Microscopic characteristics and tomography scheme of the local Chern marker

Bernhard Irgisler, Jun-Hui Zheng, and Walter Hofstetter
Institut für Theoretische Physik, Goethe-Universität Frankfurt am Main, Germany

The concept of the local Chern marker has gained a lot of attention especially in the field of ultracold quantum gases in optical lattices and artificial gauge fields. We investigate in further detail the microscopic real-space characteristics of the local Chern marker for the two-band Harper-Hofstadter-Hatsugai model and propose a tomographic scheme for the experimental detection of an approximate local Chern marker neglecting higher orders.

I. INTRODUCTION

Non-trivial topological invariants are the fundamental reason for robust edge states. The vast diversity of different topological invariants opens up a whole world of topological insulators and superconductors [1-4]. The deep connection between topology and physics originally manifests in the TKNN invariant [5] which corresponds to the first Chern number. It is defined as a $k$-space integral of the Berry curvature and thus solely a bulk property. For the case of disorder real-space versions of the Chern number have been successfully applied [6, 7]. A global topological invariant based on real-space representation is the Bott index [8]. Bianco and Resta [9], however, derived a real-space expression by Fourier transforming the Chern number and omitting a final real-space integration. The obtained local quantity is called local Chern marker. Since its introduction it enjoys great popularity in condensed matter theory as well as in the field of cold atomic gases. Applications range from heterojunctions [9] and quasicrystals [10] to interacting fermions [11] and non-interacting, non-homogeneous due to confining laser potentials which exhibits a topological phase transition at the critical stage [12].

In the last years enormous progress has been made implementing topological phases in cold atom experiments [13-15]. In these clean and highly controllable setups it was possible to measure the Chern number [16] as well as probing the Berry curvature [17]. In contrast to solid state materials, cold atom setups are intrinsically inhomogeneous due to confining laser potentials which makes local topological invariants especially interesting for cold atomic gases. Very recently non-interacting, non-equilibrium dynamics have been studied and showed an intriguing current of the local Chern marker [18]. In our work we investigate the microscopic characteristics of the local Chern marker by analyzing its contributions. One way to write the local Chern marker is

$$C(r) = -4 \pi \text{Im} \sum_{r', r''} \rho(r, r') x' \rho(r', r'') y'' \rho(r'', r).$$

This expression, as it is derived from the Chern number, is gauge invariant. Contributions, where any pair of $r$, $r'$ or $r''$ is equal, vanish since they are purely real. This leads us to the notion of triangles. The local Chern marker is thus a sum of contributions from all possible triangles of off-diagonal density matrices with one corner at lattice site $r$. In this paper, we study quantitatively these contributions.

The paper is structured as follows. In Sec. II we introduce the Harper-Hofstadter-Hatsugai model which we use to investigate the local Chern marker. In Sec. III we discuss the contributions to the local Chern marker and its finite size scaling. In Sec. IV we apply our theory to the harmonically trapped system and a topological interface as examples for inhomogeneous cold atom setups. In Sec. V we introduce a tomography scheme to measure the most prominent contribution to the local Chern marker. In Sec. IV we conclude.

II. HARPER-HOFSTADTER-HATSUGAI MODEL

To study the contributions explicitly we focus on the Harper-Hofstadter-Hatsugai (HHH) model [19] which exhibits a gapped topologically non-trivial phase at half filling, in contrast to the Hofstadter model [13], and has some similarity to the Haldane model [15]. The Hamiltonian of the HHH model reads

$$\hat{H} = \lambda \sum_{r} (-1)^{x+\hat{x}} \hat{c}_{r}^{\dagger} \hat{c}_{r} - t \sum_{r} \left[ \hat{c}_{r+\hat{x}}^{\dagger} \hat{c}_{r} + e^{2\pi i \alpha x} \hat{c}_{r+y}^{\dagger} \hat{c}_{r} + e^{2\pi i \alpha (x+1/2)} \hat{c}_{r+\hat{x}+\hat{y}}^{\dagger} \hat{c}_{r+\hat{x}} + \text{h.c.} \right],$$

where $t = 1$ is the hopping energy, $\hat{x}$ ($\hat{y}$) denotes unit vectors in the $x$ ($y$) direction, $\alpha$ is the square plaquette flux, and $\lambda$ is an additional staggering potential. We focus on the case $\alpha = 1/2$ since it yields a simple two-band model and restrict ourselves to the half-filled case. The system exhibits a topological phase transition at the critical staggered potential $\lambda_c = 2$. For $\lambda < 2$ the system shows a non-trivial phase with Chern number $1$, for $\lambda > 2$ it has a
FIG. 1. Contributions $c_l$ to the local Chern marker as function the triangle’s area and circumference in log scale in color for different values of the staggering potential $\lambda$. The sign of each contribution is marked in the figure by $\{+,-\}$. The inset shows the highest (blue) and lowest (orange) contributions in normal scale. $C$ refers to the bulk value of the local Chern marker. Results were obtained for a $40 \times 40$ lattice.

trivial phase with Chern number 0. This can be seen from the Fourier-transformed Hamiltonian $\vec{H}(k) = v(k) \cdot \vec{\sigma}$, where $v(k) = (2 \cos(k_x), 4 \sin(k_x) \sin(k_y), \lambda - 2 \cos(k_y))$ [20]. Here, $\vec{\sigma}$ refers to the Pauli vector in the pseudospin representation of the A,B-sublattice structure. If $|\lambda| < 2$ the two-dimensional surface of $v(k)$ encloses the origin and the phase is topologically non-trivial.

III. CONTRIBUTIONS TO THE LOCAL CHERN MARKER

As discussed above the local Chern marker is a sum of off-diagonal density matrices connected to triangles. If we characterize the triangle by its circumference and its area combined in a tuple $l = (\text{circumference}, \text{area})$ the summation can be rewritten as $C(r) = \sum_l c_l$. Here $c_l$ is the sum of contributions of triangles having the same circumference and area $l$. Fig. 1 shows $c_l$ of the first few triangles as function of their area and circumference for different staggering potentials $\lambda$ in a)-i) computed in a $40 \times 40$ lattice. Triangles with vanishing area do not contribute to the local Chern marker. We observe that the contribution of the triangles with area 0.5 and circumference $2 + \sqrt{2}$ is the largest. For $\lambda = 0$ in Fig. 1a) it exceeds by three orders of magnitude the second largest contribution, we call it first-order contribution. It amounts to roughly 0.87, i.e., 87% of the quantized value of 1. The missing 13% seem to stem from long-range entanglement over the lattice. Note that this value is model-specific. The bulk average of the local Chern marker $C$ is computed as the A,B-sublattice average in the center of the system $C = [C(r_A) + C(r_B)] / 2$. Since we consider a two-band model, the average has two contributions. For $\lambda = 0$ it is quantized and corresponds to the Chern number. For increasing $\lambda$, we observe that it is not quantized anymore. Also the first-order contribution decreases and simultaneously more positive-valued higher-order contributions emerge. This is best seen right before the phase transition in Fig. 1d) for $\lambda = 1.5$. Right after the phase transition for $\lambda = 2.5$, in Fig. 1f), we observe roughly as many higher-order contributions as in Fig. 1d) but with
FIG. 2. a) Labeling of the density matrices for the first-order contribution to the local Chern marker in the A,B-sublattice model. Off-diagonal density matrices are uniquely labeled by a number and a direction shown as an arrow, and are connected to triangles. b) finite-size scaling of the bulk average of the local Chern marker, c) collapse of curves after rescaling, and d) power law relation between the width of the transition region $\Delta \lambda$ and the system size $N$.

We now focus on the described first-order contribution and derive a simple expression for the local Chern marker for the two-band HHH model. The first-order contribution only contains terms with density matrices connecting nearest-neighboring and next-nearest-neighboring sites. This already contains contributions of 48 different triangles. We exploit the symmetries of the model in order to reduce this number. In Fig. (2a) we show schematically all possible triangles for the first-order contribution to the local Chern marker. The numbering as well as the arrow direction label a specific density matrix, e.g. $\rho_1 = \rho(r_A, r_A + \hat{x} + \hat{y})$. We assume local translational invariance in this small region. Evaluating Eq. (2) for the triangles in Fig. 2 and performing the A-B-sublattice average $C$ we find

$$C = -4\pi(\rho_7 - \rho_3)(2\rho_5\rho_8 + 2\rho_6\rho_8^* - \rho_1\rho_4 - \rho_2\rho_4^*).$$

The first-order contribution is thus reduced from a sum of 48 different triangles to a formula containing just eight off-diagonal density matrices.

In Fig. (2b) we look at finite size scaling of the two-band HHH model. We show the bulk average $C$ of the local Chern marker for different sizes of the lattice $N \times N$ as a function of $\lambda$ and observe that the transition at $\lambda = \lambda_c$ becomes steeper with increasing lattice size. We also show the first-order contribution as dash-dotted lines and observe that it is scale invariant showing that it is purely local. In Fig. (2c) and d) we perform a scaling analysis of the two-band HHH model according to the scaling analysis of the Haldane model in Ref. [18]. To this end, we define the width of the transition region $\Delta \lambda$ as the difference of the value of $\lambda$ where $C = 0.05$ and the value of $\lambda$ where $C = 0.95$ both represented as dashed gray lines in Fig. (2b). We further assume that the bulk correlation length $\xi$ scales as $\xi \approx (\Delta \lambda)^{-\nu}$. Since $\xi$ directly scales width the system size, we find $\Delta \approx N^{-1/\nu}$. From Fig. (2d) we compute $\nu \approx 1.02$. We assume the scaling form of the bulk value of the local Chern marker $C \sim f(\xi/N)$ which with the considerations made before becomes $C \sim \hat{f}(\lambda - \lambda_c N^{1/\nu})$. After the rescaling we observe a collapse of curves in Fig. (2c) like in Ref. [18]. In contrast to the result for the Haldane model [18], we observe a rescaled curve in Fig. (2c) for the HHH model exhibiting the symmetry $\hat{f}(-(\lambda - \lambda_c)N^{1/\nu}) = 1 - \hat{f}(\lambda - \lambda_c)N^{1/\nu})$.

IV. TRAP

As it is an example for an inhomogeneous system in cold atom setups we check our theory on a system with a harmonic trap. Fig. 3) shows the local Chern marker as well as the first-order contribution according to Eq. (3). We observe good qualitative agreement between both approaches. The bulk value deviates as discussed before by about 13%.
A second example is the topological interface \cite{12, 21, 22} which is used to create an in-situ topological phase separation. In Fig. 2a), we show the local Chern marker resulting from a system with staggering potential \( \lambda(x) = 5\pi/100(1-1)^2 \). Applying a hyperbolic tangent fit \( a - b \tanh(c(\lambda - d)) \) the local Chern marker predicts a phase transition point at \( x \approx 40 \), shown as dashed red line, which translates to a critical staggering potential of \( 2 \) as expected. The first-order contribution is rather smooth and predicts a phase transition point at \( x \approx 36 \), shown as dashed green line, which translates to a critical staggering potential of \( 1.8 \). The first-order contribution can thus estimate the phase transition point up to an error of 10\%. On the other hand, in Fig. 2b), we showed that the first-order contribution is scale invariant and always takes the value 0.4 at the phase transition. With this consideration, the phase transition point is determined as \( x \approx 40 \), corresponding to a critical staggering potential of \( 2 \) as expected. Of course this value is also model dependent.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Quench Hamiltonians realized by superlattice structures for a) nearest-neighbor and b) next-nearest-neighbor couplings. Dark regions correspond to minima at the lattice sites. c) Time evolution of the local density \( n(r, t) \) for all eight density matrices defined in Fig. 2a) after the quench with Eq. (4) for \( \theta = \pi/6 \).}
\end{figure}

VI. TOMOGRAPHY

As we have shown, the dominant contribution to the local Chern marker comes from off-diagonal terms of nearest-neighbor and next-nearest-neighbor density matrices. These should be observable in experiments. Higher-order contributions from far distant density matrices are in principle measurable as proposed in Ref. \cite{23}. For the measurement of the first-order contribution we propose a tomographic scheme \cite{17, 23, 24} which measures nearest-neighbor and next-nearest-neighbor density matrices in real space and does not rely on an additional coupling channel as in Ref. \cite{23}. Our scheme can measure the eight different density matrices defined in Fig. 2a) individually. To this end, the system is quenched with the two-level Hamiltonian

\[ H_Q = J \left( \cos \theta \sigma_x + \sin \theta \sigma_z \right) . \] (4)

The two levels correspond to the two sites \( r, r' \) of the respective density matrix. The Hamiltonian in Eq. (4) corresponds to hopping between the sites with energy \( J \cos \theta \) and an energy offset of \( J \sin \theta \) between the levels. For \( \theta = \pi/6 \) the ratio between these energy scales is about 1.7 which is a reasonable value to make a measurement. The quench can be performed by suddenly switching on a lattice potential which we schematically show in Fig. 4a) for nearest-neighbor coupling and in b) for next-nearest-neighbor coupling. These potentials are created by superponing two retroreflected laser beams with wavevectors \( \kappa \) and \( 2\kappa \). This yields a periodic double-well potential. The two lasers should exhibit a small phase difference in order to obtain an energy offset between the coupled sites. A third laser beam with wavevector \( 2\kappa \) determines the direction along which the two sites of the density matrix should be coupled. It points orthogonal for nearest-neighbor coupling and in 45° in next-nearest-neighbor coupling with respect to the direction of the aforementioned lasers. We parametrize the two-level density matrix as

\[ \tilde{\rho} = \begin{pmatrix} n(r) & \rho(r, r') \\ \rho^*(r, r') & n(r') \end{pmatrix} \] (5)

where \( n(r) \) is the on-site density of site \( r \). The time evolution of Eq. (5) follows as

\[ \tilde{\rho}(t) = e^{iH_Q t} \tilde{\rho} e^{-iH_Q t}. \] (6)

we find

\[ n(r, t) = n(r) - [n(r) - n(r')] \cos^2(\theta) \sin^2(t J) + \text{Re} \tilde{\rho}(r, r') \sin(2\theta) \sin^2(t J) + \text{Im} \tilde{\rho}(r, r') \cos(\theta) \sin(2t J). \] (7)

By measuring the local densities \( n(r, t) \) and \( n(r', t) \) as functions of time by means of a quantum gas microscope Eq. (7) can be used as a fit function to determine the off-diagonal part \( \rho(r, r') \) of the density matrix \( \tilde{\rho} \). In Fig. 4 we show the time evolution \( n(r, t) \) for all eight density matrices defined in Fig. 2. The local density matrices can then be measured for the whole lattice at once. A measurement for \( \theta = 0 \), i.e., without an energy offset, has been performed \cite{27} giving access to \( \text{Im} \tilde{\rho}(r, r') \) in Eq. (7).

VII. CONCLUSION

In conclusion, we investigated the contributions to the local Chern marker in terms of off-diagonal density matrices for the instance of the two-band HHH model. We
find that first-order contribution is by orders of magnitudes the highest and purely local. Since topological properties are of course global properties, this gives only an indicator for topological non-trivial phases. At the topological phase transition the long-range entanglement becomes large due to the gap closing. We propose a tomographic measurement scheme for the first-order contribution which consists of measuring nearest-neighbor and next-nearest-neighbor correlations by means of a non-equilibrium superlattice quench and a quantum gas microscope. The two-band HHH model serves here as an example model. Applications to other models as well as extensions to multiple bands are straightforward. The generalization to the interacting case would require the many-body derivation of the original idea by Bianco and Resta \cite{Bianco2011} on many-body Chern numbers \cite{Ishikawa1986,Wang2012}.

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