The vanishing discount problem for monotone systems of Hamilton–Jacobi equations: part 2—nonlinear coupling

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Abstract
We study the vanishing discount problem for a nonlinear monotone system of Hamilton–Jacobi equations. This continues the first author’s investigation on the vanishing discount problem for a monotone system of Hamilton–Jacobi equations. As in part 1, we introduce by the convex duality Mather measures and their analogues for the system, which we call respectively Mather and Green–Poisson measures, and prove a convergence theorem for the vanishing discount problem. Moreover, we establish an existence result for the ergodic problem.

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Contents

1 Introduction ............................................... 2
2 Preliminaries ............................................... 4
3 Green–Poisson measures: in a regular case ....................... 8
4 Green–Poisson measures: the general case ....................... 15
5 A convergence result for the vanishing discount problem .......... 20
6 Ergodic problem ............................................. 24
References ................................................................ 26

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1 Introduction

We consider the $m$-system of Hamilton–Jacobi equations

$$\lambda v^i_t(x) + H_i(x, Dv^i_t(x), v^i(x)) = 0 \quad \text{in } \mathbb{T}^n, \quad i \in \mathbb{I},$$

(P$_\lambda$)

where $\mathbb{I} := \{1, \ldots, m\}$ with $m \in \mathbb{N}$, $\lambda$ is a nonnegative constant, called the discount factor in terms of optimal control. Here $\mathbb{T}^n$ denotes the $n$-dimensional flat torus and $H = (H_i)_{i \in \mathbb{I}}$ is a family of continuous Hamiltonians. The unknown in (P$_\lambda$) is an $\mathbb{R}^m$-valued function $v^\lambda = (v^\lambda_i)_{i \in \mathbb{I}}$ on $\mathbb{T}^n$ and the above system can be written in the vector form as follows:

$$\lambda v^\lambda + H[v^\lambda] = 0 \quad \text{in } \mathbb{T}^n.$$

(P$_\lambda$)

We have used here the abbreviated expression $H[v^\lambda]$ to denote $(H_i(x, Dv^\lambda_i(x), v^\lambda(x)))_{i \in \mathbb{I}}$. The system is weakly coupled in the sense that every $i$th equation depends on $Dv^\lambda_j$ only through $Dv^\lambda_i$ but not on $Dv^\lambda_j$, with $j \neq i$.

We are concerned with the vanishing discount problem for (P$_\lambda$), that is, the asymptotic behavior of the solution $v^\lambda$ of (P$_\lambda$) as $\lambda \to 0+$. Notably, the main concern is the convergence of the whole family $(v^\lambda)_{\lambda > 0}$ as $\lambda \to 0+$.

Recently, there has been a great interest in the vanishing discount problem concerned with Hamilton–Jacobi equations and, furthermore, fully nonlinear degenerate elliptic PDEs. We refer to [1,6,8,12,13,16,23,24,28–31,41] for relevant work. The asymptotic analysis in these papers relies heavily on Mather measures or their generalizations and, thus, it is considered part of Aubry–Mather and weak KAM theories. For the development of these theories we refer to [18,20,21] and the references therein. We refer to [7,9,14,15,19,37–40] for the recent development in the asymptotic analysis and weak KAM theory for systems of Hamilton–Jacobi equations.

We are here interested in the case of systems of Hamilton–Jacobi equations. Davini and Zavidovique [16] have established a general convergence result for the vanishing discount problem for (P$_\lambda$) when the coupling is linear and the coupling coefficients are constant. Adapting the convex duality argument in [28] to the system, the first author of this paper has treated the case of linear coupling, with the coupling coefficients depending on the space variable. In this paper, we extend the scope of the previous work [26] and discuss the case of the system with nonlinear coupling. Our argument is pretty much parallel to that in [26]. We refer for further references to [16,26].

Under our hypotheses described later, the limit function $v^0$ of the solution $v^\lambda$ of (P$_\lambda$) satisfies the system $H[v^0] = 0$ and the principal difficulty in the asymptotic analysis lies in the fact that the system $H[u] = 0$, or (P$_0$), usually has multiple solutions of which the structure is not simple in general. The critical role of Mather measures is indeed to identify the limit function $v^0$ from the solutions of $H[u] = 0$.

We assume throughout [see (H2) below] that the functions: $\mathbb{R}^n \times \mathbb{R}^m \ni (p, u) \mapsto H_i(x, p, u)$ are convex for $(x, i) \in \mathbb{T}^n \times \mathbb{I}$. We have chosen to make the convexity requirement on $H$ in $u$ simply because of our technical limitations for studying the vanishing discount problem. The non-convexity issue for the vanishing discount problem has already been addressed in [8,24,45] in the case of scalar equations. The paper [45] illustrates by examples that, in general, the convergence of the whole family $(v^\lambda)_{\lambda > 0}$ does not hold without the convexity of $p \mapsto H_i(x, p, u)$, while [8,24] indicate some possible generalizations beyond the convexity of $u \mapsto H_i(x, p, u)$.

On the other hand, under the convexity assumption, the system (P$_\lambda$) may be regarded as the dynamic programming equation of optimal control of random evolutions, where the
state \((x(t), i(t))\) at time \(t\) is in \(\mathbb{T}^n \times \mathbb{I}\), \(x(t)\) is governed by a controlled ordinary differential equation, and \(i(t)\) is a Markov process with controlled transition probability matrix. See [5] for this application.

In this paper, we adopt the notion of viscosity solution to \((P_\lambda)\), for which the reader may consult [2,4,10,11,33].

Now, we give our main assumptions on the system \((P_\lambda)\). Throughout we implicitly assume that the functions \(H_i\) are continuous in \(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m\).

We assume that \(H\) is coercive, that is, for any \(i \in \mathbb{I}\) and \(R > 0\),

\[
\lim_{|p| \to \infty} \inf_{(x,u) \in \mathbb{T}^n \times B_R^m} H_i(x, p, u) = \infty,
\]

where \(B_R^m\) denotes the \(m\)-dimensional open ball with center at the origin and radius \(R\).

This is a standard assumption, under which any upper semicontinuous subsolution of \((P_\lambda)\) is Lipschitz continuous on \(\mathbb{T}^n\).

We next assume that \(H\) is convex in the variables \((p, u)\), that is, for any \((x, i) \in \mathbb{T}^n \times \mathbb{I}\), the function \((p, u) \mapsto H_i(x, p, u)\) is convex on \(\mathbb{R}^n \times \mathbb{R}^m\). \((H2)\)

We assume that the Hamiltonian \(H\) is monotone in the variable \(u\), that is, it satisfies

\[
\begin{cases}
\text{for any} \ (x, p) \in \mathbb{T}^n \times \mathbb{R}^n \text{ and } u = (u_i)_{i \in \mathbb{I}}, \ v = (v_i)_{i \in \mathbb{I}} \in \mathbb{R}^m, \ &\text{if } u_k - v_k = \max_{i \in \mathbb{I}} (u_i - v_i) \geq 0, \ \\
&\text{then } H_k(x, p, u) \geq H_k(x, p, v).
\end{cases}
\]

This is a natural assumption implying that \((P_\lambda)\) should possess the comparison principle between a subsolution and a supersolution.

When the coupling is linear, that is, when \(H\) has the form

\[
H_i(x, p, u) = G_i(x, p) + \sum_{j \in \mathbb{I}} b_{ij}(x) u_j \quad \text{for } i \in \mathbb{I},
\]

condition \((H3)\) is valid if and only if for each \(x \in \mathbb{T}^n\), the matrix \(B(x) = (b_{ij}(x))_{i,j \in \mathbb{I}}\) is monotone matrix in the following sense

\[
b_{ij}(x) \leq 0 \quad \text{if } i \neq j \quad \text{and} \quad \sum_{j \in \mathbb{I}} b_{ij}(x) \geq 0 \quad \text{for all } i \in \mathbb{I},
\]

which is equivalent to that for each \(x \in \mathbb{T}^n\),

\[
\begin{align*}
&\text{if } u = (u_i) \in \mathbb{R}^m, \ k \in \mathbb{I}, \ \text{and } u_k = \max_{i \in \mathbb{I}} u_i \geq 0, \ &\text{then } (B(x)u)_k \geq 0, \\
&\text{where } (B(x)u)_k \text{ denotes the } k\text{th component of the } m\text{-vector } B(x)u. \ \text{We refer, for instance, to [26, Lemma 3] for the equivalence of the last two conditions (2) and (3), while it is obvious in the linear coupling case (1) that (H3) and (3) are equivalent each other.}
\end{align*}
\]

When we deal with problem \((P_0)\), we use the assumption that \((P_0)\) has a solution in \(C(\mathbb{T}^n)^m\). \((H4)\)

In the scalar case, that is, the case when \(m = 1\) and the case when \(H(x, p, u)\) is independent of \(u\), a natural problem which replaces \((P_0)\) is the so-called ergodic problem that seeks a pair of a constant \(c \in \mathbb{R}\) and a function \(u \in C(\mathbb{T}^n)\) such that \(u\) is a solution of

\[
H(x, Du) = c \quad \text{in } \mathbb{T}^n.
\]

This problem is well-posed under \((H1)\), which means that there exists such a pair \((c, u) \in \mathbb{R} \times \text{Lip}(\mathbb{T}^n)\) and the constant \(c\) is unique. See for this [34]. For such \((c, u)\), if we set
$H_c = H - c$, then $u$ is a solution of (P$_0$), with $H$ replaced by $H_c$. If $v^\lambda$ is a solution of (P$_\lambda$), with current scalar Hamiltonian $H(x, p)$, then the function $v^\lambda + \lambda^{-1}c$ is a solution of (P$_\lambda$), with $H$ replaced by $H_c$. This way, if $m = 1$ and $H(x, p, u)$ is independent of $u$, then the vanishing discount problem can be transferred to the case where the limit problem (P$_0$) admits compactness of the domain of $\lambda$. In Sect. 3, under additional hypotheses on the continuity of the Lagrangian $\lambda$, are crucial in our asymptotic analysis. Section 4 establishes the compactness of the support of $\lambda$, the Green–Poisson measures and gives a representation theorem for the solution of (P$_\lambda$). In Sect. 5, we establish an existence theorem for the ergodic discount problem in Sect. 5. In Sect. 6, we establish an existence theorem for the ergodic problem.

2 Preliminaries

We use the symbol $u \leq v$ (resp., $u \geq v$) for $m$-vectors $u, v \in \mathbb{R}^m$ to indicate $u_i \leq v_i$ (resp., $u_i \geq v_i$) for all $i \in \mathbb{I}$. Let $e_i$ denote the unit vector in $\mathbb{R}^m$ with unity as its $i$th entry and let $1$ denote the $m$-vector $(1, \ldots, 1) \in \mathbb{R}^m$.

Concerning the monotonicity of $H$, we give a basic lemma.

Lemma 1 Assume that $H$ satisfies (H3). Let $\alpha \geq 0$.

(i) For all $i \in \mathbb{I}$ and $(x, p, u) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m$,

$$H_i(x, p, u + \alpha 1) \geq H_i(x, p, u).$$

(ii) For all $i, j \in \mathbb{I}$ and $(x, p, u) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m$, if $i \neq j$, then

$$H_j(x, p, u + \alpha e_i) \leq H_j(x, p, u)$$

Proof (i) Fix $k \in \mathbb{I}$ and $(x, p, u) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m$. Set $v = u + \alpha 1$ and note that

$$(v - u)_k = \alpha = \max_{i \in 1}(v - u)_i > 0.$$ 

By the monotonicity of $H$, we have

$$H_k(x, p, v) \geq H_k(x, p, u).$$

That is,

$$H_k(x, p, u + \alpha 1) \geq H_k(x, p, u).$$

(ii) Fix $i, j \in \mathbb{I}$ so that $i \neq j$. Let $(x, p, u) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m$. Note that $u_k - (u + \alpha e_i)_k = 0$ if $k \neq i$ and $= -\alpha < 0$ if $k = i$, which can be stated that $u_j - (u + \alpha e_i)_j = \max_{k \in 1}[u_k - (u + \alpha e_i)_k] = 0$. By (H3), we have

$$H_j(x, p, u) \geq H_j(x, p, u + \alpha e_i).$$

\hfill \Box
The following theorem is well-known: see [17,27] for instance for a general background and [26, the proof of Theorem 1] for some details how to adapt general results to \((P_\lambda)\).

**Theorem 2** Assume \((H1)\) and \((H3)\). Let \(\lambda > 0\). Then there exists a unique solution \(v^\lambda \in \text{Lip}(\mathbb{T}^n)^m\) of \((P_\lambda)\). Also, if \(v = (v_i)\), \(w = (w_i)\) are, respectively, upper and lower semicontinuous on \(\mathbb{T}^n\) and a subsolution and a supersolution of \((P_\lambda)\), then \(v \leq w\) on \(\mathbb{T}^n\).

We remark that if \(H\) satisfies \((H1)\) and \((H3)\), then so does the Hamiltonian \(H_\varepsilon(x, p, u) := H(x, p, u) + \varepsilon u\), with \(\varepsilon > 0\) and that if our problem \((P_\lambda)\) has \(H_\varepsilon\) in place of \(H\), then the limit problem \((P_0)\) reads \(\varepsilon u + H[u] = 0\) in \(\mathbb{T}^n\) and has a unique solution due to Theorem 2. Thus, the asymptotic analysis for the vanishing discount problem in such cases is fairly easy.

With reference to [27], we outline the proof of the theorem above.

**Outline of proof** We choose a constant \(C > 0\) so that

\[
\max_{(x, i) \in \mathbb{T}^n \times I} |H_i(x, 0, 0)| \leq C.
\]

Note by Lemma 1 that

\[
H_i(x, 0, -\lambda^{-1} C \mathbf{1}) \leq H_i(x, 0, 0) \leq H_i(x, 0, \lambda^{-1} C \mathbf{1}) \quad \text{for } x \in \mathbb{T}^n.
\]

By using this, it is easily checked that the functions \(f(x) = \lambda^{-1} C \mathbf{1}\) and \(g(x) = -\lambda^{-1} C \mathbf{1}\) are a subsolution and subsolution of \((P_\lambda)\) and satisfy \(f \geq g\) on \(\mathbb{T}^n\).

Our assumption \((H3)\) implies the quasi-monotonicity of \(H\) in [27] as was shown in [27, Lemma 4.8]. By [27, Theorem 3.3], the function \(z = (z_i)_{i \in I} \in \mathbb{T}^n\) given by

\[
z_i(x) = \sup_{\xi \in \mathbb{T}^n} \{\xi_i(x) : g \leq \xi \leq f\}
\]

is a solution of \((P_\lambda)\), in the sense that \(z^n = (z^n_i)_{i \in I}\) and \(z^s = (z^s_i)_{i \in I}\), where each \(z^n_i\) and \(z^s_i\) are respectively the upper and lower semicontinuous envelope of \(z_i\), are respectively a subolution and a supersolution of \((P_\lambda)\).

By the definition of \(z\), it is easy to infer that for all \(i \in I\), \(z_i = z^n_i\) and the function \(z_i\) is upper semicontinuous on \(\mathbb{T}^n\). By \((H1)\) (the coercivity of \(H\)), we deduce that the function \(z\) is Lipschitz continuous on \(\mathbb{T}^n\).

To see that the comparison between \(v\) and \(w\), we apply [27, Theorem 4.7] to \(v\) and \(z\) as well as \(z\) and \(w\), to conclude that \(v \leq z\) and \(z \leq w\) on \(\mathbb{T}^n\), which implies that \(v \leq w\) on \(\mathbb{T}^n\). Here, the comparison theorem [27, Theorem 4.7] requires the regularity of \(H\) (see [27, (A.2)]), which can be reduced just to the continuity of \(H\) since \(z\) is Lipschitz continuous on \(\mathbb{T}^n\). This reduction of regularity of \(H\) is a standard observation and we leave it to the interested reader to adapt the proof of [27, Theorem 4.7] to this case. \(\square\)

Setting

\[
L_i(x, \xi, \eta) := \sup_{(p, u) \in \mathbb{R}^n \times \mathbb{R}^m} \{\xi \cdot p + \eta \cdot u - H_i(x, p, u)\}
\]

for \((x, i, \xi, \eta) \in \mathbb{T}^n \times I \times \mathbb{R}^n \times \mathbb{R}^m\),

by the convex duality we have

\[
H_i(x, p, u) = \sup_{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m} \{\xi \cdot p + \eta \cdot u - L_i(x, \xi, \eta)\}
\]

for \((x, i, p, u) \in \mathbb{T}^n \times I \times \mathbb{R}^n \times \mathbb{R}^m\).

We call \(L_i\) (resp., \((L_i)_{i \in I}\)) the Lagrangian of \(H_i\) (resp., the Lagrangian of \((H_i)_{i \in I}\)). Similarly, we call \(H_i\) (resp., \((H_i)_{i \in I}\)) the Hamiltonian of \(L_i\) (resp., the Hamiltonian of \((L_i)_{i \in I}\)).

It should be remarked that the functions \(L_i\) are lower semicontinuous on \(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m\).
We examine the Lagrangian in the linear coupling case (1). By the definition of $L_i$ and a simple manipulation, we deduce that

$$L_i(x, \xi, \eta) = \sup_{p \in \mathbb{R}^n} (\xi \cdot p - G_i(x, p)) + \sup_{u \in \mathbb{R}^m} (\eta \cdot u - (B(x)u)_i)$$

where $G_i(x, p) = \sum_{i=1}^{m} b_i(x)$.

$$L_i(x, \xi, \eta) = \sup_{p \in \mathbb{R}^n} (\xi \cdot p - G_i(x, p)) + 0_{(b_i(x))}(\eta),$$

where for $x \in T^n$, $b_i(x) := (b_{ij})_{j \in I} \in \mathbb{R}^m$ and, for any sets $X \subset Y$, $0_X$ denotes the indicator function of $X$ on $Y$ given by

$$0_X(y) = \begin{cases} 0 & \text{if } y \in X, \\ +\infty & \text{if } y \in Y \setminus X. \end{cases}$$

**Lemma 3** Assume (H1)–(H2).

(i) We have

$$L_i(x, \xi, \eta) \geq -H_i(x, 0, 0) \text{ for } (x, i, \xi, \eta) \in T^n \times I \times \mathbb{R}^n \times \mathbb{R}^m.$$  (6)

(ii) For any $A > 0$ there exists a constant $C_A$ such that

$$L_i(x, \xi, \eta) \geq A|\xi| - C_A \text{ for } (x, i, \xi, \eta) \in T^n \times I \times \mathbb{R}^n \times \mathbb{R}^m.$$  (7)

**Proof** Fix $i \in I$. We have

$$L_i(x, \xi, \eta) = \sup_{(p,u) \in \mathbb{R}^n \times \mathbb{R}^m} (\xi \cdot p + \eta \cdot u - H_i(x, p, u)) \geq -H_i(x, 0, 0),$$

and

$$L_i(x, \xi, \eta) = \sup_{(p,u) \in \mathbb{R}^n \times \mathbb{R}^m} (\xi \cdot p + \eta \cdot u - H_i(x, p, u)) \geq A|\xi| - H_i(x, A\xi/|\xi|, 0) \text{ if } \xi \neq 0.$$

Hence, setting

$$C_A = \sup_{(x,i,\xi) \in T^n \times I \times B^n_0} H_i(x, \xi, 0),$$

we obtain

$$L_i(x, \xi, \eta) \geq A|\xi| - C_A.$$  \qed

Lemma 3, (ii) asserts that the functions $L_i(x, \xi, \eta)$ have a superlinear growth as $|\xi| \to \infty$.

We give a characterization of the monotonicity (H3) of $(H_i)_{i \in I}$ through $(L_i)_{i \in I}$. For $k \in I$, we write

$$Y_k := \{(\eta_i)_{i \in I} \in \mathbb{R}^m : \eta_i \leq 0 \text{ if } i \neq k, \sum_{i \in I} \eta_i \geq 0\},$$

and $\text{dom } L_k = \{(x, \xi, \eta) : L_k(x, \xi, \eta) < \infty\}$.

**Proposition 4** Assume (H1)–(H2). Then $(H_i)_{i \in I}$ satisfies (H3) if and only if

$$\text{dom } L_i \subset T^n \times \mathbb{R}^n \times Y_i \text{ for all } i \in I.$$  \qed
One can check directly that, in the linear coupling case (1), if (H1)–(H3) hold, then the inclusion above is valid. Indeed, in this case, the coupling matrix $B(x) = (b_{ij}(x))$ satisfies (2), which implies that $(b_{ij}(x))_{i,j} \in Y_{i}$ for all $(x, i) \in T^{n} \times \mathbb{I}$, and, by (5), we conclude that $\text{dom} \ L_{i} \subset T^{n} \times \mathbb{R}^{n} \times Y_{i}$ for all $i \in \mathbb{I}$.

**Proof** We assume first that $(H_{1})_{i \in \mathbb{I}}$ satisfies (H3). Fix any $(x, k, \xi, \eta) \in T^{n} \times \mathbb{I} \times \mathbb{R}^{n} \times \mathbb{R}^{m}$ and suppose that $\eta = (\eta_{i})_{i \in \mathbb{I}} \notin Y_{k}$. We have either $\eta_{j} > 0$ for some $j \neq k$ or $\sum_{i \in \mathbb{I}} \eta_{i} < 0$.

Consider the case when $\eta_{j} > 0$ for some $j \neq k$. Let $t > 0$, and, by (ii) of Lemma 1, we have

$$H_{k}(x, p, 0) \geq H_{k}(x, p, te_{j}) \text{ for } p \in \mathbb{R}^{n},$$

and hence,

$$\xi \cdot p + \eta \cdot te_{j} - H_{k}(x, p, te_{j}) \geq \xi \cdot p + t \eta_{j} - H_{k}(x, p, 0) \text{ for } p \in \mathbb{R}^{n},$$

which implies that $L_{k}(x, \xi, \eta) = \infty$.

Consider next the case when $\sum_{i \in \mathbb{I}} \eta_{i} < 0$. For $t < 0$, we observe by (i) of Lemma 1 that $H_{k}(x, p, 0) \geq H_{k}(x, p, t \mathbf{1})$ for all $p \in \mathbb{R}^{n}$. Consequently,

$$\xi \cdot p + \eta \cdot t \mathbf{1} - H_{k}(x, p, t \mathbf{1}) \geq \xi \cdot p + t \sum_{i \in \mathbb{I}} \eta_{i} - H_{k}(x, p, 0) \text{ for } p \in \mathbb{R}^{n},$$

which shows that $L_{k}(x, \xi, \eta) = \infty$. We thus conclude that $\text{dom} \ L_{k} \subset T^{n} \times \mathbb{R}^{n} \times Y_{k}$.

Next, we assume that $\text{dom} \ L_{i} \subset T^{n} \times \mathbb{I} \times \mathbb{R}^{n} \times Y_{i}$ for all $i \in \mathbb{I}$. It is obvious that for any $(x, i, p, u) \in T^{n} \times \mathbb{I} \times \mathbb{R}^{n} \times \mathbb{R}^{m},$

$$H_{i}(x, p, u) = \sup_{(\xi, \eta) \in \mathbb{R}^{n} \times Y_{i}} [\xi \cdot p + \eta \cdot u - L_{i}(x, \xi, \eta)].$$

Fix any $(x, p) \in T^{n} \times \mathbb{R}^{n}$ and $u, v \in \mathbb{R}^{m}$. Assume that for some $k \in \mathbb{I},$

$$(u - v)_{k} = \max_{i \in \mathbb{I}} (u - v)_{i} \geq 0,$$

which can be stated as

$$(u - v)_{k} - (u - v)_{i} \geq 0 \text{ for } i \in \mathbb{I} \text{ and } (u - v)_{k} \geq 0.$$

Let $\eta = (\eta_{i})_{i \in \mathbb{I}} \in Y_{k}$. Multiplying the first inequality above by $\eta_{i}$, with $i \neq k$, we get

$$0 \geq \sum_{i \neq k} \eta_{i} [(u - v)_{k} - (u - v)_{i}] = \sum_{i \in \mathbb{I}} \eta_{i} [(u - v)_{k} - (u - v)_{i}]$$

$$= (u - v)_{k} \sum_{i \in \mathbb{I}} \eta_{i} - \sum_{i \in \mathbb{I}} \eta_{i} (u - v)_{i}.$$

Since $(u - v)_{k} \geq 0$ and $\sum_{i \in \mathbb{I}} \eta_{i} \geq 0$, we infer from the above that $\eta \cdot u \geq \eta \cdot v$. Thus, we have

$$H_{k}(x, p, u) = \sup_{(\xi, \eta) \in \mathbb{R}^{n} \times Y_{k}} [\xi \cdot p + \eta \cdot u - L_{k}(x, \xi, \eta)]$$

$$\geq \sup_{(\xi, \eta) \in \mathbb{R}^{n} \times Y_{k}} [\xi \cdot p + \eta \cdot v - L_{k}(x, \xi, \eta)] = H_{k}(x, p, v).$$

This shows that $(H_{i})_{i \in \mathbb{I}}$ is monotone, which completes the proof. \qed
3 Green–Poisson measures: in a regular case

In what follows, given a topological space $X$, $B(X)$ denotes the $\sigma$-algebra of Borel sets in $X$, and $\mathcal{M}(X)$ and $\mathcal{M}_+(X)$ denote, respectively, the spaces of Borel measures on $X$ having bounded variation and of nonnegative finite Borel measures on $X$. Also, $C_b(X)$ denotes the space of bounded continuous functions on $X$.

For any $\nu \in \mathcal{M}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m)$ and integrable function $\phi$ on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m$ with respect to $\nu$, we write

$$\langle \nu, \phi \rangle = \int_{\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m} \phi(x, \xi, \eta) \, d\nu(x, \xi, \eta).$$

Similarly, for any $\nu = (\nu_i)_{i \in I} \in \mathcal{M}(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m)^I$ and Borel function $\phi = (\phi_i)_{i \in I}$ on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m$, we write

$$\langle \nu, \phi \rangle = \sum_{i \in I} \langle \nu_i, \phi_i \rangle$$

if $\phi_i$ is integrable with respect to $\nu_i$ for every $i \in I$.

For $\lambda > 0$, we define the function $S^\lambda : \mathbb{R}^m \to \mathbb{R}$ by $S^\lambda(\eta) = \lambda + \sum_{i \in I} \eta_i$ for $\eta = (\eta_i)_{i \in I} \in \mathbb{R}^m$, and, we write $\mathbb{P}^\lambda$ for the set of all $\mu = (\mu_i)_{i \in I} \in \mathcal{M}_+(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m)^I$ such that

$$\langle \mu_i, |\xi| + |\eta| \rangle < \infty \quad \text{for all } i \in I \quad \text{and} \quad \langle \mu, S^\lambda 1 \rangle = 1, \quad (8)$$

where $|\xi| + |\eta|$ denotes the function $\mathbb{R}^n \times \mathbb{R}^m \ni (\xi, \eta) \mapsto |\xi| + |\eta| \in \mathbb{R}$. Note that $S^\lambda 1$ is the function $\mathbb{R}^m \ni \eta \mapsto S^\lambda(\eta) 1 \in \mathbb{R}^m$, which can be regarded as a function of $(\chi, \xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^{n+m}$. We write $\mathbb{P}^0$ for the set of all $\mu = (\mu_i)_{i \in I} \in \mathcal{M}_+(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m)^I$ such that

$$\langle \mu_i, |\xi| + |\eta| \rangle < \infty \quad \text{for all } i \in I \quad \text{and} \quad \langle \mu, 1 \rangle \leq 1. \quad (9)$$

Similarly to the standard definition [12,16] (see also [35,36]) of Mather measures, we introduce the closed measures as follows. We call any $\mu \in \mathbb{P}^0$ a closed measure (associated with $\lambda = 0$) provided it satisfies

$$\langle \mu, \xi \cdot D\psi + \eta \cdot \psi \rangle = 0 \quad \text{for all } \psi \in C^1(\mathbb{T}^n) \quad \text{if } \lambda = 0, \quad (10)$$

and denote by $\mathcal{C}(0)$ the set of all such closed measures associated with $\lambda = 0$. In (10) above, and henceforth, we use the notation that the functions $\mathbb{T}^n \times \mathbb{R}^n \ni (x, \xi) \mapsto (\xi \cdot D\psi_i(x))_{i \in I} \in \mathbb{R}^m$ and $\mathbb{T}^n \times \mathbb{R}^m \ni (x, \eta) \mapsto \eta \cdot \psi(x) \in \mathbb{R}$ are denoted by $\xi \cdot D\psi$ and $\eta \cdot \psi$, respectively. For $(z, k, \lambda) \in \mathbb{T}^n \times I \times (0, \infty)$, we call any $\mu \in \mathbb{P}^\lambda$ a closed measure (associated with $(z, k, \lambda)$) provided it satisfies

$$\langle \mu, \xi \cdot D\psi + \eta \cdot \psi \rangle = \psi_k(z) \quad \text{for all } \psi = (\psi_i)_{i \in I} \in C^1(\mathbb{T}^n) \quad \text{if } \lambda > 0. \quad (11)$$

The set of all closed measures associated with $(z, k, \lambda)$ is denoted by $\mathcal{C}(z, k, \lambda)$.

We now introduce the following working hypothesis.

\begin{equation}
\begin{cases}
\text{There exist nonempty, compact, convex sets } K_1 \subset \mathbb{R}^n \text{ and } K_2 \subset \mathbb{R}^m \text{ such that } K_1 \\
\text{is a neighborhood of the origin and such that for } i \in I, \\
L_i \in C(\mathbb{T}^n \times K_1 \times (K_2 \cap Y_i)), \\
H_i(x, p, u) = \sup_{(\xi, \eta) \in K_1 \times (K_2 \cap Y_i)} [\xi \cdot p + \eta \cdot u - L_i(x, \xi, \eta)] \quad \text{for } (x, p, u) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m.
\end{cases}
\end{equation}

\begin{equation}
\text{(H5)}
\end{equation}
Assuming (H5) in addition, we have

\[ L_i(x, \xi, \eta) = L_i(x, \xi, \eta) + 0_{K_1 \times (K_2 \cap Y_i)}(\xi, \eta) \quad \text{for} \ (x, \xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m. \]

We remark that under (H5), the functions \( H_i(x, p, u) \) grows at most linearly as \(|(p, u)| \to \infty \).

**Theorem 5** Assume (H1)–(H3) and (H5). Let \((z, k, \lambda) \in \mathbb{T}^n \times \mathbb{I} \times (0, \infty)\) and let \( v^\lambda = (v^\lambda_i)_{i \in \mathbb{I}} \in C(\mathbb{T}^n)_m\) be the solution of \((P_\lambda)\). Then there exists \( \mu \in \mathcal{C}(z, k, \lambda) \) such that

\[ v^\lambda_i(z) = \langle \mu, L \rangle = \min_{v \in \mathcal{C}(z, k, \lambda)} \langle v, L \rangle. \quad (12) \]

Remark that, thanks to Proposition 4, if \((H_i)_{i \in \mathbb{I}}\) satisfies (H1), (H2) and (H5), then it has the property (H3) as well.

The theorem above and Theorem 12, stated later, are generalizations of the previous results in [6,12,16,23,24,26,28,29,41].

Our proof of Theorem 5 is close to the ones in [28,29] in the technicality and is similar to the ones in [22,30] in the use of duality. It depends crucially on a minimax theorem, and, in the application of the minimax theorem, it is essential to make the set compact on which which we mean that \( \text{supp} \mu \) realizes all such measures \( \mu \). The condition (H5) realizes all such measures \( \mu \) to be supported on the compact set \( \prod_{i \in \mathbb{I}} \mathbb{T}^n \times K_1 \times (K_2 \cap Y_i) \), which we mean that \( \text{supp} \mu_i \subset \mathbb{T}^n \times K_1 \times (K_2 \cap Y_i) \) for all \( i \in \mathbb{I} \).

In the next section, we remove the restriction (H5) on \( L \) adopted in Theorem 5 by appealing the fact that the solution \( v^\lambda \) of \((P_\lambda)\) is Lipschitz continuous.

We call a Green–Poisson measure any measure \( \mu \in \mathcal{C}(z, k, \lambda) \) that is a minimizer of the most right hand of (12).

For the proof of Theorem 5, we assume henceforth (H1)–(H3) and (H5), and introduce the following notation. Set \( Z_i = K_1 \times (K_2 \cap Y_i) \) for \( i \in \mathbb{I} \) and note that for any \( i \in \mathbb{I} \), \( Z_i \) is a compact convex subset of \( \mathbb{R}^n \times \mathbb{R}^m \). Let \( \lambda > 0 \) and \( \mathcal{F}(\lambda) \) denote the set of all \((\phi, u) \in \prod_{i \in \mathbb{I}} C(\mathbb{T}^n \times Z_i) \times C(\mathbb{T}^n)_m\) such that for any \((x, i) \in \mathbb{T}^n \times \mathbb{I}, \phi_i(x, \xi, \eta)\) is convex in the variable \((\xi, \eta)\) on \( Z_i \) and such that \( u \) is a subsolution of \( \lambda u + H_{\phi}[u] = 0 \) in \( \mathbb{T}^n \), where \( H_{\phi} = (H_{\phi,i})_{i \in \mathbb{I}} \) is given by

\[ H_{\phi,i}(x, p, u) = \max_{(\xi, \eta) \in Z_i} (\xi \cdot p + \eta \cdot u - \phi_i(x, \xi, \eta)). \quad (13) \]

Here, we note by the compactness of \( Z_i \) that \( H_{\phi,i} \) is continuous on \( \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m \). Also, if we identify \( \phi_i \) with the function \( \tilde{\phi}_i \) on \( \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m \) given by

\[ \tilde{\phi}_i(x, \xi, \eta) = \begin{cases} \phi_i(x, \xi, \eta) & \text{if} \ (\xi, \eta) \in Z_i, \\ +\infty & \text{otherwise,} \end{cases} \quad (14) \]

then (13) reads

\[ H_{\phi,i}(x, p, u) = \max_{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m} (\xi \cdot p + \eta \cdot u - \tilde{\phi}_i(x, \xi, \eta)). \]

Notice that the functions \( \tilde{\phi}_i(x, \xi, \eta) \) defined by (14) are lower semicontinuous in the variable \((x, \xi, \eta)\) and convex in the variable \((\xi, \eta)\). By the convex duality, we see that \( \tilde{\phi}_i \) is the Lagrangian of \( H_{\phi,i} \). Arguing similarly to the proof of (ii) of Lemma 3, with \((H_{\phi,i}, \tilde{\phi}_i)\) in place of \((H_i, L_i)\), we see that \( H_{\phi} \) satisfies (H1). By Proposition 4, we easily see that \( H_{\phi} \) satisfies (H3). Moreover, it is clear that \( H_{\phi} \) satisfies (H5).
In what follows, for any \((\phi, u) \in \mathcal{F}(\lambda)\), we identify the function \(\phi_i\) on \(\mathbb{T}^n \times Z_i\) with \(\tilde{\phi}_i\) according to the situation. In particular, if \(v^\lambda \in C(\mathbb{T}^n)\) is the solution of \((P_\lambda)\), then we have \((L, v^\lambda) \in \mathcal{F}(\lambda)\). Note in addition that if \(\phi = 0 \in \prod_{i \in \mathbb{I}} C(\mathbb{T}^n \times Z_i)\), then \(\phi_i(x, \xi, \eta)\) is convex in \((\xi, \eta)\) for all \(i \in \mathbb{I}\) and \(H_\phi(\cdot, 0, 0) = 0\). Hence, \((0, 0) \in \mathcal{F}(\lambda)\).

For \((z, k, \lambda) \in \mathbb{T}^n \times \mathbb{I} \times (0, \infty)\), we set
\[
\mathcal{G}(z, k, \lambda) = \{\phi - u_k(z)S^\lambda(\eta)1: (\phi, u) \in \mathcal{F}(\lambda)\},
\]
\[
\mathbb{P}^\lambda(K_1, K_2) = \{\mu \in \mathbb{P}^\lambda: \text{supp } \mu_i \subset \mathbb{T}^n \times Z_i \text{ for all } i \in \mathbb{I}\},
\]
\[
\mathcal{G}'(z, k, \lambda) = \{\mu \in \mathbb{P}^\lambda(K_1, K_2): \langle \mu, f \rangle \geq 0 \text{ for all } f \in \mathcal{G}(z, k, \lambda)\}.
\]

We recall that, by definition, the support of measure \(\mu \in \mathbb{M}(X)\) for a topological space \(X\) is defined as the closed set
\[
\text{supp } \mu = X \setminus \bigcup\{U \subset X: |\mu|(U) = 0, U \text{ is open}\},
\]
where \(|\mu|\) denotes the total variation of the measure \(\mu\). Accordingly, if \(X\) has a countable basis of its topology, we have
\[
\mu(X \setminus \text{supp } \mu) = 0,
\]
and
\[
\int_X \phi(x) \mu(dx) = \int_{\text{supp } \mu} \phi(x) \mu(dx) \quad \text{for all } \phi \in C_b(X), \ \mu \in \mathbb{M}(X).
\]
In what follows, when \(Q \subset X\) is a closed subset of a topological space \(X\) with a countable basis, we may identify \(\mu \in \mathbb{M}(X)\) satisfying \(\text{supp } \mu \subset Q\) with its restriction \(\mu|_Q\) to \(Q\), defined by
\[
\mu|_Q(A) = \mu(A) \quad \text{for all } A \in \mathcal{B}(Q).
\]
Then we have
\[
\int_X \phi(x) \mu(dx) = \int_Q \phi(x) \mu|_Q(dx) \quad \text{for all } \phi \in C_b(X).
\] (15)

Recalling that \((L, v^\lambda) \in \mathcal{F}(\lambda)\), where \(v^\lambda\) is a solution of \((P_\lambda)\), we easily infer that
\[
v^\lambda_i(z) \leq \langle \mu, L \rangle \quad \text{for all } \mu \in \mathcal{G}'(z, k, \lambda).
\]

Since the functions \(L_i\) are bounded from below, for any \(\mu \in \mathbb{P}^\lambda\), the inequality \(\langle \mu, L \rangle < \infty\) always makes sense and, by \((H5)\), we have \(\langle \mu, L \rangle < \infty\) if and only if \(\text{supp } \mu_i \subset \mathbb{T}^n \times Z_i\) for all \(i \in \mathbb{I}\). Hence, we have
\[
\mathbb{P}^\lambda(K_1, K_2) = \{\mu \in \mathbb{P}^\lambda: \langle \mu, L \rangle < \infty\}.\] (16)

**Lemma 6** The set \(\mathcal{F}(\lambda)\) is a convex cone in \(\prod_{i \in \mathbb{I}} C(\mathbb{T}^n \times Z_i) \times C(\mathbb{T}^n)^m\) with vertex at the origin.

**Proof** Recall [3, Remark 2.5] that for any \(u \in \text{Lip}(\mathbb{T}^n)^m\), \(u\) is a subsolution of
\[
\lambda u + H[u] = 0 \quad \text{in } \mathbb{T}^n
\]
if and only if for all \(i \in \mathbb{I}\),
\[
\lambda u_i(x) + H_i(x, Du_i(x), u(x)) \leq 0 \quad \text{a.e. in } \mathbb{T}^n,
\]
and by the coercivity (H1) that for any \((\phi, u) \in \mathcal{F}(\lambda)\), we have \(u \in \text{Lip}(\mathbb{T}^n)^m\).

Fix \((\phi, u), (\psi, v) \in \mathcal{F}(\lambda)\) and \(t, s \in [0, \infty)\). Fix \(i \in \mathbb{I}\) and observe that
\[
\begin{align*}
\lambda u_i(x) + H_{\phi, i}(x, Du_i(x), u(x)) &\leq 0 \text{ a.e. in } \mathbb{T}^n, \\
\lambda v_i(x) + H_{\psi, i}(x, Dv_i(x), v(x)) &\leq 0 \text{ a.e. in } \mathbb{T}^n,
\end{align*}
\]
which imply that there is a set \(N \subset \mathbb{T}^n\) of Lebesgue measure zero such that
\[
\begin{align*}
\lambda u_i(x) + \xi \cdot Du_i(x) + \eta \cdot u(x) &\leq \phi_i(x, \xi, \eta) \text{ for all } (x, \xi, \eta) \in (\mathbb{T}^n \setminus N) \times Z_i, \\
\lambda v_i(x) + \xi \cdot Dv_i(x) + \eta \cdot u(x) &\leq \psi_i(x, \xi, \eta) \text{ for all } (x, \xi, \eta) \in (\mathbb{T}^n \setminus N) \times Z_i.
\end{align*}
\]
Multiplying the first and second by \(t\) and \(s\), respectively, adding the resulting inequalities and setting \(w = tu + sv\), we obtain
\[
\lambda w_i(x) + \xi \cdot Dw_i(x) + \eta \cdot w(x) \leq (t \phi_i + s \psi_i)(x, \xi, \eta) \text{ for all } (x, \xi, \eta) \in (\mathbb{T}^n \setminus N) \times Z_i,
\]
which implies that \(t(\phi, u) + s(\psi, v) \in \mathcal{F}(\lambda)\). \(\square\)

**Lemma 7** Let \((z, k, \lambda) \in \mathbb{T}^n \times \mathbb{I} \times (0, \infty)\) and \(\mu = (\mu_i)_{i \in \mathbb{I}} \in \mathbb{P}^{K_1}(K_1, K_2)\). Then, we have \(\mu \in \mathcal{C}(z, k, \lambda)\) if and only if \(\mu \in \mathcal{G}^i(z, k, \lambda)\).

**Proof** Assume first that \(\mu \in \mathcal{C}(z, k, \lambda)\). Fix any \((\phi, u) \in \mathcal{F}(\lambda)\) and recall that, in the viscosity sense,
\[
\lambda u + H_\phi[u] \leq 0 \text{ in } \mathbb{T}^n.
\]
Thanks to the coercivity property (H1) of \(H_\phi, u\) is Lipschitz continuous on \(\mathbb{T}^n\). In view of the continuity of \(H_\phi\) and the convex property (H2) of \(H_\phi\), mollifying \(u\), we may choose, for each \(\varepsilon > 0\), a function \(u^\varepsilon \in C^1(\mathbb{T}^n)^m\) such that \(\lambda u^\varepsilon + H_\phi[u^\varepsilon] \leq \varepsilon 1\) in \(\mathbb{T}^n\) and \(\|u - u^\varepsilon\|_\infty < \varepsilon\). Hence, we have
\[
\lambda u^\varepsilon + \xi \cdot Du^\varepsilon(x) + \eta \cdot u^\varepsilon(x) 1 \leq \phi(x, \xi, \eta) + \varepsilon 1.
\]
Using (11) and integrating the inequality above with respect to \(\mu\), we obtain
\[
u^\varepsilon_k(z) = \langle \mu, \xi \cdot Du^\varepsilon + \eta \cdot u^\varepsilon 1 + \lambda u^\varepsilon \rangle \leq \langle \mu, \phi + \varepsilon 1 \rangle
\]
and, after sending \(\varepsilon \to 0\),
\[
0 \leq \langle \mu, \phi \rangle - u_k(z) = \langle \mu, \phi - u_k(z) S^{\lambda} 1 \rangle,
\]
which implies, together with the assumption that \(\text{supp} \mu_i \subset \mathbb{T}^n \times Z_i\) for all \(i \in \mathbb{I}\), that \(\mu \in \mathcal{G}^i(z, k, \lambda)\).

Next, we assume that \(\mu \in \mathcal{G}^i(z, k, \lambda)\). Fix any \(\psi \in C^1(\mathbb{T}^n)^m\), set \(\phi = \xi \cdot D\psi + \eta \cdot \psi 1 + \lambda \psi\), which is a function on \(\mathbb{T}^n \times \mathbb{R}^{n+m}\), and observe that \(\lambda \psi + H_\phi[\psi] \leq 0\) in \(\mathbb{T}^n\), i.e., \((\phi, \psi) \in \mathcal{F}(\lambda)\). Hence, by the definition of \(\mathcal{G}^i(z, k, \lambda)\), we have
\[
0 \leq \langle \mu, \phi - \psi_k(z) S^{\lambda} 1 \rangle = \langle \mu, \xi \cdot D\psi + \eta \cdot \psi 1 + \lambda \psi \rangle - \psi_k(z).
\]
The inequality above holds also for \(-\psi\) in place of \(\psi\), which reads
\[
0 \geq \langle \mu, \xi \cdot D\psi + \eta \cdot \psi 1 + \lambda \psi \rangle - \psi_k(z).
\]
Thus, we have
\[
(\mu, \xi \cdot D\psi + \eta \cdot \psi 1 + \lambda \psi) = \psi_k(z),
\]
and conclude that \(\mu \in \mathcal{C}(z, k, \lambda)\). \(\square\)
Lemma 8 Let $i \in \mathbb{I}, \lambda > 0$, and $(\bar{x}, \bar{\xi}, \bar{\eta}) \in \mathbb{T}^n \times Z_i$, and let $\delta_{(\bar{x}, \bar{\xi}, \bar{\eta})}$ denote the Dirac measure at $(\bar{x}, \bar{\xi}, \bar{\eta})$. Then, $(S^\lambda)^{-1} \delta_{(\bar{x}, \bar{\xi}, \bar{\eta})} e_i$ is a member of $\mathbb{P}^\lambda(K_1, K_2)$.

Proof Note first that $S^\lambda(\eta) \geq \lambda > 0$ for all $\eta \in K_2 \cap Y_i$. It follows immediately that

$$(S^\lambda)^{-1} \delta_{(\bar{x}, \bar{\xi}, \bar{\eta})} e_i \leq (|\bar{\xi}| + |\bar{\eta}|) 1 = (S^\lambda(\eta))^{-1} (|\bar{\xi}| + |\bar{\eta}|) < \infty,$$

and $$(S^\lambda)^{-1} \delta_{(\bar{x}, \bar{\xi}, \bar{\eta})} e_i, S^\lambda 1 = 1.$$ Thus, we see that $(S^\lambda)^{-1} \delta_{(\bar{x}, \bar{\xi}, \bar{\eta})} e_i \in \mathbb{P}^\lambda(K_1, K_2). \quad \square$

For the reader’s convenience, we state a minimax theorem [44, Corollary 2].

Proposition 9 Let $K$ and $Y$ be convex subsets of vector spaces. Assume in addition that $K$ is a compact space. Let $f : K \times Y \rightarrow \mathbb{R}$ be a function satisfying:

(i) For each $y \in Y$, the function: $x \mapsto f(x, y)$ is lower semicontinuous and convex on $K$.

(ii) For each $x \in K$, the function: $y \mapsto f(x, y)$ is concave on $Y$.

Then

$$\sup_{y \in Y} \min_{x \in K} f(x, y) = \min_{x \in K} \sup_{y \in Y} f(x, y).$$

We remark that in [44, Corollary 2], it is assumed that $K$ is a convex compact subset of a topological vector space $X$, but in its proofs, the compatibility of the linear structure and the topological structure (i.e., the continuity of addition and scalar multiplication) of $X$ is not used and the proposition above is valid.

In the application below of Proposition 9, we take $K$ to be a bounded subset of the Banach space $\mathbb{M}(\Sigma)$ with the total variation norm, where $\Sigma$ is a compact subset of $\mathbb{T}^n \times \mathbb{R}^{n+m}$.

Let $\Sigma$ be a compact subset of $\mathbb{T}^n \times \mathbb{R}^{n+m}$. By the Riesz representation theorem, for each $F \in C(\Sigma)^*$, there exists a unique (regular) Borel measure $\mu$ on $\Sigma$ such that for all $\phi \in C(\Sigma)$,

$$F(\phi) = \int_{s \in \Sigma} \phi(s) \mu(ds) = \langle \mu, \phi \rangle.$$

The mapping $\iota_{\Sigma} : F \in C(\Sigma)^* \mapsto \mu \in \mathbb{M}(\Sigma)$, given above, is an isomorphism between two Banach spaces. Through the mapping $\iota_{\Sigma} : C(\Sigma)^* \rightarrow \mathbb{M}(\Sigma)$, the weak star convergence corresponds to the weak convergence of measures.

Thanks to the Banach–Alaoglu theorem, we know that any closed ball $B$ (in the strong topology) of $C(\Sigma)^*$, equipped with the weak star topology, is a compact metrizable space. Moreover, if $B$ is such a ball and $N$ is a closed subset (in the weak star topology) of $B$, then $N$ is a compact subset of $B$. These say that if $D$ is a closed ball (in the total variation norm) of $\mathbb{M}(\Sigma)$, then $D$ is a compact metrizable space with the topology of the weak convergence of measures and so is any $K \subset D$ that is sequentially closed in the topology of the weak convergence of measures.

The following lemma is a simple consequence of the discussion above.

Lemma 10 Let $a > 0$ and $\Sigma = (\Sigma_i)_{i \in \mathbb{I}}$ be a collection of compact subsets $\Sigma_i$ of $\mathbb{T}^n \times \mathbb{R}^{n+m}$. Let $\mathbb{P}(\Sigma, a)$ denote the collection of $\mu = (\mu_i) \in \mathbb{M}_+(\mathbb{T}^n \times \mathbb{R}^{n+m})^m$ such that $\supp \mu_i \subset \Sigma_i$ for all $i \in \mathbb{I}$ and such that $\langle \mu, 1 \rangle \leq a$. Then, $\mathbb{P}(\Sigma, a)$ is a compact metrizable space with the topology of weak convergence of measures.

It is to be noticed that in the lemma above, $\mathbb{P}(\Sigma, a)$ is sequentially compact.
Proof In view of (15), it is clear that any sequence of measures \( \mu^q = (\mu^q_i) \in \mathbb{P}(\Sigma, a) \) on \( \mathbb{T}^n \times \mathbb{R}^{n+m} \), with \( q \in \mathbb{N} \), converges to \( \mu = (\mu_i) \) weakly in the sense of measures (i.e., in the topology of weak convergence of measures) if and only if, for each \( i \in I \), the sequence of measures \( \mu^q_i|_{\Sigma_i} \) converges to \( \mu_i|_{\Sigma_i} \) and \( \mu_i|_{\mathbb{T}^n \times \mathbb{R}^{n+m}\setminus \Sigma_i} = 0 \). Note also that \( \mu_i|_{\mathbb{T}^n \times \mathbb{R}^{n+m}\setminus \Sigma_i} = 0 \) if and only if \( \supp \mu_i \subset \Sigma_i \).

Thus, we need only to prove that the set \( \widetilde{\mathbb{P}} := \{ \mu|_{\Sigma} : \mu \in \mathbb{P}(\Sigma, a) \} \), where \( \mu|_{\Sigma} := (\mu_i|_{\Sigma_i})_{i \in I} \), is a compact metrizable space with the topology of weak convergence of measures.

As noted prior to the lemma, it is enough to prove that \( \mathbb{P} \) is a subset of a closed ball of \( \prod_{i \in I} M(\Sigma_i) \) in the norm topology and it is closed in the weak convergence of measures.

Recall that the Banach space \( \prod_{i \in I} M(\Sigma_i) \) has the total variation norm \( \|v\| = \sum_{i \in I} |v_i|\Sigma_i \) for \( v = (v_i) \). If \( \mu = (\mu_i) \in M_+(\mathbb{T}^n \times \mathbb{R}^{n+m}) \) and \( \supp \mu_i \subset \Sigma_i \) for all \( i \in I \), then

\[
\langle \mu, 1 \rangle = \sum_{i \in I} \mu_i(\Sigma_i) = \| \mu|_{\Sigma} \|.
\]

This shows that the closed ball \( B := \{ v = (v_i) \in \prod_{i \in I} M(\Sigma_i) : \|v\| \leq a \} \) contains \( \widetilde{\mathbb{P}} \).

It remains to show that \( \mathbb{P} \) is closed in the weak convergence of measures. For this, as noted at the beginning, we need only to prove that \( \mathbb{P}(\Sigma, a) \) is closed in the weak convergence of measures.

Now, let \( \mu^j = (\mu^j_i) \in \mathbb{P}(\Sigma, a) \) for all \( j \in \mathbb{N} \) and assume that the sequence of \( \mu^j \) converges weakly in the sense of measures to \( \mu = (\mu_i) \in M(\mathbb{T}^n \times \mathbb{R}^{n+m}) \). We already know that \( \supp \mu_i \subset \Sigma_i \) for all \( i \in I \) and that the sequence of \( \mu^j_i|\Sigma \) converges to \( \mu_i|\Sigma \in B \) weakly in the sense of measures. It follows that

\[
a \geq \| \mu|_{\Sigma} \| = \sum_{i \in I} |\mu_i|\Sigma_i,
\]

and, moreover, that for any \( i \in I \) and nonnegative function \( \psi \in C_b(\mathbb{T}^n \times \mathbb{R}^{n+m}) \),

\[
\langle \mu_i, \psi \rangle = \lim_{j \to \infty} \left( \mu^j_i, \psi \right) \geq 0.
\]

From these, we see that \( \mu \in M_+(\mathbb{T} \times \mathbb{R}^{n+m}) \) and \( \langle \mu, 1 \rangle \leq a \), and conclude that \( \mu \in \mathbb{P}(\Sigma, a) \).

\( \Box \)

**Lemma 11** Let \( \lambda > 0 \) and \( \Sigma = (\Sigma_i)_{i \in I} \) be a collection of compact subsets \( \Sigma_i \) of \( \mathbb{T}^n \times \mathbb{R}^{n} \times (\mathbb{R}^{m} \cap Y_i) \). Let \( \mathbb{P}^\lambda(\Sigma) \) denote the collection of all \( \mu = (\mu_i) \in \mathbb{P}^\lambda \) such that \( \supp \mu_i \subset \Sigma_i \) for every \( i \in I \). Then, \( \mathbb{P}^\lambda(\Sigma) \) is a compact metrizable space with the topology of weak convergence of measures.

**Proof** Let \( \mathbb{P}(\Sigma, a) \) denote the set defined in Lemma 10 for \( a > 0 \). For \( \mu = (\mu_i) \in \mathbb{P}^\lambda(\Sigma) \), since \( \supp \mu_i \) are compact for all \( i \in I \), it is clear that \( \langle \mu, (|\xi| + |\eta|)1 \rangle < \infty \). Since \( S^\lambda(\eta) \geq \lambda \) for all \( \eta \in \bigcup_{i \in I} Y_i \), if \( \mu = (\mu_i) \in \mathbb{P}^\lambda(\Sigma) \), then

\[
1 = \langle \mu, S^\lambda 1 \rangle = \sum_{i \in I} \int_{\Sigma_i} S^\lambda(\eta) \mu_i(d\xi d\eta) \geq \lambda \sum_{i \in I} \mu_i(\Sigma_i) = \lambda \langle \mu, 1 \rangle,
\]

which implies that \( \mathbb{P}^\lambda(\Sigma) \subset \mathbb{P}(\Sigma, 1/\lambda) \).

It remains to prove that \( \mathbb{P}^\lambda(\Sigma) \) is a closed subset of \( \mathbb{P}(\Sigma, 1/\lambda) \). Let \( \mu^j = (\mu^j_i) \in \mathbb{P}^\lambda(\Sigma) \) for \( j \in \mathbb{N} \). Assume that the sequence \( (\mu^j) \) converges weakly in the sense of measures to \( \mu = (\mu_i) \in \mathbb{P}(\Sigma, 1/\lambda) \).

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For the proof of the lemma, we need only to show that $\langle \mu, S^k 1 \rangle = 1$. We easily check that
\[
\langle \mu, S^k 1 \rangle = \sum_{i \in I} \int_{\Sigma_i} S^k(\eta) \mu_i(d\xi d\eta) = \lim_{j \to \infty} \sum_{i \in I} \int_{\Sigma_i} S^k(\eta) \mu_i^j(d\xi d\eta)
\]
\[
= \lim_{j \to \infty} \big\langle \mu^j, S^k 1 \big\rangle = 1,
\]
which finishes the proof. \hfill \Box

It is a consequence of the lemma above that $P^\lambda(K_1, K_2)$ is a compact metrizable space with the topology of weak convergence of measures.

**Proof of Theorem 5** In view of (16) and Lemma 7, it is enough to prove that
\[
v_k^\lambda(z) = \min_{\mu \in G'(z,k,\lambda)} \langle \mu, L \rangle.
\]
(17)

We intend to show that
\[
\sup_{(\phi, u) \in F(\lambda)} \inf_{v \in P^\lambda(K_1, K_2)} \big\{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^k 1 \big\} = 0.
\]
(18)

We postpone the proof of (18) and, assuming temporarily that (18) is valid, we prove that (17) holds.

To this end, we see easily that $P^\lambda(K_1, K_2)$ is a convex subset of a vector space $M(\mathbb{T}^n \times \mathbb{R}^{n+m})$ and that, by Lemma 6, $F(\lambda)$ is a convex subset of $\prod_{i \in I} C(\mathbb{T}^n \times Z_i) \times C(\mathbb{T}^m)^m$. Observe as well that the functional:
\[
P^\lambda(K_1, K_2) \ni v \mapsto \{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^k 1 \} \in \mathbb{R}
\]
is convex and continuous, in the topology of weak convergence of measures for any $(\phi, u) \in F(\lambda)$, and the functional:
\[
F(\lambda) \ni (\phi, u) \mapsto \{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^k 1 \} \in \mathbb{R}
\]
is concave, as well as continuous, for any $v \in P^\lambda(K_1, K_2)$.

By Lemma 11, the set $P^\lambda(K_1, K_2)$ is a compact space with the topology of weak convergence of measures. Hence, we may apply the minimax theorem (Proposition 9 or [43,44]), to deduce from (18) that
\[
0 = \sup_{(\phi, u) \in F(\lambda)} \min_{v \in P^\lambda(K_1, K_2)} \{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^k 1 \}
\]
\[
= \min_{v \in P^\lambda(K_1, K_2)} \sup_{(\phi, u) \in F(\lambda)} \{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^k 1 \}.
\]
(19)

Observe by using the cone property of $F(\lambda)$ that
\[
\sup_{(\phi, u) \in F(\lambda)} \{ v, u_k(z)S^k 1 - \phi \} = \begin{cases} 
0 & \text{if } v \in G'(z,k,\lambda), \\
\infty & \text{if } v \in P^\lambda(K_1, K_2) \setminus G'(z,k,\lambda).
\end{cases}
\]

This and (19) yield
\[
0 = \min_{v \in P^\lambda(K_1, K_2)} \sup_{(\phi, u) \in F(\lambda)} \{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^k 1 \}
\]
\[
= \min_{v \in G'(z,k,\lambda)} \{ v, L - v_k^\lambda(z)S^k 1 \} = \min_{v \in G'(z,k,\lambda)} \{ v, L - v_k^\lambda(z) \},
\]
which proves (17).

It remains to show (18). Note that

\[
\sup_{(\phi, u) \in \mathcal{F}(\lambda)} \inf_{v \in \mathbb{P}^\lambda(K_1, K_2)} \left\{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^\lambda 1 \right\}
\]

\[
\geq \inf_{v \in \mathbb{P}^\lambda(K_1, K_2)} \left\{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^\lambda 1 \right\}|_{(\phi, u) = (L, v^\lambda)} = 0.
\]

Hence, we only need to show that

\[
\sup_{(\phi, u) \in \mathcal{F}(\lambda)} \inf_{v \in \mathbb{P}^\lambda(K_1, K_2)} \left\{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^\lambda 1 \right\} \leq 0. \tag{20}
\]

For this, we argue by contradiction and thus suppose that (20) does not hold. Accordingly, we have

\[
\sup_{(\phi, u) \in \mathcal{F}(\lambda)} \inf_{v \in \mathbb{P}^\lambda(K_1, K_2)} \left\{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^\lambda 1 \right\} > \varepsilon
\]

for some \( \varepsilon > 0 \). We may select \((\phi, u) \in \mathcal{F}(\lambda)\) so that

\[
\inf_{v \in \mathbb{P}^\lambda(K_1, K_2)} \left\{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^\lambda 1 \right\} > \varepsilon.
\]

That is, for any \(v \in \mathbb{P}^\lambda(K_1, K_2)\), we have

\[
\left\{ v, L - \phi + (u_k(z) - v_k^\lambda(z))S^\lambda 1 \right\} > \varepsilon = \left\{ v, \varepsilon S^\lambda 1 \right\}.
\]

According to Lemma 8, the measure \((S^\lambda)^{-1}\delta(x, \xi, \eta)e_i\) is in \(\mathbb{P}^\lambda(K_1, K_2)\) for every \((x, \xi, \eta) \in \mathbb{T}^n \times Z_i\) and \(i \in \mathbb{I}\). Plugging all such \(v = (S^\lambda)^{-1}\delta(x, \xi, \eta)e_i \in \mathbb{P}^\lambda(K_1, K_2)\) into the above, we find that

\[
(L_i - \phi_i)(x, \xi, \eta) + (u_k(z) - v_k^\lambda(z) - \varepsilon)S^\lambda(\eta) > 0 \quad \text{for all} \quad (x, \xi, \eta) \in \mathbb{T}^n \times Z_i, \quad i \in \mathbb{I}.
\]

Hence, setting \(w := u - (u_k(z) - v_k^\lambda(z) - \varepsilon)1\), we have

\[
\lambda w_i(x) + \xi \cdot p + \eta \cdot w(x) - L_i(x, \xi, \eta)
\]

\[
= \lambda u_i(x) + \xi \cdot p + \eta \cdot u(x) - (u_k(z) - v_k^\lambda(z) - \varepsilon)S^\lambda(\eta) - L_i(x, \xi, \eta)
\]

\[
< \lambda u_i(x) + \xi \cdot p + \eta \cdot u(x) - \phi_i(x, \xi, \eta)
\]

for all \((x, p, \xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n \times Z_i\) and \(i \in \mathbb{I}\). This ensures that \(w\) is a subsolution of

\[
\lambda w + H[w] = 0 \quad \text{in} \quad \mathbb{T}^n.
\]

By Theorem 2, we get \(u(x) - (u_k(z) - v_k^\lambda(z) - \varepsilon)1 \leq v^\lambda(x)\) for all \(x \in \mathbb{T}^n\). The \(k\)th component of the last inequality, evaluated at \(x = z\), yields an obvious contradiction, which proves that (20) holds.

\[
\square
\]

4 Green–Poisson measures: the general case

We now remove the hypothesis (H5) in Theorem 5 and establish the following theorem.

Theorem 12 Assume (H1)–(H3). Let \((z, k, \lambda) \in \mathbb{T}^n \times \mathbb{I} \times (0, \infty)\) and \(v^\lambda \in C(\mathbb{T}^n)^m\) be the solution of \((P_\lambda)\). Then there exists \(\mu \in \mathcal{C}(z, k, \lambda)\) such that

\[
v_k^\lambda(z) = \langle \mu, L \rangle = \min_{v \in \mathcal{C}(z, k, \lambda)} \langle v, L \rangle. \tag{21}
\]
The theorem above guarantees the existence of a Green–Poisson measure associated with any \((z, k, \lambda) \in \mathbb{T}^n \times \mathbb{I} \times (0, \infty)\).

In what follows we fix \((z, k, \lambda) \in \mathbb{T}^n \times \mathbb{I} \times (0, \infty)\). According to Theorem 2, the unique solution of \((P_\lambda)\) is Lipschitz continuous on \(\mathbb{T}^n\). With this in mind, we fix a constant \(C > 0\) and consider the condition that
\[
|v^\lambda(x)| + |Dv^\lambda(x)| \leq C \quad \text{a.e. } x \in \mathbb{T}^n. \tag{22}
\]
We choose a function \(h \in C^1(\mathbb{R}^n \times \mathbb{R}^m)\) so that
\[
\begin{cases}
  h \text{ is nonnegative and convex on } \mathbb{R}^n \times \mathbb{R}^m, \\
  h(p, u) = 0 \text{ if and only if } |p| + |u| \leq C, \\
  \lim_{|p| + |u| \to \infty} (|p| + |u|)^{-1} h(p, u) = \infty.
\end{cases}
\tag{23}
\]
Also, we choose a compact convex set \(Q \subset \mathbb{R}^{n+m}\) such that for all \((x, i, p, u) \in \mathbb{T}^n \times \mathbb{I} \times \mathbb{R}^{n+m}\),
\[
\partial_{(p, u)}H_i(x, p, u) \subset Q \text{ if } |p| + |u| \leq C, \tag{24}
\]
where \(\partial_{(p, u)}H_i\) denotes the subdifferential of the convex function: \((p, u) \mapsto H_i(x, p, u)\).

**Theorem 13** Assume (H1)–(H3). Let \(v^\lambda\) be the solution of \((P_\lambda)\) and assume that (22) is satisfied for some constant \(C > 0\). Let \(Q\) be a compact convex subset of \(\mathbb{R}^{n+m}\) such that (24) holds. Assume that there exists \(\mu \in \mathcal{E}(z, k, \lambda)\) such that
\[
v^\lambda_k(z) = \langle \mu, L \rangle.
\]
Then
\[
\text{supp } \mu_i \subset \mathbb{T}^n \times [Q \cap (\mathbb{R}^n \times Y_i)] \text{ for } i \in \mathbb{I}.
\]

We recall some basic properties related to the subdifferentials of \(H\) and \(L\).

**Lemma 14** Assume (H2). Let \((x, i) \in \mathbb{T}^n \times \mathbb{I}\).

(i) We have
\[
\partial_{(p, u)}H_i(x, p, u) \neq \emptyset \text{ for } (p, u) \in \mathbb{R}^n \times \mathbb{R}^m.
\]

(ii) Let \((p, u), (\xi, \eta) \in \mathbb{R}^{n+m}\). The following three statements are equivalent each other.

(a) \((\xi, \eta) \in \partial_{(p, u)}H_i(x, p, u)\).

(b) \((p, u) \in \partial_{(\xi, \eta)}L_i(x, \xi, \eta)\).

(c) \(H_i(x, p, u) + L_i(x, \xi, \eta) = \xi \cdot p + \eta \cdot u\).

**Proof** (i) Since \((p, u) \mapsto H_i(x, p, u)\) is continuous and convex in \(\mathbb{R}^{n+m}\), it is locally Lipschitz continuous (see [25, Theorem B.3]) and hence almost everywhere differentiable (see [25, Theorem F.1]) in \(\mathbb{R}^{n+m}\). Fix any \((p, u) \in \mathbb{R}^{n+m}\) and choose a sequence of points \((p^k, u^k) \in \mathbb{R}^{n+m}\) converging to \((p, u)\) such that \((p, u) \mapsto H_i(x, p, u)\) is differentiable at \((p^k, u^k)\) for all \(k \in \mathbb{N}\). Set \((\xi^k, \eta^k) = D_{(p, u)}H_i(x, p^k, u^k)\) for \(k \in \mathbb{N}\). The local Lipschitz continuity of \(H_i(x, \cdot, \cdot)\) allows us to assume that \((\xi^k, \eta^k)\) is bounded and, moreover, convergent to some \((\xi^0, \eta^0) \in \mathbb{R}^{n+m}\) after passing to a subsequence. Since
\[
H_i(x, p^k + q, u^k + r) \geq H_i(x, p^k, u^k) + \xi^k \cdot q + \eta^k \cdot r \quad \text{for } (q, r) \in \mathbb{R}^{n+m},
\]

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sending $k \to \infty$ yields

$$H_i(x, p + q, u + r) \geq H_i(x, p, u) + \xi^0 \cdot q + \eta^0 \cdot r \quad \text{for} \quad (q, r) \in \mathbb{R}^{n+m},$$

which shows that $(\xi^0, \eta^0) \in \partial_{(p,u)} H_i(x, p, u)$ and $\partial_{(p,u)} H_i(x, p, u) \neq \emptyset$.

We here skip to prove (ii) and leave it to the reader to consult [42, Theorem 23.5] or [25, Theorem B.2].

**Lemma 15** Assume (H1)–(H3). Let $\lambda \in [0, \infty)$, and let $u \in \text{Lip}(\mathbb{T}^n)$ be a subsolution of (P$\lambda$). If $\lambda > 0$, then $u_k(z) \leq \langle \mu, L \rangle$ for all $(z, k) \in \mathbb{T}^n \times I$ and $\mu \in \mathcal{C}(z, k, \lambda)$, and, if $\lambda = 0$, then $0 \leq \langle \mu, L \rangle$ for all $\mu \in \mathcal{C}(0)$.

We remark that, in the above, $\langle \mu, L \rangle$ can be $+\infty$.

The proof below is almost identical to the first part of the proof of Lemma 7.

**Proof** In view of the continuity and convex property of $H$, mollifying $u$, we may choose, for each $\varepsilon > 0$, a function $u^\varepsilon \in C^1(\mathbb{T}^n)^m$ such that $\lambda u^\varepsilon + H[u^\varepsilon] \leq \varepsilon 1$ in $\mathbb{T}^n$ and $\|u - u^\varepsilon\|_\infty \leq \varepsilon$. Hence, we have

$$\lambda u^\varepsilon + \xi \cdot Du^\varepsilon(x) + \eta \cdot u^\varepsilon(x) \leq L(x, \xi, \eta) + \varepsilon 1.$$

When $\lambda > 0$, fixing $(z, k) \in \mathbb{T}^n \times I$, recalling (11), and integrating the inequality above with respect to $\mu \in \mathcal{C}(z, k, \lambda)$, we obtain

$$u_k^\varepsilon(z) = \langle \mu, \xi \cdot Du^\varepsilon + \eta \cdot u^\varepsilon \rangle + \lambda u^\varepsilon \leq \langle \mu, L + \varepsilon 1 \rangle.$$

Similarly, if $\lambda = 0$, then we get for any $\mu \in \mathcal{C}(0)$,

$$0 = \langle \mu, \xi \cdot Du^\varepsilon + \eta \cdot u^\varepsilon \rangle \leq \langle \mu, L + \varepsilon 1 \rangle.$$

Taking the limit as $\varepsilon \to 0$, we finish the proof. \hfill \square

In the proof below, an essential step is to construct a new Hamiltonian, say, $\tilde{H}$ satisfying (H1)–(H3) such that $v^h$ is a solution of (P$\lambda$), with $H$ replaced by $\tilde{H}$, and such that, if $|p| + |u| > C$, then $\tilde{H}(x, p, u) > H(x, p, u)$. Notice that the function $H(x, p, u) + h(p, u)$, with $h$ satisfying (23), on $\mathbb{T}^n \times \mathbb{R}^{n+m}$ does not satisfy the monotonicity (H3).

**Proof of Theorem 13** Let $h \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$ be a function having the properties in (23). We set $G^h(x, p, u) = H(x, p, u) + h(p, u)$ for $(x, p, u) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m$. Let $K^h = (K_i^h)_{i \in I}$ be the Lagrangian of $G^h$, and, since $G^h$ grows superlinearly as $|p| + |u| \to \infty$, we see that $K^h \in C(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m)^m$. Note that

$$G^h \geq H \quad \text{and} \quad K^h \leq L \quad \text{on} \quad \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m.$$

According to Proposition 4, $G^h$ does not satisfy (H3), and we need to modify $G^h$, to remove the drawback. We note by Proposition 4 that

$$L(x, \xi, \eta) + 0_Y(\eta) = L(x, \xi, \eta) \quad \text{for} \quad (x, \xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m,$$

where $0_Y := (0_Y(i))_{i \in I}$. Hence, we have

$$L^h(x, \xi, \eta) := K^h(x, \xi, \eta) + 0_Y(\eta) \leq L(x, \xi, \eta) \quad \text{for} \quad (x, \xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m.$$

Let $H^h = (H_i^h)_{i \in I}$ be the Hamiltonian of $L^h$, and note that

$$H \leq H^h \leq G^h \quad \text{on} \quad \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m.$$
In particular, we have
\[ H(x, p, u) = H^h(x, p, u) = G^h(x, p, u) \quad \text{if } |p| + |u| \leq C, \]
which shows, together with (22), that \( v^h \) is a solution of \( \lambda u + H^h[u] = 0 \) in \( \mathbb{T}^n \). It is clear that \( H^h \) satisfies (H1) and (H2). Moreover, \( H^h \) satisfies (H3) due to Proposition 4.

Now, since \( L \geq L^h \) on \( \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m \), it follows immediately that
\[ v^h_k(z) = \langle \mu, L \rangle \leq \langle \mu, L^h \rangle. \]
Combining these yields
\[ v^h_k(z) = \langle \mu, L \rangle = \langle \mu, L^h \rangle. \]
Consequently, we have
\[ \langle \mu, L - L^h \rangle = 0 \quad \text{and} \quad L \geq L^h. \] (25)

Noting that for all \( i \in \mathbb{I} \), \( Y_i \) is a closed subset of \( \mathbb{R}^m \) and \( L_i = \infty \) on \( \mathbb{T}^n \times \mathbb{R}^n \times (\mathbb{R}^m \setminus Y_i) \), and \( L_i - L^i \) is lower semicontinuous on \( \mathbb{T}^n \times \mathbb{R}^n \times Y_i \), we easily deduce from (25) that
\[ \text{supp} \mu_i \subset \{ (x, \xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n \times Y_i : L_i(x, \xi, \eta) = L^i_h(x, \xi, \eta) \}. \]
It remains to show that for all \( i \in \mathbb{I} \),
\[ \{ (x, \xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n \times Y_i : L_i(x, \xi, \eta) = L^i_h(x, \xi, \eta) \} \subset \mathbb{T}^n \times Q. \] (26)
To do this, we fix \( i \in \mathbb{I} \) and
\[ (x, \xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n \times Y_i \]
such that \( L_i(x, \xi, \eta) = L^i_h(x, \xi, \eta) \), set \( \zeta = (\xi, \eta) \) and show that \( \zeta \in Q \). We argue by contradiction and thus suppose that \( \zeta \notin Q \).

Note that, since \( \zeta \in \mathbb{R}^n \times Y_i \),
\[ K^i_h(x, \zeta) = L^i_h(x, \zeta) = L_i(x, \zeta). \] (27)
In view of Lemma 14, (i) applied to \( K^i_h \), we can select \( q_{\zeta} = (p_{\zeta}, u_{\zeta}) \in \partial(\xi, \eta)K^i_h(x, \zeta) \), which implies by the convex duality (Lemma 14, (ii)) that \( \zeta \in \partial(p_u, u)G^i_h(x, q_{\zeta}) \) and
\[ K^i_h(x, \zeta) + G^i_h(x, q_{\zeta}) = \zeta \cdot q_{\zeta}. \] (28)
We claim that \( h(q_{\zeta}) > 0 \). Indeed, if, to the contrary, \( h(q_{\zeta}) = 0 \), then we have \( |p_{\zeta}| + |u_{\zeta}| \leq C \) by (23) and, by (24), (27), and (28),
\[ \partial(p_u, u)H_i(x, q_{\zeta}) \subset Q \quad \text{and} \quad \zeta \cdot q_{\zeta} = K^i_h(x, \zeta) + G^i_h(x, q_{\zeta}) = L_i(x, \zeta) + H_i(x, q_{\zeta}), \]
which imply by Lemma 14, (ii) that
\[ \zeta \in \partial(p_u, u)H_i(x, q_{\zeta}) \subset Q. \]
This contradicts the choice of \( \zeta \), which confirms that \( h(q_{\zeta}) > 0 \).
Now, we observe that
\[
L_i(x, \xi) \geq \xi \cdot q_\xi - H_i(x, q_\xi) = \xi \cdot q_\xi - G_i^h(x, q_\xi) + h(q_\xi) = K_i^h(x, \xi) + h(q_\xi) > K_i^h(x, \xi) = L_i(x, \xi),
\]
which is a contradiction, and we conclude that (26) is valid. The proof is complete. □

In the following proof of Theorem 12, we approximate the Hamiltonian \( H(x, p, u) \) by Hamiltonians which satisfy (H1)–(H3) and (H5). In the first step of the approximation of \( H \), we follow the argument in the proof above, with \( h \) replaced by \( h/r \), with \( r \in \mathbb{N} \). In the proof above, the function \( G^h(x, p, u) \) has the superlinear growth in \((p, u)\) because of the addition of \( h \) and its nice effect is the continuity of \( K^h \) on \( \mathbb{T}^n \times \mathbb{R}^{n+m} \). The continuity on \( \mathbb{T}^n \times \mathbb{R}^{n+m} \) of the Lagrangians of the approximating Hamiltonians, obtained in the first step, is important for the second and final step of building the approximating Hamiltonians, which have at most the linear growth due to (H5).

**Proof of Theorem 12** We choose a constant \( C > 0 \) and a compact convex set \( Q \subset \mathbb{R}^{n+m} \) so that (22) and (24) hold. We may assume that \( Q = Q_1 \times Q_2 \) for some \( Q_1 \subset \mathbb{R}^n \) and \( Q_2 \subset \mathbb{R}^m \), where, moreover, \( Q_1 \) is a neighborhood of the origin of \( \mathbb{R}^n \). Let \( h \in C^1(\mathbb{R}^{n+m}) \) be a function satisfying (23). As in the proof of Theorem 13, we define sequences \((H^r)_r\in\mathbb{N}, (L^r)_r\in\mathbb{N}, (G^r)_r\in\mathbb{N}, (K^r)_r\in\mathbb{N}\) of functions, with \( h \) replaced by \( h/r \). That is, \( G^r = (G^r_i)_i \in \mathbb{I} \) is defined by

\[
G^r_i(x, p, u) = H_i(x, p, u) + \frac{1}{r} h(p, u) \quad \text{for } (x, p, u) \in \mathbb{T}^n \times \mathbb{R}^{n+m},
\]

\( K^r \) is the Lagrangian of \( G^r \), \( L^r \) is given by

\[
L^r(x, \xi, \eta) = K^r(x, \xi, \eta) + 0_Y(\eta) \quad \text{for } (x, \xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^{n+m},
\]

and \( H^r \) is the Hamiltonian of \( L^r \). We have already checked in the proof of Theorem 13 that \( H^r \) satisfies (H1)–(H3), \( v^h \) is a solution of \( \lambda v^h + H^r[v^h] = 0 \) in \( \mathbb{T}^n \), and \( L^r \in \prod_{i \in \mathbb{I}} C(\mathbb{T}^n \times \mathbb{R}^n \times Y_i) \). Moreover, it is easily seen that for \((x, p, u) \in \mathbb{T}^n \times \mathbb{R}^{n+m} \) and \( i \in \mathbb{I} \), if \(|p| + |u| \leq C\),

\[
H(x, p, u) = H^r(x, p, u) = G^r(x, p, u) \quad \text{and} \quad \partial_{(p,u)} H^r_i(x, p, u) \subset Q.
\]

Next we define function \( H^r_Q = (H^r_{Q,i})_i \in \mathbb{I} \) as the Hamiltonian of the function

\[
L^r_Q(x, \xi, \eta) := L^r(x, \xi, \eta) + \mathbf{0}_Q(\xi, \eta).
\]

Note by Lemma 14, (ii) that for \((x, i, p, u) \in \mathbb{T}^n \times \mathbb{I} \times \mathbb{R}^{n+m} \) and \( \xi \in \mathbb{R}^{n+m} \), if

\[
\xi \in \partial_{(p,u)} H^r_{Q,i}(x, p, u),
\]

then

\[
(p, u) \in \partial_{(\xi,\eta)} L^r_{Q,i}(x, \xi),
\]

and hence, by the definition of \( L^r_{Q,i} \), we have \( \xi \in Q \). That is, we have

\[
\partial_{(p,u)} H^r_{Q,i}(x, p, u) \subset Q \quad \text{for } (x, i, p, u) \in \mathbb{T}^n \times \mathbb{I} \times \mathbb{R}^{n+m}.
\]

It is now easy to see that \( H^r_Q \) satisfies (H1), (H2) and (H5). Note also by the inclusion in (29) that if \(|p| + |u| \leq C\),

\[
H^r_i(x, p, u) = \max_{(\xi,\eta) \in Q} (p \cdot \xi + u \cdot \eta - L^r_i(x, \xi, \eta)) = \max_{(\xi,\eta) \in (\mathbb{R}^n \times Y_i) \cap Q} (p \cdot \xi + u \cdot \eta - L^r_i(x, \xi, \eta)) = H^r_{Q,i}(x, p, u).
\]
We may now invoke Theorem 5, to conclude that there is $\mu \in \mathcal{C}(z, k, \lambda)$ such that
\[
v^\lambda_k(z) = \langle \mu, L^r_Q \rangle = \min_{v \in \mathcal{C}(z, k, \lambda)} \langle v, L^r_Q \rangle.
\] (30)

Theorem 13 and (29) ensure that for any minimizer $v = (v_i)_{i \in I} \in \mathcal{C}(z, k, \lambda)$ of the optimization in (30), we have the property
\[
supp v_i \subset T^n \times [((R^n \times Y_i) \cap Q].
\]

For each $r \in \mathbb{N}$, we select a minimizer $\mu^r = (\mu^r_i)_{i \in I} \in \mathcal{C}(z, k, \lambda)$ of the optimization in (30). Since $supp \mu^r_i \subset T^n \times Q$, we see immediately from (30) that $v^\lambda_k(z) = \langle \mu^r, L^r \rangle$.

In view of Lemma 11, we may assume that $\mu^r = (\mu^r_i)_{i \in I}$, after passing to a subsequence which is denoted again by the same symbol, converges weakly in the sense of measures to a measure $\mu = (\mu_i) \in \mathbb{P}_{\lambda}$ having the property that $supp \mu_i \subset T^n \times [((R^n \times Y_i) \cap Q]$. The weak convergence of $(\mu^r)$ implies that
\[
\left\{ \begin{array}{l}
\langle \mu, S^\lambda \rangle = 1, \\
\psi^\lambda_k(z) = \langle \mu, \xi \cdot D\psi + \eta \cdot \psi + \lambda \psi \rangle \quad \text{for all } \psi = (\psi_i) \in C^1(T^m)^m.
\end{array} \right\
\]

These ensure that $\mu \in \mathcal{C}(z, k, \lambda)$.

It is easily checked that, as $r \to \infty$, $K^r(x, \xi, \eta) \to L(x, \xi, \eta)$ monotonically pointwise. Since $K^r \leq K^{r+1}$ and $K^r \leq L^r$ for $r \in \mathbb{N}$, we obtain from (31),
\[
v^\lambda_k(z) \geq \langle \mu^r, K^q \rangle \geq \langle \mu^r, j \wedge K^q \rangle \quad \text{if } r \geq q, \quad \text{for all } j, q \in \mathbb{N},
\]
where $j \wedge K^q := (\min \{j, K^q_i\})_{i \in I} \in C_b(T^m \times \mathbb{R}^{n+m})$. Sending $r \to \infty$ yields
\[
v^\lambda_k(z) \geq \langle \mu, j \wedge K^q \rangle \quad \text{for all } j, q \in \mathbb{N}.
\]

By the monotone convergence theorem, after sending $j, q \to \infty$, we obtain
\[
v^\lambda_k(z) \geq \langle \mu, L \rangle.
\]

while, by Lemma 15, we have
\[
v^\lambda_k(z) \leq \inf_{v \in \mathcal{C}(z, k, \lambda)} \langle v, L \rangle.
\]

Thus, we conclude that
\[
v^\lambda_k(z) = \langle \mu, L \rangle = \min_{v \in \mathcal{C}(z, k, \lambda)} \langle v, L \rangle.
\]

\hfill $\square$

5 A convergence result for the vanishing discount problem

We study the asymptotic behavior of the solution $v^\lambda$ of $(P_\lambda)$, with $\lambda > 0$, as $\lambda \to 0$.

**Theorem 16** Assume (H1)–(H4). Let $v^\lambda$ be the solution of $(P_\lambda)$ for $\lambda > 0$. Then there exists a solution $v^0$ of $(P_0)$ such that the functions $v^\lambda$ converge to $v^0$ in $C(T^m)^m$ as $\lambda \to 0+$.
Lemma 17 Under the hypotheses of Theorem 16, there exists a constant $C_0 > 0$ such that for any $\lambda > 0$,

$$|v^\lambda_i(x)| \leq C_0 \quad \text{for } (x, i) \in \mathbb{T}^n \times I.$$  \hfill (32)

**Proof** Let $v_0 = (v_{0,i})_{i \in I} \in \text{Lip}(\mathbb{T}^n)^m$ be a solution of $(P_0)$. Choose a constant $C_1 > 0$ so that

$$|v_{0,i}(x)| \leq C_1 \quad \text{for } (x, i) \in \mathbb{T}^n \times I,$$

and observe by Lemma 1 that the functions $v_0 + C_11$ and $v_0 - C_11$ are a supersolution and a subsolution of $(P_0)$, respectively. Noting that $v_0 + C_11 \geq 0$ and $v_0 - C_11 \leq 0$, we deduce that $v_0 + C_11 \geq 0$ and $v_0 - C_11 \leq 0$ are a supersolution and a subsolution of $(P_\lambda)$, respectively, for any $\lambda > 0$. By comparison (Theorem 2), we see that, for any $\lambda > 0$,

$$v_0 - C_11 \leq v^\lambda \leq v_0 + C_11$$

on $\mathbb{T}^n$ and, moreover, $-2C_11 \leq v^\lambda \leq 2C_11$ on $\mathbb{T}^n$. Thus, (32) holds with $C_0 = 2C_1$.

\hfill \Box

Lemma 18 Under the hypotheses of Theorem 16, the family $(v^\lambda)_{\lambda \in (0,1)}$ is equi-Lipschitz continuous on $\mathbb{T}^n$.

**Proof** According to Lemma 17, we may choose a constant $C_0 > 0$ so that

$$|v^\lambda_i(x)| \leq C_0 \quad \text{for } (x, i, \lambda) \in \mathbb{T}^n \times I \times (0, \infty).$$

Hence, as $v^\lambda$ is a solution of $(P_\lambda)$, we deduce by (H1) that there exists a constant $C_1 > 0$ such that the functions $v^\lambda_i$, with $\lambda \in (0, 1)$, are subsolutions of $|Du| \leq C_1$ in $\mathbb{T}^n$. As is well-known, this implies that the $v^\lambda_i$ are Lipschitz continuous on $\mathbb{T}^n$ with $C_0$ as their Lipschitz bound.

\hfill \Box

We remark that one can show, with a slightly more elaboration, the equi-Lipschitz property of $(v^\lambda)_{\lambda > 0}$ in the above lemma.

**Theorem 19** Let $(z, k) \in \mathbb{T}^n \times I$. Assume (H1)–(H4). For any $\lambda > 0$, let $v^\lambda$ be the solution of $(P_\lambda)$ and $\mu^\lambda \in C(z, k, \lambda)$ a minimizer in (21). Then, for any sequence $(\lambda_i)_{i \in \mathbb{N}}$ of positive numbers converging to zero, there exists a subsequence of $(\lambda_i)$, which is denoted again by the same symbol, such that, as $i \to \infty$,

$$\lambda_i \mu^\lambda_i \to v^0$$

weakly in the sense of measures for some $v^0 = (v^0_i)_{i \in I} \in C(0)$, and $v^0$ satisfies

$$0 = \langle v^0, L \rangle = \min_{v \in C(0)} \langle v, L \rangle.$$  \hfill (33)

We call any minimizing measure $v^0 \in C(0)$ in (33) a Mather measure. The set of all Mather measures $v^0 \in C(0)$ is denoted by $\mathcal{M}(L)$. See, for example, [12,16,35,36] for some work related to Mather measures. Notice that the limit measure $v^0$ in Theorem 19 is a Mather measure. It should be noted that, in our formulation, the existence of a Mather measure is trivial since $0 \in C(0)$.

**Proof** We fix $(z, k) \in \mathbb{T}^n \times I$. By Theorem 12, for each $\lambda > 0$ there exists $\mu^\lambda \in C(z, k, \lambda)$ such that

$$\lambda_i v^\lambda_i(z) = \langle \lambda \mu^\lambda, L \rangle.$$  \hfill (34)
By Lemmas 17 and 18, there is a constant $C > 0$ such that for any $\lambda \in (0, 1),$

$$|v^\lambda(z)| + |Dv^\lambda(x)| \leq C \quad \text{a.e. } x \in \mathbb{T}^n.$$  

We choose a closed ball $Q \subset \mathbb{R}^{n+m}$ so that (24) holds with $C$ given above. Thanks to Theorem 13, we find that

$$\text{supp } \mu_i^\lambda \subset \mathbb{T}^n \times \{Q \cap (\mathbb{R}^n \times Y_i)\} \quad \text{for } (i, \lambda) \in I \times (0, 1).$$

Noting that $S^\lambda(\eta) \geq \lambda$ for $\eta \in Y_i$ and supp $\mu_i^\lambda \subset \mathbb{T}^n \times \mathbb{R}^n \times Y_i$ for all $i \in I$, we observe that

$$\langle \lambda \mu_i^\lambda, 1 \rangle = \langle \mu_i^\lambda, \lambda \rangle \leq \langle \mu_i^\lambda, S^\lambda \rangle = 1.$$

Hence, by applying Lemma 10 and passing to a subsequence, we may assume that the sequence $(\lambda_i\mu_i^\lambda)_{i \in \mathbb{N}} \subset M_+(\mathbb{T}^n \times \mathbb{R}^{n+m})$ converges weakly in the sense of measures to some $\nu^0 = (v^0_i) \in M_+(\mathbb{T}^n \times \mathbb{R}^{n+m})$ having properties $\langle \nu^0_i, 1 \rangle \leq 1$ and supp $\nu^0_i \subset \mathbb{T}^n \times \{Q \cap (\mathbb{R}^n \times Y_i)\}$ for all $i \in I$. It is an immediate consequence that $\nu^0 \in \mathbb{P}^0$.

Since $\mu_i^\lambda \in \mathcal{C}(z, k, \lambda_i)$, we have

$$\lambda_i \psi_k(z) = \langle \lambda_i \mu_i^\lambda, \xi \cdot D\psi + \eta \cdot \psi 1 + \lambda_i \psi \rangle \quad \text{for all } \psi = (\psi_i) \in C^1(\mathbb{T}^n)^m.$$  

Sending $\lambda_i \to 0$ along the sequence $(\lambda_i)$ yields

$$0 = \langle \nu^0_i, \xi \cdot D\psi + \eta \cdot \psi 1 \rangle \quad \text{for all } \psi = (\psi_i) \in C^1(\mathbb{T}^n)^m,$$

which concludes that $\nu^0 \in \mathcal{C}(0)$.

For any function $\phi \in C_b(\mathbb{T}^n \times \mathbb{R}^{n+m})$ such that $\phi \leq L$, we see from (34) that

$$0 \geq \langle \nu^0, \phi \rangle.$$

Moreover, by approximating $L$ monotonically from below by bounded continuous functions and applying the monotone convergence theorem, we deduce that

$$0 \geq \langle \nu^0, L \rangle.$$

By Lemma 15, we have $0 \leq \langle v, L \rangle$ for all $v \in \mathcal{C}(0)$. It is now clear that (33) holds.$\square$

Let $\mathcal{V}$ denote the set of accumulation points $v = (v_i)_{i \in I} \in C(\mathbb{T}^n)^m$ of $(v^\lambda)_{\lambda, i > 0}$ in the space $C(\mathbb{T}^n)^m$ as $\lambda \to 0$. Note by the stability of the viscosity property under uniform convergence that any $v \in \mathcal{V}$ is a solution of (P0). Let $\mathcal{W}$ denote the set of those solutions $w \in C(\mathbb{T}^n)^m$ of (P0) which satisfy

$$\langle v, w \rangle \leq 0 \quad \text{for all } v \in \mathcal{M}(L).$$  

(35)

**Proof of Theorem 16** In view of the Ascoli–Arzela theorem, Lemmas 17 and 18 assure that the family $(v^\lambda)_{\lambda \in (0, 1)}$ is relatively compact in $C(\mathbb{T}^n)^m$. In particular, the set $\mathcal{V}$ is nonempty.

If $\mathcal{V}$ is a singleton, then it is obvious that the whole family $(v^\lambda)_{\lambda > 0}$ converges to the unique element of $\mathcal{V}$ in $C(\mathbb{T}^n)^m$ as $\lambda \to 0$.

We need only to show that $\mathcal{V}$ is a singleton. For this, we first show that

$$\mathcal{V} \subset \mathcal{W}. \quad (36)$$

To see this, let $v \in \mathcal{V}$ and $v \in \mathcal{M}(L)$. Choose a sequence $(\lambda_i)_{i \in \mathbb{N}}$ of positive numbers converging to zero such that $(v_{\lambda_i})_{i \in \mathbb{N}}$ converges to $v$ in $C(\mathbb{T}^n)^m$. Since $(L - \lambda v^\lambda, v^\lambda) \in \mathcal{F}(0)$, $v \in \mathcal{C}(0)$, and $\langle v, L \rangle = 0$, using Lemma 15, we get

$$0 \leq \langle v, L - \lambda v^\lambda \rangle = \langle v, L \rangle - \langle v, \lambda v^\lambda \rangle = -\lambda \langle v, v^\lambda \rangle,$$

$$\mathcal{V} \subset \mathcal{W}.$$

(36)
which yields, after dividing by \( \lambda > 0 \) and then sending \( \lambda \to 0 \) along \( \lambda = \lambda_j \),

\[
\langle v, v \rangle \leq 0.
\]

This proves (35), which ensures the inclusion (36).

Next, we show that

\[
w \leq v \text{ for all } w \in \mathcal{W}, v \in \mathcal{V}.
\]  \hspace{1cm} (37)

To check this, it is enough to show that for any \( v \in \mathcal{V} \), \( w \in \mathcal{W} \) and \((z, k) \in T^n \times I\), the inequality \( w_k(z) \leq v_k(z) \) holds.

Fix any \( v \in \mathcal{V} \) and \( w \in \mathcal{W} \) and \((z, k) \in T^n \times I\). Select a sequence \((\lambda_j)_{j \in \mathbb{N}} \subseteq (0, \infty)\) converging to zero so that

\[
v^{\lambda_j} \to v \text{ in } C(T^n)^m \text{ as } j \to \infty.
\]

By Theorem 12, there exists a sequence \((\mu^j)_{j \in \mathbb{N}}\) such that for \( j \in \mathbb{N}, \)

\[
\mu^j \in \mathcal{C}(z, k, \lambda_j) \quad \text{and} \quad v^{\lambda_j}_k(z) = \langle \mu^j, L \rangle.
\]  \hspace{1cm} (38)

In view of Theorem 19, we may assume by passing to a subsequence if necessary that, as \( j \to \infty, \)

\[
\lambda_j \mu^j \to v \text{ weakly in the sense of measures}
\]

for some \( v = (v_i)_{i \in \mathbb{I}} \in \mathcal{M}(L). \)

Now, note that \( (L + \lambda_j w, w) \in \mathcal{F}(\lambda_j) \) and infer by Lemma 15 and (38) that

\[
w_k(z) \leq \langle \mu^j, L + \lambda_j w \rangle = v^{\lambda_j}_k(z) + \lambda_j \langle \mu^j, w \rangle.
\]

Sending \( j \to \infty \) now yields

\[
w_k(z) \leq v_k(z) + \langle v, w \rangle.
\]

This together with (35) shows that \( w_k(z) \leq v_k(z) \), which ensures that (37) holds. Noting that (37) combined with (36) shows that \( w \leq v \) for all \( w, v \in \mathcal{V} \), that is, \( \mathcal{V} \) is a singleton. The proof is complete. \( \square \)

Reviewing the proof above, we conclude easily the following proposition, which is a generalization of [12, Theorem 3.8] (see also [16, Proof of Theorem 1]).

**Corollary 20** Under the assumptions and notation of Theorem 16, the limit function \( v^0 = (v_i^0)_{i \in \mathbb{I}} \) can be represented as

\[
v^0_i(x) = \max\{w_i(x); w = (w_i) \in \mathcal{W}\} \text{ for } x \in T^n.
\]

The proof of Corollary 20, with \( \mathcal{W} \) replaced by

\[
\mathcal{W}^- = \{w \in C(T^n)^m; w \text{ is a subsolution and satisfies (35)}\}
\]

shows also that, under the hypotheses and notation of Corollary 20,

\[
v^0_i(x) = \max\{w_i(x); w = (w_i) \in \mathcal{W}^-\} \text{ for } x \in T^n.
\]
6 Ergodic problem

Remark that, given a Hamiltonian $H$, condition (H4) is not satisfied in general. We consider the problem of finding an $m$-vector $c = (c_i)_{i \in \mathbb{I}} \in \mathbb{R}^m$ and a function $u = (u_i)_{i \in \mathbb{I}} \in C(\mathbb{T}^n)^m$ such that $u$ is a solution of the $m$-system

$$H[u] = c \quad \text{in } \mathbb{T}^n,$$

which is stated componentwise as

$$H_i(x, Du_i(x), u(x)) = c_i \quad \text{in } \mathbb{T}^n \quad \text{for } i \in \mathbb{I}.$$

We call this problem the ergodic problem for $H$.

If the ergodic problem has a solution $c \in \mathbb{R}^m$ and $u \in C(\mathbb{T}^n)^m$, then we may apply the main convergence result (Theorem 16) to $(P_\lambda)$, with $H$ replaced by $H_c := H - c$. As noted in the introduction, this change of Hamiltonians, in general, does not help analyze the vanishing discount problem for the original system $(P_\lambda)$.

However, if $H$ satisfies a certain additional condition, then the argument of switching from the Hamiltonian $H$ to $H_c$ makes sense for the vanishing discount problem for $(P_\lambda)$. For instance, given a solution $(c, u) \in \mathbb{R}^m \times C(\mathbb{T}^n)^m$ of (39), assume that the equality

$$H(x, p, v + tc) = H(x, p, v) \quad \text{(40)}$$

holds for all $t \in \mathbb{R}$ and $(x, p, v) \in \mathbb{T}^n \times \mathbb{R}^{n+m}$. It is easily seen that if $v^\lambda \in C(\mathbb{T}^n)^m$ is a solution of $(P_\lambda)$, then $w^\lambda := v^\lambda + \lambda^{-1}c$ is a solution of $(P_\lambda)$, with $H_c$ in place of $H$. This is a situation where one can apply Theorem 16, to observe the convergence of $v^\lambda + \lambda^{-1}c$ as $\lambda \to 0$.

In the next result, we do not need the convexity or monotonicity of $H$, and we assume only (H1).

For $R > 0$ and $r > 0$, we set

$$\alpha_R(r) = \inf \{H_i(x, p, u) : (x, i) \in \mathbb{T}^n \times \mathbb{I}, \ u \in B_R^m, \ p \in \mathbb{R}^n \setminus B_r^m\},$$

$$\beta_R = \sup \{H_i(x, 0, u) : (x, i) \in \mathbb{T}^n \times \mathbb{I}, \ u \in B_R^m\}.$$

The constants $\alpha_R(r)$ and $\beta_R$ are finite by the continuity of $H_i$ and (H1). It is clear that for any $R > 0$, the function $r \mapsto \alpha_R(r)$ is nondecreasing in $(0, \infty)$ and diverges to infinity as $r \to \infty$.

**Theorem 21** Assume (H1) and that there exists a constant $R > 0$ such that

$$\beta_R < \alpha_R \left(\frac{2R}{\sqrt{n}}\right). \quad \text{(41)}$$

Then problem (39) has a solution $(c, u) \in \mathbb{R}^m \times C(\mathbb{T}^n)^m$.

We remark that a result similar to the above has been established in [32, Theorem 1.2] in the case of a scalar Hamilton–Jacobi equation.

Roughly speaking, the condition (41) in the theorem above is satisfied for a large $R > 0$ if the growth of $H_i(x, p, u)$ in $p$ is higher in a certain sense than that in $u$ as $|(p, u)| \to \infty$. In the case of linear coupling [and hence, $H_i$ have the form of (1)], it is obvious that if for all $i \in \mathbb{I}$, the functions $G_i(x, p)$ have the superlinear growth, i.e., satisfy

$$\lim_{R \to \infty} \inf_{(x, p) \in \mathbb{T}^n \times (\mathbb{R}^n \setminus B_R^m)} \frac{G_i(x, p)}{|p|} = \infty,$$
then condition (41) is valid. We refer to [15, Theorem 2.12], [26, Theorem 17] for results, in the linear coupling case, similar to but more subtle than the theorem above.

Condition (41) is also valid when, as a direct generalization of the linear coupling case, \( H_i, \ i \in \mathbb{I} \) have superlinear growth in \( p \), i.e., for any \( r > 0 \),

\[
\lim_{R \to \infty} \inf_{(x, p, u) \in \mathbb{T}^n \times (\mathbb{R}^n \setminus B_R^n) \times B_R^n} \frac{H_i(x, p, u)}{|p|} = \infty,
\]

and uniformly Lipschitz dependence in \( u \), i.e., there exists \( \Theta > 0 \) such that for any \( u, v \in \mathbb{R}^m \)

\[
|H_i(x, p, u) - H_i(x, p, v)| \leq \Theta |u - v|.
\]

**Proof** We choose \( R > 0 \) so that (41) holds and select \( \lambda > 0 \) so that

\[
\beta_R + \lambda R < \alpha_R \left( \frac{2R}{\sqrt{n}} \right). \tag{42}
\]

Let \( u \in C(\mathbb{T}^n)^m \) and consider the uncoupled \( m \)-system for \( v = (v_i)_{i \in \mathbb{I}} \):

\[
\lambda(v_i(x) - u_i(x)) + H_i(x, Dv_i(x), u(x)) = 0 \quad \text{in} \ \mathbb{T}^n \quad \text{for} \ i \in \mathbb{I}. \tag{43}
\]

The functions \((x, p) \mapsto H_i(x, p, u(x))\) are continuous and coercive and, hence, the standard theory of viscosity solutions (also, Theorem 2 applied to each single equations) guarantees that (43) has a unique solution \( v = (v_i)_{i \in \mathbb{I}} \) and the functions \( v_i \) are Lipschitz continuous on \( \mathbb{T}^n \).

For any \( u \in C(\mathbb{T}^n)^m \), let \( v = (v_i)_{i \in \mathbb{I}} \in C(\mathbb{T}^n)^m \) be the solution of (43). We set

\[
Tu := v - \min_{\mathbb{T}^n} v,
\]

where

\[
\min_{\mathbb{T}^n} v := (\min_{x \in \mathbb{T}^n} v_i(x))_{i \in \mathbb{I}} \in \mathbb{R}^m,
\]

which gives a mapping \( T \) from \( C(\mathbb{T}^n)^m \) to \( C(\mathbb{T}^n)^m \). Because of the stability of viscosity solutions under the uniform convergence and the uniqueness of solution of (43), we easily deduce that \( T \) is a continuous mapping on the Banach space \( C(\mathbb{T}^n)^m \), with norm \( \|u\|_\infty := \max_{x \in \mathbb{T}^n} |u(x)| \).

Now, fix \( u \) so that

\[
\|u\|_\infty \leq R \quad \text{and} \quad u(x) \geq 0 \quad \text{for all} \ x \in \mathbb{T}^n,
\]

and observe that the function \( w(x) := -\lambda^{-1} \beta_R 1 \) is a subsolution of (43). Indeed, we have

\[
\lambda(w_i(x) - u_i(x)) + H_i(x, Dw_i(x), u(x)) \\
\leq -\beta_R + H_i(x, 0, u(x)) \leq 0 \quad \text{for} \ (x, i) \in \mathbb{T}^n \times \mathbb{I}.
\]

By the standard comparison theorem, we have

\[
-\frac{\beta_R}{\lambda} 1 \leq v.
\]

Noting that \( v_i \) is Lipschitz continuous and hence it is almost everywhere differentiable, we compute at any point \( x \) of differentiability of \( v_i \) that, if \( Dv_i(x) \neq 0 \),

\[
0 \geq \lambda \left( -\frac{\beta_R}{\lambda} - u_i(x) \right) + \alpha_R (|Dv_i(x)|) \geq -\beta_R - \lambda R + \alpha_R (|Dv_i(x)|).
\]
and observe by the choice of $\lambda$ that, if $Dv_i(x) \neq 0$,

$$\alpha_R(|Dv_i(x)|) \leq \beta_R + \lambda R < \alpha_R \left(\frac{2R}{\sqrt{n}}\right),$$

which yields

$$|Dv_i(x)| < \frac{2R}{\sqrt{n}} \text{ a.e. in } \mathbb{T}^n \text{ for } i \in \mathbb{I},$$

and moreover

$$0 \leq v_i(x) - \min_{\mathbb{T}^n} v_i \leq R \text{ for all } (x, i) \in \mathbb{T}^n \times \mathbb{I}.$$

Thus, we conclude that

$$\|D(Tu)_i\|_{L^\infty(\mathbb{T}^n)} := \text{ess sup}_{\mathbb{T}^n} |D(Tu)_i| \leq R \text{ for } i \in \mathbb{I},$$

$$(Tu)(x) \geq 0 \text{ for } x \in \mathbb{T}^n \text{ and } \|Tu\|_{\infty} \leq R.$$

We set

$$K = \{u \in C(\mathbb{T}^n)^m : u \geq 0, \|u\|_{\infty} \leq R, \|Dui\|_{L^\infty(\mathbb{T}^n)} \leq R \text{ for all } i \in \mathbb{I}\},$$

and note that $K$ is a compact convex subset of $C(\mathbb{T}^n)^m$. The above observations show that $T$ maps $K$ into $K$. The Schauder fixed point theorem guarantees that there is a fixed point $u \in K$ of $T$. Let $v$ be the a solution of (43), with the fixed point $u$. By the definition of $T$, we have

$$u = Tu = v - \min_{\mathbb{T}^n} v,$$

and $u$ solves

$$\lambda \min_{\mathbb{T}^n} v_i + H_i(x, Du_i, u) = 0 \text{ in } \mathbb{T}^n \text{ for } i \in \mathbb{I}.$$

That is, the pair $(-\lambda \min_{\mathbb{T}^n} v, u)$ is a solution of (41).

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