OBSERVABLE CONCENTRATION OF MM-SPACES INTO SPACES WITH DOUBLING MEASURES

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Abstract. The property of measure concentration is that an arbitrary 1-Lipschitz function $f : X \to \mathbb{R}$ on an mm-space $X$ is almost close to a constant function. In this paper, we prove that if such a concentration phenomenon arise, then any 1-Lipschitz map $f$ from $X$ to a space $Y$ with a doubling measure also concentrates to a constant map. As a corollary, we get any 1-Lipschitz map to a Riemannian manifold with a lower Ricci curvature bounds also concentrates to a constant map.

1. Introduction

Let $\mu_n$ be the volume measure on the $n$-dimensional unit sphere $S^n$ in $\mathbb{R}^{n+1}$ normalized as $\mu_n(S^n) = 1$. In 1919, P. Lévy proved that for any 1-Lipschitz function $f : S^n \to \mathbb{R}$ and any $\varepsilon > 0$, the inequality

$$\mu_n\left(\{x \in S^n \mid |f(x) - m_f| \geq \varepsilon\}\right) \leq 2 e^{-(n-1)\varepsilon^2/2}$$

holds, where $m_f$ is some constant determined by $f$. For any fixed $\varepsilon > 0$ the right-hand side of the above inequality converges to 0 as $n \to \infty$. This means that any 1-Lipschitz function on $S^n$ is almost close to a constant function for sufficient large $n$. In 1999, M. Gromov introduced the notion of the observable diameter in [3]. Let us recall its definition.

Definition 1.1. Let $Y$ be a metric space and $\nu_Y$ a Borel measure on $Y$ such that $m := \nu_Y(Y) < +\infty$. We define for any $\kappa > 0$

$$\text{diam}(\nu_Y, m - \kappa) := \inf\{\text{diam} Y_0 \mid Y_0 \subseteq Y \text{ is a Borel subset such that } \nu_Y(Y_0) \geq m - \kappa\}$$

and call it the partial diameter of $\nu_Y$.

An mm-space is a triple $(X, d, \mu)$, where $d$ is a complete separable metric on a set $X$ and $\mu$ a Borel measure on $(X, d)$ with $\mu(X) < +\infty$.

Definition 1.2 (Observable diameter). Let $(X, d, \mu)$ be an mm-space and $Y$ a metric space. For any $\kappa > 0$ we define the observable diameter of $X$ by

$$\text{diam}(X \xrightarrow{\text{Lip}_1} Y, m - \kappa) := \sup\{\text{diam}(f_*(\mu), m - \kappa) \mid f : X \to Y \text{ is an 1-Lipschitz map}\},$$

where $f_*(\mu)$ stands for the push-forward measure of $\mu$ by $f$.

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The target metric space $Y$ is called the \textit{screen}. The idea of the observable diameter came from the quantum and statistical mechanics, that is, we think of $\mu$ as a state on a configuration space $X$ and $f$ is interpreted as an observable. Suppose that $\text{diam}(\xrightarrow{\text{Lip}_1} \mathbb{R}, m - \kappa) < \varepsilon$ for sufficiently small $\varepsilon, \kappa > 0$. By the definition, for any 1-Lipschitz function $f : X \to \mathbb{R}$, there exists a Borel subset $A_f \subseteq \mathbb{R}$ such that $\text{diam} A_f < \varepsilon$ and $f_*(\mu)(A_f) \geq m - \kappa$. If we pick a point $m_f \in A_f$ and fix it, then we have
\begin{equation}
\mu(\{x \in X \mid |f(x) - m_f| \geq \varepsilon\}) \leq \mu(f^{-1}(\mathbb{R} \setminus A_f)) \leq \kappa.
\end{equation}
Since $\varepsilon$ and $\kappa$ are sufficiently small positive numbers, the above inequality means that any 1-Lipschitz function $f$ on $X$ is almost close to the constant function $m_f$. On the basis of this fact, we define a sequence $\{X_n\}_{n=1}^{\infty}$ of mm-spaces is a \textit{Lévy family} if $\text{diam}(\xrightarrow{\text{Lip}_1} \mathbb{R}, m_n - \kappa) \to 0$ as $n \to \infty$ for any $\kappa > 0$, where $m_n$ is the total measure of the mm-space $X_n$. Gromov proved in \cite{Gr} that if a sequence $\{X_n\}_{n=1}^{\infty}$ of mm-spaces is a Lévy family, then $\text{diam}(\xrightarrow{\text{Lip}_1} \mathbb{R}^k, m_n - \kappa) \to 0$ as $n \to \infty$ for any $\kappa > 0$ and $k \in \mathbb{N}$. He also discussed the case that the dimension of $\mathbb{R}^k$ goes to $\infty$. Our paper \cite{FF} tackles this problems in the case that the screens are the real hyperbolic spaces. Gromov treated in \cite{Gr} the case that the screen $Y$ moves around all elements of a family $C_0$ of compact metric spaces which is precompact with respect to the Gromov-Hausdorff distance. In particular, he proves the following theorem. For any $m \in \mathbb{N}$, $\kappa_1 \in \mathbb{R}$ and $D > 0$, we denote by $\mathcal{M}_{m,\kappa_1,D}$ the set of all complete Riemannian manifolds $M$ of dimension $m$ such that $\text{Ric}_M \geq \kappa_1$ and $\text{diam} M \leq D$.

\textbf{Theorem 1.3 (Gromov \cite{Gr}).} Let $\{X_n\}_{n=1}^{\infty}$ be a Lévy family. Then we have
\begin{equation}
\sup\{\text{diam}(\xrightarrow{\text{Lip}_1} M, m_n - \kappa) \mid M \in \mathcal{M}_{m,\kappa_1,D}\} \to 0
\end{equation}
as $n \to \infty$ for any $\kappa > 0$.

In this paper, we consider more large class of screens. We treat the case that screen $Y$ has a doubling measure. Let $Y$ be a metric space. Given $x \in Y$ and $r > 0$, we indicate by $B_Y(x, r)$ the closed ball centered at $x$ with radius $r$. For a number $R > 0$ and a function $C : (0, R] \to [1, +\infty)$, we denote by $D_{C,R}$ the set of all pairs $(Y, \nu_Y)$ satisfying the following properties: $\nu_Y$ is a Borel measure on $Y$ such that $0 < \nu_Y(B_Y(x, r)) < +\infty$ and
\begin{equation}
\nu_Y(B_Y(x, 2r)) \leq C(r)\nu_Y(B_Y(x, r))
\end{equation}
for all $x \in Y$ and $r > 0$ with $r \leq R$. A main theorem of this paper is the following:

\textbf{Theorem 1.4.} Let $\{X_n\}_{n=1}^{\infty}$ be a Lévy family and $C : (0, R] \to [1, +\infty)$ an arbitrary function. Then we have
\begin{equation}
\sup\{\text{diam}(\xrightarrow{\text{Lip}_1} Y, m_n - \kappa) \mid (Y, \nu_Y) \in D_{C,R}\} \to 0
\end{equation}
as $n \to \infty$ for any $\kappa > 0$. 

Theorem [1,4] together with Bishop-Gromov volume comparison theorem proves that Theorem [1,3] holds even without the diameter bound $D$ for the screens.

2. Preliminaries

2.1. Basics of doubling measures. Although the following lemma and corollary are somewhat standard, we prove them for the completeness of this paper.

**Lemma 2.1** (cf. [1]). Suppose that $(Y, \nu_Y) \in D_{C,R}$. Then for any $r_1, r_2 > 0$ with $2r_1 \leq 2r_2 \leq R$, there exists a number $C(r_1, r_2)$ depending only on $r_1$ and $r_2$ such that

$$\frac{\nu_Y(B_Y(x, r_1))}{\nu_Y(B_Y(y, r_2))} \geq \frac{1}{C(r_1, r_2)^2} \left(\frac{r_1}{r_2}\right)^{C(r_1, r_2)}$$

for any $x, y \in Y$ with $x \in B_Y(y, r_2)$.

**Proof.** Put $j := \min\{i \in \mathbb{N} \mid B_Y(y, r_2) \subseteq B_Y(x, 2^i r_1)\}$ and $\tilde{C}(r_1, r_2) := \max\{C(2^i r_1) \mid 1 \leq i \leq \log_2(2r_2/r_1)\}$. Since $B_Y(y, r_2) \nsubseteq B_Y(x, 2^{j-1} r_1)$ and $B_Y(y, r_2) \subseteq B_Y(x, 2r_2)$, we have $2^{j-1} r_1 \leq 2r_2$. Iterating (1.1) $j$ times yields

$$\nu_Y(B_Y(y, r_2)) \leq \nu_Y(B_Y(x, 2^j r_1)) \leq \tilde{C}(r_1, r_2)^j \nu_Y(B_Y(x, r_1)).$$

As a result, we obtain

$$\frac{\nu_Y(B_Y(x, r_1))}{\nu_Y(B_Y(y, r_2))} \geq \tilde{C}(r_1, r_2)^{-j} \geq \tilde{C}(r_1, r_2)^{-\log_2\frac{4r_2}{r_1}} = \frac{1}{\tilde{C}(r_1, r_2)^2} \left(\frac{r_1}{r_2}\right)^{\frac{\tilde{C}(r_1, r_2)}{\log_2 \tilde{C}(r_1, r_2)}}.$$

This completes the proof. \hfill \Box

**Corollary 2.2.** Suppose that $(Y, \nu_Y) \in D_{C,R}$ for some $R > 0$ and $C : (0, R] \to [1, +\infty)$. Then $B_Y(x, r)$ is compact for any $x \in Y$ and $r > 0$ with $2r \leq R$.

**Proof.** The proof is by contradiction. Suppose that $B_Y(x, r)$ is not compact. Then, there exist $\varepsilon > 0$ with $\varepsilon \leq \min\{R - r, r\}$ and infinite $3\varepsilon$-separated set $\{x_i\}_{i=1}^{\infty} \subseteq B_Y(x, r)$. By using Lemma 2.1 we have

$$+\infty > \nu_Y(B_Y(x, r)) \geq \sum_{i=1}^{\infty} \nu_Y(B_Y(x_i, \varepsilon)) \geq \sum_{i=1}^{\infty} \frac{1}{C(\varepsilon, r)^2} \left(\frac{\varepsilon}{r}\right)^{C(\varepsilon, r)} \nu_Y(B_Y(x, r)) = +\infty,$$

which implies a contradiction. This completes the proof. \hfill \Box

2.2. Separation and concentration. In this subsection, we prove several results in [3] because we find no proof anywhere.

Let $(X, d)$ be a metric space. For $x \in X$, $r > 0$, and $A, B \subseteq X$, we put

$$d(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\}, \quad d(x, A) := d(\{x\}, A), \quad A_r := \{x \in X \mid d(x, A) \leq r\}.$$
Definition 2.3. Let \((X, d, \mu)\) be an mm-space. For any \(\kappa_0, \kappa_1, \ldots, \kappa_N \in \mathbb{R}\), we define

\[
\text{Sep}(X; \kappa_0, \ldots, \kappa_N) = \text{Sep}(\mu; \kappa_0, \ldots, \kappa_N) \\
:= \sup \{ \min_{i \neq j} d(X_i, X_j) \mid X_0, \ldots, X_N \text{ are Borel subsets of } X \text{ such that } \mu(X_i) \geq \kappa_i \text{ for any } i \},
\]

and call it the separation distance of \(X\).

The proof of the following lemma is easy and we omit the proof.

Lemma 2.4 (cf. [3]). Let \((X, d_X, \mu_X)\) and \((Y, d_Y, \mu_Y)\) be two mm-spaces. Suppose that an 1-Lipschitz map \(f : X \to Y\) satisfies \(f_\ast(\mu_X) = \mu_Y\). Then we have

\[
\text{Sep}(Y; \kappa_0, \ldots, \kappa_N) \leq \text{Sep}(X; \kappa_0, \ldots, \kappa_N).
\]

Lemma 2.5 (cf. [3]). Let \((X, d, \mu)\) be an mm-space and \(\kappa, \kappa' > 0\) with \(\kappa > \kappa'\). Then we have

\[
\text{diam}(X, \mathbb{L}^1, \mathbb{R}, m - \kappa') \geq \text{Sep}(X; \kappa, \kappa).
\]

Proof. Let \(X_1, X_2 \subseteq X\) be two closed subsets such that \(\mu(X_1), \mu(X_2) \geq \kappa\). We define a function \(f : X \to \mathbb{R}\) by \(f(x) := d(x, X_1)\). Let us show \(\text{diam}(f_\ast(\mu), m - \kappa') \geq d(X_1, X_2)\) by contradiction. Suppose \(\text{diam}(f_\ast(\mu), m - \kappa') < d(X_1, X_2)\). There exists a closed subset \(X_0 \subseteq [0, +\infty)\) such that \(\text{diam}(X_0) < d(X_1, X_2)\) and \(\mu(f^{-1}(X_0)) \geq m - \kappa'\). If \(f^{-1}(X_0) \cap X_1 = \emptyset\), we have a contradiction since

\[
\mu(f^{-1}(X_0) \cup X_1) = \mu(f^{-1}(X_0)) + \mu(X_1) \geq (m - \kappa') + \kappa > m.
\]

In the same way we have \(f^{-1}(X_0) \cap X_2 \neq \emptyset\). Take a point \(x_1 \in f^{-1}(X_0) \cap X_1\), \(f(x_1) = d(x_1, X_1) = 0 \in X_0\) implies that \(X_0 \subseteq [0, \text{diam}(X_0)]\). Therefore, we have \(f^{-1}(X_0) \subseteq (X_1)_{\text{diam}(X_0)}\), which yields \(f^{-1}(X_0) \cap X_2 = \emptyset\) since \(\text{diam}(X_0) < d(X_1, X_2)\). This is a contradiction.

Remark 2.6. In [3], Lemma 2.5 is stated as \(\kappa = \kappa'\), but that is not true in general. For example, let \(X := \{x_1, x_2\}, d(x_1, x_2) := 1, \text{ and } \mu(\{x_1\}) = \mu(\{x_2\}) := 1/2\). Putting \(\kappa = \kappa' = 1/2\), we have \(\text{diam}(X, \mathbb{L}^1, \mathbb{R}, 1/2) = 0\) and \(\text{Sep}(X; 1/2, 1/2) = 1\).

We denote by \(\text{Supp} \mu\) the support of a Borel measure \(\mu\).

Lemma 2.7 (cf. [3]). Suppose that \(\text{Supp} \mu\) is connected. Then, for any \(\kappa > 0\) we have

\[
\text{diam}(X, \mathbb{L}^1, \mathbb{R}, m - \kappa) \geq \text{Sep}(X; \kappa, \kappa).
\]

Proof. Let \(X_1, X_2 \subseteq X\) be two closed subsets such that \(\mu(X_1) \geq \kappa\) and \(\mu(X_2) \geq \kappa\). Define a function \(f : X \to \mathbb{R}\) by \(f(x) := d(x, X_1)\). We will show \(\text{diam}(f_\ast(\mu), m - \kappa) \geq d(X_1, X_2)\) by contradiction. Supposing that \(\text{diam}(f_\ast(\mu), m - \kappa) < d(X_1, X_2)\), there exists a closed
subset \( X_0 \subseteq [0, +\infty) \) such that \( \text{diam} X_0 < d(X_1, X_2) \) and \( \mu(f^{-1}(X_0)) \geq m - \kappa \). If \( f^{-1}(X_0) \cap X_1 = \emptyset \), we have
\[
\mu(f^{-1}(X_0) \cup X_1) = \mu(f^{-1}(X_0)) + \mu(X_1) \geq (m - \kappa) + \kappa = m,
\]
which implies \( \text{Supp} \mu \subseteq f^{-1}(X_0) \cup X_1 \). This is a contradiction since \( \text{Supp} \mu \) is connected. In the same way, we have \( f^{-1}(X_0) \cap X_2 \neq \emptyset \). Picking \( x_1 \in f^{-1}(X_0) \cap X_1 \), we get \( f(x_1) = d(x_1, X_1) = 0 \in X_0 \), which yields \( X_0 \subseteq [0, \text{diam} X_0] \). Hence, we have \( f^{-1}(X_0) \subseteq (X_1)_{\text{diam} X_0} \), which implies \( f^{-1}(X_0) \cap X_2 = \emptyset \) since \( (X_1)_{\text{diam} X_0} \cap X_2 = \emptyset \). This is a contradiction because \( f^{-1}(X_0) \cap X_2 \neq \emptyset \). As a result, we obtain \( \text{diam}(f_*(\mu), m - \kappa) \geq d(X_1, X_2) \), which completes the proof of the lemma. \( \square \)

**Lemma 2.8 (cf. [3]).** Let \( \nu \) be a Borel measure on \( \mathbb{R} \) with \( m := \nu(\mathbb{R}) < +\infty \). Then, for any \( \kappa > 0 \) we have
\[
\text{diam}(\nu, m - 2\kappa) \leq \text{Sep}(\nu; \kappa, \kappa).
\]

**Proof.** Put \( a_0 := \sup \{ a \in \mathbb{R} \mid \nu((-\infty, a)) \leq \kappa \} \) and \( b_0 := \inf \{ b \in \mathbb{R} \mid \nu((b, +\infty)) \leq \kappa \} \). Then, we have \( a_0 \leq b_0 \) and
\[
\kappa \leq \lim_{\varepsilon \downarrow 0} \nu\left((\varepsilon, a_0 + \varepsilon]\right) = \nu\left((-\infty, a_0]\right),
\]
\[
\kappa \leq \lim_{\varepsilon \downarrow 0} \nu\left((b_0 - \varepsilon, +\infty]\right) = \nu\left([b_0, +\infty]\right).
\]
\( \nu((-\infty, a_0]) \leq \kappa \) and \( \nu((b_0, +\infty]) \leq \kappa \) imply \( \nu([a_0, b_0]) \geq m - 2\kappa \). Therefore, indicating by \( d_\mathbb{R} \) the usual Euclidean distance, we obtain
\[
\text{diam}(\nu, m - 2\kappa) \leq \text{diam}([a_0, b_0]) = b_0 - a_0 = d_\mathbb{R} \left((-\infty, a_0], [b_0, +\infty]\right) \leq \text{Sep}(\nu; \kappa, \kappa).
\]
This completes the proof. \( \square \)

**Corollary 2.9 (cf. [3]).** For any \( \kappa > 0 \), we have
\[
\text{Sep}(X; \kappa, \kappa) \geq \text{diam}(X \xrightarrow{\text{Lip}_1} \mathbb{R}, m - 2\kappa).
\]

**Proof.** Let \( f : X \to \mathbb{R} \) be an arbitrary 1-Lipschitz function. From Lemma 2.4 and Lemma 2.8 we have \( \text{Sep}(X; \kappa, \kappa) \geq \text{Sep}(f_*(\mu); \kappa, \kappa) \geq \text{diam}(f_*(\mu), m - 2\kappa) \). This completes the proof. \( \square \)

Combining Lemma 2.3 and Corollary 2.9 we obtain the following corollary.

**Corollary 2.10 (cf. [3]).** A sequence \( \{X_n\}_{n=1}^\infty \) of mm-spaces is a Lévy family if and only if \( \text{Sep}(X_n; \kappa, \kappa) \to 0 \) as \( n \to \infty \) for any \( \kappa > 0 \).

3. Proof of the Main Theorem

**Proof of Theorem 1.4.** Let \( \{(Y_n, \nu_n)\}_{n=1}^\infty \) be any sequence of \( D_{C, R} \) and \( \{f_n : X_n \to Y_n\}_{n=1}^\infty \) any sequence of 1-Lipschitz maps. Given any \( \varepsilon > 0 \) with \( 32\varepsilon \leq 3R \), it suffices to show that \( \text{diam}(f_n_*(\mu_n), m_n - \kappa) \leq 6\varepsilon \) for any \( n \) by choosing a subsequence. The claim obviously
holds in the case of $\limsup_{n\to\infty} m_n = 0$, so we assume that $\inf_{n\in\mathbb{N}} m_n > 0$. Take a maximal $\varepsilon$-separated set $\{\xi^n_\alpha\}_{\alpha\in\mathcal{A}_n} \subseteq Y_n$ for each $n \in \mathbb{N}$.

**Claim 3.1.** For any $n \in \mathbb{N}$ and $\alpha \in \mathcal{A}_n$, we have

$$\text{Card}(\{\beta \in \mathcal{A}_n \mid \xi^n_\beta \in B_{Y_n}(\xi^n_\alpha, 5\varepsilon)\}) \leq 2^{4C(\varepsilon/3,16\varepsilon/3)} C\left(\frac{\varepsilon}{3}, \frac{16\varepsilon}{3}\right)^2.$$

**Proof.** By Corollary \ref{corollary}, the set $\{\beta \in \mathcal{A}_n \mid \xi^n_\beta \in B_{Y_n}(\xi^n_\alpha, 5\varepsilon)\}$ is finite. Let $\{\beta_1, \beta_2, \ldots, \beta_k\} := \{\beta \in \mathcal{A}_n \mid \xi^n_\beta \in B_{Y_n}(\xi^n_\alpha, 5\varepsilon)\}$ and take $j \in \{1, 2, \ldots, k\}$ such that $\nu_{Y_n}(B_{Y_n}(\xi^n_{\beta_j}, \varepsilon/3)) = \min\{\nu_{Y_n}(B_{Y_n}(\xi^n_{\beta_j}, \varepsilon/3)) \mid \ell = 1, 2, \ldots, k\}$. Since

$$\nu_{Y_n}(B_{Y_n}(\xi^n_\alpha, 5\varepsilon)) = \sum_{\ell=1}^{k} \nu_{Y_n}(B_{Y_n}(\xi^n_{\beta_j}, \varepsilon/3)) \geq k \nu_{Y_n}(B_{Y_n}(\xi^n_{\beta_j}, \varepsilon/3)),$$

combining this and Lemma \ref{lemma}, we have

$$k \leq \frac{\nu_{Y_n}(B_{Y_n}(\xi^n_\alpha, 16\varepsilon/3))}{\nu_{Y_n}(B_{Y_n}(\xi^n_{\beta_j}, \varepsilon/3))} \leq \left(16\varepsilon/3\right) C(\varepsilon/3,16\varepsilon/3) C\left(\frac{\varepsilon}{3}, \frac{16\varepsilon}{3}\right)^2 = 2^{4C(\varepsilon/3,16\varepsilon/3)} C\left(\frac{\varepsilon}{3}, \frac{16\varepsilon}{3}\right)^2.$$

This completes the proof of Claim \ref{claim}.

By Claim \ref{claim}, for each $n \in \mathbb{N}$ there exists $\alpha_n \in \mathcal{A}_n$ such that

$$k_n := \text{Card}(\{\beta \in \mathcal{A}_n \mid \xi^n_\beta \in B_{Y_n}(\xi^n_\alpha, 5\varepsilon)\}) = \sup_{\alpha\in\mathcal{A}_n} \text{Card}(\{\beta \in \mathcal{A}_n \mid \xi^n_\beta \in B_{Y_n}(\xi^n_\alpha, 5\varepsilon)\}).$$

By taking a subsequence, we get $k_n \equiv k$ for any $n$. Put $\{\beta^n_1, \beta^n_2, \ldots, \beta^n_k\} := \{\beta \in \mathcal{A}_n \mid \xi^n_\beta \in B(\xi^n_\alpha, 5\varepsilon)\}$. We take $J^n_1 \subseteq \{\xi^n_\alpha\}_{\alpha\in\mathcal{A}_n}$ which is maximal with respect to the properties that $J^n_1$ is 5-\varepsilon-separated and $\xi^n_{\beta^n_1} \in J^n_1$, $\xi^n_{\beta^n_2} \not\in J^n_1$, $\ldots$, $\xi^n_{\beta^n_k} \not\in J^n_1$. Next, we take $J^n_2 \subseteq \{\xi^n_\alpha\}_{\alpha\in\mathcal{A}_n} \setminus J^n_1$ which is maximal with respect to the properties that $J^n_2$ is 5-\varepsilon-separated and $\xi^n_{\beta^n_1} \not\in J^n_2$, $\xi^n_{\beta^n_2} \in J^n_2$, $\ldots$, $\xi^n_{\beta^n_k} \not\in J^n_2$. In the same way, we pick $J^n_3 \subseteq \{\xi^n_\alpha\}_{\alpha\in\mathcal{A}_n} \setminus (J^n_1 \cup J^n_2)$, $\ldots$, $J^n_k \subseteq \{\xi^n_\alpha\}_{\alpha\in\mathcal{A}_n} \setminus (J^n_1 \cup J^n_2 \cup \ldots \cup J^n_{k-1})$. Then we have

**Claim 3.2.** $\{\xi^n_\alpha\}_{\alpha\in\mathcal{A}_n} = J^n_1 \cup J^n_2 \cup \cdots \cup J^n_k$ for each $n \in \mathbb{N}$.

**Proof.** The proof is by contradiction. Let us suppose that $\xi^n_\alpha \not\in J^n_1 \cup J^n_2 \cup \cdots \cup J^n_k$. Since $J^n_i$ is maximal for each $i = 1, 2, \ldots, k$, there exists $\xi^n_\alpha \in J^n_i$ such that $d_{Y_n}(\xi^n_\alpha, \xi^n_\beta) < 5\varepsilon$ and $\xi^n_\alpha \not\in \xi^n_\beta$. Therefore, we have

$$k + 1 \leq \text{Card}(\{\xi^n_\alpha, \xi^n_1, \xi^n_2, \ldots, \xi^n_k\}) \leq \text{Card}(\{\beta \in \mathcal{A}_n \mid \xi^n_\beta \in B_{Y_n}(\xi^n_\alpha, 5\varepsilon)\}) \leq k,$$

which is a contradiction.

By Claim \ref{claim}, we have $Y_n = \bigcup_{i=1}^{k} \bigcup_{\xi^n_\alpha \in J^n_i} B_{Y_n}(\xi^n_\alpha, \varepsilon)$. Therefore, by taking a subsequence, there exists $j_0 \in \mathbb{N}$ such that $1 \leq j_0 \leq k$ and

$$f_{n^*}(\mu_n)\left(\bigcup_{\xi^n_\alpha \in J^n_{j_0}} B_{Y_n}(\xi^n_\alpha, \varepsilon)\right) \geq \frac{m_n}{k}$$
Claim 3.4. For any sufficiently large $n \in \mathbb{N}$ there exists $\xi_n^{\gamma_n} \in J_n^{\gamma_n}$ such that

$$f_{n^*}(\mu_n)(B_{Y_n}(\xi_n^{\gamma_n}, 2\varepsilon)) \geq \frac{1}{6} \inf_{n \in \mathbb{N}} m_n.$$  

Proof. Let us prove the claim by contradiction. Suppose that

$$f_{n^*}(\mu_n)(B_{Y_n}(\xi_n^{\gamma_n}, 2\varepsilon)) < \frac{1}{6} \inf_{n \in \mathbb{N}} m_n$$

for infinitely many $n \in \mathbb{N}$ and any $\xi_n^{\gamma_n} \in J_n^{\gamma_n}$. By Claim 3.3, there exist $n_0 \in \mathbb{N}$ such that

$$f_{n^*}(\mu_n)\left( \bigcup_{\xi_n^{\gamma_n} \in J_n^{\gamma_n}} B_{Y_n}(\xi_n^{\gamma_n}, 2\varepsilon) \right) \geq \frac{5}{6} \inf_{n \in \mathbb{N}} m_n.$$ 

for any $n \in \mathbb{N}$ with $n \geq n_0$. From the assumption, if $n \geq n_0$ we have $J_n' \subseteq J_n$ such that

$$\frac{1}{6} \inf_{n \in \mathbb{N}} m_n \leq f_{n^*}(\mu_n)\left( \bigcup_{\xi_n^{\gamma_n} \in J_n'} B_{Y_n}(\xi_n^{\gamma_n}, 2\varepsilon) \right) \leq \frac{1}{3} \inf_{n \in \mathbb{N}} m_n.$$ 

Hence, by putting $J_n'' := J_n \setminus J_n'$ we have

$$\varepsilon \leq d_{Y_n} \left( \bigcup_{\xi_n^{\gamma_n} \in J_n''} B_{Y_n}(\xi_n^{\gamma_n}, 2\varepsilon), \bigcup_{\xi_n^{\gamma_n} \in J_n''} B_{Y_n}(\xi_n^{\gamma_n}, 2\varepsilon) \right) \leq \text{Sep}\left(f_{n^*}(\mu_n); \frac{1}{6} \inf_{n \in \mathbb{N}} m_n, \frac{1}{2} \inf_{n \in \mathbb{N}} m_n\right)$$

$$\leq \text{Sep}\left(\mu_n; \frac{1}{6} \inf_{n \in \mathbb{N}} m_n, \frac{1}{2} \inf_{n \in \mathbb{N}} m_n\right),$$

which is a contradiction since the right-hand side of the above inequality converges to 0 as $n \to \infty$. \qed
Claim 3.5. We have
\[ f_{n^*}(\mu_n)(Y_n \setminus B_{Y_n}(\xi_{\gamma_n}^n, 3\varepsilon)) \to 0 \]
as \( n \to \infty \).

Proof. The claim immediately follows from the same proof of Claim 3.3. \( \square \)

By Claim 3.5, for any sufficient large \( n \in \mathbb{N} \) we have
\[ f_{n^*}(\mu_n)(Y_n \setminus B_{Y_n}(\xi_{\gamma_n}^n, 3\varepsilon)) \leq \kappa, \]
which implies \( \text{diam}(f_{n^*}(\mu_n), m_n - \kappa) \leq \text{diam} B_{Y_n}(\xi_{\gamma_n}^n, 3\varepsilon) \leq 6\varepsilon \). This completes the proof of Theorem 1.4. \( \square \)

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