GLOBAL EULER OBSTRUCTION AND POLAR INVARIANTS

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Abstract. Let \( Y \subset \mathbb{C}^N \) be a purely dimensional, complex algebraic singular space. We define a global Euler obstruction \( \text{Eu}(Y) \) which extends the Euler-Poincaré characteristic in case of a nonsingular \( Y \). Using Lefschetz pencils, we express \( \text{Eu}(Y) \) as an alternating sum of global polar invariants.

1. Introduction

The local Euler obstruction, introduced by R. MacPherson in [MP], is an invariant of complex analytic varieties that plays an essential role for studying the Chern classes of singular varieties. It appeared in the construction of MacPherson’s cycle, as the constructible function \( \text{Eu}_X \). Roughly speaking, if \( (X, x_0) \) is a possibly singular germ, the local Euler obstruction of \( X \) at \( x_0 \), \( \text{Eu}_X(x_0) \), is the obstruction for extending a continuous stratified radial vector field around \( x_0 \) in \( X \) to a non-zero section of the Nash bundle over the Nash blow up \( \tilde{X} \) of \( X \) [MP, Du1, BS, LT].

We define in this paper a global Euler obstruction \( \text{Eu}(Y) \) for an affine singular variety \( Y \subset \mathbb{C}^N \) of pure dimension \( d \), in a similar manner, i.e. as the obstruction to extend a radial vector field, defined on the link at infinity of \( Y \), to a non-zero section of the Nash bundle. In case \( Y \) is non-singular, \( \text{Eu}(Y) \) equals the Euler-Poincaré characteristic \( \chi(Y) \). Our obstruction can be actually regarded as the top dimensional Chern-Mather class of \( Y \).

We show here that \( \text{Eu}(Y) \) can be expressed as an alternating sum:

\[
\text{Eu}(Y) = (-1)^d \alpha_Y^{(1)} + \cdots - \alpha_Y^{(d)} + \alpha_Y^{(d+1)},
\]

where the invariants \( \alpha_Y^{(i)} \) are defined as follows: \( \alpha_Y^{(1)} \) is the number of Morse points on the regular part \( Y_{\text{reg}} \) of a Lefschetz pencil on \( Y \) and the following ones are similar numbers defined on successive generic hyperplane slices of \( Y \). These invariants can be viewed as global polar multiplicities. Formula (1) may therefore look analogous to Lê-Teissier’s one for the local Euler obstruction [LT], in which local polar multiplicities enter. Our proof of (1) has nevertheless different flavour. It relies on the repeated use of the Lefschetz method of slicing by pencils and on a construction, inspired from the local setting of [BLS, BMPS], which is, roughly, as follows: start from a stratified radial-at-infinity vector field.

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field and extend it first over a generic hyperplane section, then find the obstruction to extend it further.

We shall explain in §5 how the known local formulas \cite{LT, BLS}, as well as our global formulas, can be viewed as consequences, in appropriate settings, of the Lefschetz principle and of Dubson’s definition of Euler obstruction relative to a non-characteristic open set \cite{Du1, Du2}.

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2. Global Euler obstruction

Let $Y \subset \mathbb{C}^N$ be an algebraic space of pure dimension $d$. We consider the analytic closure $\bar{Y}$ of $Y$ in the complex projective space $\mathbb{P}^N$ and denote by $H_\infty$ the hyperplane at infinity of the embedding $\mathbb{C}^N \subset \mathbb{P}^N$. One may endow $\bar{Y}$ with a semi-analytic Whitney stratification $\mathcal{W}$ such that the part “at infinity” $\bar{Y} \cap H_\infty$ is a union of strata. Since $\bar{Y}$ is projective and since the stratification $\mathcal{W}$ is locally finite, it follows that $\mathcal{W}$ has finitely many strata. The regular part $Y_{\text{reg}}$ is the stratum (possibly not connected) of highest dimension $d$.

Let $\tilde{Y}$ denote the Nash blow-up of $Y$, that is:

$$
\tilde{Y} = \text{closure}\{(x, T_xY_{\text{reg}}) \mid x \in Y_{\text{reg}}\} \subset Y \times G(d, N),
$$

where $G(d, N)$ is the Grassmannian of complex $d$-planes in $\mathbb{C}^N$. Let $\nu : \tilde{Y} \to Y$ denote the (analytic) natural projection. Let $\bar{T}$ denote the Nash bundle over $\tilde{Y}$, i.e. the restriction over $\tilde{Y}$ of the bundle $\mathbb{C}^N \times U(d, N) \to \mathbb{C}^N \times G(d, N)$, where $U(d, N)$ is the tautological bundle over $G(d, N)$. We consider a continuous, stratified vector field $v$ on a subset $V \subset Y$. The restriction of $v$ to $V$ has a well-defined canonical lifting $\bar{v}$ to $\nu^{-1}(V)$ as a section of the Nash bundle $\tilde{T} \to \tilde{Y}$ (see e.g. \cite{BS}, Prop.9.1).

We say that the stratified vector field $v$ on $\bar{Y}$ is radial-at-infinity if it is defined on the restriction to $Y$ of the complement of a sufficiently large ball $B_M$ centered at the origin of $\mathbb{C}^N$, and if $v$ is transversal to $S_R$, pointing outwards, for any $R > M$. The “sufficiently large” radius $M$ is furnished by the following well-known result.

**Lemma 2.1.** There exists $M \in \mathbb{R}$ such that, for any $R \geq M$, the sphere $S_R$ centered at the origin of $\mathbb{C}^N$ and of radius $R$ is stratified transversal to $Y$, i.e. transversal to all strata of the stratification $\mathcal{W}$. \hfill $\square$

This follows essentially from Milnor’s finiteness result \cite[Cor. 2.8]{Mil} applied to the strata of $Y$. We shall call $K_\infty(Y) := Y \cap S_R$ the link at infinity of $Y$. By Lemma 2.1 and standard isotopy arguments, $K_\infty(Y)$ does not depend on the radius $R$, provided that $R > M$. Actually the link at infinity $K_\infty(Y)$ does not depend on the choice of the center of the sphere either, since for two spheres $S', S''$ centered at two different points, the links $Y \cap S'$ and $Y \cap S''$ are isotopic, provided the radii are large enough.
An important consequence of Lemma 2.1 is that every vector field \( \mathbf{w} \) defined on \( Y \cap S_R \) without zeros, can be extended to the exterior of the ball \( Y \cap B_R \) without zeros, via the fibration provided by Lemma 2.1. Therefore the obstruction to extend \( \mathbf{w} \) to the whole \( Y \) subsists only inside the ball \( Y \cap B_R \). This motivates the following definition, yet related to Dubson’s definition of Euler obstruction [Du1]:

**Definition 2.2.** Let \( \tilde{v} \) be the lifting to a section of the Nash bundle \( \tilde{T} \) of a radial-at-infinity, stratified vector field \( \mathbf{v} \) over \( K_\infty(Y) = Y \cap S_R \). We call global Euler obstruction of \( Y \), and denote it by \( \text{Eu}(Y) \), the obstruction for extending \( \tilde{v} \) as a nowhere zero section of \( \tilde{T} \) within \( \nu^{-1}(Y \cap B_R) \).

To be precise, the obstruction to extend \( \tilde{v} \) as a nowhere zero section of \( \tilde{T} \) within \( \nu^{-1}(Y \cap B_R) \) is in fact a relative cohomology class \( o(\tilde{v}) \in H^{2d}(\nu^{-1}(Y \cap B_R), \nu^{-1}(Y \cap S_R)) \). The Euler obstruction of \( Y \) is the evaluation of \( o(\tilde{v}) \) on the fundamental class of the pair \( (\nu^{-1}(Y \cap B_R), \nu^{-1}(Y \cap S_R)) \). Thus \( \text{Eu}(Y) \) is an integer. By the preceding discussion, \( \text{Eu}(Y) \) does not depend on the radius of the sphere defining the link at infinity \( K_\infty(Y) \). Since two radial vector fields are homotopic as stratified vector fields, it does not depend on the choice of \( \mathbf{v} \) either. Elementary obstruction theory tells us that \( o(\tilde{v}) \) is also independent of the way we extend the section \( \tilde{v} \) to \( \nu^{-1}(Y \cap B_R) \), see [St]. Moreover, as shown in [MP] or [Du1], this is also independent on the blow-up \( \nu \) (one can work with any blow-up which extends the tangent bundle over \( Y_{\text{reg}} \)).

2.1. **Properties of** \( \text{Eu}(Y) \). Let us recall that the local Euler obstruction is defined at each point of \( Y \), and it is constant on each stratum of a Whitney stratification (see for instance [Du1], [BS], [LT]). Thus, given a stratum \( W_i \subset Y \) of \( \mathcal{W} \), one denotes by \( \text{Eu}_Y(W_i) \) the local Euler obstruction at some point of \( W_i \). Our just defined global Euler obstruction has the following properties.

(a) If \( Y \) is non-singular, then \( \text{Eu}(Y) = \chi(Y) \), the Euler-Poincaré characteristic of \( Y \).

(b) \( \text{Eu}(Y) = c_{\text{top}}^d(Y) \), the top degree Chern-Mather class of \( Y \).

(c) \( \text{Eu}(Y) = \sum_{W_i \subset Y} \chi(W_i)\text{Eu}_Y(W_i) \).

The property (a) is clear from the definition. For property (b), we notice that our obstruction \( o(\tilde{v}) \) in cohomology is evaluated on the homology orientation class. This gives a class in \( H_0(\tilde{Y}) \), so it is an integer. The map \( \nu_* : H_0(\tilde{Y}) \to H_0(Y) \) is obviously an isomorphism and takes \( \text{Eu}(Y) \) into the top Chern-Mather class \( c_{\text{top}}^d(Y) \).

To explain the equality (c), let us recall the relation to Dubson’s definition of the Euler obstruction of some analytic variety relative to a non-characteristic open set. In the language of [Du2], our ball \( B_R \) is a non-characteristic open set with respect to the Whitney stratification, which just means that \( S_R \) is transversal to strata. The property (c) tells that \( \text{Eu}(Y) \) is an Euler-Poincaré characteristic weighted by the constructible function \( \text{Eu}_Y \). The equality follows immediately by Dubson’s [Du2, Theorem 1] applied to our setting. In terms of constructible functions, the equality follows from a direct image argument and a complete proof in this spirit can be found in [Sch2, (5.65)]. A different proof can be derived from [BS, Theorem 4.1]. One may imagine still another variant of proof,
based on extending vector fields, along the lines of the proof in [BLS], see our comments in §5.

3. Affine Lefschetz pencils and Main Theorem

In the local setting, various authors have proved “Lefschetz type” formulas for the local Euler obstruction, see for instance [Du1, LT, BLS, Sch1, BMPS]; we refer to the bibliography for background on this topic.

We now introduce global affine Lefchetz pencils and we define the set of invariants that enter in our formula for $\text{Eu}(Y)$. Let us assume that the coordinates of $\mathbb{C}^N$ are fixed and recall that $H_\infty$ is the hyperplane at infinity $\mathbb{P}^N \setminus \mathbb{C}^N$. An affine pencil of hyperplanes $H_t$, $t \in \mathbb{C}$, is defined by a linear function $l : \mathbb{C}^N \to \mathbb{C}$, where $H_t := l^{-1}(t)$. The intersection $A := H_t \cap H_\infty$ is the same for all $t$ and it is called axis of the pencil. We need to work with generic pencils and therefore recall below a well-known result on the existence of such pencils, see e.g. [La], [GM] or [Ti2] for more general statements.

Lemma 3.1. There exists a Zariski open dense subset $\Omega$ of linear functions on $\mathbb{C}^N$ such that, for any $l \in \Omega$:

(a) the axis $A$ of the pencil defined by $l$ is transversal to all the strata of $W$ which are contained in the hyperplane at infinity $H_\infty$.

(b) there exists a finite set $B \subset \mathbb{C}$ such that, for all $t \in \mathbb{C} \setminus B$, the hyperplane $H_t$ cuts transversally all the strata of $Y$. For $t \in B$, the hyperplane $H_t$ is transversal to the strata of $Y$ at all points except finitely many, which are stratified Morse singularities of the function $l$. $\square$

Definition 3.2. The pencil of hyperplanes defined by $l \in \Omega$ will be called a Lefschetz pencil with respect to $Y$.

Let us also recall the definition of complex stratified Morse singularities, used in Lemma 3.1 since this is important in the proof of our main result.

Definition 3.3. (Lazzeri '73, Benedetti '77, Pignoni '79, Goresky-MacPherson '83 [GM] p.52.) Let $W$ be a local Whitney stratification of a germ $(X, x_0) \subset (\mathbb{C}^N, x_0)$ of a complex analytic space. Let $f : (X, x_0) \to \mathbb{C}$ be a holomorphic function germ and let $F : (\mathbb{C}^N, x_0) \to \mathbb{C}$ denote some extension of it. We say that $f$ is a general function at $x_0$ if $dF_{x_0}$ does not vanish on any limit of tangent spaces to $W_i$, $\forall i \neq 0$, and to $W_0 \setminus \{x_0\}$, where $W_0$ denotes the stratum to which $x_0$ belongs. One says that $f : (X, x_0) \to \mathbb{C}$ is a stratified Morse function germ if: $\dim W_0 \geq 1$, $f$ is general with respect to the strata $W_i$, $i \neq 0$ and the restriction $f|_{W_0} : W_0 \to \mathbb{C}$ has a Morse point at $x_0$.

3.1. The global polar invariants. It appears that the polar invariants play a key role in studying the topology of spaces, sometimes in connection to the Lefschetz slicing method. Local polar multiplicities enter in the local Euler obstruction formula of Lé-Teissier [LT (5.1.2)]. Polar classes (cf [LT, Pi]) and Chern-Mather classes determine each other via a Todd type formula proved by Ragni Piene [Pi].
We now define the global invariants which enter in our formula. The first number \( \alpha^{(1)}_Y \) is by definition the number of Morse points on the regular part \( Y_{\text{reg}} \) of a Lefschetz pencil on \( Y \). Next step, consider a general hyperplane \( H_t \cap Y \) of the Lefschetz pencil and take a new Lefschetz pencil of \( H_t \cap Y \): we get the second number \( \alpha^{(2)}_Y := \# \) Morse points of the second Lefschetz pencil on \( H_t \cap Y_{\text{reg}} \). This continues by induction and we get a sequence of non-negative integers:

\[
\alpha^{(1)}_Y, \alpha^{(2)}_Y, \ldots, \alpha^{(d)}_Y,
\]

to which we attach the last one \( \alpha^{(d+1)}_Y := \# \) points of the intersection of \( Y_{\text{reg}} \) with a generic codimension \( d \) plane in \( \mathbb{C}^N \). This is in fact just the degree of \( Y \).

All these numbers are well-defined invariants of \( Y \), by the connectivity of the Zariski open sets of generic slices and of pencils which we use. In fact, these invariants can be interpreted as global polar multiplicities, similar to the local ones used by Lê-Teissier [LT], see also Piene [Pi].

Global polar invariants have been also used by the second named author in order to characterize a certain equisingularity at infinity of families of affine hypersurfaces [T1].

Our result can be stated as follows:

**Theorem 3.4.** If \( Y \subset \mathbb{C}^N \) is an algebraic variety of pure dimension \( d \), then its global Euler obstruction is \( \text{Eu}(Y) = \sum_{i=1}^{d+1} (-1)^{d-i+1} \alpha^{(i)}_Y \).

Let us stress that only Morse points on the regular part (or the regular part of repeated slices) come into the description of the Euler obstruction \( \text{Eu}(Y) \), in other words the Morse stratified singularities on other strata are ignored. Consequently, the stratification \( \mathcal{W} \) does not appear in the statement at all. Furthermore, the formula resembles to an Euler characteristic formula, by attaching cells via Lefschetz slicing method. In general, it is not the Euler characteristic of \( Y \) or of \( Y_{\text{reg}} \). Nevertheless, in case \( Y \) is non-singular, i.e. if \( Y = Y_{\text{reg}} \), we clearly have \( \text{Eu}(Y) = \chi(Y) \).

4. **Proof of the Main Theorem**

Let us fix a Lefschetz pencil with respect to \( Y \), of axis \( A \), given by a linear function \( l \in \Omega \) (Lemma 3.1, Definition 3.2). Its restriction to \( Y \) is a stratified submersion away from a finite set of points \( \Sigma = \{ y_1, \ldots, y_k \} \). At each such point the function germ \( l_{|Y} : (Y, y_i) \to \mathbb{C} \) has a stratified Morse singularity. Let us then take a large enough disk \( D \subset \mathbb{C} \) such that the finite set \( B \) of critical values of \( l_{|Y} \) is contained in \( D \).

We fix a generic slice of \( H_t := l^{-1}(t) \), i.e. \( t \in D \setminus B \). Notice that the intersections of \( H_t \) with the strata of \( \mathcal{W} \) constitutes a Whitney stratification of \( Y \cap H_t \), to which we refer in the following. We apply Lemma 2.1 to \( Y \) and then to \( Y \cap H_t \). This yields a positive real \( M \) such that the sphere \( S_R \) is stratified transversal to \( Y \) and to \( Y \cap H_t \), for all \( R \geq M \). We then fix an \( R \geq M \).

We remark that the critical set \( \Sigma \) is contained in \( Y \cap B_R \cap l^{-1}(D) \). We shall construct a special continuous stratified vector field \( v \) on \( Y \cap B_R \cap l^{-1}(D) \) which is radial on \( Y \cap S_R \cap l^{-1}(D) \) and points outwards the tube \( l^{-1}(\partial D) \). We notice that the procedure used in the local case in [BLS, BMPS] applies to our situation, when replacing a small Milnor
ball by our big ball $B_R$. Our function will be the global function $l$, instead of a germ, and this function has several stratified critical points.

As a matter of fact, we start with a radial-at-infinity vector field $v$ on $Y \cap S_R \cap l^{-1}(D)$ which in addition is radial-at-infinity with respect to the strata of the fixed slice $Y \cap H_t$. Next we extend this vector field over $Y \cap H_t \cap B_R$ (one can always do this such that the extension has isolated zeros). Then, as shown in loc. cit., one may extend this to a continuous stratified vector field without zeroes, outside $Y \cap H_t \cap B_R$ and within a tubular neighbourhood of $Y \cap H_t \cap B_R$, by plugging in the lifting by $l$ of a vector field on the disk $D$ which is radial from the point $t$. We may further extend and deform this without zeros outside the tubular neighbourhood such that it becomes the gradient vector field $\text{grad}_Y l$ (defined below) in the neighbourhood of any point $y_i \in \Sigma$. See Figure 1.

Let us first give the precise definition of $\text{grad}_Y l$. Take the complex conjugate of the gradient of $l$ and project it to the tangent spaces of the strata of $Y$ into a stratified vector field on $Y$. This may be not continuous, but one can make it continuous by “tempering” it in the neighbourhood of “smaller” strata. One then gets a continuous stratified vector field, well-defined up to stratified homotopy, which is by definition $\text{grad}_Y l$ and which we have called above the gradient vector field.

Note that the only zeros of $\text{grad}_Y l$ occur precisely at the points $y_i \in \Sigma$. If $\nu: \tilde{Y} \to Y$ is the Nash blow-up of $Y$ and $\tilde{T}$ is the Nash bundle over $\tilde{Y}$, then $\text{grad}_Y l$ lifts canonically to a never zero section $\text{grad}_{\tilde{Y}} l$ of $\tilde{T}$ restricted to $\nu^{-1}(Y \cap S_\varepsilon)$, where $S_\varepsilon$ is a small Milnor sphere around some fixed $y_i$. In our problem, we are interested in the obstruction to extend $\text{grad}_{\tilde{Y}} l$ without zeros throughout $\nu^{-1}(Y \cap B_\varepsilon)$. One considers a more general problem in [BMPS]: given a holomorphic function germ $f: (X, x_0) \to \mathbb{C}$, compute the obstruction of extending $\text{grad}_X f$ without zeros from $\nu^{-1}(X \cap S_\varepsilon)$ throughout $\nu^{-1}(X \cap B_\varepsilon)$. This is called \textit{local Euler obstruction of $f$}, or “defect of $f$”, and is denoted by $\text{Eu}_f(X, x_0)$. In case of Morse singular points, one may directly compute this obstruction as follows.
Lemma 4.1. Let \( l \) be a holomorphic function germ on \((Y, y)\) with a stratified Morse singularity at \( y \in W_0 \). Then the local Euler obstruction of \( l \) is 0 if \( \dim W_0 < \dim Y \), and is \((-1)^{\dim_Y} \) if \( W_0 = Y_{\text{reg}} \).

Proof. Take a small enough ball \( B_{\varepsilon} \) in \( Y \subset \mathbb{C}^N \), centered at \( y \) and of radius \( \varepsilon > 0 \). Let \( v \) be the gradient vector field \( \text{grad}_Y l \) restricted to the sphere \( Y \cap \partial B_{\varepsilon} \) and consider the tautological lift \( \tilde{v} \) to the Nash blow-up \( \tilde{Y} \). From the definition of a Morse function we have that the form \( dl \) does not vanish on any limit of tangent spaces at points in the regular stratum \( Y_{\text{reg}} \). Since \( \tilde{Y} \) is obtained by attaching to \( TY_{\text{reg}} \) all limits of tangent spaces of points in \( Y_{\text{reg}} \), one has that if \( y \notin Y_{\text{reg}} \), then, by the Definition 3.3 of stratified Morse points, the section \( \tilde{v} \) of \( \tilde{T}^* \) can be extended over \( \nu^{-1}(Y \cap B_{\varepsilon}) \) without zeros. This proves the first part of our statement, since the extension can be done as follows. Let us think of \( \tilde{T} \) as being a subset of \((\mathbb{C}^N \times G(d, N)) \times \mathbb{C}^N \). At each point \((x, H) \in \tilde{Y} \) one adds to \( \tilde{v} \) the tautological lift of the vector

\[
\gamma(||x||) \cdot \text{proj}_H(\text{grad}l),
\]

where \( \text{proj}_H \) denotes the projection to \( H \) and \( \gamma \) denotes a continuous non-negative real function defined on \([0, \varepsilon]\) with values in \([0, 1]\), such that \( \gamma(\varepsilon) = 0 \) and \( \gamma(0) = 1 \) (for instance \( \gamma \) can be taken linear.)

The proof of the second part of our statement, for \( y \in Y_{\text{reg}} \), goes as follows. Locally, the Nash bundle is the usual tangent bundle of \( Y_{\text{reg}} \) and \( \text{Eu}(Y, y) \) is by definition the Poincaré-Hopf index of \( \text{grad}_Y l \) at \( y \). One deduces from [Mi Th.7.2] that \( \text{Eu}(Y, y) = (-1)^{\dim_Y} \) since \( y \) is just a complex Morse point in the classical sense. \( \square \)

The above lemma is natural since the Euler obstruction is defined via the Nash blow-up and the latter only takes into account the closure of the tangent bundle over the regular part \( Y_{\text{reg}} \).

We pursue the proof of Theorem 3.4. We have shown how one may extend the radial-at-infinity vector field \( v \) to a vector field over \( Y \cap B_R \cap l^{-1}(D) \). In the same manner as in [BLS, BMPS] for the local case, one may extend this further without zeros on \((Y \cap B_R) \setminus (Y \cap B_R \cap l^{-1}(D))\).

The construction we have done shows, firstly, that the global Euler obstruction \( \text{Eu}(Y) \) is exactly the obstruction to extend \( \tilde{v} \) inside the bounded tube \( \nu^{-1}(Y \cap B_R \cap l^{-1}(D)) \) and secondly, that this obstruction is the sum of two terms:

(a) the obstruction to extend \( \tilde{v} \) within the slice \( \nu^{-1}(Y \cap H_t \cap B_R) \), as a lift of a stratified vector field with respect to the strata of \( Y \cap H_t \).

(b) the obstructions due to the isolated zeros of the gradient vector field \( \text{grad}_Y l \).

The obstruction (a) is \( \text{Eu}(Y \cap H_t) \) essentially by definition, since the Whitney strata of \( Y \cap H_t \) are precisely the intersections of the strata of \( Y \) with \( H_t \), by the assumed transversality of \( H_t \) to the Whitney stratification \( \mathcal{W} \) of \( Y \). For (b), by Lemma 4.1, the local obstruction at some stratified Morse point \( y_i \) is zero if \( y_i \) is not on \( Y_{\text{reg}} \) and it is \((-1)^d\) for \( y_i \in Y_{\text{reg}} \). So the sum of all such local obstructions is equal to \((-1)^d\) times the number of Morse points. This number is just the global invariant \( \alpha^{(1)}_Y \) defined at 3.1. We
therefore get:

\[ (2) \quad \text{Eu}(Y) = \text{Eu}(Y \cap H_t) + (-1)^d \alpha_Y^{(1)} \]

We may now apply the same procedure to \( Y \cap H_t \) instead of \( Y \), get a formula similar to (2) for \( Y \cap H_t \), i.e. \( \text{Eu}(Y \cap H_t) = \text{Eu}(Y \cap H_t \cap H_t') + (-1)^d \alpha_Y^{(2)} \), and so on by induction. Altogether this leads to formula (1). The proof of Theorem 3.4 is now complete.

5. Further remarks

We have seen that our formula (1), proved via (2), belongs to the vein of Lefschetz type formulas and in the same time it is a Lê-Teissier type one. Let us see how the term \( \text{Eu}(Y \cap H_t) \) in formula (2) can be refined in another direction. In the local setting, the obstruction to extend \( \tilde{v} \) within the slice \( \nu^{-1}(X \cap H_t \cap B) \), as a lift of a stratified vector field with respect to the strata of \( X \cap H_t \), has been looked up in [BLS, BMPS]. The proof of the main formula in [BLS] can be rephrased as follows: one first shows that actually \( \text{Eu}_X(x_0) = \text{Eu}(X \cap B \cap H_t) \) and next that \( \text{Eu}(X \cap B \cap H_t) \) can be expressed as a weighted Euler characteristic. In the global setting, the latter fact has the following interpretation:

\[ (3) \quad \text{Eu}(Y \cap H_t) = \sum_{W_i \subset Y} \chi(Y \cap B \cap H_t \cap W_i) \cdot \text{Eu}_Y(W_i), \]

By the choice of the radius \( R \) in the beginning of the proof of our Theorem 3.4, we have that \( Y \cap B \cap H_t \cap W_i \) is diffeomorphic to \( Y \cap H_t \cap W_i \), hence the sum (3) is in turn equal to \( \sum_{W_i \subset Y} \chi(Y \cap H_t \cap W_i) \cdot \text{Eu}_Y(W_i) \). Formula (2) becomes therefore:

\[ (4) \quad \text{Eu}(Y) = \sum_{W_i \subset Y} \chi(H_t \cap W_i) \cdot \text{Eu}_Y(W_i) + (-1)^d \alpha_Y^{(1)}. \]

One may alternatively prove formula (3) by using property (c) in §2.1.

Actually, Dubson’s definition of Euler obstruction relative to a non-characteristic real analytic open set [Du1, Du2] together with the Lefschetz slicing method allows one to understand in a unitary way the local or global formulas for the Euler obstruction. For instance, the formula proved in [BLS] could be rephrased saying that the local Euler obstruction \( \text{Eu}_X(x_0) \) equals the Euler obstruction of the complex link of \( X \) at \( x_0 \) to which one applies property (c) in §2.1.

Alternatively, instead of applying property (c), one may use local Lefschetz slicing in the similar manner as in the global setting (proof of Theorem 3.4). By applying formula (2) repeatedly to the Euler obstruction of the complex link of \( X \) at \( x_0 \), we get exactly Lê-Teissier’s formula for the local Euler obstruction [LT].

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GLOBAL EULER OBSTRUCTION AND POLAR INVARIANTS

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