On deterministic nature of the intermittent geodesic acoustic mode observed in L- mode discharge near tokamak edge

Part II    Zonal flow-GAM driven by electron drift wave turbulence

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Abstract

This paper is part II of a possible series of papers that derive/investigate the salient features of the Zonal Flow-Geodesic Acoustic Mode (ZF-GAM) phenomenon observed, routinely, in the edge regions of L Mode discharges in tokamaks. Utilizing the techniques and methodologies developed in Part I, we work here with a different source of turbulence - the electron drift wave (EDW) in the well-known δe model- than Part I where the turbulence originated in the ITG; the two sources of turbulence, propagating, respectively, in the electron (ion) diamagnetic directions, are highly distinguished in their basic characteristics. Though the ITG generated ZF-GAM system is widely studied, this is, perhaps, the first attempt at exploring the system when driven by modes rotating in the electron direction. Several structural features of ZF-GAM appear to be quite robust, and are preserved through this drastic change in the underlying source of turbulence. Although this very idealized δe model calculation, may not be fully representative of other electron drift waves, our initial investigations, however, do expose important differences with the ITG approach; the most striking is that the turbulence levels required for GAM excitation can be an order of magnitude higher than for ITG turbulence. Investigations involving other varieties of drift waves could finally settle the generality of this interesting result.

Keywords: zonal flow, drift wave, intermittent, GAM, L-H mode

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I Introduction

Lacking definite experimental information about the rotational direction of the micro-instabilities responsible for generating zonal flows, the standard ITG dominated approach (Part I) may not be enough for an understanding of the phenomenon of the Zonal Flow-Geodesic Acoustic Mode (ZF-GAM) excitation. Here we will build a competing scenario in which the underlying background micro-turbulence originates in modes that rotate in the electron (diamagnetic) direction. Such an exercise, by providing an alternative to the conventional picture, will, surely, extend the scope of the “theory”, and may even lead to some experimentally significant predictions.

One must confront at the outset that the best-known drift waves in the electron direction—namely the collisionless trapped electron mode, (CTEM or simply TEM) and dissipative trapped electron mode (DTEM)\[^{1,2}\]—are not likely to be major constituents of the L mode turbulence because of high collisionality in the edge region where GAM activity is observed (See Appendix A in ‘Part I’ for details). In this paper, we will resort to the instability associated with the relatively simple but well-known $\delta_e$-model\[^{3,4,5}\]; it will be examined as a candidate for edge turbulence. We chose this model for its relative simplicity even though more realistic models, such as impurity driven drift waves, do exist in literature\[^6\]. Notice that the description of the latter requires the simultaneous treatment of several fields: the perturbed density, temperature and parallel velocities for electrons, main ions and impurities. A two-dimensional (2D) theory (for example, in the 2D ballooning representation) for such a system is quite complicated. No doubt that these complexities could be significant, but a first attempt with a simpler model is quite appropriate. We expect that the fuller models may be essential to predict linear growth rates and marginal stability criteria, but quantities like Reynolds stress and group velocity (determined by the structure of the mode potential) may be well approximated in the simpler model. The last but not the least reason for the simpler approach is that a lack of experimental knowledge on impurities (at the L-H transition threshold) would be a serious impediment to the analysis of the impurity driven mode. For brevity, the toroidal electron drift wave in $\delta_e$-model will be called $\delta_e$-mode.

In this paper, the $\delta_e$-mode will be treated as a 2D system. In order to obtain the 2D structure of the mode it is important to keep all the translational symmetry breaking (TSB) terms\[^{5,7}\]. When TSB terms are neglected, the resulting 1D equation yields two branches: the slab-like Pearlstein-Berk branch\[^8\], stabilized by the magnetic shear\[^{9,10}\], and the Chen-Cheng branch\[^{4,11}\], an intrinsic toroidal branch that disappears in the slab limit. For the latter branch the shear...
stabilization is greatly reduced owing to ‘tunneling’ \[^{[4]}\]; consequently the finite non-adiabatic part of electron density can render the mode unstable. The growth rate of the latter branch is, likely, greater than the former one for same $\delta$. Therefore, in this paper $\delta$-mode solely refers to the intrinsic toroidal branch.

The remaining part of the paper is organized as follows. In section II the multiple scale derivative expansion method in spatiotemporal configuration is applied to $\delta$-mode; the zeroth order (microscale) equation defines the relevant drift wave. The first order (mesoscale) yields an equation for drift wave energy modulated by zonal flow. In section III the microscale equation is solved by making use of the 2D weakly asymmetric ballooning theory (WABT). The solution is then chosen to be the initial guess of an iterative finite difference method to acquire more accurate results in section IV. Use is made of the two types of solutions to obtain the spatiotemporal structure of Reynolds stress and group velocity in section V. This sets the stage for substituting the principal ingredients into the zonal flow-drift wave energy set - Equations (18-20) in ‘Part I’. Results of the intermittent excitation of GAM generated by $\delta$-mode are given in section VI, spatiotemporal structure of GAM moving inwardly and outwardly are displayed in detail. Discussions and conclusions are given in section VII. In Appendix A, we derive the ‘extended’ $\delta$-model by including TSB terms in \((x,l)\) representation (the relevant notation is also explained). The conditions for a valid WABT are discussed in Appendix B.

**II Multiple scale derivative expansion method for $\delta$-mode**

The $\delta$-mode of this paper is built on a large-aspect-ratio, up-down symmetric tokamak equilibrium with concentric circular magnetic surfaces. We start from the first two linear moment equations – the continuity equation of warm ion fluid, and ion parallel momentum equation under modulation of zonal flow

\[
\left( \frac{\partial}{\partial t} + \bar{u} \cdot \nabla \right) \left( 1 - i\delta - \rho_s^2 \nabla_{\perp}^2 \right) \varphi + i \omega_c \rho_s^2 \nabla_{\perp}^2 \varphi + i \omega_e \varphi - 2i \left( 1 + \delta \right) \tilde{\omega}_d \varphi + \nabla_{\parallel} u_{\parallel} = 0 \tag{1}
\]

\[
\left( \frac{\partial}{\partial t} + \bar{u} \cdot \nabla \right) u_{\parallel} + \left( 1 + \delta \right) c_s^2 \nabla_{\parallel} \varphi = 0 \tag{2}
\]

where $\bar{u} \equiv \rho_s c_s b \times \nabla \tilde{\varphi}$ is the zonal flow, $\tilde{\varphi}$ is the dimensionless zonal electric potential normalized to electron thermal energy, $u_{\parallel}$ is the ion parallel velocity (see Appendix A for detailed notations).
The methodology of Ref. 12 begins with the multiple scale expansion method in which the derivatives take the form [13]

\[
\frac{\partial}{\partial t} + \frac{\partial}{\partial \tilde{r}} + \varepsilon \frac{\partial}{\partial \tilde{r}} - i \omega + \frac{\partial}{\partial \tilde{r}} \right) + \frac{\partial}{\partial \tilde{r}} \right) \varphi(t, \tilde{r}) = \tilde{\varphi}(t, \tilde{r}) \varphi(\tilde{t}, \tilde{r})
\]

(3)

where \( \tilde{t}, \tilde{r}(t, r) \) denotes fast (slow) scale variables, \( \varepsilon \ll 1 \) is introduced for bookkeeping. The \( \delta \)-mode potential \( \varphi(\tilde{t}, \tilde{r}) \) varies on fast scale [usually with high frequency \( \omega \): \( \varphi(\tilde{t}, \tilde{r}) = \varphi(\tilde{r}) \exp(-i \omega \tilde{t}) \)], while the zonal flow modulated \( \tilde{\varphi}(t, r) \) has slow variation. The zonal flow in slow scale is treated to be small quantity, \( \tilde{b} \cdot \nabla \rightarrow \varepsilon \tilde{b}(\partial / \partial \tilde{y}) \). Here \( y \equiv r_j \theta \) and \( \partial / \partial \tilde{y} \rightarrow -i (m + l) / r_j \approx -i k_\theta \). The differential operators of interest, calculated to order \( \varepsilon \)

\[
\nabla_\perp^2 = \frac{\partial^2}{\partial \tilde{r}^2} + \frac{\partial^2}{\partial \tilde{y}^2} + 2\varepsilon \left( \frac{\partial}{\partial \tilde{r}} \frac{\partial}{\partial \tilde{r}} + \frac{\partial}{\partial \tilde{y}} \right) \equiv \nabla_\perp^{2,0} + 2\varepsilon \left( \frac{\partial}{\partial \tilde{r}} \frac{\partial}{\partial \tilde{r}} + \frac{\partial}{\partial \tilde{y}} \right)
\]

(4)

\[
\tilde{\omega}_c = -i \rho_c L_n \left( \frac{\partial}{\partial \tilde{y}} + \varepsilon \frac{\partial}{\partial \tilde{y}} \right) \equiv \hat{\omega}_{c,0} + \varepsilon \hat{\omega}_{c,1}, \quad \tilde{\omega}_s = \hat{\omega}_{s,0} + \varepsilon \hat{\omega}_{s,1}
\]

(5)

\[
\hat{\omega}_{de} = -i \rho_c \left[ \sin \theta \left( \frac{\partial}{\partial \tilde{r}} + \varepsilon \frac{\partial}{\partial \tilde{r}} \right) + \cos \theta \left( \frac{\partial}{\partial \tilde{y}} + \varepsilon \frac{\partial}{\partial \tilde{y}} \right) \right] \equiv \hat{\omega}_{de,0} + \varepsilon \hat{\omega}_{de,1}
\]

(6)

and

\[
\nabla_{\parallel}^2 = \left( 1 + \varepsilon \frac{1}{i \omega} \frac{\partial}{\partial \tilde{t}} + \tilde{b} \frac{\partial}{\partial \tilde{y}} \right) \left( 1 + \delta_t \right) \frac{c_s^2}{i \omega} \nabla_{\parallel}^2 \varphi,
\]

(7)

when substituted into Eq. (1), yield

\[
\left[ -i \omega + \varepsilon \left( \frac{\partial}{\partial \tilde{t}} + \tilde{b} \frac{\partial}{\partial \tilde{y}} \right) \right] \left[ 1 - i \delta_t \rho_s^2 \left( \nabla_{\perp}^{2,0} + 2\varepsilon \left( \frac{\partial}{\partial \tilde{r}} \frac{\partial}{\partial \tilde{r}} + \frac{\partial}{\partial \tilde{y}} \right) \right) \right] \varphi
\]

\[
+ i \rho_s^2 \left( \hat{\omega}_{s,0} + \varepsilon \hat{\omega}_{s,1} \right) \left[ \frac{\nabla_{\perp}^{2,0} + 2\varepsilon \left( \frac{\partial}{\partial \tilde{r}} \frac{\partial}{\partial \tilde{r}} + \frac{\partial}{\partial \tilde{y}} \right) \right] \varphi + i \left( \hat{\omega}_{c,0} + \varepsilon \hat{\omega}_{c,1} \right) \varphi
\]

\[
- 2i \left( 1 + \delta_t \right) \left( \hat{\omega}_{de,0} + \varepsilon \hat{\omega}_{de,1} \right) \varphi + \left[ -i \omega + \varepsilon \left( \frac{\partial}{\partial \tilde{t}} + \tilde{b} \frac{\partial}{\partial \tilde{y}} \right) \right] \left[ 1 + \delta_t \right] \frac{c_s^2}{i \omega} \nabla_{\parallel}^2 \varphi = 0
\]

(8)

For the multiple scale wave function \( \varphi(t, r) \rightarrow \varphi(\tilde{r}) \exp(-i \omega \tilde{t}) \rightarrow \tilde{\varphi}(t, r) \varphi(\tilde{r}) \) (assuming stationary background micro-turbulence), the zeroth order equation

\[
\left[ \left( 1 + \frac{\hat{\omega}_{s,0}}{\omega} \right) \rho_s^2 \nabla_{\perp}^{2,0} - (1 - i \delta_t) + \frac{\hat{\omega}_{s,0}}{\omega} - 2 \left( 1 + \delta_t \right) \frac{\partial}{\partial \tilde{t}} \frac{\partial}{\partial \tilde{t}} - \left( 1 + \delta_t \right) \frac{c_s^2}{c_t^2} \nabla_{\parallel}^2 \right] \varphi(\tilde{r}) = 0
\]

(9)

corresponds to Eq. (A1) in Appendix A. From the first order equation
\[ \frac{\partial \Phi(t, r, \theta)}{\partial t} + \nu_\varphi(\theta) \frac{\partial \Phi(t, r, \theta)}{\partial r} + \nu_\varphi(\theta) \frac{\partial \Phi(t, r, \theta)}{r \partial \theta} = i k_j \bar{\Phi}(t, r) \Phi(t, r, \theta) \]  

(10)

we calculate: the radial group velocity

\[ \nu_\varphi(\theta) \equiv -\frac{2(\omega - \nu_j \bar{k}_j) \rho_i^2 \{ K_r \} + 2(1 + \delta_i) \frac{\rho_i \omega}{R} \sin \theta}{1 - i \delta_i + \rho_i^2 k_j^2 + \rho_i^2 \{ K_r^2 \} - (1 + \delta_i) \frac{\rho_i \omega}{\omega^2} \rangle \langle V^2_\theta} \]

(11)

and the poloidal group velocity

\[ \nu_\varphi(\theta) \equiv \frac{\nu_i \left[ 1 - \rho_i \rho_i^2 \left( \langle K_r^2 \rangle + 3k_j^2 \right) \right] + 2 \omega \rho_i^2 k_j - 2(1 + \delta_i) \frac{\rho_i \omega}{R} \cos \theta}{1 - i \delta_i + \rho_i^2 k_j^2 + \rho_i^2 \langle K_r^2 \rangle - (1 + \delta_i) \frac{\rho_i \omega}{\omega^2} \rangle \langle V^2_\theta} \]

(12)

where \( K_r \equiv -i \left( \partial / \partial \bar{r} \right) \), \( \langle \ldots \rangle \equiv \int \bar{d} \varphi^* (\bar{r}) \ldots \varphi(\bar{r}) / \int \bar{d} \varphi^* (\bar{r}) \varphi(\bar{r}) \) denotes average over the radial fast scale \( \bar{r} \).

III WABT for solving \( \delta_i \) -mode

In the toroidal coordinates \( (r, \theta, \zeta) \) corresponding to the radial, poloidal, and toroidal directions respectively, the 2D mode can be expressed in the \( (x, l) \) representation near the rational surface \( r_j \)

\[ \varphi(r, \theta, \zeta) = \exp \left[ i \left( n \zeta - m \theta \right) \right] \sum_i \varphi_i(r) \exp(-il\theta) \]

(13)

where \( n \) is the toroidal number (a good quantum number in this paper), \( m = nq(r_j) \) in Eq.(13); the integer \( m \) is the central Fourier number, and \( l \) is known as the sideband number. The extended \( \delta \) -model \(^{3,4,5} \), including the translational symmetric breaking (TSB) terms, comes from the zeroth order equation Eq. (9). In \( (x, l) \) representation, where \( x \equiv n \left( dq / dr \right) (\bar{r} - r_j) \), it reads

\[ \hat{k}_j^2 \frac{\hat{\omega}}{\omega_j} \left[ 1 + \frac{\hat{k}_j}{\omega_j} \right] \Phi_j - \frac{\hat{\omega}}{\omega_j} \left( 1 - i \delta_j + \hat{k}_j - \frac{1}{\omega_j} \hat{k}_j \right) \Phi_j - (1 + \delta_j) \frac{\hat{\omega}}{\omega_j} \left[ \frac{\partial}{\partial x} \left( \varphi_{i+1} - \varphi_{i-1} \right) + (\varphi_{i+1} + \varphi_{i-1}) \right] \]

\[ + (1 + \delta_j) \frac{\hat{\omega}}{\omega_j} (x - l)^2 \Phi_j + \left[ \hat{k}_j^2 \frac{\partial}{\partial x} \left[ -3 \frac{1}{\hat{w}_j} + \left( \frac{1}{\hat{w}_j} \right) \frac{\partial}{\partial x} \right] \Phi_j - \frac{\hat{\omega}}{\omega_j} \left[ 2 \hat{k}_j^2 + (1 - i \delta_j) \frac{r_j}{\hat{w}_j} \right] \Phi_j + \right] \]

\[ \frac{1}{\omega_j} \left( 1 + \frac{1}{\hat{w}_j} \right) \Phi_j + \frac{1}{\omega_j} \left( 1 - i \delta_j \right) \frac{\partial}{\partial x} (\varphi_{i+1} + \varphi_{i-1}) \left( \frac{l}{m} \right) + \left[ \frac{1}{\omega_j} \left( \frac{1}{\hat{w}_j} \right) \frac{\partial}{\partial x} \right] \Phi_j \]

\[ + \hat{k}_j^2 \frac{\partial}{\partial x} \left[ -3 \frac{1}{\hat{w}_j} \right] \Phi_j \left( \frac{l}{m} \right)^2 = 0 \]

(14)
Eq. (14) is derived in Appendix A where the notation, and normalizations are spelled out. We will attempt to write the remaining part of the paper so that it is accessible to readers who are not familiar with WABT and/or the 2D ballooning theory; some known facts may need to be repeated for greater readability.

For a high $n$ local mode pertaining to a given rational surface $r_j$, the monotonic safety factor $q(r)$ can be expanded up to the first order: 

$$q(r) \approx q(r_j) + (dq/dr)(r-r_j) \equiv q(r_j) + x/n.$$ 

Then, one may develop the 2D theory by invoking the 2D Fourier-ballooning transform \cite{14,5,15}:

$$\varphi_i(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda \int_{-\infty}^{\infty} dke^{i(kx) - i\lambda t} \varphi(k, \lambda)$$

For WABT, we may adopt the Ansatz $\varphi(k, \lambda) = \psi(\lambda) \chi(k, \lambda)$, where $\psi(\lambda)$ is a fast varying function in $\lambda$ while $\chi(k, \lambda)$, a slowly varying envelope function (in the parameter $\lambda$), is termed the ballooning solution. $\psi(\lambda)$ is also called the Floquet phase distribution (FPD), and the preceding ansatz is warranted by a general theorem that controls the solutions of differential equations with periodic coefficients. $\psi(\lambda)$ is localized around some $\lambda_*$ known as Floquet phase. In the limit $\psi(\lambda) \to \delta(\lambda - \lambda_*)$, Eq. (15) reduces to the Lee-Van Dam representation \cite{16}. Notice that Eq. (15) is a ‘proper’ mathematical transform and the 2D mode structure can be uniquely constructed from a knowledge of $\psi(\lambda)$. The 2D ballooning equation for the $\delta_e$-model (in Fourier-ballooning representation) takes the rather complicated form

$$\left[ L_0(k, \lambda; \hat{\varphi}) + \frac{iL_1(k, \lambda; \hat{\varphi})}{\omega} \frac{\partial}{\partial \lambda} + \frac{L_2(k, \lambda; \hat{\varphi})}{\omega^2 n^2} \frac{\partial^2}{\partial \lambda^2} + \cdots - \Omega(\hat{\varphi}) \right] \varphi(k, \lambda) = 0$$

with

$$\Omega(\hat{\varphi}) = -\frac{\hat{\varphi}}{\omega_k} \left[ 1 - i\delta_e + \hat{k}_\phi^2 k^2 - \frac{1}{\omega} + \frac{\hat{\eta}_k}{\omega} \hat{k}_\phi^2 \right]$$

$$L_0(k, \lambda; \hat{\varphi}) = (1 + \delta_e) \frac{\hat{\varphi}}{\omega_k} \frac{\partial^2}{\partial k^2} + \frac{\hat{\varphi}}{\omega_k} \left[ 1 + \frac{\hat{\eta}_k}{\omega} \right] \hat{k}_\phi^2 \sin^2 k^2 + 2(1 + \delta_e) \frac{\hat{\varphi}}{\omega_k} \left[ \hat{k}_\phi \sin(k + \lambda) + \cos(k + \lambda) \right]$$

$$L_4(k, \lambda; \hat{\varphi}) = \frac{1}{q} \left\{ \hat{k}_\phi^2 \frac{\hat{\eta}_k}{\omega_k} \left[ -3 - \frac{1}{s} \frac{1}{L_n} - \left( 1 + \frac{1}{s} \frac{1}{L_n} \right) \hat{k}_\phi^2 \right] + 2(1 + \delta) \frac{\hat{\varphi}}{\omega_k} \cos(k + \lambda) \right\}$$
\[ L_2(k, \lambda; \hat{\omega}) = \frac{1}{q^2} \left( \frac{\vec{\nabla}^2}{\hat{\omega}_0} + \frac{3}{\hat{s} L_n} \left[ \frac{1}{\hat{s} L_n} \right] - \frac{1}{\hat{s} L_m} \left[ \frac{1}{\hat{s} L_m} \right] \frac{r_j}{\hat{s}} + \left( \frac{1}{\hat{s}^2} L_n^2 \right) k^2 \right) \]

\begin{align}
&+ \frac{1}{\hat{s} L_n} \left[ \frac{1}{\hat{s} L_n} \frac{r_j}{\hat{s}} \right] - \frac{\hat{\omega}_0}{\hat{s} L_n} \left[ \hat{\omega}_0 \right] \left( \frac{1}{\hat{s}^2} L_n^2 \right) \left( \frac{r_j^2}{\hat{s}} \right) \right] \}
\end{align}

where the coefficients \( L_{1,2}(k, \lambda; \hat{\omega}) \) multiply terms higher order in \( \epsilon_n \equiv (1/n)(d/d\lambda) \).

Notice that Eq. (16) reveals an expansion in terms of a small ‘parameter’ \( \epsilon_n \equiv (1/n)(d/d\lambda) \) in large \( n \) limit. The inclusion of the differential operator \( (d/d\lambda) \) simply means that the scale length of \( \psi(\lambda) \), along with \( n \), controls the smallness of the expansion parameter. For instance, if \( \psi(\lambda) \rightarrow \delta(\lambda) \), the expansion will be totally invalid. This issue will be discussed in Appendix B. We will find that \( L_{1,2}(k, \lambda; \hat{\omega}) \) are crucial elements in a singular perturbation theory that we will develop to solve Eq.(16).

The lowest order Eq. (16)

\[ \left[ L_0(k, \lambda; \hat{\omega}) - \Omega(\lambda) \right] \chi(k, \lambda) = 0 \]

is, traditionally, called the ballooning equation with \( \Omega(\lambda) \) representing the (\( \lambda \)-parameterized) effective local eigenvalue. The ballooning equation has two salient features: (a) its counter-part in the \((x,l)\) representation (calculated after removing all TSB terms from Eq.(A4) such as \( l/m \rightarrow 0 \) and \( f(x) \rightarrow f(0) \)) is translational invariant under the transform \((x,l) \rightarrow (x+1,l+1)\), and (b) the combined parity (CP) conservation, exhibited in \( L_n(k, \lambda; \hat{\omega}) = L_n(-k, -\lambda; \hat{\omega}) \) forces the “eigenvalue” \( \Omega(\lambda) \) to be an even function of \( \lambda \).

Substituting the solution of Eq. (21) into Eq. (16), and taking average over the first ballooning variable \( k \), yields the differential equation governing \( \psi(\lambda) \) (the 2D system is being solved in a series of two 1D equations)

\[ \left[ i \bar{L}_1(\lambda; \hat{\omega}) \frac{d}{n d\lambda} + \bar{L}_2(\lambda; \hat{\omega}) \frac{d^2}{n^2 d\lambda^2} + \cdots \Omega(\lambda) - \Omega(\hat{\omega}) \right] \psi(\lambda) = 0 \]

where \( \bar{L}_s (s = 1, 2, \ldots) \) are \( k \)-averages of \( L_{1,2}(k, \lambda; \hat{\omega}) \).

For the WABT approximation, the Floquet phase function obeys
\[
\frac{\partial^2 \psi(\lambda)}{\partial \lambda^2} + P(\lambda) \frac{\partial \psi(\lambda)}{\partial \lambda} + Q(\lambda) \psi(\lambda) = 0 \tag{23}
\]

with

\[
P(\lambda) \equiv \frac{n}{L_2^{(0)}(\lambda; \omega)} \left[ i L_4^{(0)}(\lambda; \omega) + \frac{2 L_2^{(1)}(\lambda; \omega)}{n} \right] \tag{24}
\]

\[
Q(\lambda) \equiv \frac{n^2}{L_2^{(0)}(\lambda; \omega)} \left[ \Omega(\lambda) - \Omega(\omega) + \frac{i L_4^{(1)}(\lambda; \omega)}{n} + \frac{L_2^{(2)}(\lambda; \omega)}{n^2} \right] \tag{25}
\]

\[
L_s^{(j)}(\lambda; \omega) \equiv \frac{\int_0^\infty dk \chi^*(k, \lambda) L_s(k, \lambda; \omega) \frac{\partial^j \chi(k, \lambda)}{\partial \lambda^j}}{\int_0^\infty dk \chi^*(k, \lambda) \chi(k, \lambda)} (s = 1, 2, j = 1, 2) \tag{26}
\]

The two coupled 1D ordinary differential equations, namely the ballooning Eq. (21), and Eq. (23), will be solved by an iterative method in the next section.

To the lowest order of \( \varepsilon_B \equiv (1/n)(d/d \lambda) \ll 1 \), the 2D eigenvalue problem Eq.(16) reduces to a 1D equation with a \( \lambda \) dependent eigenvalue. The ballooning equation Eq. (18) can be cast into the form

\[
\left[ \frac{\partial^2}{\partial k^2} + V(k, \lambda) - \frac{1}{(1 + \delta_s)} \frac{\partial}{\partial \omega_s} \Omega(\omega) \right] \chi(k, \lambda) = 0 \tag{27}
\]

where

\[
V(k, \lambda) = \frac{1}{(1 + \delta_s)} \frac{\partial^2}{\partial \omega_s^2} k_s^2 + 2 \frac{\partial}{\partial \omega_s} \left[ \hat{s} k \sin(k + \lambda) + \cos(k + \lambda) \right] \tag{28}
\]

At large \( k \), Eq. (27) becomes a Weber-Hermite equation allowing the asymptotic boundary condition for an outgoing wave \[^8\].

\[
\chi(k) \xrightarrow{} \exp \left[ i \sqrt{\frac{(1 + \delta_s)}{(1 + \delta_s)}} \frac{\partial k_s \hat{s} k_s^2}{2} \right] \tag{29}
\]

For the numerical calculation, we will use the edge parameters corresponding to an L-mode discharge in the DIII-D tokamak \[^7\], the basic equilibrium parameters are listed in Table I.

**Table I: Basic equilibrium parameters**

| \( a \) [m] | \( R \) [m] | \( r_i \) [cm] | \( L_n \) [cm] | \( L_T \) [cm] | \( B \) [T] | \( n_e \) \( \times 10^{19} \) m\(^{-3} \) | \( T_e \) [eV] | \( q \) | \( \hat{s} \) | \( n \) | \( \tau_i \) | \( \delta_e \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.6 | 1.7 | 54 | 12 | 6 | 1.8 | 1.2 | 140 | 3 | 1.5 | -8 | 0.5 | 0.1 |
For a fixed value of $\lambda$, Eq. (27) can be solved by the standard shooting code. For $\lambda = 0$, we find the eigenvalue $\hat{\omega} = 0.793 + 0.073i$ associated with the wave function $\chi(k,0)$ and the potential $-V(k,0)$; the solution is displayed in Fig.1 (the blue and red lines stand for the real and imaginary parts respectively).

In general, the two coupled 1D ordinary differential Eq.(21) and Eq.(23) will be solved by an iterative method, because the ballooning operator $L_0(k,\lambda;\hat{\omega})$, and the averaged $\bar{L}_1(k;\hat{\omega})$ and $\bar{L}_2(k;\hat{\omega})$ all contain the (2D global) eigenvalue $\hat{\omega}$ which remains unknown until the $\psi(\lambda)$ equation Eq.(23) has been solved too. The steps in the iterative procedure (same as in Ref.7) are listed below:

(I) Begin with an initial guess, $\hat{\omega} \rightarrow \hat{\omega}^{(0)}$ to solve Eq.(21),

$$
\left[ L_0(k,\lambda;\hat{\omega}^{(0)}) - \Omega^{(0)}(\lambda) \right] \chi^{(0)}(k,\lambda) = 0 \quad (30)
$$

imposing the boundary condition Eq. (29) for all $\lambda$.

(II) Substitute $\chi^{(0)}(k,\lambda)$ into Eq. (26) to compute $\bar{L}_1^{(0)}(\lambda;\hat{\omega}^{(0)})$ and $\bar{L}_2^{(0)}(\lambda;\hat{\omega}^{(0)})$, and consequently obtain an equations for $\Phi^{(0)}(\lambda)$ (see Appendix B)

$$
\frac{d^2\Phi^{(0)}(\lambda)}{d\lambda^2} + \frac{n^2}{\bar{L}_2^{(0)}(\lambda;\hat{\omega}^{(0)})} \left[ \Omega^{(0)}(\lambda) - \Omega^{(0)}(\hat{\omega}^{(0)}) + \frac{\left[ \bar{L}_1^{(0)}(\lambda;\hat{\omega}^{(0)}) \right]^2}{4\bar{L}_2^{(0)}(\lambda;\hat{\omega}^{(0)})} \right] \Phi^{(0)}(\lambda) = 0 \quad (31)
$$

to be solved with evanescent boundary conditions.

(III) The global eigenvalue $\hat{\omega}^{(1)}$ follows from Eq.(17) by substituting the eigenvalue of Eq.(31) $\Omega(\hat{\omega}^{(0)})$. 

---

Fig.1. (a) potential structure given by $-V(k,0)$, (b) wave function $\chi(k,0)$
Repeat the procedures (I-III) to obtain $\hat{\omega}^{(i+1)}$ from $\hat{\omega}^{(i)}$ until $\left|1 - \hat{\omega}^{(i+1)} / \hat{\omega}^{(i)}\right| < \varepsilon$ with $\varepsilon \equiv 10^{-4}$ as the convergence condition.

The convergence was usually achieved after 5 iterations. The 2D eigenvalue $\hat{\omega}_{\text{WABT}} = 0.758 + 0.068i$, came out to be close to the 1D ballooning solution at $\lambda = 0$ (both the real frequency and the growth rate). The ballooning wave function $\chi(k, \lambda)$ at $\lambda = 0$ and $\lambda = \pm 0.74$ are displayed in Fig.2.

![Fig.2. ballooning wave function $\chi(k, \lambda)$ at (a) $\lambda = -0.74$, (b) $\lambda = 0$, and (c) $\lambda = 0.74$](image)

In Fig.3 the averaged $\bar{L}_1^{(0)}(\lambda)$, $\bar{L}_2^{(0)}(\lambda)$, and $\Omega(\lambda)$ are presented, the blue and red lines stand, respectively, for the real and imaginary parts. One can see that, $\bar{L}_1^{(0)}(\lambda)$ is almost a constant, $\bar{L}_2^{(0)}(\lambda)$ varies in $\lambda$, but only slightly. The second small parameter of WABT $\Xi \equiv \bar{L}_1 / 2\bar{L}_2 \approx 0.06$ is small enough to warrant the validity of WABT (see the end of Appendix B). The potential $\Omega(\lambda)$ is an even function of $\lambda$. 
The Floquet phase $\psi(\lambda)$ is shown in Fig.4. The real part of $\psi(\lambda)$ is a Gaussian located at $\lambda = 0$. The imaginary part looks like a dipole with two peaks around $\lambda = \pm 0.74$ and breaks the $\lambda$-inversion symmetry (resulting from the small parameter $\Xi$).

The two small parameters required by the WABT structure (defined at the end of Appendix B) are, a posteriori, shown to be small: $\sqrt{\Omega_1 / 2n L_2} \approx 0.08 \ll 1$ and $\Xi \sqrt{n} \approx 0.18 \ll \pi/2$ ($L_1 \approx -0.3$, $L_2 \approx -2.2$ and $\Omega_1 \approx -0.25$).

**IV Iterative finite difference method for $\delta$ mode**
In order to expose the spatial structure of the $\delta_e$ mode, we must convert the 2D wave function $\varphi(k,\lambda) = \chi(k,\lambda)\psi(\lambda)$ (obtained in ballooning space) into the $(x,l)$ representation by making use of the 2D Fourier-ballooning transform Eq. (15),

$$\tilde{\chi}(x-l,\lambda) = \int_{-\infty}^{\infty} dk \exp\left[ ik(x-l)\right] \chi(k,\lambda)$$ \hspace{1cm} (32)

and

$$\varphi_l(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda \exp(-il\lambda)\psi(\lambda) \tilde{\chi}(x-l,\lambda)$$ \hspace{1cm} (33)

On substituting $\tilde{\chi}(x-l,\lambda)$ into Eq.(33) and integrating over $\lambda$, we get the WABT wave functions $\varphi_l(x)$ corresponding to the poloidal mode number $l$; $\varphi_l(x)$ for various $l = -4,-3,...,4,5$ are displayed in Fig.5(a). This solution can be used as an initial guess in the shifted inverse power method \cite{18} to solve the 2D eigenvalue problem Eq.(14) on the $(x,l)$ grid by the iterative finite difference method. In fact, the WABT solution provides not only the radial boundary of each rational surface to be outgoing waves, but also the phase relation between neighboring rational surfaces, as natural boundary condition for the 2D local eigenmode \cite{19}. 

![Fig.5. wave functions $\varphi_l(x)$ ($l = -4,-3,...,4,5$) of (a) WABT and (b) iterative finite difference solution, the solid and dashed lines are real and imaginary part respectively](image)

The 2D eigenvalue problem Eq.(14) is put in the form as $T\left[ \partial / \partial x, l, \hat{\omega}\right] \varphi_l(x) = \Omega(\hat{\omega})\varphi_l(x)$, where $T$ is a differential operator with derivative of $x$, it also depends on $l$ and the eigenvalue $\hat{\omega}$. The spatial discrete grids are
\(x_k \equiv k \cdot h \quad (k = -K, -K + 1, \ldots, -1, 1, K),\) the step size is \(h = \left(x_r - x_l\right) / 2K,\) \(x_l (x_r)\) is the left (right) boundary. The \(l\) grids are \(l = -L, -L + 1, \ldots, -1, L,\) and \(\phi_l (x)\) is cut-off at large \(|l| > L.\) Use is made of the central difference for derivative of \(x\) to yield the matrix equation

\[
M(\hat{\omega}) \cdot \Phi = \Omega(\hat{\omega}) \Phi
\]  

(34)

where \(\Phi = (\Phi_{-L}, \Phi_{-L+1}, \ldots, \Phi_{L-1}, \Phi_L)^T,\) \(\Phi_i = (\phi_{i-K}, \phi_{i-K+1}, \ldots, \phi_{i, K-1}, \phi_{i, K})^T,\) \(\phi_{i, k} = \phi_i (x_k)\). \(M\) is a block tri-diagonal matrix and its dimension is \((2L + 1)(2K + 1) \times (2L + 1)(2K + 1).\)

The eigenvalue problem Eq.(34) can only be solved by an iterative method because it’s nonlinear in \(\hat{\omega}.\) The iterative procedure is listed below:

(I) It begins with the eigenvalue of WABT solution as initial guess, \(\hat{\omega}_{WABT} \rightarrow \hat{\omega}^{(0)}\) to compute the matrix

\[
M^{(0)} = M(\omega^{(0)})
\]

and eigenvalue \(\Omega^{(0)} = \Omega(\omega^{(0)})\) (see Eq.(17)).

(II) Wave functions \(\phi_i (x)\) of WABT solution at discrete grids \(\Phi^{(0)}\) are normalized, \(\Phi^{(0)} = \Phi^{(0)} / \left\| \Phi^{(0)} \right\|_2,\) the matrix equation \((M^{(0)} - \Omega^{(0)}) \cdot \Phi^{(1)} = \Phi^{(0)}\) is solved for \(\Phi^{(1)}\) by LU decomposition.

(III) Compute \(\Delta \Omega^{(1)} = (\Phi^{(1)})^T \cdot (M^{(0)} - \Omega^{(0)}) \cdot \Phi^{(1)} / (\Phi^{(1)})^T \cdot \Phi^{(1)}\) and go back to step (II), \(\Phi^{(0)} \rightarrow \Phi^{(1)}\) until

\[
\delta = \left\| \Delta \Omega^{(1)} \cdot \Phi^{(1)} - \Phi^{(0)} \right\|_\infty < 10^{-6}.
\]

(IV) Substitute the new eigenvalue \(\Omega^{(1)} = \Omega^{(0)} + \Delta \Omega^{(1)}\) into Eq.(17) to obtain the global eigenvalue \(\hat{\omega}^{(1)}\).

In the preceding lines, \(\Phi^T\) stands for the Hermitian conjugate of \(\Phi,\) \(\left\| \Phi \right\|_2 = \sqrt{\sum_i \Phi_i^2}\) and \(\left\| \Phi \right\|_\infty = \max_i |\Phi_i|\) are the 2-norm and \(\infty\)-norm of vector \(\Phi\) respectively.

(V) Repeat the steps (I-IV) to obtain \(\hat{\omega}^{(i+1)}\) from \(\hat{\omega}^{(i)}\) until \(\left| 1 - \hat{\omega}^{(i+1)} / \hat{\omega}^{(i)} \right| < \varepsilon,\) with \(\varepsilon = 10^{-4}\) as the convergence condition.

After 5 iterations, the eigenvalue converged to \(\hat{\omega}_{FD} = 0.745 + 0.065i\). The difference of iterative finite difference solution and WABT solution, \(\left| \hat{\omega}_{FD} - \hat{\omega}_{WABT} \right| / \left| \hat{\omega}_{WABT} \right| \approx 1.7\%\), was within the expected error (~ \(1 / n = 12.5\%\)). The wave function \(\phi_i (x)\) of the iterative finite difference solution, displayed in Fig.5(b), are in good agreement with the WABT solution.
The wave function in \((r, \theta)\) representation, \(\varphi(r, \theta) = \exp(-im\theta)\sum_i \varphi_i(r)\exp(-il\theta)\), is shown in Fig.6; the 2D mode structure \(\varphi(r, \theta)\) and the close-up in bad curvature region for both WABT and iterative finite difference solution are highlighted. The mode structure, embodied in the wave function, is the crucial input to compute Reynolds stress and group velocity of \(\delta_e\) mode in section V.

![Fig.6. 2D mode structure of (a) WABT solution, (b) its close-up in bad curvature regime, and (c) iterative finite difference solution, (d) its close-up in bad curvature regime, where \(\rho = r / a\) is the normalized minor radius](image)

V Reynolds stress and group velocity for \(\delta_e\)-mode
The poloidal torque induced by Reynolds stress is defined as \( \mathbf{R}_\phi \equiv \langle \mathbf{u}, \mathbf{u}_\phi \rangle \), where \( \langle \ldots \rangle \) stands for average over stochastic fluctuations, which is equivalent to average over the poloidal angle \(^{[19,20]}\). For electrostatic potential fluctuations \( \mathbf{u} = \rho_j c_j \mathbf{b} \times \nabla \varphi \), the (poloidal) Reynolds stress pertaining to the rational surface \( r_j \) is

\[
\mathbf{R}_{\phi j}(r) = -\rho_j c_j^2 \int d^2 \varphi \frac{\partial \varphi(r_\rho, \varphi)}{r_j \partial \varphi} \frac{\partial \varphi(r_\rho, \varphi)}{\partial r}.
\]

(35)

The intensity of drift wave for \( \delta_e \)-model is defined as \( I_m(r) \equiv \int d^2 \varphi(r_\rho, \varphi) \varphi(r_\rho, \varphi) \), here \( I_m(r) \) describes the radial distribution of drift wave energy. When computing the physical quantity, we only retain the real part of \( \varphi(r_\rho, \varphi) \).

The Reynolds stress and intensity of drift wave, computed by the 2D mode structure of WABT and iterative finite difference solution in section IV by direct numerical integration, are displayed in Fig.7, where \( I(r) \equiv I_m(r) / I_m(r_m) \) and \( R(r) \equiv \mathbf{R}_{\phi j}(r) / \rho_j^2 c_j^2 k_\rho^2 I_m(r_m) \), \( r_m \) is the peak position of \( I_m(r) \). One can see that \( I(r) \) is a Gaussian envelop plus fine-scale radial structure (due to sideband coupling). The Reynolds stress \( R(r) \) has the shape of monopole instead of dipole obtained earlier in the fluid ITG model \(^{[19]}\).

---

![Fig.7](image.png)

Fig.7. (a) the intensity of drift wave, (b) Reynolds stress computed by WABT solution (red solid lines) and iterative finite difference solution (blue dashed lines)

As depicted in the section II of ‘Part I’, the toroidal effect induces poloidal moments of Reynolds stress for different harmonics,
where $K(\theta) = 1, \sin \theta, \cos \theta$ and $\sin 2\theta$. Direct numerical integration is performed on Eq. (36) by making use of the 2D mode structure of WABT, the normalized Reynolds stress $R(r)$ of the four moments are displayed in Fig.8.

Note that the Monopole shaped $R[\sin \theta]$ is much larger than $R[1]$, which indicates the significant role of geodesic curvature in determining the Reynolds stress.

\[
\tilde{R}_{g,i}[K] = -\rho^2 c_s^2 \oint d\theta \frac{\partial \varphi(r, \theta)}{r \partial r} \partial \varphi(r, \theta) K(\theta)
\] (36)

Fig.8. Poloidal moments of Reynolds stress for different harmonics, (a) $K(\theta) = 1, \cos \theta$, (b) $\sin \theta, \sin 2\theta$

The radial and poloidal group velocities are functions only of the poloidal angle because the drift wave in a tokamak is a standing (travelling) wave radially (poloidally). The characteristic line in poloidal direction of Eq. (10) is given by $d\theta / dt = v_{g\theta}(\theta) / r_j$, that yields mapping time to poloidal angle $\theta(t)$. As a result, the radial group velocity becomes a periodic function of time, $v_{g_{\rho}}(t) = v_{g_{\rho}}(\theta(t))$. To explore the structure of group velocity, the average of three operators over fast scale must be computed:

\[
\langle K_i \rangle = -i k_{\rho} \delta \int d\phi^* (x, \theta) \partial_x \varphi(x, \theta)
\] (37)

\[
\langle K_i^2 \rangle = -k_{\rho}^2 \delta^2 \int d\phi^* (x, \theta) \partial_x^2 \varphi(x, \theta)
\] (38)

\[
\langle \nabla_{\parallel} \rangle = -\frac{1}{q^2 R^2} \int d\phi^* (x, \theta) \sum (x-1)^2 \phi_i(x) \exp(-i l \theta)
\] (39)
It is done by substituting the 2D mode structure of WABT solution and iterative finite difference solution into Eqs. (37-39); the three averaged normalized operators $\hat{K}_r \equiv \langle K_r \rangle / k_\beta$, $\hat{K}_r^2 \equiv \langle K_r^2 \rangle / k_\beta^2$ and $\hat{K}_z^2 \equiv \langle \nabla_z^2 \rangle / k_\beta^2$ are displayed in Fig.9.

![Fig.9. three normalized operators](image)

For real and regular group velocities in stationary background micro-turbulence, the drift wave amplitude $|\bar{\phi}|$ is invariant \cite{12}, it is therefore convenient to take the eikonal form $\bar{\phi} \sim \exp(-i\Theta)$ with real $\Theta$. The imaginary part $i\delta_r$ and the growth rate (Im $\omega$) in Eqs. (11-12) are neglected. In Fig.10, the $\vartheta$ dependence of the radial and poloidal group velocity is displayed. Using $t = \int_0^{\vartheta} r_d \vartheta' / \nu_r \vartheta' \, (\vartheta')$ to generate the mapping of $\vartheta(t)$, the radial group velocity is plotted versus $t$ in Fig.10(c). One can see that the period of radial group velocity is about 7ms. Each period consists of two consecutive distinct phases: a long slowly varying and a short sudden spike crossing zero. There is a little difference with the results of ITG in ‘Part I’. For ITG group velocity, both up-zero-crossing and down-zero-crossing are rapid, but for $\delta_r$-mode, the rapidity of up-zero-crossing is high, but is much lower for down-zero-crossing.
VI The intermittent excitation of GAM generated by $\delta_e$-mode

The zonal flow-drift wave energy set Eqs. (18-20) in ‘Part I’ is numerically solved in this section, however, with the Reynolds stress and group velocity of $\delta_e$-mode obtained in section V. Partial mode related parameters are listed in Table II. The numerical methods and boundary condition are same as given in section III of ‘Part I’. On the left side an extended absorbing layer is set for $\rho = r / a < 0.75$, the ingoing pre-GAM is Landau damped in the low $q$ region; on the right side the zero Dirichlet condition is chosen at $\rho = 1$ for the reason that GAM cannot propagate outside the last closed magnetic surface. Step sizes of spatiotemporal grids are $\Delta \rho = 1.25 \times 10^{-3}$ and $\Delta t = 2 \mu s$ respectively.

| $\rho_i$ [cm] | $V_i$ [kHz] | $\omega_i$ [kHz] | $I_m(\tau_m)$ | $a_\mu$ | $D(\tau_e)$ |
|---------------|-------------|-----------------|----------------|----------|-------------|
| 0.067         | 14          | 13.6            | $5 \times 10^{-4}$ | 3        | 1           |

The intermittent excitation of GAM can be clearly seen in Fig.11, as TLFZF grows up to a certain level, GAM (or solely pre-GAM) appears suddenly at the moment that radial group velocity crosses zero, the GAM amplitudes also increase as the level of TLFZF being higher.

In Fig.12 displayed are the magnified graphs in 5.5-11.5ms of Fig.11. One can see that a downward (upward) zero-crossing case of the radial group velocity results in an ingoing (outgoing) GAM, the same results as ITG driven GAM reported in ‘Part I’.
Fig. 11 (a) Temporal evolution of radial group velocity and (b-f) zonal flow with intermittent excitation of GAM at five radial positions.

Fig. 12 The magnified graphs in 5.5-11.5 ms of Fig. 11 with low frequency portion filtered out.
Fig. 13 The magnified graphs in the last 6ms of Fig.11 with low frequency portion filtered out.

Fig. 14 (a) Frequency-time spectrogram at $\rho = 0.9$, the red dashed line denotes radial group velocity, (b) spatiotemporal evolution of zonal flow in all frequency range, and (c) in (5-35 kHz).
In Fig.13 displayed are the magnified graphs in the last 6ms of Fig.11. The upward zero-crossing triggers stronger outgoing GAM, while the downward zero-crossing induces an ingoing but short lasting pre-GAM with rather lower frequency ~10 kHz, however, could not be GAM. It may be attributed to the lower rapidity of down-zero-crossing. The variation of phase $\Theta$ is too slow to reach the GAM frequency $\omega_G$ for the resonant excitation.

The frequency-time spectrogram at $\rho = 0.9$ is displayed in Fig.14 (a), the radial group velocity is also displayed by the red dashed line. At the moment of zero-crossing of the radial group velocity, a temporal structure with frequency ~13-15 kHz jumps out, which matches the local GAM frequency $\omega_G = 13.6\text{kHz}$.

The spatiotemporal evolution of the zonal flow $\mathbf{u}$ in all frequency range is displayed in Fig.14 (b). One can clearly see the dipole structure of dominating TLFZF emerging gradually- note the dark blue (negative) and dark red (positive) stream as driven by the derivative of monopole-like Reynolds stress. The spatiotemporal evolution of the zonal flow, within the frequency range 5-35 kHz (more relevant to GAM activity), is displayed in Fig.14 (c), where four events are illustrated (numbers 1, 2, 3, and 4). These correspond, respectively, to the time of downward zero-crossing (1,3) and upward zero-crossing (2,4). One can readily see the pattern differences of pre-GAM induced by ingoing (outgoing) instantons right after the time of 1,3 (2,4) from Fig.14 (c) as further described below.

Let us define four periods: $P_1$ (6.9-10.4ms), $P_2$ (10.5-11.5ms), $P_3$ (55.6-57.2ms) and $P_4$ (58.9-60ms) just right after the four zero-crossing events 1, 2, 3 and 4. In period $P_1$, right after 6.9ms, the ingoing instantons induce pre-GAM moving inwardly (Fig.15(a)), then the pre-GAM reaches the WKB turning point lying within $\rho \sim 0.78-0.86$ (Fig.15(b)), partly absorbed by Landau damping further inward, and partly reflected back and moves outwardly. The reflected portion has a wide spatial spread and is long lived. In period $P_2$, right after 10.5ms, the outgoing instantons induce outward moving GAM (Fig.15(c)), overlapping the existing outgoing GAM in period $P_1$. Such an interference results in a rather complicated pattern. The scenario in periods $P_1$ and $P_2$ is very similar to the results in ‘Part I’; the corresponding spatiotemporal evolution is represented by the movie ‘GAM spatiotemporal evolution 1’ spelled out in the caption of Fig.15 (Multimedia view).
Fig. 15 Snapshots of 3 time points in the movie. The time evolution for 5.5-11.5 ms can be seen via the link\(^1\) ‘GAM \textit{s}patiotemporal evolution 1’ (Multimedia view)

In period P3, right after 55.6 ms, the ingoing instantons induce pre-GAM moving inwardly (Fig. 16(a)) with frequency ~10 kHz, fairly lower than that of GAM. Then the pre-GAM penetrates into the absorbing layer further inward, with little reflected back (Fig.16(b)). In contrast to the pattern in period P1 and P2, the signal appears spatially contracted and weaker. In period P4, right after 59 ms, the outgoing instantons drive GAM moving outwardly (Fig. 16(c)). The signal appears much stronger than that in period P2. The spatiotemporal evolution in period P3 and P4 is represented by the movie - ‘GAM \textit{s}patiotemporal evolution 2’ in the caption of Fig.16 (Multimedia view).

Fig. 16 Snapshots of 3 time points in the movie. The time evolution for 54-60 ms can be seen via the link\(^2\) ‘GAM \textit{s}patiotemporal evolution 2’ (Multimedia view)

\(^1\) http://home.ustc.edu.cn/~lzy0928/GAM%20spatiotemporal%20evolution%201.mp4
\(^2\) http://home.ustc.edu.cn/~lzy0928/GAM%20spatiotemporal%20evolution%202.mp4
The auto-correlation function for two spatial positions is presented in Fig. 17, a long tail after non-exponential quick fall-off accompanied by residue oscillations supported by TLFZF can be clearly seen. The result is same as reported in ‘Part I’, the intermittency of GAM is deterministic.

\[ f(\tau) = Ae^{-\tau/\tau_{ac}} \]
\[ A = 0.976 \]
\[ \tau_{ac} = 0.136 \text{ ms} \]

\[ f(\tau) = Ae^{-\tau/\tau_{ac}} \]
\[ A = 1.033 \]
\[ \tau_{ac} = 0.129 \text{ ms} \]

Fig. 17 Auto-correlation function for different radial position at (a) \( \rho = 0.89 \) and (b) \( \rho = 0.9 \)

VII Conclusions and discussions

This paper (Part II) extends the methods and techniques of Part I to explore another pathway to study the twin system of Zonal Flow and Geodesic Acoustic Mode observed in L-mode discharge near tokamak edge. One of the aims is to establish the deterministic nature of the intermittent GAM if it were driven by turbulence in electron direction. In Part II, the background turbulence responsible for ZF-GAM is created by the instability in the \( \delta_e \)-model, while in Part I, we exploited the ITG fluid model turbulence. It is rather significant that despite a rather drastic difference in the underlying turbulence, Part I (I) and Part II (II) share three high level features:
The deterministic GAM system reproduces many of the characteristics associated with the intermittent excitation of GAM observed in experiments.

GAM is observable only if TLFZF grows beyond a certain level; the generated GAM is an intrinsic nonlinear spatiotemporal structure (near GAM frequency) accompanying TLFZF.

The causal relationship between the phase transition of drift wave energy (from a caviton pair into instantons) and GAM generation.

The ZF-GAM systems in I and II, however, admit two major differences:

1. The first is somewhat technical - the ITG driven Reynolds stress looks like a dipole, while the $\delta_e$-mode driven Reynolds stress looks more like a monopole. As a result, the ITG driven TLFZF looks like a quadrupole, while $\delta_e$-mode driven TLFZF looks like a dipole.

2. In general, for the same machine parameters, the GAM excitation threshold is (much) higher for $\delta_e$-mode turbulence in contrast to ITG mode. For example, the critical value is around $I_m(r_m) \approx 5 \times 10^6$ for ITG mode, but $I_m(r_m) \approx 5 \times 10^4$ for $\delta_e$-mode. The minimum value of TLFZF required for onset of GAM is found close in either ITG or $\delta_e$-mode scenario, say $\approx 10 \text{ m/s}$. This suggests that in order to have the $\approx 10 \text{ m/s}$ TLFZF, the strength of $\delta_e$-mode driven turbulence must be one order of magnitude higher than that for ITG mode ($I_m \sim (\tilde{n}/n)^2$). If this feature were common to other possible models of electron drift waves, it would be a result of considerable significance.

Appendix A Derivation of $\delta_e$-model in $(x,l)$ representation

The $\delta_e$-model equation (the zeroth order equation derived in section II) is

$$
\left[ \left( 1 + \frac{\omega_e}{\omega} \right) \rho_e^2 \nabla_l^2 - (1 - i \delta_e) \right] \nabla \| \mathbf{v} (r, \theta, \zeta) = 0 \tag{A1}
$$

where $\varphi$ is the dimensionless electric potential normalized to electron thermal energy $e\delta \varphi / T_e \rightarrow \varphi$, $\rho_s \equiv \sqrt{m_{Te}^{(0)} / eB}$ is the ion Larmor radius at electron temperature, $c_s \equiv \sqrt{\frac{T_{Te}^{(0)}}{m_i}}$ is the ion sound speed, $\hat{\omega}_{\delta_e} \equiv i \rho_s c_s \mathbf{b} \times \nabla \ln n^{(0)} \cdot \nabla \left( \hat{\omega}_{\eta} = \hat{\eta}_{\delta_e} \right)$ is the electron (ion) diamagnetic frequency operator,
\[ \hat{\omega}_{de} = -i \rho_c \kappa \times \mathbf{b} \cdot \nabla \] is the curvature frequency operator, \( \mathbf{b} \) is the unit vector along the equilibrium magnetic field, \( \kappa \) is the magnetic curvature. Warm ion effect is considered here, \( \tilde{\eta}_i \equiv \tau_i (1 + \eta_i) \), \( \eta_i \equiv d \ln T_i^{(0)} / d \ln n_i^{(0)}, \) \( \tau_i \equiv T_i^{(0)} / T_e^{(0)}, \) where \( n_i^{(0)} \) is the equilibrium density, \( T_i^{(0)} (T_e^{(0)}) \) is the equilibrium electron (ion) temperature. \( \delta_e \equiv \tau_e (1 - i \delta_e), \) \( \delta_e \) represents the non-adiabatic response of electron density, \( \delta n_e / n_e^{(0)} = (1 - i \delta_e) \varphi, \) which is modeled to be a small number in this paper.

In the \((x, l)\) representation, substituting Eq.(13) into Eq.(A1) the differential operators are

\[ \nabla_i^2 \varphi \to k_g^2 \left[ \frac{\partial^2}{\partial x^2} + \left(1 + \frac{1}{m}\right)^2 \right] \varphi_l(x), \quad \nabla_i \varphi = \frac{1}{q R} \left[ \frac{\partial}{\partial \theta} + q(r) \frac{\partial}{\partial \varphi} \right] \varphi \to \frac{i}{q R} (x - l) \varphi_l(x) \] (A2)

The three frequency operators are

\[ \hat{\omega}_{e,i} \varphi \to \omega_{e,i} \left(1 + \frac{1}{m}\right) \varphi_l(x), \quad \hat{\omega}_{de} \varphi \to \omega_{de} \left[ i \bar{s} \sin \theta \frac{\partial}{\partial \varphi} + \cos \theta \left(1 + \frac{1}{m}\right) \right] \varphi_l(x) \] (A3)

where the radial variable \( x \equiv k_g \bar{s} (r - r_j) \) is introduced, \( k_g \equiv m / r_j, \) \( \omega_{e,i} \equiv -k_g T_e^{(0)} / eB_{n_i}, \) \( \omega_{i} \equiv -k_g T_i^{(0)} / eB_{n_i}, \) \( \omega_{de} \equiv -k_g T_{e,i}^{(0)} / eB_{r_j} \). Use is made of the linear density profile, \( f_n(x) \equiv n_i^{(0)} / n_j = 1 - (r_j / L_n)(x / m \bar{s}), \) \( L_n \equiv \left(d \ln n_i^{(0)} / dr \right)_r^{-1}, \) temperature profile \( f_T(x) \equiv T_i^{(0)} / T_{e,i} = 1 - (r_j / L_T)(x / m \bar{s}), \) \( L_T \equiv \left(d \ln T_i^{(0)} / dr \right)_r^{-1}, \) \( \rho_s^2 = \rho_{s,j}^2 f_s(x), \) and \( c_i^2 = c_{s,j}^2 f_s(x) \) also depend on electron temperature, subscript \( j \) denotes equilibrium quantities on rational surface \( r_j \). The sideband \( \ell / m \) and slow variation of equilibrium profile in radial direction violate the translational invariance.

After defining the dimensionless parameters, \( \hat{\omega} \equiv \omega / \omega_{e,i}, \) \( \hat{k}_g \equiv \rho_{s,j} k_g, \) \( \hat{\omega}_i \equiv \rho_{s,j} \omega_{i}, \) \( \hat{\omega}_{de} \equiv \omega_{de,i} / \omega_{e,i}, \) the equation for \( \varphi_l(x) \) is

\[ \hat{k}_g^2 \frac{\hat{\omega}_i}{\hat{\omega}_s} \left[ 1 + \frac{\eta}{\hat{\omega}} \right] f_T(x) \left(1 + \frac{1}{m}\right) \left[ \frac{\partial^2}{\partial x^2} + \left(1 + \frac{1}{m}\right)^2 \right] \varphi_l - \frac{1}{\hat{\omega}_s} \left(1 - i \hat{\omega}_s \right) \frac{\hat{\omega}_i}{f_T(x)} \frac{1}{f_n(x)} \left(1 + \frac{1}{m}\right) \varphi_l \\
-\left(1 + \delta_e \right) \frac{\hat{\omega}_{de}}{\hat{\omega}_s} \left[ \frac{\partial}{\partial x} (\varphi_{-1} - \varphi_{+1}) + \left(1 + \frac{1}{m}\right) (\varphi_{-1} + \varphi_{+1}) \right] + \left(1 + \delta_e \right) \frac{\hat{\omega}_i}{\hat{\omega}_s} (x - l)^2 \varphi_l = 0 \] (A4)
To simplify the derivation, we assume derivatives are applied only on the wave functions, not on the equilibrium quantities, and the TSB terms $x$ in the equilibrium profile can be replaced by $l$, $f(x) \approx f(l)$. Since translational invariance is the leading symmetry, all TSB terms are small enough to be retained up to the second order $O(l^2/m^2)$. Then the $\delta_x$-model expanded to include TSB terms in the $(x,l)$ representation is obtained as Eq.(14).

Appendix B  The conditions for valid WABT

Let us starting with Eq.(23) and introducing the transform

$$\psi(\lambda) \equiv \Phi(\lambda) \exp\left[-\frac{1}{2} \int \lambda' P(\lambda')\right]$$  \hspace{1cm}  (B1)

Then the equation for $\Phi(\lambda)$ follows from Eq.(23) and Eq.(B1)

$$\frac{d^2 \Phi}{d\lambda^2} + \left[Q(\lambda) - \frac{P^2}{4} - \frac{1}{2} \frac{dP}{d\lambda}\right] \Phi = 0$$  \hspace{1cm}  (B2)

In the large $n$ limit

$$P(\lambda) \approx in \frac{L_1^{(0)}(\lambda;\hat{\omega})}{L_2^{(0)}(\lambda;\hat{\omega})} \rightarrow in \frac{\bar{L}_1}{\bar{L}_2}, \quad Q(\lambda) \equiv \frac{n^2}{L_2^{(0)}(\lambda;\hat{\omega})} \left[\Omega(\lambda) - \Omega(\hat{\omega})\right] \rightarrow \frac{n^2}{\bar{L}_2} \left[\Omega(\lambda) - \Omega(\hat{\omega})\right]$$  \hspace{1cm}  (B3)

In the second step we neglect the $\lambda$ dependence in $L_1^{(0)}$ and $L_2^{(0)}$, i.e. $\bar{L}_1$ and $\bar{L}_2$ are constants. Further approximation is

$$\Omega(\lambda) \approx \Omega_0 + \Omega_1 \cos \lambda \approx \Omega_0 + \Omega_1 \left(1 - \frac{\lambda^2}{2}\right)$$  \hspace{1cm}  (B4)

Then we obtain the following analytic solution for eigenvalue and wave function

$$\Omega(\hat{\omega}) = \Omega_0 + \Omega_1 + \frac{\bar{L}_1^2}{4\bar{L}_2}$$  \hspace{1cm}  (B5)

$$\psi(\lambda) = \Phi(\lambda) \exp\left(-in \frac{\bar{L}_1}{2\bar{L}_2} \lambda\right), \quad \Phi(\lambda) = \exp\left(-\sqrt{\frac{\Omega_1}{2\bar{L}_2}} \frac{n\lambda^2}{2}\right)$$  \hspace{1cm}  (B6)

This is the WABT solution. For $\bar{L}_1 = 0$ it reduces to horizontal ballooning theory (HBT) \cite{7}. It exhibits fast variation of FPD with Gaussian width $\Delta \lambda \sim 1/\sqrt{n}$. Therefore, the true small parameter in large $n$ limit is

$$\epsilon_{ \text{B}} \equiv (1/n)(d / d\lambda) \sim 1/\sqrt{n}$$. For finite $\bar{L}_1 \neq 0$ (WABT) the exponential factor in the first equation of Eq.(B6)
introduces a fast variation unless $\Xi \equiv \overline{L}_1 / 2 \overline{L}_2 \ll 1$ otherwise the higher order derivative would contribute, and WABT becomes invalid. In case such a parameter regime comes true, the observed flavor could be vertical ballooning theory (VBT) [7].

Practically, two functions in Eq.(B6) yields two criterions, because the general criterion $(1/n)(d / d \lambda) \ll 1$ consists of two conditions. For very small $\Xi$, $(1/n)(d / d \lambda) \approx \sqrt{\Omega_1 / 2n\overline{L}_2}$, the same as HBT. However, there exists restriction on $\Xi$. The estimate for $\Delta \lambda \approx 1/\sqrt{n}$ leads to $\Xi \sqrt{n} \ll \pi/2$. Both should be satisfied for validity of WABT.

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Data availability

All the data that support the findings of this study are available from the corresponding author.

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