Synthesis of asymmetric parallel mechanism with multiple 3-DOF motion modes

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Abstract
A method for designing asymmetric multi-mode parallel mechanism is proposed. Parallel mechanisms with 2R1T and 2T1R motion modes are synthesized by using displacement manifold theory. Based on the parallel mechanism, a kind of mechanism with hybrid variable DOF kinematic chain is proposed. The DOF characteristics of the mechanism in the process of motion mode transformation are analyzed by using screw theory, and the rationality of the selection of driving pairs in different motion modes is verified. The results show that the mechanism with hybrid variable DOF kinematic chain has 3R, 2T1R, and 2R1T motion modes. When the mechanism is in the general configuration of the four motion modes mentioned above, three driving pairs can control the mechanism. When the mechanism is in the transformation configuration having 2R2T or 3R1T instantaneous DOF, an additional auxiliary driving pair is needed to control the mechanism. The mechanism can realize multiple motion modes by using fewer driving pairs. It does not need to reassemble the mechanism when the motion mode is changed.

Keywords
Motion mode, screw theory, displacement manifold, variable DOF kinematic chain, hybrid variable DOF kinematic chain

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Introduction
Parallel mechanisms with multiple motion modes are also called parallel mechanisms with multiple operation modes, variable displacement subgroups, bifurcation motion, or reconfigure ability. Its characteristics are as follows: Fewer driving pairs can realize multiple motion modes; There is no need to reassemble the mechanism when the motion mode is transformed, so the mechanism can be reconstructed quickly. Some of these parallel mechanisms need to pass the singular configuration of the mechanism when the motion mode is transformed.

DOMY² parallel mechanism has four different 3-DOF motion modes. This mechanism contains special kinematic chains, and Hunt³ firstly analyzed its characteristics when he studied the universal hubs. Geometric algebraic method is used to analyze the types of motion modes of this kind of mechanism with bifurcation of motion.⁴,⁵ The bifurcation characteristics of single-loop mechanism are analyzed in the literature.⁶ The coaxial or parallel of rotation axis and moving direction is considered as the critical condition for the transformation of mechanism motion modes. Based on this, some mechanisms with multiple motion modes are designed. In the literature,⁷ the singularities of bifurcations of mechanisms are classified, and some
parallel mechanisms with multiple modes of motion are designed. The principles of designing reconfigurable mechanisms are classified into three categories: variable topology, variable geometry, and bifurcation of motion. Lie group theory is used to analyze the bifurcation characteristics of 6R mechanism. This kind of single-loop mechanism with bifurcation characteristics is used to design parallel mechanism with multiple motion modes. Literature synthesizes parallel mechanisms which could realize the transformation of 3-DOF motion modes of 3T, 3R, 2R1T, and 2T1R. The bifurcation of motion of some single-loop mechanisms is discussed by using geometric method in the literature. Inspired by the above ideas, a large number of single-loop mechanisms with motion bifurcation are designed by using the intersection of geometric surfaces to generate bifurcated curves. Literature synthesizes the parallel mechanism of schoenflies motion with two different rotation directions by using Lie group theory. A kind of 3T1R parallel mechanism with variable rotation axis is designed by using finite displacement screw theory. Zeng classified the kinematic chains, sorted out some kinematic chains with variable DOF. The screw theory is used to analyze the kinematic bifurcation of the 3-PUP parallel mechanism. Based on the inspiration of carton forming, some single-loop mechanisms with motion bifurcation are designed, and based on this, some parallel mechanisms with multiple motion modes are designed. Matrix is used to represent the structure of variable topology mechanism. Some kinematic pairs which can transform the motion mode are proposed. These kinematic pairs can be used to design multiple mode parallel mechanism. A fully symmetrical parallel mechanism with 2R1T and 3R motion modes is designed by using reconfigurable rotation pairs. In the literature, virtual kinematic chains are used to synthesize the kinematic chains of parallel mechanisms with different motion modes. The common kinematic chains of two motion modes are selected as the optional kinematic chains. Combined with the geometric conditions of kinematic chains assembly with different motion modes, the parallel mechanisms with multiple kinematic modes are synthesized. The equivalent pair of spherical pairs is made up of lockable RRR spherical kinematic chains, which realizes the transformation of motion modes. In the literature, the over-constrained 6R mechanism is used as a moving platform, and the motion mode of the mechanism is changed by locking and releasing the kinematic pair. In the literature, some parallel mechanism with multiple motion modes are designed by using double-loop mechanism with linear driving pairs. A kind of redundant parallel mechanism is designed with two platforms. When the motion mode of the mechanisms changes, the number of DOF changes. In the literature, a redundant kinematic pair is used to design parallel mechanisms with 2T1R and 2R1T modes. The motion modes of this kind parallel mechanisms is limited. When it implements different tasks or realizes motion mode transformation, it actually needs more driving pairs. Literature suggests ways to realize motion mode transformation. Literature designs a 6-DOF Stewart parallel mechanism Using multiple SPS kinematic chains with overlapping spherical joints, which could change the stiffness of 6-DOF Stewart parallel mechanism by locking the driving pair. Literature studies multiple motion modes of mechanisms based on high-order kinematics, and finds some mechanisms with new motion characteristics.

The research on intelligent reconfigurable mechanisms and robots with the ability to actively adapt to changeable environment and passively adapt to emergencies is of great significance to the innovation and development of advanced manufacturing field and a new generation of robots. There are few existing multi-mode, reconfigurable parallel mechanisms. The innovative design of multi-mode parallel mechanism is an important research content of mechanism. The synthesis of parallel mechanisms with multiple motion modes is still one of the hot spots of mechanism research. Generally, the performance of symmetrical parallel mechanism is better. However, when the structure of multi-mode parallel mechanism is symmetrical, the number of degrees of freedom increases more during motion mode transformation, so multiple driving pairs are required to realize motion mode transformation.

In this paper, a method for designing asymmetric multi-mode parallel mechanism is proposed. This kind of mechanism only increases one degree of freedom during the transformation of motion mode, so that after adding an auxiliary driving pair, it can realize the transformation between multiple motion modes with other three driving pairs, which fully reflects the characteristics of multi-mode mechanism using fewer driving pairs to realize multiple tasks.

The paper is organized as follows. In Section “Synthesis of kinematic chain,” The variable DOF kinematic chain which construct the mechanism that have 2R1T and 2T1R motion mode is synthesized by using displacement manifold theory. In Section “Parallel mechanism with 2T1R and 2R1T motion-modes,” one fix DOF kinematic chain, one 6-DOF kinematic chain, and one 5-DOF kinematic chain with variable DOF are constructed to from a parallel mechanism which has 2R1T and 2T1R motion modes. The DOF characteristics of parallel mechanism with 2R1T and 2T1R motion modes that are analyze by using screw theory. In Section “Parallel mechanisms with 2T1R, 3R, and two 2R1Tmotion modes,” the DOF of mechanism with 2T1R, 3R, and 2R1T motion modes is analyzed by using screw theory, and the
rationality of the selection of driving pairs in different motion modes is verified. In Section “2R1T motion modes,” the two 2R1T motion modes are distinguished. The Section “Conclusion” drawn two conclusions.

**Synthesis of kinematic chain**

The coordinate system is represented by \((O, X, Y, Z)\). Its origin is \(O\) and its orthogonal basis is \((X, Y, Z)\). \((A, Z)\) is used to represent the line parallel to the \(Z\)-axis through a point \(A\) in space, and the type of the kinematic pair are represented by the direction of the kinematic pair and the points on the axis of the kinematic pair. For example, the displacement subgroup corresponding to the rotation pair parallel to the \(Z\)-axis and cross point \(N\) is represented by \(\{R(N, Z)\}\). The rotation pair parallel to the axis \(Z\) is represented as \(ZR\). A series kinematic chain is composed of links 1, 2, ..., \(i - 1, i\). The permissible displacement of link \(i\) can be expressed as the product of the displacement subgroup represented by the pairs in the kinematic chain. For example, a component is connected to a fixed platform through a \(ZR^2RZ^2R\) kinematic chain, which generates a planar three-dimensional displacement subgroup \(\{G(Z)\}\). If the motion generated by the kinematic chain does not have a group structure relative to the fixed platform, it is called a displacement manifold. \(\{G(Z)\}\) denotes the subgroup of plane displacements, and \(Z\) denotes the normal direction of the plane in which the plane displacements move. \(\{G(Z)\}\) can be decomposed into the following seven product forms.

\[
\{G(Z)\} = \{T_i(X)\} \{T_i(Y)\} \{R(N, Z)\} \\
= \{R(N, Z)\} \{T_i(X)\} \{T_i(Y)\} \\
= \{T_i(X)\} \{R(N, Z)\} \{T_i(Y)\} \\
= \{R(A, Z)\} \{R(B, Z)\} \{T_i(Y)\} \\
= \{R(A, Z)\} \{R(B, Z)\} \{R(C, Z)\}
\]  

(1)

There are seven equivalent kinematic chains of \(\{G(Z)\}\). They are \(X^pY^pZ^pR\), \(Z^pR^pY^pP\), \(X^pZ^pR^pP\), \(Z^pR^pX^pP\), \(Z^pX^pR^pP\), \(X^pZ^pR^pZ^pR\), \(Z^pR^pZ^pZ^pR\).

**Constructing a kinematic chain with fixed DOF**

The literature\(^{37}\) gives the detailed calculation results of intersection and multiplication of two displacement subgroups, and points out that the calculation of the intersection of the displacement subgroups satisfies the intersection rule in the general set theory. In more cases, the rigid body motion generated by the kinematic chain (composed of multiple pairs of motion) can not satisfy the algebraic structure of the displacement group, only is the displacement sub manifolds.

Kinematic pairs and chains can be directly used as generators of displacement subgroups or displacement sub manifolds. Correspondingly, any motion at the output link of the kinematic chain can be determined by the product of the displacement subgroups generated by all the pairs in the kinematic chain. The product of these displacement subgroups may still be displacement subgroups, but in most cases this product does not have the algebraic structure of groups, is only a displacement sub manifold.

\[M = M_1 \cdots M_k\]

represents a parallel mechanism consisting of \(K\) series kinematic chains \(M_1 \cdots M_k\). They are all connected with common moving and fixed platforms. Symbol \(C\) is used to represent the motion of the output link of the mechanism, then the motion of the moving platform of the parallel mechanism can be expressed as:

\[C_M = C_{M_1} \cap C_{M_2} \cap \cdots \cap C_{M_k}\]

(2)

Formula (2) shows that the motion of the moving platform should be the intersection of motion of the kinematic chain of the parallel mechanism.

Theorem 1. The product motion \(A-B\) of an arbitrary displacement subgroup may constitute a displacement subgroup, or it may do not have the algebraic structure of the group, is only a displacement sub manifold.

Generally speaking, the commutation law is not satisfied, that is \(A-B \neq B-A\). But if the law \(A-B = B-A\) is satisfied, then \(A-B\) is a displacement subgroup. Since the motion mode of the moving platform of the parallel mechanism will change from 2T1R to 2R1T, the calculation of the intersection of equation (2) shows that the kinematic chain of the parallel mechanism should realize at least 2R2T motion. 2T1R is a plane displacement subgroup \(\{G(Z)\}\). \(\{R(X)\}\) is a one-dimensional rotational subgroup. 2R2T does not belong to all 12 displacement subgroups in the rigid body motion enumerated by Hervé. Therefore, 2R2T is a displacement manifold. Kinematic chain with 2R2T displacement manifold can be obtained by equivalent substitution of kinematic chain with 2T1R plane displacement subgroups, namely \(X^pY^pZ^pR^p\), \(Z^pR^pX^pP^p\), \(X^pR^pY^pP^p\), \(Z^pR^pX^pP^p\), \(Z^pX^pR^pP^p\), \(X^pZ^pR^pZ^pR\), and \(Z^pR^pZ^pR^pX^p\).

Since the 6-DOF kinematic chain does not acts constraint on the moving platform, it can also be used to construct parallel mechanism to realize the transformation from 2R1T motion mode to 2T1R motion mode. Gougu uses evolutionary morphology theory to synthesize 6-DOF kinematic chains,\(^{38}\) such as:
Selection of kinematic chains with variable DOF based on displacement manifold theory

Generally, the kinematic chain structure of parallel mechanisms can be obtained from the generators of displacement subgroups or manifolds. The kinematic chain displacement manifold with fixed DOF is as follows:

\[ \{R(O, Z)\} \{T_1(X)\} \{T_1(Y)\} \{R(N, X)\} \]  \hspace{1cm} (3)

According to the motion mode of the moving platform, which has 2R1T and 2T1R motion modes, the variable DOF kinematic chain generator is constructed.

\[ \{R(O, Z)\} \{T_1(X)\} \{T_1(Y)\} \{R(N, X)\} \cap \{\{R(A, Z)\} \{T_1(X)\}\} \]
\[ \{T_1(Z)\} \{R(C, Y)\} \{R(B, Z)\} \cup \{R(A, Z)\} \{T_1(Y)\} \{T_1(Z)\}\]
\[ \{R(C, X)\} \{R(B, Z)\} \]  \hspace{1cm} (4)

Since \( \{R(O, Z)\} \{T_1(X)\} \{T_1(Y)\} \), \( \{T_1(Y)\} \{T_1(Z)\}\)
\( \{R(C, X)\} \) are planar displacement subgroups that the rotation center can be arbitrarily selected in the XOY and YOZ planes when the displacement manifolds intersect, formula (4) can be written in the form of formula (5):

\[ \{\{R(A, Z)\} \{T_1(X)\}\} \{T_1(Y)\}\{R(N, X)\} \cap \{R(A, Z)\} \{T_1(X)\}\}
\[ \{T_1(Z)\} \{R(N, Y)\} \{R(B, Z)\} \cup \{\{R(A, Z)\} \{T_1(X)\}\} \{T_1(Y)\}\]
\[ \{R(N, X)\} \cap \{R(A, Z)\} \{T_1(Y)\} \{T_1(Z)\}\{R(N, X)\} \{R(B, Z)\} \]  \hspace{1cm} (5)

Since the intersection of \( \{R(A, Z)\} \{T_1(X)\}\{T_1(Y)\}\) and \( \{R(A, Z)\} \{T_1(X)\}\{T_1(Z)\}\{R(N, Y)\}\) \( \{R(B, Z)\} \) does not contain \( \{R(N, Y)\}\), formula (5) equals:

\[ \{\{R(A, Z)\} \{T_1(X)\}\} \{T_1(Y)\}\{R(N, X)\} \cap \{R(A, Z)\} \{T_1(X)\}\}
\[ \{T_1(Z)\} \{R(N, X)\} \cup \{R(A, Z)\} \{T_1(Y)\}\{R(N, X)\} \]  \hspace{1cm} (6)

Since \( \{T_1(Z)\} \{R(B, Z)\} \) constitutes a displacement subgroup, then

\[ \{R(A, Z)\} \{T_1(X)\} \{T_1(Z)\}\{R(B, Z)\} \]
\[ = \{R(A, Z)\} \{T_1(X)\} \{R(B, Z)\} \{T_1(Z)\} \]  \hspace{1cm} (7)

\[ \{R(A, Z)\} \{T_1(X)\} \{R(B, Z)\} \] which is in formula (7)

constitutes a plane displacement subgroup, the result of formula (7) substitution (6) obtains:

\[ \{\{R(A, Z)\} \{T_1(X)\}\} \{T_1(Y)\}\{R(N, X)\} \cap \{R(A, Z)\} \{T_1(X)\}\]
\[ \{T_1(Y)\}\{T_1(Z)\}\cup \{R(A, Z)\} \{T_1(Y)\}\{R(N, X)\} \]  \hspace{1cm} (8)

Formula (8) yields:

\[ \{R(A, Z)\} \{T_1(X)\} \{T_1(Y)\} \cup \{R(A, Z)\} \{T_1(X)\}\{R(N, X)\} \]  \hspace{1cm} (9)

According to formulas (4) and (9), the displacement manifold generator which can realize the transition from 2R1T motion mode to 2T1R motion mode should have the following form:

\[ \{R(A, Z)\} \{T_1(X)\} \{T_1(Z)\}\{R(C, Y)\} \{R(B, Z)\} \]
\[ \cup \{R(A, Z)\} \{T_1(Y)\} \{T_1(Z)\}\{R(C, X)\} \{R(B, Z)\} \]  \hspace{1cm} (10)

Generator in formula (10) correspond to kinematic chains \( Z_R X_R P^X R X_R Z_R^X Z_R X_R P^X Z_R \), \( Z_R X_R P^X Z_R^X P^X Z_R \)
\( Z_R X_R P^X Z_R^X P^X Z_R \), \( Z_R X_R P^X R X_R Z_R \), \( Z_R X_R R X_R Z_R \).

\( Z_R X_R R X_R \) is a typical kinematic chain with variable DOF, as shown in Figure 1, its displacement manifold can be expressed as

\[ \{R(A, Z)\} \{T_1(Y)\} \{T_1(Z)\}\{R(A, v)\} \]  \hspace{1cm} (11)

kinematic chain shown in Figure 1 enable the parallel mechanism with the kinematic chain to realize the transformation of motion modes.

Parallel mechanism with 2T1R and 2R1T motion modes

2T1R motion mode configuration

According to the displacement manifold in the equation (3), kinematic chain \( Z_R X_R P^X R X_R \) is selected as the fixed DOF kinematic chain, namely the first kinematic chain \( B_1A_1 \). According to the equations (4), (9), and (10), \( Z_V U^Y V^Y U \) is selected as the kinematic chain of variable DOF, namely the second kinematic chain \( B_2A_2 \).

The structure of third kinematic chain \( B_3A_3 \) is SPS, which does not act constraint on the moving platform. After assembling these three kinematic chains, moving platforms, and fixed platforms, the mechanism sketch in Figure 2 is obtained.

The platform \( A_1A_2A_3 \) in Figure 2 has 3 DOF. It shows that the platform \( A_1A_2A_3 \) has two moving DOF.
along fixed coordinates \(X\) and \(Y\), and one rotation DOF around the axis parallel to the fixed coordinates \(Z\) axis, that is, it has \(2T1R\) motion mode. According to the method in reference\(^{39}\), the instantaneity of the DOF of the mechanism can be determined. For each DOF of the corresponding mechanism, a small enough displacement from the initial position of the mechanism is given in turn, and then determine whether the DOF of the mechanism changes. If it changes, the DOF of the mechanism is instance; If it does not change, the DOF of the mechanism is not instance. It can be verified that the \(2T1R\) DOF of the mechanism at the configuration of Figure 2 is not instantaneous.

The process of transforming motion mode

The rotation shafts of kinematic chain 2 are co-linear. The mechanism at the configuration shown in Figure 2 makes finite displacement at the XOY plane, and then two shafts \(Z\) in the second kinematic chain are coaxial, that the mechanism is at configuration shown in Figure 3. The second kinematic chain has a passive DOF to rotate through \(B_2\) around the \(z_2\) axis.

The moving platform in Figure 3 has 2 DOF. It shows that at the mechanism configuration shown in Figure 3, the platform \(A_1A_2A_3\) has one moving DOF along the \(X\) axis of the fixed coordinate system and 1 DOF of rotation around the axis \(z_2\). When the platform \(A_1A_2A_3\) makes limited displacement along the \(X\) axis of the fixed coordinate system, the platform \(A_1A_2A_3\) is in the motion mode of the mechanism shown in Figure 2, and the DOF of the mechanism at the configuration of Figure 2 is not instantaneous, so the DOF of the mechanism at the configuration of Figure 3 is instantaneous. At the configuration shown in Figure 3, the position and posture of the platform \(A_1A_2A_3\) remain...
unchanged. After the second kinematic chain rotates 90° around the axis $z_2$, the planar displacement subgroup chain $Y^R Y^R Y^R$ in the second kinematic chain can be changed into $X^R X^R X^R$, and the mechanism configuration in Figure 4 can be obtained.

**Motion mode transforming configuration.** At the mechanism configuration shown in Figure 4, when the platform $A_1 A_2 A_3$ moves along the $Y$ axis of the fixed coordinate system, 1 DOF will be added in the second kinematic chain. It shown that the moving platform has a 2R2T motion mode; when the platform $A_1 A_2 A_3$ rotates around the axis parallel to the $X$ axis of the fixed coordinate system, the moving platform has a 2R1T motion mode. Therefore, the motion mode at the configuration of the mechanism shown in Figure 4 is instantaneous.

**2R1T motion mode configuration**

When the platform $A_1 A_2 A_3$ rotates finitely around the axis parallel to the $X$ axis of the fixed coordinate system in the configuration shown in Figure 4, the configuration of mechanism is shown in Figure 5.

The platform $A_1 A_2 A_3$ in Figure 5 has 3 DOF. It shows that the platform $A_1 A_2 A_3$ has one moving DOF along the $Y$ axis of the fixed coordinate system and two rotation DOF parallel to the $X$ and $Z$ axis of the fixed coordinate system. That is, when the mechanism is in the configuration shown in Figure 5, the platform $A_1 A_2 A_3$ has 2R1T motion mode. It can be verified that the 2R1T DOF of the mechanism at the configuration of Figure 5 is not instantaneous.

**Parallel mechanisms with 2T1R, 3R, and two 2R1T motion modes**

When the parallel mechanism has 3R, 2R1T, and 2T1R motion modes, the instantaneous DOF of the mechanism will increase when the mechanism is in the transformation configuration of the motion mode. In order to use fewer driving pairs realize the transformation of the above motion modes, it is necessary to avoid the transformation of the motion mode of the mechanism when it has three motion modes at the same time. When the motion mode of the mechanism changes from one to another, and only 1 DOF is changed, the DOF of the mechanism is increased by one at the motion mode transformation configuration of the mechanism. If the DOF of the mechanism changes more than one, the DOF of the mechanism at transformation configuration will also increase more than one. It is necessary to add several auxiliary driving pairs to realize the transformation of the mechanism motion modes, which increases the difficulty of the control of the mechanism and the cost of machine equipment. Therefore, the motion mode of the mechanism can be transformed from 2T1R motion mode to 3R motion mode by following six steps. The first step: the motion mode of the mechanism can be transformed from 2T1R motion mode to 2R2T instantaneous DOF; The second step: the instantaneous 2R2T DOF of the mechanism can be transformed to 2R1T motion mode; The third step: the motion mode of the mechanism can be transformed from 2R1T motion mode to 3R1T instantaneous DOF; The fourth step: the instantaneous 3R1T DOF of the mechanism can be transformed to another 2R1T motion mode which is different from the 2R1T motion mode in second step; The fifth step: the motion mode of the mechanism can be transformed from 2R1T motion mode to 3R1T instantaneous DOF; The sixth step: the instantaneous 3R1T DOF of the mechanism can be transformed to 3R motion mode. At the first and second steps, the motion mode of the mechanism is transformed from the original 2T1R motion mode to 2R1T motion mode. One moving DOF of the mechanism in the 2T1R motion mode is transformed into one rotation DOF. The motion mode of the mechanism is transformed at the 2R2T instantaneous configuration of the mechanism, and the DOF of the mechanism is
increased by one. At the third and fourth steps, the motion mode of the mechanism is transformed from 2R1T motion mode to another 2R1T motion mode. The direction of one rotation DOF of the mechanism is transformed into another direction that makes this two 2R1T motion modes different. The motion mode of the mechanism is transformed at the 3R1T instantaneous configuration of the mechanism, and the DOF of the mechanism is increased by one. At the fifth and sixth steps, the motion mode of the mechanism is transformed from 2R1T motion mode to 3R motion mode. One moving DOF of the mechanism in the 2R1T motion mode is transformed into one rotation DOF. The motion mode of the mechanism is transformed at the 3R1T instantaneous configuration of the mechanism, and the DOF of the mechanism is increased by one. In theory, only one auxiliary driving pair is needed to realize the transformation of the motion mode of the mechanism, which is helpful to reduce the difficulty of the control of the transformation of the motion mode of the mechanism and the manufacturing cost of the machinery and equipment.

In order to design mechanisms with 3R, 2R1T, and 2T1R motion modes, a hybrid variable DOF kinematic chain and a variable DOF kinematic chain can be used to construct the mechanisms. In Section “Parallel mechanism with 2T1R and 2R1T motion modes,” the mechanism shown in Figure 2 has 2R1T and 2T1R motion modes. A rotation pair is connected in series on its platform to form a hybrid variable DOF kinematic chain, whose output link has 3R1T and 2T2R motion modes. As shown in Figure 6, the fixed platform of the mechanisms is \(B_1B_2B_3\), and the fixed coordinate system \(OXYZ\) consolidated with it. The moving platform is \(A_1A_4\), and the moving coordinate system \(oxyz\) consolidated with it. The kinematic chain structure connecting \(B_1\) and \(A_1\) is \(Z^RZ^RZ^RZ^R\), and the two rotation pairs \(Z^R\) are respectively connected with the fixed platform point \(B_1\) and the point \(A_1\) of the link \(A_1A_2A_4\). The kinematic chain structure connecting \(B_2\) and \(A_2\) is \(Z^RZ^RZ^RZ^R\). The rotation pair \(Z^R\) is connected with the point \(B_2\) of the fixed platform. The rotation pair \(X^R\) is connected with the point \(A_2\) on the link \(A_1A_2A_4\). The link \(A_1A_2A_4\) is connected with the moving platform \(A_3A_4\) through the rotation pair \(Y^R\). The structure of the kinematic chain connecting \(B_3\) and \(A_3\) is \(Y^RZ^RZ^RZ^R\). The rotation pair \(Y^R\) is connected with the point \(B_3\) on the fixed platform, the rotation pair \(Y^R\) is connected with the point \(A_3\) on the moving platform \(A_3A_4\), and the rotation pair \(Y^R\) in the hybrid kinematic chain and the rotation pair \(Z^R\) in the kinematic chain \(B_1A_1\) are vertically intersected at the origin of the fixed coordinate system \(OXYZ\).

2T1R motion mode

**DOF analysis of the mechanism at 2T1R motion mode configuration.** As shown in Figure 6, the coordinate system \(B_{iX_iY_iZ_i}\) parallel to the coordinate system \(OXYZ\) is established at \(B_i\) point and the rotation pair \(R_4\) is connected in series with the link \(A_1A_2A_4\). The kinematic chain \(B_1A_1, B_2A_2,\) the link \(A_1A_2A_4\), and the rotation pair \(R_4\) can be regarded as the hybrid kinematic chain. The DOF of the mechanism shown in Figure 6 can be

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**Figure 6.** Mechanism at 2T1R motion mode: (a) a sketch of parallel mechanism and (b) 3-D model of parallel mechanism.
analyzed by the following two methods. The first method is that regarding $A_3A_4$ as the moving platform, and then calculates the constraint screws of the kinematic chain $B_1A_1$ and $B_2A_2$ act on the link $A_1A_2A_4$.

Calculating the kinematic screws of the link $A_1A_2A_4$, and then obtains the kinematic screws of the hybrid kinematic chain. Finally calculating the constraint screws group acted on the moving platform $B_3A_3$ by the kinematic chain $B_3A_3$. The first method can also be used to calculate the driving screw produced by the driving pair. However, this method need a large amount of calculation. The second method, the link $A_1A_2A_4$ in Figure 2 can be regarded as the moving platform. The kinematic chain $B_1A_1$ is the first kinematic chain, the kinematic chain $B_2A_2$ is the second kinematic chain, and the kinematic chain $B_3A_3$ connects a rotation pair $R_4$ that constructs the third kinematic chain. The DOF of the mechanism shown in Figure 6 after locking the driving pairs can be calculated. By using this method, the mechanism with hybrid kinematic chain can be transformed into parallel mechanism, which is convenient for constraint screws and driving screws analysis. The first method is used to analyze the motion mode of mechanism. The second method is used to calculate the DOF of the mechanism after locking driving pairs. Combining with the 2T1R motion mode of the mechanism shown in Figure 2, the kinematic screws group of the hybrid kinematic chain in coordinate system $B_1x_1y_1z_1$ in Figure 6 can be expressed as follows:

\[
\begin{align*}
S_{11} &= (0 \ 0 \ 0 \ 0 \ 0 \ 0) \\
S_{12} &= (0 \ 0 \ 1 \ a_1 \ b_1 \ 0) \\
S_{13} &= (0 \ 0 \ 1 \ c_1 \ d_1 \ 0) \\
S_{14} &= (e_1 \ f_1 \ 0 \ g_1 \ h_1 \ i_1)
\end{align*}
\]  

(12)

The screw $S_{ij}$ in formula (12) denotes the kinematic screw of the $j$th pair of kinematic chain $i$ in the local coordinate system. The $S_{11}, S_{12},$ and $S_{13}$ denote the kinematic screws of the link $A_1A_2A_4$ in Figure 6, and the $S_{14}$ denotes the kinematic screw of rotation pair $R_4$ which is connected on the link $A_1A_2A_4$. The four kinematic screws in formula (12) are linearly independent, and the constraint screws group acted on the platform $A_3A_4$ by the hybrid kinematic chain in coordinate system $B_1x_1y_1z_1$ are:

\[
\begin{align*}
S_{11}' &= (0 \ 0 \ 1 \ j_1 \ k_1 \ 0) \\
S_{12}' &= (0 \ 0 \ 0 \ -f_1 \ e_1 \ 0) \\
j_1 \cdot e_1 + k_1 \cdot f_1 + i_1 &= 0
\end{align*}
\]  

(13)

Similarly, the constraint screws acted on the platform $A_3A_4$ by the kinematic chain $B_3A_3$ in the coordinate system $B_3x_3y_3z_3$ can be expressed as

\[
S_{31}' = (0 \ 0 \ 1 \ a_3 \ b_3 \ 0) \quad (14)
\]

The constraint screws acted on the platform $A_3A_4$ by the hybrid kinematic chain $B_1A_1B_2A_2R_4$ and $B_3A_3$ can be transformed from the local coordinate system $B_1$ and $B_3$ to the fixed coordinate system OXYZ for overall analysis. In the fixed coordinate system OXYZ, it can be expressed as follows:

\[
\begin{align*}
O^S_{11}' &= (0 \ 0 \ 1 \ l_1 \ m_1 \ 0) \\
O^S_{12}' &= (0 \ 0 \ 0 \ -f_1 \ e_1 \ 0) \\
O^S_{31}' &= (0 \ 0 \ 1 \ c_3 \ d_3 \ 0)
\end{align*}
\]

(15)

$O^S_{ij}'$ represents the $j$th constraint screw symbol in the $i$th kinematic chain in the fixed coordinate system OXYZ. In formula (15), $O^S_{11}'$ and $O^S_{31}'$ denote two forces parallel to $Z$ axis, $O^S_{12}'$ denotes a couple parallel to XOY plane. At the general configuration of the mechanism, there is no redundant constraint in the constraints group acted on the platform $A_3A_4$. The modified Kutzbach-Grübler formula is used to calculate the DOF of the mechanism.

\[
M = 6(n - g - 1) + \sum f_i + v - \zeta \quad (16)
\]

In formula (16), $M$: DOF of mechanism, $n$: total number of links, $g$: number of kinematic pairs of mechanism, $f_i$: number of freedom of the kinematic pair, $v$: number of redundant constraints, $\zeta$: passive DOF. It is worth noting that in Figure 6, the hybrid kinematic chain $B_1A_1B_2A_2R_4$ can be equivalent to $R_2R_2R_2R_4R$ (vector $u$ coincides with the axis of the rotation pair $R_4$), which could get an equivalent mechanism, so the value of $n, g, f_i$ in equation (16) could be get from the equivalent mechanism. So the equivalent mechanism in Figure 6 has six links, six joints, and nine single DOF kinematic pairs. The number of redundant constraints is 0. There is no passive DOF.

\[
M = 6 \times (6 - 6 - 1) + 9 + 0 = 3
\]

The mechanism has 3 DOF. Calculating by reciprocal product operation on the constraints on the platform $A_3A_4$, it can be seen that the mechanism has 2T1R motion mode at the configuration shown in Figure 6, that is two moving DOF along the XOY plane and one rotation DOF around the axis parallel to $Z$ axis. It can be verified that the 2T1R DOF of the mechanism at the configuration of Figure 6 is not instantaneous.
Rationality analysis of driving pairs selection at 2T1R motion mode configuration of the mechanism. Whether the selection of driving pairs of parallel mechanism are reasonable can be verified by calculating the DOF of the mechanism after driving pairs are locked in kinematic chains. When the DOF of the mechanism is zero with driving pairs in kinematic chains are locked, the selection of the driving pairs are correct. Otherwise, the driving pairs selection are wrong. At the mechanism configuration shown in Figure 6, the rotation pair $R_{13}$ in the kinematic chain $B_1A_1$, the rotation pair $R_{21}$ in the kinematic chain $B_2A_2$, and the rotation pair $R_{33}$ in the kinematic chain $B_3A_4$ are used as the driving pairs.

At the mechanism configuration shown in Figure 6, three driving pairs selected can realize the control of mechanism in 2T1R motion mode.

\section*{2R2T DOF configuration}

DOF analysis of the mechanism at 2R2T DOF configuration. The mechanism can moves in plane from the configuration shown in Figure 6 to the configuration shown in Figure 7. At this configuration, the axes $l_1$ of the rotation pair $R_{31}$ in the kinematic chain $B_3A_3$, the axes $l_1$ of the rotation pair $R_{13}$ in the kinematic chain $B_1A_1$, and the axes $l_1$ of the rotation pair $R_{41}$ intersect at point $N$. The constraint screws group acted on the moving platform $A_3A_4$ by the hybrid kinematic chain and the kinematic chain $B_3A_3$ can be expressed in the fixed coordinate system OXYZ as follows

\begin{align*}
\mathbf{O}_{11}' = (0, 0, a_1') &= (b_1')
\mathbf{O}_{21}' = (0, 0, a_2') &= (b_2')
\mathbf{O}_{32}' = (0, 0, a_2') &= (b_2')
\end{align*}

In formula (19), $\mathbf{O}_{11}'$ and $\mathbf{O}_{31}'$ represent two forces parallel to the XOY plane, and $\mathbf{O}_{32}'$ represents one force parallel to the Z axis. After locking the three driving pairs of the three kinematic chains, the driving force screws group acted on the link $A_1A_2A_4$ are as follows:

\begin{align*}
\mathbf{O}_{11}^{a} &= (c_1', d_1', f_1')
\mathbf{O}_{21}^{a} &= (c_2', d_2', f_2')
\mathbf{O}_{31}^{a} &= (c_3', d_3', f_3')
\end{align*}

In formula (19), $\mathbf{O}_{11}'$ and $\mathbf{O}_{31}'$ represent two forces. Because the linear vector of the constraint screw $\mathbf{O}_{31}'$ produced by the hybrid kinematic chain is parallel to the Z axis and intersects with the axis $l_2$ of the rotation pair $R_4$ at point $N$, the linear vector of the constraint screw $\mathbf{O}_{31}'$ produced by the kinematic chain $B_3A_3$ is parallel to the Z axis and intersects with the axis $l_1$ of the rotation pair $R_4$ at point $N$, the line vectors of $\mathbf{O}_{11}'$ and $\mathbf{O}_{31}'$ coincide. $\mathbf{O}_{12}'$ represents a couple parallel to the XOY plane. The constraints group acted on the moving platform $A_3A_4$ shown in Figure 7 has one
redundant constraint. The modified Kutzbach-Grübler formula is used to calculate the DOF of the mechanism. It is worth noting that in Figure 7, the hybrid kinematic chain $B_1A_1B_2A_2R_4$ can be equivalent to $^A\mathcal{R}_z^{2}\mathcal{R}_z^{2}\mathcal{R}_z^{1}\mathcal{R}_z^{2}$, so $n, g,$ and $f_i$ should refer to the structure value of substituting the equivalent kinematic chain $^A\mathcal{R}_z^{2}\mathcal{R}_z^{2}\mathcal{R}_z^{1}\mathcal{R}_z^{2}$ for the hybrid kinematic chain $B_1A_1B_2A_2R_4$. So the equivalent mechanism in Figure 7 has six links, six joints, and nine single DOF kinematic pairs. The number of redundant constraints is 1. There is no passive DOF.

$$M = 6 \times (6 - 6 - 1) + 9 + 1 = 4$$

The mechanism has 4 DOF. Calculating by reciprocal product operation on the constraints group acted on moving platform $A_3A_4$, it can be seen that the moving platform $A_3A_4$ has two rotation DOF parallel to the axis $l_1$ and $l_2$, and two moving DOF along the XOY plane at the configuration shown in Figure 7. Considering that the mechanism moves from the configuration shown in Figure 6 to the configuration shown in Figure 7, the DOF of the mechanism changes from 2T1R to 2R2T, and the DOF of the mechanism in the configuration shown in Figure 6 is non-instantaneous, so the 4 DOF of the mechanism shown in Figure 7 are instantaneous.

_Rationality analysis of driving pairs selection at instantaneous 2R2T DOF configuration of the mechanism._ Locking the rotation pair $R_{13}$ in the kinematic chain $B_1A_1$, $R_{21}$ in the kinematic chain $B_2A_2$, and $R_{33}$ in the kinematic chain $B_3A_4$. The driving force screws acted on the link $A_1A_2A_4$ by the kinematic chains $B_1A_1$, $B_2A_2$ and $B_3A_4$ are as follows:

$$\begin{align*}
\mathcal{O}S_{1}^{-} & = (0 \ k_1' \ l_1' \ m_1' \ 0 \ 0) \\
\mathcal{O}S_{21}^{-} & = (g_2' \ h_2' \ 0 \ 0 \ 0 \ l_2') \\
\mathcal{O}S_{3}^{-} & = (f_3' \ g_3' \ 0 \ 0 \ 0 \ h_3')
\end{align*}$$

In formula (20), $\mathcal{O}S_{11}^{-}$ denotes a driving force parallel to the YOZ plane, $\mathcal{O}S_{21}^{-}$ and $\mathcal{O}S_{31}^{-}$ denote two driving forces parallel to the XOY plane. The constraint screws group acted on the link $A_1A_2A_4$ by the kinematic chains $B_1A_1$, $B_2A_2$, and $B_3A_4$ are:

$$\begin{align*}
\mathcal{O}S_{11} & = (0 \ 0 \ 0 \ 0 \ 0 \ 0) \\
\mathcal{O}S_{21} & = (0 \ 0 \ 0 \ f_2' \ k_2' \ 0) \\
\mathcal{O}S_{22} & = (0 \ 0 \ 1 \ l_2' \ 0 \ 0) \\
\mathcal{O}S_{31} & = (0 \ 0 \ 0 \ 0 \ 0 \ 0)
\end{align*}$$

Because the linear vector of the constraint screw $\mathcal{O}S_{22}$ produced by the kinematic chain $B_2A_2$ is parallel to the $Z$ axis and intersects with the axis $l_1$ of the rotation pair $R_4$ at point $N$, the linear vector of the constraint screw $\mathcal{O}S_{31}$ produced by the kinematic chain $B_3A_4$ is parallel to the $Z$ axis and intersects with the axis $l_2$ of the rotation pair $R_4$ at point $N$, the line vectors of $\mathcal{O}S_{22}$ and $\mathcal{O}S_{31}$ coincide. The screws group consisting of the screws in formulas (20) and (21) shows that the seven screws in the screws group are linearly correlated at the mechanism configuration shown in Figure 7, the screws group has one redundant constraint. The DOF of the mechanism is calculated by using the modified Kutzbach-Grübler formula. The mechanism in Figure 7 has 7 links, 8 joints, and 12 single DOF kinematic pairs after three driving pairs are locked. The number of redundant constraints is 1. There is no passive DOF.

$$M = 6 \times (7 - 8 - 1) + 12 + 1 = 1$$

The mechanism has 1 DOF. It can be seen that at the configuration shown in Figure 7, the three selected driving pairs can not realize the control of the mechanism at the 2R2T instantaneous DOF configuration.

_Selection of auxiliary driving pair for mechanisms at 2R2T instantaneous DOF configuration._ According to the analysis results of section “Rationality analysis of driving pairs selection at instantaneous 2R2T DOF configuration of the mechanism,” after locking the rotation pair $R_{13}$ in the kinematic chain $B_1A_1$, $R_{21}$ in the kinematic chain $B_2A_2$, and $R_{33}$ in the kinematic chain $B_3A_4$, the control of the mechanism at the instantaneous DOF configuration of 2R2T can not be realized. Therefore, it is necessary to increase one auxiliary driving pair and select $R_{31}$ as the auxiliary driving pair in the kinematic chain $B_3A_4$. After locking the auxiliary driving pair, the driving force is not acted on the link $A_1A_2A_4$. The modified Kutzbach-Grübler formula is used to calculate the DOF of the mechanism. The mechanism in Figure 7 has 7 links, 8 joints, and 11 single DOF kinematic pairs after driving pairs are locked. The number of redundant constraints is 1. There is no passive DOF.

$$M = 6 \times (7 - 8 - 1) + 11 + 1 = 0$$

It can be seen that at the configuration of the mechanism shown in Figure 7, three driving pairs and one auxiliary driving pair $R_{31}$ can realize the control of the mechanism in the 2R2T instantaneous DOF configuration.

**2R1T DOF configuration**

 ĐoF analysis of the mechanism at 2R1T motion mode configuration. When three driving pairs and one...
A mechanism is in the configuration shown in Figure 8.

In formula (24), \( \mathbf{O}_1^{s_{11}} \) and \( \mathbf{O}_1^{s_{31}} \) denote two forces, \( \mathbf{O}_1^{s_{11}} \) and \( \mathbf{O}_1^{s_{21}} \) denote two couples, and the screws group composed of formulas (23) and (24) shows that the six screws in the screws group are linearly independent at the general configuration of mechanism in 2R1T motion mode. The constraints group acting on the link \( A_2A_4 \) does not have redundant constraint. The DOF of the mechanism is computed by using the modified Kutzbach-Grübler formula. The mechanism in Figure 8 has 7 links, 8 joints, and 12 single DOF kinematic pairs after driving pairs are locked. The number of redundant constraints is 0. There is no passive DOF.

\[
M = 6 \times (7 - 8 - 1) + 12 + 0 = 0
\]

The mechanism has 0 DOF. It can be seen that at the configuration shown in Figure 8, the three driving pairs selected can control the 2R1T motion mode of mechanism.

**3R1T DOF configuration**

In Figure 8, the hybrid kinematic chain \( B_1A_1, B_2A_2 \) and \( B_3A_3 \) can be expressed in the fixed coordinate system OXYZ.

\[
O_3^{s_{11}} = (0 \ 0 \ 1 \ h'_1 \ 0 \ 0)
O_3^{s_{12}} = (0 \ 0 \ 0 \ j'_1 \ j'_1 \ 0) \quad (22)
O_3^{s_{11}} = (0 \ j'_2 \ k'_2 \ 0 \ 0)
\]

In formula (22), \( O^{s_{11}} \) and \( O^{s_{12}} \) represent two forces, \( O^{s_{12}} \) represents a couple parallel to the XOY plane. At the general configuration of the mechanism, there is no redundant constraint acted on the moving platform \( A_3A_4 \). It is worth noting that in Figure 8, the hybrid kinematic chain \( B_1A_1B_2A_2R_4 \) can be equivalent to \( Z^2R^2Z^2R^2R \). Therefore, \( n, g, \) and \( f_i \) should refer to the structure value of equivalent kinematic chain \( Z^2R^2Z^2R^2R \) instead of hybrid kinematic chain \( B_1A_1A_2A_3R_4 \). The modified Kutzbach-Grübler formula is used to calculate the DOF of the mechanism. The equivalent mechanism in Figure 8 has six links, six joints, and nine single DOF kinematic pairs. The number of redundant constraints is 0. There is no passive DOF.

\[
M = 6 \times (6 - 6 - 1) + 9 + 0 = 3
\]

The mechanism has 3 DOF. Calculating by reciprocal product operation on the constraints group on the moving platform \( A_3A_4 \), it can be seen that the moving platform \( A_3A_4 \) has 2R1T motion mode at the configuration shown in Figure 8, that is, one moving DOF along the \( Y \) axis and two rotation DOF around the rotation pair \( R_4 \) and around the axis parallel to the \( Z \) axis. It can be verified that the 2R1T DOF of the mechanism at the configuration of Figure 8 is not instantaneous.

**Rationality analysis of driving pairs selection of the mechanism at 2R1T motion mode configuration.** Locking the driving pair \( R_{33} \) does not act the driving force on the link \( A_1A_2A_4 \). Locking the rotation pairs \( R_{13} \) and \( R_{21} \) in the kinematic chains that acts the following driving forces on the link \( A_1A_2A_4 \):

\[
O^{s_{11}} = (0 \ 0 \ 0 \ 1 \ 0)
O^{s_{21}} = (0 \ 0 \ 0 \ j'_2 \ k'_2 \ 0)
O^{s_{22}} = (0 \ 0 \ 1 \ l'_2 \ m'_2 \ 0)
O^{s_{31}} = (c'_3 \ d'_3 \ e'_3 \ 0 \ l'_3)
\]

In formula (24), \( O^{s_{22}} \) and \( O^{s_{31}} \) denote two forces, \( O^{s_{11}} \) and \( O^{s_{21}} \) denote two couples, and the screws group composed of formulas (23) and (24) shows that the six screws in the screws group are linearly independent at the general configuration of mechanism in 2R1T motion mode. The constraints group acting on the link \( A_1A_2A_4 \) does not have redundant constraint. The DOF of the mechanism is computed by using the modified Kutzbach-Grübler formula. The mechanism in Figure 8 has 7 links, 8 joints, and 12 single DOF kinematic pairs after driving pairs are locked. The number of redundant constraints is 0. There is no passive DOF.

\[
M = 6 \times (7 - 8 - 1) + 12 + 0 = 0
\]

The mechanism has 0 DOF. It can be seen that at the configuration shown in Figure 8, the three driving pairs selected can control the 2R1T motion mode of mechanism.
shown in Figure 9. Because the mechanism configuration shown in Figure 9 is obtained by the rotation of the mechanism configuration in Figure 8 around the rotation axis R13, and the 3 DOF of the mechanism in the configuration shown in Figure 8 are non-instantaneous. Therefore, the 4 DOF of the mechanism in the Figure 9 are instantaneous.

Rationality analysis of driving pairs selection of the mechanism at instantaneous 3R1T DOF configuration. At the mechanism configuration shown in Figure 9, after locking the rotation pair R13, the rotation pair R21, and the rotation pair R33, the driving forces acted on the link A1A2A4 are as follows:

$$
\begin{align*}
\text{oS}_{11}' &= (0 \ 0 \ 0 \ 1 \ 0 \ 0) \\
\text{oS}_{21}' &= (n_2' \ p_2' \ 0 \ 0 \ 0 \ q_2') \\
\text{oS}_{31}' &= (k_3' \ l_3' \ m_3' \ p_3' \ q_3' \ r_3')
\end{align*}
$$

(26)

The constraint screws group acted on the link A1A2A4 by the kinematic chains B1A1, B2A2, and B3A4 are as follows:

$$
\begin{align*}
\text{oS}_{11}' &= (0 \ 0 \ 0 \ 0 \ 1 \ 0) \\
\text{oS}_{21}' &= (0 \ 0 \ 0 \ 0 \ 1 \ 0) \\
\text{oS}_{22}' &= (0 \ 0 \ 1 \ u_2' \ 0 \ 0) \\
\text{oS}_{31}' &= (s_3' \ 0 \ t_3' \ u_3' \ 0 \ v_3')
\end{align*}
$$

(27)

The screws group consisting of the screws in formulas (26) and (27) shows that seven screws in the screws group are linearly correlated at the instantaneous 3R1T DOF configuration of the mechanism, and there is one redundant constraint in the constraints group acted on the link A1A2A4. The modified Kutzbach-Grübler formula is used to calculate the DOF of mechanism. The mechanism in Figure 9 has 7 links, 8 joints, and 12 single DOF kinematic pairs after driving pairs are locked. The number of redundant constraints is 1. There is no passive DOF.

$$
M = 6 \times (7 - 8 - 1) + 12 + 1 = 1
$$

The mechanism shown in Figure 9 has 1 DOF after locking the driving pairs. It can be seen that at the configuration shown in Figure 9, the three selected driving pairs can not control the mechanism at the instantaneous 3R1T DOF configuration.

Selection of driving pairs of the mechanism at instantaneous 3R1T DOF configuration. According to the analysis results of section “Rationality analysis of driving pairs selection of the mechanism at instantaneous 3R1T DOF configuration,” the control of the mechanism at the
instantaneous 3R1T DOF configuration can not be realized by locking the rotation pair R_{13}, the rotation pair R_{21}, and the rotation pair R_{33}. Therefore, it is necessary to increase the number of driving pairs. The rotation pair R_{31} in kinematic chain \(B_1A_1\) is selected as the auxiliary driving pair. Locking this driving pair that do not act the driving force on the link \(A_1A_2A_4\). When the mechanism has 3R1T instantaneous DOF, the seven screws in the screws group are linearly correlated. The modified Kutzbach-Grübler formula is used to calculate the DOF of the mechanism. The mechanism in Figure 9 has 7 links, 8 joints, and 11 single DOF kinematic pairs after the driving pairs are locked. The number of redundant constraints is 1. There is no passive DOF.

\[
M = 6 \times (7 - 8 - 1) + 11 + 1 = 0
\]

It can be seen that at the configuration of the mechanism shown in Figure 9, using the three driving pairs and the auxiliary driving pair R_{31} can realize the control of the mechanism at the 3R1T instantaneous DOF configuration.

### 2R1T DOF configuration

**DOF analysis of the mechanism at 2R1T motion mode configuration.** The link \(A_1A_2A_4\) rotates from the Figure 9 around the axis I\(_3\) parallel to the rotation pair R_{24} to the mechanism configuration in Figure 10. The constraint screws group acted on the moving platform \(A_3A_4\) by the hybrid kinematic chain and kinematic chain \(B_1A_1\) can be expressed in the fixed coordinate system OXYZ.

\[
\begin{align*}
O_{S_1}^t &= (0 \ 0 \ 1 \ s'_1 \ 0 \ 0) \\
O_{S_2}^t &= (1 \ 0 \ 0 \ 0 \ 0 \ t'_1) \\
O_{S_3}^t &= (w'_2 \ 0 \ x''_3 \ y''_3 \ 0 \ z''_3)
\end{align*}
\]

In formula (28), \(O_{S_1}^t\), \(O_{S_2}^t\), and \(O_{S_3}^t\) represent three forces. At the configuration shown in Figure 10, there is no redundant constraint in the constraints group acted on the moving platform \(A_3A_4\). The modified Kutzbach-Grübler formula is used to calculate the DOF of the mechanism. According to the content of section “2R1T motion mode configuration,” it is known that the link \(A_1A_2A_4\) has 2R1T motion mode at the configuration shown in Figure 10. Therefore, the hybrid kinematic chain \(B_1A_1B_2A_2\) in Figure 10 can be equivalent to \(Z\beta\gamma\beta\alpha\gamma\beta\) (the axis of vector \(u\) coincides with the axis of the rotation pair R_{4}). The n, g, and \(f_i\) in formula should be referred to the structure value of the equivalent kinematic chain \(Z\beta\gamma\beta\alpha\gamma\beta\).

\[M = 6 \times (6 - 6 - 1) + 9 + 0 = 3\]

The mechanism has 3 DOF. Calculating by reciprocal product operation on the constraints group acted on the moving platform \(A_3A_4\), it can be seen that the mechanism has two rotation DOF around the axis I\(_3\) and \(I_2\) and one moving DOF along the \(Y\) axis under the mechanism configuration shown in Figure 10. That is to say, the mechanism has 2R1T mode of motion at the mechanism configuration shown in Figure 10. It can be verified that the 2R1T DOF of the mechanism at the configuration of Figure 10 is not instantaneous.

**Rationality analysis of driving pairs selection of the mechanism at 2R1T motion mode configuration.** At the mechanism configuration shown in Figure 10, locking rotation pair R_{33} does not act the driving screws on the link \(A_1A_2A_4\). The driving screws group acted on the link \(A_1A_2A_4\) by the hybrid kinematic chains \(B_1A_1\), \(B_2A_2\) and \(B_3A_4\) are

\[
\begin{align*}
O_{S_1}^a &= (0 \ u'_2 \ v'_2 \ w'_1 \ 0 \ 0) \\
O_{S_2}^a &= (v'_2 \ w'_2 \ 0 \ 0 \ 0 \ x'_2)
\end{align*}
\]
The screws group consisting of the screws (29) and (30) shows that the six screws in the screw group are linearly independent in the 2R1T motion mode configuration shown in mechanism Figure 10. The constraints group imposed on the link $A_1A_2A_4$ does not have redundant constraint. The DOF of the mechanism is calculated by using the modified Kutzbach-Grübler formula. The mechanism in Figure 10 has 7 links, 8 joints, and 12 single DOF kinematic pairs after the driving pairs are locked. The number of redundant constraints is 0. There is no passive DOF.

$$M = 6 \times (7 - 8 - 1) + 12 + 0 = 0$$

Locking the rotation pair $R_{13}$, $R_{21}$, and $R_{33}$ in Figure 10, the mechanism has 0 DOF. It can be seen that at the configuration shown in Figure 10, the three selected driving pairs can realize the control of the mechanism 2R1T motion mode.

### 3R1T DOF configuration

**DOF analysis of the mechanism at instantaneous 3R1T DOF configuration.** The moving platform $A_3A_4$ moves along the $Y$ axis from the configuration shown in Figure 10 to the configuration shown in Figure 11. At this time, the rotation axis $k_3$ of the rotation pair $R_{11}$ in Figure 11 passes through the moving coordinate origin of the moving platform $A_3A_4$, and the origin of the moving coordinate system coincides with the origin of the fixed coordinate system. The constraint screws group acted on the moving platform $A_3A_4$ by the hybrid kinematic chain and the kinematic chain $B_3A_2$ can be expressed in the fixed coordinate system OXYZ.

$$O^S_{11} a = (0 1 0 0)$$
$$O^S_{12} a = (1 0 1 0 0)$$
$$O^S_{31} a = (0 0 0 0 0)$$

In formula (31), $O^S_{11} a$, $O^S_{12} a$, and $O^S_{31} a$ represent three forces. At the configuration shown in Figure 11, the constraints group acted on the moving platform $A_3A_4$ has one redundant constraint. The modified Kutzbach-Grübler formula is used to compute the DOF of the mechanism. According to the content of Section “2R1T motion mode configuration,” it can be seen that the link $A_1A_2A_4$ has a 2R1T motion mode in Figure 11. The hybrid kinematic chain $B_3A_1B_2A_2$ has an equivalent mechanism configuration shown in Figure 11. Because the mechanism configuration shown in Figure 11 is obtained by the moving of the mechanism configuration in Figure 10 along the $Y$ axis of the fixed coordinate system OXYZ, and the DOF of mechanism in Figure 10 is non-instantaneous, the 4 DOF of mechanism in Figure 11 are instantaneous.

Rationality analysis of driving pairs selection of the mechanism at 3R1T instantaneous DOF configuration.** At the mechanism configuration shown in Figure 11, locking rotation pair $R_{33}$ does not act the driving force on the link $A_1A_2A_4$. The driving forces group acted on the link $A_1A_2A_4$ by locking the rotation pair $R_{13}$, the rotation pair $R_{21}$, and the rotation pair $R_{33}$ are as follows:

$$O^S_{11} a = (0 1 0 0 0)$$
$$O^S_{21} a = (0 0 0 0 0)$$

### Figure 11.

**Instantaneous 3R1T DOF configuration of the mechanism.**

Figure 11 shows the instantaneous 3R1T DOF configuration of the mechanism. The screws group consisting of the screws (29) and (30) shows that the six screws in the screw group are linearly independent in the 2R1T motion mode configuration shown in mechanism Figure 10. The constraints group imposed on the link $A_1A_2A_4$ does not have redundant constraint. The DOF of the mechanism is calculated by using the modified Kutzbach-Grübler formula. The mechanism in Figure 10 has 7 links, 8 joints, and 12 single DOF kinematic pairs after the driving pairs are locked. The number of redundant constraints is 0. There is no passive DOF.

$$M = 6 \times (7 - 8 - 1) + 12 + 0 = 0$$

Locking the rotation pair $R_{13}$, $R_{21}$, and $R_{33}$ in Figure 10, the mechanism has 0 DOF. It can be seen that at the configuration shown in Figure 10, the three selected driving pairs can realize the control of the mechanism 2R1T motion mode.

### 3R1T DOF configuration

**DOF analysis of the mechanism at instantaneous 3R1T DOF configuration.** The moving platform $A_3A_4$ moves along the $Y$ axis from the configuration shown in Figure 10 to the configuration shown in Figure 11. At this time, the rotation axis $k_3$ of the rotation pair $R_{11}$ in Figure 11 passes through the moving coordinate origin of the moving platform $A_3A_4$, and the origin of the moving coordinate system coincides with the origin of the fixed coordinate system. The constraint screws group acted on the moving platform $A_3A_4$ by the hybrid kinematic chain and the kinematic chain $B_3A_2$ can be expressed in the fixed coordinate system OXYZ.

$$O^S_{11} a = (0 1 0 0 0)$$
$$O^S_{12} a = (1 0 1 0 0)$$
$$O^S_{31} a = (0 0 0 0 0)$$

In formula (31), $O^S_{11} a$, $O^S_{12} a$, and $O^S_{31} a$ represent three forces. At the configuration shown in Figure 11, the constraints group acted on the moving platform $A_3A_4$ has one redundant constraint. The modified Kutzbach-Grübler formula is used to compute the DOF of the mechanism. According to the content of Section “2R1T motion mode configuration,” it can be seen that the link $A_1A_2A_4$ has a 2R1T motion mode in Figure 11. The hybrid kinematic chain $B_3A_1B_2A_2$ has an equivalent mechanism configuration shown in Figure 11. Because the mechanism configuration shown in Figure 11 is obtained by the moving of the mechanism configuration in Figure 10 along the $Y$ axis of the fixed coordinate system OXYZ, and the DOF of mechanism in Figure 10 is non-instantaneous, the 4 DOF of mechanism in Figure 11 are instantaneous.

Rationality analysis of driving pairs selection of the mechanism at 3R1T instantaneous DOF configuration.** At the mechanism configuration shown in Figure 11, locking rotation pair $R_{33}$ does not act the driving force on the link $A_1A_2A_4$. The driving forces group acted on the link $A_1A_2A_4$ by locking the rotation pair $R_{13}$, the rotation pair $R_{21}$, and the rotation pair $R_{33}$ are as follows:

$$O^S_{11} a = (0 1 0 0 0)$$
$$O^S_{21} a = (0 0 0 0 0)$$

### Figure 11.

**Instantaneous 3R1T DOF configuration of the mechanism.**

Figure 11 shows the instantaneous 3R1T DOF configuration of the mechanism. The screws group consisting of the screws (29) and (30) shows that the six screws in the screw group are linearly independent in the 2R1T motion mode configuration shown in mechanism Figure 10. The constraints group imposed on the link $A_1A_2A_4$ does not have redundant constraint. The DOF of the mechanism is calculated by using the modified Kutzbach-Grübler formula. The mechanism in Figure 10 has 7 links, 8 joints, and 12 single DOF kinematic pairs after the driving pairs are locked. The number of redundant constraints is 0. There is no passive DOF.

$$M = 6 \times (7 - 8 - 1) + 12 + 0 = 0$$

Locking the rotation pair $R_{13}$, $R_{21}$, and $R_{33}$ in Figure 10, the mechanism has 0 DOF. It can be seen that at the configuration shown in Figure 10, the three selected driving pairs can realize the control of the mechanism 2R1T motion mode.

### 3R1T DOF configuration

**DOF analysis of the mechanism at instantaneous 3R1T DOF configuration.** The moving platform $A_3A_4$ moves along the $Y$ axis from the configuration shown in Figure 10 to the configuration shown in Figure 11. At this time, the rotation axis $k_3$ of the rotation pair $R_{11}$ in Figure 11 passes through the moving coordinate origin of the moving platform $A_3A_4$, and the origin of the moving coordinate system coincides with the origin of the fixed coordinate system. The constraint screws group acted on the moving platform $A_3A_4$ by the hybrid kinematic chain and the kinematic chain $B_3A_2$ can be expressed in the fixed coordinate system OXYZ.

$$O^S_{11} a = (0 1 0 0 0)$$
$$O^S_{12} a = (1 0 1 0 0)$$
$$O^S_{31} a = (0 0 0 0 0)$$

In formula (31), $O^S_{11} a$, $O^S_{12} a$, and $O^S_{31} a$ represent three forces. At the configuration shown in Figure 11, the constraints group acted on the moving platform $A_3A_4$ has one redundant constraint. The modified Kutzbach-Grübler formula is used to compute the DOF of the mechanism. According to the content of Section “2R1T motion mode configuration,” it can be seen that the link $A_1A_2A_4$ has a 2R1T motion mode in Figure 11. The hybrid kinematic chain $B_3A_1B_2A_2$ has an equivalent mechanism configuration shown in Figure 11. Because the mechanism configuration shown in Figure 11 is obtained by the moving of the mechanism configuration in Figure 10 along the $Y$ axis of the fixed coordinate system OXYZ, and the DOF of mechanism in Figure 10 is non-instantaneous, the 4 DOF of mechanism in Figure 11 are instantaneous.

Rationality analysis of driving pairs selection of the mechanism at 3R1T instantaneous DOF configuration.** At the mechanism configuration shown in Figure 11, locking rotation pair $R_{33}$ does not act the driving force on the link $A_1A_2A_4$. The driving forces group acted on the link $A_1A_2A_4$ by locking the rotation pair $R_{13}$, the rotation pair $R_{21}$, and the rotation pair $R_{33}$ are as follows:

$$O^S_{11} a = (0 1 0 0 0)$$
$$O^S_{21} a = (0 0 0 0 0)$$
The constraint screws group acted on the link $A_1A_2A_4$ by the kinematic chains $B_1A_1, B_2A_2$ and $B_3A_3$ are
\[
\begin{align*}
O\mathbf{s}_{11}^r &= (1 \ 0 \ 0 \ 0 \ 0 \ 0) \\
O\mathbf{s}_{21}^r &= (0 \ 1 \ 0 \ 0 \ 0 \ 0) \\
O\mathbf{s}_{22}^r &= (0 \ 0 \ 1 \ 0 \ 0 \ 0) \\
O\mathbf{s}_{31}^r &= (l''_3 \ 0 \ m''_3 \ 0 \ 0 \ 0)
\end{align*}
\]
(33)

According to the screws group composed of the screws in formulas (32) and (33), six screws in the screws group are linearly correlated at the 3R1T instantaneous DOF mechanism configuration shown in Figure 11. There is one redundant constraint in the constraints group acted on the link $A_1A_2A_4$. The DOF of the mechanism is calculated by using the modified Kutzbach-Grübler formula. The mechanism in Figure 11 has 7 links, 8 joints, and 11 single DOF kinematic pairs after locking the driving pairs. The number of redundant constraints is 1. There is no passive DOF.

\[
M = 6 \times (7 - 8 - 1) + 12 + 1 = 0
\]

It can be seen that at the configuration shown in Figure 11, the mechanism can be controlled at the instantaneous DOF configuration of 3R1T by adding the rotation pair $R_{31}$ of kinematic chain $B_3A_3$ as the auxiliary rotation driving pair.

### 3R DOF configuration

**DOF analysis of the mechanism at 3R motion mode configuration.** The mechanism moving platform $A_3A_4$ rotates around the Z axis from the configuration in Figure 11 to the configuration in Figure 12. The constraint screws group acted on the moving platform $A_3A_4$ by the hybrid kinematic chain and kinematic chain $B_3A_3$ can be expressed in the fixed coordinate system OXYZ.

\[
\begin{align*}
O\mathbf{s}_{11}^r &= (0 \ 0 \ 1 \ 0 \ 0 \ 0) \\
O\mathbf{s}_{12}^r &= (b''_1 \ c''_1 \ 0 \ 0 \ 0 \ 0) \\
O\mathbf{s}_{31}^r &= (u''_3 \ v''_3 \ 0 \ 0 \ 0 \ 0)
\end{align*}
\]
(35)

In formula (35), $O\mathbf{s}_{11}^r$, $O\mathbf{s}_{12}^r$, and $O\mathbf{s}_{31}^r$ denote the three forces which intersect at origins $O$ of the fixed coordinate system OXYZ. At the configuration shown in Figure 12, there is no redundant constraint in the constraints group acted on the moving platform $A_3A_4$. The modified Kutzbach-Grübler formula is used to compute the DOF of the mechanism. It is worth noting that the hybrid kinematic chain $B_1A_1B_2A_2$ in Figure 12 can be equivalent to $ZR^YP^XR^R$ (where the vector $u$ coincides with the axis of the rotation pair $R_4$). Therefore, $n$, $g$, and $f_i$ should refer to the equivalent kinematic chain $ZR^YP^XR^R$ (“Y” indicates the moving direction...
“Y’” of the hybrid kinematic chain rotate around the Z axis). So the equivalent mechanism in Figure 12 has six links, six joints, and nine single DOF kinematic pairs. The number of redundant constraints is 0. There is no passive DOF.

\[
M = 6 \times (6 - 6 - 1) + 9 + 0 = 3
\]

The mechanism has 3 DOF. Calculating by reciprocal product operation on the constraints group acted on the moving platform \(A_1A_4\), it can be seen that the mechanism has three rotation DOF around point \(O\) at the mechanism configuration shown in Figure 12. It can be verified that the 3R DOF of the mechanism at the configuration of Figure 12 is not instantaneous.

**Rationality analysis of driving pairs selection of the mechanism at 3R motion mode configuration.** At the mechanism configuration shown in Figure 12, locking rotation pair \(R_{33}\) does not act the driving force on the link \(A_1A_2A_4\). The driving forces acted on the link \(A_1A_2A_4\) by locking the rotation pair \(R_{13}\), the rotation pair \(R_{21}\), and the rotation pair \(R_{33}\) are as follows:

\[
\begin{align*}
\bar{0}s_{11}^a &= \begin{pmatrix} a'' & e'' & f'' & g'' & h'' & 0 \end{pmatrix} \\
\bar{0}s_{21}^a &= \begin{pmatrix} f'' & 0 & 0 & 0 & h'' \end{pmatrix}
\end{align*}
\]

The constraint screws group acted on the link \(A_1A_2A_4\) by the kinematic chains \(B_1A_1, B_2A_2, \) and \(B_3A_4\) are

\[
\begin{align*}
\bar{0}s_{11}^r &= \begin{pmatrix} \iota_1'' & \iota_1' & 0 & 0 & 0 \end{pmatrix} \\
\bar{0}s_{21}^r &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\
\bar{0}s_{22}^r &= \begin{pmatrix} 0 & 0 & 0 & \iota_2'' & \iota_2' \end{pmatrix} \\
\bar{0}s_{31}^r &= \begin{pmatrix} w_3'' & 0 & x_3' & 0 & 0 \end{pmatrix}
\end{align*}
\]

The screws group consisting of the screws in formulas (36) and (37) shows that the six screws in the screw system are linear independent at the 3R motion mode configuration of the mechanism shown in Figure 12. The constraints group acted on the link \(A_1A_2A_4\) does not have redundant constraint. The DOF of the mechanism is calculated by using the modified Kutzbach-Grübler formula. The mechanism in Figure 12 has 7 links, 8 joints, and 12 single DOF kinematic pairs after the driving pairs are locked. The number of redundant constraints is 0.

\[
M = 6 \times (7 - 8 - 1) + 12 + 0 = 0
\]

Locking the rotation pair \(R_{13}, R_{21}, \) and \(R_{33}\) in Figure 12, the mechanism has 0 DOF. It can be seen that at the configuration shown in Figure 12, the three selected driving pairs can realize the control of 3R motion mode of the mechanism.

To sum up, using three driving pairs and one auxiliary driving pair that can make the motion mode of the mechanism transfer from 2T1R to 2R1T or 3R motion mode. In the above 3 DOF motion modes, the mechanism can be controlled by using three driving pairs. When the mechanism is in the 4 DOF configuration of the transformation of the motion mode, besides the three driving pairs, the auxiliary driving pair is also needed to realize the transformation of the motion mode of the mechanism.

**2R1T DOF configurations**

It is noteworthy that, according to the contents of sections “2R1T DOF configuration” and “2R1T DOF configuration,” the mechanism passes the 2R1T motion mode configuration twice at the process of the changing of the motion mode. According to sections “DOF analysis of the mechanism at 2R1T motion mode configuration” and “Rationality analysis of driving pairs selection of mechanism at 2R1T motion mode configuration,” the DOF of 2R1T motion mode of mechanism are analyzed. At the configuration of two 2R1T motion modes in Figure 13, the moving platform \(A_1A_4\) has the moving DOF along the Y axis. At the mechanism configuration of Figure 13(a), the moving platform \(A_3A_4\) has two rotation DOF around the axis \(l_9\) and \(l_{10}\) respectively. At the mechanism configuration of Figure 13(a), the motion mode is \(Y\)PR3R. At the mechanism configuration of Figure 13(b), the moving platform \(A_3A_4\) has two rotation DOF around the axis \(l_{11}\) and \(l_{12}\). At the mechanism configuration of Figure 13(b), the motion mode is \(Y\)PR3R. Therefore, combined with the contents of sections “2T1R motion mode,” “2R1T motion modes” and 3R DOF configuration, it can be seen that the mechanism has two 2R1T motion modes, 2T1R motion mode and 3R motion modes, a total of four motion modes.

**Conclusions**

(1) The displacement manifold theory is used to synthesize parallel mechanisms with 2R1T and 2T1R motion modes. The moving platform is connected with the rotation pair in series that constructs a hybrid variable DOF kinematic chain. The new mechanism is composed of the hybrid variable DOF kinematic chain and a variable 5 DOF kinematic chain. The screw theory is used to analyze the mechanism. It is shown that the mechanism has 3R, 2T1R, and
two 2R1T motion modes. When analyzing the DOF of the mechanism and the rationality of the selection of the driving pairs, through selecting different link as moving platform, the hybrid kinematic chain can be transformed into series kinematic chain, which will simplify the above analysis process. The mechanism is in a singular configuration of 4-DOF when the motion mode of the mechanism is transferred. Moreover, when the motion mode of the mechanism is transferred, the type or direction of only 1 DOF is changed.

(2) At the general configuration of four motion modes, the mechanism can be controlled by three driving pairs. When the motion mode of the mechanism is transferred, it is necessary to add one auxiliary driving pair to control the mechanism. The instantaneous DOF of the transformed configuration of the mechanism is the combination of two transformed motion modes. When the mechanism transforms its motion mode, the mechanism needs pass the instantaneous 4-DOF of 3R1T or 2R2T configuration. The selected three driving pairs and one auxiliary driving pair can realize the transformation of four motion modes of the mechanism, which is verified by using screw theory.

(3) Hybrid multi-mode kinematic chain combined with multi-mode kinematic chain can be used to design mechanisms with multiple kinematic modes. The assembly geometric relationship of the branch chain can be adjusted according to the target motion mode, so that when multiple motion modes are only transformed between two motion modes, and then the degree of freedom of the mechanism is only increased by 1, so the number of auxiliary drive pair could be minimized. It is feasible to set the path of mechanism motion mode transformation and use one auxiliary driving pair to realize the transformation of more than two motion modes.

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