\Sigma^{E_6}_{0, \mathbb{C}^{27}}\). If we ignore \(C_0\), then we recover the \(A_5\) case as in §4.1.
Lemma 41. \( \mathcal{L}_0^{E_6} = \mathcal{L}_0^{A_5} \oplus (\wedge^2 \mathcal{L}_0^{A_5})^* (H) \oplus (\wedge^5 \mathcal{L}_0^{A_5}) (2H) \), where \( H = 3C_0^{11} + 3C_1 + 3C_2 + 3C_3 + 2C_4 + C_5 + C_6. \)

**Proof.** \( E_6 \) has \( A_5 \) as a Lie subalgebra, the branching rule is \( 27 = 6 + 15 + 6 \), i.e., \( \mathbb{C}^{27} = \mathbb{C}^6 \oplus \wedge^2 (\mathbb{C}^6)^* \oplus \wedge^5 (\mathbb{C}^6)^* \). The first 6 \((-1\)-curves in \( I:\ l_1 = C_0^{11}, l_2 = C_0^{11} + C_1, \ldots, l_6 = C_0^{11} + C_1 + C_2 + C_3 + C_4 + C_5 \) form the standard representation \( \mathbb{C}^6 \) of \( A_5 \). The next 15 \((-1\)-curves are given by \( H - l_i \) with \( i \neq j \in \{1, 2, \ldots, 6\} \).

The remaining 6 \((-1\)-curves are given by \( 2H - l_1 - l_2 - \cdots - l_i - \cdots - l_6 \).

From the above lemma and direct computations, for any \( C_i \),

\[
\mathcal{L}_0^{E_6} |_{C_i} \cong O_{\mathbb{P}^{15}}^{\oplus 5} \oplus (O_{\mathbb{P}^1} (1) \oplus O_{\mathbb{P}^1} (-1))^{\oplus 6}.
\]

From Lemma 41, we can easily determine the configuration of these 27 \((-1\)-curves. Fix any \((-1\)-curve, there are exactly 10 \((-1\)-curves intersect it, together with the fixed \((-1\)-curve, they form 5 triangles. A triple \( l_i, l_j, l_k \) is called a triangle if \( l_i + l_j + l_k = K' \), where \( K' = 3C_0^{11} + 4C_1 + 5C_2 + 6C_3 + 4C_4 + 2C_5 + 3C_6. \)

From the configuration of these 27 \((-1\)-curves in \( Y \), we can define a cubic form \( c \) on the vector space \( V_0 = \mathbb{C}^j = \bigoplus_{i \leq j} \mathbb{C}(v_i) \) spanned by \((-1\)-curves,

\[
c : V_0 \otimes V_0 \otimes V_0 \to \mathbb{C}, \quad (v_i, v_j, v_k) \mapsto \begin{cases} 
\pm 1 & \text{if } l_i + l_j + l_k = K' \\
0 & \text{otherwise.}
\end{cases}
\]

The signs above can be determined explicitly such that \( E_6 = aut(V_0, c) \).

Correspondingly, we have a fiberwise cubic form \( c \) on the bundle \( \mathcal{L}_0^{E_6} \),

\[
c : \mathcal{L}_0^{E_6} \otimes \mathcal{L}_0^{E_6} \otimes \mathcal{L}_0^{E_6} \to O(K').
\]

Proposition 42. There exists \( \eta \) with \( \overline{\Omega}_\eta = 0 \) such that \( \overline{\Omega}_\eta c = 0 \).

**Proof.** Note \( \overline{\Omega}_\eta c = 0 \) if and only if

\[
c(\overline{\Omega}_\eta s_i, s_j, s_k) + c(s_i, \overline{\Omega}_\eta s_j, s_k) + c(s_i, s_j, \overline{\Omega}_\eta s_k) = 0 \quad (\ast)
\]

for any \( s_i \in H^0(Y, O(l_i)) \), \( s_j \in H^0(Y, O(l_j)) \) and \( s_k \in H^0(Y, O(l_k)) \). From the definition of \( c \), if \( l_i + l_j + l_k = K' \), then the above equation (\( \ast \)) holds automatically.

If \( l_i + l_j + l_k \neq K' \), without loss of generality, we assume \( l_i \cdot l_j = 0 \), then we have the following four cases.

- **Case (i),** if \( l_i \cdot l_k = 0 \) and \( l_j \cdot l_k = 0 \), then (\( \ast \)) holds automatically.
- **Case (ii),** if \( l_i \cdot l_k = 0 \) and \( l_j \cdot l_k = 1 \), then (\( \ast \)) holds if \( \eta_{\overline{l}_i, K'-l_j-l_k} = 0 \).
- **Case (iii),** if \( l_i \cdot l_k = 1 \) and \( l_j \cdot l_k = 0 \), then (\( \ast \)) holds if \( \eta_{\overline{l}_j, K'-l_i-l_k} = 0 \).
- **Case (iv),** if \( l_i \cdot l_k = 1 \) and \( l_j \cdot l_k = 1 \), then (\( \ast \)) holds if \( \eta_{l_i, K'-l_j-l_k} \pm \eta_{l_j, K'-l_i-l_k} = 0 \), here the sign is determined by the signs of cubic form.

In conclusion, for any \( l_i, l_j \in \mathcal{I}^{(E_6, \mathbb{C}^{27})} \), if \( l_i \cdot l_j \neq 0 \), then \( \eta_{l_i, l_j} = 0 \). If \( l_i \cdot l_j = 0 \), then \( l_i - l_j = \alpha (j > i) \) for \( \alpha \in \Phi^+ \), i.e., \( \eta_{l_i, l_j} \in \Omega^{0,1}(Y, O(\alpha)) \). And for any other \( \eta_{\alpha, \beta} \in \Omega^{0,1}(Y, O(\alpha)) \), we have \( \eta_{l_i, l_j} \pm \eta_{\alpha, \beta} = 0 \). From the signs of the cubic form \( c \), we know that given any positive root \( \alpha \), there exists 6 \( \eta_{l_i, l_j} \)’s in \( \Omega^{0,1}(Y, O(\alpha)) \), where 3 of them are the same and the other 3 different to the first three by a sign. We use computer to prove we can find such \( \eta_{l_i, l_j} \)’s satisfying \( \overline{\Omega}_\eta = 0 \).
Until now, we have proved $\Xi^E_Y$ is not empty.

**Proposition 43** The bundle $\Sigma^E_Y$ over $Y$ with $\eta \in \Xi^E_Y$ can descend to $X$ if and only if for every $C_k$ and $\eta\cdot j \in \Omega^{0,1}(Y, O(C_k))$, $[\eta\cdot j|_{C_k}] \neq 0$, i.e. $\eta \in \Xi^E_X$.

**Proof.** From Lemma 11, Proposition 42 and the order of $I^{(E_6, \mathbb{C}^{27})}$, for $\eta \in \Xi^E_Y$, $\Sigma^E\eta$ can be constructed from $\Sigma^{A_8}_\eta$ for some $\eta' \in \Xi^E_{Y'}$ with $\eta' \subset \eta$. Under the (non-holomorphic) direct sum decomposition $\Sigma^E\eta = \Sigma^{A_8}_\eta \oplus (\wedge^2 \Sigma^{A_8}_\eta)^* (H) \oplus (\wedge^5 \Sigma^{A_8}_\eta)^* (2H)$, $\overline{\partial}_\eta$ for $\Sigma^E\eta$ has the following block decomposition:

$$
\begin{pmatrix}
\overline{\partial} (\wedge^6 \Sigma^E_{\eta'})^* (2H) & \ldots & \pm \beta & \ast & \cdots & \ast \\
\vdots & \vdots & \vdots & \ast & \cdots & \ast \\
0 & \pm \beta & \ast & \cdots & \ast \\
0 & 0 & \pm \beta & \cdots & \ast \\
0 & 0 & 0 & \pm \beta & \cdots & \ast \\
0 & 0 & 0 & 0 & \pm \beta & \cdots & \ast \\
\end{pmatrix}
$$

Here $\pm \beta \in \Omega^{0,1}(Y, O(C_k))$, it is because the corresponding two $(-1)$-curves $l$ and $l'$ satisfying $l - l' = C_6$. The signs of $\beta$ can be determined by $\overline{\partial}_\eta c = 0$.

From above, we know that $\Sigma^E\eta|_{C_k} (k \neq 6)$ is trivial if and only if $\Sigma^{A_8}_\eta|_{C_k} (k \neq 6)$ is trivial. From Proposition 27, we have the theorem for $k \neq 6$. For $C_6$, from Lemma 27, $\Sigma^E\eta|_{C_6}$ is trivial if and only if these $\pm \beta$'s satisfy $[\beta|_{C_6}] \neq 0$.

Note that $\mathfrak{c}_E \eta := aut(\Sigma^E \eta, c)$ is an $E_6$ Lie algebra bundle over $Y$. In order for $\mathfrak{c}_E \eta$ to descend to $X$ as a Lie algebra bundle, we need to show that $c|_{C_i} : \Sigma^E\eta|_{C_i} \otimes \Sigma^E\eta|_{C_i} \otimes \Sigma^E\eta|_{C_i} \rightarrow OC_i (K')$ is a constant map for every $C_i$. This follows from the fact that both $\Sigma^E\eta$ and $O(K')$ are trivial on all $C_i$’s and $\overline{\partial}_{\eta'} c = 0$. From the construction, $\Sigma^E\eta$ is a representation bundle of $\mathfrak{c}_E \eta$.

The only other minuscule representation $\mathfrak{c}_E \eta$ of $E_6$ is the dual of the standard representation $\mathfrak{c}^{27}$, therefore $\mathfrak{c}^{(E_6, \mathbb{C}^{27})} = \left( \mathfrak{c}_E \eta \right)^*$. 

### 6.2 $E_7$ case

We recall that $\mathfrak{c}_E = aut(\mathbb{C}^{56}, t)$ for a non-degenerate quartic form $t$ on the standard representation $\mathbb{C}^{56}$. There is no other minuscule representation of $E_7$.

We consider a surface $X$ with an $E_7$ singularity $p$ and a $(-1)$-curve $C_0$ passing through $p$ with multiplicity $C_1$. By Proposition 10, $I^{(E_7, \mathbb{C}^{56})}$ has cardinality 56. For any two distinct $(-1)$-curves $l_1$ and $l_2$ in $I$, we have $l_1 \cdot l_2 = 0, 1$ or 2.

Define $\mathfrak{c}_E^0 := \bigoplus_{l_i} O(l_i)$ over $Y$, for simplicity, we write it as $\mathfrak{c}_E^0$. If we ignore $C_7$, we recover the $A_6$ case as in §4.1.
Lemma 44 \( \mathfrak{L}_0^{(E_7, C^3)} = \mathfrak{L}_0^{A_6} \oplus (\wedge^2 \mathfrak{L}_0^{A_6})^* (H) \oplus (\wedge^3 \mathfrak{L}_0^{A_6})^* (2H) \oplus (\wedge^6 \mathfrak{L}_0^{A_6})^* (3H) \), where \( H = 3C_0^1 + 3C_1^1 + 3C_2^1 + 3C_3^1 + 3C_4^1 + 2C_5^1 + C_6^1 + C_7^1 \).

Proof. Similar to \( E_6 \) case. 

From the above lemma and direct computations, for any \( C_i \),
\[ \mathfrak{L}_0^{E_7} |_{C_i} \cong \mathfrak{O}_2^{\oplus 32} \oplus (\mathfrak{O}_2 \oplus 1) \oplus (\mathfrak{O}_2 \oplus (-1))^{\oplus 12} \]

The configuration of these 56 \((1)-\)curves is as follows: Fix any \((1)-\)curve, there are exactly 27 \((1)-\)curves intersect it once, 1 \((1)-\)curve intersects it twice. If \( l_i + l_j + p_i + q_i = 2K' \), \( K' = 2C_0^1 + 3C_1^1 + 4C_2^1 + 5C_3^1 + 6C_4^1 + 4C_5^1 + 2C_6^1 + 3C_7^1 \), the four \((1)-\)curves \( l_i, l_j, p_i \) and \( q_i \) will form a quadrangle.

From this configuration, we can define a quartic form \( t \) on the vector space \( V_0 = \mathbb{C}^l = \bigoplus_{E \in I} \mathbb{C}(v_l) \) spanned by all the \((1)-\)curves,
\[ t : V_0 \otimes V_0 \otimes V_0 \otimes V_0 \rightarrow \mathbb{C}, \quad (v_{l_1}, v_{l_2}, v_{p_1}, v_{q_1}) \mapsto \begin{cases} \pm 1 & \text{if } l_i + l_j + p_i + q_i = 2K' \\ 0 & \text{otherwise.} \end{cases} \]

The signs above can be determined explicitly such that \( E_7 = aut(V_0, t) \).

Correspondingly, we have a fiberwise quartic form \( t \) on the bundle \( \mathfrak{L}_0^{E_7} \),
\[ t : \mathfrak{L}_0^{E_7} \otimes \mathfrak{L}_0^{E_7} \otimes \mathfrak{L}_0^{E_7} \otimes \mathfrak{L}_0^{E_7} \rightarrow \mathfrak{O}(2K'). \]

Proposition 45 There exists \( \eta \) with \( \partial^{\eta}_0 = 0 \) such that \( \overline{\partial}_\eta t = 0 \).

Proof. Similar to \( E_6 \) case, but even more calculations involved. We will omit the calculations here and only list the conditions for \( \overline{\partial}_\eta t = 0 \). From \( \partial^{\eta}_0 t = 0 \) we have when \( l_i \cdot l_j = 0 \) we have \( \eta_{i,j} = 0 \). That means all the nonzero \( \eta_{i,j} \)’s are corresponding to \( l_i \cdot l_j = 0 \), then \( l_i - l_j = \alpha \) for some root \( \alpha \), i.e. \( \eta_{i,j} \in \Omega^{0,1}(Y, O(\alpha)) \). Conversely, given any positive root \( \alpha \), there exists 12 \( \eta_{i,j} \)’s in \( \Omega^{0,1}(Y, O(\alpha)) \), where 6 of them are the same and the other 6 different to the first 6 by a sign.

We use computer to prove we can find such \( \eta_{i,j} \)’s satisfying \( \overline{\partial}_\eta = 0 \). 

Until now, we have proved \( \Xi^{E_7}_Y \) is not empty.

Proposition 46 The bundle \( \mathfrak{L}_0^{E_7} \) over \( Y \) with \( \eta \in \Xi^{E_7}_Y \) can descend to \( X \) if and only if for every \( C_k \) and \( \eta_{i,j} \in \Omega^{0,1}(Y, O(C_k)) \), \( [\eta_{i,j} |_C_k] \neq 0 \), i.e. \( \eta \in \Xi^{E_7}_X \).

Proof. Similar to \( E_6 \) case (Proposition [43]).

Note that \( \mathcal{C}^{E_7} = aut(\mathfrak{L}_0^{E_7}, t) \) is an \( E_7 \) Lie algebra bundle over \( Y \). In order for \( \mathcal{C}^{E_7} \) to descend to \( X \) as a Lie algebra bundle, we need to show that \( t|_{C_i} : \mathfrak{L}_0^{E_7} |_{C_i} \otimes \mathfrak{L}_0^{E_7} |_{C_i} \otimes \mathfrak{L}_0^{E_7} |_{C_i} \rightarrow O_{C_i} (2K') \) is a constant map for every \( C_i \). This follows from the fact that both \( \mathfrak{L}_0^{E_7} \) and \( O(2K') \) are trivial on all \( C_i \)’s and \( \overline{\partial}_\eta t = 0 \). It is obvious that \( \mathfrak{L}_0^{E_7} \) is a representation bundle of \( \mathcal{C}^{E_7} \).
6.3 $E_8$ case

Though $E_8$ has no minuscule representation, the fundamental representation corresponding to $C_1$ is the adjoint representation of $E_8$.

We consider a surface $X$ with an $E_8$ singularity $p$ and a $(-1)$-curve $C_0$ passing through $p$ with multiplicity $C_1$. By direct computations, $|I| = 240$. In this case, $l \in I$ if and only if $l - K' \in \Phi$, where $K' = C_0^3 + 2C_1 + 3C_2 + 4C_3 + 5C_4 + 6C_5 + 4C_6 + 2C_7 + 3C_8$. So $E_8^{\Phi}$ defined in §2 can be written as follows:

$$E_8^{\Phi} := O(\mathbb{R}^8) \oplus \bigoplus_{l \in I} O(l) = (O(K')^{S^8} \oplus \bigoplus_{l \in I} O(l))(-K') \oplus O(\mathbb{R}^8) \oplus (O(K')^{S^8} \oplus \bigoplus_{l \in I} O(l)).$$

We will prove that $(E_8^{\Phi}, \mathcal{D}_\varphi)$ with $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Psi_X$ descends to $X$ in §7.

7 Proof of main results

In the above three sections, we have constructed and studied the Lie algebra bundles and minuscule representation bundles in $A_n$, $D_n$ and $E_n$ ($n \neq 8$) cases separately. We will prove the holomorphic structures on these bundles can be expressed by forms in the positive root classes and the representation actions.

Proof. (of Theorem 23 and 24) Recall when $\rho : g \to End(V)$ is the standard representation, $\mathcal{L}_{\eta}^{(g,V)} (\eta \in \Xi^\Phi)$ admits a holomorphic fiberwise symmetric multi-linear form $f$. And $\mathcal{D}_\eta f = 0$ implies that $\eta_{i,j} = 0$ unless $l_i - l_j = \alpha$ ($j > i$) for some $\alpha \in \Phi^+$. Thus $\eta_{i,j} = \varphi_\alpha \in \Omega^{1,1}(Y, O(\alpha))$. Furthermore, if $\eta_{i,j}$ and $\eta'_{i,j'}$ are in $\Omega^{1,1}(Y, O(\alpha))$, then they are the same up to sign. Thus we can write $\eta_{i,j} = \eta_{\alpha,w_i} \varphi_\alpha$, where $\eta_{\alpha,w_i}'s$ are as in §3 since $\rho$ preserves $f$. Namely, $\mathcal{D}_\eta = \mathcal{D}_\varphi + \sum_{\alpha \in \Phi^+} c_\alpha \rho(x_\alpha) = \mathcal{D}_\varphi + \sum_{\alpha \in \Phi^+} \rho(\varphi_\alpha)$ with $\varphi_\alpha = c_\alpha x_\alpha$.

The holomorphic structure on the bundle $\zeta_\varphi := \text{aut}(\mathcal{L}_{\eta}^{(g,V)}, f)$ is $\mathcal{D}_\varphi = \mathcal{D}_\varphi + \sum_{\alpha \in \Phi^+} c_\alpha \text{ad}(x_\alpha)$, which is the same as $\mathcal{D}_\varphi$ for $E_8^\Phi$ in §2, i.e. $\zeta_{\varphi}^\Phi = E_8^\Phi$.

The only minuscule representations $(g, V)$ besides standard representations are $(A_n, \wedge^n \mathbb{C}^{n+1})$, $(D_n, S^k)$ and $(E_6, \mathbb{C}^{27})$. We denote corresponding actions as $\rho$ as usual. In each case, for $E_8^\Phi$ to act holomorphically on the corresponding vector bundle, the holomorphic structure on $\mathcal{L}_{\eta}^{(g,V)}$ can only be $\mathcal{D}_\varphi$.

The filtration of $\mathcal{L}_{\eta}^{(g,V)}$ gives one on $E_8^\Phi$ since it is constructed from extensions using elements in $I_i \setminus I_{i+1}$ (§3.3).

We note that all the above Lie algebra bundles and representation bundles over $Y$ can descend to $X$ if and only if $0 \neq [\varphi_{C_i}]_{C_i} \in H^1(Y, O_{C_i}(C_i))$ for all $C_i$’s, i.e. $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Psi_X$. ■

From the above arguments, Theorem 9 holds true for $ADE$ except $E_8$ case.

Proof. (of Theorem 9) It remains to prove the $E_8$ case.

$$E_8^{\Phi} := O(\mathbb{R}^8) \oplus \bigoplus_{\alpha \in \Phi} O(\alpha) = (O(K')^{S^8} \oplus \bigoplus_{l \in I} O(l))(-K').$$

25
We want to show that the bundle \( (\mathcal{E}_{\varphi}^{E_8}, \overline{\vartheta}_\varphi) \) with \( \varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Phi_X \) can descend to \( X \), i.e. \( \mathcal{E}_{\varphi}^{E_8}|_{C_1} \) is trivial for \( i = 1, 2, \cdots, 8 \). Note \( O(K')|_{C_i} \) is trivial for every \( i \), but \( O(l)|_{C_3} \) can be \( O_{\varphi}(\pm 2) \), hence Lemma 28 is not sufficient. However, if we ignore \( C_8 \) (resp. \( C_7 \)) in \( Y \), then we recover the \( A_7 \) case (resp. \( D_7 \) case). Our approach is to reduce the problem of trivializing \( \mathcal{E}_{\varphi}^{E_8}|_{C_1} \) to one for a representation bundle of \( A_7 \) (resp. \( D_7 \)).

Step one, as \( A_7 \) is a Lie subalgebra of \( E_8 \), the adjoint representation of \( E_8 \) decomposes as a sum of irreducible representations of \( A_7 \). The branching rule is \( 248 = 8 + 28 + 56 + 64 + 56 + 28 + 8 \), correspondingly, we have the following decomposition of \( \mathcal{E}_{\varphi}^{E_8} \) over \( Y \),

\[
\mathcal{E}_{\varphi}^{E_8} = \mathcal{L}_{\varphi}^{A_7}(-K') \oplus \Lambda^2(\mathcal{L}_{\varphi}^{A_7})^*(H - K') \oplus \Lambda^5(\mathcal{L}_{\varphi}^{A_7})^*(2H - K') \oplus \\
\mathcal{L}_{\varphi}^{A_7} \otimes (\mathcal{L}_{\varphi}^{A_7})^* \oplus \Lambda^3(\mathcal{L}_{\varphi}^{A_7})^*(H) \oplus \Lambda^6(\mathcal{L}_{\varphi}^{A_7})^*(2H) \oplus (\mathcal{L}_{\varphi}^{A_7})^*(K'),
\]

where \( H = 3C_1^0 + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3C_5 + 2C_6 + C_7 + C_8 \) and \( K' = C_1^0 + 2C_1 + 3C_2 + 4C_3 + 5C_4 + 6C_5 + 4C_6 + 2C_7 + C_8 \).

Step two, instead of \( \mathcal{L}_{\varphi}^{A_7} \), we use \( \mathcal{L}_{\varphi}^{E_8} \) which is trivial on \( C_i \) for \( i \neq 8 \). We consider the bundle

\[
\mathcal{E}'_{\varphi}^{E_8} = \mathcal{L}_{\varphi}^{A_7}(-K') \oplus \Lambda^2(\mathcal{L}_{\varphi}^{E_8})^*(H - K') \oplus \Lambda^5(\mathcal{L}_{\varphi}^{E_8})^*(2H - K') \oplus \\
\mathcal{L}_{\varphi}^{A_7} \otimes (\mathcal{L}_{\varphi}^{E_8})^* \oplus \Lambda^3(\mathcal{L}_{\varphi}^{E_8})^*(H) \oplus \Lambda^6(\mathcal{L}_{\varphi}^{E_8})^*(2H) \oplus (\mathcal{L}_{\varphi}^{A_7})^*(K').
\]

We have \( \overline{\mathcal{E}}_{\varphi}^{E_8} = \overline{\mathcal{E}}_0 + \sum_{\alpha \in \Phi^+_A} \text{ad}(\varphi_\alpha) \). Since \( O(K') \) and \( O(H) \) are both trivial on \( C_i \) for \( i \neq 8 \), \( \mathcal{E}'_{\varphi}^{E_8} \) is trivial on \( C_i \) for \( i \neq 8 \).

Step three, we compare \( \mathcal{E}'_{\varphi}^{E_8} \) with \( \mathcal{E}_{\varphi}^{E_8} \). Topologically they are the same. Holomorphically,

\[
\overline{\mathcal{E}}_{\varphi}^{E_8} = \overline{\mathcal{E}}_0 + \sum_{\alpha \in \Phi^+_E \setminus \Phi^+_A} \text{ad}(\varphi_\alpha) = \overline{\mathcal{E}}_{\varphi}^{E_8} + \sum_{\alpha \in \Phi^+_E \setminus \Phi^+_A} \text{ad}(\varphi_\alpha).
\]

If we write the holomorphic structure of \( \mathcal{E}_{\varphi}^{E_8} \) as a \( 248 \times 248 \) matrix, then \( \varphi_\alpha \) with \( \alpha \in \Phi^+_E \setminus \Phi^+_A \) must appear at those positions \( (\beta, \gamma) \) with \( \beta - \gamma = \alpha \), where \( \beta \) has at least one more \( C_8 \) than \( \gamma \). That means, after taking extensions between the summands of \( \mathcal{E}_{\varphi}^{E_8} \), we can get \( \mathcal{E}_{\varphi}^{E_8} \). Since \( \mathcal{E}'_{\varphi}^{E_8} \) is trivial on \( C_i \) for \( i \neq 8 \) and \( \text{Ext}^{1}_{A_7}(O, O) \cong 0 \), we have \( \mathcal{E}_{\varphi}^{E_8} \) trivial on \( C_i \) for \( i \neq 8 \).

Similarly, if we consider the reduction of \( E_8 \) to \( D_7 \), from the branching rule \( 248 = 14 + 64 + 1 + 91 + 64 + 14 \), we have the following decomposition of \( \mathcal{E}_{\varphi}^{E_8} \),

\[
\mathcal{E}_{\varphi}^{E_8} = \mathcal{L}_{\varphi}^{D_7}(-K') \oplus \mathcal{L}_{\varphi}^{(D_7, S^+)}(C_7 - C_1^0) \oplus O \oplus \mathcal{L}_{\varphi}^{E_8} \oplus (\mathcal{L}_{\varphi}^{E_8})^*(C_1^0 - C_7) \oplus (\mathcal{L}_{\varphi}^{D_7})^*(K').
\]

Instead of \( \mathcal{L}_{\varphi}^{D_7} \), we consider \( \mathcal{L}_{\varphi}^{E_8} \). Similar to the reduction to \( A_7 \) case as above, we will get for \( (\mathcal{E}_{\varphi}^{E_8}, \overline{\vartheta}_\varphi) \), if we take \( [\varphi_i]|_{C_i} \neq 0 \), then \( \mathcal{E}_{\varphi}^{E_8} \) is trivial on \( C_i \) for \( i \neq 7 \). Hence we have proved Theorem 23 for type \( E_8 \).

**Proof.** (of Theorem 23) We only need to find a divisor \( B \) in \( Y \) such that (i) \( B \) is a combination of \( C_i \)'s and \( C_0 \) with the coefficient of \( C_0 \) not zero, and
(ii) $O(B)$ can descend to $X$. Then if we take $k$ to be the coefficient of $\widetilde{C}_0$ in $B$, $\mathbb{L}^{(g,V)} := S^k \mathbb{L}^{(g,V)} \otimes O(-B)$ with $\varphi \in \Psi_X$ can descend to $X$ and does not depend on the existence of $C_0$.

$(A_n, C^{n+1})$ case, $B = (n+1)\widetilde{C}_0 + nC_1 + (n-1)C_2 + \cdots + C_n$.

$(A_n, \wedge^k C^{n+1})$ case, $B = (n+1)\widetilde{C}_0 + (n-k+1)C_1 + \cdots + (k-1)(n-k-1)C_{k-1} + k(n-k)C_{k+1} + \cdots + kC_n$.

$(D_n, C^{2n})$ case, $B = F = 2\widetilde{C}_0 + 2C_1 + \cdots + 2C_{n-2} + C_{n-1} + C_n$.

$(D_n, S^+) \text{ case, } B = 4\widetilde{C}_0 + 2C_1 + 4C_2 + \cdots + 2(n-2)C_{n-2} + (n-2)C_{n-1} + nC_n$.

$(E_6, C^{27})$ case, $B = 3\widetilde{C}_0 + 4C_1 + 5C_2 + 6C_3 + 4C_4 + 2C_5 + 3C_6$.

$(E_7, C^{56})$ case, $B = 2\widetilde{C}_0 + 3C_1 + 4C_2 + 5C_3 + 6C_4 + 4C_5 + 2C_6 + 3C_7$. \hfill \blacksquare

**Remark 47** We can determine Chern classes of the Lie algebra bundles and minuscule representation bundles. For any minuscule representation bundle $\mathcal{L}^{(g,V)}$, 

$$c_1(\mathcal{L}^{(g,V)}) = \sum_{l \in \mathfrak{l}(\varphi)} [l] \in H^2(Y, \mathbb{Z}).$$

For any Lie algebra bundle $\mathcal{E}^{(g)}$, we have 

$$c_1(\mathcal{E}^{(g)}) = 0$$

and 

$$c_2(\mathcal{E}^{(g)}) = \sum_{\alpha \neq \beta \in \Phi} c_1(O(\alpha))c_1(O(\beta)) = \sum_{\alpha \in \Phi^+} c_1(O(\alpha))c_1(O(-\alpha)) = \dim(\mathfrak{g}) - \text{rank}(\mathfrak{g}).$$

In particular, the bundles we defined above are not trivial.

**Remark 48** There are choices in the construction of our Lie algebra bundles and minuscule representation bundles, we will see that these bundles are not unique. Take $\mathfrak{g}^{A_2}$ ($\varphi = (\varphi_\alpha)_{\alpha \in \Phi_{A_2}} \in \Psi_X$) as an example. The holomorphic structure on $\mathfrak{g}^{A_2}$ is as follows:

$$\overline{\mathfrak{g}} = \begin{pmatrix} \overline{\varphi}_{C_2} & \varphi_{C_1+C_2} \\ 0 & \overline{\varphi}_{C_1} \\ 0 & 0 & \overline{\varphi} \end{pmatrix}$$

with $[\varphi_{C_1}|C_1|] \neq 0$ and $[\varphi_{C_2}|C_2] \neq 0$. We replace $\varphi_{C_1+C_2}$ by $\varphi_{C_1+C_2} + \psi$, where $\psi \in H^1(Y, O(C_1 + C_2)) \neq 0$. If $[\psi] \neq 0$, then $\overline{\mathfrak{g}}_{\varphi+\psi}$ is not isomorphic to $\overline{\mathfrak{g}}_{\varphi}$.

**Remark 49** Our $\mathfrak{g}$-bundle $\mathcal{E}^{\mathfrak{g}}$ over $Y$ is given by $\text{aut}(\mathcal{L}^{\mathfrak{g},V})$ with $f : \bigotimes^r \mathcal{L}^{\mathfrak{g},V} \rightarrow O_Y(D)$. If $O(D) = O(rD')$ for some divisor $D'$, then 

$$f : \bigotimes^r \mathcal{L}^{\mathfrak{g},V}(-D') \rightarrow O_Y.$$ 

And $\text{Aut}(\mathcal{L}^{\mathfrak{g},V})(-D'), f$ is a Lie group bundle over $Y$ lifting $\mathcal{E}^{\mathfrak{g}}$. In general, we only have a $G \times \mathbb{Z}_r$-bundle, or so-called conformal $G$-bundle in $\mathfrak{g}$. 

27
8 Appendix

We now construct examples of surfaces with an $ADE$ singularity $p$ of type $g$ and a $(-1)$-curve $C_0$ passing through $p$ with minuscule multiplicity $C_k$. We call its minimal resolution a surface with minuscule configuration of type $(g, V)$, where $V$ is the fundamental representation corresponding to $-C_k$.

First we consider the standard representation $V \simeq \mathbb{C}^{n+1}$ of $A_n = sl(n+1)$. When we blowup a point on any surface, the exceptional curve is a $(-1)$-curve $E$. If we blowup a point on $E$, the strict transform of $E$ becomes a $(-2)$-curve. By repeating this process $n+1$ times, we obtain a chain of $(-2)$-curves with a $(-1)$-curve attached to the last one. Namely we have a surface with a minuscule configuration of type $(A_n, \mathbb{C}^{n+1})$.

Suppose that $D$ is a smooth rational curve on a surface with $D^2 = 0$. By blowing up a point on $D$, we obtain a surface with a chain of two $(-1)$-curves. If we blowup their intersection point and iterative blowing up points in exceptional curves, then we obtain a surface with minuscule configuration of type $(D_n, \mathbb{C}^{2n})$.

Given a surface together with a smooth rational curve $C$ with $C^2 = 1$ on it. We could obtain every minuscule configuration by the following process. If we blow up three points on $C$, then the strict transform of $C$ is an $(-2)$-curve. By the previous construction of iterated blowups of points in these three exceptional curves $E_i$’s, we could obtain many minuscule configurations. Let us denote the number of iterated blowups of the exceptional curve $E_i$ as $m_i$ with $i \in \{1, 2, 3\}$. Then we can obtain minuscule configuration of type $(g, V)$ by taking suitable $m_i$’s as follows.

| minuscule configuration of type $(g, V)$ | $(m_1, m_2, m_3)$ |
|-----------------------------------------|------------------|
| $(A_n, \Lambda^k \mathbb{C}^{n+1})$ for any $k$ | $(k-1, 0, n-k)$ |
| $(D_n, \mathbb{C}^{2n})$, $(D_n, S^+)$ and $(D_n, S^-)$ | $(n - 3, 1, 1)$ |
| $(E_6, 27)$, $(E_6, \overline{27})$ | $(2, 1, 2)$ |
| $(E_7, 56)$ | $(3, 1, 2)$ |

Note that we could obtain such a configuration for every adjoint representation of $E_6$ this way. We remark that surfaces in this last construction are necessarily rational surfaces because of the existence of $C$ with $C^2 = 1$.

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