Representations of the twisted quantized enveloping algebra of type $C_n$

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Abstract

We prove a version of the Poincaré–Birkhoff–Witt theorem for the twisted quantized enveloping algebra $U'_{q}(\mathfrak{sp}_{2n})$. This is a subalgebra of $U_{q}(\mathfrak{gl}_{2n})$ and a deformation of the universal enveloping algebra $U(\mathfrak{sp}_{2n})$ of the symplectic Lie algebra. We classify finite-dimensional irreducible representations of $U'_{q}(\mathfrak{sp}_{2n})$ in terms of their highest weights and show that these representations are deformations of the finite-dimensional irreducible representations of $\mathfrak{sp}_{2n}$.

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1 Introduction

Let $\mathfrak{g}$ denote the orthogonal or symplectic Lie algebra $\mathfrak{o}_N$ or $\mathfrak{sp}_{2n}$ over the field of complex numbers. There are at least two different $q$-analogues of the universal enveloping algebra $U(\mathfrak{g})$. These are the quantized enveloping algebra $U_q(\mathfrak{g})$ introduced by Drinfeld [3] and Jimbo [11], and the twisted (or nonstandard) quantized enveloping algebra $U'_q(\mathfrak{g})$, introduced by Gavrilik and Klimyk [4] for $\mathfrak{g} = \mathfrak{o}_N$ and by Noumi [19] for $\mathfrak{g} = \mathfrak{sp}_{2n}$. If $q$ is a complex number which is nonzero and not a root of unity then the representation theory of $U_q(\mathfrak{g})$ is very much similar to that of the Lie algebra $\mathfrak{g}$; see e.g. Chari and Pressley [2, Chapter 10]. The description of the finite-dimensional irreducible representations of $U'_q(\mathfrak{o}_N)$ given by Iorgov and Klimyk [10] both exhibits similarity with the classical theory and reveals some new phenomena specific to the quantum case.

In this paper we are concerned with the description of finite-dimensional irreducible representations of the twisted quantized enveloping algebra $U'_q(\mathfrak{sp}_{2n})$. We introduce a class of highest weight representations for this algebra and show that any finite-dimensional irreducible representation of $U'_q(\mathfrak{sp}_{2n})$ is highest weight. Then we give necessary and sufficient conditions on a highest weight representation to be finite-dimensional. By the main theorem (Theorem 6.3), the finite-dimensional irreducible representations of $U'_q(\mathfrak{sp}_{2n})$ are naturally parameterized by the $n$-tuples

$$\lambda = (\sigma_1 q^{m_1}, \ldots, \sigma_n q^{m_n}),$$

where the $m_i$ are positive integers satisfying $m_1 \leq m_2 \leq \cdots \leq m_n$ and each $\sigma_i$ equals 1 or $-1$. Regarding this parametrization, the algebra $U'_q(\mathfrak{sp}_{2n})$ appears to be closer to the quantized enveloping algebra $U_q(\mathfrak{sp}_{2n})$ than to its orthogonal counterpart $U'_q(\mathfrak{o}_N)$.

We work with the presentation of $U'_q(\mathfrak{sp}_{2n})$ introduced in [18] which is a slight modification of Noumi’s definition [19]. The defining relations are written in a matrix form as a reflection equation for the matrix of generators. A version of the Poincaré–Birkhoff–Witt theorem for this algebra over the field $\mathbb{C}(q)$ was proved in [18]. Here we consider $q$ to be a nonzero complex number such that $q^2 \neq 1$ and prove the corresponding theorem for $U'_q(\mathfrak{sp}_{2n})$ (Theorem 3.6), relying on the Poincaré–Birkhoff–Witt theorem for the quantized enveloping algebra $U_q(\mathfrak{gl}_N)$.

Note that a similar reflection equation presentation exists for the algebra $U'_q(\mathfrak{o}_N)$ as well. These presentations were derived in [19] by regarding $U'_q(\mathfrak{o}_N)$ and $U'_q(\mathfrak{sp}_{2n})$ as subalgebras of $U_q(\mathfrak{gl}_N)$ for appropriate $N$; see also Gavrilik, Iorgov and Klimyk [6]. These subalgebras are coideals with respect to the coproduct on $U_q(\mathfrak{gl}_N)$. A more general description of the coideal subalgebras of the quantized enveloping algebras was given by Letzter [15, 16]. We outline a new proof of the Poincaré–Birkhoff–Witt
theorem for \( U'_q(\mathfrak{o}_N) \) analogous to the symplectic case (see Remark 3.8 below); cf. Iorgov and Klímyk \[9\].

The algebra \( U'_q(\mathfrak{o}_N) \) and its representations were studied by many authors. In particular, it plays the role of the symmetry algebra for the \( q \)-oscillator representation of the quantized enveloping algebra \( U_q(\mathfrak{sp}_{2n}) \); see Noumi, Umeda and Wakayama \[20\].

A quantum analogue of the Brauer algebra associated with \( U'_q(\mathfrak{o}_N) \) was constructed in \[17\]. Some families of Casimir elements were produced by Noumi, Umeda and Wakayama \[20\], Havlíček, Klímyk and Pošta \[7\] and by Gavrilik and Iorgov \[5\] for the algebra \( U'_q(\mathfrak{o}_N) \), and by Molev, Ragoucy and Sorba \[18\] for both \( U'_q(\mathfrak{o}_N) \) and \( U'_q(\mathfrak{sp}_{2n}) \). The paper \[18\] also provides a construction of certain Yangian-type algebras associated with the twisted quantized enveloping algebras. Their applications to spin chain models were discussed in Arnaudon \textit{et al.} \[1\]. The algebra \( U'_q(\mathfrak{sp}_{2n}) \) appears to have received much less attention in the literature as compared to its orthogonal counterpart, which we hope to remedy by this work.

## 2 Definitions and preliminaries

Fix a complex parameter \( q \) which is nonzero and \( q^2 \neq 1 \). Following Jimbo \[12\], we introduce the \( q \)-analogue \( U_q(\mathfrak{gl}_N) \) of the universal enveloping algebra \( U(\mathfrak{gl}_N) \) as an associative algebra generated by the elements \( t_1, \ldots, t_N, t_1^{-1}, \ldots, t_N^{-1}, e_1, \ldots, e_{N-1} \) and \( f_1, \ldots, f_{N-1} \) with the defining relations

\[
t_i t_j = t_j t_i, \quad t_i t_i^{-1} = t_i^{-1} t_i = 1, \\
t_i e_j t_i^{-1} = e_j q^{\delta_{ij} - \delta_{i,j+1}}, \quad t_i f_j t_i^{-1} = f_j q^{-\delta_{ij} + \delta_{i,j+1}}, \\
[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \quad \text{with} \quad k_i = t_i t_{i+1}^{-1}, \\
[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if} \quad |i - j| > 1, \\
e_i^2 = (q + q^{-1}) e_i e_i + e_i e_i^2 = 0 \quad \text{if} \quad |i - j| = 1, \\
f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_i f_j f_i = 0 \quad \text{if} \quad |i - j| = 1.
\]

The \( q \)-analogues of the root vectors are defined inductively by

\[
e_{i,i+1} = e_i, \quad e_{i+1,i} = f_i, \\
e_{i,j} = e_{i,k} e_{k,j} - q e_{k,j} e_{i,k} \quad \text{for} \quad i < k < j, \\
e_{i,j} = e_{i,k} e_{k,j} - q^{-1} e_{k,j} e_{i,k} \quad \text{for} \quad i > k > j, \tag{2.1}
\]

and the elements \( e_{i,j} \) are independent of the choice of values of the index \( k \).
Following [12] and [21], consider the R-matrix presentation of the algebra $U_q(\mathfrak{gl}_N)$. The R-matrix is given by

$$R = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i<j} E_{ij} \otimes E_{ji} \quad (2.2)$$

which is an element of $\text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N$, where the $E_{ij}$ denote the standard matrix units and the indices run over the set $\{1, \ldots, N\}$. The R-matrix satisfies the Yang–Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \quad (2.3)$$

where both sides take values in $\text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N$ and the subindices indicate the copies of $\text{End} \mathbb{C}^N$, e.g., $R_{12} = R \otimes 1$ etc.

The quantized enveloping algebra $U_q(\mathfrak{gl}_N)$ is generated by elements $t_{ij}$ and $\bar{t}_{ij}$ with $1 \leq i, j \leq N$ subject to the relations

$$t_{ij} = \bar{t}_{ji} = 0, \quad 1 \leq i < j \leq N; \quad t_{ii} \bar{t}_{ii} = \bar{t}_{ii} t_{ii} = 1, \quad 1 \leq i \leq N; \quad RT_1 T_2 = T_2 T_1 R, \quad R \bar{T}_1 \bar{T}_2 = \bar{T}_2 \bar{T}_1 R, \quad R \bar{T}_1 T_2 = T_2 \bar{T}_1 R. \quad (2.4)$$

Here $T$ and $\bar{T}$ are the matrices

$$T = \sum_{i,j} t_{ij} \otimes E_{ij}, \quad \bar{T} = \sum_{i,j} \bar{t}_{ij} \otimes E_{ij}, \quad (2.5)$$

which are regarded as elements of the algebra $U_q(\mathfrak{gl}_N) \otimes \text{End} \mathbb{C}^N$. Both sides of each of the R-matrix relations in (2.4) are elements of $U_q(\mathfrak{gl}_N) \otimes \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N$ and the subindices of $T$ and $\bar{T}$ indicate the copies of $\text{End} \mathbb{C}^N$ where $T$ or $\bar{T}$ acts; e.g. $T_1 = T \otimes 1$. In terms of the generators the defining relations between the $t_{ij}$ can be written as

$$q^{\delta_{ij}} t_{ia} t_{jb} - q^{\delta_{ab}} t_{jb} t_{ia} = (q - q^{-1}) (\delta_{b<a} - \delta_{i<j}) t_{ja} t_{ib}, \quad (2.6)$$

where $\delta_{i<j}$ equals 1 if $i < j$ and 0 otherwise. The relations between the $\bar{t}_{ij}$ are obtained by replacing $t_{ij}$ by $\bar{t}_{ij}$ everywhere in (2.6) and the relations involving both $t_{ij}$ and $\bar{t}_{ij}$ have the form

$$q^{\delta_{ij}} \bar{t}_{ia} t_{jb} - q^{\delta_{ab}} t_{jb} \bar{t}_{ia} = (q - q^{-1}) (\delta_{b<a} \bar{t}_{ja} t_{ib} - \delta_{i<j} \bar{t}_{ja} t_{ib}). \quad (2.7)$$

An isomorphism between the two presentations is given by the formulas

$$t_i \mapsto t_{ii}, \quad t_i^{-1} \mapsto \bar{t}_{ii}, \quad e_{ij} \mapsto -\frac{\bar{t}_{ij} t_{ii}}{q - q^{-1}}, \quad e_{ji} \mapsto \frac{t_{ii} t_{ji}}{q - q^{-1}}. \quad (2.8)$$
for \( i < j \); see e.g. [14 Section 8.5.5]. We shall identify the corresponding elements of
\( U_q(\mathfrak{gl}_N) \) via this isomorphism.

For any \( N \)-tuple \( \varsigma = (\varsigma_1, \ldots, \varsigma_N) \) with \( \varsigma_i \in \{-1,1\} \) the mapping

\[
t_{ij} \mapsto \varsigma_i \bar{t}_{ij}, \quad \bar{t}_{ij} \mapsto \varsigma_i t_{ij},
\]

defines an automorphism of the algebra \( U_q(\mathfrak{gl}_N) \).

By the Poincaré–Birkhoff–Witt theorem for the \( A_n \) type quantized enveloping algebra, the monomials

\[
e_{k_{N,N-1}} e_{k_{N,N-2}} e_{k_{N-1,N-2}} \cdots e_{k_{N2}} e_{k_{N1}} \cdots e_{k_{21}}
\times t_{l_1}^{t_{11}} \cdots t_{l_N}^{t_{1N}} e_{k_{12}} \cdots e_{k_{1N}} e_{k_{23}} \cdots e_{k_{2N}} \cdots e_{k_{N-1,N}},
\]

where the \( k_{ij} \) run over non-negative integers and the \( l_i \) run over integers, form a basis of the algebra of \( U_q(\mathfrak{gl}_N) \); see [22, 2, Proposition 9.2.2] (this basis corresponds to the reduced decomposition

\[
w_0 = s_{N-1} s_{N-2} s_{N-1} \cdots s_2 s_3 \cdots s_{N-1} \cdots s_{N-1}
\]
of the longest element of the Weyl group). Using the isomorphism (2.9), we may conclude that the monomials

\[
t_{ii} t_{jb} = q^{ij - \delta_{ij}} t_{jb} t_{ii} \quad \text{and} \quad t_{ii} \bar{t}_{jb} = q^{ij - \delta_{ij}} \bar{t}_{jb} t_{ii}.
\]

Let \( U^- \) denote the subalgebra of \( U_q(\mathfrak{gl}_N) \) generated by the elements \( t_{ii} \) with \( i = 1, \ldots, N \) and \( t_{ij} \) with \( 1 \leq j < i \leq N \). Fix a permutation \( \pi \) of the indices \( 2,3,\ldots,N \) and consider a linear ordering \( < \) on the set of elements \( t_{ij} \) with \( 1 \leq j < i \leq N \) such that \( \pi(i) < \pi(k) \) implies \( t_{ij} < t_{kl} \) for all possible \( j \) and \( l \).

**Proposition 2.1.** For the linear ordering defined as above, the ordered monomials of the form

\[
\prod_{i > j} t_{ij}^{k_{ij}} \prod_i t_{ii}^{m_i},
\]

where the \( k_{ij} \) and \( m_i \) are non-negative integers, form a basis of \( U^- \).
Proof. For any non-negative integer $l$ consider the subspace $U_l^-$ of $U^-$ of elements of degree at most $l$ in the generators. Due to the Poincaré–Birkhoff–Witt theorem, a basis of $U_l^-$ is formed by the monomials

$$t_{N,N-1}^{k_N} t_{N-1,N-2}^{k_{N-1}} \cdots t_{11}^{k_1} t_{12}^{m_1} \cdots t_{NN}^{m_N},$$

with the sum of all powers not exceeding $l$. Hence, it will be sufficient to show that the ordered monomials (2.13) span $U^-$. The statement will then follow by counting the number of the ordered monomials of degree not exceeding $l$.

By the defining relations (2.6), we have

$$t_{ia} t_{jb} = t_{jb} t_{ia} + (q - q^{-1}) t_{ja} t_{ib}, \quad t_{ia} t_{ib} = t_{ib} t_{ja}$$

for $i > j > a > b$, while

$$t_{ia} t_{ab} = t_{ab} t_{ia} + (q - q^{-1}) t_{aa} t_{ib}, \quad t_{ia} t_{ib} = q t_{ib} t_{ia}$$

for $i > a > b$. Given a monomial $t_{i_1 a_1} \cdots t_{i_p a_p}$ with $i_r > a_r$ for all $r$, an easy induction on the degree $p$ shows that modulo elements of degree < $p$, this monomial can be written as a linear combination of monomials of the form $t_{j_1 b_1} \cdots t_{j_p b_p}$ where $\pi(j_1) \leq \cdots \leq \pi(j_p)$. By the second relation in (2.14), this monomial coincides with an ordered monomial up to a factor which is a power of $q$. The proof is completed by taking the first relation in (2.12) into account.

We shall also use an extended quantized enveloping algebra $\hat{U}_q(\mathfrak{gl}_N)$. This is an associative algebra generated by elements $t_{ij}$ and $\bar{t}_{ij}$ with $1 \leq i, j \leq N$ and elements $t_{ii}^{-1}$ and $\bar{t}_{ii}^{-1}$ with $1 \leq i \leq N$ subject to the relations

$$t_{ij} = \bar{t}_{ji} = 0, \quad 1 \leq i < j \leq N,$$

$$t_{ii} \bar{t}_{ii} = \bar{t}_{ii} t_{ii}, \quad t_{ii} t_{ii}^{-1} = t_{ii}^{-1} t_{ii} = 1, \quad \bar{t}_{ii} \bar{t}_{ii}^{-1} = \bar{t}_{ii}^{-1} \bar{t}_{ii} = 1, \quad 1 \leq i \leq N,$$

$$RT_1 T_2 = T_2 T_1 R, \quad R\overline{T}_1 \overline{T}_2 = \overline{T}_2 \overline{T}_1 R, \quad R\overline{T}_1 T_2 = T_2 \overline{T}_1 R,$$

where we use the notation of (2.4). Obviously, we have a natural epimorphism $\hat{U}_q(\mathfrak{gl}_N) \rightarrow U_q(\mathfrak{gl}_N)$ which takes the generators $t_{ij}$ and $\bar{t}_{ij}$ of $\hat{U}_q(\mathfrak{gl}_N)$ respectively to the elements of $U_q(\mathfrak{gl}_N)$ with the same name. More generally, for any nonzero complex numbers $\rho_i$ with $i = 1, \ldots, N$ the mapping

$$\varrho : t_{ij} \mapsto \rho_i t_{ij}, \quad \bar{t}_{ij} \mapsto \rho_i \bar{t}_{ij}$$

(2.15)

defines an epimorphism $\hat{U}_q(\mathfrak{gl}_N) \rightarrow U_q(\mathfrak{gl}_N)$. For any $i \in \{1, \ldots, N\}$ the element $t_{ii} \bar{t}_{ii}$ belongs to the center of $\hat{U}_q(\mathfrak{gl}_N)$ while $t_{ii} \bar{t}_{ii} - \rho_i^2$ is contained in the kernel of $\varrho$. 

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Let $\hat{U}^0$ denote the (commutative) subalgebra of $\hat{U}_q(\mathfrak{g}_N)$ generated by the elements $t_{ii}, t_{ii}^{-1}, \bar{t}_{ii}$ and $\bar{t}_{ii}^{-1}$ with $i = 1, \ldots, N$, and let $\hat{U}^-$ denote the subalgebra of $\hat{U}_q(\mathfrak{g}_N)$ generated by $\hat{U}^0$ and all elements $t_{ij}$ with $i > j$.

Fix a permutation $\pi$ of the indices 2, 3, ..., $N$ and consider a linear ordering $\prec$ on the set of elements $t_{ij}$ with $1 \leq j < i \leq N$ such that $\pi(i) < \pi(k)$ implies $t_{ij} \prec t_{kl}$ for all possible $j$ and $l$.

**Corollary 2.2.** The subalgebra $\hat{U}^0$ is isomorphic to the algebra of Laurent polynomials in the variables $t_{ii}$ and $\bar{t}_{ii}$. Moreover, for the linear ordering defined as above, the ordered monomials in the elements $t_{ij}$ with $1 \leq j < i \leq N$, form a basis of the right $\hat{U}^0$-module $\hat{U}^-$. 

**Proof.** Obviously, the subalgebra $\hat{U}^0$ is spanned by the Laurent monomials in the elements $t_{ii}$ and $\bar{t}_{ii}$. We need to verify that the monomials are linearly independent. Suppose that

$$\sum_{m,l} c_{m,l} t_{11}^{m_1} t_{11}^{-l_1} \cdots t_{NN}^{m_N} t_{NN}^{-l_N} = 0,$$

summed over $N$-tuples of integers $m = (m_1, \ldots, m_N)$ and $l = (l_1, \ldots, l_N)$, where only a finite number of the coefficients $c_{m,l} \in \mathbb{C}$ is nonzero. Apply an epimorphism of the form (2.13) to both sides of this relation. This gives a relation in $U^0$,

$$\sum_{m,l} c_{m,l} \rho_1^{m_1+l_1} \cdots \rho_N^{m_N+l_N} t_{11}^{m_1-l_1} \cdots t_{NN}^{m_N-l_N} = 0.$$

Since the monomials $t_{11}^{r_1} \cdots t_{NN}^{r_N}$ with $r_i \in \mathbb{Z}$ are linearly independent, we get

$$\sum_{m,l} c_{m,l} \rho_1^{m_1+l_1} \cdots \rho_N^{m_N+l_N} = 0,$$

for any fixed integer differences $m_i - l_i$ with $i = 1, \ldots, N$. Varying the values of the parameters $\rho_i$ we conclude that $c_{m,l} = 0$ for all $m$ and $l$. This proves the first part of the corollary.

Arguing as in the proof of Proposition 2.1 we obtain that the ordered monomials in the elements $t_{ij}$ with $i > j$ span $\hat{U}^-$ as a right $\hat{U}^0$-module. It remains to show that the ordered monomials are linearly independent over $\hat{U}^0$. Suppose that a linear combination of the ordered monomials is zero,

$$\sum_{k} t_{i_1a_1}^{k_{i_1a_1}} \cdots t_{i_ma_m}^{k_{i_ma_m}} d_k = 0, \quad d_k \in \hat{U}^0,$$

where $\{(i_1a_1), \ldots, (i_ma_m)\} = \{(2,1), (3,1), (3,2), \ldots, (N,N-1)\}$ and $k$ runs over a finite set of tuples of non-negative integers $k_{i,a}$. Apply an epimorphism of the form...
to both sides of this relation. By Proposition 2.1, we get $\rho(d_k) = 0$ for all $k$.

As in the first part of the proof, varying the parameters $\rho_i$ we conclude that $d_k = 0$
for all $k$.

Note that a similar argument can be used to demonstrate that the ordered monomials of the form

$$t_{i_1a_1}^{k_1} \cdots t_{i_na_m}^{k_m} t_{a_m}^{k_{a_1}} \cdots t_{a_1}^{k_1}$$

form a basis of the left or right $\hat{\mathfrak{U}}_\varnothing(q_{\mathfrak{g}}_{\mathfrak{l}}_N)$.

Now we reproduce some results from [18] and [19] concerning the twisted quantized enveloping algebra $U'_q(\mathfrak{sp}_{2n})$. This is an associative algebra generated by elements $s_{ij}$, $i, j \in \{1, \ldots, 2n\}$ and $s_{-1}^{-1}$, $i = 1, 3, \ldots, 2n - 1$. The generators $s_{ij}$ are zero for $j = i + 1$ with even $i$, and for $j \geq i + 2$ and all $i$. We combine the $s_{ij}$ into a matrix $S$ as in (2.5),

$$S = \sum_{i,j} s_{ij} \otimes E_{ij}, \quad (2.16)$$

so that $S$ has a block-triangular form with $n$ diagonal $2 \times 2$-blocks,

$$S = \begin{pmatrix}
  s_{11} & s_{12} & 0 & 0 & \cdots & 0 & 0 \\
  s_{21} & s_{22} & 0 & 0 & \cdots & 0 & 0 \\
  s_{31} & s_{32} & s_{33} & s_{34} & \cdots & 0 & 0 \\
  s_{41} & s_{42} & s_{43} & s_{44} & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  s_{2n-1,1} & s_{2n-1,2} & s_{2n-1,3} & s_{2n-1,4} & \cdots & s_{2n-1,2n-1} & s_{2n-1,2n} \\
  s_{2n,1} & s_{2n,2} & s_{2n,3} & s_{2n,4} & \cdots & s_{2n,2n-1} & s_{2n,2n}
\end{pmatrix}.$$

Consider the transposed to the $R$-matrix (2.2) given by

$$R' = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i < j} E_{ji} \otimes E_{ji}, \quad (2.17)$$

The defining relations of $U'_q(\mathfrak{sp}_{2n})$ have the form of a reflection equation

$$RS_1R'R_2 = S_2R'S_1R, \quad (2.18)$$

together with

$$s_{i,i+1}^{-1}s_{i,i+1}^{-1} = s_{i,i+1}^{-1}s_{i,i+1}^{-1} = 1 \quad (2.19)$$

and

$$s_{i+1,i+1}s_{i} - q^2 s_{i+1,i}s_{i,i+1} = q^3 \quad (2.20)$$

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for \( i = 1, 3, \ldots, 2n - 1 \). In terms of the generators \( s_{ij} \) the relation (2.18) is written as

\[
q^\delta_{a,j} s_{ia} s_{jb} - q^\delta_{a,b} s_{ia} s_{jb} = (q - q^{-1}) q^\delta_{ai} (\delta_{b<ia} - \delta_{i<j}) s_{ja} s_{ib} \tag{2.21}
\]

\[
+ (q - q^{-1}) (q^\delta_{ab} \delta_{b<ia} s_{ji} s_{ba} - q^\delta_{ij} \delta_{a<j} s_{ij} s_{ab})
\]

\[
+ (q - q^{-1})^2 (\delta_{b<ia} - \delta_{a<i<j}) s_{ji} s_{ib},
\]

where \( \delta_{i<j} \) or \( \delta_{i<j<k} \) equals 1 if the subindex inequality is satisfied and 0 otherwise.

For any \( 2n \)-tuple \( \varsigma = (\varsigma_1, \ldots, \varsigma_{2n}) \) with \( \varsigma_i \in \{-1, 1\} \) the mapping

\[
s_{ij} \mapsto \varsigma_i \varsigma_j s_{ij}, \tag{2.22}
\]

defines an automorphism of the algebra \( U'_q(\mathfrak{sp}_{2n}) \). This is verified directly from the defining relations of the algebra.

Introduce the \( 2n \times 2n \) matrix \( G \) by

\[
G = q \sum_{k=1}^{n} E_{2k-1,2k} - \sum_{k=1}^{n} E_{2k,2k-1} \tag{2.23}
\]

so that

\[
G = \begin{pmatrix}
0 & q & \cdots & 0 & 0 \\
-1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & q \\
0 & 0 & \cdots & -1 & 0 
\end{pmatrix}.
\]

The mapping

\[
S \mapsto T G T^t \tag{2.24}
\]

defines a homomorphism \( U'_q(\mathfrak{sp}_{2n}) \to U_q(\mathfrak{gl}_{2n}) \). Explicitly, in terms of generators it is written as

\[
s_{ij} \mapsto q \sum_{k=1}^{n} t_{i,2k-1} \bar{t}_{j,2k} - \sum_{k=1}^{n} t_{i,2k} \bar{t}_{j,2k-1}. \tag{2.25}
\]

When \( q \) is regarded as a variable, the mapping (2.24) is an embedding of the algebras over the field \( \mathbb{C}(q) \); see [13, Theorem 2.8]. This is implied by the fact that the algebra \( U'_q(\mathfrak{sp}_{2n}) \) specializes to \( U(\mathfrak{sp}_{2n}) \) as \( q \to 1 \). More precisely, set \( \mathcal{A} = \mathbb{C}[q, q^{-1}] \) and consider the \( \mathcal{A} \)-subalgebra \( U'_q(\mathfrak{sp}_{2n}) \) of \( U'_q(\mathfrak{sp}_{2n}) \) generated by the elements

\[
\sigma_{i,i+1} = \frac{s_{i,i+1} - q}{q - 1}, \quad \sigma_{i+1,i} = \frac{s_{i+1,i} + 1}{q - 1}, \quad i = 1, 3, \ldots, 2n - 1,
\]

and

\[
\sigma_{ij} = \frac{s_{ij}}{q - q^{-1}}, \quad i \geq j, \quad \text{excluding } i = j + 1, j \text{ odd.}
\]
Then we have an isomorphism
\[ U'_A \otimes_A \mathbb{C} \cong \mathbb{U}(\mathfrak{sp}_{2n}), \]  
(2.26)
where the action of \( A \) on \( \mathbb{C} \) defined via the evaluation \( q = 1 \); see [13]. The symplectic Lie algebra \( \mathfrak{sp}_{2n} \) is defined as the subalgebra of \( \mathfrak{gl}_{2n} \) spanned by the elements
\[ F_{ij} = \sum_{k=1}^{2n} (E_{ik} g_{kj} + E_{jk} g_{ki}), \]
where the \( g_{ij} \) are the matrix elements of the matrix \( G^\circ = [g_{ij}] \) obtained by evaluating \( G \) at \( q = 1 \),
\[ G^\circ = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}. \]
The images of the elements \( \sigma_{ij} \) under the isomorphism (2.26) are respectively the elements \( F_{ij} \) of \( \mathfrak{sp}_{2n} \).

In the next section we prove that if \( q \) is specialized to a nonzero complex number such that \( q^2 \neq 1 \), then the mapping (2.24) defines an embedding of the respective algebras over \( \mathbb{C} \).

3 Poincaré–Birkhoff–Witt theorem

Define the extended twisted quantized enveloping algebra \( \hat{\mathbb{U}}'_q(\mathfrak{sp}_{2n}) \) as follows. This is an associative algebra generated by elements \( s_{ij}, i,j \in \{1, \ldots, 2n\} \) and \( s_{i,i+1}^{-1} \), \( i = 1, 3, \ldots, 2n - 1 \), where \( s_{ij} = 0 \) for \( j = i + 1 \) with even \( i \), and for \( j \geq i + 2 \) and all \( i \).
The defining relations are given by (2.19) and (2.21). As with the algebra \( \mathbb{U}'_q(\mathfrak{sp}_{2n}) \), we combine the \( s_{ij} \) into a matrix \( S \) which a block-triangular form with \( n \) diagonal \( 2 \times 2 \)-blocks.

Consider the algebra \( \hat{\mathbb{U}}_q(\mathfrak{gl}_{2n}) \) introduced in the previous section, and denote by \( \hat{\mathbb{U}}_q(\mathfrak{gl}_{2n}) \) its quotient by the central elements \( t_{ii} \bar{t}_{ii} - 1 \) for all even \( i \). We keep the same notation for the images of the generators of \( \hat{\mathbb{U}}_q(\mathfrak{gl}_{2n}) \) in \( \hat{\mathbb{U}}_q(\mathfrak{gl}_{2n}) \). The mapping given by
\[ \psi : S \mapsto T G \bar{T}^t \]  
(3.1)
defines a homomorphism \( \hat{\mathbb{U}}'_q(\mathfrak{sp}_{2n}) \rightarrow \hat{\mathbb{U}}_q(\mathfrak{gl}_{2n}) \). This is verified by the same calculation as with the homomorphism (2.24); see [19] and [18]. In particular,
\[ s_{i,i+1} \mapsto q t_{ii} \bar{t}_{i+1,i+1} \quad \text{and} \quad s_{i,i+1}^{-1} \mapsto q^{-1} t_{ii}^{-1} \bar{t}_{i+1,i+1} \]
for \( i = 1, 3, \ldots, 2n - 1 \).

**Theorem 3.1.** The mapping (3.1) defines an embedding \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \to \tilde{U}_q(\mathfrak{gl}_{2n}) \).

**Proof.** We only need to show that the kernel of the homomorphism (3.1) is zero. We shall use a weak form of the Poincaré–Birkhoff–Witt theorem for the algebra \( \tilde{U}_q' \) provided by the following lemma.

**Lemma 3.2.** The algebra \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \) is spanned by the monomials

\[
\begin{align*}
& s_{2n,1}^{k_{2n,1}} s_{2n,2}^{k_{2n,2}} s_{2n-1,1}^{k_{2n-1,1}} s_{2n-2,1}^{k_{2n-2,1}} \cdots \times
\end{align*}
\]

\[
\begin{align*}
& s_{2n,1}^{k_{2n,2}} s_{2n,2}^{k_{2n-1,2}} s_{2n-1,2}^{k_{2n-3,1}} s_{2n-3,1}^{k_{2n-3,2}} \cdots s_{2n,2n-1}^{k_{2n,2n}} s_{2n,2n-2}^{k_{2n,2n-2}} \cdots s_{2n,1}^{k_{2n,1}} s_{22}^{k_{22}}, 
\end{align*}
\]

(3.2)

where \( k_{12}, k_{34}, \ldots, k_{2n-1,2n} \) run over all integers while the remaining \( k_{ij} \) run over non-negative integers.

**Proof.** We follow the argument of [18], where a similar statement was proved (see Lemma 2.1 there). We shall be proving that any monomial in the generators can be written as a linear combination of monomials of the form (3.2). The defining relations (2.21) imply that

\[
q^{-\delta_{i,k}+\delta_{i+1,k}} s_{i,i+1} s_{i,i+1} = q^{\delta_{i,k}-\delta_{i+1,k}} s_{i,i+1} s_{i,i+1},
\]

(3.3)

for any \( i = 1, 3, \ldots, 2n - 1 \). Therefore, it suffices to consider monomials where the generators \( s_{i2}, \ldots, s_{2n-1,2n} \) occur with non-negative powers. For any monomial

\[
s_{i_1 a_1} \cdots s_{i_p a_p}
\]

(3.4)

we introduce its weight \( w \) by \( w = i_1 + \cdots + i_p \). We shall use induction on \( w \). The defining relations (2.21) for \( \tilde{U}_q' \) imply that, modulo products of weight less than \( i + j \), we have

\[
q^{\delta_{i+j,j}} s_{i a} s_{j b} s_{i a} = q^{\delta_{i+j,b}} s_{j b} s_{i a} + (q - q^{-1}) q^{\delta_{i+1,1}} (\delta_{b < a} - \delta_{i < j}) s_{j a} s_{i b}.
\]

(3.5)

Let \( \pi \) denote the permutation of the indices 1, 2, \ldots, 2n such that \( \pi^{-1} \) is given by \( (2n, 2n-2, \ldots, 2, 2n-1, 2n-3, \ldots, 1) \). Relation (3.5) allows us to represent (3.1), modulo monomials of weight less than \( w \), as a linear combination of monomials \( s_{j_1 b_1} \cdots s_{j_p b_p} \) of weight \( w \) such that \( \pi(j_1) \leq \cdots \leq \pi(j_p) \). Consider a sub-monomial \( s_{i c_1} \cdots s_{i c_r} \) containing generators with the same first index. By (3.5) we have

\[
s_{i a} s_{i b} = q^{\delta_{i,b}-\delta_{i,a}+1} s_{i b} s_{i a}
\]

(3.6)

for \( a > b \). Using this relation repeatedly we bring the sub-monomial to the required form.  

\( \square \)
Lemma 3.3. The monomials

\[ k_{2n,1} s_{2n,1} \ldots k_{2n,2n-2} s_{2n,2n-2} k_{2n,2n} s_{2n,2n} \times k_{2n-1,1} \ldots s_{2n-1,2n-2} \ldots s_{31} k_{32} \times k_{2n,2n-1} s_{2n-1,2n} \ldots s_{21} s_{12} s_{21} s_{2n-1,2n-1} \ldots s_{11}, \tag{3.7} \]

where \( k_{12}, k_{34}, \ldots, k_{2n-1,2n} \) run over all integers while the remaining \( k_{ij} \) run over non-negative integers, are linearly independent in \( \check{U}_q(\mathfrak{sp}_{2n}) \).

Proof. Let \( \mu_i \) and \( \bar{\mu}_i \) with \( i = 1, \ldots, 2n \) be arbitrary nonzero complex numbers such that \( \mu_i = \bar{\mu}_i = 1 \) for all even \( i \). Consider the corresponding Verma module \( M(\mu, \bar{\mu}) \) over the algebra \( \check{U}_q(\mathfrak{gl}_{2n}) \) which is defined as the quotient of \( \check{U}_q(\mathfrak{gl}_{2n}) \) by the left ideal generated by the elements \( t_{ij} \) with \( i < j \) and \( t_{ii} - \mu_i, \bar{t}_{ii} - \bar{\mu}_i \) with \( i = 1, \ldots, 2n \).

Corollary 2.2 implies that the elements

\[ t_{i_1 q_1}^{k_{i_1 q_1}} \ldots t_{i_m q_m}^{k_{i_m q_m}} \xi \]

form a basis of the Verma module \( M(\mu, \bar{\mu}) \), where \( \xi \) denotes its highest vector and the generators \( t_{i_r a_r} \) with \( i_r > a_r \) are written in accordance with a certain linear ordering determined by the permutation \( \pi \) defined in the proof of Lemma 3.2.

Suppose now that a nontrivial linear combination of monomials (3.7) is zero. For any odd \( i \) the image of the power \( s_{ii}^k \) under the homomorphism (3.1) is given by

\[ \psi : s_{ii}^k \mapsto q^{k^2 - k} \bar{t}_{ii}^{k^{ii+1}}. \tag{3.8} \]

Amongst the monomials which occur in the linear combination, pick up a monomial such that each power \( k_{ii} \) takes a minimal possible value \( \kappa_i \) for each odd \( i \). Now take the image of the linear combination under (3.1) and apply this image to the vector

\[ t_{21}^{k_{21}} \ldots t_{2n,2n-1}^{k_{2n,2n-1}} \xi \in M(\mu, \bar{\mu}). \]

By the choice the parameters \( \kappa_i \), the nonzero contribution to the resulting expression can only come from the monomials (3.7) with \( k_{ii} = \kappa_i \) for all odd \( i \). However, by (3.8)

\[ \psi(s_{ii}^\kappa) t_{i+1,i}^{k_{i+1,i}} \xi = c_i \xi, \]

where \( c_i \) is a constant depending on the \( \mu_i \) and \( \bar{\mu}_j \) which can easily be calculated. Choosing the parameters \( \mu_j \) and \( \bar{\mu}_j \) in such a way that \( c_i \neq 0 \) for each odd \( i \), we conclude that the image under \( \psi \) of a certain nontrivial linear combination of monomials of the form

\[ s_{2n,1}^{k_{2n,1}} \ldots s_{2n,2n-2}^{k_{2n,2n-2}} s_{2n,2n}^{k_{2n,2n}} s_{2n,2n}^{k_{2n,2n}} \times s_{2n-1,1}^{k_{2n-1,1}} \ldots s_{2n-1,2n-2}^{k_{2n-1,2n-2}} \ldots s_{31}^{k_{31}} s_{32}^{k_{32}} \times s_{2n,2n-1}^{k_{2n,2n-1}} s_{2n-1,2n}^{k_{2n-1,2n}} \ldots s_{21}^{k_{21}} s_{12}^{k_{12}} \tag{3.9} \]
acts as zero when applied to the highest vector $\xi$ of the Verma module $M(\mu, \bar{\mu})$. As the image of $s_{ij}$ under the homomorphism $\psi$ is given by (2.25), we find that

$$\psi(s_{i,i+1}) \xi = q \mu_i \xi, \quad \psi(s_{i+1,i}) \xi = -\bar{\mu}_i \xi$$

for any odd $i$. Moreover, using (2.25) again, we come to the formulas

$$\psi(s_{ij}) \xi = \begin{cases} qt_{i,j-1} \xi, & \text{if } j \text{ is even, } i \geq j, \\ -\bar{\mu}_j t_{i,j+1} \xi, & \text{if } j \text{ is odd, } i \geq j + 2. \end{cases}$$

Varying the parameters $\mu_i$ and $\bar{\mu}_i$, we conclude that the elements $t_{k_{2n,1}}^{k_{2n,1}} \ldots t_{2n,2}^{k_{2n,2}} t_{2n,2n-2}^{k_{2n,2n-2}} \ldots t_{2n,2n-1}^{k_{2n,2n-1}} t_{42}^{k_{42}} t_{41}^{k_{41}} t_{21}^{k_{22}} \times t_{2n-1,2}^{k_{2n-1,1}} \ldots t_{2n-1,2n-3}^{k_{2n-1,2n-2}} \ldots t_{32}^{k_{32}} t_{31}^{k_{32}} (3.10)$$

of the algebra $\hat{U}_q(\mathfrak{gl}_{2n})$ are linearly dependent. This contradicts Corollary 2.2, completing the proof of the lemma.

Now, let us denote by $\hat{U}_q^+$ the subalgebra of $\hat{U}_q'(\mathfrak{sp}_{2n})$ generated by the elements $s_{i,i+1}$ for all odd $i$ and $s_{ij}$ for all $i \geq j$. For any non-negative integer $m$ consider the subspace $\hat{U}_q^+_m$ of $\hat{U}_q^+$ of elements of degree at most $m$ in the generators. Lemma 3.2 implies that $\hat{U}_q^+_m$ is spanned by the monomials (3.2) of total degree $\leq m$ with non-negative powers of the generators. On the other hand, by Lemma 3.3 the monomials (3.7) of total degree $\leq m$ with non-negative powers of the generators are linearly independent. Since the numbers of both types of monomials of total degree $\leq m$ coincide, we conclude that each of these families of monomials forms a basis of $\hat{U}_q^+_m$. This implies that each family of monomials (3.2) and (3.7) forms a basis of $\hat{U}_q'(\mathfrak{sp}_{2n})$. In particular, any element $u$ of $\hat{U}_q'(\mathfrak{sp}_{2n})$ can be written as linear combination of monomials (3.7). However, the proof of Lemma 3.3 shows that $\psi(u) = 0$ implies $u = 0$, thus proving that the kernel of $\psi$ is zero.

The following version of the Poincaré–Birkhoff–Witt theorem for the algebra $\hat{U}_q'(\mathfrak{sp}_{2n})$ was already noted in the proof of Theorem 3.1.

**Corollary 3.4.** Each family of monomials (3.2) and (3.7) constitutes a basis of the algebra $\hat{U}_q'(\mathfrak{sp}_{2n})$. □

For any $i = 1, 3, \ldots, 2n - 1$ set

$$\vartheta_i = s_{i+1,i+1} s_{i,i} - q^2 s_{i+1,i} s_{i,i+1}.$$

As was observed in [18, Section 2.2], the elements $\vartheta_i$ belong to the center of $\hat{U}_q'(\mathfrak{sp}_{2n})$. 

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Proposition 3.5. The elements

\[ s_{2n,1} \ldots s_{2n,2n-2} s_{2n,2n} \varrho_{k_{2n-2,1}} \varrho_{k_{2n-2,1,2n-1}} \]
\[ \times \cdots \times s_{41}^k s_{42}^k s_{44}^k s_{34}^3 s_{33}^3 s_{22}^k \varrho_{k_{12}}^1 \varrho_{k_{11}}^1 \]
\[ \times s_{31}^k s_{32}^k \cdots s_{2n-1,1}^k s_{2n-1,2n-2}^k, \] (3.11)

where the \( k_{i,i+1} \) for odd \( i \) run over all integers while \( k_i \) and the remaining \( k_{rj} \) run over non-negative integers, form a basis of the algebra \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \).

Proof. First, we prove that a basis of \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \) is comprised by the monomials

\[ s_{2n,1}^k \ldots s_{2n,2n-2}^k s_{2n,2n}^k \varrho_{k_{2n-2,1}}^k \varrho_{k_{2n-2,1,2n-1}}^k \]
\[ \times \cdots \times s_{41}^{k_1} s_{42}^{k_2} s_{44}^{k_4} s_{34}^{k_3} s_{33}^{k_3} s_{22}^k \varrho_{k_{12}}^1 \varrho_{k_{11}}^1 \]
\[ \times s_{31}^{k_1} s_{32}^{k_2} \cdots s_{2n-1,1}^{k_1} s_{2n-1,2n-2}^{k_1}, \] (3.12)

where the \( k_{i,i+1} \) for odd \( i \) are integers and the remaining \( k_{rj} \) are non-negative integers. Indeed, the argument used in the proof of Lemma 3.2 together with (3.3) implies that monomials (3.12) span the algebra \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \); the only additional observation required is that

\[ s_{ii} s_{jb} = s_{jb} s_{ii} \] (3.13)

for all odd \( i \) such that \( i \geq j \geq b \); see (2.21). Then, as in the proof of Theorem 3.1, we conclude that monomials (3.12) form a basis of \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \).

Now we show that the elements (3.11) span the algebra \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \). Since the defining relations between the generators \( s_{ii}, s_{i,i+1}, s_{i+1,i} \) and \( s_{i+1,i+1} \) do not involve any other generators, it is sufficient to consider the particular case \( n = 1 \). We have

\[ s_{22}^k s_{21}^l s_{12}^m s_{11}^r = -q^{-2} s_{22}^k \varrho_{1}^l s_{21}^{m-1} s_{11}^r + \sum_{a=1}^{l} c_a s_{22}^{k+1} s_{21}^{a-1} s_{12}^{m+l-a-1} s_{11}^{r+1} \]

for some complex coefficients \( c_a \). Arguing by the induction on \( l \), we find that

\[ s_{22}^k s_{21}^l s_{12}^m s_{11}^r = (-1)^l q^{-2l} s_{22}^{k'} \varrho_{1}^{l'} s_{12}^{m-l} s_{11}^r \]
\[ + \text{ a linear combination of } s_{22}^{k'} \varrho_{1}^{l'} s_{12}^m s_{11}^{r'} \text{ with } l' < l. \] (3.14)

Thus, the elements (3.11) span \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \). The relations (3.14) can obviously be inverted to get similar expressions of the elements \( s_{22}^k \varrho_{1}^{l'} s_{12}^m s_{11}^{r'} \) in terms of monomials \( s_{22}^k s_{21}^l s_{12}^m s_{11}^r \). This implies that the elements (3.11) are linearly independent. \( \Box \)
By the definition of the algebra $U'_q(\mathfrak{sp}_{2n})$, we have a surjective homomorphism $\tilde{U}'_q(\mathfrak{sp}_{2n}) \to U'_q(\mathfrak{sp}_{2n})$ which takes the generators $s_{ij}$ to the elements of $U'_q(\mathfrak{sp}_{2n})$ with the same name. In other words, we have an isomorphism

$$U'_q(\mathfrak{sp}_{2n}) \cong \tilde{U}'_q(\mathfrak{sp}_{2n})/I,$$

where $I$ is the ideal of $\tilde{U}'_q(\mathfrak{sp}_{2n})$ generated by the central elements $\vartheta_i - q^3$ for all odd $i = 1, 3, \ldots, 2n - 1$. Hence, the following Poincaré–Birkhoff–Witt theorem for the algebra $U'_q(\mathfrak{sp}_{2n})$ follows from Proposition 3.5 and the relations (3.3) and (3.13).

**Theorem 3.6.** The elements

$$s_{2n,1}^{k_{2n,1}} \cdots s_{2n,2n-2}^{k_{2n,2n-2}} s_{2n,2n}^{k_{2n,2n}} \cdots s_{41}^{k_{41}} s_{42}^{k_{42}} s_{44}^{k_{44}} s_{22}^{k_{22}} \times s_{11}^{k_{11}} s_{12}^{k_{12}} s_{31}^{k_{31}} s_{32}^{k_{32}} s_{33}^{k_{33}} s_{34}^{k_{34}} \cdots s_{2n-1,1}^{k_{2n-1,1}} \cdots s_{2n-1,2n}^{k_{2n-1,2n}}, \quad (3.15)$$

where the $k_{i,i+1}$ for odd $i$ run over all integers and the remaining $k_{rj}$ run over non-negative integers, form a basis of the algebra $U'_q(\mathfrak{sp}_{2n})$.

**Corollary 3.7.** The mapping (3.1) defines an embedding $U'_q(\mathfrak{sp}_{2n}) \hookrightarrow U_q(\mathfrak{gl}_{2n})$.

**Proof.** By Theorem 3.1, the algebra $\tilde{U}'_q(\mathfrak{sp}_{2n})$ can be identified with a subalgebra of $\tilde{U}_q(\mathfrak{gl}_{2n})$. Then, for each $i = 1, 3, \ldots, 2n - 1$ the relation $\vartheta_i = q^3$ is equivalent to $t_{ii}t_{ii} = 1$. However, the quotient of $\tilde{U}'_q(\mathfrak{sp}_{2n})$ by the relations $t_{ii}t_{ii} = 1$ for all odd $i$ is isomorphic to $U_q(\mathfrak{gl}_{2n})$. Hence, the claim follows from Theorem 3.6.

Using Corollary 3.7, we shall regard $U'_q(\mathfrak{sp}_{2n})$ as a subalgebra of $U_q(\mathfrak{gl}_{2n})$.

**Remark 3.8.** An analogue of the Poincaré–Birkhoff–Witt theorem for the algebra $U'_q(\mathfrak{o}_N)$ was proved in [9] with the use of the Diamond Lemma. If $U'_q(\mathfrak{o}_N)$ is regarded as an algebra over $\mathbb{C}(q)$, the theorem can also be proved by a specialization argument; see [18, Corollary 2.3]. If $q$ is a nonzero complex number such that $q^2 \neq 1$, a proof can be given in a way similar to the above arguments with some simplifications. Indeed, recall that $U'_q(\mathfrak{o}_N)$ is generated by elements $s_{ij}$ with the defining relations

$$s_{ij} = 0, \quad 1 \leq i < j \leq N,$$

$$s_{ii} = 1, \quad 1 \leq i \leq N,$$

$$RS_1R'S_2 = S_2R'S_1R,$$

using the same notation as for the symplectic case. The monomials

$$s_{21}^{k_{21}} s_{31}^{k_{31}} s_{32}^{k_{32}} \cdots s_{N1}^{k_{N1}} s_{N2}^{k_{N2}} \cdots s_{N,N-1}^{k_{N,N-1}} \quad (3.16)$$
span the algebra $U'_q(o_N)$; see [18] Lemma 2.1. In order to prove the linear independence of these monomials, consider their images under the homomorphism $\phi : U'_q(o_N) \to U_q(\mathfrak{g}l_N)$ defined by $S \mapsto \overline{T T T}$. Consider the Verma module $M(\mu)$ over $U_q(\mathfrak{g}l_N)$ with $\mu = (1, \ldots, 1)$ and the highest vector $\xi$ so that

\[
\bar{t}_{ij} \xi = 0, \quad i < j \quad \text{and} \quad t_{ii} \xi = 1, \quad i = 1, \ldots, N.
\]

Applying the image of the monomial (3.16) to the highest vector $\xi$ we get

\[
t_{21}^{k_2} t_{31}^{k_3} \cdots t_{N1}^{k_N} t_{N2}^{k_{N-1}} \cdots t_{N,N-1}^{k_{N,N-1}} \xi.
\]

The proof is completed by using Proposition 2.1. This argument also shows that the homomorphism $\phi$ is an embedding.

4 Highest weight representations

Here we introduce highest weight representations of $U'_q(\mathfrak{sp}_{2n})$ and prove that every finite-dimensional irreducible representation of this algebra is highest weight. The arguments are quite standard; cf. [8], [2, Chapter 10]. From now on, $q$ will denote a complex number which is nonzero and not a root of unity.

We shall be using the following notation. For any two $n$-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ we shall denote by $\alpha \cdot \beta$ the $n$-tuple $(\alpha_1\beta_1, \ldots, \alpha_n\beta_n)$. Also, $q^\alpha$ will denote the $n$-tuple $(q^{\alpha_1}, \ldots, q^{\alpha_n})$ in the case where $\alpha$ is an $n$-tuple of integers.

A representation $V$ of $U'_q(\mathfrak{sp}_{2n})$ will be called a highest weight representation if $V$ is generated by a nonzero vector $v$ such that

\[
\begin{align*}
s_{ij} v &= 0 \quad \text{for} \quad i = 1, 3, \ldots, 2n - 1, \quad j = 1, 2, \ldots, i, \\
s_{i,i+1} v &= \lambda_i v \quad \text{for} \quad i = 1, 3, \ldots, 2n - 1,
\end{align*}
\]

for some complex numbers $\lambda_i$. These numbers have to be nonzero due to the relation (2.18). The $n$-tuple $\lambda = (\lambda_1, \lambda_3, \ldots, \lambda_{2n-1})$ will be called the highest weight of $V$. Observe that due to the relations (2.21), the elements $s_{i,i+1}$ with odd $i$ pairwise commute. The commutative subalgebra of $U'_q(\mathfrak{sp}_{2n})$ generated by these elements will play the role of a Cartan subalgebra.

Consider the root system $\Delta$ of type $C_n$ which is the subset of vectors in $\mathbb{R}^n$ of the form

\[
\pm 2 \varepsilon_i \quad \text{with} \quad 1 \leq i \leq n \quad \text{and} \quad \pm \varepsilon_i \pm \varepsilon_j \quad \text{with} \quad 1 \leq i < j \leq n,
\]

where $\varepsilon_i$ denotes the $n$-tuple which has 1 on the $i$-th position and zeros elsewhere. Partition this set into positive and negative roots $\Delta = \Delta^+ \cup (-\Delta^+)$, where the set of...
positive roots $\Delta^+$ consists of the vectors
\[2 \varepsilon_i \quad \text{with} \quad 1 \leq i \leq n \quad \text{and} \quad \varepsilon_i + \varepsilon_j, \quad -\varepsilon_i + \varepsilon_j \quad \text{with} \quad 1 \leq i < j \leq n.
\]

For any $n$-tuple of nonzero complex numbers $\mu = (\mu_1, \mu_3, \ldots, \mu_{2n-1})$ define the corresponding weight subspace of $V$ by
\[V_\mu = \{ w \in V \mid s_{i,i+1} w = \mu_i w \quad \text{for} \quad i = 1, 3, \ldots, 2n - 1 \}.
\]
Any nonzero vector $w \in V_\mu$ is called a weight vector of weight $\mu$.

Given an $n$-tuple of nonzero complex numbers $\lambda = (\lambda_1, \lambda_3, \ldots, \lambda_{2n-1})$, the corresponding Verma module $M(\lambda)$ over $U'_q(\mathfrak{sp}_{2n})$ is defined as the quotient of $U'_q(\mathfrak{sp}_{2n})$ by the left ideal generated by the elements
\[s_{ij} \quad \text{with} \quad i = 1, 3, \ldots, 2n - 1, \quad j = 1, 2, \ldots, i \quad \text{(4.1)}
\]
and
\[s_{i,i+1} - \lambda_i \quad \text{with} \quad i = 1, 3, \ldots, 2n - 1.
\]
The Verma module is obviously a highest weight representation. The image $\xi$ of the element $1 \in U'_q(\mathfrak{sp}_{2n})$ in $M(\lambda)$ is the highest vector and $\lambda$ is the highest weight. By the Poincaré–Birkhoff–Witt theorem for the algebra $U'_q(\mathfrak{sp}_{2n})$ (see Theorem 3.6), a basis of $M(\lambda)$ is comprised by the elements
\[\prod_{i=2,4,\ldots,2n}^{k_{i}} s_{i1}^{k_{i}} s_{i2}^{k_{i}} \cdots s_{i,i-2}^{k_{i}} s_{i,i}^{k_{i}} \xi,
\]
where the $k_{k_{i}}$ run over non-negative integers. Due to (3.3), we have the weight space decomposition
\[M(\lambda) = \bigoplus_{\mu} M(\lambda)_\mu.
\]
The weight subspace $M(\lambda)_\mu$ is nonzero if and only if $\mu$ has the form $\mu = q^{-\omega} \cdot \lambda$, where $\omega$ is a linear combination of elements of $\Delta^+$ with non-negative integer coefficients.
The dimension of $M(\lambda)_\mu$ is given by the same formula as in the classical case; see e.g. [8]. In particular, the weight space $M(\lambda)_\lambda$ is one-dimensional and spanned by $\xi$.

Every highest weight module $V$ with the highest weight $\lambda$ is a homomorphic image of $M(\lambda)$. So, $V$ is the direct sum of its weight subspaces, $V = \bigoplus V_\mu$. By a standard argument, $M(\lambda)$ contains a unique maximal submodule which does not contain the vector $\xi$. The quotient of $M(\lambda)$ by this submodule is, up to an isomorphism, the unique irreducible highest weight module with the highest weight $\lambda$. We shall denote this quotient by $L(\lambda)$.
Proposition 4.1. Every finite-dimensional irreducible representation $V$ of $U'_q(\mathfrak{sp}_{2n})$ is isomorphic to $L(\lambda)$ for a certain highest weight $\lambda$.

Proof. This is verified by a standard argument. We need to show that $V$ contains a weight vector annihilated by all operators (4.1). Since the operators $s_{i,i+1}$ with $i = 1, 3, \ldots, 2n - 1$ on $V$ pairwise commute, the space $V$ must contain a weight vector $w$ of a certain weight $\mu$. If $w$ is not annihilated by the operators (4.1) then applying these operators to $w$ we can get other weight vectors with weights of the form $q^\omega \cdot \mu$, where $\omega$ is a linear combination of elements of $\Delta^+$ with non-negative integer coefficients. As $\dim V < \infty$, the proof is completed by the classical argument; see e.g. [8].

Due to Proposition 4.1, in order to describe the finite-dimensional irreducible representations of $U'_q(\mathfrak{sp}_{2n})$ it suffices to find the necessary and sufficient conditions on $\lambda$ for the representation $L(\lambda)$ to be finite-dimensional. As in the classical theory, the case $n = 1$ plays an important role.

5 Representations of $U'_q(\mathfrak{sp}_2)$

Using (2.20), we can regard $U'_q(\mathfrak{sp}_2)$ as the algebra with generators $s_{11}, s_{22}, s_{12}, s_{12}^{-1}$. The defining relations take the form of (2.19) with $i = 1$ together with

$$s_{11} s_{22} = q^{-2} s_{22} s_{11} - (q - q^{-1})(s_{12}^2 - q^2)$$

and

$$s_{12} s_{11} = q^2 s_{11} s_{12}, \quad s_{12} s_{22} = q^{-2} s_{22} s_{12}.$$  (5.1)

For a nonzero complex number $\lambda$, the corresponding Verma module $M(\lambda)$ has the basis

$$s_{22}^k \xi, \quad k \geq 0.$$  

Using (5.1) and (5.2) we obtain

$$s_{11} s_{22}^k \xi = q^3 (1 - \lambda^2 q^{-2k})(1 - q^{-2k}) s_{22}^{k-1} \xi.$$  

Hence, the module $M(\lambda)$ is reducible if and only if $\lambda = \sigma q^m$ for a positive integer $m$ and $\sigma \in \{-1, 1\}$. Thus, we have the following.

Proposition 5.1. The irreducible highest weight module $L(\lambda)$ over $U'_q(\mathfrak{sp}_2)$ is finite-dimensional if and only if $\lambda = \sigma q^m$ for a positive integer $m$ and $\sigma \in \{-1, 1\}$.
In this case $L(\lambda)$ has a basis $\{v_k\}, k = 0, 1, \ldots, m - 1$, with the action of $U'_q(\mathfrak{sp}_2)$ given by

$$s_{12} v_k = \sigma q^{m-2k} v_k,$$
$$s_{22} v_k = v_{k+1},$$
$$s_{11} v_k = q^2 (1 - q^{2m-2k})(1 - q^{-2k}) v_{k-1},$$

where $v_{-1} = v_m = 0$.

Note also that all finite-dimensional irreducible $U'_q(\mathfrak{sp}_2)$-modules can be obtained by restriction from $U_q(\mathfrak{gl}_2)$-modules. Indeed, consider the irreducible highest weight module over $U_q(\mathfrak{gl}_2)$ generated by a vector $w$ satisfying

$${\bar{t}}_{12} w = 0, \quad t_{11} w = \varsigma_1 q^{\mu_1} w, \quad t_{22} w = \varsigma_2 q^{\mu_2} w,$$

where $\varsigma_1, \varsigma_2 \in \{-1, 1\}$ and $\mu_1, \mu_2 \in \mathbb{Z}$. If $\mu_1 - \mu_2 \geq 0$ then this module has dimension $m = \mu_1 - \mu_2 + 1$ and its restriction to the subalgebra $U'_q(\mathfrak{sp}_2)$ is isomorphic to $L(\lambda)$ with $\lambda = \sigma q^m$, where $\sigma = \varsigma_1/\varsigma_2$. This is easily derived by using (2.25).

6 Classification theorem

Consider an arbitrary irreducible highest weight representation $L(\lambda)$ of $U'_q(\mathfrak{sp}_{2n})$. Our first aim is to find necessary conditions on $\lambda = (\lambda_1, \lambda_3, \ldots, \lambda_{2n-1})$ for $L(\lambda)$ to be finite-dimensional. So, suppose that $\dim L(\lambda) < \infty$.

For any index $k = 1, 2, \ldots, n$ the subalgebra of $U'_q(\mathfrak{sp}_{2n})$ generated by the elements $s_{2k-1,2k-1}, s_{2k,2k}, s_{2k-1,2k}$ and $s_{2k-2,2k}$ is isomorphic to $U'_q(\mathfrak{sp}_2)$. The cyclic span of the highest vector $\xi$ of $L(\lambda)$ with respect to this subalgebra is a highest weight module with the highest weight $\lambda_{2k-1}$. The irreducible quotient of this module is finite-dimensional and so, by Proposition 5.1, we must have $\lambda_{2k-1} = \sigma_k q^{m_k}$ for some positive integer $m_k$ and $\sigma_k \in \{-1, 1\}$. Thus, the highest weight $\lambda$ of $L(\lambda)$ must have the form

$$\lambda = (\sigma_1 q^{m_1}, \ldots, \sigma_n q^{m_n}).$$

Consider the composition of the action of $U'_q(\mathfrak{sp}_{2n})$ on $L(\lambda)$ with the automorphism (2.22), where

$$\varsigma_i = \sigma_i, \quad i = 1, 3, \ldots, 2n - 1 \quad \text{and} \quad \varsigma_i = 1, \quad i = 2, 4, \ldots, 2n.$$

This composition is isomorphic to an irreducible finite-dimensional highest weight module with the highest weight $(q^{m_1}, \ldots, q^{m_n})$. Thus, without loss of generality, we may only consider the modules $L(\lambda)$, where the highest weight has the form $\lambda = (q^{m_1}, \ldots, q^{m_n})$ for some positive integers $m_i$.  

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Now observe that for any $k = 1, 2, \ldots, n - 1$, if we restrict the range of indices of the generators of $U'_q(\mathfrak{sp}_{2n})$ to the subset $\{2k - 1, 2k, 2k + 1, 2k + 2\}$ then the corresponding elements will generate a subalgebra of $U'_q(\mathfrak{sp}_{2n})$ isomorphic to $U'_q(\mathfrak{sp}_4)$. The cyclic span of the highest vector $\xi$ of $L(\lambda)$ with respect to this subalgebra is a highest weight module with the highest weight $(q^{m_k}, q^{m_{k+1}})$. The irreducible quotient of this module is finite-dimensional. Hence, considering irreducible highest modules over $U'_q(\mathfrak{sp}_4)$ we can get necessary conditions on the $m_i$.

The generators of $U'_q(\mathfrak{sp}_4)$ are the nonzero entries of the matrix

$$S = \begin{pmatrix} s_{11} & s_{12} & 0 & 0 \\ s_{21} & s_{22} & 0 & 0 \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}$$

together with the elements $s_{12}^{-1}$ and $s_{34}^{-1}$. The highest vector $\xi$ of $L(\lambda)$ with the highest weight $\lambda = (q^{m_1}, q^{m_2})$ is annihilated by $s_{11}, s_{31}, s_{32}, s_{33}$ and we have

$$s_{12} \xi = q^{m_1} \xi, \quad s_{34} \xi = q^{m_2} \xi,$$

where $m_1$ and $m_2$ are positive integers. Consider the subspace $L^0$ of $L(\lambda)$ defined by

$$L^0 = \{ v \in L(\lambda) \mid s_{11} v = s_{31} v = s_{33} v = 0 \}.$$

Note that $\xi \in L^0$ and so $L^0 \neq 0$.

**Lemma 6.1.** The subspace $L^0$ is stable under the action of each of the operators $s_{32}, s_{41}, s_{12}, s_{34}, s_{12}^{-1}, s_{34}^{-1}$. Moreover, these operators on $L^0$ satisfy the relation

$$s_{32} s_{41} - s_{41} s_{32} = (q^2 - 1)(s_{12}^{-1} s_{34} - s_{12} s_{34}^{-1}).$$

**Proof.** The first statement is immediate from (3.3) for the elements $s_{12}, s_{34}$ and their inverses. The defining relations (2.21) imply

$$s_{33} s_{32} = s_{32} s_{33}, \quad s_{11} s_{32} = s_{32} s_{11} + (q^{-2} - 1) s_{12} s_{31},$$

and

$$s_{31} s_{32} = q^{-1} s_{32} s_{31} + (q - q^{-1})(q^{-1} s_{21} s_{33} - s_{12} s_{33}),$$

which implies the statement for the operator $s_{32}$. Furthermore, we have

$$s_{11} s_{41} = s_{41} s_{11}, \quad s_{33} s_{41} = s_{41} s_{33} + (q^{-1} - q) s_{43} s_{31},$$

and

$$s_{31} s_{41} = q^{-1} s_{41} s_{31} + (q - q^{-1})(q^{-1} s_{43} s_{11} - s_{34} s_{11}),$$

and

$$s_{32} s_{41} - s_{41} s_{32} = (q^2 - 1)(s_{12}^{-1} s_{34} - s_{12} s_{34}^{-1}).$$
completing the proof of the first statement.

Now, by (2.21),

\[ s_{32} s_{41} = s_{41} s_{32} + (q - q^{-1})(s_{12} s_{43} - s_{34} s_{21}). \]

However, (2.20) gives

\[ s_{21} = q^{-2} s_{12}^{-1} s_{22} s_{11} - q s_{12}^{-1} \]

so that \( s_{21} \) coincides with \( -q s_{12}^{-1} \), as an operator on \( L^0 \). Similarly, \( s_{43} \) coincides with the operator \( -q s_{34}^{-1} \) thus yielding the desired relation. \( \square \)

Lemma 6.1 implies that \( L^0 \) is a representation of the quantized enveloping algebra \( U_q(\mathfrak{sl}_2) \). Indeed, the action is defined by setting

\[ e \mapsto \frac{s_{32}}{q - q^{-1}}, \quad f \mapsto \frac{s_{41}}{q(q - q^{-1})}, \quad k \mapsto s_{12}^{-1} s_{34}, \]

where \( e, f, k, k^{-1} \) are the standard generators of \( U_q(\mathfrak{sl}_2) \) satisfying

\[ ke = q^2 ek, \quad kf = q^{-2} fk, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}. \]

Since

\[ e \xi = 0 \quad \text{and} \quad k \xi = q^{-m_1 + m_2} \xi, \]

the cyclic span of \( \xi \) with respect to \( U_q(\mathfrak{sl}_2) \) is a highest weight module. Since this module is finite-dimensional, we must have \( m_2 - m_1 \geq 0 \) due to the classification theorem for the finite-dimensional irreducible representations of the algebra \( U_q(\mathfrak{sl}_2) \); see e.g. [2, Chapter 10]. Thus, we have proved the following.

Proposition 6.2. Suppose that \( \lambda = (q^{m_1}, \ldots, q^{m_n}) \), where \( m_1, \ldots, m_n \) are positive integers. If the representation \( L(\lambda) \) of the algebra \( U'_q(\mathfrak{sp}_{2n}) \) is finite-dimensional then \( m_1 \leq m_2 \leq \cdots \leq m_n \). \( \square \)

Our aim now is to show that these conditions are also sufficient for the representation \( L(\lambda) \) to be finite-dimensional. We shall regard \( U'_q(\mathfrak{sp}_{2n}) \) as a subalgebra of \( U_q(\mathfrak{gl}_{2n}) \) and use a version of the Gelfand–Tsetlin basis for representations of \( U_q(\mathfrak{gl}_N) \); see [13]. Let \( \nu = (\nu_1, \ldots, \nu_N) \) be an \( N \)-tuple of integers such that \( \nu_1 \geq \cdots \geq \nu_N \). The corresponding finite-dimensional irreducible representation \( V(\nu) \) of \( U_q(\mathfrak{gl}_N) \) is generated by a nonzero vector \( \xi \) such that

\[ \bar{t}_{ij} \xi = 0 \quad \text{for} \quad 1 \leq i < j \leq N, \]

\[ t_{ii} \xi = q^{\nu_i} \xi \quad \text{for} \quad 1 \leq i \leq N. \]
For any integer $m$ set

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$ 

Define the Gelfand–Tsetlin pattern $\Omega$ (associated with $\nu$) as an array of integer row vectors of the form

$$\begin{array}{cccc}
\nu_{N1} & \nu_{N2} & \cdots & \nu_{NN} \\
\nu_{N-1,1} & \cdots & \nu_{N-1,N-1} \\
\cdots & \cdots & \cdots \\
\nu_{21} & \nu_{22} \\
\nu_{11}
\end{array}$$

where $\nu_{Ni} = \nu_i$ for $i = 1, \ldots, N$, so that the top row coincides with $\nu$, and the following conditions hold

$$\nu_{k+1,i} \geq \nu_{ki} \geq \nu_{k+1,i+1}$$

for $1 \leq i \leq k \leq N - 1$. There exists a basis $\{\zeta_{\Omega}\}$ of $V(\nu)$ parameterized by the patterns $\Omega$ such that the action of the generators of $U_q(\mathfrak{gl}_N)$ is given by

$$t_k \zeta_{\Omega} = q^{w_k} \zeta_{\Omega}, \quad w_k = \sum_{i=1}^{k} \nu_{ki} - \sum_{i=1}^{k-1} \nu_{k-1,i},$$

$$e_k \zeta_{\Omega} = -\sum_{i=1}^{k} \frac{[l_{k+1,1} - l_{ki}] \cdots [l_{k+1,k+1} - l_{ki}]}{[l_{k1} - l_{ki}] \cdots [l_{kk} - l_{ki}]} \zeta_{\Omega+\delta_{ki}},$$

$$f_k \zeta_{\Omega} = \sum_{i=1}^{k} \frac{[l_{k-1,1} - l_{ki}] \cdots [l_{k-1,k-1} - l_{ki}]}{[l_{k1} - l_{ki}] \cdots [l_{kk} - l_{ki}]} \zeta_{\Omega-\delta_{ki}},$$

where $l_{ki} = \nu_{ki} - i + 1$ and the symbol $\wedge$ indicates that the zero factor in the denominator is skipped. The array $\Omega \pm \delta_{ki}$ is obtained from $\Omega$ by replacing $\nu_{ki}$ with $\nu_{ki} \pm \delta_{ki}$. The vector $\zeta_{\Omega}$ is considered to be zero if the array $\Omega$ is not a pattern.

Now let $m_1, \ldots, m_n$ be positive integers satisfying $m_1 \leq m_2 \leq \cdots \leq m_n$. Consider the representation $V(\nu)$ of $U_q(\mathfrak{gl}_{2n})$ where the highest weight $\nu$ is defined by

$$\nu = (r_n, \ldots, r_1, 0, \ldots, 0), \quad r_i = m_i - 1.$$
Introduce the pattern \( \Omega^0 \) associated with \( \nu \) which is given by

\[
\begin{array}{cccccccc}
  r_n & r_{n-1} & \cdots & r_1 & 0 & \cdots & 0 & 0 & 0 \\
  r_n & r_{n-1} & \cdots & r_1 & 0 & \cdots & 0 & 0 & \cdot \cdot \cdot \\
  r_{n-1} & \cdots & r_1 & 0 & \cdots & 0 & 0 & \cdot \cdot \cdot \\
  r_{n-1} & \cdots & r_1 & 0 & \cdots & 0 & \cdot \cdot \cdot \\
  \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \cdot \cdot \\
  \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
  r_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\
  r_1 \\
\end{array}
\]

For each \( k = 1, 2, \ldots, n \) the row \( 2k - 1 \) from the bottom is \((r_k, r_{k-1}, \ldots, r_1, 0, \ldots, 0)\) with \( k - 1 \) zeros, while the row \( 2k \) from the bottom is \((r_k, r_{k-1}, \ldots, r_1, 0, \ldots, 0)\) with \( k \) zeros. By the above formulas for the action of the generators of \( U_q(\mathfrak{gl}_{2n}) \) in the Gelfand–Tsetlin basis, for any \( i = 1, 3, \ldots, 2n - 1 \) we have the relations

\[
\bar{t}_{i,i+1} \zeta_{\Omega^0} = 0 \quad \text{and} \quad t_{i,i-1} \zeta_{\Omega^0} = 0,
\]

where the value \( i = 1 \) is excluded for the latter. The defining relations of \( U_q(\mathfrak{gl}_{2n}) \) (see Section 2) imply that the vector \( \zeta_{\Omega^0} \) is also annihilated by all generators \( t_{jk} \) with \( j - k \geq 2 \) and odd \( j \), as well as by \( \bar{t}_{kj} \) with \( j - k \geq 2 \) and even \( j \).

Consider the restriction of the \( U_q(\mathfrak{gl}_{2n}) \)-module \( V(\nu) \) to the subalgebra \( U'_q(\mathfrak{sp}_{2n}) \). Using (2.7) and (2.25) we then derive that

\[
s_{ij} \zeta_{\Omega^0} = 0 \quad \text{for} \quad i = 1, 3, \ldots, 2n - 1, \quad j = 1, 2, \ldots, i
\]

and

\[
s_{2k-1,2k} \zeta_{\Omega^0} = q^{r_{k+1}} \zeta_{\Omega^0} = q^{m_k} \zeta_{\Omega^0} \quad \text{for} \quad k = 1, 2, \ldots, n.
\]

Hence, the cyclic span of the vector \( \zeta_{\Omega^0} \) with respect to the subalgebra \( U'_q(\mathfrak{sp}_{2n}) \) is a highest weight module with the highest weight \( \lambda = (q^{m_1}, \ldots, q^{m_n}) \). Since the representation \( V(\nu) \) is finite-dimensional, we conclude that the representation \( L(\lambda) \) of \( U'_q(\mathfrak{sp}_{2n}) \) is also finite-dimensional.

Thus, combining this argument with Propositions 4.1 and 5.2 we get the following theorem.

**Theorem 6.3.** Every finite-dimensional irreducible representation of the algebra \( U'_q(\mathfrak{sp}_{2n}) \) is isomorphic to a highest weight representation \( L(\lambda) \) where the highest weight \( \lambda \) is an \( n \)-tuple of the form

\[
\lambda = (\sigma_1 q^{m_1}, \ldots, \sigma_n q^{m_n}),
\]

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with positive integers $m_i$ satisfying $m_1 \leq m_2 \leq \cdots \leq m_n$ and each $\sigma_i$ is 1 or $-1$. In particular, the isomorphism classes of finite-dimensional irreducible representations of $U'_q(\mathfrak{sp}_{2n})$ are parameterized by such $n$-tuples. 

It looks plausible that the structure of $L(\lambda)$ is very much similar to that of the representation of the Lie algebra $\mathfrak{sp}_{2n}$ with the highest weight $(r_n,\ldots,r_1)$, where $r_i = m_i - 1$. In particular, these representations should have the same dimensions and characters. This can be proved for the case where $U'_q(\mathfrak{sp}_{2n})$ is regarded as an algebra over $\mathbb{C}(q)$ by following the arguments of [2, Section 10.1]. Indeed, recall the $\mathcal{A}$-subalgebra $U'_A$ of $U'_q(\mathfrak{sp}_{2n})$ introduced in Section 2. Let $\xi$ denote the highest vector of the $U'_q(\mathfrak{sp}_{2n})$-module $L(\lambda)$ with $\lambda = (q^{m_1},\ldots,q^{m_n})$, where the positive integers $m_i$ satisfy $m_1 \leq m_2 \leq \cdots \leq m_n$. Set

$$L(\lambda)_A = U'_A \xi.$$ 

Then $L(\lambda)_A$ is a $U'_A$-submodule of $L(\lambda)$ such that

$$L(\lambda)_A \otimes_A \mathbb{C}(q) \cong L(\lambda)$$

in an isomorphism of vector spaces over $\mathbb{C}(q)$. Moreover, $L(\lambda)_A$ is the direct sum of its intersections with the weight spaces of $L(\lambda)$, and each intersection is a free $\mathcal{A}$-module; cf. [2 Proposition 10.1.4]. Now set

$$\overline{L}(\lambda) = L(\lambda)_A \otimes_A \mathbb{C},$$

where the $\mathcal{A}$-action on $\mathbb{C}$ is defined by the evaluation at $q = 1$. Due to the specialization isomorphism [2,20], $\overline{L}(\lambda)$ is a module over the Lie algebra $\mathfrak{sp}_{2n}$. By the specialization formulas of Section 2 we find

$$F_{2k-1,2k} \overline{\xi} = (m_k - 1) \overline{\xi}, \quad k = 1,2,\ldots,n,$$

where $\overline{\xi}$ denotes the image of $\xi$ in $\overline{L}(\lambda)$. Moreover,

$$F_{ij} \overline{\xi} = 0, \quad \text{for} \quad i = 1,3,\ldots,2n-1, \quad j = 1,2,\ldots,i.$$ 

So, taking the weights with respect to the basis $(F_{2n-1,2n},\ldots,F_{12})$ of the Cartan subalgebra of $\mathfrak{sp}_{2n}$, we conclude that $\overline{L}(\lambda)$ is a highest weight module over $\mathfrak{sp}_{2n}$. Since $\dim \overline{L}(\lambda) < \infty$, the module $\overline{L}(\lambda)$ must be irreducible. Thus we come to the following result.

**Theorem 6.4.** The $\mathfrak{sp}_{2n}$-module $\overline{L}(\lambda)$ is isomorphic to the finite-dimensional irreducible module with the highest weight $(r_n,\ldots,r_1)$, $r_i = m_i - 1$. In particular, the character of $U'_q(\mathfrak{sp}_{2n})$-module $L(\lambda)$ is given by the Weyl formula and its dimension over $\mathbb{C}(q)$ is the same as the dimension of $\overline{L}(\lambda)$ over $\mathbb{C}$. 

□
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