On the instability of some k-essence space-times

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We study the stability properties of static, spherically symmetric configurations in k-essence theories with the Lagrangians of the form

$$F(X), \quad X \equiv \phi, \phi^{\alpha}, \rho, \rho$$

The instability under spherically symmetric perturbations is proved for two recently obtained exact solutions for

$$F(X) = F_0 X^{1/3} \quad \text{and for} \quad F(X) = F_0 X^{1/2} - 2\Lambda,$$

where $F_0$ and $\Lambda$ are constants. The first solution describes a black hole in an asymptotically singular space-time, the second one contains two horizons of infinite area connected by a wormhole. It is argued that spherically symmetric k-essence configurations with $n < 1/2$ are generically unstable because the perturbation equation is not of hyperbolic type.

1 Introduction

General Relativity (GR) theory has been very successful when tested at local scales [1]. However, GR must face, at same time, many problems. One of them is the presence of singularities in its applications to cosmology and compact objects, especially black holes. Moreover, the standard cosmological model (SCM), in spite of fitting remarkably well the observational data, requires exotic, undetected forms of matter which composes the dark sector of the universe. The SCM requires also an initial phase of accelerated expansion, the inflationary phase, in order to explain the very particular initial conditions leading to the presently observed universe. The mechanism behind this inflationary phase is still an object of debate, leading to many speculative models.

Among the modifications introduced in the GR framework in order to account for an inflationary phase in the primordial universe, a particularly interesting one is the class of k-essence models [2,3] which consists in a scalar field minimally coupled to the Einstein-Hilbert term but with a nonstandard kinetic term. Such a structure can be related to more fundamental theories like string theories. One example of this connection is the Dirac-Born-Infeld action [4]. Even if the initial motivation for the k-essence model was the inflationary cosmological phase, subsequently it has also been used also for the description of the present accelerated expansion phase of the universe [5].

In a previous paper [6], we have analyzed the possible black hole type structures in the context of k-essence theories. The k-essence class of theories is described by the Lagrangian

$$\mathcal{L} = \sqrt{-g} [R - F(X, \phi)],$$

with

$$X = \eta \phi, \phi^{\mu},$$

$$\eta = \pm 1.$$  In general, $F(X, \phi)$ can be any analytical function of $X$ and $\phi$.

In Ref. [6] the function $F(X, \phi)$ has been fixed as

$$F(X, \phi) = F_0 X^\alpha - 2V(\phi), \quad F_0 = \text{const.}$$

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The emphasis was in obtaining complete analytical solutions. Two cases have been analyzed in detail. The first one consists of a pure kinetic term, with $n = 1/3$ and $V(\phi) = 0$, leading to a Schwarzschild type black hole immersed in a singular space-time, the singularity placed at spatial infinity. The other solution is obtained by fixing $n = 1/2$ and $V(\phi) = \Lambda = \text{constant}$ leading to a non-asymptotically flat regular space-time, with a degenerate horizon, a structure locally similar to the cold black holes that are present in scalar-tensor theories of gravity [7,8]:

In both cases, the horizons have an infinite area and zero Hawking temperature.

The goal of the present paper is to study the stability of those static, spherically symmetric configurations found in the context of the class of k-essence theories described above. The study of stability of black hole-type structures with scalar fields is a controversial subject with many conceptual and technical issues, see, e.g., [9–13] and references therein. In performing the stability analysis for the k-essence black hole-type structures found in [6], we closely follow the approach used in [10]. We conclude that the solutions found in Ref. [6] are unstable with respect to radial linear perturbations.

The paper is organized as follows. In the next section we set up the relevant perturbed equations. In Section 3 we apply the perturbation analysis to the solutions found in Ref. [6], and in Section 4 we present our conclusions.

2 Perturbation equations

Varying the Lagrangian (1) with respect to the metric and the scalar field, we obtain the field equations

\[ G^\mu_{\nu} = -T^\mu_{\nu}[\phi], \]  

\[ T^\mu_{\nu}[\phi] \equiv \eta F_X \phi \phi_{\mu} \phi_{\nu} - \frac{1}{2} \delta^\mu_{\nu} F, \]  

\[ \eta \nabla_{\alpha}(F_X \phi^\alpha) - \frac{1}{2} F_{\phi} = 0, \]  

(4) \hspace{1cm} (5) \hspace{1cm} (6)

where $G^\mu_{\nu}$ is the Einstein tensor, $F_X = \partial F / \partial X$, $F_{\phi} = \partial F / \partial \phi$, and $\phi_{\mu} = \partial_{\mu} \phi$.

We consider a spherically symmetric space-times with the metric

\[ ds^2 = e^{2\gamma(u,t)} dt^2 - e^{2\alpha(u,t)} du^2 - e^{2\beta(u,t)} d\Omega^2, \]  

(7)

(where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the metric on a unit sphere) containing only small time-dependent perturbations from static ones. Accordingly, we assume

\[ \alpha(u,t) = \alpha(u) + \delta \alpha(u,t) \]  

with small $\delta \alpha$, and similarly for $\beta(u,t)$ and $\gamma(u,t)$. The radial coordinate $u$ is arbitrary, admitting any reparametrization $u \rightarrow \tilde{u}(u)$. The nonzero Ricci tensor components can be written in the form (preserving only linear terms with respect to time derivatives)

\[ R_t^t = e^{-2\gamma}(\ddot{\alpha} + 2\ddot{\beta}) \]  

\[ - e^{-2\alpha}[\dddot{\gamma} + \gamma'(\gamma' - \alpha' + 2\beta')], \]  

\[ R_u^u = e^{-2\gamma} \ddot{\alpha} - e^{-2\alpha}[\dddot{\gamma} + 2\dddot{\beta} + \gamma'^2 + 2\beta'^2 - \alpha' (\gamma' + 2\beta')], \]  

\[ R_{\theta}^\theta = R_{\varphi}^\varphi = e^{-2\beta} + e^{-2\gamma} \ddot{\beta} \]  

\[ - e^{-2\alpha}[\dddot{\beta} + \beta'(\gamma' - \alpha' + 2\beta')], \]  

\[ R_{tu} = 2[\dot{\beta}' + \dot{\beta}' - \ddot{\alpha} \beta' - \dot{\beta} \gamma'], \]  

(8) \hspace{1cm} (9) \hspace{1cm} (10) \hspace{1cm} (11)

where dots and primes stand for $\partial / \partial t$ and $\partial / \partial u$, respectively.

In a similar way, we assume $\phi = \phi(u,t) = \phi(u) + \delta \phi(u,t)$ with small $\delta \phi$. An accord with Eq. (2), for static (and slightly nonstatic) scalar fields, to keep $X$ positive, we assume $\eta = -1$, so that, preserving only linear terms in $\delta \phi$, we have

\[ X = e^{-2\alpha} \phi'^2, \]  

\[ \delta X = 2e^{-2\alpha}(\phi' \delta \phi' - \phi'^2 \delta \alpha), \]  

(12)

Then we obtain the following expressions for the nonzero stress-energy tensor (SET) components:

\[ T_t^t = T_\theta^\theta = T_\varphi^\varphi = -F/2, \]  

\[ T_u^u = -F/2 + FX, \]  

\[ T_{tu} = -FX \phi \phi', \]  

(13)

where, taking into account the perturbations, we should understand $F$ as $F(X) + FX \delta X + F_\phi \delta \phi$ and $FX_\phi \delta \phi$. In what follows, we will consider a more narrow class of k-essence Lagrangians: instead of $F(X, \phi$, we take simply $F(X)$. Then the scalar field equation and the nontrivial components of the Einstein
equations can be written as follows:
\[
F_X e^{\alpha+2\beta-\gamma \phi'} - \left(F_X e^{-\alpha+2\beta+\gamma \phi'}\right)' = 0, \tag{14}
\]
\[
e^{-2\alpha-\beta} + \beta(\beta' + 2\gamma') = \left(\frac{1}{2} F - X F_X\right) e^{2\alpha}, \tag{15}
\]
\[
e^{-2\alpha-2\beta \gamma + 3\beta'^2 - 2\alpha' \beta'} = \frac{1}{2} F e^{2\alpha}, \tag{16}
\]
\[
e^{2\alpha-2\beta + e^{2\alpha-2\beta} \gamma - \beta'' - \beta'(\gamma' - \alpha' + 2\beta') = -\frac{1}{2} e^{2\alpha}(F - X F_X), \tag{17}
\]
\[
\dot{\beta}' + \dot{\beta} - \dot{\alpha} - \dot{\beta}' - \dot{\gamma}' = \frac{1}{2} F_X \dot{\phi} \phi'. \tag{18}
\]
Equations (15)-(18) are, respectively, the equations that substantially simplify the calculations, and, excluding the quantities \(\delta \sigma, \delta \beta, \delta \gamma\) from the scalar equation.

In this problem statement, as in many similar problems [], we have two kinds of arbitrariness, the radial coordinate choice (so far to be left arbitrary), and the perturbation coordinate choice that corresponds to fixing a reference frame in perturbed space-time. As in [9,10], it is helpful to choose the gauge \(\delta \beta = 0\) that substantially simplifies the calculations, and, after obtaining the final form of the perturbation equation, it is necessary to make sure that it is gauge-invariant.

With \(\delta \beta = 0\), the equation for \(\delta \phi\) reads
\[
e^{-2\alpha-2\gamma} \delta \phi' + \delta \phi'' + \phi' \delta \sigma' + \sigma' \delta \phi' + \frac{F_y}{F_X} \delta \phi' + \phi' \delta \left(\frac{F_y}{F_X}\right) = 0, \tag{19}
\]
where
\[
\sigma = 2\beta + \gamma - \alpha, \quad \delta \sigma = \delta \gamma - \delta \alpha. \tag{20}
\]
Since \(X = e^{-2\alpha} \phi'^2\), from the quantities \(\delta X\) and \(\delta X'\) we contain combinations of \(\delta \phi', \delta \phi'', \delta \alpha\) and \(\delta \alpha'\). In particular, if we assume
\[
F(X) = F_0 X^n - 2\Lambda, \quad n, \Lambda = \text{const}, \tag{21}
\]
equation (19) takes the form
\[
e^{-2\alpha-2\gamma} \dot{\delta \phi'} + (2n-1) \delta \phi'' + \delta \phi'[\sigma' - 2(n-1)\alpha'] + \phi'[\delta \gamma' - (2n-1)\delta \alpha'] = 0. \tag{22}
\]
The quantities \(\delta \gamma', \delta \alpha\) and \(\delta \alpha'\) can be expressed in terms of \(\delta \phi\) and its derivatives and the background quantities using the Einstein equations. More specifically, Eq. (15) gives an expression for \(\delta \gamma'\) and Eq. (17) that for \(\delta \gamma' - \delta \alpha'\), containing \(\delta \alpha\); the latter is found from Eq. (18) as
\[
\delta \alpha = -\frac{n}{2\beta} X^{n-1} \phi' \delta \phi. \tag{23}
\]
As a result, the field equation for \(\delta \phi\) takes the form
\[
-e^{-2\alpha-2\gamma} \delta \phi' + (2n-1) \delta \phi'' + \delta \phi'[\sigma' - 2(n-1)\alpha'] - \frac{n^2}{\beta^2} \left(\frac{e^{-2\beta} - \Lambda}{e^{4\alpha}} F_0 X^n \delta \phi = 0. \tag{24}
\]
After the standard substitutions
\[
\begin{align*}
\frac{du}{dz} &= e^{\gamma - \alpha}, \\
\delta \phi &= \Psi(z) e^{i\omega t}, \quad \Psi(z) = f(z) \psi(z), \quad f(z) = \exp \left[-\beta + \frac{(1 - n)\gamma}{2n - 1}\right],
\end{align*}
\]
we obtain a Schrödinger-type equation for \(\psi\),
\[
(2n-1) \frac{d^2 \psi}{dz^2} + [\omega^2 - V(z)] \psi(z) = 0, \tag{27}
\]
where \(V(z)\) is a certain effective potential for scalar perturbations, whose explicit form in terms of the background functions is rather bulky.

We can state that the background static solution is unstable in our linear approximation if Eq. (24) has a solution growing with time and satisfying certain physically relevant boundary conditions. This happens when the corresponding boundary-value problem for Eq. (27) has solutions with \(\omega^2 < 0\) since in this case there are physically meaningful solutions to Eq. (24) growing as \(e^{i\omega t}\).

The exact boundary-value problem for Eq. (27) cannot be posed without invoking particular background solutions. However, one general observation can be made immediately.

Indeed, if \(n < 1/2\), Eq. (27) may be rewritten as follows:
\[
\frac{d^2 \psi}{dz^2} + \frac{\Omega^2 + V(z)}{1 - 2n} \psi(z) = 0, \tag{28}
\]
where $\Omega^2 = -\omega^2$. This is a usual form of the Schrödinger equation for the “energy level” $\Omega^2/(1-2n)$ with the potential $W(z) = -V/(1-2n)$. Depending on the particular form of $V$ and on the boundary conditions, the corresponding boundary-value problem has a certain spectrum of eigenvalues, and in a majority of situations (in a quantum-mechanical analogy, if $-V$ does not form potential walls on both ends of the $z$ range), there is a continuous spectrum $\Omega^2 > K = \text{const}$. Even if there is only a discrete spectrum, in most cases it is not bounded above. But all this means that $\omega^2$ is not bounded below, hence the background system is catastrophically unstable and decays, in the linear approximation, infinitely rapidly — which actually means that perturbations very rapidly become large and must be considered in a nonlinear mode.

Whether or not this really happens, should be verified for specific solutions and relevant boundary conditions for $\delta\phi$ and $\psi$. Nevertheless, we can conclude that solutions with $n < 1/2$ are generically unstable.

3 Two special solutions and their instability

3.1 Solution 1: $n = 1/3$, $\Lambda = 0$

The first exact solution, obtained in [6], corresponds to the case $n = 1/3$, $\Lambda = 0$. It is conveniently written in terms of the so-called quasi-global coordinate $u = x$ defined by the condition $\alpha(x) + \gamma(x) = 0$. The metric has the form

$$ds^2 = \frac{B(x)}{k^2x}dt^2 - \frac{k^2x}{B(x)}dx^2 - \frac{1}{k^2x}d\Omega^2,$$

$$B(x) = B_0 - \frac{1}{2}k^2x^4,$$ (29)

where $B_0$ and $k$ are integration constants. The scalar field $\phi$ is determined by the relation

$$\frac{d\phi}{dx} = \phi_0 \left( \frac{B_0}{x^4} - \frac{1}{2}k^2 \right), \quad \phi_0 = \text{const.}$$ (30)

If $B_0 \leq 0$, the metric function $A(x) = g_{tt} = -g^{xx} < 0$, and we are dealing with a special case of a Kantowski-Sachs cosmological model. Therefore, in our stability study, we put $B_0 > 0$, in which case the metric is static at $0 < x < x_h = (2B_0)^{1/4}/k$, and at $x = x_h$ there is a Killing horizon beyond which there is a cosmological region. At $x = 0$, where $r^2 = -g_{\theta\theta} = \infty$ (so it may be called a spatial infinity), there is a singularity where both $\phi$ and the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ are infinite. The Carter-Penrose diagram for this spacetime looks the same as for the de Sitter metric, although here a nonstatic T-region corresponds to radii $r(x) < r(x_h)$, as it happens for black holes. It was therefore concluded [6] that this solution describes a black hole in an asymptotically singular space-time.

The perturbation equation (24) for $n = 1/3$ $\alpha + \gamma = 0$ and $\Lambda = 0$ reads

$$-3e^{-4\gamma}\delta\phi'' - \delta\phi'' + 2(3\beta' + \gamma')\delta\phi' - \frac{F_0}{3\beta'^2}e^{-2\beta + \gamma}X^{1/3}\delta\phi = 0,$$ (31)

where, according to (29) and (30), we should substitute

$$e^{2\gamma} = e^{-2\alpha} = \frac{B(x)}{k^2x^4}, \quad e^{2\beta} = \frac{1}{k^2x^4},$$

$$X^{1/3} = \frac{B(x)\phi_0^{2/3}}{k^2/3x^3}.$$ (32)

Further on, we consider a Fourier mode, $\delta\phi = \Psi(x)e^{i\omega t}$. To obtain a Schrödinger-like equation for this mode, we first get rid of the factor $e^{-4\gamma}$ before $\delta\delta = -\omega^2\delta\phi$ by passing over to the “tortoise” coordinate $z$, such that

$$\frac{dx}{dz} = e^{\gamma - \alpha} = e^{2\gamma},$$ (33)

which results in

$$\Psi_{zz} - 2(3\beta_z + 2\gamma_z)\Psi_z - 3\omega^2\Psi + F_1 B(x)\Psi = 0,$$ (34)

where the subscript $z$ stands for $d/dz$, and $F_1 > 0$ is a constant whose particular value is irrelevant. Lastly, we get rid of the term with $\Psi_z$ by substituting

$$\Psi(z) = e^{3\beta + 2\gamma}\psi(z),$$ (35)

and with the functions (32) we obtain the equation

$$\psi_{zz} - 3\omega^2\psi - V(z)\psi = 0,$$

$$V(z) = -F_1 B(x) + \frac{5B_0^2}{4k^4x^4} + \frac{31B_0}{4k^2} - \frac{3x^4}{16}.$$ (36)
With this Schrödinger-like equation we can pose a boundary-value problem where, which is unusual, the role of an energy level is played by the quantity $E = -3\omega^2$. This means that if the spectrum of $E$ is not restricted above, then $\omega^2$ is not restricted below, and which leads to a possible growth of perturbations with arbitrarily large increments.

To find out whether it is really so, let us look at the behavior of the potential at the extremes of the range of $z$ and formulate the boundary conditions for $\psi$.

The coordinate $x$ ranges from $x = 0$ (singularity) to $x = x_h$ where $B(x) = 0$ (the horizon). It is easy to find that $x = 0$ corresponds to a finite $z$, and we can put there $z = 0$, and then

$$z \propto x^2 \text{ as } x \to 0, \quad V(z) \approx \frac{5B_0}{4k^4}x^2 \propto \frac{1}{x^2}. \quad (37)$$

On the other hand, near $x = x_h$,

$$z \propto -\ln(x_h - x) \to \infty, \quad V(z) \to -\frac{8B_0}{k^2}. \quad (38)$$

Thus $z \in \mathbb{R}_+$, and $V(z)$ is smoothly changing from an infinitely tall “wall” at $z = 0$ to a negative constant as $z \to \infty$.

What about boundary conditions for $\psi$? Let us require that $\delta \phi$ at the boundaries does not grow faster than $\phi$ itself, which, according to (30), means that $\delta \phi$ may grow as $x^{-3}$ near $x = 0$ and should be finite as $x \to x_h$. Taking into account the substitutions (33) and (35), we conclude that $\psi$ is allowed to grow on both boundaries not faster than

$$\psi \sim e^z \text{ as } z \to \infty, \quad \psi \sim z^{-1/4} \text{ as } z \to 0. \quad (39)$$

We see that these requirements are much milder than could be imposed on a quantum-mechanical wave function, therefore the spectrum of $E = -3\omega^2$ is manifestly not restricted above (actually, a continuous spectrum should begin with $-8B_0/k^2$ and extend to $+\infty$). Therefore, $\omega^2$ can take negative values arbitrarily large in absolute value, and thus our static configuration is catastrophically unstable.

### 3.2 Special solution 2: $n = 1/2, \Lambda > 0$

This solution has been obtained [6] using the harmonic coordinate condition [14]

$$\alpha = 2\beta + \gamma, \quad (40)$$

Then Eq. (14) leads to

$$\left[n e^{2(1-n)\alpha} \phi'^{2n-1}\right]' = 0, \quad (41)$$

which, for $n = 1/2$, implies $e^\alpha = a = \text{const}$. Next, denoting $\Lambda = 3/b^2$ and choosing the length scale so that $a = b^2$, we obtain from the Einstein equations [6]

$$ds^2 = \frac{9}{\cosh^4 bu}dt^2 - b^4 du^2 - \frac{b^2 \cosh^2 bu}{3}d\Omega^2, \quad (42)$$

$$\sqrt{X} = e^{-\alpha} |\phi'| = \frac{4}{F_0} \left(\Lambda - \frac{2}{b^2 \cosh^2 bu}\right) = \frac{4}{F_0 b^2} (1 + 2 \tanh^2 bu), \quad (43)$$

$$\phi = \pm \frac{4}{F_0 b^2} \left(3u - \frac{2}{b} \tanh bu\right) + \phi_0, \quad (44)$$

$\phi_0 = \text{const}$. In terms of the quasiglobal coordinate $x = 3b \tanh bu$, the metric reads

$$ds^2 = \frac{(9b^2 - x^2)^2}{9b^4} dt^2 - \frac{9b^4}{(9b^2 - x^2)^2} dx^2 - \frac{3b^4}{9b^2 - x^2} d\Omega^2, \quad (45)$$

from which it is clear that there are two second-order (degenerate) horizons at $x = \pm 3b$. The scalar field $\phi$ in the whole range of $x, x \in \mathbb{R}$, is found as

$$\phi = \pm \frac{4}{3F_0 b^4} \left(-2x + \frac{9b}{2} \ln \left|x + 3b\right|\right) + \phi_0 \quad (46)$$

and is singular on the horizons $x = \pm 3b$, while $X = A\phi'^2$ is infinite only as $x \to \pm \infty$ and is finite on the horizons.

Considering the stability of the static region and applying the perturbation equations to our case, we notice that in (42) or (44) we have $2\beta + \gamma = 2 \ln b = \text{const}$, hence $2\beta' + \gamma' = 0$, and Eq. (24) now takes the form

$$\delta \ddot{\phi} = h(u) \delta \phi, \quad (47)$$

where

$$h(u) = \frac{27}{b^4 \cosh^4 \frac{1}{b} + 2 \tanh^2 bu} (1 + 2 \tanh^2 bu) > 0 \quad (47)$$

Equation (46) is solved explicitly. Its first integral reads

$$\delta \dot{\phi}^2 = \delta \phi^2 h(u) + C_1(u), \quad C_1(u) \text{ arbitrary.} \quad (48)$$
Let us put $C_1(u) \equiv 0$ and suppose that $h(u) > 0$ at least in some range of $u$. We then obtain

$$\delta \phi(t, u) = e^{\pm \sqrt{h(u) + C_2(u)}}, \quad C_2(u) \text{ arbitrary.} \quad (49)$$

Evidently, in the solution with the plus sign, $\delta \phi$ grows with time. On the other hand, the arbitrariness of $C_2(u)$ makes it possible to satisfy any boundary conditions of the form $\delta \phi \leq q(u)$ where $q(u)$ is specified on each boundary from some physical requirements like finite perturbation energy, etc.

We conclude that a background solution to the k-essence equations with $n = 1/2$ is unstable if the function $h(u)$ is positive in some range of the coordinate $u$.

This obviously applies to our solution since the function (47) is positive at all $u$, that is, in the whole static region. It means that our static solution is unstable.

### 4 Conclusion

In this work we have analyzed the stability of black hole-type configurations found previously in the framework of the k-essence theory [6]. These solutions were obtained in the case where the k-essence function $F(X, \phi)$ is a power law given by $F(X, \phi) = F_0 X^n - 2V(\phi)$. The special cases $n = 1/3$ (with $V(\phi) = 0$) and $n = 1/2$ (with $V(\phi) = \text{const}$) admit analytical solution for a static, spherically symmetric configuration. The case $n = 1/2$ leads to a consistent solution in the presence of a cosmological constant. Both solutions are not asymptotically flat. For $n = 1/3$ the spatial infinity is singular, while the geometry of the case $n = 1/2$ is regular, with a degenerate horizon similar to cold black holes existing in scalar-tensor theories [7,8].

Linear radial perturbations were considered. The analysis was performed using the gauge condition $\delta \beta = 0$, where $\beta$ is the logarithm of the radius of coordinate two-spheres. This choice is consistent with the gauge-invariant approach for perturbations in static, spherically symmetric configurations [9-11]. Both black hole-type solutions found in the k-essence framework described above turn out to be unstable: it is possible to obtain solutions for the perturbed equations which grow unboundedly with time, satisfying at the same time the required boundary conditions.

Moreover, it is argued that, at least generically, all static, spherically symmetric k-essence configurations with $n < 1/2$ must be unstable: this happens, in fact, because the master equation for perturbations loses its hyperbolic nature. This is, however, not proved in a general form because it is necessary to take into account physically motivated boundary conditions for each particular solution.

A point of interest is that for the intermediate value of $n$, namely, $n = 1/2$ we see a rare case where the perturbation equation can be directly analytically integrated without a decomposition into Fourier modes; this analytical solution explicitly shows the instability of the background static solution.

The above results may be compared, among others, with those reported in [10], where the classes of scalar-vacuum static, spherically symmetric solutions known as the Fisher (ordinary scalar field) and anti-Fisher (phantom scalar field) solutions, which exhibit some features similar to the structures studied here, have been analyzed using a similar method. This analysis has led to a conclusion on the instability of those scalar-vacuum solutions. It seems hard to make a statement on the generality of the stability issue of scalar-tensor black holes, but the results obtained until now may lead to some hints on this question. In particular, to our knowledge, there are only two examples of spherically symmetric black hole solutions with scalar fields which are stable under linear spherical perturbations. One such example is a black hole with a massless conformal scalar field [15, 16], whose stability was proved in [13]. The other is a “black universe” configuration with a phantom self-interacting scalar field [17, 18], which proved to be stable in the case where the black hole horizon coincided with the minimum of the spherical radius [11]. Both examples are exceptional, while in generic cases the solutions exhibit instabilities.

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