DYNAMICAL TORSION IN VIEW OF A DISTINGUISHED CLASS OF DIRAC OPERATORS

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Abstract. In this paper we discuss geometric torsion in terms of a distinguished class of Dirac operators. We demonstrate that from this class of Dirac operators a variational problem for torsion can be derived similar to that of Yang-Mills gauge theory. As a consequence, one ends up with propagating torsion even in vacuum as opposed to Einstein-Cartan theory.

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1. Introduction

A connection on the frame bundle of any smooth manifold $M$ is known to yield the two independent geometrical concepts of curvature and torsion. There are various (but equivalent) approaches to the torsion of a connection, depending on the geometrical setup. For instance, the torsion of a connection $\nabla \equiv \nabla^{TM}$ on the tangent bundle of $M$ may be defined by

$$\tau_{\nabla} := d_{\nabla} \mathfrak{J} \in \Omega^{2}(M, TM).$$

Here, $d_{\nabla}$ denotes the exterior covariant derivative with respect to $\nabla^{TM}$. The canonical one-form $\mathfrak{J} \in \Omega^{1}(M, TM)$ is defined as $\mathfrak{J}_{x}(v) := v$ for all tangent vectors $v \in T_{x}M$ and $x \in M$. This canonical one-form corresponds to the soldering form on the frame bundle of $M$.

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In general relativity a field equation for the torsion is obtained by the so-called “Palatini formalism”, where the metric and the connection on the tangent bundle are regarded as being independent from each other (c.f. [13]). The resulting field equation for torsion in Einstein’s theory of gravity is known to be given by

$$\tau^\nu = \lambda_{\text{grav}} j_{\text{spin}},$$

where the real coupling constant $\lambda_{\text{grav}}$ is proportional to the gravitational constant. The so-called “spin-current” $j_{\text{spin}} \in \Omega^2(M, TM)$ is obtained by the variation of the action functional that dynamically describes matter fields with respect to the metric connection. Usually, ordinary bosonic matter does not depend on the metric connection as, for instance, described by the Standard Model. According to (2) the connection is thus provided by the Levi-Civita connection. This holds true, in particular, when matter is disregarded. When matter is defined in terms of spinor fields, as in the case of the Dirac action, then the right-hand side of (2) may be non-vanishing. This is usually rephrased by the statement

“Spin is the source of torsion”.

However, even in this case torsion is not propagating in space-time since (2) is purely algebraic relation between torsion and matter. Furthermore, a torsion-free connection provides a sufficient condition for the ordinary Dirac action to be real. More precisely, with respect to the identification $\Omega^2(M, TM) \cong \Omega^1(M, \text{End}(TM))$ a necessary and sufficient condition for the realness of the ordinary Dirac action is given by (c.f. [5])

$$tr \tau^\nu(v) = 0 \quad (v \in TM).$$

In this work, we discuss torsion within the framework of general relativity which is different from the ordinary Palatini formalism. We obtain field equations for the torsion from a functional that looks similar to the Einstein-Hilbert-Yang-Mills-Dirac action. This functional is derived from a certain class of Dirac operators. The geometrical background of these Dirac operators is basically dictated by the reality condition imposed on the action including (full) torsion. Furthermore, this class of Dirac operators fits well with those giving rise to Einstein’s theory of gravity, ordinary Yang-Mills-Dirac theory and non-linear $\sigma-$models, as discussed in [18].

From a physics point of view torsion provides an additional degree of freedom to Einstein’s theory of gravity. In the latter the action of a gravitational field is described in terms of the curvature of a smooth four dimensional manifold $M$. Even more, this curvature is assumed to be uniquely determined by the Levi-Civita connection on the tangent bundle of $M$ with fiber metric $g_M$. In other words, the geometrical model behind Einstein’s theory of gravity is known to be given by a smooth (orientable) Lorentzian four-manifold $(M, g_M)$ of signature $s = \pm 2$ (resp. a diffeomorphism class thereof). For a “space-time” $(M, g_M)$ to be physically admissible the metric field $g_M$ has to fulfill the Einstein equation of gravity (and maybe topological restriction on $M$, like global hyperbolicity), whereby the source of gravity is given by the energy-momentum current of matter. The latter is either phenomenologically described by a mass density or in terms of matter fields (i.e. sections of certain vector bundles over space-time). This holds true, especially, if the spin of matter is taken into account. In this case, matter is geometrically described by (Dirac) spinor fields. According to the above mentioned statement about the relation between spin and torsion a huge variety of generalizations of Einstein’s theory of gravity including torsion has been proposed over the last decades, going under the name “Einstein-Cartan theory”, “Poincare gauge
gravity”, “teleparallel gravity”, etc. (see, for instance, [8], as well as the more recent essay [9] and the references cited therein). For higher order gravity with (propagating) torsion discussed within the realm of Connes’ non-commutative geometry we refer to [7], [14] and the references cited therein. In these references only the totally anti-symmetric part of the torsion tensor is proposed to propagate according to a Procca like field equation. In contrast, in this work we discuss Yang-Mills like equations for the complete torsion tensor within Einstein’s theory of gravity. Moreover, in the approach discussed here the (so-called “strong”) equivalence principle still holds because torsion is assumed to couple only to the “internal degrees of freedom” of the fermions (see below).

In some of the above mentioned approaches torsion does not propagate, whereas other approaches propose only torsion but no curvature. In any case, spin is considered to be the source of torsion. Since spin is fundamental torsion plays also a prominent role in (super-)string theory (see, for instance, [3] and [11]). For an overview about the role of torsion in theoretical physics we refer to [6]. Also, we refer to [20] as a reasonable source of references to the issue. From a physics perspective it is speculated that torsion might contribute to dark energy, whose existence seems experimentally confirmed by the observed acceleration of the universe.

The approach to a dynamical torsion presented in this work is different, for it starts out with Rarita-Schwinger fermions to which torsion minimally couples. These fermions are geometrically modeled by sections of a twisted spinor bundle, where the “inner degrees of freedom” are generated by the (co-)tangent bundle of the underlying manifold. As a consequence, the resulting coupling to torsion completely parallels that of spinor-electrodynamics. Since the known matter is geometrically described by Dirac spinors, the Rarita-Schwinger fermions may serve as a geometrical model to physically describe dark matter (or parts thereof). Accordingly, the energy momentum of torsion (the underlying gauge field) may contribute to dark energy. The coupling constant between the Rarita-Schwinger and the torsion field, however, is a free parameter like in ordinary gauge theory, although the underlying action is derived from “first (geometrical) principles”. Hence, this gauge coupling constant is an additional free parameter as opposed to ordinary Einstein-Cartan theory.

This paper is organized as follows: We start out with a summary of the necessary geometrical background of Dirac operators in terms of general Clifford module bundles. Afterwards, we discuss torsion in the context of a distinguished class of Dirac operators which give rise to field equations similar to Einstein-Dirac-Yang-Mills equations.

2. Geometrical background

The geometrical setup presented fits well with that already discussed in [18] for non-linear $\sigma-$ models and Yang-Mills theory. For the convenience of the reader we briefly summarize the basic geometrical background. In particular, we present the basic features of Dirac operators of simple type. This class of Dirac operators will play a fundamental role in our discussion. For details we refer to [17] (or [18]) and [2], as well as to [1] which serves as a kind of “standard reference” for what follows.

In the sequel, $(M,g_M)$ always denotes a smooth orientable (semi-)Riemannian manifold of finite dimension $n \equiv p + q$. The index of the (semi-)Riemannian metric $g_M$ is $s \equiv p - q \not\equiv 1 \mod 4$. The bundle of exterior forms of degree $k \geq 0$ is denoted by $\wedge^k T^*M \to M$ with its canonical projection. Accordingly, the Grassmann bundle is given by $\Lambda T^*M \equiv \bigoplus_{k \geq 0} \wedge^k T^*M \to M$. It naturally inherits a metric denoted by $g_{\Lambda M}$,
such that the direct sum is orthogonal and the restriction of \( g_M \) to degree one equals to the fiber metric \( g_M^1 \) of the cotangent bundle \( T^*M \to M \).

The mutually inverse musical isomorphisms in terms of \( g_M \) (resp. \( g_M^1 \)) are denoted by \( g_M : T^*M \cong T^*M \), such that, for instance, \( g_M(u,v) = g_M^1(u^\flat,v^\flat) \) for all \( u,v \in TM \).

The Clifford bundle of \((M,g_M)\) is denoted by \( \pi_{Cl} : Cl_M = Cl^+_M \oplus Cl^-_M \to M \). Its canonical “even/odd” grading involution is \( \tau_{Cl} \in \text{End}(Cl_M) \). The hermitian structure \( \langle \cdot, \cdot \rangle_{Cl} \) is the one induced by the metric \( g_M \) due to the canonical linear isomorphism between the Clifford and the Grassmann bundle (c.f. (9), below). We always assume the Clifford and the Grassmann bundle to be generated by the cotangent bundle of \( M \). Notice that in the sequel the notion of “hermitian structure” does not necessarily imply a positive definite fiber metric. In fact, the signature of the fiber metric may depend on the signature of the underlying metric \( g_M \), like in the case of the Clifford bundle associated to \((M,g_M)\).

Throughout the present work we always identify on \((M,g_M)\) the vector bundle \( \pi_{\Lambda T^*M} : \Lambda^2 T^*M \to M \) with the (Lie algebra) bundle \( \pi_{so} : so(TM) \to M \) of the \( g_M \) skew-symmetric endomorphisms on the tangent bundle \( \pi_{TM} : TM \to M \) due to the canonical linear (bundle) isomorphism (over the identity on \( M \))

\[
\Lambda^2 T^*M \cong so(TM) \subset \text{End}(TM)
\]

where for all \( u,v \in TM : g_M(u,\Omega(v)) := \omega(u,v) \). Accordingly, we always take advantage of the induced isomorphism

\[
\Lambda^2 T^*M \otimes TM \cong T^*M \otimes so(TM)
\]

A smooth complex vector bundle \( \pi_E : E \to M \) of rank \( rk(E) \geq 1 \) is called a Clifford module bundle, provided there is a Clifford map. That is, there is a smooth linear (bundle) map (over the identity on \( M \))

\[
\gamma_E : T^*M \longrightarrow \text{End}(E)
\]

\[
\alpha \mapsto \gamma_E(\alpha),
\]

satisfying \( \gamma_E(\alpha)^2 = \epsilon g_M^1(\alpha,\alpha)\text{Id}_E \). Here, \( \epsilon \in \{ \pm 1 \} \) depends on how the Clifford product is defined. That is, \( \alpha^2 := \pm g_M^1(\alpha,\alpha)1_{Cl} \in Cl_M \), for all \( \alpha \in T^*M \subset Cl_M \) and \( 1_{Cl} \in Cl_M \) denotes the unit element.

To emphasize the module structure we write

\[
\pi_E : (E,\gamma_E) \longrightarrow (M,g_M).
\]

The bundle (7) is called an odd hermitian Clifford module bundle, provided it is \( \mathbb{Z}_2 \)-graded, with grading involution \( \tau_E \), and endowed with an hermitian structure \( \langle \cdot, \cdot \rangle_E \), such that \( \gamma_E \circ \tau_E = -\tau_E \circ \gamma_E \) and both the grading involution and Clifford action are either hermitian or skew-hermitian. In what follows, (7) always means an odd hermitian Clifford module. Notice that by “hermitian” we simply mean a fiber metric of arbitrary signature. The archetype example of an odd hermitian Clifford module over \((M,g_M)\) is provided by the corresponding Clifford bundle.

The linear map

\[
\delta_\gamma : \Omega(M,\text{End}(E)) \longrightarrow \Omega^0(M,\text{End}(E))
\]

\[
\omega \equiv \alpha \otimes \mathcal{B} \mapsto \psi \equiv \gamma_E(\sigma_{Cl}^{-1}(\alpha)) \circ \mathcal{B}
\]
is called the “quantization map”. It is determined by the linear isomorphism called symbol map:

\[ \sigma_{\text{ch}} : Cl_M \xrightarrow{\sim} \Lambda T^*M \]
\[ a \mapsto \Gamma_{\text{ch}}(a)1_\Lambda. \]  

(9)

Here, \(1_\Lambda \in \Lambda T^*M\) is the unit element. The homomorphism \(\Gamma_{\text{ch}} : Cl_M \to \text{End}(\Lambda T^*M)\) is given by the canonical Clifford map:

\[ \gamma_{\text{Cl}} : T^*M \to \text{End}(\Lambda T^*M) \]
\[ v \mapsto \begin{cases} \Lambda T^*M & \mapsto \Lambda T^*M \\ \omega & \mapsto \epsilon \text{int}(v)\omega + \epsilon \text{ext}(v^b)\omega, \end{cases} \]

(10)

where, respectively, “int” and “ext” indicate “interior” and “exterior” multiplication.

When restricted to \(\Omega^1(M, \text{End}(E))\) the quantization map \([5]\) has a canonical right-inverse given by

\[ \text{ext}_\Theta : \Omega^0(M, \text{End}(E)) \to \Omega^1(M, \text{End}(E)) \]
\[ \Phi \mapsto \Theta \Phi, \]

(11)

where the canonical one-form \(\Theta \in \Omega^1(M, \text{End}(E))\) is given by \(\Theta(v) := \tilde{\gamma}_E(v^b)\), for all \(v \in TM\). The associated projection operators are \(p \equiv \text{ext}_\Theta \circ \delta_{\gamma}|_{\Omega^1}\) and \(q \equiv \text{Id}_{\Omega^1} - p\), such that

\[ \Omega^1(M, \text{End}(E)) = p(\Omega^1(M, \text{End}(E))) \oplus q(\Omega^1(M, \text{End}(E))). \]

(12)

Notice that for any connection on a Clifford module bundle the first order operator \(\mathcal{T}_\psi \equiv q(\nabla^E) : \Omega^0(M, E) \to \Omega^1(M, E), \psi \mapsto \nabla^E \psi - \Theta(\nabla^E \psi)\) is the associated twistor operator. Here, \(\nabla^E \equiv \delta_{\gamma}(\nabla^E)\) denotes the Dirac operator associated to the connection (see below).

A (linear) connection on a Clifford module bundle is called a Clifford connection if the corresponding covariant derivative \(\nabla^E\) “commutes” with the Clifford map \(\gamma_E\) in the following sense:

\[ [\nabla^E_X, \gamma_E(\alpha)] = \gamma_E(\nabla^E_X \alpha) \quad (X \in \text{Sec}(M, TM), \alpha \in \text{Sec}(M, T^*M)). \]

(13)

Here, \(\nabla^{T^*M}\) is the Levi-Civita connection on the co-tangent bundle with respect to \(g^*_M\).

Equivalently, a connection on a Clifford module bundle is a Clifford connection if and only if it fulfills:

\[ \nabla^E_X \Theta = 0 \quad (X \in \text{Sec}(M, TM)). \]

(14)

Apparently, Clifford connections provide a distinguished class of connections on any Clifford module bundle.

We denote Clifford connections by \(\partial_{\gamma}\). This notation is used because Clifford connections are parametrized by a family of locally defined one-forms \(A \in \Omega^1(U, \text{End}_{\gamma}(E))\). Here, \(U \subset M\) will always denote a connected open subset and \(\text{End}_{\gamma}(E) \subset \text{End}(E)\) is the total space of the algebra bundle of endomorphisms which commute with the Clifford action provided by \(\gamma_E\).

We call in mind that a Dirac operator \(\mathcal{D}\) on a Clifford module bundle is a first order differential operator acting on sections \(\psi \in \text{Sec}(M, E)\), such that \([\mathcal{D}, df]\psi = \gamma_E(df)\psi\) for all smooth functions \(f \in C^\infty(M)\). The set of all Dirac operators on a given Clifford module bundle is denoted by \(\mathfrak{Dir}(\mathcal{E}, \gamma_E)\). It is an affine set over the vector space \(\Omega^0(M, \text{End}(E))\). Moreover, Dirac operators are defined to be odd operators on
odd (hermitian) Clifford module bundles: \( \mathcal{D}_τ = -τ_ε \mathcal{D} \). In this case, the underlying vector space reduces to \( Ω^0(M, \text{End}(\mathcal{E})) \).

We call the Dirac operator \( \nabla^\mathcal{E} \equiv \nabla^\mathcal{E} \) the “quantization” of a connection \( \nabla^\mathcal{E} \) on a Clifford module bundle \( [7] \). Let \( e_1, \ldots, e_n \in \text{Sec}(U, TM) \) be a local frame and \( e^1, \ldots, e^n \in \text{Sec}(U, T^*M) \) its dual frame. For \( \psi \in \text{Sec}(M, \mathcal{E}) \) one has

\[
\nabla^\mathcal{E} \psi := \sum_{k=1}^n \delta_\gamma(e^k) \nabla_{e_k}^\mathcal{E} \psi = \sum_{k=1}^n \gamma_\mathcal{E}(e^k) \nabla_{e_k}^\mathcal{E} \psi ,
\]

where the canonical embedding \( Ω(M) \hookrightarrow Ω(M, \text{End}(\mathcal{E})) \), \( \omega \mapsto \omega \equiv \omega \otimes \text{Id}_\mathcal{E} \) is taken into account.

Every Dirac operator has a canonical first-order decomposition:

\[
\mathcal{D} = \phi_B + \Phi_D .
\]

Here, \( \partial_B \) denotes the (covariant derivative of the) Bochner connection that is defined by \( \mathcal{D} \) as

\[
2ev_g(df, \partial_B \psi) := \epsilon([\mathcal{D}^2, f] - \partial_B df) \psi \quad (\psi \in \text{Sec}(M, \mathcal{E})) ,
\]

with \( ev_g \) being the evaluation map with respect to \( g_M \) and \( \partial_B \) the dual of the exterior derivative (see \([4]\)).

The zero-order section \( \Phi_D := \mathcal{D} - \phi_B \in \text{Sec}(M, \text{End}(\mathcal{E})) \) is thus also uniquely determined by \( \mathcal{D} \). We call the Dirac operator \( \phi_B \) the “quantized Bochner connection”.

Since the set \( \text{Dir}(\mathcal{E}, \gamma_\mathcal{E}) \) is an affine space, every Dirac operator can be written as

\[
\mathcal{D} = \phi_B + \Phi_D .
\]

However, this decomposition is far from being unique. The section \( \Phi \in \text{Sec}(M, \text{End}(\mathcal{E})) \) depends on the chosen Clifford connection \( \phi_B \). In general, a Dirac operator does not uniquely determine a Clifford connection.

**Definition 2.1.** A Dirac operator \( \mathcal{D} \in \text{Dir}(\mathcal{E}, \gamma_\mathcal{E}) \) is said to be of “simple type” provided that \( \Phi_D = \mathcal{D} - \phi_B \in \text{Sec}(M, \text{End}(\mathcal{E})) \) anti-commutes with the Clifford action:

\[
\Phi_D \gamma_\mathcal{E}(\alpha) = -\gamma_\mathcal{E}(\alpha) \Phi_D \quad (\alpha \in T^*M) .
\]

It follows that a Dirac operator of simple type uniquely determines a Clifford connection \( \phi_B \) together with a zero-order operator \( \phi_D \in \text{Sec}(M, \text{End}_{\gamma}(\mathcal{E})) \), such that (c.f. \([17]\))

\[
\mathcal{D} = \phi_B + \tau_ε \phi_D .
\]

These Dirac operators play a basic role in the geometrical description of the Standard Model (c.f. \([17]\)). They are also used in the context of the family index theorem (see, for instance, \([4]\)). Apparently, Dirac operators of simple type provide a natural generalization of quantized Clifford connections. Indeed, they build the biggest class of Dirac operators such that the corresponding Bochner connections are also Clifford connections. Notice that \( \phi_D \in \text{Sec}(M, \text{End}_{\gamma}(\mathcal{E})) \) for odd (hermitian) Clifford modules.

Every Dirac operator is known to have a unique second order decomposition

\[
\mathcal{D}^2 = \triangle_B + V_D ,
\]

where the Bochner-Laplacian (or “trace Laplacian”) is given in terms of the Bochner connection as \( \triangle_B := \epsilon ev_g(\partial_B^T \circ \phi_B) \) (c.f. \([4]\)).

The trace of the zero-order operator \( V_D \in \text{Sec}(M, \text{End}(\mathcal{E})) \) explicitly reads (c.f. \([17]\)):

\[
tr_\mathcal{E} V_D = tr_{\gamma}(\text{curv}(\mathcal{D}) - \epsilon ev_g(\omega_D^2)) - \delta_\gamma(tr_\mathcal{E} \omega_D) ,
\]
where \( \text{curv}(\mathcal{D}) \in \Omega^2(M, \text{End}(\mathcal{E})) \) denotes the curvature of the Dirac connection of \( \mathcal{D} \in \text{Dir}(\mathcal{E}, \gamma_\mathcal{E}) \) and \( \text{tr}_\gamma := \text{tr}_\gamma \circ \delta_\gamma \) the “quantized trace”. The Dirac connection of \( \mathcal{D} \) is defined by \( \partial_\mathcal{D} := \partial + \omega_\mathcal{D} \), with the “Dirac form” \( \omega_\mathcal{D} \equiv \text{ext}_\Theta \Phi_\mathcal{D} \in \Omega^1(M, \text{End}^+(-\mathcal{E})) \). The Dirac connection has the property that it is uniquely determined by \( \mathcal{D} \), as opposed to the decomposition (18) in terms of quantized Clifford connections. Furthermore, one has \( \partial_\mathcal{D} = \partial \). Notice that in general \( \partial_\mathcal{D} \neq \partial \). Indeed, \( \partial_\mathcal{D} = \partial \) holds true if and only if \( \mathcal{D} \) is a quantized Clifford connection. Only in this case, the three notions of Dirac-, Bochner- and Clifford connection coincide.

Let \( M \) be closed compact. We call the canonical mapping

\[
\mathcal{I}_\mathcal{D} : \text{Dir}(\mathcal{E}, \gamma_\mathcal{E}) \to \mathbb{C}
\]

the “universal Dirac action” and

\[
\mathcal{I}_{\mathcal{D}, \text{tot}} : \text{Dir}(\mathcal{E}, \gamma_\mathcal{E}) \times \text{Sec}(M, \mathcal{E}) \to \mathbb{C}
\]

the “total Dirac action”. Here, “*” is the Hodge map with respect to \( g_M \) and a chosen orientation of \( M \).

When (22) is taken into account, the universal Dirac action

\[
\mathcal{I}_\mathcal{D}(\mathcal{D}) = \int_M *\text{tr}_\gamma \left( \text{curv}(\mathcal{D}) - \epsilon \text{ev}_g(\omega^2_\mathcal{D}) \right)
\]

is seen to provide a natural generalization of the Einstein-Hilbert functional with a cosmological constant:

\[
\mathcal{I}_{\text{EHC}}(g_M) := \int_M *\left( \text{scal}(g_M) + \Lambda \right)
\]

\[
\sim \int_M *\text{tr}_\gamma \left( \text{curv}(\nabla^\mathcal{E}) - \epsilon \text{ev}_g(\omega^2) \right)
\]

(26)

Here, \( \nabla^\mathcal{E} \) is a Clifford connection on a Clifford module bundle (7). For instance, one may take the Clifford bundle and the lifted Levi-Civita connection. Furthermore, we set \( \omega \equiv \sqrt{-4\pi \Lambda} \Theta \in \Omega^1(M, \text{End}^+(\mathcal{E})) \). Notice, however, that this form is odd on odd hermitian Clifford module bundles. In contrast, the Dirac form \( \omega_\mathcal{D} \in \Omega^1(M, \text{End}^+(\mathcal{E})) \), which appears in (25), is always even on odd hermitian Clifford module bundles. The smooth function \( \text{scal}(g_M) \) is the scalar curvature of the Levi-Civita connection of \( g_M \) and \( \Lambda \in \mathbb{R} \) is the “cosmological constant”.

Indeed, when \( \mathcal{D} \) is a quantized Clifford connection, then the universal Dirac action (25) reduces to the Einstein-Hilbert functional. This follows from the Schrödinger-Lichnerowicz decomposition of the square of a quantized Clifford connection (c.f. [15], [12]). In contrast, for Dirac operators of simple type the universal Dirac action becomes

\[
\mathcal{I}_\mathcal{D}(\partial + \tau^\mathcal{E} \phi_\mathcal{D}) = \int_M *\left( -\epsilon \text{re}(\epsilon) \text{scal}(g_M) + \text{tr}_\gamma \phi_\mathcal{D}^2 \right)
\]

(27)

This explicit formula is a direct consequence of Lemma 4.1 and the Corollary 4.1 of Ref. [17] (see also Sec. 6 in loc. site), which generalizes the Schrödinger-Lichnerowicz formula to arbitrary Dirac operators. Therefore, restriction of the universal Dirac...
action (23) to Dirac operators of simple type (20) corresponds to (26), where (up to numerical factors)
\[ \Lambda = \text{tr}_E \phi_D^2. \] (28)

Hence, for \( \phi_D^\dagger = \pm \phi_D \) the cosmological constant is given basically by the length of the section \( \phi_D \in \text{Sec}(M, \text{End}_\gamma(E)) \) that is associated with the simple type Dirac operator \( \mathcal{D} = \pm \mathcal{D}_\dagger \in \text{Dir}(M, E) \). Indeed, whenever the section \( \phi_D \) does not depend on the metric, the variation of (27) with respect to \( g_M \) will yield that \( \phi_D \) has to be of constant length. Yet, this does not imply that the section \( \phi_D \) itself has to be constant. Merely this means that \( \phi_D \) has to be a section of the sphere sub-bundle of radius \( \Lambda \) of the hermitian vector bundle \( \text{End}_\gamma(E) \to M \). This is similar to what is encountered in the Higgs sector of the Standard Model, actually. When an underlying gauge group is supposed to act transitively on this sphere bundle, then the gauge symmetry becomes “spontaneously broken” like in the Standard Model but without the need for a Higgs potential.

We stress that the universal (total) Dirac action is fully determined by the Dirac operator, like the Einstein-Hilbert functional is fully determined by the metric. In contrast, for example, the Yang-Mills action is not a canonical functional on the set of connections for it also depends on the chosen metric. Also, we call in mind that nowadays the functional (26) is considered to be more fundamental (at least on the cosmological scale) than the original Einstein-Hilbert action. In fact, the functional (26) is mathematically well-motivated due to Lovelock’s Theorem (c.f. [10]). Similarly, Dirac operators of simple type seem to be more profound than quantized Clifford connections.

3. Dirac operators with torsion

To this end let \((M, g_M)\) be a (semi-)Riemannian spin-manifold of even dimension \( n = p + q \) and signature \( p - q \neq 1 \) mod 4. Let \( \pi_S : S = S^+ \oplus S^- \to M \) be a (complexified) spinor bundle with grading involution \( \tau_S \in \text{End}(S) \). The hermitian structure is denoted by \( \langle \cdot, \cdot \rangle_S \). The Clifford action is provided by the canonical Clifford map \( \gamma_S : T^*M^C \to \text{End}_C(S) \simeq \text{Cl}_M^C \). The induced Clifford action is supposed to be anti-hermitian. The Clifford action also anti-commutes with the grading involution. The grading involution is assumed to be either hermitian or anti-hermitian. Let us call in mind that by abuse of notation “hermitian” does not necessarily mean in what follows positive definiteness of the fiber metric\(^1\).

We consider the twisted spinor bundle
\[ \pi_{\mathcal{E}_1} : \mathcal{E}_1 := S \otimes_M TM \to M \] (29)
with the grading involution \( \tau_{\mathcal{E}_1} := \tau_S \otimes \text{Id}_TM \) and Clifford action \( \gamma_{\mathcal{E}_1} := \gamma_S \otimes \text{Id}_TM \). The hermitian structure reads: \( \langle \cdot, \cdot \rangle_{\mathcal{E}_1} := \langle \cdot, \cdot \rangle_S g_M \).

The Clifford extension of (29) is denoted by (c.f. [18])
\[ \pi_{\mathcal{E}} : \mathcal{E} := \mathcal{E}_1 \otimes_M \text{Cl}_M \to M \] (30)

Here, the grading involution and Clifford action, respectively, are given by \( \tau_{\mathcal{E}} := \tau_{\mathcal{E}_1} \otimes \tau_{\text{Cl}} \) and \( \gamma_{\mathcal{E}} := \gamma_{\mathcal{E}_1} \otimes \text{Id}_\text{Cl} \). The hermitian structure is \( \langle \cdot, \cdot \rangle_{\mathcal{E}} := \langle \cdot, \cdot \rangle_{\mathcal{E}_1} \langle \cdot, \cdot \rangle_{\text{Cl}} \).

\[^1\]The author would like to thank Ch. Bär (University of Potsdam/Germany) for a corresponding remark.
In what follows all vector bundles are regarded as complex vector bundles, though we do not explicitly indicate complexifications.

We denote the covariant derivative of the spin connection by $\nabla^S$. The corresponding spin-Dirac operator is $\nabla^S$.

For $A \in \Omega^1(M, \Lambda^2 T^*M)$, the most general metric connection on the tangent bundle is known to be given by the covariant derivative

$$\nabla^g := \nabla^{LC} + A.$$  \hfill (31)

Here, $\nabla^{LC}$ is the covariant derivative of the Levi-Civita connection on the tangent bundle with respect to $g_M$.

Accordingly, the torsion $\tau_{\nabla^g} \in \Omega^2(M, TM)$ of a metric connection (31) can be expressed by the torsion form (also called “torsion tensor”)

$$\tau_A(u, v) \equiv A(u)v - A(v)u \quad (u, v \in TM).$$  \hfill (32)

Indeed, the definitions (31) and (32) imply that for all smooth tangent vector fields $X, Y \in \mathcal{Sec}(M, TM)$:

$$\tau_A(X, Y) = \nabla^g_X Y - \nabla^{LC}_X Y - \nabla^g_Y X + \nabla^{LC}_Y X$$

$$= \nabla^g_X Y - \nabla^g_Y X - [X, Y]$$  \hfill (33)

Hence, we do not make a distinction between the torsion $\tau_{\nabla^g} \in \Omega^2(M, TM)$ of a metric connection $\nabla^g$ and the torsion form $\tau_A \in \Omega^2(M, TM)$ of $A \in \Omega^1(M, \Lambda^2 T^*M)$.

Notice that (32) may also be inverted (c.f. Prop. 2.1 in [1] and the corresponding references cited there; In particular, see also [19]). In fact, one has for all $u, v, w \in TM$:

$$2g_M(A(u)v, w) = g_M(\tau_A(u)v, w) - g_M(\tau_A(v)w, u) + g_M(\tau_A(w)u, v).$$  \hfill (34)

**Definition 3.1.** Let $\nabla^g$ be the covariant derivative of a metric connection on the tangent bundle of $(M, g_M)$. We call the section

$$A := \nabla^g - \nabla^{LC} \in \Omega^1(M, \Lambda^2 T^*M)$$  \hfill (35)

the “torsion potential” of $\tau_A = d_{\nabla^g} \omega$.

Consider the following hermitian Clifford connection on the Clifford module bundle that is provided by the following covariant derivative:

$$\partial_A := \nabla^{S \otimes TM} \equiv \nabla^S \otimes \text{Id}_{TM} + \text{Id}_S \otimes \nabla^g$$

$$= \nabla^S \otimes \text{Id}_{TM} + \text{Id}_S \otimes \nabla^{LC} + \text{Id}_S \otimes A$$

$$=: \nabla^{\xi_1} + \text{Id}_S \otimes A$$  \hfill (36)

Clearly, (36) is but the gauge covariant derivative of a twisted spin connection on (29) which is defined by the lift of (31).

Of course, every metrical connection (31) on $(M, g_M)$ can be lifted to the spinor bundle $\pi_S : S \to M$. However, in this case the resulting spin connection is neither a Clifford connection in the sense of our definition (13), nor is the ordinary Dirac action real-valued.

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\[2\] The author would like to thank Ch. Stephan (University of Potsdam/Germany) for making him aware of these references.
With respect to an oriented orthonormal frame $e_1, \ldots, e_n \in \sec(U, TM)$, with the dual frame being denoted by $e^1, \ldots, e^n \in \sec(U, T^*M)$, the corresponding twisted spin-Dirac operator reads:

$$\slashed{\partial}_A = \nabla^S \otimes \text{Id}_{TM} + \sum_{k=1}^n \gamma_S(e^k) \otimes \nabla^S_{e_k} = \nabla^S \otimes \text{Id}_{TM} + \sum_{k=1}^n \gamma_S(e^k) \otimes \nabla^L_{e_k} + \sum_{k=1}^n \gamma_S(e^k) \otimes A(e_k)$$

(37)

It looks similar to the usual gauge covariant Dirac operator encountered in ordinary electrodynamics on Minkowski space-time.

Accordingly, on the Clifford extension (30) we consider the Clifford connection

$$\tilde{\nabla}^e := \partial_A \otimes \text{Id}_{Cl} + \text{Id}_{e_1} \otimes \nabla^Cl$$

$$\equiv \nabla^e + \text{Id}_{S} \otimes A \otimes \text{Id}_{Cl},$$

(38)

with $\nabla^Cl$ being the induced Levi-Civita connection on the Clifford bundle.

With regard to the canonical embedding $\pi_E|_S : E_1 \hookrightarrow E \rightarrow M$, $z \mapsto z \equiv z \otimes 1$ one obtains for $\psi \equiv \psi \otimes 1 \in \sec(M, \mathcal{E})$ the equality

$$\tilde{\nabla}^e \psi = \partial_A \psi \otimes 1.$$

(39)

**Definition 3.2.** Let $d_{\nabla^LC}$ be the exterior covariant derivative induced by the Levi-Civita connection with the (Riemannian) curvature denoted by $F_{\nabla^LC}$. Also, let $F_{\nabla^S} \in \Omega^2(M, \Lambda^2 T^*M)$ be the curvature of $\nabla^S$. We call the relative curvature

$$F_A := F_{\nabla^S} - F_{\nabla^LC} = d_{\nabla^LC} A + A \wedge A$$

$$= d_{\nabla^LC} A + \frac{1}{2} [A, A] \in \Omega^2(M, \Lambda^2 T^*M)$$

(40)

the “torsion field strength” associated to the torsion potential $A = \nabla^S - \nabla^LC$.

Notice that on a metrical flat manifold $(M, g_M)$ the torsion field strength fulfills a Bianchi identity and therefore becomes a true curvature that is defined by torsion. In some approaches to dynamical torsion (“teleparallel gravity”) this curvature is used to describe gravity on metrical flat space-time manifolds, sometimes called “Weitzenböck space-times”.

Let again $e_1, \ldots, e_n \in \sec(U, TM)$ be a local (oriented orthonormal) frame with the dual frame being denoted by $e^1, \ldots, e^n \in \sec(U, T^*M)$. Also, let

$$\Sigma := \sum_{b=1}^n e^b \otimes \Sigma_b \in \Omega^1(M, \text{End}_\gamma(\mathcal{E})), $$

$$\Sigma_b := \sum_{a=1}^n \text{Id}_S \otimes F_a(e_b, e_a) \otimes e^a \in \mathcal{C}^\infty(U, \text{End}_\gamma(\mathcal{E})).$$

(41)

Again, $\text{End}_\gamma(\mathcal{E}) \subset \text{End}(\mathcal{E})$ denotes the sub-algebra of the (odd) endomorphisms which commute with the Clifford action provided by $\gamma_\mathcal{E}$.

We consider the Dirac operator of simple type

$$\slashed{D} := \tilde{\nabla}^e + \tau e^i \phi_D$$

(42)
acting on sections of the Clifford twist (for this notion see [18])

\[ \pi_{\varepsilon'} : \mathcal{E}' := \mathcal{E} \otimes_{M} Cl_{M} \to M \]  

of (30) (i.e. the double twist of [29]). The grading involution, the Clifford action and the hermitian Clifford product are defined, respectively, by

\[ \tau_{\varepsilon'} := \tau_{\varepsilon} \otimes \text{Id}_{Cl}, \quad \gamma_{\varepsilon'} := \gamma_{\varepsilon} \otimes \text{Id}_{Cl}, \quad \langle \cdot, \cdot \rangle_{\varepsilon'} := \langle \cdot, \cdot \rangle_{\varepsilon} \langle \cdot, \cdot \rangle_{Cl}. \]  

Also,

\[ \hat{\nabla}_{\varepsilon'} := \hat{\nabla}_{\varepsilon} \otimes \text{Id}_{Cl} + \text{Id}_{\varepsilon} \otimes \nabla_{Cl} \]  

is the covariant derivative of the induced Clifford connection on the Clifford twist of (30). Also, we put

\[ \phi_{D} := - \sum_{b=1}^{n} \Sigma_{b} \otimes e_{b} \]

\[ = \sum_{a,b=1}^{n} \text{Id}_{M} \otimes F_{\lambda}(e_{a}, e_{b}) \otimes e^{a} \otimes e^{b} \in \mathcal{S}\text{ec}(M, \text{End}_{\varepsilon'}(\mathcal{E}')). \]  

Notice that the Dirac operator of simple type [42] is fully determined by the most general hermitian Clifford connection [36] on the twisted spinor bundle [29]. In contrast, the quantization of the lift of \( \nabla^{g} \) to the spinor bundle itself is neither a quantized Clifford connection, nor a Dirac operator of simple type.

**Proposition 3.1.** Let \( M \) be closed compact. When restricted to the class of simple type Dirac operators [42] and to sections \( \psi \in \mathcal{S}\text{ec}(M, \mathcal{E}_{1}) \subset \mathcal{S}\text{ec}(M, \mathcal{E}'), \) the total Dirac action decomposes as

\[ \mathcal{I}_{D, \text{tot}}(\mathcal{D}, \psi) = \int_{M} \left( - \epsilon \mathcal{E}(\varepsilon') \, \text{scal}(g_{M}) + \langle \psi, \mathcal{D}_{\psi} \rangle_{\mathcal{E}_{1}} - 2^{2n} r \mathcal{E}(S) \| F_{\lambda} \|^2 \right), \]  

where \( \| F_{\lambda} \|^2 \equiv -g_{M}^{**}(e^{a}, e^{c}) g_{M}^{**}(e^{b}, e^{d}) \text{ tr}(F_{\lambda}(e_{a}, e_{b})F_{\lambda}(e_{c}, e_{d})) \equiv -\text{tr} F_{ab} F^{ab} \in C^{\infty}(M). \)

The variation of (47) with respect to the torsion potential \( A \in \Omega^{1}(M, \Lambda^{2}T^{*}M) \) yields a Yang-Mills like equation for the torsion field strength:

\[ \delta_{\nabla^{g}} F_{\lambda} = - \lambda_{0} \text{Re} \langle \psi, \Theta \psi \rangle_{S}, \]  

where \( \lambda_{0} \equiv 2^{-(n+1)} \epsilon n / r \mathcal{E}(S) \) is a constant and \( \delta_{\nabla^{g}} \equiv (-1)^{n(k+1)+q+1} \ast d_{\nabla^{g}} \ast \) is the formal adjoint of the exterior covariant derivative induced by \( \nabla^{g}. \)

When an oriented orthonormal frame is used, the right-hand side of (48) explicitly reads:

\[ \text{Re} \langle \psi, \Theta \psi \rangle_{S} := \frac{\epsilon}{n} \sum_{i,j,k,l,m=1}^{n} g_{M}(e_{i}, e_{j})g_{M}(e_{l}, e_{m}) \text{Re} \langle \psi^{k}(e^{j}) \psi^{m} \rangle_{S} e^{i} \otimes e^{l} \otimes e_{m} \]

\[ = \frac{\epsilon}{n} \sum_{i,j,k=1}^{n} \text{Re} \langle \psi_{i}, \gamma_{k} \psi_{j} \rangle_{S} e^{k} \otimes e^{i} \otimes e_{j} \]

\[ = \frac{\epsilon}{n} \sum_{i,j,k=1}^{n} \langle \psi_{i}, \gamma_{k} \psi_{j} \rangle_{S} e^{k} \otimes e^{i} \wedge e_{j} \]

\[ \equiv \frac{\epsilon}{n} \delta_{\text{spin}} \in \Omega^{1}(M, \Lambda^{2}T^{*}M), \]
whereby $\psi =: \sum_{k=1}^{n} \psi^k \otimes e_k \in \mathcal{S}ec(M, \mathcal{E}_1)$ and $\psi_i \equiv \sum_{j=1}^{n} g_M(e_i, e_j) \psi^j \in \mathcal{S}ec(U, S)$ are locally defined spinor fields for all $i = 1, \ldots, n$.

Before we prove the Proposition (3.1) it might be worthwhile adding some general comments first: When the embedding $\text{End}(\mathcal{T}M) \cong \text{End}_\tau(\mathcal{E}_1) \hookrightarrow \text{End}(\mathcal{E}_1)$, $\mathcal{B} \mapsto \text{id}_S \otimes \mathcal{B}$ is taken into account, one may replace (18) by

$$\delta_{\Phi} F_{\lambda} = -\lambda_{\Phi} \text{Re} \langle \psi, \Theta \psi \rangle_S,$$

with $\delta_{\Phi}$ being the formal adjoint of the exterior covariant derivative $d_\lambda$ of the defined by connection (36). In the form (50), the field equation (48) for the torsion looks even more similar to the ordinary (non-abelian) Yang-Mills equation, although $d_\lambda F_{\lambda} \neq 0$ unless $(M, g_M)$ is flat.

Clearly, the functional (17) looks much like the usual Dirac-Yang-Mills action including gravity. However, the energy-momentum current not only depends on the metric but also on its first derivative. This additional dependence, however, can be always (point-wise) eliminated by the choice of normal coordinates (nc.), such that at $x \in M$:

$$F_{\lambda} \big|_{x} \overset{\text{nc.}}{=} \left( dA + \frac{1}{2}[A, A] \right) \big|_{x}. \quad (51)$$

Such a choice of local trivialization of the frame bundle does not affect the torsion potential (in contrast to the action of diffeomorphisms on $M$).

In order to end up with the field equation (48) the sections $\psi \in \mathcal{S}ec(M, \mathcal{E}_1)$ are twisted fermions of spin $3/2$, as opposed to ordinary Dirac-Yang-Mills theory. In the presented approach to torsion the additional spin-one degrees of freedom of matter are regarded as “internal gauge degrees” that couple to torsion. We stress that the functional (17) is real-valued, indeed. This is because, the Dirac operator $\partial_{\Phi}$ is symmetric and the twisted spin-connection provided by $\nabla^\mathcal{E}_1$ is torsion-free. If the spinor bundle were not twisted with the tangent bundle in (29), then one has to use (31) instead of $\nabla^\mathcal{E}_1$ to define the ordinary Dirac action. In this case, however, the functional (17) would be complex, in general, as mentioned already. The demand to derive a real action including (full) torsion from Dirac operators of simple type basically dictates the geometrical setup presented that eventually leads to (17).

**Proof.** The statement of (3.1) is a special case of the Proposition 6.2 in [18]. To prove the statement we consider the $\mathbb{Z}_2$-graded hermitian vector bundle $\pi_E : E := TM \otimes_M Cl_{M} \otimes_M Cl_{M} \rightarrow M$. The grading involution and hermitian structure are given, respectively, by $\tau_E := \text{Id}_{TM} \otimes \tau_{Cl} \otimes \text{Id}_{Cl}$ and $\langle \cdot, \cdot \rangle_E := g_M(\cdot, \cdot)_{Cl} \langle \cdot, \cdot \rangle_{Cl}$. Hence, $\pi_{E'} : \mathcal{E}' = S \otimes_M E \rightarrow M$ is an odd twisted hermitian spinor bundle, with the grading involution $\tau_{E'} := \tau_S \otimes \tau_E$ and the Clifford action provided by $\gamma_{E'} := \gamma_S \otimes \text{Id}_E$. The hermitian structure is $\langle \cdot, \cdot \rangle_{E'} := \langle \cdot, \cdot \rangle_S \langle \cdot, \cdot \rangle_E$. Furthermore, the twisted spinor bundle carries the canonical Clifford connection that is provided by $\nabla_{E'} = \nabla^S \otimes E$, where $\nabla^E := \nabla^TM \otimes \text{Cl}_{Cl}$ and $\nabla^TM \equiv \nabla^{LC}$.

From the general statement concerning the universal Dirac action restricted to Dirac operators of simple type it follows that

$$\mathcal{I}_D(\mathcal{D}) = \int_{M} \ast tr_\gamma (\text{curv}(\nabla_{E'}^\mathcal{D}) + tr_{E'} \phi^2_{\mathcal{D}}). \quad (52)$$

One infers from the ordinary Lichnerowicz-Schrödinger formula of twisted spin-Dirac operators (c.f. [12] and [15]) that

$$tr_\gamma (\text{curv}(\nabla_{E'}^\mathcal{D})) = - \frac{2}{2} \text{scal}(g_M). \quad (53)$$
Furthermore, it is straightforward to check that
\[ \text{tr}_{\mathcal{E}'} \phi_D^2 \sim \| F_\alpha \|^2. \] (54)

Finally, when restricting to sections \( \psi \in \mathcal{S}ec(M, \mathcal{E}_1) \subset \mathcal{S}ec(M, \mathcal{E}') \) one obtains
\[
\langle \psi, \mathcal{D} \psi \rangle_{\mathcal{E}'} = \langle \psi, \hat{\nabla} \psi \rangle_{\mathcal{E}_1}
= \langle \psi, \nabla^{\mathcal{E}_1} \psi \rangle_{\mathcal{E}_1} + \langle \psi, A \psi \rangle_{\mathcal{E}_1}.
\] (55)

The first equality holds because \( \bigoplus_{k \geq 0} \Lambda^k T^* M \) is an orthogonal sum. Hence,
\[
\langle \psi, \tau_{\mathcal{E}'} \phi_D \psi \rangle_{\mathcal{E}'} = \sum_{a,b=1}^n \langle \psi, (\tau_S \otimes F_\alpha(e_a, e_b)) \psi \rangle_{\mathcal{E}_1} \langle 1, e^a \rangle_{\text{Cl}} \langle 1, e^b \rangle_{\text{Cl}} = 0.
\] (56)

This proves (17). To also prove (48) we remark that up to a boundary term
\[ \| F_\alpha \|^2 \sim \langle A, \delta \nabla g F_\alpha \rangle_{\text{AT}^* M \otimes \text{A}^2 T^* M}. \] (57)

This is similar to the usual Yang-Mills-Lagrangian. The basic difference is that the torsion potential \( A \in \Omega^1(M, \Lambda^2 T^* M) \) itself does not define a connection, in general. As already mentioned, if \( (M, g_M) \) is flat, then the torsion field strength \( F_\alpha \) has the geometrical meaning of the curvature of a general metric connection parametrized by torsion potentials. In any case, together with the solutions of the coupled Einstein-Dirac equation, the solutions of (48) determine the torsion of a general metric connection. \[ \square \]

We remark that by an appropriate re-definition of the sections \( \phi \) and \( \psi \) one may always recast the functional (17) into the even more suggestive form
\[
\mathcal{I}_{D, \text{tot}}(\mathcal{D}, \psi) \sim \int_M * \text{scal}(g_M) + \int_M * \langle \psi, \hat{\nabla} \psi \rangle_{\mathcal{E}_1} - \frac{1}{4\kappa^2} \int_M \text{tr} (F_\alpha \wedge * F_\alpha),
\] (58)

with \( \kappa > 0 \) being an arbitrary (coupling) constant like in ordinary non-abelian Yang-Mills theory. Furthermore, by re-scaling the torsion potential \( A \), which is admissible since the torsion potential belongs to a vector space as opposed to true gauge potentials, the field equation (50) changes to
\[ \delta_\Lambda F_\alpha = -g J_{\text{spin}} \] (59)

with the spin-current \( J_{\text{spin}} \in \Omega^1(M, \Lambda^2 T^* M) \) being defined by (49). The Dirac equation becomes
\[ \hat{\nabla}_\alpha \psi = 0 \Leftrightarrow \nabla^{\mathcal{E}_1} \psi = -g A \psi. \] (60)

As in ordinary general relativity the metric (and thus the connection \( \nabla^{\mathcal{E}_1} \)) is determined by the Einstein equation with the energy-momentum current similarly defined as in the usual Dirac-Yang-Mills theory. The Einstein equation together with the Dirac-Yang-Mills like equation thus provide a closed system to dynamically describe all classically admissible degrees of freedom introduced by the derived action (58). In particular, the solutions of these equations determine the Dirac operator \( \mathcal{D} \) as an extremum of the universal Dirac action, when the latter is restricted to the class of simple type Dirac operators (42). Notice that in the case considered the Dirac operators (42) are fully determined by the metric and the torsion, that is to say by a general metric connection \( \nabla^g \) on the tangent bundle.
4. Concluding remarks and outlook

On the twisted spinor bundle \((29)\) there exists no Dirac operator of simply type other than the quantized spin connection. The (double) twist of \((29)\) not only guarantees the existence of more general Dirac operators of simple type but also that the (relative) curvature added to the quantized connection \((36)\) does not change the Dirac equation. In fact, it is well-known that the coupling of the fermions to curvature like terms may spoil renormalizability. Though the action \((58)\) is derived by the distinguished class of Dirac operators \((42)\), the inner degrees of freedom of matter only “minimally couple” to torsion similar to ordinary Dirac-Yang-Mills gauge theory. Accordingly, the Yang-Mills like field equation \((59)\) for torsion should be contrasted with the field equation \((2)\) of ordinary Einstein-Cartan theory. The coupling constant \(g > 0\) determines the coupling strength of the inner degrees of freedom of the fermions to torsion similar to ordinary Dirac-Yang-Mills theory. Like in Yang-Mills theory, the coupling constant is dimensionless in four (space-time) dimensions. In particular, it is independent of the gravitational constant. This is in strong contrast to the coupling constant obtained by the Palatini formalism of general relativity. Therefore, the assumption \(0 < g << 1\) makes the geometrical description \((29)\) of matter in terms of Rarita-Schwinger fields phenomenological acceptable. Technically, such a weak coupling also allows to treat the field equations perturbatively. In particular, one also obtains non-trivial solutions of \((59)\), even if the coupling to the fermions and the self-coupling of the torsion potential is neglected (e.g. whenever all other interactions are assumed to be much stronger than the interaction with the torsion field). This is also in strong contrast to \((2)\). In the weak coupling case one gets back ordinary Einstein-Dirac theory whereby matter propagates in a non-trivial background provided by solutions of the linearized wave equation \(\Box g A - d\delta g A = 0\), with the (space-time) metric \(g_M\) being determined as in ordinary Einstein-Dirac theory. Hence, the energy of the propagating torsion \(\tau_A\) is seen to contribute effectively to the cosmological constant and thus to the dark energy in the universe. We stress that all of these considerations make physically sense even if gravitational effects are supposed to be negligible, such that \((M, g_M)\) is Minkowski space-time.

In the scheme presented torsion couples to matter similar to Yang-Mills gauge theory. Hence, when Minkowski space-time is presumed one may apply an analogous procedure to quantize the torsion potential (and thus torsion) as for Yang-Mills gauge potentials. However, in four dimensions and signature \(s = \mp 2\), the “gauge group” is provided by the Lorentz group and hence is non-compact. This is known to cause serious trouble, for instance, when one tries to make use of the path integral procedure widely used to (perturbatively) quantize non-abelian gauge theories.

In a forthcoming work we shall discuss the influence of the torsion within the physical frame provided by the Standard Model when the latter is geometrically described by Dirac operators of simple type similar to the scheme presented here. Especially, we shall discuss a possible influence on the Higgs mass when torsion is assumed to be massive. In [16] it has been shown how the Standard Model allows to make a prediction of the Higgs mass if the Standard Model is geometrically described by Dirac operators of simple type. Though the derived Higgs mass is within the range allowed by the Standard Model it is yet too big than experimentally confirmed by the LHC. The coupling to torsion, however, introduces a possible additional degree of freedom and
therefore may allow to lower the predicted value of the Higgs mass in [16] within the geometrical frame presented here, which slightly differs from that in loc. site.

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