Modified Chaplygin Gas with Variable $G$ and $\Lambda$

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In this work, we have considered modified Chaplygin gas with variable $G$ and $\Lambda$. The trivial solution describes decelerating phase to accelerating phase of the universe. The non-static with constant equation of state describes the inflationary solution. For static universe, $G$ and $\Lambda$ must be forms arbitrary and for static universe with constant equation of state $G$ and $\Lambda$ should be constant.

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I. INTRODUCTION

The Einstein field equation has two parameters - the gravitational constant $G$ and the cosmological constant $\Lambda$. The Newtonian constant of gravitation $G$ plays the role of a coupling constant between geometry and matter in the Einstein field equations. In an evolving Universe, it appears natural to look at this “constant” as a function of time. Numerous suggestions based on different arguments have been proposed in the past few decades in which $G$ varies with time [1]. Dirac [2] proposed a theory with variable $G$ motivated by the occurrence of large numbers discovered by Weyl, Eddington and Dirac himself. Many other extensions of Einstein’s theory with time dependent $G$ have also been proposed in order to achieve a possible unification of gravitation and elementary particle physics or to incorporate Mach’s principle in general relativity [3].

From the point of view of incorporating particle physics into Einstein’s theory of gravitation, the simplest approach is to interpret the cosmological constant $\Lambda$ in terms of quantum mechanics and the Physics of vacuum [4]. The $\Lambda$ term has also been interpreted in terms of the Higgs scalar field [5]. Linde [6] proposed that $\Lambda$ is a function of temperature and related it to the process of broken symmetries. Gaspirini [7] in this regard argues that $\Lambda$ can also be interpreted as a measure of temperature of a vacuum which should decrease like the radiation temperature with cosmic acceleration. By considering the conservation of the energy-momentum tensor of matter and vacuum take together, many authors have invoked the idea of a decreasing vacuum energy and hence a varying cosmological constant $\Lambda$ with cosmic expansion in the frame work of Einstein’s theory.

$\Lambda$ as a function of time has also been considered by several authors in various variable $G$ theories in different contexts [8]. Investigating the distance dependence of gravity under very general conditions, Wilkins [9] found that the gravity field at a distance $r$ from a point mass has two components: one varying as $r^{-2}$, the other as $r$ (Hookian field). The latter component is identifiable with the weak field limit of the $\Lambda$ term in Einstein’s equation. His analysis allows one to consider both the gravity fields - the Hookian field, coupled to $\Lambda$ and the Newtonian one coupled to $G$ - on an equal footing. With this in view, several authors [10, 11] have proposed linking the variation of $G$ with that of $\Lambda$ in the framework of general relativity. This approach preserves conservation of the energy-momentum tensor of matter and leaves the form of the Einstein field equations unchanged. Though this approach is non-covariant, it is worth studying because it may be a limit of some higher dimensional fully covariant theory [10, 12].

Recent observations of the luminosity of type Ia Supernovae indicate [13, 14] an accelerated expansion of the Universe and lead to the search for a new type of matter which violates the strong energy condition i.e., $\rho + 3p < 0$. The matter consent responsible for such a condition to be satisfied at a certain stage of evolution of the universe is referred to as a dark energy. There are different candidates to play the role of the dark energy. The type of dark energy represented by a scalar field is often called Quintessence. The simplest candidate for dark energy is Cosmological Constant $\Lambda$. In particular one can try another type of dark energy, the so-called Chaplygin gas which obeys an equation of state like [15] $p = -B/\rho$, $(B > 0)$, where $p$ and $\rho$ are respectively the pressure and energy density. Subsequently the above equation was generalized to the form [16] $p = -B/\rho^n$, $0 \leq n \leq 1$. There are some works on modified Chaplygin Gas obeying equation of state [17, 18].

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\[ p = A\rho - B/\rho^n, \quad (A > 0) \]  
\text{(1)}

At all stages it shows a mixture. This is described from radiation era to ΛCDM model.

**II. EINSTEIN’S FIELD EQUATIONS**

We consider the homogeneous and isotropic space-time given by FRW metric

\[ ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \]  
\text{(2)}

where \( k = 0, \pm 1 \) is the curvature parameter.

The energy-momentum tensor for perfect fluid is

\[ T_{ij} = (\rho + p)u_i u_j + p g_{ij} \]  
\text{(3)}

where \( \rho \) and \( p \) are energy density and isotropic pressure respectively and \( c = 1 \).

The Einstein field equations with variable \( G \) and \( \Lambda \) is given by

\[ R_{ij} - \frac{1}{2} R g_{ij} - \Lambda(t) g_{ij} = -8\pi G(t) T_{ij} \]  
\text{(4)}

i.e., we have two equations as

\[ \frac{\dot{a}^2}{a^2} = \frac{8\pi G(t)\rho}{3} + \frac{\Lambda(t)}{3} - \frac{k}{a^2} \]  
\text{(5)}

and

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G(t)}{3}(\rho + 3p) + \frac{\Lambda(t)}{3} \]  
\text{(6)}

From (5) and (6) we have

\[ \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) + \rho \frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0 \]  
\text{(7)}

We assume the law of conservation of energy \( (T_{ij}^n = 0) \) giving

\[ \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \]  
\text{(8)}

From (7) and (8) we have

\[ \dot{\Lambda} = -8\pi \dot{G}\rho \]  
\text{(9)}

This implies \( \dot{G} > 0 \) or \( < 0 \) according as \( \dot{\Lambda} < 0 \) or \( > 0 \) i.e., \( G \) increases or decreases according to whether \( \Lambda \) decreases or increases.

Now for modified Chaplygin gas (1) we obtain
\[
\rho = \left[ \frac{B}{1 + A} + \frac{C}{a^{3(1+A)(1+n)}} \right]^{\frac{1}{1+n}}
\]

(10)

Now from equation of state (1), field equation (5) and conservation equation (8) we have

\[
\frac{\dot{\rho}^2}{\rho^3} = 9 \left( 1 + A - \frac{B}{\rho^n+1} \right)^2 \left( \frac{8\pi G}{3} + \frac{\Lambda - \frac{k}{a^2}}{3} \rho \right)
\]

(11)

Differentiating w.r.t \( t \), we obtain

\[
\dot{\rho} \left( \frac{2\ddot{\rho}}{\rho} - \frac{3\dot{\rho}^2}{\rho^2} \right) - \frac{3B(n+1)\rho^3}{\rho^{n+3}} \left( 1 + A - \frac{B}{\rho^n+1} \right) = 3\dot{\rho} \left[ 1 + A - \frac{B}{\rho^n+1} \right] \left[ 3 \left( 1 + A - \frac{B}{\rho^n+1} \right) \left( \frac{k}{a^2} - \frac{\Lambda}{3} \right) - \frac{2k}{a^2} \right]
\]

(12)

Now we consider \( \dot{\rho} \neq 0 \) (the case \( \dot{\rho} = 0 \) i.e., \( \rho = \) constant is discussed later). Since the above differential equation of \( \rho \) is highly non-linear, so it cannot be solve analytically. Now for simplicity of calculation, we choose

\[
3 \left( 1 + A - \frac{B}{\rho^n+1} \right) \left[ 3 \left( 1 + A - \frac{B}{\rho^n+1} \right) \left( \frac{k}{a^2} - \frac{\Lambda}{3} \right) - \frac{2k}{a^2} \right] = 0
\]

(13)

and

\[
\left( \frac{2\ddot{\rho}}{\rho} - \frac{3\dot{\rho}^2}{\rho^2} \right) - \frac{3B(n+1)\rho^2}{\rho^{n+3}} \left( 1 + A - \frac{B}{\rho^n+1} \right) = 0
\]

(14)

From equation (13) we find the trivial solution of \( \Lambda \) i.e.,

\[
\Lambda = \frac{(1 + 3A)k}{(1 + A)a^2} - \frac{2kB}{2C(1 + A)^2} \frac{a^{3(1+A)(1+n)-2}}{a^{3(1+A)(1+n)-1}}
\]

(15)

Also from equation (14) we obtain (after manipulation)

\[
a^{\frac{3(1+A)}{2(1+n)}} \cdot 2F_1 \left[ \frac{1}{2(1 + A)}, \frac{1}{2(1 + n)}, 1 + \frac{1}{2(1 + n)} - \frac{Bk^{3(1+A)(1+n)}}{C(1 + A)} \right] = \frac{\rho_0}{2} C^{\frac{1}{1+n}}, t
\]

(16)

For \( k > 0 \), \( \Lambda \) decreases with \( t \) upto certain stage of the evolution of the universe and for \( k < 0 \), \( \Lambda \) increases with \( t \) after certain stage of the evolution of the universe (see figure 1) and for \( k = 0 \), we must have \( \Lambda = 0 \).

From (9) and (15) we have

\[
4\pi G = \frac{\frac{(1 + 3A)k}{(1 + A)a^2} + \frac{kB(3(1+A)(1+n)-2)}{C(1+A)^2}}{\left[ \frac{\rho}{1 + A} + \frac{C}{a^{3(1+A)(1+n)}} \right]^{\frac{1}{1+n}}}
\]

(17)

Note that here \( G_0 \) depends on the value of \( k \) i.e., \( k = 0 \) implies \( G_0 = 0 \). From figure 2 we see that \( G \) increases with the evolution of the universe for \( k = 1 \).

For early universe i.e., for \( a \approx 0 \), we get

\[
4\pi G \approx \frac{(1 + 3A)k}{(1 + A)C^{\frac{1}{1+n}}} + 4\pi G_0, \quad (G_0 > 0)
\]

(18)
Fig. 1, 2 and 3 show the variations of $\Lambda$, $G$ and $q$ against $t$ for $A = 1/3$, $B = 1$, $C = 1$, $n = 1/2$.

i.e., $G \to G_0$ as $a \to 0$.

Also for late universe i.e., for $a \approx \infty$, we have

$$4\pi G \approx \frac{kB}{C(1 + A)^2} \left( \frac{1 + A}{B} \right)^{1 + n} q^{3(1 + A)(1 + n) - 2} + 4\pi G_0$$

(19)

i.e., $G \to \infty$ as $a \to \infty$.

The deceleration parameter has the expression

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{4\pi G(\rho + 3p) - \Lambda}{8\pi G\rho + \Lambda - 3ka^{-2}}$$

(20)

From figure 3 we see that $q$ decreases from $+1$ to $-1$ for $A = 1/3$, $k = \pm 1$. This implies the universe has early deceleration and late acceleration. For $k = 0$, we have $\Lambda = 0$ and $G = 0$. So the field equations (5) and (6) yield $a = \text{constant}$, which is static solution.

The above discussions are valid for $\dot{\rho} \neq 0$ i.e., $\rho \neq \text{constant}$. Now we will discuss in the case $\dot{\rho} = 0$ i.e., $\rho = \text{constant} = \rho_0$ (say).
Now equation (8) reduces to

\[(\rho + p) \frac{\dot{a}}{a} = 0 \tag{21}\]

So we have possibilities: \(\rho + p = 0\) or \(\dot{a} = 0\).

(i) \(\dot{a} \neq 0\) and \(\rho + p = 0\): This implies \(p = -\rho = -\rho_0 = p_0\) (say). From equation (1) we have \(\rho_0 = \left(\frac{B}{1+A}\right)^{\frac{1}{n+1}}\). So from field equations (5) and (6), we get

\[a = \sqrt{\frac{k}{C_1}} \cosh(\sqrt{C_1} t)\]

From equation (9) we have

\[\Lambda = \Lambda_0 - 8\pi G \left(\frac{B}{1+A}\right)^{\frac{1}{n+1}}, \quad (\Lambda_0 = 3C_1) \tag{22}\]

There are no other equations, so \(\Lambda\) and \(G\) can not be calculated. We also see that in this case, \(G\) increases or decreases according as \(\Lambda\) decreases or increases. \(\Lambda\) and \(G\) are arbitrary functions of time in this case.

The field equations (5) and (6) and equation (22), we have

\[t = \sqrt{\frac{3}{\Lambda_0}} \log \left(\sqrt{\frac{\Lambda_0}{\Lambda_0 a_0 + \sqrt{\Lambda_0 a_0^2 - 3k}}} \right)\]

i.e.,

\[a = a_0 \cosh(\sqrt{\frac{\Lambda_0}{3}} t) + \sqrt{a_0^2 - 3k} \Lambda_0 \sinh(\sqrt{\frac{\Lambda_0}{3}} t) \tag{23}\]

For \(k = 0\) we have \(a = a_0 e^{\sqrt{\frac{\Lambda_0}{3}} t}\) which is the inflationary solution.

(ii) \(\dot{a} = 0\) and \(\rho + p \neq 0\): This implies \(a = \text{constant} = a_0\) i.e., we have static universe. In this case, from field equations we have the values of \(G\) and \(\Lambda\) as

\[G = \frac{k}{4\pi a_0^2 \left(1 + A\right)\rho_0 - \frac{B}{\rho_0}} = \text{constant} \tag{24}\]

and

\[\Lambda = \frac{k}{a_0^2} \left[\frac{(1 + 3A)\rho_0^{n+1} - 3B}{(1 + A)\rho_0^{n+1} - B} \right] = \text{constant} \tag{25}\]

where \(\rho_0 \neq \left(\frac{B}{1+A}\right)^{\frac{1}{n+1}}\).

(iii) \(\dot{a} = 0\) and \(\rho + p = 0\): This implies \(a = \text{constant} = a_0\) i.e., we also have static universe and \(p = -\rho = -\rho_0 = p_0\) (say). From equation (1) we have \(\rho_0 = \left(\frac{B}{1+A}\right)^{\frac{1}{n+1}}\). From field equations, \(G\) and \(\Lambda\) satisfies:

\[\Lambda + 8\pi G \rho_0 = \frac{3k}{a_0^2} \tag{26}\]
\[ \Lambda + 8\pi G\rho_0 = 0 \]  
which are consistent only for \( k = 0 \). This implies

\[ \Lambda = -8\pi G\rho_0 = 8\pi G \left( \frac{B}{1 + A} \right)^{\frac{1}{n+1}} \]  

The above relation (28) shows that \( G \) and \( \Lambda \) are arbitrary functions of time \( t \).

### III. CONCLUDING REMARKS

From the field equations and the conservation equation, an equation is obtained in \( \rho \), \( a \) and \( \Lambda \) which suggests two trivial solutions. The trivial case with \( \Lambda \) is given in equation (15) leads to a model which starts from big bang with non-zero gravitational constant \( G_0 \) \( (k \neq 0) \) and has a positive constant deceleration parameter \( q = 1 \) and leads to accelerating universe \( (q = -1) \) with infinite gravitational constant \( G \) (see figure 3). For \( k = 0 \) we have \( \Lambda = 0 \) and \( G = 0 \) and \( a = constant \), which is static solution. The model describes the evolution of the universe with early deceleration and late acceleration. For constant density \( \rho_0 \) \( \neq \left( \frac{B}{1 + A} \right)^{\frac{1}{n+1}} \) universe, the cosmological constant \( \Lambda \) and gravitational constant \( G \) are arbitrary functions of time in which for static universe describes \( k = 0 \). But for static universe with non-constant density, \( G \) and \( \Lambda \) are constant where constant density \( \rho_0 \) \( \neq \left( \frac{B}{1 + A} \right)^{\frac{1}{n+1}} \).

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