A note on a standard family of twist maps

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Abstract

We investigate the break up of the last invariant curve for analytic families of standard maps

\[ S_\lambda : \begin{cases} 
 y' = \lambda g(x) + y \\
 x' = x + y' \mod 1
\end{cases} \]

where \( g : S^1 \rightarrow \mathbb{R} \) is an analytic function such that \( \int_{S^1} g(x) dx = 0 \). Our main result is another evidence of how hard this problem is. We give an example of a particular function \( g \) as above such that the mapping \( S_\lambda \) associated to it has a "pathological" behavior.

Key words: twist maps, rotational invariant curves, topological methods, vertical rotation number, piecewise linear standard maps

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1 Introduction and statement of the main result

In this paper, we investigate the following problem:

Let $\tilde{g} : \mathbb{R} \to \mathbb{R}$ be an analytic, non-zero, periodic function, $\tilde{g}(x + 1) = \tilde{g}(x)$, such that $\int_0^1 \tilde{g}(x) \, dx = 0$. We define the following one parameter family $(\lambda)$ of analytic diffeomorphisms of the annulus:

$$ S_\lambda : \begin{cases} y' = \lambda g(x) + y \\ x' = x + y' \mod 1 \end{cases} ,$$

where $g : S^1 \to \mathbb{R}$ is the map induced by $\tilde{g}$.

For all $\lambda \in \mathbb{R}$, $S_\lambda$ is an area-preserving twist mapping, because $\partial_y x' = 1$, for any $(x, y) \in S^1 \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ and $\det[DS_\lambda] = 1$. Also, the fact that $\int_0^1 \tilde{g}(x) \, dx = 0$ implies that $S_\lambda$ is an exact mapping, which means that given any homotopically non-trivial simple closed curve $C \subset S^1 \times \mathbb{R}$, the area above $C$ and below $S_\lambda(C)$ is equal the area below $C$ and above $S_\lambda(C)$. Another obvious fact about this family is that $S_0$ is an integrable mapping, that is, the cylinder is foliated by invariant curves $y = y_0$.

So, KAM theory applies to $S_\lambda$ and we can prove that there is a parameter $\lambda_0 > 0$, such that for any $\lambda \in [0, \lambda_0]$ $S_\lambda$ has at least one rotational invariant curve. On the other hand, if we choose $x_0 \in S^1$ such that $g(x) \leq g(x_0)$ for all $x \in S^1$, we get that $S_\lambda$ does not have rotational invariant curves for all $\lambda \geq \lambda^* = \frac{1}{g(x_0)} > 0$. The proof of this classical fact is very simple, so we present it here:

Given $\lambda \geq \lambda^*$, choose $x_\lambda \in S^1$ such that $\lambda = \frac{1}{g(x_\lambda)}$. A computation shows that $S^n_\lambda(x_\lambda, 0) = (x_\lambda, n)$, for all $n \in \mathbb{Z}$. So there can be no rotational invariant curves.

A result due to Birkhoff implies that the set

$$ A_g = \{ \lambda \geq 0 : \text{S}_\lambda \text{ has at least one rotational invariant curve} \} \quad (2)$$

is closed. So a very "natural" conjecture would be the following (see [5]):

**Conjecture 1** : $A_g = [0, \lambda_{cr}]$, for some $\lambda_{cr} > 0$.

Another interesting one parameter family is the following:

$$ T_\lambda : \begin{cases} y' = g(x) + y + \lambda \\ x' = x + y' \mod 1 \end{cases} \quad (3)$$

Of course $T_\lambda$ is also an area-preserving twist mapping, the difference is that it is exact if and only if $\lambda = 0$, so when $\lambda \neq 0$ there is no rotational invariant curve.

It can be proved (see section 2) that there is a closed interval $\rho_V = [\rho_V^{\min}, \rho_V^{\max}]$ associated to $S_\lambda$ (and to $T_\lambda$) with the following property: Given $\omega \in \rho_V$, there is a point $X \in S_1 \times \mathbb{R}$ such that

$$ \lim_{n \to +\infty} \frac{p_2 \circ S^n_\lambda(X) - p_2(X)}{n} = \omega,$$
where \( p_1(x, y) = x \) and \( p_2(x, y) = y \). From the exactness of \( S_\lambda \) we get that \( 0 \in \rho(V(S_\lambda)) \) for all \( \lambda \in \mathbb{R} \), something that may not hold for \( T_\lambda \).

In section 3 we prove a result that implies that \( \rho_{\text{max}}^V(\rho_{\text{min}}^V) \) is a continuous functions of the parameter \( \lambda \). A first difference between \( S_\lambda \) and \( T_\lambda \) is that \( \rho_{\text{max}}^V(S_\lambda) = 0 \) for any \( \lambda \in [0, \lambda_0] \) while \( \rho_{\text{max}}^V(T_\lambda) \neq 0 \) for all \( \lambda \neq 0 \). In fact, in a certain sense, the behavior of the function \( \lambda \rightarrow \rho_{\text{max}}^V(T_\lambda) \) is similar to the one of the rotation number of certain families of homeomorphisms of the circle.

Given a circle homeomorphism \( f : S^1 \rightarrow S^1 \), a well studied family (see for instance [6]) is the one given by translations of \( f \):

\[
x' = f_\lambda(x) = f(x) + \lambda
\]

In this case it is easy to prove that the rotation number of \( f_\lambda \) is a non-decreasing function of the parameter. We have a similar result for \( T_\lambda \):

**Lemma 1** : \( \rho_{\text{max}}^V(T_\lambda) \) is a non-decreasing function of \( \lambda \).

As the proof will show, this fact is an easy consequence of proposition 3, page 466 of [6].

If we had a similar result for \( S_\lambda \), then conjecture 1 would trivially be true, because \( A_g = (\rho_{\text{max}}^V)^{-1}(0) \) and this set is an interval if \( \rho_{\text{max}}^V(S_\lambda) \) is a non-decreasing function.

The main result of this note goes in the opposite direction; we present an example in the analytic topology such that we do not know whether or not \( A_g \) is a closed interval (although we believe it is not), but for this example \( \rho_{\text{max}}^V(S_\lambda) \) is not a non-decreasing function of \( \lambda \). More precisely, we have:

**Theorem 1** : There exists an analytic function \( g^* \) as above such that \( \rho_{\text{max}}^V(S_\lambda) \) is not a non-decreasing function of \( \lambda \).

The proof of the theorem implies that we can choose \( g^*(x) = \sum_{n=1}^{N} a_n \cdot \cos(2\pi n x) \).

Although this choice of \( g^* \) is a finite sum of cosines obtained as the truncation of a certain Fourier series of a continuous function, it is still possible that for \( g_\lambda(x) = \cos(2\pi x) \), \( \rho_{\text{max}}^V(S_\lambda) \) is in fact a non-decreasing function, as numerical experiments suggest. Nevertheless, this shows how subtle the problem is.

The proof of this theorem is based on a result previously obtained by the author, on a paper due to S.Bullett [4] on piecewise linear standard maps and on some consequences of results from [9].

## 2 Basic tools

First we present a theorem which is a consequence of some results from [9]. Before we need to introduce some definitions:

1) \( D_0(T^2) \) is the set of torus homeomorphisms \( T : T^2 \rightarrow T^2 \) of the following form:

\[
T : \begin{cases}
y' = g(x) + y \mod 1 \\
x' = x + y' \mod 1
\end{cases}
\]
where \( g : S^1 \to \mathbb{R} \) is a Lipschitz function such that \( \int_{S^1} g(x)dx = 0. \)

2) \( D_0(S^1 \times \mathbb{R}) \) is the set of lifts to the cylinder of elements from \( D_0(T^2) \), the same for \( D_0(\mathbb{R}^2) \). Given \( T \in D_0(T^2) \) as in (4), its lifts \( \hat{T} \in D_0(S^1 \times \mathbb{R}) \) and \( \tilde{T} \in D_0(\mathbb{R}^2) \) write as (\( \hat{g} \) is a lift of \( g \))

\[
\hat{T} : \begin{cases} y' = g(x) + y \\ x' = x + y \mod 1 \end{cases} \quad \text{and} \quad \tilde{T} : \begin{cases} y' = \hat{g}(x) + y \\ x' = x + y \end{cases}
\]

3) We say that \( T \in D_0(T^2) \) has a \( \frac{p}{q} \)-vertical periodic orbit (set) if there is a point \( A \in S^1 \times \mathbb{R} \) such that \( \hat{T}^q(A) = A + (0, p) \). It is clear that \( T^q(\pi_2(A)) = \pi_2(A) \), where \( \pi_2 : S^1 \times \mathbb{R} \to T^2 \) is given by \( \pi_2(x, y) = (x, y \mod 1) \). The periodic orbit that contains \( \pi_2(A) \) is said to have vertical rotation number \( \rho_V = \frac{p}{q} \).

4) Given an irrational number \( \omega \), we say that \( T \in D_0(T^2) \) has an \( \omega \)-vertical quasi-periodic set if there is a compact \( T \)-invariant set \( X_\omega \subset T^2 \), such that for any \( X \in X_\omega \) and any \( Z \in \pi_2^{-1}(X) \),

\[ \rho_V(X_\omega) = \lim_{n \to \infty} \frac{p_2 \circ \hat{T}^n(Z) - p_2(Z)}{n} = \omega \]

5) We say that \( T \in D_0(T^2) \) has a rotational invariant curve if there is a homotopically non-trivial simple closed curve \( \gamma \subset S^1 \times \mathbb{R} \), such that \( \tilde{T}(\gamma) = \gamma \).

Now we have the following:

**Theorem 2**: Given \( T \in D_0(T^2) \), there exists a closed interval \( 0 \in [\rho_V^{\min}, \rho_V^{\max}] \) such that for any \( \omega \in [\rho_V^{\min}, \rho_V^{\max}] \), there is a periodic orbit or quasi-periodic set \( X_\omega \) with \( \rho_V(X_\omega) = \omega \), depending on whether \( \omega \) is rational or not. Moreover, \( \rho_V^{\min} < 0 < \rho_V^{\max} \) if and only if, \( T \) does not have any rotational invariant curve.

When \( \omega \in [\rho_V^{\min}, \rho_V^{\max}] \) a standard argument in ergodic theory (see the discussion below) proves that there is an orbit with that rotation number. In fact, much more can be said, see my forthcoming paper [3].

Following Misiurewicz and Ziemann [11], we can define another set that is equal to the limit of all the convergent sequences

\[
\left\{ \frac{p_2 \circ \hat{T}^n(Z)}{n} : Z \in S^1 \times \mathbb{R}, \ n \to \infty \right\},
\]

which we call \( \rho_V(T)^* \). In the following we present a sketch of the proof that \( \rho_V(T) = \rho_V(T)^* \).

First note that the definition of \( \rho_V(T)^* \) implies \( \rho_V(T) \subseteq \rho_V(T)^* \). Now if we define \( \omega^- = \inf \rho_V(T)^* \) and \( \omega^+ = \sup \rho_V(T)^* \), theorem 2.4 of [11] gives two ergodic \( T \)-invariant measures \( \mu_- \) and \( \mu_+ \) with vertical rotation numbers \( \omega^- \) and \( \omega^+ \), respectively. This means that

\[
\int_{T^2} [p_2 \circ T(X) - p_2(X)] d\mu_{-(+)} = \omega^{-(+)}.
\]
Therefore from the Birkhoff ergodic theorem, there are points \( Z^+ \) and \( Z^- \) with \( \rho_V(Z^+) = \omega^+ \) and \( \rho_V(Z^-) = \omega^- \). Finally, applying theorem 6 of the appendix of [3], we get that \([\omega^-, \omega^+] \subseteq \rho_V(T)\), so \( \rho_V(T) = \rho_V(T)^* \).

In the following we recall some topological results for twist maps essentially due to Le Calvez (see [7] and [8] for proofs), that are used in some proofs contained in this paper. Let \( \hat{T} \in D_0(S^1 \times \mathbb{R}) \) and \( \tilde{T} \in D_0(\mathbb{R}^2) \) be its lifting. For every pair \((s, q), s \in \mathbb{Z}\) and \( q \in \mathbb{N}^* \) we define the following sets:

\[
\tilde{K}(s, q) = \{ (x, y) \in \mathbb{R}^2 : p_1 \circ \tilde{T}(x, y) = x + s \} \\
and \quad K(s, q) = \pi_1 \circ \tilde{K}(s, q),
\]

where \( \pi_1 : \mathbb{R}^2 \to S^1 \times \mathbb{R} \) is given by \( \pi_1(x, y) = (x \mod 1, y) \).

Then we have the following:

**Lemma 2**: For every \( s \in \mathbb{Z} \) and \( q \in \mathbb{N}^* \), \( K(s, q) \supseteq C(s, q) \), a connected compact set that separates the cylinder.

Now let us define the following functions on \( S^1 \):

\[
\mu^-(x) = \min \{ p_2(Q) : Q \in K(s, q) \text{ and } p_1(Q) = x \} \\
\mu^+(x) = \max \{ p_2(Q) : Q \in K(s, q) \text{ and } p_1(Q) = x \}
\]

We also have have similar functions for \( \hat{T}^q(K(s, q)) \):

\[
\nu^-(x) = \min \{ p_2(Q) : Q \in \hat{T}^q \circ K(s, q) \text{ and } p_1(Q) = x \} \\
\nu^+(x) = \max \{ p_2(Q) : Q \in \hat{T}^q \circ K(s, q) \text{ and } p_1(Q) = x \}
\]

The following are important results:

**Lemma 3**: Defining \( \text{Graph} \{ \mu^\pm \} = \{(x, \mu^\pm(x)) : x \in S^1 \} \) we have:

\( \text{Graph} \{ \mu^- \} \cup \text{Graph} \{ \mu^+ \} \subseteq C(s, q) \)

So for all \( x \in S^1 \) we have \( (x, \mu^\pm(x)) \in C(s, q) \).

**Lemma 4**: \( \hat{T}^q(x, \mu^-(x)) = (x, \nu^+(x)) \) and \( \hat{T}^q(x, \mu^+(x)) = (x, \nu^-(x)) \).

Now we remember some ideas and results from [3].

Given a triplet \((s, p, q), s \in \mathbb{Z}^2 \times \mathbb{N}^* \), if there is no point \((x, y) \in \mathbb{R}^2 \) such that \( \hat{T}^q(x, y) = (x + s, y + p) \), it can be proved that the sets \( \hat{T}^q \circ K(s, q) \) and \( K(s, q) + (0, p) \) can be separated by the graph of a continuous function from \( S^1 \) to \( \mathbb{R} \), essentially because from all the previous results, either one of the following inequalities must hold:

\[
\nu^-(x) - \mu^+(x) > p
\]
for all $x \in S^1$, where $\nu^+, \nu^-, \mu^+, \mu^-$ are associated to $K(s, q)$.

Following Le Calvez [9], we say that the triplet $(s, p, q)$ is positive (resp. negative) for $\bar{T}$ if $\bar{T} \circ K(s, q)$ is above (resp. below) the graph. Given $\bar{T} \in D_0(\mathbb{R}^2)$, we have:

$$\bar{T}(x, y) = (x', y') \leftrightarrow y = m(x, x') \text{ and } y' = m'(x, x'),$$

where $m$ and $m'$ are continuous maps from $\mathbb{R}^2$ to $\mathbb{R}$ with some especial properties.

If $\bar{T}, \bar{T}^* \in D_0(\mathbb{R}^2)$, we say that $\bar{T} \leq \bar{T}^*$ if $m^* \leq m$ and $m^* \leq m'^*$, where $(m, m')$ is associated to $\bar{T}$ and $(m^*, m'^*)$ to $\bar{T}^*$.

**Proposition 1**: If $(s, p, q)$ is a positive (resp. negative) triplet of $\bar{T}$ and if $\bar{T} \leq \bar{T}^*$ (resp. $\bar{T} \geq \bar{T}^*$), then $(s, p, q)$ is a positive (resp. negative) triplet of $\bar{T}^*$.

Now we present an amazing example of a twist homeomorphism from $D_0(T^2)$. First, let $g'^* : S^1 \rightarrow \mathbb{R}$ be given by $g'(x) = \left| x - \frac{1}{2} \right| - \frac{1}{2}$ and so the lift $\bar{g}' : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $\bar{g}'(x+1) = \bar{g}'(x)$, $\int_0^1 \bar{g}'(x)dx = 0$, Lip($\bar{g}'$) = 1 and $\bar{g}'(x) = \bar{g}'(-x)$. Also, $\bar{g}'$ is differentiable everywhere, except at points of the form $\frac{n}{2}, n \in \mathbb{Z}$. The one parameter family $S'_\lambda \in D_0(T^2)$ is given by:

$$S'_\lambda : \begin{cases} y' = \lambda g'(x) + y \mod 1 \\ x' = x + y' \mod 1 \end{cases} \quad (8)$$

In [4] this family is studied in detail and among other things, the following theorem is proved:

**Theorem 3**: There are no rotational invariant curves for $S'_\lambda$ when

$\lambda \in ]0, 0.918, 1[\cup]4/3, \infty[ \quad \text{and for } \lambda = 4/3 \text{ there are "lots" of rotational invariant curves.}$

### 3 Proofs

#### 3.1 Preliminary results

**Proof of lemma [4]**

This result is a trivial consequence of proposition [4]. Given $\lambda_1 < \lambda_2$, we get from expression [3] that $T_{\lambda_1} \leq T_{\lambda_2}$. So if $\rho^\max V(T_{\lambda_1}) < p/q < \rho^\max V(T_{\lambda_2})$ for a certain rational number $p/q$, then for any $s \in \mathbb{Z}$ the triplet $(s, p, q)$ is negative for $T_{\lambda_2}$, which implies by proposition [4] that it is also negative for $T_{\lambda_1}$, which contradicts the fact that $\rho^\max V(T_{\lambda_1}) > p/q$.

Now we prove the following theorem that has its own interest. It is easy to see from the proof that it is valid in a more general context.

**Theorem 4**: The functions $\rho^\min V, \rho^\max V : D_0(T^2) \rightarrow \mathbb{R}$ are continuous.
Remark: The proofs are analogous, so we do it only for $\rho_v^\text{max}$.

Proof:
Suppose that there is a $T_0 \in D_0(T^2)$ such that $\rho_v^\text{max}$ is not continuous at $T_0$. This means that there is an $\epsilon > 0$ and a sequence $D_0(T^2) \ni T_n \to T_0$ in the $C^0$ topology, such that either:

1) $\rho_v^\text{max}(T_n) > \rho_v^\text{max}(T_0) + \epsilon$, for all $n$, or
2) $\rho_v^\text{max}(T_n) < \rho_v^\text{max}(T_0) - \epsilon$, for all $n$.

The first possibility means that there exists a rational number $p/q$ such that $\rho_v^\text{max}(T_n) > p/q > \rho_v^\text{max}(T_0)$. This implies that for any $s \in \mathbb{Z}$, the triplet $(s, p, q)$ is non-negative for $T_n$ (as the value of $s$ is irrelevant in this setting, we fix $s = 0$). But as $\rho_v^\text{max}(T_n) < p/q < \rho_v^\text{max}(T_0)$, we get from the upper semi-continuity in the Hausdorff topology of the maps

$$T \to K(0, q) \text{ and } T \to \hat{T}^q(K(0, q))$$

that $(0, p, q)$ is a negative triplet for all mappings sufficiently close to $T_0$, which is a contradiction.

In the same way, the second possibility means that there exists a rational number $p/q$ such that $\rho_v^\text{max}(T_n) < p/q < \rho_v^\text{max}(T_0)$. This implies that there exists $Q \in C(0, q)$ such that

$$p_2 \circ \hat{T}_0^q(Q) - p_2(Q) > p.$$  \hspace{1cm} (10)

Now we prove the following claim, which implies the theorem:

Claim: Any mapping $T \in D_0(T^2)$ sufficiently close to $T_0$ will satisfy an inequality similar to (10).

Proof:
First of all, let us define $P_0 = (x_Q, \mu^-(x_Q))$, where $x_Q = p_1(Q)$. From lemma $\text{[3]}$ and the definition of $\mu^-$ and $\nu^+$, we get that $\nu^+(x_Q) = p_2 \circ \hat{T}_0^q(P_0) > p_2(P_0) + p = \mu^-(x_Q) + p$. So there exists $\delta > 0$ such that for any $Z \in \mathcal{B}_\delta(P_0)$ we have

$$p_2 \circ \hat{T}_0^q(Z) > p_2(Z) + p.$$ 

Therefore, there exists a neighborhood $T_0 \in U \subset D_0(T^2)$ in the $C^0$ topology such that for any $T \in U$, we get $p_2 \circ \hat{T}^q(Z) > p_2(Z) + p$, for all $Z \in \mathcal{B}_\delta(P_0)$. Now defining $\overline{AB} = \{x_Q \times \mathbb{R}\} \cap \mathcal{B}_\delta(P_0)$, lemma $\text{[3]}$ implies that if we choose a sufficiently small neighborhood $V$ of $C(0, q)$, then for all homotopically non-trivial simple closed curves $\gamma \subset V$, we get that $\gamma \cap \overline{AB} \neq \emptyset$. By the upper semi-continuity in the Hausdorff topology of the maps in $\text{[3]}$, if we choose a sufficiently small sub-neighborhood $U' \subset U$ we get for any $T \in U'$ that the set $C(0, q)$ associated to $T$ is also contained in $V$. Therefore it must cross $\overline{AB}$.

So given any mapping $T \in U' \subset U$, there is a point $Q' \in C(0, q) \cap \overline{AB}$ which therefore satisfies $p_2 \circ \hat{T}^q(Q') > p_2(Q') + p$.

Finally, the above claim implies that $\rho_v^\text{max}(T_n) \geq p/q$ for sufficiently large $n$, which is a contradiction.
3.2 Main theorem

In this section we prove theorem (1).

First of all we note that from theorem (3), the mapping \( S'_\lambda \in D_0(T^2) \) (see (8)) has no rotational invariant curve for \( \lambda = 0.95 \) and has "lots" of rotational invariant curves for \( \lambda = 4/3 \). Using theorem (2) one gets that \( \rho^{\text{max}}_V(S'_{0.95}) = \epsilon > 0 \) and \( \rho^{\text{max}}_V(S'_{4/3}) = 0 \). A classical result in Fourier analysis implies that the Fourier series \( \tilde{g}'_N(x) = \sum_{n=1}^{N} a_n \cos(2\pi nx) \) of \( \tilde{g}' \) converges uniformly to \( \tilde{g}' \). So if we choose \( N > 0 \) sufficiently large, we get from theorem (3) that \( \rho^{\text{max}}_V(S'_{N,0.95}) > \epsilon/2 \) and \( \rho^{\text{max}}_V(S'_{N,4/3}) < \epsilon/10 \), where \( S'_{N,\lambda} \) is the twist mapping associated to \( g'_N \).

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