The AdS/CFT Correspondence and Logarithmic Corrections to Braneworld Cosmology and the Cardy-Verlinde Formula

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ABSTRACT

The AdS/CFT correspondence is employed to derive logarithmic corrections to the Cardy-Verlinde formula when thermal fluctuations in the Anti-de Sitter black hole are accounted for. The qualitative effect of these corrections on the braneworld cosmology is investigated. The role of such terms in enabling a contracting universe to undergo a bounce is demonstrated. Their influence on the stability of black holes in AdS space and the Hawking-Page-Witten phase transitions is also discussed.

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1 Introduction

The holographic principle implies that the number of degrees of freedom associated with gravitational dynamics is determined by the boundary of the spacetime rather than by its bulk \[1\]. The anti-de Sitter/conformal field theory (AdS/CFT) correspondence represents a realization of this principle by providing a duality between classical $d$–dimensional gravity in AdS space and a quantum CFT located on its boundary \([2, 3, 4]\). A further connection between higher–dimensional classical gravity and lower–dimensional quantum physics was uncovered by Verlinde \([5]\), who showed that the standard Friedmann–Robertson–Walker (FRW) equations for a (spatially closed) radiation dominated universe can be rewritten in a form that is equivalent to the Cardy \([6]\) formula for the entropy of a two–dimensional CFT.

Insight into the origin of such a Cardy–Verlinde formula can be gained from the braneworld scenario. It has recently become clear that braneworld cosmology is closely related to the AdS/CFT correspondence and, in particular, the Randall–Sundrum braneworld \([7]\) may be viewed as a specific manifestation of this framework, where the CFT–dominated universe is interpreted as a co–dimension one brane representing the (dynamical) boundary of Schwarzschild–Anti–de Sitter (SAdS) space \([8, 9]\). The motion of the brane away from the black hole is interpreted by an observer on the brane in terms of Hubble expansion and the classical dynamics of the expansion is formally equivalent to that of a radiation–dominated FRW universe \([10]\). Following Witten’s identification of the entropy of the AdS black hole with the entropy of the dual CFT \([8]\), it can be shown that the FRW equations correspond to the Cardy entropy formula of the CFT when the brane passes through the black hole event horizon \([9]\). Furthermore, such a form for the FRW equation defines a new dynamical bound on the cosmological entropy \([8]\). (For a recent review and list of references, see \([11]\)).

In the present letter, we take into account corrections to the entropy of the five–dimensional AdS black hole that arise due to thermal fluctuations around its equilibrium state. It has been known for some time that such fluctuations result in logarithmic corrections to the black hole entropy \([12, 13, 14, 15, 16, 17, 18, 19]\) and this is also the case for AdS black holes. Previous studies of the CV formula (or the corresponding FRW equation and cosmological entropy bound) in an AdS/CFT context have thus far neglected this sub–dominant, but important, contribution. In this letter, it is shown that these effects result in logarithmic corrections to the CV formula and the brane FRW equations. Moreover, the Hawking–Page phase transitions \([20]\), interpreted by Witten as confinement–deconfinement transitions in the dual CFT \([8]\), may also be influenced by these corrections. The modified CV formula is derived in Section 2 and we proceed to investigate the cosmological implications of

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5Of course, the precise relation is dependent on the form of bulk space and, in particular, whether it is a pure or asymptotically AdS space, an AdS black hole, or even de Sitter space. In practice, each possibility should be considered independently.
the logarithmic corrections in Section 3. We conclude with a discussion in Section 4.

2 Logarithmic Corrections to the Cardy–Verlinde Formula

The metric of the five–dimensional SAdS black hole is given by

\[ ds_5^2 = -e^{2\rho} dt^2 + e^{-2\rho} da^2 + a^2 \sum_{ij} g_{ij} dx^i dx^j, \]

\[ e^{2\rho} = \frac{1}{a^2} \left( -\mu + \frac{k}{2} a^2 + \frac{a^4}{l^2} \right), \] (1)

where \( l \) represents the curvature radius of the SAdS bulk space. The three–metric, \( g_{ij} \), is the metric of an Einstein space with Ricci tensor given by \( r_{ij} = kg_{ij} \), where the constant \( k = \{-2, 0, +2\} \) in our conventions. The induced metric on the brane is then negatively–curved, spatially flat, or positively–curved depending on the sign of \( k \).

The mass of the black hole is parametrized by the constant, \( \mu \), and may be expressed directly in terms of the horizon radius, \( a_H \), of the black hole:

\[ \mu = a_H^2 \left( \frac{a_H^2}{l^2} + \frac{k}{2} \right), \] (2)

where

\[ a_H^2 = \frac{-kl^2}{2} + \frac{1}{2} \sqrt{l^4 + 4\mu l^2} \] (3)

is the largest solution to the constraint equation \( \exp[2\rho(a_H)] = 0 \).

The CV formula is derived from the thermodynamical properties of the five–dimensional black hole. The free energy, \( F \), the entropy, \( S \), the thermodynamical energy, \( E \), and the Hawking temperature, \( T_H \), of the black hole are given, respectively, by

\[ F = -\frac{W_3}{16\pi G_5} a_H^2 \left( \frac{a_H^2}{l^2} - \frac{k}{2} \right), \quad S = \frac{W_3 a_H^3}{4G_5}, \] (4)

\[ E = F + T_H S = \frac{3W_3 \mu}{16\pi G_5}, \quad T_H = \frac{k}{4\pi a_H} + \frac{a_H}{\pi l^2}, \] (5)

where \( W_3 \) is the volume of the unit three–sphere and \( G_5 \) denotes the five–dimensional Newton constant.

In this paper, we are interested primarily in the corrections to the entropy (4) that arise due to thermal fluctuations. An expression for the leading–order correction
has been found for a generic thermodynamic system \[18\]. The entropy is calculated in terms of a grand canonical ensemble, where the corresponding density of states, \( \rho \), is determined by performing an inverse Laplace transformation of the partition function\[6\]. The integral that arises in this procedure is then evaluated in an appropriate saddle–point approximation. The correction to the entropy follows by assuming that the scale, \( \epsilon \), defined such that \( S \equiv \ln(\epsilon \rho) \), varies in direct proportion to the temperature, since this latter parameter is the only parameter that provides a physical measure of scale in the canonical ensemble. The final result is then of the form \[19\]:

\[
S = S_0 - \frac{1}{2} \ln C_v + \ldots ,
\]

(6)

where \( C_v \) is the specific heat of the system evaluated at constant volume and \( S_0 \) represents the uncorrected entropy. In the case of the AdS black hole, this is given by Eq. (4). The specific heat of the black hole is determined in terms of this entropy:

\[
C_v \equiv \frac{3 W_3}{16 \pi G_5} \frac{d \mu}{dT_H} = \frac{3}{2} a_H^2 + \frac{k l^2}{2 a_H^2} S_0
\]

(7)

and for consistency, the condition \( a_H > l^2 k/4 \) must be satisfied to ensure that the specific heat is positive. In this limit, \( C_v \approx 3 S_0 \), and this implies that \[19\]

\[
S = S_0 - \frac{1}{2} \ln S_0 + \ldots .
\]

(8)

Given the form of the logarithmic correction (8) to the entropy, it is now possible to derive the corresponding corrections to the CV formula. We begin by recalling that the four–dimensional energy, derivable from the FRW equation of motion for a brane propagating in a SAdS background, is given by

\[
E_4 = \frac{3 W_3 l \mu}{16 \pi G_5 a}
\]

(9)

and is related to the five–dimensional energy (3) of the bulk black hole such that \( E_4 = (l/a) E \). This implies that the temperature, \( T \), associated with the brane should differ from the Hawking temperature (5) by a similar factor (3):

\[
T = \frac{l}{a} T_H = \frac{a_H}{\pi a} + \frac{lk}{4 \pi a a_H}.
\]

(10)

In determining the corrections to the entropy, a crucial physical parameter is the Casimir energy, \( E_C \), defined in terms of the four–dimensional energy, \( E_4 \), pressure, \( p \), volume, \( W \), temperature, \( T \), and entropy, \( S \):

\[
E_C = 3 \left( E_4 + pW - TS \right) .
\]

(11)

\[6\]The reader is referred to Refs. [18, 19] for details.
This quantity vanishes in the special case where the energy and entropy are purely extensive, but in general, this condition does not hold. For the present discussion, the total entropy is assumed to be of the form (8), where the uncorrected entropy, $S_0$, corresponds to that associated with the black hole in Eq. (4) (due to the AdS/CFT correspondence). It then follows by employing (9) and (10) that the Casimir energy (11) can be expressed directly in terms of the uncorrected entropy:

$$E_C = \frac{3\alpha^2 W_3 k}{16\pi G S} + \frac{3}{2} T \ln S_0,$$

where the direct dependence on the pressure has been eliminated by means of the relation $p = E_4/(3W)$.

Moreover, in the limit where the logarithmic correction in Eq. (12) is small, it can be shown, after substitution of Eqs. (4), (9), and (12), that the four–dimensional and Casimir energies are related to the uncorrected entropy by (13):

$$S_0 = \frac{4\alpha a}{3\sqrt{|k|}} \left| E_C \left( E_4 - \frac{1}{2} E_C \right) \right| + \frac{\pi a l}{ka_H^2} T \left( \frac{a_H^2}{l^2} - \frac{k}{2} \frac{a^2_H}{2} \right) \ln S_0.$$

The coefficient of the logarithmic term on the right–hand side of Eq. (13) is constant. In the limit where the correction is small, this constant can be expressed directly in terms of the four–dimensional and Casimir energies through the relationship:

$$\frac{\pi a l}{ka_H^2} T \left( \frac{a_H^2}{l^2} - \frac{k}{2} \frac{a^2_H}{2} \right) = \frac{(4E_4 - E_C)(E_4 - E_C)}{2(2E_4 - E_C) E_C},$$

where we have substituted in Eq. (10) for the temperature and have also employed the relation

$$\frac{E_4 - \frac{1}{2} E_C}{E_C} = \frac{a^2_H}{k l^2}.$$

We may conclude, therefore, that in the limit where the logarithmic corrections are sub–dominant, Eq. (13) can be rewritten to express the entropy directly in terms of the four–dimensional and Casimir energies (corrected Cardy–Verlinde formula (14)):

$$S_0 = \frac{4\alpha a}{3\sqrt{|k|}} \left| E_C \left( E_4 - \frac{1}{2} E_C \right) \right| - \frac{(4E_4 - E_C)(E_4 - E_C)}{2(2E_4 - E_C) E_C} \ln \left( \frac{4\alpha a}{3\sqrt{|k|}} \left| E_C \left( E_4 - \frac{1}{2} E_C \right) \right| \right),$$

and, consequently, the total entropy Eq. (8), to first–order in the logarithmic term, is given by (16):

$$S \simeq \frac{4\alpha a}{3\sqrt{|k|}} \left| E_C \left( E_4 - \frac{1}{2} E_C \right) \right|$$
\[-\frac{E_4 (4E_4 - 3E_C)}{2 (2E_4 - E_C) E_C} \ln \left( \frac{4\pi a}{3\sqrt{|k|}} \sqrt{E_C \left( E_4 - \frac{1}{2} E_C \right)} \right). \tag{17} \]

It then follows that the logarithmic corrections to CV formula are given by the second term on right–hand–side of Eq. (17) and the magnitude of this correction can be deduced by taking the logarithm of the original CV formula. As we saw in above discussion these corrections are caused by thermal fluctuations of the AdS black hole.

Finally, the four–dimensional FRW equation also receives corrections as a direct consequence of the logarithmic correction arising in Eq. (17). In general, the Hubble parameter, \( H \), is related to the four–dimensional (Hubble) entropy by

\[ H^2 = \left( \frac{2G_4}{W} \right)^2 S^2, \tag{18} \]

where the effective four–dimensional Newton constant, \( G_4 \), is related to the five–dimensional Newton constant, \( G_5 \), by \( G_4 = 2G_5/l \). Hence, by substituting Eq. (17) into Eq. (18), it can be shown by employing Eqs. (2), (4), (10), (12), (16) and (17), that the four–dimensional FRW equation is given by

\[
H^2 = \left( \frac{2G_4}{W} \right)^2 \left[ \frac{4\pi a}{3\sqrt{|k|}} \right]^2 \left| 
E_C \left( E_4 - \frac{1}{2} E_C \right) \right| - \frac{4\pi a}{3\sqrt{|k|}} \frac{E_4 (4E_4 - 3E_C)}{(2E_4 - E_C) E_C} \\
\times \sqrt{E_C \left( E_4 - \frac{1}{2} E_C \right)} \ln \left( \frac{4\pi a}{3\sqrt{|k|}} \sqrt{E_C \left( E_4 - \frac{1}{2} E_C \right)} \right) \right] \\
= -\frac{k}{2a_H^2} + \frac{8\pi G_4}{3} \rho - \frac{2G_4}{W} \ln S_0, \tag{19} \]

where logarithmic corrections have been included up to first–order in the logarithmic term, the effective energy density is defined by \( \rho = E_4/W \) and \( W = a_H^3 V_3 \) parametrizes the spatial volume of the world–volume of the brane. Since the first term on the right–hand–side of Eq. (19) is identical to the standard (radiation–dominated) FRW equation in the limit where the scale factor, \( a \), of the brane coincides with the horizon radius, \( a_H \), of the black hole, the logarithmic corrections for the FRW equation are given by the second term on the right–hand–side in terms of the uncorrected entropy \( S_0 \) of the black hole.

### 3 Qualitative Dynamics of the Brane Cosmology

In this section we investigate the asymptotic behaviour of the FRW brane cosmology when the logarithmic corrections to the CV formula are included. Formally, the FRW equation (19) holds precisely at the instant when the brane crosses the black hole horizon. Here we extend the analysis to consider an arbitrary scale factor, \( a \), where the
world–volume of the brane is given by the line–element $ds^2 = -d\tau^2 + a^2(\tau)g_{ij}dx^i dx^j$. Thus, the FRW equation is given by

$$H^2 = -\frac{k}{2a^2} + \frac{8\pi G_4}{3} \rho - \frac{2G_4}{Wl} \ln S_0,$$

where $W = a^3 W_3$ and $S_0 = W_3 a^3/(4G_5)$. Eq. (20) can be rewritten in such a way that it represents the conservation of energy of a point particle moving in a one–dimensional effective potential, $V(a)$:

$$\left(\frac{da}{d\tau}\right)^2 = -\frac{k}{2} - V(a)$$

$$V(a) \equiv -\frac{8\pi G_4}{3} a^2 \rho + \frac{2G_4 a^2}{Wl} \ln S_0,$$

where, in this interpretation, the variable $a$ represents the position of the particle. Since $\rho \propto a^{-4}$, the first term in the effective potential (22) redshifts as $a^{-2}$ as the brane moves away from the black hole horizon. This term is often referred to as the ‘dark radiation’ term.

To proceed, let us briefly recall the behaviour of the standard FRW cosmology, whose effective potential includes only the first term on the right–hand side of Eq. (22). The behaviour of this potential is illustrated in Fig. 1. The brane exists only in regions where the line $V(a) \leq -k/2$ (so that $H^2 > 0$). For the case of $k = 2$, the spherical (de-Sitter) brane expands from an initial state at $a = 0$ and reaches a maximal size $a_{\text{max}}$ before re-collapsing. On the other hand, the brane expands to infinity for the spatially flat ($k = 0$) and hyperbolic ($k = -2$) geometries.

![Figure 1: The standard effective potential for the evolution of FRW universe. For the case of $k = 2$, the spherical (de-Sitter) brane starts from $a = 0$ and reaches its maximal size $a_{\text{max}}$ and then it re-collapses.](image)
Figure 2: The effective potential (22) with logarithmic corrections included. The continuous line represents the potential for the case where it crosses the line $V(a) = 1$ at a finite value of the scale factor. The dotted line corresponds to the region of parameter space where $V(a) < 1$ for all values of the scale factor. As we discuss in the text, the dotted potential arises if $l_P/l \ll 1$, which is the limit expected from the point of view of the AdS/CFT correspondence.

The logarithmic correction to the effective potential (22) is proportional to $(\ln a)/a$ and the corrected form of the potential is shown in Fig. 2. The asymptotic behaviour of the brane is now as follows:

- $k = 2$: The behaviour of the positively curved brane is qualitatively similar to that of the uncorrected scenario. The brane expands from an initial state of vanishing spatial volume, $a = 0$, and reaches a point of maximal expansion, $a_{\text{max}}$, where this latter quantity is defined by the constraint equation $V(a_{\text{max}}) = -1$ in Eq. (22). Re-collapse to zero volume then ensues.

- $k = 0$: As in the closed case, the spatially flat brane has vanishing spatial volume initially. However, the late–time behaviour of this model is radically altered by the logarithmic correction term. The brane reaches a maximal size, $a_{\text{max}}$, where $V(a_{\text{max}}) = 0$, and then undergoes a recollapse. This change in behaviour arises as a direct result of the logarithmic term.

- $k = -2$: There are a number of possible outcomes for the hyperbolic brane depending on the choice of parameters. In the case where the condition $V(a) < 1$ is always satisfied, the brane expands to infinity. On the other hand, if the potential exceeds the critical value of unity, the brane reverses its direction of motion through the bulk space. Moreover, the constraint equation $V(a) = 1$ admits two solutions $a_1, a_2$, ($a_1 < a_2$). The brane may either expand from $a = 0$ to reach a maximal size, $a_1$, before re-collapsing, or alternatively, it may initially have infinite spatial volume and undergo a collapse to a minimal size.
$a_2$, re-expanding to infinity. This latter behaviour is an example of a ‘bouncing’ cosmology. Such behaviour is not possible within the context of the standard FRW equation.

We now proceed to examine the qualitative behaviour outlined above in more detail. It proves convenient to define a rescaled scale factor

$$b \equiv \left( \frac{W_3}{2lG_4} \right)^{1/3} a$$

and rescaled parameters

$$\tilde{k} = \frac{k}{2} \left( \frac{W_3}{2lG_4} \right)^{2/3}, \quad \tilde{\mu} = \left( \frac{W_3}{2lG_4} \right)^{4/3} \mu.$$ \hspace{1cm} (24)

The Friedmann equation (20) may then be expressed in the form

$$\frac{\dot{b}^2}{b^2} = -\tilde{k} \frac{b^2}{b^2} + \tilde{\mu} \frac{3 \ln b}{b^2} - 3 \ln b,$$ \hspace{1cm} (25)

where we have employed Eq. (9) to re–introduce the black hole mass parameter, $\mu$.

The qualitative dynamics of the brane can be determined in terms of the function $f \equiv -b \ln b$ by a further rewriting of the Friedmann equation to

$$H^2 = \frac{1}{b^4} \left[ \tilde{\mu} - \tilde{k} b^2 + 3 \ln b f(b) \right].$$ \hspace{1cm} (26)

The first and third terms on the right–hand–side of Eq. (26) are semi positive–definite in the region $0 \leq b \leq 1$. The importance of the function $f(b)$ is that it becomes negative for $b > 1$. Thus, a recollapse of the brane may ensue if the magnitude of the third term of the right hand side of Eq. (26) comes to dominate. This function exhibits only one turning point, a maximum at $b = e^{-1}$, such that $f_{\text{max}} = e^{-1}$ and this implies that $f \to 0$ as $b \to 0$. Thus, as the scale factor tends to zero, the logarithmic correction dominates the curvature term, but both these terms are themselves dominated by the dark radiation term. Consequently, the asymptotic behaviour of the scale factor in the limit of small spatial volume corresponds to that of a radiation/CFT–dominated universe, $a \propto t^{1/2}$.

Let us now focus on the spatially flat model, $k = 0$. Since $0 \leq f \leq e^{-1}$ for $0 \leq b \leq 1$, the logarithmic term can never dominate the dark radiation if $\tilde{\mu} > 3/(el^2)$. Conversely, it may dominate for a finite time if $\tilde{\mu} < 3/(el^2)$ in the range $0 < b < 1$. When the correction term does dominate, introducing a new variable $y \equiv (-\ln b)^{1/2}$, implies that the Friedmann equation can be written as

$$\dot{y} = \frac{\sqrt{3}}{2l} \exp \left[ \frac{3y^2}{2} \right]$$ \hspace{1cm} (27)
and can therefore be integrated in terms of the Error function:

\[ t = \frac{\sqrt{2\pi l}}{3} \text{erf} \left[ \left( -\frac{3}{2} \ln b \right)^{1/2} \right]. \]  

(28)

It is of interest to determine whether the logarithmic correction can result in an epoch of inflationary, accelerated expansion. A necessary and sufficient condition for inflation is that \( \dot{H} + H^2 > 0 \). When the logarithmic term dominates, it follows from the Friedmann equation (26) that

\[ \dot{H} + H^2 = -\frac{3}{2l^2 b^3} (1 - \ln b) \]  

(29)

and since the logarithmic term can only dominate for \( b < 1 \) (without causing recollapse), Eq. (29) implies that the expansion rate is always decelerating and, consequently, the brane does not exhibit inflationary expansion.

Finally, as remarked above, since \( f < 0 \) for all \( b > 1 \), it follows that the logarithmic term inevitably comes to dominate the Friedmann equation as the brane expands, thereby causing the brane to recollapse. This is radically different to the behaviour of the standard, spatially flat universe.

In the case of negative spatial curvature, \( k = -2 \), the logarithmic term may dominate in the region \( b < 1 \), depending on the choice of parameter values. The interesting question in this case, however, is whether the effects of the curvature are sufficient to prevent a recollapse of the brane. A necessary and sufficient condition from Eq. (26) for the brane to expand to infinity is that

\[ \bar{\mu} + |\bar{k}|b^2 > \frac{3}{l^2} b \ln b \]  

(30)

for all \( b > 1 \). Since \( \bar{\mu} > 0 \), it is sufficient to show that the stronger constraint

\[ b > m \ln b, \quad m \equiv 3 \left( \frac{2}{W_3} \right)^{2/3} \left( \frac{l_P}{l} \right)^{4/3} \]  

(31)

is satisfied, where \( l_P^2 = G_4 \) is the four–dimensional Planck length and we have employed Eq. (24). If \( m < 1 \), the gradient of the function \( h = b \) always exceeds the gradient of the function \( h = m \ln b \) for \( b \geq 1 \). Consequently, since the condition (31) is trivially satisfied at \( b = 1 \), it implies that it is always satisfied for \( b > 1 \). This implies that the curvature term always dominates the logarithmic term when \( b > 1 \). Hence, the Hubble parameter never passes through zero, and recollapse does not take place. Instead, the brane asymptotes to the Milne universe, where \( a \propto t \) at late times. It is interesting that this conclusion is independent of the relative magnitude of the spatial curvature term in the FRW equation (20) at any given epoch. The brane may move an infinite distance from the bulk black hole, and thereby effectively escape.
from its gravitational influence, even if the contribution of the curvature term to the right-hand-side of Eq. (20) is arbitrarily small at very early times ($b \ll 1$).

The above analysis indicates that the numerical value of the parameter, $m$, determines which of the different types of behaviour for the hyperbolic brane are physically realistic. A turning point in the expansion (or contraction) is possible only if $m > 1$. This parameter is specified by the ratio of the AdS length parameter, $l$, and the four-dimensional Planck length, $l_P$, and these parameters can in turn be determined from the five-dimensional Newton constant, $G_5$, and the bulk cosmological constant, $\Lambda$. In principle, $m > 1$ is therefore possible since it is defined in terms of the ratio of two independent parameters. However, from the perspective of the AdS/CFT correspondence in the large $N$ limit, it is expected that $l/l_P \sim N$ and, consequently, small values for $m$ are favoured from a physical point of view.

4 Discussion

In summary, we have considered the FRW dynamics of a brane propagating in an AdS bulk space containing a black hole. Taking into account thermal fluctuations of the black hole entropy and employing the AdS/CFT correspondence then leads to the appearance of logarithmic corrections to the Cardy–Verlinde formula and the associated brane FRW equations. A qualitative analysis of the role of such logarithmic terms in brane cosmology was performed and the regions of parameter space where the early– or late–time cosmology is altered were highlighted. In particular, an open universe that initially collapses and undergoes a bounce is allowed, in contrast to the standard FRW cosmology.

We now conclude with a discussion how these corrections may influence the stability of AdS black holes. It is well known that black holes in AdS space undergo a ‘Hawking–Page’ phase transition when the temperature of the black hole reaches a critical value [20]. The transition between the AdS black hole and pure AdS space occurs because the former is unstable at low temperatures and the energetically preferred state corresponds to pure AdS space. At high temperatures, on the other hand, the black hole is stable and consequently does not decay into AdS space. This implies that the stability of a specific AdS space is determined in terms of the corresponding black hole dynamics.

Such a phase transition has been interpreted by Witten within the context of the AdS/CFT correspondence in terms of the confinement–deconfinement transition in the large $N$–limit of the boundary super Yang–Mills theory [3]. Specifically, deconfinement occurs when the expectation value of the temporal Wilson loop operator for the super Yang–Mills theory is non–zero. (This corresponds to the phase where the AdS black hole is stable). By contrast, confinement occurs when this expectation value vanishes (the global AdS space is stable). Of course, higher–derivative terms originating from next–to–leading corrections in the large–$N$ expansion may influence the structure of such an AdS phase transition [21]. Moreover, as we now demonstrate,
the logarithmic terms discussed above that arise in the FRW brane cosmology may also alter the stability of the AdS black hole and consequently may introduce new features into the nature of the Hawking–Page phase transition.

Indeed, let us consider an AdS black hole with a temperature and energy given by Eq. (5) and a corrected entropy given by Eq. (8), where $S_0$ corresponds to the entropy in Eq. (4), i.e., $S_0 = W_3 a_H^3 / (4 G_5)$. The free energy of the black hole with the logarithmic correction included then follows immediately from the definition, $F \equiv E - T_H S$:

$$F = -\frac{W_3}{16\pi G_5} a_H^2 \left(\frac{a_H^2}{l^2} - 1\right) + \frac{1}{2} \left(\frac{1}{2\pi a_H} + \frac{a_H}{\pi l^2}\right) \ln \frac{W_3 a_H^3}{4 G_5} ,$$

(32)

where we have assumed implicitly that $k = 2$. In the absence of any correction term, the free energy is negative (positive) if $a_H^2 > l^2$ ($a_H^2 < l^2$). Consequently, a large black hole (large $a_H$) is stable whereas small ones are unstable. In the case of a large black hole, the correction term to the free energy in Eq. (32) can be neglected. On the other hand, this term becomes dominant for small $a_H$ and this implies that for sufficiently small black holes, the free energy becomes negative. As a result, (very) small black holes become stable when the correction term is included. This conclusion may also hold for small black holes even in a flat background, where the AdS length parameter, $l$, tends to infinity. In this case, small primordial black holes may be stabilized by the logarithmic correction and this indicates that the lifetime of primordial black holes that may have formed in the early universe would be enhanced relative to that inferred from a standard analysis.

In view of this, it is important to derive a quantitative estimate of the role of these logarithmic terms in the Hawking–Page–Witten phase transition [3, 22]. When $k = 2$, regularization of the action when the black hole solution is substituted follows by subtracting the action of the AdS vacuum. In the case where $k = 0$, however, it has been argued that it is more straightforward to subtract the action of the AdS soliton [23] instead of the AdS vacuum [24]. When $k = 0$, the AdS black hole metric is given by

$$ds_{BH}^2 = -e^{2\mu_{BH}}(r) dt_{BH}^2 + e^{-2\mu_{BH}}(r) dr^2 + r^2 \left(d\phi_{BH}^2 + \sum_{i=1,2} (dx_i)^2\right) e^{2\mu_{BH}(r)} = \frac{1}{r^2} \left\{-\mu_{BH} + \frac{r^4}{l^2}\right\}.$$  

(33)

We assume for simplicity that the three–dimensional Einstein manifold spanned by the coordinates $\{\phi_{BH}, x^1, x^2\}$ is a torus, where $\phi_{BH}$ has a period of $\eta_{BH}$, i.e., we identify $\phi_{BH} \sim \phi_{BH} + \eta_{BH}$. The AdS soliton solution can then be obtained by exchanging the signature of $t_{BH}$ and $\phi_{BH}$ such that $t_{BH} \rightarrow i\phi$ and $\phi_{BH} \rightarrow it$. One then obtains the
free energy, $F$, and the entropy, $S$, in the following forms:

$$ F = F_0 \equiv -\frac{\eta_{BH} W_2 l^6}{\kappa^2} \left\{ \left( \pi T_{BH} \right)^4 - \left( \frac{\pi}{l \eta_{BH}} \right)^4 \right\}, \quad S = S_0 = \frac{4 \eta_{BH} W_2 l^6 \pi^4}{\kappa^2} T_{BH}^3, \quad (34) $$

where $T_{BH}$ is the Hawking temperature of the black hole and $W_2$ is the volume of the two–torus spanned by $\{x^i\}$. Eq. (34) implies that there is a phase transition when $T_{BH} = (l \eta_{BH})^{-1}$. It then follows that the black hole is stable when $T_{BH} > (l \eta_{BH})^{-1}$, but unstable when $T_{BH} < (l \eta_{BH})^{-1}$. In this latter case, the AdS soliton is the preferred state.

When the logarithmic correction is included and $F \to F_0 + \frac{1}{2} T_{BH} \ln S_0$, we find that

$$ F = -\frac{\eta_{BH} W_2 l^6}{\kappa^2} \left\{ \left( \pi T_{BH} \right)^4 - \left( \frac{\pi}{l \eta_{BH}} \right)^4 \right\} + \frac{1}{2} \ln \left( \frac{4 \eta_{BH} W_2 l^6}{\kappa^2} T_{BH}^3 \right). \quad (35) $$

Consequently, when $T_{BH}$ is large, the correction does not alter the behaviour of the free energy, $F$, but when $T_{BH}$ is small, it becomes dominant and makes the free energy negative, thereby indicating that the black hole becomes stable at low temperatures. It would be very interesting to investigate in more detail the role of these logarithmic terms in the dual CFT confinement–deconfinement phase transitions.

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