Do weak values capture the complete truth about the past of a quantum particle?

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Weak values inferred from weak measurements have been proposed as a tool to investigate trajectories of pre- and post-selected quantum systems. Are the inferences drawn from the weak values about the past of a quantum particle fully true? Can the two-state vector formalism predict everything that the standard formalism of quantum mechanics can? To investigate these questions we present a “which-path” gedanken experiment in which the information revealed by a pre- and post-selected quantum system is surprisingly different from what one would expect from the weak values computed using the two-state vector formalism. In our gedanken experiment, a particle reveals its presence in locations where the weak value of the projection operator onto those locations was vanishingly small. Therefore our predictions turn out to be in contradistinction to those made based on the non-vanishing weak values as the presence indicators of the quantum particle. We propose a six port photon-based interferometer setup as a possible physical realization of our gedanken experiment.

I. INTRODUCTION

The value of an observable does not hold a meaning prior to the measurement in the standard formalism of quantum mechanics [1–4]; however, in the time-symmetric two-state vector formalism (TSVF) of quantum mechanics such a meaning is alluded to via “weak values” [5–7]. TSVF and the concept of weak values were introduced to validate the retrodiction formula introduced by Aharonov, Bergmann and Lebowitz (ABL rule) to calculate probabilities of counterfactual-measurement outcomes of an observable for a pre- and post selected ensemble [8]. In TSVF, the weak values fully determine the properties of pre- and post-selected quantum system at all intermediate times. The weak values can lie outside the range of the values of an observable allowed by standard quantum mechanics and they need not even be real.

Experimental observations of weak values have been carried out [9–11], even in the absence of a consensus on their interpretation [12–15]. The concept of weak values has been used in understanding optical telecom networks [16], superluminal and slow light phenomena in birefringent photonic crystal [17, 18], studying optical cross-phase modulation jump [19], quantum process tomography [20], ultrasensitive quantum measurements using weak value amplification [21–23], and a review is available in reference [24]. Weak values have also been used in direct measurements of wavefunctions and in providing an operational definition to the wavefunction [25, 26].

Apart from their applications, weak values have been thought to provide insights into a number of fundamental issues in quantum mechanics, which include Hardy’s paradox [27–30], quantum tunneling time [31], the Legget-Garg inequality [32], Bohmian trajectories [33, 34] and quantum contextuality [35]. These studies are firmly based on the straightforward interpretation that the weak value is the value of an observable between two successive measurements of a quantum system. The weak values have also been called the weak-measurement elements of reality (WMER) [36] and it has been proposed that the trace a particle leaves at a location is proportional to the weak value of the projection operator onto that particular location [37].

The use of straightforward interpretation of weak value in few experimental schemes has resulted in inception of new quantum paradoxes: the paradox of negative number of particles and negative pressure [7, 38], the paradox of discontinuous trajectories of a photon [37, 39–43], and the paradox of quantum Cheshire cat [44–47]. The last two have been at the center stage of the discussion for researchers working on quantum foundations. The most surprising and ‘common sense’ defying claim made by Danan et al. [40] is that a pre- and post-selected photon in a nested Mach-Zehnder interferometer (NMZI) takes discontinuous trajectories to reach the detector. The photon visits a region in the NMZI without entering and exiting it. Another ‘common sense’ defying claim is made by Aharonov et al. [44] and Denkmayr et al. [45] that the internal degree of freedom of a quantum system can be separated from its wavefunction. Many comments and papers have been published in criticism as well as defense of these claims [48–56].

In this paper, using a gedanken experiment involving time varying Hamiltonians, we show that weak values do not capture the complete truth about the past of a quantum particle. This is in contradistinction to the claim that TSVF is a complete description of a pre- and post-selected quantum system. The signatures of the Hamiltonian evolution and the time dependence of the Hamiltonian which are present in the quantum state just before measurement, are typically lost in the postselection process for the individual particle. However, these signatures can be present in the probability distribution measured over time. In our gedanken experiment, we use this tech-
nique to investigate the past of a post-selected quantum system. Certain time dependent elements in the Hamiltonian at certain locations which oscillate at fixed frequencies are carefully inserted. We then use the presence of these frequencies in the measured probabilities as indicators for the passage of the particle through locations where such time dependent elements were installed. In our analysis we arrive at a conclusion that the weak values of the TSVF formalism at times fail to capture the presence of the quantum particles.

The material in this paper is arranged as follows: Section II gives a brief review of Aharonov-Bergmann-Lebowitz (ABL) rule, TSVF, Weak Values and their interconnections. Section III gives background of leading weak value paradoxes. Section IV describes our gedanken experiment aimed at providing a counterexample to the weak value based interpretation of the past of a quantum particle. In Section V we compare the predictions of our nalysis with those of TSVF to demonstrate the mismatch. In Section VC we provide a consistent histories (CH) analysis of our six-port interferometer. Section VI provides conclusions and discussion.

II. ABL RULE, TSVF, WEAK VALUES AND WEAK VALUE PARADOXES

According to the ABL rule, the measurement of an observable $A$ of a quantum system at time $t$ which is pre-selected in state $|\psi_1\rangle$ at time $t_1 < t$ and post-selected in state $|\psi_2\rangle$ at time $t_2 > t$ would yield eigenvalue $a_n$ with probability $[8, 57]$:

$$P_t(a_n|\psi_1, \psi_2) = \frac{|\langle \psi_2(t)|\Pi_{a_n} |\psi_1(t)\rangle|^2}{\sum_a |\langle \psi_2(t)|\Pi_a |\psi_1(t)\rangle|^2}$$

where $\Pi_{a_i} = \sum_a |a_{i,a}\rangle \langle a_{i,a}|$ with $\{|a_{i,a}\rangle\}$ being a complete set of eigenstates of $A$ labeled by eigenvalues $a_i$, and $|\psi_j(t)\rangle = \exp\left(-i \int_{t_j}^t H dt\right) |\psi_j\rangle$ with $j = 1, 2$. It can be seen that the state $|\psi_1(t)\rangle$ evolves forward while the state $|\psi_2(t)\rangle$ evolves backward in time both being on equal footing in the TSVF formalism. Application of ABL rule in counterfactual reasoning results in surprising and paradoxical situations. For example consider the three box problem [57]: given that a particle is prepared in superposition of being in three non-overlapping boxes $A, B$ and $C$ with state $|\psi_1\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle)$ and post-selected in state $|\psi_2\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle - |C\rangle)$, the probability of finding the particle in box $A$ or $B$ upon opening the respective box at any intermediate time is one according to the ABL rule. In other words if either of the boxes $A$ and $B$ had been opened at an intermediate time, one would always find the particle there.

Application of the ABL rule in time symmetric counterfactual reasoning faced a serious refutation from Kastner [58], Miller [59], Cohen [60] and others on philosophical grounds. Giving an alternative interpretation of the ABL rule as being the probability of the outcome of an actual measurement of the observable between pre-selection and post-selection measurements, these authors pinpointed that the paradoxes arise only when one uses the ABL rule for calculating probabilities of possible outcomes of observables which actually have not been measured at intermediate times. While the debate about the interpretation of the ABL rule was thought to be settled with experimental realizations of counterfactual paradoxes using weak values introduced as a witness for the ABL rule, as we shall see the predictions of the ABL rule remains questionable. According to the measurement postulate of quantum theory, performing an actual measurement on a quantum system at the intermediate time would destroy the prepared state making counterfactual interpretation no longer valid. Then the question is how to experimentally witness the counterfactualness of the ABL rule. The answer is that the ABL probabilities given by Equation (1) can be inferred using weak values $\Pi_{a_i}(t)$ of the projection operators $\{\Pi_{a_i}\}$ at the intermediate time $t$, given as:

$$\Pi_{a_i}(t) = \frac{\langle \psi_2(t)|\Pi_{a_i}|\psi_1(t)\rangle}{\langle \psi_2(t)|\psi_1(t)\rangle}$$

which can be experimentally determined without collapsing the wave function. The concept of weak values, has thus been claimed to have the potential to provide a ground for an experimental realization of the ABL rule without performing a projective measurement at intermediate times and thereby giving the counterfactual interpretation an operational meaning.

Let us for a moment revisit, the three box problem. If any of the boxes $A$ and $B$ had been opened, according to counterfactual ABL rule, the particle would have been found with certainty (probability one) in the respective box. This raises a serious and natural question: how can a single particle be present in more than one boxes with certainty? The concept of weak values resolve this problem. It has been hypothesized that, the weak values are values of corresponding observables and fulfill the conditions of being elements of reality of weak measurements (WMER) [36, 57]. Let us call it the weak value hypothesis (WVH). The validity of WVH naturally leads one to conclude that the weak value of a projection operator $|\eta\rangle \langle \eta|$ is the number of quantum systems present in the state $|\eta\rangle$. Therefore, the number of particles present in boxes $A, B$ and $C$ are $1, 1$ and $-1$ respectively keeping the total number of particles one at any intermediate time. As one can see one has to accept the concept of negative number particles in this explanation!

A natural consequence of WVH is the truthfulness of the following statement:

S-A: If the weak value of the projection operator $\Pi_x = |x\rangle \langle x|$ at an intermediate time is zero, where $|x\rangle$ is a position eigenstate; then the particle was not present at position $x$ at that time.

The above statement is just a codification of the counterfactual statement: that if $P_t(a_n|\psi_1, \psi_2) = 0$ then the measurement of observable $A$ on system at the interme-
The weak traces can be experimentally measured by a complete state tomography of the pointer state after the post-selection of the system state. Since the post-selection measurement leaves the state of the system and the pointer separable, a further measurement on the pointer state will definitely not affect the past of the system in a retro-causality manner.

III. WEAK VALUE PARADOXES

We describe here situations where WVH leads to paradoxical situations. Therefore, a closer examination of the WVH and of experiments which may contradict this hypothesis is important and is the focus of our work.

A. Photon with discontinuous trajectories

An intriguing example of the weak value paradox is the past of a photon in a nested Mach-Zehnder interferometer (NMZI), as investigated by Lev Vaidman [37, 39], where photons take discontinuous trajectories to reach the detector. The experimental setup is shown in Figure 1. A single photon is pre-selected at source S and post-selected at detector D. If the phase shifter (PS) is tuned in such a way that there is a completely destructive interference near mirror F, then the weak values of projection operators near mirrors A, B, C, E, and F are \( \Pi^A_w = 1, \Pi^B_w = 1, \Pi^C_w = -1, \Pi^E_w = 0, \) and \( \Pi^F_w = 0 \) respectively. Therefore according to S-A the photon was never present near mirror E and F but it was present near mirror A, B and C leading us to conclude that the photon took discontinuous trajectories to reach the detector D. In the language of counterfactual ABL rule, if one had placed a position measurement device near mirror E and F to find whether the photon passed through the regions, one would have never found a photon.

The experiments of Danan et al. [40] and Zhou et al. [41] were aimed at experimental realisation of Vaidman’s predictions using weak measurements in the context of NMZI. Vaidman’s claims and the experimental results of Danan et al. have faced serious criticism including from Englert et al. [48] using ‘unambiguous which-path information’ (UWI), and R. B. Griffiths [51] using the consistent histories approach argued that the paradox of discontinuous trajectories arises from discarding second and higher order perturbation terms in the interaction strength \( \kappa \). Li et al. [61] and D. Sokolovski [62] have also been critical of the TSVF interpretation and they too rely on arguments based on neglecting the higher order terms. The critique of the TSVF interpretation of the Danan experiment by these authors is based primarily on discarding certain higher order terms. This we think is insufficient to reject the main claims because an experimental realization of the counterfactual ABL rule requires negligible disturbance to the system, a limit in which the contribution of the higher order terms may be ignorable. According to counterfactual ABL rule, in the absence of a weak measurement device, (coupling constant \( \kappa = 0 \)) no photon would have been detected if a photon detector were placed near E or F. However, in the presence of a weak measuring device \( (\kappa \neq 0) \) a fraction proportional to \( \kappa^2 \) of the pre- and post-selected ensemble would have been detected near E and F. Griffiths has also agreed to this point in his consistent histories analysis of Danan experiment [51]. As we will see, while not disagreeing with these observations, we intend to analyse the situation from a different point of view and show that the conclusions based on the ABL rule can be in contradiction to those based on standard quantum mechanics in a concrete experimental NMZI.

B. Quantum Cheshire Cat

Aharonov et al. [44] predicted that the internal degree of freedom of a photon (grin) can be separated from its wavefunction (cat) which has been experimentally supported by Denkmayr et al. [45] and Ashby et al. [63]. The claims are firmly based on truthfulness of S-A. The proposed experimental setup, shown in Figure 2(a), is a modified Mach-Zehnder interferometer in which a photon source S and a beam-splitter BS1 are used to preselect a single photon in the state \( |\psi\rangle = (|A\rangle + i|B\rangle)/\sqrt{2} \) and a half wave plate (HWP), a phase shifter (PS), a beam splitter (BS2), a polarizing beam splitter (PBS)
and a single photon detector (D) are used to post-select the photon in the state \( |\phi\rangle = (|A\rangle |V\rangle + |B\rangle |H\rangle)/\sqrt{2} \). Here states \(|A\rangle\) and \(|B\rangle\) are spatial state vectors of photon being in arm A and B respectively and, \(|H\rangle\) and \(|V\rangle\) are horizontal and vertical polarization states respectively. States with circular polarization are \(|\pm\rangle = (|H\rangle \pm i |V\rangle)/\sqrt{2} \). Let us now ask a question: which arm did photon pass through to reach detector D and which of the two circular polarization (determined by measuring \( \sigma_z = |+\rangle \langle +| - \rangle \langle -| \) ) it was possessing in each arm? To answer the question we calculate weak values of operators \( \Pi_A = |A\rangle \langle A|, \Pi_B = |B\rangle \langle B|, \sigma_z^A = \Pi_A \otimes \sigma_z \) and \( \sigma_z^B = \Pi_B \otimes \sigma_z \) as:

\[
\begin{align*}
(\Pi_A)^w &= 0; & (\Pi_B)^w &= 1 \\
(\sigma_z^A)^w &= 1; & (\sigma_z^B)^w &= 0
\end{align*}
\] (3)

VWH implies that the circular polarization of the photon in arm A was non-zero but it did not pass through arm A while photon passed through arm B but the circular polarization in that arm was zero. This is how Aharonov et al. concluded that photon wavefunction (cat) was separated from polarization (grin).

C. Weak value version of Hardy’s Paradox

Hardy [27] used a gedanken experiment involving a bipartite system to give a logical proof against local realism and to show a contradiction between quantum mechanics and any realistic theory which has Lorentz invariant elements-of-reality. The paradox arises when one uses counterfactual reasoning about the past of a system in Hardy’s setup [28]. The setup consists of two Mach-Zehnder interferometers as shown in Figure 2(b), one \( (I_e^-) \) for an electron and the other \( (I_p^-) \) for a simultaneously produced positron. X is an overlapping region of inner arms of both the interferometers \( (I_{e\pm}) \) such that if positron and electron encounter each other the annihilation occurs with probability one. Arms of the two interferometers are adjusted in such a way that there is no detection in \( D_{e\pm} \) when the two interferometers are separated so that there is no overlapping region. When there is an overlap of \( I_{e\pm}, e^- \) and \( e^+ \) annihilation in region X act as a Eluterz-Vaidman bomb [64] and disturbs the interference causing coincident detection in \( D_{e\pm} \) with probability \( 1/6 \). Let us now ask a question: which arms did \( e^- \) and \( e^+ \) travel through when there was a coincidence count \( i.e. \) both \( D_{e\pm} \) detected a particle simultaneously? A counterfactual reasoning leads to the following paradox: if there is a coincident detection in \( D_{e\pm} \) then \( e^\pm \) must have travelled through region X in order to disturb the interference, but there is no annihilation! Aharonov et al. [28] made the paradox even weirder when they use the ABL rule and VWH. They conclude: (a): \( e^\pm \) always passed through region X. (b): \( e^- \) always passed through region X. (c): \( e^- \) and \( e^+ \) together never passed through the region X. While there have been experiments on the weak value version of Hardy’s paradox based on the measurement of weak values [30, 65], the interpretation requires one to accept VWH. Therefore, it is important to scrutinise VWH further.

IV. THE GEDANKEN EXPERIMENT

We now describe our main results where we will consider a situation where the predictions of weak values can be seen to come in contradiction with descriptions based on standard quantum mechanics. Consiser A quantum system with a six-dimensional Hilbert space \( \mathcal{H} \). For the purpose of the gedanken experiment, we can think of a quantum particle being in six non-overlapping boxes. If the particle is found in the \( i^{th} \) box with certainty, the state vector of the particle is written as \( |i\rangle \). In the absence of interactions these states are orthogonal to each other. The boxes are designed in such a way that the interactions can be switched on so that the particle can tunnel between any pair of boxes in a controlled manner. The boxes \( i \) and \( j \) can be made to interact instantaneously at time \( t' \) via the interaction Hamiltonian \( H' = g\delta(t-t')\sigma^{ij}_y \). Here \( \sigma^{ij}_y = i(-|i\rangle\langle j| + |j\rangle\langle i|) \) is Pauli matrix and \( \delta(t-t') \) is a Dirac delta function of time \( t \). The tunable parameter \( g \) represents the tunneling strength and we call the process as a leaky process when \( g \) is sufficiently small. Further, the operational condition \( g^2N \approx 1 \) has to be satisfied where \( N \) a large number representing ensemble size being considered by the experimenter. Therefore, we need to retain only the terms linear in \( g \) unless it is multiplied by \( N \). In the rest of the paper, whenever we neglect the contribution of higher powers of some quantity, it is understood that we are assuming that the operational condition is satisfied.

Consider a quantum state \( |\psi(t_0)\rangle \) of the particle at \( t_0 \) which undergoes time evolution according to the Hamil-
\[ H = \sum_{i=1}^{9} H_i, \quad \text{With} \]
\[ H_1 = -\hbar \{ \sin^{-1}(\sqrt{2/3}) \delta(t-t_4)(\sigma_y^{(13)} + \sigma_y^{(24)}) \} \]
\[ H_2 = -\hbar \cos(\omega t) \delta(t-t_2)(\sigma_y^{(24)}) \]
\[ H_3 = -\frac{\pi}{4} \hbar \delta(t-t_3)(\sigma_y^{(34)} + \sigma_y^{(46)}) \]
\[ H_4 = -\hbar \{ \cos(\omega t)(\sigma_y^{(12)} + \cos(\omega t))\sigma_y^{(24)} \}
+ \cos(\omega t)\sigma_y^{(56)} \}
\]
\[ H_5 = -\frac{\pi}{2} \hbar \delta(t-t_5)(\sigma_y^{(65)} - \sigma_y^{(56)}) \]
\[ H_6 = -\frac{\pi}{4} \hbar \delta(t-t_6)(\sigma_y^{(35)} + \sigma_y^{(46)}) \]
\[ H_7 = -\hbar \cos(\omega t)(\sigma_y^{(23)} + \sigma_y^{(34)} + \sigma_y^{(46)}) \]
\[ H_8 = -\hbar \{ \sin^{-1}(\sqrt{2/3}) \delta(t-t_8)(\sigma_y^{(13)} + \sigma_y^{(24)}) \}
\]
\[ H_9 = -\frac{\pi}{4} \hbar \delta(t-t_9)\sigma_y^{(12)} \]  
\hspace{0.5cm} (4)

where \( I^{(ij)} = \ket{i} \bra{j} \) and \( \sigma_y^{(ij)} = \ket{i} \bra{j} \). The impulsive interaction occurs at moments of time \( t_0 < t_1 < t_2 < \cdots < t_8 < t_9 \). The parameter \( \epsilon \ll 1 \) is such that the contributions of higher powers of \( \epsilon \) in the experimental observations are negligible. Therefore, \( H_2, H_4, \) and \( H_7 \) generate leakage processes between certain boxes. The time intervals between \( t_i \)'s are kept fixed for repeated runs of the experiment. Since all the transformations generated by \( H_1 \) are momentary and well separated in time, the state of the particle at time \( t > t_9 \) is given for infinitesimally small \( \Delta \) as

\[ |\psi(t)\rangle = \exp \left[ -\frac{i}{\hbar} \int_{t_{0-\Delta}}^{t_{0+\Delta}} H_9 dt \right] \exp \left[ -\frac{i}{\hbar} \int_{t_{1-\Delta}}^{t_{1+\Delta}} H_8 dt \right] \cdots
\]
\[ \cdots \exp \left[ -\frac{i}{\hbar} \int_{t_{2-\Delta}}^{t_{2+\Delta}} H_2 dt \right] \exp \left[ -\frac{i}{\hbar} \int_{t_{3-\Delta}}^{t_{3+\Delta}} H_1 dt \right] |\psi(t_0)\rangle \]  
\hspace{0.5cm} (5)

The sequence of momentary interactions presented in Equation (4) and time evolution of the system shown in Equation (5) can be understood as a sequence of unitary operations \( U_1, U_2, \cdots, U_9 \) acting on the system at times \( t_1, t_2, \cdots, t_9 \) respectively, where

\[ U_j = \exp \left[ -\frac{i}{\hbar} \int_{t_j-\Delta}^{t_j+\Delta} H_j dt \right] \]  
\hspace{0.5cm} (6)

Unitary operations \( \{ U_i \} \) are \( 6 \times 6 \) matrices:

\[ U_3 = U_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2} I}{2} & \frac{\sqrt{2} I}{2} \\ 0 & -\frac{\sqrt{2} I}{2} & \frac{\sqrt{2} I}{2} \end{bmatrix} ; U_4 = \begin{bmatrix} L_2 & 0 & 0 \\ 0 & L_3 & 0 \\ 0 & 0 & L_4 \end{bmatrix} 
\]
\[ U_5 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} ; U_7 = \begin{bmatrix} I & 0 & 0 \\ 0 & L_5 & 0 \\ 0 & 0 & I \end{bmatrix} ; U_9 = \begin{bmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \]

where

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ; R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \]
\[ L_1 = \begin{bmatrix} \cos(\epsilon \cos \omega t_2) & \sin(\epsilon \cos \omega t_2) \\ -\sin(\epsilon \cos \omega t_2) & \cos(\epsilon \cos \omega t_2) \end{bmatrix} \]
\[ L_5 = \begin{bmatrix} \cos(\epsilon \cos \omega t_7) & \sin(\epsilon \cos \omega t_7) \\ -\sin(\epsilon \cos \omega t_7) & \cos(\epsilon \cos \omega t_7) \end{bmatrix} \]

and

\[ L_4 = \begin{bmatrix} \cos(\epsilon \cos \omega t_4) & \sin(\epsilon \cos \omega t_4) \\ -\sin(\epsilon \cos \omega t_4) & \cos(\epsilon \cos \omega t_4) \end{bmatrix} \]

for \( i = 2, 3, 4 \).

For the purpose of possible experimental realization of the sequence of unitaries on a state in six-dimensional Hilbert space, one can think of a single photon interferometer with six-ports as detailed in Figure. (3). In this setup a single photon inside the interferometer can be in a superposition of six non-overlapping ports forming a six-dimensional Hilbert space. The single photon prepared in superposition of being present in six ports at time \( t_0 < t_1 \) undergoes the sequence of unitaries \( U_1, U_2, \cdots, U_9 \) at moments of time \( t_1, t_2, \cdots, t_9 \) respectively. The first two zero-loss beam-splitters (BS) having transmission and reflection coefficients of one-third and two-thirds respectively act on pairs of ports-1, 3, and ports-2, 4 to generate the unitary time transformation \( U_1 \) at time \( t_1 \). \( U_2 \) is generated by an element \( L_1 \), which is also a beam-splitter but with a time varying reflectivity, acting on pair of port-3 and port-4. The reflectivity of \( L_1 \) is so small that the probability amplitude in either of the input ports (3 or 4) remains unaffected but at the same time it transfers a very tiny amplitude between the ports in either direction as a leakage process so that it can make a contribution in providing information about the past of the photon in interferometer. The role of time dependence of \( L_1 \) will be become clear from the discussion that follows. The unitary \( U_9 \) is generated by a phase shifter \( \eta \) which produces a phase shift of \( \pi \) in the probability amplitude of photon in the port-6. Rest of the unitaries can be easily related to processes presented in Figure (3) which are either BS or leakage processes.
As we shall see, the leakage processes described above are engineered to provide us with a tool to investigate the past of the particle. A leakage process between two completely empty boxes will definitely not make any contribution to the time evolution of the state of the particle and hence will not have any measurable effects. Therefore, the measurable effect of such a leakage process, between any two boxes in the state of the particle is an evidence that the amplitude of the particle was not zero at least in one of the boxes involved in the leakage interaction. It is easy to see that due to the leakage process, the change in the probability amplitude of the particle being in one box is proportional to the probability amplitude of it being in the other box.

The initial state of the particle is prepared in $|1\rangle$ (a single photon enters the interferometer through port-1) at time $t_0$. The probability of finding the particle in state $|1\rangle$ (the photon in output port-1) at time $t' > t_0$, retaining only terms linear in $\epsilon$, is calculated using Equation (5), according to Born rule, as:

$$P = \frac{1}{18} \{1+2\epsilon(2\cos\omega_1 t_2 - \cos\omega_2 t_4 + \cos\omega_3 t_4 + \cos\omega_4 t_4)\}$$

(7)

As one would expect, the probability $P$ depends on the reflectivity of the time dependent beam-splitters $\{L_i\}$ at the moments of time when the localized wave packet of photon passes through them. Looking carefully at experimental setup shown in Figure (3), we can say $t_1, t_2, \cdots, t_9$ are not independent variables present in unitaries but are dependent on the time when the photon enters the interferometer. To make latter point clearer we emphasize the fact that the optical path-length of the photon travelling from one optical device to another is fixed over time. In other words one can say that time difference between the unitaries is fixed. Consider the optical path-length from source to optical device generating $U_j$ to be $L_j$, then one has $t_j = t_0 + l_j/c$ where $t_0$ is the time when photon leaves source. The time of detection of the photon at port-1 is $t'$ and $t' - t_0 = \tau$ is constant across various repetitions of the experiment. Physically it means that the photon takes $\tau$ time to travel from source to detector each time the experiment is carried out. Using the relation $t_j = t' + (l_j/c - \tau)$, Equation (7) can be re-written as a function of $t'$ as follows:

$$P = \frac{1}{18} \{1+2\epsilon(2\cos(\omega_1 t' - \theta_1) - \cos(\omega_2 t' - \theta_2) + \cos(\omega_3 t' - \theta_3) + \cos(\omega_4 t' - \theta_4))\}$$

(8)

Where $\theta_1 = \omega_1 (\tau - l_1/c), \theta_i = \omega_i (\tau - l_4/c)$ for $i = 2, 3, 4$ depend on oscillation frequencies of various time varying beam splitters and the geometry (optical path-length) of the interferometer and hence are constant phases. Further assuming the condition $\omega_j^{-1} \gg \tau$ with $j = 1, 2, 3, 5$, which gives $\theta_i \ll 1$ for $i = 1, 2, 3, 4$; Equation (8) can be simplified as:

$$P \approx \frac{1}{18} \{1+2\epsilon(2\cos\omega_1 t' - \cos\omega_2 t' + \cos\omega_3 t' + \cos\omega_4 t')\}$$

(9)

Probability $P$ depends on reflectivities of various $L_j$’s at the time when photon passes through them. Equations (8) and (9) are our main results. We use Equation (9) in drawing operational inferences about the past of the photon. It is to be emphasised here that these inferences can be drawn by using Equation 8 also to avoid any misunderstanding regarding the approximation $\omega_j^{-1} \gg \tau$, however, Equation 9 is simpler and more convenient to use.

Experiments with a single particle cannot reveal any information about the time dependency of probability $P$, but experimental runs over ensembles with varying time can provide information about the frequencies present in the modulated probability $P$. As we describe next, the experimental realisation of Equation (9) can be achieved if we sample a sufficient number of photons in a time window in which the time dependent optical elements in the circuit do not vary appreciably.

A. Sampling protocol

The probability $P$ can be experimentally measured by repeating the experiment a large number of times at a
certain rate. We need to have the frequencies $\omega_i$ sufficiently small so that we can measure over a sufficiently large number of particles before the time varying elements $L_i$ changes appreciably. Suppose at each time $t = t_0 + 2n\tau$ where $n = 0, 1, 2, \cdots, N_s$, a particle is pre-selected which will undergo a post-selection measurement at time $t = t' + 2n\tau$. $N_s$ is the number of particles pre-selected in one sample-run. Note that not all pre-selected particles get post-selected. A particle found in state $|1\rangle$ is counted, otherwise it is discarded. Right after each post-selection measurement the experimental setup is kept ready to perform pre-selection on a new particle. The sampling time period $T_s = 2\tau N_s$ is time taken to run experiment on a sample of $N_s$ particles. $N_s$ and $\epsilon$ must be chosen in such a way that the operational condition $\epsilon^2 N_s \approx 1$ is satisfied.

As we shall see this can be easily achieved with photons. For a precise measurement of modulations, the change in the number of post-selected particles in each consecutive sample is required to be smooth, hence $1 \gg T_s \omega_i$ is necessary. The number of post-selected particles in the $k^{th}$ sample is:

$$N_k = \frac{N_s}{18} + \frac{\epsilon N_s}{9} \left[ 2\cos\{\omega_1(2k-1)\frac{T_s}{2}\} - \cos\{\omega_2(2k-1)\frac{T_s}{2}\} + \cos\{\omega_3(2k-1)\frac{T_s}{2}\} \right]$$

Due to $\epsilon N_s \gg 1$, the (co)sinusoidal oscillations can be observed. The Fourier analysis of the best fit of (10) reveals the frequencies $\omega_i$.

In case of photon: $\tau = \frac{t}{2}$, here $t$ is the optical path-length - the distance each photon travel from source to detector in the interferometer. The requirements for weakness of $\epsilon$ and sampling are: $\epsilon^2 N_s \approx 1$ and $1 \gg T_s \omega_i$. That gives $\omega_i \ll \frac{n}{\tau^2}$ (here we have used $N_s = \frac{t}{2\tau}$). For an interferometer of length one meter and $\epsilon \approx 10^{-2}$, $\omega_i \ll 15000$. The choice of $\omega_i \approx 100$ is reasonable. For photons with well localized wave-packets, one can increase $N_s$ (hence decrease $\epsilon$) by sending a train of photons with a small spacing between the two successive photons into the interferometer.

B. Where was the photon?

We make use of Equation (9) to draw inferences about the presence of the photon at various locations inside the interferometer. The appearance of any observable signature of a localized-device in the post-selection probability is considered as an indicator of the presence of the particle at that location. In an experimental setup involving (co)sinusoidal time varying leakage processes $L_i$’s with various frequencies $\omega_j$’s, our operational definition of the past says: it cannot be possible that the particle was not present at the location where $L_i$ is installed if frequency $\omega_i$ corresponding to device $L_i$ is present in the modulated probability $P$ of post-selection. Therefore, we interpret the past using the following principle:

S-C: A quantum particle cannot carry information about a localized object without interacting with it. In particular, if the particle is a photon inside an interferometer, it cannot not visit the location of a localized optical device and still gain information about it.

Let us now look at Equation (9) and draw valid inferences about the past of a photon inside the interferometer under discussion. Appearances of frequencies $\omega_1, \omega_2, \omega_3$, and $\omega_4$ tell a story about the past of the photon: one cannot say with certainty that the photon, pre- and post-selected at entrance and exit of port-1 respectively, has not been at anyone or more of the locations where time varying beam-splitters $L_1, L_2, L_3$ and $L_4$ are installed. The key result of this section to be emphasized for further use is that one cannot claim with certainty that the photon entered the interferometer through port-1 and detected at output port-1 was not present at $L_1$ at any intermediate time.

V. TSVF ANALYSIS OF THE GEDANKEN EXPERIMENT

Let us now explore the predictions of the TSVF of quantum mechanics for our gedanken experiment. In order to answer the question whether the particle was present in at least one box of the pair of boxes right before the leakage took place, we perform weak measurements on both boxes. For the particle pre-selected in state $|\psi\rangle = |1\rangle$ at time $t_0$ and post-selected in the state $|\phi\rangle = |1\rangle$ at time $t'$, we calculate the weak values of the projection operators at those boxes at the corresponding times. The weak value of projection at box $k$ (port-$k$ of interferometer in case of photon) right before the time $t_j$ is written $\Pi^w_k(t_j)$. The weak values of projections $\Pi^w_k = |k\rangle \langle k|$ right before all the leakage processes come out to be:

$$\begin{align*}
\Pi^w_1(t_2) &= \epsilon (2\cos \omega_1 t' + \cos \omega_3 t' + \cos \omega_4 t') \\
\Pi^w_1(t_4) &= 1 - \epsilon (2\cos \omega_1 t' + \cos \omega_3 t' + \cos \omega_4 t') \\
\Pi^w_3(t_4) &= -1 + \epsilon (2\cos \omega_1 t' - \cos \omega_2 t' + 2\cos \omega_3 t' + \cos \omega_4 t' + \cos \omega_5 t') \\
\Pi^w_3(t_3) &= \epsilon \cos \omega_1 t' \\
\Pi^w_2(t_4) &= 1 - \epsilon (2\cos \omega_1 t' - \cos \omega_2 t' + \cos \omega_3 t' + \cos \omega_5 t') \\
\Pi^w_2(t_7) &= \epsilon \cos \omega_1 t' \\
\Pi^w_4(t_7) &= \epsilon (2\cos \omega_1 t' + \cos \omega_3 t' + \cos \omega_4 t') \\
\Pi^w_4(t_2) &= \Pi^w_1(t_2) = \Pi^w_2(t_4) = 0
\end{align*}$$

(11)
The weak interaction leads to a small shift in the center of the Gaussian state, which is the measure of the weak trace that the photon leaves on the ancillary system.

The experimenter in a weak measurement scenario has complete control over the size of pre-and post selected ensemble, state of the pointer and the weak measurement interaction strength and can tune these parameters suitably so that weak values can be measured up to a desired precision. Weak nature of the measurement implies that the effects of higher powers of coupling strength $\kappa$ are not recordable experimentally. In an ideal weak measurement scenario, the choice of $\kappa$ and the size of the ensemble $N$ should be such that $N\kappa^2 \to 1$ when $N \to \infty$. The ideal condition $N \to \infty$ is not feasible, therefore, the experimenter can choose $\kappa$ and $N < \infty$ such that $N\kappa^2 \approx 1$ while $0 < \kappa \ll 1$ in all practical scenarios.

B. Where was the photon according to TSVF?

The story told by weak values is surprisingly different. For a single photon, pre-selected in input port-1 and post selected in output port-1, the weak values of projection operators at locations of weak measurements $w_1, w_2, \ldots, w_{10}$ shown in Figure (4) are detailed in Equation (11). The values reveal that the presence of the particle was of the order of 1 at $w_3, w_5$, and $w_7$ and of the order of first or higher powers of $\epsilon$ at the rest of the locations. Particularly, for port-3 and port-4, between $t_1$ and $t_2$, at least one of the forward and backward evolving wave-functions vanishes to order $\epsilon$ (see Figure (4) for pictorial representation). To see contradiction between the conclusion drawn in subsection ?? and the prediction (retroduction) of TSVF, let us consider following two cases:

Case 1: The parameter $\epsilon$ of the interferometer is tuned in such a way that $N\epsilon^2 \to 1$ when $N \to \infty$. If experiment (as described in section IV) is performed on infinitely large ensemble ($N \to \infty$), there will be no traces of $\epsilon^2$ or higher orders in the final probability but at the same time one can record deviations of the order $\epsilon$. Once the pre-and post-selected ensemble (which is defined by pre-and post-selection states and all the unitaries including time varying beam-splitters $L_k$) is fixed, the experimenter can deploy weak measurement schemes to investigate past of the photons according to TSVF. The most optimal weak measurement setup requires $N\kappa^2 \to 1$ when $N \to \infty$. This amounts to $\kappa^2 \approx 0$ and we already have $\epsilon^2 \approx 0$, therefore, we conclude that $\kappa \epsilon \approx 0$, which implies that the weak traces corresponding to weak values of the order $\epsilon$ are too small to be observed (even ideally) in this case. In other words operational condition, $N\epsilon^2 \to 1$ and $N\kappa^2 \to 1$ when $N \to \infty$, implies $N\kappa\epsilon \to 1$. Since the weak values of order $\epsilon$ (in this case) are not experimentally measurable, according to Equation (11); the photon leaves weak traces only at ports 1, 3 and 5 with nonzero weak values $\Pi^w_1(t_4), \Pi^w_3(t_4)$, and $\Pi^w_5(t_4)$ respectively. The information about the presence of the photon in the pair of ports 3 and 4 just before $L_1$ is completely absent from the weak signal, which leads us to draw a
conclusion on the basis of weak value based operational definition of the past of a quantum particle: the photon has not been in the vicinity of time varying beam-splitter $L_1$. This prediction is in direct contradiction with our earlier conclusions based on standard quantum mechanical analysis under same approximations.

Case 2: Consider a case where the parameter $\epsilon$ of the interferometer is tuned in such a way that $N\epsilon^2 \approx 1$ for some finite $N$ and $0 < \epsilon \ll 1$. Under these conditions, the experimenter can choose arbitrary large ensemble $N' \gg N$ in weak measurement setup and can easily record weak values of order $\epsilon$. Now, for time being, imagine a situation where experimenter chooses to perform experiment with $N$ systems. Although, this is not an optimal weak measurement setup but one can draw certain inferences based on TSVF retrodiction using ABL rule. As we have discussed earlier, TSVF goes hand in hand with ABL rule. ABL rule can be expressed in terms of weak values, using equations (1) and (2), as:

$$P_t(a_n|\psi_1, \psi_2) = \frac{[\Pi^w_{a_n}|^2}{\sum_i [\Pi^w_i|^2}$$

(13)

Now we ask the following question: given that the experimenter performs experiment on finite number of systems $N$ such that $N\epsilon^2 \approx 1$, how many systems would have been found in if the box $i$ were opened at some intermediate time $t'$? The answer, according to ABL rule, is $NP_t(i)$, where

$$P_t(i) = \frac{[\Pi^w_i|^2}{|1-\Pi^w_i|^2 + [\Pi^w_i|^2}$

(14)

Weak values $\{\Pi^w_i(t)\}$ presented in equation 11 dictate us to conclude that less than one out of $N$ systems would have been found in box 3 and 4 at $t_2$ (in ports 3 and 4 just before $L_1$) if the respective boxes were opened. Which leads us further to conclude that no photon was present in vicinity of $L_1$ if ensemble size was $N$. On the other hand, equation (9) (more explicitly equation (10) suggests that given ensemble of $N$ photons or a significant fraction of it (at least of order $\epsilon N$) does carry information about the time varying element $L_1$. In light of S-C, one can safely conclude that one cannot claim with certainty that out of the $N$ photons which entered the interferometer through port-1 and were detected at output port-1 no (or only at most of order one) photons were present at $L_1$.

C. CH Analysis of the Gedanken Experiment

Along with TSVF approach, Consistent Histories (CH) formalism has also been used to analyze the past of a quantum particle. Therefore, we consider the CH formalism for our gedanken experiment. In CH formalism, one can talk about the trajectories of a quantum system only when the conditions of single framework and consistency are satisfied. A family of consistent histories represents the past of a quantum system. CH formalism considers inconsistent histories of a quantum system ‘meaningless’. To analyze the past of a photon in our interferometer using CH formalism, let us consider the family of consistent histories (see ref. [51] for notations):

$$\mathcal{F} : \Pi_{1}^{(t_{1})} \circ \{\Pi_{2}^{(t_{2})}, \Pi_{3}^{(t_{3})}, \Pi_{4}^{(t_{4})}, \Pi_{5}^{(t_{5})}, \Pi_{6}^{(t_{6})}\}$$

(15)

Here $\Pi_k = |k\rangle\langle k|$ and $\Pi_k^{(t)}$ represents the physical property ‘that photon is in $|k\rangle$ at time $t$’. $\mathcal{F}$ can be divided into two families as per our interest:

$$\mathcal{F}_1 : \Pi_{1}^{(t_{1})} \circ \Pi_{2}^{(t_{2})} = \Pi_{3}^{(t_{3})} \circ I(t_2) \circ I(t_3) \cdots \circ I(t_6) \Pi_{1}^{(t_{1})}$$

(16)

and

$$\mathcal{F}_2 : \Pi_{1}^{(t_{1})} \circ \{\Pi_{2}^{(t_{2})}, \Pi_{3}^{(t_{3})}, \Pi_{4}^{(t_{4})}, \Pi_{5}^{(t_{5})}, \Pi_{6}^{(t_{6})}\}$$

(17)

Only consistent histories of family $\mathcal{F}_1$ are of our interest in which photon enters port-1 and detected in port-1. One can easily verify that family $\mathcal{F}_1$ can only be refined further if all $L$ elements are identities simultaneously, which is possible only if $\epsilon = 0$ or $\cos \omega t = \cos \omega t = \cos \omega t = \cos \omega t = \cos \omega t = 0$ which is momentary during the experiment run. In that case photon remains in port-1 throughout the time according to CH formalism. Since $\epsilon \neq 0$ in the case of our gedanken experiment, CH formalism provide no information about the past of the photon other than saying that there is no consistent history corresponding to the trajectory being considered.

VI. CONCLUSIONS AND DISCUSSION

Truthfulness of S-C asserts that it cannot be the case that the photon did not pass through $L_1$ with certainty while S-B asserts that it did not have a passage through $L_1$ with certainty given that the operational condition $N\epsilon^2 \approx 1$ and $N\epsilon^2 \approx 1$ with $N \gg 1$ is satisfied. Even when the operational condition is not satisfied, a clear difference in the quantitative presence of photon inside the interferometer at various locations can be seen in the two different approaches. For instance, quantification of presence in TSVF is in terms of weak values of the position projection operators, according to which presence of photon near $L_1$ is very smaller than those of near $L_2, L_3$ and $L_4$; while any possible quantification of the presence based on amplitudes of oscillating terms present in equation (9) suggests it was of an equal order.

In the language of counterfactual ABL rule, less than one (which is zero) photons would have been detected if one had tried to detect $N$ pre- and post selected photons in entrance ports of $L_1$ indicating no presence of a photon near $L_1$. The contradictory conclusions inferred from two assertions imply: at least one of S-C and S-B is false. Since S-C is based on the fact that all the interactions in nature are local and the operational definition of past based on weak values itself is implicitly based on S-C, one is forced to forgo S-B. This further leads us to
conclude that the S-A is false i.e. if weak value of a projection operator $|x\rangle \langle x|$ is zero, then it is not necessary that the particle is not present at location $x$. This invalidates the WVH that the weak value of an observable is the value of that observable i.e. if weak value is zero then the system does not carry the corresponding property.

As we have seen in section III, all weak value (TSVF) paradoxes are based on correctness and rationality of WVH more specifically truthfulness of S-A, our results therefore, have a bearing on these paradoxes. As per our conclusions, absence of certain traces in Danan et al. experiment does not imply that photon does not pass through those regions taking discontinuous trajectories to reach the detector. Similarly, zero weak values of certain observables does not imply circular polarization of a photon is separated from the wave function in quantum Cheshire cat paradox. Same is applicable to weak value version of Hardy’s paradox.

A natural question arises: what are weak values if not properties of systems? What do weak values tell about the properties of systems between two successive measurements? A plausible answer is given by D. Sokolovski [15]: weak value of an observable is the transformation generated by weak measurement unitaries on pre-selected state which reaches the post-selection. If the observable is projection operator $|a\rangle \langle a|$ then the weak value is relative transition amplitude of pre-selected state $|\psi\rangle$ to post-selected state $|\phi\rangle$ through state $|a\rangle$.

We have shown in our analysis that the conclusions based on TSVF and the related weak values (WVH) may not always be correct in the context of the past of a quantum system. We clearly obtain nonzero signals in our measurable probability distribution from the regions of the interferometer where TSVF claims that the photon never entered or the presence of photon was not measurable. It will be interesting to explore the possibilities of carrying out such interferometric experiments. More detailed investigations are required to pin down the exact role of weak traces and circumstances where they play a significant role in providing information about the particle trajectories.

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