ON THE MANY DIRICHLET LAPLACIANS ON A NON-CONVEX POLYGON AND THEIR APPROXIMATIONS BY POINT INTERACTIONS

ANDREA POSILICANO

Abstract. By Birman and Skvortsov it is known that if $\Omega$ is a plane curvilinear polygon with $n$ non-convex corners then the Laplace operator with domain $H^2(\Omega) \cap H^1_0(\Omega)$ is a closed symmetric operator with deficiency indices $(n, n)$. Here, by providing all self-adjoint extensions of such a symmetric operator, we determine the set of self-adjoint non-Friedrichs Dirichlet Laplacians on $\Omega$ and, by a corresponding Krein-type resolvent formula, show that any element in this set is the norm resolvent limit of a suitable sequence of Friedrichs-Dirichlet Laplacians with $n$ point interactions.

1. Introduction.

Since their rigorous mathematical definition by Berezin and Faddeev \cite{2} as self-adjoint extensions of the Laplacian restricted to smooth functions with compact support disjoint from a finite set of points in $\mathbb{R}^d$, $d \leq 3$, point perturbations of the Laplacian have attracted a lot of attention and have been used in a wide range of applications, as the huge list of references provided in \cite{1} shows. Successively point perturbations of the Dirichlet Laplacian on a bounded domain have been defined in a similar way, see \cite{5}, \cite{4}, \cite{7}. In this case, since functions in the domain of the Dirichlet Laplacian vanish at the boundary, points perturbations can not be placed there. Nevertheless one could try to put point-like perturbations at the boundary by moving the points supporting the perturbation towards the boundary while increasing the interactions strengths, so to compensate the vanishing of the functions. However it is not clear how to implement this procedure, because there is no universal behavior for the functions in the operator domain in a neighborhood of the boundary. For example if $\Omega \subset \mathbb{R}^2$ is a plane bounded domain which either has a regular (i.e. $C^{1,1}$) boundary or has a Lipschitz boundary and is convex, then the self-adjoint Friedrichs-Dirichlet Laplacian on $L^2(\Omega)$ has domain $H^2(\Omega) \cap H^1_0(\Omega)$. Thus, by the (dense) inclusions $C^\infty(\Omega) \subset H^2(\Omega) \cap H^1_0(\Omega) \subset C_0(\Omega)$, there is no minimal vanishing rate for $u(x)$ as $x$ approaches the boundary. The
situation changes if one considers a plane non-convex polygon. Indeed in this case the Friedrichs-Dirichlet Laplacian has a domain that is strictly larger than $H^2(\Omega) \cap H_0^1(\Omega)$ and for any function $u$ in such a domain one has $u(x) \sim \xi_u s_v(x)\|x - v\|^{\pi/\omega}$ when $\|x - v\| \ll 1$, where $v$ is any vertex at a non-convex corner, $\omega > \pi$ is the measure of the interior angle at $v$, and $0 < s_v(x) \leq 1$. This indicates that it should be possible to renormalize the value of $u$ at $v$ by considering the limit of $\|x - v\|^{-\pi/\omega} s_v(x)^{-1}u(x)$ as $x \to v$. Indeed such a procedure works and in the case of an arbitrary point perturbation of the Friedrichs-Dirichlet Laplacian on a plane polygon $\Omega$ with $n$ non-convex corners, the limit operator, as the $n$ points supporting the perturbations converge to the $n$ non-convex vertices, turns out to be a well defined self-adjoint operator: it coincides with a self-adjoint extension of the closed symmetric operator (which by [3] has deficiency indices $(n, n)$ ) given by the Laplace operator on $H^2(\Omega) \cap H_0^1(\Omega)$.

The proof we provide in this paper follows the reverse path. At first in Section 2 we determine all the self-adjoint extensions of the Laplace operator on $H^2(\Omega) \cap H_0^1(\Omega)$, $\Omega$ a bounded non-convex curvilinear polygon, and provide a corresponding Krein’s resolvent formula (unknown to the author, some similar results had been given in last section of the unpublished paper [6]; we thank Mark Malamud for the communication). The operator domain of any of such extensions is contained in the kernel of the unique continuous extension of the trace (evaluation) operator along the boundary to the domain of the maximal Laplacian, and the functions in the operator domains still satisfy the Dirichlet’s boundary condition $u(x) = 0$, provided $x$ is not the vertex at a non-convex corner. Thus such family of self-adjoint extensions forms a set of non-Friedrichs Dirichlet Laplacians, the Friedrichs-Dirichlet Laplacian being the only one satisfying Dirichlet’s boundary conditions also at the vertices of the non-convex corners.

Then, in Section 3, we define an arbitrary $n$-point perturbation of the Friedrichs-Dirichlet Laplacian $\Delta_F^\Omega$, again together with a corresponding Krein’s resolvent formula (see [4] and [7] for similar results), and we show that, if the points supporting the perturbations converge to the non-convex vertices of $\Omega$, while the coupling strength is renormalized according to the vanishing rate of the functions in $\mathcal{D}(\Delta_F^\Omega)$, these self-adjoint operators converge in norm resolvent sense to the self-adjoint extensions provided in Section 2 (see Theorem 3.6).

In the Appendix, we collect some results about self-adjoint extensions of symmetric operators that we need in the proofs.

We conclude the introduction by giving some notations:
• \( \mathcal{D}(L) \), \( \mathcal{H}(L) \), \( \mathcal{R}(L) \), \( \rho(L) \) denote the domain, kernel, range and resolvent set of a closed linear operator \( L \) on an Hilbert space \( \mathcal{H} \);
• \( \| \phi \|_L = (\| L\phi \|_\mathcal{H} + \| \phi \|_\mathcal{H})^{1/2} \) denotes the graph norm on \( \mathcal{D}(L) \);
• \( L|\mathcal{V} \) denotes the restriction of \( L \) to \( \mathcal{V} \subset \mathcal{D}(L) \);
• \( L^2(\Omega) \) denotes the Hilbert space of square-integrable functions on the open domain \( \Omega \subset \mathbb{R}^2 \) with scalar product \( \langle u, v \rangle_{L^2(\Omega)} = \int_{\Omega} \bar{u}(x)v(x) \, dx \);
• The dot \( \cdot \) denotes the scalar product on \( \mathbb{C}^n \), i.e. \( \xi \cdot \zeta = \sum_{k=1}^{n} \bar{\xi}_k \zeta_k \);
• \( \mathcal{C}_H^n \equiv \mathcal{R}(\Pi) \) denotes the subspace corresponding to the orthogonal projector \( \Pi : \mathbb{C}^n \rightarrow \mathbb{C}^n \). By a slight abuse of notation we use the same symbol \( \Pi \) also to denote the injection \( \Pi|_{\mathcal{C}_H^n} : \mathcal{C}_H^n \rightarrow \mathbb{C}^n \) and the surjection \( (\Pi|_{\mathcal{C}_H^n})^* : \mathbb{C}^n \rightarrow \mathcal{C}_H^n \);
• \( E(\mathbb{C}^n) \) denotes the bundle \( p : E(\mathbb{C}^n) \rightarrow P(\mathbb{C}^n) \), where \( P(\mathbb{C}^n) \) is the set of orthogonal projectors on \( \mathbb{C}^n \) and \( p^{-1}(\Pi) \) is the set of symmetric operators on \( \mathcal{C}_H^n \);
• \( c \) denotes a generic strictly positive constant which can change from line to line.

2. Dirichlet Laplacians on a non-convex plane polygon.

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open Lipschitz domain. This means that in the neighborhood of any of its point \( \Omega \) is below the graph of a Lipschitz function and such a graph coincides with its boundary \( \Gamma \).

We denote by \( \Delta_\Omega \) the distributional Laplace operator on \( \Omega \) and we define
\[
\Delta_\Omega^{\text{max}} : \mathcal{D}(\Delta_\Omega^{\text{max}}) \subset L^2(\Omega) \rightarrow L^2(\Omega) , \quad \Delta_\Omega^{\text{max}} u := \Delta_\Omega u ,
\]
where
\[
\mathcal{D}(\Delta_\Omega^{\text{max}}) := \{ u \in L^2(\Omega) : \Delta_\Omega u \in L^2(\Omega) \} .
\]
We denote by \( C^\infty(\bar{\Omega}) \) the set of functions on \( \bar{\Omega} \), the closure of \( \Omega \), which are restriction to \( \bar{\Omega} \) of smooth functions with compact support on \( \mathbb{R}^d \) and we denote by \( H^k(\Omega) \) the Sobolev-Hilbert space given by closure of \( C^\infty(\bar{\Omega}) \) with respect to the norm defined by
\[
\| u \|_{H^k(\Omega)}^2 = \sum_{0 \leq \alpha_1 + \alpha_2 \leq k} \| \partial_1^{\alpha_1} \partial_2^{\alpha_2} u \|_{L^2(\Omega)}^2 .
\]
By Sobolev embedding theorem one has, for any \( \alpha \in (0, 1) \),
\[(2.1) \quad H^2(\Omega) \subseteq C^\alpha(\Omega) \]
and
\[(2.2) \quad \forall u \in H^2(\Omega) , \forall x, y \in \bar{\Omega} , \quad |u(x) - u(y)| \leq c \| u \|_{H^2(\Omega)} \| x - y \|^\alpha .\]
Analogously $H_0^0(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$, the set of smooth function with compact support on $\Omega$, with respect to the same norm. The space $H_0^1(\Omega)$ can be equivalently defined by

$$H_0^1(\Omega) := \{ u \in H^1(\Omega) : \gamma_0 u = 0 \},$$

where

$$\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$$

is the unique continuous linear map such that

$$\forall u \in C^\infty(\bar{\Omega}), \quad \forall x \in \Gamma, \quad \gamma_0 u(x) = u(x).$$

There is a standard, well known way to define a self-adjoint Dirichlet Laplacian on $L^2(\Omega)$: since the symmetric sesquilinear form

$$F_\Omega : H_0^1(\Omega) \oplus H_0^1(\Omega) \subset L^2(\Omega) \oplus L^2(\Omega) \to \mathbb{C}, \quad F_\Omega(u, v) := \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}$$

is closed and positive, by Friedrichs’ extension theorem there exists an unique positive self-adjoint operator

$$\Delta^F_\Omega : \mathcal{D}(\Delta^\text{max}_\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega), \quad \Delta^F_\Omega u = \Delta_\Omega u,$$

such that

$$\forall u \in \mathcal{D}(\Delta^\text{max}_\Omega) \cap H_0^1(\Omega), \quad \forall v \in H_0^1(\Omega), \quad F_\Omega(u, v) = -\langle \Delta^F_\Omega u, v \rangle_{L^2(\Omega)}.$$ 

Moreover

$$\mathcal{D}(\Delta^F_\Omega) \equiv \mathcal{D}(\Delta^\text{max}_\Omega) \cap H_0^1(\Omega)$$

is dense in $H_0^1(\Omega)$, $0 \in \rho(\Delta^F_\Omega)$, $-\Delta^F_\Omega$ has a compact resolvent, and its spectrum consists of an infinite sequence

$$\lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \ldots$$

of strictly positive eigenvalues each having finite multiplicity, $\lambda_1(\Omega)$ being simple. We call $\Delta^F_\Omega$ the Friedrichs-Dirichlet Laplacian.

In the case $\Omega$ is piecewise regular, in particular is a plane curvilinear polygon, there is another way to produce a self-adjoint Dirichlet Laplacians on $L^2(\Omega)$.  

From now on we suppose that $\Omega \subset \mathbb{R}^2$ is a plane bounded open curvilinear polygon (cups points are not allowed) which coincides with a plane polygon in the neighborhood of any (eventual) non-convex corner.

Let us recall the following Caccioppoli-type regularity estimate: for any $u \in C_\infty^\infty(\Omega) := \{ u \in C^\infty(\Omega) : u(x) = 0, \ x \in \Gamma \}$ one has

$$\|u\|_{H^2(\Omega)} \leq c \|\Delta_\Omega u\|_{L^2(\Omega)}. \quad (2.3)$$

The proof of such an estimate, for general elliptic second order differential operator on a class of bounded open sets which includes curvilinear polygons, can be found in [12], Chapter 3, Section 8. For the Laplace operator on polygons a simpler proof is given in [10], Theorem 2.2.3.
By (2.3), since \( C_0^\infty(\Omega) \) is dense in \( H^2(\Omega) \cap H_0^1(\Omega) \), (see e.g. [10], Theorem 1.6.2), the restriction of \( \Delta_\Omega \) to \( C_0^\infty(\Omega) \) is closable and its closure is given by

\[
\Delta_\Omega^\circ : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega), \quad \Delta_\Omega^\circ u := \Delta_\Omega u.
\]

By Green’s formula for curvilinear polygons (see [9], Lemma 1.5.3.3) \( \Delta_\Omega^\circ \) is a closed symmetric operator. Thus a natural question arises: is \( \Delta_\Omega^\circ \) self-adjoint? Equivalently: does \( \Delta_\Omega^\circ \) coincide with \( \Delta_\Gamma^F \)? If \( \Omega \) had a regular boundary with no corners then \( \mathcal{D}(\Delta_\Omega^{\max}) \cap H_0^1(\Omega) = H^2(\Omega) \cap H_0^1(\Omega) \), i.e. \( \Delta_\Omega^\circ = \Delta_\Gamma^F \). Otherwise the answer depends on the shape of \( \Omega \). Indeed if \( \Omega \) is a curvilinear polygon then, as it has been proven in [3], the deficiency indices of \( \Delta_\Omega^\circ \) are both equal to \( n \), the number of non-convex corners of \( \Omega \). In this case \( \mathcal{D}(\Delta_\Omega^{\max}) \cap H_0^1(\Omega) \neq H^2(\Omega) \cap H_0^1(\Omega) \) is an immediate consequence of the fact that the function

\[
u(r, \theta) = r^\beta \sin \beta \theta, \quad \beta := \frac{\pi}{\omega},
\]

belongs \( H^1(W) \), where \( W \) is the wedge

\[
W = \{ x \equiv (r \cos \theta, r \sin \theta) : 0 < r < 1, \ 0 < \theta < \omega \},
\]

is in \( \mathcal{D}(\Delta_W^{\max}) \), since \( \Delta_W u = 0 \), but fails to be in \( H^2(W) \) when \( \omega > \pi \).

From now on we will suppose that \( n > 0 \) so that

\[
\mathcal{D}(\Delta_\Omega^{\max}) \equiv H^2(\Omega) \cap H_0^1(\Omega) \not\subseteq \mathcal{D}(\Delta_\Omega^{\max}) \cap H_0^1(\Omega) \equiv \mathcal{D}(\Delta_\Gamma^F).
\]

Since \( \Delta_\Omega^{\max} \) is the adjoint of the restriction of \( \Delta_\Omega \) to \( C_c^\infty(\Omega) \), one has \( (\Delta_\Omega^\circ)^* \subset \Delta_\Omega^{\max} \) and so any self-adjoint extension of \( \Delta_\Omega^\circ \) acts on the functions in its domain as the distributional Laplacian.

Let us at first characterize \( \mathcal{K}(\Delta_\Omega^\circ)^* \). To this end we need the extension \( \hat{\gamma}_0 \) of \( \gamma_0 \) to \( \mathcal{D}(\Delta_\Omega^{\max}) \) provided in [9], Theorem 1.5.3.4, and [10], Theorem 1.5.2: there exits an unique continuous map

\[
\hat{\gamma}_0 : \mathcal{D}(\Delta_\Omega^{\max}) \to \oplus_{i=1}^m \tilde{H}^{-\frac{1}{2}}(\Gamma_i),
\]

which coincides with \( \gamma_0 \) on \( \mathcal{D}(\Delta_\Omega^{\max}) \cap H^1(\Omega) \). Here \( \tilde{H}^{-\frac{1}{2}}(\Gamma_i) \) denote Hilbert spaces of distributions on the smooth curves \( \Gamma_i, i = 1, \ldots, m \), which union, together with their endpoints (i.e. the vertices of \( \Omega \)), give \( \Gamma \). We do not need here the precise definition of \( \tilde{H}^{-\frac{1}{2}}(\Gamma_i) \), see the quoted references for the details. Then, by Lemma 2.3.1 and Theorem 2.3.3 in [10], one has the following

**Theorem 2.1.**

\[
\mathcal{K}(\Delta_\Omega^\circ)^* = \mathcal{K}(\Delta_\Omega^{\max}) \cap \mathcal{K}(\hat{\gamma}_0).
\]
As we already said before, contrarily to the case of a domain $\Omega$ either convex or with a regular boundary, the kernel of $(\Delta^0_\Omega)^*$ is not trivial. Indeed (see \cite{3} and \cite{10})
\begin{equation}
\dim \mathcal{H}((\Delta^0_\Omega)^*) = \text{number of non-convex corners of } \Omega.
\end{equation}

In order to better characterize $\mathcal{H}((\Delta^0_\Omega)^*)$ we introduce some more definitions. Let $V = \{v_1, \ldots, v_n\}$ the set of vertices at the non-convex corners of $\Omega$ and, for any $v_k \in V$, let $\omega_k > \pi$ denote the measure of the corresponding interior angle. We define the wedge
\[
W^R_k := \Omega \cap D^R_k = \{x_k \equiv (r_k \cos \theta_k, r_k \sin \theta_k) \in \mathbb{R}^2 : 0 < r_k < R, \ 0 < \theta_k < \omega_k\},
\]
where $D^R_k$ denotes the disk of radius $R$ centered at $v_k$; we choose $R$ small enough to have $W^R_i \cap W^R_k = \emptyset, i \neq k$. On any disk $D_k$ centered at $v_k$ we consider the functions $u^\pm_k \in \mathcal{H}(\Delta^\text{max}_{D_k})$ defined by
\[
u^\pm_k(r_k, \theta_k) = 1 \sqrt{\pi \sin \beta_k \theta_k}, \quad \beta_k := \frac{\pi}{\omega_k},
\]
and we take $f \in C^{1,1}(\mathbb{R}_+), i.e. f$ is differentiable with a Lipschitz derivative, such that $0 \leq f \leq 1$, $f(r) = 1$ if $0 < r \leq R/3$ and $f(r) = 0$ if $r \geq 2R/3$. With such a choice we have $fu^\pm_k \in L^2(\Omega)$ and, since $\text{supp}(fu^\pm_k) \subset W^R_k$, the functions $fu^\pm_k$ are $L^2(\Omega)$-orthogonal and thus linearly independent.

**Lemma 2.2.** Let us define
\[
s_k := fu^+_k, \quad \sigma_k := fu^-_k, \quad g_k := \sigma_k + (-\Delta^F_\Omega)^{-1} \Delta_\Omega \sigma_k.
\]
Then
1) $s_k \in \mathcal{D}(\Delta^F_\Omega)$, $\sigma_k \in \mathcal{D}((\Delta^0_\Omega)^*) \cap \mathcal{H}(\gamma_0)$;
2) $g_k$ is the unique function in $\mathcal{H}((\Delta^0_\Omega)^*)$ such that
\[
g_k - \sigma_k \in \mathcal{D}(\Delta^F_\Omega);
\]
3) the $g_k$'s are linearly independent;
4) \[
\langle g_i, (-\Delta^F_\Omega)s_k \rangle_{L^2(\Omega)} = \delta_{ik}.
\]

**Proof.** By $u^+_k \in C^\infty(W^R_k) \cap H^1(W^R_k)$ one has $s_k \in \mathcal{D}(\Delta^\text{max}_{D_k}) \cap H^1_0(\Omega)$. By $\sigma_k \in \mathcal{H}(\gamma_0)$ there follows $g_k \in \mathcal{D}(\Delta^\text{max}_{D_k}) \cap \mathcal{H}(\gamma_0)$. Hence $g_k \in \mathcal{H}((\Delta^0_\Omega)^*)$ by Theorem 2.1. This also shows that $\sigma_k \in \mathcal{D}((\Delta^0_\Omega)^*)$. Proof of point 2 is then completed by $\mathcal{H}(\Delta^F_\Omega) = \{0\}$. 

Take $c_1, \ldots, c_n$ such that $\sum_{k=1}^n c_k g_k = 0$. Then
\[
(\Delta^F)_{-1} \Delta \sum_{k=1}^n c_k \sigma_k = \sum_{k=1}^n c_k \sigma_k.
\]
This gives $c_1 = \cdots = c_n = 0$, since the $\sigma_k$’s are linearly independent and do not belong to $\mathcal{D}(\Delta^F)$. Thus point 3 is proven.

As regards point 4, let us pose $W_k := W_k^{2R/3} \setminus W_k^{R/3}$. Then
\[
\langle g_k, \Delta^F \sigma_k \rangle_{L^2(\Omega)} = \langle \sigma_k, \Delta^F \sigma_k \rangle_{L^2(k)} - \langle \Delta^F \sigma_k, \sigma_k \rangle_{L^2(k)}
\]
\[
= \int_{W_k} f u_k \left( f'' u_k^+ + \left( 1 + \frac{2\pi}{\omega_k} \right) \frac{1}{r} f' u_k^+ \right) dx
\]
\[- \int_{W_k} \left( f'' u_k^- + \left( 1 - \frac{2\pi}{\omega_k} \right) \frac{1}{r} f' u_k^- \right) f u_k^+ dx
\]
\[
= \frac{2}{\omega_k} \int_{0}^{2R/3} 2 f f \int \omega_k \sin^2 \frac{\pi}{\omega_k} \theta d\theta = -2 \frac{1}{\pi} \int_{0}^{\pi} \sin^2 \theta d\theta = -1.
\]
\[
\square
\]
We define the $\mathbb{C}^n$-valued functions
\[
s \equiv (s_1, \cdots, s_n), \quad \sigma \equiv (\sigma_1, \cdots, \sigma_n), \quad g \equiv (g_1, \cdots, g_n).
\]
By Lemma 2.3.6 and (the proof of) Theorem 2.3.7 in [10], (2.7) can be precised:

**Theorem 2.3.** For any $u \in \mathcal{H}((\Delta^o)\ast)$ there exist an unique $\xi_u \in \mathbb{C}^n$ such that $u = g \cdot \xi_u$.

In order to use the results given in the Appendix we need a more precise characterization of $\mathcal{D}(\Delta^o) \cap H_0^1(\Omega)$ i.e. of $\mathcal{D}(\Delta^F)$:

**Theorem 2.4.**
\[
\mathcal{D}(\Delta^F) = \{ u \in L^2(\Omega) : u = u_0 + s \cdot \xi_u, \; u_0 \in \mathcal{D}(\Delta^o), \; \xi_u \in \mathbb{C}^n \}.
\]
**Proof.** By point 4 in Lemma 2.2 the linearly independent functions $\Delta^o \sigma_k$ are not orthogonal to $\mathcal{H}((\Delta^o)\ast)$. Thus, given $u \in \mathcal{D}(\Delta^F)$, the decomposition $L^2(\Omega) = \mathcal{H}(\Delta^o) \oplus \mathcal{H}((\Delta^o)\ast)$ implies that there exist unique $u_0 \in \mathcal{D}(\Delta^o)$ and $\xi_u \in \mathbb{C}^n$ such that $\Delta^F u = \Delta^o u_0 + \Delta^F s \cdot \xi_u$. \square

Next we introduce a convenient map $\tau^\ast_{\Omega}$ such that $\mathcal{D}(\Delta^o) = \mathcal{H}(\tau^\ast_{\Omega})$:

**Lemma 2.5.** Let
\[
\tau^\ast_{\Omega} : \mathcal{D}(\Delta^F) \to \mathbb{C}^n, \quad (\tau^\ast_{\Omega} u)_k := \frac{\sqrt{\pi^3}}{4} (2 + \beta_k) \lim_{R \downarrow 0} \frac{1}{R^{2\beta_k}} \langle u \rangle_{W_k^R},
\]
Lemma 2.6. \( \langle u \rangle_{W^k} \) denotes the mean of \( u \) over the wedge \( W^k \). Then \( \tau^V_\Omega \) is well defined, continuous, surjective and \( \mathcal{H}(\tau^V_\Omega) = \mathcal{B}(\Delta^F_\Omega) \).

Proof. By Theorem 2.4 \( \mathcal{D}(\Delta^F_\Omega) = \mathcal{D}(\Delta_\Omega) + \mathcal{V} \), where both \( \mathcal{D}(\Delta_\Omega) \) and \( \mathcal{V} \) are closed subspaces of \( \mathcal{D}(\Delta^F_\Omega) \), and \( \mathcal{D}(\Delta^F_\Omega) \cap \mathcal{V} = \{0\} \). Therefore the map

\[ P_0 : \mathcal{D}(\Delta^F_\Omega) \to \mathbb{C}^n, \quad P_0 u = \zeta_u, \]

given by the composition of the continuous projection onto \( \mathcal{V} \) with the map giving \( \mathcal{V} \simeq \mathbb{C}^n \), is continuous. To conclude we show that \( \tau^V_\Omega = P_0 \), i.e. \( \tau^V_\Omega u = \zeta_u \). By Theorem 2.4 one has \( u = u_0 + s \cdot \zeta_u \). Thus, by using (2.2) with \( \alpha \in (\beta_k, 1) \), one has

\[ |\tau^V_\Omega u_0| \leq c \lim_{R \to 0} \frac{1}{R^{\beta_k}} \left( \frac{\omega_k}{2} R^2 \right)^{-1} \int_{W^k} |u_0(x)| \, dx \leq c \lim_{R \to 0} R^{a - \beta_k} = 0, \]

while

\[ (\tau^V_\Omega s \cdot \zeta_u)_k = \frac{\sqrt{\pi}}{4} (2 + \beta_k) (\zeta_u)_k \lim_{R \to 0} \frac{1}{R^{\beta_k}} \langle s_k \rangle_{W^k} \]

\[ = (\zeta_u)_k \frac{\beta_k}{2} \int_0^{\omega_k} \sin \beta_k \theta \, d\theta \lim_{R \to 0} \frac{2 + \beta_k}{R^{2 + \beta_k}} \int_0^R r^{1 + \beta_k} \, dr = (\zeta_u)_k, \]

and the proof is done. \( \square \)

By Theorem 2.4 and Lemma 2.5 we can determine all self-adjoint extensions of \( \Delta_\Omega \), together with their resolvents, by the methods provided in Appendix with \( A = \Delta^F_\Omega \) and \( \tau = \tau^V_\Omega \). To this end we give the following

Lemma 2.6. Let

\[ G_z : \mathbb{C}^n \to L^2(\Omega), \quad G_z^* : L^2(\Omega) \to \mathbb{C}^n, \quad z \in \rho(\Delta^F_\Omega), \]

be defined by

\[ G_z := (\tau^V_\Omega(-\Delta^F_\Omega + z)^{-1})^*. \]

Then

\[ G_z \xi = \sigma \cdot \xi - (-\Delta^F_\Omega + z)^{-1}(-\Delta_\Omega + z) \sigma \cdot \xi \]

and

\[ G_z^* u = \langle \sigma, u \rangle_{L^2(\Omega)} - \langle (-\Delta^F_\Omega + z)^{-1}(-\Delta_\Omega + z) \sigma, u \rangle_{L^2(\Omega)}. \]

Proof. By Lemma 2.2, Theorem 2.4 and Lemma 2.5 one has

\[ \langle g, -\Delta^F_\Omega u \rangle_{L^2(\Omega)} = \langle g, -\Delta^F_\Omega u_0 - \Delta^F_\Omega s \cdot \zeta_u \rangle_{L^2(\Omega)} \]

\[ = \langle (-\Delta^F_\Omega)^* g, u_0 \rangle_{L^2(\Omega)} + \langle g, -\Delta^F_\Omega s \rangle_{L^2(\Omega)} \cdot \zeta_u \]

\[ = \zeta_u = \tau^V_\Omega u. \]
Thus
\[ \langle G_0 \xi, u \rangle_{L^2(\Omega)} = \xi \cdot \tau_\Omega (-\Delta_\Omega^F)^{-1} u = \xi \cdot \langle g, u \rangle_{L^2(\Omega)}, \]
i.e.
\[ G_0 \xi = g \cdot \xi = \sigma \cdot \xi + (-\Delta_\Omega^F)^{-1} \Delta_\Omega \sigma \cdot \xi. \]
By (4.1) one has then
\[ G_\tau \xi = (1 - z(-\Delta_\Omega^F + z)^{-1}) G_0 \xi = (g - z(-\Delta_\Omega^F + z)^{-1} g) \cdot \xi \]
\[ = (\sigma - (-\Delta_\Omega^F + z)^{-1}(-\Delta_\Omega + z) \sigma) \cdot \xi. \]

Notice that in next theorem we use the extension, denoted by the same symbol, of \((\tau^\nabla_\Omega)\) to functions coinciding away from \(\nu_k\) with functions in \(\mathcal{D}(\Delta_\Omega^F)\).

**Theorem 2.7.** Any self-adjoint extension of \(\Delta_\Omega^\nabla\) is of the kind
\[ \Delta_\Omega^\nabla \in \mathcal{D}(\Delta_\Omega^\nabla) \subset L^2(\Omega) \rightarrow L^2(\Omega), \quad \Delta_\Omega^\nabla u := \Delta_\Omega u, \]
\[ \mathcal{D}(\Delta_\Omega^\nabla) := \{ u \in L^2(\Omega) : u = u_0 + g \cdot \xi_u, \quad u_0 \in \mathcal{D}(\Delta_\Omega^0), \quad \xi_u \in \mathcal{C}_n, \quad \Pi \tau^\nabla_\Omega u = \Theta \xi_u \}, \]
where \((\Pi, \Theta) \in \mathcal{E}(\mathbb{C}^n)\) and
\[ (\tau^\nabla_\Omega u)_{(\nu)} = (\tau^\nabla_\Omega (u - (\xi_u)_{(\nu)} g_k))_{(\nu)}. \]
Moreover
\[ (-\Delta_\Omega^\nabla + z)^{-1} = (-\Delta_\Omega^F + z)^{-1} + G_z \Pi (\Theta + \Pi \Gamma_z \Pi)^{-1} \Pi G_z^*, \]
where
\[ (\Gamma_z)_{ij} = \left( \| \sigma_{ij} \|_{L^2(\Omega)}^2 + \| (-\Delta_\Omega^F)^{-\frac{1}{2}} \Delta_\Omega \sigma_{ij} \|_{L^2(\Omega)}^2 \right) \delta_{ij} \]
\[ - \langle (-\Delta_\Omega^F + z)^{-1}(-\Delta_\Omega + z) \sigma_i, (-\Delta_\Omega + z) \sigma_j \rangle_{L^2(\Omega)}. \]

**Proof.** By Theorem 4.1 any \(u = u_0 + G_0 \xi_u = u_0 + g \cdot \xi_u\) in the domain of a self-adjoint extension has to satisfy the boundary condition \(\Pi \tau^\nabla_\Omega u_0 = \tilde{\Theta} \xi_u\), for some \((\Pi, \tilde{\Theta}) \in \mathcal{E}(\mathbb{C}^n)\). Thus
\[ (\tau^\nabla_\Omega u_0)_{(i)} = (\tau^\nabla_\Omega (u - g_i(\xi_u)_{(i)}))_{(i)} - \sum_{j \neq i} (\tau^\nabla_\Omega (g_j(\xi_u)_{(j)}))_{(i)} = (\tau^\nabla_\Omega u)_{(i)} - (\Lambda \xi_u)_{(i)}, \]
where, by Lemma 2.6
\[ (2.5) \quad \Lambda_{ij} = \langle (-\Delta_\Omega^F)^{-1} \Delta_\Omega \sigma_i, \Delta_\Omega \sigma_j \rangle_{L^2(\Gamma)} (1 - \delta_{ij}). \]
Moreover, by \((\ref{4.2})\) and Lemma \(\ref{2.6}\), one has
\[
z(G_0^*G_z)_{ij} = (\bar{	au}_\Omega^\nu(\tau_\Omega^\nu G_0 - G_z))_{ij}
\]
\[
= (\bar{	au}_\Omega^\nu((-\Delta_\Omega^F)^{-1}\Delta_\Omega \sigma + (-\Delta_\Omega^F + z)^{-1}(-\Delta_\Omega + z)\sigma))_{ij}
\]
\[
= (G_0^*\Delta_\Omega \sigma)_{ij} + (G_0^*(-\Delta_\Omega + z)\sigma)_{ij}
\]
\[
= \lambda_{ij} + (\sigma_i,\Delta_\Omega \sigma_i)_{L^2(\Omega)} + \langle(-\Delta_\Omega^F)^{-1}\Delta_\Omega \sigma_i,\Delta_\Omega \sigma_i\rangle_{L^2(\Omega)} \delta_{ij}
\]
\[
+ \langle\sigma_i,(-\Delta_\Omega + z)\sigma_i\rangle_{L^2(\Omega)} \delta_{ij}
\]
\[
- \langle(-\Delta_\Omega^F + z)^{-1}(-\Delta_\Omega + z)\sigma_i,(-\Delta_\Omega + z)\sigma_j\rangle_{L^2(\Omega)}
\]
\[
= \lambda_{ij} + (\Gamma z)_{ij}.
\]

The proof is then concluded by taking $\tilde{\Theta} = \Theta - \Pi\Lambda\Pi$. \hfill \Box

Since $g_k - \sigma_k \in \mathcal{D}(\Delta_{\Omega}^F)$, Theorem \(\ref{2.7}\) admits an alternative version:

**Theorem 2.8.** Any self-adjoint extension of $\Delta_{\Omega}^F$ is of the kind
\[
\tilde{\Delta}_{\Omega}^{\Pi,\Theta} : \mathcal{D}(\tilde{\Delta}_{\Omega}^{\Pi,\Theta}) \subset L^2(\Omega) \to L^2(\Omega), \quad \tilde{\Delta}_{\Omega}^{\Pi,\Theta} u := \Delta_{\Omega} u,
\]
\[
\mathcal{D}(\tilde{\Delta}_{\Omega}^{\Pi,\Theta}) := \{u \in L^2(\Omega) : u = u_0 + \sigma \cdot \xi_u, \ u_0 \in \mathcal{D}(\Delta_{\Omega}^0), \ \xi_u \in \mathbb{C}_n, \ \Pi \tau_\Omega^\nu u_0 = \Theta \xi_u\},
\]
where $(\Pi, \Theta) \in \mathbb{E}(\mathbb{C}_n)$. Moreover
\[
(-\tilde{\Delta}_{\Omega}^{\Pi,\Theta} + z)^{-1} = (-\Delta_{\Omega}^F + z)^{-1} + G_2^* \Pi (\Theta + \Pi \tilde{\Gamma} z\Pi)^{-1} \Pi G_z^*,
\]
where
\[
(\tilde{\Gamma} z)_{ij} = \left(z||\sigma_i||^2_{L^2(\Omega)} - \langle\sigma_i,\Delta_\Omega \sigma_i\rangle_{L^2(\Omega)}\right) \delta_{ij}
\]
\[
- \langle(-\Delta_\Omega^F + z)^{-1}(-\Delta_\Omega + z)\sigma_i,(-\Delta_\Omega + z)\sigma_j\rangle_{L^2(\Omega)}
\]

**Proof.** By Theorem \(\ref{4.1}\) and Lemma \(\ref{2.6}\), any $u$ in the domain of a self-adjoint extension of $\mathcal{D}(\Delta_{\Omega}^F)$ is of the kind $u = \tilde{u}_0 + g \cdot \xi_u$, $\tilde{u}_0 \in \mathcal{D}(\Delta_{\Omega}^F)$, $\xi_u \in \mathbb{C}_n$, $\Pi \tau_\Omega^\nu \tilde{u}_0 = \tilde{\Theta} \xi_u$, for some $(\Pi, \tilde{\Theta}) \in \mathbb{E}(\mathbb{C}_n)$. By the definition of $g$ (see Lemma \(\ref{2.2}\)), one has $u = u_0 + \sigma \cdot \xi_u$, $u_0 = \tilde{u}_0 + (-\Delta_{\Omega}^F)^{-1}\Delta_\Omega \sigma \cdot \xi_u$ and, by Lemma \(\ref{2.6}\),
\[
\tau_\Omega^\nu \tilde{u}_0 = \tau_\Omega^\nu u_0 - \tau_\Omega^\nu (-\Delta_{\Omega}^F)^{-1}\Delta_\Omega \sigma \cdot \xi_u = \tau_\Omega^\nu u_0 - G_0^* \Delta_\Omega \sigma \cdot \xi_u
\]
\[
= \tau_\Omega^\nu u_0 - \hat{\Lambda} \xi_u,
\]
where
\[
\tilde{\lambda}_{ij} = \langle\sigma_i,\Delta_\Omega \sigma_i\rangle_{L^2(\Omega)} \delta_{ij} + \langle(-\Delta_{\Omega}^F)^{-1}\Delta_\Omega \sigma_i,\Delta_\Omega \sigma_j\rangle_{L^2(\Omega)}
\]
\[
= \tilde{\lambda}_{ij} \delta_{ij} + \lambda_{ij}.
\]
By noticing that $\tilde{\Gamma} z_{ij} = \Gamma z_{ij} - \hat{\lambda}_{ij} \delta_{ij}$, the proof is then concluded by taking $\tilde{\Theta} = \Theta - \Pi\hat{\Lambda}\Pi$. \hfill \Box
Remark 2.9. Since both \(\sigma_k\) and \(g_k\), \(1 \leq k \leq n\), belong to \(\mathcal{H}(\gamma_0)\) (see Lemma 2.2), all the self-adjoint extensions of \(\Delta^0_\Omega\) have domains contained in \(\mathcal{H}(\hat{\gamma}_0)\), the only one with domain contained in \(\mathcal{H}(\gamma_0)\) being the Friedrichs’ Laplacian \(\Delta^F_\Omega\). Thus we can interpret the set of all self-adjoint extensions of \(\Delta^0_\Omega\) as the set of self-adjoint, non-Friedrichs’ Dirichlet Laplacians on \(L^2(\Omega)\).

Remark 2.10. Notice that if both \(\Pi\) and \(\Theta\) are diagonal, then both the boundary conditions \(\Pi^V_\Omega u = \Theta^V u\) and \(\Pi^0_\Omega u = \Theta^V u\) appearing in the previous theorems are local, i.e. they do not couple values of \(u\) at different vertices.

Example 2.11. The prototypical example is provided by \(\Omega = W\), where \(W\) denotes the non-convex wedge

\[ W = \{ x \equiv (r \cos \theta, r \sin \theta) : 0 < r < R, \ 0 < \theta < \omega \}, \ \omega \in (\pi, 2\pi). \]

In this case \(\mathcal{H}((\Delta^0_W)^*)\) is one dimensional and by Theorem 2.1 \(g\) is the unique (up to the multiplication by a constant) solution of the boundary value problem

\[
\begin{cases}
\Delta^\text{max}_W g(r, \theta) = 0, \\
g(r, 0) = g(r, \omega) = g(R, \theta) = 0, \ r \neq 0.
\end{cases}
\]

Thus

\[
g(r, \theta) = \frac{1}{\sqrt{\pi}} \left( \frac{1}{r^\beta} - \frac{r^\beta}{R^{2\beta}} \right) \sin \beta \theta, \ \beta = \frac{\pi}{\omega}.
\]

Similarly \(G_z : \mathbb{C} \to L^2(W)\) acts as the multiplication by the function \(g_z\) which solves the boundary value problem

\[
\begin{cases}
\Delta^\text{max}_W g_z(r, \theta) = z g_z(r, \theta), \\
g(r, 0) = g(r, \omega) = g(R, \theta) = 0, \ r \neq 0.
\end{cases}
\]

Thus

\[
g_z(r, \theta) = \frac{1}{\sqrt{\pi}} \left( \frac{\sqrt{z}}{2} \right)^\beta \Gamma(1 - \beta) \left( J_{-\beta}(\sqrt{z}r) - \frac{J_{-\beta}(\sqrt{z}R)}{J_{\beta}(\sqrt{z}R)} J_{\beta}(\sqrt{z}r) \right) \sin \beta \theta,
\]

where \(\text{Re}(\sqrt{z}) > 0\), \(J_{\pm \beta}\) denotes the Bessel function of order \(\pm \beta\). Here the constants are chosen in order to have \(g_z \to g\) as \(z \to 0\). Then

\[
\Gamma_z = z \langle g, g_z \rangle_{L^2(W)} = \frac{1}{R^{2\beta}} + \left( \frac{z}{4} \right)^\beta \frac{\Gamma(-\beta) J_{-\beta}(\sqrt{z}R)}{\Gamma(\beta) J_{\beta}(\sqrt{z}R)}.
\]
where $\Gamma(\pm \beta)$ denotes Euler’s gamma function at $\pm \beta$. By Theorem 2.7, the set of self-adjoint extensions of $\Delta^\omega_\omega$ different from $\Delta^F_\omega$ is parametrized by $\theta \in \mathbb{R}$. Any of such extensions has resolvent $R^\theta_\omega$ with kernel

$$R^\theta_\omega(x,y) = R^F_\omega(x,y) + \left( \theta + \left( \frac{z}{4} \right)^{\frac{\beta}{2}} \frac{\Gamma(-\beta)}{\Gamma(\beta)} \frac{J_{-\beta}(\sqrt{z} R)}{J_{\beta}(\sqrt{z} R)} \right)^{-1} g_\omega(x)g_\omega(y),$$

where $R^F_\omega$ denotes the resolvent of the Friedrichs-Dirichlet Laplacian.

3. Approximation by Friedrichs-Dirichlet Laplacians with point interactions

3.1. Laplacians with point interactions. Let $\Delta^F_\Omega$ be the Friedrichs-Dirichlet Laplacian on $\Omega$ as defined in the previous section and, given the discrete set $Y = \{y_1, \ldots, y_n\} \subset \Omega$, we define the linear map $\tau^\Omega_Y : D(\Delta^F_\Omega) \to \mathbb{C}^n$, $(\tau^\Omega_Y u)_k := u(y_k), \quad k = 1, \ldots, n$.

By $u(y_k) = u(x_k) + s(y_k) \zeta_i u$ and (2.3) such a linear map is continuous with respect with the graph norm of $\Delta^F_\Omega$, is evidently surjective and has a dense (in $L^2(\Omega)$) kernel. Thus we can apply the results provided in the Appendix to write down all the self-adjoint extensions of the symmetric operator $\Delta^\omega_\omega$, given by restricting $\Delta^F_\Omega$ to the functions that vanish at $Y$.

Let us denote by $g_\Omega(z; \cdot, \cdot)$ the Green’s function of $-\Delta^F_\Omega + z$, so that

$$g_\Omega(z; x, y) = g(z; x, y) - h_\Omega(z; x, y),$$

where

$$g(0; x, y) = \frac{1}{2\pi} \ln \frac{1}{||x - y||},$$

$$g(z; x, y) = \frac{1}{2\pi} K_0(\sqrt{z} ||x - y||), \quad \text{Re}(\sqrt{z}) > 0,$$

$K_0$ the Macdonald (or modified Hankel) function, and $h_\Omega(z; \cdot, y)$ solves the inhomogeneous Dirichlet boundary value problem

$$\begin{cases}
(-\Delta^F_\Omega + z) h_\Omega(z; x, y) = 0, & x \in \Omega \\
h_\Omega(z; x, y) = g(z; x, y), & x \in \Gamma.
\end{cases}$$

Remark 3.1. One has (see e.g. [8], formula 8.447.3)

$$K_0(\sqrt{z} ||x - y||) = \ln \frac{1}{||x - y||} - \ln \frac{\sqrt{z}}{2} + \psi(1) + o(\sqrt{z} ||x - y||),$$

where $\psi$ is Euler’s psi function.

Since $\Omega$ satisfies the exterior cone condition and $g(z; \cdot, \cdot)$ is continuous outside the diagonal, by regularity of solutions of boundary value
problems for elliptic equations with continuous boundary data (see e.g. [11], Corollary 7.4.4), one has $h_\Omega(z; \cdot, y) \in C^\infty(\Omega) \cap C(\Omega)$ for any $y \in \Omega$.

By Theorem 21 in [17] one has

$$\int_1 \frac{1}{c} g_\Omega(0; x, y) \leq \ln \left( 1 + \frac{w(x, y)}{\|x - y\|^2} \right) \leq c g_\Omega(0; x, y),$$

where

$$w(x, y) := \begin{cases} \min \left( \frac{d(x)}{u_1(x)} \right)^2, \frac{d(y)}{u_1(y)} \right)^2, \tilde{d}(x) < R, \tilde{d}(y) < R, \\
\max \left( \frac{d(x)}{u_1(x)} \right)^2, \frac{d(y)}{u_1(y)} \right)^2, \tilde{d}(x) < R, \tilde{d}(y) < R, \\
1, \quad \text{otherwise}, \end{cases}$$

$u_1$ is the first eigenfunction of $-\Delta_\Omega^E$, $d$ is the distance form the boundary, $\tilde{d}$ the distance from the set of the vertices at the convex corners and $\hat{d}$ is the distance from the set of the vertices at the non-convex corners.

Defining

$$G_2^\gamma : \mathbb{C}^n \to L^2(\Omega), \quad (G_2^\gamma)^* : L^2(\Omega) \to \mathbb{C}^n, \quad z \in \rho(\Delta_\Omega^E)$$

by

$$G_2^\gamma := \left( \tau_\Omega^\gamma (-\Delta_\Omega^E + \tilde{z})^{-1} \right)^*,$$

one has

$$(G_2^\gamma \xi)(x) = \sum_{i=1}^n g_\Omega(z; x, y_i) \xi_i$$

and

$$((G_2^\gamma)^* u)_k = \langle g_\Omega(z; \cdot, y_k), u \rangle_{L^2(\Omega)}.$$

By the results provided in the Appendix one obtains the following

**Theorem 3.2.** Any self-adjoint extension of $\Delta_{Y,\Omega}^E$ is of the kind

$$\Delta_{Y,\Omega}^{\Pi, \Theta} : \mathcal{D}(\Delta_{Y,\Omega}^{\Pi, \Theta}) \subset L^2(\Omega) \to L^2(\Omega), \quad \Delta_{Y,\Omega}^{\Pi, \Theta} u := \Delta_\Omega^E u_0,$$

$$\mathcal{D}(\Delta_{Y,\Omega}^{\Pi, \Theta}) := \{ u \in L^2(\Omega) : u = u_0 + G_0^\gamma \xi_u, \ u_0 \in \mathcal{D}(\Delta_\Omega^E), \ \xi_u \in \mathbb{C}^n, \ \Pi \tilde{\tau}_\Omega^\gamma u = \Theta \xi_u \},$$

where $(\Pi, \Theta) \in E(\mathbb{C}^n)$ and

$$\left( \tilde{\tau}_\Omega^\gamma u \right)_k := \lim_{x \to y_k} \left( u(x) - \frac{\xi_u}_k \right) g_\Omega(0; x, y_k).$$

Moreover

$$(-\Delta_{Y,\Omega}^{\Pi, \Theta} + z)^{-1} = (-\Delta_\Omega^E + z)^{-1} + G_2^\gamma \Pi (\Theta + \Pi \tilde{\tau}_\Omega^\gamma \Pi)^{-1} \Pi (G_2^\gamma)^*,$$
where
\[
(\Gamma^Y_i)_ij := \left( \frac{1}{2\pi} \left( \ln \frac{\sqrt{z}}{2} - \psi(1) \right) - h_\Omega(0; y_i, y_i) + h_\Omega(z; y_i, y_i) \right) \delta_{ij} \\
- g_\Omega(z; y_i, y_j) (1 - \delta_{ij}) .
\]

Proof. By Theorem 4.1 any \( u = u_0 + G^Y_0 \cdot \xi_u \) in the domain of a self-adjoint extension of \( \mathcal{D}(\Delta^Y_\Omega) \) has to satisfy the boundary conditions
\[
\Pi \tau^Y_\Omega u_0 = \tilde{\Theta} \xi_u , \quad \text{for some } (\Pi, \tilde{\Theta}) \in \mathcal{E}(\mathbb{C}^n) .
\]
Since
\[
(\tau^Y_\Omega u_0)_i = \lim_{x \to y_i} (u(0; x, y_i) (\xi_u)_i) - \sum_{j \neq i} g_\Omega(0; y_i, y_j) (\xi_u)_j ,
\]
one has
\[
\tau^Y_\Omega u_0 = \tau^Y_\Omega u - \Lambda^Y_\xi_u ,
\]
where
\[
(3.2) \quad \Lambda^Y_i := g_\Omega(0; y_i, y_j) (1 - \delta_{ij}) .
\]
Moreover
\[
(z(G^Y_0)^* G^Y_0)_ij = (\tau^Y_\Omega (G^Y_0 - G^Y_0))_ij \\
= \left( \lim_{x \to y_i} (g(0; x, y_i) - g(z; y_i, y_i)) - h_\Omega(0; y_i, y_i) + h_\Omega(z; y_i, y_i) \right) \delta_{ij} \\
+ (g_\Omega(0; y_i, y_j) - g_\Omega(z; y_i, y_j)) (1 - \delta_{ij}) \\
= \Lambda^Y_i + (\Gamma^Y_i)_ij ,
\]
Thus the proof is concluded by posing \( \tilde{\Theta} = \Theta - \Pi \Lambda^Y \Pi \).

By the decomposition \( g_\Omega = g + h_\Omega \) the previous theorem admits (the proof being of the same kind) an alternative version which provides results analogous to the ones given in [4] and [7].

**Theorem 3.3.** Any self-adjoint extension of \( \Delta^Y_\Omega \) is of the kind
\[
\Delta^{\Pi, \Theta}_{Y, \Omega} : \mathcal{D}(\Delta^{\Pi, \Theta}_{Y, \Omega}) \subset L^2(\Omega) \to L^2(\Omega) , \quad \Delta^{\Pi, \Theta}_{Y, \Omega} u := \Delta^F_\Omega u_0 ,
\]
\[
\mathcal{D}(\Delta^{\Pi, \Theta}_{Y, \Omega}) := \\
\{ u \in L^2(\Omega) : u = u_0 + G^Y_0 \xi_u , \quad u_0 \in \mathcal{D}(\Delta^F_\Omega), \quad \xi_u \in \mathcal{C}^n, \quad \Pi \tilde{\tau}^Y_\Omega u = \Theta \xi_u \},
\]
where \( (\Pi, \Theta) \in \mathcal{E}(\mathbb{C}^n) \) and
\[
(\tilde{\tau}^Y_\Omega u)_k := \lim_{x \to y_k} \left( u(x) - \frac{1}{2\pi} \ln \frac{1}{\|x - y_k\|} \right) .
\]
Moreover
\[
(-\Delta^{\Pi, \Theta}_{Y, \Omega} + z)^{-1} = (-\Delta^F_\Omega + z)^{-1} + G^Y_\Omega (\Theta + \Pi \tilde{\tau}^Y_\Omega)^{-1} \Pi (G^Y_\Omega)^* ,
\]
where
\[
(\tilde{\tau}_z)_{ij} := \left(\frac{1}{2\pi} \left( \ln \left( \frac{\sqrt{z}}{2} \right) - \psi(1) \right) + h_\Omega(z; y_i, y_j) \right) \delta_{ij}
- g_\Omega(z; y_i, y_j) (1 - \delta_{ij}).
\]

**Remark 3.4.** Notice that if both $\Pi$ and $\Theta$ are diagonal, then both the boundary conditions $\Pi \tilde{\tau}_z^\nu u = \Theta z$ and $\Pi \tilde{\tau}_z^\nu u = \Theta z$ are local, i.e. they do not couple values of $u$ at different points of $Y$.

### 3.2. Approximating non-Friedrichs Dirichlet Laplacians by point perturbations.

Let $\{Y_N\}_1^\infty$ denote a sequence of discrete sets $Y_N = \{y_N^i\}_1^n \subset \Omega$ such that, for any $1 \leq k \leq n$,

\[
y_N^k \equiv (r_N^k \cos \theta_N^k, r_N^k \sin \theta_N^k) \in \mathcal{W}_{k}^{R/3},
\]

and

\[
\inf_N \sin \beta_k \theta_N^k = c > 0
\]

Posing
\[
\tilde{\tau}_N^Y : \mathcal{D}(\Delta^F_\Omega) \to \mathbb{C}^n, \quad \left(\tilde{\tau}_N^Y u\right)_k := \frac{u(y_N^k)}{s_k(y_N^k)},
\]

i.e.

\[
\tilde{\tau}_N^Y = M_N \tilde{\tau}_N^Y, \quad (M_N)_{ij} := s_i(y_N^j) \delta_{ij},
\]

one has the following

**Lemma 3.5.** There exist $c > 0$ and $0 < \alpha_k < 1 - \beta_k$ such that

\[
\left| \left(\tilde{\tau}_N^Y u - \tilde{\tau}_N u\right)_k \right| \leq c \|y_N^k - v_k\|^\alpha_k \|u\|_{\Delta^F_\Omega}.
\]

**Proof.** By

\[
u(y_N^k) = u_0(y_N^k) + s(y_N^k), \quad : u_0(y_N^k) = u_0(y_N^k) + s_k(y_N^k),
\]

by $\tau_N^Y = \zeta_u$, by (2.2) and (2.3), and denoting by $Q_0$ the continuous projection $Q_0 : \mathcal{D}(\Delta^F_\Omega) \to \mathcal{D}(\Delta^F_\Omega)$, one obtains

\[
\left| \left(\tilde{\tau}_N^Y u - \tilde{\tau}_N u\right)_k \right| = \left| \frac{u_0(y_N^k)}{s_k(y_N^k)} \right| \leq \frac{1}{c} \|y_N^k - v_k\|^{\beta_k}
\]

\[
\leq \frac{1}{c} \|y_N^k - v_k\|^{\alpha - \beta_k} \|u_0\|_{H^2(\Omega)} \leq c \|y_N^k - v_k\|^{\alpha - \beta_k} \|\Delta_\Omega u_0\| L^2(\Omega)
\]

\[
\leq c \|y_N^k - v_k\|^{\alpha - \beta_k} \|Q_0 u\|_{\Delta^F_\Omega} \leq c \|y_N^k - v_k\|^{\alpha - \beta_k} \|u\|_{\Delta^F_\Omega}.
\]

In conclusion we get the following
Theorem 3.6. Let \((\Pi, \Theta) \in \mathcal{E}(\mathbb{C}^n)\) and define \((\Pi_N, \Theta_N) \in \mathcal{E}(\mathbb{C}^n)\) by
\[
\mathcal{C}_{\Pi_N} = (M_N)^{-1}(\mathcal{C}_{\Pi}), \quad \Theta_N = \Pi_N M_N \Theta M_N \Pi_N.
\]
Then \(\Delta_{\mathcal{V}, \mathcal{W}}(\Pi_N, \Theta_N)\) converges in norm resolvent sense to \(\Delta_{\mathcal{V}, \mathcal{W}}(\Pi, \Theta)\) as \(N \uparrow \infty\).

Proof. Let \(\tilde{\Delta}_{\mathcal{V}, \mathcal{W}}(\Pi, \Theta)\) be the self-adjoint operator obtained by proceeding as in the proof of Theorem 3.2 with \(\tau_{\mathcal{V}}\) substituted by \(\tau_{\mathcal{W}}\). Then \(\Delta_{\mathcal{V}, \mathcal{W}}(\Pi_N, \Theta_N) = A_{\Pi_N, \Theta_N}^N\), where \(\tilde{\Theta}_N = \Theta - \Pi M_N^{-1} A_{\mathcal{W}} M_N^{-1} \Pi, A_{\mathcal{W}}\) is defined in (3.2) and the operator sequence \(\Pi\) converges in norm resolvent sense to \(\Pi\). Therefore, by Lemma 3.5 and Lemma 4.5, \(\Delta_{\mathcal{V}, \mathcal{W}}(\Pi_N, \Theta_N)\) converges in norm resolvent sense to \(\Delta_{\mathcal{V}, \mathcal{W}}(\Pi, \Theta)\). The proof is then concluded by noticing that, by Lemma 4.4, \(\Delta_{\mathcal{V}, \mathcal{W}}(\Pi_N, \Theta_N) = \Delta_{\mathcal{V}, \mathcal{W}}(\Pi_N, \Theta_N)\).

4. Appendix

For the reader’s convenience in this section we collect some results about the self-adjoint extension of a closed symmetric operator \(S\) with
deficiency indices \((n, n)\). Since it suffices for the purposes of this paper we suppose \(n < +\infty\) and \(-S > 0\). For the general case, as well as for the connection with alternative approaches, we refer to \([16]\) and references therein.

Let

\[ S : \mathcal{D}(S) \subseteq \mathcal{H} \to \mathcal{H} \]

be a closed symmetric linear operator on the Hilbert space \(\mathcal{H}\) such that \(-S > 0\). Then by Friedrichs’ theorem \(S\) has a self-adjoint extension

\[ A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H} \]

with the same bound.

Suppose now that \(S\) has finite deficiency indices \(n_+ = \dim \mathcal{N}_+ > 0, n_- := \mathcal{N}(-S^* \pm i)\). By von Neumann’s theory of self-adjoint extensions, there exists an unitary operator \(U : \mathcal{N}_+ \to \mathcal{N}_-\) such that

\[ D(A) = D(S) \oplus_A \mathcal{G}(U_A), \]

where \(\mathcal{G}(U_A)\) is the graph of \(U_A\) and \(\oplus_A\) denotes the orthogonal sum corresponding to the scalar products inducing the graph norm on \(D(A)\). Therefore \(S = A|\mathcal{N}(P)\), where \(P : \mathcal{D}(A) \to \mathcal{N}_+\) denotes the orthogonal projection onto \(\mathcal{N}_+\). Thus, since this gives some advantages in applications, we will look for the self-adjoint extensions of \(S\) by considering the equivalent problem of the search of the self-adjoint extensions of the restriction of \(A\) to the kernel, which we suppose to be dense in \(\mathcal{H}\), of a surjective bounded linear operator

\[ \tau : \mathcal{D}(A) \to \mathbb{C}^n. \]

Typically \(A\) is an elliptic differential operator and \(\tau\) is some restriction operator to a discrete set with \(n\) points. In the case of infinite defect indices typically \(\tau\) is the restriction operator along a null subset and \(\mathbb{C}^n\) is substituted by a (fractional order) Sobolev-Hilbert space (see \([13]\), \([16]\) and references therein).

By \([13]\), \([14]\) and \([16]\) one has the following

**Theorem 4.1.** The set of all self-adjoint extensions of \(S\) is parametrized by the bundle \(p : E(\mathbb{C}^n) \to P(\mathbb{C}^n)\); if \(A^{\Pi, \Theta}\) denotes the self-adjoint extension corresponding to \((\Pi, \Theta) \in E(\mathbb{C}^n)\) then

\[ A^{\Pi, \Theta} : \mathcal{D}(A^{\Pi, \Theta}) \subseteq \mathcal{H} \to \mathcal{H}, \quad A^{\Pi, \Theta} \phi := A\phi_0, \]

\[ \mathcal{D}(A^{\Pi, \Theta}) := \{ \phi = \phi_0 + G_0 \xi_\phi, \phi_0 \in \mathcal{D}(A), \xi_\phi \in \mathbb{C}^n_\Pi, \Pi \tau \phi_0 = \Theta \xi_\phi \}, \]

where

\[ G_z : \mathbb{C}^n \to \mathcal{H}, \quad G_z := (\tau(-A + \bar{z})^{-1})^*, \quad z \in \rho(A). \]
Moreover the resolvent of $A_{\Pi,\Theta}$ is given, for any $z \in \rho(A) \cap \rho(A_{\Pi,\Theta})$, by the Krein’s type formula
\[
(-A_{\Pi,\Theta} + z)^{-1} = (-A + z)^{-1} + G_z \Pi(\Theta + z\Pi G_0^* G_z \Pi)^{-1} \Pi G_z^*.
\]

**Remark 4.2.** One can easily check that the density hypothesis about $\mathcal{K}(\tau)$ gives, for any $z \in \rho(A)$,
\[
\mathcal{R}(G_z) \cap \mathcal{D}(A) = \{0\},
\]
thus the decomposition appearing in $\mathcal{D}(A_{\Pi,\Theta})$ is well defined. Moreover, since by first resolvent identity
\[
(z - w)(-A + w)^{-1}G_w = G_w - G_z,
\]
one has
\[
\mathcal{R}(G_w - G_z) \subset \mathcal{D}(A)
\]
and
\[
zG_0^* G_z = \tau(G_0 - G_z)
\]

**Remark 4.3.** Notice that the knowledge of the adjoint $S^*$ is not required. However it can be readily calculated: by [15], Theorem 3.1, one has
\[
S^* : \mathcal{D}(S^*) \subseteq \mathcal{H} \to \mathcal{H}, \quad S^* \phi = A\phi_0,
\]
\[
\mathcal{D}(S^*) = \{\phi \in \mathcal{H} : \phi = \phi_0 + G_0 \xi_\phi, \ 0 \in \mathcal{D}(A), \ \xi_\phi \in \mathbb{C}^n\}.
\]
Moreover
\[
\mathcal{D}(A_{\Pi,\Theta}) = \{\phi \in \mathcal{D}(S^*) : \rho_0 \phi \in \mathbb{C}^n, \ \Pi \tau_0 \phi = \Theta \rho_0 \phi\}.
\]
where the regularized trace operators $\tau_0$ and $\rho_0$ are defined by
\[
\tau_0 : \mathcal{D}(S^*) \to \mathbb{C}^n, \quad \tau_0 \phi := \tau \phi_0
\]
and
\[
\rho_0 : \mathcal{D}(S^*) \to \mathbb{C}^n, \quad \rho_0 \phi := \xi_\phi.
\]
By [15], Theorem 3.1, $(\mathbb{C}^n, \tau_0, \rho_0)$ is a boundary triple for $S^*$, with corresponding Weyl function $zG_0^* G_0$, and the Green’s-type formula
\[
\langle \phi, S^* \psi \rangle - \langle S^* \phi, \psi \rangle = \tau_0 \phi \cdot \rho_0 \psi - \rho_0 \phi \cdot \tau_0 \psi
\]
holds true. Also notice that $G_z \xi$ solves the boundary value type problem
\[
\begin{cases}
S^* G_z \xi = zG_z \xi, \\
\rho_0 G_z \xi = \xi.
\end{cases}
\]
Since $S = A|\mathcal{H}(\tau) = A|\mathcal{H}(\tau_M)$, where $\tau_M := M\tau$ and $M : \mathbb{C}^n \to \mathbb{C}^n$ is any bijective linear map, the group $GL(\mathbb{C}^n)$ acts on the parametrizing bundle $E(\mathbb{C}^n)$ by

$$\alpha : GL(\mathbb{C}^n) \times E(\mathbb{C}^n) \to E(\mathbb{C}^n), \quad \alpha(M, (\Pi, \Theta)) = (\Pi_M, \Theta_M),$$

where $\alpha$ is defined in such a way that

$$A^{\Pi_M, \Theta_M} = A^{\Pi, \Theta},$$

with $A^{\Pi, \Theta}$ the extension corresponding to $(\Pi, \Theta) \in E(\mathbb{C}^n)$ provided by Theorem 4.1 in the case one uses the map $\tau_M$. The action $\alpha$ is explicitly given in the following

**Lemma 4.4.**

$$\mathbb{C}^n_{\Pi_M} = (M^*)^{-1}(\mathbb{C}^n_{\Pi}), \quad \Theta_M = \Pi_M M \Theta M^* \Pi_M.$$

**Proof.** By the definitions of $\tau_M$ and $G_0$ one has that any $\phi \in \mathcal{D}(A^{\Pi, \Theta})$ is of the kind $\phi = \phi_0 + G_0 M^* \zeta$, where $\Pi_M M \tau \phi_0 = \Theta_M \zeta$, $\zeta \in \mathbb{C}^n_{\Pi_M}$, i.e. $\phi = \phi_0 + G_0 \xi$, where $\Pi_M M \tau \phi_0 = \Theta_M (M^*)^{-1} \xi$, $\xi \in \mathbb{C}^n_{\Pi}$. Since $\mathcal{D}(A^{\Pi_M, \Theta_M}) = M((\mathcal{D}(A^{\Pi}))^\perp)$, $\Pi_M M \Pi : \mathbb{C}^n_{\Pi} \to \mathbb{C}^n_{\Pi_M}$ is a bijection. Thus $\Pi_M M \tau \phi_0 = \Theta_M (M^*)^{-1} \xi$ if and only if $\Pi \tau \phi_0 = (\Pi_M M \Pi)^{-1} \Theta_M (M^*)^{-1} \xi$. □

We conclude by providing a simple convergence result:

**Lemma 4.5.** Let the sequence $\tau_N : \mathcal{D}(A) \to \mathbb{C}^n$ be converging, with respect to the norm on bounded linear operators, to $\tau : \mathcal{D}(A) \to \mathbb{C}^n$ as $N \uparrow \infty$. Given $\Pi \in \mathcal{P}(\mathbb{C}^n)$, let the sequence $\Theta_N : \mathbb{C}^n_{\Pi} \to \mathbb{C}^n_{\Pi}$ be converging to $\Theta : \mathbb{C}^n_{\Pi} \to \mathbb{C}^n_{\Pi}$ as $N \uparrow \infty$. Let $A^{\Pi, \Theta_N}$ and $A^{\Pi, \Theta}$ denote the corresponding self-adjoint extensions of $S_N = A|\mathcal{H}(\tau_N)$ and $S = A|\mathcal{H}(\tau)$ respectively, as given by Theorem 4.1. Then $A^{\Pi, \Theta_N}$ converges in norm resolvent sense to $A^{\Pi, \Theta}$ as $N \uparrow \infty$.

**Proof.** By our hypothesis on $\tau_N$, $G^{*}_{N,z} := \tau_N(-A+z)^{-1}$ and $G_{N,z}$ norm-converge to $G^*_z$ and $G_z$ respectively. This implies that $zG^*_{0,0} G_{N,z}$ norm-converge to $zG^*_0 G_z$ and hence $(\Theta_N + z\Pi G^*_{N,0} G_{N,z} \Pi)^{-1}$ norm-converge to $(\Theta + z\Pi G^*_0 G_z \Pi)^{-1}$. The thesis then follows by the resolvent formula provided in Theorem 4.1. □

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**Dipartimento di Scienze Fisiche e Matematiche, Università dell’Insubria, I-22100 Como, Italy**

*E-mail address: posilicano@uninsubria.it*