A UNIQUENESS RESULT FOR SELF-EXPANDERS WITH SMALL ENTROPY

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Abstract. In this short note, we prove a uniqueness result for small entropy self-expanders as-
ymptotic to a fixed cone. This is a direct consequence of the mountain-pass theorem and the integer
degree argument proved by J. Bernstein and L. Wang.

1. Introduction

A properly embedded \(n\)-dimensional submanifold \(\Sigma\) in \(\mathbb{R}^{n+1}\) is called a self-expander if it satisfies

\[
\mathbf{H}_\Sigma = \frac{x^\perp}{2}
\]

where \(\mathbf{H}_\Sigma\) is the mean curvature vector of \(\Sigma\), and \(x^\perp\) is the normal component of the position vector.

Self-expanders are self similar solutions of mean curvature flow, that is, the family of hypersurfaces

\[
\{\Sigma_t\}_{t>0} = \{\sqrt{t}\Sigma\}_{t>0}
\]

satisfying

\[
\left(\frac{\partial x}{\partial t}\right)^\perp = \mathbf{H}_{\Sigma_t}
\]

Self-expanders are important as they model the behavior of a mean curvature flow coming out of
a conical singularity \([1]\), and also model the long time behaviors of the flows starting from entire
graphs \([10]\).

For a hypersurface \(\Sigma\) in \(\mathbb{R}^{n+1}\), Colding-Minicozzi \([8]\) introduced the entropy on \(\Sigma\)

\[
\lambda[\Sigma] = \sup_{y \in \mathbb{R}^{n+1}, \rho > 0} F[\rho \Sigma + y]
\]

where \(F[\Sigma]\) is the Gaussian surface area of \(\Sigma\)

\[
F[\Sigma] = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} e^{-\frac{|x|^2}{4}} d\mathcal{H}^n(x)
\]

Obviously, this quantity is invariant under dilations and translations. And Huisken’s monotonicity
formula \([12]\) shows that this quantity is non-increasing along the mean curvature flow.

Next, we talk about the space \(\mathcal{ACH}_{n,\alpha}^k\) introduced by J. Bernstein and L. Wang \([2]\). A hyper-
surface \(\Sigma \in \mathcal{ACH}_{n,\alpha}^k\), if it is a \(C^{k,\alpha}\) properly embedded codimension-one submanifold and \(C^{k,\alpha}\)-asymptotic to a \(C^{k,\alpha}\) regular cone \(C = C(\Sigma)\). We refer to \([2]\) Section 2] for technical details.

As pointed out in \([3]\) and \([5]\), there may exist more than one self-expanders asymptotic to some
specific cones. While the main theorem here states that for a small entropy cone, there’s only one
(stable) self-expander asymptotic to it. More precisely,

**Theorem 1.1.** There exists a constant \(\delta = \delta(n)\), so that for a given \(C^{k,\alpha}\)-regular cone \(C\) with \(\lambda[C] < 1 + \delta\), there is a unique stable self-expander \(\Sigma \in \mathcal{ACH}_{n,\alpha}^k\) with \(C(\Sigma) = C\).

**Remark 1.2.** As pointed out in \([7]\) Theorem 8.21], the outermost flows of any hypercone are self-
expanders. Hence, it follows from our result that for a low entropy cone, the inner and outer flows
coincide, so the cone doesn’t fatten.
2. Some regularity results

In [6], the authors defined a space $\mathcal{RMC}_n$, consisting of all regular minimal cones in $\mathbb{R}^{n+1}$ and let $\mathcal{RMC}_n^*$ be the non-flat elements in $\mathcal{RMC}_n$. For any $\Lambda > 1$, let

$$\mathcal{RMC}_n(\Lambda) = \{ C \in \mathcal{RMC}_n : \lambda[C] < \Lambda \}$$

and $\mathcal{RMC}_n^*(\Lambda) = \mathcal{RMC}_n^* \cap \mathcal{RMC}_n(\Lambda)$. Since all regular minimal cones in $\mathbb{R}^n$ consist of the unions of rays and great circles are the only geodesics in $\mathbb{S}^2$, $\mathcal{RMC}_n^*(\Lambda) = \mathcal{RMC}_n^*(\lambda) = \emptyset$ for all $\Lambda > 1$. Now fix the dimension $n \geq 3$ and a value $\Lambda > 1$. Consider the following hypothesis:

$$\tag{2.1} \text{For all } 3 \leq l \leq n, \mathcal{RMC}_n^*(\Lambda) = \emptyset$$

**Lemma 2.1.** There is a constant $\Lambda = \Lambda_n > 1$, so that the hypothesis (2.1) holds.

**Proof.** We first show that any regular minimal cone $C$ has generalized mean curvature 0 near the origin. Let $X$ be a smooth vector field compactly supported in $\mathbb{R}^{n+1}$. Define a smooth cut-off function $\eta$, so that $\eta = 0$ in $B_{\frac{1}{2}}(0)$ and $\eta = 1$ outside $B_1(0)$. Let $\eta_r = \eta(\frac{r}{r})$, then

$$\int_C \eta_r(\operatorname{div}_C X) d\mathcal{H}^n = \int_C \operatorname{div}_C(\eta_r X) d\mathcal{H}^n - \int_C X \cdot \nabla_C \eta_r d\mathcal{H}^n$$

$$= -\int_C X \cdot \nabla_C \eta_r d\mathcal{H}^n \tag{2.2}$$

The construction of $\eta_r$ gives

$$\operatorname{Spt} \nabla \eta_r \subset B_r(0) \setminus B_{\frac{r}{2}}(0)$$

and $|\nabla_C \eta_r| \leq |\nabla \eta_r| \leq \frac{C}{r}$

where $C = C(n)$ is a constant. Let $L(C) = C \cap S^n$ be the regular codimension-1 submanifold in $S^n$, then $\mathcal{H}^{n-1}(L(C)) < \infty$. This gives a upper bound for the last term in (2.2)

$$|\int_C X \cdot \nabla_C \eta_r d\mathcal{H}^n| \leq \|X\|_{\infty} \left| \int_{C \cap B_r(0) \setminus B_{\frac{r}{2}}(0)} \frac{C}{r} d\mathcal{H}^n \right|$$

$$\leq \|X\|_{\infty} \frac{C}{r} \mathcal{H}^{n-1}(L(C)) \int_{\frac{r}{2}}^r s^{n-1} ds$$

$$\leq C \mathcal{H}^{n-1}(L(C)) \|X\|_{\infty} r^{n-1}$$

So it goes to 0 as $r \to 0$. Letting $r \to 0$ in (2.2), we get

$$\int_C \operatorname{div}_C X d\mathcal{H}^n = 0 \tag{2.3}$$

for any vector field compactly supported in $\mathbb{R}^{n+1}$, which means $C$ has generalized mean curvature near the origin and it vanishes.

Next, we relate the entropy to the density at origin. Observe that

$$\frac{1}{(4\pi)^{\frac{n}{2}}} \int_C e^{-\frac{|x|^2}{4}} d\mathcal{H}^n = \frac{1}{(4\pi)^{\frac{n}{2}}} \mathcal{H}^{n-1}(L(C)) \int_0^{\infty} r^{n-1} e^{-\frac{r^2}{4}} dr$$

$$= \frac{1}{2\pi^{\frac{n}{2}}} \mathcal{H}^{n-1}(L(C)) \Gamma\left(\frac{n}{2}\right)$$

$$= \frac{\mathcal{H}^{n-1}(L(C))}{\omega_n}$$

$$= \frac{\mathcal{H}^n(B_{\rho}(0))}{\omega_n \rho^n}$$

$$= \Theta(C, 0)$$

where $\omega_n = \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$ is the volume of the unit ball in $\mathbb{R}^n$. So the density $\Theta(C, 0) = F[C] \leq \lambda[C]$. Thus, by Allard’s regularity theorem [13], if $\Lambda_n$ is sufficiently small, then $C$ is smooth at the origin, and it has to be a hyperplane. \qed
Remark 2.2. In fact, if we replace the regular cone above by a general stationary integral varifold $C$ with $\eta_{0,\rho}\# C = C$ for all $\rho > 0$, the result still holds. Indeed, we get the smoothness near the origin in the same way and the dilation invariance implies it is smooth everywhere.

The following is a lemma from [3]. For the sake of completeness, we include a proof here.

**Lemma 2.3.** For $k \geq 2$ and $\alpha \in (0, 1)$, let $C$ be a $C^{k,\alpha}$-regular cone in $\mathbb{R}^{n+1}$ and assume $\lambda[\Sigma] < \Lambda_n$. If $V$ is an $E$-stationary integral varifold with tangent cone at infinity equal to $C$, then $V = V_\Sigma$ for an element $\Sigma \in \mathcal{ACH}^{k,\alpha}_n$ satisfying (1.7).

**Proof.** For every point $x \in Spt\mu_V$, we will show that there is a tangent plane $\mathcal{P}_x$. If that is the case, $\Theta^\alpha(V, x) = \Theta^\alpha(\mathcal{P}_x, 0) = 1$. Together with the fact that $V$ has locally bounded mean curvature, the Allard’s regularity theorem applies and $V$ is smooth near $x$.

Suppose a sequence of positive number $\{\lambda_i\} \to 0$, and $\eta_{x,\lambda_i} V$ converges to a tangent varifold $C_x$, where $\eta_{x,\lambda_i}(y) = \frac{|y - x|}{\lambda_i}$. By the nature of convergence and Huisken’s monotonicity formula [12],

$$\lambda[C_x] \leq \lambda[V] \leq \lambda[C] < \Lambda_n$$

On the other hand,

$$\rho^{1-n} \| \delta V \| (B_{\rho}(x)) \leq \rho^{1-n} \int_{V \cap B_{\rho}(x)} |H_V| dH^n \leq \| H_V \|_\infty \rho \cdot \frac{\mathcal{H}^n(V \cap B_{\rho}(x))}{\rho^n}$$

It goes to 0 as $\rho \to 0^+$. Then it follows from the knowledge of varifold in [13] that $\eta_{0,\rho}\# C_x = C_x$ for $\rho > 0$, and that $C_x$ is stationary. The previous lemma indicates $C_x$ must be a plane. Thus $V = V_\Sigma$ for some smooth self-expander $\Sigma$. Following [1], $\Sigma$ is $C^{k,\alpha}$-asymptotic to $C$. 

We also need the notion of partial ordering. Roughly speaking, $\Sigma_1 \preceq \Sigma_2$, if the hypersurface $\Sigma_1$ is “above” $\Sigma_2$. For the detailed explanation, we refer to [6, Section 4]. The following theorem from that paper is a useful tool to construct self-expanders:

**Theorem 2.4.** For $k \geq 2$ and $\alpha \in (0, 1)$, let $C$ be a $C^{k,\alpha}$-regular cone in $\mathbb{R}^{n+1}$ and assume either $2 \leq n \leq 6$ or $\lambda[C] < \Lambda_n$. For any two $\Sigma_1$ and $\Sigma_2 \in \mathcal{ACH}^{k,\alpha}_n$, with $C(\Sigma_1) = C(\Sigma_2) = C$, there exist $\Sigma_{\pm}$ stable self-expanders asymptotic to $C$ with $\Sigma_{\pm} \preceq \Sigma_i \preceq \Sigma_{\pm}$ for $i = 1, 2$.

3. The Relationship Between Entropy and Stability

As observed by S. Guo [11], if the entropy of the cone is sufficiently small, then we have the curvature bound for the self-expander. More precisely,

**Theorem 3.1.** Given $\kappa > 0$, there exists a constant $\epsilon > 0$ depending on $n$ and $\kappa$ with the following property.

If $\Sigma$ is a self-expander that is asymptotic to a regular cone $C$ with $\lambda[C] < 1 + \epsilon$. Then we have $\| A_\Sigma \|_{L^\infty} \leq \kappa$.

Guo pointed out that when $\kappa < \frac{1}{\sqrt{2}}$, all self-expanders are stable. In the following lemma, we use a Poincaré type inequality from [13] to get a slightly strengthening by improving the bound to $\kappa \leq \sqrt{\frac{n+1}{2}}$.

**Lemma 3.2.** Let $\Sigma$ be a self-expander in $\mathcal{ACH}^{k,\alpha}_n$ satisfying $|A_\Sigma|^2 \leq \frac{4\pi}{\kappa}$. Then $\Sigma$ is strictly stable in the sense that for all $u \in C^\infty_c(\Sigma) \setminus \{0\}$,

$$ \langle -\mathcal{L}_\Sigma u, u \rangle = \int_\Sigma \left[ |\nabla_\Sigma u|^2 + \left( \frac{1}{2} - |A_\Sigma|^2 \right) u^2 \right] e^{\frac{1}{2} |x|^2} dH^n > 0$$

(3.1)
Proof. Following [3, Appendix A], since $(\Delta_\Sigma + \frac{1}{2}x \cdot \nabla_\Sigma)(|x|^2 + 2n) = |x|^2 + 2n$, integrating by parts gives

$$\int_\Sigma (2n + |x|^2)u^2 e^{\frac{1}{4}|x|^2}dH^n = \int_\Sigma [(\Delta_\Sigma + \frac{1}{2}x \cdot \nabla_\Sigma)(|x|^2 + 2n)]u^2 e^{\frac{1}{4}|x|^2}dH^2$$

(3.2)

$$= -\int_\Sigma \nabla_\Sigma(|x|^2 + 2n) \cdot \nabla_\Sigma(u^2) e^{\frac{1}{4}|x|^2}dH^n = -4 \int_\Sigma x^\top \cdot \nabla_\Sigma u e^{\frac{1}{4}|x|^2}dH^n$$

$$\leq \int_\Sigma (|x|^2 u^2 + 4|\nabla_\Sigma u|^2)e^{\frac{1}{4}|x|^2}dH^n.$$ 

So moving $|x|^2 u^2$ to the left hand side,

$$\int_\Sigma (2n + |x|^2)u^2 e^{\frac{1}{4}|x|^2}dH^n \leq 4 \int_\Sigma |\nabla_\Sigma u|^2 e^{\frac{1}{4}|x|^2}dH^n$$

Together with $|A_\Sigma|^2 \leq \frac{n+1}{2}$,

$$\langle -L_\Sigma u, u \rangle = \int_\Sigma \left[ |\nabla_\Sigma u|^2 + \left( \frac{1}{2} - |A_\Sigma|^2 \right)u^2 \right] e^{\frac{1}{4}|x|^2}dH^n$$

$$\geq \int_\Sigma (|\nabla_\Sigma u|^2 - \frac{n}{2}u^2) e^{\frac{1}{4}|x|^2}dH^n$$

$$\geq \frac{1}{4} \int_\Sigma |x|^2 u^2 e^{\frac{1}{4}|x|^2}dH^n \geq 0$$

If there were $u \in C^\infty_0(\Sigma) \setminus \{0\}$ satisfying $\langle -L_\Sigma u, u \rangle = 0$, then the inequality (3.2) above should be an equality, which means

$$\langle -L_\Sigma u, u \rangle = 0,$$

Fix $p \in \{ u > 0 \}, r = \sup \{ s > 0 : B^\Sigma_s(p) \subset \{ u > 0 \} \}$ and define $v = \log u$ in $B^\Sigma_r(p)$. $u$ being compactly supported implies $r < \infty$ is well-defined and that there is a $p_0 \in \partial B_r(p) \cap \{ u = 0 \}$. From (3.3) we know that $\nabla_\Sigma(v + \frac{1}{4}|x|^2) = 0$, which means $v = -\frac{1}{4}|x|^2 + constant$. However, this contradicts to the fact that $v \to -\infty$ as $q \to p_0$. Hence, $\langle -L_\Sigma u, u \rangle > 0$ for all non-trivial $u$. □

4. PROOF OF THEOREM 1.1

In this section we use the mountain-pass theorem proved by J. Bernstein and L. Wang to prove Theorem 1.1.

Proof of Theorem 1.1. The existence result follows from [3, Theorem 6.3] and [3, Proposition 3.3]. Hence, we only need to show the uniqueness. Letting $\kappa = \sqrt{\frac{n+1}{2}}$, Theorem 3.1 ensures the existence of an $\epsilon = \epsilon(n)$ so that if $\lambda|C| < 1 + \epsilon$, then $|A_\Sigma| \leq \kappa$ for all self-expanders asymptotic to $C$. Then by Lemma 3.2, all self-expanders asymptotic to $C$ are strictly stable. That is, $C$ is a regular value in the sense of [3].

Let $\delta = \min\{\epsilon(n), \Lambda_n - 1\}$. As an application of [3, Theorem 1.3], $\Pi^{-1}(C)$ is a finite set, where $\Pi$ assigns each element in $A\mathcal{C}_{\Pi}^{k, a}$ to the trace at infinity.

Now, let us argue by contradiction. Suppose there were two self-expanders $\Sigma_1$ and $\Sigma_2$ both asymptotic to $C$. Following Theorem 2.4, we can produce two distinct self-expanders $\Sigma_\pm$ with $\Sigma_- \leq \Sigma_i \leq \Sigma_+$ for $i = 1, 2$. Applying [5, Theorem 1.1], there is a self-expander $\Sigma_0 \neq \Sigma_\pm$ with possibly codimension-7 singular set and $\Sigma_- \leq \Sigma_0 \leq \Sigma_+$. Notice that Huisken’s monotonicity formula tells us $\lambda|\Sigma_0| \leq \lambda|C| < 1 + \delta \leq \Lambda_n$. Thus, by Lemma 2.3, $\Sigma_0$ is actually smooth. Now, replace $\Sigma_\pm$ by $\Sigma_0$ and $\Sigma_-$. and iterate the preceding argument. So we produce as many self-expanders as we can. And this contradicts the fact that $\Pi^{-1}(C)$ is finite. □

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