ON TOPOLOGICALLY TRIVIAL AUTOMORPHISMS OF COMPACT KÄHLER MANIFOLDS AND ALGEBRAIC SURFACES

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Abstract. In this paper, we investigate automorphisms of compact Kähler manifolds with different levels of topological triviality. In particular, we provide several examples of smooth complex projective surfaces \( X \) whose groups of \( C^\infty \)-isotopically trivial automorphisms, resp. cohomologically trivial automorphisms, have a number of connected components which can be arbitrarily large.

Dedicated to the memory of the ‘red’ Bishop of Italian Mathematics, Edoardo Vesentini (1928-2020).

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1. Introduction

Let $X$ be a compact connected complex manifold. Bochner and Montgomery [BM46, BM47] showed that the automorphism group $\text{Aut}(X)$ (the group of biholomorphic maps $g : X \to X$, i.e., the group of diffeomorphisms $g \in \text{Diff}(X)$ which preserve the complex structure of $X$) is a finite dimensional complex Lie Group, possibly with infinitely many connected components, whose Lie Algebra is the space $H^0(X, \Theta_X)$ of holomorphic vector fields on $X$.

Denote by $\text{Aut}_0(X)$ the identity component of $\text{Aut}(X)$ and define the group of $C^\infty$-isotopically trivial automorphisms as:

$$\text{Aut}_\ast(X) := \{ \sigma \in \text{Aut}(X) \mid \sigma \in \text{Diff}_0(X) \},$$

where $\text{Diff}_0(X)$ denotes the identity component of the group of diffeomorphisms.

In other words, $\text{Aut}_\ast(X)$ consists of the automorphisms that are $C^\infty$-isotopic to the identity. This group plays an important role ([Cat13]) in the construction of the Teichmüller space of $X$, and Meersseman recently constructed the Teichmüller stack $\mathcal{T}(X)$ of complex structures on the underlying differentiable manifold of $X$ ([M17b]).

The holonomy of $\mathcal{T}(X)$ turns out to be the quotient group $\Gamma_\ast(X) := \text{Aut}_\ast(X)/\text{Aut}_0(X)$, a subgroup of the group of (connected) components $\Gamma(X) := \text{Aut}(X)/\text{Aut}_0(X)$.

The group of (connected) components $\Gamma(X) := \text{Aut}(X)/\text{Aut}_0(X)$ is at most countable, and here is an easy example where it is countable:

**Example 1.1.** Let $E$ be an elliptic curve, and let $X = E^n$.

Then $\text{Aut}_0(X) = E^n$, while the group $\Gamma(X)$ contains $\text{GL}(n, \mathbb{Z})$, acting in the obvious way:

$$g \in \text{GL}(n, \mathbb{Z}), x = (x_1, \ldots, x_n) \mapsto gx = (\sum_j g_{1j}x_j, \ldots, \sum_j g_{nj}x_j).$$

Since the condition for two automorphisms to be isotopic is not directly tractable by algebro-geometrical methods, the strategy is to first consider the action of $\text{Aut}(X)$ on the cohomology groups $H^*(X; R)$, where $R$ is a coefficient ring. We denote by

$$\text{Aut}_R(X) := \{ \sigma \in \text{Aut}(X) \mid \sigma \text{ induces the trivial action on } H^*(X; R) \}$$

In practice, we choose $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. One more equivalence relation among automorphisms is the homotopy equivalence, so we define the group of homotopically trivial automorphisms as:

$$\text{Aut}_\sharp(X) = \{ \sigma \in \text{Aut}(X) \mid \sigma \text{ is homotopic to } \text{id}_X \},$$

It is clear that

$$\text{Aut}_0(X) \subset \text{Aut}_\ast(X) \subset \text{Aut}_\sharp(X) \subset \text{Aut}_\mathbb{Z}(X) \subset \cdots \subset \text{Aut}_\mathbb{Q}(X) = \text{Aut}_\mathbb{R}(X) = \text{Aut}_\mathbb{C}(X) \subset \text{Aut}(X)$$

where $\text{Aut}_\mathbb{R}(X) = \text{Aut}_\mathbb{C}(X)$ and $\text{Aut}_\mathbb{Q}(X) = \text{Aut}_\mathbb{R}(X) = \text{Aut}_\mathbb{C}(X)$.
so that it suffices to consider the smaller ladder

\[ \text{Aut}_0(X) \triangleleft \text{Aut}_\ast(X) \triangleleft \text{Aut}_Z(X) \triangleleft \text{Aut}_Q(X) \triangleleft \text{Aut}(X). \]

The case where \( X \) is a cKM = compact Kähler Manifold, with a Kähler metric \( \omega \), was considered around 1978 by Lieberman [Li78] and Fujiki [Fuj78], in particular Lieberman [Li78] proved:

**Theorem 1.2** (Lieberman). \( \text{Aut}_0(X) \) is a finite index subgroup of the group of automorphisms preserving the cohomology class of the Kähler form,

\[ \text{Aut}_\omega(X) = \{ \sigma \in \text{Aut}(X) \mid \sigma^*[\omega] = [\omega]\}. \]

In particular, the quotient group

\[ \Gamma_Q(X) := \text{Aut}_Q(X)/\text{Aut}_0(X) \]

is a finite group.

For complex dimension \( n = 1 \), it is well known that everything simplifies, in fact for \( n = 1 \) \( \text{Aut}_0(X) = \text{Aut}_Q(X) \). But already for \( n = 2 \) the situation is extremely delicate, hence this paper is dedicated to the case of complex dimension \( n = 2 \), which provides examples where the different groups of components can have arbitrarily high cardinality.

For surfaces of general type, essentially by the Bogomolov–Miyaoka–Yau inequality (final result in [Miy83]), there is a constant \( C \) such that \( |\text{Aut}_Q(X)| < C \) for any surface of general type (see [Cai04]). For surfaces of general type the important open question is whether they are rigidified in the sense of [Cat13], that is, \( \text{Aut}_\ast(X) \) is a trivial group (see the work of Cai–Liu–Zhang [CLZ13] and of the second author [CL18, Liu18] for recent results in the study of the group \( \text{Aut}_Q(X) \)).

For surfaces not of general type the aim is to describe the group \( \Gamma_Q(X) := \text{Aut}_Q(X)/\text{Aut}_0(X) \).

In 1975 Burns and Rapoport [BR75] proved that, for a K3 surface \( X \), \( \text{Aut}_Q(X) \) is a trivial group. Peters [Pet79], [Pet80] began the study of \( \text{Aut}_Q(X) \) for compact Kähler surfaces. Automorphisms of surfaces were also investigated by Ueno [Ue76] and Maruyama [Ma71] in the 70’s, then by Mukai and Namikawa [MN84].

The main results of this paper can be summarized in the following main theorem, which is obtained from several more precise theorems.

**Main Theorem.** The indices \([\text{Aut}_Q(X) : \text{Aut}_Z(X)]\), \([\text{Aut}_Z : \text{Aut}_\ast(X)]\), and \([\text{Aut}_\ast(X) : \text{Aut}_0(X)]\) can be arbitrarily large for smooth projective surfaces not of general type.

The first result which we prove (contradicting earlier assertions of other authors) answers two questions raised by Meersseman in 2017 [M17a]:

**Theorem 5.1.** For each positive integer \( m \) there exists a rational surface \( X \), blow up of \( \mathbb{P}^2 \), such that \( \text{Aut}_Q(X) = \text{Aut}_\ast(X) \cong \mathbb{Z}/m\mathbb{Z} \).
A similar example can be constructed for other non-minimal uniruled surfaces: just blowing up appropriately any smooth projective surface with an (effective) $C^*$-action. Surprisingly, the same unboundedness phenomenon for 

$$\Gamma_*(X) := \text{Aut}_r(X)/\text{Aut}_0(X)$$

can happen also for minimal ruled surfaces:

**Theorem 1.3.** Let $E$ be an elliptic curve and let $X := \mathbb{P}((\mathcal{O}_E \oplus \mathcal{O}_E(D))$ where $D$ is a divisor of even positive degree $d = 2m$. Then $\Gamma_*(X) := \text{Aut}_r(X)/\text{Aut}_0(X)$ surjects onto $(\mathbb{Z}/m\mathbb{Z})^2$.

In the rest of the paper we consider more general results concerning the various subgroups of the ladder in terms of the Enriques-Kodaira classification of compact Kähler surfaces.

We first consider rational surfaces which are blow-ups of $\mathbb{P}^2$, using Principle 5, and then we pass to consider minimal surfaces, starting from $\mathbb{P}^1$-bundles over curves.

We describe then the situation for non ruled surfaces: the case of Kodaira dimension $\kappa(X) = 0$ is pretty well understood, down here a summary of the results.

- $\text{Aut}_\#(X) = \text{Aut}_0(X)$ holds for each surface $X$ with $\kappa(X) = 0$.
- For complex tori and their blow ups $X$, $\text{Aut}_0(X) = \text{Aut}_\mathbb{Q}(X)$.
- For K3 surfaces (hence on their blow ups) $\text{Aut}_\mathbb{Q}(X) = \{\text{id}_X\}$.
- For Enriques surfaces Mukai and Namikawa [MNS84] proved that $|\text{Aut}_\mathbb{Q}(X)| \leq 4$, and that there are examples with $\text{Aut}_\mathbb{Z}(X) = \mathbb{Z}/2\mathbb{Z}$.
- For hyperelliptic surfaces $X$, we show that $\text{Aut}_\mathbb{Z}(X) = \text{Aut}_0(X)$ is isogenous to $\text{Alb}(X)$, while the group $\Gamma_\mathbb{Q} = \text{Aut}_\mathbb{Q}(X)/\text{Aut}_\mathbb{Z}(X)$ can be described in each case.

$\Gamma_\mathbb{Q}$ is a group of order $\leq 12$, and the case of order 12 occurs precisely with the alternating group $A_4$.

We postpone to the sequel to this paper the full treatment of the more delicate case where the Kodaira dimension $\kappa(X) = 1$.

In this paper we use examples of surfaces in this class in order to show unboundedness also for other quotients of the ladder

$$\text{Aut}_0(X) \vartriangleleft \text{Aut}_r(X) \vartriangleleft \text{Aut}_r(X) \vartriangleleft \text{Aut}_\mathbb{Z}(X) \vartriangleleft \text{Aut}_\mathbb{Q}(X).$$

Combining theorems 3.1 and 4.1 we get:

**Theorem 1.4.** i) For each positive integer $n$ there exists a minimal surface $X$ of Kodaira dimension 1 such that $|\text{Aut}_\mathbb{Q}(X) : \text{Aut}_\mathbb{Z}(X)| \geq n$.

ii) For each positive integer $n$ there exists a (non minimal) surface $X$ of Kodaira dimension 1 such that $\text{Aut}_r(X) = \{\text{id}_X\}$, and

$$\text{Aut}_\mathbb{Z}(X) = \mathbb{Z}/n\mathbb{Z}.$$
Question 1. Is $[\text{Aut}_0(X) : \text{Aut}_1(X)]$ uniformly bounded?

2. Elementary observations and basic principles

**Principle 1.** Let $\sigma \in \text{Aut}_Q(X)$, where $X$ is a (compact complex connected) surface, and let $C$ be an irreducible curve with $C^2 < 0$.

Then $\sigma(C) = C$.

*Proof:* Assume in fact that the irreducible curve $\sigma(C)$ is different from $C$: then $C \cdot \sigma(C) \geq 0$.

But since $\sigma(C)$ has the same rational cohomology class of $C$, we have $C \cdot \sigma(C) = C^2 < 0$, a contradiction.

**Principle 2.** Let $f : X \to B$ be a fibration of the surface $X$ onto a curve, and $\sigma \in \text{Aut}_Q(X)$: then $\sigma$ preserves the fibration, that is, there is an action of $\sigma$ on $B$ such that $\sigma \circ f = f \circ \sigma$. Moreover, if $F''_{\text{red}}$ is a reducible fibre, then $f(F'') = F''$.

*Proof:* Let $F$ be an irreducible fibre of $f$. Then $\sigma(F) \cdot F = 0$, hence $\sigma(F)$ is contained in another fibre of $f$; since $\sigma(F)$ is irreducible, it is another fibre of $f$.

The second assertion follows from Principle 1 and Zariski’s lemma (the components of $F''$ have negative self-intersection).

**Principle 3.** Let $f : X \to B$ be a fibration of the surface $X$ onto a curve, $\sigma \in \text{Aut}_Z(X)$, and let $F'' = mF'$ be a multiple fibre of $f$ with $F'$ irreducible. Then $f(F'') = F''$, unless possibly if $m = 2$, there are only two multiple fibres with multiplicity 2, they are isomorphic to each other, and all the other multiple fibres have odd multiplicity.

*Proof.* By Principle 2, $\sigma$ acts on the fibration, in particular fixing the reducible fibres, and permuting the multiple fibres having the same multiplicity. Let $g$ be the genus of $B$.

We have [Cat03], [Cat08] the exact sequence of the orbifold fundamental group of the fibration $f$:

$$\pi_1(F) \to \pi_1(X) \to \pi_1^{\text{orb}}(f) \to 1,$$

where $F$ is a smooth fibre, and, letting $\{P_1, \ldots, P_r\}$ be the set of points whose inverse images are the multiple fibres of $f$, $f^{-1}(P_j) = m_jF'_j$, then

$$\pi_1^{\text{orb}}(f) := \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, c_1, \ldots, c_r | \Pi [\alpha_i, \beta_i] c_1 \cdots c_r = 1, c_j^{m_j} = 1 \rangle.$$

Taking the Abelianization, we have a surjection

$$H_1(X, \mathbb{Z}) \to (\oplus \mathbb{Z}^g \mathbb{Z}^{\alpha_j} \oplus \mathbb{Z}^{\beta_j} \oplus (\mathbb{Z}/m_i \mathbb{Z})c_i) / \langle (\sum \sigma_i c_i) \rangle.$$

Since $\sigma$ acts trivially on homology, it acts trivially on the quotient group. Assume that $F''$ corresponds to the point $P_1$. Clearly $f$ can only send $F''$ to a multiple fibre of the same multiplicity, and isomorphic to $F''$. Assume
that there is such a fibre, and that it corresponds to the point \( P_2 \). There remains to see whether \( c_1 \) and \( c_2 \) can have the same image in the quotient group.

This means that the vector \( e_1 - e_2 \in \oplus_1^r \mathbb{Z} e_i \) is an integral linear combination of the relation vectors

\[
m_1 e_1, \ldots, m_r e_r, \ e := \sum_1^r e_i.
\]

Since \( m_1 = m_2 \), this implies that there is an integer \( a \) such that \( 1-a, -1-a \) are divisible by \( m_1 \), hence \( 2 \) is divisible by \( m_1 \), hence \( m_1 = 2 \). Since \( a \) is then odd, and \( m_j \) divides \( a \) for all \( j \geq 3 \), this is possible if and only if all the other \( m_j \) for \( j \geq 3 \) are odd, and then one can take \( a \) as the least common multiple of \( m_3, \ldots, m_r \). \( \square \)

The next Principle 4 is a special case of a more general result, and is based on the technique of surfaces isogenous to a product and of unmixed type (\[Cat00\]).

**Definition 2.1.** A surface \( X \) is said to be a SIP = Surface Isogenous to a Product if \( X \) is the quotient of a product of curves of genus \( \geq 1 \) by the free action of a finite group \( G \):

\[ X = (C_1 \times C_2)/G. \]

We speak of a higher product if both curves \( C_1, C_2 \) have genus at least 2.

The action of \( G \) is said to be UNMIXED if \( G \) acts on each factor \( C_i \), and diagonally on the product, \( g(x, y) := (gx, gy) \), so that more precisely \( X = (C_1 \times C_2)/\Delta_G \), where \( \Delta_G \subset G \times G \) is the diagonal subgroup and \( G \times G \) acts on \( C_1 \times C_2 \).

We can assume that we have a minimal realization, that is, \( G \) acts faithfully on each factor \( C_i \).

Observe that \( \text{Aut}(S) \) contains the quotient \( N_{\Delta_G}/\Delta_G \), where \( N_{\Delta_G} \) is the normalizer of \( \Delta_G \) inside \( \text{Aut}(C_1) \times \text{Aut}(C_2) \) (and the two groups are equal if we have a higher product with \( C_1, C_2 \) not isomorphic).

**Principle 4.** Let \( X \) be a SIP of unmixed type, that is, a surface isogenous to a product of unmixed type, with a minimal realization \( X = (C_1 \times C_2)/\Delta_G \).

Then \( \text{Aut}_E(X) \) is the subgroup of \( N_{\Delta_G}/\Delta_G \) corresponding to automorphisms \( h(x, y) = (h_1(x), h_2(y)) \) acting trivially on

\[ H^*(C_1 \times C_2, \mathbb{Q})^{\Delta_G}. \]

In particular, \( h_1 \) acts trivially on \( H^1(C_i, \mathbb{Q}) = H^1(C_i/G, \mathbb{Q}) \), and

(I) if \( C_2 =: E \) has genus 1 and \( G \) acts freely on it, then \( h_2 \) is a translation;

(II) if \( C_1 \) has genus \( \geq 2 \) and \( G \) acts freely on it, then we may represent an element in \( \text{Aut}_E(X) \) by such an automorphism \( h \) with \( h_1 = Id_{C_1} \) and \( h_2 \in Z_G \), where \( Z_G \) is the center of \( G \) inside \( \text{Aut}(C_2) \).

(III) Assume that \( C_2 =: E \) has genus 1, and \( G \) acts freely on \( E \). Assume moreover that \( C_1 \) has either genus \( g_1 \geq 2 \), or \( g_1 = 1 \) and \( g(C_1/G) = 0 \),
and the orders of the stabilizers of points of $C_1$ are not of the form: twice 2, and all others odd numbers. Then all automorphisms in $\text{Aut}_Z(X)$ are represented by pairs $(h_1, h_2) \in \text{Aut}(C_1) \times \text{Aut}(E)$ with $h_1 = \text{Id}_{C_1}$ and $h_2$ a translation. In this case $\text{Aut}_Z(X) = \text{Aut}_0(X) \cong E$.

**Proof.** For each $i = 1, 2$ we have a fibration $f_i : S \to C_i/G$, and, by Principle 3, $H := \text{Aut}_2(X)$ acts equivariantly on $X$ and $C_i/G$.

We have [Cat03], [Cat08] the orbifold fundamental group of the fibration $f_i$:

$$1 \to \pi_1(C_j) \to \pi_1(X) \to \pi_1^{\text{orb}}(f_i) \to 1,$$

on which $H$ acts (here $\{i, j\} = \{1, 2\}$).

Hence the elements of $H$ preserve the characteristic subgroup $\pi_1(C_1) \times \pi_1(C_2)$ of $\pi_1(X)$ and lift to $C_1 \times C_2$, preserving the horizontal and vertical leaves. Therefore these are represented by automorphisms in $\text{Aut}(C_1) \times \text{Aut}(C_2)$. Since such lift $h = (h_1, h_2)$ induces an action on $(C_1 \times C_2)/\Delta_G$ we see that $h \in N_{\Delta_G}$ and $H$ is then a subgroup of $N_{\Delta_G}/\Delta_G$.

Observing that

$$H^*(X, \mathbb{Q}) \cong H^*(C_1 \times C_2, \mathbb{Q})^{\Delta_G},$$

we obtain the first assertion.

The second follows since

$$H^1(X, \mathbb{Q}) \cong H^1(C_1, \mathbb{Q})^G \oplus H^1(C_2, \mathbb{Q})^G = H^1(C_1/G, \mathbb{Q}) \oplus H^1(C_2/G, \mathbb{Q}).$$

(II): then $H^1(C_2, \mathbb{Q}) = H^1(C_2/G, \mathbb{Q})$ and since $h_2$ acts trivially on it, it is a translation.

(III): since $h \in \text{Aut}_Z(X)$, by Principle 3 $h$ acts on the fibration $f_1$ preserving its multiple fibres. Therefore $h_1$ acts on $C_1 \to C_1/G$ fixing the branch points in $C_1/G$. Since $h_1$ acts as the identity on the cohomology of $C_1/G$, $h_1$ acts as the identity on $C_1/G$ if the quotient has genus $g'_1 \geq 2$, or if $g'_1 = 1$ and there is a branch point, or if $g'_1 = 0$ and there are at least 3 branch points. Since $C_1$ has genus $\geq 2$, one of the three possibilities must occur, and $h_1 \in G$. Multiplying by an element in $\Delta_G$, we may assume that $h_1 = \text{Id}_{C_1}$.

(II) shows that $h_2$ is a translation. The group $\{(\text{Id}_{C_1}, h_2) | h_2 \in E\} \cong E$ has an action which clearly descends to $X$. Therefore we have shown that $\text{Aut}_Z(X) = \text{Aut}_0(X) \cong E$. \hfill \square

Directly from Principle 1 follows the next Principle 5.

**Principle 5.** Let $X$ be a compact complex surface, and let $X = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0$ be a sequence of blow-downs of $(-1)$-curves. For $0 \leq k \leq n - 1$, let $P_k \in X_k$ be the blown-up point. Then, if $\text{Bir}(X)$
denotes the group of bimeromorphic self maps of $X$, then:

1. $\text{Aut}_Q(X) = \{ \sigma \in \text{Aut}_Q(X) \subset \text{Bir}(X) \mid \text{for any } 0 \leq k \leq n-1, \sigma_k := \sigma|_{X_k} \in \text{Aut}_Q(X_k) \text{ is such that } \sigma_k(P_k) = P_k \}$

2. $\text{Aut}_Z(X) = \{ \sigma \in \text{Aut}_Z(X_0) \subset \text{Bir}(X) \mid \text{for any } 0 \leq k \leq n-1, \sigma_k := \sigma|_{X_k} \in \text{Aut}_Z(X_k) \text{ is such that } \sigma_k(P_k) = P_k \}$

**Proof.** For $\sigma \in \text{Aut}_Q(X)$, $\sigma$ acts trivially on

$$H^2(X, \mathbb{Q}) = \pi_*^n H^2(X_{n-1}, \mathbb{Q}) \oplus \mathbb{Q}[E_n],$$

where $E_n$ is the exceptional divisor of $\pi_n$, so it preserves $E_n$ by Principle[1]. It follows that $\sigma$ descends to a cohomologically trivial automorphism of $X_{n-1}$ preserving $P_{n-1}$. Conversely, any $\sigma \in \text{Aut}_Q(X_{n-1})$ fixing $P_{n-1}$ lifts to a cohomologically trivial automorphism of $X_n$. By induction on $k$, we obtain the equality[11].

Exactly the same argument yields[2].

**Theorem 2.2 ([Har67], [EL78, (5.5)]).** Let $M$ and $N$ be compact Riemannian manifolds, such that $N$ has nonpositive sectional curvature. If $\phi_0, \phi_1 : M \to N$ are homotopic harmonic maps, then they are smoothly homotopic through harmonic maps $\{ \phi_t \}_{0 \leq t \leq 1}$ such that each path $x \mapsto \phi_t(x)$ is a geodesic segment (parametrized proportionally to arc length) with length independent of $x \in M$. As a consequence, if $\phi_0$ and $\phi_1$ coincide at one point, then $\phi_0 = \phi_1$.

Note that holomorphic maps between Kähler manifolds are harmonic with respect to the Kähler metrics.

**Principle 6.** Let $X$ a compact Kähler manifold with topological Euler number $e(X) \neq 0$. Suppose that there is a generically finite proper holomorphic map $\rho : X \to Y$ onto a compact Kähler manifold $Y$ with nonpositive sectional curvature. Then

$$\text{Aut}_z(X) = \{ \text{id}_X \}. $$

**Proof.** Let $\sigma \in \text{Aut}_z(X)$ be a homotopically trivial automorphism, with homotopy $\Sigma : X \times [0, 1] \to X$ from $\sigma$ to $\text{id}_X$. By Theorem 2.2, we can assume that $\sigma_t(x) := \Sigma(x, t)$ is harmonic in $x$ for each fixed $t \in [0, 1]$.

For each $t \in [0, 1]$, we have $[\Gamma_{\sigma_t}] = [\Delta_X] \in H^4(X \times X)$, where $\Gamma_{\sigma_t}$ denotes the graph of $\sigma_t$ and $\Delta_X \subset X \times X$ is the diagonal. Then

$$[\Delta_X] \cdot [\Gamma_{\sigma_t}] = [\Delta_X]^2 = \chi_{\text{top}}(X) > 0,$$

and it follows that $\Delta_X \cap \Gamma_{\sigma_t} \neq \emptyset$. In other words, there exists a point $x_0(t) \in X$ fixed by $\sigma_t$.

The map $\Sigma$ gives a homotopy from $\rho \circ \sigma_t$ to $\rho$ for each $t \in [0, 1]$.

Since $\rho \circ \sigma_t(x_0(t)) = \rho(x_0(t))$, we have $\rho \circ \sigma_t = \rho$ by Theorem 2.2. Since $\rho$ is generically finite, for a general point $y \in Y$, the inverse image $\rho^{-1}(y)$ is
a finite set. It follows that \( \sigma_t(x) = x \) for each \( t \in [0, 1] \), \( x \in \rho^{-1}(y) \). In other words \( \sigma_t = \text{id}_X \) for \( t \in [0, 1] \). In particular, \( \sigma = \sigma_0 = \text{id}_X \). □

3. Unbounded \([\text{Aut}_Q(X) : \text{Aut}_Z(X)]\)

In this section, we construct a series of examples where \([\text{Aut}_Q(X) : \text{Aut}_Z(X)]\) is unbounded.

Recall that surfaces with Kodaira dimension \( \kappa(X) = 1 \) are canonically elliptic, there is a fibration \( f : X \to B \) over a curve \( B \) and with general fibre a smooth elliptic curve, such that \( \text{Aut}(X) \) acts equivariantly on \( X, B \).

**Theorem 3.1.** For each positive integer \( n \) there exists a minimal surface \( X \) of Kodaira dimension \( 1 \) such that \([\text{Aut}_Q(X) : \text{Aut}_Z(X)]\) \( \geq n \).

**Proof.** Let \( B \) be the hyperelliptic curve, compactification of the affine curve of equation:
\[
y^2 = x^n - 1
\]
where \( n = 2g(B) + 2 \geq 6 \) is an even integer. Let \( F \) be an elliptic curve. The group \( G = \langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z} \) acts on \( B \) by
\[
\tau(x, y) = (x, -y)
\]
and we make it act on \( F \) by translations
\[
\tau(z) = z + \epsilon, \quad \forall \ z \in F
\]
where \( \epsilon \) is a torsion point of order precisely 2. Consider the surface isogenous to a product \( X = (B \times F)/\Delta_G \), where \( \Delta_G \) is the diagonal of \( G \times G \) acting naturally on \( B \times F \).

Since the action is free, the invariants of \( X \) are as follows:
\[
e(X) = \chi(O_X) = p_g(X) = 0, \ q(X) = 1, \ b_2(X) = \rho(X) = 2
\]
and
\[
\kappa(X) = \begin{cases} 1 & \text{if } n \geq 6 \\ 0 & \text{if } n = 4 \end{cases}
\]
The rational cohomology groups of \( X \) are as follows:
\[
(3) \quad H^1(X, \mathbb{Q}) = q^*H^1(F/G, \mathbb{Q}), \quad H^2(X, \mathbb{Q}) = p^*H^2(B/G, \mathbb{Q}) \oplus q^*H^2(F/G, \mathbb{Q})
\]
where \( p : X \to B/G \) and \( q : X \to F/G \) are the induced fibrations.

Consider the following action of \( \langle \tilde{\sigma} \rangle \cong \mathbb{Z}/n\mathbb{Z} \) on \( B \):
\[
(x, y) \mapsto (\xi x, y).
\]
Then \( \tilde{\sigma} \times \text{id}_F \in \text{Aut}(B \times F) \) commutes with \( \tau \), and hence it descends to an automorphism \( \sigma \in \text{Aut}(X) \), of order \( n \). One sees immediately from Principle [4] or directly from (3) that \( \sigma \) acts trivially on \( H^*(X, \mathbb{Q}) \), that is, \( \sigma \in \text{Aut}_Q(X) \). On the other hand, \( \sigma \) permutes the \( n \) double fibres of \( p : X \to B/G \). Since \( n \geq 3 \), \( \langle \sigma \rangle \) acts faithfully on \( H^*(X, \mathbb{Z}) \) in view of Principle [3].

It follows that
\[
[\text{Aut}_Q(X) : \text{Aut}_Z(X)] \geq |\sigma| = n.
\]
Thus $\text{Aut}_Q(X) : \text{Aut}_Z(X)$ is not bounded, as $n$ goes to infinity. □

4. Unbounded $[\text{Aut}_Z(X) : \text{Aut}_*(X)]$

In this section, we construct a series of examples where $[\text{Aut}_Z(X) : \text{Aut}_*(X)]$ is not bounded.

**Theorem 4.1.** For each positive integer $n$ there exists a (non minimal) surface $X$ of Kodaira dimension 1 such that $\text{Aut}_*(X) = \{\text{id}_X\}$, and

$\text{Aut}_Z(X) \cong \mathbb{Z}/n\mathbb{Z}$.

**Proof.** Let $C$ and $E$ be two smooth projective curves with $g(C) \geq 2$ and $g(E) = 1$. Suppose that $G = \langle \sigma \rangle \cong \mathbb{Z}/n\mathbb{Z}$ acts faithfully on $C$ and $E$ in such a way that

- $C^\sigma \neq \emptyset$ and $g(C/G) \geq 1$;
- $\sigma$ acts on $E$ by translations, that is, $\sigma(y) = y + a$ for some torsion element $a \in E$ of order exactly $n$.

The diagonal $\Delta_G < G \times G$ acts freely on $C \times E$, so we take the SIP of unmixed type $Y := (C \times E)/\Delta_G$.

We have shown in (III) of Principle 4 that $\text{Aut}_Z(Y) = \text{Aut}_0(Y)$, and it consists of automorphisms that lift to an automorphism $\tilde{\gamma}$ of $C \times E$ of the form $\tilde{\gamma}(x, y) = (x, y + a)$ for some $a \in E$.

Now let $t \in C/G$ and $X_t = \text{Bl}_P(Y)$ be the blow-up of a point $P \in F_t$, where $F_t$ denotes the fibre of the induced fibration $Y \to C/G$ over $t$. Then by Principle 5 we have

$\text{Aut}_Z(X_t) = \{\gamma \in \text{Aut}_Z(Y) \mid \gamma(P) = P\} \cong \text{Stab}_G(t)$. 

Note that there exists a point $t_0$ with $\text{Stab}_G(t_0) = G = \langle \sigma \rangle$ by the assumption that $C^\sigma \neq \emptyset$.

The situation is illustrated in the following picture, where $E$ denotes the exceptional divisor of the blow-up at a point of the fibre $F_{t_0}$: Letting $|G|$ go to infinity, we see that $|\text{Aut}_Z(X_{t_0})|$ is unbounded.

Now the proof is completed by applying Principle [8] that the group $\text{Aut}_*(X_t)$ is trivial for any $t$.

Corollary 4.2. Let $X$ be as in Theorem 4.1. Then $\text{Aut}_*(X) = \{\text{id}_X\}$.

As a consequence

$$[\text{Aut}_Z(X) : \text{Aut}_*(X)] = |\text{Aut}_Z(X)|$$

is unbounded.

5. **Unbounded $[\text{Aut}_*(X) : \text{Aut}_0(X)]$**

5.1. **Construction of groups of $C^\infty$ isotopically trivial automorphisms with unbounded number of components.** In this section, we give examples of smooth projective surfaces admitting groups of $C^\infty$-isotopically trivial automorphisms with number of connected components which is not bounded. A classification of these would be desirable.

**Theorem 5.1.** For each positive integer $m$ there exists a rational surface $X_m$, blow up of $\mathbb{P}^2$, such that $\text{Aut}_Q(X_m) = \text{Aut}_*(X_m) \cong \mathbb{Z}/m\mathbb{Z}$.

A simple idea lies behind the construction: if we blow up a point $P$ in a complex manifold $Y$, the differentiable manifold we obtain does not depend
on the choice of the given point, as two simple arguments show: the first is that the diffeomorphism group Diff(Y) acts transitively on the manifold Y, the second is that, varying the point P, we get a family with base Y of blow ups of Y, and they are all diffeomorphic by Ehresmann’s theorem.

Proof. Let $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, $P_3 = (0 : 0 : 1)$ be the coordinate points of $P_2$, and let $P_4 = (1 : 1 : 0)$. Denote the lines connecting these points by $L_1 = P_2P_3$, $L_2 = P_1P_3$, $L_3 = P_1P_2$ and $L_4 = P_3P_4$. Let $\pi : X_4 \to \mathbb{P}^2$ be the blow-up of the four points $P_i, 1 \leq i \leq 4$.

Lemma 5.2. Let $G_4 = \{\sigma \in \text{PGL}(3) | \sigma(P_i) = P_i \text{ for } 1 \leq i \leq 4\}$. Then

$$\text{Aut}_\mathbb{Q}(X_4) \cong G_4 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \bigg| a \in \mathbb{C}^* = \mathbb{C} - \{0\} \right\}$$

Proof. The isomorphism is by Principle 5. Since $G_4$ fixes three distinct points $P_1, P_2, P_3$ on the line $L_3 = (x_3 = 0)$, it acts as the identity on $L_3$. It follows that any element $\sigma \in G_4$ takes the form

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix}$$

where * denotes entries to be determined. Since $\sigma$ fixes $P_3 = (0 : 1 : 0)$, one sees then easily that

$$\sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

for some $a \in \mathbb{C}^*$.

Blow up now $P_5$, infinitely near $P_4$, in the direction of the line $L_4 = P_3P_4 = (x_0 - x_1 = 0)$. Since $G_4$ fixes the point $P_5$, the action of $G_4$ lifts to $X_5 = \text{Bl}_{P_5}(X_4)$. Continue to blow up inductively $P_{n+1} \in E_n \cap L_{4,n} \subset X_n$.

**Figure 3.** Blow-up $X_4$ of the plane: for each $1 \leq i \leq 4$, $L_{i,4}$ on the left picture denotes the strict transform of $L_i$ on $X_4$, and $E_i$ denotes the exceptional curve over $P_i$. 
for $n \geq 4$, where $L_{4,n}$ is the strict transform of $L_4$ on $X_n$ so that we get a chain of surfaces

$$X_n \to \cdots \to X_6 \to X_5 \to X_4$$

such that $G_4 = G_5 = G_6 = \cdots = G_n$ acts on them equivariantly. The situation is illustrated by the above picture where $L_{i,n}$ denote the strict transform of $L_i$ on $X_n$ for $1 \leq i \leq 4$, and $E_{k,n}$, $1 \leq k \leq n$, is the strict transform of the exceptional curve $E_k$ of $X_k \to X_{k-1}$ on $X_n$.

Now, let us look at the blown-up point $P_{n+1} \in X_n$, $n \geq 4$. An element of $G_n = G_4$ can be written as

$$\sigma_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}, \quad a \in \mathbb{C} - \{0\}$$

It fixes the curve $L_{4,n}$ and preserves the exceptional curve $E_n$ on $X_n$ for $n \geq 4$. At $P_4 \in \mathbb{P}^2$ there are local coordinates $(x, y)$ such that

$$\sigma_a(x, y) = (x, ay).$$

At $P_6 \in X_4$ there are local coordinates $(x/y, y)$, with $L_{4,4} = (x/y = 0)$ and $E_4 = (y = 0)$, such that

$$\sigma_a(x/y, y) = ((1/a)(x/y), ay)$$

In general, at the point $P_{n+5} \in X_{n+4}$ with $n \geq 1$ there are local coordinates $(x/y^n, y)$ such that $L_{4,n+4} = (x/y^n = 0)$ and $E_{n+4} = (y = 0)$, and

$$\sigma_a(x/y, y) = ((1/a^n)(x/y), ay)$$

Observe that the local coordinate of $P_{n+5} \in E_{n+4}$ is $x/y^n$, so $\sigma_a$ acts as the identity on $E_{n+4}$ if and only if $a^n = 1$; the two intersection points $P_{n+5}$, $P'_{n+5}$ of $E_{n+4}$ with $L'_{n+4}$ and the strict transform of $E_{n+3}$ are fixed by $\sigma_a$ for any $a \in \mathbb{C} - \{0\}$.

For any $P \in E_{n+4}$, we have by Principle 5

$$\text{Aut}_Q(\text{Bl}_P(X_{n+5})) = \{ \sigma \in G_{n+4} \mid \sigma(P) = P \} = \begin{cases} G_4 & \text{if } P = P_{n+5} \text{ or } P'_{n+5} \\ \mu_n & \text{otherwise} \end{cases}$$
where $\mu_n$ is the cyclic group of order $n$. As $P$ varies, we obtain a family of surfaces $\Phi: X \to E_{n+4} \cong \mathbb{P}^1$ such that the fibre $\Phi^* P = \text{Bl}_P(X_{n+5})$.

These are all diffeomorphic, and under a diffeomorphism by which

$$\text{Aut}_0(\text{Bl}_P(X_{n+5}))$$

is always a subgroup of the same group $G_4$. Hence

$$\text{Aut}_0(\text{Bl}_P(X_{n+5})) = \text{Aut}_0(X_{n+5}).$$

For $P \in E_{n+4} - \{P_{n+5}, P'_{n+5}\}$ we have $\text{Aut}_0(\text{Bl}_P(X_{n+5})) = \{\text{id}\}$, and

$$[\text{Aut}_0(\text{Bl}_P(X_{n+5})): \text{Aut}_0(\text{Bl}_P(X_{n+5}))] = |\text{Aut}_0(\text{Bl}_P(X_{n+5}))| = n$$

which can be arbitrarily large.

\[\blacksquare\]

**Remark 5.3.** (1) The above construction, exploring the difference between $\text{Aut}_0(\text{Bl}_P(X))$ and $\text{Aut}_0(X)$, shows that the statement in the fourth paragraph of [Pet80, page 251] is wrong.

Completely similar examples can be constructed for other non minimal ruled surfaces, blowing up appropriately any decomposable $\mathbb{P}^1$-bundle over any curve $C$ of arbitrary genus.

Indeed, generalizing Theorem 5.1, we obtain a recipe for constructing surfaces with $[\text{Aut}_s(X) : \text{Aut}_0(X)]$ arbitrarily large:

1. Choose a $\mathbb{C}^*$-surface $Z$ that is, a projective smooth surface with $\mathbb{C}^* \cong G < \text{Aut}_0(Z)$. For example, we can take $Z$ to be any smooth toric surface or any decomposable $\mathbb{P}^1$-bundle over a curve.

2. Let $Y \to Z$ be a composition of blow-ups of $G$-fixed points such that $\text{Aut}_0(Y) = G$. These blow-ups kill the automorphisms in $\text{Aut}_0(Z) \setminus G$ while preserving the action of $G$.

3. Blow up a point $P \in Y$ and its infinitely near points that is fixed by the whole $G$, in the same way as in the proof of Theorem 5.1

\[X_n \to \cdots \to X_2 \to X_1 \to Y\]

The exceptional curve $E_n$ of the $n$-th blow-up $X_n \to X_{n-1}$ is then invariant under $G$ and is fixed by and only by a finite cyclic subgroup $\mathbb{Z}/n\mathbb{Z} \cong H < G$.

4. Let $X \to X_n$ be the blow-up of a general point $P \in E_n$. Then it holds

$$\text{Aut}_0(X) = \{\text{id}_X\}, \text{ and } \text{Aut}_s(X) \cong \mathbb{Z}/n\mathbb{Z}.$$ 

The method can also be used to construct $X$ such that $\dim \text{Aut}_0(X) > 0$ and $[\text{Aut}_s(X) : \text{Aut}_0(X)]$ is arbitrarily large.

Surprisingly, however, the same unboundedness phenomenon for $\Gamma_s(X) := Aut_s(X)/\text{Aut}_0(X)$ can happen also for minimal ruled surfaces:

**Theorem 5.4.** Let $E$ be an elliptic curve and let $X := \mathbb{P}((O_E \oplus O_E(D))$ where $D$ is a divisor of even positive degree $d = 2m > 0$. 


Then $\Gamma_*(X)$ surjects onto a subgroup of index at most 4 inside $(\mathbb{Z}/d\mathbb{Z})^2$, in particular

$$|\Gamma_*(X)| \geq m^2.$$  

**Proof.** Consider the vector bundle $V := \mathcal{O}_E \oplus \mathcal{O}_E(D)$, so that $X := \mathbb{P}(V)$.

Any $\gamma \in \text{Aut}(X)$ preserves the fibration $f : X \to E$, in particular $\gamma$ acts on $E$. If $\gamma \in \text{Aut}_Q(X)$, then necessarily $\gamma$ acts on $E$ by a translation $\tau_a$, $a \in E$. Since $\tau_a^*(X) \cong X$, it must be that $\tau_a^*(V) \cong V \otimes L$, for a suitable line bundle $L$ on $E$.

Because from the degrees equality (they are in increasing order) $(0, d) = (l, d + l) \Rightarrow l = 0$, we get that $L$ is trivial, and $\tau_a^*(\mathcal{O}_E(D)) \cong \mathcal{O}_E(D)$, hence

$$a \in \mathcal{K} := \text{Ker}(\Phi_D : E \to E) \cong (\mathbb{Z}/d\mathbb{Z})^2,$$

where $\Phi_D(a) = t_a^*(D) - D \in E$.

We have the exact sequence

$$1 \to \mathbb{P}GL(V) \to \text{Aut}_Q(X) \to \mathcal{K} \to 1$$

where the group $\mathbb{P}GL(V)$ is connected, since it consists of the linear maps

$$(v_1, v_2) \mapsto (v_1, \beta v_1 + \lambda v_2), \beta \in H^0(E, \mathcal{O}_E(D)), \lambda \in \mathbb{C}^*.$$  

There remains to show that a subgroup of index at most 4 of $\text{Aut}_Q(X)$ is contained in $\text{Aut}_*(X)$, so that this subgroup maps via $\text{Aut}_Q(X) \to \mathcal{K}$ to a subgroup of index at most 4 in $\mathcal{K}$.

To prove this, observe that, differentiably, $V$ is classified by $c_1(V)$ (it is the pull back of a classifying map to a Grassmannian of 2-planes). Hence $V$ is topologically equivalent to $\mathcal{O}_E(D') \oplus \mathcal{O}_E(D')$, where $D'$ is a divisor of degree $m$.

Hence $X$ is differentiably equivalent to the product $\mathbb{P}^1 \times E$.
And $\text{Aut}_Q(X)$ maps to the group of vector bundle diffeomorphisms, so that $\gamma(v, t) = (A(t)v, t + a)$, with $a \in K$, and $A : E \to \mathbb{P}GL(2, \mathbb{C})$.

Hence $\gamma$ is isotopic to $\gamma_0(v, t) := (A(t)v, t)$. Now,

$$A : E \to \mathbb{P}GL(2, \mathbb{C}) = \mathbb{P}SU(2) = SO(3)/\pm 1$$

lifts to $SO(3) \cong S^3$ if and only if $\pi_1(A) : \pi_1(E) \to \pi_1(\mathbb{P}SU(2)) = \mathbb{Z}/2\mathbb{Z}$ is trivial.

The group $\mathcal{G} := \text{Hom}(\pi_1(E), \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ has exactly 4 elements, and we have therefore a homomorphism $\pi : \text{Aut}_Q(X) \to \mathcal{G}$. For elements $\gamma \in \text{Ker}(\pi)$, we get diffeomorphisms such that the corresponding map $A$ has now $\pi_1(A) = 0$.

Therefore $\gamma \in \text{Ker}(\pi)$ yields a map $A$ which lifts to a differentiable map $A'$ to the three-sphere $S^3$. This, by Sard’s lemma, omits one point $P_0$. Since $S^3 \setminus \{P\} \cong \mathbb{R}^3$, which is contractible, we obtain that $A$ and $\gamma$ are isotopic to the identity.

In the next sections we shall consider the case of more general $\mathbb{P}^1$-bundles over curves $C$.

6. COHOMOLOGICALLY TRIVIAL AUTOMORPHISMS OF RATIONAL SURFACES

In a sense, we are now going first to explain the philosophy behind the construction of Theorem 5.1.

Moreover, we sketch how a similar procedure leads to examples where $\text{Aut}_0(X)$ is nontrivial and the group of components $\Gamma_s(X)$ can be arbitrarily large.

Let $X$ be a blow up of the projective plane $\mathbb{P}^2$, obtained by first blowing up $P_1, \ldots, P_r \in \mathbb{P}^2$, and then infinitely near points $P_{r+1}, \ldots, P_k$. We assume that $k \geq 2$, else we have the plane or the $\mathbb{P}^1$-bundle $F_1 := \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(1))$, and for these $\text{Aut}(X) = \text{Aut}_0(X)$.

Then we define inductively:

$$G_0 := \mathbb{P}GL(3, \mathbb{C}), G_1 := \{g \in G_0 \mid g(P_1) = P_1\}, G_{i+1} := \{g \in G_i \mid g(P_{i+1}) = P_{i+1}\},$$

and $X_0 := \mathbb{P}^2, X_1, \ldots, X_k = X$.

Here $G_i$ acts on $X_i$ and $G_{i+1}$ is the fibre over $P_{i+1}$ of $G_i \to G_i P_{i+1}$; notice that stabilizers of points lying in a fixed orbit are conjugate, hence all fibres are smooth and isomorphic, in particular they are connected if the orbit is 1-connected, a fortiori if $G_i$ is 1-connected.

Clearly $G_{i+1} = G_i$ if the orbit has dimension 0, and $G_{i+1}$ is connected if the orbit is isomorphic to $\mathbb{C}$ or $\mathbb{P}^1$. But $G_{i+1}$ may not be connected if the orbit is isomorphic to $\mathbb{C}^*$.

Notice that $G_1$ is isomorphic to $\text{Aff}(2, \mathbb{C})$, while if $r \geq 2$, then $G_2$ is isomorphic to the subgroup of the affine group $\text{Aff}(2, \mathbb{C})$ of transformations $v \to Av + w$ such that $e_1$ and $e_2$ are eigenvectors of the matrix $A$ (put the two points at infinity).
If \( r \geq 3 \), then \( G_3 \) is isomorphic to \( \mathbb{C}^* \times \mathbb{C}^* \) if the three points are not collinear, else we get the group \( \mathbb{C}^2 \times \mathbb{C}^* \) of dilations \( v \rightarrow \lambda v + w \) (\( \lambda \in \mathbb{C}^*, w \in \mathbb{C}^2 \)).

We can briefly summarize the situation as follows. If \( X \) is a blow up of \( \mathbb{P}^2 \), then \( G_k = \text{Aut}_\mathbb{Q}(X) \) is:

1. trivial if \( r \geq 4 \) and \( \{P_1, \ldots, P_r\} \) contains a projective basis,
2. a subgroup of \( \mathbb{C}^* \times \mathbb{C}^* \) if \( r \geq 3 \) and \( \{P_1, \ldots, P_r\} \) contains three non-collinear points, here \( G_k \) does not need to be connected if \( k \geq 5 \);
3. a subgroup of \( \mathbb{C}^2 \times \mathbb{C}^* \) if all the points \( P_1, \ldots, P_r \) are collinear and \( r \geq 3 \), again here \( G_k \) does not need to be connected if \( k \geq 5 \);
4. \( G_k \) is connected if \( k \in \{1, 2\} \), and for \( k = 3 \) (both \( r = 2 \) and \( r = 1 \));
5. \( G_4 \) may be disconnected for \( k = 4, r = 1 \).

To conclude with these general observations, we observe that if \( X \) is a blow up of \( \mathbb{P}^2 \), then \( \text{Aut}_\mathbb{Q}(X) = \text{Aut}_\mathbb{C}(X) \). Indeed the same statement holds for all rational surfaces, and a proof could be done by induction on \( k \) for \( \text{Aut}_\mathbb{Q}(X) = G_k \) provided one could show the following general assertion:

**Question 6.1.** Let \( X \) be a compact complex manifold and \( \sigma \in \text{Aut}(X) \) an automorphism which admits a fixed point \( P \) and which is differentiably isotopic to the identity. Let \( Z \) be the blow up of \( X \) at \( P \), and \( \sigma' \) the induced automorphism of \( Z \). Then \( \sigma' \) is differentiably isotopic to the identity.

Following the ideas we have introduced, we shall give a proof of the next theorem, which can be viewed as a cohomological characterization of \( C^\infty \)-isotopically trivial automorphisms for smooth projective rational surfaces.

**Theorem 6.2.** Let \( X \) be a smooth projective rational surface. Then

\[
\text{Aut}_\mathbb{C}(X) = \text{Aut}_\mathbb{Z}(X) = \text{Aut}_\mathbb{Q}(X).
\]

We need the following lemma for the proof of Theorem 6.2.

**Lemma 6.3.** Let \( G \) be a connected linear algebraic group, defined over \( \mathbb{C} \). Let \( H \) be an algebraic subgroup of \( G \) and \( H_0 \) its identity component. Then, for any \( \sigma \in H \setminus H_0 \), there is an element \( \sigma' \in \sigma H_0 \) such that \( \sigma' \) has finite order, and \( \sigma' \) is contained in a 1-dimensional multiplicative subgroup \( T \cong \mathbb{C}^* \) of \( G \).

**Proof.** Let \( \sigma = \sigma_s \sigma_u \) be the Jordan decomposition in \( H \), with \( \sigma_s \) semisimple and \( \sigma_u \) unipotent (\cite{Spr98} Theorem 2.4.8), \cite{Hump75} 15.3, page 99). Necessarily \( \sigma_u \in H_0 \), because the Zariski closure of the subgroup generated by \( \sigma_u \) is an additive group \( \mathbb{G}_a \cong \mathbb{C} \) (see \cite{Hump75}, 15.5 exercise 9, page 101).

Hence \( \sigma_s \in \sigma G_0 \). We can replace \( \sigma \) by \( \sigma_s \), and assume that \( \sigma \) is semisimple. Since \( H \) is algebraic, there exists an \( n \in \mathbb{Z}_{>0} \) such that \( \gamma := \sigma^n \in H_0 \).

Then (see \cite{Hump75}, 22.2, Cor. A, page 140, and Prop. 19.2, pages 122-123) there is a maximal torus \( T_H \) of \( H_0 \) containing \( \gamma \). Note that \( T_H \) is divisible and commutative. Therefore, there exists an \( \tau \in T_H \) such that \( \sigma^n = \tau^n \) and \( \sigma \tau = \tau \sigma \) follows since \( \sigma \) commutes with the whole 1-parameter subgroup generated by \( \gamma \). Then \( \sigma' := \sigma \tau^{-1} \in \sigma T_H \subset \sigma H_0 \) is of finite order.
Since $G$ is connected, there is a maximal torus $T_G$ of $G$ containing $\sigma'$. Since $\sigma'$ has finite order, one sees easily that there is a 1-dimensional multiplicative group $T \cong \mathbb{C}^*$ of $T_G$ containing $\sigma'$.

**Proof of Theorem 6.2.** It suffices to show that $\text{Aut}_s(X) = \text{Aut}_Q(X)$.

Let $\sigma \in \text{Aut}_Q(X)$ be a numerically trivial automorphism. Without loss of generality, we can assume that $\sigma \notin \text{Aut}_0(X)$.

Let $f: X \to Y$ be a birational morphism to a smooth minimal model. Since $f$ is a composition of blow-downs of $(-1)$-curves, which are preserved by numerically trivial automorphisms (Principle [1]), the automorphism $\sigma$ descends all along to an automorphism $\sigma_Y \in \text{Aut}_Q(Y)$. In fact, we have $\text{Aut}_Q(X) \subset \text{Aut}_Q(Y)$, viewed as subgroups of the birational automorphism group $\text{Bir}(X) = \text{Bir}(Y)$. Note that $Y$ is either $\mathbb{P}^2$ or one of the Segre-Hirzebruch surfaces, and it holds $\text{Aut}_Q(Y) = \text{Aut}_0(Y)$.

By Lemma [6.3] after replacing $\sigma$ by an automorphism $\sigma'$ in the same component of $\text{Aut}_Q(X)$, there is a subgroup $T \cong \mathbb{C}^*$ of $\text{Aut}_Q(Y)$ containing $\sigma$. Factor $f: X \to Y$ into

$$f: X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} X_1 \xrightarrow{f_1} X_0 = Y$$

so that $f_{i+1}: X_{i+1} \to X_i$ blows up an $\text{Aut}_Q(X)$-fixed point $P_i \in X_i$. There is some $0 \leq i \leq n$ such that

- $T < \text{Aut}(X_i)$, but
- $P_i \in X_i$ is not fixed by the whole $T$,

so the action of $T$ does not lift to $X_{i+1}$. The orbit of $P_i$ under the action of $T$ is $TP_i \cong \mathbb{C}^*$. The closure $\overline{TP_i}$ is an irreducible rational curve with $T$ acting on it, and the normalization $\nu: \mathbb{P}^1 \to \overline{TP_i}$ is $T$-equivariant. Let $\Gamma_\nu \subset \mathbb{P}^1 \times X_i$ be the graph of $\nu: \mathbb{P}^1 \to \overline{TP_i} \hookrightarrow X_i$. Note that $T$ acts diagonally on $\mathbb{P}^1 \times X_i$ and $\Gamma_\nu$ is $T$-invariant. Blowing up $\mathbb{P}^1 \times X_i$ along $\Gamma_\nu$, we obtain a $T$-equivariant family

$$\Phi_{i+1}: \mathcal{X}_{i+1} \xrightarrow{\Phi_{i+1}} \mathbb{P}^1 \times X \to \mathbb{P}^1$$

The fibre of $\Phi_{i+1}$ over $P_i \in TP_i \subset \mathbb{P}^1$ is just $X_{i+1}$, and the action of $T$ lifts to the fibres over 0 and $\infty$.

Now let us look at the blow-up $f_{i+2}: X_{i+2} \to X_{i+1}$. Recall that the blow-up point is $P_i \in X_{i+1} \subset \mathcal{X}_{i+1}$. The orbit $TP_i \subset \mathcal{X}_{i+1}$ is a section of $\Phi_{i+1}$ over $\mathbb{P}^1 \setminus \{0, \infty\}$, and it readily extends to a section $S_{i-1}$ over the whole $\mathbb{P}^1$. Let $F_{i+2}: \mathcal{X}_{i+2} \to \mathcal{X}_{i+1}$ be the blow-up along $S_{i-1} \cong \mathbb{P}^1$, which is $T$-equivariant, extending $f_{i+2}: X_{i+2} \to X_{i+1}$ to a family of blow-ups.

We can continue this way, extending each blow-up $f_{j+1}: \mathcal{X}^{(j+1)} \to \mathcal{X}^{(j)}$, $j \geq i$, to a $T$-equivariant family of blow-ups $F_{j+1}: \mathcal{X}^{(j+1)} \to \mathcal{X}^{(j)}$. In the
end, we get a commutative diagram of $T$-equivariant morphisms:

$$
\begin{array}{ccccccc}
X_n & \xrightarrow{F_n} & X_{n-1} & \xrightarrow{F_{n-1}} & \cdots & \xrightarrow{F_{i+1}} & X_i \\
\downarrow^{\Phi_n} & & & & & & \downarrow^{\Phi_i} \\
\mathbb{P}^1 & & & & & & \\
\end{array}
$$

The $T$-equivariant family $\Phi_n: X_n \to \mathbb{P}^1$ satisfies the following properties:

- $X$ is the fibre of $\Phi_n$ over the point $P_i \in \mathbb{P}^1$;
- The subgroup $H = \langle \text{Aut}_0(X), \sigma \rangle$ of Aut$_Q(X)$ acts fibrewise on $X_n$ and extends to an action of $T$ on the two fibres over 0 and $\infty$.

Since $T \cong \mathbb{C}^*$ is connected, we infer that $\sigma$ is $C^\infty$-isotopic to id$_X$. □

7. Cohomologically trivial automorphisms of surfaces according to Kodaira dimension

In this section, we begin to investigate more systematically the boundedness of $[\text{Aut}_Q(X) : \text{Aut}_0(X)]$, looking through the Enriques–Kodaira classification of surfaces.

7.1. Case $\kappa(X) = -\infty$. In this subsection we study the case $\kappa(X) = -\infty$ and $X \neq \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$.

In fact, for $\mathbb{P}^2$, Aut$(X) = \text{Aut}_0(X)$, while, for $X = \mathbb{P}^1 \times \mathbb{P}^1$, Aut$_0(X) < \text{Aut}(X)$ has index 2 and Aut$_0(X) = \text{Aut}_Q(X)$.

Let $B$ be a smooth projective curve and $f: X \to B$ be a $\mathbb{P}^1$-bundle. Then $X \cong \mathbb{P}(E)$ is the projectivization of some rank two vector bundle $E \to B$. We denote by $E$ the sheaf of holomorphic sections of $E$ and often do not distinguish between $E$ and $\mathcal{E}$. We shall say that $X$ is decomposable as a ruled surface over $B$ if $E$ is so. We have $\pi_1(X) \cong \pi_1(B)$ and $H_1(X, \mathbb{Z}) \cong H_1(B, \mathbb{Z})$ which is torsion free, while $H^2(X, \mathbb{Z}) = \mathbb{Z}F \oplus \mathbb{Z}\Sigma$, where $F$ is a fibre and $\Sigma$ is a section, hence Aut$_Q(X) = \text{Aut}_\mathbb{Z}(X)$.

Moreover, topologically and differentially the bundle $E$ is determined by its first Chern class (which determines the class of the map to the classifying space, the infinite Grassmannian $Gr_C(2, \infty)$).

Hence from the differentiable view point there are only two cases:

1. deg$(E)$ even: $X$ is diffeomorphic to $\mathbb{P}^1 \times B$
2. deg$(E)$ odd: $X$ is diffeomorphic to $\mathbb{P}(\mathcal{O}_B \oplus \mathcal{O}_B(P))$, where $P$ is a point of $B$, and it is obtained from $\mathbb{P}^1 \times B$ via an elementary transformation blowing up a point and then blowing down the strict transform of the fibre through the point.
3. deg$(E)$ odd and $B$ has genus $\geq 1$: then there is an étale double covering $B' \to B$ such that the pull back $X'$ is diffeomorphic to $\mathbb{P}^1 \times B'$.

The following facts about Aut$(X)$ can be found for instance in [Ma71]:
(1) If $X \neq \mathbb{P}^1 \times \mathbb{P}^1$ then there is exactly one $\mathbb{P}^1$-bundle structure on $X$, so we have an exact sequence

$$1 \to \text{Aut}_B(X) \to \text{Aut}(X) \to \text{Aut}(B).$$

where

$$\text{Aut}_B(X) = \{ \sigma \in \text{Aut}(X) \mid \sigma \text{ preserves every fibre of } f: X \to B \}.$$ 

The image of $\text{Aut}(X) \to \text{Aut}(B)$ is the group of automorphisms $\gamma$ of $B$ such that $\gamma^*(\mathcal{E}) \cong \mathcal{E} \otimes L$ for a suitable line bundle $L \in \text{Pic}^0(B)$.

(2) The exact sequence of sheaves of Lie groups on $B$

$$0 \to \mathcal{O}_B^* \to \text{GL}(2)_{\mathcal{O}_B} \to \text{PGL}(2)_{\mathcal{O}_B} \to 1$$

induces a short exact sequence

$$1 \to \text{Aut}_B(\mathcal{E})/\mathbb{C}^* \to \text{Aut}_B(X) \to \Delta \to 1$$

where $\text{Aut}_B(\mathcal{E})$ denotes the automorphism group of the vector bundle $\mathcal{E}$ over $B$, $\mathbb{C}^*$ acts on the fibres of $E \to B$ by scalar multiplication, and $\Delta := \{ L \in \text{Pic}^0(B) \mid \mathcal{E} \otimes L \cong \mathcal{E} \}$.

Note that the fact that the cokernel is $\Delta$, contained in $\text{Pic}^0(B)[2]$, the 2-torsion part of $\text{Pic}^0(B)$, follows since every automorphism of $X$ preserves the class of the relative canonical divisor.

(3) Let $e := \max \{ 2 \deg \mathcal{L} - \deg \mathcal{E} \mid \mathcal{L} \subset \mathcal{E} \text{ invertible subsheaf} \}$. Observe that $-e$ is the minimal self intersection of a section of $X \to B$, and that $\mathcal{E}$ is stable if $e < 0$.

Then there are the following possibilities for $\text{Aut}_B(\mathcal{E})$:

(a) If $e < 0$, that is, $\mathcal{E}$ is stable, then $\text{Aut}_B(\mathcal{E}) = \mathbb{C}^*$.

(b) If $\mathcal{E}$ is indecomposable, $\mathcal{L} \subset \mathcal{E}$ is the (unique) invertible subsheaf of maximal degree, $r = h^0(B, \mathcal{L}^\otimes 2 \otimes (\det \mathcal{E})^{-1})$, then $\text{Aut}_B(\mathcal{E}) \cong H_r$, where

$$H_r := \left\{ \left( \begin{array}{cc} \alpha & 0 \\
0 & \alpha \end{array} \right), \left( \begin{array}{cc} \alpha & t_1 \\
0 & \alpha \end{array} \right), \ldots, \left( \begin{array}{cc} \alpha & t_r \\
0 & \alpha \end{array} \right) \right\} \in \text{GL}(2, \mathbb{C})^{r+1} \mid \alpha \in \mathbb{C}^*, t_i \in \mathbb{C} \}$$

(c) If $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ with $\mathcal{L}_1 \neq \mathcal{L}_2$ and $\deg \mathcal{L}_1 \geq \deg \mathcal{L}_2$, then $\text{Aut}_B(\mathcal{E}) \cong H'_r$, where

$$H'_r := \left\{ \left( \begin{array}{cc} \alpha & 0 \\
0 & \beta \end{array} \right), \left( \begin{array}{cc} \alpha & t_1 \\
0 & \beta \end{array} \right), \ldots, \left( \begin{array}{cc} \alpha & t_r \\
0 & \beta \end{array} \right) \right\} \in \text{GL}(2, \mathbb{C})^{r+1} \mid \alpha, \beta \in \mathbb{C}^*, t_i \in \mathbb{C} \}$$

where $r = h^0(B, \mathcal{L}_1^\otimes 2 \otimes (\det \mathcal{E})^{-1})$.

(d) If $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}$, then $\text{Aut}_B(\mathcal{E}) \cong \text{GL}(2, \mathbb{C})$.

**Remark 7.1.** The cardinality of the group $\Delta$ defined above is at most two. $\Delta$ is a trivial group if $\mathcal{E}$ is indecomposable, since in this case there is a unique invertible subsheaf of $\mathcal{E}$ of maximal degree. If $\mathcal{E}$ is decomposable, and $\Delta$ is non trivial, then $\mathcal{E} = \mathcal{L}_1 \oplus (\mathcal{L} \otimes \mathcal{L}_1)$, and $\Delta$ consists of only two elements $\Delta = \{ \mathcal{O}_B, \mathcal{L} \}$. 
Corollary 7.2. Let \( f : X = \mathbb{P}(E) \to B \) be a \( \mathbb{P}^1 \)-bundle over a smooth curve \( B \) with \( g(B) \geq 2 \). Then \( \text{Aut}_q(X) = \text{Aut}_Z(X) = \text{Aut}_B(X) \), \( \text{Aut}_0(X) \cong \text{Aut}_B(\mathbb{P}(E))/\mathbb{C}^* \) and \( \text{Aut}_Z(X)/\text{Aut}_0(X) \cong \Delta \), where \( \Delta \) is as in (2) above.

Proof. The automorphism group \( \text{Aut}_q(X) = \text{Aut}_Z(X) \) induces a trivial action on \( H^1(X,\mathbb{Z}) = H^1(B,\mathbb{Z}) \). Since \( g(B) \geq 2 \), we infer that \( \text{Aut}_Z(X) \) induces a trivial action on \( B \). Thus \( \text{Aut}_Z(X) \subset \text{Aut}_B(X) \). The inclusion in the other direction is clear. The isomorphism \( \text{Aut}_0(X) \cong \text{Aut}_B(\mathbb{P}(E))/\mathbb{C}^* \) and \( \text{Aut}_Z(X)/\text{Aut}_0(X) \cong \Delta \) come from (2) above.

The next theorem shows that the index of \( \text{Aut}_0(X) \) inside \( \text{Aut}_Z(X) \) can be unbounded only if the genus of the base curve equals 1 (and we have seen in theorem 5.4 that this indeed happens).

Theorem 7.3. Let \( X \cong \mathbb{P}(E) \) be a \( \mathbb{P}^1 \)-bundle over a curve \( B \) of genus \( q \geq 2 \). Then \( \text{Aut}_q(X) = \text{Aut}_Z(X) = \text{Aut}_B(X) \), and

\[
\Gamma_\ast(X) := \text{Aut}_\ast(X)/\text{Aut}_0(X) = \text{Aut}_Z(X)/\text{Aut}_0(X) =: \Gamma_\sharp(X)
\]
equals \( \Delta := \{ \mathcal{L} \in \text{Pic}^0(B) \mid \mathcal{E} \otimes \mathcal{L} \cong \mathcal{E} \} \).

In particular \( \Gamma_\sharp(X) \) has cardinality at most 2.

Proof. The first assertion is from the previous corollary, and we have that \( \text{Aut}_q(X) = \text{Aut}_B(X) \) maps onto \( \Delta \) with Kernel \( \text{Aut}_0(X) \). \( \text{Aut}_B(X) \) consists of sections \( \sigma \) of the sheaf \( \mathbb{P}GL(2,\mathcal{O}_B) = \mathbb{P}SL(2,\mathcal{O}_B) \) and \( \Delta \) measures the obstruction to lifting to a section of \( \text{SL}(2,\mathcal{O}_B) \).

This obstruction is topological. \( \Delta \) is a group of line bundles of 2-torsion, hence it is a group of maps \( \pi_1(B) \to \mathbb{Z}/2\mathbb{Z} \).

Assume first that \( \mathcal{E} \) is of even degree, hence \( X \) is differentiably trivial, so \( X \) is diffeomorphic to \( \mathbb{P}^1 \times B \). From the differentiable viewpoint, \( \sigma \) is a differentiable map \( \sigma : B \to \mathbb{P}SL(2,\mathbb{C}) \), and this map is liftable to \( \sigma' : B \to \text{SL}(2,\mathbb{C}) \) if and only if

\[
\pi_1(\sigma) : \pi_1(B) \to \pi_1(\mathbb{P}SL(2,\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}
\]
is trivial. This homomorphism is exactly \( \delta(\sigma) \), as it is easy to verify.

We already saw that, once \( \delta(\sigma) = 0 \), we can lift to \( \sigma' : B \to \text{SL}(2,\mathbb{C}) \), and since \( \text{SL}(2,\mathbb{C}) \) is homotopically equivalent to \( \text{SU}(2,\mathbb{C}) = S^3 \), \( \sigma' \) is isotopic to the identity, as \( \delta \), hence also \( \sigma \) is such, so that this another proof that \( \sigma \in \text{Aut}_\ast(X) \).

Letting \( f \) be the automorphism corresponding to \( \sigma \), topologically we have \( f : B \times \mathbb{P}^1 \to B \times \mathbb{P}^1 \), which, via the second projection, gives a continuous map \( h : B \to \text{ContMaps}(\mathbb{P}^1,\mathbb{P}^1)_1 \) of \( B \) into the space of continuous selfmaps of degree 1 on \( \mathbb{P}^1 \).

By a result of Graeme Segal, [Seg79], this space is homotopically equivalent up to dimension 1 to \( \mathbb{P}SL(2,\mathbb{C}) \).

Hence, if two maps \( f_1, f_2 \) are homotopic, also \( h_1, h_2 \) are homotopic, hence they induce the same homomorphism of fundamental groups \( \pi_1(B) \to \mathbb{Z}/2\mathbb{Z} \).
Theorem 7.4 \cite[Theorem 3 and Remark 6]{Ma71}. Let $f : X = \mathbb{P}(\mathcal{E}) \to B$ be a $\mathbb{P}^1$-bundle over a smooth curve $B$ of genus $q(X) \leq 1$. Then the following holds.

1. Suppose that $q(X) = 0$. If $X = \mathbb{F}_e$ with $e > 0$, then $\text{Aut}(X) = \text{Aut}_0(X)$, and, more precisely, we have an exact sequence

$$1 \to \tilde{H}_{e+1} \to \text{Aut}(X) \to \text{PGL}(2, \mathbb{C}) \to 1,$$

where $\tilde{H}_{e+1} := H_{e+1}/\mathbb{C}^\times$.

2. Suppose that $B$ is elliptic and $X$ is decomposable. Then we have an exact sequence:

$$1 \to \text{Aut}_B(X) \to \text{Aut}(X) \to H \to 1$$

where $H = \{ \sigma \in \text{Aut}(B) \mid \sigma^*(\mathcal{L}^\otimes 2 \otimes \det(\mathcal{E})^{-1}) \cong \mathcal{L}^\otimes 2 \otimes \det(\mathcal{E})^{-1} \}$. Moreover, the following holds.

(a) If $e > 0$ then $\text{Aut}_B(X) = \text{Aut}_0(X) \cong \tilde{H}_r'$, where $r = h^0(\mathcal{L}^\otimes 2 \otimes \det(\mathcal{E})^{-1})$ and $\tilde{H}_r' = H_r'/\mathbb{C}^\times$ (see above for the definition of $H_r'$), and we have an exact sequence

$$1 \to \text{Aut}_B(X) \to \text{Aut}_Z(X) \to \text{Aut}_0(B)[e] \to 1$$

where $\text{Aut}_0(B)[e] \cong (\mathbb{Z}/e\mathbb{Z})^2$ denotes the $e$-torsion part of $\text{Aut}_0(B)$.

(b) If $e = 0$ and $X$ has only one minimal section, then $\text{Aut}_0(X) = \text{Aut}_Z(X)$, $\text{Aut}_B(X) \cong \tilde{H}_r'$, and we have an exact sequence

$$1 \to \text{Aut}_B(X) \to \text{Aut}_0(X) \to \text{Aut}_0(B) \to 1$$

(c) If $e = 0$ and $X$ has exactly two minimal sections $C_1$ and $C_2$, then $\text{Aut}_0(X) \cong X \setminus (C_1 \cup C_2)$, where the latter is with the natural algebraic group structure, and we have $\text{Aut}_B(X) = \text{Aut}_0(X) \times \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ interchanges the two sections $C_1$ and $C_2$.

(d) If $X = \mathbb{P}^1 \times B$ then $\text{Aut}(X) = \text{Aut}(\mathbb{P}^1) \times \text{Aut}(B) = \text{PGL}(2, \mathbb{C}) \times \text{Aut}(B)$, and $\text{Aut}_Z(X) = \text{Aut}_0(X) = \text{PGL}(2, \mathbb{C}) \times \text{Aut}_0(B)$.

(e) If $e = 0$ and $X$ is indecomposable then we have an exact sequence

$$1 \to \mathbb{C}^\times \to \text{Aut}(X) \to \text{Aut}(B) \to 1.$$
In this case, \( \text{Aut}_Z(X) = \text{Aut}_0(X) \cong X \setminus C \) where \( C \subset X \) is the unique minimal section, it is a nontrivial extension of \( \text{Aut}_0(B) \) by \( \mathbb{C}^* \).

(f) If \( e = 1 \) and \( X \) is indecomposable then we have an exact sequence

\[
1 \to \Delta \to \text{Aut}(X) \to \text{Aut}(B) \to 1.
\]

where \( \Delta = \text{Pic}^0(B)[2] \cong (\mathbb{Z}/2\mathbb{Z})^2 \). Furthermore, \( \text{Aut}_Z(X) = \text{Aut}_0(X) \) and \( \Delta \) is contained in \( \text{Aut}_0(X) \), so there is an exact sequence

\[
1 \to \Delta \to \text{Aut}_0(X) \to \text{Aut}_0(B) \to 1.
\]

7.2. Case \( \kappa(X) = 0 \). In this section, we treat the surfaces \( X \) with \( \kappa(X) = 0 \).

The following is a list of known facts:

1. If \( X \) is K3 surface, then \( \text{Aut}_Q(X) = \{\text{id}_X\} \) by [BR75].

2. If \( X \) is an Enriques surface, then \( |\text{Aut}_Q(X)| \leq 4 \) and \( |\text{Aut}_Z(X)| \leq 2 \), and both bounds are sharp [MN84]. The fact that \( \text{Aut}_Z(X) \) can be nontrivial for an Enriques surface contradicts the last statement of [Pet80, Theorem 2.2].

3. If \( X \) is an Abelian surface, then \( \text{Aut}_Q(X) = \text{Aut}_0(X) \cong X \).

Suppose now that \( X \) is a hyperelliptic surface (bielliptic surface in the notation of [Be96]). These are the prototype examples of a SI PU, and were classified by Bagnera and de Franchis, resp. by Enriques-Severi ([BdF908, E-S09, E-S10]). These are \( X = (F \times E)/\Delta_G \) with \( E \) and \( F \) elliptic curves and \( G \) acting freely on \( E \), while \( g(F/G) = 0 \).

We can apply Principle 4 and obtain the following theorem.

**Theorem 7.5.** Let \( X = (F \times E)/G \) be an hyperelliptic surface in the above notation.

Then \( \text{Aut}_Z(X) \cong E = \text{Aut}_0(X) \) and \( \text{Aut}_Q(X)/\text{Aut}_Z(X) \) is isomorphic to one of following groups:

\[
1, \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, S_3, A_4
\]

where \( S_3 \) denotes the symmetric group on three elements and \( A_4 \) is the alternating group on 4 elements.

Moreover, \( \text{Aut}_Q(X)/\text{Aut}_Z(X) \cong A_4 \) if and only if \( F = F_\omega \) and \( G \cong \mathbb{Z}/2\mathbb{Z} \). In particular, \( |\text{Aut}_Q(X)/\text{Aut}_Z(X)| \leq 12 \) and the equality is attained if and only if \( F = F_\omega \), the equianharmonic (Fermat) elliptic curve, and \( G \cong \mathbb{Z}/2\mathbb{Z} \) \( (F_\omega = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \omega)) \) with \( \omega \) a primitive 3rd root of unity.

**Proof.** By Proposition 4 (I) and (III), \( \text{Aut}_Q(X) \subset \Delta_G \) corresponds to the automorphisms \( h = (h_1, h_2) \) such that \( h_2 \) is a translation, and \( h_1 \) acts trivially on \( H^1(F, \mathbb{Z})^G \).

While \( \text{Aut}_Z(X) \subset \Delta_G/G \) corresponds to the subgroup

\[
G \times E < \text{Aut}(F) \times \text{Aut}(E).
\]
Since $H^1(F/G, \mathbb{Z}) = 0$, $\text{Aut}_\mathbb{Q}(X)/\text{Aut}_\mathbb{Z}(X) \cong N_G/G$, where $N_G$ is the normalizer of $G$ in $\text{Aut}(F)$.

From the identification

$$\text{Aut}_\mathbb{Q}(X)/\text{Aut}_\mathbb{Z}(X) \cong \{ \gamma \in \text{Aut}(F) \mid \gamma G = G\gamma \}/G = N_G/G.$$ 

we proceed as follows to determine $N_G/G$: consider the split short exact sequence

$$0 \to \text{Aut}_0(F) \cong F \to \text{Aut}(F) \overset{\varphi}{\to} A \to 0$$

where $A \subset \text{Aut}(F)$ is the subgroup preserving the group structure of $F$.

For convenience of notation we write $\varphi$ instead of $\text{Aut}_0(F)$. Restricting to $G$ and $N_G$ we obtain short exact sequences

$$0 \to G \cap F \to G \to \varphi(G) \to 1$$

and

$$0 \to N_G \cap F \to N_G \to \varphi(N_G) \to 1.$$

Therefore, we have a short exact sequence

$$0 \to (N_G \cap F)/(G \cap F) \to N_G/G \to \varphi(N_G)/\varphi(G) \to 1.$$ 

Next we divide the discussion into cases according to [Be96, List VI.20].

1. $G \cong \mathbb{Z}/2\mathbb{Z}$ acting on $F$ by $x \mapsto -x$. In this case, $N_G \cap F \cong (\mathbb{Z}/2\mathbb{Z})^2$ consists of translations by 2-torsion points, and

$$\varphi(N_G) = \begin{cases} 
\langle x \mapsto -x \rangle \cong \mathbb{Z}/2\mathbb{Z} & \text{if } F \neq F_i, F_\omega \\
\langle x \mapsto ix \rangle \cong \mathbb{Z}/4\mathbb{Z} & \text{if } F = F_i \\
\langle x \mapsto -\omega x \rangle \cong \mathbb{Z}/6\mathbb{Z} & \text{if } F = F_\omega 
\end{cases}$$

For $G$ we have

$$G \cap F = \{0\} \text{ and } \varphi(G) = G = \langle x \mapsto -x \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$ 

We have thus a split short exact sequence

$$0 \to N_G \cap F \to N_G/G \to \varphi(N_G)/G \to 0$$

and it is now easy to determine the corresponding semidirect product

$$N_G/G \cong \begin{cases} 
(\mathbb{Z}/2\mathbb{Z})^2 & \text{if } F \neq F_i, F_\omega \\
D_8 & \text{if } F = F_i \\
\mathfrak{A}_4 & \text{if } F = F_\omega 
\end{cases}$$

where $D_8$ denotes the dihedral group of order 8 and $\mathfrak{A}_4$ is the alternating group on 4 elements.

2. $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ acting on $F$ by $x \mapsto -x, x \mapsto x + \epsilon$ with $\epsilon$ a nontrivial 2-torsion point of $F$. In this case, $N_G \cap F \cong (\mathbb{Z}/2\mathbb{Z})^2$ consists of the 2-torsion points, and $\varphi(N_G) = \varphi(G) = \langle x \mapsto -x \rangle$ if $F \neq F_i$, while $\varphi(N_G) = A$ if $F = F_i$ and $\epsilon = \frac{1}{2}(1 + i)$.

Thus $N_G/G \cong \mathbb{Z}/2\mathbb{Z}$ or it may be $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for $F = F_i$. 

(3) $G \cong \mathbb{Z}/4\mathbb{Z}$ acting on $F = F_i = C/(\mathbb{Z} \oplus i\mathbb{Z})$ by $x \mapsto ix$. In this case, $N_G \cap F = \langle x \mapsto x + \frac{1+i}{2} \rangle \cong \mathbb{Z}/2\mathbb{Z}$ and $\varphi(N_G) = \varphi(G) = \langle x \mapsto ix \rangle$.

Thus $N_G/G \cong \mathbb{Z}/2\mathbb{Z}$.

(4) $G \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, acting by $x \mapsto ix, x \mapsto x + (1+i)$. In this case, $N_G \cap F \cong (\mathbb{Z}/2\mathbb{Z})^2$ is the group of 2-torsion point of $F$, and $\varphi(N_G) = \varphi(G) = \langle x \mapsto ix \rangle$. Thus $N_G/G \cong \mathbb{Z}/2\mathbb{Z}$.

(5) $G \cong \mathbb{Z}/3\mathbb{Z}$ acting on $F = F_\omega$ by $x \mapsto \omega x$. In this case, $N_G \cap F = \langle x \mapsto x + \frac{1-\omega}{2} \rangle \cong \mathbb{Z}/3\mathbb{Z}$, and $\varphi(N_G) = \langle x \mapsto -\omega x \rangle \cong \mathbb{Z}/6\mathbb{Z}$.

It follows that $N_G/G \cong \mathbb{S}_3$, the symmetric group on three elements.

(6) $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ acting by $x \mapsto \omega x, x \mapsto x + (1-\omega)$. In this case, $N_G \cap F \cong (\mathbb{Z}/3\mathbb{Z})^2$ is the group of 3-torsion points of $F$, and $\varphi(N_G) = \langle x \mapsto -\omega x \rangle \cong \mathbb{Z}/6\mathbb{Z}$. It follows that $N_G/G \cong \mathbb{S}_3$.

(7) $F = F_\omega$ and $G \cong \mathbb{Z}/6\mathbb{Z}$ acting by $x \mapsto -\omega x$. In this case, $N_G \cap F = \{0\}$ and $\varphi(N_G) = \varphi(G) = G = \langle x \mapsto -\omega x \rangle$. It follows that $N_G/G$ is trivial.

\[\square\]

**Theorem 7.6.** Let $X$ be a smooth projective surface with $\kappa(X) = 0$. Then $[\text{Aut}_Q(X) : \text{Aut}_0(X)] \leq 12$.

**Proof.** If $X$ is minimal, the assertion follows from the listed facts above and Theorem 7.5.

If $X$ is not minimal, then $\text{Aut}_Q(X) \subset \text{Aut}_Q(X_{\text{min}})$ by Principle 5. So, if $\dim \text{Aut}_0(X_{\text{min}}) = 0$, then

$$|\text{Aut}_Q(X)| \leq |\text{Aut}_Q(X_{\text{min}})| \leq 12.$$  

In case $\dim \text{Aut}_0(X_{\text{min}}) > 0$, $X_{\text{min}}$ is either an abelian surface or a hyperelliptic surface and $\text{Aut}_0(X_{\text{min}})$ is an abelian variety of dimension 2 or 1. The subgroup $\text{Aut}_g(X) \subset \text{Aut}_Q(X_{\text{min}})$ fixes the points $p \in X_{\text{min}}$ over which $X \to X_{\text{min}}$ is not an isomorphism. But an element of $\text{Aut}_0(X_{\text{min}})$ fixes a point if and only if it is the identity. It follows that

$$\text{Aut}_Q(X) \cap \text{Aut}_0(X_{\text{min}}) = \text{id}_{X_{\text{min}}}$$

and hence the natural homomorphism $\text{Aut}_Q(X) \to \text{Aut}_Q(X_{\text{min}})/\text{Aut}_0(X_{\text{min}})$ is injective. Therefore,

$$|\text{Aut}_Q(X)| \leq |\text{Aut}_Q(X_{\text{min}})/\text{Aut}_0(X_{\text{min}})| \leq 12.$$  

\[\square\]

For $\text{Aut}_4(X)$ we have a sharper result:

**Theorem 7.7.** Let $X$ be a smooth projective surface with $\kappa(X) = 0$. Then $\text{Aut}_4(X) = \text{Aut}_0(X)$.

**Proof.** Let $X_{\text{min}}$ be the minimal model of $X$. Then by Principle 5, $\text{Aut}_Z(X) \subset \text{Aut}_Z(X_{\text{min}})$.

If $X_{\text{min}}$ is a K3 surface, then $\text{Aut}_Z(X_{\text{min}})$ is trivial. It follows that $\text{Aut}_Z(X)$ and hence $\text{Aut}_4(X)$ is trivial.
If $X_{\min}$ is an Enriques surface, then its universal cover $\tilde{X}_{\min}$ is a K3 surface and is the minimal model of the universal cover $\pi: \tilde{X} \to X$. Suppose that $\sigma \in \text{Aut}_{\sharp}(X)$ and that $\Sigma: X \times I \to X$ is a homotopy from $\text{id}_X$ to $\sigma$. Then, by the homotopy lifting property, there is a homotopy $\tilde{\Sigma}: \tilde{X} \times [0,1] \to \tilde{X}$ from $\text{id}_{\tilde{X}}$ to $\tilde{\sigma}$, where $\tilde{\sigma} \in \text{Aut}(\tilde{X})$ is a lifting of $\sigma$:

$$
\begin{array}{c}
\tilde{X} \times [0,1] \\
\downarrow_{\pi \times \text{id}} \\
X \times [0,1]
\end{array} \xrightarrow{\Sigma} \begin{array}{c}
\tilde{X} \\
\downarrow_{\pi}
\end{array}
$$

Since $\text{Aut}_{\sharp}(\tilde{X})$ is trivial, $\tilde{\sigma} = \text{id}_{\tilde{X}}$. It follows that $\sigma = \text{id}_X$.

If $X = X_{\min}$ is an Abelian surface or a bielliptic surface, then by Principle 4 and the discussion above we know that $\text{Aut}_{\sharp}(X) = \text{Aut}_0(X)$. It follows a fortiori that $\text{Aut}_{\sharp}(X) = \text{Aut}_0(X)$.

Finally, suppose that $X_{\min}$ is an Abelian surface or a bielliptic surface but $\rho: \tilde{X} \to X_{\min}$ is not an isomorphism. Then the topological Euler characteristics satisfy $\chi_{\text{top}}(X) > \chi_{\text{top}}(X_{\min}) = 0$.

Now observe that there is a flat metric on $X_{\min}$, and by Principle 6 we have $\text{Aut}_{\sharp}(X) = \{\text{id}_X\}$. $\square$

7.3. **Case** $\kappa(X) = 2$. For a surface $X$ of general type, we have that $|\text{Aut}_Q(X)|$ is bounded and, in fact, $|\text{Aut}_Q(X)| \leq 4$ if $\chi(O_X) \geq 189$ ([Cai04]). Subsequently, examples have been found with $|\text{Aut}_Q(X)| = 4$ and $\chi(O_X)$ arbitrarily large ([CL18]).

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In the study of algebraic geometry, automorphisms of surfaces play a significant role. They are bijective maps from a surface to itself that preserve the algebraic structure. Understanding the properties of these automorphisms, especially when they act trivially in cohomology, is crucial for many applications in geometry and topology.

For instance, Jin-Xing Cai, Wenfei Liu, and Lei Zhang in their paper [CLZ13] explore automorphisms of surfaces of general type with $q \geq 2$ acting trivially in cohomology. Their work is published in Compos. Math. 149 (2013), no. 10, 1667–1684.

Similarly, Jin-Xing Cai and Wenfei Liu also investigate automorphisms of surfaces of general type with $q = 1$ acting trivially in cohomology in their paper [CL18], which appears in Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 18 (2018), no. 4, 1311–1348.

Fabrizio Catanese's work, presented in Fibred surfaces, varieties isogenous to a product and related moduli spaces [Cat00], is a seminal contribution. His studies on fibred Kähler and quasi-projective groups [Cat03] and differentiable and deformation type of algebraic surfaces, real and symplectic structures [Cat08] have also been influential.

Other notable works include Enriques and Severi’s Memoir on hyperelliptic surfaces [E-S09, E-S10], which delved into the properties of these surfaces in detail. Akira Fujiki’s work on automorphism groups of compact Kähler manifolds [Fuj78] and Masaki Maruyama’s study on automorphisms of ruled surfaces [Mar71] have also contributed significantly to the field.

Lectures and handbooks, such as the work by Fabrizio Catanese [Cat13], provide guidance to researchers on deformations and moduli of algebraic surfaces. Other notable works by Chris A. M. Peters, Yoichi Miyaoka, and Shigeru Mukai on automorphisms and deformations of Kähler manifolds have also been pivotal.

The study of automorphisms and their actions in cohomology is a rich area of research that continues to evolve with new findings and methodologies. This field not only enriches our understanding of algebraic surfaces but also has implications in various areas of mathematics and physics.
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