Fig. 1
Fig. 2
Fig. 3
Fig. 4

$\Gamma_{G \rightarrow \eta \eta'}$ (GeV) vs. $M$ (GeV)
The Decays of Glueballs to two light mesons

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Abstract

We solve the B.S. equation of glueballs under instantaneous approximation. With the B.S wave function obtained we calculate the decay width of glueballs to two pseudoscalar mesons, $\Gamma(\pi\pi)$, $\Gamma(kk)$ and $\Gamma(\eta\eta)$. $\Gamma(\eta'\eta')$ from QCD anomaly is also estimated.

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Keywords: glueball, B.S. equation, QCD anomaly, decay
1. Introduction The existence of the glueball states is the prediction of QCD, whose discovery would be a direct confirmation of the non-abelian character of QCD. A lot of investigation both in theory [1–5] and in experiment [6–8] have been made in the glueball searches. At present there are several experimental glueball candidates, such as $f_0(1500)$, $f_J(1700)$ and $\xi(2230)$, but none is identified as glueball definitely. It’s difficult to identify a glueball because so far we haven’t understood the glueball clearly in theory: we don’t know clearly its mass, its production rates and its decay width. Therefore, more investigations are necessary.

In this paper, we focus on $0^{++}$ glueball’s decays. Up to now, $f_0(1500)$ is considered as the lowest-lying $0^{++}$ glueball candidate. One of the most important reason is that the branching ratio of decay $f_0(1500) \to \eta\eta'$ is surprisingly large, i.e $\Gamma(\eta\eta')/\Gamma(\eta\eta') \sim 1$. Since not only $\eta\eta'$ final state is almost on the kinematic threshold but also it is suppressed by SU(3) flavor symmetry, the naive quark model gives $\Gamma(\eta\eta')/\Gamma(\eta\eta') \approx 0$. Therefore $\eta'$ must be produced by another mechanism. One of the possible mechanism is the famous QCD anomaly. However, it is well known that QCD anomaly is large $N_c$ suppressed, in order to produce enough rate of $\eta'$ production, $f_0(1500)$ should have a high gluon density, i.e., it is a glueball. On this assumption, the further theoretical calculation is still necessary to check the rate $\Gamma(\eta\eta')/\Gamma(\eta\eta)$ and $\Gamma(\eta\eta')/\Gamma(\pi\pi)$. We notice there have already been some works on this topic from lattice QCD approach, however, an model-dependent calculation is still useful and may give some helpful hints.

In an early paper [5], we studied the glueball spectrum in the framework of B.S. equation, and the results are in good agreement with that of Lattice calculation. This encourages us to do more investigation. By solving the B.S. equation about gluon-gluon bound state, we can obtain not only the glueball spectrum but also the glueball B.S. wave function. In this paper we try to use these wave functions to estimate the decay width of glueballs to light pseudoscalar mesons in the framework of perturbative QCD.

This paper is arranged as follows. In the next section we briefly review the construction of the B.S. equation for gluons bound state. In section three, first we calculate
the glueball width to two pions. Then we give the estimations of width to two K’s and two \( \eta \)’s. In section four, we give the estimation of \( \Gamma(\eta\eta') \) from QCD anomaly. In the last section a discussion is presented.

2. The Glueball B.S. Equation  The details of the construction of the glueball B.S. equation has been given in ref. [5]. Here we only present a brief review.

Let \( A^a_\mu(x_1) \) and \( A^b_\nu(x_2) \) be the gluon fields at points \( x_1 \) and \( x_2 \), \( |G\rangle \) the bound state of two gluons with mass \( M_G \) and momentum \( P_\mu \). Then the B.S. wave function for a bound state is defined as

\[
\chi^{ab}_{\mu\nu}(P, x_1, x_2) = \langle 0|T(A^a_\mu(x_1)A^b_\nu(x_2))|G\rangle,
\]

(1)

where \( \mu, \nu \) are Lorentz indices, and \( a, b \) color indices. The glueballs are color singlet states, so we have

\[
\chi^{ab}_{\mu\nu}(P, x_1, x_2) = \delta^{ab}\chi_{\mu\nu}(P, x_1, x_2).
\]

(2)

With a standard method we obtain the B.S. equation for a color singlet glueball state,

\[
\chi_{\mu\nu}(P, q) = \Delta_{\mu\alpha}(p_1)\Delta_{\nu\beta}(p_2) \int \frac{d^4k}{(2\pi)^4} G^{\alpha\beta\rho\sigma}(P, q, k)\chi_{\rho\sigma}(P, k),
\]

(3)

where \( \chi_{\mu\nu}(P, q) \) is the Fourier transformation of \( \chi^{ab}_{\mu\nu}(P, x_1, x_2) \).

The tensor kernel \( G^{\alpha\beta\rho\sigma}(P, q, k) \), in which the color indices are suppressed, is defined as the sum of all two-particle irreducible graphs, and \( \Delta_{\mu\alpha}(p_i) \) is the full propagator of the constituent gluons with momentum \( p_i \).

\[
P = p_1 + p_2, \quad 2q = p_1 - p_2,
\]

(4)

Under instantaneous approximation, we ignore the \( q_0 \) and \( k_0 \) dependence of the kernel \( G^{\alpha\beta\rho\sigma}(P, q, k) \). Further we replace \( \Delta_{\mu\alpha}(p_i) \) with a free propagator. Then in Coulomb gauge we obtain a three dimensional equation

\[
E(M_G^2 - 4E^2)\varphi_{ij}(P, q) = \int \frac{d^3k}{(2\pi)^3} G_{i'j'kl}(P, q, k)\varphi_{kl}(P, k).
\]

(5)
where $E = \sqrt{q^2 + m^2}$ (m is the constituent mass of gluons, which will be chosen as zero in the present calculations), and $\varphi_{ij}(i, j = 1, 2, 3)$ is the three dimensional wave function, which is related to $\chi_{ij}$ by the following equation

$$\varphi_{\mu\nu}(P, k) \equiv \int dk_0 \chi_{\mu\nu}(P, k).$$

(6)

The kernel is divided into two parts: the short distance part $G^{(s)}_{ijkl}$ and the long distance part $G^{(l)}_{ijkl}$. $G^{(s)}_{ijkl}$ is obtained approximately by calculating the three lowest order diagrams shown in fig.1.

Besides equations (4), we also have

$$P = p_3 + p_4, \quad 2k = p_3 - p_4.$$  

(7)

Calculating the diagrams a, b and c shown in fig.1, $G^{(s)}_{\mu\nu\rho\sigma}(P, q, k)$ can be expressed explicitly as

$$G^{(s)}_{\mu\nu\rho\sigma}(P, q, k) = 3i(4\pi\alpha_s) \left\{ 2C_{\mu\rho\tau}(p_1, p_3)C_{\nu\sigma\tau'}(p_2, p_4) \left[ g^{\tau0}g^{\tau'0} \frac{l^2}{I^2} + g^{\tau i}g^{\tau'j} \frac{l^2}{I^2} (\delta_{ij} - \frac{l_i l_j}{l^2}) \right] 
- (2g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \right\},$$

(8)

where $l = q - k$ is the momentum exchanged between the two constituent gluons, and

$$C_{\mu\rho\tau}(p_1, p_3) = (p_1 - 2p_3)_{\mu}g_{\rho\tau} + (p_1 + p_3)_{\tau}g_{\mu\rho} + (p_3 - 2p_1)_{\rho}g_{\mu\tau},$$

(9)

$$C_{\nu\sigma\tau'}(p_2, p_4) = (p_2 - 2p_4)_{\nu}g_{\sigma\tau'} + (p_2 + p_4)_{\tau'}g_{\nu\sigma} + (p_4 - 2p_2)_{\sigma}g_{\nu\tau'}.$$  

(10)

The factor 3 on the right-hand side of equation (8) is the color factor which is $\frac{4}{3}$ in the case of $q\bar{q}$ bound state. Diagram a and diagram b make the same contribution to physical states, so there is a factor 2 in equation (8). The strong coupling constant $\alpha_s$ is chosen as a running one,

$$\alpha_s = \frac{12\pi}{27} \frac{1}{\ln(a + \frac{l^2}{\Lambda_{QCD}^2})},$$

(11)

where $a$ is a parameter introduced to avoid the infrared divergence. We take $a=4.0$, which implies that the largest value of the running coupling constant is 1.0, and we choose $\Lambda_{QCD} = 200MeV$.  

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As for the long distance part $G_{ijkl}^{(l)}$, we only choose in the three-gluon-vertex (9,10) the terms containing tensor $g_{\mu\rho}g_{\nu\sigma}$ as the spin dependence of the confining part, because such terms have nothing to do with spin effect. Therefore we assume

$$G_{\mu\rho\nu\sigma}^{(l)} = 2i(p_1 + p_3) \cdot (p_2 + p_4)g_{\mu\rho}g_{\nu\sigma}G(l),$$

where $G(l)$ is spatial dependence of confining part of the kernel. We choose

$$G(l) = \frac{8\pi\lambda}{l^4},$$

which corresponds to a linearly growing potential. $\lambda$ is the string tension, which in quark potential model takes the value $0.18(\text{GeV})^2$. However, here we chose $\lambda = 0.36(\text{GeV})^2$. We expect that confining force between two gluons is one times stronger than that between two quarks because each gluon acts like a $q\bar{q}$ pair[4]. Of course, such a potential is very singular at the point $l = 0$. Some form of regularization is necessary. Our method is given in ref.[5]. For $0^{++}$ glueball, the B.S. wave function can be expressed as

$$\varphi_{ij}(P, q) = f_s(q)(\delta_{ij} - \frac{q_i q_j}{q^2}).$$

(14)

With the kernel and the form of the wave functions chosen as above, the equation can be solved numerically. The glueball spectrum obtained are in good agreement with that of lattice calculation. At the same time the wave functions (14) and (15) are also obtained. Such wave functions should be normalized before being used in the calculation of the decays. If we neglect the $P_0$ dependence of the kernel, then the three dimensional wave function $\varphi_{ij}(P, k)$ is subjected to the normalization condition

$$\int \frac{d^3q}{(2\pi)^3} E\varphi_{ij}(q)\varphi_{kl}(q)(\delta_{ik} - \frac{q_i q_k}{q^2})(\delta_{jl} - \frac{q_j q_l}{q^2}) = (2\pi)^2$$

(15)

3. The Width of $0^{++}$ Glueball to Two Light Mesons

i. The decay to two pions In the framework of perturbative QCD, the decays of the glueball to two pions can be expressed by the diagram shown in fig.2. The amplitude is expressed as

$$M(G(0^{++}) \rightarrow \pi^+\pi^-) = -\frac{4\pi\alpha_s f_{\pi}^2}{9} \int_{-1}^{1} d\xi_1 d\xi_2 \varphi_\pi(\xi_1)\varphi_\pi(\xi_2)(2k_{1i}k_{2j} + k_1 \cdot k_2 \delta_{ij})\chi_{ij}(q).$$

(16)
where $\bar{\alpha}_s$ is effective coupling constant which will be discussed below, $k_1$, $k_2$ are the momentum of pions. $\varphi_\pi(\xi)$ is the pion wave function or distribution amplitude. $\chi_{ij}$ is the four dimensional glueball B.S. wave function defined in section 1., which can be replaced by the three dimensional wave function $\varphi_{ij}$ through

$$\chi_{ij} = \frac{|\mathbf{q}|(M_G^2 - 4\mathbf{q}^2)}{2\pi i p_1^2 p_2^2} \varphi_{ij}$$

(17)

$\varphi_{ij}$ is obtained by solving the B.S. equation.

The mass of pions can be neglected, and thus we have

$$k_1 \cdot k_2 = \frac{M_G^2}{2}.$$  

(18)

$$|\mathbf{q}| = \frac{|\xi_1 + \xi_2|}{4} M_G.$$  

(19)

With these relations, equation (16) is reduced to

$$M(G(0^{++}) \to \pi^+\pi^-) = \frac{4i\bar{\alpha}_s M_G f_\pi^2}{9} \int_{-1}^1 d\xi_1 d\xi_2 \varphi_\pi(\xi_1) \varphi_\pi(\xi_2)$$

$$\frac{|\xi_1 + \xi_2|}{(1 - \xi_2)(1 - \xi_2)} \sum_i \varphi_{ii}(|\mathbf{q}|)$$

(20)

There are several forms of the pion wave function. Here we will use the form given by Chernyak by QCD sum rule[9].

$$\varphi_\pi(\xi) = \frac{15}{4} \xi^2(1 - \xi^2).$$

(21)

In equations (16) and (20), $\bar{\alpha}_s$ is the effective coupling constant. In fact, we should have taken $\sqrt{\bar{\alpha}_s \cdot \alpha_s}$ at the points corresponding to the gluon virtualities: $\alpha_s(p_1^2) = \alpha_s(\frac{1 - \xi_1}{2} \frac{1 - \xi_2}{2} M_G^2)$ and $\alpha_s(p_2^2) = \alpha_s(\frac{1 + \xi_1}{2} \frac{1 + \xi_2}{2} M_G^2)$. However, for simplicity, we replace $p_{1,2}$ by their character values $\bar{p}_{1,2}$, and then extract $\alpha_s$ out of the integrals over $\xi_1$ and $\xi_2$. $\bar{p}_{1,2}$ can be determined by the forms of pion wave functions. Therefore, $\bar{\alpha}_s$ in formula (16) is chosen as $\frac{\alpha_s(\bar{p}_{1,2}^2) \alpha_s(\bar{p}_{1,2}^2)}{\alpha_s(M_G^2/4)}$. For Chernyak’s form, the pion wave function have maximum at $\bar{\xi} = 0.7$, and $\bar{p}_{1,2} = \frac{1 \mp \xi_1 1 \mp \xi_2}{2} M_G^2$. Therefore, $\bar{\alpha}_s = 0.69$ for $0^{++}$ glueball.
Substituting expression (14) into formula (20), we get

\[ M(G(0^{++}) \rightarrow \pi^+\pi^-) = \frac{8i\alpha_s f_p^2 M_G I}{9} \]  

with

\[ I = \int d\xi_1 d\xi_2 \frac{\xi_1 + \xi_2}{(1 - \xi^2)(1 - \xi^2)} |f_s(q)| \]

Finally, after making the phase space integration, we obtain

\[ \Gamma(G(0^{++}) \rightarrow \pi^+\pi^-) = \frac{4\alpha_s^2 M_G f_p^4 I^2}{81\pi} \]  

The numerical results is shown in table 1.

ii. The decays to two K’s and two η’s

To estimate the glueball decays to two k and two η mesons, we construct the following chiral Lagrangian in the lowest order

\[ L_{eff} = g\phi tr(\partial_\mu \Sigma^+ \partial^\mu \Sigma) \]  

where \( \phi \) is the field of the scalar meson, and \( \Sigma = e^{2i\bar{\pi}} \),

\[ \bar{\pi} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & k^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & k^0 \\ k^- & \bar{k}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \]

\( g \) is a coupling constant which can be determined from the result of two pions decay obtained above. Neglecting the factor of the phase space, we have

\[ \Gamma(\pi^+\pi^-) = \Gamma(k^+k^-) = \Gamma(k^0\bar{k}^0) = \frac{1}{2} \Gamma(\pi^0\pi^0) = \frac{1}{2} \Gamma(\eta\eta) \]

Taking the factor of the phase space into consideration, We give the numerical results in table 1.

4. The decay of glueball to η, η’ form QCD anomaly

As mentioned in the introduction, η, η’ final state cannot be produced enough only from naive quark model, there should be another mechanism to produce them, namely the QCD anomaly (see fig.3)

\[ \partial_\mu j^\mu_5 = \frac{3\alpha_s}{4\pi} G_{\mu\nu} G^{\mu\nu} + 2i \sum_{q=uds} m_q \bar{q} \gamma^5 q \]  

(25)
where \( j_5^\mu \) is the singlet axial current \([10]\). Indeed it is the anomalous contribution to the divergence of the axial current that makes the \( SU(3) \) singlet state much heavier than the Octet pseudo-Goldstone state and distinguish \( \eta \) from \( \eta' \). Obviously, keeping \( SU(3)_L \otimes SU(3)_R \) flavor symmetry unbroken, \( \eta \) cannot be produced from two gluons. Actually, physical \( \eta, \eta' \) states are the mixing states of flavor eigenstate \( \eta_1 \) and \( \eta_8 \)

\[
\eta = \cos \theta_\eta \eta_8 - \sin \theta_\eta \eta_1, \tag{26}
\]

\[
\eta' = \sin \theta_\eta \eta_8 + \cos \theta_\eta \eta_1, \tag{27}
\]

where the mixing angle \( \theta_\eta \) is estimated in the range of \(-10^\circ \) to \(-20^\circ \), so \( \eta' \) is dominantly \( \eta_1 \). The effective coupling of \( \eta_1 gg \) can be written in the form

\[
H(q_1^2, q_2^2, q_\eta^2) \delta^{ab} \epsilon^{\mu\nu\alpha\beta} q_1^\mu q_2^\nu \epsilon_1^\alpha \epsilon_2^\beta / \cos \theta_\eta \tag{28}
\]

where \( q_1, q_2 \) and \( \epsilon_1, \epsilon_2 \) are the 4-momenta and polarizations of two gluons and \( a, b \) are color indices. \( H \) is a form factor that is a general function of momenta, \( q_1^2, q_2^2 \) and \( q_\eta^2 \). For simplicity, we set \( H \) as a constant, then, from the experimental data \( \Gamma(J/PSi \rightarrow \gamma\eta')/\Gamma(J/\Psi \rightarrow e^+e^-) = 7 \times 10^{-2} \), one may obtain \( H \sim 1.8 \text{ GeV}^{-1} \) \([11]\).

Corresponding to fig.3, the amplitude dominated by on-shell gluons can be written as

\[
M(G \rightarrow \eta\eta') = 3\sqrt{8} \cos \theta_\eta \sin \theta_\eta \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} \cdot (2\pi)^4 \delta^4(\frac{1}{2}P + q - q_1 - q_2)(2\pi)^4 \delta^4(k_1 - q_1 - q_3)(2\pi)^4 \delta^4(k_2 - q_2 - q_4) \cdot \chi_{\mu\nu}(P,q)C^{\rho\sigma\mu}(-q_1, q_2)C^{\alpha\beta\nu}(-q_3, q_4) H^2 \epsilon_{\rho\sigma\gamma\delta} \epsilon_{\sigma\beta\lambda\tau} q_1^\gamma q_2^\delta q_3^\lambda q_4^\tau \tag{29}
\]

where \( C^{\rho\sigma\mu}(-q_1, q_2) \) and \( C^{\alpha\beta\nu}(-q_3, q_4) \) are the three-gluon-vertex defined as eq.(9), and \( 3\sqrt{8} \) is a color factor. With the the glueball B.S. wave function obtained in section 2, the calculation of the amplitude is straight forward. The corresponding width is list in table 1.

Because the sum of \( m_\eta \) and \( m_{\eta'} \) is quite near the mass of glueball, \( M \), we expect the above result from QCD anomaly will depend on \( M \). On the other hand, the glueball mass is not exactly known either from experiment or from theory. Therefore, we show
the dependence of $\Gamma_{G\to \eta\eta'}$ on glueball mass $M$ in fig.4. The value of $\Gamma(\eta\eta)$ in table 1 corresponds to $M = 1.6\text{GeV}$.

5. Discussion and Conclusions

In this paper, we study the decays of $0^{++}$ glueball to two light mesons by using the constituent gluon model. The glueball wave function is obtained by solving B-S equation, the pion is light so that it can described by the light cone wave function. $\text{SU}(3)$ chiral symmetry is used to predict decay widths $\Gamma(\bar{K}K)$ and $\Gamma(\eta\eta)$. We also study the $\eta\eta'$ production from QCD anomaly, which almost is zero from the naive quark model prediction. We are more interested in the ratios of the different decay channel, since the ratios is less sensitive to the glueball wave function we use. From table 1, we get

$$\frac{\Gamma(\eta\eta')}{\Gamma(\eta\eta)} = 0.82 \quad \frac{\Gamma(\pi^0\pi^0)}{\Gamma(\eta\eta)} = 1.5$$

Such ratios have values in experiment for $f_0(1500)$. The Crystal Barrel Collab.\cite{12} gives

$$\frac{\Gamma(\eta\eta')}{\Gamma(\eta\eta)} = 0.29 \pm 0.1 \quad \frac{\Gamma(\pi^0\pi^0)}{\Gamma(\eta\eta)} = 1.45 \pm 0.61(3\pi^0\text{channel}) \quad \frac{\Gamma(\pi^0\pi^0)}{\Gamma(\eta\eta)} = 2.12 \pm 0.81(\text{Coupled - channel}), \quad (30, 31, 32)$$

GAM group gave a rather large ratio \cite{13}

$$\frac{\Gamma(\eta\eta')}{\Gamma(\eta\eta)} = 2.7 \pm 0.8 \quad (33)$$

We see that our results agree with the experiment data of $f_0(1500)$. 

\[8\]
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Figuer Caption

Fig.1: The diagrams contributing to the short distance part of the B.S. kernel.

Fig.2: The decays of glueball to two pions in perturbative framework.

Fig.3: The decays of glueball to \( \eta \eta' \) from QCD anomaly

Fig.4: The dependence of \( \Gamma(\eta \eta') \) on glueball mass.
Table Caption

Table 1: The partial width of the glueball to two light mesons (MeV).
Table 1. The partial width of the glueballs to two and four pions (MeV).

| $\Gamma(\pi^+\pi^-)$ | $\Gamma(\pi^0\pi^0)$ | $\Gamma(k^+k^-)$ | $\Gamma(k^0\bar{k}^0)$ | $\Gamma(\eta\eta)$ | $\Gamma(\eta\eta')$ |
|----------------------|-----------------------|------------------|-------------------------|-------------------|-------------------|
| 2.4                  | 1.2                   | 0.91             | 0.91                    | 0.83              | 0.68              |