Center of Mass Momentum Dependent Interaction Between Ultracold Atoms

Jianwen Jie\textsuperscript{1} and Peng Zhang\textsuperscript{1,2,3, \textsuperscript{*}}

\textit{1Department of Physics, Renmin University of China, Beijing, 100872, China}
\textit{2Beijing Computational Science Research Center, Beijing, 100084, China}
\textit{3Beijing Key Laboratory of Opto-electronic Functional Materials & Micro-nano Devices (Renmin University of China)}

We show that a new type of two-body interaction, which depends on the momentum of the center of mass (CoM) of these two particles, can be realized in ultracold atom gases. In some other systems, \textit{e.g.} ultracold atom gases with interatomic scattering length being modulated by an inhomogeneous magnetic or laser field \cite{1}, the two-body interaction can be dependent on the CoM position. In this letter we show that a new type of interaction, which depends on the two-body CoM momentum, can be realized in ultracold atom gases, while the one-body free Hamiltonian remains the simple kinetic energy. Explicitly, we propose an approach to realizing an ultracold atom gas where the Hamiltonian of every two atoms can be formally expressed as

\begin{equation}
H_{2b} = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V_{\text{eff}}(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{p}_1 + \mathbf{p}_2), \quad (1)
\end{equation}

with $m_i$, $\mathbf{p}_i$ and $\mathbf{r}_i$ ($i = 1, 2$) being the mass, momentum and position of the \textit{i}th atom, and $V_{\text{eff}}$ being the CoM-momentum dependent effective two-atom interaction. To our knowledge, this type of interaction, which couples the two-body CoM motion and the relative motion without breaking the translational symmetry and changing the one-body dispersion relation, has not been discovered in any quantum system.

Our proposal is based on the magnetic Feshbach resonance (MFR) modulated by two Raman laser beams propagating along different directions, which couple the two-body bound states in the closed channel and are far off resonant for one-body transitions. When the interatomic interaction of an ultracold gas is controlled by such an approach, the two-atom scattering length is determined by the frequencies of these two laser beams. Furthermore, due to the Doppler effect, when the atoms are moving the frequency of the laser beams can be effectively shifted, and the magnitude of the frequency shift depends on the two-atom CoM momentum. As a result, in this system the two-atom scattering length, which describes the effective two-atom interaction, becomes CoM-momentum dependent. In addition, the two-body collisional loss induced by spontaneous emission from excited state atoms can be significantly suppressed by the molecular dark state effect \cite{2}.

In all the previous research for the optical control of interaction between ultracold atoms \cite{1--21}, the Doppler effect has always been ignored. This can be explained as follows. In these control processes the laser beams should be far off resonant to the two-body transitions so that the collisional loss induced by atomic spontaneous emission can be suppressed. As a result, the Doppler shift of the laser frequency is much smaller than the detuning of the laser-induced two-body transitions, and thus the Doppler effect is negligible. However, in our system the two-atom scattering length depends on not only the one-photon detuning but also the two-photon detuning of the laser-induced two-body Raman transition. Since the two-photon detuning can be very small, it can be significantly changed by the Doppler frequency shift of the Raman laser beams. Therefore, the Doppler effect can be very
Three-dimensional (3D) s-wave scattering length. We consider the s-wave scattering of two ultracold alkali atoms in the ground electronic orbital state (i.e., the S-state). As shown in Fig. 1, we assume these two atoms are incident from the open channel O corresponding to one specific two-atom hyperfine state. The threshold of this channel is near resonant to a bound state $|\phi_\alpha\rangle$ in the closed channel C, which corresponds to another hyperfine state of these two S-state atoms. The energy difference between $|\phi_\alpha\rangle$ and the threshold of channel O can be controlled by the magnetic field. Thus, a MFR [22] can be induced by the hyperfine coupling $V_{hf}$ between the open channel O and $|\phi_\alpha\rangle$. We further assume that a laser beam $\alpha$ with wave vector $k_\alpha$ is applied to couple $|\phi_\alpha\rangle$ to another two-body bound state $|\phi_c\rangle$ in an excited channel F where one atom is in the S-state and another atom is in the excited electronic orbital state (i.e., the P-state). The excited molecular state $|\phi_c\rangle$ can decay via the atomic spontaneous emission process. In addition, $|\phi_c\rangle$ is also coupled to a bound state $|\phi_\beta\rangle$ in the closed channel C, by another laser beam $\beta$ with wave vector $k_\beta$. As mentioned above, we assume the laser beams $\alpha$ and $\beta$ are far off resonant for all the one-atom transition processes. As a result, they only induce two-body transitions and do not change the one-atom Hamiltonian.

In our system the scattering length $a$ can be controlled by both the magnetic field and the laser beams $\alpha$ and $\beta$. As we will show below, when these two beams propagate along the same direction, the Doppler effect is negligible and the scattering length is still independent of the two-atom CoM momentum. The control of the interatomic interaction for this case was proposed by H. Wu and J. Thomas in Refs. [2, 3] in 2011, where the wave vectors $k_{\alpha,\beta}$ were reasonably ignored. It was experimentally realized by A. Jagannathan et al. in 2016 [4].

In this letter we consider the case where the two laser beams are propagating along different directions, i.e. $k_{\alpha} \neq k_{\beta}$. We will show that in this case the Doppler effect can be very important. As a result, the scattering length becomes significantly dependent on the two-atom CoM momentum.

Now we calculate the scattering length $a$. Here we first ignore the spontaneous decay of the excited molecular state $|\phi_c\rangle$ and illustrate the approach of our calculation. Then we will take this decay into account and derive the explicit expression for $a$. In the absence of the spontaneous decay, in the Schrödinger picture the two-body Hamiltonian can be written as ($\hbar = 1$)

$$H_S = \frac{P^2}{2M} + H_{\text{MFR}} + E_\beta|\phi_\beta\rangle \langle \phi_\beta| + E_c|\phi_c\rangle \langle \phi_c|,$$

$$+ \sum_{l=\alpha,\beta} \Omega_l e^{i(k_l \cdot R - \omega_l t)}|\phi_c\rangle \langle \phi_l| + h.c.,$$ (2)

with $H_{\text{MFR}}$ being defined as

$$H_{\text{MFR}} = \left[ \frac{P^2}{2\mu} + V_{bg}(r) \right] |\phi_\alpha\rangle \langle \phi_\alpha| + E_\alpha|\phi_\alpha\rangle \langle \phi_\alpha|,$$

$$+ V_{hf}(r)|\phi_\alpha\rangle \langle \phi_\alpha| + h.c..$$ (3)

Here $|j\rangle_1 (j = O, C, F)$ denote the two-body internal state corresponding to channel $j$ [23], $M$ is the total mass and $\mu$ is the two-atom reduced mass, $E_l$ ($l = \alpha, \beta$) is the energy of the bound state $|\phi_l\rangle$, $\omega_l$ ($l = \alpha, \beta$) is the frequency of laser beam $l$, and $\Omega_l$ ($l = \alpha, \beta$) is the strength of the laser-induced coupling between $|\phi_l\rangle$ and $|\phi_c\rangle$. We have chosen the zero-energy point as the threshold of the open channel O. In Eqs. (2, 3) $R$ and $P$ are the coordinate and momentum of the CoM, while $r$ and $p$ are those of the two-atom relative motion. The interaction in the open channel O is $V_{bg}(r)$, and the hyperfine coupling between channels O and C is described by $V_{hf}(r)$.

We can simplify our problem and remove the phase factor $e^{i(k_l \cdot R - \omega_l t)}$ ($l = \alpha, \beta$) by introducing a rotated frame (interaction picture) where quantum state $|\Psi\rangle_{\text{rot}}$ is related to the state $|\Psi\rangle_S$ in the Schrödinger picture via the relation $|\Psi\rangle_{\text{rot}} = U(t)|\Psi\rangle_S$. Here the unitary transformation $U$ is given by

$$U = e^{i(\omega_\alpha t - k_\alpha \cdot R)}|\phi_\alpha\rangle \langle \phi_\alpha| e^{i((\omega_\alpha - \omega_\beta)t - (k_\alpha - k_\beta) \cdot R)}|\phi_\beta\rangle \langle \phi_\beta|.$$

(4)

In this rotated frame, the two-body Hamiltonian becomes

$$H_{\text{rot}} = \frac{P^2}{2M} + H_{\text{MFR}} + \sum_{l=\alpha,\beta} \Omega_l |\phi_c\rangle \langle \phi_l| + h.c.$$

$$+ \Delta_{1p}(P)|\phi_c\rangle \langle \phi_c| + \Delta_{2p}(P)|\phi_\beta\rangle \langle \phi_\beta|,$$ (5)

where

$$\Delta_{1p}(P) = \frac{\Delta_{1p}^{(0)}}{1p} + \frac{|k_\alpha|^2}{2M} + \frac{k_\alpha \cdot P}{M};$$

$$\Delta_{2p}(P) = \frac{\Delta_{2p}^{(0)}}{2p} + \frac{(|k_\alpha - k_\beta|^2)}{2M} + \frac{(k_\alpha - k_\beta) \cdot P}{M},$$ (6)

with $\Delta_{1p}^{(0)} = E_c - \omega_\alpha$ and $\Delta_{2p}^{(0)} = E_\beta - (\omega_\alpha - \omega_\beta)$. The physical meaning of $\Delta_{1p}(P)$ and $\Delta_{2p}(P)$ can be understood as follows. Scattering length $a$ is determined by the multi-order transition process from the open channel O to the bound state $|\phi_\alpha\rangle$ (induced by hyperfine coupling) and then to the excited molecular state $|\phi_c\rangle$ (induced by laser $\alpha$), and finally to the bound state $|\phi_\beta\rangle$ (induced by laser $\beta$). This is essentially a Raman process induced by the laser beams $\alpha$ and $\beta$. $\Delta_{1p}(P)$ is given by Eq. (6) is the one-photon detuning of this Raman process (i.e., the detuning of the transition $O \rightarrow |\phi_\alpha\rangle \rightarrow |\phi_c\rangle$). Similarly, $\Delta_{2p}(P)$ is the two-photon detuning of the complete Raman process from O to $|\phi_\beta\rangle$. Moreover, in Eqs. (6) and (7) $\Delta_{1p}^{(0)}$ and $\Delta_{2p}^{(0)}$ can be understood as the bare value of these detuning, $|k_\alpha|^2/(2M)$ and $|k_\alpha - k_\beta|^2/(2M)$ are the shifts induced by the momentum-recoil effects, and the $P$-dependent terms are the Doppler shifts.
Explicitly, in the absence of the Raman laser beams we have \( a = a_{bg} [1 - \Delta_B / (B - B_0)] \), \( \delta \mu \) is the magnetic moment difference between the channels \( O \) and \( C \), and \( \gamma \) is the spontaneous decay rate of \( |\phi_+\rangle \). The energy of two-body bound state can also be calculated via the same approach.

It is clear that the scattering length \( a(P) \) depends on the CoM momentum \( P \) via the one-photon and two-photon detuning \( \Delta_{1p}(P) \) and \( \Delta_{2p}(P) \). In a realistic system, to suppress the collisional loss induced by the spontaneous decay of \( |\phi_+\rangle \) one usually sets the bare value \( \Delta_{1p}^{(0)} \) of the one-photon detuning to be very large. As a result, the \( P \)-dependence of \( \Delta_{1p}(P) \) is usually negligible. However, the bare two-photon detuning \( \Delta_{2p}^{(0)} \) can be very small, and thus the variation of \( \Delta_{2p}(P) \) with \( P \) can be very significant. Therefore, \( a(P) \) can be strongly dependent on \( P \) via \( \Delta_{2p}(P) \).

Furthermore, according to Eq. (7), when the two Raman beams are propagating along the same direction (i.e., \( k_a \approx k_b \)), \( \Delta_{2p}(P) \) takes a \( P \)-independent value \( \Delta_{2p}^{(0)} \) and thus the Doppler effect can be ignored. For this case it was shown that the two-body loss can be suppressed by the molecular dark state effect, provided that \( \Delta_{2p}^{(0)} \) is small enough [2]. Since we can re-obtain the scattering length for this case [2] by replacing \( \Delta_{2p}(P) \) in Eq. (8) with \( \Delta_{2p}^{(0)} \), for our system the two-body loss can also be suppressed when \( \Delta_{2p}(P) \) is small enough. This suppression can also be understood from the fact \( \lim_{\Delta_{2p}(P) \to 0} \text{Im}[a(P)] \propto O(\Delta_{2p}(P)^2) \).

3D ultracold Fermi gas. As an example, we consider a 3D ultracold gas of two-component 40K atoms in the lowest two hyperfine states \( |\uparrow\rangle \equiv |F = 9/2, m_F = -9/2\rangle \) and \( |\downarrow\rangle \equiv |F = 9/2, m_F = -7/2\rangle \). Here we focus on the MFR with \( B_0 = 202.2 \text{ G}, \Delta_p = 8 \text{ G} \), \( B_0 = 202.2 \text{ G} \), \( \Delta_p = 8 \text{ G} \), \( a_{bg} = 174 \text{ a}_0 \), \( a_{bg} = 174 \text{ a}_0 \), with \( a_0 \) being the Bohr radius, and assume that this MFR modulated by two Raman beams as discussed above. We take these two beams to be counter-propagating along the \( x \)-axis. Thus, the scattering length \( a \) only depends on the \( x \)-component \( P_x \) of \( P \). In the ultracold Fermi gas \( P_x \) is mainly in the region between \( -2k_F \) and \( 2k_F \), with \( k_F \) being the Fermi momentum. We consider an ultracold gas with Fermi temperature \( T_F = 0.5 \mu \text{K} \) (corresponding to \( k_F = 9.1 \times 10^3 \text{ m}^{-1} \)).

In Fig. 2 (a, b) we illustrate the real part of \( \text{Re}[a(P)] = 0.07 \Delta_B \). It is shown that in these cases \( \text{Re}[\alpha] \) is always positive or always negative for \( P_x \in [-2k_F, 2k_F] \), and can change by about 2500 \( a_0 \) with \( P_x \), ranging from the region with \( 1/(k_F |\text{Re}[\alpha]|) \) < 1 to the region with \( 1/(k_F |\text{Re}[\alpha]|) > 1 \). Direct calculations show that these results are robust with respect to the uncertainties of \( B_0 \) and \( \Delta_B \). For the case of Fig. 2 (a) we further calculate the energy \( E_0 \) of the most shallow two-body bound state, as function of the CoM momentum \( P_0 \).

In Fig. 2 (c) we show the total dimer energy \( E_{\text{dim}}(P_x) \equiv \frac{P_x^2}{2M} + \text{Re}[E_b(P_x)] \) as a function of \( P_x \), i.e.
the dispersion relation of the shallow dimer. It is clear
that if the $E_b$ were independent of $P_x$, the minimum point
of $E_{\text{dim}}$ appears at $P_x = 0$. As shown in Fig. 2(c), due
to the $P_x$-dependence of $E_b$, in our system $E(P_x)$ takes
its minimum value when $P_x = -0.66k_F$.

Now we investigate the two-atom collisional loss. If
the scattering length $a$ was independent of the CoM
momentum, the two-body collisional loss rate is $K_2 \equiv
-8\pi\hbar\text{Im}[a(P)]/m$. As shown in Fig. 2(d), $K_2(P_x)$ is below $2 \times 10^{-13}\text{cm}^3/\text{s}$ for the systems studied in Fig. 2 (a, b). Using the atom density
$n_0 = 1.28 \times 10^{13}/\text{cm}^3$ corresponding to $T_F = 0.5\mu\text{K}$,
we can obtain $1/[K_2(P_x)n_0] > 0.04a$. In addition, in
Fig. 2(d) we illustrate the lifetime $\tau_0 \equiv 1/(|\text{Im}(E_0(P_x))|)$
of the shallow dimer. It is shown that $\tau_0$ is about 0.1s at
the minimum point of the dimer energy.

Quasi one dimensional (quasi-1D) ultracold Fermi gas.

Now we consider an ultracold two-component Fermi gas
in an axially symmetric two-dimensional harmonic poten-
tial in the $y-z$ plane, with trapping frequency $\omega_L$. When
the atomic transverse motion is frozen in the ground state
of this harmonic potential, the ultracold gas becomes a
quasi-1D system. For this system the effective low-energy
1D interaction between two atoms in different compo-
enents can be expressed as

$$V_{1D} = g_{1D}(x) \equiv -\frac{1}{\mu a_{1D}} \delta(x). \quad (9)$$

Here $a_{1D}$ is the effective 1D scattering length. It can be
controlled by both the 3D scattering length and the transverse trapping frequency $\omega_L$ via the confinement-
induced resonance (CIR) effect [26]. As shown above,
when the two-atom interaction is controlled by a MFR
modulated by two Raman beams counter-propagating along
the $x$-axis, the 3D scattering length becomes a function of the CoM momentum $P_x$. As a result, both
$a_{1D}$ and the effective 1D interaction intensity $g_{1D}$ become
$P_x$-dependent.

In particular, when $a_{1D}(P_x = 0)$ is tuned to the CIR
point, i.e. when $a_{1D}(P_x = 0) = 0$, one can even obtain a
dramatic quasi-1D system with

$$g_{1D}(P_x) = \begin{cases} < 0 & \text{for } P_x > 0 \\ = \infty & \text{for } P_x = 0 \\ > 0 & \text{for } P_x < 0 \end{cases}. \quad (10)$$

Namely, in this system the atoms have attractive and
repulsive 1D interactions when the CoM momentum is
zero. In Fig. 3 we illustrate such a case for a quasi-1D
ultracold gas of $^{40}\text{K}$ atoms. We consider
the system where the one-body momentum $k_x$ along the $x$-
direction is in the region between $\pm k_F^{(1D)} \equiv \pm 3/(4b_1)$ with $b_1 = \sqrt{2/(m\omega_L)}$. Thus we have $P_x \in [-2k_F^{(1D)}, 2k_F^{(1D)}]$. In
our calculation we take $\Delta_p^{(0)} + |k_\alpha|^2/(2M) = 2\pi 	imes 100\text{MHz}$,
$\Delta_p^{(0)} + |k_\alpha - k_\beta|^2/(2M) = 2\pi \times 1.1 \times 10^9\text{Hz}$, $\omega_p = 2\pi \times 50\text{MHz}$,
$\omega_\perp = 2\pi \times 3.5\text{MHz}$, $b_\perp = 4000a_0$ and $B - B_0 = -0.07\Delta_B$.

Other parameters are the same as in Fig. 2.

Summary and discussion. In common quantum sys-
tems, the two-body CoM and relative motion can be coupled
by a non-harmonic confinement potential, a spin-orbit coupling or a CoM-position-dependent two-body
interaction. Nevertheless, these approaches either destroy
the translational symmetry or change the one-body dis-
ersion relation. Here we show that a CoM-momentum
dependent interaction between ultracold atoms can be re-
alized via a laser-modulated MFR. This interaction can
couple the CoM and relative motion without breaking
translational symmetry or changing the one-body disper-
sion relation. Thus, various new effects can be induced by
this interaction. For instance, for a 3D two-component
Fermi gas with CoM-momentum dependent positive scat-
tering length $a(P)$, the dimers are possible to condense in
a superfluid state with nonzero momentum. When $a(P)$ is
negative, it is possible that the minimum energy of a
Cooper pair appears in the region with nonzero CoM mo-
mentum, and thus the Fulde-Ferrell phase [27] can be
induced. In addition, the single-atom momentum distribu-
tion or contact relation can also be qualitatively modified by
a CoM-momentum dependent interaction [28]. Fur-
thermore, this type of interaction can be used to study
the 1D anyon model [29] or other high-order nonlinear
Schrödinger models [30–32].

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* Electronic address: pengzhang@ruc.edu.cn

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[23] In our two-body problem the Hilbert space is $H_{\text{CoM}} \otimes H_{\text{rel}} \otimes H_{\text{internal}}$, where $H_{\text{internal}}$ being the space for the two-atom internal state, while $H_{\text{CoM}}$ and $H_{\text{rel}}$ are the spaces for the spatial states of CoM motion and relative motion, respectively. Here we use $|I\rangle$ to denote the states in $H_{\text{internal}}$ and $|\ell\rangle$ for those in $H_{\text{rel}} \otimes H_{\text{internal}}$.
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SUPPLEMENTARY MATERIAL

In this supplementary material we will first prove Eqs. (5-7) in our main text, and then calculate the three-dimensional (3D) scattering length for our system, with a model in which the spontaneous decay of the excited molecule state is included. We will further calculate the energy and lifetime of the most shallow two-body bound state for our system, and discuss the robustness of our results with respect to the uncertainty of the magnetic Feshbach resonance (MFR) point and width. In the end of this supplementary material we calculate the effective one-dimensional (1D) scattering length and interaction intensity.

I. PROOF OF Eqs. (5-7)

As shown in our main text, the rotated frame introduced in our problem is defined via the relation

$$|\Psi\rangle_{rot} = U|\Psi\rangle_S,$$  \hspace{1cm} (S1)

where $|\Psi\rangle_S$ and $|\Psi\rangle_{rot}$ are the states in the Schrödinger picture and the rotated frame, respectively, and $U$ is the unitary transformation given by Eq. (4) of the main text. Substituting the relation (S1) into the Schrödinger equation

$$i\frac{d}{dt}|\Psi\rangle_S = H_S|\Psi\rangle_S,$$  \hspace{1cm} (S2)

with $H_S$ being given by Eq. (2) of the main text, we obtain the Schrödinger equation satisfied by the state in the rotated frame:

$$i\frac{d}{dt}|\Psi\rangle_{rot} = H_{rot}|\Psi\rangle_{rot},$$  \hspace{1cm} (S3)

where $H_{rot}$ is the Hamiltonian in the rotated frame, and is given by

$$H_{rot} = U H_S U^\dagger + i \left( \frac{d}{dt} U \right) U^\dagger.$$  \hspace{1cm} (S4)

Substituting Eq. (2) and (4) of our main text into Eq. (S4), we can obtain Eqs. (5-7) of the main text. Here we have used the relations

$$e^{i\mathbf{k} \cdot \mathbf{R}_{\phi_j}}|\phi_j\rangle P e^{-i\mathbf{k} \cdot \mathbf{R}_{\phi_j}}|\phi_j\rangle = P + \mathbf{k}|\phi_j\rangle \langle \phi_j|;$$  \hspace{1cm} (S5)

$$e^{i\mathbf{k} \cdot \mathbf{R}_{\phi_j}}|\phi_j\rangle (P^2) e^{-i\mathbf{k} \cdot \mathbf{R}_{\phi_j}}|\phi_j\rangle = (P + \mathbf{k}|\phi_j\rangle \langle \phi_j|)^2 = P^2 + (\mathbf{k}^2 + 2\mathbf{k} \cdot P)|\phi_j\rangle \langle \phi_j|$$  \hspace{1cm} (S6)

for $j = e, \beta$, where $\mathbf{k}$ is a constant vector (c-number) while $P$ and $\mathbf{R}$ are the operators for CoM momentum and position, respectively. They satisfy $[P_i, R_j] = i\hbar \delta_{i,j}$ for $i, j = x, y, z$, with $P_i$ and $R_j$ being the components of $P$ and $\mathbf{R}$ in the directions $i$ and $j$.

II. EXPLICIT EXPRESSION FOR 3D SCATTERING LENGTH

Now we derive the explicit expression for 3D scattering length, i.e., Eq. (8) of our main text, with a model where the spontaneous decay of the excited molecule state is taken into account.

A. Model and Rotated Frame

As shown in Ref. [23] of the main text, in our problem the Hilbert space is $\mathcal{H}_{\text{CoM}} \otimes \mathcal{H}_{\text{rel}} \otimes \mathcal{H}_{\text{internal}}$, where $\mathcal{H}_{\text{internal}}$ being the space for the two-atom internal state, while $\mathcal{H}_{\text{CoM}}$ and $\mathcal{H}_{\text{rel}}$ are the spaces for the spatial states of CoM motion and relative motion, respectively. Here we use $|\rangle\rangle$ and $|\rangle_\text{rel}$ to denote the states in $\mathcal{H}_{\text{internal}}$ and $\mathcal{H}_{\text{rel}}$ respectively, and use $|\rangle$ to denote the states in $\mathcal{H}_{\text{rel}} \otimes \mathcal{H}_{\text{internal}}$.

As mentioned in the main text, we can theoretically take into account the spontaneous decay of the excited molecular state $|\phi_e\rangle$ by introducing an auxiliary scattering channel $D$ which is coupled to the channel $F$ (Fig. S1) [1]. The two-atom internal state corresponding to this channel can be formally denoted as $|D\rangle_I$. Accordingly, in the Schrödinger picture the total Hamiltonian for the two atoms in the 3D space is given by ($\hbar = 1$)

$$H_S^{(\text{tot})} = H_S + \left[ \frac{P^2}{2\mu} + E_D + V_D(r) \right]|D\rangle_I\langle D| + V_{FD}(r)|F\rangle_I\langle D| + \text{h.c.},$$  \hspace{1cm} (S7)
FIG. S1: (color online) The model used in our calculation.

where $H_S$ is defined in Eq. (2) of the main text, $E_D$ and $V_D(r)$ are the threshold energy and potential for the auxiliary channel $D$, respectively, and $V_{FD}(r)$ describes the coupling between channel $D$ and the excited channel $F$. Other notations are defined in the main text. The energy gap $E_e - E_D$ between the excited molecular state and the threshold of the auxiliary channel $D$ is on the order of optical transition (i.e., $\sim 2\pi \times 10^{14}$Hz).

As shown in the main text, we can simplify our problem and remove the phase factor $e^{\pm i(k_i \cdot R - \omega_0 t)}$ ($l = \alpha, \beta$) in the Hamiltonian by introducing the rotated frame (i.e., the interaction picture). In the presence of the auxiliary channel $D$, quantum state $|\Psi\rangle_{\text{rot}}$ is related to the state $|\Psi\rangle_S$ in the Schrödinger picture via the relation $|\Psi\rangle_{\text{rot}} = U|\Psi\rangle_S$, with the unitary transformation $U$ being given by

$$U = U_S e^{i(\omega_0 t - k_\alpha \cdot R)} |D_t\rangle \langle D | = e^{i(\omega_0 t - k_\alpha \cdot R)} e^{i|\phi_\alpha\rangle \langle \phi_\alpha| + |D_t\rangle \langle D |} e^{i(\omega_0 t - (k_\alpha - k_\beta) \cdot R)} |\phi_\beta\rangle \langle \phi_\beta|, \quad (S8)$$

where $U$ is defined in Eq. (4) of the main text. In the rotated frame, the total Hamiltonian is given by

$$H_{\text{rot}}^{(\text{tot})} = \frac{P^2}{2M} + H_{\text{rel}}(P). \quad (S9)$$

Here $H_{\text{rel}}(P)$ is defined as

$$H_{\text{rel}}(P) = H_F(P) + Y + Z, \quad (S10)$$

with

$$H_F(P) = \left[ \frac{P^2}{2\mu} + V_{\text{bg}}(r) \right] |O_t\rangle \langle O | + \left[ \frac{P^2}{2\mu} + \Delta_1 P(P) + E_D - E_e + V_{FD}(r) \right] |D_t\rangle \langle D | + E_\alpha |\phi_\alpha\rangle \langle \phi_\alpha| + \Delta_1 P(P) |\phi_\beta\rangle \langle \phi_\beta|, \quad (S11)$$

and

$$Y = V_{\text{inf}}(r) |O_t\rangle \langle C | + V_{FD}(r) |F_t\rangle \langle D | + h.c.; \quad (S12)$$

$$Z = \sum_{l=\alpha,\beta} \Omega_l |\phi_l\rangle \langle \phi_l| + h.c.. \quad (S13)$$

In Eq. (S11) the one-photon detuning $\Delta_1 P(P)$ and two-photon detuning $\Delta_2 P(P)$ and other notations are all defined in the main text.

It is clear that for a given CoM momentum $P$, the two-body relative motion is governed by $H_{\text{rel}}(P)$. In the following we calculate the scattering length and two-body bound state energy by solving the two-body problem with Hamiltonian $H_{\text{rel}}(P)$. 
B. 3D Scattering State

To calculate the scattering length and bound-state energy, here we first calculate the 3D scattering state and scattering amplitude of two atoms incident from the open channel $O$ with incident momentum $k$. To this end we first derive the out-going scattering state $|\Psi_k^{(+)}\rangle$ for these two atoms which can be expanded as

$$|\Psi_k^{(+)}\rangle = |\psi_k^{(O)}\rangle_r|O\rangle_I + |\psi_k^{(D)}\rangle_r|D\rangle_I + \sum_{l=\alpha,\beta}b_{k}^{(l)}|\phi_l\rangle$$

(S14)

and satisfies the equation [2]

$$|\Psi_k^{(+)}\rangle = \lim_{\varepsilon\to0^+} \frac{ie}{E_k+i\varepsilon-H_{rel}(P)}|k\rangle_r|O\rangle_I.$$  

(S15)

Here $E_k = k^2/(2\mu)$ is the scattering energy, $|k\rangle_r$ is the eigen state of the relative momentum operator $p$, and thus $|k\rangle_r|O\rangle_I$ is the incident state of these two atoms. Using the relation

$$\frac{1}{E_k+i\varepsilon-H_{rel}(P)} = \frac{1}{E_k+i\varepsilon-H_F} + \frac{1}{E_k+i\varepsilon-H_F} \frac{1}{E_k+i\varepsilon-H_{rel}(P)},$$

(S16)

we can re-write Eq. (S15) as [3]

$$|\Psi_k^{(+)}\rangle = |\psi_k^{bg(+)}\rangle_r|O\rangle_I + G_F(E_k)(Y+Z)|\Psi_k^{(+)}\rangle,$$  

(S17)

with

$$|\psi_k^{bg(+)}\rangle_r = \lim_{\varepsilon\to0^+} \frac{ie}{E_k+i\varepsilon-(P^2/2\mu+V_{bg}(r))}|k\rangle_r.$$  

(S18)

being the outgoing scattering state for the open channel $O$ itself and

$$G_F(E) = \frac{1}{E+i0^+-H_F}.$$  

(S19)

Substituting Eq. (S14) into Eq. (S17), we derive the equation for $|\Phi_k\rangle$ and the coefficients $b_{k}^{(l)}$ ($l = \alpha,\beta$):

$$|\Phi_k\rangle = |\psi_k^{bg(+)}\rangle_r|O\rangle_I + G_F(E_k)Y \left[ \sum_{l'=\alpha,\beta} b_{k}^{(l')}|\phi_{l'}\rangle \right];$$

(S20)

$$b_{k}^{(l)} = \frac{1}{E_k - \Lambda_l(P)} \left( \langle \phi_l|Y|\Phi_k\rangle + \sum_{l'=\alpha,\beta} \langle \phi_l|Z|\phi_{l'}\rangle b_{k}^{(l')} \right) \text{ for } l = \alpha,\beta,$$  

(S21)

with $\Lambda_\alpha(P) = E_\alpha$, $\Lambda_\beta(P) = \Delta_{1p}(P)$ and $\Lambda_\beta(P) = \Delta_{2p}(P)$. As shown in the main text, we assume that the MFR is due to the coupling between the open channel $O$ and the bound state $|\phi_\alpha\rangle$ in the closed channel $C$, and the state $|\phi_\beta\rangle$ is far-off resonant to channel $O$. Thus, the direct coupling between $|\phi_\beta\rangle$ and $O$ can be neglected. Explicitly, we can make the approximation $\langle \phi_\beta|H_{rel}(P)|O\rangle_I = 0$ which yields $\langle \phi_\beta|Y|O\rangle_I = 0$. Substituting Eq. (S20) into (S21), we obtain the linear equations for $b_{k}^{(\alpha,\beta,\beta)}$ which gives

$$b_{k}^{(l)} = \sum_{l'=\alpha,\beta} \{[ET - \Sigma(E_k,P)]^{-1}\}_{l,l'} \langle \phi_{l'}|Y|\psi_k^{bg(+)}\rangle_r|O\rangle_I.$$

(S22)

Here $\{[ET - \Sigma(E_k,P)]^{-1}\}_{l,l'}$ denotes the $(l,l')$-th element of the inverse matrix of the $3 \times 3$ matrix $ET - \Sigma(E_k,P)$, where $I$ is the $3 \times 3$ identical matrix and $\Sigma(E,P)$ is the self-energy matrix which can be expressed as (in the basis $\alpha,\beta$)

$$\Sigma(E,P) = \begin{bmatrix} E_\alpha + \langle \phi_\alpha|V_{hi}G_{bg}(E)V_{hi}|\phi_\alpha\rangle & \Omega_\alpha^* & 0 \\ \Omega_\alpha & \Delta_{1p}(P) + \langle \phi_e|V_{FD}G_D(E)V_{FD}|\phi_e\rangle & \Omega_\beta^* \\ 0 & \Omega_\beta & \Delta_{2p}(P) \end{bmatrix},$$  

(S23)
with
\[ V_{\text{hf}} = V_{\text{hf}}(r)|O\rangle_I\langle C| + h.c.; \]
\[ V_{\text{FD}} = V_{\text{FD}}(r)|F\rangle_I\langle D| + h.c.; \]
\[ G_{\text{bg}}(E) = \frac{1}{E + i0^+ - \left(\frac{p^2}{2\mu} + V_{\text{bg}}(r)\right)}; \]
\[ G_D(E) = \frac{1}{E + i0^+ - \left(\frac{p^2}{2\mu} + V_D(r)\right)}. \]

Substituting Eq. (S22) into Eqs. (S20, S14), we can obtain the expression for \(|\Phi_k\rangle\) and the scattering state \(|\Psi^{(+)}\rangle\).

C. Low-Energy Expression for \(\Sigma(E, P)\)

In this paper we consider the cases where \(|E|\) is much smaller than the characteristic energy of \(V_{\text{bg}}(r)\), i.e., the van der Waals energy \(E_{\text{vdW}}\). For these cases, the expression (S23) of \(\Sigma(E, P)\) can be significantly simplified.

First, for real \(E\), the factor \(\langle \phi_{\alpha}|V_{\text{hf}}G_{\text{bg}}(E)V_{\text{hf}}|\phi_{\alpha}\rangle\) can be re-expressed as

\[
\langle \phi_{\alpha}|V_{\text{hf}}G_{\text{bg}}(E)V_{\text{hf}}|\phi_{\alpha}\rangle = \int dq |\langle \phi_{\alpha}|V_{\text{hf}}|q^{bg(+)}\rangle_r|O\rangle_I|^2 \frac{1}{E + i0^+ - E_q - \frac{1}{E_q}} + \int dq |\langle \phi_{\alpha}|V_{\text{hf}}|q^{bg(+)}\rangle_r|O\rangle_I|^2 \frac{1}{E + i0^+ - E_q - \frac{1}{E_q}} ;
\]

(S24)

In the region where \(E_q \gg |E|\), we have \(\frac{1}{E-E_q} - \frac{1}{E_q} \ll \frac{1}{E_q}\). Thus, in this region the contribution of the to-be-integrated function in the second term of the r.h.s. of Eq. (S24) can be neglected. Therefore, we can only do the second integration in the r.h.s. of Eq. (S24) in the region where \(E_q\) is not much larger than \(E\). On the other hand, in realistic ultracold atom systems, the inter-channel coupling \(V_{\text{hf}}(r)\) is negligible in the region \(r > r_{vdW}\) with \(r_{vdW}\) being the van der Waals radius. Thus, the the factor \(|\langle \phi_{\alpha}|V_{\text{hf}}|q^{bg(+)}\rangle_r|O\rangle_I|^2\) is determined by the behavior of the wave function \(\langle r|q^{bg(+)}\rangle_r\) in the region \(r < r_{vdW}\), where \(V_{\text{bg}}(r)\) is a deep potential well and thus \(\langle r|q^{bg(+)}\rangle_r\) is almost independent of \(q\) for \(E_q < E_{vdW}\). Therefore, in our calculation we make the approximation

\[
\int dq |\langle \phi_{\alpha}|V_{\text{hf}}|q^{bg(+)}\rangle_r|O\rangle_I|^2 \left(\frac{1}{E + i0^+ - E_q - \frac{1}{E_q}} \right) \approx |\langle \phi_{\alpha}|V_{\text{hf}}|q^{bg(+)}\rangle_r|O\rangle_I|^2 \int dq \left(\frac{1}{E + i0^+ - E_q - \frac{1}{E_q}} \right) ;
\]

(S25)

where

\[
\eta(E) = \begin{cases} 
-i\sqrt{E}, & \text{for } E > 0 \\
\sqrt{-E}, & \text{for } E < 0 \text{ or } \text{Im}[E] \neq 0
\end{cases} ;
\]

(S26)

with \(\sqrt{z} = \sqrt{|z|e^{i\arg[z]/2}}\) and \(\arg[z] \in (-\pi, +\pi]\). Thus, we have

\[
\langle \phi_{\alpha}|V_{\text{hf}}G_{\text{bg}}(E)V_{\text{hf}}|\phi_{\alpha}\rangle \approx E'_{\alpha} + \chi(E),
\]

(S27)

where

\[
E'_{\alpha} = \int dq \frac{|\langle \phi_{\alpha}|V_{\text{hf}}|q^{bg(+)}\rangle_r|O\rangle_I|^2}{-E_q} ;
\]

(S28)

\[
\chi(E) = \eta(E)|\langle \phi_{\alpha}|V_{\text{hf}}|q^{bg(+)}\rangle_r|O\rangle_I|^2 4\pi^2 \mu \sqrt{2} ;
\]

(S29)

Second, as shown above, in our system \(E_D - E_o\) is much larger then both \(\Delta_{1p}(P)\) and \(E\). As a result, the factor \(\langle \phi_{e}|V_{\text{FD}}G_D(E)V_{\text{FD}}|\phi_{e}\rangle\) is almost independent of \(E\), and satisfies [1]

\[
\text{Im}[\langle \phi_{e}|V_{\text{FD}}G_D(E)V_{\text{FD}}|\phi_{e}\rangle] = -\frac{\gamma}{2} ;
\]

(S30)
with \( \gamma \) being the decay rate of \( |\phi_e\rangle \). In addition, \( \text{Re} \left[ \langle \phi_e | V_{FDG}(E) | V_{FD} | \phi_e \rangle \right] \) is the Lamb shift of the energy of \( |\phi_e\rangle \), which is induced by the coupling between \( |\phi_e\rangle \) and the channel \( D \). It can be absorbed in the definition of \( E_c \). Explicitly, we can make the notation replacement
\[
E_c + \text{Re} \left[ \langle \phi_e | V_{FDG}(E) | V_{FD} | \phi_e \rangle \right] \to E_c. \tag{S31}
\]
Substituting Eqs. (S27,S30,S31) into Eq. (S23), we obtain
\[
\Sigma(E, P) \approx \begin{bmatrix}
E_\alpha + E'_\alpha + \chi(E) & \Omega_\alpha^* & 0 \\
\Omega_\alpha & \Delta_t(P) - i \frac{\gamma}{2} & \Omega_\beta \\
0 & \Omega_\beta & \Delta_{2p}(P)
\end{bmatrix}, \tag{S32}
\]
Our following calculation are based on this low-energy expression of \( \Sigma(E, P) \).

### D. 3D Scattering Amplitude

Now we consider the elastic scattering amplitude in the open channel \( O \). According to the formal scattering theory, for our system this scattering amplitude can be expressed as
\[
f = -(2\pi)^2 \mu \sum_{l=\alpha,\epsilon,\beta} i \langle O | \langle k' | (V + Y + Z) | \psi_{k'}^{(+)}) \rangle, \tag{S33}
\]
where \( k \) and \( k' \) are the incident and output momentum, respectively, and satisfies \(|k| = |k'| = k\), and the operator \( V \) is defined as
\[
V = V_{bg}(r) O | 1 \rangle \langle 1 | + V_D(r) | D \rangle \langle D |. \tag{S34}
\]
Substituting the expression of the scattering state \( |\psi_{k'}^{(+)}\rangle \) derived in Sec. II.B into Eq. (S33) and doing some straightforward calculations, we obtain
\[
f = f^{(bg)} - (2\pi)^2 \mu \sum_{l=\alpha,\epsilon,\beta} i \langle O | \langle k' | V_{bg}(r) | \psi_{k'}^{bg(-)} \rangle | Y | \phi_l \rangle b^{(l)}_k, \tag{S35}
\]
with \( b^{(l)}_k \) being given by Eq. (S22). Here \( f^{(bg)} = -(2\pi)^2 \mu \left[ i \langle k' | V_{bg}(r) | \psi_{k'}^{bg(+)} \rangle \right] \) is the background scattering amplitude for the open channel itself, and \( |\psi_{k'}^{bg(-)} \rangle \) is the in-coming scattering state for the open channel itself, which is defined as
\[
|\psi_{k'}^{bg(-)} \rangle = \lim_{\varepsilon \to 0^+} \frac{i \varepsilon}{E_{k'} + i \varepsilon - \left( \frac{E^2}{2\mu} + V_{bg}(r) \right)} |k' \rangle, \tag{S36}
\]
and satisfies
\[
\langle k' | V_{bg}(r) G_{bg}(E_{k'}). \tag{S37}
\]
Substituting Eq. (S22) into Eq. (S35), we can further derive
\[
f = f^{(bg)} - (2\pi)^2 \mu e^{2i \delta^{(bg)}_{E_k}} \langle \phi_\alpha | V_{hf} | \psi_{k'}^{bg(+)} \rangle | O \rangle I |^2 \left( [E_{k} I - \Sigma(E_{k}, P)]^{-1} \right)_{\alpha \alpha}. \tag{S38}
\]
Here \( \delta^{(bg)}_{E_k} \) is the s-wave phase shift for the open channel \( O \) itself and we have used the relation \( \langle k' | V_{bg}(r) | \psi_{k'}^{bg(+)} \rangle = e^{2i \delta^{(bg)}_{E_k}} r_{\psi_{k'}^{bg(+)} | \psi_{k'}^{bg(+)} \rangle} r_{\psi_{k'}^{bg(+)} | \psi_{k'}^{bg(+)} \rangle} \), with \( |r \rangle \) being the eigen-state of the two-atom relative position operator. This relation can be proved via the Schrödinger equation in the spatial space as well as the out-going and in-coming boundary conditions satisfied by \( r_{\psi_{k'}^{bg(+)} | \psi_{k'}^{bg(+)} \rangle} r \). Furthermore, as shown above, in our problem the factor \( \langle \phi_\alpha | V_{hf} | \psi_{k'}^{bg(+)} \rangle | O \rangle I |^2 \) changes very slowly with \( k \) for \( E_k \ll E_{vdw}. \) Therefore, we can make the approximation
\[
|\phi_\alpha | V_{hf} | \psi_{k'}^{bg(+)} \rangle | O \rangle I |^2 \approx |\phi_\alpha | V_{hf} | \psi_{k'}^{bg(+)} \rangle | O \rangle I |^2 \tag{S39}
\]
in the r.h.s. of Eq. (S38), and obtain
\[
f = f^{(bg)} - (2\pi)^2 \mu e^{2i \delta^{(bg)}_{E_k}} \langle \phi_\alpha | V_{hf} | \psi_{k'}^{bg(+)} \rangle | O \rangle I |^2 \left( [E_{k} I - \Sigma(E_{k}, P)]^{-1} \right)_{\alpha \alpha}, \tag{S40}
\]
with \( \Sigma(E_{k}, P) \) given by Eq. (S32). In the following we use the result (S40) to calculate the scattering length.
E. Proof of Eq. (8)

Now we calculate the scattering length $a$ for zero scattering energy and proof Eq. (8) of the main text. The scattering length is defined as

$$a = -f|_{k=k'=0}. \tag{S41}$$

We first consider the case without Raman beams (i.e., $Z = 0$). Taking $k = k' = 0$ for Eq. (S40) and using the fact $\delta \phi_{E_k=0} = 0$, we obtain

$$a = a_{bg} - (2\pi)^2 \mu |\langle \phi_0|V_{hf}|\phi_{k=0}^{bg(+)}\rangle I|^2 \over E_\alpha + E'_\alpha, \tag{S42}$$

where $a_{bg} = -f^{(bg)}|_{k=k'=0}$ is the background scattering length, and $E'_\alpha$ is defined in Eq. (S28). Comparing this result with the usual expression for the scattering length near a MFR

$$a = a_{bg} \left( 1 - \frac{\Delta B}{B - B_0} \right), \tag{S43}$$

we obtain the relations

$$(2\pi)^2 \mu |\langle \phi_0|V_{hf}|\phi_{k=0}^{bg(+)}\rangle I|^2 = (\delta \mu) a_{bg} \Delta B; \tag{S44a}$$

$$E_\alpha + E'_\alpha = (\delta \mu)(B - B_0), \tag{S44b}$$

where $\delta \mu$ is the magnetic moment difference between the open channel $O$ and the closed channel $C$, as defined in the main text.

Now we consider the case with Raman beams. In this case we can also calculate the scattering length $a$ by taking $k = k' = 0$ for Eq. (S40). Using Eqs. (S44a, S44b, S30, S32), we finally obtain

$$a(P) = a_{bg} - \frac{(\delta \mu)a_{bg}\Delta B}{(\delta \mu)(B - B_0)} \left[ \Delta_{1p}(P) - i \frac{\Omega_\alpha}{2} - \frac{|\Omega_\alpha|^2}{\Delta_{2p}(P)} \right]. \tag{S45}$$

That is Eq. (8) in the main text.

III. 3D BOUND-STATE ENERGY

Now we calculate the energy $E_b$ of the 2-body bound state $|\Phi_b\rangle$. Here we consider the case with $|E_b| \ll E_{vdW}$. Since we have assumed that the threshold energy of the auxiliary channel $D$ is much lower than $-E_{vdW}$, precisely speaking there is no bound state with $|E_b| \ll E_{vdW}$. Nevertheless, our system may have approximate bound state (quasi bound state) with complex energy $E_b$ which satisfies

$$\text{Re}[E_b] < 0, \quad |\text{Re}[E_b]| < E_{vdW}, \quad \text{Im}[E_b] < 0.$$  

Here $\text{Re}[E_b]$ and $|\text{Im}[E_b]|$ describe the energy and decay rate of this approximate bound state, respectively. Similar as above, for our system both $|\Phi_b\rangle$ and $E_b$ are functions of the of the CoM momentum $P$.

We can obtain the approximate bound state via the equation

$$H_{eff}(P)|\Phi_b\rangle = E_b(P)|\Phi_b\rangle. \tag{S46}$$

Here $H_{eff}(P)$ is the effective Hamiltonian for our system and can be expressed as

$$H_{eff}(P) = H_F(P) + V_{hf} + Z - i \frac{\gamma}{2} |\phi_e\rangle \langle \phi_e|. \tag{S47}$$

We can further express $|\Phi_b\rangle$ as $|\Phi_b\rangle = \langle \phi_0| I + \sum_{l=\alpha,e,\beta} c_l |\phi_l\rangle$. Substituting this expression into Eq. (S46), we can derive the equations for $|\phi_0\rangle$ and the coefficients $c_l$ ($l = \alpha, e, \beta$). Eliminating $|\phi_0\rangle$ from these equations and using Eq. (S27, S28, S29), we obtain

$$[E_b I - \Sigma(E_b, P)] \begin{bmatrix} c_\alpha \\ c_e \\ c_\beta \end{bmatrix} = 0. \tag{S48}$$
with \( \Sigma(E_b, P) \) being given by Eq. (S32). Eq. (S48) yields that the energy \( E_b \) is determined by the equation

\[
\det [E_b I - \Sigma(E_b, P)] = 0.
\]  

(S49)

We derive \( E_b \) by numerically solving Eq. (S49).

IV. ROBUSTNESS OF OUR RESULTS

Now we consider the robustness of our result with respect to the uncertainty of the MFR point \( B_0 \) and width \( \Delta_B \). We first study the effect induced by the uncertainty of \( B_0 \) for the ultracold \( ^{40}\text{K} \) gas discussed in our main text. For this system the main value of \( B_0 \) is 202.2G and the uncertainty of \( B_0 \) is 0.02G [4]. Below in Fig. S2(a) we show the behavior of \( \text{Re}[a(P_x)] \) for the cases where \( B_0 = 202.2G \) and \( 202.2 \pm 0.02G \) while \( B \) is fixed at \( 202.2G - 0.07\Delta_B = 201.64G \) (corresponding to the case in Fig. 2(a) of the main text). Similarly, in Fig. S2(b) and Fig. S2(c) we show the results for \( B = 202.2G + 0.07\Delta_B = 202.76G \) (corresponding to the case in Fig. 2(b) of the main text) and \( B = 202.2G \), respectively. These figures shows that our results are robust with respect to the uncertainty of the MFR point.

![Figure S2: (color online) \( \text{Re}[a(P_x)] \) of ultracold \( ^{40}\text{K} \) gases with MFR point \( B_0 = 202.2G \) (black solid line), \( B_0 = (202.2 \pm 0.02)G \) (red squares) and \( B_0 = (202.2 \pm 0.02)G \) (blue circles). Here we consider the cases with fixed value of magnetic field \( B \). (a): \( B = 202.2G - 0.07\Delta_B = 201.64G \) (corresponding to the case in Fig. 2(a) of the main text), (b): \( B = 202.2G + 0.07\Delta_B = 202.76G \) (corresponding to the case in Fig. 2(b) of the main text) and (c): \( B = 202.2G \). Other parameters of Fig. (a) and Fig. (b, c) are same as the ones in Fig. 2(a) and Fig. 2(b) of the main text, respectively.](image)

We can also understand our above conclusion with Fig. S3, where we illustrate variation of the scattering length \( a(P_x) \) with the magnetic field \( B \), for the case in Fig. 2(b) of our main text. It is shown that for different CoM momentum \( P_x \) the scattering length has different resonant point in the \( B \)-axis. Therefore, loosely speaking, in our system the Doppler effect induces a CoM momentum dependent shift of the MFR point, and make the scattering length to be CoM momentum dependent. Fig. S3 clearly show that, when \( P_x \) is modified from \(-2k_F \) to \( 2k_F \) the MFR point is shifted by about 0.6G, which is much larger than the uncertainty of the MFR point (0.02G). Therefore, our results are quite robust for that uncertainty.

![FIG. S3: (color online) The scattering length \( a(P_x) \) as a function of \( B \), for \( P_x = 0 \) (black solid line), \( P_x = -2k_F \) (violet dashed-dotted-dotted line), \( P_x = -k_F \) (red dashed-dotted line), \( P_x = +k_F \) (blue dashed line) and \( P_x = +2k_F \) (green dotted line). The parameters in this figure are same as in Fig. 2(b) of our main text.](image)
Now we investigate the effect of the uncertainty of the resonance width $\Delta B$. To this end we consider two cases with resonance width $\Delta B$ and $\Delta B + \delta B$, where $\delta B$ is the uncertainty of the resonance width. Eq. (8) of our main text clearly show that the scattering lengths for these two cases satisfy the relation

$$\frac{a(P, \Delta B + \delta B) - a_{bg}}{a(P, \Delta B) - a_{bg}} = \frac{\Delta B + \delta B}{\Delta B}. \quad (S50)$$

When the magnetic field is close to the resonance point, e.g., in the cases in Figs. 2(a, b) in the main text, we have $|a(P, \Delta B)| \gg |a_{bg}|$ and $|a(P, \Delta B + \delta B)| \gg |a_{bg}|$, which yields

$$\frac{a(P, \Delta B + \delta B) - a(P, \Delta B)}{a(P, \Delta B)} = \frac{\delta B}{\Delta B}, \quad (S51)$$

i.e., the relative error of the scattering length is just the relative uncertainty $\delta B/\Delta B$ of the resonance width. In realistic systems we usually have $|\delta B/\Delta B| \ll 1$. Nevertheless, the Doppler-effect-induced relative variation of scattering length with center-of-mass momentum (for our example of $^{40}K$ atoms it is $|a(P_\perp = 2\kappa F) - a(P_\perp = -2\kappa F)|/|a(P_\perp = 2\kappa F)|$) can be of the order of 1 or even larger, as shown in Figs 2(a, b). Thus, the Doppler effect is robust with respect to the uncertainty of the resonance width.

V. QUASI-1D SYSTEM

In this subsection we consider an two-component Fermi gas in an quasi-1D confinement along the $x$-direction, as described in the main text. In this system the effective low-energy 1D interaction between two atoms in different components can be expressed as

$$V_{1D} = g_{1D}(x) \equiv -\frac{1}{\mu a_{1D}} \delta(x), \quad (S52)$$

where the 1D scattering length $a_{1D}$ is given by [5]

$$a_{1D} = -\frac{b_0^2}{2a_{\perp}} \left(1 - C a^{(\perp)} / b_\perp \right). \quad (S53)$$

Here $b_\perp = \sqrt{1/(\mu \omega_\perp)}$ is the characteristic length of the transverse trap and $C = -\zeta(1/2) = 1.4603...$. In Eq. (S53) $a^{(E)}$ is the “energy-dependent” 3D $s$-wave scattering length of these two atoms, which is defined as

$$a^{(E)} = \frac{-1}{\cot \delta_E \sqrt{2\mu E}} \quad (S54)$$

with $\delta_E$ being the $s$-wave phase shift corresponding to scattering energy $E$. It relates to the scattering length $a$ we studied before via the relation $a^{(E=0)} = a$. When the two-atom interaction is controlled by a MFR modulated by two Raman beams, the scattering length $a^{(E)}$ becomes a function of the CoM momentum $P$. This scattering length can be calculated as follows. Using Eq. (S40) and the relation $f^{(bg)} = -1/[ik - k \cot \delta_{E_k}^{(bg)}]$, we can re-express Eq. (S40) as

$$f = \frac{-1}{ik - k \cot \delta_{E_k}^{(bg)}} - \frac{e^{2i\delta_{E_k}^{(bg)}}}{ik + A_k(P)} \quad (S55)$$

where

$$A_k(P) = \frac{1}{(2\pi)^2 \mu \langle \phi_{\alpha} | W_{\psi_{kz=\theta}}^{(+)} | O \rangle^2 \{ [E_k - \Sigma(E_k, P)]^{-1} \}_{\alpha \alpha} - ik. \quad (S56)$$

with $\Sigma(E_k, P)$ given by Eq. (S32). On the other hand, Eq. (S57) yields $f = -1/[ik + 1/a^{(E_k)}]$. Comparing this result and Eq. (S55) and using the relation $-k \cot \delta_{E_k}^{(bg)} = 1/a_{bg}$, we obtain the expression of $a^{(E_k)}$:

$$a^{(E_k)} = \frac{a_{bg} A_k(P) + 1}{A_k(P) - k^2 a_{bg}}. \quad (S57)$$
Here we consider the case in which the Raman beams are counterpropagating along the $x$-axis. It is clear that in this case $a(E_k)$ only depends on the value of $E_k$ and the $x$-component of $P$ (i.e., $P_x$), and are independent of the direction of $k$ and the values of $P_y$ and $P_z$.

Substituting Eq. (S57) into Eqs. (S53, S52), we can obtain the expressions of the effective 1D scattering length $a_{1D}$ and the effective 1D interaction intensity $g_{1D}$, which are both $P_x$-dependent. In particular, according to Eqs. (S52) and (S53), when $a(\omega_\perp)(P_x) = a/\mathcal{C}$ we can obtain an interesting quasi-1D system with $g_{1D}(P_x = 0) = \infty$, $g_{1D}(P_x > 0) < 0$ and $g_{1D}(P_x < 0) > 0$, as shown in Fig. 3 of the main text.

Finally, we consider the two-body collisional loss of the quasi-1D gas of two-component $^{40}$K atoms studied in Fig. 3 of the main text, where the one-body momentum $k_x$ along the $x$-direction is in the region between $\pm k_F^{(1D)} = \pm 3/(4b_\perp)$ and thus we have $P_x \in [-2k_F^{(1D)}, 2k_F^{(1D)}]$. For one quasi-1D scattering process, the two-body collisional loss probability $P_{\text{loss}}$ is given by $P_{\text{loss}} = 1 - |r|^2 - |t|^2$, where $r = -1/(1 + ik_x a_{1D})$ and $t = 1 + r$ are the reflection and transmission amplitude of this scattering process, respectively, with $k_x$ being the incident momentum along the $x$-direction. For our system $P_{\text{loss}}$ also depends on the CoM momentum $P_x$. In Fig. S4 we illustrate $P_{\text{loss}}$ for our system with $k_x = k_F^{(1D)}$. It is shown that for this system the loss probability is at most on the order of $10^{-3}$.

With this result, we can further estimate the life time of our system. First, the density $n_{1D}$ of a quasi-1D ultracold gas is related to the 1D Fermi momentum $k_F^{(1D)}$ via $n_{1D} = k_F^{(1D)}/\pi$. Thus, the average distance of two atoms is $d = 1/n_{1D} = \pi/k_F^{(1D)}$. Second, the maximum relative velocity of two atoms is $v_{\text{max}} = k_F^{(1D)}/m$, with $m$ being the single-atom mass. Thus, the frequency of the collision between a given atom and other atoms is at most $\nu = v_{\text{max}}/d$. Therefore, the life time for each atom in the ultracold gas can be estimated as $\tau_{1D} = 1/(\nu P_{\text{loss}}) = m\pi/(\hbar k_F^{(1D)2} P_{\text{loss}})$. Substituting the parameters of Fig. 3 of the main text into this formula, we obtain $\tau_{1D} > 0.04s$.

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