Pre-Stability of Fixed Point

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Abstract. In this paper, we introduced certain types of stability of the fixed points in discrete dynamical systems which are pre-stability, pre-c-stability, and pre-ic-stability. We studied the relationships among these types of stability, also the relationships among these types of stability and certain types of stability which are stability, c-stability, and ic-stability

Keywords: pre-open, Fixed point, Pre-stable fixed point, Orbit, and dynamical System.

1. Introduction

A discrete dynamical system consists of a non-empty set $X$ which is called the phase space and compositions $f^t$, $t \in \mathbb{N}$ of a map $f: X \to X$ where $f^t = f \circ f \circ \ldots \circ f$ (n-times). These iterates form a group or semi group. A dynamical system could be a measure space and a function that preserves measure; a metric space with an isometry; or a topological space and a continuous function, etc. In this paper, we considered phase spaces which are topological spaces [3]. A strong concept of stability for dynamical system was first formulated by N.E. Zhukovskii [10]. He introduced in 1882 a strong orbital stability concept which is based on a reparametrisation of the time variable [11]. On the 12 October 1892 (by modern calendar) Alexander Mikhailoich Lyapunov defined his doctoral thesis the general problem of the stability of motion (at Moscow university) [13]. Lyapunov defined a fixed point $x_0$ to be stable if for every neighborhood $U$ of $x_0$, there is a neighborhood $V \subseteq U$ such that every solution $x(t)$ starting in $V (x(t))$ remains in $U$ for all $t \geq 0$. Otherwise, $x_0$ is unstable [12]. In 2014 Mohammed F. Al-Ali and A.M. Hamza introduced and studied new types of stability which are c-stability and ic-stability of the fixed points [9].

In this paper, $\mathbb{P}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\overline{\mathbb{R}}$ and $\overline{\mathbb{Z}}$ will denote the family of (p-o) sets, the set of real numbers, the set of integer numbers, the set of non-negative integers, the closure of the interior of $\mathbb{R}$ and the interior of the closure of $\mathbb{R}$, respectively. For any non-empty set $X$, we denote by $\tau_{UR}$, $\tau_{ID}$, $\tau_{IND}$ and $\tau_{C}$, the usual topology on $\mathbb{R}$, the discrete topology, the indiscrete topology and the cofinite topology respectively. Finally we denote by $A_{c}$ and $O(x)$, the complement of the set $A$ and the orbit of $x$. We used space, map, and DDS to refer to a topological space, continuous function and discrete dynamical system, respectively.

2. Preliminaries

2.1 Definition [3]

A DDS consists of a phase space $X$ and iterates $f^t$, where $t$ belong to $\mathbb{N}$ of a map $f: X \to X$, the nth iterate of $f$ is the $t$-fold composition $f^t = f \circ f \circ \ldots \circ f$; we define $f^0$ to be the identity map. If $f$ satisfy the invertible properties then $f^{-t} = f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1} (n$ times). Since $f^{t+m} = f^t \circ f^m$, these iterates form a group if $f$ is invertible, and semi group otherwise.

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Although we have defined DDS in a completely abstract setting, where $X$ is simply a set, in practice $X$ usually has additional structure that is preserved by the map $f$. For example, $(X, f)$ could be a measure space and a measure-preserving map; a space and a continuous map; a metric space and an isometry; or a smooth manifold and a differentiable map.

2.2 Definition [1]
Let $(X, \tau)$ be a space, and $f: X \rightarrow X$ be a function. A point $x \in X$ is said to be fixed point of $f$ if $f(x) = x$.

2.3 Definition [1]
Let $(X, \tau)$ be a space, and $f: X \rightarrow X$ be a map. For all $x \in X$, the orbit of $x$ under $f$ is the set \{ $x, f(x), f^2(x), \ldots, f^n(x), \ldots$ \}, and it is denoted by $O(x)$, where $O(x) \subseteq X$.

2.4 Definition [5]
A subset $A$ of a space $X$ is called a pre-open (p-o) set if and only if $A \subseteq \overline{A}$. $A$ is called a pre-closed if and only if $A^c$ is (p-o) and it's easy to see that $A$ is pre-closed if and only if $\overline{A} \subseteq A$.

2.5 Remark [6]
If $A$ is a dense subset in $X$, Then it is a (p-o) set.

2.6 Theorem [7]
Let $X$ be a space. If $A$ is a (p-o) set in $X$, Then $A = U \cap B$, where $U$ is an open set in $X$ and $B$ is a dense set in $X$.

2.7 Theorem [8]
The arbitrary union of (p-o) set is also (p-o).

2.8 Definition [1]
Let $(X, \tau)$ be a space, $f: X \rightarrow X$ be a map, $x_0 \in X$ is called stable if for every open set $U \subseteq X$ containing $x_0$, there exists an open set $V \subseteq U$ containing $x_0$ such that $O(x) \subseteq U$, $\forall x \in V$.

Otherwise, $x_0$ is called unstable fixed point.

2.9 Theorem [9]
Let $(X, \tau)$ be a space, $B_\tau$ is a basis for $\tau$, $f: X \rightarrow X$ be a map, and $x_0$ be a fixed point of $f$. If $x_0$ is stable point with respect to $B_\tau$, then $x_0$ is stable point with respect to $\tau$.

2.10 Definition [9]
Let $(X, \tau)$ be a space, $f: X \rightarrow X$ be a map. A fixed point $x_0$ of $f$ is called c-stable if for any open set $U$ containing $x_0$, there exists an open set $V \subseteq U$ containing $x_0$ such that $O(x) \subseteq U$, $\forall x \in V$.

Otherwise, we say that $x_0$ is not c-stable fixed point.
2.11 Theorem [9]

Let \((X, \tau)\) be a space, \(B_{\tau}\) is a basis for \(\tau\), \(f: X \rightarrow X\) be a map and \(x_0\) is a fixed point of \(f\). If \(x_0\) is c-stable with respect to \(B_{\tau}\), then \(x_0\) is c-stable with respect to \(\tau\).

2.12 Example

Consider the space \((\mathbb{R}, \tau_u)\), and \(f: \mathbb{R} \rightarrow \mathbb{R}\) be a function defined by \(f(x) = \frac{4}{3}x\). The DDS is \(\left\{\left(\frac{1}{3}\right)^n \cdot x\right\}_{n \in \mathbb{N}}\), and 0 is the fixed point of \(f\). \(B_{\tau_u} = \{(a, b); a, b \in \mathbb{R}\}\) is a basis for \(\tau_u\).

Let \(U = (x_0, x_1) \in B_{\tau_u}\), where \(0 \in U\). Choose \(V = (-a, a) \in B_{\tau_u}\), where \(0 \in V \subseteq U\), \(a = \min\{|x_0|, |x_1|\}\). Note that \(O(x) \subseteq V \subseteq \overline{U}, \forall x \in V\). Then, 0 is c-stable.

2.13 Example

Consider the space \((\mathbb{R}, \tau_u)\) and \(f : \mathbb{R} \rightarrow \mathbb{R}\) is the function defined by \(f(x) = -5x\).

\(B_{\tau_u} = \{(a, b); a, b \in \mathbb{R}\}\) is a basis for \(\tau_u\). The DDS is \(\{(-5)^n x\}_{n \in \mathbb{N}}, \) and 0 is the fixed point of \(f\).

Let \(U = (-1, 1) \in B_{\tau_u}\). Note that, for any open subset \(V\) of \(U\) containing 0, and for any \(x \in V\), \(O(x) \not\subseteq U\).

Hence, 0 is not c-stable fixed point.

2.14 Theorem [9]

Let \((X, \tau)\) be a space, \(f : X \rightarrow X\) be a map and \(x_0\) is a fixed point of \(f\). If \(x_0\) is stable, then it is c-stable.

2.15 Definition [9]

Let \((X, \tau)\) be a space, \(f: X \rightarrow X\) be a map. \(x_0 \in X\) is called ic-stable if for every open set \(U \subseteq X\) containing \(x_0\), there exists an open set \(V \subseteq U\) containing \(x_0\) such that, \(O(x) \subseteq U\), \(\forall x \in V\).

Otherwise, we say that \(x_0\) is not ic-stable fixed point.

2.16 Theorem [9]

Let \((X, \tau)\) be a space, \(B_{\tau}\) is a basis for \(\tau\), \(f: X \rightarrow X\) be a map and \(x_0\) is a fixed point of \(f\). If \(x_0\) is ic-stable with respect to \(B_{\tau}\), then \(x_0\) is ic-stable with respect to \(\tau\).

2.17 Theorem [9]

Let \((X, \tau)\) be a space, \(f : X \rightarrow X\) be a map and \(x_0\) is a fixed point of \(f\). If \(i\)- \(x_0\) is stable, then it is ic-stable.

\(ii\)- \(x_0\) is ic-stable, then it is c-stable.
3. Main Results

3.1 Definition

Let \((X, \tau)\) be a space in a DDS \(\{f^n\}_{n\in\mathbb{N}}\), and let \(x_0\) be a fixed point of \(f\). We say that \(x_0\) is pre-stable if for any (p-o) set \(U \subseteq X\) containing \(x_0\), there exists a (p-o) set \(V \subseteq U\) containing \(x_0\) such that \(O(x) \subseteq U; \forall x \in V\).

Otherwise, \(x_0\) is called not pre-stable fixed point.

3.2 Example

Consider the space \((\mathbb{R}, \tau)\), \(\tau = \{R, \emptyset, Z, R\setminus Z\}\) and \(f : R \to R\) is the function defined by

\[
f(x) = \begin{cases} x^2, & x \in Z \\ x + 1, & \text{o.w.} \end{cases}
\]

The fixed points of \(f\) are 0 and 1.

0 is pre-stable:

Let \(U\) be any (p-o) set containing 0. Choose \(V = \{0\} \subseteq U\). \(V\) is (p-o) subset of \(U\) containing 0 with \(O(0) \subseteq U\).

So, 0 is pre-stable.

Similarly, 1 is pre-stable.

3.3 Example

Consider the space \((\mathbb{R}, \tau_c)\), and \(f : R \to R\) is the function defined by \(f(x) = \frac{1}{5}x\). The fixed point of \(f\) is 0, and the DDS is \(\{(\frac{1}{5})^n x\}_{n\in\mathbb{N}}\).

\(U = \{0\} \cup [10,15]\) is a (p-o) set in \((\mathbb{R}, \tau_c)\) containing 0. Let \(V\) be any (p-o) subsets of \(U\) containing 0. \(O(x) \subseteq U, \forall x \in V\).

Hence, 0 is not pre-stable fixed point.

3.4 Remark

A stable fixed point needs not be pre-stable. (Example 3.5)

3.5 Example

Let \((X, \tau)\) be a space, \(X = \{a, b, c\}\), \(\tau = \{X, \emptyset, \{a,c\}\}\) and \(f : X \to X\) is the function defined by, \(f(a) = c, f(b) = b\) and \(f(c) = a\).

The fixed point of \(f\) is \(b\) and the DDS is given by the following table.
The First International Conference of Pure and Engineering Sciences (ICPES2020)

The only open set containing \(b\) is \(U = X \in \tau\).

The only open subset of \(U\) that containing \(b\) is \(U\) itself, i.e. \(V = U\).

\[O(x) \subseteq U, \forall x \in V.\] So, \(b\) is stable fixed point.

\(U = \{a, b\}\) is a (p-o) set and \(b \in U\).

The only (p-o) subset of \(U\) that containing \(b\) is \(U\) itself, i.e. \(V = U\).

\[O(a) = \{a, c, a, c, ...\} \not\subseteq U.\]

So, \(b\) is not pre-stable.

Hence, stability \(\not\Rightarrow\) pre-stability.

In the following theorem, we shall give a condition that make stability implies pre-stability and pre-stability implies stability.

### 3.6 Theorem

Let \((X, \tau)\) be a space, \((f^n)_{n \in \mathbb{N}}\) be a DDS with a fixed point \(x_0\) such that every open set containing \(x_0\). Then \(x_0\) is pre-stable if and only if it is stable.

\[\Rightarrow\text{Proof:}\] Let \(x_0\) be a pre-stable fixed point and \(U\) be any open set containing \(x_0\). Then \(U\) is (p-o) set and \(x_0 \in U\), so there exists (p-o) set \(V\); \(x_0 \in V \subseteq U\), and \(O(x) \subseteq U, \forall x \in V\).

\(V^*\) is open set containing \(x_0\) and \(V^* \subseteq V \subseteq U\) with \(O(x) \subseteq U; \forall x \in V^*\). Hence, \(x_0\) is stable.

\[\Leftarrow\text{Proof:}\] Let \(U\) be any (p-o) set containing \(x_0\). Note that \(U^*\) is open set. Since \(x_0\) is stable, then there exists an open set \(V, V \subseteq U^*\) such that \(O(x) \subseteq U^*, \forall x \in V\). Now, \(V\) is (p-o) set with \(O(x) \subseteq U^* \subseteq U, \forall x \in V\).

Hence, \(x_0\) is pre-stable fixed point.

### 3.7 Theorem

Let \((X, \tau)\) be a space. In any DDS with the topology \(\tau = \{X, \emptyset, U, U^c\}\), \(U \subseteq X\), every fixed point is a pre-stable.

\[\text{Proof:}\] In such space, every non-empty subset \(A\) of \(X\) is (p-o):
\[ \overrightarrow{A} = \begin{cases} U, & \text{if } A \subseteq U \\ U^c, & \text{if } A \subseteq U^c \\ X, & \text{if } A = A_1 \cup A_2, \ A_1 \subseteq U \text{ and } A_2 \subseteq U^c \end{cases} \]

So,

\[ \overrightarrow{A} = \begin{cases} U, & \text{if } A \subseteq U \\ U^c, & \text{if } A \subseteq U^c \\ X, & \text{if } A = A_1 \cup A_2, \ A_1 \subseteq U \text{ and } A_2 \subseteq U^c \end{cases} \]

So, in such DDS, every fixed point \( x_0 \) is pre-stable, for \( V = \{ x_0 \} \) is a (p-o) subset of any (p-o) set \( U \) containing \( x_0 \) and \( O(x_0) \subseteq U \). Hence, \( x_0 \) is pre-stable.

**3.8 Theorem**

If the phase space \( X \) of a DDS has a basis of pairwise disjoint basic open sets, then every fixed point is pre-stable.

**Proof:** Let \( B = \{ A_a \}_{a \in A} \) be a basis for the topology of the phase space \( X \) in a DDS with \( A_i \cap A_j = \emptyset, \forall \ i \neq j \).

Let \( x_0 \in X \) be a fixed point. Then \( x_0 \in A_{a_0}, \ A_{a_0} \in B \).

Now, let \( U \) be any (p-o) set containing \( x_0 \). Put \( A = \{ x_0 \} \). Then \( \overrightarrow{A} \subseteq A_{a_0} \), so \( A \) is (p-o) set. We have, \( x_0 \in A \subseteq U \) with \( O(x_0) \subseteq U \).

So, \( x_0 \) is pre-stable fixed point. \( \square \)

**3.9 Theorem**

If \( \{ f^n \}_{n \in \mathbb{N}} \) is a DDS with \( \tau = \{ X, \emptyset, A \}, A \subseteq X, \) then any fixed point in \( A \) is pre-stable.

**Proof:** Let \( x_0 \in A \) be a fixed point and \( U \) be any (p-o) set containing \( x_0 \). Then \( V = \{ x_0 \} \) is (p-o) set containing \( x_0 \) and \( V \subseteq U \) with \( O(x_0) \subseteq U \).

Hence, \( x_0 \) is pre-stable fixed point. \( \square \)

**3.10 Definition**

Let \((X, \tau)\) be a space in a DDS \( \{ f^n \}_{n \in \mathbb{N}} \), and let \( x_0 \) be a fixed point of \( f \). \( x_0 \) is called pre-c-stable if for any (p-o) set \( U \subseteq X \) containing \( x_0 \), there exists a (p-o) set \( V \subseteq U \) containing \( x_0 \) such that \( O(x) \subseteq \overline{U}; \ \forall x \in V \).

Otherwise, \( x_0 \) is called not pre-c-stable fixed point.

**3.11 Example**

Consider the space \((R, \tau_{ind})\), and \( f : R \rightarrow R \) is the function defined by \( f(x) = 4x - 1 \). The fixed point of \( f \) is \( \frac{1}{3} \), and the DDS is \( \{ 4^n x - 4^{n-1} + 4^{n-2} + \cdots + 1 \}_{n \in \mathbb{N}} \).

Let \( U \) be any (p-o) sets in \((R, \tau_{ind})\) contains \( \frac{1}{3} \). \( V = \{ \frac{1}{3} \} \), is a (p-o) subset of \( U \); \( \frac{1}{3} \in V \). Note that, \( O(x) \subseteq \overline{U} = R, \forall x \in V \).
Hence, $\frac{1}{3}$ is pre-c-stable

### 3.12 Example

Let $(X, \tau)$ be a space and $\tau = \{X, \emptyset, \{1,3\}, \{2,4\}, \{1,3,4\}\}$. Let $\tau' = \tau \cup \{\{1\}, \{3\}, \{4\}, \{1,2\}, \{1,4\}, \{3,4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$.

The fixed point of $f$ is 2 and the DDS is given by the following table.

| $x$ | $f(x)$ | $f^2(x)$ | ... | $f^n(x)$ | ... |
|-----|--------|----------|-----|----------|-----|
| 1   | 4      | 2        | ... | 2        | ... |
| 2   | 2      | 2        | ... | 2        | ... |
| 3   | 4      | 2        | ... | 2        | ... |
| 4   | 2      | 2        | ... | 2        | ... |

The (p-o) set $U = \{1, 2\}$ is containing 2. The only (p-o) subset of $U$ containing 2 is $V = U$. $O(1) \not\subseteq \overline{U}$.

Hence, 2 is not pre-c-stable fixed point.

### 3.13 Theorem

Let $(X, \tau)$ be a space, $(\{f^n\}_{n \in \mathbb{N}})$ be a DDS with a fixed point $x_*$ such that every open set containing $x_*$. Then $x_*$ is pre-c-stable if and only if it is c-stable.

**⇒Proof:** Let $x_*$ be a pre-c-stable fixed point and $U$ be any open set containing $x_*$. Then $U$ is (p-o) set and $x_* \in U$. So, there exists (p-o) set $V$; $x_* \in V \subseteq U$, and

$O(x) \subseteq \overline{U}, \ \forall x \in V$.

$V^*$ is open set containing $x_0$ with $V^* \subseteq V \subseteq U$.

So, $O(x) \subseteq \overline{U}, \ \forall x \in V^*$.

Hence, $x_*$ is c-stable.

**⇐ proof:** Let $U$ be any (p-o) set containing $x_0$. Note that, $U^*$ is open set containing $x_0$. Since $x_0$ is c-stable, then there exists an open set $V$, $V \subseteq U^*$ such that $O(x) \subseteq \overline{U^*}, \ \forall x \in V$. Now, $V$ is (p-o) set with $O(x) \subseteq \overline{U} \subseteq \overline{U}, \ \forall x \in V$.

Hence, $x_0$ is pre-c-stable fixed point.

### 3.14 Theorem

Let $(X, \tau)$ be a space, $(\{f^n\}_{n \in \mathbb{N}})$ be a DDS with a fixed point $x_0$. If $x_0$ is pre-stable, then it is pre-c-stable.

**Proof:** Let $U$ be a (p-o) set; $x_0 \in U$. Since $x_0$ is pre-stable, then there exists (p-o) set $V$; $x_0 \in V \subseteq U$ such that, $O(x) \subseteq U, \ \forall x \in V$. So, $O(x) \subseteq \overline{U}, \ \forall x \in V$. 

\[\square\]
Hence, $x_0$ pre-c-stable.

The converse of above theorem isn't true generally.

### 3.15 Example

Consider the space $(R, \tau_c)$, and $f : R \to R$ be a function define by,

$$f(x) = \begin{cases} 2x, & x < 1 \\ x + 2, & x \geq 1 \end{cases}$$

The fixed point of $f$ is 0.

0 is not pre-stable fixed point. Let $U = (-7, 5)$ is a (p-o) set in $(R, \tau_c)$ containing 0.

Let $V$ be any (p-o) sub set of $U$ containing 0. Then, $O(x) \not\subseteq U$, for some $x \in V$.

Hence, 0 is not pre-stable fixed point.

But 0 is pre-c-stable:

Let $U$ be any (p-o) set of $R$ containing 0. $V = (-2, 5)$ is (p-o) set and $0 \in V \subseteq U$, Thus, $O(x) \subseteq \overline{U} = R$. Hence, 0 is pre-c-stable fixed point.

### 3.16 Theorem

If the phase space $X$ of a DDS has a basis of pairwise disjoint basic open sets, then every fixed point is pre-c-stable.

**Proof:** it is clear [Theorem 3.8] and[Theorem 3.14].

### 3.17 Definition

Let $(X, \tau)$ be a space in a DDS $\{f^n\}_{n \in \mathbb{N}}$, and let $x_0$ be a fixed point of $f$. $x_0$ is called pre-ic-stable if for any (p-o) set $U \subseteq X$ containing $x_0$, there exists a (p-o) set $V \subseteq U$ containing $x_0$, such that $O(x) \subseteq \overline{U} ; \forall x \in V$.

Otherwise, $x_0$ is called not pre-ic-stable fixed point.

### 3.18 Example

Let $(X, \tau)$ be a space and $\tau = \{1, 2, 3, 4\}$, $\tau = \{X, \emptyset, \{1,3\}, \{2,4\}\}$ and $f : X \to X$ be a function defined by, $f(1) = f(3) = 1$ and $f(2) = f(4) = 3$.

The fixed point of $f$ is 1 and the DDS is given by the following table.

| $x$ | $f(x)$ | $f^2(x)$ | ... | $f^n(x)$ | ... |
|-----|--------|----------|-----|----------|-----|
| 1   | 1      | 1        | ... | 1        | ... |
| 2   | 3      | 1        | ... | 1        | ... |
| 3   | 1      | 1        | ... | 1        | ... |
| 4   | 3      | 1        | ... | 1        | ... |

$\tau^p = P(X)$

Let $U$ be any (p-o) set containing 1. $V = \{1\}$ is a (p-o) set and $1 \in V \subseteq U$ with $O(1) \subseteq U$. So, $O(1) \subseteq \overline{U}^p$. 


Hence, 1 is pre-ic-stable fixed point.

3.19 Example

Let \((X, \tau)\) be a space and \(X = \{a, b, c\}\), \(\tau = \{X, \emptyset, \{a\}, \{b\}, \{c\}\}\) and \(f : X \to X\) be a function defined by, \(f(a) = f(b) = b, f(c) = a\).

The fixed point of \(f\) is \(b\) and the DDS is given by the following table.

|   | \(f(x)\) | \(f^2(x)\) | \(f^3(x)\) | \(f^n(x)\) |
|---|-----------|------------|------------|------------|
| \(a\) | \(b\)     | \(b\)     | \(b\)     | \(b\)     |
| \(b\) | \(b\)     | \(b\)     | \(b\)     | \(b\)     |
| \(c\) | \(a\)     | \(b\)     | \(b\)     | \(b\)     |

\(\tau^p = \tau\).

Let \(U = \{b, c\}\) is a \((p-o)\) set with \(b \in U\). The only \((p-o)\) subset of \(U\) contains the fixed point \(b\) is \(U\) itself.

\(O(c) = \{c, a, b, \ldots\}\)

So, \(O(c) \notin \overline{U}\).

Hence, \(b\) is not pre-ic-stable fixed point.

3.20 Theorem

Let \((X, \tau)\) be a space, \(\{f^n\}_{n \in \mathbb{N}}\) be a DDS with a fixed point \(x_s\) such that every open set containing \(x_s\). Then \(x_s\) is pre-ic-stable if and only if it is ic-stable.

\(\Rightarrow\) Proof: Let \(x_s\) be a pre-ic-stable fixed point and \(U\) be any open set containing \(x_s\). Then \(U\) is a \((p-o)\) set and \(x_s \in U\). So, there exists \((p-o)\) set \(V; x_s \in V \subseteq U\), and

\[O(x) \subseteq \overline{U}, \forall x \in V.\]

\(V^*\) is open set containing \(x_0\) with \(V^* \subseteq V \subseteq U\).

So, \(O(x) \subseteq \overline{U}, \forall x \in V^*\).

Hence, \(x_s\) is ic-stable fixed point.

\(\Leftarrow\) Proof: Let \(U\) be any \((p-o)\) set containing \(x_0\). Note that, \(U^*\) is open set. Since \(x_0\) is ic-stable, then there exists an open set \(V\) containing \(x_0\), \(V \subseteq U^*\) such that \(O(x) \subseteq \overline{U^*}, \forall x \in V\). Now, \(V\) is \((p-o)\) set with \(O(x) \subseteq \overline{U^*} \subseteq \overline{U}, \forall x \in V\).

Hence, \(x_0\) is pre-ic-stable fixed point.

3.21 Theorem

Let \((X, \tau)\) be a space, \(\{f^n\}_{n \in \mathbb{N}}\) be a DDS with a fixed point \(x_0\). If \(x_0\) pre-stable, then it is pre-ic-stable.
Let \( U \) be a \((p-o)\) set; \( x_0 \in U \). Since \( x_0 \) is pre-stable, then there exists a \((p-o)\) set \( V \); \( x_0 \in V \subseteq U \) such that, \( O(x) \subseteq U \), \( \forall x \in V \). Since \( U \) is \((p-o)\) set, then \( U \subseteq \overline{U} \), and so \( O(x) \subseteq \overline{U}^o \), \( \forall x \in V \). Hence, \( x_0 \) pre-ic-stable.

**Proof** : Let \( U \) be a \((p-o)\) set; \( x_0 \in U \). Since \( x_0 \) is pre-stable, then there exists a \((p-o)\) set \( V \); \( x_0 \in V \subseteq U \) such that, \( O(x) \subseteq U \), \( \forall x \in V \). Since \( U \) is \((p-o)\) set, then \( U \subseteq \overline{U} \), and so \( O(x) \subseteq \overline{U}^o \), \( \forall x \in V \).

3.22 **Theorem**

Let \((X, \tau)\) be a space, \( \{f^n\}_{n \in \mathbb{N}} \) be a DDS with a fixed point \( x_0 \). If \( x_0 \) pre-ic-stable, then it is pre-c-stable.

**Proof** : Let \( U \) be a \((p-o)\) set; \( x_0 \in U \). Since \( x_0 \) is pre-ic-stable, then there exists a \((p-o)\) set \( V \); \( x_0 \in V \subseteq U \) such that, \( O(x) \subseteq U \), \( \forall x \in V \). Since \( \overline{U} \subseteq U \), then \( O(x) \subseteq \overline{U} \), \( \forall x \in V \).

Then, \( x_0 \) pre-c-stable.

3.23 **Theorem**

If the phase space \( X \) of a DDS has a basis of pairwise disjoint basic open sets, then every fixed point is pre-ic-stable.

**Proof** : it is clear [Theorem 2.8] and [Theorem 2.21]

3.24 **Theorem**

If \( \{f^n\}_{n \in \mathbb{N}} \) is a DDS with \( \tau = \{X, \emptyset, A\}, A \subseteq X \), then any fixed point in \( A \) is pre-ic-stable, and so it is pre-c-stable.

**Proof** : Let \( x_0 \) be a fixed point. Let \( x_0 \in A \), and \( U \) be any \((p-o)\) set containing \( x_0 \). Then \( V = \{x_0\} \) is a \((p-o)\) set containing \( x_0 \) and \( V \subseteq U \) with \( O(x_0) \subseteq U \) with \( O(x_0) \subseteq \overline{U}^o \).

Hence, \( x_0 \) is pre-c-stable.

Now, if \( x_0 \in A^c \), then any \((p-o)\) set containing \( x_0 \) is of the form \( U = A^c \cup B \), where \( \emptyset \neq B \subseteq A \).

Choose \( V = A^c \cup \{x_0\} \). Then \( V \) is a \((p-o)\) set containing \( x_0 \), \( V \subseteq U \).

Then, \( O(x) \subseteq \overline{U}^o = X \), \( \forall x \in V \). Hence, \( x_0 \) is pre-ic-stable.

4. **Conclusion**

Certain types of stability which depend on the pre-open sets had been discussed. Since every open set is \((p-o)\) set, so these types of stability had been discussed the stability in phase spaces in which the collection of \((p-o)\) sets is at most finer than the collection of open sets. This means that we gave a stability in larger phase spaces.
Figure (1): Relationships among certain types of pre-stability of a fixed point $x_0$.

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