NON-DIFFERENTIABILITY OF $\alpha$ FUNCTION AT THE BOUNDARY OF FLAT

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Abstract. With the variational method introduced by J Mather, we construct a mechanical Hamiltonian system whose $\alpha$ function has a flat $F$ and is non-differentiable at the boundary $\partial F$. In the case of two degrees of freedom, we prove this phenomenon is stable under perturbations of Mañe's.

1. Introduction

Let $M$ be a smooth closed manifold with $TM$ as tangent bundle. We call such a function $L(x,v) \in C^r(TM, \mathbb{R})$ $(r \geq 2)$ Tonelli Lagrangian if it satisfies:

- **convexity**: For all $x \in M$, $v \in T_xM$ the Hessian matrix $\frac{\partial^2 L}{\partial v_i \partial v_j}(x,v)$ is positive definiteness;
- **superlinearity**: $\lim_{\|v\| \to \infty} \frac{L(x,v)}{\|v\|} = \infty$ uniformly on $(x,v) \in TM$;
- **completeness**: All solutions of the corresponding Euler-Lagrangian equation are well defined for $t \in \mathbb{R}$.

Here the Euler-Lagrangian equation is given by:

$$\frac{d}{dt} \frac{\partial L}{\partial v}(x,v) = \frac{\partial L}{\partial x}(x,v), \quad (x,v) \in TM. \quad \text{(E-L)}$$

**Remark 1.1.** In the autonomous case, the completeness is natural under the first two assumptions. That’s because we can get the Hamiltonian as

$$H(x,v) = \frac{\partial L}{\partial v}(x,v) - L.$$ 

From [Car] we know that along each orbit $(\gamma, \dot{\gamma})$ of Euler-Lagrangian equation $H(\gamma, \dot{\gamma})$ is constant. The superlinearity implies that the level sets of Hamiltonian are compact. This in turn assure the completeness of flow.

Usually we take $M = \mathbb{T}^n$. Adding a closed 1-form $\eta_c$ with the cohomology class $[\eta_c] = c \in H^1(M, \mathbb{R})$, we get a new Tonelli Lagrangian $L - \eta_c$, denoted by $L - c$ for short. From [Mat] we know that the E-L flow $(\gamma, \dot{\gamma})$ of $L - \eta_c$ also satisfies the E-L equation of $L$. So we can define a c-minimal curve $\gamma \in C^1(\mathbb{R}, M)$ if it satisfies:

$$A_c(\gamma) = \min_{\xi(a) = \gamma(a), \xi(b) = \gamma(b)} \int_a^b (L - \eta_c)(\xi(t), \dot{\xi}(t)) dt, \forall a < b \in \mathbb{R}, \xi \in C^ac(\mathbb{R}, M).$$

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All of the $c$-minimal orbit $(\gamma, \dot{\gamma})$ form a set denoted by $\tilde{G}(c)$, which is invariant under the Euler-Lagrangian flow $\Phi^t$.

Let $\mathcal{M}_{inv}$ be the set of $\Phi^t$-invariant probability measures on $TM$. We define the $\alpha$ function as follow:

$$\alpha(c) = -\min_{\mu \in \mathcal{M}_{inv}} \int_{TM} L - \eta_c \, d\mu, \quad c \in H^1(M, \mathbb{R}).$$

As is showed in [Mat], there exists at least one measure $\mu_c$ such that the minimum attains. We call this measure $\mu_c$ a $c$-minimal measure. The union of the supports of all $c$-minimal measures is called Mather set, denoted by $\tilde{M}(c)$.

Since we know that $\alpha(c)$ is convex, finite everywhere and superlinear [Mat], we can define its conjugate function in the sense of convex analysis [R] as:

$$\beta(h) = \min_{\rho(\mu) = h} \int_{TM} L \, d\mu, \quad h \in H^1(M, \mathbb{R}),$$

here $\rho(\mu)$ is defined via the De Rham inner product:

$$\langle \rho(\mu), c \rangle = \int \eta_c \, d\mu.$$

$\beta(h)$ is also a convex, finite everywhere and superlinear function. From [Man] and [Zhi] we can get the following properties:

**Theorem 1.2.**

- If $\mu$ is a $c$-minimizing measure, we have $\rho(\mu) \in D\alpha(c)$.
- The maximal connected domain on which $\alpha$ function isn't strict convex is called a flat $\mathbb{F}$. For $c, c' \in \text{int}\mathbb{F}$ we have $\tilde{M}(c) = \tilde{M}(c')$.
- For each non-differential point $c$ of $\alpha$ function, $\tilde{M}(c)$ corresponds to at least two ergodic components with different rotation vectors.
- If $h$ is a strict convex point of $\beta$ function, then there must exist one ergodic minimal measure $\mu$ with $\rho(\mu) = h$.

We also need to define another two sets called Aubry set $\tilde{A}(c)$ and Mañe set $\tilde{N}(c)$. First we define:

$$h^t_c(x, y) = \min_{\gamma \in C^1([0, t], M), \gamma(0) = x, \gamma(t) = y} \int_0^t (L - \eta_c)(\gamma(s), \dot{\gamma}(s)) + \alpha(c) \, ds,$$

$$\Phi_c(x, y) = \inf_{t \in \mathbb{R}} h^t_c(x, y),$$

and

$$h^\infty_c(x, y) = \lim_{t \to +\infty} h^t_c(x, y).$$

We call a curve $\gamma \in C^1(\mathbb{R}, M)$ $c$-semi-static if

$$\Phi_c(\gamma(a), \gamma(b)) = \int_a^b (L - \eta_c)(\gamma(t), \dot{\gamma}(t)) + \alpha(c) \, dt, \quad \forall a < b \in \mathbb{R},$$

and a curve $c$-static if

$$\Phi_c(\gamma(b), \gamma(a)) + \int_a^b (L - \eta_c)(\gamma(t), \dot{\gamma}(t)) + \alpha(c) \, dt = 0, \quad \forall a < b \in \mathbb{R}.$$
A c-static orbit must be a c-semistatic orbit \([\text{Ber}]\). We call the union of all global c-semistatic orbits Mañé set denoted by \(\tilde{\mathcal{M}}(c)\), and the union of all global c-static orbits Aubry set denoted by \(\tilde{\mathcal{A}}(c)\). From \([\text{Ber}]\) we have the following inclusions:

**Theorem 1.3.**

- \(\tilde{\mathcal{M}}(c) \subseteq \tilde{\mathcal{A}}(c) \subseteq \tilde{\mathcal{N}}(c) \subseteq \tilde{\mathcal{G}}(c)\).
- We denote the projected set of \(\tilde{\mathcal{M}}(c)\) and \(\tilde{\mathcal{A}}(c)\) from \(TM\) to \(M\) by \(\mathcal{M}(c)\) and \(\mathcal{A}(c)\). Then the inverse of the project map \(\pi^{-1}: \mathcal{A}(c) \to \tilde{\mathcal{A}}(c)\)

is a Lipschitz graph.
- For an autonomous Lagrangian we have \(\tilde{\mathcal{N}}(c) = \tilde{\mathcal{G}}(c)\).
- \(\forall c, c' \in \text{int} \mathcal{F}\) we have \(\tilde{\mathcal{A}}(c) = \tilde{\mathcal{A}}(c')\), and \(\tilde{\mathcal{A}}(\text{int} \mathcal{F}) \subseteq \mathcal{A}(\partial \mathcal{F})\).

In \([\text{Man}]\), Mañé raised the problem: Is it true that for a generic Lagrangian, its \(\alpha\) function is of \(C^1\) smoothness? Here the ‘generic’ means there is a residue subset \(\mathcal{G} \subset C^\infty(M, \mathbb{R})\) such that the property holds for all the Tonelli Lagrangian \(L - u, u \in \mathcal{O}\). Negative answer is firstly given in \([S]\), but we don’t know whether there would be a flat \(\mathcal{F}\) coming out, not to mention the stability.

Moreover, we know \(\text{int} \mathcal{F}\) share the same Mather set but there may be new measure coming out at \(c \in \partial \mathcal{F}\). If so, we can construct heteroclinic orbits between these different measures and explore much interesting dynamic behavior. This phenomenon was discovered by Zheng Yong in \([Zh]\). Based on such a premise:

\[\exists c \in \partial \mathcal{F}, \ s.t. \ \inf_{g \in H^{1}(\tilde{\mathcal{M}}(c^*))} h^\infty_c(g) = \delta > 0,\]

\[h_c^\infty(g) = \inf_{z \in M} h^\infty_c(x, x + g).\]

Here \(\tilde{\mathcal{M}}\) is the universal cover space of \(M\). He gets the following conclusion in \(\tilde{\mathcal{M}}\).

**Theorem 1.4.** \([Zh]\) There exist infinitely many \(\tilde{\mathcal{M}}\)-minimal homoclinic orbits to \(\tilde{\mathcal{A}}(\text{int} \mathcal{F})\) which are not of multi-bump type.

Actually, under this premise, we can assure one new minimal measure’s coming out at \(\partial \mathcal{F}\) (sec.2, lemma2.1 in \([Zh]\)). This new measure forms a mechanism to construction these \(\tilde{\mathcal{M}}\)-minimal homoclinic orbits. But he didn’t give the answer whether if there exists one system satisfies his premise. Our construction gives a positive answer and verifies the rationality of the premise. Also we can use his Theorem to get infinitely many \(\tilde{\mathcal{M}}\)-minimal homoclinic orbits for our example.

In this paper we can construct the following example:

\[(1.1) \quad L(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle_g + u_1(x_n) + u_2(x_n) \sum_{i=1}^{n-1} (1 - \cos x_i),\]

here the precise form of the Riemannian metric \(g\) and the potential function will be given in the later section. Then we get our main result as following:

**Theorem 1.5.** In the case of \(n\) degrees of freedom, there exist \(u_1,u_2\) and Riemannian metric \(g\) such that the \(\alpha\) function of \((L,\tilde{\mathcal{M}})\) has a flat \(\mathcal{F}\) of full dimension at the
lowest energy level. There are $2^n - 1$ points $c_i (i = 1, 2, \ldots, 2^n - 1) \in \partial F$ of which the Mather set $\mathcal{M}(c_i)$ supports at $n - 1$ ergodic minimizing measures except the one supported on the minimal fixed point. Besides, all of the $n - 1$ new minimizing measures have non-vanishing rotation vectors different from each other.

Remark 1.6. We can see that in the following section this phenomenon also happens at a higher energy level flat.

Definition 1.7. We say a property of $L$ is stable in the sense of Mañé if there exists a small enough neighborhood $\mathcal{O}$ of zero in $C^\infty(M, \mathbb{R})$ such that the property holds for all the Tonelli Lagrangian $L - u, u \in \mathcal{O}$.

Theorem 1.8. In the case of two degrees of freedom, the properties in Theorem 1.5 is stable in the sense of Mañé.

This two degrees of freedom phenomenon can be applied to a-priori stable Arnold Diffusion problem [Ch]. Recently, Cheng proved the existence of normally hyperbolic invariant cylinder near double resonance frequency ([Ch] sec.5.2), which can be divided into two different cases. Actually, our example just satisfies the case that normally hyperbolic invariant cylinder can reach the lowest energy level with a periodic orbit as the bottom. We also know this case is stable of Mañé’s sense (Theorem 1.8).

Similar construction can be found in [Ban] from a geometrical viewpoint. Here we use a variational method which is known as Aubry Mather Theory nowadays and verify the existence of $\alpha$ function’s flat. Besides, we supply Zheng Yong’s paper and get infinitely many homoclinic orbits. So our construction has extra significance in dynamic systems other than geometry.

This paper is outlined as follows. In Section 2, we give the construction of the examples of different cases. We give the proof of stability in the case of two degrees of freedom and some remarks in Section 3.

2. Construction of the example

Now we construct the example in details and verify the properties satisfied in Theorem 1.5.

2.1. the lowest energy level case. We construct the Lagrangian as follows.

$$L(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle_g + u_1(x_n) + u_2(x_n) \sum_{i=1}^{n-1} (1 - \cos x_i)$$

where $(x, \dot{x}) \in \mathbb{T}^n \times \mathbb{R}^n$ and $x = (x_1, x_2, \ldots, x_n)$. $u_1(x_n), u_2(x_n)$ and the Riemannian metric $g$ are constructed as follows.

First, we mark several channels on the $\mathbb{T}^n$ along the $n$-th coordinate $x_n$.

(1) channel $A$: $\mathbb{T}^{n-1} \times [-\frac{\pi}{4n}, \frac{\pi}{4n}], \mathbb{T} = \mathbb{R}/2\pi$,
(2) channel $B_i$: $\mathbb{T}^{n-1} \times [\frac{2i\pi}{n} - \frac{\pi}{4n}, \frac{2i\pi}{n} + \frac{\pi}{4n}], i = 1, 2, \ldots, n - 1$,
(3) channel $C_j$: $\mathbb{T}^{n-1} \times [\frac{(2j-1)\pi}{n} - \frac{\pi}{4n}, \frac{(2j-1)\pi}{n} + \frac{\pi}{4n}], j = 1, 2, \ldots, n$. 

Obviously, channel $B_i$ and $C_j$ are just constructed by shifting channel $A$ along the $x_n$ coordinate. It is easy to see that $A$ and $B_i$ are separated by $C_j$.

\[(2.1) \quad u_1(x_n) = \begin{cases} 
0, & x_n \text{ in channel } A, \\
K \gg 1, & x_n \text{ in channel } C_j \text{ and } j = 1, 2, \ldots n, \\
0 < \delta_i \ll K, & x_n \text{ in channel } B_i \text{ and } i = 1, 2, \ldots n - 1,
\end{cases} \]

where the values of $\delta_i$ will be given later (in fact it is enough to take $0 < \delta_i \leq \frac{1}{4}$).

\[(2.2) \quad u_2(x_n) = \begin{cases} 
1, & x_n \text{ in channel } A, \\
0, & x_n \text{ out of some small neighborhood of channel } A.
\end{cases} \]

Both $u_1$ and $u_2$ can be smoothly extended to the whole space $\mathbb{T}^n$. Later you will see that we needn’t give their precise evaluation because only qualitative proof is cared.

At last, we construct the Riemannian metric $\langle \dot{x}, \dot{x} \rangle_g = \sum_{i=1}^n a_i(x)\dot{x}_i^2$ as follows:

\[
G = \begin{pmatrix}
a_1^2(x) & 0 & \cdots & 0 \\
0 & a_2^2(x) & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & a_n^2(x)
\end{pmatrix}
\]

is a diagonal metric matrix.

\begin{enumerate}
\item $G = \text{Id}_{n \times n}$, for $x_n$ in channel $A$,
\item $G = \begin{pmatrix}
K & 0 & \cdots & 0 \\
0 & K & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & K
\end{pmatrix}_{n \times n}$ for $x_n$ in channel $C_j (j = 1, 2, \ldots, n)$,
\item $G = \begin{pmatrix}
\frac{1}{8} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \frac{1}{8} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \ddots & 0 & \cdots & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{8} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & \frac{1}{8} & 0 & 0 \\
0 & \cdots & \cdots & 0 & 0 & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \frac{1}{8} & 0
\end{pmatrix}_{n \times n}$ for $x_n$ in channel $B_k (k = 1, 2, \ldots, n - 1)$, where $\frac{1}{8}$ is in the $k$-th row and $k$-th column.
\end{enumerate}

We take a subspace of $H^1(\mathbb{T}^n, \mathbb{R})$ with the $n$-th coordinate $c_n = 0$, which is denoted by $\mathcal{H}$. We also denote the restriction of Lagrangian $L$ in the channels $A, B_i$ and $C_j$ by $L_A, L_{B_i}$ and $L_{C_j}$. $\alpha_A(c), \alpha_{B_i}(c)$ and $\alpha_{C_j}(c)$ as the restricted $\alpha$ function. It’s remarkable that $\alpha_A(c), \alpha_{B_i}(c)$ and $\alpha_{C_j}(c)$ are well defined for all the $c \in H^1(\mathbb{T}^n, \mathbb{R})$ but only $c \in \mathcal{H}$ we care about.
Moreover, we could calculate \( \alpha_A(c) \), \( \alpha_B(c) \) and \( \alpha_{C_j}(c) \) when \( c \in \mathcal{H} \).

\[
\alpha_A(c) = -\min_{\mu \in M_{inv}} \int L_A - cd\mu,
\]

\[
= -\min_{\mu \in M_{inv}} \int \langle v, v \rangle_g + \delta_i - \sum_{i=1}^{n-1} c_i v_i d\mu,
\]

\[
= -\min_{\mu \in M_{inv}} \sum_{i=1}^{n-1} (v_i^2 + (1 - \cos x_i) - c_i v_i) d\mu.
\]

It is easy to see that \( L_A \) consists of \( n-1 \) independent pendulum systems. Hence, \( \alpha_A \) has a flat \( [-\sqrt{2\pi}, \sqrt{2\pi}] \times \{0\} \subseteq \mathcal{H} \).

\[
\alpha_B(c) = -\min_{\mu \in M_{inv}} \int L_B - cd\mu
\]

\[
= -\min_{\mu \in M_{inv}} \int \langle v, v \rangle_g + \delta_i - \sum_{i=1}^{n-1} c_i v_i d\mu
\]

\[
= -\delta_i + \sum_{k=1}^{n-1} 16c_k^2 - 12c_i^2, \quad i = 1, 2, \ldots, n-1.
\]

We get the last equality because this is a geodesic integral system. If we take \( 0 < \delta_i < \frac{1}{2} \), the level set \( \alpha_B^{-1}(0) \subseteq [-\sqrt{2\pi}, \sqrt{2\pi}] \times \{0\} \subseteq \mathcal{H} \).

The following lemma gives the relationship between \( \alpha \) function on \( \mathbb{T}^n \) and \( \alpha \) function restricted to channel \( A \) and \( B_i \).

**Lemma 2.1.**

\[
\alpha_L(c) = \max_{i=1,2,\ldots,n-1} \{\alpha_A, \alpha_{B_i}\},
\]

where \( c = (c_1, c_2, \ldots, c_{n-1}, 0) \in \mathcal{H} \). This result is stable in the sense of Mañé.

**Proof.** First, we have \( \alpha_L(c) \geq \alpha_A(c) \), and \( \alpha_L(c) \geq \alpha_{B_i}(c) \). It follows since the action of the local minimizer is bigger than the global minimizer. Hence,

\[
\alpha_L(c) \geq \max_{i=1,2,\ldots,n-1} \{\alpha_A, \alpha_{B_i}\}.
\]

On the other hand, from [Man] we know that every minimizing measure can be approximated by a sequence of probability measures supported on absolutely continuous closed curves which are not necessarily minimal, so we just need to certify that every closed curve \( \gamma \) in \( \mathbb{T}^n \) has a larger action of the Lagrangian \( L - c \) than its projection \( \gamma^* \) to \( \mathbb{T}^{n-1} \) in some channel \( A \) or \( B_i \), here \( c = (c_1, c_2, \ldots, c_{n-1}, 0) \in \mathcal{H} \).

It follows from

\[
\int_{\gamma} L - c - \int_{\gamma^*} L - c^* \geq \int a_n^2 \dot{x}_n^2 + u_1(x_n)
\]

\[
\geq 0.
\]

Hence,

\[
\alpha_L(c) \leq \max_{i=1,2,\ldots,n-1} \{\alpha_A, \alpha_{B_i}\}.
\]
Now we prove the stability. If another Lagrangian $L'$ is $C^1$ close to $L$, i.e. $L' = L + V(x)$ and $\|V\|_r \leq \epsilon \ll \delta_i$, $i = 1, 2, \cdots, n - 1$. Now the values of $u_1$ and $u_2$ in channels $A$, $B_i$ and $C_j$ will be deformed with a magnitude not bigger than $\epsilon$. Without loss of generality, we can denote these channels’ $\epsilon$ neighborhoods by $A_\epsilon$, $B_{i,\epsilon}$ and $C_{j,\epsilon}$ and avoid the influence of deformation. For convenience we still use $A$, $B_i$ and $C_j$ without confusion.

We claim that for an arbitrary closed curve $\gamma$, we could find $\xi$ in channel $A$ or $B_i$ closed, and has a smaller action, i.e. $A_{c,L}(\xi) \leq A_{c,L'}(\gamma)$. Once this claim proved, the stability we get.

**Case 1.** Once the closed curve $\gamma$ in channel $A$, we have $A_{c,L'}(\gamma) = A_{c,L'}|_{A}(\gamma)$.

**Case 2.** If the $n$-th component of the homology type of $\gamma$ is not zero, it must cross $A$ and at least one $B_i$, then we project the part outside of channel $A$ to channel $A$ with the $n$-th coordinate keeping constant, which is denoted by $(\gamma \cap A^c)^*$ and:

$$
\int_{\gamma \cap A^c} L' - c^* - \int_{(\gamma \cap A^c)^*} L' - c^* \\
\geq \int_{\gamma \cap A^c} K_1 \dot{\gamma}_n^2 + K_2 \\
\geq K_2(T_2 - T_1) + \frac{K_1}{T_2 - T_1} \int_{T_1}^{T_2} \dot{\gamma}_n^2 dt \int_{T_1}^{T_2} 1 dt \\
\geq K_2(T_2 - T_1) + \frac{K_1}{T_2 - T_1} (\int_{T_1}^{T_2} | \dot{\gamma}_n | dt)^2 \\
\geq 2\sqrt{K_1 K_2 \cdot \text{width}(C_i)} > 0,
$$

here $K_1$ and $K_2$ could be both positive and sufficient large. In fact, $K_1 = K^2$ and $K_2 = K - \epsilon \gg \frac{1}{K}$ if we take $\epsilon$ sufficiently small. The width of some channel $C_i$ is here because $A$ and $B_i$ are separated by at least one $C_i$. We denote the two endpoints of $(\gamma \cap A^c)^*$ by $a$, $b$. Obviously these two points are both in channel $A$. Then we connect these two points with a geodesic curve $\zeta$ and $(\gamma \cap A^c)^* \cup \zeta \cup (\gamma \cap A) \ni \xi$. 

![Figure 1](image.png)
forms a closed curve (see figure(1)). But we know \( \text{dist}(a, b) \leq \sqrt{n} \), and
\[
\int_0^1 L'(\zeta, \dot{\zeta}) - \langle c^*, \dot{\zeta} \rangle dt \leq \epsilon + 1 + \int_0^1 |\dot{\zeta}|^2 dt \leq n + 1 + \epsilon.
\]

Then we could take sufficient large \( K \) such that \( n + 1 + \epsilon \leq 2K_{1,K_2} \cdot \text{width}(C_i) \). So \( \xi \) has smaller action than \( \gamma \).

With a similar approach we can also project \( \gamma \) into each channel \( B_i \) with the \( n \)-th coordinate constant and make it closed. Since all the possible cases are finitely many, we can always take a sufficient large \( K \) to make all these cases satisfy the claim. Therefore, the lemma is proved.

**Remark 2.2.** The stability property under perturbation in this lemma will be used in the next section.

Now, we have constructed our first example of \( n \) degrees of freedom. \( \alpha_L^{-1}(0) \) is a lowest flat of \( n - 1 \) dimensions restricted to \( \mathcal{H} \). This is because \( \alpha_L^{-1}(0) \subseteq [-\frac{4\sqrt{2}}{\pi}, \frac{4\sqrt{2}}{\pi}]^{n-1} \times \{0\} \subseteq \mathcal{H} \). At the boundary of this flat, we could find \( 2^{n-2} \) pairs of points which are diametrical with each other. We can see that at these points \( \alpha_L(c) \) is not differentiable along at least \( n - 1 \) different directions, since \( \alpha_B(c) \) intersects the flat \( [-\frac{4\sqrt{2}}{\pi}, \frac{4\sqrt{2}}{\pi}]^{n-1} \times \{0\} \) transversally and \( \alpha_L(c) = \max_{i=1,2,\ldots,n-1}\{\alpha_A,\alpha_B\} \).

More precisely, the coordinates of these points in \( \mathcal{H} \) are easy to write down when we take \( \delta_i = \frac{1}{i} \).
\[
\{(c_1, c_2, \ldots, c_{n-1}, 0) \in H^1(T^n, \mathbb{R}) \mid c_i = \pm \frac{1}{\sqrt{8(4n-7)}}\}
\]

At each point, there are \( n - 1 \) new measures coming out except the one supported on the fix point in channel \( A \). Also we can see that the extra \( n - 1 \) new measures are all Lagrangian invariant torus consist of periodic orbits, but we can change the Rice-mannian metric in the channel \( B_i \) to make all the new measures into quasi-periodic KAM torus. What we just need to do is change the metric vector \( (a_1, a_2, \ldots, a_n) \) to non-resonant one.

At last, we show that the lowest level flat \( \tilde{F} \) is indeed of full dimension. That’s because
\[
\tilde{\mathcal{M}}(0) = \tilde{\mathcal{N}}(0),
\]
and we also knows that \( \tilde{\mathcal{M}}(0) \) just consists of the fixed points (simple calculation of a pendulum system). So we could make use of the upper semi-continuous property of the \( \tilde{\mathcal{N}}(c) \) and expand the flat along the directions of \( H_1(M, \mathcal{N}(0), \mathbb{R}) \).

**Lemma 2.3.** If \( c \in \mathcal{H}^1 \subseteq H_1(T^n, \mathbb{R}) \), there exists a sufficient small \( \lambda \ll 1 \), such that there exists a flat \( [-\lambda c, \lambda c] \) along the direction of \( c \).

**Proof.** By the upper semi-continuity of set-valued function \( c \rightarrow \tilde{\mathcal{N}}(c) \), we know for sufficiently small \( \lambda \), \( \tilde{\mathcal{N}}(\lambda c) \) is still in channel \( A \) because \( \mathcal{M}(0) = \mathcal{A}(0) = \mathcal{N}(0) = \{0\} \in \mathbb{R}^n \). Then:
\[
-\alpha(\lambda c) = \int L - \lambda c d\mu_{\lambda c} = \int Ld\mu_{\lambda c} - \langle \lambda c, [\mu_{\lambda c}] \rangle = \int Ld\mu_{\lambda c} \geq \int Ld\mu_0 = -\alpha(0),
\]
so we have $\alpha(\pm \lambda c) \leq \alpha(0)$, where the $-\lambda c$ case is the same with $\lambda c$. We get $\alpha(\pm \lambda c) = \alpha(0)$ because of the convexity, and then $\alpha_L(c)$ has a full dimensional flat at the lowest energy level. □

We also recall that $\forall h \in \mathbb{R}$ the sublevel set $\{c | \alpha(c) \leq h\}$ is a convex set. Using the previous Lemma we get a full dimensional flat at the lowest level.

### 2.2. the higher energy level case.

Now, we show that the same phenomenon can happen at a higher energy level.

For the sake of simplicity, we set $n = 3$ and it’s easy to generalize the example to a higher dimensional case ($n \geq 3$). We could take the Lagrangian as:

$$L(x, \dot{x}) = \frac{1}{2} a_1(x)(\dot{x}_1 - 1)^2 + \frac{1}{2} a_2(x)\dot{x}_2^2 + \frac{1}{2} a_3(x)\dot{x}_3^2 + u_1(x_3) + u_2(x_3)(1 - \cos x_2)$$

We just need to set two C-type channels, one A-type channel and one B-type channel. The settings of $u_1$ and $u_2$ is the same as the previous section. Since we still restrict the cohomology to $H$, so we have $\dot{x}_3 \equiv 0$ and:

$$L_C \geq \frac{1}{2}(\dot{x}_1 - 1)^2 + K, \quad i = 1, 2,$$

$$L_B = \frac{1}{2}(\dot{x}_1 - 1)^2 + \dot{x}_2^2 + \delta,$$

and

$$L_A = \frac{1}{2}(\dot{x}_1 - 1)^2 + \frac{1}{2} \dot{x}_2^2 + (1 - \cos x_2).$$

Then Lemma 2.1 is still valid:

$$\alpha_L(c)|_{c_3 = 0} = \max\{\alpha_A, \alpha_B\}.$$ 

Based on our calculation and $c_3 = 0$, the set $\{\alpha_A = 0\}$ is $\{0\} \times [-\sqrt{\delta}, \sqrt{\delta}] \times \{0\}$, which is a flat of one dimension. And $\{\alpha_B = 0\}$ is an elliptical curve with an expression as

$$\frac{1}{2}(c_1 + 1)^2 + \frac{1}{4}c_2^2 = \frac{1}{2} + \delta.$$ 

Restrict to $c_1 = 0$ and we find that the elliptical curve go across $\{\alpha_A = 0\}$ from inner with the intersection points’ coordinates $(0, \pm 2\sqrt{\delta}, 0)$. In order to show that $\alpha_L$ is not differentiable at these two points (in fact we just need show that for one point), we need to calculate the directional derivative along $\vec{e}_2$.

$$\partial^+_{\vec{e}_2} \alpha_L(0, 2\sqrt{\delta}, 0) = \partial_{\vec{e}_2} \alpha_B(0, 2\sqrt{\delta}, 0) = \sqrt{\delta},$$

and on the other hand

$$\partial^-_{\vec{e}_2} \alpha_L(0, 2\sqrt{\delta}, 0) = \partial_{\vec{e}_2} \alpha_A(0, 2\sqrt{\delta}, 0) = 0.$$ 

So $\alpha_L(0, 2\sqrt{\delta}, 0)$ is not differentiable along the direction $\vec{e}_2$, which means at this point $(0, 2\sqrt{\delta}, 0)$, there exists an extra minimizing measure of a different homology besides the one supported on closed curve.

**Remark 2.4.** With the upper semi-continuous property of the $\tilde{N}(c)$, we can see that in the case of three degrees of freedom, the flat is in fact of two dimensions. The following graph shows the exact situation of the flat.
3. Stability of the Two Degrees of Freedom Case

In this section we prove Theorem 1.8 in the case of two degrees of freedom.

Proof. We take \( \epsilon \ll \tau^2 \ll \delta \ll 1 \), and shrink the width of channel \( A \) and \( B \) to the order of \( O(\epsilon) \) for the sake of simplity, and \( \tau \) is used to control the value of potential function perturbation. Later we will evaluate them precisely.

A neighborhood of 0 in \( C^r(M, \mathbb{R}) \) with a radius \( \epsilon \) is denoted by \( B(0, \epsilon) \). Any perturbation function \( V(x) \) in it can’t break the intersection property of \( \alpha_A \) and \( \alpha_B \) but just deform them of a \( \epsilon \) order change (see the following graphs). This is because the continuity of \( \alpha \)-function with respect to Lagrangian function [Zh].

So in a small neighborhood of point \( a \), there is still an intersection point \( a' \) of \( \alpha_{L',A} \) and \( \alpha_{L',B} \), here \( L' = L + V \). We just need to deal with \( a' \) point and get the same conclusion of \( b' \), this is because the Hamiltonian \( H' \) conjugated to \( L' \) is still a quadratic mechanical system with \( \alpha \) function axial symmetry. As is known to us, the Mañé critical value is equal to \( \alpha_{L',A}'(0) \), which is also the minimizing value of \( \alpha_{L',A}' \) function [Car]. Without lose of generality, we suppose this value is still 0. If not, we can add a constant to \( L + V \) without influencing the intersection property of restricted \( \alpha \) functions.

Still restricted to \( \mathcal{H} \), the curve \( \hat{o}o \) may be no longer a straight line, we will consider two different cases separately in the following and prove the stability. From Lemma 2.1, we know \( \mathcal{M}(a') \) have at least two ergodic minimizing measures,
one in channel $A$ and the other in channel $B$. We denote the one in channel $A$ by $\mu_{a',A}$ and the other $\mu_{a',B}$. From the construction of channel $A$, the support of this measure can only be a periodic orbit with a rotation vector $(h_1,0)$ because of the restriction of homology.

**Case 1.** If $h_1 = 0$, we could see that the curve $\hat{a'}$ is in fact a straight line. This is because

$$\alpha(a') = -\int L - a'd\mu_{a',A} = -\int Ld\mu_{a',A} \leq \int \alpha(0),$$

and $\alpha(0)$ is the minimal value. But $\epsilon \ll \delta$, so $\mu_{a',B}$ couldn’t have a trivial homology because of the convexity of the $\alpha_L$. So we get that not only $\hat{a'}$ is straight, but also $\alpha_L$ is not differential along $H$ direction. So we get the stability of this case.

**Case 2.** If $h_1 > 0$ (the same with $h_1 < 0$), we let $a' \rightarrow a''$, here $a''$ is the first point with $\alpha_L(a'') = 0$ ($a'' = 0$ is possible). Then we know that $[a'',0]$ is a flat and $\tilde{M}(0) \subseteq \tilde{A}(a'')$, and $\tilde{A}(a'') \setminus \tilde{M}(0) \neq \emptyset$. Then there must exist a homoclinic orbit or a periodic orbit in $\tilde{A}(a'')$.

![Figure 4](image-url)

**Figure 4.**

**Case 2.1** If there exists a homoclinic orbit to $\tilde{A}(a'')$, because the maximal points of the potential are contained in a neighborhood of $(0,0)$ of a radius of order $O(\epsilon)$ (that’s why we shrink the width of channel $A,B$), the minimizing homoclinic is asymptotic to this neighborhood $B(0,\epsilon)$. For any $c \in [a',a'']$, $\tilde{M}(c)$ can only be

![Figure 5](image-url)

**Figure 5.**

made up of periodic orbits in channel $A$. Under the weak topology of probability measures, there exists a sequence $c_n \rightarrow a''$ and $\mu_{L',c_n} \rightarrow \mu_{L',0} \in \tilde{M}(0)$. Moreover, the minimizing homoclinic orbit $\gamma$ is contained in the Hausdorff limit of $\text{supp}\{\mu_{c_n}\}$. As $\gamma$ is a static orbit in $\tilde{A}(a'')$, we have

$$\int_{-\infty}^{+\infty} L'(\gamma,\dot{\gamma}) dt = \langle a'',[\gamma]\rangle,$$
here the $\langle \cdot, \cdot \rangle$ is the inner product induced by de Rham. In fact, $\langle a'', [\gamma] \rangle$ is really a scalar product because the homology of channel A and $a'' \in \mathcal{H}$. As $\epsilon \ll \tau^2 \ll \delta \ll 1$, we have

$$\int_{-\infty}^{+\infty} L'(\gamma, \dot{\gamma}) dt = \int_{-\infty}^{+\infty} \langle \dot{\gamma}, \dot{\gamma} \rangle_g + u_1(\gamma_2(t)) + u_2(\gamma_2(t))(1 - \cos(\gamma_1(t))) + V(\gamma(t)) dt$$

$$\geq \int_{-T}^{+T} (\gamma_1'^2 + \gamma_2'^2) + u_1(\gamma_2(t)) + u_2(\gamma_2(t))(1 - \cos(\gamma_1(t))) + V(\gamma(t)) dt,$$

here $\gamma|_{[-T,T]}$ is the part of $\gamma$ outside the $O(\tau)$ neighborhood of $(0,0)$. Recall that $\epsilon \ll \tau^2 \ll \delta \ll 1$, we can get a positive lower bound of potential function out of $O(\tau)$ neighborhood of $(0,0)$. As we have supposed

$$\max_{x \in A} -u_1(x) - u_2(x)(1 - \cos(x)) - V(x) = 0$$

and the width of channel A is of order $O(\epsilon)$, we have:

$$\int_{-T}^{+T} (\gamma_1'^2 + \gamma_2'^2) + u_1(\gamma_2(t)) + u_2(\gamma_2(t))(1 - \cos(\gamma_1(t))) + V(\gamma(t)) dt$$

$$\geq \int_{-T}^{+T} \gamma_1'^2 + 1 - \cos \tau - \epsilon dt$$

$$\geq 2(2\pi - 2\tau) \sqrt{1 - \cos \tau - \epsilon},$$

here $(\gamma_1, \gamma_2)$ is the coordinate of $\gamma$.

At the same time, we have:

$$\langle a'', [\gamma] \rangle = a'' \pi_1([\gamma]) = a''$$

$$\geq 2(2\pi - 2\tau) \sqrt{1 - \cos \tau - \epsilon},$$

because $[\gamma] \equiv (1, 0)$. We already have $a'' < a'$. Specially we take $\tau = \frac{1}{4}\pi$, then $2(2\pi - 2\tau) \sqrt{1 - \cos \tau - \epsilon} > \frac{\pi}{2}$. But the Riemannian metric can be modified in channel B to make $a' < \frac{\pi}{2}$ and there will be a contradiction to our assumption.

**Case 2.2** For the case of periodic orbit coming out, we can give a same proof as above. We just need to modify the integral lower and upper bounds to finite in (3.1) and (3.2) to get a same contradiction.  

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NON-DIFFERENTIABILITY OF $\alpha$ FUNCTION AT THE BOUNDARY OF FLAT

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Abstract. Using the variational method introduced by J Mather, we construct a mechanical Hamiltonian systems whose $\alpha$ function has a flat $F$ and is non-differentiable at the boundary $\partial F$. In the two degrees of freedom case, we prove this phenomenon is stable under Mañé’s perturbation.

1. Introduction

Let $M$ be a smooth closed manifold with $TM$ as tangent bundle. We call such a function $L(x,v) \in C^r(TM, \mathbb{R})$ ($r \geq 2$) **Tonelli Lagrangian** if it satisfies:

- **convexity**: For all $x \in M$, $v \in T_x M$ the Hessian matrix $\frac{\partial^2 L}{\partial v_i \partial v_j}(x,v)$ is positive definiteness;
- **superlinearity**: $\lim_{\|v\| \to \infty} \frac{L(x,v)}{\|v\|} = \infty$ uniformly on $(x,v) \in TM$;
- **completeness**: All solutions of the corresponding Euler-Lagrangian equation are well defined for $t \in \mathbb{R}$.

Here the Euler-Lagrangian equation is given by:

\begin{equation}
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(x, \dot{x}) = \frac{\partial L}{\partial x}(x, \dot{x}), \quad (x, \dot{x}) \in TM.
\end{equation}

**Remark 1.1.** In the autonomous case, the completeness is natural under the first two assumptions. That’s because we can get the Hamiltonian as $H(x,v) = \frac{\partial L}{\partial v}(x,v) - L$.

From [Car] we know that along each orbit $(\gamma, \dot{\gamma})$ of Euler-Lagrangian equation $H(\gamma, \dot{\gamma})$ is constant. The superlinearity implies that the level sets of Hamiltonian are compact. These in turn assure the completeness of flow.

Usually we take $\mathbb{T}^n$ as the manifold $M$. Adding a closed 1-form $\eta_c$ with the cohomology class $[\eta_c] = c \in H^1(M, \mathbb{R})$, we get a new Tonelli Lagrangian $L - \eta_c$. From [Mat] we know that the E-L flow of $L(x,v) - \eta_c(x,v)$ is the same as that of $L$. So we can define a c-minimal curve $\gamma \in C^1(\mathbb{R}, M)$ if it satisfies:

$$ A_c(\gamma) = \min_{\xi(a) = \gamma(a)} \int_a^b L - \eta_c(\xi(t), \dot{\xi}(t))dt, \forall a < b \in \mathbb{R}, \xi \in C^{ac}(\mathbb{R}, M). $$

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All of the c-minimal orbit \((\gamma, \dot{\gamma})\) form a set denoted by \(\tilde{G}\), which is invariant for the Euler-Lagrangian flow \(\Phi^t\).

Let \(\mathcal{M}_{\text{inv}}\) be the set of \(\Phi^t\)-invariant probability measures on \(TM\). We define the \(\alpha\) function by:

\[\alpha : \alpha(c) = -\min_{\mu \in \mathcal{M}_{\text{inv}}} \int_{TM} L - \eta_c \, d\mu, \quad c \in H^1(M, \mathbb{R}).\]

As is showed in [Mat], there exists at least one measure \(\mu_c\) such that the minimum attains. We call this measure \(\mu_c\) c-minimal measure. The union of the supports of all c-minimal measures is Mather set, denoted by \(\tilde{M}(c)\).

Since we know that \(\alpha(c)\) is convex, finite everywhere and superlinear on \(H^1(M, \mathbb{R})\), we can denote its conjugate function in the sense of convex analysis as:

\[\beta(h) = \min_{\rho(\mu) = h} \mu \in \mathcal{M}_{\text{inv}} \int_{TM} L \, d\mu, \quad h \in H_1(M, \mathbb{R}),\]

here \(\rho(\mu)\) is defined via the De Rham inner product:

\[\langle \rho(\mu), c \rangle = \int_{TM} \eta_c \, d\mu.\]

\(\beta(h)\) is also a convex, finite everywhere and superlinear function. From [Man] and [Zh] we can get the following properties:

**Theorem 1.2.**

- If \(\mu\) is a c-minimizing measure, \(\rho(\mu) \in D^- \alpha(c)\).
- The maximal connected domain on which \(\alpha\) function isn’t strict convex is called a flat \(\mathcal{F}\). \(\forall c, c' \in \text{int} \mathcal{F}\) we have \(\tilde{M}(c) = \tilde{M}(c')\).
- For each non-differential point \(c\) of a function, \(\mathcal{M}(c)\) corresponds to at least two ergodic components with different rotation vectors.
- If \(h\) is a strict convex point of \(\beta\) function, then there must exist a ergodic minimal measure \(\mu\) with \(\rho(\mu) = h\).

We also need another two functions to define Aubry set \(\tilde{A}(c)\) and Mañé set \(\tilde{N}(c)\). We let

\[h_c^c(x, y) = \min_{\gamma \in C^1([0, t], M)} \int_0^t (L - \eta_c)(\gamma(s), \dot{\gamma}(s)) + \alpha(c) \, ds,\]

\[\Phi_c(x, y) = \inf_{t \in \mathbb{R}} h_c^c(x, y),\]

and

\[h^\infty_c(x, y) = \liminf_{t \to +\infty} h_c^c(x, y).\]

We call a curve \(\gamma \in C^1(\mathbb{R}, M)\) c-semistatic if

\[\Phi_c(\gamma(a), \gamma(b)) = \int_a^b (L - \eta_c)(\gamma(t), \dot{\gamma}(t)) + \alpha(c) \, dt, \quad \forall a < b \in \mathbb{R},\]

and c-static if

\[\Phi_c(\gamma(b), \gamma(a)) + \int_a^b (L - \eta_c)(\gamma(t), \dot{\gamma}(t)) + \alpha(c) \, dt = 0, \quad \forall a < b \in \mathbb{R}.\]

A c-static orbit must be a c-semistatic orbit. We call the union of all global c-semistatic orbits Mañé set \(\tilde{N}(c)\) and the union of all global c-static orbits Aubry set \(\tilde{A}(c)\). From [Ber] we have the following inclusions:
Theorem 1.3. • $\tilde{M}(c) \subseteq \tilde{A}(c) \subseteq \tilde{N}(c) \subseteq \tilde{G}(c)$.
• We denote the projected set of $M(c)$ and $A(c)$ from $TM$ to $M$ by $\tilde{M}(c)$ and $\tilde{A}(c)$. Then the inverse of the project map $\pi$
$$\pi^{-1} : A(c) \to \tilde{A}(c)$$
is a Lipschitz graph.
• For an autonomous Lagrangian, $\tilde{N}(c) = \tilde{G}(c)$ and is an upper-semicontinuous set-value function of $c$.
• If $c$-minimal measure is uniquely ergodic, then we have $\tilde{N}(c) = \tilde{A}(c)$.

In $[\text{Man}]$, Mañé raised the problem: Is it true that for a generic Lagrangian, it’s $\alpha$ function is of $C^1$ smoothness? Here the 'generic' means there is an residue subset $G \subset C^\infty(M, \mathbb{R})$ such that the property holds for all the Tonelli Lagrangian $L - u, u \in O$. Negative answer is firstly given in $[\text{S}]$, but we don’t know whether there would be a flat $F$ coming out, not to mention the stability.

Moreover, we know $\text{int}F$ share the same Mather set but there may be new measure come out at $c \in \partial F$. If so, we could construct heteroclinic orbits between these different measures and explore much interesting dynamic behavior. This phenomenon was discovered by Zheng Yong in $[\text{Zh}]$. Based on such a premise:
$$\inf_{g \in H_1(M, \text{int}F, \mathbb{Z})} h_c^\infty(g) = \delta > 0, \exists c \in \partial F,$$
he get the following conclusion in the universal cover space $\tilde{M}$ of $M$:

Theorem 1.4. (Zheng) There exist infinitely many $\tilde{M}$-minimal homoclinic orbits to $\tilde{A}(\text{int}F)$ which are not of multi-bump type.

Actually, he raised this premise in order to assure new measure’s existence at $\partial F$ (sec.2, lemma2.1 in $[\text{Zh}]$). Our construction verifies the rationality of the premise and gets the same existence of infinitely many homoclinic orbits as above.

In an arbitrary degrees of freedom case we can construct the following example:

(1.2) 
$$L(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle_g + u_1(x_n) + u_2(x_n) \sum_{i=1}^{n-1} (1 - \cos x_i),$$

here the precise form of the Riemannian metric $g$ and potential function will be given in the later section. With the help of this example we get our main result as following:

Theorem 1.5. In the case of $n$ degrees of freedom, there exist $u_1, u_2$ and Riemannian metric $g$ such that the $\alpha$ function of $(1.2)$ has a flat $F$ of full dimension at the lowest energy level. There are $2^{n-1}$ points $c_i (i = 1, 2, \ldots, 2^{n-1}) \in \partial F$ of which the Mather set $\mathcal{M}(c_i)$ supports at $n - 1$ ergodic minimizing measures except the one supported on the minimal fixed point”. Besides, all of the $n - 1$ new minimizing measures have non-vanishing rotation vectors different from each other.

Remark 1.6. Besides, we can see that in the following section this phenomenon also happens at a higher energy level flat.

Theorem 1.7. In the case of two degrees of freedom, this example is stable in the sense of Mañé.
We say a property is stable in the sense of Mañé if there is a small enough neighborhood $O$ of zero $O \subset C^\infty(M, \mathbb{R})$ such that the property holds for all the Tonelli Lagrangian $L - u, u \in \mathcal{O}$. Then we can easily get the following conclusion:

**Corollary 1.8.** The following property is open in the sense of Mañé of a two degrees of freedom case:
There is a full-dimensional flat $\mathcal{F}$ corresponding to Mañé critical value at the boundary of which the $\alpha$ function is not differentiable.

This two degree of freedom phenomenon appears recently in a priori stable Arnold Diffusion problem\cite{Ch}. In Cheng’s paper, our example is a typical case of a double resonance point. Actually, we can conclude that normally hyperbolic invariant torus can reach the lowest energy level with a periodic orbit as bottom.

The paper is outlined as follows. In Section 2, we state the main background information and tools. In Section 3, we give the construction of the examples. We also give the proof of stability in the case of two degrees of freedom and some remarks in Section 4.

## 2. Construction of the example

Use the following example we can get our result in Theorem 1.8.

### 2.1. the lowest energy level case.

We construct the Lagrangian as follows.

$$L(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle_g + u_1(x_n) + u_2(x_n) \sum_{i=1}^{n-1} (1 - \cos x_i)$$

where $(x, \dot{x}) \in \mathbb{T}^n \times \mathbb{R}^n$ and $x = (x_1, x_2, \ldots, x_n)$. $u_1(x_n), u_2(x_n)$ and the Riemannian metric $g$ are constructed as follows.

First, we make several channels on the $\mathbb{T}^n$ along the $n$-th coordinate $x_n$.

$$\begin{cases}
\text{channel } A : \mathbb{T}^n \times \left[ -\frac{\pi}{4n}, \frac{\pi}{4n} \right], & T = \mathbb{R}/2\pi, \\
\text{channel } B_i : \mathbb{T}^n \times \left[ \frac{2\pi i}{n}, \frac{2\pi i}{n} + \frac{\pi}{4n}, \frac{2\pi i}{n} + \frac{\pi}{4n} \right] , & i = 1, 2, \ldots n - 1, \\
\text{channel } C_j : \mathbb{T}^n \times \left[ \frac{(2j-1)\pi}{n}, \frac{(2j-1)\pi}{n} + \frac{\pi}{4n}, \frac{(2j-1)\pi}{n} + \frac{\pi}{4n} \right], & j = 1, 2, \ldots n,
\end{cases}$$

obviously, channel $B_i$ and $C_j$ are just constructed by shifting channel $A$ along the $x_n$ coordinate. It is easy to see that $A$ and $B_i$ are separated by $C_j$.

(2.1) \quad u_1(x_n) = \begin{cases} 0, & x_n \text{ in channel } A, \\
K \gg 1, & x_n \text{ in channel } C_j \text{ and } j = 1, 2, \ldots n, \\
0 < \delta_i \ll K, & x_n \text{ in channel } B_i \text{ and } i = 1, 2, \ldots n - 1,
\end{cases}

where the values of $\delta_i$ will be given later (in fact it is enough to take $0 < \delta_i \leq \frac{1}{2}$) and $u_1(x_n)$ can be made smooth enough.

(2.2) \quad u_2(x_n) = \begin{cases} 1, & x_n \text{ in channel } A, \\
0, & x_n \text{ out of some small neighborhood of channel } A,
\end{cases}

and we can also make $u_2(x_n)$ smooth enough.
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At last, we construct the Riemannian metric $\langle \dot{x}, \dot{x} \rangle_g = \sum_{i=1}^{n} a_i(x)\dot{x}_i^2$ as follows:

$$G = \begin{pmatrix}
a_1^2(x) & 0 & \cdots & 0 \\
0 & a_2^2(x) & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & a_n^2(x)
\end{pmatrix}_{n \times n}$$

is the metric matrix which is diagonal. For $x_n$ in channel $A$,

$$G = \text{Id}_{n \times n}$$

For $x_n$ in channel $C_j$, $j = 1, 2, \ldots, n$,

$$G = \begin{pmatrix}
K & 0 & \cdots & 0 \\
0 & K & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & K
\end{pmatrix}_{n \times n}$$

For $x_n$ in channel $B_k$, $k = 1, 2, \ldots, n-1$,

$$G = \begin{pmatrix}
\frac{1}{4} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \frac{1}{8} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \ddots & 0 & \cdots & \cdots & 0 \\
0 & \cdots & 0 & \frac{1}{8} & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & \frac{1}{8} & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \frac{1}{8} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{8}
\end{pmatrix}_{n \times n},$$

where $\frac{1}{4}$ is in the $k$-th row and $k$-th column.

The projection along the $n$-th coordinate $p: \mathbb{T}^n \to \mathbb{T}^{n-1}$ induces a homomorphism of the cohomology map $p^*: H^1(\mathbb{T}^{n-1}, \mathbb{R}) \to H^1(\mathbb{T}^n, \mathbb{R})$. Let $\mathcal{H} = \text{Im}(p^*)$ as the subset of the $H^1(\mathbb{T}^n, \mathbb{R})$ with the $n$-th coordinate $c_n = 0$.

We denote the restriction of Lagrangian $L$ on the channels $A$, $B_i$ and $C_j$ by $L_A$, $L_{B_i}$ and $L_{C_j}$. Also we could use $\alpha_A(c)$, $\alpha_{B_i}(c)$ and $\alpha_{C_j}(c)$ to denote the $\alpha$ function of $L$ restricted to $A$, $B_i$, $C_j$ separately, here $c \in \mathcal{H}$. 
Moreover, we could calculate \( \alpha_A(c) \), \( \alpha_{B_i}(c) \) and \( \alpha_{C_j}(c) \).

\[
\alpha_A(c) = - \min \int L_A - c d\mu, \\
= - \min \int \langle v, v \rangle_g + \sum_{i=1}^{n-1} (1 - \cos x_i) - \sum_{i=1}^{n-1} c_i v_i d\mu, \\
= - \min \sum_{i=1}^{n-1} \int (v_i^2 + (1 - \cos x_i) - c_i v_i) d\mu, \\
= - \sum_{i=1}^{n-1} \min \int (v_i^2 + (1 - \cos x_i) - c_i v_i) d\mu.
\]

It is easy to see that \( L_A \) consists of \( n - 1 \) independent pendulum systems. Hence, \( \alpha_A \) has a flat at \([-\frac{4\sqrt{2}}{\pi}, \frac{4\sqrt{2}}{\pi}]^{n-1} \subseteq \mathcal{H} \).

\[
\alpha_{B_i}(c) = - \min \int L_{B_i} - c d\mu \\
= - \min \int \langle v, v \rangle_g + \delta_i - \sum_{i=1}^{n-1} c_i v_i d\mu \\
= -\delta_i + \sum_{k=1}^{n-1} 16c_k^2 - 12c_i^2, \quad i = 1, 2, \cdots, n-1.
\]

In fact, if we take \( \delta_i < \frac{1}{2} \), the level set \( \alpha_{B_i}^{-1}(0) \subseteq [-\frac{4\sqrt{2}}{\pi}, \frac{4\sqrt{2}}{\pi}]^{n-1} \subseteq \mathcal{H} \).

The following lemma gives the relationship between \( \alpha \) function of \( L \) on \( \mathbb{T}^n \) and \( \alpha \) function of \( L \) restricted to channel \( A \) and \( B_i \).

**Lemma 2.1.**

\[
\alpha_L(c) = \max_{i=1, 2, \cdots, n-1} \{ \alpha_A, \alpha_{B_i} \},
\]

where \( c = (c_1, c_2, \cdots, c_{n-1}) \in \mathcal{H} \). This Lemma is still right under small perturbations of potential functions.

**Proof.** First, we have \( \alpha_L(c) \geq \alpha_A(c) \), and \( \alpha_L(c) \geq \alpha_{B_i}(c) \). It follows since the action of the local minimizer is bigger than the global minimizer. Hence,

\[
\alpha_L(c) \geq \max_{i=1, 2, \cdots, n-1} \{ \alpha_A, \alpha_{B_i} \}.
\]

On the other hand, from \([Mn]\) we know that every minimizing measure can be approximated by a sequence of measures supported on absolutely continuous closed curves, so we just need to certify that every closed curve \( \gamma \) in \( \mathbb{T}^n \) has a larger action of the Lagrangian \( L - c^* \) than its projection \( \gamma^* \) to \( \mathbb{T}^{n-1} \) in some channel \( A \) or \( B_i \),
here $c^* = (c_1, c_2, \cdots, c_{n-1}, 0) \in \mathcal{H}$. It follows from
\[
\int_{\gamma} L - c^* - \int_{\gamma^*} L - c^* \geq \int a_n^2 \gamma_n^2 + u_1(x_n)
\geq 0.
\]
Hence,
\[
\rho_L(c) \leq \max_{i=1,2,\cdots,n-1} \{ \alpha_A, \alpha_B \}.
\]

If another Lagrangian $L'$ is $C^1$ close to $L$, i.e. $L' = L + V(x)$ and $\|V\|_r \leq \epsilon \ll \delta_i$, $i = 1, 2, \cdots, n - 1$. Now the channels $A, B_i$ and $C_j$ will be deformed with an magnitude not bigger than $\epsilon$. So these new channels denoted by $A', B'_i$ and $C'_j$ have the same homology group with $A, B_i$ and $C_j$ each. Without loss of generality, we can denote these channels’ $\epsilon$ neighborhoods by $A, B_i$ and $C_j$ and avoid the deformation.

If the closed curve $\gamma$ is just in channel $A$, this Lemma is still right. If the $n$-th component of the homology type of $\gamma$ is not zero, it must cross $A$ and at least one $B_i$, then we project the part outside of channel $A$ to channel $A$ with the $n - \text{th}$ coordinate keeping constant, which is denoted by $(\gamma \cap A^c)^*$ and:
\[
\int_{\gamma \cap A^c} L' - c^* - \int_{(\gamma \cap A^c)^*} L' - c^* \geq \int_{\gamma \cap A^c} K_1 \gamma_n^2 + K_2
\geq K_2(T_2 - T_1) + \frac{K_1}{T_2 - T_1} \int_{T_1}^{T_2} \gamma_n^2 dt \int_{T_1}^{T_2} 1 dt
\geq K_2(T_2 - T_1) + \frac{K_1}{T_2 - T_1} (\int_{T_1}^{T_2} \gamma_n^2 dt)^2
\geq 2 \sqrt{K_1 K_2} \cdot \text{width}(C_i)
\geq 0,
\]
here $K_1$ and $K_2$ could be both positive and sufficient large. In fact, $K_1 = K^2$ and $K_2 = K - \epsilon \gg \frac{K}{\epsilon}$ if we take $\epsilon$ sufficiently small. The width of some channel $C_i$ is here because $A$ and $B_i$ are separated by at least one $C_i$. We denote the two endpoints of $(\gamma \cap A^c)^*$ by $a, b$. Obviously these two points are both in channel $A$. Then we connect these two points with a geodesic curve $\zeta$ and $(\gamma \cap A^c)^* \cup \zeta \cup (\gamma \cap A) \equiv \zeta$ form a closed curve. But we know $\text{dist}(a, b) \leq \sqrt{n}$, and
\[
\int_0^1 L'(\zeta, \dot{\zeta}) - \langle c^*, \dot{\zeta} \rangle dt \leq \epsilon + 1 + \int_0^1 |\dot{\zeta}|^2 dt
\leq n + 1 + \epsilon.
\]
Here we could always choose a proper $\zeta$ satisfying $\int_0^1 \langle c^*, \dot{\zeta} \rangle \geq 0$ because $\mathbb{T}^n$ is a closed manifold. Then we could take sufficient large $K$ such that $n + 1 + \epsilon \leq 2 \sqrt{K_1 K_2} \cdot \text{width}(C_i)$. So $\zeta$ has smaller action value than $\gamma$, that means there exists a measure in channel $A$ with smaller action value. Then $\gamma$ must have a homology type with zero $n$-th component.

With the help of this, we can directly project $\gamma \cap A^c$ to channel $A$ with the $n - \text{th}$ coordinate keeping constant and we needn’t extra curve to make it closed. Obviously this curve in $A$ has a smaller action value. So if $\mu$ is the minimizing
measure of $L$ at $C^*$ and $\gamma$ is the orbit in its support, we must have either $\gamma \subset A$ or $\gamma \subset A^c$.

For the other case $\gamma \subset A^c$, we can give a similar proof and project $\gamma$ into some channel $B_i$ with $n$-th coordinate constant. Since all the possible cases are finite many, we can always take a sufficient large $K$. Therefore, the lemma is proved. □

**Remark 2.2.** The stability property under perturbation in this lemma will be used in the next section. Also we can see [S] for more details about this Lemma.

Now, we have constructed our first example of $n$ degrees of freedom. First, $\alpha_L^{-1}(0)$ is a lowest flat of $n - 1$ dimensions, here $c \in \mathcal{H}$. This is because $\alpha_{B_i}(c) = -\delta_i + \sum_{k=1}^{n-1} 16c_k^2 - 12c_i^2$ and $\alpha_{B_i}^{-1}(0) \subseteq \left[-\frac{4\sqrt{2}}{\pi}, \frac{4\sqrt{2}}{\pi}\right]^{n-1} \subseteq \mathcal{H}$.

Second, we could find $2^{n-2}$ pairs of points which are diametrical with each other at the boundary of the flat. Also at these points $\alpha_L(c)$ is not differentiable along at least $n - 1$ different directions, since $\alpha_{B_i}(c)$ cross the flat $\left[-\frac{4\sqrt{2}}{\pi}, \frac{4\sqrt{2}}{\pi}\right]^{n-1}$ from inner and $\alpha_L(c) = \max_{i=1,2,\ldots,n-1}\{\alpha_{A_i}, \alpha_{B_i}\}$. More precisely, the coordinates of these points in $\mathcal{H}$ are easy to write down when ($\delta_i = \frac{1}{2}$):

$$\left(\pm \frac{1}{\sqrt{8(4n-7)}}, \ldots, \pm \frac{1}{\sqrt{8(4n-7)}}\right).$$

At each point, there are $n - 1$ more measures besides the fix points in channel $A$. Notice that there is not only one fixed point in the channel $A$, but we can shrink channel $A$ to just one line $\{x_n = 0\}$ since $x_n$ ia a free variable. Also we can see that the extra $n - 1$ new measures are all Lagrangian invariant torus consist of periodic orbits, but we can change the Riemannian metric in the channel $B_i$ to make all the new measures quasi-periodic KAM torus.

At last, we show that the lowest level flat is indeed of full dimension. That’s because in the interior of the flat, we have:

$$\tilde{\mathcal{M}}(c) = \tilde{\mathcal{N}}(c), \ \forall c \in \text{intF},$$

and we also knows that $\tilde{\mathcal{M}}(c)$ just consists of the fixed points. So we could make use of the upper semi-continuous property of the $\tilde{\mathcal{N}}(c)$ and expand the flat along the directions of $H_1(M, A(0), \mathbb{R})$. We also recall that all the sublevel set $\{c|\alpha(c) \leq h\}$ is a convex set. So in fact we get a full dimensional flat at the lowest level. See [Zh] for more details, but for the sake of completeness we also give our proof below:

**Lemma 2.3.** If $c \in \mathcal{H}^\perp \subseteq H_1(T^n, \mathbb{R})$, there exists a sufficient small $\lambda \ll 1$, such that there exists a flat $[-\lambda c, \lambda c]$ along the direction of $c$.

**Proof.** By the upper semi-continuity of set-valued function $c \rightarrow \tilde{\mathcal{N}}(c)$, we know for sufficiently small $\lambda$, $\mathcal{N}(\lambda c)$ is still in channel $A$ because $\mathcal{M}(0) = A(0) = N(0) = \{0\} \in \mathbb{R}^n$. Then:

$$-\alpha(\lambda c) = \int L - \lambda c d\mu_{\lambda c} = \int L d\mu_{\lambda c} - \langle \lambda c, \mu_{\lambda c} \rangle = \int L d\mu_{\lambda c} \geq \int L d\mu_0 = -\alpha(0),$$
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so we have $\alpha(\pm \lambda c) \leq \alpha(0)$, where the $-\lambda c$ case is the same with $\lambda c$. We get $\alpha(\pm \lambda c) = \alpha(0)$ because of the convexity, and then $\alpha_L(c)$ has a full dimensional flat at the lowest energy level. □

2.2. the higher energy level case. Now, we show that the same phenomenon can happen at a higher energy level.

For the sake of simplicity, we set $n = 3$ and it’s easy to generalize the example to a higher dimensional case ($n \geq 3$). We could take the Lagrangian as:

$$L(x, \dot{x}) = \frac{1}{2}a_1(x)(\dot{x_1} - 1)^2 + \frac{1}{2}a_2(x)\dot{x_2}^2 + \frac{1}{2}a_3(x)\dot{x_3}^2 + u_1(x_3) + u_2(x_3)(1 - \cos x_2)$$

We just need set two C-type channels, one A-type channel and one B-type channel. The settings of $u_1$ and $u_2$ is the same with the previous subsection. Since we still restrict the cohomology to $H$, so we have $\dot{x_3} \equiv 0$ and:

$$L_{C_i} \geq \frac{1}{2}(\dot{x_1} - 1)^2 + K, \quad i = 1, 2,$$

$$L_B = \frac{1}{2}(\dot{x_1} - 1)^2 + \dot{x_2}^2 + \delta,$$

and

$$L_A = \frac{1}{2}(\dot{x_1} - 1)^2 + \frac{1}{2}\dot{x_2}^2 + (1 - \cos x_2).$$

Then Lemma 2.1 is still valid:

$$\alpha_L(c)|_{c_3 = 0} = \max\{\alpha_A, \alpha_B\}.$$  \hspace{1cm}

Based on our calculation and $c_3 = 0$, the set $\{\alpha_A = 0\}$ is $(0, [-\frac{4\sqrt{2}}{\pi}, \frac{4\sqrt{2}}{\pi}], 0)$, which is a flat of one dimension. And $\{\alpha_B = 0\}$ is an elliptical curve with expression as

$$\frac{1}{2}(c_1 + 1)^2 + \frac{1}{4}c_2^2 = \frac{1}{2} + \delta.$$  \hspace{1cm}

Restrict to $c_1 = 0$ and we find that the elliptical curve go across $\{\alpha_A = 0\}$ from inner with the intersection points’ coordinates $(0, \pm 2\sqrt{\delta}, 0)$. For the purpose of showing that $\alpha_L$ is not differentiable at this two points (in fact we just need show that for one point), we need to calculate the directional derivative along $\vec{e}_2$.

$$\partial_{\vec{e}_2}^+ \alpha_L(0, 2\sqrt{\delta}, 0) = \partial_{\vec{e}_2}^+ \alpha_B(0, 2\sqrt{\delta}, 0) = \sqrt{\delta},$$  \hspace{1cm}

and on the other hand

$$\partial_{\vec{e}_2}^- \alpha_L(0, 2\sqrt{\delta}, 0) = \partial_{\vec{e}_2}^- \alpha_A(0, 2\sqrt{\delta}, 0) = 0.$$  \hspace{1cm}

So $\alpha_L(0, 2\sqrt{\delta}, 0)$ is not differentiable along the direction $\vec{e}_2$. That means at this point $(0, 2\sqrt{\delta}, 0)$, there exists an extra minimizing measure besides the closed curves $T \times \{0\} \times \{0\}$ which is of a different homology.

Remark 2.4. With the upper semi-continuous property of the $\tilde{N}(c)$, we can see that in the case of three degrees of freedom the flat is in fact of two dimensions. The following graph shows the exact situation of the flat.
In this section we prove the conclusion of Theorem 1.7 in the case of two degrees of freedom.

We take \( \epsilon \ll \tau^2 \ll \delta \ll 1 \), and shrink the width of channel \( A \) and \( B \) to the order of \( O(\epsilon) \) for the sake of simplicity, and \( \tau \) is used to control the value of potential function. Later we will explain their usage precisely. A ball in \( C^r(M, \mathbb{R}) \) centered at “0” with a radius of \( \epsilon \) is denoted by \( B(0, \epsilon) \). Any perturbation function \( V(x) \) in it can’t break the intersection property of \( \alpha_A \) and \( \alpha_B \) but just deform them of a \( \epsilon \) order change (see the following graphs). This is because the continuity of \( \alpha \)-function with respect to Lagrangian \( L \).

So in a small neighborhood of point \( a \), there is still an intersection point \( a' \) of \( \alpha_{L',A} \) and \( \alpha_{L',B} \), here \( L' = L + V \). With the help of Legendre transformation, the Hamiltonian \( H' \) dual to \( L' \) is still a quadratic mechanical system, so the \( \alpha \) function is axial symmetry. So we just deal with \( a' \) point and we can also get the same conclusion of \( b' \). As is known, the maximal potential value is equal to \( \alpha'_{L',A}(0) \), which is the minimizing value of \( \alpha'_{L',A} \) function[Car]. Without lose of generality, we suppose this value is still 0. If not, we can add a constant to \( L + V \) without influencing the property of \( \alpha \) function.

Still restrict to \( \mathcal{H} \), the curve \( \hat{a'o} \) may be no longer a straight line, we will consider the two different cases separately in the following and disprove both of them. From Lemma 2.1 we know \( \hat{\mathcal{M}}(a') \) have at least two ergodic minimizing measures, one in channel \( A \) and the other in channel \( B \). We denote the one in channel \( A \) by
µa′,A and the other µa′,B. From the construction of channel A, the support of this measure can only be a periodic orbit with a rotation vector (h₁, 0) because of the restriction of homology.

If h₁ = 0, we could see that the curve \( \hat{a}o \) is in fact a straight line and from [Mas] we know \( \hat{A}(0) \subseteq \hat{A}(a') \). But \( \epsilon \ll \delta \), so \( \mu_{a',B} \) couldn’t have a trivial homology because of the convexity of the \( \alpha_{L',B} \). So we get that not only \( \hat{a}ob' \) is straight, but also \( \alpha_{L'} \) is not differential along \( H \) direction. So we get the stability of the example.

If h₁ > 0 (the same with h₁ < 0), we let \( a' \rightarrow a'' \), here \( a'' \) is the first point with \( \alpha_{L',A}(a'') = 0 \). Here the case \( a'' = 0 \) is possible. Then we know that \( [a'',0] \) is a flat and \( \hat{M}(0) \subseteq \hat{A}(a'') \), and \( \hat{A}(a'') \backslash \hat{M}(0) \neq \emptyset \) [Zh]. Then there must exist a homoclinic orbit or a periodic orbit in \( \hat{A}(a'') \).

\[\text{Figure 3.}\]

First we consider the case homoclinic orbit comes out in \( \hat{A}(a'') \). Because the maximal points of potential value are contained in a neighborhood of (0, 0) of a radius of order \( O(\epsilon) \) (that’s why we shrink the width of channel A,B), the minimizing homoclinic is asymptotic to this neighborhood \( B(0,\epsilon) \). For any \( c \in [a',a''] \), \( \hat{M}(c) \) can only be made up of periodic orbits in channel A. Under the weak topology of probability measures, there exists a sequence \( c_n \rightarrow a'' \) and \( \mu_{L',c_n} \rightarrow \mu_{L',0} \in \hat{M}(0) \). And the minimizing homoclinic orbit \( \gamma \) is contained in the Hausdorff limit of \( supp\{\mu_c\} \). As \( \gamma \) is a static orbit in \( \hat{A}(a'') \), so

\[\text{(3.1)}\]

\[\int_{-\infty}^{+\infty} L'(\gamma,\dot{\gamma})\,dt = \langle a'', [\gamma] \rangle,\]

here the \( (,) \) is the inner product induced by de Rham. In fact, \( \langle a'', [\gamma] \rangle \) is really a scalar product because the homology of channel A and \( a'' \in H \). As \( \epsilon \ll \tau^2 \ll \delta \ll 1 \),

\[\text{Figure 4.}\]
we have
\begin{align}
\int_{-\infty}^{+\infty} L'(\gamma, \dot{\gamma}) \, dt &= \int_{-\infty}^{+\infty} \langle \dot{\gamma}, \dot{\gamma} \rangle_g + u_1(\gamma_2(t)) \\
&\quad + u_2(\gamma_2(t))(1 - \cos(\gamma_1(t))) + V(\gamma(t)) \, dt \\
&\geq \int_{-T}^{+T} (\dot{\gamma}_1^2 + \dot{\gamma}_2^2) + u_1(\gamma_2(t)) \\
&\quad + u_2(\gamma_2(t))(1 - \cos(\gamma_1(t))) + V(\gamma(t)) \, dt,
\end{align}
here $\gamma |_{[-T,T]}$ is the part of $\gamma$ outside the $O(\tau)$ neighborhood of $(0,0)$. Recall that $\epsilon \ll \tau^2 \ll \delta \ll 1$, we can get a positive lower bound of potential function out of $O(\tau)$ neighborhood of $(0,0)$. As we have supposed $\max_{x \in A} -u_1(x) - u_2(x)(1 - \cos(x)) - V(x) = 0$, and the width of channel A is of order $O(\epsilon)$, we have:
\begin{align}
\int_{-T}^{+T} (\dot{\gamma}_1^2 + \dot{\gamma}_2^2) + u_1(\gamma_2(t)) + u_2(\gamma_2(t))(1 - \cos(\gamma_1(t))) + V(\gamma(t)) \, dt \\
&\geq \int_{-T}^{+T} \dot{\gamma}_1^2 + 1 - \cos \tau - \epsilon dt \\
&= \int_{\tau}^{2\pi - \tau} \dot{\gamma}_1 + \frac{1 - \cos \tau - \epsilon}{\dot{\gamma}_1} d\gamma_1 \\
&\geq 2(2\pi - 2\tau)\sqrt{1 - \cos \tau - \epsilon},
\end{align}
here $(\gamma_1, \gamma_2)$ is the coordinate of $\gamma$.

At the same time, we have:
\begin{align}
\langle a''(\gamma) \rangle &= a''(\pi_1(\gamma)) \\
&= a'' \\
&\geq 2(2\pi - 2\tau)\sqrt{1 - \cos \tau - \epsilon},
\end{align}

because $\langle \gamma \rangle \equiv (1,0)$. We already have $a'' < a'$. Specially we take $\tau = \frac{1}{8}\pi$, then $2(2\pi - 2\tau)\sqrt{1 - \cos \tau - \epsilon} > \frac{\pi}{9}$. But the Riemannian metric can be modified in channel B to make $a' < \frac{\pi}{9}$ and there will be a contradiction to our assumption.

For the case periodic orbit coming out, we can give a same proof as above. And we just need to modify the integral lower and upper bounds to finite in (3.1) and (3.2) to get a same contradiction. So we get the stability property of the example in the case of two degrees of freedom.

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