A CHARACTERIZATION OF THE ANGLE DEFECT AND
THE EULER CHARACTERISTIC IN DIMENSION 2
PRELIMINARY DRAFT

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Abstract. The angle defect, which is the standard way to measure curvature at the vertices of polyhedral surfaces, goes back at least as far as Descartes. Although the angle defect has been widely studied, there does not appear to be in the literature an axiomatic characterization of the angle defect. We give a characterization of the angle defect for simplicial surfaces, and we show that variants of the same characterization work for two known approaches to generalizing the angle defect to arbitrary 2-dimensional simplicial complexes. Simultaneously, we give a characterization of the Euler characteristic on 2-dimensional simplicial complexes in terms of being geometrically locally determined.

1. Introduction

This paper is concerned with two related questions. The first issue concerns the curvature at a vertex \( v \) of a triangulated polyhedral surface \( M \), which is given by \( d(v, M) = 2\pi - \sum_{\alpha \ni v} \alpha \), where the \( \alpha \) are the angles at \( v \) of the triangles containing \( v \). This curvature function, which we refer to as the “classical angle defect,” goes back at least as far as Descartes (see [Fed82]). The classical angle defect satisfies various properties one would expect a curvature function on polyhedra to satisfy. For example, the angle defect is locally defined; it is invariant under simplicial local isometries (that is, functions that preserve the lengths of edges); it is zero at a vertex that has a flat star; it is invariant under subdivision; and it satisfies the polyhedral Descartes-Gauss-Bonnet Theorem, which says \( \sum_v d(v, M) = 2\pi \chi(M) \), where the summation is over all the vertices of \( M \), and where \( \chi(M) \) is the Euler characteristic of \( M \). (We refer to this theorem as the “Descartes-Gauss-Bonnet Theorem” rather than just “Descartes’ Theorem” because Descartes’ version was for convex polytopes only, and did not explicitly involve the Euler characteristic for 2-dimensional polyhedra; there appears to be some dispute in the literature as to whether or not Descartes was implicitly aware of the Euler characteristic.)

The angle defect (also known as the angle deficiency), and related constructs involving sums of angles in polyhedra, have been widely studied,
both in the classical situation, as well as in higher dimensions. It has been studied in the case of convex polytopes from a combinatorial approach, for example in [She68] and [Grü68]; more generally, for the wider study of angle sums in convex polytopes, see for example [Grü67, Chapter 14], [She67], [PS67] and [McM75]. In [GS91] a Descartes-Gauss-Bonnet Theorem is proved for the angle defect in polytopes with underlying spaces that are manifolds. This approach has been generalized to arbitrary simplicial complexes in [Blo98], and further studied in [Blo04] and [Blo06]. A different approach to generalizing the classical angle defect has been studied extensively from a differential geometric point of view; see, among others, [Ban67], [Win], [Bud89], [Che83], [CMS84] and [Zäh90]. One treatment of curvature of polyhedra that has some of the advantages of all the approaches cited above is in [Ban83], which uses curvatures functions based on critical points (similarly to [Ban67]), but this time using projection maps \( \mathbb{R}^n \to \mathbb{R}^m \), which leads to curvature functions related to the Grassman angles of [Grü68], and which are located at all simplices, and which directly generalizes standard curvature; moreover, an angle defect type formula for curvature is obtained using projection maps \( \mathbb{R}^n \to \mathbb{R}^{n-1} \). (The angle defect and its generalizations treated in the above references, and which we discuss in this note, are geometric in nature, depending upon the measurement of angles; we will not be discussing the combinatorial approach to curvature of simplicial complexes found in [For03].)

Although the angle defect has been widely studied, there does not appear to be in the literature an axiomatic characterization of the angle defect. Such a characterization would be useful not only for gaining insight into the angle defect, but also to help distinguish between different generalizations of the classical angle defect to arbitrary polyhedra. In the present paper we give a characterization of the classical angle defect for simplicial surfaces, and we show that variants of the same characterization work for the two approaches to generalizing the classical angle defect to arbitrary 2-dimensional simplicial complexes, as found in [Blo98] on the one hand, and in [Ban67] et al. on the other.

The second issue we discuss concerns the Euler characteristic, which is intimately connected to the classical angle defect by the Descartes-Gauss-Bonnet Theorem. In [For00], which follows [Lev92], it is shown that the Euler characteristic is the unique locally determined numerical invariant of finite simplicial complexes that assigns the same number to every cone, where in this context simplicial complexes are considered to be the same if they are combinatorially equivalent. A real-valued function \( \rho \) on the set of all finite simplicial complexes is “locally determined” if there is another real-valued function \( h \) on the set of all finite simplicial complexes such that for each simplicial complex \( K \), we have \( \rho(K) = \sum_v h(\text{link}(v,K)) \), where \( \text{link}(v,K) \) denotes the link of \( v \) in \( K \), and where the summation is over all the vertices of \( K \). This last condition certainly has some resemblance
to the Descartes-Gauss-Bonnet Theorem, but there is a substantial difference between the results of [Lev92] and [For00] on the one hand, and the Descartes-Gauss-Bonnet Theorem on the other: the nature of being locally determined in [Lev92] and [For00] is purely combinatorial, whereas in the Descartes-Gauss-Bonnet Theorem the Euler characteristic is locally determined by a geometric quantity, namely the classical angle defect, which depends not only on the combinatorial nature of the link of each vertex, but on the geometry of the embedding of the star of each vertex. One might therefore think of the results of [Lev92] and [For00] as characterizing the Euler characteristic among those functions that are “combinatorially locally determined.” In the present paper we will given an analogous characterization of the Euler characteristic in the 2-dimensional case among those functions that are “geometrically locally determined,” a term that will be defined below. Not surprisingly, our characterizations of both the angle defect and the Euler characteristic in dimension 2 are simply different ways of looking at the same result.

2. Statement of Results

We start with some assumptions, notation and definitions. Throughout this paper we will restrict our attention to simplicial complexes, all of which are 2-dimensional, finite, and are embedded in Euclidean space (which we will not name when it is not necessary). Different embeddings of combinatorially equivalent simplicial complexes will be considered as different simplicial complexes (in contrast to [Lev92] and [For00], whose approach is combinatorial rather than geometric). We use the term simplicial surface to mean a 2-dimensional simplicial complex the underlying space of which is a compact surface without boundary.

Let $K$ be a simplicial complex. We let $|K|$ denote the underlying space of $K$. We write $K^{(0)}$ to denote the set of vertices of $K$, and $f_i(K)$ to denote the number of $i$-simplices of $K$ for each possible value of $i$. If $v, w \in K^{(0)}$, and if there is a 1-simplex of $K$ with vertices $v$ and $w$, we let $\langle v, w \rangle$ denote this 1-simplex, and we let $O(v, w)$ denote the number of 2-simplices of $K$ that contain $\langle v, w \rangle$. If $\sigma \in K$, we use link ($\sigma, K$) and star ($\sigma, K$) to denote the link and star of $\sigma$ in $K$ respectively. (For basic definitions in PL topology, see for example [Gla70] vol. I and [Hud69], although the latter uses the notation star instead of star.)

For the sake of convenience, we adopt the convention that all angles in 2-simplices are normalized so that the circumference of the unit circle is 1. For any 2-simplex $\sigma^2$ in Euclidean space, and any vertex $v$ of $\sigma^2$, we let $\alpha(v, \sigma^2)$ denote the (interior) angle in $\sigma^2$ at $v$, where by normalization such an angle is always a number in $[0, \frac{1}{2}]$. Hence the $2\pi$ will drop out of our statement of the Descartes-Gauss-Bonnet Theorem.
The following definition gives the two general types of functions of which the Euler characteristic and the classical angle defect are examples, respectively.

**Definition.** Let $\mathcal{T}$ be a set of 2-dimensional simplicial complexes. A **simplicial-complex-supported function** on $\mathcal{T}$ is a function $\Lambda$ that assigns to every 2-dimensional simplicial complex $K \in \mathcal{T}$ a real number $\Lambda(K)$. A **vertex-supported function** on $\mathcal{T}$ is a function $\phi$ that assigns to every 2-dimensional simplicial complex $K \in \mathcal{T}$, and every vertex $v \in K^{(0)}$, a real number $\phi(v, K)$.

We now consider various properties of vertex-supported functions. These properties, defined below, are all satisfied by the classical angle defect, as can be verified easily. The first property involves subdivision.

**Definition.** Let $\mathcal{T}$ be a set of 2-dimensional simplicial complexes, and let $\phi$ be a vertex-supported function on $\mathcal{T}$. We say that $\phi$ is **invariant under subdivision** if the following condition holds. Let $K, J \in \mathcal{T}$, and let $v \in K^{(0)}$. If $J$ is a subdivision of $K$, then $\phi(v, K) = \phi(v, J)$.

For our next property, which involves isometries, we need the following terminology. Let $K$ and $L$ be 2-dimensional simplicial complexes. We say that $K$ and $L$ are simplicially isometric if there is a simplicial homeomorphism $|K| \to |L|$ that preserves the lengths of edges; such a map is called a simplicial isometry. Moreover, let $v \in K^{(0)}$ and let $w \in L^{(0)}$. We say that star $(v, K)$ and star $(w, L)$ are simplicially isometric if there is a simplicial isometry $|\text{star}(v, K)| \to |\text{star}(w, L)|$ that takes $v$ to $w$; any simplicial isometry between star $(v, K)$ and star $(w, L)$ will always be assumed to take $v$ to $w$.

**Definition.** Let $\mathcal{T}$ be a set of 2-dimensional simplicial complexes, and let $\phi$ be a vertex-supported function on $\mathcal{T}$. We say that $\phi$ is **invariant under simplicial isometries of stars** if the following condition holds. Let $K, L \in \mathcal{T}$, let $v \in K^{(0)}$ and let $w \in L^{(0)}$. If star $(v, K)$ and star $(w, L)$ are simplicially isometric, then $\phi(v, K) = \phi(w, L)$.

Our next property involves continuity. Suppose that $K$ and $\{K_n\}_{n=1}^\infty$ are combinatorially equivalent 2-dimensional simplicial complexes, and all are embedded in the same Euclidean space. We can think of all these 2-dimensional simplicial complexes as embeddings of the same abstract simplicial complex. We write $\lim_{n \to \infty} K_n = K$ to denote pointwise convergence of these embeddings; it suffices to verify convergence at the vertices of the abstract simplicial complex.

**Definition.** Let $\mathcal{T}$ be a set of 2-dimensional simplicial complexes, and let $\phi$ be a vertex-supported function on $\mathcal{T}$. We say that $\phi$ is **continuous** if the following condition holds. Let $\{K_n\}_{n=1}^\infty$ and $K$ be combinatorially equivalent 2-dimensional simplicial complexes in $\mathcal{T}$, all embedded in the same Euclidean space. Suppose $\lim_{n \to \infty} K_n = K$. If $v \in K^{(0)}$, and if we
label the corresponding vertices of the $K_n$ as $v_n$, then $\lim_{n \to \infty} \phi(v_n, K_n) = \phi(v, K)$. 

Our final property is the analog of the Descartes-Gauss-Bonnet Theorem.

**Definition.** Let $\mathcal{T}$ be a set of 2-dimensional simplicial complexes, let $\phi$ be a vertex-supported function on $\mathcal{T}$, and let $\Lambda$ be a simplicial-complex-supported function on $\mathcal{T}$. We say that $\phi$ satisfies the **Descartes-Gauss-Bonnet Theorem with respect to $\Lambda$** if the following condition holds. If $K \in \mathcal{T}$, then $\sum_{v \in K} \phi(v, K) = \Lambda(K)$, where the sum is over all the vertices of $K$.

We can now state our results, which take place in two contexts, namely simplicial surfaces on the one hand, and the set of all finite 2-dimensional simplicial complexes on the other hand. We first consider things from the point of view of the classical angle defect and its generalizations; shortly we will turn to the point of view of the Euler characteristic and its generalizations. For simplicial surfaces, we have the following characterization of the classical angle defect.

**Theorem 2.1.** Let $\phi$ be a vertex-supported function on the set of all simplicial surfaces. Then $\phi$ is the classical angle defect iff $\phi$ is invariant under simplicial isometries of stars and under subdivision, is continuous, and satisfies the Descartes-Gauss-Bonnet Theorem with respect to the Euler characteristic.

It would be interesting to know whether the four hypotheses in Theorem 2.1 are all needed. We do not have a complete answer to this question, though the following examples partially answer this question.

**Example 2.2.**

(1). It is clear that the Descartes-Gauss-Bonnet Theorem cannot be dropped in Theorem 2.1 because the constantly zero vertex-supported function on the set of all simplicial surfaces satisfies the three other criteria of the theorem.

(2). If $K$ is a simplicial surface, and if $v \in K^{(0)}$, we let $e(v, K)$ denote the number of edges of $K$ that contain $v$. Define the vertex-supported function $\psi$ on the set of all simplicial surfaces by letting $\psi(v, K) = 1 - \frac{1}{3}e(v, K)$ for any simplicial surface $K$ and any vertex $v$ of $K$. It is evident that $\psi$ is invariant under simplicial isometries of stars and is continuous, and it can also be verified that $\psi$ satisfies the Descartes-Gauss-Bonnet Theorem with respect to the Euler characteristic (using the fact that $\sum_{v \in K} e(v, K) = 2f_1(K) = 3f_2(K)$, where the summation is over all vertices of $K$). However, it is evident that $\psi$ is not invariant under subdivision, and therefore the invariance under subdivision criteria cannot be dropped from Theorem 2.1.

(3). If $K$ is a simplicial surface, we let $n(K)$ denote the number of vertices $v$ of $K$ for which the sum of the angles at $v$ is not equal to 1; observe that
Define the vertex-supported function \( \mu \) on the set of all simplicial surfaces by letting

\[
\mu(v, K) = \begin{cases} 
\chi(K)/n(K), & \text{if the sum of the angles at } v \text{ is not } 1, \\
0, & \text{otherwise}, 
\end{cases}
\]

for any simplicial surface \( K \) and any vertex \( v \) of \( K \). It is clear that \( \mu \) satisfies the Descartes-Gauss-Bonnet Theorem with respect to the Euler characteristic, and it can also be seen that \( \mu \) is invariant under subdivision (because any new vertex in a subdivision of a 2-dimensional simplicial complex \( K \) is in the relative interior of a 1-simplex or a 2-simplex of \( K \), and hence has angle sum equal to 1). We leave it to the reader to verify that \( \mu \) is not invariant under simplicial isometries of stars and is not continuous. Hence, these last two criteria cannot both be dropped from Theorem 2.1.

\[ \diamond \]

Theorem 2.1 holds unchanged for the class of all 2-dimensional simplicial pseudomanifolds (without boundary), and we omit the details. The situation becomes more interesting when we go beyond pseudomanifolds, and look at the class of all 2-dimensional simplicial complexes, because for non-pseudomanifolds, there are (at least) two generalizations of the classical angle defect, both of which are equal to the classical angle defect when restricted to pseudomanifolds.

Both of these generalizations of the classical angle defect work in all dimensions. The first of these generalizations, which I refer to as “standard curvature,” is discussed, among others, in [Ban67], [Win], [Bud89], [Che83], [CMS84] and [Zäh90]. It is very simple to define (though it does not directly resemble the classical angle defect), and it satisfies our four properties. In all dimensions, this type of curvature is concentrated at the vertices of simplicial complexes.

The second generalization of the classical angle defect, which I refer to as the “generalized angle defect,” was defined in [Blo98], and further studied in [Blo04] and [Blo06]. This type of curvature, which more closely resembles the classical angle defect than standard curvature, is a generalization of the higher dimensional angle defect for convex polytopes and manifolds studied, among others, in [She68], [Gri68] and [GS91]. In dimensions higher than 2, the generalized angle defect is not concentrated at the vertices of simplicial complexes, but rather is defined for each simplex of codimension at least 2. (A word on our terminology. In order to compare our approach with standard curvature, we originally somewhat artificially concentrated our curvature at the vertices in Section 3 of [Blo98], and called it “stratified curvature.” In Section 4 of [Blo98] we took the more natural approach that we are using at present, and referred to this approach by the unfortunate name “modified stratified curvature,” which misses the point that in this approach we are really working with a pure angle defect. Hence, in
the present paper, we will use the better name “generalized angle defect,” which we also use in [Blo04] and [Blo06]. A detailed comparison of standard curvature with both stratified curvature and the generalized angle defect may be found in [Blo98, Section 4].

It turns out that in the present paper we will not ever need the actual definitions of standard curvature and the generalized angle defect in the present paper—all we need is their properties. Like standard curvature, the generalized angle defect is invariant under simplicial isometries of stars and under subdivision, and it is continuous. It also satisfies a Descartes-Gauss-Bonnet Theorem, though not with respect to the Euler characteristic, but with respect to a variant of the Euler characteristic, called the stratified Euler characteristic. See [Blo98] for the definition of the stratified Euler characteristic, and a proof of the Descartes-Gauss-Bonnet Theorem for the generalized angle defect.

The following theorem characterizes both these types of curvatures on the set of all 2-dimensional simplicial complexes.

**Theorem 2.3.** Let \( \phi \) be a vertex-supported function on the set of all 2-dimensional simplicial complexes. Then \( \phi \) is standard curvature (respectively the generalized angle defect) iff \( \phi \) is invariant under simplicial isometries of stars and under subdivision, is continuous, and satisfies the Descartes-Gauss-Bonnet Theorem with respect to the Euler characteristic (respectively the stratified Euler characteristic).

Theorem 2.3 sheds light on how similar standard curvature and the generalized angle defect are in dimension 2. These two types of curvature are less similar in higher dimensions, because standard curvature is concentrated at vertices and the generalized angle defect is not. A better understanding of the difference between these two types of curvature awaits characterization of standard curvature and the generalized angle defect in higher dimensions. Unfortunately, our proof of Theorem 2.3 (which is really a corollary to Theorem 2.7 stated below) does not generalize beyond the 2-dimensional case. It would be interesting to know whether the higher dimensional analogs of our results are nonetheless true, using a different method of proof.

Also, we note that Theorem 2.3 does not imply Theorem 2.1, because more is being assumed about \( \phi \) in Theorem 2.3 than in Theorem 2.1.

We now turn our attention to the point of view of the Euler characteristic and its generalizations. The following definition gives our geometric analog of the notion of being locally determined as discussed in [Lev92] and [For00].

**Definition.** Let \( \mathcal{T} \) be a set of 2-dimensional simplicial complexes, and let \( \Lambda \) be a simplicial-complex-supported function on \( \mathcal{T} \). We say that \( \Lambda \) is **geometrically locally determined** if there is a vertex-supported function \( \phi \) on \( \mathcal{T} \) such that \( \phi \) is invariant under simplicial isometries of stars and under subdivision, is continuous, and satisfies the Descartes-Gauss-Bonnet Theorem with respect to \( \Lambda \). If we need to specify \( \phi \), we will say that \( \Lambda \) is geometrically locally determined by \( \phi \). \( \triangle \)
It is reasonable to expect that not every arbitrary simplicial-complex-supported function will be geometrically locally determined, and hence we should restrict our attention to those simplicial-complex-supported functions that are well-behaved in some appropriate way. In [For00], as seen in the title of that paper, the condition of being constant on the set of cones is used; this condition is weaker than the condition of being a homotopy invariant, which is used in [Lev92]. Because the stratified Euler characteristic of [Blo98, Section 2] is not a homotopy invariant (though it is a homeomorphism invariant), we adopt the approach of [For00], and will consider simplicial-complex-supported functions that are constant on a number of different sets of 2-dimensional simplicial complexes, as we now discuss.

We will use the following standard terminology. Let $D$ be a simplicial disk in $\mathbb{R}^2$. A pyramid on $D$ is the simplicial surface obtained by coning on $D$ from a point in $\mathbb{R}^3$ (called the apex of the pyramid) that is not in $\mathbb{R}^2$, and then taking the boundary. Let $R$ be a polygonal disk in $\mathbb{R}^2$ (not necessarily subdivided into simplices). A bipyramid on $R$ is the simplicial surface obtained by suspending $D$ from two points in $\mathbb{R}^3$ (called the apices of the bipyramid) that are not in $\mathbb{R}^2$, and then taking the boundary. We also need the following terms.

**Definition.** A planar fan is a simplicial disk in $\mathbb{R}^2$ with no interior vertices that is triangulated as a cone from one of its boundary vertices. Let $n \in \mathbb{Z}$ be such that $n \geq 0$. An $n$-flap with end-vertices $v$ and $w$ is a 2-dimensional simplicial complex $K$ containing vertices $v$ and $w$ such that the following conditions hold: (1) $\langle v, w \rangle$ is an edge of $K$; (2) $K$ has precisely $n$ 2-simplices, each of which has $\langle v, w \rangle$ as an edge; and (3) $K$ has no edges other than the 1-faces of these $n$ 2-simplices. △

Observe that there can be $n$-flaps for any non-negative integer $n$. A 0-flap is a single edge, and a 1-flap is a single 2-simplex. It is straightforward to verify that a 2-dimensional simplicial complex $L$ is an $n$-flap with end-vertices $v$ and $w$ iff $L = \text{star} \,(v, L) = \text{star} \,(w, L)$. Also, if $K$ is a 2-dimensional simplicial complex and $v \in K^{(0)}$, and if $N = \text{star} \,(v, K)$, then for any vertex $x$ of link $(v, K)$, it is seen that $\text{star} \,(x, N)$ is a $O(v, x)$-flap.

As mentioned above, we will consider simplicial-complex-supported functions that are constant on various sets of 2-dimensional simplicial complexes. In particular, if $\Lambda$ is a simplicial-complex-supported function, we will consider the case where $\Lambda$ is constant on the set of all planar fans, and we will write $\Lambda(\text{fan})$; the case where $\Lambda$ is constant on the set of all $n$-flaps, and we will write $\Lambda(n$-flap), for each $n \in \mathbb{N}$; the case where $\Lambda$ is constant on the set of all pyramids and bipyramids, and we will write $\Lambda(\text{pyramid})$; and the case where $\Lambda$ is constant on the set of all cones, and we will write $\Lambda(\text{cone})$.

For simplicial surfaces, our main technical result is the following theorem.

**Theorem 2.4.** Let $S$ be a set of simplicial surfaces that contains all pyramids and bipyramids. Let $\Lambda$ be a simplicial-complex-supported function on $S$. Suppose that $\Lambda$ is geometrically locally determined by a vertex-supported
function $\phi$ on $S$, and that $\Lambda$ is constant on the set of all pyramids and bipyramids. Then for each $K \in S$ and each $v \in K^{(0)}$, we have

$$\phi(v, K) = \frac{1}{2} \Lambda(\text{pyramid}) \left[ 1 - \sum_{\sigma^2 \ni v} \alpha(v, \sigma^2) \right],$$

(1)

where the summation is over all 2-simplices of $K$ containing $v$.

The proof of Theorem 2.4 will be given in Section 3. It is straightforward to see that Theorem 2.1 follows immediately from Theorem 2.4. Moreover, we have the following corollaries to Theorem 2.4, which characterize the Euler characteristic on the set of simplicial surfaces. The first of these corollaries follows immediately from Theorem 2.4 because of the Descartes-Gauss-Bonnet Theorem for the classical angle defect, and the second corollary follows from the first.

**Corollary 2.5.** Let $S$ be a set of simplicial surfaces that contains all pyramids and bipyramids. Let $\Lambda$ be a simplicial-complex-supported function on $S$. Suppose that $\Lambda$ is geometrically locally determined and is constant on the set of all pyramids and bipyramids. Then $\Lambda$ equals the Euler characteristic multiplied by $\frac{1}{2} \Lambda(\text{pyramid})$.

**Corollary 2.6.** Let $S$ be a set of simplicial surfaces that contains all pyramids and bipyramids. The Euler characteristic is the unique simplicial-complex-supported function on $S$ that is geometrically locally determined and has value 2 on all pyramids and bipyramids.

We now turn to the analogs of the above results for more general sets of 2-dimensional simplicial complexes, starting with the following definition, which is based on [For00].

**Definition.** Let $T$ be a set of 2-dimensional simplicial complexes. The set $T$ is **star-closed** if for every $K \in T$, and every vertex $v$ of $K$, we have $\text{star} (v, K) \in T$.

Clearly, the set of all finite 2-dimensional simplicial complexes is star-closed.

We can now state the analog for arbitrary 2-dimensional simplicial complexes of Theorem 2.4.

**Theorem 2.7.** Let $T$ be a star-closed set of 2-dimensional simplicial complexes that contains all planar fans. Let $\Lambda$ be a simplicial-complex-supported function on $T$. Suppose that $\Lambda$ is geometrically locally determined by a vertex-supported function $\phi$ on $T$, that $\Lambda$ is constant on the set of all planar fans, and for each $n \geq 0$ the function $\Lambda$ is constant on the set of all $n$-flaps...
in $T$. Then for each $K \in T$ and each $v \in K^{(0)}$, we have

$$
\phi(v, K) = \Lambda(\text{star}(v, K)) - \frac{1}{2} \sum_{w \in \text{link}(v, K)} \Lambda(O(v, w) \text{-flap})
+ \frac{1}{2} \Lambda(\text{fan}) f_1(\text{link}(v, K)) - \Lambda(\text{fan}) \sum_{\sigma^2 \ni v} \alpha(v, \sigma^2),
$$

(2)

where the first summation is over all vertices $w$ of $\text{link}(v, K)$, and the second summation is over all 2-simplices of $K$ containing $v$.

One could view Theorem 2.7 as the geometric, 2-dimensional analog of the uniqueness stated in [Lev92, Theorem B]. The following result is an immediate consequence of Theorem 2.7.

**Corollary 2.8.** Let $T$ be a star-closed set of 2-dimensional simplicial complexes that contains all disks. Let $\Lambda$ be a simplicial-complex-supported function on $T$. Suppose that $\Lambda$ is geometrically locally determined that $\Lambda$ is constant on the set of all planar fans, and for each $n \geq 0$ the function $\Lambda$ is constant on the set of all $n$-flaps in $T$. Then there is a unique vertex-supported function $\phi$ on $T$ such that $\Lambda$ is geometrically locally determined by $\phi$.

We note that Theorem 2.7 implies not only that any simplicial-complex-supported function $\Lambda$ that is geometrically locally determined, and is constant on the set of all planar fans and is constant on the set of all $n$-flaps for each $n \geq 0$, is geometrically locally determined by a unique vertex-supported function $\phi$, but that such $\phi$ necessarily has the form of an angle defect, in that the first three terms in the right hand side of Equation 2 depends only upon $\text{link}(v, K)$ up to homeomorphism, and hence the right hand side of Equation 2 has the form of a measure of flatness (which is 1 in the case of simplicial surfaces, and in general depends only upon the topology of a neighborhood of $v$) minus the sum of the angles at $v$ (once the term $\Lambda(\text{fan})$ has been factored out).

It is straightforward to see that Theorem 2.3 follows immediately from Corollary 2.8.

The following two corollaries to Theorem 2.7 characterize the Euler characteristic on the set of simplicial surfaces. The first of these corollaries will be proved in Section 3, and the second corollary follows from the first.

**Corollary 2.9.** Let $T$ be a star-closed set of 2-dimensional simplicial complexes that contains all planar fans. Let $\Lambda$ be a simplicial-complex-supported function on $T$. Suppose that $\Lambda$ is geometrically locally determined, and that $\Lambda$ is constant on the set of all cones in $T$. Then $\Lambda$ equals the Euler characteristic multiplied by $\Lambda(\text{cone})$.

**Corollary 2.10.** Let $T$ be a star-closed set of 2-dimensional simplicial complexes that contains all planar fans. The Euler characteristic is the unique
3. Proofs

We will prove Theorem 2.7, Corollary 2.9 and Theorem 2.4.

Proof of Theorem 2.7. We start with the following preliminary observation. Let \( K \) and \( L \) be 2-dimensional simplicial complexes, let \( v \in K(0) \) and let \( u \in L(0) \). Suppose that link \((v, K)\) and link \((u, L)\) are both polygonal arcs, and that the sum of the angles at \( v \) equals the sum of the angles at \( u \). Clearly there is a subdivision \( K' \) of \( K \) and a subdivision \( L' \) of \( L \) such that star \((v, K')\) and star \((u, L')\) are simplicially isometric. It then follows from the invariance of \( \phi \) under subdivision and under simplicial isometries of stars that \( \phi(v, K) = \phi(v, K') = \phi(u, L') = \phi(u, L) \).

Our proof has a number of steps, first looking at some special cases, and then proving the result in general in the last step. In each step, we will let \( K \) be a 2-dimensional simplicial complex, and we will let \( v \in K(0) \) subject to certain stated conditions; we will then find a formula for \( \phi(v, K) \) in the given case. The arguments in many of the steps are similar to each other, and we will omit some of the details.

Step 1. Suppose that link \((v, K)\) is a polygonal arc, and that the sum of the angles at \( v \) is a number \( \epsilon \) that has the form \( \epsilon = \frac{n-2}{2n} \) for some \( n \in \mathbb{N} \) such that \( n \geq 3 \). We will show that \( \phi(v, K) = \Lambda(\text{fan}) \left[ \frac{1}{2} - \epsilon \right] \). (It can be verified that this last equation is a special case of Equation 2, though we will not need this fact, and will not give the details.)

Let \( S \) be a disk in \( \mathbb{R}^2 \), the boundary of which is a regular polygon with \( n \) vertices, say \( a_1, \ldots, a_n \). The angle at each \( a_i \) equals \( \frac{n-2}{2n} \) (recall that we have normalized angles so that a complete circle has angle 1). The disk \( S \) can be triangulated as a planar fan. Then the link of each \( a_i \) is a polygonal arc, and the sum of the angles at each \( a_i \) is \( \frac{n-2}{2n} \). By our preliminary observation, we know that all the \( \phi(a_i, S) \) are equal to each other and to \( \phi(v, K) \). Applying the Descartes-Gauss-Bonnet Theorem to the disk \( S \), we deduce that \( \sum_{i=1}^{n} \phi(a_i, S) = \Lambda(S) \), and hence \( n \phi(v, K) = \Lambda(S) \), which implies \( \phi(v, K) = \frac{1}{n} \Lambda(\text{fan}) \). However, because \( \epsilon = \frac{n-2}{2n} \), we have \( n = \frac{1}{2} - \epsilon \), and we deduce that \( \phi(v, K) = \Lambda(\text{fan}) \left[ \frac{1}{2} - \epsilon \right] \).

Step 2. Suppose that link \((v, K)\) is a polygonal arc, and that the sum of the angles at \( v \) is a rational number \( \delta \) such that \( 0 < \delta < \frac{1}{2} \). We will show that \( \phi(v, K) = \Lambda(\text{fan}) \left[ \frac{1}{2} - \delta \right] \).

Let \( \delta = \frac{p}{q} \) for some \( p, q \in \mathbb{N} \). Because \( \delta < \frac{1}{2} \), then \( q \geq 2 \). Let \( D \) be a convex polygonal disk in \( \mathbb{R}^2 \) with \( p + 3 \) vertices, labeled \( s, x, t, a_1, \ldots, a_p \) in order around the boundary of \( D \), such that the angles at \( s \) and \( t \) are \( \frac{1}{4} \), the angle at \( x \) is \( \delta \), and the angle at each \( a_i \) is \( \frac{q-2}{2q} \). That such a convex polygon exists is due to the fact that all the angles are less than \( \frac{1}{2} \), and the sum of
the exterior angles is precisely 1 (as can easily be verified). The disk $D$ can be triangulated as a planar fan. Observe that $\frac{1}{4} = \frac{4-2}{4} = 1 - \frac{2}{4}$, and hence all of the angles in $D$ other than the angle at $x$ satisfy the hypothesis of Step 1.

Applying the Descartes-Gauss-Bonnet Theorem to the disk $D$, and solving for $\phi(x, D)$, we deduce that

$$
\phi(s, D) + \phi(x, D) + \phi(t, D) + \sum_{i=1}^{p} \phi(a_i, D) = \Lambda(D).
$$

By Step 1 we conclude that

$$
\phi(x, D) = \Lambda(D) - \phi(s, D) - \phi(t, D) - \sum_{i=1}^{p} \phi(a_i, D)
$$

and hence all of the angles in $D$ other than the angle at $x$ satisfy the hypothesis of Step 1.

Step 3. Suppose that link $(v, K)$ is a polygonal arc, and that the sum of the angles at $v$ is a real number $\gamma$ such that $0 < \gamma < \frac{1}{2}$. We will show that $\phi(v, K) = \Lambda(\text{fan}) \left[ \frac{1}{2} - \gamma \right]$.

There exists a sequence of positive rational numbers $\{\delta_n\}_{n=1}^{\infty}$ such that $0 < \delta_n < \frac{1}{2}$ for all $n$, and $\lim_{n \to \infty} \delta_n = \gamma$. Let $T = \langle x, y, z \rangle$ be a triangle in $\mathbb{R}^2$ such that the angle at $x$ is $\gamma$. Clearly there is a sequence of triangles $\{T_n\}_{n=1}^{\infty}$ in $\mathbb{R}^2$, where for each $n$ we have $T_n = \langle x_n, y_n, z_n \rangle$ with the angle at $x_n$ equal to $\delta_n$, and such that $\lim_{n \to \infty} T_n = T$, with $\lim_{n \to \infty} x_n = x$. By Step 2 we know that $\phi(x_n, T_n) = \Lambda(\text{fan}) \left[ \frac{1}{2} - \delta_n \right]$ for all $n$. By the continuity of $\phi$, we know that $\phi(x, T) = \lim_{n \to \infty} \phi(x_n, T_n) = \lim_{n \to \infty} \Lambda(\text{fan}) \left[ \frac{1}{2} - \delta_n \right] = \Lambda(\text{fan}) \left[ \frac{1}{2} - \gamma \right]$. Hence $\phi(v, K) = \Lambda(\text{fan}) \left[ \frac{1}{2} - \gamma \right]$.

Step 4. Suppose that link $(v, K)$ is a polygonal arc, and that the sum of the angles at $v$ is a real number $\beta$ such that $0 < \beta < 1$. We will show that $\phi(v, K) = \Lambda(\text{fan}) \left[ \frac{1}{2} - \beta \right]$.

Let $Q$ be a quadrilateral in $\mathbb{R}^2$ with vertices $e, b_1, b_2, b_3$, such that the angle at $e$ is $\beta$, and the other three angles are less than $\frac{1}{2}$. The quadrilateral $Q$ can be triangulated as a planar fan. The desired result can be obtained by applying the Descartes-Gauss-Bonnet Theorem to the disk $Q$, solving for $\phi(e, Q)$, and then using Step 3 and the fact that the sum of the angles in a quadrilateral is 1.

Step 5. Suppose that $K$ is an $n$-flap for some $n \geq 0$, and that $v$ is one of the end-vertices of $K$. We will show that

$$
\phi(v, K) = \frac{1}{2} \Lambda(n\text{-flap}) - \Lambda(\text{fan}) \sum_{\sigma : \exists \nu} a(v, \sigma^2),
$$

(3)
where the summation is over all 2-simplices of $K$ containing $v$. (Again, it can be verified that Equation 3 is a special case of Equation 2, though we will not need this fact.)

Let $w$ be the other end-vertex of $K$. We have four subcases.

**Subcase 1.** Suppose that $n = 0$. Then $K$ consists of a single edge $\langle v, w \rangle$ together with its vertices. Clearly $v$ and $w$ have simplicially isometric stars, and hence $\phi(v, K) = \phi(w, K)$. Applying the Descartes-Gauss-Bonnet Theorem to $\phi$, we deduce that $\phi(v, K) = \frac{1}{2} \Lambda(\langle v, w \rangle)$. Because $\langle v, w \rangle$ is a 0-flap, and $v$ is contained in no 2-simplices, clearly Equation 3 holds in this case.

**Subcase 2.** Suppose that $n = 1$. Then $K$ is a single 2-simplex. Let $\omega$ be the angle at $v$ in $K$. We deduce from Step 3 that $\phi(v, K) = \Lambda(\text{fan}) \left[ \frac{1}{2} - \omega \right]$. This last equation is a special case of Equation 3, using the fact that a 1-flap is a planar fan.

**Subcase 3.** Suppose that $n \geq 2$. Assume that $\alpha(v, \sigma^2) < \frac{1}{4}$ for all 2-simplices $\sigma^2$ of $K$ containing $v$.

First, observe that because $K$ is an $n$-flap, we can find an embedding of $K$ in $\mathbb{R}^3$ that is simplicially isometric with $K$. Hence, by the invariance of $\phi$ under simplicial isometries of stars, we may assume without loss of generality that $K$ is in $\mathbb{R}^3$.

Choose a plane $\Pi$ in $\mathbb{R}^3$ that intersects the relative interior of the edge $\langle v, w \rangle$ and is perpendicular to it, and such $v$ is on one side of $\Pi$ and all the other vertices of $K$ are on the other side. Let $V$ be the intersection of $|K|$ with the closed half-space in $\mathbb{R}^3$ that has $\Pi$ as its boundary and contains $v$. Let $V'$ be the the result of reflecting $V$ in $\Pi$, and let $Y = V \cup V'$. The fact that $Y$ is an $n$-flap follows from the hypothesis concerning the angles in the 2-simplices of $K$ that contain $v$, and the choice of $\Pi$. Let $x$ denote the vertex of $Y$ that is the mirror image of $v$, let $\tau_1, \ldots, \tau_n$ denote the $n$ 2-simplices in $Y$, and for each $i \in \{1, \ldots, n\}$ let $d_i$ denote the vertex in $\tau_i$ that is not $v$ or $x$.

By Step 4 and making use of the symmetry of $Y$, we see that $\phi(d_i, Y) = \Lambda(\text{fan}) \left[ \frac{1}{2} - \alpha(d_i, Y) \right] = 2\Lambda(\text{fan})\alpha(v, \tau_i)$ for each $i \in \{1, \ldots, n\}$. By the invariance of $\phi$ under subdivision and under simplicial isometries of stars, we see that $\phi(v, K) = \phi(v, Y) = \phi(x, Y)$. The Descartes-Gauss-Bonnet Theorem applied to $Y$ then implies that

$$2\phi(v, Y) + \sum_{i=1}^{n} \phi(d_i, Y) = \Lambda(Y),$$

and hence that

$$\phi(v, K) = \phi(v, Y) = \frac{1}{2}\left\{\Lambda(Y) - \sum_{i=1}^{n} \phi(d_i, Y)\right\}$$

$$= \frac{1}{2}\Lambda(n\text{-flap}) - \Lambda(\text{fan}) \sum_{i=1}^{n} \alpha(v, \tau_i),$$
which is equivalent to Equation 3.

**Subcase 4.** Suppose that $n \geq 2$, but we make no assumptions regarding the angles in $K$. (Note that the argument used in Subcase 3 will not work in the general case, because $Y$ as constructed would not always be an $n$-flap, though it would have underlying space homeomorphic to one.) As in Subcase 3 we may assume without loss of generality that $K$ is in $\mathbb{R}^3$.

We now modify $K$ as follows. Let $\zeta_1, \ldots, \zeta_n$ denote the $n$ 2-simplices in $K$, and for each $i \in \{1, \ldots, n\}$ let $e_i$ denote the vertex in $\zeta_i$ that is not $v$ or $w$. Let $i \in \{1, \ldots, n\}$. If $\alpha(w, \zeta_i) < \frac{1}{4}$, then we leave $\zeta_i$ unchanged. If $\alpha(w, \zeta_i) \geq \frac{1}{4}$, then we modify $\zeta_i$ by moving $e_i$ along the line segment $\langle v, e_i \rangle$ toward $v$ until $\alpha(w, \zeta_i)$ becomes less than $\frac{1}{4}$. This modification will not change $\alpha(v, \zeta_i)$. After we modify all triangles as needed, we call the new $n$-flap $Z$. Similarly to previous arguments, we note that $\phi(v, K) = \phi(v, Z)$. Observe also that the vertex $w$ in $Z$ satisfies the hypotheses of Subcase 3, and so $\phi(w, Z)$ satisfies Equation 3.

By Step 4, for each $i \in \{1, \ldots, n\}$ we have

$$\phi(e_i, Z) = \Lambda(fan) \left[ \frac{1}{2} - \alpha(e_i, Z) \right] = \Lambda(fan) \left[ \alpha(v, \zeta_i) + \alpha(w, \zeta_i) \right].$$

The Descartes-Gauss-Bonnet Theorem applied to $Z$, and then solving for $\phi(v, Z)$, we see that

$$\phi(v, K) = \phi(v, Z) = \Lambda(Z) - \phi(w, Z) - \sum_{i=1}^{n} \phi(e_i, Z)$$

$$= \Lambda(n\text{-flap}) - \left\{ \frac{1}{2} \Lambda(n\text{-flap}) - \Lambda(fan) \sum_{i=1}^{n} \alpha(w, \zeta_i) \right\}$$

$$- \Lambda(fan) \sum_{i=1}^{n} \left[ \alpha(v, \zeta_i) + \alpha(w, \zeta_i) \right],$$

which implies Equation 3.

**Step 6.** We now prove our theorem. Suppose that $K$ is a 2-dimensional simplicial complex, and that $v \in K^{(0)}$.

Let $N = \text{star}(v, K)$. Clearly $\text{link}(v, K) = \text{link}(v, N)$, and by the invariance of $\phi$ under simplicial isometries of stars we know that $\phi(v, K) = \phi(v, N)$. Hence we can rewrite Equation 2 as

$$\phi(v, N) = \Lambda(N) - \frac{1}{2} \sum_{w \in \text{link}(v, N)} \Lambda(\mathcal{O}(v, w)\text{-flap}) + \frac{1}{2} \Lambda(fan) f_1(\text{link}(v, N))$$

$$- \Lambda(fan) \sum_{\sigma \ni v} \alpha(v, \sigma)$$,

(4)
where the first summation is over all vertices $w$ of link $(v, N)$, and the second summation is over all 2-simplices of star $(v, N)$ containing $v$. We will prove Equation 4.

If link $(v, N) = \emptyset$, then $N = \{v\}$, and Equation 4 is trivially true, because the Descartes-Gauss-Bonnet Theorem applied to $N$ yields $\phi(v, N) = \Lambda(N)$, and link $(v, N) = \emptyset$ implies that all the terms in the right hand side of Equation 4 other than $\Lambda(N)$ are zero. Hence we suppose that link $(v, N) \neq \emptyset$.

Let $r$ be a vertex in link $(v, N)$. Let $M = \text{star}(r, N)$. As remarked after the definition of $n$-flaps, we know that $M$ is a $O(v, r)$-flap with end-vertices $v$ and $r$. By the invariance of $\phi$ under simplicial isometries of stars, we know that $\phi(r, N) = \phi(r, M)$. It then follows from Step 5 that

$$\phi(r, N) = \phi(r, M) = \frac{1}{2} \Lambda(O(v, r)\text{-flap}) - \Lambda(\text{fan}) \sum_{\sigma^2 \ni r} \alpha(r, \sigma^2), \quad (5)$$

where the summation is over all 2-simplices of $M$ containing $r$.

Recall that all the vertices of $N$ other than $v$ are in link $(v, N)$. Applying the Descartes-Gauss-Bonnet Theorem to $N$, and then solving for $\phi(v, N)$, we see that

$$\phi(v, N) = \Lambda(N) - \sum_{w \in \text{link}(v, N)} \phi(w, N)$$

$$= \Lambda(N) - \sum_{w \in \text{link}(v, N)} \left\{ \frac{1}{2} \Lambda(O(v, w)\text{-flap}) - \Lambda(\text{fan}) \sum_{\sigma^2 \ni w} \alpha(w, \sigma^2) \right\}$$

by Equation 5

$$= \Lambda(N) - \frac{1}{2} \sum_{w \in \text{link}(v, N)} \Lambda(O(v, w)\text{-flap}) + \sum_{w \in \text{link}(v, N)} \Lambda(\text{fan}) \sum_{\sigma^2 \ni w} \alpha(w, \sigma^2)$$

$$= \Lambda(N) - \frac{1}{2} \sum_{w \in \text{link}(v, N)} \Lambda(O(v, w)\text{-flap}) + \sum_{\sigma^2 \in \text{star}(v, N)} \Lambda(\text{fan}) \sum_{y \in \sigma^2, y \neq v} \alpha(y, \sigma^2)$$

because every 2-simplex in $N$ contains $v$

$$= \Lambda(N) - \frac{1}{2} \sum_{w \in \text{link}(v, N)} \Lambda(O(v, w)\text{-flap}) + \sum_{\sigma^2 \in \text{star}(v, N)} \Lambda(\text{fan}) \left[ \frac{1}{2} - \alpha(v, \sigma^2) \right]$$

because the sum of the angles in a triangle is $\frac{1}{2}$

$$= \Lambda(N) - \frac{1}{2} \sum_{w \in \text{link}(v, N)} \Lambda(O(v, w)\text{-flap}) + \frac{1}{2} \Lambda(\text{fan}) f_1(\text{link}(v, N))$$

$$- \Lambda(\text{fan}) \sum_{\sigma^2 \ni v} \alpha(v, \sigma^2),$$
where the last equation holds because \( f_2(\text{star}(v,N)) = f_1(\text{link}(v,N)) \). Hence Equation 4 holds.

For our next proof, we will need the following notation. For any simplex \( \eta \) of dimension 0, 1 or 2 in Euclidean space, and any vertex \( v \) of \( \eta \), we let \( \alpha^*(v,\eta) \) denote the exterior angle of \( \eta \) at \( v \). If \( \eta \) is a 2-simplex, then \( \alpha^*(v,\eta) = \frac{1}{2} - \alpha(v,\eta) \); if \( \eta \) is a 1-simplex, then \( \alpha^*(v,\eta) = \frac{1}{2} \); if \( \eta \) is a 0-simplex (so that \( \sigma = v \)), then \( \alpha^*(v,\eta) = 1 \).

Proof of Corollary 2.9. Note that every planar fan, and every star of every vertex of a simplicial complex in \( T \), and in particular every \( O(v,w) \)-flap for appropriate vertices \( v \) and \( w \), are cones.

Let \( K \in T \), and let \( v \in K(0) \). It now follows from Equation 2 that

\[
\phi(v,K) = \Lambda(\text{cone}) \left\{ 1 - \frac{1}{2} f_0(\text{link}(v,K)) + \frac{1}{2} f_1(\text{link}(v,K)) - \sum_{\sigma^2 \ni v} \alpha(v,\sigma^2) \right\}.
\]

As given in a number of sources, for example [Ban67], we know that the standard curvature of \( K \) at \( v \) is given by

\[
S(v,K) = \sum_{i=0}^{2} (-1)^i \sum_{\eta^i \ni v} \alpha^*(v,\eta^i),
\]

where the inner summation is over all \( i \)-simplices of \( K \) containing \( v \). We then compute

\[
S(v,K) = \sum_{i=0}^{2} (-1)^i \sum_{\eta^i \ni v} \alpha^*(v,\eta^i)
\]

\[
= \alpha^*(v,v) - \sum_{\eta^1 \ni v} \alpha^*(v,\eta^1) + \sum_{\eta^2 \ni v} \alpha^*(v,\eta^2)
\]

\[
= 1 - \sum_{\eta^1 \ni v} \frac{1}{2} + \sum_{\eta^2 \ni v} \left[ \frac{1}{2} - \alpha(v,\eta^2) \right]
\]

\[
= 1 - \frac{1}{2} f_0(\text{link}(v,K)) + \frac{1}{2} - \sum_{\eta^2 \ni v} \alpha(v,\eta^2)
\]

\[
= 1 - \frac{1}{2} f_0(\text{link}(v,K)) + \frac{1}{2} f_1(\text{link}(v,K)) - \sum_{\eta^2 \ni v} \alpha(v,\eta^2).
\]

It follows that \( \phi(v,K) = \Lambda(\text{cone}) S(v,K) \).

Because standard curvature satisfies the Descartes-Gauss-Bonnet Theorem with respect to the Euler characteristic, and because \( \phi \) satisfies the Descartes-Gauss-Bonnet Theorem with respect to \( \Lambda \), it follows that \( \Lambda \) equals the Euler characteristic multiplied by the constant \( \Lambda(\text{cone}) \).  \( \square \)
We now turn to the proof of Theorem 2.4. The proof of this theorem is not identical to the proof of Theorem 2.7 because in the former theorem we assume that $\phi$ is defined only on simplicial surfaces, whereas in the proof of the latter theorem we make use of various simplicial complexes that are not simplicial surfaces. We will show, however, how the proof of Theorem 2.7 can be modified to work in the case of simplicial surfaces. (It would be easier to prove Theorem 2.4 if we allowed non-embedded simplicial surfaces, but that would add unnecessarily to the hypotheses of the theorem, so we will not do so.)

**Proof of Theorem 2.4.** This proof has a number of steps, most of which are similar to some of the steps of the proof of Theorem 2.7. We start with some observations.

(a). Because all simplicial complexes under consideration are simplicial surfaces, we know that the link of every vertex is a polygonal circle.

(b). Let $K, L \in S$, let $v \in K^{(0)}$ and let $w \in L^{(0)}$. Suppose that the sum of the angles at $v$ equals the sum of the angles at $w$. Clearly there is a subdivision $K'$ of $K$ and a subdivision $L'$ of $L$ such that star ($v, K'$) and star ($w, L'$) are simplicially isometric. It then follows from the invariance of $\phi$ under subdivision and under simplicial isometries of stars that $\phi(v, K) = \phi(w, L)$.

Let $\omega \in (0, \infty)$. Then it is possible to draw a polygonal spiral ribbon $R$ in $\mathbb{R}^2$, as in Figure 1, so that an appropriately chosen bipyramid $B$ on $R$ has angle sum $\omega$ at each of the apices, and has angle sum less than 1 at all other vertices (this latter condition will be used later in the proof). Observe that $B \in S$, because $S$ contains all bipyramids.

![Figure 1](image)

We can therefore think of $\phi$ as a function $(0, \infty) \to \mathbb{R}$, where for each $\alpha \in (0, \infty)$ we define $\phi(\alpha)$ by $\phi(\alpha) = \phi(v, K)$ for any $K \in S$ that has a vertex $v$ for which the sum of the angles at $v$ is $\alpha$.

(c). The continuity of $\phi$ as originally assumed implies that $\phi$ is continuous when thought of as a function $(0, \infty) \to \mathbb{R}$, as described in Observation (b).

Given the above observations, in order to prove the theorem as originally stated it suffices to show that

$$\phi(\omega) = \frac{1}{2}A(\text{pyramid}) \left[ 1 - \omega \right], \quad (6)$$
for all $\omega \in (0, \infty)$.

For the rest of this proof, let $\omega \in (0, \infty)$. We have a number of cases.

**Case 1.** Suppose that $\omega = 1$.

Let $T$ be a triangle in $\mathbb{R}^3$, with vertices $d_1, d_2, d_3$. Let $\Delta$ be a pyramid on $T$ with apex $b$. Let $\delta_i$ denote the sum of the angles in $\Delta$ at $d_i$, and let $\beta$ denote the sum of the angles in $\Delta$ at $b$. Let $\{L_k\}_{k=1}^{\infty}$ in $\mathbb{R}^3$ be a sequence of bipyramids on $T$, where all the $L_k$ have $b$ as one of their cone apices, where the other apex in $L_k$ is denoted $e_k$ for each $k$, and where the sequence $\{e_k\}_{k=1}^{\infty}$ converges to the centroid of $T$. For each $k$, let $\delta^k_i$ denote the sum of the angles in $L_k$ at $d_i$, and let $\gamma^k$ denote the sum of the angles in $L_k$ at $e_k$; observe that the sum of the angles in each $L_k$ at $b$ is $\delta$.

Clearly, we see that $\lim_{k \to \infty} \delta^k_i = \delta_i$ for all $i$, and $\lim_{k \to \infty} \gamma^k = 1 = \omega$.

Let $k \in \mathbb{N}$. By applying the Descartes-Gauss-Bonnet Theorem to $L_k$ we obtain

$$\phi(b, L_k) + \sum_{i=1}^{3} \phi(d_i, L_k) + \phi(e_k, L_k) = \Lambda(L_k).$$

Because $\Lambda$ is constant on the set of all pyramids and bipyramids, we know that $\Lambda(L_k) = \Lambda(\Delta)$ for all $k$. It then follows that

$$\phi(\beta) + \sum_{i=1}^{3} \phi(\delta^k_i) + \phi(\omega) = \Lambda(\Delta).$$

Taking the limit as $k \to \infty$, and using the continuity of $\phi$ as stated in Observation (c), we see that

$$\phi(\beta) + \sum_{i=1}^{3} \phi(\delta_i) + \phi(\omega) = \Lambda(\Delta). \quad (7)$$

On the other hand, applying the Descartes-Gauss-Bonnet Theorem to $\Delta$ yields

$$\phi(b, \Delta) + \sum_{i=1}^{3} \phi(d_i, \Delta) = \Lambda(\Delta),$$

which implies

$$\phi(\beta) + \sum_{i=1}^{3} \phi(\delta_i) = \Lambda(\Delta). \quad (8)$$

Comparing Equation $7$ with Equation $8$ shows that $\phi(\omega) = 0$, which is equivalent to Equation $6$ in the present case.

**Case 2.** Suppose that $\omega = \frac{n-2}{n}$ for some $n \in \mathbb{N}$ such that $n \geq 3$.

Let $S$ and $a_1, \ldots, a_n$ be as in Step 1 of the proof of Theorem $2.7$. Let $\{C_k\}_{k=1}^{\infty}$ in $\mathbb{R}^3$ be a sequence of pyramids on $S$, where the apex in $C_k$ is denoted $x_k$ for each $k$, where the 2-simplices of $C_k$ containing $x_k$ are all congruent isosceles triangles, and where the sequence $\{x_k\}_{k=1}^{\infty}$ converges to the centroid of $S$. For each $k$, let $\omega^k_i$ denote the sum of the angles in $C_k$ at $a_i$,
and let $\beta^k$ denote the sum of the angles in $C_k$ at $x_k$. Clearly $\lim_{k \to -\infty} \omega^k_i = \omega$ for all $i$, and $\lim_{k \to -\infty} \beta^k = 1$.

We now proceed similarly to Case 1. Let $k \in \mathbb{N}$. By applying the Descartes-Gauss-Bonnet Theorem to $C_k$ we obtain

$$\sum_{i=1}^n \phi(\omega^k_i) + \phi(\beta^k) = \Lambda(C_k).$$

By construction we know that the $\omega^k_i$ are all equal to each other for all $i$, and hence

$$\phi(\omega^k) = \frac{1}{n} \left[ \Lambda(\text{pyramid}) - \phi(\beta^k) \right].$$

Taking the limit as $k \to \infty$, and using the continuity of $\phi$, as well as Case 1, we see that

$$\phi(\omega) = \frac{1}{n} \left[ \Lambda(\text{pyramid}) - \phi(1) \right] = \frac{1}{n} \Lambda(\text{pyramid}). \quad (9)$$

Because $\omega = \frac{n-2}{n}$, we have $n = \frac{2}{1-\omega}$. Substituting this formula for $n$ into Equation (9) we see that Equation (6) holds in this case.

**Case 3.** Suppose that $\omega$ is a rational number such that $0 < \omega < 1$.

There are $p, q \in \mathbb{N}$ such that $\omega = \frac{p}{q}$. Because $\omega < 1$, then $q \geq 2$. The argument used to show that $\phi(\omega)$ is given by Equation (6) is similar to the argument in Case 2, except that we take pyramids on the polygon $D$ in Step 2 of the proof of Theorem 2.7; we omit the details.

**Case 4.** Suppose that $\omega \in (0, 1)$.

Equation (6) follows immediately from Case 3 and the continuity of $\phi$.

**Case 5.** Suppose that $\omega \in [1, \infty)$.

As remarked in Observation (b), it is possible to draw a polygonal spiral ribbon $R$ in $\mathbb{R}^2$ so that an appropriately chosen bipyramid $B$ on $R$ has angle sum $\omega$ at each of the apices, and has angle sum less than 1 at all other vertices. Suppose that the vertices of $R$ are denoted $b_1, \ldots, b_m$, and the apices of $B$ are denoted $x$ and $y$. Suppose further that for each $i$, the sum of the angles in $B$ at $b_i$ is $\beta_i$. By Case 4 we know that $\phi(\beta_i) = \frac{1}{2} \Lambda(\text{pyramid}) [1 - \beta_i]$ for each $i$.

Clearly the classical angle defect at $b_i$ is $1 - \beta_i$, and the classical angle defect at each of $x$ and $y$ is $1 - \omega$. Using the Descartes-Gauss-Bonnet Theorem for the classical angle defect applied to $B$, we have

$$2 = \chi(B) = \sum_{i=1}^m [1 - \beta_i] + 2 [1 - \omega],$$

which implies that

$$\omega = \frac{1}{2} \sum_{i=1}^m [1 - \beta_i].$$
Applying the Descartes-Gauss-Bonnet Theorem to $B$ yields

$$\Lambda(B) = \sum_{i=1}^{m} \phi(b_i, B) + \phi(x, B) + \phi(y, B),$$

and using arguments similar to those used previously in this proof, we deduce that

$$\phi(\omega) = \frac{1}{2} \left\{ \Lambda(\text{pyramid}) - \sum_{i=1}^{m} \phi(\beta_i) \right\}$$

$$= \frac{1}{2} \Lambda(\text{pyramid}) \left\{ 1 - \frac{1}{2} \sum_{i=1}^{m} [1 - \beta_i] \right\} = \frac{1}{2} \Lambda(\text{pyramid}) [1 - \omega].$$

\[\square\]

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