Boundaries in relativistic quantum field theory *

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Abstract
Boundary conditions in relativistic QFT can be classified by deep results in the theory of braided or modular tensor categories.

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1 The physics problem
We study the behaviour of relativistic quantum field theory (QFT) in the presence of a spacelike boundary (= a hypersurface with spacelike normal). The local fields on both sides of the boundary are defined on the same Hilbert space. The boundary is assumed to be “transparent” for certain quantum fields, including the stress-energy tensor (SET). This means that the fields on both sides share the same SET. (In two-dimensional conformal QFT, this property is equivalent to conservation of energy and momentum at the boundary [3].)

Because the SET provides the generators for translations, it can be used to extend the fields, a priori supported only on one side of the boundary, to all of Minkowski spacetime. One therefore has two QFTs on the same Hilbert space, called $\mathcal{B}^L$ and $\mathcal{B}^R$, that share a common subtheory $\mathcal{A}$, and a common covariance.

The principle of causality only requires that the original local observables commute when they are spacelike separated. In two dimensions, this implies that the extended “left” fields commute with the “right” fields whenever the former are localized in the spacelike left of the latter (“one-sided locality”) – but not vice versa. In four dimensions, using Lorentz covariance, they must be relatively local.

Because the interesting new feature is one-sided locality, we restrict to two dimensions. The question to be addressed is therefore: how can a given subtheory $\mathcal{A}$ be embedded into a pair of local extensions $\mathcal{B}^L$ and $\mathcal{B}^R$ such that $\mathcal{B}^L$ is left-local w.r.t. $\mathcal{B}^R$. Yet another way

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to look at the problem is to consider the covariant QFT $\mathcal{C}$ generated by $\mathcal{B}^L$ and $\mathcal{B}^R$, which is in general non-local, but relatively local w.r.t. $\mathcal{A}$. Then $\mathcal{B}^L$ and $\mathcal{B}^R$ are intermediate extensions, such that the diagram commutes:

$$
\mathcal{A} \hookrightarrow \mathcal{B}^L \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{B}^R
$$

and the embedded $\mathcal{B}^L$ is left-local w.r.t. the embedded $\mathcal{B}^R$, and both generate $\mathcal{C}$.

When $\mathcal{A} \hookrightarrow \mathcal{B}^L$ and $\mathcal{A} \hookrightarrow \mathcal{B}^R$ are given, a “boundary condition” between $\mathcal{B}^L$ and $\mathcal{B}^R$ is a realization of this diagram. It is called irreducible if $\mathcal{A}' \cap \mathcal{C} = \mathcal{C} \cdot \mathbf{1}$. If $\mathcal{A} \hookrightarrow \mathcal{B}^L$ and $\mathcal{A} \hookrightarrow \mathcal{B}^R$ are isomorphic, the trivial solution is to identify $\mathcal{B}^L = \mathcal{B}^R$. Nontrivial solutions can be studied in terms of representation theory of $\mathcal{A}$.

Because boundary conditions must be algebraically consistent with commutation relations required by causality, they cannot be imposed as in classical field theory. Instead, a highly nontrivial classification emerges in the case of completely rational conformal QFT.

## 2 The mathematical setup

A QFT is described in terms of its (local and covariant) net of local algebras

$$
O \mapsto \mathcal{A}(O).
$$

Here $O$ are bounded open spacetime regions (it is sufficient to consider “double-cones” which are the intersections of a forward and a backward lightcone), and $\mathcal{A}(O)$ is the von Neumann algebra of observables accessible in the region $O$. Thus, we are looking for boundary conditions as covariant simultaneous realizations of

$$
\mathcal{A}(O) \hookrightarrow \mathcal{B}^L(O) \hookrightarrow \mathcal{B}^R(O) \hookrightarrow \mathcal{C}(O)
$$

for all double-cones $O \subset \mathbb{R}^2$, such that $\mathcal{B}^L(O_1)$ commutes with $\mathcal{B}^R(O_2)$ whenever $O_1$ is in the left component of the causal complement of $O_2$.

The vacuum representation of $\mathcal{C}$ is a reducible positive-energy representation of $\mathcal{A}$, containing the vacuum representations of $\mathcal{B}^L$ and $\mathcal{B}^R$. Thus, boundary conditions are an issue of positive-energy representations.

Positive-energy representations are efficiently described by the DHR theory [5], which realizes them as localized endomorphisms $\rho$ of the quasilocal algebra $\mathcal{A}$. These are the objects of a $\text{C}^*$ tensor category equipped with a unitary braiding, called DHR($\mathcal{A}$). Intertwiners $t \in \text{Hom}(\rho, \sigma)$ are elements of $\mathcal{A}$ satisfying $tp(a) = \sigma(a)t$ for all $a \in \mathcal{A}$. The monoidal product of endomorphisms is the composition $\rho \sigma$, which canonically induces the monoidal product of intertwiners. The braiding is a collection of intertwiners $\varepsilon_{\rho, \sigma} \in \text{Hom}(\rho \sigma, \sigma \rho)$ defining a natural isomorphism, as functors DHR($\mathcal{A}$) $\times$ DHR($\mathcal{A}$) $\to$ DHR($\mathcal{A}$), between the monoidal product and its reversed. The braiding was originally designed to describe
the statistics of scattering states in massive QFT \cite{6}. In two-dimensional conformal QFT, its relation with the exchange properties of conformal blocks was established in \cite{8}.

Its presence is due to the fact that by locality of $\mathcal{A}$, DHR endomorphisms commute whenever they are localized at spacelike distance. Thus, putting

$$\varepsilon_{\rho,\sigma} \equiv 1$$

whenever $\rho$ is localized in the spacelike right of $\sigma$ (in two dimensions), consistently defines $\varepsilon_{\rho,\sigma}$ in the general case by demanding naturality \cite{6}. (The choice of “right” is just a matter of convention, cf. \cite{8}; the choice of “left” would define the opposite braiding $\varepsilon^{\text{opp}}_{\rho,\sigma} = (\varepsilon_{\sigma,\rho})^*$.)

In two dimensions, double-cones are of the form $O = I \times J$ in lightlike coordinates $t \pm x$. We specify $\mathcal{A}$ in Eq. (2) to be a conformal QFT with local algebras $A(O) = A^+(I) \otimes A^-(J)$, i.e., the common subtheory consists only of local chiral observables. Then one has

$$\text{DHR}(\mathcal{A}) = \text{DHR}(A^+) \boxtimes \text{DHR}(A^-)^{\text{opp}},$$

i.e., the objects of $\text{DHR}(\mathcal{A})$ are (equivalent to) direct sums of tensor products of DHR endomorphisms of $A^+$ and $A^-$, equipped with the tensor product $\varepsilon \otimes \varepsilon^{\text{opp}}$ of chiral braidings (defined analogously by replacing “right” with “lightlike future”). The opposite braiding $\varepsilon^{\text{opp}}$ of $A^-$ arises because $\rho^+ \otimes \rho^-$ is localized in the spacelike right of $\sigma^+ \otimes \sigma^-$ iff $\rho^+$ is localized in the future of $\sigma^+$ and $\rho^-$ in the past of $\sigma^-$. 

### 3 Extensions, Q-systems, and boundary conditions

Relatively local covariant extensions $\mathcal{A} \hookrightarrow \mathcal{B}$ of a local quantum field theory $\mathcal{A}$ were classified in \cite{12} in terms of $\text{DHR}(\mathcal{A})$. They are in one-to-one correspondence (up to equivalence) with Q-systems = triples $(\Theta, W, X)$ where $\Theta$ is a DHR endomorphism equivalent to the vacuum representation of $\mathcal{B}$ regarded as a representation of $\mathcal{A}$, and $W \in \text{Hom}(\text{id}, \Theta)$ and $X \in \text{Hom}(\Theta, \Theta^2)$ are a pair of intertwiners satisfying the relations of a Frobenius algebra in the $\mathbb{C}^*$ tensor category $\text{DHR}(\mathcal{A})$. $\mathcal{A} \hookrightarrow \mathcal{B}$ is irreducible iff $\Theta$ contains $\text{id}$ (the vacuum representation of $\mathcal{A}$) with multiplicity one. $\mathcal{B}$ is local iff $\varepsilon_{\Theta, \Theta} X = X$ (i.e., the Q-system is commutative). The algebraic relations of $\mathcal{B}$ as well as its local subalgebras $\mathcal{B}(O)$ are encoded in the Q-system.

For the problem at hand, one looks for Q-systems for $\mathcal{A} \hookrightarrow \mathcal{C}$ which (a) contain “intermediate” Q-systems (see \cite{4} Ch. 4.4]) for $\mathcal{A} \hookrightarrow \mathcal{B}^Y \hookrightarrow \mathcal{C}$ ($Y = L, R$), and (b) whose algebraic relations ensure left locality of $\mathcal{B}^L$ w.r.t. $\mathcal{B}^R$. Our first result is

**Proposition 3.1.** (\cite{3} Prop. 5.1]) Given two irreducible Q-systems for $\mathcal{A} \hookrightarrow \mathcal{B}^L$, $\mathcal{A} \hookrightarrow \mathcal{B}^R$, there is a “universal construction” of a Q-system for $\mathcal{A} \hookrightarrow \mathcal{C}$, such that the central decomposition of $\mathcal{C}$ gives all inequivalent irreducible boundary conditions.

The universal construction is the braided product of extensions $\mathcal{C} = \mathcal{B}^L \times \mathcal{B}^R$. This is the extension defined by the braided product of Q-systems (cf. \cite{9} Sec. 3.2]) $Q^L \times Q^R \equiv (\Theta, W, X)$, which is defined by $\Theta = \Theta^L \Theta^R$, $W = W^L \times W^R$, and

$$X := (1_{\Theta^L} \otimes \varepsilon^{\text{opp}}_{\Theta^L, \Theta^R} \otimes 1_{\Theta^R}) \cdot (X^L \times X^R).$$
The requirement of left locality dictates the choice of the braided product $\times^-$ involving the opposite braiding: the algebraic relations of this product $Q$-system include the commutation relations

$$\psi^R_\sigma \psi^L_\rho = \varepsilon^{\text{opp}}_{\rho,\sigma} \psi^L_\rho \psi^R_\sigma,$$

and the opposite braiding is trivial, by Eq. (3), precisely when $\rho$ is localized in the spacelike left of $\sigma$. Here, $\psi^Y_\rho$ ($Y = L, R$) are charged generators of $B^Y$ carrying irreducible charge $\rho \prec \Theta^Y$ in $\text{DHR}(\mathcal{A})$, and inheriting the localization of $\rho$.

The universality of this construction follows from the fact that the commutation relations Eq. (1) are the only independent algebraic relations defining $\mathcal{C}$, besides the (given) algebraic relations of the generators $\psi^Y$ within $B^Y$ ($Y = L, R$); whereas left locality – the only apriori condition for the intermediate embeddings of $B^L$ and $B^R$ – requires Eq. (4) whenever $\rho$ is localized to the left of $\sigma$, and naturality of the braiding implies Eq. (4) also in the general case.

We write $\iota^Y : \mathcal{A} \to B^Y$ ($Y = L, R$) and $\iota : \mathcal{A} \to \mathcal{C}$ for the inclusion homomorphisms, and $\bar{\iota}^Y : B^Y \to \mathcal{A}$, $\bar{\iota} : \mathcal{C} \to \mathcal{A}$ for their conjugates, such that $\bar{\iota}^Y \iota^Y = \Theta^Y$, and $\bar{\iota} \iota = \Theta = \Theta^L \Theta^R$ [12]. The irreducible decomposition of $\mathcal{A} \hookrightarrow \mathcal{C}$ is given by the minimal projections in the relative commutant $\mathcal{A}' \cap \mathcal{C} \equiv \text{Hom}(\iota, \iota)$. But one has

**Lemma 3.2. ([4, Prop. 4.33])** If $\mathcal{C}$ is the braided product of two local extensions $\mathcal{A} \hookrightarrow B^Y$ ($Y = L, R$), then $\mathcal{A}' \cap \mathcal{C} = \mathcal{C}' \cap \mathcal{C}$. In particular, the irreducible boundary conditions are classified by the minimal central projections of $\mathcal{C}$. Moreover, the centre $\mathcal{C}' \cap \mathcal{C} = \text{Hom}(\iota, \iota)$ and $\text{Hom}(\iota^L \bar{\iota}^R, \iota^L \bar{\iota}^R)$ are isomorphic as algebras.

For the last statement, notice that the two spaces have the same images under the embeddings by monoidal units into $\text{Hom}(\Theta^L \Theta^R, \Theta^L \Theta^R)$ (using [4, Lemma 3.16]). Thus, the minimal projections in $\mathcal{C}' \cap \mathcal{C}$ (= boundary conditions) correspond to the irreducible subhomomorphisms of $\iota^L \bar{\iota}^R : B^R \to B^L$. Because subhomomorphisms of $\iota^L \rho \bar{\iota}^R : B^R \to B^L$, where $\rho \in \text{DHR}(\mathcal{A})$, are $Q^L-Q^R$-bimodules [4, Ch. 3.6], we conclude that boundary conditions are special $Q^L-Q^R$-bimodules.

The centre of $\mathcal{C}$ is spanned by operators $B_\rho \equiv \psi^L_\rho \psi^R_\rho$ such that $\rho \prec \Theta^L$ and $\rho \prec \Theta^R$ (suppressing possible multiplicities). Every minimal projection assigns a numerical value to $B_\rho$.

To determine the minimal central projections of $\mathcal{C}$, one has to compute and diagonalize the algebra of the generators $B_\rho$. One has

**Lemma 3.3. ([4, Ch. 4.12])** Let $Q^Y = (\Theta^Y, W^Y, X^Y)$ ($Y = L, R$) be two commutative $Q$-systems. Define the commutative $*$-product on $\text{Hom}(\Theta^R, \Theta^L)$

$$T_{\rho} * T_{\sigma} = X^L_{\rho} \cdot (T_{\rho} \times T_{\sigma}) \cdot X^R_{\sigma}.$$

There is a linear bijection $\chi : \text{Hom}(\Theta^R, \Theta^L) \to \mathcal{C}' \cap \mathcal{C}$, taking (appropriate multiples of) matrix units $T_\rho \in \text{Hom}(\rho, \Theta^L) : \text{Hom}(\Theta^R, \rho) \subset \text{Hom}(\Theta^R, \Theta^L)$ to $B_\rho$, such that

$$\chi(T_1) \chi(T_2) = \chi(T_1 * T_2).$$
Thus, diagonalizing the ∗-product, diagonalizes \( C’ \cap C \). If \( I_m \) are the minimal projections w.r.t. ∗, then \( E_m := \chi(I_m) \) are the minimal projections in \( C’ \cap C \). Expanding

\[
T_\rho = \sum_m c_{\rho,m} I_m \quad \Leftrightarrow \quad B_\rho = \sum_m c_{\rho,m} E_m,
\]

it follows that \( B_\rho \equiv \psi^L_\rho \psi^R_\rho \) take the numerical values \( c_{\rho,m} \in \mathbb{C} \) in the subrepresentation \( \pi_m \) of the universal construction given by the range of \( E_m \). These sesquilinear relations among the charged fields \( \psi^L \) and \( \psi^R \) are the desired boundary conditions.

### 4 Classification of irreducible boundary conditions: modular case

The diagonalization of the ∗-product in Lemma 3.3 can be achieved in some special cases. The most remarkable instance is derived under the following assumptions.

(i) The chiral subtheories \( A^+ \) and \( A^- \) have isomorphic DHR categories with finitely many irreducible objects of finite dimension.

(ii) The braiding of DHR(\( A^\pm \)) is non-degenerate.

(iii) Both \( B^L \) and \( B^R \) are maximal local extensions of \( A = A^+ \otimes A^- \).

The dimension in (i) is the statistical dimension [6, 8]. By (i), there is a canonical commutative Q-system \( R_{\text{can}} \) in DHR(\( A^+ \)) \( \otimes \) DHR(\( A^- \))\text{opp} with

\[
\Theta_{\text{can}} \simeq \bigoplus \rho \otimes \overline{\rho}
\]

of dimension \( \mu := \dim(\Theta_{\text{can}}) = \sum_\rho \dim(\rho)^2 \), where the sums run over the equivalence classes of irreducible objects (sectors) of DHR(\( A \)), cf. [12].

(ii) is an automatic consequence if the chiral theories \( A^\pm \) are completely rational [10]. By (i) and (ii), DHR(\( A^\pm \)) is a modular tensor category [8, 10]. This implies that the trace w.r.t. \( \rho \) of the monodromy \( \varepsilon_{\sigma,\rho,\varepsilon_{\rho,\sigma}} \in \text{Hom}(\rho \sigma, \rho \sigma) \), summed over all sectors \( \rho \) of DHR(\( A^\pm \)), is the projection onto \( \text{id} \prec \sigma \) in \( \text{Hom}(\sigma, \sigma) \), for every \( \sigma \in \text{DHR}(A^\pm) \). This feature is known as the “killing ring” trick (cf. [5]). Still by (i) and (ii), the maximal irreducible commutative Q-systems are of the form

\[
Q = Z[q] := C^+[(q \otimes \mathbf{1}) \times^+ R_{\text{can}}],
\]

called the full centre of \( q \) [11]. Here, \( q \) is an irreducible chiral Q-system in DHR(\( A^+ \)), and \( \mathbf{1} \) the trivial Q-system in DHR(\( A^- \)). The (left and right) centres \( C^\pm[\cdot] \) of a Q-system [9] are maximal commutative intermediate Q-systems. The full centre is a Morita invariant of Q-systems in modular categories [9, 11]. The first characterization of maximal commutative Q-systems in terms of the so-called \( \alpha \)-induction construction [13] was later recognized to coincide with the full centre [2]. This also means that their local algebras \( B(O) \) are relative commutants of nested wedge algebras of the nonlocal braided product extensions \((q \otimes \mathbf{1}) \times^+ R_{\text{can}} \) [3, Cor. 4.18].

By (i)–(iii), \( B^L \) and \( B^R \) are full centre extensions of \( A^+ \otimes A^- \), induced by chiral Q-systems \( q^L \) and \( q^R \). To classify the boundary conditions, we first use...
Proposition 4.1. ([9, 4]) Let $q^i = (\theta^i, w^i, x^i)$ be $Q$-systems in a braided tensor category. Then $q^i$-$q^j$-bimodules $m$ specify intertwiners $D_m \in \text{Hom}(\theta^j, \theta^i)$ such that the map $m \mapsto D_m$ is invariant under equivalences of bimodules and respects direct sums, conjugation and bimodule products, normalized as $w^i \cdot D_m \cdot w^j = \text{dim}(m)$.

The precise formulation of the statements can be found in [9, 4].

Next, the full centre construction “lifts” $q^i$-$q^j$-bimodules $m$ to $Z[q^i]$-$Z[q^j]$-bimodules $Z[m]$. We write $Z[q^i] = (\Theta^i, W^i, X^i)$. Then one has

Theorem 4.2. ([9, 11, 4]) For irreducible $Q$-systems $q^i$ and $q^j$ in a modular $C^*$ category, let $m$ run over the irreducible equivalence classes of $q^i$-$q^j$-bimodules. Then

$$I_m := \frac{\text{dim}(m)}{\mu \cdot \text{dim}(\theta^i) \text{dim}(\theta^j)} \cdot D_{Z[m]}$$

diagonalize the $\ast$-product in $\text{Hom}(\Theta^j, \Theta^i)$ (cf. Lemma 3.3).

Combining this result with Lemma 3.2 and Lemma 3.3, we conclude:

Corollary 4.3. The minimal central projections of the universal construction for two full centre extensions are given by $E_m = \chi(I_m)$, where $m$ runs over the irreducible equivalence classes of chiral $q^L$-$q^R$-bimodules. In other words: the boundary conditions, viewed as $Z[q^L]$-$Z[q^R]$-bimodules, are induced by the chiral bimodules.

In the case $q^L = q^R = 1$ (or Morita equivalent), hence $Z[q^L] = Z[q^R] = R_{\text{can}}$, the chiral bimodules are given by $\sigma \in \text{DHR}(\mathcal{A}^\pm)$, and one finds the numerical values

$$\pi_\sigma \left[ \psi_{\rho \otimes \sigma}^L \psi_{\rho \otimes \sigma}^R \right] = \frac{\mu^{1/2}}{\text{dim}(\sigma) \text{dim}(\rho)} \cdot S_{\sigma, \rho} = \frac{S_{\text{id}, \text{id}} S_{\sigma, \rho}}{S_{\text{id}, \sigma} S_{\text{id}, \rho}},$$

given by the entries of the Verlinde matrix $S$ (if $\psi_{\rho \otimes \sigma}$ are normalized as isometries).

5 Other cases

In the general case, one cannot benefit from properties of modular categories. One may directly read off the $\ast$-product $T_{\rho} \ast T_{\sigma}$ from the coefficients of the intertwiners $X^Y$ ($Y = L, R$), but a general formula for its minimal projections is not known.

Special cases can be treated with group theory. If $\text{DHR}(\mathcal{A})$ is a symmetric category (e.g., in four dimensions), then it is equivalent to the dual of a (finite) group, and there is an extension $\mathcal{F}$ with a faithful action of $G$ such that $\mathcal{A} = \mathcal{F}^G$, see [7]. The $\mathcal{F}$-$\mathcal{F}$-boundary conditions are classified by the elements $g \in G$, and

$$\pi_g \left[ \psi_{\rho \otimes \sigma}^L \psi_{\rho \otimes \sigma}^R \right] = u^\rho(g)_{ij} \quad \Rightarrow \quad \pi_g \left[ \psi_{\rho \otimes \sigma}^R \right] = \sum_i \pi_g \left[ \psi_{\rho \otimes \sigma}^L \right] \cdot u^\rho(g)_{ij},$$

e.i., the boundary conditions are gauge transformations [3].

If, in the two-dimensional case, $\text{DHR}(\mathcal{A}^+) \simeq \text{DHR}(\mathcal{A}^-)$ contains a symmetric subcategory (i.e., if the chiral observables admit an orbifold construction $\mathcal{A}^\pm = (\mathcal{B}^\pm)^G$ with the
faithful action of a finite group), and if $\mathcal{B}^L \simeq \mathcal{B}^R$ are given by the canonical Q-system of this subcategory, then one finds the $*$-product

$$\dim(\rho)T_{\rho \otimes \bar{\rho}} * \dim(\sigma)T_{\sigma \otimes \bar{\sigma}} = \sum_\tau N^\tau_{\rho,\sigma} \cdot \dim(\tau)T_{\tau \otimes \bar{\tau}},$$

where $N^\tau_{\rho,\sigma}$ are the fusion rules of the dual of $G$. This is diagonalized by

$$\dim(\rho)T_{\rho \otimes \bar{\rho}} = \sum_C \langle \rho|C \rangle \cdot I_C,$$

where the sum runs over the conjugacy classes of $G$ and $\langle \rho|C \rangle$ is the character of $C$ in the representation $\rho$. Namely, $\langle \rho|C \rangle \langle \sigma|C \rangle = \sum_\tau N^\tau_{\rho,\sigma} \langle \tau|C \rangle$. Thus, we have

$$\pi_C \left[ \psi^{L*}_{\rho \otimes \bar{\rho}} \psi^R_{\rho \otimes \bar{\rho}} \right] = \frac{\langle \rho|C \rangle}{\dim(\rho)}. \quad (7)$$

6 Juxtaposition of boundaries

For the juxtaposition of two boundaries between three QFTs $\mathcal{B}^1$, $\mathcal{B}^2$, $\mathcal{B}^3$, one expects a composition of boundary conditions. Two options may be considered.

Since boundary conditions are (special) $Q^i$-$Q^j$-bimodules (Sect. 3), the first option is the bimodule tensor product, or equivalently the composition of homomorphisms $\alpha^{12} \prec \iota^2 : \mathcal{B}^2 \to \mathcal{B}^1$ and $\alpha^{23} \prec \iota^3 : \mathcal{B}^3 \to \mathcal{B}^2$. In general, this does not close among boundary conditions, because $\beta^{13} \prec \alpha^{12}\alpha^{23}$ are a priori only subhomomorphisms of $\iota^1\Theta^2\iota^3$. Instead, the bimodule tensor product closes among defects, cf. [1, 3], that relax the condition that $\mathcal{C}$ in Eq. (1) is generated by $\mathcal{B}^L$ and $\mathcal{B}^R$.

Another option is to define the composition of boundary conditions as the composition of intertwiners $I_m \in \text{Hom}(\Theta^j, \Theta^i)$. This closes among boundary conditions, but fails to give rise to a tensor category. One may expect only a fusion ring, like the product of conjugacy classes in the second example of Sect. 5.

In the case of full centres of a modular CFT (Sect. 4), both options coincide.

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