FRACTIONAL $p\&q$ LAPLACIAN PROBLEMS IN $\mathbb{R}^N$ WITH CRITICAL GROWTH

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ABSTRACT. We deal with the following nonlinear problem involving fractional $p$ and $q$-Laplacians:

$$(-\Delta)_p^s u + (-\Delta)_q^s u + |u|^{p-2}u + |u|^{q-2}u = \lambda f(u) + |u|^{q^*_s-2}u \text{ in } \mathbb{R}^N$$

where $s \in (0,1)$, $1 < p < q$, $N > sq$, $q^*_s = \frac{Nq}{N-sq}$, $\lambda > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Here the nonlocal operator $(-\Delta)_p^s$, with $1 < p < \infty$, is the fractional $p$-Laplacian which is defined, up to a normalizing constant, by

$$(-\Delta)_p^s u(x) = P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} dy$$

for any $u : \mathbb{R}^N \to \mathbb{R}$ sufficiently smooth; see [8] for more motivations on this operator.

When $s = 1$, the equation (1.1) becomes a $p\&q$ elliptic problem of the form

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = f(x, u) \text{ in } \mathbb{R}^N.$$  (1.2)

As explained in [10], the study of equation (1.2) is motivated by the more general reaction-diffusion system:

$$u_t = \text{div}(D(u)\nabla u) + c(x, u) \text{ and } D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2},$$

which finds applications in biophysics, plasma physics and chemical reaction design. In these contexts, $u$ represents a concentration, $\text{div}(D(u)\nabla u)$ is the diffusion with diffusion coefficient $D(u)$, and the reaction term $c(x, u)$ relates to source and loss processes. We recall that classical $p\&q$ Laplacian problems in bounded domains and in the whole of $\mathbb{R}^N$ have been widely investigated by many authors; see for instance [6,10,11,16,17,22,23,27] and the references therein.

On the other hand, in the last years a great attention has been devoted to the study of the fractional $p$-Laplacian operator. For instance, fractional $p$-eigenvalue problems have been considered in [19,24]. Some interesting regularity results for weak solutions can be found in [13,14,21]. Several existence and multiplicity results for problems set in bounded domains or in the whole of $\mathbb{R}^N$ have been established in [2,3,18,20,28,31,32]. For more details on fractional operators and the corresponding nonlocal problems, we refer the interested reader to [12,29].

Motivated by the above papers, in this work we are interested in the existence of nontrivial solutions for a fractional $p\&q$ Laplacian problem involving the critical exponent. To our knowledge, only one
result for fractional $p$&$q$ problems is present in literature [9]. The aim of this paper is to give a further result for this interesting class of fractional problems.

Before stating our main result, we introduce the assumptions on the nonlinearity $f$. We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(t) = 0$ for $t < 0$ and

$(f_1)$ \( \lim_{|t| \to 0} \frac{|f(t)|}{|t|^{p-1}} = 0; \)

$(f_2)$ there exists $r \in (q, q^*_s)$ such that \( \lim_{|t| \to \infty} \frac{|f(t)|}{|t|^q} = 0; \)

$(f_3)$ there exists $\theta \in (q, q^*_s)$ such that $0 < \theta F(t) \leq f(t)t$ for all $t > 0$, where $F(t) = \int_0^t f(\tau) d\tau$.

In order to find weak solutions to (1.1), we look for critical points of the following Euler-Lagrange functional $J : X \to \mathbb{R}$ defined as

$$J(u) = \frac{1}{p} \|u\|_{s,p}^p + \frac{1}{q} \|u\|_{s,q}^q - \lambda \int_{\mathbb{R}^N} F(u) dx - \frac{1}{q^*_s} \|u\|_{q^*_s}^{q^*_s}. $$

where

$$\|u\|_{s,r} = ([u]_{s,r}^r + |u|^r_r)^{\frac{1}{2}}$$

$$[u]_{s,r}^r = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^r}{|x - y|^{N + sr}} \, dx \, dy \quad \text{and} \quad |u|^r_r = \int_{\mathbb{R}^N} |u|^r \, dx.$$ Here we denote by $X = W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$ endowed with the norm

$$\|u\| = \|u\|_{s,p} + \|u\|_{s,q}.$$ Then we give the following definition:

**Definition 1.1.** We say that $u \in X$ is a weak solution to (1.1) if for any $v \in X$ we have

$$\iint_{\mathbb{R}^2} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N + sp}} (v(x) - v(y)) \, dx \, dy + \int_{\mathbb{R}^N} |u|^{p-2}uv \, dx$$

$$+ \iint_{\mathbb{R}^2} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x - y|^{N + sq}} (v(x) - v(y)) \, dx \, dy + \int_{\mathbb{R}^N} |u|^{q-2}uv \, dx = \lambda \int_{\mathbb{R}^N} f(u) \, dx + \int_{\mathbb{R}^N} |u|^{q^*_s-2}uv \, dx.$$ The main result of this paper can be stated as follows:

**Theorem 1.1.** Assume that $(f_1)$-$(f_3)$ hold. Then there exists $\lambda^* > 0$ such that the problem (1.1) admits a nontrivial solution for all $\lambda \geq \lambda^*$.

We point out that Theorem 1.1 can be seen as the fractional analogue of Theorem 1 in [16]. Indeed, to prove our main result, we will borrow some ideas developed in [16]. Anyway, the presence of two fractional Laplacian operators and the lack of compactness due to the critical exponent make our analysis more delicate and intriguing; see the proof of Lemma 3.3.

More precisely, to overcome these difficulties we prove a variant of the Concentration-Compactness Lemma [26] for tight sequences in $W^{s,p}(\mathbb{R}^N)$ and we show that weak limits of Palais-Smale sequences of $J$ are weak solutions to (1.1) in a different and more technical way with respect to the proof of Lemma 2.4 in [16] which is based on some "local" arguments inspired by [22].

The paper is organized as follows: in Section 2 we recall some useful lemmas which we will use along the paper. In particular, we give a variant of the Concentration-Compactness Lemma [26] for the fractional $p$-Laplacian. In Section 3 we show that (1.1) admits a nontrivial solution for $\lambda$ big enough by applying the Mountain Pass Theorem [1] and a suitable version of the Lions’ compactness result [25].
2. PRELIMINARIES

In this section we recall some useful facts on fractional Sobolev spaces. For more details we refer the interested reader to [12, 29].

Let us define $D^{s,p}(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$[u]_{s,p}^p = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy\right)^{\frac{1}{p}}.$$ 

We denote by $W^{s,p}(\mathbb{R}^N)$ the set of functions $u : \mathbb{R}^N \to \mathbb{R}$ in $L^p(\mathbb{R}^N)$ such that $[u]_{s,p} < \infty$. Let us recall the following fundamental embeddings:

**Theorem 2.1.** [12] Let $N > sp$. Then there exists a constant $S_s = S_s(N,s,p) > 0$ such that

$$[u]_{p_s}^p \leq S_s^{-1}[u]_{s,p}^p \quad \forall u \in D^{s,p}(\mathbb{R}^N).$$

Moreover, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded into $L^t(\mathbb{R}^N)$ for any $t \in [p,p_s]$, and compactly into $L^t(\mathbb{R}^N)$ for any $t \in [1,p_s^*)$.

A simple adaption of the arguments in [26] allows us to deduce the following useful result:

**Lemma 2.1.** Let $R > 0$ and $r \in [p,p_s^*)$. For any bounded sequence $(u_n) \subset W^{s,p}(\mathbb{R}^N)$, if

$$\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n| \, dx \to 0 \quad \text{as } n \to \infty,$$

then $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (r,p_s^*)$.

**Proof.** Applying Hölder inequality we can see that

$$\int_{B_r(y)} |u_n|^q \, dx \leq \left( \int_{B_r(y)} |u_n|^r \, dx \right)^{\frac{1-p}{r}} \left( \int_{B_r(y)} |u_n|^{p_s^*} \, dx \right)^{\frac{p}{p_s^*}},$$

where $\frac{1}{r} + \frac{\lambda}{p_s} = \frac{1}{q}$. Then, covering $\mathbb{R}^N$ by balls with radius $r$ in such a way that each point of $\mathbb{R}^N$ is contained in at most $N + 1$ balls and using Theorem 2.1 we have

$$\int_{\mathbb{R}^N} |u_n|^q \, dx \leq C(N + 1) \sup_{y \in \mathbb{R}^N} \left( \int_{B_r(y)} |u_n|^r \, dx \right)^{\frac{1-p}{r}}$$

which implies the conclusion. \hfill \Box

Now, we prove the following technical lemma (see Lemma 2.1 in [5] for the case $p = 2$):

**Lemma 2.2.** Let $(u_n) \subset D^{s,p}(\mathbb{R}^N)$ be a bounded sequence and $\eta \in C_c^\infty(\mathbb{R}^N)$ such that $\eta = 0$ if $|x| < 1$ and $\eta = 1$ if $|x| > 2$, and we set $\eta_R(x) = \eta(x/R)$. Then

$$\lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} |u_n(x)|^p \, dx \, dy = 0.$$ 

**Proof.** Firstly, we note that

$$\mathbb{R}^{2N} = (\mathbb{R}^N \setminus B_{2R}) \times (\mathbb{R}^N \setminus B_{2R}) \cup ((\mathbb{R}^N \setminus B_{2R}) \times B_{2R}) \cup (B_{2R} \times \mathbb{R}^N) =: X_R \cup X_R^2 \cup X_R^3.$$ 

As a consequence

$$\int_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} |u_n(x)|^p \, dx \, dy = \int_{X_R^1} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} |u_n(x)|^p \, dx \, dy + \int_{X_R^2} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} |u_n(x)|^p \, dx \, dy + \int_{X_R^3} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} |u_n(x)|^p \, dx \, dy. \quad (2.1)$$
Since \( \eta_R = 1 \) in \( \mathbb{R}^N \setminus B_{2R} \), we can see that
\[
\int_{X_R^2} \frac{|u_n(x)|^p|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} \, dx \, dy = 0. \tag{2.2}
\]
Now, fix \( k > 4 \). Then
\[
X_R^2 \subset (\mathbb{R}^N \setminus B_{2R}) \times B_{2R} \subset ((\mathbb{R}^N \setminus B_{kR}) \times B_{2R}) \cup ((B_{kR} \setminus B_{2R}) \times B_{2R})
\]
Let us note that if \( (x,y) \in (\mathbb{R}^N \setminus B_{kR}) \times B_{2R} \), then
\[
|x-y| \geq |x| - |y| \geq |x| - 2R > \frac{|x|}{2}.
\]
Therefore, using \( 0 \leq \eta_R \leq 1 \), \( |\nabla \eta_R| \leq \frac{C}{R} \) and applying Hölder inequality we obtain
\[
\int_{X_R^2} \frac{|u_n(x)|^p|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} \, dx \, dy
\]
\[
= \int_{\mathbb{R}^N \setminus B_{kR}} \int_{B_{2R}} \frac{|u_n(x)|^p|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + \int_{B_{kR} \setminus B_{2R}} \int_{B_{2R}} \frac{|u_n(x)|^p|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} \, dx \, dy
\]
\[
\leq 2^{p+N+sp} \int_{\mathbb{R}^N \setminus B_{kR}} \int_{B_{2R}} \frac{|u_n(x)|^p}{|x-y|^{N+sp}} \, dx \, dy + C \int_{B_{kR} \setminus B_{2R}} \int_{B_{2R}} \frac{|u_n(x)|^p}{|x-y|^{N+sp}} \, dx \, dy
\]
\[
\leq CR^N \left( \int_{\mathbb{R}^N \setminus B_{kR}} \frac{|u_n(x)|^2}{|x|^{N+sp}} \, dx + \frac{C}{R^2} (kR)^p (1-s) \int_{B_{kR} \setminus B_{2R}} |u_n(x)|^p \, dx \right)^{\frac{p}{p^*}} + \frac{C k^p (1-s)}{R^p} \int_{B_{kR} \setminus B_{2R}} |u_n(x)|^p \, dx
\]
\[
\leq C \frac{k^N}{N+sp} \int_{B_{kR} \setminus B_{2R}} |u_n(x)|^p \, dx.
\] (2.3)

On the other hand, we have
\[
\int_{X_R^2} \frac{|u_n(x)|^p|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} \, dx \, dy
\]
\[
\leq \int_{B_{2R} \setminus B_{R/k}} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + \int_{B_{R/k}} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} \, dx \, dy. \tag{2.4}
\]
Since
\[
\int_{B_{2R} \setminus B_{R/k}} \int_{\mathbb{R}^N \setminus \{y: |y| < R\}} \frac{|u_n(x)|^p|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \leq \frac{C}{R^p} \int_{B_{2R} \setminus B_{R/k}} |u_n(x)|^p \, dx
\]
and
\[
\int_{B_{2R} \setminus B_{R/k}} \int_{\mathbb{R}^N \setminus \{y: |y| \geq R\}} \frac{|u_n(x)|^p|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \leq \frac{C}{R^p} \int_{B_{2R} \setminus B_{R/k}} |u_n(x)|^p \, dx
\]
we get
\[
\int_{B_{2R} \setminus B_{R/k}} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \leq \frac{C}{R^p} \int_{B_{2R} \setminus B_{R/k}} |u_n(x)|^p \, dx. \tag{2.5}
\]
Now, from the definition of $\eta_R$ and $0 \leq \eta_R \leq 1$ we obtain
\[
\int_{B_R} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} \, dx \, dy = \int_{B_{R/k}} \int_{\mathbb{R}^N \setminus B_R} \frac{|u_n(x)|^p |\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} \, dx \, dy
\[
\leq 2^p \int_{B_{R/k}} \int_{\mathbb{R}^N \setminus B_R} \frac{|u_n(x)|^p}{|x - y|^{N+sp}} \, dx \, dy
\leq C \int_{B_{R/k}} |u_n|^p \, dx \int_{(1 - \frac{1}{k})^N \setminus B_{R/k}} \frac{1}{r^{1+sp}} \, dr
\leq \frac{C}{[(1 - \frac{1}{k})^N \setminus B_{R/k}\; sp]} \int_{B_{R/k}} |u_n|^p \, dx.
\]
(2.6)

where we use the fact that if $(x, y) \in B_{R/k} \times (\mathbb{R}^N \setminus B_R)$, then $|x - y| > (1 - \frac{1}{k})R$.

Then (2.4), (2.5) and (2.6) yield
\[
\int \int_{X_R^3} \frac{|u_n(x)|^p |\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} \, dx \, dy
\leq \frac{C \int_{B_{2R} \setminus B_{2R/k}} |u_n(x)|^p \, dx + \frac{C}{(1 - \frac{1}{k})^R sp}}{B_{R/k}} \int_{B_{R/k}} |u_n(x)|^p \, dx.
\]
(2.7)

In view of (2.1), (2.2), (2.3) and (2.7) we can infer
\[
\int \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} \, dx \, dy
\leq \frac{C + \frac{Ck^p(1-s)}{Rsp}}{B_{R/k}} \int_{B_{2R/k} \setminus B_{2R/k}} |u_n(x)|^p \, dx + \frac{C}{(1 - \frac{1}{k})^R sp} \int_{B_{R/k}} |u_n(x)|^p \, dx
\]
\[
+ \frac{C}{B_{R/k}} |u_n(x)|^p \, dx.
\]
(2.8)

Since $(u_n)$ is bounded in $D^{s,p}(\mathbb{R}^N)$, by using Theorem 2.1 we may assume that $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$ for some $u \in D^{s,p}(\mathbb{R}^N)$. Then, taking the limit as $n \to \infty$ in (2.8) we get
\[
\limsup_{n \to \infty} \int \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} \, dx \, dy
\leq \frac{C + \frac{Ck^p(1-s)}{Rsp}}{B_{R/k}} \int_{B_{2R} \setminus B_{2R/k}} |u(x)|^p \, dx + \frac{C}{(1 - \frac{1}{k})^R sp} \int_{B_{R/k}} |u(x)|^p \, dx
\leq \frac{C + \frac{Ck^p}{Rsp}}{B_{R/k}} \int_{B_{2R} \setminus B_{2R/k}} |u(x)|^p \, dx + C \left( \int_{B_{2R} \setminus B_{2R/k}} |u(x)|^{sp} \, dx \right)^{\frac{1}{sp}} + C \left( \frac{1}{k - 1} \right) sp \left( \int_{B_{2R} \setminus B_{2R/k}} |u(x)|^{sp} \, dx \right)^{\frac{1}{sp}},
\]

where in the last passage we have used Hölder inequality.

Since $u \in L^p_{loc}(\mathbb{R}^N)$ and $k > 4$ we can see that
\[
\limsup_{R \to \infty} \int_{B_{R/k} \setminus B_{2R/k}} |u(x)|^{p^*_s} \, dx = \limsup_{R \to \infty} \int_{B_{2R} \setminus B_{R/k}} |u(x)|^{p^*_s} \, dx = 0,
\]
which yields

\[
\limsup_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p |\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} \, dx dy \\
\leq \limsup_{k \to \infty} \limsup_{R \to \infty} \left[ \frac{C}{kN} + C k^p \left( \int_{B_{Rk} \setminus B_{R}} |u(x)| \psi^p \right)^{\frac{1}{p}} + C \left( \int_{B_{2R} \setminus B_{R/k}} |u(x)| \psi^p \right)^{\frac{1}{p}} \right] \\
+ C \left( \frac{1}{k - 1} \right)^{sp} \left( \int_{B_{R/k}} |u(x)| \psi^p \right)^{\frac{1}{p}} \\
\leq \lim_{k \to \infty} \frac{C}{kN} + C \left( \frac{1}{k - 1} \right)^{sp} \left( \int_{\mathbb{R}^N} |u(x)| \psi^p \right)^{\frac{1}{p}} = 0.
\]

\[\square\]

Arguing as in the previous lemma one can prove the following result (see proofs of Lemma 4.3 in [4] and Lemma 3.4 in [33] when \( p = 2 \):

**Lemma 2.3.** Let \((u_n) \subset D^{s,p}(\mathbb{R}^N)\) be a bounded sequence and \(\psi \in C^\infty_c(\mathbb{R}^N)\) such that \(0 \leq \psi \leq 1\), \(\psi = 1\) in \(B_1\), \(\psi = 0\) in \(B_2^c\) and \(|\nabla \psi| \leq 2\). Set \(\psi_\rho(x) = \psi(\frac{x - x_i}{\rho})\) where \(x_i \in \mathbb{R}^N\) is a fixed point. Then we have

\[
\lim_{\rho \to 0} \limsup_{n \to \infty} \int_{\mathbb{R}^{2N}} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x - y|^{N+sp}} \, dx dy = 0. \tag{2.9}
\]

**Proof.** It is clear that

\[
\mathbb{R}^{2N} = ((\mathbb{R}^N - B_{2\rho}(x_i)) \times (\mathbb{R}^N - B_{2\rho}(x_i))) \cup (B_{2\rho}(x_i) \times \mathbb{R}^N) \cup ((\mathbb{R}^N - B_{2\rho}(x_i)) \times B_{2\rho}(x_i)) \\
=: X_1^\rho \cup X_2^\rho \cup X_3^\rho.
\]

Hence

\[
\int_{\mathbb{R}^{2N}} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x - y|^{N+sp}} \, dx dy \\
= \int_{X_1^\rho} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x - y|^{N+sp}} \, dx dy + \int_{X_2^\rho} |u_n(x)|^p \frac{2|\psi_\rho(x) - \psi_\rho(y)|^p}{|x - y|^{N+sp}} \, dx dy \\
+ \int_{X_3^\rho} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x - y|^{N+sp}} \, dx dy. \tag{2.10}
\]

In what follows, we estimate each integral in (2.10). Since \(\psi = 0\) in \(\mathbb{R}^N \setminus B_2\), we have

\[
\int_{X_1^\rho} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x - y|^{N+sp}} \, dx dy = 0. \tag{2.11}
\]
Being $0 \leq \psi \leq 1$, we obtain

\[
\int \int_{X^2} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x-y|^{N+sp}} \, dx \, dy
= \int_{B_{2\rho}(x_i)} dx \int_{\{y \in \mathbb{R}^N : |x-y| \leq \rho\}} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x-y|^{N+2s}} \, dy \\
+ \int_{B_{2\rho}(x_i)} dx \int_{\{y \in \mathbb{R}^N : |x-y| > \rho\}} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x-y|^{N+sp}} \, dy
\leq \rho^{-p} \|
\n\int_{B_{2\rho}(x_i)} dx \int_{\{y \in \mathbb{R}^N : |x-y| \leq \rho\}} |u_n(x)|^p \frac{|u_n(x)|^2}{|x-y|^{N+ps-p}} \, dy \\
+ 2^p \int_{B_{2\rho}(x_i)} dx \int_{\{y \in \mathbb{R}^N : |x-y| > \rho\}} |u_n(x)|^p \, dy
\leq c_1 \rho^{-sp} \int_{B_{2\rho}(x_i)} |u_n(x)|^p \, dx + c_2 \rho^{-sp} \int_{B_{2\rho}(x_i)} |u_n(x)|^p \, dx
= c_3 \rho^{-sp} \int_{B_{2\rho}(x_i)} |u_n(x)|^p \, dx,
\]

for some $c_1, c_2, c_3 > 0$ independent of $\rho$ and $n$. On the other hand

\[
\int \int_{X^2} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^2}{|x-y|^{N+sp}} \, dx \, dy
= \int_{\mathbb{R}^N \setminus B_{2\rho}(x_i)} dx \int_{\{y \in B_{2\rho}(x_i) : |x-y| \leq \rho\}} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x-y|^{N+sp}} \, dy \\
+ \int_{\mathbb{R}^N \setminus B_{2\rho}(x_i)} dx \int_{\{y \in B_{2\rho}(x_i) : |x-y| > \rho\}} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x-y|^{N+sp}} \, dy =: A_{\rho,n} + B_{\rho,n}.
\]

Let us note that $|x-y| < \rho$ and $|y-x_i| < 2\rho$ imply $|x-x_i| < 3\rho$, and then

\[
A_{\rho,n} \leq \rho^{-p} \|
\leq \rho^{-p} \|
\leq c_4 \rho^{-sp} \int_{B_{2\rho}(x_i)} |u_n(x)|^p \, dx,
\]

for some $c_4 > 0$ independent of $\rho$ and $n$. Let us observe that for all $K > 4$ it holds

\[
(\mathbb{R}^N \setminus B_{2\rho}(x_i)) \times B_{2\rho}(x_i) \subset (B_{K\rho}(x_i) \times B_{2\rho}(x_i)) \cup (\mathbb{R}^N \setminus B_{K\rho}(x_i)) \times B_{2\rho}(x_i)).
\]
Therefore, we can see that

\[
\int_{B_{K\rho}(x_i)} dx \int_{\{ y \in B_{2\rho}(x_i) : |y-x| > \rho \} } \frac{|u_n(x)|^p |\psi_\rho(x) - \psi_\rho(y)|^p}{|x-y|^{N+sp}} dy \\
\leq 2^p \int_{B_{K\rho}(x_i)} dx \int_{\{ y \in B_{2\rho}(x_i) : |y-x| > \rho \} } \frac{|u_n(x)|^p}{|x-y|^{N+sp}} dy \\
\leq 2^p \int_{B_{K\rho}(x_i)} dx \int_{\{ z \in \mathbb{R}^N : |z| > \rho \} } |u_n(x)|^p \frac{1}{|z|^{N+sp}} dz \\
= c_5 \rho^{-sp} \int_{B_{K\rho}(x_i)} |u_n(x)|^p dx, \tag{2.15}
\]

for some $c_5$ independent of $\rho$ and $n$.

On the other hand, if $|x - x_i| \geq K\rho$ and $|y - x_i| < 2\rho$ then

\[|x - y| \geq |x - x_i| - |y - x_i| \geq \frac{|x - x_i|}{2} + \frac{K\rho}{2} - 2\rho > \frac{|x - x_i|}{2}.\]

As a consequence

\[
\int_{\mathbb{R}^N \setminus B_{K\rho}(x_i)} dx \int_{\{ y \in B_{2\rho}(x_i) : |y-x| > \rho \} } |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x-y|^{N+sp}} dy \\
\leq 2^{N+sp+p} \int_{\mathbb{R}^N \setminus B_{K\rho}(x_i)} dx \int_{\{ y \in B_{2\rho}(x_i) : |y-x| > \rho \} } |u_n(x)|^p \frac{1}{|x-x_i|^{N+sp}} dy \\
\leq 2^{N+sp+p} (2\rho)^N \int_{\mathbb{R}^N \setminus B_{K\rho}(x_i)} |u_n(x)|^p \frac{1}{|x-x_i|^{N+sp}} dx \\
\leq 2^{2N+sp+p} N \left( \int_{\mathbb{R}^N \setminus B_{K\rho}(x_i)} |u_n(x)|^{p_\ast} dx \right) \frac{2}{p_\ast} \left( \int_{\mathbb{R}^N \setminus B_{K\rho}(x_i)} |x-x_i|^{-((N+sp) \frac{p_\ast}{p_\ast - p})} dx \right) \frac{p_\ast - p}{p_\ast} \\
\leq c_6 K^{-N} \left( \int_{\mathbb{R}^N \setminus B_{K\rho}(x_i)} |u_n(x)|^{p_\ast} dx \right) \frac{2}{p_\ast} \frac{p_\ast - p}{p_\ast}, \tag{2.16}
\]

for some $c_6 > 0$ independent of $\rho$ and $n$. Putting together (2.15) and (2.16), and the fact that $(u_n)$ is bounded in $L^{p_\ast}(\mathbb{R}^N)$, we can find $c_7 > 0$ independent of $\rho$ and $n$ such that

\[B_{\rho, n} \leq c_5 \rho^{-sp} \int_{B_{K\rho}(x_i)} |u_n(x)|^p dx + c_7 K^{-N}. \tag{2.17}\]

Then, (2.10)-(2.14) and (2.17) yield

\[
\int \int_{\mathbb{R}^{2N}} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x-y|^{N+sp}} \, dx dy \leq c_8 \rho^{-sp} \int_{B_{K\rho}(x_i)} |u_n(x)|^p dx + c_9 K^{-N}, \tag{2.18}
\]

for some $c_8, c_9 > 0$ independent of $\rho$ and $n$. Recalling that $u_n \to u$ strongly in $L^p_{\text{loc}}(\mathbb{R}^N, \mathbb{R})$ we have

\[
\lim_{n \to \infty} c_8 \rho^{-sp} \int_{B_{K\rho}(x_i)} |u_n(x)|^p dx + c_9 K^{-N} = c_8 \rho^{-sp} \int_{B_{K\rho}(x_i)} |u(x)|^p dx + c_9 K^{-N}.
\]
By using Hölder inequality we can see that

\[ c_8 \rho^{-sp} \int_{B_{K\rho}(x_i)} |u(x)|^p \, dx + c_9 K^{-N} \]

\[ \leq c_8 \rho^{-sp} \left( \int_{B_{K\rho}(x_i)} |u(x)|^p \, dx \right)^{\frac{p}{sp}} |B_{K\rho}(x_i)|^{1-\frac{p}{sp}} + c_9 K^{-N} \]

\[ \leq c_{10} K^{sp} \left( \int_{B_{K\rho}(x_i)} |u(x)|^p \, dx \right)^{\frac{p}{sp}} + c_9 K^{-N} \rightarrow c_9 K^{-N} \text{ as } \rho \rightarrow 0. \]

Hence

\[ \lim_{\rho \rightarrow 0} \limsup_{n \rightarrow \infty} \iint_{\mathbb{R}^{2N}} |u_n(x)|^p \frac{\left| \psi_{\rho}(x) - \psi_{\rho}(y) \right|^p}{|x-y|^{N+sp}} \, dx \, dy = 0. \]

Before proving a variant of the well-known Concentration-Compactness Lemma due to Lions [26] we give the following definition:

**Definition 2.1.** We say that a sequence \((u_n) \subset D^{s,p}(\mathbb{R}^N)\) is tight if for every \(\varepsilon > 0\) there exists \(R > 0\) such that

\[ \int_{|x| > R} |D^s u|^p \, dx \leq \varepsilon \quad \forall n \in \mathbb{N}, \]

where we used the notation

\[ |D^s u|^p(x) = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dy. \]

Now, we give the proof of the following result (see also [4,15,18,30] for more details):

**Lemma 2.4.** Let \((u_n)\) be a bounded tight sequence in \(D^{s,p}(\mathbb{R}^N)\) such that \(u_n \rightharpoonup u\) in \(D^{s,p}(\mathbb{R}^N)\). Let us assume that

\[ |D^s u_n|^p \rightharpoonup \mu \]

\[ |u_n|^p \rightharpoonup \nu \]

in the sense of measure, where \(\mu\) and \(\nu\) are two non-negative measures on \(\mathbb{R}^N\). Then, there exists an at most a countable set \(I\), a family of distinct points \((x_i)_{i \in I} \subset \mathbb{R}^N\) and \((\mu_i)_{i \in I}, (\nu_i)_{i \in I} \subset (0, \infty)\) such that

\[ \nu = |u|^p \]

\[ \mu \geq |D^s u|^p + \sum_{i \in I} \mu_i \delta_{x_i}. \]

Moreover, it holds the following relation

\[ \mu_i \geq S_s p \nu_i^{\frac{p}{s}} \quad \forall i \in I. \]

**Proof.** We follow the arguments in [4]. In order to prove (2.20), we aim to pass to the limit in the following relation which holds in view of Brezis-Lieb Lemma [7]:

\[ \int_{\mathbb{R}^N} |\psi|^p \nu \, dx = \int_{\mathbb{R}^N} |\psi|^{p^*_s} u_n|^{p^*_s} \, dx + \int_{\mathbb{R}^N} |\psi|^{p^*_s} |u_n - u|^{p^*_s} \, dx + o_n(1), \]
where \( \psi \in C^\infty_c(\mathbb{R}^N) \).

Set \( \tilde{u}_n = u_n - u \). Then, by Theorem 2.1, we can see that \( \tilde{u}_n \to 0 \) in \( L^p_{\text{loc}}(\mathbb{R}^N) \) and a.e. on \( \mathbb{R}^N \). Fix \( \psi \in C^\infty_c(\mathbb{R}^N) \). By using the definition of \( S_* \) we get

\[
\left[ \int_{\mathbb{R}^N} |\psi|^{p^*_s} |u_n - u|^{p^*_s} \, dx \right]^{\frac{1}{p^*_s}} \leq S_*^{-1} \int_{\mathbb{R}^N} (|D^s(\psi \tilde{u}_n)|^p + |D^s(\psi \tilde{v}_n)|^p) \, dx \\
= S_*^{-1} \left[ \int_{\mathbb{R}^{2N}} \frac{|(\psi \tilde{u}_n)(x) - (\psi \tilde{v}_n)(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right].
\]

Now, we observe that

\[
\int_{\mathbb{R}^{2N}} \frac{|(\psi \tilde{u}_n)(x) - (\psi \tilde{v}_n)(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \\
\leq 2^{p-1} \int_{\mathbb{R}^{2N}} |\psi(y)|^p |\tilde{u}_n(x) - \tilde{u}_n(y)|^p + |\tilde{u}_n(x)|^p |\psi(x) - \psi(y)|^p \, dx \, dy.
\]

It is easy to show that

\[
\int_{\mathbb{R}^{2N}} |\psi(x) - \psi(y)|^p \, dx \, dy = o_n(1).
\]

Indeed, arguing as in the proof of Lemma 2.3 (with \( x_i = 0 \) and \( \rho = 1 \)), if \( \psi = 1 \) in \( B_1 \) and \( \psi = 0 \) in \( B_2^c \) we have

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{N+sp}} |\tilde{u}_n(x)|^p \, dx \, dy \\
= \int_{B_2} \int_{\mathbb{R}^N} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{N+sp}} |\tilde{u}_n(x)|^p \, dx \, dy + \int_{\mathbb{R}^N \setminus B_2} \int_{B_2} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{N+sp}} |\tilde{u}_n(x)|^p \, dx \, dy \\
\leq C \int_{B_K} |\tilde{u}_n(x)|^p \, dx + CK^{-N} \quad \forall K > 4,
\]

and taking the limit as \( n \to \infty \) and then as \( K \to \infty \) we get the thesis.

Therefore, if we assume that \( |D^s \tilde{u}_n|^p \rightharpoonup \tilde{\mu} \) and \( |\tilde{u}_n|^{p^*_s} \rightharpoonup \tilde{\nu} \) in the sense of measures, from the above facts and by passing to the limit in (2.24) we have that

\[
\left[ \int_{\mathbb{R}^N} |\psi|^{p^*_s} d\tilde{\nu} \right]^{\frac{1}{p^*_s}} \leq C \left[ \int_{\mathbb{R}^N} |\psi|^p d\tilde{\mu} \right]^{\frac{1}{p^*_s}} \quad \text{for all } \psi \in C^\infty_c(\mathbb{R}^N).
\]

Then, by using Lemma 1.2 in [26], there exist at most a countable set \( I \), families \( (x_i)_{i \in I} \subset \mathbb{R}^N \) and \( (\nu_i)_{i \in I} \subset (0, \infty) \) such that

\[
\tilde{\nu} = \sum_{i \in I} \nu_i \delta_{x_i}.
\]

In view of (2.23), we deduce that \( \nu = |u|^{p^*_s} + \tilde{\nu} \) which together with (2.25), implies that

\[
\nu = |u|^{p^*_s} + \sum_{i \in I} \nu_i \delta_{x_i},
\]

that is (2.20) holds.

Now, we pass to prove (2.22). Take \( \psi_\rho = \eta(\frac{x}{\rho}) \), where \( \eta \in C^\infty_c(\mathbb{R}^N) \), \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) in \( B_1 \) and
\( \eta = 0 \) in \( B_2^0 \). Then, recalling the definition of \( S_* \) and the inequality
\[
(x + y)^p \leq x^p + C_p y^p, \quad \text{for all } x, y \geq 0, p > 1
\]
we can deduce that
\[
S_* \left[ \int_{\mathbb{R}^N} |\psi_\rho|^p |u_n|^p \, dx \right]^{\frac{p}{p_*}} \leq \int_{\mathbb{R}^N} |D^s(\psi_\rho u_n)|^p \, dx \\
\leq C_p \left( \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^N} |u_n(x)|^p \left( |\psi_\rho(x) - \psi_\rho(y)|^p \right) \frac{dxdy}{|x - y|^{N + sp}} \right) \right) \\
+ \left( \int_{\mathbb{R}^2} |\psi_\rho(y)|^p \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N + sp}} \, dxdy \right). \quad (2.26)
\]
Now, taking into account (2.19) and (2.20) we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\psi_\rho|^p |u_n|^p \, dx = \int_{B_\rho(x_j)} |\psi_\rho|^p |u|^p \, dx + \nu_i.
\]
Since \( 0 \leq \psi_\rho \leq 1 \) implies
\[
\left| \int_{B_\rho(x_j)} |\psi_\rho|^p |u|^p \, dx \right| \leq C \int_{B_\rho(x_j)} |u|^p \, dx \to 0 \quad \text{as } \rho \to 0,
\]
we can deduce that
\[
\lim_{\rho \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} |\psi_\rho|^p |u_n|^p \, dx = \nu_i. \quad (2.27)
\]
On the other hand, (2.19) gives
\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} |\psi_\rho(y)|^p \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N + sp}} \, dxdy = \int_{\mathbb{R}^N} |\psi_\rho(y)|^p \, d\mu
\]
and using Lemma 2.3 we can see that
\[
\lim \limsup_{\rho \to 0} \int_{\mathbb{R}^2} |u_n(x)|^p \frac{|\psi_\rho(x) - \psi_\rho(y)|^p}{|x - y|^{N + sp}} \, dxdy = 0. \quad (2.28)
\]
Then, putting together (2.26), (2.27) and (2.28) we get
\[
S_* \nu_i^\frac{p}{p_*} \leq \lim_{\rho \to 0} \mu(B_\rho(x_i)).
\]
Setting \( \mu_i = \lim_{\rho \to 0} \mu(B_\rho(x_i)) \) we deduce that (2.22) holds.

Finally we can note that
\[
\mu \geq \sum_{i \in I} \mu_i \delta_{x_i}
\]
and that the weak convergences implies that \( \mu \geq |D^s u|^p \). Then, due to the fact that \( |D^s u|^p \) is orthogonal to \( \sum_{i \in I} \mu_i \delta_{x_i} \), we can infer that (2.21) holds. \( \square \)
3. PROOF OF THEOREM 1

This section is devoted to the proof of Theorem 1.1. Let us introduce the functional $J : X \to \mathbb{R}$ associated to the problem (1.1):

$$J(u) = \frac{1}{p} \|u\|_{s,p}^p + \frac{1}{q} \|u\|_{s,q}^q - \lambda \int_{\mathbb{R}^N} F(u) dx - \frac{1}{q_s^*} |u|_{q_s^*}^{q_s^*}.$$  

From the assumptions $(f_1)-(f_2)$ we know that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon |t|^{p-1} + C_\varepsilon |t|^{r-1} \text{ for all } t \in \mathbb{R}. \quad (3.1)$$

By using (3.1) and Theorem 2.1 it is easy to check that $J$ is well-defined on $X$ and $J \in C^1(X, \mathbb{R})$. Now, we prove that $J$ possesses a Mountain Pass geometry [1]:

**Lemma 3.1.** For each $\lambda > 0$ the functional $J$ satisfies the following conditions:

(i) there exist $\alpha, \beta > 0$ such that $J(u) \geq \beta$ if $\|u\| = \alpha$

(ii) there exists $e \in X$ such that $\|e\| > \alpha$ and $J(e) < 0$.

**Proof.** In view of (3.1) we can see that for all $\varepsilon \in (0, 1)$ there exists $C_\varepsilon > 0$ such that

$$J(u) \geq \frac{1}{p} \|u\|_{s,p}^p + \frac{1}{q} \|u\|_{s,q}^q - \lambda \frac{\varepsilon}{p} |u|_p^r - \frac{C_\varepsilon}{r} |u|^r - \frac{1}{q_s^*} |u|_{q_s^*}^{q_s^*} \geq \frac{1}{p} \|u\|_{s,p}^p + (1 - \varepsilon) |u|_p^p + \frac{1}{q} |u|_{s,q}^p + \frac{1}{q} \|u\|_{s,q}^q - \lambda \frac{C_\varepsilon}{r} |u|_r^r - \frac{1}{q_s^*} |u|_{q_s^*}^{q_s^*} \geq \frac{1}{p} \|u\|_{s,p}^p + \|u\|_{s,q}^q - \lambda \frac{C_\varepsilon}{r} |u|_r^r - \frac{1}{q_s^*} |u|_{q_s^*}^{q_s^*},$$

where

$$C_1 = \min \left\{ \frac{1}{p} \min \{1, 1 - \varepsilon\}, \frac{1}{q} \right\}.$$ 

If $\|u\| < 1$ we obtain that $\|u\|_{s-p}^{q-p} < 1$ so we deduce that

$$J(u) \geq C_1 (\|u\|_{s,p}^p + \|u\|_{s,q}^q) - \lambda \frac{C_\varepsilon}{r} |u|_r^r - \frac{1}{q_s^*} |u|_{q_s^*}^{q_s^*} \geq C_2 \|u\|^{\|q\| - \lambda \frac{C_\varepsilon}{r} |u|_r^r - \frac{1}{q_s^*} |u|_{q_s^*}^{q_s^*}.$$ 

Then, from Theorem 2.1 it follows that

$$J(u) \geq C_2 \|u\|^{\|q\| - \lambda \frac{C_\varepsilon}{r} |u|_r^r - \lambda C_4 \|u\|^{r_q} - C_5 \|u\|^{q_q} = \|u\|^{\|q\| - \lambda C_4 \|u\|^{r_q} - C_5 \|u\|^{q_q}}.$$ 

Since $r \in (q, q_s^*)$ there exist $\alpha, \beta > 0$ such that $J(u) \geq \beta$ for all $u \in X$ such that $\|u\| = \alpha$. Fix $v \in C^\infty_c(\mathbb{R}^N)$ such that $v > 0$ in $\mathbb{R}^N$. We recall that $(f_3)$ implies that

$$F(x, t) \geq At^\theta - B \text{ for all } t > 1. \quad (3.2)$$

Then, using (3.2) and $\theta \in (q, q_s^*)$ we can see that

$$J(tv) \leq \frac{1}{p} \|v\|_{s,p}^p + \frac{1}{q} \|v\|_{s,q}^q - \lambda \frac{\theta^p}{p} |v|^\theta + \frac{B\lambda |\text{supp}(v)|}{q_s^*} - \frac{1}{q_s^*} |v|_{q_s^*}^{q_s^*} \to -\infty \text{ as } t \to \infty.$$ 

Then we can find $\tau > 0$ sufficiently large such that $\|\tau v\| > \alpha$ and $J(\tau v) < 0$. \hfill \square
In view of Lemma 3.1 we can define

\[ c_* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)), \]

where

\[ \Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } J(\gamma(1)) < 0 \}. \]

Our purpose is to prove that \( c_* \) is achieved by some nontrivial function \( u \in X \). The next result shows that we are able to compare \( c_* \) with a suitable constant which involves \( S_* \):

**Lemma 3.2.** There exists \( \lambda_* > 0 \) such that \( c_* \in \left( 0, \left( \frac{1}{\theta} - \frac{1}{q_*} \right) S_*^{\frac{N}{q_*^*}} \right) \) for all \( \lambda \geq \lambda_* \).

**Proof.** Take \( v \in C_0^\infty(\mathbb{R}^N) \) such that \( v > 0 \) in \( \mathbb{R}^N \). Then there exists \( t_\lambda > 0 \) such that \( J(t_\lambda v) = \max_{t \geq 0} J(tv) \). As a consequence \((J'(t_\lambda v), t_\lambda v) = 0\) that is

\[ t_\lambda^p \|v\|_{s,p}^p + t_\lambda^q \|v\|_{s,q}^q = \lambda \int_{\mathbb{R}^N} f(t_\lambda v)t_\lambda v + t_\lambda^{q_*} |v|_{q_*^*}^{q_*} \]  

(3.3)

which combined with \((f_3)\) yields

\[ t_\lambda^p \|v\|_{s,p}^p + t_\lambda^q \|v\|_{s,q}^q \geq t_\lambda^{q_*} |v|_{q_*^*}^{q_*}. \]

Since \( p \leq q < q_*^* \) we can infer that \( t_\lambda \) is bounded and that there exists a sequence \( \lambda_n \to \infty \) such that \( t_{\lambda_n} \to t_0 \geq 0 \). Let us observe that if \( t_0 > 0 \) then we have

\[ t_{\lambda_n}^p \|v\|_{s,p}^p + t_{\lambda_n}^q \|v\|_{s,q}^q \to L \in (0, \infty) \]

and

\[ \lambda_n \int_{\mathbb{R}^N} f(t_{\lambda_n} v)t_{\lambda_n} v + t_{\lambda_n}^{q_*} |v|_{q_*^*}^{q_*} \to \infty \]

which gives a contradiction in view of (3.3). Therefore \( t_0 = 0 \). Let us define \( \gamma(t) = tv \) with \( t \in [0,1] \).

Then \( \gamma \in \Gamma \) and we get

\[ 0 < c_* \leq \max_{t \in [0,1]} J(tv) = J(t_\lambda v) \leq t_\lambda^p \|v\|_{s,p}^p + t_\lambda^q \|v\|_{s,q}^q. \]  

(3.4)

Taking \( \lambda \) sufficiently large we can infer that

\[ t_\lambda^p \|v\|_{s,p}^p + t_\lambda^q \|v\|_{s,q}^q < \left( \frac{1}{\theta} - \frac{1}{q_*} \right) S_*^{\frac{N}{q_*^*}}, \]

which yields

\[ 0 < c_* < \left( \frac{1}{\theta} - \frac{1}{q_*} \right) S_*^{\frac{N}{q_*^*}}. \]

In particular, since \( t_\lambda \to 0 \) as \( \lambda \to \infty \), it follows from (3.4) that \( c_* \to 0 \) as \( \lambda \to \infty \). \( \square \)

Next we give a suitable variant of the Lions result in [25] for problems with fractional p&Q Laplacians:

**Proposition 3.1.** Let \((u_n) \subset X \) be a \((PS)_{c_*} \) sequence for \( J \) with \( u_n \to 0 \) in \( X \). Then we have

(i) \( u_n \to 0 \) in \( X \) or

(ii) there exist \((y_n) \subset \mathbb{R}^N \) and \( R, \gamma > 0 \) such that

\[ \liminf_{n \to \infty} \int_{B_R(y_n)} |u_n|^q dx \geq \gamma. \]
Proof. Assume that (ii) does not hold. Then we can use Lemma 2.1 to see that
\[ u_n \to 0 \text{ in } L^t(\mathbb{R}^N) \quad \forall t \in (q, q_*) \text{.} \] (3.5)
Since \((J'(u_n), u_n) = o_n(1)\) we can see that
\[ \|u_n\|_{s,p}^p + \|u_n\|_{s,q}^q = \lambda \int_{\mathbb{R}^N} f(u_n)u_n \, dx + |u_n|_{q_*}^{q_*} + o_n(1) \text{.} \]
On the other hand (3.5) and (3.1) imply that
\[ \int_{\mathbb{R}^N} F(u_n) \, dx = \int_{\mathbb{R}^N} f(u_n)u_n \, dx = o_n(1) \text{,} \] (3.6)
so we get
\[ \|u_n\|_{s,p}^p + \|u_n\|_{s,q}^q = |u_n|_{q_*}^{q_*} + o_n(1) \text{.} \] (3.7)
Thus, up to a subsequence, we may assume that
\[ \|u_n\|_{s,p}^p + \|u_n\|_{s,q}^q \to L_\lambda \text{ and } |u_n|_{q_*}^{q_*} \to L_\lambda \text{.} \]
If \(L_\lambda \to L > 0\), then there exists \(M > 0\) independent of \(\lambda\) such that \(L_\lambda \geq M\). Hence, using (3.6) we get
\[ \frac{1}{p}\|u_n\|_{s,p}^p + \frac{1}{q}\|u_n\|_{s,q}^q - \frac{1}{q_*}|u_n|_{q_*}^{q_*} = o_n(1) + c_* \]
which together with \(p \leq q\) yields
\[ o_n(1) + c_* \geq \left( \frac{1}{q} - \frac{1}{q_*} \right) L_\gamma \geq \left( \frac{1}{q} - \frac{1}{q_*} \right) M > 0 \text{.} \]
The above relation leads to a contradiction since from the last part of the proof of Lemma 3.2 we know that \(c_* \to 0\) as \(\lambda \to \infty\). Therefore \(L_\lambda \to 0\) as \(\lambda \to \infty\) and we can infer that \(\|u_n\| \to 0\). \(\square\)

In the lemma below we will make use of the Concentration-Compactness Lemma established in Section 2:

**Lemma 3.3.** Let \((u_n) \subset X\) be a sequence such that \(J(u_n) \to c_*\) and \(J'(u_n) \to 0\). Then \(u_n \to u\) in \(X\) and \(J'(u) = 0\) for all \(\lambda \geq \lambda_*\).

**Proof.** We begin proving that \((u_n)\) is bounded in \(X\). Since \(J(u_n) \to c_*\) and \(J'(u_n) \to 0\) we have
\[ C(1 + \|u_n\|) \geq J(u_n) - \frac{1}{\theta}(J'(u_n), u_n) \]
\[ = \frac{1}{p}\|u_n\|_{s,p}^p + \frac{1}{q}\|u_n\|_{s,q}^q - \lambda \int_{\mathbb{R}^N} F(u_n) \, dx - \frac{1}{q_*}|u_n|_{q_*}^{q_*} \]
\[ - \frac{1}{\theta}\left[ \|u_n\|_{s,p}^p + \|u_n\|_{s,q}^q - \lambda \int_{\mathbb{R}^N} f(u_n)u_n \, dx - \int_{\mathbb{R}^N} |u_n|^{q_*} \, dx \right] \]
\[ \geq \left( \frac{1}{q} - \frac{1}{\theta} \right) (\|u_n\|_{s,p}^p + \|u_n\|_{s,q}^q) + \frac{\lambda}{\theta} \int_{\mathbb{R}^N} [f(u_n)u_n - \theta F(u_n)] \, dx + \left( \frac{1}{q} - \frac{1}{q_*} \right) \int_{\mathbb{R}^N} |u_n|^{q_*} \, dx \text{.} \]
Then, by using \((f_3)\) we can see that
\[ C(1 + \|u_n\|) \geq \left( \frac{1}{q} - \frac{1}{\theta} \right) (\|u_n\|_{s,p}^p + \|u_n\|_{s,q}^q) \text{.} \] (3.8)
Assume by contradiction that \(\|u_n\| \to \infty\) and we distinguish the following three cases:

**Case 1:** \(\|u_n\|_{s,p} \to \infty\) and \(\|u_n\|_{s,q} \to \infty\).
For \( n \) big enough, we get \( \|u_n\|_{s,q}^p \geq 1 \) and \( \|u_n\|_{s,q} \geq \|u_n\|_{s,p} \). In view of (3.8) and \((a + b)^p \leq C_p(a^p + b^p)\) for all \( a, b \geq 0 \), we can deduce that

\[
C(1 + \|u_n\|) \geq \left( \frac{1}{q} - \frac{1}{\theta} \right) (\|u_n\|_{s,p}^p + \|u_n\|_{s,q}^p) \geq C_p^{-1} \left( \frac{1}{q} - \frac{1}{\theta} \right) (\|u_n\|_{s,p} + \|u_n\|_{s,q})^p =: C_1 \|u_n\|^p
\]

which implies that \( \|u_n\| \) is bounded, that is a contradiction.

Case 2: \( \|u_n\|_{s,p} \to \infty \) and \( \|u_n\|_{s,q} \) is bounded.

From (3.8) we have

\[
C(1 + \|u_n\|_{s,p} + \|u_n\|_{s,q}) = C(1 + \|u_n\|) \geq \left( \frac{1}{q} - \frac{1}{\theta} \right) \|u_n\|_{s,p}^p
\]

which yields

\[
C \left( \frac{1}{\|u_n\|_{s,p}^p} + \frac{1}{\|u_n\|_{s,q}^{p-1}} + \|u_n\|_{s,q} \right) \geq \left( \frac{1}{q} - \frac{1}{\theta} \right).
\]

Taking the limit as \( n \to \infty \) we get \( 0 \geq \left( \frac{1}{q} - \frac{1}{\theta} \right) > 0 \) that is a contradiction.

Case 3: \( \|u_n\|_{s,p} \) is bounded and \( \|u_n\|_{s,q} \to \infty \).

The proof is similar to the previous one.

Summing up, \( (u_n) \) is bounded in \( X \) and we may assume, up to a subsequence, that \( u_n \to u \) in \( X \) and \( u_n \to u \) in \( L^1_{loc}(\mathbb{R}^N) \) for all \( t \in [1, q^*_s) \). In what follows, we aim to prove that \( \langle J'(u), \varphi \rangle = 0 \) for all \( \varphi \in X \). Firstly, we show that for all \( \varepsilon > 0 \) there exists \( R > 0 \) such that

\[
\limsup_{n \to \infty} \int_{|x| > R} |D^su_n|^p + |D^su_n|^qdx + \int_{|x| > R} |u_n|^p + |u_n|^qdx \leq \varepsilon. \tag{3.9}
\]

Take \( \eta \in C^\infty(\mathbb{R}^N) \) such that \( \eta = 0 \) if \( |x| < 1 \) and \( \eta = 1 \) if \( |x| > 2 \), and we set \( \eta_R(x) = \eta(x/R) \). Since \( (u_n\eta_R) \) is bounded, we can see that \( \langle J'(u_n\eta_R), \varphi \rangle = o_n(1) \) that is

\[
\int_{\mathbb{R}^N} \left( |u_n(x) - u_n(y)\right)^{p-2} (u_n(x) - u_n(y)) (u_n(x)\eta_R(x) - u_n(y)\eta_R(y)) dx dy
\]

\[
+ \int_{\mathbb{R}^N} \left( |u_n(x) - u_n(y)\right)^{q-2} (u_n(x) - u_n(y)) (u_n(x)\eta_R(x) - u_n(y)\eta_R(y)) dx dy
\]

\[
+ \int_{\mathbb{R}^N} (|u_n|^p + |u_n|^q) \eta_R dx = \lambda \int_{\mathbb{R}^N} f(u_n)u_n\eta_R dx + \int_{\mathbb{R}^N} |u_n|^q \eta_R dx + o_n(1). \tag{3.10}
\]

Then we have

\[
\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \eta_R(x) dx dy + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^q}{|x - y|^{N+sq}} \eta_R(x) dx dy + \int_{\mathbb{R}^N} (|u_n|^p + |u_n|^q) \eta_R dx
\]

\[
= - \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y)) u_n(x) dx dy
\]

\[
- \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{q-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y)) u_n(x) dx dy
\]

\[
+ \lambda \int_{\mathbb{R}^N} f(u_n)u_n\eta_R dx + \int_{\mathbb{R}^N} |u_n|^q \eta_R dx + o_n(1). \tag{3.11}
\]
Let us note that the boundedness of \((u_n)\) in \(X\) and Hölder inequality give

\[
\left| \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}} (\eta_R(x) - \eta_R(y)) u_n(x) dx dy \right| \\
\leq [u_n]_{p,p}^{-1} \left( \int_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} |u_n(x)|^p dx dy \right)^{\frac{1}{p}} \\
\leq C \left( \int_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} |u_n(x)|^p dx dy \right)^{\frac{1}{p}} \\
\leq C \left( \int_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^q}{|x-y|^{N+sp}} |u_n(x)|^q dx dy \right)^{\frac{1}{q}}. 
\]

(3.12)

and in similar way

\[
\left| \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}} (\eta_R(x) - \eta_R(y)) u_n(x) dx dy \right| \\
\leq C \left( \int_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^q}{|x-y|^{N+sp}} |u_n(x)|^q dx dy \right)^{\frac{1}{q}}. 
\]

(3.13)

Now, using Lemma 2.2 we deduce that

\[
\lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x-y|^{N+sp}} |u_n(x)|^p dx dy = 0 
\]

(3.14)

and

\[
\lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \frac{|\eta_R(x) - \eta_R(y)|^q}{|x-y|^{N+sp}} |u_n(x)|^q dx dy = 0. 
\]

(3.15)

On the other hand the strong convergence in \(L^1_{\text{loc}}(\mathbb{R}^N)\)-norm and the definition of \(\eta_R\) imply that

\[
\lim_{R \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} f(u_n) u_n \eta_R dx = 0 
\]

(3.16)

and

\[
\lim_{R \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^q \eta_R dx = 0. 
\]

(3.17)

Putting together (3.10)-(3.17) we can deduce that (3.9) holds. In particular we can see that \((u_n)\) is a bounded tight sequence in \(D^{s,q}(\mathbb{R}^N)\). Then, assuming that

\[
|D^s u_n|^q \to \mu \text{ and } |u_n|^q \to \nu, 
\]

(3.18)

we can apply Lemma 2.4 to see that there exist an at most countable index set \(I\), sequences \((x_i) \subset \mathbb{R}^N, (\mu_i), (\nu_i) \subset (0, \infty)\) such that

\[
\nu = |u|^q + \sum_{i \in I} \mu_i \delta_{x_i}, \quad \mu \geq |D^s u|^q + \sum_{i \in I} \mu_i \delta_{x_i}, \quad \text{and} \quad S_{s\rho_i}^{q/q_i} \leq \mu_i \quad \forall i \in I. 
\]

(3.19)

We aim to show that \(\nu_i = 0\) for all \(i \in I\). Assume by contradiction that \(x_i\) is a singular point of measures \(\mu\) and \(\nu\). For any \(\rho > 0\), we set \(\psi(x) = \psi\left(\frac{x-x_i}{\rho}\right)\), where \(\psi \in C_c^\infty(\mathbb{R}^N)\) such that \(0 \leq \psi \leq 1, \psi = 1 \text{ in } B_1 \text{ and } \psi = 0 \text{ in } B_2^c \text{ and } |\nabla \psi|_{\infty} \leq 2\). Since \((u_n \psi_\rho)\) is bounded in \(X\), we get \((J'(u_n), u_n \psi_\rho) = o_n(1)\) that is

\[
\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}} (u_n(x) \psi_\rho(x) - u_n(y) \psi_\rho(y)) dx dy + \int_{\mathbb{R}^N} |u_n|^p \psi_\rho dx \\
+ \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}} (u_n(x) \psi_\rho(x) - u_n(y) \psi_\rho(y)) dx dy + \int_{\mathbb{R}^N} |u_n|^q \psi_\rho dx \\
= \lambda \int_{\mathbb{R}^N} f(u_n) u_n \psi_\rho dx + \int_{\mathbb{R}^N} |u_n|^q \psi_\rho dx + o_n(1).
\]
Let us note that
\[ \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(u_n(x)\psi_\rho(x) - u_n(y)\psi_\rho(y))dxdy}{|x-y|^{N+sp}} \]
\[ = \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} \psi_\rho(x)dxdy + \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}} u_n(y)(\psi_\rho(x) - \psi_\rho(y))dxdy \]
so we can rewrite the above identity as follows
\[ \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}} u_n(y)(\psi_\rho(x) - \psi_\rho(y))dxdy + \int_{\mathbb{R}^N} |u_n|^p \psi_\rho dx \]
\[ + \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}} u_n(y)(\psi_\rho(x) - \psi_\rho(y))dxdy + \int_{\mathbb{R}^N} |u_n|^q \psi_\rho dx \]
\[ = - \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} \psi_\rho(x)dxdy - \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^q}{|x-y|^{N+sp}} \psi_\rho(x)dxdy \]
\[ + \lambda \int_{\mathbb{R}^N} f(u_n)u_n\psi_\rho dx + \int_{\mathbb{R}^N} |u_n|^{q^*} \psi_\rho dx + o_n(1). \]
(3.20)

Since
\[ \left| \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}} u_n(y)(\psi_\rho(x) - \psi_\rho(y))dxdy \right| \]
\[ \leq [u_n]_{s,p}^{p-1} \left( \int_{\mathbb{R}^2} \frac{\psi_\rho(x) - \psi_\rho(y)}{|x-y|^{N+sp}} |u_n(y)|^p dxdy \right)^{1/p} \]
\[ \leq C \left( \int_{\mathbb{R}^2} \frac{\psi_\rho(x) - \psi_\rho(y)}{|x-y|^{N+sp}} |u_n(y)|^p dxdy \right)^{1/p}, \]
we can use Lemma 2.3 to see that
\[ \lim_{\rho \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}} u_n(y)(\psi_\rho(x) - \psi_\rho(y))dxdy = 0 \]
(3.21)
and
\[ \lim_{\rho \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^2} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))}{|x-y|^{N+sp}} u_n(y)(\psi_\rho(x) - \psi_\rho(y))dxdy = 0. \]
(3.22)
We also have
\[ \lim_{\rho \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^p \psi_\rho dx = 0 = \lim_{\rho \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{q^*} \psi_\rho dx, \]
(3.23)
and using the fact that \( f \) has subcritical growth we get
\[ \lim_{\rho \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} f(u_n)u_n\psi_\rho dx = 0. \]
(3.24)
Putting together (3.20)-(3.24) and using (3.18) we can deduce that \( \nu_i \geq \mu_i \) which together (3.19) yields \( \nu_i \geq S_{\infty}^{\frac{N}{q^*}} \). Then, observing that
\[ c_* = J(u_n) - \frac{1}{\theta} \langle J'(u_n), u_n \rangle + o_n(1) \]
\[ \geq \left( \frac{1}{\theta} - \frac{1}{q^*_s} \right) \int_{\mathbb{R}^N} |u_n|^{q^*} dx + o_n(1) \]
\[ \geq \left( \frac{1}{\theta} - \frac{1}{q^*_s} \right) \int_{B_{\rho}(x_i)} \psi_\rho |u_n|^{q^*} dx + o_n(1), \]
and taking the limit as $n \to \infty$ we find
\[ c_s \geq \left( \frac{1}{\theta} - \frac{1}{q_s^*} \right) \sum_{i \in I} \psi_p(x_i) \nu_i = \left( \frac{1}{\theta} - \frac{1}{q_s^*} \right) \sum_{i \in I} \nu_i \geq \left( \frac{1}{\theta} - \frac{1}{q_s^*} \right) S_{q_s}^N \]
which gives a contradiction. Therefore, we can deduce that
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{{q_s}^*} dx = \int_{\mathbb{R}^N} |u|^{{q_s}^*} dx. \]  
(3.25)

From Brezis-Lieb Lemma [7] we get $u_n \to u$ in $L^{{q_s}^*}(\mathbb{R}^N)$. Moreover, by interpolation, we can see that
\[ u_n \to u \text{ in } L^t(\mathbb{R}^N) \quad \forall t \in (q, q_s^*). \]  
(3.26)

Then, from the Dominated Convergence Theorem it follows that
\[ \int_{\mathbb{R}^N} |u_n|^{{q_s}^* - 2} u_n u dx \to \int_{\mathbb{R}^N} |u|^{{q_s}^*} dx. \]  
(3.27)

Now, from (3.1), Hölder inequality and the boundedness of $(u_n)$ in $X$ (and then in $L^p(\mathbb{R}^N)$ and $L^r(\mathbb{R}^N)$) we can see that
\[ \left| \int_{\mathbb{R}^N} f(u_n)(u_n - u) dx \right| \leq \varepsilon |u_n|^{p-1} |u_n - u|_p + C \varepsilon |u_n|^{r-1} |u_n - u|_r \leq \varepsilon C + C \varepsilon |u_n - u|_r. \]

Since $r \in (q, q_s^*)$ we can use (3.26) to deduce that
\[ \limsup_{n \to \infty} \left| \int_{\mathbb{R}^N} f(u_n)(u_n - u) dx \right| \leq \varepsilon C \quad \forall \varepsilon > 0 \]
which gives
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} f(u_n)(u_n - u) dx = 0. \]  
(3.28)

Now we set
\[ A_n = \langle J'(u_n), u_n \rangle + \int_{\mathbb{R}^N} f(u_n) u_n dx + \int_{\mathbb{R}^N} |u_n|^{{q_s}^*} dx \]
\[ - \langle J'(u_n), u \rangle - \int_{\mathbb{R}^N} f(u_n) u dx - \int_{\mathbb{R}^N} |u_n|^{{q_s}^* - 2} u_n u dx \]
and
\[ B_n = \|u\|^{p}_{s,p} + \|u\|^{q}_{s,q} - \int_{\mathbb{R}^N} |u|^{p-2} u u_n dx - \int_{\mathbb{R}^N} |u|^{q-2} u u_n dx \]
\[ - \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} (u_n(x) - u_n(y)) dx dy \]
\[ - \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x - y|^{N+sq}} (u_n(x) - u_n(y)) dx dy. \]

Taking into account $J'(u_n) \to 0$, $(u_n)$ is bounded in $X$, (3.27) and (3.28) we can deduce that
\[ A_n = o_n(1). \]  
(3.29)

On the other hand, using the fact that $u_n \to u$ in $X$ we also have
\[ B_n = o_n(1). \]  
(3.30)
Let us note that

\[ A_n + B_n = \left[ \|u_n\|_{s,p}^p + \|u_n\|_{s,q}^q - \int_{\mathbb{R}^N} |u_n|^{p-2}u_nudx - \int_{\mathbb{R}^N} |u_n|^{q-2}u_nudx \right. \]

\[ \left. - \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} \cdot \frac{(u(x) - u(y))}{|x-y|^{N+sp}} dxdy \right. \]

\[ \left. - \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^q}{|x-y|^{N+sp}} \cdot \frac{(u(x) - u(y))}{|x-y|^{N+sp}} dxdy \right] \]

\[ + \left[ \|u\|_{s,p}^p + \|u\|_{s,q}^q - \int_{\mathbb{R}^N} |u|^{p-2}uudx - \int_{\mathbb{R}^N} |u|^{q-2}uudx \right. \]

\[ \left. - \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} (u_n(x) - u_n(y)) dxdy \right. \]

\[ \left. - \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x-y|^{N+sp}} (u_n(x) - u_n(y)) dxdy \right] \]

which can be rewritten as

\[ A_n + B_n = \int_{\mathbb{R}^N} \left[ \frac{|v_n(x,y)|^{p-2}v_n(x,y)}{|x-y|^{N+sp}} - \frac{|v(x,y)|^{p-2}v(x,y)}{|x-y|^{N+sp}} \right] dxdy \]

\[ + \int_{\mathbb{R}^N} \left[ \frac{|v_n(x,y)|^{q-2}v_n(x,y)}{|x-y|^{N+sp}} - \frac{|v(x,y)|^{q-2}v(x,y)}{|x-y|^{N+sp}} \right] dxdy \]

\[ + \int_{\mathbb{R}^N} (|u_n|^{p-2}u_n - |u|^{p-2}u)dx + \int_{\mathbb{R}^N} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx, \tag{3.31} \]

where

\[ v_n(x,y) := u_n(x) - u_n(y) \quad \text{and} \quad v(x,y) := u(x) - u(y). \]

Now, we recall the following useful inequalities:

\[ C_p |x-y|^p \leq (|x|^{p-2}x - |y|^{p-2}y, x-y) \quad \forall x, y \in \mathbb{R}^N, p \geq 2 \tag{3.32} \]

and

\[ C_p \frac{|x-y|^2}{(|x| + |y|)^2 - p} \leq (|x|^{p-2}x - |y|^{p-2}y, x-y) \quad \forall x, y \in \mathbb{R}^N, 1 < p < 2. \tag{3.33} \]

When \( p \geq 2 \) (and then also \( q > 2 \)), we can deduce that (3.31) and (3.32) yield

\[ A_n + B_n \geq C_p \left[ \int_{\mathbb{R}^{2N}} \frac{|(u_n - u)(x) - (u_n - u)(y)|^p}{|x-y|^{N+sp}} dxdy + \int_{\mathbb{R}^N} |u_n - u|^p dx \right] \]

\[ + C_q \left[ \int_{\mathbb{R}^{2N}} \frac{|(u_n - u)(x) - (u_n - u)(y)|^q}{|x-y|^{N+sp}} dxdy + \int_{\mathbb{R}^N} |u_n - u|^q dx \right]. \tag{3.34} \]

Putting together (3.29), (3.30) and (3.34) we deduce that as \( n \to \infty \)

\[ \|u_n - u\|_{s,p} \to 0 \quad \text{and} \quad \|u_n - u\|_{s,q} \to 0, \]

that is \( u_n \to u \) in \( X \) as \( n \to \infty \).

Let us consider the case \( 1 < p < 2 \). Then (3.33) and \( (|x| + |y|)^p \leq C_p(|x|^p + |y|^p) \) imply that

\[ [(|x|^{p-2}x - |y|^{p-2}y, x-y)]^2 \geq C_p \frac{|x-y|^p}{(|x|^p + |y|^p)^{\frac{p}{2}}}. \tag{3.35} \]
Taking into account (3.35) and applying Hölder inequality with exponents \( \frac{2}{2-p} \) and \( \frac{2}{p} \) we get

\[
C_p'' \int_{\mathbb{R}^N} \frac{|v_n(x, y) - v(x, y)|^p}{|x - y|^{N+sp}} \, dx \, dy
\]

\[
\leq \left[ \int_{\mathbb{R}^N} \frac{|v_n(x, y)|^p}{|x - y|^{N+sp}} + \frac{|v(x, y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right]^{\frac{2}{2-p}} \times \left[ \int_{\mathbb{R}^N} \left[ \frac{v_n(x, y)^{p-2}v_n(x, y)}{|x - y|^{N+sp}} - \frac{|v(x, y)|^{p-2}v(x, y)}{|x - y|^{N+sp}} \right] [v_n(x, y) - v(x, y)] \, dx \, dy \right]^{\frac{2}{p}},
\]

which gives

\[
C_p'' |u_n - u|_{s,p}^2 \leq ([u_{n}]^p_{s,p} + |u|^p_{s,p})^{\frac{(2-p)}{p}} \times \left[ \int_{\mathbb{R}^N} \left[ \frac{v_n(x, y)^{p-2}v_n(x, y)}{|x - y|^{N+sp}} - \frac{|v(x, y)|^{p-2}v(x, y)}{|x - y|^{N+sp}} \right] [v_n(x, y) - v(x, y)] \, dx \, dy \right],
\]

where we have used the fact that \((u_n)\) is bounded in \(X\). Similar arguments show that

\[
C_p'' |u_n - u|_p^2 \leq ([u_{n}]^p + |u|^p)^{\frac{(2-p)}{p}} \left[ \int_{\mathbb{R}^N} (|u_{n}|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \, dx \right]
\]

\[
\leq C \left[ \int_{\mathbb{R}^N} (|u_{n}|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \, dx \right].
\]

Putting together (3.37) and (3.38) we obtain

\[
C_p \|u_n - u\|_{s,p}^2
\]

\[
\leq \int_{\mathbb{R}^N} \left[ \frac{v_n(x, y)^{p-2}v_n(x, y)}{|x - y|^{N+sp}} - \frac{|v(x, y)|^{p-2}v(x, y)}{|x - y|^{N+sp}} \right] [v_n(x, y) - v(x, y)] \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^N} (|u_{n}|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \, dx.
\]

Now, if \(1 < p < 2 \leq q\) then we can argue as in (3.34) to see that

\[
C_q \|u_n - u\|_{s,q}^q
\]

\[
\leq \int_{\mathbb{R}^N} \left[ \frac{v_n(x, y)^{q-2}v_n(x, y)}{|x - y|^{N+sq}} - \frac{|v(x, y)|^{q-2}v(x, y)}{|x - y|^{N+sq}} \right] [v_n(x, y) - v(x, y)] \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^N} (|u_{n}|^{q-2}u_n - |u|^{q-2}u)(u_n - u) \, dx.
\]

Therefore, putting together (3.29), (3.30), (3.31), (3.39) and (3.40) we get

\[
o_n(1) = A_n + B_n \geq C_p \|u_n - u\|_{s,p}^2 + C_q \|u_n - u\|_{s,q}^q
\]

that is \(u_n \to u\) in \(X\) as \(n \to \infty\).

If \(1 < p < q < 2\), from the above arguments we deduce that

\[
o_n(1) = A_n + B_n \geq C_p \|u_n - u\|_{s,p}^2 + C_q \|u_n - u\|_{s,q}^2.
\]

In conclusion, \(u_n \to u\) in \(X\) as \(n \to \infty\) and this gives \(J'(u) = 0\).

Now, we give the proof of the main result of this work.
Proof of Theorem 1.1. In view of Lemma 3.3 there exists \( u \in X \) such that \( J'(u) = 0 \). If \( u \equiv 0 \) then \( \|u_n\| \to 0 \) otherwise we have \( c_* = 0 \). Hence, using Proposition 3.1 there exist \( (y_n) \subset \mathbb{R}^N \) and \( R, \gamma > 0 \) such that

\[
\liminf_{n \to \infty} \int_{B_R(y_n)} |u_n|^qdx \geq \gamma.
\]

Set \( v_n(x) = u_n(x + y_n) \). Then \( v_n \rightharpoonup v \neq 0 \) in \( X \). Moreover, it is easy to check that \( \|v_n\| = \|u_n\| , J(v_n) = J(u_n) \) and \( J'(v_n) = v_n(1) \). Therefore \( J'(v) = 0 \) and \( v \) is a nontrivial solution to (1.1). Finally, we prove that \( v \geq 0 \) in \( \mathbb{R}^N \). Let us observe that \( \langle J'(v), v^- \rangle = 0 \), where \( v^- = \min\{v, 0\} \), that is

\[
\begin{align*}
\iint_{\mathbb{R}^N} & \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x-y|^{N+sp}}(v^-(x) - v^-(y))dxdy + \int_{\mathbb{R}^N} |v|^{p-2}vv^- dx \\
& + \iint_{\mathbb{R}^N} \frac{|v(x) - v(y)|^{q-2}(v(x) - v(y))}{|x-y|^{N+sq}}(v^-(x) - v^-(y))dxdy + \int_{\mathbb{R}^N} |v|^{q-2}vv^- dx \\
& = \lambda \int_{\mathbb{R}^N} f(v)v^- dx + \int_{\mathbb{R}^N} |v|^{q^* -2}vv^- dx.
\end{align*}
\]

Then, recalling that \( f(t) = 0 \) for \( t \leq 0 \) and that

\[ |x-y|^{r-2}(x-y)(x^- - y^-) \geq |x-y|^r \quad \forall x, y \in \mathbb{R} \quad \forall r \geq 1, \]

we can deduce that

\[ \|v^-\|_{s,p}^p + \|v^-\|_{s,q}^q \leq 0, \]

which ends the proof of Theorem 1.1. \( \square \)

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