CANONICAL THURSTON OBSTRUCTIONS FOR
SUB-HYPERBOLIC SEMI-RATIONAL BRANCHED
COVERINGS

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Abstract. We prove that the canonical Thurston obstruction
for a sub-hyperbolic semi-rational branched covering exists if the
branched covering is not CLH-equivalent to a rational map.

1. Introduction

Let $S^2$ be the two-sphere. We use $\hat{C}$ to denote the Riemann sphere
which is $S^2$ equipped with the standard complex structure. All maps
in this paper are orientation-preserving.

Let $f : S^2 \to S^2$ be a branched covering of degree $d \geq 2$. Let
\[ C_f = \{ x \in S^2 \mid \deg_x f \geq 2 \} \]
denote the set of the critical points of $f$ and
\[ P_f = \bigcup_{k \geq 1} f^k(C_f) \]
denote the post-critical set of $f$.

We say $f$ is critically finite if $\sharp P_f$ is finite. We say $f$ is geometrically
finite if $\sharp P_f$ is infinite but the accumulation set $P'_f$ of $P_f$ is a finite set.

Definition 1. Suppose $f, g : S^2 \to S^2$ are two branched coverings of
degree $d \geq 2$. They are said to be combinatorially equivalent if there
exists a pair of homeomorphisms $\phi, \varphi : S^2 \to S^2$ such that
a) $\phi$ is isotopic to $\varphi$ rel $P_f$ and
b) $\phi \circ f = g \circ \varphi$.

Note that in a) of Definition 1 the statement that $\phi$ is isotopic to $\varphi$
rel $P_f$ means that there is a continuous map $H(x, t) : S^2 \times [0, 1] \to S^2$
such that

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(1) for each $t \in [0,1]$, $H_t(x) = H(x, t) : S^2 \to S^2$ is a homeomorphism;
(2) $H_0 = \phi$ and $H_1 = \varphi$;
(3) for any point $y \in P_f$ and any $t \in [0,1]$, $H_t(y) = \phi(y) = \varphi(y)$.

Suppose $f : S^2 \to S^2$ is critically finite. Then there is an orbifold structure on $S^2$ associated to $f$ as follows. Define the signature $\nu_f : S^2 \to \mathbb{Z}_+\cup\{\infty\}$ as that $\nu_f(x)$ is the least common multiple of local degrees $\deg_y f^n$ over $y \in f^{-n}(x)$ for all $n \geq 1$. The orbifold associated to $f$ is $\Omega_f = (S^2, \nu_f)$. The Euler characteristic of $\Omega_f$, by definition, is

$$\chi(\Omega_f) = 2 - \sum_{x \in S^2} \left(1 - \frac{1}{\nu_f(x)}\right).$$

It is known that $\chi(\Omega_f) \leq 0$ (see Proposition 9.1 (i) in [DH]). Moreover, the orbifold $\Omega_f$ is called hyperbolic if $\chi(\Omega_f) < 0$ and parabolic if $\chi(\Omega_f) = 0$.

**Theorem A** ([Th, DH]). Suppose $f$ is a critically finite branched covering with a hyperbolic orbifold $\Omega_f$. Then $f$ is combinatorially equivalent to a rational map $R$ if and only if $f$ has no Thurston obstructions. Moreover, the rational map $R$ is unique up to conjugations by automorphisms of the Riemann sphere.

The reader can refer to §2 for the definition of Thurston obstruction. If a critically finite branched covering $f$ is not equivalent to a rational map, then there must exist Thurston obstructions. The canonical Thurston obstruction is the most interesting one among all Thurston obstructions. The reader can refer to §2 for the term non-peripheral and §4 for the definition of $l(\gamma, x)$.

**Theorem B** ([Pi]). Suppose $f$ is a critically finite branched covering with a hyperbolic orbifold $\Omega_f$, and let $\Gamma_c$ denote the set of all homotopy class of non-peripheral curves $\gamma$ in $S^2 \setminus P_f$ such that $l(\gamma, x_n) \to 0$ as $n \to \infty$. Then

(1) $\Gamma_c$ is empty, and $f$ is combinatorially equivalent to a rational map;
(2) Otherwise, $\Gamma_c$ is a Thurston obstruction and hence is a canonically defined Thurston obstruction to the existence of a rational map.

The nonexistence of Thurston’s obstruction condition is essentially true for any rational map.
Theorem C ([Mc]). Suppose $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map. Let $\Gamma$ be a multi-curve on $\hat{\mathbb{C}} - P_R$. It can be a Thurston obstruction only in the following cases:

1) $R$ is critically finite with $\#P_R = 4$ and the orbifold $\Omega_R = (S^2, (2, 2, 2, 2))$ is parabolic. And, moreover, $R$ is a double covered by an integral torus endomorphism (it is a special case of a Lattés map).

2) $P_R$ is an infinite set and $\Gamma$ includes the essential curves in a finite system of annuli permuted by $R$. These annuli lie in Siegel disks or Herman rings for $R$ and each annulus is a connected component of $\hat{\mathbb{C}} - P_R$.

The reader can refer to [Mi] for a definition of a Lattés map and for definitions of a Siegel disk and a Herman ring.

For a geometrically finite branched covering $f$, the situation is much more complicated. It was first studied in a manuscript [CJS], which was divided into two parts [CJS1] and [CJS2]. The first part was eventually completed and published in [CJ] as follows.

Theorem D ([CJ]). There is a geometrically finite branched covering such that it has no Thurston obstruction and it is not combinatorially equivalent to any rational map.

Due to this theorem, a semi-rational branched covering and a sub-hyperbolic semi-rational branched covering are introduced in [CJ] among the space of all geometrically finite branched coverings:

Definition 2. Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a geometrically finite branched covering of degree $d \geq 2$. We say $f$ is semi-rational if

1) $f$ is holomorphic in a neighborhood of $P_f$;

2) each cycle $< p_0, \ldots, p_{k-1} >$ of period $k \geq 1$ in $P_f$ is either attractive, that is, $0 < |(f^k)'(p_0)| < 1$, or super-attractive, that is, $(f^k)'(p_0) = 0$, or parabolic, that is, $|(f^k)'(p_0)| = 1$ and $((f^k)'(p_0))^q = 1$ for some integer $q \geq 1$; and

3) each attracting petal associated with a parabolic cycle in $P_f$ contains a point in the post-critical set $P_f$.

Furthermore, if all cycles in $P_f$ are either attractive or super-attractive, we call $f$ a sub-hyperbolic semi-rational branched covering.

Clearly, every geometrically finite rational map is a semi-rational branched covering. Furthermore, we have that
Theorem E ([CJ]). A semi-rational branched covering $f$ is always combinatorially equivalent to a sub-hyperbolic semi-rational branched covering $g$.

Thus, to study the combinatorial classification in the space of all semi-rational geometrically finite branched coverings, it is enough to study all sub-hyperbolic semi-rational branched coverings. Therefore, the CLH (combinatorially and locally holomorphically) equivalence was introduced in [CJ] in the space of all sub-hyperbolic semi-rational branched coverings.

Definition 3. Suppose $f$ and $g$ are two sub-hyperbolic semi-rational branched coverings. We say that they are CLH-equivalent if there exists a pair of homeomorphisms $\phi, \varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

1. $\phi$ is isotopic to $\varphi$ rel $P_f$,
2. $\phi \circ f = g \circ \varphi$, and
3. $\phi|U_f = \varphi|U_f$ is holomorphic on some open set $U_f \supset P'_f$.

We then completed the second part of study.

Theorem F ([CJS2],[JZ]). Suppose $f$ is a sub-hyperbolic semi-rational branched covering. Then $f$ is CLH-equivalent to a rational map $R$ if and only if $f$ has no Thurston obstructions. In this case, the rational map $R$ is unique up to conjugations by automorphisms of the Riemann sphere.

Thus, the study of canonical Thurston obstructions for sub-hyperbolic semi-rational branched coverings becomes our final goal to have a complete understanding of combinatorial structures for geometrically finite branched coverings. In this paper we will complete our final goal.

For the critically finite case, the Teichmüller space of the Riemann sphere minus several points was considered in [DH]. A crucial technical theorem they proved is that a sequence $\{x_n\}$ in the Teichmüller space converges if and only if its projection is a precompact set in the moduli space. The Mumford compactness theorem applies. But neither of them applies for the geometrically finite case. Therefore, we turn to study the bounded geometry property, which makes our approach different from the method applied in [DH]. Roughly speaking, our main result in this paper is that if a sub-hyperbolic semi-rational branched covering $f$ is not CLH-equivalent to any rational map, then there exists a canonical Thurston obstruction. To have a more precise statement

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1This paper was rewritten by Cui and Tan recently [CT].
of our main result, let us first give an idea of a proof of Theorem F by using bounded geometry as follows.

Suppose $f$ is a sub-hyperbolic semi-rational branched covering. Let $P_f' = \{a_i\}$ be the set of accumulation points of $P_f$. Then every $a_i$ is periodic. There exists a collection of finite number of open disks

$$\Lambda = \{D_i\}$$

centered at $\{a_i\}$ and a collection of finite number of annuli $\{A_i\}$ (we call them the shielding rings) such that

1. $\overline{A}_i \cap P_f = \emptyset$;
2. $A_i \cap D_i = \emptyset$, but one components of $\partial A_i$ is the boundary of $D_i$;
3. $(D_i \cup A_i) \cap (D_j \cup A_j) = \emptyset$ for $i \neq j$;
4. $f$ is holomorphic on $D_i \cup A_i$; and
5. every $f(D_i \cup A_i)$ is contained in $D_{i+1}$ for $1 \leq i \leq k - 1$ and $f(D_k \cup A_k)$ is contained in $D_1$ where $k$ is the period of $a_i$.

Denote $D = \cup_i D_i$ and

$$P_1 = P_f \setminus D.$$

Since $a_i$ are accumulation point of $P_f$, it follows that $\#P_1$ is finite. Without loss of generality, we assume that $0$, $1$, and $\infty$ belong to $P_1$.

Define

$$Q = P_1 \cup \overline{D} \text{ and } X = \partial Q = P_1 \cup \partial D.$$

We associate with $f$ the Teichmüller space $T_f = T(\hat{\mathbb{C}} \setminus Q, X)$ which is the Teichmüller space of Riemann surface $\hat{\mathbb{C}} \setminus Q$ whose boundary is $X$. Note that $T_f$ is also the Teichmüller space $T_0(Q)$ which is the space of all $Q$-equivalent classes of all Beltrami coefficients $\mu$ on $\hat{\mathbb{C}}$ such that $\mu|Q = 0$. (Two Beltrami coefficients $\mu$ and $\nu$ are $Q$-equivalent if the normalized quasiconformal maps $w^\mu$ and $w^\nu$ are isotopic rel $Q$.) The space $T_f$ is a complex manifold. The Teichmüller metric and the Kobayashi metric on $T_f$ are also equal (refer to, for example, [EM, GJW, JMW]).

The map $f$ induces a holomorphic map $\sigma_f$ from $T_f$ into itself and $\sigma_f$ weakly contracts the Teichmüller metric. An equivalent statement of Theorem F is that $\sigma_f$ has a unique fixed point if and only if $f$ has no Thurston obstruction.

Every point $x$ in $T_f$ determines a complex structure on $\hat{\mathbb{C}} \setminus Q$ up to homotopy. Then $(\hat{\mathbb{C}} \setminus Q, x)$ is a Riemann surface $R_x$. We embed $R_x$ into the Riemann sphere $\hat{\mathbb{C}}$ by a quasiconformal map $\phi_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ fixing
0, 1, ∞. Then \( \hat{C} \setminus \phi_x(Q) \) is a representative of \( R_x \). The reader can refer to §4.

Let \( d(\cdot, \cdot) \) mean the spherical distance on \( \hat{C} \). We define a subspace \( \mathcal{T}_{f,b} \) of \( T_f \) for each \( b > 0 \) as follows:

**Definition 4.** Let \( b > 0 \) be a constant. Let \( \mathcal{T}_{f,b} \) be the subspace of \( x = [\mu] \in T_f \) satisfying the following conditions:

1. For all \( z_i \neq z_i' \in P_1 \), \( d(\phi_x(z_i), \phi_x(z_i')) \geq b \);
2. For all \( z_j \in P_1 \) and all \( D_i \in \Lambda \), \( d(\phi_x(z_j), \phi_x(D_i)) \geq b \);
3. For all \( D_i \neq D_i' \in \Lambda \), \( d(\phi_x(D_i), \phi_x(D_i')) \geq b \);
4. Every \( D_i \in \Lambda \), \( \phi_x(D_i) \) contains a round disk of radius \( b \) centered at \( \phi_x(c_i) \).

We call \( \mathcal{T}_{f,b} \) the subspace having the bounded geometry property determined by \( b \).

Take an arbitrary \( x_0 \in T_f \) and let \( x_n = \sigma_f^nx_0 \). If \( f \) has no Thurston obstructions, we can prove that \( \{x_n\}_n^{\infty}_{n=0} \subseteq \mathcal{T}_{f,b} \) for some \( b > 0 \). This implies that the sequence \( \{x_n\}_n^{\infty}_{n=0} \) converges in \( T_f \). Thus \( \sigma_f \) has a unique fixed point, and \( f \) is CLH-equivalent to a unique sub-hyperbolic rational map.

For a non-peripheral curve \( \gamma \) in \( \hat{C} \setminus Q \), let \( l(\gamma, x) \) denote the hyperbolic length of the unique geodesic in \( R_x \) which is homotopic to \( \gamma \) in \( \hat{C} \setminus Q \). If \( \{x_n\}_n \subseteq \mathcal{T}_{f,b} \) for some \( b > 0 \), then there is a \( \delta > 0 \) such that \( l(\gamma, x_n) \geq \delta \) for any non-peripheral curve \( \gamma \) in \( \hat{C} \setminus Q \) and any \( n \geq 0 \). Therefore, if \( f \) is not CLH-equivalent to a sub-hyperbolic rational map, then there is a sequence of non-peripheral curves \( \gamma_n \) in \( \hat{C} \setminus Q \) such that \( l(\gamma_n, x_n) \to 0 \) as \( n \to \infty \).

**Question:** Suppose \( f \) is not CLH-equivalent to a rational map. Does there exist a non-peripheral curve \( \gamma \), such that for any \( x_0 \in T_f \) and \( x_n = \sigma_f^n(x_0) \), \( n > 0 \), \( l(\gamma, x_n) \to 0 \) as \( n \to \infty \)?

We gives an affirmative answer to this question. The positive answer to this question gives a way how \( x_n \) tends to the boundary of \( T_f \). More precisely, we will prove a stronger result as follows.

**Theorem 1 (Main Theorem).** Suppose \( f \) is a sub-hyperbolic semi-rational branched covering. Let \( \Gamma_c \) denote the set of all homotopy classes of non-peripheral curves \( \gamma \) in \( \hat{C} \setminus Q \) such that \( l(\gamma, x_n) \to 0 \) as \( n \to \infty \) for any initial \( x_0 \in T_f = T_0(Q) \). Then we have that either
(a) $\Gamma_c = \emptyset$, then $f$ is CLH-equivalent to a sub-hyperbolic rational map, or 
(b) $\Gamma_c \neq \emptyset$ is a Thurston obstruction for $f$ and $f$ is not CLH-equivalent to a rational map. In this case, we call $\Gamma_c$ the canonical Thurston obstruction for $f$.

The paper is organized as follows. In §2, we define Thurston obstructions for sub-hyperbolic semi-rational branched coverings. In §3, we review non-negative matrices and study some properties for an irreducible non-negative matrices. In §4, we study the Teichmüller space associated with a sub-hyperbolic semi-rational branched covering and short geodesics. For any Thurston obstruction $\Gamma$, we can decompose it into $\Gamma_0$ and $\Gamma_\infty$ (see Definition 7). We estimate the upper bound for $\Gamma_\infty$ in §5 and the lower bound for $\Gamma_0$ in §6. Finally, we prove Theorem 1 in §7.

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2. Thurston obstructions

Suppose $f$ is a sub-hyperbolic semi-rational branched covering. Let $Q$ be the set as we defined in (3). Then

$$f : \hat{\mathbb{C}} \setminus f^{-1}(Q) \rightarrow \hat{\mathbb{C}} \setminus Q$$

is a covering map of finite degree. If $\gamma$ is a simple closed curve in $\hat{\mathbb{C}} \setminus Q$, then all the components of $f^{-1}(\gamma)$ are simple closed curves in $\hat{\mathbb{C}} \setminus f^{-1}(Q)$, which is a subset of $\hat{\mathbb{C}} \setminus Q$. Thus all the components of $f^{-1}(\gamma)$ are simple closed curves in $\hat{\mathbb{C}} \setminus Q$.

A simple closed curve $\gamma$ is said to be non-peripheral if each component of $\hat{\mathbb{C}} \setminus \gamma$ contains at least two points of $Q$. A multi-curve

$$\Gamma = \{\gamma_1, \cdots, \gamma_n\}$$

(4)
is a set of finitely many pairwise disjoint, non-homotopic, and non-peripheral curves in \( \hat{\mathbb{C}} \setminus Q \). For each multi-curve \( \Gamma \) in (3), let 
\[ \mathbb{R}^\Gamma = \langle \gamma_1, \cdots, \gamma_n \rangle \]
be the real vector space of dimension \( n \) with a basis \( \Gamma \). We define a linear transformation 
\[ f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma \]
as follows: For each \( \gamma_j \in \Gamma \), let \( \gamma_{i,j,\alpha} \) denote the components of \( f^{-1}(\gamma_j) \) homotopic to \( \gamma_i \) in \( \hat{\mathbb{C}} \setminus Q \) and \( d_{i,j,\alpha} \) be the degree of \( f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j \). Define 
\[ f_\Gamma(\gamma_j) = \sum_i \left( \sum_\alpha \frac{1}{d_{i,j,\alpha}} \right) \gamma_i. \]
Let \( A_\Gamma \) be the corresponding matrix, that is 
\[ f_\Gamma v = A_\Gamma v, \quad v \in \mathbb{R}^\Gamma. \]

Since the matrix \( A_\Gamma \) is non-negative, by the Perron-Frobenius Theorem, there exists a maximal non-negative eigenvalue \( \lambda(A_\Gamma) \) which is the spectral radius of \( A_\Gamma \).

A multi-curve \( \Gamma \) is said to be \( f \)-stable if for any \( \gamma \in \Gamma \), every non-peripheral component of \( f^{-1}(\gamma) \) is homotopic to an element of \( \Gamma \) rel \( Q \).

**Definition 5.** A stable multi-curve \( \Gamma \) is called a Thurston obstruction for \( f \) if \( \lambda(A_\Gamma) \geq 1 \).

**Remark 1.** The definition of a Thurston obstruction for the critically finite case is similar by replacing \( Q \) by \( P_f \).

### 3. Non-negative matrices

Since a Thurston obstruction is determined by a non-negative matrix, we give a brief review of some results in the matrix theory about non-negative matrices. We use [Ga] as a reference.

A non-negative \( n \times n \) matrix \( A \) is called irreducible, if no permutation of the indices places the matrix in a block lower-triangular form. More precisely, there is no permutation matrix \( P \), which is a matrix consisting of 0 and 1 such that each row or each column contains one and only one 1, such that 
\[ PAP^{-1} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \]
where $A_{11}$ and $A_{22}$ are square matrices. An equivalent definition of irreducibility is that for any $1 \leq i, j \leq n$, there exists a $0 \leq q = q(i,j) \leq n$ such that the $ij$-th entry of $A^q$ is positive.

For the $n$-dimensional vector space $V$, we will use the norm

\[
\|v\| = \max_{1 \leq i \leq n} |v_i|, \quad v = (v_1, \cdots, v_n) \in V
\]

in the rest of this paper. For any linear map $L : V \to V$, let $A$ be the corresponding matrix for $L$, define

\[
\|A\| = \sup_{\|v\|=1} \|Av\|.
\]

The spectral radius $\lambda(A)$ of $A$ can be calculated as

\[
\lambda(A) = \lim_{n \to \infty} \sqrt[n]{\|A^n\|} \geq 0.
\]

If $A$ is a non-negative, the Perron-Frobenius Theorem implies that $\lambda(A)$ is an eigenvalue of $A$. Thus it is a maximal eigenvalue of $A$. If $A$ is irreducible, $\lambda(A)$ is a simple, positive, maximal eigenvalue with a positive eigenvector $v = (v_1, \cdots, v_n)$, i.e., $v_i > 0$ for all $1 \leq i \leq n$. However, there may exist another eigenvalue $\mu \neq \lambda(A)$ but $|\mu| = \lambda(A)$. For example, consider

\[
A = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

It is an irreducible matrix. The spectral radius is 1 which is a simple, positive, maximal eigenvalue with an eigenvector $v_1 = (1, 1)$. However, $-1$ is also an eigenvalue with an eigenvector $v = (1, -1)$. But if $A$ is positive, that is, every entry is a positive number, the Perron-Frobenius theorem states that $\lambda(A)$ is a unique, simple, positive, maximal eigenvalue with a positive eigenvector $v = (v_1, \cdots, v_n)$, i.e., $v_i > 0$ for all $1 \leq i \leq n$. Here the term “unique” means that all other eigenvalues $\mu$ of $A$ satisfy that $|\mu| < \lambda(A)$.

**Definition 6.** We say that a multi-curve $\Gamma$ is irreducible if the corresponding matrix $A_{\Gamma}$ of the linear map $f_{\Gamma} : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is irreducible.

For any non-negative matrix $A$, we can rearrange the order of the basis such that

\[
A = \begin{pmatrix}
A_{11} & 0 & \cdots & 0 \\
A_{21} & A_{22} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
A_{s1} & A_{s2} & \cdots & A_{ss}
\end{pmatrix}
\]
and all the blocks $A_{jj}$ on the diagonal are either irreducible or $0$ matrices. It is not hard to calculate that
\[ \lambda(A) = \max_j \lambda(A_{jj}). \]

Now we consider $A = A_\Gamma$ as the corresponding matrix of the linear map $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ for an $f$-stable multi-curve $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$. We assume that $A$ is in the form of (6). Then we can use $\Gamma_j$ to denote the subset of curves in $\Gamma$ corresponding to the $j$-th block in $A$. That is, $A_{jj} = A_{\Gamma_j}$. It is clear that
\[ \Gamma = \bigcup_j \Gamma_j. \]
We call $\{\Gamma_j\}$ irreducible decomposition of $\Gamma$. The reader should note that $\Gamma_j$ may not be $f$-stable.

Denote
\[ \Gamma_{Ob} = \bigcup_j \Gamma_j \]
where the union runs over all $j$ such that $\lambda(A_{jj}) \geq 1$. We have the following definition to relate every element in $\Gamma$ to $\Gamma_{Ob}$ if it is not empty.

**Definition 7.** Suppose $\Gamma$ is an $f$-stable multi-curve. For every $\gamma \in \Gamma$, if there exists a $\gamma_{ob} \in \Gamma_{Ob}$ and an integer $k \geq 0$ such that $\gamma$ is homotopic to a component $f^{-k}(\gamma_{ob})$, then we define the depth of $\gamma$ with respect to $\Gamma$ to be the least such integer $k$. Otherwise, we define the depth as $\infty$. The set of all elements in $\Gamma$ with finite depth is denoted by $\Gamma_0$. The set of all elements with infinite depth is denoted by $\Gamma_\infty$.

Then
\[ \Gamma = \Gamma_0 \cup \Gamma_\infty. \]
It is clear that if $\Gamma$ is a Thurston obstruction, then $\Gamma_0$ is non-empty. Moreover, we have

**Lemma 1.** If $\Gamma$ is a Thurston obstruction, then $\Gamma_0$ is also a Thurston obstruction. In particular, under a permutation of the basis, we can write
\[ A_\Gamma = \begin{pmatrix} A_{\Gamma_\infty} & 0 \\ \star & A_{\Gamma_0} \end{pmatrix} \]
where $\lambda(A_{\Gamma_\infty}) < 1$ and $\lambda(A_{\Gamma}) = \lambda(A_{\Gamma_0}) \geq 1$.

**Proof.** First, for every curve $\gamma \in \Gamma_0$, there exists an integer $k \geq 0$ and an element $\gamma_{ob} \in \Gamma_{Ob}$ such that $\gamma$ is homotopic to a component of $f^{-k}(\gamma_{ob})$. It follows that any non-peripheral component $\tilde{\gamma}$ of $f^{-1}(\gamma)$ is
homotopic to a component of $f^{-(k+1)}(\gamma_{ob})$. Since $\Gamma$ is $f$-stable, then there exists an element $\gamma_i \in \Gamma$ which is homotopic to $\tilde{\gamma}$. Therefore, any non-peripheral component of $f^{-1}(\gamma)$ is homotopic to an element $\gamma_i \in \Gamma$ whose depth is at most $k + 1$. This implies that $\gamma \in \Gamma_0$. Thus $\Gamma_0$ is $f$-stable.

Let us write $\Gamma_\infty = \{\gamma_1, \cdots, \gamma_s\}$. Then $\Gamma_0 = \{\gamma_{s+1}, \cdots, \gamma_n\}$. Since $\Gamma_0$ is $f$-stable, $A_{\Gamma}$ must be of the form of $[\mathbf{D}]$. Furthermore, since $\Gamma_{Ob} \subset \Gamma_0$, we have that

$$\lambda(A_{\Gamma_\infty}) < 1 \quad \text{and} \quad \lambda(A_{\Gamma_0}) = \lambda(A_{\Gamma}) \geq 1.$$ 

□

Now we study the associated matrix $A$ for a sub-hyperbolic semi-rational branched covering $f$. For each disk $D_i$ in $\Lambda$, we take a point $b_i$ on the boundary $\partial D_i$. Set

$$(8) \quad E = P_1 \cup \cup_i \{a_i, b_i\}.$$ 

Let $p = \#E$. It is obvious that every multi-curve $\Gamma$ in $\hat{\mathbb{C}} \setminus Q$ is a multi-curve in $\hat{\mathbb{C}} \setminus E$. It follows that there are only finite number of possible matrices for all linear transformations $f_{\Gamma}$ (refer to [DH, Lemma 1.2]). (There are infinitely many possible multi-curves $\Gamma$.) Therefore, we have that

**Proposition 1.** There is a number $0 < \beta \leq 1$ depending only on the degree $d$ of $f$ and the cardinality $p$ of $E$ such that for any irreducible multi-curve $\Gamma$ in $\hat{\mathbb{C}} \setminus Q$ (not necessarily $f$-stable) with $\lambda(A_{\Gamma}) \geq 1$, let $v$ be the unique positive eigenvector of $A_{\Gamma}$ corresponding to $\lambda(A_{\Gamma}) \geq 1$ with $\|v\| = 1$, then the smallest coordinate of $v$ is bounded below by $\beta$.

**Proof.** Since there are only finitely many possible matrices for all irreducible multi-curves, there are finitely many simple, positive, maximal eigenvalues. Thus there are finitely many positive eigenvectors $v$ with $\|v\| = 1$. This gives the proposition. □

**Proposition 2.** There exists a positive integer $m$ such that for any non-empty $f$-stable multi-curve $\Gamma$, if it is a Thurston obstruction,

$$\|A_{\Gamma_\infty}^m\| < 1/2.$$ 

**Proof.** Since there are only finitely many matrices $A_{\Gamma}$ corresponding to all $\Gamma$, there are only finitely many $A_{\Gamma_\infty}$. For each $A_{\Gamma_\infty}$, $\lambda(A_{\Gamma_\infty}) < 1$.
Thus we have an integer $m > 0$ such that
\[ \| A^m_{\Gamma_{\infty}} \| < 1/2. \]

□

Every multi-curve $\Gamma$ can contain at most $p - 3$ curves, so we have that

**Proposition 3.** There is a positive integer $M$ depending on $p$ such that for any $f$-stable multi-curve $\Gamma$ in $\hat{C} \setminus Q$, the depth of every $\gamma \in \Gamma_0$ is less than or equal to $M$.

4. Teichmüller space and short geodesics.

Suppose $f$ is a sub-hyperbolic semi-rational branched covering. Recall $Q$ and $P^1$ defined in (2) and (3) and the assumption that $0, 1, \infty \in P^1$. Let $\mathcal{M}(\mathbb{C})$ be the unit ball of the space $L^\infty(\mathbb{C})$. That is, it is the set of all measurable functions $\mu$ on $\mathbb{C}$ such that essential supremum norm $\| \mu \|_{\infty} < 1$. Each element $\mu \in \mathcal{M}(\mathbb{C})$ is called a Beltrami coefficient since the measurable Riemann mapping theorem [AB] says that the Beltrami equation

\[ \phi \circ \mu \phi = \mu \phi \]

has a unique quasiconformal self-map $\phi^\mu$ of $\hat{\mathbb{C}}$ fixing $0, 1, \text{and } \infty$ as a solution, which depends on $\mu \in \mathcal{M}(\mathbb{C})$ holomorphically. The map $\phi^\mu$ is called the normalized solution.

**Definition 8.** The Teichmüller space $T_f$ is the equivalence class $[\mu]$ for $\mu \in \mathcal{M}(\mathbb{C})$ satisfying that $\mu|Q = 0$ a.e., where $\mu_1$ and $\mu_2$ are equivalent if and only if $\phi^{\mu_1}$ is isotopic to $\phi^{\mu_2}$ rel $Q$. Furthermore, we can define the Teichmüller distance between two points $x = [\mu]$ and $y = [\nu]$ in $T_f$ as

\[ d_T(x, y) = \frac{1}{2} \min_{\tilde{\mu} \in [\mu], \tilde{\nu} \in [\nu]} \log K[\phi^{\tilde{\mu}} \circ (\phi^{\tilde{\nu}})^{-1}] \]

where $K[\phi]$ is the maximal dilation of the quasiconformal map $\phi$.

From [Li] (or refer to [JZ]), we knew that $T_f$ is the Teichmüller space $T(\hat{\mathbb{C}} \setminus Q)$ of Riemann surface $\hat{\mathbb{C}} \setminus Q$ with boundary $\partial Q$. It is a complex manifold and the projective map

\[ \Phi : \mathcal{M}(\mathbb{C}) \to T_f \]

is a holomorphic split submersion.
Define the self-map $\sigma_f$ of the Teichmüller $T_f$ by

$$\sigma_f([\mu]) = [f^*(\mu)].$$

In formula,

$$(f^*\mu)(z) = \frac{\mu_f(z) + \mu(f(z))\theta(z)}{1 + \mu_f(z)\mu(f(z))\theta(z)},$$

where $\theta(z) = \overline{f_z}/f_z$ and $\mu_f(z) = f_{\overline{z}}/f_z$, is the pull-back of $\mu$ by $f$. Since

$$\sigma_f = \Phi \circ f^* \circ \Phi^{-1},$$

where $\Phi^{-1}$ means a local holomorphic section of $\Phi$. Thus

$$\sigma_f : T_f \to T_f$$

is a holomorphic map. Since the Teichmüller metric $d_T$ coincides with the Kobayashi metric on the complex manifold $T_f$ and $\sigma_f$ is holomorphic, we have that

$$d_T(\sigma_f(x), \sigma_f(y)) \leq d_T(x, y), \quad \forall x, y \in T_f.$$

From [JZ], we also know that

(9) $$d_T(\sigma_f(x), \sigma_f(y)) < d_T(x, y), \quad \forall x, y \in T_f.$$

We need more definitions and lemmas from [JZ] as follows.

Let $Z$ be a subset of $Q$ with $\sharp(Z) \geq 4$. Let $x = [\mu] \in T_f$ and let $\gamma \in \hat{C} \setminus Z$ be a simple closed and non-peripheral curve. We use $l_Z(\gamma, x)$ to denote the hyperbolic length of the unique simple closed geodesic which is homotopic to $\phi^\mu(\gamma)$ in the hyperbolic Riemann surface $\hat{C} \setminus \phi^\mu(Z)$. We say $\gamma$ is a $(\mu, Z)$-simple closed geodesic if $\phi^\mu(\gamma)$ is a simple closed geodesic in $\hat{C} \setminus \phi^\mu(Z)$.

**Remark 2.** From the definition of the Teichmüller space $T_f$, we know that the definition of $l_Z(\gamma, x)$ is independent of the choice of $\mu$ in $x$.

For $x_0 \in T_f$, let $x_n = \sigma_f^n(x_0), n = 1, \cdots$ be a sequence in $T_f$. Recall our definition of $E$ in [S].

**Lemma 2.** If there is a real number $a > 0$ such that there is a point $x_0 \in T_f$ and every $(x_n, E)$-simple closed geodesic $\gamma \subset \hat{C} \setminus Q$ has hyperbolic length greater than or equal to $a$, then the sequence $\{x_n\}_{n=0}^\infty$ is convergent in $T_f$ and the limiting point is the fixed point of $\sigma_f$ in $T_f$.

**Remark 3.** This lemma implies that if there exists an $x_0 \in T_f$ such that the length of the shortest geodesics on all the $x_n$ has a uniform lower bound, then $f$ has no Thurston obstructions.
Lemma 3. There exists an \( \eta > 0 \) such that for any point \( x = [\mu] \in T_f \) with \( \mu(z) = 0 \) on \( \cup_i A_i \) and for any \((x, E)\)-simple geodesic \( \gamma \subset \hat{C} \setminus E \) with \( l_E(\gamma, x) < \eta \), we have \( \gamma \subset \hat{C} \setminus Q \). Moreover, for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
l_E(\gamma, x) > (1 - \epsilon)l_Q(\gamma, x)
\]

whenever \( l_E(\gamma, x) < \delta \).

Remark 4. The above lemma implies that for any \( x = [\mu] \in T_f \) with \( \mu(z) = 0 \) for all \( z \in \cup_i A_i \), a sufficiently short geodesics in \( \hat{C} \setminus \phi^\mu(E) \) are homotopic to the sufficiently short geodesics in \( \hat{C} \setminus \phi^\mu(Q) \). More precisely, we can find a constant \( \delta_0 > 0 \) such that

\[
\frac{1}{e} l_Q(\gamma, x) < l_E(\gamma, x) < l_Q(\gamma, x) \quad \text{whenever} \quad l_E(\gamma, x) < \delta_0.
\]

Suppose \( x = [\mu] \in T_f \) and \( Z \subset Q \). Define

\[
w_Z(\gamma, x) = -\log l_Z(\gamma, x).
\]

Consider the set

\[
L_{Z,x} = \{w_Z(\gamma, x)\}
\]

where \( \gamma \) ranges over all the non-peripheral simple closed curves in \( \hat{C} \setminus Q \). Define

\[
w_Z(x) = \sup \{w_Z(\gamma, x)\}
\]

and

\[
w_Z(\Gamma, x) = \max_{\gamma \in \Gamma} w_Z(\gamma, x).
\]

The following lemma is a general result for hyperbolic Riemann surfaces (refer to \[DH, JZ\]). We just state it in our case.

Lemma 4. Let \( Z \subset Q \) be a finite subset with \( \#Z \geq 4 \) and \( \gamma \subset \hat{C} \setminus Q \) be a non-peripheral simple closed curve. Then the function

\[
x \mapsto w_Z(\gamma, x) : T_f \to \mathbb{R}
\]
is Lipschitz with Lipschitz constant 2.

Let

\[
A = \max \{- \log \log(2\sqrt{2} + 3), - \log \delta_0\}
\]

where \( \delta_0 \) is the number in Remark 4. Note that \( \log(2\sqrt{2} + 3) \) is the magic number in the theory of hyperbolic Riemann surfaces such that
for any hyperbolic Riemann surface $S$, any two simple closed geodesics $\gamma$ and $\gamma'$ in $S$ are disjoint whenever the hyperbolic lengths of $\gamma$ and $\gamma'$ are less than $\log(2\sqrt{2} + 3)$. This implies that for any point $x \in T_f$, there are at most $p - 3$ curves $\gamma$ with $l_{E}(\gamma, x) \leq \log(2\sqrt{2} + 3)$.

For any $J > 0$, let $(a, b)$ be the lowest interval in $\mathbb{R} \setminus L_{E, x}$ such that $a \geq A$ and $b - a = J$. For any $x = [\nu] \in T_f$, define

$$
\Gamma_{J,x} = \{\gamma | \gamma \text{ is a simple closed geodesic on } R_x \text{ and } w_{E}(\gamma, x) \geq b\}.
$$

Then $\Gamma_{J,x}$ is a multi-curve consisting of the geodesics which are sufficiently short on $\hat{C} \setminus \phi^\mu(E)$. This is equivalent saying that they are all the simple closed curves in $\hat{C} \setminus \phi^\mu(Q)$ which are homotopic to sufficiently short simply closed geodesics on $\hat{C} \setminus \phi^\mu(Q)$. There are at most $p - 3$ elements in $\Gamma_{J,x}$ for any $x$ and they are pairwise disjoint.

For any $x \in T_f$, let $D = d_T(x, \sigma_f(x))$.

**Lemma 5.** If $J \geq \log d + 2D + 1$ and $\Gamma_{J,x} \neq \emptyset$, then $\Gamma_{J,x}$ is an $f$-stable multi-curve.

See Lemma 7.3 in [JZ].

5. **Upper bound for $\Gamma_{\infty}$**

We still keep the notations in the previous sections. Suppose $x_0 \in T_f$ and $x_n = \sigma_f^n(x_0)$ for all $n \geq 1$. Then we have a sequence $\{x_n\}_{n=0}^\infty$ in $T_f$.

For all $n > 0$ and all $z \in \cup_i A_i$, we have that $\mu_n(z) = 0$, where $[\mu_n] = x_n$, since $f(\cup_i A_i) \subset \cup_i D_i$ as we constructed $\{A_i\}$ as the shielding rings.

Recall the definition of $E = P_1 \cup \cup_i \{a_i, b_i\}$ in (8) and $m$ in Proposition 2. Let

$$
P_2 = E \cup f^m(E) \cup \cup_{1 \leq j \leq m} f^j(\Omega_f) \subset Q.
$$

The following lemma is also from [JZ].

**Lemma 6.** There exists an $\epsilon_0 > 0$, such that for any $x = [\mu] \in T_f$ with $\mu(z) = 0$ for all $z \in \cup_i A_i$, and for any $(\mu, P_2)$-simple closed geodesic $\gamma'$, if $l_{P_2}(\gamma', x) < \epsilon_0$, then there is a $(\mu, E)$-simple closed geodesic $\gamma$ such that $\gamma'$ is homotopic to $\gamma$ in $\hat{C} \setminus P_2$.

The following lemma is also a general result in the theory of hyperbolic Riemann surfaces and the reader can find a proof in [DH].
Lemma 7. Let $X$ be a hyperbolic Riemann surface, $P \subset X$ is a finite subset, and $\sharp P < p$. Let $X' = X \setminus P$ and $L < \log(3 + 2\sqrt{2})$. Let $\gamma$ be a simple closed geodesic on $X$, and let $\gamma_1', \ldots, \gamma_k'$ be all the geodesics on $X'$ homotopic to $\gamma$ in $X$ whose hyperbolic length on $X'$ is less than $L$. Set $l = l_X(\gamma)$ and $l_i' = l_{X'}(\gamma_i')$. Then:

1. $k \leq p + 1$;
2. for all $i$, $l_i' \geq l$;
3. $\frac{1}{T} - \frac{1}{T} - \frac{(p+1)}{T} < \sum_{i=1}^k \frac{1}{l_i'} < \frac{1}{T} + \frac{(p+1)}{\pi}$.

The next proposition is essential for our proof.

Proposition 4. Let $m$ be the constant in Proposition 2. Let $x_0 \in \Gamma_f$ and $x_n = \sigma_f^n(x_0)$ for $n > 0$. There exists a constant $C(J) > 0$ depending on $p, d, \epsilon_0, D = d_T(x_0, x_1)$ and $J \geq m(\log d + 2D + 1)$ such that if $w_E(x_0) > C(J)$, then $\Gamma = \Gamma_{J,x_0} \neq \emptyset$ is a stable multi-curve. Moreover, if $\Gamma_{\infty} \neq \emptyset$, then

$$w_E(\Gamma_{\infty}, x_m) \leq w_E(\Gamma_{\infty}, x_0).$$

Proof. If $w_E(x_0) \geq A + (p-3)J$, then $\Gamma_{J,x_0}$ is non-empty, since $R_{x_0}$ has at most $(p-3)$ simple closed geodesics with hyperbolic length less than $e^{-A}$ (they are not homotopic to each other). From Lemma 5, $\Gamma = \Gamma_{J,x_0}$ is also $f$-stable.

Suppose $\Gamma_{\infty} \neq \emptyset$ and $A_{\Gamma}$ is in the form of (7). From Proposition 2 $||A_{\Gamma, \infty}|| < 1/2$.

For each $\gamma_j \in \Gamma_{J,x_0}$, let $\gamma_{i,j,\alpha}$ be any component of $f^{-m}(\gamma_j)$ homotopic to $\gamma_i$ in $\hat{C} \setminus Q$. Then $\gamma_{i,j,\alpha}$ is also homotopic to $\gamma_i$ in $\hat{C} \setminus E$. Let $g = \phi^\mu \circ f^m \circ (\phi^\nu)^{-1}$, where $[\mu] = x_0$ and $[\nu] = x_m$. Then $g$ is a rational map and

$$g : \hat{C} \setminus \phi^\nu(f^{-m}(P_2)) \rightarrow \hat{C} \setminus \phi^\mu(P_2)$$

is a holomorphic covering map. Therefore

$$l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m) = d_{i,j,\alpha}l_{P_2}(\gamma_j, x_0),$$

where $d_{i,j,\alpha}$ is the degree of $f^m : \gamma_{i,j,\alpha} \rightarrow \gamma_j$. We get

$$\sum_{\alpha} \frac{1}{l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m)} = \left(\sum_{\alpha} \frac{1}{d_{i,j,\alpha}}\right) \frac{1}{l_{P_2}(\gamma_j, x_0)} = b_{ij} \frac{1}{l_{P_2}(\gamma_j, x_0)},$$

where $b_{ij}$ is the $ij$-entry of $A_{\Gamma}^m$.

Since $E \subset P_2$, the inclusion

$$i : \hat{C} \setminus P_2 \leftrightarrow \hat{C} \setminus E$$
decreases the hyperbolic distances. So we have that \( l_{P_2}(\gamma_j, x_0) > l_E(\gamma_j, x_0) \) for any \( \gamma_j \). It follows that

\[
\sum_\alpha \frac{1}{l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m)} < b_{ij} \frac{1}{l_E(\gamma_j, x_0)}.
\]

From the definitions of \( P_2 \) and \( E \), we know that \( E \subset f^{-m}(P_2) \). Let \( C = C(d, m, p) = \sharp(f^{-m}(P_2) \setminus E) \), where \( p = \sharp E \).

We claim that for any \((\nu, f^{-m}(P_2))\)-simple closed geodesic \( \gamma \) which is homotopic to \( \gamma_i \) in \( \hat{C} \setminus E \), either \( \gamma \) is homotopic to some \( \gamma_{i,j,\alpha} \) in \( \hat{C} \setminus f^{-m}(P_2) \) or

\[
l_{f^{-m}(P_2)}(\gamma, x_m) > \min\{e^{-A+PJ}, \epsilon_0\},
\]

where \( \epsilon_0 \) is the constant in Lemma 6.

We prove the claim. In fact, if \( \gamma \) is not homotopic in \( \hat{C} \setminus f^{-m}(P_2) \) to some \( \gamma_{i,j,\alpha} \), then \( f^{-m}(\gamma) \) is a \((\mu, P_2)\)-simple closed geodesic which is not homotopic to any \( \gamma_j \) in \( \hat{C} \setminus P_2 \). Then there are two cases: either (1) \( f^{-m}(\gamma) \) is homotopic in \( \hat{C} \setminus P_2 \) to some \( (\mu, E) \)-simple closed geodesic \( \xi \) which does not belong to \( \Gamma_{J,x_0} \), then we have

\[
l_{P_2}(f^{-m}(\gamma), x_0) > l_E(f^{-m}(\gamma), x_0) = l_E(\xi, x_0) > e^{-a} > e^{-(A+PJ)}
\]

or (2) \( f^{-m}(\gamma) \) is not homotopic in \( \hat{C} \setminus P_2 \) to any \( (\mu, E) \)-simple closed geodesic, then by Lemma 6 we have

\[
l_{P_2}(f^{-m}(\gamma), x_0) > \epsilon_0.
\]

Thus we have

\[
l_{f^{-m}(P_2)}(\gamma, x_m) \geq l_{P_2}(f^{-m}(\gamma), x_0) > \min\{e^{-(A+PJ)}, \epsilon_0\}.
\]

This proves the claim.

From the left hand of the inequality given by (3) in Lemma 7, for each \( \gamma_i \in \Gamma \), we have

\[
\frac{1}{l_E(\gamma_i, x_m)} - \frac{1}{\pi} \min\{e^{-(A+PJ)}, \epsilon_0\} \leq \sum_{j,\alpha} \frac{1}{l_{f^{-m}(P_2)}(\gamma_{i,j,\alpha}, x_m)} \leq \sum_j b_{ij} \frac{1}{l_E(\gamma_j, x_0)}.
\]

Suppose \( \Gamma_\infty = \{\gamma_1, \cdots, \gamma_s\} \subset \Gamma \). Then for each \( \gamma_i \in \Gamma_\infty \), from the form 7 of \( A_{\Gamma} \),

\[
\frac{1}{l_E(\gamma_i, x_m)} \leq \sum_{j=1}^s b_{ij} \frac{1}{l_E(\gamma_j, x_0)} + \frac{1}{\pi} + \frac{C + 1}{\min\{e^{-(A+PJ)}, \epsilon_0\}}.
\]
Let
\[ v_1 = \begin{pmatrix} \frac{1}{l_E(\gamma_1, x_m)} \\ \vdots \\ \frac{1}{l_E(\gamma_s, x_m)} \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} \frac{1}{l_E(\gamma_1, x_0)} \\ \vdots \\ \frac{1}{l_E(\gamma_s, x_0)} \end{pmatrix}. \]

Since \( \|A_m\|_\infty < 1/2 \),
\[ \|v_1\| < \frac{1}{2} \|v\| + \frac{C + 1}{\min\{e^{-(A+PJ)}, \epsilon_0\}}. \]

Define
\[ C(J) = \max\left\{ 2\left( \frac{1}{\pi} + \frac{C + 1}{\min\{e^{-(A+PJ)}, \epsilon_0\}} \right), A + (p - 3)J \right\}. \]

If \( w_E(\Gamma_\infty, x_0) \geq C(J) \), then we have
\[ w_E(\Gamma_\infty, x_m) < w_E(\Gamma_\infty, x_0). \]

\begin{lemma}
Let \( J \geq m(\log d + 2D + 1) \). Suppose \( w_E(x_0) < C(J) \) and suppose \( \Gamma = \Gamma_{J,x_k} \neq \emptyset \) for some \( k \geq 0 \). Let \( E(J) = C(J) + 2mD \). If \( \Gamma_\infty \neq \emptyset \), then for all \( n \),
\[ w_E(\Gamma_\infty, x_n) < E(J). \]
Moreover, if \( w_E(\gamma, x_k) \geq E(J) \), then \( \gamma \in \Gamma_0 \).
\end{lemma}

\begin{proof}
We prove the first inequality by contradiction. Suppose there is an \( n > 0 \) such that \( w_E(\Gamma_\infty, x_n) \geq C(J) + 2mD \). Suppose \( n_0 \) is the first integer having this property. Then we have \( w_E(\Gamma_\infty, x_{n_0 - m}) \geq C(J) \). Then by Proposition 4 and the fact that \( n_0 \) is the first integer such that \( w_E(\Gamma_\infty, x_{n_0}) \geq C(J) + 2mD \), we have
\[ w_E(\Gamma_\infty, x_{n_0}) \leq w_E(\Gamma_\infty, x_{n_0 - m}) < C(J) + 2mD. \]
This is a contradiction.

If \( w_E(\gamma, x_k) \geq E(J) > C(J) \geq A + (p - 3)J \), then \( \gamma \in \Gamma_{J,x_k} = \Gamma \) since there are at most \( p - 3 \) simple closed curves in \( R_{x_k} \) such that \( w_E(\gamma, x_k) > A \). But \( \gamma \notin \Gamma_\infty \) because of the first conclusion and the assumption. Therefore, \( \gamma \in \Gamma_0 \).
\end{proof}
6. Lower bound for $\Gamma_0$

In order to get the lower bound for $\Gamma_0$, we need the following definition.

**Definition 9.** Let $\kappa$ be a real number. A sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers is called $\kappa$-quasi-nondecreasing if for all $n_1 < n_2$ we have $a_{n_2} - a_{n_1} \geq \kappa$. A sequence is called quasi-nondecreasing if it is $\kappa$-quasi-nondecreasing for some $\kappa$.

It is easy to check that the following two properties are true.

**Property 1.** Suppose $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences. If $\{a_n\}_{n=0}^{\infty}$ is $\kappa$-quasi-nondecreasing and if $|a_n - b_n| < r$ for all $n$, then $\{b_n\}$ is $(\kappa - 2r)$-quasi-nondecreasing.

**Property 2.** Suppose $\{a_n\}$ is quasi-nondecreasing and unbounded. Then $a_n \to +\infty$ as $n \to +\infty$.

Recall that any $x = [\mu] \in T_f$ represents a complex structure on $\hat{C} \setminus Q$, which makes $\hat{C} \setminus Q$ a hyperbolic Riemann surface $R_x$. For any simple closed geodesic $\gamma$ on $R_x$, let $A(\gamma, x)$ be the Riemann surface, conformally isomorphic to an annulus, obtained by taking the unit disk $\mathbb{D}$ modulo a $\mathbb{Z}$-subgroup of the fundamental group of $R_x$ generated by $\gamma$. It is a covering space of $R_x$. The core curve of $A(\gamma, x)$ is a geodesic of length $l_Q(\gamma, x)$ and

$$\text{(10)} \quad \text{mod}(A(\gamma, x)) = \frac{\pi}{l_Q(\gamma, x)},$$

where $\text{mod}(A)$ means the modulus of an annulus $A$.

If $\gamma$ is a simple closed geodesic of hyperbolic length $l$ on the Riemann surface $R_x$, then there is an embedding annulus $a(\gamma, x)$ of modulus $m(l)$ which is continuous and decreasing and satisfies

$$\frac{\pi}{l} - 1 < m(l) < \frac{\pi}{l}.$$

Thus for all $x \in T_f$, we have

$$\text{(11)} \quad \text{mod}(A(\gamma, x)) - 1 < \text{mod}(a(\gamma, x)) < \text{mod}(A(\gamma, x)).$$

We need the following technical lemma:

**Lemma 9.** If $t \geq 1$, then $\log(t + 1) - 1 < \log t$. 

Proof. For $t \geq 1$,
\[
\log(t + 1) - \log t = \log\left(\frac{t + 1}{t}\right) \leq \log 2 < 1.
\]
\[\square\]

If $w_Q(\gamma, x) \geq \log\frac{2}{\pi} = -0.451582705 \cdots$, then we have $\mod(A(\gamma, x)) \geq 1 \geq 1$. By taking logarithms on all terms of Inequality (11) and by applying Lemma 9 and Equation (10), we have
\[
\log \pi - 1 + w_Q(\gamma, x) < \log \mod(a(\gamma, x)) < \log \pi + w_Q(\gamma, x).
\]
It follows that, if $w_Q(\gamma, x) \geq \log\frac{2}{\pi}$, then
\[
\left| \log \mod(a(\gamma, x)) - w_Q(\gamma, x) \right| < \log \pi.
\]

(12) Given a multi-curve $\Gamma$, we denote vectors of moduli $(\mod(a(\gamma, x)))$ and $(\mod(a(\gamma, x)))$ respectively. Define
\[
\mod(A(\Gamma, x)) = \min_{\gamma \in \Gamma} \{\mod(A(\gamma, x))\}
\]
and
\[
\mod(a(\Gamma, x)) = \min_{\gamma \in \Gamma} \{\mod(a(\gamma, x))\}.
\]

Lemma 10. Let $\beta$ be the constant in Proposition 1. Let $\Gamma$ be an irreducible multi-curve. Suppose the leading eigenvalue of the matrix $A_\Gamma$ is greater than or equal to 1. Then for any $x_0 \in T_f$ and $x_n = \sigma^n_f(x_0)$, $n > 0$,
\begin{enumerate}
\item $\mod(A(\Gamma, x_n)) \geq \beta \mod(a(\Gamma, x_0))$ and
\item $\mod(a(\Gamma, x_n)) \geq \beta \mod(a(\Gamma, x_0)) - 1$.
\end{enumerate}

Proof. Since for any $n$, $f^n : \hat{C} \rightarrow \hat{C}$ is a branched covering, we can similarly define the linear map $f^n_\Gamma : \mathbb{R}^\Gamma \rightarrow \mathbb{R}^\Gamma$. Let $B$ be the corresponding matrix for the linear map $f^n_\Gamma$ with the basis $\Gamma$. It is easy to see that $B \geq A^n_\Gamma$.

Let $v$ be the unique positive eigenvector of $A_\Gamma$ with $\|v\| = 1$. Let $1$ denote the vector whose coordinates are all equal to 1. Then
\[
\mod(a(\Gamma, x_0)) \geq \mod(a(\Gamma, x_0))1 \geq \mod(a(\Gamma, x_0))v.
\]

For any $n \geq 1$, let $\gamma_{ij,\alpha}$ be the components of $f^{-n}(\gamma_j)$ homotopic to $\gamma_i$, and $a^n_{ij,\alpha}$ be the components of $f^{-n}(a(\gamma_j, x_0))$ homotopic to $\gamma_i$. Then
\[
\mod(a^n_{ij,\alpha}) = \mod(a(\gamma_j, x_0))/d^n_{ij,\alpha}.
\]
where \( d_{n,i,j,\alpha} \) is \( \deg f^{|\gamma^n_{i,j,\alpha}} \). Since \( a_{n,i,j,\alpha} \) are disjoint annuli homotopic to the curve \( \gamma_i \), we have
\[
\sum_{\alpha,j} \text{mod}(a_{n,i,j,\alpha}) \leq \text{mod}(A(\gamma_i, x_n)).
\]
(One can obtain this inequality by lifting them to the covering space \( A(\gamma_i, x_n) \) of \( R_{x_n} \) and then by using Grötzsch’s inequality.) Consequently we get
\[
\text{mod}(A(\Gamma, x_n)) \geq \text{mod}(a(\Gamma, x_n)) \geq B \text{mod}(a(\Gamma, x_0)) \\
\geq A^\beta \text{mod}(a(\Gamma, x_0)) \geq A^\beta \text{mod}(a(\Gamma, x_0)) \nu \\
\geq \text{mod}(a(\Gamma, x_0)) \nu \geq \beta \text{mod}(a(\Gamma, x_0)) \mathbf{1}
\]
Hence for all \( \gamma \in \Gamma \), we have \( \text{mod}(A(\gamma, x_n)) \geq \beta \text{mod}(a(\Gamma, x_0)) \). The second conclusion follows the first one and Inequality (11). \( \square \)

**Lemma 11.** If \( a, b > 0, \beta > 0, \text{and } e^a \geq \beta e^b - 1, \text{then } a - b \geq \log \beta - 1. \)

**Proof.** If \( \beta \exp b - 1 \geq 1 \), then by Lemma 9 we have
\[
\log(\beta \exp b - 1) \geq \log(\beta e^b) - 1 \geq \log \beta + b - 1.
\]
Hence by the assumption, we have \( a - b \geq \log \beta - 1 \).

If \( \beta \exp b - 1 < 1 \), then \( b < \log 2 - \log \beta \). Since \( a > 0 \),
\[
a - b > 0 - b = -b > \log \beta - \log 2 > \log \beta - 1.
\]
\( \square \)

For any \( x \in T_f \) and any multi-curve \( \Gamma \), define
\[
\overline{w}(\Gamma, x) = \min_{\gamma \in \Gamma} w_Q(\gamma, x).
\]

**Lemma 12.** Suppose \( \Gamma \) is an irreducible multi-curve and suppose the leading eigenvalue of the matrix \( A_\Gamma \) is greater than or equal to 1. For any \( x_0 \in T_f \), if \( \overline{w}(\Gamma, x_0) \geq \log(3/\beta) + \log \pi \), then the sequence \( \{\overline{w}(\Gamma, x_n)\}_{n=0}^\infty \), where \( x_n = \sigma^n_f(x_0) \), is \( (\log \beta - 1 - 2 \log \pi) \)-quasiconvex-decreasing.

**Proof.** For \( \overline{w}(\Gamma, x_0) \geq \log(3/\beta) + \log \pi > \log \frac{3}{\beta} \), by Inequality (12), we have
\[
\log \text{mod}(a(\Gamma, x_0)) \geq \log \frac{3}{\beta}.
\]
That is, \( \text{mod}(a(\Gamma, x_0)) \geq 3/\beta \). So \( \beta \text{mod}(a(\Gamma, x_0)) - 1 \geq 2 \). By Lemma 10 we have that for all \( n \geq 0 \),
\[
\text{mod}(a(\Gamma, x_n)) \geq 2. \quad (13)
\]
Now consider the sequence \( y_n = \log \text{mod}(a(\Gamma, x_n)) \). Choose arbitrarily \( n_2 > n_1 \geq 0 \), and let \( a = y_{n_2}, b = y_{n_1} \) and \( n = n_2 - n_1 \). By Lemma 10 we have \( e^a \geq \beta e^b - 1 \). Applying Lemma 11 we have \( a - b \geq \log \beta - 1 \), so the sequence \( \{y_n\} \) is a \((\log \beta - 1)\)-quasi-nondecreasing.

By Inequalities (11) and (13), we have \( \text{mod}(A(\Gamma, x)) \geq 2 \). This implies that \( \log \pi + w(\Gamma, x_n) \geq \log 2 \). That is, \( w(\Gamma, x_n) \geq \log(2/\pi) \).

Since \( \text{mod}(a(\gamma, x_n)) \) is continuous and decreasing with \( l_Q(\gamma, x_n) \), we obtain

\[
\text{mod}(a(\Gamma, x_n)) = \text{mod}(a(\gamma, x_n)) \quad \text{and} \quad w(\Gamma, x_n) = w_Q(\gamma, x_n)
\]
at the same \( \gamma \in \Gamma \). This further implies that \( |y_n - w(\Gamma, x_n)| < \log \pi \).

From Property 1 we finally have that \( w(\Gamma, x_n) \) is \((\log \beta - 1 - 2 \log \pi)\)-quasi-nondecreasing. \( \square \)

**Lemma 13.** Let \( k \geq 1 \) be an integer. For any \( x_0 \in T_f \), let \( x_n = \sigma_f^n(x_0) \) for \( n > 0 \). Let \( D = d_T(x_0, x_1) \). If \( \gamma_1, \gamma_2 \) are non-peripheral curves in \( \hat{C} \setminus Q \) such that some component of \( f^{-k}(\gamma_1) \) is homotopic to \( \gamma_2 \), then

\[
w_Q(\gamma_2, x_k) \geq w_Q(\gamma_1, x_0) - k(\log d + 2D).
\]

**Proof.** Let \( Y = f^{-k}(R_{x_0}) \). Then \( Y \subset R_{x_k} \) is a Riemann surface and \( f^k : Y \to R_{x_0} \) is a holomorphic covering map of degree \( d^k \). Then

\[
l_Y(\gamma_2) \leq d^k l_Q(\gamma_1, x_0).
\]

Since the inclusion map \( \iota : Y \hookrightarrow R_{x_k} \) decreases the hyperbolic lengths,

\[
l_Q(\gamma_2, x_k) \leq d^k l_Q(\gamma_1, x_0).
\]

It follows that

\[
w_Q(\gamma_2, x_k) > w_Q(\gamma_1, x_0) - k \log d.
\]

Since \( \sigma_f \) decreases the Teichmüller distance \( d_T \),

\[
d_T(x_i, x_{i+1}) \leq d_T(x_0, x_1) = D.
\]

The map \( \gamma \mapsto w_Q(\gamma, x) \) for any \( x \in T_f \) is a Lipschitz function with Lipschitz constant 2 (see Lemma 4), so we have that

\[
w_Q(\gamma_2, x_0) \geq w_Q(\gamma_2, x_k) - 2kD \geq w_Q(\gamma_1, x_0) - k(2D + \log d).
\]

\( \square \)
Lemma 14. Suppose $\Gamma$ is an irreducible multi-curve. Then for all $\gamma_i, \gamma_j \in \Gamma$ and all $x \in T_f$,
$$|w_Q(\gamma_i, x) - w_Q(\gamma_j, x)| \leq (p - 3)(\log d + 2D).$$

Proof. Since $\Gamma$ is irreducible, there is an integer $q \leq \sharp \Gamma \leq p - 3$ such that $\gamma_i$ is homotopic to a preimage of $f^{-q}(\gamma_j)$. By Lemma 13, we see that $w_Q(\gamma_i, x) \geq w_Q(\gamma_j, x) - (p - 3)(\log d + 2D)$. By exchanging $i$ and $j$, we complete the proof. \hfill \square

Proposition 5. Suppose $\Gamma$ is an $f$-stable multi-curve satisfying $\Gamma = \Gamma_0$. Let $x_0 \in T_f$ and $x_n = \sigma^n(x_0)$, $n > 0$. Let $D = d_T(x_0, x_1)$. Suppose $\min_{x} w_Q(\gamma, x) \geq \log(3/\beta) + \log \pi$, where $\beta$ is the number in Proposition 3. Write $\Gamma = \Gamma' \sqcup \Gamma''$, where $\Gamma' = \Gamma_{Ob}$ is the union of the irreducible component $\Gamma_j$ of $\Gamma$ for which $\lambda(A_{\gamma_j}) \geq 1$. Then

1. for all $\gamma \in \Gamma'$, $\{w_Q(\gamma, x_n)\}_{n \geq 0}$ is $\kappa$-quasi-nondecreasing, where $\kappa = \log \beta - 1 - 2\log \pi - 2(p - 3)(\log d + 2D)$;
2. for all $\gamma \in \Gamma''$ and all $n \geq 0$,
$$w_Q(\gamma, x_n) \geq \min_{\gamma' \in \Gamma'}\{w_Q(\gamma', x_n)\} - M(\log d + 2D),$$
where $M$ is the constant in Proposition 3.
3. Suppose $\min_{\gamma \in \Gamma} w_Q(\gamma, x_0) \geq J_A - 1$, where $J_A = \max\{\log(3/\beta) + \log \pi, \Lambda\} + \kappa + M(\log d + 2D) + 1$.

Then for all $\gamma \in \Gamma$ and for all $n \geq 0$, we have
$$w_Q(\gamma, x_n) \geq A.$$

Proof. Let $\Gamma_j$ be an irreducible component of $\Gamma$ for which $\lambda(\Gamma_j) \geq 1$. By the assumption that $w(\Gamma, x_0) \geq \log(3/\beta) + \log \pi$, we have $\{w(\Gamma_j, x_n)\}$ is log $\beta - 1 - 2\log \pi$-quasi-nondecreasing.

Since $\Gamma_j$ is an irreducible multi-curve, by Lemma 13 and Property 1 we have for each $\gamma \in \Gamma_j$, the sequence $\{w_Q(\gamma, x_k)\}_{k \geq 0}$ is a $\kappa = \log \beta - 1 - 2\log \pi - 2(p - 3)(\log d + 2D)$-quasi-nondecreasing. This completes (1).

By Lemma 12 and (1), we have for all $\gamma \in \Gamma''$ and all $n \geq 0$
$$w_Q(\gamma, x_n) \geq \min_{\gamma' \in \Gamma'}\{w_Q(\gamma', x_n)\} - M(\log d + 2D).$$
This is (2).

(3) follows from (1) and (2) immediately. \hfill \square
Proposition 6. Suppose $\Gamma$ is an $f$-stable multi-curve satisfying $\Gamma = \Gamma_0$. Let $x_0 \in T_f$ and $x_n = \sigma^n_f(x), n > 0$, and $D = d_T(x_0, x_1)$. Suppose 

$$\min_{\gamma \in \Gamma} w_E(\gamma, x_0) \geq J_A.$$ 

Write $\Gamma = \Gamma' \cup \Gamma''$, where $\Gamma' = \Gamma_{Oh}$ is the union of the irreducible component $\Gamma_j$ of $\Gamma$ for which $\lambda(A_{F_j}) \geq 1$. Then

1) For all $\gamma \in \Gamma'$, $w_E(\gamma, x_n) \geq A$ for any $n \geq 0$.

2) For all $\gamma \in \Gamma''$, $\{w_E(\gamma, x_n): n \geq 0\}$ is $(\kappa - 2)$-quasi-non-decreasing.

3) For all $\gamma \in \Gamma''$ and all $n \geq 0$,

$$w_E(\gamma, x_n) \geq \min_{\gamma' \in \Gamma'} \{w_E(\gamma', x_n)\} - 2 - M(\log d + 2D).$$

Proof. From Lemma 3 we have, for any $x \in T_f$,

$$w_Q(\gamma, x) \leq w_E(\gamma, x) \leq w_Q(\gamma, x) + 1$$

if $w_E(\gamma, x) \geq A$. If $\min_\gamma w_E(\gamma, x_0) \geq J_A$, then $\min_\gamma w_Q(\gamma, x_0) \geq J_A - 1$, then by Proposition 5 for any $n \geq 0$ and $\gamma \in \Gamma$, $w_Q(\gamma, x_n) \geq A$. Consequently, $w_E(\gamma, x_n) \geq A$. We get 1).

From 1), we have $|w_E(\gamma, x_n) - w_Q(\gamma, x_n)| < 1$ for all $n \geq 0$. Then by Property 1 and Proposition 5 we have 2) and 3). $\square$

7. Proof of Theorem 4

Choose any $x_0 \in T_f$, we can find a $J \geq J_A$ such that $w_E(x_0) < C(J)$. Without loss of generality, we assume that $J = J_A$. Since $C(J)$ is an increasing function of $J$, we have $w_E(x_0) < C(J)$ for all $J \geq J_A$. Let $x_n = \sigma^n_f(x_0), n > 0$, and $D = d_T(x_0, x_1)$.

Suppose that $f$ is not equivalent to a rational map. By Lemma 2 the sequence $\{w_E(x_n): n \geq 0\}$ is unbounded. Thus there exists $\gamma_k$ and $x_{n_k}$ with $w_E(\gamma_k, x_{n_k}) \to \infty$, as $k \to \infty$.

Fix $J > J_0 = J_A + |A|$. Then $w_E(\gamma_k, x_{n_k}) > E(J) = C(J) + 2mD$ for some $k$. So by Lemma 8 the set of the finite depth curves in $\Gamma_{J, x_{n_k}}$, denoted by $\Gamma_{J, x_{n_k}, 0}$, is nonempty.

Moreover, if for some $n_0, \gamma \in \Gamma_{J, n_0, 0}$, then $w_E(\gamma, x_{n_0}) > a + J \geq J_A$, which implies $w_E(\gamma, x_n) \geq A$ for all $n \geq n_0$ by Proposition 6. This implies that $\Gamma_J = \cup_n \Gamma_{J, x_n, 0}$ and $\mathcal{G} = \cup_{J \geq J_0} \Gamma_J$ are multi-curves, since $\gamma \in \Gamma_J$ satisfies $w_E(\gamma, x_n) \geq A$ for all $n$ sufficiently large.

Since $w_E(\gamma_k, x_{n_k}) \to \infty$, as $k \to \infty$, given any fixed $J \geq J_0$, $w_E(\gamma_k, x_{n_k}) \geq E(J)$ for infinitely many $k$. Hence $\gamma_k \in \Gamma_J \subset \mathcal{G}$ infinitely often. Since $\mathcal{G}$ is finite, for some $\gamma \in \mathcal{G}$, we have $\gamma_k = \gamma$ for infinitely many $k$. Hence the set

$$\Gamma_u = \{\gamma | \{w_E(\gamma, x_n): n \geq 0\} \text{ is unbounded}\}$$
is nonempty.

**Proposition 7.** \(\Gamma_u = \bigcap_{J \geq J_0} \Gamma_J\).

**Proof.** The inclusion \(\bigcap_{J > J_0} \Gamma_J \subset \Gamma_u\) is clear. To see the other inclusion, let \(\gamma \in \Gamma_u\). Given \(J\), there exists some \(n\) such that \(w(\gamma, x_n) > E(J)\). By Lemma 8, \(\gamma \in \Gamma_{J, x_n, 0}\). Thus \(\bigcap_{J > J_0} \Gamma_J \supset \Gamma_u\). This proves the proposition. \(\square\)

**Proposition 8.** \(\Gamma_u = \Gamma_{J_c}\) for some \(J_c \geq J_A\).

**Proof.** We prove it by contradiction. Since \(\Gamma_u = \bigcup_{n} \Gamma_{J_c, x_n, 0}\), the inclusion \(\Gamma_{J_c, x_n, 0} \subset \Gamma_u \subset \mathcal{G}\) holds for all \(n\).

Since there are finitely many elements in \(\mathcal{G}\), there exists an \(n_c\) such that for all \(\gamma \in \Gamma_u\),

\[w_E(\gamma, x_{n_c}) > E(J_c)\]

By Lemma 8, we have \(\gamma \in \Gamma_{J_c, x_{n_c}, 0}\). Thus \(\Gamma_u = \Gamma_{J_c, n_c, 0}\). \(\square\)

From Proposition 3, \(\Gamma_u\) is a Thurston obstruction. Furthermore, \(\Gamma_u\) depends only on \(f\) and is independent of the initial point \(x_0\), since for any \(\gamma\), the map \(x \mapsto w_E(\gamma, x)\) is a Lipschitz map with Lipschitz constant 2 (see Lemma 5) and since \(\sigma_f\) decreases the Teichmüller distance \(d_T\).

Finally, since

\[w_Q(\gamma, x) \leq w_E(\gamma, x) \leq 1 + w_Q(\gamma, x)\]
if \( w_E(\gamma, x) \geq A \) (refer to Remark 4.3), we have that
\[
\Gamma_c = \left\{ \gamma \mid w_Q(\gamma, x_n) \to \infty \text{ as } n \to \infty \right\} = \left\{ \gamma \mid w_E(\gamma, x_n) \to \infty \text{ as } n \to \infty \right\} = \Gamma_u.
\]
Therefore, \( \Gamma_c \) is a Thurston obstruction. This completes the proof of Theorem 1.

References

[AB] L. V. Ahlfors and L. Bers, Riemann’s mapping theorem for variable metrics. Ann. of Math. 72 (1960), 385-404.

[CJS] G. Cui, Y. Jiang and D. Sullivan, On geometrically finite branched covering maps. Manuscript, 1994.

[CJS1] G. Cui, Y. Jiang and D. Sullivan, On geometrically finite branched covering maps-I. Locally combinatorial attracting. Complex Dynamics and Related Topics, New Studies in Advanced Mathematics, 2004, The International Press, 1-14.

[CJS2] G. Cui, Y. Jiang and D. Sullivan, On geometrically finite branched covering maps-II. Realization of rational maps. Complex Dynamics and Related Topics, New Studies in Advanced Mathematics, 2004, The International Press, 15-29.

[CJ] G. Cui and Y. Jiang, Geometrically finite and semi-rational branched coverings of the two-sphere. Transactions of the American Mathematical Society, Article electronically published on December 10, 2010.

[CT] G. Cui and L. Tan, A characterization of hyperbolic rational maps, preprint, arXiv:math:DS/0703380.

[DH] A. Douady and John Hubbard, A proof of Thurston’s topological characterization of rational functions. Acta. Math., 171 (1993), No. 2, 263-297.

[EM] C. Earle and S. Mitra, Variation of moduli under holomorphic motions. Contemporary Mathematics, 256 (2000) 39–67.

[GJW] F. Gardiner, Y. Jiang, and Z. Wang, Holomorphic motions and related topics. Geometry of Riemann Surfaces, London Math Society Lecture Notes series 2009.

[Ga] F. R. Gantmacher, Theory of matrices. Chelsea, 1959.

[JM] Y. Jiang, S. Mitra, and Z. Wang, Liftings of holomorphic maps into Teichmüller spaces. Kodai Mathematical Journal, 32 (2009), No. 3, 544-560.

[Li] G. Lieb, Holomorphic motions and Teichmüller space. Ph.D. dissertation, Cornell University, 1990.

[Mc] C. McMullen, Complex Dynamics and Renormalization. Ann. of Math. Studies, 79, 1994.

[Mi] J. Milnor, Dynamics in One Complex Variable, Introductory Lectures, Vieweg, 2nd Edition, 2000.

[Pi] K. M. Pilgrim, Canonical Thurston obstruction. Advances in Mathematics, 158 (2001), 154-168

[Th] W. Thurston, The combinatorics of iterated rational maps. Preprint Princeton University, Princeton, N.J. 1983.
[JZ] G. Zhang and Y. Jiang, *Combinatorial characterization of sub-hyperbolic rational maps*. *Advances in Mathematics*, 221 (2009), 1990-2018.

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