The β—Flatness Condition in CR Spheres
Multiplicity Results

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Abstract

We give multiplicity results for the problem of prescribing the scalar curvature on Cauchy-Riemann spheres under β—flatness condition. To give a lower bound for the number of solutions, we use Bahri’s methods based on the theory of critical points at infinity and a Poincare’-Hopf type formula.

1 Introduction

In an earlier paper we discussed existence results for the problem of prescribing the Webster scalar curvature on the 3-Cauchy-Riemann sphere, under β- flatness condition, 2 ≤ β < 4. The purpose of the present paper, is to study multiplicity results for this problem.

Let $S^3$ be the unit sphere of $\mathbb{C}^2$ endowed with its standard contact form $\theta_1$, and $K : S^3 \to \mathbb{R}$ be a given $C^2$ positive function. The problem of finding a contact form $\theta$ on $S^3$ conformal to $\theta_1$ admitting the function $K$ as Webster scalar curvature, is equivalent to the resolution of the following semi-linear equation:

\[
\begin{cases}
L_{\theta_1} u = K u^3 & \text{on } S^3 \\
u > 0
\end{cases}
\]  

where $L_{\theta_1} = 4\Delta_{\theta_1} + R_{\theta_1}$, is the conformal laplacian of $S^3$.

Using the CR equivalence $F$ induced by the Cayley Transform (see Definition 2.1 below) between $S^3$ minus a point and the Heisenberg group $\mathbb{H}^1$, equation (1.1) is equivalent up to an influent constant to

\[
\begin{cases}
4\Delta_{\mathbb{H}^1} u = \tilde{K} u^3 & \text{on } \mathbb{H}^1 \\
u > 0
\end{cases}
\]  

where $\Delta_{\mathbb{H}^1}$ is the sub laplacian of $\mathbb{H}^1$ and $\tilde{K} = K \circ F^{-1}$.

In order to give our new multiplicity results for problem (1.1), where the prescribed function $K$ satisfies a β—flatness condition near its critical points. We will use the same techniques displayed in [13] which are based on an adaptation to the Cauchy-Riemann settings of Bahri’s work. These techniques were first introduced by Bahri and Coron in [3]; we have to study the critical points at infinity of the associated variational problem, by computing their total Morse index. Then, we compare this total index to the Euler characteristic of the space of variation.
To state our results, we set up the following conditions and notations. Let \( G(a, \lambda) \) be a Green’s function for \( L \) at \( a \in S^3 \).

We denote by
\[
\mathcal{K} = \{ (\xi_i)_{1 \leq i \leq \tau}, \text{ such that } \nabla K(\xi_i) = 0 \}
\]
the set of all critical points of \( K \). We say that \( K \) satisfies the \( \beta \)-flatness condition if for all \( \xi_i \in \mathcal{K} \), there exist

\[
\beta = \beta(\xi_i) \quad \text{and} \quad b_1 = b_1(\xi_i), \ b_2 = b_2(\xi_i), \ b_0 = b_0(\xi_i) \in \mathbb{R}^n
\]
such that in some pseudo hermitian normal coordinates system centered at \( \xi_i \), we have
\[
K(x) = K(0) + b_1|x_1|^\beta + b_2|x_2|^\beta + b_0|t|^{\frac{\beta}{2}} + \mathcal{R}(x).
\]  
(1.3)

Where \( \sum_{k=1}^{2} b_k + \kappa b_0 \neq 0 \), \( \sum_{k=1}^{2} b_k + \kappa' b_0 \neq 0 \) with

\[
\kappa = \frac{\int_{\mathbb{H}^1} |t|^\beta \frac{1 - |z|^2 - it|^2}{1 + |z|^2 - it}|^2 \theta_0 \wedge d\theta_0}{\int_{\mathbb{H}^1} |x_1|^{\frac{\beta}{2}} \frac{1 - |z|^2 - it|^2}{1 + |z|^2 - it}|^2 \theta_0 \wedge d\theta_0}, \quad \kappa' = \frac{\int_{\mathbb{H}^1} |t|^\beta \frac{1 - |z|^2 - it|^2}{1 + |z|^2 - it}|^2 \theta_0 \wedge d\theta_0}{\int_{\mathbb{H}^1} |x_1|^{\frac{\beta}{2}} \frac{1 - |z|^2 - it|^2}{1 + |z|^2 - it}|^2 \theta_0 \wedge d\theta_0}
\]

The function \( \sum_{p=0}^{n} \nabla^p \mathcal{R}(x) \|x\|^{-\beta - r} = o(1) \) as \( x \) approaches \( \xi_i \), \( \nabla^r \) denotes all possible partial derivatives of order \( r \) and \( \lfloor \beta \rfloor \) the integer part of \( \beta \).

In this work, we will focus on the case where a collection of the critical points of \( K \) satisfy \( \beta = \beta(\xi_i) = 2 \). This case was not covered in the results of [11, 12, 13]. So, here we suppose \( 2 \leq \beta = \beta(\xi_i) < 4 \). Let

\[
\mathcal{K}_1 := \{ \xi_i \in \mathcal{K} \text{ such that } (1.3) \text{ is satisfied with } \beta = \beta(\xi_i) = 2 \text{ and } \sum_{k=1}^{2} b_k + \kappa b_0 < 0 \}
\]

\[
\mathcal{K}_2 := \{ \xi_i \in \mathcal{K} \text{ such that } (1.3) \text{ is satisfied with } \beta = \beta(\xi_i) > 2 \text{ and } \sum_{k=1}^{2} b_k + \kappa' b_0 < 0 \}
\]

The index of the function \( K \) at \( \xi_i \in \mathcal{K} \), denoted by \( m(\xi_i) \), is the number of strictly negative coefficients \( b_k(\xi_i) \):

\[
m(\xi_i) = \# \{ b_k(\xi_i) ; b_k(\xi_i) < 0 \}.
\]

For each \( p \)-tuple \( (\xi_{i_1}, \ldots, \xi_{i_p}) \in (\mathcal{K}_1)^p \) (\( \xi_{i_l} \neq \xi_{i_j} \) if \( l \neq j \)), we associate the matrix \( M(\xi_{i_1}, \ldots, \xi_{i_p}) = (M_{st})_{1 \leq s, t \leq p} \)

\[
M_{ss} = -c \frac{\sum_{k=1}^{2} b_k + \kappa' b_0}{2K^2(\xi_0)G(\xi_0, \xi_0)} \frac{1}{[K(\xi_0)K(\xi_0)]^{\frac{\beta}{2}}}, \quad \text{for } s \neq t
\]

\[
M_{st} = -c' \frac{1}{[K(\xi_0)K(\xi_0)]^{\frac{\beta}{2}}}, \quad \text{for } s \neq t
\]  
(1.4)
where \( c = \frac{|x|^{2}}{|1+|x||^{2}} \) and \( c' = 2\pi\omega_{3} \), \( \omega_{3} \) is the volume of the unit Koranyi’s ball.

We say that \( K \) satisfies condition \((C)\) if:

for each \( p \)-tuple \((\xi_{i1}, \ldots, \xi_{ip}) \in (K_1)^p \) the corresponding matrix \((M_{ij})\) is non degenerate. \( (1.5) \)

In this case, we denote by \( g(\xi_{i1}, \ldots, \xi_{ip}) \) the least eigenvalue of the matrix \( M(\xi_{i1}, \ldots, \xi_{ip}) \).

Next, we define the sets

\[
\mathcal{K}_{i}^{+} := \bigcup_{p} \{ (\xi_{i1}, \ldots, \xi_{ip}) \in (K_1)^p, \; g(\xi_{i1}, \ldots, \xi_{ip}) > 0 \}
\]

and

\[
\mathcal{I}^{+} := \max \{ p \in \mathbb{N} \; \text{s.t.} \; (\xi_{i1}, \ldots, \xi_{ip}) \in \mathcal{K}_{i}^{+} \}.
\]

For \((\xi_{i1}, \ldots, \xi_{ip}) \in \mathcal{K}_{i}^{+}\), let \( i(\xi_{i1}, \ldots, \xi_{ip}) := 4p - 1 - \sum_{j=1}^{p} m(\xi_{ij}) \)

and

\[
L_{0} := \max \left\{ \left\{ i(\xi_{i1}, \ldots, \xi_{ip}); \; (\xi_{i1}, \ldots, \xi_{ip}) \in \mathcal{K}_{i}^{+} \right\} \cup \left\{ 3 - m(\xi); \; \xi \in \mathcal{K}_{2} \right\} \right\} \quad (1.6)
\]

The main results of this paper are

**Theorem 1.1** Let \( K \) be a \( C^2 \) positive function on \( S^3 \) satisfying the \( \beta-\text{flatness condition} \) and condition \((C)\), if there exists a positive integer \( k \) such that:

1) \[
\sum_{\xi \in \mathcal{K}_{2}} (-1)^{m(\xi)+1} + \sum_{p=1}^{i^{+}} \sum_{(\xi_{i1}, \ldots, \xi_{ip}) \in \mathcal{K}_{i}^{+}} (-1)^{i(\xi_{i1}, \ldots, \xi_{ip})} \neq 1
\]

2) \( \forall (\xi_{i1}, \ldots, \xi_{ip}) \in \mathcal{K}_{i}^{+}, \sum_{j=1}^{p} m(\xi_{ij}) \neq 4p - (k + 1) \) and \( \forall \xi \in \mathcal{K}_{2}, \; 3 - m(\xi_{i}) \neq k. \)

Then, there exists a solution \( \omega \) to the problem \((1.1)\) such that

\[ m(\omega) \leq k, \]

where \( m(\omega) \) is the Morse index of \( \omega \), defined as the dimension of the space of negativity of the linearized operator \( L(\delta) := L_{0}(\delta) - 3\omega_{3}\delta. \)

Under the hypothesis of Theorem 1.1 if we denote \( \mathcal{S}_{k} \) the set of solutions of \((1.1)\) having their Morse indices less than or equal to \( k \). We have

**Theorem 1.2**

\[
\# \mathcal{S}_{k} \geq \left| \sum_{\xi \in \mathcal{K}_{2}} (-1)^{m(\xi)} - \sum_{p=1}^{i^{+}} \sum_{(\xi_{i1}, \ldots, \xi_{ip}) \in \mathcal{K}_{i}^{+}} (-1)^{i(\xi_{i1}, \ldots, \xi_{ip})} \right| \]

\[
\left| \sum_{\xi \in \mathcal{K}_{2}} (-1)^{m(\xi)} - \sum_{p=1}^{i^{+}} \sum_{(\xi_{i1}, \ldots, \xi_{ip}) \in \mathcal{K}_{i}^{+}} (-1)^{i(\xi_{i1}, \ldots, \xi_{ip})} \right|
\]

\[
\left| \sum_{\xi \in \mathcal{K}_{2}} (-1)^{m(\xi)} - \sum_{p=1}^{i^{+}} \sum_{(\xi_{i1}, \ldots, \xi_{ip}) \in \mathcal{K}_{i}^{+}} (-1)^{i(\xi_{i1}, \ldots, \xi_{ip})} \right|
\]
The proofs of Theorems $\ref{thm:1}$ and $\ref{thm:2}$ will be obtained by a contradiction argument. Therefore, we assume that equation $\eqref{eq:1}$ has no solution. Our approach involves a Morse lemma at infinity, it relies on the construction of a suitable pseudo gradient for the functional $J$. The Palais-Smale condition is satisfied along the decreasing flow lines of this pseudo gradient, as long as these flow lines do not enter the neighborhood of a finite number of critical points of $K$ where the related matrix given in $\eqref{eq:4}$ is positive definite.

This paper is organized as follows: in section 2, we recall the local structure of the Heisenberg group, the extremals for the Yamabe functional on $\mathbb{H}^1$, and the Cayley transform. In section 3, we give the expansion of the new functional $J$ near its critical points at infinity. Section 4 is devoted to the construction of a pseudo gradient for the functional $J$. The Morse lemma is based on the construction of a pseudo gradient for $J$ near its critical points at infinity, using an appropriate change of variables. The proofs of our main results, Theorems $\ref{thm:1}$ and $\ref{thm:2}$ will be the purpose of section 5. The last section is an appendix, where some technical estimates are given.

## 2 Preliminary Tools:

The Heisenberg group $\mathbb{H}^1$ is the Lie group whose underlying manifold is $\mathbb{C} \times \mathbb{R}$, with coordinates $g = (z,t)$ and group law given by: $g \cdot g' = (z,t) \cdot (z',t') = (z + z', t + t' + 2Im \, z \, z')$. We define a norm in $\mathbb{H}^1$ by $\|g\|_{\mathbb{H}^1} = \|z\|^2 + t^2$, and dilations by $g = (z,t) \rightarrow \lambda g = (\lambda z, \lambda^2 t)$, $\lambda > 0$. The Cauchy Riemann structure on $\mathbb{H}^1$ is given by the left invariant vectors fields: $Z = \frac{\partial}{\partial t} + iz \frac{\partial}{\partial z}$, $\bar{Z} = \frac{\partial}{\partial t} - iz \frac{\partial}{\partial \bar{z}}$, which are homogenous of degree $-1$ with respect to the dilations, the associated contact form is $\theta_0 = dt + i(zd\bar{z} - \bar{z}dz)$. We denote by $\Delta_{\theta_0}$ the sublaplacian operator, $\Delta_{\theta_0} = -\frac{i}{2}(ZZ + \bar{Z}Z)$ and since the Webster scalar curvature $R_{\theta_0}$ is zero, the conformal laplacian $L_0$ is a multiple of the sublaplacian operator, $L_0 = (2 + \frac{4}{n})\Delta_{\theta_0}$.

In $\cite{14}$, Jerison and Lee showed that all solutions of $\eqref{eq:2}$ are obtained from

$$w_{(0,1)}(z,t) = \frac{c_0}{|1 + |z|^2 - it|}, \quad c_0 > 0,$$

by left translations and dilatations on $\mathbb{H}^1$. That is for $g_0 = (z_0,t_0)$, $g = (z,t)$ in $\mathbb{H}^1$ and $\lambda > 0$, we have

$$w_{(\lambda g_0,\lambda)}(z,t) = \frac{\lambda}{|1 + \lambda^2 |z - z_0|^2 - i\lambda^2 (t - t_0 - 2Im \, z_0\bar{z})|}.$$

Next, we will introduce the Cayley transform. Let $B^2 = \{z \in \mathbb{C}^2 \mid |z| < 1\}$ be the unit ball in $\mathbb{C}^2$ and $D_2 = \{(z,w) \in \mathbb{C} \times \mathbb{C} \mid Im(w) > |z|^2\}$ be the Siegel domain. The boundary of the Siegel domain is: $\partial D_2 = \{(z,w) \in \mathbb{C} \times \mathbb{C} \mid Im(w) = |z|^2\}$.

**Definition 2.1** $\cite{17}$ The Cayley transform is the correspondence between the unit ball $B^2$ in $\mathbb{C}^2$ and the Siegel domain $D_2$, given by

$$\mathcal{C}(\zeta) = \left( \frac{\zeta_1}{1 + \zeta_2}, i \frac{1 - \zeta_2}{1 + \zeta_2} \right); \quad \zeta = (\zeta_1, \zeta_2), \quad 1 + \zeta_2 \neq 0.$$

The Cayley transform gives a biholomorphism of the unit ball $B^2$ in $\mathbb{C}^2$ onto the Siegel domain $D_2$. Moreover, when restricted to the sphere minus a point, $\mathcal{C}$ gives a $CR$ diffeomorphism.

$$\mathcal{C} : S^3 \setminus (0,-1) \rightarrow \partial D_2.$$
Let us recall the CR diffeomorphism

\[ f : \mathbb{H}^1 \to \partial D_2 \]

\[ (z, t) \mapsto f(z, t) = (z, t + i|z|^2) \]

with the obvious inverse \( f^{-1}(z, w) = (z, Re(w)), z \in \mathbb{C}, w \in \mathbb{C} \). We obtain the CR equivalence via this mapping:

\[ F : S^3 \setminus (0, -1) \to \mathbb{H}^1 \]

\[ \zeta = (\zeta_1, \zeta_2) \mapsto (z, t) = \left( \frac{\zeta_1}{1 + \zeta_2}, i \frac{2i|\zeta_2|}{1 + \zeta_2} \right) \]

with inverse

\[ F^{-1} : \mathbb{H}^1 \to S^3 \setminus (0, -1) \]

\[ (z, t) \mapsto \zeta = \left( \frac{2z}{1 + |z|^2 - it}, i \frac{1 - |z|^2 + it}{1 + |z|^2 - it} \right). \]

With the following choice of contact form on \( S^3 \) (the standard one)

\[ \theta_1 = i \sum_{j=1}^{2} (\zeta_j d\bar{\zeta}_j - \bar{\zeta}_j d\zeta_j). \]

We obtain \( F^* (4(c_0^{-1} w_{(0,1)})^2 \theta_0) = \theta_1 \).

Let us differentiate and take into account that \( w_{(0,1)}(F(\zeta)) = c_0|1 + \zeta_2|^2 \), we obtain

\[ d\theta_1 = \left( \frac{d\zeta_2}{1 + \zeta_2} + \frac{d\bar{\zeta}_2}{1 + \bar{\zeta}_2} \right) \wedge \theta_1 + |1 + \zeta_2|^2 F^*(d\theta_0) \]

and

\[ \theta_1 \wedge d\theta_1 = |1 + \zeta_2|^4 F^*(\theta_0 \wedge d\theta_0) \]

We introduce the following function for each \((\zeta_0, \lambda)\) on \( S^3 \times ]0, +\infty[\)

\[ \delta_{(\zeta_0, \lambda)}(\zeta) = |1 + \zeta_2|^{-1} w_{(F(\zeta_0), \lambda)} \circ F(\zeta) \tag{2.1} \]

We have \( L_{\theta_1} \delta_{(\zeta_0, \lambda)} = \delta_{(\zeta_0, \lambda)} \), i.e \( \delta_{(\zeta_0, \lambda)} \) is a solution of the Yamabe problem on \( S^3 \).

We also have

\[ \int_{S^3} L_{\theta_1} \delta_{(\zeta_0, \lambda)} \delta_{(\zeta_0, \lambda)} \wedge d\theta_1 = \int_{\mathbb{H}^1} L_{\theta_0} w_{(g_0, \lambda)} w_{(g_0, \lambda)} \theta_0 \wedge d\theta_0 \tag{2.2} \]

and

\[ \int_{S^3} |\delta_{(\zeta_0, \lambda)}|^4 \wedge d\theta_1 = \int_{\mathbb{H}^1} |w_{(g_0, \lambda)}|^4 \theta_0 \wedge d\theta_0, \tag{2.3} \]

where \( g_0 = F(\zeta_0) \), and \( g = F(\zeta) \).

As a consequence, the variational formulation for \((1.1)\) is equivalent to the one for \((1.2)\).
2.1 Cauchy Riemann Functional

Problem (1.1) has a nice variational structure, with associated Euler functional:

\[ J(u) = \int_{\mathbb{S}^3} L_{\theta_1} u \, d\theta_1 \wedge d\theta_2, \quad u \in S^2_1(\mathbb{S}^3) \]

where \( S^2_1(\mathbb{S}^3) \) is the completion of \( C^\infty(\mathbb{S}^3) \) by means of the norm \( \|u\|^2 = \int_{\mathbb{S}^3} L_{\theta_1} u \, d\theta_1 \wedge d\theta_2 \).

Let \( \Sigma = \{ u \in S^2_1(\mathbb{S}^3) / \|u\| = 1 \} \) and \( \Sigma^+ = \{ u \in \Sigma / u \geq 0 \} \).

The functional \( J \) fails to satisfy the Palais-Smale condition denoted by \((P.S)\) on \( \Sigma^+ \), that is: there exist noncompact sequences along which the functional \( J \) is bounded and its gradient goes to zero. A complete description of sequences failing to satisfy \((P.S)\) is given in [7]. A solution \( u \) of (1.1) is a critical point of \( J \) subject to the constraint \( u \in \Sigma^+ \).

2.2 Characterization of the sequences failing to satisfy the \((P.S)\) condition

In the case we study, we have the presence of multiple blow-up points. We begin by defining the sets of potential critical points at infinity of the functional \( J \).

For any \( \varepsilon > 0 \) and \( p \in \mathbb{N}^+ \), let:

\[
V(p, \varepsilon, \omega) = \left\{ u \in \Sigma^+ ; \exists (a_1, \ldots, a_p) \in \mathbb{S}^3, \alpha_1, \ldots, \alpha_p > 0 \text{ and } (\lambda_1, \ldots, \lambda_p) \in (\varepsilon^{-1}, \infty)^p \text{ such that} \right. \\
\left. \begin{array}{l}
\left\| u - \sum_{i=1}^{p} \alpha_i \delta_{a_i, \lambda_i} \right\|_{S^2_1(\mathbb{S}^3)} < \varepsilon, \; \varepsilon_{ij} < \varepsilon, \quad \frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} - 1 < \varepsilon, \quad \forall \; 1 \leq i \neq j \leq p \\
\varepsilon_{ij} = \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j (d(a_i, a_j)^2)^{-1} \right) \right. \\
\left. \quad \forall \; 1 \leq i \neq j \leq p \right. \\
\left. u - \alpha_0 \omega \in V(p, \varepsilon) \text{ and } |\alpha_0^2 J(u)| < \varepsilon \right\}
\]

(2.4)

For \( \omega \) a solution of (1.1) we also define the set

\[
V(p, \varepsilon) = \left\{ u \in \Sigma^+ ; \exists \alpha_0 > 0 / u - \alpha_0 \omega \in V(p, \varepsilon) \text{ and } |\alpha_0^2 J(u)| < \varepsilon \right\}.
\]

We then proceed as in [7] Proposition 8 to characterize the sequences which violate the \((P.S)\) condition as follows:

**Proposition 2.2** ([7]) Let \( \{u_k\} \) be a sequence such that \( \partial J(u_k) \to 0 \) and \( J(u_k) \) is bounded. There exist an integer \( p \in \mathbb{N}^+ \), a sequence \( \varepsilon_k \to 0 \) \( (\varepsilon_k > 0) \) and an extracted subsequence of \( \{u_k\} \), again denoted by \( \{u_k\} \), such that \( u_k \in V(p, \varepsilon_k) \).

Then, we consider the following minimization problem for a function \( u \in V(p, \varepsilon) \), with \( \varepsilon \) small

\[
\min_{\alpha_i > 0, \lambda_i > 0, a_i \in \mathbb{S}^3} \|u - \sum_{i=1}^{p} \alpha_i \delta_{a_i, \lambda_i}\|_{S^2_1(\mathbb{S}^3)}
\]

(2.5)

We obtain as showed in [2] and [8], the following parametrization of the set \( V(p, \varepsilon) \):

**Proposition 2.3** ([8]) For any \( p \in \mathbb{N}^+ \), there exists \( \varepsilon_p > 0 \) such that, for any \( 0 < \varepsilon < \varepsilon_p \), \( u \in V(p, \varepsilon) \), the minimization problem (2.5) has a unique solution \( (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_p, \bar{a}_1, \ldots, \bar{a}_p) \) (up to permutation on the set of indices \( \{1, \ldots, p\} \)). In particular, we can write \( u \in V(p, \varepsilon) \) as follows
The flatness-multiplicity results:

\[ u = \sum_{i=1}^{p} \alpha_i \delta a_i, \lambda_i + v, \]

where \( v \in S^2_1(S^3) \) satisfies:

\[
\begin{align*}
(\text{V}_0) \left\{ 
\langle v, \delta a_i, \lambda_i \rangle_{S^2_1(S^3)} &= 0, \\
\langle v, \frac{\partial \delta a_i}{\partial \alpha_i} \rangle_{S^2_1(S^3)} &= 0, \\
\langle v, \frac{\partial \delta a_i}{\partial \lambda_i} \rangle_{S^2_1(S^3)} &= 0.
\end{align*}
\]

Here \( \langle, \rangle \) denotes the \( L \)-scalar product defined on \( S^2_1(S^3) \) by

\[
\langle u, v \rangle = \int_{S^3} L_{\theta_1} u v \theta_1 \wedge d\theta_1.
\]

Next, we will focus on the behavior of the functional \( J \) with respect to the variable \( v \). We will prove the existence of a unique \( \bar{v} \) which minimizes \( J(\sum_{i=1}^{p} \alpha_i \delta a_i, \lambda_i) + v \) with respect to \( v \in H^p_\varepsilon(a, \lambda) \), where

\[
H^p_\varepsilon(a, \lambda) = \{ v \in S^2_1(M) \mid v \text{ satisfies } (V_0) \text{ and } \|v\| < \frac{\varepsilon}{p} \}.
\]

**Proposition 2.4** [8] There exists a \( C^1 \)-map which associates to each \( u \in V(p, \varepsilon) \), \( \varepsilon \) small, \( \bar{v} = \bar{v}(\alpha, a, \lambda) \) such that \( \bar{v} \) is unique and minimizes \( J(\sum_{i=1}^{p} \alpha_i \delta a_i, \lambda_i) + v \), with respect to \( v \in H^p_\varepsilon(a, \lambda) \). We have the following estimate

\[
\|\bar{v}\| \leq c_1 \left( \sum_{i \leq p} \frac{\|K(a_i)\|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{k \neq r} \varepsilon_{kr} \sqrt{\log(\frac{1}{\varepsilon_{kr}})} \right).
\]

For \( \omega \) a solution of (1.1), we obtain a parametrization of the set \( V(p, \varepsilon, \omega) \) as follows

**Proposition 2.5** There is \( \varepsilon_0 > 0 \) such that if \( \varepsilon \leq \varepsilon_0 \) and \( u \in V(p, \varepsilon, \omega) \), the problem

\[
\min_{\alpha_i > 0, \lambda_i > 0, a_i \in S^3, h \in T_\omega(W_u(\omega))} \|u - \sum_{i=1}^{p} \alpha_i \delta a_i, \lambda_i - \alpha_0(\omega + h)\|
\]

has a unique solution \((\bar{\alpha}, \bar{\lambda}, \bar{h})\). Thus, we write \( u \) as:

\[
u = \sum_{i=1}^{p} \bar{\alpha}_i \delta a_i, \lambda_i \bar{\lambda}_i + \bar{\alpha}_0(\omega + \bar{h}) + v,
\]

where \( v \) belongs to \( S^2_1(S^3) \cap T_\omega(W_u(\omega)) \) and satisfies \((V_0)\), \( T_\omega(W_u(\omega)) \) and \( T_\omega(W_s(\omega)) \) are respectively, the tangent spaces at \( \omega \) to the unstable and stable manifolds of \( \omega \).

**Proof:** The proof is similar to the one given in [2].
3 Asymptotic Analysis of the Functional

3.1 Domination Property: Hierarchy of the Critical point at infinity

We first introduce some definitions and notations due to Bahri [1, 2]. Let $\partial J$ denotes the gradient of the functional $J$.

**Definition 3.1** A critical point at infinity of $J$ on $\Sigma^+$ is a limit of a flow line $u(s)$ of the equation:

$$
\begin{cases}
\frac{\partial u}{\partial s} = -\partial J(u) \\
u(0) = u_0
\end{cases}
$$

such that $u(s)$ remains in $V(p, \varepsilon, \omega)$ for $s \geq s_0$, $\omega$ is zero or a solution of \([\mathbb{L}]\) and $\varepsilon(s)$ satisfies $\lim_{s \to \infty} \varepsilon(s) = 0$. One can write $u(s) = \sum_{i=1}^p \alpha_i(s)\delta_{(\alpha_i, \lambda_i)} + \alpha_0(s)(\omega + h(s)) + v(s)$. Let $\alpha_i := \lim_{s \to \infty} \alpha_i(s)$ and $\alpha := \lim_{s \to \infty} \alpha_i(s)$, we denote such a critical point at infinity by

$$
(\alpha, \omega) \quad \text{or} \quad \sum_{i=1}^p \alpha_i \delta_{(\alpha_i, \lambda_i)} + \alpha_0 \omega.
$$

A critical point at infinity is called of $\omega$-type if $\omega \neq 0$.

As for a usual critical point, to a critical point at infinity $\xi_\infty$ are associated stable and unstable manifolds which we denote by $W_s(\xi_\infty)$ and $W_u(\xi_\infty)$. These manifolds allow to compare critical points at infinity by what we call a "domination property", one can see [2, 8], where a detailed description of theses manifolds is given.

**Definition 3.2** A critical point at infinity $\xi_\infty$ is said to be dominated by another critical point at infinity $\xi'_\infty$, if

$$
W_s(\xi_\infty) \cap W_\omega(\xi'_\infty) \neq \emptyset
$$

and we write $\xi_{\infty'} > \xi_\infty$. □

If we assume that the intersection $W_s(\xi_\infty) \cap W_\omega(\xi'_\infty)$ is transverse, then we obtain $\text{index}(\xi_{\infty'}) \geq \text{index}(\xi_\infty) + 1$.

3.2 Asymptotic Analysis of the functional in the set $V(p, \varepsilon, \omega)$, $\omega \neq 0$

In this section, we expand the functional $J$ in $V(p, \varepsilon, \omega)$, for $\omega$ a non null solution of $\mathbb{L}$ in the aim to detect the critical points or critical points at infinity of $J$ in this set and we prove that:

for any $p \in \mathbb{N}^*$, there are no critical point or critical point at infinity of $J$ in the set $V(p, \varepsilon, \omega)$. More precisely, using Proposition 2.4, we will write $u \in V(p, \varepsilon, \omega)$ as $u = \sum_{i=1}^p \alpha_i \delta_{(\alpha_i, \lambda_i)} + \alpha_0(\omega + h) + v$, one obtain the following expansion of $J$:

**Proposition 3.3** There exists $\varepsilon_0 > 0$ such that for any $u = \sum_{i=1}^p \alpha_i \delta_{(\alpha_i, \lambda_i)} + \alpha_0(\omega + h) + v \in V(p, \varepsilon, \omega)$, $\varepsilon < \varepsilon_0$

$$
J(u) = \frac{S \sum_{i=1}^p \alpha_i^2 + \alpha_0^2\|w\|^2}{(S \sum_{i=1}^p \alpha_i^4 K(a_i) + \alpha_0^4\|w\|^2)^2} \left[ 1 - \frac{c_2 \alpha_0}{\gamma_1} \sum_{i=1}^p \frac{w(a_i)}{\lambda_i} - \frac{1}{\gamma_1} \sum_{i \neq j} \alpha_i \alpha_j c_{ij} \varepsilon_{ij} + f_1(v) \right. \\
+ Q_1(v, v) + f_2(h) + \alpha_0^2 Q_2(h, h) \left. + o \left( \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^p \frac{1}{\lambda_i} \|v\|^2 + \|h\|^2 \right) \right]
$$
It follows from [15] and elementary computations that

\[
f_1(v) = -\frac{1}{\eta_1} \int_{S^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_{ai,\lambda_i} \right)^3 v,
\]

\[
f_2(h) = \frac{\alpha_0}{\gamma_1} \sum_{i=1}^{p} \alpha_i < \delta_{ai,\lambda_i}, h > L_\theta - \frac{\alpha_0}{\eta_1} \int_{S^3} \left( \sum_{i=1}^{p} \alpha_i \delta_{ai,\lambda_i} + \alpha_0 w \right)^3 h,
\]

\[
Q_1(v, v) = \frac{||v||^2}{\gamma_1} - \frac{3}{\eta_1} \int_{S^3} K \left( \sum_{i=1}^{p} (\alpha_i \delta_{ai,\lambda_i})^2 + (\alpha_0 w)^2 \right) v^2,
\]

\[
Q_2(h, h) = \frac{||h||^2}{\gamma_1} - \frac{3}{\eta_1} \int_{S^3} K (\alpha_0 w)^2 h^2,
\]

\[
c_2 = c_0^3 \int_{\mathbb{H}^1} \frac{1}{1 + |z|^2 - it} \eta_0 \wedge d\theta_0, \quad S = \frac{4}{\gamma_1} \int_{\mathbb{H}^1} \frac{1}{1 + |z|^2 - it} \eta_0 \wedge d\theta_0,
\]

\[
\eta_1 = S \sum_{i=1}^{p} \alpha_i^4 K(ai) + \alpha_0^4 \|w\|^2, \quad \gamma_1 = S \sum_{i=1}^{p} \alpha_i^2 + \alpha_0^2 \|w\|^2
\]

and \(c_{ij}\) are bounded positive constants.

**Proof:** we need to estimate \(N = \|u\|^2\) and \(D^2 = \int_{S^3} K u^4 \theta \wedge d\theta\).

Expanding \(N\), we get

\[
N := \sum_{i=1}^{p} \alpha_i^2 \|\delta_i\|^2 + \alpha_i \alpha_0 < \delta_i, w + h > L_\theta + \alpha_0 (||h||^2 + \|w\|^2) + \|v\|^2 + \sum_{i \neq j} \alpha_i \alpha_j < \delta_i, \delta_j > L_\theta.
\]

It follows from [15] and elementary computations that

\[
||\delta_i||^2 = S, \quad \quad \quad \quad < \delta_i, \delta_j > L_\theta = c_{ij} \varepsilon_{ij} (1 + o(1)) \text{ for } i \neq j \text{ and } < \delta_i, w > L_\theta = c_2 \frac{w(a_i)}{\lambda_i} + o \left( \frac{1}{\lambda_i} \right).
\]

Therefore

\[
N = \gamma_1 + 2 \alpha_0 c_2 \sum_{i=1}^{p} \alpha_i \frac{w(a_i)}{\lambda_i} + \alpha_i < \delta_i, h > L_\theta + \sum_{i \neq j} \alpha_i \alpha_j + \alpha_0^2 \|h\|^2 + \|v\|^2 + o \left( \sum_{i=1}^{p} \frac{1}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right).
\]

For the denominator \(D\), we compute it as follows

\[
D^2 = \int_{S^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_i \right)^4 \theta \wedge d\theta + \int_{S^3} K (\alpha_0 w)^4 \theta \wedge d\theta + 4 \alpha_0 \int_{S^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_i \right)^3 w \theta \wedge d\theta
\]

\[
+ 4 \alpha_0^3 \int_{S^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_i \right) w^3 \theta \wedge d\theta + 4 \int_{S^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_i + \alpha_0 w \right)^3 (\alpha_0 h + v) \theta \wedge d\theta
\]

\[
+ 12 \int_{S^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_i + \alpha_0 w \right)^2 (\alpha_0^2 h^2 + v^2 + 2 hv) \theta \wedge d\theta + O \left( \sum_{i=1}^{p} \int_{S^3} w^2 \alpha_i^2 \delta_i^2 \right)
\]

\[
+ O(||v||^3 + ||h||^3).
\]
Next, we focus on the linear form in $v$

**Proof:** The proof of this lemma is similar to the one given in [2], for more details one can see the appendix of [8], where necessary modifications are given.

Using the Lemma above one can perform the expansion of the functional $J$ given in Proposition 3.3 after an adequate change of variables. More precisely, we obtain

Where

$$\int_{\mathbb{S}^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_i \right)^4 \theta \land d\theta = \sum_{i=1}^{p} \alpha_i^4 K(a_i) S + 4 \sum_{i \neq j} \alpha_i^2 \alpha_j K(a_i) c_{ij} \varepsilon_{ij} + O \left( \sum_{i=1}^{p} \frac{1}{\lambda_i^3} \right).$$

$$\int_{\mathbb{S}^3} K w^4 \theta \land d\theta = \|w\|^2,$$

$$\int_{\mathbb{S}^3} K w^3 \delta_i \theta \land d\theta = c_2 \frac{w(a_i)}{\lambda_i} + o\left( \frac{1}{\lambda_i} \right),$$

$$\int_{\mathbb{S}^3} (w^2 \alpha_i^2 \delta_i^2 + w^2 \alpha_0^2 \delta_i^2) \theta \land d\theta = o\left( \frac{1}{\lambda_i} \right),$$

$$\int_{\mathbb{S}^3} \left( \sum_{i=1}^{p} \alpha_i \delta_i + \alpha_0 w \right)^2 h v \land d\theta = O \left( \int_{\mathbb{S}^3} \left( \sum_{i=1}^{p} \delta_i^2 + w^{-1} \sum_{i=1}^{p} \delta_i \right) \|h\| |v| \theta \land d\theta \right)$$

$$= O \left( ||h||^2 + \|v\|^2 + \sum_{i=1}^{p} \frac{1}{\lambda_i^3} \right)$$

where we have used that $v \in T_w(W_{\alpha}(w))$ and $h \in T_w(W_{\alpha}(w))$.

Next, we focus on the linear form in $v \in T_w(W_{\alpha}(w))$, we obtain

$$\int_{\mathbb{S}^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_i + \alpha_0 w \right)^3 v \theta \land d\theta = \int_{\mathbb{S}^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_i \right)^3 v \theta \land d\theta + O \left( \sum_{i=1}^{p} \int_{\mathbb{S}^3} (\alpha_i^2 \alpha_0 \delta_i^2 w + \alpha_i \alpha_0^2 \delta_i w^2) |v| \right)$$

$$= f_1(v) + O \left( \sum_{i=1}^{p} \|v\| \lambda_i \right).$$

Finally, for the partial quadratic forms in $v$ and $h$, we obtain

$$\int_{\mathbb{S}^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_i + \alpha_0 w \right)^2 h^2 \theta \land d\theta = \alpha_0^2 \int_{\mathbb{S}^3} K w^2 h^2 \theta \land d\theta + o(\|h\|^2)$$

$$\int_{\mathbb{S}^3} K \left( \sum_{i=1}^{p} \alpha_i \delta_i + \alpha_0 w \right)^2 v^2 \theta \land d\theta = \sum_{i=1}^{p} \int_{\mathbb{S}^3} K(\alpha_i \delta_i)^2 v^2 \theta \land d\theta + \alpha_0^2 \int_{\mathbb{S}^3} K w^2 v^2 \theta \land d\theta + o(\|v\|^2)$$

Combining these results and the fact that $\frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} = 1 + o(1)$ the proposition follows. □

Next, we state the following result

**Lemma 3.4** [2]

a- $Q_1(v, v)$ is a positive definite quadratic form on $E_v = \{ v \in S^3(M) \text{ such that } v \in T_w(W_{\alpha}(w)) \text{ and } v \text{ satisfies}(V_0) \}.$

b- $Q_2(h, h)$ is a negative definite quadratic form on $T_w(W_{\alpha}(w)).$

**Proof:** The proof of this lemma is similar to the one given in [2], for more details one can see the appendix of [8], where necessary modifications are given. □
Proposition 3.5 Let \( u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0(\omega + h) + v \in V(p, \varepsilon, \omega) \). There is an optimal \((\mathfrak{v}, \mathfrak{h})\) and a change of variables \( v - \mathfrak{v} \rightarrow V \) and \( h - \mathfrak{h} \rightarrow H \) such that

\[
J(u) = J \left( \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + \alpha_0 \omega + \mathfrak{h} + \mathfrak{v} \right) + \|V\|^2 - \|H\|^2.
\]

Furthermore, we have the following estimates:

\[
\|\mathfrak{h}\| \leq c \sum_{i=1}^{p} \frac{1}{\lambda_i} \quad \text{and} \quad \|\mathfrak{v}\| \leq c \left( \sum_{i=1}^{p} \left( \frac{\|\nabla K(a_i)\|}{\lambda_i} + \frac{1}{\lambda_i^2} \right) + \sum_{k \neq r} \varepsilon_{kr} \sqrt{\text{Log} \left( \frac{1}{\varepsilon_{kr}} \right)} \right),
\]

\[
J(u) = \frac{S \sum_{i=1}^{p} \alpha_i^2 + \alpha_0^2 \|\omega\|^2}{\sqrt{S \sum_{i=1}^{p} \alpha_i^4 K(a_i) + \alpha_0^4 \|\omega\|^2}} \left[ 1 - \frac{c_2 \alpha_0}{\gamma_1} \sum_{i=1}^{p} \alpha_i \frac{\omega(a_i)}{\lambda_i} \right]
- \frac{1}{\gamma_1} \sum_{i \neq j} \alpha_i \alpha_j c_{ij} \varepsilon_{ij} + o \left( \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{1}{\lambda_i} \right) + \|V\|^2 - \|H\|^2.
\]

Proof: As done in [8] the proof is based on performing the expansion of the functional \( J \) in the set \( V(p, \varepsilon, \omega) \), to obtain self interactions and interactions between the bubbles, a linear form \( f_1 \) in \( v \) (respectively \( f_2 \) in \( h \)) and a positive definite quadratic form \( Q_1 \) in \( v \) (respectively a negative definite quadratic form \( Q_2 \) in \( h \)) as leading terms. Hence there is a a unique minimum \( \mathfrak{v} \) in the space of \( v \)'s (respectively a unique maximum \( \mathfrak{h} \) in the space of \( h \)'s. Furthermore, we derive \( \|\mathfrak{v}\| \leq c\|f_2\| \) and \( \|\mathfrak{h}\| \leq c\|f_1\| \). The estimate of \( \mathfrak{v} \) follows from Proposition 2.5 while the estimate of \( \mathfrak{h} \) is derived from the equivalence of the norms \( \|\cdot\|_{\infty} \) and \( \|\cdot\| \) in \( T_w(W_u(\omega)) \), since it is a space of finite dimension. We also derive that \( \|f_2\| = O(\sum \frac{1}{\lambda_i}) \), hence the result follows. □

For the sake of completeness of the proof one can see [2] and [8].

A direct consequence of the above proposition is:

Corollary 3.6 Let \( K \) be a \( C^2 \) positive function and let \( \omega \) be a non degenerate critical point of \( J \) in \( \Sigma^+ \). Then, for each \( p \in \mathbb{N}^* \), there is no critical points or critical points at infinity in the set \( V(p, \varepsilon, \omega) \), that means we can construct a pseudo gradient of \( J \) so that the Palais-Smale condition is satisfied along its decreasing flow lines.

The proof follows immediately from Proposition 3.3 and the fact that \( \omega \) is a solution of (1.1), hence strictly positive on \( M \).

4 Morse Lemma at infinity

The Morse lemma at infinity establishes near the set of critical points at infinity of the functional \( J \) a change of variables in the space \( (a_i, \alpha_i, \lambda_i, v), 1 \leq i \leq p \) to \( (\tilde{a}_i, \tilde{\alpha}_i, \tilde{\lambda}_i, V) \), \( (\tilde{\alpha}_i = \alpha_i) \), where \( V \) is a variable completely independent of \( \tilde{a}_i \) and \( \lambda_i \) such that \( J(\sum \alpha_i \delta_{(a_i, \lambda_i)}) \) behaves like \( J(\sum \alpha_i \delta_{(\tilde{a}_i, \tilde{\lambda}_i)}) + \|V\|^2 \).

We define also a pseudo-gradient for the \( V \) variable in the aim to make this variable disappear by setting \( \frac{\partial V}{\partial s} = -\nu V \) where \( \nu \) is taken to be a very large constant. Then at \( s = 1, V(s) = \exp(-\nu s) V(0) \) will be as small as we wish. This shows that, in order to define our deformation, we can work as if \( V \) was zero. The deformation will be extended immediately with the same properties to a neighborhood of zero in the \( V \) variable.
We begin by characterizing the critical points at infinity of $J$ in the sets $V(p, \varepsilon)$, $p \geq 1$ under condition (1.3). This characterization is obtained through the construction of a suitable pseudogradient at infinity for the functional $J$ for which the Palais-Smale condition is satisfied along the decreasing flow lines as long as these flow lines do not enter in the neighborhood of a finite number of critical points $\xi_i$; $1 \leq i \leq p$ in $\mathcal{K}_2$ or such that $(\xi_i, \ldots, \xi_p) \in \mathcal{K}_1^+$.  

### 4.1 Construction of the pseudo gradient

This subsection is devoted to the construction of the pseudo gradient for the functional $J$. It was extracted from [10], where a complete and detailed description of the construction of the pseudo gradient is given.

- In the set $V(1, \varepsilon)$, we have the following result:

**Proposition 4.1** Assume that $K$ satisfies the $\beta$-flatness condition and condition (C) and let $\beta := \max\{\beta(\xi_i), \xi_i \text{ verifying (1.3)}\}$. Then, there exists a pseudo gradient $W$ and a constant $c > 0$ independent of $u = \alpha \delta(\alpha, \lambda) \in V(1, \varepsilon)$, $\varepsilon$ small enough such that, if we denote $\overline{u} = u + \overline{v}$, we have

\[
1) \quad -J'(u)(W) \geq c\left(\frac{\|\nabla K(a)\|}{\lambda} + \frac{1}{\lambda^\beta}\right).
\]

\[
2) \quad -J'(\overline{u})(W + \frac{\partial \overline{v}}{\partial(a, \alpha, \lambda)}(W)) \geq c\left(\frac{\|\nabla K(a)\|}{\lambda} + \frac{1}{\lambda^\beta}\right).
\]

3) $|W|$ is bounded. Furthermore, $\lambda$ is an increasing function along the flow lines generated by $W$, only if $a$ is close to a critical point $\xi_i \in \mathcal{K}_1 \cup \mathcal{K}_2$.

- In the set $V(p, \varepsilon)$, $p \geq 2$, we obtain:

**Proposition 4.2** Assume that $K$ satisfies the $\beta$-flatness condition and condition (C) and let $\beta := \max\{\beta(\xi_i), \xi_i \text{ verifying (1.3)}\}$. For any $p \geq 2$, there exists a pseudo gradient $W$ so that the following hold:

there is a positive constant $c$ independent of $u = \sum_{i=1}^p \alpha_i \delta_{a_i, \lambda} \in V(p, \varepsilon)$, $\varepsilon$ small enough such that, if we denote $\overline{u} = u + \overline{v}$, we have

\[
1) \quad -J'(u)(W) \geq c\left(\sum_{i=1}^p \frac{\|\nabla K(a_i)\|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i^\beta} + \sum_{i \neq j} \varepsilon_{ij}\right)
\]

\[
2) \quad -J'(\overline{u})(W + \frac{\partial \overline{v}}{\partial(a, \alpha, \lambda)}(W)) \geq c\left(\sum_{i=1}^p \frac{\|\nabla K(a_i)\|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i^\beta} + \sum_{i \neq j} \varepsilon_{ij}\right)
\]

3) $|W|$ is bounded. Furthermore, the only cases where the maximum of the $\lambda'_s$ is not bounded is when the concentration points $(a_1, \ldots, a_p)$ satisfy: each point $a_j$ is close to a critical point $\xi_{ij}$ of $K$ in the set $\mathcal{K}_1$ with $i_j \neq i_k$ for $j \neq k$ and $g(\xi_{i_1}, \ldots, \xi_{i_p}) > 0$, where $g(\xi_{i_1}, \ldots, \xi_{i_p})$ is the least eigenvalue of $M(\xi_{i_1}, \ldots, \xi_{i_p})$.

### 4.2 Morse Lemma

Once the pseudo gradient is constructed, following [2] and [8], we establish our Morse Lemma at infinity: we can find a change of variables which gives the normal form of the functional $J$ on the subsets $V(p, \varepsilon)$. We obtain the following result:
Proposition 4.3  For $\xi \in K_1 \cup K_2$, there exists a change of variables in the set $\{\alpha \delta_{(a,\lambda)} + v : a \text{ is close to } \xi\}$, $v - \bar{\nabla} \rightarrow V$ and $(a,\lambda) \mapsto (\tilde{a},\tilde{\lambda})$ such that in these new variables the functional $J$ behaves as

$$J(\alpha \delta_{(a,\lambda)} + v) = \frac{S}{K(\tilde{a})^2} \left(1 + c(1 - \mu) \frac{\Gamma(\xi)}{\lambda \gamma(\xi)}\right) + \|V\|^2$$

where $\mu$ is a small positive constant and

$$\gamma(\xi) = \begin{cases} 
2 & \text{if } \xi \in K_1 \\
\beta & \text{if } \xi \in K_2
\end{cases} \quad \Gamma(\xi) = -\sum_{k=1}^{2} b_k + \kappa' b_0 \quad \text{if } \xi \in K_1 \cup K_2$$

The proof is similar to the one given in [2, 5, 8], so we omit it here.

As a consequence of Proposition 4.1, we obtain:

Corollary 4.4  Let $K$ be a positive function on $S^3$ satisfying the $\beta$-flatness condition and condition (C). The only critical points at infinity in $V(1,\varepsilon)$ are $\xi_\infty$ where $\xi \in K_1 \cup K_2$. The Morse index $i(\xi_\infty)$ of such a critical point is equal to

$$i(\xi_\infty) = 3 - m(\xi)$$

If $p \geq 2$, we have the following result:

Proposition 4.5  [8]  For any $u = \sum_{i=1}^{p} \alpha_i \delta_{a_i,\lambda_i} \in V(p,\varepsilon_1)$, $(\varepsilon_1 < \frac{\xi}{2})$, each $a_i$ close to a critical point $\xi \in K_1$, we find a change of variables in the space $(a_i, \alpha_i, \lambda_i, v)$, $1 \leq i \leq p$ to $(\tilde{a}_i, \tilde{\alpha}_i, \tilde{\lambda}_i, V)$, $(\tilde{a}_i = \alpha_i)$, such that

$$J(\sum_{i=1}^{p} \alpha_i \delta_{a_i,\lambda_i} + \bar{\nabla}(\alpha, a, \lambda)) = J(\sum_{i=1}^{p} \alpha_i \delta_{\tilde{a}_i,\tilde{\lambda}_i})$$

with

$$\sum_{i \neq j} \varepsilon_{ij} + \sum_{i} \frac{1}{\lambda_i^2} \rightarrow 0 \Leftrightarrow \sum_{i \neq j} \varepsilon_{ij} + \sum_{i} \frac{1}{\lambda_i^2} \rightarrow 0. \quad (4.1)$$

and

$$\|\tilde{a}_i - a_i\| \rightarrow 0 \quad \text{as} \quad \sum_{i \neq j} \varepsilon_{ij} + \sum_{i} \frac{1}{\lambda_i^2} \rightarrow 0. \quad (4.2)$$

As a consequence of proposition 4.2 we obtain:

Corollary 4.6  The only critical points at infinity in $V(p,\varepsilon)$, $p \geq 2$ are: $\xi_\infty = (\xi_{i_1}, \ldots, \xi_{i_p})_\infty$ such that the matrix $M(\xi_{i_1}, \ldots, \xi_{i_p})$ defined in (1.4) is positive definite, where the $\xi_{i}$'s are critical points of $K$ in the set $K_1$ and $i_j \neq i_k$ for $j \neq k$. Such a critical point at Infinity has a Morse index equal to

$$i(\xi_\infty) = i(\xi_{i_1}, \ldots, \xi_{i_p})_\infty = 4p - 1 - \sum_{j=1}^{p} m(\xi_{i_j})$$

$\square$
5 Proofs of Theorem [1.1] and Theorem [1.2]

Proof of Theorem [1.1]

Following [9] and [13], let $\mathcal{K}_\infty$ be the set of all critical points at infinity of $J$ and $L_0$ be their maximal Morse index given in [1.0]. We define for $0 \leq l \leq L_0$ the following sets:

$$X_l^{\infty} = \bigcup_{\xi \in \mathcal{K}_\infty \cap M(\xi) \leq l} W_u^{\infty}(\xi)$$

where $W_u^{\infty}(\xi)$ is the unstable manifold associated to the critical point at infinity $\xi$. By a theorem of Bahri and Rabinowitz [4], we have:

$$W_u^{\infty}(\xi) = W_u^{\infty}(\xi) \cup \bigcup_{\xi' < \xi} W_u^{\infty}(\xi') \cup \bigcup_{\omega < \xi} W_u(\omega),$$  \quad (5.1)

where $\xi'$ is a critical point at infinity dominated by $\xi$ and $\omega$ is a solution of (1.1) dominated by $\xi$. Hence,

$$X_l^{\infty} = \bigcup_{\xi \in \mathcal{K}_\infty \cap M(\xi) \leq l} \left( W_u^{\infty}(\xi) \cup \bigcup_{\omega < \xi} W_u(\omega) \right)$$

It follows that $X_l^{\infty}$ is a stratified set of top dimension $\leq l$. Without loss of generality, we may assume it equal to $l$. Now, we consider the cone based on $X_l^{\infty}$ of vertex $(\xi_0)_{\infty}$ where $\xi_0$ is a global maximum of $K$ on $S^3$:

$$C(X_l^{\infty}) := X_l^{\infty} \times [0, 1) / (x, 1) \sim (y, 1), \quad x, y \in X_l^{\infty}. \quad (5.2)$$

The cone $C(X_l^{\infty})$ is a stratified set of top dimension $l + 1$. Next, we use the vector field $-\partial J$ to deform $C(X_l^{\infty})$. During this deformation and based on transversality arguments, we assume that we can avoid the stable manifolds of all critical points as well as critical points at infinity having their Morse indices greater than $l + 2$. It follows, by a theorem of Bahri and Rabinowitz [4], that $C(X_l^{\infty})$ retracts by deformation on the set

$$U^{\infty} := X_l^{\infty} \cup \bigcup_{M(\xi) = l + 1} W_u^{\infty}(\xi) \cup \bigcup_{\omega, \omega \text{ dominated by } C(X_l^{\infty})} W_u(\omega). \quad (5.3)$$

Now, taking $l = k - 1$ and using the assumption that there are no critical points at infinity with index $k$, we derive that $C(X_{k-1}^{\infty})$ retracts by deformation onto

$$Z_k^{\infty} := X_{k-1}^{\infty} \cup \bigcup_{\omega, \omega \text{ dominated by } C(X_{k-1}^{\infty})} W_u(\omega). \quad (5.4)$$

Using the deformation above, problem [1.1] has necessary a solution $\omega$ with $\text{morse}(\omega) \leq k$. Otherwise it follows from [5.4] that

$$1 = \sum_{\xi \in \mathcal{K}_\infty \cap M(\xi) \leq k - 1} (-1)^{M(\xi)} = \sum_{\xi \in \mathcal{K}_2 \cap m(\xi) \geq 4 - k} (-1)^{m(\xi) + 1} + \sum_{p=1}^{l+} \sum_{(\xi_1, \ldots, \xi_p) \in \mathcal{K}_+ \cap \text{morse}(\xi_1, \ldots, \xi_p) \leq k - 1} (-1)^{i(\xi_1, \ldots, \xi_p)}$$

Obviously this formula contradicts the first assumption of the theorem.

$\square$
Proof of Theorem 1.2 Let us denote by $S_k$ the set of solutions of problem (1.1) having their morse indices less than or equal to $k$. We derive from (5.4), taking the Euler characteristic of its both sides, that:

$$1 = \sum_{\xi_{\infty} \in K_{\infty}: M(\xi_{\infty}) \leq k - 1} (-1)^{M(\xi_{\infty})} + \sum_{\omega < C(X_{k-1}^\infty)} (-1)^{\text{morse}(\omega)}$$

It follows that

$$\left| 1 - \sum_{\xi_{\infty} \in K_{\infty}: M(\xi_{\infty}) \leq k - 1} (-1)^{M(\xi_{\infty})} \right| \leq \# S_k.$$

The result follows.

If we let $k = L_0 + 1$, in Theorem 1.1 the second assumption of this theorem is obviously satisfied and we obtain under this condition the following

Corollary 5.1 Let $K$ be as in Theorem 1.1 such that:

$$\sum_{\xi \in K_{2}} (-1)^{m(\xi)} + \sum_{p=1}^{l^+} \sum_{(\xi_1, \ldots, \xi_p) \in K_1^+} (-1)^{i(\xi_1, \ldots, \xi_p)} \neq 1$$

Then, there exists at least one solution of (1.1).

This result generalizes the existence results due to Gamara and Riahi in [12] and the multiplicity results due to the same authors in [13] and finally recovers the existence results of Gamara and Hafassa in [10]. Moreover, if we denote $S$ the set of all the solutions of (1.1), we obtain the following lower bound for $S$

Corollary 5.2

$$\# S \geq \left| 1 + \sum_{\xi \in K_{2}} (-1)^{m(\xi)} - \sum_{p=1}^{l^+} \sum_{(\xi_1, \ldots, \xi_p) \in K_1^+} (-1)^{\sum_{j=1}^{p} m(\xi_j)} \right|$$

6 Appendix

Without loss of generality, we can assume for $p \geq 2$ that $\lambda_1 \leq \cdots \leq \lambda_p$. Given $N$ a large positive constant, we define:

$$I_1 := \{1\} \cup \{i \leq p : \lambda_k \leq N\lambda_{k-1} \ \forall k \leq i\}, \quad I_2 := \{i \in I_1 : a_i \text{ is close to a critical point } \xi_{k_i} \text{ satisfying (1.3) with } \beta > 2\}.$$
The set $I_1$ contains the indices $i$ such that $\lambda_i$ and $\lambda_1$ are of the same order.

We denote by $V(p, \varepsilon)_1$ the subset of $V(p, \varepsilon)$ composed of the functions $u = \sum_{i=1}^{p} \alpha_i \delta_{a_i, \lambda_i}$ such that $\forall i \in I_1$, $\lambda_i |\nabla_\theta K(a_i)| \leq 2C$, $\sum_{j \neq k} \varepsilon_{jk} \leq \frac{C}{\lambda_1}$, $C$ and $C'$ are positive constants and $I_2 = \emptyset$. Following the work done in [3], we obtain the following expansion of the functional $J$ in $V(p, \varepsilon)_1$.

**Proposition 6.1** There exists $\varepsilon_0 > 0$ such that, for any $u = \sum_{j=1}^{p} \alpha_i \delta_{a_i, \lambda_i} + v \in V(p, \varepsilon)_1$, $\varepsilon < \varepsilon_0$, $v$ satisfying $(V_0)$, we have

\[
J(u) = -\frac{1}{\gamma_1} \int_{S^3} K(\sum_{i=1}^{p} \alpha_i \delta_{a_i, \lambda_i})^3 v \theta_1 \wedge d\theta_1,
\]

\[
Q(v, v) = \frac{1}{\gamma_2} ||v||^2_{L_1} - \frac{3}{\gamma_1} \int_{S^3} K \sum_{i=1}^{p} \alpha_i^2 \delta_{a_i, \lambda_i}^2 v_1 \wedge d\theta_1,
\]

\[
\gamma_1 = S^2 \sum_{i=1}^{p} \alpha_i^4 K(a_i), \quad \gamma_2 = S^2 \sum_{i=1}^{p} \alpha_i^2.
\]

Furthermore $\|f\|_{\theta_1}$ is bounded

\[
\|f\|_{\theta_1} = O\left(\sum_{i=1}^{p} \left(\frac{\nabla K(a_i)}{\lambda_i} + \frac{1}{\lambda_i^2}\right)^2 + \sum_{i \neq j} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{\frac{1}{2}}\right), \text{ if } K \text{ satisfies } (1.3).
\]

For a proof we refer to [10].

Next, we will give the expansions of the gradient of the functional $J$ which is the key of the Morse Lemma. Since the vector field $W$ is a variation of $\sum_{i=1}^{p} \alpha_i \delta_i \in V(p, \varepsilon)$ ($p \geq 2$), hence we will expand $J'(u)(\lambda_j \frac{\partial \delta_j}{\partial \lambda_j})$, $J'(u)(\frac{1}{\lambda_j} \frac{\partial \delta_j}{\partial a_j})$ and $J'(u)(\frac{1}{\lambda_j} (D_j)_k \delta_j)$, for $k = 1, 2$ and $J'(u)(\frac{1}{\lambda_j} (D_j)_0 \delta_j)$ in the case where
If there exists a point \( a \) close to a critical point \( \xi \) of \( K \) verifying (1.3), then the estimates in the above proposition can be improved see [10] and we obtain:

**Proposition 6.3**

1) For \( k \in \{1, 2\} \)

\[
J'(u)\left(\frac{1}{\lambda_j} (D_j) k \delta_j\right) = -4J(u)^3 \alpha_j^4 \rho_j^\beta b_j \int_{\mathbb{H}^1} \frac{|x_k + \lambda_j(a_j) k|^2}{1 + |z|^2 - it} x_k (1 + |z|^2) \theta_0 \wedge d\theta_0 \\
+ b_0 \int_{\mathbb{H}^1} \frac{|t + \lambda_j^2(a_j) 0 + 2\lambda_j (x_2(a_j) 1 - x_2(a_j) 1)|^2}{1 + |z|^2 - it} \left(x_k (1 + |z|^2) + (-1)^k' x_k' \right) \theta_0 \wedge d\theta_0 \\
+ o \left( \frac{1}{\lambda_j^\beta} \right) + O \left( \sum_{i \neq j} \epsilon_{ij} \right)
\]

and

\[
J'(u)\left(\frac{1}{\lambda_j^2} (D_j) 0 \delta_j\right) = -4J(u)^3 \alpha_j^4 \rho_j^\beta b_j \int_{\mathbb{H}^1} \frac{|t + \lambda_j^2(a_j) 0 + 2\lambda_j (x_2(a_j) 1 - x_2(a_j) 1)|^2}{1 + |z|^2 - it} \theta_0 \wedge d\theta_0 \\
+ o \left( \frac{1}{\lambda_j^\beta} \right) + O \left( \sum_{i \neq j} \epsilon_{ij} \right).
\]

2) If we assume that \( \lambda_j | a_j | \leq \mu \), where \( \mu \) is a small positive constant, then

\[
J'(u)\left(\frac{1}{\lambda_j} \frac{\partial \delta_j}{\partial \lambda_j}\right) = -2c_4 J(u) \sum_{k \neq r} \alpha_j \lambda_j \frac{\partial \epsilon_{ij}}{\partial \lambda_j} + c_5 \sum_{i = 1}^2 b_i + \epsilon_{0} \left( \sum_{k \neq r} \epsilon_{kr} + \frac{1}{\lambda_j^\beta} \right)
\]

\( \Box \)
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