From bypass transition to flow control and data-driven turbulence modeling: An input-output viewpoint

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Abstract

Transient growth and resolvent analyses are routinely used to assess non-asymptotic properties of fluid flows. In particular, resolvent analysis can be interpreted as a special case of viewing flow dynamics as an open system in which free-stream turbulence, surface roughness, and other irregularities provide sources of input forcing. We offer a comprehensive summary of the tools that can be employed to probe the dynamics of fluctuations around a laminar or turbulent base flow in the presence of such stochastic or deterministic input forcing and describe how input-output techniques enhance resolvent analysis. Specifically, physical insights that may remain hidden in the resolvent analysis are gained by detailed examination of input-output responses between spatially-localized body forces and selected linear combinations of state variables. This differentiating feature plays a key role in quantifying the importance of different mechanisms for bypass transition in wall-bounded shear flows and in explaining how turbulent jets generate noise. We highlight the utility of a stochastic framework, with white or colored inputs, in addressing a variety of open challenges including transition in complex fluids, flow control, and physics-aware data-driven turbulence modeling. Applications with time- or spatially-periodic base flows are discussed and future research directions are outlined.

Keywords

input-output analysis, flow modeling and control, physics-aware data-driven modeling, stochastic dynamics, frequency responses, transition to turbulence, turbulent flows, convex optimization
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1. INTRODUCTION

Hydrodynamic stability theory focuses on spectral analysis of the dynamical generator in the linearized Navier-Stokes (NS) equations while seeking the critical Reynolds number at which exponentially growing modes emerge (Schmid & Henningson 2001). Although in many flows predictions agree well with experiments, in wall-bounded shear flows both the critical Reynolds number and the spatial structure of the least-stable or unstable modes are at odds with experimental observations. A broader viewpoint, based on nonmodal analysis of the linearized NS equations, provides reconciliation with experiments and identifies mechanisms for the early stages of subcritical transition (Schmid 2007).

In the words of Trefethen & Embree (2005), the eigenvalue decomposition gives a square matrix, or an operator, a personality. However, this “personality test” is conclusive only for normal (i.e., unitarily diagonalizable) operators. For non-normal operators, it is the singular value decomposition (SVD) that offers a robust predictor of “personality” (Trefethen & Embree 2005). In wall-bounded shear flows, non-normality of the linearized dynamical operator introduces coupling of exponentially decaying modes which explains high sensitivity of the laminar flow (Schmid 2007). The high sensitivity degrades the accuracy of analytical and computational predictions that do not explicitly account for modeling imperfections. These are typically difficult to model and may arise from a variety of sources, including surface roughness, thermal fluctuations, and irregularities in the incoming stream.

The study of dynamical systems with input forcing has a rich history in several branches of electrical engineering including circuit theory, communications, signal processing, and control. In this, dynamical systems are decomposed into essential pieces and represented as interconnections of input-output “blocks”. This input-output viewpoint facilitates the analysis, design, and optimization of complex systems, since they can be viewed as simpler sub-systems placed in cascade, parallel, and feedback arrangements with one another. It also allows us to quantify the influence of modeling imperfections (e.g., background noise or experimental uncertainty that is unavoidable in physical systems) on quantities of interest.

In fluid mechanics, input-output analysis addresses the influence of deterministic as well as stochastic inputs on transient and asymptotic properties of fluid flows. It offers a complementary viewpoint to transient growth (Butler & Farrell 1992) and resolvent (Trefethen et al. 1993) analyses and brings in an appealing robustness interpretation. Specifically, additional insight about the dynamics is gained by carrying out SVD of the operator that maps excitation sources (i.e., inputs such as body forcing fluctuations) to the quantities of interest (i.e., outputs such as velocity fluctuations). In contrast to the resolvent, this operator is not necessarily a square object; it captures the effect of different inputs to particular physical quantities and thereby reveals finer physical aspects (Jovanović & Bamieh 2005). In wall-bounded shear flows, input-output analysis exposes large amplification of disturbances and high sensitivity of the laminar flow to uncertainty in the geometry or base velocity (Trefethen et al. 1993, Farrell & Ioannou 1993, Bamieh & Dahleh 2001, Jovanović 2004), and provides insights into structural features of turbulent flows (McKeon & Sharma 2010, Hwang & Cossu 2010a,b). Additional successful applications range from discovering mechanisms for transition to elastic turbulence in viscoelastic fluids (Hoda et al. 2008, 2009, Jovanović & Kumar 2011), to revealing how turbulent jets generate noise (Jeun et al. 2016), and explaining the origin of reattachment streaks in hypersonic flows (Dwivedi et al. 2019).

This review highlights the merits, effectiveness, and versatility of the input-output framework for modeling, analysis, and control of fluid flows. We offer a comprehensive summary of the tools that can be used to probe the dynamics of infinitesimal fluctuations.
around a given laminar or turbulent base flow, and explain how the framework augments resolvent analysis (Trefethen et al. 1993). We illustrate how the componentwise input-output approach (Jovanović & Bamieh 2005) identifies key mechanisms for bypass transition in channel flows of Newtonian and viscoelastic fluids. We then describe how periodic base flow modifications, induced by streamwise traveling waves and spanwise wall-oscillations, can be designed to, respectively, control the onset of turbulence (Moarref & Jovanović 2010) and identify the optimal period of oscillation for turbulent drag reduction (Moarref & Jovanović 2012). The input-output framework is also well-suited for data-driven turbulence modeling; in contrast to physics-agnostic machine learning techniques, the tools from control theory and convex optimization allow for strategic use of data in order to capture second-order statistics of turbulent flows via first-principle models (Zare et al. 2017b, 2020).

2. INPUT-OUTPUT VIEWPOINT: BEYOND RESOLVENT ANALYSIS

We first review the tools that can be used to probe the dynamics of infinitesimal fluctuations around a given base flow. While this framework can be utilized in a variety of flow regimes and geometries, we resort to a channel flow with homogeneous wall-parallel directions to illustrate the key concepts; see Figure 1(a) Even in this simple setup a variety of non-trivial fundamental questions can be addressed by employing an input-output viewpoint, including transition in complex fluids, flow control, and data-driven turbulence modeling.

Evolution Model

A 1st order (in time) differential equation governs the evolution of the state \( \psi_k(t) \) and a static-in-time output equation relates \( \psi_k(t) \) to the output \( \xi_k(t) = C_k \psi_k(t) \). Apart from the boundary conditions, no additional constraints are imposed on \( \psi_k(t) \).

The linearized NS equations govern the dynamics of infinitesimal fluctuations around a given base flow. Fluctuations can arise from a variety of sources, including surface roughness, imperfections in the incoming stream, acoustics, vibrations, particles, and impurities. In turbulent flows, nonlinear interactions between different length scales can also provide forcing that sustains fluctuations. The linearized NS equations, with an input forcing \( d_k(t) \) and an output of interest \( \xi_k(t) \), can be brought to an evolution form,

\[
\begin{align*}
\frac{d\psi_k(t)}{dt} &= A_k \psi_k(t) + B_k d_k(t), \\
\xi_k(t) &= C_k \psi_k(t),
\end{align*}
\]

where \( \psi_k(t) \) is the state and \( k \) is the vector of wavenumbers. The operator \( A_k \) characterizes dynamical interactions between the states, \( B_k \) specifies the way the input \( d_k(t) \) enters into the dynamics, and \( C_k \) maps the state \( \psi_k(t) \) to the output \( \xi_k(t) \). Equation 1 is a standard state-space model in the controls literature, and it provides a convenient starting point for modal and nonmodal analysis, system identification, turbulence modeling, and flow control.

2.1. From evolution model to input-output representation

A pressure-driven channel flow (a) between two parallel infinite walls with base flow \( (U(y), 0, 0) \), inhomogeneous wall-normal \( y \), and homogeneous streamwise and spanwise \( (x, z) \) directions; (b) subject to blowing and suction along the walls; and (c) subject to spanwise wall-oscillations.

![Figure 1](image-url)
For a pressure-driven channel flow of an incompressible Newtonian fluid, the base flow \( \bar{u} \) is either given by the laminar parabolic profile (Poiseuille flow) or the turbulent mean velocity. In both cases, the flow is fully-developed and \( \bar{u} \) only depends on the wall-normal distance \( y \), \( \bar{u} = (U(y), 0, 0) \). Thus, the linearized NS equations are translationally-invariant in wall-parallel directions and in time and fluctuations can be decomposed in terms of the normal modes in \( x \) and \( z \) as \( \psi(x, y, z, t) = \psi_k(y, t) e^{i(k_x x + k_z z)} \). Here, \( k := (k_x, k_z) \) denotes the vector of wall-parallel wavenumbers and \( A_k \) is the Orr-Sommerfeld/Squire operator (Schmid & Henningson 2001). In addition to \( k \), System 1 is parameterized by the base flow \( \bar{u} \) and the Reynolds number \( Re \). For any \((k, t)\), the state \( \psi_k(t) \), input \( d_k(t) \), and output \( \xi_k(t) \) are functions of \( y \) but, for notational convenience, we suppress this dependence.

**Derivation of Equation 1.** The linearized model is obtained by expressing the flow as the sum of the base and fluctuation components and by neglecting the quadratic fluctuation terms. In incompressible flows of Newtonian fluids, the velocity obeys a continuity equation (Schmid & Henningson 2001). In addition to \( k \), System 1 is parameterized by the base flow \( \bar{u} \) and the Reynolds number \( Re \). For any \((k, t)\), the state \( \psi_k(t) \), input \( d_k(t) \), and output \( \xi_k(t) \) are functions of \( y \) but, for notational convenience, we suppress this dependence.

**Derivation of Equation 1.** The linearized model is obtained by expressing the flow as the sum of the base and fluctuation components and by neglecting the quadratic fluctuation terms. In incompressible flows of Newtonian fluids, the velocity obeys a continuity equation and a Poisson equation for the pressure \( p \) is obtained by applying the divergence operator to the linearized NS equations. The Orr-Sommerfeld equation is obtained by acting with the Laplacian \( \Delta \) on the wall-normal velocity equation and using the expression for \( \Delta p \) to eliminate \( p \). The Squire equation is obtained by taking the curl of the linearized NS equations. This yields an evolution model, in the form of two PDEs, for the wall-normal velocity and vorticity [Kim et al. 1987]. \( \psi := (v, \eta) \). All other velocity and vorticity components can be expressed in terms of \((v, \eta)\) via kinematic relations [Jovanović & Bamieh 2005].

Standard stability analysis of a laminar Poiseuille flow predicts modal instability for \( Re = 5772 \). The discrepancy with experiments, in which transition occurs for \( Re \approx 1000 \), can be explained using nonmodal analysis [Schmid 2007] which reveals significant transient growth of fluctuations [Gustavsson 1991] [Butler & Farrell 1992] and strong amplification of disturbances [Trefethen et al. 1993] [Farrell & Ioannou 1993] [Bamieh & Dahleh 2001].

**2.1.1. Resolvent, transfer function, impulse and frequency response operators.** While the governing equations and geometry determine the dynamical generator \( A_k \), there is flexibility in selecting the operators \( B_k \) and \( C_k \) and different choices can reveal different aspects of flow physics [Jovanović & Bamieh 2005]. All of these operators play a role in the response of System 1 which arises from the initial condition \( \psi_k(0) \) and the exogenous input \( d_k(t) \).

\[
\xi_k(t) = C_k e^{A_k t} \psi_k(0) + \int_0^t C_k e^{A_k (t - \tau)} B_k d_k(\tau) \, d\tau,
\]

where \( e^{A_k t} \) is the state-transition operator associated with \( A_k \). The Laplace transform can be utilized to rewrite Equation 2 as,

\[
\hat{\xi}_k(s) = C_k (sI - A_k)^{-1} \psi_k(0) + C_k (sI - A_k)^{-1} B_k \hat{d}_k(s),
\]

where \( s \) is the complex number, \( I \) is the identity operator, \( \hat{\xi}_k(s) \) is the Laplace transform of \( \xi_k(t) \), and \( (sI - A_k)^{-1} \) is the resolvent operator. Equations 2 and 3 determine responses of System 1 and provide the basis for quantifying important dynamical features of the linearized flow equations. As the blue terms demonstrate, the natural (i.e., unforced) responses are characterized by the state-transition \( e^{A_k t} \) and resolvent \( (sI - A_k)^{-1} \) operators. On the other hand, the forced response is obtained by convolving an input \( d_k(t) \) with the impulse...
response operator $T_k(t)$; equivalently, the transfer function $T_k(s)$ specifies an input-output mapping in the complex domain, i.e., $\tilde{\xi}_k(s) = T_k(s)\hat{d}_k(s)$, where

\[
\text{impulse response} \quad T_k(t) := C_k e^{\lambda_k t} B_k \quad \text{Laplace transform} \quad T_k(s) := C_k (sI - A_k)^{-1} B_k.
\]

For flows over perfectly smooth walls and in noise-free environments, study of natural responses aids in understanding the fundamental fluid mechanics. Specifically, the eigenvalue decomposition of $A_k$ and the singular value decomposition of $e^{\lambda_k t}$, respectively, offer insights into modal and nonmodal aspects of the flow \cite{Schmid2007}. While such insights are valuable, engineering flows seldom exist in isolation and understanding the forced responses is equally important. In particular, input-output analysis examines forced responses with the objective of quantifying amplification of disturbances and impact of modeling imperfections on fluctuations’ dynamics. In contrast to natural responses, study of forced responses requires specifying how disturbances enter into System 1, through the operator $B_k$.

In the special case when the input excites all degrees of freedom equally and the output is the entire state, $B_k$ and $C_k$ are the identity operators and the resolvent completely determines the transfer function. However, it is often of interest to confine the inputs to certain spatial regions and to examine outputs that are given by a linear combination of certain state variables. In such cases, the transfer function is determined by a “compressed resolvent” and its analysis can uncover important dynamical aspects that may be obscured by only paying attention to the “standard resolvent”. This distinction played a key role in understanding how turbulent jets generate noise. \cite{Jeun2016} utilized “compressed resolvent” analysis by restricting inputs to the vicinity of the jet turbulence and selecting far-field pressure as the output. In contrast to a standard resolvent analysis, which provides links to jet hydrodynamics but does not explain noise generation \cite{Garnaud2013}, this approach identifies acoustic sources to be wavepackets that are in excellent agreement with experiments \cite{Jordan2013} and reveals mechanisms for noise generation.

**Singular Value Decomposition.** In transient growth analysis, SVD identifies spatial structure of initial conditions that maximize energy at a given time. SVD also provides the tool for quantifying responses to unsteady deterministic as well as stochastic inputs $d_k(t)$ that neither grow nor decay in time (on average). This allows us to set $s = \omega$, and the frequency response $T_k(i\omega)$ is obtained by evaluating the transfer function $T_k(s)$ along the imaginary axis; see the sidebar FREQUENCY RESPONSE OPERATOR.

SVD of $T_k(i\omega)$ identifies fundamental input-output features across $(k, \omega)$,

\[
\hat{\xi}_k(i\omega) = T_k(i\omega)\hat{d}_k(i\omega) = \sum_{j=1}^{\infty} \sigma_{k,j}(\omega) \hat{v}_{k,j}(\omega) \langle \hat{v}_{k,j}(\omega), \hat{d}_k(i\omega) \rangle.
\]

The left and right singular functions, $\hat{v}_{k,j}(\omega)$ and $\hat{u}_{k,j}(\omega)$, provide orthonormal bases of the input and output spaces, the singular value $\sigma_{k,j}(\omega)$ determines the corresponding amplification, and $\langle \cdot, \cdot \rangle$ is the inner product. SVD requires computation of the adjoint $T_k^\dagger(i\omega)$,

\[
\left\langle T_k^\dagger(i\omega)\hat{\xi}_k(i\omega), \hat{d}_k(i\omega) \right\rangle = \left\langle \hat{\xi}_k(i\omega), T_k(i\omega)\hat{d}_k(i\omega) \right\rangle,
\]

and the eigenvalue decomposition of $TT^\dagger$ and $T_i^\dagger T$, $T_k(i\omega)T_k^\dagger(i\omega)\hat{u}_{k,j}(\omega) = \sigma_{k,j}(\omega)\hat{u}_{k,j}(\omega)$. $T_k^\dagger(i\omega)T_k(i\omega)\hat{v}_{k,j}(\omega) = \sigma_{k,j}(\omega)\hat{v}_{k,j}(\omega)$). Unless noted otherwise, the $L_2$ inner product $\langle \cdot, \cdot \rangle$, which induces energy norm, is taken over inhomogeneous spatial directions in Equation 4.
For a harmonic input in \((x, z, t)\), \(d_k(y, \omega) e^{i(k_x x + k_z z + \omega t)}\), is determined by \(\hat{\xi}_k(y, \omega) e^{i(k_x x + k_z z + \omega t)}\), where \(\hat{\xi}_k(y, \omega) = \|T_k(\omega)\| \hat{d}_k(\cdot, \omega)(y)\). Spatial structures of (a) spanwise forcing fluctuations; and (b) resulting streamwise velocity fluctuations at one time instant in Poiseuille flow with \(Re = 2000\) for 4 combinations of \((k_x, k_z, \omega) = (1, ±1, -0.385); (-1, ±1, 0.385)\).

### 2.2. Amplification of deterministic inputs

For a harmonic input \(d_k(t) = d_k(i\omega)e^{i\omega t}\) with \(d_k(i\omega) = v_{k,j}(\omega)\), where \(v_{k,j}(\omega)\) is the \(j\)th left singular function of \(T_k(\omega)\), the steady-state output \(\xi_k(t) = \hat{\xi}_k(i\omega)e^{i\omega t}\) of System 1 in the direction of the \(j\)th right singular function, \(\xi_k(i\omega) = \sigma_{k,j}(\omega)\hat{u}_{k,j}(\omega)\), and its energy is given by \(\|\xi_k(i\omega)\|^2_2 := (\xi_k(i\omega), \xi_k(i\omega)) = \sigma_{k,j}^2(\omega)\). The principal singular value, \(\sigma_{k,1}(\omega) := \sigma_{\text{max}}(T_k(\omega))\), determines the largest amplification at any \((k, \omega)\) and the smallest upper bound over \(\omega\) determines the \(H_\infty\) norm of System 1 [Zhou et al. 1996]. \(G_k := \sup_\omega \sigma_{k,1}^2(\omega)\). This measure of input-output amplification has several appealing interpretations for any \(k\).

(a) The \(H_\infty\) norm represents the worst-case amplification of harmonic (in homogeneous directions and in time) deterministic (in inhomogeneous directions) inputs. This worst-case input-output gain is obtained by maximizing over spatial profiles (largest singular value of \(T_k\)) and temporal frequency (supremum over \(\omega\)); see Figure 3(a).

(b) The \(H_\infty\) norm determines the induced gain from finite energy inputs to outputs, \(G_k = \sup_{\|\xi_k\|_2 \leq 1} (E_k^{\text{out}}/E_k^{\text{in}})\), where \(E_k^{\text{in}}\) and \(E_k^{\text{out}}\) denote the \(k\)-parameterized energy of input and output, e.g., \(E_k^{\text{in}} := \int_0^\infty \|d_k(t)\|^2 dt\), with \(\|d_k(t)\|^2 = (d_k(t), d_k(t))\). For a unit-energy input \(d_k(t)\) to stable System 1, \(G_k\) quantifies the largest possible energy of the output \(\xi_k(t)\) across the spatial wavenumber \(k\).

(c) The \(H_\infty\) norm quantifies robustness to modeling imperfections; see Figure 3(b).

### 2.3. Amplification of stochastic inputs

A common criticism of transient growth and resolvent analyses is difficulty of implementing the worst-case initial conditions or inputs in the lab. An alternative approach introduces a random excitation to the NS equations that can account for background noise. It identifies almost identical dominant flow structures and opens the door to turbulence modeling.

Control-theoretic tools can be utilized to exploit the structure of the linearized Model 1, avoid costly stochastic simulations, and offer insight into amplification mechanisms. For 20 realizations of persistent channel-wide temporally and spatially uncorrelated stochastic input \(d_k(t)\) to Equation 1, Figure 4 shows the variance of the velocity fluctuation vector \(v_k(t) := (u_k(t), v_k(t), w_k(t))\) in Poiseuille flow with \(Re = 2000\). Although individual simulations display different responses, their average (marked by a thick black line) reaches the steady-state limit. In the absence of modal instability, viscosity asymptotically dissipates natural responses but a persistent excitation source maintains fluctuations for all times.
FREQUENCY RESPONSE OPERATOR

Time-invariant systems. The natural response of a stable Linear Time-Invariant (LTI) System asymptotically decays to zero. The frequency response operator determines the steady-state response to harmonic inputs with frequency \( \omega \) and is obtained by evaluating the transfer function along the imaginary axis,

\[
T_k(\omega) := T_k(s)
\mid_{s = \omega} = C_k (i\omega I - A_k)^{-1} B_k. \tag{FR1}
\]

For \( d_k(t) = \hat{d}_k(\omega) e^{i\omega t} \), the steady-state response of a stable System is harmonic with the same frequency but with different amplitude and phase, i.e., \( \xi_k(t) = \hat{\xi}_k(\omega) e^{i\omega t} \). The frequency response \( T_k(\omega) \) is an operator (in inhomogeneous spatial directions) that maps a spatial input profile \( \hat{d}_k(\omega) \) into the output \( \hat{\xi}_k(\omega) \), thereby determining how amplitude and phase change across \( k \) and \( \omega \).

Time-periodic systems. If the operator \( A_k \) in Equation has time-periodic coefficients, i.e., \( A_k(t) = A_k(t + 2\pi/\omega_k) \), the steady-state response to a harmonic input with frequency \( \omega \) contains an infinite number of harmonics separated by integer multiplies of \( \omega_k \), i.e., \( \omega + n\omega_k, \ n \in \mathbb{Z} \). The proper normal modes for frequency response analysis are no longer purely harmonic, \( e^{i\omega t} \). Rather, they are determined by the Bloch waves (Odeh & Keller 1964), i.e., by a product of \( e^{i\theta t} \) and the \( 2\pi/\omega_k \) periodic function in \( t \),

\[
d_k(t) = \sum_{n = -\infty}^{\infty} \hat{d}_{k,n}(\theta) e^{i(\theta + n\omega_k)t}, \quad \theta \in [0, \omega_k), \tag{BW}
\]

where \( \theta \) is the angular frequency and \( \theta = 0 \) and \( \theta = \omega_k/2 \) identify the fundamental and subharmonic modes, respectively. The steady-state response of a stable linear time-periodic system to a Bloch wave input \( BW \) is also a Bloch wave, \( \xi_k(t) = \sum_n \hat{\xi}_{k,n}(\theta) e^{i(\theta + n\omega_k)t} \), and, for any \( (k, \theta) \), the frequency response operator \( T_k(\theta) \) maps \( d_k(\theta) := \text{col} \{d_{k,n}(\theta)\}_{n \in \mathbb{Z}} \) to \( \xi_k(\theta) := \text{col} \{\xi_{k,n}(\theta)\}_{n \in \mathbb{Z}} \), i.e., \( \hat{\xi}_k(\theta) = T_k(\theta) \hat{d}_k(\theta) \). If the operators \( B_k \) and \( C_k \) in Equation are time-invariant, for a system with \( A_k(t) = \sum_m A_{k,m} e^{im\omega_k t} \) we have

\[
T_k(\theta) = C_k (E(\theta) - A_k)^{-1} B_k, \tag{FR1}
\]

where \( E(\theta) := \text{diag} \{i(\theta + n\omega_k)\}_{n \in \mathbb{Z}} \), \( B_k \) and \( C_k \) are the block-diagonal operators with \( B_k \) and \( C_k \) on the main diagonal, and \( A_k := \text{toep} \{\ldots, A_{k,1}, A_{k,0}, A_{k,-1}, \ldots\} \) is the block-Toeplitz operator (Jovanović 2008).

**Figure 3**

(a) The \( H_\infty \) norm is determined by the peak value of \( \sigma_{\max}(T_k(\omega)) \) over \( \omega \). (b) A large \( H_\infty \) norm of the linearized dynamics signals low robustness margins; modeling imperfections, captured by the operator \( \Gamma_k \), with the \( H_\infty \) norm \( 1/\sqrt{G_k} \) can trigger instability of \( A_k + B_k \Gamma_k C_k \). This interpretation is related to the pseudo-spectra of linear operators (Trefethen & Embree 2005).

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The Reynolds-Orr equation. In channel flow with stochastic forcing, the kinetic energy $E_k(t) \equiv \mathbb{E}(\langle \psi_k(t) \psi_k(t) \rangle)$ of fluctuations $\psi_k(t)$ around $(U(y), 0, 0)$ obeys,

$$\frac{1}{2} \frac{dE_k(t)}{dt} = \mathbb{E} \left( \frac{1}{Re} \langle \psi_k(t), \Delta \psi_k(t) \rangle - \langle u_k(t), U' \psi_k(t) \rangle + \langle \psi_k(t), d_k(t) \rangle \right) .$$

Here, $\mathbb{E}$ is the expectation operator, $\langle \cdot, \cdot \rangle$ is the $L_2[-1, 1]$ inner product, $U'(y) \equiv dU(y)/dy$, and the terms on the right-hand side denote the viscous energy dissipation, the energy exchange with the base shear, and the work done by the body forces, respectively. The nonlinear terms in the NS equations are conservative and the Reynolds-Orr equation takes the same form for nonlinear and linearized dynamics (Schmid & Henningson 2001). Since it is driven by the terms that need to be determined by solving the equations for flow fluctuations, it is not in the form which allows for direct determination of its solution. For the linearized NS equations, both the kinetic energy and the terms on the right-hand side of the Reynolds-Orr equation can be computed using the solution to differential Lyapunov equation (DL), that we present below. This avoids the need for costly stochastic simulations and provides an alternative way for solving an important equation in fluid mechanics.

Time-invariant systems. Let System 1 with the output $\xi_k(t) = v_k(t)$, be driven by a stochastic input $d_k(t)$ with the spectral density $\Omega_k(i\omega)$. The spectral density operator $S_k(i\omega) = T_k(i\omega) \Omega_k(i\omega) T_k^*(i\omega)$ quantifies the two-point correlations of $v_k(t)$ across the wavenumber $k$ and the frequency $\omega$, where $T_k(i\omega)$ is the frequency response given in Equation 4. The inverse Fourier transform of $S_k(i\omega)$ yields the lagged covariance operator,

$$P_k(\tau) \equiv \lim_{t \to \infty} \mathbb{E}(v_k(t) \otimes v_k(t + \tau)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_k(i\omega) e^{i\omega \tau} d\omega,$$

where $\otimes$ denotes the tensor product. Furthermore, the integration of $S_k(i\omega)$ over $\omega$ yields the steady-state two-point correlation (i.e., covariance) operator $V_k$ of the output $v_k(t)$,

$$V_k \equiv P_k(0) = \lim_{t \to \infty} V_k(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_k(i\omega) d\omega,$$

where $V_k(t) \equiv \mathbb{E}((v_k(t) \otimes v_k(t))$ is the time-dependent covariance operator of velocity fluctuations. For System 1, $V_k(t) = C_k X_k(t) C_k^*$, where $X_k(t) \equiv \mathbb{E}((\psi_k(t) \otimes \psi_k(t))$ is the covariance operator of the state $\psi_k(t)$ and $C_k^*$ is the adjoint of the operator $C_k$. In channel flow, for any $k$, $V_k$ is an operator in the wall-normal direction, $g_k(y_1) \equiv [V_k f_k(\cdot)](y_1), \quad (1/2 \pi \int_{-\infty}^{+\infty} S_k(i\omega) e^{i\omega \tau} d\omega)$. 

Figure 4

The variance of velocity fluctuations, $(1/t) \int_0^t \|v_k(\tau)\|^2 d\tau$, for 20 realizations of stochastic forcing to the linearized NS equations in Poiseuille flow with $Re = 2000$, $k = (0, 1.78); (1, 1);$ and $(1, 0)$. The variance averaged over all simulations is shown by a thick black line.
whose kernel representation determines all stationary two-point correlations of $v_k(t)$.

\[
g_k(y_1) = \int_{-1}^{1} V_k^e(y_1, y_2) f_k(y_2) \, dy_2 = \int_{-1}^{1} \lim_{t \to \infty} E(v_k(y_1, t) v_k^*(y_2, t)) f_k(y_2) \, dy_2.
\]

One- and two-point correlations in $y$ are obtained for $y_1 = y_2$ and $y_1 \neq y_2$, respectively; $V_k^e(y_1, y_2)$ determines the two-point spectral density tensor and its inverse Fourier transform gives the two-point correlation tensor in $x$ and $z$ (Moin & Moser 1989).

**Lyapunov equation.** For a zero-mean temporally uncorrelated input $d_k(t)$ with the covariance operator $W_k$, i.e., $E(d_k(t)) = 0$, $E(d_k(t) \otimes d_k(\tau)) = W_k \delta(t - \tau)$, the input spectral density $\Omega_k(\omega)$ is constant across $\omega$, i.e., $\Omega_k(\omega) = W_k$. In this case, as described in the sidebar **LYAPUNOV EQUATION: TWO-POINT CORRELATIONS**, the covariance operator $X_k(t)$ of the state $\psi_k(t)$ in System 1 satisfies the differential Lyapunov equation,

\[
\frac{dX_k(t)}{dt} = A_k X_k(t) + X_k(t) A_k^\dagger + B_k W_k B_k^\dagger.
\]

For System 1 with the input covariance $W_k$ and the initial condition $X_k(0)$, this *operator-valued differential equation* can be used to compute $X_k(t)$ and determine energy of fluctuations via $E_k(t) = \text{trace} (C_k X_k(t) C_k^\dagger)$. For linearly unstable flows, the steady-state limit of $X_k(t)$ is either unbounded or it does not exist. However, the solution of Equation DL can still be computed, e.g., by forward marching in time or via the following formula,

\[
X_k(t) = e^{A_k t} X_k(0) e^{A_k^\dagger t} + \left[ I \quad 0 \right] \exp \left( \begin{bmatrix} A_k & B_k W_k B_k^\dagger \\ 0 & -A_k \end{bmatrix} t \right) \begin{bmatrix} 0 \\ I \end{bmatrix} e^{A_k^\dagger t}.
\]

In the absence of modal instability, $X_k := \lim_{t \to \infty} X_k(t)$ is well-defined and the steady-state limit of Equation DL is given by,

\[
A_k X_k + X_k A_k^\dagger = -B_k W_k B_k^\dagger.
\]

In this case, $X_k(t)$ can be computed from the solution $X_k$ to the algebraic Lyapunov equation AL and the initial condition $X_k(0)$ via $X_k(t) = X_k - e^{A_k t} (X_k - X_k(0)) e^{A_k^\dagger t}$, and the steady-state limit of $E_k(t)$ determines the energy amplification $E_k := \lim_{t \to \infty} E_k(t) = \text{trace} (C_k X_k C_k^\dagger)$; see the sidebar **POWER SPECTRAL DENSITY AND ENERGY AMPLIFICATION**. Finally, for colored-in-time input $d_k(t)$, $X_k$ satisfies,

\[
A_k X_k + X_k A_k^\dagger = -(B_k H_k^1 + H_k B_k^1)
\]

where the operator $H_k$ determines the stationary cross-correlation between the input $d_k(t)$ and the state $\psi_k(t)$ in Equation 1 (Zare et al. 2017b, Appendix B).

Departure from the white-in-time restriction removes sign-definiteness requirement on the right-hand-side in Equation AL while the operator $B_k W_k B_k^\dagger$ in Equation AL has non-negative eigenvalues, $B_k H_k^1 + H_k B_k^1$ in Equation ALc is allowed to be sign-indefinite which provides additional flexibility. Furthermore, for a zero-mean white input $w_k(t)$ with the covariance operator $W_k$, the stationary covariance operator of $\psi_k(t)$ in the system

\[
\frac{d}{dt} \begin{bmatrix} \psi_k(t) \\ \phi_k(t) \end{bmatrix} = \begin{bmatrix} A_k & -B_k K_k \\ 0 & A_k - B_k K_k \end{bmatrix} \begin{bmatrix} \psi_k(t) \\ \phi_k(t) \end{bmatrix} + \begin{bmatrix} B_k \\ B_k \end{bmatrix} w_k(t),
\]

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The Lyapunov equation can be used to propagate two-point correlations of the white stochastic input \( d(t) \) into colored statistics of the state \( \psi(t) \) of a linear systems ([Bamieh & Dahleh 2001, Appendix A]). Herein, we derive the Lyapunov equation for a finite-dimensional discrete-time LTI system,

\[
\psi(t+1) = A \psi(t) + B d(t),
\]

where time \( t \) is a non-negative integer and \( A, B \) are constant matrices. The derivation for continuous-time systems is standard (Kwakernaak & Sivan 1972, Chapter 1.11) but is more involved and less intuitive. Let \( X(t) := E(\psi(t)\psi^*(t)) \) be the covariance matrix of the state at time \( t \), where \( E \) is the expectation operator and \( \psi^*(t) \) is the complex-conjugate transpose of the vector \( \psi(t) \). Then, Equation [DT] can be used to write,

\[
X(t+1) = E \left( (A\psi(t) + Bd(t))(\psi^*(t)A^* + d^*(t)B^*) \right)
\]

\[
= A E(\psi(t)\psi^*(t))A^* + B E(d(t)\psi^*(t))A^* + A E(\psi(t)d^*(t))B^* + B E(d(t)d^*(t))B^*.
\]

If the stochastic input is white-in-time with the covariance matrix \( W \), i.e., \( E(d(t)d^*(\tau)) = W\delta(t-\tau) \), where \( \delta \) is the Kronecker delta, the cross terms in Equation [5] disappear and we obtain the Lyapunov equation,

\[
X(t+1) = AX(t)A^* + BWB^*, \quad X(0) = X_0.
\]

If the matrices \( A \) and \( B \) in Equation [DT] as well as the matrices \( W \) and \( X_0 \) are known, this deterministic equation can be propagated forward in time to obtain the covariance matrix \( X(t) \). Even though the above derivation holds irrespective of stability properties of System [DT] the steady-state limit, \( X := \lim_{t \to \infty} X(t) \), only exists for stable systems. In this case, Equation [5] converges to the algebraic Lyapunov equation,

\[
AXA^* = X = -BWB^*,
\]

which is linear in \( X \) and it is typically used to compute the stationary covariance matrix \( X \) for given \( A, B, \) and \( W \). For colored in-time stochastic inputs \( d(t) \), the cross terms in Equation [5] are non-zero and introducing the matrix \( H(t) := A E(\psi(t)d^*(t)) + \frac{1}{2} B E(d(t)d^*(t)) \) in Equation [5] allows us to write it as,

\[
X(t+1) = AX(t)A^* + BH^* + H(t)B^*, \quad X(0) = X_0.
\]

For stable System [DT] Equation [8] converges asymptotically to the algebraic Lyapunov-like equation,

\[
AXA^* = X = -(BH^* + HB^*),
\]

where \( H := \lim_{t \to \infty} H(t) \). For continuous-time systems, Equation [6] takes the form of the differential Lyapunov equation [DL] which, for stable systems, converges to the algebraic Lyapunov equation [AL]. While Equations [5, 6, and 8] also hold for systems in which the matrices \( A(t) \) and \( B(t) \) depend on time, their steady-state limits may not be well-defined. Finally, for infinite-dimensional systems, the complex-conjugate transpose of a matrix, e.g., \( A^* \), should be replaced with an adjoint of an operator, e.g., \( A^\dagger \).
Equation 10 can be represented via,
\[
\frac{d\psi_k(t)}{dt} = (A_k - B_k K_k) \psi_k(t) + B_k w_k(t),
\]
and the algebraic Lyapunov Equation 11 can be used to verify that the stationary two-point correlation operator of \(\psi_k(t)\) is indeed given by \(X_k\). Thus, the impact of a colored-in-time input can be interpreted as a state-feedback modification of the operator \(A_k\) in Equation 7.

For a stable stochastically-forced System 1 algebraic Relation 11 identifies admissible steady-state covariance operators. This fundamental relation was recently utilized for low-complexity stochastic dynamical modeling of turbulent flows (Zare et al. 2017b,a, 2020).

**Time-periodic systems.** The response of a linear time-periodic System 1 to a stationary stochastic input is a cyclo-stationary process (Gardner 1990): the covariance operator of the state is \(2\pi/\omega_1\) periodic, i.e., \(X_k(t) := \mathbb{E}(\psi_k(t) \otimes \psi_k(t)) = \sum_n X_{k,n} e^{i n \omega_1 t}\), with \(X_{k,n} = X_{k,n}\), and the effect of the stationary input, over one period \(T := 2\pi/\omega_1\), is determined by \((1/T) \int_0^T X_k(t) \, dt = X_{k,0}\). If the stochastic input \(d_k(t)\) is white-in-time with the spatial covariance \(W_k\), the harmonic Lyapunov equation,
\[
(A_k - \mathbb{E}(\delta(0))) X_k + X_k (A_k - \mathbb{E}(\delta(0)))^T = -B_k W_k B_k^T,
\]
HLE.

can be used to compute the Fourier series coefficients \(X_{k,n}\) of \(X_k(t)\). Here, \(A_k, B_k\), and \(\mathbb{E}\) are defined in the sidebar FREQUENCY RESPONSE OPERATOR, \(W_k\) is the block-diagonal operator with \(W_k\) on the main diagonal, and \(X_k\) is the self-adjoint block-Toeplitz operator whose elements are determined by \(X_{k,n}\) (Jovanović 2008, Jovanović & Fardad 2008).

3. UNCOVERING MECHANISMS IN WALL-BOUNDED SHEAR FLOWS

We next illustrate how the input-output approach provides insights into the physics of transitional and turbulent wall-bounded shear flows of Newtonian and viscoelastic fluids. In addition to offering a computational framework that quantifies impact of modeling imperfections on relevant flow quantities, a control-theoretic viewpoint also reveals influence of dimensionless groups on amplification of deterministic as well as stochastic disturbances and uncovers mechanisms that may initiate bypass transition. In Section 3.1 we highlight how streamwise streaks, oblique waves, and Orr-Sommerfeld modes are identified as input-output resonances of the operator that maps forcing fluctuations to different velocity components in Newtonian fluids. In Section 3.2 we demonstrate how a control-theoretic approach uncovers a viscoelastic analogue of the familiar inertial lift-up mechanism, thereby identifying mechanisms that may trigger transition to elastic turbulence in rectilinear flows of viscoelastic fluids. Finally, in Section 3.3 we offer a brief overview of the merits and the effectiveness of the input-output analysis in turbulent channel and pipe flows of Newtonian fluids.

3.1. Bypass transition in channel flows of Newtonian fluids

For Poiseuille flow with \(Re = 2000\), Figure 4 shows that the streamwise constant flow structures with \(k_x = 1.78\) are much more energetic than the oblique waves (\(k_x = k_z = 1\) and the Orr-Sommerfeld modes (\(k_x = 1, k_z = 0\)). We next illustrate how the tools of Section 2.3 offer insights into the physics of transitional flows while avoiding need for stochastic simulations. Figure 5 displays the joint impact of forcing fluctuations in all three spatial directions on the individual velocity components. For a channel-wide forcing \(d_k(t)\),
**POWER SPECTRAL DENSITY AND ENERGY AMPLIFICATION**

The power spectral density quantifies the energy of the output $\xi_k(t)$ of stochastically-forced System 1 across the wavenumber $k$ and temporal frequency $\omega$,

$$\Pi_k(\omega) := \text{trace} \left( S_k(i\omega) \right) = \text{trace} \left( T_k(i\omega) \Omega_k(i\omega) T_k^*(i\omega) \right),$$

where $T_k(i\omega)$ is the frequency response and $\Omega_k(i\omega)$ is the spectral density of $d_k(t)$. At any $k$, the temporal-average of $\Pi_k(\omega)$ determines the energy (variance) amplification of harmonic (in homogeneous spatial directions) stochastic (in inhomogeneous directions and time) disturbances to the linearized NS equations,

$$E_k := \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi_k(\omega) \, d\omega.$$ 

This quantity is also known as the *ensemble-average energy density* of the statistical steady-state, and it is hereafter referred to as the (steady-state) *energy amplification* (or energy density). For white-in-time inputs $d_k(t)$ with $\Omega_k(i\omega) = W_k$, the solution to the algebraic Lyapunov equation EA can be used to compute $E_k$,

$$E_k = \text{trace} \left( C_k X_k C_k^\dagger \right),$$

thereby avoiding integration over $\omega$. When the input is uncorrelated in inhomogeneous spatial directions with $W_k = I$, the sum of squares of the singular values of $T_k(i\omega)$ gives the power spectral density, i.e., the Hilbert-Schmidt norm of $T_k(i\omega)$. In this case, $E_k$ determines the $H_2$ norm of System 1 and Parseval’s identity yields,

$$E_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^{\infty} \sigma_{k,j}^2(\omega) \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left( T_k(i\omega) T_k^*(i\omega) \right) \, d\omega = \int_{-\infty}^{\infty} \text{trace} \left( T_k(t) T_k^*(t) \right) \, dt.$$ 

Thus, in addition to quantifying the steady-state variance of System 1 subject to spatially and temporally uncorrelated stochastic inputs, the $H_2$ norm also determines the $L_2$-norm of the impulse response and the same control-theoretic quantity enjoys both stochastic and deterministic interpretations.

**Comparison of $H_2$ and $H_\infty$ norms.** For flows without homogeneous directions, these two quantities compress the dynamics into a single positive number; otherwise, they are parameterized by the wavenumber $k$. Section 2.2 offers interpretations of the $H_\infty$ norm and this sidebar discusses the $H_2$ norm. Herein, we highlight how these measures of input-output amplification of System 1 compress information in inhomogeneous directions and in time; while the $H_\infty$ norm maximizes over both spatial profiles and frequency by computing the temporal supremum of $\sigma_{\text{max}}(T_k(i\omega))$, the $H_2$ norm quantifies the aggregate effect of inputs by integrating the sum of squares of the singular values of $T_k(i\omega)$ over $\omega$.

---

we utilize Equation EA to evaluate the impact of the wavenumbers $k_z$ and $k_x$ on the steady-state variance of $u$, $v$, and $w$. The streamwise velocity component $u$ contains most energy and the strongest amplification occurs in the dark red region that corresponds to small values of $k_z$ and $O(1)$ values of $k_x$. The oblique modes (i.e., the flow structures with $O(1)$ values of $k_x$ and $k_z$) emerge as input-output resonances in the response of the spanwise velocity $w$ and they are significantly less amplified than the streamwise elongated flow structures with $k_x \approx 0$. On the other hand, the least-stable Orr-Sommerfeld mode, which is the dominant source of amplification for the wall-normal velocity $v$, creates only a local peak around $(k_x, k_z) \approx 1, 1 =
Figure 5
Energy amplification of streamwise (left), wall-normal (middle), and spanwise (right) velocity fluctuations for the linearized NS equations subject to channel-wide stochastic forcing in Poiseuille flow with \( Re = 2000 \). The largest value in each plot is marked by a black dot and a logarithmic scaling with the same color map is employed. The streamwise velocity contains most energy and the dominant flow structures are given by the streamwise elongated spanwise periodic streaks.

0) in the response of \( u \). Thus, the flow structures that are deemed important in classical hydrodynamic stability play a marginal role in amplification of stochastic disturbances. This identifies shortcomings of modal stability theory, highlights the utility of componentwise input-output analysis (Jovanović & Bamieh 2005), and demonstrates that significant insight can be gained by examining linearized dynamics in the presence of modeling imperfections (in this case, additive stochastic disturbances).

### 3.1.1. Streamwise constant model: lift-up mechanism

In addition to computational advantages, a control-theoretic viewpoint also uncovers mechanisms for subcritical transition and quantifies impact of the Reynolds number on amplification of deterministic as well as stochastic disturbances (Jovanović 2004, Jovanović & Bamieh 2005). By considering how the disturbances propagate through the linearized dynamics, important insight can be gained without any computations. Since the streamwise constant fluctuations experience the largest amplification (see Figure 5), we examine System 1 for \( k := (k_x,k_z) = (0,k_z) \),

\[
\begin{align*}
\frac{d}{dt} & \begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{Re} A_{os} & 0 \\ \frac{1}{Re} A_{cp1} & \frac{1}{Re} A_{sq} \end{bmatrix} \begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} 0 & B_2 & B_3 \\ B_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \end{bmatrix}, \\
\begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix} &= \begin{bmatrix} 0 & C_u \\ C_v & 0 \\ C_w & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ \eta(t) \end{bmatrix},
\end{align*}
\]

where we suppress the dependence on the spanwise wavenumber \( k_z \). Here, \( v \) and \( \eta \) denote the wall-normal velocity and vorticity fluctuations, whereas \( (d_1, d_2, d_3) \) and \( (u, v, w) \) are the forcing and velocity fluctuations in \( (x, y, z) \). The Orr-Sommerfeld, Squire, and Coupling operators are given by \( A_{os} := \Delta^{-1} \Delta^2 \), \( A_{sq} := \Delta \), and \( A_{cp1} := -ik_z U'(y) \), where \( \Delta = \partial_{yy} - k_z^2 I \) is a Laplacian with homogeneous Dirichlet boundary conditions, \( \Delta^{-1} \) is the inverse of the Laplacian, \( \Delta^2 = \partial_{yyy} - 2k_z^2 \partial_{yy} + k_z^4 I \) with homogeneous Dirichlet and Neumann boundary conditions, and \( U'(y) = dU(y)/dy \). We refer the reader to Jovanović & Bamieh (2005) Section 4 for a definition of the input and output operators \( B \) and \( C \).

As described in the sidebar BLOCK DIAGRAMS, this control-theoretic tool decomposes complex systems into essential pieces, abstract unnecessary details, and highlight the flow of information. A graphical representation of the frequency response operator in Figure 6(a) illustrates that the wall-normal and spanwise forcing fluctuations \( (d_2, d_3) \) produce \( O(Re) \)

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**Normal operator:** an operator is normal if it commutes with its adjoint. A normal operator is unitarily diagonalizable (i.e., it has a complete set of orthogonal eigenfunctions).
fluctuations in \( v \) and \( w \). Although these are dissipated by viscosity, the resulting spanwise variations in \( v \), \( ikzv \), tilt the spanwise vorticity of the laminar base flow, \(-U'(y)\), in the wall-normal direction \( y \), thereby triggering \( O(Re^2) \) fluctuations in \( \eta \) and, consequently, in \( u = \eta/(ikz) \). This lift-up mechanism \cite{Landahl1975} is a dominant source of amplification in wall-bounded shear flows of Newtonian fluids. The operator \( A_{\phi t} \) acts as a source in the wall-normal vorticity equation and it accounts for vortex tilting which arises from linearization of the convective terms in the NS equations. Since \( A_{\phi t} \) and \( A_{\alpha t} \) are self-adjoint, in the absence of vortex tilting the dynamics are characterized by viscous dissipation.

### 3.2. Early stages of transition to elastic turbulence: viscoelastic lift-up

In complex fluids and complex flows, it is even more important to explicitly account for modeling imperfections by quantifying their influence on transient and asymptotic dynamics. Herein, we illustrate how input-output analysis discovers mechanisms that may initiate bypass transition in channel flows of viscoelastic fluids in the absence of inertia. Transition in fluids that contain polymer chains can impact polymer processing and enhance micro-fluidic mixing. In contrast to Newtonian fluids, viscoelastic liquids can deviate from laminar profiles even when inertia is negligible \cite{Groisman2000} and, in curvilinear flows, a purely elastic instability triggers transition \cite{Larson1990}. In low inertial regimes, rectilinear flows are asymptotically stable but the dynamics associated with polymer stress fluctuations can still induce complex responses \cite{Qin2019a}. Since no single constitutive equation fully describes the range of phenomena in viscoelastic fluids, it is important to understand how modeling imperfections may adversely affect their dynamics.

Newtonian fluids are characterized by a static-in-time linear relation between stresses and velocity gradients. In viscoelastic fluids, constitutive equations determine the influence of velocity gradients on the dynamics of stress tensor. For dilute polymer solutions, polymer molecules are treated as springs that connect spherical beads \cite{Bird1987}; the Oldroyd-B (infinitely extensible linear spring) and the FENE-type (finitely extensible nonlinear elastic) models are most commonly used. In the absence of inertia we can set \( Re = 0 \) and the Weissenberg number, \( We = \lambda \bar{u}/h \), and the viscosity ratio, \( \beta = \mu_s/(\mu_s + \mu_p) \), characterize channel flows of Oldroyd-B fluids. The Weissenberg number quantifies the ratio between the elastic and viscous forces and it is given by the product of the polymer relaxation time \( \lambda \) and the velocity gradient \( \bar{u}/h \). The steady solution determines the laminar base flow \((\bar{u}, \bar{\tau})\), where \( \bar{u} = (U(y), 0, 0) \), \( U(y) = 1 - y^2 \) in pressure-driven Poiseuille flow, \( U(y) = y \) in shear-driven Couette flow, and the non-zero components of the base polymer stress tensor \( \bar{\tau} \) are \( \bar{\tau}_{11} = 2We(U'(y))^2 \) and \( \bar{\tau}_{12} = \bar{\tau}_{21} = U'(y) \). Equations for infinitesimal velocity, pressure, and stress fluctuations are obtained by linearization around \((\bar{u}, \bar{\tau})\).

Hoda et al. \cite{Hoda2008, Hoda2009} were the first to investigate nonmodal amplification of disturbances in channel flows of viscoelastic fluids and demonstrate high sensitivity of the laminar flow in both inertia- and elasticity-dominated regimes. Jovanović & Kumar \cite{Jovanovic2010} showed that velocity and stress fluctuations experience significant transient growth even in the absence of inertia. Jovanović & Kumar \cite{Jovanovic2011} identified a new slow-fast decomposition of the governing equations and used singular-perturbation techniques to analytically establish unfavorable scaling of the energy amplification with the Weissenberg number in weakly-inertial flows. Lieu et al. \cite{Lieu2013} quantified the role of finite extensibility of polymers on the worst-case amplification of disturbances in FENE-type models and Hariharan et al. \cite{Hariharan2018} studied amplification of localized body forces. The combined effects of inertia and elasticity on streak evolution was examined in Page & Zaki \cite{Page2014}, Agarwal et al. \cite{Agarwal2014}.
For streamwise-constant flows of Oldroyd-B fluids the block diagram in Figure 6(b) reflects the structure of the frequency response operator that maps disturbances to the momentum equation (inputs) to the velocity fluctuations (outputs) and eliminates all unnecessary variables. Apart from the operator $A_{cp2} := ik_z(U'(y)Δ + 2U''(y)∂y)$, which accounts for stretching of polymer stress fluctuations by a base shear, all other operators are the same as in Newtonian fluids; see Section 3.1.1. The block diagrams reveal striking structural similarity between streamwise-constant inertial flows of Newtonian fluids and inertialess flows of viscoelastic fluids. In the absence of base shear $U'(y)$ and spanwise variations in fluctuations, the responses of viscoelastic fluids are governed by viscous dissipation and all velocity components are $\text{We}$-independent. However, in contrast to Newtonian fluids, spanwise variations in fluctuations and their interactions with $U'(y)$ provide a source in the vorticity equation even in the absence of inertia. In particular, the influence of $d_2$ and $d_3$ on $u$ can be understood by analyzing the wall-normal vorticity equation (Jovanović & Kumar 2011),

$$\beta \Delta \eta(t) = -\Delta \eta(t) - (1 - \beta) We(U'(y)Δ + 2U''(y)∂y) ik_z \vartheta(t),$$

where $\vartheta$ denotes a low-pass version of the wall-normal velocity $v$, $\hat{\vartheta} := \hat{v}/(i\omega + 1)$. The source term arises from stretching of polymer stress fluctuations by a base shear and it introduces a lift-up of fluctuations in a similar way as vortex tilting in inertia-dominated flows of Newtonian fluids. Thus, the wall-normal and spanwise inputs give rise to an energy transfer from the base flow to fluctuations and generate streamwise velocity fluctuations that are proportional to the Weissenberg number. Responses from all other inputs to all other velocities are $\text{We}$-independent and they are governed by viscous dissipation. Jovanović & Kumar (2011) also demonstrated that $d_2$ and $d_3$ induce a quadratic scaling with the Weissenberg number of the streamwise component of the polymer stress tensor, $\tau_{11}$.

**Summary.** Elementary control-theoretic analysis identifies key physical mechanisms and demonstrates that the wall-normal and spanwise body forces have the largest impact on the streamwise velocity fluctuation in inertia-dominated channel flows of Newtonian fluids and elasticity-dominated flows of viscoelastic fluids. These conclusions are derived without any computations by examining the frequency responses of streamwise constant fluctuations and showing that $d_2$ and $d_3$ induce a quadratic scaling of $u$ with the Reynolds number (in Newtonian fluids) and a linear scaling of $u$ with the Weissenberg number (in inertialess Oldroyd-B fluids). At $k_x = 0$, $d_1$ does not influence $v$ and $w$ and the mappings from all other forcing to all other velocity components are proportional to the Reynolds number (in Newtonian fluids) and are $\text{We}$-independent (in viscoelastic fluids); see Jovanović & Bamieh (2005, Section 4) and Jovanović & Kumar (2011) for additional details. In spite of these structural similarities, amplification in Newtonian and viscoelastic fluids originates from different physical mechanisms; vortex tilting and polymer stretching, respectively.

**3.3. Turbulent channel and pipe flows of Newtonian fluids**

Lee et al. (1990) used DNS of homogeneous turbulence to demonstrate that the linear amplification of eddies that interact with large mean shear induces streamwise streaks even in the absence of a solid boundary. This study also employed linear rapid distortion theory (Pope 2000) to predict the lack of isotropy and the structure of turbulence at high shear rate. Furthermore, Kim & Lim (2000) used DNS of a turbulent channel flow to show decay of near-wall turbulence in the absence of the linear vortex tilting term.

In contrast to the laminar base flow, the time-averaged turbulent mean velocity is not
BLOCK DIAGRAMS: A TOOL FOR REVEALING STRUCTURE W/O COMPUTATIONS

Block diagrams decompose complex systems into essential pieces, abstract unnecessary details, and highlight the flow of information. This control-theoretic tool reveals structure without any computations and allows to make useful analogies. The circles denote summation of input signals and the boxes represent different parts of the system. Inputs into each box/circle are represented by lines with arrows directed toward the box/circle, and outputs are represented by lines with arrows leading away from the box/circle. The inputs specify the signals affecting subsystems, and the outputs specify the signals of interest or signals affecting other parts of the system. In streamwise-constant channel flows of Newtonian and inertialess viscoelastic fluids, the block diagrams in Figure 6 illustrate influence of disturbances ($d_1, d_2, d_3$) to the momentum equation on the velocity fluctuations ($u, v, w$). The blue boxes represent resolvent operators associated with $A_{os}$ and $A_{sq}$, and the red boxes represent the coupling operators $A_{cp1} = -ik_zU'(y)\Delta + 2U''(y)\partial_y$ and $A_{cp2} = ik_zU'(y)\Delta$. In Newtonian fluids, $\Omega := \omega Re$ is the frequency scaled with the diffusive time scale, $h^2/\nu$, and in viscoelastic fluids, $\omega$ is the frequency scaled with the polymer relaxation time, $\lambda$.

Figure 6

Block diagrams of the frequency response operators that map the forcing to the velocity fluctuations in streamwise-constant channel flows of (a) Newtonian fluids; and (b) inertialess Oldroyd-B fluids. The thick black lines indicate the part of the system responsible for large amplification. In Newtonian fluids amplification originates from vortex tilting, i.e., the operator $A_{cp1}$ in Equation 2D3C and in viscoelastic fluids it originates from polymer stretching, i.e., the operator $A_{cp2}$. In Newtonian fluids, singular values of the frequency responses from $d_l$ to $r$ are proportional to $Re^2$, for $r = u$ and $l = \{2, 3\}$; to $Re$, for $\{r = u, l = 1\}$; $r = \{v, w\}$, $l = \{2, 3\}$; and are equal to zero for $\{r = \{v, w\}, l = 1\}$; in inertialess flows of viscoelastic fluids, they are proportional to $We$, for $\{r = u, l = \{2, 3\}\}$; all other singular values are $We$-independent.
a solution of the NS equations and even the question of what to linearize around can be contentious (Beneddine et al. 2016). Since the linearized NS equations around the turbulent mean flow are stable (Malkus 1956, Reynolds & Tiederman 1967), they are well-suited for input-output analysis. Butler & Farrell (1993) utilized transient growth analysis over a horizon determined by the eddy turnover time to show that the streak spacing of approximately 100 wall units represents the optimal response of the NS equations linearized around the turbulent mean flow. McKeon & Sharma (2010) employed a gain-based decomposition of fluctuations around mean velocity in turbulent pipe flow to characterize energetic structures in terms of their convection speeds and wavelengths. This study highlighted the role of critical layers in wall-normal localization of experimentally identified energetic modes and related the wave speed, \( c := \omega/k \), to the wall-normal localization of the dominant flow structures. Moarref et al. (2013) leveraged the role of wave speed to formally determine three different scalings for the most amplified modes; showed that these scales are consistent with inner, logarithmic, and outer layers in the turbulent mean velocity; and established dependence of the dominant resolvent modes on the spatial coordinates.

Other classes of linearized models have also been utilized to identify the spatio-temporal structure of the most energetic fluctuations in turbulent flows. In particular, the turbulent mean flow can be obtained as the steady-state solution of the NS equations in which molecular viscosity is augmented with turbulent eddy-viscosity (Reynolds & Tiederman 1967, Reynolds & Hussain 1972, del Álamo & Jiménez 2006, Cossu et al. 2009, Pujals et al. 2009, Hwang & Cossu 2010a,b) demonstrated that transient growth and input-output analyses of the resulting linearization qualitatively capture features of turbulent flows.

For a turbulent channel flow with \( Re = 547 \) and \( k_x = 0 \), Figure 7 demonstrates the emergence of channel-wide and near-wall streaks in a stochastically-forced eddy-viscosity-enhanced linearized model. The values of \( k_z \), where the two peaks in the premultiplied energy spectrum \( k_z E_{k_z} \) emerge determine the spanwise length scales of the most energetic response of velocity fluctuations to stochastic forcing (left plot). Streamwise velocity fluctuations that contain the most variance are harmonic in \( z \) and their wall-normal shapes are determined by the principal eigenfunctions of the stationary covariance operator \( V_k \) (middle and right plots). Pairs of counter-rotating streamwise vortices (contour lines) distribute momentum in the \((y,z)\)-plane and promote amplification of high (hot colors) and low (cold colors) speed streamwise streaks. The most energetic flow structures occupy the entire channel width and the second set of strongly amplified fluctuations is determined by near-wall streaks.

4. CONTROL OF TRANSITIONAL AND TURBULENT FLOWS

Flow control by sensor-less means is often inspired by the desire to bring the efficiency of birds and fish to engineering systems. Control of conductive fluids using the Lorentz force, periodic blowing and suction, wall oscillations, and geometry modifications (e.g., riblets, superhydrophobic surfaces, and jet-engine chevrons) are characterized by the absence of sensing capabilities and implementation of control without measurement of the relevant flow quantities or disturbances. Rather, as illustrated in Figure 8(a), the dynamics are impacted by spatio-temporal oscillations through geometry or base velocity modifications. Min et al. (2006) used DNS to show that a blowing and suction in the form of an upstream traveling wave can provide a sustained sub-laminar drag in a fully-developed turbulent channel flow. This paper inspired other researchers to examine fundamental limitations of streamwise traveling waves for control of transitional and turbulent flows. Marusic et al. 18 M. R. Jovanović
Furthermore, simulations and experiments showed that spanwise wall oscillations can reduce skin-friction drag by as much as 45% (Jung et al. 1992, Laadhar et al. 1994, Choi et al. 1998, Choi 2002, Ricco 2004). While these and other studies (e.g., Fransson et al. 2006) demonstrate the potential of sensor-less periodic strategies, until recently a model-based design for transitional and turbulent flows remained elusive.

In Section 4.1, we highlight the utility of the input-output framework in the design of traveling waves for controlling the onset of turbulence while achieving positive net efficiency (Moarref & Jovanović 2010); and, in Section 4.2, we describe how stochastic analysis in conjunction with control-oriented turbulence modeling quantifies the effect of control on turbulent flow dynamics and identifies the optimal period of oscillations for drag reduction (Moarref & Jovanović 2012). Apart from demonstrating the merits of the input-output approach in the design of periodic strategies for controlling laminar and turbulent flows, we also illustrate how to overcome challenges that arise in “secondary receptivity analysis”, i.e., in nonmodal analysis of the dynamics associated with spatially- or time-periodic base flows.

Figure 7
(a) Premultiplied energy spectrum, $k_z E_{k_z}$; and (b,c) dominant flow structures resulting from stochastically-forced eddy-viscosity enhanced linearization around the turbulent mean flow with $Re = 547$ (based on friction velocity) and $k_z = 0$. Color plots display most energetic streamwise velocity fluctuations $u(z, y)$ and contour lines show streamfunction fluctuations with the spanwise wavelength determined by (b) $2\pi/k_{z1}$ (channel-wide streaks); and (c) $2\pi/k_{z2}$ (near-wall streaks).

Figure 8
Block diagrams of (a) a modification to the dynamics introduced by spatio-temporal oscillations which introduce a sensor-less feedback by changing a base flow $U_0(y)$ to a periodic profile; and (b) a simulation-free approach for determining the influence of control on skin-friction drag in turbulent flows. The hollow arrows represent coefficients into the mean-flow and linearized equations. In Moarref & Jovanović 2012, the turbulent mean velocity is updated once.
4.1. Controlling the onset of turbulence by streamwise traveling waves

Let channel flow be subject to a uniform pressure gradient and a zero-net-mass-flux blowing and suction along the walls, \( V(y = \pm 1) = \mp 2\alpha \cos (\omega_z (x - c t)) \); see Figure 1(b). Here, \( \alpha \), \( \omega_z \), and \( c \) denote amplitude, frequency, and speed of the wave that travels in the streamwise direction \( \bar{x} \). Positive/negative values of \( c \) identify downstream/upstream waves, and \( c = 0 \) gives a standing wave. The Galilean transformation, \( x := \bar{x} - ct \), eliminates the time dependence in \( V(\pm 1) \) and the steady-state solution of the NS equations, \( \bar{u} = (U(x, y), V(x, y), 0) \), does not depend on \( t \) in the frame of reference that travels with the wave. The new laminar base flow \( \bar{u} \) is no longer a parabola: it is periodic in \( x \), with frequency \( \omega_z \), and it contains both streamwise and wall-normal components, \( U(x, y) \) and \( V(x, y) \).

4.1.1. Net efficiency of modified base flow. Blowing and suction induces a bulk flux in the direction opposite to the direction in which the wave travels \( \text{[Höpffner & Fukagata 2009]} \). This pumping mechanism occurs even in the absence of the pressure gradient and a weakly-nonlinear analysis explains it. For small amplitude \( \alpha \), \( U(x, y) \) is given by

\[
U(x, y) = U_0(y) + \alpha^2 U_{20}(y) + \alpha (U_{1p}(y) e^{i\omega z x} + U_{1m}(y) e^{-i\omega z x}) + \alpha^2 (U_{2p}(y) e^{i2\omega z x} + U_{2m}(y) e^{-i2\omega z x}) + O(\alpha^3).
\]

In addition to an oscillatory \( O(\alpha) \) correction to \( U_0(y) \) with frequency \( \omega_z \), both the second harmonic \( 2\omega_z \) and the mean flow correction \( U_{20}(y) \) are induced by the quadratic nonlinearity in the NS equations at the level of the order \( \alpha^2 \). For the fixed pressure gradient, the skin-friction drag coefficient of the base flow is inversely proportional to the square of the bulk flux. Since the integral of \( U_{20}(y) \) is positive for the upstream and negative for the downstream waves \( \text{[Höpffner & Fukagata 2009]} \) \( \text{[Moarref & Jovanović 2010]} \), upstream/downstream waves reduce/increase skin-friction drag coefficient relative to the laminar uncontrolled flow.

The net efficiency of wall-actuation is given by the difference of the produced and required powers \( \text{[Quadrio & Ricco 2004]} \). These two quantities, respectively, determine increase/decrease in bulk flux relative to the flow with no control and the control effort exerted at the walls. Compared to laminar uncontrolled flow, any strategy based on blowing and suction reduces net efficiency \( \text{[Bewley 2009]} \) \( \text{[Fukagata et al. 2009]} \). However, if uncontrolled flow becomes turbulent, both upstream and downstream waves of small enough amplitudes can improve net efficiency \( \text{[Moarref & Jovanović 2010]} \) \text{Section 2.4}. \( \text{[Moarref & Jovanović 2010]} \) also demonstrated that, apart from the net efficiency, the dynamics of fluctuations around the modified base flow have to be evaluated when designing the traveling waves.

4.1.2. Dynamics of velocity fluctuations. The laminar base flow induced by the traveling waves is periodic in \( x \), with frequency \( \omega_z \), and the resulting linearization is not translationally-invariant in the streamwise direction. The normal modes in \( z \) are still harmonic, \( e^{ik_z z} \), but in \( x \) they are given by the Bloch waves, which are determined by a product of \( e^{i\theta z} \) and the \( 2\pi/\omega_z \) periodic function in \( x \), \( \bar{d}_k(x, y, t) = \bar{d}_k(x + 2\pi/\omega_z, y, t) \),

\[
d(x, y, z, t) = \bar{d}_k(x, y, t) e^{i(\theta z + k_z z)} = \sum_{n = -\infty}^{\infty} \bar{d}_{k,n}(y, t) e^{i((\theta + n\omega_z) z + k_z z), \quad \theta \in [0, \omega_z),
\]

where \( k := (\theta, k_z) \) and \( \bar{d}_{k,n}(y, t) \) are the coefficients in the Fourier series expansions of \( \bar{d}_k(x, y, t) \). In this case, signals in Equation \( \bar{d}_k(x, y, t) \) are the \( k \)-parameterized bi-infinite column vectors whose components are determined by the corresponding Fourier series coefficients,
e.g., $d_k(t) := \col\{\hat{d}_{k,n}(y,t)\}_{n \in \mathbb{Z}}$, and similarly for $\psi_k(t)$ and $\xi_k(t)$. Thus, for each $k$, $A_k$, $B_k$, and $C_k$ in Equation 1 are bi-infinite matrices whose entries are operators in the wall-normal direction $y$ (Moarref & Jovanović 2010), and the frequency response operator $T_k(i \omega)$ in Equation 1 maps $d_k(i \omega) := \col\{\hat{d}_{k,n}(y,i \omega)\}_{n \in \mathbb{Z}}$ to $\hat{\xi}_k(i \omega) := \col\{\hat{\xi}_{k,n}(y,i \omega)\}_{n \in \mathbb{Z}}$.

Since modal stability does not capture the early stages of transition, Moarref & Jovanović (2010) utilized input-output analysis of a linearization around $(U(x,y),V(x,y),0)$ to quantify the effect of control on amplification of stochastic disturbances and identify waves that reduce receptivity relative to the flow without control. A discretization in $y$ and truncation of bi-infinite matrices in Equation 1 yield a large-scale Lyapunov Equation 11 computing its solution to assess impact of control parameters $(\alpha, \omega_z, c)$, wavenumbers $(\theta, k_z)$, and the Reynolds number $Re$ on the energy amplification is demanding. Motivated by the observation that large values of $\alpha$ introduce high cost of control, Moarref & Jovanović (2010) employed a perturbation analysis to efficiently compute the solution to Equation 11. This approach offers significant advantages relative to the approach based on truncation: impact of small amplitude waves on energy amplification can be assessed via computations that are of the same complexity as computations in the uncontrolled flow.

In particular, for small amplitude waves, the following explicit formula,

$$
\frac{\text{energy amplification with control}}{\text{energy amplification without control}} = 1 + \alpha^2 g_k(\omega_z, c, Re) + O(\alpha^4),
$$

12.

offers insights into the impact of control on energy amplification. For $\alpha \ll 1$, the analysis amounts to examining the dependence of the function $g_k$ in Equation 12 on $k = (\theta, k_z)$, the frequency/speed $\omega_z, c$ of the wave, and the Reynolds number $Re$. Positive (negative) values of $g_k$ identify parameters that increase (decrease) energy amplification. For channel flow with $Re = 2000$ and fixed values of $\omega_z$ and $c$, we use a sign-preserving logarithmic scale to visualize the $k$-dependence of the function $g_k$ in Figure 9. While the downstream waves with $(\omega_z = 2, c = 5)$ reduce amplification for all values of $\theta$ and $k_z$, the upstream waves with $(\omega_z = 0.5, c = -2)$ promote amplification for a broad range of $\theta$ and $k_z$. Thus, in addition to guaranteeing positive net efficiency relative to the uncontrolled flow that becomes turbulent (see Section 4.1.1), the downstream waves also suppress energy of 3D fluctuations. On the other hand, the upstream waves with the parameters that provide favorable skin-friction coefficient of the modified laminar flow (Min et al. 2006) increase amplification of the most energetic modes of the uncontrolled flow. In fact, since, at best, they exhibit receptivity similar to that of the uncontrolled flow (Moarref & Jovanović 2010) and can even induce modal instability of the modified laminar flow (Lee et al. 2008), they are not suitable for controlling the onset of turbulence. In contrast, properly designed downstream waves can substantially reduce production of fluctuations’ kinetic energy (Moarref & Jovanović 2010) and they are an excellent candidate for preventing transition to turbulence.

**DNS verification.** All theoretical predictions resulting from a simulation-free approach of Moarref & Jovanović (2010) were verified by Lieu et al. (2010). Their DNS confirmed that the downstream waves indeed provide an effective means for controlling the onset of turbulence and that the upstream waves promote transition even when the uncontrolled flow stays laminar. This demonstrates considerable predictive power of the input-output framework and suggests that reducing receptivity is a viable approach to controlling transition.
4.2. Turbulent drag reduction by spanwise wall oscillations

Spanwise wall oscillations can reduce turbulent drag by as much as 45%. This observation was made in both simulations and experiments and theoretical studies (Dhanak & Si 1999, Bandyopadhyay 2006, Ricco & Quadrio 2008) focused on explaining the effectiveness of this sensor-less strategy. Herein, we describe how input-output analysis in conjunction with a control-oriented turbulence modeling identifies the optimal period of oscillations for turbulence suppression in channel flow; see Moarref & Jovanović (2012) for details.

Modified mean flow. In pressure-driven channel flow, the steady-state solution of the NS equations in which the molecular viscosity is augmented with the turbulent viscosity \( \nu_T \) is determined by the Reynolds-Tiederman profile, \( U_0(y) \). If the flow is also subject to \( W(y = \pm 1, t) = 2\alpha \sin (\omega t) \), the steady-state solution is given by \( (U_0(y), 0, W_0(y, t)) = \alpha (W_0(y) e^{i\omega t} + W^*(y) e^{-i\omega t}) \).

Here, \( U_0(y) \) approximates the mean streamwise velocity of the uncontrolled turbulent flow and the wall oscillations induce the time-periodic spanwise component \( W_0(y, t) \) under the assumption that the turbulent viscosity is not modified by control. If this were the case, the oscillations would have no impact on \( U_0 \), which is at odds with simulations/experiments. In contrast, the estimates of the required power exerted by wall oscillations resulting from the use of \( W_0 \) closely match the DNS results of Quadrio & Ricco (2004) over a broad range of oscillation periods (Moarref & Jovanović 2012, Section 2.2).

Turbulence modeling. The inability of the above approach to predict drag reduction arises from the fact that the wall oscillations change the turbulent viscosity of the flow with no control. Moarref & Jovanović (2012) pioneered a method based on the stochastically-forced eddy-viscosity-enhanced NS equations linearized around \( (U_0(y), 0, W_0(y, t)) \) to capture the influence of control on turbulent viscosity. The approach utilizes the Boussinesq hypothesis but, in contrast to standard practice, the turbulent kinetic energy \( k \) and its rate of dissipation \( \epsilon \) are computed using the second-order statistics of velocity fluctuations in the linearized model. Using analogy with homogenous isotropic turbulence, Moarref & Jovanović (2012, Section 3.1) selected spatial correlations of white-in-time forcing to provide equivalence between the 2D energy spectra of the uncontrolled turbulent flow and the flow governed by the stochastically-forced linearization around \( (U_0(y), 0, 0) \). This approach was the first to utilize available DNS data (del Álamo & Jiménez 2003, del Álamo et al. 2004) of the uncontrolled turbulent flow to guide control-oriented modeling for design purposes; it takes advantage of the turbulent viscosity and the energy spectrum of the uncontrolled flow and determines the effect of control on the turbulent flow using a model-based approach.

![Figure 9](image-url)

The second order correction to the energy amplification in Equation 12 visualized using a sign-preserving logarithmic scale, \( \text{sign}(g_k) \log_{10}(1 + |g_k|) \), in channel flow with \( Re = 2000 \) (based on the centerline velocity of the parabolic laminar profile and the channel half-height). While the downstream waves with selected parameters reduce amplification for all values of \( \theta \) and \( k_z \), the upstream waves promote amplification for a broad range of \( \theta \) and \( k_z \).
Dynamics of velocity fluctuations. Linearization around \((U_0(y), 0, W_0(y, t))\) yields a time-periodic model with \(A_k(t) = A_k,0 + \alpha (A_{k,-1} e^{-\omega t} + A_{k,1} e^{\omega t})\), and the solution to Equation [HLE] provides two-point correlations. For small amplitude oscillations, [Moarref & Jovanović 2012] utilized perturbation analysis to efficiently solve this equation and identify the oscillation periods that yield the largest drag reduction and net efficiency. This approach quantifies the influence of velocity fluctuations on the turbulent viscosity in the flow with control, \(\nu_T(y) = \nu_{T0}(y) + \alpha^2 \nu_{T2}(y) + O(\alpha^4)\), where \(\nu_{T0}(y)\) is the turbulent viscosity of the uncontrolled flow and \(\nu_{T2}(y)\) is determined by the second-order corrections (in \(\alpha\)) to the kinetic energy \(k_2(y)\) and its rate of dissipation \(\epsilon_2(y)\). These quantities are obtained by averaging the second-order statistics resulting from a stochastically-forced linearization around \((U_0(y), 0, W_0(y, t))\) over the wall-parallel directions and one period of oscillations.

The solution to Equation [HLE] and the above expression for \(\nu_T\) are used to assess the influence of small amplitude oscillations on the dynamics of velocity fluctuations and to identify the optimal period of oscillations for drag reduction. For the controlled flow with constant bulk flux and the friction Reynolds number \(Re = 186\), solid curve in Figure 10 shows the second-order correction to the skin-friction coefficient \%\(C_{f2}(T^+)\), normalized by its largest value, and symbols display normalized DNS results at \(Re = 200\) [Quadrio & Ricco 2004]. A close agreement between a theoretical prediction for the optimal period resulting from input-output analysis \((T^+ = 102.5)\) and DNS results \((T^+ \approx 100)\) is observed. Middle and right plots in Figure 10 show the premultiplied 2D energy spectrum of the uncontrolled flow, \(k_z k_z E_{k,0}\), and the second-order correction, \(k_z k_z E_{k,2}\), triggered by small amplitude oscillations with the optimal period \(T^+ = 102.5\). The most energetic modes of the uncontrolled flow occur at \((k_z \approx 2.5, k_z \approx 6.5)\). The red region in the right plot shows that the wall oscillations increase amplification of the modes with small streamwise wavelengths, and the blue region indicates suppression of energy of large streamwise wavelengths. This observation agrees with the study of the impact of wall oscillations on free-stream vortices in a pre-transitional boundary layer [Ricco 2011]. [Moarref & Jovanović 2012] also showed that the optimal wall-oscillations minimize the turbulent viscosity near the interface of the buffer and log-law layers and that oscillations are less effective at higher Reynolds numbers.
The NS equations can be viewed as a feedback interconnection of the linearized dynamics with the nonlinear term; (b) Stochastically-driven linearized NS equations with low-rank state-feedback modification. At the level of second-order statistics, the two representations can be made equivalent by proper selection of $B_k$ and $K_k$; cf. Equation 11.

Summary. Traveling waves and wall oscillations introduce a sensor-less feedback via periodic modifications to the dynamics (see Figure 8(a)) by changing a base flow $U_0(y)$ to a spatially- or time-periodic profile. Depending on the actuation waveform and the parameters, the properties can be improved or made worse relative to the flow without control. In contrast to a standard approach, which employs DNS and experiments to assess sensor-less periodic strategies, Moarref & Jovanović (2010, 2012) developed a model-based framework for determining the influence of control on transitional and turbulent flows. These references demonstrate the critical importance of the dynamics associated with the modified base flows for the design of effective strategies for controlling the onset of turbulence and drag reduction. The developed simulation-free method enables computationally-efficient design by merging receptivity analysis and control-oriented turbulence modeling with techniques from control theory and its utility goes beyond the case studies presented here. Recently, input-output approach was used to quantify the effect of riblets on kinetic energy and turbulent drag in channel flow (Chavarin & Luhar 2020, Ran et al. 2020) and it is expected to enable optimal design of periodic strategies for control of transitional and turbulent flows. Input-output framework is also at the heart of the optimal and robust $H_2$ and $H_\infty$ feedback control strategies (Zhou et al. 1996) and it has recently found use in the model-based design of opposition control (Luhar et al. 2014, Toedtli et al. 2019).

5. PHYSICS-AWARE DATA-DRIVEN TURBULENCE MODELING

The advances in high-performance computing and measurement techniques provide abundance of data for a broad range of flows. Thus, turbulence modeling can be formulated as an inverse problem where the objective is to identify a parsimonious model that explains available and generalizes to unavailable data. Techniques from machine learning and statistical inference were recently employed to reduce uncertainty and improve predictive power of models based on the Reynolds-Averaged NS (RANS) equations (Duraisamy et al. 2019). Large data sets can also be exploited to develop reduced dynamical representations (Rowley & Dawson 2017) but an exclusive reliance on data makes such models agnostic to physical constraints and can yield subpar performance in regimes that are not contained in the training data set. Moreover, sensing and actuation can significantly change the identified model, thereby making its use for flow control challenging (Noack et al. 2011, Tadmor & Noack 2011). A compelling alternative for model-based optimization and control is to leverage data in conjunction with a prior model that arises from first principles.

As demonstrated in Section 3, the linearized NS equations in the presence of stochastic excitation can be used to qualitatively predict structural features of transitional and turbulent shear flows. In most prior studies excitations were restricted to white-in-time stochastic processes but this assumption is often too narrow to fully capture observed statistics of turbulent shear flows.
flows (Jovanović & Georgiou 2010). To overcome these limitations, Zare et al. (2017b,a) developed a framework to allow for colored-in-time inputs to the linearized NS equations.

We next briefly summarize how strategic use of data enhances predictive power of the linearized NS equations in order to capture second-order statistics of turbulent flows (Zare et al. 2017b,a, 2020). Since machine learning tools are physics-agnostic, the power spectrum of stochastic forcing is identified by merging tools from control theory and convex optimization. The resulting stochastic model, given by Equation 11, accounts for neglected nonlinear interactions via a low-rank perturbation to the original dynamics; see Figure 11.

5.1. Completion of partially available channel flow statistics

Herein, we examine linearization around mean velocity in turbulent channel flow and highlight the utility of the framework developed in Zare et al. (2017b,a). A pseudo-spectral method (Weideman & Reddy 2000) yields a finite-dimensional approximation of the operators in $y$ and a change of variables (Zare et al. 2017b Appendix A) leads to an evolution model in which the kinetic energy at any $k$ is determined by the Euclidean norm of the state vector $\psi_k$. For given $(A_k, B_k)$ and input statistics $(W_k$ or $H_k)$, algebraic Relation ALc and $X_k$ can be used to compute the stationary covariance matrix $X_k$ of the state $\psi_k$ in System 1. However, in turbulence modeling, the converse question arises: starting from Model 1 and the covariance matrix $X_k$ (resulting from DNS or experiments), can we identify the power spectrum of the stochastic input $d_k(t)$ that generates such statistics? For the NS equations linearized around turbulent mean velocity with white-in-time stochastic forcing, the answer to this question is negative (Zare et al. 2017b Figure 6). This limitation can be overcome by allowing for colored-in-time stochastic inputs to the linearized Model 1.

The positive-definite matrix $X_k$ is the stationary covariance of the state $\psi_k(t)$ of linear time-invariant System 1 with controllable pair $(A_k, B_k)$ if and only if (Georgiou 2002a,b)

$$\text{rank} \left[ \begin{array}{cc} A_k X_k + X_k A_k^* B_k & B_k \\ B_k^* & 0 \end{array} \right] = \text{rank} \left[ \begin{array}{c} 0 & B_k \\ B_k^* & 0 \end{array} \right].$$

This fundamental condition guarantees that, for given $A_k$, $B_k$, and $X_k$, Equation ALc can be solved for $H_k$. It also implies that any $X_k = X_k^* > 0$ is admissible as a stationary covariance of $\psi_k(t)$ in Equation 1 if the input matrix $B_k$ is full rank. In particular, for $B_k = I$, Equation ALc is satisfied with $H_k = -A_k X_k$ and stochastically-forced System 11 which for this choice of $H_k$ and $W_k = I$ simplifies to $\dot{\psi}_k(t) = -(1/2) X_k^{-1} \psi_k(t) + w_k(t)$, can be used to generate $X_k$. This implies that a colored-in-time input which excites all degrees of freedom in Equation 1 can completely cancel relevant physics contained in $A_k$. Thus, in data-driven turbulence modeling, it is critically important to restrict the rank of the matrix $B_k$ which specifies the number of inputs to the linearized NS equations.

To address this challenge, Zare et al. (2017b,a) formulated and solved the problem of completing a subset of entries $V^{\text{dns}}_{k,ij}$ of the stationary covariance matrix $V_{k,ij}^{\text{dns}}$ of velocity fluctuations using stochastically-forced linearization around the turbulent mean velocity. The approach utilizes algebraic Relation ALc with $Z_k := B_k H_k + H_k B_k$ and a maximum entropy formalism along with a convex surrogate for rank minimization to limit the number of inputs to the linearized model and identify spectral content of the colored-in-time forcing, minimizing

$$-\log \det (X_k) + \gamma \sum_i \sigma_i (Z_k)$$

subject to $A_k X_k + X_k A_k^* + Z_k = 0$ (physics CC) and $$(C_k X_k C_k^*)_ij = V_{k,ij}^{\text{dns}}, \ (i,j) \in \mathcal{I}.$$ (available data)
The Hermitian matrices $X_k \succ 0$ and $Z_k$ are the optimization variables, whereas the matrices $(A_k, C_k)$, the available entries $V_{k,i,j}^{\text{dns}}$ of $V_k^{\text{dns}}$ for a selection of indices $(i, j) \in \mathcal{I}$, and $\gamma > 0$ are known problem parameters. The first constraint in CC comes from physics and it imposes the requirement that second-order statistics are consistent with linearization around turbulent mean velocity; and the second constraint requires that the available elements of the matrix $V_k^{\text{dns}}$ are exactly reproduced by the linearized model. The logarithmic barrier function is introduced to ensure positive-definiteness of $X_k$ (Boyd & Vandenberghe 2004) and the sum of singular values of the matrix $Z_k$, which reflects contribution of the stochastic input, is used as a convex proxy to restrict the rank of $Z_k$ (Fazel 2002, Recht et al. 2010).

The convexity of the objective function and the linearity of the constraint set in CC imply the existence of a unique globally optimal solution $(X_k^\star, Z_k^\star)$. This solution reproduces all available entries of the stationary covariance matrix $V_k^{\text{dns}}$ resulting from DNS (or experiments) and completes unavailable second-order statistics via low-complexity stochastic dynamical model given by Equation 11. In particular, the factorization of $Z_k^\star$ can be used to determine $B_k^\star$ and $H_k^\star$, which along with $X_k^\star$ yield a low-rank modification $B_k^\star K_k^\star$ to $A_k$ in Equation 11. This approach provides a model which refines predictive power of the linearized NS equations by employing data while preserving relevant physics of turbulent flows.

Figure 12 shows covariance matrices $V_{k,uu}$ and $V_{k,uv}$ resulting from DNS of turbulent channel flow with $Re = 186$ (left plots) and the solution to optimization problem CC with $\gamma = 300$ (right plots) for $k = (2.5, 7)$. Black lines along the main diagonals mark the one-point correlations (in $y$) that are used as available data in CC and are perfectly matched. Using a Frobenius norm measure, $\|V_k^\star - V_k^{\text{dns}}\|_F/\|V_k^{\text{dns}}\|_F$, approximately 60% of $V_k^{\text{dns}}$ can be recovered by the stationary covariance matrix $V_k^\star = C_k^\star X_k^\star C_k^\star$ of velocity fluctuations resulting from the solution of problem CC (Zare et al. 2017b). The high-quality recovery of two-point correlations is attributed to the Lyapunov-like structural constraint in CC which keeps physics in the mix and enforces consistency between data and the linearized dynamics.

Alternative formulations. Covariance completion problem CC can be cast as an optimal control problem aimed at establishing a trade-off between control energy and the number of feedback couplings that are required to modify $A_k$ in System 11 and achieve consistency with available data (Zare et al. 2019, 2020). Depending on modeling purpose and available data, many alternative turbulence modeling formulations are possible. Jovanović & Bamieh (2001) showed that portions of one-point correlations in $(x, y, z)$, resulting from the integration of DNS-based $V_k^{\text{dns}}(y, y)$ over $k$, can be approximated by the appropriate choice of covariance of white-in-time forcing to the NS equations linearized around turbulent mean velocity. This early success inspired the development of optimization algorithms for approximation of full covariance matrices using stochastic dynamical models (Hœpffner 2005, Lin & Jovanović 2009). For the eddy-viscosity enhanced linearization, Moarref & Jovanović (2012) demonstrated that white-in-time forcing with variance proportional to the turbulent energy spectrum can be used to reproduce the DNS-based energy spectrum of velocity fluctuations. Hwang & Eckhardt (2020) determined the wave-number dependence of the variance of stochastic forcing, which is uncorrelated in $t$ and $y$, that minimizes the difference between the Reynolds shear stresses resulting from the mean and the linearized eddy-viscosity enhanced NS equations. Several recent efforts were aimed at matching individual entries of the spectral density matrix $S_k(i\omega)$ at given frequencies (Beneddine et al. 2016, 2017, Towne et al. 2020) or at capturing the spectral power, $\text{trace}(S_k(i\omega))$. (Morra et al. 2019). Finally, compared to the standard resolvent analysis (Moarref et al. 2014), an optimization-based approach that utilizes a componentwise approach (Rosenberg & McKeon 2019) offers con-
Figure 12
(a, b) Streamwise $V_{k,uu}$ and (c,d) streamwise/wall-normal $V_{k,uv}$ covariance matrices resulting from DNS of turbulent channel flow with $Re = 186$ (left plots); and the solution to optimization problem $\mathbf{CC}$ with $\gamma = 300$ (right plots) for $k = (2.5, 7)$. Black lines along the main diagonals mark the one-point correlations that are used as available data in $\mathbf{CC}$ and are perfectly matched.

Figure 12 illustrates considerable improvement in matching spectra and co-spectra in turbulent channel flow (McMullen et al. 2020). This further exemplifies the power and versatility of the componentwise input-output viewpoint of fluid flows that was introduced in Jovanović & Bamieh (2005).

SUMMARY POINTS

1. The following quote is attributed to Eric Eady: \textit{“It is not the process of linearization that limits insight. It is the nature of the state that we choose to linearize about.”} In addition, this review demonstrates that the tools that we use to study the linearized equations are as important as the base flow that we choose to linearize about.

2. Componentwise input-output analysis uncovers mechanisms for subcritical transition and identifies streamwise streaks, oblique waves, and Orr-Sommerfeld modes as input-output resonances from forcing to velocity components.

3. Input-output analysis discovers a viscoelastic analogue of the familiar inertial lift-up mechanism. This mechanism arises from stretching of polymer stress fluctuations by a base shear and, even in the absence of inertia, it induces significant amplification that can trigger transition to elastic turbulence in rectilinear flows.

4. Input-output analysis quantifies impact of forcing and energy-content of velocity components. It reveals influence of dimensionless groups (e.g., Reynolds and Weis-
5. Input-output viewpoint provides a model-based approach to vibrational flow control, where the dynamics are impacted by zero-mean oscillations. Effective strategies for controlling the onset of turbulence and turbulence suppression can be designed by examining the dynamics of fluctuations around the base flow induced by oscillations.

6. Linearized NS equations with stochastic forcing qualitatively predict structural features of turbulent shear flows and provide sufficient flexibility to account for two-point correlations of fully-developed turbulence via low-complexity models.

7. Input-output framework provides a data-driven refinement of a physics-based model, guarantees statistical consistency, and captures complex dynamics of turbulent flows in a way that is tractable for analysis, optimization, and control design.

8. Tools and ideas from control theory and convex optimization overcome shortcomings of physics-agnostic machine learning algorithms and enable the development of theory and techniques for physics-aware data-driven turbulence modeling.

FUTURE ISSUES

1. Complex fluids and complex flows. Among other emerging applications, input-output analysis is expected to clarify the importance of different physical mechanisms in the presence of surface roughness and free-stream disturbances, and to quantify the impact of modeling uncertainties that arise from chemical reactions and gas surface interactions in hypersonic flows (Candler 2019).

2. Computational complexity. For an evolution model with n degrees of freedom, the tools presented in this review require $O(n^3)$ computations. Such computations are routine for canonical flows, but the large-scale nature of spatially-discretized models in complex geometries induces significant computational overhead.

3. Nonlinear interactions. Precise characterization of the interplay between high flow sensitivity and nonlinearity in order to capture later stages of disturbance development, identify possible routes for transition, and design effective control strategies for the nonlinear NS equations remains a grand challenge.

4. Data-driven techniques. In spite of the apparent promise of machine and reinforcement learning, a number of challenges have to be addressed, including the development of methods that respect physical constraints, generalize to flow regimes that are not accounted for in the available data, and offer convergence, performance, and robustness guarantees on par with model-based approaches to flow control.

5. Feedback control. A host of challenges including estimation using noisy measurements, optimal sensor and actuator placement, efficient computation of optimal and robust controllers, structured and distributed control synthesis, as well as convergence and sample complexity of data-driven reinforcement learning strategies have to be addressed to enable a successful feedback control at high Reynolds numbers.
6. DISCUSSION

Herein, we expand on future issues and provide an overview of outstanding challenges.

**Complex fluids and complex flows.** In addition to parallel flows, input-output analysis was utilized to quantify the influence of deterministic [Sipp & Marquet 2013] and stochastic [Ran et al. 2019] inputs as well as base flow variations [Brandt et al. 2011] on spatially-evolving boundary layers. In high-speed compressible flows, there is a coupling of inertial and thermal effects, and experiments suggest a significant impact of exogenous disturbances on transition [Fedorov 2011]. Traditional receptivity is based on a local spatial analysis [Mick 1989; Bertolotti & Herbert 1991], and is not applicable to most bodies of aerodynamic interest. For hypersonic vehicles with complex geometry or shock interactions with control surfaces, transition is poorly understood, and empirical testing is typically used to characterize their behavior. Linearization around spatially-evolving base flows in the presence of sharp gradients involves multiple inhomogeneous directions, and even modal stability analysis becomes challenging and computationally demanding [Hildebrand et al. 2018; Sidharth et al. 2018]. Recently, Dwivedi et al. (2019) employed a global input-output analysis to quantify the amplification of exogenous disturbances and explain the appearance of experimentally observed steady reattachment streaks in a hypersonic flow over a compression ramp. For the laminar shock-boundary layer interaction, this study showed that upstream counter-rotating vortices trigger streaks with a preferential spanwise length scale. Input-output analysis is expected to clarify the importance of different physical mechanisms in the presence of surface roughness and free-stream disturbances, and to quantify the impact of modeling uncertainties that arise from chemical reactions and gas surface interactions on hypersonic flows [Candler 2019].

**Computational complexity.** For an evolution model with \( n \) degrees of freedom, the tools presented in this review require \( O(n^3) \) computations. Such computations are routine for canonical flows, but the large-scale nature of spatially-discretized models in complex geometries induces significant computational overhead. Dominant singular values of the state-transition and frequency response operators can be computed iteratively [Schmid 2007] or via randomized techniques [Halko et al. 2011]. Such computations have been used to conduct nonmodal analysis of complex flows [Jean et al. 2016; Dwivedi et al. 2019]. While in general it is challenging to efficiently solve large-scale Lyapunov equations, efficient iterative algorithms (both in terms of memory and computations) exist for systems with a small number of inputs and outputs and sparse dynamic matrices [Benner et al. 2008]. These are expected to bring the utility of stochastic analysis from canonical channels [Jovanovic & Bamieh 2005] and boundary layers [Ran et al. 2019] to flows in complex geometries.

**Nonlinear interactions.** Large amplification of disturbances in conjunction with nonlinear interactions can induce secondary instability of streamwise streaks, their breakdown, and transition [Waleffe 1997]. An alternative self-sustaining mechanism shows that turbulence can be triggered by the streamwise-constant NS equations in feedback with a stochastically-forced streamwise-varying linearization [Farrell & Ioannou 2012; Thomas et al. 2014]. Nonlinear nonmodal stability analysis identifies initial conditions of a given amplitude that maximize energy at a fixed time [Kerswell 2018]. Dissipation inequalities [Ahmadi et al. 2019] and a harmonic balance approach [Rigas et al. 2020] were recently utilized to extend input-output analysis to the nonlinear NS equations, and the theory of integral quadratic constraints [Megretski & Rantzer 1997] was used to study of a phenomenological model of transition [Kalur et al. 2020]. However, it is still an open challenge to precisely characterize the interplay between high flow sensitivity and nonlinearity in order to capture later stages.
of transition routes and design effective control strategies.

**Data-driven techniques.** Machine learning has revolutionized many disciplines, e.g., image recognition and speech processing, and is increasingly used in modeling and decision making based on available data. While the NS equations are often too complex for model-based optimization and control, DNS provides data for reduced-order dynamical modeling (Rowley 2005, Lumley 2007, Schmid 2010, Jovanović et al. 2014, Towne et al. 2018). Capitalizing on the availability of such data, machine and reinforcement learning have recently been used for flow modeling, optimization, and control (Brunton et al. 2020) and this trend will continue. In spite of the apparent promise, a number of challenges have to be addressed, including the development of methods that respect physical constraints, generalize to flow regimes that were not accounted for in the available data sets, and offer convergence, performance, and robustness guarantees on par with model-based approaches to flow control.

**Feedback control** offers a more viable approach than sensor-less control for dealing with uncertainties that impact the operation of engineering flows. The scale and complexity of the problem introduce significant challenges for modeling, sensor and actuator placement, and control design. These necessitate the development of model-based and data-driven techniques. In wall-bounded flows at low Reynolds numbers, model-based feedback control has shown significant promise (Joshi et al. 1997, Bewley & Liu 1998, Högberg et al. 2003a, b Kim & Bewley 2007). Since sensing and actuation are typically restricted to the surface, the flow field needs to be estimated using limited noisy measurements in order to form a control action. Hopfner et al. (2005), Chevalier et al. (2006) demonstrated the importance of statistics of disturbances in the design of estimation gains. Alternatively, the data-refined model, given by Equation 11, that matches statistics of turbulent flows can be readily embedded into a Kalman filter estimation framework. Alongside estimation, challenges associated with the optimal sensor and actuator placement (Chen & Rowley 2011, Zare et al. 2019), efficient computation of optimal and robust controllers (Bewley et al. 2016), structured and distributed control synthesis (Lin et al. 2013), as well as convergence and sample complexity of data-driven reinforcement learning strategies (Mohammadi et al. 2019) have to be addressed to enable a successful feedback control at high Reynolds numbers.

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