Notes on Inhomogeneous Quantum Walks

Yutaka Shikano and Hosho Katsura

1Department of Physics, Tokyo Institute of Technology, Meguro, Tokyo, 152-8551, Japan
2Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
3Department of Physics, Gakushuin University, Toshima, Tokyo 171-8588, Japan
4Kavli Institute for Theoretical Physics, University of California Santa Barbara, CA 93106, USA

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We study a class of discrete-time quantum walks with inhomogeneous coins defined in [Y. Shikano and H. Katsura, Phys. Rev. E 82, 031122 (2010)]. We establish symmetry properties of the spectrum of the evolution operator, which resembles the Hofstadter butterfly.

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Throughout this paper, we focus on a one-dimensional discrete time quantum walk (DTQW) with two-dimensional coins. The DTQW is defined as a quantum-mechanical analogue of the classical random walk. The Hilbert space of the system is a tensor product $\mathcal{H}_p \otimes \mathcal{H}_c$, where $\mathcal{H}_p$ is the position space of a quantum walker spanned by the complete orthonormal basis $|n\rangle$ ($n \in \mathbb{Z}$) and $\mathcal{H}_c$ is the coin Hilbert space spanned by the two orthonormal states $|L\rangle = (1, 0)^T$ and $|R\rangle = (0, 1)^T$. Here, the superscript $T$ denotes matrix transpose. A one-step dynamics is described by a unitary operator $U = WC$ with

$$C = \sum_n [(a_n|n, L\rangle + c_n|n, R\rangle)\langle n, L| + (d_n|n, R\rangle + b_n|n, L\rangle)\langle n, R|],$$

$$W = \sum_n (|n - 1, L\rangle\langle n, L| + |n + 1, R\rangle\langle n, R|),$$

where $|n, \xi\rangle = |n\rangle \otimes |\xi\rangle \in \mathcal{H}_p \otimes \mathcal{H}_c$ ($\xi = L, R$) and the coefficients at each position satisfy the following relations: $|a_n|^2 + |c_n|^2 = 1$, $a_n\overline{a}_n + c_n\overline{c}_n = 0$, $c_n = -\Delta_n\overline{a}_n$, $d_n = \Delta_n\overline{c}_n$, where $\Delta_n = a_nb_n - b_nc_n$ with $|\Delta_n| = 1$. Two operators $C$ and $W$ are called coin and shift operators, respectively. The probability distribution at the position $n$ at the $t$th step is then defined by

$$\text{Pr}(n; t) = \sum_{\xi \in \{L, R\}} |\langle n, \xi|U^t|0, \phi\rangle|^2.$$  

A homogeneous version of this DTQW was first introduced in Ref. [1]. Suppose that the coin operator is given by

$$C(\alpha, \theta) = \sum_n [(\cos(2\pi\alpha + 2\pi\theta)|n, L\rangle + \sin(2\pi\alpha + 2\pi\theta)|n, R\rangle)\langle n, L|$$

$$+ (\cos(2\pi\alpha + 2\pi\theta)|n, R\rangle - \sin(2\pi\alpha + 2\pi\theta)|n, L\rangle)\langle n, R|]$$

$$:= \sum_n |n\rangle\langle n| \otimes \tilde{C}_n(\alpha, \theta),$$

where $\alpha$ and $\theta$ are constant real numbers. Then this class of DTQW is called an inhomogeneous quantum walk (QW). This model is based on the idea of the Aubry-André model [2], which provides a solvable example of metal-insulator transition in a one-dimensional incommensurate system. In this class of DTQW, we have obtained the weak limit theorem as follows.

**Theorem 1** (Shikano and Katsura [3]). Fix $\theta = 0$. For any irrational $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and any special rational $\alpha = \frac{P}{Q \eta} \in \mathbb{Q}$ with relatively prime $P$ (odd integer) and $Q$, the limit distribution of the inhomogeneous QW is given by

$$\lim_{t \to \infty} \frac{X_t}{\sqrt{t}} \Rightarrow I,$$  

where $I$ is the identity operator.
FIG. 1: Probability distribution of the inhomogeneous QW at 300th step with $\alpha = 1/3$. From simple algebra, it can be easily shown that the inhomogeneous QW is finitely confined when $\theta = (2m - 1)/12$ ($m \in \mathbb{Z}$).

where $X_t$ is the random variable for the position at the $t$ step, “$\Rightarrow$” means the weak convergence, and $\eta > 0$ is an arbitrary positive parameter. Here, the limit distribution $I$ has the probability density function $f(x) = \delta(x)$ ($x \in \mathbb{R}$), where $\delta(\cdot)$ is the Dirac delta function. This is called a localization for the inhomogeneous QW.

However, in the case of the other rational $\alpha$, it has not yet been clarified whether the inhomogeneous QW is localized or not. This is still an open question. The situation becomes more complicated when we consider a nonzero $\theta$. As seen in Figure 1, the reflection points for the quantum walker (see more details in Ref. [3, Lemma 1 and Figure 2]) are changed by the parameter $\theta$. In the rest of the paper, we will establish symmetry properties of the eigenvalue distribution of the one-step evolution operator ($U = WC$) at $\theta = 0$.

**Theorem 2.** For the eigenvalues of the one-step evolution operator $WC$, the following properties hold:

(P1) All the eigenvalues at $\alpha$ are identical to those at $1 - \alpha$.

(P2) For every eigenvalue $\lambda$, there is an eigenvalue $\lambda^*$.

(P3) For every eigenvalue $\lambda$, there is an eigenvalue $-\lambda$.

(P4) All the eigenvalues are simple, i.e., nondegenerate.

(P5) There are four eigenvalues $\lambda = \pm 1, \pm i$ for any $\alpha = \frac{P}{4Q} \in \mathbb{Q}$.

(P6) Every eigenvalue $\lambda$ at $\alpha = \frac{P}{4Q} \in \mathbb{Q}$ corresponds to an eigenvalue $i\lambda$ at $\alpha + 1/2$.

**Proof.** The proofs of properties (P1) – (P5) can be found in Ref. [3]. Here, we give a proof of (P6). According to Ref. [3, Theorem 3], the eigenvalues of $WC$ and $WC$ are identical. Therefore, we only study the eigenvalues of $CW$. First, we can express the wavefunction at the $t$th step evolving from the state $|0, \tilde{\phi}\rangle$ by $CW$:

$$
(CW)^t|0, \tilde{\phi}\rangle := \sum_{n \in \mathbb{Z}, \xi \in \{L,R\}} \varphi_t(n, \xi)|n, \xi\rangle.
$$

The one-step time evolution of the coefficients $\varphi_t(n, \xi)$ is given by

$$
\begin{pmatrix}
\varphi_{t+1}(n; L) \\
\varphi_{t+1}(n; R)
\end{pmatrix} = C_n(\alpha, 0)
\begin{pmatrix}
\varphi_t(n + 1; L) \\
\varphi_t(n - 1; R)
\end{pmatrix}.
$$

Here, we define $\tilde{\varphi}_t$ by $\varphi_t(n, \xi)$ and a square matrix of order $4Q$, denoted as $CW$, as

$$
\tilde{\varphi}_{t+1} = CW\tilde{\varphi}_t,
$$
FIG. 2: Eigenvalue distribution of the one-step operator for the inhomogeneous QW (U). Arguments of the eigenvalues of WC (vertical axis) are plotted as a function of the parameter $\alpha = \frac{P}{Q}$ (horizontal axis) with $Q \leq 60$. Here, $P$ (odd number) and $Q$ are relatively prime.

see more details in Ref. [2]. Let $\vec{\varphi} = (\varphi(-Q; R), \varphi(-Q + 1; L), \varphi(-Q + 1; R), ..., \varphi(Q; L))^T$ be the eigenvector of $CW$ at $\alpha$ with the eigenvalue $\lambda$ and $\vec{\tilde{\varphi}} = (\tilde{\varphi}(-Q; R), \tilde{\varphi}(-Q + 1; L), \tilde{\varphi}(-Q + 1; R), ..., \tilde{\varphi}(Q; L))^T$ be at $\alpha + 1/2$ with the eigenvalue $\tilde{\lambda}$. Then, according to Eq. (7), we obtain

$$\begin{align*}
\lambda \varphi(-Q; R) &= (-1)^{\frac{P+1}{2}} \varphi(-Q + 1; L), \\
\lambda \begin{pmatrix} \varphi(n; L) \\ \varphi(n; R) \end{pmatrix} &= \hat{C}_n(\alpha, 0) \begin{pmatrix} \varphi(n + 1; L) \\ \varphi(n - 1; R) \end{pmatrix}, (n \in (-Q, Q)) \\
\lambda \varphi(Q; L) &= (-1)^{\frac{P+1}{2}} \varphi(Q - 1; R)
\end{align*}$$

and

$$\begin{align*}
\tilde{\lambda} \tilde{\varphi}(-Q; R) &= (-1)^{-Q}(-1)^{\frac{P+1}{2}} \tilde{\varphi}(-Q + 1; L), \\
\tilde{\lambda} \begin{pmatrix} \tilde{\varphi}(n; L) \\ \tilde{\varphi}(n; R) \end{pmatrix} &= (-1)^n \hat{C}_n(\alpha, 0) \begin{pmatrix} \tilde{\varphi}(n + 1; L) \\ \tilde{\varphi}(n - 1; R) \end{pmatrix}, (n \in (-Q, Q)) \\
\tilde{\lambda} \tilde{\varphi}(Q; L) &= (-1)^Q(-1)^{\frac{P+1}{2}} \tilde{\varphi}(Q - 1; R),
\end{align*}$$

where we have used the fact $\hat{C}_n(\alpha + 1/2) = (-1)^n \hat{C}_n(\alpha, 0)$. Now we apply the following local unitary transformation to Eq. (10):

$$\tilde{\varphi}(n; \xi) = \begin{cases} 
\varphi'(n; \xi) & \text{when } n \text{ is even,} \\
\i \varphi'(n; \xi) & \text{when } n \text{ is odd.}
\end{cases}$$

According to Eq. (9), $\vec{\tilde{\varphi}}$ defined by Eq. (11) can be taken as the eigenvector $CW$ at $\alpha$ with the eigenvalue $\tilde{\lambda} = i\lambda$.

Figure 2 shows the numerically obtained spectrum of $CW$ as a function of $\alpha$, which is quite similar to the Hofstadter butterfly [1]. By combining all the properties of (P1)-(P6), the smallest fundamental domain of this diagram is identified as the triangular region shown in Figure 2. Therefore, we have rigorously established all the symmetries in Figure 2.

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