THE GENERALISED q-WRONSKIAN SOLUTIONS OF THE
q-DEFORMED CONSTRAINED MODIFIED KP HIERARCHY

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December 30, 2021

Abstract: In this paper, we give the form of the q-cmKP hierarchy generated by the gauge transformation operator $T_{n+k}$. We show a necessary and sufficient condition to reduce the generalised q-Wronskian solutions from the q-mKP hierarchy to generalised the q-Wronskian solutions of $M$-component q-cmKP hierarchy.

Keywords: q-mKP hierarchy, q-cmKP hierarchy, generalised q-Wronskian solutions

1. INTRODUCTION

The quantum-calculus [1–4] was used by some mathematicians to prove identities, such as the Ramanujan identity given in terms of $q$. In recent years, $q$-calculus has been used to construct quantum groups, quantum integrability and so on [5–7]. The $q$-deformed integrable system is an important branch of integrable system. Many quantum integrable systems have been relatively complete, such as the $q$-AKNS-D hierarchy, the $q$-KdV hierarchy [8–12], the $q$-KP hierarchy [13–17], sub-hierarchies of the $q$-KP hierarchy and its generalizing cases [18–24]. The wave functions, $\tau$ functions, additional symmetries and Virasoro constraints of these hierarchies have been studied.

The reduction of the $q$-Wronskian solutions of the $q$-mKP hierarchy to its constrained case has been studied. By using the eigenfunction symmetry constraints of the $q$-mKP hierarchy [25], we show a necessary and sufficient condition to reduce the $q$-Wronskian solutions of the $q$-mKP hierarchy to the $q$-cmKP hierarchy. On the basis of the narrow $q$-Wronskian solutions of the $q$-cmKP hierarchy, it is extended to the generalized $q$-Wronskian solutions of the $q$-cmKP hierarchy, which plays a good supplementary role in exploring the physical significance of $q$-deformation. We can obtain the generalized $q$-Wronskian solutions of the $q$-mKP hierarchy via the gauge transformation of the $q$-mKP hierarchy [20–31]. Hence, we will continue to study the generalized $q$-Wronskian solutions of the $q$-cmKP hierarchy.

The main purpose of this paper is to discuss how to reduce the $\tau$ function in generalised $q$-Wronskian forms of the $q$-mKP hierarchy to the solutions of the $M$-component $q$-cmKP hierarchy. We need to think about what the form of the $l$-constrained $q$-mKP hierarchy generated by the gauge transformation operator $T_{n+k}(n \geq k)$ is. We think about whether there is a necessary and sufficient condition for reducing the generalised $q$-Wronskian solutions of the $q$-mKP hierarchy to the $q$-cmKP hierarchy, and whether the results can be returned to the classical cmKP hierarchy as $q \to 1$. In this paper, we have given the $l$-constrained mKP hierarchy
generated by the gauge transformation operator $T_{n+k}(n \geq k)$. Finally, we provide an example to illustrate our results.

This paper is organized as follows. In Section 2, we introduce some basic results of $q$-mKP hierarchy. We give the $q$-cmKP hierarchy generated by the gauge transformation operator $T_{n+k}$. In Section 3, the necessary and sufficient condition reducing the $q$-Wronskian solutions of the $q$-mKP hierarchy to the solutions of the $q$-cmKP hierarchy is given which is the main result. In Section 4, we give a example with $T_{2+1}$. In Section 5, the conclusions are given.

2. THE $q$-DEFORMED CONSTRAINED MODIFIED KP HIERARCHY GENERATED BY $T_{n+k}$

The $q$-derivative $\partial_q$ is defined by its actions on a function $f(x)$ as

$$\partial_q(f(x)) = \frac{f(qx) - f(x)}{x(q - 1)}.$$  \hspace{1cm} (1)

When $q \to 1$, the $q$-derivative $\partial_q$ defined by the above equation is reduced to the derivative $\partial_x(f(x))$ in classical calculus theory. The $q$-shift operator is defined as

$$\theta(f(x)) = f(qx).$$  \hspace{1cm} (2)

The $q$-shift operator $\theta$ and $\partial_q$ are not commutative and they satisfy

$$q^m \theta^n(\partial_q^k(f(x))) = \partial_q^k(\theta^m(f(x))), \text{ } k, m \in \mathbb{Z}.$$  \hspace{1cm} (3)

Let $\partial_q^{-1}$ be the formal inverse operator of $\partial_q$, the algebraic multiplication of $\partial_q^n$ with the multiplication operator $f$ is given by the $q$-deformed Leibnitz rule

$$\partial_q^n \circ f = \sum_{k \geq 0} \binom{n}{k}_q \theta^{n-k}(\partial_q^k(f))\partial_q^{n-k}, \text{ } n \in \mathbb{Z},$$  \hspace{1cm} (3)

where the $q$-number and $q$-binomial are defined by

$$\binom{n}{k}_q = \frac{(n)_q(n-1)_q \cdots (n-k+1)_q}{(1)_q(2)_q \cdots (k)_q},$$

$$\binom{n}{0}_q = 1, \text{ } (n)_q = \frac{q^n - 1}{q - 1}.$$  \hspace{1cm} (4)

For a $q$-pseudo-differential operator $A = \sum_i a_i \partial_q^i$, we divide operator $A$ into two parts. One part is $A_{\geq k} = \sum_{i \geq k} a_i \partial_q^i$, the other part is $A_{< k} = \sum_{i < k} a_i \partial_q^i$. The symbol $A(f)$ means the action of $A$ on $f$, whereas $Af$ or $A \circ f$ denotes the operator multiplication of $A$ and $f$. The conjugate operation $*$ has the rules $(AB)^* = B^* A^*$, $\partial_q^* = -\partial_q \theta^{-1} = -\frac{1}{q} \partial_{\frac{1}{q}}$, $(\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta^2 \partial_q^{-1}$ and $f^* = f$.

The $q$-exponent $e_q(x)$ is defined as

$$e_q(x) = \sum_{k=1}^{\infty} \frac{x^n}{(n)_q!} = \exp(\sum_{k=1}^{\infty} \frac{(1-q)k}{k(1-q^k)} x^k).$$  \hspace{1cm} (5)
The $e_q(x)$ satisfies $\partial^k_q(e_q(xz)) = z^k e_q(xz)$, $k \in \mathbb{Z}$.

**Lemma 1** For any $q$-pseudo-differential operator $A$ and arbitrary functions $f$, $g$ and $h$, here are some of operator identities used in this paper.

\[
(A \circ f \circ q^{-1})_{<0} = A_{\geq 0}(f) \circ q^{-1} + A_{<0} \circ f \circ q^{-1},
\]
\[
(\partial_q^{-1} \circ f \circ A)_{<0} = \partial_q^{-1} \circ A_{\geq 0}(f) + \partial_q^{-1} \circ f \circ A_{<0},
\]
\[
f \partial_q^{-1} = \sum_{k=1}^{M} \frac{1}{q^k} \partial_{q^{-k-1}} \circ (\partial_{q}^{k}(\theta(f))),
\]
\[
(\partial_q^{-1} \circ g \circ \partial_q^{-1} \circ h) = \partial_q^{-1}(g) \circ \partial_q^{-1} \circ h - \partial_q^{-1} \circ \theta(\partial_q^{-1}(g)) \circ h,
\]
\[
(\partial_q^{-1} \circ f \circ \partial_q^{j} \circ g \circ \partial_q^{-1} \circ h)_{<0} = (\partial_q^{-1} \circ f \circ \partial_q^{j})_{<0} \circ g \circ \partial_q^{-1} \circ h + (\partial_q^{-1} \circ f \circ \partial_q^{j})_{\geq 0}(g) \circ \partial_q^{-1} \circ h,
\]
\[
(\partial_q^{-1} \circ f \circ \partial_q^{j})_{<0} = (-1)^{i} q^{-(1+2+\cdots+l)} \partial_q^{-1} \circ \theta^{-i}(\partial_q^{l}(f)),
\]
\[
(\partial_q^{-1} \circ f \circ \partial_q^{j})_{\geq 0}(g) = \sum_{i=0}^{l-1} (-1)^{i} q^{-(1+2+\cdots+l)} \theta^{-i-1}(\partial_q^{-i}(f)) \partial_q^{l-i-1}(g),
\]

so we have

\[
(\partial_q^{-1} \circ f \circ \partial_q^{j} \circ g \circ \partial_q^{-1} \circ h)_{<0} = (-1)^{l} q^{-(1+2+\cdots+l)} \partial_q^{-1} \circ \theta^{-l}(\partial_q^{l}(f)) \circ g \circ \partial_q^{-1} \circ h
\]
\[
+ \sum_{i=0}^{l-1} (-1)^{i} q^{-(1+2+\cdots+l)} \theta^{-i-1}(\partial_q^{-i}(f)) \partial_q^{l-i-1}(g) \circ \partial_q^{-1} \circ h
\]
\[
= (-1)^{l} q^{-(1+2+\cdots+l)} \partial_q^{-1} \circ \theta^{-l}(\partial_q^{l}(f)) \circ g \circ \partial_q^{-1} \circ h - (-1)^{l} q^{-(1+2+\cdots+l)} \partial_q^{-1} \circ \theta(\partial_q^{-1}(\theta^{-l}(\partial_q^{l}(f))) \circ h
\]
\[
+ \sum_{i=0}^{l-1} (-1)^{i} q^{-(1+2+\cdots+l)} \theta^{-i-1}(\partial_q^{-i}(f)) \partial_q^{l-i-1}(g) \circ \partial_q^{-1} \circ h
\]
\[
= \partial_q^{-1}(f \partial_q^{l}(g)) \circ \partial_q^{-1} \circ h - (-1)^{l} q^{-(1+2+\cdots+l)} \partial_q^{-1} \circ \theta(\partial_q^{-1}(\theta^{-l}(\partial_q^{l}(f))) \circ h.
\]

The Lax equation of the $q$-mKP hierarchy is given by

\[
\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 2, 3, \ldots,
\]

where $B_n = (L^n)_{\geq 1}$. The Lax operator $L$ is defined by

\[
L = u_1 \partial_q + u_0 + u_{-1} \partial_q^{-1} + u_{-2} \partial_q^{-2} + \cdots,
\]

in which $u_s = u_s(x,t) = u_s(x,t_1,t_2,\cdots)$. 

3
The Lax operator $L$ of the $q$-mKP hierarchy can be expressed in terms of the dressing operator $Z$ as

$$L = Z \circ \partial_q \circ Z^{-1}$$

with the dressing operator $Z = z_0 + z_1 \partial_q^{-1} + z_2 \partial_q^{-2} + \cdots$ ($z_0^{-1}$ exists). The Lax equation (9) is equivalent to the Sato equation

$$\frac{\partial Z}{\partial t_n} = -(Z \circ \partial_q^n \circ Z^{-1})_{\leq 0} \circ Z. \quad (11)$$

Similar to the $q$-KP hierarchy, the definition of the $q$-wave function $W_q(x, t)$ and the $q$-adjoint wave function $W_q^*(x, t)$ can be given as

$$W_q(x, t; \lambda) = Ze_q(x\lambda)exp\left(\sum_{i=1}^{\infty} t_i \lambda^i\right),$$

$$W_q^*(x, t; \lambda) = (Z^{-1} \partial_q^{-1})^*|_q e_1 e_1(-x\lambda)exp(-\sum_{i=1}^{\infty} t_i \lambda^i),$$

where

$$P|_q = \sum_i p_i(\frac{x}{t})t^i \partial_q^i$$

for a $q$-pseudo-differential operator $P = \sum_i p_i(x) \partial_q^i$. $W_q(x, t)$ satisfies the following equations

$$LW_q = \lambda W_q,$$

$$\frac{\partial W_q}{\partial t_n} = B_n W_q.$$

**Lemma 2** [32] Let $P$ and $Q$ be $q$-pseudo-differential operators, then one have

$$\text{res}_\lambda(Pe_q(x\lambda)Q^*|_q e_1(-x\lambda)) = \text{res}_{\partial_q}(PQ). \quad (12)$$

By using the above lemma, the bilinear identity of the $q$-mKP hierarchy can be established.

**Lemma 3** [25] The bilinear identity

$$\text{res}_\lambda \left((\partial_q^n \partial_{t_1}^{\alpha_1} \cdots \partial_{t_m}^{\alpha_m} W_q)W_q^*\right) = 1 \quad (13)$$

holds for any $n \in \mathbb{Z}_+$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{Z}_+^m$.

The eigenfunction $s$ and the adjoint eigenfunction $r$ of the $q$-mKP hierarchy satisfy the following equations

$$\frac{\partial s}{\partial t_n} = (L^n)_{\geq 1}(s), \quad (14)$$

$$\frac{\partial r}{\partial t_n} = -(L^n)_{\geq 1}^*(r). \quad (15)$$
The solution of the Sato equation can be represented by a simple $\tau$ function, let $L^l = (L^l)_{<0} + (L^l)_{>0}$, we know that the Lax operator (10) and generalized $l$-constraints

$$(L^l)_{<0} = \sum_{i=1}^{M} s_i \circ \partial_q^{-1} \circ r_i \circ \partial_q^{-1}$$

are compatible. And $L^l$ is the Lax operator of the $M$-component $q$-cmKP hierarchy.

The gauge transformation operators $T_{n+k}$ [23] can be given by the following lemma.

**Lemma 4** When $k < n$,

$$T_{n+k} = \frac{1}{IW_{k,n+1}^q(\Psi_k, \ldots, \Psi_1; \Phi_1, \ldots, \Phi_n, 1)}$$

$$\times \begin{vmatrix}
\partial_q^{-1}(\Phi_1 \psi_k) & \partial_q^{-1}(\Phi_2 \psi_k) & \cdots & \partial_q^{-1}(\Phi_n \psi_k) & \partial_q^{-1} \circ \psi_k \\
\partial_q^{-1}(\Phi_1 \psi_{k-1}) & \partial_q^{-1}(\Phi_2 \psi_{k-1}) & \cdots & \partial_q^{-1}(\Phi_n \psi_{k-1}) & \partial_q^{-1} \circ \psi_{k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\partial_q^{-1}(\Phi_1) & \partial_q^{-1}(\Phi_2) & \cdots & \partial_q^{-1}(\Phi_n) & \partial_q \\
\partial_q^l(\Phi_1) & \partial_q^l(\Phi_2) & \cdots & \partial_q^l(\Phi_n) & \partial_q^l
\end{vmatrix}$$

$$T_{n+k}^{-1} = \frac{(-1)^{n-1} q^{-k} IW_{k,n+1}^q(\Psi_k, \ldots, \Psi_1; \Phi_1, \ldots, \Phi_n, 1)}{IW_{k,n}^q(\Psi_k, \ldots, \Psi_1; \Phi_1, \ldots, \Phi_n) \theta(IW_{k,n}^q(\Psi_k, \ldots, \Psi_1; \Phi_1, \ldots, \Phi_n))}$$

$$\times \begin{vmatrix}
\Phi_1 \partial_q^{-1} & \theta(\partial_q^{-1}(\psi_k \Phi_1)) & \cdots & \theta(\partial_q^{-1}(\psi_1 \Phi_1)) & \theta(\Phi_1) & \cdots & \theta(\partial_q^{(n-k-2)}(\Phi_1)) \\
\Phi_2 \partial_q^{-1} & \theta(\partial_q^{-1}(\psi_k \Phi_2)) & \cdots & \theta(\partial_q^{-1}(\psi_1 \Phi_2)) & \theta(\Phi_2) & \cdots & \theta(\partial_q^{(n-k-2)}(\Phi_2)) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\Phi_n \partial_q^{-1} & \theta(\partial_q^{-1}(\psi_k \Phi_n)) & \cdots & \theta(\partial_q^{-1}(\psi_1 \Phi_n)) & \theta(\Phi_n) & \cdots & \theta(\partial_q^{(n-k-2)}(\Phi_n))
\end{vmatrix}$$

and

$$(T_{n+k}^{-1})^* = \frac{(-1)^n q^{-k} IW_{k,n+1}^q(\Psi_k, \ldots, \Psi_1; \Phi_1, \ldots, \Phi_n, 1)}{IW_{k,n}^q(\Psi_k, \ldots, \Psi_1; \Phi_1, \ldots, \Phi_n) \theta(IW_{k,n}^q(\Psi_k, \ldots, \Psi_1; \Phi_1, \ldots, \Phi_n))}$$
The determinant of $T_{n+k}$ in Lemma 4 is expanded by the last column and the functions are on the left-hand side. The determinant of $T_{n+k}^{-1}$ is expanded by the first column and the functions are on the right-hand side, and the coefficient function before the determinant should be placed after the operators $\Phi \partial^{-1}$. The determinant of $(T_{n+k}^{-1})^*$ is expanded by the first column and the functions are on the left-hand side.

The generalized Wronskian determinant is defined in the following form

$$
\tau_{k,n}^q = IW_{k,n}^q(\Psi_k, \cdots, \Psi_1; \Phi_1, \cdots, \Phi_n)
$$

$$
= \begin{vmatrix}
\partial_q^{-1}(\Phi_1 \Psi_k) & \partial_q^{-1}(\Phi_2 \Psi_k) & \cdots & \partial_q^{-1}(\Phi_n \Psi_k) \\
\vdots & \vdots & \cdots & \vdots \\
\partial_q^{-1}(\Phi_1 \Psi_1) & \partial_q^{-1}(\Phi_2 \Psi_1) & \cdots & \partial_q^{-1}(\Phi_n \Psi_1) \\
\Phi_1 & \Phi_2 & \cdots & \Phi_n \\
\partial_q(\Phi_1) & \partial_q(\Phi_2) & \cdots & \partial_q(\Phi_n) \\
\vdots & \vdots & \cdots & \vdots \\
\partial_q^{n-k-1}(\Phi_1) & \partial_q^{n-k-1}(\Phi_2) & \cdots & \partial_q^{n-k-1}(\Phi_n)
\end{vmatrix}.
$$

**Remark 1** When $k = 0$,

$$
IW_{0,n}^q = W_n^q(\Phi_1, \cdots, \Phi_n) = \begin{vmatrix}
\Phi_1 & \Phi_2 & \cdots & \Phi_n \\
\partial_q(\Phi_1) & \partial_q(\Phi_2) & \cdots & \partial_q(\Phi_n) \\
\vdots & \vdots & \cdots & \vdots \\
\partial_q^{n-1}(\Phi_1) & \partial_q^{n-1}(\Phi_2) & \cdots & \partial_q^{n-1}(\Phi_n)
\end{vmatrix}.
$$

Thus $T_{n+k}$ and $T_{n+k}^{-1}$ have the following forms

$$
T_{n+k} = \sum_{p=0}^{n-k} a_p \partial_q^p + \sum_{p=-1}^{-k} a_p \partial_q^{-1} \circ \Psi_{|p|}, \quad (18)
$$

$$
T_{n+k}^{-1} = \sum_{j=1}^{n} \Phi_j \partial_q^{-1} \circ b_j. \quad (19)
$$

**Remark 2** When $k = n$,

$$
T_{n+n} = \frac{1}{IW_{n,n}^q(\Psi_n, \cdots, \Psi_1; \Phi_1, \cdots, \Phi_n)}
$$
Thus and we have
\[
\begin{vmatrix}
\partial_q^{-1}(\Phi_1\Psi_n) & \partial_q^{-1}(\Phi_2\Psi_n) & \cdots & \partial_q^{-1}(\Phi_n\Psi_n) & \partial_q^{-1} \circ \Psi_n \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\partial_q^{-1}(\Phi_1\Psi_1) & \partial_q^{-1}(\Phi_2\Psi_1) & \cdots & \partial_q^{-1}(\Phi_n\Psi_1) & \partial_q^{-1} \circ \Psi_1 \\
\Phi_1 & \Phi_2 & \cdots & \Phi_n & 1
\end{vmatrix},
\]
and
\[T_{n+n} = 1 + \sum_{p=-1}^{-k} a_q \partial_q^{-1} \circ \Psi_{|p|}.
\]
In other words, \(T_{n+k}\) can still be expressed by the formula (17).

Remark 3 There is a sufficient condition \(IW_{k,n}^q \neq 0\) for the existence of these operators, in addition, the eigenfunctions \(\Phi_i, (i = 1, 2, \cdots, n)\) and the adjoint eigenfunctions \(\Psi_j, (j = 1, 2, \cdots, k)\) of the \(q\)-mKP hierarchy defined by the free operators \(L^0 = \partial_q\) satisfy
\[
\frac{\partial \Phi_i}{\partial t_n} = B_n^0(\Phi_i) = \partial_q^n(\Phi_i),
\]
(20)
\[
\frac{\partial \Psi_j}{\partial t_n} = -(\partial_q B_n^0 \partial_q^{-1})^*(\Psi_j).
\]
(21)
Let’s first prove that \(T_{n+k}\) is dressing operator of the \(q\)-mKP hierarchy.

Proposition 1 \(T_{n+k}\) satisfies Sato equation
\[(T_{n+k})_{t_n} = -(T_{n+k} \circ \partial_q^n \circ T_{n+k}^{-1}) \leq_0 T_{n+k}.
\]
(22)

Proof. Let \(l = 1\), and let \(L = T_{n+k} \circ \partial_q \circ T_{n+k}^{-1} = u_1 \partial_q + u_0 + u_{-1} \partial_q^{-1} + u_{-2} \partial_q^{-2} + u_{-3} \partial_q^{-3} + \cdots\), \(L\) satisfies
\[
\frac{\partial L}{\partial t_n} = [B_n, L] = [L, (L^n) \leq_0].
\]
Thus
\[
\frac{\partial L}{\partial t_n} = \frac{\partial}{\partial t_n}(T_{n+k} \circ \partial_q \circ T_{n+k}^{-1})
\]
\[
= (T_{n+k})_{t_n} \circ \partial_q \circ T_{n+k}^{-1} + T_{n+k} \circ \partial_q \circ (T_{n+k})_{t_n}
\]
\[
= (T_{n+k})_{t_n} \circ \partial_q \circ T_{n+k}^{-1} - T_{n+k} \circ \partial_q \circ T_{n+k}^{-1} \circ (T_{n+k})_{t_n} \circ T_{n+k}
\]
\[
= (T_{n+k})_{t_n} \circ T_{n+k}^{-1} \circ T_{n+k} \circ \partial_q \circ T_{n+k}^{-1} - T_{n+k} \circ \partial_q \circ T_{n+k}^{-1} \circ T_{n+k} \circ (T_{n+k})_{t_n} \circ T_{n+k}
\]
\[
= -[L, (T_{n+k})_{t_n} T_{n+k}^{-1}],
\]
and we have
\[(T_{n+k})_{t_n} = -(T_{n+k} \circ \partial_q^n \circ T_{n+k}^{-1}) \leq_0 T_{n+k}.
\]
□

According to [18], the \(\tau\) function of \(q\)-mKP hierarchy generated by \(L = T_{n+k} \circ \partial_q \circ T_{n+k}^{-1}\) is generalized Wronskian \(IW_{k,n}^q\). In addition, since \(L^l = T_{n+k} \circ \partial_q^l \circ T_{n+k}^{-1}\) and the determinant of the gauge transformations, there is the following theorem.
Theorem 1 The $q$-cmKP hierarchy generated by the gauge transformation $T_{n+k}$ is as follow

$$(L^i)_{\leq 0} = - \sum_{j=1}^{n} T_{n+k}(\partial_q^j(\Phi_j)) \circ \partial_q^{-1} \circ \theta(\partial_q^{-1}(b_j)) \circ \partial_q$$

$$+ (-1)^i q^{-(1+2+\cdots+l)} \sum_{p=-l}^{-k} a_p \circ \partial_q^{-1} \circ (T_{n+k}^{-1}\partial_q^{-1})^*(\theta^{-l}(\partial_q^j(\Psi_{|p|}))) \partial_q$$

$$+ \sum_{p=-l}^{-k} \sum_{j=1}^{n} a_p \circ \partial_q^{-1}(\Psi_{|p|}\partial_q^j(\Phi_j\partial_q^{-1}(b_j))) \ . \ (23)$$

Proof. Using the identities in Lemma 1, we have

$$(T_{n+k} \circ \partial_q^j \circ T_{n+k}^{-1})_{\leq 0}$$

$$= ((T_{n+k})_+ \circ \partial_q^j \circ T_{n+k}^{-1} + (T_{n+k})_- \circ \partial_q^j \circ T_{n+k}^{-1})_{\leq 0}$$

$$= ((T_{n+k})_+ \circ \partial_q^j \circ \sum_{j=1}^{n} \Phi_j \circ \partial_q^{-1} \circ b_j \circ \partial_q^{-1} + \sum_{p=-l}^{-k} a_p \circ \partial_q^{-1} \circ \Psi_{|p|} \circ \partial_q^j \circ \sum_{j=1}^{n} \Phi_j \circ \partial_q^{-1} \circ b_j \circ \partial_q^{-1})_{\leq 0} \partial_q$$

$$= - \sum_{j=1}^{n} (T_{n+k})_+(\partial_q^j(\Phi_j)) \circ \partial_q^{-1} \circ \theta(\partial_q^{-1}(b_j)) \partial_q$$

$$+ \sum_{p=-l}^{-k} \sum_{j=1}^{n} (a_p \circ \partial_q^{-1} \circ \Psi_{|p|} \circ \partial_q^j \circ \Phi_j \circ \partial_q^{-1}(b_j)) \partial_q$$

$$- \sum_{p=-l}^{-k} \sum_{j=1}^{n} (a_p \circ \partial_q^{-1} \circ \Psi_{|p|} \circ \partial_q^j \circ \Phi_j \circ \partial_q^{-1} \circ \theta(\partial_q^{-1}(b_j))) \partial_q$$

$$= - \sum_{j=1}^{n} T_{n+k}(\partial_q^j(\Phi_j)) \circ \partial_q^{-1} \circ \theta(\partial_q^{-1}(b_j)) \circ \partial_q$$

$$+ (-1)^i q^{-(1+2+\cdots+l)} \sum_{p=-l}^{-k} a_p \circ \partial_q^{-1} \circ (T_{n+k}^{-1}\partial_q^{-1})^*(\theta^{-l}(\partial_q^j(\Psi_{|p|}))) \partial_q$$

$$+ \sum_{p=-l}^{-k} \sum_{j=1}^{n} a_p \circ \partial_q^{-1}(\Psi_{|p|}\partial_q^j(\Phi_j\partial_q^{-1}(b_j))).$$

It is similar for the case of $k = n$. \[\square\]

3. REDUCED TO THE $M$-COMPONENT $q$-CMKP HIERARCHY

According to Theorem 1, from the equation (23), the $M$-component $q$-cmKP hierarchy can be given, then $(L^i)_{\leq 0}$ can be expressed as

$$(L^i)_{\leq 0} = \sum_{i=1}^{M} s_i \circ \partial_q^{-1} \circ r_i \circ \partial_q$$
\[\begin{align*}
\sum_{i=1}^{\alpha} s_i \circ \partial_q^{-1} \circ r_i \circ \partial_q + \sum_{i=\alpha+1}^{\alpha+\beta=M-1} s_i \circ \partial_q^{-1} \circ r_i \circ \partial_q + s_M \circ \partial_q^{-1} \circ r_M \circ \partial_q
&= (L_\alpha)_{\leq 0} + (L_\beta)_{\leq 0} + (L_\gamma)_{\leq 0},
\end{align*}\]

here \((L_\gamma)_{\leq 0} = s_M \circ \partial_q^{-1} \circ r_M \circ \partial_q = \sum_{p=1}^{n} \sum_{j=1}^{\alpha} a_p \partial_q^{-1}(\Psi_{|p|}\partial_q^j(\Phi_{j}(\partial_q^{-1}(b_j))))\), thus \(r_M = 1, s_M = \sum_{p=1}^{n} \sum_{j=1}^{\alpha} a_p \partial_q^{-1}(\Psi_{|p|}\partial_q^j(\Phi_{j}(\partial_q^{-1}(b_j))))\). \((L_\alpha)_{\leq 0}\) and \((L_\beta)_{\leq 0}\) are obtained by reducing the first and second parts of the right-hand side of the equation (23), respectively.

Now, let’s reduce \((L_\alpha)_{\leq 0}\) and \((L_\beta)_{\leq 0}\) respectively.

**Proposition 2** For the above Lax operator \(L\), the following two conclusions are true.

(I). For \((L_\alpha)_{\leq 0}\), if the Wronskian determinant \(W_\alpha = (1, s_1, \ldots, s_\alpha) \neq 0\), there is a unique \(\alpha\)-order \(q\)-differential operator

\[A = \partial_q^\alpha + a_{\alpha-1}\partial_q^{\alpha-1} + \cdots + a_1\partial_q + 1,\]

such that \((A(L_\alpha)_{\leq 0})_{\leq 0} = 0\).

(II). For \((L_\beta)_{\leq 0}\), if the Wronskian determinant \(W_{\alpha+1} = (1, r_{\alpha+1}, \ldots, r_{\alpha+\beta}) \neq 0\), there exists \(\beta\)-order \(q\)-differential operator

\[B = \partial_q^\beta + d_{\beta-1}\partial_q^{\beta-1} + \cdots + d_1\partial_q + 1,\]

such that \((L_\beta)_{\leq 0}B)_{\leq 0} = 0\).

**Proof.** (I) By using Lemma 1,

\[(A(L_\alpha)_{\leq 0})_{\leq 0} = (A \sum_{i=1}^{\alpha} s_i \circ \partial_q^{-1} \circ r_i \circ \partial_q)_{\leq 0}\]

\[= \sum_{i=1}^{\alpha} A(s_i) \circ \partial_q^{-1} \circ r_i \circ \partial_q = 0.\]

thus

\[\begin{align*}
& a_1\partial_q(s_1) + a_2\partial_q^2(s_1) + \cdots + a_\alpha\partial_q^\alpha(s_1) = -s_1, \\
& a_1\partial_q(s_2) + a_2\partial_q^2(s_2) + \cdots + a_\alpha\partial_q^\alpha(s_2) = -s_2, \\
& \vdots \\
& a_1\partial_q(s_\alpha) + a_2\partial_q^2(s_\alpha) + \cdots + a_\alpha\partial_q^\alpha(s_\alpha) = -s_\alpha.
\end{align*}\]

By solving this liner equations, we know that when \(W_{\alpha+1}^q(1, s_1, s_2, \ldots, s_\alpha) \neq 0\), \(A\) is unique. There is the similar way for the conclusion (II). ∎

**Theorem 2** The \(q\)-mKP hierarchy of \(L = T_{n+k}\partial_q T_{n+k}^{-1}\) with solutions given by \(\tau_0\)-functions \(\tau_0 = W_{\alpha+1}(1, s_1, \ldots, s_\alpha) \neq 0\), if and only if

\[(W_{\alpha+1}^q(T_{n+k}(\partial_q^j(\Phi_{j_1})), T_{n+k}(\partial_q^j(\Phi_{j_2})), \ldots, T_{n+k}(\partial_q^j(\Phi_{j_{\alpha+1}}))) = 0,\]

we have \((L_\alpha)_{\leq 0} = \sum_{i=1}^{\alpha} s_i \circ \partial_q^{-1} \circ r_i \circ \partial_q.\)
Proof. When $\alpha \leq n$, if $(L_\alpha)_{\leq 0} = -\sum_{j=1}^{n} T_{n+k}(\partial_q^j(\Phi_j)) \circ \partial_q^{-1} \circ \theta(\partial_q^{-1}(b_j)) \partial_q = \sum_{i=1}^{\alpha} s_i \circ \partial_q^{-1} \circ r_i \circ \partial_q$ holds, there exists an $\alpha$-order differential operator $A$ such that

$$0 = (A \circ (L_\alpha)_{\leq 0}) = -\sum_{j=1}^{n} A(T_{n+k}(\partial_q^j(\Phi_j))) \circ \partial_q^{-1} \circ \theta(\partial_q^{-1}(b_j)) \circ \partial_q,$$

so

$$A(T_{n+k}(\partial_q^j(\Phi_j))) = 0, \quad j = 1, 2, \cdots, n,$$

we know that $T_{n+k}(\partial_q^j(\Phi_j)) \in \ker(A)$. Since the kernel of $A$ is $\alpha$ dimensional space, at most $\alpha$ of these functions $T_{n+k}(\partial_q^j(\Phi_j))$ are linearly independent. Thus

$$W_{\alpha+1}^q(T_{n+k}(\partial_q^j(\Phi_{j_1})), T_{n+k}(\partial_q^j(\Phi_{j_2})), \cdots, T_{n+k}(\partial_q^j(\Phi_{j_{\alpha+1}}))) = 0.$$  

Conversely, when equation (27) holds, there are at most $\alpha$ functions in $T_{n+k}(\partial_q^j(\Phi_j))$ that are linearly independent. Then we can find suitable $\alpha$ functions $s_1, s_2, \cdots, s_\alpha$, such that

$$T_{n+k}(\partial_q^j(\Phi_j)) = -\sum_{i=1}^{\alpha} c_{ji}s_i, \quad j = 1, 2, \cdots, n. \quad (28)$$

thus

$$(L_\alpha)_{\leq 0} = -\sum_{j=1}^{n} T_{n+k}(\partial_q^j(\Phi_j)) \circ \partial_q^{-1} \circ \theta(\partial_q^{-1}(b_j)) \circ \partial_q$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{\alpha} c_{ji}s_i \circ \partial_q^{-1} \circ \theta(\partial_q^{-1}(b_j)) \circ \partial_q$$

$$= \sum_{i=1}^{\alpha} s_i \circ \partial_q^{-1} \circ (\sum_{j=1}^{n} c_{ji}\theta(\partial_q^{-1}(b_j))) \circ \partial_q$$

$$= \sum_{i=1}^{\alpha} s_i \circ \partial_q^{-1} \circ r_i \circ \partial_q.$$  

In this way we achieve the goal of reduction, and $r_i = \sum_{j=1}^{n} c_{ji}\theta(\partial_q^{-1}(b_j))$.

Using $q$–Wronskian determinant [6], we have

$$IW_{k,n+1+\alpha+1} = (\Psi_k, \Psi_{k-1}, \cdots, \Psi_1; \Phi_1, \cdots, \Phi_n, 1, \partial_q^j(\Phi_{j_1}), \cdots, \partial_q^j(\Phi_{j_{\alpha+1}})) = 0. \quad (29)$$

Theorem 3 For $(L_\beta)_{\leq 0}$, there is a $\beta$-order differential operator $B = \partial_q^\beta + d_{\beta-1}\partial_q^{\beta-1} + \cdots + d_1\partial_q + 1$, and $\tau_1 = W_{\beta+1}^q(1, r_{\alpha+1}, \cdots, r_{\alpha+\beta}) \neq 0$, if and only if

$$W_{\beta+1}^q(\tilde{r}_{i_1}, \cdots, \tilde{r}_{i_{\beta+1}}) = 0, \quad n + 1 \leq i_1, \cdots, i_{\beta+1} \leq n + k, \quad (30)$$

where $\tilde{r}_{ijj} = (T_{n+1}(\partial_q^{-1})^*(\theta^{-1}(\partial_q^{-1}(\Psi_{ij}))$, we have $(L_\beta)_{\leq 0} = \sum_{i=\alpha+1}^{\alpha+\beta=M-1} s_i \circ \partial_q^{-1} \circ r_i \circ \partial_q.$

Proof. When $\beta \leq k$, if

$$(L_\beta)_{\leq 0} = (-1)^{\beta}q^{-(1+2+\cdots+k)} \sum_{p=-1}^{k} a_p \partial_q^{-1} \circ (T_{n+k}(\partial_q^{-1})^*(\theta^{-1}(\partial_q^{-1}(\Psi_{ij}))) \circ \partial_q$$
holds, we can get by Proposition 2

\[ 0 = ((L_\beta)_{\leq 0} B)_{\leq 0} \]

\[ = (-1)^l q^{-((1+2+\cdots+l)-1)} \sum_{p=-1}^{k} a_p \partial_q^{-1} \circ (T_{n+k}^{-1} \partial_q^{-1})^*(\theta^{-l}(\partial_q^k(\Psi_{[p]}))) \circ \partial_q \circ B \leq 0 \]

\[ = (-1)^l q^{-((1+2+\cdots+l)-1)} \sum_{p=-1}^{k} a_p \partial_q^{-1} \circ \tilde{B}^* (T_{n+k}^{-1} \partial_q^{-1})^*(\theta^{-l}(\partial_q^k(\Psi_{[p]}))) \circ \partial_q. \]

From the above equation, we have

\[ \tilde{B}^* ((T_{n+k}^{-1} \partial_q^{-1})^*(\theta^{-l}(\partial_q^k(\Psi_{[p]})))) = 0. \]

let \( \tilde{r}_{n+|p|} = (T_{n+k}^{-1} \partial_q^{-1})^*(\theta^{-l}(\partial_q^k(\Psi_{[p]}))), \) we get that

\[ \tilde{B}^* (\tilde{r}_{n+1}) = 0, \quad \tilde{B}^* (\tilde{r}_{n+2}) = 0, \quad \cdots, \quad \tilde{B}^* (\tilde{r}_{n+k}) = 0, \]

so \( \tilde{r}_{n+|p|} \) is the kernel of \( \tilde{B}^* \). The the kernel of \( \tilde{B}^* \) is \( \beta \) dimensions, at most \( \beta \) functions in \( \tilde{r}_{n+|p|} \) are linearly independent. We know \( W^q_{\alpha_1+1}(\tilde{r}_{1}, \cdots, \tilde{r}_{\beta+1}) = 0, \quad n + 1 \leq i_1, \cdots, i_{\beta+1} \leq n + k \) is true.

When equation (30) holds, function \( \tilde{r}_{n+|p|} \) can be expressed by the \( \beta \) functions \( r_{\alpha+1}, r_{\alpha+2}, \cdots r_{\alpha+\beta}, \)

\[ \tilde{r}_{n+|p|} = \sum_{i=\alpha+1}^{\alpha+\beta-M-1} v_{n+|p|, i} r_i, \quad p = -1, -2, \cdots, -k, \quad (31) \]

thus

\[ (L_\beta)_{\leq 0} = (-1)^l q^{-((1+2+\cdots+l)-1)} \sum_{p=-1}^{k} a_p \circ \partial_q^{-1} \circ (T_{n+k}^{-1} \partial_q^{-1})^*(\theta^{-l}(\partial_q^k(\Psi_{[p]}))) \circ \partial_q \]

\[ = \sum_{i=\alpha+1}^{\alpha+\beta-M-1} (-1)^l q^{-((1+2+\cdots+l)-1)} \sum_{p=-1}^{k} a_p v_{n+|p|, i} \circ \partial_q^{-1} \circ r_i \circ \partial_q \]

\[ = \sum_{i=\alpha+1}^{\alpha+\beta-M-1} s_i \circ \partial_q^{-1} \circ r_i \circ \partial_q, \]

here \( s_i = (-1)^l q^{-((1+2+\cdots+l)-1)} \sum_{p=-1}^{k} a_p v_{n+|p|, i}. \) \( \square \)

4. EXAMPLE

Let’s take the \( q \)-cmKP hierarchy generated by \( T_{2+1} \) as an example. When

\[ IW^q_{1,2}(\Psi_1; \Phi_1, \Phi_2) \neq 0, \]

\[ \begin{vmatrix} \partial_q^{-1}(\Phi_1 \Psi_1) & \partial_q^{-1}(\Phi_2 \Psi_1) \\ \Phi_1 & \Phi_2 \end{vmatrix} \]

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we have gauge transformations

\[ T_{2+1} = \frac{1}{IW_{1,2+1}^q} \begin{vmatrix} \partial_q^{-1}(\Phi_1) & \partial_q^{-1}(\Phi_2) & \partial_q^{-1}(\Psi_1) \\ \Phi_1 & \Phi_2 & \mathbf{1} \\ \partial_q(\Phi_1) & \partial_q(\Phi_2) & \partial_q \end{vmatrix} \]

\[ = a_{-1} \circ \partial_q^{-1} \circ \Psi_1 + a_0 + a_{-1} \circ \partial_q, \quad (32) \]

\[ T_{2+1}^{-1} = -\frac{q^{-1}IW_{1,2+1}^q}{IW_{1,2}^q \theta(IW_{1,2}^q)} \begin{vmatrix} \Phi_1 \partial_q^{-1} & \partial_q^{-1} \theta(\Psi_1) \Phi_1 \\ \Phi_2 \partial_q^{-1} & \partial_q^{-1} \theta(\Psi_1) \Phi_2 \\ \end{vmatrix} \]

\[ = \Phi_1 \partial_q^{-1} \circ b_1 + \Phi_2 \partial_q^{-1} \circ b_2. \quad (33) \]

Using the Theorem 1, we can get the \( q \)-cmKP hierarchy generated by \( T_{2+1} \)

\[ (L^1)^{\leq 0} = (T_{2+1} \circ \partial_q^j \circ T_{2+1}^{-1})^{\leq 0} \]

\[ = -\sum_{j=1}^2 T_{2+1}(\partial_q^j(\Phi_j)) \circ \partial_q^{-1} \circ \theta(\partial_q^{-1}(b_j)) \circ \partial_q \]

\[ + (-1)^j q^{-(1+2+\cdots+j)} a_1 \circ \partial_q^{-1} \circ [(T_{2+1}^{-1} \partial_q^{-1})(\theta^{-1}(\partial_q^j(\Psi_1)))] \circ \partial_q \]

\[ + \sum_{j=1}^2 a_1 \partial_q^{-1}(\Psi_1 \partial_q^j(\Phi_j) \partial_q^{-1}(b_j)), \]

\[ (L^I)^{\leq 0} = (L_\alpha)^{\leq 0} + (L_\beta)^{\leq 0} + (L_\gamma)^{\leq 0}, \]

here \( \alpha = 1, \beta = 1 \). We have

\[ IW_{k,n+1+\alpha+1}^q = IW_{1,5}^q(\Psi_1, \Phi_1, \Phi_2, \partial_q^j(\Phi_1), \partial_q^j(\Phi_2), 1) \]

\[ \begin{vmatrix} \partial_q^{-1}(\Phi_1) & \partial_q^{-1}(\Phi_2) & \partial_q^{-1}(\Psi_1) \\ \Phi_1 & \Phi_2 & \partial_q^j(\Phi_1) \\ \partial_q(\Phi_1) & \partial_q(\Phi_2) & \partial_q^j+1(\Phi_1) \\ \partial_q^2(\Phi_1) & \partial_q^2(\Phi_2) & \partial_q^j+2(\Phi_1) \\ \partial_q^3(\Phi_1) & \partial_q^3(\Phi_2) & \partial_q^j+3(\Phi_1) \\ \end{vmatrix} \]

\[ = 0, \]

here we let

\[ \Phi_1 = e_q(\mu_1 x)e^{\theta_1} + e_q(\mu_2 x)e^{\theta_2}, \]

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\[
\Phi_2 = e_q(\mu_3 x) e^{\theta_3} + e_q(\mu_4 x) e^{\theta_4}, \\
\Psi_1 = e_q(-\lambda_1 x) e^{\eta_1} + e_q(-\lambda_2 x) e^{\eta_2},
\]

and

\[
\theta_i = c_i + \mu_i t_1 + \mu_i^2 t_2 + \mu_i^3 t_3 + \cdots + \mu_i^n t_n + \cdots, \quad i = 1, 2, 3, 4, \\
\eta_j = \rho_j - \frac{1}{q} \lambda_j t_1 - \frac{1}{q^2} \lambda_j^2 t_2 - \frac{1}{q^3} \lambda_j^3 t_3 - \cdots - \frac{1}{q^n} \lambda_j^n t_n - \cdots, \quad j = 1, 2,
\]

where \( c_i, \rho_j \) are constants. We have

\[
IW_{1,5}^n(\Psi_1, \Phi_1, \Phi_2, \partial_q^j(\Phi_1), \partial_q^j(\Phi_2), 1) \\
= \begin{vmatrix}
\partial_q^{-1}(\Phi_1 \Psi_1) & \partial_q^{-1}(\Phi_2 \Psi_1) & \partial_q^{-1}(\Psi_1 \partial_q^j(\Phi_1)) & \partial_q^{-1}(\Psi_1 \partial_q^j(\Phi_2)) & \partial_q^{-1}(\Psi_1)
\end{vmatrix}
\]

\[
= - \begin{vmatrix}
\partial_q^{-1}(\Phi_1 \Psi_1) & \partial_q^{-1}(\Phi_2 \Psi_1) & \partial_q^{-1}(\Psi_1 \partial_q^j(\Phi_1)) & \partial_q^{-1}(\Psi_1 \partial_q^j(\Phi_2))
\end{vmatrix}
\]

\[
+ \partial_q^{-1}(\Psi_1) \begin{vmatrix}
\Phi_1 & \Phi_2 & \partial_q^j(\Phi_1) & \partial_q^j(\Phi_2)
\end{vmatrix}
\]

\[
- \partial_q^{-1}(\Phi_1 \Psi_1) \begin{vmatrix}
\partial_q(\Phi_2) & \partial_q(\Phi_2) & \partial_q^{j+1}(\Phi_1) & \partial_q^{j+1}(\Phi_2)
\end{vmatrix}
\]

\[
+ \partial_q^{-1}(\Phi_2 \Psi_1) \begin{vmatrix}
\partial_q^2(\Phi_2) & \partial_q^2(\Phi_2) & \partial_q^{j+2}(\Phi_1) & \partial_q^{j+2}(\Phi_2)
\end{vmatrix}
\]

\[
+ \partial_q^{-1}(\Phi_1) \begin{vmatrix}
\partial_q^3(\Phi_1) & \partial_q^3(\Phi_1) & \partial_q^{j+3}(\Phi_1) & \partial_q^{j+3}(\Phi_2)
\end{vmatrix}
\]

\[
+ \partial_q^{-1}(\Phi_2) \begin{vmatrix}
\partial_q^4(\Phi_2) & \partial_q^4(\Phi_2) & \partial_q^{j+4}(\Phi_1) & \partial_q^{j+4}(\Phi_2)
\end{vmatrix}
\]

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\[-\vartheta_1^{-1}(\Psi_1 \vartheta_q^l(\Phi_1))\]

\[
\begin{vmatrix}
\vartheta_q(\Phi_1) & \vartheta_q(\Phi_2) & \vartheta_q^{l-1}(\Phi_1) \\
\vartheta_q^2(\Phi_1) & \vartheta_q^2(\Phi_2) & \vartheta_q^{l+2}(\Phi_1) \\
\vartheta_q^3(\Phi_1) & \vartheta_q^3(\Phi_2) & \vartheta_q^{l+3}(\Phi_1)
\end{vmatrix}
\]

\[
\begin{vmatrix}
\vartheta_q(\Phi_1) & \vartheta_q(\Phi_2) & \vartheta_q^{l-1}(\Phi_1) \\
\vartheta_q^2(\Phi_1) & \vartheta_q^2(\Phi_2) & \vartheta_q^{l+2}(\Phi_1) \\
\vartheta_q^3(\Phi_1) & \vartheta_q^3(\Phi_2) & \vartheta_q^{l+3}(\Phi_1)
\end{vmatrix}
\]

and

\[
\begin{vmatrix}
\vartheta_q(\Phi_2) & \vartheta_q^{l+1}(\Phi_1) & \vartheta_q^{l+1}(\Phi_2) \\
\vartheta_q^2(\Phi_2) & \vartheta_q^{l+2}(\Phi_1) & \vartheta_q^{l+2}(\Phi_2) \\
\vartheta_q^3(\Phi_2) & \vartheta_q^{l+3}(\Phi_1) & \vartheta_q^{l+3}(\Phi_2)
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & 1 & 1 \\
\mu_1 & \mu_3 & \mu_4 \\
\mu_1^2 & \mu_3^2 & \mu_4^2
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & 1 & 1 \\
\mu_2 & \mu_3 & \mu_4 \\
\mu_2^2 & \mu_3^2 & \mu_4^2
\end{vmatrix}
\]

when \(\mu_3 = \mu_4, c_3 = c_4\), we know \(\theta_3 = \theta_4\), thus

\[
\begin{vmatrix}
\vartheta_q(\Phi_2) & \vartheta_q^{l+1}(\Phi_1) & \vartheta_q^{l+1}(\Phi_2) \\
\vartheta_q^2(\Phi_2) & \vartheta_q^{l+2}(\Phi_1) & \vartheta_q^{l+2}(\Phi_2) \\
\vartheta_q^3(\Phi_2) & \vartheta_q^{l+3}(\Phi_1) & \vartheta_q^{l+3}(\Phi_2)
\end{vmatrix} = 0.
\]
Similarly, when \( \mu_1 = \mu_2, c_1 = c_2 \), we know \( \theta_1 = \theta_2 \), thus

\[
\begin{vmatrix}
\partial_q(\Phi_1) & \partial_{q+1}(\Phi_1) & \partial_{q+1}(\Phi_2) \\
\partial_q^2(\Phi_1) & \partial_{q+2}(\Phi_1) & \partial_{q+2}(\Phi_2) \\
\partial_q^3(\Phi_1) & \partial_{q+3}(\Phi_1) & \partial_{q+3}(\Phi_2)
\end{vmatrix} = 0,
\]

and

\[
\begin{vmatrix}
\partial_q(\Phi_1) & \partial_q(\Phi_2) & \partial_{q+1}(\Phi_2) \\
\partial_q^2(\Phi_1) & \partial_q^2(\Phi_2) & \partial_{q+2}(\Phi_2) \\
\partial_q^3(\Phi_1) & \partial_q^3(\Phi_2) & \partial_{q+3}(\Phi_2)
\end{vmatrix} = 0,
\]

\[
\begin{vmatrix}
\partial_q(\Psi_1) & \partial_q(\Phi_2) \\
\Phi_1 & \Phi_2
\end{vmatrix} = 0.
\]

From the above equations, we have let \( \theta_1 = \theta_2, \theta_3 = \theta_4 \), so we can get that

\[
\tau_{2+1}^q = IW_{1,2}^q,
\]

\[
= \begin{vmatrix}
\partial_q^{-1}(\Phi_1) & \partial_q^{-1}(\Phi_2) \\
\Phi_1 & \Phi_2
\end{vmatrix}
= 2\partial_q^{-1}(\Phi_1)\phi_q(\mu_3 x)e^{\theta_3} - 2\partial_q^{-1}(\Phi_2)\phi_q(\mu_1 x)e^{\theta_1}.
\]

After a series of calculations, the following expressions \( T_{2+1}(\partial_q^j(\Phi_1)), T_{2+1}(\partial_q^j(\Phi_2)), b_1, b_2, a_{-1}, (T_{n+k}^{-1}\partial_q^{-1})^*(\theta^{-1}(\partial_q^j(\Psi_1))) \) are obtained,

\[
T_{2+1}(\partial_q^j(\Phi_1)) = \frac{4}{IW_{1,3}^q}(\mu_1^j\partial_q^{-1}(\Phi_1) + \partial_q^{-1}(\Psi_1\partial_q^j(\Phi_1)))\phi_q(\mu_1 x)e_q(\mu_3 x)e^{\theta_1 + \theta_3} \begin{vmatrix} 1 & 1 \\ \mu_1 & \mu_3 \end{vmatrix},
\]

\[
T_{2+1}(\partial_q^j(\Phi_2)) = \frac{4}{IW_{1,3}^q}(\mu_3 \partial_q^{-1}(\Phi_2) + \partial_q^{-1}(\Psi_1\partial_q^j(\Phi_2)))\phi_q(\mu_1 x)e_q(\mu_3 x)e^{\theta_1 + \theta_3} \begin{vmatrix} 1 & 1 \\ \mu_1 & \mu_3 \end{vmatrix},
\]

\[
b_1 = -\frac{q^{-1}IW_{1,3}^q(\Psi_1, \Phi_1, \Phi_2)\partial_q^{-1}(\theta(\Phi_1 \Phi_2))}{IW_{1,2}^q(\Psi_1, \Phi_1, \Phi_2)\theta(IW_{1,2}^q(\Psi_1, \Phi_1, \Phi_2))},
\]

\[
b_2 = -\frac{q^{-1}IW_{1,3}^q(\Psi_1, \Phi_1, \Phi_2)\partial_q^{-1}(\theta(\Phi_1 \Phi_2))}{IW_{1,2}^q(\Psi_1, \Phi_1, \Phi_2)\theta(IW_{1,2}^q(\Psi_1, \Phi_1, \Phi_2))},
\]

\[
a_{-1} = \frac{1}{IW_{1,3}^q} \begin{vmatrix} \Phi_1 & \Phi_2 \\ \partial_q(\Phi_1) & \partial_q(\Phi_2) \end{vmatrix}
= \frac{4}{IW_{1,3}^q}(\mu_3 - \mu_1)\phi_q(\mu_1 x)e_q(\mu_3 x)e^{\theta_1 + \theta_3}.
\]
The goal is to construct \((L^l)_{\leq 0} = s_1 \partial^{-1}_q \circ r_1 \circ \partial_q + s_2 \partial^{-1}_q \circ r_2 \circ \partial_q + s_3 \partial^{-1}_q \circ r_3 \circ \partial_q\). Due to (24),

\[
\begin{align*}
  r_3 &= 1, \\
  s_3 &= \sum_{j=1}^2 a_{ij} \partial^{-1}_q (\Psi_i \partial^j_q(\Phi_j \partial^{-1}_q(b_j))).
\end{align*}
\]

The formula (28) reads as

\[
\begin{align*}
  T_{2+1}(\partial^l_q(\Phi_1)) &= -c_{11}s_1, \\
  T_{2+1}(\partial^l_q(\Phi_2)) &= -c_{21}s_1.
\end{align*}
\]

Then we have

\[
s_1 = -\frac{1}{c_{11}} T_{2+1}(\partial^l_q(\Phi_1)) = -\frac{1}{c_{21}} T_{2+1}(\partial^l_q(\Phi_1)),
\]

here

\[
\begin{align*}
  c_{11} &= \frac{1}{\mu_3 \partial^{-1}_q(\Phi_2 \Psi_1) - \partial^{-1}_q(\Psi_1 \partial^l_q(\Phi_2))}, \\
  c_{21} &= \frac{1}{\mu_1 \partial^{-1}_q(\Phi_1 \Psi_1) - \partial^{-1}_q(\Psi_1 \partial^l_q(\Phi_1))}.
\end{align*}
\]

We can get

\[
r_1 = c_{11} \theta(\partial^{-1}_q(b_1)) + c_{21} \theta(\partial^{-1}_q(b_2))
\]

by the last equation of the formula (28). \(r_2\) and \(s_2\) are obvious, because before the group reduction, it only has one component, we have

\[
\begin{align*}
  s_2 &= (-1)^l q^{-(1+2+\cdots+l)} a_{-1} \\
  &= (-1)^l q^{-(1+2+\cdots+l)} \frac{4}{IW_{1,3}} (\mu_3 - \mu_1) c_q(\mu_3 x) c_q(\mu_3 x) e^{\theta_1 + \theta_3}, \\
  r_2 &= (T_{n+k}^{-1} \partial^{-1}_q)^* (\theta^{-1}(\partial^l_q(\Psi_1))) \\
  &= -\theta \partial^{-1}_q \left( \frac{q^{-1} IW_{1,2+1}^{\mu_1} \left[ \theta \partial^{-1}_q(\Phi_1 \theta^{-1}(\partial^l_q(\Psi_1))) \right] \partial^{-1}_q \theta(\Psi_1 \Phi_1)}{IW_{1,2}^{\mu_1} \theta[IW_{1,2}^{\mu_1}]} \left[ \theta \partial^{-1}_q(\Phi_2 \theta^{-1}(\partial^l_q(\Psi_1))) \right] \right).
\end{align*}
\]

5. CONCLUSION

In this paper, the \(l\)-constrained \(q\)-mKP hierarchy generated by the gauge transformation operator \(T_{n+k}(n \geq k)\) is divided into three parts. Therefore, we divide the Lax operator of the \(M\)-component \(q\)-cmKP hierarchy into three groups by grouping reduction

\[
(L^l)_{\leq 0} = \sum_{i=1}^{\alpha} s_i \partial^{-1}_q r_i \partial_q + \sum_{i=\alpha+1}^{\alpha+\beta=M-1} s_i \partial^{-1}_q r_i \partial_q + s_M \partial^{-1}_q r_M \partial_q.
\]

In Proposition 8, we give the necessary and sufficient condition for reducing the generalised \(q\)-Wronskian solutions of the \(q\)-mKP hierarchy to the \(q\)-cmKP hierarchy. Finally, we give a concrete example by letting \(T_{n+1}\) and \(M = 3\).
We give the form of the $l$-constrained mKP hierarchy generated by gauge transformation operator $T_{n+k}$,

$$(L^l)_{\leq 0} = -\sum_{j=1}^{n} T_{n+k}(\Phi_j^{(l)}) \circ \partial^{-1} \circ b_j dx \circ \partial + (-1)^l \sum_{p=-1}^{-k} a_p \circ \partial^{-1} \circ [(T_{n+k}^{-1} \partial^{-1})^*(\Psi_{[p]}^{(l)})] \circ \partial$$

$$+ \sum_{j=1}^{n} \sum_{p=-1}^{-k} a_p \int (\Phi_j \int b_j dx)^{(l)} \Psi_{[p]} dx.$$

When $q \to 1$, the results can be returned to the classical cmKP hierarchy. This further verifies the correctness of this paper.

Acknowledgements: This work is supported by the National Natural Science Foundation of China under Grant Nos.12171133, 12171132 and 11871446, and the Anhui Province Natural Science Foundation (No. 2008085MA05).

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