Generalized Hamilton-Jacobi theory of Nambu Mechanics

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Abstract

We develop a Hamilton-Jacobi-like formulation of Nambu mechanics. The Nambu mechanics, originally proposed by Nambu more than four decades ago, provides a remarkable extension of the standard Hamilton equations of motion in even-dimensional phase space with a single Hamiltonian to a phase space of three (and more generally, arbitrary) dimensions with two Hamiltonians ($n$ Hamiltonians in the case of $(n + 1)$-dimensional phase space) from the viewpoint of the Liouville theorem. However, it has not been formulated seriously in the spirit of Hamilton-Jacobi theory. The present study is motivated to suggest a possible direction towards quantization from a new perspective.
1. Introduction

In 1973, Nambu\(^\text{b}\) proposed the following system of equations of motion for the flows of a point \((\xi^1, \xi^2, \xi^3)\) in a three-dimensional phase space \(\mathbb{R}^3\):

\[
\frac{d\xi^i}{dt} = \{H, G, \xi^i\} \equiv X^i, \tag{1.1}
\]

where the bracket notation (the Nambu bracket) on the r.h.side is defined for an arbitrary triplet of three functions \((K, L, M)\) on the phase space in terms of three-dimensional Jacobian:\(^b\)

\[
\{K, L, M\} = \frac{\partial(K, L, M)}{\partial(\xi^1, \xi^2, \xi^3)} = \epsilon^{ijk} \partial_i K \partial_j L \partial_k M. \tag{1.2}
\]

Therefore the vector field \(X^i\) defined in (1.1), generating the lines of flows in the phase space, is equal to

\[
X^i = \epsilon^{ijk} \partial_j H \partial_k G, \tag{1.3}
\]

in which two conserved functions \(H\) and \(G\), or “Hamiltonians”, govern the time evolution on an equal footing. The phase-space coordinates \(\xi^i\) satisfy a “canonical” Nambu bracket relation,

\[
\{\xi^i, \xi^j, \xi^k\} = \epsilon^{ijk}. \tag{1.4}
\]

Thus we have a natural extension of the Hamilton equations \((i = 1, \ldots, n)\) of motion,

\[
\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} = \{H, q^i\}, \tag{1.5}
\]

\[
\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} = \{H, p_i\}, \tag{1.6}
\]

in ordinary phase spaces, which are intrinsically of even \((2n)\) dimensions; we are liberated from the restriction of paired sets \((p_i, q^i)\) of independent canonical variables, satisfying the canonical Poisson bracket relations \(\{p_i, q^j\} = \delta^j_i\).

Once this generalization is given, it is obvious that similar systems in the phase spaces of arbitrary dimensions \(n+1\), irrespectively of odd or even dimensions, can be constructed by replacing 3-dimensional Jacobian in (1.1) by a general \((n+1)\)-dimensional Jacobian with \(n\) Hamiltonians. Let us call this general case the Nambu mechanics of order \(n\).

\(^\text{b}\)We assume the usual summation convention for coordinate indices in phase space.
The proposal would be quite suggestive from the standpoint of exploring new methods to express fundamental dynamical laws. Nambu himself was motivated by the Liouville theorem of ordinary Hamilton mechanics \((\partial_t X^i = 0)\), aiming at an extension of statistical mechanics. He stressed in particular that the Euler equations of motion for a free rigid rotator can be cast in the form \((1.1)\) by identifying the component \(\ell_i\) of angular momentum in the body-fixed frame to be the canonical coordinates \(\xi^i\), with two Hamiltonians, \(G = \frac{1}{2}((\xi^1)^2/I_1 + (\xi^2)^2/I_2 + (\xi^3)^2/I_3)\) and \(H = \frac{1}{2}((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2)\).

During the first two decades after its proposal, this remarkable concept had been a matter of interest only for a relatively small circle of mathematical physicists. An important basis for further developments was laid, among others, by Takhtajan\(^2\), who found a crucial identity clarifying the canonical structure of Nambu mechanics, now called the fundamental identity (FI) for the Nambu bracket, and also formulated an action principle. In the sequel to this long initial period, we gradually came to recognize the relevance of these ideas to physics, especially, to string/membrane theory (or M-theory)\(^2\) and the interests in Nambu mechanics and Nambu bracket have been renewed and broadened fruitfully in the past two decades. The purpose of the present paper, however, is not to pursue further such new directions for applications, but rather to fill a missing aspect of classical Nambu mechanics by going back to the spirit of the original proposal.

What we address is whether and how some sort of Hamilton-Jacobi (HJ)-like formalism is possible in this system. To the author’s knowledge, this question has never been pursued in appropriate depth in the past literature.\(^3\) From a purely technical point of view, the system of the equations \((1.1)\) of order \(n\) may be regarded as a special new class of integrable systems in which we have \(n\) independent conserved quantities for \(n + 1\) variables obeying ordinary differential equations of first order with respect to time. The latter property ensures that we can solve the system, in principle, by reducing it to a single quadrature directly at the level of the equations of motion, without higher apparatus such as HJ-like formalism. Conceptually, however, we can take a different attitude. The whole endeavors of physicists for developing Nambu mechanics have been devoted to uncovering possible deeper structures behind its surface, as suggested, e. g., from its stringent and higher symmetry properties, in hopes of utilizing them for fundamental physics. From

\(^3\)For these developments, interested readers can refer to Refs. e. g. \(^3\) and \(^4\), and references therein.

\(^d\)The only work of which the present author is aware in connection with this is Ref. \(^5\) where the problem of HJ theory in Nambu mechanics is briefly mentioned without any concrete formulation.
this viewpoint, it must be worthwhile exploring the possibility of HJ-like formalism by focusing its implications towards quantization of Nambu mechanics, a subject that has been quite elusive even to this day. We should recall here the well-known significance that classical HJ theory had given to the creation of quantum mechanics. This is not a mere accident: it reflects the important fact that the idea of the HJ formalism already contains some essential elements of quantum mechanics.

In fact, it turns out that the procedure towards an HJ-like formulation of Nambu mechanics is not straightforward. To achieve our goal, it will be useful to reformulate the basic ideas of the HJ formalism in ordinary Hamilton mechanics in a manner that does not rely too much on the standard textbook formulation. For this reason and also for the purpose of making the present paper accessible as widely as possible to readers of various backgrounds in a reasonably self-contained manner, we start in the next preliminary section by recounting ordinary HJ theory from a slightly nonstandard but physical viewpoint. That will guide us in constructing HJ-like formulations of Nambu mechanics in Sects. 3 and 4, where we restrict ourselves to the simplest case of three-dimensional \((n = 2)\) phase space. In Sect. 5, as an application of our formalism, we demonstrate some concrete computations to solve our generalized HJ equations and to rederive the Nambu equations of motion from them, taking the example of the (free) Euler top. In Sect. 6, we propose a possible new standpoint towards quantization, on the basis of our main results. Appendix A is devoted to a discussion on the extension of the present formalism to general Nambu mechanics of higher orders.

2. Some preliminaries

Let us first recall briefly what the standard HJ theory is. The usual textbook accounts presuppose the existence of an action functional and also the general formula of finite canonical transformations in terms of the action function \(S(q, t; Q)\) as a generating function defined on the configuration space of generalized coordinates \(q^i, (i = 1, \ldots, n)\) joined with time \(t\). The “action function” means that the action integral as a functional of arbitrary trajectories is now evaluated for particular trajectories that solve the equations of motion, by specifying an arbitrary point \(q^i\) at time \(t\) as the endpoint condition, together with the initial conditions, \(q^i = Q^i, \text{ at } t = 0\). If we fix the \(Q^i\) as constants, the solutions of the equations of motion, Eqs. (1.5) and (1.6), are uniquely fixed since we now have \(2n\)
conditions. These conditions in turn determine, at least locally, the action as a function (or a field), \( S(q, t; Q) \), of \( n \) independent variables \( q^i \) with fixed parameters \( Q^i \)'s. Then, the momenta as the canonical conjugates of the \( q^i \) and hence the Hamiltonian \( H = H(p, q) \) also become fields, and are expressed in terms of the action function as

\[
p_i(q, t) = \frac{\partial S}{\partial q^i}, \quad (2.1)
\]

\[
H(q, p(q, t)) + \frac{\partial S}{\partial t} = 0, \quad (2.2)
\]

which constitute the HJ equation as a partial differential equation of first order for the action function or field \( S \). These equations are usually derived on the basis of familiar variational principles associated with the action integral. The parameters \( Q^i \) emerge as constants of integration for the HJ equation; such a solution with \( n \) independent integration constants is called a “complete solution”. Given a complete solution, we obtain general solutions, now with \( 2n \) integration constants, of the Hamilton equations of motion by simple quadratures, after imposing on it the conditions (called “Jacobi conditions” for convenience in the present paper),

\[
\frac{\partial S}{\partial Q^i} = -P_i, \quad (2.3)
\]

by introducing the \( P_i \) as new and additional constant parameters, and solving them for \( q^i = q^i(t; P, Q) \) and \( p_i = p_i(t; P, Q) \). The solvability is guaranteed by requiring

\[
\text{det} \left( \frac{\partial^2 S}{\partial Q^i \partial q^j} \right) \neq 0. \quad (2.4)
\]

Equation (2.1) together with their counterpart equations (2.3) at \( t = 0 \) are interpreted as defining relations for a canonical transformation from \((p, q; t)\) to \((P, Q; 0)\) which unfolds the time development by sending \( H \) to zero as signified by (2.2). Finally, condition (2.3) can be rephrased in the following form. Since the HJ equation involves the action field through its derivatives, we can always shift it by adding a constant. In particular, by choosing a shift in the form \( S \to \tilde{S} = S + P_i Q^i \), condition (2.3) is \( \partial \tilde{S} / \partial Q_i = 0 \). This interpretation is appropriate to Schrödinger’s wave mechanical quantization: the condition \( \partial \tilde{S} / \partial Q_i = 0 \) naturally arises in deriving classical trajectories in the limit \( \hbar \to 0 \) from a wave function whose phase is \( \tilde{S} / \hbar \).

Now, in the case of Nambu mechanics, we have an action functional proposed in Ref.\( ^2 \). However, it is a functional not of one-dimensional trajectories, but of \( n \)-dimensional
(hyper) surfaces, represented say by $\xi^i(t, s_1, \ldots, s_{n-1})$, which consist of continuous families of one-dimensional trajectories, parametrized by spatial world-hypersurface coordinates $(s_1, \ldots, s_{n-1})$, as if we were treating objects extending in $(n-1)$ dimensions. This is not convenient for our purpose. Although nothing prevents us from treating arbitrary families of trajectories, such an approach forces us to introduce too many inessential and unphysical degrees of freedom, caused by the presence of the additional parameters $s_i$ that are not necessary for describing the true dynamical degrees of freedom: $s_i$ are simply redundant, at least for the present purpose, since they do not correspond to any “energetic” couplings among trajectories, a property that is related to the existence of $n$ independent “Hamiltonians”, in spite of the fact that we have only a single time for the dynamical evolution.

To make things worse, we do not know an appropriate and useful characterization for finite canonical transformations, to a similar extent that we are familiar with in ordinary Hamilton mechanics. The origin of this difficulty will be discussed in section 3. Thus it is not at all straightforward to proceed if we try to mimic the above procedure. We therefore start with a different root, due originally to Einstein (Ref. [6]), that does not presuppose any knowledge of an action functional, nor of canonical transformations.

Since, under the above conditions on the initial points and endpoints, the trajectories are uniquely determined, we can follow the time development of the $p_i$ as functions the $q^i$ on the configuration space, $p_i = p_i(q, t)$. Then we rewrite the l.h.side of (1.6) as

$$\frac{\partial p_i}{\partial t} + \frac{\partial p_i}{\partial q^j} \frac{dq^j}{dt} = \frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial p_i}{\partial q^j},$$

by using (1.5). We then obtain

$$\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial p_j} \frac{\partial p_i}{\partial q^j} = 0.$$ (2.5)

These are partial differential equations of first order for the vector field $p_i(q, t)$ on the configuration space of independent variables $(q, t)$, since $\partial H/\partial p_i$ and $\partial H/\partial q^i$ are known.

\*Historical remark: Einstein’s original intention was to extend the Sommerfeld quantum condition to non-separable cases, giving a coordinate-independent formulation of the semiclassical quantum condition, the significance of which is now well known in connection with the theory of quantum chaos. This work (the second of Ref. [6]) played an influential role in a forming period of quantum mechanics and was cited by Schrödinger and also by de Broglie in their monumental works. In this attempt, Einstein gave a simple descriptive formulation of the HJ formalism, which is, according to him, “free of surprising tricks of the trade” [6]. Unfortunately, this small but useful observation is almost forgotten now. The present author could not find any appropriate reference that explicitly mentioned his observation.
algebraic functions of the canonical variables. We call this type of partial differential equation an “Euler-Einstein (EE) equation” for later convenience, since they are analogous to the standard Euler equations in fluid mechanics where the role of the vector field \( p_i \) is played by the velocity field of the fluid.

Here, following Einstein, it is convenient to introduce the notation

\[
\bar{H} = \bar{H}(q, t) = H(p(q, t), q),
\]

which helps us to make clear the difference in independent variables between \( H \) and \( \bar{H} \), before performing partial differentiation. Now let us further require that the motion of fluid has no vorticity:

\[
\frac{\partial p_i}{\partial q^j} - \frac{\partial p_j}{\partial q^i} = 0,
\]

which guarantees the existence of the “velocity” potential \( J \) such that

\[
p_i = \frac{\partial J}{\partial q^i}.
\]

Then the sum of the second and third terms in the l.h.side of (2.5) simply takes the form \( \partial \bar{H}/\partial q_i \).

The EE equations now take the form,

\[
\frac{\partial}{\partial q^i} \left( \frac{\partial J}{\partial t} + \bar{H} \right) = 0.
\]

Thus we arrived at a single equation

\[
\frac{\partial J}{\partial t} + \bar{H} = f(t),
\]

where \( f \) is an arbitrary function of time only. Obviously, the arbitrariness of \( f \) does not affect the dynamics, since we can always redefine a new potential function \( S \) such that \( \partial J/\partial t - f = \partial S/\partial t \) and \( p_i = \partial S/\partial q_i \). Hence we obtain the HJ equation

\[
\frac{\partial S}{\partial t} + \bar{H} = 0. \tag{2.6}
\]

Conversely, we can also reproduce the original Hamilton equations of motion, in a manner analogous to the way we make transitions from the Euler picture to the Lagrange picture in fluid mechanics. Suppose we know the trajectories in configuration space as
functions, \( q^i = q^i(t) \), satisfying (1.5). Then the motions of the momenta as functions of time automatically satisfy (1.6) as a consequence of the HJ equation:

\[
\frac{dp_i}{dt} = \frac{\partial p_i}{\partial t} + \frac{\partial p_i}{\partial q^j} \frac{dq^j}{dt} = \frac{\partial^2 S}{\partial q^i \partial t} + \frac{\partial^2 S}{\partial q^j \partial q^i} \frac{dq^j}{dt} = -\frac{\partial H}{\partial q^i} - \frac{\partial H}{\partial p_j} \frac{\partial^2 S}{\partial q^j \partial q^i} \frac{dq^j}{dt} = -\frac{\partial H}{\partial q^i}.
\]

This can be regarded as a version of the integrability condition for the Hamilton-Jacobi equation if \( p_i = \partial S/\partial q^i \) is separately treated as differential equations. Its validity is actually guaranteed by our derivation, since this calculation merely inverts the process from Eq. (1.6) to Eq. (2.6). This is one of the merits of the Einstein approach.

On the other hand, Eq. (1.5) itself is obtained by the Jacobi condition (2.3), given a complete solution to (2.6): by a total differentiation of (2.3) by \( t \), we obtain

\[
0 = \frac{\partial^2 S}{\partial Q^i \partial t} + \frac{\partial^2 S}{\partial Q^j \partial q^i} \frac{dq^j}{dt} = \frac{\partial H}{\partial Q^i} + \frac{\partial^2 S}{\partial Q^i \partial q^j} \frac{dq^j}{dt} = -\frac{\partial \bar{H}}{\partial q^i} - \frac{\partial^2 S}{\partial Q^j \partial q^i} \frac{dq^j}{dt} \frac{dq^i}{dt},
\]

which gives Eq. (1.5) under condition (2.4).

The above procedures actually fit well into an abstract but modern language of differential forms. First define a closed (and exact) 2-form in the phase space \((p, q, t)\) adjoined by a time variable,

\[
\omega^{(2)} = dp_i \wedge dq^i - dH \wedge dt = d\omega^{(1)}
\]

where

\[
\omega^{(1)} = p_i dq^i - H dt.
\]

The EE equations together with the vortex-free condition are equivalent to a demand that the 2-form \( \omega^{(2)} \) vanishes when it is evaluated under the projection to the configuration space \((q^i, t)\), by assuming \( p_i = p_i(q, t) \) and hence \( dp_i = \frac{\partial p_i}{\partial q^j} dq^j + \frac{\partial p_i}{\partial t} dt \):

\[
\tilde{\omega}^{(2)} \equiv \omega^{(2)}|_{(q,t)} = \frac{1}{2} \left( \frac{\partial p_j}{\partial q^j} - \frac{\partial p_i}{\partial q^i} \right) dq^i \wedge dq^j = \left( \frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q^i} \right) dq^i \wedge dt = 0.
\]

Obviously, the vortex-free condition is nontrivial only for \( n \geq 2 \). The requirements of vortex-free flow and consequently of the HJ equation for potential function \( S \) as the vanishing condition for \( \tilde{\omega}^{(2)} \) are formulated equivalently to the condition that the 1-form \( \omega^{(1)} \) in the phase space becomes exact after the projection to the configuration space: namely, the equality

\[
\tilde{\omega}^{(1)} \equiv \omega^{(1)}|_{(q,t)} = dS = \frac{\partial S}{\partial q^i} dq^i + \frac{\partial S}{\partial t} dt
\]
is nothing but the HJ equation.

It is also useful, though not essential to our development, to note the following in understanding the connection of $\omega^{(2)}$ to the action principle. If we do not make the projection by treating $p_i$ and $q^j$ as independent variables, we can characterize it by (Ref. 7)

$$i_{\tilde{V}}(\omega^{(2)}) = 0,$$  \hspace{1cm} (2.11)

where the symbol $i_L(\cdot)$ denotes in general the operation of internal multiplication of a vector differential operator $L$ on differential forms abbreviated as “.”. In the present case,

$$L = \tilde{V} = V + \frac{\partial}{\partial t}, \quad V \equiv \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i},$$  \hspace{1cm} (2.12)

corresponding to the Hamilton equations of motion. In this sense, $\tilde{V}$, whose component form is $(-\partial H/\partial q^i, \partial H/\partial p_i, 1)$, is called a “null vector” for the form $\omega^{(2)}$. This property is akin to the above formulation with projection. The null condition (2.11), which is an analogue of our vanishing condition, is essentially equivalent, due to the Stokes theorem, to the usual variational principle for the action integral $S[p, q; t] = \int \omega^{(1)}$ in the phase space. A null vector in this sense is often called a “line field”, alternatively, reflecting the fact that its properties are similar to those associated with the Faraday magnetic lines of forces in electromagnetism: the null condition is analogous to an obvious property that the circulation or rotation (corresponding to $\omega^{(2)}$) of vector potential (corresponding to $\omega^{(1)}$) along the boundary of an infinitesimal square containing magnetic lines of forces is always zero. For more details about this, see Ref. 7. A moral here is that the HJ theory can also be interpreted as a counterpart of the null condition (2.11) under the Einstein projection from phase space to the base configuration space. Since the null condition in phase space includes the action principle as one of its consequences, the vanishing condition can be regarded as amounting to replacing the action principle without the action integral explicitly.

For guiding our later development, it is convenient to regard the HJ theory as consisting of three steps. We call the initial process to obtain the EE equations from the equations of motion “step I”. Here, in terms of the language of fiber bundles, we first decompose phase space into the configuration space as the base space and consider the
space parametrized by the momentum vector $p_i$ (cotangent vector) as the fiber lying on each point of the base space. In this sense, the ordinary phase space is usually called the “cotangent bundle”. In step I, we describe the motions in phase space by regarding the section specified by functions $p_i = p_i(q,t)$ as a dynamical (covariant) vector field on the base space. The next step from the EE equations to the HJ equation by demanding the vanishing of the 2-form under the projection is called “step II”. The remaining and final step in which we re-derive the equations of motion from the HJ equations is called “step III”. Here we constrain complete solutions for the HJ equations by designing a device with a particular (Jacobi) prescription through which we can determine the trajectories with respect to the base space coordinates directly as functions of time. It will turn out that this step, being actually a decisive part of the HJ formalism as a preliminary to quantization, is a hurdle in establishing the generalized HJ formalism for Nambu mechanics.

With this basic understanding of the nature of HJ formalism, we can now proceed to our main objective, the construction of an HJ-like formalism for Nambu mechanics. First we have to determine the decomposition of the phase space $(\xi^1, \xi^2, \xi^3)$ into a base space and fibers lying on it. Initially, there are two possibilities: designating the dimensions of fibers and the base spaces by a symbol (fiber/base),

- (1/2) decomposition: $(\xi^1, \xi^2, \xi^3) \rightarrow (\xi^1, \xi^2, \xi^3(\xi_1, \xi_2, t))$,
- (2/1) decomposition: $(\xi^1, \xi^2, \xi^3) \rightarrow (\xi^1, \xi^2(\xi_1, t), \xi^3(\xi_1, t))$,

where, in the first case, the base space has two dimensions with coordinates $(\xi^1, \xi^2)$ and the fibers are one-dimensional spaces of $\xi^3$, whose sections are described by one component field $\xi^3(\xi^1, \xi^2, t)$. In the second case, the base space is just a line whose coordinate is $\xi^1$, and the fibers have two dimensions whose sections are described by two-component fields $(\xi^2(\xi^1, t), \xi^3(\xi^1, t))$. From the viewpoint of the equations of motions whose solutions can be specified by three independent parameters, the first case corresponds to specifying the endpoint (at the time $t$) of the trajectories in the base-configuration space at $(\xi^1, \xi^2)$ and the initial condition is assigned a single implicit parameter, say $Q_1$, such that the trajectories are uniquely determined, in principle, and the sections of the one-dimensional fibers are described by $\xi^3$ as a field $\xi^3(\xi^1, \xi^2, t; Q_1)$. In the second case, we specify the endpoint at $\xi^1$ on the one-dimensional base space, and the initial condi-
tion by two implicit constant parameters, say \((Q_1, Q_2)\). Then the two-component field \((\xi^2(\xi^1, t; Q_1, Q_2), \xi^3(\xi^1, t; Q_1, Q_2))\) gives a section of the two-dimensional fibers. We stress that, apart from questions on the global existence of these functions, the odd dimensionality of phase space does not obstruct us at all: in the present approach, the specification of the conditions for determining trajectories uniquely is most essential and sufficient for our purpose, irrespectively of the dimensions of phase space.

3. \((1/2)\)-formalism

3.1 Steps I and II

Let us start from the case of \((1/2)\) decomposition. We define

\[
H = H(\xi^1, \xi^2, t) = H(\xi^1, \xi^2, \xi^3(\xi^1, \xi^2, t)),
\]

\[
\bar{G} = \bar{G}(\xi^1, \xi^2, t) = G(\xi^1, \xi^2, \xi^3(\xi^1, \xi^2, t)).
\]

The EE equation for the field \(\xi^3 = \xi^3(\xi^1, \xi^2, t)\) is derived by following the procedures explained in the previous section. The partial differentiations will be abbreviated as \(\frac{\partial}{\partial \xi^i} = \partial_i, \frac{\partial}{\partial t} = \partial_t = \partial_0\). We first have

\[
\frac{\partial (H, G)}{\partial (\xi^1, \xi^2)} = \frac{d\xi^3}{dt} = \partial_3 \xi^3 + \partial_i \xi^3 \frac{d\xi^i}{dt} = \partial_3 \xi^3 + \partial_1 \xi^3 \frac{\partial (H, G)}{\partial (\xi^2, \xi^3)} + \partial_2 \xi^3 \frac{\partial (H, G)}{\partial (\xi^3, \xi^1)}. \tag{3.3}
\]

Here and throughout this section it should be understood, unless stated otherwise, that, for \((H, G)\) without bars, \(\xi^3 = \xi^3(\xi^1, \xi^2, t)\) is substituted after operating partial differentiations by treating all three coordinates of the phase space independently: e. g., \(\partial_t \bar{H} = \partial_1 H + \partial_2 H \partial_3 \xi^3\). Thus we have

\[
\frac{\partial (H, G)}{\partial (\xi^2, \xi^3)} = (\partial_2 \bar{H} - \partial_3 H \partial_2 \xi^3) \partial_3 G - (\partial_2 \bar{G} - \partial_3 G \partial_2 \xi^3) \partial_3 H = \partial_2 \bar{H} \partial_3 G - \partial_2 \bar{G} \partial_3 H,
\]

\[
\frac{\partial (H, G)}{\partial (\xi^3, \xi^1)} = \partial_3 H (\partial_1 \bar{G} + \partial_1 G \partial_1 \xi^3) - \partial_3 G (\partial_1 \bar{H} + \partial_1 H \partial_1 \xi^3) = \partial_3 H \partial_1 \bar{G} - \partial_3 G \partial_1 \bar{H},
\]

\[
\frac{\partial (H, G)}{\partial (\xi^1, \xi^3)} = \partial_1 \bar{H} \partial_2 \bar{G} - \partial_1 \bar{G} \partial_2 \bar{H} - \partial_3 \xi^3 (\partial_3 H \partial_2 \bar{G} - \partial_3 G \partial_2 \bar{H}) - \partial_2 \xi^3 (\partial_1 \bar{H} \partial_3 G - \partial_1 \bar{G} \partial_3 H).
\]

When these results are put together, we find that the terms proportional to \(\partial_i \xi^3 (i = 1, 2)\) cancel between r.h. and l.h. sides of (3.3), and we obtain

\[
\partial_i \xi^3 = \partial_1 \bar{H} \partial_2 \bar{G} - \partial_1 \bar{G} \partial_2 \bar{H} = \partial_1 (\bar{H} \partial_2 \bar{G}) - \partial_2 (\bar{H} \partial_1 \bar{G}). \tag{3.4}
\]

\(^4\)For the integration constants of Nambu mechanics, no discrimination is assigned regarding the upper (contravariant) or lower (covariant) positions of their indices.
Since the r.h.side is a known (algebraic) function of \( \xi^1, \xi^2, \xi^3(\xi^1, \xi^2, t) \) and \( \partial_i\xi^3(\xi^1, \xi^2, t) \), this is the desired EE equation. We have completed step I.

There is now a natural way to step II. The form (3.4) exhibited in the last equality suggests itself to represent, without losing generality, the field \( \xi^3 \) as the vorticity of a two-component vector field \((S_1, S_2)\) as

\[
\xi^3 \equiv \epsilon^{3ij}\partial_i S_j = \partial_1 S_2 - \partial_2 S_1.
\]

(3.5)

Then, Eq. (3.4) can be expressed as (paired) partial differential equations for \( S_i \), being supplemented with an arbitrary scalar field \( S_0 \) that does not contribute to the vorticity of \( \partial_t S_i \) on the l.h. side of Eq. (3.4):

\[
\partial_t S_i = \bar{H}\partial_i \bar{G} + \partial_i S_0
\]

(3.6)

\[
\bar{H} = H(\xi^1, \xi^2, \partial_1 S_2 - \partial_2 S_1), \quad \bar{G} = G(\xi^1, \xi^2, \partial_1 S_2 - \partial_2 S_1).
\]

(3.7)

We call this system of equations “generalized HJ equations” in the (1/2)-formalism.

The emergence of \( S_0 \) can be understood from a gauge symmetry of the Nambu equations of motion, which was stressed by Nambu himself, but has often been discarded by later workers. The system of the equations (1.1) is invariant under transformations \((H, G) \rightarrow (H', G')\) of the two Hamilton functions such that

\[
H\delta G - H'\delta G' = \delta \Lambda,
\]

(3.8)

where a generating function \( \Lambda \) is an arbitrary function of \( G \) and \( G' \), satisfying

\[
\frac{\partial \Lambda}{\partial G} = H, \quad \frac{\partial \Lambda}{\partial G'} = -H'.
\]

(3.9)

This property ensures that \( \frac{\partial (H', G')}{\partial (H, G)} = 1 \) or equivalently, in terms of the partial derivatives with respect to \( \xi^i \),

\[
\partial_i H \partial_j G - \partial_j H \partial_i G = \partial_i H' \partial_j G' - \partial_j H' \partial_i G',
\]

(3.10)

and consequently the r.h.sides of (1.1) are invariant under \((H, G) \rightarrow (H', G')\). Now, if the pair \((H, G)\) is replaced by \((H', G')\), the r.h.side of (3.6) is equal to

\[
\bar{H}'\partial_i \bar{G}' + \partial_i S_0 = \bar{H}\partial_i \bar{G} - \partial_i \Lambda + \partial_i S_0.
\]
Therefore, the generalized HJ equations are invariant if we simultaneously shift $S_0$ by $S_0 \rightarrow S_0' = S_0 + \Lambda$. It should be kept in mind here that, though this gauge symmetry is analogous to that of electromagnetism, the degree of gauge freedom is weaker than the latter. The reason is that the form $H \partial_i G$ is not sufficiently general for representing an arbitrary given vector field in the form $A_i = H \partial_i G$: in order to exhaust the whole range of a vector field in this way, we would have to introduce multiple pairs of $(H_a, G_a)$ by extending to $A_i = \sum_a H_a \partial_i G_a$. But this weaker gauge symmetry is sufficient, at least, for the purpose of ensuring the symmetrical roles of two Hamiltonians in Nambu mechanics.

In addition to this, the system of generalized HJ equations itself has a gauge symmetry of its own with fixed $(H, G)$, under $(\mu = 1, 2, 0, \partial_t = \partial_0)$

$$S_\mu \rightarrow S_\mu + \partial_\mu \lambda$$

with an arbitrary scalar function $\lambda = \lambda(\xi_1, \xi_2, t)$. The above shift of $S_0$ associated with the gauge transformation of $(H, G)$ can be compensated by the second gauge transformation with $\partial_t \lambda = -\Lambda$, which in turn induces the shift $\partial_i \lambda$ of the spatial components $S_i$. In order to distinguish these two gauge symmetries, we call the first one “N”-gauge symmetry, and the second one “S”-gauge symmetry.

Now, we emphasize that one of prerequisites for step III is that, if $(\xi_1(t), \xi_2(t))$ are chosen to be the solutions for the Nambu equations of motion as functions of time and are substituted into the generalized HJ equations, $\xi_3$ also becomes a function of time and automatically satisfies the Nambu equation as a consequence of (3.6). As in the case of ordinary HJ formalism, evidently, this is guaranteed in our case, because its derivation is attained merely by tracing back the above procedure conversely from the generalized HJ equations via the EE equations to the starting equation (3.3).

Let us recast the generalized HJ equations in terms of differential forms. We have a natural 1-form on the base space $(\xi_1, \xi_2, t)$,

$$\bar{\Omega}^{(1)} \equiv S_i d\xi^i + S_0 dt \equiv S_\mu d\xi^\mu.$$  

(3.12)

Then, the generalized HJ equations are expressed by the following equality on taking its exterior derivative:

$$\Omega^{(2)} \equiv d\bar{\Omega}^{(1)} = (\partial_1 S_2 - \partial_2 S_1) d\xi^1 \wedge d\xi^2 + (\partial_1 S_0 - \partial_0 S_1) d\xi^1 \wedge dt$$

$$= \xi_3 d\xi^1 \wedge d\xi^2 - \bar{H}(\partial_1 G d\xi^1 + \partial_2 G d\xi^2) \wedge dt.$$  

(3.13)
The EE equation (3.4) is equivalent to the vanishing condition for the 3-form $\bar{\Omega}^{(3)} = d\bar{\Omega}^{(2)}$:

$$0 = \Omega^{(3)} = \partial_i \xi^3 d\xi^1 \wedge d\xi^2 \wedge dt - (\partial_1 H \partial_2 G - \partial_2 H \partial_1 G) d\xi^1 \wedge d\xi^2 \wedge dt. \quad (3.14)$$

It is to be noted that, unlike ordinary Hamilton mechanics, the fiber here is not directly related to tangent planes of the base space: the phase space in the present (1/2)-formalism may rather be called a “vorticity bundle” instead of a cotangent bundle in the usual case.

On the other hand, since $\partial_i \bar{G} = \partial_i G + \partial_i \xi^3 \partial_3 G$ and $d\xi^3 = \partial_1 \xi^3 d\xi^1 + \partial_2 \xi^3 d\xi^2$, the 2-form $\bar{\Omega}^{(2)}$ can be regarded as the projection, from the three-dimensional phase space to the two-dimensional base space, each adjoined with time, of the following 2-form on the phase space $(\xi^1, \xi^2, \xi^3, t)$:

$$\Omega^{(2)} \equiv \xi^3 d\xi^1 \wedge d\xi^2 - H dG \wedge dt, \quad (3.15)$$

where $dG$ is regarded as a differential 1-form on the phase space treating all of $\xi^1, \xi^2, \xi^3$ and $t$ as independent variables. This coincides with the 2-form that was used for defining the action integral (Ref. [2]). Then the closed (and exact) 3-form $d\Omega^{(2)}$

$$\Omega^{(3)} \equiv d\Omega^{(2)} = d\xi^1 \wedge d\xi^2 \wedge d\xi^3 - dH \wedge dG \wedge dt \quad (3.16)$$

has a null field associated with the Nambu equations of motion:

$$\dot{X} = \sum_{i=1}^{3} X^i \partial_i + \frac{\partial}{\partial t} \cdot i_X (\Omega^{(3)}) = 0. \quad (3.17)$$

Thus, up to step II of the present (1/2)-formalism, the structures and the relation between the EE equation and the HJ equations are almost parallel to the case of ordinary Hamilton mechanics, if the orders of the corresponding differential forms are increased by 1: $(\omega^{(1)}, S) \rightarrow (\bar{\Omega}^{(2)}, \bar{\Omega}^{(1)}), (\omega^{(2)}, \omega^{(1)}) \rightarrow (\Omega^{(3)}, \Omega^{(2)})$.

### 3.2 A difficulty: From infinitesimal to finite canonical transformations

At this juncture, let us examine the property of $\Omega^{(2)}$ in three-dimensional phase space under a general time-dependent canonical transformation, a problem that, to the author’s knowledge, has not been studied in the literature. Our purpose is to consider whether we

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\*The closed 3-form $\Omega^{(3)}$ was first considered in [5]. Note that the apparent emergence of the world “surface” $\xi(t, s_1)$ as mentioned in the previous section arises when we construct an integral invariant (Ref. [2]) corresponding to (3.15).
can expect that the similarities mentioned at the end of the previous subsection continue to the next step III. The general form of the infinitesimal canonical transformation for canonical coordinates is

\[ \delta \xi^i = \{ K, L, \xi^i \}, \quad (3.18) \]

where \( K \) and \( L \) are arbitrary time-dependent functions. By a straightforward calculation, we find

\[ \delta (\xi^3 d\xi^1 \wedge d\xi^2) = d\Sigma - \partial_t LdK \wedge dt + \partial_t KdL \wedge dt, \quad (3.19) \]

\[ \Sigma \equiv KdL + \xi^3 \left( \frac{\partial(K, L)}{\partial(\xi^2, \xi^3)} d\xi^2 - \frac{\partial(K, L)}{\partial(\xi^3, \xi^1)} d\xi^1 \right). \quad (3.20) \]

This implies that the variation \( \delta \Omega^{(2)} \) can be exact such that \( \delta(\Omega^{(3)}) = 0 \):

\[ \delta (\xi^3 d\xi^1 \wedge d\xi^2 - HdG \wedge dt) = d\Sigma, \]

provided that we choose, say, \( L = G \) and \( \delta H = \partial_t K, \delta G = 0 \), which lead to

\[ -\delta(HdG) \wedge dt - \partial_t LdK \wedge dt + \partial_t KdL \wedge dt = -(\delta H)dG \wedge dt + \partial_t KdG \wedge dt = 0. \]

In other words, the infinitesimal time development described by the Nambu equations can be unfolded to initial time, as in the case of the ordinary Hamilton equations of motion. This can also be checked directly by performing the same canonical transformation on the Nambu equations themselves.

Since a finite-time evolution can be generated by successive infinitesimal developments, we are justified in assuming that there exists a finite canonical transformation that unfolds the time development by sending \( H \) to zero, with some finite 1-form \( \Sigma' \) satisfying

\[ \xi^3 d\xi^1 \wedge d\xi^2 - \bar{H}d\bar{G} \wedge dt = d\Sigma' + Q_3dQ_1 \wedge dQ_2, \quad (3.21) \]

where \( (Q_1, Q_2, Q_3) \) are appropriate canonical variables corresponding to the initial conditions. Note that here we have to regard \( \xi^3 \) as being projected to the space \( (\xi^1, \xi^2, t) \), since we are implicitly assuming that conditions are appropriately given such that the trajectory is uniquely determined. This is analogous to the corresponding property of ordinary Hamilton mechanics

\[ p_i dq^i - \bar{H}dt = dS + P_i dQ^i, \]
which implies that we must regard the generating function \( S = S(q, Q) \) as a function of \( 2n \) independent variables \( q^i \) and \( Q^i \). This is legitimate, since there are \( 2n \) independent canonical variables. Hence for a given \( S \) satisfying the HJ equation \( \partial_t S = -\bar{H} \), we obtain the standard relations that characterize the unfolding canonical transformation in concrete form:

\[
\frac{\partial S}{\partial q^i} = p_i, \quad \frac{\partial S}{\partial Q^i} = -P_i,
\]

the latter of which is nothing but the Jacobi condition, Eq. (2.3).

Now going back to the case of Nambu mechanics, we notice a critical difference that there are only three independent canonical variables. Therefore we cannot treat both \( dQ_1 \) and \( dQ_2 \) as differentials that are independent of \( d\xi^\mu \) \((\mu = 1, 2, 0)\). Thus, one, say \( Q_2 \), out of the pair \((Q_1, Q_2)\), must be regarded as a dependent variable, namely, as a function of \((\xi^1, \xi^2, t)\), involving one constant parameter that we denote by \( P \). Note that the last parameter is still necessary for representing the number (=3) of degrees of freedom for initial conditions for determining the trajectory uniquely as a function of time. Then, by defining

\[
\Sigma' = \Sigma'_\mu d\xi^\mu + \Sigma'_Q dQ_1,
\]

\[
d\Sigma' = (\partial_1 \Sigma'_2 - \partial_2 \Sigma'_1) d\xi^1 \wedge d\xi^2 - (\partial_i \Sigma'_i - \partial_i \Sigma'_0) d\xi^i \wedge dt
\]

\[
+ \left( \partial_i \Sigma'_{Q_1} - \frac{\partial \Sigma'_i}{\partial Q_1} \right) d\xi^i \wedge dQ_1 + \left( \partial_i \Sigma'_{Q_1} - \frac{\partial \Sigma'_0}{\partial Q_1} \right) dt \wedge dQ_1
\]

we obtain, formally,

\[
\partial_1 \Sigma'_2 - \partial_2 \Sigma'_1 = \xi^3, \quad (3.22)
\]

\[
\partial_i \Sigma'_i - \partial_i \Sigma'_0 = \bar{H} \partial_i \bar{G}, \quad (3.23)
\]

\[
\partial_i \Sigma'_{Q_1} - \frac{\partial \Sigma'_i}{\partial Q_1} = Q_3 \partial_i Q_2, \quad \partial_i \Sigma'_{Q_1} - \frac{\partial \Sigma'_0}{\partial Q_1} = Q_3 \partial_i Q_2. \quad (3.24)
\]

Comparing Eqs. (3.22) and (3.23) with the generalized HJ equations (3.5) and (3.6) previously obtained, respectively, we can identify that \( \Sigma'_\mu = S_\mu \) up to gauge degrees of freedom. On the other hand, Eq. (3.24) specifies the dependence on the initial conditions, giving an implicit characterization of the required finite canonical transformation. Therefore these two relations replace the Jacobi conditions of the ordinary case. Using the S-gauge symmetry that is now extended to \( \Sigma'_{Q_1} \rightarrow \Sigma'_{Q_1} + \partial \lambda / \partial Q_1 \), we can set \( \Sigma'_{Q_1} = 0 \) without losing generality. Then the residual gauge symmetry is the previous S-gauge symmetry.
By a further redefinition $\Sigma'_\mu = S_\mu \rightarrow S_\mu - Q_1Q_3\partial_\mu Q_2(\xi^1, \xi^2, t)$, the r.h.sides of (3.24) can be sent to zero. This is analogous to our remarks on the condition (2.3) in the previous section, concerning the shift $S \rightarrow S + P_iQ^i$ of the action function in the ordinary case. However, the situation here is quite different in the sense that $Q_2$ is in general an unknown function. This exhibits a difficulty in characterizing finite canonical transformations in Nambu mechanics.

A possible way out is to choose $Q_2$ from the beginning to be a known function. Since we have two conserved fields ($\bar{H}, \bar{G}$), a natural choice would be $Q_2 = \bar{G}$ in view of our choice $\delta G = 0$ for the infinitesimal transformation discussed at the beginning of this subsection. In this case, the additional constant $P$ should be taken as the numerical value of $\bar{G}$ itself. However, a difficulty remains. Remember that $\bar{G}$ depends, in general, on $\xi^3 = \partial_1S_2 - \partial_2S_1$. This means that the r.h.sides of the first relation in Eq. (3.24) involve, in general, second derivatives of $S_i$. But the generalized HJ equation (3.23) ($\Sigma'_\mu = S_\mu$) does not directly dictate their property: to use them for deriving the Nambu equation, we are led to perform partial derivatives on both sides. Then the r.h.sides of (3.23) involve higher derivatives than second. Thus we are led to performing another partial derivative on both sides, thereby inducing yet higher derivatives, ad infinitum. In contrast to this, (2.3) in the ordinary case does not involve derivatives $\partial_iS$ at all.

Conclusion: The relations (3.24) are still too implicit to characterize finite canonical transformations for the purpose of regaining the equations of motion, unfortunately, in completing the Jacobi procedure in general.

3.3 Step III

Consideration of the previous subsection suggests that we should look at solving the generalized HJ equation in order to obtain explicit results more suitable to step III than (3.24) regarding integration constants. We will present two approaches. The first one assumes that by suitable N-gauge and/or canonical transformation, $G$ is chosen to be independent of $\xi^3$: $\partial_3G = 0$. In principle, this requirement is not restrictive, but practical applicability may be confined to relatively simpler theories. The second approach applies to the general case without any restriction at the beginning, but requires us, instead, to make a further fiber-decomposition of the two-dimensional base space to (1/1).
**Approach (i):** $\partial_3 G = 0$ case

A drastic simplification occurs when $G$ is independent of $\xi^3$: $\partial_3 G = 0$, or equivalently $\bar{G} = G$ identically, which implies that $\partial_t \bar{G} = 0$ since the whole dependence on time arises through $\xi^3$. Let us choose the gauge $S_0 = 0$ using the $S$-gauge symmetry of the generalized HJ equations. Then, if we set

$$S_i = -S \partial_i G + g_i,$$  \hfill (3.25)

by introducing a function $S = S(\xi^i, t)$ that satisfies

$$\partial_t S = -\bar{H},$$  \hfill (3.26)

Eq. (3.26) reduce to

$$\partial_t g_i = 0,$$

and we have $\xi^3 = \partial_1 G \partial_2 S - \partial_2 G \partial_1 S + \partial_1 g_2 - \partial_2 g_1$. Using the residual (time-independent) S-gauge symmetry, furthermore, we can set $g_i = \epsilon^{ij} \partial_j g$ in terms of a time-independent scalar function $g$: $\partial_1 g_2 - \partial_2 g_1 = -\partial_i \partial^i g$. Then, $g_i$ can be absorbed into $S$ by making a redefinition

$$S \rightarrow S + F$$

where $F$ is defined such that $F \partial_i G = g_i = \epsilon^{ij} \partial_j g$, which is solved as $d_G F = \partial^2 g$ with

$$d_G \equiv \partial_1 G \partial_2 - \partial_2 G \partial_1,$$  \hfill (3.27)

assuming that at least one $\partial_i G$ is not zero. Therefore we can actually set $g_i = 0$ without losing generality, and obtain

$$\xi^3 = d_G S.$$  \hfill (3.28)

We have arrived almost at the usual HJ equation: the only difference is that, instead of the projected generalized momenta as just plain partial derivatives of $S$, we now have Eq. (3.28) corresponding to an infinitesimal shift $\delta \xi^i = -\epsilon^{ij} \partial_j G$ in the base (configuration) space, which preserves $G$: $d_G G = 0$, identically.

We can now complete step III for this system. Suppose we have a solution to the generalized HJ equation (3.26), supplemented by Eq. (3.28), which has one arbitrary
integration constant $Q_1$. Then in analogy with the case of ordinary HJ equations, we impose

$$\frac{\partial S}{\partial Q_1} = Q_3, \quad (3.29)$$

by introducing an additional constant $Q_3$. If we further require that $G$ is preserved and hence impose $G = \bar{G} = Q_2$ with $Q_2$ being an additional time-independent constant, we can solve these two conditions to obtain $\xi^i$ as functions, $\xi^i(t; Q_1, Q_2, Q_3)$, of time with three parameters. Solvability is ensured by requiring that (3.29) is not invariant under the action of $d_G$, namely,

$$\frac{\partial^2 S}{\partial \xi^1 \partial Q_1} \partial_2 G - \frac{\partial^2 S}{\partial \xi^2 \partial Q_1} \partial_1 G \neq 0, \quad (3.30)$$

for it to be independent of the condition $G = Q_2$. Condition (3.29) can also be derived from consideration of the canonical transformation of the previous subsection, due to the simplification that now $G = Q_2$ does not depend on $\xi^3$, since then $\frac{\partial S}{\partial Q_1} = -\partial_3 G \frac{\partial S}{\partial Q_1}$ and consequently the first of (3.24), with $\Sigma'_{Q_1} = 0$, reduces to (3.29), while the second becomes trivial in the present gauge $S_0 = \Sigma'_0 = 0$.

Then we can derive the Nambu equations of motion by extending the method that is explained in the ordinary case: by taking a (total) time derivative of (3.29) and $G = Q_2$, we obtain

$$\frac{\partial^2 S}{\partial Q_1 \partial t} + \frac{\partial^2 S}{\partial \xi^1 \partial Q_1} \frac{d \xi^1}{dt} + \frac{\partial^2 S}{\partial \xi^2 \partial Q_1} \frac{d \xi^2}{dt} = 0, \quad (3.31)$$

$$\partial_1 G \frac{d \xi^1}{dt} + \partial_2 G \frac{d \xi^2}{dt} = 0. \quad (3.32)$$

On the other hand, since (3.26) is satisfied for an arbitrary constant $Q_1$ by its definition, we also have

$$\frac{\partial^2 S}{\partial Q_1 \partial t} = -\frac{\partial H}{\partial \xi^3} \left( \partial_1 G \frac{\partial^2 S}{\partial \xi^2 \partial Q_1} - \partial_2 G \frac{\partial^2 S}{\partial \xi^1 \partial Q_1} \right),$$

remembering that the dependence on $Q_1$ of $\bar{H}$ resides only in $\xi^3$. Using this result and (3.32), (3.31) reduces, due to condition (3.30), to

$$\frac{d \xi^1}{dt} = -\partial_3 H \partial_2 G, \quad (3.33)$$

$$\frac{d \xi^2}{dt} = \partial_3 H \partial_1 G, \quad (3.34)$$
which are nothing but the Nambu equations under the condition \( \partial_3 G = 0 \).

It remains to see to what extent the condition \( \partial_3 G = 0 \) is attainable. Consider first the possibility of an N-gauge transformation alone. Since the general N-gauge transformation \( (H, G) \to (H', G') \) must satisfy (3.10), the necessary and sufficient condition for the “axial gauge” \( \partial_3 G' = 0 \) can be stated as

\[
\partial_3 H \partial_1 G - \partial_1 H \partial_3 G = \partial_3 H' \partial_1 G', \quad \partial_3 H \partial_2 G - \partial_2 H \partial_3 G = \partial_3 H' \partial_2 G'
\]

for some \( H' \). This shows that the ratio

\[
R \equiv \frac{\partial_3 H \partial_1 G - \partial_1 H \partial_3 G}{\partial_3 H \partial_2 G - \partial_2 H \partial_3 G}
\]

must be independent of \( \xi^3 \). Obviously, there are an infinite number of possibilities for \( (H, G) \) for which this is not satisfied. Note that this exemplifies the weakness of the N-gauge symmetry compared with the case of electromagnetism. Thus, from the viewpoint of the N-gauge transformation, the attainability is restricted to a special class. For example, the case of the Euler top belongs to this class: \( R = (\xi^1(1 - I_1/I_3))/(\xi^2(1 - I_1/I_2)) \).

Indeed by an N-gauge transformation with the generator \( \Lambda = H^2/2I_3 \), giving \( (H', G') = (H, G - H/I_3) \), we have \( \partial_3 G' = 0 \).

On the other hand, it is evident, in principle, that we can use a finite canonical (or volume-preserving) coordinate transformation \( (\xi^1, \xi^2, \xi^3) \to (\xi'^1, \xi'^2, \xi'^3) \) in order to bring \( G \) to \( G' \) such that \( \partial_3 G' = 0 \), at least, locally. If a concrete form of \( G \) is given, we will be able, in general, to combine N-gauge and canonical coordinate transformations in bringing the system to fit our requirement, unless it is too complicated.

**Approach (ii): (1/1/1)-formalism**

There is another possibility for simplification without any special requirement for \( H, G \). In ordinary HJ theory, we can reduce the HJ equation to a time-independent one by setting the Hamiltonian to be a constant \( H = E \) at the beginning; the HJ equation then reduces to \( H(\partial_q \tilde{S}, q) = E \) for a reduced function \( \tilde{S}(q) \) that is related to \( S \) by \( S(q, t) = -Et + \tilde{S}(q) \) where only the first term involves time. By choosing \( E \) as one of the integration constants, the Jacobi conditions in step III necessary involve the condition

\[
t - t_0 = \frac{\partial \tilde{S}}{\partial E}
\]  

(3.35)
from the Jacobi condition $\partial S/\partial E = t_0$ with $t_0$ being one of the additional constants. Together with other Jacobi conditions associated with the other integration constants, equation (3.35) determines the trajectories as functions of time, satisfying the equations of motion.

In the present case, by setting $H = E$, the generalized HJ equations reduce to

$$\partial_t S_i - \partial_i S_0 = E \partial_i G$$

which are trivially solved as

$$S_0 = -EG + f(t), \quad S_i = g_i$$

where $f$ is an arbitrary function of time and the $g_i$ are time-independent functions. In terms of the latter, $\xi^3$ is given by

$$\xi^3 = \partial_1 g_2 - \partial_2 g_1$$

which is constrained by

$$\bar{H} = H(\xi^1, \xi^2, \xi^3(\xi^1, \xi^2; E)) = E.$$  

Obviously, the S-gauge symmetry actually allows us to set $f = 0$ without losing generality. Therefore in comparison with the case of ordinary time-independent HJ equations, there is a crucial difference that time is now completely eliminated from the scene. Of course, it is still guaranteed that if $\xi^1(t)$ and $\xi^2(t)$ are given as functions of time satisfying the Nambu equations of motion, $\xi^3$ determined through these relations automatically obeys the Nambu equation

$$\frac{d\xi^3}{dt} = \partial_1 H\partial_2 G - \partial_2 H\partial_1 G.$$

In this situation, we introduce further decomposition of the base space $(\xi^1, \xi^2)$ into $(1/1)$ fiber bundle in which $\xi^2$ is regarded as a one-dimensional fiber and $\xi^1$ as a base space, respectively, in order to regain the time $t$ on the scene. In other words, the two-dimensional base space $(\xi^1, \xi^2, t)$ regarded as the configuration space up to this point is now treated as the phase space. Then by making the projection $\xi^2 = \xi^2(\xi^1, t)$ we can introduce a further HJ structure, involving time, such that the Nambu equations of motion are obtained by the Jacobi-like prescription. What we do is expressed symbolically by $(1/2) \rightarrow (1/1/1)$.  

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This procedure itself should be regarded as a part of step III in establishing prescriptions to obtain the equations of motion from our generalized HJ equations.

For this purpose, we must have an appropriate 1-form, which we denote by \( \Omega^{(1)} \), on \( (\xi^1, t) \) such that the requirement \( \Omega^{(1)} = dT \) with some 0-form \( T \) gives an HJ-like equation in the base space \( (\xi^1, t) \). Now the EE equation in the present problem is obtained by the same method as before

\[
\partial_t \xi^2 = \partial_3 H \partial_1 \bar{G},
\]  

(3.40)

by rewriting the Nambu equation of motion

\[
\frac{d\xi^2}{dt} = \partial_3 H \partial_1 G - \partial_1 H \partial_3 G
\]

a a field equation for \( \xi^2(\xi^1, t) \) using

\[
\frac{d\xi^1}{dt} = \partial_2 H \partial_3 G - \partial_3 H \partial_2 G.
\]

Here and in what follows, double bars over symbols mean, e.g.,

\[
\bar{G}(\xi^1, t) = G(\xi^1, \xi^2(\xi^1, t)) = G\left(\xi^1, \xi^2(\xi^1, t), \xi^3(\xi^1, \xi^2(\xi^1, t))\right).
\]

(3.41)

[proof of (3.40)]: We note that for \( i = 1 \) and \( i = 2 \), because of constraint (3.39),

\[
0 = \partial_i \bar{H} = \partial_i H + \partial_3 H \partial_1 \xi^3.
\]

(3.42)

and also

\[
1 = \partial_3 H \frac{\partial \xi^3}{\partial E}.
\]

(3.43)

Then, we have

\[
\partial_i \bar{G} = \partial_i G + \partial_3 G \partial_i \xi^3 = \frac{1}{\partial_3 H} (\partial_3 H \partial_i G - \partial_i H \partial_3 G).
\]

(3.44)

Using this result, we rewrite the equation of motion for \( \xi^2 \) with a further projection \( \xi^2 = \xi^2(\xi^1, t) \),

\[
\frac{d\xi^2}{dt} = \partial_t \xi^2 + \partial_1 \xi^2 \frac{d\xi^1}{dt} = \partial_t \xi^2 - \partial_1 \xi^2 \partial_3 H \partial_2 \bar{G}.
\]

(3.45)

On the other hand, this is also rewritten as

\[
\frac{d\xi^2}{dt} = \partial_3 H \partial_1 \bar{G}.
\]

(3.46)
Hence,
\[
\frac{\partial}{\partial t} \xi^2 = \left[ \partial_3 H (\partial_1 \bar{G} + \partial_1 \xi^2 \partial_2 \bar{G}) \right]_{\xi^2 = \xi^2(\xi^1, t)} = \partial_3 H \partial_1 \bar{G}.
\]

[Q.E.D.]

These results show that we can define the following closed 2-form on \((\xi^1, \xi^2, t)\):
\[
\bar{\Omega}^{(2)}(2) \equiv \frac{\partial \xi^3}{\partial E} d\xi^1 \wedge d\xi^2 - d\bar{G} \wedge dt = \frac{1}{\partial_3 H} d\xi^1 \wedge d\xi^2 - d\bar{G} \wedge dt.
\] (3.47)

Namely, we now consider a symplectic structure \((\partial \xi^3 / \partial E) d\xi^1 \wedge d\xi^2 = (1 / \partial_3 H) d\xi^1 \wedge d\xi^2\) in the phase space \((\xi^1, \xi^2)\). The EE equation is then nothing but the vanishing condition for \(\bar{\Omega}^{(2)}\) under the projection by \(\xi^2 = \xi^2(\xi^1, t)\) from \((\xi^1, \xi^2, t)\) to \((\xi^1, t)\):
\[
\bar{\Omega}^{(2)}(2) \equiv \bar{\Omega}^{(2)}|_{\xi^2 = \xi^2(\xi^1, t)} = \left( \frac{1}{\partial_3 H} \partial_1 \xi^2 - \partial_1 \bar{G} \right) d\xi^1 \wedge dt = 0.
\] (3.48)

The desired 1-form \(\Omega^{(1)}\) is then obtained as
\[
d\Omega^{(1)} = \bar{\Omega}^{(2)},
\] (3.49)
\[
\Omega^{(1)} = p_1 d\xi^1 - \bar{G} dt,
\] (3.50)

where
\[
p_1 = p_1(\xi^1, \xi^2) = - \int_{\xi^2}^{\xi^2} \frac{dx}{\partial_3 H(\xi^1, x)},
\] (3.51)

which satisfies \(\partial_t p_1 = - \partial_1 \xi^2 / \partial_3 H\). Thus we have a generalized HJ equation for the 0-form \(T\) by requiring \(dT = \Omega^{(1)}\) under the projection \(\xi^2 = \xi^2(\xi^1, t)\):
\[
\partial_t T + \bar{G} = 0,
\] (3.52)
\[
p_1 = \partial_1 T,
\] (3.53)

the latter of which enables us to express \(\xi^2\) in terms of \(\partial_1 T\) through (3.51). Thus what we have achieved is that the base space \((\xi^1, \xi^2)\) is interpreted as a quasi-cotangent bundle on the one-dimensional base space \(\xi^1\) equipped with fibers represented by \(p_1\). In this way, we have recovered time on the scene.

We can now finish step III in the present case. Suppose we have a complete solution of (3.52) with one integration constant \(Q_1\). Then in order to extract the solution \(\xi^1 = \xi^1(t; Q_1, P, E)\) with three integration constants to the equation of motion, we impose a Jacobi-type condition
\[
\frac{\partial T}{\partial Q_1} = P
\] (3.54)
by introducing one additional constant $P$. The solubility of $\xi^1$ from this condition
\[ \frac{\partial T}{\partial Q_1 \partial \xi^1} \neq 0 \]
must be assumed here. Note also that the constant $E$ is inherited already from the constraint (3.39). Then, using Eqs. (3.52) and (3.53), we have
\[ 0 = \frac{d}{dt} \frac{\partial T}{\partial Q_1} = \frac{\partial^2 T}{\partial Q_1 \partial t} + \frac{\partial^2 T}{\partial Q_1 \partial \xi^1} \frac{d \xi^1}{dt} = -\frac{\partial \tilde{G}}{\partial Q_1} - \frac{\partial^2 T}{\partial Q_1 \partial \xi^1} \frac{d \xi^1}{dt} , \]
where $\partial \tilde{G}$ should be understood as being substituted $\xi^2 = \xi^2(\xi^1, t)$ after performing the designated partial differentiation. Thus,
\[ \frac{d \xi^1}{dt} = -\frac{\partial^2 \tilde{H}}{\partial Q_1 \partial \xi^1} \frac{d \xi^1}{dt} = \frac{\partial^2 H}{\partial Q_1 \partial \xi^1} \frac{d \xi^1}{dt} - \frac{\partial \tilde{G}}{\partial Q_1} \frac{d \xi^1}{dt} , \]
\[ \frac{d \xi^2}{dt} = -\frac{\partial \tilde{H}}{\partial \xi^1} - \frac{\partial \tilde{G}}{\partial \xi^1} \frac{d \xi^1}{dt} , \]
due to the relation (3.44). As before, the equation for $\xi^2$ is a consequence of the EE equation (3.40) for $\xi^2$, and the equation for $\xi^3$ is a consequence of the generalized HJ equation obtained in step II. The last procedure, in the present case, is essentially equivalent to using the constraint (3.39) directly through $dH/dt = 0$:
\[ \frac{d \xi^3}{dt} = -\frac{1}{\partial \tilde{H}} \left( \frac{\partial \tilde{H}}{\partial \xi^1} \frac{d \xi^1}{dt} + \frac{\partial \tilde{H}}{\partial \xi^2} \frac{d \xi^2}{dt} \right) = \frac{\partial \tilde{H}}{\partial \xi^1} \frac{d \xi^1}{dt} - \frac{\partial \tilde{H}}{\partial \xi^2} \frac{d \xi^2}{dt} , \]
\[ \frac{d \xi^3}{dt} = \frac{\partial \tilde{H}}{\partial \xi^1} \frac{d \xi^1}{dt} - \frac{\partial \tilde{H}}{\partial \xi^2} \frac{d \xi^2}{dt} , \]
\[ \bar{\Omega}^{(2)} = \xi^3 d \xi^1 \wedge d \xi^2 - \bar{H} d \bar{G} \wedge dt = E \left( \frac{\xi^3}{E} d \xi^1 \wedge d \xi^2 - d \bar{G} \wedge dt \right) , \]
Thus $\bar{\Omega}^{(2)}$, (3.47), is obtained from $\bar{\Omega}^{(2)}/E$ by substituting $\partial \xi^3/\partial E$ in place of $\xi^3/E$. There is also a similar relation for 1-forms. In the present case, $\bar{\Omega}^{(1)}$ takes the form
\[ \bar{\Omega}^{(1)} = g_1 d \xi^1 - E \bar{G} dt = E \left( \frac{g_1}{E} d \xi^1 - \bar{G} dt \right) , \]
where we have set $g_2 = 0$ by using the S-gauge symmetry without losing generality. In this gauge, we have
\[ g_1(\xi^1, \xi^2) = -\int^{\xi^2} dx \xi^3(\xi^1, x) , \]
which apparently corresponds to (3.51). Thus, again, $\Omega^{(1)}$ is obtained from $\bar{\Omega}^{(1)}/E$ by the same replacement $\xi^3/E \to \partial \xi^3/\partial E$. We interpret these relations as part of the analogue, for
the case of the integration constant $E$, of the prescription (3.35) for restricting complete solutions of the ordinary HJ equations. It would be interesting to clarify more about these correspondences in geometrical terms.

4. (2/1)-formalism

We turn to the second possibility of decomposing phase space: one-dimensional base space with the coordinate $\xi^1$ and two-dimensional fibers described by the two-component fields $(\xi^2(\xi^1, t), \xi^3(\xi^1, t))$. Throughout this section, the convention for denoting independent variables is

$$\hat{H}(\xi^1, t) \equiv H(\xi^1, \xi^2(\xi^1, t), \xi^3(\xi^1, t)),$$

so that

$$\partial_1 \hat{H} = \partial_1 H + \partial_2 H \partial_1 \xi^2 + \partial_3 H \partial_1 \xi^3,$$

where (and in what follows unless otherwise stated) the substitution of $\xi^2 = \xi^2(\xi^1, t)$ and $\xi^3 = \xi^3(\xi^1, t)$ should be understood on the r.h.side after performing partial differentiations on functions without the hat symbol.

4.1 Step I

By following the method explained in Sect. 2, we derive the EE equations for the field $(\xi^2(\xi^1, t), \xi^3(\xi^1, t))$:

$$\partial_t \xi^2 = \partial_3 H \partial_1 \hat{G} - \partial_3 G \partial_1 \hat{H}, \quad (4.1)$$

$$\partial_t \xi^3 = \partial_1 \hat{H} \partial_2 G - \partial_1 \hat{G} \partial_2 H. \quad (4.2)$$

It is guaranteed that if we substitute a solution $\xi^1 = \xi^1(t)$ of the equation of motion into the EE equations, they automatically give the equations of motion for $\xi^2$ and $\xi^3$. For example,

$$\frac{d\xi^2}{dt} = \partial_t \xi^2 + \partial_t \xi^2 \frac{d\xi^1}{dt} = \partial_3 H \partial_1 \hat{G} - \partial_3 G \partial_1 \hat{H} + \partial_1 \xi^2 (\partial_2 H \partial_3 G - \partial_3 H \partial_2 G)$$

$$= \partial_3 H \partial_1 G - \partial_3 G \partial_1 H.$$

Of course, as we have already emphasized in Sect. 2, this merely traces back the derivation of the EE equations in reverse order.
It is also easy to recast the EE equations in terms of differential forms. We first define two independent 2-forms \( \Omega^{(2)}_2, \Omega^{(2)}_3 \) in the original phase space \((\xi^1, \xi^2, \xi^3, t)\):

\[
\Omega^{(2)}_2 \equiv d\xi^2 \wedge d\xi^1 + (\partial_3 H dG - \partial_3 G dH) \wedge dt = d\xi^2 \wedge d\xi^1 + (\partial_3 H \partial_1 G - \partial_3 G \partial_1 H) d\xi^1 \wedge dt + (\partial_3 H \partial_2 G - \partial_3 G \partial_2 H) d\xi^2 \wedge dt, \tag{4.3}
\]

\[
\Omega^{(2)}_3 \equiv d\xi^3 \wedge d\xi^1 - (\partial_2 H dG - \partial_2 G dH) \wedge dt = d\xi^3 \wedge d\xi^1 - (\partial_2 H \partial_1 G - \partial_2 G \partial_1 H) d\xi^1 \wedge dt - (\partial_2 H \partial_3 G - \partial_2 G \partial_3 H) d\xi^3 \wedge dt. \tag{4.4}
\]

By making a projection to the base space \((\xi^1, t)\) with substitutions \(\xi^2 = \xi^2(\xi^1, t)\) and \(\xi^3 = \xi^3(\xi^1, t)\), these 2-forms are given, respectively, as

\[
\hat{\Omega}^{(2)}_2 \equiv -\partial_t \xi^2 d\xi^1 \wedge dt + (\partial_3 H \partial_1 \hat{G} - \partial_3 G \partial_1 \hat{H}) d\xi^1 \wedge dt, \tag{4.5}
\]

\[
\hat{\Omega}^{(2)}_3 \equiv -\partial_t \xi^3 d\xi^1 \wedge dt - (\partial_2 H \partial_1 \hat{G} - \partial_2 G \partial_1 \hat{H}) d\xi^2 \wedge dt. \tag{4.6}
\]

Thus the EE equations (4.1) and (4.2) coincide with the vanishing conditions for these projected 2-forms.

Furthermore, when \(\xi^2\) and \(\xi^3\) are treated as independent variables without projection, it is straightforward to confirm by explicit calculation that the vector differential operator \(\tilde{X}\) defined in Sect. 3 satisfies the null condition both for these 2-forms simultaneously:

\[
i_{\tilde{X}}(\hat{\Omega}^{(2)}_2) = i_{\tilde{X}}(\hat{\Omega}^{(2)}_3) = 0. \tag{4.7}
\]

These properties are almost parallel to those of the 3-form \(\Omega^{(3)}\) of the \((1/2)\)-formalism, with decreased orders of the corresponding forms. One difference is that, in the present case, vanishing (or null) conditions that lead to the EE equations are now characterized by two 2-forms instead of a single 3-form in the \((1/2)\)-formalism. In the latter case, the line field corresponding to the vector \(X^i\) is contained as the bounding edges of infinitesimal cubes associated with \(\Omega^{(3)}\), while in the present \((2/1)\) decomposition it is contained simultaneously in the boundaries of two infinitesimal squares associated with \((\Omega^{(2)}_2, \Omega^{(2)}_3)\). Another (and crucial) difference is that the 2-forms \(\Omega^{(2)}_2\) and \(\Omega^{(2)}_3\) are not closed, due to the presence of the second terms in their definitions. This shows that we cannot associate these 2-forms directly to variational principles in the three-dimensional (plus time) phase space \((\xi^1, \xi^2, \xi^3, t)\). It would be very interesting to clarify further the geometrical meaning of \((\Omega^{(2)}_2, \Omega^{(2)}_3)\).
4.2 Step II and III: Reduction to the (1/1/1)-formalism

Due to the fact that $\Omega_2^{(2)}$ and $\Omega_3^{(2)}$ are not closed, the next steps II and III must necessarily be modified, compared with the previous section, in the present (2/1)-formalism. Since the base space is one-dimensional, it is natural to find connection with the (1/1/1)-formalism. Let us first examine whether the EE equations (4.1) and (4.2) themselves allow $H$ (and $G$, if both necessary) as integration constants. We find that

$$
\partial_t \hat{H} = \partial_2 H \partial_t \xi^2 + \partial_3 H \partial_t \xi^3 = \partial_t \hat{H}(\partial_3 H \partial_2 G - \partial_2 H \partial_3 G), \quad (4.8)
$$

$$
\partial_t \hat{G} = \partial_2 G \partial_t \xi^2 + \partial_3 G \partial_t \xi^3 = \partial_t \hat{G}(\partial_3 H \partial_2 G - \partial_2 H \partial_3 G). \quad (4.9)
$$

Thus we can indeed choose $\hat{H}$ (and/or $\hat{G}$) as an integration constant separately for the partial differential equations (4.1) and (4.2). In order to connect this system to the (1/1/1)-formalism we set only the first one,

$$
\hat{H} = H(\xi^1, \xi^2(\xi^1, t), \xi^3(\xi^1, t)) = E \quad (4.10)
$$

with constant $E$. Then the EE equations are rewritten as

$$
\partial_t \xi^2 = \partial_3 H \partial_1 \hat{G}, \quad (4.11)
$$

$$
\partial_t \xi^3 = -\partial_2 H \partial_1 \hat{G} \quad (4.12)
$$

using $\partial_1 \hat{H} = \partial_1 \hat{G} = 0$. It is sufficient to consider the first one $\partial_t \xi^2$, noticing that it coincides with (3.40), when $\hat{G}$ is identified with $\hat{G}$ as it should be since the definition (3.41) amounts, after all, to making $G$ a field on one and the same base space $(\xi^1, t)$. In connection with this, it is to be noted that the closed 2-form $\Omega_2^{(2)}$ of the (1/1/1) formalism is naturally obtained from (4.3) after the projection from $(\xi^1, \xi^2, \xi^3, t)$ to $(\xi^1, \xi^2, t)$ under the constraint $H = E$ as

$$
\tilde{\Omega}_2^{(2)} = -\frac{1}{\partial_3 H} \Omega_2^{(2)} \bigg|_{\xi^3 = \xi^3(\xi^1, \xi^2, E)},
$$

on using $\partial_1 H = -\partial_3 H \partial_\xi^3$ and $\partial_1 \tilde{\hat{G}} = \partial_1 \hat{G} + \partial_3 G \partial_\xi^3$. We can thus repeat the same arguments as in the (1/1/1)-formalism, finishing steps II and III simultaneously. Thus the equation of motion for $\xi^1(t)$ is derived in exactly the same way: the existence of three independent constants $Q_1, P$ and $E$ ensures that we have general solutions for the Nambu equations of motion.

Finally, for a better appreciation of the necessity of the (1/1/1)-formalism in the present context, we note the following. If we choose from the beginning both $\hat{H}$ and
$\dot{G}$ simultaneously as integration constants for the EE equations, the latter reduces to
$\partial_t \xi^2 = 0 = \partial_t \xi^3$. Thus, time is apparently eliminated from the scene again. Although the
origin is somewhat different from what happens in the time-independent solution in the
case of the (1/2)-formalism, this motivates us to the (1/1/1)-formalism.

5. Examples

We have established a generalized Hamilton-Jacobi theory for Nambu mechanics. Two
different formalisms were presented: the first one under the requirement $\partial_3 G = 0$ is called
the (1/2)-formalism, and the second which is something analogous to the ordinary time-
independent Hamilton-Jacobi theory is called the (1/1/1)-formalism; the case of the (2/1)
decomposition was also reduced to the (1/1/1) formalism. We have discussed only the
simplest case of three (+time)-dimensional phase space. The basic ideas, in principle,
can be extended straightforwardly to Nambu mechanics of higher orders. \[\text{The resultant}
formalisms, however, become increasingly complicated since we have various possibilities
of decomposing phase spaces into fibers and base spaces of different combinations with
respect to their dimensions. In Appendix, we will give a partial description of general HJ
theory for Nambu mechanics of $(n + 1)$-dimensional phase space with $n$ Hamiltonians as
a natural extension of the case $n = 2$. In the present section, instead of discussing such
formal extensions of our formalism, we present some concrete computations by taking the
example of the Euler top, $G = \frac{1}{2} \left( \xi_1^2 \frac{I_1}{I_1} + \xi_2^2 \frac{I_2}{I_2} + \xi_3^2 \frac{I_3}{I_3} \right)$ and $H = \frac{1}{2} (\xi_1^2 + \xi_2^2 + \xi_3^2)$. It would help
us to understand more deeply the meaning and working of our general constructions.

Let us start from the (1/2)-formalism. By an N-gauge transformation as discussed in
Sect. 3, we can replace $G$ by

\[ G - H/I_3 \rightarrow G = \frac{\alpha}{2} (\xi^1)^2 + \frac{\beta}{2} (\xi^2)^2 \]  
(5.1)

where

\[ \alpha = \frac{I_3 - I_1}{I_3 I_1}, \quad \beta = \frac{I_3 - I_2}{I_3 I_2}. \]  
(5.2)

\[ \text{It is to be noted here that the extension to } 3n \ (n > 1) \text{ dimensions by introducing } n \ \text{coupled triplets as originally suggested by Nambu is not feasible, for the reason that such "canonical" structures cannot be preserved by the equations of motion and canonical transformations defined by the corresponding brackets, as signified by the violation of the so-called fundamental identity. This situation is in marked contrast to that of the usual Hamilton mechanics. The negative comment given in Ref. 5 about the possibility of HJ formalism is also related to this difficulty.} \]
Thus the generalized HJ equations for the 1-form $\bar{\Omega}^{(1)} = S_\mu d\xi^\mu$ are reduced, with $S_i = -S \partial_i G$ and $S_0 = 0$, to

$$\frac{\partial S}{\partial t} = -\bar{H} = -\frac{1}{2}((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2) \quad (5.3)$$

where

$$\xi^3 = \partial_1 G \partial_2 S - \partial_2 G \partial_1 S = \alpha \xi^1 \partial_2 S - \beta \xi^2 \partial_1 S. \quad (5.4)$$

In view of the elliptical form of $G$, it is convenient to change the variables, $(\xi^1, \xi^2) \to (G, u)$,

$$\xi^1 = \sqrt{2G/\alpha} \text{sn} u, \quad \xi^2 = \sqrt{2G/\beta} \text{cn} u, \quad (5.5)$$

by using Jacobi’s elliptic functions satisfying $\text{sn}^2 u + \text{cn}^2 u = 1, k^2 \text{sn}^2 u + \text{dn}^2 u = 1$ and $\text{sn}' u = \text{cn} u \text{dn} u, \text{cn}' u = -\text{sn} u \text{dn} u, \text{dn}' u = -k^2 \text{sn} u \text{cn} u$, where the modulus parameter $k$ is a constant to be fixed later such that the generalized HJ equation takes a simple form that is most convenient for integration. Of course, we should expect that when $\alpha = \beta$, corresponding to a symmetrical top, the above coordinate transformation would reduce to the usual polar coordinates with $k = 0$. Then,

$$\xi^3 = -\sqrt{\frac{\alpha \beta}{\text{dn} u}} \partial_u S. \quad (5.6)$$

By setting

$$S = -E(t - t_0) + \bar{S}(\xi^1, \xi^2) \quad (5.7)$$

where $E$ and $t_0$ are constants, the equation is reduced to a time-independent one:

$$\frac{\alpha \beta}{2 \text{dn}^2 u} (\partial_u \bar{S})^2 = E - \frac{G}{\alpha} \text{sn}^2 u - \frac{G}{\beta} \text{cn}^2 u = E + \frac{G(\alpha - \beta)}{\alpha \beta} \text{sn}^2 u - \frac{G}{\beta}. \quad (5.8)$$

We fix the parameter $k$ by

$$k^2 = \frac{G(\beta - \alpha)}{\alpha \beta E_0}, \quad E_0 = E - \frac{G}{\beta},$$

1Note that here we have used abbreviated notation for the elliptic functions $\text{sn}(u; k), \text{cn}(u; k)$ and $\text{dn}(u; k)$ by suppressing implicit dependencies on the parameter $k$.
which allows us to integrate $\bar{S}$ as

$$\bar{S} = \sqrt{\frac{2E_0}{\alpha \beta}} \int_0^u du \, \text{dn}^2 u \equiv \sqrt{\frac{2E_0}{\alpha \beta}} = \frac{A}{\alpha \beta} \mathcal{E}(u),$$  \hspace{1cm} (5.9)

where we have chosen $\bar{S}|_{u=0} = 0$ without losing generality and also redefined a constant $A = \sqrt{2\alpha \beta E_0}$ for later convenience. Thus $\xi^3$ is given by

$$\xi^3 = -\sqrt{2E_0} \, \text{dn} u. \hspace{1cm} (5.10)$$

The function $\mathcal{E}(u)$ is known as the fundamental elliptic integral of the second kind (or Jacobi’s epsilon function; see, e.g., Ref. [8]). This function can also be expressed as

$$\mathcal{E}(u) = \int_0^{\text{sn} u} dx \sqrt{\frac{1 - k^2 x^2}{1 - x^2}},$$ \hspace{1cm} (5.11)

which can easily be proven by noting that

$$\frac{d\mathcal{E}}{du} = \text{dn}^2 u = \text{cn} u \, \text{dn} u \frac{du}{\text{cn} u} = \frac{\sqrt{1 - k^2 \text{sn}^2 u}}{\sqrt{1 - \text{sn}^2 u}} \frac{d\text{sn} u}{du}. $$

Thanks to this formula, we now have the desired form of complete solution, which is expressed in terms of the original independent variables $(\xi^1, \xi^2, t)$:

$$S = -E(t - t_0) + A \alpha \beta \int_0^{\xi^1/\sqrt{(\xi^1)^2 + \xi^2}} \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} \, dx. \hspace{1cm} (5.12)$$

We can then follow the general prescription given in Sect. 3 to derive the equations of motion. We first impose the Jacobi condition for the above solution with respect to the integration constant $E$:

$$\frac{\partial S}{\partial E} = \text{const.} \hspace{1cm} (5.13)$$

Since we have already introduced $t_0$ corresponding to the shift $S \rightarrow S + E t_0$, the constant on the r.h.s. can actually be absorbed in $t_0$. According to our general formalism, this condition, together with the constraint $dG/dt = 0$, allows us to obtain the general solution of the Nambu equations of motion. By taking a time-derivative, we obtain

$$\frac{d}{dt} \left( \frac{\partial \bar{S}}{\partial E} \right) = \frac{du}{dt} \frac{d}{du} \left( \frac{\partial \bar{S}}{\partial E} \right) = 1. \hspace{1cm} (5.14)$$
It should be kept in mind that the derivative with respect to \( E \) must be taken while keeping \((\xi^1, \xi^2, t)\) fixed. Using \( dA/dE = \alpha\beta/A \) and \( dk^2/dE = -2\alpha\beta k^2/A^2 \),

\[
\frac{\partial \bar{S}}{\partial E} = \frac{\alpha\beta}{A} \frac{\partial \bar{S}}{\partial A} - \frac{2\alpha\beta k^2}{A^2} \frac{\partial \bar{S}}{\partial k^2} = \frac{1}{A} \left( \epsilon - 2k^2 \frac{\partial \epsilon}{\partial k^2} \right),
\]

where, for the first equality, we treated \( A \) and \( k^2 \) as independent variables. Using the integral representation (5.11) and the properties of elliptic functions, we derive

\[
\frac{d}{dt} \frac{\partial}{\partial k^2} \epsilon = -\frac{1}{2k^2} (1 - \text{dn}^2 u) \frac{du}{dt}
\]

and from the definition of \( \epsilon \) we also have

\[
\frac{d}{dt} \epsilon = \text{dn}^2 u \frac{du}{dt}.
\]

Putting these results together, we finally arrive at

\[
\frac{du}{dt} = A,
\]

and hence \( u = At \) up to an arbitrary choice of the origin of time \( t \). Substituting this result into the expressions for \((\xi^1, \xi^2, \xi^3)\) immediately gives a standard form of the general solution (see, e.g., Ref. [9]) in terms of the elliptic functions, which is usually obtained by directly integrating the equations of motion using the conservation of \( H \) and \( G \). As a check of these results, we can derive, e.g., from Eq. (5.10),

\[
\frac{d\xi^3}{dt} = \sqrt{2E_0 Ak^2} \text{sn} u \text{cn} u = \frac{I_1 - I_2}{I_1I_2} \xi^1 \xi^2.
\]

Next let us treat the same system by applying the (1/1/1)-formalism. In this case, using \( \partial_3 H = \xi^3 \), we first obtain an expression for \( p_1 \):

\[
p_1 = -\int \xi^2 \frac{dx}{\sqrt{2E - (\xi^1)^2 - x^2}} = -\arcsin \frac{\xi^2}{\sqrt{2E - (\xi^1)^2}}
\]

or equivalently

\[
\xi^2 = -\sqrt{2E - (\xi^1)^2} \sin(\partial_1 T).
\]

We then have to solve

\[
\partial_1 T = -\tilde{G} = -\frac{1}{2} (\alpha(\xi^1)^2 + \beta(\xi^2)^2).
\]

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By introducing a constant $F$ such that

$$T = -F t + \bar{T}(\xi^1)$$

the equation is reduced to

$$\sin\left(\frac{d\bar{T}}{d\xi^1}\right) = \sqrt{\frac{2F - \alpha(\xi^1)^2}{\beta(2E - (\xi^1)^2)}}.$$  \hspace{1cm} (5.18)

and we obtain a complete solution

$$\bar{T} = \int_{\xi^1}^d \arcsin\sqrt{\frac{2F - \alpha x^2}{\beta(2E - x^2)}} \, dx$$  \hspace{1cm} (5.19)

with $F$ being the integration constant.

The general solution for the trajectory $\xi^1(t)$ is derived by imposing the Jacobi condition

$$-t_0 = \frac{\partial T}{\partial F} = -t + \frac{1}{\sqrt{2\alpha \beta (E - F/\beta)}} \int_{\sqrt{\alpha \xi^1}/\sqrt{2F}}^\sqrt{\alpha \xi^1/\sqrt{2F}} \frac{dx}{\sqrt{1 - x^2} \sqrt{1 - k^2 x^2}}$$

$$= -t + \frac{1}{\sqrt{2\alpha \beta (E - F/\beta)}} u\left(\frac{\sqrt{\alpha \xi^1}}{\sqrt{2F}}\right)$$  \hspace{1cm} (5.20)

where the integration variable is rescaled $x \to \sqrt{2F x}/\sqrt{\alpha}$, and $u = u(x)$ is the inverse of the elliptic function: $x = \text{sn} (u; k)$ with the modulus parameter $k^2 = (\beta - \alpha)/\alpha(\beta E - F)$.

We thus obtain

$$\xi^1(t) = \frac{\sqrt{2F}}{\alpha} \text{sn} \left(\frac{\sqrt{2\alpha \beta (E - F/\beta)}(t - t_0)}{2F}, k\right),$$  \hspace{1cm} (5.21)

which coincides with the previous result of the (1/2)-formalism after renaming the integration constant as $F \to G$. Other components are automatically satisfied by our general arguments.

Comparing with the well-known and traditional Hamilton-Jacobi treatments (see, e.g., Ref. 9) of the Euler top in terms of Euler angles and the separation of variables, the new methods illustrated here on the basis of our generalized HJ theory of Nambu mechanics are much more direct and elegant, in the sense that the components of angular momentum themselves are treated as canonical coordinates.
6. Towards quantization

So far, we have not emphasized the relevance to our development of the canonical structure associated with the Nambu bracket. There is in fact a natural interpretation of our formulation of generalized HJ theory from the viewpoint of its connection with the Nambu bracket. It will lead us to a new standpoint towards quantization of Nambu mechanics.

As has already been pointed out in Ref. 2, for any realization of the Nambu bracket satisfying the Fundamental identity (FI), we can define subordinated Poisson brackets that are intrinsically associated with the Nambu bracket. For example, if a function $G = G(\xi^1, \xi^2, \xi^3)$ is fixed, we can define

$$\{A, B\}_G \equiv \{A, G, B\}. \quad (6.1)$$

The Jacobi identity is automatically satisfied because of the FI,

$$\{A, G, \{B, F, C\}\} = \{\{A, G, B\}, F, C\} + \{B, \{A, G, F\}, C\} + \{B, F, \{A, G, C\}\} \quad (6.2)$$

which reduces to the Jacobi identity for (6.1) by setting $F = G$. If we assume $\partial_3 G = 0$, we have

$$\{\xi^1, \xi^2\}_G = 0, \quad \{\xi^3, \xi^1\}_G = -\partial_2 G, \quad \{\xi^3, \xi^2\}_G = \partial_1 G, \quad (6.3)$$

and the Nambu equations of motion take the standard Hamiltonian form,

$$\frac{d\xi^i}{dt} = \{H, \xi^i\}_G. \quad (6.4)$$

This shows that the $(1/2)$-formalism with $\partial_3 G = 0$ can be regarded as the classical limit of a quantized Nambu mechanics; it is obtained by introducing a wave function that we denote by $\langle \xi^1, \xi^2 | 1(t) \rangle$ in the representation where $\xi^1, \xi^2$ are diagonalized, corresponding to the first component of Eq. (6.3), and $\xi^3$ is replaced by a differential operator acting on the wave function,

$$\xi^3 \rightarrow -i\hbar(\partial_1 G \partial_2 - \partial_2 G \partial_1) = -i\hbar d_G, \quad (6.5)$$

which is consistent with the above subordinated Poisson brackets (the last two of them) and the generalized HJ equations in the reduced form given in Sect. 3.3, if we assume the
usual relationship between Poisson brackets and commutators. The Schrödinger equation is then

\[ i\hbar \partial_t \langle \xi^1, \xi^2 | 1(t) \rangle = H(\xi^1, \xi^2, -i\hbar d_G) \langle \xi^1, \xi^2 | 1(t) \rangle, \] (6.6)

which leads us to the \((1/2)\)-formalism under an ansatz \( \langle \xi^1, \xi^2 | 1(t) \rangle \sim e^{iS(\xi^1, \xi^2, t)}/\hbar \). The ket symbol \( |1(t)\rangle \) is meant to imply that it contains information on the initial conditions corresponding classically to a single integration constant \( Q_1 \).

Similarly, if we choose another Poisson bracket by exchanging \( G \) for \( H \) and assuming \( \partial_3 H \neq 0 \),

\[ \{A, B\}_H \equiv \{A, H, B\}, \] (6.7)

we have

\[ \{\xi^1, \xi^2\}_H = -\partial_3 H, \quad \{\xi^3, \xi^1\}_H = -\partial_2 H, \quad \{\xi^3, \xi^2\}_H = \partial_1 H. \] (6.8)

Then, the first bracket can be interpreted as defining the symplectic 2-form \( d\xi^1 \wedge d\xi^2 / \partial_3 H \) which we have defined in the phase space \((\xi^1, \xi^2, t)\) of the \((1/1/1)\)-formalism, under the constraint \( H = E \): the latter constraint, in principle, enables us to express \( \xi^3 = \xi^3(\xi^1, \xi^2, E) \) in terms of \( \xi^1 \) and \( \xi^2 \). Indeed, once the first bracket (6.8) is given, the remaining two are consequences of this constraint. This is due to the following identities:

\[ 0 = \partial_2 H(\xi^1, \xi^2, \xi^3(\xi^1, \xi^2, E)) = \partial_2 H + \partial_3 H \partial_2 \xi^3, \]
\[ 0 = \partial_1 H(\xi^1, \xi^2, \xi^3(\xi^1, \xi^2, E)) = \partial_1 H + \partial_3 H \partial_1 \xi^3. \] (6.9)

In this case, the wave function is a function of \((\xi^1, t)\), denoted by \( \langle \xi^1 | 2(t) \rangle \) and the Schrödinger equation is

\[ i\hbar \partial_t \langle \xi^1 | 2(t) \rangle = \hat{G}(\xi^1, \hat{\xi}^2) \langle \xi^1 | 2(t) \rangle, \] (6.10)

where \( \hat{\xi}^2 \) is a differential operator, formally given by

\[ -i\hbar \partial_1 = -\int \hat{\xi}^2 \frac{dx}{\partial_3 H(\xi^1, x)}. \] (6.11)

The \((1/1/1)\)-formalism is then obtained in the classical limit with an ansatz \( \langle \xi^1 | 2(t) \rangle \sim e^{iT(\xi^1, t)}/\hbar \). The ket \( |2(t)\rangle \) is meant to imply two integration constants \((Q_1, Q_2)\) classically as initial conditions.
Rigorously speaking, these formal constructions (especially, the second one) of quantum theory are not in general well defined as they stand, due to the ambiguity of operator orderings, since both $H$ and $G$ can be arbitrarily complicated functions. However, they are meaningful at least in a semiclassical limit. More importantly, these quantized theories, which seem to be entirely different from each other with different Hilbert spaces of wave functions and different Schrödinger (or Heisenberg) equations, are guaranteed to give one and the same Nambu equations of motion in the classical limit. Classically, they are connected by N-gauge transformations and/or canonical coordinate transformations in the sense of the original phase space $(\xi_1, \xi_2, \xi_3, t)$. From this viewpoint, one possible standpoint towards more general and rigorous formulations of the quantum theory of Nambu mechanics seems to

(1) quantize the subordinated Poisson brackets defined by usual commutators algebras with variable choices of two Hamiltonians $(H, G)$ with respect to N-gauge transformations;

(2) enlarge the usual framework of quantum mechanics to a new extended scheme, allowing (infinitely) many different Hilbert spaces corresponding to different choices of Poisson brackets and different Hamiltonians;

(3) construct a transformation theory by which we can transform systems among the sets of Hilbert spaces and corresponding Hamiltonians in some covariant fashion, such that it gives the N-gauge and canonical coordinate transformations in the classical limit;

(4) find probabilistic interpretations of the formalism, in such a manner that different and allowed choices of Hilbert spaces and Hamiltonians in the framework of transformation theory give physically unique results.

The most challenging and imaginative parts of this program would be (3) and (4), by which we should expect that various possible ambiguities associated with transition from classical theory to quantum theory would be removed or restricted appropriately.

We emphasize that the above program is certainly a possible route towards quantization, though, to the author’s knowledge, such a viewpoint has scarcely been stressed in
We hope that this new viewpoint and our generalized HJ theory would be useful ultimately for further applications of Nambu mechanics and the Nambu bracket in the arena of fundamental physics.

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Appendix: Generalized HJ formalism for Nambu mechanics of order $n$

Here we extend the $(1/2)$-formalism of Sect. 3 to a $(1/n)$-formalism in the case of $(n + 1)$-dimensional phase space $(\xi^1, \ldots, \xi^{n+1})$. We denote one of $n$ Hamiltonians by $H$ and the remaining ones by $G_a$ ($a = 1, \ldots, n-1$). The equations of motion are

$$\frac{d\xi^i}{dt} = \{H, G_1, \ldots, G_{n-1}, \xi^i\} \equiv X^i. \quad (A.1)$$

The EE equations are obtained from an $n$-form

$$\Omega^{(n)} = (-1)^n \xi^{n+1} d\xi^1 \wedge \cdots \wedge d\xi^n - H dG_1 \wedge \cdots \wedge dG_{n-1} \wedge dt \quad (A.2)$$

as the vanishing condition $d\bar{\Omega}^{(n)} = 0$ after the projection $\Omega^{n} \to \bar{\Omega}^{n}$ by setting $\xi^{n+1} = \xi^{n+1}(\xi^1, \ldots, \xi^n, t)$, corresponding to the property (Ref. 23) that the vector differential operator $\bar{X} = X^i \partial_i + \partial_t$ is a line field for the exact $(n+1)$-form $\Omega^{(n+1)} \equiv d\Omega^n$. The generalized HJ equations are then obtained by demanding

$$\bar{\Omega}^{n} = dS^{(n-1)} \quad (A.3)$$

for an $(n-1)$-form $S^{(n-1)}$ on the base space,

$$S^{(n-1)} = \frac{1}{(n-1)!} \sum_{i_1, \ldots, i_{n-1} = 1}^{n} S_{i_1, \ldots, i_{n-1}} d\xi^{i_1} \wedge \cdots \wedge d\xi^{i_{n-1}}$$

3For a review on quantization and related matters, we refer the interested readers to 10 which contains an extensive list of references. It is also to be noted that an example of quantized Nambu mechanics based on the reduced Poisson bracket (18) is discussed in 11. The present author thanks to the authors of the last reference for bringing their works to his attention.
\[ + \frac{1}{(n-2)!} \sum_{\substack{i_1,\ldots,i_{n-2}=1}}^{n} S_{i_1,\ldots,i_{n-2},0} d\xi^{i_1} \wedge \cdots \wedge d\xi^{i_{n-2}} \wedge dt, \quad (A.4) \]

where \( S_{i_1,\ldots,i_{n-1}} \) and \( S_{i_1,\ldots,i_{n-2},0} \) are completely antisymmetric tensors with respect to spatial indices:

\[
-\bar{H} \frac{\partial}{\partial(\xi^{i_1},\ldots,\xi^{i_{n-1}})} = (-1)^{n-1} (\partial_i S_{i_1,\ldots,i_{n-1}} - \frac{1}{(n-2)!} \sum_{P(i_1,\ldots,i_{n-1})} (-1)^{\epsilon(P)} \partial_{i_{n-1}} S_{i_1,\ldots,i_{n-2},0}), \quad (A.5)
\]

\[
\xi^{n+1} = -\frac{1}{(n-1)!} \sum_{i_1,\ldots,i_{n}=1}^{n} \left( \sum_{P(i_1,\ldots,i_{n})} (-1)^{\epsilon(P)} \partial_{i_{n}} S_{i_1,\ldots,i_{n-1}} \right), \quad (A.6)
\]

where, as in the case of \( n = 2 \), we defined projected Hamiltonians such as

\[
H = H(\xi^1,\ldots,\xi^n,\xi^{n+1}(\xi^1,\ldots,\xi^n,t)). \quad (A.7)
\]

Also note that \((-1)^{\epsilon(P)}\) is the parity of permutations \( P \) of the set of indices \((1,2,\ldots,n)\) as indicated for summation symbols: depending on even or odd permutations, \( \epsilon(P) = 0 \) or 1. The S-gauge transformations as a symmetry of this set of equations are, with \( \lambda_{i_1,\ldots,i_{n-2}} \) being an arbitrary completely antisymmetric \((n-2)\)-tensor,

\[
S_{i_1,\ldots,i_{n-1}} \rightarrow S_{i_1,\ldots,i_{n-1}} - \frac{1}{(n-2)!} \sum_{P(i_1,\ldots,i_{n-1})} (-1)^{\epsilon(P)} \partial_{i_{n-1}} \lambda_{i_1,\ldots,i_{n-2}}, \quad (A.8)
\]

\[
S_{i_1,\ldots,i_{n-2},0} \rightarrow S_{i_1,\ldots,i_{n-2},0} - \partial_t \lambda_{i_1,\ldots,i_{n-2}}. \quad (A.9)
\]

In what follows, we choose the gauge condition

\[
S_{i_1,\ldots,i_{n-2},0} = 0. \quad (A.10)
\]

On the other hand, the N-gauge symmetry of the equations of motion is generated by transformations \((H,G_1,\ldots,G_{n-1}) \rightarrow (H',G'_1,\ldots,G'_{n-1})\) of Hamiltonians such that

\[
\frac{\partial(H',G'_1,\ldots,G'_{n-1})}{\partial(H,G_1,\ldots,G_{n-1})} = 1. \quad (A.11)
\]

As in the \((1/2)\)-formalism, we first assume, utilizing the N-gauge symmetry and/or canonical coordinate transformations, that we can choose such that the \( G_a \) do not depend on \( \xi^{n+1}:

\[
\partial_{n+1} G_a = 0. \quad (A.12)
\]
Then, by using $\partial_t \bar{G}_a = \partial_t G_a = 0$ and defining a scalar $S$,

$$S_{i_1, \ldots, i_{n-1}} = (-1)^{n-1} \frac{\partial(G_1, \ldots, G_{n-1})}{\partial(\xi_{i_1}, \ldots, \xi_{i_{n-1}})} S$$  \hspace{1cm} (A.13)

the generalized HJ equations are reduced to

$$\partial_t S = -H(\xi^1, \ldots, \xi^n, \xi^{n+1}),$$

$$\xi^{n+1} = \frac{(-1)^n}{(n-1)!} \sum_{P(i_1, \ldots, i_n)} (-1)^P \frac{\partial(G_1, \ldots, G_{n-1})}{\partial(\xi_{i_1}, \ldots, \xi_{i_{n-1}})} \partial_{i_n} S.$$  \hspace{1cm} (A.14)

To make things more concrete, let us collect some relevant expressions for $n = 3$, since the above general expressions are not very elegant to deal with. The 2-form is

$$S^{(2)} = S_{12} d\xi^1 \wedge d\xi^2 + S_{23} d\xi^2 \wedge d\xi^3 + S_{31} d\xi^3 \wedge d\xi^1$$

$$+ S_{10} d\xi^1 \wedge dt + S_{20} d\xi^2 \wedge dt + S_{30} d\xi^3 \wedge dt.$$  \hspace{1cm} (A.15)

The generalized HJ equations are

$$-\bar{H} \frac{\partial(G_1, G_2)}{\partial(\xi^1, \xi^2)} = \partial_t S_{12} - \partial_2 S_{10} + \partial_1 S_{20},$$  \hspace{1cm} (A.16)

$$-\bar{H} \frac{\partial(G_1, G_2)}{\partial(\xi^2, \xi^3)} = \partial_t S_{23} - \partial_3 S_{20} + \partial_2 S_{30},$$  \hspace{1cm} (A.17)

$$-\bar{H} \frac{\partial(G_1, G_2)}{\partial(\xi^3, \xi^1)} = \partial_t S_{31} - \partial_3 S_{10} + \partial_1 S_{30},$$

$$- \xi^4 = \partial_3 S_{12} + \partial_1 S_{23} + \partial_2 S_{31}.$$  \hspace{1cm} (A.18)

The S-gauge transformations are

$$S_{ij} \rightarrow S_{ij} + \partial_i \lambda_j - \partial_j \lambda_i, \quad S_{i0} \rightarrow S_{i0} - \partial_i \lambda_i.$$  \hspace{1cm} (A.20)

The generalized HJ equations under the assumption $\partial_4 G_1 = \partial_4 G_2 = 0$ and the gauge condition $S_{i0} = 0$ reduce to

$$\partial_t S + H(\xi^1, \xi^2, \xi^3, \xi^4) = 0,$$  \hspace{1cm} (A.21)

$$\xi^4 = -\partial_1 S_{23} - \partial_2 S_{31} - \partial_3 S_{12} = -\frac{\partial(G_1, G_2, S)}{\partial(\xi_1, \xi_2, \xi_3)} \equiv -d_G S$$  \hspace{1cm} (A.22)

with

$$S_{ij} = \frac{\partial(G_1, G_2)}{\partial(\xi_i, \xi_j)} S,$$  \hspace{1cm} (A.23)

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where we used
\[
\frac{\partial}{\partial t} \left( \frac{\partial (G_1, G_2)}{\partial (\xi^2, \xi^3)} \right) + \frac{\partial}{\partial t} \left( \frac{\partial (G_1, G_2)}{\partial (\xi^3, \xi^1)} \right) + \frac{\partial}{\partial t} \left( \frac{\partial (G_1, G_2)}{\partial (\xi^1, \xi^2)} \right) = 0.
\]

For this case, let us consider the converse problem, namely, the derivation of the equations of motion from these equations. We first consider the derivation of the equation of motion for \( \xi^4 \) from the integrability condition for our generalized HJ equations. Taking a partial derivative of (A.19) with respect to time and using (A.16)–(A.18), we obtain
\[
\frac{\partial}{\partial t} \left( \frac{\partial (G_1, G_2)}{\partial (\xi^4)} \right) = \frac{\partial}{\partial t} \left( \frac{\partial (G_1, G_2)}{\partial (\xi^1)} \right) + \frac{\partial}{\partial t} \left( \frac{\partial (G_1, G_2)}{\partial (\xi^2)} \right) + \frac{\partial}{\partial t} \left( \frac{\partial (G_1, G_2)}{\partial (\xi^3)} \right) = 0.
\]

Substituting these expressions into the integrability condition, we find that the terms proportional to \( \partial H \) on both sides just cancel each other, and the final result is, as promised,
\[
\frac{d}{dt} \left( \frac{\partial (G_1, G_2)}{\partial (\xi^2, \xi^3)} \right) + \frac{\partial}{\partial t} \left( \frac{\partial (G_1, G_2)}{\partial (\xi^3, \xi^1)} \right) + \frac{\partial}{\partial t} \left( \frac{\partial (G_1, G_2)}{\partial (\xi^1, \xi^2)} \right) = \{H, G_1, G_2, \xi_4\}.
\]

Next we have to derive the equations of motion for the base coordinates \((\xi_1, \xi_2, \xi_3)\) themselves, by imposing Jacobi-type conditions appropriately. As a generalization of the
case \( n = 2 \), we suppose that we are given a complete solution that has three integration constants, and pick up one of them, which is denoted by \( Q \). Then we demand that the derivative \( \frac{\partial S}{\partial Q} \) is a constant that is independent of time,

\[
0 = \frac{d}{dt} \frac{\partial S}{\partial Q} = \frac{\partial^2 S}{\partial Q \partial t} + \sum_{i=1}^{3} \frac{\partial^2 S}{\partial \xi^i \partial Q} \frac{d\xi^i}{dt}, \tag{A.31}
\]

together with the conditions

\[
0 = \frac{dG_1}{dt} = \sum_{i=1}^{3} \frac{\partial G_1}{\partial \xi^i} \frac{d\xi^i}{dt}, \quad 0 = \frac{dG_2}{dt} = \sum_{i=1}^{3} \frac{\partial G_2}{\partial \xi^i} \frac{d\xi^i}{dt}. \tag{A.32}
\]

We have to require that the choice of \( Q \) is such that these three conditions enable us to solve \((\xi_1, \xi_2, \xi_3)\) as functions of time \( t \). Now using the generalized HJ equations, we have

\[
\frac{\partial^2 S}{\partial Q \partial t} = -\frac{\partial H}{\partial \xi^4} \frac{\partial^2 S}{\partial \xi^4 \partial Q} = \partial_4 H \left( \frac{\partial (G_1, G_2)}{\partial (\xi^1, \xi^2)} \frac{\partial S}{\partial (\xi^3) \partial Q} + \frac{\partial (G_1, G_2)}{\partial (\xi^2, \xi^3) \partial Q} \frac{\partial S}{\partial \xi^1} \frac{\partial S}{\partial \xi^1} \right), \tag{A.33}
\]

Thus the condition (A.31) is rewritten as

\[
0 = \sum_{i=1}^{3} \frac{\partial^2 S}{\partial \xi^i \partial Q} \left( \frac{d\xi^i}{dt} + \partial_4 H \frac{\partial (G_1, G_2)}{\partial (\xi^j, \xi^k)} \right), \tag{A.34}
\]

where the set of indices \((i, j, k)\) should be understood as a cyclic permutation of the ordered set of indices \((1, 2, 3)\).

In order to be able to conclude from (A.34) that it gives the equations of motion, we further require that the two conditions for the conservation of \( G_1, G_2 \) are independent of each other. This implies that, out of three time derivatives \( d\xi^1/\partial t, d\xi^2/\partial t, d\xi^3/\partial t \), we can solve two of them in terms of a single one.

\[
\begin{pmatrix}
\partial_1 G_1 \\
\partial_1 G_2
\end{pmatrix}
\begin{pmatrix}
\dot{\xi}^1 \\
\dot{\xi}^2
\end{pmatrix}
= - \begin{pmatrix}
\partial_3 G_1 \\
\partial_3 G_2
\end{pmatrix} \dot{\xi}^3, \tag{A.35}
\]

Thus

\[
\begin{pmatrix}
\dot{\xi}^1 \\
\dot{\xi}^2
\end{pmatrix}
= \Delta^{-1} \begin{pmatrix}
\partial_2 G_2 & -\partial_2 G_1 \\
-\partial_1 G_2 & \partial_1 G_1
\end{pmatrix}
\begin{pmatrix}
\partial_3 G_1 \\
\partial_3 G_2
\end{pmatrix} \dot{\xi}^3, \tag{A.36}
\]

where

\[
\Delta = \partial_1 G_1 \partial_2 G_2 - \partial_2 G_1 \partial_1 G_2 = \frac{\partial (G_1, G_2)}{\partial (\xi^1, \xi^2)}. \tag{A.37}
\]
Then we substitute this result into the above condition \((A.34)\) and obtain

\[
0 = \frac{\partial^2 S}{\partial \xi^3 \partial Q} \left( \frac{d\xi^3}{dt} + \partial_4 H \frac{\partial (G_1, G_2)}{\partial (\xi^1, \xi^2)} \right) \\
+ \frac{\partial^2 S}{\partial \xi^1 \partial Q} \left( \Delta^{-1} (\partial_2 G_2 \partial_3 G_1 - \partial_2 G_1 \partial_3 G_2) \frac{d\xi^3}{dt} + \partial_4 H \frac{\partial (G_1, G_2)}{\partial (\xi^2, \xi^3)} \right) \\
+ \frac{\partial^2 S}{\partial \xi^2 \partial Q} \left( \Delta^{-1} (-\partial_1 G_2 \partial_3 G_1 + \partial_1 G_1 \partial_3 G_2) \frac{d\xi^3}{dt} + \partial_4 H \frac{\partial (G_1, G_2)}{\partial (\xi^3, \xi^1)} \right) \\
= \Delta^{-1} \left( \sum_{i=1}^{3} \frac{\partial^2 S}{\partial Q \partial \xi^i} \frac{\partial (G_1, G_2)}{\partial (\xi^j, \xi^k)} \right) \left[ \frac{d\xi^3}{dt} + \partial_4 H \frac{\partial (G_1, G_2)}{\partial (\xi^1, \xi^2)} \right]. \tag{A.38}\]

The expression appearing in the round bracket as the coefficient in the last line does not vanish if we require that \(\partial S/\partial Q\) is not \(d_G\) invariant:

\[
d_G \left( \frac{\partial S}{\partial Q} \right) = \sum_{i=1}^{3} \frac{\partial^2 S}{\partial Q \partial \xi^i} \frac{\partial (G_1, G_2)}{\partial (\xi^j, \xi^k)} \neq 0. \tag{A.39}\]

This requirement is sufficient to ensure the equation of motion for \(\xi^3\),

\[
\frac{d\xi^3}{dt} = -\partial_4 H \frac{\partial (G_1, G_2)}{\partial (\xi^1, \xi^2)}, \tag{A.40}\]

which also implies immediately the equations of motion for \(\xi^1, \xi^2\) using the above formula \((A.36)\), expressing \(\dot{\xi}^1, \dot{\xi}^2\) in terms of \(\dot{\xi}^3\). Thus we conclude that sufficient conditions for deriving equations of motion are

\[
\Delta \neq 0, \tag{A.41}\]

and the requirement that \(\partial S/\partial Q\) is not \(d_G\) invariant. It is to be noted that if \(\Delta = 0\) we can turn to other choices of independent \(\dot{\xi}_i\). Hence in general it is sufficient to require at least one of

\[
\frac{\partial (G_1, G_2)}{\partial (\xi^i, \xi^j)}
\]

is nonzero out of the three possible ones.

These conditions are naturally extended to general \(n\). One is

\[
\sum_{P(i_1, \ldots, i_n)} (-1)^{\ell(P)} \frac{\partial^2 S}{\partial Q \partial \xi_{i_n}} \frac{\partial (G_1, \ldots, G_{n-1})}{\partial (\xi_{i_1}, \ldots, \xi_{i_{n-1}})} \neq 0 \tag{A.42}\]

and also that

\[
\frac{\partial (G_1, G_2, \ldots, G_{n-1})}{\partial (\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_{n-1}})} \neq 0 \tag{A.43}\]
for at least one of the possible combinations of the set \((i_1, \ldots, i_n)\) of indices.

Finally, let us briefly consider an extension of the \((1/1/1)\)-formalism to the general case: \((1/n) \rightarrow (1/1/n - 1)\). We first set the condition

\[
\bar{H}(\xi^1, \ldots, \xi^n) \equiv H(\xi^1, \ldots, \xi^n, \xi^{n+1}(\xi^1, \ldots, \xi^n)) = E \tag{A.44}
\]

with a constant \(E\), which implies

\[
1 = \partial_{n+1}H \frac{\partial\xi^{n+1}}{\partial E}, \quad 0 = \partial_i H + \partial_{n+1}H \partial_i \xi^{n+1}, \quad (i = 1, \ldots, n). \tag{A.45}
\]

As has always been the case, for the partial derivatives without bar symbols it should be understood that \(\xi^{n+1} = \xi^{n+1}(\xi^1, \ldots, \xi^n, t)\) is substituted after differentiation. Then, we have a completely time-independent solution to the generalized HJ equations:

\[
S_{i_1, \ldots, i_{n-1}} = g_{i_1, \ldots, i_{n-1}}, \tag{A.46}
\]

\[
S_{i_1, \ldots, i_{n-2}, 0} = (-1)^{n-1}E G_1 \frac{\partial (G_2, \ldots, G_{n-1})}{\partial (\xi^{i_1}, \ldots, \xi^{i_{n-2}})}, \tag{A.47}
\]

\[
\xi^{n+1} = -\frac{1}{(n-1)!} \sum_{i_1, \ldots, i_{n-1}} (-1)^{\epsilon(P)} \partial_{n} g_{i_1, \ldots, i_{n-1}}, \tag{A.48}
\]

where \(g_{i_1, \ldots, i_{n-1}}\) is a time-independent and completely antisymmetric \((n - 1)\)-tensor. The disappearance of time motivates us to decompose the \(n\)-dimensional base space into a fiber bundle with \((n - 1)\)-dimensional base space and one-dimensional fiber parametrized by \(\xi^n\), and define a closed (and exact) \(n\)-form \(\bar{\Omega}'(n)\) by regarding the base space \((\xi^1, \ldots, \xi^n, t)\) now as an \(n\)-dimensional phase space:

\[
\bar{\Omega}'(n) = (-1)^{n} \frac{\partial \xi^{n+1}}{\partial E} d\xi^1 \wedge \cdots \wedge d\xi^n - d\bar{G}_1 \wedge \cdots \wedge d\bar{G}_{n-1} \wedge dt = d\Omega^{n-1}, \tag{A.49}
\]

with

\[
\Omega'^{(n-1)} = -\left(\int^{\xi^n} \frac{dx}{\partial_{n+1}H(\xi^1, \ldots, \xi^{n-1}, x)}\right) d\xi^1 \wedge \cdots \wedge d\xi^{n-1} - \bar{G}_1 d\bar{G}_2 \wedge \cdots \wedge d\bar{G}_{n-1} \wedge dt. \tag{A.50}
\]

On the other hand, the Nambu equations of motion for \(\xi^i (i = 1, \ldots, n)\) are rewritten as

\[
\frac{d\xi^i}{dt} = (-1)^{n} \partial_{n+1}H \{\bar{G}_1, \ldots, \bar{G}_{n-1}, \xi^i\} \equiv \bar{X}^i, \tag{A.51}
\]
where \(\{\cdots\}_n\) denotes the \(n\)-dimensional Nambu bracket with respect to \((\xi^1, \ldots, \xi^n)\), and use has been made of

\[
\partial_i G_a = \partial_i \bar{G}_a - \partial_i \xi^{n+1} \partial_{n+1} G_a = \partial_i \bar{G}_a + \frac{\partial_i H}{\partial_{n+1} H} \partial_{n+1} G_a.
\]

The vector differential operator \(\tilde{X} = \bar{X}^i \partial_i + \partial_t\) is a null vector for the closed \(n\)-form \(\Omega\). Correspondingly, the EE equations, as partial differential equations in \((n-1)\)-dimensional (+time) base space \((\xi^1, \ldots, \xi^{n-1}, t)\), are obtained as the vanishing condition

\[
0 = \bar{\Omega}^n \equiv \bar{\Omega}^n|_{\xi^n = \xi^n(\xi^1, \ldots, \xi^{n-1}, t)}.
\]

Similarly, reduced generalized HJ equations are obtained by requiring that the \((n-1)\) form \(\Omega^{(n-1)}\) is equal to the exterior derivative \(d\Omega^{(n-2)}\) of an \((n-2)\)-form \(\Omega^{(n-1)}\) after the same projection to \((\xi^1, \ldots, \xi^{n-1}, t)\). In this way, we can, in principle, continue reductions to lower-dimensional base spaces, recursively: we have a “nested” structure of generalized HJ formalisms, ranging all the way from \((1/n), (1/1/n-1), \ldots, (1/1/\ldots/1)\).

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