Separation of variables in the generalized 4th Appelrot class*

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Abstract

We consider the analogue of the 4th Appelrot class of motions of the Kowalevski top for the case of two constant force fields. The trajectories of this family fill the four-dimensional surface $O$ in the six-dimensional phase space. The constants of three first integrals in involution restricted to this surface fill one of the sheets of the bifurcation diagram in $R^3$. We point out the pair of partial integrals to obtain the explicit parametric equations of this sheet. The induced system on $O$ is shown to be Hamiltonian with two degrees of freedom having the thin set of points where the induced symplectic structure degenerates. The region of existence of motions in terms of the integral constants is found. We provide the separation of variables on $O$ and the algebraic formulae for the initial phase variables.

Key words and phrases: Kowalevski top, double field, Appelrot classes, separation of variables

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1 Preliminaries

The equations of motion of the Kowalevski top in two constant fields, expressed in the reference system $Oe_1e_2e_3$ of the principal axes of inertia at the fixed point $O$,

\begin{align}
2\dot{\omega}_1 &= \omega_2\omega_3 + \beta_3, \quad 2\dot{\omega}_2 = -\omega_1\omega_3 - \alpha_3, \quad \dot{\omega}_3 = \alpha_2 - \beta_1, \\
\dot{\alpha}_1 &= \alpha_2\omega_3 - \alpha_3\omega_2, \quad \dot{\beta}_1 = \beta_2\omega_3 - \beta_3\omega_2, \\
\dot{\alpha}_2 &= \alpha_3\omega_1 - \alpha_1\omega_3, \quad \dot{\beta}_2 = \beta_3\omega_1 - \beta_1\omega_3, \\
\dot{\alpha}_3 &= \alpha_1\omega_2 - \alpha_2\omega_1, \quad \dot{\beta}_3 = \beta_1\omega_2 - \beta_2\omega_1,
\end{align}

\noindent can without loss of generality be restricted to the phase space $P^6 \subset R^9(\omega, \alpha, \beta)$ defined by the geometric integrals

\begin{align}
\alpha_1^2 + \alpha_2^2 + \alpha_3^2 &= a^2, \quad \beta_1^2 + \beta_2^2 + \beta_3^2 = b^2, \quad \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0
\end{align}
This system is completely integrable due to the existence of three integrals in involution \cite{2,3}:

\[
H = \omega_1^2 + \omega_2^2 + \frac{1}{2} \omega_3^2 - (\alpha_1 + \beta_2), \\
K = (\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2, \\
G = \frac{1}{4}(M_2^2 + M_3^2) + \frac{1}{2} \omega_2 M_\gamma - b^2\alpha_1 - a^2\beta_2,
\]

where

\[
M_\alpha = 2\omega_1\alpha_1 + 2\omega_2\alpha_2 + \omega_3\alpha_3, \\
M_\beta = 2\omega_1\beta_1 + 2\omega_2\beta_2 + \omega_3\beta_3, \\
M_\gamma = 2\omega_1(\alpha_2\beta_3 - \alpha_3\beta_2) + 2\omega_2(\alpha_3\beta_1 - \alpha_1\beta_3) + \omega_3(\alpha_1\beta_2 - \alpha_2\beta_1).
\]

For the general case

\[ a > b > 0 \]

the explicit integration has not been found yet. It is natural to study first the invariant submanifolds in \(P^6\) such that the induced system has only two degrees of freedom. It is proved in \cite{4,11} that there exist only three submanifolds \(\mathcal{M}, \mathcal{N}, \mathcal{O}\) of this type. The union \(\mathcal{M} \cup \mathcal{N} \cup \mathcal{O}\) coincides with the set of critical points of the integral map

\[
H \times K \times G : P^6 \to \mathbb{R}^3.
\]

If \(b = 0\) (the classical Kowalevski case \cite{5}), then the critical set of the map (1.6) consists of the motions that belong to the so-called four Appelrot classes \cite{6}. The set \(\mathcal{M}\), first found in \cite{2} as the zero level of the integral \(K\), generalizes the 1st Appelrot class. The phase topology of the system induced on \(\mathcal{M}\) was studied in \cite{7}. The dynamical system on \(\mathcal{N}\) generalizing the 2nd and 3rd Appelrot classes was explicitly integrated in \cite{8}. The present work considers the restriction of the system (1.1) to the invariant subset \(\mathcal{O}\).

Introduce the complex phase variables \cite{9} \((i^2 = -1)\):

\[
x_1 = (\alpha_1 - \beta_2) + i(\alpha_2 + \beta_1), \quad x_2 = (\alpha_1 - \beta_2) - i(\alpha_2 + \beta_1), \\
y_1 = (\alpha_1 + \beta_2) + i(\alpha_2 - \beta_1), \quad y_2 = (\alpha_1 + \beta_2) - i(\alpha_2 - \beta_1), \\
z_1 = \alpha_3 + i\beta_3, \quad z_2 = \alpha_3 - i\beta_3, \\
w_1 = \omega_1 + i\omega_2, \quad w_2 = \omega_1 - i\omega_2, \quad w_3 = \omega_3.
\]

The system (1.1) takes the form

\[
x_1' = -x_1w_3 + z_1w_1, \quad x_2' = x_2w_3 - z_2w_2, \\
y_1' = -y_1w_3 + z_2w_1, \quad y_2' = y_2w_3 - z_1w_2, \\
2z_1' = x_1w_2 - y_2w_1, \quad 2z_2' = -x_2w_1 + y_1w_2, \\
2w_1' = -(w_1w_3 + z_1), \quad 2w_2' = w_2w_3 + z_2, \quad 2w_3' = w_3 - y_1.
\]

Here the prime stands for \(d/d(it)\).

The set \(\mathcal{O}\), by the definition given in the work \cite{11}, includes as a proper subset the following points

\[
w_1 = w_2 = 0, \quad z_1 = z_2 = 0.
\]

The invariant relations (1.9) lead to the family of pendulum motions first found in \cite{4}

\[
\alpha = a(e_1 \cos \theta - e_2 \sin \theta), \quad \beta = \pm b(e_1 \sin \theta + e_2 \cos \theta), \\
\alpha \times \beta \equiv \pm ab e_3, \quad \omega = \frac{d\theta}{dt} e_3, \quad \frac{d^2\theta}{dt^2} = -(a \pm b) \sin \theta.
\]
The corresponding values of the integrals (1.3) satisfy one of the following

\[ g = \pm ab h, \quad k = (a \mp b)^2, \quad h \geq -(a \pm b). \] (1.11)

In the sequel we use functions and expressions having singularities at the points (1.9). Therefore, by default, we exclude the trajectories (1.10). The remaining part of \( O \) can be described by the pair of equations

\[ R_1 = 0, \quad R_2 = 0, \] (1.12)

where

\[
R_1 = \frac{w_2x_1 + w_1y_2 + w_3z_1}{w_1} - \frac{w_1x_2 + w_2y_1 + w_3z_2}{w_2},
\]

\[
R_2 = (w_2z_1 + w_1z_2)w_3^2 + \left[ \frac{w_2z_1^2}{w_1} + \frac{w_1z_2^2}{w_2} + w_1w_2(y_1 + y_2) + x_1w_2^2 + x_2w_1^2 \right] w_3 +
\]
\[ + \frac{w_2^2x_1z_1}{w_1} + \frac{w_1^2x_2z_2}{w_2} + x_1z_2w_2 + x_2z_1w_1 + (w_1z_2 - w_2z_1)(y_1 - y_2). \] (1.13)

For the derivatives we have, in virtue of (1.8),

\[ R'_1 = \kappa_2 R_2, \quad R'_2 = \kappa_1 R_1, \]

\[ \kappa_1 = \frac{1}{2w_1w_2} \left[ (w_1w_2w_3 + z_2w_1 + z_1w_2)^2 + w_1w_2(x_2w_1^2 + x_1w_2^2) \right], \quad \kappa_2 = \frac{1}{2w_1w_2}. \] (1.14)

The equations (1.14) straightforwardly prove that the set (1.12) is preserved by the phase flow (1.1). Besides, it follows that the Poisson bracket \( \{ R_1, R_2 \} \) is a partial integral on \( O \). The subset in \( O \) defined by the equation

\[ \{ R_1, R_2 \} = 0 \] (1.15)

is the set of points at which the induced symplectic structure is degenerate. Below we calculate \( \{ R_1, R_2 \} \) using the appropriate integrals on \( O \) and show that the set (1.15) is of measure zero. Thus, the dynamical system induced on \( O \) by the system (1.1) is almost everywhere a Hamiltonian system with two degrees of freedom. In particular, almost all its integral manifolds consist of two-dimensional Liouville tori.

### 2 Partial integrals

Recall that in the classical Kowalevski case (\( \beta = 0 \)) there exists the momentum integral

\[ L = \frac{1}{2} I \omega \cdot \alpha \quad (I = \text{diag } \{2, 2, 1\}). \] (2.1)

Then the integral \( G \) becomes equal to \( L^2 \). Let \( \ell \) denote the constant of the integral (2.1). In Appelrot’s notation the 4th class of especially remarkable motions is defined by the following conditions.

(i) The second polynomial of Kowalevski has a multiple root. One of the Kowalevski variables remains constant and equal to the multiple root \( s \) of the corresponding Euler resolvent \( \varphi(s) = s(s - h)^2 + (a^2 - k)s - 2\ell^2 \):

\[ \varphi(s) = 0, \quad \varphi'(s) = 0. \] (2.2)
Two equatorial components of the angular velocity are constant: $\omega_1 \equiv -\ell/s$, $\omega_2 \equiv 0$. Given $\beta = 0$, this fact can be written in the form
\[
\frac{I\omega \cdot \alpha}{I\omega \cdot e_1} = -s, \quad I\omega \cdot \beta = 0, \quad I\omega \cdot e_2 = 0.
\] (2.3)

The next statement establishes the conditions similar to (2.3) for the generalized top.

**Theorem 1.** On each trajectory belonging to $O$ the ratios
\[
\frac{I\omega \cdot \alpha}{I\omega \cdot e_1}, \quad \frac{I\omega \cdot \beta}{I\omega \cdot e_2}
\]
are constant and equal to each other.

**Proof.** Let
\[
M = I\omega, \quad M_j = I\omega \cdot e_j \quad (j = 1, 2, 3).
\] (2.4)
Due to (1.4) we also have $M_\alpha = I\omega \cdot \alpha$, $M_\beta = I\omega \cdot \beta$. Then the first equation (1.12) yields
\[
\frac{M_\alpha}{M_1} - \frac{M_\beta}{M_2} = 0.
\] (2.5)
Introduce the function
\[
S = -\frac{M_\alpha M_1 + M_\beta M_2}{M_1^2 + M_2^2}
\]
and calculate its time derivative:
\[
\frac{dS}{dt} = -\frac{(M_1^2 + M_2^2)\omega_3 + 4\alpha_3 M_1 + 4\beta_3 M_2}{2(M_1^2 + M_2^2)^2}(M_\alpha M_2 - M_\beta M_1).
\]
In virtue of (2.5) the right hand side is identically zero. Therefore $S$ is a partial integral on $O$. Denote by $s$ the corresponding constant:
\[
\frac{M_\alpha M_1 + M_\beta M_2}{M_1^2 + M_2^2} = -s.
\] (2.6)
From (2.5), (2.6) we obtain
\[
M_\alpha = -sM_1, \quad M_\beta = -sM_2
\] (2.7)
with the constant value $s$.

Note that according to (2.7) the function $S$ can be written in either of the representations
\[
S = -\frac{1}{4} \left( \frac{M_\alpha + iM_\beta}{\omega_1 + i\omega_2} + \frac{M_\alpha - iM_\beta}{\omega_1 - i\omega_2} \right) = -\frac{1}{2} \frac{M_\alpha + iM_\beta}{\omega_1 + i\omega_2} = -\frac{1}{2} \frac{M_\alpha - iM_\beta}{\omega_1 - i\omega_2}.
\] (2.8)

**Theorem 2.** On the set $O$ the system (1.1) has the partial integral
\[
T = \frac{1}{2}(M_\alpha M_1 + M_\beta M_2) - 2(\alpha_1\beta_2 - \alpha_2\beta_1) + a^2 + b^2.
\] (2.9)

**Proof.** The time derivative of $T$
\[
\frac{dT}{dt} = \frac{1}{4} \omega_3 (M_\alpha M_2 - M_\beta M_1)
\]
vanishes on $O$ due to (2.5). □
Denote by $\tau$ the constant of the integral $T$.

**Remark 1.** In the work [1] equations similar to (1.12) were derived from the condition that the function

$$\quad 2G + (\tau - a^2 - b^2)H + sK$$

with Lagrange’s multipliers $s, \tau$ has a critical point on $P^6$. Using the coordinates (1.7) we have from (2.8), (2.9)

$$S = -\frac{w_1(x_2w_1 + y_1w_2 + z_2w_3) + w_2(y_2w_1 + x_1w_2 + z_1w_3)}{4w_1w_2} = -\frac{x_2w_1 + y_1w_2 + z_2w_3}{2w_2} = -\frac{y_2w_1 + x_1w_2 + z_1w_3}{2w_1},$$

$$T = \frac{1}{2}[w_1(x_2w_1 + y_1w_2 + z_2w_3) + w_2(y_2w_1 + x_1w_2 + z_1w_3)] + x_1x_2 + z_1z_2 = -2Sw_1w_2 + x_1x_2 + z_1z_2.$$

Comparing (2.11), (2.12) with the expressions for $s, \tau$ given in [1] we see that on $O$ Lagrange’s multipliers in (2.10) coincide with the constants of the integrals $S, T$ introduced here.

Supposing (1.5), introduce parameters $p, r$ ($p > r > 0$) such that

$$p^2 = a^2 + b^2, \quad r^2 = a^2 - b^2. \quad (2.13)$$

Let $h, k, g$ denote the constants of the general integrals (1.3). Then according to Remark [1] from the results of the work [1] we obtain that on $O$

$$h = \frac{p^2 - \tau}{2s} + s, \quad k = \frac{\tau^2 - 2p^2\tau + r^4}{4s^2} + \tau, \quad g = \frac{p^4 - r^4}{4s} + \frac{1}{2}(p^2 - \tau)s. \quad (2.14)$$

These equations can be considered as parametric equations of the corresponding bifurcation sheet of the integral map (1.6). Eliminating $\tau$ we have

$$\psi(s) = 0, \quad \psi'(s) = 0, \quad (2.15)$$

where

$$\psi(s) = s^2(s - h)^2 + (p^2 - k)s^2 - 2gs + \frac{p^4 - r^4}{4}.$$  

If $\beta = 0$ ($p^2 = r^2 = a^2$), then $\psi(s) = s \varphi(s)$ and the conditions (2.15) turn to (2.2). Therefore, the set of trajectories belonging to $O$ is the generalization of the set of the especially remarkable motions of the 4th Appelrot class.

### 3 Parametric equations of integral manifolds

Due to the equations (2.14) the functions $S, T$ form a complete system of first integrals on $O$. In particular, the equations of the integral manifold

$$\{\zeta \in P^6 : H(\zeta) = h, K(\zeta) = k, G(\zeta) = g\}$$

in this class of motions are equivalent to the relations (1.12) and the equations

$$S = s, \quad T = \tau. \quad (3.1)$$
Using (2.5), (2.11) we replace the equations \( R_1 = 0, S = s \) by
\[
(y_2 + 2s)w_1 + x_1w_2 + z_1w_3 = 0, \\
x_2w_1 + (y_1 + 2s)w_2 + z_2w_3 = 0. 
\tag{3.2}
\]
At the same time, from (1.13) and (2.12) the system \( R_2 = 0, T = \tau \) is equivalent to
\[
x_2z_1w_1 + x_1z_2w_2 + (\tau - x_1x_2)w_3 = 0, \\
2s w_1w_2 - (x_1x_2 + z_1z_2) + \tau = 0. 
\tag{3.3}
\]
To obtain a closed system of equations we must add the geometric integrals (1.2). In terms of the variables (1.7) we get
\[
z_1^2 + x_1y_2 = r^2, \\
z_2^2 + x_2y_1 = r^2, \\
x_1x_2 + y_1y_2 + 2z_1z_2 = 2p^2. 
\tag{3.5}
\]
The variables (1.7) by definition satisfy
\[
x_2 = x_1, \quad y_2 = y_1, \quad z_2 = z_1, \quad w_2 = w_1, \quad w_3 \in \mathbb{R}, 
\tag{3.7}
\]
thus forming a space of real dimension 9. Seven relations (3.2) – (3.6) define the integral manifold. In the case of independency of the integrals \( S, T \) this manifold is two-dimensional and consists of Liouville tori filled with quasi-periodic motions.

Let
\[
x = \sqrt{x_1x_2}, \quad z = \sqrt{z_1z_2}, \quad \xi = 2s w_1w_2. 
\tag{3.8}
\]
Then from (3.4)
\[
\xi = x^2 + z^2 - \tau. 
\tag{3.9}
\]
From (3.5), (3.8) we obtain
\[
(r^2 - x_1y_2)(r^2 - x_2y_1) = z^4, 
\tag{3.10}
\]
and the equation (3.6) yields
\[
y_1y_2 = 2p^2 - x^2 - 2z^2. 
\tag{3.11}
\]
Hence
\[
r^2(x_1y_2 + x_2y_1) = r^4 + 2p^2x^2 - (x^2 + z^2)^2. 
\tag{3.12}
\]
Rewrite (3.5) in the form
\[
(z_1 + z_2)^2 = 2r^2 - (x_1y_2 + x_2y_1) + 2z^2, \\
(z_1 - z_2)^2 = 2r^2 - (x_1y_2 + x_2y_1) - 2z^2 
\tag{3.13}
\]
and substitute (3.12) to obtain
\[
r^2(z_1 + z_2)^2 = \Phi_1, \quad r^2(z_1 - z_2)^2 = \Phi_2, 
\tag{3.14}
\]
where
\[
\Phi_1 = (x^2 + z^2 + r^2)^2 - 2(p^2 + r^2)x^2 = (\xi + r^2)^2 - 2(p^2 + r^2)x^2, \\
\Phi_2 = (x^2 + z^2 - r^2)^2 - 2(p^2 - r^2)x^2 = (\xi - r^2)^2 - 2(p^2 - r^2)x^2. 
\tag{3.15}
\]

Note that the equilibria \( \omega \equiv 0 \) of the system (1.1) are included in the family of motions (1.10). On the rest of the trajectories in \( \mathcal{O} \) the determinant of the equations (3.2), (3.3) in \( w_j \)
(j = 1, 2, 3) vanishes identically. Calculate this determinant and eliminate \( z_1^2, z_2^2 \) from (3.5) and the product \( y_1y_2 \) from (3.11) to obtain

\[
2s[(r^2x_1 - \tau y_1) + (r^2x_2 - \tau y_2)] = -r^2(x_1y_2 + x_2y_1) + 2[2s^2(\tau - x^2) + \rho^2(\tau + x^2) - \tau(x^2 + z^2)].
\]  

(3.16)

On the other hand, the direct calculation in view of (3.8), (3.11) gives

\[
(r^2x_1 - \tau y_1)(r^2x_2 - \tau y_2) = r^4x^2 + \tau(2\rho^2 - x^2 - 2z^2) - r^2\tau(x_1y_2 + x_2y_1).
\]  

(3.17)

Denote

\[
\sigma = \tau^2 - 2\rho^2\tau + r^4, \quad \chi = \sqrt{k} \geq 0.
\]  

(3.18)

From the second relation (2.14) we have the identity

\[
4s^2\chi^2 = \sigma + 4s^2\tau.
\]  

(3.19)

Introduce the complex conjugate variables

\[
\mu_1 = r^2x_1 - \tau y_1, \quad \mu_2 = r^2x_2 - \tau y_2.
\]  

(3.20)

Eliminating \( x_1y_2 + x_2y_1 \) from (3.16), (3.17) with the help of (3.12) we come to the system

\[
2s(\mu_1 + \mu_2) = \xi^2 - 4s^2(x^2 - \tau) - \sigma, \quad \mu_1\mu_2 = \tau\xi^2 + \sigma x^2 - \tau\sigma.
\]  

(3.21)

Choose

\[
\mu_1^* = \sqrt{2s\mu_1 - 4s^2\tau}, \quad \mu_2^* = \sqrt{2s\mu_2 - 4s^2\tau}
\]  

(3.22)

to be complex conjugate. Then the system (3.21) takes the form

\[
(\mu_1^* + \mu_2^*)^2 = \Psi_1, \quad (\mu_1^* - \mu_2^*)^2 = \Psi_2,
\]  

(3.23)

where

\[
\Psi_1 = \xi^2 - 4s^2(x + \chi)^2, \quad \Psi_2 = \xi^2 - 4s^2(x - \chi)^2.
\]  

(3.24)

As was to be expected from dimensional reasoning, all phase variables could be expressed in terms of two almost everywhere independent auxiliary variables. The above formulae emphasize the special role of the pair \((x, \xi)\).

From (3.22), (3.23) we find

\[
\mu_1 = 2s\tau + \frac{1}{8s}(\sqrt{\Psi_1} + \sqrt{\Psi_2})^2, \quad \mu_2 = 2s\tau + \frac{1}{8s}(\sqrt{\Psi_1} - \sqrt{\Psi_2})^2.
\]  

(3.25)

Further on the calculation sequence is as follows. From (3.14) we find \( z_1, z_2 \). Multiplying the equations (3.20) by \( x_2, x_1 \) respectively and using (3.5) we get

\[
x_2\mu_1 = r^2x^2 - \tau x_2y_1 = r^2(x^2 - \tau) + \tau z_2^2, \quad x_1\mu_2 = r^2x^2 - \tau x_1y_2 = r^2(x^2 - \tau) + \tau z_1^2,
\]  

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whence $x_1, x_2$ are found. Substituting these values back to (3.20) we find $y_1, y_2$. As a result, after some obvious transformations we obtain the following expressions for the configuration variables:

\[
x_1 = \frac{2s}{r^2} 4r^4(x^2 - \tau) + \tau(\sqrt{\Phi_1} + \sqrt{\Phi_2})^2,
\]

\[
x_2 = \frac{2s}{r^2} 4r^4(x^2 - \tau) + \tau(\sqrt{\Phi_1} - \sqrt{\Phi_2})^2,
\]

\[
y_1 = \frac{2s}{16s^2r + (\sqrt{\Psi_1} - \sqrt{\Psi_2})^2} 4[2\tau \xi - \tau(x^2 - \tau) + \sigma] - (\sqrt{\Phi_1} - \sqrt{\Phi_2})^2,
\]

\[
y_2 = \frac{2s}{16s^2r + (\sqrt{\Psi_1} + \sqrt{\Psi_2})^2} 4[2\tau \xi - \tau(x^2 - \tau) + \sigma] - (\sqrt{\Phi_1} + \sqrt{\Phi_2})^2,
\]

\[
z_1 = \frac{1}{2r}(\sqrt{\Phi_1} + \sqrt{\Phi_2}), \quad z_2 = \frac{1}{2r}(\sqrt{\Phi_1} - \sqrt{\Phi_2}). \tag{3.28}
\]

Note that all radicals are algebraic. The formal choice of signs in (3.26)–(3.28) is defined by the initial choice in the expressions for $\mu_1, \mu_2, z_1, z_2$. The polynomials $\Phi_j, \Psi_j$ ($j = 1, 2$) obviously split to multipliers linear with respect to $x, \xi$. Then, typically, the projection of an integral manifold onto the $(x, \xi)$-plane has the form of a quadrangle. Fix any inner point $(x, \xi)$ of such projection. Then the expressions (3.26)–(3.28) define eight points of the configuration space with different set of signs of the radicals $\sqrt{\Phi_1}, \sqrt{\Phi_2}, \sqrt{\Psi_1}, \sqrt{\Psi_2}$.

To find $w_3$, use the energy integral. On account of (2.14) it takes the form

\[
2s w_3^2 + 4s w_1 w_2 - 2s(y_1 + y_2) = 4s^2 + 2p^2 - 2\tau.
\]

Substitute $2s w_1 w_2$ from (3.4) and replace $2p^2$ by its expression from (3.6) to obtain

\[
2s w_3^2 = D, \tag{3.29}
\]

where

\[
D = (y_1 + 2s)(y_2 + 2s) - x_1 x_2 \tag{3.30}
\]

is the determinant of (3.2) with respect to $w_1, w_2$. In particular, this determinant vanishes along with $w_3$. Due to this fact we can express $w_1, w_2$ either as linear functions of $w_3$, or in inverse proportion to $w_3$:

\[
w_1 = \frac{x_1 z_2 - (y_1 + 2s)z_1}{(y_1 + 2s)(y_2 + 2s) - x^2} w_3 = \frac{x_1 z_2 - (y_1 + 2s)z_1}{2s w_3},
\]

\[
w_2 = \frac{x_2 z_1 - (y_2 + 2s)z_2}{(y_1 + 2s)(y_2 + 2s) - x^2} w_3 = \frac{x_2 z_1 - (y_2 + 2s)z_2}{2s w_3}. \tag{3.31}
\]

With two possibilities of the sign choice for $w_3$ in (3.29) we have that the inverse image in the integral manifold $\{S = s, T = \tau\} \cap \mathcal{D}$ of a generic point $(x, \xi)$ contains 16 points.

We also need the explicit formula for $w_3$ in terms of $x, \xi$. From (3.27) we write

\[
y_1 + 2s = \frac{4[2\tau \xi - \tau(x^2 - \tau) + 4s^2 \chi^2]}{16s^2r + (\sqrt{\Psi_1} - \sqrt{\Psi_2})^2} - (\sqrt{\Phi_1} - \sqrt{\Phi_2})^2,
\]

\[
y_2 + 2s = \frac{4[2\tau \xi - \tau(x^2 - \tau) + 4s^2 \chi^2]}{16s^2r + (\sqrt{\Psi_1} + \sqrt{\Psi_2})^2} - (\sqrt{\Phi_1} + \sqrt{\Phi_2})^2. \tag{3.32}
\]
Note that
\[ [16s^2\tau + (\sqrt{\Psi_1} - \sqrt{\Psi_2})^2][16s^2\tau + (\sqrt{\Psi_1} + \sqrt{\Psi_2})^2] = 64s^2(\tau \xi^2 + \sigma x^2 - \tau \sigma). \] (3.33)

Then
\[ D = \frac{\sqrt{\Phi_1 \Phi_2 \Psi_1 \Psi_2} - P}{2(\tau \xi^2 + \sigma x^2 - \tau \sigma)}, \]
where
\[ P(x, \xi) = \xi^4 + 2\tau \xi^3 + 2[(\tau - 2s^2 - p^2)x^2 - \tau(2s^2 - p^2) - r^4]\xi^2 - 8s^2[(\tau - 2\chi^2)x^2 + 2\chi^2]\xi - 4s^2(x^2 - \chi^2)[2(\tau - p^2)x^2 - (\tau^2 - r^4)]. \]

Let
\begin{align*}
Q(x, \xi) &= (\xi + \tau + 2s^2 - p^2)^2 - 4s^2x^2 + r^4 - (2s^2 - p^2)^2, \\
P_1(x, \xi) &= P(x, \xi) + 2xQ(x, \xi)\sqrt{\tau \xi^2 + \sigma x^2 - \tau \sigma}, \\
P_2(x, \xi) &= P(x, \xi) - 2xQ(x, \xi)\sqrt{\tau \xi^2 + \sigma x^2 - \tau \sigma}.
\end{align*}

Direct calculation proves the identity \( P_1P_2 \equiv \Phi_1 \Phi_2 \Psi_1 \Psi_2 \). On the other hand, \( P_1 + P_2 \equiv 2P \). Therefore,
\[ D = -\frac{(\sqrt{P_1} - \sqrt{P_2})^2}{4(\tau \xi^2 + \sigma x^2 - \tau \sigma)}, \]
whence
\[ w_3 = i\frac{\sqrt{P_1} - \sqrt{P_2}}{2\sqrt{2s(\tau \xi^2 + \sigma x^2 - \tau \sigma)}}. \] (3.34)

Below we use one more representation of \( w_1, w_2 \) not depending on \( w_3 \). The system (3.2) yields
\[ x_2w_1^2 = -z_2w_1w_3 - (y_2 + 2s)w_1w_2, \quad x_1w_2^2 = -z_1w_2w_3 - (y_1 + 2s)w_1w_2. \]

Substitute the expressions for \( w_1w_3, w_2w_3 \) found from the inverse proportion in (3.31) and eliminate \( w_1w_2 \) by (3.4) to obtain
\[ 2s x_2w_1^2 = -(\mu_1 - 2s\tau) - 2sx^2, \quad 2s x_1w_2^2 = -(\mu_2 - 2s\tau) - 2sx^2. \]

Then from (3.25) we find
\[ w_1 = \frac{i}{4s\sqrt{x_2}}(\sqrt{\Theta_1} + \sqrt{\Theta_2}), \quad w_2 = \frac{i}{4s\sqrt{x_1}}(\sqrt{\Theta_1} - \sqrt{\Theta_2}), \] (3.35)

where
\[ \Theta_1(x, \xi) = (\xi - 2sx)^2 - 4s^2\chi^2, \quad \Theta_2(x, \xi) = (\xi + 2sx)^2 - 4s^2\chi^2. \] (3.36)

Note that the products \( \Theta_1 \Theta_2 \) and \( \Psi_1 \Psi_2 \) coincide. The choice of the signs in (3.35) is determined by the condition \((\sqrt{x_2w_1})(\sqrt{x_1w_2}) = x\xi/2s\) following from (3.8). The signs of the complex conjugate values \( \sqrt{x_1}, \sqrt{x_2} \) must be chosen in such a way that the expressions (3.35), (3.36) satisfy one of the equations (3.2), (3.3) (then the other two hold automatically).

Thus, all phase variables are algebraically expressed in terms of two auxiliary variables \( x, \xi \); the domain of the latter depends on the constants of the first integrals.
4 Singularities of the induced symplectic structure

The Hamiltonian structure of the system (1.1) is provided by the Poisson brackets on $\mathbb{R}^9(\omega, \alpha, \beta)$. In notation (2.1), these brackets are [2]

$$\{M_j, M_k\} = \varepsilon_{jki}M_i, \quad \{M_j, \alpha_k\} = \varepsilon_{jkl}\alpha_l, \quad \{M_j, \beta_k\} = \varepsilon_{jkl}\beta_l, \quad \{\alpha_j, \alpha_k\} = \{\alpha_j, \beta_k\} = \{\beta_j, \beta_k\} = 0. \quad (4.1)$$

Being restricted to $P^6$ they correspond to Kirillov's symplectic form $\lambda \in \Lambda^2(P^6)$. Recall the following well-known facts. Let $N \subset P^6$ be a submanifold defined in $P^6$ by two independent equations

$$f_1 = 0, \quad f_2 = 0 \quad (4.2)$$

and let $\mathfrak{x}_1, \mathfrak{x}_2$ be the Hamiltonian vector fields with the Hamilton functions $f_1, f_2$. Then the span of $\mathfrak{x}_1, \mathfrak{x}_2$ at each point $\zeta \in N$ is skew orthogonal to the tangent space $T_\zeta N$. Therefore the restriction of $\lambda$ to $N$ is non-degenerate at the point $\zeta$ if and only if

$$\{f_1, f_2\}(\zeta) = \lambda(\mathfrak{x}_1, \mathfrak{x}_2)(\zeta) \neq 0. \quad (4.3)$$

Let us calculate the bracket $\{R_1, R_2\}$ of the functions (1.13). As $R_1$ is pure imaginary denote

$$R = \frac{1}{i}\{R_1, R_2\}. \quad (4.4)$$

The change of variables (1.7) is linear with constant coefficients, so the rules (4.1) are easily transformed for new coordinates. Omitting technical details we present $R$ in the form

$$R = \frac{F_1}{w_1^2w_2^2w_3^2}[w_3(w_3F_2 + F_3)^2 - F_4], \quad (4.5)$$

where

$$F_1 = z_1z_2w_3 + x_2z_1w_1 + x_1z_2w_2, \quad F_2 = w_1w_2w_3 + z_2w_1 + z_1w_2, \quad F_3 = 2w_1^2w_2^2 + x_2w_1^2 + x_1w_2^2, \quad F_4 = 4w_1^2w_2^2[(w_1^2w_2^2 + x_2w_1^2 + x_1w_2^2)w_3 + x_2z_1w_1 + x_1z_2w_2]. \quad (4.6)$$

The variables $y_1, y_2$ have been eliminated as the solution of the system (1.12)

$$y_1 = -\frac{w_1w_3(x_2w_1 + z_2w_3) + F_1}{w_1w_2w_3}, \quad y_2 = -\frac{w_2w_3(x_1w_2 + z_1w_3) + F_1}{w_1w_2w_3}. \quad (4.7)$$

Fixing the values $h, k, g, s, \tau$ of the functions (1.3), (2.11), (2.12) we have

$$w_3F_2 + F_3 = 2w_1w_2h - 2(x_1x_2 + z_1z_2 - \tau), \quad F_4 = 4w_1^2w_2^2w_3(\tau - k). \quad (4.8)$$

Then from (2.14), (3.3), (3.4), (3.8), (3.9) we obtain

$$w_3F_2 + F_3 = \frac{\xi^2}{s}(\frac{p^2 - \tau}{2s} - s), \quad F_4 = -\frac{\sigma\xi^2w_3}{4s^4}, \quad F_1 = \xi w_3, \quad w_1^3w_2^3 = \frac{\xi^3}{8s^3}. \quad (4.9)$$

Finally, the expression (4.5) takes the form

$$R = \frac{8}{s} \left[ s^4 - (p^2 - \tau)s^2 + \frac{p^4 - r^4}{4} \right]. \quad (4.10)$$
Obviously, \( R \) is a first integral of the induced system, and the equation

\[
s^4 - (p^2 - \tau)s^2 + \frac{p^4 - r^4}{4} = 0
\]

(4.11)
defines the set of the integral constants \( s, \tau \) such that on the corresponding integral manifolds the 2-form induced on \( \mathcal{O} \) by the symplectic structure \( \lambda \) degenerates. As the functions (2.11), (2.12) have an algebraic structure, the set (4.11) has codimension 1 in \( \mathcal{O} \). Thus, \( \lambda|_{\mathcal{O}} \) is almost everywhere non-degenerate.

To locate the set (4.11) on the surface (2.14) in \( \mathbb{R}^3(h, k, g) \) notice that, in virtue of (2.14),

\[
\frac{\partial}{\partial s}(h, k, g) \times \frac{\partial}{\partial \tau}(h, k, g) = \left[ s^4 - (p^2 - \tau)s^2 + \frac{p^4 - r^4}{4}\right]\left(\frac{\tau - p^2}{2s^4}, \frac{1}{2s^3}, \frac{1}{s^4}\right).
\]

Therefore, the equation (4.11) defines cuspidal edges of the surface (2.14).

5 Bifurcation diagram and the existence of motions

Introduce the integral map \( J \) of the dynamical system on \( \mathcal{O} \)

\[
J(\zeta) = (S(\zeta), T(\zeta)) \in \mathbb{R}^2, \quad \zeta \in \mathcal{O}.
\]

(5.1)
The bifurcation diagram \( \Sigma(J) \) of this map is, by definition, the set of pairs \((s, \tau)\) over which \( J \) is not locally trivial. In our case all integral manifolds \((3.1)\) are compact. Therefore, \( \Sigma(J) \) is the set of critical values of \( J \).

Define the admissible region as the image of the map (5.1), i.e., the set of values \((s, \tau)\) such that the integral manifold \((3.1)\) is not empty. Obviously, \( \partial J(\mathcal{O}) \subset \Sigma(J) \).

Due to the above results the existence of a trajectory on \( \mathcal{O} \) with given \( s, \tau \) is equivalent to the existence of a point on the \((x, \xi)\)-plane for which the values (3.26)–(3.28), (3.35), (3.35) satisfy (3.7). It easily follows from (3.25), (3.28), (3.8) that the conditions (3.7), in turn, are equivalent to the system of inequalities

\[
\Phi_1(x, \xi) \geq 0, \quad \Phi_2(x, \xi) \leq 0,
\]

(5.2)

\[
\Psi_1(x, \xi) \geq 0, \quad \Psi_2(x, \xi) \leq 0
\]

(5.3)

considered in the quadrant \( x \geq 0, s \xi \geq 0 \).

Theorem 3. The bifurcation diagram \( \Sigma(J) \) consists of the following subsets of the \((s, \tau)\)-plane:

1°) \( \tau = (a + b)^2, \ s \in [-a, 0) \cup [b, +\infty) \);
2°) \( \tau = (a - b)^2, \ s \in [-a, -b] \cup (0, +\infty) \);
3°) \( s = -a, \ \tau \geq (a - b)^2 \);
4°) \( s = -b, \ \tau \geq (a - b)^2 \);
5°) \( s = b, \ \tau \leq (a + b)^2 \);
6°) \( s = a, \ \tau \leq (a + b)^2 \);
7°) \( \tau = 0, \ s \in (0, +\infty) \);
8°) \( \tau = (\sqrt{a^2 - s^2} + \sqrt{b^2 - s^2})^2, \ s \in [-b, 0) \);
9°) \( \tau = (\sqrt{a^2 - s^2} - \sqrt{b^2 - s^2})^2, \ s \in (0, b) \);
10°) \( \tau = -(\sqrt{s^2 - a^2} - \sqrt{s^2 - b^2})^2, \ s \in [a, +\infty) \).
Theorem 4. The solutions of the system (1.1) under the conditions (1.12) exist iff the constants of the first integrals (2.11), (2.12) satisfy one of the following:

1°) \(-a \leq s \leq -b, \tau \geq (a-b)^2\);
2°) \(-b \leq s < 0, \tau \geq (\sqrt{a^2 - s^2} + \sqrt{b^2 - s^2})^2\);
3°) \(0 < s \leq b, \tau \leq (\sqrt{a^2 - s^2} - \sqrt{b^2 - s^2})^2\);
4°) \(b \leq s \leq a, \tau \leq (a+b)^2\);
5°) \(s \geq a, -(\sqrt{s^2 - b^2} - \sqrt{s^2 - a^2})^2 \leq \tau \leq (a+b)^2\).

The complete proof of these statements is purely technical (see [10]) and contains the scrupulous analysis of the regions on the \((x, \xi)\)-plane defined by (5.2), (5.3) and the cases of their structural transformations. Another approach is suggested in [11] for the pair \((S,H)\). It is based on the classification of the trajectories in \(\mathcal{D}\) satisfying the condition \(\text{rank}(H \times K \times G) < 2\).

The admissible region is shaded in Fig.1. The dense lines and curves represent the equations of the bifurcation diagram, the dashed curve illustrates the equation (4.11), i.e., the first integrals constants such that the symplectic structure is degenerate on the corresponding integral manifolds.

6 Separation of variables

In the sequel we suppose that \(\tau \sigma \neq 0\). In fact, if \(\tau = 0\), then from (2.14) we obtain the relation \((2g - p^2h)^2 = r^4k\) characteristic for the critical manifold \(\mathcal{M} [9]\). The equations of motion on \(\mathcal{M}\) were explicitly integrated in [8]. If \(\sigma = 0\), then the equations (2.14) yield one of the relations (1.11). This case corresponds to the set of points (1.19). At these points the manifold \(\mathcal{D}\) fails to be smooth. The corresponding trajectories are the pendulum motions (1.10).

Considering the second equation (3.21) denote \(\mu = |r^2 x_1 - \tau y_1| = |r^2 x_2 - \tau y_2|\). Then

\[\mu^2 = \tau \xi^2 + \sigma x^2 - \tau \sigma.\] (6.1)

This equation defines the second-order surface \(\mathcal{M}\) in three-dimensional space \(\mathbb{R}^3(x, \xi, \mu)\). Each trajectory in \(\mathcal{D}\) is in a natural way represented by a curve on \(\mathcal{M}\).
Theorem 5. Supposing $\tau \sigma \neq 0$, introduce the variables

$$U = \frac{\tau \xi + x \mu}{\sqrt{\sigma (\tau - x^2)}}, \quad V = \frac{\tau \xi - x \mu}{\sqrt{\sigma (\tau - x^2)}}. \quad (6.2)$$

Then the equations of motion on $\mathcal{Q}$ separate

$$\frac{dU}{\sqrt{Q(U)}} - \frac{dV}{\sqrt{Q(V)}} = 0, \quad \frac{dU}{\sqrt{Q(U)}} - \frac{dV}{\sqrt{Q(V)}} = \frac{dt}{\sqrt{2s\tau \sigma}}. \quad (6.3)$$

Here

$$Q(w) = (w^2 - 1)(\sigma w^2 - 4s^2 \chi^2)[(\sqrt{\sigma} w + \tau - r^4]. \quad (6.4)$$

Proof. Consider the local coordinates $u, v$ on the surface $\mathcal{M}$:

$$\xi = \sqrt{\sigma} \frac{uv + 1}{u + v}, \quad x = \sqrt{\tau} \frac{u - v}{u + v}, \quad \mu = \sqrt{\tau \sigma} \frac{uv - 1}{u + v}. \quad (6.5)$$

In addition to (3.19) note the following two identities for the constants introduced above,

$$\sigma + 2\tau (p^2 + r^2) = (\tau + r^2)^2. \quad (6.6)$$

The polynomials (3.15), (3.24), (3.36) become

$$\Phi_1 = \kappa \varphi_1(u) \varphi_1(v), \quad \Phi_2 = \kappa \varphi_2(u) \varphi_2(v),$$

$$\Psi_1 = \kappa \psi_1(u) \psi_2(v), \quad \Psi_2 = \kappa \psi_2(u) \psi_1(v),$$

$$\Theta_1 = \kappa \theta_2(u) \theta_1(v), \quad \Theta_2 = \kappa \theta_1(u) \theta_2(v),$$

where $\kappa = 1/(u + v)^2$ and

$$\varphi_1(w) = \sqrt{\sigma}(1 + w^2) + 2(\tau + r^2)w, \quad \varphi_2(w) = \sqrt{\sigma}(1 + w^2) + 2(\tau - r^2)w,$$

$$\psi_1(w) = 2s[\chi + \sqrt{\tau} w^2 - (\chi - \sqrt{\tau})], \quad \psi_2(w) = 2s[\chi - \sqrt{\tau} w^2 - (\chi + \sqrt{\tau})],$$

$$\theta_1(w) = \sqrt{\sigma}(1 - w^2) + 4s \sqrt{\tau} w, \quad \theta_2(w) = \sqrt{\sigma}(1 - w^2) - 4s \sqrt{\tau} w.$$

Then from (3.26)–(3.28) we find the expressions for the configuration variables

$$x_1 = \frac{2s \tau}{r^2} \left[ \frac{\sqrt{\varphi_1(u) \varphi_2(v)} + \sqrt{\varphi_2(u) \varphi_1(v)}}{\sqrt{\theta_1(u) \theta_1(v)} - \sqrt{\theta_2(u) \theta_2(v)}} \right]^2,$$

$$x_2 = \frac{2s \tau}{r^2} \left[ \frac{\sqrt{\varphi_1(u) \varphi_2(v)} - \sqrt{\varphi_2(u) \varphi_1(v)}}{\sqrt{\theta_1(u) \theta_1(v)} + \sqrt{\theta_2(u) \theta_2(v)}} \right]^2,$$

$$y_1 = 2s \left[ \frac{\sqrt{\varphi_1(u) \varphi_2(v)} + \sqrt{\varphi_2(u) \varphi_1(v)}}{\sqrt{\theta_1(u) \theta_1(v)} - \sqrt{\theta_2(u) \theta_2(v)}} \right]^2 - 4\sigma(uv - 1)^2,$$

$$y_2 = 2s \left[ \frac{\sqrt{\varphi_1(u) \varphi_2(v)} - \sqrt{\varphi_2(u) \varphi_1(v)}}{\sqrt{\theta_1(u) \theta_1(v)} + \sqrt{\theta_2(u) \theta_2(v)}} \right]^2 - 4\sigma(uv - 1)^2,$$

$$z_1 = \frac{1}{2r(u + v)} \sqrt{\varphi_1(u) \varphi_1(v) + \sqrt{\varphi_2(u) \varphi_2(v)}},$$

$$z_2 = \frac{1}{2r(u + v)} \sqrt{\varphi_1(u) \varphi_1(v) - \sqrt{\varphi_2(u) \varphi_2(v)}}. \quad (6.9)$$
Hence, in particular,

\[
\sqrt{x_1} = \frac{\sqrt{2s}}{r} \left( \sqrt{\varphi_1(u)\varphi_2(v)} + \sqrt{\varphi_2(u)\varphi_1(v)} \right),
\]

\[
\sqrt{x_2} = \frac{\sqrt{2s}}{r} \left( \sqrt{\varphi_1(u)\varphi_2(v)} - \sqrt{\varphi_2(u)\varphi_1(v)} \right).
\]

Here the arbitrary choice of sign is provided by the algebraic value \(\sqrt{2s}\). From (3.33) we have

\[
\sqrt{x_2}w_1 = \frac{i}{4s} \frac{\sqrt{\varphi_2(u)\varphi_1(v) + \varphi_1(u)\varphi_2(v)}}{u + v},
\]

\[
\sqrt{x_1}w_2 = \frac{i}{4s} \frac{\sqrt{\varphi_2(u)\varphi_1(v) - \varphi_1(u)\varphi_2(v)}}{u + v}.
\]

Substitute (6.10) into (6.11) to obtain the expressions for the variables defining the equatorial velocity

\[
w_1 = \frac{ir}{4s\sqrt{2s}} \left( \frac{\sqrt{\varphi_2(u)\varphi_1(v) + \varphi_1(u)\varphi_2(v)}}{u + v} \right) \left[ \sqrt{\varphi_1(u)\varphi_2(v)} + \sqrt{\varphi_2(u)\varphi_1(v)} \right]
\]

\[
w_2 = \frac{ir}{4s\sqrt{2s}} \left( \frac{\sqrt{\varphi_2(u)\varphi_1(v) - \varphi_1(u)\varphi_2(v)}}{u + v} \right) \left[ \sqrt{\varphi_1(u)\varphi_2(v)} + \sqrt{\varphi_2(u)\varphi_1(v)} \right].
\]

The axial component is found from (3.33),

\[
w_3 = \frac{i}{2\sqrt{2s\pi\sigma}} \frac{\varphi_2(u)\varphi_2(v) - \varphi_2(u)\varphi_2(u)\varphi_1(v)}{(u + v)(uv - 1)}.
\]

Thus we have expressed all phase variables in terms of \(u, v\).

To obtain the differential equations for \(u, v\) consider the following variables

\[
s_1 = \frac{x^2 + z^2 + r^2}{2x}, \quad s_2 = \frac{x^2 + z^2 - r^2}{2x}.
\]

The derivatives, in virtue of the system (1.8), are

\[
s'_1 = \frac{r^2}{4x^3}(z_1 + z_2)(x_1w_2 - x_2w_1), \quad s'_2 = \frac{r^2}{4x^3}(z_1 - z_2)(x_1w_2 + x_2w_1).
\]

On the other hand, from (6.14), (6.5) we have

\[
s_1 = \frac{\sqrt{\sigma(uv + 1) + (\tau + r^2)(u + v)}}{2\sqrt{\tau(u - v)}}, \quad s_2 = \frac{\sqrt{\sigma(uv + 1) + (\tau - r^2)(u + v)}}{2\sqrt{\tau(u - v)}},
\]

whence

\[
\frac{\partial s_1}{\partial u} = -\frac{\varphi_1(v)}{2\sqrt{\tau(u - v)^2}}, \quad \frac{\partial s_1}{\partial v} = \frac{\varphi_1(u)}{2\sqrt{\tau(u - v)^2}},
\]

\[
\frac{\partial s_2}{\partial u} = -\frac{\varphi_2(v)}{2\sqrt{\tau(u - v)^2}}, \quad \frac{\partial s_2}{\partial v} = \frac{\varphi_2(u)}{2\sqrt{\tau(u - v)^2}}.
\]
Therefore,
\[
\frac{du}{dt} = \frac{2\sqrt{\tau(u-v)^2}}{\varphi_1(u)\varphi_2(v) - \varphi_2(u)\varphi_1(v)}[\varphi_2(u)\frac{ds_1}{dt} - \varphi_1(u)\frac{ds_2}{dt}],
\]
\[
\frac{dv}{dt} = \frac{2\sqrt{\tau(u-v)^2}}{\varphi_1(u)\varphi_2(v) - \varphi_2(u)\varphi_1(v)}[\varphi_2(v)\frac{ds_1}{dt} - \varphi_1(v)\frac{ds_2}{dt}].
\]
(6.16)

Substitute the values (6.9)–(6.11) into (6.15) and the resulting expressions into (6.16). We obtain
\[
f(u, v)\frac{du}{dt} = \frac{\sqrt{\varphi_1(u)\varphi_2(u)}\theta_1(u)\theta_2(u)}{2u\sqrt{2s\tau\sigma}}, \quad f(u, v)\frac{dv}{dt} = \frac{\sqrt{\varphi_1(v)\varphi_2(v)}\theta_1(v)\theta_2(v)}{2v\sqrt{2s\tau\sigma}},
\]
where
\[
f(u, v) = \frac{(u-v)(1-uv)}{uv} = (v + \frac{1}{v}) - (u + \frac{1}{u}).
\]

The variables (6.2) in terms of \(u, v\) have the form
\[
U = \frac{1}{2}\left(u + \frac{1}{u}\right), \quad V = \frac{1}{2}\left(v + \frac{1}{v}\right).
\]
(6.18)

To be definite choose \(u = U - \sqrt{U^2 - 1}, v = V - \sqrt{V^2 - 1}\). Then the equations (6.17) yield
\[
(U - V)\frac{dU}{dt} = \frac{1}{\sqrt{2s\tau\sigma}}\sqrt{Q(U)}, \quad (U - V)\frac{dV}{dt} = \frac{1}{\sqrt{2s\tau\sigma}}\sqrt{Q(V)}
\]
(6.19)

with the polynomial (6.4) of degree 6. This system is obviously equivalent to the system (6.3); the latter have the standard form for hyperelliptic quadratures.

To reveal the connection of Theorem 5 with the bifurcation diagram suppose that the polynomial (6.4) has a multiple root. The resultant of \(Q(w)\) and \(Q'(w)\) is
\[
2^{32}a^4b^4(a^2 - b^2)^2s^{10}\tau^{12}\sigma^8\chi^2(a^2 - s^2)^2(b^2 - s^2)^2.
\]
(6.20)

According to (2.14), \(s \neq 0\) on \(\mathcal{D}\). The equation \(\sigma = 0\) gives \(\tau = (a \pm b)^2\). The case \(\chi = 0\) provides the values \(\tau = (\sqrt{a^2 - s^2} \pm \sqrt{b^2 - s^2})^2\). Thus, the bifurcation diagram of \(J = S \times T\) belongs to the discriminant set of the polynomial (6.4). This fact is typical for systems with separating variables.

Note that all roots of the polynomial (6.4) are explicitly expressed in terms of the integral constants along with the roots of the polynomial
\[
\varphi_1(w)\varphi_2(w)\theta_1(w)\theta_2(w)
\]
(6.21)

defining the solutions of the system (6.17). The variables introduced in this section are real only in the case \(\tau > 0\) and \(\sigma > 0\). For other combinations of signs the definition (6.5) of the local coordinates \(u, v\) needs obvious modifications. Then the intervals of oscillations for the variables \(U, V\) and \(u, v\) are easily determined for any given values of \(s, \tau\) or, more exactly, for each connected component of \(\mathbb{R}^3 \setminus \Sigma(J)\). The obtained differential equations and the explicit formulae for the phase variables in terms of \(u, v\) provide the possibility to investigate the phase topology of the case considered and to calculate analytically the numerical invariants of the corresponding Liouville foliation.
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