TERMINAL TORIC FANO THREE-FOLDS WITH CERTAIN NUMERICAL CONDITIONS

HIROSHI SATO AND RYOTA SUMIYOSHI

Abstract. We completely classify the $\mathbb{Q}$-factorial terminal toric Fano three-folds such that the sum of the squared torus invariant prime divisors is non-negative.

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1. INTRODUCTION

A smooth projective variety $X$ is a Fano manifold if its first Chern character $c_1(X) = -K_X$ is an ample divisor. Although the definition of Fano manifolds is very simple, there are a lot of important properties for them.

In order to investigate the existence of rational surfaces on a Fano manifold, the notion of 2-Fano manifolds was introduced in [5]:

Definition 1.1. A Fano manifold $X$ is a 2-Fano manifold if the second Chern character $c_2(X)$ is nef, that is, $c_2(X) \cdot S \geq 0$ for any surface $S$ on $X$.

For recent results for the classification of 2-Fano manifolds, see [1]. Few examples of 2-Fano manifolds are known so far. This means that the condition that $c_2(X)$ is nef strongly determines the structure of a Fano manifold.

For the case of toric manifolds, we can see this phenomenon without the condition that $X$ is Fano. Namely, it seems that projective toric manifolds whose second Chern characters are nef have common geometric properties (see [8], [10] and [11]).

In this paper, we generalize these researches for singular cases. Let $X$ be a $\mathbb{Q}$-factorial projective toric $d$-fold and $D_1, \ldots, D_n$ the torus invariant prime divisors on $X$. Put

$$\gamma_2(X) := D_1^2 + \cdots + D_n^2.$$ 

If $X$ is smooth, it is well known that $2c_2(X) = \gamma_2(X)$. So, we investigate a singular toric variety $X$ such that $\gamma_2(X)$ is positive or nef (see Definition 3.1). In particular, we completely determine which $\mathbb{Q}$-factorial terminal toric Fano 3-fold is $\gamma_2$-nef. There exist exactly 23 $\mathbb{Q}$-factorial terminal $\gamma_2$-nef toric Fano 3-folds (see Theorem 5.3). To prove the main theorem, we introduce the useful formulas for the calculation of intersection numbers.

This paper is organized as follows: In Section 2 we collect basic results for toric varieties. In Section 3 we introduce the notion of $\gamma_2$-positive and $\gamma_2$-nef toric varieties which are...
main objects of this paper. Moreover, we introduce the formulas for the calculation of intersection numbers which are useful for checking whether a given toric variety is $\gamma_2$-nef or not. In Section 3 we determine which Gorenstein toric del Pezzo surface is $\gamma_2$-nef. This classification is useful to understand the results of Section 4. In Section 5, we complete the classification of $\mathbb{Q}$-factorial terminal $\gamma_2$-nef toric Fano 3-folds. One can see that they have some common geometric properties.

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2. Preliminaries

In this section, we introduce some basic results and notation of toric varieties and intersection numbers on them. For the details, please see [3], [4] and [9].

Let $X = X_\Sigma$ be the toric $d$-fold associated to a fan $\Sigma$ in $N = \mathbb{Z}^d$ over an algebraically closed field $k$ of arbitrary characteristic. Put $N_\mathbb{R} := N \otimes \mathbb{R}$. It is well known that there exists a one-to-one correspondence between the $r$-dimensional cones in $\Sigma$ and the torus invariant subvarieties of dimension $d - r$ in $X$. Let $G(\Sigma)$ be the set of primitive generators for 1-dimensional cones in $\Sigma$. Thus, for $v \in G(\Sigma)$, we have the torus invariant prime divisor corresponding to $\mathbb{R} \geq 0v \in \Sigma$.

For an $r$-dimensional simplicial cone $\sigma \in \Sigma$, let $N_\sigma \subset N$ be the sublattice generated by $\sigma \cap N$ and let $\sigma \cap G(\Sigma) = \{v_1, \ldots, v_r\}$. Put

$$\text{mult}(\sigma) := [N_\sigma : Zv_1 + \cdots + Zv_r],$$

which is the index of the subgroup $Zv_1 + \cdots + Zv_r$ in $N_\sigma$. The following property for intersection numbers on a toric variety is fundamental.

**Proposition 2.1.** Let $X$ be a $\mathbb{Q}$-factorial toric $d$-fold, and let $\sigma, \tau \in \Sigma$, $V_\sigma$ and $V_\tau$ the torus invariant subvarieties associated to $\sigma$ and $\tau$. If $\sigma$ and $\tau$ span $\lambda \in \Sigma$ with $\dim \lambda = \dim \sigma + \dim \tau$, then

$$V_\sigma \cdot V_\tau = \frac{\text{mult}(\sigma) \cdot \text{mult}(\tau)}{\text{mult}(\lambda)} V_\lambda,$$

where $V_\lambda$ is the torus invariant subvariety associated to $\lambda$. On the other hand, if $\sigma$ and $\tau$ are contained in no cone of $\Sigma$, then $V_\sigma \cdot V_\tau = 0$.

Let $X$ be a projective toric $d$-fold. For $1 \leq i \leq d$, we put

$$Z_i(X) := \{\text{the } i\text{-cycles on } X\}, \text{ while } Z^i(X) := \{\text{the } i\text{-cocycles on } X\}.$$

We introduce the numerical equivalence $\equiv$ on $Z_i(X)$ and $Z^i(X)$ as follows: For $C \in Z_i(X)$, we define $C \equiv 0$ if $D \cdot C = 0$ for any $D \in Z^i(X)$, while for $D \in Z^i(X)$, we define $D \equiv 0$ if $D \cdot C = 0$ for any $C \in Z_i(X)$. We put

$$N_i(X) := (Z_i(X) \otimes \mathbb{R}) / \equiv, \text{ while } N^i(X) := (Z^i(X) \otimes \mathbb{R}) / \equiv.$$

3. Formulas for intersection numbers

In this section, we introduce the notion of $\gamma_r$-positive toric varieties, which are main objects of this paper. Also, we prepare the formulas for intersection numbers which are useful for our calculations in Section 5.
Definition 3.1. Let $X$ be a $\mathbb{Q}$-factorial projective toric $d$-fold. For any integer $1 \leq r \leq d$, put
\[
\gamma_r = \gamma_r(X) := D_1^r + \cdots + D_n^r \in N^r(X),
\]
where $D_1, \ldots, D_n$ are the torus invariant prime divisors.

If $\gamma_r \cdot Y > 0$ (resp. $\geq 0$) for any $r$-dimensional torus invariant subvariety $Y \subset X$, then we say that $X$ is $\gamma_r$-positive (resp. $\gamma_r$-nef).

Remark 3.2. For the case where $X$ is smooth, it is well known that
\[
\frac{1}{r!} \gamma_r(X) = \text{ch}_r(X)
\]
which is the $r$-th Chern character.

The main purpose of this paper is to determine which $\mathbb{Q}$-factorial terminal toric Fano 3-fold is $\gamma_2$-positive or $\gamma_2$-nef (see Section 5).

For the classification, we introduce the following notion about intersection numbers on $X$. This is convenient for the calculation of intersection numbers.

Definition 3.3. Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric $d$-fold, and $S \subset X$ a torus invariant subsurface on $X$. Let $G(\Sigma) = \{x_1, \ldots, x_n\}$, and we consider the polynomial ring $\mathcal{R}(X) := \mathbb{Q}[X_1, \ldots, X_n]$, where $X_1, \ldots, X_n$ are the independent variables of polynomials corresponding to $x_1, \ldots, x_n$, respectively. Let $D_1, \ldots, D_n$ be the torus invariant prime divisors corresponding to $x_1, \ldots, x_n$, respectively. Then, we define the quadric homogeneous polynomial $I_{S/X} \in \mathbb{Q}[X_1, \ldots, X_n]$ as follows:
\[
I_{S/X} = I_{S/X}(X_1, \ldots, X_n) := \sum_{1 \leq i, j \leq n} (D_i \cdot D_j \cdot S)X_iX_j.
\]

For the cases where $\rho(S) = 1$ or 2, we can determine $I_{S/X}$ explicitly as follows.

Proposition 3.4. Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric $d$-fold, and $S \subset X$ a torus invariant subsurface on $X$. If $\rho(S) = 1$, then $\gamma_2 \cdot S > 0$.

Proof. Let $\tau \in \Sigma$ be a $(d-2)$-dimensional cone associated to $S$ and $\tau \cap G(\Sigma) = \{x_1, \ldots, x_{d-2}\}$. There exist exactly 3 maximal cones
\[
\mathbb{R}_{\geq 0}x_{d-1} + \mathbb{R}_{\geq 0}x_d + \tau, \quad \mathbb{R}_{\geq 0}x_d + \mathbb{R}_{\geq 0}x_{d+1} + \tau, \quad \mathbb{R}_{\geq 0}x_{d+1} + \mathbb{R}_{\geq 0}x_{d-1} + \tau
\]
in $\Sigma$, where $\{x_{d-1}, x_d, x_{d+1}\} \subset G(\Sigma)$. There exists a linear relation
\[
a_1x_1 + \cdots + a_{d-2}x_{d-2} + a_{d-1}x_{d-1} + a_dx_d + a_{d+1}x_{d+1} = 0,
\]
where $a_1, \ldots, a_{d+1} \in \mathbb{Q}$ and $a_{d-1}, a_d, a_{d+1} > 0$. We remark that this relation is unique up to multiplication by a positive rational number. Since $\{x_1, \ldots, x_d\}$ is an $\mathbb{R}$-basis for $N_\mathbb{R}$, we have $d$ relations
\[
D_i - \frac{a_i}{a_{d+1}}D_{d+1} + F_i = 0 \quad \text{for } 1 \leq i \leq d,
\]
in $N^1(X)$, where $D_1, \ldots, D_{d+1}$ are torus invariant prime divisors corresponding to $x_1, \ldots, x_{d+1}$, respectively, while $F_1, \ldots, F_d$ are torus invariant divisors which do not intersect $S$. So, we can ignore $F_i$'s in the following calculation. For any $1 \leq i, j \leq d$,
\[
D_i \cdot D_j \cdot S = \left(\frac{a_i}{a_{d+1}} \times a_{d+1} \cdot D_{d-1}\right) \cdot \left(\frac{a_j}{a_{d+1}} \times a_{d+1} \cdot D_d\right) \cdot S = \frac{a_ia_j}{a_ia_{d+1}a_d}D_{d-1}D_d \cdot S.
\]
This means that there exists a rational positive number $\alpha$ such that for any $1 \leq i, j \leq d+1$, we have $\alpha D_i \cdot D_j \cdot S = a_ia_j$. In particular, we have
\[
\gamma_2 \cdot S = \frac{a_1^2 + \cdots + a_{d+1}^2}{\alpha} > 0.
\]
\[\square\]
Remark 3.5. The calculation in the proof of Proposition 3.4 tells us that
\[ \alpha I_{S/X} = (a_1 X_1 + \cdots + a_{d+1} X_{d+1})^2, \]
where \( X_1, \ldots, X_{d+1} \) are the independent variables of polynomials corresponding to \( x_1, \ldots, x_{d+1} \), respectively.

Although the case where \( \rho(S) = 2 \) is more complicated, we can determine \( I_{S/X} \) as follows:

Let \( X = X_\Sigma \) be a \( \mathbb{Q} \)-factorial projective toric \( d \)-fold, and \( S \subset X \) a torus invariant subsurface of \( \rho(S) = 2 \). Let \( \tau \in \Sigma \) be a \( (d-2) \)-dimensional cone associated to \( S \) and \( \tau \cap G(\Sigma) = \{ x_1, \ldots, x_{d-2} \} \). Then, there exist exactly 4 maximal cones
\[ \mathbb{R}_{\geq 0} y_1 + \mathbb{R}_{\geq 0} y_3 + \tau, \mathbb{R}_{\geq 0} y_2 + \mathbb{R}_{\geq 0} y_3 + \tau, \mathbb{R}_{\geq 0} y_1 + \mathbb{R}_{\geq 0} y_4 + \tau, \mathbb{R}_{\geq 0} y_2 + \mathbb{R}_{\geq 0} y_4 + \tau \]
in \( \Sigma \), where \( \{ y_1, y_2, y_3, y_4 \} \subset G(\Sigma) \). For \( (d-1) \)-dimensional cones \( \mathbb{R}_{\geq 0} y_3 + \tau \) and \( \mathbb{R}_{\geq 0} y_1 + \tau \), we have linear relations
\[ b_1 y_1 + b_2 y_2 + c_3 y_3 + a_1 x_1 + \cdots + a_{d-2} x_{d-2} = 0 \]
and
\[ b_3 y_3 + b_4 y_4 + c_1 y_1 + e_1 x_1 + \cdots + e_{d-2} x_{d-2} = 0, \]
respectively, where \( a_1, \ldots, a_{d-2}, b_1, b_2, b_3, b_4, c_1, c_3, e_1, \ldots, e_{d-2} \in \mathbb{Q} \) and \( b_1, b_2, b_3, b_4 > 0 \). Then, the following holds.

Proposition 3.6. Under the above setting, we have
\[ \alpha I_{S/X} = -b_3 c_1 \left( b_1 y_1 + b_2 y_2 + c_3 y_3 + \sum_{i=1}^{d-2} a_i X_i \right)^2 + 2b_1 b_3 \left( b_1 y_1 + b_2 y_2 + c_3 y_3 + \sum_{i=1}^{d-2} a_i X_i \right) \left( b_3 y_3 + b_4 y_4 + c_1 y_1 + \sum_{i=1}^{d-2} e_i X_i \right) - b_1 c_3 \left( b_3 y_3 + b_4 y_4 + c_1 y_1 + \sum_{i=1}^{d-2} e_i X_i \right)^2, \]
for some positive rational number \( \alpha \), where \( X_1, \ldots, X_{d-2}, Y_1, Y_2, Y_3, Y_4 \) are the independent variables of polynomials corresponding to \( x_1, \ldots, x_{d-2}, y_1, y_2, y_3, y_4 \), respectively. In particular,
\[ \alpha \gamma_2 \cdot S = -b_3 c_1 \left( b_1^2 + b_2^2 + c_3^2 + \sum_{i=1}^{d-2} a_i^2 \right) + 2b_1 b_3 \left( b_1 c_1 + b_3 c_3 + \sum_{i=1}^{d-2} a_i e_i \right) - b_1 c_3 \left( b_3^2 + b_4^2 + c_1^2 + \sum_{i=1}^{d-2} e_i^2 \right). \]

Proof. \( \{ x_1, \ldots, x_{d-2}, y_1, y_3 \} \) is an \( \mathbb{R} \)-basis for \( N_\mathbb{R} \), and by using this basis, \( y_2 \) and \( y_4 \) are expressed as
\[ y_2 = -\frac{1}{b_2} \left( b_1 y_1 + c_3 y_3 + \sum_{i=1}^{d-2} a_i x_i \right) \]
and
\[ y_4 = -\frac{1}{b_4} \left( b_3 y_3 + c_1 y_1 + \sum_{i=1}^{d-2} e_i x_i \right), \]
respectively. Thus, we obtain \( d \) relations
\[ \left( D_i - \frac{a_i}{b_2} E_2 - \frac{e_i}{b_4} E_4 \right) \cdot S = 0 \]
for \( 1 \leq i \leq d-2 \),
\[ (1) \left( E_1 - \frac{b_1}{b_2} E_2 - \frac{c_1}{b_4} E_4 \right) \cdot S = 0 \]
and
\[ (2) \left( E_3 - \frac{c_3}{b_2} E_2 - \frac{b_3}{b_4} E_4 \right) \cdot S = 0 \]
in \(N_1(X)\), where \(D_1, \ldots, D_{d-2}, E_1, E_2, E_3, E_4\) are the torus invariant prime divisors corresponding to \(x_1, \ldots, x_{d-2}, y_1, y_2, y_3, y_4\), respectively. On the other hand, \(E_1E_2 \cdot S = E_3E_4 \cdot S = 0\) and \(E_1E_3 \cdot S, E_2E_4 \cdot S > 0\). So, we can express the intersection numbers by \(E_2E_4 \cdot S\).

By multiplying the equality (1) by \(E_1, E_2, E_3, E_4\), we obtain
\[
E_1^2 \cdot S = \frac{c_1}{b_4} E_1 E_4 \cdot S, \quad E_2^2 \cdot S = -\frac{b_2c_1}{b_1 b_4} E_2 E_4 \cdot S,
\]
\[
E_1E_3 \cdot S = \frac{b_1}{b_2} E_2 E_3 \cdot S, \quad E_1E_4 \cdot S = \frac{b_1}{b_2} E_2 E_4 \cdot S + \frac{c_1}{b_4} E_3^2 \cdot S,
\]
while by multiplying the equality (2) by \(E_1, E_2, E_3, E_4\), we obtain
\[
E_1^2 \cdot S = \frac{b_3}{b_4} E_1 E_4 \cdot S, \quad E_2E_3 \cdot S = \frac{c_3}{b_2} E_2^2 \cdot S + \frac{b_3}{b_4} E_2 E_4 \cdot S,
\]
\[
E_3^2 \cdot S = \frac{c_3}{b_2} E_2 E_3 \cdot S, \quad E_1^2 \cdot S = -\frac{b_4c_3}{b_2 b_3} E_2 E_4 \cdot S.
\]
Thus, we have the equalities
\[
E_1E_4 \cdot S = \left( \frac{a_i}{b_2} - \frac{c_1c_3}{b_2b_3} \right) E_2 E_4 \cdot S, \quad E_2^2 \cdot S = \frac{1}{b_4} \left( \frac{a_i}{b_2} - \frac{c_1c_3}{b_2b_3} \right) E_2 E_4 \cdot S,
\]
\[
E_1E_3 \cdot S = \frac{b_1}{b_4} \left( \frac{a_i}{b_2} - \frac{c_1c_3}{b_2b_3} \right) E_2 E_4 \cdot S, \quad E_2E_3 \cdot S = \left( \frac{a_i}{b_4} - \frac{c_1c_3}{b_2b_3} \right) E_2 E_4 \cdot S,
\]
\[
E_3^2 \cdot S = \frac{c_3}{b_2} \left( \frac{b_3}{b_4} - \frac{c_1c_3}{b_1b_4} \right) E_2 E_4 \cdot S.
\]

On the other hand, for any \(1 \leq i, j \leq d - 2\), we have
\[
D_i D_j \cdot S = \left( \frac{a_i}{b_2} E_2 + \frac{e_i}{b_4} E_4 \right) \cdot \left( \frac{a_j}{b_2} E_2 + \frac{e_j}{b_4} E_4 \right) \cdot S
\]
\[
= \frac{a_i a_j}{b_2^2} E_2^2 \cdot S + \frac{e_i e_j}{b_4^2} E_4^2 \cdot S + \frac{a_i e_j + a_j e_i}{b_2 b_4} E_2 E_4 \cdot S
\]
\[
= \left( \frac{a_i a_j c_1}{b_1 b_2 b_4} - \frac{e_i e_j c_2}{b_2 b_3 b_4} + \frac{a_i e_j + a_j e_i}{b_2 b_4} \right) E_2 E_4 \cdot S,
\]
while
\[
D_i E_1 \cdot S = \frac{e_i}{b_4} E_1 E_4 \cdot S = \frac{e_i}{b_4} \left( \frac{b_1}{b_2} - \frac{c_1 c_3}{b_2 b_3} \right) E_2 E_4 \cdot S,
\]
\[
D_i E_2 \cdot S = \frac{a_i}{b_2} E_2 \cdot S + \frac{e_i}{b_4} E_2 E_4 \cdot S = \left( \frac{e_i}{b_4} - \frac{a_i c_1}{b_2 b_4} \right) E_2 E_4 \cdot S,
\]
\[
D_i E_3 \cdot S = \frac{a_i}{b_2} E_2 E_3 \cdot S = \frac{a_i b_3}{b_2} \left( \frac{b_3}{b_4} - \frac{c_1 c_3}{b_1 b_4} \right) E_2 E_4 \cdot S,
\]
\[
D_i E_4 \cdot S = \frac{a_i}{b_2} E_2 E_4 \cdot S + \frac{e_i}{b_4} E_4^2 \cdot S = \left( \frac{a_i}{b_2} - \frac{c_3 e_i}{b_2 b_4} \right) E_2 E_4 \cdot S,
\]
for \(1 \leq i \leq d - 2\). Put \(\beta := D_2 D_4 \cdot S\). By multiplying these intersection numbers by the positive rational number \(b_1 b_2 b_3 b_4\), we have the following tables of intersection numbers:

| \(\frac{b_1 b_2 b_3 b_4}{\beta}E_i E_j \cdot S\) | \(E_1\) | \(E_2\) | \(E_3\) | \(E_4\) |
|---|---|---|---|---|
| \(E_1\) | \(b_1^2 b_3 c_1 - b_1 c_1^2 c_3\) | \(0\) | \(b_2^2 b_3^2 - b_1 b_3 c_1 c_3\) | \(b_2^2 b_4 b_1 - b_1 b_4 c_1 c_3\) |
| \(E_2\) | \(-b_2^2 b_3 c_1\) | \(b_1 b_2 b_3^2 - b_2 b_3 c_1 c_3\) | \(b_1 b_2 b_3 b_4\) | \(b_1 b_2 b_3 c_3 - b_3 c_1 c_3^2\) |
| \(E_3\) | \(b_1 b_2^2 c_3 - b_3 c_1 c_3^2\) | \(0\) | \(b_2^2 c_3 - b_3 c_1 c_3^2\) | \(-b_1 b_2^2 c_3\) |
| \(E_4\) | \(b_1 b_2^2 c_3\) | \(b_1 b_2^2 c_3\) | \(-b_1 b_2^2 c_3\) | \(0\) |
for $1 \leq i \leq d-2$.

$$\frac{b_1 b_2 b_3 b_4}{D_i} D_i E_j \cdot S = -a_i a_j b_3 c_1 - e_i e_j b_1 c_3 + a_i e_j b_1 b_3 + a_j e_i b_1 b_3$$

for $1 \leq i, j \leq d-2$.

Put

$$f_1 := b_1 Y_1 + b_2 Y_2 + c_3 Y_3, \quad f_2 := b_3 Y_3 + b_4 Y_4 + c_1 Y_1,$$

$$g_1 := \sum_{i=1}^{d-2} a_i X_i \text{ and } g_2 := \sum_{i=1}^{d-2} e_i X_i \in \mathcal{R}(X).$$

Then,

$$-b_3 c_1 \left( b_1 Y_1 + b_2 Y_2 + c_3 Y_3 + \sum_{i=1}^{d-2} a_i X_i \right)^2 +$$

$$2b_1 b_3 \left( b_1 Y_1 + b_2 Y_2 + c_3 Y_3 + \sum_{i=1}^{d-2} a_i X_i \right) \left( b_3 Y_3 + b_4 Y_4 + c_1 Y_1 + \sum_{i=1}^{d-2} e_i X_i \right)$$

$$-b_1 c_3 \left( b_3 Y_3 + b_4 Y_4 + c_1 Y_1 + \sum_{i=1}^{d-2} e_i X_i \right)^2$$

$$= -b_3 c_1 (f_1 + g_1)^2 + 2b_1 b_3 (f_1 + g_1)(f_2 + g_2) - b_1 c_3 (f_2 + g_2)^2$$

$$= (-b_3 c_1 f_1^2 + 2b_1 b_3 f_1 f_2 - b_1 c_3 f_2^2) + (-b_3 c_1 g_1^2 + 2b_1 b_3 g_1 g_2 - b_1 c_3 g_2^2)$$

$$+ (-2b_1 c_3 f_1 g_1 + 2b_1 b_3 f_1 g_2 + 2b_1 b_3 f_2 g_1 - 2b_1 c_3 f_2 g_2).$$

Thus, one can easily check that every coefficient of this polynomial coincides with the intersection number calculated above. \qed

In Proposition [3.6] if $c_1 = c_3 = a_1 e_1 = \cdots = a_{d-2} e_{d-2} = 0$, we have $\gamma_2 \cdot S = 0$. In particular, the following holds:

**Proposition 3.7.** Let $X$ be a $\mathbb{Q}$-factorial projective toric $d$-fold. If there exists a toric finite morphism $\pi : X' \to X$ such that $X'$ is a direct product of lower-dimensional $\mathbb{Q}$-factorial projective $\gamma_2$-nef toric varieties, then $X$ is also $\gamma_2$-nef (but not $\gamma_2$-positive).

**Proof.** It is sufficient to prove for the case where $X$ itself is a direct product. Suppose $X = X_1 \times X_2$. For any torus invariant curves $C_1 \subset X_1$ and $C_2 \subset X_2$, put $S := C_1 \times C_2$. Then, for this $S$, the set of elements in $G(\Sigma)$ which appear in the left-hand side of

$$b_1 y_1 + b_2 y_2 + c_3 y_3 + a_1 x_1 + \cdots + a_{d-2} x_{d-2} = 0$$

and the one of

$$b_3 y_3 + b_4 y_4 + c_1 y_1 + e_1 x_1 + \cdots + e_{d-2} x_{d-2} = 0$$

have no common element, because $X$ is a direct product. Since $b_1 \neq 0$ and $b_3 \neq 0$, consequently, $c_1 = c_3 = a_1 e_1 = \cdots = a_{d-2} e_{d-2} = 0$. Thus, we have $\gamma_2 \cdot (C_1 \times C_2) = 0$ as seen above. Therefore, $X$ is $\gamma_2$-nef (but not $\gamma_2$-positive). \qed
4. Gorenstein del pezzo surfaces

In order to understand the 3-folds studied in Section 5 more deeply, we classify the Gorenstein $\gamma_2$-nef toric del Pezzo surfaces in this section. In this case, $\gamma_2$ is a rational number.

In general, the case where $\rho(X) = 1$ for any dimension can be determined easily as follows:

**Proposition 4.1.** If $X = X_\Sigma$ is a $\mathbb{Q}$-factorial projective toric $d$-fold of $\rho(X) = 1$, then $X$ is $\gamma_2$-positive.

**Proof.** Any torus invariant subsurface of $X$ is of Picard number 1. Therefore, this proposition is an immediate consequence from Proposition 3.4. □

For a projective toric surface $S$, the polynomial $I_{S/\tilde{S}}$ defined in Definition 3.3 can be calculated inductively:

Let $S = S_\Sigma$ be a projective toric surface. For a maximal cone $\sigma = \mathbb{R}_{\geq 0}x_1 + \mathbb{R}_{\geq 0}x_2 \in \Sigma$ and a primitive lattice point $p, q, r, s$ where $p, q, r, s \in \mathbb{Z}$ and $q > 0, s > 0, qr - ps > 0$. The linear relation

$$qy = (qr - ps)x_1 + sx_2$$

holds. We prepare two additional elements $x_3 = (a_3, b_3), x_4 = (a_4, b_4) \in G(\Sigma) \subset G(\tilde{\Sigma})$ such that $\mathbb{R}_{\geq 0}x_1 + \mathbb{R}_{\geq 0}x_3$ and $\mathbb{R}_{\geq 0}x_2 + \mathbb{R}_{\geq 0}x_4$ are maximal cones in $\Sigma$ (the case where $x_3 = x_4$ is admitted). Let $D_1, D_2, D_3, D_4$ be the torus invariant prime divisors on $S$ associated to $x_1, x_2, x_3, x_4$, respectively, while $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3, \tilde{D}_4, E$ be the torus invariant prime divisors on $\tilde{S}$ associated to $x_1, x_2, x_3, x_4, y$, respectively.

**Proposition 4.2.** Under the above setting, we have

$$I_{\tilde{S}/\tilde{\Sigma}} = I_{S/\tilde{S}} - \frac{1}{qs(qr - ps)}((qr - ps)x_1 + sx_2 - qy)^2.$$

**Proof.** We calculate the intersection numbers of $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3, \tilde{D}_4, E$.

On $S$, we have

$$D_1D_2 = \frac{1}{\det \begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}} = \frac{1}{q}$$

by Proposition 2.1. The relations

$$D_1 + pD_2 + a_3D_3 + a_4D_4 + \cdots = 0$$

and

$$qD_2 + b_3D_3 + b_4D_4 + \cdots = 0$$

tells us that

$$D_1^2 = -\frac{p}{q} - a_3D_1D_3$$

and

$$D_2^2 = -\frac{b_4}{q}D_2D_4.$$

On the other hand, on $\tilde{S}$,

$$\tilde{D}_1E = \frac{1}{\det \begin{pmatrix} 1 & r \\ 0 & s \end{pmatrix}} = \frac{1}{s} \quad \text{and} \quad \tilde{D}_2E = \frac{1}{\det \begin{pmatrix} r & p \\ s & q \end{pmatrix}} = \frac{1}{qr - ps}$$

for $s \neq qr - ps$.
The relations
\[ \tilde{D}_1 + p\tilde{D}_2 + a_3\tilde{D}_3 + a_4\tilde{D}_4 + rE + \cdots = 0 \]
and
\[ q\tilde{D}_2 + b_3\tilde{D}_3 + b_4\tilde{D}_4 + sE + \cdots = 0 \]
tells us that
\[ \tilde{D}_1^2 = -a_3\tilde{D}_1\tilde{D}_3 - \frac{r}{s}\tilde{D}_1^2 - \frac{s}{q(qr-ps)} \]
and
\[ E^2 = -\frac{q}{s(qr-ps)}. \]
Additionally, we remark that
\[ D_1D_3 = \tilde{D}_1\tilde{D}_3 \]
and
\[ D_2D_4 = \tilde{D}_2\tilde{D}_4. \]
Thus, we obtain
\[ I_{\tilde{S}/\tilde{S}} - I_{S/S} = \frac{2}{q}X_1X_2 + \left( \frac{p}{q} - \frac{r}{s} \right)X_1^2 - \frac{s}{q(qr-ps)}X_2^2 \]
\[ + \frac{2}{s}X_1Y + \frac{2}{qr-ps}X_2Y - \frac{q}{s(qr-ps)}Y^2, \]
where \(X_1, X_2, X_3, X_4, Y\) are the independent variables of polynomials corresponding to \(x_1, x_2, x_3, x_4, y\), respectively.

The following is an immediate consequence of Proposition 4.2:

**Corollary 4.3.** Let \( \varphi : \tilde{S} \to S \) be a toric birational morphism between projective toric surfaces \( \tilde{S} \) and \( S \). Then, \( \gamma_2(\tilde{S}) < \gamma_2(S) \).

By Corollary 4.3, for the classification, it is sufficient to investigate Gorenstein toric del Pezzo surfaces of lower Picard numbers. The Gorenstein toric del Pezzo surfaces are classified by Koelman [7]. There exist exactly 16 Gorenstein toric del Pezzo surfaces, and we can get the list of them on the internet (see [6]).

The case of Picard number 1 is done by Proposition 4.1. For the next case where the Picard number is two, we show the following easy lemma:

**Lemma 4.4.** Let \( S = S_\Sigma \) be a projective toric surface of \( \rho(S) = 2 \). If there exist \( x_1, x_2 \in G(\Sigma) \) such that \( x_1 + x_2 = 0 \), then \( S \) is \( \gamma_2 \)-nef.

Proof. Let \( G(\Sigma) = \{x_1, x_2, x_3, x_4\} \). Then, we may assume that there exists another relation
\[ ax_3 + bx_4 - cx_1 = 0 \]
for \( a, b, c \in \mathbb{Z}_{\geq 0} \) (\( a, b > 0 \)). By Proposition 3.6, there exists a positive rational number \( \alpha \) such that
\[ \alpha \gamma_2 = ac \times (1 + 1) + 2a \times (-c) = 0. \]

There exists exactly one Gorenstein \( \gamma_2 \)-nef (but not \( \gamma_2 \)-positive) toric del Pezzo surface \( S = S_\Sigma \) of \( \rho(S) = 2 \) such that there is no centrally symmetric pair in \( G(\Sigma) \) (see ID 9 in Table 1 below). Therefore, by Corollary 4.3 if \( \rho(S) \geq 3 \), then \( S \) is not \( \gamma_2 \)-nef.

Thus, we obtain the following list of Gorenstein \( \gamma_2 \)-nef toric del Pezzo surfaces. In the first column of the table, we describe ID of Kasprzyk’s classification list:

\[ \text{http://www.grdb.co.uk/forms/toricldp} \]

We describe a toric surface \( S_\Sigma \) by giving all the elements in \( G(\Sigma) \) in the second column.
Table 1. Gorenstein $\gamma_2$-nef toric del Pezzo surfaces

| ID | $\text{G}(\Sigma)$          |
|----|-----------------------------|
| 12 | (0, 1), (3, 1), (-3, -2)    |
| 13 | (0, 1), (4, 1), (-2, -1)    |
| 14 | (0, 1), (2, 1), (-3, -2)    |
| 15 | (0, 1), (2, 1), (-1, -1)    |
| 16 | (0, 1), (1, 1), (-1, -2)    |
| 6  | (0, 1), (2, 1), (0, -1), (-2, -1) |
| 8  | (0, 1), (1, 1), (0, -1), (-2, -1) |
| 9  | (0, 1), (1, 1), (-1, -2), (-1, 0) |
| 10 | (0, 1), (1, 1), (0, -1), (-1, 0) |
| 11 | (0, 1), (1, 1), (0, -1), (-1, -1) |

5. Terminal Fano 3-folds

In this final section, we give the classification of $\mathbb{Q}$-factorial terminal $\gamma_2$-nef toric Fano 3-folds.

**Definition 5.1.** A $\mathbb{Q}$-factorial projective toric variety $X$ is a Fano variety if its anticanonical divisor $-K_X$ is an ample divisor.

**Remark 5.2.** A $\gamma_1$-positive toric variety is nothing but a toric Fano variety.

Terminal toric Fano 3-folds are classified by Kasprzyk [6], and the classification list is available on the internet (see [6]). There exist exactly 233 $\mathbb{Q}$-factorial terminal toric Fano 3-folds up to isomorphism. We checked the non-negativities of $\gamma_2$ for these 233 $\mathbb{Q}$-factorial terminal toric Fano 3-folds mainly by hand calculation. Partially, we used the software polymake (see [2]). Thus, the following is the main theorem of this paper:

**Theorem 5.3.** There exist exactly 23 $\mathbb{Q}$-factorial terminal $\gamma_2$-nef toric Fano 3-folds.

The following is the classification table. In the first column of the table, we describe ID of Kasprzyk’s classification list:

[http://www.grdb.co.uk/forms/toricf3t](http://www.grdb.co.uk/forms/toricf3t)

We describe a toric variety $X_\Sigma$ by giving all the elements in $\text{G}(\Sigma)$ in the second column. In the 3rd column, if there exists a Fano contraction $\varphi : X \to S$ such that $S$ is a Gorenstein $\gamma_2$-nef toric del Pezzo surface, then we describe ID of $S$ in Table 1 in Section [4].
Table 2. \(\mathbb{Q}\)-factorial terminal \(\gamma_2\)-nef toric Fano 3-folds

| ID | \((1,0,0),(0,1,0),(1,-3,5),(-2,2,-5)\) |
|----|------------------------------------------|
| 2  | \((1,0,0),(0,1,0),(1,-2,7),(-1,1,-5)\)   |
| 3  | \((1,0,0),(0,1,0),(-2,2,7),(1,-2,-5)\)   |
| 4  | \((1,0,0),(0,1,0),(0,0,1),(-1,-1,-1)\)   |
| 5  | \((1,0,0),(0,1,0),(-1,-2,5),(-1,1,-4)\)   |
| 6  | \((1,0,0),(0,1,0),(-2,1,5),(1,-1,-3)\)   |
| 7  | \((1,0,0),(0,1,0),(1,1,2),(-1,-1,-1)\)   |
| 8  | \((1,0,0),(0,1,0),(-1,-2,3),(-1,1,-2)\)   |
| 9  | \((1,0,0),(0,1,0),(1,2,3),(-1,-1,0),(-1,-2,3)\) | 12 |
| 34 | \((1,0,0),(0,1,0),(0,0,1),(-1,-1,0),(0,0,-1)\) | 16 |
| 35 | \((1,0,0),(0,1,0),(-1,-2,3),(-1,1,-2),(-1,0,0)\) | 14 |
| 36 | \((1,0,0),(0,1,0),(1,1,2),(-1,-1,-1),(1,1,1)\) | 16 |
| 38 | \((1,0,0),(0,1,0),(-1,2,3),(-1,1,-2),(-1,0,0)\) | 14 |
| 43 | \((1,0,0),(0,1,0),(-1,2,3),(-1,1,-2),(-1,1,1)\) | 15 |
| 45 | \((1,0,0),(0,1,0),(1,1,2),(-1,1,-1),(-1,0,0)\) | 15 |
| 47 | \((1,0,0),(0,1,0),(1,1,2),(-1,0,0),(0,-1,0),(-1,-1,-2)\) | 6, 6, 6 |
| 62 | \((1,0,0),(0,1,0),(0,0,1),(-1,0,0),(0,-1,0),(0,0,-1)\) | 11, 11, 11 |
| 105| \((1,0,0),(0,1,0),(1,1,1),(-1,1,0),(0,0,-1),(1,1,0)\) | 11 |
| 110| \((1,0,0),(0,1,0),(1,1,2),(-1,-1,-1),(-1,0,0),(-1,0,0)\) | 8, 8 |
| 123| \((1,0,0),(0,1,0),(0,0,1),(-1,-1,0),(0,0,-1),(-1,0,0)\) | 10, 11 |
| 131| \((1,0,0),(0,1,0),(0,0,1),(-1,1,-1),(-1,0,0),(-1,1,0)\) | 10 |
| 140| \((1,0,0),(0,1,0),(0,0,1),(-1,-1,-1),(-1,0,0),(-1,-1,-2)\) | 9 |

Remark 5.4. In the above table, every \(\mathbb{Q}\)-factorial terminal \(\gamma_2\)-nef toric Fano 3-fold \(X\) of \(\rho(X) \geq 2\) is not \(\gamma_2\)-positive.

So, we suggest the following question:

Question 5.5. Does there exist a \(\mathbb{Q}\)-factorial terminal projective \(\gamma_2\)-positive toric variety \(X\) of \(\rho(X) \geq 2\)?

The following example tells us that the answer to Question 5.5 is positive, if \(X\) is not terminal:

Example 5.6. Let \(\Sigma\) be a complete fan in \(N = \mathbb{Z}^2\) such that \(G(\Sigma) = \{x_1 := (1,0), x_2 := (1,2), x_3 := (-1,2), x_4 := (-1,-1)\}\), and \(S = S_2\) the associated projective toric surface. One can easily find the two relations
\[
2x_1 + x_3 - x_2 = 0 \quad \text{and} \quad 3x_2 + 4x_4 - x_3 = 0.
\]
By Proposition 3.6, for a positive number \(\alpha\), we have
\[
\alpha \gamma_2 = 3 \times (1 + 2^2 + 1) + 2 \times 3 \times (-1 - 3) + (3^2 + 4^2 + 1) = 20 > 0.
\]
Therefore, \(S\) is \(\gamma_2\)-positive. We remark that \(S\) has two canonical singular points and one log terminal singular point.

On the other hand, for every \(\gamma_2\)-nef toric Fano 3-fold \(X\) of \(\rho(X) \geq 2\) in Table 2, there exists a Fano contraction \(\varphi : X \to S\) such that \(\dim S = 2\) and \(S\) is \(\gamma_2\)-nef. So, the following is also a natural question.
**Question 5.7.** For any \(\mathbb{Q}\)-factorial terminal projective \(\gamma_2\)-nef toric \(d\)-fold of \(\rho(X) \geq 2\), does one of the following hold?

1. There exists a Fano contraction \(\varphi : X \to \overline{X}\) such that \(\overline{X}\) is a \(\gamma_2\)-nef toric \((d-1)\)-fold.
2. There exists a toric finite morphism \(\pi : X' \to X\) such that \(X'\) is a direct product of lower-dimensional \(\gamma_2\)-nef toric varieties (see Proposition 3.7).

**Remark 5.8.** Without the assumption that \(X\) is terminal, the Gorenstein \(\gamma_2\)-nef toric del Pezzo surface of ID 9 in Table 1 in Section 4 tells us that the answer to Question 5.7 is negative.

**Remark 5.9.** There exists a \(\mathbb{Q}\)-factorial terminal toric 3-fold such that it has a Fano contraction to a \(\gamma_2\)-nef surface, but is not \(\gamma_2\)-nef. For example, let \(X = X_\Sigma\) be the smooth toric Fano 3-fold such that
\[
G(\Sigma) = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1), (-1, -1, 0), (1, 1, 0)\}.
\]
Then, \(X\) is a \(\mathbb{P}^1\)-bundle over \(\mathbb{P}^1 \times \mathbb{P}^1\), but not \(\gamma_2\)-nef.

We end this section by giving an example of calculations for a terminal toric Fano 3-fold.

**Example 5.10.** Let \(X = X_\Sigma\) be the \(\mathbb{Q}\)-factorial terminal toric Fano 3-fold of ID 34 in the above table. Put \(v_1 := (1, 0, 0), v_2 := (0, 1, 0), v_3 := (-2, 1, 5), v_4 := (-1, -1, -3), v_5 := (-1, 1, 3)\), and put \(D_1, \ldots, D_5\) be the torus invariant divisors corresponding to \(v_1, \ldots, v_5\), respectively. Then, we have a 3 relations
\[
D_1 - 2D_3 + D_4 - D_5 = 0, \quad D_2 + D_3 - D_4 + D_5 = 0, \quad 5D_3 - 3D_4 + 3D_5 = 0
\]
in \(\text{N}^1(X)\). There exist exactly 6 maximal cones generated by
\[
\{v_1, v_2, v_4\}, \quad \{v_2, v_3, v_4\}, \quad \{v_1, v_3, v_4\}, \quad \{v_1, v_2, v_5\}, \quad \{v_2, v_3, v_5\}, \quad \{v_1, v_3, v_5\}.
\]
The equalities \(3D_4 = 5D_3 + 3D_5, 3D_1 = D_3\) and \(2D_1 = D_2\) say that it is sufficient to check the non-negativities for two torus invariant surfaces \(S_1\) and \(S_5\) corresponding to \(1\)-dimensional cones \(\mathbb{R}_{\geq 0} v_1\) and \(\mathbb{R}_{\geq 0} v_5\), respectively. Since \(\rho(S_5) = 1\), we have \(\gamma_2 \cdot S_5 > 0\) by Proposition 3.7. On the other hand, since \(\rho(S_1) = 2\), we can apply Proposition 3.6. One can easily calculate the relations
\[
2v_2 + 3v_3 - 5v_5 + v_1 = 0 \quad \text{and} \quad v_4 + v_5 = 0
\]
corresponding to \(2\)-dimensional cones
\[
\mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_5 \quad \text{and} \quad \mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2,
\]
respectively. By Proposition 3.6, there exists a positive rational number \(\alpha\) such that
\[
\alpha I_{S_1/X} = 4(2V_2 + 3V_2 - 5V_5 + V_1)(V_4 + V_5) + 10(V_4 + V_5)^2,
\]
where \(V_1, \ldots, V_5\) are the independent variables of polynomials corresponding to \(v_1, \ldots, v_5\), respectively. In particular, \(\alpha \gamma_2 \cdot S_1 = 4 \times (-5) + 10 + 10 = 0\). Therefore, \(X\) is \(\gamma_2\)-nef (not \(\gamma_2\)-positive).

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Department of Applied Mathematics, Faculty of Sciences, Fukuoka University, 8-19-1, Nanakuma, Jonan-ku, Fukuoka 814-0180, Japan
E-mail address: hirosato@fukuoka-u.ac.jp

Department of Applied Mathematics, Faculty of Sciences, Fukuoka University, 8-19-1, Nanakuma, Jonan-ku, Fukuoka 814-0180, Japan
E-mail address: sd170002@cis.fukuoka-u.ac.jp