Holonomic quantum logic gates

Marie Ericsson
Department of Quantum Chemistry,
Uppsala University, Box 518, Se-751 20 Uppsala, Sweden
Email: marie.ericsson@kvac.uu.se

This is a brief overview of quantum holonomies in the context of quantum computation. We choose an adequate set of quantum logic gates, namely, a phase gate, the Hadamard gate, and a conditional-phase gate and show how they can be implemented by purely geometric means. Such gates may be more resilient to certain types of errors.

1 Introduction

Any quantum computation can be build out of simple operations involving only one or two quantum bits (qubits). Such operations are called quantum logic gates and a finite number of them suffices to construct any quantum boolean network, and therefore any quantum computer. Moreover, efficient and reliable quantum-coherent computation of arbitrarily long duration is possible, even with faulty components. That is, errors can be corrected faster than they occur, even if the error correction machinery is faulty. Such fault tolerance is only possible if the size of the systematic error in the quantum logic gates is not greater than 10^{-8} (possibly a pessimistic estimate). Motivated by this precision requirement we here consider quantum logic gates which are of a geometric / holonomic rather than dynamical origin and therefore are resilient to certain types of errors. They may offer the potential of an intrinsically fault-tolerant computation.

As a set of adequate (universal) quantum gates we choose a phase gate, the Hadamard gate, and a conditional-phase gate. In the standard notation, in which the computational basis is formed out of the two orthogonal states of a quantum bit (qubit) labelled as $|0\rangle$ and $|1\rangle$, the action of the gates is defined as follows:

- The phase gate: $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto e^{i\alpha} |1\rangle$, for some prescribed $\alpha$.
- The Hadamard gate: $|0\rangle \mapsto \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, $|1\rangle \mapsto \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$.
- Two-qubit conditional phase gate: $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |01\rangle$, $|10\rangle \mapsto e^{i\beta} |10\rangle$, $|11\rangle \mapsto e^{i\beta} |11\rangle$, again for some prescribed $\beta$.

In the following we provide a very brief description of the theory behind holonomic quantum logic gates.
2 Abelian and Non-Abelian Holonomy

If an energy eigenstate of a quantal system depends on a set of external parameters \( \lambda = \{ \lambda_1, \cdots, \lambda_n \} \) then an adiabatic cyclic variation of these parameters returns the system to its original state. The final state vector turns out to be related to the initial state vector via the product of a dynamical phase factor, which depends on the speed of the curve in the parameter space, and a geometric phase factor \( \exp(i\gamma) \), which depends only on the shape of path in parameter space,

\[
|\psi(t)\rangle = \exp(i\gamma) |\psi(0)\rangle \exp \left[ -\frac{i}{\hbar} \int_0^t E(\tau) \, d\tau \right].
\]  

(1)

If the state \( |\psi\rangle \) belongs to a degenerate subspace it remains in the degenerate subspace during an adiabatic evolution. However, the system returns in general to a final state \( |\psi(t)\rangle \) related to the initial state \( |\psi(0)\rangle \) by some unitary operator \( U \)

\[
|\psi(t)\rangle = U |\psi(0)\rangle \exp \left[ -\frac{i}{\hbar} \int_0^t E(\tau) \, d\tau \right],
\]  

(2)

where \( U \) depends only on the shape of the path in parameter space. Matrix \( U \) is a generalization of the geometric phase (multiplication by the \( \exp(i\gamma) \) factor) into non-Abelian cases.

In order to evaluate \( U \) let us choose \( N \)-fold energy degenerate reference states \( |n(\lambda)\rangle \), being local in parameter space. Then at any point on the adiabatic path in the parameter space \( \lambda(\tau) \) we can write

\[
|\psi_l(\tau)\rangle = \sum_n U_{ln}(\tau) |n(\lambda(\tau))\rangle .
\]  

(3)

In particular, for a closed loop the final state \( |\psi_l(t)\rangle = \sum_n U_{ln}(t) |n\rangle \) is related to the initial state, chosen to be \( |l\rangle \), via \( U_{ln}(t) \). These matrix elements can be evaluated as the path ordered line integral

\[
U = \mathcal{P} \exp \left[ i \int_0^t A_\lambda \frac{d\lambda}{d\tau} d\tau \right]
\]  

(4)

with the gauge potential defined as

\[
A_{ab,\lambda} = i \langle a | \frac{\partial}{\partial \lambda} | b \rangle .
\]  

(5)

In the Abelian case, i.e. the geometric phase, the path order is not necessary and the integral reduces to a regular linear integral.
3 Holonomic phase gate

The simplest case is a single qubit phase gate. The reference state \( |n(\theta, \phi)\rangle \) can be written as
\[
|n(\theta, \phi)\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle,
\] (6)
where \( \lambda = \{\theta, \phi\} \) are the spherical polar angles of the Bloch vector. The gauge potential reads
\[
A_\theta = i \langle n | \frac{\partial}{\partial \theta} | n \rangle = 0,
\] (7)
\[
A_\phi = i \langle n | \frac{\partial}{\partial \phi} | n \rangle = -\frac{1}{2} (1 - \cos \theta).
\] (8)

Suppose that a qubit, for example a spin half nucleus in a slowly varying magnetic field, undergoes a cyclic conical evolution with cone angle \( \theta \). Then the line integral of the gauge potential gives geometric phase \( \gamma = \pm \frac{\pi}{2} \Omega = \pm \pi (1 - \cos \theta) \), where the \( \pm \) signs depend on whether the system is in the eigenstate aligned with or against the field, and \( \Omega \) is the solid angle subtended by the conical circuit. Thus the two qubit states \( |0\rangle \) and \( |1\rangle \) may end up with geometric phases of the opposite sign, which gives a phase gate with shift \( 2\gamma \) between the two states.

The most common experimental realization is a qubit coupled to an oscillating electromagnetic field. If \( \omega_0 \) is the transition frequency of the qubit, \( \omega \) is the frequency of the oscillating field, and \( \omega_1 \) is the amplitude of the oscillating field then by controlling \( \omega \) and \( \omega_1 \) one can effectively implement the conical circuit equivalent to that of slowly varying magnetic field with \( \theta \) given by
\[
\cos \theta = \frac{\omega_0 - \omega}{\sqrt{(\omega_0 - \omega)^2 + \omega_1^2}}.
\] (9)

For details see, for example, Ekert et al.\[3\]. Note that any deformation of the path of the spin which preserves this solid angle leaves the phase unchanged. Thus the phase is not affected by the speed with which the path is traversed; nor is it very sensitive to random fluctuations about the path. For an experimental realization of this scheme see, for example, Suter et al.\[4\].

4 Holonomic conditional phase gate

Consider to begin with a system of two non-interacting spin-half particles \( S_a \) and \( S_b \). In a reference frame aligned with the static field, the Hamiltonian reads
\[
H_0 = \hbar \omega_a S_{az} \otimes 1_b + \hbar \omega_b 1_a \otimes S_{bz},
\] (10)
where the frequencies $\omega_a/2\pi$ and $\omega_b/2\pi$ are the transition frequencies of the two spins and we have used the scaled Pauli operators $S_i = \sigma_i/2$. (From now on we assume that $\omega_a$ and $\omega_b$ are very different with $\omega_a > \omega_b$.)

If the two particles are sufficiently close to each other, they will interact, creating additional splittings between the energy levels. In the case of two spin-half particles, the magnetic field of one spin may directly or indirectly affect the energy levels of the other spin; the energy of the system is increased by $\frac{\pi\hbar J}{2}$ if the spins are parallel and decreased by $\frac{\pi\hbar J}{2}$ if the spins are antiparallel.

The Hamiltonian of the system taking into account this interaction reads

$$H = H_0 + 2\pi\hbar JS_a z \otimes S_b z.$$  \hfill (11)

Figure 1 illustrates the energy levels of the system. When spin $S_b$ is in state $|\uparrow\rangle$, the transition frequency of the spin $S_a$ is

$$\omega_+ = \omega_a + \frac{\pi J}{2},$$  \hfill (12)

whereas when spin $S_b$ is in state $|\downarrow\rangle$, the transition frequency of the spin $S_a$ is

$$\omega_- = \omega_a - \frac{\pi J}{2}. $$  \hfill (13)

Now suppose that in addition to the static field, we apply a rotating field that is slowly varied as mentioned in the previous section. We have seen
that the Berry phase acquired by a spin depends on its transition resonance frequency as given by Eq. (9). Therefore, at the end of a cyclic evolution, the Berry phase acquired by the spin $S_a$ will be different for the two possible states of spin $S_b$. Indeed, when spin $S_b$ is in state $|\uparrow\rangle$, the Berry phase acquired by the spin $S_a$ is $\gamma_+ = \mp \pi (1 - \cos \theta_+)$, with the sign negative or positive depending on whether spin $S_a$ is up or down, respectively, and

$$\cos \theta_+ = \frac{\omega_+ - \omega}{\sqrt{(\omega_+ - \omega)^2 + \omega_1^2}}. \tag{14}$$

Similarly, when spin $S_b$ is in state $|\downarrow\rangle$, the Berry phase acquired by the spin $S_a$ is $\gamma_- = \mp \pi (1 - \cos \theta_-)$ where

$$\cos \theta_- = \frac{\omega_- - \omega}{\sqrt{(\omega_- - \omega)^2 + \omega_1^2}}. \tag{15}$$

The geometric phase difference $\gamma_+ - \gamma_-$ depends on the amplitude of the oscillating magnetic field $\hbar \omega_1$ in such a way that it has a maximum for a nonvanishing value of $\omega_1$. Thus, if $\omega_1$ is chosen to be close to this value, fluctuation errors are of second order and the implementation of the conditional phase gate is intrinsically fault tolerant.

This mechanism effectively implements the conditional phase gate such as the one demonstrated experimentally by Jones et al using the NMR technique.

5 Holonomic hadamard gate

The Hadamard gate requires non-Abelian holonomies. They have been analyzed from a theoretical point of view in using the path ordered integration. However, probably the simplest, experimentally viable construction has been presented by Bergmann et al and Duan et al. In this construction two degenerate qubit states $|0\rangle$ and $|1\rangle$ and two ancilla states $|a\rangle$ and $|b\rangle$ are used together with an interaction Hamiltonian

$$H = \hbar [b (\omega_0 |0\rangle + \omega_1 |1\rangle + \omega_a |a\rangle) + h.c.] \tag{16}$$

with the degenerate eigenvalues $\lambda_{1,2} = 0$ and the nondegenerate eigenvalues $\lambda_{3,4} = \pm \hbar \Omega$, where $\Omega = (\omega_0^2 + \omega_1^2 + \omega_a^2)^{1/2}$. If we parameterize $\omega_0 = B \sin \theta \cos \phi$, $\omega_1 = B \sin \theta \sin \phi$, and $\omega_a = B \cos \theta$ the eigenvectors for the degenerate eigenvalue read

$$\chi_1 = \sin \phi |0\rangle - \cos \phi |1\rangle$$
$$\chi_2 = \cos \theta \cos \phi |0\rangle + \cos \theta \sin \phi |1\rangle - \sin \theta |a\rangle. \tag{17}$$
A cyclic adiabatic evolution in the parameter space, starting and ending at \( \phi = 0 \) and with fixed \( \theta \), generates a non-Abelian holonomy of the form

\[
U = \exp \left[ i \sigma_y \int_0^t \dot{\phi} \cos \theta \, d\tau \right] = \begin{pmatrix} \cos \gamma - \sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix},
\]

where \( \gamma = \int_0^t \dot{\phi} \cos \theta \, d\tau \) is given by the swept solid angle in the space of parameters \( \{\theta, \phi\} \). For \( \gamma = \pi/4 \) the non-Abelian phase matrix will be the Hadamard gate.

6 Discussion

In all experimental realizations, in addition to the geometric phases there will also be dynamical phases, which depend on experimental details. In principle these could be calculated and corrected for using, for example, a conventional spin echo technique.\(^5\) We have implicitly assumed adiabatic schemes for implementations of the holonomic gates. This does not have to be the case for a nonadiabatic scheme of the conditional see, for example, Wang et al.\(^9\). The phase shifts gates, both the single qubit and the conditional phase shift gates have been implemented. The Hadamard gate, which requires non-Abelian holonomies is more difficult to implement by purely geometric means. However, the Duan et al. proposal\(^8\) gives hope that it will be implemented in a not too distant future.

Acknowledgments

I wish to thank Artur Ekert, Björn Hessmo, and Erik Sjöqvist for discussions. Financial support from the European Science Foundation Quantum Information Theory programme and the Swedish Research Council (NFR) is acknowledged.

References

1. M.V. Berry, Proc. R. Soc. London Ser. A 392, 45 (1984).
2. F. Wiczek and A. Zee, Phys. Rev. Lett. 52, 2111 (1984).
3. A. Ekert, M. Ericsson, P. Hayden, H. Inamori, J.A. Jones, D.K.L. Oi, and V. Vedral, J. Mod. Opt. 47, 2501 (2000).
4. D. Suter, G. Chingas, R. Harris, and A. Pines, Mol. Phys. 61, 1327 (1987).
5. J.A. Jones, V. Vedral, A. Ekert, and G Castagnoli, Nature 403, 869 (1999).
6. P. Zanardi and M. Rasetti, Phys. Lett. A 264, 94, (1999), J. Pachos, P. Zanardi, and M. Rasetti, Phys. Rev. A 61, 010305(R), (2000), J. Pachos and P. Zanardi, Int. J. Mod. Phys. B15 1257 (2001), and J. Pachos, \url{http://xxx.lanl.gov/abs/quant-ph/0003150}.
7. R.G. Unanyan, B.W. Shore, and K. Bergmann, Phys. Rev. A 59, 2910 (1999).
8. L.-M. Duan, J.I. Cirac, P. Zoller, Science 292, 1695 (2001).
9. W. Xiang-Bin and M. Keiji, Phys. Rev. Lett. 87, 097901 (2001).