High-Dimensional Diffusive Growth

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We consider a model of aggregation, both diffusion-limited and ballistic, based on the Cayley tree. Growth is from the leaves of the tree towards the root, leading to non-trivial screening and branch competition effects. The model exhibits a phase transition between ballistic and diffusion-controlled growth, with non-trivial corrections to cluster size at the critical point. Even in the ballistic regime, cluster scaling is controlled by extremal statistics due to the branching structure of the Cayley tree; it is the extremal nature of the fluctuations that enables us to solve the model.

I. INTRODUCTION

Diffusion-limited aggregation (DLA) \cite{1} is a model of central importance in the field of fractal growth. For the theorist, it presents a model with very simple rules that gives rise to complex, branching patterns; for the experimentalist, it provides a model relevant for dielectric breakdown \cite{2}, viscous fingering \cite{3}, electrodeposition \cite{4}, and dendritic growth \cite{5}, among other physical processes.

Recent progress in this field has been primarily restricted to \( d = 2 \) dimensions, due to the availability of powerful conformal techniques in the plane \cite{3}. However, it remains important to elucidate the basic physics of DLA, branch competition and screening, for all dimensions \( d \). A phenomenological model of branch competition was put forth by Halsey and coworkers \cite{6}. In this study, we present a high-dimensional model in which many of the branched growth ideas of Halsey may become exact. We find that the branched growth recursion equations are controlled by extremal fluctuations. Finally, we comment on whether the extremal fluctuations are important in DLA as well as in our model.

For a fractal cluster of dimension \( D \), the number of particles \( N(k) \) on a length scale \( k \) obeys

\[ N(k) \propto k^D. \tag{1} \]

It is known that for DLA, \( D \geq d - 1 \) \cite{4}; it is believed that \( D \to d - 1 \) as \( d \to \infty \). In order to study this high-dimensional limit, we choose to study diffusive and ballistic growth on a Cayley tree. Unlike models of diffusive growth on Cayley trees in which the cluster grows from the root towards the leaves \cite{1}, our clusters grow from the leaves towards the root on a Cayley tree with \( k \) generations of \( 2^k - 1 \) sites. The ballistic limit of this model has been considered previously \cite{10}. There, the scaling behavior of \( N(k) \) was found only for a modification of the model, for which \( N(k) \) behaves differently.

We observe non-trivial competition between the growing branches, due to both geometrical and diffusive screening effects. We find that extremal fluctuations dominate the growth, leading to \( N(k) \) much less than the compact result of \( 2^k \). \( N(k) \) still behaves exponentially in \( k \), so that \( D \) is infinite. By examining the behavior of \( N(k) \) in different regimes, we find power law corrections to the exponential behavior, which may be interpreted as codimensions.

II. THE MODEL

Consider a Cayley tree with a constant branching factor of \( 2 \) and a specified number of levels, \( k \). We choose to initially occupy all the leaves of the tree, leaving all the other nodes empty. Therefore, a tree of \( k \) levels will have a total of \( 2^k - 1 \) nodes, of which \( 2^{k-1} - 1 \) are initially empty. Random walkers will be released sequentially from the root of the tree to perform a random walk on the tree. When a random walker moves onto an occupied site, it is added to the cluster at the site which it occupied previous to the last random walk step.

Since each node has two daughters and one parent, there is an entropic bias to the random walk, causing it to drift towards the leaves. We offset this bias by defining a parameter \( \gamma \) such that the walker moves towards a given daughter node with probability \( 1/(2 + \gamma) \), and moves towards the parent node with probability \( \gamma/(2 + \gamma) \). At \( \gamma = 2 \), the walker is twice as likely to move to the parent as to any given daughter node, so the total probability of moving towards the leaves is equal to the probability of moving towards the root. At \( \gamma = 0 \), the walker only moves towards the daughter nodes, so the growth is ballistic, not diffusive. Growing branches can screen each other geometrically for all \( \gamma \), but diffusive screening is present only for \( \gamma > 0 \) (see figure 1).

Let us define the “absorbency” of a node. This is the probability that a random walker starting on that node will be absorbed into the cluster without first moving to that node’s parent. The advantage of the absorbency is that it hides some of the non-local effects in the calculation of random walk probabilities, and is useful for both analytical and numerical calculations. By definition the absorbency of an occupied node is equal to 1.

The absorbency, \( a \), of a node can be obtained solely from the absorbencies \( a_l, a_r \) of its left and right daughter nodes. We have

\[ a = \frac{a_l + a_r}{\gamma + a_l + a_r}. \tag{2} \]
For $\gamma < 2$, we find that for a tree in the initial configuration, the absorbency of the root tends to a constant for large $k$. For $\gamma = 2$ the initial absorbency of the root is equal to $1/k$. For $\gamma > 2$, the absorbency of the root scales as $(2/\gamma)^k$. This behavior of the absorbency suggests that $\gamma = 2$ is a critical point; we confirm this suspicion below.

Given the absorbencies, we can simulate the random walk as follows: start with the random walker at the root. Move the random walker towards the leaves at each step, until it hits an occupied node, with the relative probability of moving to the left or right daughters given by the relative absorbencies.

### III. Behavior at $\gamma = 0$

We first consider the $\gamma = 0$ model, in which the random walker only moves towards the leaves, with equal probability of moving left and right at each stage. Due to the branching nature of the Cayley tree, we can construct recursion relations to describe the growth. (These equations will be very similar to those obtained phenomenologically in the branched growth model.)

Define $p_0(N, k)$ to be the probability that the root is unoccupied after $N$ particles are added to a tree of $k$ levels. Define $p_1(N, k)$ to be the probability that the root of the tree is occupied by the addition of the $N$-th particle. Define $p_2(N, k)$ to be the probability that the root was occupied before the $N$-th particle was added (note that this means that some of the last particles must overflow from the tree). Clearly, $p_0(N, k) + p_1(N, k) + p_2(N, k) = 1$.

These probabilities obey simple recursion relations. Let $N$ particles be added to a tree of $j + 1$ levels, and assume that the root of the tree is unoccupied at this point. All the particles lie below the root in the two subtrees of the tree, and with probability $2^{-N}\binom{N}{N'}$ we find that $N'$ particles were added to the left subtree and $N - N'$ were added to the right subtree. The root of a given subtree is either empty, or became occupied only as a result of the last particle to be added to that subtree. Therefore,

\[
p_0(N, j + 1) = \sum_{N' = 0}^{N} 2^{-N}\binom{N}{N'} \times (1 - p_2(N', j))(1 - p_2(N - N', j)). \tag{3}
\]

Similarly, we can derive an equation for $p_1$. We find

\[
p_1(N, j + 1) = \sum_{N' = 0}^{N} 2^{-N}\binom{N}{N'} \times 2(p_1(N' - 1, j))(1 - p_2(N - N', j)). \tag{4}
\]

Combining the equations for $p_0$ and $p_1$, we obtain an equation for $p_2$.

Let us make a naive guess as to the solution of these equations. The guess, while incorrect, will be instructive. The number of particles which go into each subtree will fluctuate. If we imagine that for $N$ large these fluctuations become negligible, and further assume that $p_1 << p_0$, we find that

\[
p_0(N, j + 1) = p_0(N/2, j)^2. \tag{5}
\]

The solution to this equation is $p_0(N, j) = f(N/2^j)$ such that $f(x) = 1$ for $x < x_c$ and $f(x) = 0$ for $x > x_c$. This means that the root of the tree is always empty if less than $x_c2^k$ particles are added and after this the root is always occupied.

Indeed, for $j$ large, the noise in the recursion relation is exponentially small, so eq. (3) is not so different from eq. (5). However, if we perturb $f(x)$ to $f(x) = 1 - \epsilon(x)$ for $x < x_c$, we find that $\epsilon$ grows exponentially under eq. (5). From this we can guess that extremal statistics will become important. Since each node has an exponentially large number of nodes below it on the tree, even exponentially rare events may become important. In fact, using an argument based on extremal statistics, we now show that the average number of particles that must be added to a tree of $k$ levels in order to occupy the root is given by $N(k) \approx 2k^2\sqrt{2\pi k}$. Consider a tree of $k$ levels. Let $k = l + m$, and assume $l >> 1$, $m >> 1$. We can imagine the tree as being made up of $2^l$ subtrees, each of $m$ levels. Let us add $(m - 1)2^l$ particles to the tree. At this point, each of the subtrees has received roughly $m - 1$ particles (later, we will consider the effects of fluctuations in this number of particles). It is impossible for a subtree to complete its growth with only $m - 2$ particles added, since the subtree is $m$ levels deep, with the leaves initially seeded. However, it is possible for one of these subtrees to complete its growth after only $m - 1$ particles are added. To compute this probability we note that the first particle added to the subtree can go anywhere; the second particle must be added to the parent node of the first particle, which occurs with probability $2^{-(m-2)}$; the third particle must go to the parent node of the second particle, with probability $2^{-(m-3)}$; and so on. This yields the result that the filling probability, which is $p_1(m - 1, m)$, is $2^{-(m-1)(m-2)/2} \approx 2^{-m^2/2}$.

Thus, if $2^{l-2^{m^2/2}}$ is of order unity, it is likely that one of the subtrees will have been filled after only $(m - 1)2^l$ particles are added to the main tree. At this point, the next particle added from the root has probability $2^{-l}$ of attaching to the parent of the filled subtree, so that it will take approximately $2^l$ further particles added before the parent node of the subtree is occupied. At that point, it will take approximately $2^{l-1}$ more particles before the grandparent of the subtree is occupied. Eventually, after $2^l + 2^{l-1} + 2^{l-2} + \ldots = 2^{l+1}$ particles are added, we occupy the root of the whole tree. At this point, the total number
of particles added is \((m - 1)2^l + 2^{l+1} \approx m2^l\).

If we pick \(l\) and \(m\) correctly, we will thus take only \(m2^l\) particles to fill the tree. Since \(2^{2\gamma - m^2/2}\) must be of order unity, \(l \approx m^2/2\). For \(k >> 1\), we find \(m \approx \sqrt{2k}\) and \(l \approx k - \sqrt{2k}\). Once this number \((m2^l)\) particles has been added, we expect the tree to be filled so that

\[
N(k) \lesssim m2^l \approx 2^{k2 - \sqrt{2k}}\sqrt{2k}.
\]

Had we chosen \(l, m\) such that \(l > m^2/2\) we would find many filled subtrees, and a weaker inequality than eq. \((6)\). For \(l < m^2/2\), the argument does not work, because no subtree is filled.

Next, we have used eq. \((5)\) to argue that for \(l, m\), each subtree receives roughly \(N(k)/2^l\) particles. The number of particles arriving in a subtree forms an atypical structure are much more likely to be occupied. Using eq. \((3)\) instead of eq. \((2)\), one can show that the probability of

\[
\sum_{n=1}^{\infty} p_1(n, m) \frac{n_0^n}{m^n} e^{-n_0},
\]

with \(n_0 = N(k)/2^l\). The number of particles arriving in each subtree is still of order \(N(k)/2^l\) and the inclusion of these fluctuations does not change the estimate of the order of magnitude of \(N(k)\); the rare events considered in which a typical number of particles arrive in a subtree and form an atypical structure are much more likely than events in which an exponentially larger number of particles arrive and form a typical structure.

To justify the neglect of geometrical screening, we note that for \(l, m\), such that \(l < m^2/2\), we expect at most one subtree to be occupied. Thus, the argument is self-consistent.

Finally, we will express these estimates in terms of the recursion relations for \(p_0, p_1\). First, we have found that \(p_0(N, m) = 1\) for \(N < m - 1\). However,

\[
p_0(m - 1, m) = 1 - O(2^{-m^2/2}).
\]

Next, we have used eq. \((3)\) to argue that \(p_0(2^l(m-1), m + l) = 1 - O(2^{2\gamma - m^2/2})\) so that for \(2^{2\gamma - m^2/2} \approx 1\), the probability \(p_0(2^l(m-1), m + l) << 1\), which indicates that the root is likely to be occupied. Using eq. \((3)\) instead of eq. \((5)\) corresponds to including the fluctuations expressed by eq. \((6)\).

\section{IV. Behavior at \(\gamma > 0\)}

At \(\gamma = 0\), the only screening effect is the geometrical screening effect. We now repeat the argument based on extremal statistics for the case \(\gamma > 0\), when diffusive effects begin to increase the competition between subtrees.

For \(\gamma > 0\) it is impossible to write closed recursion relations for the probabilities \(p_0, p_1, p_2\). We can, however, write more general recursion relations, which include the absorbencies of the nodes.

Still, for any \(\gamma\), it is impossible to reach the root of an \(m\) level tree with \(m - 2\) particles. The probability of reaching the root of an \(m\) level tree with \(m - 1\) particles can be computed as in the \(\gamma = 0\) case, only taking the absorbencies into account. For \(\gamma < 2\), the result is still given by eq. \((6)\), while for \(\gamma = 2\), we find \(p_0((m-1), m) = 1 - O(m!2^{-m^2/2})\), and for \(\gamma > 2\), \(p_0((m-1), m) = 1 - O(2^{-m^2/2\log_2\gamma})\).

Using the same extremal argument as above, we find that, for \(\gamma < 2\),

\[
N(k) \lesssim 2^{k2 - \sqrt{2k}}\sqrt{2k},
\]

for \(\gamma = 2\),

\[
N(k) \lesssim 2^{k2 - \sqrt{2k}},
\]

and for \(\gamma > 2\),

\[
N(k) \lesssim 2^{k2 - \sqrt{2k\log_2\gamma}}\sqrt{2k\log_2\gamma}.
\]

In deriving this, we assume that each of the \(2^l\) subtrees continues to receive roughly \(N/2^l\) particles. However, now the number of particles arriving in a given subtree becomes non-Poissonian due to the effects of diffusive screening. Physically, this effect must reduce \(N(k)\).

We still conjecture that eqs. \((3)\) are estimates, and not just inequalities. Suppose we add only \(N(k) << m2^l\) particles to the main tree, with \(m, l\) chosen to obtain the most stringent inequality. Let us show that this is not enough particles to grow to the root of the tree. We again do this by showing that it is justified to assume that each subtree receives roughly \(N(k)/2^l\) particles, so that none of the subtrees can be completely occupied. To justify this assumption, we now need to show that diffusive, as well as geometrical, screening effects are negligible. When a subtree has received roughly \(N(k)/2^l\) particles, the top \(O(m)\) levels of each subtree are still likely to be completely empty. As a result, the absorbency of the subtree is is only slightly changed from that of a completely empty subtree and so the assumption of Poissonian statistics is reasonable.

Interestingly, the \(\gamma = 2\) result for \(N(k)\) is reduced from the \(\gamma < 2\) result by an amount of order \(\sqrt{k}\), implying a codimension of \(1/2\) at the critical point. For \(\gamma > 2\), we find an infinite codimension, as the corrections to the
\( \gamma < 2 \) results are exponential, not power law. These results should be compared to the expected co-dimension of 1 for DLA in high dimensions.

V. COMPARISON TO NUMERICS

Using the absorbency technique, we have performed numerical calculations on the model for trees up to \( k = 24 \) for \( \gamma = 0, 1, 2, 4 \). In figure 2 we show \( \log_2(N(k)/2^k) \) averaged over 100 runs for each data point. As can be seen, \( N(k) \ll 2^k \). Next, in figure 3, we show \( N(k) \) divided by the appropriate theoretical result for the scaling of \( N(k) \). For all curves, we see that the plot decreases initially, and then appears to stabilize at some given value, indicating that the theoretical result is in agreement with numerics. The theoretical calculation is only able to produce the asymptotic scaling of \( N(k) \). However, the prefactor for \( N(k) \) appears to be close to unity for \( \gamma = 0 \).

VI. CONCLUSION

We have introduced a model that provides an interesting realization of some ideas of branched growth. We find in eq. (5) a simple recursion equation for the branch competition. However, the solution of this equation is unstable to noise. On the Cayley tree, the noise grows indefinitely leading to a situation controlled by extremal statistics, not unlike the “infinite disorder” critical points known in statistical mechanics [11].

In infinite dimensional DLA, we still expect that branch competition will “self-generate” noise, causing small perturbations of a noiseless solution to grow, leading to a regime with non-vanishing fluctuations. Extremal statistics have been used to estimate the fractal dimension of DLA by Turkevich and Scher [12]. The relative importance of large fluctuations as a function of dimensionality remains an unsolved problem.

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FIG. 3. \( N(k)/(2^{k + \sqrt{2k}}) \) as a function of \( k - 1 \) for \( \gamma = 0, 1 \); \( N(k)/(2^{k + \sqrt{2k}}) \) for \( \gamma = 2 \); \( N(k)/(2^{k + \sqrt{2k} \log_2 \gamma}) \) for \( \gamma = 4 \). All curves asymptote to a constant.