New aspects of absorption line formation in intervening turbulent clouds – I. General principles

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ABSTRACT
We study the formation of absorption lines in intervening clouds with stochastic velocity fields, accounting for the fact that actually only one line of sight is observed. Our results show that the introduction of the finite velocity correlation length leads to a new type of absorption line profiles which are asymmetric in general, may have different shifts of the centres of gravity, and look like barely resolved blends, i.e. could be interpreted in a standard Voigt fitting analysis as being caused by several independent clouds with different physical parameters. Numerical results are presented for the H\textsc{i} Ly\alpha line with \(N_{\text{H}\textsc{i}} = 10^{12}, 10^{14}, 10^{15}\) and \(10^{16}\) cm\(^{-2}\), \(T_{\text{kin}} = 10^4\) K, and different sets of turbulent parameters. The intensity fluctuations within the line profile caused by ‘turbulent noise’ are investigated and the confidence belts for the absorption lines are calculated. We conclude that an exact measurement of the column densities of the absorbing atoms \(N_{\text{A}}\) from the observed values of the optical depths \(\tau_{\text{A}}\) is actually impossible for the case of the correlated velocity field. One can only determine a range of values within which \(N_{\text{A}}\) is to be found with a certain probability.

Key words: line: formation – line: profiles – ISM: clouds – quasars: absorption lines – ultraviolet: ISM.

1 INTRODUCTION
Spectral lines are broadened by radiation damping, by collisions, and by the Doppler effect caused by the motion of the absorbing and emitting particles. Besides the thermal motion of the particles, we find in most astrophysical objects hydrodynamical flows. If the hydrodynamical velocity field is known exactly, it is in principle straightforward to account for it in radiative transfer problems. In this case the local absorption coefficient \(k_\rho\Phi\) [with \(\Phi = \Phi(\lambda, \nu)\) being the profile function] has to be Doppler shifted according to the local velocity component \(\nu\) parallel to the line of sight.

In many astrophysical objects, however, the velocity field is of a more or less stochastic nature and the radiative transfer problem becomes very complex in detail. The simplest and most widely used approximation to handle the problem is to neglect all correlations in the velocity field (\textit{microturbulence}). In this approximation the local absorption coefficient \(k_\rho\) is substituted by its expectation value \(\langle k_\rho \rangle\), where the averaging procedure applies to the profile function, while the transfer equation is kept in its original form (see e.g. Spitzer 1978). This approximation is valid only if the mean free path of the photons is large compared with the correlation length of the velocity field and if there are very many turbulent elements along each line of sight.

It has been shown in recent years, however, that accounting for the correlations in the velocity field (\textit{mesoturbulence}) may change the quantitative results of radiative transfer calculations substantially. A new parameter introduced when one considers the velocity as a stochastic variable is the ratio of the mean free path of the photons to the correlation length, \(\lambda\), of the velocity field. This parameter varies from line to line and across the profile of each line.

A theory accounting for the correlation effects in some approximation was developed by G. Traving and collaborators (see e.g. Gail et al. 1974; Gail, Sedlmayr & Traving 1975b; Traving 1975). The most general formulation has been given by Gail, Sedlmayr & Traving (1980). A somewhat different approach to the problem was independently developed by H. Frisch and collaborators (see e.g. Auvergne et al. 1973; Frisch & Frisch 1976; Audic & Frisch 1993). Both formalisms lead to the conclusion that a finite correlation length of the turbulent velocity field may strongly affect line profiles and equivalent widths. Since different lines are affected to a different degree, the curve of growth is also different for different lines.

In the following we use the formalism developed by Gail et al. (1974). In this theory the hydrodynamic velocity \(\nu\) and the intensity \(I_\lambda\) are considered to be stochastic variables (random functions of the coordinate along the line of sight). If the medium is characterized by its statistical properties only, it is not possible to determine \(I_\lambda\) exactly. One can calculate only its statistical properties; in particular one may calculate the expectation value \(\langle I_\lambda \rangle\) of the intensity which by definition is an ensemble average or space (or time)
average. If one wants to compare theoretical calculations with observations, one has to ask what the relation between \( l_h \) and an observed intensity is. The answer is quite different for different problems.

The theory was originally developed with the interpretation of stellar spectra in mind (see e.g. Travins 1975). In this case the turbulent velocity field is related to the convective motion in the atmosphere. An observed stellar spectrum is an average over the total stellar surface containing a very large number of turbulent elements. Therefore the theoretical expectation value \( \langle I_h \rangle \) should very closely correspond to the observed spectrum since this case is a good approximation to the mathematical averaging procedure.

Later the theory was applied to the formation of maser lines (Gail, Kegel & Sedlmayr 1975a) and interstellar emission lines from molecular clouds (Albrecht & Kegel 1987; Kegel, Pelieier & Albrecht 1993; Piehler & Kegel 1995). In single-dish radio observations the beam covers an essential part of the molecular cloud and \( I_h \) should still reasonably well correspond to the observations.

Finally we have applied the theory to interstellar absorption lines (Levshakov & Kegel 1994; hereafter LK94), and to QSO absorption lines (Levshakov 1995; Levshakov & Kegel 1996; Levshakov & Takahara 1996). In this case the situation is quite different, since in this context a star (or a quasar) may be considered as a point-like light source. The observed absorption spectra therefore correspond to only one line of sight, i.e. to one realization of the one-dimensional velocity distribution. If there are many turbulent velocity elements along the line of sight, the theoretical expectation value \( \langle I_h \rangle \) should still be a fair approximation to the observations. (In spectra with infinite resolution, one would expect to find fluctuations around the expectation value. These will be smeared out with decreasing resolution.) If, however, the correlation length is not very small compared with the cloud's thickness \( L \), substantial deviations of \( \langle I_h \rangle \) from the observed profile are to be expected.

There are two natural reasons which can explain these deviations qualitatively. First, the turbulent structure of the velocity field is changing very slowly in comparison with a normal exposure time which can range from minutes to hours. (A typical correlation length of 0.2 pc and a typical rms turbulent velocity of about 10 km s\(^{-1}\) would correspond to a period of the turbulent structure intermixing of about 2 \times 10\(^4\) yr.) Therefore the observed profile cannot be considered as a time average in the statistical sense. Secondly, if the ratio \( L/l \) is not very large, the observed profile is just the resultant of the contributions from a small number of turbulent cells randomly distributed along the line of sight at a given instant. Thus significant intensity deviations can arise because the statistical ensemble along the line of sight under consideration is not complete.

It is the purpose of the present paper to investigate the problem of line formation in turbulent intervening clouds with a finite velocity correlation length for the case of incomplete statistical samples. We will calculate absorption line profiles and the confidence regions for them, and will consider the dependence of their shapes on different turbulent parameters. The theory describes the process of line formation in the absorbing clouds more adequately than the commonly used microturbulent approximation. Thus the results obtained will be of great importance for the interpretation of modern high-resolution absorption line spectra.

## 2 BASIC EQUATIONS

Following Gail et al. (1974) we consider the probability \( P(s; v, l_h) \) for finding at point \( s \) along a given line of sight the velocity component parallel to the ray between \( v \) and \( v + dv \) and the intensity between \( I_h \) and \( I_h + dI_h \). Assuming the one-point distribution function of \( v \) to be a Gaussian with rms turbulent velocity \( \sigma_v \),

\[
W_i(v) = \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left(-\frac{v^2}{2\sigma_v^2}\right),
\]

and the variations of \( v \) to be governed by a Markov process, we have for the two-point correlation function

\[
f(\Delta s) = \frac{\langle \delta I_h(s)\delta I_h(s + \Delta s) \rangle}{\sigma_I^2},
\]

a unique exponential form (Doob 1942):

\[
f(\Delta s) = \exp\left(-\frac{\Delta s}{l}\right).
\]

With these assumptions about \( v \), Gail et al. (1974) derived a Fokker–Planck equation for \( P \):

\[
\frac{\partial P}{\partial s} = \frac{1}{I} \frac{\partial}{\partial v} \left( \nu P + \sigma_v^2 \frac{\partial P}{\partial v} \right) + \frac{\partial}{\partial I} \left[ \kappa_0 \Phi P(I_h - S_h) \right],
\]

where \( S_h \) is the source function. Multiplying (4) by \( I_h \) and integrating over \( I_h \) leads to

\[
\frac{\partial Q_h}{\partial s} = \frac{1}{I} \frac{\partial}{\partial v} \left( \nu Q_h + \sigma_v^2 \frac{\partial Q_h}{\partial v} \right) - \kappa_0 \Phi (Q_h - S_h) W_i(v),
\]

where \( Q_h \) is the first moment of \( P \) with respect to \( I_h \):

\[
Q_h = \int_0^\infty I_h P(s; v, I_h) \, dI_h.
\]

Mainly in order to facilitate the numerical solution of (5), Gail & Sedlmayr (1974) have introduced a new quantity \( q_h(s, v) \) by writing

\[
q_h(s, v) W_i(v) = Q_h(s, v).
\]

Now \( q_h(s, v) \) is a new product of the energy in the beam of radiation which is transferred under the condition that at the point \( s \) the velocity \( v \) lies between \( v \) and \( v + dv \). We therefore call \( q_h(s, v) \) the \textit{conditional} intensity hereafter.

Inserting (7) into (5) leads to the by now standard generalized radiative transfer equation

\[
\frac{\partial q_h}{\partial s} = \frac{1}{l} \frac{\partial}{\partial v} \left[ \frac{\partial q_h}{\partial v} - \frac{\partial q_h}{\partial u} \right] - \kappa_0 \Phi (q_h - S_h),
\]

where the dimensionless velocity \( u = v/\sigma_v \) has been introduced.

The partial differential equation (8) has to be solved with the initial condition

\[
q_h(s = 0, u) = I_0,
\]

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Absorption line formation

\[ u_0 = 1, \quad \sigma / \nu_{th} = 2, \quad N_{HI} = 10^{12} \, \text{cm}^{-2} \]

Figure 1. Synthetic mesourbulent conditional (\( q \), solid lines) and unconditional (\( I \), dashed lines) spectra of \( \text{H} I \) Ly\( \alpha \) as functions of \( N_{HI} \) and \( \sigma / \nu_{th} \). \( T_{kin} = 10000 \, \text{K} \), \( N_{HI} = 10^{12} \, \text{cm}^{-2} \), \( u_0 = 2 \), and \( u_0 = 1 \) and 5 for the upper and lower panels, respectively.

\( I_0 \) being the continuum intensity of the incident radiation, and the boundary conditions

\[
\lim_{u \to \pm \infty} q_{\lambda}(s, u) = I_0(s),
\]

where \( I_0 \) is the local continuum intensity.

After solving (8) one obtains the expectation value for the local intensity \( \langle I_\lambda(s) \rangle \) by a simple quadrature (see LK94, for details of numerical solution):

\[
\langle I_\lambda(s) \rangle = \int_{-\infty}^{\infty} q_{\lambda}(s, u) W_\lambda(u) \, du.
\]

In the following we call \( \langle I_\lambda(s) \rangle \) the ‘unconditional intensity’ in order to distinguish it from \( q_{\lambda}(s, u) \).

If one considers an interstellar absorption line arising in an intervening cloud of a finite thickness \( L \), \( \langle I_\lambda(s) \rangle \), the expectation value of the intensity after having passed the cloud, may deviate substantially from the actually observed intensity although the model describes the statistical properties of the cloud under consideration correctly, as pointed out in the introduction. To study these deviations one may ask for the probability distribution function of \( I_\lambda \). There are in principle two ways to determine this. One of them is to compute \( P(s; v, I_\lambda) \) directly from (4) and integrate the result over \( v \). However, this would require a very large numerical effort since (4) contains one more independent variable than (8). An alternative is to compute higher moments of \( P \) with respect to \( I_\lambda \), which can easily be done. In our case the set of moments determines the distribution uniquely since the range of variation of \( I_\lambda \) is finite:

\[ 0 \leq I_\lambda I_0 \leq 1 \] (see Appendix A).

Multiplying (4) by \( I_\lambda \) and integrating over \( I_\lambda \) leads to

\[
\frac{\partial Q_{\lambda j}}{\partial s} = \frac{1}{L} \frac{\partial}{\partial v} \left( v Q_{\lambda j} + \sigma^2 \frac{\partial Q_{\lambda j}}{\partial v} \right) - j_{\lambda 0} \Phi (Q_{\lambda j} - S_{\lambda} Q_{\lambda j-1}) ,
\]

with

\[
Q_{\lambda j} = \int_{0}^{\infty} I_\lambda P(s; v, I_\lambda) \, dI_\lambda .
\]

Thus we obtain a system of coupled differential equations (12). It may be solved with the same techniques as used for the solution of (5). The situation is further simplified if one considers pure absorption lines, as we shall do in the following, i.e. if one assumes the source function to be zero. In this case the higher moments can be computed using the same code as for calculation of the first moment. [One only has to substitute \( \kappa_0 \) by \( j_{\lambda 0} \) in (8).] Thus it is straightforward to compute \( \langle I_\lambda \rangle \).

From the higher moments we may reconstruct the cumulative distribution function for \( I_\lambda \) (see Appendix A). Once the distribution function is known, we can calculate an intensity interval around \( \langle I_\lambda \rangle \) in which the observed intensity is expected to lie with a certain probability. As stressed in Appendix A, the obtained probability
density function is rather wide and asymmetric in certain parameter ranges so that it becomes difficult to make a meaningful comparison between theoretical results and observations.

Considering the distribution function of $I_\lambda$ in the present case, however, does not make use of all information that we have. If we consider interstellar absorption lines, i.e. a situation in which the observations correspond to only one line of sight, we know that only one realization of the velocity field determines the observed spectrum, since over a normal exposure time any bulk motions within the cloud are to be considered as frozen, and we therefore observe neither a time average nor a spatial average. At any point along this line of sight (and in particular at the cloud edges) the velocity has an (a priori unknown) unique value, although the velocity field in the cloud as a whole may be described by its statistical properties only. It therefore appears more appropriate to compare with observations the conditional intensity function $q_\nu(L, u_0)$ rather than $q_\nu(L)$. In doing so, we have introduced an additional free parameter, $u_0$, the velocity at the cloud edge (i.e. the velocity of the first turbulent cell along the line of sight), which in principle is to be determined from the observations. $q_\nu(L, u_0)$ is the expectation value of the intensity under the condition that at the cloud edge we have $u = u_0$.

If we calculate the profile of a spectral line as given by $q_\nu(L, u_0)$, this profile will in general be asymmetric if $u_0 \neq 0$. It is relatively easy to understand this asymmetry qualitatively. With the assumptions (1) and (3) – or more accurately with the underlying assumption of a Markov process – the conditional probability of finding, at the point $s + \Delta s$, the value $v + \Delta v$ for the velocity, provided the value at $s$ is $v$, is given by (Gail et al. 1980)

$$P_\nu(s + \Delta s, v + \Delta v|s, v) = \frac{1}{\sigma \sqrt{2\pi(1-f^2)}} \exp \left\{ -\frac{[\Delta v + \tau(1-f)]^2}{2\sigma^2(1-f^2)} \right\}$$

with $f$ given by (3). It is obvious from (14) that the expectation value for $\Delta v$ is

$$\langle \Delta v \rangle = -\sigma [1 - \exp(-|\Delta s|/\ell)] ,$$

and the variance is

$$\text{Var}(\Delta v) = \sigma^2 [1 - \exp(-2|\Delta s|/\ell)] .$$

It follows that the conditional expectation value for the velocity at $s + \Delta s$ lies between zero and $v$, while the unconditional expectation value, derived with the distribution (1), is of course zero. This effect causes an asymmetry in the profile of $q_\nu(u_0) = q_\nu(L, u_0)$. The dependence of this asymmetry on the various parameters will be discussed in the next section. Only after performing the integration in (11) does the profile become symmetric.

Since $q_\nu(u_0)$ is an (conditional) expectation value, an observed spectrum may still deviate from the theoretical one based on a
The distribution function for directly. To study these deviations one has to determine the cumulative distribution function of the conditional intensity. This can be done as described before, i.e. by computing higher moments \( q_{\lambda} \). The distribution function for \( q_{\lambda}(L, u_0) \) is narrower than that for \( I_q(L) \). This is due to the fact that in considering \( q_{\lambda}(L, u_0) \) we assume that, in addition to the general statistical properties of the velocity field, \( u_0 \) is also known.

An alternative approach to studying the possible deviations of an observed spectrum from the expectation value would be to calculate spectra that correspond to individual realizations of the velocity field. Such an approach, based on a Monte Carlo technique, is followed in an accompanying paper by Levshakov, Kegel & Mazets (1997).

3 NUMERICAL RESULTS

Considering only pure absorption (\( S_\lambda = 0 \)), we have calculated line profiles \( q_{\lambda}(u_0) \) for different parameter sets (using the techniques outlined in LK94) in order to illustrate the effects discussed above.

To be specific, we performed these calculations for the H\textsc{i} Ly\alpha line which is frequently observed in interstellar and QSO absorption spectra. The results may be scaled, however, to any other line. As in LK94, we considered a cloud model which – aside from the turbulent velocity field – is homogeneous. The free parameters specifying the model are the column density \( N_{\text{H}I} \), the ratio of the cloud thickness \( L \) to the correlation length \( l \), the dispersion of the turbulent velocity \( \sigma_\lambda \), and the kinetic temperature \( T_{\text{kin}} \). (If one chooses as abscissa \( \Delta \lambda \) in units of the Doppler width instead of the absolute wavelength, one has to specify instead of the last two parameters only the ratio \( \sigma_\lambda/v_{\text{th}} \).) In addition to the parameters specifying the general cloud model, we have to specify \( u_0 \), the hydrodynamical velocity at the cloud edge for one particular line of sight. For all calculations we choose \( T_{\text{kin}} = 10^4 \) K. All other parameters are varied over a wide range.

Fig. 1 shows our simulated spectra for \( N_{\text{H}I} = 10^{12} \) cm\(^{-2} \), \( \sigma_\lambda/v_{\text{th}} = 2 \), \( u_0 = 1 \) and 5 and various values of \( L/l \). In order to illustrate the differences, the figure shows beside the profiles of the conditional intensity \( q_{\lambda} \) those for the unconditional intensity \( I_q \) (solid and dashed lines, respectively). \( N_{\text{H}I} = 10^{12} \) cm\(^{-2} \) corresponds to an optically thin cloud. We see that for moderate values of \( L/l \) the two profiles differ substantially: \( q_{\lambda}(u_0) \) is shifted with respect to \( I_q \) and the profile is asymmetric. For large values of \( L/l \) the two profiles are identical, as expected.

Figs 2 and 3 show the corresponding results for \( N_{\text{H}I} = 10^{14} \) and \( 10^{16} \) cm\(^{-2} \), respectively. Since optical depth effects are non-linear, the relative line profiles change with increasing column density. Furthermore, for large column densities the profile of \( q_{\lambda}(u_0) \) may differ from that of \( I_q \) even for rather large values of \( L/l \).

In Figs 1–3 we varied \( N_{\text{H}I} \) and kept \( \sigma_\lambda/v_{\text{th}} \) constant. In Figs 4 and 5 we keep \( N_{\text{H}I} \) constant \( (10^{12} \) cm\(^{-2} \) \) and vary \( \sigma_\lambda/v_{\text{th}} \) to show the dependence of the asymmetries on this parameter.

\[ u_0 = 1, \quad \sigma_\lambda/v_{\text{th}} = 2, \quad N_{\text{H}I} = 10^{16} \text{ cm}^{-2} \]

\[ u_0 = 5 \]

\[ \sigma_\lambda/v_{\text{th}} = 2, \quad N_{\text{H}I} = 10^{16} \text{ cm}^{-2} \]
Figure 4. Synthetic mesoturbulent conditional (\( q \), solid lines) and unconditional (\( I \), dashed lines) spectra of H\textsc{I} Ly\( \alpha \) as functions of \( \sigma /v_{\text{th}} \) and \( U_0 \). \( T_{\text{kin}} = 10,000 \) K, \( N_{\text{HI}} = 10^{15} \) cm\(^{-2} \), \( U_0 = 1 \).

We emphasize that our model cloud has been assumed to be homogeneous. The asymmetries come about by considering the conditional intensity \( q_u(U_0) \) with \( U_0 \neq 0 \), reflecting the fact that in interstellar (or quasar) absorption spectra we observe only a single line of sight. The asymmetries shown in Figs 1–5 are such that if they occurred in observed data, they would be interpreted as blends, i.e. being caused by several (independent) cloudlets. However, our computations refer to a single cloud with a constant density and kinetic temperature and continuous variations of \( \nu(s) \) along the line of sight. Therefore the only reasons for this 'blending' effect are the finite thickness of the absorbing cloud and the frozen structure of the stochastic velocity field over the exposure time. So when \( U_0 \neq 0 \) the absorption line profiles are skew, in general.

The second main topic of this paper is the calculation of the confidence regions for the conditional expectation value \( q_u(U_0) \) which are important for the interpretation of data of very high resolving power when 'turbulent noise' becomes pronounced. This noise should affect strongly those points in the line profile where the absorption coefficient has its steepest gradient, whereas points in the wings and in the saturated cores should be less affected by the turbulence (see also Traving 1975). The
latter is to be expected since, in the optically thin limit and for very large optical depths, the microturbulent and mesoturbulent solutions of the radiative transfer equation tend to coincide; thus for a given \( \sigma \) the variance

\[
\text{Var}(q) = \sigma_{q,2} - \sigma_{q,1}^2
\]  

vanishes in the microturbulent limit. This means that the probability density function for \( q \) tends to the Dirac delta function in the case of microturbulence (see Appendix A). Therefore for \( q \), of the order of unity or for \( q \ll 1 \), the cumulative distribution function is the unit-step function.

Let us now consider the points in the flank of the line profile where the conditional intensity \( q \) has a steep slope. In this case the corresponding probability density function is asymmetric and bell-shaped, or J-shaped, or U-shaped (see examples shown in Fig. A3 in Appendix A).

Once the cumulative distribution function \( \Psi_\lambda \) has been determined, and some quantity \( \alpha < 1 \) is given, one may find some interval \( x_1(q) \leq x \leq x_2(q) \) for which we have

\[
\Psi_\lambda(x_2) - \Psi_\lambda(x_1) = \int_{x_1}^{x_2} \varphi_\lambda(x) \, dx = \alpha .
\]  

Thus, for a given value of \( \alpha \), a confidence region can be found for which the chance \( \mathcal{P}(x_1 < q < x_2) \) has the value \( \alpha \). It is clear that the confidence region is not defined uniquely. One may choose \( x_1 \) arbitrarily (within certain limits) and determine \( x_2 \) so as to fulfil...
$u_0 = 1, \quad \sigma/\nu_{th} = 2, \quad N_{HI} = 10^{15} \text{ cm}^{-2}$

$u_0 = 5$

**Figure 6.** Conditional intensity spectra (solid lines) and confidence regions at the level of 90 per cent (restricted by dotted lines) for H I Lyα as functions of $u_0$ and $L/l$. $T_{kin} = 10000$ K, $N_{HI} = 10^{15}$ cm$^{-2}$, $\sigma/\nu_{th} = 2$, and $u_0 = 1$ and 5 for the upper and lower panels, respectively.

Condition (18). To solve this problem we have used an additional condition: a confidence belt must be of the lowest thickness.

Fig. 6 demonstrates our results for $N_{HI} = 10^{15}$ cm$^{-2}$, $\sigma/\nu_{th} = 2$, $u_0 = 1$ and 5 and various values of $L/l$. Dotted lines restrict confidence regions for $\alpha = 0.9$. We see that one may expect very strong intensity fluctuations in the case of moderate $L/l$ values, the highest uncertainties occurring in the steepest slopes of the line profile where intensity fluctuations from $q_A = 0$ to $q_A = 1$ may be found in the idealized case of an infinitely high spectral resolution observation. If, however, the resolving power is not sufficiently high, these fluctuations will be smeared out and the expectation value of the conditional intensity will be more appropriate to describe observations. Under this condition, a rough estimate of the actual confidence interval $\Delta x_c = x_c(q_{A_1}) - x_c(q_{A_2})$ at a given $\lambda$ may be obtained by dividing $\Delta x_c$ by $\sqrt{M}$, where $M$ is the number of points used in the convolution interval (it is approximately equal to the ratio of the width of the instrumental function to the correlation length of the intensity fluctuations in $\lambda$-space; see Appendix B). The instrumental function of the spectrograph is to act as a filter to smooth fluctuations of the turbulent noise. We may imagine, however, that there remain small fluctuations in the profile. It is difficult to assess by taking the observational data from a single line of sight whether these remaining fluctuations are caused by separated cloudlets or are just a low-frequency component of the turbulent noise which could not be smoothed. It is therefore worth emphasizing that in general absorption line observations do not give the exact value of $\tau_A$, thus the exact column density of the absorbing atoms is a function of $u$. The straightforward calculation of the column density from the measured value of $\tau_A$ is possible only in the special case of a completely uncorrelated velocity field when $\tau_A = k_0(\Phi)N_A$, otherwise one can only calculate the probability $P[N_A < N_A(v) < N_A] = \alpha$.

### 4 Discussion

Besides the total column density $N_A$, the fundamental parameters of the theory are (i) $\tau_A$, the ratio of the correlation length to the mean free path of photons of wavelength $\lambda$ at $u = 0$, (ii) $\sigma/\nu_{th}$, the ratio of the rms turbulent velocity to the thermal velocity, and (iii) $u_0$, the value of $u$ at the cloud edge. The variations of these dimensionless parameters within certain intervals give rise to a variety of absorption line profiles shown in Figs 1–5. Their main characteristic is asymmetry which leads to the shifted line centres of gravity with respect to the unconditional intensity profiles (h). Figs 4 and 5 show that these relative shifts are very sensitive to the second fundamental parameter $\sigma/\nu_{th}$. For example, at moderate values of $L/l \sim 3$ and $u_0 \sim 1$ the conditional intensity profiles shown in Figs
Table 1. Interstellar lines from the 281 km s$^{-1}$ absorption system towards SN1987A (Blades et al. 1988).

| Atom | $\lambda_{\text{obs}}$, Å | $\lambda_{\text{abs}}$, Å | $v_r$ | $(v_r)$ | $\xi$ |
|------|----------------|----------------|------|--------|------|
| $^{12}$C$^0$ | 1350.07 | 1328.83 | 279 | 277.2 | 1.0 |
| | 1346.75 | 1326.30 | 277 | | |
| | 1658.48 | 1656.92 | 281 | | |
| | 1657.78 | 1656.26 | 274 | | |
| | 1562.87 | 1561.43 | 275 | | |
| $^{24}$Mg$^0$ | 2858.74 | 2852.12 | 279 | 279 | 1.4 |
| | 1856.44 | 1854.71 | 279 | 278.5 | 1.5 |
| | 1864.52 | 1862.79 | 278 | | |
| $^{28}$Si$^+$ | 1395.06 | 1393.75 | 281 | 278.5 | 1.5 |
| | 1404.06 | 1402.77 | 276 | | |
| $^{32}$S$^+$ | 1255.00 | 1253.81 | 284 | 284 | 1.6 |
| | 1348.48 | 1347.24 | 276 | 276 | 1.7 |
| $^{55}$Cr$^+$ | 2057.54 | 2055.59 | 284 | 290.0 | 2.1 |
| | 2067.50 | 2065.46 | 296 | | |
| $^{57}$Fe$^+$ | 2578.59 | 2576.10 | 289 | 288.0 | 2.1 |
| | 2596.22 | 2593.73 | 281 | | |
| $^{39}$Fe$^+$ | 2068.19 | 2065.69 | 287 | | |
| | 2063.97 | 2062.01 | 284 | | |
| $^{56}$Fe$^+$ | 2063.97 | 2062.01 | 284 | 284 | 2.3 |

Notes: $\xi = \left[a/(v_u)_{\text{(atom)}}/a/(v_u)_{\text{(Cr)}} \right]$, $v_r$ and $(v_r)$ are in km s$^{-1}$.

4 (a), (b), (i) and (j) have velocity shifts $\Delta v$ of about 25 km s$^{-1}$; the corresponding value is $\Delta v \sim 100$ km s$^{-1}$, if $v_u \sim 5$ [compare Figs 5(a) and (b) with 5(i) and (j)]. It follows that absorption lines of different species located in the same volume of space may show different radial velocities since each atom has its individual value of $a/(v_u)$.

One may prove the existence of this effect by comparing radial velocities $v_u$ of different species with preferably large differences of the atomic masses $M_A$. If correlation effects are of importance one should find a tendency for the heavier species to be shifted by a few km s$^{-1}$ with respect to the light species. As an example, let us consider the spectral atlas of the UV lines towards SN1987A in the Large Magellanic Cloud (Blades et al. 1988), where radial velocities of unblended absorption lines were determined with an accuracy of about 1.5–2 km s$^{-1}$. The authors identified 10 different velocity groups along this line of sight. The dominant LMC component (consisting of C, O, Mg, Al, Si, S, Cl, Cr, Mn, Fe, Ni and Zn absorption lines) occurs at about 281 km s$^{-1}$. We list unblended lines from this absorption system in Table 1. We use only unblended lines to minimize possible systematic shifts caused by rarely resolved (in accord with the authors’ designations) absorption features. In column 5 of Table 1, the mean values of radial velocities for each atom are given, and column 6 lists relative values of $a/(v_u)_{\text{(atom)}}$ with respect to $a/(v_u)_{\text{(Cr)}}$ = parameter $\xi$. These columns show that indeed there appears to be a weak dependence of $v_r$ on $\xi$; the mean value of the radial velocity for the group of atoms from C to Cl is $v_r = 279$ km s$^{-1}$, and that for the group from Cr to Zn is $v_r = 287$ km s$^{-1}$.

This is an indication that the physical effects discussed before are of relevance for the interpretation of interstellar lines. We are aware, however, that our model is rather idealized. A detailed comparison of theoretical and observational results requires observational data of higher quality.

5 CONCLUSIONS

We have described a method of calculating the shape of absorption lines formed in an intervening turbulent cloud. The random velocity field is assumed to be correlated in general. For real interstellar (or quasar) absorption line spectra it appears to be appropriate to regard the turbulent structure of the cloud as frozen over the exposure time, and the observed profile to be the result of a point-like light source. Several examples of simulated H I Ly$\alpha$ profiles are presented. Comparing these profiles with those based on the averaging procedure over the complete statistical ensemble, we find large deviations. Our main results may be summarized as follows.

(i) We have advanced the theory of line formation in turbulent intervening clouds for the case of a point-like light source.

(ii) It is shown that in general (when the correlation length $l \neq 0$) the absorption line profile is symmetric (skew). The extent of the asymmetry depends sensitively on turbulent characteristics of the gas and especially on the parameter $v_u$ – the hydrodynamic velocity at the cloud edge in units of $v_u$.

(iii) The asymmetry leads to a shift of the line centre of gravity which is different for different species. This effect is illustrated by a sample of data from the 281 km s$^{-1}$ absorption system towards SN1987A, where a shift of about 8 km s$^{-1}$ between radial velocities of light atoms (from C to Cl) and heavy atoms (from Cr to Zn) has been revealed.

(iv) In Appendix A we describe how to restore the cumulative distribution function of the intensity $I_\lambda$ through its first moments. This technique has been used to calculate the confidence regions for $q_\lambda$. It is found that in high-resolution observational data very large intensity fluctuations ($\theta < q_\lambda < 1$) may be observed in the line profile where the slope is steepest. These fluctuations are caused by the 'turbulent noise' which prevents an accurate measurement of the column densities from the observed values of $q_\lambda$. In general, one can only estimate the probability that $N_\lambda$ lies within certain limits.

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APPENDIX A: RESTORING THE DISTRIBUTION THROUGH ITS MOMENTS

Let $x_\lambda$ denote the observed normalized intensity $x_\lambda = I_\lambda / I_0$. In Section 2 we have shown that any moment of $P(s; v, x_\lambda)$ with respect to $x_\lambda$ can be found by solving the system (12) of coupled differential equations in which $j$ is an integer ($j = 0, 1, 2, \ldots$), and $Q_\lambda(s, v) = 1$.

It follows that the moment of order $j$ for the unconditional intensity is given by

$$
\langle x_\lambda^j(s) \rangle = \int_{-\infty}^{\infty} Q_\lambda(s, v) \, dv .
$$

(A1)

To simplify notations we define $x = x_\lambda$, the intensity within the line profile at wavelength $\lambda$; $\theta(x)$, the probability density function (p.d.f.) and $\Theta(x)$, the cumulative distribution function (c.d.f.) for the unconditional intensity.

If one compares the unconditional intensity with observations, then

$$
\theta(x) = \int_{-\infty}^{x} P(s; v, x) \, dv ,
$$

(A2)

and

$$
\Theta(x) = \int_{0}^{x} \theta(t) \, dt .
$$

(A3)

If we consider, however, interstellar absorption lines, it will be more appropriate to compare with observations the conditional intensity $q_\lambda(s, u)$, as pointed out in Section 2. To calculate moments for the conditional intensity, similar differential equations can be used. With the assumption $S_\lambda = 0$, for the case of pure absorption

$$
N_{\text{HI}} = 10^{15} \, \text{cm}^{-2}, \quad \sigma_\lambda / v_{\text{th}} = 1, \quad L/l = 1
$$

Figure A1. HI Lyα data: (a) first four moments for the unconditional intensity; (b) the measures of the skewness of the distribution (the asymmetry $As$ and the excess $Ex$) as functions of wavelength; (c) an example of the probability density function for $\lambda = 1215.80$ Å; (d) the corresponding cumulative distribution functions $\Theta$ and $\Theta_1$ (continuous and step-like curves, respectively).
we have
\[ \frac{\partial q_{s}}{\partial s} = \frac{1}{i} \left( \frac{\partial^{2} q_{s}}{\partial u^{2}} - u \frac{\partial q_{s}}{\partial u} \right) - j\phi q_{s}, \]  
(A4)

where
\[ q_{s}(s, u) = Q_{s}(s, u)/W_{s}(u). \]  
(A5)

In this case the conditional probability density function has the form
\[ \psi(x) = P(s; u, x)/W_{s}(u), \]  
(A6)

and the conditional cumulative distribution function is given by
\[ \Psi(x) = \int_{0}^{x} \psi(t) \, dt. \]  
(A7)

At a given column density and kinetic temperature we have, for the normalized intensity \( x \), the following inequality:
\[ 0 < \xi \leq x \leq 1, \]  
(A8)

with
\[ \xi = \exp\{-\kappa_{0}L\Phi(0)\}. \]  
(A9)

Therefore the distribution functions \( \Psi(x) \) and \( \Theta(x) \) have to be exactly zero for \( 0 \leq x < \xi \). Since we know the distribution function in the interval \( 0 \leq x < \xi \), it appears more appropriate to consider for the restoring procedure only the interval \( \xi \leq x \leq 1 \). We normalize this interval by transforming the stochastic variable \( x \) to
\[ y = \frac{x - \xi}{1 - \xi}. \]  
(A10)

From the condition
\[ \dot{\psi}(y)dy = \psi(x)dx \]  
we find with (A10)
\[ \dot{\psi}(y) = (1 - \xi)\psi(x). \]  
(A11)

In order to restore the distribution function from its moments, we now need the moments with respect to the \( y \)-system, while those computed from (A1) and (A5) refer to the \( x \)-system. The relation between the two sets of moments is obtained by observing that
\[ m_{j} = \int_{0}^{1} x^{j} \psi(x) \, dx = \int_{0}^{1} x^{j} \psi(y) \, dy. \]  
(A13)

(A13) implies that
\[ m_{j} = \sum_{i=0}^{j} (-1)^{i} C_{j} (1 - \xi)^{j-i} \hat{m}_{j-i} \xi^{i}, \]  
(A14)

or
\[ \hat{m}_{j} = \frac{1}{(1 - \xi)^{j}} \sum_{i=0}^{j} (-1)^{i} C_{j} m_{j-i} \xi^{i}, \]  
(A15)

where \( C_{j} = j! / (j-i)! \) (binomial coefficients), and \( m_{0} = \hat{m}_{0} = 1 \).

\[ N_{HI} = 10^{15} \text{ cm}^{-2}, \quad \sigma_{\nu/v_{\text{th}}} = 2, \quad L/l = 100 \]

\[ \sigma_{\nu/v_{\text{th}}} = 2, \quad L/l = 100 \]

\[ \lambda = 1215.92 \text{ Å} \]

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Figure A2. As Fig. A1 but for different \( \sigma_{\nu/v_{\text{th}}} \) and \( L/l \); (c) and (d) are shown for \( \lambda = 1215.92 \text{ Å} \).
A set of moments $m_1, m_2, \ldots$ may be found for the case of pure absorption either from (12) for the unconditional intensity or from (A4) for the conditional intensity.

Figs A1(a) and A2(a) show the first four moments for the HI Ly$\alpha$ half-profiles (the origin corresponds to the line centre $\lambda_0 = 1215.67$ Å) which were calculated from (12) and (A1). For all calculations we used $T_{\text{kin}} = 10^4$ K; the values of the other parameters are shown at the top of the figures. The first four moments may be used to characterize the skewness of the distribution. The usual measures of the skewness are the asymmetry, $As$, and the excess, $Ex$,

$$As = \frac{\mu_3}{\mu_2^2} \quad \text{and} \quad Ex = \frac{\mu_4}{\mu_2^3} - 3,$$  \hfill (A16)

which vanish if the distribution is symmetrical. Here $\mu_j$ are the central moments defined by

$$\mu_j = \int (x - m_1)^j \psi(x) \, dx.$$  \hfill (A17)

These characteristics of the skewness for the case of Ly$\alpha$ are shown in Figs A1(b) and A2(b). As expected, the probability density functions for the essential part of the profile are asymmetrical (they tend to become singular when $y_0 \to 0$ or $y_0 \to 1$).

Let us now consider the restoring procedure itself. We will restore the conditional cumulative distribution function through its first moments. The random variable $y$ lies in the finite interval $[0, 1]$. This implies that the set of moments determines the distribution uniquely (see e.g. Kendall & Stuart 1963). One may define a general type of the distribution by using Pearson's criterion. If, for example,

$$\beta_1(\beta_2 + 3)^2 > 0$$  \hfill (A18)

with $\beta_1 = As^2$, and $\beta_2 = Ex + 3$, then the distribution defined on the finite interval belongs to Pearson's Type I (Kendall &

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**Figure A3.** HI Ly$\alpha$ data: (a) and (b) different types of unconditional probability density functions as one goes from the line centre to the wing; (c) the corresponding unconditional cumulative distribution functions.
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Stuart 1963):

\[ \tilde{\psi}(y) = \frac{1}{B(\gamma_1, \gamma_2)} y^{\gamma_1-1} (1-y)^{\gamma_2-1} , \]  

(A19)

with \( \gamma_1 > 0, \gamma_2 > 0 \). Here \( B(\gamma_1, \gamma_2) \) is the complete beta function (also known as Euler’s integral of the first kind):

\[ B(\gamma_1, \gamma_2) = \int_0^1 t^{\gamma_1-1} (1-t)^{\gamma_2-1} \, dt . \]  

(A20)

It is related to the gamma function through

\[ B(\gamma_1, \gamma_2) = \frac{\Gamma(\gamma_1) \Gamma(\gamma_2)}{\Gamma(\gamma_1 + \gamma_2)} . \]  

(A21)

In all our examples we found (A18) to be satisfied. So we may assume that a rough approximation to the exact distribution of \( y \) is the curve of Pearson’s Type I, i.e. the cumulative distribution function may be approximated by the incomplete beta function

\[ \tilde{\psi}(y) = \frac{B(\gamma_1, \gamma_2; y)}{B(\gamma_1, \gamma_2)} , \]  

(A22)

where

\[ B(\gamma_1, \gamma_2; y) = \int_0^y t^{\gamma_1-1} (1-t)^{\gamma_2-1} \, dt , \quad 0 \leq y < 1 . \]  

(A23)

This implies in particular that at a given \( \lambda \) within the line profile

\[ \bar{m}_1 = \frac{\gamma_1}{\gamma_1 + \gamma_2} , \quad \bar{m}_2 = \frac{\gamma_1(\gamma_1+1)}{(\gamma_1+\gamma_2)(\gamma_1+\gamma_2+1)} , \]  

(A24)

i.e. the first two moments \( \bar{m}_1 \) and \( \bar{m}_2 \) define uniquely the parameters \( \gamma_1, \gamma_2 \) of the distribution:

\[ \gamma_1 = \frac{\bar{m}_1 (\bar{m}_1 - \bar{m}_2)}{\bar{m}_2} , \quad \gamma_2 = \frac{(\bar{m}_1 - \bar{m}_2)(1 - \bar{m}_1)}{\bar{m}_2} . \]  

(A25)

If both \( \gamma_1 \) and \( \gamma_2 \) exceed unity, the probability density function \( \tilde{\psi}(y) \) is bell-shaped with its maximum at \((\gamma_1 - 1)/(\gamma_1 + \gamma_2 - 2), 0 \) and \( \gamma = 1 \). If either \( \gamma_1 \) or \( \gamma_2 \) is between 0 and 1, \( \tilde{\psi}(y) \) is infinite at \( y = 0 \) or \( y = 1 \), respectively, and the distribution is J-shaped. If \( \gamma_1 \) and \( \gamma_2 \) are both between 0 and 1, \( \tilde{\psi}(y) \) is infinite at both limits, and the curve is U-shaped.

These different types of distribution are shown in Figs A1(c), A2(c), and A3(a) and (b). The J-shaped distributions occur for the points in the line profile where \( y \sim 0 \) or \( y \sim 1 \), i.e. at the terminals where \( \tilde{\psi}(y) \) tends to become singular. The U-shaped probability distribution functions occur (according to our calculations) for the points in the line profile where the absorption coefficient has its steepest slope. In this wavelength range a small change in the random Doppler shift leads to a large shift in the intensity, making it more probable to find the intensity either at large values (near unity) or at very low values, rather than at the value corresponding to the expectation value. This tendency of the probability density to be concentrated at the end-points of the interval is reflected in the U-shaped curves.

The next step in the calculation of the exact probability density function for \( y \) is to expand \( \tilde{\psi}(y) \) into a series

\[ \tilde{\psi}(y) = \omega(y) \sum_{j=0}^n a_j G_j(y) , \]  

(A26)

where the \( G_j \) are the Jacobi polynomials \((G_0 = 1, a_0 = 1)\) which are orthogonal on the interval \((0, 1)\) with respect to the weight function \( \omega(y) \). This weight function is chosen to be the density of the beta distribution (A22):

\[ \omega(y) = y^{\gamma_1-1}(1-y)^{\gamma_2-1} \]  

(A27)

The Jacobi polynomials are defined by (see e.g. Abramowitz & Stegun 1964)

\[ G_j(y) = \sum_{i=0}^j (-1)^i C_i^j g_{i,j} y^i , \]  

(A28)

where \( g_{i,j} = \int_0^1 y^i (1-y)^{\gamma_1 + \gamma_2 - 1} \, dy \). \( G_j(y) \) and integrating over the range \((0, 1)\) leads to

\[ a_k = \frac{1}{h_k} \sum_{j=0}^k (-1)^j C_j^k g_{j,k} h_i , \]  

(A30)

where

\[ h_k = \int_0^1 \omega(y) G_k^2(y) \, dy = \frac{h_k^2}{B(\gamma_1, \gamma_2)} \]  

(A31)

It is easy to show that if the weight function is of the form (A27), the coefficients \( a_1 \) and \( a_2 \) are zero. We further define an approximation to \( \tilde{\psi}(y) \) of order \( n \) as

\[ \tilde{\psi}_n(y) = \omega(y) \sum_{j=0}^n a_j G_j(y) . \]  

(A32)

So the lowest order approximation is the weight function \( \omega(y) \). [This procedure was originally suggested by Durbin & Watson (1951).] In our calculations we used approximations of order \( n = 4 - 6 \).

(A30) allows us to determine \( a_k \) if the moments \( \bar{m}_j \) are known. Inserting (A32) with (A30) into

\[ \tilde{m}_j^* = \int_0^1 \tilde{\psi}_n(y) \, dy \]  

(A33)

gives a new set of moments \( \tilde{m}_j^* \) for any order \( j \) \((j = 1, 2, \ldots, n, n + 1, \ldots)\) in terms of the \( a_k \):

\[ \tilde{m}_j^* = \frac{1}{B(\gamma_1, \gamma_2)} \sum_{k=0}^n a_k \sum_{i=0}^k (-1)^i C_i^j g_{i,j} \beta(j + \gamma_1, \gamma_2) . \]  

(A34)

To check the validity of the approximation for a given value of \( n \), we can compare the \( \tilde{m}_j^* \) with the \( \bar{m}_j \) as determined from (A15). We find that the relative errors \( \tilde{m}_j^* - \bar{m}_j/\bar{m}_j \) are less than 1 per cent for \( j = 8 - 10 \), if \( n = 4 - 6 \).

Using (A27)–(A32), we finally obtain, after some algebra,

\[ \tilde{\psi}_n(y) = \frac{1}{B(\gamma_1, \gamma_2)} \sum_{j=0}^n a_j \sum_{i=0}^k (-1)^i C_i^j g_{i,j} \beta(j + \gamma_1, \gamma_2) . \]  

(A35)

The same restoring procedure can be used to calculate the unconditional distribution function \( \tilde{\psi}(y) \), of course. There is a possibility of testing this distribution function independently. One may determine the distribution through a step function \( \tilde{\psi}(y) \). This step function is determined by \( 2i \) constants (which are the first \( 2i \) moments of the distribution): the \( i \) abscissae of the steps and the \( i \) heights of the jumps. It may be proven (see von Mises 1964) that if \( 2i \) quantities are such that an \( i \) step function exists with these moments, then either \( \tilde{\psi}(y) = \tilde{\psi}_i(y) \), or \( \tilde{\psi}(y) \) intersects the step
line in 2r − 1 points, one on each horizontal and one on each vertical segment.

We have calculated the step functions Θk(y) using the technique described by von Mises (1964). Two examples of these calculations are shown in Figs A1(d) and A2(d) for the points λ = 1215.80 and 1215.92 Å, respectively. Different types of cumulative distribution functions Θ(y) are shown in Fig. A3(c) for various points in the line profile as one goes from the line core (the upper curve) to the wing (the lower curve). Similar curves were used to calculate the confidence regions as described in Section 3.

Finally we note that in the microturbulent limit (l → 0) the probability density function may be determined analytically. In this limit the solution of equation (5) with $S_{A} = 0$ yields (see Gail et al. 1974)

$$\lim_{l \to 0} (I_{A}) = I_{0} \exp(-\langle \tau_{A} \rangle),$$  \hfill (A36)

with

$$\langle \tau_{A} \rangle = \kappa_{0} L \int \Phi(\lambda, v) W_{1}(v) \, dv.$$  \hfill (A37)

Correspondingly, we obtain from (12), with $S_{A} = 0$,

$$\lim_{l \to 0} (I_{A}) = I_{0} \exp(-j(\tau_{A})).$$  \hfill (A38)

$$= (\lim_{l \to 0} (I_{A}))^{j}.$$  \hfill (A39)

The relation (A39) implies that the corresponding distribution function is the Dirac δ-function:

$$\theta(x) = \delta(x - \langle x \rangle).$$  \hfill (A40)

In the limit $l \to \infty$ (macroturbulence) one may in principle also derive an analytic expression for $\theta(x)$. This, however, is rather complicated since it contains the inverse of the profile function $\Phi(\lambda, v)$.

**APPENDIX B: THE CORRELATION FUNCTION OF THE INTENSITY FLUCTUATIONS**

The fluctuations of the intensity around the expectation value are characterized by the distribution function for the intensity at a given wavelength and by the correlation function as function of $(\lambda_{1}, \lambda_{2})$. The correlation function $C(\lambda_{1}, \lambda_{2})$ for the intensity fluctuations is defined by

$$C(\lambda_{1}, \lambda_{2}; u) = \langle \langle I_{\lambda_{1}} - q_{A}(u)I_{\lambda_{2}} - q_{A}(u) \rangle \rangle_{u} \rangle,$$  \hfill (B1)

where the expectation value is to be taken for a given value of $u_{0}$.

It should be noted that, since $q_{A}$ depends on $\lambda$, $C$ depends on $\lambda_{1}$ and $\lambda_{2}$ and not only on $\Delta \lambda$ as in the case of a homogeneous background. With the definitions (A14) and (A17) for the moments, (B1) may be written in the form

$$C(\lambda_{1}, \lambda_{2}; u) = \frac{\langle I_{\lambda_{1}}I_{\lambda_{2}} \rangle_{u} - q_{A}(u)q_{A}(u)}{\sqrt{\langle I_{\lambda_{1}}^{2} \rangle_{u} - q_{A}(u)q_{A}(u)}}.$$  \hfill (B2)

Thus the only new quantity to be calculated is $\langle I_{\lambda_{1}}I_{\lambda_{2}} \rangle_{u}$. For pure absorption we have

$$\frac{d(I_{\lambda_{1}}I_{\lambda_{2}})}{ds} = (-\kappa_{A} + \kappa_{A})I_{\lambda_{1}}I_{\lambda_{2}}.$$  \hfill (B3)

We may quote along the same line as did Gail et al. (1974), i.e. we may define the joint probability $P_{c}(s|v, I_{\lambda_{1}}, I_{\lambda_{2}})$ for finding at $s$ the values $v$ for the velocity and $I_{\lambda_{1}}, I_{\lambda_{2}}$ for the product of the intensities at $\lambda_{1}$ and $\lambda_{2}$. For $P_{c}$ one than derives a Fokker–Planck equation analogous to equation (4). Introducing the quantity $q_{c}$ defined by

$$q_{c}(\lambda_{1}, \lambda_{2}; u)W_{1}(u) = \int_{0}^{\infty} (I_{\lambda_{1}}I_{\lambda_{2}})P_{c} \, d(I_{\lambda_{1}}I_{\lambda_{2}}),$$  \hfill (B4)

one obtains

$$\frac{d q_{c}}{ds} = \frac{1}{l} \left( \frac{\partial^{2} q_{c}}{\partial u^{2}} - u \frac{\partial q_{c}}{\partial u} \right) - \kappa_{0}[\Phi(\lambda_{1}) + \Phi(\lambda_{2})]q_{c}.$$  \hfill (B5)

(B5) corresponds to equation (A4) for the second moment, the essential difference being that instead of $2\Phi(\lambda, u)$ one has to take in the last term on the right-hand side the sum of the profile at two different wavelengths. According to the derivation we have

$$q_{c}(\lambda_{1}, \lambda_{2}; u_{0}) = \langle I_{\lambda_{1}}I_{\lambda_{2}} \rangle_{u_{0}}.$$  \hfill (B6)

In writing (B1) we considered intensity fluctuations with respect to $q_{A}$. One may, of course, also define fluctuations with respect to $I_{\lambda_{1}}$. In this case one has to use in (B1) the unconditional expectation value, and we have

$$\langle I_{\lambda_{1}}I_{\lambda_{2}} \rangle = \int q_{c}(u)W_{1}(u) \, du.$$  \hfill (B7)

As an illustration Fig. B1 shows the autocovariance for the Lyα line for $N_{H} = 10^{14}$ cm$^{-2}$, $T_{\text{kin}} = 10000$ K, $\sigma / v_{\text{th}} = 2$, $u_{0} = 5$, and

\footnote{The autocovariance is given by the numerator of (B2). We show the autocovariance instead of the correlation function $C$ for numerical reasons. As $q_{A}$ approaches unity, $\mu_{2}(\lambda_{2})$ – which appears in the denominator of (B2) – goes to zero.}

![Figure B1. Autocovariance of the intensity fluctuations of the H1 Lyα line (for details, see text).](https://www.astro-ph.org/figs/a0288006.jpg)
different values of $L/l$ (corresponding to Figs 2e–g). For $\lambda_1$ we chose that value of $\lambda$ at which $(\lambda_0 - q_{\lambda})$ reaches half its maximum value at the blue side of the line. In Fig. B1, $\lambda_1$, which is kept fixed for each curve, corresponds to the wavelength at which the autocovariance has its maximum. This maximum corresponds to the second central moment $\mu_2(\lambda_1)$. From Fig. B1 we see that the autocovariance becomes smoother and its amplitude decreases as $l$ decreases. In the microturbulent limit ($l \rightarrow 0$), the amplitude of the fluctuations, of course, goes to zero, which formally follows from

$$\lim_{l \rightarrow 0} \mu_2(\lambda_1) = (q_{\lambda_1})^2.$$  \hfill (B8)

(B8) corresponds to (A39).

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