The Coppersmith-Tetali-Winkler Identity for Mechanical Systems

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Abstract

Using a mixture of linear algebra and statics, we derive what can be viewed as a slight generalization of the Coppersmith-Tetali-Winkler Identity $H(i, j) + H(j, k) + H(k, i) = H(j, i) + H(k, j) + H(i, k)$ for hitting times of a random walk.

1 Linear Algebra

In this paper, the bold-faced characters are reserved for denoting vectors; the dimension of the vectors is $n \geq 3$, except when the context suggests otherwise. For example, $f$ is a vector with coordinates $f_1, \ldots, f_n$, while $0$ and $1$ are the all-zero and all-one vectors, respectively.

Consider the system of linear equations

$$\begin{equation}
Ax = f
\end{equation}$$

where $x$ is the vector of the unknowns, $f$ is a fixed vector, and $A$ is an $n \times n$ matrix such that:

(i) $A$ is symmetric;

(ii) the off-diagonal entries of $A$ are non-positive;

(iii) $A$ is irreducible, which is to say that the simple graph whose adjacency matrix has the same off-diagonal zero pattern as $A$ is connected (throughout the paper, we refer to this graph as "the underlying graph $G$"");

(iv) $A1 = 0$.

Applied to a matrix with properties (i)–(iii), the Perron-Frobenius Theorem states that its least eigenvalue has multiplicity one, and the corresponding eigenvector can be chosen to have all its coordinates positive. Hence, by (iv), $A$ is positive semidefinite and of rank $n - 1$. Consequently, (1) has no solution if $f \cdot 1 \neq 0$ and a solution unique up to a shift by a multiple of $1$ if $f \cdot 1 = 0$. For $i \in \{1, \ldots, n\}$, set $f_i := (f_1, \ldots, f_i - f \cdot 1, \ldots, f_n)$ (so that $f_i \cdot 1 = 0$) and let $x_i = (x_{i1}, \ldots, x_{in})$ denote the [unique] solution to the system of linear equations

$$\begin{equation}
Ax = f_i
\end{equation}$$

such that $x_{ii} = 0$. Our goal is to demonstrate the identity

$$\begin{equation}
x_{12} + x_{23} + x_{31} = x_{21} + x_{32} + x_{13}.
\end{equation}$$
Suppose first that \( n > 3 \) and let us apply [symmetric] Gaussian elimination to (2), to eliminate the non-zero off-diagonal entries of \( A \) in the last column and row. The reader can easily see that the first \( n - 1 \) unknowns are not affected by such an elimination, while the upper left \( (n - 1) \times (n - 1) \) corner, \( A' \), of the matrix obtained from \( A \), and the \( (n - 1) \)-dimensional vector, \( f'_i \), whose entries are the same as the first \( n - 1 \) entries of the vector obtained from \( f_i \), have the same properties as \( A \) and \( f_i \), respectively. In other words, the system of \( n - 1 \) linear equations
\[
A'x' = f'_i
\]
with \( n - 1 \) unknowns \( x_1, \ldots, x_{n-1} \) can be considered in lieu of (2), and if \( x_i = (x_{i1}, \ldots, x_{in}) \) is a solution to (2) then \( x'_i := (x_{i1}, \ldots, x_{i,n-1}) \) is a solution to (4). Hence, in order to establish (3), we may assume that \( n = 3 \).

Still, to verify (3) for \( n = 3 \) directly is a rather tedious task. After all, one would have to solve three systems of three equations each in general. Instead, we prefer the following detour. In the next section, we will see that (3) is trivially true if the underlying graph \( G \) is the star \( K_{1,n-1} \). If \( n = 3 \) then \( G \) is either the star \( K_{1,2} \) or the triangle \( K_3 \). In the latter case, the matrix \( A \) has the form
\[
\begin{bmatrix}
\alpha_2 + \alpha_3 & -\alpha_3 & -\alpha_2 \\
-\alpha_3 & \alpha_1 + \alpha_3 & -\alpha_1 \\
-\alpha_2 & -\alpha_1 & \alpha_1 + \alpha_2
\end{bmatrix}
\]
where \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) are some positive numbers. Set \( c := \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 \) and \( \beta_i := c/\alpha_i, \) \( i \in \{1, 2, 3\} \). Then, the matrix
\[
B := \begin{bmatrix}
\beta_1 & 0 & 0 & -\beta_1 \\
0 & \beta_2 & 0 & -\beta_2 \\
0 & 0 & \beta_3 & -\beta_3 \\
-\beta_1 & -\beta_2 & -\beta_3 & \beta_1 + \beta_2 + \beta_3
\end{bmatrix}
\]
has properties (i)–(iv) and it is not difficult to verify that \( A \) is the upper left \( 3 \times 3 \) corner of the matrix obtained from \( B \) by symmetric Gaussian elimination. Hence, (3) holds for \( A \) if it holds for \( B \). Which it does, since the underlying graph for \( B \) is the star \( K_{1,3} \).

## 2 Mechanical Model

The systems (1) and (2) have the following mechanical interpretation: \( n \) masses labelled 1 through \( n \) are submerged in water, so that each mass \( j \) is subject to a (positive or negative or zero) weight, \( f_j \). In addition, every two masses \( j \) and \( k \) are connected by a rubber band of resistance \( a_{jk} = a_{kj} \geq 0 \) where \(-a_{jk}\) is the \( jk \) entry of \( A \). (A rubber band is of resistance at most \( a \geq 0 \) if and only if a force of magnitude \( a\ell \) is sufficient to stretch it to length \( \ell \).) If mass \( i \) is nailed to the origin, such a system of masses has a unique equilibrium. Let \( x_{ij} \) denote the altitude of mass \( j \) in this equilibrium. (So that \( x_{ii} = 0 \.) The fact that the system is in an equilibrium is expressed by equating the sum of the forces acting on every mass \( j \) to zero. In an equilibrium, these forces are the weight \( f_j \), the reaction \(-f \cdot 1 = -(f_1 + \ldots + f_n)\)
of the nail if \( j = i \), and the resistance forces \( a_{ij}(x_{i1} - x_{ij}), \ldots, a_{nj}(x_{in} - x_{ij}) \). In other words, the vector \((x_{i1}, \ldots, x_{in})\) is the unique solution \( x_i \) to the system (2) such that \( x_{ii} = 0 \).

Using this physical interpretation of (2), we see that (3) is trivially true if the underlying graph \( G \) is the star \( K_{1,n-1} \) with say center \( k \). This is because \( x_{ij} = x_{ik} + x_{kj} \) in this case.

This completes the proof of (3).

Remark 1: In the mechanical model, we could replace the weights \( f_i \) by arbitrary \( d \)-dimensional forces and, correspondingly, the altitudes \( x_{ij} \) by the \( d \)-dimensional position vectors. Then, the “\( d \)-dimensional generalization” of (3) would be proved coördinate-wise.

Remark 2: The identity \( x_{i_1i_2} + x_{i_2i_3} + \ldots + x_{i_{k-1}i_k} = x_{i_1i_2} + x_{i_3i_2} + \ldots + x_{i_1i_k} \) can be shown for any sequence \( i_1, \ldots, i_k \) of indices using (3); we leave this to the reader.

3 Random Walks

Let \( G \) be a connected graph on \( n \geq 3 \) vertices. To obtain a random walk out of vertex \( j \) in \( G \), one recursively builds a sequence \( i_0i_1i_2 \ldots \) of vertices, in which \( i_0 = j \) and, for \( t \geq 0 \), the vertex \( i_{t+1} \) is chosen among the neighbors of \( i_t \) with the uniform probability. The hitting time \( H(j, i) \) is the expectation of the least number \( t \) such that \( i_t = i \). (So that \( H(i, i) = 0 \).)

If \( i \neq j \) then, clearly,

\[
H(j, i) = 1 + \frac{1}{\deg(j)} \sum_{\{k : jk \in G\}} H(k, i).
\]

Multiplying both sides of (5) by \( \deg(j) \), we see that the vector \( x_i := (H(1, i), \ldots, H(n, i)) \) is a solution to the system of linear equations

\[
A x = g_i,
\]

where \( g_i \) is some vector coinciding with the vector \( f := (\deg(1), \ldots, \deg(n)) \) in all the coordinates except perhaps \( i \), and \( A \) is the difference between the diagonal matrix \( \text{diag}\{f\} \) and the adjacency matrix of \( G \). Clearly, \( A \) satisfies (i)–(iv), whence \( g_i \) must be the vector \( f_i \) obtained from \( f \) as in Section 1. Hence (3), taking in this context the form of the Coppersmith-Tetali-Winkler Identity \( \sum_{\{k : ik \in G\}} H(k, i) = 2m - \deg(i) \)

where \( 2m := \deg(1) + \ldots + \deg(n) \) is twice the number of edges in \( G \). It follows that the return time \( R(i) \), i.e. the expectation of the smallest natural \( t \) such that \( i_t = i \) in a random walk out of \( i \), is

\[
1 + \frac{1}{\deg(i)}(2m - \deg(i)) = \frac{2m}{\deg(i)}.
\]

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References

[1] D. Coppersmith, P. Tetali, and P. Winkler: Collisions among random walks on a graph, *SIAM J. Disc. Math.* 6 #3 (Aug. 1993), 363–374, also available at http://cm.bell-labs.com/who/pw