ON THE NONAUTONOMOUS HOPF BIFURCATION PROBLEM

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Abstract. Under well-known conditions, a one-parameter family of two-dimensional, autonomous ordinary differential equations admits a supercritical Andronov-Hopf bifurcation. Let such a family be perturbed by a non-autonomous term. We analyze the sense in which and some conditions under which the Andronov-Hopf pattern persists under such a perturbation.

1. Introduction. There is an important and well-developed theory, due to Andronov and Hopf, which describes the genesis of an asymptotically stable periodic solution in a certain class of one-parameter families of ordinary differential equations in the plane. In fact consider

\[ \frac{dx}{dt} = f(x, \varepsilon) \quad x \in \mathbb{R}^2, \quad \varepsilon \in \mathbb{R} \quad (1.1) \]

where \( f(0, \varepsilon) \equiv 0 \) and \( f \) is sufficiently regular. In the supercritical Andronov-Hopf theory one imposes hypotheses which ensure that:

a): the origin \( x = 0 \) is an exponentially asymptotically stable equilibrium of \((1.1)\) for \( \varepsilon < 0 \);

b): there is an exponentially asymptotically orbitally stable periodic solution of \((1.1)\), with non-zero minimal period, at distance \( O(\sqrt{\varepsilon}) \) from the origin, for small positive values of \( \varepsilon > 0 \).

When \( \varepsilon \) is small and positive, the periodic solution of \( b) \) is exponentially attracting at rate \( O(\varepsilon) \), and has an annulus of attraction of width \( O(\varepsilon^d) \) for any \( d > 1/2 \). It is unique in some neighbourhood of the origin in \( \mathbb{R}^2 \), so one concludes that the

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asymptotic stability is transferred from the origin \( x = 0 \) to a limit cycle, as \( \varepsilon \) increases through 0.

The supercritical Andronov-Hopf theory can be adapted to the case when \( f \) depends periodically on \( t \). Consider the one-parameter family of periodic differential systems

\[
\frac{dx}{dt} = f(t, x, \varepsilon) \quad x \in \mathbb{R}^2, \varepsilon \in \mathbb{R} \tag{1.2}
\]

where \( f(t, 0, \varepsilon) \equiv 0 \) and \( f(t + 1, x, \varepsilon) = f(t, x, \varepsilon) \) for all \( t \in \mathbb{R}, x \in \mathbb{R}^2, \varepsilon \in \mathbb{R} \). In this case one introduces the period map \( \Pi \) which sends a point \( x_0 \in \mathbb{R}^2 \) to the position at time \( t = 1 \) of the solution \( x(t, x_0) \) of (1.2) which satisfies the initial condition \( x(0, x_0) = x_0 \). In this case one can write down conditions which are sufficient so that

**A):** the origin \( x = 0 \) is an exponentially asymptotically stable fixed point of \( \Pi \) for \( \varepsilon < 0 \);

**B):** there exists a \( \Pi \)-invariant orbitally asymptotically stable closed curve surrounding the origin in \( \mathbb{R}^2 \), for small positive values of \( \varepsilon \) ([35, 38, 37]).

In the case where \( f \) depends on \( t \) in a non-periodic way, the very notion of “Hopf bifurcation” becomes problematic. In the context of differential systems with quasi-periodic coefficients, Broer and his coauthors have proved substantial results using KAM methods, see e.g. [8, 9, 10]. In this theory, the function \( f \) typically depends on several parameters: \( \varepsilon \in \mathbb{R}^s \) where \( s > 1 \). In this context it is also typical that there is a measure-theoretically large subset of the parameter space \( \mathbb{R}^s \) where an analogue of the Andronov-Hopf bifurcation takes place, and also a complementary subset of \( \mathbb{R}^s \) where one has little or no information.

Other ideas in the subject of “non-autonomous Hopf bifurcation” were presented in the early 1990s, in [3] and [29]. The Arnold “two-step” pattern [3] has been observed in numerical works. It is presumably due to the passage of a full interval of the “dynamical spectrum” through the origin as the parameter \( \varepsilon \) in (1.2) increases from zero to a finite positive value. This pattern has been discussed in [28], [2].

The ansatz in [29] goes as follows. Assume that \( f \) in (1.2) depends quasi-periodically on \( t \), and that \( f(t, 0, \varepsilon) \equiv 0 \). Then linearize (1.2) around \( x = 0 \) to obtain a 1-parameter family of quasi-periodic linear differential systems:

\[
\frac{dx}{dt} = A(t, \varepsilon)x \quad x \in \mathbb{R}^2, \varepsilon \in \mathbb{R} \tag{1.3}
\]

One can show that a generic family of curves \( \varepsilon \to A(\cdot, \varepsilon) \) has the property that (1.3) admits a \((1, 1)\) exponential separation for an open dense set \( \mathcal{E} \) of parameter values. This means that, for \( \varepsilon \in \mathcal{E} \), equation (1.3) is projectively hyperbolic. So one can expect that, under appropriate supplementary hypotheses, equation (1.2) will admit an “intermittently existing” attractive invariant set which preserves the recurrence properties of \( f \) - hence the setting is not analogous to that of a quasi periodically forced limit cycle. This scenario can be realized in certain examples [29].

The purpose of this paper is to introduce and discuss another natural non-autonomous analogue of the Andronov-Hopf bifurcation scenario. Namely, assume that the family (1.1) exhibits a supercritical A.-H. bifurcation as \( \varepsilon \) increases through \( \varepsilon = 0 \). Suppose that (1.1) is subject to a time-dependent perturbation:

\[
\frac{dx}{dt} = f(x, \varepsilon) + \mu g(t, x, \varepsilon, \mu) \tag{1.4}
\]
where $g$ is regular in all arguments, and $\mu$ is a second parameter which may be a function of $\varepsilon$. The time-dependence of the function $g$ may be, for example, almost periodic in the sense of Bohr, or recurrent in the sense of Birkhoff [17]. One asks for information concerning the fate of the A.-H. limit cycle for positive values of $\varepsilon$ and non-zero values of $\mu$: is it conserved in some reasonable sense?

When $g$ and its derivatives with respect to $(x,\varepsilon)$ are uniformly bounded and uniformly continuous functions of $t$, one can apply a Bebutov-type construction to embed equation (1.4) in a family of equations of the same form, which is parametrized by the elements $p$ of the phase space $P$ of a real dynamical system $(P, \{\phi_t\})$. We will discuss the Bebutov construction in section 2. Here we simply pass to the following reformulation of the system (1.4): it will be the main object of study of this paper.

Let $P$ be a compact metric space, and let $\{\phi_t\}$ be a real flow on $P$. This means that each $\phi_t : P \to P$ is a homeomorphism, and that the group property is valid: $\phi_t \circ \phi_s = \phi_{t+s}$ for all $t, s \in \mathbb{R}$. In addition it is required that the map $\phi : P \times \mathbb{R} \to P : (t, p) \to \phi_t(p)$ is continuous. Let $g : P \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be a continuous function which is $C^\infty$ as a function of $(x,\varepsilon,\mu)$. Actually finite smoothness is all that will ever be required at any given moment, and at various points we will state how much is sufficient. We will study the family of differential equations

$$\frac{dx}{dt} = f(x,\varepsilon) + \mu g(\phi_t(p), x, \varepsilon, \mu).$$

(1.5)

This is the reformulation of equation (1.4) referred to above.

Let us remark that the family of equation (1.5) is formally similar to the random differential system

$$\frac{dx}{dt} = f(x,\varepsilon) + \mu \xi_t$$

which was studied in [6]. Here $\xi_t$ represents a generic path of some noise process. If the function $g$ in (1.5) does not depend on $x$, then it defines a process by varying $p \in P$ (for fixed values of $\varepsilon$ and $\mu$); this process differs from that of [6] in that, for the functions $g$ we consider, it will be “less random”. Note that we are mainly interested in the case where the origin is a fixed point for the perturbed system, while this assumption does not hold in [6]. However we give a result in this setting too, Lemma 2.2, which allows to start a comparison of the two approaches: a detailed discussion of this topic may be the object of future investigations.

Let us sketch the main steps of our analysis of the family (1.5). The starting point is of course the supercritical Andronov-Hopf pattern which holds when $\mu = 0$. We first fix a sufficiently small value of $\varepsilon$, then state and prove an integral manifold theorem for equation (1.5) when $\mu$ is small. For this we adopt the point of view of Yi [40]. We refine his results in two ways. First, we give an estimate on $\mu$ as a function of $\varepsilon$, as to when the integral manifold $M = M_{\varepsilon,\mu}$ exists. Second, we will want to prove smoothness of the integral manifold (assuming that $P$ itself is a smooth manifold). It will turn out that the integral manifold is the graph of a function $v : P \times S^1 \to \mathbb{R}^2$, where $S^1 = \mathbb{R}/[0, 2\pi]$ is the unit circle in $\mathbb{R}^2$.

The next step is to study the flow $\{\psi_t\}$ on the integral manifold $M$. It turns out that, by construction, the flow $(M, \{\psi_t\})$ is a circle extension of $(P, \{\phi_t\})$. This property of $M$ may be viewed as indicating the presence of a generalized Andronov-Hopf bifurcation, for those values of $(\varepsilon,\mu)$ for which $M$ exists. Now, the flow of a
circle extension may exhibit behavior which is quite a bit more complex than the
flow on $P$ itself.

Nevertheless there are certain limits on this extra complexity. For example if
the notion of “generalized period map” to analyze

Diliberto [15, 16] who in turn was influenced by Levinson [34]. In fact we will adapt
this analysis will proceed. Consider the positive

$x \in \mathbb{R}$ and each $p \in P$, let $\tau_\mu(p; x_1) = \tau_\mu(p; x_1, x_2)$ be the first return time of a given
point $x_1 \in I$ to $I$, via the corresponding solution of (1.5). Now if one restricts $\tau_\mu$
to points in the integral manifold $M$, then one obtains a map $p \to \tau_\mu(p) : P \to \mathbb{R}$
because $M \subset P \times \mathbb{R}^2$ turns out to be the graph of a function $v : P \times S^1 \to \mathbb{R}$,
and the function $v$ turns out to be transversal to $I$ in an appropriate sense). We will

call $\tau_\mu$ the Diliberto map for the flow $\{\psi_t\}$.

According to the situation we will use either the notation $\phi_t(p)$, or the notation

$\phi(t, p)$. Next introduce the map

$$T_\mu : P \to P : p \to \phi(\tau_\mu(p), p)$$

One can show that, if $\mu$ is small, then $T_\mu$ is a homeomorphism, and moreover the
flow $\{\psi_t\}$ on $M$ is flow-isomorphic to the suspension of $T_\mu$ with roof function $\tau$ [31].

So the dynamical properties of the flow $\{\psi_t\}$ on $M$ are determined by those of this
suspension flow. In certain cases one can make substantial statements concerning
the flow $\{\psi_t\}$ as we now indicate.

First of all the concept of bounded mean motion is emphasized by Huang-Yi [25].
When it is present in a circle extension, then the corresponding flow can be described
as indicated in [25]. It turns out that the bounded mean motion property holds
for $(M, \{\psi_t\})$ if and only if the function $\tau_\mu$ is a so-called coboundary with respect
to the homeomorphism $T$. In some cases this condition can be checked, especially
when $T$ is conjugated to a homeomorphism with simpler structure (see section 4).

When the bounded mean motion property holds, one has available a semiconjugacy
result of Huang-Yi (see also [1]) which generalizes the classical Poincare’–Denjoy
result for flows on the 2-torus. To state this result one makes use of the rotation
number $\rho_\nu$ of the flow on $M$, with respect to a given ergodic measure $\nu$ on $P$.

Second, as was anticipated above, it sometimes happens that $T$ is conjugate to a
homeomorphism $R$ with simpler structure. In this case the suspension flow can be
studied in more detail. Suppose for example that $P = T^d = \mathbb{R}^d / \mathbb{Z}^d$ is the $d$-torus,
and that $\{\phi_t\}$ is a Kronecker winding on $P$. Then, roughly speaking, one can prove
results of the following sort: for “most” small $\varepsilon > 0$, it is the case that for “most”
small $\mu$, the map $T_\mu$ is conjugate to a rigid rotation on $P$. This can be obtained
via KAM techniques [21, 22]. At this point one might expect that the flow $\{\psi_t\}$ is
itself quasi-periodic on the $d + 1$ dimensional torus $M$. Though this may happen,
it seems that other phenomena are possible as well. For example, Fayad [18] writes down conditions under which the flow \( \{ \psi_t \} \) is weakly mixing.

Apart from the suspension technique, we will also apply the notion of time-change to obtain information about the flow \( \{ \psi_t \} \) on \( M \). Thus we will write down a simple condition on \( \tau_\mu \) which ensures that the flow \( \{ \psi_t \} \) can be reparametrized in such a way as to obtain a flow \( \{ \hat{\phi}_s \} \) on \( P \), which has the same orbits as \( \{ \phi_t \} \), and which has the property that \( \tau_\mu = \hat{\phi}_1 \). That is, \( \tau_\mu \) is the time 1 map of the reparametrized flow \( \{ \hat{\phi}_s \} \). The condition on \( \tau_\mu \) can be verified in an explicit way in certain concrete cases, when \( \{ \phi_t \} \) is quasi-periodic. We will state consequences of the existence of a time-change as above; in particular it turns out that the flow \((M, \{ \psi_t \})\) is isomorphic to a Furstenberg flow [20].

We indicate the main points of emphasis of this paper.  
1) The use of integral manifold theory permits to study the bifurcation theory of the family (1.5) in terms of a circle extension of \( P \).  
2) The circle extension can be studied using standard tools of topological dynamics, because of some particular properties enjoyed by the solutions of (1.5) when \( \mu \) is small.

The paper is organized as follows. In section 2 we present some facts which are part of the standard Andronov-Hopf bifurcation theory ([33], [30]). In particular [33] contains information sufficient for our needs. In section 3 we work out the integral manifold theory of equation (1.5), making use of the methods of Yi [40] and those of the standard reference [24]. Finally in section 4 we present our analysis of the flow on the integral manifold.

Our message might be summarized as follows: in the non-autonomous setting we envisage, the Andronov-Hopf limit cycle “blows up” to a circle extension of \( P \), which can be studied by adapting the Diliberto technique and by using methods of topological dynamics.

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2. Preliminaries. In this section we first present some standard facts from the field of topological dynamics which will be needed later. Let \( P \) be a metric space. A (real) flow on \( P \) is a family \( \{ \phi_t \mid t \in \mathbb{R} \} \) of homeomorphisms of \( P \) with the following properties:

1. \( \phi_0(p) = p \) for all \( p \in P \);  
2. \( \phi_t \circ \phi_s = \phi_{t+s} \) for all \( t, s \in \mathbb{R} \);  
3. \( \phi : P \times \mathbb{R} \to P : (p, t) \to \phi(p, t) \) is continuous.

Suppose now that \( P \) is a compact metric space, and let \( \{ \phi_t \} \) be a flow on \( P \). A regular Borel probability measure \( \mu \) on \( P \) is said to be \( \{ \phi_t \} - \text{invariant} \) if \( \mu(\phi_t(B)) = \mu(B) \) for each Borel set \( B \subset P \) and any \( t \in \mathbb{R} \). An invariant measure \( \mu \) is said to be \textit{ergodic} if, in addition, the following condition holds: if \( B \subset P \) is a Borel set, and if \( \mu(B \Delta \phi_t(B)) = 0 \) for each \( t \in \mathbb{R} \), then \( \mu(B) = 0 \) or \( \mu(B) = 1 \). Here \( \Delta \) is the usual symmetric difference of sets: \( A \Delta B = (A \setminus B) \cup (B \setminus A) \).

A famous result by Krylov and Bogoliubov ([32], [36]) states that, if \( P \) is a compact metric space, and if \( \phi_t \) is a flow on \( P \), then there exists at least one \( \phi_t \)-ergodic measure on \( P \). One can thus apply a basic theorem of Birkhoff to the triple \( (P, \{ \phi_t \}, \mu) \), together with a useful refinement, obtaining the following.
Theorem 2.1. Let \( P \) be a compact metric space and \((P, \{\phi_t\})\) be a flow, and let \( \mu \) be a \( \{\phi_t\} \)-ergodic measure on \( P \). If \( h \in L^1(P, \mu) \), then

\[
\lim_{|t| \to \infty} \frac{1}{t} \int_0^t h(\phi_s(p)) \, ds = \int_P h \, d\mu
\]

for \( \mu \)-almost any \( p \in P \). Suppose that \( \mu \) is the only \( \{\phi_t\} \)-ergodic measure on \( P \) (in this case one says that the flow \((P, \{\phi_t\})\) is uniquely ergodic). Then if \( h : P \to \mathbb{R} \) is a continuous function, one has

\[
\lim_{|t| \to \infty} \frac{1}{t} \int_0^t h(\phi_s(p)) \, ds = \int_P h \, d\mu
\]

for all \( p \in P \), and the limit is uniform in \( P \). That is, given \( \varepsilon > 0 \), there exists \( T > 0 \) such that if \( |t| \geq T \), then

\[
|\frac{1}{t} \int_0^t h(\phi_s(p)) \, ds - \int_P h \, d\mu| \leq \varepsilon
\]

Of course the first part of Theorem 2.1 can be stated and proved in a more general context of measurable flows, see e.g. [36]. The second part of the theorem is specific to flows with a compact phase space \( P \).

Let us recall that a flow \( \{\phi_t\} \) on a (nonempty) compact metric space \( P \) is called minimal if for each \( p \in P \), the orbit \( \{\phi_t(p) \mid t \in \mathbb{R}\} \) is dense in \( P \). A flow \((P, \{\phi_t\})\) is said to be strictly ergodic if it is minimal and admits a unique ergodic measure \( \mu \).

Again let \( P \) be a compact metric space. A flow \( \{\phi_t\} \) on \( P \) is said to be Bohr almost periodic or isometric if there is a metric \( d \) on \( P \), which is compatible with the topology on \( P \), such that

\[
d(\phi_t(p_1), \phi_t(p_2)) = d(p_1, p_2)
\]

for all \( p_1, p_2 \in P \) and all \( t \in \mathbb{R} \). If \((P, \{\phi_t\})\) is Bohr almost periodic/isometric and if \( p \in P \), then the orbit closure \( \text{cls}\{\phi_t(p) \mid t \in \mathbb{R}\} \) is strictly ergodic (hence minimal), and in fact \( P \) is equal to the union of its minimal subsets. We note that, though a minimal almost periodic flow is strictly ergodic, the converse is not true, and it is demonstrated by e.g. the Furstenberg flows ([20]; see below).

We consider a concrete example of a minimal almost periodic flow, namely a Kronecker flow on the \( d \)-torus \( \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \) \((d \geq 2)\). Let \( \alpha_1, \ldots, \alpha_d \) be \( \mathbb{Q} \)-independent real numbers, let \( p_1, \ldots, p_d \) be \( 1 \)-periodic angular coordinates on \( \mathbb{T}^d \), and set

\[
\phi_t(p_1, \ldots, p_d) = (p_1 + \alpha_1 t, \ldots, p_d + \alpha_d t).
\]

Then the Kronecker flow \( \{\phi_t\} \) is minimal and almost periodic. More generally, if \( (\alpha_1, \ldots, \alpha_d) \) satisfies \( k \in \{1, \ldots, d-1\} \) independent \( \mathbb{Q} \)-linear relations, then \((\mathbb{T}^d, \{\phi_t\})\) turns into a disjoint union of almost periodic minimal flows, each of which is isomorphic in the natural sense to a \( d-k \)-dimensional Kronecker flow.

A Furstenberg flow is defined as follows. Let \( P = \mathbb{T}^d \), and let \((P, \{\phi_t\})\) be a Kronecker flow. Let \( \nu \) be the normalized Lebesgue measure on \( P \), so that \( \nu \) is the unique \( \{\phi_t\} \)-ergodic measure. Set \( \hat{P} = P \times S^1 \), and let \( \theta \) be a \( 2\pi \)-periodic angular coordinate on \( S^1 \). Let \( r : \hat{P} \to \mathbb{R} \) be a continuous function. For \( t \in \mathbb{R}, p \in P, \theta \in S^1 \) set \( \hat{\phi}_t(p, \theta) = (\phi_t(p), \theta + \int_0^t r(\phi_s(p)) \, ds) \). Then \((\hat{P}, \{\hat{\phi}_t\})\) is a flow, with remarkable properties. Suppose for example that \( \int_P r \, d\nu = 0 \), or more generally that \( \int_P r \, d\nu \) belongs to the frequency module \( \mathcal{F} = \{\sum_{j=1}^d n_j \alpha_j \mid n_1, \ldots, n_d \in \mathbb{Z}\} \) of \((P, \{\phi_t\})\).
Then \((\hat{P}, \{\hat{\phi}_t\})\) is strictly ergodic if and only if there does not exist a measurable solution \(R\) of the so-called cohomology equation

\[
R(\phi_t(p)) - R(p) = \int_0^1 r(\phi_s(p))ds.
\]

(2.1)

If there exists a measurable but non-continuous solution of the cohomology equation (2.1), then \((\hat{P}, \{\hat{\phi}_t\})\) is minimal but admits uncountably many distinct ergodic measures. See [20] for details; one can determine whether or not \(R\) exists using Fourier series arguments. Finally, if \(\int_P r dv\) does not lie in the tensor product \(\mathcal{F} \otimes Z \mathbb{Q} = \{\sum_{i=1}^d q_i \alpha_i \mid q_1, \ldots, q_d \in \mathbb{Q}\}\), then \((\hat{P}, \{\hat{\phi}_t\})\) is strictly ergodic.

Next consider a time-dependent differential system

\[
\dot{x} = f(t, x), \quad x \in \mathbb{R}^d.
\]

(2.2)

Under appropriate assumptions, one can apply the methods of topological dynamics to study the solutions of (2.2) by introducing a flow of Bebutov type [4]. We explain briefly how this is done.

Introduce the space

\[
\mathcal{C} := C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d) = \{g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \mid g \text{ continuous}\}.
\]

We put on \(\mathcal{C}\) the topology of uniform convergence on compact sets. We introduce the Bebutov (translation) flow \(\{\phi_t\}\) on \(\mathcal{C}\), defined by \(\phi_t(g)(\cdot, \cdot) = g(\cdot + t, \cdot)\). Suppose now that \(f \in \mathcal{C}\) has the property that, for each compact subset \(K \subset \mathbb{R}^d\), the restriction of \(f\) to \(\mathbb{R} \times K\) is uniformly continuous and bounded. Then one concludes that \(P = \text{cls}\{\phi_t(f) \mid t \in \mathbb{R}\} \subset \mathcal{C}\) is a compact translation invariant set. It is useful to modify this construction as follows. If \(r \geq 1\), let \(\mathcal{C}' := C^{0,r}(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)\), i.e. if \(g \in \mathcal{C}'\) then each derivative \(D_{x_1}^{s_1} \cdots D_{x_d}^{s_d} g(\cdot, \cdot)\) is continuous on \(\mathbb{R} \times \mathbb{R}^d\) whenever \(0 \leq s_1, \ldots, s_d \leq r\), and \(s_1 + \cdots + s_d \leq r\). One topologizes \(\mathcal{C}'\) by requiring that a sequence \(\{g_k\} \subset \mathcal{C}'\) converges to \(g\) if \(D_{x_1}^{s_1} \cdots D_{x_d}^{s_d} g_k \to D_{x_1}^{s_1} \cdots D_{x_d}^{s_d} g\) uniformly on compact subsets of \(\mathbb{R} \times \mathbb{R}^d\), (and \(0 \leq s_1, \ldots, s_d \leq r\), and \(s_1 + \cdots + s_d \leq r\)). Again there is a Bebutov flow on \(\mathcal{C}'\). If \(f\) is \(C^r\) differentiable in \(x\), and \(f\) together with all its derivatives of order less or equal to \(r\) are uniformly continuous and bounded on \(\mathbb{R} \times K\) for each compact subset \(K \subset \mathbb{R}^d\), then \(P = \text{cls}\{\phi_t(f)\} \subset \mathcal{C}'\) is compact and \(\{\phi_t\}\) invariant. In this case it is convenient to define \(f : P \times \mathbb{R}^d \to \mathbb{R}^d\) so that \(\hat{f}(p, x) = p(0, x)\); thus \(\hat{f}(\phi_t(p), x) = p(t, x)\) for any \((t, x) \in \mathbb{R} \times \mathbb{R}^d\). In particular if \(p\) coincides with \(f\) then, \(\hat{f}(\phi_t(f), x) = f(t, x)\) and we, so to say, recuperate the original function \(f\). We will usually abuse the notation and write \(f\) instead of \(\hat{f}\).

The result of the Bebutov construction which begins with equation (2.2), is then a family of O.D.E.

\[
\dot{x} = f(\phi_t(p), x) \quad p \in P, \ x \in \mathbb{R}^d
\]

(2.2p)

where \(P\) is a compact metric space, \(\{\phi_t\}\) is a flow on \(P\), and \(f : P \times \mathbb{R}^d \to \mathbb{R}^d\) is a continuous function whose derivatives \(D_{x_1}^{s_1} \cdots D_{x_d}^{s_d} f(\cdot, \cdot)\) are all continuous on \(P \times \mathbb{R}^d\) (for \(0 \leq s_1 \leq \cdots \leq s_d \leq r, s_1 + \cdots + s_d \leq r, r \geq 0\)).

Next we review some standard facts concerning the Andronov-Hopf bifurcation pattern. Let us return to equation (1.1):

\[
\frac{dx}{dt} = f(x, \varepsilon) \quad x \in \mathbb{R}^2, \ \varepsilon \in \mathbb{R}.
\]

To simplify matters we assume that \(f\) is a \(C^\infty\) function of \((x, \varepsilon) \in \mathbb{R}^2 \times \mathbb{R}\), although finite regularity will be all that will be actually needed at any given moment. We
assume that (1.1) satisfies the standard conditions for a supercritical Hopf bifurcation for \( \varepsilon = 0 \). We follow Kuznetsov [33] and Kocak-Hale [30] in discussing these conditions and in deriving consequences of them.

Since \( f(0, \varepsilon) = 0 \) for all \( \varepsilon = 0 \), we can consider \( f(x, \varepsilon) \) to be the sum of its linear part \( L \) in \( x = 0 \), and its higher order part \( N \), so that (1.1) takes the form

\[
\frac{dx}{dt} = f(x, \varepsilon) = L(\varepsilon)x + N(x, \varepsilon).
\]  

(2.3)

Here \( L \) is a \( 2 \times 2 \) real matrix, and the derivative \( D_x N(0, \varepsilon) = 0 \) for all \( \varepsilon \in \mathbb{R} \). Both \( L \) and \( N \) are \( C^\infty \) in their arguments. Assume that \( L(0) \) has complex conjugate pure imaginary eigenvalues \( \pm ib(0) \); then choose \( E \subset \mathbb{R} \) to be an open subinterval containing \( \varepsilon = 0 \) so that, if \( \varepsilon \in E \), then \( L(\varepsilon) \) has complex conjugate eigenvalues \( a(\varepsilon) \pm ib(\varepsilon) \). The functions \( a(\cdot) \), \( b(\cdot) \) are of class \( C^\infty \). We assume that \( \frac{dE}{d\varepsilon}(0) > 0 \), so that \( \text{sign}[a(\varepsilon)] = \text{sign}[\varepsilon] \) for \( \varepsilon \in E \setminus \{0\} \). Following Kuznetsov [33], one can determine a \( C^\infty \) change of variables \( x = h(y, \varepsilon) \), which is in fact a polynomial in \( y \), such that, in some neighborhood \( V \) of \( y = 0 \) and in some perhaps smaller interval \( E \subset \mathbb{R} \) containing \( \varepsilon = 0 \), one has

\[
\frac{dy}{dt} = \begin{pmatrix}
a(\varepsilon) & -b(\varepsilon) \\
b(\varepsilon) & a(\varepsilon)
\end{pmatrix} y - \begin{pmatrix}
c(\varepsilon) & d(\varepsilon) \\
d(-\varepsilon) & c(\varepsilon)
\end{pmatrix} |y|^2 y + R(y, \varepsilon)
\]  

(2.4)

where \( c(\cdot), d(\cdot) \) are \( C^\infty \) functions of \( \varepsilon \), and the \( C^\infty \) remainder term \( R \) is of order \( O(|y|^5) \) as \( y \to 0 \) in \( V \). Note that there are neither second-order nor fourth-order terms in (2.4).

Let us now assume that \( c(0) > 0 \). Then there is no loss of generality in assuming \( c(\varepsilon) > 0 \) for all \( \varepsilon \in E \). In this situation, the one-parameter family (2.4) admits a supercritical Andronov-Hopf bifurcation in \( \varepsilon = 0 \), i.e. it satisfies a) and b) of the Introduction. We assume that these statements hold on \( E \). There is no loss of generality in assuming that the limit cycle is, for \( \varepsilon \in E \), the only periodic solution of (2.4) in \( V \).

We will need some facts which follow from the theory outlined above, and which are collected in Lemma 2.4 below. To prove them, it is convenient to make some changes of variables in equation (2.4). First we scale the time variable by a factor \( b(\varepsilon) \), and write \( \tau = b(\varepsilon)t \). Then we introduce a new parameter \( e = \frac{a(\varepsilon)}{b(\varepsilon)} \); this is permissible because \( \frac{dE}{d\varepsilon}(0) = \frac{a'(0)}{b(0)^2} > 0 \). Next we scale the state variable by a factor \( (\frac{\xi}{\varepsilon})^{1/2} \), and write \( \xi = (\frac{\xi}{\varepsilon})^{1/2} y \). Also we write \( \varpi = \frac{\xi}{\varepsilon} \). Finally we abuse the notation by returning to the old variables and parameter: we write \( t \) for \( \tau \), \( \varepsilon \) for \( e \), and \( y \) for \( \xi \). In this way we obtain the following system, which is equivalent to (2.4) on some neighborhood \( V \times E \) of \((0,0)\) in \( \mathbb{R}^2 \times \mathbb{R}^2 \):

\[
\frac{dy}{dt} = \begin{pmatrix}
\varepsilon & -1 \\
1 & \varepsilon
\end{pmatrix} y - \begin{pmatrix}
1 & \varpi(\varepsilon) \\
-\varpi(\varepsilon) & 1
\end{pmatrix} |y|^2 y + O(|y|^5)
\]  

(2.5)

Let us now introduce polar coordinates \((r, \theta)\) in the \( y = \begin{pmatrix}y_1 \\ y_2\end{pmatrix}\) plane: \( r^2 = y_1^2 + y_2^2 \) and \( \theta = \arctan(y_2/y_1) \). Then equation (2.5) takes the form

\[
\begin{align*}
\frac{dr}{d\tau} &= \varpi r - r^3 + \gamma_1(r, \theta, \varepsilon) \\
\frac{d\theta}{d\tau} &= 1 + \varpi(\varepsilon) r^2 + \gamma_2(r, \theta, \varepsilon)
\end{align*}
\]  

(2.6)

where \( \gamma_1 \) and \( \gamma_2 \) are \( 2\pi \)-periodic in \( \theta \) and are \( C^\infty \) in all variables. Also \( \gamma_1 = O(r^3) \) and \( \gamma_2 = O(r^4) \) as \( r \to 0 \), uniformly in \( \theta \in [0, 2\pi] \) and \( \varepsilon \in E \).
The stable limit cycle guaranteed by the Andronov-Hopf theorem is supported on a smooth curve $\Gamma$ in the $(r, \theta)$-plane which is parametrized by $\theta$; that is $\Gamma$ is the graph \{$(r(\theta), \theta)$ | $0 \leq \theta \leq 2\pi$\} of a smooth $2\pi$-periodic function $r(\cdot) = \Gamma(\cdot) = \sqrt{c} + O(\varepsilon)$.

We now go back to the non-autonomous case, so we consider equation (1.5): we will need some detailed information concerning the solutions of eq. (1.5) in a neighbourhood of the limit cycle $\Gamma$.

We perform the changes of variables just described, so that (1.5) becomes

\[
\begin{align*}
\frac{dr}{dt} &= \varepsilon r - r^3 + \gamma_1(r, \theta, \varepsilon) + \mu \gamma_1(\phi_t(p), r, \theta, \varepsilon, \mu) \\
\frac{d\theta}{dt} &= 1 + \omega(\varepsilon) r^2 + \gamma_2(r, \theta, \varepsilon) + \mu \gamma_2(\phi_t(p), r, \theta, \varepsilon, \mu)
\end{align*}
\]

Even if the main focus of the paper will be on the case where the origin is a critical point for (1.5) for any $\mu$, we consider briefly the case where $g(t, 0, \varepsilon, \mu)$ may be different from 0. This way we can compare the results obtained via our deterministic but non-autonomous approach, with those obtained via a stochastic analysis in [6]. Note that if $g(t, 0, \varepsilon, \mu) \equiv 0$ then $\gamma_1(\phi_t(p), 0, \theta, \varepsilon, \mu) \equiv 0$ and $\gamma_2(\phi_t(p), r, \theta, \varepsilon, \mu)$ is bounded. However, if we let $g(t, 0, \varepsilon, \mu) \neq 0$ then $\gamma_1(\phi_t(p), 0, \theta, \varepsilon, \mu)$ is just bounded and $\gamma_2(\phi_t(p), r, \theta, \varepsilon, \mu) \sim 1/r$ so it is not defined for $r = 0$.

We have the following.

**Lemma 2.2.** Assume that there exists $q > 1/2$ such that $g(p, 0, \varepsilon, \mu) = O(\varepsilon^q)$, uniformly in all the variables, and that $\mu = o(\varepsilon)$. Then for any $s \in (1/2, q]$ there exist $\varepsilon_0 = \varepsilon_0(q, s), c = c(q, s) > 0$ such that the annulus $\mathcal{A}$ and the disc $\mathcal{D}$

\[
\mathcal{A} := \{(r, \theta) \mid \sqrt{c} - c\varepsilon^s \leq r \leq \sqrt{c} + c\varepsilon^s, 0 \leq \theta \leq 2\pi\}
\]

\[
\mathcal{D} := \{(r, \theta) \mid r \leq c\varepsilon^s, 0 \leq \theta \leq 2\pi\}
\]

are respectively positively and negatively invariant for (2.7), for any $0 < \varepsilon \leq \varepsilon_0$.

**Proof.** From a straightforward computation we see that for a point $(r_0, \theta)$ on the boundary of $\mathcal{D}$ it is the case that $r(\cdot, \theta) = c\varepsilon^{1+s} + O(\varepsilon^{3s} + \mu^2)$. Thus we see that $\mathcal{D}$ is negatively invariant.

We introduce the variable $\mathcal{R} = -1 + r/\sqrt{c}$ so that (2.7) is turned into

\[
\begin{align*}
\dot{\mathcal{R}} &= -2c\mathcal{R} - 3c\mathcal{R}^2 + \varepsilon^2 A(\theta, \mathcal{R}) + \mu \dot{B}(\phi_t(p), \theta, \mathcal{R}) \\
\dot{\theta} &= 1 + \varepsilon \omega(\varepsilon)(1 + \mathcal{R})^2 + \varepsilon^2 \dot{C}(\theta, \mathcal{R}) + \mu \dot{D}(\phi_t(p), \theta, \mathcal{R})
\end{align*}
\]

where the functions $A, B, C = C\varepsilon^{1/2}, \tilde{D} = D\varepsilon^{1/2}$ in fact depend on $\varepsilon$ and $\tilde{B}, \tilde{D}$ on $\mu$ too, and they are all smooth in $\theta$ and $\mathcal{R}$. Note also that $\tilde{D} = O(\varepsilon^{3s-1/2})$. Let us set $\delta = \delta(\varepsilon) = c(s)\varepsilon^{s-1/2}$, so that $\delta$ is small but $\varepsilon^{1/2} < \delta$, and $\tilde{D} = O(\varepsilon^{3s-1/2}) < \delta$ are smaller. Then observe that if $\mathcal{R}_2 = \delta$, then $\dot{\mathcal{R}}(\mathcal{R}_2, \theta)$ is negative for any $\theta$, while if $\mathcal{R}_1 = -\delta$, then $\dot{\mathcal{R}}(\mathcal{R}_1, \theta)$ is positive for any $\theta$, so we find that $\mathcal{A}$ is positively invariant.

With this set of assumptions it can probably be shown that $\mathcal{D}$ contains a repeller (an attracting set in negative time), while $\mathcal{A}$ contains an attractor. The repeller may well be a copy of the base (i.e. its dynamics is isomorphic to $\{\phi_t\}$), while it is difficult to get information on the shape of the attractor. It likely can be retracted onto a circle bundle over $P$, but its dynamical properties remain somewhat of a mystery. These questions might form the object of future investigations.

For the present paper we go back to consider the case where the origin is a critical point for any $\mu$, i.e. $g(\phi_t(p), 0, \varepsilon, \mu) \equiv 0$. Consequently $\gamma_1(\phi_t(p), 0, \varepsilon, \mu) \equiv 0$, $\gamma_2(\phi_t(p), 0, \varepsilon, \mu)$ is well defined and bounded; moreover $\dot{B}(\phi_t(p), \theta, 0) = O(\varepsilon^{1/2})$. 

and \( \bar{D} \) is well defined and bounded. Using these observations and repeating the proof of Lemma 2.2, we get the following.

**Lemma 2.3.** Assume \( g(p, 0, \varepsilon, \mu) \equiv 0 \) for (1.5) and consequently \( \gamma_1(p, 0, \varepsilon, \mu) \equiv 0 \) for (2.7). For any \( s \in (1/2, 1] \) there exist numbers \( \varepsilon_0(s) \) and \( c = c(s) > 0 \) such that the annulus \( \mathfrak{A} \) and the disc \( \mathcal{D} \) are respectively positively and negatively invariant for (2.7) for any \( 0 < \varepsilon < \varepsilon_0(s) \).

Note that, with this new setting, the restriction on the smallness of \( \delta \) is due to the presence of the \( \theta \)-dependent term \( \bar{A} \). In fact for \( \mu = 0 \), the annulus \( \mathfrak{A} \) contains the manifold \( \Gamma_\varepsilon \), which lives in a strip of size \( \varepsilon \). We can improve slightly the argument with a further change of variable: let us set \( \rho = \rho(r, \theta) = |r - \Gamma_\varepsilon|/\sqrt{\varepsilon} = \mathfrak{A} + K(\theta) \), and observe that \( K(\theta) = O(\sqrt{\varepsilon}) \) uniformly in \( \theta \). Denote by \( A(r, \theta) = A_0(\theta) + \bar{A}_1(r, \theta) \), with this new variable the first equation in (2.7), for \( \mu = 0 \) becomes

\[
\dot{\rho} = \mathfrak{B} + K'(\theta) \dot{\theta} = -2\varepsilon [\rho + K(\theta)] - 3\varepsilon [\rho + K(\theta)]^2 + \varepsilon^2 [\bar{A}_0(\theta) + \bar{A}_1(\theta, \rho + K(\theta))] + K'(\theta) \dot{\theta}
\]

Recalling that \( \Gamma_\varepsilon \) is invariant for \( \mu = 0 \) we see that \( \rho = 0 \) is invariant for \( \mu = 0 \). Hence the the term of order 0 in \( \rho \) in the right hand side of (2.9) is identically zero. Therefore we can rewrite (2.7) as follows

\[
\dot{\rho} = -2\varepsilon \rho - 3\varepsilon \rho^2 + \varepsilon^{3/2} \rho A(\rho, \theta) + \mu B(\phi_1(p), \rho, \theta) \quad \dot{\theta} = 1 + \alpha \varepsilon(\varepsilon)(1 + \rho)^2 + \varepsilon^{3/2} C(\rho, \theta) + \mu D(\phi_1(p), \rho, \theta)
\]

where the remainder term \( A(\rho, \theta) \) is bounded and its leading term is contained in \( K'(\theta) \dot{\theta} \).

As already specified in Lemmas 2.2 and 2.3, for our purposes we need a smallness condition on \( \mu \) which is not consistent with a linear relationship between the parameters: let \( \delta = \delta(\varepsilon) > 0 \) be a value which measures the size of the annulus \( \mathfrak{A} \). In fact \( \delta > 0 \) can be any continuous increasing function such that \( \delta(0) = 0 \). Then we set

\[
\mu = O(\varepsilon \delta^2)
\]

With these new coordinates we can rephrase Lemma 2.3 as follows:

**Lemma 2.4.** Assume \( \gamma_1(p, 0, \theta, \varepsilon, \mu) \equiv 0 \) (i.e. \( g(p, 0, \varepsilon, \mu) \equiv 0 \) for (1.5)). For any \( \delta = \delta(\varepsilon) > 0 \) there is \( \varepsilon_0 = \varepsilon_0(\delta) \) such that the strip

\[
\mathfrak{S} = \{(\rho, \theta) \mid |\rho| \leq \delta \}
\]

is positively invariant for (2.10) for any \( 0 < \varepsilon < \varepsilon_0 \) and any \( \mu \) satisfying (2.11).

Consequently, the annulus

\[
\bar{\mathfrak{A}} = \{(r + r_1, \theta) \mid r = \Gamma_\varepsilon(\theta), \; |r_1| \leq \delta \sqrt{\varepsilon} \}
\]

is positively invariant for (2.7) for any \( 0 < \varepsilon < \varepsilon_0 \) and any \( \mu \) satisfying (2.11).

Again the proof is a repetition of that of Lemma 2.2 so it is omitted.

The object of Lemma 2.4 is to show that we can find a positively invariant set inside \( \sqrt{\varepsilon} - c\varepsilon \leq \bar{r} \leq \sqrt{\varepsilon} + c\varepsilon \), whose size depends on the ratio \( \mu/\varepsilon \), and therefore gets smaller and smaller as \( \mu \to 0 \).

We close this section with a discussion of a concept which will be found useful in section 4, namely that of rotation number. This quantity is defined for a circle extension of a compact metric base flow, once an ergodic measure has been fixed on that base flow. Let \( P \) be a compact metric space, and let \( \{\phi_t\} \) be a flow on \( P \). Let \( S^1 \) be the unit circle in the complex plane, let \( M = P \times S^1 \), and let...
π : M → P : (p, z) → p be the natural projection. A circle extension of \{ϕ_t\} is by definition a flow \{ψ_t\} on M with the following property: π ◦ ψ_t(p, z) = ϕ_t ◦ π(p, z) for each m = (p, z) ∈ M.

Let θ be the polar angle in C, which we identify both as a coordinate on S^1 and as a coordinate on the covering space R of S^1. We will consider circle extensions which are defined by differential equations. More precisely let Θ : P × S^1 → R : (p, θ) → Θ(p, θ) be a continuous function which is Lipschitz in θ, uniformly in p ∈ P. Consider the family of differential systems

\[
\frac{dθ}{dt} = Θ(ϕ_t(p), θ(t))
\]

Then the solutions of equation (2.12) define a flow \{ψ_t\} on M which is a circle extension of \{ϕ_t\}. In fact define ψ_t(p, θ_0) = (ϕ_t(p), θ(t)) where θ(t) is the unique solution of equation (2.12) which satisfies θ(0) = θ_0. Then one can check that \{ψ_t\} is a circle extension of \{ϕ_t\}.

Let ν be a \{ϕ_t\}-ergodic measure on P.

**Proposition 2.5.** There is a set P_1 ⊂ P with ν(P_1) = 1 such that, if p ∈ P_1, θ_0 ∈ R, and θ(t) = θ(t, θ_0, p) is the solution of equation (2.12) such that θ(0) = θ_0, then the limit

\[
\lim_{|t|→∞} \frac{θ(t)}{t} := ρ_ν
\]

exists and does not depend on the choice of p ∈ P_1, θ_0 ∈ R. If ν is the unique \{ϕ_t\}-ergodic measure on P, then

\[
ρ_ν = \lim_{|t|→∞} \frac{θ(t, θ_0, p)}{t}
\]

where the limit is uniform in (p, θ) ∈ M = P × S^1.

The number ρ_ν is called the ν-rotation number of the circle extension \{ψ_t\}. For the reader’s convenience we sketch the proof of Proposition 2.5. We can write

\[
θ(t) := θ(θ_0, t) = θ_0 + \int_0^t Θ(ϕ_s(p), θ(s))ds = θ_0 + \int_0^t Θ(ψ_s(p, θ_0))ds
\]

for each solution θ(t) of equation (2.12). Let \hat{ν} be a \{ϕ_t\}-ergodic measure such that \hat{ν}(π^{-1}(B)) = ν(B) for each Borel set B ⊂ P. Such an “ergodic” lift exists. By the Birkhoff theorem 2.1, there is a set M_1 ⊂ M, of \hat{ν}-measure 1 such that if (p, θ_0) ∈ M_1, then \lim_{|t|→∞} \frac{θ(t)}{t} = \lim_{|t|→∞} \frac{1}{|t|} \int_0^t Θ(ψ_s(p, θ_0))ds = ∫_M Θd\hat{ν}. Fix p ∈ P; using the 2π-periodicity of the θ-variable, one shows that \lim_{|t|→∞} \frac{θ(t)}{t} exists for one value θ_0 ∈ R if and only if it exists for all θ_0 ∈ R. So in fact ∫_M Θd\hat{ν} depends only on ν and not on the choice of the ergodic lift \hat{ν} of ν, and ∫_M Θd\hat{ν} = p_ν as defined in (2.13). This proves the first part of Proposition 2.5. The second part of Proposition 2.5 follows from what has been said and the second part of Theorem 2.1.

3. Integral manifold theory. In this section, we work out an integral manifold theory which is appropriate for our purposes. We will make use of ideas and techniques of Yi [40] and of Hirsch-Pugh-Shub [24]. The methods of Yi are appropriate when the phase space P of the driving flow \(P, \{ϕ_t\}\) is a general compact metric space, while those of [24] are appropriate when P is a smooth manifold and one requires the integral manifold M to be smooth.
Definition 3.1. We say that a surface $S$ in $\mathbb{R}^n \times \mathbb{R}$ is an integral manifold for $\dot{z} = P(z,t)$ if, for any $p = (z_0,t) \in S$, the solution $z(t)$ of the equation through $p$ (i.e. such that $z(\tau) = z_0$) satisfies $(z(t), t) \in S$ for all $t$ in the domain of definition of $z(t)$.

Our purpose is to construct an integral manifold $\Psi_{\mu}(p, \theta, \varepsilon)$ made up by trajectories of (1.5) which are bounded for any $t \in \mathbb{R}$, bifurcating from the limit cycle $\Gamma_\zeta$, i.e. $\Psi_{\mu}(p, \theta, \varepsilon) = \Gamma_\varepsilon(\theta)$ for any $\theta \in \mathbb{R}$ and any $\varepsilon \in \mathbb{E}$.

Let us first consider the case in which $P$ is assumed only to be a compact metric space with metric $d$.

Proposition 3.2. Let $s > 1$ be a real number. Let the functions $B$ and $D$ of (2.10) be Lipschitz continuous in $p$, uniformly for all relevant values of $\rho$ and $\theta$. Suppose that the metric $d$ satisfies the following condition:

$$\sup_{p_1 \neq p_2 \in P} \left\{ \limsup_{t \to \infty} \frac{1}{t} \ln \left( \frac{d(\phi_t(p_1), \phi_t(p_2))}{d(p_1, p_2)} \right) \right\} = 0$$

There exist numbers $\delta_0 = \varepsilon_0(s) > 0$ and $c_0 = c_0(s) > 0$ such that, if $0 < \varepsilon \leq \varepsilon_0$ and if $|\mu| \leq c_0 \varepsilon^s$, then there is a Lipschitz function $v_\mu : P \times S^1 \to \mathbb{R}$ such that

$$M_\mu = \{ (p, v_\mu(p, \theta), \theta) \mid p \in P, 0 \leq \theta \leq 2\pi \} \subset P \times \mathbb{R}^2$$

is invariant with respect to the flow on $P \times \mathbb{R}^2$ generated by equations (2.10). Let $r \geq 0$ be an integer, and let $C = C(\rho, \theta)$ be the coefficient in (2.10). If

$$\sup_{0 \leq \theta \leq 2\pi} \left| \frac{\partial C}{\partial \theta}(0, \theta) \right| \leq \frac{1}{(r + 1)\sqrt{\varepsilon_0}},$$

then $v_\mu$ is of class $C^r$ in $(\theta, \varepsilon, \mu)$.

Proof. (Sketch) We use Theorem 6.1 of [40] together with a well-known roughness result of Coppel ([14], Prop. 1 Chapter 4). Let us write the $\rho$-equation in (2.10) as

$$\dot{\rho} = -2\varepsilon \rho - 3\varepsilon \rho^2 + \varepsilon^{3/2} \rho A(\rho, \theta) + \mu B(\phi_t(p), \rho, \theta)$$

We view the linear part of this equation as a perturbation of $\dot{\rho} = -2\varepsilon \rho$. Set

$$\delta_0 = \delta_0(\varepsilon) = \varepsilon^{3/2} \sup_{0 \leq \theta \leq 2\pi} |A(0, \theta)| + \hat{c}_0 \varepsilon^s \sup_{p \in P, 0 \leq \theta \leq 2\pi} \left| \frac{\partial B}{\partial \rho}(p, 0, \theta) \right|$$

and choose $\varepsilon_0, \hat{c}_0$ so that $\delta_0(\varepsilon) < \varepsilon/2$ for all $0 < \varepsilon \leq \varepsilon_0$. According to Proposition 1 in Chapter 4 of [14], for each $0 < \varepsilon \leq \varepsilon_0$ the linearized $\rho-$ equation admits an exponential dichotomy for each orbit $\phi_{\nu}(p)$ of the flow $\{\phi_t\}$ on $P$ and each solution of the $\theta-$ equation in (2.10). Moreover the dichotomy constants $(K, \alpha)$ are such that $\alpha \geq \varepsilon$ for $0 < \varepsilon \leq \varepsilon_0$.

Now use the proof of ([40], Theorem 6.1) to check that, if $r \geq 0$ is an integer, and if

$$\varepsilon^{3/2} \sup_{0 \leq \theta \leq 2\pi} \left| \frac{\partial C}{\partial \theta}(0, \theta) \right| \leq \frac{\varepsilon}{r + 1} \quad (0 < \varepsilon \leq \varepsilon_0),$$

then there exists $c_0 = c_0(r) \leq \hat{c}_0$ such that, if $|\mu| \leq c_0 \varepsilon^s$, then there exists a map $v_\mu$ with the indicated properties.

Next we assume that $P$ is a $C^\infty-$smooth manifold, and the dynamics on $P$ is determined by a $C^\infty$ vector field $h = h(p)$. Then the differential equation $\dot{p} = h(p)$ gives rise to the solution $t \to \phi_t(p)$, and $\{\phi_t \mid t \in \mathbb{R}\}$ is a flow on $P$. 
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We start again from eq. (2.10), so the variable \( \rho = |r - \Gamma_{\varepsilon}(\theta)|/\sqrt{\varepsilon} \) measures the radial distance from the attractive cycle obtained for \( \mu = 0 \) and \( \varepsilon > 0 \). We also embed (2.10) in a higher dimensional system where we have added the dynamics on the base \( P \): for \( p \in P \) we consider the differential equation \( \dot{p} = h(p) \) whose solution is \( \phi_t(p) \). We get

\[
\begin{align*}
\dot{p} &= -2\varepsilon \rho - 3\varepsilon \rho^2 + \varepsilon^{3/2}\rho A(\rho, \theta) + \mu B(\phi_t(p), \rho, \theta) \\
\dot{\theta} &= 1 + \varepsilon \varpi(\varepsilon)(1 + \rho)^2 + \varepsilon^{3/2}C(\rho, \theta) + \mu D(\phi_t(p), \rho, \theta) + .
\end{align*}
\tag{3.1}
\]

Note that (3.1) is autonomous, and has a skew-product structure, i.e. the dynamics in \( P \) is not influenced by the flow on \( (\rho, \theta) \). Note further that for \( \mu = 0 \) the manifold \( V_0 = \{(\rho, \theta, p) \mid \rho = 0, \theta \in S^1, p \in P\} \) is an invariant centre manifold for the flow of (3.1), corresponding to the invariant cycle \( \Gamma_{\varepsilon} \). We want to show that for \( \mu \neq 0 \) this invariant manifold perturbs to an invariant manifold \( V_\mu \) for (3.1), which is \( O(\mu) \) close to \( V_0 \). We recall that the manifold \( V_\mu \) in the variable \( \rho \) of eq. (3.1) corresponds to the invariant manifold \( M = M_{\mu}(p, \theta, \varepsilon) \) in the original variables.

We introduce the following assumption:

**H**: Let \( p(t) \) be a solution of \( \dot{p} = h(p) \), and let \( A(t) \) be the solution of the variational equation \( \dot{P} = \partial^2_{pp}(p(t))P \), such that \( A(0) = I \). Then all the eigenvalues of \( A(\tau) \) have modulus 1.

Assumption **H** is satisfied by any Kronecker flow, which is the main class of examples we are interested in, when \( P \) is a smooth.

Our argument is a translation for this context of [24], see in particular section 4, following the outline of [27]. In particular we need to apply theorem 4.1 point “f” in [24], which guarantees the persistence under small perturbation of a centre manifold for a system with discrete time. Then we pass from discrete to continuous time using the ideas in Theorem 2.16 of [27].

We briefly sketch the argument to help the reader to understand the role played by the two small parameters involved.

We need to introduce some notation. From now on we take \( \varepsilon > 0 \) to be fixed.

We set

\[ E = \{ (\rho, \theta, p) \mid |\rho| \leq \delta, (\theta, p) \in S^1 \times P \}. \]

If \( (\theta, p) \in S^1 \times P \) we set \( \|(\theta, p)\| = \max\{|\theta|, ||p||_P\} \). From Lemma 2.4 we know that \( E \) is positively invariant for the flow of (3.1). We denote by \( P, P_\theta, P_p, P_\rho \) the projection \( P(\rho, \theta, p) = (\theta, p), P_{\theta}(\rho, \theta, p) = \theta, P_{\rho}(\rho, \theta, p) = \rho \). Fix \( \tau > 0 \); we need to introduce a discretization; i.e. let \( \Phi(\theta_0, \rho_0, p_0) \) be the solution of (3.1) such that \( \Phi_0(\theta_0, \rho_0, p_0) = (\theta_0, \rho_0, p_0) \) and let \( F_\mu : E \to E, F_\mu = \Phi_\tau \). Observe that \( V_0 \) is an invariant centre manifold for \( F_0 \), i.e for \( \mu = 0 \).

Let \( \sigma : (S^1 \times P) \to E \) be a \( C^r \) function (a section), i.e. \( \sigma(v) = (v, s(v)) \) and \( s : (S^1 \times P) \to [-\delta, \delta] \) is \( C^r \). We define the slope of a section \( \sigma \) as

\[ ||\sigma||_{sl} := \sup \left| \frac{\partial s(\theta, p)}{\partial \theta, \partial p} \right| \mid (\theta, p) \in S^1 \times P \]

and we consider the set \( \Sigma := \{ \sigma : (S^1 \times P) \to E \mid ||\sigma||_{sl} \leq \delta \} \). We endow \( \Sigma \) with the \( C^0 \) norm, which makes it complete, see [24] for details.

We are ready to state the following result on the discrete map \( F_\mu \), rephrased from [24].
Proposition 3.3. Assume $H$ and that $f$ and $g$ are $C^r$ in all their variables. Then there is a $C^r$ function $v_\mu : S^1 \times P \to \mathbb{R}$, such that the manifold

$$V_\mu = \{(v_\mu(\theta, p), \theta, p) \mid \theta \in S^1 \ p \in P\}$$

is invariant for forward and backward iterates of $F_\mu$. Moreover $\|v_\mu\| = O(\delta)$ and $\|\partial v_\mu / \partial p\| \leq \delta$.

Proof. The plan of the proof is as follows. Fix $\sigma \in \Sigma$ and consider the map $H_\mu : (S^1 \times P) \to (S^1 \times P)$ defined by $H_\mu = \Pi \circ F_\mu \circ \sigma$: it is easy to check that this map is surjective since $F_\mu$ is a diffeomorphism. Following the proof of Theorem 4.1 in [24], in particular point (v) on pag. 44, it can be shown that, for any fixed $\sigma \in \Sigma$, $H_\mu$ is injective too, so it admits a proper inverse, say $H_\mu^{-1}$.

Next we follow again Theorem 4.1 in [24] and we define the map $F_\mu^\sharp : \Sigma \to \Sigma$ as $F_\mu^\sharp(\sigma) = F_\mu \circ \sigma \circ H_\mu^{-1}$. For $\mu = 0$, such a map is a contraction and has the null section as unique fixed point, which corresponds to the unperturbed centre manifold $\rho = 0$, that is $\Gamma_\varepsilon$ in the original coordinates. Then, following the ideas of the proof of Theorem 4.1 point f) in [24] (pagg. 49-51), we see that $F_\mu^\sharp$ is a contraction for $\mu \neq 0$ too, and its unique fixed point $\sigma^\sharp$ parametrizes the integral manifold $V_\mu$. The argument is developed in several steps.

Step 1. Using the fact that $\mu = o(\varepsilon)$ we see that the map $H_\mu = \Pi \circ F_\mu \circ \sigma$ is bijective for any fixed $\sigma$ and we denote by $H_\mu^{-1}$ its inverse.

To prove this claim we need to evaluate the derivatives of the functions $F_\mu$, and $H_\mu = \Pi \circ F_\mu \circ \sigma$. By construction $H_\mu(\theta, p(0)) = (\theta', p(\tau))$ for a certain $\theta' \in \mathbb{R}$, so it is clearly invertible in its $p$ component. In fact

$$\frac{\partial}{\partial \theta} \Pi_p[F_\mu(p, \theta, p)] = 0$$

due to the skew-product nature of the flow.

Observe further that $F_\mu(0, \theta, p(0)) = (0, \theta + \tau + O(\varepsilon), p(\tau))$. Moreover

$$\Pi_\theta \circ F_\mu \circ \sigma(\theta, p) = \Pi_\theta[F_\mu(s(\theta, p), \theta, p)]$$

$$= \theta + \tau[1 + \varepsilon \varpi(\varepsilon)] + 2\varepsilon \varpi(\varepsilon)O(\delta) + O(\varepsilon \delta^2 + \varepsilon^{3/2} + \mu).$$

Differentiating (3.1) we get

$$\frac{\partial}{\partial \theta} (\Pi_\theta \circ F_\mu \circ \sigma) = \frac{d\Pi_\theta F_\mu}{d\theta}(s(\theta, p), \theta, p) =$$

$$= \frac{\partial \Pi_\theta F_\mu}{\partial \theta}(s(\theta, p), \theta, p) + \frac{\partial F_\mu}{\partial \rho}(s(\theta, p), \theta, p) \frac{\partial s(\theta, p)}{\partial \theta} =$$

$$= 1 + \varepsilon^{3/2} + O(\mu) + 2\varepsilon \varpi(\varepsilon)(1 + O(\delta)) \frac{\partial s(\theta, p)}{\partial \theta} \geq 1 - 4\varpi(\varepsilon)\varepsilon \delta > 0$$

where we used (3.1). Reasoning similarly, differentiating (3.1) with respect to $p$ we have $\frac{\partial}{\partial p} H_\mu(\theta, p) = O(\mu)$. Note that $\frac{\partial}{\partial p} \Pi_p H_\mu(\theta, p) = A(\tau)$ where the matrix $A(\tau)$ has been defined in assumption $H$. So summing up we have

$$\begin{pmatrix}
\frac{\partial}{\partial \theta} \Pi_\theta H_\mu \\
\frac{\partial}{\partial \rho} \Pi_\rho H_\mu \\
\frac{\partial}{\partial p} \Pi_p H_\mu
\end{pmatrix} = \begin{pmatrix}
1 + O(\varepsilon) & O(\mu) \\
0 & A(\tau)
\end{pmatrix}.$$

Then, using assumption $H$, it follows that $H_\mu$ is invertible.

For later purposes we need to estimate several quantities associated to $H_\mu$ and its inverse.
Remark 3.4. The estimate (3.3) holds for $H^{-1}_\mu$ too.

Proof. Observe that the map $H^{-1}_\mu$ is obtained simply reversing time (once proved its uniqueness), i.e. if $(\rho(t), \theta(t), p(t))$ is the solution of (3.1) such that $(\rho(0), \theta(0), p(0)) = \sigma(\theta(0), p(0))$, then

$$H^{-1}_\mu(\theta(0), p(0)) = (\theta(\tau), p(\tau))$$

and $H^{-1}_\mu(\theta(0), p(0)) = (\theta(-\tau), p(-\tau))$.

Hence (3.3) holds for $H^{-1}_\mu$ too.

We need a Gronwall-type argument. Let us denote by $y(t)$ the solution of the following scalar non-autonomous O.D.E.

$$\begin{cases}
\dot{y} = f(t, y) \\
y(0) = y_0
\end{cases} \quad (3.4)$$

Denote respectively by $\bar{v}(t)$ and $\bar{v}(t)$ the solutions of the following autonomous problems:

$$\begin{cases}
\dot{\bar{v}} = \bar{A}(t)\bar{v} + \bar{B}(t) \\
\bar{v}(0) = y_0
\end{cases} \quad \text{and} \quad \begin{cases}
\dot{\bar{v}} = \bar{A}(t)\bar{v} + \bar{B}(t) \\
\bar{v}(0) = y_0
\end{cases} \quad (3.5)$$

Lemma 3.5. Assume $f$ is such that local uniqueness of the solutions of (3.4) is ensured, e.g. it is locally Lipschitz, and that $\bar{A}(t), \bar{B}(t), \bar{A}(t), \bar{B}(t)$ are bounded and locally Lipschitz, too. Suppose further that $\bar{A}(t)g + \bar{B}(t) < f(t, y) < \bar{A}(t)\bar{y} + \bar{B}(t)$ for any $y \in \mathbb{R}$ and any $t \geq 0$. Then $\bar{v}(t) < y(t) < \bar{v}(t)$ for any $t > 0$.

Proof of Lemma 3.5. We just prove that $y(t) < \bar{v}(t)$ for any $t > 0$, the other inequality is proved in an analogous way. Observe first that $\dot{y}(0) < \dot{v}(0)$, hence there is $\delta > 0$ such that $y(t) < v(t)$ whenever $0 < t < \delta$; let us set

$$T = \sup\{\tau > 0 \mid y(t) < v(t), \text{ for any } t \in [0, \tau]\},$$

and assume for contradiction that $T < \infty$. Then $y(T) = \bar{v}(T)$, and

$$\dot{v}(T) = \bar{A}(T)\bar{v}(T) + \bar{B}(T) > f(T, \bar{v}(T)) = \dot{y}(T).$$

From a continuity argument we see that there is $\tilde{\delta} > 0$ such that $\bar{v}(t) - y(t) > 0$ and $\dot{v}(t) - \dot{y}(t) > 0$ for $t \in (T - \tilde{\delta}, T)$; but this is a contradiction with $y(T) = \bar{v}(T)$, so the Lemma is proved.

In fact we need this result just in the case where $\bar{A}(t), \bar{B}(t), \bar{A}(t), \bar{B}(t)$ are constants.

Notice further that $H^{-1}_\mu$ and its inverse depend on $\sigma$.

Lemma 3.6. Let us consider $\sigma''$, $\sigma' \in \Sigma$, and the associated functions $H^{-1}_\mu$ denoted respectively by $H''_\mu$ and $H'\mu$. Then

$$\|H''_\mu - H'\mu\| \leq c_1 \varepsilon \|\sigma'' - \sigma'\|,$$

$$\|H''_\mu - [H''_\mu]^{-1}\| \leq c_1 \varepsilon \|\sigma'' - \sigma'\|,$$

where $c_1 = 3\tau \varpi(0) > 0$.

Proof. Observe first that the action of $H^{-1}_\mu$ on its $p$ component is in fact independent of $\sigma$ since the third equation in (3.1) is not influenced by the others. Hence $\Pi_p[H''_\mu - H'\mu] = 0$. Then, differentiating the first and the second equations in (3.1) with respect to the initial condition $\rho(0)$, we get

$$\frac{d}{dt} \frac{\partial \rho(t)}{\partial \rho(0)} = [-2\varepsilon + o(\varepsilon)] \frac{\partial \rho(t)}{\partial \rho(0)} + O(\mu + \varepsilon^{4/2} \delta) \frac{\partial \theta(t)}{\partial \rho(0)} \quad (3.6)$$

and

$$\frac{d}{dt} \frac{\partial \theta(t)}{\partial \rho(0)} = O(\varepsilon^2 + \mu) \frac{\partial \theta(t)}{\partial \rho(0)} + \left[2\varepsilon \varpi(\varepsilon)(1 + \frac{\mu}{\varpi(\varepsilon)}) + O(\varepsilon^2)\right] \frac{\partial \rho(t)}{\partial \rho(0)} \quad (3.7)$$
Next, using the fact that $\frac{\partial \rho(t)}{\partial \rho(0)}$ is bounded for $t \in [0, \tau]$, and Lemma 3.5 with $y(t) = \frac{\partial \rho(t)}{\partial \rho(0)}$, $y_0 = 1$, we find the following relation:

$$\left\| \frac{\partial \rho(\tau)}{\partial \rho(0)} \right\| = 1 + O(\varepsilon \tau). \tag{3.8}$$

Then observing that $\frac{\partial \theta(0)}{\partial \rho(0)} = 0$, and using again Lemma 3.5 we find the following:

$$\left\| \frac{\partial \theta(\tau)}{\partial \rho(0)} \right\| = \tau \left[ 2\varepsilon \omega(\varepsilon)(1 + O(\delta + \sqrt{\varepsilon})) + O(\mu) \right] \frac{\partial \rho(\tau)}{\partial \rho(0)} \leq c_1 \varepsilon. \tag{3.9}$$

Thus $\|\Pi_{\theta}(H_\mu - H_\mu)\| \leq c_1 \varepsilon \|\sigma' - \sigma'\|$: the analogous estimate for $H_\mu$ follows from Remark 3.4.

Now we consider the map $F_\mu^2(\sigma) = F_\mu \circ \sigma \circ H_\mu^{-1}$.

**Step 2.** $F_\mu^2$ maps $\Sigma$ into itself.

Let $H_\mu^{-1}(\theta, p) = (\theta', p')$, then

$$\begin{align*}
[F_\mu^2(\sigma)](\theta, p) &= F_\mu(s(H_\mu^{-1}(\theta, p)), H_\mu^{-1}(\theta, p)) = F_\mu(s(\theta', p'), \theta', p') = \\
&= (\rho, \Pi[F_\mu(\sigma[H_\mu^{-1}(\theta, p)])]) = (\rho, \theta, p)
\end{align*} \tag{3.10}$$

for a certain $\rho \in \mathbb{R}$. Here we used the fact that $\Pi F_\mu(\sigma[H_\mu^{-1}(\theta, p)]) = (\theta, p)$ for any $(\theta, p) \in P \times S^1$, by step 1. So we define $\rho = \delta^2(\theta, p)$ and $\sigma^2(\theta, p) = (\sigma^2(\theta, p), \theta, p)$. From Lemma 2.4 we see that $F_\mu$ maps the annulus $|\rho| \leq \delta$ into itself; therefore $|\sigma^2(\theta, p)| \leq \delta$.

We need to show that the derivative of $s^2(\theta, p) = \Pi_{\theta}(F_\mu(\sigma[H_\mu^{-1}(\theta, p)]))$ is uniformly smaller than $\delta$ so that $\sigma^2$ has slope smaller than $\sigma$. So, using (3.6), integrating for $t \in [0, \tau]$, and using the Gronwall inequality we get

$$\left\| \frac{\partial}{\partial \rho}[\Pi_{\theta} F_\mu(\theta, p)] \right\| \leq e^{-\frac{2}{2} \tau} + O(\varepsilon^{3/2} + \mu) \leq 1 - \varepsilon \tau. \tag{3.11}$$

Therefore, using (3.10), (3.11), (3.3) we find

$$\begin{align*}
\left\| \frac{\partial s^2(\theta, p)}{\partial \theta} \right\| &= \frac{d}{d \theta} \Pi_{\theta}(F_\mu(\sigma[H_\mu^{-1}(\theta, p)])) = \left( \frac{\partial(F_\mu(\sigma[H_\mu^{-1}(\theta, p)])}{\partial \rho} \right) \left( \frac{\partial s}{\partial \theta} \frac{\partial \Pi_{\theta}[H_\mu^{-1}]}{\partial \theta} \right) \\
&= (1 - \varepsilon \tau)\delta(1 + O(\varepsilon \delta)) \leq \delta \left( 1 - \frac{\varepsilon \tau}{2} \right).
\end{align*} \tag{3.12}$$

Similarly, again from (3.10), (3.11), (3.3) and assumption $H$ we find

$$\begin{align*}
\frac{\partial s^2(\theta, p)}{\partial p} &= \left\| \frac{\partial \Pi_{\theta}(F)}{\partial \rho} \left[ \frac{\partial s}{\partial \theta} \frac{\partial \Pi_{\theta}[H_\mu^{-1}]}{\partial \theta} + \frac{\partial s}{\partial \rho} \frac{\partial \Pi_{\theta}[H_\mu^{-1}]}{\partial \theta} \right] \right\| \\
&= (1 - \varepsilon \tau)\delta(1 + O(\mu)) \leq \delta \left( 1 - \frac{\varepsilon \tau}{2} \right).
\end{align*} \tag{3.13}$$

Hence $|\sigma^2| \leq \delta \left( 1 - \frac{\varepsilon \tau}{2} \right) \leq \delta$, therefore $\sigma^2 = F_\mu^2(\sigma) \in \Sigma$.

**Step 3.** $F_\mu^2$ is a contraction in $\Sigma$ of factor $K = 1 - \frac{\varepsilon \tau}{2}$, i.e

$$\|F_\mu^2(\sigma_2) - F_\mu^2(\sigma_1)\| \leq K\|\sigma_2 - \sigma_1\|. \tag{3.14}$$

Observe first that $F_\mu^2$ is the identity in its $\theta$ and $p$ coordinates, and therefore

$$F_\mu^2(\sigma_2) - F_\mu^2(\sigma_1) = (s^2_2(\theta, p) - s^2_1(\theta, p), 0, 0).$$
We recall that the function $H_\mu$ depends on $s$ and we denote by $H''_\mu$ respectively $H'_\mu$ the function constructed via $s_2$ respectively $s_1$.

\[
\|s_2^t(\theta, p) - s_1^t(\theta, p)\| = \|\Pi_\rho \{ F_\mu(\sigma_2([H''_\mu])^{-1}(\theta, p)) - F_\mu(\sigma_1([H''_\mu])^{-1}(\theta, p))\}\| \leq I_1 + I_2.
\]

(3.15)

where using (3.11) we find

\[
I_1 = \|\Pi_\rho \{ F_\mu(\sigma_2([H''_\mu])^{-1}(\theta, p)) - F_\mu(\sigma_1([H''_\mu])^{-1}(\theta, p))\}\| \leq (1 - \varepsilon \tau)\|\sigma_2 - \sigma_1\|.
\]

(3.16)

and using Lemma 3.6 we find

\[
I_2 = \|\Pi_\rho \{ F_\mu(\sigma_1([H''_\mu])^{-1}(\theta, p)) - F_\mu(\sigma_1([H''_\mu])^{-1}(\theta, p))\}\| \leq (1 - \varepsilon \tau)\|\sigma_2 - \sigma_1\|.
\]

(3.17)

Putting together (3.15), (3.16), (3.17) we find (3.14).

From Step 3 we obtain the existence and the uniqueness of a fixed point for $F^\sharp_\mu$, which is denoted by $\sigma^F(\theta, p) = (s^F(\theta, p), \theta, p)$.\]

**Remark 3.7.** Observe that the image of $\sigma^F(\theta, p)$ is invariant for the action of $F_\mu$.

**Proof.** Let $Q(t) = (\rho(t), \theta(t), p(t))$ be a solution of (3.1), and assume that $\rho(0) = s(\theta(0), p(0))$. Since $\sigma^F$ is a fixed point for $F^\sharp_\mu$, we find that

\[
\sigma^F(\theta(\tau), p(\tau)) = F^\sharp_\mu[\sigma^F(\theta(\tau), p(\tau))] = F_\mu[\sigma^F(\theta(0), p(0))].
\]

Therefore $Q(\tau) = \sigma^F(\theta(\tau), p(\tau))$ if and only if $Q(0) = \sigma^F(\theta(0), p(0))$. So $\sigma^F$ is invariant for the action of $F_\mu$ and its graph is an invariant manifold for $F_\mu$.\]

To prove that $\sigma^F$ is of class $C^r$ we need to use the argument of point d) in Theorem 4.1 of [24] which is in fact based on the fiber contraction theorem for $C^r$ maps, i.e. Theorem 3.5 in [24]. To apply this theorem we need to know that all the functions involved are of class $C^r$ together with the following key estimate.

**Step 4.** $F_\mu$ is a fiber contraction of sharpness $r$. I.e. there are $K > 0$ and $\alpha > 0$ such that $K\alpha^r < 1$ and

\[
\begin{align*}
\|\Pi_\rho[F_\mu(\rho_2, \theta, p)] - \Pi_\rho[F_\mu(\rho_1, \theta, p)]\| &\leq K\|\rho_2 - \rho_1\| \quad (3.18) \\
\|\Pi[H''_\mu^{-1}(\theta_2, p_2) - H''_\mu^{-1}(\theta_1, p_1)]\| &\leq \alpha\|\theta_2, p_2 - (\theta_1, p_1)\|.
\end{align*}
\]

In fact repeating the proof of step 3 we see that the first inequality of (3.18) is satisfied with $K = 1 - \varepsilon \tau / 2$, and from (3.3) and assumption $\mathbf{H}$ we see that there exists $c > 0$ (independent of $\varepsilon$ and $\mu$) such that in the second inequality in (3.18) we can choose $\alpha = 1 + c\varepsilon$. Therefore

\[
K\alpha^r = (1 - \frac{\varepsilon \tau}{2})(1 + c\varepsilon)^r = 1 - \varepsilon \frac{r}{2} - cr\delta + o(\varepsilon \delta) < 1 - \frac{\varepsilon \tau}{3}.
\]

Now we are in the position to apply Theorem 3.5 in [24] and conclude that the unique fixed section $\sigma^F_\mu(\theta, p) = (s^F_\mu(\theta, p), \theta, p)$ for $F^\sharp_\mu$ is of class $C^r$. So the proof of Proposition 3.3 is completed.\]

**Remark 3.8.** We emphasize that $V_\mu$ is a codimension 1 invariant manifold for the map $F_\mu$, and that it is asymptotically stable since in the transversal direction $F_\mu$ is uniformly contracting for any $t$ and any $(\rho, \theta, p) \in V_\mu$, see (3.11). Thus there is a neighbourhood $Y_1$ of $V_\mu$ such that $F^N_\mu(\rho, \theta, p) \in Y_1$ for any $N > 0$ if and only if $(\rho, \theta, p) \in V_\mu$, see Theorem 4.1 c) in [24]. Moreover let $(\rho(t), \theta(t), p(t)) = \Phi_t(\rho_0, \theta_0, p_0)$ be a solution of (3.1). Then, from the arbitrariness of $\tau$ and from
From our argument it follows that \( (\rho(T),\theta(T),p(T)) \in Y_2 \subset Y_1 \), and assume for contradiction that \( (\rho(T),\theta(T),p(T)) \notin V_\mu \). Then there is \( N > 0 \) such that \( F_{\mu}^{-N}(\rho(T),\theta(T),p(T)) = \Phi_{T-N\tau}(\rho_0,\theta_0,p_0) \notin Y_1 \). But then, from the second part of Remark 3.8, we see that \( \Phi_{\tau}(\rho_0,\theta_0,p_0) \notin Y_1 \) for any \( t < T - N\tau \); in particular \( F_{\mu}^{-n}(\rho_0,\theta_0,p_0) = \Phi_{-n\tau}(\rho_0,\theta_0,p_0) \notin Y_1 \) for any \( n \geq N + 1 \), a contradiction. So \( (\rho(T),\theta(T),p(T)) \in V_\mu \) for any \( T \in (0,\tau) \), hence \( V_\mu \) is invariant for the flow of (3.1), too. \( \square \)

We conclude the section by going back to the original coordinates.

**Remark 3.10.** If we consider (1.5) and its polar version (2.7), we see that the system admits a \( C^r \) attracting integral manifold \( M = M_\mu \) contained in the annulus \( \mathcal{A} \) of Lemma 2.3. In fact, thanks to Lemma 2.4, we see that \( M \) is contained in the positively invariant annulus

\[
\mathcal{A} := \{(\rho_1 + \rho_2,\theta) \mid \rho_1 = \Gamma_\varepsilon(\theta), |\rho_2| \leq \sqrt{\varepsilon}\delta\}
\]

From our argument it follows that \( \delta = O(\sqrt{\varepsilon}/\varepsilon) \), consequently the distance between \( M_\mu \) and \( \Gamma_\varepsilon \) is of order \( O(\sqrt{\varepsilon}/\sqrt{|\varepsilon|}) \). In fact, looking more carefully at our computation we can see that we could replace (2.11) by \( \mu = K(\varepsilon^s) \) for any \( s > 1 \) and some \( K > 0 \). In this case the distance between \( M_\mu \) and \( \Gamma_\varepsilon \) is of order \( o(\sqrt{\varepsilon}/\varepsilon^q) \) for \( q = 1/s > 1 \).

**Remark 3.11.** We emphasize that in this entire section the parameter \( \delta = \delta(\varepsilon) \) can be any continuous and increasing function such that \( \delta(0) = 0 \). In particular we can set \( \delta = \varepsilon^s \), or even \( \delta = |\ln(\varepsilon)|^{-s} \) for any \( s > 0 \); therefore the ratio \( \mu/\varepsilon \) is infinitesimal but can approach 0 as slowly as we wish. However setting \( \delta = \varepsilon^s \) we get accordingly \( \varepsilon_0 = \varepsilon_0(\sigma) \) and \( \varepsilon_0(s) \to 0 \) as \( s \to 0 \); thus the phenomena described exist for a range of parameters which can hardly be detected. For this reason in the following section we fix \( \mu = O(\varepsilon^{3/2}) \) (i.e. \( \delta = \varepsilon^{1/4} \)), so that the value of \( \varepsilon_0 \) is fixed: this power seems to be an “appropriate one”, taking into account that the distance between attractor and repeller is of order \( \sqrt{\varepsilon} \).
4. The flow on $M$. We begin by stating and proving some results which will be of use in describing the flow $\{\psi_t\}$ on the integral manifold $M_\iota$ in certain cases. We adopt a method of Diliberto [15, 16], who was motivated by Levinson [34].

Let us return to equation (2.4), which is a valid normal form for (1.1) on some neighborhood $W$ of $y = 0$. Let $\varepsilon_0 > 0$ be such that $[-\varepsilon_0, \varepsilon_0] \subset E$. If $0 < \varepsilon \leq \varepsilon_0$ we can make a constant time-change $s = \alpha(\varepsilon)t$ in such a way that the period of the asymptotically stable periodic solution $\Gamma_\varepsilon$ of (2.4) is exactly 1. We incorporate this time-change in the flow $\{\phi_t\}$ on $P$, so that the “new” flow $\{\hat{\phi}_t\}$ on $P$ has the form $\hat{\phi}_t = \phi_{\alpha(\varepsilon)t}$ ($t \in \mathbb{R}$). This is not an entirely innocuous move: if for example $P$ is a torus $P = \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, and if $\{\hat{\phi}_t\}$ is a Kronecker flow determined by a frequency vector $\omega_0 = (\omega_0^1, \ldots, \omega_0^d)$ of Diophantine type, then the “new” frequency vector $\omega_\varepsilon = \omega_0 \cdot \alpha(\varepsilon)$ may be no longer of Diophantine type. We will deal with this issue below, in the appropriate place.

Next recall that there exists $\varepsilon_0 > 0$ and a positive function $\mu = \mu_0(\varepsilon)$, defined for $0 < \varepsilon \leq \varepsilon_0$, such that the statement of Theorem 3.9 holds for $|\mu| \leq \mu_0(\varepsilon)$. That is, if $0 < \varepsilon \leq \varepsilon_0$ and if $|\mu| \leq \mu_0(\varepsilon)$, then equation (1.5) admits an integral manifold $M$ which is homeomorphic to $P \times S^1$. If $\varepsilon_0$ is sufficiently small we can take, for instance, $\mu_0(\varepsilon) = \varepsilon^{3/2}$ (i.e. $\delta = \varepsilon^{1/4}$). If $\{\psi_t\}$ is the flow on $M$ and if $\pi : M \to P$ is the natural projection, then $\pi \circ \psi_t = \phi_t \circ \pi$.

Fix $\varepsilon \in (0, \varepsilon_0]$ and $\mu \in [-\mu_0(\varepsilon), \mu_0(\varepsilon)]$. We will define a map $\tau : P \to P$, which has the nature of a “first return map” and which will be useful in the study of $\{\psi_t\}$. For this recall that $M \subset P \times \mathbb{R}^2$ can be parametrized by a map $v : P \times S^1 \to \mathbb{R}$ in such a way that $M = \{(p, v(p, \theta), \theta) \mid p \in P, 0 \leq \theta \leq 2\pi\}$. Let $M_0 = \{(p, v(p, 0), 0) \mid p \in P\} \subset M$, so that $M_0$ is the intersection of $M$ with the Cartesian product $P \times X$ of $P$ with the positive $x_1$-axis in the $(x_1, x_2)$-space $\mathbb{R}^2$.

Note that, if $m_0 \in M_0$, then $m_0$ is uniquely determined by its projection $\pi(m_0)$ to $P$, and vice-versa if $p \in P$, then there is a unique $m_0 \in M_0$ such that $\pi(m_0) = p$.

**Definition 4.1.** For each $p \in P$ let $m_0$ be the unique point in $M_0$ such that $\pi(m_0) = p$. Then let $\tau(p) = \tau_{\varepsilon, \mu}(p)$ be the first positive time such that $\psi(\tau(p), m_0) \in M_0$. In other words if we write $\psi_t(m_0) = (\phi_t(p), r(t), \theta(t))$, then $\tau(p)$ is the least positive time such that $\theta(\tau(p)) = 2\pi$.

It is clear that the map $\tau : P \to \mathbb{R}$ is continuous. Since $M$ is transversal to $X$, the map $\tau$ inherits any smoothness properties which $M$ and $\{\phi_t\}$ may have. Because of the normalization of the time variable, we have that $\tau(p) \equiv 1$ if $\mu = 0$. We call $\tau = \tau_\mu$ the Diliberto map for the flow $\{\phi_t\}$.

**Definition 4.2.** Let $T : P \to P$ be the map defined by $T(p) = \phi(\tau(p), p)$.

It is clear that $T$ is continuous, and that it inherits any smoothness properties which $P$ and $\{\phi_t\}$ may possess.

**Proposition 4.3.** If $0 < \varepsilon \leq \varepsilon_0$ and if $|\mu| < \mu_0(\varepsilon)$, then $T$ is a homeomorphism.

**Proof.** First note that, if $m = (p, v(p, \theta), \theta) \in M$ and if $\psi_t(m) = (\phi_t(p), r(t), \theta(t))$ then $(r(t))$ is bounded below by a positive constant which does not depend on $m$.

Let us show that $T$ is injective. Suppose that $p_1, p_2 \in P$ and that $T(p_1) = T(p_2) = p_* \in P$. Then $p_1$ and $p_2$ both lie on the $\{\phi_t\}$-trajectory in $P$ which contains $p_*$. Let $m_* = (p_*, v(p_*, 0), 0) \in M_0$, and write $\psi_t(m_*) = (\phi_t(p_*), r_*(t), \theta_*(t))$. Then $t \to \theta_*(t)$ is strictly monotone increasing so there exists exactly one $t_* \in \mathbb{R}$
such that 

\[-2\pi = \theta_*(t_*)\,.

It is clear that \(\tau(p_1)\) and \(\tau(p_2)\) both equal \(-t_*\), and so \(p_* = \phi(-t_*, p_1) = \phi(-t_*, p_2)\). This implies that \(p_1 = p_2\), and so \(T\) is injective.

The map \(T\) is thus a homeomorphism onto its range. To show that \(T\) is surjective, let \(p \in P\), and consider the point \(m = (p, v(p, 2\pi), 2\pi) \in M_0\). We follow the \(\{\psi_t\}\)-orbit of this point backward in time, to arrive at a point \(m_* = (p_*, v(p_*, 0), 0) = \psi(-t_*, m)\) where \(t_* > 0\). Then \(\tau(p_*) = t_*\) and \(T(p_*) = p\). This proves that \(T\) is surjective, and we can conclude that \(T\) is a homeomorphism of \(P\). This completes the proof of Proposition 4.3. \(\square\)

Let us observe that the proof of Proposition 4.3 requires only the existence of the integral manifold \(M\) and the strict positivity of \(\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial t}(m)\) as \(m\) ranges over \(M\). The next step in the analysis is to identify the flow on \(M\) in terms of a standard construction in topological dynamics.

**Proposition 4.4.** The flow \(\{\psi_t\}\) on the integral manifold \(M\) is isomorphic to the suspension flow (special flow) \(\{\sigma_t\}\) of \(T\) with roof function \(\tau\).

**Proof.** The terminology is standard in topological dynamics. For the reader’s convenience we review the construction of the suspension flow in question. Consider the product space \(P \times \mathbb{R}\), subject to the equivalence relation \(\sim\) defined as follows:

\[(p, s + \tau(p)) \sim (T(p), s) \quad (p \in P, s \in \mathbb{R})\]

One defines a flow \(\{\sigma_t\}\) on the quotient space \(\Sigma = P \times \mathbb{R}/\sim\) by setting

\[\sigma_t[p, s] = [p, s + t]\]

where \([p, s]\) denotes the equivalence class (i.e., element of \(\Sigma\)) which contains \((p, s) \in P \times \mathbb{R}\). It can be checked that \(\Sigma\) is a compact metrizable space, and that \(\{\sigma_t\}\) is a flow on \(\Sigma\).

If \(p \in P\) write \(v_0(p) = v(p, 0)\) and \(m_0 = (p, v_0(p), 0) \in M_0\). Further write \(\psi_t(m_0) = (\phi_t(p), r(t), \theta(t))\), where \(r(t) = r(t, v_0(p), 0)\) and \(\theta(t) = \theta(t, v_0(p), 0)\) solve equation (3.1) with the initial values \(r(0) = v_0(p), \theta(0) = 0\). Define \(h : P \times \mathbb{R} \to M\):

\[h(p, t) = (\phi_t(p), r(t), \theta(t))\]

with notation as above. We want to check that \(h\) induces a homeomorphism of \(\Sigma\) onto \(M\), and that \(h \circ \sigma_t = \psi_t \circ h\) for all \(t \in \mathbb{R}\).

To see this, note that \(\Sigma\) can be viewed as the set \(\{(p, s) \in P \times \mathbb{R} \mid 0 \leq s \leq \tau(p)\}\) modulo the equivalence relation \((p, \tau(p)) \sim (T(p), 0)\) \((p \in P)\). That is, the topological cylinder with lower face \(\{(p, 0) \mid p \in P\}\) and upper face \(\{(p, \tau(p)) \mid p \in P\}\) undergoes the above identification along the faces. We see that \(h(p, \tau(p)) = (T(p), v_0(\tau(p)), \theta(\tau(p))) = (T(p), v_0(\tau(p)), 2\pi), \) which equals \((T(p), v_0(T(p)), 0)\) in \(M\). On the other hand, \(h(T(p), 0)) also equals \((T(p), v_0(T(p)), 0)\) in \(M\). So in fact \(h\) descends to a map of \(\Sigma\) into \(M\), and it can be checked that this map - which we also call \(h\) - is a homeomorphism of \(\Sigma\) onto \(M\).

Next note that, if \(p \in P\) and \(0 \leq t, s \leq t + s < \tau(p)\), then \(h \circ \sigma_t[p, s] = h[p, t + s] = \psi_{t+s}(m_0)\) where \(m_0 = (p, v_0(p), 0)\) in \(M\). But \(\psi_{t+s}(m_0) = \psi_t \circ \psi_s(m_0) = \psi_t h[p, s]\) for the indicated values of \(t, s\). From this and some additional calculations it follows that \(\psi_t \circ h = h \circ \sigma_t\) for all \(t \in \mathbb{R}\). This concludes the proof of Proposition 4.4. \(\square\)
Let us denote by $\Sigma_T$ the suspension of $T$ with roof function $\tau$. Proposition 4.4 states that the circle extension $(T, \{\psi_t\})$ of $(P, \{\phi_t\})$ can be identified with $(\Sigma_T, \{\sigma_t\})$. Let us recall that the roof function $\tau$ is “close to 1”. Our objective now is to use these facts to study the flow $\{\psi_t\}$.

First we consider the bounded mean motion property of $\{\psi_t\}$. The significance of this property has been emphasized by Huang-Yi [25]; see also [1]. Let $\nu$ be a $\{\phi_t\}$ ergodic measure on $P$, and let $\rho_\nu$ be the $\nu$-rotation number of $\{\psi_t\}$; see Proposition 2.5.

**Definition 4.5.** We say that the flow $\{\psi_t\}$ on $M$ has the bounded mean motion property (bmm for short) if there exists a fixed constant $c$ such that, for each $m = (p, v(p, \theta), \theta) \in M$ with $\psi_t(m) = (\phi_t(p), r(t), \theta(t))$, it is the case that
\[
|\theta(t) - \rho_\nu t| \leq c \quad (t \in \mathbb{R}, \ p \in P).
\]

**Proposition 4.6.** Suppose that $|\tau(p) - 1| \leq c_0 < 1$ for all $p \in P$. The flow $\{\psi_t\}$ on $M$ has the bounded mean motion property if and only if there exists a real number $\tau_\nu$ and a constant $c_1 > 0$ such that, for all $p \in P$ and all $n \geq 1$, one has
\[
|\sum_{j=0}^{n-1} \tau(T^j(p)) - n\tau_\nu| \leq c_1.
\]

If (4.2) holds then $\rho_\nu = \frac{2\pi}{\tau_\nu}$.

**Proof.** Let us suppose that (4.2) holds and prove (4.1). According to Proposition 2.5, there is a set $P_\nu \subset P$ with $\nu(P_\nu) = 1$ such that, if $p \in P_\nu$, $m = (p, v(p, \theta), \theta) \in M$, and $\psi_t(m) = (\phi_t(m), r(t), \theta(t))$, then
\[
\lim_{t \to +\infty} \frac{\theta(t)}{t} = \rho_\nu.
\]

Let us examine $\theta(t)$ at the times $t_1 = \tau(p)$, $t_2 = \tau(p) + \tau(T(p))$, $t_3 = \sum_{j=1}^{n-1} t_j + \tau(T^{n-1}(p))$, and so on. Clearly $\theta(t_1) = 2\pi$, $\theta(t_2) = 4\pi$, $\theta(t_3) = 2n\pi$, and so on. By hypothesis $t_n = n\tau_\nu + b_n$ where the sequence $\{b_n\}$ is uniformly bounded for $p \in P$. Let us compare $\theta(t_1) = 2n\pi$ with $t_n$: we have that $\theta(t_n) = \frac{2\pi}{\tau_\nu} t_n + b_n^* \equiv \theta(t_n) - \frac{2\pi}{\tau_\nu} t_n = 2n\pi - \frac{2\pi}{\tau_\nu} (n\tau_\nu + b_n) = -\frac{2\pi}{\tau_\nu} b_n$ ($n = 1, 2, \ldots$). The sequence $\{b_n^*\}$ is uniformly bounded. Furthermore $|\tau(p) - 1| \leq c_0$ for all $p \in P$, so $0 < t_{n+1} - t_n < 2$ for all $n \geq 1$, and this together with the fact that $|\frac{\theta}{t}|$ is uniformly bounded for points $(p, v(p, \theta), \theta) \in M$ is enough to show that
\[
\lim_{t \to +\infty} \frac{\theta(t)}{t} = \frac{2\pi}{\tau_\nu}
\]
for each $m \in M$. By Proposition 2.5, we must have that $\rho_\nu = \frac{2\pi}{\tau_\nu}$, and moreover the bounded mean motion property holds because $\{b_n^*\}$ is uniformly bounded.

Suppose now that (4.1) holds for all $p \in P$, $t \in \mathbb{R}$. Define $t_1, \ldots, t_n$, as above, and note that $t_n = \frac{2\pi}{\rho_\nu} n - \frac{b_n}{\rho_\nu}$ where $|b_n| \leq c$ for all $n \geq 1$. This implies that $\sum_{j=1}^{n-1} \tau(T^j(p)) = \frac{2\pi}{\rho_\nu} t_n + b_n^*$ where $b_n^* = -\frac{b_n}{\rho_\nu}$ defines a uniformly bounded sequence. This holds for all $p \in P$, which means that (4.2) holds with $\tau_\nu = \frac{2\pi}{\rho_\nu}$. This completes the proof of Proposition 4.6.

**Remark 4.7.**

1. Suppose that $P_* \subset P$ is minimal with respect to $T$: this means that, for each $p \in P_*$, the orbit $\{T^n(p) \mid -\infty < n < \infty\}$ is dense in $P_*$. By
As a first consequence of Proposition 4.8, suppose that \( \eta : P_\ast \to \mathbb{R} \) such that
\[
\eta(T^n(p)) - \eta(p) = \sum_{j=0}^{n-1} \tau(T^j(p)) - n\tau_\nu
\]
for all \( p \in P_\ast \). This basic fact is proved, e.g. in [23]. So if \( \{ \psi_t \} \) has the bounded mean motion property, then (4.3) holds for all \( T \)-minimal subsets of \( P \). One says that \( \tau - \tau_\nu \) is a \( T \)-coboundary.

2. The bmm property has strong consequences as regards the structure of the flow \( \{ \psi_t \} \). We indicate some of them. Let \( (P, \{ \phi_t \}) \) be an almost periodic minimal set, and let \( \nu \) be the unique \( \{ \phi_t \} \)-ergodic measure on \( P \). Let \( \rho_\nu \) be the \( \nu \)-rotation number of \( \{ \psi_t \} \). Then each minimal subset of \( M \) is almost automorphic, and the frequency module of \( M \) is generated by \( \rho_\nu \) and the frequency module of \( \{ \phi_t \} \). See Huang-Yi ([25], Theorem 8.3). If \( P \) is a torus and \( \{ \phi_t \} \) is a Kronecker flow on \( P \), then further information is available ([25], Theorem 8.7): if \( \rho_\nu \) does not belong to the frequency module of \( \{ \phi_t \} \), then \( M \) contains a unique minimal set, which is either equal to \( M \) itself or is a “Cantorus”, see [25].

Another result is contained in [1], and is valid in the more general case where \( (P, \{ \phi_t \}) \) is a minimal flow. Namely, let \( (\hat{P}, \{ \hat{\phi}_t \}) \) be the flow on \( \hat{P} = P \times S^1 \) defined as follows: \( \hat{\phi}_t(p, \theta) = (\phi_t(p), \theta + \rho_\nu t \mod 2\pi) \). If \( \{ \hat{\phi}_t \} \) is minimal and if the bmm property holds, then \( \{ \hat{\phi}_t \} \) is semiconjugate to \( \{ \phi_t \} \). See [1, Corollary 4.1].

3. If \( P \) is the circle \( \mathbb{R}/\mathbb{Z} \) and if \( \{ \phi_t \} \) is a periodic flow on \( P \), then any circle extension \( (M, \{ \psi_t \}) \) of \( (P, \{ \phi_t \}) \) has the bmm property, see [13]. This is no longer true when \( (P, \{ \phi_t \}) \) is not periodic, although it does not seem to be known “how often” \( \{ \psi_t \} \) has this property. One suspects that the bmm property holds infrequently in some sense.

As we will see, it is sometimes possible to conjugate \( T \) to a “simpler” homeomorphism \( R : P \to P \). One wants to know what the effect of such a conjugation will be on the suspension flow \( \Sigma_T \), and also when the existence of such a conjugation might permit to determine the presence or the absence of the bmm property. We formulate some simple results in this regard.

**Proposition 4.8.** Let \( h : P \to P \) be a homeomorphism which conjugates \( T \) to another homeomorphism \( R \) of \( P \); that is \( h^{-1} \circ T \circ h = R \). Then \( h \) induces a flow isomorphism from \( \Sigma_T \) to the suspension flow \( \Sigma_R^\rho \) where \( \rho = \tau \circ h \).

**Proof.** Let us define \( \hat{h} : P \times \mathbb{R} \to P \times \mathbb{R} : \hat{h}(p, t) = (h(p), t) \). We show that \( \hat{h} \) maps points \( (p, s + \rho(p)) \) and \( (R(p), s) \) which are equivalent in \( \Sigma_R^\rho \) to points which are equivalent in \( \Sigma_T^\tau \); that is, we show that \( (h(p), s + \tau(h(p))) \) and \( (h \circ R(p), s) \) define the same equivalence class in \( \Sigma_T^\tau \). But \( (h(p), s + \rho(p)) = (h(p), s + \tau(h(p))) \sim (T(h(p)), s) = (h \circ R(p), s) \), which is what we wanted to prove.

Caution: it is not generally true that \( R(p) = \phi(\rho(p), p) \) in the above situation. What is true is that \( \phi(\rho(p), h(p)) = T(h(p)) = h(R(p)) \), or \( R(p) = \hat{\phi}_p(p) \) where \( \hat{\phi} \) is the flow \( \{ \hat{\phi}_t \} = \{ h^{-1} \hat{\phi}_t h \} \).

**Remarks 4.9.** 1) As a first consequence of Proposition 4.8, suppose that \( P = \mathbb{R}^d / \mathbb{Z}^d = T^d \) is a torus. Suppose that there is a homeomorphism \( h : P \to P \) which
conjugates $T$ to an irrational rotation $R = R_\omega$: that is, $\omega = (\omega^1, \ldots, \omega^d)$ is a vector with rationally independent components, and $R_\omega(p) := p + \omega$ with $p = (p^1, \ldots, p^d) \mod \mathbb{Z}^d$. Let $\rho = \tau \circ h$, and let $\tilde{\rho} = \int_P \rho \, d\bar{m}$ where $\bar{m}$ is the normalized Haar measure on $P = \mathbb{T}^d$. By Propositions 4.6 and 4.8, the flow $(M, \{\psi_t\})$ has the bmm property 

if and only if $\sum_{j=0}^{n-1} \theta(R_j(p)) - n\tilde{\rho}$ is uniformly bounded in $p \in P$, $n \geq 1$. In turn this relation holds if and only if there is a continuous function $\chi : P \to \mathbb{R}$ such that

$$
\chi(R_n^\rho(p)) - \chi(p) = \sum_{j=0}^{n-1} \rho(R_j^\rho(p)) - n\tilde{\rho} \quad (p \in P, n \geq 1),
$$

\noindent which holds if and only if $\chi(R_n^\rho(p)) - \chi(p) = \rho(p) - \tilde{\rho}$ for all $p \in P$. The existence/nonexistence of such a function $\chi$ can usually be checked by analyzing Fourier coefficients. In fact, write

$$
\rho(p) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i <p, n>}
$$

where $<p, n> = \sum_{j=1}^d p^j n_j$. If $\chi$ exists, then

$$
\chi(p) = \sum_{n \in \mathbb{Z} \setminus \{0\}} b_n e^{2\pi i <p, n>}, \quad b_n = \frac{a_n}{e^{2\pi i <\omega, n>} - 1}
$$

where $\chi$ has been normalized so that its mean value $\bar{\chi} = \int_P \chi \, d\bar{m} = 0$. It is clear that the relation (4.5) permits to study the so-called cohomology equation (4.4) in considerable detail.

2) If there is a continuous solution $\chi$ of (4.4), then one can define a flow isomorphism from $\Sigma^\rho_{R\omega}$ to the suspension flow over $R$ with constant roof function $\tilde{\rho}$; see e.g. [30]. So in this case $(M, \{\psi_t\})$ is actually flow isomorphic to an almost periodic flow on $\mathbb{T}^{d+1} = \mathbb{R}^{d+1} / \mathbb{Z}^{d+1}$ with frequency vector $(\omega, \tilde{\rho})$. This flow is either itself minimal, or it laminates into almost periodic minimal sets. On the other hand, if (4.4) does not hold, then one can show that the flow $\Sigma^\rho_{R\omega}$ may have various properties; for example it can be “rigid” [19], weakly mixing [18], or mixing [19]. It is to be emphasized that, even if $T$ can be conjugated to a rotation, it needs not to be the case that (4.4) holds. In fact there are various situations in which (4.4) fails to hold for a “generic” function $\rho$.

After these preliminary remarks, we study in more detail the question of the existence of a conjugacy between $T$ and an irrational rotation $R_\omega$. So we continue to set $P = \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ with angular coordinates $p^1, \ldots, p^d \mod 1$. Suppose that the flow $\{\phi_t\}$ is of Kronecker type with frequency vector $\omega_0 = (\omega_0^1, \ldots, \omega_0^d)$. Thus the components $\omega_0^1, \ldots, \omega_0^d$ are rationally independent, and $\phi_t(p) = (p^1 + \omega_0^1 t, \ldots, p^d + \omega_0^d t)$ $(t \in \mathbb{R})$. Suppose further that the coefficient functions in (2.4) are of class $C^\infty$ in $(p, x, \varepsilon, \mu) \in P \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$. Then, as explained in section 3, for a given differentiability class $l$, we can determine a positive function $\mu_l = \mu_l(\varepsilon)$ so that the integral manifold $M$ is of class $C^l$ for $|\mu| \leq |\mu_l(\varepsilon)|$ and $0 < \varepsilon \leq \varepsilon_0$.

Fix $0 < \varepsilon \leq \varepsilon_0$. We introduce a constant time-change $t \to \alpha(\varepsilon) t$ as at the beginning of this section, so that the period of the periodic solution $\Gamma(\varepsilon)$ of (1.4) is exactly 1. This means that the frequency vector of the time-changed flow on $P = \mathbb{T}^d$ is $\omega_\varepsilon = \omega_0 \cdot \alpha(\varepsilon)$. Let us assume that $\omega_\varepsilon$ is diophantine: that is, there exist constants $\kappa > 0$, $\sigma \geq d$ such that

$$
|m \omega_\varepsilon, n > - m| \geq \frac{\kappa}{|m|^\sigma} \quad (m \in \mathbb{Z}^d, n \in \mathbb{Z}^d \setminus \{0\})
$$


where $|n| = |n_1| + \ldots + |n_d|$ and $<\omega, n> = \sum_{j=1}^{d} \omega_j n_j$. For $|\mu| \leq |\mu_0(\varepsilon)|$ we pose the following questions: when is the map $T = T_{\varepsilon, \mu}$ conjugate via a homeomorphism/diffeomorphism $h : P \to P$ to a rotation $R = R_\omega$ on $\mathbb{T}^d$? If such a conjugacy exists, what can be said about the dynamics of $(M, \{\psi_t\})$?

To respond to the first question, we use KAM theory as explained by Gonzalez-Enriquez and Gonzalez-Enriquez/Vano. We summarize some of their results in a way which is adapted to the present context. See Theorem 3 and 4 in [21], and Theorem 5.1 in [22].

We fix some notation. For constants $\kappa > 0$, $\sigma > d$ let $D_{\kappa, \sigma} = \{\omega \in \mathbb{R}^d \mid \text{the relation (4.6) holds}\}$. If $l \geq 1$ is fixed and if $0 < \varepsilon \leq \varepsilon_0$, let $J_\varepsilon$ denote a closed neighbourhood of the origin $\mu = 0$ in $\mathbb{R}$, to be more precisely determined in the following.

**Proposition 4.10.** Let $0 < \varepsilon \leq \varepsilon_0$, and suppose that $\omega_\varepsilon \in D_{\kappa, \sigma}$. Let $l \geq 2(\sigma + 2)$. Let $T : J_\varepsilon \times P \to P$ be a $C^l$ map such that, for each fixed $\mu \in J_\varepsilon$, $T_\mu$ is a diffeomorphism in $P$. Let $|\cdot|$ denote the $C^l$-norm on $C^l$-mappings of $P = \mathbb{T}^d$ into $P$.

There is a constant $c > 0$ with the following property. Suppose that $|T_\mu - R_\omega|_l \leq c$ for all $\mu \in J_\varepsilon$. Let

$$F_\varepsilon := \{(\mu_0, \omega) \in \mathbb{R} \times \mathbb{R}^d \mid \mu \in J_\varepsilon, |\omega - \omega_\varepsilon| \leq c, \omega \in D_{\kappa, \sigma}\}.$$ 

Then for each $(\mu_0, \omega) \in F_\varepsilon$, there exists a diffeomorphism $h = h_{\mu_0, \omega} : P \to P$ of class $C^{l-\sigma-1}$ and a function $\lambda = \lambda(\mu_0, \omega) : F_\varepsilon \to \mathbb{R}^d$ such that $\lambda$ is of Whitney class $C^2$ on $F_\varepsilon$, and such that

$$T_\mu \circ h = h \circ R_\omega \circ \lambda(\mu_0, \omega) \quad \text{for } (\mu_0, \omega) \in F_\varepsilon.$$ 

The notion of Whitney differentiability is reviewed in [21] and it is discussed in detail in Chapter 6 of the excellent text [39]. We refer the reader to ([21], [22]) for the proof of Proposition 4.10. Actually the full proof is given only for the case of analytic maps $T_\mu$, but the methods sufficient to prove the $C^l$ version are present in [22].

Proposition 4.10 is our point of departure for the discussion of the question of conjugating $T = T_{\varepsilon, \mu}$ to a rotation. At times we will be rather synthetic and will take for granted certain points which are explained in [21]. Let us return to equation (1.5). We fix $\varepsilon \in (0, \varepsilon_0)$ and $l \geq 1$, and write $T_\mu = T_{\varepsilon, \mu}$ for $|\mu| \leq \mu_0(\varepsilon)$. First note that, if $\mu = 0$, then $T_0 = R_{\omega_\varepsilon}$, and so if $h$ is the identity diffeomorphism of $P$, then $T_0 \circ h = h \circ R_{\omega_\varepsilon}$. This means that $\lambda(0, \omega_\varepsilon) = 0$. Now the Whitney extension theory can be applied to obtain a $C^2$ function $\hat{\lambda}$, defined on $\mathbb{R} \times \mathbb{R}^d$, which agrees with $\lambda$ on $F_\varepsilon$. It turns out that the Frechet derivatives $\frac{\partial \hat{\lambda}}{\partial \omega}$ is invertible at $(0, \omega_\varepsilon)$; in fact it is minus the identity map on $\mathbb{R}^d$. By the implicit function theorem, there is a $C^2$ map $\Omega$ which is defined in some neighbourhood of $\mu = 0$, such that $\hat{\lambda}(\mu, \Omega(\mu)) = 0$ for all $\mu$ in this neighbourhood, which we identify with $J_\varepsilon$. Set $C_\varepsilon = \{\mu \in J_\varepsilon \mid \Omega(\mu) = D_{\kappa, \sigma}\}$, and write $\Omega(\mu) = \Omega(\mu)$ for $\mu \in C_\varepsilon$. If $\mu \in C_\varepsilon$ and $h_\mu = h_{\mu, \Omega(\mu)}$ then one has

$$T_\mu \circ h_\mu = h_\mu \circ R_{\Omega(\mu)}. \quad (4.7)$$

So $T_\mu$ is $C^{l-\sigma-1}$- conjugated to the rotation $R_{\Omega(\mu)}$; the frequency vector $\Omega(\mu)$ lies in $D_{\kappa, \sigma}$.

Now we must deal with the following difficulty: it is not clear that $C_\varepsilon$ contains points other than $\mu = 0$. If that is true, then the conjugation result (4.7) is of no
interest. Let us discuss this issue, together with the related issues of how \( \kappa \) and \( \sigma \) should be chosen, and when their components differ by an integer. Thus we view a given frequency vector \( \omega = (\omega_1, \ldots, \omega_d) \) as an element of \( \Phi = \mathbb{R}^d / \mathbb{Z}^d \). Let \( m \) be the normalized Lebesgue measure on \( \Phi \). Let \( \delta > 0 \) be a (small) number. It is well known that one can choose \( \kappa > 0, \sigma > d \) such that the set \( D_{\kappa, \sigma} \) has measure \( m(D_{\kappa, \sigma}) \geq 1 - \delta \); we suppose that this has been done.

Let \( a > 0 \), and consider the product space \( \Phi_* = [1 - a, 1 + a] \times \Phi \). Define

\[
D_\alpha = \{ (\alpha, \omega) \in \Phi_* | \omega \cdot \alpha \in D_{\kappa, \sigma} \}.
\]

One can show that, for each fixed \( \alpha \in [1 - a, 1 + a] \), the section \( D_{\alpha,*} = \{ \omega \in \Phi | \omega \cdot \alpha \in D_{\kappa, \sigma} \} \) has measure \( m(D_{\alpha,*}) \geq (1 - \delta)(1 + a)^{-d} \). Let us write \( |A| \) for the one-dimensional Lebesgue measure of a Borel set \( A \subset \mathbb{R} \). Let us set \( A_\alpha = \{ \alpha \in [1 - a, 1 + a] | \omega \cdot \alpha \in D_{\kappa, \sigma} \} \); applying Fubini’s theorem to the characteristic function of \( D_\alpha \), we obtain for each \( N \geq 2 \):

\[
m\{ \omega \in \Phi | |A_\alpha| \geq 2a(1 - N \delta) \} \geq (N - 1)\delta[1 + N\delta - (1 + a)^{-d}]^{-1}.
\]

The point of this slightly complicated relation is the following. Suppose that \( \alpha(\cdot) \) is a nonconstant function whose image contains the interval \( [1 - a, 1 + a] \). If \( \alpha \) is small and if \( \delta \) and \( N \) are chosen in a convenient way, then for “most” \( \omega \in \Phi \), it will be the case that \( \omega \cdot \alpha \in D_{\kappa, \sigma} \) for “most” \( \alpha \in [1 - a, 1 + a] \). So if, for example, \( \alpha(\cdot) \) has an everywhere nonzero derivative, then \( \omega \cdot \alpha(\cdot) \in D_{\kappa, \sigma} \) for “most” \( \varepsilon \) near 0.

Now let us consider the following situation. Suppose that the image \( \text{Im}\alpha(\cdot) \) contains the interval \( [1 - a, 1 + a] \) where \( a > 0 \). Let \( \Delta > 0 \) be a small number. Perhaps decreasing \( a \), and choosing \( \kappa \) and \( \sigma \) appropriately, we can arrange that there is a set \( \Phi_0 \subset \Phi \) of measure \( m(\Phi_0) \geq 1 - \Delta \) such that, if \( \omega \in \Phi_0 \), then

\[
|\{ \alpha \in [1 - a, 1 + a] : \omega \cdot \alpha \in D_{\kappa, \sigma} \}| \geq 2a(1 - \Delta).
\]

Having so determined \( a > 0, \kappa > 0, \sigma > d \) and \( \Phi_0 \subset \Phi \), we fix \( \omega_0 \in \Phi_0 \) and set

\[
A_0 := A_{\omega_0} = \{ \alpha \in [1 - a, 1 + a] : \omega_0 \cdot \alpha \in D_{\kappa, \sigma} \}.
\]

We also set

\[
I = \{ \omega_0 \cdot \alpha | 1 - a \leq \alpha \leq 1 + a \} \quad \text{and} \quad A = \{ \omega_0 \cdot \alpha \in A_0 \},
\]

so that \( A \subset I \).

Let \( \varepsilon \in (0, \varepsilon_0) \) be a point such that \( \alpha(\varepsilon) \) is a Lebesgue density point of \( A_0 \); this means that

\[
\lim_{r \to 0} \frac{1}{2r} |A_0 \cap [\alpha(\varepsilon) - r, \alpha(\varepsilon) + r]| = 1.
\]

It is well known that almost all elements of \( A_0 \) have this property. Let \( \omega_\varepsilon = \omega_0 \cdot \alpha(\varepsilon) \), so that \( \omega_\varepsilon \in A \subset I \) and \( \omega_\varepsilon \) is a Lebesgue density point of \( A \). Now return to equation (1.5) and to the families of maps \( \tau_{\varepsilon, \mu} = \tau_\mu : P \to \mathbb{R} \) and \( T_{\varepsilon, \mu} = T_\mu : P \to P \) such that \( T_\mu(p) = \phi(\tau_\mu(p), p) \). Let \( J_\varepsilon \subset \mathbb{R} \) and \( c > 0 \) be as in Proposition 4.10. Assume that if \( |\omega - \omega_\varepsilon| \leq c \) then \( \omega \in I \). With reference to the discussion following Proposition 4.10, let us write \( \lambda(\mu, \alpha) \) instead of \( \lambda(\mu, \omega_\varepsilon, \alpha) \) where \( |\alpha - 1| \leq a \). Due to the special form of \( T_\mu : T_\mu(p) = \phi(\tau_\mu(p), p) \), it can be shown that

\[
\lambda(\mu, \alpha) = \omega_\varepsilon \cdot \ell(\mu, \alpha) \quad \mu \in J_\varepsilon, \alpha \in A_0.
\]

In fact, if \( \tilde{\omega} = \omega_0, \alpha \in D_{\kappa, \sigma}, |\tilde{\omega} - \omega_\varepsilon| \leq c \), and \( T_\mu \circ h_\mu = h_\mu \circ R_\omega + \lambda(\mu, \tilde{\omega}) \), then \( h_\mu(p) = \phi(1 + \eta_\mu(p), p) \) where \( \eta_\mu : P \to \mathbb{R} \) is a \( C^{1-\sigma} \)-function. The function \( \ell(\mu, \alpha) \) is of class Whitney \( C^2 \) on \( J_\varepsilon \times A_0 \), and \( \frac{\partial \ell}{\partial \alpha}(0, \alpha(\varepsilon)) = -1 \). This all means that the map \( \Omega(\mu) \), which is defined for \( \mu \in J_\varepsilon \), must take values in \( I \).
One can go further and determine the derivative \( \frac{d\hat{\Omega}}{d\mu}(0) \) via a Lindstedt-type analysis [12], in the present case, the result is

\[
\frac{d\hat{\Omega}}{d\mu}(0) = \int_P \frac{\partial r_p}{\partial \mu}(0,p) \nu(dp)
\]

(4.8)

where \( \nu \) is the normalized Lebesgue measure on \( P = \mathbb{T}^d \). If this derivative is non-zero, then the set \( C_\varepsilon \) does not reduce to \( \mu = 0 \); indeed the image set \( \hat{\Omega}(J_\varepsilon) \) contains a Cantor-type subset of \( I \) because \( \omega_\varepsilon \) is a point of density of \( A \). So the conjugacy relation (4.7) holds for \( \mu \) in a Cantor-type subset \( C_\varepsilon \) of \( J_\varepsilon \) which contains \( \mu = 0 \) as a Lebesgue density point.

To summarize this discussion: if the derivative in (4.8) is non-zero, one can conjugate \( T_\mu \) to a rotation for “most” small \( \varepsilon \) and \( \mu \), if \( \omega_0 \) lies in a “large” set of frequency vectors. Let us now discuss the flow \( \{\psi_t\} \) on \( M \), under conditions in which Proposition 4.10 can be applied. First of all, the conjugation \( h \) is smooth of class \( C^{l-s-1} \), so \( \rho = \tau \circ h \) is also smooth of class \( C^{l-s-1} \). Write \( \bar{\rho} = \int_P \rho d\nu \) for the mean value of \( \rho \). Since \( l \geq 2(\sigma + 2) \) the cohomology equation

\[
\chi(R_\omega(p)) - \chi(p) = \rho(p) - \bar{\rho}
\]

admits a continuous solution, so the flow \( \{\psi_t\} \) is isomorphic to an almost periodic flow on \( P \times S^1 \simeq \mathbb{T}^{d+1} \).

Having said this, let us increase \( |\mu| \), so that, generally speaking, the smoothness of \( M \) will decrease. One suspects that, if \( \mu \) increases beyond some threshold value \( \mu_* \), the flow \( \{\psi_t\} \) will cease to be quasi periodic. It is not immediately clear what properties \( \{\psi_t\} \) might then have. One possible scenario is suggested by a result of B. Fayad ([18], Theorem 3). Namely, if \( l \) is less than \( \sigma + d \), then for a “generic” roof function \( \rho \), the suspension flow \( \Sigma_{R_\omega}^\rho \) is weakly mixing. So one might conjecture that, in some sense, a quasi-periodic regime gives way to a weakly mixing regime.

We will not go into this (apparently delicate) question here. Instead we consider another way in which one might attempt to obtain information about the flow \( \{\psi_t\} \): we discuss the concept of time-change in the context of suspension flows of the type \( \Sigma_T \).

Let us return to the situation in which \( P \) is a compact metric space and \( \{\phi_t\} \) is a continuous flow on \( P \). Consider the equation (1.5): for \( 0 < \varepsilon \leq \varepsilon_0 \) and \( |\mu| \leq \mu_0(\varepsilon) \) we construct the return map \( \tau : P \to \mathbb{R} \) and the homeomorphism \( T : P \to P \) given by \( T(p) = \phi(\tau(p),p) \). The goal is to introduce a new time variable \( t \), in a way which may depend on \( p \), so that the reparametrized flow \( \{\hat{\phi}_t\} \) on \( P \) has the property that \( \hat{\phi}_1 = T \). When this is possible, certain conclusions can be drawn concerning \( \{\psi_t\} \), as we will see.

To begin the discussion, suppose that there exists a strictly positive continuous function \( a : P \to \mathbb{R} \) such that

\[
\int_0^{\tau(p)} a(\phi_u(p)) du = 1, \quad (p \in P).
\]

(4.9)

In this case, we define a function \( t = t(s,p) : \mathbb{R} \times P \to \mathbb{R} \) by

\[
t = \int_0^s a(\phi_u(p)) du,
\]

(4.10)

then define a new flow \( \{\hat{\phi}_t \mid t \in \mathbb{R}\} \) on \( P \) by the formula

\[
\hat{\phi}_t(p) = \phi_{s}(p) \quad (p \in P).
\]
Using the strict positivity and continuity of \( a(\cdot) \), one checks that \( \{ \dot{\phi}_t \mid t \in \mathbb{R} \} \) is indeed a flow on \( P \). Observe that \( \dot{\phi}_1(p) = \phi(\tau(p), p) \) for all \( p \in P \), if such a function can be found.

In general we do not know how to formulate necessary and sufficient conditions on \( \tau \) so that a function \( a \) satisfying (4.9) can be found. We do offer some remarks concerning the problem. Let \( \mathcal{C} \) be the set of real-valued continuous functions on \( P \) with the usual sup-norm \( \| \cdot \|_\infty \). Let \( 1 \in \mathcal{C} \) be the constant function, and let \( \Psi = \{ a \in \mathcal{C} \mid |a - 1|_0 < 1 \} \). Define \( H : \Psi \to \mathcal{C} \) by setting \( H(a) = \tau \) where \( \tau \) is the unique element of \( \mathcal{C} \) for which (4.9) is valid. Clearly \( H(1) = 1 \); the question we ask is: what is the image \( H(\Psi) \) of \( \Psi \)?

As already stated, we do not know how to answer this question. Let us note that \( H \) is Frechet differentiable in \( a = 1 \), with derivative \( L = D_1 H \) given by

\[
L(\xi)(p) = -\int_0^1 \xi(\phi_u(p))du . \tag{4.11}
\]

An element \( \eta \) of \( \mathcal{C} \) lies in the image \( \text{Im}L \) if and only if there exists \( \xi \in \mathcal{C} \) for which (4.11) holds, which implies that \( \eta \) is differentiable along \( \{ \phi_t \} \)-orbits; in fact

\[
\eta(\phi_t(p)) = -\int_0^1 \xi(\phi_{u+t}(p))du = -\int_t^{1+t} \xi(\phi_s(p))ds
\]

Hence

\[
\dot{\eta}(p) := \frac{d}{dt} \eta(\phi_t(p))|_{t=0} = \xi(p) - \xi(\phi_1(p)) . \tag{4.12}
\]

So \( \dot{\eta} \) must be well-defined, and must be a coboundary with respect to the time-one map \( \phi_1 \). It turns out that if the flow \((P, \{ \phi_t \})\) is minimal, almost periodic and non-periodic, then the image of \( L \) is dense in \( \mathcal{C} \). This can be proved using a Fourier series argument. However, \( L \) is not surjective in this case, so we cannot hope to prove that \( H(\Psi) \) contains a neighbourhood of \( \tau = 1 \) in \( \mathcal{C} \).

In any case, when one can find a function \( a \) such that (4.9) holds, then certain conclusions can be drawn which we collect here.

**Remark 4.11.** Suppose that equation (1.5) gives rise to the map \( \tau : P \to \mathbb{R} \). Suppose that there exists a continuous function \( a : P \to \mathbb{R} \) such that (4.9) holds. In this case, the time-changed flow \( \{ \dot{\phi}_t \} \) has the same orbits as the \( \{ \phi_t \} \)-flow, so \((P, \{ \dot{\phi}_t \})\) is minimal if and only if \((P, \{ \phi_t \})\) is so. A regular Borel measure \( \nu \) is \( \{ \phi_t \} \)-invariant (ergodic) if and only if \( \hat{\nu} = (\hat{a})^{-1}a\nu \) is \( \{ \dot{\phi}_t \} \)-invariant (ergodic) where \( \hat{a} = \int_\mu a d\nu \). On the other hand, \( \{ \dot{\phi}_t \} \) needs not to be almost periodic when \( \{ \phi_t \} \) is so; for example it can be weakly mixing [18].

These remarks are actually beside the point, as we now show. Let \( \tau : P \to \mathbb{R} \) be a continuous function such that there exists a continuous positive function \( a : P \to \mathbb{R} \) for which the following modified form of (4.9) holds:

\[
\int_0^{\tau(p)} a(\phi_u(p))du = 2\pi . \tag{4.13}
\]

We can relate \( a \) to an explicit realization of equation (1.5), as follows. Passing to polar coordinates \((r, \theta)\), we can write the generic family (1.5) in the form:

\[
\frac{dr}{d\theta} = \varepsilon r - 3 + \gamma_1(r, \theta, \varepsilon) + \mu \gamma_1(\phi_t(p), r, \theta, \mu, \varepsilon)
\]

\[
\frac{d\theta}{d\theta} = 2\pi + \varphi(\varepsilon)r^2 + \gamma_2(r, \theta, \varepsilon) + \mu \gamma_2(\phi_t(p), r, \theta, \mu, \varepsilon)
\]
Let us set \( \gamma_1 \equiv 0, \gamma_2 \equiv 0, \tilde{\gamma}_1 \equiv 0, \tilde{\gamma}_2 = \tilde{\gamma}_2(p, r) \). Let \( 0 < \varepsilon \leq \varepsilon_0, 0 < |\mu| \leq \mu_0(\varepsilon) \). Then the integral manifold \( M = M_{\varepsilon, \mu} \) takes the simple form \( M = \{(p, v(p, \theta), \theta) \mid p \in P, \, 0 \leq \theta \leq 2\pi \} \) where \( v(p, \theta) \) is the constant function which assumes the value \( \sqrt{\varepsilon} \); then if \( a = 2\pi + \varpi(\varepsilon) \varepsilon + \mu \tilde{\gamma}_2(p, \sqrt{\varepsilon}) \), and if \((p, r = \sqrt{\varepsilon}, \theta_0) \in M\) we have

\[
\theta(t) = \theta_0 + \int_0^t a(\phi_u(p))du.
\]  

(4.14)

It is important to notice that \( a \) depends on \( p \) but not on \( \theta \).

Formula (4.14) shows that the flow \( \{\psi_t\} \) is explicitly realized as a Furstenberg extension of the flow \( \{\phi_t\} \) on \( P \). These flows are, on the one hand, relatively simple to study, and on the other hand have remarkable properties. Suppose that \((P, \{\phi_t\})\) is an almost periodic minimal flow with frequency module \( \mathcal{F} \), and let \( \bar{a} = \int_P \text{adv} \).

Then if

\[
\int_0^t a(\phi_u(p))du - \bar{a}t
\]  

(4.15)

is bounded for some \( - p \in P \), the flow \( \{\psi_t\} \) is almost periodic; it is minimal iff, in addition, \( \bar{a} \) does not belong to \( \mathcal{F} \). On the other hand, the function in (4.15) is unbounded for one \( - p \in P \) iff \( \{\psi_t\} \) is minimal but not almost periodic, see [20] and Section 2 above.

The point now is the following.

**Proposition 4.12.** Consider a family (1.5) of equations which gives rise to an integral manifold \( M = M_{\varepsilon, \mu} \) for \( 0 < \varepsilon \leq \varepsilon_0, |\mu| \leq \mu_0(\varepsilon) \). Let \( \tau = \tau_{\varepsilon, \mu} \) be the corresponding Diliberto map. Suppose that, for some fixed \( \varepsilon \) and \( \mu \neq 0 \), there exists a continuous positive function \( \bar{a} \) such that (4.13) holds. Then the flow \((M, \{\psi_t\})\) is the Furstenberg extension on \((P, \{\phi_t\})\) defined by \( \bar{a} \).

**Proof.** The flow \((M, \{\psi_t\})\) is isomorphic to the suspension flow \( \Sigma^\tau_{\bar{a}} \), which is determined by \( \{\phi_t\} \) and \( \tau \). On the other hand, the suspension flow \( \Sigma^\tau_{\bar{a}} \) is isomorphic to the Furstenberg extension of \((P, \{\phi_t\})\) defined by \( \bar{a} \), as the above discussion shows. \[\square\]

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