Research Article

Sine Inverse Lomax Generated Family of Distributions with Applications

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This paper introduces a new family of distributions by combining the sine produced family and the inverse Lomax generated family. The new proposed family is very interesting and flexible more than some old and current families. It has many new models which have many applications in physics, engineering, and medicine. Some fundamental statistical properties of the sine inverse Lomax generated family of distributions as moments, generating function, and quantile function are calculated. Four special models as sine inverse Lomax-exponential, sine inverse Lomax-Rayleigh, sine inverse Lomax-Fréchet and sine inverse Lomax-Lomax models are proposed. Maximum likelihood estimation of model parameters is proposed in this paper. For the purpose of evaluating the performance of maximum likelihood estimates, a simulation study is conducted. Two real life datasets are analyzed by the sine inverse Lomax-Lomax model, and we show that providing flexibility and more fitting than known nine models derived from other generated families.

1. Introduction

Authors have recently been interested in produced families of distributions. Among them are the beta-G [1], Kumaraswamy (Ku)-G [2], T-X [3], Type-II half-logistic-G [4], [0, 1] truncated Fréchet-G [5], extended odd Fréchet-G [6], truncated Lomax-G [7], Ku Type-I half-logistic-G [8], transmuted (Tr) odd Fréchet-G [9], truncated Burr X-G [10], and exponentiated transmuted G [11].

According to [12], the inverse Lomax (IL)-G has the distribution function (CDFu) and density function (PDFu) shown below (PDFu):

\[ R(x; \xi) = \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha}, x \in \mathbb{R}, \alpha > 0 \]  

(1)

\[ r(x; \xi) = \frac{\alpha g(x; \xi)}{[G(x; \xi)]^{\alpha}} \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha-1}, x \in \mathbb{R}, \alpha, \beta > 0. \]  

(2)

By entering the scale parameter \( \beta = 1 \), we are interested. Then, the PDFu and CDFu of IL-G have the following:

\[ r(x; \xi) = \frac{\alpha g(x; \xi)}{[G(x; \xi)]^{\alpha}} \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha-1}, x \in \mathbb{R}, \alpha > 0 \]  

(3)

and

\[ R(x; \xi) = \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha}. \]  

(4)

The sine generated (S-G) family of distributions studied by [13], and it has the following CDFu and PDFu:

\[ F(x; \xi) = \sin[\pi/2R(x; \xi)], x \in \mathbb{R}, \]  

(5)

and

\[ f(x; \xi) = \frac{\pi}{2} r(x; \xi) \cos \left[ \frac{\pi}{2} R(x; \xi) \right], x \in \mathbb{R}. \]  

(6)

Indeed, the characteristics of the models generated from the S-G family have inspired numerous comprehensive families of continuous distributions that are likewise centered on trigonometric functions, such as Beyond the S-G [14] and sine TL-G [15]. The bulk of these families are focused on the S-G structure and do not include any extra tuning parameters or modifications.
The primary goal of this work is to introduce and explore a novel family of probability distributions, based on the S-G and IL-G families. The sine IL family is the name given to the new family (SIL-G). Thus, the SIL-G family differs from other modified S-G families in terms of general simplicity and the advantage of adjustable skewness. As a result, the SIL-G can give intriguing models for a variety of fitting purposes. This work develops this practical element, as well as key theoretical conclusions.

This work is divided into seven sections as follows. We will look at the new family and some of the statistical features of the sine inverse Lomax produced family of distributions in Section 2. Section 3 examines four new models that are unique to the SIL-G family. Maximum likelihood (ML) estimation of the family’s parameters is described in Section 4. SIL-Lomax simulation results are shown in Section 5. Two real-world data sets are utilized in Section 6 to test the SIL-Lomax model. Section 7 concludes with a few words.

2. The New SIL-G Family

A new family of continuous probability distributions known as the sine inverse Lomax generated (SIL-G) family will be introduced in this section. It is possible to obtain the new family’s CDF by substituting (3) into (5):

\[ F(x; \xi) = \sin \left( \frac{\pi}{2} \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha} \right), \quad x \in R, \alpha > 0. \]  

(7)

The PDF of the SIL-G family is

\[ f(x; \xi) = \frac{\pi \alpha g(x; \xi)}{2[G(x; \xi)]^2} \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha-1} \]

(8)

The survival function (SF), hazard rate function (HRF), reversed HRF, and cumulative HRF are

\[ F(x; \xi) = 1 - \sin \left( \frac{\pi}{2} \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha} \right), \]

\[ h(x; \xi) = \frac{\pi \alpha g(x; \xi)(1 + (G(x; \xi)/G(x; \xi)))^{-\alpha-1}}{2[G(x; \xi)]^2 \left[ 1 - \sin \left( \frac{\pi}{2} \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha} \right) \right]}, \]

\[ \tau(x; \xi) = \frac{\pi \alpha g(x; \xi)}{2[G(x; \xi)]^2} \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha-1} \cot \left[ \frac{\pi}{2} \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha} \right], \]

and

\[ H(x; \xi) = -\ln \left( 1 - \sin \left( \frac{\pi}{2} \left( 1 + \frac{G(x; \xi)}{G(x; \xi)} \right)^{-\alpha} \right) \right), \]

(10)

respectively. The quantile function of SIL-G, say \( Q(u) \) of \( X \), is obtained by inverting (7) as

\[ Q(u) = G^{-1} \left( \frac{1}{(2/\pi \text{Arc} \sin(u))^{-1/\alpha} - 1} \right). \]

(11)

The following expansion is used to get PDF (8) expansion:

\[ \cos[G(x)] = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} G(x)^{2i}. \]

(12)

4. SIL-Lomax simulation results are shown in Section 5. Two real-world data sets are utilized in Section 6 to test the SIL-Lomax model. Section 7 concludes with a few words.
\[ f(x; \xi) = \frac{\pi a g(x; \xi)}{2[G(x; \xi)]^2} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(2i)!} \left( \frac{\alpha(2i + 1) + j}{j} \right) \left( \frac{\pi/2}{G(x; \xi)} \right)^j. \] (15)

Alternatively, we may express the last equation as follows:

\[ f(x; \xi) = \frac{\pi a g(x; \xi)}{2[G(x; \xi)]^2} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(2i)!} \left( \frac{\alpha(2i + 1) + j}{j} \right) \left( \frac{\pi/2}{1 - [G(x; \xi)]} \right)^j. \] (16)

Again, using the following binomial theory,

\[ (1 - Z)^{-a} = \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} Z^k, \quad a > 0, |Z| < 1. \] (17)

By inserting (13) in (12) then, the PDF of SIL-G is given by

\[ f(x; \xi) = \frac{\pi a g(x; \xi)}{2[G(x; \xi)]^2} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(2i)!} \left( \frac{\alpha(2i + 1) + j}{j} \right) \left( \frac{j + k - 1}{k} \right) \left( \frac{\pi/2}{1 - [G(x; \xi)]} \right)^j. \] (18)

Again, using the following binomial theory,

\[ (1 - Z)^{-a} = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} Z^m, \quad a > 0, |Z| < 1. \] (19)

By inserting (15) in (14), we obtain

\[ f(x; \xi) = \sum_{r=0}^{\infty} \eta_m g(x; \xi) G(x; \xi)^{m-1}, \] (20)

where \( \eta_m = \frac{\pi a \alpha}{2} \sum_{i,j=0}^{\infty} (-1)^{i+j} m! / (2i)! \left( \frac{\alpha(2i + 1) + j}{j} \right) \binom{j + k - 1}{k} \left( \frac{\pi/2}{m!} \right)^j. \)

The \( r \)th moment of the SIL-G family is calculated as

\[ \mu_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x; \xi) dx = \sum_{m=0}^{\infty} \eta_m \int_{-\infty}^{\infty} x^r g(x; \xi) G(x; \xi)^{m-2} dx. \] (21)

3.3. SIL-Exponential Distribution. For \( g(x; \theta) = \theta e^{-\theta x}, x > 0 \), and \( G(x; \theta) = 1 - e^{-\theta x} \), the PDF of SIL-exponential (SILE) is

\[ f(x) = \frac{\pi a \theta e^{-\theta x}}{2[1 - e^{-\theta x}]} \left( 1 + \left( e^{\theta x} - 1 \right)^{-1} \right)^{-a-1} \cos \left[ \pi/2 \left( 1 + \left( e^{\theta x} - 1 \right)^{-1} \right)^{-a} \right]. \] (24)

The corresponding CDF is

\[ F(x) = \sin \left[ \pi/2 \left( 1 + \left( e^{\theta x} - 1 \right)^{-1} \right)^{-a} \right]. \] (25)
\[ f(x) = \frac{\pi \alpha \delta \beta x^{-\beta-1} e^{-\alpha x^\beta}}{2} \cdot \cos \left( \frac{\pi}{2} \cdot \frac{\mu}{x} \right). \] (26)

The corresponding CDF is
\[ F(x) = \sin \left[ \frac{\pi}{2} \left( 1 + \left( e^{\alpha x^\beta} - 1 \right)^{-1} \right)^{-a} \right]. \] (27)

### 3.3. SIL-\(F\)-rêchet Distribution

The \(F\) rêchet (F) model with PDF \(g(x; \mu, \delta) = \delta x^{-\delta-1} e^{-x^\delta}, x, \mu, \delta > 0\), and CDF \(G(x; \mu, \delta) = e^{-x^\delta}\); hence, the SIL-\(F\) (SILF) PDF is as follows:
\[ f(x) = \frac{\pi \alpha \delta \beta x^{-\beta-1} e^{-\alpha x^\beta}}{2} \cdot \cos \left( \frac{\pi}{2} \cdot \frac{\mu}{x} \right). \] (28)

The CDF of the SILF distribution is
\[ F(x; \alpha, \mu, \delta) = \sin \left[ \frac{\pi}{2} \cdot \frac{\mu}{x} \right]. \] (29)

### 3.4. TIL-Lomax Distribution

For \(g(x; a, b) = \frac{b}{a} (1 + x/a)^{-(b+1)}\) and \(G(x; a, b) = 1 - (1 + x/a)^{-b}\), we obtain the SIL-Lomax (SILL) PDF as
\[ f(x) = \frac{\pi \alpha \delta \beta x^{-\beta-1}}{2a} \cdot \left( 1 + \left( 1 + \frac{x}{a} \right)^{-1} \right)^{-a} \cdot \cos \left( \frac{\pi}{2} \cdot \frac{\mu}{x} \right). \] (30)

### 4. ML Method of Parameter Estimation

Suppose \(X_1, X_2, \ldots, X_n\) be a series of observed values from the SIL - \(G\) family with the set of parameter \(\Phi = (\alpha, \xi)^T\). According to the following formula, the log-likelihood function (LLF) is
\[ \ln(L, \Phi) = n \ln \left( \frac{\pi}{2} \right) + n \ln \alpha + n \sum_{i=1}^{n} \ln g(x_i; \xi) - 2 \sum_{i=1}^{n} \ln \left( G(x_i; \xi) \right) - (\alpha + 1) \sum_{i=1}^{n} \ln(Z_i) + \sum_{i=1}^{n} \ln(\cos(\pi/2(Z_i)^{-a})). \] (32)

where \(Z_i = 1 + \frac{G(x_i; \xi)}{g(x_i; \xi)}\). Score vector components \(U_\Phi = (U_\alpha, U_\xi)^T\) are used to calculate the first partial derivatives of the LLF with respect to \(\alpha\) and \(\xi\) as
\[ U_\alpha = n \ln 2 + \frac{n}{\alpha} \sum_{i=1}^{n} \ln(Z_i) \]
\[ + \frac{\pi}{2} \sum_{i=1}^{n} (Z_i)^{-a} \ln(Z_i) \tan(\pi/2(Z_i)^{-a}) \] (33)

and
\[ U_\xi = \sum_{i=1}^{n} \frac{g(x_i; \xi)}{g(x_i; \xi)} - 2 \sum_{i=1}^{n} \frac{G(x_i; \xi)}{G(x_i; \xi)} - (\alpha + 1) \sum_{i=1}^{n} \frac{\partial Z_i/\partial \xi_k}{Z_i} - \frac{\pi \alpha}{2} \sum_{i=1}^{n} (Z_i)^{-a-1} \frac{\partial Z_i}{\partial \xi_k} \tan(\pi/2(Z_i)^{-a}) \] (34)
where \( g'_k(x_i; \xi) = \partial g(x_i; \xi)/\partial \xi_k \) and \( G'_k(x_i; \xi) = \partial G(x_i; \xi)/\partial \xi_k \).

5. Simulation Study

The ML estimators’ performance is measured according to the number of sample size \( n \). Evaluation based on numerical information of the performance of ML estimations for the SILL model is performed. The mean square errors (A1), lower bound (A2), upper bound (A3), average length (A4) of confidence interval, and ML estimations are assessed. The simulation method is carried out with the help of the MATHEMATICA (9) package:

1. The SILL distribution is used to create a random sample \( X_1, X_2, \ldots, X_n \) of sizes \( n = 30, 50, \) and 100.
2. Nine sets of parameter values evaluated.
3. The SILL model’s ML estimate is assessed using parameter values and sample sizes.
4. Repeat this procedure 5000 times to obtain the means and A1 of the ML estimations for model parameter values. Table 1 summarizes empirical findings. Furthermore, the following observations have been made.
5. From Tables 1–9, we can note that when \( n \) increases the values of ML and A4 are decreased.
Table 1: MLE, M1, M2, M3, and M4 of the SILL model at $(\alpha = 0.3, b = 0.5, a = 0.5)$.  

| $n$ | ML  | A1   | A2 (%) | A3 (%) | A4 (%) | A2 (%) | A3 (%) | A4 (%) |
|-----|------|------|--------|--------|--------|--------|--------|--------|
| 30  | 0.32141 | 0.03838 | 0.04057 | 0.63825 | 0.63368 | −0.05610 | 0.69892 | 0.75503 |
| 50  | 0.29965 | 0.02133 | 0.03589 | 0.56341 | 0.52753 | −0.00146 | 0.61392 | 0.62854 |
| 100 | 0.33423 | 0.01181 | 0.17221 | 0.49624 | 0.39829 | 0.14118 | 0.52727 | 0.38609 |

Table 2: MLE, M1, M2, M3, and M4 of the SILL model at $(\alpha = 0.5, b = 0.5, a = 0.5)$.  

| $n$ | ML  | A1   | A2 (%) | A3 (%) | A4 (%) | A2 (%) | A3 (%) | A4 (%) |
|-----|------|------|--------|--------|--------|--------|--------|--------|
| 30  | 0.46501 | 0.02528 | 0.05660 | 0.87442 | 0.81882 | −0.02280 | 0.95282 | 0.97562 |
| 50  | 0.57574 | 0.09949 | 0.28438 | 0.86710 | 0.58272 | 0.22859 | 0.92289 | 0.69430 |
| 100 | 0.51253 | 0.01609 | 0.31406 | 0.71100 | 0.39694 | 0.27606 | 0.74901 | 0.47295 |

Table 3: MLE, M1, M2, M3, and M4 of the SILL model at $(\alpha = 0.7, b = 0.5, a = 0.5)$.  

| $n$ | ML  | A1   | A2 (%) | A3 (%) | A4 (%) | A2 (%) | A3 (%) | A4 (%) |
|-----|------|------|--------|--------|--------|--------|--------|--------|
| 30  | 0.66814 | 0.05167 | 0.14173 | 1.19455 | 1.05282 | 0.04093 | 1.29535 | 1.25443 |
| 50  | 0.67232 | 0.01675 | 0.30203 | 1.04260 | 0.74057 | 0.23113 | 1.11350 | 0.82383 |
| 100 | 0.62888 | 0.00828 | 0.14069 | 0.50606 | 0.39829 | 0.27606 | 0.74901 | 0.47295 |

Table 4: MLE, M1, M2, M3, and M4 of the SILL model at $(\alpha = 0.3, b = 0.3, a = 0.5)$.  

| $n$ | ML  | A1   | A2 (%) | A3 (%) | A4 (%) | A2 (%) | A3 (%) | A4 (%) |
|-----|------|------|--------|--------|--------|--------|--------|--------|
| 30  | 0.49196 | 0.02335 | 0.11903 | 0.86488 | 0.74585 | 0.04762 | 0.93629 | 0.88867 |
| 50  | 0.56873 | 0.02194 | 0.28892 | 0.84853 | 0.55961 | 0.23534 | 0.90211 | 0.66677 |
| 100 | 0.48152 | 0.00491 | 0.31466 | 0.64838 | 0.33372 | 0.28271 | 0.68033 | 0.39762 |

| $n$ | ML  | A1   | A2 (%) | A3 (%) | A4 (%) | A2 (%) | A3 (%) | A4 (%) |
|-----|------|------|--------|--------|--------|--------|--------|--------|
| 30  | 0.34941 | 0.02958 | 0.03205 | 0.66677 | 0.63472 | −0.02872 | 0.72754 | 0.75626 |
| 50  | 0.50774 | 0.00507 | 0.37343 | 0.64205 | 0.26863 | 0.37471 | 0.66777 | 0.32007 |
| 100 | 0.48448 | 0.00065 | 0.42865 | 0.54031 | 0.11167 | 0.41795 | 0.55100 | 0.13305 |

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Table 5: MLE, M1, M2, M3, and M4 of the SILL model at \((\alpha = 0.5, b = 0.3, a = 0.5)\).

| n  | ML   | A1     | A2     | A3     | A4     | A2   | A3     | A4     |
|----|------|--------|--------|--------|--------|------|--------|--------|
| 30 | 0.26173 | 0.03201 | -8.33302 | 8.85648 | 17.18950 | -9.97883 | 10.50230 | 20.48110 |
| 50 | 0.25386 | 0.02093 | -3.18902 | 2.90874 | 5.29776 | -2.89625 | 3.41597 | 6.31223 |
| 100| 0.32841 | 0.01036 | 0.12947 | 0.40816 | 0.27869 | 0.14579 | 0.33206 | 0.351190 |

Table 6: MLE, M1, M2, M3, and M4 of the SILL model at \((\alpha = 0.8, b = 0.8, a = 0.8)\).

| n  | ML   | A1     | A2     | A3     | A4     | A2   | A3     | A4     |
|----|------|--------|--------|--------|--------|------|--------|--------|
| 30 | 0.82318 | 0.10258 | 0.25466 | 1.39170 | 1.13705 | 0.14579 | 1.50057 | 1.35478 |
| 50 | 0.68312 | 0.06063 | 1.30560 | 1.24497 | 0.05857 | 1.42480 | 1.48337 | 1.01214 |
| 100| 0.75937 | 0.00820 | 0.41332 | 0.55356 | 0.39989 | 0.56699 | 0.16709 | 0.80944 |

Table 7: MLE, M1, M2, M3, and M4 of the SILL model at \((\alpha = 0.5, b = 0.8, a = 0.8)\).

| n  | ML   | A1     | A2     | A3     | A4     | A2   | A3     | A4     |
|----|------|--------|--------|--------|--------|------|--------|--------|
| 30 | 0.53499 | 0.06024 | -0.19520 | 1.26518 | 1.46039 | -0.33503 | 1.40501 | 1.74033 |
| 50 | 0.42841 | 0.00797 | 0.01249 | 0.84432 | 0.83183 | -0.06715 | 0.93297 | 0.99111 |
| 100| 0.75937 | 0.00820 | 0.63763 | 0.88111 | 0.24347 | 0.61432 | 0.29010 | 0.80944 |

Table 8: MLE, M1, M2, M3, and M4 of the SILL model at \((\alpha = 0.5, b = 1.2, a = 0.5)\).

| n  | ML   | A1     | A2     | A3     | A4     | A2   | A3     | A4     |
|----|------|--------|--------|--------|--------|------|--------|--------|
| 30 | 0.62992 | 0.12515 | 0.04928 | 1.21057 | 1.16129 | -0.01691 | 1.32176 | 1.38367 |
| 50 | 0.43954 | 0.07393 | 0.07381 | 0.80527 | 0.73146 | 0.00378 | 0.87530 | 0.87152 |
| 100| 0.59828 | 0.04365 | 0.31395 | 0.88262 | 0.56867 | 0.25950 | 0.93706 | 0.67757 |

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Table 9: MLE, M1, M2, M3, and M4 of the SILL model at \((\alpha = 0.5, a = 1.5, b = 0.5)\).

| n  | MLE | A1 | A2 | A3 | A4 | A2 | A3 | A4 |
|----|-----|----|----|----|----|----|----|----|
| 30 | 0.67610 | 0.31649 | 0.06943 | 1.28277 | 1.21334 | −0.04674 | 1.39894 | 1.44568 |
| 50 | 1.40532 | 0.54789 | 0.54646 | 2.26417 | 1.71771 | 0.38200 | 2.42863 | 2.04663 |
| 100 | 0.55815 | 0.02585 | 0.38167 | 0.73463 | 0.35296 | 0.34787 | 0.76842 | 0.42055 |
| 30 | 0.48180 | 0.03807 | 0.08026 | 0.88333 | 0.80307 | 0.00337 | 0.96022 | 0.95685 |
| 50 | 1.46630 | 0.07518 | 0.86646 | 2.06614 | 1.19968 | 0.75160 | 2.18100 | 1.42940 |
| 100 | 0.51568 | 0.00314 | 0.39752 | 0.63384 | 0.23632 | 0.37489 | 0.65646 | 0.28157 |

Table 10: Estimates and SErs for the first data.

| Fitted models | Estimates and SErs |
|---------------|-------------------|
| SILL \((a, a, b)\) | 3.132 (0.489) 797.744 (2694) 416.678 (1404) |
| ExL \((a, a, b)\) | 3.630 (0.624) 20074.510 (2041.826) 26257.680 (99.742) |
| TrMOF \((a, \alpha, a, b)\) | 200.747 (87.275) 1.952 (0.125) 0.102 (0.017) |
| TrLL \((a, a, b)\) | 2.060 (0.280) 3.103 (0.341) |
| IL \((a, b)\) | 4.002 (1.290) 1.981 (0.750) |
| MW \((c, a, \beta, a)\) | 16.721 (9.622) 33.640 (19.994) |
| LL \((a, b)\) | 2.391 (0.140) 3.224 (0.297) |
| BeW \((\beta, a, b)\) | 0.298 (0.060) 1.360 (1.002) |
| KuW \((a, b, \alpha)\) | 34.660 (17.530) 81.850 (52.014) |

Table 11: Estimates and SErs for the second data.

| Fitted models | MLEs and SErs |
|---------------|---------------|
| SILL \((a, a, b)\) | 1.517 (0.279) 15.419 (9.989) 1.933 (0.743) |
| TIITLPoL \((\theta, a, a, b)\) | 8.430 (42.442) 0.785 (3.350) 0.984 (0.128) 8.557 (15.373) |
| GeTrW \((\lambda, a, \beta, a)\) | 0.002 (1.770) 0.013 (7.214) 2.800 (11.177) 0.654 (0.121) 0.299 (0.151) |
| TrMW \((a, a, \lambda, \beta)\) | 0.1208 (0.024) 0.0002 (0.011) 0.407 (0.410) 0.8955 (0.626) |
| KuExBXII \((b, c, a, k, \beta)\) | 67.636 (104.728) 0.340 (0.390) 2.780 (44.510) 0.840 (1.723) 3.083 (49.353) |
| BeExBXII \((b, a, k, \beta, c)\) | 20.278 (17.300) 22.190 (21.960) 1.310 (1.080) 1.780 (1.076) 0.224 (0.144) |
| ExTrGeR \((a, \beta, \lambda, \delta)\) | 7.376 (5.390) 0.050 (0.004) 0.120 (0.260) 0.050 (0.036) |
| BeF \((\beta, a, b)\) | 0.169 (0.104) 27.753 (71.510) 33.342 (36.350) |
| TrCoWG \((a, b, \alpha)\) | 106.070 (124.800) 1.712 (0.100) 0.217 (0.610) 0.010 (0.007) |

Table 12: flZ_he E1, E2, E3, E4, and E5 statistics for the first data.

| Fitted models | E1 | E2 | E3 | E4 | E5 |
|---------------|----|----|----|----|----|
| SILL | 277.235 | 277.535 | 280.166 | 1.128 | 0.125 |
| ExL | 288.800 | 289.100 | 291.328 | 1.744 | 0.220 |
| TrMOF | 309.472 | 309.980 | 319.200 | 2.404 | 0.320 |
| L | 333.977 | 334.123 | 335.663 | 1.398 | 0.167 |
| TrLL | 284.772 | 285.0720 | 287.703 | 1.492 | 0.184 |
| IL | 360.950 | 361.100 | 362.634 | 4.384 | 0.538 |
| MW | 283.900 | 284.670 | 296.053 | 1.591 | 0.199 |
| LL | 283.163 | 283.311 | 285.117 | 1.520 | 0.187 |
| BeW | 305.028 | 305.534 | 314.751 | 3.220 | 0.465 |
| KuW | 281.434 | 281.941 | 291.160 | 1.506 | 0.185 |
6. Real Data Analysis

SILL distribution as an example for the family is exemplified in this section based on two datasets. A number of authors have used this data to illustrate the usefulness of competing models. Also, the models’ fit is evaluated formatively and compared to that of other distributions. The Akaike IC (E1), Consistent Akaike IC (E2), Hannan–Quinn IC (E3), Anderson–Darling (E4), and Cramér-von Mises IC (E5) are used to evaluate the fitted models. This is because, in general, smaller values of these statistics indicate a better fit to the data.

Tahir et al. [16] investigated the first dataset, which represented the 84 failure times for a certain windshield device. For these data, we will compare the SILL distribution’s fits to the exponentiated Lomax (ExL) [17], Lomax (L), inverse Lomax (IL), Ku Weibull (KuW) [18], McDonald Weibull (MW) [19], beta Weibull (BeW) [20], Tr Marshall–Olkin F (TrMOF) [21], Tr log-logistic (TrLL) [13], and log-logistic (LL) models.

The second dataset (in months): a random sample of 128 bladder cancer patients was investigated by [22]. The SILL is compared to other distributions such as the type-II TL power Lomax (THTLPoL) [23], Tr complementary Weibull geometric (TrCoWG) [24], Ku exponentiated Burr XII (KuExBXII) [25], beta exponentiated Burr XII (BeExBXII) [26], beta F (BeF) [27], exponentiated Tr generalized Rayleigh (ExTrGeR) [28], Tr modified Weibull (TrMW) [29], and generalized Tr Weibull (GeTrW) [30] models.

Tables 10 and 11 show estimated parameters and associated standard errors (SErs). Tables 12 and 13 provide the statistics for the fitted models. E1, E2, E3, E4, and E5 values for SILL are the lowest among the competitor models, as shown in Tables 12 and 13. Both datasets are best fit by SILL distribution. See Figures 2 and 3, for further information on this topic.
7. Concluding Remarks

In this manuscript, we propose the SIL-G family, a novel one parameter produced family of distributions. There are four new unique models specified. We investigate many essential characteristics of the SIL-G family, such as PDF, expansions, explicit formulations for moments, generating function, and quantile. We estimate the parameters using ML estimation techniques. To examine the limited sample behavior of the ML estimates, we undertake a Monte Carlo simulation analysis on one SILL model as an example of the new family. The new family’s relevance is demonstrated through applications to two real-world datasets.

In our future works, we plan to study some new generating families of distributions depending on trigonometric functions as sine function. Also, some new models depending on the sine function and statistical inference by using different methods of parameter and various algorithms and comparing each other will be studied. Applications for many fields as physics, economics, engineering, and medicine will be studied.

Data Availability

The data used to support the findings of the study can be obtained from corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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