A counterexample to $L^\infty$-gradient type estimates for Ornstein-Uhlenbeck operators

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Abstract

Let $(\lambda_k)$ be a strictly increasing sequence of positive numbers such that $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$. Let $f$ be a bounded smooth function and denote by $u = u^f$ the bounded classical solution to $u(x) - \frac{1}{2} \sum_{k=1}^{m} D^2_{kk} u(x) + \sum_{k=1}^{m} \lambda_k x_k D_k u(x) = f(x)$, $x \in \mathbb{R}^m$. It is known that the following dimension-free estimate holds:

$$\int_{\mathbb{R}^m} \left( \sum_{k=1}^{m} \lambda_k (D_k u(y))^2 \right)^{p/2} \mu_m(dy) \leq (c_p)^p \int_{\mathbb{R}^m} |f(y)|^p \mu_m(dy), \quad 1 < p < \infty;$$

here $\mu_m$ is the “diagonal” Gaussian measure determined by $\lambda_1, \ldots, \lambda_m$ and $c_p > 0$ is independent of $f$ and $m$. This is a consequence of generalized Meyer’s inequalities [3]. We show that, if $\lambda_k \sim k^2$, then such estimate does not hold when $p = \infty$. Indeed we prove

$$\sup_{f \in C^2_b(\mathbb{R}^m), \|f\|_{\infty} \leq 1} \left\{ \sum_{k=1}^{m} \lambda_k (D_k u^f(0))^2 \right\} \rightarrow \infty \quad \text{as} \quad m \rightarrow \infty.$$

This is in contrast to the case of $\lambda_k = \lambda > 0$, $k \geq 1$, where a dimension-free bound holds for $p = \infty$.

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1 Introduction and main result

Let us recall dimension-free $L^p$-gradient estimates involving Ornstein-Uhlenbeck operators (cf. [19, 22, 6, 3, 4]). Let $(\lambda_k)$ be a strictly increasing sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty.$$ (1)

For any $m \geq 1$ we denote by $A_m$ the $m \times m$ diagonal matrix with negative eigenvalues $-\lambda_k$, $k = 1, \ldots, m$.

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a bounded $C^2$-function with all first and second bounded derivatives, i.e., $f \in C^2_b(\mathbb{R}^m)$, and denote by $u \in C^2_b(\mathbb{R}^m)$ the unique bounded classical solution to

$$u(x) - \left( \frac{1}{2} \Delta_m u(x) + \langle A_m x, D u(x) \rangle \right) = u(x) - \frac{1}{2} \sum_{k=1}^{m} D^2_{kk} u(x) + \sum_{k=1}^{m} \lambda_k x_k D_k u(x) = f(x),$$ (2)

where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^m$. See, for instance, [5]. Here, $D^2_{kk}$ and $D_k$ are first and second partial derivatives with respect to the canonical basis $(e_k)$.
in \( \mathbb{R}^m \). The operator we consider is an \( m \)-dimensional Ornstein-Uhlenbeck operator, namely \( L_m = \frac{1}{2} \Delta_m + (A_m, x, D) \),

Then, introduce the Gaussian measure \( \mu_m = N(0, (-2A_m)^{-1}) \) with mean 0 and covariance matrix \((-2A_m)^{-1}\), with density as in (16). Note that \( L_m \) is a self-adjoint operator on \( L^2(\mathbb{R}^m, \mu_m) \) which is the usual \( L^2 \)-space with respect to \( \mu_m \). See, for instance, [12, 6, 3, 7]. It is known that if \( 1 < p < \infty \) there exists a constant \( c_p \) (independent of \( f \) and the dimension \( m \)) such that the following sharp gradient estimate holds:

\[
\int_{\mathbb{R}^m} \left( \sum_{k=1}^{m} \lambda_k (Du(y))^2 \right)^{p/2} \mu_m(dy) \leq (c_p)^p \int_{\mathbb{R}^m} |f(y)|^p \mu_m(dy).
\]

The result follows by the general estimates (11) given in Theorem 5.3 of [3] which extends Proposition 3.5 in [22] (see also the references therein). Note that (3) can be rewritten as

\[
\|(-A_m)^{1/2} Du\|_{L^p(\mathbb{R}^m, \mu_m)} \leq c_p \|f\|_{L^p(\mathbb{R}^m, \mu_m)},
\]

where \((-A_m)^{1/2} Du(x) = \sum_{k=1}^{m} \sqrt{\lambda_k} D_k u(x) e_k \).

Our main result (cf. Theorem 2.2 below) shows that, when \( p = \infty \), the dimension-free estimate (4) in general fails to hold. Indeed, we prove the following stronger assertion. Writing \( u = u^f \) to stress the dependence of the solution \( u \) on \( f \), we show that if \( \lambda_k \sim k^2 \) as \( k \to \infty \), then, choosing \( x = 0 \), we have

\[
\sup_{\|f\|_{\infty} \leq 1} \|(-A_m)^{1/2} Du^f(0)\|_{\mathbb{R}^m} = \sup_{\|f\|_{\infty} \leq 1} \left\{ \sum_{k=1}^{m} \lambda_k (D_k u^f(0))^2 \right\} \to \infty \text{ as } m \to \infty.
\]

We point out that in contrast to (5) when \( A_m = -\lambda I_m \) with \( \lambda > 0 \) and \( I_m \) the \( m \times m \) identity matrix then the dimension-free \( L^\infty \)-gradient estimates

\[
\|\lambda^{1/2} Du^f\|_\infty = \sup_{x \in \mathbb{R}^m} |(\lambda^{1/2} Du^f(x))|_{\mathbb{R}^m} \leq \frac{\pi}{\sqrt{2}} \sup_{x \in \mathbb{R}^m} |f(x)|, \quad f \in C^2_b(\mathbb{R}^m)
\]

holds true; see Proposition 2.1.

Let us comment on the previous dimension-free \( L^p \)-estimate (4). This can be deduced by known results for infinite dimensional Ornstein-Uhlenbeck operators. To introduce this setting, we replace \( \mathbb{R}^m \) by a real separable Hilbert space \( H \) with orthonormal basis \((e_k)_{k \geq 1}\) and inner product \((\cdot, \cdot)\). Then, we consider the unbounded self-adjoint operator \( A : D(A) \subset H \to H \) such that

\[
D(A) = \left\{ x \in H : \sum_{k \geq 1} (\langle x, e_k \rangle^2 \lambda_k^2 < \infty \right\}, \quad Ae_k = -\lambda e_k, \quad k \geq 1
\]

(cf. [8, 9, 1, 20]). Our condition (1) is equivalent to require that the inverse operator \( A^{-1} : H \to H \) is a trace class operator. The operator \( A \) generates a strongly continuous semigroup \( (e^{tA}) \) on \( H \), given by \( e^{tA}e_k = e^{-t\lambda} e_k, \quad t \geq 0, \quad k \geq 1 \). We can define the corresponding Ornstein-Uhlenbeck semigroup \( (P^1) \):

\[
P_t f(x) = \int_{H} f(e^{tA}x + \sqrt{t} - e^{2iA}y) N(0, -(2A)^{-1}) (dy), \quad f \in B_b(H), \quad x \in H, \quad t \geq 0
\]

where \( f : H \to \mathbb{R} \) is a Borel, bounded function, and \( N(0, -(2A)^{-1}) \) stands for the centered Gaussian measure defined on the Borel \( \sigma \)-algebra of \( H \) (see Chapter 1 in [7], [9] and Section 2.2); \( I \) is the identity.

Formula (8) is an extension of a well-known formula used in finite dimension. From the probabilistic point of view \((P^1)\) is the transition Markov semigroup of the OU stochastic process \( (X_t^f) \) which solves \( dX_t = AX_t dt + dW_t, \quad X_0 = x \) where \( W \) is a cylindrical Wiener process on \( H \); cf. [8, 7, 13]. When \( f \in C^2_b(H) \), i.e., \( f \) is bounded, twice Fréchet-differentiable with first and second bounded and continuous derivatives, we consider \( u : H \to \mathbb{R} \),

\[
u(x) = R(1, L)f(x) = \int_0^\infty e^{-t}(P^1 I)(x)dt, \quad x \in H.
\]

Following Chapter 6 in [7], \( u \) is the generalized bounded solution to \( u - Lu = f \), where \( L \) is formally given by \( \mathcal{A} \text{Tr}(D^2) + (x, AD) \). Here, we only note that if \( f \) is also cylindrical, i.e., there exists \( m \geq 1 \) and \( \tilde{f} \in C^2_b(\mathbb{R}^m) \) such that

\[
f(x) = \tilde{f}((x, e_1), \ldots, (x, e_m)), \quad x \in H,
\]

as in (9).
then \( u \) given in (9) depends only on a finite number of variables, i.e., \( u(x) = \tilde{u}((x,e_1),\ldots,(x,e_m)), x \in H \) (cf. Section 2.2). Moreover, \( \tilde{u} \) solves (2) with \( f \) replaced by \( \tilde{f} \). In addition, if \( f \in C^1_b(H) \), we have that 
\[ u = R(1,L)f \in C^1_b(H), \text{ and } Du(x) \in D((-A)^{1/2}), \quad x \in H \] (cf. [9] for stronger results).

By Theorem 5.3 of [3] (see also Corollary 5.4 in [3] and Remark 1.1), there exists a constant \( c_p \) (independent of \( f \)) such that
\[
\|(-A)^{1/2}Du\|_{L^p(H,\mu)} \leq c_p \|f\|_{L^p(H,\mu)}, \quad 1 < p < \infty,
\]
where \( \mu = N(0,-(2A)^{-1}) \). Moreover, we have \( \|D^2u\|_{L^p(H,\mu)} \leq c_p \|f\|_{L^p(H,\mu)}, \) i.e.,
\[
\int_H \left( \sum_{k=1}^\infty (D_{kk}u(y))^2 \right)^{p/2} \mu(dy) \leq (c_p)^p \int_H |f(y)|^p \mu(dy).
\]

It is not difficult to show that (11) implies (4) using cylindrical functions \( f \) as in (10); see Section 2.2.

Estimates (11) and (12) are part of the generalized Meyer’s inequalities proved in [3] using the elliptic Littlewood-Paley-Stein inequalities associated with the OU semigroup \((P_t)\). For applications of the classical Meyer’s inequalities to the Malliavin Calculus we refer to [19, 18, 14] (see also Remark 1.2). The results given in [3] give a characterization of the domain of the generator of \((P_t)\) in \( L^p(H,\mu); \) see also [4] (the case \( p = 2 \) was obtained earlier in [6]). We also mention the characterization of the domain of non self-adjoint Ornstein-Uhlenbeck generators given in [15, 17, 16]. Estimates (12) have been used to prove strong uniqueness for a class of SPDEs in [10]. For related results on Ornstein-Uhlenbeck operators in Gaussian harmonic analysis we refer to [12, 2] and the references therein.

Our main result implies that (11) fails to hold for \( p = \infty \), i.e., it is not true that there exists \( C > 0 \), independent of \( f \), such that
\[
\sup_{x \in H} \|(-A)^{1/2}DR(1,L)f(x)\| H \leq C \sup_{x \in H} |f(x)|, \quad f \in C^1_b(H),
\]
where we have used \( u = R(1,L)f \) as in (9). This estimate is stated in [20, Theorem 7] which is based on [20, Lemma 6]. However, there is a mistake in the proof of such lemma. In particular we show that [20, Theorem 7] cannot hold.

**Remark 1.1.** Let us recall the notation used in [3] to study general symmetric Ornstein-Uhlenbeck semigroups in Hilbert spaces. For the sake of notational clarity, the operator \( C \) used in [3] corresponds to our \(-2A^{-1}\), while our semigroup \((e^{tA})\) corresponds to \((e^{-tA})\) in [3]. They use the Malliavin gradient \( D_t = C^{1/2}D \) (where \( D \) is the Fréchet derivative) and \( D_A = \frac{1}{\sqrt{2}}D \). Moreover, the symbol \( D_{A^2} = AD_t \), which is used in the definition of the Sobolev space \( W^{1,p}_{A^2} \) (see Corollary 5.4 in [3]) corresponds to our operator \( \frac{1}{\sqrt{2}}(-A)^{1/2}D \).

**Remark 1.2.** Let us recall the classical Ornstein-Uhlenbeck semigroup \((S_t)\)
\[
S_tf(x) = \int_H f(e^{-t}x + \sqrt{1-\frac{e^{-2t}}{2}}y) \nu(dy), \quad f \in B_b(H), \quad x \in H,
\]
where \( \nu \) is a centered Gaussian measure on \( H \) (see Section 2.2). The classical Meyer’s inequalities give a complete characterization of the domains of \( (I - N_p)^{m/2} \) in \( L^p(H,\nu) \) for all \( p \in (1,\infty) \) and \( m = 1,2,\ldots \) in terms of Gaussian Sobolev spaces related to \( \nu \). Here \( N_p \) denotes the generator of \((S_t)\) in \( L^p(H,\nu) \) (see [18], [19] and [14]).

**Remark 1.3.** Estimates like (11) and (12) holds also in Hölder spaces (see [9, 21] for more details). In particular, for any \( \theta \in (0,1) \), there exists an absolute constant only depending on \( \theta \) such that
\[
\|(-A)^{1/2}DR(1,L)f\|_{C^\theta_b(H,H)} \leq c_\theta \|f\|_{C^\theta_b(H,H)}.
\]

## 2 Notations and preliminary results

Let \( Q \) be a symmetric and positive definite \( m \times m \) matrix; we denote by \( N(0,Q) \) the Gaussian measure with mean 0 and covariance matrix \( Q \); it has density
\[
(2\pi)^{-m/2}(\det Q)^{-1/2}e^{-\|Q^{-1/2}x\|^2/2}
\]
(16)
with respect to the \(m\)-dimensional Lebesgue measure. We first consider for \(\lambda > 0\) the equation
\[
v(x) - \left( \frac{1}{2} \Delta_m v(x) - \lambda \langle x, Dv(x) \rangle \right) = v(x) - M_m v(x) = f(x), \quad x \in \mathbb{R}^m,
\] (17)
with \(M_m = \frac{1}{2} \Delta_m - \lambda \langle x, D \rangle\). We assume that \(f \in C_b^2(\mathbb{R}^m)\). Equation (17) is similar to (2) with \(A_m\) replaced by \(-\lambda I_m\). Using the following Ornstein-Uhlenbeck semigroup \((S^m_t)\):
\[
S^m_t f(x) = \int_{\mathbb{R}^m} f(e^{-\lambda t} x + \sqrt{1 - e^{-2\lambda t}} y) N\left(0, \frac{1}{2} I_m\right) (dy), \quad x \in \mathbb{R}^m, \quad t \geq 0,
\] (18)
we find (cf. (9), and [5, 7])
\[
v(x) = R(1, M_m) f(x) = \int_0^\infty e^{-t}(S^m_t f)(x) dt, \quad x \in \mathbb{R}^m.
\]

Then, we have the following

**Proposition 2.1.** For any \(\lambda > 0\) it holds:
\[
\sup_{x \in \mathbb{R}^m} (\lambda)^{1/2} |DR(1, M_m) f(x)|_{\mathbb{R}^m} \leq \frac{\pi}{\sqrt{2}} \sup_{x \in \mathbb{R}^m} |f(x)|, \quad f \in C_b^2(\mathbb{R}^m).
\] (19)

**Proof.** Let \(v(x) = R(1, M_m) f \in C_b^2(\mathbb{R}^m)\). We set \(v(x) = u(\sqrt{\lambda} x)\) and so, for \(y \in \mathbb{R}^m\), we get
\[
u(y) - \frac{\lambda}{2} \Delta u(y) + \lambda \langle y, Du(y) \rangle = f(y/\sqrt{\lambda})
\]
and
\[
\frac{1}{\lambda} \nu(y) - \frac{1}{2} \Delta u(y) + \langle y, Du(y) \rangle = \frac{1}{\lambda} f(y/\sqrt{\lambda}) = \tilde{f}(y).
\]
We have
\[
u(x) = \int_0^\infty e^{-\frac{1}{\lambda} t} dt \int_{\mathbb{R}^m} \tilde{f}(e^{-t} x + y) N\left(0, \frac{1 - e^{-2t}}{2} I_m\right) (dy)
\]
and, considering the directional derivative \(\langle Du(x), h \rangle = D_h u(x), h \in \mathbb{R}^m, |h| = 1\), we get
\[
D_h u(x) = 2 \int_0^\infty e^{-\frac{1}{\lambda} t} \int_{\mathbb{R}^m} \tilde{f}(e^{-t} x + y) \frac{e^{-t}}{1 - e^{-2t}} h(y) N\left(0, \frac{1 - e^{-2t}}{2} I_m\right) (dy)
\]
(cf. Theorem 6.2.2 in [7], [9] or page 101 in [5]). Then, changing variable in the integral over \(\mathbb{R}^m\) and differentiating under the integral sign, we obtain
\[
\|D_h u\| \leq 2 \|\tilde{f}\| \int_0^\infty \int_{\mathbb{R}^m} \frac{e^{-\frac{1}{\lambda} t} e^{-t}}{1 - e^{-2t}} dt \int_{\mathbb{R}^m} |h(y) (\frac{1 - e^{-2t}}{2})^{1/2} y| N\left(0, I_m\right) (dy)
\]
\[
\leq \frac{\sqrt{2}}{\lambda} \|f\| \int_0^\infty \int_{\mathbb{R}^m} \left|\frac{1 - e^{-2t}}{2}\right|^{1/2} dt \int_{\mathbb{R}^m} |h(y)| N(0, I_m) dy \leq \frac{\pi}{\lambda \sqrt{2}} \|f\|.
\]
Since \(D_h u(y) = \frac{1}{\sqrt{\lambda}} D_h v\left(\frac{y}{\sqrt{\lambda}}\right)\) we have \(\|D_h u\| = \frac{1}{\sqrt{\lambda}} \|D_h v\|\) and (19) follows. \(\square\)

Let us start the proof of the main estimate (5) concerning equation (2) involving the Ornstein-Uhlenbeck operator \(L_m\). Similarly to the proof of Proposition 2.1 the solution \(u \in C_b^2(\mathbb{R}^m)\) to (2) is given by
\[
u(x) = R(1, L_m) f(x) = \int_0^\infty e^{-t}(P^m_t f)(x) dt
\] (20)
with
\[
P^m_t f(x) = \int_{\mathbb{R}^m} f(e^{tA_m} x + \sqrt{1 - e^{2tA_m}} y) N\left(0, \frac{1}{2} A_m^{-1}\right) (dy)
\]
\[
= \int_{\mathbb{R}^m} f(e^{tA_m} x + y) N\left(0, Q^m_t\right) (dy), \quad f \in C_b^2(\mathbb{R}^m), \quad x \in \mathbb{R}^m,
\]
where

\[ Q^m_t = \int_0^t e^{2sA_m} ds = (-2A_m)^{-1}(I_m - e^{2tA_m}), \quad t \geq 0 \]

\((Q^m_t)\) is a diagonal matrix with positive eigenvalues. Let \( \mu^m_t = N(0, Q^m_t) \). The following formula holds for the directional derivative of \( P^m_t f \) along \( h \in \mathbb{R}^m \):

\[
D_h P^m_t f(x) = \langle DP^m_t f(x), h \rangle = \int_{\mathbb{R}^m} \langle A^m h, (Q^m_t)^{-\frac{1}{2}} y \rangle f(e^{tA_m} x + y) \mu^m_t(dy), \quad x \in \mathbb{R}^m, \quad t > 0, \tag{21}
\]

where \( A^m_t = (Q^m_t)^{-1/2} e^{tA_m} \); cf. Theorem 6.2.2 in [7] or page 101 in [5]. Hence

\[
(-A_m)^{1/2} D_P f(0) = (-A_m)^{1/2} D_R(1, L_m) f(0) \in \mathbb{R}^m
\]

appearing in (5) has components

\[
\langle (-A_m)^{1/2} D_P f(0), e_k \rangle = \int_0^\infty e^{-t} dt \int_{\mathbb{R}^m} \langle (-A_m)^{1/2} A^m_t e_k, (Q^m_t)^{-\frac{1}{2}} y \rangle f(y) \mu^m_t(dy)
\]

\[
= \int_0^\infty e^{-t} dt \int_{\mathbb{R}^m} \langle (-A_m)^{1/2} A^m_t e_k, y \rangle f((Q^m_t)^{\frac{1}{2}} y) N(0, I_m)(dy), \quad k = 1, \ldots, m.
\]

An easy calculation shows that

\[
|(-A_m)^{1/2} D_R(1, L_m) f(0)|^2 = \sum_{k=1}^m \left( \int_0^\infty \frac{\lambda_k e^{-t\lambda_k}}{(1 - e^{-2\lambda_k t})^{1/2}} \frac{1}{\sqrt{(2\pi t)^m}} \int_{\mathbb{R}^m} f(c_1(t)x_1, \ldots, c_m(t)x_m) x_k e^{-|x|^2/2} dx dt \right)^2, \tag{22}
\]

where, for \( k \in \{1, \ldots, m\} \) and \( t \geq 0 \),

\[
c_k(t) = \left( 1 - e^{-2\lambda_k t} \right)^{1/2} \quad \text{and} \quad (Q^m_t)^{1/2} = \text{diag}[c_1(t), \ldots, c_m(t)].
\]

We will prove the following result.

**Theorem 2.2.** Let \( \{\lambda_k\} \) be a strictly increasing sequence of positive numbers, such that \( \lambda_k \sim k^2 \) as \( k \to +\infty \). Then, assertion (5) is in force, i.e., taking into account (22), there holds

\[
\sup_{m \in \mathbb{N}} \sup_{\|f\|_\infty \leq 1} \sum_{k=1}^m \left( \int_0^\infty \frac{\lambda_k e^{-t\lambda_k}}{(1 - e^{-2\lambda_k t})^{1/2}} \frac{1}{\sqrt{(2\pi t)^m}} \int_{\mathbb{R}^m} f(c_1(t)x_1, \ldots, c_m(t)x_m) x_k e^{-|x|^2/2} dx dt \right)^2 = +\infty.
\]

The proof of the theorem is given in Section 3. Next, we discuss an application of Theorem 2.2 to infinite dimensions, see Corollary 2.3.

### 2.1 An infinite dimensional Ornstein-Uhlenbeck semigroup

Let \( H \) be a real separable Hilbert space with inner product \( \langle \cdot \rangle \). Let \( Q : H \to H \) be a symmetric non-negative definite trace class operator. The centered Gaussian measure \( \mu = N(0, Q) \) is the unique probability measure on the Borel \( \sigma \)-algebra of \( H \) such that

\[
\int_H e^{\langle x, h \rangle} \mu(dx) = e^{-\frac{1}{2} \langle Qh, h \rangle}, \quad h \in H
\]

(cf. [10]). We denote by \( B_b(H) \) the Banach space of all Borel and bounded real functions endowed with the supremum norm \( \| \cdot \|_\infty \). Moreover, \( C^2_b(H) \subset B_b(H) \) is the space of all functions which are bounded and Fréchet differentiable on \( H \) up to the second order with all the derivatives \( D^j f \) bounded and continuous on \( H, j = 1, 2 \). According to Chapter 1 in [7] we can rewrite the OU semigroup \( P_t \) in (8) as follows

\[
P_t f(x) = \int_H f(e^{tA} x + y) N(0, Q_t) (dy), \quad f \in B_b(H), \quad x \in H,
\]

\[
Q_t = \int_0^t e^{2sA} ds = (2A)^{-1}(I - e^{2tA}), \quad t \geq 0, \quad \text{and} \ A \text{ is given in (7)}.
\]

Suppose that \( f \in C^2_b(H) \) is also **cylindrical**, i.e., there exists \( m \geq 1 \) and \( \tilde{f} \in C^2_b(\mathbb{R}^m) \) such that (10) holds. This is equivalent to require that \( f = f \circ \pi_m \), using the finite dimensional approximations \( \pi_m = \sum_{j=1}^m \epsilon_j \otimes \epsilon_j \).
Identifying $H$ with $l^2$, we have: $f(e^{tA}x + y) = \tilde{f}(e^{tA}x^{(m)} + y^{(m)})$ using the notation

$$h^{(m)} = ((h, e_1), \ldots, (h, e_m)) \in \mathbb{R}^m,$$

while $A_m$ is the same matrix given in (2) and (20). Moreover, $N(0, -(2A)^{-1}) = N(0, -(2A_m)^{-1}) \times \nu_m$ where $\nu_m = \prod_{k=m+1}^{\infty} N(0, (2A_k)^{-1})$; see Theorem 1.2.1 in [7]. It follows that, for any $x \in H$,

$$P_t f(x) = P_t^m(\tilde{f})(x^{(m)}) = \int_{\mathbb{R}^m} \tilde{f}(e^{tA}x^{(m)} + \sqrt{I_m - e^{2tA_m}} y) N(0, -(2A_m)^{-1}) (dy);$$

while

$$u(x) = R(1, L) f(x) = \int_0^\infty e^{-t}(P_t f)(x) dt = \tilde{u}(x, e_1, \ldots, (x, e_m));$$

$$\tilde{u}(z) = \int_0^\infty e^{-t} P_t^m \tilde{f}(z), z \in \mathbb{R}^m.$$ 

Recall that $P_t^m$ is given in (20). Setting $\mu_m = N(0, -(2A_m)^{-1})$ and using that $C^2_b(H)$ contains particular cylindrical functions as in (10) we infer, for any $m \geq 1$,

$$\sup_{f \in C^2_b(\mathbb{R}^m), \|f\|_{\infty} \leq 1} \|(-A_m)^{1/2} D \tilde{u}\|_{L^p(\mathbb{R}^m, \mu_m)} \leq \sup_{f \in C^2_b(H), \|f\|_{L^p(H, \mu)} \leq 1} \|(-A)^{1/2} Du\|_{L^p(H, \mu)}, 1 < p < \infty, \quad (26)$$

and

$$\sup_{f \in C^2_b(\mathbb{R}^m), \|f\|_{\infty} \leq 1} \|(-A_m)^{1/2} D \tilde{u}\|_{\infty} \leq \sup_{f \in C^2_b(H), x \in H, \|f\|_{\infty} \leq 1} \sup_{x \in H} |(-A)^{1/2} Du(x)|_{H}. \quad (27)$$

As a consequence of Theorem 2.2 we obtain (see (9))

**Corollary 2.3.** Under the same assumptions of Theorem 2.2, there holds

$$\sup_{f \in C^2_b(H), \|f\|_{\infty} \leq 1} \|(-A)^{1/2} D(R(1, L) f)(0)\|_{H} = \infty.$$ 

### 3 Proof of Theorem 2.2

Let $\delta \in (0, +\infty)$. Then, put

$$S_m = S_m(\delta) = \sup_{f \in C^2_b(\mathbb{R}^m), \|f\|_{\infty} \leq 1} \sum_{k=1}^{m} \left( \int_0^{\delta} \frac{\lambda_k e^{-\lambda_k t}}{(1 - e^{-2\lambda_k t})^{1/2}} \int_{\mathbb{R}^m} f(c_1(t)x_1, \ldots, c_m(t)x_m) x_k e^{\frac{|x|^2}{\sqrt{(2\pi)^m}} dx} \right)^2.$$ 

If we show that

$$\sup_{m \geq 2} S_m = \infty \quad (28)$$

holds under the assumption that $\lambda_k \sim k^2$ as $k \to +\infty$, then the validity of Theorem 2.2 will follow.

#### 3.1 Two useful lemmas

The following identity will be important. Recall that $x_k = (x, e_k), k = 1, \ldots, m$ where $(e_j)$ denotes the canonical basis in $\mathbb{R}^m$.

**Lemma 3.1.** For any $m \geq 2$, $k \in \{1, \ldots, m\}$, $c = (c_1, \ldots, c_m) \in \mathbb{R}^m \setminus \{0\}$ and $F \in B_b(\mathbb{R})$, it holds

$$I_{m, k}(c) = \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} F((c, x)) x_k e^{\frac{|x|^2}{2\pi}} dx$$

$$= \frac{2\pi (\sqrt{\pi})^{m-3} c_k}{(2\pi)^{m/2} \Gamma \left( \frac{m-1}{2} \right)} \int_0^{\infty} \int_0^{\infty} e^{-\frac{\rho^2}{2\pi}} \rho^{m-2} \cos \theta (sin \theta)^{m-2} F(|c| \rho \cos \theta) d\rho d\theta.$$

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Proof. We provide additional details for the sake of completeness. Let us first consider \( m = 2 \). We introduce the unitary vectors \( \gamma_1 = c / |c| \) and \( \gamma_2 \in \mathbb{R}^2 \) such that \( (\gamma_1, \gamma_2) \) is an orthonormal basis in \( \mathbb{R}^2 \). Using the polar coordinates with respect to such basis we can write

\[
x = \rho \cos \theta \gamma_1 + \rho \sin \theta \gamma_2,
\]

which entails that

\[
I_{2,k}(F) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi} \rho^2 F(|\rho \cos \theta|) (\cos \theta \langle \gamma_1, e_k \rangle + \sin \theta \langle \gamma_2, e_k \rangle) e^{-\frac{\rho^2}{2}} d\rho d\theta
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} \int_0^{\pi} \rho^2 F(|\rho \cos \theta|) \cos \theta \langle \gamma_1, e_k \rangle e^{-\frac{\rho^2}{2}} d\rho d\theta, \quad k = 1, 2
\]

since \( \int_0^{2\pi} F(|\rho \cos \theta|) \sin \theta d\theta = 0 \). We get easily (29) for \( m = 2 \) recalling that \( \Gamma(1/2) = \sqrt{\pi} \).

In the general case of \( m \geq 3 \), we consider an orthonormal basis \( (\gamma_k) \) of \( \mathbb{R}^m \) where \( \gamma_1 = c / |c| \). Then, we introduce polar coordinates with respect to \( (\gamma_k) \). Let \( \rho = |x| \). Proceeding similarly to [11, Section 5.9], we have, for \( x \neq 0 \),

\[
x = \rho \cos \theta_1 \gamma_1 + \rho \sin \theta_1 \cos \theta_2 \gamma_2 + \ldots + \rho \sin \theta_1 \cdots \sin \theta_{m-2} \sin \theta_{m-1} \gamma_m,
\]

where \( \rho > 0 \) (radial distance), \( \theta_1, \ldots, \theta_{m-2} \in [0, \pi] \) (latitudes; \( \theta_1 \) is the angle between \( x \) and \( \gamma_1 \)) and \( \theta_{m-1} \in [0, 2\pi] \) (longitude). Let \( \theta = (\theta_1, \ldots, \theta_{m-1}) \). Denote by

\[
J(\rho, \theta) = \rho^{m-1} (\sin \theta_1)^{m-2} (\sin \theta_2)^{m-3} \cdots (\sin \theta_{m-2})
\]

the Jacobian determinant. Moreover, set \( \gamma_i^{(k)} = (\gamma_i, e_k) \), for \( i, k = 1, \ldots, m \). Let

\[
\xi_1(\theta) = \cos \theta_1, \quad \xi_2(\theta) = \sin \theta_1 \cos \theta_2, \ldots, \xi_{m-1}(\theta) = \sin \theta_1 \cdots \sin \theta_{m-2} \sin \theta_{m-1}.
\]

For instance, for \( m = 4 \), we have: \( \xi_1(\theta) = \cos \theta_1, \quad \xi_2(\theta) = \sin \theta_1 \cos \theta_2, \quad \xi_3(\theta) = \sin \theta_1 \sin \theta_2 \cos \theta_3, \quad \xi_4(\theta) = \sin \theta_1 \sin \theta_2 \sin \theta_3 \), with \( \theta_1, \theta_2 \in [0, \pi] \) and \( \theta_3 \in [0, 2\pi] \). We infer that

\[
I_{m,k}(F) = \frac{1}{\sqrt{(2\pi)^m}} \int_0^\infty \int_{[0,\pi]^{m-2} \times [0,2\pi]} \rho e^{-\frac{\rho^2}{2}} F(|\rho \cos \theta_1|) \left( \sum_{i=1}^m \xi_i(\theta) \gamma_i^{(k)} \right) J(\rho, \theta) d\rho d\theta
\]

\[
= \frac{1}{\sqrt{(2\pi)^m}} \int_0^\infty \int_{[0,\pi]^{m-2} \times [0,2\pi]} \rho e^{-\frac{\rho^2}{2}} F(|\rho \cos \theta_1|) \xi_1(\theta) \gamma_1^{(k)} J(\rho, \theta) d\rho d\theta
\]

\[
(30)
\]

using that

\[
\frac{1}{\sqrt{(2\pi)^m}} \int_0^\infty \int_{[0,\pi]^{m-2} \times [0,2\pi]} \rho e^{-\frac{\rho^2}{2}} F(|\rho \cos \theta_1|) \left( \sum_{i=2}^m \xi_i(\theta) \gamma_i^{(k)} \right) J(\rho, \theta) d\rho d\theta = 0.
\]

(31)

In order to prove (31) we check that if \( \rho > 0 \) then

\[
\int_{[0,\pi]^{m-2} \times [0,2\pi]} F(|\rho \cos \theta_1|) \xi_i(\theta) J(\rho, \theta) d\theta = 0, \quad 2 \leq i \leq m.
\]

(32)

If \( i = m \), we find that

\[
\int_{[0,\pi]^{m-2} \times [0,2\pi]} F(|\rho \cos \theta_1|) \xi_m(\theta) (\sin \theta_1)^{m-2} (\sin \theta_2)^{m-3} \cdots (\sin \theta_{m-2}) d\theta
\]

\[
= \int_0^\pi F(|\rho \cos \theta_1|) (\sin \theta_1)^{m-2} \sin \theta_1 d\theta_1 \times
\]

\[
\times \int_{[0,\pi]^{m-3} \times [0,2\pi]} \sin \theta_2 \cdots \sin \theta_{m-1} (\sin \theta_2)^{m-3} \cdots (\sin \theta_{m-2}) d\theta_2 \cdots d\theta_{m-1} = 0
\]
by the Fubini theorem, since \( \int_0^{2\pi} \sin \theta_{m-1} d\theta_{m-1} = 0 \). Similarly we obtain that (32) holds with \( i = m - 1 \). Note that up to now we have already proved (32) when \( m = 3 \). Let \( m \geq 4 \). We check (32) when \( 2 < i \leq m - 2 \). We have

\[
\begin{align*}
\int_{[0,\pi]^m} F(|c| \rho \cos \theta_1) \xi_i(\theta) (\sin \theta_1)^{m-2} (\sin \theta_2)^{m-3} \cdots (\sin \theta_{m-2}) d\theta \\
= \int_0^{\pi} F(|c| \rho \cos \theta_1)(\sin \theta_1)^{m-2} \sin \theta_1 d\theta_1 \times \\
\times \int_{[0,\pi]^{m-3}} \sin \theta_2 \cdots \sin \theta_i (\sin \theta_2)^{m-3} \cdots (\sin \theta_{m-2}) d\theta_2 \cdots d\theta_{m-1} = 0,
\end{align*}
\]

because \( \int_0^{\pi} \cos \theta_i (\sin \theta_i)^{m-1-i} d\theta_i = 0 \). Similarly, for \( i = 2 \), we get

\[
\begin{align*}
\int_0^{\pi} F(|c| \rho \cos \theta_1)(\sin \theta_1)^{m-2} \sin \theta_1 d\theta_1 \times \\
\times \int_{[0,\pi]^{m-3}} \cos \theta_2 (\sin \theta_2)^{m-3} \cdots (\sin \theta_{m-2}) d\theta_2 \cdots d\theta_{m-1} = 0.
\end{align*}
\]

We have verified (32) and so (30) holds. We rewrite (30) as follow

\[
I_{m,k}(F) = R_m \frac{\gamma_1^{(k)}}{\sqrt{2\pi}^m} \int_0^{\infty} \int_0^{\pi} \rho^m e^{-\frac{\rho^2}{2}} F(|c| \rho \cos \theta_1) \cos \theta_1 (\sin \theta_1)^{m-2} d\rho d\theta_1, \quad \gamma_1^{(k)} = \frac{c_k}{|c|}, \quad (33)
\]

where \( R_m = 2\pi \) if \( m = 3 \) and if \( m > 3 \)

\[
R_m = \int_{[0,\pi]^{m-3}} (\sin \theta_1)^{m-3} (\sin \theta_3)^{m-4} \cdots \sin \theta_{m-2} d\theta_2 \cdots d\theta_{m-1}
\]

\[
= 2\pi \prod_{j=1}^{m-3} \int_0^{\pi} (\sin \phi)^j d\phi = 2\pi \prod_{j=1}^{m-3} B\left(\frac{j+1}{2}, \frac{1}{2}\right) = 2\pi \prod_{j=1}^{m-3} \frac{\Gamma\left(\frac{j+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m-1}{2}\right)}.
\]

We have used the Beta function \( B(\cdot, \cdot) \) (cf. page 103 of [24]). Hence since \( \Gamma(1/2) = \sqrt{\pi} \), we get

\[
R_m = 2\pi (\sqrt{\pi})^{m-3} \left(\frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{-1}.
\]

Inserting \( R_m \) in (33) we obtain (29), i.e.,

\[
I_{m,k}(F) = \frac{2\pi (\sqrt{\pi})^{m-3}}{(2\pi)^{m/2} \Gamma\left(\frac{m-1}{2}\right)} \frac{c_k}{|c|} \int_0^{\infty} \int_0^{\pi} e^{-\frac{\rho^2}{2}} \rho^m \cos \theta (\sin \theta)^{m-2} F(|c| \rho \cos \theta) d\rho d\theta.
\]

\[\square\]

**Lemma 3.2.** If \( F \in B_b(\mathbb{R}) \) verifies \( F(x) = -F(-x) \) for any \( x \in \mathbb{R} \), then we have, for any \( m \geq 2 \), \( k \in \{1, \ldots, m\} \), \( c = (c_1, \ldots, c_m) \in \mathbb{R}^m \setminus \{0\} \),

\[
I_{m,k}(F) = \frac{4\pi (\sqrt{\pi})^{m-3}}{(2\pi)^{m/2} \Gamma\left(\frac{m-1}{2}\right)} \frac{c_k}{|c|} \int_0^{\infty} e^{-\frac{\rho^2}{2}} \rho^m d\rho \int_0^1 x(1-x^2)^{m-3} F(|c| \rho x) dx \quad (34)
\]

(cf. (29)). In the special case of \( F = F_0 := \mathbb{1}_{(0, \infty)} - \mathbb{1}_{(-\infty, 0)} \), we obtain

\[
I_{m,k}(F_0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^m} F_0(x) x_k e^{-\frac{\|x\|^2}{2}} dx = \frac{\sqrt{2} c_k}{\sqrt{\pi} |c|} \quad (35)
\]

**Proof.** By changing variable \( x = \cos \theta \) and using that \( F(x) = -F(-x) \), \( x \neq 0 \), we have

\[
\int_0^{\pi} \cos \theta (\sin \theta)^{m-2} F(|c| \rho \cos \theta) d\theta = 2 \int_0^1 x(1-x^2)^{m-3} F(|c| \rho x) dx.
\]
Whence,
\[
I_{m,k}(F) = \frac{2 \cdot 2\pi (\sqrt{\pi})^{m-3} c_k}{(2\pi)^{m/2} \Gamma \left(\frac{m}{2}\right) |c|} \int_0^{+\infty} e^{-\frac{\rho^2}{2}} \rho^m d\rho \int_0^1 x(1-x^2)^{\frac{m-3}{2}} F(|c| \rho x) dx.
\]
Let us assume that \( F = F_0 = \mathbb{I}_{(0,\infty)} - \mathbb{I}_{(-\infty,0)} \). We find
\[
I_{m,k}(F_0) = \frac{4\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2} \Gamma \left(\frac{m}{2}\right) |c|} \int_0^{+\infty} e^{-\frac{\rho^2}{2}} \rho^m d\rho \int_0^1 x(1-x^2)^{\frac{m-3}{2}} dx.
\]
Using that \( \int_0^1 x(1-x^2)^{\frac{m-3}{2}} dx = \frac{1}{m-1} \) and
\[
\int_0^{+\infty} e^{-\frac{\rho^2}{2}} \rho^m d\rho = \Gamma \left( \frac{m+1}{2} \right) 2^{\frac{m-1}{2}}
\]
we find
\[
I_{m,k}(F_0) = \frac{4\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2} \Gamma \left(\frac{m}{2}\right) |c|} \Gamma \left( \frac{m+1}{2} \right) 2^{\frac{m-1}{2}} \frac{1}{m-1} c_k
\]
\[
= \frac{4\pi(\sqrt{\pi})^{m-3}}{(\pi)^{m/2}} \left( \frac{m-1}{2} \right) 2^{-\frac{1}{2}} \frac{1}{m-1} c_k \sqrt{\frac{2}{\pi}} |c|,
\]
since \( x\Gamma(x) = \Gamma(x+1) \), \( x > 0 \) and this finishes the proof. \( \square \)

### 3.2 Proof of assertion (28)

Recall that \( c_k(t) := \left(1 - e^{-2\lambda_k t}\right)^{1/2} \) for \( k \in \{1, \ldots, m\} \) and \( t \geq 0 \). Set \( c(t) = (c_1(t), \ldots, c_m(t)) \in \mathbb{R}^m \).

Fix \( m \geq 2 \) and \( \delta > 0 \), and put \( S_m := S_m(\delta) \). Then, for \( m \geq 2 \), define
\[
A_m := \frac{2}{\pi} \sum_{k=1}^{m} \left( \int_0^\delta \frac{\lambda_k e^{-\lambda_k t}}{(1 - e^{-2\lambda_k t})^{1/2} |c(t)|} dt \right)^2.
\]
We prove that \( \lim_{m \to \infty} S_m = \infty \) in two steps.

**I step.** We prove
\[
S_m \geq A_m, \quad \forall \ m \geq 2.
\]

We start by constructing an approximating sequence of smooth functions for \( F_0 := \mathbb{I}_{(0,\infty)} - \mathbb{I}_{(-\infty,0)} \).

For any \( n \geq 1 \), consider a non-decreasing \( F_n \in C_0^\infty(\mathbb{R}_+) \) such that \( F_n(y) = 0 \) if \( 0 \leq y \leq 1/(n+1) \) and \( F_n(y) = 1 \) if \( y \geq 1/n \). Then, extend each \( F_n \) to an odd function on \( \mathbb{R} \) by the rule \( F_n(x) = -F_n(-x) \) if \( x < 0 \), and define
\[
f_n(x_1, \ldots, x_m) = F_n(x_1 + \ldots + x_m), \quad x_1, \ldots, x_m \in \mathbb{R}.
\]

It is clear that each \( f_n \in C_0^\infty(\mathbb{R}^m) \) and \( \|f_n\|_\infty \leq 1 \). Whence,
\[
S_m \geq \sup_{n \geq 1} \sum_{k=1}^{m} \left( \int_0^\delta \frac{\lambda_k e^{-\lambda_k t}}{(1 - e^{-2\lambda_k t})^{1/2} \sqrt{2\pi} m} \int_{\mathbb{R}^m} f_n(c_1(t)x_1, \ldots, c_m(t)x_m) x_k e^{-\frac{|x|^2}{2}} dx dt \right)^2
\]
\[
= \sup_{n \geq 1} \sum_{k=1}^{m} \left( \int_0^\delta \frac{\lambda_k e^{-\lambda_k t}}{(1 - e^{-2\lambda_k t})^{1/2} \sqrt{2\pi} m} \int_{\mathbb{R}^m} f_n((c(t),x)) x_k e^{-\frac{|x|^2}{2}} dx dt \right)^2.
\]

Moreover, combining the fact that each \( F_n \) is an odd functions with (34), with \( c \) replaced by \( c(t) \), yields
\[
\sup_{n \geq 1} \sum_{k=1}^{m} \left( \int_0^\delta \frac{\lambda_k e^{-\lambda_k t}}{(1 - e^{-2\lambda_k t})^{1/2} \sqrt{2\pi} m} \int_{\mathbb{R}^m} F_n((c(t),x)) x_k e^{-\frac{|x|^2}{2}} dx dt \right)^2
\]
\[
= \sup_{n \geq 1} \frac{4\pi(\sqrt{\pi})^{m-3}}{(2\pi)^{m/2} \Gamma \left(\frac{m}{2}\right)} \sum_{k=1}^{m} \left( \frac{c_k(t)}{|c(t)|} \right) \int_0^{+\infty} e^{-\frac{\rho^2}{2}} \rho^m d\rho \int_0^1 x(1-x^2)^{\frac{m-3}{2}} F_n((c(t)|\rho x) dx)^2.
\]

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Then, using that both $F_n(x) \leq F_{n+1}(x)$ and $F_n(x) \to F_0(x)$ hold for any $x \geq 0$, apply the monotone convergence theorem to get

$$S_m \geq \sup_{n \geq 1} \frac{4\pi \left(\frac{m}{2}\right)^{m-3}}{(2\pi)^{m/2} \Gamma \left(\frac{m+1}{2}\right)} \sum_{k=1}^{m} \frac{c_k(t)}{|c(t)|} \int_0^{+\infty} e^{-\frac{t^2}{2} \rho^2} \rho^{m} d\rho \int_0^1 x(1-x^2) \frac{\sqrt{x}}{1-x^2} F_n(|c(t)|x) dx = A_m,$$

for $m \geq 2$. In the last line we have used both (34) and (35) with $c$ replaced by $c(t)$. This proves (37).

II step. We prove that

$$\lim_{m \to \infty} A_m = \infty,$$

thus completing the proof of (28). Recalling the definition of $c_k(t)$, we have

$$A_m = \frac{2}{\pi} \sum_{k=1}^{m} \left( \int_0^{+\infty} \frac{\sqrt{\lambda_k e^{-\lambda_k t}}}{\sqrt{2 \pi}} \frac{1}{|c(t)|} \frac{\sqrt{k}}{\pi^{1/2}} \right)^2 \geq \frac{1}{\pi} \sum_{k=1}^{m} \lambda_k \left( \int_0^t \left( \frac{e^{-\lambda_k t}}{|c(t)|} \right) dt \right)^2, \quad m \geq 2. \tag{38}$$

To bound (39) from below, note that

$$|c(t)| = \left( \sum_{k=1}^{m} \frac{1-e^{-2\lambda_k t}}{2 \lambda_k} \right)^{1/2} \leq \left( \sum_{k=1}^{+\infty} \frac{1-e^{-2\lambda_k t}}{2 \lambda_k} \right)^{1/2} = \left( \int_0^t \left( \sum_{k=1}^{+\infty} e^{-2\lambda_k s} \right) ds \right)^{1/2} \tag{39}$$

holds for any $t \geq 0$. Now, if there is a positive constant $c_0$ such that $\lambda_k \geq c_0 k^2$ for any $k \geq 1$, then

$$\int_0^{+\infty} e^{-2c_0 k^2 s} ds = \sqrt{\frac{\pi}{2c_0}}, \quad s > 0,$

yielding

$$|c(t)| \leq \left( \int_0^t \frac{\pi}{\sqrt{2c_0 s}} ds \right)^{1/2} = \left( \frac{2\pi t}{c_0} \right)^{1/4}. \tag{40}$$

Up to now we have found that

$$A_m \geq \frac{1}{\pi} \sum_{k=1}^{m} \lambda_k \left( \int_0^t e^{-\lambda_k t} \left( \frac{c_0}{2\pi t} \right)^{1/4} dt \right)^2, \quad m \geq 2. \tag{41}$$

Now, exploit that

$$\int_0^t t^{\frac{2}{4}} e^{-\lambda t} dt = \left( \frac{1}{\lambda} \right)^{\frac{3}{2}} \int_0^{\lambda t} s^{\frac{1}{2}} e^{-s} ds \geq \left( \frac{1}{\lambda} \right)^{\frac{3}{2}} \int_0^{c_0 \delta} s^{\frac{1}{2}} e^{-s} ds \tag{42}$$

holds for every $\lambda \geq c_0$, to get (after recalling that, in particular, $\lambda_k \geq c_0$, for any $k \geq 1$)

$$A_m \geq \frac{1}{\pi} \sqrt{\frac{c_0}{2\pi}} \sum_{k=1}^{m} \lambda_k \left( \int_0^t t^{\frac{1}{4}} e^{-\lambda_k t} dt \right)^2 \geq \frac{1}{\pi} \sqrt{\frac{c_0}{2\pi}} \sum_{k=1}^{m} \lambda_k \left( \frac{1}{\lambda_k} \right)^{\frac{3}{2}} \left( \int_0^{c_0 \delta} s^{\frac{1}{2}} e^{-s} ds \right)^2 \tag{43}$$

$$= \frac{1}{\pi} \sqrt{\frac{c_0}{2\pi}} \sum_{k=1}^{m} \frac{1}{\lambda_k} \left( \int_0^{c_0 \delta} s^{\frac{1}{2}} e^{-s} ds \right)^2 \sum_{k=1}^{m} \lambda_k .$$

Thus, if $\lambda_k \sim k^2$ as $k \to +\infty$, then $\sum_{k=1}^{m} \frac{1}{\sqrt{\lambda_k}} \sim \log m$ as $m \to +\infty$, and (38) holds. This finishes the proof.
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